RANDOM ATTRACTORS FOR NON-AUTONOMOUS FRACTIONAL STOCHASTIC GINZBURG-LANDAU EQUATIONS ON UNBOUNDED DOMAINS

Ji Shu\textsuperscript{1} and Jian Zhang\textsuperscript{2,†}

Abstract This paper deals with the dynamical behavior of solutions for non-autonomous stochastic fractional Ginzburg-Landau equations driven by additive noise with $\alpha \in (0,1)$. We prove the existence and uniqueness of tempered pullback random attractors for the equations in $L^2(\mathbb{R}^3)$. In addition, we also obtain the upper semicontinuity of random attractors when the intensity of noise approaches zero. The main difficulty here is the noncompactness of Sobolev embeddings on unbounded domains. To solve this, we establish the pullback asymptotic compactness of solutions in $L^2(\mathbb{R}^3)$ by the tail-estimates of solutions.

Keywords Non-autonomous stochastic fractional Ginzburg-Landau equation, random dynamical system, random attractor, additive noise, upper semicontinuity.

MSC(2010) 37L55, 60H15, 35Q56.

1. Introduction

In this paper, we consider the following non-autonomous stochastic fractional Ginzburg-Landau equation with additive noise

\[
\frac{\partial u}{\partial t} + (1 + i\lambda)(-\Delta)^\alpha u + \gamma u = f(u) + g(t, x) + \delta h(x) \frac{dW}{dt}, \quad x \in \mathbb{R}^3, t > \tau, \tag{1.1}
\]

with initial condition

\[
u(\tau, x) = u_\tau(x), \quad x \in \mathbb{R}^3, \tag{1.2}
\]

where $u(t, x)$ is an unknown complex-valued function, $i$ is the imaginary unit, $\lambda \in \mathbb{R}$, $\alpha \in (0, 1)$, $\gamma \in \mathbb{R}$, the nonlinear term $f(u)$ is a complex-valued function, $g(t, x) \in L^\infty(\mathbb{R}, L^2(\mathbb{R}^3))$, $\delta > 0$, $h \in H^{2\alpha}(\mathbb{R}^3) \cap W^{2\alpha, 4}(\mathbb{R}^3)$, $W$ is a two-sided real-valued Wiener process on a probability space, $\tau \in \mathbb{R}$.

\textsuperscript{†}The corresponding author.

Email: shujisicnu.edu.cn(J. Shu), zhangjiancdv@sina.com(J. Zhang)
\textsuperscript{1}School of Mathematical Sciences and V.C. & V.R. Key Lab, Sichuan Normal University, Chengdu 610068, China
\textsuperscript{2}School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu 611731, China
\textsuperscript{*}The authors were supported by National Natural Science Foundation of China (No.1187138), joint research project of Laurent Mathematics Center of Sichuan Normal University and National-Local Joint Engineering Laboratory of System Credibility Automatic Verification.
Recently, fractional partial differential equations arise in a wide range of fields within in physics, biology, chemistry, etc., and some classical equations of mathematical physics have been postulated with fractional derivative to better describe complex phenomena, including the fractional Schrödinger equation [10,16,17], fractional Landau-Lifshitz equation [20], fractional Landau-Lifshitz-Maxwell equation [35] and fractional Ginzburg-Landau equation [18,36,45].

When $\alpha \in (0,1)$, we call the operator $(-\Delta)\alpha$ a fractional Laplacian. There are different definitions for the fractional Laplace operator on bounded domain $U$, including the integral fractional operator and the spectral fractional operator. These two kinds of fractional operators are distinct, and specially they have different eigenfunctions and eigenvalues as expressed in [40]. When $\alpha = 1$, it becomes the standard Laplace operator $(-\Delta)$. The concept of pullback random attractor, which is a generalization of global attractor in deterministic systems (see [21,37,39,46]), was introduced in [1,6–8,13,38], and characterizes the long-time behavior of random dynamical systems perfectly. The random attractor for stochastic equations have been widely discussed by many authors, see, e.g., [3–5,14,22,31,32,42,43,49–51,56,57,62] in the autonomous stochastic partial differential equations, and [12,52–55,58,59,63] in the non-autonomous case. In [4], Bates etc. discussed the random attractor of stochastic reaction-diffusion equation on unbounded domains. In recent years, there are some results on the random attractors for stochastic equations with the fractional Laplacian $(-\Delta)^\alpha$ with $\alpha \in (\frac{1}{2},1)$ in [28–30,41,44]. However, there are few results in the fractional case of $\alpha \in (0,1)$. In [55], Wang discussed the asymptotic behavior of fractional reaction-diffusion equation with $\alpha \in (0,1)$.

The generalized complex Ginzburg-Landau equation is one of the most important equations in mathematical physics, which can describe turbulent dynamics and has a long history in physics as a generic amplitude equation near the onset of instabilities in fluid mechanical systems, as well as in the theory of phase transitions and superconductivity [2,9]. The fractional Ginzburg-Landau equation describes the dynamical processes in a medium with fractal dispersion and the fractional generalization of Ginzburg-Landau equation from variational Euler-Lagrange equation for the fractal media is derived in [45]. The global existence and long behavior of Ginzburg-Landau equation were studied in [9,11,19,25,26]. In fractional case, the well-posedness and dynamical behavior were proved in [27,36]. For stochastic fractional Ginzburg-Landau equation with $\alpha \in (\frac{1}{2},1)$, the existence of the random attractor in $L^2$ and $H^1$ were respectively discussed in [28,41,44]. We note that in [28], the authors derived the estimates of solutions for $H^{1+\alpha}$ and the tail-estimates of solutions in $H^1$ instead of $H^\alpha$. This may be caused by the definition of fractional Laplace operator. In this paper, we will furthermore consider uniform a priori estimates of solutions in $H^\alpha$ and the tail-estimates of solutions in $L^2$ by introducing another definition of fractional Laplace operator, which is different from [28].

As we know, there are some results on random attractors for stochastic fractional Ginzburg-Landau equation, but few results for the fractional case with $\alpha \in (0,1)$(see [23,24,60]). In this paper, motivated by [4,55], we explore the random attractors for non-autonomous stochastic fractional Ginzburg-Landau equation with additive noise for $\alpha \in (0,1)$. However, there are several difficulties to overcome. Firstly, the fractional Laplace operator $(-\Delta)^\alpha$ is non-local and thus deriving uniform estimates on the solutions of (1.1) is much more involved than the standard Laplacian $-\Delta$. Secondly, the domain is unbounded, so the Sobolev embedding $H^\alpha(\mathbb{R}^3) \hookrightarrow L^2(\mathbb{R}^3)$
with $\alpha \in (0, 1)$ is not compact. Thirdly, since the Ginzburg-Landau equation is a complex equation, the condition of nonlinearity and uniform estimates of solutions in $L^2$ are slightly different from the real equation such as reaction-diffusion equation [55], thus we need to develop some different technologies to solve these problems. Lastly, due to $\alpha \in (0, 1)$ instead of $\alpha \in (\frac{1}{2}, 1)$, the methods in [28] are not suitable, so we have to deal with (1.1) by some different methods. We mention that, in this paper, since considering fractional Laplacian operator with $\alpha \in (0, 1)$ and estimates of solutions for $H^\alpha$, the result of random attractors is a generalization in some sense for the results of [28] with $\alpha \in (\frac{1}{2}, 1)$.

When we prove the existence and uniqueness of random attractors in $H^\alpha(\mathbb{R}^3)$, the nonlinearity $f(u)$ is special form, i.e., $f(u) = -(1 + i\mu)\|u\|^2u$ with $\mu \in \mathbb{R}$, which is consistent with the general physical background for the Ginzburg-Landau equation [2, 9]. In addition, we apply equivalent representations of the fractional Laplace operator $(-\Delta)^\alpha$ to derive the uniform estimates on solutions of (1.1) in $H^\alpha(\mathbb{R}^3)$ and carefully treat all terms involved. To overcome the lack of compactness of Sobolev embeddings on unbounded domains, we apply the idea of uniform estimates on the tails of solutions and prove the solutions are asymptotically null when $t$ and $x$ tend to infinity, which is slightly different from the fractional case [28] with $\alpha \in (\frac{1}{2}, 1)$, and the standard Laplacian case [4] in $H^1(\mathbb{R}^3)$.

This paper is organized as follows. In Section 2, the working function space, some basic concepts related to the non-autonomous random dynamical system, upper semicontinuity of random attractors, the fractional derivative and Sobolev space are introduced. In Section 3, we transform the stochastic equation into a random equation which solutions generate a random dynamical system, then give the existence and uniqueness of solutions for non-autonomous stochastic fractional Ginzburg-Landau equation. In Section 4, we derive uniform estimates for solutions and the pullback asymptotic compactness, then the existence of a pullback random attractor is proved. In Section 5, we establish the upper semicontinuity of random attractors when the coefficient $\delta$ approaches zero.

2. Preliminaries

In this section, we first present some basic notions about random attractors and non-autonomous random dynamical systems, which can be found in [1, 6, 38].

Let $(X, \| \cdot \|_X)$ be a separable Hilbert space with the Borel $\sigma$-algebra $\mathcal{B}(X)$, $(\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}})$ be an ergodic metric dynamical system, $H = L^2(U)$ with the usual scalar product and norm $\{\cdot, \cdot\}, \| \cdot \|_2$ and $L^p(U)$ be the $p$-times integrable functions space on $\mathbb{D}$ with norm denoted by $\| \cdot \|_p$.

**Definition 2.1.** A continuous random dynamical system on $\Omega$ over $(\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}})$ is a $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable mapping:

$$\varphi: \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X, (\cdot, \tau, \cdot, \cdot) \mapsto \varphi(\cdot, \tau, \cdot, \cdot)$$

such that the following properties hold:

1. $\varphi(0, \tau, \omega, \cdot)$ is the identity on $X$;
2. $\varphi(t + s, \tau, \omega, \cdot) = \varphi(t, \tau + s, \theta_s \omega, \varphi(s, \tau, \omega, \cdot))$ for all $s, t \geq 0$;
3. $\varphi(t, \tau, \omega, \cdot): X \to X$ is continuous for all $t \geq 0$.

**Definition 2.2.** (1) A set-valued mapping $\{D(\tau, \omega)\} : \Omega \to 2^X, \omega \to D(\tau, \omega)$, is said to be a random set if the mapping $\omega \to d(u, D(\tau, \omega))$ is measurable for any
Moreover, random closed (compact) set. A random set \( \{u \in X : \|u - u_0\|_X \leq R(\tau, \omega)\} \) is called a random closed (compact) set. A random set \( \{D(\tau, \omega)\} \) is said to be bounded if there exist \( u_0 \in X \) and a random variable \( R(\tau, \omega) > 0 \) such that

\[
D(\tau, \omega) \subset \{u \in X : \|u - u_0\|_X \leq R(\tau, \omega)\}, \quad \text{for all } \omega \in \Omega.
\]

(2) A random set \( \{D(\tau, \omega)\} \) is called tempered provided for P-a.e. \( \omega \in \Omega \)

\[
\lim_{t \to +\infty} e^{-\beta t} d(D(t + t, \theta t, \omega)) = 0, \quad \text{for all } \beta > 0,
\]

where \( d(D) = \sup\{\|b\|_X : b \in D\} \).

(3) A random set \( \{B(\tau, \omega)\} \) is said to be a random absorbing set if for any tempered random set \( \{D(\tau, \omega)\} \), and P-a.e. \( \omega \in \Omega \), there exists \( t_0 \) such that

\[
\varphi(t, \tau - t, \theta_{-t} \omega, D(\theta_{-t} \omega)) \subset B(\omega), \quad \text{for all } t \geq t_0.
\]

(4) A random set \( \{B_1(\tau, \omega)\} \) is said to be a random attracting set if for any tempered random set \( \{D(\tau, \omega)\} \), pull-back attractor and P-a.e. \( \omega \in \Omega \), we have

\[
\lim_{t \to +\infty} d_H(\varphi(t, \tau - t, \theta_{-t} \omega, D(\theta_{-t} \omega)), B_1(\tau - t, \omega)) = 0,
\]

where \( d_H \) is the Hausdorff semi-distance given by \( d_H(E, F) = \sup_{u \in E} \inf_{v \in F} \|u - v\|_X \) for any \( E, F \subset X \).

(5) \( D \) is called inclusion-closed if \( D = \{D(\tau, \omega)\}_{\omega \in \Omega} \in D \) and if \( \hat{D} = \{\hat{D}(\tau, \omega)\}_{\omega \in \Omega} \)

is a random subset of \( X \) with \( \hat{D}(\tau, \omega) \subseteq D(\tau, \omega) \) for all \( \omega \in \Omega \) then \( \hat{D} \in D \).

(6) Let \( D \) be a collection of random subsets of \( X \). Then \( \varphi \) is said to be \( D \)-pullback asymptotically compact in \( X \) if for P-a.e. \( \omega \in \Omega \), \( \{\varphi(t_n, \tau - t_n, \theta_{-t_n} \omega, X_n)\}_{n=1}^\infty \) has a convergent subsequence in \( X \) whenever \( t_n \to \infty \), and \( x_n \in B(\tau - t_n, \theta_{-t_n} \omega) \) with \( \{\varphi(t_n, \tau - t_n, \theta_{-t_n} \omega, X_n)\} \in D \).

**Definition 2.3.** Let \( D \) be a collection of random subsets of \( X \) and \( \{A(\tau, \omega)\}_{\omega \in \Omega} \in D \). Then \( \{A(\tau, \omega)\}_{\omega \in \Omega} \) is called a \( D \)-random attractor (or \( D \)-pullback attractor) for \( \varphi \) if the following conditions are satisfied, for P.a.e. \( \omega \in \Omega \),

(1) \( \{A(\tau, \omega)\} \) is compact, and \( \omega \to d(X, A(\tau, \omega)) \) is measurable for every \( X \in X \);

(2) \( \{A(\tau, \omega)\}_{\omega \in \Omega} \) is strictly invariant, i.e., \( \varphi(t, \tau, \omega, A(\tau, \omega)) = A(t + \tau, \theta t, \omega) \), and for a.e. \( \omega \in \Omega \);

(3) \( \{A(\tau, \omega)\}_{\omega \in \Omega} \) attracts all sets in \( D \), i.e., for all \( B \in D \) and a.e. \( \omega \in \Omega \), we have

\[
\lim_{t \to +\infty} d_H(\varphi(t, \tau - t, \theta_{-t} \omega, B(\tau - t, \theta_{-t} \omega)), A(t, \omega)) = 0.
\]

From [52], we have the existence and uniqueness theorem of random attractors for non-autonomous random dynamical system.

**Theorem 2.4.** Let \( \varphi \) be a continuous random dynamical system on \( X \) over \( (\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}}) \). If there exists a closed random tempered absorbing set \( \{B(\tau, \omega)\} \)

of \( \varphi \) and \( \varphi \) is asymptotically compact in \( X \), then \( \{A(\tau, \omega)\} \) is a random attractor of \( \varphi \), where

\[
A(\tau, \omega) = \bigcap_{s \geq 0} \bigcup_{t \geq s} \varphi(t, \tau - t, \theta_{-t} \omega, B(\tau - t, \theta_{-t} \omega)), \quad \omega \in \Omega.
\]

Moreover, \( \{A(\tau, \omega)\} \) is the unique random attractor of \( \varphi \).
In the following, we give a theorem on upper semicontinuity of random attractors (see [54]).

**Theorem 2.5.** Let $I$ be an interval of $\mathbb{R}$ and given $a \in I$. Let \( \Phi^{a}(t, \tau, \omega)_{a \in I} \) be a family of continuous RDSs on $X$ over $\mathbb{R}$ and \((\Omega, \mathcal{F}, P, (\theta_{t})_{t \in \mathbb{R}})\). Given $a_{0} \in I$, \( \Phi^{a_{0}}(t, \tau) \) is a continuous process over $\mathbb{R}$ independent of $\omega \in \Omega$. Suppose that

1. \( \Phi^{a_{0}}(t, \tau) \) has a pullback attractor \( A^{a_{0}}(\tau)_{\tau \in \mathbb{R}} \) with properties:
   
   (a) \( A^{a_{0}}(\tau) \) is compact for $\tau \in \mathbb{R}$;
   
   (b) \( \Phi^{a_{0}}(t, \tau)A^{a_{0}}(\tau) = A^{a_{0}}(t) \) for $t \geq \tau$;
   
   (c) for any bounded set $B \subset X$, \( \lim_{t \to +\infty} d_{H}(\Phi^{a_{0}}(\tau, \tau-t)B, A^{a_{0}}(\tau)) = 0; \)
   
2. \( \Phi^{a_{0}}(t, \tau) \) has a uniform pullback absorbing set \( B^{a_{0}} = \{ x \in X : \| x \|_{X} \leq R^{a_{0}} \} \subset X \) and for each $a \in I$, \( \Phi^{a} \) has a $D(X)$-pullback random attractor \( A^{a}(\tau, \omega) \in D(X) \) and a $D(X)$-pullback random absorbing set \( K^{a}(\tau, \omega) \in D(X) \) such that for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$, \( \sup_{a \to a_{0}} \| K^{a}(\tau, \omega) \|_{X} \leq R^{a_{0}}; \)

3. for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, \( \bigcup_{a \in I} A^{a}(\tau, \omega) \) is precompact in $X$;

4. for every $t \in \mathbb{R}^{+}$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, $a_{n} \in I$ with $a_{n} \to a_{0}$ and $x_{n}, x \in X$ with $x_{n} \to x$, it holds \( \lim_{n \to \infty} \Phi^{a_{n}}(t, \tau-t, \theta_{-\tau}(\omega), x_{n}) = \Phi^{a_{0}}(t, \tau-t, x). \)

Then for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, \( d_{H}(A^{a}(\tau, \omega), A^{a_{0}}(\tau)) \to 0 \) as $a \to a_{0}$.

At last, we review some concepts and notations of the fractional derivative and fractional Sobolev space (see [34] for details). Let $\mathcal{S}$ be the Schwartz space of rapidly decaying $C^{\infty}$ functions on $\mathbb{R}^{3}$, then for $0 < \alpha < 1$, the fractional Laplace operator \( (-\Delta)^{\alpha} \) is given by, for $u \in \mathcal{S}$,

\[
(-\Delta)^{\alpha}u(x) = -\frac{1}{2}C(\alpha) \int_{\mathbb{R}^{3}} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{3+2\alpha}} dy, \quad x \in \mathbb{R}^{3},
\]

(2.1)

where $C(\alpha)$ is a positive constant depending on $\alpha$ as given by

\[
C(\alpha) = \int_{\mathbb{R}^{3}} \frac{1 - \cos(\xi_{1})}{|\xi|^{3+2\alpha}} d\xi, \quad \xi = (\xi_{1}, \xi_{2}, \xi_{3}) \in \mathbb{R}^{3}.
\]

(2.2)

In particular, it follows from [34] that for any $u \in \mathcal{S}$,

\[
(-\Delta)^{\alpha}u = \mathcal{F}^{-1}(|\xi|^{2\alpha}(|\xi|^{2\alpha} u)), \quad \xi \in \mathbb{R}^{3},
\]

(2.3)

where $\mathcal{F}$ is the Fourier transform defined by

\[
(\mathcal{F}u)(\xi) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^{3}} e^{-ix \cdot \xi} u(x) dx, \quad u \in \mathcal{S},
\]

and $\mathcal{F}^{-1}$ is the inverse Fourier transform. Let $H^{\alpha}(\mathbb{R}^{3})$ be the fractional Sobolev space defined by

\[
H^{\alpha}(\mathbb{R}^{3}) = \{ u \in L^{2}(\mathbb{R}^{3}) : \int_{\mathbb{R}^{3}} \int_{\mathbb{R}^{3}} \frac{|u(x) - u(y)|^{2}}{|x-y|^{3+2\alpha}} dx dy < \infty \}
\]

which is equipped with the norm

\[
\| u \|_{H^{\alpha}(\mathbb{R}^{3})} = \left( \int_{\mathbb{R}^{3}} |u(x)|^{2} dx + \int_{\mathbb{R}^{3}} \int_{\mathbb{R}^{3}} \frac{|u(x) - u(y)|^{2}}{|x-y|^{3+2\alpha}} dx dy \right)^{1/2}.
\]
From now on, we write the norm and the inner product of $L^2(\mathbb{R}^3)$ as $\|\cdot\|$ and $(\cdot, \cdot)$, respectively. We also write the Gagliardo semi-norm of $H^\alpha(\mathbb{R}^3)$ as $\|\cdot\|_{H^\alpha(\mathbb{R}^3)}$, i.e.,

$$\|u\|_{H^\alpha(\mathbb{R}^3)} = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x-y|^{3+2\alpha}} \, dx \, dy, \quad u \in H^\alpha(\mathbb{R}^3).$$

Then for all $u \in H^\alpha(\mathbb{R}^3)$ we have $\|u\|_{H^\alpha(\mathbb{R}^3)}^2 = \|u\|^2 + \|u\|_{H^\alpha(\mathbb{R}^3)}^2$. Note that $H^\alpha(\mathbb{R}^3)$ is a Hilbert space with inner product given by

$$(u, v)_{H^\alpha(\mathbb{R}^3)} = \int_{\mathbb{R}^3} u(x)v(x) \, dx + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{3+2\alpha}} \, dx \, dy, \quad u, v \in H^\alpha(\mathbb{R}^3).$$

In terms of (2.3), one can verify (see [35]):

$$\|u\|^2_{H^\alpha(\mathbb{R}^3)} = \|u\|^2_{L^2(\mathbb{R}^3)} + \frac{2}{C(\alpha)} \|(-\Delta)^{\frac{\alpha}{2}} u\|^2_{L^2(\mathbb{R}^3)}, \quad \text{for all } u \in H^\alpha(\mathbb{R}^3), \quad (2.4)$$

and hence $(\|u\|^2_{L^2(\mathbb{R}^3)} + \|(-\Delta)^{\frac{\alpha}{2}} u\|^2_{L^2(\mathbb{R}^3)})^{\frac{1}{2}}$ is an equivalent norm of $H^\alpha(\mathbb{R}^3)$.

### 3. The stochastic fractional Ginzburg-Landau equation with additive noise

In this section, we will give the existence and uniqueness of solutions of problem (1.1)-(1.2) which generates a continuous random dynamical system.

The standard probability space $(\Omega, \mathcal{F}, P)$ will be used in this paper where $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$, and $\mathcal{F}$ is the Borel $\sigma$-algebra induced by the compact-open topology of $\Omega$, and $P$ is the Wiener measure on $(\Omega, \mathcal{F})$. Given $t \in \mathbb{R}$, define $\theta_t : \Omega \to \Omega$ by

$$\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \omega \in \Omega.$$ 

Then $(\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}})$ is a parametric dynamical system. Let $y : \Omega \to \mathbb{R}$ be a random variable given by: $y(\omega) = -\gamma \int_{-\infty}^{0} e^{\gamma \tau} \omega(\tau) \, d\tau$ for $\omega \in \Omega$. Then $y(t)$ is the unique stationary solution of the stochastic equation

$$dy + \gamma y \, dt = dW. \quad (3.1)$$

In addition, it follows from [1], that there exists a $\theta_t$-invariant set of full measure (still denoted by $\Omega$) such that $y(\theta_t \omega)$ is pathwise continuous for each fixed $\omega \in \Omega$ and there exists a tempered function $r(\omega) > 0$ such that

$$|y(\omega)|^2 + |y(\omega)|^4 \leq r(\omega), \quad (3.2)$$

where $r(\omega)$ satisfies, for $P.a.e. \omega \in \Omega$,

$$r(\theta_t \omega) \leq e^{2|t|} r(\omega), \quad t \in \mathbb{R}.$$ 

From above, we obtain that, for $P - a.e. \omega \in \Omega$,

$$|y(\theta_t \omega)|^2 + |y(\theta_t \omega)|^4 \leq e^{2|t|} r(\omega), \quad t \in \mathbb{R}. \quad (3.3)$$
We now transform the stochastic equation (1.1) into a pathwise deterministic one by using the random variable \( z \). Put \( z(\theta_t \omega) = h(x)g(\theta_t \omega) \). Given \( \tau \in \mathbb{R}, t \geq \tau, \omega \in \Omega \) and \( u_\tau \in L^2(\mathbb{R}^3) \), if \( u = u(t, \tau, \omega, u_\tau) \) is a solution of (1.1)-(1.2), then we introduce a new variable \( v = v(t, \tau, \omega, v_\tau) \) by

\[
v(t, \tau, \omega, v_\tau) = u(t, \tau, \omega, u_\tau) - \delta z(\theta_t \omega) \quad \text{with} \quad v_\tau = u_\tau - \delta z(\theta_t \omega).
\]

(3.4)

From (1.1)-(1.2) and (3.4) we obtain, for \( t > \tau \),

\[
\frac{dv}{dt} + (1 + i\lambda)(-\Delta)^\alpha v + \delta(1 + i\lambda)(-\Delta)^\alpha z(\theta_t \omega) + \gamma v = f(u) - (1 + i\mu)|u|^2u + g(t, x), \quad x \in \mathbb{R}^3,
\]

(3.5)

and initial condition

\[
v(t, \tau, \omega, v_\tau) = v_\tau(x), \quad x \in \mathbb{R}^3,
\]

(3.6)

where \( f(u) = -(1 + i\mu)|u|^2u \).

Next we will first give the existence and uniqueness of solutions for problem (3.5)-(3.6), and then obtain the solutions of (1.1)-(1.2) by the transform (3.4). Recall that \( V \) is a Hilbert space given by \( V = \{ u \in H^\alpha(\mathbb{R}^3) \} \). The dual space of \( V \) is denoted by \( V^\ast \). To give the existence of solutions, we also need the space \( H = \{ u \in L^2(\mathbb{R}^3) \} \).

By the standard Galerkin method and compactness argument, as shown in [24], we can prove that in the case of a bounded domain with Dirichlet boundary conditions, for \( P.a.e. \omega \in \Omega \) and for all \( v_\tau \in L^2(\mathbb{R}^3) \), equation (3.5) has a unique solution \( v(t, \tau, \omega, v_\tau) \in C([0, \infty), L^2(\mathbb{R}^3)) \cap L^2((0, T), H^\alpha(\mathbb{R}^3)) \) with \( v(t, \tau, \omega, v_\tau) = v_\tau \) for every \( T > \tau \). This is similar to [9, 36, 55]. Then, following the approach in [33], we can take the domain to be a sequence of balls with radii approaching \( \infty \) to deduce the existence of a weak solution of equation (3.5) on \( \mathbb{R}^3 \). Furthermore, we can get that \( v(t, \tau, \omega, v_\tau) \) is unique and continuous with respect to \( v_\tau \) in \( H^\alpha(\mathbb{R}^3) \) for all \( t \geq \tau \).

Now by the solution \( v \) of (3.5)-(3.6) and the transform (3.4), we get a solution \( u \) of the stochastic equation (1.1)-(1.2) which is given by

\[
u(t, \tau, \omega, u_\tau) = v + \delta z(\theta_t \omega)
\]

with \( u_\tau = v_\tau + \delta z(\theta_t \omega) \). We note that \( u(t, \tau, \omega, u_\tau) \) is both continuous in \( t \in [\tau, \infty) \) and in \( u_\tau \in H \). Moreover, \( u(t, \tau, \omega, u_\tau) : \Omega \to H \) is measurable. Then we can define a continuous cocycle in \( H \) associated with the solutions of problem (1.1)-(1.2). Let \( \Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times H \to H \) be a mapping given by, for every \( t \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega \) and \( u_\tau \in H \),

\[
\Phi(t, \tau, \omega, u_\tau) = u(t + \tau, \tau, \theta_{-\tau} \omega, u_\tau) = v(t + \tau, \tau, \theta_{-\tau} \omega, v_\tau) + \delta z(\theta_t \omega), \quad (3.7)
\]

where \( v_\tau = u_\tau - \delta z(\theta_t \omega) \). In later sections, we will prove the existence and upper semicontinuity of tempered random attractors for \( \Phi \) in \( H \).

Let \( B = \{ B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \) be a family of bounded nonempty subsets of \( H \). Such a family \( B \) is called tempered if for every \( c > 0, \tau \in \mathbb{R} \) and \( \omega \in \Omega \),

\[
\lim_{t \to -\infty} e^{ct} \| B(\tau + t, \theta_t \omega) \| = 0,
\]

where the norm \( \| B \| \) of set \( B \) in \( H \) is given by \( \| B \| = \sup_{u \in B} \| u \| \). From now on, we will use \( \mathcal{D} \) to denote the collection of all tempered families of bounded nonempty subsets of \( H \):

\[
\mathcal{D} = \{ B = \{ B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} : B \text{ is tempered in } H \}.
\]
When deriving uniform estimates, for simplicity, we assume that $\gamma > 0$, and also assume that for every $\tau \in \mathbb{R}$,

$$
\int_{-\infty}^{0} e^{\gamma s}\|g(s + \tau, \cdot)\|^2 ds < \infty. \quad (3.8)
$$

Sometimes, we also assume $g$ is tempered in the following sense: for every $c > 0$,

$$
\lim_{r \to \infty} e^{-cr} \int_{-\infty}^{0} e^{\gamma s}\|g(s - r, \cdot)\|^2 ds = 0. \quad (3.9)
$$

Note that these conditions do not require $g$ to be bounded in $H$ when $t \to \infty$. Through this paper, $c$ denotes a positive constant which may be different from the context.

4. Random attractors

In this section, we will derive uniform estimates on the solutions of non-autonomous stochastic fractional Ginzburg-Landau equation in $H$ and $V$. These estimates are necessary for proving the existence of random attractors. Then we prove the existence and uniqueness of pullback random attractors.

We first derive uniform estimates of solutions in $H$.

**Lemma 4.1.** Suppose (3.8) holds. Then for every $\delta_0 > 0$, $r \in \mathbb{R}$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $B = \{B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D$, there exists $T(\tau, \omega, B, r, \delta_0) > 0$ such that for all $t \geq T$ and $0 < \delta \leq \delta_0$, the solution $v$ of problem (3.5)-(3.6) satisfies

$$
\|v(r, \tau - t, \theta - t, v_{\tau - t})\|^2 + \int_{-\infty}^{r - \tau} e^{\gamma(s - \tau + \tau)}\|v(s + \tau, \tau - t, \theta - \tau, v_{\tau - t})\|^2_{H^\alpha(\mathbb{R}^3)} ds
$$

$$
+ \int_{-\tau}^{r - \tau} e^{\gamma(s - \tau + \tau)}\|u(s + \tau, \tau - t, \theta - \tau, v_{\tau - t})\|^4_{L^4(\mathbb{R}^3)} ds
$$

$$
\leq c + c\int_{-\infty}^{r - \tau} e^{\gamma(s + \tau)}(e^{\frac{1}{2}\gamma|s|})r(\omega) + \|g(s + \tau)\|^2 ds,
$$

where $v_{\tau - t} + \delta z(\theta - \tau, \omega) \in B(\tau - t, \theta - \tau, \omega)$.

**Proof.** Taking the inner product of (3.5) with $v$, and taking the real part, we have

$$
\frac{d}{dt}\|v\|^2 + C(\alpha)\|v\|^2_{H^\alpha} + 2\gamma\|v\|^2 = -2\delta Re((1 + i\lambda)(-\Delta)^{\alpha}z(\theta, \omega), v)
$$

$$
- 2Re(1 + i\mu)\int_{\mathbb{R}^3}|u|^2 \bar{v} dx + 2Re\int_{\mathbb{R}^3} g(t, x)\xi dx. \quad (4.1)
$$

We now estimate all terms in (4.1). For the last term, we have

$$
2Re\int_{\mathbb{R}^3} g(t, x)\bar{v} dx \leq \frac{\gamma}{2}\|v\|^2 + 2\gamma^{-1}\|g(t, x)\|^2. \quad (4.3)
$$

On the other hand, we find

$$
-2Re(\delta(1 + i\lambda)(-\Delta)^{\alpha}z(\theta, \omega), v)
$$
Replacing $\omega$

\[
\leq \frac{2\delta^2 (1 + \lambda^2)}{\gamma} \|(-\Delta)^{\alpha} z(\theta_t \omega)\|^2 + \frac{\gamma}{2} \|v\|^2 \tag{4.4}
\]

\[
- 2 \text{Re}(1 + i\mu) \int_{\mathbb{R}^3} |u|^2 \bar{u} \overline{\alpha} dx
\]

\[
= - 2 \text{Re}(1 + i\mu) \int_{\mathbb{R}^3} |u|^2 \bar{u} \overline{\alpha} dx + 2 \text{Re}(1 + i\mu) \int_{\mathbb{R}^3} |u|^2 \bar{u} \overline{\alpha} z(\theta_t \omega) dx
\]

\[
\leq - 2 \int_{\mathbb{R}^3} |u|^4 dx + 2(1 + i\mu) |\delta| \int_{\mathbb{R}^3} |u|^3 |z(\theta_t \omega)| dx
\]

\[
\leq - \|u\|^4_4 + c \|z(\theta_t \omega)\|^4_4. \tag{4.5}
\]

It follows from (4.1)-(4.5) that

\[
\frac{d}{dt} \|v\|^2 + \gamma \|v\|^2 + C(\alpha) \|v\|^2_{H^{2\alpha}(\mathbb{R}^3)} + \|u\|^4_{L^1(\mathbb{R}^3)} 
\leq c(\|z(\theta_t \omega)\|^4_4 + \|z(\theta_t \omega)\|^2) + 2\gamma^{-1} \|g(t, x)\|^2. \tag{4.6}
\]

Note that $z(\theta_t \omega) = h(x)g(\theta_t \omega)$, $h(x) \in H^{2\alpha} \cap W^{2\alpha, 4}$. Therefore, the first term on the right-hand side of (4.6) can be bounded by

\[
c(\|z(\theta_t \omega)\|^4_4 + \|z(\theta_t \omega)\|^2) = p_1(\theta_t \omega).
\]

By (3.3), we find that for $P - a.e. \omega \in \Omega$

\[
p_1(\theta_t \omega) \leq ce^{\frac{1}{2} |\tau-t\rangle r(\omega)}, \text{ for all } \tau \in \mathbb{R}. \tag{4.7}
\]

Multiplying (4.6) with $e^{\gamma t}$, then integrating the inequality on $(\tau - t, \sigma)$ with $\sigma > \tau - t$, we get

\[
\|v(\sigma, \tau - t, \omega, v_{\tau - t})\|^2 + C(\alpha) \int_{\tau - t}^{\sigma} e^{\gamma(s-\sigma)} \|v(s, \tau - t, \omega, v_{\tau - t})\|^2_{H^{2\alpha}} ds
\]

\[
+ \int_{\tau - t}^{\sigma} e^{\gamma(s-\sigma)} \|u(s, \tau - t, \omega, v_{\tau - t})\|^4_{L^4} ds
\]

\[
\leq e^{\gamma(\tau - t - \sigma)} \|v_{\tau - t}\|^2 + \int_{\tau - t}^{\sigma} e^{\gamma(s-\sigma)} p_1(\theta_{s - \tau} \omega) ds + 2\gamma^{-1} \int_{\tau - t}^{\sigma} e^{\gamma(s-\sigma)} \|g(s)\|^2 ds.
\]

Replacing $\omega$ by $\theta_{-\tau} \omega$ we have

\[
\|v(\sigma, \tau - t, \theta_{-\tau} \omega, v_{\tau - t})\|^2 + C(\alpha) \int_{\tau - t}^{\sigma} e^{\gamma(s-\sigma)} \|v(s, \tau - t, \theta_{-\tau} \omega, v_{\tau - t})\|^2_{H^{2\alpha}} ds
\]

\[
+ \int_{\tau - t}^{\sigma} e^{\gamma(s-\sigma)} \|u(s, \tau - t, \theta_{-\tau} \omega, v_{\tau - t})\|^4_{L^4} ds
\]

\[
\leq e^{\gamma(\tau - t - \sigma)} \|v_{\tau - t}\|^2 + \int_{\tau - t}^{\sigma} e^{\gamma(s-\sigma)} p_1(\theta_{s - \tau} \omega) ds
\]

\[
+ 2\gamma^{-1} \int_{\tau - t}^{\sigma} e^{\gamma(s-\sigma)} \|g(s)\|^2 ds.
\]

After change of variables, we obtain

\[
\|v(r, \tau - t, \theta_{-\tau} \omega, v_{\tau - t})\|^2 + C(\alpha) \int_{\tau - t}^{\tau} e^{\gamma(s-r)} \|v(s, \tau - t, \theta_{-\tau} \omega, v_{\tau - t})\|^2_{H^{2\alpha}} ds
\]
By (4.8)-(4.10) we obtain, for all $t$

\[
\leq e^{\gamma(t-t-\tau)}\|v_{t-\tau}\|^2 + \int_{-t}^{t-\tau} e^{\gamma(s-t+\tau)}p_1(\theta_s\omega)ds
\]

\[
+ 2\gamma^{-1}\int_{-t}^{t-\tau} e^{\gamma(s-t+\tau)}\|g(s+\tau)\|^2ds. \tag{4.8}
\]

Now we estimate the first term on the right-hand side of (4.8). Since $B = \{B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is tempered, $v_{t-\tau} \in B(\tau-t, \theta_{-\tau}\omega)$, there exists $T = T(\tau, \omega, B, r, \delta_0) > 0$ such that for all $t \geq T$, $0 < \delta < \delta_0,
\]

\[
e^{\gamma(t-t-\tau)}\|v_{t-\tau}\|^2 \leq 1. \tag{4.9}
\]

For the second and last term we have

\[
\int_{-t}^{t-\tau} e^{\gamma(s-t+\tau)}p_1(\theta_s\omega)ds \leq \int_{-\infty}^{t-\tau} e^{\gamma(s-t+\tau)}p_1(\theta_s\omega)ds,
\]

\[
\int_{-t}^{t-\tau} e^{\gamma(s-t+\tau)}\|g(s+\tau)\|^2ds \leq \int_{-\infty}^{t-\tau} e^{\gamma(s-t+\tau)}\|g(s+\tau)\|^2ds. \tag{4.10}
\]

By (4.8)-(4.10) we obtain, for all $t \geq T$,

\[
\|v_{t, t, \tau-\theta_{-\tau}\omega, v_{t-\tau}}\|^2+C(\alpha)\int_{-t}^{t-\tau} e^{\gamma(s-t+\tau)}\|v(s+\tau, \tau-t, \theta_{-\tau}\omega, v_{t-\tau})\|^2_Hds
\]

\[
+ \int_{-t}^{t-\tau} e^{\gamma(s-t+\tau)}\|u(s+\tau, \tau-t, \theta_{-\tau}\omega, v_{t-\tau})\|^2_{L^2}ds
\]

\[
\leq 1 + \int_{-\infty}^{t-\tau} e^{\gamma(s-t+\tau)}(ce^{\frac{1}{2}t}t^{1/2})y_r(\omega) + 2\gamma^{-1}\|g(s+\tau)\|^2)ds. \tag{4.11}
\]

From (4.11), the desired estimates follow immediately.

Based on Lemma 4.1, we claim the solution operator of problem (3.5)-(3.6) has a random pullback absorbing set in $H$ as stated below.

**Lemma 4.2.** Suppose (3.8) holds. Let $B_1 = \{B_1(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ be a random set given by

\[B_1(\tau, \omega) = \{v \in H : \|v\|^2 \leq R_1(\tau, \omega)\},\]

where $R_1(\tau, \omega)$ is defined by

\[
R_1(\tau, \omega) = c + c\int_{-\infty}^{0} (e^{\frac{1}{2}\gamma t}r(\omega) + e^{\gamma t}\|g(s+\tau)\|^2)ds. \tag{4.12}
\]

Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $B = \{B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(\tau, \omega, B, \delta) > 0$ such that the solution $v$ of (3.5)-(3.6) with $v_{t-\tau} + \delta z(\theta_{-\tau}\omega) \in B(\tau-t, \theta_{-\tau}\omega)\) satisfies, for all $t \geq T$,

\[
v(\tau-t, \theta_{-\tau}\omega, v_{t-\tau}) \in B_1(\tau, \omega). \tag{4.13}
\]

In addition, the random variable $R_1$ as in (4.12) is tempered, i.e., for any $c > 0$,

\[
\lim_{t \to \infty} e^{-ct}R_1(\tau-t, \theta_{-\tau}\omega) = 0. \tag{4.14}
\]
Lemma 4.3. Suppose we now prove (4.13). From (4.12) we have

\[ R_1(\tau - t, \omega) = c + c \int_{-\infty}^{0} (e^{\frac{r}{2}} r(\omega) + e^{\frac{r}{2}} g(s + \tau - t)) ds. \] (4.15)

Note that \( r(\omega) \) is a tempered function. By (3.9) we get

\[ \lim_{t \to \infty} e^{-ct} R_1(\tau - t, \omega) \]

\[ \leq \lim_{t \to \infty} e^{-ct} \int_{-\infty}^{0} (e^{\frac{r}{2}} r(\omega) + e^{\frac{r}{2}} g(s + \tau - t)) ds \]

\[ \leq \lim_{t \to \infty} e^{-ct} e^{-cr} \int_{-\infty}^{0} (e^{\frac{r}{2}} r(\omega) + e^{\frac{r}{2}} g(s - r)) ds = 0. \]

We now derive uniform estimates of solutions in \( V \).

Lemma 4.3. Suppose (3.8) holds. Then for every \( \delta_0 > 0, \sigma \in \mathbb{R}, \tau \in \mathbb{R}, \omega \in \Omega \) and \( B = \{ B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \) in \( B \), there exists \( T(\tau, \omega, B, \sigma, \delta_0) \) such that for all \( t \geq T \) and \( 0 < \delta \leq \delta_0 \), the solution \( v \) of problem (3.5)-(3.6) satisfies

\[ \| v(\sigma, \tau - t, \theta - \tau \omega, v_{\tau - t}) \|^2_{H^2(\mathbb{R}^3)} ds \]

\[ \leq c + c \int_{-\infty}^{\sigma - \tau} e^{(s - \sigma + \tau)(e^{\frac{r}{2}} r(\omega) + \| g(s + \tau) \|)} ds, \]

where \( v_{\tau - t} + \delta z(\theta - \tau \omega) \in B(\tau - t, \theta - \tau \omega) \).

Proof. Taking the inner product of (3.5) with \( (-\Delta)^{\alpha} v \), and taking the real part, we have

\[ \frac{d}{dt} \| (-\Delta)^{\frac{r}{2}} v \|^2 + 2 \| (-\Delta)^{\alpha} v \|^2 + 2 \gamma \| (-\Delta)^{\frac{r}{2}} v \|^2 \]

\[ = -2 \text{Re}(1 + i\mu)|u|^2 u, (-\Delta)^{\alpha} v + 2 \text{Re}(g(t, x), (-\Delta)^{\alpha} v) \]

\[ -2\text{Re}(\delta(1 + i\lambda)(-\Delta)^{\alpha} z(\theta \omega), (-\Delta)^{\alpha} v). \] (4.16)

We now estimate the terms on the right-hand side of (4.16). For the nonlinear term, from Taylor’s formula and Lemma 4.1, we have

\[ -2 \text{Re}(1 + i\mu)|u|^2 u, (-\Delta)^{\alpha} v \]

\[ = -C(\alpha) \text{Re}(1 + i\mu)(|u|^2 u, (-\Delta)^{\alpha} v) + 2 \text{Re}(1 + i\mu)(|u|^2 u, (-\Delta)^{\alpha} \delta z(\theta \omega)) \]

\[ \leq -C(\alpha) \text{Re}(1 + i\mu) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (|u(x)|^2 u(x) - |u(y)|^2 u(y))(u(x) - u(y)) \frac{dx dy}{|x - y|^{3 + 2\alpha}} \]

\[ + 2\delta |1 + i\mu| \int_{\mathbb{R}^3} |u|^2 |(-\Delta)^{\alpha} z(\theta \omega)| dx \]

\[ \leq c\| (-\Delta)^{\frac{r}{2}} u \|^2 + \| u \|^4 + c\| (-\Delta)^{\alpha} z(\theta \omega) \|^4. \] (4.17)

For the second term, we obtain

\[ 2 \text{Re}(g(t, x), (-\Delta)^{\alpha} v) \leq \frac{1}{2} \| (-\Delta)^{\alpha} v \|^2 + 2 \| g(t) \|^2. \] (4.18)
For the last term, we have
\[
-2\text{Re}(\delta(1+i\lambda)(-\Delta)^\alpha z(\theta_t \omega)), (-\Delta)^\alpha v)
\leq \frac{1}{2} \|(-\Delta)^\alpha v\|^2 + 2\delta^2 (1+\lambda^2) \|(-\Delta)^\alpha z(\theta_t \omega)\|^2.
\]
(4.19)

It follows from (4.16)-(4.19) that
\[
\frac{d}{dt} \|(-\Delta)^{\frac{3}{2}} v\|^2 + \|(-\Delta)^\alpha v\|^2 + \gamma \|(-\Delta)^{\frac{1}{2}} v\|^2
\leq c(\|(-\Delta)^{\frac{3}{2}} u\|^2 + \|u\|_4^4 + c(\|(-\Delta)^\alpha z(\theta_t \omega)\|^2 + \|(-\Delta)^\alpha z(\theta_t \omega)\|_4^4) + 2\|g(t)\|^2.
\]
(4.20)

Note that \(z(\theta_t \omega) = h(x)g(\theta_t \omega), h(x) \in H^{2\alpha} \cap W^{2\alpha,2}\). Therefore, the second term on the right-hand side of (4.20) can be bounded by
\[
c(\|z(\theta_t \omega)\|_4^4 + \|z(\theta_t \omega)\|_2^2) = p_2(\theta_t \omega).
\]

By (3.3), we find that for \(P - a.e. \omega \in \Omega\)
\[
p_2(\theta_t \omega) \leq ce^{\frac{t}{2}|\gamma|} r(\omega), \text{ for all } \tau \in \mathbb{R}.
\]
(4.21)

Given \(t \in \mathbb{R}^+, \tau \in \mathbb{R}\) and \(\omega \in \Omega\), let \(\sigma \in (\tau - 1, \tau)\) and \(\sigma \in (\tau - 2, \tau - 1)\). Multiplying (4.20) with \(e^{\gamma t}\), first integrating with respect to \(t\) on \((r, \sigma)\) we obtain
\[
\|(-\Delta)^{\frac{3}{2}} v(\sigma, \tau - t, \omega, \nu_{\tau - t})\|^2
\leq e^{\gamma (r-t)} \|(-\Delta)^{\frac{1}{2}} v(r, \tau - t, \omega, \nu_{\tau - t})\|^2dr
+ \int_r^\sigma e^{\gamma (\zeta - \sigma)} (c(\|(-\Delta)^{\frac{3}{2}} u(\zeta, \tau - t, \omega, \nu_{\tau - t})\|^2 + c\|u(\zeta, \tau - t, \omega, \nu_{\tau - t})\|_4^4)
+ ce^{\frac{t}{2}|\gamma|} r(\omega)) + 2\|g(\zeta + \tau)\|^2) d\zeta dr.
\]
(4.22)

Then integrating with respect to \(r\) on \((\tau - 2, \tau - 1)\), we get
\[
\|(-\Delta)^{\frac{3}{2}} v(\sigma, \tau - t, \omega, \nu_{\tau - t})\|^2
\leq \int_{\tau - 2}^{\tau - 1} e^{\gamma (r-t)} \|(-\Delta)^{\frac{1}{2}} v(r, \tau - t, \omega, \nu_{\tau - t})\|^2dr
+ \int_{\tau - 2}^{\tau - 1} \int_r^\sigma e^{\gamma (\zeta - \sigma)} (c(\|(-\Delta)^{\frac{3}{2}} u(\zeta, \tau - t, \omega, \nu_{\tau - t})\|^2 + c\|u(\zeta, \tau - t, \omega, \nu_{\tau - t})\|_4^4)
+ ce^{\frac{t}{2}|\gamma|} r(\omega)) + 2\|g(\zeta + \tau)\|^2) d\zeta dr.
\]
(4.23)

Replacing \(\omega\) by \(\theta_{-\tau} \omega\) we have
\[
\|(-\Delta)^{\frac{3}{2}} v(\sigma, \tau - t, \theta_{-\tau} \omega, \nu_{\tau - t})\|^2
\leq \int_{\tau - 2}^{\tau - 1} e^{\gamma (r-t)} \|(-\Delta)^{\frac{1}{2}} v(r, \tau - t, \theta_{-\tau} \omega, \nu_{\tau - t})\|^2dr
+ \int_{\tau - 2}^{\tau - 1} \int_r^\sigma e^{\gamma (\zeta - \sigma)} (c(\|(-\Delta)^{\frac{3}{2}} u(\zeta, \tau - t, \theta_{-\tau} \omega, \nu_{\tau - t})\|^2 + c\|u(\zeta)\|_4^4)
+ ce^{\frac{t}{2}|\gamma|} r(\omega)) + 2\|g(\zeta + \tau)\|^2) d\zeta dr.
\]
We now prove (4.27). From (4.26) we have

\begin{align*}
& \leq \int_{-2}^{-1} e^{\gamma (r-\sigma + \tau)} \| (-\Delta)^{\frac{\sigma}{2}} v(r+\tau, \tau-t, \theta_{-\tau} \omega, v_{\tau-t}) \|^2 dr \\
& + \int_{-2}^{-1} \int_{-\tau}^{0} e^{\gamma (r-\sigma + \tau)} (c \| (-\Delta)^{\frac{\sigma}{2}} u(s+\tau, \tau-t, \theta_{-\tau} \omega, u_{\tau-t}) \|^2 + c \| u(s+\tau) \|^4) \\
& + ce^{\frac{1}{2} |\gamma r(\omega) + 2 \| g(s+\tau) \|^2} d\zeta dr \\
& \leq \int_{-2}^{-1} e^{\gamma (r-\sigma + \tau)} \| (-\Delta)^{\frac{\sigma}{2}} v(r+\tau, \tau-t, \theta_{-\tau} \omega, v_{\tau-t}) \|^2 dr \\
& + \int_{-2}^{-1} e^{\gamma (s-\sigma + \tau)} (c e^{\frac{1}{2} |s| \gamma r(\omega) + 2 \| g(s+\tau) \|^2} ds.
\end{align*}

(4.24)

Let $T$ be the constant in Lemma 4.1, and $T_0 = \max\{2, T\}$. Note that $u = v + \delta z(\theta_{-1} \omega)$ and (3.3), from $\sigma \in (\tau - 1, \tau)$ and Lemma 4.1, then for all $t \geq T_0$, we obtain

\begin{align*}
\| (-\Delta)^{\frac{\sigma}{2}} v(\sigma, \tau-t, \theta_{-\tau} \omega, v_{\tau-t}) \|^2 & \leq (1+c) \int_{-2}^{-1} e^{\gamma (r-\sigma + \tau)} \| (-\Delta)^{\frac{\sigma}{2}} v(r+\tau, \tau-t, \theta_{-\tau} \omega, v_{\tau-t}) \|^2 dr \\
& + \int_{-2}^{-1} e^{\gamma (s-\sigma + \tau)} (c e^{\frac{1}{2} |s| \gamma r(\omega) + 2 \| g(s+\tau) \|^2} ds.
\end{align*}

(4.25)

Based on Lemma 4.3, we claim the solution operator of problem (3.5)-(3.6) has a random pullback absorbing set in $V$ as stated below.

**Lemma 4.4.** Suppose (3.8) holds. Let $B_2 = \{ B_2(\tau, \omega) : \tau \in \mathbf{R}, \omega \in \Omega \}$ be a random set given by

\[ B_2(\tau, \omega) = \{ v \in H : \| v \|^2_{H^\sigma} \leq R_2(\tau, \omega) \}, \]

where $R_2(\tau, \omega)$ is defined by

\[ R_2(\tau, \omega) = c + c \int_{-\infty}^{0} (e^{\frac{1}{2} \gamma s r(\omega)} + e^{\gamma s \| g(s+\tau) \|^2} ds. \]

(4.26)

Then for every $\tau \in \mathbf{R}$, $\omega \in \Omega$ and $B = \{ B(\tau, \omega) : \tau \in \mathbf{R}, \omega \in \Omega \} \in \mathcal{D}$, there exists $T = T(\tau, \omega, B, \delta) > 0$ such that the solution $v$ of (3.5)-(3.6) with $v_{\tau-t} + \delta z(\theta_{-1} \omega) \in B(\tau-t, \theta_{-1} \omega)$ satisfies, for all $t \geq T$,

\[ v(\tau, \tau-t, \theta_{-\tau} \omega, v_{\tau-t}) \in B_1(\tau, \omega). \]

(4.27)

In addition, the random variable $R_2$ as in (4.26) is tempered, i.e., for any $c > 0$,

\[ \lim_{t \to -\infty} e^{-ct} R_2(\tau-t, \theta_{-1} \omega) = 0. \]

(4.28)

**Proof.** As a special case of Lemma 4.3 with $\sigma = \tau$, we obtain (4.27) immediately. We now prove (4.27). From (4.26) we have

\[ R_2(\tau-t, \omega) = c + c \int_{-\infty}^{0} (e^{\frac{1}{2} \gamma s r(\omega)} + e^{\gamma s \| g(s+\tau-t) \|^2} ds. \]

(4.29)
Note that \( r(\omega) \) is a tempered function. By (3.9) we get

\[
\limsup_{t \to \infty} e^{-ct} R_2(\tau - t, \omega) \\
\leq \limsup_{t \to \infty} e^{-ct} \int_{-\infty}^{0} (e^{\gamma s} r(\omega) + e^{\gamma s} (\|g(s + \tau - t)\|^2)) ds \\
\leq \limsup_{r \to \infty} e^{-cr} e^{-ct} \int_{-\infty}^{0} (e^{\gamma s} r(\omega) + e^{\gamma s} (\|g(s - r)\|^2)) ds = 0.
\]

In this following, we will derive the uniform priori estimates on the tail of solutions. Firstly, we introduce a smooth function \( \rho(s) \) defined for \( 0 \leq s < \infty \) such that \( 0 \leq \rho(s) \leq 1 \) for \( s \geq 0 \) and \( \rho(s) = \begin{cases} 
0, & \text{if } 0 \leq s < \frac{1}{2}, \\
1, & \text{if } s \geq 1.
\end{cases} \) \( (4.30) \)

Note that there exists a positive constant \( c \) such that \( |\rho'(s)| \leq c \) for all \( s \geq 0 \). From the definition of the cut-off function \( \rho(s) \) and fractional Laplace operator \((-\Delta)^\alpha\), we can easily get the following properties of \( \rho(s) \).

**Lemma 4.5** \((15)\). Let \( \rho(s) \) be the smooth function defined by \( (4.30) \) and \( \alpha \in (0, 1) \). For every \( x, y \in \mathbb{R}^3 \) and \( l \in \mathbb{N} \), then we have

\[
\int_{\mathbb{R}^3} \frac{|\rho(|x|) - \rho(|y|)|^2}{|x - y|^{3 + 2\alpha}} dx \leq \frac{\nu_1}{l^{2\alpha}}, \quad (4.31)
\]

where \( \nu_1 \) is a positive constant, \( \rho_l(.) = \rho(\frac{|.|}{l}) \).

We now give uniform estimates on the tails of solutions in \( L^2(\mathbb{R}^3) \).

**Lemma 4.6.** Suppose (3.8) holds. Then for every \( \varepsilon > 0, \tau \in \mathbb{R}, \omega \in \Omega \) and \( B = \{B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \subset \mathcal{D} \), there exists \( T(\tau, \omega, B, \delta, \varepsilon) \geq 0 \) and \( L = L(\tau, \omega, \varepsilon) \geq 1 \) such that for all \( t \geq T \) and \( l \geq L \), the solution \( v \) of problem (3.5)-(3.6) satisfies

\[
\int_{|x| \geq l} |v(\tau, \tau - t, \theta_{-\tau} \omega, v_{\tau - t})|^2 dx \leq \varepsilon,
\]

and

\[
\int_{-t}^{0} e^{\gamma s} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\rho(|x|)(v(\zeta + \tau, \tau - t, \theta_{-\tau} \omega, v_{\tau - t}))^2}{|x - y|^{3 + 2\alpha}} dxdy d\zeta \leq \varepsilon,
\]

where \( \rho \) is defined by \( (4.30) \).

**Proof.** Taking the inner product of (3.5) with \( \rho(|x|) v \) and taking the real part, we have

\[
\frac{d}{dt} \int_{\mathbb{R}^3} \rho(|x|)|v|^2 dx + 2\gamma \int_{\mathbb{R}^3} \rho(|x|)|v|^2 dx \\
= - 2 \Re(1 + i\lambda) \int_{\mathbb{R}^3} (-\Delta)^\alpha v \rho(|x|) \bar{v} dx
\]
Now we estimate each term on the right side of (4.32). For the first term, from (4.31), we get

\[-2\delta \text{Re}(1+i\lambda) \int_{\mathbb{R}^3} (-\Delta)^\alpha z(\theta_1\omega)\rho\left(\frac{|x|}{l}\right)\bar{v}dx\]

\[= -C(\alpha)\text{Re}(1+i\lambda) \int_{\mathbb{R}^3} \frac{(\rho\left|\frac{|x|}{l}\right|v(x) - \rho\left|\frac{|y|}{l}\right|v(y))(v(x) - v(y))}{|x-y|^{3+2\alpha}} \, dxdy\]

\[\leq -C(\alpha) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(\rho\left|\frac{|x|}{l}\right|v(x) - v(y))^2}{|x-y|^{3+2\alpha}} \, dxdy\]

\[+ \sqrt{1 + \lambda^2 C(\alpha)} \|v\|_H^2 \left(\int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} \frac{|(\rho\left|\frac{|x|}{l}\right|) - \rho\left|\frac{|y|}{l}\right||v(x) - v(y)|}{|x-y|^{3+2\alpha}} \, dx \right)^2 dy \right)^{\frac{1}{2}}\]

\[\leq -C(\alpha) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(\rho\left|\frac{|x|}{l}\right|v(x) - v(y))^2}{|x-y|^{3+2\alpha}} \, dxdy + \sqrt{1 + \lambda^2 C(\alpha)} d^{-\alpha} \|v\|_H^2 \|v\|_{H^{-\alpha}(\mathbb{R}^3)}.\]

(4.33)

For the second term, we have

\[-2\delta \text{Re}(1+i\lambda) \int_{\mathbb{R}^3} (-\Delta)^\alpha z(\theta_1\omega)\rho\left(\frac{|x|}{l}\right)\bar{v}dx\]

\[\leq \frac{2\delta^2 (1 + \lambda^2)}{\gamma} \int_{\mathbb{R}^3} \rho\left(\frac{|x|}{l}\right)|(-\Delta)^\alpha z(\theta_1\omega)|^2 dx + \gamma \int_{\mathbb{R}^3} \rho\left(\frac{|x|}{l}\right)|v|^2 dx.\]

(4.34)

For the last term, we get

\[2\text{Re} \int_{\mathbb{R}^3} g(t, x)\rho\left(\frac{|x|}{l}\right)\bar{v}dx \leq \frac{\gamma}{2} \int_{\mathbb{R}^3} \rho\left(\frac{|x|}{l}\right)|v|^2 dx + \frac{\gamma}{2} \int_{\mathbb{R}^3} \rho\left(\frac{|x|}{l}\right)|g(t, x)|^2 dx.\]

(4.35)

For the nonlinear term, we find

\[2\text{Re} \int_{\mathbb{R}^3} f(t, x, v + \delta z(\theta_1\omega))\rho\left(\frac{|x|}{l}\right)\bar{v}dx\]
\begin{align*}
&= 2\Re \int_{\mathbb{R}^3} f(t, x, u) \rho \left(\frac{|x|}{k}\right) \tilde{u} \, dx - 2\delta \Re \int_{\mathbb{R}^3} f(t, x, u) \rho \left(\frac{|x|}{l}\right) z(\theta t, \omega) \, dx \\
&\leq -2 \int_{\mathbb{R}^3} \rho \left(\frac{|x|}{k}\right) |u|^4 \, dx + 2\delta \int_{\mathbb{R}^3} \rho \left(\frac{|x|}{l}\right) |f(t, x, u)| |z(\theta t, \omega)| \, dx \\
&\leq -2 \int_{\mathbb{R}^3} \rho \left(\frac{|x|}{l}\right) |u|^4 \, dx + 2\delta |1 + i\mu| \int_{\mathbb{R}^3} \rho \left(\frac{|x|}{l}\right) |u|^3 |z(\theta t, \omega)| \, dx \\
&\leq - \int_{\mathbb{R}^3} \rho \left(\frac{|x|}{l}\right) |u|^4 \, dx + c \int_{\mathbb{R}^3} \rho \left(\frac{|x|}{l}\right) |z(\theta t, \omega)|^4 \, dx. \quad (4.36)
\end{align*}

It follows from (4.32)-(4.36) that for all \(t \geq L_1(\omega) \geq 1\)

\begin{align*}
&\frac{d}{dt} \int_{\mathbb{R}^3} \rho \left(\frac{|x|}{l}\right) |v|^2 \, dx + \gamma \int_{\mathbb{R}^3} \rho \left(\frac{|x|}{l}\right) |v|^2 \, dx + C(\alpha) \int_{\mathbb{R}^3} \rho \left(\frac{|x|}{l}\right) |v(x) - v(y)|^2 \, dx \\
&\leq ct^{-\alpha} \|v\|_{H^s} + \int_{\mathbb{R}^3} \rho \left(\frac{|x|}{l}\right) (2\delta |z(\theta t, \omega)|^4 + \frac{2}{\gamma} (|(-\Delta)^{\alpha} z(\theta t, \omega)|^2) + \frac{2}{\gamma} \int_{\mathbb{R}^3} \rho \left(\frac{|x|}{l}\right) |g(t)|^2 \, dx \\
&\leq c \|v\|_{H^s} + c \int_{|x| \geq \frac{1}{2l}} (|z(\theta t, \omega)|^4 + |(-\Delta)^{\alpha} z(\theta t, \omega)|^2) \, dx + c \int_{|x| \geq \frac{1}{2l}} |g(t, x)|^2 \, dx. \quad (4.37)
\end{align*}

Given \(t \in \mathbb{R}^+, \tau \in \mathbb{R}\) and \(\omega \in \Omega\), multiplying (4) with \(e^{\gamma t}\), then integrating the result on \((\tau - t, \tau)\), we get for all \(l \geq L_1(\omega)\)

\begin{align*}
&\int_{\mathbb{R}^3} \rho \left(\frac{|x|}{l}\right) |v(\tau, \tau - t, \omega, v_{\tau - t})|^2 \, dx \\
&+ C(\alpha) \int_{\tau - t}^{\tau} e^{\gamma(t-\tau)} \int_{\mathbb{R}^3} \rho \left(\frac{|x|}{l}\right) (|v(\zeta, \tau - t, \omega, v_{\tau - t})| - v(y))^2 \, dx \, dy \, d\zeta \\
&\leq e^{-\gamma t} \int_{\mathbb{R}^3} \rho \left(\frac{|x|}{l}\right) |v_{\tau - t}|^2 \, dx + c \int_{\tau - t}^{\tau} e^{\gamma(s-\tau)} \|v(s, \tau - t, \omega, v_{\tau - t})\|_{H^s}^2 \, ds \\
&+ c \int_{\tau - t}^{\tau} \int_{|x| \geq \frac{1}{2l}} e^{\gamma(s-\tau)} |g(s, x)|^2 \, dx \\
&+ c \int_{\tau - t}^{\tau} \int_{|x| \geq \frac{1}{2l}} e^{\gamma(s-\tau)} (|z(\theta t, \omega)|^4 + |(-\Delta)^{\alpha} z(\theta t, \omega)|^2) \, dx.
\end{align*}

Then replacing \(\omega\) by \(\theta_{\tau - t} \omega\) we have for all \(l \geq L_1(\omega)\)

\begin{align*}
&\int_{\mathbb{R}^3} \rho \left(\frac{|x|}{l}\right) |v(\tau, \tau - t, \theta_{\tau - t} \omega, v_{\tau - t})|^2 \, dx \\
&+ C(\alpha) \int_{\tau - t}^{\tau} e^{\gamma(t-\tau)} \int_{\mathbb{R}^3} \rho \left(\frac{|x|}{l}\right) (|v(\zeta, \tau - t, \theta_{\tau - t} \omega, v_{\tau - t})| - v(y))^2 \, dx \, dy \, d\zeta \\
&\leq e^{-\gamma t} \int_{\mathbb{R}^3} \rho \left(\frac{|x|}{l}\right) |v_{\tau - t}|^2 \, dx + c \int_{\tau - t}^{\tau} e^{\gamma(s-\tau)} \|v(s, \tau - t, \theta_{\tau - t} \omega, v_{\tau - t})\|_{H^s}^2 \, ds \\
&+ c \int_{\tau - t}^{\tau} \int_{|x| \geq \frac{1}{2l}} e^{\gamma(s-\tau)} |g(s, x)|^2 \, dx \\
&+ c \int_{\tau - t}^{\tau} \int_{|x| \geq \frac{1}{2l}} e^{\gamma(s-\tau)} (|z(\theta t, \omega)|^4 + |(-\Delta)^{\alpha} z(\theta t, \omega)|^2) \, dx.
After change of variables, for all \( l \geq L_1(\omega) \geq 1 \) we obtain
\[
\int_{\mathbb{R}^3} \rho\left(\frac{|x|}{l}\right) |v(\tau, \tau - t, \theta_{-\tau \omega}, v_{\tau - t})|^2 dx \leq e^{-\gamma t} \int_{\mathbb{R}^3} \rho\left(\frac{|x|}{l}\right) |v_{\tau - t}|^2 dx + c \int_{t}^{0} e^{\gamma(s-t)} \|v(s + \tau, \tau - t, \theta_{-\tau \omega}, v_{\tau - t})\|_{H^s}^2 ds 
+ c \int_{t}^{0} \int_{|x| \geq \frac{1}{l}} e^{\gamma(s-t)} |g(s + \tau, x)|^2 dx ds
+ c \int_{t}^{0} \int_{|x| \geq \frac{1}{l}} e^{\gamma(s-t)} (|z(\theta_{s} \omega)|^4 + |(-\Delta)^\alpha z(\theta_{s} \omega)|^2) dx ds.
\] (4.39)

Now we estimate the first term on the right-hand side of (4.38). Since \( B = \{ B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \) is tempered, \( v_{\tau - t} \in B(\tau - t, \theta_{-\tau \omega}) \), there exists \( T_1 = T_1(\tau, \omega, B, \varepsilon) > 0 \) such that for all \( t \geq T_1 \),
\[
e^{-\gamma t} \int_{\mathbb{R}^3} \rho\left(\frac{|x|}{l}\right) |v_{\tau - t}|^2 dx \leq \varepsilon.
\] (4.40)

For the second term, from Lemma 4.1 with \( \sigma = \tau \), there exists \( T_2 = T_2(\tau, \omega, B, \varepsilon) > 0 \) such that for all \( t \geq T_2 \), we have
\[
c \int_{t}^{0} e^{\gamma(s-t)} \|v(s + \tau, \tau - t, \theta_{-\tau \omega}, v_{\tau - t})\|_{H^s}^2 ds \leq \varepsilon C(\tau, \omega).
\] (4.41)

For the third term, from (3.8)-(3.9) we find
\[
c \int_{-\infty}^{0} \int_{\mathbb{R}^3} e^{\gamma s} |g(s + \tau, x)|^2 dx ds < \infty.
\] (4.42)

Hence, there exists \( L_2 = L_2(\tau, \omega, \varepsilon) \geq L_1 \) such that for all \( l \geq L_2 \),
\[
c \int_{-\infty}^{0} \int_{|x| \geq \frac{1}{l}} e^{\gamma s} |g(s + \tau, x)|^2 dx ds < \varepsilon.
\] (4.43)

For the last term, from (3.3) we have
\[
c \int_{-\infty}^{0} \int_{\mathbb{R}^3} e^{\gamma s} (|z(\theta_{s} \omega)|^4 + |(-\Delta)^\alpha z(\theta_{s} \omega)|^2) dx ds < \infty.
\] (4.44)

Hence, there exists \( L_3 = L_3(\tau, \omega, \varepsilon) \geq L_2 \) such that for all \( l \geq L_3 \),
\[
c \int_{-\infty}^{0} \int_{|x| \geq \frac{1}{l}} e^{\gamma s} (|z(\theta_{s} \omega)|^4 + |(-\Delta)^\alpha z(\theta_{s} \omega)|^2) dx ds < \varepsilon.
\] (4.45)

By (4.38)-(4.45) we obtain, for all \( l \geq L_3 \) and \( t \geq T_2 \),
\[
\int_{\mathbb{R}^3} \rho\left(\frac{|x|}{l}\right) |v(\tau, \tau - t, \theta_{-\tau \omega}, v_{\tau - t})|^2 dx
\]
\[
+ C(\alpha) \int_{-t}^{0} e^c \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \beta(\frac{|x|}{r})(v(x) - v(y))^2 \frac{dxdydz}{|x-y|^{3+2\alpha}} \leq \varepsilon(3 + C(\tau, \omega)). \tag{4.46}
\]

From (4.46), the desired estimates follow immediately.

In this following, we give the existence of tempered pullback absorbing set in \(H\), and the asymptotic compactness of (1.1)-(1.2) in \(H\).

**Lemma 4.7.** Suppose (3.8) holds. Given \(\delta > 0, \tau \in \mathbb{R}\) and \(\omega \in \Omega\), let

\[
K_\delta(\tau, \omega) = \{u \in H : \|u\|^2 \leq c(\tau, \omega) + R_1(\tau, \omega)\},
\]
where \(R_1(\tau, \omega)\) is the same number as in (4.12). Then \(K_1\) is a closed measurable tempered pullback absorbing set of cocycle \(\Phi\) in \(H\).

**Proof.** We first prove that \(K_\delta\) absorbs every member \(B\) of \(D\). By (3.7) we have

\[
u(\tau, \tau - t, \theta - t, \omega, u_{\tau-t}) = v(\tau, \tau - t, \theta - t, \omega, u_{\tau-t}) + \delta \tau \theta - t, \omega).
\tag{4.47}
\]

If \(u_{\tau-t} \in B(\tau - t, \theta - t, \omega)\), then by (4.47) we get \(v_{\tau-t} + \delta \tau \theta - t, \omega) \in D(\tau - t, \theta - t, \omega)\) which together with Lemma 4.2 implies that there exists \(T = T(\tau, \omega, B, \delta) > 0\) such that for all \(t \geq T\),

\[
v(\tau, \tau - t, \theta - t, \omega, u_{\tau-t}) \in B_1(\tau, \omega),
\tag{4.48}
\]

where \(B_1(\tau, \omega)\) is the same as in (4.13). It follows from (4.47)-(4.48) and (4.13)-(4.14) that for all \(t \geq T\),

\[
\|u(\tau, \tau - t, \theta - t, \omega, u_{\tau-t})\|^2 \leq c(\tau, \omega) + R_1(\tau, \omega)).
\tag{4.49}
\]

On the other hand, by (3.7) we have

\[
\Phi(t, \tau - t, \theta - t, \omega, u_{\tau-t}) = u(\tau, \tau - t, \theta - t, \omega, u_{\tau-t}),
\tag{4.50}
\]

which along with (4.50) shows that \(\Phi(t, \tau - t, \theta - t, \omega, u_{\tau-t}) \in K_\delta\) for all \(t \geq T\), and hence \(K_\delta\) absorbs all elements of \(D\). We now prove \(K_\delta\) is tempered, i.e., \(K_\delta \in D\).

Note that \(r(\omega)\) is tempered, by (4.47) we find that for every \(c > 0\)

\[
\lim_{t \to \infty} e^{-ct}\|K_\delta(\tau - t, \omega, u_{\tau-t})\| = c \lim_{t \to \infty} e^{-ct}(r(\tau) + R_1(\tau, \omega))^\frac{1}{2} = 0,
\]

which implies that \(K_\delta \in D\). Note that \(R_1(\tau, \omega)\) is measurable in \(\omega \in \Omega\) and so in \(K_\delta(\tau, \omega)\), which completes the proof.

Next we first give the \(D\)-pullback asymptotic compactness of the solutions of problem (3.5)-(3.6) in \(L^2(\mathbb{R}^3)\).

**Lemma 4.8.** Suppose (3.8) holds. Then for every \(\tau \in \mathbb{R}, \omega \in \Omega\) and \(B = \{B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D\), the sequence \(\{v(\tau, \tau - t_n, \theta - t_n, \omega, v_{t_n, \omega})\}_{n=1}^\infty\) of solutions of problem (3.5)-(3.6) has a convergent subsequence in \(L^2(\mathbb{R}^3)\) provided \(t_n \to \infty\) and \(v_{t_n, \omega} \in B(\tau - t_n, \theta - t_n, \omega)\).

**Proof.** Let \(t_n \to \infty\), \(B = \{B(\tau, \omega)\} \in D\) and \(v_{t_n, \omega} \in B(\tau - t_n, \theta - t_n, \omega)\). Applying Lemma 4.1, for all \(\omega \in \Omega\), we have \(\{v(\tau, \tau - t_n, \theta - t_n, \omega, v_{t_n, \omega})\}_{n=1}^\infty\) is bounded in \(L^2(\mathbb{R}^3)\). Therefore, there exists \(\eta \in L^2(\mathbb{R}^3)\) and a subsequence, still denoted by \(\{v(\tau, \tau - t_n, \theta - t_n, \omega, v_{t_n, \omega})\}_{n=1}^\infty\), such that

\[
v(\tau, \tau - t_n, \theta - t_n, \omega, v_{t_n, \omega}) \to \eta \text{ weakly in } L^2(\mathbb{R}^3), \text{ for some } \eta = \eta(\omega) \in L^2(\mathbb{R}^3).
\]
By Lemma 4.6, there is $T^* = T_B^*(\omega)$ and $t^* = l^*(\omega, \varepsilon)$ such that for all $t \geq T^*$,

$$
\|v(\tau, \tau - t, \theta_{-\tau} \omega, v_0)\|_{L^2(\{x|\geq t^*\})} \leq \varepsilon. \tag{4.51}
$$

Since $t_n \to \infty$, there exists $N_1 = N_1(B, \omega, \varepsilon)$ such that $t_n \geq T_B^*$ for all $n \geq N_1$. By (4.51), for all $n \geq N_1$, one has

$$
\|v(\tau, \tau - t_n, \theta_{-\tau} \omega, v_{0,n})\|_{L^2(\{x|\geq t^*\})} \leq \varepsilon.
$$

By Lemma 4.2 and 4.4, there exists $T_{1_B} = T_{1_B}(\omega)$ such that for all $t \geq T_{1_B}$,

$$
\|v(\tau, \tau - t, \theta_{-\tau} \omega, v_0)\|_{H^\alpha(R^3)}^2 \leq c(R_2(\tau, \omega) + r(\omega)). \tag{4.52}
$$

Let $N_2 = N_2(B, \omega, \varepsilon)$ be large enough such that $t_n \geq T_{1_B}$ and $n \geq N_2$. It follows (4.52) that, for all $n \geq N_2$,

$$
\|v(\tau, \tau - t_n, \theta_{-\tau} \omega, v_{0,n})\|_{H^\alpha(R^3)}^2 \leq c(R_2(\tau, \omega) + r(\omega)), \tag{4.53}
$$

and so a further subsequence converges weakly in $H^\alpha(R^3)$, again to $\eta$. Thus, $\eta \in H^\alpha(R^3)$. Let $B_{r^*} = \{X \in R^3 : |x| \leq r^*\}$. By the compactness of the embedding $H^\alpha(B_{r^*}) \hookrightarrow L^2(B_{r^*})$, together with (4.53), we obtain, up to a subsequence depending on $l^*$,

$$
v(\tau, \tau - t_n, \theta_{-\tau} \omega, v_{0,n}) \to \eta \text{ strongly in } L^2(B_{r^*}),
$$

which implies that for given $\varepsilon > 0$, there exists $N_3 = N_3(B, \omega, \varepsilon)$ such that for all $n \geq N_3$,

$$
\|v(\tau, \tau - t_n, \theta_{-\tau} \omega, v_{0,n}) - \eta\|_{L^2(B_{r^*})} \leq \varepsilon. \tag{4.54}
$$

Since $\eta \in L^2(R^3)$, there exists $l^{**} = l^{**}(\omega) > 0$ such that

$$
\int_{|x| \geq t^{**}} |\eta(x)|^2 dx \leq \varepsilon. \tag{4.55}
$$

Let $l' = \max\{l^*, l^{**}\}$ and $N' = \max\{N_1, N_3\}$. Then, from (4.53)-(4.55), we obtain that for all $n \geq N'$,

$$
\|v(\tau, \tau - t_n, \theta_{-\tau} \omega, v_{0,n}) - \eta\|_{L^2(R^3)}^2 \leq \|v(\tau, \tau - t_n, \theta_{-\tau} \omega, v_{0,n}) - \eta\|_{L^2(B_{r^*})}^2 + \|v(\tau, \tau - t_n, \theta_{-\tau} \omega, v_{0,n}) - \eta\|_{L^2(B_{r^*})}^2
\leq 5\varepsilon,
$$

which implies $v(\tau, \tau - t_n, \theta_{-\tau} \omega, v_{0,n}) \to \eta$ strongly in $L^2(R^3)$. This completes the proof.

From Lemma 4.8, we immediately get the $D$-pullback asymptotic compactness of the solutions of problem (1.1)-(1.2) in $L^2(R^n)$.

**Lemma 4.9.** Suppose (3.8) holds, for every $\tau \in R, \omega \in \Omega$ and $B = \{B(\tau, \omega) : \tau \in R, \omega \in \Omega\}$, the sequence $\{\Phi(t_n, \tau - t_n, \theta_{-\tau} \omega, u_{0,n})\}_{n=1}^\infty$ of solutions of problem (1.1)-(1.2) has a convergent subsequence in $L^2(R^n)$ provided $t_n \to \infty$, and $u_{0,n} \in B(\tau - t_n, \theta_{-t_n} \omega)$. 
Proof. By (3.4) and (3.7) we obtain
\[
\Phi(t_n, \tau - t_n, \theta_{-t_n} \omega, u_{0,n}) = u(\tau, \tau - t_n, \theta_{-\tau} \omega, u_{\tau-t_n})
\]
\[
= v(\tau, \tau - t_n, \theta_{-\tau} \omega, v_{\tau-t_n}) + \delta z(\omega)
\]
with \(u_{\tau-t_n} = v_{\tau-t_n} + \delta z(\theta_{-t_n} \omega)\)

which along with Lemma 4.8 implies Lemma 4.9 directly.

Now we give the existence of tempered pullback random attractors of \(\Phi\) in \(H\).

Theorem 4.10. Suppose (3.8) and \(\alpha \in \left(\frac{3}{4}, 1\right)\) hold. Then the cocycle \(\Phi\) of problem (1.1)-(1.2) has a unique \(D\)-pullback random attractor \(A_\delta = \{A_\delta(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}\) in \(H\).

Proof. From [52, 54], based on Lemma 4.7, Lemma 4.9 and Theorem 2.4, the existence and uniqueness of the \(D\)-pullback attractor \(A_\delta\) follows immediately.

5. Upper semicontinuity of random attractors

In this section, we discuss the limiting behavior of random pullback attractors \(A_\delta\) of fractional stochastic Ginzburg-Landau equation (1.1) as the intensity of noise \(\delta \to 0\). Throughout this section, we assume \(\delta \in [0, 1]\), and write the cocycle of problem (1.1)-(1.2) as \(\Phi_\delta\) to indicate its dependence on \(\delta\). Then has a tempered pullback attractor \(A_\delta\) by Theorem 4.9, and has a tempered pullback absorbing set \(K_\delta\) by Lemma 4.7. Given \(\tau \in \mathbb{R}, \omega \in \Omega\), let
\[
\tilde{R}(\tau, \omega) = c + c \int_0^0 (e^{\frac{1}{2} \gamma s \tau} r(\omega) + e^{\gamma s} (\|g(s + \tau)\|^2)) ds,
\]
and
\[
K(\tau, \omega) = \{u \in H : \|u\|^2 \leq c(r(\omega) + \tilde{R}(\tau, \omega))\}.
\]

Thus we have
\[
\|K(\tau - t, \theta_{-t} \omega)\|^2 \leq c(r(\omega) + \tilde{R}(\tau - t, \omega)),
\]
and
\[
\|\Phi(\tau - t, \theta_{-t} \omega)\|^2 \leq c(r(\omega) + \tilde{R}(\tau - t, \omega)).
\]

By (3.7) and \(r(\omega)\) is tempered, for every \(\varepsilon > 0, \tau \in \mathbb{R}\) and \(\omega \in \Omega\), there exists \(T = T(\tau, \omega, \varepsilon) > 0\) such that for all \(t > T\),
\[
e^{-ct} \|K(\tau - t, \theta_{-t} \omega)\|^2 \leq \varepsilon.
\]

From (5.4), we now prove the following uniform estimates on the tails of functions in random attractors.

Lemma 5.1. Suppose (3.8) holds, then for every \(\varepsilon > 0, \tau \in \mathbb{R}\) and \(\omega \in \Omega\), there exists \(L = L(\tau, \omega, \varepsilon) \geq 1\) such that for all \(l \geq L\),
\[
\int_{|x| \geq l} |\xi(x)|^2 dx \leq \varepsilon \text{ for all } \xi \in \bigcup_{0 < \delta \leq 1} A_\delta(\tau, \omega).
\]
Proof. From the proof of Lemma 4.1 and 4.3, by (5.4), we can verify that for every 
\( \varepsilon > 0 \), \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \), there exists \( T = T(\tau, \omega, \varepsilon) > 0 \) and \( L = L(\tau, \omega, \varepsilon) \geq 1 \) such that for all \( t > T, l \geq L \) and for all \( \delta \in (0, 1] \), the solution \( u_\delta \) of (1.1)-(1.2) satisfies

\[
\int_{|x| \geq l} |u_\delta(\tau, \tau - t, \theta_{-t}\omega, u_\delta, \tau - t)|^2 dx \leq \varepsilon, \tag{5.5}
\]

where \( u_{\delta, \tau - t} \in K(\tau - t, \theta_{-t}\omega) \) with \( K \) given by (5.2). By (5.2)-(5.5) we have

\[
\bigcup_{0 < \delta \leq 1} A_\delta(\tau, \omega) \subseteq \bigcup_{0 < \delta \leq 1} K_\delta(\tau, \omega) \subseteq K(\tau, \omega). \tag{5.6}
\]

Let \( \xi \in A_\delta(\tau, \omega) \) for some \( \delta \in (0, 1] \). By the invariance of \( A_\delta \), there exists \( \zeta \in A_\delta(\tau - T, \theta_{-T}\omega) \) such that \( \xi = u_\delta(\tau, \tau - T, \theta_{-T}\omega, \zeta) \), which with (5.5) and (5.6) together implies that for all \( l \geq L \),

\[
\int_{|x| \geq l} |\xi(x)|^2 dx = \int_{|x| \geq l} |u_\delta(\tau, \tau - T, \theta_{-T}\omega, \zeta)|^2 dx \leq \varepsilon.
\]

The proof is completed. \( \square \)

The limiting equation of (1.1) with \( \delta = 0 \) is given by

\[
\frac{du}{dt} + (1 + i\lambda)(-\Delta)^{\alpha} u + \gamma u = -(1 + i\mu)|u|^2 u + g(t, x), \quad x \in \mathbb{R}^3, t > \tau, \tag{5.7}
\]

and initial condition

\[
u(\tau, x) = u_\tau(x), \quad x \in \mathbb{R}^3. \tag{5.8}
\]

Similar to (1.1)-(1.2), one can prove that problem (5.7)-(5.8) generates a continuous cocycle \( \Phi_0 \) in \( H \). Moreover, has a unique tempered pullback attractor \( A_0 = \{A_0(\tau), \tau \in \mathbb{R}\} \) in \( H \) and has a tempered pullback absorbing set \( K_0 = \{K_0(\tau), \tau \in \mathbb{R}\} \) where \( K_0(\tau) \) is given by

\[
K_0(\tau) = \{u \in H : \|u\|^2 \leq R_0(\tau)\}, \tag{5.9}
\]

and

\[
R_0(\tau) = c + c \int_{-\infty}^{0} e^{\gamma s} ||g(s + \tau)||^2 ds. \tag{5.10}
\]

By Lemma 4.7 and (5.9)-(5.10) we get, for all \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \),

\[
\lim_{\delta \to 0} \sup \|K_\delta(\tau, \omega)\| \leq \|K_0(\tau)\|, \tag{5.11}
\]

which will be used for proving the upper semicontinuity of \( A_\delta \). We also need the convergence of solutions of (1.1)-(1.2) as \( \delta \to 0 \).

**Lemma 5.2.** Let \( u_\delta(t, \tau, \omega, u_{\delta, \tau}) \) and \( u(t, \tau, u_\tau) \) be the solutions of (1.1)-(1.2) and (5.7)-(5.8) with initial data \( u_{\delta, \tau} \) and \( u_\tau \), respectively. If \( \lim_{\delta \to 0} u_{\delta, \tau} = u_\tau \), then for any \( t \geq \tau \) and \( \omega \in \Omega \),

\[
\lim_{\delta \to 0} u(t, \tau, \omega, u_{\delta, \tau}) = u(t, \tau, u_\tau).
\]
It follows from (5.13) and (5.15) that there exists \( \delta \) and \( t \) such that \( \varepsilon > 0 \) after simple calculations, we find from (5.14) that for any \( \delta \) and \( t \),

\[
\frac{dW}{dt} + (1 + i\lambda)(-\Delta)\alpha W + \rho W = -\delta(1 + i\lambda)(-\Delta)\alpha z(\theta_t\omega) - (1 + i\mu)\|v_\delta + \delta z(\theta_t\omega)\|^2(v_\delta + \delta z(\theta_t\omega)) + (1 + i\mu)|u|^2 u.
\]

Multiplying the above equation with \( W \) and taking the real part, we obtain

\[
\frac{1}{2} \frac{d}{dt} \|W\|^2 + \frac{1}{2} C(\alpha)\|W\|_{H^\alpha}^2 + \rho \|W\|^2 = -Re(1 + i\lambda)((-\Delta)\alpha z(\theta_t\omega), W)
\]

\[+ Re \int_{\mathbb{R}^3} (-1 + i\mu)(v_\delta + \delta z(\theta_t\omega))^2(v_\delta + \delta z(\theta_t\omega)) + (1 + i\mu)|u|^2 u)W dx. \quad (5.12)
\]

Now we will estimate the terms on the right-hand side of (5.12). For the first term, we have

\[-Re(1 + i\lambda)((-\Delta)\alpha z(\theta_t\omega), W) \leq \frac{\gamma}{4} \|W\|^2 + \frac{\delta^2}{\gamma}((-\Delta)\alpha z(\theta_t\omega))^2. \quad (5.13)
\]

For the nonlinear term we obtain

\[-Re(1 + i\mu) \int_{\mathbb{R}^3} (v_\delta + \delta z(\theta_t\omega))^2(v_\delta + \delta z(\theta_t\omega)) - |u|^2 u)W dx
\]

\[= -Re(1 + i\mu) \int_{\mathbb{R}^3} (v_\delta + \delta z(\theta_t\omega))^2(v_\delta + \delta z(\theta_t\omega)) - |u + \delta z(\theta_t\omega)|^2(u + \delta z(\theta_t\omega))W dx
\]

\[-Re(1 + i\mu) \int_{\mathbb{R}^3} (u + \delta z(\theta_t\omega))^2(u + \delta z(\theta_t\omega)) - |u|^2 u)W dx, \quad (5.14)
\]

After simple calculations, we find from (5.14) that for any \( \tau \in \mathbb{R}, \omega \in \Omega, \tau + T > 0 \) and \( \varepsilon > 0 \), there exists \( \delta_0 = \delta_0(\tau, \omega, T, \varepsilon) > 0 \) such that for all \( \delta \in (0, \delta_0) \) and \( t \in [\tau, \tau + T] \),

\[-Re(1 + i\mu) \int_{\mathbb{R}^3} (v_\delta + \delta z(\theta_t\omega))^2(v_\delta + \delta z(\theta_t\omega)) - |u|^2 u)W dx
\]

\[\leq c\|W\|^2 + c\varepsilon + c\varepsilon \int_{\mathbb{R}^3} (|u|^4 + |v_\delta|^4)dx. \quad (5.15)
\]

It follows from (5.13) and (5.15) that there exists \( \delta_1 > 0 \) such that for all \( \delta \in (0, \delta_1) \) and \( t \in [\tau, \tau + T] \),

\[
\frac{d}{dt} \|W\|^2 \leq c\|W\|^2 + c\varepsilon + c\varepsilon \int_{\mathbb{R}^3} (|u|^4 + |v_\delta|^4)dx. \quad (5.16)
\]

Solving (5.16) on \([\tau, \tau + T]\) we have, for all \( \delta \in (0, \delta_1) \) and \( t \in [\tau, \tau + T] \),

\[
\|W(t)\|^2 \leq c\|W(\tau)\|^2 + c\varepsilon + c\varepsilon \int_{\tau}^{t} (\|u(s, \tau, u_\tau)\|^4_{L^4(\mathbb{R}^3)} + \|v_\delta(s, \tau, \omega, v_\delta, \tau)\|^4_{L^4(\mathbb{R}^3)})ds. \quad (5.17)
\]

On the other hand, by (4.11) we find that for all \( \delta \in (0, 1) \) and \( t \in [\tau, \tau + T] \),

\[
\|v_\delta(t, \tau, \omega, v_\delta, \tau)\|^2 + \int_{\tau}^{t} \|v_\delta(s, \tau, \omega, v_\delta, \tau)\|^4_{L^4(\mathbb{R}^3)}ds \leq c + c\|v_\delta, \tau\|^2. \quad (5.18)
\]
Similarly, we also get for all \( t \in [\tau, \tau + T] \),

\[
\|u(t, \tau, u_\tau)\|^2 + \int_{\tau}^{t} \|u(s, \tau, u_\tau)\|_{L^2(\mathbb{R}^3)}^4 ds \leq c + c\|u_\tau\|^2. \tag{5.19}
\]

Let \( \delta_2 = \min\{1, \delta_1\} \). By (5.18)-(5.19) we have, for all \( \delta \in (0, \delta_2) \) and \( t \in [\tau, \tau + T] \),

\[
\|v_\delta(t, \tau, \omega, v_\delta, \tau) - u(t, \tau, u_\tau)\|^2 \leq c\|v_\delta, \tau - u_\tau\|^2 + c\varepsilon(\|v_\delta, \tau\|^2 + \|u_\tau\|^2). \tag{5.20}
\]

Due to \( v_\delta, \tau = u_\delta, \tau - \delta z(\theta, \omega) \), we have from (5.20) that if \( \lim_{\delta \to 0} v_\delta, \tau = u_\tau \) then for all \( t \in [\tau, \tau + T] \),

\[
\lim_{\delta \to 0} v_\delta(t, \tau, \omega, v_\delta, \tau) = u(t, \tau, u_\tau)
\]

which along with (3.4) indicates \( \lim_{\delta \to 0} u_\delta(t, \tau, \omega, v_\delta, \tau) = u(t, \tau, u_\tau) \).

**Lemma 5.3.** Suppose (3.8) holds. Let \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \) be fixed. If \( \delta_n \to 0 \) and \( u_n \in \mathcal{A}_{\delta_n}(\tau, \omega) \), then the sequence \( \{u_n\}_{n=1}^{\infty} \) is precompact in \( H \).

**Proof.** The proof is standard based on argument of [55]. Let \( K \) be the family of subsets of \( L^2(\mathbb{R}^3) \) given by (5.2). Then it follows from Lemma 5.1 that there exist \( T = (\tau, \omega, \varepsilon) \geq 1 \) and \( l = (\tau, \omega, \varepsilon) \geq 1 \) such that for all \( t \geq T \) and \( \delta \in (0, 1] \)

\[
\int_{|x| \geq l} |u_\delta(\tau, \tau - t, \theta, \omega, u_\delta, \tau - t) (x)|^2 dx \leq \frac{1}{2} \varepsilon, \tag{5.21}
\]

where \( u_\delta, \tau - t \in K(\tau - t, \theta, \omega) \). By (5.6) and (5.21), we get from the invariance of \( \mathcal{A}_{\delta} \) that, for each \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \),

\[
\int_{|x| \geq l} |u(x)|^2 dx \leq \frac{1}{2} \varepsilon, \text{ for all } u \in \mathcal{A}_{\delta}(\tau, \omega) \text{ with } 0 < \delta \leq 1. \tag{5.22}
\]

By (5.3), we see that the set \( \bigcup_{0 < \delta \leq 1} \mathcal{A}_{\delta}(\tau, \omega) \) is bounded in \( H^0(U) \) with \( U = \{ x \in \mathbb{R}^3 : |x| < l \} \). Then the compactness of embedding \( H^0(U) \hookrightarrow L^2(U) \) implies that the set \( \bigcup_{0 < \delta \leq 1} \mathcal{A}_{\delta}(\tau, \omega) \) has a finite covering of balls of radii less than \( \frac{1}{2} \varepsilon \) in \( L^2(U) \), which along with (5.22) completes the proof.

Now we are in position to present the upper semicontinuity of random attractors \( \mathcal{A}_{\delta} \) as \( \delta \to 0 \).

**Theorem 5.4.** Assume that (3.8) holds. Then for every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \),

\[
\lim_{\delta \to 0} \text{dist}_H(\mathcal{A}_{\delta}(\tau, \omega), \mathcal{A}_0(\tau)) = 0. \tag{5.23}
\]

**Proof.** This is an direct consequence of Theorem 3.2 in [52] from (5.11), Lemma 5.2 and 5.3.

**Acknowledgement**

The authors would like to thank the reviewers for their helpful comments.

**References**

[1] L. Arnold, *Random Dynamical Systems*, Springer-Verlag, New York, 1998.
[2] M. Bartuccelli, P. Constantin, C. Doering, J. Gibbon and M. Gisselfält, *On the possibility of soft and hard turbulence in the complex Ginzburg-Landau equation*, Phys. D, 1990, 44, 421–444.

[3] P. W. Bates, H. Lisei and K. Lu, *Attractors for stochastic lattice dynamical system*, Stoch. Dyn., 2006, 6, 1–21.

[4] P. W. Bates, K. Lu and B. Wang, *Random attractors for stochastic reaction-diffusion equations on unbounded domains*, J. Differential Equations, 2009, 246, 845–869.

[5] Z. Brzeźniak and Y. Li, *Asymptotic compactness and absorbing sets for 2D stochastic Navier-Stokes equations on some unbounded domains*, Trans. Amer. Math. Soc., 2006, 358, 5587–5629.

[6] I. Chueshov, *Monotone Random Systems Theory and Applications*, Springer, Berlin, 2002.

[7] H. Crauel and F. Flandoli, *Attractors for random dynamical systems*, Probab. Theory Related Fields, 1994, 100, 365–393.

[8] H. Crauel, A. Debussche and F. Flandoli, *Random attractors*, J. Dynam. Differential Equations, 1997, 9, 307–341.

[9] C. Doering, J. Gibbon and C. Levermore, *Weak and strong solutions of the complex Ginzburg-Landau equation*, Phys. D, 1994, 71, 285–318.

[10] J. Dong and M. Xu, *Space-time fractional Schrödinger equation with time-independent potentials*, J. Math. Anal. Appl., 2008, 344, 1005–1017.

[11] J. Duan, P. Holme and E. S. Titi, *Global existence theory for a generalized Ginzburg-Landau equation*, Nonlinearity, 2009, 5, 1303–1314.

[12] X. Fan and Y. Wang, *Attractors for a second order nonautonomous lattice dynamical systems with nonlinear damping*, Phys. Lett. A, 2007, 365, 17–27.

[13] F. Flandoli and B. Schmalfuss, *Random attractors for the 3D stochastic Navier-Stokes equation with multiplicative white noise*, Stoch. Stoch. Rep., 1996, 59, 21–45.

[14] M. Garrido-Atienza, K. Lu and B. Schmalfuss, *Random dynamical systems for stochastic equations driven by a fractional Brownian motion*, Discrete Contin. Dyn. Syst. Ser. B, 2010, 14, 473–493.

[15] A. Gu, D. Li, B. Wang and H. Yang, *Regularity of random attractors for fractional stochastic reaction-diffusion equations on $\mathbb{R}^n$*, J. Differential Equations, 2018, 264, 7094–7137.

[16] B. Guo, Y. Han and J. Xin, *Existence of the global smooth solution to the period boundary value problem of fractional nonlinear Schrödinger equation*, Appl. Math. Comput., 2008, 204, 458–477.

[17] B. Guo and Z. Huo, *Global well-posedness for the fractional nonlinear Schrödinger equation*, Commun. Partial Differential Equations, 2011, 36, 247–255.

[18] B. Guo, X. Pu and F. Huang, *Fractional Partial Differential Equations and their Numerical Solutions*, Science Press, Beijing, 2011.

[19] B. Guo and X. Wang, *Finite dimensional behavior for the derivative Ginzburg-Landau equation in two spatial dimensions*, Phys. D, 1995, 89, 83–99.
[20] B. Guo and M. Zeng, *Solutions for the fractional Landau-Lifshitz equation*, J. Math. Anal. Appl., 2010, 361, 131–138.

[21] J. K. Hale, *Asymptotic Behavior of Dissipative Systems*, AMS, Providence, 1988.

[22] X. Han, W. Shen and S. Zhou, *Random attractors for stochastic lattice dynamical system in weighted space*, J. Differential Equations, 2011, 250, 1235–1266.

[23] Y. Lan and J. Shu, *Fractal dimension of random attractors for non-autonomous fractional stochastic Ginzburg-Landau equations with multiplicative noise*, Dyn. Syst., 2019, 34(2), 274–300.

[24] Y. Lan and J. Shu, *Dynamics of non-autonomous fractional stochastic Ginzburg-Landau equations with multiplicative noise*, Commun. Pure Appl. Anal., 2019, 18(5), 2409–2431.

[25] D. Li, Z. Dai and X. Liu, *Long time behavior for generalized complex Ginzburg-Landau equation*, J. Math. Anal. Appl., 2007, 330, 938–948.

[26] D. Li and B. Guo, *Asymptotic behavior of the 2D generalized stochastic Ginzburg-Landau equation with additive noise*, Appl. Math. Mech., 2009, 30, 883–894.

[27] H. Lu, P. W. Bates, S. Lu and M. Zhang, *Dynamics of 3-D fractional complex Ginzburg-Landau equation*, J. Differential Equations, 2015, 259, 5276–5301.

[28] H. Lu, P. W. Bates, S. Lu and M. Zhang, *Dynamics of the 3D fractional Ginzburg-Landau equation with multiplicative noise on a unbounded domain*, Commu. Math. Sci., 2016, 14, 273–295.

[29] H. Lu, P. W. Bates, J. Xin and M. Zhang, *Asymptotic behavior of stochastic fractional power dissipative equations on $\mathbb{R}^n$*, Nonlinear Anal., 2015, 128, 176–198.

[30] H. Lu and S. Lv, *Random attractor for fractional Ginzburg-Laudau equation with multiplicative noise*, Taiwanese J. Math., 2014, 18, 435–450.

[31] Y. Lv and J. Sun, *Asymptotic behavior of stochastic discrete complex Ginzburg-Landau equations*, Phys. D, 2006, 221, 157–169.

[32] B. Maslowski and B. Schmalfuss, *Random dynamical systems and stationary solutions of differential equations driven by the fractional Brownian motion*, Stoch. Anal. Appl., 2004, 22, 1577–1607.

[33] F. Morillas and J. Valero, *Attractors for reaction-diffusion equations in $\mathbb{R}^n$ with continuous nonlinearity*, Asymptot. Anal., 2005, 44, 111–130.

[34] E. D. Nezza, G. Palatucci and E. Valdinoci, *Hitchhiker’s guide to the fractional Sobolev spaces*, Bull. Sci. Math., 2012, 136, 521–573.

[35] X. Pu and B. Guo, *Global weak Solutions of the fractional Landau-Lifshitz-Maxwell equation*, J. Math. Anal. Appl., 2010, 372, 86–98.

[36] X. Pu and B. Guo, *Well-posedness and dynamics for the fractional Ginzburg-Landau equation*, Appl. Anal., 2013, 92, 318–334.

[37] J. C. Robinson, *Infinite-Dimensional Dynamical Systems*, Cambridge Univ. Press, Cambridge, 2001.
[38] B. Schmalfuss, "Backward cocycle and attractors of stochastic differential equations," in V. Reitmann, T. Riedrich, N. Koksch (Eds.), International Seminar on Applied Mathematics-Nonlinear Dynamics: Attractor Approximation and Global Behavior, Technische Universität, Dresden, 1992, pp. 185–192.

[39] G. Sell and Y. You, Dynamics of Evolutional Equations, Springer-Verlag, New York, 2002.

[40] R. Servadei and E. Valdinoci, "On the spectrum of two different fractional operators," Proc. Roy. Soc. Edinburgh Sect. A, 2014, 144, 831–855.

[41] T. Shen and J. Huang, "Well-posedness and dynamics of stochastic fractional model for nonlinear optical fiber materials," Nonlinear Anal., 2014, 110, 33–46.

[42] Z. Shen, S. Zhou and W. Shen, "One-dimensional random attractor and rotation number of the stochastic damped sine-Gordon equation," J. Differential Equations, 2010, 248, 1432–1457.

[43] J. Shu, "Random attractors for stochastic discrete Klein-Gordon-Schrödinger equations driven by fractional Brownian motions," Discrete Contin. Dyn. Syst. Ser. B, 2017, 22, 1587–1599.

[44] J. Shu, P. Li, J. Zhang and O. Liao, "Random attractors for the stochastic coupled fractional Ginzburg-Landau equation with additive noise," J. Math. Phys., 2015, 56, 102702.

[45] E. Tarasov Vasily and M. Zaslavsky George, "Fractional Ginzburg-Landau equation for fractal media," Phys. A, 2005, 354, 249–261.

[46] R. Temam, Infinite Dimension Dynamical Systems in Mechanics and Physics, Springer-Verlag, New York, 1997.

[47] M. J. Vishik and A. V. Fursikov, Mathematical Problems of Statistical Hydrodynamics, Kluwer Academic Publishers, Boston, 1988.

[48] P. Walters, Introduction to Ergodic Theory, Springer-Verlag, New York, 2000.

[49] B. Wang, "Random attractors for the stochastic FitzHugh-Nagumo system on unbounded domains," Nonlinear Anal., 2009, 71, 2811–2828.

[50] B. Wang, "Upper semicontinuity of random for non-compact random systems," J. Differential Equations, 2009, 139, 1–18.

[51] B. Wang, "Asymptotic behavior of stochastic wave equations with critical exponents on $\mathbb{R}^3$," Trans. Amer. Math. Soc., 2011, 363, 3639–3663.

[52] B. Wang, "Sufficient and necessary criteria for existence of pullback attractors for non-compact random dynamical systems," J. Differential Equations, 2012, 253, 1544–1583.

[53] B. Wang, "Random attractors for non-autonomous stochastic wave equations with multiplicative noise," Discrete Contin. Dyn. Syst. Ser. A, 2014, 34, 269–300.

[54] B. Wang, "Existence and upper-semicontinuity of attractors for stochastic equations with deterministic non-autonomous terms," Stoch. Dyn., 2014, 14(4), 1450009(1–31).

[55] B. Wang, "Asymptotic behavior of non-autonomous fractional stochastic reaction-diffusion equations," Nonlinear Anal., 2017, 158, 60–82.
[56] X. Wang, S. Li and D. Xu, Random attractors for second-order stochastic lattice dynamical systems, Nonlinear Anal., 2010, 72, 483–494.

[57] W. Yan, S. Ji and Y. Li, Random attractors for stochastic discrete Klein-Gordon-Schrödinger equations, Phys. Lett. A, 2009, 373, 1268–1275.

[58] J. Yin, Y. Li and A. Gu, Backwards compact attractors and periodic attractors for non-autonomous damped wave equations on an unbounded domain, Comput. Math. Appl., 2017, 74, 744–758.

[59] F. Yin and L. Liu, D-pullback attractor for a non-autonomous wave equation with additive noise on unbounded domains, Comput. Math. Appl., 2014, 68, 424–438.

[60] J. Zhang and J. Shu, Existence and upper semicontinuity of random attractors for non-autonomous fractional stochastic Ginzburg-Landau equations, J. Math. Phys., 2019, 60, 042702.

[61] W. Zhao, Existence and upper-semicontinuity of pullback attractors in $H^1(\mathbb{R}^n)$ for non-autonomous reaction-diffusion equations perturbed by multiplicative noise, Electronic J. Differential Equations, 2016, 2016, 1–28.

[62] C. Zhao and S. Zhou, Sufficient conditions for the existence of global random attractors for stochastic lattice dynamical systems and applications, J. Math. Anal. Appl., 2009, 354, 78–95.

[63] S. Zhou and M. Zhao, Random attractors for damped non-autonomous wave equations with memory and white noise, Nonlinear Anal., 2015, 120, 202–226.