L\textsuperscript{p}-VALUED STOCHASTIC CONVOLUTION INTEGRAL DRIVEN BY VOLTERRA NOISE

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Abstract. Space-time regularity of linear stochastic partial differential equations is studied. The solution is defined in the mild sense in the state space \( L^p \). The corresponding regularity is obtained by showing that the stochastic convolution integrals are Hölder continuous in a suitable function space. In particular cases, this allows to show space-time Hölder continuity of the solution. The main tool used is a hypercontractivity result on Banach-space valued random variables in a finite Wiener chaos.

1. Introduction

The paper is devoted to the study of mild solutions to linear stochastic differential equations in the Lebesgue \( L^p(D, \mu) \) space perturbed by additive noise of Volterra type. Sufficient conditions for the existence and Hölder continuity of the solutions are given. This allows to show that solutions to particular SPDEs are Hölder continuous random fields.

More precisely, we consider the stochastic evolution equation which takes the form
\[
dX_t = AX_t + \Phi dB_t, \quad X_0 = x, \tag{1.1}
\]
where \( B \) is an infinite-dimensional \( \alpha \)-regular Volterra process which belongs to a finite Wiener chaos and \( A \) is a generator of a strongly continuous, analytic semigroup \( (S(t), t \geq 0) \) of operators acting on the space \( L^p(D, \mu) \) with \( 1 < \frac{2}{1+2\alpha} \leq p < \infty \). The mild solution is given by the stochastic convolution integral
\[
X_t = S(t)x + \int_0^t S(t-r)\Phi dB_r, \quad t \geq 0
\]
and we give sufficient conditions for its existence and Hölder continuity in the domain of a fractional power of \( A \). Canonical examples of SPDEs to which our theory may be applied are the heat equation on bounded domain \( \mathcal{O} \subset \mathbb{R}^d \) with pointwise noise, formally given by
\[
\partial_t u(t, \xi) = \Delta u(t, \xi) + \eta(t)\delta_\xi(\xi) \quad \text{on} \quad \mathbb{R}_+ \times \mathcal{O},
\]
where \( \delta_\xi \) is the Dirac distribution; or the heat equation with distributed noise
\[
\partial_t u(t, \xi) = \Delta u(t, \xi) + \eta(t, \xi) \quad \text{on} \quad \mathbb{R}_+ \times \mathcal{O}.
\]

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In both these examples, our results allow to make use of the embedding of Sobolev-Slobodeckii spaces into the spaces of Hölder continuous functions and hence, by taking sufficiently large $p$, to obtain space-time Hölder continuity for $d = 1, 2, 3$.

The scalar Volterra process is a continuous stochastic process which might not be Markov, Gaussian or a semimartingale but which admits a certain covariance structure instead. In particular, there is a kernel $K$ which satisfies suitable regularity conditions (see section 2.1) such that the covariance function can be written as

$$R(s, t) = \int_0^{s \wedge t} K(s, r)K(t, r)\,dr, \quad s, t \geq 0.$$  

The most notable examples which satisfy the definition are the fractional Brownian motion (fBm) with the Hurst parameter $H > 1/2$, which is Gaussian and lives in the first Wiener chaos (see e.g. [2, 10, 17] for its definition and further properties), and the Rosenblatt process, which is non-Gaussian and lives in the second Wiener chaos (see e.g. [23, 24]). Note, however, that the class of $\alpha$-regular Volterra processes in a finite Wiener chaos is not restricted only to these two processes (see [6] for other examples). See also [1, 3, 13, 14, 15].

Stochastic convolution integral with respect to $\alpha$-regular Volterra processes has already been considered in [6] in the Hilbert space setting. In particular, it has been shown that the integral admits a version with Hölder continuous sample paths of very small order which can be improved by $1/2$ if the driving process is Gaussian. In the present paper, we further develop this idea by assuming that the Volterra process lives in a finite Wiener chaos which allows us to prove hypercontractivity of the $L^p$-valued stochastic integrals. This in turn yields the same regularity in the non-Gaussian case as in the Gaussian case. Other works on evolution equations driven by Volterra processes are [7, 12].

The paper is organized as follows.

In Section 2, we collect the tools needed in the following sections. In particular, definition of an $\alpha$-regular Volterra process is given and one-dimensional stochastic integration of deterministic real-valued functions is recalled. This part closely follows the papers [1, 6]. We then proceed to the definition of the $n$-th Wiener chaos and give a hypercontractivity result which states that Banach-space valued linear combinations of elements in $n$-th Wiener chaos have equivalent moments (Proposition 2.3). Then definition of $\gamma$-radonifying operators follows together with their basic properties. Finally, we collect the main assumptions used in the paper.

Section 3 is devoted to stochastic integration of operator-valued functions with respect to cylindrical Volterra process in the space $L^p(D, \mu)$. We give characterization of admissible integrands (Proposition 3.2) and identify sufficient conditions for integrability on the scale of Lebesgue spaces (Corollary 3.4).

Section 4 contains the main results of the paper. In particular, sample path measurability of the mild solution to (1.1) is shown under certain natural conditions on the semigroup $S$ (Proposition 4.1) and then, factorization method is used to prove Hölder continuity of the solution under slightly stronger conditions (Proposition 4.2).

The paper is concluded with two examples contained in Section 5 – the stochastic heat equation with pointwise Volterra noise and the stochastic parabolic equation of the $2m$-th order with distributed noise which is Volterra in time and can be both white or correlated in space.
2. Preliminaries

This section collects the main tools needed in the next sections. Throughout the paper, $A \lesssim B$ means that there exists a positive constant $C$ such that $A \leq CB$. Similarly, the symbol $A \sim B$ means that there exist positive constants $C_1, C_2$ such that $C_1B \leq A \leq C_2B$.

2.1. Volterra processes. Consider a measurable function $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ which is

- Volterra, i.e.
  
  (i) $K(0,0) = 0$ and $K(t,r) = 0$ on $\{0 \leq t < r < \infty\}$,
  
  (ii) $\lim_{t \to r^+} K(t,r) = 0$ for all $r \geq 0$,

- square integrable and $\alpha$-regular, i.e.
  
  (iii) $K(t,\cdot) \in L^2(0,t)$ for all $t \geq 0$,
  
  (iv) $K(\cdot,r) \in C^1(r,T)$ for all $r \geq 0, T > 0$. Furthermore, there is an $\alpha \in (0, \frac{1}{2})$ such that

$$\left| \frac{\partial K}{\partial u}(u,r) \right| \lesssim (u-r)^{\alpha-\frac{1}{2}} \left( \frac{u}{r} \right)^{\alpha}$$

on $\{0 < r < u \leq T\}$.

Such a function $K$ is called an $\alpha$-regular Volterra kernel in the sequel.

**Definition 2.1.** We say that a centered, continuous stochastic process $b = (b_t, t \geq 0)$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is an $\alpha$-regular Volterra process if $b_0 = 0$ and its covariance function takes the form

$$\mathbb{E}(b_s b_t) = \int_0^{\wedge \wedge} K(s,r)K(t,r)dr, \quad s, t \geq 0,$$

for some $\alpha$-regular Volterra kernel $K$.

**Remark 2.2.** The fractional Brownian motion (fBm) with the Hurst parameter $H \in (1/2, 1)$ is an $\alpha$-regular Volterra process with $\alpha = H - \frac{1}{2}$. The fBm is defined as the centered Gaussian process with continuous sample paths whose covariance function is

$$R^H(s,t) := \frac{1}{2} \left( |s|^{2H} + |t|^{2H} - |t-s|^{2H} \right), \quad s, t \geq 0.$$

$R^H(s,t)$ can indeed be written as (2.1) with

$$K^H(t,r) = C_H \int_r^t \left( \frac{u}{r} \right)^{H-\frac{1}{2}} (u-r)^{H-\frac{3}{2}} du,$$

(2.2)

where $C_H$ is a suitable constant (see [2]). Although the fBm is Gaussian, in the present paper, Gaussianity is not assumed. The Rosenblatt process $Z$ is an example of a non-Gaussian Volterra process. In particular, $Z$ is defined as

$$Z_t := C \int_\mathbb{R} \int_\mathbb{R} \left( \int_0^t (u-y_1)_+^{\frac{2-H'}{2}} (u-y)_+^{\frac{2-H'}{2}} du \right) dW_{y_1} dW_y, \quad t \geq 0,$$

where $C$ is a normalizing constant such that $\mathbb{E}Z_t^2 = 1, H' \in (1/2, 1)$ and $W$ is the two-sided standard Wiener process. Then $Z$ satisfies Definition 2.1 with the kernel $K^{H'}$ which takes
the form (2.2) and hence, $Z$ is an $\alpha$-regular Volterra process with $\alpha = H' - \frac{1}{2}$ (see [23] and [24]). Other examples of $\alpha$-regular Volterra processes include the multifractional Liouville Brownian motion (mLFBm) where the Hurst parameter $H = H(t)$ may be a function of $t$. Its definition and the assumptions on $H(\cdot)$ which ensure that the mLFBm is an $\alpha$-regular Volterra process are given in [6, Example 2.14].

2.2. Stochastic integral. Since Volterra processes are not necessarily semimartingales, the standard Itô approach to a stochastic integral is not applicable. The (already rather standard) definition of Wiener-type integrals (i.e. for deterministic integrands) driven by Volterra processes is given below (cf. [1] and [4]).

Let $T > 0$ and consider the linear space of ($\mathbb{R}$-valued) deterministic step functions $\mathcal{E}$, i.e.

$$
\mathcal{E} := \left\{ \varphi : [0,T] \to \mathbb{R}, \varphi = \sum_{i=1}^{n-1} \varphi_i 1_{[t_i, t_{i+1})} + \varphi_n 1_{[t_n, T]} \right\},
$$

$$
\varphi_i \in \mathbb{R}, i \in \{1, \cdots, n\}, 0 = t_1 < t_2 < \cdots < t_{n+1} = T, n \in \mathbb{N}.
$$

Define an operator $\mathcal{K}_T^* : \mathcal{E} \to L^2(0, T)$ by

$$
(\mathcal{K}_T^* \varphi)(r) := \int_0^T \varphi(u) \frac{\partial K}{\partial u}(u, r) du, \quad \varphi \in \mathcal{E}. \tag{2.3}
$$

Let $b = (b_t, t \geq 0)$ be a Volterra process with the kernel $K$. Consider the linear operator

$$
i_T : \mathcal{E} \to L^2(\Omega) \quad \text{given by}
$$

$$
\varphi := \sum_{i=1}^{n-1} \varphi_i 1_{[t_i, t_{i+1})} + \varphi_n 1_{[t_n, T]} \quad \mapsto \quad \sum_{i=1}^{n-1} \varphi_i (b_{t_{i+1}} - b_{t_i}) =: i_T(\varphi).
$$

Using (2.1) and (2.3), it can be shown that

$$
\|i_T(\varphi)\|_{L^2(\Omega)} = \|\mathcal{K}_T^* \varphi\|_{L^2(0, T)} \tag{2.4}
$$

which is an Itô-type isometry for Volterra processes. For $f, g \in \mathcal{E}$, set

$$
\langle f, g \rangle_{\mathcal{E}} := \langle \mathcal{K}_T^* f, \mathcal{K}_T^* g \rangle_{L^2(0, T)}.
$$

Without loss of generality, we assume that $\mathcal{K}_T^*$ is injective and thus, the function $\langle \cdot, \cdot \rangle_{\mathcal{E}}$ defines an inner product on $\mathcal{E}$. If this is not the case, we consider the quotient space $\tilde{\mathcal{E}} := \mathcal{E} / \ker \mathcal{K}_T^*$ and we lift $\mathcal{K}_T^*$ to $\tilde{\mathcal{E}}$. Completing $\mathcal{E}$ under $\langle \cdot, \cdot \rangle_{\mathcal{E}}$ yields the Hilbert space $\mathcal{D}$ and extends $\mathcal{K}_T^*$ to $\mathcal{D}$. This, in turn, extends $i_T$ to an operator from $\mathcal{D}$ to $L^2(\Omega)$ by (2.4). $\mathcal{D}$ is the space of admissible integrands with respect to $b$ and $i_T : \mathcal{D} \to L^2(\Omega)$ is the Wiener-type integral. The usual notation is $i_T(\varphi) = \int_0^T \varphi db$. The space $\mathcal{D}$ can be very large and thus, it is important to identify certain function spaces which can be embedded into $\mathcal{D}$. By [6, Proposition 2.9] and its proof, we have that

$$
\|i_T(\varphi)\|^2_{L^2(\Omega)} \lesssim \int_0^T \int_0^T \varphi(u)\varphi(v)|u - v|^{2\alpha-1} du dv \tag{2.5}
$$

from which it follows that

$$
L^{2\alpha} \to \mathcal{D}.
$$
2.3. Wiener chaos. Let further $H$ be a real separable Hilbert space and let $W$ be an $H$-isomormal Gaussian process, i.e. $W = (W(h), h \in H)$ is a centred Gaussian family such that

$$
\mathbb{E}W(h_1)W(h_2) := \langle h_1, h_2 \rangle_H , \quad h_1, h_2 \in H.
$$

Denote by $H_n$ the $n$-th Hermite polynomial, i.e.

$$
H_n(x) := \frac{(-1)^n}{n!} e^{x^2} \frac{d^n}{dx^n} \left( e^{-x^2} \right).
$$

The $n$-th Wiener chaos (of $W$), $\mathcal{H}_n$, is the closed linear subspace of $L^2(\Omega)$ generated by the linear span $\{H_n(W(h)), h \in H, \|h\|_H = 1\}$. In particular, the space $\mathcal{H}_0$ consists of constant random variables and $\mathcal{H}_1$ contains zero-mean Gaussian random variables which can be interpreted as stochastic integrals (w.r. to $W$). For further references see e.g. [21]. We shall use the following feature of the spaces $\mathcal{H}_n$:

**Proposition 2.3.** Let $p, q \in [1, \infty)$ and $n \geq 0$. Then there exists a number $C_{p,q,n}$ such that

$$
\left\| \sum_{j=1}^{m} \xi_j x_j \right\|_{L^q(\Omega; X)} \leq C_{p,q,n} \left\| \sum_{j=1}^{m} \xi_j x_j \right\|_{L^p(\Omega; X)}
$$

holds for every Banach space $X$, $m \in \mathbb{N}$ and every $\{\xi_j\}_{j \leq m} \subseteq \mathcal{H}_n$ and $\{x_j\}_{j \leq m} \subseteq X$.

**Proof.** One can prove the inequality (2.6) either by a decoupling argument and the Kahane-Khintchine inequality as in [16] Proposition 3.1 or by the scalar Neveu inequality [19] applied to the vector-valued Mehler’s formula for the Ornstein-Uhlenbeck semigroup, see [21] Theorem 1.4.1 and the remark on page 62 in [21]. The latter approach yields the inequality (2.6) only for $p, q \in (1, \infty)$ which then extends to the general range by the Hölder inequality, see the remark following [9] Theorem 3.2.2, p. 113-114. \qed

2.4. $\gamma$-Radonifying operators. Let $U$ be a (real separable) Hilbert space and $E$ be a (real separable) Banach space. A bounded linear operator $R \in \mathcal{L}(U, E)$ is $\gamma$-radonifying provided that there exists a centered Gaussian probability measure $\nu_R$ on $E$ such that

$$
\int_E \varphi^2(x) \nu_R(dx) = \|R^* \varphi\|^2_U, \quad \varphi \in E^*.
$$

Such a measure is at most one, therefore we set

$$
\|R\|_{\gamma(U,E)}^2 := \int_E \|x\|_E^2 \nu_R(dx),
$$

and denote by $\gamma(U,E)$ the space of $\gamma$-radonifying operators. It is well-known that $\gamma(U,E)$ equipped with the norm $\|\cdot\|_{\gamma(U,E)}$ is a separable Banach space (see [18] or [22]).

**Proposition 2.4.** Let $R \in \mathcal{L}(U, E), \{e_n\}$ be an orthonormal basis of $U$ and $\{\delta_n\}$ a sequence of independent standard centered Gaussian random variables. Denote

$$
S_n := \sum_{k=1}^{n} \delta_k R e_k.
$$

The following claims are equivalent:

- $R$ is $\gamma$-radonifying,
• the sequence \( \{S_n\} \) is convergent almost surely,
• the sequence \( \{S_n\} \) is convergent in probability,
• the sequence \( \{S_n\} \) is convergent in every \( L^q(\Omega; E) \), \( 1 \leq q < \infty \).

Proof. This is a consequence of Theorem 2.3, Chapter V.2.4 and Theorem 5.3 in Chapter V.5.3 in [25]. Alternatively, the proof may be inferred from the Itô-Nisio theorem and the Fernique theorem. \( \Box \)

Remark 2.5. Separability of a measure space \((D, \mu)\) means that there exists a countable system \( \{V_n\} \) of measurable sets satisfying \( \mu(V_n) < \infty \) such that, for every \( \varepsilon > 0 \) and every measurable set \( C \) satisfying \( \mu(C) < \infty \), there is \( n \in \mathbb{N} \) such that \( \mu([C \setminus V_n] \cup (V_n \setminus C)) < \varepsilon \).

The following conditions are equivalent:
• the measure space \((D, \mu)\) is separable,
• there exists \( 1 \leq p < \infty \) such that \( L^p(D, \mu) \) is a separable Banach space,
• \( L^p(D, \mu) \) is a separable Banach space for every \( 1 \leq p < \infty \).

The symbol \( L^p \) is used to denote the space \( L^p(D, \mu) \) in the rest of the paper.

Proposition 2.6 ([5, Theorem 2.3]). Let \( 1 \leq p < \infty \), and let \( R \in \mathcal{L}(U, L^p) \). Then \( R \in \gamma(U, L^p) \) if and only if there exists a measurable function \( r: D \to U \) satisfying
\[
\int_D \|r(x)\|_U^p \mu(dx) < \infty
\]
and \( (Ru)(x) = \langle r(x), u \rangle_U \) holds \( \mu \)-almost everywhere for every \( u \in U \). There also exists a constant \( c > 1 \) independent of \( R \) such that
\[
\frac{1}{c} \|R\|_{\gamma(U, L^p)} \leq \left[ \int_D \|r(x)\|_U^p \mu(dx) \right]^\frac{1}{p} \leq c \|R\|_{\gamma(U, L^p)}.
\]

2.5. Hypotheses and further notation. In the rest of the paper, we assume:
• \((D, \mu)\) is a separable \( \sigma \)-finite measure space (see Remark 2.5).
• \( U \) is a real, separable Hilbert space and \( B = (B_t, t \geq 0) \) is an \( \alpha \)-regular \( U \)-cylindrical Volterra process with \( \alpha \in (0, \frac{1}{2}) \) defined on \((\Omega, \mathcal{F}, \mathbb{P})\), i.e. for an orthonormal basis \( \{e_n\} \) of \( U \) there exists a sequence of \( \alpha \)-regular Volterra processes \( \{b^{(n)}\} \) such that:
(i) It holds that
\[
\mathbb{E} b^{(n)}_s b^{(m)}_t = R(s, t) \delta_{m,n} \tag{2.7}
\]
where \( R(s, t) \) is given by (2.1) and \( \delta_{m,n} \) is the Kronecker delta.
(ii) There exists \( m \in \mathbb{N}_0 \) such that for every \( n \),
\[
b^{(n)}_t \in \bigoplus_{i=0}^m \mathcal{H}_i =: \mathcal{H}, \quad t \geq 0.
\]
(iii) \( B_t \) satisfies
\[
B_t = \sum_n e_n b^{(n)}_t. \tag{2.8}
\]
Remark 2.7. As in the classical case of the standard cylindrical Brownian motion, the series (2.8) does not converge in $L^2(\Omega; U)$, however, the integral with respect to $B$ introduced below is well-defined as an $L^p$-valued random variable.

3. Stochastic integration with respect to Volterra processes in $L^p$

In this section, a stochastic integral $I_T(G)$ with respect to the cylindrical Volterra process $B$ is defined and characterization of integrable operators $G$ is given.

**Definition 3.1.** Let $T > 0$. An operator $G \in \mathcal{L}(U, L^p(D; \mathcal{D}))$ is called elementary if

$$[Gu](x)(t) = \sum_{k=1}^{m} g_k(t) \langle u, e_k \rangle_U f_k(x)$$

holds for every $u \in U$, every $t \in [0, T]$ and $\mu$-almost every $x \in D$ with some $\{g_k\} \subset C^1(0, T)$ and $\{f_k\} \subset L^p$.

For $G$ elementary, let $I_T$ be the linear operator given by

$$I_T(G) := \sum_{k=1}^{m} \left( \int_{0}^{T} g_k(r) dB_r^{(k)} \right) f_k.$$

As usual, we have to extend the operator $I_T$ to a large space of integrable functions. The next proposition shows that the natural space of integrands is $\gamma(U, L^p(D; \mathcal{D}))$.

**Proposition 3.2.** Let $1 \leq p < \infty$. A bounded linear operator $G : U \to L^p(D; \mathcal{D})$ is stochastically integrable if and only if $G \in \gamma(U, L^p(D; \mathcal{D}))$. In this case,

$$\|I_T(G)\|_{L^q(\Omega; L^p)} \simeq \|G\|_{\gamma(U, L^p(D; \mathcal{D}))}$$

holds for every $1 \leq q < \infty$.

**Proof.** Let $G \in \mathcal{L}(U, L^p(D; \mathcal{D}))$ be elementary. Let $\{\delta_k\}$ be independent standard centered Gaussian random variables. Using successively Proposition 2.4, Proposition 2.6, the definition of the $L^p(D; \mathcal{D})$, Proposition 2.3 (real centered Gaussian random variables belong to $\mathcal{H}_1$) and the independence of $\delta_k$ and $\delta_l$ for $k \neq l$, we obtain that $G$ is $\gamma$-radonifying and

$$\|G\|_{\gamma(U, L^p(D; \mathcal{D}))}^p \simeq \mathbb{E} \left\| \sum_{k=1}^{m} \delta_k g_k f_k \right\|_{L^p(D; \mathcal{D})}^p \simeq \int_D \mathbb{E} \left\| \sum_{k=1}^{m} \delta_k g_k f_k(x) \right\|_{\mathcal{D}}^p \mu(dx) \simeq \int_D \left( \mathbb{E} \left\| \sum_{k=1}^{m} \delta_k g_k f_k(x) \right\|_{\mathcal{D}}^2 \right)^{\frac{p}{2}} \mu(dx) \simeq \int_D \left( \sum_{k=1}^{m} \|g_k\|_{\mathcal{D}}^2 |f_k(x)|^2 \right)^{\frac{p}{2}} \mu(dx).$$
On the other hand, using the definition of $I_T(G)$ and the definition of the $L^p$ norm, we obtain
\[
\mathbb{E} \left\| I_T(G) \right\|^p_{L^p} = \mathbb{E} \left\| \sum_{k=1}^{m} \left( \int_{0}^{T} g_k(r) \text{d}b_r^{(k)} \right) f_k \right\|^p_{L^p} = \int_D \mathbb{E} \left\| \sum_{k=1}^{m} \left( \int_{0}^{T} g_k(r) \text{d}b_r^{(k)} \right) f_k(x) \right\|^p \mu(dx).
\]

Let us denote
\[
\varepsilon_k := \int_{0}^{T} g_k(r) \text{d}b_r^{(k)}, \quad I(x) := \mathbb{E} \left\| \sum_{k=1}^{m} \varepsilon_k f_k(x) \right\|^p.
\]

Since $b_t \in \mathcal{H}$ for all $t \geq 0$, we have that every one-dimensional elementary integral of the form $\sum_{i=1}^{n} \varphi_i(b_{t_{i+1}} - b_{t_i}) \in \mathcal{H}$. If $\varphi \in \mathcal{D}$ and $\{\varphi_k\}$ is a sequence of step functions such that $\varphi_k \to \varphi$ in $\mathcal{D}$, then $\mathcal{H} \ni i_T(\varphi_k) \to i_T(\varphi)$ in $L^2(\Omega)$ and hence, $i_T(\varphi) \in \mathcal{H}$. This means, that $\varepsilon_k \in \mathcal{H}$ for every $k = 1, \ldots, m$ and also $\sum_{k=1}^{m} \varepsilon_k f_k(x) \in \mathcal{H}$. Hence, we obtain
\[
I(x) = \mathbb{E} \left\| \sum_{k=1}^{m} \varepsilon_k f_k(x) \right\|^p \approx \left( \mathbb{E} \left\| \sum_{k=1}^{m} \varepsilon_k f_k(x) \right\|^2 \right)^{\frac{p}{2}} \mu(dx)
\]

by Proposition 2.3. Using the fact that $\varepsilon_k$ and $\varepsilon_l$ for $k \neq l$ are uncorrelated (see formula (2.7)), it follows that
\[
\mathbb{E} \left\| I_T(G) \right\|^p_{L^p} \approx \int_D \left( \sum_{k=1}^{m} \mathbb{E}(\varepsilon_k^2) |f_k(x)|^2 \right)^{\frac{p}{2}} \mu(dx) = \int_D \left( \sum_{k=1}^{m} \|g_k\|^2_{L^2} |f_k(x)|^2 \right)^{\frac{p}{2}} \mu(dx)
\]

by (2.4). Proposition 2.3 yields
\[
\| I_T(G) \|_{L^q(\Omega; L^p)} \approx \| I_T(G) \|_{L^p(\Omega; L^p)} \approx \| G \|_{\gamma(U, L^p(D; \mathcal{D}))}
\]

for any $1 \leq q < \infty$. Let now $G \in \gamma(U, L^p(D; \mathcal{D}))$ be arbitrary and let $\{G_k\}$ be a sequence of elementary operators such that $G_k \to G$ in $\gamma(U, L^p(D; \mathcal{D}))$. By (3.1), we have that
\[
\| I_T(G_k) - I_T(G_l) \|_{L^2(\Omega; L^p)} = \| I_T(G_k - G_l) \|_{L^2(\Omega; L^p)} \approx \| G_k - G_l \|_{\gamma(U, L^p(D; \mathcal{D}))}
\]

which tends to zero as $k, l \to \infty$. Hence, $\{I_T(G_k)\}$ is a Cauchy sequence in $L^2(\Omega; L^p)$ and since this is a Banach space, there must be a limit $Z$. The stochastic integral of $G$ is then defined as the map $I_T : G \mapsto Z$. Applying Proposition 2.3 again yields the claim. □

Remark 3.3. The stochastic integral of $G \in \gamma(U, L^p(D; \mathcal{D}))$ with respect to $B$ can be written as
\[
\int_{0}^{T} G(r) \text{d}B_r := I_T(G) = \sum_{n} \int_{0}^{T} G_{e_n} \text{d}b^{(n)}.
\]

Corollary 3.4. Let $\frac{2}{1+2\alpha} \leq p < \infty$. Then the space $L^{\frac{2}{1+2\alpha}}([0, T]; \gamma(U, L^p))$ is continuously embedded in $\gamma(U, L^p(D; \mathcal{D}))$ and
\[
\| I_T(G) \|_{L^q(\Omega; L^p)} \lesssim \| G \|_{L^{\frac{2}{1+2\alpha}}([0, T]; \gamma(U, L^p))}
\]

holds for every $1 \leq q < \infty$ and $G \in L^{\frac{2}{1+2\alpha}}([0, T]; \gamma(U, L^p))$. 

\[
E \left\| I_T(G) \right\|^p_{L^p} = E \left\| \sum_{k=1}^{m} \left( \int_{0}^{T} g_k(r) \text{d}b_r^{(k)} \right) f_k \right\|^p_{L^p} = \int_D E \left\| \sum_{k=1}^{m} \left( \int_{0}^{T} g_k(r) \text{d}b_r^{(k)} \right) f_k(x) \right\|^p \mu(dx).
\]
Proof. Let $G$ be elementary. From the proof of Proposition 3.2 we have that
\[
\|I_T(G)\|_{L^q(\Omega; L^p)} \approx \left( \int_D \left( \sum_{k=1}^m \| g_k \|_2^2 \| f_k(x) \|_2^2 \right)^q \mu(dx) \right)^{\frac{1}{q}}.
\]
By (2.5) and the Cauchy-Schwarz inequality it follows that
\[
\sum_{k=1}^m \| g_k \|_2^2 \| f_k(x) \|_2^2 \lesssim \int_0^T \int_0^T \| J(u, x) \|_U \| J(v, x) \|_U |u - v|^{2\alpha - 1} \mu(dx) \mu(dy)
\]
where
\[J(u, x) := \sum_{k=1}^m g_k(u) f_k(x) e_k.\]
Using (3.3), the Hardy-Littlewood inequality and the Minkowski inequality successively yields
\[
\|I_T(G)\|_{L^q(\Omega; L^p)} \lesssim \left( \int_D \left( \int_0^T \int_0^T \| J(u, x) \|_U \| J(v, x) \|_U |u - v|^{2\alpha - 1} \mu(dx) \mu(dy) \right)^{\frac{q}{2}} \mu(dx) \right)^{\frac{1}{q}}
\]
\[
\lesssim \left( \int_D \left( \int_0^T \| J(u, x) \|_U^{2\alpha} \mu(dx) \right)^{\frac{q}{2p}} \mu(dx) \right)^{\frac{1}{p}}
\]
\[
\lesssim \left( \int_0^T \left( \int_D \| J(u, x) \|_U^p \mu(dx) \right)^2 \mu(dx) \right)^{\frac{\alpha + \frac{1}{2}}{p}}
\]
\[
\lesssim \left( \int_0^T \| G(u) \|_U^{\frac{2\alpha}{2\alpha + 1}} \mu(dx) \right)^{\alpha + \frac{1}{2}}.
\]
The claim for $G \in L^{\frac{2}{1+2\alpha}}([0, T]; \gamma(U, L^p))$ by a standard approximation argument. $\square$

Remark 3.5. Let us mention that in the case of fBm with $H > 1/2$ or the Rosenblatt process, the condition $\frac{2}{1+2\alpha} \leq p < \infty$ reads as $1 \leq pH < \infty$.

4. Stochastic evolution equation in $L^p$

In the rest of the paper, we assume that $\frac{2}{1+2\alpha} \leq p < \infty$. Consider the following stochastic differential equation
\[
\begin{align*}
dX_t &= AX_t dt + \Phi dB_t, \quad t \geq 0, \\
X_0 &= x,
\end{align*}
\]
where $x \in L^p$, $A : \text{Dom}(A) \to L^p$, $\text{Dom}(A) \subset L^p$, is an infinitesimal generator of an analytic, strongly continuous semigroup of linear operators $(S(t), t \geq 0)$ acting on $L^p$, $\Phi \in \mathcal{L}(U, L^p)$ and $B = (B_t, t \geq 0)$ is a $U$-cylindrical Volterra process satisfying the hypotheses from subsection 2.5. The solution to (4.1) is considered in the mild form, i.e.
\[
X_t = S(t)x + \int_0^t S(t-r)\Phi dB_r, \quad t \geq 0.
\]
Proposition 4.1. Assume that $S(u) \Phi \in \gamma(U, L^p)$ for every $u > 0$ and that there is a $T_0 > 0$ such that

$$
\int_0^{T_0} \|S(r) \Phi\|_{\gamma(U, L^p)}^{\frac{2}{1+2\alpha}} dr < \infty.
$$

Then the solution $X$, given by (1.2), is well-defined, $L^p$-valued, and mean-square right continuous. In particular, $X$ admits a version with measurable sample paths.

Proof. Existence: By the same arguments as in the proof of Corollary 3.4 we need to show that the following is finite for every $t > 0$:

$$
\mathbb{E} \left\| \int_0^t S(t-r) \Phi dB_r \right\|_{L^p}^2 \lesssim \left( \int_0^t \|S(t-r) \Phi\|_{\gamma(U, L^p)}^{\frac{2}{1+2\alpha}} dr \right)^{1+2\alpha} = \left( \int_0^t \|S(r) \Phi\|_{\gamma(U, L^p)}^{\frac{2}{1+2\alpha}} dr \right)^{1+2\alpha}.
$$

First assume that $t \in [0, T_0]$. Then, we have that

$$
\int_0^t \|S(r) \Phi\|_{\gamma(U, L^p)}^{\frac{2}{1+2\alpha}} dr \leq \int_0^{T_0} \|S(r) \Phi\|_{\gamma(U, L^p)}^{\frac{2}{1+2\alpha}} dr < \infty.
$$

If $t \in (T_0, \infty)$, then we can write

$$
\int_0^t \|S(r) \Phi\|_{\gamma(U, L^p)}^{\frac{2}{1+2\alpha}} dr = \int_0^T \|S(r) \Phi\|_{\gamma(U, L^p)}^{\frac{2}{1+2\alpha}} dr + \int_T^t \|S(r) \Phi\|_{\gamma(U, L^p)}^{\frac{2}{1+2\alpha}} dr.
$$

The last integral is finite for every $t \in (T_0, \infty)$ by the following semigroup property: If $\omega > 0$ is such that $\|S(u)\|_{\mathcal{L}(L^p)} \lesssim e^{\omega u}$ for $u > 0$, then for $r > T_0$ we have that

$$
\|S(r) \Phi\|_{\gamma(U, L^p)} = \|S(r-T_0) S(T_0) \Phi\|_{\gamma(U, L^p)} \lesssim e^{\omega(r-T_0)} \|S(T_0) \Phi\|_{\gamma(U, L^p)}.
$$

Mean-square right continuity: Notice first that we can write

$$
\left\| \int_s^t S(t-r) \Phi dB_r - \int_s^s S(s-r) \Phi dB_r \right\|_{L^2(\Omega; L^p)}^2 = 
= \left\| \int_s^t S(t-r) \Phi dB_r + \int_0^s (S(t-s) - I) S(s-r) \Phi dB_r \right\|_{L^2(\Omega; L^p)}^2 
\leq \left( \mathbb{E} \left\| \int_s^t S(t-r) \Phi dB_r \right\|_{L^p}^2 \right)^{\frac{1}{2}} + \left( \mathbb{E} \left\| \int_0^s (S(t-s) - I) S(s-r) \Phi dB_r \right\|_{L^p}^2 \right)^{\frac{1}{2}}.
$$

for $0 < s < t$. For the first integral, write

$$
I_1(s, t) := \mathbb{E} \left\| \int_s^t S(t-r) \Phi dB_r \right\|_{L^p}^2 \lesssim \left( \int_s^t \|S(t-r) \Phi\|_{\gamma(U, L^p)}^{\frac{2}{1+2\alpha}} dr \right)^{1+2\alpha} = \left( \int_0^{t-s} \|S(r) \Phi\|_{\gamma(U, L^p)}^{\frac{2}{1+2\alpha}} dr \right)^{1+2\alpha} = \left( \int_0^{T_0} 1_{(0,t-s]}(r) \|S(r) \Phi\|_{\gamma(U, L^p)}^{\frac{2}{1+2\alpha}} dr \right)^{1+2\alpha}
$$

for $t - s < T_0$. For every $r \in [0, T_0]$ we have that

$$
1_{(0,t-s]}(r) \|S(r) \Phi\|_{\gamma(U, L^p)}^{\frac{2}{1+2\alpha}} \rightarrow 0
$$
as \( t \to s^+ \) and
\[
1_{(0,t-s)}(r)\|S(r)\phi\|_{\gamma(U,L^p)}^2 \lesssim \|S(r)\phi\|_{\gamma(U,L^p)}^2
\]
which is integrable on \([0,T_0]\) by (4.3). Hence, by the Lebesgue Dominated Convergence Theorem (DCT), \( I_1(s,t) \) tends to 0 as \( t \to s^+ \). The second integral can be estimated as follows:
\[
I_2(s,t) := \mathbb{E}\left|\int_0^s (S(t-s) - I)S(s-r)\phi dB_r\right|^2_{L^2}
\]
\[
\lesssim \left(\int_0^s \|S(t-s) - I)S(s-r)\phi\|_{\gamma(U,L^p)}^2 \, dr\right)^{1+2\alpha}.
\]
By the definition of the \( \gamma \)-radonifying norm and Proposition 2.4, if we take a centered sequence \( \{\delta_n\} \) of independent, standard Gaussian random variables, we have that
\[
\|S(t-s) - I)S(s-r)\phi \|_{\gamma(U,L^p)} \approx \mathbb{E}\left(\sum_{n=1}^\infty \delta_n (S(t-s) - I)S(s-r)\phi \cdot e_n\right)_{L^p}^2 = \left(\sum_{n=1}^\infty \mathbb{E}\|\delta_n (S(t-s) - I)S(s-r)\phi \cdot e_n\|_{L^p}^2\right)^{1/2}.
\]
For every \( r \in [0,s] \) and every \( n \in \mathbb{N} \), we have that
\[
\mathbb{E}\|\delta_n (S(t-s) - I)S(s-r)\phi \cdot e_n\|_{L^p}^2 \to 0
\]
as \( t \to s^+ \) by the strong continuity of the semigroup \( S \) and furthermore,
\[
\mathbb{E}\|\delta_n (S(t-s) - I)S(s-r)\phi \cdot e_n\|_{L^p}^2 \leq \mathbb{E}\|S(t-s) - I\|_{L^p} \|S(s-r)\phi \cdot e_n\|_{L^p}^2
\]
\[
\leq \mathbb{E}\delta_n \|S(s-r)\phi \cdot e_n\|_{L^p}^2
\]
which is a summable sequence for every \( r \in [0,s] \) since
\[
\sum_{n=1}^\infty \mathbb{E}\delta_n \|S(s-r)\phi \cdot e_n\|_{L^p}^2 = \|S(s-r)\phi\|_{\gamma(U,L^p)}^2 < \infty.
\]
We thus have, by DCT, that
\[
\|S(t-s) - I)S(s-r)\phi\|_{\gamma(U,L^p)}^2 \to 0
\]
as \( t \to s^+ \) for every \( r \in [0,s] \). Furthermore, we have that
\[
\|S(t-s) - I)S(s-r)\phi\|_{\gamma(U,L^p)} \lesssim \|S(t-s) - I\|_{L^p} \|S(s-r)\phi\|_{\gamma(U,L^p)}
\]
and
\[
\int_0^s \|S(s-r)\phi\|_{\gamma(U,L^p)}^2 \, dr < \infty
\]
by the first part of the proof. This yields \( I_2(s,t) \to 0 \) as \( t \to s^+ \) by DCT. Hence, we have proved the mean-square right continuity of the process \( X \). The existence of a version with measurable sample paths follows by Theorem 2.6 and Theorem 2.7 from [11] adapted to the Banach-space setting. \( \square \)
Continuity of the solution $X$ to (4.1) is discussed now. Since the semigroup $S$ is analytic, there is $\lambda \in \mathbb{R}$ such that the operator $(\lambda I - A)$ is strictly positive. Let us thus denote, for $\delta \geq 0$,

$$V_{\delta,p} := \text{Dom}((\lambda I - A)^\delta) \subset L^p.$$  

Equipped with the graph norm topology, the space $V_{\delta,p}$ is a Banach space. The main result follows.

**Proposition 4.2.** Assume that $S(u)\Phi \in \gamma(U, L^p)$ for all $u > 0$ and that there are $T_0 > 0$, $\delta \geq 0$ and $\beta > 0$ such that $x \in V_{\delta,p}$,

$$\beta + \delta < \alpha + \frac{1}{2}$$

and

$$\int_0^{T_0} \left( r^{-\beta-\delta} \| S(r)\Phi \|_{\gamma(U, L^p)} \right)^{\frac{2}{1+2\alpha}} dr < \infty. \quad (4.5)$$

Then $X$ has a version which belongs to $\mathcal{C}^\nu([0, T]; V_{\delta,p})$, $\mathbb{P}$-a.s. for every $\nu \in [0, \beta)$.

**Proof.** Step 1: We show that the integral

$$\int_0^t (\lambda I - A)^\delta S(t-r)\Phi dB_r$$

exists for all $t > 0$. First we have to notice that since $S(u)\Phi \in \gamma(U, L^p)$ for each $u > 0$, it follows that also $(\lambda I - A)^\delta S(u)\Phi \in \gamma(U, L^p)$ for all $u > 0$. In the same way as in Corollary 3.4, we can obtain

$$\mathbb{E} \left\| \int_0^t (\lambda I - A)^\delta S(t-r)\Phi dB_r \right\|_{L^p}^2 \lesssim \left( \int_0^t \left\| (\lambda I - A)^\delta S(t-r)\Phi \right\|_{\gamma(U, L^p)}^\frac{2}{1+2\alpha} dr \right)^{1+2\alpha} \lesssim \left( \int_0^t \left( r^{-\delta} \| S(r)\Phi \|_{\gamma(U, L^p)} \right)^{\frac{2}{1+2\alpha}} dr \right)^{1+2\alpha}$$

since the integrand can be estimated by

$$\| (\lambda I - A)^\delta S(u)\Phi \|_{\gamma(U, L^p)} \leq \left\| (\lambda I - A)^\delta S \left( \frac{u}{2} \right) \right\|_{L^\infty(V)} \left\| S \left( \frac{u}{2} \right) \Phi \right\|_{\gamma(U, L^p)} \lesssim u^{-\delta} \left\| S \left( \frac{u}{2} \right) \Phi \right\|_{\gamma(U, L^p)}$$

for $u > 0$. Now, if $t < 2T_0$, we can only enlarge the integration bounds and use (4.3). For $t > 2T_0$, we use the semigroup property (4.4). 

Step 2: Since $(\lambda I - A)^\delta$ is a closed operator, it follows that $\int_0^t S(t-r)\Phi dB_r \in V_{\delta,p}$ and, moreover,

$$(\lambda I - A)^\delta \int_0^t S(t-r)\Phi dB_r = \int_0^t (\lambda I - A)^\delta S(t-r)\Phi dB_r \quad \mathbb{P}$$.a.s.$$

for every $t > 0$. 


Step 3: We use the factorization technique to show that the solution admits a $V_{\delta,p}$-valued version with Hölder continuous sample paths if $x \in V_{\delta,p}$. Fix $T > 0$. Similarly as above, we can show that

$$
\mathbb{E}\left\|\int_0^u (u-r)^{-\beta}(\lambda I - A)^{\delta} S(u-r) \Phi dB_r \right\|_{L^p}^2 \lesssim \left( \int_0^u (t-r)^{-(\beta+\delta)} \|S(r)\Phi\|_{\gamma(U,L^p)}^{2+2\alpha} \, dr \right)^{1+2\alpha}
$$

and the assumption (4.5) assures that this is finite for all $u > 0$. Since again, $(\lambda I - A)^{\delta}$ is closed, we have that

$$
\int_0^u (u-r)^{-\beta} S(u-r) \Phi dB_r \in V_{\delta,p} \quad \mathbb{P}\text{-a.s.}
$$

for all $u > 0$ and the integral commutes with $(\lambda I - A)^{\delta}$. This means that we can define an $L^p$-valued process $Y^\delta = (Y^\delta_u, u \in [0,T])$ by

$$
Y^\delta_u := \int_0^u (u-r)^{-\beta}(\lambda I - A)^{\delta} S(u-r) \Phi dB_r.
$$

(4.6)

$Y^\delta$ has a version with measurable sample paths which can be shown similarly as in the proof of Proposition 4.1 using (4.5). Moreover, since

$$
\sup_{t \in [0,T]} \left\|Y^\delta_t\right\|_{L^q(\Omega;L^p)} \lesssim \sup_{t \in [0,T]} \left( \int_0^t (r-u)^{-(\beta+\delta)} \|S(r)\Phi\|_{\gamma(U,L^p)}^{2+2\alpha} \, dr \right)^{1+\frac{1}{\alpha}} < \infty,
$$

for every $1 \leq q < \infty$ by (3.2) and (4.5), we infer that $Y^\delta \in L^q([0,T];L^p)$, $\mathbb{P}$-almost surely for every $1 \leq q < \infty$.

Recall that

$$
\int_r^t (t-u)^{\beta-1}(u-r)^{-\beta} \, du = \frac{\pi}{\sin \pi \beta} = \frac{1}{\Lambda}
$$

for $r \in [0,t]$. Using this fact, we can write,

$$
\int_0^t (\lambda I - A)^{\delta} S(t-r) \Phi dB_r =
$$

$$
= \Lambda \sum_{n=1}^{\infty} \int_0^t (t-u)^{\beta-1}(u-r)^{-\beta}(\lambda I - A)^{\delta} S(t-u)S(u-r) \Phi \epsilon_n \, du \, db_r^{(n)}
$$

$$
= \Lambda \sum_{n=1}^{\infty} \int_0^t (t-u)^{\beta-1} S(t-u) \left( \int_0^u (u-r)^{-\beta}(\lambda I - A)^{\delta} S(u-r) \Phi \epsilon_n \, db_r^{(n)} \right) \, du
$$

$$
=: Y^\delta_n(u)
$$

$$
= \Lambda \lim_{N \to \infty} \int_0^t (t-u)^{\beta-1} S(t-u) \left( \sum_{n=1}^{N} Y^\delta_n(u) \right) \, du.
$$

The interchange of the order of integration is possible due to the fact that the function

$$
f(u, r) := 1_{(r,t]}(u)(t-u)^{\beta-1}(u-r)^{-\beta}(\lambda I - A)^{\delta} S(t-r) \Phi z
$$

(here $z \in L^p$) belongs to the mixed Lebesgue space $L^{1+2\alpha}(\Omega;L^p)$ and $f^*(u, r) := f(r, u)$ belongs to $L^{1+2\alpha}(\Omega;L^p)$ since $0 < \beta + \delta < \alpha + \frac{1}{2}$ (see [4]). Therefore, a suitable stochastic
Fubini theorem can be proved for this particular function (see [6, Lemma 4.4] for a similar result). In order to interchange the limit $N \to \infty$ and the integral, consider the following:

$$\left\| \int_0^t (t-u)^{\beta-1} S(t-u) Y^N(u) du - \int_0^t (t-u)^{\beta-1} S(t-u) Y_u^\delta du \right\|_{L^1(\Omega; L^p)} \lesssim \int_0^t (t-u)^{\beta-1} \|S(t-u)\|_{L^p} \|Y^N(u) - Y_u^\delta\|_{L^p(\Omega; L^p)} du.$$  

The norm inside the last integral can be estimated by

$$\sup_N \sup_{u \in (0,t]} \|Y^N(u) - Y_u^\delta\|_{L^p(\Omega; L^p)} \lesssim \sup_N \sup_{u \in (0,t]} \left( \int_0^u \left( \int_{\frac{\beta}{1+2\alpha}} \left( \sum_{n=N+1}^{\infty} |[S(r)\Phi e_n](x)|^2 \right)^\frac{\gamma}{2} \mu(dx) \right)^{\frac{2}{p(1+2\alpha)}} dr \right) \alpha + \frac{1}{2}$$

by similar computations as in the proof of Corollary 3.4. Hence, we can infer by DCT that

$$\int_0^t (\lambda I - A)^\delta S(t-u)\Phi dB_r = \Lambda \int_0^t (t-u)^{\beta-1} S(t-u) Y_u^\delta du, \quad \mathbb{P} \text{- a.s.}$$

holds for every $t > 0$. Consider the operator

$$R_{\beta,T}(Z)(t) := \int_0^t (t-u)^{\beta-1} S(t-u) Z(u) du, \quad t \in [0,T].$$

By [3, Theorem 5.14, (ii)], the operator $R_{\beta,T}$ is bounded from $L^q([0,T]; L^p)$ to $\mathcal{C}^\nu([0,T]; L^p)$ for every $\nu \in [0,\beta - \frac{1}{q})$. Taking $q$ sufficiently large yields the claim. \hfill \Box

**Corollary 4.3.** Assume that $S(u)\Phi \in \mathcal{C}(U,L^p)$ and that there is a $\gamma \in [0,\alpha + \frac{1}{2})$ such that

$$\|S(u)\Phi\|_{\mathcal{C}(U,L^p)} \lesssim u^{-\gamma} \quad (4.7)$$

for all $u > 0$. Then $X$ has a version which belongs to $\mathcal{C}^\nu([0,T]; V_{\delta,p})$ for every $\nu, \delta \geq 0$ such that $x \in V_{\delta,p}$ and

$$\nu + \delta < \alpha + \frac{1}{2} - \gamma.$$  

**Proof.** Let $\delta \in [0,\alpha + \frac{1}{2} - \gamma)$ be arbitrary but fixed. Now we can choose $\beta > 0$ such that

$$0 < \beta + \delta + \gamma < \alpha + \frac{1}{2}.$$

Then (4.5) holds since

$$\int_0^T \left( r^{-(\beta+\delta)} \|S(r)\Phi\|_{\mathcal{C}(U,L^p)} \right)^{\frac{2}{1+2\alpha}} dr \lesssim \int_0^T r^{-\frac{\beta}{1+2\alpha}} \|S(r)\Phi\|_{\mathcal{C}(U,L^p)} dr < \infty.$$  

Hence, $X \in \mathcal{C}^\nu([0,T]; V_{\delta,p})$ for every $\nu \in [0,\beta)$ by Proposition 4.2. Taking the supremum over all such $\beta$'s yields the claim. \hfill \Box
Remark 4.4. For the fBm or the Rosenblatt process (\( H > \frac{1}{2} \) for both), we obtain that if there is \( \gamma \in [0, H) \) such that \( \| S(u) \Phi \|_{(U,L^p)} \lesssim u^{-\gamma} \) for all \( u > 0 \), then for every \( 0 \leq \delta + \nu < H - \gamma \) with \( x \in V_{\delta,p} \), the solution has a version in \( \mathcal{C}^{\nu}([0,T]; V_{\delta,p}) \).

5. Examples

5.1. **Parabolic equations with Volterra pointwise noise.** Consider the following parabolic equation

\[ \partial_t u = \Delta u + \delta z \eta, \quad \text{on} \quad \mathbb{R}_+ \times \mathcal{O} \]

with the initial condition \( u(0, \cdot) = f \) on \( \mathcal{O} \) and the Dirichlet boundary condition \( u|_{\mathbb{R}_+ \times \partial \mathcal{O}} = 0 \).

Given a point \( z \in \mathcal{O}, \mathcal{O} \subset \mathbb{R}^d \) an open bounded domain with \( \mathcal{C}^1 \) boundary \( \partial \mathcal{O} \), \( \delta z \) denotes the Dirac distribution at \( z \).

This formal system can be rewritten as the stochastic evolution equation (4.1). The noise process \( \eta \) is the formal derivative of a scalar (i.e. \( \mathbb{R} \)-cylindrical) \( \alpha \)-regular Volterra process \( b = (b_t, t \geq 0) \) which belongs to a finite Wiener chaos \( \mathcal{H} \). We assume that \( p \geq \frac{2}{1+2\alpha} \) and take \( x := f \in L^p(\mathcal{O}); A := \Delta_{|\text{Dom}(A)} \) with \( \text{Dom}(A) := W_{2,p}^{2}(\mathcal{O}) \cap W_{0}^{1,p}(\mathcal{O}) \), which is a generator of an analytic semigroup on \( L^p(\mathcal{O}) \); and \( \Phi \) which is given by \( \Phi a = a\delta z \) for \( a \in \mathbb{R} =: U \). By the Sobolev embedding (see e.g. [20, Theorem 8.2]), for every \( \varepsilon \in (0, 1 - \frac{d}{2p}) \) we have that \( \Phi \in \mathcal{L}(\mathbb{R}, V_{\varepsilon-1,p}) \) since \( V_{\delta,p} \subset W_{2,p}^{2}(\mathcal{O}) \).

Note that

\[ \| S(r) \Phi \|_{(R,L^p)} \lesssim \| S(r) \|_{\mathcal{L}(V_{\varepsilon-1,p},L^p)} \| \Phi \|_{\mathcal{L}(R,V_{\varepsilon-1,p})} \lesssim r^{\varepsilon-1} \]

for \( r > 0 \). Thus, we can apply Corollary 4.3 with \( \gamma := 1 - \varepsilon \).

If

\[ \alpha > \frac{d}{2p} - \frac{1}{2}, \quad (5.1) \]

then we can choose \( \varepsilon \) such that

\[ \varepsilon \in \left( \frac{1}{2} - \alpha, 1 - \frac{d}{2p} \right) \]

so that, by Corollary 4.3, there is a version of the solution \( X \) which belongs to \( \mathcal{C}^{\nu}([0,T]; V_{\delta,p}) \) for every \( \nu, \delta \geq 0 \) such that \( x \in V_{\delta,p} \) and \( \nu + \delta < \alpha + \varepsilon - \frac{1}{2} \). Taking the supremum over all such \( \varepsilon \) yields that \( X \) has a version \( \bar{X} \) such that

\[ \bar{X} \in \mathcal{C}^{\nu}([0,T]; V_{\delta,p}), \quad \text{for all} \quad \nu + \delta < \alpha + \frac{1}{2} - \frac{d}{2p}, \]

Note that (5.1) does not pose additional constraints on \( \alpha \) if \( d = 1 \); it excludes the case \( p = \frac{2}{1+2\alpha} \) if \( d = 2 \) and; finally, for \( d \geq 3 \), (5.1) can only be satisfied if \( 2p > d \).

If, however, the stronger condition

\[ \alpha > \frac{d}{p} - \frac{1}{2} \quad (5.2) \]

is satisfied, then by the Sobolev embedding, we have that \( X \) has a version \( \tilde{X} \) such that

\[ \tilde{X} \in \mathcal{C}^{\nu}([0,T]; \mathcal{C}^{2d-\frac{d}{p}}(\mathcal{O})) \quad (5.3) \]
for every $\nu \geq 0$ and $\delta > \frac{d}{2p}$ such that $x \in V_{\delta,p}$, $\nu + \delta < \alpha + \frac{1}{2} - \frac{d}{2p}$. Note that if $d = 1$, the condition (5.2) excludes the case $p = \frac{2}{1+2\alpha}$, and, in higher dimensions ($d \geq 2$), it can only be satisfied if $p > d$.

### 5.2. $2m$-th order parabolic equations with Volterra noise

Let $m \in \mathbb{N}$ and consider the following parabolic equation

$$\partial_t u = L_{2m} u + \eta \quad \text{on} \quad \mathbb{R}_+ \times \mathcal{O}$$

with the initial condition $u(0, \cdot) = f$ which belongs to the space $L^p(\mathcal{O})$; and the Dirichlet boundary condition

$$\frac{\partial^k u}{\partial v^k}\bigg|_{\mathbb{R}_+ \times \partial \mathcal{O}} = 0$$

for $k \in \{0, \ldots, m - 1\}$ where $\frac{\partial}{\partial v}$ denotes the conormal derivative. Here, $\mathcal{O} \subset \mathbb{R}^d$ is an open bounded domain with smooth boundary and $L_{2m}$ is a differential operator of order $2m$, i.e.

$$L_{2m} = \sum_{|k| \leq 2m} a_k(\cdot) \partial^k,$$

with $a_k \in \mathcal{C}_b^\infty(\mathcal{O})$ which is assumed to be uniformly elliptic. The considered noise $\eta$ is Volterra in time and can be both white or correlated in space. This system can be rewritten as the stochastic evolution equation (4.1) in $L^p(\mathcal{O})$. Indeed, let $U := L^2(\mathcal{O})$ and $B = (B_t, t \geq 0)$ be a $U$-cylindrical $\alpha$-regular Volterra process which satisfies the hypotheses of section 2.5. Then the noise $\eta$ is formally given by

$$\eta(t, \cdot) = \Phi \frac{d}{dt} B_t$$

where $\Phi \in \mathcal{L}(U, L^p(\mathcal{O}))$ determines the space correlation of the noise process $\eta$. Assume that $\frac{2}{1+2\alpha} \leq p < \infty$ and take $x = f \in L^p(\mathcal{O})$ and $A := L_{2m}\nabla\text{Dom}(A)$ where

$$\text{Dom}(A) := \left\{ f \in W^{2m,p}(\mathcal{O}) : \frac{\partial^k f}{\partial v^k} = 0 \text{ on } \partial \mathcal{O} \text{ for } k \in \{0, \ldots, m - 1\} \right\}.$$ 

The operator $A$ generates an analytic semigroup $(S(t), t \geq 0)$ on $L^p(\mathcal{O})$. By standard estimates on the Green function, we have that

$$\|S(r)\Phi\|_{\gamma(U, L^p(\mathcal{O}))} \lesssim r^{-\frac{d}{4m}}$$

for $r > 0$ then we can use Corollary [4.3] with $\gamma := \frac{d}{4m}$.

Thus, if

$$\alpha > \frac{d}{4m} - \frac{1}{2}, \quad (5.4)$$

then by Corollary [4.3] we have that for every $\nu, \delta \geq 0$ such that $x \in V_{\delta,p}$ and $\nu + \delta < \alpha + \frac{1}{2} - \frac{d}{4m}$, the solution $X$ has a version in $\mathcal{C}^{\nu}(\mathcal{O})$. If, moreover,

$$\alpha > \frac{d}{4m} - \frac{1}{2} - \frac{d}{2mp}, \quad (5.5)$$

then by the Sobolev embedding, we have that for every $\nu \geq 0$ and $\delta > \frac{d}{2mp}$ such that $x \in V_{\delta,p}$ and $\nu + \delta < \alpha + \frac{1}{2} - \frac{d}{4m}$, the solution has a version in the space $\mathcal{C}^{\nu}(\mathcal{O})$. 


Note that the condition \((5.4)\) can be only satisfied if \(d < 4m\) and the condition \((5.5)\) can only be satisfied if \(d < \frac{4mp}{p+2}\). In the particular case of the stochastic heat equation (i.e. \(m = 1\)), we have that if \(d < 4\), it is possible to take sufficiently smooth Volterra noise so that the solution is time (Hölder) continuous in the space \(V_{δ,p}\) and if, moreover, we have that \(d < \frac{4p}{p+2}\), then we may even obtain (Hölder) continuity in the spatial variable. If the initial condition is regular (i.e. \(p > 6\)), the space-time continuity may be obtained in dimensions \(d = 1, 2, 3\).

References

[1] E. Alòs, O. Mazet, and D. Nualart, Stochastic calculus with respect to Gaussian processes, Ann. Probab., 29 (2001), pp. 766–801.
[2] E. Alòs and D. Nualart, Stochastic integration with respect to the fractional Brownian motion, Stochastics and Stochastics Reports, 75 (2003), pp. 129–152.
[3] F. Baudoin and D. Nualart, Equivalence of Volterra processes, Stoch. Proc. Appl., 107 (2003), pp. 327–350.
[4] A. Benedek and R. Panzone, The space \(L^p\), with mixed norm, Duke Math. J., 28 (1961), pp. 301–324.
[5] Z. Brzeźniak and J. van Neerven, Space-time regularity for linear stochastic evolution equations driven by spatially homogeneous noise, J. Math. Kyoto Univ., 43 (2003), pp. 261–303.
[6] P. Čoupek and B. Maslowski, Stochastic evolution equations with Volterra noise, Stoch. Proc. Appl., 127 (2017), pp. 877–900.
[7] P. Čoupek, B. Maslowski, and J. Šnupárová, SPDEs with Volterra noise, Proceedings of Stochastic Partial Differential Equations and Related Fields - In Honor of Michael Röckner, (2017). Submitted.
[8] G. Da Prato and J. Zabczyk, Stochastic equations in infinite dimensions, Encyclopedia of Mathematics and its Applications, Oxford University Press, 2. ed., 2014.
[9] V. c. H. de la Peña and E. Giné, Decoupling, Probability and its Applications (New York), Springer-Verlag, New York, 1999. From dependence to independence, Randomly stopped processes. \(U\)-statistics and processes. Martingales and beyond.
[10] L. Decreusefond and A. S. Üstünel, Stochastic analysis of the fractional Brownian motion, Potential Anal., 10 (1999), pp. 177–214.
[11] J. Doob, Stochastic processes, Wiley, 1990.
[12] T. E. Duncan, B. Maslowski, and B. Pasik-Duncan, Stochastic linear-quadratic control for bilinear evolution equations driven by Volterra processes, (2016). Submitted.
[13] M. Erraoui and E. H. Essaky, Canonical representation for Gaussian processes, Springer Berlin Heidelberg, Berlin, Heidelberg, 2009, pp. 365–381.
[14] T. Hida, Canonical representations of Gaussian processes and their applications, Memoirs of College of Science, University of Kyoto, XXXIII (1960), pp. 109–155.
[15] H. Hult, Approximating some Volterra type stochastic integrals with applications to parameter estimation, Stoch. Proc. Appl., 105 (2003), pp. 1–32.
[16] J. Maas, Malliavin calculus and decoupling inequalities in Banach spaces, J. Math. Anal. Appl., 363 (2010), pp. 383–398.
[17] B. B. Mandelbrot and J. W. Van Ness, Fractional Brownian motion, fractional noises and applications, SIAM Rev., 10 (1968), pp. 422–437.
[18] A. L. Neidhardt, Stochastic integrals in 2-uniformly smooth Banach spaces, University of Wisconsin, 1978. Ph.D. dissertation.
[19] J. Neveu, Sur l’espérance conditionnelle par rapport à un mouvement brownien, Ann. Inst. H. Poincaré Sect. B (N.S.), 12 (1976), pp. 105–109.
[20] E. D. Nezza, G. Palatucci, and E. Valdinoci, Hitchhiker’s guide to the fractional sobolev spaces, Bulletin des Sciences Mathématiques, 136 (2012), pp. 521–573.
[21] D. Nualart, The Malliavin calculus and related topics, Probability and its applications, Springer, 2006.
[22] M. Ondreját, Uniqueness for stochastic evolution equations in Banach spaces, Dissertationes Math., 426 (2004), p. 63.
[23] M. S. Taqqu, The Rosenblatt process, in Selected works of Murray Rosenblatt, R. A. Davis, K.-S. Lii, and D. N. Politis, eds., Springer New York, 2011, pp. 29–45.
[24] C. A. Tudor, Analysis of the Rosenblatt process, ESAIM: Probability and Statistics, 12 (2008), pp. 230–257.
[25] N. N. Vakhania, V. I. Tarieladze, and S. A. Chobanyan, Probability distributions on Banach spaces, vol. 14 of Mathematics and its Applications, Reidel Publishing Co., Dordrecht, 1987.

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