Applied lattice gauge calculations: diquark content of the nucleon

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As an example of an application of lattice QCD we describe a computation of four-quark operators in the nucleon. The results are interpreted in a diquark language.

1. INTRODUCTION

Lattice gauge theory offers the possibility to perform non-perturbative computations in a strongly coupled theory without any model assumptions. Lattice QCD, in particular, allows us to investigate low-energy properties of QCD, like, e.g., the hadron mass spectrum, decay constants, form factors, and moments of structure functions. In this article, we shall discuss as an example the computation of four-quark operators in the nucleon recently performed by our group (the QCDSF collaboration). Since four-quark operators are related to higher-twist effects in nucleon structure functions, a subject of considerable theoretical and experimental importance, they are a particularly interesting object for lattice-QCD studies. For more technical details we refer to Ref.\cite{1}.

2. WHAT CAN WE COMPUTE ON THE LATTICE?

The basic observables in lattice QCD are Euclidean $n$-point correlation functions. Since space-time has been discretised (with lattice spacing $a$) the path integral has become a high-dimensional ordinary integral over a discrete set of field variables, which can be evaluated by Monte Carlo methods. As the (Grassmann valued) quark fields appear bilinearly in the action, they can be integrated out analytically leaving behind the determinant of

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the lattice Dirac operator and products of quark propagators. In the quenched approximation, which will be employed throughout the whole paper, this determinant is replaced by 1. The quenched approximation saves a lot of computer time, but it is hardly possible to estimate its accuracy.

Let us briefly sketch the computation of hadronic matrix elements. First, one has to choose suitable interpolating fields for the particle to be studied. For a proton with momentum $\vec{P}$ one may take

$$B_\alpha(t, \vec{P}) = \sum_{x;x_4=t} e^{-i\vec{P} \cdot \vec{x}} \epsilon_{ijk} u_i^\alpha(x) u_j^\beta(x) (C\gamma_5)_{\beta\gamma} d_k^\gamma(x)$$  \hspace{1cm} (1)

and the corresponding $\bar{B}$, where $i, j, \ldots$ are colour indices and $\alpha, \beta, \ldots$ are Dirac indices.

As the time extent $T$ of our lattice tends to $\infty$, the two-point correlation function becomes the vacuum expectation value of the corresponding Hilbert space operators with the Euclidean evolution operator $e^{-Ht}$ in between, i.e. we have, omitting Dirac indices and momenta for simplicity:

$$\langle B(t) \bar{B}(0) \rangle_{T \to \infty} \equiv \langle 0 | Be^{-Ht} \bar{B} | 0 \rangle.$$  \hspace{1cm} (2)

If in addition the time $t$ gets large, the ground state $|p\rangle$ of the proton will dominate the sum over intermediate states between $B$ and $\bar{B}$, and the two-point function will decay exponentially with a decay rate given by the proton energy $E_p$:

$$\langle B(t) \bar{B}(0) \rangle_{T \to \infty} \equiv \langle 0 | Be^{-Ht} \bar{B} | 0 \rangle_{t \to \infty} \equiv \langle 0 | B | p \rangle e^{-E_p t} \langle p | \bar{B} | 0 \rangle + \cdots$$  \hspace{1cm} (3)

Of course, if the momentum vanishes, we have $E_p = m_p$, the proton mass.

Similarly we have for a three-point function with the operator $O$ whose matrix elements we want to calculate:

$$\langle B(t) O(\tau) \bar{B}(0) \rangle_{T \to \infty} \equiv \langle 0 | Be^{-H(t-\tau)} O e^{-H\tau} \bar{B} | 0 \rangle = \langle 0 | B | p \rangle e^{-E_p(t-\tau)} \langle p | O | p \rangle e^{-E_p \tau} \langle p | \bar{B} | 0 \rangle + \cdots$$  \hspace{1cm} (4)

if $t > \tau > 0$. Hence the ratio

$$R \equiv \frac{\langle B(t) O(\tau) \bar{B}(0) \rangle}{\langle B(t) \bar{B}(0) \rangle} = \langle p | O | p \rangle + \cdots$$  \hspace{1cm} (5)

should be independent of the times $\tau$ and $t$, if all time differences are so large that excited states can be neglected.

The proton three-point function for a two-quark operator contains quark-line connected as well as quark-line disconnected pieces. In the quark-line connected contributions the operator is inserted in one of the quark lines of the nucleon propagator, while in the disconnected pieces the operator is attached to an additional closed quark line which communicates with the valence quarks in the proton only via gluon exchange. In the limit of exact isospin invariance as it is considered in this paper, the disconnected contributions cancel in the case of non-singlet two-quark operators. Moreover, one might try to argue that it would be more consistent with the quenched approximation to drop quark-line disconnected contributions generally.
3. WHAT DO WE WANT TO COMPUTE ON THE LATTICE?

The operator product expansion (OPE) relates moments of structure functions to nucleon matrix elements of local operators. In the deep-inelastic limit, where the momentum transfer $Q^2$ becomes large, one can express Nachtmann moments of, e.g., $F_2$ in the following form:

$$\int_0^1 dx x^{n-2} F_2(x, Q^2) |_{\text{Nachtmann}} = c_n^{(2)} A_n^{(2)}(\mu) + \frac{c_n^{(4)}}{Q^2} A_n^{(4)}(\mu) + O\left(\frac{1}{Q^4}\right).$$

(6)

Here $n = 2, 4, 6, \ldots$, the (reduced) matrix elements $A_n^{(t)}$ of twist $t$ and spin $n$ are normalised at the scale $\mu$, and $c_n^{(t)} = c_n^{(t)}(Q^2/\mu^2, g(\mu))$ are the Wilson coefficients, calculated in perturbation theory. Note that we go back to Minkowski space in this section.

Besides the leading twist-2 contribution we have included in (6) the twist-4 corrections. These are suppressed by $1/Q^2$ and have been the subject of intensive investigations both experimentally and theoretically. In the following we shall describe what lattice computations can tell us about twist-4 effects.

An important class of twist-4 operators consists of four-quark operators. In particular, the twist-4, spin-2 matrix element $A_2^{(4)}$ is defined by the following expectation value in a proton state with 4-momentum $P$ ($\{\cdots\}$ indicates symmetrisation)

$$\langle P|A_{\mu\nu} - \text{traces}|P\rangle = 2A_2^{(4)}(P \mu P \nu - \text{traces})$$

(7)

with the four-quark operator

$$A_{\mu\nu} = \bar{\psi}G\gamma_\mu\gamma_5 t^a \psi \bar{G}\gamma_\nu\gamma_5 t^a \psi.$$  

(8)

Here the quark field $\psi$ carries flavour, colour, and Dirac indices, the $t^a$ are the usual generators of colour SU(3), and $G$ is a matrix in flavour space, the charge matrix

$$G = \text{diag}(e_u, e_d) = \text{diag}(2/3, -1/3).$$

(9)

The corresponding leading-twist matrix element $A_2^{(2)}$ is defined by

$$\langle P|O_{\mu\nu} - \text{traces}|P\rangle = 2A_2^{(2)}(P \mu P \nu - \text{traces})$$

(10)

in terms of the two-quark operator

$$O_{\mu\nu} = \frac{i}{2}\bar{\psi}G^2\gamma_\mu \gamma_5 \gamma_\nu \psi.$$  

(11)

This operator has dimension 4, while the four-quark operator (8) has dimension 6. Hence four-quark operators may mix with operators of lower dimension (two-quark operators). This is a rather general phenomenon for operators of higher twist, because for given spin, higher twist means higher dimension. Unfortunately, the mixing with operators of lower dimension is hard to deal with in perturbation theory. For the time being we do not attempt a non-perturbative calculation of the renormalisation and mixing coefficients. Instead we look for four-quark operators for which mixing with two-quark operators is prohibited by flavour symmetry and apply perturbative (one-loop) renormalisation.
4. FOUR-QUARK OPERATORS IN THE PROTON

Assuming SU(2) flavour symmetry (isospin symmetry) we find that two-quark operators can have isospin $I = 0$ or $1$, whereas four-quark operators can have $I = 0, 1, \text{or } 2$. Hence four-quark operators with $I = 2$ cannot mix with two-quark operators. Unfortunately, the expectation value of any $I = 2$ operator in the proton vanishes. So we have to enlarge the flavour symmetry from SU(2)$_F$ to SU(3)$_F$, i.e., we assume three quarks of the same mass, and the flavour structure of the operator in the OPE is now

$$O = (e_u \bar{u} u + e_d \bar{d} d + e_s \bar{s} s)(e_u \bar{u} u + e_d \bar{d} d + e_s \bar{s} s).$$

(12)

Whereas two-quark operators transform under SU(3)$_F$ according to $3 \otimes 3 = 1 \oplus 8$, we have for four-quark operators:

$$(3 \otimes 3) \otimes (3 \otimes 3) = 2 \cdot 1 \oplus 4 \cdot 8 \oplus 10 \oplus 10 \oplus 27.$$

Four-quark operators with $I = 0, 1, I_3 = 0$, and hypercharge $Y = 0$ from the 10, 10, 27 multiplets do not mix with two-quark operators and can be used in a proton expectation value, e.g., the $I = 1$ operator

$$O_{27}^{I=1} = \frac{1}{10} [(\bar{u} u)(\bar{u} u) - (\bar{d} d)(\bar{d} d) - (\bar{s} s)(\bar{s} s) - (\bar{s} u)(\bar{u} s) + (\bar{d} s)(\bar{s} d) + (\bar{s} d)(\bar{d} s)]$$

(13)

belongs to the 27 multiplet. Such a flavour component can be projected out by suitable linear combinations of the matrix elements of the octet baryons $p, n, \Lambda, \Sigma, \Xi$, e.g.

$$\langle \Sigma^+ | O | \Sigma^+ \rangle - 2 \langle \Sigma^0 | O | \Sigma^0 \rangle + \langle \Sigma^- | O | \Sigma^- \rangle = \langle p | O_{27}^{I=1} | p \rangle.$$

(14)

Note that the flavour combination in four-quark operators which do not mix with two-quark operators (like $O_{27}^{I=1}$) is such that all quark-line disconnected contributions cancel. However, mixing with other four-quark operators (with different Dirac or colour structures) remains and is taken into account using perturbatively calculated mixing coefficients.

After a chiral extrapolation (linear in the quark mass) we obtain in our special flavour channel

$$\int_0^1 dx F_2(x, Q^2)_{\text{Nachtmann}}^{27, I=1} = -0.0005(5) \frac{m_p^2 \alpha_s(Q^2)}{Q^2} + O(\alpha_s^2).$$

(15)

The $O(\alpha_s^2)$ contribution arises from the fact that we know only the tree level Wilson coefficient $c_2^{(4)} = g^2 (1 + O(g^2))$. For reasonable values of $Q^2$ the result is rather small using, e.g., $\alpha_s(Q^2) \approx 0.3$ for $Q^2 = 4 \text{GeV}^2$. But it is interesting to note that the bag model estimates the prefactor in (13) to be $\propto B/m_p^4 \approx 0.0006$, where $B \approx (145 \text{MeV})^4$ is the bag constant.

5. DIQUARKS

The four-quark operators can be rewritten to look like diquark densities. We have computed matrix elements of operators of the following form:

$$\frac{1}{10} (\bar{u}_a \Gamma \gamma_5 C \bar{u}_b^\Gamma)(u_a^\Gamma C^{-1} \gamma_5 \Gamma' u_b')(\delta_{aa'} \delta_{bb'} - \delta_{aa'} \delta_{bb'}).$$

(16)
\[
\frac{1}{10} (\bar{u}_a \Gamma \gamma_5 C \bar{u}_b ^T)(u_{a'} ^T C^{-1} \gamma_5 \Gamma' u_{b'}) (\delta_{a b'} \delta_{b a'} + \delta_{a a'} \delta_{b b'}) ,
\]
(17)

where \(a, b, a', \text{ and } b'\) are colour indices. The Dirac structure of the operators is represented by the matrices \(\Gamma\) and \(\Gamma'\). In (16) the diquark is in a \(3\) of colour and thus anti-symmetric in the colour indices. In (17) it is in a \(6\) and symmetric in the colour indices. Strictly speaking, we are again studying operators from the \(27\) representation of \(SU(3)_F\) whose \(\bar{u}uu\) component is as given above. But at least within the quenched approximation the interpretation as valence diquark densities seems reasonable. In order to interpret our results we have combined the operators such that they correspond to diquarks of spin zero and spin one. For an operator \(O_{\mu \nu}\) with two space-time indices (in Euclidean notation), e.g. for the Dirac structure \(\Gamma \otimes \Gamma' = \gamma_\mu \otimes \gamma_\nu\), we take the expectation value of \(O_{44}\) in a state with vanishing momentum to represent a spin-zero diquark and the expectation value of \(\sum_{i=1}^{3} O_{ii}\) to correspond to a spin-one diquark.

In Fig. 1 we show our results for the diquark densities. They are plotted versus \(1/\kappa\) and linearly extrapolated to the chiral limit at \(1/\kappa_{cr}\), where the variable \(1/\kappa\) determines the bare quark mass \(m_q = (1/\kappa - 1/\kappa_{cr})/(2a)\). In physical units these bare quark masses are approximately 100, 190, and 250 MeV.

The pattern of the results can tentatively be understood in a non-relativistic quark picture. When the diquark is in the \(3\) of \(SU(3)_c\) it is anti-symmetric in the colour indices, and therefore the symmetric (in the spin indices) spin-one state is favoured over the anti-symmetric spin-zero state. On the other hand, when the diquark is in the (symmetric) \(6\) of colour one might at first sight expect the anti-symmetric spin-zero state to dominate over the symmetric spin-one state. Although the spin-zero contribution is indeed less suppressed than in the \(3\) case it is not really dominating. This is probably related to the fact that a diquark in the \(6\) of \(SU(3)_c\) must be accompanied by (at least) one gluon if it is to form a colour singlet with the remaining quark. The coupling to the gluon, mixing “large” and “small” components of the quark spinors, would invalidate the above arguments which worked reasonably well for diquarks in the \(3\) of colour.

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Figure 1. Diquark densities with the Dirac structures (from top to bottom) $\Gamma \otimes \Gamma' = \gamma_\mu \gamma_5 \otimes \gamma_\nu \gamma_5$ (3 of colour), $\sigma_\mu \sigma_\alpha \otimes \sigma_\nu \sigma_\alpha$ (3 of colour), $\gamma_\mu \otimes \gamma_\nu$ (6 of colour).