Abstract

Using real-space renormalisation techniques we analyse the Ising model on a Sierpiński gasket with anisotropic microscopic couplings, and observe a restoration of isotropy on macroscopic scales. In particular, via use of a decimation procedure directly on the fractal lattice, we calculate explicitly the exponential anisotropy decay coefficients near the isotropic regime for both ferromagnetic and antiferromagnetic systems. The results suggest the universality of the phenomenon in lattice field theories on fractals.

I. INTRODUCTION

Restoration of macroscopic isotropy in a system which has an underlying microscopic anisotropy is a phenomenon which, while absent in uniform media, appears almost universal in fractals. Recently, Barlow et al. [1] considered this phenomenon in the form of resistor networks. In particular, they considered a resistor network constructed on the Sierpiński gasket with locally anisotropic resistance elements. Then, by successive use of star-triangle transformations they obtained recursion relations for the resistance elements from one length scale to another, and observed a restoration of isotropy at the macroscopic scale. The extensions to other fractal objects were also considered and the results suggest the universality of the phenomenon at least on exactly self–similar fractals. A question which naturally arises in this regard is whether the same mechanism is present for systems in statistical mechanics, or quantum field theories.

*Electronic address: d.brody@damtp.cam.ac.uk

†Electronic address: a.ritz@ic.ac.uk
The statistical mechanics of spin systems on fractal lattices, on the other hand, has been a subject of considerable interest for which a number of models has been studied. In particular, for exactly self–similar fractals with a finite degree of ramification, exact real-space renormalisation techniques can be pursued for simple spin models, and as a consequence the Ising spin model with isotropic couplings has been solved exactly on various fractals. See, for example, Gefen et. al. [2,3], Luscombe and Desai [4] or Stosić et. al. [5] for further details. The physical motivation behind these investigations is in part that the results obtained on fractals may have some relevance for real random systems, or crystals with defects. Also, it is interesting to note that this analysis indicates properties of spin systems in nonintegral dimensions which are not apparent in more formal analyses such as the $\epsilon$–expansion, and also provides the possibility for considering how this physical behaviour crosses over to uniform media [5].

In the present Letter, we study the Ising model on a Sierpiński gasket with locally anisotropic coupling configurations. As a comparison to the formulation presented in [1], we first analyse this model with a successive use of star-triangle transformations and one-dimensional decimation, in order to study the level of anisotropy at larger length scales. The results obtained show enormously rapid recovery of isotropy at larger scales. However, unlike the case of resistance networks, the use of the star-triangle transformation in the present context is not necessarily advantageous in the sense that frustrated spin systems cannot be studied within this methodology. Nevertheless, as we noted above, in the case of the Ising model, exact real-space renormalisation transformations can also be performed in the frustrated system (see for example Stinchcombe [6] and Grillon & Brady Moreira [7]), and as a consequence we are able to study the large scale structure for both ferromagnetic and antiferromagnetic Ising models. In both cases, we observe recovery of isotropy at an exponential rate. More specifically, the technique used allows us to rigorously prove the convergence of the system to isotropy and to investigate the rate at which homogeneity is restored. In particular in the near isotropic regime the scaling to isotropy is exponential with a coefficient given by $\rho = 1 \pm \tanh 2$ with the upper(lower) sign corresponding to the ferro(antiferro)-magnetic system.

II. RESULTS FROM STAR-TRIANGLE TRANSFORMATIONS

The model is constructed as follows. We embed the Sierpiński gasket in two Euclidean dimensions and place Ising spins $\sigma_i = \pm 1$ at each vertex. Geometrically the Ising model is then effectively carried by a space of Hausdorff dimension $d = \ln 3/\ln 2 = 1.585$ with a finite degree of ramification $R_{\text{max}} = 4$. We consider the standard nearest neighbour Hamiltonian given by

$$\mathcal{H} = -\sum_{(ij)} K_{ij} \sigma_i \sigma_j,$$

where $K_{ij} = J_{ij}/kT$ are anisotropic ferromagnetic couplings which depend upon the orientation of the spins $\sigma_i$ and $\sigma_j$, and the summation is taken over all the nearest neighbour sites on the fractal lattice.
FIG. 1. A stage in the renormalisation group procedure for the Ising model on the SG is indicated. The interior spins $\mu_1, \mu_2, \mu_3$ are integrated out resulting in course grained couplings $L_{n+1}$ and $K_{n+1}$. This is achieved in (A): via a combination of star-triangle transformations and 1D decimation, the dashed lines indicating the transformation used at each stage (the various couplings being defined in the text); while in (B): direct decimation on the fractal lattice is used.

Motivated by the technique used in [1] we can construct a real-space renormalisation group procedure as indicated in Fig. 1 [part (A)] where after the $n^{th}$ iteration of the process we denote the horizontal coupling by $K_{n}^{\text{hor}} = K_n$, and the two 'vertical' couplings as $K_{n}^{\text{vert1}} = K_{n}^{\text{vert2}} = L_n = r_n K_n$ so that $r_n$ represents the degree of anisotropy after the $n^{th}$ iteration.

The process involves the use of an alternate sequence of star-triangle transformations and one-dimensional decimation so that if we define the following functions

$$\begin{align*}
F_1(\alpha, \beta) &= \frac{1}{2} \arccosh \left[ e^{2\alpha} \cosh 2\beta \right] \\
F_2(\alpha, \beta) &= \arctanh \left[ \frac{e^{2\alpha} \sinh 2\beta}{\sinh(2F_3(\alpha, \beta))} \right] \\
F_3(\alpha, \beta) &= \frac{1}{2} \ln \left[ \frac{\alpha + \beta}{\alpha - \beta} \right] \\
F_4(\alpha, \beta) &= \frac{1}{4} \ln \left[ \frac{2\alpha + \beta}{2\alpha - \beta} \right] \\
F_5(\alpha, \beta) &= \frac{1}{2} \ln \left[ \cosh^2 2\beta - \sinh^2 2\beta \tanh^2 \alpha \right],
\end{align*}$$

(2)

then the couplings at each stage of the process indicated in Fig. 1 are given by
\[
\begin{align*}
g &= F_1(K_n, L_n), & h &= F_2(K_n, L_n) \\
M &= F_3(g, h), & N &= F_3(h, h) \\
W &= F_1(N, M), & V &= F_2(N, M) \\
x &= F_3(g, V), & y &= F_3(h, W) \\
K_{n+1} &= F_4(x, y), & L_{n+1} &= F_5(x, y).
\end{align*}
\]  

This procedure may be iterated numerically, and with the initial conditions \(K_0 = 0.01, L_0 = 1\), i.e. an initial microscopic anisotropy level of \(r_0 = 100\), the results we obtain are shown in Fig. 2. The simulation indicates a rapid restoration of isotropy although the actual structure of the anisotropy oscillates after each iteration, due to the fact that an odd number of star-triangle transformations are used during each iteration process, as indicated in Fig. 1.

One can easily observe, from the form of the functions \(F_1\) and \(F_2\), that a ‘cut’ along the negative real axis of the coupling space prevents the consideration of negative couplings and therefore a frustrated system cannot be treated via this procedure. In other words, if the initial couplings for each triangle are all negative (antiferromagnetic), then the resulting couplings on the ‘star’ become pure imaginary.

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{fig2}
\caption{The level of anisotropy \(r_n\) plotted against the order of the coarse–graining iteration of type (A). Starting from initial ratio \(r_0 = 100\), we recover \(r \sim O(1)\) after only three iterations.}
\end{figure}

Physically, this reflects the fact that, while on the triangular lattice there are frustrations associated with each plaquette, these frustrations cannot be realised for any real-valued couplings on the ‘star’, or equivalently, a hexagonal lattice.

### III. FERROMAGNETIC CASE

In order to allow us to analyse more general situations, and also consider the possibility of obtaining analytic results for the asymptotic behaviour, we now consider a different method, based on the direct use of real-space renormalisation transformations. As mentioned above, due to the finite degree of ramification, exact renormalisation group analysis
is easily accomplished analytically via decimation, as reported in [2–4]. A course graining transformation of this form is represented in part (B) of Fig. 1.

Consideration of the partition function allows the recursion relations for the couplings \( L \) and \( K \) to be determined as follows,

\[
e^{4K_{n+1}} = \frac{2 \cosh 2K_n + \cosh 2L_n + e^{4K_n} \cosh 6L_n}{2 \cosh^2 2L_n (1 + e^{4K_n} \cosh 2L_n)}
\]

(4)

\[
e^{4L_{n+1}} = \frac{2 \cosh 2K_n + \cosh 2L_n + e^{4K_n} \cosh 6L_n}{2 \cosh 2K_n (1 + e^{4K_n} \cosh 2L_n)},
\]

(5)

which, in the isotropic limit \( L_n = K_n \), clearly reduce to the result of Gefen et al. [2,3],

\[
e^{4K_{n+1}} = \frac{e^{8K_n} - e^{4K_n} + 4}{e^{4K_n} + 3}.
\]

(6)

After \( n \) iterations of the renormalisation group transformation the level of anisotropy in the interaction is again characterised by \( r_n = L_n/K_n \). Since we are concerned only with this ratio of the couplings we can rescale both the horizontal and ‘vertical’ couplings after each iteration by an arbitrary factor. A convenient choice of this factor proves to be \( 1/K_n \), and as a consequence the couplings at each order now take on the values \( K_n = 1 \) and \( L_n = r_n \). Clearly this rescaling has no effect on the observable behaviour of the system with regard to changes in the level of anisotropy. However, the benefit we gain by using this technique is that by taking the ratio of (4) and (5) we can construct a recursion relation purely for the level of anisotropy in the system given by

\[
\frac{e^{4K_{n+1}}}{e^{4K_n}} = \frac{e^{8K_n} - e^{4K_n} + 4}{e^{4K_n} + 3}.
\]

(7)

Before we analyse this relation we should first consider the possible initial conditions. i.e., the initial choices one can make for \( r_0 \). Ignoring the trivial isotropic case there are clearly two; either \( r_0 > 1 \) or \( r_0 < 1 \). With reference to Fig.1 we see that these conditions are not equivalent. The first corresponds physically to the situation where couplings in two of the directions on the lattice are larger than the third, while the latter case corresponds to the situation where two of the couplings are weaker than the third. To avoid confusion we shall always differentiate these particular initial conditions and refer to each by the corresponding values of \( r_0 \).

The structure of the relation (7) indicates that it possesses a two element fixed point set \( \{1, \infty \} \), with the first point \( r^* = 1 \), corresponding to isotropy. In order to determine which of these fixed points is stable we analyse the recursion relation for each of the possible initial conditions. Making use of the trivial inequality, \((a + b)/(c + d) < a/c\) for any \( a, b, c, d \in \mathbb{R}_+ \), we find

\[
r_{n+1} < 1 + \frac{1}{2} \ln \frac{e^{2r_n}}{e^2} = r_n, \quad \text{for } r_n > 1, \quad \text{(8)}
\]

\[
r_{n+1} > 1 + \frac{1}{2} \ln \frac{e^{2r_n}}{e^2} = r_n, \quad \text{for } r_n < 1. \quad \text{(9)}
\]
FIG. 3. The deviation from isotropy $R_n$ of the interaction plotted against the renormalisation group iteration $n$ in the ferromagnetic case. The upper plot uses the initial condition $r_0 = 10$ while the lower curve uses $r_0 = 0.1$.

Thus the deviation from isotropy at each order $\delta_n \equiv |r_n - 1|$ satisfies the inequality $\delta_{n+1} < \delta_n$ unless $r_n = 1$ and thus $r^* = 1$ is the only stable fixed point of the recursion relation and, as a consequence, restoration of isotropy on macroscopic scales is assured.

The actual behaviour of the system as one carries out the course–graining transformations may be observed by iterating (7) numerically. Specifically, in Fig. 3 we present the scaled deviation from isotropy

$$R_n = \left| 1 - \frac{1}{r_n} \right|$$

for examples of the possible initial microscopic conditions $r_0 = 10$ and $r_0 = 0.1$. It is clear that the presence of the unstable fixed point at $r^* = \infty$ will only affect the restoration of isotropy in the first of these cases. With the initial condition $r_0 = 0.1$, recovery of isotropy is smooth and quite rapid, although approximately an order of magnitude slower than the corresponding result using star–triangle transformations (see Section 2). For later reference it is useful to characterise this regime as having ‘positive curvature’ (i.e., $r''(n) > 0$ where we symbolically treat $r_n$ as a smooth function). In contrast to the case $r_0 = 0.1$, with the initial condition $r_0 = 10$ restoration of isotropy is quite slow for several iterations and appears to be significantly affected by the fixed point at $r^* = \infty$. We characterise this regime as having ‘negative curvature’ (symbolically $r''(n) < 0$). We observe from Fig. 3 that after sufficient iterations there is a crossover from this negative curvature regime to the positive curvature regime, corresponding to dominance of the $r^* = 1$ fixed point, at approximately $r_n \approx 1.3$. As a consequence we can conclude that the negative curvature regime is only present if the initial configuration is such that $r_0 > 1.3$.

As we have mentioned above, the phenomenon of macroscopic restoration of isotropy appears to be an almost universal phenomenon in fractal media. In order to test this conjecture in a more quantitative manner it would be helpful to obtain a measure of the
scaling near the isotropic fixed point \( r^* = 1 \). A numerical study of the recursion relation (\( \mathbb{H} \)) in this regime, for an arbitrary choice of initial conditions, indicates that the rate of restoration of isotropy appears to be approximately exponential. That this is indeed the case may be verified analytically as follows. If we treat \( r_n \) as a function, \( r = r(n) \), we have in general \( r(n + dn) = r(n) + r'(n)dn \) where the derivative is taken with respect to \( n \). In the regime where \( r_n \approx 1 \) the variation in \( r \) between iterations is very small and thus we may approximate the recursion relation (\( \mathbb{H} \)) by the truncated Taylor series, i.e., \( r(n + 1) \approx r(n) + r'(n) \). As a consequence, we obtain the differential equation

\[
\frac{dr}{dn} + r - \ln \left( \frac{\cosh 2r}{\cosh 2} \right) - 1 = 0,
\]  
(11)

which, for arbitrary initial conditions, is valid in the regime near isotropy where \( r \) is only weakly dependent on \( n \). Recalling the definition of \( \delta = |r - 1| \), in the limit \( \delta \to 0 \) where the differential equation is valid, we may expand the nonlinear term in (11) to obtain the following simple differential equation for \( \delta \),

\[
\frac{d\delta}{dn} + (1 + \tanh 2)\delta = 0,
\]  
(12)

with the solution given by

\[
\delta(n) = \exp(-\rho n),
\]  
(13)

where the decay coefficient for the level of anisotropy near the isotropic regime is given by \( \rho = 1 + \tanh 2 \approx 1.96 \).

### IV. ANTIFERROMAGNETIC CASE

A significant advantage of the use of direct decimation on the fractal lattice is that, in contrast to the results obtained using star–triangle transformations, there are no ‘cuts’ in the recursion relations (\( \mathbb{H} \) and \( \mathbb{I} \)) for negative values of the couplings. This allows us to adapt the technique to consider a frustrated system with negative couplings \( K_0 \) and \( L_0 \).

By rescaling the couplings after each renormalisation group iteration, as discussed in Section 3, we can derive a recursion relation for the level of anisotropy in a manner analogous to that described for the ferromagnetic case, obtaining

\[
r_{n+1} = 1 - \frac{1}{2} \ln \left( \frac{\cosh 2r_n}{\cosh 2} \right).
\]  
(14)

The possible initial conditions for this relation again fall into the same two classes, \( r_0 > 1 \) or \( r_0 < 1 \), which were relevant in the previous discussion. However, for this model the fixed point set for the recursion relation (\( \mathbb{I} \)) reduces to a single point \( \{1\} \) and from this fact alone one might expect that the two \textit{a priori} different initial conditions will not lead to significantly different behaviour of the system under renormalisation. In other words, the existence of only one fixed point should lead to a single form of scaling behaviour.

Before we test this conjecture numerically we can verify the stability of the isotropic fixed point via arguments similar to those presented in Section 3. Treating the possible initial conditions separately we obtain
\[ r_{n+1} > 1 - \frac{1}{2} \ln \frac{e^{2r_n}}{e^2} = 2 - r_n, \quad \text{for } r_n > 1, \tag{15} \]
\[ r_{n+1} < 1 - \frac{1}{2} \ln \frac{e^{2r_n}}{e^2} = 2 - r_n, \quad \text{for } r_n < 1. \tag{16} \]

Thus the behaviour of \( r_n \) in the frustrated case is oscillatory about \( r = 1 \) between iterations, which clearly accounts for the loss of the fixed point at \( \infty \) in this case. From these relations one may again derive the inequality \( \delta_{n+1} < \delta_n \) unless \( r_n = 1 \) and thus \( r^* = 1 \) is again a stable fixed point of the recursion relation and the system will tend to isotropy on macroscopic scales. It is of interest to note that this is similar to the behaviour observed using star–triangle transformations in the ferromagnetic system.

As anticipated, numerical iteration of the recursion relation (14) indicates that for the frustrated case there is no qualitative difference in the behaviour of the deviation \( R_n \) from isotropy due to a different choice of initial conditions. Specifically, in Fig. 4 the scaled deviation \( R_n \) is plotted for the two initial configurations \( r_0 = 10 \) and \( r_0 = 0.1 \) indicating explicitly that the behaviour is qualitatively the same in each case. This is again a consequence of the loss of the second fixed point at \( r = \infty \).

![FIG. 4. The deviation from isotropy \( R_n \) of the frustrated interaction plotted against the renormalisation group iteration \( n \). The upper plot uses the initial condition \( r_0 = 10 \) while the lower curve uses \( r_0 = 0.1 \).](image)

Note that in this case, due to the fact that the oscillatory behaviour is not quite symmetric about \( r = 1 \), we plot every second iteration for clarity.

The physical consequence of the loss of the second fixed point at infinity can easily be observed from Figures 3 and 4. In the situation where at microscopic scales we have \( r_0 >> 1 \), or in physical terms where the interaction of the Ising spins in one direction on the lattice is much weaker than the other two, the restoration of isotropy in the frustrated system is significantly more rapid than the standard ferromagnetic system. Thus in this case the frustration appears to relieve the anisotropy on macroscopic scales.

However, if we consider the regime near isotropy we obtain a rather different conclusion.
In particular if we analyse the scaling of the ‘function’ $r(n)$ in a manner analogous to that presented in Section 3 we obtain a similar differential equation to (11), that is,

$$\frac{dr}{dn} + r + \ln \frac{\cosh 2r}{\cosh 2} - 1 = 0.$$  \hspace{1cm} (17)

The deviation $\delta_f = |r - 1|$ in this case also satisfies the scaling relation (13), with a smaller decay coefficient $\rho = 1 - \tanh 2 \approx 0.036$. Thus it appears that in this regime the frustration inhibits homogenisation. As a consequence, one can envisage that at least in the situation described by an initial condition $r_0 \gg 1$ the presence of a frustration has a markedly different effect on the level of isotropy at different length scales when compared to the ferromagnetic system.

V. DISCUSSION

In conclusion, we have examined the phenomenon of restoration of macroscopic isotropy for a simple interacting system, a microscopically anisotropic Ising model on a finitely ramified fractal, the Sierpiński gasket. This is an interesting example of a phenomenon that is absent in uniform media.

The results for the Ising spin model show an exponential recovery of isotropy at large scales, for both ordered (ferromagnetic) and disordered (antiferromagnetic) systems. In particular, it is interesting to note that for disordered systems, the phenomenon is independent of the initial conditions, while for ordered systems the behaviour depends sensitively on the initial conditions. Furthermore, we have also observed that, in the strongly frustrated regime, the restoration of isotropy is considerably slower than the corresponding unfrustrated cases.

The above results are also suggestive that further studies of models on spaces which are explicit geometric objects of nonintegral Hausdorff dimensions could uncover other subtle characteristics which may be of relevance to various physical systems. It could also be conjectured that the scaling behaviour which was analytically determined near isotropy may in fact be universal in such magnetic systems.

The authors express their gratitude to B. Meister and N. Rivier for stimulating discussions. The financial support of D.B. by PPARC and A.R by the Commonwealth Scholarship Commission and the British Council is gratefully acknowledged.
REFERENCES

[1] M. T. Barlow, K. Hattori, T. Hattori, and H. Watanabe, Phys. Rev. Lett. 75 (1995) 3042.
[2] Y. Gefen, B. B. Mandelbrot, and A. Aharony, Phys. Rev. Lett. 45 (1980) 855.
[3] Y. Gefen, A. Aharony, Y. Shapir, and B. B. Mandelbrot, J. Phys. A 17 (1984) 435.
[4] J. H. Luscombe and R. C. Desai, Phys. Lett. A 108 (1985) 39; Phys. Rev. B 32 (1985) 1614.
[5] T. Stosić, B. Stocić, S. Milosević, and H. E. Stanley, Phys. Rev. A 37 (1988) 1747.
[6] R. B. Stinchcombe, Physica D 38 (1989) 345.
[7] M. P. Grillon and F. G. Brady Moreira, Phys. Lett. A 142 (1989) 22.