Convex Neighbourhoods and Complete Finsler Spaces *

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Abstract

In this paper, it is shown that a large set of connections on a suitable sub–bundle of the tangent bundle of a Finsler Manifold can be used to study all the properties of convex neighbourhoods with respect to the Finsler Metric, which are needed to see that any Complete Finsler Space is Geodesically Connected.

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1 Introduction

Let $M$ be a $C^\infty$–differentiable $n$–dimensional manifold endowed with a Finslerian metric function $F : TM \to \mathbb{R}$, being $TM$ the tangent bundle of $M$.

The following properties of the convex neighbourhoods of $M$ with respect to $F$ are well known (see, [1], [2], see also [3]). In order to quote them, let us denote by

$$B_\rho(0_x) = \{ X \in T_x(M) \mid F(x, X) < \rho \}$$

the open indicatrix having $\rho$ as its radius and the zero vector $0_x \in T_x(M)$ as its center, for any $\rho > 0$ and $x \in M$.

Moreover, we denote by $\exp_x$ the exponential mapping of the Finsler metric and $\exp_x$ is defined on an open neighbourhood of $0_x \in T_x(M)$ into $T_x(M)$, with $x \in M$. We recall that the mapping $\exp_x$ is defined by setting $\exp_x(X) = c_X(1)$, being $c_X$ the geodesic of $F$ defined by the initial conditions $c_X(0) = x$ and $\dot{c}_X(0) = X$, with $X$ belonging to a suitable open neighbourhood of $0_x$ in $T_xM$.

With these notations, we have:

**Proposition 1.1.** For each $x \in M$, there exist two positive real numbers $\varepsilon = \varepsilon(x)$ and $\eta = \eta(x)$ with $0 < \varepsilon < \eta$ such that:

i). $\exp_x : B_\varepsilon(0_x) \to \exp_x(\exp_x(B_\varepsilon(0_x))) = B_\varepsilon(x)$ is a diffeomorphism defined on $B_\varepsilon(0_x)$, whose degree of differentiability is $C^\infty$ on $B_\varepsilon(0_x) - \{0_x\}$ and $C^1$ on $B_\varepsilon(0_x)$. Moreover, $B_\varepsilon(x)$ is an open neighbourhood of $x$.

ii). For each $y, z \in B_\varepsilon(x)$, there exists a unique geodesic $c : [0, 1] \to M$ lying entirely in $B_\eta(y) = \exp_y(\exp_y(0_y))$ having length lesser than $\eta$ and such that $c(0) = y$ and $c(1) = z$.

iii). For $y \in B_\varepsilon(x)$, the mapping $\exp_y : B_\eta(0_y) \to B_\eta(y)$ is a diffeomorphism defined on $B_\eta(0_y) - \{0_y\}$ and $C^1$ on $B_\eta(0_y)$ and $B_\eta(y)$ is an open neighbourhood of $y$ containing $B_\varepsilon(x)$.

As in the Riemannian case (see, e.g., [4]), we consider the canonical identification $T_xT_xM = T_xM$, for any $X \in T_xM$ and $x \in M$ and we fix an element $x$ of $M$. Then, $T_x'M = T_xM - \{0_x\}$ is endowed by the Riemannian Metric $g_p(X, Y) = g(x, p)(X, Y)$, for each $p \in T_x'M$ and $X, Y \in T_pT_xM = T_xM$, being $g$ the metric tensor which is induced by $F$. Moreover, let $\tilde{T}M \subseteq TM$ be the open neighbourhood of the zero section where the exponential map of the Finsler Metric is defined. Then, we have:
Proposition 1.2. Let $x \in M$ and $X \in \tilde{U}_x = \tilde{T}M \cap T_xM$. Moreover, let $\tilde{b} : [0, s_1] \to \tilde{U}_x$ be any differentiable curve such that $\tilde{b}(0) = 0_x$ and $\tilde{b}(s_1) = X$. Finally let us put $b(s) = \text{Exp}_x \tilde{b}(s)$, for any $s \in [0, s_1]$ and $c(t) = \text{Exp}_x (tX)$, for any $t \in [0, 1]$. Then:

i). $L(c) \leq L(b)$, where $L$ denotes the Finslerian length of curves.

ii). If $\tilde{b}(s) = \tau(s)X$, for any $s \in [0, s_1]$, being $\tau : [0, s_1] \to [0, 1]$ a strictly increasing differentiable mapping, then $L(c) = L(b)$.

iii). If $L(c) = L(b)$ and the total differential $(\text{DExp}_x)_{\tilde{b}(s)}$ has maximal rank, for any $s \in [0, s_1]$ and $t \in [0, 1]$, then there exists a differentiable map $\tau : [0, s_1] \to [0, 1]$ such that $\tilde{b}(s) = \tau(s)X$, for each $s \in [0, s_1]$ and $\tau$ is strictly increasing.

Proposition 1.3. Let $\tilde{\varepsilon} = \varepsilon/3$, and $\varepsilon$ as in Proposition 1.1. Then, every $y, z \in B_{\tilde{\varepsilon}}(x)$ can be joined by a geodesic $c$, lying entirely in $B_{\tilde{\varepsilon}}(x)$ and having Finslerian length equal to the Finslerian distance between $y$ and $z$. Moreover, any further geodesic joining $y$ and $z$ (if it exists) has points outside of $B_{\eta}(y)$.

Finally, we recall the following definition: we say that $F$ is a Complete Finsler Metric, if and only if the distance which is induced by $F$ on $M$ is complete. Then it results:

Proposition 1.4. Let $F$ be a Finslerian metric on $M$. Then the following properties are equivalent:

i). $F$ is a complete Finslerian metric.

ii). There exists a point $p \in M$ such that any geodesic starting from $p$ can be extended to the whole $\mathbb{R}$.

iii). Assertion ii) holds, for any $p \in M$.

Moreover, it results:

iv). If Assertion i) holds, any two points of $M$ can be joined by a geodesic of $F$.

In 1967, B. T. Hassan (see [3]) proved Proposition 1.1 by using the Cartan Connection relative to $F$. Proofs of more or less complete versions of Proposition 1.1 are given by several authors by means of connections different from the Cartan one. In 1993, D. Bao and S. S. Chern proved the same proposition by using a new connection, called the Chern Connection (see [5]), which coincides with the Rund Connection (see [6]).

Finally, in [1] the previous proposition was proved by means of the Cartan Form. The proofs of all the previous propositions can be found in this book.
Generally, all these proofs are different from the corresponding proofs of the Riemannian case, more difficult than them and this fact explains the number of the proofs of Proposition 1.1.

Some questions arise for the previous observations. For example: Why there exist so much proofs for Proposition 1.1? Why they are so different from the proof needed in the Riemannian case? Does the used connection play any role?

In [6] and [7], by using the bundle $T'M = TM - \sigma(M)$, where $\sigma : M \rightarrow TM$ is the zero section, we determined all the connections by means of which the Euler–Lagrange assumes the simplest form. These connections were called Finslerian Connections and they are a large class of connections.

In this paper we prove:

**Proposition 1.5.** Any Finslerian Connection can be used to prove all the previous propositions.

We also show that all the proofs of the previous propositions can follow in the closest way the corresponding proofs used in the Riemannian case and this fact can be useful for the proof of further results. We choose to follow [4].

In our opinion, the problems previously listed raise because classical methods used in this context mix up the Finsler Geometry with the geometry of the tangent bundle of the manifold under consideration.

By using the previous considerations and the methods introduced here, in [9] we determine all the properties corresponding to the ones considered here for a large class of Lagrangian Functions and in [10] we show that the results of [9] give a new sufficient condition for the geodesic connectedness of the generalized Bolza problem.

## 2 Definitions and Proofs

Let $M$ be a $C^\infty$–differentiable $n$–dimensional manifold.

The notations, which are more frequently used in the following, are:

- $\alpha). \ TM = \bigcup_{x \in M} T_x M$ is the tangent bundle of $M$ and $\pi : TM \rightarrow M$ its natural projection. Moreover, for each $x \in M$, either $X_x$ or equivalently $(x, X)$ will denote the same element of $T_x M$ according to this element is considered as a tangent vector at $x \in M$ or as a point of $TM$; hence $X_x = (x, X)$. 

\( \beta \). \( \mathcal{I}(M) = \bigoplus_{(r,s) \in \mathbb{N}^2} \mathcal{I}^r_s(M) \) is the algebra of tensor fields on \( M \), where \( \mathcal{T}_0^0(M) = \mathcal{X}(M) \) and \( \mathcal{T}_0^0(M) = \mathcal{F}(M) \) are the Lie algebra of vector fields and the ring of \( C^\infty \)-differentiable real valued functions, respectively.

\( \gamma \). Let \( \sigma : M \to TM \) be the zero section and we set \( T'M = TM - \sigma(M) \). \( T'M \) is the open sub-bundle of \( TM \) of non-zero tangent vectors of \( M \) and we shall denote by \( \pi' : T'M \to M \) its canonical projection.

\( \delta \). \( \mathcal{I}^r_s = \bigoplus \mathcal{I}^r_s(M) \) is the \( \mathcal{F}(T'M) \)-module (\( \mathcal{F}(M) \)-module) of differentiable tensor fields along \( \pi' \), with \( \mathcal{T}_0^1 = \mathcal{X}_{\pi'} \) and \( \mathcal{T}_0^0 = \mathcal{F}(T'M) \).

\( \theta \). If \((U,\varphi)\) is a local chart of \( M \), we denote by \((TU,T\varphi)\) the local chart canonically induced on \( TM \), with \( TU = \varphi^{-1}U \). We set \( \varphi = (x^1,\ldots,x^n) \), \( T\varphi = (x^1,\ldots,x^n,\dot{x}^1,\ldots,\dot{x}^n) \), \( e_i = \frac{\partial}{\partial x^i}, \varepsilon_i = \frac{\partial}{\partial \dot{x}^i} \), \( e^j = dx^j, \varepsilon^j = d\dot{x}^j \), for any \( i,j,i',j' \in \{1,\ldots,n\} \).

We recall that:

A non-linear connection \( \nabla \) (with three indices) can be regarded as an \( \mathbb{R} \)-bilinear mapping \( \nabla : \mathcal{X}_{\pi'} \times \mathcal{I}(M) \to \mathcal{I}_{\pi'} \) such that:

i). \( \nabla_{fX}(kY) = fX(k)Y + k\nabla_X Y \), \( \forall X \in \mathcal{X}_{\pi'}, \forall Y \in \mathcal{X}(M), \forall f,k \in \mathcal{F}(M) \);

ii). \( \nabla \) commutes with all the contractions;

iii). for each \( X \in \mathcal{X}_{\pi'} \), with \( X \) positively homogeneous of degree \( \rho \), \( \nabla_X Y \) is positively homogeneous of degree \( \rho \), for any \( Y \in \mathcal{X}(M) \).

Moreover, \( \nabla \) defines a horizontal lift \( h : \mathcal{X}_{\pi'} \to \mathcal{X}(T'M) \) which can be extended in a trivial way to the whole tensor algebra \( \mathcal{I}_{\pi'} \). This extension is called Matsumoto lift (see, e.g., [3]). We also recall that the identity \( id : T'M \to T'M \) can be considered as a vector field along \( \pi' \) \((id \) is the so-called fundamental vector field along \( \pi' \)). The vector field \( id_h \) will be said to be associated to \( \nabla \) and one of its properties is:

**Proposition 2.1.** \( id_h \) can be extended to a spray defined on the whole \( M \), which will be denoted again with \( id_h \). This extension is \( C^1 \)-differentiable on \( TM \) and \( C^\infty \)-differentiable on \( T'M \). Moreover, a curve \( \gamma : [0,1] \to M \) is a non constant geodesic of \( id_h \) if and only if \( \gamma \) is a path of \( \nabla \).

The spray \( id_h \) defines an exponential map \( \text{Exp} : \tilde{T}M \to M \), being \( \tilde{T}M \) an open neighbourhood of \( \sigma(M) \). We have \( \text{Exp}(x,X) = c_{(x,X)}(1) \), being \( c_{(x,X)} : [0,\varepsilon] \to M \), with \( \varepsilon > 1 \), the geodesic of \( id_h \) having \( (x,X) \) as initial condition, for all \( (x,X) \in \tilde{T}M \). The exponential map verifies the following properties:
1. $\Exp$ is $C^1$–differentiable on $\tilde{T}M$ and $C^\infty$–differentiable on $\tilde{T}M – \sigma(M)$ (see [11], pg. 72).

2. $\Exp$ has maximal rank in the zero vector $0_x \in T_x M$, for any $x \in M$ (see [12], 2.8, Satz. (a), pg. 61).

3. The map $(\pi, \Exp) : \tilde{T}M \to M \times M$ has maximal rank in $0_x \in T_x M$, for any $x \in M$ (see [12], 2.8, Satz. (c), pg. 61).

4. The total differential of $\Exp$, when it is restricted to $\tilde{T}M \cap T_x M$ and it is calculated in $0_x$, coincides with the identity map, for any $x \in M$ (see [11], Th. 8, pg. 72).

Moreover, as in the Riemannian case (see, e.g., [4]) from the previous properties, it follows:

**Proposition 2.2.** There exists an open neighbourhood $\tilde{W}$ of $\sigma(M)$, such that $\Exp(\tilde{W}) = W$ is open and:

i). For each $y, z \in W$ there exists a unique geodesic $c : [0, 1] \to M$ such that $c(0) = y$, $c(1) = z$, $\dot{c}(0) \in \tilde{W}$ and $c(t) \in W$, for any $t \in [0, 1]$.

ii). For each $y \in W$, the map $\Exp_y : \tilde{W} \cap T_y M \to \Exp(\tilde{W} \cap T_y M) = W(y)$ is a diffeomorphism of class $C^1$, of class $C^\infty$ on $(\tilde{W} \cap T_y M) – \{0_y\}$ and if $y \in W(x)$, with $x \in M$, then $W(x) \subseteq W(y)$.

iii). The mapping $(\pi, \Exp)_{\tilde{W}} : \tilde{W} \to W \times W$ is a diffeomorphism of class $C^1$ and of class $C^\infty$ on $\tilde{W} – \sigma(M)$.

Taking into account the previous proposition, the set $W(0_x) = \tilde{W} \cap T_x M$ is an open neighbourhood of $0_x$ in $T_x M$ and $W_x = \Exp_x(W(0_x))$ is an open neighbourhood of $x$, for any $x \in M$.

Now, let $F : TM \to \mathbb{R}$ be a Finsler metric and $g$ its metric tensor. Then $g$ is a family of Riemannian metrics on $M$ depending on $(x, X) \in TM$ and such that $g_{(x,X)}(X_x, X_x) = F^2(x, X)$, for any $x \in M$ and $(x, X) = X_x \in TM$. Locally, we set $g = g_{ij}e^i \otimes e^j$, with

$$g_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial X^i \partial X^j}.$$

Let $V = V(TM)$ be the vertical sub bundle of $TTM$; i.e., $V$ is the subset of $TTM$ containing all the vectors tangent to the fibres of the tangent bundle of $M$. We fix a further sub bundle $H$ of $TTM$ such that $T_{(x,X)} M = H_{(x,X)} \oplus V_{(x,X)}$, for any $(x, X) \in TM$. Two canonical projections $P : TTM \to H$ and $Q : TTM \to V$ are associated to the previous splitting into direct sum of
These projections are determined by two tensor fields of type (1,1) denoted, by an abuse of notation, again by $P$ and $Q$. Locally, $P$ and $Q$ are defined by:

$$P = \delta^i_j e_i \otimes e^j - P_j^i \varepsilon_i \otimes e^j \quad\text{and}\quad Q = \delta^i_j \varepsilon_i \otimes \varepsilon^j + P_j^i \varepsilon_i \otimes e^j .$$

We recall that the distribution $H$ is said to be a non linear–connection with two indices if the functions $P_j^i$ are positively homogeneous of degree one (see, e.g.,[2]). Moreover, any connection on $M$ determines a couple of tensor fields of the previous kind.

Now, we turn to the general case and we set

$$G = g^v = g_{ij} e_i \otimes e^j ,$$

being $v$ the extension to the tensor fields along $\pi'$ of the usual vertical lift of tensors (see [13]).

Then, from [7], it follows:

**Proposition 2.3.** There exists a connection $\nabla$ on $TM$ such that

i). $C_1^2(P \otimes T) = 0$, where $T$ is the torsion tensor field of $\nabla$ and $C_1^2$ is the contraction of the first contra–variant index with the second covariant index of $P \otimes T$.

ii). Let us denote by $\pi_* : TTM \to TM$ the total differential of the canonical projection $\pi : TM \to M$ (see [13]). Then, for each vector field $X \in \mathcal{X}(T'(M))$, having $\pi_*(X(x,z)) = Z_x$, for any $Z_x = (x,z) \in T'(M)$ it results:

$$(\nabla_X G)(X,Y) = (\nabla_Y G)(X,X) = 0 , \ \forall Y \in \mathcal{X}(T'M) .$$

iii). The mapping $\nabla' : \mathcal{X}_{\pi'} \times \mathcal{I}(M) \to \mathcal{I}_{\pi'}$ defined by

$$\nabla'_X Y = \pi_*(\nabla_X Y_c) , \ \forall X,Y \in \mathcal{X}(M)$$

being $c$ the complete lift (see [13]), is a non–linear connection with three indices.

A connection $\nabla$ verifying i), ii) and iii) of the previous proposition is called Finslerian Connection. Moreover, the non–linear connection $\nabla'$, defined by iii) of the same proposition, is called Berwald Connection deduced from $\nabla$.

From [7], we also get:
Proposition 2.4. Let us denote $\nabla$ by a Finslerian Connection and let $\nabla'$ be the Berwald Connection deduced from $\nabla$. Let $Y = (\gamma, \dot{\gamma}) : [a, b] \to T'M$ be a curve. Then, $\gamma$ is a path of $\nabla'$ if and only if there exists a vector field $Z \in \mathcal{X}(T'M)$ such that:

$$(\nabla_Y \dot{Y})(t) = Q_{(\gamma(t),\dot{\gamma}(t))}(Z_{(\gamma(t),\dot{\gamma}(t))}), \quad \forall t \in [a, b].$$

In [7], it was also proved that the set of all the Finslerian Connections can be obtained in the following way.

Let $\tilde{\nabla}$ be any connection on $T'M$ and $(U, \varphi)$ be a chart of $M$. With respect to the chart $(T'U, T'\varphi)$, induced by $(U, \varphi)$ on $T'M$, we set:

$$\nabla_{e_j} e_k = \tilde{\Gamma}^{1i}_{jk} e_i + \tilde{\Gamma}^{5i}_{jk} \varepsilon_i, \quad \nabla_{e_j} e_k = \tilde{\Gamma}^{2i}_{jk} e_i + \tilde{\Gamma}^{6i}_{jk} \varepsilon_i,$$

$$\nabla_{e_j} e_k = \tilde{\Gamma}^{3i}_{jk} e_i + \tilde{\Gamma}^{7i}_{jk} \varepsilon_i, \quad \nabla_{e_j} e_k = \tilde{\Gamma}^{4i}_{jk} e_i + \tilde{\Gamma}^{8i}_{jk} \varepsilon_i.$$

The $8n^3$ functions $(\tilde{\Gamma}^{1i}_{jk}, \tilde{\Gamma}^{2i}_{jk}, \tilde{\Gamma}^{3i}_{jk}, \tilde{\Gamma}^{4i}_{jk}, \tilde{\Gamma}^{5i}_{jk}, \tilde{\Gamma}^{6i}_{jk}, \tilde{\Gamma}^{7i}_{jk}, \tilde{\Gamma}^{8i}_{jk})$ are the local components of $\tilde{\nabla}$. By using the local components of $G$ and $P$ and the previous $8n^3$ functions, we obtain the following new $8n^3$ functions defined on $T'U$:

$$2\Gamma^{1i}_{jk} = g^{im}(\partial_f g_{mk} + \partial_k g_{mj} - \partial_m g_{jk} + P^h_m \partial_h g_{jk} - P^h_j \partial_h g_{mk} - P^h_k \partial_h g_{mj}),$$

$$\Gamma^{2i}_{jk} = 0, \quad \Gamma^{3i}_{jk} = 0, \quad \Gamma^{4i}_{jk} = 0,$$

$$\Gamma^{5i}_{jk} = P^i_l \tilde{\Gamma}^{1l}_{jk} - P^i_l \tilde{\Gamma}^{1l}_{jk} + \tilde{\Gamma}^{5i}_{jk},$$

$$\Gamma^{6i}_{jk} = P^i_l \tilde{\Gamma}^{2l}_{jk} + \tilde{\Gamma}^{6i}_{jk}, \quad \Gamma^{7i}_{jk} = P^i_l \tilde{\Gamma}^{3l}_{jk} + \tilde{\Gamma}^{7i}_{jk}, \quad \Gamma^{8i}_{jk} = P^i_l \tilde{\Gamma}^{4l}_{jk} + \tilde{\Gamma}^{8i}_{jk},$$

where we used the notations $\partial f = \frac{\partial f}{\partial x^i}$ and $\partial_i f = \frac{\partial f}{\partial x^i}$, for any $f \in \mathcal{F}(T'M)$ and any $i, \hat{i} \in \{1, \ldots, n\}$.

The above functions are the local components of a connection $\nabla$ of $T'M$. Since $\nabla$ verifies i), ii) and iii) of Proposition 2.3, it is a Finslerian Connection. The connection $\nabla$ can be used in order to determine all the Finslerian Connections. In fact, let $\nabla^1$ be a further connection and let us denote by $N \in \mathcal{T}^1_2(T'M)$ the tensor field defined by setting:

$$N(X, Y) = \nabla_X Y - \nabla^1_X Y, \quad \forall X, Y \in \mathcal{X}(T'M).$$

There exists a unique tensor field, $\tilde{N}$, of type $(1, 2)$ along $\pi'$, such that:

$$\tilde{N}(X, Y) = \pi_a N(X^a, Y^a), \quad \forall X, Y \in \mathcal{X}(M).$$
\( \tilde{N} \) is called the tensor field associated to \( N \). Now, we denote by \( d' \) the so-called "derivation along the fibres", which is defined by setting \( d'f = (\partial_i f) e^i \), for any \( f \in F(T'M) \) and is extended to the whole tensor algebra \( \mathcal{L}_x' \) in the well-known way. Then, we have:

**Proposition 2.5.** Under the previous assumptions, the connection \( \nabla^1 \) is a Finslerian Connection, if and only if the following conditions hold for \( N \):

i). \( \tilde{N}(x,Z)(Z_x,Z_x) = 0 \), \( \forall (x,Z) = Z_x \in T'M \).

ii). \( C_1^1(Z_x \otimes C_1^1((d'F \otimes \tilde{N})(x,Z))) = 0 \), \( \forall (x,Z) = Z_x \in T'M \).

iii). \( PN(X,QY) = 0 \), \( \forall X,Y \in \mathcal{X}(T'M) \).

iv). \( PN \) is symmetric with respect to the two lower indices and it is positive homogeneous of degree 0.

In the literature on Finsler spaces, one can find a lot of tensor fields having the properties i)–iv) (see, e.g., [15] and [2]) and by following the known examples many other tensors of the same kind can be constructed. From \( \tilde{N} \) one can obtain many tensors like \( N \) by following methods, which are analogous to the ones previously used for the construction of the connection \( \nabla^1 \). Hence, we omit them for the sake of brevity.

Now, let us consider a point \( x \in M \). Then the (closed) indicatrix having 0\(_x\) as its center and \( \rho > 0 \) as its radius is the set \( \overline{B}_\rho(0_x) = \{(x,X) \in T_xM | F(x,X) \leq \rho\} \) and it is the closure of the open indicatrix \( B_\rho(0_x) \), defined in the Introduction. \( \overline{B}_\rho(0_x) \) and \( B_\rho(0_x) \) are both convex, that is for any \( X_x,Y_x \) elements of \( \overline{B}_\rho(0_x) \) \( (B_\rho(0_x)) \) the vector \( tX_x + (1-t)Y_x \) belongs to \( \overline{B}_\rho(0_x) \) \( (B_\rho(0_x)) \), for any \( t \in [0,1] \).

Finally, it is easy to see that:

**Proposition 2.6.** Let \( x \in M \), the following assertions are true:

i). The family \( \{B_\rho(0_x)\}_{\rho > 0} \) is a basic system of open neighbourhoods of 0\(_x\) in \( T_xM \).

ii). The sets \( \tilde{A} = \cup_{x \in A} B_\rho(0_x) \), with \( A \) open neighbourhood of \( x \) in \( M \) and \( \rho > 0 \) form a basic system of open neighbourhoods of 0\(_x\) in \( TM \).

Now, we are in position to prove Proposition 1.1:

**Proof.** Let \( \nabla \) be a Finslerian Connection and let \( x \in M \). Since the Berwald Connection \( \nabla' \) induced by \( \nabla \) induces a spray, by Proposition 2.2, there exists an open neighbourhood \( \tilde{W} \) of the zero section \( \sigma(M) \) in \( TM \) and an open subset \( W \) of \( M \) such that \( (\pi,\text{Exp}) : \tilde{W} \to W \times W \) is a diffeomorphism of class
$C^1$, where $\text{Exp}$ is the exponential map of the spray defined by $\nabla'$. Moreover, ii) of Proposition 2.6 ensures us that there exists a positive real number $\eta$ and an open neighbourhood $A$ of $x$ in $M$, such that $\tilde{A} = \cup_{z \in A} B_\eta(0_z) \subseteq \tilde{W}$. Then, by using ii) of Proposition 2.3 and i) of Proposition 2.6, there also exists a real number $\varepsilon$, with $0 < \varepsilon \leq \eta$ such that $B_\varepsilon(x) \times B_\varepsilon(x) \subseteq (\pi, \text{Exp})(\tilde{A}) \subseteq W \times W$, where $B_\varepsilon(x) = \text{Exp}_y(B_\varepsilon(0_y))$ is an open neighbourhood of $x$. Finally, i) follows, since $\text{Exp}_x : B_\varepsilon(0_y) \to B_\varepsilon(x)$ is a $C^1$–diffeomorphism.

To prove ii), we consider $y, z \in B_\varepsilon(x)$ with $y \neq z$. It results:

$$0_y \neq (\pi, \text{Exp})^{-1}(y, z) = X^z_y \in B_\eta(0_y) \subseteq \tilde{W}.$$ 

Then, i) of Proposition 1.1, implies that the curve $c(t) = \text{Exp}_y(tX^z_y)$, with $t \in [0, 1]$, is the unique geodesic lying entirely in $W$ and joining $y$ and $z$. The curve $c$ is also the unique geodesic that joins $y$ and $z$ lying entirely in $B_\eta(y)$. In fact, we shall see that $c$ has length lesser than $\eta$. Let $\beta = (c, \dot{c}) : [0, 1] \to TM$ the complete lift of $c$ to $TM$, for any $t \in [0, 1]$. Then, it results $\dot{\beta} = \dot{c} e_i + \ddot{c} \varepsilon_i$.

Hence it follows:

$$g_{c(t), \dot{c}(t)}(\dot{c}(t), \dot{c}(t)) = G_{\beta(t)}(\dot{\beta}(t), \dot{\beta}(t)) , \quad \forall t \in [0, 1] .$$

Moreover, being $c$ a path of $\nabla'$, we have:

$$(\nabla_{\dot{\beta}} \dot{\beta})(t) = Q_{\beta(t)}(Z_{\beta(t)}) , \quad \forall t \in [0, 1]$$

where $Z$ is a suitable vector field defined on an open neighbourhood of $\beta([0, 1])$ in $TM$. From the previous identities and ii) of the previous proposition we obtain:

$$\frac{d}{dt} g_{c(t), \dot{c}(t)}(\dot{c}(t), \dot{c}(t)) = (\nabla_Y G)_{\beta(t)}(\dot{\beta}(t), \dot{\beta}(t)) + 2G_{\beta(t)}(\beta(t), Q_{\beta(t)}(Z_{\beta(t)})) = 0 , \quad \forall t \in [0, 1] .$$

Hence, we have:

$$F^2(c(t), \dot{c}(t)) = g_{c(t), \dot{c}(t)}(\dot{c}(t), \dot{c}(t)) = F^2(y, X^z_y) < \eta^2 , \quad \forall t \in [0, 1] ; \quad (2.2)$$

that is $c$ has length lesser than $\eta$. Moreover, for $y = z \in B_\varepsilon(x)$ the unique geodesic joining $y$ and $z$ in $W$ is the constant curve. Now, we observe that, the mapping $\text{Exp}_y : B_\eta(0_y) \to B_\eta(y)$ is a $C^1$–diffeomorphism, because of ii) of Proposition 2.2, by using the inclusion $B_\eta(0_y) \subseteq \tilde{W} \cap T_y M$. Finally, we have iii), since the inclusion $(\pi, \text{Exp})^{-1}(\{y\} \times B_\eta(x)) \subseteq B_\eta(0_y)$ implies $B_\varepsilon(x) \subseteq B_\eta(y)$.
Now, we recall that the standard identification \( T_X T_x M = T_x M \), induces a Riemannian Metric on \( T'_x M \), for any \( X \in T_x M \) and any \( x \in M \). By means of this identification, we prove the corresponding of the Gauss Lemma.

**Proposition 2.7.** Let \( x \in M \) and \( X \in B_\varepsilon(0_x) \), with \( \varepsilon \) as in Proposition 1.1.

Then:

i). It results \( F(x, X) = F(\text{Exp}_x(X), (D\text{Exp}_x)_X X) \), where we denoted by \( (D\text{Exp}_x)_X : T_X T_x M = T_x M \rightarrow T_{\text{Exp}_x(x)} M \) the total differential of \( \text{Exp}_x \).

ii). For any \( Y \in T_x M \), such that \( g(x, X)(X, Y) = 0 \), it results:

\[
g(\text{Exp}_x(X), (D\text{Exp}_x)_X X)((D\text{Exp}_x)_X X, (D\text{Exp}_x)_X Y) = 0.
\]

**Proof.** If \( X = 0_x \), then the assertion is trivially true. Hence, we suppose \( X \neq 0_x \).

In order to prove i) let us denote by \( c_X : [0, 1] \rightarrow M \) the geodesic of the Berwald connection \( \nabla' \), then we have:

\[
(D\text{Exp}_x)_tX = \dot{c}_X(t) = \beta(t) \in T' M , \quad \forall t \in [01] ;
\]

and

\[
\frac{d}{dt} g(c_X(t), \dot{c}_X(t)) (\dot{c}_X(t), \dot{c}_X(t)) = 0
\]

Hence, from Equation (2.2), it follows:

\[
g(c_X(t), \dot{c}_X(t)) (\dot{c}_X(t), \dot{c}_X(t)) = F^2(x, X) , \quad \forall t \in [0, 1] .
\]

Consequently

\[
F(c_X(t), \dot{c}_X(t)) = F(\text{Exp}_x(tX), (D\text{Exp}_x)_tX X) = F(x, X), \\
\forall t \in [0, 1] ; \quad (2.3)
\]

and the assertion follows for \( t = 1 \).

Now, we prove ii). Let \( Y \in T'_x M \), such that \( g(x, X)(X, Y) = 0 \). Consider a curve, denoted again by \( X : (-a, a) \rightarrow B_\varepsilon(0_x) \), with \( a > 0 \) and \( \varepsilon \) as in Proposition 1.1, having \( X(0) = X, \dot{X}(0) = Y \) and \( F(x, X(s)) = F(x, X) = r < \varepsilon \), for each \( s \in (-a, a) \). Being \( tX(s) \in B_\varepsilon(0_x) \), we can define the mapping:

\[
\lambda(s, t) = \text{Exp}_x(tX(s)) = c_X(s)(t) , \quad (s, t) \in (-a, a) \times [0, 1] .
\]
Then:
\[ \left( \frac{\partial \lambda}{\partial s} \right)_{(s,t)} = (D\text{Exp}_x)_{tX(s)} t\dot{X}(s) \]
and
\[ \left( \frac{\partial \lambda}{\partial t} \right)_{(s,t)} = (D\text{Exp}_x)_{tX(s)} X(s) = \dot{c}_{X(s)}(t) , \]
for any \((s, t) \in (-a, a) \times [0, 1] .\)

Hence, it results:
\[ F(\lambda(s, t), \left( \frac{\partial \lambda}{\partial t} \right)_{(s,t)}) = F(x, X(s)) = r , \quad \forall (s, t) \in (-a, a) \times [0, 1] . \quad (2.4) \]

Now, we set:
\[ \beta(s, t) = (\lambda(s, t), \left( \frac{\partial \lambda}{\partial t} \right)_{(s,t)}) \in T^\prime M, \quad \forall (s, t) \in (0, a) \times [0, 1] . \]

Then, we have:
\[
\left( \frac{\partial}{\partial t} g_{(\lambda, \lambda)} \left( \frac{\partial \lambda}{\partial s} , \frac{\partial \lambda}{\partial t} \right) \right)_{(s,t)} = G_{\beta} \left( \nabla_{\frac{\partial \beta}{\partial s}} \frac{\partial \beta}{\partial t} , \frac{\partial \beta}{\partial t} \right)_{(s,t)} =
\frac{1}{2} \left( \frac{\partial}{\partial s} G_{\beta} \left( \frac{\partial \beta}{\partial t} , \frac{\partial \beta}{\partial t} \right) \right)_{(s,t)} - \frac{1}{2} \left( \nabla_{\frac{\partial \beta}{\partial s}} G \left( \frac{\partial \beta}{\partial t} , \frac{\partial \beta}{\partial t} \right) \right)_{(s,t)}. \]

Hence, because of ii) of Proposition 2.3, recalling the definition of \( \beta \), we have:
\[ \frac{\partial}{\partial s} F^2(\lambda(s, t), \left( \frac{\partial \lambda}{\partial t} \right)_{(s,t)}) = 0 , \quad \forall (s, t) \in (-a, a) \times [0, 1] . \]

Consequently:
\[ g_{(X\text{Exp}_x(tX(s)), (D\text{Exp}_x)_{tX(s)} X(s))}((D\text{Exp}_x)_{tX(s)} t\dot{X}(s), (D\text{Exp}_x)_{tX(s)} X(s)) = 0 \]
\[ \forall (s, t) \in (-a, a) \times [0, 1] . \quad (2.5) \]

Finally, we obtain ii) by putting \( t = 1 \) and \( s = 0 \) in the previous identity. \( \square \)

For the sequel, we need the following lemma, which is proved in [3].

**Lemma 2.1.** Let \( x \in M \) and \( Z \in T^\prime_x M \). Then for each \( Y \in T_x M \) such that \( g_{(x, Z)}(Z, Y) = 0 \) it results \( F(x, Y + Z) \geq F(x, Z) \) and equality holds if and only if \( Y = 0_x \).
Then, we can prove the Proposition 1.2.

**Proof.** First, we prove i).

We can suppose $\tilde{b}(s) \neq 0_x$, for any $s \in (0, s_1]$. Then, the function $r(s) = F(x, \tilde{b}(s))$ is not zero, for each $s \in (0, s_1]$. Hence, we can put:

$$X(s) = \frac{1}{r(s)} \tilde{b}(s), \quad \forall s \in (0, s_1].$$

With these notations, it is easy to see that:

$$\tilde{b}(s) = r(s)X(s), \quad (2.6)$$

$$g(x, X(s))(X(s), X(s)) = 1 \quad (2.7)$$

$$\dot{b}(s) = (DExp_x)\tilde{b}(s)\dot{r}(s)X(s) + (DExp_x)\tilde{r}(s)X(s) \dot{X}(s), \quad (2.8)$$

$$g(x, X(s))(X(s), \dot{X}(s)) = 0 \quad (2.9)$$

for any $s \in (0, s_1]$.

Then, from ii) of Proposition 2.7 and from (2.4), it follows:

$$g((b(s), (DExp_x)\tilde{b}(s))(DExp_x)\tilde{r}(s)X(s), (DExp_x)\tilde{b}(s)r(s)X(s)) = 0 \quad (2.10)$$

for any $s \in (0, s_1]$. Moreover, i) of Proposition 2.7 and equations (2.4) and (2.5) imply:

$$F^2(b(s), (DExp_x)\tilde{b}(s)\dot{r}(s)X(s)) = \dot{r}^2(s), \quad \forall s \in (0, s_1] \quad (2.11)$$

Then, from (2.7), (2.9), (2.10) and from the Lemma 2.1, we obtain:

$$F^2(b(s), \dot{b}(s)) \geq \dot{r}^2(s), \quad \forall s \in (0, s_1], \quad (2.12)$$

hence:

$$L(b) = \int_{\rho}^{s_1} |\dot{r}(s)| ds \geq |r(s_1) - r(\rho)|, \quad \forall \rho \in (0, s_1]$$

13
and i) is true.

Assertion ii) trivially holds.

Under the assumptions of iii), by using the previous notations, we have

\[ L(c_X) = L(b) = r(s_1). \]

Moreover, being i) true, it results \( L(b) \geq L(c_{\tilde{b}(s)}) = r(s) \), with \( b_s = b|_{[0,s]} \), for any \( s \in (0, s_1] \). Hence, it follows:

\[
\int_{s_1}^{s_1} (F(b(s), \dot{b}(s)) - |\dot{r}(s)|) ds = 0, \quad \forall \rho \in (0, s_1].
\]

Since (2.12) holds, the previous inequality implies:

\[
\int_{s_1}^{s_1} (F(b(s), \dot{b}(s)) - |\dot{r}(s)|) ds = 0, \quad \forall \rho \in (0, s_1]
\]

and, for continuity reasons, we have

\[
F(b(s), \dot{b}(s)) = |\dot{r}(s)|, \quad \forall s \in [0, s_1].
\]

Consequently, by using (2.8), (2.10), (2.11) and the previous Lemma we have:

\[
(\text{DExp}_x)_{\tilde{b}(s)}(r(s)) \dot{X}(s) = 0, \quad \forall s \in [0, s_1].
\]

Then, \( \dot{X}(s) = 0 \), for any \( s \in [0, s_1] \), because \( \text{Exp}_x \) has maximal rank along the curve \( \tilde{b} \) and \( r(s) \neq 0 \) for any \( s \in (0, s_1] \). Hence, iii) is true.

From the above proposition, it trivially follows:

**Proposition 2.8.** Let \( x \in M \). Moreover, let us consider \( \varepsilon = \varepsilon(x) \) and \( \eta = \eta(x) \) as in Proposition 1.1. Then, a geodesic of length lesser than \( \eta \) starting from an arbitrary point \( y \in B_\varepsilon(x) \) is a curve of minimal length between its end points.

Let \( x \in M \) and \( B_\rho(0_x) \subseteq \tilde{W} \cap T_xM \). Moreover, let \( u = (u_i)_{1 \leq i \leq n} \) be a basis of \( T_xM \). Then, \( u \) defines a mapping, which by an abuse of notations, we denote again by \( u : \mathbb{R}^n \to T_xM \) obtained by putting \( u(\xi) = \xi^i u_i \), for any \( \xi = (\xi^i)_{1 \leq i \leq n} \in \mathbb{R}^n \). Furthermore, the mapping \( \psi = u^{-1} \circ \text{Exp}_x^{-1} : B_\rho(x) \to u^{-1}(B_\rho(0_x)) \) is a \( C^\infty \)-diffeomorphism on \( B_\rho(x) \) \( \setminus \{x\} \) and \( C^1 \)-differentiable on \( x \). Consequently, \( (B_\rho(x), \psi) \) is a chart of the \( C^1 \)-differentiable manifold structure canonically induced on \( M \) by the considered structure of \( C^\infty \)-differentiable manifold on \( M \).

Then, we can prove:
Proposition 2.9. Let \( x \in M \) and \( \varepsilon = \varepsilon(x) \) as in Proposition 1.1. There exists \( \varepsilon_0 = \varepsilon_0(x) \in (0, \eta) \) such that for any non constant geodesic \( c : [0, a] \to M \), with \( a > 0 \), satisfying the conditions \( c(t_0) \in \partial B_\varepsilon((x) = \text{Exp}_x(\partial B_\varepsilon(0_x)) \) and \( \dot{c}(t_0) \in T\partial B_\varepsilon(x) \), for some \( t_0 \in (0, a) \) and \( \varepsilon \in (0, \varepsilon_0) \) and for each \( \mu \in (0, 1) \) there exists \( \rho = \rho(\mu, \varepsilon) \) with the following property:

\[
d(x, c(t)) \geq d(x, c(t_0)) + \mu(t - t_0)^2, \quad \forall t \in [t_0 - \rho, t_0 + \rho],
\]

where \( d \) is the Finslerian distance function.

Proof. We fix a basis of \( T_xM \) and consider the \( C^1 \)-differentiable chart \( (B_\eta(x), \psi) \), with \( \psi = u^{-1} \circ \text{Exp}_x^{-1} \). Moreover, we can suppose \( x \neq c(t) \in B_\eta(x) \), for any \( t \in (0, a) \). Then all the derivatives of the function \( f(t) = d^2(x, c(t)) = F^2(x, \gamma(t)) \), being \( \gamma(t) = \psi(c(t)) \), for any \( t \in (0, a) \), are defined. Hence by using the homogeneity conditions, we get:

\[
\begin{align*}
   f(t_0) &= \varepsilon^2; \quad \dot{f}(t_0) = g(x, \gamma(t_0))(\gamma(t_0), \dot{\gamma}(t_0)); \\
   \ddot{f}(t_0) &= g(x, \gamma(t_0))(\dot{\gamma}(t_0), \ddot{\gamma}(t_0)) + g(x, \gamma(t_0))(\gamma(t_0), \dddot{\gamma}(t_0)).
\end{align*}
\]

Therefore, the proof follows as in [4] (cf. [3], too).

The proof of Proposition 1.3 follows from the previous proposition in a trivial way.

We can also omit the proof of the following Lemma, because it needs only the compactness of boundary of convex neighbourhoods, which holds, because an indicatrix is always compact and the exponential map is a diffeomorphism near to any point. Hence, its proof follows as in [4].

Lemma 2.2. Let \( p \in M \) and \( \rho > 0 \) be a positive real number such that \( \text{Exp}_p \) is defined on the ball \( B_\rho(0_p) \subseteq T_pM \). Then, every \( q \in M \), with \( d(p, q) < \rho \), can be joined to \( p \) by a minimizing geodesic.

Finally, the proof of Proposition 1.4 is the same as in the Riemannian case (see, e.g., [4]).

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