On the second homotopy group of the classifying space for commutativity in Lie groups

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Abstract
In this note we show that the second homotopy group of $B(2, G)$, the classifying space for commutativity for a compact Lie group $G$, contains a direct summand isomorphic to $\pi_1(G) \oplus \pi_1([G, G])$, where $[G, G]$ is the commutator subgroup of $G$. It follows from a similar statement for $E(2, G)$, the homotopy fiber of the canonical inclusion $B(2, G) \hookrightarrow BG$. As a consequence of our main result we obtain that if $E(2, G)$ is 2-connected, then $[G, G]$ is simply-connected. This last result completes how the higher connectivity of $E(2, G)$ resembles the higher connectivity of $[G, G]$ for a compact Lie group $G$.

Keywords Compact Lie groups · Classifying spaces · Commutator subgroup · Spaces of commuting elements

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Introduction

Let $G$ be a Lie group. A. Adem, F. Cohen and E. Torres-Giese [2] studied a filtration of the classifying space $BG$ associated to the descending central series of a group. The space $B(2, G)$ sitting in the first term of this filtration arises by assembling together the spaces of commuting tuples in $G$. The homotopy fiber of the inclusion $B(2, G) \hookrightarrow BG$, denoted $E(2, G)$ can be thought of as the difference between $B(2, G)$ and $BG$, and in some sense measures how far $G$ is from being abelian. In this note we study the second homotopy group $\pi_2(E(2, G))$ for a compact Lie group $G$. The homotopy classes in $\pi_2(E(2, G))$ for the orthogonal matrix groups $G = O(n)$, were previously used in [16] to produce non-standard classes in the reduced commutative orthogonal $K$-theory of closed connected surfaces. This cohomology theory is a variant of orthogonal $K$-theory (as defined in [3] and further studied in [4]), whose classes are represented by orthogonal vector bundles equipped with transition functions that point-wise commute. In general for a Lie group $G$, a principal $G$-bundle equipped with such transition functions is said to have a transitionally commutative structure.
Let \([G, G]\) denote the commutator subgroup of \(G\). To prove our main result we combine the commutator map \(\epsilon : E(2, G) \to B[G, G]\) introduced in [6] with the techniques developed in [16] of producing new classifying maps from old ones by inverting cocycles that point-wise commute. For instance, in the case of the special orthogonal groups \(SO(n)\), this procedure can be translated into \(\epsilon_\ast : \pi_2(E(2, SO(n))) \to \pi_2(BSO(n))\) being an isomorphism, for every \(n \geq 3\). Our main result is to extend these ideas to any compact Lie group.

**Theorem 1** Let \(G\) be a compact Lie group. Then:

1. \(\pi_2(E(2, G))\) contains a direct summand isomorphic to \(\pi_2(B[G, G])\).
2. If \(G\) is connected, then \(\epsilon_\ast : \pi_2(E(2, G)) \to \pi_2(B[G, G])\) is an isomorphism.

As an application of part 2 of Theorem 1, in Corollary 4 we show that when \(G\) is a connected compact Lie group, every principal \(G\)-bundle over a CW complex of dimension \(\leq 3\), with a reduction to \([G, G]\), possesses a transitively commutative structure.

In a slightly different context, when \(G\) is discrete, \(E(2, G)\) is homotopy equivalent to the coset poset of abelian subgroups of \(G\). Coset posets of families of subgroups were studied by Abels and Holz [1], who were interested in their higher connectivity and the algebraic implications over \(G\). For instance, the coset poset of the family of abelian subgroups is simply connected if and only if \(G\) is abelian (see [15]).

When \(G\) is a compact Lie group, we can also treat \(E(2, G)\) as the Lie group analogue of the coset poset of abelian subgroups. In this scenario, in [6, Theorem 1] it is shown that \(E(2, G)\) is contractible if and only if \(G\) is abelian, and it further asserts that the latter holds if and only if \(E(2, G)\) is 4-connected. As for compact Lie groups, \(G\) abelian is equivalent to \([G, G]\) 3-connected, our motivation is then to understand the relation between the higher connectivity of \(E(2, G)\) and the higher connectivity of \([G, G]\).

As a consequence of part 1 of our main result we obtain:

**Corollary 1** Let \(G\) be a compact Lie group. If \(E(2, G)\) is 2-connected, then \([G, G]\) is 1-connected.

By [6, Proposition 9], the vanishing of the fundamental group \(\pi_1(E(2, G))\) implies that \([G, G]\) is connected. Combining this with Corollary 1, [6, Theorem 1] and the fact that \(\pi_2([G, G])\) is always trivial, yields for a compact Lie group, that the higher connectivity of \(E(2, G)\) completely determines how highly connected is the commutator subgroup. To spell this out precisely: if \(E(2, G)\) is \(n\)-connected, then \([G, G]\) is \((n - 1)\)-connected for \(n = 1, 2\) and 4. In this context we then obtain a full interpretation of how \(E(2, G)\) measures how far \(G\) is from being abelian.

Perhaps it is worth mentioning that the analogy does not work with non-compact Lie groups. For example, consider \(SL(2, \mathbb{R})\) with its maximal compact subgroup \(SO(2)\), which is abelian. By [3, Theorem 3.1] there is a homotopy equivalence \(E(2, SL(2, \mathbb{R})) \simeq ESO(2)\) making \(E(2, SL(2, \mathbb{R}))\) contractible, but \([SL(2, \mathbb{R}), SL(2, \mathbb{R})] = SL(2, \mathbb{R})\), which is not simply-connected.

We finish the introduction with the following observation. In [6, Question 21] the authors asked if the looped commutator map \(\Omega \epsilon : \Omega E(2, G) \to [G, G]\) splits, up to homotopy, for a compact Lie group \(G\). Our techniques allow us to answer this in the affirmative for a family of extensions of finite groups by tori (see Corollary 3), which for instance contains every normalizer \(N(T)\) of a maximal torus \(T\) in a semisimple connected compact Lie group.

The paper is organized as follows. In Section 1 we briefly recall simplicial models of \(B(2, G)\) and \(E(2, G)\), and argue that, for a connected compact Lie group, the homotopy
type of $E(2, G)$ only depends on the semisimple part of $G$. In Section 2, for semisimple connected compact Lie groups, we prove the existence of an isomorphism $\pi_2(E(2, G)) \cong \pi_1(G)$ by analyzing the Moore complex associated to the simplicial abelian group $(n \mapsto H_0(C_n(G) ; \mathbb{Z}))$, where $C_n(G)$ is the space of commuting $n$-tuples in $G$. In Section 3 we study the fundamental group of the commutator subgroup of extensions of finite groups by a torus, and describe its generators via the commutator map. In Section 4 we prove Theorem 1, and the applications mentioned above.

Some conventions and notation. Throughout this paper we will use the following conventions:

- For a pair of elements $x, y$ in a group $G$, the commutator $[x, y]$ will mean $x^{-1}y^{-1}xy$.
- For a compact Lie group $G$, $G_0$ will denote the connected component of the identity, and $Z(G)$ will denote the center of $G$.

1 A few preliminaries

Let $G$ be a Lie group. Consider the space of commuting $k$-tuples

$$C_k(G) = \{(g_1, \ldots, g_k) \in G^k : [g_i, g_k] = 1\},$$

with its subspace inclusion $i_k : C_k(G) \hookrightarrow G^k$. A simplicial model $B_*G$ of the classifying space $BG$ with $B_kG = G^k$, is the nerve of $G$, where $G$ is taken as a topological category with a single object. In this model the simplicial structure of $B_*G$ is compatible with every inclusion $i_k$, so that defining $C_0(G) = pt$, the family $\{C_k(G)\}_{k \geq 0}$ assembles into a simplicial space $C_\bullet(G)$ where the induced inclusion $i_* : C_\bullet(G) \rightarrow B_*G$ becomes a simplicial map. Taking geometric realization the classifying space for commutativity is defined as

$$B(2, G) := |C_\bullet(G)|,$$

and is equipped with a canonical inclusion $i := |i_*| : B(2, G) \rightarrow BG$. (See [2] for more details). The space $E(2, G)$ is defined as the homotopy fiber of $i$, that is, the pullback of the universal principal $G$-bundle $EG \rightarrow BG$ along $i$. A simplicial model for $E(2, G)$ is as follows. For every $k \geq 0$ let

$$E_k(2, G) = \{(g_0, \ldots, g_k) : \langle g_0^{-1}g_1, \ldots, g_{k-1}^{-1}g_k \rangle \text{ is abelian}\},$$

and $E(2, G) = |E_\bullet(2, G)|$. It comes with a projection $p : E(2, G) \rightarrow B(2, G)$, given as the realization of the simplicial map $E_\bullet(2, G) \rightarrow C_\bullet(G)$ where for every $k \geq 1$, $(g_0, \ldots, g_k) \mapsto (g_0^{-1}g_1, \ldots, g_{k-1}^{-1}g_k)$. See [6, Section 2] for more details.

We record 2 properties of these spaces. First, if $G$ is abelian clearly $B(2, G) = BG$, and hence $E(2, G) = EG$ which is contractible. Secondly, by construction $B(2, G)$ is connected, and the first term in the skeletal filtration of $E(2, G)$ is homotopy equivalent to the suspension $\Sigma(G \wedge G)$, so that $E(2, G)$ is connected, as well.

Now we will argue that the problem of studying the homotopy type of $E(2, G)$ for connected compact Lie groups $G$ can be reduced to semisimple connected compact Lie groups.

Lemma 1 Suppose $H$ and $K$ are Lie groups or discrete groups, where $H$ is non-trivial and abelian. Suppose further that $Z$ is a common central subgroup, and consider the central product $G = H \times_Z K$ where $Z \subset H \times K$ via $z \mapsto (z, z^{-1})$. Then the inclusion $K \hookrightarrow G$ induces a homotopy equivalence

$$E(2, K) \tilde{\rightarrow} E(2, G).$$
Let $\pi : H \times K \to G$ be the quotient map. We claim that the commutative diagram
\[
\begin{array}{ccc}
H^n \times C_n(K) & \xrightarrow{\pi^n} & (H \times K)^n \\
\downarrow{\pi^n} & & \downarrow{\pi^n} \\
C_n(G) & \xrightarrow{} & G^n,
\end{array}
\]
is a pullback square. Notice that if $(\pi(h_1, k_1), \ldots, \pi(h_n, k_n)) \in G^n$ is a commuting $n$-tuple, then $[(h_i, k_i), (h_j, k_j)] \in Z$ for every $1 \leq i, j \leq n$, but since the $h_i$'s pairwise commute, the only possible value of these commutators is $(1, 1) \in Z$. In particular $[k_i, k_j] = 1$ for every $1 \leq i, j \leq n$ and the claim now follows. The restriction $\pi^n|_C$ is then a principal $Z^n$-bundle since $\pi^n$ is a principal bundle, as well.

Upon taking geometric realization of (1) we again obtain a pullback square and hence a principal $BZ$-bundle $BH \times B(2, K) \to B(2, G)$. Now the inclusion $B(2, G) \to BG$ being natural induces a map of fiber sequences
\[
\begin{array}{ccc}
BZ & \to & BH \times B(2, K) \to B(2, G) \\
\downarrow & & \downarrow \\
BZ & \to & BH \times BK \to BG,
\end{array}
\]
and taking vertical homotopy fibers gives the claimed homotopy equivalence. \qed

Let $G$ be a compact Lie group and $H, K \subset G$ closed subgroups. Let us briefly recall that the commutator subgroup $[H, K] \subset G$ is the topological closure of the algebraic commutator subgroup $[H, K]_{\text{alg}}$, but since $G$ is compact, $[H, K] = [H, K]_{\text{alg}}$.

**Proposition 1** Let $G$ be a connected compact Lie group. Then the inclusion $[G, G] \hookrightarrow G$ induces a homotopy equivalence $E(2, [G, G]) \tilde{\to} E(2, G)$.

**Proof** Let $Z(G)_0$ denote the path connected component of the identity of the center of $G$. By [9, Chapter IX, Section 1.4, Corollary 1] the homomorphism
\[
p : Z(G)_0 \times [G, G] \to G
\]
\[(x, y) \mapsto xy\]
is a finite covering. The kernel of $p$ can be identified with the elements $(z, z^{-1}) \in Z(G)_0 \times [G, G]$ such that $z \in Z(G)_0 \cap [G, G]$, which is a central subgroup. Then $G$ is isomorphic to the central product $Z(G)_0 \times_{\ker p} [G, G]$. The result now follows from Lemma 1. \qed

### 2 Calculation of $\pi_2(E(2, G))$ for connected compact Lie groups

Recall that when $G$ is a connected compact Lie group, $[G, G]$ is a semisimple connected compact Lie group, and if further $G$ is semisimple, then the commutator subgroup $[G, G] = G$. In light of Proposition 1, to study $E(2, G)$, hereon we may assume that $G$ is a semisimple connected compact Lie group, if necessary.

The next step is to understand the (un-normalized) Moore complex associated to the simplicial abelian group $n \mapsto H_0(C_n(G); \mathbb{Z})$. See [11, Chapter 3, Section 2] for the precise definition of Moore complex.

We will need the following two lemmas. The first one is a standard group homology result, for example see [17, Lemma 9.51].
Lemma 2 Let $\pi$ be a discrete group and let $W$ be the augmentation ideal of the group ring $\mathbb{Z}[\pi]$. Then there is an isomorphism of abelian groups $W/W^2 \cong \pi/\langle \pi, \pi \rangle$.

Let $K$ be a subgroup $K \subset Z(G)$, where $Z(G)$ is the center of $G$. We denote the space of almost commuting $n$-tuples relative to $K$ by $B_n(G, K) = \{(g_1, \ldots, g_n) : [g_i, g_j] \in K\}$.

Lemma 3 Let $G$ be a semisimple simply-connected compact Lie group, and $K \subset Z(G)$. Then for every pair of elements $c_1, c_2 \in K$, there is a triple $(g_1, g_2, g_3) \in B_3(G, K)$ such that $[g_1, g_3] = c_1, [g_2, g_3] = c_2$ and $[g_1, g_2] = 1$.

Proof Suppose $K$ is cyclic and let $c$ be a generator. We can find a pair $(g, h) \in G^2$ such that $[g, h] = c$, for example by [14, Appendix A] which shows how to construct Heisenberg pairs in $G$. Then $(1, c^i, c^j)$ can be realized for instance by the associated triple of commutators of $(g^i, g^j, h) \in G^3$.

Among simple Lie groups, the only ones that have non-cyclic center are the spin groups Spin$(4k)$, for $k \geq 2$, whose center is $\mathbb{Z}/2 \times \mathbb{Z}/2$. Consider Spin$(4) \cong SU(2) \times SU(2)$ (see [7, Theorem 8.9.8]). One can readily check that every triple in $Z(SU(2) \times SU(2)) = \mathbb{Z}/2 \times \mathbb{Z}/2$ can be realized by almost commuting triples of $SU(2) \times SU(2)$. Then the diagonal block matrix inclusion $SO(4) \hookrightarrow SO(4k)$ induces a map $\delta : \text{Spin}(4) \to \text{Spin}(4k)$, which is an isomorphism restricted to the center. Thus any triple of $Z(\text{Spin}(4k))$ can be realized by a triple of commutators of elements in the image of $\delta$.

Without loss of generality suppose $G$ is a product of two simple groups $G = G_1 \times G_2$. Pick $(a_1, b_1), (a_2, b_2) \in Z(G) = Z(G_1) \times Z(G_2).$ The previous argument gives us triples $(g_1, g_2, g_3) \in B_3(G_1, Z(G_1))$ and $(h_1, h_2, h_3) \in B_3(G_2, Z(G_2))$ that realize $(a_1, b_1) \times (1, b_2)$, respectively. Then $(g_1, h_1), (g_2, h_2), (g_3, h_3)) \in B_3(G, Z(G))$ realizes $((1, 1), (a_1, b_1), (a_2, b_2)) \in Z(G)^3$. \qed

Let $(D, \partial)$ be the Moore complex associated to the simplicial group $H_0(C^*_G); \mathbb{Z})$. In [16, Lemma 4.2] it is shown that in our context $H_2(B(2, G); \mathbb{Z}) \cong H_1(G; \mathbb{Z}) \oplus H_2(D, \partial)$. Since both $B(2, G)$ and $E(2, G)$ are simply connected when $G$ is connected (e.g. [4, Lemma 4.3]), and $\pi_2(B(2, G)) \cong \pi_2(BG) \oplus \pi_2(E(2, G))$ (e.g. [2, Theorem 6.3]), we conclude that $\pi_2(E(2, G)) \cong H_2(D, \partial)$.

Proposition 2 Let $G$ be a semisimple connected compact Lie group. Then $\pi_2(E(2, G)) \cong \pi_1(G)$.

Proof We will show that $H_2(D, \partial) \cong \pi_1(G)$. We are interested in the following portion of the chain complex $(D, \partial)$:

$$H_0(C_3(G); \mathbb{Z}) \xrightarrow{\partial_3} H_0(C_2(G); \mathbb{Z}) \xrightarrow{\partial_2} H_0(G; \mathbb{Z}) \to 0.$$ 

A result of Kac and Smilga [14, Remark A1] asserts that the path-connected components of $C_2(G)$ are in bijection with the elements of $\pi_1(G)$. Moreover, this bijection can be realized as follows. Let $\tilde{G}$ be the universal cover of $G$, then $\pi_1(G) \cong Z(\tilde{G})$. The bijection $\rho : \pi_0(C_2(G)) \to Z(\tilde{G})$ assigns to a commuting pair $(g_1, g_2)$ representing a path-connected component, the commutator of a lift $([\tilde{g}_1, \tilde{g}_2]) \in B_2(\tilde{G}, Z(G))$. Under this correspondence we have an isomorphism and henceforth an identification

$$H_0(C_2(G); \mathbb{Z}) \xrightarrow{\tilde{\omega}} \mathbb{Z} \pi_1(G).$$

As $G$ is connected, the differential $\partial_2$ is the augmentation map with respect to the group ring $\mathbb{Z}\pi_1(G)$. Hence ker $\partial_2$ is the augmentation ideal.
To analyze $\partial_3$, we need to understand what is the image of the path-connected components of $C_3(G)$ under the face maps $d_i : C_3(G) \to C_2(G)$, which are given by $d_0(g_1, g_2, g_3) = (g_2, g_3)$, $d_1(g_1, g_2, g_3) = (g_1, g_2, g_3)$, $d_2(g_1, g_2, g_3) = (g_1, g_2, g_3)$ and $d_3(g_1, g_2, g_3) = (g_1, g_2)$. Similarly, we have face maps $d_i : B_3(\tilde{G}, Z(\tilde{G})) \to B_2(\tilde{G}, Z(\tilde{G}))$ which are defined verbatim. Since the universal cover $\tilde{G} \to G$ is a homomorphism, for every $0 \leq i \leq 3$ we have the commutative diagram

$$
\pi_0(B_3(\tilde{G}, Z(\tilde{G}))) \xrightarrow{\pi_0(d_i)} \pi_0(B_2(\tilde{G}, Z(\tilde{G}))) \to Z(\tilde{G})
$$

where $\pi_0(B_2(\tilde{G}, Z(\tilde{G}))) \to Z(\tilde{G})$ is induced by the commutator and the vertical maps are induced by the universal cover $\tilde{G} \to G$.

Let $c$ be a path-connected component in $C_3(G)$. Choose an element $(g_1, g_2, g_3)$ in $c$, and a lift $(\tilde{g}_1, \tilde{g}_2, \tilde{g}_3)$. Then

- $\rho \pi_0(d_0)(c) = [\tilde{g}_2, \tilde{g}_3]$;
- $\rho \pi_0(d_1)(c) = [\tilde{g}_1, \tilde{g}_2, \tilde{g}_3]$;
- $\rho \pi_0(d_2)(c) = [\tilde{g}_1, \tilde{g}_2, \tilde{g}_3]$;
- $\rho \pi_0(d_3)(c) = [\tilde{g}_1, \tilde{g}_2]$.

Now let $(\eta_1, \eta_2, \eta_3) = ([\tilde{g}_1, \tilde{g}_2], [\tilde{g}_1, \tilde{g}_3], [\tilde{g}_2, \tilde{g}_3])$. As the commutators are central, one can readily verify that under correspondence (2) we obtain

$$
\partial_3(c) = \eta_3 - \eta_2 \eta_3 + \eta_1 \eta_2 - \eta_1 = (\eta_3 - \eta_1)(1 - \eta_2).
$$

Therefore $\text{im} \ \partial_3 \subset (\ker \partial_2)^2$.

We claim that $\text{im} \ \partial_3$ generates $(\ker \partial_2)^2$. By hypothesis $\tilde{G}$ is a semisimple simply-connected compact Lie group, hence for every triple of the form $(1, \eta_1, \eta_2) \in Z(\tilde{G})^3$, Lemma 3 asserts that we can find a triple $(g_1, g_2, g_3) \in C_3(G)$ and a lift $(\tilde{g}_1, \tilde{g}_2, \tilde{g}_3) \in B_3(\tilde{G}; Z(\tilde{G}))$ such that its associated triple of commutators is $([\tilde{g}_1, \tilde{g}_2] = 1, [\tilde{g}_1, \tilde{g}_3] = \eta_1, [\tilde{g}_2, \tilde{g}_3] = \eta_2)$. Let $c$ be the path-connected component containing $(g_1, g_2, g_3)$. Then $\partial_3(c) = (\eta_2 - 1)(1 - \eta_1)$. Since the elements $1 - \eta$ form a basis of the augmentation ideal $\ker \partial_2$, we obtain that $\text{im} \ \partial_3 = (\ker \partial_2)^2$, which proves our claim.

The proposition now follows from Lemma 2, as $H_2(D, \partial) = \ker \partial_2/(\ker \partial_2)^2 \cong \pi_1(G)$.

$\square$

**Remark 1** Proposition 1 and Proposition 2 then yield an isomorphism $\pi_2(E(2, G)) \cong \pi_1([G, G])$, for every connected compact Lie group $G$.

**Remark 2** When $G$ is not connected, there is no isomorphism between $\pi_2(E(2, G))$ and $\pi_2(B[G, G])$. For instance consider $G = O(2)$, where $[G, G] = SO(2)$. In [5, Theorem 1.5] it is shown that $E(2, O(2)) \cong S^2 \vee S^2 \vee S^3$, and it follows that $\pi_2(E(2, O(2))) \cong \mathbb{Z}^2$, whereas $\pi_2(BSO(2)) = \mathbb{Z}$. However, we will see that $\pi_2(E(2, G))$ splits off $\pi_2(B[G, G])$.

As mentioned before, by [2, Theorem 6.3], the following corollary is an immediate consequence of Remark 1.

**Corollary 2** Let $G$ be a connected compact Lie group. Then $\pi_2(B(2, G)) \cong \pi_1(G) \oplus \pi_1([G, G])$.  

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Remark 3 Let $G$ be a semisimple simply-connected compact Lie group. Proposition 2 then asserts that $\pi_2(E(2, G)) = 0$, and hence $\pi_2(B(2, G))$ is trivial, as well. As $\pi_3(BG) = 0$, it follows that $H_3(B(2, G); \mathbb{Z}) \cong \pi_3(B(2, G)) \cong \pi_3(E(2, G))$. By inspection of the spectral sequence $E^2_{p,q} = H_p H_q(C_*(G)) \implies H_{p+q}(B(2, G))$, we can conclude that $\pi_3(E(2, G)) \cong H_3(B_0(C_*(G); \mathbb{Z}))$. For the classical simple Lie groups $SU(n)$ and $Sp(m)$, it is known that $\pi_3(E(2, G)) = 0$, as the spaces $C_k(G)$ are path-connected. But this is not known in general. For instance the space of commuting triples $C_3(Spin(7))$ is not path-connected, so $\pi_3(E(2, Spin(7)))$ could potentially be non-trivial.

3 $\pi_2(E(2, G))$ for extensions of a finite group by a torus

Let $G$ be an extension of a finite group $Q$ by a torus $T = (S^1)^k$. Pick an element $q \in Q$, and let $\tilde{q} \in G$ be a lift of $q$ under the quotient map. Recall that we use the commutator convention $[x, y] = x^{-1}y^{-1}xy$. We can write the commutator $[\tilde{q}, -] : T \to T$ as a product of maps $[c_q \circ (-)^{-1}] \cdot I d$, where $c_q : T \to T$ is conjugation by $\tilde{q}$ and $(-)^{-1}$ is inversion in $T$. As $T$ is abelian, conjugation is independent of the choice of a lift, hence

$$\psi(q) := [\tilde{q}, -]$$

is a well defined group homomorphism $T \to T$ for every $q \in Q$. The induced homomorphism in $\pi_1$ is then

$$\psi(q)_* : \mathbb{Z}^k \to \mathbb{Z}^k$$

$$x \mapsto x - \pi_1(c_{\tilde{q}})(x)$$

Lemma 4 Let $G$ be an extension of a finite group $Q$ by a torus $T$. Let $[G, G]_0$ be the connected component of the identity of the commutator subgroup. Then

1. Every element in $[G, G]_0$ is a product of the form $\psi(q_1)(t_1) \cdots \psi(q_n)(t_n)$, where $q_i \in Q$ and $t_i \in T$.

2. The subgroup $\sum_{q \in Q} \text{im } \psi(q)_* \subset \pi_1([G, G]_0)$ is of maximal rank.

Proof Borel and Serre [8, Lemma 5.1, footnote p. 152] noticed that when $G$ is a compact Lie group that is not connected there exists a finite group $F \subset G$ such that $G = FG_0$. That is, we can choose the lifts $\tilde{q}$ to be of finite order. As $G$ is compact, the elements in $[G, G]$ are finite products of commutators. We can use the identity $t \tilde{q} = \tilde{q}c_{\tilde{q}}(t)$ to write any single commutator as

$$[\tilde{p}s, \tilde{q}t] = [\tilde{p}, \tilde{q}]c_{[\tilde{p}, \tilde{q}]}(s^{-1})c_{\tilde{q}}^{-1}(t^{-1})c_{\tilde{q}}(s)t$$

$$= [\tilde{p}, \tilde{q}]\psi((p, q))(s)\psi(q^{-1}pq)(t)\psi(q)(s^{-1}),$$

where $p, q \in Q$ and $s, t \in T$. Any finite product of these can be rearranged similarly as in (3) to an expression of the form $[\tilde{q}_1, \tilde{p}_1] \cdots [\tilde{q}_m, \tilde{p}_m]c_{s_1, t_1, \ldots, s_m, t_m}$, where $c_{s_1, t_1, \ldots, s_m, t_m}$ is a product of conjugates of $s_i, t_i \in T$ by words in terms of lifts of $p_i, q_i \in Q$. Now since $[G, G]_0 \subset T$ and $c_{s_1, t_1, \ldots, s_m, t_m}$ can be path-connected to the identity we see that $r = [\tilde{q}_1, \tilde{p}_1] \cdots [\tilde{q}_m, \tilde{p}_m] \in T$ has finite order. Thus we have an exhaustive description of the elements of $[G, G]_0$. Finally, similarly as in (4), we see that they are all products of the form $r\psi(q_1)(t_1) \cdots \psi(q_n)(t_n)$ for some $q_i \in Q$ and $t_i \in T$.

Now given a group decomposition of $[G, G]_0$ into $S^1$’s, we can choose $r\psi(q_1)(t_1) \cdots \psi(q_m)(t_m) \in S^1_j$ of infinite order, where $S^1_j$ is the $j$-th $S^1$-factor of
[G, G]₀. Then as each ψ(qᵢ) is a homomorphism, taking N = ord(r), we have ψ(q₁)(t₁ᴺ) ⋯ ψ(qₘ)(tₘᴺ) ∈ S_j¹, and since it is still of infinite order, ψ(q₁)(−) ⋯ ψ(qₘ)(−) must cover S_j¹. Indeed, let Y ⊂ T × ⋯ × T be a circle subgroup, such that (t₁ᴺ, ⋯, tₘᴺ) ∈ Y (for example take Y as the closure of a 1-parameter subgroup through (t₁ᴺ, ⋯, tₘᴺ)). Then

\[ Y \xrightarrow[ψ(q₁)(−) ⋯ ψ(qₘ)(−)]{} S_j¹ \]

is a covering map, and part 1 follows.

For part 2, consider the homomorphism

\[ T × ⋯ × T \xrightarrow{\prod_{q ∈ Q} ψ(q)} [G, G]₀ \]

which by part 1 is surjective. Since it is a continuous homomorphism between products of S¹’s, and rank([G, G]₀) ≤ rank(T), the induced map in fundamental groups is of maximal rank, as desired. □

**Remark 4** In general ψ(q)ₖ may not be surjective. For example consider G = O(2), where T = SO(2) = [G, G], and Q = Z/2. If τ ∈ Z/2 is a generator, then [T, r₀] = r₂τ, where r₀ ∈ SO(2) is rotation by τ. It follows that ψ(τ)ₖ : Z → Z is multiplication by 2.

### 3.1 Cocycles and associated clutching function.

Let \( C = \{C₁, C₂, C₃\} \) be the closed cover of \( S^2 \) given by \( C₁ = \{x ∈ S^2 : x₀ ≤ 0\} \), \( C₂ = \{x ∈ S^2 : x₀ ≥ 0 \text{ and } x₂ ≥ 0\} \) and \( C₃ = \{x ∈ S^2 : x₀ ≥ 0 \text{ and } x₂ ≤ 0\} \). Let G be a topological group. A G-valued commutative cocycle over C is a cocycle

\[ α = \{α₁₂ : C₁ ∩ C₂ → G, \ α₁₃ : C₁ ∩ C₃ → G, \ α₂₃ : C₂ ∩ C₃ → G\} \]

satisfying the usual cocycle equation with the added condition that the commutators

\[ [α₁₂(x), α₂₃(x)] = [α₁₂(x), α₁₃(x)] = [α₂₃(x), α₁₃(x)] = 1 ∈ G \]

for any \( x ∈ C₁ ∩ C₂ ∩ C₃ \) (of course one can generalize this definition to any cover). Note that each non-degenerate double intersection is a semicircle and the triple intersection consists of 2 points — ‘front’ (0, 1, 0) and ‘back’ (0, −1, 0).

By definition, a commutative cocycle α will give rise to a simplicial map after applying the nerve functor \( Nα : N(C) → B_*G \), where \( N(C) \) is the Čech complex as considered in [10]. It will factor through the inclusion \( i_* : B_*G → B_*G \). That is, we obtain a map

\[ fα : S^2 \xrightarrow{\sim} |N(C)| \xrightarrow{|N(α)|} B(2, G) \]

Note that when α is commutative we can point-wise invert α in G and still get a cocycle which we may call the inverse cocycle and will denote it by \( α^{-1} \). Then we have the identity \( φ^{-1}fα = fα^{-1} \).

To a G-valued cocycle α over C one can associate a G-bundle \( E_α → S^2 \). By [16, Lemma 3.3] \( i_*fα \) classifies \( E_α \). Moreover, by [16, Lemma 3.2] \( E_α \) is clutching by the map \( φ_α : S^1 → G \) given by

\[ φ_α(x) = \begin{cases} α₁₂(x)α₂₃(x) & \text{if } x ∈ C₁ ∩ C₂ \\ α₁₃(x) & \text{if } x ∈ C₁ ∩ C₃ \end{cases} \]
where we are identifying \((C_1 \cap C_2) \cup (C_1 \cap C_3)\) with \(S^1\). We refer to \(\varphi_a\) as the clutching function induced by \(\alpha\).

As both \(\varphi_a\) and \(i_{f_a}\) classify the same bundle there is a relation between their homotopy classes. When considered as pointed maps, this relation is attained by the natural isomorphism \(\pi_2(BG) \cong \pi_1(G, 1_G) \cong \pi_1(G, \varphi_a(1_{S^1}))\).

### 3.2 The commutator map via commutative cocycles

Let us assume \(G\) is any Lie group. Recall the simplicial model of \(E(2, G)\) whose \(n\)-simplices are \(E_n(2, G) = \{(g_0, \ldots, g_n) : (g_0^{-1}g_1, \ldots, g_{n-1}^{-1}g_n)\) is abelian\} described in Section 1. Such an \((n + 1)\)-tuple in \(E_n(2, G)\) is called affinely commutative. This model is particularly useful because it allows us to construct the commutator map which has proven to be a powerful tool to study homotopical properties of \(E(2, G)\). Precisely, the commutator map \(c\) is defined simplicially as

\[
E_k(2, G) \overset{c_k}{\to} B_k[G, G]
\]

\[
(g_0, \ldots, g_k) \mapsto ([g_0, g_1], \ldots, [g_{k-1}, g_k])
\]

for every \(k \geq 1\), and \(c := |c_\bullet| : E(2, G) \to B[G, G]\) (see [6, Section 3] for more details).

**Proposition 3** Let \(G\) be an extension of a finite group \(Q\) by a torus \(T\). Then the image of the commutator map

\[
c_\bullet : \pi_2(E(2, G)) \to \pi_2(B[G, G])
\]

is of maximal rank.

**Proof** Consider the cover \(C = \{C_1, C_2, C_3\}\) of \(S^2\) as in Subsection 3.1. We parametrize the semi-circles \(C_r \cap C_s\) by \(Cr(t)\) where \(t \in [0, 1]\). For every \(q \in Q\) and every loop \(x : [0, 1] \to T\) based at the identity \(1 \in T\), let us now define the 0-cocycles \(q.x\varphi_r : C_r \to G\) by

\[
q.x\varphi_1 = h
\]

\[
q.x\varphi_2 = \bar{q}
\]

\[
q.x\varphi_3 = 1,
\]

where \(h\) is an extension to \(C_1\) obtained from the null map \(x \ast \bar{x}\) which applies \(x\) over \(C_{12}(t)\) and \(x\) in the opposite direction over \(C_{13}(t)\). We claim that these maps induce a simplicial map \(q.x\varphi_\bullet : N(C) \to E_\bullet(2, G)\). To see this, we need to analyse \(q.x\varphi_1^{-1}q.x\varphi_j\), with \(i < j\). On double intersections we obtain

\[
(q.x\varphi_1^{-1}q.x\varphi_2)(C_{12}(t)) = x(t)^{-1}\bar{q}
\]

\[
(q.x\varphi_2^{-1}q.x\varphi_3)(C_{23}(t)) = \bar{q}^{-1},
\]

and on triple intersections we get \(\bar{q}\) and \(\bar{q}^{-1}\). Thus \((q.x\varphi_1, q.x\varphi_2, q.x\varphi_3)\) is affinely commutative on triple intersection points. Therefore the maps \(q.x\varphi_r\) induce a simplicial map \(q.x\varphi_\bullet\) as claimed. Consider the composite

\[
N(C) \xrightarrow{q.x\varphi_\bullet} E_\bullet(2, G) \xrightarrow{c_\bullet} B_\bullet[G, G],
\]

where \(c_\bullet\) is the simplicial model of the commutator map. It can be seen that the composite is induced by the cocycle over \(C\) that is constant over \(C_1 \cap C_3\) and \(C_2 \cap C_3\) with value 1, and
t \mapsto [x(t), \tilde{q}]$ over the parametrized intersection $C'_{12}(t)$, where $t \in [0, 1]$. One can verify that after taking geometric realization $\epsilon \circ q_x \psi : S^2 \to B[G, G]$ classifies the same bundle as the clutching map

$$S^1 \xrightarrow{\chi} T \xrightarrow{\psi(q)} [G, G]_0 \xrightarrow{(-)^{-1}} [G, G]_0,$$

for example by [16, Lemma 3.3]. In other words, the class of $\epsilon \circ q_x \psi$ is mapped to the class of $(-)^{-1} \circ \psi(q) \circ x$ under the isomorphism $\pi_2(B[G, G]) \xrightarrow{\sim} \pi_1([G, G]_0)$. Now let $\{x_i : S^1 \to T\}$ be a set of generators of $\pi_1(T)$. Since $\sum_i \text{im} \psi(q)_* \circ x_i = \text{im} \psi(q)_*$ for every $q \in Q$, our previous argument and Lemma 4 part 2 imply that

$$\text{im} \ c_* \circ \left( \sum_{q \in Q, x} (q, x, \psi)_* \right)$$

is of maximal rank and hence $\text{im} \ c_*$ is of maximal rank, as well. \hfill \square

The space $B(2, G)$ is equipped with an involution $\phi^{-1} : B(2, G) \to B(2, G)$ given by the realization of the maps $\phi_k^{-1} : C_k(G) \to C_k(G)$ that invert each coordinate. This involution is closely related to the commutator map (see [6, Remark 11]), which we will need to prove the following.

**Proposition 4** Let $G$ be an extension of a finite group $Q$ by a torus $T$, in which $[G, G]_0$ consists of single commutators. Then $c_* : \pi_2(E(2, G)) \to \pi_2(B[G, G])$ is surjective.

**Proof** Choose a group decomposition of $[G, G]_0$ into a product of $S^1$'s. Similarly as in the proof of Lemma 4 part 2, we can write any single commutator as a product of two $\psi(q)$'s. Thus, for each $i$-th projection $\text{proj}_i : [G, G]_0 \to S^1$ there is a product of commutators $\psi(q_i)(-\psi(p_i)(-)$ and some circle subgroup $S^1_i \subset T \times T$ such that the image $\psi(q_i)(-\psi(p_i)(-)(S^1_i) = \text{proj}_i([G, G]_0)$. In particular the composition

$$S^1_i \hookrightarrow T \times T \xrightarrow{\psi(q_i)(-\psi(p_i)(-)} [G, G]_0 \xrightarrow{\text{proj}_i} S^1$$

yields a degree $n$ map for some $n \in \mathbb{Z}$, as it is a non-trivial continuous endomorphism of $S^1$. By post composing (5) with $(-)^{-1}$ if necessary, we may assume $n > 0$. Then by an appropriate parametrization of $S^1 \subset T \times T$, say $z \mapsto (x(z), y(z))$, the image of the $n$-th root of unity $\xi_n = e^{\frac{2\pi n}{n}}$ will satisfy $[px(\xi_n), qy(\xi_n)] = 1$. Parametrize $C_r \cap C_s$ by $C_{rs}(t)$, $t \in [0, 1/n]$. Then define $a$ as

$$a_{12}(C_{12}(t)) = px(t)qy(t)$$

$$a_{23}(C_{23}(t)) = (qy(t))^{-1}$$

$$a_{13}(C_{13}(t)) = px(t)$$
which is a commutative cocycle by construction. We can represent the associated clutching functions as in the following pictures:

\[
\begin{align*}
px(t) & \quad t = \frac{1}{n} \\
\varphi_\alpha & \quad \text{and} \\
px(t) & \quad t = 0
\end{align*}
\]

It follows that \( \varphi_\alpha \) is null-homotopic, and \( \varphi_{\alpha^{-1}} \) is homotopic to (5). Now since \( f_\alpha : S^2 \to B(2, G) \) is null, it induces a class in \( \pi_2(E(2, G)) \).

Consider the diagram

\[
\begin{array}{ccc}
E(2, G) & \xrightarrow{c} & B[\mathcal{G}, \mathcal{G}] \\
\downarrow & & \downarrow j \\
E(2, G) & \xrightarrow{\phi^{-1}} & B(2, G) \xrightarrow{i} BG,
\end{array}
\]

which commutes up to homotopy (16, Remark 11).

Since \( T/[G, G]_0 \) and \( T \) are tori, the inclusion \( [G, G]_0 \to T \) induces a splitting \( \pi_1(T) \cong \pi_1([G, G]_0) \oplus \pi_1(T/[G, G]_0) \). We claim that \( \pi_2(i\phi^{-1})(f_\alpha) = \pi_2(if_{\alpha^{-1}}) \) lies in \( j_*\pi_2(\mathcal{G}[\mathcal{G}, \mathcal{G}]) \). Let \( \varphi_{\alpha^{-1}} : S^1 \to G \) be the associated clutching map of \( \alpha^{-1} \) depicted in (6). By construction \( (-)^n \circ \varphi_{\alpha^{-1}} \) is homotopic to (5), thus \( n[\varphi_{\alpha^{-1}}] \in \pi_1([G, G]_0) \), which implies \( [\varphi_{\alpha^{-1}}] \in \pi_1([G, G]_0) \) as \( \pi_1([G, G]_0) \) is a direct summand and \( \pi_1(T/[G, G]_0) \) is torsion free. Our claim now follows.

We conclude that the composite \( \pi_2(E(2, G)) \xrightarrow{c_*} \pi_2(B[\mathcal{G}, \mathcal{G}]) \xrightarrow{\text{proj}_*} \pi_2(BS^1) \) is surjective for every \( i \)-th projection, so \( c_* \) is surjective, as well.

**Example 1** A generic example of an extension where \([G, G]_0\) consists of single commutators is when \( G \) is an extension by \( T = S^1 \) (e.g. \( G = O(2) \)). Recall that \( \text{Aut}(S^1) = \mathbb{Z}/2 \), then since \( T \) is normal in \( G \), either \( T \) is fixed by any lift of \( \pi_0(G) \) in \( G \), in which case \( T \subset Z(G) \) and \([G, G]_0\) is finite, or there exists a lift \( a \) that acts by inversion on \( T \). In the latter case we can write \( z \in T \) as \( [\sqrt{-1}, a], \) and \([G, G]_0 = T \).

**Example 2** Let \( G \) be a semisimple connected compact Lie group, and \( T \) a maximal torus. It is known that the normalizer \( N_G(T) \) satisfies the hypothesis of Proposition 4. To see this, let \( \text{Lie}(T) \) be the Lie algebra of \( T \). Choose a basis of \( \text{Lie}(T)^* \) consisting of simple roots \( \{\alpha_i\} \). To each \( \alpha_i \) there is an associated coroot \( \alpha_i^\vee \) in \( \text{Lie}(T) \). These elements induce a Lie group homomorphism

\[
\exp(\alpha_i^\vee) : \text{span}(\alpha_i^\vee) \hookrightarrow \text{Lie}(T) : S^1 \to T
\]

such that \( \prod_i \exp(\alpha_i^\vee) : (S^1)^m \to T \) is an isomorphism, where \( m \) is the rank of \( T \). The Weyl group of \( G \) is then generated by the reflections \( s_i \) through the hyperplane \( \ker(\alpha_i) \) which is also orthogonal to \( \alpha_i^\vee \) for all \( 1 \leq i \leq m \). Therefore for each \( i \) the action of \( s_i \) on \( \exp(\alpha_i^\vee)(S^1) \) is given by inversion, and as in Example 1 we can now write any element in \( T \) as a single commutator. In particular \( T = [N_G(T), N_G(T)]_0 \). This is a way of proving that any element in a compact connected semisimple Lie group consists of single commutators [12].
We finish this subsection with an application concerning the splitting of the commutator map after looping [6, Question 21].

**Corollary 3** Let $G$ be an extension of a finite group by a torus, where $[G, G]_0$ consists of single commutators. Then $\Omega c : \Omega E(2, G) \to [G, G]$ splits, up to homotopy.

**Proof** The commutator subgroup is a disjoint union of copies of $[G, G]_0$ indexed by representatives $a$ of each element in $\pi_0(G, G)$, that is,

$$[G, G] = \bigsqcup_{[a] \in \pi_0(G, G)} a[G, G]_0.$$  

To define a splitting $s_1$ at the component of the identity we only need to specify maps $s_1^i : S^1 \to \Omega_0 E(2, G)$ that composed with

$$\Omega_0 E(2, G) \xrightarrow{\Omega c} [G, G]_0 \xrightarrow{\text{proj}} S^1$$

are the identity, up to homotopy. But by Proposition 4, $\pi_1(\Omega c)$ is surjective, hence we have a section $\sigma_1$ (which in fact can be chosen as a group homomorphism since $\pi_1([G, G]_0)$ is free abelian). Thus we can define $s_1^i := \sigma_1[i_i]$ for each $i$-th generator $S^1 \xrightarrow{i_i} [G, G]_0$ in $\pi_1([G, G]_0)$.

To extend $s_0$ to $\Omega E(2, G)$ we use [6, Proposition 9] which asserts that $\pi_0(\Omega c)$ has a section $\sigma$. Then we extend on each $a[G, G]$ by picking an element $\sigma_0(a) \in \sigma[a]$ for every $[a] \in \pi_0(G, G)$. Then we can set $s = \sigma_0 s_0$. \qed

### 4 Proof of Theorem 1 and applications

We have all the ingredients to prove our main result, but we still have to pin down generators of $\pi_2(B[G, G])$. For that we require the following lemma.

**Lemma 5** Let $G$ be a compact Lie group. Then there exists a finite subgroup $F \subset G$, such that $[G, G]_0$ is isomorphic to the central product

$$[G, G]_0 \cong [F, Z(G_0)_0] \times_Z [G_0, G_0],$$

where $Z = [F, Z(G_0)_0] \cap [G_0, G_0]$. In particular there is a split short exact sequence

$$0 \to \pi_1([G_0, G_0]) \to \pi_1([G, G]_0) \to \pi_1([F, Z(G_0)_0]/Z) \to 0$$

induced by the inclusion $[G_0, G_0] \to [G, G]_0$ and the projection $[G, G]_0 \to [F, Z(G_0)_0]/Z$.

**Proof** As in the proof of Lemma 4 part 1, we can find a finite group $F \subset G$ such that $G = FG_0$, and [9, Chapter IX, Section 1.4, Corollary 1] applied to $G_0$ then yields $G = FZ(G_0)_0[G_0, G_0]$. Since $[G_0, G_0]$ is normal in $G$ we obtain that

$$[G, G] = [F, F][F, Z(G_0)_0][G_0, G_0].$$

As $Z(G_0)_0$ is a characteristic subgroup of $G_0$ it is normal in $G$, hence $[F, Z(G_0)_0] \subset Z(G_0)_0$ is a closed and connected subgroup, and therefore a torus. Moreover, since $[G_0, G_0] \subset [G, G]_0$ is also a connected subgroup, we obtain $[G, G]_0 = [F, Z(G_0)_0][G_0, G_0]$. Then as in the proof of Lemma 1 we have an isomorphism $[G, G]_0 \cong [F, Z(G_0)_0] \times Z [G_0, G_0]$, where $Z = Z([G_0, G_0]) \cap [F, Z(G_0)_0]$. In particular we have a short exact sequence $0 \to \pi_1([G_0, G_0]) \to \pi_1([G, G]_0) \to \pi_1([F, Z(G_0)_0]/Z) \to 0$ obtained from the fibration.
sequence $[G_0, G_0] \rightarrow [F, Z(G_0)_0] \times Z [G_0, G_0] \rightarrow [F, Z(G_0)_0]/Z$. Since $[F, Z(G_0)_0]/Z$ is a quotient of a torus by a closed subgroup, it is a torus, as well. As $\pi_1([F, Z(G_0)_0]/Z)$ is a free abelian group and $\pi_1([G, G]_0)$ is abelian, the sequence splits. 

Thus we can study separately the elements in $\pi_1([G_0, G_0])$ which are all torsion (as $[G_0, G_0]$ is semisimple), and the elements in $\pi_1([F, Z(G_0)_0]/Z)$ which is a finitely generated free abelian group. In both cases these elements arise from extensions of finite groups by tori.

**Proof of Theorem 1** We divide the proof of the theorem into 3 claims.

**Claim 1** $\pi_2(c)$ is an isomorphism when $G$ is connected.

**Proof** The existence of an isomorphism $\pi_2(E(2, G)) \cong \pi_1([G, G])$ has already been showed in Proposition 2, thus to conclude $\pi_2(c)$ is an isomorphism, we only need to show it is surjective, as in this situation $[G, G]$ is semisimple, so $\pi_1([G, G])$ is finite. We have the following commutative diagram

$$
\begin{array}{ccc}
E(2, [G, G]) & \xrightarrow{c} & B[G, G] \\
\cong & & \\
E(2, G) & \xrightarrow{c} & B[G, G],
\end{array}
$$

where the left vertical map is a homotopy equivalence by Lemma 1. So we can further assume that $G$ is semisimple.

Recall that if $T$ is a maximal torus of $G$, the flag manifold $G/T$ is simply-connected (as it has a CW structure with only even dimensional cells). Then the long exact sequence of homotopy groups for the fiber sequence $G/T \rightarrow BT \rightarrow BG$ gives a surjection $\pi_2(BT) \rightarrow \pi_2(BG)$.

Let $N = N_G(T)$. We have the following commutative diagram:

$$
\begin{array}{ccc}
E(2, N) & \xrightarrow{c} & B[N, N] \\
\downarrow & & \\
E(2, G) & \xrightarrow{c} & BG.
\end{array}
$$

As explained in Example 2, when $G$ is semisimple, $T = [N, N]_0$ and $T$ consists of single commutators. We can apply Proposition 4 to see that the top horizontal map is surjective on $\pi_2$. Since the inclusion $BT \rightarrow BG$ factors through $B[N, N] \rightarrow BG$, the right vertical arrow is surjective in $\pi_2$, implying the bottom horizontal arrow is surjective in $\pi_2$, as well. Then by Proposition 2, $\pi_2(c)$ must be an isomorphism, as $\pi_1(G)$ is a finite group when $G$ is semisimple.

In the general case, Lemma 5 gives us the splitting $\pi_2(B[G, G]) \cong \pi_2(B[G_0, G_0]) \oplus A$ where $A$ is a finitely generated free abelian group. Since $[G_0, G_0]$ is semisimple, $\pi_2(B[G_0, G_0])$ is precisely the torsion part of $\pi_2(B[G, G])$.

**Claim 2** $\pi_2(c) : \pi_2(E(2, G)) \rightarrow \pi_2(B[G, G])$ is a split surjection onto $\pi_2(B[G_0, G_0])$. 

\[ \square \]
\textbf{Proof} Consider the inclusion \(i: G_0 \rightarrow G\), and let \(\epsilon_0: E(2, G_0) \rightarrow B[G_0, G_0]\) be the commutator map specialized to \(G_0\). We have the commutative diagram

\[
\begin{array}{ccc}
\pi_2(E(2, G_0)) & \xrightarrow{\pi_2(\epsilon_0)} & \pi_2(B[G_0, G_0]) \\
\pi_2(E(2, i)) & \downarrow & \pi_2(B[i, i]) \\
\pi_2(E(2, G)) & \xrightarrow{\pi_2(c)} & \pi_2(B[G, G]).
\end{array}
\]

By Claim 1 \(\pi_2(\epsilon_0)\) is an isomorphism. Hence by Lemma 5, the composite \(\pi_2(B[i, i]) \circ \pi_2(\epsilon_0)\) is an isomorphism to its image \(\pi_2(B[G_0, G_0]) \subset \pi_2(B[G, G])\). It follows that \(\pi_2(E(2, i))\) is injective and \(\pi_2(c)\) is split surjective onto \(\pi_2(B[G_0, G_0])\), as claimed. \(\square\)

\textbf{Claim 3} The image of \(\pi_2(c)\) is of maximal rank.

\textbf{Proof} Let \(F \subset G\) be a finite subgroup as in Lemma 5, so that \([G, G]_0 \cong [F, Z(G_0)_0] \times Z[G_0, G_0]\). From the long exact sequence of homotopy groups of the fiber sequence \([F, Z(G_0)_0] \rightarrow [G, G]_0 \rightarrow [G_0, G]/Z\) we can conclude that \(\pi_1([F, Z(G_0)_0]) \hookrightarrow \pi_1([G, G]_0)\) is a subgroup of maximal rank, since \(\pi_1([G_0, G]/Z) = 0\) and \(\pi_1([G_0, G]/Z)\) is finite. Consider the extension

\[1 \rightarrow Z(G_0)_0 \rightarrow FZ(G_0)_0 \rightarrow F/F \cap Z(G_0)_0.\]

Let \(j: FZ(G_0)_0 \rightarrow G\) be the inclusion. As the path-connected component of the identity of the commutator subgroup of \(FZ(G_0)_0\) is \([F, Z(G_0)_0]\), we have the commutative diagram

\[
\begin{array}{ccc}
\pi_2(E(2, FZ(G_0)_0)) & \xrightarrow{\pi_2(\epsilon_F)} & \pi_2(B[F, Z(G_0)_0]) \\
\pi_2(E(2, j)) & \downarrow & \pi_2(B[j, j]) \\
\pi_2(E(2, G)) & \xrightarrow{\pi_2(c)} & \pi_2(B[G, G]).
\end{array}
\]

where \(\epsilon_F\) is the commutator map specialized to \(FZ(G_0)_0\). By Proposition 3 \(\text{im} \pi_2(\epsilon_F)\) is of maximal rank and as noticed above \(\text{im} \pi_2(B[j, j])\) is also of maximal rank. From commutativity of the diagram it follows that \(\text{im} \pi_2(c)\) is of maximal rank, as well. \(\square\)

Now we complete the proof of the theorem. Claim 1 is precisely part 2 of the theorem. By claims 2 and 3, the image of \(\pi_2(c)\) is a subgroup of the form \(A' \oplus \pi_2(B[G_0, G_0]) \subset \pi_2(B[G, G])\), where \(A'\) is a free abelian subgroup of maximal rank, and \(\pi_2(B[G_0, G_0])\) is the torsion part of \(\pi_2(B[G, G])\). In particular \(A' \oplus \pi_2(B[G_0, G_0])\) is isomorphic to \(\pi_2(B[G, G])\). Combining Claim 2 and the fact that \(A'\) is free abelian, we can guarantee the existence of a split surjective homomorphism \(\pi_2(E(2, G)) \rightarrow A' \oplus \pi_2(B[G_0, G_0])\), and part 1 of the theorem now follows. \(\square\)

We record some applications of Theorem 1. First we prove Corollary 1.

\textbf{Proof of Corollary 1} Suppose \(E(2, G)\) is 2-connected. In [6, Proposition 9] it is shown that \(\pi_2(c)\) is surjective, thus if \(\pi_1(E(2, G)) = 1, [G, G]\) is connected. Part 1 of Theorem 1 implies that \(\pi_2(B[G, G]) = \pi_1([G, G]) = 0\). \(\square\)

An equivalent formulation of a transitionally commutative (tc) structure on a principal \(G\)-bundle, as defined in [13], is that its classifying map is a factorization, up to homotopy, through the inclusion \(i: B(2, G) \rightarrow BG\). Consider the involution \(\phi^{-1}: B(2, G) \rightarrow B(2, G)\). Then any principal \(G\)-bundle with a tc structure has an associated involution. Let us denote \(p: E(2, G) \rightarrow B(2, G)\) the pullback of \(EG \rightarrow BG\) along \(i\).
Lemma 6 Let $G$ be a well based topological group and suppose $E$ is a principal $G$-bundle over a space $X$ with a reduction to the commutator subgroup of $G$. If its classifying map $f: X \to B[G, G]$ factors through the commutator map $c$, up to homotopy, then $E$ has a transitionally commutative structure whose involution is a trivial bundle.

Proof Consider the diagram

$$
\begin{array}{cccc}
E(2, G) & \phi^{-1}p & B(2, G) \\
\downarrow f' & & \downarrow \\
X & \phi & B[G, G] \downarrow j & BG.
\end{array}
$$

By [6, Remark 11] the square on the right commutes up to homotopy. The composite $\phi^{-1}pf'$ is a tc structure with associated involution $pf''$ which after composing with $i$ is null-homotopic.

\[\Box\]

Part 2 of Theorem 1 in particular implies that when $G$ is connected, the homotopy fiber of $c$ is 2-connected. A standard obstruction theory argument and Lemma 6 show the following:

Corollary 4 Let $X$ be a CW complex of dimension $\leq 3$, and let $G$ be a connected compact Lie group. Then every principal $G$-bundle over $X$ with a reduction to the commutator subgroup has a transitional commutative structure whose involution is a trivial bundle.

For example, any oriented vector bundle over a CW complex of dimension $\leq 3$, will posses a tc structure.

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