DIVERGENCE OF MORSE GEODESICS

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Abstract. Behrstock and Drutu raised one question about the existence of Morse geodesics in CAT(0) spaces with divergence strictly greater than \( r^n \) and strictly less than \( r^{n+1} \), where \( n \) is an integer greater than 1. In this paper, we answer the question of Behrstock and Drutu by showing that there is a CAT(0) space \( X \) with a proper, cocompact action of some finitely generated group such that for each \( s \) in (2,3) there is a Morse geodesic in \( X \) with divergence \( r^s \).

1. Introduction

The divergence of two geodesic rays \( \alpha \) and \( \beta \) with the same initial point \( x_0 \) in a geodesic space \( X \), denoted \( \text{Div}_{\alpha,\beta} \), is a function \( g : (0, \infty) \to (0, \infty) \) which for each positive number \( r \) the value \( g(r) \) is the infimum on the lengths of all paths outside the open ball with radius \( r \) about \( x_0 \) connecting \( \alpha(r) \) and \( \beta(r) \). Consequently, the divergence of a bi-infinite geodesic \( \gamma \), denoted \( \text{Div}_{\gamma} \), is the divergence of the two geodesic rays obtained from \( \gamma \) with the initial point \( \gamma(0) \). The divergence of \( X \) is the supremum of \( \text{Div}_{\gamma} \) over all bi-infinite geodesics \( \gamma \) of \( X \). We define the divergence of a finitely generated group to be the divergence of its Cayley graph. A geodesic \( \gamma \) is Morse if for any constants \( K > 1 \) and \( L > 0 \), there is a constant \( M = M(K,L) \) such that every \( (K,L) \)-quasi-geodesic \( \sigma \) with endpoints on \( \gamma \) lies in the \( M \)-neighborhood of \( \gamma \). When investigating the divergence of CAT(0) spaces, Behrstock and Drutu asked two questions:

Question 1.1. (see Question 1.3, [BD]) Are there examples of CAT(0) groups whose divergence is strictly between \( r^n \) and \( r^{n+1} \) for some \( n \)?

Question 1.2. (see Question 1.5, [BD]) If \( X \) is a CAT(0) space, can the divergence of a Morse geodesic be greater than \( r^n \) and less than \( r^{n+1} \) for some \( n \geq 2 \)?

If the answer for the Question 1.1 is positive, the positive answer for Question 1.2 follows easily. However, Question 1.1 is still open, which makes it harder to construct Morse geodesics for the answer of Question 1.2. Motivated by these questions, we show the following theorem:

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Main Theorem. There is a CAT(0) space $X$ with a proper, cocompact action of some finitely generated group such that for each $s$ in $(2, 3)$ there is a Morse geodesic in $X$ with the divergence $r^s$.

The main theorem gives a positive answer for Question 1.2 and we hope that the positive answer for this question can shed a light for the positive answer for Question 1.1. The CAT(0) space we examine in this paper was constructed by Dani-Thomas [DT]. They showed that the divergence of this space is cubic and they constructed some periodic geodesics with quadratic and cubic divergence. However, the existence of Morse geodesic with divergence greater than $r^2$ and less than $r^3$ was still unknown.

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2. Right-angled Coxeter groups

Definition 2.1. Given a finite, simplicial graph $\Gamma$, the associated right-angled Coxeter group $G_\Gamma$ has generating set $S$ the vertices of $\Gamma$, and relations $s^2 = 1$ for all $s$ in $S$ and $st = ts$ whenever $s$ and $t$ are adjacent vertices.

Definition 2.2. Given a nontrivial, connected, finite, simplicial, triangle-free graph $\Gamma$ with the set $S$ of vertices, we may define the Davis complex $\Sigma = \Sigma_\Gamma$ to be the Cayley 2–complex for the presentation of the Coxeter group $G_\Gamma$, in which all disks bounded by a loop with label $s^2$ for $s$ in $S$ have been shrunk to an unoriented edge with label $s$. Then the vertex set of $\Sigma$ is $G_\Gamma$ and the 1-skeleton of $\Sigma$ is the Cayley graph $C_\Gamma$ of $G_\Gamma$ with respect to the generating set $S$. Since all relators in this presentation other than $s^2 = 1$ are of the form $stst = 1$, $\Sigma$ is a square complex. The Davis complex $\Sigma_\Gamma$ is a CAT(0) space and the group $G_\Gamma$ acts properly and cocompactly on the Davis complex $\Sigma_\Gamma$ (see [Dav08]).

Definition 2.3. Given a nontrivial, connected, finite, simplicial, triangle-free graph $\Gamma$ and let $\Sigma = \Sigma_\Gamma$ be the associated Davis complex. We observe that each edge of $\Sigma$ is on the boundary of a square. We define a midline of a square in $\Sigma$ to be a geodesic segment in the square connecting two midpoints of its opposite edges. We define a hyperplane to be a connected subspace that intersects each square in $\Sigma$ in empty set or a midline. Each hyperplane divides the square complex $\Sigma$ into two components. We define the support of a hyperplane $H$ to be the union of squares which contain edges of $H$.

Since each square in $\Sigma$ has the label of the form $stst$, then each midline in each square of $\Sigma$ connects two midpoints of edges with the same label. Thus, each hyperplane is a graph and vertices are the midpoints of edges with the same label. Therefore, we define the type of a hyperplane $H$ to be the label of edges containing vertices of $H$. Obviously, if two hyperplanes with the types $a$ and $b$ intersect, then $a$ and $b$ commute.
Remark 2.4. The length of a path $\alpha$ in $C_\Gamma$ is equal to its number of hyperplane-crossings. A path is a geodesic if and only if it does not cross any hyperplane twice (see Lemma 3.2.14 \cite{Dav08}).

Lemma 2.5. Let $\alpha$ be a geodesic in $C_\Gamma$. Let $H_1$ and $H_2$ be two hyperplanes that cross $\alpha$. If $H_1$ and $H_2$ intersect, then we can find two hyperplanes that cross two consecutive edges of $\alpha$ which intersect. In particular, $\alpha$ contains two consecutive edges such that their labels commute.

The proof of the above lemma is quite obvious by using the fact that each hyperplane divides the square complex $\Sigma$ into two components.

3. Proof of the Main Theorem

Let $\Gamma$ be the graph and $\Gamma_1$ a full subgraph as drawn in Figure 1. We see that $\Gamma$ and $\Gamma_1$ are finite, simplicial and triangle-free graphs. Denote $S$ to be the set of all vertices of $\Gamma$. Let $w$ be the word $b_1a_1a_2b_2b_0a_0b_2a_2a_3$ of length 9. Observe that each pair of consecutive generators of $w$ does not commute. Moreover, the first and last generators of $w$ do not commute and neither commutes with $c$.

For each $d > 1$ let $\gamma_d$ be the bi-infinite path in $C_\Gamma$ which passes through $e$ and labeled by $\cdots wwwwcw[cw[cw[cw[cw[\cdots$, where $k = d - 1$, such that $\gamma_d(0) = e$, $\gamma_d(1) = c$ and $\gamma_d(-9) = w^{-1}$. We observe that the labels of two consecutive edges of $\gamma_d$ do not commute. Thus, $\gamma_d$ is a bi-infinite geodesic (see Theorem 3.4.2, \cite{Dav08}). We define a function $f_d$ on the set of positive integers as follow:

$$f_d(n) = [1^{d-1}] + [2^{d-1}] + [3^{d-1}] + [4^{d-1}] + [5^{d-1}] + \cdots + [n^{d-1}].$$

There are constants $0 < h_d \leq 1/2$ and $n_d > 0$ such that for each $n > n_d$ the following holds:

$$h_d n_d^d \leq f_d(n) \leq n_d^d.$$

We are going to use the constants $h_d$ and $n_d$ many times in the rest of the paper.

The following lemma is a direct consequence of Lemma 4.10 in \cite{DT}.

\begin{figure}[h]
\centering
\begin{tabular}{ll}
(A) & (B) \\
\end{tabular}
\caption{The graph $\Gamma$ and a full subgraph $\Gamma_1$}
\end{figure}
Lemma 3.1. Let $\alpha$ be an arbitrary geodesic ray emanating from $e$ that travels along the support of the hyperplane labeled by $c$ and let $\beta$ be a path emanating from $e$ consisting of a geodesic segment labeled $cw^i$ followed by an arbitrary geodesic ray emanating from $cw^j$ that travels along the support of hyperplane labeled by $c$. Then $\beta$ is a geodesic, and for any $r > 20i$,

$$\operatorname{Div}_{\alpha, \beta}(r) \geq \frac{1}{16} r^2$$

The following lemma is obtained from Proposition 4.7 in [DT] and the fact that $\Sigma_{\Gamma_1}$ embeds isometrically in $\Sigma_{\Gamma}$ (see [Dav08]).

Lemma 3.2. Let $\alpha$ and $\beta$ be two geodesic rays with the same initial point $x_0$ and edges labeled by the vertices of $\Gamma_1$. There is a positive constant $M$ such that the following holds. For each positive number $r$, there is a path $\eta$ outside the ball $B(x_0, r)$ in $C_{\Gamma}$ connecting $\alpha(r)$ and $\beta(r)$ such that the length of $\eta$ is bounded above by $Mr^2$.

Proposition 3.3. For each $d > 1$, the divergence $\operatorname{Div}_{\gamma_d} \leq r^{2+\frac{1}{4}}$.

Proof. For each number $r$ large enough, we can choose an integer $n > n_d$ such that

$$r \leq h_d n^d \leq f_d(n) \leq n^d \leq \left( \frac{2}{h_d} \right) r.$$ 

Let $x = cw^{[2^k]}cw^{[3^k]}cw^{[4^k]} \cdots cw^{[n^k]}$, where $k = d - 1$. Then

$$|x|_S = n + 9\left( |1^{d-1}| + |2^{d-1}| + |3^{d-1}| + |4^{d-1}| + |5^{d-1}| + \cdots + |n^{d-1}| \right) = n + 9 f_d(n).$$

Thus, $5 f_d(n) \leq |x|_S \leq 10 f_d(n)$. Therefore, we can connect $x$ and $\gamma_d(r)$ by a path $\beta_1$ outside $B(e, r)$ such that

$$\ell(\beta_1) \leq 10 f_d(n) - r \leq \left( \frac{20}{h_d} - 1 \right) r.$$ 

We now try to connect $\gamma_d(-r)$ and $x$ by a path $\beta_2$ outside $B(e, r)$ such that $\ell(\beta_2) \leq Mr^{2+\frac{1}{4}}$ for some constant $M$ not depending on $r$ and which completes the proof of the proposition.

Let $k = d - 1$. Let $s_0 = e$ and $s_i = cw^{[2^k]}cw^{[3^k]}cw^{[4^k]} \cdots cw^{[i^k]}$ for $1 \leq i \leq n$. Let $H_e$ be the support of the hyperplane that crosses the edge $e$ with one endpoint $e$. For $0 \leq i \leq n$, let $u_i$ and $v_i$ be geodesic rays which run along the support $s_i H_e$ with the initial points $s_i$ and $s_i c$ respectively. We can choose $u_i$ and $v_i$ such that they have the same label for all $i$. Thus, each edges of $u_i$ and $v_i$ is labeled by $a_2$ or $b_2$. For $0 \leq i \leq n - 1$, let $m_i$ be a geodesic with the initial point $s_i c$ which runs along $w^{[i(i+1)^k]}$ followed by $u_{i+1}$. (The fact that $m_i$ is a geodesic is guaranteed by Lemma 3.1.) Moreover, the ray $\sigma$ with the initial point $e$ which runs along a geodesic segment $cw^{[2^k]}cw^{[3^k]}cw^{[4^k]} \cdots cw^{[n^k]}$ followed by $u_n$ is a geodesic since each pair of consecutive edges of $\sigma$ is labeled by two group generators which do not commute.
For $0 \leq i \leq n - 1$, we can connect $u_i(20f_d(n))$ and $v_i(20f_d(n))$ by a single edge path $\eta_i$ labeled by $c$. Let $\eta_n$ be a subsegment of $u_n$ connecting $x$ and $u_n(20f_d(n))$. Obviously, each $\eta_i$ must lie outside $B(e, r)$ for $0 \leq i \leq n$. By Lemma 3.2, we can connect $v_i(20f_d(n))$ and $m_i(20f_d(n))$ by a path $\eta'_i$ outside $B(s_i, c, 20f_d(n))$ with length bounded above by $M_1(f_d(n))^2$ for some constant $M_1$ not depending on $r$ and $n$. Thus, we can connect $m_i(20f_d(n))$ and $u_{i+1}(20f_d(n))$ by a path $\eta''_i$ outside $B(s_i, c, 20f_d(n))$ such that the length of $\eta''_i$ is bounded above by $9[(i + 1)^k]$. Thus, the length of $\eta''_i$ is bounded above by $9n^k$. By the same argument as above, we can show that $\eta''_i$ lies outside $B(e, r)$.

Let $\eta = (\eta_0\eta_1')\eta_1''(\eta_2\eta_2')\cdots(\eta_{n-1}\eta_{n-1}')\eta_{n-1}$. Thus, $\eta$ is a path outside $B(e, r)$ connecting $u_0(20f_d(n))$ and $x$. Moreover,

$$\ell(\eta) \leq n\left(1 + M_1(f_d(n))^2 + 9n^k\right) + 20f_d(n).$$

It follows that there is some constant $M_2$ not depending on $r$ and $n$, such that the length of $\eta$ is bounded above by $M_2n^{2d+1}$. Therefore, there is some constant $M_3$ not depending on $r$ and $n$, such that the length of $\eta$ is bounded above by $M_3r^{2d+\frac{1}{2}}$.

Obviously, we can connect $u_0(20f_d(n))$ and $u_0(r)$ by a path $\alpha_1$ outside $B(e, r)$ with length bounded above by $20f_d(n)$. Thus, the length of $\alpha_1$ is bounded above by $40(h_d)r$. Using Lemma 3.2 again, we can connect $\gamma_d(-r)$ and $u_0(r)$ by a path $\alpha_2$ outside $B(e, r)$ with length bounded above by $M_4r^2$ for some constant $M_4$ not depending on $r$. Let $\beta_2 = \alpha_2\alpha_1$. Then, $\beta_2$ lies outside $B(e, r)$ and connects $\gamma_d(-r)$ and $x$. Moreover, the length of $\beta_2$ is bounded above by $M_4r^{2d+\frac{1}{2}}$ for some constant $M$ not depending on $r$. □

Before working on the lower bound of $\text{Div}_{\gamma_d}$, we have a small observation.

**Remark 3.4.** Let $\alpha$ and $\beta$ be two geodesic rays in a CAT(0) space with the same initial point $x_0$. Assume that $\text{Div}_{\alpha, \beta}(r) \geq f(r)$. Using the fact that projections do not increase distances, we can show that if $\eta$ is a path outside $B(x_0, r)$ connecting two points on $\alpha$ and $\beta$, then $\ell(\eta) \geq f(r)$. These observations will be used in the following proof.

**Proposition 3.5.** For each $d > 1$, we have $r^{2d+\frac{1}{2}} \leq \text{Div}_{\gamma_d}$.

**Proof.** For each number $r$ large enough, we can choose an integer $n > n_d$ such that

$$r \leq 60h_d n^d \leq 60f_d(n) \leq 60n^d \leq \left(\frac{2}{h_d}\right)r.$$

Let $\eta$ be any path outside $B(e, r)$ connecting $\gamma_d(-r)$ and $\gamma_d(r)$. Since $\gamma_d$ restricted to $[-r, r]$ is a geodesic and $\eta$ is a path with the same endpoints, $\eta$ must cross each hyperplane crossed by $\gamma_d([-r, r])$ at least once. Let $s_0 = e$
and $s_i = cw^{[2^k]} c w^{[3^k]} c w^{[4^k]} \cdots c w^{[i^k]}$ for $1 \leq i \leq [h_d n]$, where $k = d - 1$.

Thus,

$$|s_i|_S \leq i + 9 \left( [d^{-1}] + [2d^{-1}] + [3d^{-1}] + [4d^{-1}] + [5d^{-1}] + \cdots + [i^{-1}] \right) \leq 10 h_d n^d.$$ 

Let $H_c$ be the support of the hyperplane that crosses the edge $c$ with one endpoint $e$. For $0 \leq i \leq [h_d n]$, let $(g_i, g_{i+1})$ be the edge of $C_T$ at which $\eta$ first crosses $s_i H_c$, where $g_i$ is the vertex in the component of the complement of the hyperplane in $s_i H_c$ containing $e$. Let $v_i$ denote the geodesic connecting $s_i$ and $g_i$ which runs along $s_i H_c$. For $0 \leq i \leq [h_d n] - 1$, let $\eta_i$ be a subsegment of $\eta$ connecting $g_i$ and $g_{i+1}$. Let $m_i$ be a geodesic with the initial point $s_i$ which runs along $cw^{[(i+1)^k]}$ followed by $v_i+1$. (The fact that $m_i$ is a geodesic is guaranteed by Lemma 3.1.) Since

$$d_S(g_i, s_i) \geq d_S(g_i, e) - d_S(s_i, e) \geq r - 10 h_d n^d \geq 20 h_d n^d$$

and

$$d_S(g_{i+1}, s_i) \geq d_S(g_{i+1}, e) - d_S(s_i, e) \geq r - 10 h_d n^d \geq 20 h_d n^d,$$

then

$$\ell(\eta_i) \geq \frac{1}{16} (20 h_d n^d)^2 \geq 25 h_d^2 n^{2d}$$

by Lemma 3.1 and Remark 3.4. Thus,

$$\ell(\eta) \geq 25 h_d^2 n^{2d}(h_d n - 2) \geq M r^{2+\delta}$$

for some constant $M$ not depending on $r$, which proves the proposition. □

Before showing that each bi-infinite geodesic $\gamma_d$ is Morse, we would like to mention the concept of lower divergence as follows. The lower divergence of a bi-infinite ray $\gamma$, denoted $\text{ldiv} \gamma$, is a function $h : (0, \infty) \to (0, \infty)$ which for each positive number $r$ the value $h(r) = \inf_{t} \rho_{\gamma}(r, t)$, where $\rho_{\gamma}(r, t)$ is the infimum of the lengths of all paths from $\gamma(t - r)$ to $\gamma(t + r)$ which lie outside the open ball of radius $r$ about $\gamma(t)$.

**Proposition 3.6.** For each $d > 1$, the geodesic $\gamma_d$ is Morse.

**Proof.** In this proof, we will use the result of Charney-Sultan [CS] that a bi-infinite geodesic in a CAT(0) space is Morse if its lower divergence is superlinear. For each $t$ and each $r$ large enough, let $\eta$ be any path outside $B(\gamma_d(t), r)$ connecting $\gamma_d(t - r)$ and $\gamma_d(t + r)$. Since $\gamma_{[t-r,t+r]}$ is a geodesic and $\eta$ is a path with the same endpoints, $\eta$ must cross each hyperplane crossed by $\gamma_{[t-r,t+r]}$ at least once. By the construction of $\gamma_d$, we can choose two points $u_1$ and $u_2$ on $\gamma_d$ which lie between $\gamma_d(t - r)$ and $\gamma_d(t + r)$ and satisfy the following conditions:

1. The distance between $\gamma_d(t)$ and $u_1$ is bounded above by 9, and the distance between $\gamma_d(t + r)$ and $u_2$ is also bounded above by 9
2. The subsegment $\gamma_d^t$ between $u_1$ and $u_2$ is label by $w^{n_0}$ for $n_0 \geq 1$ or
   $$w^{n_0} c w^{[n_1]} c w^{[(n_1+1)^k]} c w^{[(n_1+2)^k]} \cdots c w^{[(n_1+\ell)^k]} c w^{[n_2]}$$
   for non-negative numbers $n_0$, $n_1$, $n_2$, $t$ and $k = d - 1$. 


Thus, $\eta$ lies outside the ball $B(u_1, r - 9)$; and the length of $\gamma_d'$ is bounded below by $r - 18$ and above by $r + 18$.

Let $m$ be the number of edges of $\gamma_d'$ labeled by $b_1$ (the first generator of $w$). Then, $\ell(\gamma_d')/10 - 1 \leq m \leq \ell(\gamma_d')/9 + 1$. Thus, $(r - 28)/10 \leq m \leq (r + 27)/9$.

For $1 \leq i \leq m$, let $e_i$ be the $i$th edge of $\gamma_d'$ labeled by $b_1$. Thus, the endpoints of each $e_i$ are $s_i$ and $s_i b_1$. We assume that $s_i$ is closer to $u_1$ than $s_i b_1$. Obviously, the distance between $s_i$ and $u_1$ is bounded above by $10i$.

Let $H_{b_1}$ be the support of the hyperplane that crosses the edge $b_1$ with one endpoint $e$. For $1 \leq i \leq m$, let $(g_i, g_i b_1)$ be the edge of $C_I$ at which $\eta$ first crosses $s_i H_{b_1}$, where $g_i$ is the vertex in the component of the complement of the hyperplane in $s_i H_{b_1}$ containing $u_1$. Let $v_i$ denote the geodesic connecting $s_i$ and $g_i$ which runs along $s_i H_{b_1}$. Thus, each edge of $v_i$ is labeled by $a_0$, $b_0$ or $b_2$. For $1 \leq i \leq m - 1$, let $\eta_i$ be a subsegment of $\eta$ connecting $g_i$ and $g_{i+1}$.

For $1 \leq i \leq m - 1$, let $H_i$ be any hyperplane crossed by $v_i$, with the type $y$. Thus, $y$ is $a_0$, $b_0$ or $b_2$. By the construction, the segment of $\gamma_d'$ between $s_i$ and $s_{i+1}$ crosses a hyperplane of type $y$. Moreover, this hyperplane cannot intersect $s_i H_{b_1}$ by Lemma 2.5 and the construction of $\gamma_d'$. Therefore, it is distinct from $H_i$ and consequently separates $H_i$ from $v_{i+1}$. Similarly, $H_i$ can not intersect $\gamma_d'$. Thus, $H_i$ must separate $g_i$ and $g_{i+1}$ into distinct components, so $\eta_i$ must cross $H_i$. Therefore,

$$\ell(\eta_i) \geq \ell(v_i) \geq d_S(g_i, u_1) - d_S(s_i, u_1) \geq (r - 9) - 10i \geq r - 9 - 10i.$$  

Thus,

$$\ell(\eta) \geq \sum_{i=1}^{m-1} \ell(\eta_i) \geq (m - 1)(r - 9 - 5m) \geq \left( \frac{r - 28}{10} - 1 \right) \left( r - 9 - \frac{5(r + 27)}{9} \right)$$

Therefore, $\rho_{\gamma_d}(r, t)$ is bounded below by $f(r)$ for all $t$, where $f$ is a quadratic function not depending on $t$. Thus, $\gamma_d$ has superlinear lower divergence. \(\square\)

Thus, for each $s$ in $(2, 3)$ the geodesic $\gamma_d$ is Morse and has the divergence $r^s$, where $d = 1/(s - 2)$. This proves the Main Theorem.

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