Laguerre-Gaussian modes: entangled state representation and generalized Wigner transform in quantum optics

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Abstract

By introducing a new entangled state representation, we show that the Laguerre-Gaussian (LG) mode is just the wave function of the common eigenvector of the orbital angular momentum and the total photon number operators of 2-d oscillator, which can be generated by 50:50 beam splitter with the phase difference $\phi = \pi/2$ between the reflected and transmitted fields. Based on this and using the Weyl ordering invariance under similar transforms, the Wigner representation of LG is directly obtained, which can be considered as the generalized Wigner transform of Hermite Gaussian modes.

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1 Introduction

It has been known that a Laguerre-Gaussian (LG) beam of paraxial light has a well-defined orbital angular momentum \cite{1-5}, which is useful in studying quantum entanglement \cite{6}. In Ref. \cite{3} Nienhuis and Allen employed operator algebra to describe the Laguerre-Gaussian beam, and noticed that Laguerre-Gaussian modes are laser mode analog of the angular momentum eigenstates of the isotropic 2-d harmonic oscillator. In Ref. \cite{7} Simon and Agarwal presented a phase-space description (the Wigner function) of the LG mode by exploiting the underlying phase-space symmetry. In this Letter we shall go a step further to show that LG mode is just the wave function of the common eigenvector $|n, l\rangle$ of the orbital angular momentum operator and the total photon number operator of 2-d oscillator in the entangled state representation (ESR). The ESR was constructed \cite{8, 9} based on the Einstein-Podolsky-Rosen quantum entanglement \cite{10}. It is shown that $|n, l\rangle$ can be generated by 50:50 beam splitter with the phase difference $\phi = \pi/2$ between the reflected and transmitted fields. Then we use the Weyl ordering form of the Wigner operator and the Weyl ordering’s covariance under similar transformations to directly derive the Wigner representation of LG beams, which seems economical. The marginal distributions of Wigner function (WF) are also obtained by the entangled state representation. It is found that the amplitude of marginal distribution is just the eigenfunction of the fractional Fourier transform (FrFT). In addition, LG mode can also be considered as the generalized Wigner transform of Hermite Gaussian modes by using the Schmidt decomposition of the ESR.

2 Eigenvector corresponding to Laguerre-Gaussian mode

In Ref. \cite{3} the Bosonic operator algebra of the quantum harmonic oscillator is applied to the description of Gaussian modes of a laser beam, i.e., a paraxial beam of light is described by operators’

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eigenvector equations

\[ N |n, l\rangle = n |n, l\rangle, \quad N \equiv \left( a_1^\dagger a_1 + a_2^\dagger a_2 \right), \]
\[ L |n, l\rangle = l |n, l\rangle, \quad L \equiv X_1 P_2 - X_2 P_1, \]  
(1)
since \([N, L] = 0\), where \(a_i^\dagger\) and \(a_i\) \((i = 1, 2)\) are Bose creation operator and annihilation operator; \(L\) and \(N\) are the orbital angular momentum operator and the total photon number operator of a paraxial beam of light, respectively. Using \(X_i = \left( a_i + a_i^\dagger \right) / \sqrt{2}\) and \(P_i = \left( a_i - a_i^\dagger \right) / (i \sqrt{2})\), and \([a_i, a_j^\dagger] = \delta_{ij}\), then
\[ L = i(a_2^\dagger a_1 - a_1^\dagger a_2). \]  
(2)
Here we search for the common eigenvector of \((N, L)\) in the entangled state representation. By introducing
\[ A_+ = \frac{1}{\sqrt{2}} (a_1 - ia_2), \quad A_- = \frac{1}{\sqrt{2}i} (a_1 + ia_2), \]  
(3)
which obey the commutative relation

\[ [A_+, A_-^\dagger] = 1, \quad [A_-, A_+^\dagger] = 1, \]
\[ [A_+, A_-^\dagger] = 0, \quad [A_-, A_+^\dagger] = 0, \]
(4)
one can see
\[ N = A_+^\dagger A_+ + A_-^\dagger A_-, \quad L = A_+^\dagger A_+ - A_-^\dagger A_- \ldots \]  
(5)
Now we introduce the entangled state representation in Fock space,
\[ |\eta\rangle = \exp \left\{ \frac{1}{2} |\eta|^2 + \eta A_+^\dagger - \eta^* A_-^\dagger + A_+ A_-^\dagger \right\} |00\rangle, \]  
(6)
here \(\eta = |\eta\rangle e^{i\varphi} = \eta_1 + i\eta_2\), \(|00\rangle\) is annihilated by \(A_+\) and \(A_-\). It is not difficult to see that \(|\eta\rangle\) is the common eigenvector of operators \((X_1 - X_2 - P_1 + P_2, P_1 + P_2 - X_1 - X_2)\)
\[ (X_1 - X_2 - P_1 + P_2) |\eta\rangle = 2\eta_1 |\eta\rangle, \]
\[ (P_1 + P_2 - X_1 - X_2) |\eta\rangle = 2\eta_2 |\eta\rangle. \]  
(7)
Using the normal ordering form of vacuum projector
\[ |00\rangle \langle 00| = : \exp \left( -A_+^\dagger A_+ - A_-^\dagger A_- \right) :, \]  
(8)
(where \(: :\) denotes normal ordering) and the technique of integral within an ordered product (IWOP) of operators \([11][12]\), we can prove the completeness relation and the orthonormal property of \(|\eta\rangle\),
\[ \int \frac{d^2\eta}{\pi} |\eta\rangle \langle \eta| = 1, \quad \langle \eta | \eta'| = \pi \delta (\eta - \eta'^*) \delta (\eta^* - \eta'). \]  
(9)
Thus \(|\eta\rangle\) is qualified to make up a new representation. It follows from \((5)\) and \((8)\) that
\[ A_+ |\eta\rangle = (\eta + A_+^\dagger) |\eta\rangle, \quad A_+^\dagger |\eta\rangle = \left( \frac{\partial}{\partial \eta} + \frac{\eta^*}{2} \right) |\eta\rangle, \]  
(10)
\[ A_- |\eta\rangle = (A_-^\dagger - \eta^*) |\eta\rangle, \quad A_-^\dagger |\eta\rangle = \left( -\frac{\partial}{\partial \eta^*} - \frac{\eta}{2} \right) |\eta\rangle. \]  
(11)
which lead to (denote \(r = |\eta|\) for simplicity)
\[ \langle A_+^\dagger A_+ + A_-^\dagger A_- |\eta\rangle = \left( \frac{1}{2} r^2 - 1 - 2 \frac{\partial^2}{\partial \eta \partial \eta^*} \right) |\eta\rangle \]
\[ = \left[ \frac{r^2}{2} - 1 - \frac{1}{2} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} \right) \right] |\eta\rangle \]
\[ \langle A_+^\dagger A_+ - A_-^\dagger A_- |\eta\rangle = \left( \eta \frac{\partial}{\partial \eta} - \eta^* \frac{\partial}{\partial \eta^*} \right) |\eta\rangle = -i \frac{\partial}{\partial \varphi} |\eta\rangle. \]  
(12)
Projecting Eqs. (1) onto the $|n\rangle$ representation and using (3), (10)-(20), one can obtain the following equations

\[ \langle n| l, n, l \rangle = i \frac{\partial}{\partial \varphi} \langle n| n, l \rangle, \quad (13) \]

and

\[ n \langle n| n, l \rangle = \left[ \frac{r^2}{2} - 1 - \frac{1}{2} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} \right) \right] \langle n| n, l \rangle. \quad (14) \]

Eq. (13) indicates that $\langle n| n, l \rangle \propto e^{-il\varphi}$. From the uniqueness of wave function, $e^{-il\varphi}|\varphi=0 = e^{-il\varphi}|\varphi=2\pi$, we know $l = 0, \pm 1, \pm 2 \cdots$. So letting $\langle n| n, l \rangle = R(|r)e^{-il\varphi}$ and substituting it into (14) yields

\[ \frac{d^2R}{dr^2} + \frac{1}{r} \frac{dR}{dr} + \left( -r^2 + 2(n + 1) - \frac{l^2}{r^2} \right) R = 0. \quad (15) \]

Introducing $\xi = r^2$ such that

\[ \frac{dR}{dr} = 2\sqrt{\xi} \frac{dR}{d\xi}, \quad \frac{d^2R}{dr^2} = 2 \frac{dR}{d\xi} + 4\xi \frac{d^2R}{d\xi^2}. \quad (16) \]

Eq. (15) becomes

\[ \frac{d^2R}{d\xi^2} + \frac{1}{\xi} \frac{dR}{d\xi} + \left( \frac{1}{4} \xi + \frac{n + 1}{2} \xi - \frac{l^2}{4\xi^2} \right) R = 0. \quad (17) \]

Then make the variable transform in (17)

\[ R(\xi) = e^{-\xi/2\xi|l|/2}u(\xi), \quad (18) \]

one can obtain the equation for $u(\xi)$,

\[ \xi \frac{d^2u}{d\xi^2} + (|l| + 1 - \xi) \frac{du}{d\xi} + \frac{n - |l|}{2} u = 0. \quad (19) \]

Eq. (19) is just a confluent hypergeometric equation whose solution is the associate Laguerre polynomials, $L^{|l|}_{n\rho}(\xi)$, where $n\rho = \frac{n + \rho}{2}$, $(n\rho = 0, 1, 2, \cdots)$ [13]. Thus the wave function of $|n\rangle$ in $\langle n|$ representation is given by

\[ \langle n| n, l \rangle = C_1 e^{-il\varphi}e^{-\frac{1}{2}r^2}L^{|l|}_{n\rho}(r^2), \quad (20) \]

where $C_1$ is an integral constant. The right-hand side of Eq. (20) is just the LG mode, so we reach the conclusion that the wave function of $|n\rangle, |l\rangle$ in the entangled state representation is just the LG mode, i.e., the LG mode gets its new physical meaning in quantum optics.

Next, we further derive the explicit expression of $|n\rangle, |l\rangle$. Using the completeness relation of $\langle n| \langle \rho \rangle$ and (20), we have

\[ |n, l\rangle = \int \frac{d^2q}{\pi} |\eta\rangle \langle n| n, l \rangle \]

\[ = C_1 \int \frac{d^2q}{\pi} e^{-\frac{1}{2}r^2} |\eta\rangle e^{-il\varphi}|l\rangle L^{|l|}_{n\rho}(r^2). \quad (21) \]

Then noticing the relation between two-variable Hermite polynomial [14,15] and Laguerre polynomial,

\[ H_{m,n}(\eta, \eta^*) = m! (-1)^m \eta^{n-m}L^{|m|}_{m-n}(\eta \eta^*), \quad (22) \]

where $m < n$, and the generating function of $H_{m,n}(\eta, \eta^*)$ is

\[ H_{m,n}(x, y) = \left. \frac{\partial^{m+n}}{\partial t^m \partial t'^n} \exp \left[ -tx + t'y \right] \right|_{t=t'=0}, \quad (23) \]
as well as using the integral formula \[10\]
\[
\int \frac{d^2\zeta}{\pi} \exp\left(\zeta |z|^2 + \zeta z^*\right) = -\frac{1}{\zeta}e^{-\frac{|z|^2}{\zeta^2}}, \quad \text{Re}(\zeta) < 0,
\]
we can reform Eq.\(21\) as (without loss of the generality, setting \(l > 0\) and \(m_\rho = [n + |l|]/2\))
\[
|n, l\rangle = \frac{(-1)^{n_\rho}C_1}{n_\rho!} \int \frac{d^2\eta}{\pi} H_{n_\rho,m_\rho}(\eta, \eta^*) e^{-\frac{1}{2}|\eta|^2}|\eta\rangle
= \frac{(-1)^{n_\rho}C_1}{n_\rho!} \frac{\partial^{n_\rho+m_\rho}}{\partial H^{n_\rho} \partial \eta^{m_\rho}} \exp\left[-tt' + A_1^\dagger A_1\right]
\times \int \frac{d^2\eta}{\pi} \exp\left[-|\eta|^2 + \left(A_1^\dagger + t\right) \eta + \left(t' - A_1\right) \eta^*\right]_{t'=0} |00\rangle
= \frac{(-1)^{n_\rho}C_1}{n_\rho!} \frac{\partial^{n_\rho+m_\rho}}{\partial H^{n_\rho} \partial \eta^{m_\rho}} \exp\left[A_1^\dagger t' - tA_1\right]_{t'=0} |00\rangle
= \frac{C_1}{n_\rho!} \left(A_1^\dagger\right)^{m_\rho} \left(A_1\right)^{n_\rho} |00\rangle.
\]

\[25\]

3 Generation of \(|n, l\rangle\) by Beam Splitter

Note Eq.\(13\) and
\[
A_+ = e^{\frac{i}{2} J_x} a_1^\dagger e^{-i\frac{J_x}{2}}, \quad A_1^\dagger = e^{\frac{i}{2} J_x} a_1 e^{-i\frac{J_x}{2}},
\]
\[
J_x = \frac{1}{2} \left(a_1^\dagger a_2 + a_2^\dagger a_1\right), \quad |00\rangle = e^{\frac{i}{2} J_x} |00\rangle,
\]
thus Eq.\(25\) can be further put into the following form
\[
|n, l\rangle = \frac{C_1}{n_\rho!} e^{\frac{i}{2} J_x} \left(a_1^\dagger\right)^{m_\rho} \left(a_2^\dagger\right)^{n_\rho} |00\rangle
= \frac{C_1}{n_\rho!} \sqrt{\frac{m_\rho!}{n_\rho!}} e^{\frac{i}{2} J_x} |m_\rho, n_\rho\rangle.
\]
\[27\]

It is easy to see that the normalized constant can be chosen as \(C_1 = \sqrt{n_\rho!/m_\rho!}\), which further leads to
\[
|n, l\rangle = e^{\frac{i}{2} J_x} |m_\rho, n_\rho\rangle,
\]
where \(J_x\) can be expressed by angular momenta operators \(J_+ = a_1^\dagger a_2\) and \(J_- = a_1 a_2^\dagger\), \(J_x = \frac{1}{2} (J_+ + J_-)\). \(J_+\), \(J_-\) and \(J_x\) make up a close SU(2) Lie algebra.

On the other hand, the beam splitter is one of the few experimentally accessible devices that may act as an entangler. In fact, the role of a beam splitter operator \[17,18\] is expressed by
\[
B(\theta, \phi) = \exp\left[\frac{\theta}{2} \left(a_1^\dagger a_2 e^{i\phi} - a_1 a_2^\dagger e^{-i\phi}\right)\right],
\]
with the amplitude reflection and transmission coefficients \(T = \cos \frac{\theta}{2}, R = \sin \frac{\theta}{2}\). The beam splitter gives the phase difference \(\phi\) between the reflected and transmitted fields. Comparing Eq.\[29\] with \(e^{\frac{i}{2} J_x}\) leads us to choose \(\theta = \pi/2\) (corresponding to 50:50 beam splitter) and \(\phi = \pi/2\), thus \(B(\pi/2, \pi/2)\) is just equivalent to \(e^{\frac{i}{2} J_x}\) in form. This indicates that \(|n, l\rangle\) can be generated by acting a symmetric beam splitter with \(\phi = \pi/2\) on two independent input Fock states \(|m_\rho, n_\rho\rangle \equiv |n_\rho\rangle |n_\rho\rangle\). In addition, note that \(n = m_\rho + n_\rho\), i.e., when the total number of input photons is \(n\), so the output state becomes an \((n + 1)\)-dimensional entangled state \[19\].
4 The Wigner representation

As is well-known, the Wigner quasidistribution provides with a definite phase space distribution of quantum states and is very useful in quantum statistics and quantum optics. In this section, we evaluate the Wigner representation of $|n, l\rangle$. According Ref. [20], the Wigner representation of $|n, l\rangle$ is given by

$$W_{|n, l\rangle} = \langle n, l | \Delta_1 (x_1, p_1) \Delta_2 (x_2, p_2) | n, l \rangle$$

$$= \langle m_\rho, n_\rho | e^{-i \frac{\pi}{2} J_\rho} \Delta_1 (x_1, p_1)$$

$$\times \Delta_2 (x_2, p_2) e^{i \frac{\pi}{2} J_\rho} | m_\rho, n_\rho \rangle,$$

(30)

where $\Delta_1 (x_1, p_1)$ is the single-mode Wigner operator, whose Weyl form [21] is

$$\Delta_1 (x_1, p_1) = \hat{\delta} (p_1 - P_1) \delta (x_1 - X_1),$$

(31)

where the symbol $\hat{\delta}$ denotes Weyl ordering [22]. Note that the order of Bose operators $a$ and $a^\dagger$ within $\hat{\delta}$ can be permitted. That is to say, even though $[a, a^\dagger] = 1$, we can have $\langle a a^\dagger \rangle = \langle a^\dagger a \rangle$. According to the covariance property of Weyl ordering under similar transformations [21] and

$$e^{-i \frac{\pi}{2} J_\rho} X_1 e^{i \frac{\pi}{2} J_\rho} = \frac{1}{\sqrt{2}} (X_1 - P_2),$$

$$e^{-i \frac{\pi}{2} J_\rho} P_1 e^{i \frac{\pi}{2} J_\rho} = \frac{1}{\sqrt{2}} (P_1 + X_2),$$

$$e^{-i \frac{\pi}{2} J_\rho} X_2 e^{i \frac{\pi}{2} J_\rho} = \frac{1}{\sqrt{2}} (X_2 - P_1),$$

$$e^{-i \frac{\pi}{2} J_\rho} P_2 e^{i \frac{\pi}{2} J_\rho} = \frac{1}{\sqrt{2}} (P_2 + X_1),$$

(32)

we have

$$e^{-i \frac{\pi}{2} J_\rho} \Delta_1 (x_1, p_1) \Delta_2 (x_2, p_2) e^{i \frac{\pi}{2} J_\rho}$$

$$= \hat{\delta} \left( p_1 - \frac{P_1 + X_2}{\sqrt{2}} \right) \delta \left( x_1 - \frac{X_1 - P_2}{\sqrt{2}} \right)$$

$$\times \delta \left( p_2 - \frac{P_2 + X_1}{\sqrt{2}} \right) \delta \left( x_2 - \frac{X_2 - P_1}{\sqrt{2}} \right)$$

$$= \Delta_1 \left( \frac{x_1 + p_2 - x_2}{\sqrt{2}} \right) \Delta_2 \left( \frac{x_2 + p_1 - x_1}{\sqrt{2}} \right),$$

(33)

Since the Wigner representation of number state $|m\rangle$ is well known [23],

$$W_{|m\rangle} = \langle m | \Delta_1 (x_1, p_1) | m \rangle$$

$$= \frac{(-1)^m}{\pi} e^{-\left(\frac{x_1^2 + p_1^2}{2}\right)} L_m \left[ 2 \left( \frac{x_1^2 + p_1^2}{2} \right) \right],$$

(34)

so we directly obtain the Wigner representation of L-G mode,

$$W_{|n, l\rangle} = \langle m_\rho | \Delta_1 \left( \frac{x_1 + p_2 - x_2}{\sqrt{2}} \right) \Delta_2 \left( \frac{x_2 + p_1 - x_1}{\sqrt{2}} \right) | m_\rho \rangle$$

$$\times \langle n_\rho | \Delta_2 \left( \frac{x_2 + p_1 - x_1}{\sqrt{2}} \right) \Delta_1 \left( \frac{x_1 + p_2 - x_2}{\sqrt{2}} \right) | n_\rho \rangle$$

$$= \frac{(-1)^{m_\rho + n_\rho}}{\pi^2} e^{-Q_0 L_{m_\rho} (Q_0 + Q_2) L_{n_\rho} (Q_0 - Q_2)},$$

(35)

where $Q_0 = p_1^2 + p_2^2 + x_1^2 + x_2^2$ and $Q_2 = 2p_2x_1 - 2p_1x_2$. Eq. (35) is in agreement with the result of Ref. [5][7]. Our derivation seems economical.
5 The marginal distributions and fractional Fourier transform of $|n, l\rangle$

The fractional Fourier transform (FrFT) has been paid more and more attention within different contexts of both mathematics and physics. It is also very useful tool in Fourier optics and information optics. In this section, we examine the relation between the FrFT and the marginal distributions of $W_{[n,l]}$.

For this purpose, we recall that the two-mode Wigner operator $\Delta_1 (x_1, p_1) \Delta_2 (x_2, p_2) \equiv \Delta_1 (\alpha) \Delta_2 (\beta)$ $(\alpha = (x_1 + ip_1)/\sqrt{2}, \beta = (x_1 + ip_1)/\sqrt{2})$ in entangled state representation $|\tau\rangle$. Using the IWOP technique we have shown in [22] that $\Delta_{1,2} (\sigma, \gamma)$ is just the product of two independent single-mode Wigner operators $\Delta_1 (\alpha) \Delta_2 (\beta) = \Delta_{1,2} (\sigma, \gamma)$ i.e.,

$$\Delta_{1,2} (\sigma, \gamma) = \int \frac{d^2 \tau}{\pi!} |\sigma - \tau\rangle \langle \sigma + \tau| e^{\tau^* \gamma^* - \tau \gamma},$$

(36)

where $\sigma = \alpha - \beta^*$, $\gamma = \alpha + \beta^*$ and $|\tau = \tau_1 + i\tau_2\rangle$ can be expressed in two-mode Fock space as [12][13]

$$|\tau\rangle = \exp \left\{ -\frac{1}{2} |\tau|^2 + \tau a_1^\dagger - \tau^* a_2^\dagger + a_1^\dagger a_2^\dagger \right\} |00\rangle,$$

(37)

which is the common eigenvector of $X_1 - X_2$ and $P_1 + P_2$, which obeys the eigenvector equations $(X_1 - X_2) |\tau\rangle = \sqrt{2} \tau_1 |\tau\rangle$, $(P_1 + P_2) |\tau\rangle = \sqrt{2} \tau_2 |\tau\rangle$.

Performing the integration of $\Delta_{1,2} (\sigma, \gamma)$ over $d^2 \gamma$ ($d^2 \sigma$) leads to the projection operator of the entangled state $|\tau\rangle$ $(|\xi\rangle)$

$$\int d^2 \gamma \Delta_{1,2} (\sigma, \gamma) = \frac{1}{\pi} |\tau\rangle \langle \tau|_{\gamma=\sigma},$$

$$\int d^2 \sigma \Delta_{1,2} (\sigma, \gamma) = \frac{1}{\pi} |\xi\rangle \langle \xi|_{\xi=\gamma},$$

(38)

where $|\xi\rangle$ is the conjugate state of $|\tau\rangle$. Thus the marginal distributions for quantum states $\rho$ in $(\tau_1, \tau_2)$ and $(\xi_1, \xi_2)$ phase space are given by

$$\int d^2 \sigma W (\sigma, \gamma) = \frac{1}{\pi} \langle \xi| \rho |\xi\rangle \langle \xi=\gamma, \gamma=|,$$

$$\int d^2 \gamma W (\sigma, \gamma) = \frac{1}{\pi} \langle \tau| \rho |\tau\rangle |_{\gamma=\sigma},$$

(39)

respectively. Eq. (39) shows that, for bipartite system, the marginal distributions can be calculated by evaluating the quantum average of $\rho$ in $(|\xi\rangle, |\tau\rangle)$ representations.

Now we calculate the inner-product $(\tau |n, l\rangle$. Note that $a_1^\dagger a_2, a_1 a_2^\dagger$, and $J_z = \frac{1}{2} (a_1^\dagger a_1 - a_2^\dagger a_2)$ make up a close SU(2) Lie algebra, thus $e^{i \frac{\pi}{2} J_z}$ can be decomposed as

$$e^{i \frac{\pi}{2} J_z} = e^{i a_1 a_2^\dagger} \exp \left\{ \frac{1}{2} (a_1^\dagger a_1 - a_2^\dagger a_2) \ln 2 \right\} e^{ia_2^\dagger a_1},$$

(40)

then we have

$$(\tau |n, l\rangle = \sqrt{\prod_{\rho=0}^{m} \sum_{k=0}^{m} \sqrt{2}^{m_{\rho} - n_{\rho} - 2k}} \times \frac{(n_{\rho} + k)!}{k!(m_{\rho} - k)!} \sum_{j=0}^{k+j} \frac{(m_{\rho} - k + j)!}{j!} \sqrt{(m_{\rho} + k - j)!} \times \langle \tau |n_{\rho} - k + j, n_{\rho} + k - j\rangle.$$
Using the generating function of $H_{m,n}$, we have

$$
\langle \tau' | m, n \rangle = \frac{(-1)^n}{\sqrt{m!n!}} H_{m,n} (\tau^\alpha, \tau) e^{-|\tau'|^2/2}.
$$

(42)

Substituting Eq. (42) into Eq. (41) leads to

$$
\langle \tau | n, l \rangle = (-)^{m+1} \sqrt{2^{m-n} m! \frac{m^m}{n!}} \sum_{k=0}^{m} \frac{(n+k)!}{2^k k! (m-k)!} \frac{(-i)^k}{k!} \left[ \frac{H_{m-k,n+k-j} (\sigma^\alpha, \tau)}{j! (n+k-j)!} \right]^2.
$$

(43)

Thus the marginal distribution is

$$
\int d^2 \gamma W (\sigma, \gamma) = e^{-|\sigma|^2/2} \sum_{m,n} \frac{m!}{n!} \frac{(2m-n)!}{2^m m!} \sum_{k=0}^{m} \frac{(-i)^k}{k!} \left[ \frac{H_{m-k,n+k-j} (\sigma^\alpha, \tau)}{j! (n+k-j)!} \right]^2.
$$

(44)

Due to the presence of sum polynomial, the marginal distribution is not Gaussian. In a similar way, one can obtain the other marginal distribution in $\gamma$ direction.

Before the end of this section, we mention the relation between the FrFT and marginal distribution. In Ref. [24] we have proved that, in the context of quantum optics, the FrFT can be described as the matrix element of fractional operator $-ia_1 (a_1^\dagger a_2^\dagger + a_2 a_1^\dagger)$ between $\langle \tau' \rangle$ and $|f\rangle$, i.e.,

$$
\mathcal{F}_\alpha \left[ \langle \tau' \rangle \right] = \frac{e^{i(\alpha - \frac{\tau^\alpha}{2})}}{2\sin \alpha} \int \frac{d^2 \tau'}{\pi} \exp \left[ \frac{i(|\tau'|^2 + |\tau|^2)}{2\tan \alpha} - \frac{i(\tau^\alpha \tau' + \tau'^\alpha \tau)}{2\tan \alpha} \right] f (\tau') = \langle \tau | \exp \left[ -i\alpha \left( a_1^\dagger a_1 + a_2^\dagger a_2 \right) \right] | f \rangle,
$$

where $f (\tau') = \langle \tau' | f \rangle$. When $|f\rangle = e^{i\frac{\tau^\alpha}{2}\sigma} |m, n\rangle$, the corresponding FrFT is

$$
\mathcal{F}_\alpha \left[ \langle \tau' | n, l \rangle \right] = \langle \tau | \exp \left[ -i\alpha \left( a_1^\dagger a_1 + a_2^\dagger a_2 \right) \right] e^{i\frac{\tau^\alpha}{2}\sigma} |m, n\rangle
$$

(45)

where we have used $e^{-\alpha a_1^\dagger a_1} e^{ia_1 a_2^\dagger} = a_1 e^{i\alpha}$. Eq. (45) implies that the eigenvalues of FrFT can also be $\langle \tau | n, l \rangle$ with the eigenvalue being $e^{-i\alpha (m+n)}$, which is the superposition (11) of two-variable Hermite polynomials. In Ref. [25], we have proved that the two-variable Hermite polynomials (TVHP) is just the eigenfunction of the FrFT in complex form by using the IWOP technique and the bipartite entangled state representations. Here, we should emphasize that for any unitary two-mode operators $U$ obeying the relation $\exp(-ia_1 a_2^\dagger + a_2 a_1^\dagger) U \exp[ia_1 a_1^\dagger + a_2^\dagger a_2] = U$, the wave function $\langle \tau | U | m, n\rangle$ is the eigenfunction of FrFT with the eigenvalue being $e^{-i\alpha (m+n)}$ [2].

Combining Eqs. (45) and (50), one can obtain a simple formula connecting the FrFT and the marginal distribution of $W_{(n,l)}$,

$$
\int d^2 \gamma W (\sigma, \gamma) = \frac{1}{\pi} |\mathcal{F}_\alpha \left[ \langle \tau = \sigma | n, l \rangle \right]|^2,
$$

(46)

which is $\alpha$–independent. Thus we can also obtain the marginal distribution by the FrFT.
6 L-G mode as generalized Wigner transform of Hermit-Gaussian modes

In this section we shall reveal the relation between the L-G mode and the single variable Hermit-Gaussian (H-G) modes. Note that by taking the Fourier transformation of $|\tau\rangle$ with regard to $\tau_2$ followed by the inverse Fourier transformation, we can recover the entangled state $|\tau\rangle$. In another word, $|\tau=\tau_1+i\tau_2\rangle$ can be decomposed into

$$|\tau\rangle = \int_{-\infty}^{\infty} dx e^{ix\sqrt{\tau_2}} \left| x + \frac{\tau_1}{\sqrt{2}} \right\rangle \otimes \left| x - \frac{\tau_1}{\sqrt{2}} \right\rangle,$$

in which $|x\rangle$ ($i=1, 2$) are the coordinate eigenvectors. Eq. (47) is called the Schmidt decomposition of $|\tau\rangle$ and indicates $|\tau\rangle$ is an entangled state [26].

Eq. (47) leads to

$$\langle m, n | \tau \rangle = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dxe^{ix\sqrt{\tau_2}} \langle m | x + \frac{\tau_1}{\sqrt{2}} \rangle \langle n | x - \frac{\tau_1}{\sqrt{2}} \rangle.$$  

(48)

It is interesting to notice that the integration [...] in Eq. (48) is similar to the single-mode Wigner operator,

$$\Delta (x, p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} du e^{-iup} \left| x - \frac{u}{2} \right\rangle \langle x + \frac{u}{2} |,$$

(49)

and the left hand side of Eq. (48) corresponds to TVHP mode (L-G mode). It might be expected that the L-G can be expressed in terms of the generalized Wigner transform (GWT). Actually, after making variable replacement, Eq. (48) can be rewritten as

$$\langle m, n | \tau \rangle = \pi \langle m | \Delta \left( \frac{\tau_1}{\sqrt{2}}, \frac{\tau_2}{\sqrt{2}} \right) (-1)^n |n\rangle.$$  

(50)

If we introduce the following GWT,

$$W_g [f, v] (x, p) = \langle f | \Delta (x, p) | v \rangle,$$

(51)

which reduces to the usual Wigner transform under the condition $|v\rangle = |f\rangle$, while for $|v\rangle = (-1)^n |n\rangle$ and $|f\rangle = |m\rangle$ Eq. (51) becomes the right hand side of Eq. (50), which corresponds to the GWT.

On the other hand, note that Eqs. (22) and (42), the left hand side of Eq. (50) can be put into (without loss of the generality, letting $m < n$)

$$\langle m, n | \tau \rangle = \frac{(-1)^n}{\sqrt{m! n!}} H_{m,n} (\tau, \tau^*) e^{-|\tau|^2/2}$$

$$= (-1)^{n+m} \sqrt{\frac{n!}{m!}} \tau^{n-m} L_{m}^{n-m} (\tau^*) e^{-|\tau|^2/2},$$  

(52)

which indicates that the left hand side of Eq. (50) is just corresponding to the L-G mode, as well as

$$\langle m | x \rangle = \frac{e^{-x^2/2}}{\sqrt{2^m m! \sqrt{\pi}}} H_m (x) \equiv h_m (x),$$

(53)
where $H_m(x)$ is single variable Hermite polynomial, and $h_m(x)$ just corresponds to the H-G mode, we have

$$
\sqrt{\frac{m!}{n!}} e^{\frac{-m-n}{e^{\frac{1}{2}}} \left( \tau \tau^* \right)} e^{-|\tau|^2/2}
$$

$$
= (-1)^m \int_{-\infty}^{\infty} du e^{-iu^2/2} h_m \left( \frac{\tau_1}{\sqrt{2}} - \frac{u}{2} \right) h_n \left( \frac{\tau_1}{\sqrt{2}} + \frac{u}{2} \right)
$$

$$
= (-1)^m \pi W_g [h_m, h_n] \left( \tau_1/\sqrt{2}, \tau_2/\sqrt{2} \right). \quad (54)
$$

Thus, we can conclude that the L-G mode can be obtained by the GWT of two single-variable H-G modes. In addition, we should point out that the L-G mode can also be generated by windowed Fourier transform (which is often used in signal process) of two single-variable H-G modes by noticing the second line of Eq.(47).

In summary, we have endowed the Laguerre-Gaussian (LG) mode with new physical meaning in quantum optics, i.e., we find that it is just the wave function of the common eigenvector of the orbital angular momentum and the total photon number operators of 2-d oscillator in the entangled state representation. The common eigenvector can be obtained by using beam splitter with the phase difference $\phi = \pi/2$ between the reflected and transmitted fields. With the aid of the Weyl ordering invariance under similar transforms, the Wigner representation of LG is directly obtained. It is shown that its marginal distributions can be calculated by the FrFT. In addition, L-G mode can also be considered as the generalized Wigner transform of Hermite Gaussian modes by using the Schmidt decomposition of the entangled state representation.

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