The unimodality of the Ehrhart $\delta$-polynomial of the chain polytope of the zig-zag poset

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Abstract. We prove the unimodality of the Ehrhart $\delta$-polynomial of the chain polytope of the zig-zag poset, which was conjectured by Kirillov. First, based on a result due to Stanley, we show that this polynomial coincides with the $W$-polynomial for the zig-zag poset with some natural labeling. Then, its unimodality immediately follows from a result of Gasharov, which states that the $W$-polynomials of naturally labeled graded posets of rank 1 or 2 are unimodal.

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1 Introduction

The main objective of this paper is to prove a unimodality conjecture on $\delta$-polynomials of the chain polytope of the zig-zag poset, which was proposed by Kirillov [5] in the study of Kostka numbers and Catalan numbers. Let us first give an overview of Kirillov’s conjecture.

Let $\mathbb{Z}^m$ denote the $m$-dimensional integer lattice in $\mathbb{R}^m$, and let $\mathcal{P}$ be an $m$-dimensional lattice polytope in $\mathbb{R}^m$. A remarkable theorem due to Ehrhart [3] states that the number of lattice points that lie inside the dilated polytope $n\mathcal{P}$:

$$i(\mathcal{P};n) = |n\mathcal{P} \cap \mathbb{Z}^m|.$$  \hspace{1cm} (1.1)

is given by a polynomial in $n$ of degree $m$, called the Ehrhart polynomial of the lattice polytope $\mathcal{P}$. By a well known result about rational generating functions, see [9, Corollary 4.3.1], the generating function (called the Ehrhart series of $\mathcal{P}$)

$$J(\mathcal{P};t) = \sum_{n \geq 0} i(\mathcal{P};n) t^n$$  \hspace{1cm} (1.2)

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evaluates to a rational function:

\[ J(P; t) = \frac{\delta(P; t)}{(1 - t)^{\dim P + 1}} \]  

for some polynomial \( \delta(P; t) \) of degree at most \( \dim(P) \), which is called the Ehrhart \( \delta \)-polynomial of \( P \). If the polynomial \( \delta(P; t) \) is of the following form

\[ \delta(P; t) = \delta_0 + \delta_1 x + \cdots + \delta_m x^m, \]

then we call \((\delta_0, \delta_1, \ldots, \delta_m)\) the (Ehrhart) \( \delta \)-vector of \( P \). Stanley [7] also proved that \( \delta(P; t) \) must be a polynomial in nonnegative coefficients. For more information on the Ehrhart theory of rational polytopes, see [1].

Let \( P_n \) be a convex integral polytope in \( \mathbb{R}^n \) determined by the following inequalities

\[ x_i \geq 0, \quad \text{for } 1 \leq i \leq n, \]
\[ x_i + x_{i+1} \leq 1, \quad \text{for } 1 \leq i \leq n - 1. \]

Kirillov conjectured that the \( \delta \)-polynomial of \( P_n \) is unimodal [5]. Recall that a polynomial \( f(x) = \sum_{i=0}^{n} a_i x^i \) with real coefficients is said to be unimodal if there exists an integer \( i \geq 0 \) such that

\[ a_0 \leq \cdots \leq a_{i-1} \leq a_i \geq a_{i+1} \geq \cdots \geq a_n, \]

and symmetric if for all \( 0 \leq i \leq n \)

\[ a_i = a_{n-i}. \]

Kirillov’s conjecture is stated as follows.

**Conjecture 1.1 ([5, p.119, Conjecture 3.11])** For any \( n \geq 1 \), the \( \delta \)-polynomial \( \delta(P_n; t) \) is unimodal.

In this paper, we give a proof of Kirillov’s conjecture. Our proof is based on the theory of chain polytopes of posets, as well as the theory of \( W \)-polynomials of posets.

## 2 Preliminaries

In this section, we shall review some definitions and results on chain polytopes and \( W \)-polynomials of posets.

We begin with some definitions concerning posets. Let \((P, \leq)\) be a poset with \( d \) elements. Recall that a chain of length \( \ell \) in \( P \) is a sequence \( a_0 \prec a_1 \prec \cdots \prec a_\ell \).
\(\cdots \prec a_\ell\), and it is called maximal in \(P\) if we cannot add elements to this chain. If every maximal chain of \(P\) has the same length \(r\), then we say that \(P\) is graded of rank \(r\) and denote the rank of \(P\) by \(\text{rank}(P)\). In this case, there is a unique rank function \(\rho: P \to \{0, 1, \ldots, d\}\) such that \(\rho(x) = 0\) if \(x\) is a minimal element of \(P\), and \(\rho(y) = \rho(x) + 1\) if \(y\) covers \(x\) in \(P\). If \(\rho(x) = i\), then we say that \(x\) is of rank \(i\).

The notion of chain polytopes was introduced by Stanley [8]. Given a poset \(P\) with elements \(\{a_1, \ldots, a_d\}\), Stanley associated it with a polytope \(C(P)\) defined by the chains in \(P\), called the chain polytope of \(P\). Precisely, the chain polytope \(C(P)\) is the convex polytope consisting of those \((x_1, \ldots, x_d) \in \mathbb{R}^d\) such that

- \(x_i \geq 0\), for every \(a_i \in P\),
- \(x_{p_1} + x_{p_2} + \cdots + x_{p_k} \leq 1\), for every chain \(a_{p_1} \prec \cdots \prec a_{p_k}\) of \(P\).

Since \(C(P)\) contains the \(d\)-dimensional simplex

\[\{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_i \geq 0 \text{ for all } 1 \leq i \leq d \text{ and } x_1 + x_2 + \cdots + x_d \leq 1\},\]

we know that

\[\text{dim}(C(P)) = d = |P|\]

We would like to point out that the polytope \(P_n\) is just the chain polytope of the zig-zag poset of order \(n\). Recall that the zig-zag poset of order \(n\) is the poset \(Z_n = \{a_1, \ldots, a_n\}\) in which

\[a_1 \prec a_2 \succ a_3 \prec \cdots,\]

see [2, 9]. Note that, if \(n = 2k + 1\), the maximal chains of \(Z_n\) are \(a_{2i-1} \prec a_{2i}\) and \(a_{2i+1} \prec a_{2i}\) for \(1 \leq i \leq k\). While, if \(n = 2k + 2\), the maximal chains of \(Z_n\) are \(a_{2i-1} \prec a_{2i}\) and \(a_{2i+1} \prec a_{2i}\) for \(1 \leq i \leq k\), together with \(a_{2k+1} \prec a_{2k+2}\). By definition, it is clear that

\[P_n = C(Z_n).\]  \hfill (2.1)

Based on the above viewpoint, Conjecture 1.1 is equivalent to the statement that the \(\delta\)-polynomial of \(C(Z_n)\) is unimodal. While for the chain polytope of poset \(P\), Stanley [8] has already established a connection between the \(\delta\)-polynomial of the chain polytope and the number of order-preserving maps of \(P\). Let \(m\) be a positive integer and let \(\Omega(P; m)\) denote the number of order-preserving maps \(\eta: P \to \{1, 2, \ldots, m\}\), i.e., if \(x \preceq y\) in \(P\) then
η(x) ≤ η(y). It is known that \( \tilde{\Omega}(P; m) \) is a polynomial of degree \(|P|\) in \( m \).

Equivalently, there exists a polynomial \( \tilde{W}(P; t) \) of degree \( \leq |P| \) such that

\[
\sum_{m \geq 0} \tilde{\Omega}(P; m + 1)t^m = \frac{\tilde{W}(P; t)}{(1 - t)^{|P|+1}}. \tag{2.2}
\]

Stanley obtained the following theorem.

**Theorem 2.1 ([8, Theorem 4.1])** For any positive integer \( m \) and any poset \( P \), we have

\[
i(C(P); m) = \tilde{\Omega}(P; m + 1),
\]

or equivalently,

\[
\delta(C(P); t) = \tilde{W}(P; t). \tag{2.3}
\]

Instead of considering the number of order-preserving maps of \( P \), we may also study the number of order-reversing maps of \( P \). In fact, there is a more general theory on order-reversing maps, developed by Stanley [6] and called the theory of \( P \)-partitions. Suppose that \( P \) is a finite poset with \( d \) elements as before. A labeling \( \omega \) of \( P \) is a bijection from \( P \) to \( \{1, 2, \ldots, d\} \). The labeling \( \omega \) is called natural if \( x \preceq y \) implies \( \omega(x) \leq \omega(y) \) for any \( x, y \in P \), namely, it is an order-preserving map. A \((P, \omega)\)-partition is a map \( \sigma \) which satisfies the following conditions:

- \( \sigma \) is order reversing, namely, \( \sigma(x) \geq \sigma(y) \) if \( x \preceq y \) in \( P \); and moreover
- if \( \omega(x) > \omega(y) \), then \( \sigma(x) > \sigma(y) \).

The order polynomial \( \Omega(P, \omega; n) \) is defined as the number of \((P, \omega)\)-partitions \( \sigma \) with \( \sigma(x) \leq n \) for any \( x \in P \). It is also known that \( \Omega(P, \omega; n) \) is a polynomial of degree \(|P|\) in \( n \), or equivalently, there exists a polynomial \( W(P, \omega; t) \), called the \( W \)-polynomial of \((P, \omega)\), of degree \( \leq |P| \) such that

\[
\sum_{n \geq 0} \Omega(P, \omega; n + 1)t^n = \frac{W(P, \omega; t)}{(1 - t)^{|P|+1}}. \tag{2.4}
\]

Note that, for a natural labeling \( \omega \), we must have

\[
\Omega(P, \omega; n) = \tilde{\Omega}(P; n), \tag{2.5}
\]

since \( \Omega(P, \omega; n) \) is just the number of order-reversing maps in this case, and \( \tilde{\Omega}(P; n) \) is the number of order-preserving maps. In fact, if \( \eta : P \rightarrow \)}
\( \{1, 2, \ldots, m\} \) is order reversing, then the map \( \tilde{\eta} : P \to \{1, 2, \ldots, m\} \) defined by
\[
\tilde{\eta}(x) = m + 1 - \eta(x)
\]
is order-preserving, and vice versa. By (2.3) and (2.5), we have
\[
\delta(C(P); t) = \tilde{W}(P; t) = W(P, \omega; t).
\]

(2.6)

3 Proof

In this section, we shall give a proof of Conjecture 1.1. Our proof is based on the following result due to Gasharov [4].

**Theorem 3.1 ([4, Theorem 1.2])** If \( P \) is a graded poset with \( 1 \leq \text{rank}(P) \leq 2 \) and \( \omega \) is a natural labeling of \( P \), then \( W(P, \omega; t) \) is unimodal.

We proceed to prove Conjecture 1.1.

**Proof of Conjecture 1.1.** By (2.6), we have
\[
\delta(C(Z_n); t) = \tilde{W}(Z_n; t) = W(Z_n, \omega; t)
\]
for some natural labeling \( \omega \) of the zig-zag poset \( Z_n \). It is clear that \( Z_n \) a graded poset with \( \text{rank}(P) = 1 \). From Theorem 3.1 it follows the unimodality of \( \delta(C(Z_n); t) \), and hence that of \( \delta(P_n; t) \). This completes the proof.

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