BBGKY kinetic approach for an $e^-e^+\gamma$ plasma created from the vacuum in a strong laser-generated electric field: The one-photon annihilation channel

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In the present work a closed system of kinetic equations is obtained from the truncation of the BBGKY hierarchy for the description of the vacuum creation of an electron - positron plasma and secondary photons due to a strong laser field. This truncation is performed in the Markovian approximation for the one-photon annihilation channel which is accessible due to the presence of the strong external field. Estimates of the photon production rate are obtained for different domains of laser field parameters (frequency $\nu$ and field strength $E$). A huge quantity of optical photons of the quasiclassical laser field is necessary to satisfy the conservation laws of the energy and momentum of the constituents ($e^-, e^+$ and $\gamma$) in this channel. Since the number of these optical photons corresponds to the order of perturbation theory, a vanishingly small photon production rate results for the optical region and strongly subcritical fields $E \ll E_c$. In the $\gamma$-ray region $\nu \lesssim m$ the required number of laser photons is small and the production rate of photons from the one-photon annihilation process becomes accessible to observations for subcritical fields $E \lesssim E_c$. In the infrared region the photon distribution has a $1/k$ spectrum typical for flicker noise.

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I. INTRODUCTION

The Schwinger effect [1] of electron-positron pair (EPP) production off the vacuum under the action of strong electromagnetic fields is one of the few QED effects that has not yet been tested experimentally. This is due to the huge electric fields $E \sim E_c = m^2/e = 1.3 \cdot 10^{16}$ V/cm necessary to observe the EPP production effect in a constant external field. Such field strengths are unachievable for static fields. Therefore, the main attention was devoted to the theoretical and experimental study of pair creation by time-varying electric fields generated either in the focal spot of high-power lasers [2, 3] or ultraperipheral heavy-ion collisions [3, 4]. Below we will concentrate our attention to EPP creation in strong laser fields. The estimations made before in Refs. [2, 3, 5, 10, 12] showed that pair creation by a single optical laser pulse with $E \ll E_c$ could hardly be observed. More optimistic results have been obtained for planned X-ray free electron lasers (XFEL’s) [14, 16] and for counter-propagating laser beams in the optical range [17, 21].

It is obvious that for subcritical fields $E \ll E_c$ the electron - positron excitations have the character of short-lived quasiparticles which are not observable after the laser signal ceases. In essence, it is a vacuum polarization effect. Therefore, the $S$ - matrix methods cannot be used [22] and existing estimates [10] are not reliable. An adequate method is provided by the kinetic theory. Only on this basis the different experimental manifestations (creation and radiation of annihilation photons [23], generation of harmonics by focussed laser beams [24], birefrigency [25], etc. [21, 24, 28]) find a proper explanation. Such a kinetic approach has been introduced in [29], for a recent review see [30]. However, in these works the photon sector was not yet included consistently into the quantum kinetic approach.

In the present work we develop a general kinetic approach where both, the Dirac and the photon field are consistently included. We base our approach on the BBGKY hierarchy of kinetic equations for the electron - positron - photon system generated from the vacuum under the action of a time dependent electric field. As a first step, we will consider the one-photon annihilation process of the quasiparticle EPP, which in the presence of a strong field is not forbidden [31]. Estimates of the photon production rate for this case constitute the main contents of the present work.

The treatment of the two-photon annihilation channel is the next step in the development of the quantum kinetic description on the basis of the BBGKY hierarchy for the electron - positron - photon system. The detailed treatment of this important class of processes is delegated to a subsequent paper.

Below we will assume an external electric field with the 4 - potential (in the Hamiltonian gauge) $A^\mu(t) = (0, A(t))$ that is spatially homogeneous. It is expected that a similar field can be realized experimentally, e.g., as a standing wave in the small spatial domain of the focal spot of crossed laser beams. The kinetics of quasiparticle EPP creation in vacuum has been investigated...
in detail for the case of a linearly polarized laser field within a non-perturbative approach. In the case of a rather strong external field $A_{\text{ext}}(t)$ some internal field $A_{\text{int}}(t)$ will be generated too. The total acting field will be equal to $A(t) = A_{\text{int}}(t) + A_{\text{ext}}(t)$, and this field is quasiclassical. Fluctuations of the internal electromagnetic field on this background can lead to photon excitations that, in principle, can be registered outside the active zone of the focal spot.

This article is organized as follows. In Sect. II the system of kinetic equations (KE) in the quasiparticle representation of the electron - positron subsystem in a strong time dependent electric field are given. The KE’s for the distribution functions $f_{\gamma}(p,t)$ of the photon and ferminon KE’s (see below), in principle, the quasiparticle annihilation photons is presented in Sect. III and the first equation of the BBGKY chain is obtained. In Sect. IV the truncation of the BBGKY hierarchy of equations on the level of the one-photon annihilation process is performed with account of vacuum polarization effects. We intend to clearly state all approximation steps and therefore indicate them in the text by “Approximation 1” ... “Approximation 6” in brackets at the place of their introduction. The spectrum of annihilation photons is investigated here for the large (multiphoton processes) and small (strong fields) values of the adiabaticity parameter. Estimates for the photon radiation rate show that it is very small in the case of optical lasers with parameter. Estimates for the photon radiation rate show that it is very small in the case of optical lasers with subcritical fields $E \ll E_c$ since a huge number of optical photons from the quasiclassical laser field is necessary here to overcome the energy gap. However, the effect becomes quite observable in the domain of strong fields $E \lesssim E_c$ and high frequencies (e.g., for the XFEL domain [34–36]). Finally, we summarize the basic results of this work in Sect. V.

We use the metric $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ and natural units $\hbar = c = 1$.

II. KINETIC EQUATIONS IN THE ELECTRON - POSITRON SECTOR

In general, the complete set of equations for a consistent kinetic description of the electron - positron - photon plasma consists of

- the KE’s for the distribution functions $f_{\alpha\beta}(p,t)$ of the electron and positron quasiparticle components in the presence of the strong quasiclassical total Maxwell field $A(t) = (0, 0, A(t))$;
- the KE for the distribution function of the photon component, and
- the Maxwell equation for the quasiclassical internal field $A_{\text{int}}(t)$.

We assume that the electro-neutrality condition holds

$$f_e(p,t) = f_p(-p,t) = f(p,t). \quad (1)$$

We start from the standard QED Lagrangian $\mathcal{L} = \mathcal{L}_{\text{qc}} + \mathcal{L}_I$ taking into account the interaction of the electron-positron Dirac spinor fields with a quasiclassical electromagnetic field in $\mathcal{L}_{\text{qc}}$ and with a quantized one in $\mathcal{L}_I$,

$$\mathcal{L}_{\text{qc}} = \frac{i}{2} (\bar{\psi}\gamma^\mu D_\mu \psi - (D^\mu \bar{\psi}) \gamma^\mu \psi) - m \bar{\psi}\psi, \quad (2)$$

$$\mathcal{L}_I = -\bar{\psi} \gamma^\mu \dot{A}_\mu \psi, \quad (3)$$

where $D_\mu = \partial_\mu + ieA_\mu(t)$. Thus, the spatially homogeneous nonstationary quasiclassical Maxwell field $A(t)$ is a background to the quantized photon field $\dot{A}_\mu(x) = (0, \dot{A}(x))$. We assume that the intensity of the quantized field is rather weak and its influence on the state of the system can be neglected. In other words, the electron-positron system plays the role of a photon source. The photon in - vacuum $|\text{in}\rangle$ is defined such that for this state $\langle \text{in} | \dot{A}_\mu(x) = 0$.

Below we will not consider the backreaction problem because for subcritical fields $E \ll E_c$ the internal field is negligible, and hence $E = E_{\text{ext}}(t)$.

Thus, the idea of the work is the following: With the fermion KE’s (see below), in principle, the quasiparticle distribution function $f_{\alpha\beta}^\pm(p,t)$ can be determined, taking into account the spin degrees of freedom. The separation of the quasiclassical Maxwell field allows to formulate a non-perturbative approach to the photon kinetics (Sect. III). In the present work, the photon spectral distribution will be considered mainly by including the one-photon annihilation process only (Sect. IV).

The kinetics of electron - positron vacuum pair creation under the action of a linearly polarized electric field has been studied in several works, (see, e.g., [41, 32, 33] and the references therein). The corresponding generalization to the case of an arbitrarily polarized time-dependent electric field was obtained in [32–36]. In the present work, we will use the oscillator representation [32] which leads to the nonstationary spinor basis $|\alpha\beta\rangle$.

$$u_1^+(p,t) = B(p)[\omega_+, 0, P^3, P^-],$$

$$u_2^+(p,t) = B(p)[0, \omega_+, P_-, -P^3],$$

$$v_1^+(p,t) = B(p)[-P^3, -P_-, \omega_+, 0],$$

$$v_2^+(p,t) = B(p)[P_-, P^3, 0, \omega_+] \quad (4)$$

with the usual orthonormalization conditions

$$u_{\alpha}^+(p,t)v_{\beta}(-p,t) = \frac{1}{\omega(p,t)} \delta_{\alpha\beta},$$

$$\bar{u}_{\alpha}(p,t)u_{\beta}(p,t) = \delta_{\alpha\beta},$$

$$\bar{v}_{\alpha}(p,t)v_{\beta}(p,t) = \delta_{\alpha\beta}. \quad (5)$$

1 In this representation the diagonalization of the Hamiltonian in the presence of an external field is achieved at once, without the canonical Bogoliubov transformation procedure.
where $\omega(p, t) = \sqrt{m^2 + P^2}$, $P = p - eA(t)$, $P_\perp = P^1 \pm iP^2$, $\omega_\perp = \omega + m$ and $B(p) = [2q_\omega]^{-1/2}$. The Hamiltonian has the diagonal form in this quasiparticle representation

$$H_f(t) = \sum_\alpha \int d^3p \, \omega(p, t)[a^\dagger \alpha(p, t)a_\alpha(p, t) - b_\alpha(-p, t)b^\dagger_\alpha(-p, t)].$$ (6)

Finally, the Dirac equation in the presence of the external quasiparticle field $A_{\text{ext}}(t)$ generates the Heisenberg-like equations of motion for the time-dependent creation and annihilation operators

$$\dot{a}(p, t) = -U_{(1)}(p, t)a(p, t) - U_{(2)}(p, t)b^\dagger(-p, t) - i\omega(p, t)a(p, t),$$
$$\dot{b}(-p, t) = b(-p, t)U_{(1)}(p, t) + a^\dagger(p, t)U_{(2)}(p, t) - i\omega(p, t)b(-p, t),$$ (7)

with the matrices in the representation being

$$U_{(1)}(p, t) = i\omega C(p)|P|E\sigma = iU\sigma,$$
$$U_{(2)}(p, t) = q\sigma = C(p)|P|E\psi - E\omega_\perp\sigma.$$ (8)

Here, $E(t) = -\dot{A}(t)$ is the strength of the external electric field for $|A_{\text{int}}| \ll |A_{\text{ext}}|$ and $C(p) = e/(2q_\omega)$. The matrices $\mathbf{5}$ and $\mathbf{6}$ describe the different vacuum effects in the presence of an external electric field (polarization, spin rotation, EPP creation).

The KE's for the electron - positron component of the plasma follow from the equations of motion and the definitions of the electron and positron distribution functions in the instantaneous representation $|4|$

$$f_{\alpha\beta}(p, t) = \langle a^\dagger \beta(p, t)a_\alpha(p, t) \rangle,$$
$$f^{\alpha\beta}(p, t) = \langle b_\beta(-p, t)b^\dagger_\alpha(-p, t) \rangle,$$ (10)

and also two additional functions

$$f_{\alpha\beta}^{(+)}(p, t) = \langle a^\dagger_\beta(p, t)b^\dagger_\alpha(-p, t) \rangle,$$
$$f_{\alpha\beta}^{(-)}(p, t) = \langle b_\beta(-p, t)a_\alpha(p, t) \rangle,$$ (11)

describing vacuum polarization. The KE system in matrix notation is then

$$\dot{f} = [f, U_{(1)}] - \left( U_{(2)}f^{(+)} + f^{(-)}U_{(2)} \right),$$
$$\dot{f}^c = \left( f^c, U_{(1)} \right) + \left( f^{(+)}U_{(2)} + U_{(2)}f^{(-)} \right),$$
$$\dot{f}^{(+)} = \left( f^{(+)}, U_{(1)} \right) + \left( U_{(2)}f - fU_{(2)} \right) + 2i\omega f^{(+)}$$
$$\dot{f}^{(-)} = \left( f^{(-)}, U_{(1)} \right) + \left( fU_{(2)} - U_{(2)}f^c \right) - 2i\omega f^{(-)}.$$ (12)

If the standard decomposition in the basis of Pauli matrices is used ($k = 1, 2, 3$)

$$f = f_0 + f_k\sigma_k,$$ (13)

where $f_0 = \frac{1}{2}\text{Tr}f$, $f_k = \frac{1}{2}\text{Tr}f\sigma_k$, the KE's (12) can be rewritten in the spin representation, where the first set of equations concerns the spinless distributions

$$\dot{f}_0 = -2qu,$$
$$\dot{f}_0^c = 2qu,$$
$$u_0 = 2\omega_0 + (f - f^c)q,$$
$$v_0 = -2\omega_0,$$ (14)

and the KE's for the spin distribution functions are collected in the second set

$$\dot{f}_k = -2u_0q_k - 2[fU]_k + 2[vq]_k,$$
$$\dot{f}_k^c = 2u_0q_k - 2[f^cU]_k - 2[vq]_k,$$
$$\dot{u}_k = 2\omega_k - 2[iU]_k + (f_0 - f_0^c)q_k,$$
$$\dot{v}_k = -2\omega_k - 2[iU]_k + [(f + f^c)q]_k.$$ (15)

Some simple applications of this system of KE's can be found in [34–36].

In the simplest case of the linearly polarized photon field $A(t) = (0, 0, A(t))$ the system of equations (14) is transformed into a coupled system of three ordinary first-order differential equations

$$\dot{f} = \frac{1}{2}\lambda u,$$
$$\dot{u} = \lambda(1 - 2f) - 2\omega v,$$
$$\dot{v} = 2\omega u,$$ (16)

where $\lambda(p, t) = eE(t)\varepsilon_\perp/\omega^2$, $\varepsilon_\perp(p) = \sqrt{m^2 + p_\perp^2}$ and $p_\perp = (p_1, p_2, 0)$. The system (16) corresponds to a KE of the non-Markovian type

$$f(p, t) = \frac{1}{2}\lambda(p, t)\int_{t_0}^{t} dt'\lambda(p, t')[1 - 2f(p, t')]$$
$$\cos \left[ 2\int_{t'}^t d\tau\omega(p, \tau) \right].$$ (17)

This KE and its representation (16) have been used in numerous applications, see [36].

The further development of the kinetic theory approach to vacuum particle creation is progressing in search of different exact and approximate solutions of the problem (see, e.g., [37–40]) and also in extending the possibilities of the formalism [41].

III. PHOTON SECTOR

A. The basic equations

An external electric field that generates an instability of the vacuum with respect to electron - positron pair creation is accompanied by the appearance of internal currents and electromagnetic fields (back reaction problem). Quantum fluctuations of this internal field are interpreted
as photon excitations, which can leave the active zone (focal spot) and can be registered experimentally. Below the KE for the description of the photon component will be obtained and investigated for some simple situations. For the construction of the photon kinetics it is necessary to develop the corresponding generalization of the quasiparticle formalism developed in Sect. II using it as a nonperturbative basis.

Let us introduce at first the interaction with the quantized electromagnetic field \( A_\mu(x) \) in the fermion sector of the theory by means of the substitution \( H_f \rightarrow H_f + H_{\text{int}} \) in the Heisenberg-like equation of motion \( (7) \).

\[
\dot{a}(p, t) + U(1)(p, t)a(p, t) + U(2)(p, t)b^+(p, t) = -i[a(p, t), H_f + H_{\text{int}}]
\]

\[
b(-p, t) - b(-p, t)U(1)(p, t) + a^+(p, t)U(2)(p, t) = -i[b(-p, t), H_f + H_{\text{int}}],
\]

where \( H_f \) is the Hamiltonian \( (6) \) of the fermion field in the quasiparticle representation and \( H_{\text{int}} \) is the usual Hamiltonian of the interaction with the quantized field

\[
H_{\text{int}}(t) = e \int d^3x : \bar{\psi}(x) \gamma^\mu \hat{A}_\mu(x) \psi(x) : .
\]

The time dependence of \( H_{\text{int}}(t) \) is due to the nonstationarity of the system that is reflected in the characteristic oscillator representation of the decomposition of the field operators \( \psi(x), \psi^\dagger(x) \) in the nonstationary basis \( 41, 45 \).

The same source (external field) generates the nonstationarity of the quantized electromagnetic field. However, this does not induce an alteration of the mass shell of the photon field, \( k^\mu k_\mu = 0 \) (in contrast to the electron - positron field, \( \omega(k, t) \)), and then the standard decomposition is valid,

\[
\hat{A}_\mu(x) = (2\pi)^{-3/2} \int \frac{d^3k}{\sqrt{2k}} A_\mu(k, t) e^{-ikx},
\]

where \( A_\mu(k, t) = A_\mu^+(k, t) + A_\mu^-(k, t) \) with the condition \( \{A_\mu^-(k, t)\}^+ = A_\mu^+(k, t) \) and the standard commutation relations

\[
[A_\mu^-(k, t), A_\mu^+(k', t)] = -g_{\mu\nu} \delta(k - k') .
\]

The Hamiltonian of the interaction \( (19) \) in the oscillator representation has the form

\[
H_{\text{int}}(t) = e(2\pi)^{-3/2} \sum_{\alpha\beta} \int d^3p_1 d^3p_2 \frac{d^3k}{\sqrt{2k}} \\
\delta(p_1 - p_2 + k) \\
\{ [\tilde{u}v]_\alpha^\dagger(p_1, p_2, k; t)a_\alpha^+(p_1, t)a_\beta(p_2, t) + [\tilde{u}v]_\beta^\dagger(p_1, p_2, k; t)b_\alpha^+(p_1, t)b_\beta(p_2, t) + [\tilde{u}v]_\alpha(p_1, p_2, k; t)b_\alpha(-p_1, t)a_\beta(p_2, t) + [\tilde{u}v]_\beta(p_1, p_2, k; t)b_\alpha(-p_1, t)b_\beta(-p_2, t) \} A_\nu(k, t) .
\]

Here and below the vectors \( p_1, p_2, \ldots \) are used for notation of the canonical momenta of fermions and \( k_1, k_2, \ldots \).

\[ [\tilde{\eta}]^\dagger_{\beta\alpha}(p_1, p_2, k; t) = \tilde{\eta}_\alpha(p_1, t)\gamma^\mu \eta_\beta(p_2, t)e^\mu_\nu(k), \]

which correspond to the momenta of photons. The spinor convolutions are

\[
A_\mu^+(k, t) = \sum_{r=0}^3 e^r_{\nu} A_r^+(k, t),
\]

and has been used here.

The Hamiltonian of the free photon field is given by

\[
H_{\phi b}(t) = \sum_{r=1, 2} \int d^3k k A_r^+(k, t) A_r^-(k, t),
\]

The system of Heisenberg-like equations of motion with account of the photon subsystem \( 13 \) can be writ-
ten now in explicit form. For example,

$$\dot{a}(p, t) = -i\omega(p, t)a(p, t) - U_{(2)}(p, t)a(p, t)$$

$$-U_{(2)}(p, t)b^+(p, t)$$

$$+ie(2\pi)^{-3/2}\int d^3p_1\frac{d^3k}{\sqrt{2k}}\delta(p - p_1 + k)$$

$$\left\{a(p_1, t)[\overline{\bar{u}}]^\dagger(p_1, p_1, k; t)$$

$$+b(-p_1, t)[\overline{\bar{w}}]^\dagger(p, p_1, k; t)\right\}A_r(k, t)$$.

(26)

$$\dot{b}(-p, t) = -i\omega(p, t)b(-p, t) + b(-p, t)U_{(1)}(p, t)$$

$$+a^+(p, t)U_{(2)}(p, t)$$

$$-ie(2\pi)^{-3/2}\int d^3p_1\frac{d^3k}{\sqrt{2k}}\delta(p_1 - p + k)$$

$$\left\{[\overline{\bar{u}}]^\dagger(p_1, p, k; t)a(p_1, t)$$

$$+[\overline{\bar{w}}]^\dagger(p_1, p, k; t)b(-p_1, t)\right\}A_r(k, t)$$.

(27)

$$iA_r^{(+)}(k, t) = \mp kA_r^{(-)}(k, t)$$

$$+e(2\pi)^{-3/2}\frac{1}{\sqrt{2k}}\int d^3p_1d^3p_2\delta(p_1 - p_2 + k)$$

$$\left\{a^+(p_1, t)[\overline{\bar{u}}]^\dagger(p_1, p_2, k; t)a(p_2, t)$$

$$+a^+(p_1, t)[\overline{\bar{w}}]^\dagger(p_1, p_2, k; t)b(-p_2, t)$$

$$+b(-p_1, t)[\overline{\bar{w}}]^\dagger(p_1, p_2, k; t)a(p_2, t)$$

$$+b(-p_1, t)[\overline{\bar{u}}]^\dagger(p_1, p_2, k; t)b(-p_2, t)\right\}$$.

(28)

This is the exact system of equations with nonperturbative account of the external electric field.

B. The first equation of the BBGKY hierarchy

We start from the single - time, two - point photon correlation function in momentum space

$$F_{r,r'}(k, k', t) = \langle A_r^{(+)}(k, t)A_{r'}^{(-)}(k', t)\rangle$$.

(29)

In the spatially homogeneous case it is diagonal in the momentum k and polarization r with the photon distribution function $F_t(k, t)$ as matrix element,

$$\langle A_r^{(+)}(k, t)A_{r'}^{(-)}(k, t)\rangle = \delta_{rr'}\delta(k - k')F_t(k, t)$$.

(30)

The diagonalization in the polarization indices is an approximation here (Approximation 1).

Then from the commutation relation (21) it follows

$$\langle A_r^{(-)}(k, t)A_r^{(+)}(k', t)\rangle = \delta_{rr'}\delta(k - k')\{1 + F_t(k, t)\}.$$  

(31)

Let us write now the first equation of the BBGKY hierarchy for the correlation function (29) using the photon equation of motion (28)

$$\dot{F}_{r,r'}(k, k', t) = i\frac{e(2\pi)^{-3/2}}{\sqrt{2k}}\int d^3p_1d^3p_2\left\{-\frac{1}{\sqrt{2k}}\delta(p_1 - p_2 - k)\right\}$$

$$\left\{[\overline{\bar{u}}]^\dagger_{\beta\alpha}(p_1, p_2, k; t)a^+_\beta(p_1, t)a^\alpha_{\beta}(p_2, t)A_{r'}^{(-)}(k', t)\right\}$$

$$+[\overline{\bar{w}}]^\dagger_{\alpha\beta}(p_1, p_2, k; t)a^+_\alpha(p_1, t)b^\beta_{\alpha}(p_2, t)A_{r'}^{(-)}(k', t)\right\}$$

$$+[\overline{\bar{w}}]^\dagger_{\beta\alpha}(p_1, p_2, k; t)b^\alpha_{\beta}(p_1, t)b^\beta_{\alpha}(p_2, t)A_{r'}^{(-)}(k', t)\right\}$$

$$+[\overline{\bar{u}}]^\dagger_{\alpha\beta}(p_1, p_2, k; t)b^\alpha_{\beta}(p_1, t)b^\beta_{\alpha}(p_2, t)A_{r'}^{(-)}(k', t)\right\}$$

$$+i(k - k')F_{r,r'}(k, k', t).$$

(32)

The last term on the r.h.s. of Eq. (32) describes the quantum beating of two-photon states and can be omitted in the approximation (30).

Thus, the kinetics of the photon states is defined by the different forced processes of either one-photon scattering of electron and positron (considered as quasiparticles) or their creation and annihilation. Some processes forbidden in absence of an external field become possible here: e.g., in the lowest order of the perturbation theory it is the one-photon annihilation, the simultaneous creation of an electron - positron pair and a photon (31). Apparently, the latter processes are strongly suppressed in the region of subcritical fields. In highest orders of perturbation theory the number of this kind of forbid-
den processes increases abruptly. In the general case, the necessity of taking into account such kind of processes strongly complicates the problem.

According to the approach given in Sect. II, the interaction of the photon subsystem with the electron-positron one can be taken into account in the framework of the standard perturbation theory with the fine-structure constant $\alpha = e^2/(4\pi)$ as an expansion parameter. This justifies the truncation of the BBGKY hierarchy for the correlators occurring in Eq. (32).

IV. ONE-PHOTON ANNihilation CHANNEL

A. Annihilation channel

Below we shall restrict the discussion to the annihilation channel only. Keeping the relevant terms in Eq. (32) ('incoming - outgoing' terms, the second line in the first square bracket in r.h.s. of Eq. (32) and the third line in the second bracket), we obtain

$$
\hat{F}_{rr'}(k, k', t) = \frac{i e(2\pi)^{-3/2}}{\sqrt{2k}} \int d^3p_1 d^3p_2 \left\{ \frac{1}{\sqrt{2k'}} \delta(p_1 - p_2 - k)[\bar{u}u]_{\alpha\beta}(p_1, p_2, k; t) \langle b_\beta^+(-p_2, t)a_\alpha^+(p_1, t)A_r^{(-)}(k', t) \rangle 
+ \frac{1}{\sqrt{2k'}} \delta(p_1 - p_2 + k')[\bar{u}u]_{\alpha\beta}(p_1, p_2, k'; t) \langle b_\alpha(-p_1, t)a_\beta(p_2, t)A_r^{+(+)}(k, t) \rangle \right\} . \quad (33)
$$

For obtaining a closed photon KE it is necessary to perform some truncation procedure for the correlators occurring in this equation. The simplest truncation of the type

$$
\langle b_\alpha(-p_1, t)a_\beta(p_2, t)A_r^{(\pm)}(k, t) \rangle \simeq \langle b_\alpha(-p_1, t)a_\beta(p_2, t) \rangle \langle A_r^{(\pm)}(k', t) \rangle = 0 \quad (34)
$$

is not effective due to the definition of the photon vacuum $\langle A_r^{(\pm)}(k, t) \rangle = 0$ (Sect. II).

The equations of the second order for the correlators from Eq. (33) can be obtained easily with help of the equations of motion \cite{26,28}, e.g.,

$$
\left\{ \frac{\partial}{\partial t} + i[\omega(p_1, t) + \omega(p_2, t) - k] \right\} \langle b_\alpha(-p_1, t)a_\beta(p_2, t)A_r^{(\pm)}(k, t) \rangle = -i e(2\pi)^{-3/2} \int d^3p' d^3k' \left\{ \delta(p' - p_1 + k') 
- \delta(p_2 - p' + k') \cdot \left[ [\bar{u}u]_{\alpha\beta}(p_1, p_2, k'; t) \langle b_\alpha(-p_1, t)a_\beta(p_2, t)A_r^{(\pm)}(k', t) \rangle \right] 
+ \langle b_\alpha(-p_1, t)a_\beta(p_2, t) \rangle \langle A_r^{(\pm)}(k', t) \rangle \right\} 
+ S_{\alpha\beta}^r(p_1, p_2, k; t) + U_{\alpha\beta}^r(p_1, p_2, k; t) . \quad (35)
$$

On the r.h.s. of this equation there is a set of the terms originating from vacuum polarization effects in the presence of the quantized electromagnetic field (Sect. II), which are absent in the standard QED without a strong field. The group of terms
can be omitted because of the approximation in order to close the chain of equations at this minimal level.

The other set of terms leads to contributions of the vacuum polarization effects in the one-photon annihilation channel.

\[ S_{\alpha\beta}^{\gamma}(p_1, p_2, k; t) = -ie(2\pi)^{-3/2} \frac{1}{\sqrt{2k}} \int d^3p_1'd^3p_2'd\delta(p'_1 - p_2 - k) \]

\[ \{ [\bar{u}u]_{\alpha'\beta'}^r(p_1', p_2', k; t)\bar{b}_\alpha(-p_1, t)a_\beta(p_2, t)\} + \{ [\bar{v}u]_{\alpha'\beta'}^r(p_1', p_2', k; t)\bar{b}_\alpha(-p_1, t)a_\beta(p_2, t)\} \]

As the result, only the first and the fourth term in Eq. (35) and the second term in Eq. (37) survive in this order of perturbation theory (Approximation 2).

The next approximation is the diagonalization of all one-particle correlation functions with respect to the momentum variables and spin indices,

\[ \langle a_\alpha'(p, t)a_\beta(p', t)A_{\alpha'}(k, t)A_{\beta'}^{(+)}(k, t) \rangle \simeq \langle a_\alpha'(p', t)a_\beta(p, t)\rangle \langle A_{\alpha'}(k, t)A_{\beta'}^{(+)}(k, t) \rangle , \]

\[ \langle b_\alpha(-p, t)a_\beta(p, t)\rangle \simeq \langle a_\alpha^\dagger(p, t)b_{\beta'}^{(+)}(p, t)\rangle \langle B_{\alpha'}(p, t)A_{\beta'}^{(+)}(p, t) \rangle . \]

These relations mean that spin effects are neglected. The analogous approximation was introduced for the photon correlation function, Eq. (39).

The processes of the instantaneous radiation of two photons have been omitted here, i.e., in Eq. (35) the substitution \( A_{\alpha'}(k', t) \rightarrow A_{\alpha'}(k', t) \) has been made (Approximation 3). In order to rewrite Eq. (35) in the integral form, let us perform the intermediate transition to the interaction representation

\[ \tilde{a}(p, t) = a(p, t) \exp \left\{ i \int_{t_0}^{t} dt' \omega(p, t') \right\} , \]

\[ \tilde{b}(p, t) = b(p, t) \exp \left\{ i \int_{t_0}^{t} dt' \omega(p, t') \right\} , \]

\[ \tilde{A}(\pm)(k, t) = A(\pm)(k, t) \exp \{ \mp ik(t - t_0) \} , \]

where \( t_0 \) is some initial time.

The approximations (38)-(41) allow to rewrite the anomalous correlation functions from the l.h.s. of Eq. (35) taking into account the vacuum polarization.
contribution so that
\[
\langle b_\alpha(-p_1,t)a_\beta(p_2,t)A^{(+)}_{\tau}(k,t) \rangle = \\
-\frac{i\epsilon\delta(p_2 - p_1 + k)}{\sqrt{2k(2\pi)^{3/2}}} \int dt' [\bar{u}v]_{\alpha\beta}(p_2, p_1, k; t') \\
\times \{ [f(p_1, t') + f(p_2, t') - 1][1 + F_\tau(k, t')] \\
+ [1 - f(p_1, t')][1 - f(p_2, t')] \} e^{-i\theta(p_1, p_2, k; t', t)} \\
+ c.c.,
\]
where it was used that \( f^c = 1 - f \) due to the electric charge neutrality of the vacuum at \( t \to -\infty \) and
\[
\theta(p_1, p_2, k; t', t) = \int_{t'}^t dt \left[ \omega(p_1, \tau) + \omega(p_2, \tau) - k \right].
\]

The first group of terms in the curly brackets in Eq. (43) corresponds to the one-photon annihilation process (this contribution was investigated in the work [43]) while the second group describes the radiationless vacuum fluctuations. In the case of a strong subcritical laser field the number density of the radiated photons is small, \( F_\tau(k, t) \ll 1 \) (Approximation 4), so that the influence of the photon reservoir on the photon emissivity of the system can be neglected. Eq. (43) then takes the form
\[
\langle b_\alpha(-p_1,t)a_\beta(p_2,t)A^{(+)}_{\tau}(k,t) \rangle = \\
-\frac{i\epsilon\delta(p_2 - p_1 + k)}{\sqrt{2k(2\pi)^{3/2}}} \int dt' e^{-i\theta(p_1, p_2, k; t', t)} \\
\times [\bar{u}v]_{\alpha\beta}(p_2, p_1, k; t') f(p_1, t')f(p_2, t') + c.c. .
\]

Substituting (45) into Eq. (46), we obtain a closed expression for the photon production rate
\[
\tilde{F}(k, t) = \frac{e^2}{4k(2\pi)^3} \int dt' \int d^3r e^{-i\theta(p, p + k, k; t', t)} \\
K(p, p + k, k; t, t') f(p, t')f(p + k, t') \\
+ c.c.,
\]
where we have introduced the two-time convolution with respect to spin and polarization indices
\[
K(p, p + k, k; t, t') = [\bar{u}v]_{\alpha\beta}(p, p + k, k; t) \\
[\bar{u}v]_{\alpha\beta}(p + k, k; t') .
\]

with the definitions (4) and (28).

Additionally, it is assumed in Eq. (46) that the photons have equiprobable distributions regarding their polarizations, \( F_1 = F_2 = F \).

Thus, the photon production rate is a nonlinear (quadratic) non-Markovian function with respect to the electron-positron distribution function \( f(p, t) \). This order in the nonlinearity corresponds to the result of the S-matrix approach.

The consequent estimation procedure of the integrals on the l.h.s. of Eq. (46) (method of photon count) was presented in (33) on the basis of the methods given in [44].

The meaning of these approximations is the following. On the r.h.s. of Eq. (46) there is a high frequency multiplier \( \exp(-i\theta) \) with the phase (44). In order to select the low frequency component of the photon production rate (46) (this corresponds to the averaged, observable value), it is necessary to compensate this high frequency phase by means of the higher harmonics in the Fourier decompositions of the other functions in the integral (46).

Indeed, the integrand in the expression (44) for the phase (mismatch) is very large
\[
\omega(p, \tau) + \omega(p + k, \tau) - k \sim 2m
\]

and the energy conservation law is not fulfilled for the one-photon annihilation process. That is why the radiation of a real photon can be interpreted as a multiphoton process. The photon number \( N_\nu \) from the photon condensate of the external quasiclassical laser field with the frequency \( \nu \) can be estimated from the condition to compensate the mismatch (18) by the energy of \( N_\nu \) quasiclassical photons. This leads to the estimate \( N_\nu \sim 2m/\nu \). For optical lasers this is a huge number and therefore such kind of fluctuation event is very scarce. But in the case of \( \gamma - \) radiation the result raises hopes for the possibility of an observation. In the general case, the effect is defined by the external field parameters: the amplitude \( E_0 \) and the frequency \( \nu \).

This conclusion about the role of multiphoton processes correlates with the analysis of the absorption coefficient of the electron - positron plasma created from vacuum in the infrared region [44]. According to the structure of the time dependent \( u, v \)-spinors, the convolution (47) is a polynomial in \( eA(t) \) and hence it cannot guarantee for the necessary compensation. Below it is assumed that the laser electric field \( A(t) = A^3(t) = -(E_0/\nu) \cos(\nu t) \) is subcritical, \( E_0 \lesssim E_c \). Therefore, we use here the Markovian approximation (Approximation 5) \( K(p, p + k, k; t, t') \to K(p, p + k, k; t, t) = K_p = 4 \), see Appendix A. Thus there are two sources for compensation of the high-frequency phase (44): the time dependence of the fermion distribution functions \( f(p, t) \) and the multiphoton contributions contained in the mismatch (48).

As it can be seen from the KE's (16), (17) the fermion distribution function \( f(p, t) \) has two time scales defined by the frequency \( \nu \) of the external field and by the rest mass \( m \) (or the one-particle quasi-energy \( \omega(p, t) \)). Then the corresponding Fourier transformation can be defined by the double series
\[
f(p, t) = \sum_{n, l} f_{n, l}(p) e^{i\nu nt + imlt}.
\]

The numerical calculations have shown that the low-frequency behavior of the fermion distribution function in the presence of a linearly polarized field is defined approximately by the second harmonic (Approximation 6)
\[
f(p, t) = \tilde{f}(p) [1 - \cos(2\nu t)]/2 .
\]
The higher low-frequency harmonics and the high-frequency harmonics in the decomposition \( \alpha \) are very small compared to \( \beta \) and will be omitted in the following.

The choice of the following approximations depends on the value of the adiabaticity parameter \( \gamma \)

\[
\gamma = \frac{(E_c/E_0)}{\nu/m} .
\]

(51)

The domain \( \gamma \gg 1 \) corresponds to the multiphoton processes in a rather small external field \( E_0 \ll E_c \). In the case \( \gamma \ll 1 \) the external field is rather large, \( E_0 \approx E_c \). These two limiting cases will be considered below.

**B. Multiphoton domain \( (\gamma \gg 1) \)**

Since \( E_0 \ll E_c \), one can use the appropriate perturbation theory in order to select the constant component in the photon production rate. The unique source is now the time dependence of the mismatch \( \alpha \). For \( E_0 \ll E_c \), one can select in the quasi-energy the first order field effect (Approximation 7)

\[
\omega(p,t) = \frac{eE_0}{\nu\alpha} \cos(\nu t),
\]

(52)

where \( \omega(p,t) = \omega(p,t)|_{\lambda=0} \). Then the phase \( \theta(t,t') \) will be

\[
\Omega(t,t') = \Omega_0(p,k)(t-t') + a(p,k)\sin \nu t - \sin \nu t',
\]

(53)

where the mismatch in the absence of the external field is

\[
\ Omega_0(p,k) = \omega_0(p) + \omega_0(p + k) - k,
\]

(54)

and

\[
a(p,k) = -\frac{e}{\nu^2} \left\{ \frac{E_0p}{\omega_0(p)} + \frac{E_0(p + k)}{\omega_0(p + k)} \right\} .
\]

(55)

The fast oscillating function in Eq. (51) allows then the following representation

\[
\cos \Phi(t,t') \approx e^{\omega_0(t-t')} \sum_{n,n'} J_n(a) J_{n'}(a) e^{i\nu(nt-n't') + c.c.},
\]

(56)

which is based on the known decomposition

\[
\exp(i\nu \sin \varphi) = \sum_{n=-\infty}^{\infty} J_n(a) e^{i\nu n},
\]

(57)

where \( J_n(a) \) is the Bessel function. Thus, the representation (56) contains two high frequency harmonics \( \omega(p) \) and \( \omega_0(p + k) \) (see Eq. (51)), the set of the low frequency harmonics \( \nu \) and the \( k \) harmonic corresponding to the radiated photon.

The following steps are these: we calculate the time integral in Eq. (51) and select its time independent component corresponding to the observed value. Performing then the remaining momentum space integrations in the isotropic approximation and using the textbook formula

\[
\delta[\phi(x)] = \sum_i \{|\phi'(x_i)|^{-1} \delta(x - x_i), \quad \phi(x_i) = 0, \quad (58)
\]

we obtain then in the low frequency approximation

\[
\hat{F}(k) = \frac{\alpha K_0}{\nu^2} \frac{p_1 \omega_1 \sqrt{\omega_1^2 + k^2}}{2k \omega_1 + \sqrt{\omega_1^2 + k^2}} J_{n_0+1}(a)
\]

\[
[J_{n_0+3}(a) + J_{n_0-1}(a)] \hat{f}(p_1) \hat{f}(p_1 + k),
\]

(59)

where we took into account the lowest harmonics of the distribution function Eq. (54) and \( \omega_1 = \sqrt{m^2 + p_1^2} \). The argument of the Bessel function in the isotropic approximation is

\[
a = \frac{2\sqrt{\pi} \alpha E_0}{\nu^2} \left[ \frac{p_1}{\omega_1} + \frac{p_1 + k}{\sqrt{\omega_1^2 + k^2}} \right] .
\]

(60)

Let us explain the meaning of the momentum \( p_1 \). By \( p_0 \) we denote the positive root of the equation \( \Omega_0 - n\nu = 0 \), where

\[
\Omega_0(p,k) = \omega_0(p) + \sqrt{\omega_0^2(p) + k^2} - k,
\]

(61)

is the mismatch and \( n = [\Omega_0(p_0,k)/\nu] \) (\([x]\) is the integer part \( x \)) is the photon number necessary for overcoming the mismatch (51). Then we obtain

\[
p_0 = \left\{ \frac{(\nu^2(\nu - 2k)^2}{4(\nu - k)^2} - m^2 \right\}^{1/2} .
\]

(62)

Let now \( n_0 \) be the minimal number of quasiclassical photons, \( n_0 \sim p_0 = 0. \) The photon production rate (62) is equal to zero in this point. Let us suppose \( n_1 = n_0 + 1 \) in Eq. (62). The momentum \( p_1 \) corresponds to this number, \( p_1 = n_1 \). Thus, according to Eq. (53) we have \( I(k) \sim \alpha^{n_0+1} f^2(0) \). It is difficult to estimate the general order of the perturbation theory, because the amplitude \( f(0) \) of the distribution function is calculated nonperturbatively.

1. **The case of optical vacuum excitation \( (\nu \ll m) \)**

In the optical part of the photon spectrum \( (k \lesssim \nu) \) we have \( p_1 = \sqrt{km} \) as the first root of the equation \( \Omega_0(p_1) - (n_0 + 1)\nu = 0 \) (it corresponds to the leading contribution from the set \( n > n_0 \), \( a = 2(E_0/E_c)(m/\nu)^{3/2} \ll 1 \) and \( n_0 = [2m/\nu] \gg 1 \), i.e., the necessary number of quasiclassical photons is huge and the intensity of photon radiation a very small. Indeed, from Eq. (59) it follows that

\[
I(k) = \frac{1}{m} \frac{dF(k)}{dt} = \frac{\alpha K_0}{4} \sqrt{\frac{m}{k}} f^2(0) \left\{ \frac{1}{n_0^2} \left[ \frac{E_0}{E_c} \left( \frac{m}{\nu} \right)^{3/2} \right]^{2n_0} \right\} .
\]

(63)
The value \( \hat{f}(0) \) can be estimated as a result of the numerical solution of the KE \(^{[15, 17]}\) for \( \epsilon^– \epsilon^+ \) excitations in a laser field \(^{[16]}\). For the PW laser system Astra Gemini we have \( E_0 \sim 10^{-5} E_c \), \( \lambda = 800 \) nm and \( \hat{f}_0(0) \sim 10^{-11} \) \(^{[21, 30]}\). The spectral distribution of the radiated photons from the volume \( \lambda^3 \) of the focal spot per time interval,

\[
\frac{dN_k}{dt dk} = \frac{\lambda^3}{\pi^2} I(k) \, k^2 m = \frac{8\pi k^2 m}{\nu^2} I(k) ,
\]

will be negligibly small for the mentioned parameters. However, the term in the curly brackets on the r.h.s. of Eq. \(^{[13]}\) behaves as a \( \theta(a) \)-function with the branch point \( a_0 = 2 \) when letting \( a \to a_0 \). Then the spectral distribution \(^{[24]}\) starts to grow strongly. Unfortunately, this value \( a_0 = 2 \) lies outside of the validity range of Eq. \(^{[68]}\). Nevertheless, this gives a hint on the possible growth of the radiation intensity in this domain. Some additional analyses are necessary here.

The \( \gamma \)-ray part of the photon spectrum \( k \sim m \) can not be considered in the framework of this approximation \( (a \gg 1) \) again.

2. The case of \( \gamma \)-ray vacuum excitation \( (\nu \sim m) \)

In this case the mismatch \(^{[61]}\) can be compensated by the smallness of the photon number from the quasiclassical laser field, \( n_0 \gtrsim 1 \) \(^{[19]}\). Let \( n_0 = 1 \) (this is the hypothetical limiting case, e.g., for the planned XFEL facility with \( \lambda = 0.15 \) nm \(^{[14]}\)). The developed theory is working well in this case \( (a \ll 1) \) for the subcritical fields \( E \ll E_c \).

In the optical part of the photon band \( (k \ll m) \) we have \( p_1 = \sqrt{3}m \) and

\[
a = \sqrt{3} \frac{E_0}{E_c} \left( \frac{m}{\nu} \right)^2 .
\]

The spectral distribution according to Eq. \(^{[66]}\) is

\[
\frac{dN_k}{dt dk} = \frac{3\sqrt{3} \alpha K_0 k}{2\nu} \hat{f}^2(p_1) \left( \frac{E_0}{E_c} \right)^2 \left( m \sqrt{\nu} \right)^6 .
\]

Thus, the effect grows linearly with \( k \). For \( \nu = 1 \) MeV, \( E_0 = 10^{-5} E_c \) and \( \hat{f}(p_1) \sim 10^{-11} \) (see Fig. \(^{[1]}\)) we obtain again a negligibly small effect: the suppression factor is \( \hat{f}(0) E_0/E_c \sim 10^{-5} \) so that a very weak signal results.

The situation is slowly changing when going to higher frequencies of the excited signal (X-ray or \( \gamma \)-ray domain) at \( E/E_c = \) const. One can demonstrate this by writing \( p_1 \) \(^{[22]}\) for \( k \neq 0 \) and \( n_0 = 1 \)

\[
p_1 = m \left\{ \frac{2}{3} \left[ \frac{2 + k/m}{2 + k/m} \right]^{1/2} - 1 \right\} .
\]

The situation becomes more optimistic at \( E \to E_c \) when \( a \to 1 \). For example, for the XFEL with \( E = 0.24 \) \( E_c \) and \( \lambda = 0.15 \) nm \(^{[14]}\) the intensity \(^{[63]}\) can be accessible to observation, apparently. However, this case needs special investigation since for \( \gamma \ll 1 \) the presented approach is not valid.

C. Strong field case \( (\gamma \lesssim 1) \)

We will use here the effective mass model \(^{[31]}\) based on the approximation

\[
\omega(p, t) = \sqrt{m^2 + (p - eA(t))^2} \to \omega_*(p) ,
\]

\[
\omega_*(p) = \sqrt{m_*^2 + p^2} ,
\]

with the effective mass defined by the relation

\[
m_*^2 = m^2 + e^2 \partial A^2(t) \gtrsim m^2 + e^2 E_0^2/2\nu^2 = m^2(1 + 1/2\gamma^2) ,
\]

where \( \langle ... \rangle \) denotes the time averaging operation, and \( \gamma \) is the adiabaticity parameter \(^{[61]}\).

In this approximation the phase \(^{[6]}\) becomes monochromatic

\[
\theta(p_1, p_2, k; t', t) = \Omega_*(p_1, p_2, k)(t - t') ,
\]

\[
\Omega_*(p_1, p_2, k) = \omega_*(p_1) + \omega_*(p_2) - k ,
\]

i.e., the approximation \(^{[68]}\) leads to a suppression of multiphoton processes (it corresponds to large values of the adiabaticity parameter \( \gamma \gg 1 \)) and the mismatch \(^{[71]}\) can be compensated by the harmonics of the fermion distribution functions in Eq. \(^{[66]}\) only.

The inspection of the fermion distribution function shows, in particular, that it oscillates basically with twice the laser frequency \(^{[50]}\). The substitution of Eqs. \(^{[70]}\) and \(^{[50]}\) into the KE \(^{[46]}\) allows to perform the time integration, leading to the appearance of two harmonics in the radiation spectrum only (the 2nd and the 4th),

\[
\hat{F}(k, t) = -A^{(2)}(k) \cos(2\nu t) + A^{(4)}(k) \cos(4\nu t) ,
\]

\[
A^{(2)}(k) = \frac{\pi^2 K_0 \alpha}{2k} \int \frac{d^3p}{(2\pi)^3} \hat{f}(\hat{p}) \hat{f}(\hat{p} + k) \delta(2\nu - \Omega_*) ,
\]

\[
A^{(4)}(k) = \frac{\pi^2 K_0 \alpha}{8k} \int \frac{d^3p}{(2\pi)^3} \hat{f}(\hat{p}) \hat{f}(\hat{p} + k) \delta(4\nu - \Omega_*) .
\]

It is important that a constant component is absent here, because the mismatch \(^{[48]}\) could not be compensated in this case by other sources of the time dependence on the r.h.s. of Eq. \(^{[66]}\).

Thus, in the case of the infinite system the solution \(^{[72]}\) can be interpreted as ”breathing” of the photon subsystem. However, the situation is changed, when the generation of the \( \epsilon^– \epsilon^+ \gamma \) plasma is considered in a small spatial...
domain of the focal spot with volume $\sim \lambda^3$ due to the vacuum condition of the absence of the $e^- e^+ \gamma$ plasma in the initial moment of switching on the laser field. In this case one can expect, that all annihilation photons generated in the first half-period of the field will leave the volume of the system and therefore in the next half-period the reverse process (photon transformation to $e^- e^+$ plasma) will be impossible. Such a mechanism leads to a pulsation pattern for the photon radiation from the focal spot. It corresponds to introducing the condition of a positive definite photon production rate on the r.h.s. of Eq. (72).

For estimates of the amplitudes (73), (74) let us introduce the additional model approximation in the spirit of the model (68).

$$\omega_{\ast}(p + k) \rightarrow \omega_{\ast}(p, k) = \sqrt{\omega_{\ast}^2(p) + k^2}, \quad (75)$$

and the isotropisation condition $\bar{f}(p + k) \rightarrow \bar{f}(p + k)$. The integrals on the r.h.s of Eqs. (73), (74) can then be calculated. For example,

$$A^{(2)}(k) = \frac{\alpha}{k} \int \frac{d^3p}{(2\pi)^3} \frac{\omega_{\ast}(p_0 + k) \omega_{\ast}(p_0, k)}{\omega_{\ast}(p_0) + \omega_{\ast}(p_0, k)} p_0, \quad (76)$$

where

$$p_0 = \sqrt{\frac{4\nu^2(k + \nu)^2}{(k + 2\nu)^2} - m^2}, \quad (77)$$

is the root of the equation $\Omega_{\ast} - 2\nu = 0$. From Eq. (77) follows the threshold condition $3$

$$2\nu(k + \nu) \geq m^2. \quad (78)$$

This condition is rather nontrivial because the effective mass $(79)$ depends also on $\nu$. The minimal permissible value $\nu = 2m_{\ast}$ corresponds to $k = 0$. For the 4th harmonic the threshold value falls to $\nu = m_{\ast}$, which is close to the parameters of the XFEL (14).

The 1/k - dependence on the r.h.s. of Eq. (76) corresponds to the flicker-like noise of electrodynamical origin. This feature in the spectrum of radiated annihilation photons has been found first in Ref. (48).

The number of photons with the frequency $k$ lying in the interval $[k, k + dk]$ and radiated from the focal spot with the volume $\lambda^3 = \nu^3$ per time interval is defined by Eq. (64). The fraction on the r.h.s. of Eq. (70) is a slow function of the frequencies $k$ and $\nu$ and for the sake of a preliminary estimate it can be replaced by $m_{\ast}/2$. For the 2nd harmonic we then obtain from Eqs. (70) and (64)

$$\frac{d^2N^{(2)}}{dt dk} = \frac{2\nu \alpha K_0 m_{\ast}}{\nu^3} \int \frac{d\nu}{2\pi} \bar{f}(p_0) \bar{f}(p_0 + k)p_0. \quad (79)$$

As a representative characteristics of the effectiveness of the radiation from the focal spot domain we will consider the total photon number per time interval,

$$\dot{N}^{(2)} = \frac{2\pi \alpha K_0 m_{\ast}}{\nu^3} \int \frac{d\nu}{2\pi} \int \frac{d^3k}{(2\pi)^3} \bar{f}(0) \bar{f}(p_0 + k)p_0. \quad (80)$$

The electron and positron distribution functions entering here are defined as the solutions of the corresponding non-perturbative KE (Sect. II) describing vacuum creation of $e^- e^+$ pairs under the action of a strong, time dependent electric field of a standing wave of two counter-propagating laser beams. The cutoff parameter $k_{\max} = 2m_{\ast}$ is introduced in order to take into account the annihilation photons in the radiation spectrum.

The fermion distribution function $f(p, t)$ is a rapidly decreasing function with its maximum in the point $p = 0$ (20, 30). On this basis for a rough estimate one can put $p_0 = 0$ in the arguments of these functions on the r.h.s. of Eq. (80),

$$\dot{N}^{(2)} = \frac{2\pi \alpha K_0 m_{\ast}}{\nu^3} \int \frac{d\nu}{2\pi} \bar{f}(0) \int \frac{d^3k}{(2\pi)^3} \bar{f}(p_0 + k)p_0, \quad (81)$$

where according to Eq. (77)

$$p_0(k) = \frac{m_{\ast}^2}{k + 4m_{\ast}} \sqrt{48 + 56 \frac{k}{m_{\ast}} + 15 \frac{k^2}{m_{\ast}^2}}, \quad (82)$$

because the small $k_{\max} \ll m_{\ast}$ is effective in the integral (81). As the result, we obtain the following order of magnitude estimate

$$\dot{N}^{(2)} \sim \alpha m_{\ast} \bar{f}^2(0). \quad (83)$$

For the XFEL parameters $E_0 = 0.24 E_c$ and $\lambda = 15$ nm (14) we have according to the kinetic theory in the $e^- e^+$ sector $\bar{f}(0) \sim 10^{-2}$, see Fig. 1. From Eq. (83) then follows

$$\dot{N}^{(2)} \sim 10^{17} \text{ s}^{-1}. \quad (84)$$

For the 4th harmonic with the oscillation amplitude (74) the threshold for the generation of annihilation photons is lowered (see discussion after Eq. (78)) but the intensity of the photon radiation is also lowered so that the order of magnitude of (83) remains unchanged.

V. SUMMARY

We have studied the photon production rate resulting from the one-photon annihilation mechanism in a quasiparticle EPP created from the vacuum under the influence of a strong electric field being a necessary condition
for the possible occurrence of such a process\cite{31}. The required strong electric fields occur, e.g., in the focal spot of two counter-propagating high-intensity laser beams. The methodic basis is an appropriately developed kinetic theory constructed in the quasiparticle representation. The fermion sector of the theory has been investigated before\cite{28,30} on an essentially nonperturbative basis. The photon kinetics can be considered in the framework of the usual perturbation theory which allows to truncate the BBGKY hierarchy of KEs at the lowest order with respect to the fine-structure constant and to obtain the closed formula\cite{48} for the photon production rate including vacuum polarization effects.

We have investigated this expression for different characteristics of the laser field (frequencies $\nu$ and amplitudes $E_0$) which is conveniently described with the adiabaticity parameter $\gamma$. In order to become observable it is necessary for the photon radiation process to compensate the energy mismatch\cite{18} by a sufficient number of quasiclassical photons from photon reservoir of the external electric field. The strength of the laser field $E_0$ defines the intensity of this process. Thus, it can be understood that in the case of optical lasers the number of quasiclassical photons in the optical range needed for the compensation of the mismatch\cite{18} is huge and the intensity of such process is negligibly small.

However, the intensity of photon production increases strongly in the domain of $\gamma$–ray excitations of the vacuum (where only a small number of quasiclassical photons is necessary to overcome the mismatch) and reaches the values sufficient for experimental observation, e.g., for the projected XFEL\cite{14}. This result is qualitatively confirmed by the recent findings of Ref.\cite{49} which have been obtained within a different approach.

In the case of the two-photon annihilation channel the necessity to compensate the mismatch of the type\cite{18} is absent and therefore the photon production rate can be large enough to be observable already in the optical domain. This channel of the EPP annihilation will be investigated separately.

We would like to note also that the type of kinetic theory based on the quasiparticle representation, which has been described in the present work can also be useful for the investigation of other reaction channels in the $e^-e^+\gamma$ plasma. One of such channels is the cascade process of EPP multiplication in a strong electric field (e.g.,\cite{50}). The example of one-photon annihilation presented here shows, that strong external fields can lead to qualitative modifications of processes relative to standard QED.

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\begin{thebibliography}{59}
\bibitem{1} J. Schwinger, Phys. Rev. \textbf{82}, 664 (1951);
W. Heisenberg and H. Euler, Z. Phys. \textbf{98}, 714 (1936);
F. Sauter, Z. Phys. \textbf{69}, 742 (1931).
\bibitem{2} E. Brezin and C. Itzykson, Phys. Rev. D \textbf{2}, 1191 (1970).
\bibitem{3} V. S. Popov, Sov. J. Nucl. Phys. \textbf{34}, 709 (1972);
N. B. Narozhny and A. I. Nikishov, Sov. Phys. JETP \textbf{38}, 427 (1974);
A. I. Nikishov, Tr. Fiz. Inst. Akad. Nauk SSSR \textbf{111}, 152 (1979).
\bibitem{4} A. A. Grib, S. G. Mamaev and V. M. Mostepanenko, \textit{Vacuum Quantum Effects in Strong External Fields}, Friedmann Laboratory Publishing, St. Petersburg, 1994.
\bibitem{5} M. S. Marinov and V. S. Popov, Fortschr. Phys. \textbf{25}, 373 (1977).
\bibitem{6} S. S. Bulanov, N. B. Narozhny, V. D. Mur and V. S. Popov, Phys. Lett. A \textbf{330}, 1 (2004).
\bibitem{7} N. Tanji, Annals Phys. \textbf{324}, 1691 (2009).
\bibitem{8} G. Baur, Eur. Phys. J. D \textbf{55}, 265 (2009).
\bibitem{9} G. Baur, K. Hencken and D. Trautmann, Phys. Rept. \textbf{453}, 1 (2007).
\bibitem{10} F. V. Bunkin and I. I. Tugov, Sov. Phys. Dokl. \textbf{14}, 678 (1969).
\bibitem{11} B. Richards and E. Wolf, Proc. Roy. Soc. A (London) \textbf{253}, 358 (1959).
\bibitem{12} C. J. Troup and H. S. Perlman, Phys. Rev. D \textbf{6}, 2299 (1972).
\end{thebibliography}
As an example, we will consider here the calculation \[ K(p_1, p_2, k; t, t) = \frac{1}{2\omega(p)} \left( \hat{p}_1 + m \right) \left( \hat{p}_2 - m \right) \] in the Markovian approximation. Setting \( t = t' \), we obtain

\[ K(p_1, p_2, k; t, t) = \frac{1}{2\omega(p)} \left( \hat{p}_1 + m \right) \left( \hat{p}_2 - m \right) \]

\[ = \frac{1}{2\omega(p)} \left( \hat{p}_1 + m \right) \left( \hat{p}_2 - m \right) \]
Thus, we finally obtain

\[ K(p_1, p_2, k; t, t) = \frac{2}{\omega(p_1)\omega(p_2)} \left\{ m^2 + p_1 p_2 + (p_1 e^\tau)(p_2 e^\tau) \right\}. \quad (A5) \]

In the case \( p_1 = p_2 = 0 \) it follows from (A5)

\[ K(p_1, p_2, k; t, t) = K_0 = 4. \quad (A6) \]

The last formula is used in the qualitative estimates of the photon production rate in the different models.

For \( p_1 \neq p_2 \neq 0 \) one can obtain another estimate.

Since in the integral we have

\[ p_1^i p_2^j = \frac{1}{3} \delta_{ik} p_1^i p_2^k, \]

we obtain from Eq. (A6)

\[ K(p_1, p_2, k; t, t) = \frac{2}{\omega(p_1)\omega(p_2)} \left\{ m^2 + \omega(p_1)\omega(p_2) - \frac{1}{3} p_1 p_2 \right\}. \quad (A7) \]