A FABER–KRAHN INEQUALITY FOR INDENTED AND CUT MEMBRANES

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In 1960, Payne and Weinberger proved that, among all domains that lie within a wedge (an angle whose measure is less than or equal to $\pi$) and have a given value of a certain integral, the circular sector has the lowest fundamental eigenvalue of the Dirichlet Laplacian. We show that an analogue of this assertion is true for domains with a cut and for indented domains, i.e., for those located in a reflex angle (its measure is between $\pi$ and $2\pi$). Bibliography: 9 titles.

1 Introduction

Isoperimetric inequalities for eigenvalues of the Laplacian have its roots in the work of Lord Rayleigh presented in the first volume of his monograph *The Theory of Sound* [1]. It was found that the normal modes and proper frequencies characterizing the vibrations of a homogeneous, elastic membrane fixed along its boundary are determined by the eigenvalue problem for the Dirichlet Laplacian on a plane, bounded domain (see [2] for a review and historical remarks).

Indeed, let $D \subset \mathbb{R}^2$ be a bounded domain with a piecewise smooth boundary ($D$ coincides with the membrane at rest). If for some real $\lambda$ the boundary value problem

$$u_{xx} + u_{yy} + \lambda u = 0 \text{ in } D, \quad u = 0 \text{ on } \partial D$$

has a nontrivial solution continuous on $\overline{D}$ and belonging to $C^2(D)$, then $\lambda$ and $u$ are a Dirichlet eigenvalue of $D$ and the corresponding eigenfunction respectively. The sequence of Dirichlet eigenvalues is positive, and the squares of the membrane eigenfrequencies are proportional to them. This sequence is characterized by the max-min principle (see, for example, [3, 4]), according to which the lowest eigenvalue $\lambda_1(D)$ is simple and the corresponding eigenfunction $u_1$ can be taken to be positive in the interior of $D$.

This eigenvalue has the following isoperimetric property referred to as the Faber–Krahn inequality:

$$\lambda_1(D) \geq \pi j_{0,1}^2/|D|,$$

(1.1)
where $|D|$ is the area of $D$ and $j_{0,1} = 2.4048\ldots$ is the first zero of the Bessel function $J_0$ (the notation of [5] is used for Bessel functions and their zeros). Equality is attained in (1.1) if and only if $D$ is a disc. In other words, among all homogeneous membranes of a given area, the circular one has the lowest fundamental frequency because $j_{0,1}^2$ is the lowest eigenvalue for the unit disc. This inequality was conjectured by Lord Rayleigh [1, pp. 339–340] on the basis of numerical computations for simple domains and a variational argument for nearly circular domains. Independent proofs of (1.1) were given by Faber [6] and Krahn [7]; the last author also proved its higher-dimensional version [8].

Other isoperimetric inequalities are discussed in [3]. In particular, for domains subject to certain restrictions it is possible to improve the lower bound obtained by Faber and Krahn. For example, Payne and Weinberger [9] improved (1.1) for the so-called wedge-like domains. In terms of $S_\alpha = \{(r,\theta) : r \in (0,\infty); \theta \in (0,\pi/\alpha)\}, \quad \alpha \geq 1,$ where $(r,\theta)$ is the polar coordinate system on the $(x,y)$-plane, their result is as follows (see also [4, Chapter III, Section 2.3]).

**Theorem 1.1.** Let $D \subset S_\alpha$, and let

$$J_\alpha(D) = \int_D r^{2\alpha+1} \sin^2 \alpha \theta \, r \, d\theta$$

be fixed. Then

$$\lambda_1(D) \geq \left[4\pi^{-1}(\alpha + 1)J_\alpha(D)\right]^{-1/(\alpha+1)}j_{\alpha,1}^2,$$

(1.2)

where $j_{\alpha,1}$ is the first positive zero of the Bessel function $J_{\alpha}$. Equality is attained when $D$ is a circular sector of angle $\pi/\alpha$.

Thus, a circular sector of angle $\pi/\alpha$ (in [9], this number is misprinted as $\alpha$) has the lowest fundamental eigenvalue among all domains lying in $S_\alpha$ and having a given value of $J_\alpha(D)$.

It is natural to ask whether an analogue of this assertion is true for indented domains, i.e., for those located in a reflex angle (its radian measure is between $\pi$ and $2\pi$) or in the plane with an infinite straight cut. The goal of this note is to show how to obtain the corresponding result by modifying considerations in [9].

First, instead of $S_\alpha$ it is convenient to introduce

$$R_\beta = \{(r,\theta) : r \in (0,\infty); \theta \in (-\pi/\beta,\pi/\beta)\}, \quad \beta \in [1,2].$$

The plane cut along the negative $x$-axis corresponds to $\beta = 1$, and for $\beta \in (1,2)$ one obtains the whole family of reflex angles centered at the origin. Finally, the half-plane $\{x > 0\}$ corresponds to $\beta = 2$. Now, we are in a position to formulate the following result.

**Theorem 1.2.** Let $D \subset R_\beta$, and let

$$I_\beta(D) = \int_D r^{\beta+1} \cos^2 \frac{\beta \theta}{2} \, r \, d\theta$$

be fixed. Then

$$\lambda_1(D) \geq \left[\pi^{-1}\beta(\beta + 2)I_\beta(D)\right]^{-2/(\beta+2)}j_{\beta/2,1}^2,$$

(1.3)

where $j_{\beta/2,1}$ is the first positive zero of the Bessel function $J_{\beta/2}$. Equality is attained when $D$ is a circular sector of angle $2\pi/\beta$.
Thus, a circular sector of angle $2\pi/\beta$ has the lowest fundamental eigenvalue among all domains lying in $R_\beta$ and having a given value of $I_\beta(D)$. It should be also mentioned that the inequalities (1.3) with $\beta = 2$ and (1.2) with $\alpha = 1$ are just two different forms of the same fact. Indeed, let $D' \subset S_1$ coincide with $D \subset R_2$ rotated by $\pi/2$. Of course, $\lambda_1(D') = \lambda_1(D)$, but the same is true for the right-hand side expressions of these inequalities which follows by changing variables. Therefore, it is reasonable to consider the inequality (1.3) as a generalization of (1.2) to the case of reflex angles.

Both lower bounds (1.2) and (1.3) for particular domains depend on the choice of the origin. In this regard, it is reasonable to cite Payne and Weinberger [9, p. 186]: “There appears to be no systematic method of determining the origin to give the best lower bound. Experience and considerations of symmetry are certainly helpful.”

\section{Auxiliary Lemma}

The following lemma provides the geometric inequality analogous to that proved by Payne and Weinberger (see Lemma in [9, Section 2]).

\textbf{Lemma 2.1.} If $D \subset R_\beta$, then

$$\left[ \frac{\beta}{\pi} \int_{\partial D} r^\beta \cos^2 \frac{\beta \theta}{2} \, ds \right]^{(\beta+2)/(\beta+1)} \geq \pi^{-1} \beta(2) I_\beta(D),$$

and equality is attained when $D$ is a circular sector of angle $2\pi/\beta$.

\textbf{Proof.} Instead of the transformation [9, formula (2.3)] used by Payne and Weinberger, we apply the mapping

$$D \ni (x = r \cos \theta, y = r \sin \theta) \mapsto \left( x_1 = r^{(\beta+1)/2} \cos \frac{\beta \theta}{2}, \quad y_1 = r^{(\beta+1)/2} \sin \frac{\beta \theta}{2} \right) \in D^*. \quad (2.2)$$

It maps $R_\beta \ni D \mapsto D^* \subset \{x_1 > 0; -\infty < y_1 < +\infty\}$, and since $1 \leq \beta \leq 2$, the inequality

$$ds^2 \geq \frac{dx_1^2 + dy_1^2}{r^{\beta-1}}$$

holds for the element of arc length $ds$ measured along curves in the $(x,y)$-plane. This implies

$$r^\beta \cos^2 \frac{\beta \theta}{2} \, ds \geq 2\beta^{-1} x_1^2 (dx_1^2 + dy_1^2)^{1/2},$$

where the integrand in (2.1) stands on the left-hand side.

The rest of the proof literally repeats considerations in pp. 183–184 of [9] that follow formula (2.5) on p. 183 in [9]. However, $x$ and $y$ must be changed to $y_1$ and $x_1$ respectively. Indeed, $D^* \subset \{x_1 > 0; -\infty < y_1 < +\infty\}$ in the present case, whereas $D^*$ used in [9] lies in the upper half-plane, and so sin $\alpha \theta$ must be changed to cos $\frac{\beta \theta}{2}$.
3 Proof of Theorem 1.2

The fundamental Dirichlet eigenvalue is characterized by the variational principle based on the Rayleigh quotient

$$\lambda_1(D) = \inf_D \frac{\int_D (w_x^2 + w_y^2) \, dx \, dy}{\int_D w^2 \, dx \, dy}. \quad (3.1)$$

It is sufficient to take this infimum over all $C^2(D)$ functions which are nonnegative and vanish in a neighborhood of $\partial D$. Since $D \subset R_\beta$, any such trial function can be taken in the form

$$w = v r^{\beta/2} \cos \frac{\beta \theta}{2},$$

where $v$ belongs to the same class as $w$ itself.

Let us consider the identity

$$\int_D [(\varphi \psi)_x + (\varphi \psi)_y]^2 \, dx \, dy = \int_D \varphi^2 (\psi_x^2 + \psi_y^2) \, dx \, dy + \int_D [\varphi_x (\varphi^2)_x + \varphi_y (\varphi^2)_y] \, dx \, dy,$$

which holds for arbitrary $\varphi$ and $\psi$. Putting $\varphi = r^{\beta/2} \cos \frac{\beta \theta}{2}$, $\psi = v$ and applying the divergence theorem to the last integral, one obtains that this integral vanishes because $r^{\beta/2} \cos \frac{\beta \theta}{2}$ is harmonic and $v$ is equal to zero on $\partial D$. Thus, the equality

$$\int_D (w_x^2 + w_y^2) \, dx \, dy = \int_D (v_x^2 + v_y^2) r^{\beta+1} \cos^2 \frac{\beta \theta}{2} \, r \, d\theta$$

is valid. Manipulating with the right-hand side integral in the same way as Payne and Weinberger do with the right-hand side integral of their formula (3.4) (of course, $\alpha$ must be changed to $\beta/2$ and sin to cos) and using the inequality (2.1) instead of that proved in [9, Lemma, p. 183], one arrives at the required inequality (1.3).

4 Examples

In this section, we use subscripts to distinguish different domains.

**Example 4.1** (disc cut along a radius). Let $D_{cd}$ be the disc of radius $\rho$ centered at the origin and cut along the negative $x$-axis, i.e.,

$$D_{cd} = \{(r, \theta) : r < \rho; \quad \theta \in (-\pi, \pi)\}.$$  

In this case, $\beta = 1$ and $J_{\beta/2}(t) = J_{1/2}(t) = \sqrt{2/(\pi t)} \sin t$. Furthermore, equality is attained in formula (1.3), according to which, $\lambda_1(D_{cd}) = (\pi/\rho)^2$ because $j_{1/2,1} = \pi$. The corresponding eigenfunction is as follows:

$$u_1(D_{cd}) = J_{1/2}\left(\frac{\pi r}{\rho}\right) \cos \frac{\theta}{2} = \sqrt{\frac{2\rho}{\pi^2 r}} \sin \frac{\pi r}{\rho} \cos \frac{\theta}{2}.$$

Thus, the first eigenvalue of a half-cut disc is $(\pi/j_{0,1})^2 = 1.7066...$ times larger than the first eigenvalue of the whole disc of the same radius.
Example 4.2 (sector of an annulus). Let $D_{\text{as}} = \{(r, \theta) : r \in (\rho_1, \rho_2); \ \theta \in (-\pi/\beta, \pi/\beta)\}$ be the annular sector centered at the origin. Then $\lambda_1(D_{\text{as}}) = k^2$, where $k$ is the smallest positive root of the equation

$$J_{\beta/2}(k \rho_1)Y_{\beta/2}(k \rho_1) = J_{\beta/2}(k \rho_2)Y_{\beta/2}(k \rho_2).$$

A consequence of (1.3) is the lower bound $k \geq (\rho_2^{\beta+2} - \rho_1^{\beta+2})^{-2/(\beta+2)}j_{\beta/2,1}$ for this root. This bound is similar to formula (3.27) in [9].

Example 4.3 (square cut along a half-midline). Let us consider the domain

$$D_1 = \{(x, y) : |x| < 1; \ |y| < 1; \ \theta \neq \pm \pi\}.$$ 

The exterior sides of this square are pairwise symmetric about the $x$- and $y$-axes, its area is equal to 4 and it is cut along the negative $x$-axis. From (1.1) it follows that

$$\lambda_1(D_1) \geq \pi j_{0,1}^2 / 4 = 4.5420...$$  \hspace{1cm} (4.1)

If the square has the same exterior sides as $D_1$, but no cut, then (1.1) yields the same lower bound, i.e.,

$$\lambda_1(D_0) \geq \pi j_{0,1}^2 / 4 = 4.5420..., \quad D_0 = \{(x, y) : |x| < 1; \ |y| < 1\}.$$ 

Moreover, the last lower bound is less than 10% smaller than the exact value

$$\lambda_1(D_0) = \pi^2 / 2 = 4.9348...$$

As in the case of discs with and without a cut, it is reasonable to expect that $\lambda_1(D_1) > \lambda_1(D_0)$. Indeed, formula (1.3) with $\beta = 1$ and $j_{1/2,1} = \pi$ gives the following lower bound:

$$\lambda_1(D_1) \geq \frac{\pi^{8/3}}{[(\pi + 1)\sqrt{2} + \log(1 + \sqrt{2})]^{2/3}} = 5.9341..., \quad (4.2)$$

which is about 20% larger than the exact value for the uncut square $D_0$ and substantially better than the Faber–Krahn bound (4.1).

Example 4.4 (square cut along a half-diagonal). Let $D_2$ be as follows:

$$\{(x, y) : -\sqrt{2}/2 < y - x < \sqrt{2}/2; \ -\sqrt{2}/2 < y + x < \sqrt{2}/2; \ \theta \neq \pm \pi\}.$$ 

This square is also cut along the negative $x$-axis, but its vertices are located on the $x$- and $y$-axes so that its area is equal to 4 like that of $D_0$ and $D_1$. Therefore, the Faber–Krahn inequality (1.1) gives for $\lambda_1(D_2)$ the same lower bound as for $\lambda_1(D_1)$ and $\lambda_1(D_0)$ (see (4.1)). It occurs that formula (1.3) with $\beta = 1$ and $j_{1/2,1} = \pi$ gives the following lower bound:

$$\lambda_1(D_2) \geq \pi^2 / 2 = 4.9348... = \lambda_1(D_0).$$  \hspace{1cm} (4.3)

Comparing this lower bound with that following from the Faber–Krahn inequality, we see that (4.3) is better. However, unlike the case of square cut along a half-midline, (4.3) does not improve the bound natural from a physical point of view.
Declarations

Data availability This manuscript has no associated data.
Ethical Conduct Not applicable.
Conflicts of interest The author declares that there is no conflict of interest.

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