Formation of Cylindrical Gravastars in Modified Gravity

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Abstract

In this paper, we analyze a few physical characteristics of gravastar with cylindrical geometry in \(f(R, T)\) theory, where \(R\) is the Ricci scalar and \(T\) is the trace of energy-momentum tensor. The gravastar is generally considered to be a substitute of a black hole with three different regions. In the present work, we examine the formulation of gravastar-like cylindrical structures in \(f(R, T)\) theory. By using Darmois and Israel matching conditions, we formulate a mass function of a thin shell. We calculate the different physical characteristics of gravastar, in particular, entropy within the thin shell, proper length of the intermediate thin shell as well as energy of the shell.

Keywords: Alternative to Black Holes, Manifolds, Gravitation.

1 Introduction

Some current experimental and observational results of cosmos asserted that our universe is expanding with an acceleration. Moreover, it has been analyzed that our universe is comprised of 5% baryonic matter along with 27% and 68% as dark matter (DM) and dark energy.
energy (DE), respectively [2–4]. Apart from general relativity (GR), there are many popular ways to comprehend evolutionary phases of our cosmos in modified theories, the theories that could be formulated from the modification of GR action function. Nojiri and Odintsov [5] indicated how these gravity theories provide help in the study of cosmic structure formation. Recent attractive modified theories include $f(R)$ with $R$ being the Ricci scalar [6], $f(T)$ (here, $T$ represents the torsion scalar) [7], $f(R, \Box R, T)$ (where, $T$ being the trace of matter tensor while $\Box$ indicates the de Alember’t’s operator) [8], $f(G)$ with $G$ as the Gauss-Bonnet term [9] and $f(G, T)$ [10], are the most popular gravity theories as alternative to GR (please see [11] for details). Harko et al. [12] presented $f(R, T)$ gravity by invoking corrections of $T$ in $f(R)$ gravity. They took $T$ in their calculations in order to study quantum effects. Houndjo [13] used this theory and presented some important cosmological models that are applicable at a certain cosmic era. The concept of Gravastars was introduced through the theoretical modeling of Mazur and Mottola [14, 15] by generalizing the basic concept of condensation provided by Bose-Einstein (BEC) for which the gravastar is governed by following regions. 1. Interior cylindrical region: $p = -\rho$.
2. Transitional layer: $p = +\rho$.
3. Exterior cylindrical region: $p = \rho = 0$.

In general, some other equations of state are described by Visser and Wiltshire [16]. The gravastars’s shell is treated here to be very thin ranging $r \in (r_1, r_2)$, here $r_1$ and $r_2$ are equivalent to $D$ and $D + \epsilon$ as the interior and exterior radial values. A lot of work is available in literature about the gravastar and their physical characteristic [17].

Ghosh et al. [19] found a class of solutions under two considerations for a $d$-dimensional metric with Einstein-Maxwell corrections. They discussed few captivating outcomes from the Reissner-Nordstr"om model. Horvat et al. [20] proposed a solution of gravastar under the influence of charged field. For the existence of such structure, they studied some corresponding astrophysical results. They calculated charged equations of motion, energy conditions and equation of state by making use of de-Sitter geometry as an interior geometry and Reissner-Nordstr"om (RN) as outer metric.

DeBenedictis et al. [23] investigated the existence of gravastars with the help of qualitative analysis and presented some viability bounds through energy conditions and a particular EoS. Carter [24] also discussed the stability of such bodies with the help of few conditions in order to model thin shell. The obtained solutions were found after taking a de-Sitter geometry as an inner region and Reissner-Nordstr"om geometry as an outer metric. Chirenti and Rezzolla [25] described the criteria by which one can perform differentiation of gravastar from a black hole. They calculated quasinormal constraints in order to present a generic gravastar models. Cecilia et al. [26] described some physical properties of for the fast rotating celestial bodies and claimed that there could be no horizon for the possible formation of gravastars.

Recently, Bhatti and his collaborators [28] found different significantly crucial results
related to the collapse and instability of compact objects. Yousaf [29] calculated the wide range of parametric values under which the obtained cylindrical gravastar solutions satisfy the stability conditions. He presented various qualitative aspects of EoS with the help of these parameters. His work is then extended for $f(R, T, R_{\mu\nu}T^{\mu\nu})$ [30] and $f(R, G)$ [31] theories of gravity with spherical spacetimes. Recently, the origin as well as the existence of gravastars have been reviewed by Ray et al. [32].

This paper extended the work of [33] and [34] for cylindrically symmetric metric in $f(R, T)$ theory. The work of this paper is arranged in following sections. In section 2, we formulate $f(R, T)$ field equation and few important corresponding equations. In section 3 and 4, we calculate the hydrostatic equation and gravitational mass with interior geometry. In section 5, we shall elaborate on the relation between pressure and radius. In section 6, we calculate the mass of the shell by matching inner and outer regions of spacetime. Section 7 describes a few important characteristics for the existence of gravastars. In section 7, we will discuss the various physical characteristic of gravastar.

2 Formulation of $f(R, T)$ Gravity

This section describe basic framework of $f(R, T)$ theory as well as their conservation equation. The action of $f(R, T)$ [35] is characterized with a generic function $f(R, T)$ as follows

$$S = \frac{1}{16\pi} \int (16\pi L_m \sqrt{-g} + \sqrt{-g} f(R, T)) \, d^4x,$$

and its trace by $T = g^{\xi\eta} T_{\xi\eta}$. The matter tensor can be evaluated as [36]

$$T_{\xi\eta} = \frac{-2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} L_m)}{\delta g^{\xi\eta}},$$

where $L_m$ indicates the matter Lagrangian and depends upon the metric $g_{\xi\eta}$ as

$$T_{\xi\eta} = g_{\xi\eta} L_m - 2 \frac{\partial L_m}{\partial g^{\xi\eta}}.$$

The variation on the action of the gravity corresponding to the metric $g_{\xi\eta}$ yields

$$\delta S = \frac{1}{16\pi} \int \left[ f_R \delta R + f_T \frac{\delta T}{\delta g^{\xi\eta}} \delta g^{\xi\eta} - \frac{1}{2} g_{\xi\eta} f_T \delta g^{\xi\eta} + 16\pi \frac{1}{\sqrt{-g}} \frac{\delta(\sqrt{-g} L_m)}{\delta g^{\xi\eta}} \right] \sqrt{-g} \, d^4x.$$

The change in the Ricci scalar $R$ can be given as follows

$$\delta R = \delta(g^{\xi\eta} R_{\xi\eta}) = R_{\xi\eta} \delta g^{\xi\eta} + g^{\xi\eta}(\nabla_{\omega} \delta \Gamma_{\xi\eta}^{\omega} - \nabla_{\eta} \delta \Gamma_{\xi\omega}^{\eta}),$$
where $\nabla_\omega$ represents covariant differential operator. The variation of Christoffel symbol provides

$$\delta \Gamma^\omega_{\xi\eta} = \frac{1}{2}g^{\omega\alpha}(\nabla_\xi \delta g_{\eta\alpha} + \nabla_\eta \delta g_{\alpha\xi} - \nabla_\alpha \delta g_{\xi\eta}).$$

(6)

Using the above equations in Eq.(1), we get field equation of $f(R, T)$ theory as

$$f(R_{\xi\eta} - \nabla_\xi \nabla_\eta + g_{\xi\eta} \Box) - \frac{1}{2}fg_{\xi\eta} + f_T(T_{\xi\eta} + \Theta_{\xi\eta}) = 8\pi T_{\xi\eta},$$

(7)

where $f=f(R, T)$, $f_R$ and $f_T$ are the partial differentiations of arbitrary function corresponding to $R$ and $T$ respectively, $\Box=\nabla^2=\nabla_\eta \nabla^\eta$ and

$$\Theta_{\xi\eta} = \frac{g^{\alpha\beta}}{\partial g_{\xi\eta}} \partial T_{\alpha\beta}.$$

(8)

The covariant derivative of Eq.(7) is

$$\nabla^\xi T_{\xi\eta} = \frac{f_T(R, T)}{8\pi - f_T(R, T)}[(T_{\xi\eta} + \Theta_{\xi\eta})\nabla^\xi \ln f_T(R, T) + \nabla^\xi (\Theta_{\xi\eta} - \frac{1}{2}g_{\xi\eta} T)].$$

(9)

We want to consider the isotropic cylindrically symmetric spacetime in order to model gravastars in $f(R, T)$ theory. Thus, we take matter tensor for the isotropic fluid as

$$T_{\xi\eta} = (p + \rho)U_\xi U_\eta - pg_{\xi\eta},$$

(10)

where $U_\xi$ is the four velocity component satisfying $U_\xi U^\xi = 1$, $\rho$ indicates matter density while $p$ shows the isotropic pressure of the fluid. In this environment we took the specific choices of few quantities as $L_m = -p$ and $\Theta_{\xi\eta} = -(2T_{\xi\eta} + pg_{\xi\eta})$ along with $f(R, T) = R(1 + \frac{2\chi T}{R})$ with constant term $\chi$. By substituting these, Eq.(7) becomes

$$G_{\xi\eta} = 8\pi T_{\xi\eta} + \chi[2T_{\xi\eta} + g_{\xi\eta}(T + 2p)],$$

(11)

where $G_{\xi\eta}$ represents the Einstein tensor and $T$ the trace of matter tensor. One can write Eq.(9) as

$$\nabla^\xi T_{\xi\eta} = \frac{-\chi}{2(4\pi + \chi)} [2\nabla^\xi (pg_{\xi\eta}) + g_{\xi\eta} \nabla^\xi T].$$

(12)

Using $\chi = 0$ in Eq.(11), one can easily get the field equation of $GR$. 

4
3 Formulation of Field Equations with Cylindrically Symmetric Spacetime

We take static cylindrical metric

\[ ds^2 = -dt^2 H(r) + dr^2 K(r) + (d\phi^2 + \alpha^2 dz^2) r^2, \]  

(13)

where \( H(r) = \sqrt{\alpha^2 r^2 - 4M \alpha r} \) and \( H(r) = \frac{1}{K(r)} \). The nonzero components of Einstein tensor of cylindrical symmetric interior geometry are

\[ G_{00} = \frac{H}{K^2 r^2} (K'r - K), \]  

(14)

\[ G_{11} = \frac{1}{H r^2} (H'r + H), \]  

(15)

\[ G_{22} = \frac{r}{4H^2 K'} (2H''HK - HH'K' - KH'^2 - 2K'H^2 + 2H'KH), \]  

(16)

where prime represents the derivative with respect \( r \). Now using Eqs. (13)-(16) in Eq. (11) then we get the following relations

\[ \frac{K'r - K}{K^2} = -r^2 [8\pi \rho - \chi(p - 3\rho)], \]  

(17)

\[ \frac{H'r + H}{HK} = -r^2 [8\pi p + \chi(3p - \rho)], \]  

(18)

\[ \frac{r}{4H^2 K'} (2H''HK - HH'K' - KH'^2 - 2K'H^2 + 2H'KH) = -r^2 [8\pi p + \chi(3p - \rho)]. \]  

(19)

Here \( G_{33} = \alpha^2 G_{22} \). The hydrostatic equilibrium equation follows the nonconservation Eq. (9) as

\[ \frac{dp}{dr} + \frac{H'(\rho + p)}{2H} + \frac{\chi}{2(4\pi + \chi)} \left( \frac{dp}{dr} - \frac{d\rho}{dr} \right) = 0. \]  

(20)

If \( r \) is the gravastar inner radius and \( m \) is the mass of the gravitating source within it, then by using Eq. (17) we get

\[ \frac{1}{K} = \frac{8m}{3h} + \chi(\rho - \frac{p}{3}) r^2. \]  

(21)

Substituting Eq. (21) in Eq. (20), we obtain

\[ \frac{dp}{dr} = -\frac{(\rho + p)}{2} \left[ \frac{r^2 [8\pi \rho + \chi(\rho - 3p)]}{\frac{8m}{3h} + \chi(\rho - \frac{p}{3}) r^2 [1 + \frac{\chi}{2(4\pi + \chi)} (1 - \frac{d\rho}{dp})]} \right]. \]  

(22)
4 Interior Geometry

In this section, we will explore the inner region of gravastar. We have already discussed that the structure of gravastar is based on three different regions (a) inner region (b) transitional layer and (c) outer region. The structure of gravastar is based on the parameter of the EoS. Also, \( r_1 = D = r_2 \) as the radius of inner and outer regions with EoS \( p = -\rho \) and \( p = \rho \), respectively while \( r_1 \leq r < r_2 \) is the radius of thin shell with EoS \( p = 0 = \rho \). To find the structural form of the coordinate of metric and their relationships, we use EoS of the interior region \( p = -\rho \). Assume that density is constant within the interior region so \( \rho = Y_0 \).

\[
p = -Y_0. \tag{23}
\]

Now, using Eq.(23) in Eq.(17) we obtain

\[
K = \frac{3}{r^3} \left[ \frac{1}{4(2\pi + \chi)Y_0} \right], \quad \text{and} \quad K = BH^{-1}, \tag{24}
\]

where \( B \) is the integration constant. A relationship between the spacetime potentials \( K \) and \( H \) is defined in Eq.(24). Gravitational mass \( M(D) \) of the inner region is defined as in given below

\[
M(D) = \int_0^{r_1=D} 4\pi Y_0 r^2 dr = \frac{4}{3} \pi D^3 Y_0. \tag{25}
\]

5 Geometry of Thin Shell

This section provides the shape of shell and we will examine the influence of pressure on the radius of gravastar. The shell is composed of ultrarelativistic matter under nonvacuum region. The state equation for intermediate shell is \( p = \rho \). With the help of \( \rho = p \) state equation in Eqs.(17) and (18), one can get

\[
\frac{d}{dr} (\ln r) = \frac{2}{r}, \tag{26}
\]

The substitution of EoS \( \rho = p \) in Eqs.(17) and (18) provide

\[
r \left( \frac{r H'}{2H} + 3 \right) K' = r^2. \tag{27}
\]

The integration of Eq.(26) yields

\[
K = r^2 + C, \tag{28}
\]
where $C$ represents integration constant while $r$ is the radius with range $D \leq r \leq D + \epsilon$, where $\epsilon \ll 1$. Integrating Eq.(27), we get

$$\frac{H'}{H} = -\frac{6}{r}. \quad (29)$$

By making use of EoS $p = \rho$ in Eq.(20), it follows that

$$p = \rho = Fr^6. \quad (30)$$

### 6 Junction Conditions

The configuration of gravastar is based on three different shall, i.e interior cylindrical shell, thin shell, and exterior cylindrical shell. In this section, we shall find the condition on an intermediate thin shell by interior and exterior regions. For smooth matching, we used Darmois [39] and Israel [38] junction conditions. Here, we consider the following spacetime as

$$ds^2 = -\left(1 - \frac{r^2}{\beta^2}\right) dt^2 + \left(1 - \frac{r^2}{\beta^2}\right)^{-1} dr^2 + r^2(d\phi^2 + \alpha^2 dz^2). \quad (31)$$

One can observe the continuity of coefficients across the boundary surface, the existence of their derivative at $r = D$ may not be continuous. To overcome the effects of discontinuity,
we use Israel formulation. The Lanczos equation \[40–43\] is defined as

\[
S_\kappa^\iota = -\frac{1}{8\pi}(\xi_\kappa^\iota - \delta_\kappa^\iota \xi_\iota^\iota),
\]

(32)

where \(S_\kappa^\iota\) expression is used for stress-energy surface and \(\xi_\kappa^\iota = \Omega_\kappa^\iota + \Omega_\kappa^\iota\). Here (+) sign represents the interior region and (−) sign represents exterior region. Second fundamental \[44–46\] form of hypersurface is defined as

\[
\Omega_\pm_{ij} = -n^\pm_\nu \left[ \frac{\partial^2 y_\nu}{\partial \zeta^i \partial \zeta^j} + \Gamma^\nu_\gamma_\delta \frac{\partial y^\gamma}{\partial \zeta^i} \frac{\partial y^\delta}{\partial \zeta^j} \right]_\Sigma,
\]

(33)

where \(\zeta^i\) are obtained through the coordinates on the hypersurface, \(y^\gamma\) represents the coordinate of manifold and \(n^\pm_\nu\) is the normal over the boundary which is defined as

\[
n^\pm_\nu = \pm \frac{1}{\sqrt{g^{\alpha\beta} \frac{\partial \ell(r)}{\partial y^\alpha} \frac{\partial \ell(r)}{\partial y^\beta}}^{\frac{1}{2}}} \frac{\partial \ell(r)}{\partial y^\nu},
\]

(34)

with timelike condition. From Lanczos equation, one can obtain \(S_\kappa^\iota = diag[\vartheta, -\varrho, -\varrho, -\varrho]\). Here \(\vartheta\) denotes the surface density and \(\varrho\) denotes the isotropic pressure over the boundary surface. The values of \(\varrho\) and \(\vartheta\) are

\[
\vartheta = -\frac{1}{4\pi D} [\sqrt{\ell(r)}]_+, \\
\varrho = -\frac{\vartheta}{2} + \frac{1}{16\pi} \left[ \frac{\ell'(r)}{\sqrt{\ell(r)}} \right]_+.
\]

(35)

(36)

Using Eqs.(35) and (36) we get

\[
\vartheta = \frac{1}{4\pi D} \left[ \sqrt{\left( \frac{D^2}{\beta^2} - 1 \right) - \sqrt{\frac{4D^2}{3}(2\pi + \chi)\varrho_0}} \right],
\]

(37)

and

\[
\varrho = \frac{1}{8\pi D} \left[ \sqrt{\left( \frac{D^2}{\beta^2} - 1 \right) - \sqrt{\frac{4\varrho_0 D^2}{3}(2\pi + \chi)}} \right].
\]

(38)
The thin shell mass can be calculated via areal density as

\[ m_s = 4\pi D^2 \vartheta = D \left[ \sqrt{\frac{4D^2}{3}} (2\pi + \chi) \rho_0 - \left( \frac{D^2}{\beta^2} - 1 \right) \right], \tag{39} \]

where

\[ \beta = \sqrt{\frac{D^4}{m_s^2} + \frac{3}{2\rho_0 (2\pi + \chi)} + \frac{D^3}{2m_s} \sqrt{\frac{3}{2\rho_0 D^2 (2\pi + \chi)}} + D^2}. \tag{40} \]

7 Characteristics of Gravastars

In this segment, we discuss different physical characteristics of gravastar through graphical representation.

7.1 Length of Intermediate Shell

This subsection is devoted to find the thickness of the shell using interior and exterior radius. We assume that the ratio \( \epsilon/D \) is too much less, i.e, \( \epsilon/D \ll 1 \). The length of the intermediate shell can be calculated as \[33\]

\[ \ell = \int_D^{D+\epsilon} \sqrt{K} \, dr = \int_D^{D+\epsilon} \sqrt{r^2 + C} \, dr. \tag{41} \]

Integrating the above equation gives us

\[ \ell = \left[ \frac{1}{2} r \sqrt{r^2 + C} + \frac{1}{2} C \ln(r + \sqrt{r^2 + C}) \right]_D^{D+\epsilon}. \tag{42} \]

7.2 Energy Contents

Energy Contents within the thin shell is given as \[33\]

\[ \varepsilon = \int_D^{D+\epsilon} 4\pi r^2 \rho \, dr = \int_D^{D+\epsilon} 4\pi H r^8 \, dr, \tag{43} \]

which gives

\[ \varepsilon = \frac{4}{9} \pi H \left[ (D + \epsilon)^9 - D^9 \right]. \tag{44} \]
We can see the behavior of energy contents within the thin shell against thickness is shown in Fig. (3). Graph shows the positive and linear relationship.

### 7.3 Entropy within the Shell

By the theory of Mazur and Mottola [14, 15], gravastar has zero disorderness in the interior cylindrical region. Now we are going to determine the corresponding entropy [33] by letting $r_1 = D$ and $r_2 = D + \epsilon$

$$S = \int_{D}^{D+\epsilon} 4\pi r^2 N(r) \sqrt{K} \, dr,$$

where $N(r)$ is the entropy density and is defined as

$$N(r) = \left(\frac{\alpha K_B}{h}\right) \left(\frac{P}{2\pi}\right)^{\frac{1}{2}}, \quad K_B = h = 1.$$  \hspace{1cm} (46)

Using Eqs. (30) and (46) in Eq. (45), the entropy within the shell is

$$S = (8\pi F)^{\frac{1}{2}}\alpha \int_{D}^{D+\epsilon} r^5 \sqrt{r^2 + C} \, dr,$$

which provides

$$S = (8\pi F)^{\frac{1}{2}}\alpha \left[\frac{1}{105}(r^2 + C)^{\frac{1}{2}}(15r^4 - 12Cr^2 + 8C^2)\right]_{D}^{D+\epsilon}. \hspace{1cm} (48)$$
7.4 State Equation

The EoS explains the link between density and pressure as

$$\varrho = \omega(D) \vartheta,$$

substituting the values of $\varrho$ and $\vartheta$ from Eqs. (37) and (38) in above the above equation, it follows that

$$\omega(D) = \frac{\left( \frac{2D^2}{D^2 - 1} - \frac{8\pi D^2}{3(2\pi + \chi)\rho_0} \right)}{2 \left[ \sqrt{\frac{D^2}{D^2 - 1}} - \sqrt{\frac{4D^2}{3(2\pi + \chi)\rho_0}} \right]},$$

we used some approximation $\frac{D^2}{D^2 - 1} \ll 1$ and $\frac{4}{3}\rho_0(2\pi + \chi)D^2 \ll 1$ to obtain a real solution, thus we get

$$\omega(D) \approx -1.$$  

8 Conclusion

As a result of the gravitational collapse, the presence of a new theoretical system may be evident. It can be accomplished by carefully examining the fundamental concepts of BEC. In
BEC the boson gas is cooled to the point where infinite kinetic energy is lost by the molecules. We should consider that all boson molecules are identical with the same quantum spin, and thus do not obey the exclusive theory of Pauli. Any of the final gravitational collapsing stages may lead to gravastar, thus suggesting this as an alternative to the black hole as indicated by [14].

In this work, we have discussed the geometry of gravastar and their different characteristics under static cylindrically symmetric spacetime. The solution of the gravastar is based on these three cases, first is the interior, second is the thin shell and third is the exterior region of cylindrically. By matching the first and third regions, we obtain a mass of thin cylindrical shell. In this paper, we calculate the gravitational mass of the interior cylindrical region using EoS $p = -\rho$ under constant density.

We have also discussed the characteristics of gravastar, e.g proper length of the thin shell, entropy and energy contents within the shell. In this first hand, we have studied the proper length of a thin shell. It is found that it depends upon the radius of the interior and exterior regions. In Fig. (2), we see the behavior of the proper length of the thin shell against the thickness. Second is the energy content within the shell, Fig. (3) shows the positive and linear relationship between energy and thickness. The third that we have discussed is the entropy, which is the disorderliness in the surface of gravastar. In Fig. (4), we observe that if thickness increases then entropy will also increase. Lastly, we calculate the equation of state by putting some certain state such as $\frac{D^2}{\rho^2} \ll 1$ and $\frac{4}{3}\rho_0(2\pi + \chi)D^2 \ll 1$. 

Figure 4: Plot of the entropy of the shell against the thickness $\epsilon$ (km) of the shell.
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References

[1] D. Pietrobon, A. Balbi, and D. Marinucci, Phys. Rev. D 74, 043524 (2006); T. Gianantonio, et al., Phys. Rev. D 74, 063520 (2006); A. G. Riess, et al., Astrophys. J. 659, 98 (2007).

[2] P. A. R. Ade et al. [BICEP2 Collaboration], Phys. Rev. Lett. 112, 241101 (2014) [arXiv:1403.3985 [astro-ph.CO]].

[3] P. A. R. Ade et al. [BICEP2 and Planck Collaborations], Phys. Rev. Lett. 114, 101301 (2015) [arXiv:1502.00612 [astro-ph.CO]].

[4] P. A. R. Ade et al. [BICEP2 and Keck Array Collaborations], Phys. Rev. Lett. 116, 031302 (2016) [arXiv:1510.09217 [astro-ph.CO]].

[5] S. Nojiri and S. D. Odintsov, eConf C 0602061 (2006) 06 [Int. J. Geom. Meth. Mod. Phys. 04, (2007) 115] [hep-th/0601213].

[6] S. Capozziello, Int. J. Mod. Phys. D 11, 483 (2002); E. Elizalde, S. Nojiri, S. D. Odintsov, L. Sebastiani, and S. Zerbini Phys. Rev. D, 83, 086006 (2011).

[7] K. Bamba, C.-Q. Geng, C.-C. Lee and L.-W. Luo, J. Cosmol. Astropart. Phys. 01, 021 (2011).

[8] M. J. S. Houndjo, M. E. Rodrigues, N. S. Mazhari, D. Momeni and R. Myrzakulov, Int. J. Mod. Phys. D 26, 1750024 (2017); Z. Yousaf, M. Sharif, M. Ilyas and M. Z. Bhatti, Int. J. Geom. Meth. Mod. Phys. 15, 1850146 (2018); M. Ilyas, Z. Yousaf and M. Z. Bhatti, Mod. Phys. Lett. A. 34 1950082 (2019).

[9] S. Nojiri and S. D. Odintsov, Phys. Lett. B 631, 1 (2005); A. De Felice and S. Tsujikawa, Phys. Lett. B 675, 1 (2009); N. M García, F. S. N. Lobo and J. P. Mimosa, J. Phys. Conf. Ser. 314, 01205 (2011); K. Bamba, M. Ilyas, M. Z. Bhatti and Z. Yousaf, Gen. Relativ. Gravit. 49, 112 (2017) [arXiv:1707.07386 [gr-qc]].
M. Sharif and A. Ikram, Eur. Phys. J. C, 363, 178 (2018); Z. Yousaf, Astrophys. Space Sci. 363, 226 (2018); Z. Yousaf, Eur. Phys. J. Plus 134, 245 (2019); M. F. Shamir and M. Ahmad, Mod. Phys. Lett. A 34, 1950038 (2019).

S. Nojiri and S. D. Odintsov, Phys. Rept. 505, 59 (2011) [arXiv:1011.0544 [gr-qc]]; A. Joyce, B. Jain, J. Khoury and M. Trodden, Phys. Rept. 568, 1 (2015) [arXiv:1407.0059 [astro-ph.CO]]; S. Capozziello and V. Faraoni, Beyond Einstein Gravity (Springer, Dordrecht, 2010); S. Capozziello and M. De Laurentis, Phys. Rept. 509, 167 (2011) [arXiv:1108.6266 [gr-qc]]; K. Bamba, S. Capozziello, S. Nojiri and S. D. Odintsov, Astrophys. Space Sci. 342, 155 (2012) [arXiv:1205.3421 [gr-qc]]; K. Koyama, [arXiv:1504.04623 [astro-ph.CO]]; A. de la Cruz-Dombriz and D. Sáez-Gómez, Entropy 14, 1717 (2012) [arXiv:1207.2663 [gr-qc]]; K. Bamba, S. Nojiri and S. D. Odintsov, [arXiv:1302.4831 [gr-qc]]; Symmetry 7, 220 (2015) [arXiv:1503.00442 [hep-th]]; Z. Yousaf, K. Bamba and M. Z. Bhatti, Phys. Rev. D 95, 024024 (2017) [arXiv:1701.03067 [gr-qc]]; M. Z. Bhatti and Z. Yousaf, Int. J. Mod. Phys. D 26, 1750045 (2017); M. Sharif and Z. Yousaf, Mon. Not. R. Astron. Soc. 432, 264 (2013).

T. Harko, F. S. N. Lobo, S. Nojiri and S. D. Odintsov, Phys. Rev. D 84, 024020 (2011).

M. J. S. Houndjo, Int. J. Mod. Phys. D 21, 1250003 (2012).

P. Mazur and E. Mottola, Report No. LA-UR-01- 5067.

P. Mazur and E. Mottola, Proc. Natl. Acad. Sci. U.S.A. 101, 9545 (2004).

M. Visser, and D. L. Wiltshire, Class. Quantum Grav. 21, 1135 (2004).

B. V. Turimov, B. J. Ahmedov, and A. A. Abdujabbarov, Mod. Phys. Lett. A 24, 733 (2009); F. Rahaman, S. Ray, A. A. Usmani, and S. Islam, Phys. Lett. B 707, 319 (2012); F. S. N. Lobo and R. Garattini, J. High Energy Phys. 12, 065 (2013); A. A. Usmani, et al., Phys. Lett. B 701, 388 (2011); M. F. Shamir and M. Ahmad, Phys. Rev. D 97, 104031 (2018); M. Sharif and F. Javed, Ann. Phys. 415, 168124 (2020).

P. Bhar, Astrophys. Space Sci. 354(2014)457.

S. Ghosh, F. Rahaman, B. K. Guha, and S. Ray, Phys. Lett. B 380, 767 (2017).

D. Horvat, S. Ilijic and A. Marunovic, Class. Quantum Grav. 26, 025003 (2009).

T. Kubo and N. Sakai, Phys. Rev. D 93, 084051 (2016).

V. Cardoso, P. Pani, M. Cadoni, and M. Cavaglia, Phys. Rev. D 77(2008)124044.
[23] A. DeBenedictis, D. Horvat, S. Ilijic, S. Kloster and K. S. Viswanathan, Class. Quantum Grav. 23, 7 (2006).

[24] B. M. N Carter, Class. Quant. Grav. 22, 4551 (2005).

[25] C. B. M. H. Chirenti and L. Rezzolla, Class. Quantum Grav. 24, 4191 (2007).

[26] C. Cecilia, B. M. H. Chirenti and L. Rezzolla, Phys. Rev. D 78, 084011 (2008).

[27] C. Cattoen, T. Faber, and M. Visser, Class. Quantum Grav. 22, 4189 (2005).

[28] M. Z. Bhatti and Z. Tariq, Phys. Dark Universe 28, 100482 (2020); M. Z. Bhatti, Z. Yousaf and M. Yousaf, Phys. Dark Universe 28, 100501 (2020); Z. Yousaf, M. Z. Bhatti and T. Naseer: Phys. Dark Universe 28, 100535 (2020).

[29] Z. Yousaf, Phys. Dark Universe 28, 100509 (2020).

[30] Z. Yousaf, M. Z. Bhatti and H. Asad, Phys. Dark Universe 28, 100527 (2020).

[31] M. Z. Bhatti, Z. Yousaf and A. Rehman, Phys. Dark Universe 28, 100561 (2020).

[32] S. Ray, R. Sengupta and H. Nimesh, Int. J. Mod. Phys. D 29, 2030004 (2020).

[33] A. Das et al.: Phys. Rev. D 95, 124011 (2017).

[34] Z. Yousaf, K. Bamba, M. Z. Bhatti and U. Ghafoor, Phys. Rev. D 100, 024062 (2019) [arXiv:1907.05233 [gr-qc]].

[35] M. Sharif and A. Ikram, Eur. Phys. J. C 76, 640 (2016).

[36] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Butterworth-Heinemann, Oxford, 1998).

[37] O. J. Barrientos and G. F. Rubilar, Phys. Rev. D 90, 028501 (2014).

[38] W. Israel, Nuovo Cimento, 44, 1 (1966); 48, 463(E) (1967).

[39] G. Darmois, *Des Sciences Mathematiques XXV, Fasticule XXV* (Gauthier-Villars, Paris, France, 1927).

[40] K. Lanczos, Ann. Phys. 379, 518 (1924).

[41] N. Sen, Ann, Phys. 378, 365 (1924).
[42] G. P. Perry and R. B. Mann, Gen. Relativ. Gravit. 24, 305 (1992).

[43] P. Musgrave and K. Lake, Class. Quantum Grav. 13, 1885 (1996).

[44] F. Rahaman, M. Kalam, and S. Chakraborty, Gen. Relativ. Gravit. 38, 1687 (2006).

[45] A. A. Usmani et al, Gen. Relativ. Gravit. 42, 2901 (2010).

[46] G. A. S. Dias and J. P. S. Lemos, Phys. Rev. D 82, 084023 (2010).