Tensor Rank and Other Multipartite Entanglement Measures of Graph States

Louis Schatzki,1,2, * Linjian Ma,3,2 Edgar Solomoni,3,2 and Eric Chitambar1,2, *

1Department of Electrical and Computer Engineering, University of Illinois at Urbana-Champaign
2Illinois Quantum Information Science and Technology (IQUIST) Center, University of Illinois Urbana-Champaign
3Department of Computer Science, University of Illinois at Urbana-Champaign

Graph states play an important role in quantum information theory through their connection to measurement-based computing and error correction. Prior work has revealed elegant connections between the graph structure of these states and their multipartite entanglement content. We continue this line of investigation by identifying additional entanglement properties for certain types of graph states. From the perspective of tensor theory, we tighten both upper and lower bounds on the tensor rank of odd ring states \((R_{2n+1})\) to read \(2^n + 1 \leq \text{rank}(R_{2n+1}) \leq 3 \cdot 2^{n-1}\). Next, we show that several multipartite extensions of bipartite entanglement measures are dichotomous for graph states based on the connectivity of the corresponding graph. Lastly, we give a simple graph rule for computing the \(n\)-tangle \(\tau_n\).

**I. INTRODUCTION**

Entanglement is one of the defining properties of quantum systems [1] and has been recognized as a fundamental resource for quantum information processing [2, 3]. A pure state is considered to be entangled if it cannot be written in the form \(|\psi\rangle = \bigotimes_{i=1}^{n} |\psi^{(i)}\rangle\). Similarly, a mixed state is entangled if it cannot be written as \(\rho = \sum_k p_k \bigotimes_{i=1}^{n} \rho_k^{(i)}\).

Quantifying the amount of entanglement in a quantum state is not always straightforward. For pure bipartite systems, the Schmidt decomposition and resulting spectra fully characterize the entanglement properties and transformations under local operations and classical communication (LOCC) [2, 4, 5]. The Schmidt decomposition of a pure state takes the form \(|\psi\rangle = \sum_i \sqrt{\mu_i} |u_i\rangle |v_i\rangle\), where \(\mu = \{|\mu_i\rangle\}\) and \(\{|v_i\rangle\}\) are sets of orthogonal states. Further, a variety of entanglement measures are known, i.e. functionals \(E(\rho)\) that are non-increasing (on average) under LOCC and \(E(\rho) = 0\) if \(\rho\) is a separable state [6]. Examples include the entanglement of formation [7], distillable entanglement [8], negativity [9, 10], geometric measure [11], and concurrence [12, 13]. However, the picture grows significantly more complicated when considering multipartite entanglement, as we discuss below.

In this work we consider the amount and form of multipartite entanglement that arises in a class of quantum states known as graph states. These are of particular interest due to their application in measurement based quantum computing [14, 15], secret sharing [17], and stabilizer computation simulation [18]. Further, by studying the entanglement properties of graph states, we actually quantify the entanglement of the larger set of stabilizer states. This follows from the fact that every stabilizer state is equivalent under local unitaries to at least one graph state [19, 20]. As entanglement measures are invariant under local unitaries, one thus need only consider graph states to analyze all stabilizer states.

In this work we focus on the tensor rank and the GME concurrence, negativity, and geometric measures of entanglement. Our main contributions to the study of graph states are twofold. First, we consider ring states of \(2n+1\) qubits and sharpen the bound on tensor rank from \(2^n \leq \text{rank}(R_{2n+1}) \leq 2^n+1\) to \(2^n + 1 \leq \text{rank}(R_{2n+1}) \leq 3 \cdot 2^{n-1}\). While this may seem like incremental progress, we stress that computing the tensor rank is a very challenging problem, and any progress in this direction is noteworthy. Indeed, the analysis we employ goes beyond the bipartite bounding techniques of previous approaches. This work thus contributes to the steadily growing research on the tensor rank of multipartite entangled states [21–29]. Operationally, the improved bounds help better characterize the amount of entanglement needed to generate ring states using LOCC. Second, we study the GME concurrence, negativity, and geometric measure for general graph states. These are shown to be sharply dichotomous and having a constant value for all connected graphs. Before presenting our results, we briefly review the main concepts considered in this paper.

**A. Schmidt Measure and Tensor Decomposition**

Any \(N\)-party pure state \(|\psi\rangle \in \mathcal{H}^{(1)} \otimes \cdots \otimes \mathcal{H}^{(N)}\) can be represented as

\[
|\psi\rangle = \sum_{i=1}^{R} \mu_i |\psi^{(i)}_1 \rangle \otimes \cdots \otimes |\psi^{(i)}_N \rangle ,
\]

(1)

where each \(|\psi^{(i)}_j\rangle \in \mathcal{H}^{(j)}\). When \(|\psi\rangle\) is viewed as an \(N\)-dimensional tensor, Eq. (1) is also known as a canonical polyadic (CP) tensor decomposition [30, 31]. The CP rank \(r = \text{rank}(|\psi\rangle)\) of a tensor is defined as the smallest \(R\) such that (1) can be satisfied. The CP rank is also known as the tensor rank, and we will use both types of terminology throughout this paper. In general, finding the CP rank of a tensor is NP-hard [32].

* louisms2@illinois.edu
Different from the matrix case, for tensors the best rank-$R$ approximation may not exist. And there exists tensors that can be approximated arbitrarily well by rank-$R$ tensors where $R \leq \text{rank}(\rho)$. In this case, border rank [33, 34] is defined as the minimum number of rank-one tensors that are sufficient to approximate the given tensor with arbitrarily small error.

The tensor rank is a bona fide entanglement measure [21] that is particularly useful studying state transformations under stochastic local operations and classical communication (SLOCC). These are transformations such that $|\psi\rangle \xrightarrow{\text{SLOCC}} |\phi\rangle$ with some non-zero probability (and is thus a generalization of LOCC). It is known that if $|\psi\rangle \xrightarrow{\text{SLOCC}} |\phi\rangle$, then $\text{rank}(\psi) \geq \text{rank}(\phi)$ [22]. Note that we can characterize SLOCC equivalence (i.e. $|\psi\rangle \xrightarrow{\text{SLOCC}} |\phi\rangle$ and $|\phi\rangle \xrightarrow{\text{SLOCC}} |\psi\rangle$) via invertible operators:

$$|\psi\rangle = A_1 \otimes A_2 \otimes \ldots \otimes A_n |\phi\rangle,$$

implying that $\text{rank}(\psi) = \text{rank}(\phi)$. Lastly, we note that the tensor rank relates to entanglement cost. In particular, a generalized d-dimensional GHZ state (or any equivalent state) can be converted to an arbitrary state $|\psi\rangle$ iff $d \geq \text{rank}(\psi)$ via SLOCC [23]. This provides an operational meaning to the tensor rank in terms of the entanglement resources needed to build $|\psi\rangle$ using GHZ states in the distributed setting.

### B. Measures of Genuine Multipartite Entanglement

An $N$-partite pure state $|\psi\rangle$ is said to have genuine multipartite entanglement (GME) if it is not a product state under any bipartition $A\overline{A}$ [35, 36]; i.e. $|\psi\rangle \neq |\alpha\rangle \otimes |\beta\rangle$, where $|\alpha\rangle$ is held by parties in $A$ and $|\beta\rangle$ is held by parties $\overline{A}$. States for which $|\psi\rangle = |\alpha\rangle \otimes |\beta\rangle$ are called biseparable, and in general it may be desirable for a multipartite entanglement measure to capture how close a state is to being biseparable. Accordingly, one can define measures via minimization over all possible bipartitions. That is, given some bipartite entanglement measure $E(|\phi\rangle)$, $|\phi\rangle \in \mathcal{H}_{A} \otimes \mathcal{H}_{B}$, we take the multipartite extension to be $E_{A}(|\psi\rangle)$, where $E_{A}$ is the measure $E$ evaluated according to partition $A\overline{A}$. Note that $E(|\psi\rangle) = 0$ if $|\psi\rangle$ is biseparable according to some partition. Thus, this multipartite extension is faithful with respect to GME. In this work we consider, beyond tensor rank, the following:

- **GME concurrence** [37],

$$C_{\text{GME}}(\rho) = \min_{A} \sqrt{2(1 - \text{Tr}[\rho_{A}^{2}])},$$

- **GME negativity** [9],

$$N_{\text{GME}}(\rho) = \min_{A} \left\{ \frac{1}{2} \left( \| \rho^{T_{A}} \|_{1} - 1 \right) \right\},$$

- and GME geometric measure [36],

$$G_{\text{GME}}(\rho) = \min_{A} \left( 1 - \max_{i} \mu_{i} \right),$$

where $\| \cdot \|_{1}$ denotes the Schatten 1-norm and $\rho^{T_{A}} = I_{\overline{A}} \otimes T_{A}(\rho)$ is the partial transpose, and $\mu_{i}$ are the Schmidt coefficients from the Schmidt decomposition according to partition $A\overline{A}$. The geometric measure for bipartite systems takes this form, but this is different than the general definition [11].

Lastly, we also evaluate the $n$-tangle $\tau_{n}$ [38, 39] on graph states. For pure states of even numbers of qubits, this is defined as

$$\tau_{n}(|\psi\rangle) = \| |\psi\rangle\tilde{\psi} \|^2,$$

where $|\tilde{\psi}\rangle = \sigma_{y}^{\otimes n} |\psi^{*}\rangle$ and $|\psi^{*}\rangle$ indicates the complex conjugate. The $n$-tangle is the size of a quadratic SLOCC invariant and can thus be used to distinguish between types of multipartite entangled states [40].

### C. Graph States

Graph states are quantum states corresponding to some graph $G = (V, E)$, where $V$ is the vertex set and $E$ is the edge set with corresponding adjacency matrix $\Gamma$ [14, 41]. There are two equivalent ways to think of graph states. The first is operational in the sense that it provides a formula for preparing the state given a graph:

$$|G\rangle = \prod_{(a, b) \in E} U^{(a, b)} |+\rangle^{\otimes |V|},$$

where

$$U^{(a, b)} = |0\rangle \langle 0|^{(a)} \otimes I^{(b)} + |1\rangle \langle 1|^{(a)} \otimes \sigma_{z}^{(b)}$$

is a controlled $Z$ operation on qubits $a$ and $b$, and

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad |\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

forms the Hadamard basis. Thus, given a graph, the graph state initialize $|V\rangle$ qubits in the state $|+\rangle^{\otimes |V|}$ and, for each edge, apply a controlled $Z$ between the corresponding qubits.

Graph states can be equivalently thought of as stabilizer states [42]. Here the stabilizers are $S_{a} = \sigma_{x}^{(a)} \prod_{b \in N_{a}} \sigma_{z}^{(b)}$, where $N_{a}$ is the neighborhood of vertex $a$. As there are $|V|$ qubits and stabilizers, $|G\rangle$ is the unique state stabilized by all $S_{a}$.

Also note that a basis for $\mathcal{H} = \bigotimes_{i=1}^{n} \mathcal{H}_{2}^{(i)}$ can be constructed given a graph $G$ [15]:

$$|G_{s}\rangle = \sigma_{z}^{a} |G\rangle = \prod_{(a, b) \in E} U^{(a, b)} \bigotimes_{i=1}^{n} (\sigma_{z}^{(i)})^{s_{i}} |+\rangle^{(i)}.$$
It is clear that there are $2^n$ such orthogonal states and thus they form a basis. Further, one can think of $s$ as flipping the eigenvalues of stabilizers $S_a$ from $+1$ to $-1$. Going forward, we will denote these as graph basis states. We will later use a result from [42] that the partial trace to a subspace of dimension $\Gamma$ is applied to the red vertex, adding or removing the edge connecting the two other vertices.

While these bounds may not be tight, it is often possible to use complementation rules to find locally equivalent graphs for which these bounds improve. It is known that the full orbit of any graph state under local clifford operations can be found via local complementations [19, 42]. That is, for some vertex $a \in V$, complement the subgraph given by the neighborhood $N_a$ (Fig. 1). These rules have been used to classify all graph states of up to 8 qubits [42–44]. Further, classes of two-colorable graphs corresponding to states of maximal schmidt measure are known [45]. However, odd rings, corresponding to non two-colorable graphs, lead to loose bounds.

Line states (Fig. 2), also known as one-dimensional cluster states, are those with one-dimensional nearest neighbor connections. We will write $|L_n\rangle$ to denote a line state on $n$ qubits. We will find line states to be useful in proving an upper bound on the rank of ring states. An explicit construction of a minimal CP decomposition of line states is given in the appendix.

**Lemma I.2.** 

$$\text{rank}(|L_n\rangle) = 2^{\frac{n}{2}}.$$  

**Proof.** This readily follows from the mentioned graph theoretic tools. See [42] for details. □

For any even ring $|R_{2n}\rangle$, it’s known that the lower bound equals the upper bound, thus the CP rank is $2^n$ [42]. For any odd ring $|R_{2n+1}\rangle$, it is known that $2^n \leq \text{rank}(|R_{2n+1}\rangle) \leq 2^{n+1}$, coming from the rank of the adjacency matrix and minimal vertex cover. Any tightening of these bounds will therefore require a new type of analysis not based on the latter graph-theoretic concepts.

**II. THE TENSOR RANK OF RING STATES**

In this paper we improve the odd ring CP rank bounds as stated in the following theorem.

**Theorem II.1.** The CP rank of the graph state corresponding to any odd ring $|R_{2n+1}\rangle$, is bounded by

$$2^n + 1 \leq \text{rank}(|R_{2n+1}\rangle) \leq 3 \cdot 2^{n-1}. (15)$$

**Proof.** This readily follows from the mentioned graph theoretic tools. See [42] for details. □
We will break the proof into two propositions corresponding to the upper and lower bounds. To help clarify the arguments these are explicitly laid out for $|R_7\rangle$ in the appendix.

A. Upper Bound Analysis

In this section we provide a CP rank upper bound of $3 \cdot 2^{n-1}$ for odd ring graph states $|R_{2n+1}\rangle$. Throughout the proof, we let

$$P_0 = |0\rangle \langle 0|, P_1 = |1\rangle \langle 1|,$$

and let $U^{(a,b)}$ denote a controlled $Z$ operation where the $a$th qubit is the controlling qubit and the $b$th qubit is the controlled qubit. We have

$$U^{(a,b)} = P_0^{(a)} \otimes I^{(b)} + P_1^{(a)} \otimes \sigma_z^{(b)} = I^{(a)} \otimes \sigma_z^{(b)} + 2P_0^{(a)} \otimes P_1^{(b)}. \quad (16)$$

Below we show the main statement.

**Proposition II.2** (CP rank upper bound for odd ring graph states). The CP rank of any odd ring state $|R_{2n+1}\rangle$ is upper bounded by

$$\text{rank}(|R_{2n+1}\rangle) \leq 3 \cdot 2^{n-1}. \quad (17)$$

**Proof.** For the case with $n = 1$, we can easily verify that

$$|R_3\rangle = |+ + + \rangle + \frac{1}{\sqrt{2}} |001\rangle - \frac{1}{\sqrt{2}} |110\rangle,$$

thus satisfying the upper bound. Below we show the cases with $n \geq 2$.

Based on (16) and the fact that

$$|R_{2n+1}\rangle = U^{(1,2n+1)} |L_{2n+1}\rangle,$$

we have

$$|R_{2n+1}\rangle = U^{(1,2n+1)} |L_{2n+1}\rangle = I^{(1)} \otimes \sigma_z^{(2n+1)} |L_{2n+1}\rangle + 2P_0^{(1)} \otimes P_1^{(2n+1)} |L_{2n+1}\rangle.$$

The CP rank of the term $I^{(1)} \otimes \sigma_z^{(2n+1)} |L_{2n+1}\rangle$ is $2^n$ since the CP rank of $|L_{2n+1}\rangle$ is $2^n$. Define the state

$$|\phi_{2n+1}\rangle = P_0^{(1)} |L_{2n+1}\rangle.$$ 

(18)

Based on Lemma II.3 and Lemma II.4 below, the CP rank of the term $P_0^{(2n+1)} |\phi_{2n+1}\rangle$ for all integers $n \geq 2$ is upper-bounded by $2^{n-1}$, thus proving the statement. \qed

Below we present Lemma II.3 and Lemma II.4, which upper-bound the CP rank of $P_0^{(2n+1)} |\phi_{2n+1}\rangle$ and $P_1^{(2n+1)} |\phi_{2n+1}\rangle$ for all integers $n \geq 2$. In our analysis below, we define a generalized controlled gate

$$CZZ^{(i,j,k)} := U^{(i,j)} U^{(i,k)} = P_0^{(i)} \otimes I^{(j)} \otimes I^{(k)} + P_1^{(i)} \otimes \sigma_z^{(j)} \otimes \sigma_z^{(k)},$$

whose CP rank is also 2. The line state $|L_{2n+1}\rangle$ can be expressed as

$$|L_{2n+1}\rangle = \prod_{i=1}^{n} U^{(2i,2i-1)} U^{(2i,2i+1)} |+\rangle^{(1,\ldots,2n+1)} = \prod_{i=1}^{n} CZZ^{(2i,2i-1,2i+1)} |+\rangle^{(1,\ldots,2n+1)}. \quad (19)$$

**Lemma II.3.** When $n = 2$, the ranks of both $P_0^{(2n+1)} |\phi_{2n+1}\rangle$ and $P_1^{(2n+1)} |\phi_{2n+1}\rangle$ with $|\phi_{2n+1}\rangle$ defined in (18) are bounded by 2.

**Proof.** For $n = 2$, 

$$|L_{2n+1}\rangle = |L_5\rangle = CZZ^{(4,3,5)} CZZ^{(2,1,3)} |+\rangle^{(1,\ldots,5)} = (I \otimes P_0 \otimes I \otimes P_0 \otimes I + I \otimes P_0 \otimes \sigma_z \otimes P_1 \otimes \sigma_z) |+\rangle^{(1,\ldots,5)} + (\sigma_z \otimes P_1 \otimes \sigma_z \otimes P_0 \otimes I + \sigma_z \otimes P_0 \otimes I \otimes P_1 \otimes \sigma_z) |+\rangle^{(1,\ldots,5)} = \frac{1}{2} |+0+0+\rangle + \frac{1}{2} |+0-1+\rangle + \frac{1}{2} |-1-0+\rangle + \frac{1}{2} |-1-1-\rangle,$$

thus we have

$$|\phi_5\rangle = P_0^{(1)} |L_5\rangle = \frac{1}{2 \sqrt{2}} |0\rangle \left( |0+0+\rangle + |0-1-\rangle + |1-0+\rangle + |1+1-\rangle \right) \quad (20)$$

$$+ \frac{1}{4} |0\rangle \left( |0+0+\rangle + |0-1-\rangle + |1-0+\rangle + |1+1+\rangle \right) \left( |0\rangle - |0+0-\rangle - |0-1-\rangle - |1-0+\rangle - |1+1+\rangle \right) \left( |0\rangle + |0+0+\rangle + |0-1-\rangle + |1-0+\rangle + |1+1+\rangle \right).$$

Above expressions for $P_0^{(5)} |\phi_5\rangle$ and $P_1^{(5)} |\phi_5\rangle$ can be rewritten as follows,

$$P_0^{(5)} |\phi_5\rangle = \frac{1}{2 \sqrt{2}} |0\rangle \left( |+0+\rangle + |-1-\rangle \right) |0\rangle,$$

$$P_1^{(5)} |\phi_5\rangle = \frac{1}{2 \sqrt{2}} |0\rangle \left( |+0-\rangle + |-1+\rangle \right) |1\rangle,$$

thus the CP ranks are bounded by 2. \qed

**Lemma II.4.** When $n \geq 2$, the CP ranks of both states $P_0^{(2n+1)} |\phi_{2n+1}\rangle$ and $P_1^{(2n+1)} |\phi_{2n+1}\rangle$ are bounded by $2^{n-1}$.

**Proof.** We argue by induction on $n$. Assume that the ranks of both $P_0^{(2n+1)} |\phi_{2n+1}\rangle$ and $P_1^{(2n+1)} |\phi_{2n+1}\rangle$ are bounded by $2^{n-1}$. We will show that the CP ranks of both states $P_0^{(2n+3)} |\phi_{2n+3}\rangle$ and $P_1^{(2n+3)} |\phi_{2n+3}\rangle$ are bounded by $2^n$. 


\(|\phi_{2n+3}\rangle\) can be rewritten as follows,
\[
|\phi_{2n+3}\rangle = P_0^{(1)} |L_{2n+3}\rangle \\
= P_0^{(1)} CZZ(2n+2,2n+1,2n+3) |L_{2n+1}\rangle |+\rangle |+\rangle \\
= CZZ(2n+2,2n+1,2n+3) P_0^{(1)} |L_{2n+1}\rangle |+\rangle |+\rangle \\
= CZZ(2n+2,2n+1,2n+3) P_0^{(1)} |\phi_{2n+1}\rangle |+\rangle \\
+ CZZ(2n+2,2n+1,2n+3) P_1^{(1)} |\phi_{2n+1}\rangle |+\rangle \\
= \frac{1}{\sqrt{2}} P_0^{(2n+1)} |\phi_{2n+1}\rangle \left( |0\rangle + |1\rangle \right) \\
+ \frac{1}{\sqrt{2}} P_0^{(2n+1)} |\phi_{2n+1}\rangle \left( |0\rangle - |1\rangle \right). \tag{20}
\]

Note that the third equality comes from the commutativity of \(CZZ(2n+2,2n+1,2n+3), P_0^{(1)}\). Based on the transformation
\[
|0\rangle + |1\rangle = |+\rangle + |1\rangle,
|0\rangle - |1\rangle = |+\rangle + |1\rangle,
\]
(20) can be rewritten as
\[
|\phi_{2n+3}\rangle = \frac{1}{\sqrt{2}} P_0^{(2n+1)} |\phi_{2n+1}\rangle \left( |0\rangle + |1\rangle \right) \\
+ \frac{1}{\sqrt{2}} P_0^{(2n+1)} |\phi_{2n+1}\rangle \left( |+\rangle + |0\rangle \right) \\
= \frac{1}{\sqrt{2}} \left( P_0^{(2n+1)} |\phi_{2n+1}\rangle |+\rangle + P_1^{(2n+1)} |\phi_{2n+1}\rangle |+\rangle \right) |0\rangle \\
+ \frac{1}{\sqrt{2}} \left( P_0^{(2n+1)} |\phi_{2n+1}\rangle |+\rangle + P_1^{(2n+1)} |\phi_{2n+1}\rangle |+\rangle \right) |1\rangle.
\]

It can be easily seen that the CP ranks of both states \(P_0^{(2n+1)} |\phi_{2n+3}\rangle\) and \(P_1^{(2n+1)} |\phi_{2n+3}\rangle\) are bounded by \(2^n\). Since the rank upper bounds for the base case \((n = 2)\) has been shown in Lemma II.3, the lemma is proved.

**B. Lower Bound Analysis**

We now turn to the lower bound. Recall that for any graph \(G\) and subset of vertices \(A \subseteq V\), the graph state \((G)\) can be expressed via Eq. 10 as
\[
|G\rangle = \frac{1}{\sqrt{2^{|A|}}} \sum_{\mathbf{z} \in \mathbb{Z}^{|A|}} (-1)^{|A|} |\mathbf{z}\rangle^{(A)} U(\mathbf{z}) |G - A\rangle, \tag{21}
\]
where \(U(\mathbf{z}) = \prod_{a \in A} (\prod_{b \in N_a} \sigma_z^{(b)}) \mathbf{z}_a\). Here and later we represent \(\mathbf{z}\) as \((z_2, z_4, \ldots, z_{2n})\). Consider splitting the ring into \(n\) even and \(n + 1\) odd vertices, denoting \(A = \{2, 4, \ldots, 2n\}\) and \(\overline{A} = \{1, 3, \ldots, 2n+1\}\). Now, the density matrix of \(\overline{A}\) in \(R_{2n+1}\) is, using Eq. (21),
\[
\rho(\overline{A}) = \frac{1}{2^n} \sum_{\mathbf{z} \in \mathbb{Z}^n_{2}} U(\mathbf{z}) |R_{2n+1} - A\rangle \langle R_{2n+1} - A| U(\mathbf{z})^\dagger,
\]
since for each \(a \in A, N_a \subseteq \overline{A}\). The above density matrix decomposition is an eigendecomposition as a consequence of the following lemma.

**Lemma II.5.** The states
\[
|\epsilon_{\mathbf{z}}\rangle := U(\mathbf{z}) |R_{2n+1} - A\rangle \quad \forall \mathbf{z} \in \mathbb{Z}^n_{2}\text{ are orthonormal and thus eigenvectors of } \rho(\overline{A}).
\]

**Proof.** We can decompose the ring state into a product state and a two-qubit line state,
\[
|R_{2n+1} - A\rangle = \bigotimes_{k=2,4,\ldots,2(n-1)} |\rangle^{(k+1)} |L_2^{(1,2n+1)}\rangle,
\]
where \(|L_2^{(1,2n+1)}\rangle = U(2n+1) |\rangle^{(1)} \otimes |\rangle^{(2n+1)}\). From the definition of \(U(\mathbf{z})\), for any \((z_2, z_4, \ldots, z_{2n}) \in \mathbb{Z}^n_{2}\) we can then write
\[
|\epsilon_{\mathbf{z}}\rangle = |\phi_{\mathbf{z}}\rangle \otimes (\sigma_{z_2}^{0} \otimes \sigma_{z_4}^{0} |L_2^{(1,2n+1)}\rangle).
\]

We first show that the product states \(|\phi_{\mathbf{z}}\rangle\) are orthogonal except when \(\mathbf{z}' = \mathbf{z}\), where \(\mathbf{z}\) denotes the bitwise conjugate of \(\mathbf{z}\). Observe that,
\[
\langle \phi_{\mathbf{z}} | \phi_{\mathbf{z'}}\rangle = \prod_{k=2,4,\ldots,2(n-1)} \langle + \rangle^{\sigma_{z_k}^{0} \otimes z_{k+2}^{0} \otimes z_{k+2}'^{0} \otimes z_{k+4}^{0} \otimes z_{k+4}'^{0}} + \rangle.
\]

hence \(\langle \phi_{\mathbf{z}} | \phi_{\mathbf{z'}}\rangle = 0\), unless \(z_{k}^{0} + z_{k+2}^{0} = z_{k+2}^{0} + z_{k+4}^{0} = 1\) for all \(k = 2, 4, \ldots, 2(n-1)\). If \(z_{k}^{0} + z_{k+2}^{0} = 0\), then \(\mathbf{z}' = \mathbf{z}\), otherwise each \(z_{k}^{0} + z_{k+2}^{0} = 1\). Hence, we have established
\[
\langle \phi_{\mathbf{z}} | \phi_{\mathbf{z'}}\rangle = \begin{cases} 0 & \text{if } \mathbf{z}' \neq \mathbf{z} \\ 1 & \text{if } \mathbf{z}' = \mathbf{z} \end{cases}. \tag{23}
\]

We now complete the proof of the lemma by showing that if \(\mathbf{z}' = \mathbf{z}\), the 2-vertex line-state components of \(|\epsilon_{\mathbf{z}}\rangle\) and \(|\epsilon_{\mathbf{z'}}\rangle\) are orthogonal,
\[
\langle L_2^{(1,2n+1)} | \sigma_{z_2}^{0} \otimes \sigma_{z_4}^{0} |L_2^{(1,2n+1)}\rangle = \frac{1}{2} \left( |0\rangle + (1-|0\rangle \right) \langle \sigma_{z_2}^{0} \otimes \sigma_{z_4}^{0} |0\rangle + (1-|0\rangle \right) = \frac{1}{2} \left( |0\rangle + (1-|0\rangle \right) |0\rangle + (1-|0\rangle \right) = 0. \tag{24}
\]

can organize the $2^n$ eigenstates $|e_z\rangle$ into four sets as

\[ z_2 = 0, z_{2n} = 0 : \left\{ \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)^{1,2,n+1} |\phi_z\rangle \right\} \]

\[ z_2 = 1, z_{2n} = 1 : \left\{ \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle)^{1,2,n+1} |\phi_z\rangle \right\} \]

\[ z_2 = 0, z_{2n} = 1 : \left\{ \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)^{1,2,n+1} |\phi_z\rangle \right\} \]

\[ z_2 = 1, z_{2n} = 0 : \left\{ \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle)^{1,2,n+1} |\phi_z\rangle \right\}. \]

The crucial observation is that the states in the first two sets can be written using only two product states for qubits 2 and 3, and similarly for the states in the second two sets:

\[ \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}} (e^{-i\pi/4} |\bar{\phi} + \phi\rangle + e^{i\pi/4} |\bar{\phi} - \phi\rangle) \]

\[ \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) = \frac{1}{\sqrt{2}} (e^{i\pi/4} |\bar{\phi} + \phi\rangle + e^{-i\pi/4} |\bar{\phi} - \phi\rangle) \]

\[ \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}} (e^{i\pi/4} |\bar{\phi} + \phi\rangle + e^{-i\pi/4} |\bar{\phi} - \phi\rangle) \]

\[ \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) = \frac{1}{\sqrt{2}} (e^{i\pi/4} |\bar{\phi} + \phi\rangle + e^{-i\pi/4} |\bar{\phi} - \phi\rangle). \]

Therefore, we conclude that the support of $\rho^{A}$ is spanned by $2^n$ orthogonal product states

\[ \left\{ |\bar{\phi} + \phi\rangle |\bar{\phi} + \phi\rangle, |\bar{\phi} - \phi\rangle |\bar{\phi} - \phi\rangle \mid z \in S_{0,0} \right\} \]

\[ \bigcup \left\{ |\bar{\phi} + \phi\rangle |\bar{\phi} - \phi\rangle, |\bar{\phi} - \phi\rangle |\bar{\phi} + \phi\rangle \mid z \in S_{0,1} \right\}. \]

We next show that these are the only product states in the support of $\rho^{A}$.

Suppose that $|\Psi\rangle$ is a product state in the support of $\rho^{A}$. Then we can find coefficients such that

\[ |\Psi\rangle = \sum_{z \in S_{0,0}} (a_z |\phi_z\rangle + b_z |\bar{\phi}_z\rangle) \]

\[ + \sum_{z \in S_{0,1}} (c_z |\bar{\phi}_z\rangle + d_z |\phi_z\rangle) \]

\[ = |\bar{\phi} + \phi\rangle |\beta\rangle + |\bar{\phi} - \phi\rangle |\gamma\rangle + |\phi + \bar{\phi}\rangle |\delta\rangle, \]

where $|\phi\rangle = \sum_{z \in S_{0,0}} a_z |\phi_z\rangle$, $|\beta\rangle = \sum_{z \in S_{0,0}} b_z |\phi_z\rangle$, $|\gamma\rangle = \sum_{z \in S_{0,1}} c_z |\phi_z\rangle$, and $|\delta\rangle = \sum_{z \in S_{0,1}} d_z |\phi_z\rangle$. By Eq. (23), the states $|\alpha\rangle, |\beta\rangle$ are orthogonal to the states $|\gamma\rangle, |\delta\rangle$.

Suppose first that both $|\alpha\rangle$ and $|\beta\rangle$ are nonzero. Then we can find a vector $|v\rangle$ in the linear span of $\{|\alpha\rangle, |\beta\rangle\}$ that has nonzero overlap with both $|\alpha\rangle$ and $|\beta\rangle$. Partially contracting both sides of Eq. (29) by $|v\rangle$ yields

\[ \langle v | \Psi \rangle = x |\bar{\phi} + \phi\rangle + y |\bar{\phi} - \phi\rangle \]
with \(x, y \neq 0\). But since \(|\Psi\rangle\) is a product state, it remains a product state under partial contraction, and so the RHS must be a product state. However the only product states contained in the linear span of \(|\mp\pm\rangle\) and \(|\mp\mp\rangle\) are these themselves. We thus have a contradiction, and so it is not possible for both \(|\alpha\rangle\) and \(|\beta\rangle\) to be nonzero. A similar argument shows that both \(|\gamma\rangle\) and \(|\delta\rangle\) cannot be nonzero. Hence, there are only four possible forms of \(|\Psi\rangle\); each pairing an element in \(|\{\alpha, \beta\}\rangle\) with an element in \(|\{\gamma, \delta\}\rangle\). For example, we have could have

\[
|\Psi\rangle = |\mp\mp\rangle|\alpha\rangle + |\mp\pm\rangle|\beta\rangle = |\mp\rangle(|\alpha\rangle + |\beta\rangle),
\]

which is not a product state unless either \(|\alpha\rangle\) or \(|\beta\rangle\) are zero, since \(|\alpha\rangle|\beta\rangle = 0\). A similar argument applies for the other three possible forms of \(|\Psi\rangle\). Therefore, any product state in \(|\Psi\rangle\) must have the form \(|\mp\mp\rangle|\alpha\rangle, |\mp\pm\rangle|\beta\rangle, |\mp\pm\rangle|\gamma\rangle, \text{ or } |\mp\mp\rangle|\delta\rangle\), where \(|\alpha\rangle, |\beta\rangle, |\gamma\rangle, |\delta\rangle\) are product states in the span of \(|\{\phi_\alpha\} | \exists z \in S_{0,0}\rangle\) and \(|\gamma\rangle, |\delta\rangle\) are product states in the span of \(|\{\phi_\gamma\} | \exists z \in S_{0,1}\rangle\).

Then, it finally remains to be shown that the only product states in the span of \(|\{\phi_\alpha\} | \exists z \in S_{0,0}\rangle\) are the \(|\phi_\alpha\rangle\) themselves; likewise, the only product states in the span of \(|\{\phi_\gamma\} | \exists z \in S_{0,1}\rangle\) are the \(|\phi_\gamma\rangle\) themselves.

Suppose that \(|\varphi\rangle = \sum_{z \in S_{0,0}} a_z |\phi_z\rangle\) is a product state. If \(n = 2\), then there is only a single state \(|+\rangle^{(3)}\) in this sum. If \(n = 3\), then there are two terms,

\[
|\varphi\rangle = a_0 |++\rangle^{(3,5)} + a_1 |--\rangle^{(3,5)},
\]

which requires that either \(a_0 = 0\) or \(a_1 = 0\) in order for \(|\varphi\rangle\) to be a product state. Next consider the case when \(n > 3\). For any binary sequence \(w = (w_8, w_1, \ldots, w_{2n})\), define the \((n - 3)\)-qubit state

\[
|\omega_w\rangle^{(7,\ldots,2n-1)} := \bigotimes_{k=6,8,\ldots,2(n-1)} (\sigma_z^{(k+1)}) w_k + w_{k+2} |+\rangle^{(k+1)},
\]

where \(w_{(2(n-1))} = 0\). By Eq. (23), it follows that for any \(|\phi_z\rangle^{(3,5,\ldots,2n-1)}\) with \(z \in S_{0,0}\) we have the partial contractions

\[
\langle \omega_w | \phi_z \rangle = \begin{cases} 
|++\rangle^{(3,5)} & \text{if } w_6 = 0 \\
|--\rangle^{(3,5)} & \text{if } w_6 = 1 
\end{cases}
\]

Therefore, \(\langle \omega_w | \varphi \rangle\) is either contained in the linear span of \(|++\rangle, |--\rangle\) or \(|+-\rangle, |+-\rangle\). In both cases, there are no other product states in the respective spaces besides the given ones. Thus, if \(|\varphi\rangle = \sum_{z \in S_{0,0}} a_z |\phi_z\rangle\) is a product state, then it requires that one and only one of the \(a_z\) be nonzero. An analogous holds for the superposition states \(|\varphi\rangle = \sum_{z \in S_{0,1}} a_z |\phi_z\rangle\). This concludes the proof. \(\square\)

Lemma II.6 provides a structural analysis of the ring state \(|R_{2n+1}\rangle\) that we will use to lower bound the tensor rank of \(|R_{2n+1}\rangle\). To get to this we will also need one general fact about CP decompositions.

**Lemma II.7.** Suppose that

\[
|\psi\rangle = \sum_{i=1}^{R} \mu_i |\psi_i^{(1)}\rangle \otimes \cdots \otimes |\psi_i^{(N)}\rangle,
\]

is a CP decomposition of \(|\psi\rangle\). For any subset of parties \(A\), the states \(\bigotimes_{c \in A} |\psi_i^{(c)}\rangle\) contain the support of \(\rho(A) = \text{Tr}_A |\psi\rangle\langle\psi|\). Moreover, if \(\rho(A)\) has rank \(R\), then conversely the states \(\bigotimes_{c \in A} |\psi_i^{(c)}\rangle\) must belong to the support of \(\rho(A)\).

**Proof.** Given Eq. (32), the reduced density matrix of \(|\psi\rangle\) on \(A\) is

\[
\rho(A) = \sum_{i=1}^{R} \sum_{j=1}^{R} \mu_i \mu_j \prod_{k \in A} |\psi_i^{(k)}\rangle \langle\psi_j^{(k)}| \prod_{c \in A \setminus \{A\}} |\psi_i^{(c)}\rangle \langle\psi_j^{(c)}|. 
\]

Hence, \(\rho(A) = U M U^T\), where

\[
U = \left[\bigotimes_{c \in A} |\psi_i^{(c)}\rangle \ldots \bigotimes_{c \in A} |\psi_i^{(c)}\rangle\right].
\]

Consequently, the support (column span) of this reduced density matrix is contained in \(\text{span}(\bigotimes_{c \in A} |\psi_i^{(c)}\rangle)\). Further, if \(\text{rank}(\rho(A)) = R\), the rank of \(M\) is also \(R\), and the column span of \(\rho(A)\) is the same as that of \(U\).

\(\square\)

Now, we put everything together to obtain our desired lower bound.

**Theorem II.8.** \(\text{rank}(|R_{2n+1}\rangle) > 2^n\).

**Proof.** Lemma II.7 constructs a reduced density matrix from \(|R_{2n+1}\rangle\) of rank \(2^n\), hence \(\text{rank}(|R_{2n+1}\rangle) \geq 2^n\). Now, suppose for sake of contradiction that \(\text{rank}(|R_{2n+1}\rangle) = 2^n\). Since \(\rho(A)\) has matrix rank \(2^n\) for the subset \(A = \{2, 4, \ldots, 2n\}\), Lemma II.7 says that any CP decomposition of \(|R_{2n+1}\rangle\) of minimal length must contain product states \(\bigotimes_{c \in A} |\psi_i^{(c)}\rangle\) belonging to the support of \(\rho(A)\). However, Lemma II.6 then implies that these product states must be the ones given in (25). That is, we must be able to write

\[
|R_{2n+1}\rangle = \sum_{A \in S_{0,0}} |A_2\rangle^{(2,4,\ldots,2n)} |++\rangle^{(1,2n+1)} |\phi_z\rangle^{(3,5,\ldots,2n-1)} \\
+ \sum_{A \in S_{0,0}} |B_2\rangle^{(2,4,\ldots,2n)} |--\rangle^{(1,2n+1)} |\phi_z\rangle^{(3,5,\ldots,2n-1)} \\
+ \sum_{A \in S_{0,1}} |C_2\rangle^{(2,4,\ldots,2n)} |+-\rangle^{(1,2n+1)} |\phi_z\rangle^{(3,5,\ldots,2n-1)} \\
+ \sum_{A \in S_{0,1}} |D_2\rangle^{(2,4,\ldots,2n)} |+-\rangle^{(1,2n+1)} |\phi_z\rangle^{(3,5,\ldots,2n-1)}
\]

with \(|A_2\rangle, |B_2\rangle, |C_2\rangle, \text{ and } |D_2\rangle\) all being product states.
At the same time, from Lemma II.5 and Eq. (21), we can express the ring state as
\[ |R_{2n+1} \rangle = \frac{1}{\sqrt{2^n}} \sum_{z \in \mathbb{Z}} e^{i\pi/4} |\hat{z}\rangle^A \langle e_z |^{\mathcal{F}}, \]  
where \(|\hat{z}\rangle^A := (-1)^{|z|} |z\rangle^A\). By inverting the equalities in Eq. (25), this can be written as
\[ |R_{2n+1} \rangle = \frac{1}{\sqrt{2^n}} \sum_{z \in \mathbb{Z}} e^{i\pi/4} |\hat{z}\rangle^A \langle e_z |^{\mathcal{F}} \]
\[ + \frac{1}{\sqrt{2^n}} \sum_{z \in \mathbb{Z}} e^{-i\pi/4} |\hat{z}\rangle^A \langle e_z |^{\mathcal{F}} \]
\[ + \frac{1}{\sqrt{2^n}} \sum_{z \in \mathbb{Z}} e^{-i\pi/4} |\hat{z}\rangle^A \langle e_z |^{\mathcal{F}} \]
\[ + \frac{1}{\sqrt{2^n}} \sum_{z \in \mathbb{Z}} e^{i\pi/4} |\hat{z}\rangle^A \langle e_z |^{\mathcal{F}} \].
Comparing this with Eq. (33) shows that
\[ |A_z \rangle = \frac{1}{\sqrt{2^n}} (e^{i\pi/4} |\hat{z}\rangle^A + e^{-i\pi/4} |\hat{z}\rangle^A), \]
\[ |B_z \rangle = \frac{1}{\sqrt{2^n}} (e^{i\pi/4} |\hat{z}\rangle^A + e^{-i\pi/4} |\hat{z}\rangle^A), \]
etc., which is a contradiction because these are not product states.

\[ \Box \]

III. ENTANGLEMENT MEASURES ON GENERAL GRAPH STATES

A. Extensions of Bipartite Measures

Here we evaluate the multipartite extensions of bipartite measures previously discussed. Recall that the multipartite extension of some bipartite entanglement measure \(E\) is defined as \(\min_A E_A(|\psi\rangle)\), where \(E_A\) is the measure \(E\) evaluated according to partition \(A|A\). Surprisingly for graph states, the multipartite extensions of many standard bipartite entanglement measures are dichotomous: one of two values based on if the graph is connected. The GME concurrence, negativity, and geometric measure have been previously calculated for connected graphs [46]. Since these results were derived independently of this work, we provide in the appendix a self-contained and direct calculation of the following:

**Theorem III.1.**

\[ C_{GME}(|G\rangle) = \begin{cases} 
0 & \text{if } G \text{ is a disconnected graph} \\
1 & \text{otherwise}
\end{cases} \]  
\[ \mathcal{N}_{GME}(|G\rangle) = \begin{cases} 
0 & \text{if } G \text{ is a disconnected graph} \\
1 & \text{otherwise}
\end{cases} \]  
\[ G_{GME}(|G\rangle) = \begin{cases} 
0 & \text{if } G \text{ is a disconnected graph} \\
1 & \text{otherwise}
\end{cases} \]  

**Remark III.3.** There are \(2^{\binom{n}{2}-1}\) graphs on \(n\) (where \(n\) is even) vertices such that \(\tau_n(|G\rangle) = 1\). There are none for odd \(n\).

**Proof.** We claim that for even \(n\), given any graph on \(n-1\) vertices we can construct a graph on \(n\) of all odd degree.
This is clear from stabilizer states taking the form $|s\rangle\langle s| = \sum_g g_i$, where $\{g_i\}$ are the $n$ stabilizers. The complex conjugate of the state can readily be found via transposing the stabilizers with respect to the computational basis, $|s\rangle\langle s^*| = \sum_i T(g_i)$, which simply adds a sign of $-1$ to $\sigma_y$ terms and leaves all other stabilizer unchanged. Thus, one can continue by calculating $g' = \{g_i\} = \sigma_y g_i$ and checking if $\langle g'\rangle = \langle g\rangle$.

IV. OVERVIEW AND DISCUSSION

In this work we tightened the bounds on CP rank of odd rings to $2^n + 1 \leq \text{rank}(|R_{2n+1}|) \leq 3 \cdot 2^{n-1}$. This indicates that odd rings are, according to the Schmidt measure, more entangled than a line in the same number of qubits. Further, odd rings are thus not of particularly high rank. For $2n + 1$ qubits, the maximum CP rank is known to be on the order of $2^{2n-1}$ [48]. Based on numerical CP decomposition, we suspect rank$(|R_{2n+1}|) = 3 \cdot 2^n$, but the question remains open.

Beyond CP rank, we considered several multipartite entanglement measures on graph states based on bipartite measures. Surprisingly, these prove dichotomous: either 0 if the graph is disconnected, or a fixed value irrespective of graph structure beyond connectivity.

V. ACKNOWLEDGEMENTS

We would like to thank Yuchen Pang for helpful discussions. We are very grateful to Julio de Vicente for carefully pointing out how their work in Ref. [46] implies Theorem III.1. The authors acknowledge support from the NSF Quantum Leap Challenge Institute for Hybrid Quantum Architectures and Networks (NSF Award 2016136). Linjian Ma and Edgar Solomonik were also supported via the US NSF RAISE/TAQS program, award number 1839204.

[1] A. Einstein, B. Podolsky, and N. Rosen, “Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?” Physical Review 47, 777 (1935).
[2] Ryszard Horodecki, Paweł Horodecki, Michal Horodecki, and Karol Horodecki, “Quantum entanglement,” Reviews of Modern Physics 81, 865–942 (2009), 0702225.
[3] Artur Ekert and Richard Jozsa, “Quantum algorithms: entanglementenhanced information processing,” Philosophical Transactions of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences 356, 1769–1782 (1998).
[4] Artur Ekert and Peter L. Knight, “Entangled quantum systems and the Schmidt decomposition,” American Journal of Physics 63, 415 (1998).
[5] Michael Walter, David Gross, and Jens Eisert, “Multipartite Entanglement,” Quantum Information , 293–330 (2016).
[6] Guifré Vidal, “Entanglement monotones,” Journal of Modern Optics 47-2, 355–376 (2000).
[7] Charles H. Bennett, David P. DiVincenzo, John A. Smolin, and William K. Wootters, “Mixed-state entanglement and quantum error correction,” Physical Review A 54, 3824 (1996).
[8] E. M. Rains, “Rigorous treatment of distillable entanglement,” Physical Review A 60, 173 (1999).
[9] G. Vidal and R. F. Werner, “Computable measure of entanglement,” Physical Review A 65, 032314 (2002).
[10] Soojoon Lee, Dong Pyo Chi, Sung Dahn Oh, and Jaewan Kim, “Convex-roof extended negativity as an entanglement measure for bipartite quantum systems,” Physical Review A 68, 062304 (2003).
[48] Toshio Sumi, Toshio Sakata, and Mitsuhiro Miyazaki, “Rank of tensors with size 2 x ... x 2,” (2013), arXiv:1306.0708.
APPENDIX

MINIMAL DECOMPOSITION OF LINE STATES

We find it informative to give a recursive minimal CP decomposition for line states $|L_n\rangle$. Note that this can be used to construct the $3 \times 2^n$ term CP decomposition for $|R_{2n+1}\rangle$.

**Remark V.1.** A minimal decomposition for any $|L_n\rangle$ can be found via a simple recursive method. Define the following 2 qubit states:

$$|a\rangle = |0+\rangle, \quad |b\rangle = |1-\rangle, \quad |c\rangle = |0-\rangle, \quad |d\rangle = |1+\rangle.$$  \hfill (41)

**Proof.** We can write a line state as a decomposition of an $n \times 2$ matrix used to construct the $3 \times 2^{n+1}$ term CP decomposition of $|R_{2n+1}\rangle$. Following the proof in the main text, we can find the desired decomposition from that of $|R_5\rangle$, $|a^{(4)}\rangle = \frac{1}{\sqrt{2}} |0\rangle (|+0+\rangle + |-1-\rangle)$ and $|b^{(4)}\rangle = \frac{1}{\sqrt{2}} |0\rangle (|+0-\rangle + |-1+\rangle)$, $P_0^{(1)} P_1^{(7)} |L_7\rangle = \frac{1}{4\sqrt{2}} |0\rangle (|aa0\rangle - |ac1\rangle + |cb0\rangle - |cd1\rangle + |ba0\rangle - |bc1\rangle + |db0\rangle - |dd1\rangle) |1\rangle.$  \hfill (46)

Following the proof in the main text, we can find the desired decomposition from that of $|R_5\rangle$, $|a^{(4)}\rangle = \frac{1}{\sqrt{2}} |0\rangle (|+0+\rangle + |-1-\rangle)$ and $|b^{(4)}\rangle = \frac{1}{\sqrt{2}} |0\rangle (|+0-\rangle + |-1+\rangle)$, $P_0^{(1)} P_1^{(7)} |\phi^{(7)}\rangle = \frac{1}{\sqrt{2}} (|a^{(4)}\rangle |0-\rangle + |b^{(4)}\rangle |1+\rangle) |1\rangle$.

**UPPER BOUND FOR 7 QUBIT RING**

Here we explicitly construct the rank $3 \times 2^2 = 12$ decomposition for $|R_7\rangle$. From remark V.1 the 7 qubit line state can be written as:

$$|L_7\rangle = \frac{1}{\sqrt{2}} |+\rangle (|aa0\rangle + |acb\rangle + |cba\rangle + |cdb\rangle) + \frac{1}{\sqrt{2}} |-\rangle (|baa\rangle + |bcb\rangle + |dba\rangle + |ddb\rangle),$$  \hfill (45)

$$P_0^{(1)} P_1^{(7)} |L_7\rangle = \frac{1}{4\sqrt{2}} |0\rangle (|aa0\rangle - |ac1\rangle + |cb0\rangle - |cd1\rangle + |ba0\rangle - |bc1\rangle + |db0\rangle - |dd1\rangle) |1\rangle.$  \hfill (46)
It can be verified that these are the same states via expanding into the computational basis. As \( |R_7\rangle = U^{(1,7)} |L_7\rangle = (\sigma_z^{2n+1} + 2P_0^{(1)} \otimes P_1^{(7)}) |L_7\rangle \), we can thus write \( |R_7\rangle \) in the following 12 term decomposition

\[
|R_7\rangle = \frac{1}{2\sqrt{2}} |+\rangle (|0 + 0 + 0\rangle + |0 + 0 - 1\rangle + |0 - 1 - 0\rangle + |0 - 1 + 1\rangle) + \\
\frac{1}{2\sqrt{2}} |-\rangle (|1 - 0 + 0\rangle + |1 - 0 - 1\rangle + |1 + 1 - 0\rangle + |1 + 1 + 1\rangle) + \\
\frac{1}{2} |0\rangle (|+0+\rangle + |-1-\rangle |0-\rangle + (|+0-\rangle + |-1+\rangle |1+\rangle |1\rangle). \tag{48}
\]

### CALCULATION OF GME ENTANGLEMENT FOR GRAPH STATES

To show Theorem III.1, we first observe the following.

**Corollary V.2.** The reduced density matrix for any individual qubit party \( i \) that corresponding to vertex \( v \) is

\[
\rho_i = \begin{cases} 
  \frac{1}{2} I & \delta(v) > 0 \\
  |+\rangle \langle +| & \delta(v) = 0
\end{cases}, \tag{49}
\]

where \( \delta(v) \) is the degree of vertex \( v \).

**Proof.** This follows readily from Lem. I.1. If \( v \) is not an isolated vertex, there is at least one non-zero value in \( \Gamma_{A^2} \), where \( A = \{v\} \), and thus \( \text{rank}(\Gamma_{A^2}) = 1 \). If \( v \) is an isolated vertex, then \( \text{rank}(\Gamma_{A^2}) = 0 \). \( \Box \)

With this corollary and Lem. I.1, we now show that the measures previously introduced are either 0 or a fixed constant based on if the graph is connected.

**Theorem V.3.**

\[
\mathcal{C}_{GME}(|G\rangle) = \begin{cases} 
  0 & \text{if } G \text{ is a disconnected graph} \\
  1 & \text{otherwise}
\end{cases}. \tag{50}
\]

**Proof.** From Lem. I.1 we know that any reduced density matrix is maximally mixed on a certain subspace of dimension \( 2^d = 2^{\text{rank}(\Gamma_{A^2})} \). By finding \( \max_A \text{Tr}[\rho_A^2] \) we minimize GME-Concurrence. The purity of a \( k \)-dimensional maximally mixed state is \( \frac{1}{k} \). If there is a disconnected component \( A \), \( \text{rank}(\Gamma_{A^2}) = 0 \) and \( \mathcal{C}_{GME}(|G\rangle) = 0 \). Otherwise, the maximal purity is \( \frac{1}{2} \), which is achieved by considering any single vertex. Thus, \( \mathcal{C}_{GME}(|G\rangle) = 1 \). \( \Box \)

**Theorem V.4.**

\[
\mathcal{N}_{GME}(|G\rangle) = \begin{cases} 
  0 & \text{if } G \text{ is a disconnected graph} \\
  \frac{1}{2} & \text{otherwise}
\end{cases}. \tag{51}
\]

**Proof.** Before continuing, we note that negativity can be equivalently written as a summation of the absolute value of the negative eigenvalues of the partial transpose.

\[
\mathcal{N}(\rho^{AB}) = \frac{1}{2} (\|\rho^{T_A}|1\rangle \langle 1| - 1) = \sum_{\lambda < 0} |\lambda|, \tag{52}
\]

where \( \lambda \) are the eigenvalues of \( \rho^{T_A} \).

Next, we use Lem. I.1 to write \( |G\rangle \) in a Schmit decomposition \( |G\rangle = 2^{-d/2} \sum_{i=1}^{2^d} |u_i\rangle |v_i\rangle \), where \( d = \text{rank}(\Gamma_{A^2}) \). Thus, the partial transpose with respect to \( A \) is

\[
I \otimes T(|G\rangle \langle G|) = 2^{-d} \sum_{i,j=1}^{2^d} |u_i\rangle \langle u_j| \otimes |v_j\rangle \langle v_i|. \tag{53}
\]
This has negative eigenvalue $-2^{-d}$ with multiplicity $\binom{2d}{2}$, corresponding to eigenvectors $\frac{1}{\sqrt{2}}(|u_i\rangle|v_j\rangle - |u_j\rangle|v_i\rangle)$. Thus, the negativity according to partition $A|\overline{A}$ is

$$\mathcal{N}(\rho_{A|\overline{A}}) = \binom{2d}{2} 2^{-d} = \frac{1}{2}(2^d - 1).$$

(54)

Clearly this is increasing with $d$. Thus, the GME negativity will be minimized by a partition with the smallest $\text{rank}(\Gamma_{A|\overline{A}})$. If $G$ is disconnected, there exists a partition $A$ such that $\text{rank}(\Gamma_{A|\overline{A}}) = 0$. Otherwise, $d = 1$ for any partition into a single vertex, for which $\mathcal{N}(\rho_{A|\overline{A}}) = \frac{1}{2}$. $\Box$

**Theorem V.5.**

$$\mathcal{G}_{\text{GME}}(|G\rangle) = \begin{cases} 0 & \text{if } G \text{ is a disconnected graph} \\ \frac{1}{2} & \text{otherwise} \end{cases}.$$ (55)

*Proof.* If $G$ is disconnected there is a partition with Schmidt coefficient 1 (Cor. V.2). Otherwise, the largest Schmidt coefficient possible is always $\frac{1}{2}$ via Lem. I.1. $\Box$