Graphs, free groups and the Hanna Neumann conjecture

Brent Everitt*

Abstract. A new bound for the rank of the intersection of finitely generated subgroups of a free group is given, formulated in topological terms, and very much in the spirit of Stallings [19]. The bound is a contribution to the strengthened Hanna Neumann conjecture.

Introduction

This paper is about the interplay between graphs and free groups, with particular application to subgroups of free groups. This subject has a long history, where one approach is to treat graphs as purely combinatorial objects (as in for instance [11,20,21]), while another (for example [16]), is to treat them topologically by working in the category of 1-dimensional CW complexes.

We prefer a middle way, where to quote Stallings [19] who initiated it, graphs are “something purely combinatorial or algebraic”, but also one may apply to them topological machinery, motivated by their geometrical realizations. We use this to give a new bound for the rank of the intersection of two finitely generated subgroups of a free group (Theorems 1 and 2), and to formulate graph theoretic versions of some other classical results. The first section sets up the combinatorial-topological background: §2 studies graphs of finite rank; the topological meat of the paper is §3 and the group theoretic consequences explored in §4.

1. Preliminaries from the topology of graphs

A combinatorial 1-complex or graph [6, §1.1] (see also [2,3,17,19]) is a set \( \Gamma \) with involutory \(^{-1} : \Gamma \to \Gamma \) and idempotent \( s : \Gamma \to \Gamma \), (ie: \( s^2 = s \)) maps, where \( V_\Gamma \) is the set of fixed points of \(^{-1} \). Thus a graph has vertices \( V_\Gamma \) and edges \( E_\Gamma := \Gamma \setminus V_\Gamma \) with (i). \( s(v) = v \) for all \( v \in V_\Gamma \); (ii). \( v^{-1} = v \) for all \( v \in V_\Gamma \), \( e^{-1} \in E_\Gamma \) and \( e^{-1} \neq e = (e^{-1})^{-1} \) for all \( e \in E_\Gamma \). The edge \( e \) has start vertex \( s(e) \) and terminal vertex \( t(e) := s(e^{-1}) \); an arc is an edge/inverse edge pair; a pointed graph is a pair \( \Gamma_v := (\Gamma, v) \) for \( v \in \Gamma \) a vertex.

A map of graphs is a set map \( f : \Gamma \to \Lambda \) with \( f(V_\Gamma) \subseteq V_\Lambda \) that commutes with \( s \) and \(^{-1} \), and preserves dimension if \( f(E_\Gamma) \subseteq E_\Lambda \). An isomorphism is a dimension preserving map, bijective on the vertices and edges. A map \( f : \Gamma_v \to \Lambda_u \) of pointed graphs is a graph map \( f : \Gamma \to \Lambda \) with \( f(v) = u \).

A graph \( \Gamma \) has a functorial geometric realization as a 1-dimensional CW complex \( B\Gamma \) (see, eg: [6 §1.3]) with a graph map \( f : \Gamma \to \Lambda \) inducing a regular cellular map \( Bf : B\Gamma \to B\Lambda \) of CW complexes, in the sense of [13 §4]. Thus, one may transfer to graphs and their maps topological notions and adjectives (connected, fundamental group, homology, covering map, etc...) from their geometrical realizations.

If \( \Lambda \hookrightarrow \Gamma \) is a subgraph, we will write \( \Gamma/\Lambda \) for the resulting quotient graph and quotient map \( q : \Gamma \to \Gamma/\Lambda \). For a set \( \Lambda_i \hookrightarrow \Gamma, (i \in I) \), of mutually disjoint subgraphs, we will write \( \Gamma/\Lambda_i \) for the graph resulting from taking successive quotients by the \( \Lambda_i \). The coboundary \( \delta \Lambda \) of

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a subgraph consists of those edges \( e \in \Gamma \) with \( s(e) \in \Lambda \) and \( t(e) \notin \Lambda \); equivalently, it is those edges \( e \in \Gamma \) with \( sq(e) \) the vertex \( q(\Lambda) \) in the quotient graph \( q : \Gamma \to \Gamma/\Lambda \). The real line graph \( \mathbb{R} \) has vertices \( V_\mathbb{R} = \{v_k\}_{k \in \mathbb{Z}} \) and edges \( E_\mathbb{R} = \{e_k\}_{k \in \mathbb{Z}} \) with \( s(e_k) = v_k, s(e_k^{-1}) = v_{k+1} \).

We have the obvious notion of path and in particular, a spur is a path that successively traverses both edges of an arc, and a path is reduced when it contains no spurs. A tree is a simply connected graph and a forest a graph, all of whose connected components are trees. Any connected graph has a spanning tree \( T \to \Gamma \) with the homology \( H_1(T) \) free abelian on the set of arcs of \( \Gamma \) omitted by \( T \), and the rank \( rk \) of \( \Gamma \) (connected) defined to be \( rk_Z H_1(\Gamma) \). If \( \Gamma \) has finite rank then \( rk \Gamma - 1 = -\chi(\Gamma) \), and if \( \Gamma \) is finite, locally finite, connected, then \( 2(rk \Gamma - 1) = |E_\Gamma| - 2|V_\Gamma| \). If \( \Gamma \) is connected and \( T_i \to \Gamma \) a set of mutually disjoint trees, then the fundamental group is unaffected by their excision: \( \pi_1(\Gamma, v) \cong \pi_1(\Gamma/T_i, q(v)) \) and so \( rk \Gamma = rk \Gamma/T_i \).

If \( \Lambda \) is a connected graph and \( v \) a vertex, then the spine \( \Lambda_v \) of \( \Lambda \) at \( v \), is defined to be the union of \( \Lambda \) of all closed reduced paths starting at \( v \). It is easy to show that \( \Lambda_v \) is connected with \( rk \Lambda_v = rk \Lambda \), that every closed reduced path starting at \( u \in \Lambda_v \) is contained in \( \Lambda_v \), and an isomorphism \( \Lambda_u \to \Lambda_v \) restricts to an isomorphism \( \Lambda_u \to \Lambda_v \) (so that spines are invariants of graphs).

If \( \Lambda_1, \Lambda_2 \) and \( \Delta \) are graphs and \( f_i : \Lambda_i \to \Delta \) maps of graphs, then the pullback \( A_1 \coprod_\Delta A_2 \) has vertices (resp. edges) the \( x_1 \times x_2, x_1 \in V_{\Lambda_1} \) (resp. \( x_1 \in E_{\Lambda_1} \)) such that \( f_1(x_1) = f_2(x_2) \), and \( s(x_1 \times x_2) = s(x_1) \times s(x_2), (x_1 \times x_2)^{-1} = x_1^{-1} \times x_2^{-1} \) (see [19], page 552). Taking \( \Delta \) to be the trivial graph gives the product \( A_1 \coprod A_2 \).

Define maps \( t_i : A_1 \coprod A_2 \to A_i \) to be the compositions \( A_1 \coprod A_2 \to \Lambda_i \) with the second map the projection \( x_1 \times x_2 \to x_1 \). Then the \( t_i \) are dimension preserving maps making the diagram commute, and the pullback is universal with this property.

In general the pullback need not be connected, but if the \( f_i : \Lambda_i \to \Delta \) are pointed maps then the pointed pullback \( (A_1 \coprod A_2)_{u_1 \times u_2} \) is the connected component of the pullback containing the vertex \( u_1 \times u_2 \) (and we then have a pointed version of the diagram above).

There is a “co”-construction, the pushout, for dimension preserving maps of graphs, \( f_i : \Delta \to A_i \), although it will play a lesser role for us (see [19] page 552). The principal example for us is the wedge sum \( A_1 \vee_\Delta A_2 \).

Graph coverings \( f : \Lambda \to \Delta \) can be characterized combinatorially as dimension preserving maps such that for every vertex \( \gamma \in \Lambda \), \( f \) is a bijection from the set of edges in \( \Lambda \) with start vertex \( v \) to the set of edges in \( \Delta \) with start vertex \( f(v) \). Graph coverings have the usual path and homotopy lifting properties [19, §4], and from now on, all coverings will be maps between connected complexes unless stated otherwise, and we will write \( \deg(\Lambda \to \Delta) \) for the degree of the covering. A covering is Galois if for all closed paths \( \gamma \) at \( v \), the lifts of \( \gamma \) to each vertex of the fiber of \( v \) are either all closed or all non-closed.

**Proposition 1.** Let \( \Lambda \) be a graph and \( \Upsilon_1, \Upsilon_2 \to \Lambda \) subgraphs of the form,

\[
\Lambda = \begin{array}{c}
\Upsilon_1 \\
\circ \\
\Upsilon_2
\end{array} \quad \xymatrix{\Lambda \ar[r]_{\xi} & \Upsilon_1 \ar[r] & \Upsilon_2}
\]

(i). If \( f : \Lambda \to \Delta \) is a covering with \( \Delta \) single vertexed, then the real line is a subgraph \( g : \mathbb{R} \to \Lambda \), with \( g(e_0) = e \) and \( f(g(e_k)) = f(e) \) for all \( k \in \mathbb{Z} \).
(ii). If \( \Upsilon_1 \) is a tree, \( \Lambda \to \Delta \) and \( \Gamma \to \Delta \) coverings, and \( \Upsilon_2 \to \Gamma \) a subgraph, then there is an intermediate covering \( \Lambda \to \Gamma \to \Delta \).
(iii). If \( \Upsilon_1 \) is a tree, and \( \Psi \to \Lambda \) a covering, then \( \Psi \) has the same form as \( \Lambda \) for some subgraphs \( \Upsilon_1', \Upsilon_2' \to \Psi \) and with \( \Upsilon_1' \) a tree.

**Proof.** These are easy exercises using path lifting. For (i), build \( \mathbb{R} \to \Lambda \) by taking successive lifts of the edge \( f(e) \in \Delta \). For (ii), it suffices to find a map \( \Lambda \to \Gamma \) commuting with the two coverings given. Let it coincide with \( \Upsilon_2 \to \Gamma \) on \( \Upsilon_2 \), and on \( \Upsilon_1 \), project to \( \Delta \) and then lift to \( \Gamma \). For (iii), take \( \Upsilon_1' \) to be the union of lifts of reduced paths from \( t(e) \) to the vertices of \( \Upsilon_1 \). \( \Box \)
If $f : A \rightarrow \Delta$ is a covering and $T \hookrightarrow \Delta$ a tree, then path and homotopy lifting give that $f^{-1}(T)$ is a forest such that if $T_i \hookrightarrow A$ ($i \in I$) are the component trees, then $f$ maps each $T_i$ isomorphically onto $T$. There is then an induced covering $f' : A/T_i \rightarrow \Delta/T$, defined by $f'q = qf$ where $q, q'$ are the quotient maps, and such that $\deg(A/T_i \rightarrow \Delta/T) = \deg(A \rightarrow \Delta)$.

If $f : A_u \rightarrow \Delta_v$ is a covering then intermediate coverings $A_u \rightarrow \Gamma_x \rightarrow \Delta_v$ and $A_u \rightarrow \mathcal{T}_y \rightarrow \Delta_v$ are equivalent if and only if there is an isomorphism $\Gamma_x \rightarrow \mathcal{T}_y$ making the obvious diagram commute. Then the set $\mathcal{L}(\Lambda_u, \Delta_v)$ of equivalence classes of intermediate coverings is a lattice with join $\Gamma_{x_1} \vee \Gamma_{x_2}$ the pullback $(\Gamma \prod_{\Delta} \mathcal{T})_{x_1 \times x_2}$, meet $\Gamma_{x_1} \wedge \Gamma_{x_2}$ the pushout $(\Gamma \prod_{\Delta} \mathcal{T})_{g(x_1)}$ ($g$ the covering $A_u \rightarrow \Gamma_{x_1}$, a $0 = \Delta_v$ and a $\tilde{1} = \Lambda_u$. The incessant pointing of covers is annoying, but essential if one wishes to work with connected intermediate coverings and also have a lattice structure (both of which we do). The problem is the pullback: because it is not in general connected, we need the pointing to tell us which component to choose.

We end the preliminaries by observing that the excision of trees has little effect on the lattice $\mathcal{L}(\Lambda, \Delta)$. Let $f : A_u \rightarrow \Delta_v$ be a covering, $T \hookrightarrow \Delta$ a spanning tree, $T_i \hookrightarrow A$ the components of $f^{-1}(T)$, and $f : (A/T_i)_{q(v)} \rightarrow (\Delta/T)_{q(v)}$ the induced covering (where we have (abused $q$ for both quotients and $f$ for both coverings). One can then show (either by brute force, or using the Galois correspondence between $\mathcal{L}(\Lambda, \Delta)$ and the subgroup lattice of the group $\text{Gal}(\Lambda, \Delta)$ of covering transformations), that there is a degree and rank preserving isomorphism of lattices $\mathcal{L}(\Lambda, \Delta) \rightarrow \mathcal{L}(\Lambda/T_i, \Delta/T)$, that sends Galois coverings to Galois coverings, and the equivalence class of $A_u \rightarrow \Gamma_x$ to $\Lambda_u \rightarrow \Gamma/T_i \rightarrow \Delta/T$, with $T_i \hookrightarrow \Gamma$ the components of $r^{-1}(T)$. We will call this process lattice excision.

2. Graphs of finite rank

This section is devoted to a more detailed study of the coverings $\Lambda \rightarrow \Delta$ where $\text{rk} \Lambda < \infty$.

**Proposition 2.** Let $\Lambda$ be a connected graph, $\Gamma \hookrightarrow \Lambda$ a connected subgraph and $v \in \Gamma$ a vertex such that every closed reduced path at $v$ in $\Lambda$ is contained in $\Gamma$. Then $\Lambda$ is a wedge sum decomposition $\Lambda = \Gamma \sqcup \Phi$ with $\Phi$ a forest and no two vertices of the image of $\Theta \hookrightarrow \Phi$ lying in the same component.

**Proof.** Consider an edge $e$ of $\Lambda \setminus \Gamma$ having at least one of its end vertices $s(e)$ or $t(e)$, in $\Gamma$. For definiteness we can assume, by relabeling the edges in the arc containing $e$, that it is $s(e)$ that is a vertex of $\Gamma$. If $t(e) \in \Gamma$ then by traversing a reduced path in $\Gamma$ from $v$ to $s(e)$, crossing $e$ and a reduced path in $\Gamma$ from $t(e)$ to $v$, we get a closed reduced path not contained in $\Gamma$, a contradiction. Thus $t(e) \notin \Gamma$. Let $T_v$ be the union of all the reduced paths in $\Lambda \setminus \{e\}$ starting at $t(e)$, so we have the situation as in (a):

(a) \hspace{2cm} (b) \hspace{2cm} (c)

If $\gamma$ is a non-trivial closed path in $T_v$ starting at $t(e)$, then a path from $v$ to $t(e)$, traversing $\gamma$, and going the same way back to $v$ cannot be reduced. But the only place a spur can occur is in $\gamma$ and so $T_v$ is a tree. If $e'$ is another edge of $\Lambda \setminus \Gamma$ with $s(e') \in \Gamma$ then we claim that neither of the two situations (b) and (c) above can occur, i.e: $t(e')$ is not a vertex of $T_v$. For otherwise, a reduced closed path in $T_v$ from $t(e)$ to $t(e')$ will give a reduced closed path at $v$ not in $\Gamma$. Thus, another edge $e'$ yields a tree $T_{v'}$ defined like $T_v$, but disjoint from it. Each component of $\Phi$ is thus obtained this way. \hfill $\square$

**Corollary 1.** $\Lambda$ connected is of finite rank if and only if for any vertex $v$, the spine $\hat{\Lambda}_v$ is finite, locally finite.
Proposition 2 gives the wedge sum decomposition $\Lambda = \hat{\Lambda}_v \bigvee_{\Theta} \Phi$, and by connectedness, any spanning tree $T \hookrightarrow \Lambda$ must contain the forest $\Phi$ as a subgraph. Thus if $\Lambda$ has finite rank, then $\hat{\Lambda}_v$ is a tree with finitely many edges added, hence finite. Conversely, a finite spine has finite rank and $\text{rk} \Lambda = \text{rk} \hat{\Lambda}_v$. $\square$

Thus if $\text{rk} \Lambda < \infty$ then the decomposition of Proposition 2 becomes,

$$\Lambda = \left( \cdots \left( \left( \hat{\Lambda}_v \bigvee_{\Theta_1} T_1 \right) \bigvee_{\Theta_2} T_2 \right) \cdots \right) \bigvee_{\Theta_k} T_k ,$$

(1)

with $\hat{\Lambda}_v$ finite, the $\Theta_i$ single vertices, the $\Theta_i \hookrightarrow \hat{\Lambda}_v$, and the images $\Theta_i \hookrightarrow T_i$ incident with a single arc. Moreover, if $\Lambda \rightarrow \Delta$ is a covering with $\Delta$ single vertexed and $\Lambda$ of finite rank, then by Proposition 1(i), each tree $T_i$ realizes an embedding $\mathbb{R} \hookrightarrow \Lambda$ of the real line in $\Lambda$, and as the spine is finite, the trees are thus paired

with the $e_i$ (and indeed all the edges in the path $\mathbb{R} \hookrightarrow \Lambda$) in the same fiber of the covering. This pairing will play a key role in §3.

Corollary 2. Let $\Lambda \rightarrow \Delta$ be a covering with $\Delta$ single vertexed having non-empty edge set and $\text{rk} \Lambda < \infty$. Then $\text{deg}(\Lambda \rightarrow \Delta) < \infty$ if and only if $\Lambda = \hat{\Lambda}_v$.

Proof. If $\Lambda$ is more than $\hat{\Lambda}_v$ then one of the trees $T_i$ in the decomposition (1) is non trivial and by Proposition 1(i) we get a real line subgraph $\mathbb{R} \hookrightarrow \Lambda$, with image in the fiber of an edge, contradicting the finiteness of the degree. The converse follows from Corollary 1. $\square$

Proposition 3. Let $\Lambda \rightarrow \Delta$ be a covering with (i), $\text{rk} \Delta > 1$, (ii), $\text{rk} \Lambda < \infty$, and (iii). for any intermediate covering $\Lambda \rightarrow \Gamma \rightarrow \Delta$ we have $\text{rk} \Gamma < \infty$. Then $\text{deg}(\Lambda \rightarrow \Delta) < \infty$.

The covering $\mathbb{R} \rightarrow \Delta$ of a single vertexed $\Delta$ of rank 1 by the real line shows why the $\text{rk} \Delta > 1$ condition cannot be dropped.

Proof. By lattice excision we may pass to the $\Delta$ single vertexed case while preserving (i)-(iii). Establishing the degree here and passing back to the general $\Delta$ will give the result. If the degree of the covering $\Lambda \rightarrow \Delta$ is infinite for $\Delta$ single vertexed, then by Corollary 2 in the decomposition (1) for $\Lambda$, one of the trees is non-empty and $\Lambda$ has the form of the graph in Proposition 1 with this non-empty tree the union of the edge $e$ and $\gamma_2$. Let $\Gamma$ be a graph defined as follows: take the union of $\gamma_1$, the edge $e$ and $\alpha(\mathbb{R}) \cap \gamma_2$, where $\alpha(\mathbb{R})$ is the embedding of the real line given by Proposition 1(i). At each vertex of $\alpha(\mathbb{R}) \cap \gamma_2$ place $\text{rk} \Delta - 1$ edge loops:

![Diagram](image)

(the picture depicting the $\text{rk} \Delta = 2$ case). Then there is an obvious covering $\Gamma \rightarrow \Delta$ so that by Proposition 1(ii) we have an intermediate covering $\Lambda \rightarrow \Gamma \rightarrow \Delta$. Equally obviously, $\Gamma$ has infinite rank, contradicting (iii). Thus, $\text{deg}(\Lambda \rightarrow \Delta) < \infty$. $\square$

Proposition 4. Let $\Psi \rightarrow \Lambda \rightarrow \Delta$ be coverings with $\text{rk} \Lambda < \infty$, $\Psi \rightarrow \Delta$ Galois, and $\Psi$ not simply connected. Then $\text{deg}(\Lambda \rightarrow \Delta) < \infty$. 

The idea is that if the degree is infinite, then \( \Lambda \) has a hanging tree in its spine decomposition, and so \( \Psi \) does too. But \( \Psi \) should look the same at every point, hence is a tree.

**Proof.** Apply lattice excision to \( \mathcal{L}(\Psi, \Delta) \), and as \( \pi_1(\Psi, u) \) is unaffected by the excision of trees, we may assume that \( \Delta \) is single vertexed. As \( \text{deg}(\Lambda \to \Delta) \) is infinite, the spine decomposition for \( \Lambda \) has an infinite tree, and \( \Lambda \) has the form of Proposition \([1]\). Thus \( \Psi \) does too, by part (iii) of this Proposition, with subgraphs \( \Upsilon_1' \to \Psi \), edge \( e' \) and \( \Upsilon_1' \) a tree. Take a closed reduced path \( \gamma \) in \( \Upsilon_1' \), and choose a vertex \( u_1 \) of \( \Upsilon_1' \) such that the reduced path from \( u_1 \) to \( s(e') \) has at least as many edges as \( \gamma \). Project \( \gamma \) via the covering \( \Psi \to \Delta \) to a closed reduced path, and then lift to \( u_1 \). The result is reduced, closed as \( \Psi \to \Delta \) is Galois, and entirely contained in the tree \( \Upsilon_1' \), hence trivial. Thus \( \gamma \) is also trivial, so that \( \Upsilon_2' \) is a tree and \( \Psi \) is simply connected. \( \square \)

**Proposition 5.** Let \( \Lambda_u \to \Delta_u \) be a covering with \( \text{rk} \Lambda < \infty \) and \( \gamma \) a non-trivial reduced closed path at \( u \) lifting to a non-closed path at \( u \). Then there is an intermediate covering \( \Lambda_u \to \Gamma_w \to \Delta_u \) with \( \text{deg}(\Gamma \to \Delta) \) finite and \( \gamma \) lifting to a non-closed path at \( w \).

Stallings shows something very similar \([19]\) Theorem 6.1] starting from a finite immersion rather than a covering. As the proof shows, the path \( \gamma \) in Proposition 5 can be replaced by finitely many such paths. Moreover, for \( T \leftrightarrow \Lambda \) a spanning tree, recall that Schreier generators for \( \pi_1(\Lambda, u) \) are the homotopy classes of paths through \( T \) from \( u \) to \( s(e) \), traversing \( e \) and traveling back through \( T \) to \( u_1 \), for \( e \in \Lambda \setminus T \). Then the intermediate \( \Gamma \) constructed has the property that any set of Schreier generators for \( \pi_1(\Lambda, u) \) can be extended to a set of Schreier generators for \( \pi_1(\Gamma, w) \).

**Proof.** If \( T \leftrightarrow \Delta \) is a spanning tree and \( q : \Delta \to \Delta/T \) then \( \gamma \) cannot be contained in \( T \), and so \( q(\gamma) \) is non-trivial, closed and reduced. If the lift of \( q(\gamma) \) to \( \Lambda/T \) is closed then the lift of \( \gamma \) to \( \Lambda \) has start and finish vertices that lie in the same component \( T_i \) of \( f^{-1}(T) \), mapped isomorphically onto \( T \) by the covering, and thus implying that \( \gamma \) is not closed. Thus we may apply lattice excision and pass to the single vertexed case while maintaining \( \gamma \) and its properties. Moreover, the conclusion in this case gives the result in general as closed paths go to closed paths when excising trees. If the lift \( \gamma_1 \) of \( \gamma \) at \( u \) is not contained in the spine \( \hat{\Lambda}_u \), then its terminal vertex lies in a tree \( T_{e_0} \) of the spine decomposition (\( \dagger \)). By adding an edge if necessary to \( \hat{\Lambda}_u \cup \gamma_1 \), we obtain a finite subgraph whose coboundary edges are paired, with the edges in each pair covering the same edge in \( \Delta \), as below left:

![Diagram](image)

(if the lift is contained in the spine, take \( \hat{\Lambda}_u \) itself). In any case, let \( \Gamma \) be \( \hat{\Lambda}_u \cup \gamma_1 \) together with a single edge replacing each pair as above right. Restricting the covering \( \Lambda \to \Delta \) to \( \hat{\Lambda}_u \cup \gamma_1 \) and mapping the new edges to the common image of the old edge pairs gives a finite covering \( \Gamma \to \Delta \), and hence an intermediate covering \( \Lambda \to \Gamma \to \Delta \), with \( q(\gamma_1) \) non-closed at \( q(u) \). \( \square \)

For the rest of this section we investigate the rank implications of the decomposition \([1]\) and the pairing (\( \dagger \)) in a special case. Suppose \( \Lambda \to \Delta \) is a covering with \( \Delta \) single vertexed, \( \text{rk} \Delta = 2 \), \( \Lambda \) non-simply connected, and \( \text{rk} \Lambda < \infty \). Let \( x_i^{\pm 1}, (1 \leq i \leq 2) \) be the edge loops of \( \Delta \) and fix a spine so we have the decomposition \([1]\).

An extended spine for such a \( \Lambda \) is a connected subgraph \( \Gamma \leftrightarrow \Lambda \) obtained by adding finitely many edges to a spine, so that every vertex of \( \Gamma \) is incident with either zero or three edges in its coboundary \( \delta \Gamma \). It is always possible to find an extended spine: take the union of the spine \( \hat{\Lambda}_u \) and each edge \( e \in \delta \hat{\Lambda}_u \) in its coboundary. Observe that \( \Gamma \) is finite and the decomposition \([1]\) gives \( \text{rk} \Gamma = \text{rk} \hat{\Lambda}_u = \text{rk} \Lambda \). Call a vertex of the extended spine \( \Gamma \) interior (respectively boundary) when it is incident with zero (resp. three) edges in \( \delta \Gamma \).
We have the pairing of trees (\( \Upsilon \)) for an extended spine, so that each boundary vertex \( v_1 \) is paired with another \( v_2 \),

\[
\begin{array}{c}
T_1 \\
\downarrow \\
v_1 \\
\downarrow \\
\gamma \\
\downarrow \\
v_2 \\
\downarrow \\
T_2
\end{array}
\]

with \( e_1, e_2 \) and all the edges in the path \( \gamma = \alpha(\mathbb{R}) \cap \Gamma \) covering an edge loop \( x_i \in \Delta \). Call this an \( x_i \)-pair, \( (i = 1, 2) \).

For two \( x_i \)-pairs (fixed \( i \)), the respective \( \gamma \) paths share no vertices in common, for otherwise there would be two distinct edges covering the same \( x_i \in \Delta \) starting at such a common vertex. Moreover, \( \gamma \) must contain vertices of \( \Gamma \) apart from the two boundary vertices \( v_1, v_2 \), otherwise \( \Lambda \) would be simply connected. These other vertices are incident with at least two edges of \( \gamma \in \Gamma \), hence at most 2 edges of the coboundary \( \partial \Gamma \), and thus must be interior.

**Lemma 1.** If \( n_i, (i = 1, 2) \), is the number of \( x_i \)-pairs in an extended spine \( \Gamma \) for \( \Lambda \), then the number of interior vertices is at least \( \sum n_i \).

**Proof.** The number of interior vertices is \( |V_T| - 2 \sum n_i \) and the number of edges of \( \Gamma \) is \( 4(|V_T| - 2 \sum n_i) + 2 \sum n_i \), hence \( \text{rk} \Gamma - 1 = |V_T| - 3 \sum n_i \). As \( \Lambda \) is not simply connected, \( \text{rk} \Lambda - 1 = \text{rk} \Gamma - 1 \geq 0 \), thus \( |V_T| - 2 \sum n_i \geq \sum n_i \) as required. \( \square \)

The lemma is not true in the case \( \text{rk} \Delta > 2 \). It will be helpful in \( \S 3 \) to have a pictorial description of the quantity \( \text{rk} \Lambda - 1 \) for our graphs. To this end, a *checker* is a small plastic disk, as used in the eponymous boardgame (called *draughts* in British English). We place black checkers on some of the vertices of an extended spine \( \Gamma \) according to the following scheme: place black checkers on all the interior vertices of \( \Gamma \); for each \( x_1 \)-pair in \( (*) \), take the interior vertex on the path \( \gamma \) that is closest to \( v_1 \) (ie: is the terminal vertex of the edge of \( \gamma \) whose start vertex is \( v_1 \)) and remove its checker; for each \( x_2 \)-pair, we can find, by Lemma \( \S 1 \) an interior vertex with a checker still on it. Choose such a vertex and remove its checker also. We saw in the proof of Lemma \( \S 1 \) that \( \text{rk} \Lambda - 1 = \text{rk} \Gamma - 1 \) is equal to the number of interior vertices of \( \Gamma \), less the number of \( x_i \)-pairs \( (i = 1, 2) \). Thus,

**Lemma 2.** With black checkers placed on the vertices of an extended spine for \( \Lambda \) as above, the number of black checkers is \( \text{rk} \Lambda - 1 \).

From now on we will only use the extended spine obtained by adding the coboundary edges to some fixed spine \( \tilde{\Lambda}_u \).

Let \( p : \Lambda_u \to \Delta_v \) be a covering with \( \text{rk} \Delta = 2 \), \( \text{rk} \Lambda < \infty \) and \( \Lambda \) not simply connected. A spanning tree \( T \leftrightarrow \Delta \) induces a covering \( \Lambda/T_i \to \Delta/T \) with \( \Delta/T \) single vertexed. Let \( \mathcal{H}(\Lambda_u \to \Delta_v) \) be the number of vertices of the spine of \( \Lambda/T_i \) at \( q(u) \) and \( n_i(\Lambda_u \to \Delta_v) \) the number of \( x_i \)-pairs in the extended spine. The isomorphism class of \( \Lambda/T_i \) and the spine are independent of the spanning tree \( T \), hence the quantities \( \mathcal{H}(\Lambda_u \to \Delta_v) \) and \( n_i(\Lambda_u \to \Delta_v) \) are too.

### 3. Pullbacks

Let \( p_i : \Lambda_i := \Lambda_{u_i} \to \Delta_{v_i} \), \( (i = 1, 2) \) be coverings and \( \Lambda_1 \prod_{\Delta} \Lambda_2 \) their (unpointed) pullback. If \( \tilde{\Lambda}_{u_i} \) is the spine at \( u_i \) then we can restrict the coverings to maps \( p_i : \tilde{\Lambda}_{u_i} \to \Delta_{v_i} \) and form the pullback \( \tilde{\Lambda}_{u_1} \prod_{\Delta} \tilde{\Lambda}_{u_2} \).

**Proposition 6 (spine decomposition of pullbacks).** The pullback \( \Lambda = \Lambda_1 \prod_{\Delta} \Lambda_2 \) has a wedge sum decomposition \( \Lambda = (\tilde{\Lambda}_{u_1} \prod_{\Delta} \tilde{\Lambda}_{u_2}) \lor_{\Theta} \Phi \), with \( \Phi \) a forest and no two vertices of the image of \( \Theta \leftrightarrow \Phi \) lying in the same component.
Proof. Let $A_i = \hat{A}_{u_i} \bigvee_{\Theta_i} \Phi_i$, $(i = 1, 2)$ be the spine decomposition, $t_i : A_1 \prod_{\Delta} A_2 \to A_i$ $(i = 1, 2)$ the coverings provided by the pullback and $\Omega$ a connected component of the pullback. If $\Omega \cap (\hat{A}_{u_1} \prod_{\Delta} \hat{A}_{u_2}) = \emptyset$, then a reduced closed path $\gamma \in \Omega$ must map via one of the $t_i$ to a closed path in the forest $\Phi_i$. As the images under coverings of reduced paths are reduced, $t_i(\gamma)$ must contain a spur which can be lifted to a spur in $\gamma$. Thus $\Omega$ is a tree.

Otherwise choose a vertex $w_1 \times w_2$ in $\Omega \cap (\hat{A}_{u_1} \prod_{\Delta} \hat{A}_{u_2})$ and let $\Gamma$ be the connected component of this intersection containing $w_1 \times w_2$. If $\gamma$ a reduced closed path at $w_1 \times w_2$ then $t_i(\gamma)$, $(i = 1, 2)$ is a reduced closed path at $w_i \in \hat{A}_{u_i}$, hence $t_i(\gamma) \in \hat{A}_{u_i}$ and thus $\gamma \in \hat{A}_{u_1} \prod_{\Delta} \hat{A}_{u_2}$. Applying Proposition 2 we have $\Omega$ a wedge sum of $\Gamma$ and a forest of the required form.

Corollary 3 (Howsen-Stallings). Let $p_i : A_i \to \Delta_i (i = 1, 2)$, be coverings with $rkA_i < \infty$ and $u_1 \times u_2$ a vertex of their pullback. Then $rk(\Lambda_1 \prod_{\Delta} \Lambda_2)_{u_1 \times u_2} < \infty$.

Proof. The component $\Omega$ of the pullback containing $u_1 \times u_2$ is either a tree or the wedge sum of a finite graph and a forest as described in Proposition 6. Either case gives the result. □

The remainder of this section is devoted to a proof of an estimate for the rank of the pullback of finite rank graphs in a special case. Let $p_j : A_j := A_{u_j} \to \Delta_{v_j}$ $(j = 1, 2)$ be coverings with $rk\Delta = 2$, $rk\Lambda_j < \infty$ and the $\Lambda_j$ not simply connected. Let $\mathcal{H}_j := \mathcal{H}(A_{u_j} \to \Delta_v)$ and $n_{ji} := n_i(A_{u_j} \to \Delta_v)$ be as at the end of 2.

**Theorem 1.** For $i = 1, 2$,
\[
\sum_{\Omega}(rk\Omega - 1) \leq \prod_j (rkA_j - 1) + \mathcal{H}_1\mathcal{H}_2 - (\mathcal{H}_1 - n_{1i})(\mathcal{H}_2 - n_{2i}),
\]
the sum over all non simply connected components $\Omega$ of the pullback $A_1 \prod_{\Delta} A_2$.

Proof. Lattice excision and the description of the $\mathcal{H}_j$ and $n_{ji}$ allow us to pass to the $\Delta$ single vertexed case. Suppose then that $\Delta$ has edge loops $x_i^\pm, (1 \leq i \leq 2)$ at the vertex $v$, and extended spines $\hat{A}_{u_j} \hookrightarrow \Gamma_j \hookrightarrow \Lambda_j$. The covering $p_j : A_j \to \Delta_v$ can be restricted to maps $\Gamma_j \to \Delta_v$ and $\hat{A}_{u_j} \to \Delta_v$, and we form the three resulting pullbacks $A_1 \prod_{\Delta} A_2$, $\Gamma_1 \prod_{\Delta} \Gamma_2$ and $\hat{A}_{u_1} \prod_{\Delta} \hat{A}_{u_2}$, with $\hat{A}_{u_1} \prod_{\Delta} \hat{A}_{u_2} \hookrightarrow \Gamma_1 \prod_{\Delta} \Gamma_2 \hookrightarrow A_1 \prod_{\Delta} A_2$, and $t_j : A_1 \prod_{\Delta} A_2 \to A_j$ the resulting covering maps.

Place black checkers on the vertices of the extended spines $\Gamma_j$ as in 2 and place a black checker on a vertex $v_1 \times v_2$ of $\Gamma_1 \prod_{\Delta} \Gamma_2$ precisely when both $t_j(v_1 \times v_2) \in \Gamma_j$, $(j = 1, 2)$ have black checkers on them. By Lemma 2 and the construction of the pullback for $\Delta$ single vertexed, we get the number of vertices in $\Gamma_1 \prod_{\Delta} \Gamma_2$ with black checkers is equal to $\prod_i (rkA_i - 1)$.

Let $\Omega$ be a non simply connected component of the pullback $A_1 \prod_{\Delta} A_2$ and $\mathcal{Y} = \Omega \cap (\Gamma_1 \prod_{\Delta} \Gamma_2)$. If $v_1 \times v_2$ is the start vertex of at least one edge in the coboundary $\delta\mathcal{Y}$, then at least one of the $v_j$ must be incident with at least one, hence three, edges of the coboundary $\delta\Gamma_j$. Lifting these three via the covering $t_j$ to $v_1 \times v_2$ gives at least three edges starting at $v_1 \times v_2$ in the coboundary $\delta\mathcal{Y}$. Four coboundary edges starting here would mean that $\Omega$ was simply connected, hence every vertex of $\mathcal{Y}$ is incident with either zero or three coboundary edges.

We can thus extend the interior/boundary terminology of 2 to the vertices of $\mathcal{Y}$, and observe that a vertex of $\mathcal{Y}$ covering, via either of the $t_j$, a boundary vertex $v \in \Gamma_j$, must itself be a boundary vertex. The upshot is that $\mathcal{Y}$ is an extended spine in $\Omega$ and by Proposition 6 $rk\Omega - 1 = rk\mathcal{Y} - 1$. Now place red checkers on the vertices of $\mathcal{Y}$ as in 2 and do this for each non-simply connected component $\Omega$. The number of red checkered vertices is $\sum_{\Omega}(rk\Omega - 1)$.

The result is that $\Gamma_1 \prod_{\Delta} \Gamma_2$ has vertices with black checkers, vertices with red checkers, vertices with red checkers sitting on top of black checkers, and vertices that are completely uncheckered. Thus,
\[
\sum_{\Omega}(rk\Omega - 1) \leq \prod_i (rkA_i - 1) + N,
\]
where \( N \) is the number of vertices of \( \Gamma_1 \prod \Delta \Gamma_2 \) that have a red checker but no black checker.

It remains then to estimate the number of these "isolated" red checkers. Observe that a vertex of \( \Gamma_1 \prod \Delta \Gamma_2 \) has no black checker precisely when it lies in the fiber, via at least one of the \( t_j \), of a checkerless vertex in \( \Gamma_j \). Turning it around, we investigate the fibers of the checkerless vertices of both \( \Gamma_j \). Indeed, in an \( x_1 \)-pair, the vertices \( v_1, v_2 \) and \( u \) are checkerless, while \( v_1, v_2 \) are also checkerless in an \( x_2 \)-pair. We claim that no vertex in the fiber, via \( t_j \), of these five has a red checker. A vertex of \( \Upsilon \) in the fiber of the boundary vertices \( v_1, v_2 \) is itself a boundary vertex, hence contains no red checker. If \( v_1 \times v_2 \in \Upsilon \) is in the fiber of \( u \) and is a boundary vertex of \( \Upsilon \) then it carries no red checker either. If instead \( v_1 \times v_2 \) is an interior vertex then the lift to \( v_1 \times v_2 \) of \( e^{-1} \) cannot be in the coboundary \( \delta \Upsilon \), hence the terminal vertex of this lift is in \( \Upsilon \) also and covers \( v_1 \). Thus, this terminal vertex is a boundary vertex for an \( x_1 \)-pair of \( \Upsilon \), and \( v_1 \times v_2 \) is the interior vertex from which a red checker is removed for this pair.

The only remaining checkerless vertices of the \( \Gamma_j \) unaccounted for are those interior vertices chosen for each \( x_2 \)-pair. Let \( S_1 = \{ u_1, \ldots, u_{n_{12}} \} \subset \Gamma_1 \) and \( S_2 = \{ w_1, \ldots, w_{n_{22}} \} \subset \Gamma_2 \) be these sets of vertices. The result of the discussion above is that if \( v_1 \times v_2 \) has an isolated red checker then it must be contained in \( (S_1 \times V_1) \cup (V_1 \times S_2) \), the vertices of \( \Gamma_1 \prod \Delta \Gamma_2 \) in the fiber of a \( u_i \) or a \( w_i \). If \( u_i \times y \in S_1 \times V_2 \), then \( u_i \times y \) is a boundary vertex of \( \Gamma_2 \), hence has no red checker. Similarly a \( x \times w_i \) with \( x \) a boundary vertex of \( \Gamma_1 \) has no red checker, and so \( N \) is at most the number of vertices in the set \( (S_1 \times V_2) \cup (V_1 \times S_2) \), \( V \) the vertices of the spine \( \hat{A}_{u_i} \). As \( S_1 \subset V_1 \), the two sets in this union intersect in \( S_1 \times S_2 \), so we have

\[
N \leq |S_1 \times V_2| + |V_1 \times S_2| - |S_1 \cap S_2| = n_{12}K_2 + n_{22}K_1 - n_{12}n_{22},
\]

hence the result for \( i = 2 \). Interchanging the checker scheme for the \( x_1 \)-pairs gives the result for \( i = 1 \).

\[\square\]

### 4. Free groups and the topological dictionary

A group \( F \) is free of rank \( \text{rk} F \) if and only if it is isomorphic to the fundamental group of a connected graph of rank \( \text{rk} F \). If \( \Gamma_1, \Gamma_2 \) are connected graphs with \( \pi_1(\Gamma_1, v_1) \cong \pi_1(\Gamma_2, v_2) \), then \( H_1(\Gamma_1) \cong H_1(\Gamma_2) \) and thus \( \text{rk} \Gamma_1 = \text{rk} \Gamma_2 \).

The free groups so defined are of course the standard free groups and the rank is the usual rank of a free group. At this stage we appeal to the existing (algebraic) theory of free groups, and in particular, that by applying Nielsen transformations, a set of generators for a free group can be transformed into a set of free generators whose cardinality is no greater. Thus, a finitely generated free group has finite rank (the converse being obvious). From now on we use the (topologically more tractable) notion of finite rank as a synonym for finitely generated.

Let \( F \) be a free group and \( \varphi : F \to \pi_1(\Delta, v) \) an isomorphism for \( \Delta \) connected. We call \( \varphi \) a topological realization, and the "topological dictionary" is the loose term used to describe the correspondence between algebraic properties of \( F \) and topological properties of \( \Delta \). The non-abelian \( F \) correspond to the \( \Delta \) with \( \text{rk} \Delta > 1 \). A subgroup \( A \subset F \) corresponds to a covering \( f : \Delta_u \to \Delta_v \) with \( f_*\pi_1(\Delta, u) = \varphi(A) \), and hence \( \text{rk} A = \text{rk} \Lambda (f_* \text{ the homomorphism induced by } p \text{ using the functorality of } \pi_1) \). Thus finitely generated subgroups correspond to finite rank \( \Lambda \) and normal subgroups to Galois coverings. Inclusion relations between subgroups correspond to covering relations, indices of subgroups to degrees of coverings, trivial subgroups to simply connected coverings, conjugation to change of basepoint, and so on.

Applying the topological dictionary to the italicised results below we recover some classical facts (see also \[18,19\]).

1. \[7,12\]: If a finitely generated subgroup \( A \) of a non-abelian free group \( F \) is contained in no subgroup of infinite rank, then \( A \) has finite index in \( F \); Proposition\[5\]
2. \[7\]: If a finitely generated subgroup \( A \) of a free group \( F \) contains a non-trivial normal subgroup of \( F \), then it has finite index in \( F \); Proposition\[7\]
3. [\textbf{18}]: Let $F$ be a free group, $X$ a finite subset of $F$, and $A$ a finitely generated subgroup of $F$ disjoint from $X$. Then $A$ is a free factor of a group $G$, of finite index in $F$ and disjoint from $X$; \textit{Proposition} \textbf{3}(and the comments following it).

4. [\textbf{10}]: If $A_1, A_2$ are finitely generated subgroups of a free group $F$, then the intersection of conjugates $A_1^g \cap A_2^h$ is finitely generated for any $g, h \in F$; \textit{Corollary} \textbf{3}.

If $\Delta$ is a graph, $\text{rk} \Delta = 2$, and $A \subset F = \pi_1(\Delta, v)$, then we define $\mathcal{H}(F, A) := \mathcal{H}(A_u \to \Delta_v)$ and $n_i(F, A) := n_i(A_u \to \Delta_v)$, where $f : A_u \to \Delta_v$ is the covering with $f_*\pi_1(A, u) = A$. For an arbitrary free group $F$ realized via $\varphi : F \to \pi_1(\Delta, v)$, define $\mathcal{H}^\varphi(F, A)$ and $n_i^\varphi(F, A)$ to be $\mathcal{H}(\varphi(F), \varphi(A))$ and $n_i(\varphi(F), \varphi(A))$.

The appearance of $\varphi$ in the notation is meant to indicate that these quantities, unlike rank, are realization dependent. This can be both a strength and a weakness. A weakness because it seems desirable for algebraic statements to involve only algebraic invariants, and a strength if we have the freedom to choose the realization, especially if more interesting results are obtained when this realization is not the “obvious” one.

For example, if $F$ is a free group with free generators $x$ and $y$, and $\Delta$ is single vertexed with two edge loops whose homotopy classes are $a$ and $b$, then the subgroup $A = \langle x, y \rangle \subset F$ corresponds to the $A$ below left under the obvious realization $\varphi_1(x) = a, \varphi_1(y) = b$, and to the righthand graph via $\varphi_2(x) = a, \varphi_2(y) = a^{-1}b$:

Thus, $\mathcal{H}^{\varphi_1}(F, A) = 2, n_1^{\varphi_1}(F, A) = 1, (i = 1, 2)$, whereas $\mathcal{H}^{\varphi_2}(F, A) = 1, n_1^{\varphi_2}(F, A) = 1, n_2^{\varphi_2}(F, A) = 0$.

We now apply the topological dictionary to \textit{Theorem 1}. Let $\varphi : F \to \pi_1(\Delta, v), A_j \subset F, (j = 1, 2)$, finitely generated non-trivial subgroups, and $f_j : A_u \to \Delta_v, (j = 1, 2)$ coverings with $\varphi(A_j) = f_j, \pi_1(A, u_j)$. Each non simply-connected component $\Omega$ of the pullback corresponds to some non-trivial intersection of conjugates $A_1^g \cap A_2^h$. As observed in [\textbf{14}], these in turn correspond to the conjugates $A_1^g \cap A_2^h$ for $g$ from a set of double coset representatives for $A_2 \setminus F/A_1$.

\textbf{Theorem 2.} Let $F$ be a free group of rank two and $A_j \subset F, (j = 1, 2)$, finitely generated non-trivial subgroups. Then for any realization $\varphi : F \to \pi_1(\Delta, v)$ and $i = 1, 2$,

$$
\sum_g (\text{rk} (A_1 \cap A_2^g) - 1) \leq \prod_j (\text{rk} A_j - 1) + \mathcal{H}_1 \mathcal{H}_2 - (\mathcal{H}_1 - n_{1ij})(\mathcal{H}_2 - n_{2ij}),
$$

the sum over all double coset representatives $g$ for $A_2 \setminus F/A_1$ with $A_1 \cap A_2^g$ non-trivial, and where $\mathcal{H}_j = \mathcal{H}^{\varphi_j}(F, A_j)$ and $n_{ij} = n_{ij}^\varphi(F, A_j)$.

This theorem should be viewed in the context of attempts to prove the so-called \textit{strengthened Hanna Neumann conjecture}: namely, if $A_j, (j = 1, 2)$ are finitely generated, non-trivial, subgroups of an arbitrary free group $F$, then

$$
\sum_g (\text{rk} (A_1 \cap A_2^g) - 1) \leq \prod_j (\text{rk} A_j - 1) + \varepsilon,
$$

the sum over all double coset representatives $g$ for $A_2 \setminus F/A_1$ with $A_1 \cap A_2^g$ non-trivial, where the conjecture is that $\varepsilon$ is zero, while in the existing results, it is an error term having a long history. A selection of estimates for $\varepsilon$, in chronological order is, $\langle (\text{rk} A_1 - 1)(\text{rk} A_2 - 1) \rangle \textbf{15}$, $\max\{ (\text{rk} A_1 - 2)(\text{rk} A_2 - 1), (\text{rk} A_1 - 1)(\text{rk} A_2 - 2), \}, \textbf{11}$, $\max\{ (\text{rk} A_1 - 2)(\text{rk} A_2 - 2) - \ldots \}$.
1, 0}, \mathcal{H}_i = k \) (the original, unstrengthened conjecture \[15\]) involved just the intersection of the two subgroups, rather than their conjugates, and the first two expressions for \( \varepsilon \) were proved in this restricted sense; the strengthened version was formulated in \[14\], and the H. Neumann and Burns estimates for \( \varepsilon \) were improved to the strengthened case there. Observe that as the join \( \langle A_1, A_2 \rangle \) of two finitely generated subgroups is finitely generated, and every finitely generated free group can be embedded as a subgroup of the free group of rank two, we may replace the ambient free group in the conjecture with the free group of rank two.

It is hard to make a precise comparison between the \( \varepsilon \) provided by Theorem \[2\] and those above. Observe that if \( A_j \subset F \), with \( F \) free of rank two, then with respect to a topological realization we have \( \text{rk} A_j = \mathcal{H}_j - (n_{j1} + n_{j2}) + 1 \). It is straightforward to find infinite families \( A_{1k}, A_{2k} \subset \pi_1(\Delta, v), (k \in \mathbb{Z}^{>0}) \), for which the error term in Theorem \[2\] is less than those listed above for all but finitely many \( k \), or even for which the strengthened Hanna Neumann conjecture is true by Theorem \[2\] for instance,

\[
A_{1k} = A_{2k} = \begin{array}{c}
\text{k edge loops}
\end{array}
\]

\[
\mathcal{H}_i = k \quad n_{11} = n_{21} = 0 \quad n_{12} = n_{22} = 1
\]

but where the error terms above are quadratic in \( k \).

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