Relay a message as far as possible with local decisions in a network

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Abstract. Suppose there is a message generated at a node $v$ in a network and $v$ decides to pass the message to one of the neighbors $u$, and $u$ next decides to pass the message to one of its own neighbors, and so on. How to relay the message as far as possible with local decisions? To the best of our knowledge no general solution other than randomly picking available adjacent node exists. Here we report some progress. Our first contribution is a new framework called $[t,p]$-separate chain decomposition for studying network structures. Each $[t,p]$-separate chain induces a ranking of nodes. We then prove that the ranks can be locally and distributively computed via searching some stable states of certain dynamical systems on the network and can be used to search long paths of a guaranteed length containing any given node. Numerical analyses on a number of typical real-world networks demonstrate the effectiveness of the approach.

keywords: separate chain decomposition, network dynamical system, longest path, message relaying, fixed point

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1 Introduction

Structural properties of large-scale networks are of central importance for understanding the formation principles of said networks and the dynamics associated to them. One of the most important structural properties that has been widely studied concerns paths in networks, including determining the length of paths and searching paths, as it is of practical importance in information routing, and transportation, etc. In some scenarios, short paths are desired, while in other scenarios, long paths are more relevant. Here we are concerning with the latter. For instance, in a sensor network or alike, in certain situations, a message generated at an agent may simply need relaying as many hops as possible. In addition, there may be no one having all detailed structural information. Hence, agents have to make their own decisions with local information they can acquire. As another example, the problem may be described as: how to help a travel salesman travel as many cities as possible by providing certain limited local coordinates of the cities.
Finding the shortest paths from any fixed node to any other nodes can be solved by the well-known breadth first search algorithm and Dijkstra’s algorithm. However, searching a longest path of a network is known to be NP-hard as it is a generalization of the Hamiltonian problem. It is even shown that for any \( \epsilon < 1 \), the problem of finding a path of length \( n - n^\epsilon \) in an \( n \)-vertex Hamiltonian graph is NP-hard [16]. Algorithms for finding paths, probably with a fixed length or a prescribed performance ratio, can be found in [2, 25, 23, 24] and references therein. A number of lower bounds for the length of the longest paths or cycles, in particular in terms of the minimum degree or the number of edges of the graph in question, have been obtained, e.g., in Dirac [9], Erdős and Gallai [12], and Alon [1], etc. But, these results do not seem to be applicable to our problem.

In order to provide a solution, our first contribution is a very general new framework for analyzing network structures, called \([t, p]\)-separate chain decomposition of networks. From a \([t, p]\)-separate chain of a network, every node there is assigned a rank. We next prove that the ranks for all nodes can be computed via searching fixed points in certain dynamical systems on the network, with the worst-case cubic time complexity. Finally, we show that these ranks can help us determine the length of the longest paths starting with or containing any given node, and design algorithms to search long paths. Evaluations on a number of typical real-world networks demonstrate that our searching algorithms generally outperform natural strategies such as randomly picking neighbors or randomly picking neighbors of maximal degree.

2 Results

2.1 \([t, p]\)-separate chains

Let \( G = (V, E) \) be a simple graph, where \( V \) is the set of nodes (or vertices) and \( E \) is its edge set. A subgraph \( H = (V(H), E(H)) \) of \( G \) is denoted by \( H \leq G \). In the following, if not explicitly specified otherwise, a network of \( n \) nodes has \([n] = \{1, 2, \ldots, n\}\) as its node set. We sometimes write \( v \in G \) for \( v \in V(G) \).

Definition 2.1. Let

\[
t = (t_1, t_2, \ldots, t_n) \in \mathbb{Z}^n, \quad p = (p_1, p_2, \ldots, p_n) \in \mathbb{Z}^n.
\]

Suppose \( C : G_0, G_1, G_2, \ldots, G_m \) is a chain of subgraphs of a network \( G \) of \( n \) nodes, where \( G_0 = G \), and \( G_i \) is a node-induced subgraph of \( G_{i-1} \) for \( 1 \leq i \leq m \). We call the chain \( C \) a \([t, p]\)-separate chain if for any \( 0 \leq i \leq m \), any vertex \( v \in G_i \) has at least \( \max\{0, i + p_v\} \) neighbors in \( G_i \) and at least \( i \) neighbors in \( G_j \) where \( j = \max\{0, i + t_v\} \).

We call the number \( m \) the length of the chain \( C \), and denote it by \( L(C) = m \). The number \( k \) such that \( G_k \neq \emptyset \) while \( G_{k+1} = \emptyset \) or \( k + 1 > m \) is called the size of the chain \( C \), and denoted by \( |C| = k \). We sometimes write the chain \( C \) as \( G = G_0 \geq G_1 \geq G_2 \geq \cdots \geq G_m \).

These chains aim at providing new insight into the “positions” of nodes in the global structure of the network \( G \). Namely, we intend to characterize the mutual connection of
a “community” in the global structure and the ability of the nodes there reaching the outside of the community.

For a node $v$ of $G$, we denote by $\text{deg}_G(v)$ the degree of $v$ in $G$, and we write $\text{deg}(v)$ for short if the graph $G$ referred to is clear from the context. For some $t, p \in \mathbb{Z}^n$, there may be no $[t, p]$-separate chains for $G$ at all. The following proposition gives a sufficient and necessary condition for the existence of $[t, p]$-separate chains.

**Proposition 2.1.** Let $G = (V, E)$ and $t, p \in \mathbb{Z}^n$. Then, there exists a $[t, p]$-separate chain of $G$ if and only if $p_v \leq \text{deg}(v)$ for any $v \in V$.

Hereafter, if not explicitly specified otherwise, we assume $p_v \leq \text{deg}(v)$ for any node $v$.

**Definition 2.2.** A $[t, p]$-separate chain $C$ of $G$: $G_0 \supseteq G_1 \supseteq \cdots \supseteq G_m$ is called maximal if there does not exist a $[t, p]$-separate chain $C'$: $G'_0 \supseteq G'_1 \supseteq \cdots \supseteq G'_m$, satisfying either (i) $|C'| > |C|$, or (ii) $|C'| = |C|$ and for some $1 \leq i \leq |C|$, $G_i < G'_i$.

The $k$-core of a network $G$ is the maximal subgraph of $G$ where any vertex has degree at least $k$ [3]. There are many applications of the $k$-core decompositions to real-world network problems, see [10, 8, 15, 17, 18] for instance. A discussion on the distributed computation of $k$-core can be found in Montresor, Pellegrini and Miorandi [20] and later in Lü et al. [17]. It is left to the reader to verify that the chain consisting of the $k$-cores of $G$ is actually a maximal $[0, 0]$-separate chain of $G$, where $0 = (0, 0, \ldots, 0)$. Note that merging two $[t, p]$-chains entry-wisely gives another $[t, p]$-chain. Therefore, it is not hard to show the following uniqueness result.

**Proposition 2.2.** Let $G = (V, E)$ and $t, p \in \mathbb{Z}^n$ where $p_v \leq \text{deg}(v)$ for any $v \in V$. Then, there exists a unique maximal $[t, p]$-separate chain of $G$.

### 2.2 Computation via dynamical systems

Inspecting the definition of $[t, p]$-separate chains, it is clear how to obtain such chains. In fact, we do not know any efficient approach for obtaining all $[t, p]$-separate chains at present. However, we do find an approach of obtaining the maximal $[t, p]$-separate chain if exists. The approach is based on a surprising connection discovered between the maximal $[t, p]$-separate chain of a graph $G$ and a certain fixed point of some discrete dynamical system on $G$.

A discrete dynamical system over a network [7, 11, 14, 21, 27, 19] is concerned with the dynamics generated when nodes in the network update their states following a system update schedule and their respective local rule. Von Neumann’s cellular automata [21] are such dynamical systems.

Let $P$ denote a finite set of states a node in $G$ may have and let $x_i$ denote the state of node $i$. A function $f_i$ specifies how node $i$ updates its state $x_i$ based on the states of the neighbors of $i$ (and itself) in $G$. Usually, $f_i$ is called the local update function at node $i$. An infinite sequence of subsets of vertices $W = W_1 W_2 \cdots$, is called a fair update schedule, if for any $k \geq 1$, and any $1 \leq i \leq n$, there exists $l > k$ such that $i \in W_l$. Namely, $i$
appears an infinite number of times in theory. Suppose the initial system state at time \( t = 0 \) is \( x^{(0)} = (x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}) \). For \( j > 0 \), the system state \( x^{(j)} \) at time \( t = j \) results from that the nodes contained in \( W_j \) update their states via applying their respective local functions to the states of their respective neighbors in \( x^{(j-1)} \) while the states of the nodes not contained in \( W_j \) remain unchanged. We denote this dynamical system by \([G, f, W]\), and we denote by \([G, f, W]^{(j)}(x)\) the system state at time \( t = j \) when the system starts from the initial state \( x \) at time \( t = 0 \).

For a dynamical system \([G, f, W]\), we say a system state \( x \) is reaching a fixed point (or stable state) \( z \) if there exists \( k \geq 0 \) such that for any \( j > k \),

\[
[G, f, W]^{(j)}(x) = [G, f, W]^{(k)}(x) = z.
\]

Let \( G \) be a graph on \([n]\). Here we are interested in a particular class of systems, where each vertex of \( G \) can have a state from the set \([n]\), and the local function \( f_v \) at a vertex \( v \) that returns the maximum \( k \) such that there are at least \( k \) of the neighbors of \( v \) with (state) values at least \( k + p_v \) while at least \( \max\{0, k + p_v\} \) of them with values at least \( k \). For example, suppose \( t_v = -2 \) and \( p_v = -1 \), and \( \{2, 4, 4, 5, 3\} \) is the (multi)set of values of the neighbors of \( v \). Then, \( f_v \) returns \( 4 \). We call such a system the \([t, p]\)-system on \( G \).

It turns out that these systems are useful in computing maximal chains, and \( W \) can be arbitrary fair update schedule thus not specified.

In the maximal \([t, p]\)-separate chain, if \( v \in G_i \) and \( v \notin G_{i+1} \), then we denote \( C_{t,p}(v) = i \). Clearly, \( G_i \) is just the vertex-induced subgraph of \( G \) by the set of vertices \( v \) with \( C_{t,p}(v) = i \).

**Theorem 2.3.** Suppose \( G \) is a graph on \([n]\). Let \( d = (\deg(1), \deg(2), \ldots, \deg(n)) \) and \( t, p \in \mathbb{Z}^n \) where \( p \leq d \). Then, in the \([t, p]\)-system on \( G \), the state \( d \) is reaching a stable state \( C^{t,p} = (C_1^{t,p}, C_2^{t,p}, \ldots, C_n^{t,p}) \), and for any \( i \in G \),

\[
C_i^{t,p} = C_{t,p}(i).
\]

Moreover, if \( t \leq t' \) and \( p \leq p' \), then \( C^{t,p} \) is reaching \( C^{t',p'} \) in the \([t', p']\)-system on \( G \).

**Proof.** Here we only prove \( C_i^{t,p} = C_{t,p}(i) \), and the proof of the rest can be found in Supplementary Material. Let \( z = (C_{t,p}(1) \ldots C_{t,p}(n)) \). Then it suffices to prove that the state \( z \) is a fixed point of the \([t, p]\)-system and the state \( d \) is reaching \( z \). We shall first prove:

**Claim 1.** The state \( z \) is a fixed point of the \([t, p]\)-system.

Let \( C : G_0 \geq G_1 \geq \cdots \) be the maximal \([t, p]\)-separate chain of \( G \). First, for a vertex \( v \), suppose \( C_{t,p}(v) = i \). By definition, \( v \) belongs to \( G_i \) but not \( G_{i+1} \). It implies the following two facts:

(i) there exist at least \( i \) neighbors of \( v \) contained in \( G_j \) \((j = \max\{0, i+t_v\})\) and at least \( \max\{0, i+p_v\} \) neighbors of \( v \) contained in \( G_i \). Note that for any \( u \) among these said neighbors in \( G_j \), we have \( C_{t,p}(u) \geq i + t_v \), and for any \( u' \) among these neighbors contained in \( G_i \) we have \( C_{t,p}(u') \geq i \) by definition. Thus, among the neighbors of \( v \), there exist at least \( i \) of them with values at least \( i + t_v \) and at least \( \max\{0, i + p_v\} \) of them with values at least \( i \) in \( z \). This leads to that \( f_v(z) \geq i = C_{t,p}(v) \);
(ii) it is impossible that \(i+1\) neighbors of \(v\) are contained in \(G_{j'} (j' = \max \{0, i+1+t_v\})\) and \(i + 1 + p_v\) neighbors of \(v\) are contained in \(G_{i+1}\). Otherwise, the chain \(G_0 \geq \cdots \geq G_i \geq G_{i+1} \cup \{v\} \geq G_{i+2} \geq \cdots\) gives a \([t, p]\)-separate chain, which contradicts the maximality of \(C\). Hence, the case that at least \(i + 1\) of the neighbors of \(v\) have values at least \(i + 1 + t_v\) and at least \(i + 1 + p_v\) of the neighbors with values at least \(i + 1\) in \(z\) cannot happen. This implies \(f_i(z) < i + 1\).

From (i) and (ii), we conclude \(f_i(z) = i = C_{t, p}(v)\) for any \(v\). Therefore, \(z\) is a fixed point.

We next shall show that \(d\) is reaching \(z\). If \(z = d\), we are done. Otherwise we clearly have \(z < d\). For this case, in the light of Proposition ?? in Supplementary Information, it suffices to show that any state \(y \leq d\) such that \(y > z\) or \(y\) is uncomparable with \(z\) is not a fixed point.

We prove by contradiction. Suppose \(y\) is such a state which is a fixed point. Then there must be a coordinate indexed by some \(v\) that satisfies \(y_v > C_{t, p}(v)\). Consider the sequence of subgraphs induced by the sequence of sets of vertices \(S_0 \supseteq S_1 \supseteq \cdots \supseteq S_m\) for a sufficient large number \(m\) (say \(m > n + \max \{t_v: v \in V\}\)) which are iteratively constructed as follows:

1. set \(S_{y_v} = \{v\}\) and \(S_r = \emptyset\) for \(r \neq y_v\);
2. for \(r\) from 0 to \(m\), if \(u \in S_r\), then set

\[
S_{\max \{0, r + t_u\}} = S_{\max \{0, r + t_u\}} \cup \\
\{w: w \text{ is a neighbor of } u \text{ and } y_w \geq r + t_u\}, \\
S_r = S_r \cup \{w: w \text{ is a neighbor of } u \text{ and } y_w \geq r\},
\]

and for \(0 \leq r \leq m\), set \(S_r = \bigcup_{j=r}^m S_j\);
3. iterate (2) until the sequence \(S_0 \supseteq S_1 \supseteq \cdots \supseteq S_m\) becomes stable, i.e., \(S_r\) does not change for any \(0 \leq r \leq m\) when further executing (2).

Clearly, by construction we have \(S_r \subseteq S_{r-1}\). By abuse of notation, we denote by \(S_r\) the subgraph induced by \(S_r\) as well. Then we have a chain of graphs \(S_0 \supseteq S_1 \supseteq \cdots \supseteq S_m\). Let \(t\) be the restriction of \(t\) to that of the vertices in \(S_0\) and \(p\) be the restriction of \(p\) to that of the vertices in \(S_0\). We proceed to show that

Claim 2. The chain \(S_0 \supseteq S_1 \supseteq \cdots \supseteq S_m\) gives a \([t, p]\)-separate chain of the graph \(S_0 \leq G\). That is, for any \(u \in S_0\), if \(u \in S_r\), then there are at least \(r\) neighbors of \(u\) contained in \(S_j\) where \(j = \max \{0, r + t_u\}\) and at least \(\max \{0, r + p_u\}\) neighbors of \(u\) contained in \(S_r\).

Since \(y\) is a fixed point by assumption, there exist at least \(y_u\) neighbors of \(u\) whose corresponding values in \(y\) are at least \(y_u + t_u\) and at least \(y_u + p_u\) neighbors whose corresponding values in \(y\) are at least \(y_u\). By construction of (2), if \(u \in S_r\), then \(r \leq y_u\). Furthermore, since \(u\) has at least \(y_u\) neighbors \(w\) such that \(y_w \geq y_u + t_u \geq r + t_u\), these vertices \(w\) are contained in \(S_{\max \{0, r + t_u\}}\). Analogously, there are at least \(y_u + p_u \geq r + p_u\)
neighbors of $u$ contained in $S_r$. Accordingly, the chain $S_0 \geq S_1 \geq \cdots \geq S_m$ gives a $[t, p]$-separate chain, whence Claim 2.

In view of Claim 2, it can be easily checked that the chain

$$G_0 \bigcup S_0 \geq G_1 \bigcup S_1 \geq \cdots \geq G_y \bigcup S_y \geq \cdots$$

gives a $[t, p]$-separate chain of $G$. Therefore, we have $C_{t, p}(v) \geq y_v$, which yields a contradiction. Hence, $y$ cannot be a fixed point. According to Proposition 2.3, the maximal $[t, p]$-system is obtained.

The proof of Theorem 2.3 may be not hard to complete. However, discovering the relation between maximal chains and fixed points of certain dynamical systems in the first place is nontrivial. According to Theorem 2.3, the maximal $[t, p]$-separate chain can be immediately constructed once the fixed point of the $[t, p]$-system is obtained.

### 2.3 Long paths at a node

The numbers $C_{t, p}(i)$ encode lots of information about node $i$ including paths through $i$. A path is a sequence of distinct nodes $v_1, v_2, \ldots, v_k$ such that $v_i$ and $v_{i+1}$ ($1 \leq i < k$) are neighbors in $G$. The number $k - 1$ is the length of the path, and $v_1$ as well as $v_k$ are called the terminals of the path. A cycle is determined by a path $v_1, v_2, \ldots, v_k$ such that $v_1$ and $v_k$ are neighbors in $G$. The number $k$ is the length of the cycle.

It can be proven that if $p_v \leq -\deg(v)$, then $C_{t, p}(v)$ only depends on $t_v$ (see the Supplementary Information). For simplicity, we only utilize the numbers $C_{t, p}(v)$ for $t_v$ being a nonpositive constant $t$ and $p_v \leq -\deg(v)$, and we also simply write $C_{t, p}(v)$ as $C_t(v)$.

Let

$$\lambda(G) = \max\{|\deg(u) - \deg(v)| : u \text{ and } v \text{ are adjacent in } G\}.$$

Then, we can show that for any $t \leq -\lambda(G)$ and any node $v$, we have $C_t(v) = \deg(v)$ (See Supplementary Information).

For $t < 0$ and $z \geq 0$, we introduce the following notations:

$$q_t(z) = \left\lfloor \frac{z}{|t|} \right\rfloor, \quad 0 \leq r_t(z) = z \mod |t| < |t|, \quad \pi_t(z) = r_t(z) - t \cdot j,$$

and $\xi_t(z) = \min\{y : \pi_t(y) > 1 \land y \geq 0\}$ when such $y$ exist. Obviously, $z = r_t(z) - t \cdot q_t(z) = \pi_t(z)$. In the case of $z = C_t(v)$, we simply write $q_t(C_t(v))$ as $q_t(v)$, and other notations are simplified analogously. When the vertex $v$ in question is clear from the context, $v$ may be dropped in these quantities. We also need the concept of girth of a graph $G$ which is the minimum length of a cycle in $G$, and if otherwise stated $g$ always denotes the girth of the graph under consideration and we assume no isolated nodes exist in $G$ discussed in this paper.

Let

$$J_t(z, x) = \min\left\{q_t(z), \max\left\{-1, \left[\frac{q_t(z) + x - 1 - (g - 2)[r_t(z) - 1]}{1 - t(g - 2)}\right]\right\}\right\}.$$
For $z = C_t(v), J_t(v, x) := J_t(z, x)$. We also use the short-hands $J_t(v)$ and $J'_t(v)$ for $J_t(v, 0)$ and $J_t(v, q_t - J_t(v, 0))$, respectively.

**Theorem 2.4.** Let $G$ be a graph of girth $g \geq 3$ and $v \in G$. Suppose $q_0(v) - J_0(v) = (g - 2)[C_0(v) - 1] + 1, q_0(v) - J'_0(v) = 0$, and

$$L_e(v) = \max_{-\lambda(G) \leq t \leq 0} \{ q_t(v) - J_t(v) \},$$

$$L_m(v) = \max_{-\lambda(G) \leq t \leq 0} \{ 2q_t(v) - J_t(v) - J'_t(v) \}.$$

Then, there exists a path of length at least $L_e(v)$ that has $v$ as a terminal, and there exists a path of length at least $L_m(v)$ that contains $v$.

Let $G$ be a graph of girth $g \geq 3$ and $v \in G$. For $0 \leq k \leq J_t(v, 0) - \xi_t(v)$, let

$$x_t(v, k) = q_t - J_t(v) + k - (g - 2)(\pi_{t,J_t(v)-k} - 1).$$

Then, there exists a path of length at least $\hat{L}_e(v)$ that has $v$ as a terminal, where

$$\hat{L}_e(v) = \max_{-\lambda(G) \leq t \leq -1} \{ 2q_t - J_t(v) + \min_{0 \leq k \leq J_t(v) - \xi_t(v)} \{ k - x_t(v, k) - J_t(\pi_{t,q_t-x_t(v,k)+1}, q_t - J_t(v) + k) \} \}.$$  

We remark that there is room for further optimization and analogous optimization for $L_m(v)$ is possible too. In addition, if there is no information on the girth $g$ of the graph $G$, we can simply set $g = 3$.

### 2.4 Relay a message as far as possible

We denote by $N_G(v)$ the set of neighbors of $v$ in the graph $G$, and $G$ may be dropped if it is clear from the context.

Suppose $q_0(v) - J_0(v, x) = C_0(v) - x$, and

$$L_{end}(v, x) = \max_{-\lambda(G) \leq t \leq 0} \{ \max\{ q_t(v) - J_t(v, x), 0 \} \}.$$  

Moreover, let $A_{end}(v, x)$ denote the set of $t$’s that achieve the maximum $L_{end}(v, x)$. An algorithm (Algorithm 1) for relaying a message with local decisions can be derived from the proof of Theorem 2.4.

It is easy to see that nodes can make their own computations and decisions as long as they have certain computing power and can communicate with their neighbors. To be specific, suppose the message starting with $v$ has arrived at the current node $u$. What $u$ needs to know in order for making its own optimal decision is the index $i$ indicating it is the $i$-th in the relay and which of its neighbors are already in the race via communication with its neighbors. Nodes do not need to know the global structural information of the network, and each node is even not aware of which non-neighbors are in the race.
Algorithm 1 Finding a path starting with $v$

1: $i \leftarrow 1$, $j \leftarrow 1$
2: $v_1 \leftarrow v$, $w \leftarrow v$
3: $W \leftarrow \emptyset$
4: while $N(w) \notin W$ do
5:     pick $t \in A_{\text{end}}(w, i-1)$, and among the neighbors of $w$, pick $x$ such that $x \notin W$ and $C_t(x)$ is maximal
6:     $W \leftarrow W \cup \{v_i\}$
7:     $i \leftarrow i + 1$
8:     $v_i \leftarrow x$, $w \leftarrow x$
9: end while
10: $W \leftarrow W \cup \{v_i\}$
11: return $v_i, v_{i-1}, \ldots, v_1$

In order to evaluate our searching algorithm, we compare it with two natural candidate searching algorithms: the current node randomly adding an unused neighbor to the path (i.e., random path) and the current node randomly adding an unused neighbor of maximal degree to the path (i.e., max-degree based random path). Six representative real-world networks from disparate fields are used for evaluation, and the networks are: Email [28], USAir [29], Jazz [30], PB [31], Router [32], and Email2 [33]. In brief, Email is the e-mail interchanges between members of the University Rovira i Virgili (Tarragona), USAir is the US air transportation network, Jazz records the collaborations between jazz musicians, PB is the network of US political blogs, Router is a symmetrized snapshot of the structure of the Internet at the level of autonomous systems, and Email2 is the e-mail interchanges between members of the Computer Sciences Department of University College London. All networks are treated as undirected, and their basic topological features are shown in Table I.

| network name | $N$ | $N_E$ | $k_{\text{max}}$ | $< k >$ | $\lambda$ |
|-------------|-----|------|-----------------|--------|---------|
| Email       | 1,133 | 5,451 | 71              | 9.62   | 70      |
| Jazz        | 198  | 742  | 100             | 27.69  | 99      |
| PB          | 1,222 | 16,714 | 351            | 27.35  | 350     |
| Router      | 5,022 | 6,258 | 106            | 2.49   | 105     |
| USAir       | 332  | 2,126 | 139            | 12.80  | 138     |
| Email2      | 12,625 | 20,362 | 576          | 3.22566 | 575     |

Table 1: $N$ denotes the number of nodes, $N_E$ denotes the number of edges, $k_{\text{max}}$ the maximum degree, $< k >$ the average degree.

For each network, we randomly pick 100 nodes as the message source, and each candidate algorithm is used to search 1000 paths starting from each picked node respectively. Then, we compare the algorithms by comparing the corresponding average lengths of the paths. We actually compare two algorithms arising from our framework: Algorithm I and
the one (i.e., zero-core) obtained by considering $t = 0$ alone in Algorithm 1. At first, we use the random path length of a node $v$ as the benchmark and compute the respective differences of path lengths from the remaining three algorithms normalized by the random path length of $v$. Figure 1 shows that except for Jazz network, the remaining algorithms are generally better. In Email2 network, they are even as high as 6000% better. Our approach is generally better than max-degree based approach, and Algorithm 1 (more choices for $t$) is better than zero-core ($t = 0$ alone) as well.

Analogous algorithm searching a long path containing (instead of starting with) any given node from our framework can be derived as well.
Theorem 2.4 and Algorithm 1 rely on data attained. Algorithm 1 is overall the best. For each of the six networks, algorithms are compared based on the found paths starting with each of 100 random nodes (rearranged on the x-axis), where the y-coordinates show the normalized gains over the random path approach. Except for Jazz network, the remaining algorithms respectively have at least 20% gain. For Email2, as high as 7000% gain can be even attained. Algorithm 1 is overall the best.

3 Discussion

Theorem 2.4 and Algorithm 1 rely on data $C_t(v)$ for all $-\lambda(G) \leq t \leq 0$ and $v \in G$. Now we discuss the complexity of obtaining these data. We first compute the $C_{-\lambda(G)}$-values according to Theorem 2.3 with the initial system state being the degree sequence $d$. For $t$ from $-\lambda(G) + 1$ to 0, we can obtain the $C_t$-values in accordance with Theorem 2.3 by searching the fixed point reached by the initial system state $(C_{t-1}(1), \ldots, C_{t-1}(n))$ in the
Note that
\[0 \leq C_0 \leq C_{-1} \leq \cdots \leq C_{-\lambda(G)} \leq d.\]

In addition, in each round of update of the vertices \( v \in [n] \) (with the local functions, \( f_v \)'s, with respect to the corresponding \( t \)), the state of at least one vertex will decrease unless we have already reached \( C_t \). Since the total number of decrease steps from \( d \) to \( 0 \) is at most \( \sum_i \deg(i) \), the time complexity is at most \( n \cdot \sum_i \deg(i) = 2n|E| \). Thus, the worst-case time complexity is \( O(n^3) \). For graphs of practical interest, the expected time complexity is probably just \( O(n^2) \).

Our framework can certainly provide lower bounds on the length of the longest paths in networks. In many cases, our lower bounds outperform classical results such as Erdős-Gallai \([12]\) and Alon \([1]\).

Here we have only considered \( [t, p] \)-separate chains where \( t_v \) is a constant and \( p_v \) is not relevant. This is actually equivalent to the D-chain decomposition of networks introduced in Chen, Bura and Reidys \([6]\) where the induced D-spectra of nodes were used to effectively characterize the spreading power of nodes. However, \( [t, p] \)-separate chains are significant generalization of D-chains, and we refer to Supplementary Information for discussion. As such, further optimization is possible by exploring additional degree of freedom. Moreover, our \( [t, p] \)-separate chain framework is of independent interest and may be modified to adapt to directed networks and weighted networks, and can be applied to other network problems.

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