Energies and widths of Efimov states in the three-boson continuum

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Three-boson Efimov physics is well known in the bound-state regime, but far less in the three-particle continuum at negative two-particle scattering length where Efimov states evolve into resonances. They are studied solving rigorous three-particle scattering equations for transition operators in the momentum space. The dependence of the three-boson resonance energy and width on the two-boson scattering length is studied with several force models. The universal limit is determined numerically considering highly excited states; simple parametrizations for the resonance energy and width in terms of the scattering length are established. Decreasing the attraction, the resonances rise not much above the threshold but broaden rapidly and become physically unobservable, evolving into subthreshold resonances. Finite-range effects are studied and related to those in the bound-state regime.

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I. INTRODUCTION

Fifty years ago V. Efimov studied theoretically the three-body system with large two-body scattering lengths [1] and laid the foundations of the universal physics, also called Efimov physics. Since then a large number of theoretical and experimental works with applications to nuclear, cold atom, and molecular physics has been performed, and the properties of universal few-body systems have been determined; see recent reviews [2, 3] for summary and further references. In particular, the bound-state energy levels of three identical bosons and their dependence on the two-boson scattering length is among the best-known manifestations of the Efimov physics, even with semi-analytic parametrizations available [2, 4], the most refined version being presented in Ref. [5]. However, the universal properties of the three-boson system were studied less extensively in the regime of negative scattering length above the free-particle threshold where the trimers become unbound. Reference [4] considered only the limit of vanishing energy, while Refs. [6–8] investigated the recombination into deeply bound dimers. It is known that in this regime the trimers become resonant states [9]; their energies, widths, and their evolution with the scattering length were determined by Bringas et al. [9], but not in a strictly universal regime where the scattering length is sufficiently large such that the finite-range effects are negligible. A systematic study of finite-range effects is also still missing. The aim of the present work is to resolve this situation and to present a high-precision study of universal three-boson resonant states in the three-particle continuum with no bound dimers, neither shallow nor deep.

The study is based on exact Faddeev-type equations [10] for three-body transition operators in the version proposed by Alt, Grassberger, and Sandhas (AGS) [11], that are solved in the momentum space. The method has been used for the search for three-neutron resonances [12]. An important advantage of the direct continuum approach is its ability to estimate not only the resonance position but also its effect on scattering amplitudes that lead to observables in collision processes. Additional complications in the context of Efimov physics arise due to very different sizes of the interaction range and the characteristic interparticle distances.

Section II describes three-particle scattering equations and some details of calculations whereas Sec. III contains results. The summary is presented in Sec. IV.

II. THEORY

If the two-particle potential $v$ is tuned to yield a negative two-boson scattering length $a$ and no bound dimers, then the only possible process in the three-boson continuum is the $3 \rightarrow 3$ scattering. As given in Ref. [12], the symmetrized transition operator for this process,

$$U_{00} = (1 + P)t(1 + P) + (1 + P)tG_0UG_0t(1 + P) \quad (1)$$

contains two-body and three-body parts represented by the first and second terms, respectively; only the latter corresponds to a genuine three-particle process. All operators in Eq. (1) refer to the relative three-particle motion. They are the combination of particle permutation operators $P = P_{12}P_{23} + P_{13}P_{23}$, the free resolvent

$$G_0 = (E + i\theta - H_0)^{-1} \quad (2)$$

at the available energy $E$ with the kinetic energy operator $H_0$, and the two- and three-particle transition operators $t$ and $U$, respectively. They are related to the two-particle potential $v$ via integral equations by Lippmann-Schwinger,

$$t = v + vG_0t \quad (3)$$
and AGS
\[ U = PG_0^{-1} + PtG_0U. \] (4)

As is well known, a true bound state of the few-body system (stable with respect to decay into several clusters) corresponds to the pole of the respective transition operator at a real negative energy \( E = -E_b \) in the physical energy sheet, \( E_b \) being the binding energy. In the system of three spinless bosons, bound Efimov states have zero total angular momentum \( J = 0 \) and positive parity \( \Pi \). When this bound state, via variation of the interaction, crosses the free-particle threshold and becomes a resonance, the pole of the transition operators (both \( U \) and \( U_{0b} \)) moves to \( E = E^R - i\Gamma/2 \) in the nonphysical complex energy sheet, \( E^R \) being the resonance position and \( \Gamma \) its width. As long as \( E^R > 0 \) and \( \Gamma \) is sufficiently small, the pole is not far from the physical scattering region and manifests itself via resonant enhancement of continuum processes. This is reflected in the energy dependence of the corresponding transition operator, that can be expanded into power series near the pole, i.e.,
\[ U_{jmn} = \sum_{n_r=-1}^{\infty} \tilde{U}^{(n_r)}_{jmn} (E - E^R + i\Gamma/2)^{n_r}, \] (5)
the \( n_r = -1 \) term being the pole term and \( n_r \geq 0 \) being background terms. In this way one can determine not only the position and width of the resonance, but also its importance relative to the non-resonant background. To a very good approximation, in the vicinity of the pole the series \( \tilde{U}^{(n_r)}_{jmn} \) can be truncated after few lowest terms while higher-order terms yield negligible contribution.

The procedure of determining the resonance parameters follows closely the one of Ref. [12] for three-neutron resonances. The AGS equation \( \text{(4)} \) is solved in the momentum-space partial wave representation; for technical details, including also the treatment of kinematical singularities in the kernel of the equation, see also Ref. [13]. Various on-shell and off-shell matrix elements of three-boson transition operators \( U \) and \( T = tG_0UG_0t \) are calculated as functions of the energy \( E \), within numerical accuracy all exhibiting the same resonant behavior, much like that shown in Ref. [12] for the three-neutron system with enhanced interaction. The position and width of the resonance is determined fitting those energy-dependent matrix elements to Eq. \( \text{(5)} \). Differences between the resonance parameters obtained from transition matrix elements with different initial and final test states, as well as variations due to the different maximal power \( n_r \) in the expansion \( \tilde{U}^{(n_r)}_{jmn} \), serve as an estimation of the error for this procedure. Typically, for small \( \Gamma \) the pole is close to the physical scattering region, the resonant behavior is pronounced very clearly, and, consequently, the error is extremely small. In contrast, very broad resonances become hardly distinguishable from the background, resulting also slower convergence of the expansion \( \tilde{U}^{(n_r)}_{jmn} \) where \( n_r \) up to 5 or 6 was employed in the present work. This leads to decreasing accuracy, and renders a reliable extraction of the resonance parameters impossible when \( \Gamma \) is too large. On the other hand, the fact that resonant behavior cannot be seen in transition matrix elements indicates that the resonance cannot be distinguished from background and becomes physically unobservable.

### III. RESULTS

The present work aims to study the evolution of three-boson Efimov resonances, i.e., to establish universal relations between their energies and widths and the two-boson scattering length, similar to what is well known for bound states [2, 4, 5]. The quantity connecting the trimer bound- and resonant-state regimes is the negative two-boson scattering length \( a_\Lambda \) where the \( n \)-th Efimov trimer crosses the free-particle threshold. There is a universal relation between \( a_\Lambda \) and the binding momentum \( k_n \) of the \( n \)-th Efimov trimer in the unitary limit. Reference [3], where also the elastic three-boson scattering in the zero energy limit was considered, predicted \( k_n a_\Lambda = -1.56(5) \). However, the most accurate results are \( k_n a_\Lambda = -1.50763 \), obtained in Ref. [14] analytically, and \( k_n a_\Lambda = -1.5077(1) \), obtained numerically in Ref. [15] considering highly excited Efimov states up to \( n = 5 \). To make the connection with the latter work, the two-boson interaction model is taken from Ref. [15]. It has only the \( S \)-wave component
\[ \langle p_f | v_s | p_i \rangle = (1 + \beta p_f^2 / \Lambda^2) e^{-p_f^2 / \Lambda^2} \]
\[ \times \frac{2}{\pi m} \left\{ \frac{1}{a} - \left[ 1 + \frac{\beta}{2} \left( 1 + \frac{3\beta}{8} \right) \left( \frac{\Lambda}{\sqrt{2\pi}} \right) \right]^{-1} \right\} \times e^{-p_i^2 / \Lambda^2} (1 + \beta p_i^2 / \Lambda^2) \] (6)
where \( p_f \) (\( p_i \)) is the final (initial) momentum, \( m \) is the boson mass, and \( \Lambda \) is the momentum cutoff parameter that controls the range of the force; the \( h = 1 \) convention is used. The parameter \( \beta \) enables variations in the balance of low- and high-momentum components as will be needed later on; for the following results it is, however, chosen as \( \beta = 0 \) unless explicitly stated otherwise. For the extraction of universal relations highly excited states have to be considered with vanishing finite-range corrections. This can be characterized by the ratio of the two-boson \( S \)-wave effective range \( r_s \) and the scattering length \( a \). At the trimer and free-particle threshold intersection points these ratios \( r_s / a_n \) are 0.372, 0.0187, 8.45 × 10^-4, and 3.73 × 10^-5 for \( n = 0, 1, 2, \) and 3, respectively. The trimer ground state \( n = 0 \) is therefore expected to exhibit significant finite-range corrections. However, finite-range effects should become very small for \( n \geq 2 \). Indeed, the product \( k_n a_\Lambda \) is −2.0629, −1.5642, −1.5116, and −1.5079 for \( n = 0, 1, 2, \) and 3, respectively, in the \( n = 3 \) case being very close to the universal value of −1.50763. This suggests that \( n = 3 \) should be sufficient
also for the determination of universal Efimov resonance properties with a good accuracy.

To emphasize the universal character of the results, they will be given as dimensionless quantities. Thus, the energy \( E_n^R \) and width \( \Gamma_n \) of the \( n \)-th Efimov state will be shown in the dimensionless forms \( \varepsilon_n = E_n^R m(a_n^-)^2 / \hbar^2 \) and \( \gamma_n = \Gamma_n m(a_n^-)^2 / \hbar^2 \). In other words, \( \varepsilon_n \) (\( \gamma_n \)) is the energy (width) of the \( n \)-th Efimov resonance in units of \( \hbar^2 / m(a_n^-)^2 \). For the two-boson scattering length the most meaningful reference point in the present context is \( a_n^- \) for each \( n \). To keep consistency with the standard representation of the Efimov physics in terms of the inverse scattering length, the results will be shown as functions of the dimensionless quantity \( |a_n^-|/a \) that takes values below (above) \(-1\) in the resonance (bound state) regime. The dimensionless energies \( \varepsilon_n \) of four lowest three-boson states with \( n = 0, 1, 2, \) and \( 3 \) are shown in Fig. 1 partially including also the bound state region where the binding energy was obtained by solving the standard bound-state Faddeev equation [12].

The bound-state and continuum calculation results connect well at \( |a_n^-|/a = -1 \). The corresponding dimensionless resonance widths \( \gamma_n \) are shown in Fig. 2. In both cases the convergence with increasing \( n \) is evident, suggesting that, \( n = 2 \) and \( n = 3 \) results, being very close to each other, accurately represent the universal limit. In contrast, the ground state \((n = 0)\) results show significant deviations from the universal limit due to finite-range corrections, as can be expected given the corresponding ratio \( r_n/|a_n^-| = 0.372 \). The width of the resonance increases with decreasing \( |a_n^-|/a \), and, as a consequence, the theoretical error bars increase too. For clarity, the theoretical error bars are given only for \( n = 3 \) results, they are roughly the same for all considered \( n \) values. Beyond \( |a_n^-|/a = -2 \) a reliable extraction of resonance properties is very difficult. The resonant behavior becomes very broad and unobservable since the width exceeds the energy by a factor of 4 or more.

The interdependence of the two-boson scattering length and three-boson energy (or the binding momentum) in the bound state region \( |a_n^-|/a > -1 \) is well-known and parameterized in Refs. [2, 4, 5]. It is desirable to derive semi-analytical relations also in the resonance region for the real energy part and width. An insight into a possible form of the parametrization can be obtained by investigating the dependence of the dimensionless complex resonant momentum \( \kappa_n^r - i\kappa_n^i \), related to energy and width as \( \varepsilon_n - i\gamma_n/2 = (\kappa_n^r - i\kappa_n^i)^2 \), i.e.,

\[
\varepsilon_n = (\kappa_n^r)^2 - (\kappa_n^i)^2, \tag{7a}
\]

\[
\gamma_n = 4\kappa_n^r \kappa_n^i. \tag{7b}
\]

Note that in the bound-state region \( \kappa_n^r = k_n |a_n^-| \) simply relates to the standard binding momentum \( k_n \) while \( \kappa_n^i = 0 \). The dependence of \( \kappa_n^r \) and \( \kappa_n^i \) on the inverse scattering length is presented in panels (a)-(b) of Fig. 3.

A remarkable feature is that \( \kappa_n^i \) depends on \( |a_n^-|/a \) almost linearly. Furthermore, \( (\kappa_n^r)^2 \) displayed in the panel (c) of Fig. 3 also exhibits a nearly linear dependence on \( |a_n^-|/a \) in the resonance region \( |a_n^-|/a < -1 \), while deviations from the linearity are more visible in the bound-state region. Thus, introducing for brevity the variable \( x_a = -1 + |a_n^-|/a \), in the resonance region \( (x_a > 0) \) the behavior of the dimensionless momentum components roughly is \( \kappa_n^i \sim x_a \) and \( (\kappa_n^r)^2 \sim x_a \). Based on these observations and Eq. (7), the lowest-order (LO) approximate expressions for energy and width are postulated, i.e.,

\[
\varepsilon_n \approx \varepsilon_1^c x_a - \varepsilon_2^c x_a^2, \quad \tag{8a}
\]

\[
\gamma_n \approx 2\gamma_1^c x_a^{3/2}. \quad \tag{8b}
\]

The state label \( n \) on the right-hand side is suppressed for brevity. Assuming strictly linear dependence of \( \kappa_n^i \) and \( (\kappa_n^r)^2 \) on \( x_a \), only two out of three coefficients in Eqs. (8) would be independent. However, as there are deviations,
the most evident one being \((\kappa_n^i)^2\), the quality of the approximation is improved by considering the three coefficients as independent. The higher order (HO) approximation is obtained taking into account small deviations from linearity in \(\kappa_n^i\) and \((\kappa_n^i)^2\) at small \(x_a\), i.e., near the threshold. Proposed phenomenological relations

\[
(\kappa_n^i)^2 \approx c_1^i x_a + c_2^i x_a^{2/3}, \tag{9a}
\]

\[
\kappa_n^i \approx x_a (c_1^i - c_2^i e^{-c_1^i x_a^{1/2}}), \tag{9b}
\]

are used in Eq. 7 for HO approximations of \(\varepsilon_n\) and \(\gamma_n\). The coefficients in Eqs. 8 and 9 are obtained by fitting the results presented in Figs. 1 - 3. In order to check the predictive power of the LO and HO approximations, only the data points in the regime \(-1.67 < \left|a_n^-/a\right| < 1\) are included in the fit. This way one can estimate the quality of approximations by comparing them with data at \(\left|a_n^-/a\right| < -1.67\). For \(n = 3\), which is expected to be a very accurate approximation of the universal limit, the values of fit parameters are

\[
\begin{align*}
c_1^i &= 0.8582 \pm 0.0042, \quad (10a) \\
c_2^i &= 0.3911 \pm 0.0079, \quad (10b) \\
c_1^i &= 0.9766 \pm 0.0057, \quad (10c)
\end{align*}
\]

and

\[
\begin{align*}
c_1^i &= 0.7416 \pm 0.0018, \quad (11a) \\
c_2^i &= 0.0525 \pm 0.0014, \quad (11b) \\
c_1^i &= 0.5671 \pm 0.0009, \quad (11c) \\
c_2^i &= 0.4751 \pm 0.0131, \quad (11d) \\
c_3^i &= 4.1714 \pm 0.0900 \quad (11e)
\end{align*}
\]

The LO and HO approximations for energy and width are compared in panels (a) and (b) of Fig. 4 with the results of the direct calculation; in all cases \(n = 3\). Both LO and HO approximations seem to fit the original data very well up to \(|a_n^-/a| > -1.67\). However, a closer inspection of deviations \(\Delta \varepsilon_n = \varepsilon_n - \varepsilon_n(X0)\) and \(\Delta \gamma_n = \gamma_n - \gamma_n(X0)\), amplified by a factor of 10 and shown by thin curves near the zero line, demonstrates considerably better accuracy of the HO approximation, especially for the two points at the left that were excluded from the fit. Nevertheless, the LO still fits those data within their error bars.

For curiosity, in Fig. 4 the LO and HO results are extrapolated to \(|a_n^-/a| < -2\) region. The LO and HO predictions, though qualitatively different, qualitatively exhibit the same trend, i.e., \(\gamma_n(X0)\) increases rapidly while \(\varepsilon_n(X0)\) reaches its maximum and then decreases with decreasing \(|a_n^-/a|\), becoming a completely unobservable very broad subthreshold resonance. The trajectory of the resonance in the complex plane \(\varepsilon_n - i\gamma_n/2\) is shown in the panel (c) of Fig. 4 which uses roughly the same scale for both real and imaginary part. The shape clearly shows that the resonance does not rise much above the threshold but broadens rapidly. The lower part of the curves is an extrapolation based on Eqs. 8 and 9, but both LO and HO approximations consistently indicate the evolution into a subthreshold resonance when \(|a|\) decreases.

As mentioned, an earlier study of Efimov resonances was performed by Bringas et al. 4. They solved bound-state equation with contour deformation into the complex plane. Instead of the inverse two-boson scattering length \(1/a\) they used the square root of the virtual dimer energy, \(\sqrt{|B_2|}\). Since in the universal regime \(B_2 = -1/(ma^2)\), these two representations differ by finite-range corrections only, allowing for an easy qualitative comparison. Bringas et al. studied \(n = 0, 1,\) and 2 Efimov states, the evolution of their energies and widths is qualitatively fully consistent with present results. A perfect agreement between \(n = 1\) and 2 results in Ref. 4 was not achieved, indicating that the universal regime was not yet reached with a good accuracy. This can be quantified by ratios of virtual dimer energies \(B_2^{(n)} = B_2\big|_{a=a_n^-}\) when the \(n\)-th trimer crosses the threshold. In detail,
The dependence of the resonance energy and width on the inverse two-boson scattering length was presented also by Jonsell [8]. However, in that work deep dimer states were effectively included by using a three-body parameter with finite imaginary part. This precludes a direct comparison with the present work. Nevertheless, a very good qualitative agreement is found between the results in Figs. 1 and 2 and the ones in Fig. 3 of Ref. [8], limited to $|a_n^\ast|/a > -1.3$, the quantitative differences being of the order of 10%. This possibly indicates that deep dimers do not affect significantly the energy and dissociation width of the Efimov resonance.

Another important and interesting problem is the quantification of finite-range effects. Since the evolution of the resonance can be reasonably reproduced by a few parameters of the LO approximation [8], they are chosen for this study. Ji et al. showed that finite-range corrections to various Efimov features in the bound-state regime can be expressed by terms proportional to $r_s$ and $nr_s$ [10], and related this result to an earlier work by Kievsky et al. [17]. The connection between the two approaches was updated recently by Gattobigio et al. [3] through the running Efimov parameter. It appears that finite-range effects for the resonance parameters $c_1^\ast$ and $c_2^\ast$ determined in the present work are compatible with the pattern of Ref. [12]. For $n = 0, 1, 2,$ and 3 the approximate expressions

\[
c_1^\ast \approx 0.8581 - 0.4236 r_s/a_n - 1.1753 nr_s/a_n^3, \quad (12a)
\]

\[
c_2^\ast \approx 0.3890 - 1.1618 r_s/a_n - 1.8038 nr_s/a_n^3, \quad (12b)
\]

\[
c_1^\ast \approx 0.9729 - 0.6640 r_s/a_n - 1.3157 nr_s/a_n^3, \quad (12c)
\]

reproduce the results of direct calculations with good accuracy, the deviations being below 0.05%, 1%, and 3%, respectively. However, the above relations involve nonuniversal coefficients and cannot predict finite-range corrections for other force models.

To study finite-range effects, four additional calculations with different force models are performed for low ($n = 0$ or 1) resonances. One of them uses the scaled Malfliet-Tjon (MT) I-III $1S_0$ potential [15], that has an attractive long-range part and a strongly repulsive short-range part. However, the results turn out to be quite close to the ones based on the potential [11], i.e., $r_s/a_n^\ast = 0.369$ and $k_n^u/a_n^\ast = 2.1036$ for $n = 0$. Therefore other models are desired to have different $r_s/a_n^\ast$ and $k_n^u/a_n^\ast$ values, spanning the gap between the previously shown $n = 0$ and 1 results of the potential [6] with $\beta = 0$. This goal can be achieved by the variation of the parameter $\beta$ in the potential [6] thereby enhancing the high-momentum components and enabling the control of $r_s/a_n^\ast$ and $k_n^u/a_n^\ast$ values. With $n = 1$ and $\beta = -4, -6$, and $-8$ they become $r_s/a_n^\ast = 0.258, 0.160$, and 0.0671, and $k_n^u/a_n^\ast = 1.8206, 1.6190$, and 1.5004, respectively. Thus, in total this amounts to eight different combinations of force model and level. The $c_1^\ast$ and $c_2^\ast$ results for these eight sets do not exhibit a clear dependence on $r_s/a_n^\ast$ and/or $nr_s/a_n^\ast$, but show linear correlation with the corresponding $k_n^u/a_n^\ast$ values as demonstrated in Fig. 5. Most of the points are very close to the lines parameterized by

\[
c_1^\ast \approx 0.8604 + 0.3755(k_n^u/a_n^\ast - 1.50763), \quad (13a)
\]

\[
c_2^\ast \approx 0.3885 + 0.9949(k_n^u/a_n^\ast - 1.50763), \quad (13b)
\]

\[
c_1^\ast \approx 0.9730 + 0.6058(k_n^u/a_n^\ast - 1.50763), \quad (13c)
\]

except for the two $n = 0$ points possessing largest $r_s/a_n^\ast$ and $k_n^u/a_n^\ast$ values, for which the finite-range effects are expected to be sizable, thereby leading to larger deviations. Nevertheless, Eqs. (13) may be useful for the estimation of the resonance position and width from the features of the corresponding bound state.

**IV. SUMMARY**

The three-boson continuum study was carried out in the regime of negative two-boson scattering length with...
no shallow or deep dimers, where the three-particle Efimov states are not bound but resonant. Exact scattering equations for three-particle transition operators were solved in the momentum-space framework using several interaction models. From the energy dependence of the various on-shell and off-shell transition matrix elements the resonance properties were determined, in particular, their universal limit was accurately calculated considering highly excited Efimov states such that the finite-range effects become negligible. Universal relations between the resonance energy and width and the two-particle scattering length \( a \) were established, and simple parametrizations were proposed, based on a nearly linear dependence of the complex resonant momentum components on the inverse scattering length. It was found that decreasing \( |a| \), i.e., decreasing the attraction, the resonances rise not much above the threshold, the real part of energy reaching roughly \( \hbar^2/[2m(a_\nu)\gamma] \) or \( \hbar^2/(8ma^2) \) around \( a \approx a_\nu/2 \), while the width exceeds \( 2\hbar^2/[m(a_\nu)\gamma] \) at that point, thereby rendering the resonance practically unobservable. Extrapolation of the developed parametrizations indicates the evolution into a very broad subthreshold resonance by further decrease of \( |a| \).

In addition, finite-range effects were studied as well, and their correlation with the bound state properties at unitarity are found. This may be useful for the estimation of the resonance energy and width using the information from the bound-state regime.

The present work demonstrated the feasibility of accurately extracting the properties of three-boson Efimov resonances, and thereby encourages the future studies of four-particle Efimov resonances. This constitutes a highly challenging problem with novel aspects, as the four-boson resonances, due to the presence of lower trimers, have a finite width already at the threshold. The work into this direction is underway.

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