Locations of $L_{4,5}$ of a dust grain type II comet tail in Solar-Jupiter system in the photogravitational relativistic R3BP

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Abstract

A calculating formula when $q\in(-\infty, 1]$ for the locations of the triangular points $L_{4,5}$ in the relativistic restricted three-body problem is proposed when the bigger primary is a source of radiation with a mass reduction factor $q$. As an application, we consider a dust grain in type II comet tail in Solar-Jupiter system with a mass reduction factor $q\in(-1.2, 0.5)$. An investigation in this neighborhood reveals that the shifts due to the relativistic terms on the locations of a dust grain particle are very small with the shifts in $\xi$—coordinate smaller than those of $\eta$—coordinate. It is further observed that increasing negative values of $q$ results in moving $L_{4,5}$ further away from the line joining the primaries and towards the smaller primary. Increasing positive values of $q$, however results in a shift of $L_{4,5}$ towards both the line joining both primaries and towards the smaller primary.

1. Introduction

The restricted three-body problem (R3BP) captures the dynamics of a body of negligible mass (a test particle or an infinitesimal body) under the gravitational influence of two massive body bodies (the primaries) moving in circular orbits about their common barycenter. The infinitesimal body does not influence the motion of the primary bodies. Five equilibrium points exist in this restricted problem of three bodies denoted by $L_i(i=1, \ldots, 5)$ which $L_1, L_2$ & $L_3$ are called collinear points while $L_4$ & $L_5$ are called triangular points. The R3BP becomes relativistic R3BP when relativistic corrections to the gravitational field of the primaries is taken into consideration [1–5]. Studied more details of this problem and got most of the importance results on relativistic celestial mechanics [6–8]. All studied the existence and linear stability of $L_{4,5}$ on this problem and they all came up with different results on the region of stability.

The motion of a test particle under the influence of both gravitational and radiation repulsive forces referred to as the photogravitational R3BP has been subject of several researches [9, 10]. Considered radiation pressure in their communications to be the only significant radiation force. Following this consideration, it is stated in [11] that the difference between the gravitational force $F_g$ and the force due to radiation pressure on the small body denoted by $F_p$ is:

$$F_g - F_p = \left(1 - \frac{F_p}{F_g}\right)(1 - \beta)F_g = qF_g,$$

where $\beta = \frac{F_p}{F_g}$, $q = 1 - \beta$.

Clearly:
- If $q = 1$, the radiation pressure has no effect.
- If $0 < q < 1$, the gravitational force is greater than the force due to radiation pressure;
- If $q = 0$, the force due to radiation pressure balances with the gravitational force;
- If $q = 0$, the force due to radiation pressure overrides the gravitational force.

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Following this discovery, several studies [12–20] and references therein under different assumptions and characterizations by taking radiation pressure into account have produced vital results [21]. Studied the existence and linear stability of the triangular points in the restricted three-body with numerical applications. He proved that the positions and linear stability of the triangular points are dependent on the first and second even zonal harmonic (J₂ & J₄) of the bigger primary. Furthermore, his investigation reveals that the size of the range of stability is sometimes not affected by the presence of J₄ for some planets systems like Earth-moon, Saturn-Phoebe and Uranus-Caliban systems.

The relativistic R3BP with or without perturbing forces have been the subject of many researchers. Among them [22], obtained the approximate locations of the triangular points while [23] obtained new perturbed locations of the triangular points within the frame work of the post-Newtonian approximation when the primaries are oblate and sources of radiation, in addition [24] examined the stability of I₄₅ in the presence of a luminous more massive primary among others.

All the above studies are restricted to \( q \in (0, 1) \) and they did not discuss the case when the radiation pressure is negative. However, the studies of coplanar libration points in [11, 25, 26], in the photogravitational restricted three-body have shown that the radiation factor \( q \) could be in \((-\infty, 1]\) [26]. Have shown that for the Sun, the mass reduction factor is \( q_{0} = 1 - 5.7396 \times 10^{-5} \frac{a}{\rho} \), where \( a \) and \( \rho \) are radius and density of moving body and \( \kappa \) is the radiation pressure efficiency factor of the Sun. They have stated that for a dust grain particle, since \( a \) is a very small, it is possible that the value of \( q \) to be taken in the range \((-\infty, 1] \). They also gave as an example that the II Kind of comet tail consists of dust grain with the radii being microns, and in the present case \(-1.2 < q < 0.5 \).

The streams of dust and gas each form their own distinct tail, pointing in slightly different directions. The tail of dust is left behind in the comet’s orbit in a such manner that it often forms a curved tail called type II or dust tail. From our knowledge and from the above literature, there is no study that has been carried out when \( q \) is negative in the planar restricted three-body problem. On this premise, we present a calculating formula for the locations of the triangular solutions for \( q \in (-\infty, 1] \) and consider as an application a dust grain in II kind of comet tail in the Solar-Jupiter system.

This paper is organized as follows: in section 2, the equations governing the motion are presented; section 3 describes the perturbed locations and the new calculating formula is given; in section 4, the numerical results are given; while in sections 5 and 6 the results are analysed and conclusions highlighted respectively.

### 2. Equations of motion

The pertinent equations of motion of an infinitesimal mass in the relativistic R3BP when the bigger primary is radiating with radiation factor \( q \) in a barycentric synodic coordinate system \((\xi, \eta)\) and dimensionless variables can be written as [4, 6, 24] as follows:

\[
\begin{align*}
\ddot{\xi} - 2n\dot{\eta} &= \frac{\partial W}{\partial \xi} - \frac{d}{dt}\left(\frac{\partial W}{\partial \xi}\right), \\
\ddot{\eta} + 2n\dot{\xi} &= \frac{\partial W}{\partial \eta} - \frac{d}{dt}\left(\frac{\partial W}{\partial \eta}\right),
\end{align*}
\]

where \( W \) is the force function defined by:

\[
\begin{align*}
W &= \frac{1}{2}(\xi^2 + \eta^2) + \frac{q(1 - \mu)}{\rho_1} + \frac{\mu}{\rho_2} + \frac{1}{c^2}\left[-\frac{3}{2}(1 - \frac{1}{3}(1 - \mu))(\xi^2 + \eta^2)
\right. \\
&\quad + \frac{1}{8}(\xi^2 + \eta^2 + 2(\xi \dot{\eta} - \eta \dot{\xi}) + (\xi^2 + \eta^2))^2 \\
&\quad + \left.\frac{3}{2}\left(\frac{q(1 - \mu)}{\rho_1} + \frac{\mu}{\rho_2}\right)(\xi^2 + \eta^2 + 2(\xi \dot{\eta} - \eta \dot{\xi}) + (\xi^2 + \eta^2)) - \frac{1}{2}\left(\frac{q^2(1 - \mu)^2}{\rho_1} + \frac{\mu^2}{\rho_2} + q\mu(1 - \mu)\right)\frac{(4\eta + 7\xi)(\xi^2 + \eta^2)(\xi \dot{\eta} - \eta \dot{\xi})}{\rho_1^2} - \frac{1}{2}\left(\frac{q(1 - \mu)}{\rho_1} + \frac{\mu}{\rho_2}\right)\frac{1}{\rho_1}\frac{1}{\rho_2}
\right)
\end{align*}
\]

with the dimensionless mean motion \( n \) given as

\[
n^2 = 1 - \frac{3}{2c^2}(1 - \frac{1}{3}(1 - \mu)).
\]
\( \rho_1^2 = (\xi + \mu)^2 + \eta^2, \rho_2^2 = (\xi + \mu - 1)^2 + \eta^2, \) \hspace{1cm} (4)

Where \( \mu \left( 0 < \mu \leq \frac{1}{2} \right) \) is the ratio of the mass of the smaller primary to the total mass of the primaries; \( \rho_1 \) and \( \rho_2 \) are distances of the infinitesimal mass from the bigger and smaller primary respectively and \( c \) is the speed of light.

3. Locations of the triangular points

The equilibrium solutions of the relativistic R3BP are singularities of the manifold of components of velocity and the coordinates. The infinitesimal mass can be at rest in a rotating coordinate frame at these equilibrium points where the gravitational and centrifugal forces balance each other. They are thus the stationary solutions of the relativistic R3BP and are obtained from equation (1) after setting \( \xi = \eta = \xi_1 = \eta_1 = 0 \).

This implies that, these points are the solutions of the equations

\[ \frac{\partial \mathbf{v}}{\partial \eta} = 0 \text{ with } \xi = \eta = 0. \]

That is

\[ \xi = \frac{q(1 - \mu)(\xi + \mu)}{\rho_1^3} - \frac{\mu(\xi - 1 + \mu)}{\rho_2^3} + \frac{1}{c^2} \left[ -3\xi \left( 1 - \frac{1}{3} \mu(1 - \mu) \right) + \frac{1}{2} \xi(\xi^2 + \eta^2) \right] \]

\[ - \frac{3}{2} (\xi^2 + \eta^2) \left( \frac{q(1 - \mu)(\xi + \mu)}{\rho_1^3} + \frac{\mu(\xi - 1 + \mu)}{\rho_2^3} \right) \]

\[ + 3 \left( \frac{q(1 - \mu)}{\rho_1} + \frac{\mu}{\rho_2} \right) \xi + \frac{q^2(1 - \mu)^2(\xi + \mu)}{\rho_1^4} + \frac{\mu^2(\xi - 1 + \mu)}{\rho_2^4} \]

\[ + q\mu(1 - \mu) \left( \frac{7}{2} \left( \frac{1}{\rho_1} - \frac{1}{\rho_2} \right) + \frac{7}{2} \left( \frac{\xi + \mu}{\rho_1^3} + \frac{\xi - 1 + \mu}{\rho_2^3} \right) \right) \]

\[ + \frac{3}{2} \eta^2 \left( \frac{\mu(\xi + \mu)}{\rho_1^3} + \frac{q(1 - \mu)(\xi - 1 + \mu)}{\rho_2^3} \right) + \frac{\xi + \mu}{\rho_1^4} + \frac{\xi - 1 + \mu}{\rho_2^4} \]

\[ - \frac{(\mu - 2q(1 - \mu))(\xi + \mu)}{2\rho_1^3} - \frac{(q(1 - \mu) - 2\mu)(\xi - 1 + \mu)}{2\rho_2^3} \]

\[ = 0 \] \hspace{1cm} (5)

and

\[ \eta F = 0 \]

with

\[ F = \left( 1 - \frac{q(1 - \mu)}{\rho_1} - \frac{\mu}{\rho_2} \right) + \frac{1}{c^2} \left[ -3\left( 1 - \frac{1}{3} \mu(1 - \mu) \right) + \frac{1}{2}(\xi^2 + \eta^2) + \frac{3}{2} \left( \frac{q(1 - \mu)}{\rho_1} + \frac{\mu}{\rho_2} \right) \right] \]

\[ - \frac{3}{2} (\xi^2 + \eta^2) \left( \frac{q(1 - \mu)}{\rho_1^3} + \frac{\mu}{\rho_2^3} \right) + \frac{q^2(1 - \mu)^2}{\rho_1^4} + \frac{\mu^2}{\rho_2^4} \]

\[ + q\mu(1 - \mu) \left( \frac{7}{2} \left( \frac{1}{\rho_1} - \frac{1}{\rho_2} \right) + \frac{7}{2} \left( \frac{\xi + \mu}{\rho_1^3} + \frac{\xi - 1 + \mu}{\rho_2^3} \right) \right) \]

\[ + \frac{3}{2} \eta^2 \left( \frac{\mu(\xi + \mu)}{\rho_1^3} + \frac{q(1 - \mu)(\xi - 1 + \mu)}{\rho_2^3} \right) + \frac{\mu}{\rho_1^4} + \frac{\mu - 2q(1 - \mu)}{2\rho_1^3} \]

\[ - \frac{(q(1 - \mu) - 2\mu)}{2\rho_1^3} \]

The triangular points are the solutions of the equations (5) and (6) with \( \eta = 0 \) and the coordinate of the triangular points \((\xi, \pm \eta)\) denoted by \( L_4 \) and \( L_5 \) respectively are obtained in [24] as:

\[ \xi = -\frac{(q^2 - q) + (1 - q - 3q^2)\mu + (2 + 2q + 2q^2)\mu^2}{2(q - q^2 + (1 + q + q^2)\mu)} \]

\[ \frac{(-20 + 44q - 38q^2 - q^3)\mu + (66q^2 + 2q^3 - 66q + 28)\mu^2 + (-28q^2 + 33q - q^3 - 4)\mu^3}{8(q - q^2 + (1 + q + q^2)\mu)c^2} \] \hspace{1cm} (7)
4. Numerical results

In this section, the effect of the mass reduction factor \( q \) and relativistic terms on \( L_{4,5} \) of the locations of triangular points of a dust grain in type II comet tail in Solar-Jupiter system has been conducted numerically using equations (7) and (8). The necessary data are obtained from [7, 26]. The numerical results are shown in Table 1. It should be noted that the first entries in Table 1 correspond to the classical restricted three-body with radiation pressure of the bigger primary while the second entries correspond to relativistic restricted three-body problem with radiation.

### Table 1. Locations of a dust grain in Solar-Jupiter system with \(-1.2 < q < 0.5, \mu = 0.000 953 692 200 \& c = 22 947.35\).

| \( q \) | \( \xi \) | \( \Delta \xi = |\xi_R - \xi_C| \) | \( \pm \eta \) | \( \Delta \eta = |\eta_R - \eta_C| \) |
|---|---|---|---|---|
| 0 | -0.5009536922 | 4.72127082×10^{-9} | -402.9159470 | 1.917352114×10^{-6} |
| 0.1 | 0.4896202569 | 3.967931539×10^{-11} | -2.55998128 | 1.692089313×10^{-8} |
| 0.2 | 0.4943128325 | 1.785366299×10^{-11} | -0.6635547788 | 7.845819772×10^{-9} |
| 0.3 | 0.4958872753 | 1.096284175×10^{-11} | -0.02741401828 | 4.75285393×10^{-9} |
| 0.4 | 0.4966767661 | 7.998768314×10^{-12} | 0.2914112457 | 3.165970264×10^{-9} |
| -0.1 | 0.5086590710 | 5.133271586×10^{-11} | 5.135076944 | 1.9455864803×10^{-8} |
| -0.2 | 0.5038037389 | 2.864897208×10^{-11} | 3.183713358 | 1.012292916×10^{-8} |
| -0.3 | 0.5022134349 | 2.125288334×10^{-11} | 2.537399723 | 6.981899059×10^{-9} |
| -0.4 | 0.5014336282 | 1.763023061×10^{-11} | 2.215013792 | 5.378470558×10^{-9} |
| -0.5 | 0.5009555130 | 1.550015671×10^{-11} | 2.021828222 | 4.388330588×10^{-9} |
| -0.6 | 0.5006369958 | 1.410738193×10^{-11} | 1.893140181 | 3.704351936×10^{-9} |
| -0.7 | 0.5004095884 | 1.313049669×10^{-11} | 1.801270229 | 3.195146592×10^{-9} |
| -0.8 | 0.5002390866 | 1.240973999×10^{-11} | 1.732395133 | 2.795099263×10^{-9} |
| -0.9 | 0.5001065039 | 1.185740395×10^{-11} | 1.678841825 | 2.467785754×10^{-9} |
| -1.0 | 0.5000004550 | 1.142097528×10^{-11} | 1.636009388 | 2.191368666×10^{-9} |
| -1.1 | 0.4999136982 | 1.10676468×10^{-11} | 1.600971416 | 1.951950823×10^{-9} |
It should be noted here that \((\xi_C, \eta_C)\) correspond to coordinate in the classical restricted three-body with radiation while \((\xi_C, \eta_R)\) correspond to relativistic restricted three-body with radiation. \(\Delta \xi\) and \(\Delta \eta\) correspond to shift in \(\xi\) and in \(\eta\) respectively. Thus, for indicated values of \(q\) in figures 1 and 2, we show different positions of \(L_{4,5}\) indicating the shifts.

5. Discussion

It can be seen from table 1 that the shifts in \(\xi\) and \(\eta\) are very small. This indicates that the relativistic terms with radiation have little effects on the locations of a dust grain particle. However it is observed that the shifts in \(\xi\) coordinate are smaller than those of \(\eta\) coordinate. It is also observed from figure 1 that for increasing values of negative radiation, the triangular points \(L_{4,5}\) are moving away from the line joining the primaries and moving towards the smaller primary while for increasing values of positive radiation, the points \(L_{4,5}\) are moving towards both the line joining the primaries and the smaller primary.

6. Conclusion

By considering the bigger primary as a source of radiation, we have produced a calculating formula for the locations of triangular points when a mass reduction factor \(q \in (-\infty, 1]\). The numerical results on a dust grain in type II comet tail have shown that for increase values in negative \(q\), the triangular points move away from the line joining the primaries while for increase values of positive \(q\), the triangular points move towards both the line joining the primaries and the smaller primary.
The stability of the triangular points with locations as presented in equations (7) and (8) is an open problem in celestial mechanics.

**Data availability statement**

The data that support the findings of this study are available upon reasonable request from the authors.

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