Simulation of truncated normal variables

Christian P. Robert
LSTA, Université Pierre et Marie Curie, Paris

Abstract

We provide in this paper simulation algorithms for one-sided and two-sided truncated normal distributions. These algorithms are then used to simulate multivariate normal variables with restricted parameter space for any covariance structure.

Keywords: Accept-reject; Gibbs sampling; Markov Chain Monte-Carlo; censored models; order restricted models.
AMS Subject Classification (1991): 62–04, 62E25, 62F30.

1. Introduction

The need for simulation of truncated normal variables appears in Bayesian inference for some truncated parameter space problems. Indeed, it is rarely the case that analytical computations are possible and numerical integration can be very intricated for large dimensions. Typical examples of such setups can be found in order restricted (or isotonic) regression, as illustrated in Robertson, Wright and Dykstra (1988). For instance, one can consider a $n \times n$ table of normal random variables $x_{ij}$ with means $\theta_{ij}$ which are increasing in $i$ and $j$ ($1 \leq i, j \leq n$), as in Dykstra and Robertson (1982). When $n$ is large, both maximum likelihood and Bayesian inferences on this table can be quite cumbersome and simulation techniques are then necessary to either obtain mle’s by stochastic restoration (see Qian and Titterington, 1991) or Bayes estimators by Gibbs sampling (see Gelfand and Smith, 1990). Gibbs sampling actually provides a large set of examples where simulation from truncated distributions is necessary, for instance for censored models since the recovery of the censored observations implies simulation from the corresponding truncated distribution, as shown in details by Gelfand, Smith and Lee (1992). See also Chen and Deely (1992) who propose a new version of the Gibbs sampler for estimating the ordered coefficients of a regression model.

We first construct in Section 2 an efficient algorithm for unidimensional truncated normal variables. This algorithm is quite simple and, in the particular case of one-sided
truncated normal distributions, it slightly improves on a previous algorithm developed by Marsaglia (1964). Our multidimensional extension in Section 3 is also based on this algorithm. Actually, we propose to use Gibbs sampling to reduce the simulation problem to a sequence of one-dimensional simulations. The resulting sample, being derived from a Markov chain, is not independent, but can be used similarly for all estimation purposes.

2. The univariate case

2.1. One-sided truncation. Let us denote $N_+(\mu, \mu^-, \sigma^2)$ the truncated normal distribution with left truncation point $\mu^-$, i.e. the distribution with density

$$f(x|\mu, \mu^-, \sigma^2) = \frac{\exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right)}{\sqrt{2\pi\sigma}(1 - \Phi((\mu^- - \mu)/\sigma))} \mathbb{I}_{x \geq \mu^-}.$$

Obviously, a readily available method is to simulate from a normal distribution $N(\mu, \sigma^2)$ until the generated number is larger than $\mu^-$. This method is quite reasonable when $\mu^- < \mu$ but is of no use when $\mu^-$ is several standard deviations to the right of $\mu$. Similarly, Gelfand et al. (1992) and Chen and Deely (1992) suggest to use the classical c.d.f. inversion technique, namely to simulate $u \sim U_{[0,1]}$ and to take

$$z = \mu + \Phi^{-1}\left(\Phi\left(\frac{\mu^--\mu}{\sigma}\right) + u\left\{1 - \Phi\left(\frac{\mu^- - \mu}{\sigma}\right)\right\}\right)$$

as the simulation output, but this method calls for a simultaneous evaluation of the normal c.d.f. $\Phi$ and of its inverse $\Phi^{-1}$, and may be quite inefficient if $\mu^- - \mu$ is large, since the precision of the approximation of $\Phi$ then strongly matters. We provide below an accept-reject algorithm which is more efficient than repeatedly simulating from the normal distribution as soon as $\mu^- > \mu$. In the sequel, we will assume without loss of generality that $\mu = 0$ and $\sigma^2 = 1$, since the usual location-scale rescaling allows to standardize truncated normal variables.

Let us recall first that the general accept-reject algorithm is based on the following result (see Devroye, 1985, pp. 40-60).

**Lemma 2.1** Let $h$ and $g$ be two densities such that $h(x) \leq Mg(x)$ for every $x$ in the support of $h$. The random variable $x$ resulting from the following algorithm

1. Generate $z \sim g(z)$;
2. Generate $u \sim U_{[0,1]}$. If $u \leq h(z)/Mg(z)$, take $x = z$; otherwise, repeat from step 1.
is distributed accorded to \( h \).

In our case, a possible choice for \( g \) is the translated exponential distribution \( \mathcal{E}xp(\alpha, \mu^-) \) with density

\[
g(z|\alpha, \mu^-) = \alpha e^{-\alpha(z-\mu^-)} \mathbb{1}_{z \geq \mu^-}.
\]

Since, for \( z \geq \mu^- \), we have

\[
e^{\alpha(z-\mu^-)}e^{-z^2/2} \leq e^{\alpha^2/2-\mu^-\alpha}
\]
if \( \alpha > \mu^- \) and

\[
e^{\alpha(z-\mu^-)}e^{-z^2/2} \leq e^{-(\mu^-)^2/2}
\]
if \( \alpha \leq \mu^- \), the constant \( M \) is given by

\[
\begin{cases}
\frac{\alpha}{\sqrt{2\pi(1-\Phi(\mu^-))}}e^{\alpha^2/2-\alpha\mu^-} & \text{if } \alpha \geq \mu^-, \\
\frac{\alpha}{\sqrt{2\pi(1-\Phi(\mu^-))}}e^{-(\mu^-)^2/2} & \text{otherwise}
\end{cases}
\]

and the ratio \( h(z)/Mg(z) \) by

\[
\frac{h(z)}{Mg(z)} = \begin{cases}
e^{-z^2/2+\alpha(z-\mu^-)-\alpha^2/2+\alpha\mu^-} & \text{if } \alpha \geq \mu^-, \\
e^{-z^2/2+\alpha(z-\mu^-)+(\mu^-)^2/2} & \text{otherwise.}
\end{cases}
\]

We then derive from Lemma 2.1 the corresponding accept-reject algorithm.

**Lemma 2.2** The following algorithm

1. Generate \( z \sim \mathcal{E}xp(\alpha, \mu^-) \);
2. Compute \( \varrho(z) = \exp(-(\alpha - z)^2/2) \) if \( \mu^- < \alpha \) and \( \varrho(z) = \exp((\mu^- - \alpha)^2/2) \exp(-(\alpha - z)^2/2) \) otherwise;
3. Generate \( u \sim \mathcal{U}_{[0,1]} \) and take \( x = z \) if \( u \leq \varrho(z) \); otherwise, repeat from step 1.

leads to the generation of a random variable from \( \mathcal{N}(0, \mu^-, 1) \).

Now, noticing that the probability of acceptance in one single run is

\[
\mathbb{E}_\alpha[\varrho(z)] = \begin{cases}
\alpha e^{\alpha\mu^- - \alpha^2/2}\Phi(-\mu^-)\sqrt{2\pi} & \text{if } \mu^- < \alpha, \\
\alpha e^{(\mu^-)^2/2}\Phi(-\mu^-)\sqrt{2\pi} & \text{otherwise,}
\end{cases}
\]

we deduce that the optimal scale factor in the exponential distribution attained for

\[
\alpha^*(\mu^-) = \frac{\mu^- + \sqrt{(\mu^-)^2 + 4}}{2}
\]
in the first case and for $\alpha = \mu^-$ in the second case. Furthermore, since the corresponding probabilities are proportional to

$$\alpha^*(\mu^-) e^{\mu^- \alpha^*(\mu^-)/2} / \sqrt{\pi}$$

and $\mu^- \exp((\mu^-)^2/2)$ respectively, with the same coefficient of proportionality, it can be shown by using the reparametrization in $\alpha^*$ (i.e. $\mu^- = \alpha^* - 1/\alpha^*$) that the first probability is always greater and that the best choice of $\alpha$ is $\alpha^*(\mu^-)$. Therefore,

**Proposition 2.3** The optimal exponential accept-reject algorithm to simulate from a $\mathcal{N}_+(0, \mu^-, 1)$ when $\mu^- > 0$ is given by

1. Generate $z \sim \exp(\alpha^*, \mu^-)$;
2. Compute $\varrho(z) = \exp\{-(z - \alpha^*)^2/2\}$;
3. Generate $u \sim \mathcal{U}[0, 1]$ and take $x = z$ if $u \leq \varrho(z)$; otherwise, go back to step 1.

Table 2.1 below gives the expected probability $\mathbb{E}_{\alpha^*}[\varrho(z)]$ for several values of $\mu^-$. It shows the gain brought by using this accept-reject algorithm since the probability of accepting in one passage is 0.760 for $\mu^- = 0$, as compared with 0.5 for the repeated normal sampling alternative. The improvement increases as $\mu^-$ goes away from 0 and the probability of accepting goes to 1 as $\mu^-$ goes to infinity. Note that the probability of accepting is greater than

$$\mu^- e^{(\mu^-)^2/2} \Phi(-\mu^-) \sqrt{2\pi},$$

probability of accepting for $\alpha = \mu^-$; this is also the rate obtained by Marsaglia (1964) when proposing an accept-reject algorithm using the tail of a Raleigh distribution (see also Devroye, 1985, pp. 380-382). The improvement brought by using $\exp(\alpha^*, \mu^-)$ is significant for the moderate values of $\mu^-$. Those large probabilities also hint at likely improvements over repeated normal sampling even when $\mu^- < 0$, but such developments would call for much more elaborated algorithms and, moreover, fast normal generators can overcome the advantages of using a more complex algorithm.

| $\mu^-$ | 0   | 0.5 | 1   | 1.5 | 2   | 2.5 | 3   |
|---------|-----|-----|-----|-----|-----|-----|-----|
| $\mathbb{E}_{\alpha^*}[\varrho(z)]$ | 0.760 | 0.826 | 0.876 | 0.910 | 0.934 | 0.950 | 0.961 |

**Table 2.1** - Average probability of acceptance according to the truncation point $\mu^-$. Simulation from the right truncated normal distribution, $x \sim \mathcal{N}_-(\mu, \sigma^2)$, can be directly derived from the above algorithm since $-x \sim \mathcal{N}_+(\mu, \sigma^2)$. We consider in
the next section the simulation from the two-sided truncated normal distribution for which modifications of the above algorithm are necessary.

2.2. Two-sided truncated normal distribution. When considering the two-sided truncated normal distribution \( N^\pm(\mu, \mu^-, \mu^+, \sigma^2) \), with density

\[
f(x|\mu, \mu^-, \mu^+, \sigma) = \frac{e^{-(x-\mu)^2/2\sigma^2}}{\sqrt{2\pi\sigma} \left[ \Phi((\mu^+ - \mu)/\sigma) - \Phi((\mu^- - \mu)/\sigma) \right]},
\]

the simulation method heavily depends on the range \( \mu^+ - \mu^- \). As before, a first possibility is to simulate from a \( N(\mu, \sigma^2) \) distribution until \( z \in [\mu^-, \mu^+] \) (or even to invert the c.d.f.). However, if \( \Phi(\mu^+ - \mu) - \Phi(\mu^- - \mu) \) is small or even if \((\mu^- - \mu)(\mu^+ - \mu) > 0\), more efficient alternatives are available. We propose here to consider, in addition to the previous algorithms, an accept-reject approach based on the uniform \( U[\mu^-, \mu^+] \) distribution. Once again, we can assume without loss of generality that \( \mu = 0 \) and \( \sigma^2 = 1 \).

The accept-reject algorithm based on \( U[\mu^-, \mu^+] \) is

1. Generate \( z \sim U[\mu^-, \mu^+] \);
2. Compute

\[
\varrho(z) = \left\{ \begin{array}{ll}
\exp(-z^2/2) & \text{if } 0 \in [\mu^-, \mu^+]
\\
\exp(\{(\mu^+)^2 - z^2\}/2) & \text{if } \mu^+ < 0
\\
\exp(\{(\mu^-)^2 - z^2\}/2) & \text{if } 0 < \mu^-
\end{array} \right.
\]
3. Generate \( u \sim U[0,1] \) and take \( x = z \) if \( u \leq \varrho(z) \); otherwise, go back to step 1.

The corresponding expected probability of running the above algorithm only once is

\[
\mathbb{E}[\varrho(z)] = \int_{\mu^-}^{\mu^+} e^{-z^2/2} \, dz \frac{e^d}{\mu^+ - \mu^-}
= \sqrt{2\pi} \frac{e^d}{\mu^+ - \mu^-} (\Phi(\mu^+) - \Phi(\mu^-))
\]

where \( d = 0, (\mu^+)^2/2 \) or \((\mu^-)^2/2\) whether \( \mu^+\mu^- < 0 \), \( \mu^+ < 0 \) or \( \mu^- > 0 \). Therefore, when \( \mu^+\mu^- < 0 \), it is more efficient to use this algorithm rather than to use the repeated normal method if \( \mu^+ - \mu^- < \sqrt{2\pi} \).

We now oppose simulation from the uniform algorithm to repeated simulation from a one-sided truncated normal distribution. For instance, if \( \mu^- > 0 \), we simulate \( z \sim \)
\( \mathcal{N}_+(0, \mu^-, 1) \) until \( z < \mu^+ \). Using the optimal algorithm of Proposition 2.3, the probability of accepting in one passage is

\[
P(u \leq \varrho(z) \text{ and } z \leq \mu^+)
\]

\[
= \int_{\mu^-}^{\mu^+} e^{-(z-\alpha^*)^2/2} \alpha^* e^{-\alpha^*(z-\mu^-)} dz
\]

\[
= \alpha^* e^{\alpha^* \mu^+-\alpha^*(\mu^-)^2/2} \sqrt{2\pi} (\Phi(\mu^+) - \Phi(\mu^-))
\]

\[
= \alpha^* e^{\alpha^* \mu^-/2} \sqrt{2\pi/e} (\Phi(\mu^+) - \Phi(\mu^-)).
\]

Therefore, it is better to use the truncated \( \mathcal{N}_+(0, \mu^-, 1) \) algorithm if

\[
\alpha^* e^{\alpha^* \mu^-/2} \sqrt{e} > e^{\mu^-/2} \frac{1}{\mu^+-\mu^-}
\]

i.e. if

\[
\mu^+ > \mu^- + \frac{2\sqrt{e}}{\mu^- + \sqrt{\mu^-^2 + 4}} \exp \left\{ \frac{\mu^-^2 - \mu^- \sqrt{\mu^-^2 + 4}}{4} \right\}.
\]  

(2.1)

Figure 2.1 - Lower bound (2.1) on \( \mu^+ \) for the use of the truncated normal algorithm.

Figure 2.1. provides the lower bound of (2.1) as a function of \( \mu^- \). Note that, as \( \mu^- \) increases, the range \( \mu^+-\mu^- \) has to get smaller for uniform accept-reject sampling to be used. The corresponding decomposition is straightforward to derive when \( \mu^+ < 0 \). Table 2.2 below gives the expected probabilities of acceptance in one run for several values of \( \mu^- \) and \( \mu^+-\mu^- \).
3. The multivariate case

We consider now a multivariate normal distribution $\mathcal{N}_p(\mu, \Sigma)$ restricted to a convex subset $\mathcal{R}$ of $\mathbb{R}^p$, denoted $\mathcal{N}_T(\mu, \Sigma, \mathcal{R})$. We assume that the one-dimensional slices of $\mathcal{R}$,

$$\mathcal{R}_i(\theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_p) = \{\theta_i; (\theta_1, \ldots, \theta_{i-1}, \theta_i, \theta_{i+1}, \ldots, \theta_p) \in \mathcal{R}\},$$

are readily available, in the sense that these sets can be represented as intervals $[\theta_i^-, \theta_i^+]$, where the bounding functions $\theta_i^-$ and $\theta_i^+$, depending on $(\theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_p)$, are easily computable ($1 \leq i \leq p$).

The algorithm we propose below belongs to the class of Markov Chain Monte-Carlo methods (as referred to in Hastings (1970) and Geyer (1991)). Namely, instead of generating a sequence $\theta_k$ of i.i.d. random vectors from the distribution of interest, we provide a sequence $\theta^{(n)}$ which is a Markov chain with stationary distribution the distribution of interest. Such an approximation may seem to fall far from the mark but results like the ergodic theorem ensure that the average of any quantity of interest $f(\theta)$,

$$\frac{1}{N} \sum_{n=1}^{N} f(\theta^{(n)}), \quad (3.1)$$

is converging to the expectation $\mathbb{E}[f(\theta)]$ as $N$ goes to infinity, thus generalizing the law of large numbers. More details on the application of Markov chain theory in this setup are given in Ripley (1987, pp. 113-114), Geyer (1991) and Tierney (1991). Following the early Metropolis algorithm (Metropolis et al., 1953), Markov chain Monte-Carlo simulation methods have been used extensively in the past years in Gibbs sampling theory for Bayesian computation (see Tanner and Wong (1987), Gelfand and Smith (1990) and Tanner (1991)). The main difficulty of this approach, as opposed to usual (independent) Monte-Carlo

\[
\begin{array}{cccccc}
\mu^- & 0 & 0.5 & 1 & 1.5 & 2 \\
2 & 0.726 & 0.811 & 0.869 & 0.907 & 0.932 \\
\mu^+ - \mu^- & 1 & 0.856 & 0.687 & 0.751 & 0.826 & 0.878 \\
0.5 & 0.960 & 0.851 & 0.759 & 0.680 & 0.679 \\
0.1 & 0.998 & 0.974 & 0.950 & 0.927 & 0.905 \\
\end{array}
\]

Table 2.2 - Average probabilities of acceptance for the simulation of $\mathcal{N}_T^+(0, \mu^-, \mu^+, 1)$. 

methods, is to monitor the convergence of the chain to the stationary distribution. Apart from classical central limit theorem (see Geyer, 1991) and time-series methods (see Ripley, 1987, chap. 6), one can suggest the simultaneous estimation of several quantities until approximate stationarity of the corresponding averages (3.1) is attained for all functions. Gelman and Rubin (1991) also suggest to run several times the algorithm with drastically different starting values. In our particular setup, convergence to the stationary distribution should be particularly fast since the compactness of \( \mathcal{R} \) ensures geometric convergence (see Tierney, 1991).

In the setup of truncated normal distributions, the Markov chain \( \theta^{(n)} \) is obtained by generating successively the components of \( N^{T}(\mu, \Sigma, \mathcal{R}) \), i.e.

\[
\begin{align*}
1. \theta_{1}^{(n)} & \sim \mathcal{N}_{+}^{+}(\mathbb{E}[\theta_{1}|\theta_{2}^{(n-1)}, \ldots, \theta_{p}^{(n-1)}], \theta_{1}^{-}, \theta_{1}^{+}, \sigma_{1}^{2}) \\
2. \theta_{2}^{(n)} & \sim \mathcal{N}_{+}^{+}(\mathbb{E}[\theta_{2}|\theta_{1}^{(n)}, \theta_{3}^{(n-1)}, \ldots, \theta_{p}^{(n-1)}], \theta_{2}^{-}, \theta_{2}^{+}, \sigma_{2}^{2}) \\
& \quad \ldots \\
p. \theta_{p}^{(n)} & \sim \mathcal{N}_{+}^{+}(\mathbb{E}[\theta_{p}|\theta_{1}^{(n)}, \ldots, \theta_{p-1}^{(n)}], \theta_{p}^{-}, \theta_{p}^{+}, \sigma_{p}^{2})
\end{align*}
\]

where the expectations and variances in the above truncated normal distributions are the conditional (non-truncated) expectations and variances of the \( \theta_{i} \) given \( \theta_{\neg i} = (\theta_{1}, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_{p}) \). Namely, we have

\[
\mathbb{E}[\theta_{i}|\theta_{\neg i}] = \mu_{i} + \Sigma_{i-i}^{t} \Sigma_{\neg i-i}^{-1} (\theta_{\neg i} - \mu_{\neg i}),
\]

\[
\sigma_{i}^{2} = \sigma_{ii}^{2} - \Sigma_{i-i}^{t} \Sigma_{i-i}^{-1} \Sigma_{\neg i-i},
\]

where \( \Sigma_{\neg i-i} \) is the \( (p-1) \times (p-1) \) matrix derived from \( \Sigma = (\sigma_{ij}^{2}) \) by eliminating its \( i \)-th row and its \( i \)-th column and \( \Sigma_{i-i} \) is the \( (p-1) \) vector derived from the \( i \)-th column of \( \Sigma \) by removing the \( i \)-th row term.

Moreover, it is important to note that there is no need to invert all the matrices \( \Sigma_{\neg i-i} \) to run the algorithm. Indeed, it is possible to derive these inverses from the global inverse matrix \( \mathbf{V} = \Sigma^{-1} \) since they can be written

\[
\Sigma_{\neg i-i}^{-1} = \mathbf{V}_{\neg i-i} - \mathbf{V}_{i-i} \mathbf{V}_{i-i}^{t} / \mathbf{V}_{ii}, \tag{3.2}
\]

where \( \mathbf{V}_{\neg i-i} \) and \( \mathbf{V}_{i-i} \) are derived from \( \mathbf{V} \) the way \( \Sigma_{\neg i-i} \) and \( \Sigma_{i-i} \) are derived from \( \Sigma \). Therefore, the algorithm only requires at most one inversion of \( \Sigma \) and the computation of the submatrices \( \Sigma_{\neg i-i}^{-1} \) by (3.2).

The comparison with a classical rejection-sampling method based on the simulation of \( x \sim \mathcal{N}_{p}(\mu, \Sigma) \) until the result belongs to \( \mathcal{R} \) is quite delicate, depending on the probability
P(x ∈ R) but also on the overall purpose of the simulation. In fact, if this probability is rather large and a single observation from \( \mathcal{N}^T(\mu, \Sigma, \mathcal{R}) \) is needed, it is clear that rejection sampling is preferable. On the contrary, if a large sample is needed, as it is the case for Gibbs sampling and related maximum likelihood methods, then the Markov chain Monte-Carlo method should be superior, especially if \( \mathcal{R} \) is small, since as mentioned above, convergence of the Gibbs sampler to the stationary distribution should be fast.

As a concluding remark, let us consider the following example. The truncated distribution of interest is

\[
\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \sim \mathcal{N}^T \left( \begin{bmatrix} 0 \\ \varrho \\ 1 \end{bmatrix}, \mathcal{R} \right),
\]

with truncation space \( \mathcal{R} \) the ball \( B(\gamma, r) \) of center \( \gamma = (\gamma_1, \gamma_2) \) and radius \( r \). Therefore,

\[
\begin{align*}
\theta_1^- (\theta_2) &= \gamma_1 - \sqrt{r^2 - (\gamma_2 - \theta_2)^2}, \\
\theta_1^+ (\theta_2) &= \gamma_1 + \sqrt{r^2 - (\gamma_2 - \theta_2)^2}, \\
\theta_2^- (\theta_1) &= \gamma_2 - \sqrt{r^2 - (\gamma_1 - \theta_1)^2}, \\
\theta_2^+ (\theta_1) &= \gamma_2 + \sqrt{r^2 - (\gamma_1 - \theta_1)^2},
\end{align*}
\]

and the conditional distributions defining the Markov chain are

1. \( \theta_1^{(n)} \sim \mathcal{N}_+^+ \left( \varrho \theta_2^{(n-1)}, \theta_1^- (\theta_2^{(n-1)}), \theta_1^+ (\theta_2^{(n-1)}), 1 - \varrho^2 \right) \)
2. \( \theta_2^{(n)} \sim \mathcal{N}_-^+ \left( \varrho \theta_1^{(n)}, \theta_2^- (\theta_1^{(n)}), \theta_2^+ (\theta_1^{(n)}), 1 - \varrho^2 \right) \).

Acknowledgements

This research was performed while visiting Cornell University. The author is grateful to George Casella for his support through NSF Grant No. DMS9100839 and NSA Grant No. 90F-073 and to Charles McCulloch for pointing out the single inversion argument in the multivariate case and helpful comments. By mentioning a mistake in an earlier version, Ranjini Natarajan also led to an improvement in the efficiency of the algorithms.

References

Casella, G. and George, E.I. (1991) Explaining the Gibbs sampler. *The Amer. Statist.* (to appear).

Chen, M.H. and Deely, J. (1992) Application of a new Gibbs Hit-and-Run sampler to a constrained linear multiple regression problem. Tech. report, Purdue University, Lafayette, IN.

Devroye, L. (1985) *Non-Uniform Random Variate Generation*. Springer-Verlag, New York.
Dykstra, R.L. and Robertson, T. (1982) An algorithm for isotonic regression for two or
more independent variables. *Ann. Statist.* **10**, 708-716.

Gelfand, A.E. and Smith, A.F.M. (1990) Sampling based approaches to calculating marginal
densities. *JASA* **85**, 398-409.

Gelfand, A.E., Smith, A.F.M. and Lee, T.M. (1992) Bayesian analysis of constrained parameter
and truncated data problems using Gibbs sampling. *JASA* **87**, 523-532.

Gelman, A. and Rubin, D.B. (1991) A single series from the Gibbs sampler provides a false sense of security. In *Bayesian Statistics 4*, J.O. Berger, J.M. Bernardo, A.P. Dawid and A.F.M. Smith (Eds.). Oxford University Press.

Geyer, C.J. (1991) Markov Chain Monte Carlo Maximum Likelihood. To appear in *Computer Sciences and Statistics: Proc. 23d Symp. Interface*.

Hastings, W.K. (1971) Monte-Carlo sampling methods using Markov chains and their applications. *Biometrika* **57**, 97-109.

Marsaglia, G. (1964) Generating a variable from the tail of a normal distribution. *Technometrics* **6**, 101-102.

Metropolis, N., Rosenbluth, A.W., Rosenbluth, M.N., Teller, A.H. and Teller, E. (1953) Equations of state calculations by fast computing machines. *J. Chemical Phys.* **21**, 1087-1091.

Qian, W. and Titterington, D.M. (1991) Estimation of parameters in hidden Markov models. *Phil. Trans. Royal Soc. London A* **337**, 407-428.

Ripley, B.D. (1987) *Stochastic simulation*. J. Wiley, New York.

Robertson, T., Wright, F.T. and Dykstra, R.L. (1988) *Order Restricted Statistical Inference*. J. Wiley, New York.

Tanner, M. (1991) *Tools for Statistical Inference*. Springer-Verlag, New York.

Tanner, M. and Wong, W. (1987) The calculation of posterior distributions by data augmentation. *Journal of the American Statistical Association* **82**, 528-550.

Tierney, L. (1991) Markov chains for exploring posterior distributions. To appear in *Computer Sciences and Statistics: Proc. 23d Symp. Interface*.

LSTA, Boîte 158
Université Paris 6
4, place Jussieu
75252 Paris Cedex 5 - France

March 1992