A Zoo of Translating Solitons on a Parallel Light-like Direction in Minkowski 3-Space

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Abstract

We deal with solitons of the mean curvature flow. The definition of translating solitons on a light-like direction in Minkowski 3-space is introduced. Firstly, we classify those which are graphical, translation surfaces, obtaining space-like and time-like, entire and not entire, complete and incomplete examples. Among them, all our time-like examples are incomplete. The second family consists of those which are invariant by a 1-dimensional subgroup of parabolic motions, i.e, with light-like axis. The classification result implies that all examples of this second family have singularities.

Keywords: Translating soliton, mean curvature flow, light-like vector, Minkowski 3-space.

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1 Introduction

Hypersurfaces in Euclidean space which evolve along the mean curvature flow have been widely studied. Of particular interest are those called translating solitons, which are those whose mean curvature vector satisfies the following equation

\[ \vec{H} = \vec{K} \]

where \( \perp \) and \( \vec{K} \) denote projection on the normal bundle and a unit vector field, respectively. A much more general, but very weak, definition can be found in [1], where virtually no restriction on \( \vec{K} \) is set (see also [5]). Among many papers, we can select [1], [5], [10], which study translating solitons in Riemannian manifolds. However, the theory seems to be less developed in Lorentzian geometry, although relevant results can be found in [6] and [14]. According to them, translating solitons in Minkowski space are mainly studied when the vector \( \vec{K} \) is parallel and time-like.

In this paper, we wish to introduce a new family, namely translating solitons of the mean curvature flow such that \( \vec{K} \) is a light-like parallel vector. More precisely.
Definition 1.1 Given a parallel light-like vector \( \vec{K} \) in Minkowski 3-space \( \mathbb{L}^3 \), a non-degenerate immersion \( \psi : M \to \mathbb{L}^3 \) will be called a translating soliton on the light-like direction \( \vec{K} \) if its mean curvature vector \( \vec{H} \) satisfies \( \vec{H} = \vec{K} \perp \), where \( \perp \) means the orthogonal projection on the normal bundle.

This definition makes sense because the induced metric is not degenerate, which implies that \( \vec{H} \) will not be light-like at any point.

We write the standard flat metric in Minkowski 3-space in a suitable way for our needs, namely \( \langle , \rangle = -2dxdy + dz^2 \). That is to say, we are considering a basis \( B = \{ \vec{x}, \vec{y}, \vec{z} \} \) such that \( \vec{x}, \vec{y} \) are light-like, future pointing, and satisfying the normalizing condition \( \langle \vec{x}, \vec{y} \rangle = -1 \). It is important to recall that any two parallel light-like vectors are linked by an isometry of \( \mathbb{L}^3 \), because the light cone is invariant by rotations, boosts and a few reflections. This means that we can reduce to the case \( \vec{K} = \vec{x} \). We will study two families.

Firstly, Section 3 is devoted to studying graphical surfaces, that is to say, the ones that admit a parametrization \( \psi : \Omega \subset \mathbb{R}^2 \to \mathbb{L}^3, \psi(y,z) = (u(y,z), y, z) \), where \( u : \Omega \to \mathbb{R} \). In Theorem 3, we will classify those which, in addition, are translation surfaces, i.e., for some smooth functions \( a \) and \( b \), then \( u(y,z) = a(y) + b(z) \). Translation surfaces in Euclidean space were introduced by S. Lie (see [4], also [12]). Needless to say, the same definition can be easily set in Minkowski space. We study space-like and time-like surfaces, obtain four types, which are flat by chance. Note that [6] and [14] paid attention to entire examples. We later study the completeness of the four cases, showing entire and not entire, complete and incomplete surfaces, along Corollaries 3.2, 3.3, 3.4 and 3.5. We should remark that the standard techniques to show the completeness of space-like surfaces in Minkowski space do not work in our setting (see Remark 3.1).

Secondly, there are 1-dimensional subgroups of parabolic isometries of \( \mathbb{L}^3 \), whose axis is light-like. We will reduce to the well-adapted case when the rotation axis is spanned by the vector \( \vec{x} \) (see Section 4 for more details.) That is to say, we study surfaces obtained by letting this subgroup of isometries act on a suitable profile curve. We classify in Theorem 4.2 those translating solitons which are invariant by this subgroup of isometries.

2 Preliminaries

Given a smooth manifold \( M \), assume a family of smooth immersions in a semi-Riemannian manifold \((M,g), F_t : M \to M \times \mathbb{R}, t \in [0,\delta], \delta > 0\), with mean curvature vector \( \vec{H}_t \). The initial immersion \( F_0 \) is called a solution to the Mean Curvature Flow (MCF), up to local diffeomorphism, if the following equation holds

\[
\left( \frac{d}{dt} F_t \right) \perp = \vec{H}_t,
\]

where \( \perp \) means the orthogonal projection on the normal bundle. If an immersion \( F : M \to \mathbb{L}^3 \) satisfies the condition \( \vec{H} = \vec{K} \perp \), then it is possible to define the forever flow \( \psi : M \times \mathbb{R} \to \mathbb{L}^3, \psi(p,t) = F(p) + t\vec{K} \), and clearly,

\[
\left( \frac{d}{dt} F_t \right) \perp = \vec{K} \perp = \vec{H}.
\]

We recall the basic theory of surfaces in Minkowski 3-space. See [11] for details. Let \( \psi : M \to \mathbb{L}^3 \) be an immersion of a surface \( M \) in Minkowski 3-space. We assume that the
induced metric $I = \psi^* \langle \cdot, \cdot \rangle$ is not degenerate, that is to say, it is either Riemannian or Lorentzian. This metric is usually known as the first fundamental form. Let $N$ be a unit normal vector field on $M$. Therefore, $\varepsilon = \langle N, N \rangle = \pm 1$ is a constant function on $M$. Given $\nabla^O$ and $\nabla$ the Levi-Civita connection of $\mathbb{L}^3$ and the induced connection on $M$, respectively, we know

$$\nabla^O_X Y = \nabla_X Y + \varepsilon \langle AX, Y \rangle N, \quad X, Y \in TM,$$

where $A$ is the shape operator of $N$. The second fundamental form $\sigma$ is defined as

$$\sigma(X, Y) = \varepsilon \langle AX, Y \rangle N, \quad X, Y \in TM.$$

Our definition of the mean curvature vector is

$$\vec{H} = \varepsilon H N = \text{trace}_I(\sigma), \quad H = \text{trace}_I(A).$$

The function $H : M \to \mathbb{R}$ is called the mean curvature (function) of $M$. As in [11], if $X$ is a local parametrization $X : U \subset \mathbb{R}^2 \to \mathbb{L}^3$, $X = X(u, v)$, then $B = (X_u, X_v)$ is a local basis of the tangent plane at each point of $X(U)$. The coefficients of the first fundamental form are

$$E = \langle X_u, X_u \rangle, \quad F = \langle X_u, X_v \rangle, \quad G = \langle X_v, X_v \rangle,$$

so that the matricial expression is

$$(E \ F \ F \ G).$$

Let

$$\begin{pmatrix} e & f \\ f & g \end{pmatrix}$$

be the matricial expression of $\sigma$ with respect to $B$. Since $\langle AX, Y \rangle = \langle \nabla_X Y, N \rangle$, then

$$e = \langle AX_u, X_u \rangle = \langle N, X_{uu} \rangle, \quad f = \langle AX_u, X_v \rangle = \langle N, X_{uv} \rangle, \quad g = \langle AX_v, X_v \rangle = \langle N, X_{vv} \rangle.$$

The shape operator can be computed by

$$A \equiv \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} e & f \\ f & g \end{pmatrix}.$$

(1)

With this, the expressions of the mean and Gaussian curvatures are

$$H = \frac{Eg - 2Ff + Ge}{EG - F^2}, \quad K = \frac{eg - f^2}{EG - F^2}.$$

(2)

### 3 Graphical Translating Solitons on a Light-like Direction

Given a domain $\Omega \subset \mathbb{R}^2$, let us take a parametrization of a non-degenerate surface

$$\psi : \Omega \to \mathbb{L}^3, \; \psi(y, z) = (u(y, z), y, z).$$

The partial derivatives of $\psi(y, z)$ are $\psi_y = (u_y, 1, 0)$ and $\psi_z = (u_z, 0, 1)$. The coefficients of the first fundamental form $I = \psi^* \langle \cdot, \cdot \rangle$ are

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} -2u_y & -u_z \\ -u_z & 1 \end{pmatrix}.$$ 

In addition, it is Riemannian if $EG - F^2 = -2u_y - u_z^2 > 0$. Since we are assuming that the surface is not degenerate, the following function is constant, $\varepsilon = \text{sign}(2u_y + u_z^2) = \pm 1$. A unit normal vector field is given by

$$N = \frac{1}{W}(-u_y, 1, u_z), \text{ where } W = \sqrt{\varepsilon(2u_y + u_z^2)} > 0.$$
Case 1. A differential equation turns into the following, and the light-like direction. We finally obtain the following PDE, which characterizes our graphical translating solitons in the light-like direction $\vec{K} = \vec{x}$,

$$u_{yy} + 2u_z u_{yz} - 2uyu_{zz} + 2u_y + u_z^2 = 0. \quad (3)$$

Now, we consider $\psi$ as a translation surface, which means $u(y, z) = a(y) + b(z)$, where $a$ and $b$ are smooth functions. This is equivalent to $u_{yz} = 0$ everywhere. So, our partial differential equation turns into the following,

$$a''(y) - 2a'(y)b''(z) + 2a'(y) + (b'(z))^2 = 0. \quad (4)$$

Case 1. $a''(y) = 0$ in an interval. Then $a(y) = a_1 y + a_0$ for some $a_0, a_1 \in \mathbb{R}$. In (4), we obtain

$$-2a_1 b''(z) + 2a_1 + (b'(z))^2 = 0.$$ 

If $a_1 = 0$, then $b'(z) = 0$ in an interval, that is to say, $u(y, z) = a_0 + b_1$. But we are discarding this case because we need $2u_y + u_z^2 \neq 0$. We put $\varphi = b'$, so that

$$\frac{\varphi'(z)}{(\varphi(z))^2 + 2a_1} = \frac{1}{2a_1}. \quad \text{Case 1.1. } a_1 = 2\lambda^2 > 0: \text{ By integrating, and recalling that } b'(z) = \varphi(z), \text{ we get for some } z_0 \in \mathbb{R},$$

$$b(z) = -4\lambda^2 \log_e \left| \cos \left( \frac{z - z_0}{2\lambda} \right) \right| + b_0,$$

where $b_0$ is an integration constant. We reach to the following solution

$$u(y, z) = 2\lambda^2 y - 4\lambda^2 \log_e \left| \cos \left( \frac{z - z_0}{2\lambda} \right) \right| + b_0, \quad \lambda > 0, b_0, z_0 \in \mathbb{R}. \quad (5)$$

Case 1.2. $a_1 = -2\lambda^2 < 0$: If we integrate both sides, then we obtain the following,

$$\log_e \left| \frac{\varphi(z) - 2\lambda}{\varphi(z) + 2\lambda} \right| = \frac{z - z_0}{\lambda}.$$ 

From here, we have to discuss two cases, namely positive and negative:

$$b'(z) = \varphi(z) = 2\lambda \coth \left( \frac{z - z_0}{2\lambda} \right), \quad b'(z) = \varphi(z) = 2\lambda \tanh \left( \frac{z - z_0}{2\lambda} \right).$$
Our solutions are
\[
\begin{align*}
  u(y, z) &= -2\lambda^2 y - 4\lambda^2 \log e \left| \sinh \left( \frac{z - z_0}{2\lambda} \right) \right| + a_0, \\
  u(y, z) &= -2\lambda^2 y - 4\lambda^2 \log e \left( \cosh \left( \frac{z - z_0}{2\lambda} \right) \right) + a_0.
\end{align*}
\]

\textbf{Case 2.} \( a''(y) \neq 0 \) in an interval.

\textbf{Case 2.1.} We differentiate equation (4) with respect to \( y \):

\[
b''(z) = \frac{a'''(y) + 2a''(y)}{2a'''} = b_2 \in \mathbb{R}.
\]

Therefore, \( b(z) = b_2 z^2 / 2 + b_1 z + b_0 \) for some integration constants \( b_1, b_0 \in \mathbb{R} \). We return to (4),

\[
a''(y) - 2a'(y)b_2 + 2a'(y) + (b_2 z + b_1)^2 = 0.
\]

This must hold on some intervals. This readily implies that \( b_2 = 0 \). Next, we obtain

\[
a''(y) + 2a'(y) + b_1^2 = 0.
\]

(Note that this implies \( a'''(y) + 2a''(y) = 0 \)). The general solution to this equation is

\[
a(y) = a_1 e^{-2y} - \frac{b_1^2}{2} y + a_0, \quad a_0, a_1 \in \mathbb{R}.
\]

Therefore, a family of solutions to (3) is

\[
u : \mathbb{R}^2 \to \mathbb{R}, \quad u(y, z) = a_1 e^{-2y} - \frac{b_1^2}{2} y + b_1 z + b_0, \quad a_0, a_1 \in \mathbb{R}.
\]

\textbf{Case 2.2.} Similarly to the previous idea, we differentiate equation (4) with respect to \( z \):

\[
b'(z)b''(z) = a'(y)b'''(z).
\]

\textbf{Case 2.2.1.} If \( b'''(z) = 0 \) in an interval, then \( b(z) = b_2 z^2 + b_1 z + b_0 \), \( b_0, b_1, b_2 \in \mathbb{R} \). This is the same function as in the previous Case 2.1, so that we obtain again solution (8).

\textbf{Case 2.2.2.} If \( b'''(z) \neq 0 \) in an interval, then \( \frac{b'(z)b''(z)}{b'''(z)} = a'(y) = a_1 \), where \( a_1 \in \mathbb{R} \) is constant. If \( a_1 = 0 \), \( b'(z)b''(z) = 0 \), and we are again in Case 2.2.1. So, we assume \( a_1 \neq 0 \). Obviously,

\[
a(y) = a_1 y + a_0, \quad a_1, a_0 \in \mathbb{R}.
\]

Moreover, we have the following equality \( b'(z)b''(z) = a_1 b'''(z) \), which implies

\[
\frac{(b'(z))^2}{2} = a_1 b''(z) + b_0.
\]

We return to (4), so that \( 0 = 2a_1 + 2b_0 \), that is to say, \( b_0 = -a_1 \). This was already discussed in Case 1.
Theorem 3.1 Consider \( L^3 \) with the usual metric written as \( g = -2dxdy + dz^2 \). Let \( \psi : \Omega \subset \mathbb{R}^2 \rightarrow L^3 \), \( \psi(y, z) = (u(y, z), y, z) \) be a translation surface, i.e., we write \( u(y, z) = a(y) + b(z) \), where \( a \) and \( b \) are smooth functions. If \( \psi \) is a translating soliton on the parallel light-like direction \( K = \vec{e} = (1, 0, 0) \), then \( u \) is one of the following:

I. \( u(y, z) = -2\lambda^2 y + 4\lambda^2 \log_e \left( \cosh \left( \frac{z - z_0}{2\lambda} \right) \right) + a_0, \lambda > 0, a_0, z_0 \in \mathbb{R} \).

II. \( u(y, z) = a_1 e^{-2y} - \frac{b_0^2}{2} y + b_1 z + b_0, a_1, b_1, b_0 \in \mathbb{R}, a_1 \neq 0 \);

III. \( u(y, z) = 2\lambda^2 y - 4\lambda^2 \log_e \left( \cos \left( \frac{z - z_0}{2\lambda} \right) \right) + b_0, \lambda > 0, b_0, z_0 \in \mathbb{R} \);

IV. \( u(y, z) = -2\lambda^2 y + 4\lambda^2 \log_e \left| \sinh \left( \frac{z - z_0}{2\lambda} \right) \right| + a_0, \lambda > 0, a_0, z_0 \in \mathbb{R} \);

We will call them of type I, II, III and IV, respectively. A straightforward computation shows that \( K = 0 \) in all cases.

Corollary 3.1 All examples in Theorem 3.1 are flat.

Next, we want to study the completeness of these surfaces.

Corollary 3.2 Any inextensible solution of type I in Theorem 3.1 is space-like, entire and complete.

Proof: Given \( a_0, z_0 \in \mathbb{R}, \lambda > 0 \), we consider

\[
u(y, z) = -2\lambda^2 y + 4\lambda^2 \log_e \left( \cosh \left( \frac{z - z_0}{2\lambda} \right) \right) + a_0.\]

Needless to say, it can be extended to \( u : \mathbb{R}^2 \rightarrow \mathbb{R} \). We compute

\[
u_y = -2\lambda^2, \quad \nu_z = 2\lambda \tanh \left( \frac{z - z_0}{2\lambda} \right), \]

\[
E = -2\nu_y = 4\lambda^2, \quad F = -\nu_z = -2\lambda \tanh \left( \frac{z - z_0}{2\lambda} \right), \quad G = 1,
\]

\[
EG - F^2 = \frac{4\lambda^2}{\cosh^2 \left( \frac{z - z_0}{2\lambda} \right)} > 0.
\]

We consider \( \mathbb{R}^2 \) with the metric \( I = \psi^*(\cdot, \cdot) \). Next, let us compute the Christoffel symbols. First,

\[
\nabla_{\partial_y} \partial_y = \Gamma_{11}^1 \partial_y + \Gamma_{11}^2 \partial_z.
\]

Taking inner product with \( \partial_y, \partial_z \) we obtain:

\[
EG \Gamma_{11}^1 + FT_1^2 = I \left( \nabla_{\partial_y} \partial_y, \partial_y \right) = \frac{1}{2} \partial_y \left( I (\partial_y, \partial_y) \right) = \frac{1}{2} E_y = 0,
\]

\[
FT_1^1 + GT_1^2 = I \left( \nabla_{\partial_y} \partial_y, \partial_z \right) = \partial_y F - I \left( \partial_y, \nabla_{\partial_y} \partial_z \right) = -I (\partial_y, \nabla_{\partial_y} \partial_y)
\]

\[
= -\frac{1}{2} \partial_z E = 0.
\]
The solution to this system is $\Gamma_{11}^1 = \Gamma_{11}^2 = 0$. Similarly, we compute the other Christoffel symbols:

$$\Gamma_{12}^1 = \Gamma_{12}^2 = 0, \quad \Gamma_{22}^1 = \frac{-1}{4\lambda^2}, \quad \Gamma_{22}^2 = \frac{1}{2\lambda} \tanh \left( \frac{z - z_0}{2\lambda} \right).$$

Now the equations of a geodesic $\alpha(t) = (y(t), z(t))$ are (cf. [13, p. 67])

$$0 = y'' + \Gamma_{11}^1 (y')^2 + 2\Gamma_{12}^1 y'z' + \Gamma_{22}^1 (z')^2 = y'' - \frac{(z')^2}{4\lambda^2},$$
$$0 = z'' + \Gamma_{11}^2 (y')^2 + 2\Gamma_{12}^2 y'z' + \Gamma_{22}^2 (z')^2$$
$$= z'' + \frac{1}{2\lambda} \tanh \left( \frac{z - z_0}{2\lambda} \right) (z')^2.$$

The general solution to the second ODE is

$$z(t) = z_0 + 2\lambda \text{asinh} \left( \frac{a_1}{2\lambda} t + a_2 \right), \quad a_1, a_2 \in \mathbb{R},$$

where $\text{asinh} : \mathbb{R} \to \mathbb{R}$ is the globally defined inverse function of $\sinh$. Moreover,

$$y''(t) = \frac{a_1^2}{(a_1 t + 2\lambda a_2)^2 + 4\lambda^2}.$$

Integrating here, we obtain

$$y'(t) = \frac{a_1}{2\lambda} \arctan \left( \frac{a_1 t + 2\lambda a_2}{2\lambda} \right) + b_1.$$

As $|\arctan(x)| < \pi/2$ for any $x \in \mathbb{R}$, then for a suitable constant $A > 0$, $|y'(t)| \leq A$ for any $t \in \mathbb{R}$. Therefore, any inextensible solution $y$ is globally defined on the whole $\mathbb{R}$. □

**Remark 3.1** Recall that a properly immersed space-like hypersurface in Minkowski $n$-space whose normal vector satisfies the subaffine growth condition is complete (see [3]). Also, if a properly immersed space-like hypersurface in Minkowski $n$-space has bounded principal curvatures, then it is complete (see [7] and [9]).

Take an inextensible example of type I. This entire graph is properly embedded. The partial derivatives of the immersion are

$$\psi_y = (-2\lambda^2, 1, 0), \quad \psi_z = \left( 2\lambda \tanh \left( \frac{z - z_0}{2\lambda} \right), 0, 1 \right).$$

The coefficients of the first fundamental form are

$$E = 4\lambda^2, \quad F = -2\lambda \tanh \left( \frac{z - z_0}{2\lambda} \right), \quad G = 1.$$

The normal vector is

$$N = \left( \lambda \cosh \left( \frac{z - z_0}{2\lambda} \right), \frac{1}{2\lambda} \cosh \left( \frac{z - z_0}{2\lambda} \right), \sinh \left( \frac{z - z_0}{2\lambda} \right) \right).$$
However, this vector does not satisfy the subaffine growth condition, because its coordinates behave as the exponential map at infinity. The coefficients of the second fundamental form are
\[ e = f = 0, \quad g = \frac{-1}{2\lambda} \text{sech} \left( \frac{z - z_0}{2\lambda} \right). \]
From (1), a straightforward computation gives the principal curvatures \( \lambda_1 = 0 \) and \( \lambda_2 = \frac{-1}{2\lambda} \cosh \left( \frac{z - z_0}{2\lambda} \right). \)

Clearly, this function is not bounded. \( \square \)

**Corollary 3.3** All inextensible solutions of type III in Theorem 3.1 are time-like, never entire, and incomplete (space-like, time-like, light-like).

**Proof:** Given \( b_0, z_0 \in \mathbb{R}, \lambda > 0 \), we take \( u(y, z) = 2\lambda^2 y - 4\lambda^2 \log_e \left| \cos \left( \frac{\pi z}{2\lambda} \right) \right| + b_0 \), and \( \psi(y, z) = (u(y, z), y, z) \). Firstly, they cannot be entire because they can only be defined on horizontal strips of the form
\[ S(z_0, \lambda, k) = \left\{ (y, z) \in \mathbb{R}^2 : -\frac{\pi}{2} + k\pi < \frac{z - z_0}{2\lambda} < \frac{\pi}{2} + k\pi \right\}, \quad k \in \mathbb{Z}. \]
We recall the coefficients of the first fundamental form:
\[ \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} -2u_y & -u_z \\ -u_z & 1 \end{pmatrix} = \begin{pmatrix} -4\lambda^2 & -2\lambda \tan \left( \frac{z - z_0}{2\lambda} \right) \\ -2\lambda \tan \left( \frac{z - z_0}{2\lambda} \right) & 1 \end{pmatrix}. \]
The causal character of \( \psi^*g \) is determined by
\[ EG - F^2 = -4\lambda^2 \left( 1 + \tan^2 \left( \frac{z - z_0}{2\lambda} \right) \right) < 0, \]
namely the surface \( \psi \) is time-like. Take the metric \( I = \psi^*(\cdot, \cdot) \). Since \( \psi : (S(z_0, \lambda, k), I) \to \mathbb{L}^2 \) is an isometric embedding, the map \( \psi \) is an isometry onto its image. Then we can work on \( (S(z_0, \lambda, k), I) \). This surface is simply connected, and \( I \) is a Lorentzian metric. By Corollary 3.1, \( (S(z_0, \lambda, k), I) \) is flat, so that it is globally isometric to an open subset of the Minkowski plane, say \( \Phi : (\Omega, g_o) \to (S(z_0, \lambda, k), I), \) where \( \Omega \subset \mathbb{L}^2 \) and \( g_o \) is the standard metric on \( \mathbb{L}^2 \).

With this information, given \( k \in \mathbb{Z}, \) we consider the following curve \( \alpha : (-\pi\lambda, \pi\lambda) \to S(z_0, \lambda, k) \subset \mathbb{R}^2, \alpha(t) = (y_0, t + z_0 + 2k\lambda\pi) \). This curve is inextensible and divergent. Simple computations show \( |\alpha'(t)|^2 = 1, \) so its total length is \( L(\alpha) = 2\pi\lambda. \) But now, since \( \Phi \) is an isometry, the curve \( \beta = \Phi^{-1} \circ \alpha \) is inextensible, space-like, unit, divergent, with total length \( 2\pi\lambda. \) As \( (\Omega, g_o) \) is an open subset of \( \mathbb{L}^2, \) then \( \Omega \) is not the whole \( \mathbb{L}^2. \) This readily shows that the surface is not complete (in any sense, space-like, time-like, light-like). \( \square \)

**Corollary 3.4** Any inextensible solution of type IV in Theorem 3.1 is time-like, not entire, and not complete (space-like, time-like, light-like).

**Proof:** Given \( a_0, z_0 \in \mathbb{R}, \lambda > 0, \) let us consider \( u(y, z) \) as in case IV and \( \psi(y, z) = (u(y, z), y, z). \) Clearly, they can only be defined on horizontal strips of the form
\[ S^+(z_0) = \left\{ (y, z) \in \mathbb{R}^2 : z > z_0 \right\}, \quad S^-(z_0) = \left\{ (y, z) \in \mathbb{R}^2 : z < z_0 \right\}. \]
We compute
\[ u_y = -2\lambda^2, \quad u_z = 2\lambda \coth \left( \frac{z - z_0}{2\lambda} \right), \]
\[ E = -2u_y, \quad F = -u_z, \quad G = 1, \quad EG - F^2 = \frac{-4\lambda^2}{\sinh^2 \left( \frac{z - z_0}{2\lambda} \right)} < 0. \]

Next, the divergent curve \( \alpha : (z_0, z_0 + 1) \to S^+ (z_0), \alpha(s) = (y_0, s) \), satisfies that \( \alpha'(s) = (0, 1) \), it is inextensible at \( z_0 \), and \( |\alpha'(s)|^2 = 1 \). In particular, its total length is finite. Similarly, \( \alpha : (z_0 - 1, z_0) \to S^- (z_0), \alpha(s) = (y_0, s) \) is an inextensible, divergent curve with finite total length. By repeating the argument in Corollary 3.3, we obtain that this surface cannot be complete. □

**Corollary 3.5** Any solution \( \psi \) of \( H \) in Theorem is space-like when \( a_1 > 0 \), and time-like when \( a_1 < 0 \). Moreover, any inextensible solution \( \psi \) is entire, but not complete (space-like, time-like, light-like).

**Proof:** We consider \( \psi(y, z) = (u(y, z), y, z) \), with \( u(y, z) = a_1 e^{-2y} - \frac{b_2^2}{2} y + b_1 z + b_0 \), for \( a_1, b_1, b_0 \in \mathbb{R} \). First, it is very clear the \( u \) can always be extended to the whole \( u : \mathbb{R}^2 \to \mathbb{R} \). We compute the coefficients of the first fundamental form:
\[
\begin{pmatrix}
E & F \\
F & G
\end{pmatrix} = \begin{pmatrix}
-2u_y & -u_z \\
-u_z & 1
\end{pmatrix} = \begin{pmatrix}
b_1^2 + 4a_1 e^{-2y} & -b_1 \\
-b_1 & 1
\end{pmatrix}.
\]

Clearly, the surface is space-like for \( a_1 > 0 \), namely the metric \( I \) is Riemannian. Vice versa, the surface is time-like when \( a_1 < 0 \). Let us consider \( \mathbb{R}^2 \) with the metric \( I = \psi^* \langle \cdot, \cdot \rangle \). We compute the Christoffel symbols:
\[
\Gamma^1_{11} = -1, \quad \Gamma^2_{11} = -b_1, \quad \Gamma^1_{12} = \Gamma^2_{12} = \Gamma^1_{22} = \Gamma^2_{22} = 0.
\]
The equations of a geodesic \( \alpha(t) = (y(t), z(t)) \) are
\[
\begin{align*}
0 &= y'' + \Gamma^1_{11} (y')^2 + 2\Gamma^1_{12} y' z' + \Gamma^1_{22} (z')^2 = y'' - (y')^2, \\
0 &= z'' + \Gamma^2_{11} (y')^2 + 2\Gamma^2_{12} y' z' + \Gamma^2_{22} (z')^2 = z'' - b_1 (y')^2.
\end{align*}
\]
It is simple to check that the divergent curve
\[
\alpha : [0, 1) \to \mathbb{R}^2, \quad \alpha(t) = (\log \epsilon (1 - t), b_1 \log \epsilon (1 - t))
\]
is a geodesic. By a simple computation, we obtain \( I_{\alpha(t)} (\alpha'(t), \alpha'(t)) = 4a_1 \). This implies that the length of \( \alpha \) is \( \text{length}(\alpha) = \int_0^1 |\alpha'(t)| dt = 2\sqrt{|a_1|} \). In either case \( (a_1 > 0 \text{ or } a_1 < 0) \), there exists a divergent geodesic with finite total length. This means that this geodesic is not complete. By repeating the argument of Corollary 3.3, this surface is not complete (space-like, time-like, light-like). □

**Example 3.1** Plenty of mean curvature flows in the Minkowski 2-plane are computed in [8]. In particular, there are essentially 3 translating solitons, travelling in space-like, time-like and light-like directions. These curves can be seen as the corresponding Grim Reaper curves in
L$^2$. Those in a light-like direction are written in our coordinates as $x = e^{-2y}/2$. If we consider the projection $\pi : L^3 \to L^2$, $\pi(x,y,z) = (x,y)$, the preimage by $\pi$ of each curve provides a translating soliton in a space-like, time-like and light-like direction, respectively. We will regard them as the Grim Reaper surfaces in $\mathbb{L}^3$. If we take $a_1 = 2$, $b_1 = 0$ in case II in Theorem 3.1, we obtain the corresponding Grim Reaper surface in a light-like direction, but written our way. □

We summarize these corollaries in the following table:

| Type | Entire | Causal Character | Completeness |
|------|--------|-----------------|--------------|
| I    | yes    | space-like      | yes          |
| II   | yes    | space-like if $a_1 > 0$ | no           |
|      | yes    | time-like if $a_1 < 0$ | no           |
| III  | no     | time-like       | no           |
| IV   | no     | time-like       | no           |

Table 1: Causal character and completeness.

4 The Group of Isometries Whose Axis Is Light-like

We use the following subgroup of direct, time-orientation preserving isometries

$$A_3 = \left\{ \xi_t = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2}t^2 & 1 & t \\ t & 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\}.$$ 

The action is given by $(x,y,z) \in \mathbb{L}^3$, $\xi_t \cdot (x,y,z) = (x,y,z)\xi_t$, with the usual matrix multiplication. We will need the following regions in $\mathbb{L}^3 \setminus \langle \vec{x} \rangle$:

$$S^+ = \{(x,y,z) \in \mathbb{L}^3 : y > 0\}, \quad S^- = \{(x,y,z) \in \mathbb{L}^3 : y < 0\},$$
$$S = \{(x,y,z) \in \mathbb{L}^3 : z = 0\},$$

and inside them, the following open half planes:

$$\tilde{S}^+ = S^+ \cap S = \{(x,y,0) \in \mathbb{L}^3 : y > 0\}, \quad \tilde{S}^- = S^- \cap S = \{(x,y,0) \in \mathbb{L}^3 : y < 0\}.$$

We recall the following result from [2].

**Theorem 4.1** Let $M$ be a connected surface and $\Phi : M \to \mathbb{L}^3$ a non-degenerate immersion. Then $(M, \Phi^*(g))$ is $A_3$-invariant if and only if one of the following statements holds:

1. If $(M, \Phi^*(g))$ is Riemannian, there exists a regular space-like curve $\alpha$, immersed in either $S^+$ or $S^-$, such that $\Phi(M) = \{\xi_t (\text{trace} (\alpha)) : t \in \mathbb{R}\}$.
2. If $(M, \Phi^*(g))$ is Lorentzian, there exists a regular time-like curve $\alpha$, immersed in either $\tilde{S}^+$ or $\tilde{S}^-$, such that $\Phi(M) = \{\xi_t (\text{trace} (\alpha)) : t \in \mathbb{R}\}$.
Now we take a regular curve in $S$, $\alpha : I \subseteq \mathbb{R} \rightarrow S \subset \mathbb{L}^3$, $\alpha(s) = (x(s), y(s), 0)$, and construct the $A_3$ invariant surface $\psi(s,t)$ as:

$$\psi : I \times \mathbb{R} \rightarrow \mathbb{L}^3, \quad \psi(s,t) = \left( x(s) + \frac{t^2}{2} y(s), y(s), ty(s) \right).$$

We compute the partial derivatives of $\psi(s,t)$,

$$\psi_s = \left( x'(s) + \frac{t^2}{2} y'(s), y'(s), ty'(s) \right), \quad \psi_t = (ty(s), 0, y(s)).$$

The matricial expression of the first fundamental form $I = \psi^* \langle \cdot, \cdot \rangle$ is

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} -2x'(s)y'(s) & 0 \\ 0 & y^2(s) \end{pmatrix}.$$

Since we assume that the induced metric is not degenerate, we obtain $x'(s)y'(s) \neq 0$. Then we define

$$\varepsilon = \text{sign} \left( x'(s)y'(s) \right) = \pm 1, \quad W = \sqrt{\frac{2\varepsilon x'(s)y'(s)}{y'(s)}} > 0,$$

and we construct the unit normal vector

$$N = \frac{1}{W} \left( \frac{t^2}{2} - \frac{x'(s)}{y'(s)}, 1, t \right).$$

Note that $\langle N, N \rangle = \varepsilon$. The second partial derivatives of $\psi$ are

$$\psi_{ss} = \left( x''(s) + \frac{t^2}{2} y''(s), y''(s), ty''(s) \right),$$

$$\psi_{st} = (ty'(s), 0, y'(s)), \quad \psi_{tt} = (y(s), 0, 0).$$

The coefficients of the second fundamental form are

$$e = \langle N, \psi_{ss} \rangle = \frac{x'y'' - x''y'}{Wy'}, \quad f = \langle N, \psi_{st} \rangle = 0, \quad g = \langle N, \psi_{tt} \rangle = -\frac{y}{W}.$$

Therefore

$$H = -\frac{y x' y'' + 2x' (y')^2 - y y' x''}{2W x y' (y')^2}.$$

Next, by Definition 1.1, with $\vec{K} = \vec{x}$ the chosen parallel light-like vector field, we compute:

$$\langle \vec{x}^1, N \rangle = \langle \vec{H}, N \rangle = \langle \varepsilon H N, N \rangle = H,$$

$$\langle \vec{x}^1, N \rangle = \langle (1, 0, 0), \frac{1}{W} \left( \frac{t^2}{2} - \frac{x'(s)}{y'(s)}, 1, t \right) \rangle = -\frac{1}{W}.$$

As a result, we obtain the following ODE:

$$y x' y'' + 2x' (y')^2 - y y' x'' = 2y x'(y')^2.$$

(9)
We want to express the profile curve $\alpha$ as a graph. Firstly, let us assume $y(s) = s$ and we examine $x(s)$. Then our equation transforms into

$$2x'(s) - s x''(s) = 2s x'(s).$$

A standard computation shows the general solution to this ODE,

$$x(s) = a_0 (2s^2 + 2s + 1) e^{-2s} + a_1, \; a_1, a_0 \in \mathbb{R}. $$

However, we have to discard the case $a_0 = 0$ because $x'(s) = 0$ for any $s$. Coming back, we see that $x'(s) = -4a_0 s^2 e^{-2s}$, so that

$$\varepsilon = \text{sign}(x'(s)y'(s)) = \text{sign}(-a_0).$$

Therefore, the normal $N$ is time-like when $\varepsilon = -1$, that is to say, when $a_0 > 0$. And it is space-like when $a_0 < 0$. In other words, $\psi$ is space-like when $a_0 > 0$, and time-like when $a_0 < 0$.

Now, we assume $x(s) = s$ and let us examine $y(s)$. Our ODE transforms into

$$y y'' + 2(y')^2 = 2y (y')^2.$$

Clearly, $y(s) = y_0 \in \mathbb{R}$ is a solution to this equation, but then we get $y'(s) = 0$, and we supposed that $x'(s)y'(s) \neq 0$. If we arrange the above equality, we get

$$\frac{y''(s)}{y'(s)} = 2 \left(1 - \frac{1}{y(s)}\right) y'(s).$$

By integrating both sides, we obtain $y^2(s)e^{-2y(s)}y'(s) = b_0, \; b_0 \in \mathbb{R}, \; b_0 \neq 0$. We define the function

$$\phi : \mathbb{R} \to \mathbb{R}, \; \phi(r) = \frac{-1}{4} (2r^2 + 2r + 1) e^{-2r}. $$

Note that $\phi'(r) = r^2 e^{-2r} \leq 0$. Moreover, $\phi'(r) = 0$ if, and only if, $r = 0$. Therefore, $\phi$ is injective. Next,

$$\lim_{r \to +\infty} \phi(r) = 0, \quad \lim_{r \to -\infty} \phi(r) = -\infty, \quad \phi(0) = -\frac{1}{4}.$$

This shows $\phi : \mathbb{R} \to (-\infty, 0)$. To have a well-defined surface, the profile curve cannot get out of $\mathcal{S}^+$ or $\mathcal{S}^-$, so that we need to exclude $-1/4$ from the interval $J$. Therefore, the solutions are

$$y : J \to \mathbb{R}, \; y(s) = \phi^{-1} (b_0 s + b_1),$$

where $J \subseteq (-\infty, -b_1/b_0) \backslash \{-1/4\}$, if $b_0 > 0$, or $J = (-b_1/b_0, +\infty) \backslash \{-1/4\}$, if $b_0 < 0$. Now,

$$\varepsilon = \text{sign}(x'(s)y'(s)) = \text{sign} \left( b_0 \frac{e^{2y(s)}}{y(s)^2} \right) = \text{sign}(b_0).$$

**Theorem 4.2** Let $\psi : I \times \mathbb{R} \to \mathbb{L}^3$, $\psi = \psi(s,t)$, be an $A_3$ invariant surface, such that it is a translating soliton on the parallel light-like direction $\mathbf{K}^3 = (1, 0, 0)$. Then the profile curve $\alpha : I \subset \mathbb{R} \to \mathbb{L}^3$, $\alpha(s) = (x(s), y(s), 0)$ of $\psi(s,t)$, is one of the following:

1. For $y(s) = s (\neq 0)$, given $a_0, a_1 \in \mathbb{R}, \; a_0 \neq 0, \; x(s) = a_0 (2s^2 + 2s + 1) e^{-2s} + a_1$. In addition, $\psi$ is space-like iff $a_0 > 0$, and time-like iff $a_0 < 0$. 

2. For $x(s) = s$, given $b_2, b_3 \in \mathbb{R}$, $b_2 \neq 0$, and the diffeomorphism $\phi : \mathbb{R} \to (0, +\infty)$, 
$\phi(r) = (2r^2 - 2r + 1)e^{-2r}$, 
$y : J \to \mathbb{R}$, $y(s) = \phi^{-1}(-4b_2s + b_3)$,

where the interval $J$ is included in either $J \subseteq (-\infty, -b_1/b_0) \setminus \{1\}$, if $b_0 > 0$, or $J \subseteq (-b_1/b_0, +\infty) \setminus \{1\}$, if $b_0 < 0$. In addition, $\phi$ is space-like iff $b_0 < 0$, and $\psi$ is time-like iff $b_0 > 0$.

Remark 4.1 When the surface approaches the affine plane $S$, there are singularities. Indeed, case 1), for each $t \in \mathbb{R}$, $\lim_{s \to 0} \psi(s, t) = (a_0 + a_1, 0, 0)$.

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