HECKE MODULES AND SUPERSINGULAR REPRESENTATIONS OF U(2,1)

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Abstract. Let $F$ be a nonarchimedean local field of odd residual characteristic $p$. We classify finite-dimensional simple right modules for the pro-$p$-Iwahori-Hecke algebra $\mathcal{H}_C(G, I(1))$, where $G$ is the unramified unitary group $U(2,1)(E/F)$ in three variables. Using this description when $C = \mathbb{F}_p$, we define supersingular Hecke modules and show that the functor of $I(1)$-invariants induces a bijection between irreducible nonsupersingular representations of $G$ and nonsupersingular simple right $\mathcal{H}_C(G, I(1))$-modules. We then use an argument of Paškūnas to construct supersingular representations of $G$.

1. Introduction

This article is set in the framework of the mod-$p$ representation theory of $p$-adic reductive groups. Our motivation comes from the possibility of a mod-$p$ Local Langlands Correspondence, that is to say a matching between (packets of) smooth mod-$p$ representations of a $p$-adic reductive group and certain Galois representations. The case of $GL_2(Q_p)$ has been most extensively studied, and a mod-$p$ Local Langlands Correspondence has been established by Breuil ([5]) based on the explicit determination of the irreducible smooth mod-$p$ representations of $GL_2(Q_p)$. Moreover, this correspondence is compatible with the $p$-adic Local Langlands Correspondence established by the work of several mathematicians: see [6], [7], [11], [14], [23], [24], [27], and the references therein. The case of $GL_2(F)$ with $F \neq Q_p$ is already much more complicated. For example, when $F$ is a nontrivial unramified extension of $Q_p$, Breuil and Paškūnas ([8]) have shown that there exists an infinite family of supersingular $GL_2(F)$-representations associated to a “generic” Galois representation.

Recently, Abdellatif has classified the irreducible smooth mod-$p$ representations of $SL_2(Q_p)$ (cf. [1]) by restricting the irreducible representations of $GL_2(Q_p)$, allowing her to take the first steps towards a mod-$p$ Local Langlands Correspondence for $SL_2(Q_p)$. In addition, the results of [1] are the first to consider a mod-$p$ Local Langlands Correspondence with $L$-packets. The explicit classification of mod-$p$ representations of $p$-adic reductive groups other than $GL_2(Q_p)$ and $SL_2(Q_p)$ is not yet known, however.

In the present article, we investigate the smooth mod-$p$ representations of the unitary group $G = U(2,1)(E/F)$, where $E/F$ is an unramified quadratic extension of nonarchimedean local fields, and where the residue field of $F$ is of size $q$, a power of $p$. The irreducible subquotients of parabolically induced representations have been classified by Abdellatif ([1]). We are interested in the smooth irreducible representations that do not appear in this fashion, which we call supersingular representations (we will comment on this terminology at the end of this introduction). These representations are the ones which are expected to play a crucial role in a potential Local Langlands Correspondence. The purpose of this article is to construct such representations. We now describe the ingredients in our method, inspired by the work of Vignéras and Paškūnas.

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Let $I(1)$ be the unique pro-$p$-Sylow subgroup of the standard Iwahori subgroup $I$ of $G$, and let $C$ denote an algebraically closed field. The pro-$p$-Iwahori-Hecke algebra $\mathcal{H}_C(G, I(1))$ is the convolution algebra of compactly supported, $C$-valued functions on the double coset space $I(1) \backslash G/I(1)$. Under a mild assumption on the characteristic of $C$, we determine explicitly the structure of the algebra $\mathcal{H}_C(G, I(1))$ and describe its center. This allows us to classify all simple finite-dimensional right modules of $\mathcal{H}_C(G, I(1))$ for any field $C$ satisfying Assumption 3.5 (Section 3).

The motivation for considering modules of the algebra $\mathcal{H}_C(G, I(1))$ comes from the following observation. Attaching to a smooth representation $π$ of $G$ its space of $I(1)$-invariants $π^{I(1)}$ yields a functor with values in the category of $\mathcal{H}_C(G, I(1))$-modules. If $C$ is of characteristic $p$, then $π^{I(1)}$ is nonzero provided $π$ is nonzero; this suggests that the functor of $I(1)$-invariants is likely to give information about representations generated by their $I(1)$-invariants (though in general, we don’t expect an equivalence of categories, given the $\text{GL}_2(F)$ case, $F \neq \mathbb{Q}_p$ (cf. [25])).

Using our explicit description of finite-dimensional simple $\mathcal{H}_{\mathbb{F}_p}(G, I(1))$ modules, we establish a bijection between irreducible smooth nonsupersingular representations of $G$ and certain simple modules (Corollary 4.4). In particular, we show that the simple $\mathcal{H}_{\mathbb{F}_p}(G, I(1))$-modules not arising in this fashion are precisely those with a “zero” central character. We call these modules supersingular (Definition 3.11), and note that they are all one-dimensional (cf. Definition 4.3). Our goal is to attach an irreducible smooth supersingular representation of $G$ to every supersingular $\mathcal{H}_{\mathbb{F}_p}(G, I(1))$-module. We achieve this goal by showing the existence of such representations, and construct them explicitly in the case $q = p$. The tool we will use is (homological) coefficient systems on the semisimple Bruhat-Tits building $X$ of $G$ (Section 6).

In [28], Schneider and Stuhler introduced coefficient systems on the Bruhat-Tits building and used them to study complex representations of $p$-adic reductive groups. Coefficient systems were later used in the mod-$p$ setting by Paškūnas to construct supersingular representations of $\text{GL}_2(F)$. The use of coefficient systems in this context has proved extremely useful (cf. [26]), but so far has only been considered for the group $\text{GL}_2(F)$. We adapt this method to representations of $G = \text{U}(2, 1)(E/F)$. To this end, we define an analog of Paškūnas’ diagrams, which are easier to handle than coefficient systems (Definition 6.4). In particular, the category of diagrams is equivalent to the category of $G$-equivariant coefficient systems on $X$ (this is the content of Section 9.2).

Next, we attach to every supersingular module $M$ a diagram $D_M$. The 0-homology of the corresponding coefficient system $\mathcal{D}_M$ is naturally a smooth $G$-representation, and we show that the $I(1)$-invariants of any of its nonzero irreducible quotients contain an $\mathcal{H}_{\mathbb{F}_p}(G, I(1))$-module isomorphic to $M$. This implies that any nonzero irreducible quotient is a supersingular representation of $G$ (Corollary 4.5). We then want to produce such irreducible quotients, and for this purpose we construct an auxiliary coefficient system $\mathcal{E}_M$ of a relatively simple form, along with a morphism $\mathcal{D}_M \rightarrow \mathcal{E}_M$. Constructing $\mathcal{E}_M$ involves analyzing injective envelopes of irreducible representations of the finite groups $\Gamma = \text{U}(2, 1)(\mathbb{F}_{q^2}/\mathbb{F}_q)$ and $\Gamma' = (\text{U}(1, 1) \times \text{U}(1))(\mathbb{F}_{q^2}/\mathbb{F}_q)$ associated to $G$.

In the course of constructing $\mathcal{E}_M$, it will become necessary to have several descriptions of the irreducible mod-$p$ representations of $\Gamma$ and $\Gamma'$, and their injective envelopes. In Section 5, we recall two parametrizations of these representations: one in terms of the simple modules of the respective finite Hecke algebras, based on the work of Carter and Lusztig ([10]), and another in terms of highest weight modules. We provide a dictionary between these two
descriptions, and prove a useful decomposition when \( q = p \), which is used in determining the decomposition of certain injective envelopes.

We next specialize to the case \( q = p \). In this setting we are able to construct explicitly an auxiliary coefficient system \( \mathcal{E}_M \) along with a morphism \( \mathcal{D}_M \to \mathcal{E}_M \) (Section 7). This morphism induces a map on the 0-homology of the coefficient systems, and we consider the representation afforded by the image

\[
\pi_{\mathcal{E}_M} = \text{im}(H_0(X, \mathcal{D}_M) \to H_0(X, \mathcal{E}_M)).
\]

The result here is the following:

**Theorem (Corollary 7.13).** Assume \( q = p \). The representation \( \pi_{\mathcal{E}_M} \) is nonzero, irreducible, admissible, and supersingular. For nonisomorphic supersingular \( \mathcal{H}_{\mathbb{F}_p}(G, I(1)) \)-modules \( M, M' \), the representations \( \pi_{\mathcal{E}_M}, \pi_{\mathcal{E}_M'} \) are nonisomorphic.

We remark that while \( \mathcal{D}_M \) is uniquely determined, the choice of the coefficient system \( \mathcal{E}_M \) is in general not unique. Therefore, to every supersingular module \( M \) we attach at least one supersingular representation; in this way, we construct at least \( p^2(p + 1) \) supersingular representations of \( G \).

We next address the shortcomings of our method when \( q \neq p \). As mentioned before, our method relies on the comparison of injective envelopes for representations of the finite groups \( \Gamma \) and \( \Gamma' \). For \( q \neq p \), we demonstrate cases where the construction of Section 7 would produce a coefficient system \( \mathcal{E}_M \) which is “too big,” in the sense that we cannot guarantee irreducibility of the resulting representation. Our main tool will be Dordowsky’s Diplomarbeit ([13]), in which the dimensions of injective envelopes of representations of \( \Gamma \) are computed.

To conclude, we draw some comparisons between our results and the analogous results for the group SL\(_2\)(\( F \)), drawing on the results of Abdellatif in [1]. We first use the explicit classification of Hecke modules to determine by elimination which are supersingular. As is the case for U(2, 1)(\( E/F \)), the action of SL\(_2\)(\( F \)) on its Bruhat-Tits tree \( X_S \) partitions the set of vertices into two disjoint orbits and acts transitively on the edges. Therefore, the results of Section 6 hold equally well for SL\(_2\)(\( F \)) (the proofs carry over formally). When \( q = p \), we attach to every supersingular \( \mathcal{H}_{\mathbb{F}_p}(SL_2(F), I_S(1)) \)-module \( M \) two coefficient systems \( \mathcal{D}_{MS} \) and \( \mathcal{E}_{MS} \). There is one striking difference between this case and the case of U(2, 1)(\( E/F \)), however: when \( q = p \), there is a canonical choice of auxiliary diagram \( \mathcal{E}_{MS} \). In this way, we construct \( p \) supersingular representations of SL\(_2\)(\( F \)): exactly one such representation for every supersingular \( \mathcal{H}_{\mathbb{F}_p}(SL_2(F), I_S(1)) \)-module. We record the result here.

**Theorem (Theorems 8.4 and 8.5).** Assume \( q = p \). For each of the \( p \) nonisomorphic supersingular \( \mathcal{H}_{\mathbb{F}_p}(SL_2(F), I_S(1)) \)-modules \( M \) there is a canonical pair of associated coefficient systems (\( \mathcal{D}_{MS}, \mathcal{E}_{MS} \)). The resulting SL\(_2\)(\( F \))-representation afforded by

\[
\pi_{MS} = \text{im}(H_0(X_S, \mathcal{D}_{MS}) \to H_0(X_S, \mathcal{E}_{MS}))
\]

is nonzero, irreducible, admissible, and supersingular. For nonisomorphic \( \mathcal{H}_{\mathbb{F}_p}(SL_2(F), I_S(1)) \)-modules \( M, M' \), the representations \( \pi_{MS}, \pi_{M'S} \) are nonisomorphic. In particular, when \( F = \mathbb{Q}_p \), we recover in this way all \( p \) nonisomorphic supersingular representations of SL\(_2\)(\( \mathbb{Q}_p \)) as classified in [1].

**Remark on Terminology.** We briefly address our choice of nomenclature. The notion of supersingularity was introduced by Barthel and Livné ([3] and [4]) in their classification of
smooth, irreducible, nonsupercuspidal mod-$p$ representations of $GL_2(F)$. A smooth representation of $GL_2(F)$ is called \textit{supersingular} if a certain operator of the spherical Hecke algebra acts by zero, while it is called \textit{supercuspidal} if it is not a subquotient of a parabolically induced representation. Theorems 33, 34, and Corollary 36(1) of [3] show that a smooth representation of $GL_2(F)$ admitting a central character is supercuspidal if and only if it is supersingular. In the present case, the study of spherical Hecke algebras for the group $U(2,1)(E/F)$ has been initiated by Henniart and Vignéras in [16], and Abdellatif in [1], with an analogous notion of supersingularity defined in [1]. The arguments in [3] and [4] may be extended to the group $U(2,1)(E/F)$; these results, combined with those of [1], would show the equivalence of the notions of supercuspidality and supersingularity. Based on results contained in [17] and [18], it is very likely that these two notions coincide. In anticipation of these results, we shall assume that this is the case. We use the term \textit{supersingular representation} for what might otherwise be referred to as a \textit{supercuspidal representation}, and henceforth refer only to supersingular representations.

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\textbf{2. Notation}

2.1. \textbf{General Notation.} Fix a prime number $p$ greater than 2, and let $F$ be a nonarchimedean local field of residual characteristic $p$. Denote by $\mathfrak{o}_F$ its ring of integers, and by $p_F$ the unique maximal ideal of $\mathfrak{o}_F$. Fix a uniformizer $\varpi_F$ and the normalized valuation $\nu$ given by $\nu(\varpi_F) = 1$. Let $k_F = \mathfrak{o}_F/p_F$ denote the (finite) residue field. The field $k_F$ is a finite extension of $\mathbb{F}_p$ of size $q = p^f$. We shall identify $k_F$ with $\mathbb{F}_q$ in a fixed algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$. We fix also a separable closure $\overline{F}$ of $F$, compatible with the chosen algebraic closure of the residue field, and let $E$ denote the unique unramified extension of degree 2 in $\overline{F}$. We write $E = F(\sqrt{\varpi})$, where $\varpi \in F$ is some fixed but arbitrary nonsquare unit. We let $x \mapsto \overline{x}$ denote the nontrivial Galois automorphism of $E$ fixing $F$. The ring of integers of $E$ is denoted $\mathfrak{o}_E$, and $p_E$ is its unique maximal ideal. Since $E$ is unramified, we may and do take $\varpi_E = \varpi_F =: \varpi$ as our uniformizer. The residue field of $\mathfrak{o}_E$ is denoted $k_E = \mathbb{F}_{q^2}$, and is a degree 2 extension of $k_F$.

Denote by $G$ the $F$-rational points of the algebraic group $U(2,1)$. We perform our computations using the following realization of $G$: let $V$ denote a three-dimensional vector space over $E$. We identify $V$ with $E^3$ by a choice of basis, and for $\overline{x} = (x_1, x_2, x_3)^\top$, $\overline{y} = (y_1, y_2, y_3)^\top \in V$ we define a nondegenerate Hermitian form $\langle \cdot, \cdot \rangle$ by

$$\langle \overline{x}, \overline{y} \rangle = \overline{x_1}y_3 + \overline{x_2}y_2 + \overline{x_3}y_1.$$
Letting 
\[ s = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \]
our form is represented by \( \langle \vec{x}, \vec{y} \rangle = \vec{x}^* s \vec{y} \), where \( m^* \) denotes the conjugate transpose of a matrix \( m \). With this notation, we have \( G = \{ g \in \text{GL}_3(E) : g^* s g = s \} \).

The group \( G \) possesses, up to conjugacy, two maximal compact subgroups (cf. [32], Sections 2.10 and 3.2), given by

\[ K := \text{GL}_3(\mathcal{O}_E) \cap G \quad \text{and} \quad K' := \begin{pmatrix} \mathcal{O}_E & \mathcal{O}_E & \mathcal{O}_E \\ \mathcal{O}_E & \mathcal{O}_E & \mathcal{O}_E \\ \mathcal{O}_E & \mathcal{O}_E & \mathcal{O}_E \end{pmatrix} \cap G. \]

Let \( K_1, K'_1 \) be the following subgroups of \( G \):

\[ K_1 := \begin{pmatrix} 1 + \mathcal{P}_E & \mathcal{P}_E & \mathcal{P}_E \\ \mathcal{P}_E & 1 + \mathcal{P}_E & \mathcal{P}_E \\ \mathcal{P}_E & \mathcal{P}_E & 1 + \mathcal{P}_E \end{pmatrix} \cap G, \quad K'_1 := \begin{pmatrix} 1 + \mathcal{P}_E & \mathcal{O}_E & \mathcal{O}_E \\ \mathcal{P}_E & 1 + \mathcal{P}_E & \mathcal{O}_E \\ \mathcal{P}_E & \mathcal{O}_E & 1 + \mathcal{P}_E \end{pmatrix} \cap G. \]

The group \( K_1 \) (resp. \( K'_1 \)) is the maximal normal pro-\( p \) subgroup of \( K \) (resp. \( K' \)). We define

\[ \Gamma := K/K_1 \cong U(2,1)(\mathbb{F}_{q^2}/\mathbb{F}_q), \quad \Gamma' := K'/K'_1 \cong (U(1,1) \times U(1))(\mathbb{F}_{q^2}/\mathbb{F}_q). \]

We let \( B \) denote the upper Borel subgroup of \( \Gamma \), \( U \) its unipotent radical, and \( U^- \) the opposite unipotent; let \( B' \) denote the lower Borel subgroup of \( \Gamma' \), \( U' \) its unipotent radical, and \( U'^- \) the opposite unipotent. The groups \( U \) and \( U' \) are \( p \)-Sylow subgroups of \( \Gamma \) and \( \Gamma' \), respectively. The intersection of \( K \) and \( K' \) is the Iwahori subgroup \( I \), which we may also think of as the preimage under the reduction-modulo-\( \varpi \) map of \( B \leq U(2,1)(\mathbb{F}_{q^2}/\mathbb{F}_q) \). We denote by \( I(1) \) the unique pro-\( p \)-Sylow subgroup of \( I \), which is the preimage of \( U \).

We define the following distinguished elements of \( G \):

\[ s := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad s' := \begin{pmatrix} 0 & 0 & \varpi^{-1} \\ 0 & 1 & 0 \\ \varpi & 0 & 0 \end{pmatrix}, \]

\[ n_s := \begin{pmatrix} 0 & 0 & \sqrt{\epsilon}^{-1} \\ 0 & 1 & 0 \\ \sqrt{\epsilon} & 0 & 0 \end{pmatrix}, \quad n_{s'} := \begin{pmatrix} 0 & 0 & -\varpi^{-1}\sqrt{\epsilon}^{-1} \\ 0 & 1 & 0 \\ \varpi\sqrt{\epsilon} & 0 & 0 \end{pmatrix}, \]

\[ \alpha := s's = \begin{pmatrix} \varpi^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varpi \end{pmatrix}, \quad \alpha^{-1} := ss' = \begin{pmatrix} \varpi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varpi^{-1} \end{pmatrix}. \]

2.2. Weyl Groups. The maximal torus \( T \) of \( G \) consists of all elements of the form 

\[ \begin{pmatrix} a & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & \overline{a}^{-1} \end{pmatrix}, \]
with \( a \in \mathbb{E}^\times, \delta \in \mathbf{U}(1)(E/F) \). Note that \( T \) is not split over \( F \). Let
\[
T_0 := T \cap K = T \cap K', \quad T_1 := T \cap K_1 = T \cap K_1',
\]
\[
H := T_0/T_1 \cong I/I(1) \cong \mathbb{F}_q^\times \times \mathbf{U}(1)(\mathbb{F}_q/\mathbb{F}_q).
\]
We will identify the characters of \( H \) and those of \( I/I(1) \). We will also identify \( \mathbb{F}_q^\times \) with the image of the Teichmüller lifting map \( [\cdot] : \mathbb{F}_q^\times \to \mathfrak{o}_E \) when convenient.

Let \( N \) denote the normalizer of \( T \) in \( G \). Then the affine Weyl group \( W_{\text{aff}} \) is defined as \( N/T_0 \), and the finite Weyl group \( W \) is defined as \( N/T \). The group \( W_{\text{aff}} \) is a Coxeter group, generated by the classes of the two reflections \( s \) and \( s' \). We have a decomposition \( G = INI \), where two cosets \( INI \) and \( In'I \) are equal if and only if \( n \) and \( n' \) have the same image in \( W_{\text{aff}} \). This yields the Bruhat decomposition for the BN pair \((I,N)\):
\[
G = \bigsqcup_{w \in W_{\text{aff}}} IwI;
\]
here we engage in the standard abuse of notation, letting \( IwI \) denote \( I\dot{w}I \) for any preimage \( \dot{w} \) of \( w \) in \( N \). We will take as our double coset representatives the elements \( \alpha^n, n_s \alpha^n \), for \( n \in \mathbb{Z} \).

We let \( \ell \) denote the length of an element of \( W_{\text{aff}} \), defined by
\[
q^{\ell(w)} = [IwI : I]
\]
(cf. Section 3.3.1 in [32]). In particular, we have \( \ell(n_s) = 3, \ell(n_{s'}) = 1 \).

Let \( U \) and \( U^- \) denote the upper and lower unipotent elements of \( G \), respectively, and define
\[
u(x,y) := \begin{pmatrix} 1 & x & y \\ 0 & 1 & -x \\ 0 & 0 & 1 \end{pmatrix}, \quad \nu^-(x,y) := \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & -x & 1 \end{pmatrix},
\]
where \( x, y \in E \) are such that \( x\bar{x} + y + \bar{y} = 0 \). We have \( \nu(x,y)^{-1} = \nu(-x, \bar{y}), \nu^-(x,y)^{-1} = \nu^-(x, \bar{y}) \).

### 3. Hecke Algebras

3.1. Preliminaries. Let \( C \) denote an algebraically closed field. We shall be interested in the category \( \mathcal{R}\mathcal{E}\mathcal{P}_C(G) \) of smooth representations of \( G \) over \( C \). We briefly recall some preliminary results. Let \( J \) be a closed subgroup of \( G \), and let \( (\sigma, V) \) be a smooth \( C \)-representation of \( J \) (meaning that stabilizers are open). We denote by \( \text{ind}_J^G(\sigma) \) the space of functions \( f : G \to V \) such that \( f(jg) = \sigma(j)f(g) \) for \( j \in J, g \in G \), and such that the action of \( G \) given by right translation is smooth (meaning that there exists some open subgroup \( J' \), depending on \( f \), such that \( f(gj') = f(g) \) for every \( j' \in J', g \in G \)). We let \( \text{c-ind}_J^G(\sigma) \) denote the subspace of \( \text{ind}_J^G(\sigma) \) spanned by functions whose support in \( J \setminus G \) is compact. These functors are called induction and compact induction, respectively. We will mostly be concerned with the cases when \( J \) is a parabolic subgroup of \( G \), or when \( J \) is a compact open subgroup.

3.2. Pro-\( p \)-Iwahori-Hecke Algebra. Let \( \pi \) be a smooth \( C \)-representation of \( G \). Frobenius Reciprocity for compact induction gives
\[
\pi^{I(1)} \cong \text{Hom}_{I(1)}(1, \pi|_{I(1)}) \cong \text{Hom}_G(\text{c-ind}_{I(1)}^G(1), \pi),
\]
where $I$ denotes the trivial character of $I(1)$. The pro-$p$-Iwahori-Hecke algebra $\mathcal{H}_C(G, I(1)) = \text{End}_G(\text{c-ind}_{I(1)}^G(I))$ is the algebra of $G$-equivariant endomorphisms of the universal module $\text{c-ind}_{I(1)}^G(I)$. This algebra has a natural right action on $\text{Hom}_G(\text{c-ind}_{I(1)}^G(I), \pi)$ by precomposition, which induces a right action on $\pi^{I(1)}$. In this way, we obtain the functor of $I(1)$-invariants, $\pi \mapsto \pi^{I(1)}$, from the category of smooth $C$-representations of $G$ to the category of right $\mathcal{H}_C(G, I(1))$-modules.

By adjunction, we have a natural identification

$$\mathcal{H}_C(G, I(1)) = \text{End}_G(\text{c-ind}_{I(1)}^G(I)) \cong \text{Hom}_{I(1)}(1, \text{c-ind}_{I(1)}^G(I)|_{I(1)}) \cong \text{c-ind}_{I(1)}^G(1)^{I(1)},$$

so we may view endomorphisms of $\text{c-ind}_{I(1)}^G(I)$ as compactly supported functions on $G$ which are $I(1)$-biinvariant. This leads to the following definition.

**Definition 3.1.** Let $g \in G$. We let $T_g \in \mathcal{H}_C(G, I(1))$ denote the endomorphism of $\text{c-ind}_{I(1)}^G(I)$ corresponding by adjunction to the characteristic function of $I(1)gI(1)$; in particular, $T_g$ maps the characteristic function of $I(1)$ to the characteristic function of $I(1)gI(1)$.

From this definition it is clear that $T_g = T_{g'}$ if and only if $I(1)gI(1) = I(1)g'I(1)$; moreover, since $W_{\text{aff}} = N/T_0$ is a set of representatives for the double coset space $G/I$, the group $N/T_1$ gives a set of representatives for $I(1)\backslash G/I(1)$. We therefore only consider the operators $T_n$, where $n$ is a representative of a coset in $N/T_1$. These operators give a basis for $\mathcal{H}_C(G, I(1))$ as a vector space over $C$.

**Lemma 3.2.** Let $n \in N$. If $\pi$ is a smooth $C$-representation of $G$ and $v \in \pi^{I(1)}$, then

$$v \cdot T_n = \sum_{u \in I(1) \backslash I(1)nI(1)} u^{-1}v = \sum_{u \in I(1) \backslash I(1)c^{-1}n^{-1}I(1)n} un^{-1}v.$$

**Proof.** See [3], Proposition 6. □

**Lemma 3.3.** Let $n, n' \in N$, and assume that $n$ normalizes $I(1)$. We then have $T_nT_{n'} = T_{nn'}, T_nT_n = T_{n'n}$.

**Proof.** This follows readily from the definition of $T_n$. □

### 3.3. Decomposition of the pro-$p$-Iwahori-Hecke Algebra

Let $\widehat{H}$ denote the group of all $C^\times$-valued characters of $H = T_0/T_1$, and let $\chi : H \to C^\times$ be an element of $\widehat{H}$. We define $\zeta : \mathbb{F}_q^\times \to C^\times$ and $\eta : U(1)(\mathbb{F}_q^2/\mathbb{F}_q) \to C^\times$ by

$$\zeta(a) = \chi \left( \begin{array}{ccc} a & 0 & 0 \\ 0 & \overline{a}a^{-1} & 0 \\ 0 & 0 & \overline{a}^{-1} \end{array} \right), \quad \eta(\delta) = \chi \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & 1 \end{array} \right),$$

where $a \in \mathbb{F}_q^\times$, $\delta \in U(1)(\mathbb{F}_q^2/\mathbb{F}_q)$. We stress that the characters $\zeta$ and $\eta$ depend on $\chi$, though we will suppress this dependence from our notation, and write $\chi = \zeta \otimes \eta$ when convenient. The finite Weyl group $W$ acts on the characters $\chi$ by conjugation; we denote by $\chi^a$ the character $\chi^a : h \mapsto \chi(n^{-1}hn)$, where $h \in H$ and $n \in N \setminus T$.

**Definition 3.4.** Let $\chi : H \to C^\times$ be a character. We say $\chi$ is of trivial Iwahori type if $\chi$ factors through the determinant, $\chi$ is hybrid if $\chi^a = \chi$, but $\chi$ does not factor through the determinant, and $\chi$ is regular if $\chi^a \neq \chi$. 
Note that \( \chi = \zeta \otimes \eta \) factors through the determinant if and only if \( \zeta \) is trivial, and \( \chi^s = \chi \) if and only if \( \zeta^{q+1} \) is trivial. For a character \( \chi \), we let \( \gamma_\chi \) denote the representation of \( H \) defined by \( \gamma_\chi = \chi \) if \( \chi^s = \chi \), and \( \gamma_\chi = \chi \oplus \chi^s \) otherwise.

From this point onwards, we make the following technical assumption:

**Assumption 3.5.** The integers \( \text{char}(C) \) and \( |H| \) are relatively prime.

With this hypothesis, we will now decompose \( H_C(G, I(1)) \) into blocks indexed by \( W \)-orbits of \( C \)-characters of \( H \). For \( h \in H \), we define \( T_h \in H_C(G, I(1)) \) to be the operator \( T_{t_0} \), for any preimage \( t_0 \) of \( h \) in \( T_0 \).

**Definition 3.6.** For a \( C \)-character \( \chi \) of \( H \), we define 
\[
e_\chi = |H|^{-1} \sum_{h \in H} \chi(h)T_h,
\]
\[
e_{\gamma_\chi} = \begin{cases} 
e_\chi & \text{if } \chi^s = \chi, \\
\ne_\chi + \ne_{\chi^s} & \text{if } \chi^s \neq \chi. \end{cases}
\]

The operators \( e_\chi \) have the following properties:
- \( e_\chi e_\chi = e_\chi \),
- \( e_\chi e_{\chi'} = 0 \) for \( \chi \neq \chi' \),
- \( \text{id}_{c\text{-ind}_{I(1)}^G} = \sum_{\chi \in \hat{H}} e_\chi \).

These follow readily from the orthogonality relations of characters. Applying these relations to \( \pi_{I(1)} \) gives the following Lemma.

**Lemma 3.7.** Let \( \pi \) be a smooth \( C \)-representation of \( G \). Then \( (\pi_{I(1)}) \cdot e_\chi = \pi_{I(1)} \chi \), and \( \pi_{I(1)} \cong \bigoplus_{\chi \in \hat{H}} (\pi_{I(1)}) \cdot e_\chi = \bigoplus_{\chi \in \hat{H}} \pi_{I(1)} \chi \). Here \( \pi_{I(1)} \chi = \{ v \in \pi : i.v = \chi(i)v \text{ for every } i \in I \} \) is the \( \chi \)-isotypic subspace of \( \pi \).

**Proof.** Since \( I(1) \) is normal in \( I \) and \( I/I(1) \cong H \) is abelian and of order prime to \( \text{char}(C) \), the action of \( I \) on \( \pi_{I(1)} \) is semisimple and decomposes as a sum of characters. Since (lifts of) elements of \( H \) normalize \( I(1) \), Lemma 3.2 implies that \( (\pi_{I(1)}) \cdot e_\chi = \pi_{I(1)} \chi \). The orthogonality properties above imply the direct sum decomposition. \( \square \)

We now use the idempotents \( e_{\gamma_\chi} \) to decompose the algebra \( H_C(G, I(1)) \). Denote by \( H_C(G, \gamma_\chi) \) the algebra \( \text{End}_C(c\text{-ind}_I^G(\gamma_\chi)) \). Using this notation, we obtain the following Proposition:

**Proposition 3.8.** There is an isomorphism of \( C \)-algebras 
\[
H_C(G, I(1)) \cong \bigoplus_{\gamma_\chi} H_C(G, \gamma_\chi) \cong \bigoplus_{\gamma_\chi} H_C(G, I(1))e_{\gamma_\chi},
\]
the sums taken over all \( W \)-orbits of \( C \)-characters of \( H \).

**Proof.** Assumption 3.5 guarantees that the regular representation of \( I/I(1) \) is semisimple. Using this fact, the proof is nearly identical to that in [33], Proposition 3.1. \( \square \)

**Proposition 3.9.** The operators \( T_{n_s}, T_{n_s'}, \) and \( e_\chi \) for all \( C \)-characters \( \chi \) generate \( H_C(G, I(1)) \) as an algebra.
Proof. We first claim that $e_\chi(c\text{-ind}_{I(1)}^G(1)) \cong c\text{-ind}_{I(1)}^G(\chi)$. Indeed, since the characteristic function of $I(1)$ generates $c\text{-ind}_{I(1)}^G(1)$ as a $G$-representation, its image under $e_\chi$ will generate $e_\chi(c\text{-ind}_{I(1)}^G(1))$. Denote this image by $\varphi_\chi$. By definition of the operators $e_\chi$, we have $\text{supp}(\varphi_\chi) = I$ and $\varphi_\chi(h) = |H|^{-1}\chi(h)$ for $h \in I$ (via the isomorphism $H \cong I/I(1)$). The action of $G$ on $\varphi_\chi$ shows that the representation it generates is canonically isomorphic to $c\text{-ind}_{I(1)}^G(\chi)$.

Let $\mathcal{M}$ be the subalgebra of $\mathcal{H}_C(G, I(1))$ generated by $T_n$, $T_{n'}$, and the operators $e_\chi$ for every $\chi \in \hat{H}$. Using the decomposition of Proposition 3.8, we have that $\mathcal{M}e_{\gamma_\chi}$ is a subalgebra of $\mathcal{H}_C(G, \gamma_\chi)$. Assume first that $\chi^s = \chi$. The claim above shows that $T_n e_\chi$ and $T_{n'} e_\chi$ are elements of $\mathcal{H}_C(G, \chi)$, and Propositions 3.12 and 3.17 imply that these elements generate $\mathcal{H}_C(G, \chi)$. We therefore have $\mathcal{M}e_{\gamma_\chi} \cong \mathcal{H}_C(G, \chi)$.

Assume now that $\chi^s \neq \chi$. The claim above shows that $T_n e_{\gamma_\chi}$ and $T_{n'} e_{\gamma_\chi}$ are elements of $\mathcal{H}_C(G, \gamma_\chi)$. The algebra $\mathcal{M}e_{\gamma_\chi}$ also contains the elements $e_{\chi^s}$ and $e_{\chi^s}$, which implies that each of the elements $T_n e_{\gamma_\chi}$, $T_{n'} e_{\gamma_\chi}$, $T_n e_{\chi^s}$, $T_{n'} e_{\chi^s}$ are contained in $\mathcal{M}e_{\gamma_\chi}$. Propositions 3.23 and 3.25 show that these elements generate $\mathcal{H}_C(G, \gamma_\chi)$, so that $\mathcal{M}e_{\gamma_\chi} \cong \mathcal{H}_C(G, \gamma_\chi)$.

Combining these results with the decomposition of Proposition 3.8 shows that $\mathcal{M} = \mathcal{H}_C(G, I(1))$.

\[\square\]

**Theorem 3.10.** The algebra $\mathcal{H}_C(G, I(1))$ is a noncommutative algebra, generated by the elements $T_n$, $T_{n'}$, and $e_\chi$ for every $\chi \in \hat{H}$, subject to the following relations:

(i) 
\[
T_n e_\chi = e_\chi T_n, \quad T_{n'} e_\chi = e_\chi T_{n'}, \quad e_\chi e_{\chi'} = \begin{cases} e_\chi & \text{if } \chi' = \chi, \\ 0 & \text{if } \chi' \neq \chi. \end{cases}
\]

(ii) If $\chi$ factorizes through the determinant, then
\[
T_n^2 e_\chi = (q^3 - 1)T_n e_\chi + q^3 e_\chi, \quad T_{n'}^2 e_\chi = (q - 1)T_{n'} e_\chi + q e_\chi.
\]

If $\chi^s = \chi$, but $\chi$ does not factorize through the determinant, then
\[
T_n^2 e_\chi = (q - q^2)T_n e_\chi + q^3 e_\chi, \quad T_{n'}^2 e_\chi = (q - 1)T_{n'} e_\chi + q e_\chi.
\]

If $\chi^s \neq \chi = \zeta \otimes \eta$, then
\[
T_n^2 e_\gamma_\chi = \zeta(-1)q^3 e_\chi, \quad T_{n'}^2 e_\gamma_\chi = \zeta(-1)q e_\chi.
\]

(iii) The center $\mathcal{Z}$ of $\mathcal{H}_C(G, I(1))$ is generated by the idempotents $e_{\gamma_\chi}$, and the elements
\[
\begin{cases}
(T_n(T_{n'} - (q - 1)) + T_{n'}(T_n - (q^3 - 1)) + 1)e_\chi & \text{for } \chi = \eta \circ \det, \\
(T_n(T_{n'} - (q - 1)) + T_{n'}(T_n - (q - q^2)))e_\chi & \text{for } \chi^s = \chi, \text{ but } \chi \neq \eta \circ \det, \\
\zeta(-1)(T_{n'} T_n e_\chi + T_n T_{n'} e_{\chi^s}) & \text{for } \chi^s \neq \chi = \zeta \otimes \eta.
\end{cases}
\]

\[\text{Proof.} \text{ Part (i) follows directly from the definitions and Lemma 3.3. To prove part (ii), we may either appeal to Propositions 3.12, 3.17, and 3.25 below, or note that these results are a special case of [10], Proposition 3.18. One simply needs to use the fact that } T_n^2 e_\chi \text{ (resp. } T_{n'}^2 e_\chi) \text{ maps the characteristic function of } I(1) \text{ to a function with support contained in } K.\]
(resp. \(K'\)) and reduce the computations to those in the respective finite groups, as in [26], Lemma 2.11. Part (iii) follows from Propositions 3.12, 3.17, and Corollary 3.26.

**Remark.** Let \(h_s: \mathbb{F}_q^\times \to H\) be the homomorphism defined by

\[
h_s(y) = \begin{pmatrix} y & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & y^{-1} \end{pmatrix},
\]

and set

\[
\tau_s := (q + 1) \sum_{y \in \mathbb{F}_q^\times} T_{h_s(y)} - q \sum_{y \in \mathbb{F}_q^\times} T_{h_s(y)}, \quad \tau_{s'} := \sum_{y \in \mathbb{F}_q^\times} T_{h_s(y)}.
\]

These elements satisfy the relation \(\tau_s \tau_{s'} = \tau_{s'} \tau_s = (q - 1)\tau_s\). Using Fourier inversion and the theorem above, the quadratic relations take the form

\[
T_{n_s}^2 = T_{n_s} \tau_s + q^3 T_{h_s(-1)}
\]

\[
T_{n_{s'}}^2 = T_{n_{s'}} \tau_{s'} + q T_{h_s(-1)}.
\]

Moreover, we see that the center \(Z\) of \(H_{C}(G, I(1))\) is generated by the central idempotents \(e_{\gamma_\chi}\) and the elements

\[
T_{n_s} T_{n_s} \vartheta_1 + T_{n_s} T_{n_s} \vartheta_2 - T_{n_s} \tau_s - T_{n_{s'}} \tau_{s'} + (q - 1)\tau_s,
\]

\[
T_{n_{s'}} T_{n_s} \vartheta_2 + T_{n_s} T_{n_{s'}} \vartheta_1 - T_{n_{s'}} \tau_{s'} - T_{n_s} \tau_s + (q - 1)\tau_s.
\]

Here

\[
\vartheta_1 = \sum_{\chi^2 = \chi} e_\chi + 2 \sum_{\chi^2 \neq \chi} e_\chi, \quad \vartheta_2 = \sum_{\chi^2 = \chi} e_\chi + 2 \sum_{\chi^2 \neq \chi} e_\chi,
\]

where the sums are taken over \(W\)-orbits of \(C\)-characters, such that \(\vartheta_1 + \vartheta_2 = 2 \cdot \text{id}_{\text{c-ind}^G_{C}(I(1))}\).

In light of Theorem 3.10, we make the following definition:

**Definition 3.11.** Assume \(\text{char}(C) = p\), and let \(M\) be a nonzero simple right \(H_{C}(G, I(1))\)-module which admits a central character. We say \(M\) is supersingular if every generator of the center \(Z\) (as given in Theorem 3.10) which is not a central idempotent \(e_{\gamma_\chi}\), acts by 0.

In the subsequent sections, we describe the structures of the Hecke algebras \(H_{C}(G, \gamma_\chi)\). From the descriptions of these blocks, we obtain Proposition 3.9 and Theorem 3.10, and identify the supersingular modules of \(H_{C}(G, I(1))\) when \(\text{char}(C) = p\).

3.4. **The Trivial Case.** We first assume that \(\chi\) is “trivial,” meaning \(\chi\) factors through the determinant and \(\chi = \eta \circ \det\), for \(\eta\) a character of \(U(1)(\mathbb{F}_q^2/\mathbb{F}_q)\). In this case, we have \(H_{C}(G, \chi) \cong H_{C}(G, 1) = \text{End}_G(\text{c-ind}^G_I(1))\); this equivalence is induced by the isomorphism \(\text{c-ind}^G_I(\chi) \cong \eta \circ \det \otimes \text{c-ind}^G_I(1)\) and Frobenius Reciprocity.

Let \(1_I \in \text{c-ind}^G_I(\chi)\) denote the function with support in \(I\), taking the value 1 at the identity. We let \(T_{n_s}\) (resp. \(T_{n_{s'}}\)) denote the endomorphism of \(\text{c-ind}^G_I(\chi)\) sending \(1_I\) to the function with support \(I n_s I\) (resp. \(I n_{s'} I\)), taking the value 1 at \(n_s\) (resp. \(n_{s'}\)), on which \(I\) acts by \(\chi\). In the notation of the previous subsection, we have

\[
T_{n_s} = T_{n_s} e_\chi, \quad T_{n_{s'}} = T_{n_{s'}} e_\chi.
\]

We now arrive at the following result on the structure of \(H_{C}(G, \chi)\):
Proposition 3.12. The algebra \( \mathcal{H}_C(G, \chi) \) is a noncommutative algebra, generated by \( \mathcal{T}_{n_s} \) and \( \mathcal{T}_{n_{s'}} \), subject to the relations

\[
(\mathcal{T}_{n_s}+1)(\mathcal{T}_{n_s}-q^3) = 0 \\
(\mathcal{T}_{n_{s'}}+1)(\mathcal{T}_{n_{s'}}-q) = 0.
\]

The center \( \mathcal{Z}_\chi \) is generated by \( Z = \mathcal{T}_{n_s}(\mathcal{T}_{n_s'}-(q-1)) + \mathcal{T}_{n_{s'}}(\mathcal{T}_{n_s}-(q^3-1))+1 \). We have an isomorphism

\[ \mathcal{H}_C(G, \chi) \cong C\langle X,Y \rangle/(X^2 + (1-q^3)X - q^3, Y^2 + (1-q)Y - q), \]

sending \( \mathcal{T}_{n_s} \) to \( X \) and \( \mathcal{T}_{n_{s'}} \) to \( Y \). Here \( C\langle X,Y \rangle \) denotes the noncommutative polynomial algebra in two variables over \( C \).

Remark. Note that using the length function on \( W_{aff} \), the Hecke relations take the simple form \((\mathcal{T}_n+1)(\mathcal{T}_n-q^\ell(n)) = 0\), where \( n = n_s \) or \( n_{s'} \).

Proof. The verification of the Proposition is included in the proof of Proposition 3.17 below. \( \square \)

Given this result, we can quickly classify the finite-dimensional simple right \( \mathcal{H}_C(G, \chi) \)-modules.

Definition 3.13. (i) Let \((\theta, \theta') \in \{-1, q^3\} \times \{-1, q\}\). We define the characters \( \mu_{\theta, \theta'} : \mathcal{H}_C(G, \chi) \to C \) by

\[
\mathcal{T}_{n_s} \mapsto \theta, \quad \mathcal{T}_{n_{s'}} \mapsto \theta'.
\]

The central element \( Z \) maps to \( \theta(\theta' - q + 1) + \theta'(\theta - q^3 + 1) + 1 \).

(ii) Let \( \langle v_1, v_2 \rangle_C \) be a two-dimensional vector space over \( C \), and let \( \lambda \in C \). We define \( M(\lambda) \) to be the following right \( \mathcal{H}_C(G, \chi) \)-module:

\[
v_1 \cdot \mathcal{T}_{n_s} = -v_1, \quad v_1 \cdot \mathcal{T}_{n_{s'}} = v_2 \\
v_2 \cdot \mathcal{T}_{n_s} = (\lambda - q)v_1 + q^3v_2, \quad v_2 \cdot \mathcal{T}_{n_{s'}} = qv_1 + (q-1)v_2
\]

The central element \( Z \) acts by \( \lambda \).

One may check directly that the action of \( \mathcal{H}_C(G, \chi) \) on \( M(\lambda) \) is well-defined. This fact will also be made clear in the proof of Theorem 3.16.

Proposition 3.14. Assume \( q^3 + 1 \neq 0 \) in \( C \). Then the module \( M(\lambda) \) is reducible if and only if \( \lambda = q^3 + q + 1 \) or \( \lambda = -q^4 \). In these cases, we have the following exact sequences:

\[
0 \to \mu_{q^3,q} \to M(q^3 + q + 1) \to \mu_{-1,-1} \to 0 \\
0 \to \mu_{q^3,-1} \to M(-q^4) \to \mu_{-1,q} \to 0
\]

The sequences are not split.

Proof. Assume that \( M(\lambda) \) is reducible, so that we have some character \( \mu \subset M(\lambda) \). This means exactly that there is some vector \( v \in M(\lambda) \) which is a common eigenvector for \( \mathcal{T}_{n_s} \) and \( \mathcal{T}_{n_{s'}} \). The eigenvectors for \( \mathcal{T}_{n_s} \) are \((\lambda - q)v_1 + (q^3 + 1)v_2 \) and \( v_1 \), with eigenvalues \( q^3 \) and \(-1\), respectively; the eigenvectors for \( \mathcal{T}_{n_{s'}} \) are \( v_1 + v_2 \) and \(-qv_1 + v_2 \), with eigenvalues \( q \) and \(-1\), respectively.

We see that the only possibility for a common eigenvector is if \((\lambda - q)v_1 + (q^3 + 1)v_2 \) is a scalar multiple of \( v_1 + v_2 \) or of \(-qv_1 + v_2 \). Assume the former. We then have \( \lambda - q = q^3 + 1 \), implying \( \lambda = q^3 + q + 1 \). Thus,

\[
\langle v_1 + v_2 \rangle_C \cong \mu_{q^3,q} \subset M(q^3 + q + 1) \quad \text{and} \quad M(q^3 + q + 1)/\mu_{q^3,q} \cong \mu_{-1,-1}.
\]
If the surjection split, then there would exist a $-1$-eigenvector in $M(q^3 + q + 1)$ for both $\mathcal{T}_{n_s}$ and $\mathcal{T}_{n_s'}$, which clearly cannot happen.

Assume now that $(\lambda - q)v_1 + (q^3 + 1)v_2$ is a scalar multiple of $-qv_1 + v_2$. We then have $\lambda - q = -q^4 - q$, implying that $\lambda = -q^4$. Thus, we have

$$\langle -qv_1 + v_2 \rangle_C \cong \mu_{q^4, -1} \subset M(-q^4) \quad \text{and} \quad M(-q^4)/\mu_{q^4, -1} \cong \mu_{-1,q}.$$ 

By the same reasoning as before, the surjection cannot split. \hfill $\square$

**Proposition 3.15.** Assume $q^3 + 1 = 0$ in $C$. Then the module $M(\lambda)$ is reducible if and only if $\lambda = q$. In this case the module decomposes as $M(\lambda) \cong \mu_{-1,q} \oplus \mu_{-1, -1}$.

**Proof.** Assume that $M(\lambda)$ is reducible, so that it contains either $\mu_{-1, -1}$ or $\mu_{-1,q}$. In either case, the central element $\mathcal{T}_{n_s}((\mathcal{T}_{n_s} - (q - 1)) + \mathcal{T}_{n_s'}(\mathcal{T}_{n_s} + 2) + 1$ acts by $q$, so we must have $\lambda = q$. The action of $\mathcal{T}_{n_s}$ and $\mathcal{T}_{n_s'}$ on $M(q)$ shows that

$$\langle v_1 + v_1 \rangle_C \cong \mu_{-1,q} \quad \text{and} \quad \langle -qv_1 + v_2 \rangle_C \cong \mu_{-1, -1},$$

so that

$$M(q) = \mu_{-1,q} \oplus \mu_{-1, -1}. \hfill \square$$

We now imitate the proof of Theorem 1.2 in [33] to classify simple right $\mathcal{H}_C(G, \chi)$-modules.

**Theorem 3.16.** Every finite-dimensional simple right $\mathcal{H}_C(G, \chi)$-module is either a character $\mu_{\theta, \theta'}$, $(\theta, \theta') \in \{-1, q^3\} \times \{-1, q\}$, or a module of the form $M(\lambda), \lambda \neq q^3 + q + 1, -q^4$.

**Proof.** Assume $M$ is a nonzero simple right module which is not a character, and assume that $Z$ acts by $\lambda$. Consider the space $\ker(\mathcal{T}_{n_s} + 1)$. We claim that this is a nontrivial proper subspace of $M$. Indeed, if $\ker(\mathcal{T}_{n_s} + 1) = \{0\}$ or $M$, the element $\mathcal{T}_{n_s}$ would act by a scalar, and any nonzero eigenvector for $\mathcal{T}_{n_s'}$ would generate a one-dimensional submodule. This gives a contradiction, since $M$ was assumed simple of dimension greater than 1.

The element $\mathcal{T}_{n_s}(\mathcal{T}_{n_s} - q^3)$ maps $\ker(\mathcal{T}_{n_s} + 1)$ into itself, and therefore has an eigenvector $v$ in $\ker(\mathcal{T}_{n_s} + 1)$. We have

$$v \cdot \mathcal{T}_{n_s}(\mathcal{T}_{n_s} - q^3) = v \cdot (Z - (\mathcal{T}_{n_s} + 1)\mathcal{T}_{n_s'} + (q - 1)\mathcal{T}_{n_s} - 1) = \lambda v + (q - 1)v \cdot \mathcal{T}_{n_s} - v = (\lambda - q)v.$$

Consider now the subspace $V := \langle v \rangle_C + \langle v \cdot \mathcal{T}_{n_s}' \rangle_C$. The quadratic relations and the computation above show that $V$ is stable under $\mathcal{H}_C(G, \chi)$, and therefore must be all of $M$ by simplicity. Moreover, since $M$ was assumed to be of dimension greater than one, we have $v \cdot \mathcal{T}_{n_s'} \neq 0$, and the sum $\langle v \rangle_C + \langle v \cdot \mathcal{T}_{n_s}' \rangle_C$ is direct. Writing out the actions of $\mathcal{T}_{n_s}$ and $\mathcal{T}_{n_s'}$ on the basis $\{v, v \cdot \mathcal{T}_{n_s}'\}$ shows that $M \cong M(\lambda)$. We again use simplicity of $M$ to deduce that $\lambda \neq q^3 + q + 1, -q^4$. \hfill $\square$

**3.5. The Hybrid Case.** We now assume that $\chi^4 = \chi = \zeta \otimes \eta$, but that $\chi$ does not factor through the determinant. This condition implies that the character $\zeta$ is nontrivial. In addition, we have $\zeta(a) = \zeta(\bar{a}^{-1})$; since the map $a \mapsto a^{q+1}$ maps $\mathbb{F}_{q^2}^*$ onto $\mathbb{F}_q^*$, this implies $\zeta$ is trivial on $\mathbb{F}_q^*$.

As before, we let $1_I \in \text{c-ind}^G_I(\chi)$ denote the function with support in $I$, taking the value 1 at the identity. We let $\mathcal{T}_{n_s}$ (resp. $\mathcal{T}_{n_s'}$) denote the endomorphism of $\text{c-ind}^G_I(\chi)$ sending $1_I$ to
the function with support $I_{n_s}I$ (resp. $I_{n_s'I}$), taking the value 1 at $n_s$ (resp. $n_{s'}$), on which $I$ acts by $\chi$. In the notation of Section 3.3, we have

$$T_{n_s} = T_{n_s}e_\chi, \quad T_{n_{s'}} = T_{n_{s'}}e_\chi.$$  

**Proposition 3.17.** The algebra $\mathcal{H}_C(G, \chi)$ is a noncommutative algebra, generated by $T_{n_s}$ and $T_{n_{s'}}$, subject to the relations

$$(T_{n_s} + q^2)(T_{n_s} - q) = 0$$

$$(T_{n_{s'}} + 1)(T_{n_{s'}} - q) = 0.$$  

The center $Z_\chi$ is generated by $Z = T_{n_s}(T_{n_{s'}} - (q - 1)) + T_{n_{s'}}(T_{n_s} - (q - q^2))$. We have an isomorphism of algebras

$$\mathcal{H}_C(G, \chi) \cong C\langle X, Y \rangle / (X^2 + (q^2 - q)X - q^2, Y^2 + (1 - q)Y - q),$$  

sending $T_{n_s}$ to $X$ and $T_{n_{s'}}$ to $Y$. Here $C\langle X, Y \rangle$ denotes the noncommutative polynomial algebra in two variables over $C$.

**Proof.** We shall prove Propositions 3.12 and 3.17 simultaneously. We begin with only the assumption that $\chi^s = \chi$. By Frobenius Reciprocity, we may view elements of $\mathcal{H}_C(G, \chi)$ as functions $\varphi : G \to C$ satisfying

$$\varphi(igi') = \chi(i)\varphi(g)\chi(i')$$

for $g \in G, i, i' \in I$. If $\varphi_1, \varphi_2$ are the endomorphisms associated to $\varphi_1, \varphi_2$, respectively, then the composition product on $\mathcal{H}_C(G, \chi)$ gives $T_{\varphi_1}T_{\varphi_2} = T_{\varphi_1\varphi_2}$, where

$$\varphi_1 \ast \varphi_2(g) = \sum_{h \in G/I} \varphi_1(h)\varphi_2(h^{-1}g).$$

Assume that $\varphi$ has support in $I\hat{w}I$, where $\hat{w}$ is some representative of $w \in W_{\text{aff}}$. Let $w = s_1s_2 \cdots s_k$ be a reduced word expression for $w$, where $s_i \in \{s, s'\}$, and let $\varphi_{n_s}$ (resp. $\varphi_{n_{s'}}$) be the function with support in $I_{n_s}I$ (resp. $I_{n_{s'}}I$) taking the value 1 at $n_s$ (resp. $n_{s'}$). We claim that $\varphi$ is a scalar multiple of $\varphi_{n_{s_1}} \ast \varphi_{n_{s_2}} \cdots \ast \varphi_{n_{s_k}}$. Indeed, the definition of the convolution product shows that $\text{supp}(\varphi_1 \ast \varphi_2) \subseteq \text{supp}(\varphi_1)\text{supp}(\varphi_2)$. By induction, we have that

$$\text{supp}(\varphi_{n_{s_1}} \ast \varphi_{n_{s_2}} \cdots \ast \varphi_{n_{s_k}}) \subseteq \text{supp}(\varphi_{n_{s_1}})\text{supp}(\varphi_{n_{s_2}}) \cdots \text{supp}(\varphi_{n_{s_k}}) = I_{n_{s_1}}I_{n_{s_2}} \cdots I_{n_{s_k}} = I\hat{w}I,$$

where the last equality follows from [2], Prop. 6.36(4). An elementary inductive argument shows that $\varphi_{n_{s_1}} \ast \varphi_{n_{s_2}} \cdots \ast \varphi_{n_{s_k}} \neq 0$, which implies that $\mathcal{H}_C(G, \chi)$ is generated as an algebra by $T_{n_s}$ and $T_{n_{s'}}$.

It now suffices to determine relations for the operators $T_{n_s}$ and $T_{n_{s'}}$. We again identify these operators with $\varphi_{n_s}$ and $\varphi_{n_{s'}}$, respectively. We will make use of the following decompositions:

$$I_{n_s}I = \bigsqcup_{x, y \in \mathbb{F}_{q^2}, xT + y\bar{T} = 0} u(x, y)n_sI$$

$$I_{n_{s'}}I = \bigsqcup_{y \in \mathbb{F}_{q^2}, y + \bar{T} = 0} w^-(0, \bar{w}y)n_{s'I}I$$
Here (and henceforth) we use the notational convention that $x$ and $y$ are representatives in $\mathfrak{o}_E$ for the set $\mathbb{F}_{q^2}$, satisfying $x\mathfrak{o} + y + \mathfrak{o} = 0$.

As before, we have $\text{supp}(\varphi_{n_s} * \varphi_{n_s}) \subset In_s I_n I = In_s I \sqcup I$, and therefore it suffices to evaluate this function at 1 and $n_s$. We shall also make use of the following identity (for $y \neq 0$):

\begin{equation}
(1) \quad u^-(x, y) = u(-x, y^{-1}) n_s \text{diag}(y\sqrt{\epsilon^{-1}}, -\overline{y}\overline{\epsilon}^{-1}, -\overline{y}^{-1}\sqrt{\epsilon}) u(-x, y^{-1})
\end{equation}

Note that if $u^-$ is a nonidentity lower unipotent element, then $u^- \in Un_s TU$. We compute:

$$
\varphi_{n_s} * \varphi_{n_s}(1) = \sum_{x, y \in \mathbb{F}_{q^2}} \varphi_{n_s}(u(x, y)n_s) \varphi_{n_s}(n_s^{-1}u(x, y)^{-1})
$$

= $q^3$

$$
\varphi_{n_s} * \varphi_{n_s}(n_s) = \sum_{x, y \in \mathbb{F}_{q^2}} \varphi_{n_s}(u(x, y)n_s) \varphi_{n_s}(n_s^{-1}u(x, y)^{-1}n_s)
$$

= $\sum_{x, y \in \mathbb{F}_{q^2}} \varphi_{n_s}(u^-(-x, y))$

= $\sum_{x, y \in \mathbb{F}_{q^2}} \varphi_{n_s}(n_s \text{diag}(-\overline{y}\sqrt{\epsilon}, -\overline{y}\overline{\epsilon}^{-1}, y^{-1}\sqrt{\epsilon}^{-1}))$

= $\sum_{x, y \in \mathbb{F}_{q^2}, \overline{x} + y + \overline{y}, y \neq 0} \chi(\text{diag}(-\overline{y}\sqrt{\epsilon}, -\overline{y}\overline{\epsilon}^{-1}, y^{-1}\sqrt{\epsilon}^{-1}))$.

If $\chi$ factors through the determinant, then the last sum equals $q^3 - 1$. Assume that $\chi$ does not factor through the determinant. We then have

$$
\varphi_{n_s} * \varphi_{n_s}(n_s) = \sum_{y \in \mathbb{F}_{q^2}, y + \overline{y} = 0} \zeta(-\overline{y}\sqrt{\epsilon}) + \sum_{t \in \mathbb{F}_{q^2}} \sum_{x, y \in \mathbb{F}_{q^2}} \sum_{y \in \mathbb{F}_{q^2}} \zeta(-\overline{y}\sqrt{\epsilon})
$$

$\overset{*}{=} (q - 1) + (q + 1) \sum_{y \in \mathbb{F}_{q^2}, y + \overline{y} = 0} \zeta(-\overline{y}\sqrt{\epsilon})$

= $(q - 1) + (q + 1) \left( \sum_{y \in \mathbb{F}_{q^2}, y + \overline{y} = 0} \zeta(-\overline{y}\sqrt{\epsilon}) - \sum_{y \in \mathbb{F}_{q^2}, y + \overline{y} = 0} \zeta(-\overline{y}\sqrt{\epsilon}) \right)$

$\overset{**}{=} (q - 1) + (1 - q^2)$

= $q - q^2$.

The equality ($*$) follows from the fact that if $y + \overline{y} = 0$, then $-\overline{y}\sqrt{\epsilon} \in \mathbb{F}_q$, while ($**$) follows from the fact that $\zeta$ is a nontrivial character.

We now compute the Hecke relations for $\varphi_{n_{s'}}$. Again it suffices to evaluate $\varphi_{n_{s'}} * \varphi_{n_{s'}}$ at 1 and $n_{s'}$. 

\[ \varphi_{n,\epsilon} \ast \varphi_{n,\epsilon}(1) = \sum_{y \in \mathbb{F}_q^2} \varphi_{n,\epsilon}(u(0, w y) n, s) \varphi_{n,\epsilon}(n, s^{-1} u(0, w y)^{-1}) \]

\[ = q \]

\[ \varphi_{n,\epsilon} \ast \varphi_{n,\epsilon}(n, s) = \sum_{y \in \mathbb{F}_q^2} \varphi_{n,\epsilon}(u(0, y w^{-1} \epsilon^{-1})) \]

\[ = \sum_{y \in \mathbb{F}_q^2} \varphi_{n,\epsilon}(\text{diag}(-y \sqrt{\epsilon^{-1}}, 1, -y^{-1} \sqrt{\epsilon}) n, s) \]

\[ = \sum_{y \in \mathbb{F}_q^2} \chi(\text{diag}(-y \sqrt{\epsilon^{-1}}, 1, -y^{-1} \sqrt{\epsilon})) \]

\[ = q - 1. \]

Note that the last equality depends only on the fact that \( \chi^s = \chi \).

We again assume only that \( \chi^s = \chi \). It is an elementary computation to check that \( Z \in \mathcal{Z}_\chi \).

To verify the claim about the centers \( \mathcal{Z}_\chi \) of the algebras \( \mathcal{H}_C(G, \chi) \) in general, we first note that any central element, when viewed as a polynomial in \( \mathcal{T}_{n, s} \) and \( \mathcal{T}_{n, s'} \), must be of even degree (this follows from the Hecke relations). Moreover, the Hecke relations imply that the coefficients of the two highest even-degree terms must be equal. Let \( Y \in \mathcal{Z}_\chi \) be of degree 2k. Then there exists some \( c \in \mathbb{C} \) such that \( Y - cZ^k \) is of strictly smaller degree. Proceeding by induction, we see that \( Y \) is a polynomial in \( Z \), and therefore \( \mathcal{Z}_\chi = \mathbb{C}[Z] \).

\[ \square \]

As before, we can now classify the finite-dimensional simple right \( \mathcal{H}_C(G, \chi) \)-modules.

**Definition 3.18.** (i) Let \( (\theta, \theta') \in \{-q^2, q\} \times \{-1, q\} \). We define the characters \( \mu_{\theta, \theta'} : \mathcal{H}_C(G, \chi) \to \mathbb{C} \) by

\[ \mathcal{T}_{n, s} \mapsto \theta, \quad \mathcal{T}_{n, s'} \mapsto \theta'. \]

The central element \( Z \) maps to \( \theta(\theta' - q + 1) + \theta'(\theta - q + q^2) \).

(ii) Let \( (v_1, v_2)_C \) be a two-dimensional vector space over \( \mathbb{C} \), and let \( \lambda \in \mathbb{C} \). We define \( M(\lambda) \) to be the following right \( \mathcal{H}_C(G, \chi) \)-module:

\[ v_1 \cdot T_{n, s} = -q^2 v_1, \quad v_1 \cdot T_{n, s'} = v_2 \]

\[ v_2 \cdot T_{n, s} = (\lambda + q^2 - q^3)v_1 + qv_2, \quad v_2 \cdot T_{n, s'} = qv_1 + (q - 1)v_2 \]

The central element \( Z \) acts by \( \lambda \).

Again, the proof of Proposition 3.21 shows that the action of \( \mathcal{H}_C(G, \chi) \) on \( M(\lambda) \) is well-defined.

**Proposition 3.19.** Assume \( \text{char}(\mathbb{C}) \neq p \). Then \( M(\lambda) \) is reducible if and only if \( \lambda = q^3 + q \) or \( \lambda = -2q^2 \). In these cases, we have the following exact sequences:

\[ 0 \to \mu_{q, -1} \to M(-2q^2) \to \mu_{-q^2, q} \to 0 \]

\[ 0 \to \mu_{q, q} \to M(q^3 + q) \to \mu_{-q^2, -1} \to 0 \]

The sequences are not split.
Proof. Assume that $M(\lambda)$ is reducible, so that it contains a character $\mu$. The operator $\mathcal{T}_{n_s}$ has eigenvectors $v_1$ and $(\lambda + q^2 - q^3)v_1 + (q + q^2)v_2$, with eigenvalues $-q^2$ and $q$, respectively; the operator $\mathcal{T}_{n_{s'}}$ has eigenvectors $-qv_1 + v_2$ and $v_1 + v_2$, with eigenvalues $-1$ and $q$, respectively.

We see that the only possibility for a common eigenvector of $\mathcal{T}_{n_s}$ and $\mathcal{T}_{n_{s'}}$ is if the vector $(\lambda + q^2 - q^3)v_1 + (q + q^2)v_2$ is a scalar multiple of $-qv_1 + v_2$ or $v_1 + v_2$. Assume the former. Then $\lambda + q^2 - q^3 = -q(q + q^2)$, which implies $\lambda = -2q^2$. It is then clear that

$$\langle -qv_1 + v_2 \rangle_C \cong \mu_{q,-1} \subset M(-2q^2) \quad \text{and} \quad M(-2q^2)/\mu_{q,-1} \cong \mu_{-q^2,q}.$$ 

Since $M(-2q^2)$ does not contain a $(-q^2,q)$-eigenvector for $\mathcal{T}_{n_s}, \mathcal{T}_{n_{s'}}$, the surjection cannot split.

Assume now that $(\lambda + q^2 - q^3)v_1 + (q + q^2)v_2$ is a scalar multiple of $v_1 + v_2$. This implies $\lambda = q^3 + q$. It is clear that

$$\langle v_1 + v_2 \rangle_C \cong \mu_{q,q} \subset M(q^3 + q) \quad \text{and} \quad M(q^3 + q)/\mu_{q,q} \cong \mu_{-q^2,-1}.$$ 

By the same reasoning as above, the surjection doesn’t split. \hfill $\square$

Proposition 3.20. Assume $\text{char}(C) = p$. Then $M(\lambda)$ is reducible if and only if $\lambda = 0$. In this case the module decomposes as $M(0) \cong \mu_{0,0} \oplus \mu_{0,-1}$.

Proof. Assume that $M(\lambda)$ is reducible, so that it contains either $\mu_{0,0}$ or $\mu_{0,-1}$. In either case, the central element $Z = \mathcal{T}_{n_s}(\mathcal{T}_{n_{s'}} + 1) + \mathcal{T}_{n_{s'}}\mathcal{T}_{n_s}$ acts by 0, so we must have $\lambda = 0$. The action of $\mathcal{T}_{n_s}$ and $\mathcal{T}_{n_{s'}}$ on $M(0)$ shows that

$$\langle v_1 + v_2 \rangle_C \cong \mu_{0,0} \quad \text{and} \quad \langle v_2 \rangle_C \cong \mu_{0,-1},$$

so that

$$M(0) \cong \mu_{0,0} \oplus \mu_{0,-1}. \hfill \square$$

We may now classify the simple right $\mathcal{H}_C(G,\chi)$-modules.

Theorem 3.21. Every finite-dimensional simple right $\mathcal{H}_C(G,\chi)$-module is either a character $\mu_{\theta,\theta'}(\theta, \theta') \in \{-q^2,q\} \times \{-1,q\}$, or a module of the form $M(\lambda), \lambda \neq q^3 + q, -2q^2$.

Proof. The proof is virtually the same as the proof of 3.16, with only a few cosmetic changes. More precisely, we consider the space $\ker(\mathcal{T}_{n_s} + q^2)$ in $M$, and compute the action of $\mathcal{T}_{n_{s'}}(\mathcal{T}_{n_s} - q)$ on an eigenvector $v$ in $\ker(\mathcal{T}_{n_s} + q^2)$. The set $\{v, v \cdot \mathcal{T}_{n_{s'}}\}$ then forms a basis for $M$. \hfill $\square$

3.6. The Regular Case. We assume now that $\chi^s \neq \chi = \zeta \otimes \eta$. In this case we have nontrivial intertwining maps between $\text{c-ind}^G_I(\chi)$ and $\text{c-ind}^G_I(\chi^s)$, and we are led to consider the algebra

$$\mathcal{H}_C(G,\gamma_{\chi}) = \mathcal{H}_C(G,\chi \oplus \chi^s) = \mathcal{H}_C(G,\chi) \oplus \text{Hom}_C(\text{c-ind}^G_I(\chi), \text{c-ind}^G_I(\chi^s)) \oplus \text{Hom}_C(\text{c-ind}^G_I(\chi^s), \text{c-ind}^G_I(\chi)) \oplus \mathcal{H}_C(G,\chi^s).$$

We first determine the algebra $\mathcal{H}_C(G,\chi)$. For $n \in N$, we denote by $1_{1nI} \in \text{c-ind}^G_I(\chi)$ the function with support $1nI$, taking the value 1 at $n$, on which $I$ acts by $\chi$ or $\chi^s$ (depending on the class of $n$ in $W$). We let $\mathcal{T}_{n^{-1}}$ (resp. $\mathcal{T}_n$) denote the endomorphism of $\text{c-ind}^G_I(\chi)$ sending $1_I$ to $1_{1n^{-1}I}$ (resp. $1_{InI}$).
Proposition 3.22. The algebra \( \mathcal{H}_C(G, \chi) \) is commutative, generated by \( T_{\alpha^{-1}} \) and \( T_\alpha \), with the relations
\[
T_{\alpha^{-1}}T_\alpha = T_\alpha T_{\alpha^{-1}} = q^4.
\]
We have an isomorphism of algebras \( \mathcal{H}_C(G, \chi) \cong C[X, Y]/(XY - q^4) \), sending \( T_{\alpha^{-1}} \) to \( X \) and \( T_\alpha \) to \( Y \).

Proof. We adopt the same method as in the proof of 3.17, viewing elements of \( \mathcal{H}_C(G, \chi) \) as functions \( \phi \) on the double cosets \( I \backslash G / I \). In this case, however, the relation \( \phi(ig^i) = \chi(i)\phi(g)\chi(i') \) shows that the functions \( \phi \) associated to elements of \( \mathcal{H}_C(G, \chi) \) are supported only on cosets of the form \( I\alpha^n I, n \in \mathbb{Z} \). Once again using properties of the Bruhat decomposition for BN pairs (cf. [2]), if \( \phi \) has support in \( I\alpha^n I \) (resp. \( I\alpha^n I \)) with \( n > 0 \), then \( \phi \) is a scalar multiple of \( \varphi_{\alpha^{-1}} \ast \varphi_{\alpha^{-1}} \ast \ldots \ast \varphi_{\alpha^{-1}} \) (resp. \( \varphi_{\alpha} \ast \varphi_{\alpha} \ast \ldots \ast \varphi_{\alpha} \)), the convolution taken \( n \) times.

It therefore suffices to compute the products \( \varphi_{\alpha^{-1}} \ast \varphi_{\alpha} \) and \( \varphi_{\alpha} \ast \varphi_{\alpha^{-1}} \). We compute the first of these; the method of computation for the second is the same. We have \( \text{supp}(\varphi_{\alpha^{-1}} \ast \varphi_{\alpha}) \subset I\alpha^{-1}I \alpha I \subset I \cup I_{s}n I \cup \alpha^{-1}n_{s} I \), and since the convolution must have support on cosets of the form \( I\alpha^n I \), we actually have \( \text{supp}(\varphi_{\alpha^{-1}} \ast \varphi_{\alpha}) \subset I \). Hence, we need only evaluate this function at \( 1 \), using the decomposition
\[
I\alpha^{-1}I = \bigcup_{x \in \mathbb{F}^2, y \in \mathbb{E}/\mathbb{F}_{E}^2, x^2 + y + \mathfrak{p} = 0} u(x, y)\alpha^{-1}I.
\]
We have:
\[
\varphi_{\alpha^{-1}} \ast \varphi_{\alpha}(1) = \sum_{x \in \mathbb{F}^2, y \in \mathbb{E}/\mathbb{F}_{E}^2, x^2 + y + \mathfrak{p} = 0} \varphi_{\alpha^{-1}}(u(x, y)\alpha^{-1})\varphi_{\alpha}(\alpha u(x, y)^{-1})
\]
\[
= q^4.
\]
\[\square\]

We now turn our attention to \( \text{Hom}_G(\text{c-ind}^G_I(\chi), \text{c-ind}^G_I(\chi^s)) \). This has the structure of an \((\mathcal{H}_C(G, \chi^s), \mathcal{H}_C(G, \chi))\)-bimodule, with the action given by post-composition and precomposition, respectively. By Frobenius Reciprocity we have
\[
\text{Hom}_G(\text{c-ind}^G_I(\chi), \text{c-ind}^G_I(\chi^s)) \cong \text{Hom}_I(\chi, \text{c-ind}^G_I(\chi^s)|_I) \cong \text{c-ind}^G_I(\chi^s)|_I \chi,
\]
which has a basis consisting of the functions \( 1_{In_{a}n^{-a}}I \) with support \( In_{a}n^{-a}I \) and value 1 at \( n_{s}a^{n} \), on which \( I \) acts by \( \chi \). We let \( S_{n, \chi} \) denote the homomorphism sending \( 1_{I} \in \text{c-ind}^G_I(\chi) \) to \( 1_{In_{a}n^{-a}I} \in \text{c-ind}^G_I(\chi^s) \), and append a \( \chi \) (or \( \chi^s \)) to the parameters for the operator \( T_\alpha \) (or \( T_{\alpha^{-1}} \)) to denote the Hecke algebra to which it corresponds. In the notation of Section 3.3, we have
\[
S_{0, \chi} = T_{n_{s}}e_{\chi}, \quad S_{0, \chi^s} = T_{n_{s}}e_{\chi^s},
\]
\[
S_{-1, \chi} = T_{n_{s}e_{\chi}}, \quad S_{-1, \chi^s} = T_{n_{s}e_{\chi^s}}.
\]

We note that the set
\[
\{id_{\chi}, T_{\alpha, \chi}^m, T_{\alpha^{-1}, \chi}^m, id_{\chi^s}, T_{\alpha, \chi^s}^m, T_{\alpha^{-1}, \chi^s}^m, S_{n, \chi}, S_{n, \chi^s}\}_{m > 0, n \in \mathbb{Z}}
\]
forms a basis for \( \mathcal{H}_C(G, \gamma_{\chi}) \), where \( id_{\chi} \) (resp. \( id_{\chi^s} \)) denotes the identity element of \( \mathcal{H}_C(G, \chi) \) (resp. \( \mathcal{H}_C(G, \chi^s) \)).
Proposition 3.23. We have the following relations for the \((H_C(G, χ^s), H_C(G, χ))\)-bimodule \(\text{Hom}_G(\text{c-ind}^G_I(χ), \text{c-ind}^G_I(χ^s))\):

\[
\begin{align*}
T_{α−1,χ^s}S_{n,χ} &= S_{n,χ}T_{α,χ} = \begin{cases} 
S_{n+1,χ} & n ≥ 0 \\
qS_{n+1,χ} & n = −1 \\
q^4S_{n+1,χ} & n ≤ −2
\end{cases} \\
T_{α,χ^s}S_{n,χ} &= S_{n,χ}T_{α−1,χ} = \begin{cases} 
q^4S_{n−1,χ} & n ≥ 1 \\
q^3S_{n−1,χ} & n = 0 \\
S_{n−1,χ} & n ≤ −1
\end{cases}
\end{align*}
\]

In particular, \(\text{Hom}_G(\text{c-ind}^G_I(χ), \text{c-ind}^G_I(χ^s))\) is generated as a module by \(S_{0,χ}\) and \(S_{−1,χ}\).

Proof. We will need the following coset decompositions:

\[
\begin{align*}
IαI &= \bigsqcup_{x,y \in E/p^2E, x \neq y \neq x \parallel \neq = 0} Iαu(x, y) \\
Iα^{-1}I &= \bigsqcup_{x,y \in E/p^2E, x \neq y \neq x \parallel \neq = 0} Iα^{-1}u^−(wx, wy) \\
In_αα^nI &= \bigsqcup_{x,y \in E/p^{n+1}E, x \neq y \neq x \parallel \neq = 0} In_αα^n u(x, y) \quad \text{if } n ≥ 0 \\
In_αα^nI &= \bigsqcup_{x,y \in E/p^{n−1}E, x \neq y \neq x \parallel \neq = 0} In_αα^n u^−(wx, wy) \quad \text{if } n < 0
\end{align*}
\]

In order to compute \(S_{n,χ}T_{α,χ}\), it suffices to know its action on the function \(1_I \in \text{c-ind}^G_I(χ)\). The definitions of \(S_{n,χ}\) and \(T_{α,χ}\) show that the image will have support contained in \(In_αα^nIαI\). Using Proposition 6.36 and Exercise 6.37 of [2], we see that this product of double cosets is equal to \(In_αα^{n+1}I\) (if \(\ell(n_αα^{n+1}) = \ell(n_αα^n) + \ell(α)\)), or is contained in \(In_αα^{n+1}I \sqcup Iα^{-n}I \sqcup Iα^{-n−1}I\) (if \(\ell(n_αα^{n+1}) \neq \ell(n_αα^n) + \ell(α)\)). Since the support of \(S_{n,χ}T_{α,χ}(1_I)\) must be of the form \(In_αα^mI\), we see that in both cases the support is contained in \(In_αα^{n+1}I\), and therefore
it suffices to evaluate the function at \( n_s \alpha^{n+1} \). This gives

\[
S_{n,\chi} T_{\alpha,\chi}(1_I)(n_s \alpha^{n+1}) = S_{n,\chi} \left( \sum_{x \in \mathbb{F}_2^2, y \in \mathcal{E}/p^2_E} u(-x, \overline{y})\alpha^{-1} \cdot 1_I \right)(n_s \alpha^{n+1})
\]

\[
= \sum_{x \in \mathbb{F}_2^2, y \in \mathcal{E}/p^2_E} u(-x, \overline{y})\alpha^{-1} \cdot 1_{I_{n_s \alpha^n}} I(n_s \alpha^{n+1})
\]

\[
= \sum_{x \in \mathbb{F}_2^2, y \in \mathcal{E}/p^2_E} \mathbf{1}_{I_{n_s \alpha^n}} I(n_s \alpha^{n+1}) u(-x, \overline{y})\alpha^{-1}
\]

\[
= \begin{cases} 
1 & n \geq 0 \\
q & n = -1 \\
q^4 & n \leq -2.
\end{cases}
\]

The last line follows from (the transpose of) equation (1).

Using the same methods as above, we see that the support of \( S_{n,\chi} T_{\alpha^{-1},\chi}(1_I) \) is contained in \( I_{n_s \alpha^{n-1}} I \). This gives

\[
S_{n,\chi} T_{\alpha^{-1},\chi}(1_I)(n_s \alpha^{n-1}) = S_{n,\chi} \left( \sum_{x \in \mathbb{F}_2^2, y \in \mathcal{E}/p^2_E} u^-(-\varpi x, \varpi \overline{y})\alpha \cdot 1_I \right)(n_s \alpha^{n-1})
\]

\[
= \sum_{x \in \mathbb{F}_2^2, y \in \mathcal{E}/p^2_E} u^-(-\varpi x, \varpi \overline{y})\alpha \cdot 1_{I_{n_s \alpha^n}} I(n_s \alpha^{n-1})
\]

\[
= \sum_{x \in \mathbb{F}_2^2, y \in \mathcal{E}/p^2_E} \mathbf{1}_{I_{n_s \alpha^n}} I(n_s \alpha^{n-1}) u^-(-\varpi x, \varpi \overline{y})\alpha
\]

\[
= \begin{cases} 
q^4 & n \geq 1 \\
q^3 & n = 0 \\
1 & n \leq -1.
\end{cases}
\]

Again, the last line follows from equation (1).

The support of \( T_{\alpha,\chi} S_{n,\chi}(1_I) \) is contained in the coset \( I_{n_s \alpha^{n-1}} I \). In computing the value of \( T_{\alpha,\chi} S_{n,\chi}(1_I)(n_s \alpha^{n-1}) \), we must treat the cases \( n \geq 0 \) and \( n < 0 \) separately (based on the
coset decomposition of $I_{n_s} \alpha^n I$). For $n \geq 0$, we have

$$T_{\alpha, \chi} \mathcal{S}_{n, \chi}(1_I)(n_s \alpha^{n-1}) = T_{\alpha, \chi} \left( \sum_{x \in E/p_E^{n-1}, y \in E/p_E^{2n-1}} u(-x, y) \alpha^{-n} n_s^{-1} \cdot 1_I \right)(n_s \alpha^{n-1})$$

$$= \sum_{x \in E/p_E^{n-1}, y \in E/p_E^{2n-1}} u(-x, y) \alpha^{-n} n_s^{-1} \cdot 1_{1_{\alpha I}}(n_s \alpha^{n-1})$$

$$= 1_{1_{\alpha I}}(n_s \alpha^{n-1}) u(-x, y) \alpha^{-n} n_s^{-1}$$

$$= \begin{cases} q^4 & n \geq 1 \\ q^3 & n = 0. \end{cases}$$

For $n < 0$, we have

$$T_{\alpha, \chi} \mathcal{S}_{n, \chi}(1_I)(n_s \alpha^{n-1}) = T_{\alpha, \chi} \left( \sum_{x \in E/p_E^{n-1}, y \in E/p_E^{2n-1}} u^-(\omega x, \omega y) \alpha^{-n} n_s^{-1} \cdot 1_I \right)(n_s \alpha^{n-1})$$

$$= \sum_{x \in E/p_E^{n-1}, y \in E/p_E^{2n-1}} u^-(\omega x, \omega y) \alpha^{-n} n_s^{-1} \cdot 1_{1_{\alpha I}}(n_s \alpha^{n-1})$$

$$= 1_{1_{\alpha I}}(n_s \alpha^{n-1}) u^-(\omega x, \omega y) \alpha^{-n} n_s^{-1}$$

$$= 1.$$

Equation (1) once again shows that the element $n_s \alpha^{n-1} u^-(\omega x, \omega y) \alpha^{-n} n_s^{-1}$ is not in $I_{\alpha I}$ for $y \neq 0$. We omit the argument for $T_{\alpha^{-1}, \chi^s} \mathcal{S}_{n, \chi}$, as the computation is virtually the same as that for $T_{\alpha, \chi} \mathcal{S}_{n, \chi}$. \qed

**Corollary 3.24.** As a right $\mathcal{H}_C(G, \chi)$-module, we have

$$\text{Hom}_G(\text{c-ind}_I^G(\chi), \text{c-ind}_I^G(\chi^s)) \cong (\mathcal{H}_C(G, \chi) \oplus \mathcal{H}_C(G, \chi))/((T_{\alpha^{-1}, \chi}, -q^3), (-q, T_{\alpha, \chi})),$$

the isomorphism sending $S_{0, \chi}$ to $(1, 0)$ and $S_{-1, \chi}$ to $(0, 1)$.

Likewise, as a left $\mathcal{H}_C(G, \chi^s)$-module, we have

$$\text{Hom}_G(\text{c-ind}_I^G(\chi), \text{c-ind}_I^G(\chi^s)) \cong (\mathcal{H}_C(G, \chi^s) \oplus \mathcal{H}_C(G, \chi))/((T_{\alpha, \chi^s}, -q^3), (-q, T_{\alpha^{-1}, \chi^s})),$$

the isomorphism sending $S_{0, \chi}$ to $(1, 0)$ and $S_{-1, \chi}$ to $(0, 1)$.

In addition to the bimodule structure on $\text{Hom}_G(\text{c-ind}_I^G(\chi), \text{c-ind}_I^G(\chi^s))$, we also have a composition product between elements $\mathcal{S}_{n, \chi^s} \in \text{Hom}_G(\text{c-ind}_I^G(\chi^s), \text{c-ind}_I^G(\chi))$ and elements $\mathcal{S}_{m, \chi} \in \text{Hom}_G(\text{c-ind}_I^G(\chi), \text{c-ind}_I^G(\chi^s))$. The product of two such homomorphisms will be an element of $\mathcal{H}_C(G, \chi)$. We have the following result:
Proposition 3.25. The composition $S_{n,\chi}S_{m,\chi}$ has the following property:

$$S_{n,\chi}S_{m,\chi} = \begin{cases} 
\zeta(-1)q^{3+4 \min(n,m)}\tau_{\alpha,\chi}^{\max(0,m-n)}\tau_{\alpha-1,\chi}^{\max(n-m,0)} & n, m \geq 0 \\
\zeta(-1)q^{1+4 \min(-n-1,-m-1)}\tau_{\alpha,\chi}^{\max(n-m,0)}\tau_{\alpha-1,\chi}^{\max(0,m-n)} & n, m < 0 \\
\zeta(-1)\tau_{\alpha,\chi}^{n-m} & n < 0, m \geq 0 \\
\zeta(-1)\tau_{\alpha-1,\chi}^{-m} & m < 0, n \geq 0.
\end{cases}$$

Proof. By Proposition 3.23, it suffices to compute the four products $S_{0,\chi}S_{0,\chi}$, $S_{-1,\chi}S_{-1,\chi}$, $S_{-1,\chi}S_{0,\chi}$ and $S_{0,\chi}S_{-1,\chi}$. The method of proof is the same as in the proof of Proposition 3.23, this time using equations (4) and (5) for $n = 0$ and $n = -1$. We give the proof for the first of these products. The definition of $S_{0,\chi}S_{0,\chi}$ and $S_{0,\chi}$ shows that the function $S_{0,\chi}S_{0,\chi}(1)$ will have support contained in $I \cup I_nI$; as $S_{0,\chi}S_{0,\chi} \in \mathcal{H}_C(G, \chi)$, the support is actually contained in $I$. This gives

$$S_{0,\chi}S_{0,\chi}(1) = S_{0,\chi} \left( \sum_{x, y \in \mathbb{F}_2^2, x \neq y, x+y = 0} u(-x, y)n_s^{-1}1_I \right) (1)$$

$$= \sum_{x, y \in \mathbb{F}_2^2, x \neq y, x+y = 0} u(-x, y)n_s^{-1}1_{I_nI}(1)$$

$$= \sum_{x, y \in \mathbb{F}_2^2, x \neq y, x+y = 0} 1_{I_nI}(u(-x, y)n_s^{-1})$$

$$= \zeta(-1)q^3.$$

The other products follow similarly. \qed

Combining Propositions 3.22, 3.23, and 3.25, we now have a full description of the algebra structure of $\mathcal{H}_C(G, \gamma_\chi)$. When $\text{char}(C) = p$, there is a more elegant presentation of $\mathcal{H}_C(G, \gamma_\chi)$. We record the result here.

Corollary 3.26. Assume $\text{char}(C) = p$. We then have

$$\mathcal{H}_C(G, \gamma_\chi) \cong \left( C[X, Y]/(XY) \quad C[X] \oplus C[Y] \right),$$

where the algebra on the right is a “twisted matrix algebra.” If $(f(X), f'(Y)) \in C[X] \oplus C[Y], (g'(Y), g(X)) \in C[Y] \oplus C[X]$, then we define their product to be

$$(f(X), f'(Y))(g'(Y), g(X)) := \zeta(-1)Xf(X)g(X) + \zeta(-1)Yf'(Y)g'(Y) =: (g'(Y), g(X))(f(X), f'(Y)).$$

The isomorphism is given by
\[ T_{\alpha,\chi} \mapsto \begin{pmatrix} Y & 0 \\ 0 & 0 \end{pmatrix} \quad T_{\alpha^{-1},\chi} \mapsto \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \]
\[ T_{\alpha,\chi^s} \mapsto \begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix} \quad T_{\alpha^{-1},\chi^s} \mapsto \begin{pmatrix} 0 & 0 \\ 0 & Y \end{pmatrix} \]
\[ S_{n,\chi} \mapsto \begin{pmatrix} 0 & 0 \\ (Y^n, 0) & 0 \end{pmatrix} \quad S_{n,\chi^s} \mapsto \begin{pmatrix} 0 & (X^n, 0) \\ 0 & 0 \end{pmatrix} \text{ for } n \geq 0 \]
\[ S_{n,\chi} \mapsto \begin{pmatrix} 0 & 0 \\ (0, X^{n-1}) & 0 \end{pmatrix} \quad S_{n,\chi^s} \mapsto \begin{pmatrix} 0 & (0, Y^{n-1}) \\ 0 & 0 \end{pmatrix} \text{ for } n < 0. \]

The center \( Z_{\gamma_X} \) of \( \mathcal{H}_C(G, \gamma_X) \) consists of all elements of the form
\[ h(T_{\alpha,\chi}, T_{\alpha^{-1},\chi}) + h(T_{\alpha^{-1},\chi^s}, T_{\alpha,\chi^s}), \]
where \( h \) is a polynomial of two variables.

**Proof.** It only remains to verify the claim about the center \( Z_{\gamma_X} \) of \( \mathcal{H}_C(G, \gamma_X) \). Let \( \mathcal{Y} \) be an arbitrary element of \( Z_{\gamma_X} \). Writing \( \mathcal{Y} \) as a linear combination of the basis elements, multiplying \( \mathcal{Y} \) on the left and right by the “diagonal” elements \( T_{\alpha,\chi}, T_{\alpha^{-1},\chi}, \) etc., and using Proposition 3.23 shows that the coefficients of \( S_{n,\chi} \) and \( S_{n,\chi^s} \) in \( \mathcal{Y} \) must be zero. Subtracting an appropriate central element of the form \( h(T_{\alpha,\chi}, T_{\alpha^{-1},\chi}) + h(T_{\alpha^{-1},\chi^s}, T_{\alpha,\chi^s}) \), we may assume that \( \mathcal{Y} \) is a polynomial in \( T_{\alpha,\chi} \) and \( T_{\alpha^{-1},\chi} \) alone. Proposition 3.23 again shows that \( 0 = S_{0,\chi} \mathcal{Y} = \mathcal{Y} S_{0,\chi^s}, 0 = S_{-1,\chi^s} \mathcal{Y} = \mathcal{Y} S_{-1,\chi^s} \), which is enough to conclude that \( \mathcal{Y} = 0. \)

**Remark.** The characterization of the center \( Z_{\gamma_X} \) of \( \mathcal{H}_C(G, \gamma_X) \) is the same for the case \( \text{char}(C) \neq p \). The proof carries over without any essential change.

We may now classify finite-dimensional simple modules for the algebra \( \mathcal{H}_C(G, \gamma_X) \). We begin with the characters.

**Proposition 3.27.** Assume \( \text{char}(C) \neq p \). Then \( \mathcal{H}_C(G, \gamma_X) \) has no characters.

**Proof.** Let \( \mu \) be a character of \( \mathcal{H}_C(G, \gamma_X) \). Since \( S_{n,\chi}^2 - S_{n,\chi^s} = 0 \), we must have \( \mu(S_{n,\chi}) = \mu(S_{n,\chi^s}) = 0 \) for every \( n \in \mathbb{Z} \). Proposition 3.25 now implies that all elements of \( \mathcal{H}_C(G, \chi) \) and \( \mathcal{H}_C(G, \chi^s) \) map to 0. This gives a contradiction, since 1 = \( \mu(\text{id}_{c-ind C} \gamma_X) = \mu(\text{id}_\chi + \text{id}_{\chi^s}) = 0. \)

Assume now that \( \text{char}(C) = p \).

**Definition 3.28.** Let \( i \in \{0, 1\} \). We define \( \mu_i : \mathcal{H}_C(G, \gamma_X) \rightarrow C \) to be the character for which \( \text{id}_{\chi^s} \mapsto 1 \) and every other basis element maps to 0.

**Proposition 3.29.** Assume \( \text{char}(C) = p \). Then the characters of \( \mathcal{H}_C(G, \gamma_X) \) are exactly \( \mu_0 \) and \( \mu_1 \).

**Proof.** As in the characteristic prime-to-\( p \) case, we use Propositions 3.23 and 3.25 to conclude that every basis element besides \( \text{id}_\chi \) and \( \text{id}_{\chi^s} \) must map to zero. Since \( \text{id}_\chi + \text{id}_{\chi^s} = \text{id}_{c-ind C} \gamma_X \) and \( \text{id}_\chi \text{id}_{\chi^s} = 0 \), we see that the characters must be exactly those stated.

We now turn our attention to modules of dimension greater than one. We first assume that \( \text{char}(C) \neq p \). Let \( \sqrt{\zeta(-1)} \) denote a fixed square root of \( \zeta(-1) \), and let
\[ \mathcal{A} = \sqrt{\zeta(-1)}(S_{0,\chi} + S_{-1,\chi^s}). \]
We have that $A^2 = T_{\alpha, \chi} + T_{\alpha^{-1}, \chi^*}$, and that $H_C(G, \gamma_\chi)$ is free of rank two over $\mathbb{Z}_{\gamma_\chi}[A]$, with basis $\{id_{\chi}, id_{\chi^*}\}$.

Let $\lambda \in C^\times$, and fix a square root $\sqrt{\lambda}$. Let $\mu_{\lambda, \sqrt{\lambda}}$ denote the representation of $\mathbb{Z}_{\gamma_\chi}[A]$ spanned by $v$, with action given by

$$v \cdot (T_{\alpha, \chi} + T_{\alpha^{-1}, \chi^*}) = \lambda v, \quad v \cdot (T_{\alpha, \chi^*} + T_{\alpha^{-1}, \chi}) = q^4 \lambda^{-1} v,$$

$$v \cdot A = \sqrt{\lambda} v.$$

We consider the induced representation $\mu_{\lambda, \sqrt{\lambda}} \otimes \mathbb{Z}_{\gamma_\chi}[A] H_C(G, \gamma_\chi)$. Since the algebra $H_C(G, \gamma_\chi)$ admits no characters, this immediately implies that this module is simple.

**Lemma 3.30.** The (isomorphism class of the) representation $\mu_{\lambda, \sqrt{\lambda}} \otimes \mathbb{Z}_{\gamma_\chi}[A] H_C(G, \gamma_\chi)$ is independent of the choice of square root $\sqrt{\lambda}$.

**Proof.** Let $\langle v \rangle_C$ denote the underlying space of $\mu_{\lambda, \sqrt{\lambda}}$. Then $\mu_{\lambda, \sqrt{\lambda}} \otimes \mathbb{Z}_{\gamma_\chi}[A] H_C(G, \gamma_\chi)$ is spanned by $\{v \otimes id_\chi, v \otimes id_{\chi^*}\}$. The action of $\mathbb{Z}_{\gamma_\chi}[A]$ on the vector $v \otimes (id_\chi - id_{\chi^*})$ shows that $\mu_{\lambda, \sqrt{\lambda}}$ is contained in $\mu_{\lambda, \sqrt{\lambda}} \otimes \mathbb{Z}_{\gamma_\chi}[A] H_C(G, \gamma_\chi)|_{\mathbb{Z}_{\gamma_\chi}[A]}$. By Frobenius Reciprocity we have

$$\{0\} \neq \text{Hom}_{\mathbb{Z}_{\gamma_\chi}[A]}(\mu_{\lambda, -\sqrt{\lambda}}, \mu_{\lambda, \sqrt{\lambda}} \otimes \mathbb{Z}_{\gamma_\chi}[A] H_C(G, \gamma_\chi)|_{\mathbb{Z}_{\gamma_\chi}[A]})$$

$$\cong \text{Hom}_{H_C(G, \gamma_\chi)}(\mu_{\lambda, -\sqrt{\lambda}} \otimes \mathbb{Z}_{\gamma_\chi}[A] H_C(G, \gamma_\chi), \mu_{\lambda, \sqrt{\lambda}} \otimes \mathbb{Z}_{\gamma_\chi}[A] H_C(G, \gamma_\chi)).$$

As both modules are simple, the result follows. \hfill $\square$

With this lemma, we may unambiguously define $M(\lambda) = \mu_{\lambda, \sqrt{\lambda}} \otimes \mathbb{Z}_{\gamma_\chi}[A] H_C(G, \gamma_\chi)$. By considering central characters, we see that the modules $M(\lambda)$ are pairwise nonisomorphic for distinct values of $\lambda$.

**Theorem 3.31.** Assume $\text{char}(C) \neq p$. Every finite-dimensional simple right $H_C(G, \gamma_\chi)$-module is of the form $M(\lambda), \lambda \in C^\times$.

**Proof.** Assume $M$ is a nonzero simple right module, and assume that $M|_{\mathbb{Z}_{\gamma_\chi}[A]}$ contains a character $\mu_{\lambda, \sqrt{\lambda}}$. Frobenius Reciprocity gives

$$\{0\} \neq \text{Hom}_{\mathbb{Z}_{\gamma_\chi}[A]}(\mu_{\lambda, \sqrt{\lambda}}, M|_{\mathbb{Z}_{\gamma_\chi}[A]} \cong \text{Hom}_{H_C(G, \gamma_\chi)}(M(\lambda), M),$$

which implies $M(\lambda) \cong M$ by simplicity of $M(\lambda)$ and $M$. \hfill $\square$

Assume now that $\text{char}(C) = p$. Let $\sqrt{\zeta(-1)}$ denote a fixed square root of $\zeta(-1)$, and let

$$A_1 = \sqrt{\zeta(-1)}(S_{0,1} + S_{-1,1}), \quad A_2 = \sqrt{\zeta(-1)}(S_{0,\chi} + S_{-1,\chi}).$$

Note that $A_1A_2 = A_2A_1 = 0$, $A_1^2 = T_{\alpha, \chi} + T_{\alpha^{-1}, \chi^*}$, and $A_2^2 = T_{\alpha^{-1}, \chi} + T_{\alpha, \chi^*}$. The algebra $H_C(G, \gamma_\chi)$ is free of rank two over $\mathbb{Z}_{\gamma_\chi}[A_1, A_2]$, with basis $\{id_{\chi}, id_{\chi^*}\}$.

Let $\lambda, \lambda' \in C$ be such that $\lambda\lambda' = 0$, and fix square roots $\sqrt{\lambda}, \sqrt{\lambda'}$. We let $\mu_{\lambda, \lambda', \sqrt{\lambda}, \sqrt{\lambda'}}$ denote the representation of $\mathbb{Z}_{\gamma_\chi}[A_1, A_2]$ spanned by $v$, with action given by

$$v \cdot (T_{\alpha, \chi} + T_{\alpha^{-1}, \chi^*}) = \lambda v, \quad v \cdot (T_{\alpha^{-1}, \chi} + T_{\alpha, \chi^*}) = \lambda' v,$$

$$v \cdot A_1 = \sqrt{\lambda} v, \quad v \cdot A_2 = \sqrt{\lambda'} v.$$

We consider the induced representation $\mu_{\lambda, \lambda', \sqrt{\lambda}, \sqrt{\lambda'}} \otimes \mathbb{Z}_{\gamma_\chi}[A_1, A_2] H_C(G, \gamma_\chi)$.

**Proposition 3.32.** The module $\mu_{\lambda, \lambda', \sqrt{\lambda}, \sqrt{\lambda'}} \otimes \mathbb{Z}_{\gamma_\chi}[A_1, A_2] H_C(G, \gamma_\chi)$ is reducible if and only if $(\lambda, \lambda') = (0, 0)$. In this case, we have

$$\mu_{0,0,0,0} \cong \mu_0 \oplus \mu_1.$$
Proof. Assume that \( \mu_{\lambda,\lambda'\sqrt{k},\sqrt{k}^2} \mathcal{H}(G, \gamma) \) is reducible, so that it contains either \( \mu_0 \) or \( \mu_1 \). In either case, both \( A_1 \) and \( A_2 \) must act by 0, and therefore \( (\lambda, \lambda') = (0, 0) \). The action of \( \text{id}_\chi \) and \( \text{id}_{\chi'} \) show that if \( \langle v \rangle_C \cong \mu_{0,0,0,0} \) as \( \mathcal{Z}_\gamma[A_1, A_2] \)-module, then \( \langle v \otimes \text{id}_\chi \rangle_C \cong \mu_0 \) and \( \langle v \otimes \text{id}_{\chi'} \rangle_C \cong \mu_1 \) as \( \mathcal{H}(G, \gamma) \)-modules.

Lemma 3.33. The (isomorphism class of the) representation \( \mu_{\lambda,\lambda'\sqrt{k},\sqrt{k}^2} \mathcal{H}(G, \gamma) \) is independent of the choice of square roots \( \sqrt{k}, \sqrt{k} \).

Proof. This is obvious if \( \lambda = \lambda' = 0 \), so assume that \( \lambda' \neq 0 \). If we let \( \langle v \rangle_C \) denote the underlying space of the character \( \mu_{\lambda,\lambda'\sqrt{k},\sqrt{k}^2} \), then \( \langle v \otimes \text{id}_\chi \rangle_C \) is spanned by \( \{ v \otimes \text{id}_\chi, v \otimes \text{id}_{\chi'} \} \). Considering the action of \( \mathcal{Z}_\gamma[A_1, A_2] \) on the vector \( v \otimes \text{id}_\chi \), we see that \( \langle v \otimes \text{id}_\chi \rangle_C \cong \mu_{\lambda,\lambda'\sqrt{k},\sqrt{k}^2} \mathcal{H}(G, \gamma) \mathcal{H}(G, \gamma) \). By Frobenius Reciprocity we have

\[
\begin{align*}
\{ 0 \} & \neq \text{Hom}_{\mathcal{Z}_\gamma[A_1, A_2]}(\mu_{\lambda,\lambda'\sqrt{k},\sqrt{k}^2}, \mu_{\lambda,\lambda'\sqrt{k},\sqrt{k}^2} \mathcal{H}(G, \gamma) \mathcal{H}(G, \gamma)) \\
& \cong \text{Hom}_{\mathcal{H}(G, \gamma)}(\mu_{\lambda,\lambda'\sqrt{k},\sqrt{k}^2}, \mu_{\lambda,\lambda'\sqrt{k},\sqrt{k}^2} \mathcal{H}(G, \gamma) \mathcal{H}(G, \gamma)).
\end{align*}
\]

As both modules are simple, the result follows. The case \( \lambda = 0, \lambda' \neq 0 \) is similar.

With this lemma, we can unambiguously define \( M(\lambda, \lambda') = \mu_{\lambda,\lambda'\sqrt{k},\sqrt{k}^2} \mathcal{H}(G, \gamma) \cdot \text{module is either a character } \mu_0 \text{ or } \mu_1, \text{ or a module of the form } M(\lambda, \lambda') \) with \( \lambda \lambda' = 0, (\lambda, \lambda') \neq (0, 0) \).

Theorem 3.34. Assume \( \text{char}(C) = p \). Every finite-dimensional simple right \( \mathcal{H}(G, \gamma) \)-module is either a character \( \mu_0 \) or \( \mu_1 \), or a module of the form \( M(\lambda, \lambda') \) with \( \lambda \lambda' = 0, (\lambda, \lambda') \neq (0, 0) \).

Proof. Assume \( M \) is a nonzero simple right module which is not a character, and assume that \( M|_{\mathcal{Z}_\gamma[A_1, A_2]} \) contains a character \( \mu_{\lambda,\lambda'\sqrt{k},\sqrt{k}^2} \). If \( (\lambda, \lambda') = (0, 0) \), then \( M \) would contain either \( \mu_0 \) or \( \mu_1 \), and by simplicity would be equal to a character, giving a contradiction. Frobenius Reciprocity now gives

\[
\{ 0 \} \neq \text{Hom}_{\mathcal{Z}_\gamma[A_1, A_2]}(\mu_{\lambda,\lambda'\sqrt{k},\sqrt{k}^2}, M|_{\mathcal{Z}_\gamma[A_1, A_2]}) \cong \text{Hom}_{\mathcal{H}(G, \gamma)}(M(\lambda, \lambda'), M),
\]

which implies \( M(\lambda, \lambda') \cong M \) by simplicity of \( M(\lambda, \lambda') \) and \( M \).

We conclude with one final definition.

Definition 3.35. Let \( \chi : H \to C^\times \) be an arbitrary character, and let \( M \) be a finite-dimensional simple module for \( \mathcal{H}(G, \gamma) \). We append \( \chi \) to the list of parameters of \( M \), and use this notation to denote the corresponding module for \( \mathcal{H}(G, I(1)) \), via the decomposition of Proposition 3.8.

Remark. The isomorphism in Corollary 3.26 depends on the ordered pair \( (\chi, \chi^*) \). There is an obvious isomorphism of algebras \( \mathcal{H}(G, \chi + \chi^*) \cong \mathcal{H}(G, \chi^* + \chi) \), which identifies simple modules. In particular, the isomorphism gives \( M(\lambda, \chi) \cong M(g^4 \lambda^{-1}, \chi^*) \) if \( \text{char}(C) \neq p \), and \( \mu_{0,0} \cong \mu_{1,\chi^*} \), \( \mu_{1,\chi} \cong \mu_{0,\chi^*} \), \( M(\lambda, \lambda', \chi) \cong M(\lambda', \lambda, \chi^*) \) if \( \text{char}(C) = p \).

4. Principal Series and Supersingular Modules

4.1. Principal Series. We shall assume from this point onwards that \( C = \mathbb{F}_p \), and that all representations are smooth \( \mathbb{F}_p \)-representations. We call such representations \( \text{mod}-p \) or \( \text{modular} \) representations. We let \( \iota : \mathbb{F}_{q^2} \hookrightarrow \mathbb{F}_p \) denote a fixed embedding, and assume that
we have \( \mathbb{F}_p^\times \)-valued characters of \( H \) factors through \( \iota \). In an attempt to understand supersingular representations of \( G \) (cf. Introduction), we will make use of the functor sending a smooth representation \( \pi \) to \( \pi_I^{(1)} \), called the functor of \( I(1) \)-invariants. By Lemma 3(1) of [3], if \( \pi \) is a nonzero smooth representation of \( G \), then the module \( \pi_I^{(1)} \) will also be nonzero.

Let \( \varepsilon = \tilde{\zeta} \otimes \tilde{\eta} \) be a smooth character of the full torus \( T \) of \( G \), and consider the principal series representation \( \text{ind}^G_B(\varepsilon) \), where \( B \) is the standard upper Borel subgroup of \( G \), and \( \tilde{\zeta} \) and \( \tilde{\eta} \) are characters of \( E^\times \) and \( U(1)(E/F) \), respectively. In Proposition 4.4.9 of [1], Abdellatif has shown that the principal series representation is reducible if and only if \( \varepsilon = \tilde{\eta} \circ \text{det} \), in which case it is of length 2. More precisely, we have a nonsplit short exact sequence

\[
0 \rightarrow \tilde{\eta} \circ \text{det} \rightarrow \text{ind}^G_B(\varepsilon) \rightarrow \tilde{\eta} \circ \text{det} \otimes \text{St}_G \rightarrow 0,
\]

where \( \text{St}_G = \text{ind}^G_B(1)/1 \) is the Steinberg representation of \( G \).

The Bruhat decomposition applied to \( K \) and the Iwasawa decomposition together imply that

\[
G = BI \sqcup B_n I = BI(1) \sqcup B_n I(1).
\]

Therefore, we may take as a basis for the space of \( I(1) \)-invariants of \( \text{ind}^G_B(\varepsilon) \) the functions \( \{f_1, f_2\} \), defined by

\[
f_1(1) = 1, \quad f_1(n_s) = 0, \\
f_2(1) = 0, \quad f_2(n_s) = 1;
\]

the function \( f_1 \) is the unique \( I(1) \)-invariant function with support \( BI(1) \) taking the value 1 at the identity (likewise for \( f_2 \), supported in \( B_n I(1) \)).

For a smooth character \( \varepsilon \) of \( T \), we recall that since \( T_1 \) is a pro-\( p \) subgroup, the restriction of \( \varepsilon \) to \( T_1 \) must be trivial. Let \( \varepsilon^* \) denote the representation of \( H = T_0/T_1 \), given by restricting \( \varepsilon \) to \( T_0 \). The action of \( \mathcal{H}_{\mathbb{P}}(G, I(1)) \) on \( \text{ind}^G_B(\varepsilon)^{I(1)} \) will depend on the character \( \varepsilon^* \). If \( (\varepsilon^*)^s = \varepsilon^* \), then in the notation of Lemma 3.7 we have

\[
\text{ind}^G_B(\varepsilon)^{I(1)} = \text{ind}^G_B(\varepsilon)^{I,\varepsilon^*},
\]

and the action of \( \mathcal{H}_{\mathbb{P}}(G, I(1)) \) factors through algebra \( \mathcal{H}_{\mathbb{P}}(G, \varepsilon^*) \) (via the decomposition of Proposition 3.8). Likewise, if \( (\varepsilon^*)^s \neq \varepsilon^* \), then

\[
\text{ind}^G_B(\varepsilon)^{I(1)} = \text{ind}^G_B(\varepsilon)^{I,\varepsilon^* \oplus (\varepsilon^*)^s} = \text{ind}^G_B(\varepsilon)^{I,\varepsilon^*} \oplus \text{ind}^G_B(\varepsilon)^{I,(\varepsilon^*)^s},
\]

and the action of \( \mathcal{H}_{\mathbb{P}}(G, I(1)) \) factors through \( \mathcal{H}_{\mathbb{P}}(G, \varepsilon^* \oplus (\varepsilon^*)^s) \).

**Theorem 4.1.** The algebra \( \mathcal{H}_{\mathbb{P}}(G, I(1)) \) acts on \( (f_1, f_2)_{\mathbb{P}} \) in the following way:

(i) If \( \varepsilon^* \) factors through the determinant, then

\[
f_1 \cdot e_{\varepsilon^*} = f_1, \quad f_1 \cdot e_\chi = 0, \quad f_1 \cdot T_{n_s} = f_2, \quad f_1 \cdot T_{n_s'} = -f_1 \\
f_2 \cdot e_{\varepsilon^*} = f_2, \quad f_2 \cdot e_\chi = 0, \quad f_2 \cdot T_{n_s} = -f_2, \quad f_2 \cdot T_{n_s'} = \varepsilon(\alpha)f_1,
\]

for \( \chi \neq \varepsilon^* \).

(ii) If \( (\varepsilon^*)^s = \varepsilon^* \) but \( \varepsilon^* \) does not factor through the determinant, then

\[
f_1 \cdot e_{\varepsilon^*} = f_1, \quad f_1 \cdot e_\chi = 0, \quad f_1 \cdot T_{n_s} = f_2, \quad f_1 \cdot T_{n_s'} = -f_1 \\
f_2 \cdot e_{\varepsilon^*} = f_2, \quad f_2 \cdot e_\chi = 0, \quad f_2 \cdot T_{n_s} = 0, \quad f_2 \cdot T_{n_s'} = \varepsilon(\alpha)f_1,
\]

for \( \chi \neq \varepsilon^* \).

(iii) If \( (\varepsilon^*)^s \neq \varepsilon^* \), then

\[
f_1 \cdot e_{\varepsilon^*} = f_1, \quad f_1 \cdot e_\chi = 0, \quad f_1 \cdot T_{n_s} = f_2, \quad f_1 \cdot T_{n_s'} = 0 \\
f_2 \cdot e_{(\varepsilon^*)^s} = f_2, \quad f_2 \cdot e_{\chi^*} = 0, \quad f_2 \cdot T_{n_s} = 0, \quad f_2 \cdot T_{n_s'} = \tilde{\zeta}(-1)\varepsilon(\alpha)f_1,
\]
for $\chi \neq \varepsilon^*$, $\chi' \neq (\varepsilon^*)^s$.

Proof. See Appendix. \hfill \Box

**Corollary 4.2.** In the notation of Definition 3.35, the $H_{\mathbb{F}_p}(G, I(1))$-module $\text{ind}_{B}^{G}(\varepsilon)^{I(1)}$ is given by the following:

(i) Assume $\varepsilon^*$ factors through the determinant. Then $\text{ind}_{B}^{G}(\varepsilon)^{I(1)} \cong M(\varepsilon(\alpha), \varepsilon^*)$ as right $H_{\mathbb{F}_p}(G, I(1))$-modules.

(ii) Assume $(\varepsilon^*)^s = \varepsilon^*$, but $\varepsilon^*$ does not factor through the determinant. Then $\text{ind}_{B}^{G}(\varepsilon)^{I(1)} \cong M(\varepsilon(\alpha), \varepsilon^*)$ as right $H_{\mathbb{F}_p}(G, I(1))$-modules.

(iii) Assume $(\varepsilon^*)^s \neq \varepsilon^*$. Then $\text{ind}_{B}^{G}(\varepsilon)^{I(1)} \cong M(0, \varepsilon(\alpha), \varepsilon^*)$ as right $H_{\mathbb{F}_p}(G, I(1))$-modules.

Proof. (i) Firstly, note that by Theorem 4.1, the central element of $H_{\mathbb{F}_p}(G, \varepsilon^*)$ acts by $\varepsilon(\alpha)$. Assume that $\text{ind}_{B}^{G}(\varepsilon)^{I(1)}$ is reducible as an $H_{\mathbb{F}_p}(G, \varepsilon^*)$-module, and let $c_1 f_1 + c_2 f_2$, for $c_1, c_2 \in \mathbb{F}_p$, span a one-dimensional invariant subspace. The action of $T_{n_0}$ shows that either $c_1 = c_2$, or $c_1 = 0$. In the first case, the action of $T_{n_0}$ implies $\varepsilon(\alpha) = 1$, while in the second case it implies $\varepsilon(\alpha) = 0$, an impossibility.

If $\varepsilon(\alpha) \neq 1$, then $\text{ind}_{B}^{G}(\varepsilon)^{I(1)}$ is a simple module, with the center of $H_{\mathbb{F}_p}(G, \varepsilon^*)$ acting by $\varepsilon(\alpha)$. Therefore $\text{ind}_{B}^{G}(\varepsilon)^{I(1)} \cong M(\varepsilon(\alpha), \varepsilon^*)$. If $\varepsilon(\alpha) = 1$, then the action of the operators $T_{n_0}$ and $T_{n_0}$ on the basis $\{f_2, f_1\}$ shows that $\text{ind}_{B}^{G}(\varepsilon)^{I(1)} \cong M(1, \varepsilon^*)$. The condition $\varepsilon(\alpha) = 1$ implies $\varepsilon = \tilde{\eta} \circ \det$, in which case we have a short exact sequence

$$0 \longrightarrow \tilde{\eta} \circ \det \longrightarrow \text{ind}_{B}^{G}(\varepsilon) \longrightarrow \tilde{\eta} \circ \det \otimes \text{St}_G \longrightarrow 0.$$  

Using the same argument as in [33], we conclude that taking $I(1)$-invariants is exact in this case, and we obtain:

$$0 \longrightarrow \tilde{\eta} \circ \det \longrightarrow \text{ind}_{B}^{G}(\varepsilon)^{I(1)} \longrightarrow \tilde{\eta} \circ \det \otimes \text{St}^{I(1)} \longrightarrow 0$$

Applying the Five Lemma gives $\tilde{\eta} \circ \det \otimes \text{St}^{I(1)} \cong \mu_{-1, -1, \varepsilon^*}$.

(ii) By Theorem 4.1, the central element of $H_{\mathbb{F}_p}(G, \varepsilon^*)$ acts by $\varepsilon(\alpha)$. Assume that $\text{ind}_{B}^{G}(\varepsilon)^{I(1)}$ is reducible as an $H_{\mathbb{F}_p}(G, \varepsilon^*)$-module, so that it contains either $\mu_{0, 0, \varepsilon^*}$ or $\mu_{0, -1, \varepsilon^*}$. In either case, the central element acts as 0, which implies $\varepsilon(\alpha) = 0$, an impossibility. Thus, the module $\text{ind}_{B}^{G}(\varepsilon)^{I(1)}$ is simple, and by Theorem 3.21, we must have $\text{ind}_{B}^{G}(\varepsilon)^{I(1)} \cong M(\varepsilon(\alpha), \varepsilon^*)$ as right $H_{\mathbb{F}_p}(G, I(1))$-modules.

(iii) The action described in Theorem 4.1 shows that $\text{ind}_{B}^{G}(\varepsilon)^{I(1)}$ does not contain a character. Thus $\text{ind}_{B}^{G}(\varepsilon)^{I(1)}$ is simple as a right $H_{\mathbb{F}_p}(G, \varepsilon^* \oplus (\varepsilon^*)^s)$-module, and is therefore determined by the action of the center. Proposition 3.25 and Corollary 3.26 imply that the center is generated by

$$\tilde{\zeta}(-1)(\varepsilon, T_{n_0} \circ T_{n_0}, \varepsilon)$$

The first element acts by 0, while the second element acts by $\varepsilon(\alpha)$. Therefore, we have $\text{ind}_{B}^{G}(\varepsilon)^{I(1)} \cong M(0, \varepsilon(\alpha), \varepsilon^*)$ as right $H_{\mathbb{F}_p}(G, I(1))$-modules. \hfill \Box

**4.2. Supersingular Modules.** In light of the results of the previous section, we make the following definition:
**Definition 4.3.** Let $\chi = \zeta \otimes \eta$ be a character of the finite torus $H$. We define the following characters of $\mathcal{H}_{\mathcal{P}}(G, I(1))$:

(i) Assume that $\chi = \eta \circ \det$. We set

$$
M_{\chi,(S,0)}: \begin{align*}
e_{\chi} & \mapsto 1, \quad e_{\chi'} \mapsto 0, \quad T_{n_s} \mapsto 0, \quad T_{n',s'} \mapsto -1;
M_{\chi,(0,S')}: \begin{align*}
e_{\chi} & \mapsto 1, \quad e_{\chi'} \mapsto 0, \quad T_{n_s} \mapsto -1, \quad T_{n',s'} \mapsto 0,
\end{align*}
$$

for $\chi' \neq \chi$.

(ii) Assume that $\chi^s = \chi$, but $\chi \neq \eta \circ \det$. We set

$$
M_{\chi,(0,S')}: \begin{align*}
e_{\chi} & \mapsto 1, \quad e_{\chi'} \mapsto 0, \quad T_{n_s} \mapsto 0, \quad T_{n',s'} \mapsto 0;
M_{\chi,(0,0)}: \begin{align*}
e_{\chi} & \mapsto 1, \quad e_{\chi'} \mapsto 0, \quad T_{n_s} \mapsto 0, \quad T_{n',s'} \mapsto -1,
\end{align*}
$$

for $\chi' \neq \chi$.

(iii) Assume that $\chi^s \neq \chi$. We set

$$
M_{\chi,(0,0)}: \begin{align*}
e_{\chi} & \mapsto 1, \quad e_{\chi'} \mapsto 0, \quad T_{n_s} \mapsto 0, \quad T_{n',s'} \mapsto 0,
\end{align*}
$$

for $\chi' \neq \chi$.

The modules defined in this way are supersingular (as defined in Definition 3.11). We will denote a generic supersingular module above by $M_{\chi,J}$, where $J = (J, J')$ is an ordered pair as above with $J \subset J_0(\chi), J' \subset J'_0(\chi)$. This notation is motivated by the notation of Section 5 (cf. Definitions 5.2 and 5.3).

Note that we have $M_{\chi,J} \cong M_{\chi',J'}$ if and only if $\chi = \chi'$ and $J = J'$. The computations of the previous sections lead to the following Corollary:

**Corollary 4.4.** (i) Let $M$ be a finite-dimensional supersingular $\mathcal{H}_{\mathcal{P}}(G, I(1))$-module. Then $M \cong M_{\chi,J}$ for some $\chi$ and $J$, where $M_{\chi,J}$ is a module as in Definition 5.3.

(ii) The functor of $I(1)$-invariants induces a bijection between irreducible nonsupersingular representations of $G$ and nonsupersingular finite-dimensional simple right $\mathcal{H}_{\mathcal{P}}(G, I(1))$-modules. Moreover, if $M$ is a simple right $\mathcal{H}_{\mathcal{P}}(G, I(1))$-module such that $M \not\cong \pi^{I(1)}$ for any nonsupersingular representation $\pi$, then $M$ is a supersingular module.

**Proof.** This follows from Theorems 3.16, 3.21, 3.34, and Corollary 4.2. \qed

**Corollary 4.5.** Let $\pi$ be a smooth irreducible representation of $G$. If $\pi^{I(1)}$ contains a submodule isomorphic to a supersingular module, then $\pi$ is supersingular.

5. **Representations of the Finite Groups and Finite Hecke Algebras**

In this section, we recall results about mod-$p$ representations of the finite groups $\Gamma = U(2,1)(\mathbb{F}_{q^2}/\mathbb{F}_q)$ and $\Gamma' = (U(1,1) \times U(1))(\mathbb{F}_{q^2}/\mathbb{F}_q)$. On one hand, we have a complete description of such representations in terms of characters $\chi$ of $H$ and subsets of a certain set $J_0(\chi)$, due to Carter and Lusztig (cf. [10]); on the other hand, we have a more classical description in terms of highest weight modules. Our goal will be to provide a dictionary for matching the two sets of representations. For references on the highest weight classification, the reader is urged to consult the lecture notes of Steinberg ([30]) or Humphreys ([20]).

We fix some notation. Let $S$ and $S'$ denote the sets of Coxeter generators for the Weyl groups associated to $\Gamma$ and $\Gamma'$, respectively. In both cases, the sets $S$ and $S'$ have size 1, consisting of the class of the elements $s$ and $s'$, respectively.
5.1. **Finite Hecke Algebras.** We first describe the Hecke algebras for the finite groups \( \Gamma \) and \( \Gamma' \), and their associated simple modules.

**Definition 5.1.** We define
\[
\mathcal{H}_\Gamma := \text{End}_\Gamma(\text{ind}_\mathbb{U}^\mathbb{U}(1)), \quad \mathcal{H}_{\Gamma'} := \text{End}_{\Gamma'}(\text{ind}_\mathbb{U'}^\mathbb{U'}(1)),
\]
where \( \text{ind} \) denotes induction in the category of representations of finite groups and \( 1 \) denotes the trivial character of \( \mathbb{U} \) or \( \mathbb{U}' \).

Extending functions by zero induces the injections \( \text{ind}_\mathbb{U}^\mathbb{U}(1) \cong \text{ind}_\mathbb{U'}^{\mathbb{U'}}(1) \hookrightarrow c\text{-ind}_\mathbb{U}^{\mathbb{U}}(1) \) and \( \text{ind}_{\mathbb{U}'}^{\mathbb{U}'}(1) \cong \text{ind}_{\mathbb{U}'}^{\mathbb{U}'}(1) \hookrightarrow c\text{-ind}_{\mathbb{U}'}^{\mathbb{U}'}(1) \). Passing to \( I(1) \)-invariants, we may view the algebras \( \mathcal{H}_\Gamma \) and \( \mathcal{H}_{\Gamma'} \) as subalgebras of \( \mathcal{H}_{\mathbb{F}_p}(G, I(1)) \) by the morphisms
\[
\mathcal{H}_\Gamma \hookrightarrow \text{Hom}_K(\text{ind}_{\mathbb{U}'}^{\mathbb{U}'}(1), c\text{-ind}_{\mathbb{U}}^{\mathbb{U}}(1)) \cong \text{Hom}_G(c\text{-ind}_{\mathbb{U}'}^{\mathbb{U}'}(1), c\text{-ind}_{\mathbb{U}}^{\mathbb{U}}(1)) = \mathcal{H}_{\mathbb{F}_p}(G, I(1)),
\]
\[
\mathcal{H}_{\Gamma'} \hookrightarrow \text{Hom}_{K'}(\text{ind}_{\mathbb{U}}^{\mathbb{U}}(1), c\text{-ind}_{\mathbb{U}'}^{\mathbb{U}'}(1)) \cong \text{Hom}_{G'}(c\text{-ind}_{\mathbb{U}}^{\mathbb{U}}(1), c\text{-ind}_{\mathbb{U}'}^{\mathbb{U}'}(1)) = \mathcal{H}_{\mathbb{F}_p}(G, I(1)).
\]
We deduce from these morphisms that the algebra \( \mathcal{H}_\Gamma \) is generated by \( T_{n,s} \) and \( e_\chi \) for all characters \( \chi \) of \( H \), while \( \mathcal{H}_{\Gamma'} \) is generated by \( T_{n,s'} \) and \( e_\chi \) for all characters \( \chi \) of \( H \).

**Definition 5.2.** We define
\[
J_0(\chi) := \begin{cases} 
\{s\} & \text{if } \chi \text{ factors through the determinant,} \\
\emptyset & \text{otherwise},
\end{cases}
\]
\[
J'_0(\chi) := \begin{cases} 
\{s'\} & \text{if } \chi^s = \chi, \\
\emptyset & \text{otherwise}.
\end{cases}
\]

**Definition 5.3.** Let \( \chi : H \to \mathbb{F}_p^\times \) be a character.

(i) Let \( J \subset J_0(\chi) \), and let \( M_{\chi,J} \) denote the character of \( \mathcal{H}_\Gamma \) given by
\[
e_\chi \mapsto 1, \quad e_{\chi'} \mapsto 0, \quad T_{n,s} \mapsto \begin{cases} 
0 & \text{if } s \in J, \\
1 & \text{if } s \in J_0(\chi) \setminus J, \\
0 & \text{if } s \not\in J_0(\chi).
\end{cases}
\]
for \( \chi' \neq \chi \).

(ii) Let \( J' \subset J'_0(\chi) \), and let \( M'_{\chi,J'} \) denote the character of \( \mathcal{H}_{\Gamma'} \) given by
\[
e_\chi \mapsto 1, \quad e_{\chi'} \mapsto 0, \quad T_{n,s'} \mapsto \begin{cases} 
0 & \text{if } s' \in J', \\
1 & \text{if } s' \in J'_0(\chi) \setminus J', \\
0 & \text{if } s' \not\in J'_0(\chi).
\end{cases}
\]
for \( \chi' \neq \chi \).

With these definitions in place, we arrive at the following Proposition.

**Proposition 5.4.** Let \( \chi : H \to \mathbb{F}_p^\times \) be a character.

(i) Every simple right \( \mathcal{H}_\Gamma \)-module is isomorphic to a character \( M_{\chi,J} \) with \( J \subset J_0(\chi) \).

(ii) Every simple right \( \mathcal{H}_{\Gamma'} \)-module is isomorphic to a character \( M'_{\chi,J'} \) with \( J' \subset J'_0(\chi) \).

**Proof.** The pairs \((\mathbb{B}, (N \cap K)/(N \cap K_1))\) and \((\mathbb{B}', (N \cap K')/(N \cap K_1'))\) form “strongly split BN pairs of characteristic \( p \)” (cf. [9] Definition 2.20). The result then follows from Theorem 6.10(iii) of [9]. \( \square \)
5.2. **Carter-Lusztig Theory.** Using the results of the previous section, we may begin classifying the mod-$p$ representations of the finite groups $\Gamma$ and $\Gamma'$. The starting point of this theory relies on Proposition 26 of [29]: if $\rho$ is a nonzero mod-$p$ representation of $\Gamma$, then $\rho^U \neq \{0\}$. The latter space has an action of the Hecke algebra $\mathcal{H}_\Gamma$, so we obtain a functor from the category of mod-$p$ representations of $\Gamma$ to right $\mathcal{H}_\Gamma$-modules. We remark that this discussion holds equally well for $\Gamma'$ and $\Gamma'$. The properties of this functor are made precise by the following Proposition:

**Proposition 5.5.** (i) The functor $\rho \mapsto \rho^U$ induces a bijection between irreducible representations of $\Gamma$ and simple right $\mathcal{H}_\Gamma$-modules.

(ii) The functor $\rho' \mapsto (\rho')^U$ induces a bijection between irreducible representations of $\Gamma'$ and simple right $\mathcal{H}_{\Gamma'}$-modules.

**Proof.** Since $\mathcal{H}_\Gamma$ and $\mathcal{H}_{\Gamma'}$ are Frobenius algebras, the result follows from Proposition 1.25(ii) of [9].

In light of this Proposition, we make the following definition:

**Definition 5.6.** Let $\chi : H \rightarrow \mathbb{F}_p^\times$ be a character.

(i) For $J \subset J_0(\chi)$, we define $\rho_{\chi,J}$ to be the representation of $\Gamma$ such that $\rho_{\chi,J}^U \cong M_{\chi,J}$.

(ii) For $J' \subset J'_0(\chi)$, we define $\rho'_{\chi,J'}$ to be the representation of $\Gamma'$ such that $(\rho'_{\chi,J'})^U \cong M'_{\chi,J'}$.

The irreducible mod-$p$ representations of $\Gamma$ (resp. $\Gamma'$) have been classified by Carter and Lusztig in terms of characters $\chi$ of $H$ and certain subsets of $S$ (resp. $S'$). More precisely, given a nonzero irreducible mod-$p$ representation $\rho$ of $\Gamma$, we have $\rho^U \neq \{0\}$; by Frobenius Reciprocity for finite groups, we obtain a surjection from $\text{ind}_U^H(1)$ onto $\rho$, where 1 denotes the trivial character of $\mathbb{F}_p$. Since $\text{ind}_U^H(1)$ decomposes as a direct sum of $\text{ind}_B^H(\chi)$, we see that $\rho$ is actually a quotient of a parabolically induced representation. In [10], Carter and Lusztig show how to construct irreducible quotients of parabolic inductions $\text{ind}_B^H(\chi)$ by using the Hecke operators $e_\chi$ and $T_{n_s}$ (with analogous results holding for the group $\Gamma'$).

**Proposition 5.7.** Let $\chi : H \rightarrow \mathbb{F}_p^\times$ be a character.

(i) If $\chi$ factors through the determinant, then

\[ \rho_{\chi,S} \cong \text{im}(1 + T_{n_s} : \text{ind}_B^H(\chi) \rightarrow \text{ind}_B^H(\chi)), \]

\[ \rho_{\chi,0} \cong \text{im}(T_{n_s} : \text{ind}_B^H(\chi) \rightarrow \text{ind}_B^H(\chi)). \]

If $\chi$ does not factor through the determinant, then

\[ \rho_{\chi,0} \cong \text{im}(T_{n_s} : \text{ind}_B^H(\chi) \rightarrow \text{ind}_B^H(\chi)). \]

(ii) If $\chi^s = \chi$, then

\[ \rho'_{\chi,S'} \cong \text{im}(1 + T_{n_{s'}} : \text{ind}_{B'}^{H'}(\chi) \rightarrow \text{ind}_{B'}^{H'}(\chi)), \]

\[ \rho'_{\chi,0} \cong \text{im}(T_{n_{s'}} : \text{ind}_{B'}^{H'}(\chi) \rightarrow \text{ind}_{B'}^{H'}(\chi)). \]

If $\chi^s \neq \chi$, then

\[ \rho'_{\chi,0} \cong \text{im}(T_{n_{s'}} : \text{ind}_{B'}^{H'}(\chi) \rightarrow \text{ind}_{B'}^{H'}(\chi^s)). \]

**Proof.** Theorem 7.1 and Corollary 7.5 of [10] imply that the images of the Hecke operators are irreducible and inequivalent; it therefore suffices to match the two sets of representations. Theorem 7.1 and Proposition 6.6 of [10] give the action of $\mathcal{H}_\Gamma$ and $\mathcal{H}_{\Gamma'}$ on the $\mathbb{U}$- and $\mathbb{U}'$-invariants of the image representations. The claim then follows from Proposition 5.4 and Definition 5.6.
Lemma 5.8. Let \( \chi = \zeta \otimes \eta : H \to \overline{\mathbb{F}}_p^\times \) be a character.

(i) Assume \( \chi = \eta \circ \det \). Then \( e_\chi(1 + T_{n_s})e_\chi \) and \(-e_\chi T_{n_s}e_\chi \) are orthogonal idempotents, and induce a splitting

\[
\text{ind}^\Gamma_B(\chi) \cong \rho_{\chi, s} \oplus \rho_{\chi, \emptyset}.
\]

Moreover, we have

\[
\rho_{\chi, s} \cong \eta \circ \det, \quad \rho_{\chi, \emptyset} \cong \eta \circ \det \otimes \text{St},
\]

where \( \text{St} = \text{ind}^\Gamma_B(1)/1 \) is the Steinberg representation of \( \Gamma \).

(ii) Assume \( \chi^s = \chi \). Then \( e_\chi(1 + T_{n_s})e_\chi \) and \(-e_\chi T_{n_s}e_\chi \) are orthogonal idempotents, and induce a splitting

\[
\text{ind}^\Gamma_B(\chi) \cong \rho_{\chi, s'} \oplus \rho_{\chi, \emptyset}.
\]

Let \( \det^* : U(1, 1)(\mathbb{F}_q^2/\mathbb{F}_q) \to U(1)(\mathbb{F}_q^2/\mathbb{F}_q) \) denote the determinant map of the group \( U(1, 1)(\mathbb{F}_q^2/\mathbb{F}_q) \). Then there exists a unique character \( \zeta' : U(1)(\mathbb{F}_q^2/\mathbb{F}_q) \to \overline{\mathbb{F}}_p^\times \) such that \( \chi \cong ( (\zeta' \circ \det^* ) \boxtimes \eta ) |_H \), and we have

\[
\rho_{\chi, s'} \cong (\zeta' \circ \det^* ) \boxtimes \eta, \quad \rho_{\chi, \emptyset} \cong (\zeta' \circ \det^* \circ \text{St}' ) \boxtimes \eta,
\]

where \( \text{St}' = \text{ind}_{U(1, 1)(\mathbb{F}_q^2/\mathbb{F}_q)}(U(1, 1)(\mathbb{F}_q^2/\mathbb{F}_q)) \) is the Steinberg representation of \( U(1, 1)(\mathbb{F}_q^2/\mathbb{F}_q) \).

**Remark.** We use the notation \( \boxtimes \) to denote the external tensor product of representations of \( U(1, 1)(\mathbb{F}_q^2/\mathbb{F}_q) \) and \( U(1)(\mathbb{F}_q^2/\mathbb{F}_q) \).

**Proof.** The first claim of parts (i) and (ii) follow from Theorem 3.10. For a character \( \chi = \eta \circ \det \) that factors through the determinant, we have \( \text{ind}^\Gamma_B(\chi) \cong (\eta \circ \det ) \otimes \text{ind}^\Gamma_B(1) \), so it suffices to assume \( \chi = 1 \) is the trivial character of \( B \). Theorem 7.1 of [10] and Proposition 5.7 now imply that \( \rho_{1, s} \) is the trivial representation, which means \( \rho_{1, \emptyset} = \text{St} \). This proof also applies mutatis mutandis for representations of \( \Gamma' \).

We record one final result regarding the constituents of \( \text{ind}^\Gamma_B(\chi) \), which will be of use later.

Lemma 5.9. Let \( \chi : H \to \overline{\mathbb{F}}_p^\times \) be a character.

(i) Assume \( \chi^s = \chi \). Then we have

\[
\text{ind}^\Gamma_B(\chi) \cong \rho_{\chi, s'} \oplus \rho_{\chi, \emptyset}.
\]

(ii) Assume \( \chi^s \neq \chi \). Then the sequence

\[
0 \to \rho_{\chi, \emptyset} \to \text{ind}^\Gamma_B(\chi) \to \rho_{\chi, s'} \to 0
\]

is exact if and only if \( q = p \). In this case, the sequence is nonsplit.

**Proof.** Part (i) follows from Lemma 5.8. For part (ii), note that the representation \( \rho_{\chi, \emptyset} \) is defined by \( \rho_{\chi, \emptyset} \cong T_{n_s}(\text{ind}^\Gamma_B(\chi)) \); moreover, the endomorphism \( T_{n_s} \) maps \( \text{ind}^\Gamma_B(\chi^s) \) into \( \text{ind}^\Gamma_B(\chi) \), which implies \( \rho_{\chi, \emptyset} \subset \text{ind}^\Gamma_B(\chi) \). Since \( T_{n_s}^2 = 0 \) on the space \( \text{ind}^\Gamma_B(\chi^s) \), we have \( \rho_{\chi, \emptyset} \subset \ker (T_{n_s}) \). If \( \rho_{\chi, \emptyset} \) is isomorphic to the representation \( V'_{j, k} \boxtimes \omega^c \) defined below, Proposition 5.19 implies

\[
\dim_{\overline{\mathbb{F}}_p}(\rho_{\chi, \emptyset}) + \dim_{\overline{\mathbb{F}}_p}(\rho_{\chi, s'}) = \frac{f-1}{i=0} (j_i + 1) + \frac{f-1}{i=0} (p - j_i);
\]

this quantity is equal to \( q + 1 = \dim_{\overline{\mathbb{F}}_p}(\text{ind}^\Gamma_B(\chi)) \) if and only if \( q = p \). Theorem 7.4 of [10] implies that the sequence is nonsplit. \( \square \)
Remark. One can show, by computing dimensions and using Proposition 5.14 below, that for \( \chi \neq \eta \circ \det \) the sequence
\[
0 \to \rho_{\chi^*, \emptyset} \to \text{ind}_F^G(\chi) \to \rho_{\chi, \emptyset} \to 0
\]
is never exact, even for \( q = p \).

5.3. Highest Weight Modules: \( U(2,1) \). We now describe a classification of representations of \( \Gamma \) in terms of highest weight modules. We begin with the representations of \( \text{SL}_3(\mathbb{F}_p) \).

Let \( \chi_{j,k} \) denote the character of the maximal torus of \( \text{SL}_3(\mathbb{F}_p) \) given by
\[
\chi_{j,k}
\begin{pmatrix}
a & 0 & 0 \\
0 & a^{-1}b & 0 \\
0 & 0 & b^{-1}
\end{pmatrix} = a^j b^k,
\]
where \( a, b \in \mathbb{F}_p^\times \) and \( j, k \in \mathbb{Z} \). The characters \( \chi_{j,k} \) with \( j, k \geq 0 \) are called the dominant weights (with respect to the “standard” choice of upper Borel subgroup). The characters \( \chi_{1,0} \) and \( \chi_{0,1} \) are the fundamental dominant weights.

**Theorem 5.10.** The irreducible finite-dimensional mod-\( p \) representations of \( \text{SL}_3(\mathbb{F}_p) \) are parametrized by the set of dominant weights. For a weight \( \chi_{j,k} \), we let \( V_{j,k} \) denote the corresponding representation. If \( U \) denotes the upper unipotent elements of \( \text{SL}_3(\mathbb{F}_p) \), then \( V_{j,k}^U \) is one-dimensional, and the upper Borel subgroup of \( \text{SL}_3(\mathbb{F}_p) \) acts on \( V_{j,k}^U \) by the character \( \chi_{j,k} \).

**Proof.** This is §12, Theorem 39 in [30]. □

Given a representation \( V \) of \( \text{SL}_3(\mathbb{F}_p) \), we form a new representation \( V^{Fr} \): the underlying space of \( V^{Fr} \) is the same as that of \( V \), with the action given by first applying the map \( x \mapsto x^p \) to the entries of an element of \( \text{SL}_3(\mathbb{F}_p) \). In particular, we have \( V_{j,k}^{Fr} \cong V_{pj, pk} \). With this tool we can be more precise about the structure of the representations \( V_{j,k} \) thanks to Steinberg’s Tensor Product Theorem:

**Theorem 5.11.** Let \( j, k \in \mathbb{Z}_{\geq 0} \), and let \( j = \sum_{i \geq 0} j_i p^i \), \( k = \sum_{i \geq 0} k_i p^i \) be the \( p \)-adic expansions of \( j \) and \( k \). Then
\[
V_{j,k} \cong \bigotimes_{i=0}^{\infty} V_{j_i, k_i}^{Fr_i}
\]
as \( \text{SL}_3(\mathbb{F}_p) \)-representations.

**Proof.** This is §12, Theorem 41 in [30]. □

Remark. The theorem above shows that in order to classify the irreducible finite-dimensional representations of \( \text{SL}_3(\mathbb{F}_p) \), it suffices to understand the representations \( V_{j,k} \) with \( 0 \leq j, k < p \). The precise structure of these representations is governed by the Linkage Principle. We shall not need these results here, but for more information on this topic the reader may consult the book of Jantzen ([22]).

**Theorem 5.12.** The representations \( V_{j,k} \) of \( \text{SL}_3(\mathbb{F}_p) \) with \( 0 \leq j, k < q \) remain irreducible upon restriction to \( \text{SU}(2,1)(\mathbb{F}_q^2/\mathbb{F}_q) \). Moreover, the given representations \( V_{j,k} \) exhaust the irreducible mod-\( p \) representations of \( \text{SU}(2,1)(\mathbb{F}_q^2/\mathbb{F}_q) \).

**Proof.** This is §13, Theorem 43 in [30]. □
To obtain the irreducible representations of $\Gamma = U(2,1)(\mathbb{F}_q^2/\mathbb{F}_q)$, we proceed as follows. The subgroup
\[
\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & 1 \end{pmatrix} : \delta \in U(1)(\mathbb{F}_q^2/\mathbb{F}_q) \right\}
\]
gives a full set of coset representatives for $\Gamma/\text{SU}(2,1)(\mathbb{F}_q^2/\mathbb{F}_q)$. For an irreducible representation $V_{j,k}$ of $\text{SU}(2,1)(\mathbb{F}_q^2/\mathbb{F}_q)$, let $V_{j,k}^\delta$ denote the representation with the same underlying space as $V_{j,k}$, with the action given by first conjugating an element of $\text{SU}(2,1)(\mathbb{F}_q^2/\mathbb{F}_q)$ by \( \begin{pmatrix} 1 & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & 1 \end{pmatrix} \). Since these diagonal elements normalize $U \leq \text{SU}(2,1)(\mathbb{F}_q^2/\mathbb{F}_q)$, we have that $(V_{j,k}^\delta)^U = V_{j,k}^U$ (as vector spaces), for every $\delta \in U(1)(\mathbb{F}_q^2/\mathbb{F}_q)$. Moreover, the action of the maximal torus of $\text{SU}(2,1)(\mathbb{F}_q^2/\mathbb{F}_q)$ on these spaces is identical. This implies that $V_{j,k}^\delta \cong V_{j,k}$ for every $\delta \in U(1)(\mathbb{F}_q^2/\mathbb{F}_q)$, which means we may lift $V_{j,k}$ to a projective representation of $\Gamma$. Since
\[
H^2(\Gamma/\text{SU}(2,1)(\mathbb{F}_q^2/\mathbb{F}_q), \mathbb{F}_p^\times) = \{0\},
\]
this representation lifts to a genuine representation of $\Gamma$. After twisting by an appropriate power of the determinant, we can ensure that the element \( \begin{pmatrix} 0 & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & \pi^{-1} \end{pmatrix} \) acts by the scalar $a^\delta$ on $V_{j,k}^\delta$. We continue to denote by $V_{j,k}$ this representation of $\Gamma$.

**Corollary 5.13.** The irreducible mod-$p$ representations of $\Gamma$ are given by $V_{j,k} \otimes (\det)^c$, where $0 \leq j, k < q$ and $0 \leq c < q + 1$.

**Proof.** Restricting $V_{j,k} \otimes (\det)^c$ to $\text{SU}(2,1)(\mathbb{F}_q^2/\mathbb{F}_q)$ veriﬁes that each representation is irreducible, and by examining the action of $H$ (and dimensions of the $V_{j,k}$) we see that they are pairwise nonisomorphic. Since the number of $p$-regular conjugacy classes of $\Gamma$ is $q^2(q+1)$, we conclude that these exhaust all irreducible representations. \qed

We can now provide a dictionary between the Carter-Lusztig description of representations and the description in terms of highest weight modules.

**Proposition 5.14.** Let $\chi = \zeta \otimes \eta : H \to \mathbb{F}_p^\times$ be a character. Let $0 \leq r < q^2 - 1$ be the unique integer such that
\[
\zeta(a) = a^r
\]
for every $a \in \mathbb{F}_q^\times$, and let $0 \leq c < q + 1$ be the unique integer such that
\[
\eta(\delta) = \delta^c
\]
for every $\delta \in U(1)(\mathbb{F}_q^2/\mathbb{F}_q)$.

(i) Assume $r = 0$. Then $\chi = \eta \circ \det$ factors through the determinant, and we have
\[
\rho_{\chi,S} \cong V_{0,0} \otimes V_{0,0}^{Fr} \otimes \cdots \otimes V_{0,0}^{Fr^{-1}} \otimes (\det)^c \cong (\det)^c
\]
\[
\rho_{\chi,0} \cong V_{p-1,p-1} \otimes V_{p-1,p-1}^{Fr} \otimes \cdots \otimes V_{p-1,p-1}^{Fr^{-1}} \otimes (\det)^c \cong St \otimes (\det)^c,
\]
where $St$ denotes the Steinberg representation.

(ii) Assume $r \neq 0$. Then there exists a unique pair $(j,k)$ such that $0 \leq j,k < q$ and $j + qk = r$. Let $j = \sum_{i=0}^{f-1}jp^i$, $k = \sum_{i=0}^{f-1}kp^i$ be the $p$-adic expansions of $j$ and $k$. We have
\[
\rho_{\chi,0} \cong V_{j_0,k_0} \otimes V_{j_1,k_1}^{Fr} \otimes \cdots \otimes V_{j_{f-1},k_{f-1}}^{Fr^{-1}} \otimes (\det)^c.
\]
Proof. In each description of irreducibles, we have $q^2(q+1)$ representations; it therefore suffices to match these representations. Given a character $\chi = \zeta \otimes \eta : H \to \mathbb{F}_p^\times$ with parameters $(r, c)$, we have

$$\chi \left( \begin{array}{ccc} a & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & \overline{\alpha}^{-1} \end{array} \right) = a^r (a\overline{\alpha}^{-1}\delta)^c.$$  

(i) We see that $r = 0$ if and only if $\chi = \eta \circ \det$. In this case, Lemma 5.8 implies that

$$\rho_{\chi,S} \cong \eta \circ \det = (\det)^c \quad \text{and} \quad \rho_{\chi,\theta} \cong \St \otimes \eta \circ \det = \St \otimes (\det)^c.$$  

Of the representations $V_{j,k} \otimes (\det)^c$, the only representations on which $H$ acts by $(a\overline{\alpha}^{-1}\delta)^c$ on the $U$-invariants are $V_{0,0} \otimes (\det)^c$ and $V_{q-1,q-1} \otimes (\det)^c$. We have that the dimension of $V_{0,0} \otimes (\det)^c$ is 1, while the representation $\St$ has dimension $|U| = q^3$; hence we must have

$$(\det)^c \cong V_{0,0} \otimes (\det)^c \quad \text{and} \quad \St \otimes (\det)^c \cong V_{q-1,q-1} \otimes (\det)^c.$$  

(ii) As before, it suffices to match the action of $H$ on the $U$-invariants of each representation. Let $(j, k)$ be integers such that $0 \leq j, k < q$ and $r = j + qk$. We see that $H$ acts on $(V_{j,k} \otimes (\det)^c)^U$ by $a^{j+qk}(a\overline{\alpha}^{-1}\delta)^c = a^r (a\overline{\alpha}^{-1}\delta)^c$. Writing out the $p$-adic expansions of $j$ and $k$ implies

$$\rho_{\chi,\theta} \cong V_{j_0,k_0} \otimes V_{j_1,k_1}^{\Fr_1} \otimes \cdots \otimes V_{j_{q-1},k_{q-1}}^{\Fr_{q-1}} \otimes (\det)^c.$$  

$\square$

Remark. The results of [30] apply more generally to a split reductive algebraic group over a finite field with a semisimple derived subgroup, or one of their “twisted analogues.” In particular, we may match the representations $\Theta_{\text{new}}^J \mathcal{F}_\chi$ of Carter-Lusztig with the highest weight modules as described in [30].

5.4. Highest Weight Modules: $U(1,1) \times U(1)$. We now describe representations of $\Gamma' = (U(1,1) \times U(1))(\mathbb{F}_p^2/\mathbb{F}_q)$ in terms of highest weight modules. Every such representation is of the form $\rho' \boxtimes \eta$, where $\rho'$ is a representation of $U(1,1)(\mathbb{F}_p^2/\mathbb{F}_q)$ and $\eta$ is a character of $U(1)(\mathbb{F}_p^2/\mathbb{F}_q)$. Though we may again use the results of [30], we instead proceed in a more explicit and ad hoc manner.

Definition 5.15. Let $0 \leq j < q, 0 \leq k < q+1$, and let $j = \sum_{i=0}^{f-1} j_i p^i$ be the $p$-adic expansion of $j$. We denote by $V_{j,k}'$ the representation of $U(1,1)(\mathbb{F}_p^2/\mathbb{F}_q)$ given by

$$\Sym^{j_0}(\mathbb{F}_p^2) \otimes \Sym^{j_1}(\mathbb{F}_p^2)^{\Fr_1} \otimes \cdots \otimes \Sym^{j_{q-1}-1}(\mathbb{F}_p^2)^{\Fr_{q-1}} \otimes (\det^*)^k,$$

where $\det^*$ denotes the determinant map of $U(1,1)(\mathbb{F}_p^2/\mathbb{F}_q)$.

Definition 5.16. We let $\omega : U(1)(\mathbb{F}_p^2/\mathbb{F}_q) \hookrightarrow \mathbb{F}_p^\times \to \mathbb{F}_p^\times$ denote a fixed fundamental character of $U(1)(\mathbb{F}_p^2/\mathbb{F}_q)$. Every $\mathbb{F}_p^\times$-valued character of $U(1)(\mathbb{F}_p^2/\mathbb{F}_q)$ is of the form $\omega^c$, $0 \leq c < q+1$.

Theorem 5.17. The irreducible mod-$p$ representations of $\Gamma'$ are given by $V_{j,k}' \boxtimes \omega^c$, where $0 \leq j < q$ and $0 \leq k, c < q+1$.

Proof. One may check that the $q(q+1)^2$ representations $V_{j,k}' \boxtimes \omega^c$ are inequivalent and irreducible. Since the number of $p$-regular conjugacy classes of $\Gamma'$ is $q(q+1)^2$, we conclude that these exhaust all irreducible representations. $\square$
Remark. We remark that this theorem may be deduced from the fact that \( \text{SU}(1, 1)(\mathbb{F}_q'/\mathbb{F}_q) \) is conjugate to \( \text{SL}_2(\mathbb{F}_q) \) inside of \( \text{SL}_2(\mathbb{F}_p) \).

**Lemma 5.18.** Let \( 0 \leq j < q \) and \( 0 \leq k, c < q + 1 \). Then the action of \( H \) on \( (V'_{j,k} \boxtimes \omega^c)^{1/2} \) is given by the character 
\[
\left( \begin{array}{ccc}
0 & 0 & 0 \\
0 & \delta & 0 \\
0 & 0 & \tau^{-1}
\end{array} \right) \mapsto a^{-qj+(1-q)k} \delta^c.
\]

**Proof.** The previous theorem implies that \( V'_{j,k} \boxtimes \omega^c \) is irreducible, and Proposition 5.5 implies that \( (V'_{j,k} \boxtimes \omega^c)^{1/2} \) is one-dimensional. Let \( j = \sum_{i=0}^{j-1} j_i p^i \) be the \( p \)-adic expansion of \( j \), and let \( \{v_1, v_2\} \) be the standard basis of \( \mathbb{F}_p^2 \). The vector 
\[
v_2^{j_0} \otimes v_2^{j_1} \otimes \cdots \otimes v_2^{j_{j-1}} \otimes 1 \otimes 1
\]
is fixed by \( U' \), and therefore spans \( (V'_{j,k} \boxtimes \omega^c)^{1/2} \). The action of \( \left( \begin{array}{ccc}
0 & 0 & 0 \\
0 & \delta & 0 \\
0 & 0 & \tau^{-1}
\end{array} \right) \) on this vector is given by 
\[
(\tau^{-1})^j \delta^{-qj+(1-q)k} \delta^c = a^{-qj+(1-q)k} \delta^c.
\]

We may now provide a dictionary between the representations \( \rho'_{X,J} \) and \( V'_{j,k} \boxtimes \omega^c \).

**Proposition 5.19.** Let \( \chi = \zeta \otimes \eta : H \to \mathbb{F}_p^\times \) be a character. Let \( 0 \leq c < q + 1 \) be the unique integer such that 
\[
\eta(\delta) = \delta^c
\]
for every \( \delta \in \text{U}(1)(\mathbb{F}_q^2/\mathbb{F}_q) \), and let \( 0 \leq r < q^2 - 1 \) be the unique integer such that 
\[
\zeta(a) = a^{r+c(q-1)}
\]
for every \( a \in \mathbb{F}_q^\times \).

(i) Assume \( r \equiv 0 \) (mod \( q-1 \)). Then \( \chi^s = \chi \), and there exists a unique integer \( 0 \leq k < q+1 \) such that \( (1-q)k \equiv r \) (mod \( q^2 - 1 \)). We have 
\[
\rho'_{\chi,S^c} \cong V'_{0,k} \boxtimes \omega^c \cong (\text{det}^*)^k \boxtimes \omega^c
\]
and 
\[
\rho'_{\chi,0} \cong V'_{q-1,k+1} \boxtimes \omega^c \cong (\text{St}' \otimes (\text{det}^*)^k) \boxtimes \omega^c,
\]
where \( \text{St}' \) denotes the Steinberg representation of \( \text{U}(1,1)(\mathbb{F}_q^2/\mathbb{F}_q) \).

(ii) Assume \( r \not\equiv 0 \) (mod \( q-1 \)). Then \( \chi^s \not\equiv \chi \), and there exists a unique pair \( (j, k) \) such that \( 0 < j < q, 0 \leq k < q+1 \) and \( -aqj + (1-q)k \equiv r \) (mod \( q^2 - 1 \)). Let \( j = \sum_{i=0}^{j-1} j_i p^i \) be the \( p \)-adic expansion of \( j \). We have 
\[
\rho'_{\chi,0} \cong V'_{j,k} \boxtimes \omega^c \cong (\text{Sym}^{j_0}(\mathbb{F}_p^2) \otimes \text{Sym}^{j_1}(\mathbb{F}_p^2) \otimes \cdots \otimes \text{Sym}^{j_{j-1}}(\mathbb{F}_p^2)^{p^{j-1}} \otimes (\text{det}^*)^k) \boxtimes \omega^c.
\]

**Proof.** In each description of irreducibles, we have \( q(q+1)^2 \) representations; it suffices to match these representations. Given a character \( \chi = \zeta \otimes \eta : H \to \mathbb{F}_p^\times \) with parameters \( (r, c) \), we have 
\[
\chi \left( \begin{array}{ccc}
a & 0 & 0 \\
0 & \delta & 0 \\
0 & 0 & \tau^{-1}
\end{array} \right) = a^r \delta^c.
\]

(i) We see that \( r \equiv 0 \) (mod \( q-1 \)) if and only if \( \chi^s = \chi \). Let \( 0 \leq k < q+1 \) be the unique integer such that \( (1-q)k \equiv r \) (mod \( q^2 - 1 \)). In this case, Lemma 5.8 implies that there exists a unique \( \zeta' : \text{U}(1)(\mathbb{F}_q^2/\mathbb{F}_q) \to \mathbb{F}_p^\times \) such that 
\[
\rho'_{\chi,S^c} \cong (\zeta' \circ \text{det}^*) \boxtimes \eta = (\text{det}^*)^k \boxtimes \omega^c \quad \text{and} \quad \rho'_{\chi,0} \cong (\text{St}' \otimes \zeta' \circ \text{det}^*) \boxtimes \eta = (\text{St}' \otimes (\text{det}^*)^k) \boxtimes \omega^c.
\]
Of the representations $V'_{j,k} \boxtimes \omega^c$, the only representations on which $H$ acts by $a^{(1-q)k}\delta^c$ on the $U'$-invariants are $V'_{0,k} \boxtimes \omega^c$ and $V'_{q-1,k+1} \boxtimes \omega^c$. Since $\dim_p(V'_{0,k} \boxtimes \omega^c) = 1$ and the representation $St'$ has dimension $|U'| = q$, we see that we must have
\[(\zeta' \circ \det^*) \boxtimes \eta \cong V'_{0,k} \boxtimes \omega^c \quad \text{and} \quad (St' \otimes (\det^*)^k) \boxtimes \omega^c \cong V'_{q-1,k+1} \boxtimes \omega^c.\]

(ii) As before, it suffices to compute the action of $H$ on the $U'$-invariants of each representation. Let $(j, k)$ be a pair of integers such that $0 \leq j < q, 0 \leq k < q + 1$ and $r \equiv -qj + (1 - q)k \pmod{q^2 - 1}$; the condition $r \not\equiv 0 \pmod{q - 1}$ implies $1 \leq j \leq q - 2$ and $\chi^q \neq \chi$, which in turn implies that the pair $(j, k)$ is unique. Lemma 5.3.5 implies that $H$ acts on $(V'_{j,k} \boxtimes \omega^c)^{U'}$ by $a^{-qj+(1-q)k}\delta^c = a^r\delta^c$. Writing out the $p$-adic expansion of $j$ implies
\[\rho_{X,0} \cong V'_{j,k} \boxtimes \omega^c \cong (\Sym^j(\mathbb{F}_p^2) \otimes \Sym^j(\mathbb{F}_p^2)^{Fr} \otimes \cdots \otimes \Sym^{j-1}(\mathbb{F}_p^2)^{Fr^{-1}} \otimes (\det^*)^k) \boxtimes \omega^c.\]

\[\square\]

6. Diagrams and Coefficient Systems

6.1. Definitions and First Properties. In this section, we follow [26] closely and translate the language of coefficient systems and diagrams to the group $G$. In fact, our case is even easier to some extent, due mainly to the fact that the extended Bruhat-Tits building of $G$ coincides with the reduced Bruhat-Tits building, and therefore stabilizers of vertices are maximal compact subgroups of $G$.

Let $X$ be the reduced Bruhat-Tits building of $G$ (which is also the Bruhat-Tits building of $SU(2,1)(E/F)$). We refer the reader to [32] for an excellent exposition, particularly Sections 2.7 and 2.10. The building $X$ is a simplicial complex of dimension 1 (that is, a tree), with a natural action of $G$. We let $X_0$ denote the set of all vertices on the tree, and let $X_1$ denote the set of all edges of $X$. Given a simplex $\sigma \subset X$, we let $\mathcal{R}(\sigma) \leq G$ denote its stabilizer subgroup. We denote by $A$ the apartment corresponding to the maximal $F$-split torus of $T$. Since the group $K$ is hyperspecial, there exists a vertex $\sigma_0$ in $A$ such that $\mathcal{R}(\sigma_0) = K$. Moreover, there exists a unique vertex $\sigma'_0$ neighboring $\sigma_0$ in $A$ such that $\mathcal{R}(\sigma'_0) = K'$ (cf. [32] Section 3.1.1). The vertex $\sigma'_0$ has $q + 1$ neighboring vertices in $X$; the vertex $\sigma_0$ has $q^3 + 1$ neighboring vertices and is hyperspecial (these facts follow from Statement 3.5.4 in [32] and $|\Gamma/\mathbb{B}| = q^3 + 1, |\Gamma'/\mathbb{B}'| = q + 1$). Moreover, in any fixed apartment the vertices alternate valency (that is, the number of neighboring vertices in $X$) between $q^3 + 1$ and $q + 1$. We let $\tau_1$ denote the edge from $\sigma_0$ to $\sigma'_0$; we have $\mathcal{R}(\tau_1) = I$. We remark that we may alternatively define $X$ in terms of additive norms (cf. [32] Examples 2.10 and 3.11).

The tree $X$ has a natural (combinatorial) $G$-invariant distance function, and we denote by $X^e_0$ (resp. $X^o_0$) the set of vertices at an even (resp. odd) distance from $\sigma_0$. The group $N$ acts on $A$, and under this action the points of $A \cap X^e_0$ (resp. $A \cap X^o_0$) are all conjugate. Given any vertex $\sigma$ in $X^e_0$ (resp. $X^o_0$) and an apartment $A'$ containing $\sigma$ and $\sigma_0$, there exists an element $g \in G$ fixing $\sigma_0$ such that $g.A' = A$ (cf. Section 2.2.1 in [32]). This implies that all vertices in $X^e_0$ (resp. $X^o_0$) are conjugate. Since the action of $G$ preserves valency, we conclude that $X^e_0$ and $X^o_0$ constitute two disjoint orbits for the action of $G$ on $X_0$.

Coefficient systems over $\mathbb{C}$ were first introduced in [28] by Schneider and Stuhler, and used in the mod-$p$ setting by Paskunas in [26]. We recall the definition.
Definition 6.1. A coefficient system $\mathcal{V} = (V_\sigma)_\sigma$ on $X$ consists of $\mathbb{F}_p$-vector spaces $V_\sigma$ for every simplex $\sigma \subset X$, along with linear restriction maps $r_\sigma^\tau : V_\tau \to V_\sigma$ for every inclusion $\sigma \subset \tau$, such that $r_\sigma^\sigma = \text{id}_{V_\sigma}$ for every $\sigma$.

Definition 6.2. Let $\mathcal{V} = (V_\sigma)_\sigma$ be a coefficient system on $X$. We say the group $G$ acts on $\mathcal{V}$ if for every $g \in G$ and every simplex $\sigma \subset X$, we have linear maps $g_\sigma : V_\sigma \to V_{g.\sigma}$ satisfying the following properties:

(i) For every $g, h \in G$ and every simplex $\sigma \subset X$, we have $(gh)_\sigma = g_{h.\sigma} \circ h_\sigma$.
(ii) For every simplex $\sigma \subset X$, we have $1_\sigma = \text{id}_{V_\sigma}$.
(iii) For every $g \in G$ and every inclusion $\sigma \subset \tau$, the following diagram commutes:

\[
\begin{array}{ccc}
V_\tau & \xrightarrow{g_\tau} & V_{g.\tau} \\
\downarrow{r_\sigma^\tau} & & \downarrow{r_{g.\sigma}^{g.\tau}} \\
V_\sigma & \xrightarrow{g_\sigma} & V_{g.\sigma}
\end{array}
\]

Definition 6.3. Let $\mathcal{V} = (V_\sigma)_\sigma$ be a coefficient system on which $G$ acts. In particular, the definition above implies that $V_\sigma$ is a representation of $\mathcal{R}(\sigma)$ for every simplex $\sigma \subset X$. If this action is smooth, we call $\mathcal{V}$ a $G$-equivariant coefficient system. We denote by $\mathcal{C}O\mathcal{E}\mathcal{F}_G$ the category of all $G$-equivariant coefficient systems on $X$, with the evident morphisms.

Before going on to more details, we record the following useful fact: given a $G$-equivariant coefficient system $\mathcal{V} = (V_\sigma)_\sigma$, let $\tau = \{\sigma, \sigma'\}$ be an edge such that $\sigma \in X^c_0, \sigma' \in X^c_0$. There exists $g \in G$ such that $\tau = g.\tau_1$, meaning $\sigma = g.\sigma_0$ and $\sigma' = g.\sigma'_0$. We see that this implies

$$V_\sigma = g_{\sigma_0}.V_{\sigma_0}, \quad V_{\sigma'} = g_{\sigma'_0}.V_{\sigma'_0}, \quad V_\tau = g_{\tau_1}.V_{\tau_1}.$$ 

From these translation relations, we have the following relations on the restriction maps $r_\sigma^\tau$:

$$r_\sigma^\tau = g_{\sigma_0} \circ r_{\sigma_0}^{\tau_1} \circ (g^{-1})_\tau, \quad r_{\sigma'}^\tau = g_{\sigma'_0} \circ r_{\sigma'_0}^{\tau_1} \circ (g^{-1})_{\tau}.$$

Definition 6.4. A diagram is a quintuple $D = (D_0, D_0', D_1, r, r')$, in which $(\rho_0, D_0)$ is a smooth representation of $K$, $(\rho'_0, D_0')$ is a smooth representation of $K'$, $(\rho_1, D_1)$ is a smooth representation of $I$, and $r \in \text{Hom}_I(D_1, D_0|_I)$, $r' \in \text{Hom}_I(D_1, D_0'|_I)$.

We may represent a diagram pictorally as:

```
\begin{tikzpicture}
    \node (D0) at (0,0) {$D_0$};
    \node (D1) at (-1,-1) {$D_1$};
    \node (D0') at (-1,1) {$D_0'$};
    \draw[->] (D0) to (D1);
    \draw[->] (D1) to (D0');
    \draw[->] (D0) to [bend right] (D0');
    \draw[->] (D0) to [bend left] (D0');
\end{tikzpicture}
```

Definition 6.5. A morphism $\psi$ between two diagrams $D = (D_0, D_0', D_1, r_D, r'_D)$ and $E = (E_0, E_0', E_1, r_E, r'_E)$ is a triple $(\psi_0, \psi'_0, \eta_1)$, where $\psi_0 \in \text{Hom}_K(D_0, E_0)$, $\psi'_0 \in \text{Hom}_{K'}(D_0', E_0')$, and $\eta_1 \in \text{Hom}_I(D_1, E_1)$, such that the squares in the following diagram commute as $I$-representations:
We say $\psi$ is an embedding if the maps $\psi_0, \psi'_0,$ and $\eta_1$ are injective.

The set of diagrams with the morphisms defined above becomes a category, which we denote by $\mathcal{DLAG}$. The main result here is:

**Theorem 6.6.** The categories $\mathcal{DLAG}$ and $\mathcal{COEFG}$ are equivalent. The equivalence is induced by the functors

$$D : \mathcal{COEFG} \to \mathcal{DLAG}$$
$$C : \mathcal{DLAG} \to \mathcal{COEFG},$$

where $C$ and $D$ are as in Definitions 6.15 and 6.16.

**Proof.** See Appendix. □

6.2. **Homology.** Let $\mathcal{V} = (V_\tau)_\tau$ be a $G$-equivariant coefficient system. We denote by $C_c(X_0, \mathcal{V})$ the $\mathbb{F}_p$-vector space of all maps:

$$\omega : X_0 \to \bigoplus_{\sigma \in X_0} V_\sigma,$$

such that:

- $\omega$ has finite support;
- $\omega(\sigma) \in V_\tau$ for every vertex $\sigma$.

We call such a map $\omega$ a 0-chain.

Let $X(1)$ be the set of all oriented edges: if $\{\sigma, \sigma'\}$ is an edge, we let $(\sigma, \sigma')$ denote the oriented edge from $\sigma$ to $\sigma'$. Denote by $C_c(X(1), \mathcal{V})$ the $\mathbb{F}_p$-vector space of all maps:

$$\omega : X(1) \to \bigoplus_{(\sigma, \sigma') \in X_1} V_{\{\sigma, \sigma'\}},$$

such that

- $\omega$ has finite support;
- $\omega((\sigma, \sigma')) \in V_{\{\sigma, \sigma'\}}$;
- $\omega((\sigma', \sigma)) = -\omega((\sigma, \sigma'))$.

We call such a map $\omega$ a 1-chain.

There is an action of $G$ on the two spaces above, induced from the action of $G$ on the tree $X$ and the coefficient system $\mathcal{V}$. Explicitly, for an element $g \in G$, we have

$$\omega \mapsto g \circ \omega(g^{-1} \cdot \omega),$$

for $\omega \in C_c(X_0, \mathcal{V})$; and

$$\omega \mapsto g \circ \omega(g^{-1} \cdot \sigma, g^{-1} \cdot \sigma'),$$

for $\omega \in C_c(X(1), \mathcal{V})$.

The action on both spaces is smooth.

The boundary map $\partial$ is defined as:
\[
\partial : C_c(X(1), \mathcal{V}) \to C_c(X_0, \mathcal{V})
\]
\[
\omega \mapsto \left(\sigma \mapsto \sum_{\sigma'} r_{\sigma}^{(\sigma, \sigma')}((\omega(\sigma, \sigma')))\right),
\]
where \(\sigma'\) ranges over all neighbors of the vertex \(\sigma\). One may easily verify that \(\partial\) is a \(G\)-equivariant map. We define \(H_0(X, \mathcal{V})\) as the cokernel of \(\partial\) and \(H_1(X, \mathcal{V})\) as the kernel of \(\partial\), both of which inherit a smooth action of \(G\).

6.3. Properties of \(H_0(X, \mathcal{V})\) and \(H_1(X, \mathcal{V})\). We fix a \(G\)-equivariant coefficient system \(\mathcal{V} = (V_r)_{r}\).

**Lemma 6.7.** Let \(\omega\) be a 1-chain, supported on a single edge \(\tau = \{\sigma, \sigma'\}\). Then
\[
\partial(\omega) = \omega_\sigma - \omega_{\sigma'},
\]
where \(\omega_\sigma\) and \(\omega_{\sigma'}\) are two 0-chains, supported respectively on \(\sigma\) and \(\sigma'\). More precisely, letting \(v = \omega((\sigma, \sigma'))\), then we have
\[
\omega_\sigma(\sigma) = r_\sigma^\tau(v), \quad \text{and} \quad \omega_{\sigma'}(\sigma') = r_{\sigma'}^{\tau}(v).
\]

**Proof.** This follows directly from the definition of the boundary map \(\partial\). \(\Box\)

**Lemma 6.8.** Let \(\omega\) be a 0-chain, supported on a single vertex \(\sigma\). Suppose that the two restriction maps \(r_{\sigma_0}^{\tau_0}\) and \(r_{\sigma_0}^{\tau_1}\) are both injective. Then the image of \(\omega\) in \(H_0(X, \mathcal{V})\) is nonzero.

**Proof.** From the assumption and equation (6) above, we see every restriction map is injective. The claim then follows from [26], Lemma 5.7. \(\Box\)

**Lemma 6.9.** Suppose that the two restriction maps \(r_{\sigma_0}^{\tau_1}\) and \(r_{\sigma_0}^{\tau_1}\) are both injective. Then
\(H_1(X, \mathcal{V}) = \{0\}\).

**Proof.** Let \(\omega \in C_c(X(1), \mathcal{V})\) be a nonzero 1-chain such that \(\partial(\omega) = 0\), and let \(\sigma\) be a vertex which is contained in only one edge \(\tau = \{\sigma, \sigma'\}\) of the support of \(\omega\). We then have \(0 = \partial(\omega)(\sigma) = r_{\sigma}^{\tau}(\omega((\sigma, \sigma')))\). Injectivity of the restriction maps implies \(r_{\sigma}^{\tau}(\omega((\sigma, \sigma'))) \neq 0\), a contradiction. \(\Box\)

**Lemma 6.10.** Let \(\omega\) be a 0-chain. Suppose the two restriction maps \(r_{\sigma_0}^{\tau_1}\) and \(r_{\sigma_0}^{\tau_1}\) are both surjective. Then, for any vertex \(\sigma\), there is a 0-chain \(\omega_\sigma\), supported on the single vertex \(\sigma\), such that
\[
\omega + \partial(C_c(X(1), \mathcal{V})) = \omega_\sigma + \partial(C_c(X(1), \mathcal{V})).
\]

**Proof.** As \(r_{\sigma_0}^{\tau_1}\) and \(r_{\sigma_0}^{\tau_1}\) are both surjective, we see that every restriction map is surjective by equation (6). The claim then follows from [26], Lemma 5.8. \(\Box\)

**Proposition 6.11.** Suppose \(r_{\sigma_0}^{\tau_1}\) and \(r_{\sigma_0}^{\tau_1}\) are both isomorphisms of vector spaces. We then have \(H_0(X, \mathcal{V})|_K \cong V_{\sigma_0}\), \(H_0(X, \mathcal{V})|_{K'} \cong V_{\sigma_0}'\), and \(H_0(X, \mathcal{V})|_I \cong V_{\tau_1}\).

**Proof.** For \(\sigma = \sigma_0\) or \(\sigma'_0\), denote by \(C_c(\sigma, \mathcal{V})\) the vector space of 0-chains with support contained in \(\{\sigma\}\). We then have an evaluation map \(ev_\sigma\), which is an isomorphism of \(\mathfrak{R}(\sigma)\)-representations:
\[
ev_\sigma : C_c(\sigma, \mathcal{V}) \to V_{\sigma}
\]
\[
\omega \mapsto \omega(\sigma).
\]
Let $j_\sigma$ be the composition of the inclusion $C_c(\sigma, \mathcal{V}) \to C_c(X_0, \mathcal{V})$ and the canonical map $C_c(X_0, \mathcal{V}) \to H_0(X, \mathcal{V})$. It is easily seen to be $\mathcal{R}(\sigma)$-equivariant. Moreover, Lemma 6.8 and Lemma 6.10 imply that $j_\sigma$ is an isomorphism of vector spaces. Hence, we get a $\mathcal{R}(\sigma)$-equivariant isomorphism $j_\sigma \circ (ev_\sigma)^{-1} : V_\sigma \to H_0(X, \mathcal{V})|_{\mathcal{R}(\sigma)}$. Since the restriction maps $r_\sigma^1$ are isomorphisms of $I$-representations, and $I \subset \mathcal{R}(\sigma)$, we see that $\iota_\sigma = j_\sigma \circ (ev_\sigma)^{-1} \circ r_\sigma^1 : V_\tau \to H_0(X, \mathcal{V})|_I$ is an isomorphism of $I$-representations.

**Corollary 6.12.** Suppose $r_{\sigma_0}^1$ and $r_{\sigma_0'}^1$ are both isomorphisms of vector spaces, and let $\sigma = \sigma_0$ or $\sigma'$. Then the following diagram of $I$-representations commutes:

\[
\begin{array}{ccc}
V_\tau^1 & \xrightarrow{ev} & H_0(X, \mathcal{V}) \\
\downarrow r_\sigma^1 & & \downarrow \text{id} \\
V_\sigma & \xrightarrow{j_\sigma \circ (ev_\sigma)^{-1}} & H_0(X, \mathcal{V})
\end{array}
\]

**Proof.** This follows readily from the previous Theorem. \qed

### 6.4. Constant functor

Let $\pi$ be a smooth representation of $G$, with underlying space $W$. We define a constant coefficient system $\mathcal{K}_\pi$ as follows. Let $\sigma$ be a simplex on the tree $X$, and set

$$(\mathcal{K}_\pi)_\sigma = W.$$ 

If $\sigma \subset \tau$ are two simplices, the restriction map $r_{\sigma}^\tau$ is defined as $\text{id}_W$. For every $g \in G$, and every simplex $\sigma$ in $X$, the linear map $g_\sigma$ is defined by:

$g_\sigma : (\mathcal{K}_\pi)_\sigma \to (\mathcal{K}_\pi)_{g\sigma}$

$\omega \mapsto \pi(g)\omega.$

**Lemma 6.13.** Let $\pi$ be a smooth representation of $G$. Then

$$H_0(X, \mathcal{K}_\pi) \cong \pi$$

as $G$-representations.

**Proof.** Define an evaluation map $ev$ from $C_c(X_0, \mathcal{K}_\pi)$ to $\pi$:

$$ev : C_c(X_0, \mathcal{K}_\pi) \to \pi$$

$$\omega \mapsto \sum_{\sigma \in X_0} \omega(\sigma).$$

As the restriction maps are $\text{id}_W$, we see from Lemma 6.7 that the image of the boundary map $\partial$ is contained in $\ker(ev)$. Hence $ev$ induces a $G$-equivariant map:

$$ev : H_0(X, \mathcal{K}_\pi) \to \pi.$$ 

We need to show that this map is an isomorphism of vector spaces. Since $(\mathcal{K}_\pi)_\sigma = W$, we have

$$ev|_{C_c(\sigma, \mathcal{K}_\pi)} = ev_\sigma,$$

i.e., $ev_\sigma = ev \circ j_\sigma$. As the restriction maps are all $\text{id}_W$, Proposition 6.11 implies $j_\sigma$ is an isomorphism, which gives $ev = ev_\sigma \circ j_\sigma^{-1} : H_0(X, \mathcal{K}_\pi) \cong (\mathcal{K}_\pi)_\sigma$, as desired. \qed

**Proposition 6.14.** Let $\mathcal{V} = (\mathcal{V})_\sigma$ be a $G$-equivariant coefficient system with restriction maps $r_{\sigma}^1$, and let $(\pi, W)$ be a smooth representation of $G$. Then

$$\text{Hom}_{\text{CG}}(\mathcal{V}, \mathcal{K}_\pi) \cong \text{Hom}_{G}(H_0(X, \mathcal{V}), \pi)$$
Proof. By Lemma 6.13, \( H_0(X, K_\pi) \cong \pi \). Any morphism between \( G \)-equivariant coefficient systems induces a homomorphism between the corresponding 0-homology which is compatible with the action of \( G \); that is, there is a map
\[
\text{Hom}_{\text{COEF}_G}(V, K_\pi) \to \text{Hom}_G(H_0(X, V), \pi),
\]
and it suffices to construct an inverse to this map.

Let \( \phi \in \text{Hom}_G(H_0(X, V), \pi) \). Given a vertex \( \sigma \), and a vector \( v \) in \( V_\sigma \), let \( \omega_{\sigma, v} \) be the 0-chain such that
\[
\text{supp}(\omega_{\sigma, v}) \subset \{ \sigma \}, \quad \omega_{\sigma, v}(\sigma) = v.
\]
For this vertex \( \sigma \), we define
\[
\phi_\sigma : V_\sigma \to W, \quad v \mapsto \phi(\omega_{\sigma, v} + \partial(C_c(X(1), V))).
\]
For an edge \( \tau \) in \( X \), with endpoints \( \sigma \) and \( \sigma' \), we define:
\[
\phi_\tau : V_\tau \to W, \quad v' \mapsto \phi_\sigma(r_\tau^{\sigma'}(v')).
\]
The independence of the choice of the vertex \( \sigma \) in the definition of \( \phi_\tau \) results from Lemma 6.7.

The linear maps \((\phi_\sigma)\) constitute a morphism from \( V \) to \( K_\pi \), respecting the \( G \)-action on both. One can check that \((\phi_\sigma)\) induces \( \phi \) on the 0-homology.

\[\Box\]

6.5. The Functors \( \mathcal{C} \) and \( \mathcal{D} \). To prove the equivalence of \( \text{COEF}_G \) and \( \text{DIAG} \), we first observe that there is an obvious functor in one direction:

**Definition 6.15.** Let \( \mathcal{D} \) be the functor from \( \text{COEF}_G \) to \( \text{DIAG} \) given by:
\[
\mathcal{D} : \text{COEF}_G \to \text{DIAG}, \quad \mathcal{V} = (V_\sigma) \mapsto (V_{\sigma_0}, V_{\sigma_0}', V_{\tau_1}, r_{\sigma_0}, r_{\sigma_0}')
\]

Let \( D = (D_0, D_0', D_1, r, r') \) be a fixed diagram in \( \text{DIAG} \). We consider the following compactly induced representations:
\[
c\text{-ind}_K^G(\rho_0), \ c\text{-ind}_K^G(\rho_0'), \ c\text{-ind}_I^G(\rho_1).
\]

For a vertex \( \sigma \in X_0 \), there exists \( g \in G \) such that \( \sigma = g.\sigma_0 \) or \( \sigma = g.\sigma_0' \), depending on whether \( \sigma \in X_0^e \) or \( \sigma \in X_0^o \). We define
\[
F_\sigma = \begin{cases}
\{ f \in \text{c\text{-ind}}_K^G(\rho_0) : \text{supp}(f) \subset K g^{-1} \} & \text{if } \sigma \in X_0^e, \\
\{ f \in \text{c\text{-ind}}_K^G(\rho_0') : \text{supp}(f) \subset K' g^{-1} \} & \text{if } \sigma \in X_0^o.
\end{cases}
\]
For an edge \( \tau \in X_1 \), there exists \( g \in G \) such that \( \tau = g.\tau_1 \). We define
\[
F_\tau = \{ f \in \text{c-ind}_{G}(\rho_1) : \text{supp}(f) \subset Ig^{-1}\}.
\]
We note that these definitions are independent of the choice of \( g \).

6.5.2. Restriction maps. To define restriction maps of the coefficient system, we begin with the two given maps \( r \) and \( r' \), and extend by translations. The evaluation map \( ev_{\sigma_0} : F_{\sigma_0} \to D_0 \) is naturally an isomorphism of \( K \)-representations. Explicitly, it is defined by
\[
ev_{\sigma_0} : F_{\sigma_0} \to D_0
f \mapsto f(1),
\]
with inverse \( ev^{-1}_{\sigma_0} \) given by
\[
ev^{-1}_{\sigma_0} : D_0 \to F_{\sigma_0}
v \mapsto f_v,
\]
where \( f_v : G \to D_0 \) has support in \( K \), and \( f_v(k) = \rho_0(k)v \) for \( k \in K \) (this is the function denoted \( \tilde{f}_v \) in [3]). One defines \( ev_{\sigma_0}^\prime \) and its inverse \( ev^{-1}_{\sigma_0}^\prime \) similarly.

We also have isomorphisms \( ev_{\tau_1} \) and \( ev^{-1}_{\tau_1} \) of \( I \)-representations, given by
\[
ev_{\tau_1} : F_{\tau_1} \to D_1
f \mapsto f(1),
\]
and
\[
ev^{-1}_{\tau_1} : D_1 \to F_{\tau_1}
v \mapsto f_v,
\]
where \( f_v : G \to D_1 \) has support in \( I \), and \( f_v(i) = \rho_1(i)v \) for \( i \in I \).

Let \( r_{\sigma_0}^\tau = ev_{\sigma_0}^{-1} \circ r \circ ev_{\tau_1} ; \) this is an \( I \)-equivariant map from \( F_{\tau_1} \) to \( F_{\sigma_0} \). Explicitly, it is given by
\[
r_{\sigma_0}^\tau(f_v) = f_{r(v)},
\]
where \( v \in D_1 \). We define \( r_{\sigma_0}^\tau = ev_{\sigma_0}^{-1} \circ r' \circ ev_{\tau_1} ; \) it enjoys the same properties as \( r_{\sigma_0}^\tau \).

To summarize, given a diagram \( D = (D_0, D_0', D_1, r, r') \), we may construct a diagram \( \tilde{D} = (F_{\sigma_0}, F_{\sigma_0'}, F_{\tau_1}, r_{\sigma_0}^\tau, r_{\tau_1}^\tau) \); the diagrams \( \tilde{D} \) and \( D \) are isomorphic via \( \text{ev} = (ev_{\sigma_0}, ev_{\sigma_0'}, ev_{\tau_1}) \).

Now let \( \tau \) be an edge, containing a vertex \( \sigma \), and suppose \( \sigma \in X_0^e \). Then there exists \( g \in G \) such that \( \tau = g.\tau_1 \), and \( \sigma = g.\sigma_0 \), where the choice of \( g \) is unique up to an element of \( I = \mathfrak{R}(\sigma_0) \cap \mathfrak{R}(\tau_1) \). We define the restriction map \( r_\sigma^\tau \) from \( F_\tau \) to \( F_\sigma \) by
\[
r_\sigma^\tau : F_\tau \to F_\sigma
f \mapsto g.r_{\sigma_0}^{\tau_1}(g^{-1}.f).
\]
Note that this is independent of the choice of \( g \). In particular, we have \( r_\sigma^\tau(f) = g.f_{r(v)} \), where \( v = f(g^{-1}) \). When \( \sigma \in X_0^e \), we define \( r_\sigma^\tau(f) = g.r_{\sigma_0}^{\tau_1}(g^{-1}.f) \); it enjoys the same properties as when \( \sigma \in X_0^e \). Finally, given any simplex \( \tau \), we define \( r_\tau^\tau = \text{id}_{F_\tau} \).

6.5.3. \( G \)-action. Let \( \tau \) be a simplex of \( X_1 \), and let \( f \in F_\tau \). Since the space \( F_\tau \) is a subspace of either \( \text{c-ind}_{K}^{G}(\rho_0) \), \( \text{c-ind}_{K}^{G}(\rho_0') \), or \( \text{c-ind}_{I}^{G}(\rho_1) \), the element \( g.f \) is well-defined, and induces linear maps
\[
g_\tau : F_\tau \to F_{g.\tau}
f \mapsto g.f.
\]
We have $1_\tau = \text{id}_{F_\tau}$ and $g\tau \circ h_\tau = (gh)_\tau$ for every $g, h \in G$. It only remains to check the linear maps above are compatible with the restriction maps in 6.5.2. In other words, for an edge $\tau$ containing a vertex $\sigma$, we must verify the following diagram is commutative for all $g \in G$:

$$
\begin{array}{c}
F_\tau \xrightarrow{g_\sigma} F_{g,\tau} \\
\downarrow r^*_\tau \quad \downarrow r^*_\tau \\
F_\sigma \xrightarrow{g_\sigma} F_{g,\sigma}
\end{array}
$$

Assume $\sigma \in X_0^\psi$, and let $g'$ be such that $\tau = g', \tau_1$, $\sigma = g'. \sigma_0$. The previous subsection implies $g_\sigma \circ r^*_\sigma(f) = gg'. f_{r(v)}$, where $v = f(g'^{-1})$. On the other hand, $r^*_\sigma \circ g_\tau(f) = r^*_\sigma(g.f) = gg'. f_{r(v')}$, where $v' = g.f((gg')^{-1}) = v$. The same argument applies mutatis mutandis to the case $\sigma \in X_0^\psi$.

Combining all of these results, we see that to each diagram $D \in \mathcal{DLAG}$ we may associate a $G$-equivariant coefficient system $F = (F_\sigma)_\sigma \in \mathcal{COE}F_G$.

6.5.4. Morphisms. Let $D = (D_0, D_0', D_1, r_D, r'_D)$ and $E = (E_0, E_0', E_1, r_E, r'_E)$ be two diagrams, and $\psi = (\psi_0, \psi_0', \eta_1)$ be a morphism between them. Let $F = (F_\sigma)_\sigma$ and $F' = (F'_\sigma)_\sigma$ be the coefficient systems associated to $D$ and $E$, respectively.

Let $\sigma \in X_0^\psi$, and let $g \in G$ be such that $\sigma = g. \sigma_0$. For $f \in F_\sigma$, we let $v = f(g^{-1})$, and define

$$
\psi_\sigma : F_\sigma \to F'_\sigma,
$$

where $f_{\psi_0(v)}$ is the unique function in $F'_\sigma$ such that $f_{\psi_0(v)}(1) = \psi_0(v)$. We define $\psi_\sigma(f) = g.f_{\psi_0'(v)}$ if $\sigma \in X_0^\psi$ and $\sigma = g.\sigma_0'$.

Let $\tau$ be an edge, and let $g \in G$ be such that $\tau = g. \tau_1$. For $f \in F_\tau$, we let $v = f(g^{-1})$, and define

$$
\psi_\tau : F_\tau \to F'_\tau,
$$

where $f_{\eta_1(v)}$ is the unique function in $F'_\tau$ such that $f_{\eta_1(v)}(1) = \eta_1(v)$. Note that the definitions of $\psi_\sigma$ and $\psi_\tau$ are both independent of the choice of $g$.

This process gives a collection of linear maps $(\psi_\tau)_\tau$; we need to verify they are compatible with the restriction maps and the $G$-action. That is, we must check that the following two diagrams commute:

$$
\begin{array}{c}
F_\tau \xrightarrow{\psi_\tau} F'_\tau \\
\downarrow r^*_\tau \quad \downarrow (r')^*_\tau \\
F_\sigma \xrightarrow{\psi_\sigma} F'_\sigma
\end{array} \quad \quad
\begin{array}{c}
F_\tau \xrightarrow{\psi_\tau} F'_\tau \\
\downarrow h_\tau \\
F_{h,\tau} \xrightarrow{\psi_{h,\tau}} F'_{h,\tau}
\end{array}
$$

In the first square, $\tau$ is an edge containing a vertex $\sigma$, and in the second, $\tau$ is any simplex, $h \in G$. We begin with the first square. Suppose $\sigma \in X_0^\psi$ with $\sigma = g. \sigma_0$ and $\tau = g. \tau_1$ for some $g \in G$, and let $f \in F_\tau$. We have $\psi_\sigma \circ r^*_\sigma(f) = \psi_\sigma(g.f_{r_D(v)})$, with $v = f(g^{-1})$. As $g.f_{r_D(v)}(g^{-1}) = r_D(v)$, we get $\psi_\sigma \circ r^*_\sigma(f) = g.f_{\psi_\sigma r_D(v)}$. On the other hand, we have $(r')^*_\sigma \circ \psi_\tau(f) = (r')^*_\sigma(g.f_{\eta_1(v)})$. As $g.f_{\eta_1(v)}(g^{-1}) = \eta_1(v)$, we see $(r')^*_\sigma(g.f_{\eta_1(v)}) = g.f_{r_E(v)}$. Since $\psi$ is a morphism of diagrams, we have $\psi_0 \circ r_D(v) = r_E \circ \eta_1(v)$, and the result follows. The argument is identical for the case $\sigma \in X_0^\psi$. 


In the second diagram, we note that given \( f \in F_\tau \), we have \( h.f((hg)^{-1}) = f(g^{-1}) = v \). The commutativity then follows directly from the definitions.

We may now make the following definition:

**Definition 6.16.** Let \( C \) be the map:
\[
C : \mathcal{DLAG} \rightarrow \mathcal{COEF}_G
\]
\[
D = (D_0, D'_0, D_1, r, r') \mapsto \mathcal{F} = (F_\tau)_\tau,
\]
where \((F_\tau)_\tau\) is the coefficient system defined above.

The results of the previous subsections imply that \( C \) is a bona fide functor between the two categories.

### 7. Supersingular Representations

#### 7.1. Initial Diagrams

We begin with some general remarks. Given any irreducible mod-

\( p \) representation \( \rho \) of \( \Gamma \), we may view it as a representation of \( K \) via the projection \( K \rightarrow K/K_1 \cong \Gamma \). Conversely, any smooth irreducible representation of \( K \) must be of this form; this follows from Lemma 3(1) of [3] and the fact that \( K_1 \) is a normal pro-

\( p \) subgroup of \( K \). In light of this, we shall abuse notation and identify smooth irreducible representations of \( K \) and those of \( \Gamma \). The same statements hold for the groups \( K' \) and \( \Gamma' \), and \( I \) and \( H \).

Using the functor \( C \), we may now construct coefficient systems by defining the appropriate diagrams. In particular, to each supersingular Hecke module in Definition 4.3, we associate a diagram as follows.

**Definition 7.1.** Let \( \chi = \zeta \otimes \eta : H \rightarrow \mathbb{F}_p^\times \) be a character.

(i) Assume \( \chi = \eta \circ \det \). We associate to \( M_{\chi,(S,\emptyset)} \) and \( M_{\chi,(\emptyset,S')} \) the diagrams
\[
D_{\chi,(S,\emptyset)} = (\rho_{\chi,S}, \rho'_{\chi,\emptyset}, \chi, j, j') ;
\]
\[
D_{\chi,(\emptyset,S')} = (\rho_{\chi,\emptyset}, \rho'_{\chi,S'}, \chi, j, j') ,
\]
where \( j \) and \( j' \) are the inclusion maps.

(ii) Assume \( \chi^a = \chi \), but \( \chi \neq \eta \circ \det \). We associate to \( M_{\chi,(\emptyset,S')} \) and \( M_{\chi,(\emptyset,\emptyset)} \) the diagrams
\[
D_{\chi,(\emptyset,S')} = (\rho_{\chi,\emptyset}, \rho'_{\chi,S'}, \chi, j, j') ;
\]
\[
D_{\chi,(\emptyset,\emptyset)} = (\rho_{\chi,\emptyset}, \rho'_{\chi,\emptyset}, \chi, j, j') ,
\]
where \( j \) and \( j' \) are the inclusion maps.

(iii) Assume \( \chi^a \neq \chi \). We associate to \( M_{\chi,(\emptyset,\emptyset)} \) the diagram
\[
D_{\chi,(\emptyset,\emptyset)} = (\rho_{\chi,\emptyset}, \rho'_{\chi,\emptyset}, \chi, j, j') ,
\]
where \( j \) and \( j' \) are the inclusion maps.

If \( D_{\chi,\mathbf{J}} = (\rho, \rho', \chi, j, j') \) is a diagram as defined above, we let the underlying space of the \( I \)-representation \( \chi \) be spanned by a fixed vector \( v \), and identify \( v \) with its image in \( \rho^{I(1)} = \rho^U \) and \( (\rho')^{I(1)} = (\rho')^U \) via \( j \) and \( j' \).

For a diagram \( D_{\chi,\mathbf{J}} \), we define \( D_{\chi,\mathbf{J}} = C(D_{\chi,\mathbf{J}}) \) to be the associated \( G \)-equivariant coefficient system.

**Remark.** We note that if \( M_{\chi,\mathbf{J}} \) is a supersingular module and \( D_{\chi,\mathbf{J}} = (\rho, \rho', \chi, j, j') \) is the associated diagram, we have \( \rho^U \cong M_{\chi,\mathbf{J}}|_{\mathcal{H}_I} \) as \( \mathcal{H}_I \)-modules and \( (\rho')^U \cong M_{\chi,\mathbf{J}}|_{\mathcal{H}_{I'}} \) as \( \mathcal{H}_{I'} \)-modules.
Proposition 7.2. Let $M_{\chi,J}$ be a supersingular $H_{\mathbb{F}_p}(G,I(1))$-module, and let $\pi$ be a nonzero irreducible quotient of $H_0(X,\mathcal{D}_{\chi,J})$. Then $\pi^{I(1)}$ contains $M_{\chi,J}$, and $\pi$ is supersingular as a $G$-representation.

Proof. In the notation of Subsection 6.5.2, let $\omega_{\sigma_0,f_v}$ be the 0-chain supported on $\sigma_0$, such that $\omega_{\sigma_0,f_v}(\sigma_0) = f_v$, and let $\bar{\omega}_{\sigma_0,f_v}$ denote its image in $H_0(X,\mathcal{D}_{\chi,J})$. Recall that $f_v \in \text{c-ind}_A^G(\rho_0)$ denotes the unique function such that $\text{supp}(f) = K$ and $f_v(1) = v$, where $v$ is a fixed vector spanning the underlying space of $\chi$. By definition of the $G$-action, $\omega_{\sigma_0,f_v}$ is $I(1)$-invariant and the group $I$ acts by the character $\chi$. To proceed, we must show two things:

(i) The element $\bar{\omega}_{\sigma_0,f_v}$ generates $H_0(X,\mathcal{D}_{\chi,J})$ as a $G$-representation.
(ii) The right action of $\mathcal{H}(G,I(1))$ on $\langle \bar{\omega}_{\sigma_0,f_v} \rangle_{\mathbb{F}_p}$ yields an isomorphism onto $M_{\chi,J}$.

Assuming these two results, we let $\pi$ be a nonzero irreducible quotient of $H(X,\mathcal{D}_{\chi,J})$. Since $\bar{\omega}_{\sigma_0,f_v}$ generates $H_0(X,\mathcal{D}_{\chi,J})$, the image of $\bar{\omega}_{\sigma_0,f_v}$ in $\pi$ will be nonzero. The second result above then shows that $\pi^{I(1)}$ contains the $\mathcal{H}(G,I(1))$-module $M_{\chi,J}$ and the Proposition follows from Corollary 4.5.

It remains to prove the two claims. For the first, we note that if $\omega_{\sigma_0,f_v}$ denotes the 0-chain supported on $\sigma_0$ such that $\omega_{\sigma_0,f_v}(\sigma_0) = f_v$, then Lemma 6.7 implies $\bar{\omega}_{\sigma_0,f_v} = \omega_{\sigma_0,f_v}$ in $H_0(X,\mathcal{D}_{\chi,J})$. The Carter-Lusztig theory tells us that any irreducible representation of $K$ or $K'$ is generated by its $I(1)$-invariants, and therefore $\bar{\omega}_{\sigma_0,f_v}$ (resp. $\bar{\omega}_{\sigma_0',f_v}$) generates the image in $H_0(X,\mathcal{D}_{\chi,J})$ of the space $C_c(\sigma_0,\mathcal{D}_{\chi,J})$ (resp. $C_c(\sigma_0',\mathcal{D}_{\chi,J})$). This fact, combined with the observation that $G$ acts transitively on the sets $X_0^0$ and $X_0^0$, verifies the claim.

For the second claim, note that by Definition 5.6 and our choice of irreducible $K$- and $K'$-representations, we have

$$\langle v \rangle_{\mathbb{F}_p} = (\rho_{\chi,J})^{I(1)} \cong M_{\chi,J}$$

as $\mathcal{H}_{I(1)}$-modules,

$$\langle v \rangle_{\mathbb{F}_p} = (\rho_{\chi,J}')^{I(1)} \cong M_{\chi,J}'$$

as $\mathcal{H}_{I'(1)}$-modules,

where $J = (J,J')$. We conclude from Proposition 3.9 that $\langle \bar{\omega}_{\sigma_0,f_v} \rangle_{\mathbb{F}_p}$ is equivalent to $M_{\chi,J}$ as a right $\mathcal{H}(G,I(1))$-module. \qed

7.2. Injective Envelopes. In this section we briefly recall some definitions and notation regarding socles and injective envelopes, which will be of use in subsequent sections. For more details, we refer to [29] and [26].

Let $K$ be any finite or profinite group, and denote by $\mathcal{R}\mathcal{E}\mathcal{P}_{\mathbb{F}_p}(K)$ the category of smooth $\mathbb{F}_p$-representations of $K$. Denote by $\text{Irr}_K$ be the subcategory of smooth irreducible representations of $K$.

Definition 7.3. Let $\pi \in \mathcal{R}\mathcal{E}\mathcal{P}_{\mathbb{F}_p}(K)$ and let $\rho$ be a subrepresentation of $\pi$. We say $\pi$ is an essential extension of $\rho$ if for every nonzero subrepresentation $\pi'$ of $\pi$, we have $\pi' \cap \rho \neq \{0\}$.

Let $\rho \in \mathcal{R}\mathcal{E}\mathcal{P}_{\mathbb{F}_p}(K)$ and let $\mathcal{J}$ be an injective object of $\mathcal{R}\mathcal{E}\mathcal{P}_{\mathbb{F}_p}(K)$. We say $\mathcal{J}$ is an injective envelope of $\rho$ if there exists an injection $\rho \hookrightarrow \mathcal{J}$ such that $\mathcal{J}$ is an essential extension of the image of $\rho$. We write $\mathcal{J} = \text{inj}_K(\rho)$.

It is known that injective envelopes exist, and are unique up to isomorphism: for finite groups, see Chapter 14 of [29], and for profinite groups, see Section 3.1 of [31].

Definition 7.4. Let $\rho \in \mathcal{R}\mathcal{E}\mathcal{P}_{\mathbb{F}_p}(K)$. The socle of $\rho$, denoted $\text{soc}_K(\rho)$, is the maximal semisimple subrepresentation of $\rho$.

We now use these constructions for the groups $K, K'$ and $I$. 
Lemma 7.5. (i) Let $\rho$ be an irreducible representation of $K$ and let $\rho \hookrightarrow \text{inj}_K(\rho)$ be an injective envelope of $\rho$. Then

$$\text{inj}_K(\rho)|I \cong \bigoplus_{\chi \in \hat{H}} \text{inj}_I(\chi)^{\oplus m_{\rho,\chi}},$$

where $m_{\rho,\chi} = \dim_{\mathbb{F}_p}(\text{Hom}_H(\chi, \text{inj}_I(\rho)^U)).$

(ii) Let $\rho'$ be an irreducible representation of $K'$ and let $\rho' \hookrightarrow \text{inj}_{K'}(\rho')$ be an injective envelope of $\rho'$. Then

$$\text{inj}_{K'}(\rho')|I \cong \bigoplus_{\chi \in \hat{H}} \text{inj}_I(\chi)^{\oplus m_{\rho',\chi}},$$

where $m_{\rho',\chi} = \dim_{\mathbb{F}_p}(\text{Hom}_H(\chi, \text{inj}_{K'}(\rho')^U)).$

In particular, the integers $m_{\rho,\chi}$ and $m_{\rho',\chi}$ are finite for every character $\chi$ of $H$.

Proof. The proof is identical to the proof of Lemma 6.19 of [26].

Remark. Under the assumption $q = p$, we will determine (8) explicitly, using a simple counting argument.

Corollary 7.6. Let $K \in \{K, K'\}$, and let $\rho \in \mathcal{RP}_{\mathbb{F}_p}(K)$ be a representation such that $\text{soc}_K(\rho)$ is of finite length as a $K$-representation. Then the space of $I(1)$-invariants of $\text{inj}_K(\rho)$ is finite-dimensional and $\rho$ is admissible.

Proof. From Lemma 7.5, we have

$$\text{inj}_K(\rho)|I \cong \text{inj}_K(\text{soc}_K(\rho))|I \cong \bigoplus_{\chi \in \hat{H}} \text{inj}_I(\chi)^{\oplus m_{\chi}},$$

where the integers $m_{\chi}$ are finite. Hence, we see that

$$\rho^{I(1)} \hookrightarrow \text{inj}_K(\rho)^{I(1)} \cong \left( \bigoplus_{\chi \in \hat{H}} \text{inj}_I(\chi)^{\oplus m_{\chi}} \right)^{I(1)} \cong \bigoplus_{\chi \in \hat{H}} \left( \text{inj}_I(\chi)^{I(1)} \right)^{\oplus m_{\chi}} \cong \bigoplus_{\chi \in \hat{H}} \chi^{\oplus m_{\chi}},$$

for the last isomorphism, we use the fact that $\text{inj}_I(\chi)^{I(1)} \cong \text{inj}_{I/I(1)}(\chi) \cong \chi$ as representations of $H$. Admissibility now follows from [26], Lemma 6.18.

7.3. Pure Diagrams. In light of Proposition 7.2, it suffices to construct irreducible quotients of $H_0(X, D_{\chi,J})$ to produce supersingular representations. With this in mind, we adapt the arguments of [26] into a more formal context:

Definition 7.7. Let $M_{\chi,J}$ be a supersingular module, and let $D = (D_0, D'_0, D_1, r_D, r'_D)$ be a diagram. We say $D$ is essentially pure for $M_{\chi,J}$ if it satisfies the following conditions:

(i) There exists an embedding of diagrams:
\[ \psi : D_{\chi, J} \to D. \]

(ii) The maps \( r_D \) and \( r'_D \) induce isomorphisms \( D_0|_I \cong D'_0|_I \cong D_1 \).

Moreover, we say \( D \) is pure for \( M_{\chi, J} \), if it also satisfies the following extra condition:

(iii) Either \( \text{soc}_K(D_0) \) or \( \text{soc}_{K'}(D'_0) \) is irreducible.

With these definitions, we are able to prove a formal result, whose proof is due to Paškūnas.

**Theorem 7.8.** Let \( M_{\chi, J} \) be a supersingular module, and suppose that \( D \) is a pure diagram for \( M_{\chi, J} \). Then the image of the induced \( G \)-morphism between the 0-homology

\[ \pi_D = \text{im}(\psi_* : H_0(X, D_{\chi, J}) \to H_0(X, \mathcal{C}(D))) \]

is an irreducible admissible supersingular representation. Moreover, we have

\[ \pi_D \cong \text{soc}_G(H_0(X, \mathcal{C}(D))). \]

**Proof.** To verify the result, it suffices to show \( \pi_D \) is irreducible, admissible and nonzero, by Proposition 7.2. Let us assume that \( \text{soc}_K(D_0) \) is irreducible; the case with \( \text{soc}_{K'}(D'_0) \) irreducible is the same.

We first claim that the space \( \text{soc}_K(H_0(X, \mathcal{C}(D))|_K)^{(1)} \) generates \( \pi_D \). Let \( \psi_* (\bar{\omega}_{\sigma_0, f_v}) \) denote the image of the homology class \( \bar{\omega}_{\sigma_0, f_v} \). Claim (i) in the proof of Proposition 7.2 shows that

\[ \pi_D = \langle G. \psi_* (\bar{\omega}_{\sigma_0, f_v}) \rangle_{\overline{\mathbb{F}}_p}, \]

since \( \psi \) is an embedding, we have \( \psi_* (\bar{\omega}_{\sigma_0, f_v}) \neq 0 \) and hence \( \pi_D \neq \{0\} \). From the definition of pure diagrams and Proposition 6.11, we know that \( H_0(X, \mathcal{C}(D))|_K \cong D_0 \). The purity condition on \( D \) also implies that the \( K \)-representation of the diagram \( D_{\chi, J} \) is exactly \( \text{soc}_K(D_0) \), so that \( \psi_* (\bar{\omega}_{\sigma_0, f_v}) \in \text{soc}_K(H_0(X, \mathcal{C}(D))|_K) \). This shows

\[ \langle \psi_* (\bar{\omega}_{\sigma_0, f_v}) \rangle_{\overline{\mathbb{F}}_p} = \text{soc}_K(H_0(X, \mathcal{C}(D))|_K)^{(1)}, \]

which combined with the previous observation verifies the claim. Moreover, this shows how to define an action of \( \mathcal{H}_{\overline{\mathbb{F}}_p}(G, I(1)) \) on \( \text{soc}_K(H_0(X, \mathcal{C}(D))|_K)^{(1)} \) such that

\[ \text{soc}_K(H_0(X, \mathcal{C}(D))|_K)^{(1)} \cong M_{\chi, J} \]
as right \( \mathcal{H}_{\overline{\mathbb{F}}_p}(G, I(1)) \)-modules.

Now let \( \pi' \) be a nonzero \( G \)-invariant subspace of \( \pi_D \). Since \( K_1 \) is a pro-\( p \) group we have \( \langle \pi' \rangle_{K_1} \neq \{0\} \) and consequently \( \text{soc}_K(\pi'|_K) \neq \{0\} \). We also note that

\[ \text{soc}_K(H_0(X, \mathcal{C}(D))|_K)^{(1)} \cap (\pi'|^{K(1)} \neq \{0\}), \]
as \( \text{soc}_K(\pi'|_K) \) is contained in \( \text{soc}_K(H_0(X, \mathcal{C}(D))|_K) \). However, we have just shown that the space \( \text{soc}_K(H_0(X, \mathcal{C}(D))|_K)^{(1)} \) is simple as a right \( \mathcal{H}(G, I(1)) \)-module, which implies

\[ \text{soc}_K(H_0(X, \mathcal{C}(D))|_K)^{(1)} \subset (\pi'|^{K(1)}. \]

Note that we can deduce this simply from the fact that

\[ \dim_{\overline{\mathbb{F}}_p}(\text{soc}_K(H_0(X, \mathcal{C}(D))|_K)^{(1)}) = 1. \]

Collecting these results, we conclude that \( \pi' = \pi_D \). This argument also shows that the socle of \( H_0(X, \mathcal{C}(D)) \) is exactly \( \pi_D \).

To show admissibility, we observe that

\[ \pi_D|_K \subset H_0(X, \mathcal{C}(D))|_K \cong D_0, \]
which implies that $\text{soc}_K(\pi_D|_K)$ is of finite length. The claim then follows from Corollary 7.6.

The definitions of pure and essentially pure diagrams do not make it clear that such diagrams exist in general. We take up this question when $q = p$ in the next section, and in general propose the following:

**Conjecture 7.9.** Given a supersingular module $M_{\chi, J}$, an essentially pure diagram $D$ for $M_{\chi, J}$ exists, and the image of $H_0(X, D_{\chi, J})$ in $H_0(X, C(D))$ is a sum of supersingular representations.

### 7.4. Construction of Pure Diagrams when $q = p$

We now give an application of the formalism developed in the previous section, using results of Section 5.

**Theorem 7.10.** Suppose $q = p$. Then for every supersingular module $M_{\chi, J}$, there exists a pure diagram for $M_{\chi, J}$.

More precisely, the corresponding initial diagram

$$D_{\chi, J} = (\rho, \rho', \chi, j, j')$$

can be embedded into a pure diagram

$$E_{\chi, J} = (\text{inj}_K(P), \text{inj}_{K'}(P'), \text{inj}_I(X), j_p, j'_p)$$

where $P = \rho$, $P'$ is a semisimple representation of $K'$ having $\rho'$ as a summand, and $X$ is a semisimple representation of $I$ having $\chi$ as a summand. Furthermore, the maps $j_p$ and $j'_p$ induce isomorphisms $\text{inj}_K(P)|_I \cong \text{inj}_I(X) \cong \text{inj}_{K'}(P')|_I$.

Our main tool in proving this Theorem will be Lemma 7.5, which states that if $\rho$ and $\rho'$ are smooth irreducible representations of $K$ and $K'$, respectively, then we have

$$\text{inj}_K(\rho)|_I \cong \bigoplus_{\chi \in \hat{H}} \text{inj}_I(\chi)^{\oplus m_{\rho, \chi}}, \quad \text{inj}_{K'}(\rho')|_I \cong \bigoplus_{\chi \in \hat{H}} \text{inj}_I(\chi)^{\oplus m_{\rho', \chi}}.$$

In general, it is not clear how the multiplicities in the above equations compare with each other. We record one result in this direction, which holds for general $q$:

**Lemma 7.11.** We have $m_{\rho, \chi} = m_{\rho, \chi^*}$, and $m_{\rho', \chi} = m_{\rho', \chi^*}$.

**Proof.** The definition of the numbers $m_{\rho, \chi}$ and Frobenius Reciprocity give

$$m_{\rho, \chi} = \dim_{\mathbb{F}_p}(\text{Hom}_H(\chi, \text{inj}_\Gamma(\rho)|_\Gamma))$$

$$= \dim_{\mathbb{F}_p}(\text{Hom}_B(\chi, \text{inj}_\Gamma(\rho)|_B))$$

$$= \dim_{\mathbb{F}_p}(\text{Hom}_\Gamma(\text{ind}_B^\Gamma(\chi), \text{inj}_\Gamma(\rho))).$$

We note that given an arbitrary finite-dimensional mod-$p$ representation $V$ of $\Gamma$, the number $\dim_{\mathbb{F}_p}(\text{Hom}_\Gamma(V, \text{inj}_\Gamma(\rho)))$ is precisely the multiplicity with which $\rho$ occurs as a composition factor of $V$.

Given a $\mathbb{Q}_p$-representation $\mathcal{W}$ of $\Gamma$, there exists a $\Gamma$-stable $\mathbb{Z}_p$-lattice in $\mathcal{W}$. Reducing this lattice modulo the maximal ideal gives a mod-$p$ representation of $\Gamma$, and Theorem 32 in [29] asserts that the semisimplification of this quotient is independent of the choice of $\mathbb{Z}_p$-lattice. In our case, it is straightforward to verify that (the semisimplification of) every mod-$p$ principal series representation comes from a characteristic 0 principal series representation in this fashion.
The character tables of irreducible complex representations of $\Gamma$ (and $\Gamma'$) have been determined by Ennola in [15]. Considering the character tables with values in $\mathbb{Q}_p$, we obtain the Brauer characters of the principal series representations $\text{ind}_{B}^{F}(\chi)$. In particular, the Brauer characters of the representations $\text{ind}_{B}^{F}(\chi)$ and $\text{ind}_{B}^{F}(\chi^p)$ are identical. Since Brauer characters determine mod-$p$ representations up to semisimplification, we conclude that $m_{\rho,\chi} = m_{\rho,\chi^p}$.

The same proof holds for the numbers $m_{\rho',\chi}$. □

**Remark.** We note that this result holds in a more general context. In particular, if $G$ is a finite group with a “split BN pair $(B, N)$” of characteristic $p$, $\chi$ is a character of the maximal torus of $G$, and $w$ is an element of the Weyl group of $(B, N)$, then the induced representations $\text{ind}_{B}^{G}(\chi)$ and $\text{ind}_{B}^{G}(\chi^w)$ have the same composition factors. The proof may be found in the Remarks of Section 7.2 and Section 9.7 of [19].

**Proposition 7.12.** Assume $q = p$. We then have

\[
\begin{align*}
\text{inj}_{K'}(\rho'_{\chi,J'}) |_{I} & \cong \text{inj}_{I}(\chi) \quad \text{if } \chi^s = \chi \text{ and } J' \subseteq J'_0(\chi), \\
\text{inj}_{K'}(\rho'_{\chi,J}) |_{I} & \cong \text{inj}_{I}(\chi) \oplus \text{inj}_{I}(\chi^s) \quad \text{if } \chi^s \neq \chi.
\end{align*}
\]

**Proof.** Recall from Corollary 4.3 of [26] that we have a decomposition

\[
\mathbb{F}_p[G'] \cong \bigoplus_{\tau \in \text{Irr}_{G'}} \text{inj}_{G'}(\tau) \oplus \dim_{\mathbb{F}_p}(\tau),
\]

where $\text{Irr}_{G'}$ denotes the set of equivalence classes of irreducible mod-$p$ representations of $G'$.

Using Proposition 5.19, we translate the above decomposition into a sum over characters of $H$:

\[
\mathbb{F}_p[G'] \cong \bigoplus_{\chi \in \chi^s} \text{inj}_{G'}(\rho'_{\chi,s}) \oplus (\rho'_{\chi,0})^p \oplus \bigoplus_{\chi \neq \chi^s} \text{inj}_{G'}(\rho'_{\chi,0}) \oplus \text{inj}_{G'}(\rho'_{\chi^s,0}) \oplus \dim_{\mathbb{F}_p}(\rho'_{\chi^s,0}),
\]

where we use the fact that $\rho'_{\chi,0}$ with $\chi = \chi^s$ is a twist of the Steinberg representation, and therefore is injective of dimension $p$. A similar argument as in the proof of Lemma 4.7 of [26] implies that

\[
\dim_{\mathbb{F}_p}(\text{inj}_{G'}(\rho'_{\chi,0})) \geq 2p
\]

for any character $\chi$ satisfying $\chi \neq \chi^s$. Additionally, by Lemma 5.9, we have

\[
\dim_{\mathbb{F}_p}(\rho'_{\chi,0}) + \dim_{\mathbb{F}_p}(\rho'_{\chi^s,0}) = p + 1.
\]

These two facts allow us to evaluate the dimensions of both sides in the decomposition above:

\[
p(p + 1)^2(p^2 - 1) \geq \sum_{\chi = \chi^s} (\dim_{\mathbb{F}_p}(\text{inj}_{G'}(\rho'_{\chi,s})) + p^2) + \sum_{\chi \neq \chi^s} 2p(p + 1)
\]

The number of $\chi$ satisfying $\chi = \chi^s$ is $(p + 1)^2$, while the number of unordered pairs $\{\chi, \chi^s\}$ such that $\chi \neq \chi^s$ is $\frac{1}{2}(p + 1)^2(p - 2)$. The above inequality now reduces to

\[
(p + 1)^2 p \geq \sum_{\chi = \chi^s} \dim_{\mathbb{F}_p}(\text{inj}_{G'}(\rho'_{\chi,s})).
\]
Since the order of $V'$ is $p$, Corollary 4.6 of [26] implies that $\text{dim}_p(\text{inj}_{V'}(\rho'_\chi, V')) \geq p$. We therefore have
\[
\sum_{\chi = \chi'} \text{dim}_p(\text{inj}_{V'}(\rho'_\chi, V')) \geq (p + 1)^2 p.
\]
which forces every inequality above to be an equality. To sum up, we have
\[
\text{dim}_p(\text{inj}_{V'}(\rho'_\chi, V')) = 1 \quad \text{if } \chi' = \chi,
\]
\[
\text{dim}_p(\text{inj}_{V'}(\rho'_\chi, V')) = 2 \quad \text{if } \chi' \neq \chi.
\]

To proceed, note that the dimensions computed above tell us precisely the number of terms appearing on the right-hand side of equation (8). Taking $V'$-invariants of the exact sequence $0 \to \rho'_\chi \to \text{inj}_{V'}(\rho'_\chi)$ yields
\[
0 \to \chi \to \text{inj}_{V'}(\rho'_\chi); 
\]
combining this with Lemma 7.11 gives the result. 

\textbf{Remark.} We note that an alternate proof of this result may be obtained by explicitly computing the composition factors of the representations $\text{ind}^{V'}_V(\chi)$ using Lemma 5.9.

\textbf{Proof of Theorem 7.10.} Using Lemma 7.11, we now rewrite equation (7) as
\[
\text{inj}_K(\rho) | I = \bigoplus_{\mu = \mu^s} \text{inj}_I(\mu)^{\otimes m_{\rho, \mu}} \bigoplus_{\mu \neq \mu^s} (\text{inj}_I(\mu) \oplus \text{inj}_I(\mu^s))^{\otimes m_{\rho, \mu}},
\]
the sums being taken over $W$-orbits of characters.

We let $X$ be the representation of $I$ defined by
\[
X = \bigoplus_{\mu = \mu^s} \mu^{\otimes m_{\rho, \mu}} \bigoplus_{\mu \neq \mu^s} (\mu \oplus \mu^s)^{\otimes m_{\rho, \mu}},
\]
and let $P'$ be a representation of $K'$ of the form
\[
P' = \bigoplus_{\mu = \mu^s} (\rho'_{\mu})^{\otimes m_{\rho, \mu}} \bigoplus_{\mu \neq \mu^s} (\rho'_{\mu^s})^{\otimes m_{\rho, \mu}}.
\]
Here we choose $\rho'_{\mu^s} \in \{\rho'_{\mu, \nu}, \rho'_{\mu, \emptyset}\}$ if $\mu = \mu^s$ and $\rho'_{\mu^s} \in \{\rho'_{\mu, \emptyset}, \rho'_{\mu^s, \emptyset}\}$ if $\mu \neq \mu^s$; the only stipulation we make is that $\rho'$ be among the summands. By definition and Proposition 7.12, we have $\text{inj}_K(P) | I \cong \text{inj}_{K'}(P') | I \cong \text{inj}_I(X)$.

We now have natural injective maps from $\rho$ to $\text{inj}_K(P)$, from $\rho'$ to $\text{inj}_{K'}(P')$ and from $\chi$ to $\text{inj}_I(X)$: they are defined by first mapping each representation into its respective injective envelope, followed by the canonical inclusion into the direct sum. Moreover, one can choose the maps $j_p$ and $j'_p$ such that these injective maps induce a morphism of diagrams
\[
\psi : D_{X, J} \to E_{X, J}.
\]
It is evident that the diagram $E_{X, J}$ is pure for $M_{X, J}$.

\textbf{Corollary 7.13.} Assume $q = p$, let $M_{X, J}$ be a supersingular module, and let $E_{X, J}$ be a pure diagram for $M_{X, J}$, constructed as in the proof of the previous theorem. Set $E_{X, J} = C(E_{X, J})$.

Then the image
\[
\pi_{E_{X, J}} = \text{im}(\psi) : H_0(X, D_{X, J}) \to H_0(X, E_{X, J})
\]
is an irreducible admissible supersingular representation. Moreover, for distinct modules $M_{X, J}, M'_{X', J'}$, the representations $\pi_{E_{X, J}}, \pi_{E_{X', J'}}$ are nonisomorphic.
Proof. The first part of the Corollary follows from Theorems 7.8 and 7.10. To prove the second part, let us assume \( \phi : \pi_{E_{\chi,J}} \xrightarrow{\sim} \pi_{E_{\chi,J'}} \) is an isomorphism; we then obtain an induced isomorphism

\[
\overline{\phi} : \text{soc}_K(\pi_{E_{\chi,J}}|_K)^{(1)} \xrightarrow{\sim} \text{soc}_K(\pi_{E_{\chi,J'}}|_K)^{(1)}.
\]

The proof of Theorem 7.8 shows how to equip these spaces with an action of \( \mathcal{H}_{\overline{G}}(G, I(1)) \), which gives

\[
M_{\chi,J} \cong \text{soc}_K(\pi_{E_{\chi,J}}|_K)^{(1)} \overset{\bar{\phi}}{\to} \text{soc}_K(\pi_{E_{\chi,J'}}|_K)^{(1)} \cong M_{\chi',J'}.
\]

The claim now follows from the comments following Definition 4.3. \( \square \)

Remark. Assume \( q = p \). Given a supersingular module \( M_{\chi,J} \), our construction shows that there may be many choices of pure diagram \( E_{\chi,J} \) associated to \( M_{\chi,J} \). As a consequence, if \( E_{\chi,J} \) and \( E_{\chi, J'} \) are two such diagrams, we obtain two supersingular representations \( \pi_{E_{\chi,J}} \) and \( \pi_{E_{\chi,J'}} \) whose \( I(1) \)-invariants contain \( M_{\chi,J} \). It is not clear, however, if these representations are isomorphic.

8. Some Remarks

8.1. The Case \( q \neq p \). In this section we point out the shortcomings of our method in the case when \( q \neq p \). We assume that \( q = p^2 \) for the sake of simplicity.

Let \( 1 \) denote the trivial character of \( H \) (or, equivalently, of \( I \)), and consider the diagram \( D_{1,(\emptyset, S')} = (\rho_{1,0}, \rho_{1,S'}, 1, j, j') \). Here \( \rho_{1,0} \) is the Steinberg representation of \( K \), and \( \rho_{1,S'} \) is the trivial character of \( K' \). We claim that there does not exist a pure diagram \( D \) for \( M_{1,(\emptyset, S')} \) of the form \((\text{inj}_K(P), \text{inj}_{K'}(P'), \text{inj}_I(X), j_p, j'_p)\), where \( P \) is a semisimple representation of \( K \), \( P' \) is a semisimple representation of \( K' \), and \( X \) is a semisimple representation of \( I \).

We require some preparatory facts. Let \( \mu \) and \( \mu^* \) be two \( \overline{F}_p \)-characters of \( H \) defined by

\[
\mu \left( \begin{array}{ccc}
 a & 0 & 0 \\
 0 & \delta & 0 \\
 0 & 0 & \overline{a}^{-1}
\end{array} \right) = a^{(p^2+1)(p-1)}, \quad \mu^* \left( \begin{array}{ccc}
 a & 0 & 0 \\
 0 & \delta & 0 \\
 0 & 0 & \overline{a}^{-1}
\end{array} \right) = a^{(p^2+1)(p+1)}.
\]

A computation with Brauer characters verifies that

\[
\text{ind}^\Gamma_{\overline{G}}(\mu)^{ss} \cong V_{0,0} \boxtimes \omega^0 \oplus V_{p^2-2p+1,0} \boxtimes \omega^0 \oplus V'_{2p-2,1-p} \boxtimes \omega^0 \oplus V'_{p^2-2p-3, p+2} \boxtimes \omega^0 \cong \rho_{1,S'}^* \oplus \rho_{\mu,0}^* \oplus \rho_{\mu^*,0}^* \oplus \rho_{\mu^*,\emptyset}^*,
\]

where the superscript “ss” denotes semisimplification. Alternatively, we may obtain this decomposition from a slightly modified version of Proposition 1.1 in [12], along with the character tables computed in [15]. Using the fact that \( SU(1,1)(\overline{F}_p/\overline{F}_p) \) is conjugate to \( SL_2(\overline{F}_p) \), and modifying the arguments in Section 4.2 of [26] shows that \( \dim_{\overline{F}_p}(\text{inj}_I(\rho_{1,S'}^*)) = 3p^2 \). Combining these two facts with Lemma 7.11 shows that

\[
\text{inj}_{K'}(\rho_{1,S'})|_I \cong \text{inj}_I(1) \oplus \text{inj}_I(\mu) \oplus \text{inj}_I(\mu^*).
\]

Assume now that we have an embedding of diagrams \( D_{1,(\emptyset, S')} \to D \), with \( D \) pure:
Assume first that the $K$-representation of $D$ has simple $K$-socle, so that $P \cong \rho_{1,0}$. Since $\rho_{1,0}$ is injective as a representation of $\Gamma$, we have $\text{inj}_K(\rho_{1,0})|_I \cong \text{inj}_I(1)$. We have an injection

$$\rho'_{1,s} \hookrightarrow \text{inj}_{K'}(P')$$

and the latter representation is injective, so Lemma 6.13 of [26] implies there exists an injection

$$\text{inj}_{K'}(\rho'_{1,s}) \hookrightarrow \text{inj}_{K'}(P').$$

Restricting to $I$ and using the definition of purity gives

$$\text{inj}_I(1) \oplus \text{inj}_I(\mu) \oplus \text{inj}_I(\mu^*) \cong \text{inj}_{K'}(\rho'_{1,s})|_I \hookrightarrow \text{inj}_{K'}(P')|_I \cong \text{inj}_I(1),$$

which is absurd.

We may therefore assume that the $K'$-representation of $D$ has simple $K'$-socle, so that $P' \cong \rho'_{1,s}$ and

$$\text{inj}_{K'}(P')|_I \cong \text{inj}_I(1) \oplus \text{inj}_I(\mu) \oplus \text{inj}_I(\mu^*) \cong \text{inj}_K(P)|_I,$$

by the definition of purity. This implies that we must have

$$\text{inj}_K(P/\rho_{1,0})|_I \cong \text{inj}_I(\mu) \oplus \text{inj}_I(\mu^*);$$

the only representations for which this could potentially be true are $\rho_{\mu,0}$ and $\rho_{\mu^*,0}$. The dimensions of the injective envelopes of $SU(2,1)(\mathbb{F}_p^*/\mathbb{F}_p^2)$ have been computed explicitly by Dordowsky in his Diplomarbeit ([13]). In particular, his results show that $\dim_{\mathbb{F}_p}(\text{inj}_I(\rho_{\mu,0})) = \dim_{\mathbb{F}_P}(\text{inj}_I(\rho_{\mu^*,0})) = 12p^6$, which implies that the number of summands in the decompositions of $\text{inj}_K(\rho_{\mu,0})|_I$ and $\text{inj}_K(\rho_{\mu^*,0})|_I$ is 12. This verifies our claim.

8.2. **Comparison with $SL_2(F)$**. In the course of defining diagrams and coefficient systems for $U(2,1)(E/F)$, there are several parallels one can draw between the formalism we have used and the analogous formalism for the group $SL_2(F)$. We hope to make this connection precise here, drawing on results of Abdellatif in [1]. In this section only, the prime $p$ may be arbitrary.

We let $G_S = SL_2(F)$, $K_S = SL_2(\mathfrak{o}_F)$, and $K'_S = \alpha_S K_S \alpha_S^{-1}$, where

$$\alpha_S = \begin{pmatrix} 1 & 0 \\ 0 & \varpi_F \end{pmatrix};$$

the groups $K_S$ and $K'_S$ are representatives of the two conjugacy classes of maximal compact subgroups of $G_S$. We note that our notation differs slightly from that of [1]. Let $I_S = K_S \cap K'_S$ be the Iwahori subgroup, and $I_S(1) \leq I_S$ its unique pro-$p$-Sylow subgroup. Let

$$w_s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad w_{s'} = \begin{pmatrix} 0 & -\varpi_F^{-1} \\ \varpi_F & 0 \end{pmatrix},$$
and for \( r \in \mathbb{Z} \), let \( \omega^r \) denote the \( \mathbb{F}_p \)-character of the finite torus \( H_S \) defined by
\[
\omega^r \left( \begin{array}{cc} a & 0 \\ 0 & a^{-1} \end{array} \right) = a^r,
\]
where \( a \in \mathbb{F}_q^\times \).

As in Section 3, we denote by \( \mathcal{H}_{\mathbb{F}_p}(G_S, I_S(1)) = \text{End}_{G_S}(\text{c-ind}^{G_S}_{I_S(1)}(1)) \) the pro-\( p \)-Iwahori-Hecke algebra, and let \( T_{w_0} \) (resp. \( T_{\omega^r} \)) be the endomorphism corresponding by adjunction to the characteristic function of \( I_S(1)w_0I_S(1) \) (resp. \( I_S(1)\omega^rI_S(1) \)). For \( 0 \leq r < q - 1 \), we define
\[
e_{\omega^r} = |H_S|^{-1} \sum_{h \in H_S} \omega^r(h)T_h,
\]
where \( T_h \) denotes the endomorphism corresponding by adjunction to the characteristic function of \( I_S(1)hI_S(1) \). A proof similar to that of Proposition 3.9 shows that \( \mathcal{H}_{\mathbb{F}_p}(G_S, I_S(1)) \) is generated by \( T_{w_0}, T_{\omega^r} \) and \( e_{\omega^r} \) for \( 0 \leq r < q - 1 \).

The supersingular Hecke modules (as defined in [34]) have been classified in [1], Chapitre 6. They naturally divide into three classes, depending on the nature of the character \( \omega^r \) (or equivalently, the parameter \( r \)).

**Proposition 8.1.** The supersingular \( \mathcal{H}_{\mathbb{F}_p}(G_S, I_S(1)) \)-modules are all one-dimensional. They are given by:
\[
\begin{align*}
M_0 & : e_1 \mapsto 1, \quad T_{w_0} \mapsto 0, \quad T_{\omega^r} \mapsto -1; \\
M_{q-1} & : e_1 \mapsto 1, \quad T_{w_0} \mapsto -1, \quad T_{\omega^r} \mapsto 0; \\
M_{(q-1)/2} & : e_{\omega^{(q-1)/2}} \mapsto 1, \quad T_{w_0} \mapsto 0, \quad T_{\omega^r} \mapsto 0; \\
M_r & : e_{\omega^r} \mapsto 1, \quad T_{w_0} \mapsto 0, \quad T_{\omega^r} \mapsto 0,
\end{align*}
\]
where \( 0 < r < q - 1, r \neq \frac{q-1}{2} \), and 1 denotes the trivial character of \( H_S \). The module \( M_{(q-1)/2} \) is nonexistent if \( q \) is even, while the modules \( M_r \) are nonexistent if \( q = 2 \) or \( q = 3 \).

As is the case for \( U(2,1)(E/F) \), the Bruhat-Tits building \( X_S \) of \( G_S \) is a tree. We let \( \sigma_0 \) denote the hyperspecial vertex for which \( \mathfrak{a}(\sigma_0) = K_S \). The action of \( G_S \) partitions the vertices into two orbits, those at an even distance from \( \sigma_0 \) and those at an odd distance from \( \sigma_0 \). Since the action of \( G_S \) on the set of (nonoriented) edges is transitive, the notion of a diagram is the same as in Definition 6.4. Moreover, the results of Section 6 do not rely on any other properties of the group \( U(2,1)(E/F) \); replacing \( G \) by \( G_S \), \( K \) by \( K_S \), etc., shows that every conclusion holds equally well for \( G_S \). In particular, the categories \( \text{COEF}_{G_S} \) and \( \text{DLAG} \) are equivalent. With this analogy in mind, we define the following diagrams.

**Definition 8.2.** Let \( \rho_{\omega^r} \cdot j_S \) and \( \rho'_{\omega^r} \cdot j'_S \) denote the representations of \( K_S \) and \( K'_S \) obtained by inflation from \( \text{SL}_2(\mathbb{F}_q) \). We set
\[
\begin{align*}
D_0 & = (\rho_{1, S}, \rho'_{1, S}, 1, j, j'); \\
D_{q-1} & = (\rho_{1, \emptyset}, \rho'_{1, S}, 1, j, j'); \\
D_{(q-1)/2} & = (\rho_{\omega^{(q-1)/2}, \emptyset}, \rho'_{\omega^{(q-1)/2}, \emptyset}, \omega^{(q-1)/2}, j, j'); \\
D_r & = (\rho_{\omega^r, \emptyset}, \rho'_{\omega^r, \emptyset}, \omega^r, j, j'),
\end{align*}
\]
where \( 0 < r < q - 1, r \neq \frac{q-1}{2} \), and where \( j \) and \( j' \) are the inclusion maps.

Using the same arguments as in Section 7, one can show that given a diagram \( D_r \) of the form above, the \( I_S(1) \)-invariants of every nonzero irreducible quotient of \( H_0(X_S, C(D_r)) \) contain \( M_r \), and therefore such a quotient must be supersingular. Moreover, if we specialize to the case \( q = p \), there exists a canonical pure diagram \( E_r \) corresponding to an initial diagram \( D_r \):
Proposition 8.3. Assume $q = p$. Let $M_r$ be a supersingular $\mathcal{H}_{p}(G_S, I_S(1))$-module, and let
\[ D_r = (\rho, \rho', \omega^r, j, j') \]
be the associated diagram as in Definition 8.2. Then the diagram
\[ E_r = (\text{inj}_{K_S}(\rho), \text{inj}_{K_S}(\rho'), \text{inj}_{K_S}(\rho)|_{I_S}, \bar{j}_p, \bar{j}'_p) \]
is pure for $M_r$, where $j_p$ and $j'_p$ are isomorphisms.

Theorem 8.4. Assume $q = p$. Let $M_r$ be a supersingular $\mathcal{H}_{p}(G_S, I_S(1))$-module, let $D_r$ and $E_r$ be the diagrams constructed above, and let $\psi : D_r \to E_r$ denote the canonical embedding. Then the representation afforded by
\[ \text{im}(\psi : H_0(X, C(D_r)) \to H_0(X, C(E_r))) \]
is irreducible, admissible and supersingular. For distinct supersingular modules $M_r, M_{r'}$, the resulting representations are nonisomorphic.

Proof. The proofs of Propositions 7.2 and 7.8 hold equally well in the context of the group $G_S$, which implies the first claim. The proof of the second claim follows in a manner similar to the proof of Corollary 7.13.

In this way, we have constructed $p$ irreducible supersingular representations, corresponding to the supersingular $\mathcal{H}_{p}(G_S, I_S(1))$-modules. In particular, for $F = \mathbb{Q}_p$, we recover the following classification of supersingular representations:

Theorem 8.5. Let $M_r$ be a supersingular Hecke module for $\text{SL}_2(\mathbb{Q}_p)$, and let $D_r$ and $E_r$ be the diagrams constructed above. We then have
\[ \text{im}(\psi : H_0(X, C(D_r)) \to H_0(X, C(E_r))) \cong \pi_r, \]
where $\pi_r$ is the supersingular representation of $\text{SL}_2(\mathbb{Q}_p)$ defined in [1], Chapitre 3.

Proof. By Théorème 3.6.13 of [1], there are precisely $p$ isomorphism classes of irreducible supersingular representations of $\text{SL}_2(\mathbb{Q}_p)$, given by the representations $\pi_r, 0 \leq r \leq p - 1$. Likewise, the representations
\[ \text{im}(\psi : H_0(X, C(D_r)) \to H_0(X, C(E_r))), \]
for $0 \leq r \leq p - 1$ constitute $p$ pairwise nonisomorphic irreducible supersingular representations. It therefore suffices to match these. Since the $I_S(1)$-invariants of the image of the induced map on homology contain a Hecke module isomorphic to $M_r$, and since $\pi_{r(1)} \cong M_r$ as right $\mathcal{H}_{p}(G_S, I_S(1))$-modules, we conclude
\[ \text{im}(\psi : H_0(X, C(D_r)) \to H_0(X, C(E_r))) \cong \pi_r. \]

Remark. When $q \neq p$, the above construction fails in a manner similar to the construction for $U(2, 1)(E/F)$, meaning that pure diagrams of the form $(\text{inj}_{K_S}(P), \text{inj}_{K_S}(P'), \text{inj}_{I_S}(X), j_p, j'_p)$ do not always exist. One may translate the example of the previous section to the case of $\text{SL}_2(F)$ to produce such an example explicitly.
9. Appendix

9.1. Proof of Theorem 4.1. Here we carry out the computations for Theorem 4.1. Recall that \( \varepsilon = \bar{\zeta} \otimes \bar{\eta} \) is a character of the torus \( T \), and the space of \( I(1) \)-invariants \( \text{ind}_B^G(\varepsilon)^{I(1)} \) is two-dimensional, spanned by the functions \( f_1 \) and \( f_2 \) (cf. Section 4). The computations are split up according to the nature of \( \varepsilon^* \).

Using Proposition 6 in [3], we can compute the action of \( T_{n_s} \) and \( T_{n_{s'}} \) on \( v \in \text{ind}_B^G(\varepsilon)^{I(1)} \).

Equation (4) implies

\[
v \cdot T_{n_s} = \sum_{x,y \in \mathbb{F}_q^2, x+y \not\equiv \eta \mod q} u(-x,y)n_s^{-1}v.
\]

Substituting \( f_1 \) for \( v \) and evaluating at 1 and \( n_s \) gives

\[
f_1 \cdot T_{n_s}(1) = \sum_{x,y \in \mathbb{F}_q^2, x+y \not\equiv \eta \mod q} f_1(u(-x,y)n_s^{-1})
\]

\[
= 0
\]

\[
f_1 \cdot T_{n_s}(n_s) = \sum_{x,y \in \mathbb{F}_q^2, x+y \not\equiv \eta \mod q} f_1(n_s u(-x,y)n_s^{-1})
\]

\[
= \sum_{x,y \in \mathbb{F}_q^2, x+y \not\equiv \eta \mod q} f_1(u^*(-\sqrt{\varepsilon}, -\eta \varepsilon))
\]

\[
= 1,
\]

where the last equality follows from equation (1). Similarly,

\[
f_2 \cdot T_{n_s}(1) = \sum_{x,y \in \mathbb{F}_q^2, x+y \not\equiv \eta \mod q} f_2(u(-x,y)n_s^{-1})
\]

\[
= 0
\]

\[
f_2 \cdot T_{n_s}(n_s) = \sum_{x,y \in \mathbb{F}_q^2, x+y \not\equiv \eta \mod q} f_2(n_s u(-x,y)n_s^{-1})
\]

\[
= \sum_{x,y \in \mathbb{F}_q^2, x+y \not\equiv \eta \mod q} f_2(u^*(-\sqrt{\varepsilon}, -\eta \varepsilon))
\]

\[
= \sum_{x,y \in \mathbb{F}_q^2, x+y \not\equiv \eta \mod q} f_2(n_s \text{diag}(-y \sqrt{\varepsilon}, -y \eta^{-1}, y^{-1} \sqrt{\varepsilon}^-1))
\]

\[
= \sum_{x,y \in \mathbb{F}_q^2, x+y \not\equiv \eta \mod q} \varepsilon(\text{diag}(y^{-1} \sqrt{\varepsilon}^-1, -y \eta^{-1}, -\sqrt{\varepsilon}^-1))
\]

\[
= \begin{cases} 
-1 & \text{if } \varepsilon^* = \bar{\eta} \circ \det \\
0 & \text{if } (\varepsilon^*)^* = \varepsilon^*, \varepsilon^* \neq \bar{\eta} \circ \det \\
0 & \text{if } (\varepsilon^*)^* \neq \varepsilon^*.
\end{cases}
\]

The last equality is obtained in precisely the same manner as in the proof of Proposition 3.17 (cf. the computation of \( \varphi_{n_s} * \varphi_{n_s}(n_s) \)).
Equation (5) implies
\[ v \cdot T_{n'} = \sum_{y \in \mathbb{F}_q^2, y + \overline{y} = 0} u^{-}(0, \varpi \overline{y}) \alpha n^{-1} \cdot v \]
for \( v \in \text{ind}_B^G(\varepsilon)^{I(1)} \). Thus
\[ f_1 \cdot T_{n'}(1) = \sum_{y \in \mathbb{F}_q^2, y + \overline{y} = 0} f_1(u^{-}(0, \varpi \overline{y}) \alpha n^{-1}) \]
\[ = \sum_{y \in \mathbb{F}_q^2, y + \overline{y} = 0} f_1(u(0, \varpi^{-1}y^{-1}) \text{diag}(-y^{-1}\sqrt{\varepsilon}, 1, -y\sqrt{\varepsilon}^{-1})u^{-}(0, \varpi \varepsilon y^{-1})) \]
\[ = \sum_{y \in \mathbb{F}_q^2, y + \overline{y} = 0} \varepsilon(\text{diag}(-y^{-1}\sqrt{\varepsilon}, 1, -y\sqrt{\varepsilon}^{-1})) \]
\[ = \begin{cases} 
-1 & \text{if } \varepsilon^* = \overline{\eta} \circ \det \\
-1 & \text{if } (\varepsilon^*)^q = \varepsilon^*, \varepsilon^* \neq \overline{\eta} \circ \det \\
0 & \text{if } (\varepsilon^*)^q \neq \varepsilon^* 
\end{cases} 
\]
\[ f_1 \cdot T_{n'}(n_s) = \sum_{y \in \mathbb{F}_q^2, y + \overline{y} = 0} f_1(n_su^{-}(0, \varpi \overline{y}) \alpha n^{-1}) \]
\[ = \sum_{y \in \mathbb{F}_q^2, y + \overline{y} = 0} f_1(u(0, -\varpi \varepsilon^{-1}y) \alpha^{-1}) \]
\[ = 0. \]

The equality (*) follows from the fact that \( \widetilde{\zeta} \) is trivial on \( \mathbb{F}_q^\times \) if and only if \( \widetilde{\zeta}^q + 1 \) is trivial on \( \mathcal{O}_E^\times \). Similarly, we have
\[ f_2 \cdot T_{n'}(1) = \sum_{y \in \mathbb{F}_q^2, y + \overline{y} = 0} f_2(u^{-}(0, \varpi \overline{y}) \alpha n^{-1}) \]
\[ = \widetilde{\zeta}(-1)\varepsilon(\alpha) + \sum_{y \in \mathbb{F}_q^2, y + \overline{y} = 0} f_2(u(0, \varpi^{-1}y^{-1}) \text{diag}(-y^{-1}\sqrt{\varepsilon}, 1, -y\sqrt{\varepsilon}^{-1})u^{-}(0, \varpi \varepsilon y^{-1})) \]
\[ = \widetilde{\zeta}(-1)\varepsilon(\alpha) \]
\[ f_2 \cdot T_{n'}(n_s) = \sum_{y \in \mathbb{F}_q^2, y + \overline{y} = 0} f_2(n_su^{-}(0, \varpi \overline{y}) \alpha n^{-1}) \]
\[ = \sum_{y \in \mathbb{F}_q^2, y + \overline{y} = 0} f_2(u(0, -\varpi \varepsilon^{-1}y) \alpha^{-1}) \]
\[ = 0. \]

9.2. Proof of Theorem 6.6. In order to prove that the categories \( \mathcal{COE}_G \) and \( \mathcal{DLAG} \) are equivalent, we must verify that there is a natural transformation from \( \mathcal{D} \circ \mathcal{C} \) (resp. \( \mathcal{C} \circ \mathcal{D} \)) to \( \text{id}_{\mathcal{DLAG}} \) (resp. \( \text{id}_{\mathcal{COE}_G} \)).

Given a diagram \( D = (D_0, D'_0, D_1, r_D, r'_D) \), the definition of the functors \( \mathcal{D} \) and \( \mathcal{C} \) gives
\[ \mathcal{D} \circ C(D) = \tilde{D} = (F_{\sigma_0}, F_{\sigma_0}', F_{\tau_1}, r_{\sigma_0}^\tau, r_{\sigma_0}'^\tau), \]

and we have already shown that \( ev = (ev_{\sigma_0}, ev_{\sigma_0}', ev_{\tau_1}) \) is an isomorphism from \( \tilde{D} \) to \( D \). To show it gives a natural transformation, we let \( E = (E_0, E_0', E_1, r_E, r'_E) \) be another diagram and \( \psi = (\psi_0, \psi_0', \eta_1) \) a morphism from \( D \) to \( E \). Let

\[ \mathcal{D} \circ C(E) = \tilde{E} = (F_{\sigma_0}', F_{\sigma_0}'', F_{\tau_1}', (r')_{\sigma_0}^\tau, (r')_{\sigma_0}'^\tau). \]

We must check that the following diagram is commutative:

\[
\begin{array}{ccc}
\tilde{D} & \xrightarrow{\mathcal{D} \circ C(\psi)} & \tilde{E} \\
\downarrow{ev} & & \downarrow{ev} \\
D & \xrightarrow{\psi} & E
\end{array}
\]

Writing this out explicitly, commutativity of this diagram is equivalent to commutativity of the following three diagrams:

\[
\begin{array}{ccc}
F_{\sigma_0} & \xrightarrow{D \circ C(\psi_0)} & F_{\sigma_0}' \\
\downarrow{ev_{\sigma_0}} & & \downarrow{ev_{\sigma_0}'} \\
D_0 & \xrightarrow{\psi_0} & E_0
\end{array}
\quad
\begin{array}{ccc}
F_{\sigma_0}' & \xrightarrow{D \circ C(\psi_0')} & F_{\sigma_0}'' \\
\downarrow{ev_{\sigma_0}'} & & \downarrow{ev_{\sigma_0}''} \\
D'_0 & \xrightarrow{\psi_0'} & E'_0
\end{array}
\quad
\begin{array}{ccc}
F_{\tau_1} & \xrightarrow{D \circ C(\eta_1)} & F_{\tau_1}' \\
\downarrow{ev_{\tau_1}} & & \downarrow{ev_{\tau_1}'} \\
D_1 & \xrightarrow{\eta_1} & E_1
\end{array}
\]

We check the first of these assertions. Given \( f \in F_{\sigma_0} \), we see \( \psi_0 \circ ev_{\sigma_0}(f) = \psi_0(f(1)) \). By the definitions of \( C(\psi) \) and \( D \), we know that \( D \circ C(\psi)(f) = f_{\psi_0(f(1))} \), therefore \( ev_{\sigma_0}'(D \circ C(\psi)(f)) = \psi_0(f(1)) \). The other two follow similarly, and hence \( ev \) gives a natural transformation between \( \mathcal{D} \circ C \) and \( \text{id}_{\mathcal{D} \circ \mathcal{C}} \).

For the other direction, let \( \mathcal{V} = (V_r)_r \) be a \( G \)-equivariant coefficient system, with restriction maps \( t_\sigma \). Let \( \mathcal{F} = (F_r)_r \) be the coefficient system \( C \circ D(\mathcal{V}) \), with restriction maps \( r_\sigma \). Given an edge \( \tau \) containing a vertex \( \sigma \in X^e_0 \), let \( g \in G \) be such that \( \tau = g.\tau_1 \) and \( \sigma = g.\sigma_0 \). For this vertex \( \sigma \), we define a map \( ev_\sigma \) by

\[ ev_\sigma : F_\sigma \to V_\sigma \]

\[ f \mapsto g_{\sigma_0}.v, \]

where \( v = f(g^{-1}) \in V_{\sigma_0} \). One defines \( ev_\sigma \) similarly if \( \sigma \in X^e_0 \). For the edge \( \tau \), define \( ev_\tau \) by

\[ ev_\tau : F_\tau \to V_\tau \]

\[ f \mapsto g_{\tau_1}.v, \]

where \( v = f(g^{-1}) \in V_{\tau_1} \). Note that both definitions are independent of the choice of \( g \), and that the maps \( ev_\sigma, ev_\tau \) are isomorphisms of vector spaces.

We must now show that the system \( (ev_\tau)_r \) is compatible with the \( G \)-action. Let \( \sigma \in X^e_0 \) be such that \( \sigma = g.\sigma_0 \) for some \( g \). For an element \( h \in G \) and \( f \in F_\sigma \), we have \( ev_{h.\sigma}(h.f) = ev_{h.\sigma}(h,f) = (h.g)_{\sigma_0}.v \), where \( v = f(g^{-1}) \). On the other hand, we have \( h.\sigma \circ ev_\sigma(f) = h.\sigma(g_{\sigma_0}v) = (h.g)_{\sigma_0}.v \). The same argument applies for the case \( \sigma \in X^e_0 \) or \( \tau \in X^e_1 \).

To check that the system \( (ev_\tau)_\tau \) is compatible with the restriction maps, we need to verify the commutativity of the following diagram (for \( \tau \in X_1 \) containing \( \sigma \in X_0 \):
Let $\tau \in X_1$ contain $\sigma \in X_0^\sigma$, and let $g \in G$ be such that $\tau = g.\tau_1$, $\sigma = g.\sigma_0$. Given $f \in F_\tau$, we have $t^{\tau}_\sigma \circ ev_\tau(f) = t^{\tau}_\sigma(g_{\tau_1}, v)$, where $v = f(g^{-1})$. On the other hand, since $F$ comes from the diagram $(V_{\sigma_0}, V_{\sigma_1}, V_{\tau_1}, t^{\tau_1}_{\sigma_0})$, we have $ev_{\sigma_0} \circ t^{\tau_1}_{\sigma_0}(f) = ev_{\sigma_0}(g_{\tau_1} t^{\tau_1}_{\sigma_0}(v)) = g_{\sigma_0} \circ t^{\tau_1}_{\sigma_0}(v)$. By definition of a $G$-equivariant coefficient system, we have $t^{\tau}_\sigma \circ g_{\tau_1} = g_{\sigma_0} \circ t^{\tau_1}_{\sigma_0}$, and therefore the diagram commutes. The argument for the case $\sigma \in X_0^\sigma$ is the same.

To show that the system $(ev_\tau)_\tau$ defines a natural transformation from $C \circ D$ to $id_{\mathcal{COE}_{\mathcal{F}G}}$, it only remains to check the compatibility of $(ev_\tau)_\tau$ with morphisms in $\mathcal{COE}_{\mathcal{F}G}$. The proof is virtually identical to the proof of $D \circ C \simeq id_{\mathcal{D}T\mathcal{A}G}$, so we omit it. Collecting these results shows that the two categories are equivalent.

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