Extensions of dualities and a new approach to the de Vries duality

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Abstract

We prove a general categorical theorem for the extension of dualities. Applying it, we present new proofs of the de Vries Duality Theorem for the category $\text{CHaus}$ of compact Hausdorff spaces and continuous maps, and of the recent Bezhanishvili-Morandi-Olberding Duality Theorem which extends the de Vries duality to the category $\text{Tych}$ of Tychonoff spaces and continuous maps. In the process of doing so we obtain new duality theorems for the categories $\text{CHaus}$ and $\text{Tych}$.

1 Introduction

The celebrated Stone Duality Theorem [36] shows that the entire information about a zero-dimensional compact Hausdorff space (= Stone space) $X$ is, up to homeomorphism, contained in its Boolean algebra $\text{CO}(X)$ of all clopen (= closed and open) subsets of $X$. Likewise, all information about the continuous maps between two such

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spaces \( X \) and \( Y \) is encoded by the Boolean homomorphisms between the Boolean algebras \( \text{CO}(Y) \) and \( \text{CO}(X) \). It is natural to ask whether a similar result holds for all compact Hausdorff spaces and continuous maps between them. The first candidate for the role of the Boolean algebra \( \text{CO}(X) \) under such an extension seems to be the Boolean algebra \( \text{RC}(X) \) of all regular closed subsets of a compact Hausdorff space \( X \) (or, its isomorphic copy \( \text{RO}(X) \), which collects all regular open subsets of \( X \)), but it fails immediately since, as is well-known, \( \text{RC}(X) \) is isomorphic to \( \text{RC}(EX) \), where \( EX \) is the absolute of \( X \). However, in 1962, de Vries [15] showed that, if we regard the Boolean algebra \( \text{RC}(X) \) together with the relation \( \rho_X \) on \( \text{RC}(X) \), defined by

\[
F \rho_X G \iff F \cap G \neq \emptyset,
\]

then the pair \((\text{RC}(X), \rho_X)\) determines uniquely (up to homeomorphism) the compact Hausdorff space \( X \). Moreover, with the help of some special maps between \((\text{RC}(X), \rho_X)\) and \((\text{RC}(Y), \rho_Y)\), where \( X \) and \( Y \) are compact Hausdorff spaces, one can reconstruct all continuous maps between \( Y \) and \( X \). De Vries gave an algebraic description of the pairs \((\text{RC}(X), \rho_X)\) as pairs \((A, C)\), formed by a complete Boolean algebra \( A \) and a relation \( C \) on \( A \), satisfying some axioms, and he also described algebraically the needed special maps of such pairs. In this way he obtained the category \( \text{DeV} \) and its dual equivalence with the category \( \text{CHaus} \) of compact Hausdorff spaces and continuous maps. In fact, de Vries did not use the relation \( \rho_X \) as mentioned above, but its “dual”, that is, the relation \( F \ll_X G \), defined by \((F \ll_X G \iff F \subseteq \text{int}_X(G))\) (with \( \rho_X \) complementary to \( \rho_X \) and \( G^* \) denoting the Boolean negation of \( G \) in \( \text{RC}(X) \)) and called the non-tangential inclusion; equivalently,

\[
F \ll_X G \iff F \subseteq \text{int}_X(G).
\]

Now known as de Vries algebras, he originally called the abstract pairs \((A, \ll)\) compingent algebras. The axioms for the relation \( C \) (respectively, \( \ll_C \)) on \( A \) are precisely the axioms for Efremović proximities [23], with only one exception: instead of Efremović’s separation axiom, which refers to the points of the space in question, de Vries introduced what is now called the extensionality axiom (see [20, Lemma 2.2, p.215] for a motivation for this terminology). Since Efremović proximities are relations on the Boolean algebra \((P(X), \subseteq)\) of all subsets of a set \( X \), de Vries algebras may be regarded as point-free generalizations of the Efremović proximities.

Nowadays the pairs \((A, C)\), where \( A \) is a Boolean algebra and \( C \) is a proximity-type relation on \( A \), attract the attention not only of topologists, but also of logicians and theoretical computer scientists. Amongst the many generalizations of de Vries algebras, the most popular ones are the so-called RCC systems (Region Connection Calculus) of Randell-Cui-Cohn [34]. Their generalizations include the contact algebras (introduced in [20, 21]), which are point-free analogues of the Čech proximity spaces, and precontact algebras, defined independently and almost simultaneously, but in a completely different form, by S. Celani [12] (for the needs of logic) and by I. Düntsch and D. Vakarelov [22] (for the needs of theoretical computer science). These and the RCC systems are very useful notions in the foundations of artificial intelligence, geographic information systems, robot navigation, computer-aided design, and more
A relation \( C \) on a Boolean algebra \( A \), which satisfies the de Vries axioms (corresponding to the relation \( \rho_X \) above), is called a normal contact relation, and the pair \((A, C)\) then becomes a normal contact algebra (briefly, an NCA, \([20]\)). In other words, the de Vries algebras “in \( \rho_X \)-form” are precisely the complete NCAs. De Vries \([15]\) noted that his dual equivalence \( \Psi^a : \text{DeV} \rightarrow \text{CHaus} \) is an extension of the restriction \( T : \text{CBool} \rightarrow \text{ECH} \) of Stone’s dual equivalence \( S^a : \text{Bool} \rightarrow \text{Stone} \); here \text{Bool} denotes the category of Boolean algebras and Boolean homomorphisms, and \text{CBool} is its full subcategory of complete Boolean algebras; \text{Stone} is the category of Stone spaces and continuous maps, and \text{ECH} denotes its full subcategory of extremally disconnected compact Hausdorff spaces. Therefore, the objects of the category \text{DeV} are precisely “the structured \text{CBool}-objects \((A, C)\)”.

Using the de Vries duality, in \([7, \text{Theorem 8.1}(1)]\) Bezhanishvili proved that, if \( A \) is a complete Boolean algebra, then there exists a bijective correspondence between the set of all normal contact relations \( C \) on \( A \) and the set of all (up to homeomorphism) Hausdorff irreducible images of the Stone dual \( S^a(A) \) of \( A \). Hence, the objects of de Vries’ category \text{DeV} may be regarded as pairs \((A, p)\), where \( A \) is a \text{CBool}-object and \( p : S^a(A) \rightarrow X \) is an irreducible map onto a Hausdorff space \( X \), so that \( p \) is a special \text{CHaus}-morphism; in fact, \( p \) is a projective cover of \( X \). With the structure of the objects presented in map form, we are ready to formulate the principal problem of this paper in categorical terms.

Let \( T : A \rightarrow B \) be a dual equivalence between two categories \( A \) and \( B \), and \( B \) be a full subcategory of a category \( C \). Then it is not at all surprising that one can construct a category \( \mathcal{D} \) containing \( A \) as a full subcategory, and a dual equivalence \( \bar{T} : \mathcal{D} \rightarrow \mathcal{C} \) extending \( T \) along the inclusion functors \( I \) and \( J \), as in the diagram

\[
\begin{array}{ccc}
\mathcal{D} & \xrightarrow{\bar{T}} & \mathcal{C} \\
\downarrow{J} & & \downarrow{I} \\
A & \xleftarrow{T} & B.
\end{array}
\]

Inside \( \mathcal{C} \), one may simply replace \( B \) by \( A \) and adjust the composition using the dual equivalence \( T \) to obtain the category \( \mathcal{D} \) ! This ad-hoc procedure, however, does not make for a naturally described category \( \mathcal{D} \), since the definition of the hom-sets of \( \mathcal{D} \) changes with the two types of objects involved. The principal goal of this paper is therefore to model the objects of a suitable extension category \( \mathcal{D} \) of \( A \) dually equivalent to \( \mathcal{C} \) in a natural way, as \( A \)-objects provided with a structure that gives them a strong algebraic flavour. Our comments on the de Vries duality suggest to consider as objects of \( \mathcal{D} \) the pairs \((A, p)\), with \( A \) an \( A \)-object and \( p \) a “special” \( \mathcal{C} \)-morphism with domain \( T(A) \). Being “special” may be described as lying in a given class \( \mathcal{P} \) of \( \mathcal{C} \)-morphisms satisfying suitable axioms, which suffice to establish a category \( \mathcal{D} \) with a dual equivalence \( \bar{T} : \mathcal{D} \rightarrow \mathcal{C} \).

In \([19]\) we presented a set of axioms on the class \( \mathcal{P} \) and a construction of the category \( \mathcal{D} \), which allowed us to reproduce the Fedorchuk duality \([25]\) from the general setting. The same construction will, however, not work for the principal target of this
paper, the de Vries duality [15]. The general reason for this lies in the fact that the formation of the projective cover, or of the injective hull of an object, cannot be extended functorially, in such a way that the chosen essential morphism from the cover, or into the hull, becomes part of a natural transformation (see [2]). However, by slightly weakening the axioms on $\mathcal{P}$ and substantially modifying the construction of $\mathcal{D}$ we succeed in providing a general categorical template from which to derive the de Vries duality, and other known or new dualities.

The paper is organized as follows. Section 2 summarizes all notions and terms that are needed for our exposition. Section 3 gives the categorical extension result, as indicated in the previous paragraph. In Section 4 we give a review of mostly known facts concerning the de Vries duality, augmented by some results from [19] on which we rely heavily in our exposition. In Section 5, we derive the de Vries Duality Theorem from our general categorical Theorem 3.1. Introducing the notions of Fedorchuk homomorphism and de Vries transformation (see Definitions 5.5 and 5.6), we define a new category $\text{StoneDeV}$; it has the same objects as the category $\text{DeV}$, but its morphisms are equivalence classes of Fedorchuk homomorphisms, where two Fedorchuk homomorphisms between the same two objects are equivalent when their de Vries transformations coincide. With the help of Theorem 3.1, we prove that the category $\text{StoneDeV}$ is dually equivalent to the category $\text{CHaus}$ (see Theorem 5.9). Note that, in contrast to the de Vries category $\text{DeV}$, the morphisms of the category $\text{StoneDeV}$ are equivalence classes of Boolean homomorphisms which preserve the non-tangential inclusion $\ll$, and their composition is a very natural one; however they are sets of special Boolean homomorphisms. (Let us mention that in [18] another category dually equivalent to the category $\text{CHaus}$ was constructed; it has the same objects as the category $\text{DeV}$, and its morphisms are multi-valued maps which may be composed in a natural way.) After that, in Theorem 5.13, we show that the categories $\text{StoneDeV}$ and $\text{DeV}$ are isomorphic. Obviously, Theorems 5.13 and 5.9 imply that the categories $\text{CHaus}$ and $\text{DeV}$ are dually equivalent, obtaining in this way a new proof of de Vries’ Duality Theorem [15].

In Section 6, we apply Theorem 3.3 (a dualization of Theorem 3.1) for presenting a new proof of the Bezhanishvili-Morandi-Olberding Duality Theorem [9]. Adapting our general categorical result to the concrete situation, we first obtain a new duality theorem which extends the de Vries Duality Theorem to the category $\text{Tych}$ (see Theorem 6.6). A crucial role in this process of adaptation plays our Proposition 6.2 (a more general and still unpublished version of it was presented in [17]). It permits to define in a new way de Vries’ dual equivalence $\Psi^a$ (see Proposition 6.3). We obtain Bezhanishvili-Morandi-Olberding Duality Theorem from our new duality theorem using the Tarski duality between the category $\text{Set}$ of sets and functions and the category $\text{CaBa}$ of complete atomic Boolean algebras and suprema-preserving Boolean homomorphisms (see Theorem 6.13, Corollary 6.14 and Theorem 6.15).

Our general references for unexplained notation are [1] for category theory, [24] for topology, and [29] for Boolean algebras.
2 Preliminaries

Below we first recall the notions of contact algebra and normal contact algebra. They can be regarded as algebraic analogues of proximity spaces (see [23, 35, 11, 5, 31] for proximity spaces). Generally speaking, in this paper we work mainly with Boolean algebras with supplementary structures on them. In all cases, we will say that the structured Boolean algebra in question is complete if the underlying Boolean algebra is complete. Our standard notation for the operations of a Boolean algebra \( B \) is indicated by \( B = (B, \wedge, \vee, *, 0, 1) \); note in particular that the complement in \( B \) is denoted by \( * \), and that 0 and 1 denote the least and largest element in \( B \), not excluding the case \( 0 = 1 \).

Definition 2.1. ([20]) A Boolean contact algebra, or, simply, contact algebra (abbreviated as CA), is a structure \((B, C)\), where \( B \) is a Boolean algebra, and \( C \) a binary relation on \( B \), called a contact relation, which satisfies the following axioms:

1. If \( a \neq 0 \) then \( aCa \).
2. If \( aCb \) then \( a \neq 0 \) and \( b \neq 0 \).
3. \( aCb \) implies \( bCa \).
4. \( aC(b \lor c) \) if, and only if, \( aCb \) or \( aCc \).

Two contact algebras \((B, C)\) and \((B', C')\) are said to be isomorphic if there exists a \( CA\)-isomorphism between them, i.e., a Boolean isomorphism \( \varphi : B \rightarrow B' \) such that, for all \( a, b \in B \), \( aCb \) if and only if \( \varphi(a)C'\varphi(b) \).

With \( -C \) denoting the set complement of \( C \) in \( B \times B \), we shall consider two more properties of contact algebras:

5. If \( a(-C)b \) then \( a(-C)c \) and \( b(-C)c^* \) for some \( c \in B \).
6. If \( a \neq 1 \) then there exists \( b \neq 0 \) such that \( b(-C)a \).

A contact algebra \((B, C)\) is called a Boolean normal contact algebra or, briefly, a normal contact algebra (abbreviated as NCA) [15, 25] if it satisfies (C5) and (C6). (Note that if \( 0 \neq 1 \), then (C2) follows from the axioms (C4), (C3), and (C6).)

The notion of normal contact algebra was introduced by Fedorchuk [25] under the name Boolean \( \delta \)-algebra, as an equivalent expression of the notion of compingent Boolean algebra by de Vries (see the definition below). We call such algebras normal because they form a subclass of the class of contact algebras which naturally arise in the context of normal Hausdorff spaces (see [20]).

Definition 2.2. For a contact algebra \((B, C)\) we define a binary relation \( \preccurlyeq_C \) on \( B \), called non-tangential inclusion, by

\[ a \preccurlyeq_C b \text{ if, and only if, } a(-C)b^*. \]

If \( C \) is understood, we shall simply write \( \preccurlyeq \) instead of \( \preccurlyeq_C \).
The relations \( C \) and \( \ll \) are inter-definable. For example, normal contact algebras may be defined equivalently – and exactly in this way they were introduced under the name of *compingent Boolean algebras* by de Vries in [15] – as a pair consisting of a Boolean algebra \( B \) and a binary relation \( \ll \) on \( B \) satisfying the following axioms:

\[(\ll 1). a \ll b \text{ implies } a \leq b.\]
\[(\ll 2). 0 \ll 0.\]
\[(\ll 3). a \leq b \ll c \leq t \text{ implies } a \ll t.\]
\[(\ll 4). a \ll c \text{ and } b \ll c \text{ implies } a \lor b \ll c.\]
\[(\ll 5). \text{If } a \ll c \text{ then } a \ll b \ll c \text{ for some } b \in B.\]
\[(\ll 6). \text{If } a \neq 0 \text{ then there exists } b \neq 0 \text{ such that } b \ll a.\]
\[(\ll 7). a \ll b \text{ implies } b^* \ll a^*.\]

Indeed, if \((B, C)\) is an NCA, then the relation \( \ll_C \) satisfies the axioms \((\ll 1) – (\ll 7)\). Conversely, having a pair \((B, \ll)\), where \( B \) is a Boolean algebra and \( \ll \) is a binary relation on \( B \) which satisfies \((\ll 1) – (\ll 7)\), we define a relation \( C_\ll \) by \( aC_\ll b \) if, and only if, \( a \ll b^* \); then \((B, C_\ll)\) is an NCA. Note that the axioms (C5) and (C6) correspond to \((\ll 5)\) and \((\ll 6)\), respectively. It is easy to see that a contact algebra could be equivalently defined as a pair consisting of a Boolean algebra \( B \) and a binary relation \( \ll \) on \( B \) subject to the axioms \((\ll 1) – (\ll 4)\) and \((\ll 7)\).

The most important example of a CA is given by the regular closed sets of an arbitrary topological space \( X \). Let us start with some standard notations and conventions that we use throughout the paper. For a subset \( M \) of \( X \), we denote by \( \text{cl}_X(M) \) (or simply \( \text{cl}(M) \)) the closure of \( M \) in \( X \), and by \( \text{int}(M) \) its interior. \( \text{CO}(X) \) denotes the set of all clopen (= closed and open) subsets of \( X \); trivially, \( \text{CO}(X) \cup, \cap, \setminus, \emptyset, X \) is a Boolean algebra. \( \text{RC}(X) \) (resp., \( \text{RO}(X) \)) denotes the set of all regular closed (resp., regular open) subsets of \( X \); recall that a subset \( F \) of \( X \) is said to be regular closed (resp., regular open) if \( F = \text{cl}(\text{int}(F)) \) (resp., \( F = \text{int}(\text{cl}(F)) \)).

Note that in this paper (unlike in [24]) compact spaces are not assumed to be Hausdorff.

**Example 2.3.** For a topological space \( X \), the collection \( \text{RC}(X) \) becomes a complete Boolean algebra under the operations

\[ F \lor G \overset{\text{df}}{=} F \cup G, \quad F \land G \overset{\text{df}}{=} \text{cl}(\text{int}(F \cap G)), \quad F^* \overset{\text{df}}{=} \text{cl}(X \setminus F), \quad 0 \overset{\text{df}}{=} \emptyset, \quad 1 \overset{\text{df}}{=} X. \]

The infinite operations are given by the formulas

\[ \bigvee \{ F_\gamma \mid \gamma \in \Gamma \} = \text{cl}(\bigcup_{\gamma \in \Gamma} F_\gamma) = \text{cl}(\text{int}(\bigcup_{\gamma \in \Gamma} F_\gamma)), \]
\[ \bigwedge \{ F_\gamma \mid \gamma \in \Gamma \} = \text{cl}(\text{int}(\bigcap_{\gamma \in \Gamma} \{ F_\gamma \mid \gamma \in \Gamma \})). \]
One defines the relation $\rho_X$ on $\text{RC}(X)$ by setting, for each $F, G \in \text{RC}(X)$,

$$F \rho_X G \text{ if, and only if, } F \cap G \neq \emptyset.$$  

Clearly, $\rho_X$ is a contact relation on $\text{RC}(X)$, called the standard contact relation of $X$. The complete CA $(\text{RC}(X), \rho_X)$ is called a standard contact algebra. Note that, for $F, G \in \text{RC}(X)$,

$$F \ll_{\rho_X} G \text{ if, and only if, } F \subseteq \text{int}(G).$$

Thus, if $X$ is a normal Hausdorff space then the standard contact algebra $(\text{RC}(X), \rho_X)$ is a complete NCA.

Instead of looking at regular closed sets, we may, equivalently, consider regular open sets. The collection $\text{RO}(X)$ of regular open sets becomes a complete Boolean algebra by setting

$$U \lor V \overset{\text{df}}{=} \text{int}(\text{cl}(U \cup V)), \quad U \land V \overset{\text{df}}{=} U \cap V, \quad U^* \overset{\text{df}}{=} \text{int}(X \setminus U), \quad 0 \overset{\text{df}}{=} \emptyset, \quad 1 \overset{\text{df}}{=} X,$$

and

$$\bigwedge_{i \in I} U_i \overset{\text{df}}{=} \text{int}(\text{cl}(\bigcap_{i \in I} U_i)) \quad (= \text{int}(\bigcap_{i \in I} U_i)), \quad \bigvee_{i \in I} U_i \overset{\text{df}}{=} \text{int}(\text{cl}(\bigcup_{i \in I} U_i)),$$

see [29, Theorem 1.37]. We define a contact relation $D_X$ on $\text{RO}(X)$ as follows:

$$UD_X V \text{ if, and only if, } \text{cl}(U) \cap \text{cl}(V) \neq \emptyset.$$  

Then $(\text{RO}(X), D_X)$ is a complete CA.

The contact algebras $(\text{RC}(X), \rho_X)$ and $(\text{RO}(X), D_X)$ are CA-isomorphic via the mapping $\nu : \text{RC}(X) \rightarrow \text{RO}(X)$ defined by the formula $\nu(F) \overset{\text{df}}{=} \text{int}(F)$, for every $F \in \text{RC}(X)$.

**Example 2.4.** Let $B$ be a Boolean algebra. Then there exist a largest and a smallest contact relation on $B$; the largest one, $\rho_l$, is defined by

$$a \rho_l b \iff (a \neq 0 \text{ and } b \neq 0),$$

and the smallest one, $\rho_s$, by

$$a \rho_s b \iff a \land b \neq 0.$$  

Note that, for $a, b \in B$,

$$a \ll_{\rho_s} b \iff a \leq b;$$

hence $a \ll_{\rho_s} a$, for any $a \in B$. Thus $(B, \rho_s)$ is a normal contact algebra.

We will need the following definition and assertion from [20]:
**Definition 2.5.** ([20]) For a contact algebra \((B, C)\) one defines the relation \(R_{(B,C)}\) on the set of all filters on \(B\) by

1. \(f R_{(B,C)} g\) if, and only if, \(f \times g \subseteq C\),

for all filters \(f, g\) on \(B\).

**Proposition 2.6.** (a) ([20, Lemma 3.5, p. 222]) Let \((B, C)\) be a contact algebra. Then, for all \(a, b \in B\), one has \(aCb\) if, and only if, there exist ultrafilters \(u, v\) in \(B\) such that \(a \in u\), \(b \in v\) and \(u R_{(B,C)} v\).

(b) ([20, 22]) If \((B, C)\) is a normal contact algebra, then \(R_{(B,C)}\) is an equivalence relation.

**Definition 2.7.** For CA \((B, C)\), a non–empty subset \(\sigma\) of \(B\) is called a cluster if for all \(x, y \in B\),

- (CL1). If \(x, y \in \sigma\) then \(xCy\).
- (CL2). If \(x \lor y \in \sigma\) then \(x \in \sigma\) or \(y \in \sigma\).
- (CL3). If \(xCy\) for every \(y \in \sigma\), then \(x \in \sigma\).

The set of all clusters in an NCA \((B, C)\) is denoted by \(\text{Clust}(B, C)\)

The next theorem is used later on and may be proved exactly as Theorem 5.8 of [31]:

**Theorem 2.8.** A subset \(\sigma\) of a normal contact algebra \((B, C)\) is a cluster if, and only if, there exists an ultrafilter \(u\) in \(B\) such that

\[ \sigma = \{ a \in B \mid aCb \text{ for every } b \in u \}. \]

Moreover, given \(\sigma\) and \(a_0 \in \sigma\), there exists an ultrafilter \(u\) in \(B\) satisfying (2) and containing \(a_0\).

**Corollary 2.9.** Let \((B, C)\) be a normal contact algebra and \(u\) be an ultrafilter in \(B\). Then there exists a unique cluster \(\sigma_u\) in \((B, C)\) containing \(u\), namely

\[ \sigma_u = \{ a \in B \mid aCb \text{ for every } b \in u \}. \]

The following simple result can be proved exactly as Lemma 5.6 of [31]:

**Fact 2.10.** Let \((B, C)\) be a normal contact algebra and \(\sigma_1, \sigma_2\) clusters in \((B, C)\). If \(\sigma_1 \subseteq \sigma_2\), then \(\sigma_1 = \sigma_2\).

**Notation 2.11.** For a topological space \((X, \tau)\) and \(x \in X\), we set

\[ \sigma_x^X = \{ F \in RC(X) \mid x \in F \} \]

and often write just \(\sigma_x\).

The next assertion is obvious:
Fact 2.12. For a regular topological space $X$, $\sigma_x$ is a cluster in the CA $(RC(X), \rho_X)$, called a point-cluster.

For a category $\mathcal{C}$, we denote by $|\mathcal{C}|$ its class of objects, by $\text{Mor}(\mathcal{C})$ its class of morphisms, and by $\mathcal{C}(X,Y)$ the set of all $\mathcal{C}$-morphisms $X \to Y$.

2.13. Let us fix the notation for the Stone Duality ([36, 29]). We denote by $\text{Stone}$ the category of all zero-dimensional compact Hausdorff spaces (= Stone spaces) and their continuous mappings, and by $\text{Bool}$ the category of Boolean algebras and Boolean homomorphisms. The contravariant functors furnishing the Stone duality are denoted by

$$S^a : \text{Bool} \to \text{Stone} \quad \text{and} \quad S^t : \text{Stone} \to \text{Bool}.$$ 

Hence, for $A \in |\text{Bool}|$, $S^a(A)$ is the set $\text{Ult}(A)$ of all ultrafilters in $A$ endowed with the topology whose open base is the family $\{s^A_a \mid a \in A\}$, where

$$s^A_a \overset{df}{=} \{ u \in \text{Ult}(A) \mid a \in u \}$$

for all $a \in A$. For $X \in |\text{Stone}|$, one sets $S^t(X) \overset{df}{=} \text{CO}(X)$, and for morphisms $f \in \text{Stone}(X,Y)$ and $\varphi \in \text{Bool}(B_1,B_2)$ one puts

$$S^t(f)(F) = f^{-1}(F) \quad \text{and} \quad S^a(\varphi)(u) = \varphi^{-1}(u)$$

for all $F \in \text{CO}(Y)$ and $u \in \text{Ult}(B_2)$. Now, for every Boolean algebra $A$, the map

$$s^A_a : A \to S^t(S^a(A)), \ a \mapsto s^A_a(a),$$

is a Boolean isomorphism, and for every Stone space $X$, the map

$$t_X : X \to S^a(\text{CO}(X)), \ x \mapsto u_x,$$

is a homeomorphism; here, for every $x \in X$,

$$(5) \quad u_x \overset{df}{=} \{ P \in \text{CO}(X) \mid x \in P \}.$$ 

Moreover, $s^A_a$ and $t_X$ are natural in $A$ and $X$.

2.14. Let us recall some standard properties for a continuous map of topological spaces: $f : X \to Y$ is

- **closed** if the image of each closed set is closed;
- **perfect** if it is closed and has compact fibres;
- **quasi-open** ([30]) if $\text{int}(f(U)) \neq \emptyset$ for every non-empty open subset $U$ of $X$;
- **irreducible** if $f(X) = Y$ and if, for every proper closed subset $F$ of $X$, $f(F) \neq Y$. 


Recall that, for a regular space $X$, a space $EX$ is called an absolute of $X$ if there exists a perfect irreducible mapping $\pi_X : EX \to X$ and every perfect irreducible preimage of $EX$ is homeomorphic to $EX$ (see, e.g., [6, 33]). It is well-known that:

(a) the absolute is unique up to homeomorphism;
(b) a space $Y$ is an absolute of a regular space $X$ if, and only if, $Y$ is an extremally disconnected Tychonoff space for which there exists a perfect irreducible mapping $\pi : Y \to X$; such mappings $\pi$ are called projective covers of $X$;
(c) if $X$ is a compact Hausdorff space, then it is well-known (see, e.g., [38]) that $EX = S^a(RC(X))$ and the projective cover $\pi_X$ of $X$ is defined by

$$\pi_X(u) \overset{\text{df}}{=} \bigcap u,$$

for every $u \in \operatorname{Ult}(RC(X)) = S^a(RC(X))$ (here $S^a : \text{Bool} \to \text{Stone}$ is the Stone contravariant functor).

2.15. Let $\mathcal{C}$ be a subcategory of the category $\text{Top}$ of all topological spaces and all continuous mappings between them. Recall that a $\mathcal{C}$-object $P$ is called a projective object in $\mathcal{C}$ if for every $g \in \mathcal{C}(P, Y)$ and every perfect surjection $f \in \mathcal{C}(X, Y)$, there exists $h \in \mathcal{C}(P, X)$ such that $f \circ h = g$.

A. M. Gleason [26] proved:

In the category $\text{CHaus}$ of compact Hausdorff spaces and continuous mappings, the projective objects are precisely the extremally disconnected spaces.

3 Extensions of dualities

3.1. Given a dual equivalence $T : A \to B$ and an embedding $I$ of $B$ as a full subcategory of a category $\mathcal{C}$, we wish to give a natural construction for a category $\mathcal{D}$ into which $A$ may be fully embedded via $J$, such that $T$ extends to a dual equivalence $\tilde{T} : \mathcal{D} \to \mathcal{C}$:

$$\begin{array}{ccc}
\mathcal{D} & \xrightarrow{\tilde{T}} & \mathcal{C} \\
J \downarrow & & \downarrow I \\
A & \xrightarrow{T} & B
\end{array}$$

Our construction depends on a class $\mathcal{P}$ of morphisms of $\mathcal{C}$ satisfying certain conditions, which are closely related to certain properties of the full embedding $I$. It turns out that, when $B$ is projective in $\mathcal{C}$, such a class $\mathcal{P}$ always exists.

We call a class $\mathcal{P}$ of morphisms in $\mathcal{C}$ a weak $(\mathcal{B}, \mathcal{C})$-covering class if it satisfies the following conditions:

(P1) $\forall (p : B \to C) \in \mathcal{P} : B \in |\mathcal{B}|$;

(P2) $\forall B \in |\mathcal{B}| : 1_B \in \mathcal{P}$;
\(\mathcal{P} \circ \text{Iso}(\mathcal{B}) \subseteq \mathcal{P};\)

\(\forall C \in |\mathcal{C}| \exists (p : B \to C) \in \mathcal{P};\)

\((P5^\circ)\) for morphisms in \(\mathcal{C},\) there is an assignment

\[
\begin{array}{ccc}
B & \to & B' \\
\downarrow p & & \downarrow p' \\
C & \to & C'
\end{array}
\]

\[
\begin{array}{ccc}
B & \to & B' \\
\downarrow \hat{v} & & \downarrow \hat{v}' \\
C & \to & C'
\end{array}
\]

\(((p : B \to C) \in \mathcal{P}, v : C \to C', (p' : B' \to C') \in \mathcal{P}) \mapsto (\hat{v} : B \to B' \text{ with } v \circ p = p' \circ \hat{v}),\)

Note that in the given assignment, \(\hat{v}\) depends not only on \(v,\) but also on \(p\) and \(p'.\)

In condition \((P4)\) we tacitly assume that, for every \(C \in |\mathcal{C}|,\) we have a \textit{chosen} morphism \(p \in \mathcal{P}\) with codomain \(C.\) In the presence of \((P2),\) that morphism may be taken to be an identity morphism whenever \(C \in |\mathcal{B}|.\) To emphasize the choice, we may reformulate \((P4),\) as follows:

\((P4')\) \(\forall C \in |\mathcal{C}| \exists (\pi_C : EC \to C) \in \mathcal{P} \text{ (with } \pi_C = 1_C \text{ when } C \in |\mathcal{B}|).\)

As a precursor to the category \(\mathcal{D}\) as envisaged at the beginning of 3.1, we consider the comma category \((IT \downarrow \mathcal{P} \mathcal{C}),\) defined as follows:

- objects in \((IT \downarrow \mathcal{P} \mathcal{C})\) are pairs \((A, p)\) with \(A \in |\mathcal{A}|\) and \(p : TA \to C\) in the class \(\mathcal{P};\)

- morphisms \((\varphi, f) : (A, p) \to (A', p')\) in \((IT \downarrow \mathcal{P} \mathcal{C})\) are given by morphisms \(\varphi : A \to A'\) in \(\mathcal{A}\) and \(f : C' \to C\) in \(\mathcal{C},\) such that \(p \circ T\varphi = f \circ p':\)

\[
\begin{array}{ccc}
TA & \xrightarrow{T\varphi} & TA' \\
\downarrow p & & \downarrow p' \\
C & \xleftarrow{f} & C'
\end{array}
\]

- composition is as in \(\mathcal{A}\) and \(\mathcal{C};\) that is, \((\varphi, f)\) as above gets composed with \((\varphi', f') : (A', p') \to (A'', p'')\) by the horizontal pasting of diagrams, that is,

\[
(\varphi', f') \circ (\varphi, f) \overset{\text{df}}{=} (\varphi' \circ \varphi, f \circ f').
\]

- the identity morphism of a \((IT \downarrow \mathcal{P} \mathcal{C})\)-object \((A, p)\) is the \((IT \downarrow \mathcal{P} \mathcal{C})\)-morphism \((1_A, 1_{\text{cod}(p)}).\)

On the hom-sets of \((IT \downarrow \mathcal{P} \mathcal{C})\) we define a compatible equivalence relation by

\[
(\varphi, f) \sim (\psi, g) \iff f = g,
\]
for all \((\varphi, f), (\psi, g) : (A, p) \rightarrow (A', p')\). We denote the equivalence class of \((\varphi, f)\) by 
\([\varphi, f] \) (or \([\varphi, f]_{(A, p), (A', p')}\), if clarity demands it), and let \(\mathcal{D}\) be the quotient category 
\((IT \downarrow f \mathcal{C})/ \sim \).

Thanks to (P2), we have the functor \(J : \mathcal{A} \rightarrow \mathcal{D}\), defined by 
\((\varphi : A \rightarrow A') \mapsto( J\varphi \overset{df}{=} [\varphi, T\varphi] : (A, 1_{TA}) \rightarrow (A', 1_{TA'}) )\),

which is easily seen to be a full embedding.

Given a dual equivalence \((S, T, \eta, \varepsilon)\) with contravariant functors 
\(T : \mathcal{A} \rightarrow \mathcal{B} \quad \text{and} \quad S : \mathcal{B} \rightarrow \mathcal{A}\)

and natural isomorphisms \(\eta : \text{Id}_\mathcal{B} \rightarrow T \circ S\) and \(\varepsilon : \text{Id}_\mathcal{A} \rightarrow S \circ T\), satisfying the triangular identities 
\(T\varepsilon \circ \eta T = 1_T\) and \(S\eta \circ \varepsilon S = 1_S\),

it is now straightforward to establish a dual equivalence of \(\mathcal{D}\) with \(\mathcal{C}\), as follows:

**Theorem.** There is a dual equivalence \(\tilde{T} : \mathcal{D} \leftarrow \mathcal{C} : \tilde{S}\) extending the given dual equivalence \(T : \mathcal{A} \leftarrow \mathcal{B} : S\), in the sense that that \(\tilde{T}J = IT\) and \(\tilde{S}I \cong JS\):

\[
\begin{array}{ccc}
\mathcal{D} & \xrightarrow{\tilde{T}} & \mathcal{C} \\
\downarrow J & & \downarrow I \\
\mathcal{A} & \xrightarrow{T} & \mathcal{B}
\end{array}
\quad \quad \quad \quad \quad
\begin{array}{ccc}
\mathcal{D} & \xleftarrow{\tilde{S}} & \mathcal{C} \\
\downarrow J & \cong & \downarrow I \\
\mathcal{A} & \xleftarrow{S} & \mathcal{B}
\end{array}
\]

The unit \(\tilde{\eta} : \text{Id}_\mathcal{D} \rightarrow \tilde{T}\tilde{S}\) and the counit \(\tilde{\varepsilon} : \text{Id}_\mathcal{B} \rightarrow \tilde{S}\tilde{T}\) of the extended adjunction and the natural isomorphism \(\gamma : JS \rightarrow \tilde{S}I\) may be chosen to satisfy the identities 
\(\tilde{\eta} = 1_{\text{Id}_\mathcal{D}}, \tilde{T}\tilde{\varepsilon} = 1_{\tilde{T}}, \tilde{\varepsilon}\tilde{S} = 1_{\tilde{S}},\) and \(\tilde{T}\gamma = I\eta, \gamma T \circ J\varepsilon = \varepsilon J\).

**Proof.** \(\tilde{T}\) is given by the projection \([\varphi, f] \mapsto f\); this trivially gives a faithful functor.

With (P5°) it is easy to see that \(T\) is full since \(T\) is. To define \(\tilde{S}\) on objects, one chooses for every \(C \in |\mathcal{C}|\) a morphism \(\pi_C : EC \rightarrow C\) in \(\mathcal{P}\) (as in (P4')), with \(p_B = 1_B\) for all \(B \in |\mathcal{B}|\) (according to (P2)), and then puts \(\tilde{S}C \overset{df}{=} (SEC, \pi_C \circ \eta_{EC}^{-1})\). For a morphism \(f : C' \rightarrow C\) in \(\mathcal{C}\), again, (P5°) and the fullness of \(T\) allow one to choose a morphism \(\varphi_f : SEC \rightarrow SEC'\) in \(\mathcal{A}\) with \(\pi_C \circ \eta_{EC}^{-1} \circ T\varphi_f = f \circ \pi_{C'} \circ \eta_{EC'}^{-1}\); we then put \(\tilde{S}f \overset{df}{=} [\varphi_f, f]\). Checking that \(\tilde{S}\) is a functor with \(\tilde{T}\tilde{S} = \text{Id}_\mathcal{D}\) is straightforward.

For \((A, p : TA \rightarrow C) \in |\mathcal{D}|\) one puts \(\bar{\varepsilon}_{(A,p)} \overset{df}{=} [\varphi_{(A,p)}, 1_C]\), with any \(A\)-morphism \(\varphi_{(A,p)} : A \rightarrow SEC\) satisfying \(p \circ T\varphi_{(A,p)} = \pi_C \circ \eta_{EC}^{-1}\). Clearly, \(\bar{\varepsilon}\) is, like \(\bar{\eta} \overset{df}{=} 1_{\text{Id}_\mathcal{D}}\), a natural isomorphism satisfying the claimed identities. Also, with \(\gamma_B \overset{df}{=} [1_{SB}, \eta_B]\) for all \(B \in |\mathcal{B}|\), one obtains a natural isomorphism \(\gamma\) satisfying \(\tilde{T}\gamma = I\eta, \gamma T \circ J\varepsilon = \varepsilon J\).

\(\square\)
3.2. Recall that, for a class $\mathcal{Q}$ of morphisms in $\mathcal{C}$, an object $B \in |\mathcal{C}|$ is $\mathcal{Q}$-projective if, for all $(q : C \to D) \in \mathcal{Q}$, the map 
\[
\mathcal{C}(B, q) : \mathcal{C}(B, C) \to \mathcal{C}(B, D), \quad h \mapsto q \circ h,
\]
is surjective. Since this map is trivially bijective when $q$ is an isomorphism, without loss of generality we may assume that $\mathcal{Q}$ contain all isomorphisms and be closed under composition with them. We call a full subcategory $\mathcal{B}$ in $\mathcal{C}$ projective if there is a such a class $\mathcal{Q}$ satisfying

(Q1) $\forall C \in |\mathcal{C}| \exists (q : B \to C) \in \mathcal{Q}$ with $B \in |\mathcal{B}|$;

(Q2) $\forall B \in |\mathcal{B}| : B$ is $\mathcal{Q}$-projective.

**Proposition.** A full subcategory $\mathcal{B}$ of a category $\mathcal{C}$ is projective if, and only if, there is a weak $(\mathcal{B}, \mathcal{C})$-covering class $\mathcal{P}$.

**Proof.** Having a class $\mathcal{Q}$ containing all $\mathcal{C}$-isomorphisms, being closed under composition with them, and satisfying (Q1-2), one lets $\mathcal{P}$ be the subclass of those morphisms in $\mathcal{Q}$ whose domains lie in $\mathcal{B}$. Then, trivially (P1-3) hold, and (Q1) coincides with (P4). Given $\mathcal{C}$-morphisms $(p : B \to C) \in \mathcal{P}$, $v : C \to C'$, $(p' : B' \to C') \in \mathcal{P}$, one exploits the $\mathcal{Q}$-projectivity of $B$ (by (Q2)) to find $\hat{v} : B \to B'$ with $v \circ p = p' \circ \hat{v}$, which confirms (P5*).

Conversely, having a class $\mathcal{P}$ satisfying (P1-4), (P5*), we consider its closure $\mathcal{Q}$ under isomorphisms in $\mathcal{C}$ and trivially obtain (Q1) from (P4). To confirm (Q2), we let $B \in |\mathcal{B}|, (q : C \to D) \in \mathcal{Q}$, and $f : B \to D$ in $\mathcal{C}$, and may, for simplicity, assume $q \in \mathcal{P}$. Since $1_B \in \mathcal{P}$ by (P2), condition (P5*) provides us with a morphism $h = \hat{f}$ with $f = f \circ 1_B = q \circ h$, thus confirming the surjectivity of the map $\mathcal{C}(B, q)$.

Note that conditions (Q1-2) imply in particular that the following condition holds:

(Q1\*) $\mathcal{C}$ has enough $\mathcal{Q}$-projectives: $\forall C \in |\mathcal{C}| \exists (q : B \to C) \in \mathcal{Q}$ and $B$ is $\mathcal{Q}$-projective.

If we strengthen (Q2) to

(Q2\*) $\forall B \in |\mathcal{C}| : (B \in |\mathcal{B}| \iff B$ is $\mathcal{Q}$-projective),

then, in the presence of (Q2\*), condition (Q1\*) is a weakening of (Q1). The conjunction of (Q1\*) and (Q2\*) is equivalent to (Q1-2) if the full subcategory $\mathcal{B}$ is retractive in $\mathcal{C}$, that is: if, for all $s : C \to B$, $r : B \to C$ in $\mathcal{C}$ with $r \circ s = 1_C$, $B \in |\mathcal{B}|$ implies $C \in |\mathcal{B}|$. Since retracts of $\mathcal{Q}$-projective objects are $\mathcal{Q}$-projective, one obtains the following modification of Proposition 3.2:

**Corollary.** For a full subcategory $\mathcal{B}$ of a category $\mathcal{C}$, there is a class $\mathcal{Q}$ satisfying (Q1\*) and (Q2\*) if, and only if, $\mathcal{B}$ is retractive and $\mathcal{C}$ admits a weak $(\mathcal{B}, \mathcal{C})$-covering class $\mathcal{P}$.
3.3. In [19] we noted that \( \mathcal{B} \) is a coreflective subcategory of \( \mathcal{C} \) if, and only if, there exists a class \( \mathcal{P} \) of \( \mathcal{C} \)-morphisms satisfying properties (P1-4) and the following strengthening of (P5):

\[(P5^*) \text{ for all } v : C \to C' \text{ in } \mathcal{C} \text{ and } p : B \to C, \ p' : B' \to C' \text{ in } \mathcal{P}, \text{ there is precisely one morphism } \hat{v} : B \to B' \text{ with } v \circ p = p' \circ \hat{v}.\]

Note that if, in the notation of 3.1, the class \( \mathcal{P} \) satisfies conditions (P1-4) and \((P5^*)\), then the equivalence relation \( \sim \) is just the equality relation. Thus, in this case, the category \( \mathcal{D} \) coincides with the category \( (IT \downarrow_\mathcal{P} \mathcal{C}) \). In the sequel, we will also use the dualization of this special form of Theorem 3.1. To be able to refer to it later on, next we formulate this dualization explicitly.

Let \( \mathcal{A} \) be a full subcategory of a category \( \mathcal{D} \) with inclusion functor \( J \). We call a class \( J \) of morphisms in \( \mathcal{D} \) a strong \( (\mathcal{A}, \mathcal{D}) \)-insertion class if it satisfies the following conditions (J1-4) and \((J5^*)\):

\[(J1) \ \forall (j : D \to A) \in J : A \in |\mathcal{A}|; \]
\[(J2) \ \forall A \in |\mathcal{A}| : 1_A \in J; \]
\[(J3) \text{Iso}(\mathcal{A}) \circ J \subseteq J; \]
\[(J4) \ \forall D \in |\mathcal{D}| \ \exists (j : D \to A) \in J; \]
\[(J5^*) \text{ for all } v : D \to D' \text{ in } \mathcal{D} \text{ and } j : D \to A, \ j' : D' \to A' \text{ in } J, \text{ there is precisely one morphism } \overline{v} : A \to A' \text{ with } j' \circ v = \overline{v} \circ j.\]

Again, we point out that, in the given assignment, \( \overline{v} \) depends not only on \( v \), but also on \( j \) and \( j' \), so that, whenever needed, we will write \( \overline{v}(j, j') \) instead of just \( \overline{v} \). Next, we note that, in the presence of (J3), condition (J2) means equivalently

\[(J2') \text{Iso}(\mathcal{A}) \subseteq J.\]

In condition (J4) we tacitly assume that, for every \( D \in |\mathcal{D}| \), we have a chosen morphism \( j \in J \) with domain \( D \). In the presence of (J2), that morphism may be taken to be an identity morphism whenever \( D \in |\mathcal{A}| \). To emphasize the choice, we may reformulate (J4), as follows:

\[(J4') \ \forall D \in |\mathcal{D}| \ \exists (\rho_D : D \to FD) \in J \text{ (with } \rho_D = 1_D \text{ when } D \in |\mathcal{A}|).\]

It is now clear that \((J5^*)\) enables us to make \( F \) a functor \( \mathcal{D} \to \mathcal{A} \) and \( \rho \) a natural transformation \( \text{Id}_\mathcal{C} \to JF \).

Dualizing an observation made in [19], we obtain the following proposition:

**Proposition.** The full subcategory \( \mathcal{A} \) of \( \mathcal{D} \) is reflective in \( \mathcal{D} \) if, and only if, there is a strong \( (\mathcal{A}, \mathcal{D}) \)-insertion class \( J \) of morphisms in \( \mathcal{D} \).

In addition to the full subcategory \( \mathcal{A} \) of \( \mathcal{D} \) with inclusion functor \( J \) and a strong \( (\mathcal{A}, \mathcal{D}) \)-insertion class \( J \) (giving us the reflector \( F : \mathcal{D} \to \mathcal{A} \) and a natural transformation \( \rho : \text{Id}_\mathcal{D} \to JF \), with \( \rho_A \) an isomorphism for all \( A \in |\mathcal{A}| \)), as in Theorem 3.1 we consider again a dual equivalence \( (S, T, \eta, \varepsilon) \) with contravariant functors

\[T : \mathcal{A} \to \mathcal{B} \quad \text{and} \quad S : \mathcal{B} \to \mathcal{A}\]
and natural isomorphisms \( \eta : \text{Id}_B \to T \circ S \) and \( \epsilon : \text{Id}_A \to S \circ T \) satisfying the triangular identities. We then construct the category \( \mathcal{C} \), as follows:

- objects in \( \mathcal{C} \) are pairs \((B, j)\) with \( B \in |\mathcal{B}| \) and \( j : D \to SB \) in the class \( \mathcal{J} \);

- morphisms \((\varphi, f) : (B, j) \to (B', j')\) in \( \mathcal{C} \) are given by morphisms \( \varphi : B \to B' \) in \( \mathcal{B} \) and \( f : D' \to D \) in \( \mathcal{D} \), such that, in the notation of \((J5^*)\), \( S\varphi = \overline{f} : SB \to SB' \):

\[
\begin{array}{ccc}
SB & \xrightarrow{S\varphi = \overline{f}} & SB' \\
\downarrow{j} & & \downarrow{j'} \\
D & \xrightarrow{f} & D'
\end{array}
\]

- composition is as in \( \mathcal{B} \) and \( \mathcal{D} \); that is, \((\varphi, f)\) as above gets composed with \((\varphi', f') : (B', j') \to (B'', j'')\) by the horizontal pasting of diagrams, that is,

\[
(\varphi', f') \circ (\varphi, f) \overset{df}{=} (\varphi' \circ \varphi, f \circ f').
\]

- the identity morphism of a \( \mathcal{C} \)-object \((B, j)\) is the \( \mathcal{C} \)-morphism \((1_B, 1_{\text{dom}(j)})\).

Of course, the fact that the composition and the identity morphisms of \( \mathcal{C} \) are well defined, relies heavily on \((P5^*)\). Since \( S \) is fully faithful, we note that, for a morphism \((\varphi, f)\) in \( \mathcal{C} \), the \( \mathcal{B} \)-morphism \( \varphi \) is determined by \( f \) and, hence, by \( f, j \), and \( j' \). With \((J2)\) one obtains the full embedding \( I : \mathcal{B} \to \mathcal{C} \), defined by

\[
(\varphi : B \to B') \mapsto (I\varphi \overset{df}{=} (\varphi, S\varphi) : (B, 1_{SB}) \to (B', 1_{SB'})).
\]

A dual equivalence

\[
\overline{S} : \mathcal{C} \leftrightarrow \mathcal{D} : \overline{T}
\]

with natural isomorphisms \( \overline{\epsilon} : \text{Id}_\mathcal{D} \to \overline{S} \circ \overline{T} \) and \( \overline{\eta} : \text{Id}_\mathcal{C} \to \overline{T} \circ \overline{S} \) may now be established, as follows:

- \( \overline{S} : ((\varphi, f) : (B, j) \to (B', j)) \mapsto (f : \text{dom}(j') \to \text{dom}(j)) \);

- \( \overline{T} : (f : D' \to D) \mapsto ((\varphi_f, f) : (TFD, \epsilon_{FD} \circ \rho_D) \to (TFD', \epsilon_{FD'} \circ \rho_{D'})), \)

where \( \varphi_f : TFD \to TFD' \) is the unique \( \mathcal{B} \)-morphism to make the diagram

\[
\begin{array}{ccc}
STFD & \xrightarrow{S\varphi_f} & STFD' \\
\downarrow{\epsilon_{FD}} & & \downarrow{\epsilon_{FD'}} \\
FD & \xrightarrow{Ff} & FD' \\
\downarrow{\rho_D} & & \downarrow{\rho_{D'}} \\
D & \xrightarrow{f} & D'
\end{array}
\]

commutative (so that \( S\varphi_f = \overline{f}(j', j) \) with \( j = \epsilon_{FD} \circ \rho_D \) and \( j' = \epsilon_{FD'} \circ \rho_{D'} \)).
• $\varepsilon_D \overset{\text{df}}{=} 1_D : D \to S\mathcal{T}D = D$, for every $D \in |\mathcal{D}|$;

• $\eta_{(B,j)} \overset{\text{df}}{=} (\psi_{B,j}, 1_D) : (B, j) \to T\mathcal{S}(B, j) = (T\mathcal{F}D, \varepsilon_{FD} \circ \rho_D)$, for every $\mathcal{C}$-object $(B, j : D \to SB)$, where the $\mathcal{B}$-isomorphism $\psi_{B,j} : B \to T\mathcal{F}D$ is determined by the commutative diagram

\[
\begin{array}{ccc}
SB & \overset{S\psi_{B,j}}{\longrightarrow} & STFD \\
\downarrow \varepsilon_D & & \varepsilon_D \\
D & \overset{1_D}{\longrightarrow} & D
\end{array}
\]

The dualization of Theorem 3.1 now reads as follows:

**Theorem.** $(\mathcal{T}, \mathcal{S}, \varepsilon, \eta)$ is a dual equivalence with $\mathcal{S}\mathcal{T} = \text{Id}_{\mathcal{D}}$, extending the given dual equivalence $(\mathcal{T}, \mathcal{S}, \varepsilon, \eta)$, so that $\mathcal{S}I = JS$ and $\mathcal{T}J \cong IT$:

\[
\begin{array}{cc}
\mathcal{D} & \overset{\mathcal{S}}{\longrightarrow} \mathcal{C} \\
J & \downarrow \cong \quad \uparrow I \\
\mathcal{A} & \overset{\mathcal{T}}{\longrightarrow} \mathcal{B}
\end{array}
\]

Furthermore, with a natural isomorphism $\delta : IT \to T\mathcal{J}$, the unit and co-unit of the adjunction satisfy

\[
\varepsilon = 1_{\text{Id}_\mathcal{D}}, \quad S\eta = 1_{\mathcal{T}}, \quad \eta\mathcal{T} = 1_{\mathcal{T}}, \quad \mathcal{S}\delta = J\varepsilon, \quad \delta S \circ I\eta = \eta I.
\]

We note that, as $\mathcal{A}$ is reflective in $\mathcal{D}$, $\mathcal{B}$ is coreflective in $\mathcal{C}$, with the coreflection satisfying some easily established identities involving the reflection and the units and counits of the dual equivalences.

### 4 The de Vries duality revisited

In this section we recall and extend various facts leading up to the de Vries Duality Theorem [15]. Our alternative proof of it follows in the next section.

We will now formulate and sketch a proof of the de Vries Duality Theorem.

**Definition 4.1.** (De Vries [15]) We denote by $\text{DeV}$ the category of complete normal contact algebras (see 2.1); its morphisms $\varphi : (A, C) \to (A', C')$ are maps $A \to A'$ satisfying the conditions:

(DV1) $\varphi(0) = 0$;

(DV2) $\varphi(a \land b) = \varphi(a) \land \varphi(b)$, for all $a, b \in A$;

(DV3) If $a, b \in A$ and $a \ll_C b$, then $(\varphi(a^*))^* \ll_C \varphi(b)$;

(DV4) $\varphi(a) = \bigvee \{\varphi(b) \mid b \ll_C a\}$, for every $a \in A$;
the composition "o" of \( \varphi_1: (A_1, C_1) \rightarrow (A_2, C_2) \) with \( \varphi_2: (A_2, C_2) \rightarrow (A_3, C_3) \) in Dev is defined by

\[
\varphi_2 \circ \varphi_1 \overset{\text{df}}{=} (\varphi_2 \circ \varphi_1)^*,
\]

where, for objects \((A, C), (A', C')\) in Dev and any function \(\psi: A \rightarrow A'\), one defines \(\psi^*: (A, C) \rightarrow (A', C')\) for all \(a \in A\) by

\[
\psi^*(a) \overset{\text{df}}{=} \bigvee \{ \psi(b) \mid b \ll_C a \}.
\]

We call the morphisms of the category Dev de Vries morphisms.

**Fact 4.2.** ([15]) Let \( \varphi: (A, C) \rightarrow (A', C') \) be a de Vries morphism. Then:

(a) \( \varphi(1_A) = 1_{A'} \);

(b) for every \( a \in A \), \( \varphi(a^*) \leq (\varphi(a))^* \);

(c) for every \( a, b \in A \), \( a \ll_C b \) implies \( \varphi(a) \ll_C \varphi(b) \);

(d) if \( \varphi': (A', C') \rightarrow (A'', C'') \) is a de Vries morphism, such that \( \varphi' \) is a suprema-preserving Boolean homomorphism, then \( \varphi' \circ \varphi = \varphi' \circ \varphi \);

(e) for every \( a \in A \), \( a = \bigvee \{ b \mid b \in A, b \ll a \} \).

De Vries [15] proved the following duality theorem:

**Theorem 4.3.** ([15]) The categories CHaus and Dev are dually equivalent.

**Sketch of the proof.** One defines contravariant functors

\[ \Psi^t: \text{CHaus} \rightarrow \text{Dev}, \quad \Psi^a: \text{Dev} \rightarrow \text{CHaus}, \]

by

- \( \Psi^t(X, \tau) \overset{\text{df}}{=} (\text{RC}(X, \tau), \rho_X) \), for all \( X \in |\text{CHaus}| \);

- \( \Psi^t(f)(G) \overset{\text{df}}{=} \text{cl}(f^{-1}(\text{int}(G))) \), for all \( f \in \text{CHaus}(X, Y) \) and \( G \in \text{RC}(Y) \);

- \( \Psi^a(A, C) \overset{\text{df}}{=} (\text{Clust}(A, C), \mathcal{T}) \), for all \( (A, C) \in |\text{Dev}| \), where \( \mathcal{T} \) is the topology on \( \text{Clust}(A, C) \) having the family \( \{ v_{(A,C)}(a) \mid a \in A \} \) with \( v_{(A,C)}(a) = \{ \sigma \in \text{Clust}(A, C) \mid a \in \sigma \} \) as a base of closed sets;

- \( \Psi^a(\varphi)(\sigma') \overset{\text{df}}{=} \{ a \in A \mid \forall b \in A \ (b \ll_C a^* \implies (\varphi(b))^* \in \sigma') \}, \) for all \( \varphi \in \text{Dev}((A, C), (A', C')) \) and \( \sigma' \in \text{Clust}(A', C') \).

Then one shows that, for every \( (A, C) \in |\text{Dev}| \), \( v_{(A,C)}: (A, C) \rightarrow \Psi^t(\Psi^a(A, C)) \) is a Dev-isomorphism, producing the natural isomorphism

\[ v: \text{id}_{\text{Dev}} \rightarrow \Psi^t \circ \Psi^a. \]

Likewise,

\[ t': \text{id}_{\text{CHaus}} \rightarrow \Psi^a \circ \Psi^t, \]

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with $t'_X(x) \overset{\text{df}}{=} \sigma_x$, for every $X \in \mathcal{CHaus}$ and all $x \in X$, is a natural isomorphism. Thus, the categories $\mathcal{CHaus}$ and $\mathcal{DeV}$ are dually equivalent. 

We note that, in [15], de Vries used regular open sets, rather than regular closed sets, as we do here. Hence, above we have paraphrased his definitions in terms of regular closed sets.

Remarks 4.4. (a) As it is noted in [15], for any complete Boolean algebra $B$,

$$\Psi^a((B, \rho_s)) = S^a(B),$$

where $S^a$ is the Stone dual equivalence.

(b) If $B$ is a complete atomic Boolean algebra, then for every $x \in \text{At}(B)$,

$$\uparrow (x) \overset{\text{df}}{=} \{ b \in B \mid x \leq b \}$$

is an ultrafilter in $B$ and, thus, by (a), $\uparrow (x) \in \Psi^a(B, \rho_s)$.

(c) If $B$ is a complete atomic Boolean algebra, then the set $\{ \uparrow (x) \mid x \in \text{At}(B) \}$ is dense in $S^a(B) = \Psi^a((B, \rho_s))$.

Indeed, if $b \in B^+$ then there exists $x \in \text{At}(B)$ such that $x \leq b$. Then $b \in \uparrow (x)$ and thus $\uparrow (x) \in s_B(b)$. Therefore, the set $\{ \uparrow (x) \mid x \in \text{At}(B) \}$ is dense in $S^a(B)$.

(d) If $(A, C) \in |\mathcal{DeV}|$ and $Y \overset{\text{df}}{=} \Psi^a(A, C)$, then, for every $a \in A$,

$$\text{int}_Y(v_{(A,C)}(a)) = Y \setminus v_{(A,C)}(a^*).$$

We will need the following result as well:

Theorem 4.5. ([15]) A de Vries morphism $\alpha$ is an injection if, and only if, the mapping $\Psi^a(\alpha)$ is a surjection.

Of great importance to our investigations is the following beautiful theorem by Alexandroff [4], which follows easily from Ponomarev’s results [32] on irreducible mappings:

Theorem 4.6. ([4, Corollary, p. 346]) Let $p : X \rightarrow Y$ be a closed irreducible mapping. Then the map

$$\varphi_p : \text{RC}(X) \rightarrow \text{RC}(Y), \quad H \mapsto p(H),$$

is a Boolean isomorphism, and one has $\varphi_p^{-1}(K) = \text{cl}_X(p^{-1}(\text{int}_Y(K)))$, for all $K \in \text{RC}(Y)$.

Denote by $\mathcal{CBool}$ the category of complete Boolean algebras and Boolean homomorphisms. The following assertions, proved in [19], are very important in this paper.
Lemma 4.7. ([19]) Let $A \in \mathcal{C}_{\text{Bool}}$, $X \in \mathcal{C}_{\text{Haus}}$, $\pi : S^a(A) \to X$ be an irreducible mapping and for every $a, b \in A$, define
\[ aC_{(A,\pi)}b \iff \pi(s_A(a)) \cap \pi(s_A(b)) \neq \emptyset. \]
Then $(A, C_{(A,\pi)})$ is a complete normal contact algebra.

Clearly, the definition of the relation $C_{(A,\pi)}$ as in Lemma 4.7 may be given equivalently, as follows: for all $a, b \in A$,
\[ aC_{(A,\pi)}b \iff \exists u, v \in \text{Ult}(A) : a \in u, b \in v \text{ and } \pi(u) = \pi(v). \]

Lemma 4.8. ([19]) Let $(A, C)$ be a CNCA and $R_{(A,C)}$ be the equivalence relation of Definition 2.5 (see also Proposition 2.6(b)), i.e., for all $u, v \in S^a(A)$,
\[ uR_{(A,C)}v \iff u \times v \subseteq C. \]
Then the natural quotient mapping $\pi_{(A,C)} : S^a(A) \to S^a(A) / R_{(A,C)}$ is an irreducible mapping, and $S^a(A) / R_{(A,C)}$ is a compact Hausdorff space.

Following [8], we call a closed equivalence relation $R$ on a compact Hausdorff space $X$ irreducible if the natural quotient mapping $\pi_R : X \to X/R$ is irreducible.

Proposition 4.9. ([19]) For a complete Boolean algebra $A$, let $\text{NCRel}(A)$ be the set of all normal contact relations on $A$ and $\text{IRel}(T(A))$ the set of all closed irreducible equivalence relations on $S^a(A)$. Then the function
\[ f : \text{NCRel}(A) \to \text{IRel}(S^a(A)), \quad C \mapsto R_{(A,C)}, \]
is a bijection, and $f^{-1}(R) = C_{(A,\pi_R)}$, for every $R \in \text{IRel}(T(A))$.

Note that Lemmas 4.7 and 4.8 and Proposition 4.9 reveal the topological nature of CNCA s, i.e., of the objects of the category $\text{DeV}$. Proposition 4.9 implies also Bezhanishvili’s Theorem [7, Theorem 8.1] mentioned in the Introduction: for any complete Boolean algebra $B$ there is a bijection between the set of all normal contact relations on $B$ and the set of all (up to homeomorphism) Hausdorff irreducible images of the Stone dual $S^a(B)$ of $B$. In [7] this result is obtained with the help of the de Vries Duality Theorem, while our proof is direct and therefore topologically more informative.

Let $(A, C)$ and $(A', C')$ be contact algebras, and $\varphi : (A, C) \to (A', C')$ be a map. Following Fedorchuk [25], we consider the following condition
(F) $\varphi(a)C'\varphi(b)$ implies $aCb$, for all $a, b \in A$.
If $\varphi$ preserves the negation, we see immediately that condition (F) is equivalent to asking that
(F') $a \ll_{C'} b$ implies $\varphi(a) \ll_{C'} \varphi(b)$, for all $a, b \in A$. 

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Proposition 4.10. ([19]) For objects \((A, C), (A', C')\) in \(\text{DeV}\), a Boolean homomorphism \(\psi : A \rightarrow A'\) satisfies condition \((F)\) (or, equivalently, condition \((F')\)) if, and only if, \(u' R_{(A', C')} v'\) implies \(S^a(\psi)(u') R_{(A, C)} S^a(\psi)(v')\), for all \(u', v' \in S^a(A')\).

Proposition 4.11. ([19]) For all \((A, C) \in |\text{DeV}|\), the mapping 

\[ h_{(A,C)} : S^a(A)/R_{(A,C)} \rightarrow \Psi^a(A, C), \ [u] \mapsto \sigma_u, \]

is well-defined and is a homeomorphism (see Corollary 2.9 for \(\sigma_u\) and Lemma 4.8 for \(R_{(A,C)}\)).

Note that Proposition 4.11 clarifies the definition of the contravariant functor \(\Psi^a\) on the objects of the category \(\text{DeV}\).

5 A new approach to the de Vries duality

5.1. In view of Section 3, throughout this section we use the following notation:

\[ \mathcal{A} \overset{df}{=} \text{CBool}, \ \mathcal{B} \overset{df}{=} \text{ECH}, \ \mathcal{C} \overset{df}{=} \text{CHaus}, \]

with \(I : \mathcal{B} \hookrightarrow \mathcal{C}\) denoting the inclusion functor; \(\mathcal{P}\) denotes the class of all irreducible continuous maps between compact Hausdorff spaces with domain in \(|\mathcal{B}|\). (Recall that we denote by \(\text{CBool}\) the category of complete Boolean algebras and Boolean homomorphisms, and by \(\text{ECH}\) the category of extremally disconnected compact Hausdorff spaces and continuous maps.)

Trivially, \(\mathcal{B}\) is a full subcategory of \(\mathcal{C}\) that is closed under \(\mathcal{C}\)-isomorphisms. By the results of Gleason [26] (see 2.15), the class \(\mathcal{P}\) satisfies conditions \((P1-4), (P5^o)\) of Section 3 (and \(\mathcal{B}\) is a projective subcategory of \(\mathcal{C}\)).

With the restrictions 

\[ T \overset{df}{=} S^a |_A \quad \text{and} \quad S \overset{df}{=} S^t |_B \]

of the functors furnishing the Stone Duality, using the well-known Stone’s result [36], we obtain the contravariant functors \(T : \mathcal{A} \rightarrow \mathcal{B}\) and \(S : \mathcal{B} \rightarrow \mathcal{A}\). Together with the restrictions \(\eta \overset{df}{=} t |_B\) and \(\varepsilon \overset{df}{=} s |_A\) of Stone’s natural isomorphisms (so that one has natural isomorphisms \(\eta : \text{Id}_B \rightarrow T \circ S\) and \(\varepsilon : \text{Id}_A \rightarrow S \circ T\)), they realize a dual equivalence between the categories \(\mathcal{A}\) and \(\mathcal{B}\).

Defining the category \(\mathcal{D}\) as in Theorem 3.1, we obtain the full embedding \(J : \mathcal{A} \rightarrow \mathcal{D}\) and the dual equivalence \(\tilde{T} : \mathcal{D} \rightarrow \mathcal{C}\) which extends the dual equivalence \(T : \mathcal{A} \rightarrow \mathcal{B}\), so that \(I \circ T = \tilde{T} \circ J\), as given by Theorem 3.1. We now prove that the categories \(\text{DeV}\) and \(\mathcal{D}\) are equivalent, thus completing our alternative proof of de Vries Duality Theorem. This will be done in several steps. In one of them, we will obtain a new category dual to the category \(\text{CHaus}\).

5.2. Let us start by recalling the definition of the category \(\mathcal{D}\). In our concrete situation, following Theorem 3.1, we obtain that

\[ |\mathcal{D}| \overset{df}{=} \{(A, \pi) \mid A \in \mathcal{A}, \pi \in \mathcal{P}, \text{dom}(\pi) = T(A)\}; \]
further, for every \((A, \pi), (A', \pi') \in |\mathcal{D}|\),
\[ \mathcal{D}(\langle A, \pi \rangle, \langle A', \pi' \rangle) \overset{\text{df}}{=} \{[\varphi, f] \mid \varphi \in A(A, A'), f \in \mathcal{C}(\text{cod}(\pi'), \text{cod}(\pi)), f \circ \pi' = \pi \circ T(\varphi)\}, \]
where \([\varphi, f]\) is the equivalence class of \((\varphi, f)\) under the equivalence relation \(\simeq\) in the set \(\{(\psi, g) \mid \psi \in A(A, A'), g \in \mathcal{C}(\text{cod}(\pi'), \text{cod}(\pi)), g \circ \pi' = \pi \circ T(\psi)\}\) defined by \((\varphi, f) \simeq (\psi, g) \iff f = g\); the composition law is the following one:
\[ [\varphi', f'] \circ [\varphi, f] \overset{\text{df}}{=} [\varphi' \circ \varphi, f \circ f'], \]
where \([\varphi, f], [\varphi', f']\) are any two composable \(\mathcal{D}\)-morphisms; finally, for any \(\mathcal{D}\)-object \((A, \pi), 1_{\langle A, \pi \rangle} \overset{\text{df}}{=} [1_A, 1_{\text{cod}(\pi)}] \).

We will need the following assertion:

**Proposition.** The full subcategory \(\mathcal{D}_{\text{nqum}}\) of \(\mathcal{D}\), where
\[ |\mathcal{D}_{\text{nqum}}| \overset{\text{df}}{=} \{(A, \pi) \in |\mathcal{D}| \mid \pi \text{ is a natural quotient mapping}\}, \]
is equivalent to \(\mathcal{D}\).

**Proof.** Denote by \(J' : \mathcal{D}_{\text{nqum}} \to \mathcal{D}\) the inclusion functor. Obviously, it is full and faithful. We have to show that it is essentially surjective on objects. Let \((A, \pi) \in |\mathcal{D}|\), \(X \overset{\text{df}}{=} \text{cod}(\pi), R_\pi\) be the equivalence relation on \(X\) determined by the fibres of \(\pi\), and \(q : T(\pi) = T(A) \to T(A)/R_\pi\) be the natural quotient mapping. Since \(\pi\) is a closed mapping, the map \(f_\pi : T(A)/R_\pi \to X, \forall u \in T(A), q(u) \mapsto \pi(u)\), is a homeomorphism and \(\pi = f_\pi \circ q\). Hence, \(q\) is an irreducible mapping and \((A, q) \in |\mathcal{D}_{\text{nqum}}|\). Then, clearly, \([1_A, f_\pi] : (A, \pi) \to (A, q)\) and \([1_A, f_\pi^{-1}] : (A, q) \to (A, \pi)\) are \(\mathcal{D}\)-isomorphisms. Therefore, \(J'(A, q)\) is \(\mathcal{D}\)-isomorphic to \((A, \pi)\). Thus, \(J'\) is essentially surjective on objects. All this shows that \(J'\) is an equivalence. \(\square\)

**Lemma 5.3.** ([19]) The correspondence \(F : |\mathcal{DeV}| \to |\mathcal{D}_{\text{nqum}}|, (A, C) \mapsto (A, \pi_{(A,C)}),\)
is a bijection.

**Proof.** Let \((A, C) \in |\mathcal{DeV}|\). Then, by Lemma 4.8, \(\pi_{(A,C)} \in \mathcal{P}\) and \((A, \pi_{(A,C)}) \in |\mathcal{D}|\). This makes the correspondence \(F\) well-defined. Now, with the notation of Lemma 4.7, we consider
\[ G : |\mathcal{D}_{\text{nqum}}| \to |\mathcal{DeV}|, (A, \pi) \mapsto (A, C_{(A, \pi)}). \]
Clearly, Lemma 4.7 confirms that \(G\) is well-defined. We show that \(F\) and \(G\) are inverse to each other.

For \((A, C) \in |\mathcal{DeV}|\) one has \(G(F(A, C)) = G(A, \pi_{(A,C)}) = (A, C_{(A, \pi_{(A,C)})})\). By Proposition 4.9, \(C = f^{-1}(f(C)) = C_{(A, \pi_{R(A,C)})}\) follows. Since \(\pi_{(A,C)} = \pi_{R(A,C)}\) (see Lemma 4.8), we obtain \(G(F(A, C)) = (A, C)\).

For \((A, \pi) \in |\mathcal{D}_{\text{nqum}}|\) one has \(F(G(A, \pi)) = F(A, C_{(A, \pi)}) = (A, \pi_{(A,C_{(A, \pi)})})\). Denote by \(R_\pi\) the relation on \(T(A)\) determined by the fibers of \(\pi\); then \(R_\pi \in I\mathcal{Rel}(T(A))\). Using once more Proposition 4.9, we obtain \(R_\pi = f(g(R_\pi)) = R_{(A,C_{(A, \pi_{(A,C_{(A, \pi)})})})}\). Since \(\pi_{(A,C_{(A, \pi)})} = \pi_{R(A,C_{(A, \pi)})}\) and \(\pi = \pi_{R_\pi}\) (because \(\pi\) is a natural quotient map), we obtain \(F(G(A, \pi)) = (A, \pi)\). \(\square\)
Let us note that in this paper, by a $T_3$-space (resp., $T_4$-space) we will understand a regular (resp., normal) Hausdorff space. For proving our new duality theorem for the category $\text{CHaus}$, we will need the following lemma:

**Lemma 5.4.** Let $\pi : X \to Y$ and $\pi' : X' \to Y'$ be two closed irreducible mappings, $Y$ be a $T_4$-space, $f : Y' \to Y$ and $\hat{f} : X' \to X$ be continuous maps such that $\pi \circ \hat{f} = f \circ \pi'$. Then, for every $G \in \text{RC}(X)$,

$$\text{cl}(f^{-1}(\text{int}(\pi(G)))) = \bigvee \{\pi'(\hat{f}^{-1}(H)) \mid H \in \text{RC}(X) \text{ and } \pi(H) \subseteq \text{int}(\pi(G))\}.$$ 

**Proof.** By Alexandroff’s Theorem 4.6, the map $\varphi_{\pi} : \text{RC}(X) \to \text{RC}(Y)$, $H \mapsto \pi(H)$, is a Boolean isomorphism, and one has $\varphi_{\pi}^{-1}(K) = \text{cl}_X(\pi^{-1}(\text{int}_Y(K)))$, for all $K \in \text{RC}(Y)$. From here, using the fact that $Y$ is a $T_3$-space, we obtain that $\text{int}(\pi(G)) = \bigcup \{\pi(H) \mid H \in \text{RC}(X) \text{ and } \pi(H) \subseteq \text{int}(\pi(G))\}$. Since $\pi'$ is a surjection, we have that $f^{-1}(M) = \pi'(\hat{f}^{-1}(\pi^{-1}(M)))$, for every $M \subseteq Y$. Hence,

$$f^{-1}(\text{int}(\pi(G))) = \bigcup \{\pi'(\hat{f}^{-1}(\pi^{-1}(\pi(H)))) \mid H \in \text{RC}(X) \text{ and } \pi(H) \subseteq \text{int}(\pi(G))\}.$$ 

Since $Y$ is a $T_4$-space, the theorem of Alexandroff cited above implies that for every $H \in \text{RC}(X)$ such that $\pi(H) \subseteq \text{int}(\pi(G))$, there exists $H' \in \text{RC}(X)$ with $\pi(H) \subseteq \text{int}(\pi(H')) \subseteq \pi(H') \subseteq \text{int}(\pi(G))$. Using again Alexandroff’s theorem, we obtain that $H' = \text{cl}(\pi^{-1}(\text{int}(\pi(H'))))$. Therefore, $\pi^{-1}(\pi(H)) \subseteq \pi^{-1}(\text{int}(\pi(H'))) \subseteq H'$. On the other hand, it is obvious that $H \subseteq \pi^{-1}(\pi(H))$. Thus we obtain that

$$f^{-1}(\text{int}(\pi(G))) = \bigcup \{\pi'(\hat{f}^{-1}(H)) \mid H \in \text{RC}(X) \text{ and } \pi(H) \subseteq \text{int}(\pi(G))\}.$$ 

Then, by Example 2.3,

$$\text{cl}(f^{-1}(\text{int}(\pi(G)))) = \bigvee \{\pi'(\hat{f}^{-1}(H)) \mid H \in \text{RC}(X) \text{ and } \pi(H) \subseteq \text{int}(\pi(G))\}.$$ 

$$\square$$

The next two definitions are of great importance for our investigations.

**Definition 5.5.** Let $(A, C), (A', C')$ be two normal contact algebras. Then a Boolean homomorphism $\varphi : A \to A'$ will be called a *Fedorchuk homomorphism* (briefly, *Fed-homomorphism*) if $a \ll_c b$ implies $\varphi(a) \ll_{C'} \varphi(b)$, for all $a, b \in A$.

Note that Fedorchuk [25] defined a category $\text{Fed}$ such that $|\text{Fed}| \cong |\text{DeV}|$, the morphisms of the category $\text{Fed}$ are the complete Fedorchuk homomorphisms and their compositions are the usual set-theoretic compositions of functions; he proved that $\text{Fed}$ is a subcategory of the category $\text{DeV}$ which is dually equivalent to the category $\text{CHaus}_{\text{qop}}$ of compact Hausdorff spaces and their quasi-open mappings.

**Definition 5.6.** Let $(A, C)$ be a contact algebra, $B$ be a complete Boolean algebra and $\varphi : A \to B$ be a function. Then the function $V(\varphi) : A \to B$, defined by

$$(V(\varphi))(a) \overset{df}{=} \bigvee \{\varphi(b) \mid b \ll a\},$$

for every $a \in A$, will be called a *de Vries transformation of the function* $\varphi$. 

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We need two more lemmas. The first of them is a generalization of [15, Proposition 1.5.4].

**Lemma 5.7.** Let \((A, C), (A', C'), (A'', C'')\) be complete normal contact algebras and \(\varphi : (A, C) \to (A', C'), \psi : (A', C') \to (A'', C'')\) be Fedorchuk homomorphisms. Then \(V(\psi \circ \varphi) = V(V(\psi) \circ V(\varphi))\).

**Proof.** Set \(\alpha \overset{df}{=} V(\varphi)\) and \(\beta \overset{df}{=} V(\psi)\). Then, for every \(a \in A\),

\[
(V(\beta \circ \alpha))(a) = V\{\beta(\alpha(b)) \mid b \ll a\} \\
= V\{\beta(V\{\varphi(c) \mid c \ll b\}) \mid b \ll a\} \\
= V\{V\{\psi(d) \mid d \ll V\{\varphi(c) \mid c \ll b\}\} \mid b \ll a\}.
\]

Also,

\[
(V(\psi \circ \varphi))(a) = \bigvee \{\psi(\varphi(b)) \mid b \ll a\}.
\]

Let \(d \ll \bigvee \{\varphi(c) \mid c \ll b\}\). Then \(\varphi(c) \ll \varphi(b)\), for every \(c \ll b\). Thus, \(\bigvee \{\varphi(c) \mid c \ll b\} \leq \varphi(b)\). Hence \(d \ll \varphi(b)\). This implies that \(\psi(d) \ll \psi(\varphi(b))\). Therefore, \(\bigvee \{\psi(d) \mid d \ll \bigvee \{\varphi(c) \mid c \ll b\}\} \leq \psi(\varphi(b))\) and we obtain that

\[
(V(\beta \circ \alpha))(a) \leq (V(\psi \circ \varphi))(a)
\]

for every \(a \in A\).

Conversely, let \(a, b \in A\) and \(b \ll a\). Then there exist \(a', b' \in A\) such that \(b \ll a' \ll b' \ll a\). Now we obtain that \(\varphi(b) \ll \varphi(a') \leq \bigvee \{\varphi(c) \mid c \ll b'\}\). Set \(d \overset{df}{=} \varphi(b)\). Then \(d \ll \bigvee \{\varphi(c) \mid c \ll b'\}\). Hence, \(\psi(\varphi(b)) = \psi(d) \leq (V(\beta \circ \alpha))(a)\).

Therefore, \((V(\psi \circ \varphi))(a) \leq (V(\beta \circ \alpha))(a)\) for every \(a \in A\).

All this shows that \(V(\psi \circ \varphi) = V(V(\psi) \circ V(\varphi))\).

\(\square\)

**Lemma 5.8.** Let \(X\) be a topological space, \(Y\) be a \(T_3\)-space and \(f, g : X \to Y\) be two continuous mappings such that \(\text{cl}(f^{-1}(\text{int}(G))) = \text{cl}(g^{-1}(\text{int}(G)))\) for every \(G \in \text{RC}(Y)\). Then \(f = g\).

**Proof.** Suppose that \(f \neq g\). Then there exists \(x \in X\) such that \(f(x) \neq g(x)\). Since \(Y\) is a \(T_3\)-space, there exists \(G \in \text{RC}(Y)\) such that \(f(x) \in \text{int}(G)\) and \(g(x) \notin G\). Then \(x \in f^{-1}(\text{int}(G))\). Thus \(x \in \text{cl}(f^{-1}(\text{int}(G))) = \text{cl}(g^{-1}(\text{int}(G)))\). Then \(g(x) \in g(\text{cl}(g^{-1}(\text{int}(G)))) \subseteq \text{cl}(g^{-1}(\text{int}(G))) \subseteq \text{cl}(\text{int}(G)) = G\), a contradiction. Therefore, \(f = g\).

\(\square\)

**5.9.** We are now ready to define a new category \(\text{StoneDeV}\) and to prove that it is dually equivalent to the category \(\text{CHaus}\). We set

\[
|\text{StoneDeV}| \overset{df}{=} |\text{DeV}|.
\]

Further, for every \((A, C), (A', C') \in |\text{StoneDeV}|\), we define

\[
\text{StoneDeV}((A, C), (A', C')) \overset{df}{=} \{\langle \varphi \rangle \mid \varphi : (A, C) \to (A', C')\text{ is F-homomorphism}\}.
\]
where \( \langle \varphi \rangle \) is the equivalence class of \( \varphi \) under the equivalence relation \( \simeq \) in the set of all Fedorchuk homomorphisms between \((A, C)\) and \((A', C')\) defined by

\[
\varphi \simeq \psi \iff V(\varphi) = V(\psi).
\]

The StoneDeV-composition between two StoneDeV-morphisms \( \langle \varphi \rangle : (A, C) \rightarrow (A', C') \) and \( \langle \psi \rangle : (A', C') \rightarrow (A'', C'') \) is defined as follows:

\[
\langle \psi \rangle \circ \langle \varphi \rangle \overset{df}{=} \langle \psi \circ \varphi \rangle.
\]

Finally, for every StoneDeV-object \((A, C)\), its StoneDeV-identity is

\[
1_{(A, C)} \overset{df}{=} \langle 1_A \rangle.
\]

Let us prove that the composition in StoneDeV is well-defined. Indeed, let \( \varphi, \varphi' : (A, C) \rightarrow (A', C') \) and \( \psi, \psi' : (A', C') \rightarrow (A'', C'') \) be Fedorchuk homomorphisms, \( \varphi \simeq \varphi' \) and \( \psi \simeq \psi' \). Then \( \psi \circ \varphi \simeq \psi' \circ \varphi' \). Indeed, we have that \( V(\varphi) = V(\varphi') \) and \( V(\psi) = V(\psi') \); then, using twice Lemma 5.7, we obtain that \( V(\psi \circ \varphi) = V(V(\psi) \circ V(\varphi)) = V(V(\psi') \circ V(\varphi')) = V(\psi' \circ \varphi') \) which means that \( \psi \circ \varphi \simeq \psi' \circ \varphi' \).

Consequently, StoneDeV is a well-defined category.

**Proposition.** The categories \( \mathcal{D}_{\text{nqm}} \) and StoneDeV are isomorphic.

**Proof.** Since \( |\text{StoneDeV}| \overset{df}{=} |\text{DeV}| = |\text{Fed}| \), Corollary 5.3 shows that the correspondence \( I_V : |\text{StoneDeV}| \rightarrow |\mathcal{D}_{\text{nqm}}|, (A, C) \mapsto (A, \pi_{(A, C)}) \), is a bijection (see Lemma 4.8 for \( \pi_{(A,C)} \)). We will extend this bijection to an isomorphism

\[
I_V : \text{StoneDeV} \rightarrow \mathcal{D}_{\text{nqm}}.
\]

Let \((A, C), (A', C') \in |\text{StoneDeV}| \) and \( \langle \varphi \rangle \in \text{StoneDeV}((A, C), (A', C')) \). Then \( \varphi : (A, C) \rightarrow (A', C') \) is a Fedorchuk homomorphism. Thus \( \varphi \in \mathcal{A}(A, A') \). For \( \pi \overset{df}{=} \pi_{(A,C)}, \pi' \overset{df}{=} \pi_{(A',C')}, X \overset{df}{=} \text{cod}(\pi) \) and \( X' \overset{df}{=} \text{cod}(\pi') \), we will define a continuous function \( f_{(\varphi)} : X' \rightarrow X \) such that \( f_{(\varphi)} \circ \pi' = \pi \circ T(\varphi) \).

Since \( \varphi \) satisfies condition (F), using Proposition 4.10, we obtain that, if \( u', v' \in T(A') \) and \( \pi'(u') = \pi'(v') \), then

\[
(8) \quad \pi(T(\alpha)(u')) = \pi(T(\alpha)(v')).
\]

To define \( f_{(\varphi)} \), since \( \pi' \) is a surjection, given \( x' \in X' \), one has some \( u' \in T(A') \) such that \( x' = \pi'(u') \), and with (8) we can put

\[
f_{(\varphi)}(x') \overset{df}{=} \pi(T(\alpha)(u')).
\]

Then \( f_{(\varphi)} \circ \pi' = \pi \circ T(\varphi) \). Since \( \pi' \) is a quotient mapping, we obtain that \( f_{(\varphi)} : X' \rightarrow X \) is continuous.

We have to show that if \( \psi \in \langle \varphi \rangle \) (i.e., \( \langle \psi \rangle = \langle \varphi \rangle \)) then \( f_{(\varphi)} = f_{(\psi)} \). Set \( f \overset{df}{=} f_{(\varphi)}, g \overset{df}{=} f_{(\psi)}, \hat{f} \overset{df}{=} T(\varphi) \) and \( \hat{g} \overset{df}{=} T(\psi) \). Then \( f \circ \pi' = \pi \circ \hat{f} \) and \( g \circ \pi' = \pi \circ \hat{g} \).
Since \( T(A), T(A') \in \text{ECH} \), we have that \( \text{RC}(T(A)) = \text{CO}(T(A)) \) and \( \text{RC}(T(A')) = \text{CO}(T(A')) \). Then, by Theorem 4.6, the maps \( \varphi_\pi : \text{CO}(T(A)) \longrightarrow \text{RC}(X), \ H \mapsto \pi(H), \) and \( \varphi'_\pi : \text{CO}(T(A')) \longrightarrow \text{RC}(X'), \ H \mapsto \pi'(H), \) are Boolean isomorphisms. We will show that for every \( G \in \text{RC}(X), \)
\[
(9) \quad \text{cl}(f^{-1}(\text{int}(G))) = (\varphi'_\pi \circ s_A)(V(\varphi))((s_A^{-1}(\varphi^{-1}_\pi(G))))
\]
and
\[
(10) \quad \text{cl}(g^{-1}(\text{int}(G))) = (\varphi_\pi \circ s_A)(V(\psi))((s_A^{-1}(\varphi^{-1}_\pi(G))))
\]
(see 2.13 for \( s_A \)). It is enough to prove the first equality since the proof of the second one is analogous.

Let us first recall that, according to Proposition 4.9, Lemma 4.7 and Lemma 4.8, we have that for every \( a, b \in A, \) \( aCb \Leftrightarrow aC(A, \pi) b \Leftrightarrow \pi(s_A(a)) \cap \pi(s_A(b)) \neq \emptyset; \) thus, \( a \ll b \Leftrightarrow \pi(s_A(a)) \subseteq \text{int}(\pi(s_A(b))). \) Recall as well that, by the Stone Duality Theorem, \( s_A \circ \varphi = S(T(\varphi)) \circ s_A. \) Now, using also Lemma 5.4, we obtain that for every \( G \in \text{RC}(X), \)
\[
\text{cl}(f^{-1}(\text{int}(G))) = \bigvee \{ \pi'(\mathcal{H}) \mid \mathcal{H} \in \text{CO}(T(A)), \pi(\mathcal{H}) \subseteq \text{int}(G) \}
\]
\[
= \bigvee \{ \varphi'_\pi(S(T(\varphi))((\varphi^{-1}_\pi(F)))) \mid F \in \text{RC}(X), F \subseteq \text{int}(G) \}
\]
\[
= \varphi'_\pi(\bigvee \{ s_A((\varphi(s_A^{-1}(\varphi^{-1}_\pi(F)))) \mid F \in \text{RC}(X), F \subseteq \text{int}(G) \})
\]
\[
= (\varphi'_\pi \circ s_A)(\bigvee \{ \varphi(b) \mid b \in A, b \ll s_A^{-1}(\varphi^{-1}_\pi(G)) \})
\]
\[
= (\varphi'_\pi \circ s_A)(V(\varphi))((s_A^{-1}(\varphi^{-1}_\pi(G)))).
\]
Since \( V(\varphi) = V(\psi), \) we obtain that \( \text{cl}(f^{-1}(\text{int}(G))) = \text{cl}(g^{-1}(\text{int}(G))) \), for every \( G \in \text{RC}(X) \). According to Lemma 5.8, this implies that \( f = g \). All this shows that \( f(\varphi) \) is well-defined. We now set
\[
I_V(\langle \varphi \rangle) \overset{\text{df}}{=} [\varphi, f(\varphi)].
\]
As it follows from the above considerations, \( I_V \) is well defined on the objects and morphisms of the category \( \text{StoneDeV} \). \( I_V : \text{StoneDeV} \longrightarrow \mathcal{D}_{nqm} \) is obviously a functor. As it is bijective on objects, we need to show only that it is full and faithful. Let \( \langle \varphi \rangle, \langle \psi \rangle \in \text{StoneDeV}(\langle A, C \rangle, \langle A', C' \rangle) \) and \( I_V(\langle \varphi \rangle) = I_V(\langle \psi \rangle) \). Then \( f(\varphi) = f(\psi). \) Using (9) and (10), we obtain that \( V(\varphi) = V(\psi) \). Thus \( \langle \varphi \rangle = \langle \psi \rangle. \) Therefore, \( I_V \) is faithful. Let now \([\varphi, f] \in \mathcal{D}_{nqm}(I_V(A,C), I_V(A',C')). \) Setting \( \pi \overset{\text{df}}{=} \pi_{(A,C)} \) and \( \pi' \overset{\text{df}}{=} \pi_{(A',C')}, \) we obtain that \( f \circ \varphi' = \pi \circ T(\varphi). \) Then Proposition 4.10 implies that \( \varphi : (A,C) \longrightarrow (A',C') \) is a Fedorchuk homomorphism. Now we obtain easily that \( f = f(\varphi). \) Thus, \( I_V(\langle \varphi \rangle) = [\varphi, f] \). Therefore, \( I_V \) is full. All this shows that \( I_V \) is an isomorphism.

**Theorem.** The categories \( \text{CHaus} \) and \( \text{StoneDeV} \) are dually equivalent.

**Proof.** Composing the dual equivalence \( \tilde{T} : \mathcal{D} \longrightarrow \mathcal{E} \) from 5.1 with the equivalence \( J' : \mathcal{D}_{nqm} \leftrightarrow \mathcal{D} \) from Proposition 5.2, and with the isomorphism \( I_V : \text{StoneDeV} \longrightarrow \mathcal{D}_{nqm} \) from the above Proposition, we obtain a dual equivalence \( \tilde{T}' \overset{\text{df}}{=} \tilde{T} \circ J' \circ I_V : \text{StoneDeV} \longrightarrow \text{CHaus}. \)
In what follows, we will prove that the categories \( \text{StoneDeV} \) and \( \text{DeV} \) are isomorphic. We start with some lemmas. The first one is a particular case of [16, Lemma 3.9] but, for completeness of our exposition, we outline its proof.

**Lemma 5.10.** Let \( (A,C) \) and \( (A',C') \) be complete normal contact algebras and \( \varphi : A \rightarrow B \) be a function between them. Then:

(a) If \( \varphi \) satisfies condition (DV2), then \( V(\varphi) \) satisfies conditions (DV2) and (DV4);
(b) If \( \varphi \) satisfies condition (DV4), then \( V(\varphi) = \varphi \);
(c) If \( \varphi \) satisfies condition (DV2), then \( V(\varphi)) = V(\varphi) \);
(d) If \( \varphi \) is a monotone function then, for every \( a \in A \), \( (V(\varphi))(a) \leq \varphi(a) \).

**Proof.** Properties (b) and (d) are clearly fulfilled, and (c) follows from (a) and (b). Hence, we need to prove only (a).

Let \( a \in A \). If \( c \in A \) and \( c \ll a \) then there exists \( d_c \in A \) such that \( c \ll d_c \ll a \) and we fix such a one; hence \( \varphi(c) \leq (V(\varphi))(d_c) \). Also, by (d), for every \( a \in A \), \( (V(\varphi))(a) \leq \varphi(a) \). Now we obtain that

\[
(V(\varphi))(a) = \bigvee \{\varphi(c) \mid c \in A, c \ll a\} \\
\leq \bigvee \{(V(\varphi))(d_c) \mid c \in A, c \ll a\} \\
\leq \bigvee \{(V(\varphi))(e) \mid e \in A, e \ll a\} \\
\leq \bigvee \{\varphi(e) \mid e \in A, e \ll a\} \\
= (V(\varphi))(a).
\]

Thus, \( (V(\varphi))(a) = \bigvee \{(V(\varphi))(e) \mid e \in A, e \ll a\} \). So, \( V(\varphi) \) satisfies (DV4).

Further, let \( a, b \in A \). Then

\[
(V(\varphi))(a) \wedge (V(\varphi))(b) = \bigvee \{\varphi(d) \wedge \varphi(e) \mid d, e \in A, d \ll a, e \ll b\} \\
= \bigvee \{\varphi(d \wedge e) \mid d, e \in A, d \ll a, e \ll b\} \\
= \bigvee \{\varphi(c) \mid c \in A, c \ll a \wedge b\} \\
= (V(\varphi))(a \wedge b).
\]

So, \( V(\varphi) \) satisfies condition (DV2).

**Lemma 5.11.** Let \( (A,C) \) and \( (A',C') \) be two complete normal contact algebras and \( \varphi : (A,C) \rightarrow (A',C') \) be a Fedorchuk homomorphism. Then \( V(\varphi) \) is a de Vries morphism.

**Proof.** Clearly, \( (V(\varphi))(0) = 0 \); thus, condition (DV1) is satisfied. Since \( \varphi \) satisfies (DV2), Lemma 5.10(a) implies that \( V(\varphi) \) satisfies conditions (DV2) and (DV4). So, we need only to prove that \( V(\varphi) \) satisfies condition (DV3).

Let \( a, b \in A \) and \( a \ll b \). There exist \( c, d \in A \) such that \( a \ll c \ll d \ll b \). Now we have that

\[
((V(\varphi))(a^*))^* = (\bigvee \{\varphi(d) \mid d \in A, d \ll a^*\})^* \\
= (\bigvee \{\varphi(e^*) \mid e \in A, e^* \ll a^*\})^* \\
= (\bigvee \{(\varphi(e))^* \mid e \in A, a \ll e\})^* \\
= (\bigwedge \{\varphi(e) \mid e \in A, a \ll e\} \\
\leq \varphi(c) \ll \varphi(d) \leq (V(\varphi))(b).
\]

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Hence, $V(\varphi)$ satisfies condition (DV3).

All this shows that $V(\varphi)$ is a de Vries morphism.

\[ \square \]

**5.12.** We will now recall a result of de Vries [15]. Since de Vries works with ends and we work with clusters, we will present here a proof of his result.

Let $(A, C)$ be a complete normal contact algebra. Set $Y \overset{df}{=} \Psi^a(A, C)$ (see the proof of Theorem 4.3 for $\Psi^a(A, C)$). Setting $\pi \overset{df}{=} \pi_{(A,C)}$ and $X \overset{df}{=} T(A)/R_{(A,C)}$ (see Lemma 4.8 for the notation), Proposition 4.11 tell us that the mapping $h_{(A,C)} : X \rightarrow Y, [u] \mapsto \sigma_u$, (see (3) for $\sigma_u$) is a homeomorphism. Set $h \overset{df}{=} h_{(A,C)}$. Then, clearly, the map

$$\psi_h : RC(X) \rightarrow RC(Y), \ G \mapsto h(G),$$

is a Boolean isomorphism. Recall that for every $a \in A$, $v_{(A,C)}(a) \overset{df}{=} \{ \sigma \in Y \mid a \in \sigma \}$ (see the proof of Theorem 4.3) and the family $\{v_{(A,C)}(a) \mid a \in A\}$ is a closed base for $Y$. Hence, setting

$$\overline{v}_{(A,C)}(a) \overset{df}{=} Y \setminus v_{(A,C)}(a),$$

we obtain that the family $\{\overline{v}_{(A,C)}(a) \mid a \in A\}$ is an open base for $Y$. Further, from the proof of Proposition 4.11 we know that for every $a \in A$,

$$v_{(A,C)}(a) = h(\pi(s_A(a))).$$

Hence, with $v_{(A,C)} : A \rightarrow RC(Y), a \mapsto v_{(A,C)}(a)$, we obtain that $v_{(A,C)} = \psi_h \circ \varphi_x \circ s_A$ (see Theorem 4.6 for $\varphi_x$). Thus,

$$v_{(A,C)} : A \rightarrow RC(Y)$$

is a Boolean isomorphism. We are now ready to prove the result of de Vries mentioned above.

**Lemma.** ([15]) Let $\alpha : (A, C) \rightarrow (A', C')$ be a de Vries morphism, $Y \overset{df}{=} \Psi^a(A, C)$, $Y' \overset{df}{=} \Psi^a(A', C')$ and $g_\alpha : Y' \rightarrow Y$ be defined by the formula

$$g_\alpha(\sigma') \overset{df}{=} \{ a \in A \mid \forall b \in A, (b \preceq a^*) \Rightarrow ((a(b))^* \in \sigma') \}.$$

Then $g_\alpha$ is a continuous function and for every $a \in A$,

$$\text{cl}(g_\alpha^{-1}(\text{int}(v_{(A,C)}(a)))) = v_{(A',C')}(\alpha(a)).$$

**Proof.** We first show that the function $g_\alpha$ is well-defined. Let $\sigma' \in Y'$. We have to prove that $\sigma \overset{df}{=} g_\alpha(\sigma')$ satisfies conditions (CL1)-(CL3) of Definition 2.7. Clearly, $1 \in \sigma$, i.e., $\sigma \neq \emptyset$.

(CL1): Let $a, b \in \sigma$. Suppose that $a(-C)b$. Then $a \preceq b^*$. There exist $c, d \in A$ such that $a \preceq c \preceq d^* \preceq b^*$. Then $b \preceq d \preceq c^* \preceq a^*$ and, by the definition of $\sigma$, $(\alpha(d^*))^* \in \sigma'$ and $(\alpha(c^*))^* \in \sigma'$. Thus $(\alpha(d^*))^* C (\alpha(c^*))^*$. Since $a \preceq d^*$, condition (DV3) from
Definition 4.1 implies that $(\alpha(c^*))^* \ll \alpha(d^*)$. Therefore, $(\alpha(d^*))^*(-C)(\alpha(c^*))^*$, a contradiction. Hence, $aCb$.

(CL2): Let $a \lor b \in \sigma$. Suppose that $a \not\in \sigma$ and $b \not\in \sigma$. Then there exist $c, d \in A$ such that $c \ll a^*$, $d \ll b^*$ and $(\alpha(c))^* \not\in \sigma'$, $(\alpha(d))^* \not\in \sigma'$. We have that $c \land d \ll a^* \land b^* = (a \lor b)^*$. Then $(\alpha(c \land d))^* \in \sigma'$. Thus $(\alpha(c) \land \alpha(d))^* \in \sigma'$, i.e., $(\alpha(c))^* \land (\alpha(d))^* \in \sigma'$. This implies that $(\alpha(c))^* \in \sigma'$ or $(\alpha(d))^* \in \sigma'$, a contradiction.

(CL3): Let $aCb$ for every $b \in \sigma$. Suppose that $a \not\in \sigma$. Then there exists $c \in A$ such that $c \ll a^*$ and $(\alpha(c))^* \not\in \sigma'$. Thus $\alpha(c) \in \sigma'$. We will show that $c \in \sigma$. Indeed, let $d \in A$ and $d \ll c^*$. Then, by Fact 4.2(c), $\alpha(d) \ll \alpha(c)^*$. Hence, using Fact 4.2(b), we obtain that $\alpha(c) \leq (\alpha(c^*))^* \ll (\alpha(d))^*$. Thus $(\alpha(d))^* \in \sigma'$. Therefore, $c \in \sigma$. Since $c \ll a^*$, we have that $a(-C)c$, a contradiction.

This proves that $g_\alpha$ is well defined. Now we will show that $g_\alpha$ is continuous. Clearly, $RO(Y) = \{\text{int}(F) \mid F \in RC(Y)\}$ and $RO(Y)$ is an open base for $Y$. From the above considerations we know that $RC(Y) = \{\nu_{(A,C)}(a) \mid a \in A\}$. Thus, it is enough to show that for every $a \in A$, $g_\alpha^{-1}(\text{int}(\nu_{(A,C)}(a)))$ is an open subset of $Y'$.

Let $a \in A$ and set $G \overset{df}{=} \nu_{(A,C)}(a)$. Then, by Example 2.3 and above considerations, $\text{int}(G) = Y \setminus \text{cl}(Y \setminus G) = Y \setminus \text{cl}(Y \setminus \nu_{(A,C)}(a^*)) = \overline{\nu_{(A,C)}(a^*)} = \{\sigma \in Y \mid a^* \not\in \sigma\}$. Thus, $g_\alpha^{-1}(\text{int}(G)) = \{\sigma' \in Y' \mid a^* \not\in g_\alpha(\sigma')\}$. We have that for any $\sigma' \in Y'$, $a^* \not\in g_\alpha(\sigma') \Leftrightarrow \exists b \in A$ such that $b \ll a$ and $(\alpha(b))^* \not\in \sigma' \Leftrightarrow \exists b \in A$ such that $b \ll a$ and $\sigma' \in \overline{\nu_{(A',C')}((\alpha(b))^*]}$. Therefore,

$$g_\alpha^{-1}(\text{int}(G)) = \bigcup \{\overline{\nu_{(A',C')}((\alpha(b))^*)} \mid b \ll a\}.$$ 

This shows that $g_\alpha$ is a continuous function. Further, using Example 2.3, (DV4) and above considerations, we obtain that

$$g_\alpha^{-1}(\text{int}(\nu_{(A,C)}(a)))) = g_\alpha^{-1}(\text{int}(G)) = \bigcup \{\text{int}(\nu_{(A',C')}((\alpha(b))^*) \mid b \ll a\}$$

and thus,

$$\text{cl}(g_\alpha^{-1}(\text{int}(\nu_{(A,C)}(a)))) = \text{cl}(\bigcup \{\text{int}(\nu_{(A',C')}((\alpha(b))^*) \mid b \ll a\})$$

$$= \bigvee \{\nu_{(A',C')}(\alpha(b)) \mid b \ll a\}$$

$$= \nu_{(A',C')}(\bigvee \{\alpha(b) \mid b \ll a\})$$

$$= \nu_{(A',C')}(\alpha(a)).$$

\[\Box\]

**Theorem 5.13.** The categories StoneDeV and DeV are isomorphic.

**Proof.** We will define a functor $J_V : \text{StoneDeV} \rightarrow \text{DeV}$ and will prove that it is bijective on objects, full and faithful.

For every $\text{StoneDeV}$-object $(A, C)$, we set

$$J_V(A, C) \overset{df}{=} (A, C).$$

Further, for every $\langle \varphi \rangle \in \text{StoneDeV}((A, C), (A', C'))$, we put

$$J_V(\langle \varphi \rangle) \overset{df}{=} V(\varphi).$$

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Then Lemma 5.11 shows that $J_V$ is well-defined on the morphisms of the category $\text{StoneDeV}$. From Fact 4.2 we obtain that $J_V$ preserves identities. Let $\langle \psi \rangle \in \text{StoneDeV}((A',C'),(A'',C''))$ and $\langle \varphi \rangle \in \text{StoneDeV}((A,C),(A',C'))$. Then, using Lemma 5.7, we obtain that $J_V(\langle \psi \rangle \circ \langle \varphi \rangle) = J_V(\langle \psi \circ \varphi \rangle) = V(\psi \circ \varphi) = V(V(\psi) \circ V(\varphi)) = V(J_V(\langle \psi \rangle) \circ J_V(\langle \varphi \rangle)) = J_V(\langle \psi \rangle) \circ J_V(\langle \varphi \rangle)$, hence $J_V$ is a functor.

Since $|\text{StoneDeV}| = |\text{DeV}|$, we obtain that $J_V$ is bijective on objects.

Let $(A,C),(A',C') \in |\text{StoneDeV}|$. If $\langle \varphi \rangle, \langle \psi \rangle \in \text{StoneDeV}((A,C),(A',C'))$ and $J_V(\langle \varphi \rangle) = J_V(\langle \psi \rangle)$, then $V(\varphi) = V(\psi)$; this implies that $\langle \varphi \rangle = \langle \psi \rangle$. Thus, $J_V$ is faithful. Hence, it is only left to show that $J_V$ is full.

Let $\alpha \in \text{DeV}(J_V(A,C),J_V(A',C'))$. Then $\alpha \in \text{DeV}((A,C),(A',C'))$. Set $Y \overset{df}{=} \Psi^\alpha(A,C)$ and $Y' \overset{df}{=} \Psi^\alpha(A',C')$. Using Lemma 5.12, we obtain that the function $g_\alpha : Y' \rightarrow Y$, $\sigma' \mapsto \{a \in A \mid \forall b \in A, (b \ll a^*) \Rightarrow ((\alpha(b))^* \in \sigma')\}$, is continuous. Set $\pi \overset{df}{=} \pi_{(A,C)}$, $\pi' \overset{df}{=} \pi_{(A',C')}$, $X \overset{df}{=} \text{cod}(\pi)$ and $X' \overset{df}{=} \text{cod}(\pi')$ (see Lemma 4.8 for the notation $\pi_{(A,C)}$). Then Proposition 4.11 implies that the mappings $h : X \rightarrow Y$, $[u] \mapsto \sigma_u$, and $h' : X' \rightarrow Y'$, $[v] \mapsto \sigma_v$, (see (3) for the notation $\sigma_u$) are homeomorphisms. Put $f_\alpha \overset{df}{=} h^{-1} \circ g_\alpha \circ h'$. Then $f_\alpha : X' \rightarrow X$ is a continuous function. Now, using the Gleason Theorem 2.15, we obtain that there exists a continuous function $f : T(A') \rightarrow T(A)$ such that $\pi \circ f = f_\alpha \circ \pi'$. By the Stone Duality Theorem, there exists a unique Boolean homomorphism $\varphi : A \rightarrow A'$ such that $f = T(\varphi)$.

Proposition 4.10 shows that $\varphi$ is a Fedorchuk homomorphism. Further, recall that by the Alexandroff Theorem 4.6, the map $\varphi_\pi : \text{CO}(T(A)) \rightarrow \text{RC}(X)$, $U \mapsto \pi(U)$, is a Boolean isomorphism. Hence the map $\varphi_\pi \circ s_A : A \rightarrow \text{RC}(X)$ is a Boolean isomorphism (see 2.13 for the notation $s_A$). Clearly, the map $\psi_h : \text{RC}(X) \rightarrow \text{RC}(Y)$, $G \mapsto h(G)$, is a Boolean isomorphism. As we have shown in 5.12, the map $\psi_{(A,C)} : A \rightarrow \text{RC}(Y)$, $a \mapsto \psi_{(A,C)}(a)$, is a Boolean isomorphism (see the proof of Theorem 4.3 for $\psi_{(A,C)}(a)$) and $\psi_{(A,C)} = \psi_h \circ \varphi_\pi \circ s_A$. From Lemma 5.12 we have that for every $a \in A$, $\text{cl}(g_a^{-1}(\text{int}(\psi_{(A,C)}(a)))) = \psi_{(A',C')}(\alpha(a))$. Since $g_a = h \circ f_\alpha \circ (h')^{-1}$, we obtain that for every $M \subseteq Y$, $g_a^{-1}(M) = h'(f_\alpha^{-1}(h^{-1}(M)))$ and thus $\text{cl}((h' \circ f_\alpha^{-1} \circ h^{-1})(\text{int}(\psi_h \circ \varphi_\pi \circ s_A(a)))) = (\psi_h \circ \varphi_\pi \circ s_A)(\alpha(a))$. This implies that $h'(\text{cl}(f_\alpha^{-1}(h^{-1}(\text{int}(h(\pi(s_A(a))))))) = h'(\pi'(s_A(\alpha(a))))$. Hence,

$$\text{cl}(f_\alpha^{-1}(\text{int}(\pi(s_A(a))))) = \pi'(s_A(\alpha(a))).$$

Set $G \overset{df}{=} \pi(s_A(a))$. Then $G \in \text{RC}(X)$, $a = s_A^{-1}(\varphi_\pi^{-1}(G))$ and we obtain that $\text{cl}(f_\alpha^{-1}(\text{int}(G))) = \pi'(s_A(\alpha(a)))$. Further, by (9),

$$\text{cl}(f_\alpha^{-1}(\text{int}(G))) = (\varphi_\pi \circ s_{A'})(V(\varphi))(s_A^{-1}(\varphi_\pi^{-1}(G))).$$

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Therefore, $\varphi_{\pi}(s_{A^{'}}((V(\varphi))(a))) = \varphi_{\pi}(s_{A^{'}}(\alpha(a)))$. Thus, $\alpha(a) = (V(\varphi))(a)$ for every $a \in A$. This implies that $\alpha = J_{V}((\varphi))$. So, $J_{V}$ is full.

All this shows that $J_{V}$ is an isomorphism. \qed

In conclusion we obtain a new proof of the de Vries Duality Theorem:

**Corollary 5.14.** ([15]) The categories $\text{CHaus}$ and $\text{DeV}$ are dually equivalent.

**Proof.** By Theorem 5.9, there is a dual equivalence $\tilde{T}' : \text{StoneDeV} \rightarrow \text{CHaus}$. Composing it with the isomorphism $(J_{V})^{-1} : \text{DeV} \rightarrow \text{StoneDeV}$ from Theorem 5.13, we obtain a dual equivalence $\tilde{T}'' \overset{df}{=} \tilde{T}' \circ (J_{V})^{-1} : \text{DeV} \rightarrow \text{CHaus}$. \qed

6. **A new approach to the Bezhanishvili-Morandi-Olberding duality**

6.1. In their recent paper [9], G. Bezhanishvili, P.J. Morandi and B. Olberding described a category $\text{BMO}$ and a dual equivalence of $\text{BMO}$ with the category $\text{Tych}$ of Tychonoff spaces and continuous maps which extends de Vries’ dual equivalence $\Psi_{a} : \text{DeV} \rightarrow \text{CHaus}$. In this section we will derive the Bezhanishvili-Morandi-Olberding Duality Theorem ([9]) from our Theorem 3.3.

We set (and we will keep this notation throughout this section)

$\mathcal{A} \overset{df}{=} \text{CHaus}, \quad \mathcal{B} \overset{df}{=} \text{DeV}, \quad \mathcal{D} \overset{df}{=} \text{Tych}, \quad \mathcal{S} \overset{df}{=} \Psi_{a},$

$\mathcal{J} \overset{df}{=} \{j : X \rightarrow Y \mid X \in |\text{Tych}|, Y \in |\text{CHaus}|, j \text{ is a dense embedding, } j(X) \subseteq Y\},$

where $j(X) \subseteq Y$ means that $j(X)$ is $C^{*}$-embedded in $Y$, and we denote by

$J : \mathcal{A} \hookrightarrow \mathcal{D}$

the inclusion functor. Note that we regard as elements of the class $\mathcal{J}$ all representatives of the Stone-Čech compactifications of Tychonoff spaces. Obviously, the class $\mathcal{J}$ satisfies conditions (J1-4) and (J5*) (and $\mathcal{A}$ is a reflective subcategory of $\mathcal{D}$). Therefore, we can apply Theorem 3.3. It gives us a category $\mathcal{C}$, a dual equivalence

$\overline{S} : \mathcal{C} \rightarrow \mathcal{D}$

and a full embedding $I : \mathcal{B} \rightarrow \mathcal{C}$ such that

$\overline{S} \circ I = J \circ S.$

The plan of the section is now as follows. Adapting the category $\mathcal{C}$ to the concrete situation, we first describe a subcategory $\mathcal{C}'$ of $\mathcal{C}$ which is equivalent to $\mathcal{C}$, and after that we find a second category $\mathcal{C}''$ isomorphic to the category $\mathcal{C}'$. In this way we obtain a new duality theorem which extends de Vries’ Duality Theorem to the category $\text{Tych}$. Finally, using the Tarski Duality between the category $\text{Set}$ of sets and
functions and the category $\text{CaBa}$ of complete atomic Boolean algebras and suprema-preserving Boolean homomorphisms, we prove that our category $\mathcal{C}''$ is equivalent to the category $\text{BMO}$. All this shows that we obtain, as an application of our Theorem 3.3, a new proof of the Bezhanishvili-Morandi-Olberding Duality Theorem.

We start with the description of the category $\mathcal{C}'$ mentioned above.

**Proposition.** Let $\mathcal{C}'$ be the full subcategory of the category $\mathcal{C}$ with $|\mathcal{C}'| \overset{\text{df}}{=} \{(B, j) \in |\mathcal{C}| \mid j \text{ is an inclusion map}\}$. Then the inclusion functor $I' : \mathcal{C}' \hookrightarrow \mathcal{C}$ is an equivalence.

**Proof.** Clearly, $I'$ is a full and faithful functor. For showing that it is essentially surjective on objects, let $((A, C), j) \in |\mathcal{C}|$, i.e., $(A, C) \in |\text{DeV}|$ and $j : X \rightarrow S(A, C)$ is in $\mathcal{J}$. Let $j' : j(X) \hookrightarrow S(A, C)$ be the inclusion mapping and let $f \overset{\text{df}}{=} j \upharpoonright X$, where $j \upharpoonright X : X \rightarrow j(X)$ is the restriction of $j$. Then $f$ is a homeomorphism and $j' = j \circ f^{-1}$. Since $\mathcal{J}$ satisfies condition (J3), we obtain that $j' \in \mathcal{J}$. Hence, $((A, C), j') \in |\mathcal{C}'|$. Obviously, the map $(1_{(A, C)}, f^{-1}) : ((A, C), j) \rightarrow ((A, C), j')$ is a $\mathcal{C}$-isomorphism. Therefore, $I'$ is an equivalence. $\square$

A more general version of the next proposition was proved in [17]. Since it was not published till now and since it plays an important role in the construction of our category $\mathcal{C}''$, we will present its proof here.

**Proposition 6.2.** Let $(A, C)$ be a CNCA. Then the clusters of $(A, C)$ are precisely those subsets of $A$ which are of the form

$$\sigma^\varphi \overset{\text{df}}{=} \{a \in A \mid \varphi(a^*) = 0\},$$

where $\varphi \in \text{DeV}((A, C), (2, \rho_s))$.

**Proof.** We will show that the map

$$\xi : \text{DeV}((A, C), (2, \rho_s)) \rightarrow \text{Clust}(A, C), \quad \varphi \mapsto \sigma^\varphi$$

is a bijection. First of all, we will prove that the map $\xi$ is well defined.

Let $\varphi \in \text{DeV}((A, C), (2, \rho_s))$. We will show that $\sigma^\varphi$ is a cluster in the CNCA $(A, C)$. Clearly, $\sigma^\varphi \neq \emptyset$ because, by (DV1), $\varphi(0) = 0$ and thus $1 \in \sigma^\varphi$. We have to prove that $\sigma^\varphi$ satisfies the axioms (CL1), (CL2), (CL3).

(1): Let $a, b \in \sigma^\varphi$. Suppose that $a \not\leq b^*$. Thus, using (DV3), we obtain that $(\varphi(a^*))^* \not\leq \varphi(b^*)$, i.e., $1 \not\leq 0$, a contradiction. Hence, $aCb$.

(2): Let $a \vee b \in \sigma^\varphi$. Then, using (DV2), we obtain that $0 = \varphi((a \vee b)^*) = \varphi(a^*) \wedge \varphi(b^*)$. Hence, $\varphi(a^*) = 0$ or $\varphi(b^*) = 0$. Thus, $a \in \sigma^\varphi$ or $b \in \sigma^\varphi$.

(3): Let $aCb$, for every $b \in \sigma^\varphi$. Suppose that $a \notin \sigma^\varphi$. Then $\varphi(a^*) = 1$. Now, using (DV4), we obtain that there exists $b \in A$ such that $b \leq a^*$ and $\varphi(b) = 1$. Then $a \not\leq b$. Hence $b \notin \sigma^\varphi$. Thus $\varphi(b^*) = 1$. Since $0 = \varphi(b \wedge b^*) = \varphi(b) \wedge \varphi(b^*)$, we obtain that $\varphi(b) = 0$, a contradiction. Therefore, $a \in \sigma^\varphi$.

So, $\sigma^\varphi \in \text{Clust}(A, C)$ and thus, the map $\xi$ is well defined. Setting

$$M^* = \{b^* \mid b \in M\},$$
for every subset \( M \) of \( A \), we can rewrite the definition of \( \xi(\varphi) \), i.e., of \( \sigma^\varphi \), as follows: 
\( \xi(\varphi) = (\varphi^{-1}(0))^* \). This shows that \( \xi \) is an injection. We are now going to prove that \( \xi \) is a surjection.

Let \( \sigma \in \text{Clust}(A,C) \). Let \( \varphi_\sigma : A \rightarrow 2 \) be defined by

\[
(11) \quad \varphi_\sigma(a) = 0 \iff a^* \in \sigma.
\]

Then, clearly, \( \sigma = \{ a \in A \mid \varphi_\sigma(a^*) = 0 \} \), i.e., \( \sigma = \xi(\varphi_\sigma) \). We will show that \( \varphi_\sigma \in \text{DeV}((A,C),(2,\rho_s)) \), i.e., we will prove that \( \varphi_\sigma \) satisfies axioms (DV1)-(DV4).

(DV1): Since \( 0^* = 1 \in \sigma \), we obtain that \( \varphi_\sigma(0) = 0 \).

(DV2): Let \( \varphi_\sigma(a \land b) = 0 \). Then \( (a \land b)^* \in \sigma \), i.e., \( a^* \lor b^* \in \sigma \). Hence, by (CL2), \( a^* \in \sigma \) or \( b^* \in \sigma \). Then \( \varphi_\sigma(a) = 0 \) or \( \varphi_\sigma(b) = 0 \). Thus \( \varphi_\sigma(a) \land \varphi_\sigma(b) = 0 = \varphi_\sigma(a \land b) \).

(DV3): Let \( a,b \in A \) and \( a \ll b \). Let \( \varphi_\sigma(a^*) = 0 \). Then \( a \in \sigma \). Since \( a(-C)b^* \), we obtain that \( b^* \notin \sigma \). Therefore, \( \varphi_\sigma(b) = 1 \). Thus \( (\varphi_\sigma(a^*))^* \ll \varphi_\sigma(b) \). If \( \varphi_\sigma(a^*) = 1 \) then, clearly, \( (\varphi_\sigma(a^*))^* \ll \varphi_\sigma(b) \). Therefore, \( \varphi_\sigma \) satisfies the axiom (DV3).

(DV4): Let \( a \in A \). If \( \varphi_\sigma(a) = 0 \) then, using the facts that \( \varphi_\sigma \) is a monotone function (since, as we have shown, \( \varphi_\sigma \) satisfies (DV2)), \( 0 \ll a \) and \( \varphi_\sigma(0) = 0 \), we obtain that \( \varphi_\sigma(a) = \bigvee \{ \varphi_\sigma(b) \mid b \in A, b \ll a \} \). If \( \varphi_\sigma(a) = 1 \) then \( a^* \notin \sigma \). Thus, by (CL3), there exists \( c \in \sigma \) such that \( a^*(C)c \). Hence \( c \ll a \). Then there exists \( b \in A \) such that \( c \ll b \ll a \). Since \( c(-C)b^* \), we obtain that \( b^* \notin \sigma \). Therefore \( \varphi_\sigma(b) = 1 \). This implies that \( \varphi_\sigma(a) = \bigvee \{ \varphi_\sigma(b) \mid b \ll a \} \). Hence, \( \varphi_\sigma \) satisfies the axiom (DV4).

So, \( \varphi_\sigma \in \text{DeV}((A,C),(2,\rho_s)) \) and \( \sigma = \xi(\varphi_\sigma) \). All this shows that \( \xi \) is a bijection. Also, we have seen that

\[
\xi^{-1}(\sigma) = \varphi_\sigma,
\]

for every \( \sigma \in \text{Clust}(A,C) \).

\( \Box \)

6.3. The definition of de Vries’ dual equivalence \( \Psi^a \) is given on the language of clusters (see the proof of Theorem 4.3). The above Proposition 6.2 shows that we can use de Vries’ morphisms from a CNCA to \((2,\rho_s)\) instead of clusters. Transporting everything from clusters to morphisms via the bijection \( \xi^{-1} \) from 6.2, we will here express the definition of \( \Psi^a \) in a new much more natural and beautiful form. Although we have set above \( S \overset{\text{df}}{=} \Psi^a \), in order to distinguish between the old and new form of \( \Psi^a \), we will use the symbol \( S \) when we have in mind the new form of \( \Psi^a \).

Let us first introduce some notation. For every CNCA \((A,C)\) and each \( a \in A \), we set

\[
X_{(A,C)} \overset{\text{df}}{=} \text{DeV}((A,C),(2,\rho_s)) \quad \text{and} \quad \nu'_{(A,C)}(a) \overset{\text{df}}{=} \xi^{-1}(\nu_{(A,C)}(a)).
\]

Thus, according to Proposition 6.2, we obtain that

\[
(12) \quad \nu'_{(A,C)}(a) = \{ \varphi \in X_{(A,C)} \mid \varphi(a^*) = 0 \}
\]
and therefore,

\[(13) \ X_{(A,C)} \setminus v'_{(A,C)}(a) = \{ \varphi \in X_{(A,C)} \mid \varphi(a^*) = 1 \}\.\]

Now we can prove the following assertion:

**Proposition.** The new form \(S\) of the dual equivalence \(\Psi^a\) is the following one:

- for any \((A, C) \in |B|\),
  \[
  S(A, C) \overset{df}{=} (X_{(A,C)}, T'),
  \]
  where the topology \(T'\) is generated by the closed base \(\{v'_{(A,C)}(a) \mid a \in A\}\),
- for every \(\alpha \in B((A, C), (A', C'))\), \(S(\alpha) : S(A', C') \longrightarrow S(A, C)\) is defined by the formula
  \[
  S(\alpha)(\varphi') \overset{df}{=} \varphi' \circ \alpha,
  \]
  for any \(\varphi' \in X_{(A', C')}\).

**Proof.** The definition of \(S\) on the objects of \(B\) is obtained simply by transporting the topological structure of \(\Psi^a(A, C)\) from \(\text{Clust}(A, C)\) to \(X_{(A,C)}\) via the bijection \(\xi^{-1}\) from Proposition 6.2. Thus \(\Psi^a(A, C)\) and \(S(A, C)\) are homeomorphic topological spaces. Also, we obtain that the family

\[
\{\{\varphi \in X_{(A,C)} \mid \varphi(a^*) = 1\} \mid a \in A\}
\]

is an open base for the topology \(T'\).

If \(\alpha \in B((A, C), (A', C'))\), then we have that \(\Psi^a(\alpha)(\sigma') = \sigma\), for every \(\sigma' \in \text{Clust}(A', C')\), where \(\sigma = \{a \in A \mid \text{if } b \in A \text{ and } b \ll_C a^* \text{ then } (a(b))^* \in \sigma'\}\) (see 4.3). Thus, in the notation of Proposition 6.2, the transportation of the clusters via the bijection \(\xi^{-1}\) gives us the following formula: \(S(\alpha)(\varphi_{\sigma'}) \overset{df}{=} \varphi_{\sigma}\). Now, using again Proposition 6.2, as well as the definition of the composition \(\circ\) (see Definition 4.1), we obtain that, for every \(a \in A\), \(\varphi_{\sigma}(a) = 0 \iff a^* \in \sigma \iff \{b \in A \mid b \ll_C a \rightarrow (\varphi_{\sigma'}(a(b)) = 0]\} \iff (\varphi_{\sigma'} \circ \alpha)(a) = 0\). Hence, \(S(\alpha)(\varphi_{\sigma'}) = \varphi_{\sigma'} \circ \alpha\). Since \(\xi\) is a bijection, we can rewrite this formula as follows: \(S(\alpha)(\varphi') \overset{df}{=} \varphi' \circ \alpha\), for each \(\varphi' \in X_{(A', C')}\).

Let us also note that, setting \(Y \overset{df}{=} S(A, C)\) and using Remark 4.4(d) and (13), one has, for every \(a \in A\),

\[(14) \ \text{int}_Y(v'_{(A,C)}(a)) = \{y \in Y \mid y(a) = 1\}\.\]

**6.4.** We are now almost ready for defining our category \(\mathcal{C}''\). Let us start with the following definition:

**Definition.** If \((A, C)\) is an CNCA, \(X\) is a set and \(f \in \text{Set}(X, X_{(A,C)})\), then \(f\) is a \(t\)-injection (resp., \(t\)-inclusion) if \(f\) is an injection (resp., an inclusion) and for each
Proof. We will define a functor $a \in C''$, where $(A, C)$ is a CNCA, $X$ is t-included in $X_{(A,C)}$, and for every CNCA $(A', C')$ and every t-injection $f : X \rightarrow X_{(A',C')}$ for which

$$ (15) \quad \{ \alpha \in \mathcal{B}((A', C'), (A, C)) | f(x) = \alpha \circ x \text{ for every } x \in X \}; $$

one may define a category $C''$, as follows:

- its objects are all pairs $((A, C), X)$, where $(A, C)$ is a CNCA, $X$ is t-included in $X_{(A,C)}$, and for every CNCA $(A', C')$ and every t-injection $f : X \rightarrow X_{(A',C')}$ for which

$$ (15) \quad \{ \alpha \in \mathcal{B}((A', C'), (A, C)) | f(x) = \alpha \circ x \text{ for every } x \in X \}; $$

- its morphisms are all pairs $(\alpha, f) : ((A, C), X) \rightarrow ((A', C'), X')$ such that $\alpha \in \mathcal{B}((A, C), (A', C'))$, $f \in \text{Set}(X', X)$ and $f = S(\alpha)|X'$;

- composition is as in $\mathcal{B}$ and $\text{Set}$; that is, $(\alpha, f)$ as above gets composed with $(\alpha', f') : ((A', C'), X') \rightarrow ((A'', C''), X'')$ by the horizontal pasting of diagrams, that is,

$$ (\alpha', f') \circ (\alpha, f) \overset{\text{df}}{=} (\alpha' \circ \alpha, f \circ f'); $$

- the identity morphism of an $((A, C), X) \in |C''|$ is the $C''$-morphism $(1_{(A,C)}, 1_X)$.

For brevity, the condition (15) will be written in the following form:

$$ X \cap u'_{(A,C)}(A) = f^{-1}(v'_{(A',C')}(A')). $$

Remark. Using Proposition 6.3, one can easily obtain that $((A, C), X) \in |C''|$ if, and only if, $X$ is a dense subspace of the compact Hausdorff space $S(A, C)$ such that if $f : X \rightarrow S(A', C')$ is a compactification of $X$, then there exists a continuous map $g : S(A, C) \rightarrow S(A', C')$ with $g|X = f$; at that, condition (15) means that the subspace $X$ of $S(A, C)$ is homeomorphic to the subspace $f(X)$ of $S(A', C')$ (recall that $v'_{(A,C)}(A) = \text{RC}(S(A, C))$ and $v'_{(A',C')}(A') = \text{RC}(S(A', C'))$ (see 4.3)). In other words, if $\beta : X \rightarrow S(A, C)$ is the inclusion map, then $\beta$ is a Stone-Čech compactification of $X$ (we do not regard it here up to equivalence).

Proposition 6.5. The categories $C'$ and $C''$ are isomorphic.

Proof. We will define a functor $E' : C' \rightarrow C''$. Let $((A, C), j) \in |C'|$. Setting $X \overset{\text{df}}{=} \text{dom}(j)$, we obtain that $j : X \hookrightarrow S(A, C)$, the subset $X(= j(X))$ is dense in $S(A, C)$ and $X$ is $C''$-embedded in $S(A, C)$. Set

$$ E'((A, C), j) \overset{\text{df}}{=} ((A, C), X). $$

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Then Proposition 6.3 and Fact 6.4 show that $X$ is $t$-included in $X_{(A,C)}$; also, the fact that $X$ is $C^*$-embedded in $S(A,C)$ shows, as it is well known, that the required extensions described in Remark 6.4 can be obtained. Hence, $E'((A,C),j) \in |\mathcal{C}|$. Further, it is easy to see that setting for every $C'$-morphism $(\alpha,f)$,

$$E'(\alpha,f) \overset{df}{=} (\alpha,f),$$

we obtain that $(\alpha,f)$ is a $\mathcal{C}'$-morphism. Clearly, $E'$ is a functor which is full, faithful and injective on objects. For showing that it is surjective on objects, let $((A,C),X) \in |\mathcal{C}|$. Then, using Remark 6.4, we obtain that the inclusion $j : X \rightarrow S(A,C)$ is a dense embedding and $X$ is $C^*$-embedded in $S(A,C)$, i.e., $((A,C),j) \in |\mathcal{C}'|$. Since $E'((A,C),j) = ((A,C),X)$, we obtain that $E'$ is surjective on objects. Therefore, $E'$ is an isomorphism.

Now we obtain the following result:

**Theorem 6.6.** There is a dual equivalence between the categories $\mathcal{C}$ and $\text{Tych}$ which extends de Vries’ dual equivalence $\Psi$ between the categories $\text{DeV}$ and $\text{CHaus}$.

*Proof.* Setting $S' \overset{df}{=} S \circ I' \circ (E')^{-1}$, we obtain, using 6.1 and Propositions 6.5, 6.1, that $S'$ is a dual equivalence. Now define a functor $I'' : \text{DeV} \rightarrow \mathcal{C}$ by $I''((A,C)) \overset{df}{=} ((A,C),X_{(A,C)})$ and $I''(\alpha) \overset{df}{=} (\alpha,S(\alpha))$. Clearly, it is a full embedding and $I = I' \circ (E')^{-1} \circ I''$. Hence, $S' \circ I'' = J \circ S$. □

Now we are going to prove that the category $\mathcal{C}$ is equivalent to the category $\text{BMO}$. First we need to prove some lemmas and to recall some definitions and facts about the Tarski duality and the Bezhanishvili-Morandi-Olberding Duality Theorem.

6.7. We recall that the Tarski duality between the categories $\text{Set}$ and $\text{CaBa}$ is given by the contravariant functors

$$T^s : \text{Set} \rightarrow \text{CaBa} \quad \text{and} \quad T^a : \text{CaBa} \rightarrow \text{Set}$$

which are defined as follows. For every set $X$,

$$T^s(X) \overset{df}{=} (P(X), \subseteq).$$

If $f \in \text{Set}(X,Y)$, then $T^s(f) : T^s(Y) \rightarrow T^s(X)$ is defined by the formula

$$T^s(f)(M) \overset{df}{=} f^{-1}(M),$$

for every $M \in P(Y)$. Further, for every $B \in |\text{CaBa}|$,

$$T^a(B) \overset{df}{=} \text{At}(B);$$

if $\varphi \in \text{CaBa}(A,B)$, then $T^a(\varphi) : T^a(B) \rightarrow T^a(A)$ is defined by the formula

$$T^a(\varphi)(x) \overset{df}{=} \bigwedge\{a \in A \mid x \leq \varphi(a)\},$$

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for every $x \in T^a(B)$. For each set $X$, we have a natural isomorphism

$$\eta^T_X : X \longrightarrow T^a(T^s(X)) (= \text{At}(P(X)) = \{\{x\} \mid x \in X\}),$$

given by $\eta^T_X(x) = \{x\}$ for each $x \in X$. For each $B \in |\text{CaBa}|$ we have a natural isomorphism

$$\varepsilon^T_B : B \longrightarrow T^s(T^a(B)) (= P(\text{At}(B))),$$

given by $\varepsilon^T_B(b) \overset{\mathrm{df}}{=} \{x \in \text{At}(B) \mid x \leq b\}$ for each $b \in B$.

The following assertion is well-known (because $T^a(\varphi)$ is the restriction on $\text{At}(B')$ of the lower (or, left) adjoint for $\varphi$ (see [28, Theorem 4.2]), but we will present here its short proof.

**Lemma 6.8.** Let $\varphi \in \text{CaBa}(B, B')$. Then, for every $b \in B$ and each $x' \in \text{At}(B')$, $(x' \leq \varphi(b)) \iff (T^a(\varphi)(x') \leq b)$.

**Proof.** Since $T^a(\varphi)(x') = \bigwedge\{b \in B \mid x' \leq \varphi(b)\}$, we obtain immediately that $(x' \leq \varphi(b)) \implies (T^a(\varphi)(x') \leq b)$. Suppose now that $T^a(\varphi)(x') \leq b$. Then $\varphi(T^a(\varphi)(x')) \leq \varphi(b)$. Since $\varphi(T^a(\varphi)(x')) = \varphi(\bigwedge\{c \in B \mid x' \leq \varphi(c)\}) = \bigwedge\{\varphi(c) \mid c \in B, x' \leq \varphi(c)\} \geq x'$, we obtain that $x' \leq \varphi(b)$. \qed

**6.9.** The following assertion is well-known:

**Fact.** For every complete atomic Boolean algebra $B$, there is a bijection $m_B$ between the sets $\text{At}(B)$ and $\text{CaBa}(B, 2)$, namely,

- for every $x \in \text{At}(B)$, we set $m_B(x) \overset{\mathrm{df}}{=} u_x$, where $u_x \in \text{CaBa}(B, 2)$ is defined by $u_x(b) \overset{\mathrm{df}}{=} 1 \iff x \leq b$, for every $b \in B$;

- to every $u \in \text{CaBa}(B, 2)$ corresponds $x_u \in \text{At}(B)$ defined by $x_u \overset{\mathrm{df}}{=} \bigwedge u^{-1}(1) (= \bigwedge\{b \in B \mid u(b) = 1\})$; at that, $x = x_{u_x}$ and $u = u_{x_u}$, for every $u \in \text{CaBa}(B, 2)$ and for every $x \in \text{At}(B)$.

Note that the notation "$u_x$" in the above assertion was already used in (5), but we hope that it will be clear from the context which of the two meanings of this notation is used.

**6.10.** We now recall the Bezhanishvili-Morandi-Olberding Duality Theorem, starting with the main definitions of [9]:

- if $(A, C)$ is a CNCA, $B \in |\text{CaBa}|$ and $\gamma : (A, C) \longrightarrow (B, \rho_a)$ is an injective de Vries morphism, then $\gamma$ is called a de Vries extension provided that each atom of $B$ is a meet from $\gamma(A)$;
• two de Vries extensions \( \gamma : (A, C) \rightarrow (B, \rho_s) \) and \( \gamma' : (A', C') \rightarrow (B, \rho_s) \) are said to be compatible if \( \gamma(A) = \gamma'(A') \).

• a de Vries extension \( \gamma : (A, C) \rightarrow (B, \rho_s) \) is called maximal if for every compatible de Vries extension \( \gamma' : (A', C') \rightarrow (B, \rho_s) \) there is a de Vries morphism \( \alpha' : (A', C') \rightarrow (A, C) \) such that \( \gamma \circ \alpha' = \gamma' \).

Now we are ready to recall the definition of the category BMO:

• its objects are all maximal de Vries extensions;

• its morphisms are all pairs \((\alpha, \varsigma) : \gamma \rightarrow \gamma'\), where \( \gamma : (A, C) \rightarrow (B, \rho_s) \) and \( \gamma' : (A', C') \rightarrow (B', \rho_s) \) are de Vries extensions, \( \alpha \in \text{DeV}((A, C), (A', C')) \), \( \varsigma \in \text{CaBa}(B, B') \) and \( \varsigma \circ \gamma = \gamma' \circ \alpha \);

• composition is as in \( \text{DeV} \) and \( \text{CaBa} \); that is, \((\alpha, \varsigma)\) as above gets composed with \((\alpha', \varsigma') : \gamma' \rightarrow \gamma''\) as follows:

\[
(\alpha', \varsigma') \circ (\alpha, \varsigma) \overset{\text{def}}{=} (\alpha' \circ \alpha, \varsigma' \circ \varsigma);
\]

• the identity morphism of a BMO-object \( \gamma : (A, C) \rightarrow (B, \rho_s) \) is the BMO-morphism \((1_{(A,C)}, 1_B)\).

Let us recall the following assertions from [9]:

**Theorem.** ([9, Theorem 4.5]) Let \( c : X \rightarrow Y \) be a Hausdorff compactification of a Tychonoff space \( X \). Then the map \( \gamma : (\text{RO}(Y), D_Y) \rightarrow (P(X), \rho_s), U \mapsto c^{-1}(U) \), is a de Vries extension.

**Lemma.** ([9, Lemma 6.2]) Two de Vries extensions \( \gamma : (A, C) \rightarrow (B, \rho_s) \) and \( \gamma' : (A', C') \rightarrow (B, \rho_s) \) are compatible if, and only if, the initial topologies on the set \( X_B \overset{\text{def}}{=} \{ \uparrow(x) \mid x \in \text{At}(B) \} \) generated by the map \( \Psi^a(\gamma)|X_B \) and the map \( \Psi^a(\gamma')|X_B \), respectively, are equal.

**Proposition.** ([9, Theorem 6.4((3)\rightarrow(1))]) Let \( X \) be a Tychonoff space, \( c : X \rightarrow Y \) be a Hausdorff compactification of \( X \) which is equivalent to the Stone-Čech compactification of \( X \). Then the de Vries extension \( \gamma_c : (\text{RC}(Y), \rho_Y) \rightarrow (P(X), \rho_s) \) is maximal.

6.11. Since we work with the Boolean algebra \( \text{RC}(Y) \), we have to restate Theorem 6.10. By Example 2.3, there exists a CA-isomorphism \( \nu : (\text{RC}(Y), \rho_Y) \rightarrow (\text{RO}(Y), D_Y), F \mapsto \text{int}_Y(F) \). Thus, using Theorem 6.10, it is easy to see that the map \( \gamma \circ \nu : (\text{RC}(Y), \rho_Y) \rightarrow (P(X), \rho_s) \) is a de Vries extension. We will show, however, that even the map \( \gamma \circ \nu \) is a de Vries extension. Obviously, this will imply that \( \gamma \circ \nu = \gamma \circ \nu \).

**Proposition.** Let \( c : X \rightarrow Y \) be a Hausdorff compactification of a Tychonoff space \( X \). Then the map \( \gamma_c : (\text{RC}(Y), \rho_Y) \rightarrow (P(X), \rho_s), F \mapsto c^{-1}(\text{int}_Y(F)) \), is a de Vries extension.
Proof. Since the map \( r : \text{RO}(Y) \longrightarrow \text{RO}(X), \ U \mapsto c^{-1}(U) \) is a Boolean isomorphism (see [14, p. 271] or [37, Lemma 44]), we obtain that \( \gamma_c \) is an injection. Let us show that it is a de Vries morphism. Obviously, condition (DV1) is satisfied. For showing that (DV2) is fulfilled, let \( F, G \in \text{RC}(Y) \). We have to prove that \( \gamma_c(F \cap G) = \gamma_c(F) \cap \gamma_c(G), \) i.e., that \( c^{-1}(\text{int}_Y(F \cap G)) = c^{-1}(\text{int}_Y(F)) \cap c^{-1}(\text{int}_Y(G)) \). Obviously, it is enough to show that \( \text{int}_Y(F \cap G) = \text{int}_Y(F \cap G) \). Since \( \text{int}_Y(F \cap G) = \text{int}_Y(\text{cl}_Y(\text{int}_Y(F \cap G))) \) and, as it is well known, \( \text{int}_Y(F \cap G) \in \text{RO}(Y) \), we conclude that (DV2) is satisfied. For proving (DV3), let \( F \ll G \), i.e., \( F \subseteq \text{int}_Y(G) \). We have to show that \( (\gamma_c(F*))^* \subseteq \gamma_c(G) \), i.e., that \( X \setminus c^{-1}(\text{int}_Y(F*)) \subseteq c^{-1}(\text{int}_Y(G)) \). We have that \( \text{int}_Y(F*) = Y \setminus F \). Thus \( X \setminus c^{-1}(\text{int}_Y(F*)) \subseteq c^{-1}(\text{int}_Y(G)). \) So, (DV3) is also satisfied. Since \( Y \) is a regular space, we have that \( \text{int}_Y(F) = \bigcup \{ \text{int}_Y(G) \mid G \in \text{RC}(Y), G \subseteq \text{int}_Y(F) \} \), for every \( F \in \text{RC}(Y) \). This implies that (DV4) is fulfilled. Hence, \( \gamma_c \) is a de Vries morphism. Since \( \{ \text{int}_Y(F) \mid F \in \text{RC}(Y) \} \) is a base for \( Y \), we obtain that \( \gamma_c \) is a de Vries extension. \( \square \)

Lemma 6.12. If \( \gamma : (A, C) \longrightarrow (B, \rho_s) \) is a de Vries extension, then there exists a bijection \( \mu \) between the sets \( \{ u_x \mid x \in \text{At}(B) \} \) and \( \{ u_x \circ \gamma \mid x \in \text{At}(B) \} \) (see Fact 6.9 for notation). (When it is needed, we will write \( \mu_\gamma \) instead of \( \mu \).)

Proof. For every \( x \in \text{At}(B) \), set \( \mu(u_x) \overset{df}{=} u_x \circ \gamma \). Let \( x, y \in \text{At}(B) \) and \( u_x \neq u_y \). Then \( x \neq y \). There exists a subset \( A_x \) of \( A \) such that \( x = \bigwedge \gamma(A_x) \). Thus \( 1 = u_x(x) = u_x(\bigwedge \gamma(A_x) = \bigwedge \{ u_x(\gamma(a)) \mid a \in A_x \} \). Therefore, \( u_x(\gamma(a)) = 1 \) for every \( a \in A_x \). There exists \( a_0 \in A_x \) such that \( y \not\subseteq \gamma(a_0) \). Then \( u_y(\gamma(a_0)) = 0 \neq 1 = u_x(\gamma(a_0)) \). Hence \( \mu \) is an injection. Clearly, \( \mu \) is a surjection. Therefore, \( \mu \) is a bijection. \( \square \)

Theorem 6.13. The categories \( \text{BMO} \) and \( \mathcal{C}'' \) are equivalent.

Proof. We define a functor \( \Theta : \text{BMO} \longrightarrow \mathcal{C}'' \) as follows:

- for every de Vries' extension \( \gamma : (A, C) \longrightarrow (B, \rho_s) \), we set
  \[ \Theta(\gamma) \overset{df}{=} ((A, C), X_\gamma), \text{ where } X_\gamma \overset{df}{=} \{ u_x \circ \gamma \mid x \in \text{At}(B) \}; \]

- for every \( (\alpha, \varsigma) \in \text{BMO}(\gamma, \gamma') \), where \( \gamma : (A, C) \rightarrow (B, \rho_s) \) and \( \gamma' : (A', C') \rightarrow (B', \rho_s) \), we set
  \[ \Theta((\alpha, \varsigma)) \overset{df}{=} (\alpha, f_\varsigma), \]
  where \( f_\varsigma : X_\gamma \longrightarrow X_{\gamma'} \) is defined by
  \[ f_\varsigma(u_x \circ \gamma') \overset{df}{=} u_{T^s(\varsigma)(x')} \circ \gamma, \forall x' \in \text{At}(B'). \]

Now we have to show that the functor \( \Theta \) is well-defined. Let us start by proving that it is well-defined on objects.

First of all, note that \( (u \in \text{CaBa}(B, 2)) \Rightarrow (u \in \text{DeV}((B, \rho_s), (2, \rho_s))) \) and \( u \circ \gamma = u \circ \gamma \) (see Fact 4.2(d)). Thus, if \( \gamma : (A, C) \longrightarrow (B, \rho_s) \) is a de Vries' extension, then \( X_\gamma \subseteq X_{(A,C)} \). Moreover, \( X_\gamma \) is \( t \)-included in \( X_{(A,C)} \). Indeed, if \( a \in A^+ \) then...
\( \gamma(a) \neq 0 \) (because \( \gamma(0) = 0 \) and \( \gamma \) is an injection); hence, there exists \( x \in \text{At}(B) \) such that \( x \leq \gamma(a) \); this means that \( (u_x \circ \gamma)(a) = 1 \). Therefore, \( X_\gamma \) is t-included in \( X_{(A,C)} \).

Let now \( X_{(A,C)} \) be a t-injection for which

\[
(16) \quad X_\gamma \cap \gamma'(A) = f^{-1}(\gamma'(A')).
\]

We have to show that there exists \( \alpha \in \text{DeV}(A', C') \) such that \( f(u_x \circ \gamma) = (u_x \circ \gamma) \circ \alpha \) for every \( x \in \text{At}(B) \).

Set \( X \equiv \{ u_x \mid x \in \text{At}(B) \} \), \( Y \equiv S(A, C) \), \( Z \equiv S(A', C') \) and let \( i_\gamma : X_\gamma \hookrightarrow Y \) be the inclusion map. Note that, by Remark 4.4 and Proposition 6.3, \( X \subseteq B \), \( \rho_s = S^a(B) \).

Now we define \( c \equiv \mu \circ i_\gamma \) and \( c' \equiv f \circ \mu \).

We have, by Lemma 6.12, that \( \mu(X) = X_\gamma \). Hence, \( c(X) \) is dense in \( Y \) and if \( T_c \) is the initial topology on \( X \) generated by the map \( c \), then \( c : X \rightarrow Y \) is a compactification of \( (X, T_c) \). Further, since \( f \) is a t-injection, we obtain that \( c'(X) \) is dense in \( Z \) and \( c' \) is an injection. Thus, if \( T_{c'} \) is the initial topology on \( X \) generated by the map \( c' \), then \( c' : X \rightarrow Z \) is a compactification of \( (X, T_{c'}) \). Moreover, the topologies \( T_c \) and \( T_{c'} \) on \( X \) are equal. Indeed, \( \mu^{-1}(X_\gamma \cap \gamma'(A)) \) is a closed base for the topology \( T_c \) and \( \mu^{-1}(f^{-1}(\gamma'(A'))) \) is a closed base for the topology \( T_{c'} \). Thus, by (16), \( T_c = T_{c'} \). Also, note that Proposition 6.3 and Fact 6.9 show that the set \( X \) defined above plays the role of the set \( X_B \) from Lemma 6.10. Since, by Proposition 6.3, \( \mu(u_x) = S(\gamma)(u_x) \) for every \( x \in \text{At}(B) \), we obtain that the initial topology on \( X \) generated by the map \( S(\gamma)|X \) coincides with the topology \( T_c \).

We will now define a de Vries extension \( \gamma' : (A', C') \rightarrow (B, \rho_s) \) such that \( \gamma(A) = \gamma'(A') \).

By Proposition 6.11, \( \gamma_{c'} : (RC(Z), \rho_Z) \rightarrow (P(X), \rho_s), G \rightarrow (c')^{-1}(\text{int}_Z(G)) \), is a de Vries’ extension. Set, for brevity, \( \varepsilon_B \equiv m_B \circ \varepsilon_B \) (see Fact 6.9 and 6.7 for the notation). Now we define \( \gamma' \equiv \varepsilon_B^{-1} \circ (\gamma_c \circ v(A', C') \gamma) \).

Since \( \varepsilon_B \) can be regarded as a \( \text{DeV} \)-isomorphism from \( (B, \rho_s) \) to \( (P(X), \rho_s) \), we obtain that \( \gamma' : (A', C') \rightarrow (B, \rho_s) \) is a de Vries’ morphism. Obviously, by Theorem 4.5, \( S(\gamma') \) is a surjection (as a composition of surjections). Thus, applying once more Theorem 4.5, we obtain that \( \gamma' \) is an injection. Now it is easy to see that \( \gamma' \) is a de Vries’ extension. We will show that the initial topology on \( X \) generated by the map \( S(\gamma')|X \) coincides with the initial topology on \( X \) generated by the map \( c' \), i.e., with topology \( T_c \) on \( X \). So, we have to prove that \( S(\gamma')(u_x) = c'(u_x) \) for every \( x \in \text{At}(B) \).

We have that \( c'(u_x) = f(\mu(u_x)) = f(u_x \circ \gamma) \) and, by Proposition 6.3 and Fact 4.2(d), \( S(\gamma')(u_x) = u_x \circ \gamma = u_x \circ \varepsilon_B^{-1} \circ (\gamma_c \circ v(A', C') \gamma) \). Further, for any \( a' \in A' \), we obtain, using (14), that

\[
(17) \quad (\gamma_c \circ v(A', C') \gamma)(a') = \bigcup \{ \gamma_c \circ v(A', C') \gamma(b) \mid b \ll a' \} = \bigcup \{ \mu^{-1}(\text{int}_Z(v(A', C') \gamma(b))) \mid b \ll a' \} = \bigcup \{ \mu^{-1}(\{ u_y \mid y \in \text{At}(B), f(u_y \circ \gamma)(b) = 1 \}) \mid b \ll a' \} = \bigcup \{ \{ u_y \mid y \in \text{At}(B), f(u_y \circ \gamma)(b) = 1 \} \mid b \ll a' \}.
\]

Since, for every \( M \subseteq X, \varepsilon_B^{-1}(M) = \bigvee \{ \varepsilon_B^{-1}(\{ u_y \mid y \in \text{At}(B), f(u_y \circ \gamma)(b) = 1 \}) \mid b \ll a' \} \), we obtain that \( \varepsilon_B^{-1}(\gamma'(a')) = \bigvee \{ y \in \text{At}(B) \mid f(u_y \circ \gamma)(b) = 1, b \ll a' \} \). Now we have, by Fact 6.9, that \( u_x \circ \gamma(a') = 1 \)
$x \leq \gamma'(a') \iff x \leq \varepsilon^{-1}_{\gamma}(\gamma_\cdot \circ u_\cdot'(A',C'))(a')) \iff x \leq \bigvee\{y \in \text{At}(B) \mid f(u_y \circ \gamma)(b) = 1, b \ll a'\} \iff (\exists b \ll a' \text{ such that } f(u_x \circ \gamma)(b) = 1) \iff f(u_x \circ \gamma)(a') = 1$. Hence, $u_x \circ \gamma' = f(u_x \circ \gamma)$, for every $x \in \text{At}(B)$. Therefore, $S'(u_x) = c'(u_x)$ for every $x \in \text{At}(B)$.

All this shows that the initial topologies on $X$ generated by the maps $S(\gamma)|X$ and $S'(\gamma)|X$, respectively, are equal. Now, Lemma 6.10 implies that $\gamma(A) = \gamma'(A')$. Since $\gamma'$ is a maximal de Vries’ extension, there exists $\alpha \in \text{DeV}((A',C'),(A,C))$ such that $\gamma' = \gamma \circ \alpha$. Hence, we obtain that for every $x \in \text{At}(B)$, $f(u_x \circ \gamma) = S'(\gamma')(u_x) = u_x \circ \gamma' = u_x \circ (\gamma \circ \alpha) = (u_x \circ \gamma) \circ \alpha$. Therefore, $\Theta(\gamma) \in |\mathcal{C}|$.

We now show that $\Theta$ is well-defined on morphisms. Let $(\alpha, \varsigma) \in \text{BMO}(\gamma,\gamma')$, where $\gamma : (A,C) \to (B,\rho_s)$ and $\gamma' : (A',C') \to (B',\rho_s)$ are de Vries’ extensions. Then $\alpha \in \text{DeV}((A,C),(A',C'))$, $\varsigma \in \text{CaBa}(B,B')$, and $\varsigma \circ \gamma = \gamma' \circ \alpha$. We have that $\Theta(\gamma) = ((A,C),X_\gamma)$ and $\Theta(\gamma') = ((A',C'),X_{\gamma'})$, where $X_\gamma = \{u_x \circ \gamma \mid x \in \text{At}(B)\}$ and $X_{\gamma'} = \{u_x \circ \gamma' \mid x \in \text{At}(B')\}$. Also, $\Theta((\alpha,\varsigma)) = (\alpha, f)$, where $f : X_{\gamma'} \to X_{\gamma}$, $u_{x'} \circ \gamma' \mapsto u_{x'} \circ \gamma$, for every $x' \in \text{At}(B')$. We have to prove that $f = S(\alpha)|X_{\gamma'}$, i.e., that $u_{x'} \circ \gamma = (u_x \circ \gamma) \circ \alpha$, for every $x' \in \text{At}(B')$. Fix a $x' \in \text{At}(B')$. Note that $(u_x \circ \gamma) \circ \alpha = u_x \circ \gamma' \circ \alpha = u_x \circ (\gamma' \circ \alpha) = u_x \circ \varsigma \circ \gamma$. Also, for every $a \in A$, $u_{x'}(\varsigma(\gamma(a))) = 1 \iff x' \leq \varsigma(\gamma)(a)$ and $u_{T^s(\varsigma)(x')}(\gamma(a)) = 1 \iff T^a(\varsigma)(x') \leq \gamma(a)$. Now, applying Lemma 6.8, we obtain that $u_{x'} \circ \varsigma \circ \gamma = u_{T^s(\varsigma)(x')}(\gamma)$. Thus, $f = S(\alpha)|X_{\gamma'}$. Therefore, $\Theta((\alpha,\varsigma)) \in \mathcal{C}'(\Theta(\gamma),\Theta(\gamma'))$, i.e., $\Theta$ is well-defined on morphisms.

Now, it is easy to see that $\Theta : \text{BMO} \to \mathcal{C}$ is a functor. Let us show that $\Theta$ is full and faithful. Let $\gamma : (A,C) \to (B,\rho_s)$ and $\gamma' : (A',C') \to (B',\rho_s)$ be de Vries’ extensions. We have to prove that the restriction

$$\Theta : \text{BMO}(\gamma,\gamma') \to \mathcal{C}'(\Theta(\gamma),\Theta(\gamma'))$$

is a bijection. For proving injectivity, we let $(\alpha,\varsigma), (\alpha',\varsigma') \in \text{BMO}(\gamma,\gamma')$ and assume $(\alpha,\varsigma) \neq (\alpha',\varsigma')$. Then $\Theta((\alpha,\varsigma)) = (\alpha, f)$ and $\Theta((\alpha',\varsigma')) = (\alpha', f')$, where $\alpha, \alpha' \in \text{DeV}((A,C),(A',C'))$, $f, f' : X_{\gamma} \to X_{\gamma'}$, $X_\gamma = \{u_x \circ \gamma \mid x \in \text{At}(B)\}$, $X_{\gamma'} = \{u_x \circ \gamma' \mid x' \in \text{At}(B')\}$. Then, $f(\text{Id}_{X_{\gamma}}) = u_{\varsigma(\gamma')} \circ \gamma$ and $f'(\text{Id}_{X_{\gamma'}}) = u_{T^s(\gamma')(x')}(\gamma)$. If $\alpha \neq \alpha'$, then, clearly, $\Theta((\alpha,\varsigma)) \neq \Theta((\alpha',\varsigma'))$. Let now $\alpha = \alpha'$ and $\varsigma \neq \varsigma'$. Then, by the Tarski duality, $T^a(\varsigma) \neq T^a(\varsigma')$. Hence, there exists $x' \in \text{At}(B')$ such that $T^a(\varsigma)(x') \neq T^a(\varsigma')(x')$. Using Fact 6.9 and Lemma 6.12, we obtain that $f(\text{Id}_{X_{\gamma}}) \neq f'(\text{Id}_{X_{\gamma}})$, i.e., $f \neq f'$ and thus, $\Theta((\alpha,\varsigma)) \neq \Theta((\alpha',\varsigma'))$. So, $\Theta$ is a faithful functor.

We now prove that $\Theta$ is full, i.e., the above restriction of $\Theta$ is a surjection. Let $(\alpha,f) \in \mathcal{C}'(\Theta(\gamma),\Theta(\gamma'))$. Then $f : X_{\gamma} \to X_{\gamma'}$ and $f = S(\alpha)|X_{\gamma'}$. As Fact 6.9 and Lemma 6.12 show, the maps $\mu_x \circ m_B : \text{At}(B) \to X_{\gamma}$, $x \mapsto u_x \circ \gamma$, and $\mu_{\gamma'} \circ m_{B'} : \text{At}(B') \to X_{\gamma'}$, $x' \mapsto u_{x'} \circ \gamma'$, are bijections. Set

$$\lambda \overset{\text{df}}{=} \mu_\gamma \circ m_B, \quad \lambda' \overset{\text{df}}{=} \mu_{\gamma'} \circ m_{B'} \text{ and } \overline{f} \overset{\text{df}}{=} \lambda^{-1} \circ f \circ \lambda'.$$

Then $\overline{f} : \text{At}(B') \to \text{At}(B)$. Since $T^a$ is full (and faithful), there exists a (unique) $\varsigma \in \text{CaBa}(B,B')$ such that $T^a(\varsigma) = \overline{f}$.

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We will prove that $(\alpha, \varsigma) \in \text{BMO}(\gamma, \gamma')$ and $\Theta((\alpha, \varsigma)) = (\alpha, f)$. Let us first show that $(\alpha, \varsigma) \in \text{BMO}(\gamma, \gamma')$. We need only to check that $\varsigma \circ \gamma = \gamma' \circ \alpha$. Since $S$ is faithful, it is enough to prove that $S(\varsigma \circ \gamma) = S(\gamma' \circ \alpha)$, i.e., that $S(\gamma) \circ S(\varsigma) = S(\alpha) \circ S(\gamma')$. Since $\text{CaBa}(B', 2)$ is a dense subset of $S((B', \rho_s))$ (see Remark 4.4), we need only to prove that $S(\gamma) \circ S(\varsigma) = S(\alpha) \circ S(\gamma')$ on $\text{CaBa}(B', 2)$, i.e., on $\{ u_x, | x' \in \text{At}(B') \}$ (see Fact 6.9). So, let $x' \in \text{At}(B')$. Then $S(\gamma)(S(\varsigma)(u_x)) = S(\gamma)(u_x \circ \varsigma) = u_x \circ \varsigma \circ \gamma$ and $S(\alpha)(S(\gamma')(u_{x'})) = S(\alpha)(u_{x'} \circ \gamma') = f(u_{x'} \circ \gamma')$ (since $f = S(\alpha)|X_\gamma$). Hence, we have to show that $f(u_x \circ \gamma') = u_x \circ \varsigma \circ \gamma$. Since $f = \lambda \circ \overline{f} \circ (\lambda')^{-1}$, we obtain that $f(u_x \circ \gamma') = (\lambda \circ \overline{f} \circ (\lambda')^{-1})(u_x \circ \gamma') = \lambda(\overline{f}(x'))$. Then $\gamma(\overline{f}(x')) = \gamma(S^\alpha(x')) = \gamma(T^\alpha(\varsigma)(x')) \circ \gamma$. We also have that, for every $a \in A$, $u_{T^\alpha(\varsigma)(x')}(\gamma(a)) = 1 \Leftrightarrow T^\alpha(\varsigma)(x') \leq \gamma(a) \Leftrightarrow x' \leq \varsigma(\gamma(a)) \Leftrightarrow u_{x'}(\varsigma(\gamma(a))) = 1$ (applied Lemma 6.8 here). Therefore, $S(\gamma) \circ S(\varsigma) = S(\alpha) \circ S(\gamma')$ and, thus, $(\alpha, \varsigma) \in \text{BMO}(\gamma, \gamma')$. We will now show that $\Theta((\alpha, \varsigma)) = (\alpha, f)$, i.e. that $f(u_x \circ \gamma') = u_{T^\alpha(\varsigma)(x') \circ \gamma}$ for any $x' \in \text{At}(B')$. Since the validity of this equation was already demonstrated, we obtain that $\Theta$ is a full functor.

Finally, we prove that $\Theta$ is essentially surjective on objects. Let $((A, C), X) \in |\mathcal{E}'|$. Set $Y \overset{df}{=} S(A, C)$. Then $X \subseteq \text{DeV}(\text{At}(A, C), (2, \rho_s)) = Y$, $X$ is dense in $Y$ and if $\beta : X \hookrightarrow Y$ is the inclusion map, then $\beta$ is the Stone-Cech compactification of $X$ (we do not regard it here up to equivalence) (see Remark 6.4). Let $\gamma : (\text{RC}(Y), \rho_Y) \longrightarrow (P(X), \rho_X)$ be defined by $\gamma(G) \overset{df}{=} X \cap \text{int}_Y(G)$, for every $G \in \text{RC}(Y)$. Then Propositions 6.11 and 6.10 imply that $\gamma \in |\text{BMO}|$. We will prove that $\Theta(\gamma)$ is $\mathcal{E}''$-isomorphic to $((A, C), X)$. We have that $\Theta(\gamma) = ((\text{RC}(Y), \rho_Y), X_\gamma)$, where $X_\gamma = \{(x) \in X | x \in X \}$ (since $\text{At}(P(X)) = \{ \{x\} | x \in X \}$ and writing $u_x$ instead of $u_{(x)}$). As we already noted, the map $\lambda : X \longrightarrow X_\gamma$, $x \mapsto u_x \circ \gamma$, is a bijection. We will show that $(\lambda : (A, C), X) \longrightarrow \Theta(\gamma)$ is a $\mathcal{E}''$-isomorphism. Set $\alpha = \gamma'_{(A, C)}$. We first have to prove that $(\alpha, \lambda^{-1})$ is a $\mathcal{E}''$-morphism, i.e. that $\lambda^{-1} = S(\alpha)|X_\gamma$. We have that for every $x \in X$, $S(\alpha)(u_x \circ \gamma) = (u_x \circ \gamma) \circ \alpha$ and $\lambda^{-1}(u_x \circ \gamma) = x$. So that, we need to show that $x = (u_x \circ \gamma) \circ \alpha$. For every $a \in A$, we have that $((u_x \circ \gamma) \circ \alpha)(a) = \bigvee \{ u_x(\gamma(\alpha(b))) | b \leq a \} = \bigvee \{ u_x(\gamma(\alpha(b))) | b \leq a \} = \bigvee \{ u_x(X \cap \text{int}_Y(\alpha(b))) | b \leq a \}$. Recall that $\text{int}_Y(\alpha(b)) = \{ \varphi \in \text{DeV}(\text{At}(A, C), (2, \rho_s)) | \varphi(b) = 1 \}$. Now, we have that $u_x(X \cap \text{int}_Y(\alpha(b))) = 1 \Leftrightarrow x \in X \cap \text{int}_Y(\alpha(b)) \Leftrightarrow x \in \text{int}_Y(\alpha(b)) \Leftrightarrow x(b) = 1$. Hence, $((u_x \circ \gamma) \circ \alpha)(a) = 1 \Leftrightarrow \exists b \in A$ such that $b \leq a$ and $x(b) = 1 \Leftrightarrow x(a) = 1$. Therefore, $x = (u_x \circ \gamma) \circ \alpha$. Hence, $(\alpha, \lambda^{-1})$ is a $\mathcal{E}''$-morphism. Since $\alpha$ is a $\text{DeV}$-isomorphism and $\lambda^{-1}$ is a $\text{Set}$-isomorphism, we obtain that $(\alpha, \lambda^{-1})$ is a $\mathcal{E}''$-isomorphism. Therefore, $\Theta$ is essentially surjective on objects.

This completes the proof that $\Theta$ is an equivalence.

\begin{cor}
Corollary 6.14. ([9]) There is a dual equivalence between the categories $\text{BMO}$ and $\text{Tych}$.
\end{cor}
\begin{proof}
Setting $\overline{\mathcal{E}''} \overset{df}{=} \overline{\mathcal{E}'} \circ \Theta$, we obtain, using Theorems 6.6 and 6.13, that $\overline{\mathcal{E}''} : \text{BMO} \longrightarrow \text{Tych}$ is a dual equivalence.
\end{proof}

\begin{thm}
Theorem 6.15. There exists a full embedding $I'' : \text{DeV} \longrightarrow \text{BMO}$ such that $J \circ S = \overline{\mathcal{E}''} \circ I''$. Hence, we can say that the dual equivalence $\overline{\mathcal{E}''} : \text{BMO} \longrightarrow \text{Tych}$ extends de Vries’ dual equivalence $\Psi^a : \text{DeV} \longrightarrow \text{CHaus}$.
\end{thm}
Proof. We define a functor $I'' : \text{DeV} \rightarrow \text{BMO}$. Let $(A, C) \in |\text{DeV}|$ and set $Y \stackrel{df}{=} S(A, C)$. Then $\text{id}_Y : Y \rightarrow Y$ is a compactification of $Y$. Set, for short, $i \stackrel{df}{=} \text{id}_Y$. Then, by Proposition 6.11, the map $\gamma_i : RC(Y) \rightarrow P(Y)$, $G \mapsto \text{int}_Y(G)$, is a de Vries extension. Set $\gamma \stackrel{df}{=} \gamma_i \circ v'_{(A,C)}$. Then it is easy to see that $\gamma$ is a de Vries extension. We set $I''((A, C)) \stackrel{df}{=} \gamma$, and for every $\alpha \in \text{DeV}((A, C), (A', C'))$,

$$I''(\alpha) \stackrel{df}{=} (\alpha, T^s(S(\alpha))).$$

Clearly, $I''$ is well-defined on objects. We show that $I''$ is well-defined on morphisms, as well. Using the above notation, we have only to show that $T^s(S(\alpha)) \circ \gamma = \gamma' \circ \alpha$, where $\gamma' = I''((A', C'))$. Set, for short, $Y' \stackrel{df}{=} S(A', C')$, $\varsigma \stackrel{df}{=} T^s(S(\alpha))$, $v \stackrel{df}{=} v'_{(A,C)}$ and $v' \stackrel{df}{=} v'_{(A',C')}$. We have that for every $a \in A$, $\gamma(a) = (\gamma_i \circ v)(a) = \bigcup \{\gamma_i(v(b)) \mid b \ll a\} = \bigcup \{\text{int}_Y(v(b)) \mid b \ll a\} = \{y \in Y \mid y(a) = 1\}$. Hence, for every $a \in A$, $\varsigma(\gamma(a)) = (S(\alpha))^{-1}(\{y \in Y \mid y(a) = 1\}) = \{y' \in Y' \mid (S(\alpha)(y'))(a) = 1\} = \{y' \in Y' \mid (\gamma'(\alpha(b)) \mid b \ll a\} = \{y' \in Y' \mid \exists b \ll a \text{ such that } y'(\alpha(b)) = 1\}$. Further, as we have shown above, $\gamma'(a') = \{y' \in Y' \mid y'(a') = 1\}$ for every $a' \in A'$. Hence, for every $a \in A$, $(\gamma' \circ \alpha)(a) = \bigcup \{\gamma'(\alpha(b)) \mid b \ll a\} = \bigcup \{y' \in Y' \mid y'(\alpha(b)) = 1\} \mid b \ll a\} = \{y' \in Y' \mid \exists b \ll a \text{ such that } y'(\alpha(b)) = 1 = \varsigma(\gamma(a))$. Thus, $T^s(S(\alpha)) \circ \gamma = \gamma' \circ \alpha$. Therefore, $I''$ is well-defined on morphisms.

Now, it is easy to see that $I''$ is a functor. As it follows from Theorem 6.6 and its proof, for showing that $J \circ S = S'' \circ I''$, it suffices to prove that $I'' = \Theta \circ I''$. Let $(A, C) \in |\text{DeV}|$. Then $I''((A, C), Y)$ and $\Theta(I''((A, C))) = \Theta(\gamma) = ((A, C), X_\gamma)$, where $X_\gamma = \{u_y \circ \gamma \mid y \in Y\}$ (since $\text{At}(P(Y)) = Y$ and writing $u_y$ instead of $u_{(y)}$). Hence, for showing that $I''((A, C)) = (\Theta \circ I'')((A, C))$, it is enough to prove that $y = u_y \circ \gamma$, for every $y \in Y$. We have that for every $a \in A$, $u_y(\gamma(a)) = 1 \Leftrightarrow y \in \gamma(a) \Leftrightarrow y \in \{z \in Y \mid z(a) = 1\} \Leftrightarrow y(a) = 1$. Thus, $y = u_y \circ \gamma$, for every $y \in Y$. Therefore, $I''((A, C)) = (\Theta \circ I'')((A, C))$. Let now $\alpha \in \text{DeV}((A, C), (A', C'))$. Using the above notation, we obtain that $I''(\alpha) = (\alpha, S(\alpha))$ and $\Theta(I''(\alpha)) = \Theta((\alpha, \varsigma)) = (\alpha, f_\varsigma)$, where $f_\varsigma : Y' \rightarrow Y, y' \mapsto u_{T^s(\varsigma)(y')} \circ \gamma$. So, we have to show that $S(\alpha) = f_\varsigma$. Using the Tarski duality, we obtain that $\eta_Y^T \circ S(\alpha) = T^a(T^s(S(\alpha))) \circ \eta_Y^T$. Since $\eta_Y^T(y) = \{y\}$ for every $y \in Y$, and $\eta_Y^T(y') = \{y'\}$ for every $y' \in Y'$, we obtain that $T^a(\varsigma) = T^a(T^s(S(\alpha))) = S(\alpha)$. Thus, $f_\varsigma(y') = u_{S(\omega)(y')} \circ \gamma = S(\alpha)(y')$. Therefore, $I'' = \Theta \circ I''$. This equality, together with the facts that $\Theta$ is an equivalence and $I''$ is an embedding, imply that $I''$ is a full embedding. \hfill \Box

We note that a theorem, analogous to the previous one, was proved in [10], but in a completely different way.

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