A Quasi-Hopf algebra interpretation of quantum 3-j and 6-j symbols and difference equations.

O. Babelon † D. Bernard ∗ E. Billey †

Abstract

We consider the universal solution of the Gervais-Neveu-Felder equation in the \( U_q(sl_2) \) case. We show that it has a quasi-Hopf algebra interpretation. We also recall its relation to quantum 3-j and 6-j symbols. Finally, we use this solution to build a q-deformation of the trigonometric Lamé equation.
1 Introduction

The Gervais-Neveu-Felder equation is a deformation of the standard Yang-Baxter equation. In the \( sl_2 \) case, it reads

\[
R_{12}(x)R_{13}(xq^{H_2})R_{23}(x) = R_{23}(xq^{H_1})R_{13}(x)R_{12}(xq^{H_3})
\]

(1)

Here, \( H \) denotes a Cartan generator in \( sl_2 \) (or rather \( \mathcal{U}_q(sl_2) \)) and \( x \) is a parameter not to be confused with the spectral parameter (absent in the \( sl_2 \) case).

This equation appeared independently in several contexts. It was first discovered by J.L. Gervais and A. Neveu in their studies on Liouville theory \([4]\). It was rediscovered by G. Felder in his approach to the quantization of the Knizhnik-Zamolodchikov-Bernard equation \([4]\). Finally, it was shown to play an important role in the quantization of the Calogero-Moser models in the \( R \)-matrix formalism \([3]\). For all these reasons, we believe that this equation deserves much attention.

In this note, we analyse the universal solution \( R(x) \in \mathcal{U}_q(sl_2) \otimes \mathcal{U}_q(sl_2) \) of eq.\([2]\) obtained in \([4]\). We show that it has a nice quasi-Hopf algebra interpretation. For completeness, we recall the connection of this solution with \( q \)-analogs of 3-j and 6-j symbols. Finally, we explain how it can be used to construct a \( q \)-difference analog of the trigonometric Lamé equation (Calogero model for 2 particles).

2 A summary of universal formulae

In this section, we recall the universal formulae obtained in \([4]\) for the matrix \( R_{12}(x) \in \mathcal{U}_q(sl_2) \otimes \mathcal{U}_q(sl_2) \). We denote by \( H, E_\pm \) the generators of the quantum group \( \mathcal{U}_q(sl_2) \)

\[
[H, E_\pm] = \pm 2E_\pm, \quad [E_+, E_-] = \frac{q^H - q^{-H}}{q - q^{-1}}
\]

The coproduct is defined as

\[
\Delta(H) = H \otimes id + id \otimes H, \quad \Delta(E_\pm) = E_\pm \otimes q^H \mp q^{-H} \otimes E_\pm
\]

We have \( R^D_{12}\Delta(a) = \Delta'(a)R^D_{12} \) for any \( a \in \mathcal{U}_q(sl_2) \) where \( \Delta' \) is the opposite comultiplication and \( R^D_{12} \) Drinfeld’s universal \( R \)-matrix :

\[
R^D_{12} = q^\frac{1}{2}H \otimes H \sum_{i=0}^{\infty} (q - q^{-1})^i \frac{q^{-i(4i+1)}}{[i]!} \cdot q^\frac{1}{2}H E^i_+ \otimes q^{-\frac{1}{2}H} E^i_-
\]

As usual, \( q \)-numbers are defined as \([i] = (q^i - q^{-i})/(q - q^{-1}) \). Let us now define

\[
R_{12}(x) = F_{21}^{-1}(x) \ R^D_{12} \ F_{12}(x)
\]

(2)

with

\[
F_{12}(x) = \sum_{k=0}^{\infty} (q - q^{-1})^k \frac{(-1)^k}{[k]!} \frac{x^k}{\prod_{\nu=k}^{2k-1} (xq^\nu q^{H_2} - x^{-1}q^{-\nu}q^{-H_2})} q^{k(H_1 + H_2)} E^k_+ \otimes E^k_-
\]

(3)

\[
F_{12}^{-1}(x) = \sum_{k=0}^{\infty} (q - q^{-1})^k \frac{1}{[k]!} \frac{x^k}{\prod_{\nu=1}^{k} (xq^\nu q^{H_2} - x^{-1}q^{-\nu}q^{-H_2})} q^{k(H_1 + H_2)} E^k_+ \otimes E^k_-
\]
It follows from the construction of \cite{4} that \( R_{12}(x) \) is a solution of eq. (1).

One can check that \( F_{12}(x) \) satisfies the following “shifted cocycle” condition

\[
[(id \otimes \Delta) F] \cdot [id \otimes F] = [(\Delta \otimes id) F] \cdot [F(x q^H) \otimes id] \tag{4}
\]

This relation is proved using standard q-binomial identities. It turns out that \( F_{12}(x) \) is actually a “shifted coboundary”

\[
F_{12}(x) = \Delta M(x) \left[ id \otimes M(x) \right]^{-1} \left[ M(x q^H) \otimes id \right]^{-1} \tag{5}
\]

where the formula for the “boundary” reads

\[
M(x) = \sum_{n,m=0}^{\infty} \frac{(-1)^m x^m q^{\frac{1}{2} n(n-1)+m(n-m)}}{[n]![m]! \prod_{\nu=1}^{n} (x q^\nu - x^{-1} q^{-\nu})} E_+^n E_-^m q^{\frac{1}{2} (n+m) H}
\]

Equation (5) implies eq. (4).

### 3 Quasi-Hopf algebra interpretation

The previous construction possesses a natural quasi-Hopf interpretation. Indeed, since \( R(x) \) is defined in eq. (2) by a twisting procedure in the sense of Drinfeld \cite{5} it is canonically associated to a quasi-Hopf structure on \( U_q(sl_2) \). We shall denote it as \( U_q(sl_2) \). This quasi-Hopf algebra possesses very specific properties due to the “shifted cocycle” relation (4) satisfied by \( F(x) \). Besides Drinfeld’s construction, this gives another example of a quasi-Hopf algebra structure over \( U_q(sl_2) \).

Let us recall following ref.\cite{5} that a quasi-Hopf algebra is specified by a quadruplet \((A, \Delta, R, \Phi)\) where \( A \) is an associative algebra, \( \Delta \) is a (non-coassociative) comultiplication in \( A \), \( R \in A \otimes A \) and \( \Phi \in A \otimes A \otimes A \) are such that :

\[
R \Delta(a) = \Delta'(a) R \tag{6}
\]

\[
(id \otimes \Delta) \Delta(a) \Phi = \Phi (\Delta \otimes id) \Delta(a) \tag{7}
\]

for all \( a \in A \). There also are extra compatibility relations between \( \Delta, R \) and \( \Phi \) which we shall mention when needed. We will consider quasitriangular quasi-Hopf algebra, i.e., \( R \) is assumed to verify the conditions

\[
(\Delta \otimes id) R = \Phi_{321} R_{13} \Phi_{123}^{-1} R_{23} \Phi_{123}
\]

\[
(id \otimes \Delta) R = \Phi_{231}^{-1} R_{13} \Phi_{213} R_{12} \Phi_{123}^{-1}
\]

There exists a twisting procedure to construct quasi-Hopf algebras. Namely, if \((A, \Delta, R, \Phi)\) is a quasitriangular quasi-Hopf algebra then a new quasitriangular quasi-Hopf algebra \((A, \bar{\Delta}, \bar{R}, \bar{\Phi})\) is defined by \( \bar{\Delta}(a) = F_{12}^{-1} \Delta(a) F_{12} \), and

\[
\bar{\Phi} = F_{23}^{-1} \left( (id \otimes \Delta)(F^{-1}) \right) \Phi \left( (\Delta \otimes id)(F) \right) F_{12} \tag{8}
\]

\[
\bar{R} = F_{21}^{-1} R F_{12} \tag{9}
\]

with \( F_{12} \in A \otimes A \).
In our case, we are twisting $U_q(sl_2) \equiv (U_q(sl_2), \Delta, R^D, id)$ by $F(x)$. So we have $\tilde{R} = F_{21}^{-1}(x)R_{12}^DF_{12}(x) = R(x)$ as defined in eq. (4). We denote $\Delta$ by $\Delta_x$ with:

$$\Delta_x(a) = F_{12}^{-1}(x)\Delta(a)F_{12}(x), \quad \forall a \in U_q(sl_2)$$

(10)

It is a simple check to verify that the “shifted cocycle” condition (4) implies that:

$$(id \otimes \Delta_x)(\Delta_x(a)) = (\Delta_{xq^1} \otimes id)\Delta_x(a)$$

(11)

In other words, the shift breaks the co-associativity. We denote $\tilde{\Phi}$ by $\Phi(x)$. It possesses a simple expression in terms of $F(x)$:

$$\Phi(x) = F_{23}^{-1}(x)[(id \otimes \Delta)(F^{-1}(x))[[\Delta \otimes id](F(x))]F_{12}(x) = F_{12}^{-1}(xqH_3)F_{12}(x)$$

(12)

where in the last equality we again used the “shifted cocycle” relation (4).

We can now write all the quasi-Hopf relations in $U_{q,x}(sl_2)$ in terms of $R(x)$ or $F(x)$. For example, the general relation (7) reduces to eq. (11). Also, thanks to the following property, the quasi- Yang-Baxter equation,

$$\Phi_{321}^{-1}(x)R_{12}(x)\Phi_{312}(x)R_{13}(x)\Phi_{132}^{-1}(x)R_{23}(x) = R_{23}(x)\Phi_{231}^{-1}(x)R_{13}(x)\Phi_{213}(x)R_{12}(x)\Phi_{123}^{-1}(x)$$

valid in any quasitriangular quasi-Hopf algebra reduces to the equation (4).

Similarly, the quasitriangular property of $U_{q,x}(sl_2)$ implies that

$$(\Delta_x \otimes id)R(x) = R_{13}(xqH_2)R_{23}(x)F_{12}^{-1}(xqH_3)F_{12}(x)$$

$$(id \otimes \Delta_x)R(x) = F_{23}^{-1}(x)F_{23}(xqH_1)R_{13}(x)R_{12}(xqH_3)$$

Notice that for $x = 0$, $F_{12}(x)|_{x=0} = 1$ and therefore $R(x)|_{x=0} = R^D$. In the limit $x = \infty$, $F_{12}^{-1}(x)|_{x=\infty} = q^{-H \otimes H/2}R_{12}^D$. Thus $R_{12}(x)|_{x=\infty} = q^{-H \otimes H/2}R_{12}^Dq^{-H \otimes H/2}$, and $\Delta_{x=\infty}(a) = q^{-H \otimes H/2}\Delta'(a)q^{H \otimes H/2}$ for all $a \in U_q(sl_2)$.

4 Relation to 3-j and 6-j symbols

We now give a list of formulae expressing the matrix elements of the various objects we have considered so far in terms of standard q-analogs of the 3-j and 6-j symbols. Let $\rho^{(j)}$ denote the spin j representation of $U_q(sl_2)$. Then

$$\rho^{(j)}(H)|j, m\rangle = 2m |j, m\rangle$$

$$\rho^{(j)}(E_{\pm})|j, m\rangle = \sqrt{[j \mp m][j \pm m + 1]} |j, m \pm 1\rangle$$

The first step is to find the matrix elements of the matrix $M(x)$ in the spin-j representation. We get

$$[M^{(j)}(x)]_{\sigma_1 m_1}^{\sigma_1 m_1} = (-1)^{\sigma_1 + m_1}\sqrt{[j_1 + \sigma_1][j_1 - \sigma_1][j_1 + m_1][j_1 - m_1]} \prod_{r=1}^{j_1+\sigma_1}(1 - x^2q^{2r})^{q_{2p \sigma_r}q_{2p \sigma_r}} \sum_p [p]! [\sigma_1 - m_1 + p][j_1 - \sigma_1 - p][j_1 + m_1 - p]!$$

(13)
This formula agrees (up to normalizations) with the one found in [10].

This matrix \( M(x) \) is known to perform the vertex-IRF transformation in conformal field theory [6, 7, 8, 9].

\[
\xi^{(j_1)}_{m_1}(z) = \sum_{\sigma_1} \psi^{(j_1)}_{\sigma_1}(z)M^{(j_1)}_{\sigma_1 m_1}(x)
\]

where the \( \psi \)'s are IRF type operators and the \( \xi \)'s are vertex type operators. The braiding relations of the \( \psi \)'s are described by the matrix \( R(x) \), while those of the \( \xi \)'s are described by \( R^D \). Thus, we expect the elements \( M^{(j_1)}_{\sigma_1 m_1}(x) \) to be related to quantum 3-j symbols. The precise connexion was found in [11]. We have

\[
[M^{(j_1)}(x)]_{\sigma_1 m_1} = \frac{N^{(j_1)}_{\psi}(x, \sigma_1)}{N^{(j_1)}_{\xi}(m_1)} \lim_{m \to \infty} \left[ \begin{array}{ccc} j_1 & j(x) & j(x) + \sigma_1 \\ m_1 & m & m + m_1 \end{array} \right]_q
\]

where we have defined \( j(x) \) through the relation

\[
x = q^{2j(x)+1}
\]

Eq. (14) has to be understood as an analytic continuation in \( j(x) \) of 3-j symbols [12]. We give a sketch of the proof in the Appendix. The factors \( N^{(j_1)}_{\xi} \) and \( N^{(j_1)}_{\psi} \) can be reabsorbed into the normalizations of the fields \( \psi \) and \( \xi \) respectively. Their expression is also given in the Appendix. We represent eq. (14) by a diagram

\[
[M^{(j_1)}(x)]_{\sigma_1 m_1} = \begin{array}{c} j_1, m_1 \\ j(x) + \sigma_1 \\ j(x) \end{array}
\]

From eq. (14), it is now possible to build the complete dictionary between the matrix elements of \( F_{12}(x) \) and \( R_{12}(x) \) and standard 3-j and 6-j symbols.

We start with

\[
\langle j_1, \sigma_1 | \langle j_2, \sigma_2 | M_2^{(j_2)}(xq^H_1)M_1^{(j_1)}(x)|j_1, m_1)|j_2, m_2 \rangle = M_2^{(j_2)}(xq^{2\sigma_1})M_1^{(j_1)}(x) = \frac{N^{(j_1)}_{\psi}(x, \sigma_1)N^{(j_2)}_{\psi}(xq^{2\sigma_1}, \sigma_2)}{N^{(j_1)}_{\xi}(m_1)N^{(j_2)}_{\xi}(m_2)} \lim_{m, m' \to \infty} \left[ \begin{array}{ccc} j_2 & j(x) + \sigma_1 & j(x) + \sigma_1 + \sigma_2 \\ m_2 & m & m + m_2 \end{array} \right] \left[ \begin{array}{ccc} j_1 & j(x) & j(x) + \sigma_1 \\ m_1 & m' & m' + m_1 \end{array} \right]
\]

Notice that we have used the fact that

\[
xq^{2\sigma_1} = q^{2(j(x)+\sigma_1)+1}
\]

Hence the shift in the Gervais-Neveu-Felder equation \( x \to xq^H \) precisely corresponds to the shift of spins \( j(x) \to j(x) + \sigma \). Thus we have the diagramatic correspondance

\[
M_2^{(j_2)}(xq^H_1)M_1^{(j_1)}(x) = \begin{array}{c} j_1, \sigma_1, \sigma_2 \\ j(x) \end{array}
\]

This formula agrees (up to normalizations) with the one found in [10].
The matrix elements of $R_{12}(x)$ are computed from the formula

$$R_{12}(x)M_1(xq^{H_2})M_2(x) = M_2(xq^{H_1})M_1(x)R_{12}^D$$

or graphically

$$\sum_{\sigma_1 \sigma_2} R(x)_{j_1,j_2}^{j_1',j_2'} \delta_{j_1',j_1} \delta_{j_2',j_2} = \begin{array}{c|c|c}
\sum \cdots & \sum \cdots & \sum \cdots \\
\end{array}$$

This is equivalent to the braiding relation and relates the matrix elements of $R(x)$ to 6-j symbols:

$$\langle j_1, \sigma'_1 | j_2, \sigma'_2 | R_{12}(x) | j_1, \sigma_1 | j_2, \sigma_2 \rangle = (-1)^{\sigma'_1 - \sigma_1} q^{C(j(x)) + C(j(x) + \sigma_1 + \sigma_2) - C(j(x) + \sigma'_1) - C(j(x) + \sigma_2)}$$

$$\frac{\mathcal{N}^{(j_1)}(x, \sigma_1 \sigma_2) \mathcal{N}^{(j_2)}(xq^{2\sigma_2}, \sigma'_2)}{\mathcal{N}^{(j_1)}(xq^{2\sigma_2}, \sigma_1 \sigma_2) \mathcal{N}^{(j_2)}(x, \sigma'_2)} \left\{ \begin{array}{ccc}
\hat{j}_2 & j(x) + \sigma_1 + \sigma_2 & j(x) + \sigma_2 \\
\hat{j}_1 & j(x) & j(x) + \sigma_1 \\
\end{array} \right\}_q$$

where $C(j) = j(j + 1)$ and the last symbol represents a 6-j coefficient (see eq. 5.11 in [13]).

Finally, we give the formula for the matrix elements of $F_{12}(x)$ in terms of 3-j and 6-j symbols. We start from the formula

$$F_{12}(x)M_1(xq^{H_2})M_2(x) = \Delta M(x)$$

From the definition of the coproduct, we have

$$[\Delta M^{j_1j_2}(x)]_{\sigma_1 \sigma_2, m_1 m_2} = \sum_{j_{12}} \left[ \begin{array}{ccc}
\hat{j}_1 & \hat{j}_2 & j_{12} \\
\sigma_1 & \sigma_2 & \sigma_1 + \sigma_2 \\
\end{array} \right]_q \left[ M^{(j_{12})}(x) \right]_{\sigma_1 + \sigma_2, m_1 + m_2} \left[ \begin{array}{ccc}
\hat{j}_1 & \hat{j}_2 & j_{12} \\
m_1 & m_2 & m_1 + m_2 \\
\end{array} \right]_q$$

Using the interpretation of $M$ as a 3-j symbol and the defining relation of 6-j symbols, we can write

$$\left[ M^{(j_{12})}(x) \right]_{\sigma_1 + \sigma_2, m_1 + m_2} \left[ \begin{array}{ccc}
\hat{j}_1 & \hat{j}_2 & j_{12} \\
m_1 & m_2 & m_1 + m_2 \\
\end{array} \right]_q =$$

$$\sum_{\sigma'_1 \sigma'_2} \frac{\mathcal{N}^{(j_{12})}(x, \sigma_1 + \sigma_2)}{\mathcal{N}^{(j_1)}(xq^{2\sigma_2}, \sigma'_1) \mathcal{N}^{(j_2)}(x, \sigma'_2)} \left\{ \begin{array}{ccc}
\hat{j}_2 & j(x) + \sigma_1 + \sigma_2 & j(x) + \sigma_2 \\
\hat{j}_1 & j(x) & j(x) + \sigma_1 \\
\end{array} \right\}_q M^{(j_1)}(xq^{2\sigma_2}) M^{(j_2)}(x)_{\sigma'_1, m_1} M^{(j_2)}(x)_{\sigma'_2, m_2}$$

Hence

$$[F^{j_1j_2}(x)]_{\sigma_1 \sigma_2, \sigma'_1 \sigma'_2} =$$

$$\sum_{j_{12}} \frac{\mathcal{N}^{(j_{12})}(x, \sigma_1 + \sigma_2)}{\mathcal{N}^{(j_1)}(xq^{2\sigma_2}, \sigma'_1) \mathcal{N}^{(j_2)}(x, \sigma'_2)} \left[ \begin{array}{ccc}
\hat{j}_1 & \hat{j}_2 & j_{12} \\
\sigma_1 & \sigma_2 & \sigma_1 + \sigma_2 \\
\end{array} \right]_q \left\{ \begin{array}{ccc}
\hat{j}_1 & \hat{j}_2 & j_{12} \\
\hat{j}_1 & j(x) + \sigma'_1 + \sigma'_2 & j(x) + \sigma'_2 \\
\end{array} \right\}_q$$
5 Application to the trigonometric q-deformed Lamé equation

In [3] it was shown how solutions of eq. (11) could be used to construct a set of commuting operators corresponding to q-deformations of the quantum Calogero-Moser Hamiltonians. In the $\mathcal{U}_q(sl_2)$ case, there is only one such operator once we separate the center of mass motion.

According to the general prescription [3], we start from a Lax matrix satisfying

$$R_{12}(xq^{-\frac{1}{2}H_3})L_{13}(x)L_{23}(x) = L_{23}(x)L_{13}(x)R_{12}(xq^{\frac{1}{2}H_3}),$$

(19)

with a subscript 3 denoting the quantum space. The first Hamiltonian is the restriction of $\text{Tr}_1(L_{13}(x))$ to the subspace of zero-weight vectors.

In the representation $\rho = \rho^{(1/2)} \otimes \rho^{(1/2)}$ of $\mathcal{U}_q(sl_2) \otimes \mathcal{U}_q(sl_2)$, the following extra property is true:

$$\rho \left( \left[ (H_1 + H_2)\partial_x, R_{12}(x) \right] \right) = 0.$$ 

This condition allows to recast eq. (11) in the form (19) with a Lax operator $L(x)$ obtained by dressing $R(x)$ with suitable shift operators:

$$L_{13}(x) = q^{-(H_1 + \frac{\sqrt{2}}{2}H_3)p} R_{13}(x) q^{\frac{\sqrt{2}}{2}H_3p}, \quad \text{with} \quad p = x \frac{\partial}{\partial x}.$$ 

In the representation $\rho_j = \rho^{(1/2)} \otimes \rho^{(j)}$ of $\mathcal{U}_q(sl_2) \otimes \mathcal{U}_q(sl_2)$,

$$\rho_j(L_{13}(x)) = \begin{pmatrix} q^{-p}q^{\frac{\sqrt{2}}{2}H} & -q^{-\frac{\sqrt{2}}{2}x^{-1}}f(xq^{\frac{\sqrt{2}}{2}H})q^{-\frac{1}{2}H}E_- \\ q^{-\frac{\sqrt{2}}{2}x}f(xq^{-\frac{\sqrt{2}}{2}H+1})q^{\frac{\sqrt{2}}{2}H}E_+ & q^{p}q^{-\frac{1}{2}H} \left[ 1 - f(xq^{-\frac{1}{2}H})f(xq^{\frac{1}{2}H-1})E_+E_- \right] \end{pmatrix}$$

(20)

with $f(x) = (q - q^{-1})/(x - x^{-1})$.

Taking the trace on the first space we get

$$\text{Tr}_1(L_{13}(x)) = q^{-p}q^{\frac{\sqrt{2}}{2}H} + q^{p}q^{-\frac{1}{2}H} \left[ 1 - f(xq^{-\frac{1}{2}H})f(xq^{\frac{1}{2}H-1})E_+E_- \right].$$

We still have to restrict this operator to the space of zero-weight vectors. In the spin $j$ representation, when $j$ is integer, this subspace is one-dimensional and the resulting Hamiltonian is scalar. Using $E_+E_-|j, 0\rangle = [j][j + 1]|j, 0\rangle$, we get

$$H_j = q^{-p} + q^{p} \left( 1 - \frac{(q - q^{-1})^2[j][j + 1]}{(x - x^{-1})(q^{-1}x - qx^{-1})} \right).$$

At the first non-trivial order of $H_j$ in the limit $q \to 1$, we recover the Calogero-Moser Hamiltonian $-\partial_x^2 + j(j + 1)/\sinh^2(x)$, with $x = \exp(z)$. Notice that the coupling constant is related to the spin of the representation.

Alternatively, introducing the function

$$c_j(x) = \frac{(q^i x - q^{-j}x^{-1})(q^{-j-1}x - q^{i+1}x^{-1})}{(x - x^{-1})(q^{-1}x - qx^{-1})},$$

the Hamiltonian $H_j$ is given by

$$H_j = q^{-p} + q^{p}c_j(x).$$
The eigenfunctions $\Psi$ of $H_j$ are the solutions of the following trigonometric $q$-deformed Lamé equation:

$$\Psi(q^{-1}x) + c_j(qx)\Psi(qx) = E\;\Psi(x).$$

An elliptic version of this equation already appeared in a different context in [14].

Integrability of the system manifests itself in the fact that we can easily solve this equation by using, for instance, the following recursive procedure. For $j=0$ the Hamiltonian $H_0 = q^p + q^{-p}$ is free; its eigenfunctions are plane waves $\Psi_0(x,k) = x^k$ with the corresponding energy $E(k) = q^k + q^{-k}$. Let us now introduce the following “shift” operator

$$D_j = q^{-p} - q^p \frac{(q^{-j}x - q^jx^{-1})(q^{-j+1}x - q^{j+1}x^{-1})}{(x-x^{-1})(q^{-1}x - qx^{-1})}$$

which satisfies

$$H_j D_j = D_j H_{j-1},$$

The eigenfunction $\Psi_j(x,k)$ of $H_j$ with energy $E(k) = q^k + q^{-k}$, are obtained by the successive action of the “shift” operator:

$$\Psi_j(x,k) = D_j \Psi_{j-1}(x,k).$$

Since the energy is even in $k$, we can start the recursion with $\Psi_0(x,k) = x^k - x^{-k}$. Then we get

$$\Psi_j(x,k) = \sum_{n=0}^{j} (-1)^n \left[ \begin{array}{ccc} j \end{array} \right]_q \frac{\prod_{r=1}^{n}(q^{-j+1}x - q^{-j+1}x^{-1})}{\prod_{r=1}^{n}(q^r x - q^r x^{-1})} (q^{k(2n-j)}x^k - q^{-k(2n-j)}x^{-k})$$

This wave function has the interesting properties that the residues at the poles $x = \pm q^{-r}$ for $1 \leq r \leq j$ all vanish, and moreover, one has $\Psi_j(x,k) = 0$ for $k = -j, -j+1, \ldots, j$. This is an analogue of the generalized exclusion principle present in the Calogero-Sutherland model [15].

6 Appendix

We give an idea of the proof of eq. (14). We adopt here a naive point of vue. We refer to [11] for a more detailed discussion. We start from an avatar of van der Waerden formula for 3-j symbols (combine eq. 3.5 and eq. 3.10 in ref. [13]):

$$\left[ \begin{array}{ccc} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{array} \right]_q = \delta_{m_1+m_2,m_3} \Delta(j_1,j_2,j_3) \frac{q^{-\frac{1}{2}(j_1+j_2-j_3)(j_1+j_2+j_3+1)+j_1m_2-j_2m_1}}{\sqrt{[2j_3+1]}} \cdot \frac{[j_1+m_1]![j_1-m_1]![j_2+m_2]![j_2-m_2]![j_3+m_3]![j_3-m_3]}{[j_1+j_2+j_3+1]} \cdot \sum_p [p]![j_1+j_2-j_3-p]![j_2-m_2-p]![j_1+m_1-p]![j_3-j_1+m_2+p]![j_3-j_2-m_1+p]$$

where

$$\Delta(j_1,j_2,j_3) = (-1)^{j_1+j_2-j_3} \sqrt{\frac{[-j_1+j_2+j_3]![j_1-j_2+j_3]![j_1+j_2-j_3]}{[j_1+j_2+j_3+1]}}$$
We take a limit $m_2 \to \infty$ such that
\[
\lim_{m_2 \to \infty} q^{m_2} = 0, \quad \lim_{m_2 \to \infty} q^{-m_2} = \infty
\]

Then, one has
\[
\frac{[\alpha \pm m_2]!}{[\beta \pm m_2]!} \sim (\mp)^{\alpha-\beta} \frac{q^{\frac{1}{2}((\alpha+\beta)(\alpha+\beta+1))}}{(q-q^{-1})^{\alpha-\beta}} q^{-2(\alpha-\beta) m_2}
\]

To perform the limit, we write the terms containing $m_2$ in the following form
\[
\sqrt{\left[ \frac{[j_2 + m_2]!}{[j_3 - j_1 + m_2]!} \right] \left[ \frac{[j_3 + m_1 + m_2]!}{[j_2 - m_2]!} \right] \left[ \frac{[j_2 - m_2]!}{[j_2 - m_2]!} \right] \left[ \frac{[j_3 - m_1 - m_2]!}{[j_3 - j_1 + m_2]!} \right]}
\sim
(-1)^{j_1+\frac{1}{2}(j_2-j_3+m_1)} \frac{q^{\frac{1}{2}(-2j_3+1)} q^{-j_2(j_1+1)+j_3(j_3+1)-j_1(j_3+j_2+1)}}{(q-q^{-1})^{j_1}} q^{-j_1 m_2}
\]
and
\[
\lim_{m_2 \to \infty} \frac{[j_2 - m_2]! [j_3 - j_1 + m_2]!}{[j_2 - m_2 - p]![j_3 - j_1 + m_2 + p]!} = (-1)^{p} q^{p(j_2+j_3-j_1+1)}
\]

This decomposition is to ensure that we get the above important sign $(-1)^{p}$ correctly. Hence

\[
\lim_{m_2 \to \infty} \left[ \frac{j_1}{m_1} \frac{j_2}{m_2} \frac{j_3}{m_1+m_2} \right] = \Delta(j_1, j_2, j_3) \frac{\sqrt{[2j_3+1] \sqrt{[j_1+m_1]! [j_1-m_1]!}}}{(q-q^{-1})^{j_1}} \times \frac{(-1)^{j_1+\frac{1}{2}(j_2-j_3+m_1)} q^{-j_2(j_1+1)+j_3(j_3+1)-j_1(j_3+j_2+1)} q^{-\frac{1}{2}m_1}}{q^{2p(j_2+j_3+1)}} \sum_{p} [p]! [j_1+j_3-j_2-p]! [j_1+m_1-p]! [j_3-j_2-m_1+p]!
\]

Comparing with eq. (13) we get eq. (14) with $j_2 = j(x)$ and $j_3 = j(x) + \sigma_1$ where $j(x)$ is given by eq. (15). Moreover we find
\[
N_{\xi}^{(j_1)} (m_1) = (-1)^{-\frac{1}{2}m_1} q^{\frac{1}{2}m_1}
\]
\[
N_{\psi}^{(j_1)} (x, \sigma_1) = (-1)^{-j_1+\frac{1}{2}(m_1-\sigma_1)} \frac{\sqrt{[j_1+\sigma_1]! [j_1-\sigma_1]!}}{\Pi_{r=1}^{j_1+\sigma_1} (1-x^{2q^{2r}})} \Delta(j_1, j(x), j(x)+\sigma_1) \sqrt{[2j(x)+1+\sigma_1]}
\]

References

[1] J.L. Gervais, A. Neveu, Novel triangle relation and absence of tachyons in Liouville string field theory, Nucl. Phys. B 238 (1984) 125.

[2] G. Felder, Conformal field theory and integrable systems associated to elliptic curves, hep-th/9407154.
[3] J. Avan, O. Babelon, E. Billey *The Gervais-Neveu-Felder equation and the quantum Calogero Moser systems*. hep/th 9505091, To appear in Commun. Math. Phys.

[4] O. Babelon, *Universal exchange algebra for Bloch waves and Liouville theory*, Commun. Math. Phys. 139 (1991) 619.

[5] V.G. Drinfeld, *Quasi-Hopf algebras*. Algebra and Analysis, 1 (1989) p. 1419. *On quasitriangular quasi-Hopf algebras and a group closely connected with $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$*. Algebra and Analysis, 2 (1990) p.829.

[6] V. Pasquier, *Etiology of IRF models*. Commun. Math. Phys. 118 (1988) p.355.

[7] O. Babelon, *Extended conformal algebra and the Yang-Baxter equation*, Phys. Lett. B 215 (1988) 523.

[8] G. Moore, N. Reshetikhin, *A comment on quantum group symmetry in conformal field theory*. Nucl. Phys. B328 (1989) p. 557.

[9] G. Felder, J. Fröhlich, J. Keller, *Braid matrices and structure constants for minimal conformal models*. Commun. Math. Phys. 124 (1989) p. 646.

[10] J.L. Gervais, *The Quantum Group Structure of 2D Gravity and Minimal Models*, Commun Math Phys 130, (1990) p. 257.

[11] E. Cremmer, J.L. Gervais, J.F. Roussel, *The Quantum Group Structure of 2D Gravity and Minimal Models II: The Genus-Zero Chiral Bootstrap*. Commun. Math. Phys. 161, (1994), p. 597.

[12] J.L. Gervais, J.F. Roussel, *Solving the strongly coupled 2D gravity II. Fractional-spin operators and topological three point functions*. Nucl. Phys. B426 (1994) p.140.

[13] A.N. Kirillov, N.Yu. Reshetikhin, *Representations of the algebra $\mathcal{U}_q(sl_2)$, $q$-orthogonal polynomials and invariants of links*. Infinite dimensional Lie algebras and groups. Advanced Study in Mathematical Physics, Vol 7, Proceedings of the 1988 Marseille Conference. World Scientific.

[14] I. Krichever, A. Zabrodin, *Spin generalization of the Ruijsenaars-Schneider model, non-abelian 2D Toda chain and representations of Sklyanin algebra*, hep-th/9505039.

[15] D. Bernard, M. Gaudin, F. Haldane, V. Pasquier, *Yang-Baxter equation in spin chains with long range interactions*. J. Phys. A: Math. Gen. 26 (1993) p.5219.