THE ÉTALE FUNDAMENTAL GROUPOID AS A TERMINAL COSTACK

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Abstract. Let $X$ be a noetherian scheme. We denote by $\Pi_1(X)$ the fundamental groupoid. In this paper we prove that the assignments $U \mapsto \Pi_1(U)$ is the 2-terminal costack over the site of étale coverings of $X$.

1. Introduction

In a recent paper [6], we showed that for a good topological space $X$, the assignment $U \mapsto \Pi_1(U), U \subset X$, being the topological fundamental groupoid, was the 2-terminal costack over $X$. In this paper, we show that the analogue of this theorem holds for the étale fundamental groupoid over a noetherian scheme as well.

This paper is part of my PhD thesis at the University of Leicester under the supervision of Dr. Frank Neumann. He introduced me to the fundamental groupoid and inspired much of this paper, for which I would like to thank him.

2. Basic definitions and general results in 2-mathematics

2.1. Limits, 2-limits, colimits, 2-colimits.

First let us fix some notations. For functors and natural transformations $A \xrightarrow{F} B \xrightarrow{G_1} C \xrightarrow{T} D$, $\alpha : G_1 \Rightarrow G_2$, one denotes by $\alpha_\star F$ and $T_\star \alpha$ the induced natural transformations $G_1 F \Rightarrow G_2 F, TG_1 \Rightarrow TG_2$.

Let $I$ be a category and let $F : I \to \text{Cat}$ be a covariant 2-functor from the category $I$ to the 2-category of categories. For an element $i \in I$ we let $F_i$ be the value of $F$ at $i$. For a morphism $\psi : i \to j$ we let $\psi_\star : F_i \Rightarrow F_j$ be the induced functor. For any $i \xrightarrow{\psi} j \xrightarrow{\nu} k$, one has the natural transformation $\mu_{\psi,\nu} : \psi_\star \nu_\star \to (\nu \psi)_\star$ satisfying the coherent relations. Recall the construction of the category $\lim_{\rightarrow i} F_i$ called the 2-limit of $F$.

2.1.1. 2-Limits of categories. Objects of the category $\lim_{\rightarrow i} F_i$ are collections $(x_i, \xi_\psi)$, where $x_i$ is an object of $F_i$, while $\xi_\psi : \psi_\star(x_i) \to x_j$ for $\psi : i \to j$ is a morphism of the category $F_j$ satisfying the following condition: For any $i \xrightarrow{\psi} j \xrightarrow{\nu} k$, the following diagram

commutes.

A morphism from $(x_i, \xi_\psi)$ to $(y_i, \eta_\psi)$ is a collection $(f_i)$, where $f_i : x_i \to y_i$ is a morphism of $F_i$ such that for any $\psi : i \to j$, the following diagram

commutes.
is a commutative diagram.

2.1.2. 2-Colimits of categories. Dually, for a (covariant) 2-functor \( \mathcal{G} : I \to \text{Cat} \) one can define the 2-colimit of \( \mathcal{G} \) as follows: Let \( \psi : i \to j \) a morphism in \( I \) and \( \psi_* : \mathcal{G}_i \to \mathcal{G}_j \) be its induced functor. For a composition \( i \xrightarrow{\alpha} j \xrightarrow{\beta} k \), one has the natural transformation \( \mu_{\psi,\nu} : \nu_* \psi_* \to (\nu \psi)_* \). Then

\[
\alpha_i : \mathcal{G}_i \to \text{colim}_i \mathcal{G}_i
\]

and natural transformations \( \lambda_\psi : \alpha_j \psi_* \to \alpha_i \), satisfying the following condition: For any \( i \xrightarrow{\psi} j \xrightarrow{\nu} k \), the following diagram

\[
\begin{array}{ccc}
\alpha_k (\nu \psi)_* & \xrightarrow{\lambda_\psi} & \alpha_i \\
\alpha_k \mu_{\psi, \nu} \downarrow & & \downarrow \lambda_\psi \\
(\alpha_k \nu_*) \psi_* & \xrightarrow{\lambda_{\nu} \lambda_\psi} & \alpha_j \psi_*
\end{array}
\]

commutes. Furthermore, one requires that for any category \( G \), the canonical functor

\[
\kappa : \text{Hom}_{\text{Cat}}(2 - \text{colim}_i \mathcal{G}_i, G) \to \lim_i \text{Hom}_{\text{Cat}}(\mathcal{G}_i, G)
\]

is an equivalence of categories. Here the functor \( \kappa \) is given by \( \kappa(\chi) = (\chi \circ \alpha_i, \chi_i \star \lambda_\psi) \). It is well-known that 2-colim exists and is unique up to an equivalence of categories, see, [11] pp. 192-193.

**Proposition 2.1.** Let \( I \) be a category, \( A : I \to \text{Cat} \) a 2-functor and denote \( L = 2\text{-colim}_{i \in I} A_i \).

- Assume that finite limits exist in every category \( A_i \) and that the maps \( A_i \to A_j \) preserve finite limits. Then finite limits exist in \( L \) and the canonical maps \( L \to A_i \) respect finite limits as well.
- Assume that finite colimits exist in every category \( A_i \) and that the maps \( A_i \to A_j \) preserve finite colimits. Then finite colimits exist in \( L \) and the canonical maps \( L \to A_i \) respect finite colimits as well.

**Proof.** Let \( C \) be a finite category and \( A : C \to L \) a functor. To show that it has a limit, recall first the construction of the 2-limit. We have that any object \( A \) in \( L \) is a collection \( \{(A_i, \alpha_\psi : \psi_*(A_i) \to A_j)\} \) such that the \( \alpha_{ij} \) are compatible. So for any object \( c \in C \), the object \( A(c) \) can be seen as a compatible collection \( \{(A_{ci}, \alpha_\psi : \psi_{ac}(A_{ci}) \to A_{cj})\} \). Hence for every \( i \), we can take the limit of the \( \{A_{ci}\} \) with respect to \( c \), which we denote by \( P_i \). The fact that these are compatible and form an element in \( L \), which we denote by \( P \), comes from the universality of the \( P_i \)'s. To show the universality of \( P \), one takes an other element \( Q \in L \) and since \( Q \) is again a collection of elements \( \{Q_{ci}, \alpha_\psi : \psi_{ac}(Q_{ci}) \to Q_{cj}\} \), we get an map \( Q \to P \). Universality follows again from universality of the \( P_i \)'s. For colimits, the proof is analogous to the above one. \( \square \)

**Proposition 2.2.** Let \( I \) be a filtered category, \( A : I \to \text{Cat} \) a 2-functor and denote \( L = 2\text{-colim}_{i \in I} A_i \).

- Assume that finite limits exist in every category \( A_i \) and that the maps \( A_i \to A_j \) preserve finite limits. Then finite limits exist in \( L \) and the canonical maps \( L \to A_i \) respect finite limits as well.
- Assume that finite colimits exist in every category \( A_i \) and that the maps \( A_i \to A_j \) preserve finite colimits. Then finite colimits exist in \( L \) and the canonical maps \( L \to A_i \) respect finite colimits as well.

**Proof.** Let \( C \) be a category and \( A : C \to L \) a functor. By the definition of the 2-colimit for filtered system, every \( A_i \) can be thought to be in one of the \( A_i \)'s. Hence by the finiteness of \( C \) and the fact that \( I \) is filtered, we can find a single \( A_j \) such that the whole diagram \( A : C \to L \) can be represented in it. We then take the colimit of \( A \) in any such \( A_j \), which gives us an element in \( L \) and we denote it by \( P \). Note that it does not depend on our choice of \( A_j \). To show that it is indeed the colimit in \( L \), consider an other object \( Q \in L \) such that we have compatible maps \( Q \to A_c \) for all \( c \in C \). Again by the definition of the 2-colimit, \( Q \) is in one of the \( A_i \)'s and we can again find a
category $A_k$ such that the diagram $A : C \to L$, $P$ and $Q$, as well as all the morphisms are inside it and hence we will have a map $Q \to P$. Uniqueness follows trivially as well. The proof for the colimit is analogous to the above one. 

2.2. Stacks and Costacks.

2.2.1. Stacks. Let $X$ be a site and $\mathcal{F} : X^{op} \to \text{Cat}$ a 2-functor where $\text{Cat}$ is the 2-category of categories. This is called a fibered category over $X$. If we have two fibered categories over $X$, then a morphism between them is called a fibered functor. The following important fact holds.

**Lemma 2.3.** Let $\mathcal{F}$ be a fibered category over $X$ and let $A, B$ be objects in $\mathcal{F}(U)$. Then the assignment $V \mapsto H_{\mathcal{F}}(V^i(A), i^*(B))$ for any morphism $i : V \to U$ defines a presheaf on $X|_U$. It is denoted by $H_{\mathcal{F}}(A, B)$.

**Definition 2.4 ( Prestack).** If the presheaf $H_{\mathcal{F}}(A, B)$ is in addition a sheaf, then we say $\mathcal{F}$ is a prestack.

Let $X$ be a site and $\mathcal{F} : X^{op} \to \text{Cat}$ a 2-functor. Let $U$ be an object in $X$ and $\mathcal{U} = \{U_i \to U\}$ a covering of $U$. Then we can consider the following diagram:

\[
\prod_{i \in I} \mathcal{F}(U_i) \longrightarrow \prod_{i,j \in I} \mathcal{F}(U_{ij}) \longrightarrow \prod_{i,j,k \in I} \mathcal{F}(U_{ijk}).
\]

We denote its limit by $\lim_{\mathcal{U}}(\mathcal{F}, \mathcal{F})$ and its 2-limit by $\lim_{\mathcal{U}}^{2}(\mathcal{F}, \mathcal{F})$. Note that the last part $\prod_{i,j,k \in I} \mathcal{F}(U_{ijk})$ does not factor in the limit, only the 2-limit. Also note that the 2-limit is usually called the descent data. But since we will use 2-limits and 2-colimits throughout this paper, it is preferable to call it the 2-limit.

**Definition 2.5 (Stack).** A fibered category $\mathcal{F}$ over $X$ is called a stack if for all objects $U$ of $X$ and for all coverings $\mathcal{U}$ of $U$, the functor $\mathcal{F}(U) \to \lim_{\mathcal{U}}^{2}(\mathcal{F}, \mathcal{F})$ is an equivalence of categories.

**Definition 2.6 (Direct Stackification).** Let $\mathcal{F} : X^{op} \to \text{Cat}$ be a 2-functor. Define $\mathcal{F}'(U) := \lim_{\mathcal{U}}^{2}(\mathcal{F}, \mathcal{F})$ and then we iterate it 3 times. I.e. define $\mathcal{F}''(U) := \lim_{\mathcal{U}}^{2}(\mathcal{F}, \mathcal{F}')$ and finally $\mathcal{F}'''(U) := \lim_{\mathcal{U}}^{2}(\mathcal{F}, \mathcal{F}'')$.

It is a general result that $\mathcal{F}$ is the associated stack of $\mathcal{F}$. (See Theorem 3.8 [5])

2.2.2. Costacks. Let $X$ be a site and $\mathcal{F} : X \to \text{Cat}$ a 2-functor where $\text{Cat}$ is the 2-category of categories. We call this a cofibred category over $X$.

Let $X$ be a site and $\mathcal{F} : X \to \text{Cat}$ a 2-functor. Let $U$ be an object in $X$ and $\mathcal{U} = \{U_i \to U\}$ a covering of $U$. Then we can consider the following cosimplicial sequence:

\[
\prod_{i \in I} \mathcal{F}(U_i) \longrightarrow \prod_{i,j \in I} \mathcal{F}(U_{ij}) \longrightarrow \prod_{i,j,k \in I} \mathcal{F}(U_{ijk})
\]

where $U_{ij} = U_i \times_U U_j$. We denote its colimit by $\colim_{\mathcal{U}}(\mathcal{F})$ and its 2-colimit by $\colim_{\mathcal{U}}^{2}(\mathcal{F})$. Note that the last part $\prod_{i,j,k \in I} \mathcal{F}(U_{ijk})$ does not factor in the colimit, only the 2-limit.

**Definition 2.7 (Costack).** A cofibred category $\mathcal{F}$ over $X$ is called a costack if for all objects $U$ of $X$ and for all coverings $\mathcal{U}$ of $U$, the functor $\mathcal{F}(U) \leftarrow \colim_{\mathcal{U}}^{2}(\mathcal{F})$ is an equivalence of categories.

Alternatively, we can define a costack using stacks.

**Definition 2.8.** Let $\mathcal{F}$ be a cofibred category over $X$. We say that $\mathcal{F}$ is a costack, if for every category $C$, the assignment $U \mapsto \text{Hom}_{\text{Cat}}(\mathcal{F}(U), C)$ is a stack.

Note that if our category took values in groupoids, than it would be enough to check it for every groupoid.
3. The 2-category of Galois Categories

It should be noted that the following definition of a Galois category differs from the standard definition of a Galois category.

**Definition 3.1.** A (finite-connected) Galois category is a category $C$ together with a set of covariant functors $\{F_i : C \to FSets\}_{i \in I}$, where $I$ is a finite set, satisfying the following axioms:

1. Finite projective limits exist in $C$.
2. Finite inductive limits exist in $C$.
3. Any morphism $u : Y \to X$ in $C$ factors as $Y \xrightarrow{u'} X' \xrightarrow{u''} X''$, where $u'$ is a strict epimorphism and $u''$ is a monomorphism which is an isomorphism onto a direct summand of $X$.
4. Every $F_i$ is right exact.
5. Every $F_i$ is left exact.
6. Let $\{u : Y \to X\}$ be a morphism in $C$. Then $u$ is an isomorphism if and only if $F_i(u)$ is an isomorphism for all $i \in I$.

If $I$ can be chosen to be a one element set, then $C$ is connected. This is equivalent to the standard definition of a Galois category. However, from now on, Galois category will refer to Def. 3.1. For more on connected Galois categories, see [2].

**Definition 3.2.** Let $\{F_i : C \to FSets\}_{i \in I}$ and $\{G_j : D \to FSets\}_{j \in J}$ be two Galois categories. A morphism of galois categories consists of a map $f : J \to I$, a collection of functors $\varphi_j : C \to D, j \in J$, preserving finite limits and finite colimits and isomorphisms $\lambda_j, j \in J$, as given in the following diagram

$$
\begin{array}{ccc}
D & \xrightarrow{\varphi_j} & C \\
\downarrow{G_j} & & \downarrow{F_i(j)} \\
FSets & & FSets
\end{array}
$$

For simplicity, we will sometimes just write $\{\varphi_j : F_i(j) \to G_j\}$.

To define composition, we need to define the composition of the $\lambda_j$'s. So say we now have

$$
\begin{array}{ccc}
E & \xrightarrow{\phi_k} & D \\
\downarrow{H_k} & & \downarrow{G_{f(j)}} \\
FSets & & FSets
\end{array}
\quad \quad \quad
\begin{array}{ccc}
D & \xrightarrow{\lambda_k \varphi_k} & C \\
\downarrow{G_{f(k)}} & & \downarrow{F_i(j)} \\
FSets & & FSets
\end{array}
$$

Define $\lambda_k \varphi_k \circ \lambda_{k', \varphi_{k'}}(x) = \lambda_{k', \varphi_{k'}}(\varphi(x)) \circ \lambda_{k, \varphi_k}(x)$. In more detail we have

$$
\lambda_k \varphi_k \circ \lambda_{k', \varphi_{k'}}(x) : F(x) \xrightarrow{\lambda_k \varphi_k(x)} G(\varphi(x)) \xrightarrow{\lambda_{k', \varphi_{k'}}(\varphi(x))} E(\phi \circ \varphi(x)).
$$

It is easily verified that the above construction is strictly associative.

**Definition 3.3.** A 2-morphism between $\{\varphi_j : F_i(j) \to G_j\}$ and $\{\phi_j : F_j(j) \to G_j\}$ is a collection of natural transformations $C \xrightarrow{\phi_j} D$ such that additionally the following diagram

$$
\begin{array}{ccc}
F_x & \xrightarrow{\lambda_k \varphi_k(x)} & G(\varphi(x)) \\
\downarrow{\lambda_k \varphi_k(x)} & & \downarrow{\zeta(x)} \\
\lambda_k \varphi_k(x) & \xrightarrow{\zeta(x)} & G(\varphi(x))
\end{array}
$$

commutes.
This shows that we can talk about the (strict) 2-category of Galois categories. We will denote it by \( \mathbf{GCat} \). Hence we can now talk about 2-functors with values in \( \mathbf{GCat} \), as well as stacks, prestacks etc. A stack with values in the 2-category of Galois categories will be referred to as a Galois stack. If we have two 2-functors with values in Galois categories, then we will call a morphism between them that respects the structures a Galois functor. In the case of fibered categories, we will keep the notation and refer to a morphism between two fibered categories that preserves the Galois structures a Galois transformation, even though it would technically be a Galois functor.

Note that throughout the paper, whenever we are dealing with 2-functors in Galois categories, we will ignore the functors in \( \text{Sets} \). This is to keep the notations simple, but in actuality the functors are of course part of the structure.

Let \( \mathfrak{S} \) be a groupoid and assume that \( \pi_0(\mathfrak{S}) \) is trivial and equipped with a discrete topology. Then we say that \( \mathfrak{S} \) is a profinite groupoid if for every object \( X \) of \( \mathfrak{S} \), the group \( \text{Aut}(X) \) is a profinite group and for every morphism \( x \to y \), the associated group homomorphism \( \text{Aut}(Y) \to \text{Aut}(x) \) respects the profinite structures.

**Proposition 3.4.** The 2-category of Galois categories is equivalent to the 2-category of profinite groupoids.

**Proof.** This equivalence is given by associating to a profinite groupoid \( \mathfrak{S} \), the Galois category \( \text{Hom}_{\mathbf{GCat}}(\mathfrak{S}, \text{Sets}) \). On functors and natural transformations, the 2-functor is defined in the obvious way by composition. The fact that \( \text{Hom}_{\mathbf{GCat}}(\mathfrak{S}, \text{Sets}) \) is a Galois category and that it defines an equivalence is easy to check. \( \square \)

Let \( X \) be a site and \( \mathfrak{S} : X \to \text{Groupoids} \) a covariant 2-functor. Then we denote by \( \hat{\mathfrak{S}} : X \to \text{Groupoids} \) the contravariant 2-functor given by \( U \mapsto \text{Hom}_{\mathbf{Cat}}(\mathfrak{S}(U), \text{Sets}) \). Now let \( \mathfrak{E} : X \to \text{Groupoids} \) and \( \mathfrak{S} : X \to \text{Groupoids} \) be two covariant 2-functors and \( F : \mathfrak{E} \to \mathfrak{S} \) a natural transformation. Then it is clear that \( F_S : \mathfrak{S}_S \to \mathfrak{E}_S \) is a Galois transformation. But indeed the above proposition shows that the reverse is also true. Hence we have the following as well.

**Corollary 3.5.** Let \( X \) be a site. Then the 2-category of fibred functors over \( X \) with values in Galois categories, and morphisms and 2-morphisms preserving the Galois structure, is anti-equivalent to the 2-category of cofibered functors over \( X \) with values in Groupoids.

**Lemma 3.6.** Let \( X \) be a site and \( \hat{\mathfrak{S}} : X \to \mathbf{GCat} \) a contravariant 2-functor. Then \( \hat{\mathfrak{S}} \) again takes values in \( \mathbf{GCat} \), where \( \hat{\mathfrak{S}} \) is the stackification of \( \mathfrak{S} \).

**Proposition 3.7.** Let \( X \) be a site, \( \mathfrak{S} \) a contravariant 2-functor from \( X \) to \( \mathbf{GCat} \) and \( \mathfrak{S} \) be a Galois stack. Let \( F : \hat{\mathfrak{S}} \to \mathfrak{S} \) be a Galois transformation. Then \( F : \hat{\mathfrak{S}} \to \mathfrak{S} \) is a Galois transformation, where \( \hat{\mathfrak{S}} \) is the associated stack of \( \mathfrak{S} \) and \( F \) the associated 2-functor of \( F \).

**Proof.** To prove this, we will use definition 2.6. First observe that \( \text{Des}^{\mathbb{U}}(\mathfrak{S}) \) is a 2-limit, hence using Prop. 2.1 we know that it respects finite limits and finite colimits, hence respects Galois transformations. Since the colimit in \( 2\text{-colim}_{\mathbb{U}} \text{Des}^{\mathbb{U}}(\mathfrak{S}) \) is taken with respects to coverings, and hence a filtered system, it too respects Galois 2-transformations by 2.2. Since stackification is obtained purely through 2-limits and filtered 2-colimits, the proposition is proven. \( \square \)

4. The Van-Kampen theorem for the étale fundamental groupoid

Let \( X \) be a noetherian scheme. We denote by \( \text{FEC}(X) \) the site of finite étale coverings of \( X \) and for an étale covering \( Y \to X \), we denote by \( \text{FEC}(Y/X) \) coverings of \( Y \) such that the diagram

\[
\begin{array}{ccc}
Z & \rightarrow & Y \\
\downarrow & & \downarrow \\
X & \rightarrow & \\
\end{array}
\]

commutes. It is a well known result that the category of finite étale coverings is equivalent to a Galois category.
Proposition 4.1. The 2-functor \( \text{FEC} : \text{FEC}(X) \to \text{GCat} \), where \( \text{FEC} \) denotes the category of finite étale coverings, given by \( Y \mapsto \text{FEC}(Y/X) \), forms a stack.

This result is well known. For example [4] discusses this to some extent. Equivalently, this can be stated as the following.

Lemma 4.2. The 2-functor \( \text{FEC} : \text{FEC}(X) \to \text{GCat} \) given by \( Y \mapsto \text{Hom}_{\text{Cat}}(\Pi_1(Y), \text{Sets}) \), where \( \Pi_1(Y) \) denotes the étale fundamental groupoid of \( Y \), forms a stack.

Theorem 4.3 (Van-Kampen Theorem). Let \( X \) be a noetherian scheme. Then the assignment \( Y \mapsto \Pi_1(Y) \) defines a costack on the site of finite étale coverings of \( X \).

Lemma 4.4. Let \( \emptyset \to \mathcal{S} \) be a functor between groupoids. Assume that the functor \( \text{Hom}_{\text{Cat}}(\mathcal{S}, \text{Sets}) \to \text{Hom}_{\text{Cat}}(\emptyset, \text{Sets}) \) is an equivalence of categories. Then \( \emptyset \to \mathcal{S} \) is an equivalence of categories as well.

For the proof see the proposition on page 140 in [3].

Proof. The proof of this statement follows from Lemma [4.4]. For any covering \( Z \in \text{Cov}(Y/X) \) we have \( \text{2-colim}(Z, \Pi_1) \to \Pi_1(Z) \), where \( \text{2-colim}(Z, \Pi_1) \) denotes the 2-colimit of

\[
\Pi_1(Z \otimes_X Y) \longrightarrow \Pi_1(Z \otimes_X Y \otimes_X Y) \longrightarrow \Pi_1(Z \otimes_X Y \otimes_X Y)
\]

Hence get the associated functor \( \text{Hom}_{\text{Cat}}(\Pi_1(Z), \text{Sets}) \to \text{Hom}_{\text{Cat}}(\text{2-colim}(Z, \Pi_1), \text{Sets}) \). Since \( \text{Hom} \) is left exact, we have \( \text{Hom}_{\text{Cat}}(\text{2-colim}(Z, \Pi_1), \text{Sets}) \cong \text{2-lim}(Z, \text{Hom}_{\text{Cat}}(\Pi_1, \text{Sets})) \), where \( \text{2-lim}(Z, \text{Hom}_{\text{Cat}}(\Pi_1, \text{Sets})) \) denotes the 2-limit of \( \text{Hom}_{\text{Cat}}(\Pi_1(Z \otimes_X Y), \text{Sets}) \cong \text{Hom}_{\text{Cat}}(\Pi_1(Z \otimes_X Y \otimes_X Y), \text{Sets}) \cong \text{Hom}_{\text{Cat}}(\Pi_1(Z \otimes_X Y \otimes_X Y), \text{Sets}) \).

Since by Lemma [4.4] the functor \( \text{Hom}_{\text{Cat}}(Z, \text{Sets}) \to \text{2-lim}(Z, \text{Hom}_{\text{Cat}}(\Pi_1, \text{Sets})) \) is an equivalence of categories, by lemma [4.4] \( \text{2-colim}(Z, \Pi_1) \to \Pi_1(Z) \), where \( \text{2-colim}(Z, \Pi_1) \) is an equivalence of categories as well, proving the assertion.

5. The 2-Terminal Costack

Definition 5.1. Let \( \mathcal{C} \) be a 2-category. We say that \( \mathcal{P} \) is the 2-terminal object of \( \mathcal{C} \), if for any other object \( C \in \mathcal{C} \), \( \text{Hom}_{\text{Cat}}(C, \mathcal{P}) \) is equivalent to the 1-point category.

Theorem 5.2. Let \( X \) be a noetherian scheme. Then the assignment \( U \mapsto \Pi_1(U) \), \( U \in X \) is the 2-terminal costack over the site of étale coverings of \( X \).

To prove this theorem, we first need a few other results.

Lemma 5.3. Consider the constant 2-functor \( s : U \mapsto \text{FSets} \), with morphisms chosen in the contravariant way. Then the associated prestack of it is given by \( \overline{s} : U \mapsto \text{CS}(U) \), where \( \text{CS}(U) \) denotes the category of constant sheaves on \( U \).

Proof. First of all, observe that we can replace the category of sets, with the category of constant presheaves with values in sets. Next we consider the associated prestack of this 2-functor. To do so, we keep the objects the same and replace the presheaves \( \text{Hom}_{\text{V}}(a,b) \) by its sheafification. We claim that this is equivalent to the category of constant sheaves on \( U \). The fact that the objects of these two categories are equivalent is clear. To see that the hom sets are isomorphic, first observe that the sheafification of \( \text{Hom}_{\text{V}}(a,b) \) is \( \text{Hom}(a, \text{Hom}(a,b)) \), which is isomorphic to \( (b^a)^{\tau_0(U)} = b^{\tau_0(U)} \). Denote by \( \overline{a} \) the constant sheaf with value \( a \). We have \( \text{Hom}_{\text{Sh}(\overline{a}, \overline{b})} = \text{Hom}_{\text{Presheaf}}(\overline{a}, \overline{b}) = \text{Hom}_{\text{Presheaf}}(a, b) \). Since \( \overline{s} \) is given by \( U \mapsto \text{Hom}(U, B) \), \( \text{Hom}_{\text{Presheaf}}(a, \overline{b}) = \text{Hom}(a, \text{Hom}(a, b)) \). This now is isomorphic to \( (\overline{b}^{-\tau_0(U)})a = b^{\tau_0(U)}a \), proving the assertion.

Corollary 5.4. Consider the constant 2-functor \( s : U \mapsto \text{FSets} \), with morphisms chosen in the contravariant way. Then the associated stack \( \hat{s} \) is given by \( \hat{s}(U) = \text{LCS}(U) \), where \( \text{LCS}(U) \) denotes the category of locally constant sheaves on \( U \).

Let \( A \) be a covariant 2-functor. Recall that we denoted by \( A_s \) the contravariant 2-functor given by \( U \mapsto \text{Hom}(A(U), \text{FSets}) \).
Proof of Thm. 5.2. We have already shown that the assignment $U \to \Pi_1(U)$ forms a costack. Hence to prove this theorem, we essentially have to show that for every costack $C$ we have an essentially unique map $C \to \Pi_1$. Denote by $P$ the covariant assignment $U \mapsto \text{pt}$. It is clear that we have a map $C \to P$ and hence a map $C_S \to C_S$ which is a Galois transformation by definition. Since $C$ was a costack, $C_S$ is a stack and hence the map $P_S \to C_S$ factors through the stackification of $P_S$, which is $U \mapsto \text{LCS}(U)$, where $\text{LCS}(U)$ denotes the category of locally constant sheaves on $U$. In the case of noetherian schemes, $\text{LCS}(U)$ is equivalent to $\text{Hom}(\Pi_1(U), \text{FSets})$. Hence, using Prop. 3.6 and 3.7 we know that there exists an essentially unique map $C \to \Pi_1$, proving the result. \hfill $\square$

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