ON THE SYLVESTER-GALLAI AND THE ORCHARD PROBLEM FOR PSEUDOLINE ARRANGEMENTS

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Abstract

We study a non-trivial extreme case of the orchard problem for 12 pseudolines and we provide a complete classification of pseudoline arrangements having 19 triple points and 9 double points. We have also classified those that can be realized with straight lines. They include new examples different from the known example of Böröczky. Since Melchior’s inequality also holds for arrangements of pseudolines, we are able to deduce that some combinatorial point-line configurations cannot be realized using pseudolines. In particular, this gives a negative answer to one of Grünbaum’s problems. We formulate some open problems which involve our new examples of line arrangements.

Keywords line arrangements, pseudoline arrangements, orchard problem, Sylvester’s problem

Mathematics Subject Classification (2000) 52C30, 32S22

1 The Sylvester-Gallai problem

We begin with a few definitions. A line arrangement in the real projective plane \( \mathbb{P}^2_\mathbb{R} \) is a finite set of lines in \( \mathbb{P}^2_\mathbb{R} \). A pseudoline in \( \mathbb{P}^2_\mathbb{R} \) is a simple closed curve such that its removal does not cut \( \mathbb{P}^2_\mathbb{R} \) in two connected components. A pseudoline arrangement is a set of pseudolines in \( \mathbb{P}^2_\mathbb{R} \) such that every pair of pseudolines has precisely one point in common where the two curves intersect each other. A first book about pseudoline arrangements was written by Grünbaum [13]. It has turned out later that pseudoline arrangements are isomorphic to reorientation classes of oriented matroids in rank 3. This implies a close connection of our investigation to the theory of oriented matroids. The interested reader can find more about this relation in [2] and [3]. For a given line arrangement, or for a given pseudoline arrangement, we count the number of points that are incident with precisely \( k \) lines or \( k \) pseudolines, respectively, with \( k \geq 2 \), and we denote this number of the arrangement by \( t_k \). We call a point incident with precisely \( r \) lines or with precisely \( r \) pseudolines an \( r \)-point. We use also the notion double point for \( r = 2 \), triple point for \( r = 3 \), and quadruple point for \( r = 4 \). We speak of an essential line (pseudoline) arrangement when all lines (pseudolines) do not intersect at one point. Our article can also be seen in the spirit of Grünbaum’s book about point-line configurations, see [14]. The reader will get some benefit for understanding our paper when she/he has a look at this book. For a point-line configuration we have not only a set of lines but also a set of points together with an incidence relation between the set of points and the set of lines. It is clear that the lines can be replaced with pseudolines and we arrive at a point-pseudoline arrangement. Moreover, when we forget about any underlying geometric set of points, lines, or pseudolines, we arrive at an abstract point-line configuration. Point-line configurations with \( n \) lines and \( n \) points in which (●) the lines are incident with precisely \( k \) points and (●●) the points are incident with precisely \( k \) lines have been
called \((n_k)\)-configurations. We refer to them later on. We use the point-line duality of the projective plane, i.e., a line arrangement defines via duality a point configuration, and vice versa.

The solution of the famous problem due to Sylvester [24] says that for every finite configuration of points in the real projective plane there exists at least one ordinary line, i.e., a line passing through exactly two points from the configuration, provided that not all points lie on a line. The dual version of this problem tells us that every arrangement of lines in the real projective plane, not all intersecting at one point, contains at least one double intersection point, i.e., \(t_2 \geq 1\). Sylvester’s problem was solved by Gallai [10]. It is worth pointing out that Melchior [21] has shown in the dual situation that for essential line arrangements of at least 3 lines one has \(t_2 \geq 3\). It was natural to ask what is the maximal possible number of ordinary lines for configurations of points or, via the famous orchard problem [18], what is the maximal possible number of triple intersection points for line arrangements. Quite recently, Green and Tao [12] have shown the so-called Dirac-Motzkin conjecture which provides also the upper bound for the number of triple intersection points for arrangements of \(n \gg 0\) lines defined over the reals, namely

\[
t_3 \leq 1 + \left\lfloor \frac{n(n-3)}{6} \right\rfloor.
\]

It was well-known several years before the proof of Green and Tao that their upper-bound can be obtained using the so-called Böröczky family of line arrangements [9]. Sylvester’s problem provides also a lot of geometrical constraints, for instance it implies that the famous dual-Hesse arrangement of 9 lines and 12 triple points (see for instance [1]) cannot be realized as a straight-line arrangement in the real projective plane since it does not contain any double point. It means also, via duality, that the Hesse arrangement of 12 lines, 12 double points, and 9 quadruple points cannot be realized as a straight-line arrangement in the real projective plane.

An analogon to Sylvester’s problem can be formulated for pseudoline arrangements in which not all pseudolines intersect in a common point.

**Problem 1.1.** Let \(L\) be a pseudoline arrangement with \(n \geq 3\) pseudolines not intersecting in a common point. Is it true that \(t_2 \geq 1\)?

Probably questions around Sylvester’s problem and the orchard problem for pseudolines were considered for the first time in the paper due to Burr, Grünbaum, and Sloane [6]. However, the authors did not know whether there exists a duality result for configurations of points and arrangements of pseudolines. In 1980 Goodman [11, Theorem 2] has shown the following result.

**Theorem 1.2.** (Duality [11]) If, in \(P^2_R\), \(L\) is an arrangement of pseudolines and \(P\) a configuration of points, and if \(I\) is the set of all true statements of the form “\(P \in P\) is incident to \(L \in L\)”, then there is a configuration \(L’\) of points and an arrangement \(P’\) of pseudolines, such that the set of all incidences holding between members of \(L’\) and members of \(P’\) is the dual \(I’\) of \(I\).

This result allows to establish some bounds in the context of the orchard problem [6].

On the other hand, let us point out here explicitly that Problem [1.1] has a positive answer. Indeed, using the same proof as in the case of Melchior’s inequality, one can show the following result. In our formulation by \(p_j\) we mean the number of regions bounded by precisely \(j\) sections of pseudolines.

**Theorem 1.3.** (Melchior [21]) Let \(L\) be an essential arrangement of \(n \geq 3\) pseudolines. Then

\[
\sum_{r \geq 2} (3-r)t_r = 3 + \sum_{j \geq 3} (j-3)p_j.
\]

**Corollary 1.4.** For an essential arrangement \(L\) of \(n \geq 3\) pseudolines one has

\[
t_2 \geq 3.
\]
Remark 1.5. The fact that every configuration of pseudolines has at least one double point follows from a result by Kelly and Rottenberg [19]. However, our aim was to provide an explicit inequality which is more adequate for our purposes.

This quite natural result has some significant geometrical consequences. Before we formulate some corollaries, it is worth pointing out that for arrangements of \(n\) pseudolines we have the same combinatorial equality as in the case of straight lines, namely

\[
\binom{n}{2} = \sum_{r \geq 2} \binom{r}{2} t_r.
\]

On the left hand side of the equation, we have the number of pairwise intersections, and on the right hand side, we have the sum over all \(r\)-points in the arrangement. Using this fact we can derive the following result.

**Corollary 1.6.** There does not exist an arrangement of \(n = 9\) pseudolines with 12 triple points.

**Proof.** Using the above combinatorial equality, observe that if \(n = 9\) and \(t_3 = 12\), then \(t_r = 0\) for \(r \neq 3\). In particular, \(t_2 = 0\). \(\square\)

Now by Theorem 1.2 one has the following.

**Corollary 1.7.** There does not exist an arrangement of 12 pseudolines intersecting at 9 quadruple points and 12 double points.

At last, let us recall the following question which was formulated by Grünbaum in his book [14].

Before we proceed further, let us also recall that by a geometric realization of a given abstract point-line configuration we mean a realization with straight lines, and by a topological realization we mean a realization with pseudolines.

**Problem 1.8.** ([14, Page 254, Problem 1]) Decide whether any abstract point-line configuration of 26 triple points and 13 lines can be realized geometrically or topologically.

Now we disprove partially the above problem of Grünbaum.

**Proposition 1.9.** Abstract point-line configurations of 26 triple points and 13 lines are not realizable with pseudolines.

**Proof.** Using the combinatorial equality for pseudoline arrangements, we see that for 13 pseudolines 26 triple points is the maximal possible number of intersection points, which means that \(t_r = 0\) for \(r \neq 3\). This fact, combined with Melchior’s inequality, completes the proof. \(\square\)

Using exactly the same argument, one can show that:

1. **Klein’s arrangement [20]** consisting of 21 lines and \(t_3 = 28, t_4 = 21\);
2. **Wiman’s arrangement [24]** consisting of 45 lines and \(t_3 = 120, t_4 = 45, t_5 = 36\);
3. **Fermat’s family of line arrangements** [25 Example II.6] for \(n \geq 4\) consisting of \(3n\) lines and \(t_n = 3, t_3 = n^2\);

cannot be constructed as pseudoline arrangements. Our considerations here lead to the following very interesting question which seems to be not formulated explicitly in the literature.
Problem 1.10. Let $\mathcal{L}$ be an arrangement of $n \geq 4$ lines in the complex projective plane $\mathbb{P}^2_\mathbb{C}$ such that $t_2 = 0$. Then $\mathcal{L}$ is isomorphic to either the dual Hesse, Klein’s, Wiman’s arrangement, or to one of Fermat’s arrangements of lines.

On the other hand, as the referee kindly pointed out to us, it might be interesting to ask whether the dual Hesse, Klein’s, Wiman’s arrangement, or to one of Fermat’s arrangements of lines, can be realized as line arrangements in the three dimensional real projective space – this is a natural intermediate step between the real projective plane and the complex projective plane.

2 The orchard problem for pseudoline arrangements

2.1 Böröczky’s arrangement of 12 lines

Let us here recall the main construction due to Böröczky which motivated our research – for more details please consult [9].

We start with a regular $n$-gon inscribed in a circle $O$. We denote vertices of this $n$-gon traced in clockwise order by $P_0, \ldots, P_{n-1}$. For simplicity, we assume here that $n$ is even – we are only interested in the case where $n$ is even and thus we omit the odd case. Then we construct the first line by joining $P_0$ with $P_{n/2}$. In the next step we join $P_{n/2-2}$ with $P_1$, and we continue this procedure until we obtain exactly $n$ lines – after this moment our construction repeats and it does not provide new distinct lines (since we consider indices modulo $n$). Of course, it may happen that $P_{n/2-2}$ and $P_i$ coincide – then we draw the tangent line at $P_i$ to $O$. We obtain the arrangement $\mathcal{B}_n$ which consists of $n$ lines, $n-3+\varepsilon$ double points, and $1 + \left\lfloor \frac{n(n-3)}{6} \right\rfloor$ triple points, where $\varepsilon$ is equal to 0 if $n = 0 \mod(3)$, or 2, otherwise.

![Figure 1: Böröczky’s arrangement of 12 lines.](image)

2.2 Sweeping argument

Now we explain how we determine pseudoline arrangements with 12 pseudolines in which 19 triples of them intersect in a point and in which the remaining pairs intersect pairwise. We can assume that one pseudoline $P_1$ is incident with 5 triple points since $3 \cdot t_3/n = 3 \cdot 19/12 > 4$. We use an additional
sweeping pseudoline $P$ that is at the beginning equal to pseudoline $P_1$ and that has otherwise all the time precisely one additional point on $P_1$ between the intersection of $P_{12}$ with $P_1$ and the intersection of $P_2$ with $P_1$. We sweep $P$ through the projective plane until it finally coincides with $P_1$ again. We move $P$ across all intersection points of two or three pseudolines. In other words, we imagine all possible crossings that can occur when we have an arrangement with 12 pseudolines in which 19 triples occur and the remaining crossings occur pairwise. This sweeping process has been used heavily in [4] and we refer the reader for details of this sweeping process to this article. On this sweeping pseudoline $P$, we consider only one position of $n$-geometric ($n$)-configurations in which long standing problems are still open. One variant of the sweeping method was decisive for investigations of the case $n = 19$, see [5].

Another approach is to work towards a solution of the long standing open problem to get a complete classification of all the problem size. However, the motivation for both investigations can be seen as an equal attempt to investigate the outcome. The sweeping process that was used in [4] was more involved because of the sweeping process that was used in [5] was more involved because of the problem size. However, the motivation for both investigations can be seen as an equal attempt to work towards a solution of the long standing open problem to get a complete classification of all geometric ($n_4$)-configurations.

In Section 3, we provide a complete list of arrangements with 12 pseudolines, 19 triple points, and 9 double points – this list was obtained using machine-computations based on Haskell code. For all those that are realizable with straight lines, we provide in Section 4 corresponding pictures. We also explain why some of them cannot be represented by lines.

### 2.3 Problems and Applications

The methods we have used to classify all arrangements with 12 pseudolines and 19 triple points are useful in the context of classifying ($n_4$)-configurations in which long standing problems are still open. One variant of the sweeping method was decisive for investigations of the case $n = 19$, see [5].

Apart from providing building blocks for this area, our investigated case is as well strongly related to the so-called containment problems in algebraic geometry that we are going to describe next. Here we focus on a special case of this problem which fits to our setting – for a general introduction please consult a very recent survey [23].

Let $\mathcal{P} = \{P_1, \ldots, P_s\}$ be a finite set of mutually distinct points in the projective plane and denote by $I(P_i)$ the associated radical ideal of $P_i$, that is the ideal of rational functions vanishing on $P_i$. We define the radical ideal $I = I(\mathcal{P})$ of $\mathcal{P}$ by

$$I = I(P_1) \cap \ldots \cap I(P_s).$$
The *m*-th symbolic power of $I$ is defined as

$$I^{(m)} = I^m(P_1) \cap \ldots \cap I^m(P_s),$$

which means that the *m*-th symbolic power of $I$ can be viewed as the set of forms vanishing along points $P_i$ with multiplicity $\geq m$. Another famous result due to Ein, Lazarsfeld, and Smith [8] in characteristic 0, and Hochster and Huneke [16] in positive characteristic, tells us that one has the following containment:

$$I^{(2k)} \subset I^k.$$

Few years later, Huneke asked whether one can improve the above containment.

**Problem 2.1.** ([17 Problem 0.4]) Does the containment

$$I^{(3)} \subset I^2$$

hold?

It turned out that in general [1] does not hold and most counter-examples are based on radical ideals of points which are given by intersection points of line arrangements. In particular, in [7] the very first counter-example to [1] in the real projective plane is provided and it is given by the radical ideal of triple points of Böröczky’s arrangement of 12 lines. In order to understand better the containment problem, one can formulate the following question.

**Question 2.2.** Using singular intersection points of line arrangements from Section 4, decide whether taking radical ideals of triple points one always has the containment

$$I^{(3)} \subset I^2.$$  

This question is of interest because the mentioned configurations have the same combinatorics, i.e., the number of lines and types of singular points, thus the main problem is whether the containment problem is combinatorial in nature. We want to verify whether the following is true: If $\mathcal{L}$ and $\mathcal{L}'$ are two line configurations with the same combinatorics (with the same number of lines $n$ and with the same combinatorial structure $(t_2, \ldots, t_n)$) and if the containment [1] does not hold for the radical ideal of a certain subset of singular points of $\mathcal{L}$, then [1] does not hold for the radical ideal of corresponding singular points of $\mathcal{L}'$.

### 3 Pseudoline arrangements

All arrangements with 12 pseudolines, 19 triple points, and 9 double points can be reconstructed via its triple points and this is one of the reasons why we provide only the triple lists. We can always start our sweeping algorithm with the line at infinity (we think of the circle model of the projective plane as in [4]) with the triples $(1, 2, 3), (1, 4, 5), (1, 6, 7), (1, 8, 9), (1, 10, 11)$, and we can insert successively all triples whereby one observes that no pair of pseudolines can meet twice. Finally, the list of 13 non-isomorphic arrangements has appeared during our computer-based tests. For instance, in Figure 2 we have drawn the pseudoline arrangement $\mathcal{C}_1$ that cannot be realized with straight lines. Here you see the line at infinity as the boundary of the circle with identified antipodal points. We provide also a short explanation after the enumeration of all 13 arrangements why certain pseudoline arrangements are not realizable with straight lines. Proceeding along the same lines one can handle the remaining cases.
$\mathcal{C}_1 = \{(1, 2, 3), (1, 4, 5), (1, 6, 7), (1, 8, 9), (1, 10, 11), (2, 4, 7), (2, 6, 9), (2, 8, 11), (2, 10, 12), (3, 4, 9), (3, 5, 7), (3, 6, 11), (3, 8, 12), (4, 6, 12), (4, 8, 10), (5, 6, 10), (5, 9, 11), (7, 9, 10), (7, 11, 1)\}$,

$\mathcal{C}_2 = \{(1, 2, 3), (1, 4, 5), (1, 6, 7), (1, 8, 9), (1, 10, 11), (2, 4, 7), (2, 6, 9), (2, 8, 11), (2, 10, 12), (3, 4, 9), (3, 5, 7), (3, 6, 11), (3, 8, 12), (4, 6, 12), (4, 8, 10), (5, 9, 11), (7, 9, 10), (7, 11, 12), (5, 6, 8)\}$,

$\mathcal{C}_3 = \{(1, 2, 3), (1, 4, 5), (1, 6, 7), (1, 8, 9), (1, 10, 11), (2, 4, 7), (2, 6, 9), (2, 8, 11), (2, 10, 12), (3, 5, 7), (3, 6, 11), (3, 8, 12), (4, 6, 12), (4, 8, 10), (4, 9, 11), (5, 9, 12), (7, 9, 10), (7, 11, 12), (5, 6, 8)\}$,

$\mathcal{C}_4 = \{(1, 2, 3), (1, 4, 5), (1, 6, 7), (1, 8, 9), (1, 10, 11), (2, 4, 7), (2, 6, 9), (2, 8, 11), (2, 10, 12), (3, 4, 10), (3, 5, 7), (3, 6, 12), (3, 9, 11), (4, 6, 11), (4, 8, 12), (5, 6, 8), (5, 9, 10), (7, 11, 12), (7, 8, 10)\}$,

$\mathcal{C}_5 = \{(1, 2, 3), (1, 4, 5), (1, 6, 7), (1, 8, 9), (1, 10, 11), (2, 4, 7), (2, 6, 9), (2, 8, 11), (2, 10, 12), (3, 4, 10), (3, 5, 7), (3, 6, 12), (3, 9, 11), (4, 6, 11), (4, 8, 12), (5, 8, 10), (5, 9, 12), (7, 9, 10), (7, 11, 12)\}$,

$\mathcal{C}_6 = \{(1, 2, 3), (1, 4, 5), (1, 6, 7), (1, 8, 9), (1, 10, 11), (2, 4, 7), (2, 6, 9), (2, 8, 11), (2, 10, 12), (3, 4, 9), (3, 5, 7), (3, 6, 11), (3, 8, 12), (4, 6, 8), (4, 11, 12), (5, 6, 10), (5, 9, 12), (7, 9, 11), (7, 8, 10)\}$,

Figure 2: The pseudoline arrangement $\mathcal{C}_1$. 
Let us now explain shortly why certain pseudoline arrangements are not realizable with straight lines. Recall that a projective base in the real projective plane is a set of four points in general position, i.e., no three
points are collinear. We start with $C_1$ and we use its corresponding tripels as points. When we pick a projective base of the points $(1, 2, 3), (1, 4, 5), (3, 5, 7),$ and $(2, 4, 7),$ we have determined six lines $P_1, P_2, P_3, P_4, P_5,$ and $P_7$ because two indices occur in former triples. Now we can construct, as intersections, the points $(1, 6, 7)$ and $(3, 4, 9).$ A movable point $(2, 6, 9)$ on line $P_2$ determines the points $(1, 8, 9), (5, 9, 11), (5, 6, 10), \text{ and } (3, 6, 11).$ This defines in turn point $(2, 8, 11)$ and line $P_8$. However, line $P_8$ passes through point $(5, 6, 10)$ because of a movable projective incidence theorem, which is an obstruction. It is a movable projective incidence theorem because the incidence of line $P_8$ and point $(5, 6, 10)$ remains when you move point $(2, 6, 9)$ on line $P_2$. You can check this with any dynamic geometric software or you provide an algebraic proof as follows. In our illustration in Figure 3, we have chosen the projective base (four red points) as vertices of a square with edges of length 2 parallel to the coordinate axes and with midpoint as the origin. Thus the coordinates of all points are very easy and showing the final collinearity becomes an easy exercise:

\[
\begin{align*}
(1, 2, 3) &= (-1, -1), \\
(1, 4, 5) &= (-1, 1), \\
(3, 5, 7) &= (1, 1), \\
(2, 4, 7) &= (1, -1), \\
(3, 4, 9) &= (0, 0), \\
(2, 6, 9) &= (v, v), v \neq 1, v \neq -1, v \neq 0, \\
(5, 9, 11) &= (-v, 1), \\
(5, 6, 10) &= (v, 1), \\
(3, 6, 11) &= (v, v), \\
(2, 8, 11) &= (-v \cdot (3 + v)/(v - 1), -1).
\end{align*}
\]

$P_{11}: y = (v - 1)x/2v + (v + 1)/2, \quad P_8: y = (v - 1)x/v(1 + v) + 2/(1 + v)$

We have not drawn $P_{10}$ since it does not play any role for finding the obstruction.

The second example $C_2$ can be realized with straight lines. Here a movable projective incidence theorem works in favor of a realization. Pick again a projective base of the points $(1, 2, 3), (1, 4, 5), (3, 5, 7),$ and $(2, 4, 7)$ in a dynamical drawing software like Cinderella. After picking a movable point $(2, 6, 9)$ on line $P_2$, we see a movable projective incidence theorem when we forget about the two points $(7, 9, 10)$ and $(4, 6, 12)$. A corresponding line arrangement can be seen in Figure 4.

Let us point out here that $C_6$ corresponds to Böröczky’s arrangement of 12 lines and $C_7$ corresponds to the arrangement depicted in Figure 5. As we can see, the arrangement $C_7$ has 3 axes of symmetry.

4 New non-isomorphic line arrangements

![Figure 4: A geometric realization of the arrangement $C_2$.](image)
Figure 5: A geometric realization of the arrangement $C_7$.

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All figures in the paper are made with GeoGebra.

References

[1] M. Artebani, and I. Dolgachev, The Hesse pencil of plane cubic curves, *Enseign. Math. (2)* 55, No. 3-4 (2009) 235 – 273.
[2] A. Björner, M. Las Vergnas, B. Sturmfels, B. White, and G. M. Ziegler, Oriented Matroids, Vol. 46 of Encyclopedia Math. Appl., Cambridge, Cambridge University Press. Second edn. (1999).
[3] J. Bokowski, Computational Oriented Matroids, *Cambridge University Press* (2006).
[4] J. Bokowski and V. Pilaud, Enumerating topological $(n_k)$-configurations, *Comput. Geom.* 47 (2014) 175 – 186.
[5] J. Bokowski and V. Pilaud, Quasi-Configurations: Building blocks for point-line configurations, *ARS Mathematica* *Contemporanea* 10 (2016) 99 – 112.
[6] S. A. Burr, B. Grünbaum, and N. J. A. Sloane, The orchard problem, *Geometriae Dedicata* 2 (1974) 397 – 424.
[7] A. Czapliński, A. Główka, G. Malara, M. Lampa-Baczyńska, P. Łuszcz-Świdecka, P. Pokora, and J. Szpond, A counterexample to the containment $I^{(3)} \subset I^2$ over the reals, *Adv. Geom.* 16(1) (2016) 77 – 82.
[8] L. Ein, R. Lazarsfeld, and K. E. Smith, Uniform bounds and symbolic powers on smooth varieties, *Invent. Math.* 144 (2001) 241 – 252.
[9] Z. Füredi and I. Palásti, Arrangements of lines with a large number of triangles, *Proc. A. M. S.*, 92 (4) (1984) 561 – 566.
[10] T. Gallai, Solution to problem number 4065, *American Math. Monthly* 51 (1944) 169 – 171.
[11] J. E. Goodman, Proof of a conjecture of Burr, Grünbaum, and Sloane, *Discrete Mathematics* 32 (1980) 27 – 35.
[12] B. Green and T. Tao, On sets defining few ordinary lines, *Discrete Comput Geom* 50 (2013) 409 – 468.
[13] B. Grünbaum, Arrangements and Spreads, Regional Conf., Vol. 10, Amer. Math. Soc., Providence, RI (1972).
[14] B. Grünbaum, Configurations of Points and Lines, Graduate Studies in Mathematics, vol. 103, American Mathematical Society, Providence, RI (2009).
[15] Haskell Programming Language. [https://www.haskell.org](https://www.haskell.org)
[16] M. Hochster and C. Huneke, Comparison of symbolic and ordinary powers of ideals, Invent. Math. 147 (2002) 349 – 369.
[17] C. Huneke, Open problems on powers of ideals. [http://www.aimath.org/WWW/integralclosure/Huneke.pdf](http://www.aimath.org/WWW/integralclosure/Huneke.pdf)
[18] J. Jackson, Rational Amusement for Winter Evenings, Longman, Hurst, Rees, Orme and Brown, London (1821).
[19] L. M. Kelly and R. Rottenberg, Simple points in pseudoline arrangements. Pacific J. Math. 40(3) (1972) 617 – 622.
[20] F. Klein, Über die Transformation siebenter Ordnung der elliptischen Functionen, Math. Annalen 14 (1879) 428 – 471.
[21] E. Melchior, Über Vielseite der Projektive Ebene, Deutsche Mathematik 5 (1941) 461 – 475.
[22] J. Richter-Gebert and U. Kortenkamp, Cinderella Software. [http://cinderella.de/tiki-index.php](http://cinderella.de/tiki-index.php)
[23] T. Szemberg, and J. Szpond, On the containment problem. To appear in Rend. Circ. Mat. Palermo, [DOI:10.1007/s12215-016-0281-7](https://doi.org/10.1007/s12215-016-0281-7)
[24] J. Sylvester, Mathematical question 11851, Educational Times (1893).
[25] G. Urzúa, Arrangements of Curves on Algebraic Surfaces, University of Michigan, (2008) 166 pp.
[26] A. Wiman, Zur Theorie der endlichen Gruppen von birationalen Transformationen in der Ebene, Math. Annalen 48 (1896) 195 – 240.

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