A note on the Minimum Norm Point algorithm

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Abstract

We present a provably more efficient implementation of the Minimum Norm Point Algorithm conceived by Fujishige than the one presented in [FUJI06]. The algorithm solves the minimization problem for a class of functions known as submodular. Many important functions, such as minimum cut in the graph, have the so called submodular property [FUJI82]. It is known that the problem can also be efficiently solved in strongly polynomial time [IWAT01], however known theoretical bounds are far from being practical. We present an improved implementation of the algorithm, for which unfortunately no worst case bounds are known, but which performs very well in practice. With the modifications presented, the algorithm performs an order of magnitude faster for certain submodular functions.

Introduction

Given a base set $S$, a submodular function $F$ is such that, for any $A, B \subseteq S$ the following holds

$$F(A) + F(B) \geq F(A \cap B) + F(A \cup B)$$

(1)

It is not hard to show, that a cut in the graph is a submodular function, where $F(A) = \text{cut}(A, V \setminus A)$. The objective is to minimize the cut and which in turn enables us to find a maximum flow in a graph. It is also known that any symmetric submodular function, that is for $F(A) = F(S \setminus A)$ for all $A \subseteq S$, can be seen as a cut function in a certain graph [QUER95].

Base Polyhedra and Submodular Function Minimization

Throughout this paper we assume that for a set $E \in 2^{\{1,...,n\}}$ and a point $x \in \mathbb{R}^n$

$x(E) = \sum_{e_i \in E} x_{e_i}$ or a sum of projection on coordinates in $E$. It will also be useful
to define a base polyhedron $B(F)$ with respect to a submodular function $F$: 
Definition 1. Let $E$ be a finite nonempty set and $F$ be a submodular function $F : 2^E \mapsto \mathbb{R}$. Suppose that $F(\emptyset) = 0$, then we can define the base polyhedron:

$$P(F) = \{x | x \in \mathbb{R}^E, \forall X \in 2^E : x(X) \leq F(X)\}$$
$$B(F) = \{x | x \in P(F), x(E) = F(E)\}$$

Minimum Norm Point Algorithm

Suppose we are given a finite set $P$ of points $p_i \in \mathbb{R}^n$. The problem is to find the minimum norm point $x^*$ in the convex hull of points $p_i$ i.e. $\text{argmin} \|x\|_2, x \in \text{CH}(p_1, p_2, \ldots, p_n)$. The following theorem establishes the relationship between minimum norm point in the convex hull and the minimization of a submodular function:

**Theorem 1.** Let $x^*$ be the minimum norm point in the base polyhedron $B(F)$ as defined above. Define

$$A_+ = \{e | e \in E, x^*(e) \leq 0\}$$
$$A_- = \{e | e \in E, x^*(e) < 0\}$$

Then $A_+$ is the unique maximal minimizer of $F$ and $A_-$ is the unique minimal minimizer of $F$.

Equipped with this theorem we can find the minimum norm point in the base polyhedron $B(F)$ and find the minimum of a submodular function $F$.

Here follows the description of the minimum norm point algorithm: Throughout the runtime, the algorithm maintains a simplex of points $\in 2^P$ and a current minimum norm point $\hat{x}$. With each update of the simplex, the norm of $\hat{x}$ decreases.

**Input:** A finite set of points $P = \{p_1, p_2, \ldots, p_k\}, p_i \in \mathbb{R}^n$

**Output:** The minimum norm point $x^*$ in the convex hull $\hat{P}$ of the points $\{p_1, \ldots, p_k\}$

1. Choose any point $p \in P$ and put $S = p$ and $\hat{x} = p$.
2. Find a point $\hat{p} \in P$ that minimizes the linear function $\langle \hat{x}, p \rangle = \sum_i \hat{x}_i p_i$. If $\langle \hat{x}, p \rangle = \langle \hat{x}, \hat{x} \rangle$, return $x^* = \hat{x}$. Else go to step 3.
3. Find the minimum norm point $y$ in the affine hull of points in $S$. If $y$ lies in the relative interior of the convex hull of $S$, then put $\hat{x} = y$ and go to step 2.
4. Let $z$ be the point that is the nearest to $y$ among the intersection of the convex hull of points in $S$ and the line $[y, \hat{x}]$ between $y$ and $\hat{x}$. Additionally, let $S' \subset S$ be the unique proper subset of $S$ such that $z$ lies in the relative interior of the convex hull of $S'$. Put $S = S'$ and $\hat{x} = z$. Go to step 3.

The cycle formed by the steps 2 $\leftrightarrow$ 3 is called a major cycle and the one by steps 3 $\leftrightarrow$ 4 a minor cycle. In major/minor cycles, the simplex size increases/decreases correspondingly. In major cycle the simplex increases by 1 and in the minor decreases by at least 1.
Definition 2. A simplex is called a corral if the minimum norm point lies in the relative interior of the convex hull of the points of the simplex.

Lemma 1. Every corral uniquely determines the current minimum norm point.

Lemma 2. After at most \( n - 1 \) iterations in the minor cycle, the current simplex becomes a corral.

Proof. As there will be left at most 1 point in the simplex \( S \).

Lemma 3. After each iteration of step 2 the norm of the \( \hat{x} \) is decreasing.

Theorem 2. The described minimum norm point algorithm terminates in a finite number of steps. It is currently open to decide if the algorithm runs in polynomial time.

Implementation

Step 2 of the algorithm requires a linear optimization, which can be done by computing \( \langle \hat{x}, p \rangle = \sum \hat{x}_i p_i \) for all the points in \( P \), however the number of points can be exponential.

In the case the set \( P \) is given implicitly, such as a number of extreme points of a polytope \( Q \).

Luckily, for base polyhedra associated with submodular functions this problem can be solved greedily as was shown by Edmonds:

Input: \( w \in \mathbb{R}^E \), submodular function \( F \)
Output: An optimal \( x^* \in B(F) \) that minimizes \( \sum_{e \in E} w(e)x(e) \)

1. Find an ordering of \( e_1, e_2, \ldots, e_n \) s.t.
\[
w(e_1) \leq w(e_2) \leq \ldots w(e_n)
\]

2. Compute \( x^* \) as follows:
\[
x^*(e_i) = F(\{e_1, e_2, \ldots, e_i\}) - F(\{e_1, e_2, \ldots, e_{i-1}\}), \ (i = 1, 2, \ldots n)
\]

Lemma 4. The resulting \( x^* \) lies in the base polyhedron \( B(F) \) and minimizes \( \sum_{e \in E} w(e)x(e) \)

Step 3 requires solving the following optimization problem:

\[
\min \| x \| \quad x = \sum_{1 \leq i \leq n} \alpha_i p_i
\]
\[
\sum_{1 \leq i \leq n} \alpha_i = 1, p_i \in P, \alpha_i \in \mathbb{R}
\]

Equivalently, the problem can we rewritten as
\[
\min \|x\|
\]
\[
x = p_1 + \sum_{1 \leq i \leq n} \alpha_i (p_i - p_1) \iff p_1 + \sum_{2 \leq i \leq n} \alpha_i (p_i - p_1)
\]
\[
\sum_{1 \leq i \leq n} \alpha_i = 1, p_i \in P, \alpha_i \in \mathbb{R} \iff p_i \in P, \alpha_i \in \mathbb{R}
\]

Consider the subspace of vectors \(p_i - p_1\). Let \(p_1 = p_1^\parallel + p_1^\perp\), such that \(\langle p_1^\perp, p_i - p_1 \rangle = 0\) for all \(2 \leq i \leq n\).

**Lemma 5.** There exists a unique such decomposition \(p_1 = p_1^\parallel + p_1^\perp\)

Then it follows that

**Lemma 6.** Denote \(S_P\) the subspace of vectors \(p_i - p_1, 2 \leq i \leq n\). Also, let \(v \in \mathbb{R}^n\) belong to \(S_P\), s.t. \(v = p_1^\parallel + \sum_{2 \leq i \leq n} \alpha_i (p_i - p_1)\). For a minimum norm point \(x\),

\[
\min \|x\| = \|p_1^\parallel + p_1^\perp + \sum_{2 \leq i \leq n} \alpha_i (p_i - p_1)\| = \|p_1^\parallel\| + \|v\| \geq \|p_1^\parallel\| \quad (4)
\]

Clearly, the inequality is tight and holds for \(v = \vec{0}\), hence the optimization problem is minimized for \(x = p_1^\perp = p_1 - p_1^\parallel\). Note that the choice of \(p_1\) was completely arbitrary, and any vector \(p_i\) could be chosen.

Finding \(p_1^\parallel\) is known as projection onto the subspace and can be found as follows:

**Lemma 7.** Let \(M = \begin{bmatrix} p_2 - p_1 & p_3 - p_1 & \ldots & p_n - p_1 \end{bmatrix} \in \mathbb{R}^{n(k-1)}\), then projection of \(p_1\) onto the subspace is

\[
p_1^\parallel = M(M^T M)^{-1} M^T p_1 \quad (5)
\]

It is however inefficient to find the projection in this way, as we would need to compute the inverse of the hat matrix \((M^T M)\). Instead, we can solve the following system of equations:

\[
(M^T M)y = M^T p_1 \quad (6)
\]
\[
p_1^\parallel = My \quad (7)
\]

The system of equations above is usually solved using the Gaussian elimination process of the matrix on the left hand side and back substitution subsequently. For general matrices, straightforward Gaussian elimination requires \(O(n^3)\) operations or more precisely \(\approx 2/3n^3\) operations. Matrix vector multiplication takes \(nk\) operations where the \(n, k\) are the matrix dimensions.

The step 4 of the algorithm, we can determine the point \(z\) as follows:

**Lemma 8.** Let

\[
\hat{x} = \sum_i \lambda_i p_i, \ y = \sum_i \mu_i p_i \quad (8)
\]

then \(z\) can be determined such that

\[
z = (1 - \beta)\hat{x} + \beta y, \ (1 - \beta)\lambda_i + \beta\mu_i \geq 0, \forall i \quad (9)
\]

and \(\beta\) is large as possible.
Improvement idea

The improvement is based on the following idea known as Sherman-Morrison-Woodbury matrix inverse update:

**Lemma 9.** Let $M = \begin{bmatrix} A & U \\ V & D \end{bmatrix}$ then, $M^{-1} = \begin{bmatrix} A^{-1} + A^{-1}UC^{-1}VA^{-1} & -A^{-1}UC^{-1} \\ -C^{-1}VA^{-1} & C^{-1} \end{bmatrix}$

where $C = D - VA^{-1}U$

Now, notice that during the runtime of our algorithm, in the steps $2 \leftrightarrow 3$ we only add 1 column to the matrix $S$ and in the steps $3 \leftrightarrow 4$ we delete at least 1 column of the matrix. With the lemma above, we could update the inverse of the matrix $MTM$ and solve the system of equations (6), (7) more efficiently.

Precise formulation of updates

As was demonstrated above, it is possible to update the matrix inverse using the blockwise approach. The matrix that we are dealing with is of the form $MTM$.

Without loss of generality, suppose that a column $v$ is appended to the matrix $M$ as the last column, i.e. $M' = \begin{bmatrix} M & v \end{bmatrix}$. Let us call such an update a rank-up update. Then, $M'TM' = \begin{bmatrix} M'TM & M'Tv \\ vTM & vTv \end{bmatrix}$

Then, simply substituting $A = M'TM$, $V = vTM$, $U = M'Tv$ and $C = vTv$ we can apply the Sherman-Morrison-Woodbury matrix inverse update formula.

Now, suppose, the last $k$ columns of the matrix $M$ are removed and let the new matrix be $N$. We call such an update a rank-down update. Then $M = \begin{bmatrix} N & K \end{bmatrix}$ and $MTM = \begin{bmatrix} N'TN & N'TK \\ KTN & KTK \end{bmatrix}$.

Now, notice that, in the Sherman-Morrison-Woodbury update, the lower right, upper right and lower left blocks of the matrix multiplied in the following way:

$$(A^{-1}UC^{-1})(C^{-1})^{-1}(-C^{-1}VA^{-1}) = A^{-1}UC^{-1}VA^{-1}$$

which is exactly the term of the upper left block of the matrix

$$M^{-1} = \begin{bmatrix} A^{-1} + A^{-1}UC^{-1}VA^{-1} & -A^{-1}UC^{-1} \\ -C^{-1}VA^{-1} & C^{-1} \end{bmatrix}$$

Hence, knowing the inverse of the matrix $MTM = \begin{bmatrix} N'TN & N'TK \\ KTN & KTK \end{bmatrix}$, we can find out the inverse of the matrix $N'TN$:

Let

$$(MTM)^{-1} = \begin{bmatrix} P & Q \\ QT & R \end{bmatrix}$$

Then, the inverse of $N'TN = P - QT(R^{-1}Q)$. Note that, we are again faced with the problem of taking the inverse of a matrix $R$ and matrix multiplications. Note that a rank-down update by $k$ columns can be realized by a series of $k$ rank-down updates which remove only a single column.
Lemma 10. The running time of a single rank-up or rank-down operations that add or remove a single column is $O(n^2)$.

Proof. Let us firstly consider the rank-up update. Matrices $V = v^T M$ and $U = M^T v$ can be computed in time $O(n^2)$ as computing them corresponds to matrix-vector multiplications. The matrix $D = v^T v$ is computable in time $O(n)$ and the matrix $C$ in $O(n^2)$. With similar reasoning, the product $A^{-1} U C^{-1} V A^{-1} = (A^{-1} U C^{-1})(C^{-1})^{-1}(-C^{-1} V A^{-1})$ can be computed in $O(n^2)$, given that the multiplications are realized as suggested by the placement of brackets. Hence, overall the time to do a rank-up update is $O(n^2)$. We can apply the very same techniques to verify that a rank-down update by 1 column is implementable in $O(n^2)$ time. □

The lemma that follows implies that the efficient updates presented above make it possible to carry all the inverse updates an order of magnitude faster than in the original algorithm.

Lemma 11. The amortized cost of rank-up and rank-down in arbitrary sequence of operations is $O(n^2)$. And hence, the total running time of a sequence of length $t$ of rank-up and rank-down updates takes time $O(t n^2)$.

Proof. Recall that the running time of the algorithm is dominated by the total number of times the steps 2 and 3 are called, multiplied by the time a respective step takes. In a major cycle we add a column to the matrix $M$ and during a minor cycle a number of columns are removed. In the step 2 we would need to solve an optimization problem, which will be discussed later. In the 3-rd step the algorithm needs to solve a system of a kind $y = Ax$ for a given $y$. When an update of $A$ is readily available, this can be done in $O(n^2)$ time, however without it we would need $O(n^3)$ time.

Let the number of removed columns during the $i$-th minor cycle be $k_i$ and the number of times the major cycle is called $m_i$, then the total running time of the algorithm is $O(\sum_i m_i n^2 + \sum_i k_i n^2)$, as a rank update of any kind of a single column takes $O(n^2)$ time.

Although the number of operations during a minor cycle can be as bad as $O(n^3)$ when $k_i = O(n)$, the amortized time for every rank-up and rank-down update (independent of the number of columns that are deleted) can be shown to be $O(n^2)$. This can be shown using the accounting method of amortized analysis. Let every rank-up update bring a $O(n^2)$ to the system and another $O(n^2)$ to pay for its own update. Then when a rank-down happens, every column has funds to pay for its rank-down update. Hence, the amortized time for every update is $O(n^2)$ and any sequence of such updates is computable in time $O(t n^2)$ where $t$ is the length of the sequence.

Note that, the original algorithm also needs to solve a system of a kind $y = Ax$ for a given $y$, however without the update of $A$ readily available, which needs $O(n^3)$ time. And hence when $t$ such steps need to be performed, the total time is $O(t n^3)$, while the improved version requires only $O(t n^2)$ time. □
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