ORBITAL STABILITY OF PERIODIC TRAVELING WAVE SOLUTIONS TO THE COUPLED COMPOUND KDV AND MKDV EQUATIONS WITH TWO COMPONENTS

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Abstract. In this article, the authors consider the orbital stability of periodic traveling wave solutions for the coupled compound KdV and MKdV equations with two components

\[
\begin{align*}
    u_t + vv_x + \beta u^2 u_x + u_{xxx} - uu_x &= 0, \quad \beta > 0, \\
    v_t + (uv)_x + 2vv_x &= 0,
\end{align*}
\]

Firstly, we show that there exist a smooth curve of positive traveling wave solutions of dnoidal type with a fixed fundamental period \( L \) for the coupled compound KdV and MKdV equations. Then, combining the orbital stability theory presented by Grillakis et al., and detailed spectral analysis given by using Lamé equation and Floquet theory, we show that the dnoidal type periodic wave solution with period \( L \) is orbitally stable. As the modulus of the Jacobian elliptic function \( k \to 1 \), we obtain the orbital stability results of solitary wave solution with zero asymptotic value for the coupled compound KdV and MKdV equations from our work. In addition, we also obtain the stability results for the coupled compound KdV and MKdV equations with the degenerate condition \( v = 0 \), called the compound KdV and MKdV equation.

1. Introduction. As is well known, the coupled nonlinear equations in which a KdV structure is embedded occur naturally in shallow water wave problems. Guha-Roy et al. [7, 8, 9] have studied the coupled nonlinear partial differential equations that can be solved exactly. The following coupled version of compound KdV and MKdV equations with two components

\[
\begin{align*}
    u_t + \alpha vv_x + \beta u^2 u_x + \gamma u_{xxx} + \lambda uu_x &= 0, \\
    v_t + \alpha(uv)_x + 2vv_x &= 0,
\end{align*}
\]

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model the physical problem of describing the strong interaction of two-dimensional long internal gravity waves propagating on neighboring pycnocline in a stratified fluid, where $\alpha, \beta, \gamma, \lambda$ are arbitrary constants. When the variable $v = 0$, Eqs.(1) can reduce to the compound KdV and mKdV equation or the Garder equation

$$u_t + \beta u^2 u_x + \gamma u_{xxx} + \lambda uu_x = 0. \tag{2}$$

Eq.(2) represents a model for wave propagation in a one-dimensional nonlinear lattice and has widespread applications in the field of solid-state physics, plasma physics, fluid physics, and quantum field theory [19, 20, 5]. Recently, Eq.(2) has attracted great attention of many mathematicians and physicists. By the elementary integral method, Dai et al. [6] got the approximate solutions for the solitary waves with zero asymptotic value for Eq.(2). By using many other different methods, a various of exact solitary solution for Eq.(2) have been obtained [17, 13]. More recently, Zhang and Shi et al. [21] considered orbital stability of solitary waves with zero and nonzero asymptotic value for the compound KdV and MKdV equation (2).

Moreover, in 1981, Hirota and Satsuma [14] presented a coupled Korteweg-de Vries equation

$$\begin{cases} u_t - 6\alpha uu_x - 2\beta vv_x - \alpha u_{xxx} = 0, \\ v_t + 3uv_x + v_{xxx} = 0. \end{cases} \tag{3}$$

and indicated the equation exhibited a soliton solution and three basic conserved quantities. Eqs.(3) describe an interaction of two long waves with different dispersion relations [4]. As one of the variable $v = 0$, Eqs.(3) can be reduced to the well-known KdV equation. In the recent years, there have been many profound results on the orbital stability of solitary waves, cnoidal waves and dnooidal waves for the systems (3) and its generalization [10, 1, 2]. Guo and Chen [10] studied the orbital stability for solitary waves for Eqs.(3) by applying the abstract results of Grillakis et al. [11] and detailed spectral analysis. Angulo [1, 2] obtained the existence of non-trivial smooth curve of cnoidal and dnooidal periodic traveling waves solutions respectively, and proved the nonlinear stability of these waves solutions.

When $\alpha = 1, \lambda = -1$, the coupled compound KdV and MKdV equations (1) become

$$\begin{cases} u_t + vv_x + \beta u^2 u_x + u_{xxx} - uu_x = 0, \beta > 0, \\ v_t + (uv)_x + 2vv_x = 0. \end{cases} \tag{4}$$

As we known, even if the stability of solitary waves for the compound KdV and mKdV equation and some types of the coupled nonlinear partial differential equations have been studied, but the orbital stability of solitary wave and periodic wave of the coupled version of compound KdV and MKdV equations with two components have not been studied. In this paper, we will study the existence and orbital stability of periodic traveling wave solutions of dnooidal type for Eqs.(4). We focus on solutions for (4) of the form

$$u(x, t) = \phi_c(x - ct), \text{ and } v(x, t) = \psi_c(x - ct), \tag{5}$$

where $c \in \mathbb{R}, \xi = x - ct, \phi_c, \psi_c : \mathbb{R} \to \mathbb{R}$ are smooth functions with the same fundamental period $L > 0$. Because the stability in view here refers to perturbations of the periodic-wave profile itself, a study of the initial-value problem for Eqs.(4) is necessary. Similar to Theorem [12], we have the following general lemma regarding the existence of solutions to the initial value problem of Eqs.(4).
Lemma 1.1. Suppose that $s \geq 1$. For any fixed periodic initial values $(\varphi_0, \psi_0) \in H^s(0, L) \times H^s(0, L)$, there exists a unique periodic solution $(\varphi, \psi) \in C([0, \infty); H^s(0, L) \times H^s(0, L))$ satisfying $(\varphi(0), \psi(0)) = (\varphi_0, \psi_0)$ for Eqs.(4).

Next, based on the classical Grillakis, Shatah and Strauss theory [11], we study orbital stability of the periodic wave solutions (5). Firstly, we prove that there exist smooth periodic traveling wave solutions (5) for Eqs.(4), where $\phi_c$, $\psi_c$ are smooth function with given period $L > 0$.

Theorem 1.2. Let $c > 0$. For $L > 0$ fixed, consider $c_0 > \frac{\pi^2}{L^2}$ and the unique $\eta_{2,0} = \eta_2(c_0) \in (0, \frac{\pi^2 n^2}{L^2})$ such that $T_{\phi_{c_0}} = T_{\psi_{c_0}} = L$. Then,

(1) there exist intervals $I(c_0)$ and $B(\eta_{2,0})$ around $c_0$ and $\eta_{2,0}$ respectively, and a unique smooth function $\Pi : I(c_0) \rightarrow B(\eta_{2,0})$ such that $\Pi(c_0) = \eta_{2,0}$ and

$$\frac{2\sqrt{6}}{\sqrt{12c - \beta \eta_2^2}} K(k) = L,$$

for all $c \in I(c_0)$, $\eta_2 \in \Pi(c)$ and

$$k^2 = k^2(c) = \frac{12c - 2\beta \eta_2^2}{12c - \beta \eta_2^2} \in (0, 1).$$

(2) The dnoidal waves $\phi(\cdot; \eta_1, \eta_2)$ and $\psi(\cdot; \eta_1, \eta_2)$ in (19) and (20) determined by $\eta_1 \equiv \eta_1(c), \eta_2 \equiv \eta_2(c) = \Pi(c)$, with $\eta_1^2 + \eta_2^2 = \frac{12c}{\beta}$, have fundamental period $L$ and satisfies (14) and (15). Moreover, the mapping

$$c \in I(c_0) \mapsto (\phi(\cdot; \eta_1(c), \eta_2(c)), \psi(\cdot; \eta_1(c), \eta_2(c))) \in H_{\text{per}}^n([0, L]) \times H_{\text{per}}^n([0, L])$$

is smooth for all integer $n \geq 1$.

(3) $I(c_0)$ can be chosen as $\left(\frac{\pi^2}{L^2}, +\infty\right)$.

Then, by applying the Floquet theory related to the linear operator

$$L_1 = -\frac{d^2}{dx^2} + 2c - \beta \phi^2,$$

and the stability framework [11], we show that for $c > 0$, the orbit

$$\Theta_{\phi_c} = \{(\phi(\cdot + y), \psi(\cdot + y)) : y \in R\}$$

will be stable in the space $H_{\text{per}}^1([0, L]) \times H_{\text{per}}^1([0, L])$ by the periodic flow of the system (4).

Theorem 1.3. For any wave speed $\frac{\pi^2}{L^2} < c < \frac{6}{\pi^2}$, if $d''(c) > 0$, then periodic solitary waves $T(ct)\Phi_c(x)$ is orbitally stable.

The rest of this paper is organized as follows. In section 2, we devote to prove the existence of a smooth curve of dnoidal wave solutions for Eqs.(4). Section 3 studies the spectral analysis of one certain self-adjoint operator with a crucial role to obtain our stability result. In section 4, we show our stability result of the dnoidal waves solutions for system (4).

2. Existence of dnoidal wave solutions for the coupled compound KdV and MKdV equations. In this section, we devote to show the existence of a smooth curve of dnoidal wave solutions of the form (5) for the coupled compound KdV and MKdV equations (4).
Substituting the form of solutions in (5) into the system (4), we obtain that
\[
\begin{align*}
\phi''_{c} + \psi_{c}\psi'_{c} + \beta\phi'_{c}\psi_{c} - \phi_{c}\phi'_{c} - c\phi_{c} = 0, \\
2\psi_{c}\psi'_{c} + (\phi_{c}\psi_{c})' - c\psi'_{c} = 0.
\end{align*}
\tag{9}
\]
Integrating the system (9) with respect to \(\xi\) once, and assuming the integration constant of the second equation of (9) being zero, we obtain \(\phi = \phi_{c}, \psi = \psi_{c}\) have to satisfy the following ordinary differential system
\[
\begin{align*}
\phi'' + \frac{1}{2}\psi^2 + \frac{\beta}{3}\phi^3 - \frac{1}{2}\phi^2 - c\phi &= E_0, \\
\psi^2 + \phi\psi - c\psi &= 0,
\end{align*}
\tag{10}
\]
where \(E_0\) is an arbitrary integration constant. By the second equation in (10), we have that
\[
\psi = c - \phi,
\tag{11}
\]
for all \(\psi \neq 0\). Then, substituting (11) into the first equation of (10) and assuming \(E_0 = \frac{c^2}{2}\), we have
\[
\phi'' + \frac{\beta}{3}\phi^3 - 2c\phi = 0.
\tag{12}
\]
Next, we prove that there is an explicit periodic solution which will depend on Jacobian elliptic functions for Eq.(12). Multiplying (12) by \(\phi'\) and integrating once, we get that
\[
(\phi')^2 = \frac{\beta}{6} [-\phi^4 + \frac{12c}{\beta} \phi^2 + \frac{12}{\beta} A_{\phi}],
\tag{13}
\]
where \(A_{\phi}\) is a needed nonzero integration constant. For convenience, we make \(F(t) = -t^4 + \frac{12c}{\beta} t^2 + \frac{12}{\beta} A_{\phi}\). We know that solutions of Eq.(13) depend on the roots of the polynomial \(F(\phi)\). For \(-\frac{3c^2}{\beta^2} < A_{\phi} < 0\), we have
\[
F(t) = -(t^2 - \frac{6c}{\beta})^2 + \frac{36c^2}{\beta^2} + \frac{12}{\beta} A_{\phi}
= (t^2 - \frac{6c}{\beta} + \sqrt{\frac{36c^2}{\beta^2} + \frac{12}{\beta} A_{\phi}}) \cdot (\sqrt{\frac{36c^2}{\beta^2} + \frac{12}{\beta} A_{\phi}} + \frac{6c}{\beta} - t^2),
\]
and
\[
\frac{6c}{\beta} > \sqrt{\frac{36c^2}{\beta^2} + \frac{12}{\beta} A_{\phi}} > 0, \quad \sqrt{\frac{36c^2}{\beta^2} + \frac{12}{\beta} A_{\phi}} + \frac{6c}{\beta} > 0.
\]
Hence, \(F(t)\) has the real and symmetric roots \(\pm \eta_1\) and \(\pm \eta_2\). Without loss of generality, we assume that \(0 < \eta_2 < \eta_1\). Hence, we can write
\[
(\phi')^2 = \frac{\beta}{6} (\phi^2 - \eta_2^2)(\eta_1^2 - \phi^2).
\tag{14}
\]
Since \(\beta > 0\), the left side of (14) is not negative. Then, we obtain that \(\eta_2 \leq \phi \leq \eta_1\) and the \(\eta_i\)s satisfy
\[
\begin{cases}
\eta_1^2 + \eta_2^2 = \frac{12c}{\beta} > 0, \\
-\eta_1^2 \eta_2^2 = \frac{12}{\beta} A_{\phi} < 0.
\end{cases}
\tag{15}
\]
From the first equation of (15), we have \( c > 0 \). Define \( \rho = \frac{\phi}{n} \) and \( k^2 = \frac{n^2 - \eta^2}{\eta^2} \), then (14) becomes

\[
(r)^2 = \frac{\beta \eta^2}{6} (\rho^2 - \frac{\eta^2}{\eta^2})(1 - \rho^2).
\]

(16)

And then, define a new variable \( \chi \) through the relation \( \rho^2 = 1 - k^2 \sin^2 \chi \), from (15) and (16), we get

\[
(\chi')^2 = \frac{\beta \eta^2}{6} (1 - k^2 \sin^2 \chi).
\]

(17)

According to the definition of the Jacobi elliptic function snoidal, we get that

\[
\int_0^{\chi(\xi)} dt \sqrt{1 - k^2 \sin^2 t} = \frac{\sqrt{6\beta}}{6} \eta_1 \xi
\]

(18)

has the solution

\[
\sin(\chi(\xi)) = sn(\frac{\sqrt{6\beta}}{6} \eta_1 \xi; k).
\]

Hence, using the fact that \( k^2 \sin^2 \chi + \rho^2 = 1 \), we obtain

\[
\rho(\xi) = \sqrt{1 - k^2 \sin^2 \chi} = \sqrt{1 - k^2 \sin^2 (\frac{\sqrt{6\beta}}{6} \eta_1 \xi; k)} = dn(\frac{\sqrt{6\beta}}{6} \eta_1 \xi; k),
\]

and \( \rho(0) = 1 \). Substituting the form of \( \rho(\xi) \) to the definition \( \rho = \frac{\phi}{n} \), we get the snoidal wave solution

\[
\phi(\xi) = \eta_1 dn(\frac{\sqrt{6\beta}}{6} \eta_1 \xi; k).
\]

(19)

Substituting (19) into (11), we have

\[
\psi(\xi) = c - \eta_1 dn(\frac{\sqrt{6\beta}}{6} \eta_1 \xi; k).
\]

(20)

Since \( dn \) has fundamental period \( 2K \), namely, \( dn(u; k) = dn(u + 2K; k) \), where \( K = K(k) \) represents the complete elliptic integral of first kind, we obtain that \( \phi \) and \( \psi \) have fundamental period

\[
T_\phi = T_\psi = \frac{12}{\sqrt{6\beta \eta_1}} K(k).
\]

(21)

Then, from (15), we get \( 0 < \eta_2 < \sqrt{\frac{6\beta}{3}} < \eta_1 < \frac{2\sqrt{6\beta}}{3} \), and fundamental period \( T_\phi = T_\psi \) can be seen as a function of variable \( \eta_2 \) only, that is

\[
T_\phi(\eta_2) = T_\psi(\eta_2) = \frac{2\sqrt{6}}{\sqrt{12c - \beta \eta_2^2}} K(k(\eta_2)), \quad \text{with} \quad k^2(\eta_2) = \frac{12c - 2\beta \eta_2^2}{12c - \beta \eta_2^2}.
\]

(22)

Next, we will show that \( T_\phi = T_\psi > \frac{\pi}{\sqrt{\beta}} \). Note that if \( \eta_2 \rightarrow 0 \), we have that \( k(\eta_2) \rightarrow 1^- \), and then \( K(k(\eta_2)) \rightarrow +\infty \). Therefore, \( T_\phi, T_\psi \rightarrow +\infty \) as \( \eta_2 \rightarrow 0 \). On the other hand, if \( \eta_2 \rightarrow \frac{\sqrt{6\beta}}{3} \), we have that \( k(\eta_2) \rightarrow 0^+ \), which imply that \( K(k(\eta_2)) \rightarrow \frac{\pi}{2} \). Therefore, \( T_\phi, T_\psi \rightarrow \frac{\pi}{\sqrt{\beta}} \) as \( \eta_2 \rightarrow \frac{\sqrt{6\beta}}{3} \). Moreover, since the function \( \eta_2 \in (0, \frac{\sqrt{6\beta}}{3}) \rightarrow T_\phi(\eta_2) = T_\psi(\eta_2) \) decreases strictly (see proof of Theorem 1.2), it follows that \( T_\phi = T_\psi > \frac{\pi}{\sqrt{\beta}} \).
For $L > 0$ fixed, we choose $c > 0$ such that $\sqrt{c} > \frac{\pi}{L}$. From the analysis given above, there exists a unique $\eta_2$ such that the fundamental period of the dnoidal wave $\phi = \phi(\cdot; \eta_1(c); \eta_2(c))$ and $\psi = \psi(\cdot; \eta_1(c); \eta_2(c))$ will be $T_\phi(\eta_2) = T_\psi(\eta_2) = L$.

**Remark 2.1.** If $\eta_2 \to 0^+$, we obtain that $\eta_1 \to \frac{2\sqrt{3c\beta}}{\beta}$, $k(\eta_2) \to 1^-$. Then, on the basis of the limitation $dn(x,1) = sech(x)$, the formulae (19) and (20) lose its periodicity and we get a waveform with a single hump and with “infinity period” of the form

$$
\phi(\xi; \frac{2\sqrt{3c\beta}}{\beta}, 0) \to \frac{2\sqrt{3c\beta}}{\beta} sech(\sqrt{2c}\xi), \text{ and } \psi(\xi; \frac{2\sqrt{3c\beta}}{\beta}, 0) \to c - \frac{2\sqrt{3c\beta}}{\beta} sech^2(\sqrt{2c}\xi),
$$

which are the classical solitary wave solutions for the coupled compound KdV and MKdV equations.

In the following, by applying the implicit function theorem, we show that the Theorem 1.2 holds.

**Proof of Theorem 1.2.** Firstly, we define a function $\Lambda : \Omega \to R$ by

$$
\Lambda(\eta, c) = \frac{2\sqrt{6}}{\sqrt{12c - \beta\eta^2}} K(k) - L,
$$

in the open set

$$
\Omega = \{(\eta, c) \in R^2 : c > \frac{\pi^2}{L^2} \text{ and } \eta \in (0, \frac{\sqrt{6c\beta}}{\beta})\},
$$

where

$$
k^2(\eta, c) = \frac{12c - 2\beta\eta^2}{12c - \beta\eta^2}.
$$

From the hypotheses, we get $\Lambda(\eta_2, 0) = 0$.

Next, we show that $\partial_\eta \Lambda < 0$ in $\Omega$. Differentiating (24) with respect to $\eta$, we have

$$
\frac{\partial k}{\partial \eta} = -\frac{12\eta c\beta}{k(12c - \beta\eta^2)^2} < 0.
$$

Hence, the function $k(\eta, c)$ decreases strictly with respect to $\eta$. Then, according to the relation

$$
\frac{dK(k)}{dk} = \frac{E(k) - k^2K(k)}{kk'^2},
$$
Thus, there exist intervals $I(0, \sqrt{\frac{2}{3}})$ with respect to $k'$ in $(0, 1)$. Differentiating $f(k')$ defined in (28) with respect to $k'$ and using the relation $k' \frac{dE(k)}{dk} = E(k) - K(k)$ and $E(k) < K(k)$, we have

$$\frac{\partial f(k')}{\partial k'} = 3k'(E - K) < 0.$$  

Thus, $f(k')$ is a decreasing function. Since $f(1) = 0$, we have $f(k') > f(1) = 0$ for $k' \in (0, 1)$, which show (28) and verify $\frac{\partial \Lambda}{\partial \eta} < 0$. Hence, applying the implicit function theorem, there exist intervals $I(c_0)$ and $B(\eta_{2,0})$ around $c_0$ and $\eta_{2,0}$ respectively, and a unique smooth function, $\Pi : I(c_0) \rightarrow B(\eta_{2,0})$ such that $\Pi(c_0) = \eta_{2,0}$ and $\Lambda(\Pi(c), c) = 0$, $\forall c \in I(c_0)$. So, we can obtain (1) of Theorem 1.2.

Since $c_0$ is chosen arbitrarily in the interval $I = \left(\frac{2}{\sqrt{3}}, +\infty\right)$, from the uniqueness of the function $\Lambda$, it follows that we can extend $\Lambda$ to $(\frac{2}{\sqrt{3}}, +\infty)$. Using the smoothness of the function involved, we can immediately obtain part(2) of Theorem 1.2. □

Corollary 2.2. The map $\Pi : I(c_0) \rightarrow B(\eta_{2,0})$ is a strictly decreasing function. Therefore, from (24), $k(c)$ strictly increases with respect to $c$.

Proof. According to the proof of Theorem 1.2, we know that the function $\Lambda(\eta, c)$ is strictly decreasing with respect to $\eta$. Note that $\Lambda(\Pi(c), c) = 0$ for all $c \in I(c_0)$ in Theorem 1.2, we have to prove $\frac{\partial \Lambda}{\partial c} < 0$ in $I(c_0)$ for $\frac{\partial \Lambda}{\partial c} < 0$. Since $\eta^2 = \frac{(12c - \beta \eta^2)}{\beta} k'^2$, differentiating (23) with respect to $c$ and combining (24), (26), we have

$$\frac{\partial \Lambda(c, \eta)}{\partial c} = \frac{12\sqrt{6}}{(12c - \beta \eta^2)^{\frac{3}{2}}} \left[\frac{\beta \eta^2}{k(12c - \beta \eta^2)} \frac{dK}{dk} - K\right]$$

$$= \frac{12\sqrt{6}}{(12c - \beta \eta^2)^{\frac{3}{2}}} \left[\frac{\beta \eta^2}{k(12c - \beta \eta^2)} \frac{E - k'^2 K}{kk'^2} - K\right]$$

$$= \frac{12\sqrt{6}}{(12c - \beta \eta^2)^{\frac{3}{2}}} \left[E - K\right].$$  

(29)
Since \( E(k) < K(k) \) for any \( k \in (0, 1) \) (see section 2 in [22]), we have \( \frac{\partial L}{\partial c} < 0 \) from (29). Hence, from the relation \( \frac{\partial L}{\partial c} + \frac{\partial L}{\partial k} = 0 \), we obtain \( \frac{\partial L}{\partial k} = -\frac{\partial L}{\partial c} < 0 \), namely, \( \Pi \) is a strictly decreasing function with respect to \( c \). Next, differentiating \( k \) given by (24) with respect to \( c \), we have

\[
\frac{dk}{dc} = \frac{6\beta\eta(\gamma - 2c\frac{dL}{dc})}{k(12c - \beta\eta^2)^2} > 0.
\]

Then, we prove that the function \( k(c) \) is strictly increasing with respect to \( c \) which show that Corollary 2.2 holds.

**Corollary 2.3.** Let \( L > 0 \) and \( c \in \left( \frac{\pi}{2L}, +\infty \right) \). From the dnonoidal waves \( c \mapsto (\phi(\cdot; \eta_1(c), \eta_2(c)), \psi(\cdot; \eta_1(c), \eta_2(c))) \) given by Theorem 1.2, we have

\[
\frac{d}{dc} \int_0^L \phi_\xi^2(\xi) d\xi > 0.
\]

**Proof.** Combining \( 2\sqrt{\beta\eta} K = \eta_1 L \) and \( \int_0^K dn^2(x) dx = E(k) \), we have

\[
\int_0^L \phi_\xi^2(\xi) d\xi = \eta_1^2 \int_0^L dn^2 \left( \frac{2\beta\eta}{6} \eta_\xi, k \right) d\xi = 2\eta_1 \frac{\sqrt{\beta\eta}}{3} \int_0^K dn^2(x) dx = \frac{24}{L^2} KE.
\]

Since functions \( K(K)E(k) \) and \( k(c) \) are strictly increasing with respect to \( k \) and \( c \) respectively, we arrive at \( \frac{d}{dc} \int_0^L \phi_\xi^2(\xi) d\xi > 0 \), which prove that Corollary 2.3 holds.

3. **Spectral analysis.** Before starting the spectral analysis of a linear operator, we firstly derive the operator \( L_1 \). Differentiating (12) with respect to \( x \), we have

\[
-\partial_\xi^2 + 2c - \beta \phi^2 \phi_x = 0,
\]

then, we define the operator \( L_1 = -\partial_\xi^2 + 2c - \beta \phi^2 \), that is, \( L_1 \phi_x = 0 \).

Our main purpose in this section is devoted to consider the spectral properties related to the linear operator

\[
L_1 = -\frac{d^2}{dx^2} + 2c - \beta \phi^2,
\]

(30)

which play a key role in the proof of the orbital stability of dnoidal wave solutions, where \( \phi \) is the dnonoidal wave solution (19) with the fundamental period \( L \) and \( c \in \left( \frac{\pi}{2L}, +\infty \right) \). We write \( L_1 = L_2 + M_2 \), where \( L_2 = -\frac{\partial^2}{\partial x^2} + 2c \). Since \( M_2 \) is relatively compact with respect to \( L_2 \), we have \( \sigma_{ess}(L_1) = \sigma_{ess}(L_2) \), following from Weyl’s essential spectrum theorem. The spectrum properties for operators \( L_1 \) in the following theorem are obtained with the assistance of the periodic eigenvalue problem considered on \([0, L]\)

\[
\begin{aligned}
L_1 \chi &= \lambda \chi, \\
\chi(0) &= \chi(L), \quad \chi'(0) = \chi'(L),
\end{aligned}
\]

(31)

and a semi-periodic eigenvalue problem considered on \([0, L]\)

\[
\begin{aligned}
L_1 \chi &= \mu \chi, \\
\chi(0) &= -\chi(L), \quad \chi'(0) = -\chi'(L).
\end{aligned}
\]

(32)

(see (4.2)-(4.8) in [22]).

**Theorem 3.1.** Consider the dnonoidal wave solutions \( \phi = \phi(\cdot; \eta_1(c), \eta_2(c)) \) and \( \psi = \psi(\cdot; \eta_1(c), \eta_2(c)) \) given by Theorem 1.2. Then, the operator \( L_1 : H^2_{per}([0, L]) \to H^2_{per}([0, L]) \)
the first three eigenvalues $\lambda_0, \lambda_1$ and $\lambda_2$, where $\lambda_1 = 0$ is the second one with associated eigenfunction $\phi'$. Moreover, the remainder of the spectrum for operator $L_1$ consists of a discrete set of eigenvalues which are double.

**Proof.** Initially, From (4.7) in [22], we will proof that $0 = \lambda_1 < \lambda_2$. Differentiating (12), we have $L_1 \phi' = 0$, that is, zero is an eigenvalue of $L_1$ with associated eigenfunction $\phi'$. From (19), we know that $\phi'$ has two zeros in $[0, L)$, then the eigenvalue zero is either $\lambda_1$ or $\lambda_2$ (see (4.8) of [22]). Next, we show that the eigenvalue zero is the second one. By using the transformation $\Psi(x) = e^{\eta x}$ where $\eta^2 = \frac{a}{\beta \eta}$ and the relation $k^2 sn^2(x) + dn^2(x) = 1$, we get that problem (31) turns to the eigenvalue problem

$$\begin{align*}
\Psi'' + (\varrho - 6k^2 sn^2(x; k))\Psi &= 0, \\
\Psi(0) &= \Psi(2K), \quad \Psi'(0) = \Psi'(2K),
\end{align*}$$

(33)

where the relation between $\varrho$ and $\lambda$ is given by

$$\varrho = \frac{6}{\beta \eta_1^2} (\lambda + \beta \eta_1^2 - 2c).$$

(34)

The second order differential equation in (33) is called the Jacobian form of Lamé's equation. From Floquet theory ([18], Theorem 7.8), we have that Eq. (33) has exactly 3 intervals of instability: $(-\infty, \varrho_0)$, $(\varrho_0, \varrho_1)$, $(\varrho_1, \varrho_2)$ where $\varrho_1$ are the eigenvalues associated to the periodic problem and $\varrho_2$ are the eigenvalues associated to the semi-periodic problem determined by the Lamé’s equation for $i \geq 0$. Hence, the first three eigenvalues $\varrho_0, \varrho_1, \varrho_2$ will be simple and the rest of the eigenvalues $\varrho_3 \leq \varrho_4 \leq \varrho_5 \leq \varrho_6 \leq \cdots$ satisfy that $\varrho_3 = \varrho_4 = \varrho_5 = \cdots$.

Now, we give explicit formulas for the eigenvalues $\varrho_0, \varrho_1, \varrho_2$ and its corresponding eigenfunctions $\Psi_0, \Psi_1, \Psi_2$. First, we observe that $\varrho_1 = 4 + k^2$ satisfies $L_1 \Psi_1 = \varrho_1 \Psi_1$ with $\Psi_1(x) = sn(x)cn(x)$. Moreover, from Ince [16], we have that the functions $\Psi_0 = 1 - (1 + k^2 - \sqrt{1 - k^2 + k^2}sn^2$, and $\Psi_2 = 1 - (1 + k^2 + \sqrt{1 - k^2 + k^2})sn^2$, with period $2K$, satisfy that $L_1 \Psi_0 = \varrho_0 \Psi_0$ and $L_1 \Psi_2 = \varrho_2 \Psi_2$, with

$$\varrho_0 = 2(1 + k^2 - \sqrt{1 - k^2 + k^2}), \quad \text{and} \quad \varrho_2 = 2(1 + k^2 + \sqrt{1 - k^2 + k^2}).$$

Observe that $\Psi_0$ has no zeros in $[0, 2K]$, $\Psi_2$ has two zeros in $[0, 2K]$ and $\varrho_0 < \varrho_1 < \varrho_2$ for every $k \in (0, 1)$, then $\varrho_0$ is the first eigenvalue, $\varrho_1$ is the second eigenvalue and $\varrho_2$ is the third eigenvalue. Since the relation between $\varrho$ and $\lambda$ is given by

$$\lambda = \frac{\beta \eta_1^2}{6} (\varrho - 6) + 2c,$$

we obtain that the function $\lambda$ is increasing with respect to $\varrho$, then $\lambda_0 < \lambda_1 < \lambda_2$. Since $k^2 = 2 - \frac{12c}{\beta \eta_1^2}$ from (24) and $\lambda_0 < \lambda_1 < \lambda_2$, we have that $\lambda(\varrho_1) = 0 = \lambda_1$ and $\lambda_0 < 0$.

To complete the proof of Theorem 3.1, we need to study the semi-periodic problem (32) and get the first two eigenvalues $\varrho_0, \varrho_1$ related to problem (32). Similar to Eqs. (33), the semi-periodic problem (32) can also turn to the Lamé’s equation in (33) with the conditions $\Psi(0) = -\Psi(2K), \Psi'(0) = -\Psi'(2K)$, and the eigenvalues $\varrho_i$ associated to problem (32) are related to the $\mu_i$ by the relation between $\varrho_i$ and $\mu_i$

$$\varrho_i = \frac{6}{\beta \eta_1^2} (\mu_i + \beta \eta_1^2 - 2c).$$

(35)
4. Orbital stability of the dnoidal wave solutions for the coupled compound KdV and MKdV equations. Let \( U = (u, v)^T \). The function space in which we will work is \( X = H^1_{\text{per}}([0, L]) \times H^1_{\text{per}}([0, L]) \), with inner product

\[
(f, g) = \int_R (f_1 g_1 + f_1 x g_1 x + f_2 g_2 + f_2 x g_2 x) dx, \quad \text{for } f, g \in X. \tag{36}
\]

We know that the dual space of \( X \) is \( X^* = H^{-1}_{\text{per}}([0, L]) \times H^{-1}_{\text{per}}([0, L]) \). Define the pairing \( <\cdot, \cdot> \) between \( X \) and \( X^* \) by

\[
<f, g> = \int_0^L (f_1 g_1 + f_2 g_2) dx, \quad \text{for } f, g \in X. \tag{37}
\]

Then, there is a natural isomorphism \( I : X \to X^* \) defined by \( <If, g> = (f, g) \).

By (36) and (37), we obtain that

\[
I = \begin{pmatrix}
1 - \frac{\partial^2}{\partial x^2} & \frac{\partial^4}{\partial x^4} \\
0 & 1
\end{pmatrix}.
\]

Define one-parameter groups of unitary operator \( T \) on \( X \) by

\[
T(s)U(\cdot) = U(\cdot + s), \quad \text{for } U(\cdot) \in X, \ s \in R. \tag{38}
\]

Differentiating (38) with respect to \( s \) at \( s = 0 \), we can obtain

\[
T'(0) = \begin{pmatrix}
\frac{\partial}{\partial x} & 0 \\
0 & \frac{\partial}{\partial x}
\end{pmatrix}. \tag{39}
\]

It follows from (19), (20) that Eqs.(4) exist solitary waves \( T(ct)\Phi_c(x) \) with \( \Phi_c(x) \) defined by

\[
\Phi_c(x) = (\phi(x), \psi(x)) = (\eta_1 dn(\sqrt{\frac{6}{k}} \eta_1 x; k), c - \eta_1 dn(\sqrt{\frac{6}{k}} \eta_1 x; k)). \tag{40}
\]

In this section, we shall consider the orbital stability of solitary waves \( T(ct)\Phi_c(x) \) of (4). We will prove that Eqs.(4) are a Hamiltonian system, and satisfy the conditions of the general orbital stability theory proposed by Grillakis et al. [11]. Note that Eqs.(4) are invariant under \( T(\cdot) \), we define the orbital stability as follows:

**Definition 4.1** Let \( \Phi_c = (\phi, \psi) \in X \) be a dnoial wave solution of equation (4), where \( X = H^1_{\text{per}}([0, L]) \times H^1_{\text{per}}([0, L]) \). We say that the orbit generated by \( \Phi_c \),

\[
\Theta_{\Phi_c} := \{ (\phi(\cdot + s), \psi(\cdot + s)) : s \in R \} \tag{41}
\]
is stable in $X$ by the periodic flow generated by the coupled nonlinear wave equation (4), if for every $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that for any $U_0 = (u_0(x), v_0(x)) \in X$ satisfying
\[
\inf \| U_0 - \Phi_c \|_X < \delta,
\]
we have that the solution $U(x, t) = (u(x, t), v(x, t))$ of the system (4) with initial data $U(0) = (u(0), v(0)) = (u_0, v_0)$ satisfies
\[
\sup_{0 \leq t < +\infty} \inf_{s \in \mathbb{R}} \| U(t) - T(s) \Phi_c \|_X < \epsilon.
\]
(42)
Otherwise, we say that $T(s) \Phi_c$ is called orbitally unstable.

Define
\[
E(U) = \int_R \left( \frac{u^2}{2} + \frac{1}{6} u^3 - \frac{1}{2} uv^2 - \frac{\beta}{12} u^4 - \frac{v^3}{3} + \frac{c^2}{2} u \right) dx.
\]
(43)
By (38) and (43), we can verify that $E(U)$ is invariant under operator $T$, namely,
\[
E(T(s)U) = E(U), \quad \text{for any } s \in \mathbb{R},
\]
(44)
and $E(U)$ is conserved, namely, for any $t \in \mathbb{R}$,
\[
E(U(t)) = E(U(0)).
\]
(45)
Note that the system (4) can be written as the following Hamiltonian system:
\[
\frac{dU}{dt} = JE'(U),
\]
(46)
where $U = (u, v)^T$, $J$ is a skew-symmetrically linear operator defined by
\[
J = \begin{pmatrix}
\frac{\partial}{\partial x} & 0 \\
0 & \frac{\partial}{\partial x}
\end{pmatrix},
\]
and
\[
E'(U) = \begin{pmatrix}
-u_{xx} + \frac{1}{2} u^2 - \frac{\beta}{3} u^3 - \frac{1}{2} v^2 + \frac{c^2}{2} u \\
u v - v^2
\end{pmatrix}
\]
(47)
is the Frechet derivative of $E$.

From (47), we get the Frechet derivative of $E'(U)$
\[
E''(U) = \begin{pmatrix}
-\partial_x^2 + u - \beta u^2 & -v \\
-v & -u - 2v
\end{pmatrix}.
\]
Let
\[
B = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix},
\]
such that $T'(0) = JB$. Then, we can define the following conserved functional $Q(U)$
\[
Q(U) = \frac{1}{2} < BU, U > = \frac{1}{2} \int_R u^2 + v^2 dx.
\]
(48)
By (38) and (48), we can verify that $Q(U)$ is invariant under operator $T$, namely,
\[
Q(T(s)U) = Q(U), \quad \text{for any } s \in \mathbb{R},
\]
\[
Q(U(t)) = Q(U(0)),
\]
and
\[
Q'(U) = BU = U, \quad Q''(U) = B.
\]
(49)
Then, combining (10), (47), and (49), we have
\[ E'(\Phi_c) + cQ'(\Phi_c) = 0. \]  
(50)

Define an operator \( H_c : X \to X^* \) by
\[ H_c = E''(\Phi_c) + cQ''(\Phi_c). \]  
(51)

Since \( c \) is fixed, we write \( \Phi \) for \( \Phi_{c} \), and we have
\[ H_c = \left( \begin{array}{ccc} -\partial_x^2 + 2c - \beta \phi^2 - \psi & -\psi & -\psi \\ -\psi & -\psi & -\psi \end{array} \right). \]  
(52)

From the definition of inner product (37), we know that \( H_c \) is a self-adjoint operator in the sense of \( H^*_c = H_c \). In effect, the operator \( I^{-1}H_c \) is bounded self-adjoint on \( X \). Then, the spectrum of \( H_c \) is constituted by the real numbers \( \lambda \) such that \( H_c - \lambda I \) is not invertible.

For any \( y = (y_1, y_2) \in X \), by (52), we have
\[ \langle H_c y, y \rangle = \langle (\partial_x^2 + 2c - \beta \phi^2)y_1 - \psi(y_1 + y_2), y_1 \rangle + \langle -\psi(y_1 + y_2), y_2 \rangle \]
\[ = \langle L_1y_1, y_1 \rangle - \int_R \psi(y_1 + y_2)^2 dx, \]  
(53)

where \( L_1 = -\partial_x^2 + 2c - \beta \phi^2 \). In order to get the spectrum properties of operator \( H_c \), we make \( \psi < 0 \). Since \( \beta^2 \sigma^2 + \sigma^2 = 1 \), we have \( 1 - k^2 < \sigma^2 \leq 1 \). Then, if \( c - \sqrt{1-k^2} \eta_1 < 0 \), \( \psi < 0 \). Hence, \( c < \eta_2 < \sqrt{\frac{2c}{\sigma}} \) and \( \eta_1 < \sqrt{\frac{12c}{\sigma} - c^2} \).

So, we have \( 0 < c < \frac{6}{\beta} \). In the condition of \( 0 < c < \frac{6}{\beta} \), according to Theorem 3.1, for any \( \Psi = (y_1, y_2) \), we choose \( y_1 = \Psi_0(\frac{\sqrt{6\beta}}{\beta} \eta_1 x) \), \( y_2 = -\Psi_0(\frac{\sqrt{6\beta}}{\beta} \eta_1 x) \), and \( \Psi^- = (\Psi_0(\frac{\sqrt{6\beta}}{\beta} \eta_1 x), -\Psi_0(\frac{\sqrt{6\beta}}{\beta} \eta_1 x)) \), then
\[ \langle H_c \Psi^-, \Psi^- \rangle = \lambda_0(\Psi_0(\frac{\sqrt{6\beta}}{\beta} \eta_1 x), \Psi_0(\frac{\sqrt{6\beta}}{\beta} \eta_1 x)) < 0, \]  
(54)

where \( \lambda_0 < 0 \) defined in Theorem 3.1 is the eigenvalue of \( L_1 \). Then, operator \( H_c \) has one simple negative eigenvalue with associated eigenfunction \( \Psi^- \).

If \( \beta_1 \) denotes the second eigenvalue of operator \( H_c \), then by min-max characterization of eigenvalues, we have
\[ \beta_1 = \max_{y_1, y_2} \min_{y_1 \perp \Psi_0(\frac{\sqrt{6\beta}}{\beta} \eta_1 x), y_2 \perp -\Psi_0(\frac{\sqrt{6\beta}}{\beta} \eta_1 x)} \frac{\langle H_c \Psi, \Psi \rangle}{\langle \Psi, \Psi \rangle} \]  
(55)

From (55), we have
\[ \beta_1 \geq \min_{y_1 \perp \Psi_0(\frac{\sqrt{6\beta}}{\beta} \eta_1 x), y_2 \perp -\Psi_0(\frac{\sqrt{6\beta}}{\beta} \eta_1 x)} \frac{\langle H_c \Psi, \Psi \rangle}{\langle \Psi, \Psi \rangle}. \]  
(56)

From Theorem 3.1, we know zero is the second one and simple with associated eigenfunction \( \phi_\lambda \). Combining the above inequality, we have that \( \beta_1 = 0 \) is the second eigenvalue of operator \( H_c \) with associated eigenfunction \( T'(0)\Phi_c \), namely,
\[ H_c T'(0)\Phi_c = 0. \]  
(57)

Moreover, from Theorem 3.1, we know that the rest of spectrum for operator \( L_1 \) is positive away from zero. Let
\[ P = \{ p \in X | p = (p_1, p_2), \langle p_1, \Psi_0(\frac{\sqrt{6\beta}}{\beta} \eta_1 x) \rangle = \langle p_1, \phi_\lambda \rangle = 0 \}. \]  
(58)
Again using min-max characterization of eigenvalues, we get that the third eigenvalue of $H_c$ is strictly positive, that is, for any $\zeta \in P$ defined by (58), there exist $\delta_3 > 0$ such that

$$\langle H_c \zeta, \zeta \rangle \geq \delta_3 \| \zeta \|_X^2,$$

(59)

where $\delta_3$ is independent of $\zeta$.

Let

$$Z = \{ k_1 T'(0) \Phi_c | k_1 \in \mathbb{R} \},$$

(60)

$$N = \{ k_2 \Psi^- | k_2 \in \mathbb{R} / \{0\} \}.$$

(61)

From (57) and (60), we obtain $Z$ is contained in the kernel of $H_c$. Combining (54) and (61), we have

$$\langle H_c U, U \rangle = \langle H_c k_2 \Psi^-, k_2 \Psi^- \rangle = -k_2^2 \langle \Psi^-, \Psi^- \rangle < 0$$

for any $U \in N$.

Hence, according to the above discussion, the space $X$ in which we work can be divided into a direct sum $X = N + Z + P$, that is the assumption 3.3 in [11] holds.

Next, we define function $d(c) : \mathbb{R} \to \mathbb{R}$ by

$$d(c) = E(\Phi_c) + cQ(\Phi_c)$$

(62)

and define $d''(c)$ to be the second derivative of $d$ with respect to $c$. Then, we denote $n(H_c)$ the numbers of negative eigenvalue of $H_c$, and $p(d'')$ the numbers of positive eigenvalue of $d''$ at $c$.

According to Lemma 1.1, (50) and (62), we prove that Theorem 1.3 holds.

Combining (40), (48) with (50) and differentiating (62) once with respect to $c$, it follows that

$$d'(c) = \langle E'(\Phi_c), \Phi_c' \rangle + cQ'(\Phi_c), \Phi_c' \rangle + Q(\Phi_c)$$

$$= Q(\Phi_c) = \frac{1}{2} \int_{0}^{L} (\phi_c^2 + \psi_c^2) dx = \frac{1}{2} \int_{0}^{L} (\phi_c^2 + (c - \phi_c)c^2) dx$$

$$= \int_{0}^{L} (\phi_c^2 - c\phi + \frac{c^2}{2}) dx = \int_{0}^{L} \phi^2 dx - c\sqrt{\frac{6}{\beta}} \int_{0}^{2K} dn(x) dx + \frac{c^2}{2} L$$

$$= \int_{0}^{L} \phi^2 dx + \frac{c^2}{2} L,$$

(63)

where we have used $\int d\nu(x) dx = \arcsin(sn(x)) + C$, $sn(0) = sn(2K) = 0$. Hence, according to Corollary 2.3, we have

$$d''(c) = \frac{d}{dx} \int_{0}^{L} \phi^2 dx + cL > 0.$$

Therefore, we obtain that periodic solitary waves $T(ct) \Phi_c(x)$ is orbitally stable. This completes the proof of Theorem 1.3.

\[\Box\]

**Remark 4.2.** From Remark 2.1, the method to show Theorem 1.3 can also be applied to prove the orbital stability in $H^1(R) \times H^{-1}(R)$ of the solitary wave solution in Remark 2.1 for the coupled compound KdV and MKdV equations (4). This gives another method to show orbital stability of solitary waves with zero asymptotic value in our work. Our work improve the orbital stability results of solutions for the coupled compound KdV and MKdV equations (4). Moreover, when the variable $v = 0$, Eqs.(4) deduce to the compound KdV equation which was studied by Zhang[21]. Let $v = 0$ in our work, we can also obtain the orbital stability results of solitary wave solution for the compound KdV equation in the sense of limit.
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