OPENNESS OF VERSALITY VIA COHERENT FUNCTORS

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Abstract. We give a proof of openness of versality using coherent functors.
As an application, we streamline Artin’s criterion for algebraicity of a stack.
We also introduce multi-step obstruction theories, employing them to produce
obstruction theories for the stack of coherent sheaves, the Quot functor, and
spaces of maps in the presence of non-flatness.

Introduction

In M. Artin’s classic paper on stacks, a criterion for algebraicity is expounded
[Art74, Thm. 5.3]. In this paper, we take a novel approach to algebraicity, proving
an algebraicity criterion for stacks which is easier to apply, more widely applicable,
and admitting a substantially simpler proof.

Theorem A. Fix an excellent scheme $S$ and a category $X$, fibered in groupoids
over the category of $S$-schemes, $\text{Sch}/S$. Then, $X$ is an algebraic stack, locally of
finite presentation over $S$, if and only if the following conditions are satisfied.

1. [Stack] $X$ is a stack over the site $(\text{Sch}/S)_{\acute{e}t}$.
2. [Limit preservation] For any inverse system of affine $S$-schemes $\{\text{Spec } A_j\}_{j \in J}$
   with limit $\text{Spec } A$, the natural functor:
   $$\lim_{\rightarrow j} X(\text{Spec } A_j) \to X(\text{Spec } A)$$
   is an equivalence of categories.
3. [Homogeneity] For any diagram of affine $S$-schemes $\text{Spec } B \leftarrow \text{Spec } A \rightarrow \text{Spec } A'$, with $i$ a nilpotent closed immersion, the natural functor:
   $$X(\text{Spec } (B \times_A A')) \to X(\text{Spec } A') \times_{X(\text{Spec } A)} X(\text{Spec } B)$$
   is an equivalence of categories.
4. [Effectivity] For any local noetherian ring $(B, m)$, such that the ring $B$ is
   $m$-adically complete, with an $S$-scheme structure $\text{Spec } B \to S$ such that
   the induced morphism $\text{Spec } (B/m) \to S$ is locally of finite type, the natural functor:
   $$X(\text{Spec } B) \to \lim_{\leftarrow n} X(\text{Spec } (B/m^n))$$
   is an equivalence of categories.
5. [Conditions on automorphisms and deformations] For any affine $S$-scheme
   $T$, locally of finite type over $S$, and $\xi \in X(T)$, the functors $\text{Aut}_{X/S}(\xi, -)$,
   $\text{Def}_{X/S}(\xi, -): \text{QCoh} (T) \to \text{Ab}$ are coherent.
6. [Conditions on obstructions] For any affine $S$-scheme $T$, locally of finite
type over $S$, and $\xi \in X(T)$, there exists an integer $n$ and a coherent $n$-step
obstruction theory for $X$ at $\xi$. 

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Except for conditions (5) and (6), Theorem [A] is similar to Artin’s criterion [Art74, Thm. 5.3]. Note, however, that we have fewer conditions, and these conditions are cleaner (e.g., no deformation situations). The conditions of Theorem [A] are also stable under composition, in the sense of [Sta06].

This paper began with the realization that the homogeneity condition (3), which is stronger than the analogous condition of [Art74, (S1')], together with conditions (5) and (6), simplifies and broadens the applicability of existing results.

Our usage of the term “coherent” in conditions (5) and (6) of Theorem [A] is in a different sense than what many readers may be familiar with, so we recall the following definition of M. Auslander [Aus66]. For an affine scheme $S$, a functor $F : \text{QCoh}(S) \to \text{Ab}$ is coherent if there exists a morphism of quasi-coherent $\mathcal{O}_S$-modules $K_1 \to K_2$, such that for all $I \in \text{QCoh}(S)$, there is a natural isomorphism of abelian groups:

$$F(I) \cong \text{coker}(\text{Hom}_{\mathcal{O}_S}(K_2, I) \to \text{Hom}_{\mathcal{O}_S}(K_1, I)).$$

It is proven in [Hal12] that most functors arising in moduli are coherent.

**Relation with other work.** The idea of using the Exal functors to simplify M. Artin’s results should be attributed to H. Flenner [Fle81]. Our results and techniques are quite different, however. In particular, H. Flenner [op. cit.] does not address the relationship between formal smoothness and formal versality.

Independently, work in the Stacks Project [Stacks, 07T0] has provided a different perspective on Artin’s results. This approach, however, requires that the deformation–obstruction theory is given by a bounded complex. If there are non-flat or non-tame objects in the moduli problem, the existence of such a complex is subtle. The problems with non-tame stacks can be dealt with by [Hal12, Thm. B]. The problems with non-flatness can be handled by derived algebraic geometry [Stacks, blog:2572] or 2-step obstruction theories [c.f. §8–9].

Using the ideas of B. Töen and G. Vezzosi [HAGII, 1.4], J. Lurie has developed a criterion for algebraicity in the derived context [Lur12, Thm. 3.2.1]. Conditions (5) and (6) of Theorem [A] are related to Lurie’s requirement of the existence of a cotangent complex. As Lurie observes, his criterion is not applicable to Artin stacks, though it is a future intention to make it so [Lur12, Rem. 2]. J. Pridham has proved a criterion for Artin stacks [Pri12, Thm. 3.16], which is related to the results of Lurie’s PhD Thesis [Lur04, Thm. 7.1.6 & Thm. 7.5.1].

To prove that the Quot functors for separated Deligne-Mumford stacks are algebraic spaces, M. Olsson and J. Starr [OS03, Thm. 1.1] did not apply [Art74, Cor. 5.4], which like [op. cit., Thm. 5.3], is formulated in terms of a single-step obstruction theory. The reason for this is simple: in the presence of non-flatness, it is difficult to formulate a single-step obstruction theory with good properties.

They circumvented this predicament by the use of Artin’s original algebraicity criterion [Art69a, Thm. 5.3]. This earlier algebraicity criterion is not formulated in terms of the existence and properties of a single-step obstruction theory, but in terms of certain explicit lifting problems—making its application more complicated (note that J. Starr [Sta06, Thm. 2.15] has subsequently generalized the criteria of [Art69a, Thm. 5.3] to stacks). To solve these lifting problems, M. Olsson and J. Starr [OS03, Lem. 2.5] used a 2-step process. This 2-step process is insufficiently functorial to define a multi-step obstruction theory in the sense of this paper. It is, however, closely related, and inspired the multi-step obstruction theories we define.

M. Olsson and J. Starr [OS03, p. 4077] noted that M. Artin had incorrectly computed the obstruction theory of the Quot functor in the presence of non-flatness [Art69a, 6.4]. We have also located some other articles in the literature...
that have not observed the subtlety of deformation theory in the presence of non-flatness (see §3 and §9). We would like to emphasize that the impact of this on the main ideas of these articles is nil. Indeed, the relevant arguments in these articles are still perfectly valid in the flat case—covering most cases of interest to geometers.

In the non-flat case, the relevant statements in these articles can be shown to hold with the techniques and examples of this article.

By work of M. Olsson [Ols06, Rem. 1.7], the conditions of Theorem A are seen to be necessary. The sufficiency of the conditions of Theorem A is demonstrated by the following sequence of observations:

(i) the existence of formally versal deformations,
(ii) the existence of algebraizations of formally versal deformations, and
(iii) formal versality at a point implies smoothness in a neighbourhood.

Using the generalizations of M. Artin’s techniques [Art74] due to B. Conrad and J. de Jong [CJ02, Thm. 1.5], conditions (1)–(4) of Theorem A prove (i) and (ii). The main contribution of this paper is the usage of conditions (3), (5), and (6) of Theorem A to prove (iii).

Note that in our proof of (iii), the techniques of Artin approximation [Art69a] are not used. This is in contrast to M. Artin’s treatments [Art69b, Art74], where this technique features prominently. In a paper joint with D. Rydh [HR12], we illustrate how refinements of the homogeneity condition (3) clarify and simplify M. Artin’s results on versality.

Outline. In §1 we discuss the notion of homogeneity. Homogeneity is a generalization of the Schlessinger-Rim criteria [SGA7, Exp. VI]. This section is quite categorical, but it is the only section of the paper that is such. Morally, homogeneity provides a stack $X$ with a linear structure at every point, which we describe in §2. To be precise, for any scheme $T$, together with an object $\xi \in X(T)$, homogeneity produces an additive functor $\text{Exal}_X(\xi, -) : \text{QCoh}(T) \to \text{Ab}$ sharply controlling the deformation theory of $\xi$. The author learnt these ideas from J. Wise (in person) and his paper [Wis11], though they are likely well-known, and go back at least as far as the work of H. Flenner [Fle81]. In §3 we recall and generalize—to the relative setting—the notion of limit preserving groupoid [Art74, §1].

In §4 we recall the notions of formal versality and formal smoothness. Then, we recast these notions in terms of vanishing criteria for the functors $\text{Exal}_X(T, -)$. The central technical result of this paper is Theorem 4.5—our new proof of (iii).

In §5 we briefly review coherent functors. In §6 we formalize multi-step obstruction theories. In §7 we prove Theorem A.

The remainder of the paper is devoted to applications. In §8 we compute a 2-step obstruction theory for the stack of coherent sheaves. Finally, in §9 we compute a 2-step obstruction theory for the stack of morphisms between two algebraic stacks.

In Appendix A we prove that pushouts of algebraic stacks along nilimmersions and affine morphisms exist. This enables the verification of the homogeneity condition (3) in practice. In Appendix B we state two basic results on local Tor-functors for morphisms of algebraic stacks.

Assumptions, conventions, and notations. For a category $\mathcal{C}$, denote the opposite category by $\mathcal{C}^\circ$. A fibration of categories $Q : \mathcal{C} \to \mathcal{D}$ has the property that every arrow in the category $\mathcal{D}$ admits a strongly cartesian lift. For an object $d$ of the category $\mathcal{D}$, we denote the resulting fiber category by $Q(d)$. It will also be convenient to say that the category $\mathcal{C}$ is fibered over $\mathcal{D}$. If the category $\mathcal{C}$ is fibered over $\mathcal{D}$, and every arrow in the category $\mathcal{C}$ is strongly cartesian, then we say that the functor $Q$ is fibered in groupoids. The assumptions guarantee that if the
Definition 1.2. For an \( S \)-morphism of \( \mathcal{D} \), the fiber category \( \mathcal{Q}(d) \) is a groupoid.

For a scheme \( T \), denote by \( |T| \) the underlying topological space (with the Zariski topology) and \( \mathcal{O}_T \) the (Zariski) sheaf of rings on \( |T| \). For \( t \in |T| \), let \( \kappa(t) \) denote the residue field. Denote by \( \mathcal{Q}\text{Coh}(T) \) (resp. \( \mathcal{Coh}(T) \)) the abelian category of quasicoherent (resp. coherent) sheaves on the scheme \( T \). Let \( \mathbf{Sch}/T \) denote the category of schemes over \( T \). The big étale site over \( T \) will be denoted by \( (\mathbf{Sch}/T)_{\acute{e}t} \).

For a ring \( A \) and an \( A \)-module \( M \), denote the quasicoherent \( \mathcal{O}_{\text{Spec} \, A} \)-module associated to \( M \) by \( \mathcal{M} \). Denote the abelian category of all (resp. coherent) \( A \)-modules by \( \text{Mod}(A) \) (resp. \( \text{Coh}(A) \)).

As in [Stacks], we make no separation assumptions on our algebraic stacks and spaces. As in [Ols07], we use the lisse-étale site for sheaves on algebraic stacks.

Fix a scheme \( S \). An \( S \)-morphism of algebraic stacks \( \Phi : (Y,a_Y) \rightarrow (Z,a_Z) \) is a functor \( \Phi : Y \rightarrow Z \) that commutes strictly over \( \mathbf{Sch}/S \). We will typically refer to an \( S \)-groupoid \((X,a_X)\) just as \( X \).

Example 1.1. For any \( S \)-scheme \( T \), there is a canonical functor \( \mathbf{Sch}/T \rightarrow \mathbf{Sch}/S : (W \rightarrow T) \mapsto (W \rightarrow T \rightarrow S) \) which is faithful. In particular, we may view an \( S \)-scheme \( T \) as an \( S \)-groupoid. Thus, a morphism of \( S \)-schemes \( g : U \rightarrow V \) induces a 1-morphism of \( S \)-groupoids \( \mathbf{Sch}/g : \mathbf{Sch}/U \rightarrow \mathbf{Sch}/V \). The converse is also true: any 1-morphism of \( S \)-groupoids \( G : \mathbf{Sch}/U \rightarrow \mathbf{Sch}/V \) is uniquely isomorphic to a 1-morphism of the form \( \mathbf{Sch}/g \) for some morphism of \( S \)-schemes \( g : U \rightarrow V \).

Definition 1.2. For an \( S \)-groupoid \( X \), an \( X \)-scheme is a pair \((T, \sigma_T)\) consisting of an \( S \)-scheme \( T \) together with a 1-morphism of \( S \)-groupoids \( \sigma_T : \mathbf{Sch}/T \rightarrow X \). A morphism of \( X \)-schemes \((f, \alpha_f) : (U, \sigma_U) \rightarrow (V, \sigma_V)\) is given by a morphism of \( S \)-schemes \( f : U \rightarrow V \) together with a 2-morphism \( \alpha_f : \sigma_V \Rightarrow \sigma_V \circ \mathbf{Sch}/f \). The collection of all \( X \)-schemes forms a 1-category, which we denote as \( \mathbf{Sch}/X \).

For a 1-morphism of \( S \)-groupoids \( \Phi : Y \rightarrow Z \) there is an induced functor \( \mathbf{Sch}/\Phi : \mathbf{Sch}/Y \rightarrow \mathbf{Sch}/Z \). It is readily seen that for an \( S \)-groupoid \( X \), the category \( \mathbf{Sch}/X \) is also an \( S \)-groupoid. The content of the 2-Yoneda Lemma is essentially that the

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1. Homogeneity

Schlessinger’s conditions [Sch68], for a functor of artinian rings, are fundamental to the theory and understanding of infinitesimal deformation theory. This was generalized to groupoids by R.S. Rim [SGA7, Exp. VI], clarifying infinitesimal deformation theory in the presence of automorphisms. These conditions are instances of the notion of homogeneity, which can be traced back to A. Grothendieck [FGA, 195.II]. More recently, a generalisation of these conditions [SGA7, Exp. VI] was considered by J. Wise [Wis11, §2]. In this section, we will develop a relative formulation of homogeneity for use in this paper.

Fix a scheme \( S \). An \( S \)-groupoid is a pair \((X,a_X)\) consisting of a category \( X \) and a fibration in groupoids \( a_X : X \rightarrow \mathbf{Sch}/S \). A 1-morphism of \( S \)-groupoids \( \Phi : (Y,a_Y) \rightarrow (Z,a_Z) \) is a functor \( \Phi : Y \rightarrow Z \) that commutes strictly over \( \mathbf{Sch}/S \). We will typically refer to an \( S \)-groupoid \((X,a_X)\) just as \( X \).

Example 1.1. For any \( S \)-scheme \( T \), there is a canonical functor \( \mathbf{Sch}/T \rightarrow \mathbf{Sch}/S : (W \rightarrow T) \mapsto (W \rightarrow T \rightarrow S) \) which is faithful. In particular, we may view an \( S \)-scheme \( T \) as an \( S \)-groupoid. Thus, a morphism of \( S \)-schemes \( g : U \rightarrow V \) induces a 1-morphism of \( S \)-groupoids \( \mathbf{Sch}/g : \mathbf{Sch}/U \rightarrow \mathbf{Sch}/V \). The converse is also true: any 1-morphism of \( S \)-groupoids \( G : \mathbf{Sch}/U \rightarrow \mathbf{Sch}/V \) is uniquely isomorphic to a 1-morphism of the form \( \mathbf{Sch}/g \) for some morphism of \( S \)-schemes \( g : U \rightarrow V \).

Definition 1.2. For an \( S \)-groupoid \( X \), an \( X \)-scheme is a pair \((T, \sigma_T)\) consisting of an \( S \)-scheme \( T \) together with a 1-morphism of \( S \)-groupoids \( \sigma_T : \mathbf{Sch}/T \rightarrow X \). A morphism of \( X \)-schemes \((f, \alpha_f) : (U, \sigma_U) \rightarrow (V, \sigma_V)\) is given by a morphism of \( S \)-schemes \( f : U \rightarrow V \) together with a 2-morphism \( \alpha_f : \sigma_V \Rightarrow \sigma_V \circ \mathbf{Sch}/f \). The collection of all \( X \)-schemes forms a 1-category, which we denote as \( \mathbf{Sch}/X \).

For a 1-morphism of \( S \)-groupoids \( \Phi : Y \rightarrow Z \) there is an induced functor \( \mathbf{Sch}/\Phi : \mathbf{Sch}/Y \rightarrow \mathbf{Sch}/Z \). It is readily seen that for an \( S \)-groupoid \( X \), the category \( \mathbf{Sch}/X \) is also an \( S \)-groupoid. The content of the 2-Yoneda Lemma is essentially that the
natural 1-morphism of $S$-groupoids $\text{Sch}/X \to X$ is an equivalence. An inverse to this equivalence is given by picking a clivage for $X$.

The principal advantage of working with the fibered category $\text{Sch}/X$ is that it admits a canonical clivage. In practice, this means that given an $X$-scheme $V$, and an $S$-scheme $U$, then for a morphism of $S$-schemes $p : U \to V$, the way to make $U$ an $X$-scheme is already chosen for us: it is the composition $\text{Sch}/U \xrightarrow{\text{Sch}/p} \text{Sch}/V \to X$. It is for this reason that working with $\text{Sch}/X$ greatly simplifies proofs and definitions. Calculations, however, are typically easier to perform in $\text{Sch}/X$.

Given a class $P$ of morphisms of $S$-schemes and an $S$-groupoid $X$, then a morphism of $X$-schemes $p : U \to V$ is said to be $P$ if the underlying morphism of $S$-schemes is $P$. The following definition is a trivial generalization of the ideas of M. Olsson [Ols04, App. A], J. Starr [Sta06, §2], and J. Wise [Wis11 §2].

**Definition 1.3 ($P$-Homogeneity).** Fix a scheme $S$ and a class $P$ of morphisms of $S$-schemes. A 1-morphism of $X$-schemes $\Phi : Y \to Z$ is $P$-homogeneous if the following conditions are satisfied.

(H$_P^1$) A commutative diagram in the category of $Y$-schemes

\[
\begin{array}{ccc}
T & \xrightarrow{i} & T' \\
\downarrow{p} & & \downarrow{p} \\
V & \xrightarrow{i} & W,
\end{array}
\]

where $i$ is a locally nilpotent closed immersion and $p$ is $P$, is cocartesian in the category of $Z$-schemes if and only if it is cocartesian in the category of $Y$-schemes.

(H$_P^2$) If a diagram of $Y$-schemes $[V \xrightarrow{i} T \xrightarrow{p} T']$, where $i$ is a locally nilpotent closed immersion and $p$ is $P$, admits a colimit in the category of $Z$-schemes, then there exists a commutative diagram of $Y$-schemes:

\[
\begin{array}{ccc}
T' & \xleftarrow{i} & T \\
\downarrow{p} & & \downarrow{p} \\
V & \xrightarrow{i} & W,
\end{array}
\]

An $S$-groupoid $X$ is $P$-homogeneous if its structure 1-morphism is $P$-homogeneous.

For homogeneity, we will be interested in the following classes of morphisms:

- **Nil** – locally nilpotent closed immersions,

- **Cl** – closed immersions,

- **rNil** – morphisms $T \to V$ such that there exists $(T_0 \to T) \in \text{Nil}$ with the composition $(T_0 \to T \to V) \in \text{Nil}$,

- **rCl** – morphisms $T \to V$ such that there exists $(T_0 \to T) \in \text{Nil}$ with the composition $(T_0 \to T \to V) \in \text{Cl}$,

- **Aff** – affine morphisms.

By [EGA] IV.18.12.11 universal homeomorphisms are integral, thus affine. Hence, it is readily deduced that we have a containment of classes of morphisms of $S$-schemes:

\[
\text{Nil} \subseteq \text{Cl} \subseteq r\text{Cl} \subseteq \text{Aff}.
\]

In [HR12 App. A] it is shown that if $X$ is limit preserving, in the sense of [Art74 §1], and a stack for the Zariski topology, then $r\text{Cl}$-homogeneity is equivalent to the condition (S1') of [Art74 2.3].
J. Wise [Wis11 Prop. 2.1] has shown that every algebraic stack is \textit{Aff}-homogeneous. In Appendix A we generalize results of D. Ferrand [Fer03] and obtain techniques to prove that many “geometric” moduli problems are \textit{Aff}-homogeneous. We record for frequent future reference the following

**Lemma 1.4.** Fix a scheme $S$, a class of morphisms $P \subseteq \text{Aff}$, a $P$-homogeneous $S$-groupoid $X$, and a diagram of $X$-schemes $[V \xleftarrow{p} T \xrightarrow{i} T']$, where $i$ is a locally nilpotent closed immersion and $p$ is $P$. Then, there exists a cocartesian diagram in the category of $X$-schemes:

$$
\begin{array}{ccc}
T & \xrightarrow{i} & T' \\
p & \downarrow & \downarrow \\
V & \xrightarrow{i'} & V'.
\end{array}
$$

This diagram is also cocartesian in the category of $S$-schemes, the morphism $i'$ is a locally nilpotent closed immersion, $p'$ is affine, and the induced homomorphism of sheaves:

$$\mathcal{O}_{V'} \rightarrow i'_* \mathcal{O}_V \times_{p'_* \mathcal{O}_{T'}} p'_* \mathcal{O}_{T'}$$

is an isomorphism.

**Proof.** By Proposition A.2 (or [Fer03 Thm. 7.1]) there is a cocartesian diagram in the category of $S$-schemes:

$$
\begin{array}{ccc}
T & \xrightarrow{i} & T' \\
p & \downarrow & \downarrow \\
V & \xrightarrow{i'} & V'.
\end{array}
$$

The morphism $i'$ is a locally nilpotent closed immersion, $p'$ is affine, and the induced homomorphism of sheaves $\mathcal{O}_{V'} \rightarrow i'_* \mathcal{O}_V \times_{p'_* \mathcal{O}_{T'}} p'_* \mathcal{O}_{T'}$ is an isomorphism. By Condition $(H_P^2)$ for $X$, there is thus a commutative diagram of $X$-schemes:

$$
\begin{array}{ccc}
T & \xrightarrow{i} & T' \\
p & \downarrow & \downarrow \\
V & \xrightarrow{i'} & V'.
\end{array}
$$

Taking the image of this diagram in the category of $S$-schemes, the universal property of the colimit $V'$ in the category of $S$-schemes produces a unique $S$-morphism $V' \rightarrow W$ which makes everything commute, giving $V'$ the structure of an $X$-scheme. The $S$-morphisms $i'$ and $p'$ are promoted to $X$-morphisms, and our original diagram becomes a commutative diagram in the category of $X$-schemes. Condition $(H_P^2)$ now implies that it is cocartesian in the category of $X$-schemes. \qed

The following definition is a convenient computational tool. A 1-morphism of $S$-groupoids $\Phi : Y \rightarrow Z$ is \textit{formally étale} if for any $Z$-scheme $T'$ and any locally nilpotent closed immersion of $Z$-schemes $T \hookrightarrow T'$, then any $Y$-scheme structure on $T$ which is compatible with its $Z$-scheme structure under $\Phi$, lifts uniquely to a compatible $Y$-scheme structure on $T'$. That is, there is always a unique solution to the following lifting problem:

$$
\begin{array}{ccc}
T' & \xrightarrow{i} & Y \\
p & \downarrow & \downarrow \Phi \\
T & \xrightarrow{i'} & Z.
\end{array}
$$

**Lemma 1.5.** Fix a scheme $S$, a 1-morphism of $S$-groupoids $\Phi : Y \rightarrow Z$, and a class $P \subseteq \text{Aff}$ of morphisms of $S$-schemes.

1. If $\Phi$ is $P$-homogeneous, then for any other $P$-homogeneous 1-morphism $W \rightarrow Y$, the composition $W \rightarrow Z$ is $P$-homogeneous.
(2) If $Z$ is $P$-homogeneous, then a cocartesian diagram of $Y$-schemes:

$$
\begin{array}{ccc}
T' & \xrightarrow{i} & T'' \\
\downarrow{p'} & & \downarrow{p''} \\
V & \xrightarrow{f} & V',
\end{array}
$$

where $i$ is a locally nilpotent closed immersion and $p$ is $P$, is also cocartesian in the category of $Z$-schemes.

(3) If $Z$ is $P$-homogeneous, then the 1-morphism $\Phi$ is $P$-homogeneous if and only if for any $Z$-scheme $T$, the $T$-groupoid $Y \times_Z (\text{Sch}/T)$ is $P$-homogeneous.

(4) If $Z$ and $\Phi$ are $P$-homogeneous, then for any $P$-homogeneous 1-morphism of $S$-groupoids $\Psi : W \to Z$, the 1-morphism $Y \times_Z W \to W$ is $P$-homogeneous.

(5) If $Z$ and $\Phi$ are $P$-homogeneous, then the diagonal 1-morphism $\Delta_{\Phi} : Y \to Y \times_Z Y$ is $P$-homogeneous.

(6) If $Z$ is $P$-homogeneous and $\Phi$ is formally étale, then $\Phi$ is $P$-homogeneous.

Proof. For (2), by Lemma [4] the diagram of $Y$-schemes $[V \xrightarrow{f} T \xrightarrow{i} T'']$ fits into a cocartesian diagram of $Z$-schemes:

$$
\begin{array}{ccc}
T' & \xrightarrow{i} & T'' \\
\downarrow{p'} & & \downarrow{p''} \\
V & \xrightarrow{f} & V',
\end{array}
$$

The universal property defining this square gives a unique map of $Z$-schemes $\tilde{V} \to V'$. Since $V'$ is a $Y$-scheme, $\tilde{V}$ becomes a $Y$-scheme, and the diagram above is promoted to a commutative diagram of $Y$-schemes. We now apply the universal property defining $V'$ and obtain a unique morphism of $Y$-schemes $V' \to \tilde{V}$. The morphisms of $Y$-schemes $V' \cong \tilde{V}$ are readily seen to be mutually inverse. The remainder of the claims are straightforward. \qed

2. Extensions

The results of this section are well-known to experts, and similar to those obtained by H. Flenner [Fle81] and J. Wise [Wis11] [§2.3].

Fix a scheme $S$ and an $S$-groupoid $X$. An $X$-extension is a square zero closed immersion of $X$-schemes $i : T \hookrightarrow T'$. An obligatory observation is that the $i^{-1} \mathcal{O}_T$-module $\ker(i^{-1} \mathcal{O}_T \to \mathcal{O}_T)$ is canonically a quasicoherent $\mathcal{O}_T$-module. If the $X$-scheme $T$ is affine, so is the scheme $T'$ [EGA] I.5.1.9. A morphism of $X$-extensions $(i_1 : T_1 \hookrightarrow T'_1) \to (i_2 : T_2 \hookrightarrow T'_2)$ is a commutative diagram of $X$-schemes:

$$
\begin{array}{ccc}
T_1 & \xrightarrow{i_1} & T'_1 \\
\downarrow & & \downarrow \\
T_2 & \xrightarrow{i_2} & T'_2.
\end{array}
$$

In a natural way, the collection of $X$-extensions forms a category, which we denote as $\text{Exal}_X$. There is a natural functor $\text{Exal}_X \to \text{Sch}/X : (i : T \hookrightarrow T') \mapsto T$.

We denote by $\text{Exal}_X(T)$ the fiber of the category $\text{Exal}_X$ over the $X$-scheme $T$. An $X$-extension of $T$ is an object of $\text{Exal}_X(T)$. There is a natural functor:

$$
\text{Exal}_X(T) \to \text{QCoh}(T) : (i : T \hookrightarrow T') \mapsto \ker(i^{-1} \mathcal{O}_T \to \mathcal{O}_T).
$$

We denote by $\text{Exal}_X(T, I)$ the fiber category of $\text{Exal}_X(T)$ over the quasicoherent $\mathcal{O}_T$-module $I$. An $X$-extension of $T$ by $I$ is an object of $\text{Exal}_X(T, I)$. A morphism
Moreover, via the natural map \( \mathcal{O}_T \to \mathcal{O}_{T_2} \) in \( \text{Exal}_X(T, I) \) induces a commutative diagram of sheaves of rings on the topological space \( |T| \):

\[
\begin{array}{c}
0 \rightarrow I \rightarrow \mathcal{O}_{T_2} \rightarrow \mathcal{O}_T \rightarrow 0 \\
\end{array}
\]

The Snake Lemma implies that the morphism of \( T_1 \to T_2 \) is an isomorphism, thus the category \( \text{Exal}_X(T, I) \) is a groupoid. The following is a triviality that we record here for future reference.

**Lemma 2.1.** Fix a scheme \( S \), a formally étale 1-morphism of \( S \)-groupoids \( X \to Y \), an \( X \)-scheme \( T \), and a quasicoherent \( \mathcal{O}_T \)-module \( I \). Then, the natural functor:

\[ \text{Exal}_X(T, I) \to \text{Exal}_Y(T, I) \]

is an equivalence of categories.

Fix a scheme \( W \) and a quasicoherent \( \mathcal{O}_W \)-module \( J \). Then, the quasicoherent \( \mathcal{O}_W \)-module \( \mathcal{O}_W \oplus J \) is readily seen to be a ring: for an open subset \( U \subseteq W \) and \((w, j), (w', j') \in \Gamma(U, \mathcal{O}_W) \) we set

\[ (w, j) \cdot (w', j') = (ww', wj' + w'j). \]

Moreover, via the natural map \( \mathcal{O}_W \to \mathcal{O}_W \oplus J : w \mapsto (w, 0) \), we see that the ring \( \mathcal{O}_W \oplus J \) admits a canonical structure as an \( \mathcal{O}_W \)-algebra, which we denote as \( \mathcal{O}_W[J] \). We now set \( W[J] \) to be the \( W \)-scheme \( \text{Spec}_{\mathcal{O}_W[J]}(\mathcal{O}_W[J]) \). Corresponding to the natural surjection of \( \mathcal{O}_W \)-algebras \( \mathcal{O}_W[J] \to \mathcal{O}_W \), we obtain a canonical \( W \)-extension of \( W \) by \( J \), which we denote as \( (W, J) \) and call the trivial \( W \)-extension of \( W \) by \( J \). In particular, the structure morphism \( r_{W,J} : W[J] \to W \) is a retraction of the morphism \( r_{W,J} : W \to W[J] \).

For a morphism of \( X \)-schemes \( q : U \to V \), denote by \( \text{Ret}_X(U/V) \) the set of \( X \)-retractions to the morphism \( q : U \to V \). That is,

\[ \text{Ret}_X(U/V) = \{ r \in \text{Hom}_{\text{Sch}/X}(V, U) : rq = \text{Id}_U \}. \]

**Lemma 2.2.** Fix a scheme \( S \), an \( S \)-groupoid \( X \), an \( X \)-scheme \( T \), a quasicoherent \( T \)-module \( I \), and an \( X \)-extension \( (i : T \hookrightarrow T') \) of \( T \) by \( I \). Then, there is a natural bijection:

\[ \text{Hom}_{\text{Exal}_X(T, I)}(i : T \hookrightarrow T'), (i_T : T \hookrightarrow T[I]) \to \text{Ret}_X(T/T'). \]

**Proof.** For a morphism of \( X \)-extensions \( (T \hookrightarrow T') \to (T' \hookrightarrow T[I]) \), the composition \( T' \to T[I] \overset{r_{T, I}}{\longrightarrow} T \) defines an \( X \)-retraction to \( i \). This assignment is bijective. \( \square \)

Assuming some homogeneity really gets us something.

**Proposition 2.3.** Fix a scheme \( S \), an \( S \)-groupoid \( X \), and an \( X \)-scheme \( T \), then the functor \( \text{Exal}_X(T) \to \text{QCoh}(T)^\circ \) is a fibration in groupoids. If the \( S \)-groupoid \( X \) is \( \text{Nil} \)-homogeneous, then \( \forall I \in \text{QCoh}(T) \), \( \text{Exal}_X(T, I) \) is a Picard category.

**Proof.** Fix a morphism \( \alpha : J \to I \) in \( \text{QCoh}(T)^\circ \). This corresponds to a morphism of quasicoherent \( \mathcal{O}_T \)-modules \( \alpha : I \to J \). Also, fix an \( X \)-extension \( (i : T \hookrightarrow T'_I) \) of \( T \) by \( I \). On the topological space \( |T| \) we obtain a commutative diagram of sheaves of abelian groups with exact rows:

\[
\begin{array}{c}
0 \rightarrow I \rightarrow \mathcal{O}_{T'_I} \rightarrow \mathcal{O}_T \rightarrow 0 \\
\uparrow \alpha \\
0 \rightarrow J \rightarrow \mathcal{O}_{T'_I} \oplus_I J \rightarrow \mathcal{O}_T \rightarrow 0,
\end{array}
\]
Corollary 2.4. We note that the 0-object of the abelian group Der\(_X(T,J)\) is a sheaf of rings and that the homomorphism \(\alpha\) is a ring homomorphism. The subsheaf \(J \subseteq \sO\) defines a square zero sheaf of ideals and as \(\alpha\) is \(\sO\)-quasicoherent, one immediately concludes that the closed immersion \((|T|, \sO\) is an \(S\)-scheme, \(T\), and that we have defined an \(S\)-extension \((i_\alpha : T \hookrightarrow T')\) of \(T\) by \(J\). The morphism of \(S\)-schemes \(T' \to T\) promotes the \(S\)-extension \(i_\alpha\) to an \(X\)-extension of \(T\) by \(J\). It is immediate that the resulting arrow \(i_\alpha \to i\) in \(\text{Exal}_X(T)\) is strongly cartesian over the arrow \(\alpha : J \to I\) in \(\text{QCoh}(T)\), and we deduce the first claim.

For the second claim, the fibration \(\text{Exal}_X(T) \to \text{QCoh}(T)\) induces for \(M, N \in \text{QCoh}(T)\), a functor:

\[
\pi_{M,N} : \text{Exal}_X(T,M \times N) \to \text{Exal}_X(T,M) \times \text{Exal}_X(T,N).
\]

Note that this functor is not unique, but for any other choice of such a functor \(\pi'_{M,N}\), there is a unique natural isomorphism of functors \(\pi_{M,N} \cong \pi'_{M,N}\). This renders the Picard category structure on \(\text{Exal}_X(T,I)\) as essentially unique (on the level of isomorphism classes of objects, the abelian group structure is unique).

By [Gro68 §1.2] it is sufficient to show that the functor \(\pi_{M,N}\) is an equivalence, which we show using the arguments of [EGA 0IV.18.3]. For the essential surjectivity, we fix \(X\)-extensions \((i_M : T \hookrightarrow T'_M)\) and \((i_N : T \hookrightarrow T'_N)\) of \(T\) by \(M\) and \(N\) respectively. Since \(X\) is \(\text{Nil}\)-homogeneous, by Lemma 1.4 there is a cocartesian diagram in the category of \(X\)-schemes:

\[
\begin{array}{ccc}
T' & \xrightarrow{i_M} & T'_M \\
\downarrow{i_N} & & \downarrow{} \\
T'_N & \rightarrow & T'.
\end{array}
\]

The resulting closed immersion \(i : T \hookrightarrow T'\) defines an \(X\)-extension of \(T\) by \(M \times N\). Moreover, it is plain to see that \(\pi_{M,N}(i) \cong (i_M,i_N)\). The full faithfulness of the functor \(\pi_{M,N}\) follows from a similar argument. \(\square\)

Denote the set of isomorphism classes of the category \(\text{Exal}_X(T, I)\) by \(\text{Exal}_X(T, I)\).

By Proposition 2.3 if \(X\) is \(\text{Nil}\)-homogeneous, there are additive functors:

\[
\begin{align*}
\text{Der}_X(T,-) : & \text{QCoh}(T) \to \text{Ab} : I \mapsto \text{Aut}_{\text{Exal}_X(T, I)}(\text{ι}_{T,I}) \\
\text{Exal}_X(T,-) : & \text{QCoh}(T) \to \text{Ab} : I \mapsto \text{Exal}_X(T, I).
\end{align*}
\]

We note that the 0-object of the abelian group \(\text{Der}_X(T, I)\) corresponds to the identity automorphism, and the 0-object of the group \(\text{Exal}_X(T, I)\) corresponds to the isomorphism class containing the \(X\)-extension \((i_{T,I} : T \hookrightarrow T[I])\). Increasing the homogeneity, more structure is obtained.

Corollary 2.4. Fix a scheme \(S\), a \(\text{rNil}\)-homogeneous \(S\)-groupoid \(X\), and an \(X\)-scheme \(T\). Then, for each short exact sequence of quasicoherent \(\sO\)-modules:

\[
0 \to K \to M \to C \to 0
\]

there is a natural 6-term exact sequence of abelian groups:

\[
\begin{array}{cccccc}
0 & \to & \text{Der}_X(T,K) & \to & \text{Der}_X(T,M) & \to & \text{Der}_X(T,C) \\
& & \text{\textbf{α}} & & & \\
& & \downarrow & & & \\
& & \text{Exal}_X(T,K) & \to & \text{Exal}_X(T,M) & \to & \text{Exal}_X(T,C).
\end{array}
\]
Proof. This is actually a consequence of [Wis11] Prop. 2.3(iv)], where it was shown that the fibered category \( \text{Exal}_X(T) \to \text{Qcoh}(T)^p \) is additive and left-exact, in the sense of [Gro68]. We will not follow this route, but instead utilize arguments similar to [EGA, 0IV.20.2.2-3]. We will also only prove the exactness of the last three terms, since this is all that is necessary in this paper.

Given an \( X \)-extension \((i_M : T \to T'_M)\) of \( T \) by \( M \), suppose that its image, \((i_C : T \hookrightarrow T'_C)\), in \( \text{Exal}_X(T,C) \) is 0. By Lemma 2.2 this is equivalent to the existence of an \( X \)-retraction \( r : T'_C \to T \) of the \( X \)-morphism \( i_C \). Proposition 2.3 implies that there is an induced \( X \)-morphism \( T'_C \to T_M \). Since the \( \mathcal{O}_T \)-module homomorphism \( M \to C \) is surjective with kernel \( K \), it follows that the \( X \)-morphism \((T'_C \hookrightarrow T_M)\) defines an \( X \)-extension of \( T'_C \) by \( K \). Since \( X \) is \( \mathbb{r} \text{Nil} \)-homogeneous, Lemma 1.4 implies that there is a cocartesian diagram in the category \( X \)-schemes:

\[
\begin{array}{c}
T'_C \xrightarrow{i} T'_M \\
\downarrow \quad \quad \downarrow \quad \quad \quad \downarrow \\
T_C \xrightarrow{i} T'.
\end{array}
\]

Certainly, \((i : T \hookrightarrow T')\) defines an \( X \)-extension of \( T \) by \( K \) and the image of the \( X \)-extension \( i \) in \( \text{Exal}_X(T,M) \) is readily seen to be \( i_M \).

Strengthening our homogeneity assumption again, we see more.

**Corollary 2.5.** Fix a scheme \( S \), an \( \text{Aff} \)-homogeneous \( S \)-groupoid \( X \), and an \( X \)-scheme \( T \). For any affine and etale morphism \( p : U \to T \), and any quasicoherent \( \mathcal{O}_U \)-module \( J \), there is an equivalence of Picard categories:

\[
\text{Exal}_X(U,J) \to \text{Exal}_X(T,p_*J).
\]

**Proof.** First, we observe that given any etale morphism \( q : V \to T \) and an \( X \)-extension \( T \hookrightarrow T' \) of \( T \) by \( K \), then by [EGA, IV.18.1.2], there exists a unique \( X \)-extension \( V \hookrightarrow V' \) of \( V \) by \( q*K \) together with an etale morphism \( V' \to T \) such that \( V' \times_T T \cong V \) and the second projection is the map \( V \to T \). This describes a functor \( q^* : \text{Exal}_X(T,K) \to \text{Exal}_X(V,q*K) \). Applying this with \( K = p_*J \), we obtain a functor \( \text{Exal}_X(T,p_*J) \to \text{Exal}_X(U,p^*p_*J) \). Applying Proposition 2.3 to the \( \mathcal{O}_U \)-module homomorphism \( p^*p_*J \to J \), there is an induced functor \( \text{Exal}_X(U,p^*p_*J) \to \text{Exal}_X(U,J) \). Composing these two functors produces a functor \( \text{Exal}_X(T,p_*J) \to \text{Exal}_X(U,J) \).

Also, since the morphism \( p : U \to T \) is affine, \( \text{Aff} \)-homogeneity implies that there is a functor \( p_* : \text{Exal}_X(U,J) \to \text{Exal}_X(T,p_*J) \). Indeed, given an \( X \)-extension \((U \hookrightarrow U')\) of \( U \) by \( J \), the \( \mathbb{Aff} \)-homogeneity of \( X \), combined with Lemma 1.4 gives a cocartesian diagram of \( X \)-schemes:

\[
\begin{array}{c}
U' \xrightarrow{i} U \\
\downarrow \quad \quad \downarrow \quad \quad \quad \downarrow \\
T^\circ \xrightarrow{i} T'.
\end{array}
\]

It is readily verified that the \( X \)-morphism \((T \hookrightarrow T')\) defines an \( X \)-extension of \( T \) by \( p_*J \). The functors \( \text{Exal}_X(T,p_*J) \cong \text{Exal}_X(U,J) \) are clearly quasi-inverse. \( \square \)

3. **Limit preservation**

In this section we prove that the functors defined in [2] \( M \mapsto \text{Der}_X(T,M) \) and \( M \mapsto \text{Exal}_X(T,M) \), frequently preserve direct limits. We also relativize the notion of limit preserving \( S \)-groupoid [Art74, §1].

**Definition 3.1.** Fix a scheme \( S \). A 1-morphism of \( S \)-groupoids \( \Phi : Y \to Z \) is **limit preserving** if given an inverse system of quasicompact and quasiseparated \( Z \)-schemes with affine transition maps \( \{W_j\}_{j \in J} \), as well as a \( Y \)-scheme \( V \), such
that as a $Z$-scheme it is an inverse limit of \{\(W_j\)\}_{j \in J}, then there exists \(j_0 \in J\) and
an essentially unique $Y$-scheme structure on \(W_{j_0}\) (i.e.
for any two choices and all \(j \gg j_0\) the two induced
$Y$-scheme structures on \(W_j\) are isomorphic) such that
the induced diagram of $Y$-schemes \{\(W_j\)\}_{j \geq j_0}\ has
limit $V$. An $S$-groupoid $X$ is limit preserving.
if its structure morphism to $\text{Sch}/S$ is so. Similarly, an $X$-scheme $T$ is
limit preserving if its structure 1-morphism $\text{Sch}/T \rightarrow X$ is so.

Analogous to Lemma 1.5, we have the following easily verified lemma.

**Lemma 3.2.** Fix a scheme $S$ and a 1-morphism of $S$-groupoids $\Phi : Y \rightarrow Z$.

1. If $Z$ is a Zariski stack, then it is limit preserving if and only if for any
inverse system of affine $S$-schemes $\{\text{Spec } A_j\}_{j \in J}$ with limit $\text{Spec } A$, the
natural functor:
\[
\lim_j Z(\text{Spec } A_j) \rightarrow Z(\text{Spec } A)
\]
is an equivalence.
2. If $Z$ is an algebraic stack, then it is limit preserving if and only if it is
locally of finite presentation over $S$.
3. If $\Phi$ is limit preserving, then for any other limit preserving 1-morphism
$W \rightarrow Y$, the composition $W \rightarrow Z$ is limit preserving.
4. The 1-morphism $\Phi$ is limit preserving if and only if for any $Z$-scheme $T$,
the $T$-groupoid $Y \times_Z \text{Sch}/T$ is limit preserving.
5. If $\Phi$ is limit preserving, then for any 1-morphism of $S$-groupoids $W \rightarrow Z$,
the 1-morphism $Y \times_Z W \rightarrow W$ is limit preserving.
6. If $\Phi$ is limit preserving, then the diagonal 1-morphism $\Delta_\Phi : Y \rightarrow Y \times_Z Y$ is
limit preserving.

**Proof.** The only non-obvious point is (2), which follows from [LMB, 4.15–18].

**Example 3.3.** Fix a scheme $S$ and a limit preserving $S$-groupoid $X$. Then, an
$X$-scheme is limit preserving if and only if it is locally of finite presentation over $S$.

We now have the main result of this section.

**Proposition 3.4.** Fix a scheme $S$, a Nil-homogeneous $S$-groupoid $X$, and a quasi-
compact, quasiseparated, limit preserving $X$-scheme $T$.

1. The functor $M \mapsto \text{Der}_X(T, M)$ preserves direct limits.
2. If, in addition, $X$ is limit preserving, then the functor $M \mapsto \text{Exal}_X(T, M)$
preserves direct limits.

**Proof.** Throughout, we fix a directed system of quasicoherent $\mathcal{O}_T$-modules $\{M_j\}_{j \in J}$
with direct limit $M$. Certainly, in the category of $X$-schemes the natural map
$T[M] \rightarrow \lim_j T[M_j]$ is an isomorphism. For (1), by Lemma 2.2 we have:
\[
\text{Der}_X(T, M) = \text{Ret}_X(T/T[M]) = \lim_j \text{Ret}_X(T/T[M_j]) = \lim_j \text{Der}_X(T, M_j).
\]
For (2), we first show that the map $\lim_{\rightarrow j\in J} \text{Exal}_X(T, M_j) \rightarrow \text{Exal}_X(T, M)$ is
injective. Lemma 2.2 shows that an $X$-extension $(T \hookrightarrow T'')$ of $T$ by a quasicoherent
$\mathcal{O}_T$-module $N$ represents 0 in $\text{Exal}_X(T, N)$ if and only if $\text{Ret}_X(T/T'') \neq \emptyset$. So,
given a compatible collection of $X$-extensions $(T \hookrightarrow T'_j)$ of $T$ by $M_j$, with limit
$(T \hookrightarrow T')$, then since $\text{Ret}_X(T/T') = \lim_{\rightarrow j} \text{Ret}_X(T/T'_j)$, we deduce that the map
$\lim_{\rightarrow j} \text{Exal}_X(T, M_j) \rightarrow \text{Exal}_X(T, M)$ is injective.

We now show that the natural map $\lim_{\rightarrow j} \text{Exal}_X(T, M_j) \rightarrow \text{Exal}_X(T, S)$ is
surjective. First, we prove the result in the case where $X = S$ and $T$ are
affine. Since $T$ is affine and of finite presentation over $S$, there exists an integer
n and a closed immersion \( k : T \hookrightarrow \mathbb{A}_S^n \). By [EGA, 0IV, 20.2.3], there is a functorial surjection for every \( \mathcal{O}_T \)-module \( K : \text{Hom}_{\mathcal{O}_T}(k^*\Omega_{S/S}, K) \to \text{Exal}_S(T, K) \). Since the \( \mathcal{O}_T \)-module \( k^*\Omega_{S/S} \) is finite free, it follows that the functor \( K \mapsto \text{Hom}_{\mathcal{O}_T}(k^*\Omega_{S/S}, K) \) preserves direct limits. Direct limits are exact so we have a surjection \( \lim_{\longrightarrow j} \text{Exal}_S(T, M_j) \to \text{Exal}_S(T, M) \).

If \( S \) and \( T \) are no longer assumed to be affine, a straightforward Zariski descent argument, combined with the affine case already considered, shows that we also have a bijection \( \lim_{\longrightarrow j} \text{Exal}_S(T, M_j) \to \text{Exal}_S(T, M) \). Now for the general case: given \( (T \to T') \in \text{Exal}_X(T, M) \), by the above considerations there exists a \( j_0 \) and an \( S \)-extension of \( T \) by \( M_{j_0} \), \( (T \hookrightarrow T'_{j_0}) \), such that its pushforward along \( M_{j_0} \to M \) is isomorphic to \( (T \hookrightarrow T') \) as an \( S \)-extension. If \( j \geq j_0 \), denote the pushforward of \( (T \hookrightarrow T'_{j_0}) \) along the morphism \( M_{j_0} \to M_j \), \( (T \hookrightarrow T'_{j}) \). There is a natural morphism of \( S \)-schemes \( T_j' \to T_{j_0}' \) and the resulting inverse system \( \{T_j\}_{j \geq j_0} \) has limit \( T' \). Since \( X \) is a limit preserving \( S \)-groupoid, there exists \( j_1 \geq j_0 \) and an \( X \)-scheme structure on \( T_{j_1}' \) such that the resulting inverse system of \( X \)-schemes \( \{T_j'\}_{j \geq j_1} \) has limit \( T' \). The result follows. \( \square \)

4. Formal smoothness and formal versality

In this section we prove the main result of the paper.

**Definition 4.1.** Fix a scheme \( S \), an \( S \)-groupoid \( X \), and an \( X \)-scheme \( T \). Consider the following lifting problem: given a square zero closed immersion of \( X \)-schemes \( Z_0 \hookrightarrow Z \) fitting into a commutative diagram of \( X \)-schemes:

\[
\begin{array}{ccc}
Z_0 & \xrightarrow{g} & T \\
\downarrow & & \downarrow \\
Z & \longrightarrow & X.
\end{array}
\]

We say that the \( X \)-scheme \( T \) is

*formally smooth* if the lifting problem above can always be solved \( \acute{e} \text{tale} \) locally on \( Z \);

*formally versal at \( t \in [T] \) if the lifting problem can be solved whenever \( Z \) is a local artinian, with closed point \( z \), such that \( g(z) = t \), \( \kappa(z) \cong \kappa(t) \), and there is an isomorphism of \( \mathcal{O}_T \)-modules \( \kappa(t) \cong g_*(\mathcal{O}_Z \to \mathcal{O}_{Z_0}) \).

We certainly have the following implication:

formally smooth \( \Rightarrow \) formally versal at all \( t \in [T] \).

In general, there is no reverse implication. We will see, however, that this subtlety vanishes once the \( S \)-groupoid is \( \text{Aff} \)-homogeneous.

**Example 4.2.** Fix an \( S \)-groupoid \( X \) and an \( X \)-scheme \( T \) such that the 1-morphism \( T \to X \) is representable by algebraic spaces which are locally of finite presentation. Then, the \( X \)-scheme \( T \) is formally smooth if and only if the 1-morphism \( T \to X \) is representable by smooth morphisms of algebraic spaces.

There is a tight connection between formal smoothness (resp. formal versality) and \( X \)-extensions in the affine setting. The next result has arguments similar to those of [Fle81, Satz 3.2], but the definitions are slightly different.

**Lemma 4.3.** Fix a scheme \( S \), an \( S \)-groupoid \( X \), and an affine \( X \)-scheme \( T \).

1. If \( X \) is \( \text{Aff} \)-homogeneous and the abelian group \( \text{Exal}_X(T, M) \) is trivial for all quasicoherent \( \mathcal{O}_{\mathcal{T}} \)-modules \( M \), then the \( X \)-scheme \( T \) is formally smooth.
(2) If $X$ is $\mathfrak{Cl}_0$-homogeneous and at a closed point $t \in |T|$, $\text{Exal}_X(T, \kappa(t)) = 0$, then the $X$-scheme $T$ is formally versal at $t$.

(3) If $X$ is $\mathfrak{Cl}_0$-homogeneous and $T$ is noetherian and formally versal at a closed point $t \in |T|$, then $\text{Exal}_X(T, \kappa(t)) = 0$.

Proof. For (1), fix a square zero closed immersion $Z_0 \hookrightarrow Z$ (defined by a quasicoherent $\mathcal{O}_{Z_0}$-module $I$) of $X$-schemes, fitting into a commutative diagram:

$$
\begin{array}{ccc}
Z_0 & \xrightarrow{g} & T \\
\downarrow & & \downarrow \\
Z & \xrightarrow{\eta} & X.
\end{array}
$$

We need to construct an $X$-morphism $Z \to T$ étale locally on $Z$. Thus we easily reduce to the case where $Z_0$, $Z$, and $T$ are affine. Lemma 1.4 now gives a cocartesian diagram of $X$-schemes:

$$
\begin{array}{ccc}
Z_0 & \xrightarrow{g} & T \\
\downarrow & & \downarrow \\
Z & \xrightarrow{\eta} & T',
\end{array}
$$

where the $X$-morphism $T \to T'$ defines an $X$-extension of $T$ by $g_*I$. By hypothesis, $\text{Exal}_X(T, g_*I) = 0$, and Lemma 2.2 produces an $X$-retraction $T' \to T$. The composition $Z \to T' \to T$ gives the required lifting. The claim (2) follows from an identical argument just given for (1).

For (3), given an $X$-extension $T \hookrightarrow T'$ of $T$ by $\kappa(t)$, write $T = \text{Spec } R$, $T' = \text{Spec } R'$, $m = t \in |T|$, and $I = \ker(R' \to R) \cong R/m$. Let the ideal $m' \triangleq R'$ denote the (unique) maximal ideal induced by $m$. For $n \geq 0$ define $R_n = R/m^{n+1}$, $R'_n = R'/m'^{n+1}$, and $I_n = \ker(R'_n \to R_n)$. The following diagram commutes:

$$
\begin{array}{ccc}
\text{Spec } R_n & \xrightarrow{T} & T \\
\downarrow & & \downarrow \\
\text{Spec } R'_n & \xrightarrow{T'} & T' \\
\downarrow & & \downarrow \\
\text{Spec } R_n & \xrightarrow{\eta} & X.
\end{array}
$$

Formal versality at $t \in |T|$ gives for each $n \geq 0$ an $X$-morphism $\text{Spec } R'_n \to T$ completing the diagram. For each $n \geq 0$ there is also a cocartesian diagram of $X$-schemes (Lemma 1.4):

$$
\begin{array}{ccc}
\text{Spec } R_n & \xrightarrow{T_n} & T \\
\downarrow & & \downarrow \\
\text{Spec } R'_n & \xrightarrow{T'_n} & T' \\
\downarrow & & \downarrow \\
\text{Spec } R_n & \xrightarrow{\eta} & T_n.
\end{array}
$$

Thus, an $X$-morphism $\text{Spec } R'_n \to T$ induces a unique $X$-retraction $\tilde{T}_n \to T$ to the $X$-extension $T \hookrightarrow \tilde{T}_n$. Moreover, there is a unique morphism of $X$-extensions $\alpha : (T \hookrightarrow \tilde{T}_n) \to (T \hookrightarrow T')$. Since the $R$-module $I$ is of length 1, it follows that for $n \gg 0$ the surjective map $I \to I_n$ is an isomorphism. Thus, the morphism $\alpha$ is an isomorphism for $n \gg 0$ and the $X$-extension $T \hookrightarrow T'$ admits an $X$-retraction. By Lemma 2.2 $\text{Exal}_X(T, \kappa(t)) = 0$. \hfill \Box

Remark 4.4. With some additional work and some finiteness assumptions, it is possible to prove the converse to Lemma 1.3 (1).

Fix an affine scheme $T$ and an additive functor $F : \mathbf{QCoh}(T) \to \mathbf{Ab}$. The functor $F$ is \textit{finitely generated} if there exists a quasicoherent $\mathcal{O}_T$-module $I$ and an object $\eta \in F(I)$ such that for all $M \in \mathbf{QCoh}(T)$, the induced morphism of abelian groups $\text{Hom}_{\mathcal{O}_T}(I, M) \to F(M) : f \mapsto f_*\eta$ is surjective. The notion of finite generation of a functor is due to M. Auslander \cite{Aus66}.
The functor $F$ is half-exact if for any short exact sequence in $\text{QCoh}(T)$, $0 \to M' \to M \to M'' \to 0$, the sequence $F(M') \to F(M) \to F(M'')$ is exact.

If, in addition, the scheme $T$ is noetherian, and $F$ is half-exact, sending coherent $O_T$-modules to coherent $O_T$-modules, then A. Ogus and G. Bergman have shown Thm. 2.1] that if for all closed points $t \in |T|$ we have $F(\kappa(t)) = 0$, then $F$ is the zero functor. If $F$ is finitely generated, then this result can be refined. Indeed, it is shown in Cor. 6.7] that if $F(\kappa(t)) = 0$, then there exists an affine open subscheme $p : U \to T$ such that the composition $F \circ p_*(-) : \text{QCoh}(U) \to \text{Ab}$ is identically zero. We now use this to prove the main technical result of the paper.

**Theorem 4.5.** Fix a locally noetherian scheme $S$, an $\text{Aff}$-homogeneous and limit preserving $S$-groupoid $X$, and an affine $X$-scheme $T$, locally of finite type over $S$. If the functor $M \mapsto \text{Exal}_X(T,M)$ is finitely generated and $T$ is formally versal at a closed point $t \in |T|$, then it is formally smooth in an open neighbourhood of $t$.

**Proof.** By Lemma 4.3, $\text{Exal}_X(T,\kappa(t)) = 0$. By Corollary 2.4 the functor $M \mapsto \text{Exal}_X(T,M)$ is half-exact, and by Proposition 5.4 it commutes with direct limits. As $\text{Exal}_X(T,\_)$ is finitely generated, [Hal12 Cor. 6.7] now applies. Thus, there exists an affine open neighbourhood $p : U \to T$ of $t$ such that the functor $\text{Exal}_X(T,p_*(-)) : \text{QCoh}(U) \to \text{Ab}$ is the zero functor. By Corollary 2.5 $\text{Exal}_X(U,\_)$ is also the zero functor. By Lemma 5.3, we conclude that $U$ is a formally smooth $X$-scheme.

We will defer the proof of the following Corollary until §8 as we currently lack the necessary computational tools (e.g. the relationship between Exal and Def).

**Corollary 4.6.** Fix an excellent scheme $S$. An $S$-groupoid $X$ is an algebraic $S$-stack, locally of finite presentation over $S$, if and only if the following conditions are satisfied.

1. $X$ is a stack over the site $(\text{Sch}/S)_{\text{et}}$.
2. $X$ is limit preserving.
3. $X$ is $\text{Aff}$-homogeneous.
4. The diagonal $\Delta_X/S : X \to X \times_S X$ is representable by algebraic spaces.
5. For any local noetherian ring $(B,m)$, such that the ring $B$ is $m$-adically complete, with an $S$-scheme structure $\text{Spec} B \to S$ such that the induced morphism $\text{Spec}(B/m) \to S$ is locally of finite type, then the natural functor:
   
   $$X(\text{Spec} B) \to \varprojlim_n X(\text{Spec}(B/m^n))$$

   has dense image.
6. For any affine $X$-scheme $T$, locally of finite type over $S$, the functor $M \mapsto \text{Exal}_X(T,M)$ is finitely generated.

5. **Coherent functors**

Fix a ring $A$. An additive functor $F : \text{Mod}(A) \to \text{Ab}$ is coherent, if there exists an $A$-module homomorphism $f : I \to J$ and an element $\eta \in F(I)$, inducing an exact sequence for any $A$-module $M$:

$$\text{Hom}_A(J,M) \longrightarrow \text{Hom}_A(I,M) \longrightarrow F(M) \longrightarrow 0.$$ 

We refer to the data $(f : I \to J, \eta)$ as a presentation for $F$. For a comprehensive account of coherent functors, we refer the interested reader to [Aus66]. Some stronger results that are available in the noetherian situation are developed in [Har98]. Here we record some simple consequences of [Aus66 Prop. 2.1].
Lemma 5.1. Fix a ring $A$. For each $i = 1, \ldots, 5$, let $H_i : \text{Mod}(A) \rightarrow \text{Ab}$ be an additive functor fitting into an exact sequence:

\[ H_1 \rightarrow H_2 \rightarrow H_3 \rightarrow H_4 \rightarrow H_5. \]

1. If $H_2$, $H_4$ are finitely generated, and $H_5$ is coherent, then $H_3$ is finitely generated.
2. If $H_1$, $H_2$ are finitely generated, and $H_4$, $H_5$ are coherent, then $H_3$ is coherent.

We now have two fundamental examples.

Example 5.2. Fix a scheme $S$ and a locally noetherian algebraic $S$-stack $X$. Let $T$ be an affine and noetherian $X$-scheme, which is locally of finite type. Then, the functors $M \mapsto \text{Der}_X(T, M)$ and $M \mapsto \text{Exal}_X(T, M)$ are coherent. Indeed, by Thm. 1.1, there is a bounded above complex of $\mathcal{O}_T$-modules $L_{T/X}$, with coherent cohomology sheaves, as well as functorial isomorphisms $\text{Der}_X(T, M) \cong \text{Ext}^0_{\mathcal{O}_T}(L_{T/X}, M)$ and $\text{Exal}_X(T, M) \cong \text{Ext}^1_{\mathcal{O}_T}(L_{T/X}, M)$ for all quasicoherent $\mathcal{O}_T$-modules $M$. The claim now follows from [Hal12] Ex. 3.13.

The next example is [Hal12] Thm. C.

Example 5.3. Fix an affine scheme $S$ and a morphism of algebraic stacks $f : X \rightarrow S$ which is separated and locally of finite presentation. Let $M, N \in \text{QCoh}(X)$, with $N$ of finite presentation, flat over $S$, with support proper over $S$, then for all $i \geq 0$ the functor:

\[ \text{Ext}^i_{\mathcal{O}_X}(M, N \otimes_{\mathcal{O}_X} f^*(-)) : \text{QCoh}(S) \rightarrow \text{Ab} \]

is coherent.

6. Automorphisms, deformations, obstructions, and composition

A hypothesis in Theorem 4.5 is that the functor $M \mapsto \text{Exal}_X(T, M)$ is finitely generated. We have found the direct verification of this hypothesis to be difficult. In this section, we provide some exact sequences to remedy this situation. We also take the opportunity to formalize and relativize obstruction theories.

Fix a scheme $S$ and a 1-morphism of $S$-groupoids $\Phi : Y \rightarrow Z$. Define the category $\text{Def}_{\Phi}$ to have objects the triples $(T, J, \eta)$, where $T$ is a $Y$-scheme, $J$ is a quasicoherent $\mathcal{O}_T$-module, and $\eta$ is a $Y$-scheme structure on the trivial $Z$-extension of $T$ by $J$. A morphism $(T, J, \eta) \rightarrow (V, K, \xi)$ consists of a $Y$-scheme morphism $f : T \rightarrow V$, a morphism of quasicoherent $\mathcal{O}_T$-modules $f^*K \rightarrow J$ such that the induced morphism of trivial $Z$-extensions $(T \mapsto T[J]) \rightarrow (V \mapsto V[K])$ is a morphism of $Y$-extensions. Graphically, it is the category of completions of the following diagram:

\[ \begin{array}{ccc}
T & \rightarrow & Y \\
\downarrow \eta & & \downarrow \Phi \\
T[J] & \rightarrow & Z.
\end{array} \]

There is a natural functor $\text{Def}_{\Phi} \rightarrow \text{Sch}/Y : (T, J, \eta) \mapsto T$. We denote the fiber of this functor over the $Y$-scheme $T$ by $\text{Def}_{\Phi}(T)$. There is also a functor $\text{Def}_{\Phi}(T)^{\circ} \rightarrow \text{QCoh}(T) : (J, \eta) \mapsto J$. We denote the fiber of this functor over a quasicoherent $\mathcal{O}_T$-module $J$ as $\text{Def}_{\Phi}(T, J)$. This category is naturally pointed by the trivial $Y$-extension of $T$ by $J$. Also, if the 1-morphism $\Phi$ is fibered in setoids, then the category $\text{Def}_{\Phi}(T, J)$ is discrete. Another observation is that if $\Phi_T$ denotes the $T$-groupoid $\Phi \times_Z T$, then the natural functor

$\text{Def}_{\Phi}(T, J) \rightarrow \text{Def}_{\Phi_T}(T, J)$
is an equivalence. We record for future reference the following trivial observations.

**Lemma 6.1.** Fix a scheme $S$, 1-morphisms of $S$-groupoids $X \xrightarrow{\Psi} Y \xrightarrow{\Phi} Z$, an $X$-scheme $T$, and a quasicoherent $O_T$-module $I$. If the 1-morphism $\Psi : X \to Y$ is formally étale, then the natural functor:

$$\text{Def}_{\Phi \circ \Psi}(T, I) \to \text{Def}_{\Phi}(T, I).$$

is an equivalence of categories.

**Lemma 6.2.** Fix a scheme $S$, a class of morphisms $P \subseteq \text{Aff}$, a 1-morphism of $P$-homogeneous $S$-groupoids $\Phi : Y \to Z$, a $Y$-scheme $T$, and a quasicoherent $O_T$-module $I$. If $p \in P$, and $K \in \text{QCoh}(U)$. Then, the natural functor:

$$\text{Def}_{\Phi}(V, p^* K) \to \text{Def}_{\Phi}(U, K),$$

is an equivalence of categories.

The proof of the next result is similar to Proposition 2.3, thus is omitted.

**Proposition 6.3.** Fix a scheme $S$, a 1-morphism of $\text{Nil}$-homogeneous $S$-groupoids $\Phi : Y \to Z$, a $Y$-scheme $T$, and a quasicoherent $O_T$-module $J$. Then the category $\text{Def}_{\Phi}(T, J)$ admits a natural structure as a Picard category.

Denote the set of isomorphism classes of the Picard category $\text{Def}_{\Phi}(T, J)$ by $\text{Def}_{\Phi}(T, J)$. Thus, by Proposition 6.3, we obtain functors:

$$\text{Def}_{\Phi}(T, -) : \text{QCoh}(T) \to \text{Ab} : J \mapsto \text{Def}_{\Phi}(T, J)$$

$$\text{Aut}_{\Phi}(T, -) : \text{QCoh}(T) \to \text{Ab} : J \mapsto \text{Aut}_{\text{Def}_{\Phi}(T, J)}(T, J).$$

The proof of the next result is similar to Corollary 2.3. We will not be using this result, however, so we omit the proof.

**Corollary 6.4.** Fix a scheme $S$, a 1-morphism of $\text{rNil}$-homogeneous $S$-groupoids $\Phi : Y \to Z$, and a $Y$-scheme $T$. Then, for each short exact sequence in $\text{QCoh}(T)$:

$$0 \to K \to M \to C \to 0$$

there is a natural exact sequence of abelian groups:

$$0 \to \text{Aut}_{\Phi}(T, K) \to \text{Aut}_{\Phi}(T, M) \to \text{Aut}_{\Phi}(T, C) \to \text{Def}_{\Phi}(T, K) \to \text{Def}_{\Phi}(T, M) \to \text{Def}_{\Phi}(T, C).$$

We now have a simple result whose proof we leave to the conscientious reader.

**Proposition 6.5.** Fix a scheme $S$, a 1-morphism of $\text{Nil}$-homogeneous $S$-groupoids $\Phi : Y \to Z$, a $Y$-scheme $T$, and a quasicoherent $O_T$-module $J$. Then, there is a natural exact sequence of abelian groups:

$$0 \to \text{Aut}_{\Phi}(T, J) \to \text{Der}_{Y}(T, J) \to \text{Der}_{Z}(T, J) \to \text{Def}_{\Phi}(T, J) \to \text{Exal}_{Y}(T, J) \to \text{Exal}_{Z}(T, J).$$

We now introduce multi-step relative obstruction theories. For single-step obstruction theories, this definition is similar to [Art74 2.6] and [Ols04 A.10].
Definition 6.6. Fix a scheme $S$, a 1-morphism of Nil-homogeneous $S$-groupoids $\Phi : Y \to Z$, and an integer $n \geq 1$. For a $Y$-scheme $T$, an $n$-step relative obstruction theory for $\Phi$ at $T$ is a sequence of additive functors (the obstruction spaces):

$$O^n(T, -) : \text{QCoh}(T) \to \text{Ab} : J \mapsto O^n(T, J) \quad i = 1, \ldots, n$$

as well as natural transformations of functors (the obstruction maps):

$$o^i(T, -) : \text{Exal}_Z(T, -) \Rightarrow O^i(T, -)$$

$$o^i(T, -) : \text{ker} o^{i-1}(T, -) \Rightarrow O^i(T, -) \quad \text{for} \ i = 2, \ldots, n,$$

such that the natural transformation of functors:

$$\text{Exal}_Y(T, -) \Rightarrow \text{Exal}_Z(T, -)$$

has image $\text{ker} o^n(T, -)$. For an affine $Y$-scheme $T$, an $n$-step relative obstruction theory at $T$ is coherent if the functors $\{O^i(T, -)\}_{i=1}^n$ are all coherent.

We feel that it is important to point out that simply taking the cokernel of the last morphism in the exact sequence of Proposition 6.5 produces a 1-step relative obstruction theory at $T$.

We now recall the results on minimal obstruction theory described in [Fle81]. In practice, the minimal obstruction theory is a difficult object to explicitly describe. Now, combining Lemmata 6.1 and 2.1 we obtain

Lemma 6.7. Fix a scheme $S$, 1-morphisms of Nil-homogeneous $S$-groupoids $X \overset{\Psi}{\to} Y \overset{\Phi}{\to} Z$, an $X$-scheme $T$, and a quasicoherent $O_T$-module $I$. If $\Psi$ is formally étale, then any $n$-step relative obstruction theory for $\Phi$ at $T$ lifts to an $n$-step relative obstruction theory for $\Phi \circ \Psi$ with the same obstruction spaces.

What follows is an immediate consequence of Proposition 6.5 and Lemma 5.1.

Corollary 6.8. Fix a scheme $S$, a 1-morphism of Nil-homogeneous $S$-groupoids $\Phi : Y \to Z$, an affine $Y$-scheme $T$, and an integer $n \geq 1$. Suppose there exists a coherent $n$-step relative obstruction theory at $T$.

1. If the functor $M \mapsto \text{Exal}_Z(T, M)$ is finitely generated, then the minimal obstruction theory $(\text{obs}_\Phi, \text{Obs}_\Phi)$ is coherent at $T$.

2. If the functors $M \mapsto \text{Def}_\Phi(T, M)$, $\text{Exal}_Z(T, M)$ are finitely generated, then the functor $M \mapsto \text{Exal}_Y(T, M)$ is finitely generated.

This next result summarizes, in the conventions of this paper, some well-known results from the literature. As can be seen, the relative situation is clarifying. The result that follows also shows the stability of the conditions of Theorem A under composition, in the sense of J. Starr [Sta09].

Proposition 6.9. Fix a scheme $S$ and 1-morphisms of Nil-homogeneous $S$-groupoids $X \overset{\Psi}{\to} Y \overset{\Phi}{\to} Z$, an $X$-scheme $T$, and a quasicoherent $O_T$-module $I$.

1. There is a natural 9-term exact sequence of abelian groups:

$$0 \to \text{Aut}_\Phi(T, I) \to \text{Aut}_{\Phi \circ \Psi}(T, I) \to \text{Aut}_\Phi(T, I) \to \text{Def}_\Phi(T, I) \to \text{Def}_{\Phi \circ \Psi}(T, I) \to \text{Def}_\Phi(T, I) \to \text{Obs}_\Phi(T, I) \to \text{Obs}_{\Phi \circ \Psi}(T, I) \to \text{Obs}_\Phi(T, I) \to 0.$$
(2) There are natural isomorphisms of abelian groups:
\[ \text{Aut}_T(I) \to \text{Def}_{\phi}(T, I) \quad \text{and} \quad \text{Def}_\psi(T, I) \to \text{Obs}_{\Delta}(T, I). \]
In particular, we may realize the functor \( I \mapsto \text{Def}_\phi(T, I) \) as a 1-step relative obstruction theory for the 1-morphism \( \Delta_\phi \).

(3) Fix a \( \mathbf{N} \)-homogeneous 1-morphism of \( \mathcal{S} \)-groupoids \( W \to Y \), an \( X_W \)-scheme \( U \), and a quasicoherent \( \mathcal{O}_U \)-module \( J \). Then there is a natural injection
\[ \text{Obs}_{\phi_\psi}(U, J) \subseteq \text{Obs}_\phi(U, J). \]
In particular, we may realize the functor \( J \mapsto \text{Obs}_\phi(U, J) \) as a 1-step relative obstruction theory for the 1-morphism \( \Psi_W : X_W \to W \).

Proof. For \( \mathbb{1} \), we first apply the Snake Lemma to the commutative diagram:
\[ \begin{array}{ccc}
\text{Exal}_X(T, I) & \to & \text{Exal}_Y(T, I) \\
& \text{Exal}_Y(T, I) & \to \\
0 & \to & 0.
\end{array} \]
Combining this with Proposition \( \mathbb{6.3} \) produces an exact sequence:
\[ \text{Def}_\phi(T, I) \to \text{Obs}_\phi(T, I) \to \text{Obs}_{\phi+\phi}(T, I) \to \text{Obs}_\phi(T, I) \to 0. \]
A direct argument, as in [Ols04, A.15], produces the first 7 terms of the exact sequence. Splicing these together gives the result.

The claim \( \mathbb{2} \) follows from \( \mathbb{1} \) upon taking \( \Psi := \Delta_\phi \), \( \Phi \) the first projection \( X \times_Y X \to X \), and noting that \( \text{Aut}_{1_\phi} = \text{Def}_{1_\phi} = 0 \).

For \( \mathbb{3} \), we note that \( \mathbb{1} \) provides a natural homomorphism of abelian groups \( \text{Obs}_{\phi_\psi}(U, J) \to \text{Obs}_{X_W/Y}(U, J) \to \text{Obs}_\phi(U, J) \). To see that this composition of maps is injective, suppose that we have a \( W \)-extension \( (U \to U') \) of \( U \) by \( J \). If it lifts, as a \( Y \)-extension, to an \( X \)-extension, then the universal property of the 2-fiber product implies that it lifts to an \( X_W \)-extension. This proves the claim. \( \square \)

7. Proof of Theorem [A]

In this section we prove Theorem [A]. Before we do this, however, we prove Corollary [L6].

Proof of Corollary [L6]. Fix a morphism \( x : \text{Spec } k \to S \), where \( k \) is a field. Denote by \( \mathcal{A}_S(x) \) the category whose objects are pairs \((A, a)\), where \( A \) is a local artinian ring with residue field \( k \), and \( a : \text{Spec } A \to S \) is a morphism of schemes, such that the composition \( \text{Spec } A_{\text{red}} \to \text{Spec } A \to S \) agrees with \( x \). Morphisms \((A, a) \to (B, b)\) in \( \mathcal{A}_S(x) \) are ring homomorphisms \( A \to B \) preserving the data. For \( x \in X \), there is an induced category fibered in groupoids \( X_\xi : \xi \to \mathcal{A}_S(x)^{\circ} \). The \( \text{Aff} \)-homogeneity of the \( S \)-groupoid \( X \) implies the homogeneity (in the sense of [SGA7, Exp. VI, Defn. 2.5]) of the cofibered category \( X_\xi : \xi \to \mathcal{A}_S(x) \).

If the morphism \( x \) is locally of finite type, then by (6) and [Hal12] Lem. 6.6 the \( k \)-vector space \( \text{Exal}_X(\xi, k) \) is finite dimensional. By Example [L2] and [loc. cit.] the \( k \)-vector space \( \text{Def}_S(x, k) \) is finite dimensional, and thus by Proposition [L3] the \( k \)-vector space \( \text{Def}_{X/S}(\xi, k) \) is finite dimensional. By definition, \( \text{Def}_{X/S}(\xi, k) \) is the set of isomorphism classes of the category \( X_\xi(\xi') \).

Thus, by (5), [L12] Thm. 1.5 applies, and so for any such \( \xi \), there is a pointed and affine \( X \)-scheme \((Q_\xi, q)\), locally of finite type over \( S \), such that the \( X \)-scheme \( \text{Spec } k(q) \) is isomorphic to \( \xi \), and \( Q_\xi \) is formally versal at \( q \). We now apply Theorem [L5] to conclude that we may (by passing to an open subscheme) assume that \( Q_\xi \) is a
formally smooth $X$-scheme containing $q$. Condition (4) implies that the $X$-scheme $Q_\xi$ is representable by smooth morphisms.

Define $K$ to be the set of all morphisms $x: \text{Spec } k \to S$ which are locally of finite type, and where $k$ is a field. Set $Q := \bigcup_{\kappa \in K, \xi \in X(\kappa)} Q_\xi$. Then, we have seen that the $X$-scheme $Q$ is representable by smooth morphisms, and it remains to show that it is representable by surjective morphisms. Since the stack $X$ is limit preserving, it is sufficient to test this claim with affine $X$-schemes $V$ which are of locally of finite type over $S$. The morphism of algebraic $S$-spaces $Q \times_X V \to V$ is smooth, and by construction its image contains all the points $v \in |V|$ such that the morphism $\text{Spec } k(v) \to S$ is locally of finite type. Since, $V$ is of locally of finite type over $S$, it follows that $Q \times_X V \to V$ is surjective.

Bootstrapping, we can use Corollary 4.6 to obtain Theorem A.

**Proof of Theorem A.** Note that conditions (1) and (2), combined with Lemma 3.2(1), imply that the $S$-groupoid $X$ is limit preserving.

Suppose that the diagonal morphism $\Delta_{X/S}: X \to X \times_X X$ is representable. Conditions (5) and (6), together with Corollary 6.8, imply that for any affine $X$-scheme $V$ which is locally of finite type over $S$, the functor $M \mapsto \text{Exal}_X(V, M)$ is finitely generated. Thus, Corollary 4.6 implies that $X$ is an algebraic stack which is locally of finite presentation over $S$.

Next, will show that if the second diagonal morphism $\Delta_{\Delta_{X/S}}: X \to X \times_X X X$ is representable, then the 1-morphism $\Delta_{X/S}: X \to X \times_X X$ is representable by algebraic spaces. By Lemma 1.5(3) and 3.2(6), the diagonal 1-morphism $\Delta_{X/S}: X \to X \times_X X$ is $\text{Aff}$-homogeneous and limit preserving. By Lemma 1.5(4,1), we see that the $S$-groupoid $X \times_X X$ is $\text{Aff}$-homogeneous. Thus, by Lemma 1.5(4,1) and 3.2(5), for any $X \times_X X$-scheme $T$, the $T$-groupoid $I_{X,T} := X \times_X X (\text{Sch}/T)$ is limit preserving and $\text{Aff}$-homogeneous. Representability of $I_{X,T}$ is local on $T$ for the Zariski topology, thus we may assume that $T$ is an affine scheme. By Lemma 3.2(3), the $S$-groupoid $X \times_X X$ is limit preserving, thus any affine $X \times_X X$-scheme $X \times_X X$-scheme $T$ factors through an affine $X \times_X X$-scheme $T_0$ that is locally of finite type over $S$. Thus, we may assume henceforth that $T$ is locally of finite type over $S$, and is consequently excellent.

Let $V$ be an affine $I_{X,T}$-scheme that is locally of finite type over $T$ (thus locally of finite type over $S$). Then, given $I \in \text{QCoh}(V)$, we have natural isomorphisms:

$$\text{Def}_{I_{X,T}/T}(V, I) \cong \text{Def}_{I_{X,S,V}/V}(V, I) \cong \text{Def}_{I_{X,V}/V}(V, I) \cong \text{Def}_{X/S}(V, I).$$

By Proposition 6.5(2), we thus have $\text{Def}_{I_{X,T}/T}(V, I) \cong \text{Aut}_{X/S}(V, I)$ and so the functor $M \mapsto \text{Def}_{I_{X,T}/T}(V, M)$ is coherent. By Proposition 6.9(3) we also have

$$\text{Obs}_{I_{X,T}/T}(V, I) \subseteq \text{Obs}_{X/S}(V, I) \cong \text{Def}_{X/S}(V, I).$$

Hence, the functor $M \mapsto \text{Def}_{I_{X/S}}(V, M)$ defines a 1-step, coherent relative obstruction theory for the 1-morphism $I_{X,T} \to T$ at $V$. The $T$-groupoid $I_{X,T}$ has representable diagonal, thus satisfies the conditions of the previous analysis, hence is an algebraic stack, locally of finite presentation over $T$. The diagonal 1-morphism $\Delta_{I_{X,T}}$ is a monomorphism, thus $I_{X,T}$ is an algebraic space.

It remains to show that the hypotheses of the Theorem guarantee that the second diagonal morphism $\Delta_{\Delta_{X/S}}$ is representable. Fix an $X$-scheme $T$, which by the analysis above we may assume is locally of finite type over $S$ and excellent, then it remains to show that the $T$-groupoid $R_{X,T} := X \times_{(X \times_X X \times_X X)} (\text{Sch}/T)$ is representable by algebraic spaces. By the previous analysis, we deduce immediately that $R_{X,T}$ is a limit preserving and $\text{Aff}$-homogeneous $T$-groupoid. Also, the third diagonal 1-morphism of $S$-groupoids $\Delta_{\Delta_{X/S}}$ is an isomorphism, thus is
representable. So the diagonal 1-morphism of the $T$-groupoid $R_{X,T}$ is an isomorphism. For an affine $R_{X,T}$-scheme $V$ which is locally of finite type over $S$, and a quasicoherent $\mathcal{O}_Y$-module $I$ we have just shown that $\text{Def}_{R_{X,T}/T}(V, I) = 0$. By Proposition 6.9(2&3) we see that

$$\text{Obs}_{R_{X,T}/T}(V, I) \subseteq \text{Obs}_{\Delta_X/S}(V, I) \cong \text{Def}_{X/S}(V, I) \cong \text{Aut}_{X/S}(V, I).$$

Hence, the functor $M \mapsto \text{Aut}_{X/S}(V, M)$ defines a 1-step coherent relative obstruction theory for the $T$-groupoid $R_{X,T}$ at $V$. Applying the first analysis to this $T$-groupoid proves the result. 

8. Application I: the stack of quasicoherent sheaves

Fix a scheme $S$. For an algebraic $S$-stack $Y$ and a property $P$ of quasicoherent $\mathcal{O}_Y$-modules, denote by $\text{QCoh}^P(Y)$ the full subcategory of $\text{QCoh}(Y)$ consisting of objects which are $P$. We will be interested in the following properties $P$ of quasicoherent $\mathcal{O}_Y$-modules and their combinations:

- $\text{fp}$ – finitely presented,
- $\text{fl}$ – $Y$-flat,
- $\text{fb}$ – $S$-flat,
- $\text{prb}$ – $S$-proper support.

Fix a morphism of algebraic stacks $f : X \to S$. For any $S$-scheme $T$, consider a property $P$ of quasicoherent $\mathcal{O}_{X,T}$-modules. Define $\text{QCoh}^P_{X/S}$ to be the category with objects a pair $(T, M)$, where $T$ is an $S$-scheme and $M \in \text{QCoh}^P(X_T)$. A morphism $(a, \alpha) : (V, N) \to (T, M)$ in the category $\text{QCoh}^P_{X/S}$ consists of an $S$-scheme morphism $a : V \to T$ together with an $\mathcal{O}_{X,T}$-isomorphism $\alpha : a^*_X \mathcal{M} \to N$. Set $\text{Coh}_{X/S} = \text{QCoh}^{\text{fb}, \text{fp}, \text{prb}}_{X/S}$. In this section we will prove

**Theorem 8.1.** Fix a scheme $S$ and a morphism of algebraic stacks $f : X \to S$, which is separated and locally of finite presentation. Then, $\text{Coh}_{X/S}$ is an algebraic stack, locally of finite presentation over $S$, with affine diagonal over $S$.

A proof of Theorem 8.1 without the statement about the diagonal, appeared in [Lie06 Thm. 2.1], though was light on details. In particular, no explicit obstruction theory was given and, as we will see, the obstruction theory is subtle when $f$ is not flat (and is not a standard fact). There was also a minor error in the statement—that the morphism $f$ is separated is essential [LS08]. The statement about the diagonal of $\text{Coh}_{X/S}$ was addressed by M. Roth and J. Starr [RS09, §2]—their approach, however, is completely different, and relies on [Lie06 Prop. 2.3]. In the setting of analytic spaces, the properties of the diagonal were addressed by H. Flenner [Fle82 Cor. 3.2].

Just as in [op. cit., Prop. 2.7], an immediate consequence of Theorems 8.1 and [Hal12 Thm. D] is the existence of Quot spaces. Recall that for a quasicoherent $\mathcal{O}_X$-module $\mathcal{F}$, the presheaf $\text{Quot}_{X/S}(\mathcal{F}) : (\text{Sch}/S)^\circ \to \text{Sets}$ is defined as follows:

$$\text{Quot}_{X/S}(\mathcal{F})[T \to S] = \{r^*_X \mathcal{F} \to \mathcal{O} : \mathcal{O} \in \text{QCoh}^{\text{fb}, \text{fp}, \text{prb}}(X_T)\} / \cong .$$

**Corollary 8.2.** Fix a scheme $S$ and an algebraic $S$-stack $X$ that is separated and locally of finite presentation over $S$. Let $\mathcal{F} \in \text{QCoh}(X)$, then $\text{Quot}_{X/S}(\mathcal{F})$ is an algebraic space which is separated over $S$. If, in addition, $\mathcal{F}$ is of finite presentation, then $\text{Quot}_{X/S}(\mathcal{F})$ is locally of finite presentation over $S$.

When $\mathcal{F}$ is of finite presentation, Corollary 8.2 was proved by M. Olsson and J. Starr [OS03 Thm. 1.1] and M. Olsson [Ols05 Thm. 1.5]. When $\mathcal{F}$ is quasicoherent
and \( X \to S \) is locally projective, Corollary 8.2 was recently addressed by G. Di Brino [Di 12, Thm. 0.0.1] using different methods.

To prove Theorem 8.1 we use Theorem A. Note that there are inclusions:

\[
QCo h_{\text{flb,fp},S}^{\text{rb}} \subseteq QCo h_{\text{X,S}}^{\text{rb,fp}} \subseteq QCo h_{\text{X,S}}^{\text{rb}}
\]

The first inclusion is trivially formally étale. By Lemma A.3.3 the second inclusion is also formally étale. Thus, by Lemmata 1.6(1) if \( QCo h_{\text{X,S}}^{\text{rb}} \) is \( \text{Aff} \)-homogeneous, the same will be true of \( QCo h_{\text{X,S}}^{\text{rb}} \). Also, by Lemmata [6.1] and [6.7] it is sufficient to determine the automorphisms, deformations, and obstructions for \( QCo h_{\text{X,S}}^{\text{rb}} \).

Throughout, we fix a clivage for \( QCo h_{\text{X,S}}^{\text{rb}} \). This gives an equivalence of \( S \)-groupoids \( QCo h_{\text{X,S}}^{\text{rb}} \to \text{Sch}/QCo h_{\text{X,S}}^{\text{rb}} \), which we will use without further comment.

**Lemma 8.3.** Fix a scheme \( S \) and a morphism of algebraic stacks \( f : X \to S \). Then, the \( S \)-groupoid \( QCo h_{\text{X,S}}^{\text{rb}} \) is \( \text{Aff} \)-homogeneous.

**Proof.** First we check \((H^1_{\text{Aff}})\). Fix a commutative diagram of \( QCo h_{\text{X,S}}^{\text{rb}} \)-schemes:

\[
\begin{array}{c}
(T_0, M_0) \\ \downarrow (p, \pi) \\
(T_1, M_1) \\
\end{array}
\begin{array}{c}
(T_2, M_2) \\
\downarrow (p', \pi') \\
(T_3, M_3) \\
\end{array}
\]

where \( p \) is affine and \( i \) is a locally nilpotent closed immersion. Set \((g, \gamma) = (i', \varnothing') \circ (p, \pi) : (T_0, M_0) \to (T_3, M_3)\). Lemma 1.6[2] implies that if the diagram \((8.1)\) is cocartesian in the category of \( QCo h_{\text{X,S}}^{\text{rb}} \)-schemes, then it remains cocartesian in the category of \( S \)-schemes. Conversely, suppose that the diagram \((8.1)\) is cocartesian in the category of \( S \)-schemes. By Lemma 1.3(applied to \( X = S \)), \( i' \) is a locally nilpotent closed immersion and \( p' \) is affine. Let \((W, N)\) be a \( QCo h_{\text{X,S}}^{\text{rb}} \)-scheme, and for \( k \neq 3 \) fix \( QCo h_{\text{X,S}}^{\text{rb}} \)-scheme maps \((y_k, \psi_k) : (T_k, M_k) \to (W, N)\). Since the diagram \((8.1)\) is cocartesian in the category of \( S \)-schemes, there exists a unique \( S \)-morphism \( y_3 : T_3 \to W \) that is compatible with this data. By adjunction, we obtain unique maps of \( \mathcal{O}_{X,S} \)-modules:

\[
N \to (y_1)_* M_1 \times (y_0)_* M_0 \ni (y_2)_* M_2 \cong \{(y_3)* p'_1 M_1 \} \times \{(y_3)* i'_1 M_2 \}.
\]

The functor \((y_1)_*\) is left-exact, so there is a functorial isomorphism \( \mathcal{O}_{X,S} \)-modules:

\[
\{ (y_3)_* p'_1 M_1 \} \times \{ (y_3)_* i'_1 M_2 \} \cong (y_3)_* \{ p'_1 M_1 \times _g M_0 i'_1 M_2 \}.
\]

The commutativity of the diagram \((8.1)\) posits a uniquely induced morphism:

\[
\delta : M_3 \to p'_1 M_1 \times_g M_0 i'_1 M_2 \cong p'_1 p'^* M_3 \times_g M_0 i'_1 i'^* M_3.
\]

Thus, it is sufficient to prove that the map \( \delta \) is an isomorphism, which is local for the smooth topology. So, we immediately reduce to the affine case, where \( S = \text{Spec} A \), \( X = \text{Spec} D \), and \( f : X \to S \) is given by a ring homomorphism \( A \to D \). For each \( i \) we have \( T_1 = \text{Spec} A_i \) and we set \( D_i = D \otimes_A A_i \). Also, \( M_3 \cong M_3 \), where \( M_3 \) is a \( D_3 \)-module which is \( A_3 \)-flat. Now, we have an exact sequence of \( A_3 \)-modules:

\[
0 \to A_3 \to A_1 \times A_2 \to A_0 \to 0.
\]

Applying the exact functor \( M_3 \otimes_{A_3} - \) to this sequence produces an exact sequence:

\[
0 \to M_3 \to (M_3 \otimes_{A_3} A_1) \times (M_3 \otimes_{A_3} A_2) \to M_3 \otimes_{A_3} A_0 \to 0.
\]
Since $M_3 \otimes_{A_3} A_i \cong M_3 \otimes_{D_3} D_i$, we obtain the required isomorphism $\delta$:
\[
M_3 \cong (M_3 \otimes_{A_3} A_1) \times_{(M_3 \otimes_{A_2} A_0)} (M_3 \otimes_{A_3} A_2) \cong (M_3 \otimes_{D_3} D_1) \times_{(M_3 \otimes_{D_2} D_0)} (M_3 \otimes_{D_3} D_2).
\]

Next we check condition (H\textsuperscript{Aff}_2). Fix a diagram of $\text{QCoh}\textsubscript{X/S}^{\text{flb}}$-schemes:
\[
[(T_1, M_1) \xleftarrow{(i, \phi)} (T_0, M_0) \xrightarrow{(p, \pi)} (T_2, M_2)],
\]
where $i$ is a locally nilpotent closed immersion and $p$ is affine. Given a cocartesian square of $S$-schemes:
\[
\begin{array}{ccc}
T_0 & \xleftarrow{i} & T_1 \\
\downarrow{p} & & \downarrow{p'} \\
T_2 & \xleftarrow{i'} & T_3,
\end{array}
\]
write $g = i'p$ and set
\[
\mathcal{M}_3 = \ker \left( (p'_{X,T_2})_* \mathcal{M}_1 \times (i'_{X, T_2})_* \mathcal{M}_2 \xrightarrow{d} g_* \mathcal{M}_0 \right) \in \text{QCoh}(X_{T_1}),
\]
where $d$ is the map $(m_1, m_2) \mapsto (g_* \phi)(m_1) - (g_* \pi)(m_2)$. It remains to show that $\mathcal{M}_3$ is $T_3$-flat, that the induced morphisms of quasicoherent $O_{X,T}$-modules $\phi' : i'_{X,T_2}^* \mathcal{M}_3 \to \mathcal{M}_2$ and $\pi' : p'_{X,T_3}^* \mathcal{M}_3 \to \mathcal{M}_3$ are isomorphisms, and that the following diagram commutes:
\[
\begin{array}{ccc}
i'_{X,T_2}^* \mathcal{M}_3 & \xrightarrow{i'_{X,T_2}^* \phi'} & i'_{X,T_2}^* \mathcal{M}_2 \\
\downarrow{g^* \mathcal{M}_3} & & \downarrow{p'_{X,T_2}^* \mathcal{M}_2} \\
i'_{X,T_2}^* \mathcal{M}_1 & \xrightarrow{\phi} & \mathcal{M}_0
\end{array}
\]
Indeed, this shows that the pairs $(i', \phi')$ and $(p', \pi')$ define $\text{QCoh}\textsubscript{X/S}^{\text{flb}}$-morphisms, and that the resulting completion of the diagram (8.2) commutes.

Now, these claims may all be verified locally for the smooth topology. Thus, we reduce to the affine situation as before, with the modification that for $k \neq 3$ we have $\mathcal{M}_k \cong \tilde{\mathcal{M}}_k$, where $\mathcal{M}_k$ is a $D_k$-module which is flat over $A_k$, and $M_3 \cong \tilde{M}_3 \cong M_1 \times_{\tilde{M}_0} M_2$. The result now follows from [Fer03, Thm. 2.2].

We now determine the automorphisms, deformations, and obstructions. Let $(T, \mathcal{M})$ be a $\text{QCoh}\textsubscript{X/S}^{\text{flb}}$-scheme, and fix a quasicoherent $O_T$-module $I$. For an $S$-extension $i : T \hookrightarrow T'$ of $T$ by $I$, we have a 2-cartesian diagram:
\[
\begin{array}{ccc}
X_T & \xrightarrow{j} & X \\
\downarrow{f_T} & & \downarrow{f} \\
T' & \xrightarrow{i} & T'
\end{array}
\]
Set $J = j^* \ker(O_{X_T} \to j_* O_{X_T})$. Fix a $\text{QCoh}_{X/S}$-extension $(i, \phi) : (T, \mathcal{M}) \to (T', \mathcal{M}')$, then we obtain a commutative diagram:
\[
\begin{array}{ccc}
\mathcal{M} \otimes_{O_X} f_T^* I & \xrightarrow{\mathcal{M} \otimes_{O_X} j^*} & \mathcal{M} \otimes_{O_X} J \\
\downarrow{j^* \ker(M' \to j_* \mathcal{M})}
\end{array}
\]
By the local criterion for flatness, \(\mathcal{M}'\) is \(T'\)-flat if and only if the diagonal map is an isomorphism. Thus, if \(\mathcal{M} \in \text{QCoh} \mathcal{O}_{X/S}^{\text{flb}}\), then \(\mathcal{M} \rightarrow (T',\mathcal{M}')\) exists, the top map must be an isomorphism. This is how we will describe our first obstruction.

**Example 8.4.** This obstruction can be non-trivial when \(E\) is not flat and \(i\) is not split. Indeed, let \(S = \text{Spec } \mathbb{C}[x,y]\) and take \(0 = (x,y) \in |S|\) to be the origin. Set \(X = \text{Bl}_0 S = \text{Proj}_S \mathcal{O}_S(U,V)/(xV - yU), f : X \rightarrow S\) the induced map, and let \(E = f^{-1}(0)\) be the exceptional divisor. Now take \(\mathcal{M} = \mathcal{O}_E\) and consider the \(S\)-extension \(T = \text{Spec } \mathbb{C}(0) \hookrightarrow T' = \text{Spec } \mathbb{C}[x,y]/(x^2,y)\). A straightforward calculation shows that \(\mathcal{M} \otimes_{\mathcal{O}_X} J\) is the skyscraper sheaf supported at the point of \(E\) corresponding to the \(y = 0\) line in \(S\). Also, \(f_2^I = \mathcal{O}_{X_T}\) and so \(\mathcal{M} \otimes_{\mathcal{O}_X} f_2^I \cong \mathcal{O}_E\). The resulting map \(\mathcal{M} \otimes_{\mathcal{O}_X} f_2^I \rightarrow \mathcal{M} \otimes_{\mathcal{O}_X} J\) is not injective.

Observe that there is a short exact sequence of \(\mathcal{O}_T\)-modules:

\[
0 \rightarrow i_* I \rightarrow \mathcal{O}_T \rightarrow i_* \mathcal{O}_T \rightarrow 0.
\]

By Theorem [B.1] we obtain an exact sequence of quasicoherent \(\mathcal{O}_{X_T}\)-modules:

\[
\mathcal{F} \mathcal{O}r_{1}^{S,\tau,j}(i_* \mathcal{O}_T, \mathcal{O}_X) \rightarrow f_2^I i_* I \rightarrow j_* J \rightarrow 0.
\]

Since we have a functorial isomorphism \(f_2^*, i_* I \cong j_* f_2^I I\), by Lemma [B.2] we obtain a natural exact sequence of quasicoherent \(\mathcal{O}_{X_T}\)-modules:

\[
\mathcal{F} \mathcal{O}r_{1}^{S,\tau,j}(\mathcal{O}_T, \mathcal{O}_X) \rightarrow f_2^I I \rightarrow J \rightarrow 0.
\]

Applying the functor \(\mathcal{M} \otimes_{\mathcal{O}_X} -\) to this sequence produces another exact sequence:

\[
\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{F} \mathcal{O}r_{1}^{S,\tau,j}(\mathcal{O}_T, \mathcal{O}_X) \rightarrow \mathcal{M} \otimes_{\mathcal{O}_X} f_2^I I \rightarrow \mathcal{M} \otimes_{\mathcal{O}_X} J \rightarrow 0.
\]

Thus, we have defined a natural class

\[
o^1((T,\mathcal{M}),I)(i) \in \text{Hom}_{\mathcal{O}_{X_T}}(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{F} \mathcal{O}r_{1}^{S,\tau,j}(\mathcal{O}_T, \mathcal{O}_X), \mathcal{M} \otimes_{\mathcal{O}_X} f_2^I I),
\]

whose vanishing is necessary and sufficient for the map \(\mathcal{M} \otimes_{\mathcal{O}_X} f_2^I I \rightarrow \mathcal{M} \otimes_{\mathcal{O}_X} J\) to be an isomorphism. By functoriality of the class \(o^1((T,\mathcal{M}),I)(i)\), we obtain a natural transformation of functors:

\[
o^1((T,\mathcal{M}),-) : \text{Exal}_S(T, -) \Rightarrow \text{Hom}_{\mathcal{O}_{X_T}}(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{F} \mathcal{O}r_{1}^{S,\tau,j}(\mathcal{O}_T, \mathcal{O}_X), \mathcal{M} \otimes_{\mathcal{O}_X} f_2^I I).
\]

So, suppose that the \(S\)-extension \(i : T \hookrightarrow T'\) now has the property that the map \(\mathcal{M} \otimes_{\mathcal{O}_X} f_2^I I \rightarrow \mathcal{M} \otimes_{\mathcal{O}_X} J\) is an isomorphism. Let \(\gamma_{M,I}\) denote the inverse to this map, then \([3.1.12]\) gives a naturally defined obstruction:

\[
o^2((T,\mathcal{M}),I)(i) \in \text{Ext}_j^{2,\mathcal{O}_X}(j_* \mathcal{M}, j_* (\mathcal{M} \otimes_{\mathcal{O}_X} f_2^I I)) \cong \text{Ext}_j^{2,\mathcal{O}_X}(\mathcal{M} \otimes_{\mathcal{O}_X} f_2^I I)
\]

whose vanishing is a necessary and sufficient condition for there to exist a lift of \(\mathcal{M}\) over \(T'\). Thus, there is a natural transformation

\[
o^2((T,\mathcal{M}),-) : \ker o^1((T,\mathcal{M}),-) \Rightarrow \text{Ext}_j^{2,\mathcal{O}_X}(\mathcal{M} \otimes_{\mathcal{O}_X} f_2^I (-))
\]

such that the pair \(\{o^1((T,\mathcal{M}),-), o^2((T,\mathcal{M}),-))\) defines a 2-step obstruction theory for the \(\mathcal{S}\)-groupoid \(\text{QCoh} \mathcal{O}_{X/S}^{\text{flb}}\) at \((T,\mathcal{M})\).

In the case where \(i = \text{tr}_T : T \hookrightarrow T[I]\), the trivial \(X\)-extension of \(T\) by \(I\), then the map \(\mathcal{M} \otimes_{\mathcal{O}_X} f_2^I I \rightarrow \mathcal{M} \otimes_{\mathcal{O}_X} J\) is an isomorphism. By [3.1.12], we
obtain natural isomorphisms of abelian groups:

\[ \text{Aut}_{\mathcal{QCoh}_{X/S}}((T, \mathcal{M}), I) \cong \text{Hom}_{O_{X_T}}(j_\ast M, j_\ast (M \otimes_{O_{X_T}} f_T^\ast I)), \]

\[ \cong \text{Hom}_{O_{X_T}}(M, M \otimes_{O_{X_T}} f_T^\ast I), \]

\[ \text{Def}_{\mathcal{QCoh}_{X/S}}((T, \mathcal{M}), I) \cong \text{Ext}^1_{\mathcal{O}_{X_T}}(j_\ast M, j_\ast (M \otimes_{O_{X_T}} f_T^\ast I)), \]

\[ \cong \text{Ext}^1_{\mathcal{O}_{X_T}}(M, M \otimes_{O_{X_T}} f_T^\ast I). \]

In [Hal13], using simplicial techniques, we will exhibit a 1-step obstruction theory for \( \mathcal{QCoh}_{X/S}^{\text{fib}} \).

**Proof of Theorem A** Using standard reductions [Ryd09, App. B], we are free to assume that \( f \) is, in addition, finitely presented, and the scheme \( S \) is affine and of finite type over \( \text{Spec} \mathbb{Z} \) (in particular, it is noetherian and excellent). We now verify the conditions of Theorem A. Certainly, the \( S \)-groupoid \( \mathcal{QCoh}_{X/S}^{\text{fib}} \) is a limit preserving étale stack over \( S \). By Lemma 8.3, we know that it is also \( \text{Aff} \)-homogeneous. Consider a noetherian local ring \((B, m)\), which is \( m \)-adically complete, and a map \( \text{Spec} B \to S \), then the canonical functor:

\[ \mathcal{QCoh}_{X/S}^{\text{fib}, \text{fp-prh}}(X_{\text{Spec} B}) \to \lim_n \mathcal{QCoh}_{X/S}^{\text{fib}, \text{fp-prh}}(X_{\text{Spec} (B/m^n)}), \]

is an equivalence of categories [Ols05, Thm. 1.4]. Let \((T, \mathcal{M})\) be a \( \mathcal{QCoh}_{X/S} \)-scheme, then we have determined that:

\[ \text{Aut}_{\mathcal{QCoh}_{X/S}}((T, \mathcal{M}), -) = \text{Hom}_{O_{X_T}}(M, M \otimes_{O_{X_T}} f_T^\ast (-)), \]

\[ \text{Def}_{\mathcal{QCoh}_{X/S}}((T, \mathcal{M}), -) = \text{Ext}^1_{O_{X_T}}(M, M \otimes_{O_{X_T}} f_T^\ast (-)), \]

\[ O^1((T, \mathcal{M}), -) = \text{Hom}_{O_{X_T}}(M \otimes \mathcal{F}or_T^0, \mathcal{O}_T, \mathcal{O}_X), M \otimes_{O_{X_T}} f_T^\ast (-)), \]

\[ O^2((T, \mathcal{M}), -) = \text{Ext}^2_{O_{X_T}}(M, M \otimes_{O_{X_T}} f_T^\ast (-)), \]

where \( \{O^1((T, \mathcal{M}), -), O^2((T, \mathcal{M}), -)\} \) are the obstruction spaces for a 2-step obstruction theory. If \( T \) is assumed to be locally noetherian, then by Theorem B.1 the \( O_{X_T} \)-module \( \mathcal{F}or_T^0(\mathcal{O}_T, \mathcal{O}_X) \) is coherent. In addition, if \( T \) is affine, [Hal12, Thm. C] implies that the functors listed above are coherent. Having met the conditions of Theorem A, we see that the \( S \)-groupoid \( \mathcal{QCoh}_{X/S}^{\text{fib}, \text{fp-prh}} \) is algebraic and locally of finite presentation over \( S \).

It remains to show that the diagonal of \( \mathcal{QCoh}_{X/S} \) is affine. Let \((T, \mathcal{M}), (T, \mathcal{N})\) be \( \mathcal{QCoh}_{X/S} \)-schemes, then the commutative diagram in the category of \( T \)-presheaves:

\[ \begin{array}{ccc}
\text{Hom}_{\mathcal{QCoh}_{X/S}}((T, \mathcal{M}), (T, \mathcal{N})) & \xrightarrow{\lambda \mapsto (\lambda, \lambda^{-1})} & \text{Hom}_T(-, T) \\
\downarrow & & \downarrow (\text{Id}_T, \text{Id}_T) \\
\text{Hom}_{O_{X_T}/T}(\mathcal{M}, \mathcal{N}) \times \text{Hom}_{O_{X_T}/T}(\mathcal{N}, \mathcal{M}) & \xrightarrow{\text{Hom}_{O_{X_T}/T}(\mathcal{M}, \mathcal{M}) \times \text{Hom}_{O_{X_T}/T}(\mathcal{N}, \mathcal{N})} & \text{Hom}_{O_{X_T}/T}(\mathcal{M}, \mathcal{M}) \times \text{Hom}_{O_{X_T}/T}(\mathcal{N}, \mathcal{N}),
\end{array} \]

where the morphism along the base is \((\mu, \nu) \mapsto (\nu \circ \mu, \mu \circ \nu)\), is cartesian. By [Hal12, Thm. D] we deduce the result. \( \square \)

9. **Application II: the Hilbert stack and spaces of morphisms**

Fix a scheme \( S \) and a 1-morphism of algebraic stacks \( f : X \to S \). For an \( S \)-scheme \( T \), consider a property \( P \) of a morphism \( Z \to X_T \). Such properties \( P \) could be (but not limited to):

- \( \text{qf} \) – quasi-finite,
- \( \text{lfpb} \) – the composition \( Z \to X_T \to T \) is locally of finite presentation,
**prb** – the composition $Z \to X_T \to T$ is proper,

**flb** – the composition $Z \to X_T \to T$ is flat.

Define $\text{Mor}_{X/S}^P$ to be the category with objects pairs $(T, Z \xrightarrow{g} X_T)$, where $T$ is an $S$-scheme and $g: Z \to X_T$ is a representable morphism of algebraic $S$-stacks that is $P$. Morphisms $(p, \pi): (V, W \xrightarrow{h} X_V) \to (T, Z \xrightarrow{g} X_T)$ in the category $\text{Mor}_{X/S}^P$ are 2-cartesian diagrams:

$\begin{array}{ccc}
W & \xrightarrow{h} & X_V \\
\downarrow & & \downarrow f_V \\
Z & \xrightarrow{g} & X_T
\end{array}$

If the property $P$ is reasonably well-behaved, the natural functor $\text{Mor}_{X/S}^P \to \text{Sch}/S$ defines an $S$-groupoid. We define the Hilbert stack, $\text{HS}_{X/S}$, to be the $S$-groupoid $\text{Mor}_{X/S}^{\text{flb}, \text{lfpb}, \text{prb}, \text{qf}}$. This Hilbert stack contains A. Vistoli’s Hilbert stack [Vis91] as well as the stack of branchvarieties [AK10]. We will prove the following Theorem.

**Theorem 9.1.** Fix a scheme $S$ and a morphism of algebraic stacks $f : X \to S$, which is separated and locally of finite presentation. Then, $\text{HS}_{X/S}$ is an algebraic stack, locally of finite presentation over $S$, with affine diagonal over $S$.

Theorem 9.1 was the result alluded to in the title of M. Lieblich’s paper [Lie06], though a precise statement was not given. Theorem 9.1 was established in [op. cit.] using an auxiliary representability result [op. cit., Prop. 2.3] combined with [op. cit., Thm. 2.1] (Theorem 5.1). In the non-flat case, the obstruction theory used in [op. cit., Proof of Prop. 2.3] is incorrect (a variant of Example 8.4 can be made into a counterexample in this setting also). The stated obstruction theory can be made into the second step of a 2-step obstruction theory, however. The properties of the diagonal of $\text{HS}_{X/S}$ have not been addressed previously. We would like to reiterate what was stated in the Introduction: the just mentioned errors have no net effect on the main ideas of the articles.

**Corollary 9.2.** Fix a scheme $S$, and morphisms of algebraic stacks $f : X \to S$ and $g : Y \to S$. Let $f$ be locally of finite presentation, proper, and flat; and $g$ locally of finite presentation with finite diagonal. Then, $\text{Hom}_S(X, Y)$ is an algebraic stack, locally of finite presentation over $S$, with affine diagonal over $S$.

Corollary 9.2 can be used in the construction of the stack of twisted stable maps [AOV11, Prop. 4.2]. The original construction of the stack of twisted stable maps utilized an incorrect obstruction theory in the non-flat case [AV02, Lem. 5.3.3]. The original proof of Corollary 9.2 due to M. Aoki [Aok06a, Aok06b, §3.5], also has an incorrect obstruction theory in the case of a non-flat target. The stated obstruction theories, as before, can be realized as the second step of a 2-step obstruction theory. A variant of Example 8.4 can be made into counterexamples in these settings too. We would like to reiterate what was stated in the Introduction and above: the just mentioned errors have no net effect on the main ideas of the articles.

To prove Theorem 9.1, we will apply Theorem A directly (though as mentioned previously, this could be done as in [Lie06] using Theorem 8.1). With Theorem 9.1 proven it is easy to deduce Corollary 9.2 via the standard method of associating to a morphism its graph, thus the proof is omitted. Now, just as in §8 there are inclusions:

$$\text{Mor}_{X/S}^{\text{flb}, \text{lfpb}, \text{prb}, \text{qf}} \subseteq \text{Mor}_{X/S}^{\text{flb}, \text{lfpb}, \text{prb}} \subseteq \text{Mor}_{X/S}^{\text{flb}, \text{lfpb}} \subseteq \text{Mor}_{X/S}^{\text{flb}}.$$

The first two inclusions are trivially formally étale. By Lemma 1.6 the third inclusion is formally étale. Thus, by Lemma 1.5(1&6) they will all be $\text{Aff}$-homogeneous.
if we can show that the $S$-groupoid $\text{Mor}^{\text{flb}}_{X/S}$ is $\text{Aff}$-homogeneous. Also, by Lemmata 6.7 and 6.7, descriptions of the automorphisms, deformations, and obstructions for $\text{Mor}^{\text{flb}}_{X/S}$ descend to the subcategories listed above.

**Lemma 9.3.** Fix a scheme $S$ and a morphism of algebraic stacks $f : X \to S$. Then, the $S$-groupoid $\text{Mor}^{\text{flb}}_{X/S}$ is $\text{Aff}$-homogeneous.

**Proof.** First we check $(H^2_{\text{Aff}})$. Fix a diagram of $\text{Mor}^{\text{flb}}_{X/S}$-schemes

$$[(T_1, Z_1 \xrightarrow{g_1} X_{T_1}) \xleftarrow{(i, \phi)} (T_0, Z_0 \xrightarrow{g_0} X_{T_0}) \xrightarrow{(p, \pi)} (T_2, Z_2 \xrightarrow{g_2} X_{T_2})],$$

where $i$ is a locally nilpotent closed immersion and $p$ is affine, and a cocartesian square of $S$-schemes:

$$\begin{array}{ccc}
T_0 & \xrightarrow{i} & T_1 \\
\downarrow & & \downarrow \\
T_2 & \xrightarrow{\phi'} & T_3.
\end{array}$$

By Proposition A.2 there exists a 2-commutative diagram of algebraic $S$-stacks:

$$\begin{array}{ccc}
Z_2 & \xrightarrow{p} & Z_0 \\
\downarrow & & \downarrow \\
T_2 & \xrightarrow{i} & T_0 \\
\downarrow & & \downarrow \\
T_3 & \xrightarrow{\phi'} & T_1
\end{array}$$

where the left and rear faces of the cube are 2-cartesian, and the top and bottom faces are 2-cocartesian in the 2-category of algebraic $S$-stacks. Thus, the universal properties guarantee the existence of a unique $T_3$-morphism $Z_3 \xrightarrow{g_3} X_{T_3}$. By Lemma A.3 the morphism $Z_3 \to T_3$ is flat and all faces of the cube are 2-cartesian. In particular, the resulting $\text{Mor}^{\text{flb}}_{X/S}$-scheme diagram

$$\begin{array}{ccc}
(T_0, Z_0 \xrightarrow{g_0} X_{T_0}) & \xleftarrow{(i, \phi)} & (T_1, Z_1 \xrightarrow{g_1} X_{T_1}) \\
\downarrow & & \downarrow \\
(T_2, Z_2 \xrightarrow{g_2} X_{T_2}) & \xrightarrow{(i', \phi')} & (T_3, Z_3 \xrightarrow{g_3} X_{T_3})
\end{array}$$

is cocartesian in the category of $\text{Mor}^{\text{flb}}_{X/S}$-schemes. Condition $(H^2_{\text{Aff}})$ follows from a similar argument as that given in Lemma S.3.

Fix a $\text{Mor}^{\text{flb}}_{X/S}$-scheme $(T, Z \xrightarrow{g} X_T)$ and a quasicoherent $\mathcal{O}_T$-module $I$. Then, unravelling the definitions and applying the results of [Ols06], demonstrates that there are natural isomorphisms of abelian groups:

$$\text{Aut}_{\text{Mor}^{\text{flb}}_{X/S}}((T, Z \xrightarrow{g} X_T), I) \cong \text{Hom}_{\mathcal{O}_Z}(L_{Z/X_T}, g^* f_T^* I)$$

$$\text{Def}_{\text{Mor}^{\text{flb}}_{X/S}}((T, Z \xrightarrow{g} X_T), I) \cong \text{Ext}^1_{\mathcal{O}_Z}(L_{Z/X_T}, g^* f_T^* I).$$

Using identical ideas to those developed in [S] together with [op. cit.], we obtain a 2-term obstruction theory for the $S$-groupoid $\text{Mor}^{\text{flb}}_{X/S}$ at $(T, Z \xrightarrow{g} X_T)$:

$$\begin{align*}
o^1((T, Z \xrightarrow{g} X_T), -) & : \text{Exal}_S(T, -) \Rightarrow \text{Hom}_{\mathcal{O}_Z}(g^* \mathcal{F} \circ_{\text{aff}}^{S, f} (\mathcal{O}_T, \mathcal{O}_X), g^* f_T^*(-)) \\
o^2((T, Z \xrightarrow{g} X_T), -) & : \ker o^1((T, Z \xrightarrow{g} X_T), -) \Rightarrow \text{Ext}^2_{\mathcal{O}_Z}(L_{Z/X_T}, g^* f_T^*(-)).
\end{align*}$$

In [Hal13], using simplicial techniques, we will exhibit a 1-step obstruction theory for $\text{Mor}^{\text{flb}}_{X/S}$. 


Proof of Theorem 9.1. The proof that the $S$-groupoid $\mathbf{HS}_{X/S}$ is algebraic and locally of finite presentation is essentially identical to the proof of Theorem 8.1, thus is omitted. It remains to show that the diagonal is affine. So, let $(T, Z_1 \overset{g_1}{\rightarrow} X_T)$ and $(T, Z_2 \overset{g_2}{\rightarrow} X_T)$ be $\mathbf{HS}_{X/S}$-schemes, then the inclusion of $T$-presheaves:

$$\text{Isom}_{\mathbf{HS}_{X/S}}((T, Z_1 \overset{g_1}{\rightarrow} X_T), (T, Z_2 \overset{g_2}{\rightarrow} X_T)) \subseteq \text{Isom}_{\mathbf{QCoh}_{X/S}}((T, (g_1)_*O_{Z_1}), (T, (g_2)_*O_{Z_2})).$$

is representable by closed immersions. By Theorem 8.1 we deduce the result. □

Appendix A. Homogeneity of stacks

The results of this section are routine bootstrapping arguments. They are included so that Aff-homogeneity can be proved for moduli problems involving stacks.

Definition A.1. Fix a 2-commutative diagram of algebraic stacks

\[
\begin{array}{ccc}
X_0 & \overset{i}{\longrightarrow} & X_1 \\
\downarrow f & \swarrow & \downarrow f' \\
X_2 & \overset{i'}{\longrightarrow} & X_3,
\end{array}
\]

where $i$ and $i'$ are closed immersions and $f$ and $f'$ are affine. If the induced map:

$$O_{X_3} \rightarrow i'_*O_{X_2} \times_{(i'f)_*O_{X_1}} f'_*O_{X_1}$$

is an isomorphism of sheaves, then we say that the diagram is a geometric pushout, and that $X_3$ is a geometric pushout of the diagram $[X_2 \overset{f}{\leftarrow} X_0 \overset{i}{\rightarrow} X_1]$.

The main result of this section is the following

Proposition A.2. Any diagram of algebraic stacks $[X_2 \overset{i}{\leftarrow} X_0 \overset{i}{\rightarrow} X_1]$, where $i$ is a locally nilpotent closed immersion and $f$ is affine, admits a geometric pushout $X_3$. The resulting geometric pushout diagram is 2-cartesian and 2-cocartesian in the 2-category of algebraic stacks.

We now need to collect some results which aid with the bootstrapping process.

Lemma A.3. Fix a 2-commutative diagram of algebraic stacks:

\[
\begin{array}{ccc}
X_0 & \overset{i}{\longrightarrow} & X_1 \\
\downarrow f & \swarrow & \downarrow f' \\
X_2 & \overset{i'}{\longrightarrow} & X_3
\end{array}
\]

(1) If the diagram is a geometric pushout diagram, then it is 2-cartesian.
(2) If the diagram above is a geometric pushout diagram, then it remains so after flat base change on $X_3$.
(3) If after fppf base change on $X_3$, the above diagram is a geometric pushout diagram, then it was a geometric pushout prior to base change.

Proof. The claim (1) is local on $X_3$ for the smooth topology, thus we may assume that everything in sight is affine—whence the result follows from [Fer03, Thm. 2.2]. Claims (2) and (3) are trivial applications of flat descent. □
Lemma A.4. Consider a 2-commutative diagram of algebraic stacks:

\[
\begin{array}{ccc}
D_0 & \rightarrow & D_1 \\
\downarrow & & \downarrow \\
D_2 & \rightarrow & D_3 \\
\downarrow & & \downarrow \\
C_0 & \rightarrow & C_1 \\
\downarrow & & \downarrow \\
C_2 & \rightarrow & C_3
\end{array}
\]

where the back and left faces of the cube are 2-cartesian, the top and bottom faces are geometric pushout diagrams, and for \(i = 0, 1, 2\), the morphisms \(D_i \rightarrow C_i\) are flat. Then, all faces of the cube are 2-cartesian and the morphism \(D_3 \rightarrow C_3\) is flat.

Proof. By Lemma A.3(2), this is all smooth local on \(C_3\) and \(D_3\), thus we immediately reduce to the case where everything in sight is affine. Fix a diagram of rings \([A_2 \rightarrow A_0 \rightarrow A_1]\) where \(p : A_1 \rightarrow A_0\) is surjective. For \(i = 0, 1, 2\) fix flat \(A_i\)-algebras \(B_i\), and \(A_0\)-isomorphisms \(B_2 \otimes_{A_2} A_0 \cong B_0\) and \(B_1 \otimes_{A_1} A_0 \cong B_0\). Set \(A_3 = A_2 \times_{A_0} A_1\) and \(B_3 = B_2 \times_{B_0} B_1\), then we have to prove that the \(A_3\)-algebra \(B_3\) is flat, the natural maps \(B_3 \otimes_{A_1} A_i \rightarrow B_i\) are isomorphisms, and that these isomorphisms are compatible with the given isomorphisms. This is an immediate consequence of [Fer03, Thm. 2.2], since these are questions about modules. \(\square\)

We omit the proof of the following easy result from commutative algebra.

Lemma A.5. Fix a surjection of rings \(A \rightarrow A_0\) and let \(I = \ker(A \rightarrow A_0)\). Suppose that there is a \(k\) such that \(I^k = 0\).

(1) Given a map of \(A\)-modules \(u : M \rightarrow N\) such that \(u \otimes_A A_0\) is surjective, then \(u\) is surjective.

(2) For an \(A\)-module \(M\), if \(M \otimes_A A_0\) is finitely generated, then \(M\) is finitely generated.

(3) Given an \(A\)-algebra \(B\) and a \(B\)-module \(M\), let \(M_0 = A_0 \otimes_A M\).

(a) If \(M\) is \(A\)-flat and \(M_0\) is \(B_0\)-finitely presented, then \(M\) is \(B\)-finitely presented.

(b) If \(B_0\) is a finite type \(A_0\)-algebra, then \(B\) is a finite type \(A\)-algebra.

(c) If \(B\) is a flat \(A\)-algebra and \(B_0\) is a finitely presented \(A_0\)-algebra, then \(B\) is a finitely presented \(A\)-algebra.

Lemma A.6. Fix a morphism \(f : X \rightarrow Y\) of algebraic stacks and a locally nilpotent closed immersion \(Y_0 \hookrightarrow Y\). If \(f\) is flat, then it is locally of finite presentation (resp. smooth) if and only if the same is true of the map \(X \times_Y Y_0 \rightarrow Y_0\).

Proof. Observe that for flat morphisms which are locally of finite presentation, smoothness is a fibral condition, thus follows from the first claim. The first claim is smooth local on \(Y\) and \(X\), thus follows from Lemma A.5(3c). \(\square\)

Lemma A.7. Consider a locally nilpotent closed immersion \(X \hookrightarrow X'\) and a smooth morphism \(U \rightarrow X\) where \(U\) is an affine scheme. Then, there exists a smooth morphism \(U' \rightarrow X'\) which pulls back to \(U \rightarrow X\).

Proof. Since \(U\) is quasicompact, it is sufficient to treat the case where the locally nilpotent closed immersion \(X \hookrightarrow X'\) is square zero. Then, [Ols06, Thm. 1.4] implies that the obstruction to the existence of a flat lift lies in the abelian group \(\text{Ext}_{\mathbf{O}_{X'}}(L_{U/X}, M)\), for some quasicoherent \(\mathbf{O}_U\)-module \(M\). The morphism \(U \rightarrow X\) is smooth, \(U\) is affine, and the \(\mathbf{O}_U\)-module \(\mathcal{H}\text{om}_{\mathbf{O}_U}(\mathcal{O}_{U/X}, M)\) is quasicoherent, thus \(\text{Ext}_{\mathbf{O}_U}(L_{U/X}, M) = H^2(U, \mathcal{H}\text{om}_{\mathbf{O}_U}(\mathcal{O}_{U/X}, M)) = 0\). Finally, by Lemma A.6 any such lift that is flat, is also smooth. \(\square\)

Proof of Proposition A.3. Throughout, the following notation will be used.
(i) For \( d = 1, 2, 3, \) let \( \mathcal{C}_d \) denote the full 2-subcategory of the 2-category of algebraic stacks, with objects those algebraic stacks whose \( d \)th diagonal is affine. Note that \( \mathcal{C}_d \) is the full 2-category of algebraic stacks.

(ii) Let \( \mathcal{C}_0 \) denote the category of affine schemes.

(iii) Fix an algebraic stack \( Y \) and a collection of morphisms \( \{ Y^i \to Y \}_{i \in \Lambda} \). For \( i, j, k \in \Lambda \), set \( Y^{ij} \) (resp. \( Y^{ijk} \)) to be \( Y^i \times_Y Y^j \) (resp. \( Y^i \times_Y Y^j \times_Y Y^k \)).

Claim: Let \( d = 0, 1, \) or \( 2 \). Suppose that any diagram \( [X_2 \leftarrow X_0 \to X_1] \in \mathcal{C}_d \), where \( i \) is a locally nilpotent closed immersion and \( f \) is affine, admits a geometric pushout in \( \mathcal{C}_d \), such that the resulting geometric pushout diagram is 2-cartesian and 2-cocartesian in the 2-category \( \mathcal{C}_d \), then the same is true of \( \mathcal{C}_{d+1} \).

To see that the Claim is sufficient to prove the Proposition, observe that given a diagram \( [X_2 \leftarrow X_0 \to X_1] \in \mathcal{C}_0 \), where \( i \) is a locally nilpotent closed immersion and \( f \) is affine, the affine scheme \( X_3 = \text{Spec}(\Gamma(O_X) \times_{\Gamma(O_X)} \Gamma(O_X)) \) is a geometric pushout of this diagram. The resulting geometric pushout diagram is trivially seen to be cocartesian in \( \mathcal{C}_0 \); by Lemma A.3(1), it is also 2-cartesian. By induction, the Claim now implies that every such diagram in \( \mathcal{C}_d \) admits a geometric pushout, and the resulting geometric pushout diagram is 2-cocartesian and 2-cartesian in \( \mathcal{C}_d \). Since every algebraic stack belongs to \( \mathcal{C}_d \), we deduce the Proposition.

Proof of Claim: First, we show that geometric pushout diagrams in \( \mathcal{C}_{d+1} \) are 2-cocartesian (they are always 2-cartesian by Lemma A.3(1)). Thus, we must uniquely complete all 2-commutative diagrams:

\[
\begin{array}{ccc}
X_0 & \xymatrix{ \psi_1} & W_1 \\
\downarrow & \downarrow & \downarrow \\
X_1 & \xymatrix{ \psi_2} & W_2 \\
\downarrow & \downarrow & \downarrow \\
X_2 & \xymatrix{ \psi_3} & W_3
\end{array}
\]

where the square is a geometric pushout diagram in \( \mathcal{C}_{d+1} \), \( i \) is a locally nilpotent closed immersion, and \( W \in \mathcal{C}_{d+1} \). Observe that if \( \coprod_{i \in \Lambda} X^i_3 \to X_3 \) is a smooth cover of \( X_3 \), where each \( X^i_3 \) is affine, then \( \forall i, j, k \in \Lambda \), we have that \( X^{ij}_3, X^{ijk}_3 \in \mathcal{C}_d \). By smooth descent, it is thus sufficient to prove that the diagram above is 2-cocartesian when the \( X_i \in \mathcal{C}_d \). If \( d \neq 0 \), we may repeat this argument \( (d-1) \)-more times, to reduce to the case where \( X_1 \in \mathcal{C}_0 \). That is, they are affine.

Now we show the uniqueness of completions of the diagram. Suppose that for \( j = 1, 2 \) we have a 1-morphism \( g^j : X_3 \to W \) together with 2-morphisms \( \gamma^j_1 : \psi_1 \Rightarrow g^j \circ f' \) and \( \gamma^j_2 : g^j \circ i' \Rightarrow \psi_2 \), satisfying \( (f^* \gamma^j_2) \circ \alpha \circ (i^* \gamma^j_1) = \beta \). We claim that there is a unique 2-morphism \( \eta : g^1 \Rightarrow g^2 \) such that \( \eta \circ \gamma^1_1 = \gamma^2_1 \) and \( \gamma^2_2 \circ \eta = \gamma^1_2 \).

To see this, let \( e : E = X_3 \times_{(g^1, g^2)} W \to W \) be the equalizer of the pair of 1-morphisms \( g^1 \) and \( g^2 \). Since \( X_3 \) affine and \( W \in \mathcal{C}_{d+1} \), we have that \( E \in \mathcal{C}_d \). By the universal property defining \( E \), we obtain 1-morphisms \( f^j : X_1 \to E \) and \( \tilde{\gamma}^j : X_2 \to E \) such that \( e \circ \tilde{\gamma}^1 = f' \) and \( e \circ \tilde{\gamma}^2 = i' \), which are unique up to a unique 2-morphism. Since our square is a geometric pushout diagram, by assumption it is 2-cartesian in \( \mathcal{C}_d \). Thus, there exists a 1-morphism \( s : X_3 \to E \), which is unique up to a unique 2-morphism, that is compatible with this data. Since \( e \circ s = \text{Id}_X \), the definition of \( E \) gives a unique 2-morphism \( \eta : g^1 \Rightarrow g^2 \) with the required properties.

Now we show the existence of a completion of the diagram. Fix a smooth presentation \( \coprod_{l \in \Lambda} W^l \to W \), where each \( W^l \) is an affine scheme. For \( m \neq 3 \) and \( l \in \Lambda \) set \( X^m_l = X^m \times_W W^l \). Since \( W \in \mathcal{C}_{d+1} \) and the schemes \( X^m_l \) are affine, for \( m \neq 3 \) the stacks \( X^m_l \) all belong to \( \mathcal{C}_d \). Thus, the geometric pushout \( X^m_3 \) of the diagram \( [X^m_2 \leftarrow X^m_0 \to X^m_1] \) exists, and the resulting geometric pushout diagram is
2-cocartesian in $\mathcal{C}_d$. In particular, there is a unique map $X'^1_0 \to W'$ which is compatible with the data. Similarly, there is also a unique map $X'^3_0 \to X_3$ which is compatible with the data—by Lemmata A.4 and A.6 this map is smooth. By the uniqueness statement that we have already proven, we obtain a unique map $\Pi_{i \in A} X'^1_0 \to W$ which is compatible with the data. Since the morphism $\Pi_{i \in A} X'^3_0 \to X_3$ is smooth and surjective, smooth descent gives a map $X_3 \to W$ completing the diagram.

Finally, we show that any diagram $[X_2 \leftarrow X_0 \rightarrow X_1]$ in $\mathcal{C}_{d+1}$, where $i$ is a locally nilpotent closed immersion and $f$ is affine, admits a geometric pushout. Fix a smooth surjection $\Pi_{i \in A} X'^1_2 \to X_2$, where $X'^1_2$ is an affine scheme $\forall l \in \Lambda$. Set $X'^1_0 = X'^1_2 \times_{X_2} X_0$, then as $f$ is affine, the scheme $X'^1_0$ also affine. By Lemma A.7 the resulting smooth morphism $X'^1_0 \to X_0$ lifts to a smooth morphism $X'^3_0 \to X_1$, with $X'^3_0$ affine, and $X'^3_0 \cong X'^3_1 \times X_1 X_0$. As before, $\forall m \neq 3$ and $\forall i, j, k \in \Lambda$ we have $X'^3_m \cong X'^3_i \times X_1 X_0$. Thus, for $I = i, ij$ or $ijk$, a geometric pushout $X'^1_0$ of the diagram $[X'^1_0 \leftarrow X'^1_I \rightarrow X'^1_J]$ exists, and belongs to $\mathcal{C}_d$. We have already shown that geometric pushouts in $\mathcal{C}_{d+1}$ are 2-cartesian in $\mathcal{C}_d$, thus there are uniquely induced morphisms $X'^1_m \to X'^3_m$. For $m \neq 3$, these morphisms are clearly smooth, and by Lemmata A.4 and A.6 the morphisms $X'^1_I \to X'^3_I$ are smooth. It easily verified that the universal properties give rise to a smooth groupoid $[\Pi_{i,j} X'^1_i \cong \Pi_{i,j} X'^3_i]$. The quotient $X_3$ of this groupoid in the category of stacks is algebraic. By Lemma, A.8.3 it is also a geometric pushout of the diagram $[X_2 \leftarrow X_0 \to X_1].$ \hfill \Box

### Appendix B. Local Tor functors on algebraic stacks

The aim of the section is to state some easy generalizations of [EGA III.6.5] to algebraic stacks. We omit the proofs as they are simple descent arguments.

**Theorem B.1.** Fix a scheme $S$ and a 2-cartesian diagram of algebraic $S$-stacks:

$$
\begin{array}{ccc}
X_3 & \xrightarrow{f'_3} & X_2 \\
| \downarrow f_3 & & | \downarrow f_2 \\
X_1 & \xrightarrow{f_1} & X_0.
\end{array}
$$

Then, for each integer $i \geq 0$, there exists a natural bifunctor:

$$
\mathcal{T}or_{i}^{X_0,f_1,f_2}(\cdot,\cdot) : \text{QCoh}(X_1) \times \text{QCoh}(X_2) \to \text{QCoh}(X_3),
$$

The family of bifunctors $\{\mathcal{T}or_{i}^{X_0,f_1,f_2}(\cdot,\cdot)\}_{i \geq 0}$ forms a 0-functor in each variable. Moreover, there is a natural isomorphism for all $M \in \text{QCoh}(X_1)$ and $N \in \text{QCoh}(X_2)$:

$$
\mathcal{T}or_{i}^{X_0,f_1,f_2}(M, N) \cong f_2^* M \otimes_{f_0^* N} f_1^* N.
$$

If $M$ or $N$ is $X_0$-flat, then for all $i > 0$ we have $\mathcal{T}or_{i}^{X_0,f_1,f_2}(M, N) = 0$. In addition, if the algebraic stacks $X_1$ and $X_0$ are locally noetherian and the morphism $f_2$ is locally of finite type, then the bifunctor above restricts to a bifunctor:

$$
\mathcal{T}or_{i}^{X_0,f_1,f_2}(\cdot,\cdot) : \text{Coh}(X_1) \times \text{Coh}(X_2) \to \text{Coh}(X_3).
$$

Another result that will be useful is the following.

**Lemma B.2.** Fix a scheme $S$ and a 2-cartesian diagram of algebraic $S$-stacks

$$
\begin{array}{ccc}
W \times_Z Y & \xrightarrow{h'} & X \times_Z Y \\
| \downarrow g_w & & | \downarrow f_Y \\
W & \xrightarrow{h} & X \\
| g_x & & | f \\
& & Z.
\end{array}
$$
where the morphism $h$ is affine. Then, for any $M \in \mathbf{QCoh}(W)$, $N \in \mathbf{QCoh}(Y)$, and $i \geq 0$, there is a natural isomorphism of quasicoherent $\mathcal{O}_{X \times_{Y} Y}$-modules:
\[
\mathcal{F} \mathcal{O}_{i}^{\mathbb{Z}, f_{\phi}}(h, M, N) \cong h_{*}^{i} \mathcal{F} \mathcal{O}_{i}^{\mathbb{Z}, f_{\phi}}(M, N).
\]

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