On the moments of characteristic polynomials

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Abstract

We examine the asymptotics of the moments of characteristic polynomials of \(N \times N\) matrices drawn from the Hermitian ensembles of Random Matrix Theory, in the limit as \(N \to \infty\). We focus in particular on the Gaussian Unitary Ensemble, but discuss other Hermitian ensembles as well. We employ a novel approach to calculate asymptotic formulae for the moments, enabling us to uncover subtle structure not apparent in previous approaches.

1 Introduction

The characteristic polynomials of random matrices have received considerable attention over the past twenty years. One of the principal motivations stems from their connections to the statistical properties of the Riemann zeta-function and other families of \(L\)-functions [39, 38, 34, 16, 14, 15, 36, 26, 28]. In this context, the value distributions of the characteristic polynomials of random unitary, orthogonal and symplectic matrices have been calculated using a variety of approaches. For example, the moments have been computed in all three cases and the results used to develop conjectures for the moments of the Riemann zeta-function \(\zeta(s)\) on its critical line and for the moments of families of \(L\)-functions at the centre of the critical strip. Specifically, if \(A\) is an \(N \times N\) unitary matrix, drawn at random uniformly with respect to Haar measure on the unitary group \(U(N)\), then for \(\text{Re} \beta > -1/2\)

\[
E_{A \in U(N)} [ | \det(I - Ae^{-i\theta}) |^{2\beta} ] = \prod_{j=1}^{N} \frac{\Gamma(j) \Gamma(j + 2\beta)}{\Gamma(j + \beta)^2},
\]

(1.1)

from which one can deduce that as \(N \to \infty\)

\[
E_{A \in U(N)} [ | \det(I - Ae^{-i\theta}) |^{2\beta} ] \sim \frac{G(1 + \beta)^2}{G(1 + 2\beta)} N^{\beta^2},
\]

(1.2)

where \(G(s)\) is the Barnes \(G\)-function, and for \(k \in \mathbb{N}\)

\[
E_{A \in U(N)} [ | \det(I - Ae^{-i\theta}) |^{2k} ] \sim \left( \prod_{m=0}^{k-1} \frac{m!}{(m + k)!} \right) N^{k^2}.
\]

(1.3)
These formulae lead to the conjectures [39] that for \( \text{Re}\beta > -1/2 \), as \( T \to \infty \)
\[
\frac{1}{T} \int_0^T |\zeta(1/2 + it)|^{2\beta} dt \sim a(\beta) \frac{G(1 + \beta)^2}{G(1 + 2\beta)} (\log T)^{\beta^2}
\]
and for \( k \in \mathbb{N} \), as \( T \to \infty \)
\[
\frac{1}{T} \int_0^T |\zeta(1/2 + it)|^{2k} dt \sim a(k) \prod_{m=0}^{k-1} \frac{m!}{(m + k)!} (\log T)^{k^2}
\]
where
\[
a(s) = \prod_p \left[ (1 - \frac{1}{p^s}) s^2 \sum_{m=0}^{\infty} \left( \frac{\Gamma(m + s)}{m! \Gamma(s)} \right)^2 p^{-m} \right]
\]
with the product running over primes \( p \).

Our focus here will primarily be on the Gaussian Unitary Ensemble (GUE) of random complex Hermitian matrices. For an \( N \times N \) matrix \( M \) drawn from the GUE, the joint eigenvalue density function is
\[
\frac{1}{Z_N^{(H)}} \prod_{1 \leq j < k \leq N} |x_j - x_k|^2 \prod_{j=1}^{N} e^{-\frac{N \rho_{sc}}{2} x_j^2},
\]
where \( Z_N^{(H)} \) is a normalization constant. Brezin and Hikami [9] calculated the \( N \to \infty \) asymptotics of the moments of the associated characteristic polynomials to be
\[
\mathbb{E}_N^{(H)} [\det(t - M)^{2p}] = e^{-Np} e^{Np^2 \frac{t^2}{2}} (2\pi N \rho_{sc}(t))^{p^2} \prod_{j=0}^{p-1} \frac{j!}{(p + j)!}, \quad p \in \mathbb{N},
\]
where the asymptotic eigenvalue density is given by the Wigner semi-circle law
\[
\rho_{sc}(x) = \frac{1}{2\pi} \sqrt{4 - x^2}.
\]
This corresponds precisely to (1.3), where the mean density is constant.

Our purpose is to focus on some subtle features of the asymptotics of the moments of the characteristic polynomials of GUE matrices, and of matrices drawn from other unitarily invariant ensembles, that are not captured by (1.8). In particular, we will show that the asymptotics when \( N \) is even differs from when it is odd, and that one only recovers (1.8) when one formally averages the two cases. At the one hand, this is somewhat surprising, because (1.8) is a central pillar of the theory of the characteristic polynomials of random matrices, but on the other it is not completely expected, based on the following reasoning. A straightforward application of the method of orthogonal polynomials gives that the correlations of characteristic polynomials have a determinantal structure involving classical orthogonal polynomials. For the GUE, for example,
\[
\mathbb{E}_N^{(H)} \left[ \prod_{j=1}^{2p} \det(t_j - M) \right] = \frac{1}{\Delta(t)} \det[H_{N + 2p - j}(\sqrt{N} t_k)]_{1 \leq j, k \leq 2p},
\]
where $H_n(x)$ is a Hermite polynomial of degree $n$ and
\[ \Delta(\mathbf{t}) = \Delta(t_1, \ldots, t_{2p}) = \prod_{1 \leq j < k \leq 2p} (t_j - t_k) \tag{1.11} \]
is the Vandermonde determinant. As a consequence, the moments also take a determinantal form comprising derivatives of Hermite polynomials. But Hermite polynomials depend on the parity of the degree $n$ via
\[ H_n(-x) = (-1)^n H_n(x). \tag{1.12} \]
Therefore, the moments of characteristic polynomials also depend on the parity of the degree, which in turn depends on the parity of the dimension of the matrix $N$. As a result, one might expect that the asymptotic behaviour of the moments of characteristic polynomials should be different for even and odd dimensional matrices. The question is: at what order in the asymptotics is this important? We show below that for the GUE it influences the leading-order behaviour. This is in contrast to other Hermitian ensembles such as the Laguerre unitary ensemble (LUE) and the Jacobi unitary ensemble (JUE), for which both even and odd dimensional matrices have the same moments at leading order.

Brezin and Hikami \cite{9} used orthogonal polynomial techniques to arrive at (1.8). Other studies to-date relating to the asymptotics of the moments of characteristic polynomials have relied mainly on the orthogonal polynomial method and saddle point techniques \cite{9, 10, 3}, the Riemann-Hilbert method \cite{46}, Hankel determinants with Fisher-Hartwig symbols \cite{40, 19, 33}, and supersymmetric tools \cite{2, 31, 24, 47}. In the present paper we take a different line of attack: we express the moments in terms of certain multivariate orthogonal polynomials and take a combinatorial approach to compute the asymptotics of the moments using the properties of these polynomials. By doing so, we discover that even and odd dimensional GUE matrices give different contributions in the large $N$ limit, and that only a formal average gives formulae consistent with (1.8). In Sec. 4.2.1, this phenomenon is discussed in detail for the second moment of the characteristic polynomial.

In addition to connections with number theory, characteristic polynomials have found numerous applications in quantum chaos \cite{2}, mesoscopic systems \cite{23}, quantum chromodynamics \cite{17}, and in a variety of combinatorial problems \cite{45, 18}. The asymptotic study of negative moments and ratios of characteristic polynomials is another active area of research, see for example \cite{6, 27, 32, 3, 46, 20, 7, 8, 25, 1}. More recently, the statistics of the maximum of the characteristic polynomial are being extensively studied, motivated by the relations to logarithmically correlated Gaussian processes. For example, see \cite{26, 28, 29, 30} and references therein. We expect that the techniques developed here will have applications to those calculations as well.

This paper is structured as follows. After introducing the required tools in Sec. 2, we recall the moments of characteristic polynomials of the GUE, LUE and JUE in Sec. 3. In Sec. 4, we compute the asymptotics of moments of the GUE and illustrate how to recover the semi-circle law in the limit as the matrix size goes to infinity. In the last section Sec. 5, as an application of the results discussed, we compute the correlations of secular coefficients which are the coefficients of a characteristic polynomial when expanded as a function of the spectral variable.
2 Background

A partition $\mu$ is a sequence of integers $(\mu_1, \ldots, \mu_l)$ such that $\mu_1 \geq \cdots \geq \mu_l > 0$. Here $l$ is the length of the partition and we denote $|\mu| = \mu_1 + \cdots + \mu_l$ to be the weight of the partition. We do not distinguish partitions that only differ by a sequence of zeros at the end. For example $(4, 2)$ and $(4, 2, 0, 0)$ are equivalent with length $l = 2$ and weight 6. A partition can be represented with a Young diagram which is a left adjusted table of $|\mu|$ boxes and $l(\mu)$ rows such that the first row contains $\mu_1$ boxes, the second row contains $\mu_2$ boxes, and so on. The conjugate partition $\mu'$ is defined by transposing the Young diagram of $\mu$ along the main diagonal.

\[
\begin{array}{c|c}
\mu & \mu' \\
\hline
\begin{array}{cccc}
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\end{array} &
\begin{array}{cccc}
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\end{array}
\end{array}
\]

(2.1)

Young diagram of $\mu$  Young diagram of $\mu'$

For a partition $\mu$, let $\Phi_\mu$ be the multivariate symmetric polynomial, with leading coefficient equal to 1, that obey the orthogonality relation

\[
\int \Phi_\mu(x_1, \ldots, x_N)\Phi_\nu(x_1, \ldots, x_N) \prod_{1 \leq i < j \leq N} (x_i - x_j)^2 \prod_{j=1}^N w(x_j) \, dx_j = \delta_{\mu\nu}C_\mu
\]

(2.2)

for a weight function $w(x)$. Here the lengths of the partitions $\mu$ and $\nu$ are less than or equal to the number of variables $N$, and $C_\mu$ is a constant which depends on $N$. Polynomial $\Phi_\mu$ can be expressed as a ratio of determinants, as given in [44],

\[
\Phi_\mu(x) = \frac{1}{\Delta(x)} | \begin{array}{cccc}
\varphi_{\mu_1+N-1}(x_1) & \varphi_{\mu_1+N-1}(x_2) & \cdots & \varphi_{\mu_1+N-1}(x_N) \\
\varphi_{\mu_2+N-2}(x_1) & \varphi_{\mu_2+N-2}(x_2) & \cdots & \varphi_{\mu_2+N-2}(x_N) \\
\vdots & \vdots & \ddots & \vdots \\
\varphi_{\mu_N}(x_1) & \varphi_{\mu_N}(x_2) & \cdots & \varphi_{\mu_N}(x_N)
\end{array} |
\]

(2.3)

where $\varphi_j(x)$ is a polynomial of degree $j$ orthogonal with respect to $w(x)$. We focus in particular to the case when $w(x)$ in (2.2) is one of the weights

\[
w(x) = \begin{cases} 
 e^{-\frac{x^2}{2}}, & x \in \mathbb{R}, \quad \text{Gaussian}, \\
 x^\gamma e^{-2Nx}, & x \in \mathbb{R}_+, \quad \gamma > -1, \quad \text{Laguerre}, \\
x^{\gamma_1}(1-x)^{\gamma_2}, & x \in [0, 1], \quad \gamma_1, \gamma_2 > -1, \quad \text{Jacobi}.
\end{cases}
\]

(2.4)

The monic polynomials orthogonal with respect to these weights are

\[
h_n(x) = N^{-\frac{n}{2}} H_n(\sqrt{N}x),
\]

(2.5)

\[
l_n^{(\gamma)}(x) = \frac{(-1)^n n!}{(2N)^n} L_n^{(\gamma)}(2Nx),
\]

(2.6)

\[
j_n^{(\gamma_1, \gamma_2)}(x) = (-1)^n n! \frac{\Gamma(n + \gamma_1 + \gamma_2 + 1)}{\Gamma(2n + \gamma_1 + \gamma_2 + 1)} J_n^{(\gamma_1, \gamma_2)}(x),
\]

(2.7)
where $H_n(x)$, $L_n^{(\gamma)}(x)$ and $J_n^{(\gamma_1,\gamma_2)}(x)$ are the classical orthogonal polynomials that satisfy

$$
\int_{\mathbb{R}} H_j(x)H_k(x)e^{-x^2} \, dx = \sqrt{2\pi j!} \delta_{jk}, \quad (2.8a)
$$

$$
\int_{\mathbb{R}^+} L_m^{(\gamma)} L_n^{(\gamma)} x^\gamma e^{-x} \, dx = \frac{\Gamma(n+\gamma+1)}{\Gamma(n+1)} \delta_{nm}, \quad (2.8b)
$$

$$
\int_0^1 J_n^{(\gamma_1,\gamma_2)}(x)J_m^{(\gamma_1,\gamma_2)}(x)x^{\gamma_1}(1-x)^{\gamma_2} \, dx 
= \frac{1}{(2n+\gamma_1+\gamma_2+1)} \frac{\Gamma(n+\gamma_1+1)\Gamma(n+\gamma_2+1)}{n!\Gamma(n+\gamma_1+\gamma_2+1)} \delta_{mn}. \quad (2.8c)
$$

When $\varphi_n(x)$ in (2.3) is chosen to be one of the Hermite $h_n(x)$, Laguerre $l_n^{(\gamma)}(x)$ and Jacobi $j_n^{(\gamma_1,\gamma_2)}(x)$ polynomials of degree $n$, we get their multivariable analogues denoted by $H_\mu$, $L_\mu^{(\gamma)}$ and $J_\mu^{(\gamma_1,\gamma_2)}$. These multivariate generalisations are the eigenfunctions of differential equations called Calogero–Sutherland Hamiltonians. Several properties such as recursive relations and integration formulas extend to the multivariate case [4, 5].

Define

$$
C_\lambda(N) = \prod_{j=1}^N \frac{(\lambda_j + N - j)!}{(N-j)!},
G_\lambda(N, \gamma) = \prod_{j=1}^N \Gamma(\lambda_j + N - j + \gamma + 1).
$$

The constants $C_\lambda(N)$ and $G_\lambda(N, \gamma)$ have several interesting combinatorial interpretations [37]. The joint probability densities function for the GUE, LUE and JUE are

$$
p^{(H)}(x_1, \ldots, x_N) = \frac{1}{Z_N^{(H)}}^2 \Delta^2(x_1, \ldots, x_N) \prod_{j=1}^N e^{-\frac{N_x^2}{2}}, \quad (2.10)
$$

$$
p^{(L)}(x_1, \ldots, x_N) = \frac{1}{Z_N^{(L)}}^2 \Delta^2(x_1, \ldots, x_N) \prod_{j=1}^N x_j^\gamma e^{-2N x_j}, \quad (2.11)
$$

$$
p^{(J)}(x_1, \ldots, x_N) = \frac{1}{Z_N^{(J)}}^2 \Delta^2(x_1, \ldots, x_N) \prod_{j=1}^N x_j^{\gamma_1}(1-x_j)^{\gamma_2}, \quad (2.12)
$$

with

$$
Z_N^{(H)} = \frac{(2\pi)^{\frac{N}{2}}}{N^{\frac{N}{2}}} \prod_{j=1}^N j!,
$$

$$
Z_N^{(L)} = \frac{N!}{(2N)^{N(N+\gamma)}} G_0(N, \gamma) G_0(N, 0),
$$

$$
Z_N^{(J)} = N! \prod_{j=0}^{N-1} \frac{j! \Gamma(j + \gamma_1 + 1) \Gamma(j + \gamma_2 + 1) \Gamma(j + \gamma_1 + \gamma_2 + 1)}{\Gamma(2j + \gamma_1 + \gamma_2 + 2) \Gamma(2j + \gamma_1 + \gamma_2 + 1)}. \quad (2.15)
$$
Similarly, one has

\[
\frac{1}{2} \int_{(-\infty, \infty)^n} \mathcal{H}_\mu(x) \mathcal{H}_\nu(x) \Delta^2(x) \prod_{j=1}^n e^{\frac{x_j^2}{2}} \, dx_j = \frac{1}{N_{|\mu|}} C_{\mu}(n) \delta_{\mu\nu}, \quad (2.16)
\]

\[
\frac{1}{2} \int_{(0, \infty)^n} \mathcal{L}_\mu(x) \mathcal{L}_\nu(x) \Delta^2(x) \prod_{j=1}^n x_j^2 e^{-2N x_j} \, dx_j = \frac{1}{(2N)^{2|\mu|}} \frac{G_{\mu}(n, \gamma)}{G_0(n, \gamma)} C_{\mu}(n) \delta_{\mu\nu}, \quad (2.17)
\]

\[
\frac{1}{2} \int_{[0,1]^n} \mathcal{J}_\mu^{(\gamma_1, \gamma_2)}(x) \mathcal{J}_\nu^{(\gamma_1, \gamma_2)}(x) \Delta^2(x) \prod_{j=1}^n x_j^2 (1 - x_j)^{\gamma_2} \, dx_j \]

\[
= \prod_{j=1}^n \frac{\Gamma(2n - 2j + \gamma_1 + \gamma_2 + 1) \Gamma(2n - 2j + \gamma_1 + \gamma_2 + 2) \Gamma(2\lambda_j + 2n - 2j + \gamma_1 + \gamma_2 + 1)}{\Gamma(2\lambda_j + 2n - 2j + \gamma_1 + \gamma_2 + 2)} \times \frac{G_\mu(n, \gamma_1 + \gamma_2) G_\mu(n, \gamma_1) G_\mu(n, \gamma_2)}{G_0(n, \gamma_1 + \gamma_2) G_0(n, \gamma_1) G_0(n, \gamma_2)} C_{\mu}(n) \delta_{\mu\nu}. \quad (2.18)
\]

The Schur polynomials \( S_\lambda \), indexed by a partition \( \lambda \), are defined as

\[
S_\lambda(x_1, \ldots, x_n) = \frac{\det[x_j^{\lambda_i+n-j}]}{\det[x_j^{n-j}]} \quad (2.19)
\]

for \( l(\lambda) \leq n \), and \( S_\lambda = 0 \) for \( l(\lambda) > n \). The polynomials \( \mathcal{H}_\mu, \mathcal{L}_\mu^{(\gamma)} \) and \( \mathcal{J}_\mu^{(\gamma_1, \gamma_2)} \) form a basis for symmetric polynomials of degree \( |\mu| \). The Schur polynomials can be expanded as [35]

\[
S_\lambda(x_1, \ldots, x_n) = \sum_{\nu \subseteq \lambda} \Psi_{\lambda \nu} \Phi_\nu(x_1, \ldots, x_n), \quad (2.20)
\]

where \( \Phi_\nu(x) \) can be either \( \mathcal{H}_\nu, \mathcal{L}_\nu^{(\gamma)} \), or \( \mathcal{J}_\nu^{(\gamma_1, \gamma_2)} \). In the following the superscripts \( (H) \), \( (L) \) and \( (J) \) indicate Hermite, Laguerre and Jacobi, respectively. The coefficients in (2.20) are

\[
\Psi_{\lambda \nu}^{(H)} = \left( \frac{1}{2N} \right)^{|\lambda| - |\mu|} \frac{C_{\lambda}(n)}{C_{\nu}(n)} D_{\lambda \nu}^{(H)}, \quad (2.21)
\]

where

\[
D_{\lambda \nu}^{(H)} = \det \left[ \mathbb{1}_{\lambda_j - \nu_k - j + k = 0 \mod 2} 2^{(\lambda_j - \nu_k - j + k)!} \right]_{j, k = 1, \ldots, l(\lambda)}. \quad (2.22)
\]

Similarly, one has

\[
\Psi_{\lambda \nu}^{(L)} = \frac{1}{(2N)^{|\lambda| - |\nu|}} \frac{G_\lambda(n, \gamma) G_\lambda(n, 0)}{G_\nu(n, \gamma) G_\nu(n, 0)} D_{\lambda \nu}^{(L)}, \quad (2.23)
\]

\[
\Psi_{\lambda \nu}^{(J)} = \frac{G_\lambda(n, \gamma_1) G_\lambda(n, 0)}{G_\nu(n, \gamma_1) G_\nu(n, 0)} \left( \prod_{j=1}^n \Gamma(2\nu_j + 2n - 2j + \gamma_1 + \gamma_2 + 2) \right) D_{\lambda \nu}^{(J)}, \quad (2.24)
\]

where

\[
D_{\lambda \nu}^{(L)} = \det \left[ \mathbb{1}_{\lambda_j - \nu_j - i + j \geq 0 \mod 2} \frac{1}{(\lambda_j - \nu_j - i + j)!} \right]_{1 \leq i, j \leq l(\lambda)}, \quad (2.25)
\]

\[
D_{\lambda \nu}^{(J)} = \det \left[ \mathbb{1}_{\lambda_j - \nu_k - j + k \geq 0} \frac{1}{(\lambda_j - \nu_k - j + k)! \Gamma(2n + \lambda_j + \nu_k - j - k + \gamma_1 + \gamma_2 + 2)} \right]_{1 \leq i, j \leq n}. \quad (2.26)
\]
The polynomials \( \mathcal{H}_\mu, \mathcal{L}_\mu^{(\gamma)} \) and \( j_\mu^{(\gamma_1,\gamma_2)} \) are chosen such that the leading coefficient of these polynomials in the Schur basis is 1. More precisely,

\[ \Phi_\lambda(x_1, \ldots, x_n) = \sum_{\mu \subseteq \lambda} \Upsilon_{\lambda \mu} S_\mu(x_1, \ldots, x_n), \tag{2.27} \]

where \( \Phi_\lambda \) is one of the \( \mathcal{H}_\lambda, \mathcal{L}_\lambda^{(\gamma)}, j_\lambda^{(\gamma_1,\gamma_2)} \), and

\begin{align*}
\Upsilon_{\lambda \mu}^{(H)} & = \left( -\frac{1}{2N} \right)^{|\lambda|-|\mu|} \frac{C_\lambda(n)}{C_\mu(n)} D_{\lambda \mu}^{(H)}, \\
\Upsilon_{\lambda \mu}^{(L)} & = \left( -\frac{1}{2N} \right)^{|\lambda|-|\mu|} \frac{G_\lambda(n, \gamma) G_\lambda(n, 0)}{G_\mu(n, \gamma) G_\mu(n, 0)} D_{\lambda \mu}^{(L)}, \\
\Upsilon_{\lambda \mu}^{(J)} & = (-1)^{|\lambda|+|\mu|} \left( \prod_{j=1}^{n} \frac{1}{\Gamma(2\lambda_j + 2n - 2j + \gamma_1 + \gamma_2 + 1)} \right) \frac{G_\lambda(n, \gamma_1) G_\lambda(n, 0)}{G_\mu(n, \gamma_1) G_\mu(n, 0)} \tilde{D}_{\lambda \mu}^{(J)},
\end{align*}

with

\[ \tilde{D}_{\lambda \mu}^{(J)} = \det \left[ \frac{\Gamma(2n + \lambda_j + \nu_k - j - k + \gamma_1 + \gamma_2 + 1)}{(\lambda_j - \nu_k - j + k)!} \right]_{1 \leq i,j \leq n} \tag{2.31} \]

In this paper, the above results play an important role in studying the correlations of characteristic polynomials and secular coefficients.

### 3 Moments of characteristic polynomials

The Schur polynomials satisfy the following identity which has proven to be crucial in computing the correlations of characteristic polynomials of the unitary group [11].

**Lemma 3.1** (Dual Cauchy identity). Let \( p, N \in \mathbb{N} \). For \( \lambda \subseteq (N^p) \equiv (N, \ldots, N) \), let \( \tilde{\lambda} = (p - \lambda_1', \ldots, p - \lambda_p') \). Then [41]

\[ \prod_{i=1}^{p} \prod_{j=1}^{N} (t_i - x_j) = \sum_{\lambda \subseteq (N^p)} (-1)^{\tilde{\lambda}} S_{\lambda}(t_1, \ldots, t_p) S_{\lambda}^*(x_1, \ldots, x_N). \tag{3.1} \]

Here \( \lambda = (\lambda_1, \ldots, \lambda_p) \) is a sub-partition of \( (N^p) \) indicated by \( \lambda \subseteq (N^p) \) (each \( \lambda_j \leq N \) for \( j = 1, \ldots, p \)) and \( \lambda' \) is the conjugate partition of \( \lambda \). The \( \Phi_\mu \)'s satisfy a generalised dual Cauchy identity, which is similar to that for the Schur polynomials.

**Lemma 3.2.** With the notation introduced above, we have [35].

\[ \prod_{i=1}^{p} \prod_{j=1}^{N} (t_i - x_j) = \sum_{\lambda \subseteq (N^p)} (-1)^{\tilde{\lambda}} \Phi_{\lambda}(t_1, \ldots, t_p) \Phi_{\lambda}^*(x_1, \ldots, x_N). \tag{3.2} \]

The identity in (3.2) gives a compact way to calculate the correlation functions and moments of characteristic polynomials of unitary invariant Hermitian ensembles.
Proposition 3.1. Let $M$ be an $N \times N$ GUE, LUE or JUE matrix and $t_1, \ldots, t_p \in \mathbb{C}$. Then, using the generalised dual Cauchy identity \[35\],

\[
\begin{align*}
(a) \quad & \mathbb{E}_N^{(H)} \left[ \prod_{j=1}^{p} \det(t_j - M) \right] = \mathcal{H}_{(N^p)}(t_1, \ldots, t_p) \\
(b) \quad & \mathbb{E}_N^{(L)} \left[ \prod_{j=1}^{p} \det(t_j - M) \right] = \mathcal{L}_{(N^p)}^{(\gamma)}(t_1, \ldots, t_p) \\
(c) \quad & \mathbb{E}_N^{(J)} \left[ \prod_{j=1}^{p} \det(t_j - M) \right] = \mathcal{J}_{(N^p)}^{(\gamma_1, \gamma_2)}(t_1, \ldots, t_p)
\end{align*}
\]

The moments can be readily computed from the above formulae by taking the limit $t_j \to t$ for $j = 1, \ldots, p$. This leads to a determinantal formula for the moments involving the derivatives of orthogonal polynomials:

\[
(-1)^{p(p-1)} \prod_{j=1}^{p-1} (j-1)! \begin{vmatrix} \varphi_N(t) & \varphi_{N+1}(t) & \ldots & \varphi_{N+p-1}(t) \\ \varphi_N'(t) & \varphi_{N+1}'(t) & \ldots & \varphi_{N+p-1}'(t) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_N^{(p-1)}(t) & \varphi_{N+1}^{(p-1)}(t) & \ldots & \varphi_{N+p-1}^{(p-1)}(t) \end{vmatrix}.
\]

Here $\varphi_n(t)$ are Hermite $h_n(t)$, Laguerre $l_n^{(\gamma)}(t)$ and Jacobi $j_n^{(\gamma_1, \gamma_2)}(t)$ polynomials for the GUE, LUE and JUE, respectively. By expressing the multivariate polynomials in the Schur basis and using

\[
C_{\lambda}(n) = \prod_{j=1}^{l(\lambda)} \frac{(\lambda_j + n - j)!}{(n-j)!} = |\lambda|! \frac{\dim V_\lambda}{\dim V_\lambda},
\]

where the dimension of the irreducible representation of the symmetric group is

\[
\dim V_\lambda = |\lambda|! \frac{\prod_{1 \leq j < k \leq l(\lambda)} \lambda_j - \lambda_k - j + k}{\prod_{j=1}^{l(\lambda)} (\lambda_j + l(\lambda) - j)!},
\]

we have the following proposition.

Proposition 3.2. Let $\lambda = (N^p)$. The moments of characteristic polynomial are given by [35]

\[
\begin{align*}
\mathbb{E}_N^{(H)} [\det(t-M)^p] &= C_{\lambda}(p) \sum_{\nu \subseteq \lambda} \frac{(-1)^{\left|\nu\right|}}{2N^p} \frac{\dim V_\nu}{|\nu|!} D_{\lambda\nu}^{(H)} t^{|\nu|} \\
\mathbb{E}_N^{(L)} [\det(t-M)^p] &= \left(\frac{-1}{2N^p}\right) G_\lambda(p, \gamma) G_\lambda(p, 0) \sum_{\nu \subseteq \lambda} \frac{(-2N)^{|\nu|}}{G_\nu(p, \gamma)} \frac{\dim V_\nu}{|\nu|!} D_{\lambda\nu}^{(L)} t^{|\nu|} \\
\mathbb{E}_N^{(J)} [\det(t-M)^p] &= \left(\prod_{j=N}^{N+p-1} \frac{1}{\Gamma(2j + \gamma_1 + \gamma_2 + 1)} \right) (-1)^{Np} G_\lambda(p, \gamma_1) G_\lambda(p, 0) \\
&\times \sum_{\nu \subseteq \lambda} \frac{(-1)^{|\nu|}}{|\nu|! G_\nu(p, \gamma_1)} \frac{\dim V_\nu D_{\lambda\nu}^{(J)} t^{|\nu|}}{|\nu|!}.
\end{align*}
\]
These expansions are exact and hold for any finite $N$. In the next section we study the large $N$ asymptotics of the moments of characteristic polynomials. In the large $N$ limit, only the even moments are interesting, since the odd moments result in oscillatory behaviour.

4 Asymptotics

In this section, we consider the asymptotics of the moments of characteristic polynomials for the GUE. By exploiting the integral representation of the classical Hermite polynomials, Brezin and Hikami [9] showed that in the Dyson limit, $N \rightarrow \infty$, $t_i - t_j \rightarrow 0$ and $N(t_i - t_j)$ finite, the moments of characteristic polynomials are

$$E_N^{(H)} \left[ \det (t - M)^{2p} \right] = e^{-Np} e^{Np^2/2} (2\pi N \rho_{sc}(t))^{p^2} \prod_{j=0}^{p-1} \frac{j!}{(p+j)!}, \quad (4.1)$$

where the asymptotic eigenvalue density is

$$\rho_{sc}(x) = \frac{1}{2\pi} \sqrt{4 - x^2}. \quad (4.2)$$

Using (3.9), we show in Sec. 4.1 that

$$E_N^{(H)} \left[ \det M^{2p} \right] = e^{-Np} (2N)^{p^2} \prod_{j=0}^{p-1} \frac{j!}{(p+j)!}, \quad (4.3)$$

which coincides with (4.1) for $t = 0$. For $t \neq 0$, we discover that the asymptotic behaviour is different for even and odd dimensional GUE matrices and these contributions combine in a special way to produce the semi-circle law. These cases are discussed in Sec. 4.1 and Sec. 4.2 in more detail.

4.1 Centre of the semi-circle

Let $\lambda = (N^2p)$. For any finite $N$ we have

$$E_N^{(H)} \left[ \det M^{2p} \right] = \left( -\frac{1}{2N} \right)^{Np} C_\lambda(2p) D_{\lambda_0}^{(H)}. \quad (4.4)$$

Proposition 4.1.

$$D_{\lambda_0}^{(H)} = \prod_{j=0}^{p-1} \frac{j!^2}{(m+j)!^2}, \quad N = 2m, \ m \in \mathbb{N}, \quad (4.5)$$

$$D_{\lambda_0}^{(H)} = (-1)^p \frac{m!}{(m+p)!} \prod_{j=0}^{p-1} \frac{j!^2}{(m+j)!^2}, \quad N = 2m + 1, \ m \in \mathbb{N}.$$
Proof. The determinant \( D_{(H)0}^{(H)} \) can be evaluated as follows. Let \( N = 2m \), then
\[
D_{(H)0}^{(H)} = \det \left[ 1_{k-j = 0 \mod 2} (m + \frac{k-j}{2})! \right]_{1 \leq j, k \leq p}
\]
\[
= \prod_{j=0}^{p-1} \frac{1}{(m+j)!^2}
\]
\[
\begin{array}{ccc}
1 & 0 & m \\
0 & 1 & 0 & m \\
\vdots & & & \ddots \\
0 & 1 & 0 & m + p - 1 \\
\end{array}
\]
Perform the row operations \( R_{2j} = R_{2j} - R_{2j-2} \), \( R_{2j-1} = R_{2j-1} - R_{2j-3} \) with \( j \) running from \( p, p-1, \ldots, 2 \) in that order. Using the Pascal’s rule for binomial coefficients, we get
\[
D_{(H)0}^{(H)} = (p-1)!^2 \prod_{j=0}^{p-1} \frac{1}{(m+j)!^2}
\]
\[
\begin{array}{ccc}
1 & 0 & m \\
0 & 1 & 0 & m \\
\vdots & & & \ddots \\
0 & 0 & 0 & 1 \\
\end{array}
\]
Next perform \( R_{2j} = R_{2j} - R_{2j-2}, R_{2j-1} = R_{2j-1} - R_{2j-3} \) with \( j \) running from \( p, p-1, \ldots, 3 \) in that order. Repeat this process \( p - 2 \) more times to reach an upper triangular matrix with determinant given in (4.5). Similarly, \( D_{(H)0}^{(H)} \) can be calculated for \( N \) odd.

Define
\[
D_e(N) = \prod_{j=0}^{p-1} \frac{j!^2}{(m+j)!^2}, \quad N = 2m,
\]
\[
D_o(N) = (-1)^p \frac{m!}{(m+p)!} \prod_{j=0}^{p-1} \frac{j!^2}{(m+j)!^2}, \quad N = 2m + 1.
\]
Using this notation, (4.4) reads
\[
E_N^{(H)} \left[ \det M^{2p} \right] = \left( -\frac{1}{2N} \right)^{N_p} \times \begin{cases} 
C_{\lambda}(2p)D_e(N), & N \text{ even,} \\
C_{\lambda}(2p)D_o(N), & N \text{ odd.}
\end{cases}
\]
The functions \( C_{\lambda}(2p)D_e(N) \) and \( C_{\lambda}(2p)D_o(N), \lambda = (N^{2p}) \), can be expressed in terms of the ratios of factorials,
\[
C_{(N^{2p})}(2p)D_e(N) = \prod_{j=0}^{p-1} \frac{(2m+j)!(2m+p+j)!}{(m+j)!^2} \frac{j!}{(p+j)!}, \quad N = 2m,
\]
\[
C_{(N^{2p})}(2p)D_o(N) = (-1)^p \frac{m!}{(m+p)!} \prod_{j=0}^{p-1} \frac{(2m+1+j)!(2m+1+p+j)!}{(m+j)!^2} \frac{j!}{(p+j)!}, \quad N = 2m + 1.
\]
Denote

\[ \gamma_p = \prod_{j=0}^{p-1} \frac{j!}{(p+j)!}. \]  

(4.11)

The universal constant \( \gamma_p \) is present in the moments for any finite \( N \). To compute the large \( N \) limit, we require the asymptotic expansion of (4.10). In App. A, we compute the first few terms in this expansion. As \( N \to \infty \),

\[
C_{(N^2p)}(2p)D_e(N) \sim e^{-Np(2N)^{Np+p^2}} \gamma_p \left[ 1 + \frac{p}{6N}(4p^2 + 1) + O(N^{-2}) \right], \quad N \text{ even},
\]

(4.12)

\[
C_{(N^2p)}(2p)D_o(N) \sim (-1)^p e^{-Np(2N)^{Np+p^2}} \gamma_p \left[ 1 + \frac{p}{3N}(2p^2 - 1) + O(N^{-2}) \right], \quad N \text{ odd}.
\]

Plugging (4.12) in (4.9), the leading order behaviour of the moments for \( N \) even and \( N \) odd is

\[ e^{-Np(2N)^{p^2}} \gamma_p \]

(4.13)

which coincides with (4.1) for \( t = 0 \). On the other hand, the sub-leading behaviour depends on the parity of \( N \).

\[ \quad \]

### 4.2 Away from the centre of the semi-circle

For \( t_j = t \),

\[
\mathbb{E}_N^{(H)} \left[ \det(t - M)^{2p} \right] = C_{\lambda}(2p) \sum_{\nu \subseteq \lambda} \left(-\frac{1}{2N}\right)^{|\lambda|-|\nu|} \frac{\dim V_\nu}{|\nu|!} D_{\lambda\nu}^{(H)} t^{|\nu|}.
\]

(4.14)

To compute the asymptotics near the centre of the semi-circle, \( t \neq 0 \), we need to evaluate \( D_{\lambda\nu}^{(H)} \) for a non-empty partition \( \nu \). In Table. 1, we give the values of \( D_{\lambda\nu}^{(H)} \) when \( \nu \) is a partition of 2 and 4.

Therefore,

\[
\mathbb{E}_N^{(H)} \left[ \det(t - M)^{2p} \right] = \sum_{\nu \subseteq \lambda} \left(-\frac{1}{2N}\right)^{|\lambda|-|\nu|} \frac{\dim V_\nu}{|\nu|!} t^{|\nu|} \text{poly}_{\nu}(N, p)
\]

\[
\times \begin{cases} C_{\lambda}(2p)D_e, & N \text{ even}, \\ C_{\lambda}(2p)D_o, & N \text{ odd}, \end{cases}
\]

(4.15)

where \( \text{poly}_j(N, p) \) denotes a polynomial of degree \( j \) in variables \( N, p \), and the explicit expressions are given in Table. 1 for \( j \leq 4 \). By referring to (4.10), it is remarkable to see that the universal constant \( \gamma_p \) is a factor of the moments for any finite \( N \). The first few terms in
\[ D_{\lambda_0}^{(H)} \quad N=2m \quad D\quad N=2m+1 \]

\begin{align*}
D_{\lambda_0}^{(H)} & \quad D_e \quad D_o \\
D_{\lambda(2)}^{(H)} & \quad mpD_e \quad mpD_o \\
D_{\lambda(1^2)}^{(H)} & \quad -mpD_e \quad -(m+1)pD_o \\
D_{\lambda(4)}^{(H)} & \quad \frac{1}{2}m(m-1)p(p+1)D_e \quad \frac{1}{2}m(m-1)p(p+1)D_o \\
D_{\lambda(3,1)}^{(H)} & \quad -\frac{1}{2}m(m-1)p(p+1)D_e \quad -\frac{1}{2}m(m+1)p(p+1)D_o \\
D_{\lambda(2^2)}^{(H)} & \quad m^2p^2D_e \quad m(m+1)p^2D_o \\
D_{\lambda(2,1^2)}^{(H)} & \quad -\frac{1}{2}m(m+1)p(p-1)D_e \quad -\frac{1}{2}m(m+1)p(p-1)D_o \\
D_{\lambda(1^4)}^{(H)} & \quad \frac{1}{2}m(m+1)p(p-1)D_e \quad \frac{1}{2}(m+2)(m+1)p(p-1)D_o
\end{align*}

Table 1: The values of determinant \( D_{\lambda_0}^{(H)} \) for \( \lambda = (N^{2p}) \). Determinants \( D_e \) and \( D_o \) are given in (4.8).

The moments of characteristic polynomials are

\[
\mathbb{E}_N^{(H)}[\det(t-M)^{2p}] = \left( -\frac{1}{2N} \right)^{Np} C_\lambda(2p)D_e \\
\times \left[ 1 + \left( \frac{2^2N^2}{4!} \right) Np t^4 + \left( \frac{2^3N^3}{6!} \right) 2Np(2p-N)t^6 \\
+ \left( \frac{2^4N^4}{8!} \right) Np(4N^2-17Np+16p^2+2)t^8 \\
+ O(t^{10}) \right], \quad \text{N even.}
\]

\[
\mathbb{E}_N^{(H)}[\det(t-M)^{2p}] = \left( -\frac{1}{2N} \right)^{Np} C_\lambda(2p)D_o \left[ 1 + \left( \frac{2N}{2!} \right) pt^2 + \left( \frac{2^2N^2}{4!} \right) (p^2-Np)t^4 \\
+ \left( \frac{2^3N^3}{6!} \right) p(2N^2-3Np+p^2)t^6 \\
+ \left( \frac{2^4N^4}{8!} \right) p(-4N^3+15N^2p-6Np^2-2N+p^3-4p)t^8 \\
+ O(t^{10}) \right], \quad \text{N odd.}
\]

(4.16)

Up to a factor of \((-1)^p\), both \( C_\lambda D_e \) and \( C_\lambda D_o \) have the same leading term,

\[
\exp(-NP(2N)^{Np+p^2} \gamma_{p^*})
\]

(4.17)

but they differ at sub-leading order as shown in (4.12). In App. A, we give the asymptotic expansion of \( C_\lambda(N)D_e \) and \( C_\lambda(N)D_o \) up to \( O(N^{-6}) \). Note that the coefficients of \( t^{2j} \) in (4.16)
are polynomials in $N$, and both $C_\lambda D_e$ and $C_\lambda D_o$ have an expansion in $1/N$. Therefore for higher values of $j$, more sub-leading terms in the expansion of $C_\lambda(N)D_e$ and $C_\lambda(N)D_o$ are required to compute the correct coefficients of $t^j$. But finding the exact asymptotic expansion of $C_\lambda D_e$ and $C_\lambda D_o$ is far from trivial as it involves a sequence of ratios of factorials, whose asymptotics is only known via recurrence relations.

In the next section, we focus on the second moment and show that we recover the semi-circle law only after averaging over even and odd matrix dimensional contributions.

### 4.2.1 Second moment

The correlations of characteristic polynomials are connected to the correlation functions of random matrices [42, 21, 43]. In particular,

$$
R_1^{(N)}(t) = \frac{N!}{(N-1)!} \frac{\mathcal{Z}_N^{(H)}}{\mathcal{Z}_N^{(H)}} \exp \left( -\frac{N t^2}{2} \right) \mathcal{E}_N^{(H)} \left[ \det(t-M)^2 \right],
$$

(4.18)

where $R_1^{(N)}$ is the one-point density of eigenvalues of matrix size $N$. As the second moment of the characteristic polynomial is related to the density of states, it is natural to expect the semi-circle law in the limit $N \to \infty$ as given in (4.1).

Re-writing (4.1) for $p = 1$,

$$
\lim_{N \to \infty} \frac{1}{2N} e^{-N \left(1-\frac{t^2}{2} \right)} \mathcal{E}_N^{(H)} \left[ \det(t-M)^2 \right] = \pi \rho_{sc}(t),
$$

(4.19)

which as an expansion in $t$ reads

$$
\lim_{N \to \infty} \frac{1}{2N} e^{-N \left(1-\frac{t^2}{2} \right)} \mathcal{E}_N^{(H)} \left[ \det(t-M)^2 \right] = 1 - \frac{1}{8} t^2 - \frac{1}{128} t^4 - \frac{1}{1024} t^6 + O(t^8).
$$

(4.20)

We now show that for $p = 1$, starting with (4.14) we arrive at (4.20). Inserting the asymptotics of $C_\lambda D_e$ and $C_\lambda D_o$ in (4.16), one obtains

$$
e^{-\frac{N t^2}{2}} \mathcal{E}_N^{(H)} \left[ \det(t-M)^2 \right] = 2N e^{-N} \left[ 1 + \left( -\frac{5}{12} + \frac{1}{2} N \right) t^2 + \left( -\frac{811}{77760} + \frac{17}{216} N + \frac{19}{72} N^2 + \frac{1}{6} N^3 \right) t^4
+ \left( \frac{640879}{587865600} + \frac{799}{1749600} N - \frac{3667}{291600} N^2 - \frac{323}{6480} N^3 - \frac{31}{540} N^4 - \frac{1}{45} N^5 \right) t^6
+ O(t^8) \right], \quad N \text{ even},
$$

(4.21)

$$
e^{-\frac{N t^2}{2}} \mathcal{E}_N^{(H)} \left[ \det(t-M)^2 \right] = 2N e^{-N} \left[ 1 + \left( \frac{1}{6} + \frac{1}{2} N \right) t^2 + \left( -\frac{101}{19440} - \frac{17}{216} N - \frac{19}{72} N^2 - \frac{1}{6} N^3 \right) t^4
+ \left( -\frac{15853}{18370800} + \frac{799}{1749600} N + \frac{3667}{291600} N^2 + \frac{323}{6480} N^3 + \frac{31}{540} N^4 + \frac{1}{45} N^5 \right) t^6
+ O(t^8) \right], \quad N \text{ odd}.
$$
Treating the above expansions as a formal series in $N$ and taking their average gives (4.20). In App. B, it is shown that the average over even and odd $N$ coincides with the semi-circle law up to $O(t^{10})$. Also, a general expression for the coefficient of $t^{2j}$ in (4.14) is given for $p = 1$.

### 4.2.2 Higher moments

For higher moments, the correlations of characteristic polynomials are related to the correlation functions of eigenvalues as

$$R_p^{(N)}(t_1, \ldots, t_p) = \frac{N!}{(N-p)!} \frac{Z_{N-p}^{(H)}}{Z_N^{(H)}} \exp \left( -\frac{N}{2} \sum_{j=1}^{p} t_j^2 \right) \Delta^2(t_1, \ldots, t_p) \mathbb{E}_{N-p}^{(H)} \left[ \prod_{j=1}^{p} \det(t_j - M)^2 \right],$$

(4.22)

where $R_p^{(N)}(t_1, \ldots, t_p)$ denotes a $p$–point correlation function of a GUE matrix of size $N$. The correlations of characteristic polynomials of matrices of size $N - p$ are related to the correlation functions of eigenvalues of matrices of size $N$. The Dyson sine-kernel for the $p$–point correlation function and (4.1) for the moments of characteristic polynomials are recovered in the Dyson limit: $t_i \to t, N \to \infty$ and $N(t_i - t_j)$ is kept finite when $|t_j| < 2, j = 1, \ldots, p$.

In terms of the Schur polynomials, $\lambda = (N^{2p})$,

$$\mathbb{E}_{N}^{(H)} \left[ \prod_{j=1}^{2p} \det(t_j - M) \right] = C_\lambda(2p) \sum_{\nu \subseteq \lambda} \left( -\frac{1}{2N} \right)^{|\lambda| - |\nu|} \frac{1}{C_\nu(2p)} D_\lambda^{(H)} S_\nu(t_1, \ldots, t_{2p}).$$

(4.23)

Computing the asymptotics of moments of characteristic polynomials in the Dyson limit using (4.23) is highly non-trivial. Instead, we fix $t_j = t, j = 1, \ldots, 2p$, and give an expansion of the moments as a function of $t$ in the large $N$ limit.

As $N \to \infty$, up to $O(t^2)$,

$$\mathbb{E}_{N}^{(H)} \left[ \det(t - M)^{2p} \right] = (2N)^p t^{2p} e^{-Np^2 \gamma_p} \left[ 1 + O(t^4) \right], \quad N \text{ even},$$

$$\mathbb{E}_{N}^{(H)} \left[ \det(t - M)^{2p} \right] = (2N)^p t^{2p} e^{-Np^2 \gamma_p} \left[ 1 + t^2 \left( Np + \frac{p^2}{3} (2p^2 - 1) \right) + O(t^4) \right], \quad N \text{ odd}.$$  

(4.24)

Note that the coefficient of $t^2$ is identically zero for even $N$, where as for odd $N$ it is a polynomial in $N$ and $p$. Treating the above expansions as a formal series in $N$ and taking their average gives

$$(2N)^p t^{2p} e^{-Np^2 \gamma_p} \left( 1 + \frac{Npt^2}{2} \right) \left( 1 - \frac{t^2}{8} \right) \left( 1 + \frac{p}{12N} (8p^2 - 1) \right).$$

(4.25)

By comparing with (4.1), the terms in the first and second parenthesis of (4.25) are the expansions of $e^{\frac{Npt^2}{2}}$ and $\pi \rho_{sc}(t)$ up to $O(t^2)$ respectively. The last factor in (4.25) is sub-leading. Thus at $O(t^2)$, moments of characteristic polynomials in the Dyson limit and in the limit $t \to 0$ and $N \to \infty$ coincide.
Similarly, as $N \to \infty$, up to $O(t^4)$,
\[
\mathbb{E}_N^{(H)} \left[ \det(t - M)^{2p} \right] = (2N)^p e^{-Np\gamma_p} \left[ 1 + t^4 \frac{N^3p}{6} \left( 1 + \frac{p}{6N} (4p^2 + 1) \right. \right.
\]
\[
+ \frac{p^2}{72N^2} (16p^4 - 16p^2 - 11) + \frac{p}{6480N^3} (320p^8 - 1200p^6
\]
\[
+ 708p^4 + 1265p^2 - 756) + O(t^6) \right], \quad N \text{ even,}
\]
\[
\mathbb{E}_N^{(H)} \left[ \det(t - M)^{2p} \right] = (2N)^p e^{-Np\gamma_p} \left[ 1 + t^4 \left( Np + \frac{p^2}{3} (2p^2 - 1) \right. \right.
\]
\[
+ t^4 \frac{N^3p}{6} \left( -1 - \frac{2p}{3N} (p^2 - 2) - \frac{p^2}{18N^2} (4p^4 - 22p^2 + 13
\]
\[
- \frac{p}{405N^3} (20p^8 - 210p^6 + 483p^4 - 385p^2 + 54) \right)
\]
\[
+ O(t^6) \right], \quad N \text{ odd.}
\]

Taking average of the above series and factorising gives
\[
(2N)^p e^{-Np\gamma_p} \left[ 1 + \frac{Npt^2}{2} + \frac{N^2p^2t^4}{8} + O(t^6) \right]
\]
\[
\left( 1 - \frac{p^2t^2}{8} + \frac{t^4}{128} p^2 (p^2 - 2) + O(t^6) \right)
\]
\[
\times \left[ 1 + \frac{1}{N} \left( \frac{p}{12} (8p^2 - 1) + \frac{pt^2}{96} (13p^2 - 1) + O(t^4) \right) + \frac{1}{N^2} \left( \frac{p^2}{144} (32p^4 - 56p^2 + 17) + O(t^2) \right) \right],
\]
where the first two brackets correspond to the expansion of $e^{\frac{Np^2}{2}}$ and $\pi \rho_{sc}(t)$, respectively, up to $O(t^4)$, and the last factor is sub-leading. Thus, asymptotics calculated by letting first $t \to 0$ and then $N \to \infty$ coincides with that of Dyson limit asymptotics up to $O(t^4)$. For higher orders in $t$, mismatch between the two limits starts to appear.

### 5 Secular coefficients

Consider a matrix $M$ of size $N$. Its characteristic polynomial can be expanded as
\[
\det(t - M) = \prod_{j=1}^{N} (t - x_j) = \sum_{j=0}^{N} (-1)^j \text{Sc}_j(M) t^{N-j},
\]
where $\text{Sc}_j$ is the $j^{th}$ secular coefficient of the characteristic polynomial. We have
\[
\text{Sc}_1(M) = \text{Tr}M, \quad \text{Sc}_N(M) = \det(M).
\]

These secular coefficients are nothing but the elementary symmetric polynomials $e_j$ defined as
\[
e_j(x_1, \ldots, x_N) = \sum_{1 \leq k_1 < k_2 < \cdots < k_j \leq N} x_{k_1} x_{k_2} \cdots x_{k_j}
\]
for $j \leq N$ and $e_j = 0$ for $j > N$. 


The correlations of secular coefficients and their connections to combinatorics have been studied in the past [22, 18]. For example, the joint moments of secular coefficients of the unitary group are connected to the enumeration of magic squares: matrices with positive entries with prescribed row and column sum. In a similar way, the joint moments of secular coefficients of Hermitian ensembles, such as the GUE, are connected to matching polynomials of closed graphs. In this section, we compute these correlations and indicate their combinatorial properties.

Gaussian ensemble: Elementary symmetric polynomials can be expanded in terms of multivariate Hermite polynomials as

$$e_r = \sum_{j=0}^{\left\lfloor \frac{r}{2} \right\rfloor} \Psi_{(1^r)(1^r-2j)}^{(H)} \mathcal{H}_{(1^r-2j)},$$  \hspace{1cm} (5.4)

where

$$\Psi_{(1^r)(1^r-2j)}^{(H)} = (-1)^j \frac{(N-r+2j)!}{(2N)^{j} j! (N-r)!},$$  \hspace{1cm} (5.5)

Equivalently, we have

$$e_{2r} = \sum_{j=0}^{r} \Psi_{(2^r)(2^r)}^{(H)} \mathcal{H}_{(2^r)}, \hspace{1cm} e_{2r+1} = \sum_{j=0}^{r} \Psi_{(2^r+1)(2^r+1)}^{(H)} \mathcal{H}_{(2^r+1)},$$  \hspace{1cm} (5.6)

with

$$\Psi_{(2^r)(2^r)}^{(H)} = (-1)^{r-j} \frac{1}{(2N)^{r-j} (r-j)! (N-2j)!},$$  \hspace{1cm} (5.7)

$$\Psi_{(2^r+1)(2^r+1)}^{(H)} = (-1)^{r-j} \frac{1}{(2N)^{r-j} (r-j)! (N-2r-1)!}.$$

Because of the orthogonality of the $\mathcal{H}_\mu$,

$$E_N^{(H)}[\text{Sc}_r] = E_N^{(H)}[e_r] = \begin{cases} (-1)^{\frac{r}{2}} \frac{1}{(2N)^{r/2} (N-r)!}, & \text{if } r \text{ is even}, \\ 0, & \text{if } r \text{ is odd}. \end{cases}$$  \hspace{1cm} (5.8)

These expectations are nothing but the coefficients of Hermite polynomial of degree $N$. Thus,

$$E_N^{(H)}[\det(t-M)] = \sum_{j=0}^{\left\lfloor \frac{N}{2} \right\rfloor} E_N^{(H)}[\text{Sc}_{2j}(M)] t^{N-2j} = h_N(t),$$  \hspace{1cm} (5.9)

which coincides with (3.3) for $p = 1$. The expectation $|N|^j E_N^{(H)}[\text{Sc}_{2j}(M)]|$ is equal to the number of $2j$ matchings in the complete graphs [18, 22].

By using (5.4), the second moment of the secular coefficient can also be computed. Similar to the univariate case, multivariate Hermite polynomials $\mathcal{H}_\lambda$ corresponding to even and odd $|\lambda|$ do not mix. Hence, we obtain

$$E_N^{(H)}[\text{Sc}_{2j}(M)\text{Sc}_{2k+1}(M)] = 0,$$  \hspace{1cm} (5.10)
Proposition 5.1. Consider a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$. We have
\[
\mathbb{E}^{(H)}_N[\text{Sc}_2^{(H)}(M)\text{Sc}_2^{(H)}(M)] = \sum_{j=0}^{r} \sum_{k=0}^{s} \frac{1}{N^{2j}} \frac{1}{N^{2k}} \Psi^{(H)}_{(1^{2j})(1^{2k})} \mathbb{E}^{(H)}_N[\mathcal{H}^{(H)}_{(1^{2j})}\mathcal{H}^{(H)}_{(1^{2k})}]
\]
\[
= \sum_{j=0}^{r+s} \frac{1}{2N} \frac{2^{2j}}{(r-j)!(s-j)!(N-2r)!(N-2s)!} N!(N-2j)!
\]
Similarly, we write
\[
\mathbb{E}^{(H)}_N[\text{Sc}_2^{(H)}(M)\text{Sc}_2^{(H)}(M)] = \left( -\frac{1}{2N} \right)^{r+s} \sum_{j=0}^{r+s} \frac{2^{2j}}{(r-j)!(s-j)!(N-2r)!(N-2s)!} (N-1)!(N-2j-1)!(N-2s-1)!
\]
Computing higher order correlations requires evaluating integrals involving a sequence of multivariate Hermite polynomials. Busbridge [12, 13] calculated these integrals for the univariate case, but the results are still unknown for the multivariate generalisation. Instead, we take a different approach by first expressing the product $\prod_j \text{Sc}_2^{(H)}(M)$ in terms of the $\mathcal{H}_\mu$ and then using orthogonality for the $\mathcal{H}_\mu$.

**Proposition 5.1.** Consider a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$. We have
\[
\mathbb{E}^{(H)}_N\left[ \prod_{j=1}^{l} \text{Sc}_{\lambda_j}(M) \right] = \begin{cases} \sum_{\mu} \frac{1}{(2N)^{\frac{1}{2}}} K^\mu_{\lambda} \chi_{(2^{(\nu_0+\nu)})} \mathcal{H}_\nu(N), & \text{if } |\lambda| \text{ is even,} \\ 0, & \text{otherwise.} \end{cases}
\]
Here $K^\mu_{\lambda}$ are Kostka numbers\(^1\) and $\chi_{\mu}$ is the character of the symmetric group.

**Proof.** For a partition $\lambda$, denote
\[
e_{\lambda} = e_{\lambda_1} e_{\lambda_2} \ldots.
\]
Elementary symmetric polynomials $e_{\lambda}$ can be expanded in Schur basis as follows:
\[
e_{\lambda} = \sum_{\mu} K^\mu_{\lambda} S_{\mu},
\]
where $K^\mu_{\lambda}$ are the Kostka numbers [41] and $\mu$ is a partition of $|\lambda|$. Using (2.20),
\[
e_{\lambda} = \sum_{\mu+|\lambda|} \sum_{\nu} K^\mu_{\lambda} \chi_{\mu}^{(H)} \mathcal{H}_\nu.
\]
When $|\lambda|$ is odd, $\mathbb{E}^{(H)}_N[e_{\lambda}] = 0$ due to the orthogonality of multivariate Hermite polynomials. When $|\lambda|$ is even,
\[
\mathbb{E}^{(H)}_N[e_{\lambda}] = \mathbb{E}^{(H)}_N\left[ \prod_{j=1}^{l} \text{Sc}_{\lambda_j}(M) \right] = \mathbb{E}^{(H)}_N\left[ \sum_{\mu} \sum_{\nu} K^\mu_{\lambda} \chi_{\mu}^{(H)} \mathcal{H}_\nu \right] = K^\mu_{\lambda} \chi_{\mu}^{(H)}.
\]

\(^1\)The Kostka numbers are non-negative integers that count the number of semi-standard Young tableau of shape $\lambda$ and weight $\mu$. 

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It can be shown that
\[\Psi_{\mu_0}^{(H)} = \frac{1}{(2N)^{|\mu|/2}} \chi_{(2|\mu|/2)}^\mu C_\mu(N).\] (5.18)

Putting everything together completes the proof. ■

**Laguerre ensemble:** All the calculations discussed for the Gaussian ensemble can be extended to the Laguerre and the Jacobi ensembles.

The polynomials \(e_r\) can be expanded as
\[e_r = \sum_{j=0}^r \Psi_{(1^r)(1^j)}^{(L)} \mathcal{L}_r^{(\gamma)},\] (5.19)
where
\[\Psi_{(1^r)(1^j)}^{(L)} = \frac{1}{(2N)^{r-j}} \frac{1}{(r-j)!} \frac{(N-j)!}{(N-r)!} \frac{\Gamma(N + \gamma + 1)}{\Gamma(N + r + \gamma + 1)}.\] (5.20)

By using (2.17) we arrive at
\[\mathbb{E}_N^{(L)}[\text{Sc}_r] = \mathbb{E}_N^{(L)}[e_r] = \frac{1}{(2N)^r} \frac{1}{r! \Gamma(N + \gamma + 1)} \frac{\Gamma(N + \gamma + 1)}{(N-r)! \Gamma(N + r + \gamma + 1)} N!,\] (5.21)
which are the absolute values of the coefficients of the Laguerre polynomial of degree \(N\). For the characteristic polynomial, we have
\[\mathbb{E}_N^{(L)}[\det(t - M)] = \sum_{j=0}^N (-1)^j \mathbb{E}_N^{(L)}[\text{Sc}_j(M)] t^{N-j} = l_N^{(\gamma)}(t).\] (5.22)

The correlations of secular coefficients can be computed similar to the Gaussian case.

**Proposition 5.2.** Let \(\lambda = (\lambda_1, \ldots, \lambda_l)\), we have
\[\mathbb{E}_N^{(L)} \left[ \prod_{j=1}^l \text{Sc}_{\lambda_j}(M) \right] = \sum_{\mu \subseteq [\lambda]} \frac{1}{(2N)^{|\mu|/2}} \frac{G_\mu(N, \gamma) G_\mu(N, 0) \chi_{(1|\mu|)}^{\mu}}{G_0(N, \gamma) G_0(N, 0) \mu!} K_{\lambda^\mu}^{X_{\mu}}.\] (5.23)

**Proof.** The proof is similar to the Gaussian case. By writing
\[e_\lambda = \sum_{\mu} \sum_{\nu \subseteq [\lambda]} K_{\lambda^\mu}^{X_{\mu}} \Psi_{\mu(\nu)}^{(L)} \mathcal{L}_r^{(\gamma)},\] (5.24)
and using (2.17) along with the result [35]
\[\Psi_{\mu_0}^{(L)} = \frac{1}{(2N)^{|\mu|/2}} G_\mu(N, \gamma) G_\mu(N, 0) \chi_{(1|\mu|)}^{\mu} \mu!\] (5.25)
proves the proposition. ■
Jacobi ensemble. The $e_r$ can be expanded as

$$e_r = \sum_{j=0}^{r} \Psi_{(1')(1')}(\gamma_1, \gamma_2),$$  \hspace{1cm} (5.26)$$

where $\Psi_{(1')}$ is given in (2.24). The expected values of the $e_r$ are related to the coefficients of the Jacobi polynomial of degree $N$.

$$E_N[\det(t - M)] = \sum_{j=0}^{N} (-1)^j E_N[\text{Sc}_j(M)] t^{N-j} = j^{(\gamma_1, \gamma_2)}(t)$$  \hspace{1cm} (5.27)$$

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Appendix

A Asymptotics of ratio of factorials

The asymptotics of the ratio of factorials can be computed as follows. First we look at $C_{\lambda}(2p)D_e$ with $\lambda = (2m, \ldots, 2m)$. Consider

$$\frac{(2m+j)!(2m+p+j)!}{(m+j)!^2} = (2m)^p \frac{(2m+j)!^2}{(m+j)!^2} \prod_{a=1}^{p} \left(1 + \frac{a}{2m}\right).$$  \hspace{1cm} (1.1)$$

Now, one can see that

$$\frac{(2m+j)!}{(m+j)!} = 2^{j+1} \frac{\Gamma(2m)}{\Gamma(m)} \prod_{a=0}^{j} \frac{1 + \frac{a}{2m}}{1 + \frac{a}{m}}.$$  \hspace{1cm} (1.2)$$

Using the duplication formula for the Gamma functions

$$\Gamma(z)\Gamma\left(z + \frac{1}{2}\right) = 2^{1-2z} \sqrt{\pi} \Gamma(2z)$$  \hspace{1cm} (1.3)$$

and Stirling’s series

$$\Gamma(z+h) \sim \sqrt{2\pi e^{-z}} z^{z+h+\frac{1}{2}} \prod_{j=2}^{\infty} \exp\left(\frac{(-1)^j B_j(h)}{j(j-1)z^{j-1}}\right), \hspace{1cm} z \to \infty,$$  \hspace{1cm} (1.4)$$

the asymptotic expansion for the ratio of Gamma functions can be found. Here $B_j$ is the Bernoulli polynomial of degree $j$. Combining all the formulae, up to first order correction,

$$C_{\lambda}(2p)D_e \sim e^{-2mp_2^{4mp^2+2p^2} m^{2mp^2+p^2}} \left[1 + \frac{p}{12m}(4p^2 + 1) + O(m^{-2})\right].$$  \hspace{1cm} (1.5)$$

Similarly for the case $C_{\lambda}(2p)D_o$, we obtain

$$\frac{(2m+1+p+j)!(2m+1+j)!}{(m+j)!^2} = (2m+1)^p \frac{(2m+1+j)!^2}{(m+j)!^2} \prod_{a=1}^{p} \left(1 + \frac{j + a}{2m+1}\right).$$  \hspace{1cm} (1.6)$$
Let \( z = m + \frac{1}{2} \), then
\[
\frac{(2m + 1 + j)!}{(m + j)!} = \frac{\Gamma(2z + j + 1)}{\Gamma(z + \frac{1}{2} + j)} = 2^{j+1}z^{\frac{j}{2}} \Gamma(2z) \prod_{a=1}^{j} \frac{1 + a}{2z}, \tag{1.7}
\]
and
\[
\frac{m!}{(m+p)!} = \frac{\Gamma(z + \frac{1}{2})}{\Gamma(z + p + \frac{1}{2})} = \frac{1}{z^p} \prod_{a=1}^{p} \frac{1}{1 + \frac{2a-1}{2z}}. \tag{1.8}
\]
Combining the above formulae and using (1.3) and (1.4),
\[
C_{((2m+1)2^p)} D_o \equiv C_{((2z)2^p)} D_o \sim (-1)^p e^{-2zp^2(2^p+2)p^2+4pz} \left( \prod_{j=0}^{p-1} \frac{j!}{(p+j)!} \right) \times \left[ 1 + \frac{p}{6z} (2p^2 - 1) + O(z^{-2}) \right] \tag{1.9}
\]
Higher order corrections can also be calculated with some effort or using any commercial software like Mathematica. Writing in terms of the matrix size \( N \), as \( N \to \infty \), we have
\[
C_{(N2^p)} D_e \sim e^{-Np(2N)^{Np+p^2}} \left( \prod_{j=0}^{p-1} \frac{j!}{(p+j)!} \right) \left[ 1 + \frac{p}{6N} (4p^2 + 1) + \frac{p^2}{72N^2} (16p^4 - 16p^2 - 11) 
+ \frac{p}{6480N^3} (320p^8 - 1200p^6 + 708p^4 + 1265p^2 - 756) 
+ \frac{p^2}{155520N^4} (1280p^{10} - 10240p^8 + 25248p^6 - 6400p^4 - 56371p^2 + 51408) 
+ \frac{p}{6531840N^5} (7168p^{14} - 98560p^{12} + 499072p^{10} - 982688p^8 - 399844p^6 
+ 4606735p^4 - 5598936p^2 + 1607040) \right] 
+ \frac{p^2}{1175731200N^6} \left( 143360p^{16} - 3010560p^{14} + 25294080p^{12} - 103093760p^{10} 
+ 158864016p^8 - 298943760p^6 - 1697420809p^4 + 2663679600p^2 - 1390123296 \right) 
+ O\left( \frac{1}{N^7} \right), \quad N \text{ even.} \tag{1.10}
\]
\[ C(N_\nu)D_\nu \sim (-1)^p e^{-Np}(2N)^{N_\nu + p^2} \left( \prod_{j=0}^{p-1} \frac{j!}{(p+j)!} \right) \left[ 1 + \frac{p}{3N}(2p^2 - 1) + \frac{p^2}{18N^2}(4p^4 - 10p^2 + 7) \right] + \frac{p}{810N^3}(40p^8 - 240p^6 + 516p^4 - 455p^2 + 108) \]
\[ + \frac{p^2}{9720N^4}(80p^{10} - 880p^8 + 3828p^6 - 8356p^4 + 9509p^2 - 4320) \]
\[ + \frac{p}{204120N^5}(224p^{14} - 3920p^{12} + 28616p^{10} - 113428p^8 + 266818p^6 \]
\[ - 37127p^4 + 255528p^2 - 51840 \]}
\[ + \frac{p^2}{18370800N^6}(2240p^{16} - 57120p^{14} + 628320p^{12} - 3919160p^{10} + 15363624p^8 \]
\[ - 39481170p^6 + 65605589p^4 - 62864640p^2 + 25046496 \]}
\[ + O \left( \frac{1}{N^7} \right), \quad N \text{ odd.} \quad (1.11) \]

**B More on the second moment**

Fix \( \lambda = (N, N) \). The second moment of the characteristic polynomial is given by

\[
\mathbb{E}_N^{(H)} \left[ \det(t - M)^2 \right] = \left( \frac{-1}{2N} \right)^N C_\lambda(2) \sum_{\nu \subseteq \lambda} \frac{1}{|\nu|!} (-2N)^{|\nu|/2} D^{(H)}_{\lambda\nu} \dim V_\nu t^{|\nu|}. \quad (2.1)
\]

Let \( \nu = (\nu_1, \nu_2) \subseteq \lambda \). Since \(|\nu|\) is even, either both \( \nu_1, \nu_2 \) are even or both of them are odd.

For \( N = 2m, m \in \mathbb{N} \),

\[
D^{(H)}_{\lambda\nu} = \begin{cases} 
\frac{1}{(m-\nu_1^2)!((m-\nu_2^2)!)}, & \nu_1, \nu_2 \text{ are even,} \\
-\frac{1}{(m-\nu_1^2)!((m-\nu_2^2)!)}, & \nu_1, \nu_2 \text{ are odd.}
\end{cases} \quad (2.2)
\]

Therefore,

\[
C_\lambda(2)D^{(H)}_{\lambda\nu} = (2m)!(2m + 1)! \begin{cases} 
\frac{1}{(m-\nu_1^2)!((m-\nu_2^2)!)}, & \nu_1, \nu_2 \text{ are even,} \\
-\frac{1}{(m-\nu_1^2)!((m-\nu_2^2)!)}, & \nu_1, \nu_2 \text{ are odd.}
\end{cases} \quad (2.3)
\]

Similarly, for \( N = 2m + 1, m \in \mathbb{N} \), we have

\[
D^{(H)}_{\lambda\nu} = \begin{cases} 
\frac{1}{(m-\nu_1^2)!((m-\nu_2^2)!)}, & \nu_1, \nu_2 \text{ are even,} \\
\frac{1}{(m-\nu_1^2)!((m-\nu_2^2)!)}, & \nu_1, \nu_2 \text{ are odd.}
\end{cases} \quad (2.4)
\]
and
\[
C_\lambda(2) D_{\lambda^0}^{(H)} = (2m + 1)! (2m + 2)! \left\{ \begin{array}{ll}
\frac{1}{(m - \frac{1}{2})! (m - \frac{1}{2})!}, & \nu_1, \nu_2 \text{ are even}, \\
\frac{1}{(m - \frac{1}{2})! (m - \frac{1}{2})!}, & \nu_1, \nu_2 \text{ are odd}.
\end{array} \right.
\]

(2.5)

For a partition of length 2, \(\nu = (\nu_1, \nu_2)\),
\[
\frac{1}{|\nu|!} \dim V_\nu = \frac{\nu_1 - \nu_2 + 1}{(\nu_1 + 1)! \nu_2!}.
\]

(2.6)

Inserting (2.3), (2.5), (2.6) in (2.1), and observing that \(\nu\) runs over all partitions such that \(0 \leq |\nu| \leq 2N\) gives
\[
E_N^{(H)} \left[ \det (t - M)^2 \right] = \left( -\frac{1}{2N} \right)^N C_\lambda(2) D_{\lambda^0}^{(H)} \sum_{k=0}^{N} (-2N)^k t^{2k}
\]
\[
\times \left[ \sum_{j=0}^{\lfloor k/2 \rfloor} \left( \frac{2k + 1 - 4j}{(2k + 1 - 2j)! (2j)!} - \frac{2k - 1 - 4j}{(2k - 2j)! (2j + 1)!} \right) \frac{\left( \frac{N}{2} \right)!^2}{(\frac{N}{2} - k + j)! (\frac{N}{2} - j)!} \right.
\]
\[
+ \left. \frac{1}{k!(k + 1)!} \frac{\left( \frac{N}{2} \right)!^2}{(\frac{N}{2} - \frac{k}{2})!^2} \mathbb{1}_{k=0 \mod 2} \right].
\]

(2.7)

for \(N\) even. Similarly, for \(N\) odd, one gets
\[
E_N^{(H)} \left[ \det (t - M)^2 \right] = \left( -\frac{1}{2N} \right)^N C_\lambda(2) D_{\lambda^0}^{(H)} \sum_{k=0}^{N} (-2N)^k t^{2k}
\]
\[
\times \left[ \sum_{j=0}^{\lfloor k/2 \rfloor} \left( \frac{2k - 3 - 4j}{(2k - 2j)! (2j + 1)!} + \frac{2k - 3 - 4j}{(2k - 2j - 1)! (2j + 2)!} \right) \frac{\left( \frac{N-1}{2} \right)! \left( \frac{N+1}{2} \right)!}{\left( \frac{N+1}{2} \right) - k + j)! \left( \frac{N-1}{2} \right) - j)!} \right.
\]
\[
+ \left. \frac{1}{(2k)!} \frac{\left( \frac{N-1}{2} \right)!}{\left( \frac{N-1}{2} \right) - k)!} - \frac{1}{k!(k + 1)!} \frac{\left( \frac{N-1}{2} \right)! \left( \frac{N+1}{2} \right)!}{\left( \frac{N-1}{2} \right) - \frac{k-1}{2})!^2} \mathbb{1}_{k=0 \mod 1} \right].
\]

(2.8)

We have
\[
C_\lambda(2) D_{\lambda^0}^{(H)} = \begin{cases} 
\frac{N!(N+1)!}{\left( \frac{N}{2} \right)!^2}, & N \text{ even}, \\
-\frac{N!(N+1)!}{\left( \frac{N}{2} \right)! \left( \frac{N}{2} + 1 \right)!}, & N \text{ odd}.
\end{cases}
\]

(2.9)

The asymptotics of the ratio of the factorials are already discussed in App. A. For the sake

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of completion, here we again give the result for $p = 1$,

$$C_\lambda(2)D_{\lambda_0}^{(H)} \sim e^{-N} (2N)^{N+1} \left[ 1 + \frac{5}{6N} - \frac{11}{72N^2} + \frac{337}{2480N^3} + \frac{985}{31104N^4} - \frac{360013}{6531840N^5} ight. \\
\left. - \frac{1175731200}{889926952101377} + \frac{1410877440}{11757020981N^7} + \frac{33861058600}{N^8} \right] + O(N^{-10}), \quad N \text{ even,}$$

$$C_\lambda(2)D_{\lambda_0}^{(H)} \sim -e^{-N} (2N)^{N+1} \left[ 1 + \frac{1}{3N} + \frac{1}{18N^2} - \frac{810}{N^3} - \frac{139}{9720N^4} + \frac{9871}{204120N^5} ight. \\
\left. + \frac{324179}{8225671} - \frac{18370800}{1674981058019} - \frac{55112400}{N^5} - \frac{1322697600}{N^6} \right] + O(N^{-10}), \quad N \text{ odd.}$$

Substituting the above asymptotic series in

$$\frac{1}{2N} e^{-N} \sum_{N} \frac{1}{N} \det(t - M)^2 = \exp \left(-\frac{N t^2}{2} \right) E_N^{(H)} \left[ \det(t - M)^2 \right]$$

and taking the average over $N$ even and odd gives

$$\lim_{N \to \infty} \frac{1}{2N} e^{-N} \exp \left(-\frac{N t^2}{2} \right) E_N^{(H)} \left[ \det(t - M)^2 \right] = \exp \left(-\frac{N t^2}{2} \right) + O(t^{12}).$$

The R.H.S. in (2.12) coincides with $\pi \rho_{sc}(t)$ up to $O(t^{10}).$

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