Statistical Complexity in Traveling Densities

RICARDO LÓPEZ-RUIZ
Universidad de Zaragoza
Department of Computer Science and BIFI
Campus San Francisco, E-50009 Zaragoza
SPAIN
rilopez@unizar.es

JAIME SAÑUDO
Universidad de Extremadura
Department of Physics and BIFI
Avda. de Elvas, E-06071 Badajoz
SPAIN
jsr@unex.es

Abstract: In this work, we analyze the behavior of statistical complexity in several systems where two identical densities that travel in opposite direction cross each other. The crossing between two Gaussian, rectangular and triangular densities is studied in detail. For these three cases, the shape of the total density presenting an extreme value in complexity is found.

Key Words: Statistical complexity, traveling densities, distribution shape

1 Introduction

An interesting problem that has not been broadly investigated in the literature is the behavior of the statistical complexity in time-dependent systems. A work in this direction was done in Ref. [1] where this behavior was studied in a gas out of equilibrium decaying toward the asymptotic equilibrium state.

In this communication, we carry out the study of the statistical complexity $C$ in a time-dependent system $\rho(x, t)$ composed of two one-dimensional (variable $x$) identical densities that travel in opposite directions with the same velocity $v$, one of them, $\rho_+(x, t)$, going to the right and the other one, $\rho_-(x, t)$ going to the left. That is

$$\rho(x, t) = \frac{1}{2} \rho_+(x, t) + \frac{1}{2} \rho_-(x, t),$$

with the normalization condition $\int_{\mathbb{R}} \rho_\pm(x, t) dx = 1$ that implies the normalization of $\rho(x, t)$. In the next section, we perform the analysis of $C$ for two Gaussian, rectangular and triangular traveling densities, which verify the initial condition $\rho_+(x, 0) = \rho_-(x, 0)$. Specifically, the shape of $\rho(x, t)$ presenting the maximum and minimum $C$ is found for these three cases. The final section includes our conclusions.

2 Complexity in Traveling Densities

Let us start by recalling the definition of the statistical complexity $C$ [2], the so-called LMC complexity, that is defined as

$$C = H \cdot D,$$

where $H$ represents the information content of the system and $D$ gives an idea of how much concentrated is its spatial distribution. For our purpose, we take a version used in Ref. [3] as quantifier of $H$. This is the simple exponential Shannon entropy $\mathcal{H}$, that takes the form,

$$H = e^S,$$

where $S$ is the Shannon information entropy $\mathcal{S}$,

$$S = -\int p(x) \log p(x) \, dx,$$

with $x$ representing the continuum of the system states and $p(x)$ the probability density associated to all those states. We keep for the disequilibrium the form originally introduced in Refs. [2, 3], that is,

$$D = \int p^2(x) \, dx.$$

Now we proceed to calculate $C$ for the system above mentioned (1) in the Gaussian, rectangular and triangular cases.

2.1 Gaussian case

Here the two one-dimensional traveling densities that compose system (1) take the form:

$$\rho_\pm(x, t) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left\{ -\frac{(x \mp vt)^2}{2\sigma^2} \right\},$$

where $\sigma$ is the variance of the density distribution.

The behavior of complexity, $C_G$, as a function of the adimensional quantity $2vt/\sigma$ is given in Fig. 1. Let us observe that $C_G$ presents a minimum. The shape of system (1) for this minimum complexity case is plotted in an adimensional scale in Fig. 2.
2.2 Rectangular case

Now the two one-dimensional traveling densities that compose system (1) take the form:

\[
\rho_{\pm}(x, t) = \begin{cases} 
\frac{1}{\delta} & \text{if } -\delta/2 \leq x \mp vt \leq \delta/2, \\
0 & \text{if } |x \mp vt| > \delta/2. 
\end{cases}
\] (7)

where \(\delta\) is the width of each distribution.

For this case, the complexity, \(C_R\), can be analytically obtained. Its expression is:

\[
C_R(t) = \begin{cases} 
2^{2vt/\delta} \left(1 - \frac{vt}{\delta}\right) & \text{if } 0 \leq 2vt \leq \delta, \\
1 & \text{if } 2vt > \delta. 
\end{cases}
\] (8)

The behavior of \(C_R\) as a function of the adimensional quantity \(2vt/\delta\) is given in Fig. 3. Let us observe that \(C_R\) presents a maximum. The shape of system (1) for this maximum complexity case is plotted in an adimensional scale in Fig. 4.

2.3 Triangular case

The two one-dimensional traveling densities that compose system (1) take the form in this case:

\[
\rho_{\pm}(x, t) = \begin{cases} 
\frac{(x \mp vt)}{\epsilon^2} + \frac{1}{\epsilon} & \text{if } -\epsilon \leq x \mp vt \leq 0, \\
\frac{-(x \mp vt)}{\epsilon^2} + \frac{1}{\epsilon} & \text{if } 0 < x \mp vt \leq \epsilon, \\
0 & \text{if } |x \mp vt| > \epsilon, 
\end{cases}
\] (9)

where \(\epsilon\) is the width of each distribution (isosceles triangle whose base length is \(2\epsilon\)).

The behavior of complexity, \(C_T\), as a function of the adimensional quantity \(2vt/\epsilon\) is given in Fig. 5. Let us observe that \(C_T\) presents a maximum and a minimum. The shape of system (1) for both cases, with maximum and minimum complexity, are plotted in an adimensional scale in Figs. 6 and 7, respectively.

3 Conclusion

In this communication, we have studied the behavior of the statistical complexity as a function of time when two traveling identical densities are crossing each other. Three cases have been analyzed: Gaussian, rectangular and triangular densities. The Gaussian case presents a configuration with minimum complexity. The rectangular case displays a configuration with maximum complexity. The triangular case
Figure 3: Statistical complexity, $C_R$, vs. the adimensional separation, $2vt/\delta$, between the two traveling rectangular densities defined in Eq. (7). The maximum of $C_R$ is reached when $2vt/\delta = 0.557$. Observe that the normalized rectangular distribution has $C_R = 1$. This shows an intermediate behavior between the two former cases with a maximum complexity configuration and another one with minimum complexity.

Acknowledgements: This research was supported by the Spanish Grant with Ref. FIS2009-13364-C02-C01. J.S. also thanks to the Consejería de Economía, Comercio e Innovación of the Junta de Extremadura (Spain) for financial support, Project Ref. GRU09011.

References:
[1] X. Calbet and R. Lopez-Ruiz, Tendency toward Maximum Complexity in a Non-Equilibrium Isolated System, Phys. Rev. E 63, 2001, 066116 (9 pages).
[2] R. Lopez-Ruiz, H.L. Mancini and X. Calbet, A Statistical Measure of Complexity, Phys. Lett. A 209, 1995, 321–326.
[3] R.G. Catalan, J. Garay and R. Lopez-Ruiz, Features of the Extension of a Statistical Measure of Complexity to Continuous Systems, Phys. Rev. E 66, 2002, 011102 (6 pages).
[4] A. Dembo, T.A. Cover and J.A. Thomas, Information Theoretic Inequalities, IEEE Trans. Inf. Theory 37, 1991, pp. 1501–1518.
[5] C.E. Shannon, A Mathematical Theory of Communication, Bell. Sys. Tech. J. 27, 1948, pp. 379–423; ibid., 1948, pp. 623–656.

Figure 4: Shape of the density (7) in adimensional units that presents the maximum statistical complexity when the two traveling rectangular densities defined in (7) are crossing. Notice that the value of the adimensional separation between the centers of both rectangular distributions must be 0.557. Then, the width of the overlapping between both distributions is 0.443.

Figure 5: Statistical complexity, $C_T$, vs. the adimensional separation, $2vt/\epsilon$, between the two traveling triangular densities given in Eq. (9). The maximum and minimum of $C_T$ are reached when $2vt/\epsilon$ takes the values 0.44 and 1.27, respectively. The dashed line indicates the value of complexity for the normalized triangular distribution.
Figure 6: Shape of the density (1) in adimensional units that presents the maximum statistical complexity when the two traveling triangular densities defined in (9) are crossing. Notice that the value of the adimensional separation between the centers of both triangular distributions must be 0.44.

Figure 7: Shape of the density (1) in adimensional units that presents the minimum statistical complexity when the two traveling triangular densities defined in (9) are crossing. Notice that the value of the adimensional separation between the centers of both triangular distributions must be 1.27.