On Schubert’s Problem of Characteristics

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Abstract

The Schubert varieties on a flag manifold $G/P$ give rise to a cell decomposition on $G/P$ whose Kronecker duals, known as the Schubert classes on $G/P$, form an additive base of the integral cohomology $H^\ast(G/P)$. The Schubert’s problem of characteristics asks to express a monomial in the Schubert classes as a linear combination in the Schubert basis.

We present a unified formula expressing the characteristics of a flag manifold $G/P$ as polynomials in the Cartan numbers of the group $G$. As application we develop a direct approach to our recent works on the Schubert presentation of the cohomology rings of flag manifolds $G/P$.

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1 The problem

Schubert considered what he called the problem of characteristics to be the main theoretical problem of enumerative geometry. -S. Kleiman [40, 1987]

The existence of a finite basis for the homologies in every closed manifold implies furthermore the solvability of Schubert’s “characteristics problems” in general. -Van der Waerden [60, 1930]

Schubert calculus is the intersection theory of the 19th century, together with applications to enumerative geometry. Justifying this calculus was a major topic of the 20 century algebraic geometry, and was also the content of Hilbert’s 15th problem “Rigorous foundation of Schubert’s enumerative calculus” [33, 50].

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‡This English translation of the profound discovery of van der Waerden [60] is quoted from N. Schappacher [56].
Thanks to the pioneer works \[60, 26\] of Van der Waerden and Ehresmann, the problem of characteristics \[53\], considered by Schubert as the fundamental problem of enumerative geometry, has had a concise statement by the 1950’s.

Let \(G\) be compact connected Lie group with a maximal torus \(T\). For an one parameter subgroup \(\alpha : \mathbb{R} \to G\) its centralizer \(P\) is a parabolic subgroup on \(G\), while the homogeneous space \(G/P\) is a projective variety, called a flag manifold of \(G\). Let \(W(\mathcal{P}, G)\) be the set of left cosets of the Weyl group \(W\) of \(G\) by the Weyl group \(W(\mathcal{P})\) of \(\mathcal{P}\) with associated length function \(l: W(\mathcal{P}, G) \to \mathbb{Z}\). The following result was discovered by Ehresmann \[26, 1934\] for the Grassmannians \(G_{n,k}\) of \(k\)-dimensional linear subspaces on \(\mathbb{C}^n\), announced by Chevalley \[13, 1958\] for the complete flag manifolds \(G/T\), and extended to all flag manifolds \(G/P\) by Bernstein-Gel’fand-Gel’fand \[5, 1973\].

**Theorem 1.1.** The flag manifold \(G/P\) admits a decomposition into the cells indexed by the elements of \(W(\mathcal{P}, G)\),

\[
G/P = \bigcup_{w \in W(\mathcal{P}, G)} X_w, \quad \dim X_w = 2l(w),
\]

with each cell \(X_w\) the closure of an algebraic affine space, called the Schubert variety on \(G/P\) associated to \(w\).

Since only even dimensional cells are involved in the decomposition (1.1), the set \(\{[X_w], w \in W(\mathcal{P}, G)\}\) of fundamental classes forms an additive basis of the integral homology \(H_*(G/P)\). The cocycle classes \(s_w \in H^*(G/P)\) Kronecker dual to the basis (i.e. \(\langle s_w, [X_u] \rangle = \delta_{w,u}, w, u \in W(\mathcal{P}, G)\)) gives rise to the Schubert class associated to \(w \in W(\mathcal{P}, G)\). Theorem 1.1 implies the following result, well known as the basis theorem of Schubert calculus \[60, \S 8\].

**Theorem 1.2.** The set \(\{s_w, w \in W(\mathcal{P}, G)\}\) of Schubert classes forms an additive basis of the integral cohomology \(H^*(G/P)\).

An immediate consequence is that any monomial \(s_{w_1} \cdots s_{w_k}\) in the Schubert classes on \(G/P\) can be expressed as a linear combination of the basis elements

\[
s_{w_1} \cdots s_{w_k} = \sum_{w \in W(\mathcal{P}, G), l(w) = l(w_1) + \cdots + l(w_k)} a_{w_{1}, \ldots, w_{k}}^{w} s_{w}, \quad a_{w_{1}, \ldots, w_{k}}^{w} \in \mathbb{Z},
\]

where the coefficients \(a_{w_{1}, \ldots, w_{k}}^{w}\) are called characteristics by Schubert \[53, 60, 26\].

**The problem of characteristics.** Given a monomial \(s_{w_1} \cdots s_{w_k}\) in the Schubert classes, determine the characteristics \(a_{w_{1}, \ldots, w_{k}}^{w}\) for all \(w \in W(\mathcal{P}, G)\) with \(l(w) = l(w_1) + \cdots + l(w_k)\).

The characteristics are of particular importance in geometry, algebra and topology. They provide solutions to the problems of enumerative geometry \[51, 52, 53, 54\]; were seen by Hilbert as “the degree of the final equations and the multiplicity of their solutions” of a system \[33\]; and are requested by describing the cohomology ring \(H^*(G/P)\) in the Schubert basis \[61, \text{p.331}\]. Notably, the degree of a Schubert variety \[54\] and the multiplicative rule of two Schubert classes \[53\] are two special cases of the problem which have received
considerable attentions in literatures, see [15][16] for accounts on the earlier relevant works.

This paper summaries and simplifies our series works [15][16][19][20][21] devoted to describe the integral cohomologies of flag manifolds by a minimal system of generators and relations in the Schubert classes. Precisely, based on a formula of the characteristics \( a^w_{w_1,\ldots,w_k} \) stated in Section §2 and established in §3, we address in Sections §4 and §5 a more direct approach to the Schubert presentations [20][21] of the cohomology rings of flag manifolds \( G/P \).

2 The formula of the characteristics \( a^w_{w_1,\ldots,w_k} \)

To investigate the topology of a flag manifold \( G/P \) we may assume that the Lie group \( G \) is 1-connected and simple. Resorting to the geometry of the Stiefel diagram of the Lie group \( G \) we present in Theorem 2.4 a formula that boils down the characteristics \( a^w_{w_1,\ldots,w_k} \) to the Cartan matrix of the group \( G \).

Fix a maximal torus \( T \) on \( G \) and set \( n = \dim T \). Equip the Lie algebra \( L(G) \) with an inner product \( (, ) \) so that the adjoint representation acts as isometries of \( L(G) \). The Cartan subalgebra of \( G \) is the linear subspace \( L(T) \subset L(G) \). The restriction of the exponential map \( \exp : L(G) \to G \) on \( L(T) \) defines a set \( S(G) \) of \( \frac{1}{2}(\dim G - n) \) hyperplanes on \( L(T) \), namely, the set of singular hyperplanes through the origin in \( L(T) \) [11] p.226]. These planes divide \( L(T) \) into finitely many convex regions, each one is called a Weyl chambers of \( G \). The reflections on \( L(T) \) in these planes generate the Weyl group \( W \) of \( G \) [35] p.49].

Moreover, the map \( \exp \) on \( L(T) \) carries the normal line \( l \) (through the origin) of a hyperplane \( L \in S(G) \) to a circle subgroup on \( T \). Let \( \pm \alpha \in l \) be the non-zero vectors with minimal length so that \( \exp(\pm \alpha) = e \) (the group unit). The set \( \Phi(G) \) consisting of all those vectors \( \pm \alpha \) is called the root system of \( G \). Fixing a regular point \( x_0 \in L(T) - \bigcup_{L \in S(G)} L \) the set of simple roots relative to \( x_0 \) is

\[
\Delta(x_0) = \{ \beta \in \Phi(G) \mid (\beta, x_0) > 0 \} \quad (35 \text{ p.47}).
\]

In addition, for a simple root \( \beta \in \Delta \) the simple reflection relative to \( \beta \) is the reflection \( \sigma_\beta \) in the plane \( L_\beta \in S(G) \) perpendicular to \( \beta \). If \( \beta, \beta' \in \Delta \) the Cartan number

\[
\beta \circ \beta' := 2(\beta, \beta')/(\beta', \beta')
\]

is always an integer, and only \( 0, \pm 1, \pm 2, \pm 3 \) can occur [35] p.55).

Since the set of simple reflections \( \{ \sigma_\beta \mid \beta \in \Delta \} \) generates \( W \) (11] p.193)], every \( w \in W \) admits a factorization of the form

\[
(2.1) \quad w = \sigma_{\beta_1} \circ \cdots \circ \sigma_{\beta_m}, \quad \beta_i \in \Delta.
\]

**Definition 2.1.** The length \( l(w) \) of an element \( w \in W \) is the least number of factors in all decompositions of \( w \) in the form (2.1). The decomposition (2.1) is called reduced if \( m = l(w) \).

For a reduced decomposition (2.1) of \( w \) the \( m \times m \) (strictly upper triangular) matrix \( A_w = (a_{i,j}) \) with \( a_{i,j} = 0 \) if \( i \geq j \) and \( -\beta_j \circ \beta_i \) if \( i < j \) is called the Cartan matrix of \( w \) relative to the decomposition (2.1). \( \Box \)
Example 2.2. In [57] Stembridge asked for an approach to find a reduced decomposition (2.1) for each \( w \in W \). Resorting to the geometry of the Cartan subalgebra \( L(T) \) this task can be implemented by the following method.

Picture \( W \) as the \( W \)-orbit \( \{ w(x_0) \in L(T) \mid w \in W \} \) through the regular point \( x_0 \). For a \( w \in W \) let \( C_w \) be a line segment on \( L(T) \) from the Weyl chamber containing \( x_0 \) to \( w(x_0) \), that crosses the planes in \( S(G) \) once at a time. Assume that they are met in the order \( L_{\alpha_1}, \ldots, L_{\alpha_k}, \alpha_i \in \Phi(G) \). Then \( l(w) = k \) and \( w = \sigma_{\alpha_k} \circ \cdots \circ \sigma_{\alpha_1} \). Set

\[
\beta_1 = \alpha_1, \quad \beta_2 = \sigma_{\alpha_1}(\alpha_2), \quad \ldots, \quad \beta_k = \sigma_{\alpha_1} \circ \cdots \circ \sigma_{\alpha_{k-1}}(\alpha_k).
\]

Then, from \( \beta_i \in \Delta \) and \( \sigma_{\beta_i} = \sigma_{\alpha_1} \circ \cdots \circ \sigma_{\alpha_{i-1}} \circ \sigma_{\alpha_i} \circ \cdots \circ \sigma_{\alpha_1} \) one sees that \( w = \sigma_{\beta_1} \circ \cdots \circ \sigma_{\beta_k} \), which is reduced because of \( l(w) = k \).

Let \( \mathbb{Z}[x_1, \ldots, x_m] = \oplus_{n \geq 0} \mathbb{Z}[x_1, \ldots, x_m]^n \) be the ring of integral polynomials in \( x_1, \ldots, x_m \), graded by \( \deg x_i = 1 \).

Definition 2.3. For a \( m \times m \) strictly upper triangular integer matrix \( A = (a_{ij}) \) the triangular operator \( T_A \) associated to \( A \) is the additive homomorphism \( T_A : \mathbb{Z}[x_1, \ldots, x_m]^m \rightarrow \mathbb{Z} \) defined recursively by the following elimination rules:

i) If \( h \in \mathbb{Z}[x_1, \ldots, x_m-1]^m \) then \( T_A(h) = 0 \);

ii) If \( m = 1 \) (consequently \( A = (0) \)) then \( T_A(x_1) = 1 \);

iii) For any \( h \in \mathbb{Z}[x_1, \ldots, x_m-1]^{m-r} \) with \( r \geq 1 \),

\[
T_A(h \cdot x_m^r) = T_A(h \cdot (a_{1,m}x_1 + \cdots + a_{m-1,m}x_{m-1})^{r-1}),
\]

where \( A' \) is the \( (m-1) \times (m-1) \) strictly upper triangular matrix obtained from \( A \) by deleting both of the \( m^{th} \) column and row.

By additivity, \( T_A \) is defined for every \( h \in \mathbb{Z}[x_1, \ldots, x_m]^m \) using the unique expansion \( h = \sum_{0 \leq r \leq m} h_r x_m^r \) with \( h_r \in \mathbb{Z}[x_1, \ldots, x_{m-1}]^{m-r} \).

For a parabolic subgroup \( P \) of \( G \) the set \( W(P, G) \) of left cosets of \( W \) by \( W(P) \) can be identified with the subset of \( W \)

\[
W(P, G) = \{ w \in W \mid l(w) \leq l(ww'), w' \in W(P) \} \quad (\text{by \( [5] \ 5.1 \))},
\]

where \( l \) is the length function on \( W \). Assume that \( w = \sigma_{\beta_1} \circ \cdots \circ \sigma_{\beta_m} \in \Delta \), is a reduced decomposition of an element \( w \in W(P, G) \) with associated Cartan matrix \( A_w = (a_{i,j})_{m \times m} \). For a multi-index \( I = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, m\} \) we put \( |I| := k \) and set

\[
\sigma_I := \sigma_{\beta_{i_1}} \circ \cdots \circ \sigma_{\beta_{i_k}} \in W, \quad x_I := x_{i_1} \cdots x_{i_k} \in \mathbb{Z}[x_1, \ldots, x_m].
\]

Our promised formula for the characteristics is:

Theorem 2.4. For every monomial \( s_{w_1} \cdots s_{w_k} \) in the Schubert classes on \( G/P \) with \( l(w) = l(w_1) + \cdots + l(w_k) \) we have

\[
(2.2) \quad a_{w_1 \cdots w_k}^w = T_A_w \left( \prod_{i=1}^k \left( \sum_{\sigma_I = w_i, |I| = l(w_i), I \subseteq \{1, \ldots, m\}} x_I \right) \right).
\]

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Remarks 2.5. It is clear by the definition of the operator $T_A$ that the formula (2.2) reduces the characteristic $a_{w_1,\ldots,w_k}^w$ to a polynomial in the Cartan numbers of the group $G$.

In the case $k = 2$ the characteristic $a_{w_1,w_2}^w$ is well known as a Littlewood-Richardson coefficient, and the formula (2.2) has been obtained by Duan in [10]. In [63] M. Willems generalizes the formula of $a_{w_1,w_2}$ to the more general context of flag varieties associated to Kac-Moody groups, and for the equivariant cohomologies. Recently, A. Bernstein and E. Richmond [10] obtained also a formula expressing $a_{w_1,w_2}^w$ in the Cartan numbers of $G$. □

3 Proof of the characteristics formula (2.2)

In this paper the homologies and cohomologies are over the ring $\mathbb{Z}$ of integers. If $f : X \to Y$ is a continuous map between two topological spaces $X$ and $Y$, $f_*$ (resp. $f^*$) denote the homology (resp. cohomology) homomorphism induced by $f$. For an oriented closed manifold $M$ (resp. a connected projective variety) the notion $[M] \in H_{dim}M(M)$ stands for the orientation class. In addition, the Kronecker pairing between cohomology and homology of a space $X$ is written as $\langle , \rangle : H^*(X) \times H_*(X) \to \mathbb{Z}$. The proof of Theorem 2.4 makes use of the celebrated $K$-cycles on $G/T$ constructed by Bott and Samelson in [8, 9]. We begin by recalling the construction of these cycles, as well as their basic properties developed in [8, 9] [15, 16].

For a simple Lie group $G$ fix a regular point $x_0 \in L(T)$ and let $\Delta$ be the set of simple roots relative to $x_0$. For a $\beta \in \Delta$ let $K_\beta$ be the centralizer of the subpace $exp(L_\beta)$ on $G$, where $L_\beta \in S(G)$ is the plane perpendicular to $\beta$. Then $T \subset K_\beta$ and the quotient $K_\beta/T$ is diffeomorphic to the 2-sphere [9, p.996].

The 2-sphere $K_\beta/T$ carries a natural orientation specified as follows. The Cartan decomposition of the Lie algebra $L(K_\beta)$ relative to the maximal torus $T \subset K_\beta$ takes the form $L(K_\beta) = L(T) \oplus \vartheta_\beta$, where $\vartheta_\beta \subset L(G)$ is the root space belonging to the root $\beta$ [35, p.35]. Taking a non-zero vector $v \in \vartheta_\beta$ and letting $v' \in \vartheta_\beta$ be such that $[v, v'] = \beta$, where $[ , ]$ is the Lie bracket on $L(G)$, then the ordered base $\{v, v'\}$ furnishes $\vartheta_\beta$ with an orientation that is irrelevant to the choices of $v$. The tangent map of the quotient $\pi_\beta : K_\beta \to K_\beta/T$ at the group unit $e \in K_\beta$ maps the 2-plane $\vartheta_\beta$ isomorphically onto the tangent space to the sphere $K_\beta/T$ at the point $\pi_\beta(e)$. In this manner the orientation $\{v, v'\}$ on $\vartheta_\beta$ furnishes the sphere $K_\beta/T$ with the orientation $\omega_\beta = \{\pi_\beta(v), \pi_\beta(v')\}$.

For an ordered sequence $\beta_1, \ldots, \beta_m \in \Delta$ of $m$ simple roots (repetitions like $\beta_i = \beta_j$ may occur) let $K(\beta_1, \ldots, \beta_m)$ be the product group $K_{\beta_1} \times \cdots \times K_{\beta_m}$. With $T \subset K_{\beta_i}$ the product $T \times \cdots \times T$ ($m$-copies) acts on $K(\beta_1, \ldots, \beta_m)$ by

$$(g_1, \ldots, g_m)(t_1, \ldots, t_m) = (g_1t_1, t_1^{-1}g_2t_2, \ldots, t_{m-1}^{-1}g_mt_m).$$

Let $\Gamma(\beta_1, \ldots, \beta_m)$ be the base manifold of this principal action that is oriented by the $\omega_{\beta_i}$, $1 \leq i \leq m$. The point on $\Gamma(\beta_1, \ldots, \beta_m)$ corresponding to the point $(g_1, \ldots, g_m) \in K(\beta_1, \ldots, \beta_m)$ is called $[g_1, \ldots, g_m]$.

The integral cohomology of the oriented manifold $\Gamma(\beta_1, \ldots, \beta_m)$ has been determined by Bott and Samelson in [8]. Let $\varphi_i : K_{\beta_i}/T \to \Gamma(\beta_1, \ldots, \beta_m)$ be the embedding induced by the inclusion $K_{\beta_i} \to K(\beta_1, \cdots, \beta_m)$ onto the $i$th factor group, and put
\[ y_i = \varphi_i(\omega_{\beta_i}) \in H_2(\Gamma(\beta_1, \ldots, \beta_m)), \quad 1 \leq i \leq m. \]

Form the \( m \times m \) strictly upper triangular matrix \( A = (a_{i,j})_{m \times m} \) by setting
\[ a_{i,j} = 0 \text{ if } i \geq j, \quad \text{but } a_{i,j} = -2(\beta_j, \beta_i)/(\beta_i, \beta_i') \text{ if } i < j. \]

It is easy to see from the construction that the set \( \{y_1, \ldots, y_m\} \) forms a basis of the second homology group \( H_2(\Gamma(\beta_1, \ldots, \beta_m)) \).

**Lemma 3.1** \((\text{[13]}\)) Let \( x_1, \ldots, x_m \in H^2(\Gamma(\beta_1, \ldots, \beta_m)) \) be the Kronecker duals of the cycle classes \( y_1, \ldots, y_m \) on \( \Gamma(\beta_1, \ldots, \beta_m) \). Then
\[ (3.1) \quad H^*(\Gamma(\beta_1, \ldots, \beta_m)) = \mathbb{Z}[x_1, \ldots, x_m]/J, \]
where \( J \) is the ideal generated by \( x_j^2 - \sum_{i<j} a_{i,j}x_ix_j, \quad 1 \leq j \leq m. \quad \square \)

In view of (3.1) the map \( p_T(\beta_1, \ldots, \beta_m) \) from the polynomial ring \( \mathbb{Z}[x_1, \ldots, x_m] \) onto its quotient \( H^*(\Gamma(\beta_1, \ldots, \beta_m)) \) gives rise to the additive map
\[ \int_{\Gamma(\beta_1, \ldots, \beta_m)}: \mathbb{Z}[x_1, \ldots, x_m]^{(m)} \to \mathbb{Z} \]
evaluated by \( \int_{\Gamma(\beta_1, \ldots, \beta_m)} h = \langle p_T(\beta_1, \ldots, \beta_m)(h), \Gamma(\beta_1, \ldots, \beta_m) \rangle \). The geometric implication of the triangular operator \( T_A \) in Definition 2.3 is shown by the following result.

**Lemma 3.2** \((\text{[15]} \text{ Proposition 2})\). We have
\[ \int_{\Gamma(\beta_1, \ldots, \beta_m)} = T_A: \mathbb{Z}[x_1, \ldots, x_m]^{(m)} \to \mathbb{Z}. \]

In particular, \( \int_{\Gamma(\beta_1, \ldots, \beta_m)} x_1 \cdots x_m = 1. \quad \square \)

For a parabolic \( P \) on \( G \) we can assume, without loss of the generalities, that \( T \subseteq P \subseteq G \). For a sequence \( \beta_1, \ldots, \beta_m \) of simple roots the associated Bott-Samelson’s K-cycle on \( G/P \) is the map
\[ \varphi_{\beta_1, \ldots, \beta_m,P}: \Gamma(\beta_1, \ldots, \beta_m) \to G/P \]
defined by \( \varphi_{\beta_1, \ldots, \beta_m,P}(g_1, \ldots, g_m) = g_1 \cdots g_m P \). If \( P = T \) Hansen \([32]\) has shown that certain \( K \)-cycles are desingularizations of the Schubert varieties on \( G/T \). The following more general result allows one to translate the calculation with Schubert classes on \( G/P \) to computing with monomials in the much simpler ring \( H^*(\Gamma(\beta_1, \ldots, \beta_m)) \).

**Lemma 3.3.** With respect the Schubert basis on \( H^*(G/P) \) the induced map of \( \varphi_{\beta_1, \ldots, \beta_m,P} \) on the cohomologies is given by
\[ (3.2) \quad \varphi_{\beta_1, \ldots, \beta_m,P}^*(s_w) = (-1)^{l(w)} \sum_{x_I \in w, |I| = l(w); j \in [1, \ldots, m]} x_I, \quad w \in W(P; G). \]

**Proof.** With \( T \subseteq P \subseteq G \) the map \( \varphi_{\beta_1, \ldots, \beta_m,P} \) factors through \( \varphi_{\beta_1, \ldots, \beta_m,T} \) in the fashion
where the map $\pi$ is the fibration with fiber $P/T$. By \cite[Lemma 5.1]{55} formula (3.2) holds for the case $P = T$. According to \cite[55]{55} the induced map $\pi^*: H^*(G/P) \to H^*(G/T)$ is given by $\pi^*(s_w) = s_w$, $w \in W(P; G)$, showing formula (3.2) for the general case $T \subset P$. 

\textbf{Proof of Theorem 2.4.} For a monomial $s_{w_1} \cdots s_{w_k}$ in the Schubert classes of $G/P$ assume as in (1.2) that

\begin{equation}
(3.3) \quad s_{w_1} \cdots s_{w_k} = \sum_{w \in W(P; G), l(w) = m} a_{w_1, \ldots, w_k}^w \cdot s_w, \quad a_{w_1, \ldots, w_k}^w \in \mathbb{Z},
\end{equation}

where $m = l(w_1) + \cdots + l(w_k)$. For an element $w_0 \in W(P; G)$ with a reduced decomposition $w_0 = \sigma_{\beta_1} \circ \cdots \circ \sigma_{\beta_m}, \beta_i \in \Delta$, let $A_{w_0} = (a_{i,j})_{m \times m}$ be the relative Cartan matrix. Applying the ring map $\varphi_{\beta_1, \ldots, \beta_m}: P$ to the equation (3.3) on $H^*(G/P)$ we obtain by (3.2) the equality on the group $H^{2m}(\Gamma(\beta_1, \ldots, \beta_m))$

\begin{equation}
(-1)^{l(w_1) + \cdots + l(w_k)} \prod_{1 \leq i \leq k} \left( \sum_{x_I \in \{ \sum_{1 \leq i \leq k} x_I \leq (w_i), I \subseteq \{1, \ldots, m\} \}} x_I \right) = (-1)^m a_{w_0}^w \cdot x_1 \cdots x_m.
\end{equation}

Applying $\int_{\Gamma(\beta_1, \ldots, \beta_m)}$ to both sides we get by Lemma 3.2 that

\begin{equation}
(-1)^{l(w_1) + \cdots + l(w_k)} \cdot T_{A_w} \left( \prod_{1 \leq i \leq k} \left( \sum_{x_I \in \{ \sum_{1 \leq i \leq k} x_I \leq (w_i), I \subseteq \{1, \ldots, m\} \}} x_I \right) \right) = (-1)^m \cdot a_{w_0}^w \cdot x_1 \cdots x_m.
\end{equation}

This is identical to (2.2) because of $m = l(w_1) + \cdots + l(w_k)$. 

\section{The cohomology of flag manifolds $G/P$}

The classical Schubert calculus amounts to the determination of the intersection rings on Grassmann varieties and on the so-called "flag manifolds" of projective geometry. - A. Weil \cite[p.331]{61}.

A classical problem of topology is to express the integral cohomology ring $H^*(G/H)$ of a homogeneous space $G/H$ by a minimal system of explicit generators and relations. The traditional approach due to H. Cartan, A. Borel, P. Baum, H. Toda utilize various spectral sequence techniques \cite{2, 3, 57, 58, 65}, and the calculation encounters the same difficulties when applied to a Lie group $G$ with torsion elements in its integral cohomology, in particular, when $G$ is one of the five exceptional Lie groups \cite{38, 55, 52}.

However, if $P \subset G$ is parabolic, Schubert calculus makes the structure of the ring $H^*(G/P)$ appearing in a new light. Given a set $\{y_1, \ldots, y_k\}$ of $k$ elements let $\mathbb{Z}[y_1, \ldots, y_k]$ be the ring of polynomials in $y_1, \ldots, y_k$ with integer coefficients. For a subset $\{r_1, \ldots, r_m\} \subset \mathbb{Z}[y_1, \ldots, y_k]$ of homogeneous polynomials denote by $\langle r_1, \ldots, r_m \rangle$ the ideal generated by $r_1, \ldots, r_m$.

\textbf{Theorem 4.1.} For each flag manifold $G/P$ there exist a set $\{y_1, \ldots, y_k\}$ of Schubert classes on $G/P$, and a set $\{r_1, \ldots, r_m\} \subset \mathbb{Z}[y_1, \ldots, y_k]$ of polynomials, so that the inclusion $\{y_1, \ldots, y_k\} \subset H^*(G/P)$ induces a ring isomorphism
(4.1) $H^*(G/P) = \mathbb{Z}[y_1, \ldots, y_k]/\langle r_1, \ldots, r_m \rangle$,

where both the numbers $k$ and $m$ are minimal subject to this presentation. □

**Proof.** Let $D(H^*(G/P)) \subset H^*(G/P)$ be the ideal of the decomposable elements. Since the ring $H^*(G/P)$ is torsion free and has a basis consisting of Schubert classes, there is a set $\{y_1, \ldots, y_k\}$ of Schubert classes on $G/P$ that corresponds to a basis of the quotient group $H^*(G/P)/D(H^*(G/P))$. In particular, the inclusion $\{y_1, \ldots, y_k\} \subset H^*(G/P)$ induces a surjective ring map

$$f : \mathbb{Z}[y_1, \ldots, y_k] \to H^*(G/P).$$

Since $\ker f$ is an ideal the Hilbert basis theorem implies that there exists a finite subset $\{r_1, \ldots, r_m\} \subset \mathbb{Z}[y_1, \ldots, y_k]$ so that $\ker f = \langle r_1, \ldots, r_m \rangle$. We can of course assume that the number $m$ is minimal subject to this constraint.

As the cardinality of a basis of the quotient group $H^*(G/P)/D(H^*(G/P))$ the number $k$ is an invariant of $G/P$. In addition, if one changes the generators $y_1, \ldots, y_k$ to $y'_1, \ldots, y'_k$, then each old generator $y_i$ can be expressed as a polynomial $g_i$ in the new ones $y'_1, \ldots, y'_k$, and the invariance of the number $m$ is shown by the presentation

$$H^*(G/P) = \mathbb{Z}[y'_1, \ldots, y'_k]/\langle r'_1, \ldots, r'_m \rangle,$$

where $r'_j$ is obtained from $r_j$ by substituting $g_i$ for $y_i$, $1 \leq j \leq m$. □

A presentation of the ring $H^*(G/P)$ in the form of (4.1) will be called a Schubert presentation of the cohomology of $G/P$, while the set $\{y_1, \ldots, y_k\}$ of generators will be called a set of special Schubert classes on $G/P$. Based on the characteristic formula (2.2) we develop in this section algebraic and computational machineries implementing Schubert presentation of the ring $H^*(G/P)$. To be precise the following conventions will be adopted throughout the remaining part of this section.

i) $G$ is a 1-connected simple Lie group with Weyl group $W$, and a fixed maximal torus $T$;

ii) A set $\Delta = \{\beta_1, \ldots, \beta_n\}$ of simple roots of $G$ is given and ordered as the vertex of the Dykin diagram of $G$ pictured on [32, p.58];

iii) For each simple root $\beta_i \in \Delta$ write $\sigma_i$ instead of $\sigma_{\beta_i} \in W$; $\omega_i$ in place of the Schubert class $s_{\sigma_{\beta_i}} \in H^2(G/T)$.

Note that Theorem 1.2 implies that the set $\{\omega_1, \ldots, \omega_n\}$ is the Schubert basis of the second cohomology $H^2(G/T)$, whose elements is identical to the fundamental dominant weights of $G$ in the context of Borel and Hirzebruch [4, 17].

4.1 Decomposition

By convention iii) each $w \in W$ admits a factorization of the form

$$w = \sigma_{i_1} \circ \cdots \circ \sigma_{i_k}, \ 1 \leq i_1, \ldots, i_k \leq n, \ l(w) = k.$$
The cardinality of the Schubert basis of $G/T$ agrees with the order of the Weyl group $W$, which in general is very large. To reduce the computation costs we may take a proper subset $K \subset \{1, \ldots, n\}$ and let $P := P_K$ be the corresponding parabolic subgroup. The inclusion $T \subset P \subset G$ then induces the fibration

$$(4.3) \quad P/T \xrightarrow{i^*} G/T \xrightarrow{\pi} G/P,$$

where the induced maps $\pi^*$ and $i^*$ behave well with respect to the Schubert bases of the three flag manifolds $P/T$, $G/P$ and $G/T$ in the following sense:

**Algorithm 4.2. Decomposition.**

**Input:** The Cartan matrix $A = (a_{ij})_{n \times n}$ of $G$, and a subset $K \subset \{1, \ldots, n\}$.

**Output:** The set $W(P_K; G)$ being presented by the minimized decompositions of its elements, together with the index system (4.2) imposed by the order $\leq$.

For examples of the results coming from Decomposition we refer to [22, 1.1–7.1].

### 4.2 Factorization of the ring $H^*(G/T)$ using fibration

The cardinality of the Schubert basis of $G/T$ agrees with the order of the Weyl group $W$, which in general is very large. To reduce the computation costs we may take a proper subset $K \subset \{1, \ldots, n\}$ and let $P := P_K$ be the corresponding parabolic subgroup. The inclusion $T \subset P \subset G$ then induces the fibration

$$(4.3) \quad P/T \xrightarrow{i^*} G/T \xrightarrow{\pi} G/P,$$

where the induced maps $\pi^*$ and $i^*$ behave well with respect to the Schubert bases of the three flag manifolds $P/T$, $G/P$ and $G/T$ in the following sense:
For these reasons we can make no difference in notation between an element in $H^*(G/P)$ and its $\pi^*$ image in $H^*(G/T)$, and between a Schubert class on $P/T$ and $G/P$ and with respect to them one has the Schubert presentations

$$H^*(P/T) = \mathbb{Z}[y_1, y_2, \ldots, y_{n_1}]_{[s_1]}; \quad H^*(G/P) = \mathbb{Z}[x_1, x_2, \ldots, x_{n_2}]_{[s_2]};$$

where $h_s = \mathbb{Z}[y_1, y_2, \ldots, y_{n_1}]$, $r_t = \mathbb{Z}[x_1, x_2, \ldots, x_{n_2}]$. The following result allows one to formulate the ring $H^*(G/T)$ by the simpler ones $H^*(P/T)$ and $H^*(G/P)$.

**Theorem 4.3.** The inclusions $y_i, x_j \in H^*(G/T)$ induces a surjective ring map

$$\varphi: \mathbb{Z}[y_1, y_2, \ldots, y_{n_1}]_{[s_1]} \to H^*(G/T).$$

Furthermore, if $\{\rho_s\}_{1 \leq s \leq m_1} \subset \mathbb{Z}[y_1, y_2]$ is a system satisfying

$$\rho_s \in \ker \varphi \quad \text{and} \quad \rho_s |_{x_j = 0} = h_s;$$

then $\varphi$ induces a ring isomorphism

$$H^*(G/T) = \mathbb{Z}[y_1, y_2, \ldots, y_{n_1}]_{[s_1]} / (\rho_s, r_t)_{1 \leq s \leq m_1, 1 \leq t \leq m_2}.$$

**Proof.** By the property i) above the bundle (4.3) has the Leray-Hirsch property. That is, the cohomology $H^*(G/T)$ is a free module over the ring $H^*(G/P)$ with the basis $\{1, s_w\}_{w \in W(P)}$:

$$H^*(G/T) = H^*(G/P) \{1, s_w\}_{w \in W(P)} \quad (\text{KZ} \text{ p.} 231),$$

implying that $\varphi$ surjects. It remains to show that for any $g \in \ker \varphi$ one has

$$g \in (\rho_s, r_t)_{1 \leq s \leq m_1, 1 \leq t \leq m_2}.$$

To this end we notice by (4.5) and (4.7) that

$$g \equiv \sum w_{w \in W(P)} g_w \cdot s_w \mod (\rho_s)_{1 \leq s \leq m_1} \quad \text{with} \quad g_w \in \mathbb{Z}[x_j]_{1 \leq j \leq n_2}.$$

Thus $\varphi(g) = 0$ implies $\varphi(g_w) = 0$, showing $g_w \in (r_t)_{1 \leq t \leq m_2}$ by (4.4). \(\square\)
4.3 The generalized Grassmannians

For a topological space $X$ we set

$$H^{\text{even}}(X) := \bigoplus_{r \geq 0} H^{2r}(X), \quad H^{\text{odd}}(X) := \bigoplus_{r \geq 0} H^{2r+1}(X).$$

Then $H^{\text{even}}(X)$ is a subring of $H^*(X)$, while $H^{\text{odd}}(X)$ is a module over the ring $H^{\text{even}}(X)$.

If $P$ is a parabolic subgroup that corresponds to a singleton $K = \{i\}$, the flag manifold $G/P$ is called generalized Grassmannians of $G$ corresponding to the weight $\omega_i$ [20]. With $W^i(P; G) = \{\sigma_i\}$ consisting of a single element the basis theorem implies that $H^2(G/P) = \mathbb{Z}$ is generated by $\omega_i$. Furthermore, letting $P^s$ be the semi-simple part of $P$, then the projection $p : G/P^s \to G/P$ is an oriented circle bundle on $G/P$ with Euler class $\omega_i$. With $H^{\text{odd}}(G/P) = 0$ by the basis theorem the Gysin sequence [48, p.143]

$$\cdots \to H^r(G/P)^{\beta} \to H^r(G/P^s)^{\beta} \to H^r-1(G/P)^{\beta} \oplus H^r+1(G/P)^{\beta} \to \cdots,$$

of $p$ breaks into the short exact sequences

$$(4.8) \quad 0 \to \omega_i \cup H^{2r-2}(G/P) \to H^{2r}(G/P)^{\beta} \to H^{2r}(G/P^s) \to 0$$

as well as the isomorphisms

$$(4.9) \quad \beta : H^{2r-1}(G/P^s) \cong \ker\{H^{2r-2}(G/P)^{\omega_i} \to H^{2r}(G/P)\},$$

where $\omega_i \cup$ means taking cup product with $\omega_i$. In particular, formula (4.8) implies that

**Lemma 4.4.** If $S = \{y_1, \ldots, y_m\} \subset H^*(G/P)$ is a subset so that $p^*S = \{p^*(y_1), \ldots, p^*(y_m)\}$ is a minimal set of generators of the ring $H^{\text{even}}(G/P^s)$, then $S' = \{\omega_i, y_1, \ldots, y_m\}$ is a minimal set of generators of $H^*(G/P)$.

By Lemma 4.4 the inclusions $\{\omega_i\} \cup S \subset H^*(G/P), p^*S \subset H^*(G/P^s)$ extend to the surjective maps $\pi$ and $\overline{\pi}$ that fit in the commutative diagram

$$\begin{array}{ccc}
Z[\omega_i, y_1, \ldots, y_m]^{(2r)} & \xrightarrow{\pi} & Z[y_1, \ldots, y_m]^{(2r)} \\
\downarrow & & \downarrow \\
H^{2r-2}(G/P) & \xrightarrow{\omega_i \cup} & H^{2r}(G/P) \\
\end{array}$$

$$(4.10) \quad 0 \to \varphi(\omega_i) = 0, \varphi(y_i) = y_i; \overline{\varphi}(y_i) = p^*(y_i).$$

The next result showing in [20] Lemma 8] enables us to formulate a presentation of the ring $H^*(G/P)$ in term of $H^*(G/P^s)$.

**Theorem 4.5.** Assume that $\{h_1, \ldots, h_d\} \subset Z[y_1, \ldots, y_m]$ is a subset so that

$$(4.11) \quad H^{\text{even}}(G/P^s) = Z[p^*(y_1), \ldots, p^*(y_m)]/ (p^*(h_1), \ldots, p^*(h_d)),$$

and that $\{d_1, \ldots, d_t\}$ is a basis of the module $H^{\text{odd}}(G/P^s)$ over $H^{\text{even}}(G/P^s)$.
(4.12) $H^*(G/P) = \mathbb{Z}[\omega_i, y_1, \ldots, y_m]/\langle r_1, \ldots, r_d; \omega g_1, \ldots, \omega g_t \rangle$,

where $\{r_1, \ldots, r_d\}, \{g_1, \ldots, g_t\} \subset \mathbb{Z}[\omega_i, y_1, \ldots, y_m]$ are two sets of polynomials that satisfy respectively the following “initial constraints”

i) $r_k \in \ker \pi$ with $r_k |_{\omega_i = 0} = h_k$, $1 \leq k \leq d$;

ii) $\pi(g_j) = \beta(d_j)$, $1 \leq j \leq t$. \hfill $\blacksquare$

4.4 The Characteristics

To make Theorems 4.3 and 4.5 applicable in practical computation we develop in this section a series of three algorithms, entitled Characteristics, Null-space, Giambelli polynomials, all of them are based on the characteristic formula (2.2).

4.4.1. The Characteristics. For a $w \in W(P; G)$ with the minimized decomposition $w = \sigma_{i_1} \circ \cdots \circ \sigma_{i_m}$, $1 \leq i_1, \ldots, i_m \leq n$, $l(w) = m$, we observe in formula (2.2) that

i) The Cartan matrix $A_w$ of $w$ can be read directly from the Cartan matrix \[\text{[35] p.59}\] of the Lie group $G$;

ii) For a $u \in W(P; G)$ with $l(u) = r < m$ the solutions in the multi-indices $I = \{j_1, \ldots, j_r\} \subseteq \{i_1, \ldots, i_m\}$ to the equation $\sigma_I = u$ in $W(P; G)$ agree with the solutions to the linear system $\sigma_I(x_0) = u(x_0)$ on the vector space $L(T)$, where $x_0$ is the fixed regular point;

iii) The evaluation the operator $T_{A_w}$ on a polynomial have been programmed using different methods in \[\text{[19] 59}\].

Summarizing, granted with Decomposition, formula (2.2) indicates an effective algorithm to implement a parallel program whose function is briefed below.

**Algorithm 4.6:** Characteristics.

**Input:** The Cartan matrix $A = (a_{ij})_{n \times n}$ of $G$, and a monomial $s_{w_1} \cdots s_{w_k}$ in Schubert classes on $G/P$.

**Output:** The expansion (1.2) of $s_{w_1} \cdots s_{w_k}$ in the Schubert basis. $\blacksquare$

4.4.2. The Null-space. Let $\mathbb{Z}[y_1, \ldots, y_k]$ be the ring of polynomials in $y_1, \ldots, y_k$ graded by $\deg y_i > 0$, and let $\mathbb{Z}[y_1, \ldots, y_k]^{(m)}$ be the $\mathbb{Z}$-module consisting of all the homogeneous polynomials with degree $m$. Denote by $\mathbb{N}^k$ the set of all $k$-tuples $\alpha = (b_1, \ldots, b_k)$ of non-negative integers. Then the set of monomials basis of $\mathbb{Z}[y_1, \ldots, y_k]^{(m)}$ is

$$B(m) = \{y^\alpha = y_1^{b_1} \cdots y_k^{b_k} \mid \alpha = (b_1, \ldots, b_k) \in \mathbb{N}^k, \deg y^\alpha = m\}.$$  

(4.13) It will be considered as an ordered set with respect to the lexicographical order on $\mathbb{N}^k$, whose cardinality is called $b_m$.

Let $S = \{y_1, \ldots, y_k\}$ be a set of Schubert classes on $G/P$ that generates the ring $H^*(G/P)$ multiplicatively. Then the inclusion $S \subseteq H^*(G/P)$ induces a surjective ring map $f : \mathbb{Z}[y_1, \ldots, y_k] \to H^*(G/P)$ whose restriction to degree $2m$ is
Combining the Characteristics with the function “Null-space” in Mathematica, a basis of ker $f_m$ can be explicitly exhibited.

Let $s_{m,i}$ be the Schubert class corresponding to the element $w_{m,i} \in W(P;G)$. With respect to the Schubert basis $\{s_{m,i} | 1 \leq i \leq \beta(m)\}$ on $H^{2m}(G/P)$ every monomial $y^\alpha \in B(2m)$ has the unique expansion

$$
\pi_m(y^\alpha) = c_{\alpha,1} \cdot s_{m,1} + \cdots + c_{\alpha,\beta(m)} \cdot s_{m,\beta(m)}; c_{\alpha,i} \in \mathbb{Z},
$$

where the coefficients $c_{\alpha,i}$ can be evaluated by the Characteristics. The matrix $M(f_m) = (c_{\alpha,i})_{b(2m) \times \beta(m)}$ so obtained is called the structure matrix of $f_m$. The built-in function Null-space in Mathematica transforms $M(f_m)$ to another matrix $N(f_m)$ in the fashion

\begin{align*}
\text{In:} & = \text{Null-space}[M(f_m)] \\
\text{Out:} & = a \text{ matrix } N(f_m) = (b_{j,\alpha})_{(b(2m) - \beta(m)) \times b(2m)},
\end{align*}

whose significance is shown by the following fact.

**Lemma 4.7.** The set $\kappa_i = \sum_{y^\alpha \in B(2m)} b_{i,\alpha} \cdot y^\alpha, 1 \leq i \leq b(2m) - \beta(m)$, of polynomials is a basis of the $\mathbb{Z}$ module ker $f_m$. $\square$

### 4.4.3. The Giambelli polynomials (i.e. the Schubert polynomials)

[6] For the unitary group $G = U(n)$ of rank $n$ with parabolic subgroup $P = U(k) \times U(n-k)$ the flag manifold $G_{n,k} = G/P$ is the Grassmannian of $k$-planes through the origin on $\mathbb{C}^n$. Let $1 + c_1 + \cdots + c_k$ be the total Chern class of the canonical $k$-bundle on $G_{n,k}$. Then the $c_i$'s can be identified with appropriate Schubert classes on $G_{n,k}$ (i.e. the special Schubert class on $G_{n,k}$), and one has the classical Schubert presentation

$$
H^*(G_{n,k}) = \mathbb{Z}[c_1, \ldots, c_k]/\langle r_{n-k+1}, \ldots, r_n \rangle,
$$

where $r_j$ is the component of the formal inverse of $1 + c_1 + \cdots + c_k$ in degree $j$. It follows that every Schubert class $s_w$ on $G_{n,k}$ can be written as a polynomial $G_w(c_1, \ldots, c_k)$ in the special ones, and such an expression is afforded by the classical Giambelli formula [24, p.112].

In general, assume that $G/P$ is a flag variety, and that a Schubert presentation (4.1) of the ring $H^*(G/P)$ has been specified. Then each Schubert class $s_w$ of $G/P$ can be expressed as a polynomial $G_w(y_1, \ldots, y_k)$ in these special ones, and such an expression will be called a Giambelli polynomial of the class $s_w$. Based on the Characteristics a program implementing $G_w(y_1, \ldots, y_k)$ has been compiled, whose function is summarized below.

**Algorithm 4.8: Giambelli polynomials**

**Input:** A set $\{y_1, \ldots, y_k\}$ of special Schubert classes on $G/P$.

**Output:** Giambelli polynomials $G_w(y_1, \ldots, y_k)$ for all $w \in W(P;G)$.

We clarify the details in this program. By (4.13) we can write the ordered monomial basis $B(2m)$ of $\mathbb{Z}[y_1, \ldots, y_k]^{2m}$ as $\{y^{\alpha_1}, \ldots, y^{\alpha_{b(2m)}}\}$. The corresponding structure matrix $M(f_m)$ in degree $2m$ then satisfies
the 1-connected simple Lie groups consist of the three infinite families see the proof of Theorem 4.3. Further, according to E. Cartan [64, p.674] all $\mathcal{T} \subseteq \mathcal{G}/\mathcal{T}$ complete flag manifold in which the induced map $\pi$ provides us with two unique invertible matrices $P = P_{\beta(2m)\times\beta(2m)}$ and $Q = Q_{\beta(m)\times\beta(m)}$ that satisfy

\[ P \cdot M(f_m) \cdot Q = \left( \begin{array}{c} I_{\beta(m)} \\ C \end{array} \right)_{\beta(2m)\times\beta(m)}, \]

where $I_{\beta(m)}$ is the identity matrix of rank $\beta(m)$. The Giambelli polynomials is realized by the procedure below.

**Step 1.** Compute $M(f_m)$ using the Characteristics;

**Step 2.** Diagonalize $M(f_m)$ to get the matrices $P$ and $Q$:

\[ \begin{pmatrix} \mathcal{G}_{m,1} \\ \vdots \\ \mathcal{G}_{m,\beta(m)} \end{pmatrix} = Q \cdot [P] \begin{pmatrix} y^{\alpha_1} \\ \vdots \\ y^{\alpha_{\beta(2m)}} \end{pmatrix}, \]

where $[P]$ is formed by the first $\beta(m)$ rows of $P$. Obviously, the polynomial $\mathcal{G}_{m,j}$ so obtained depends only on the special Schubert classes $\{y_1, \ldots, y_k\}$ on $\mathcal{G}/\mathcal{P}$, and is a Giambelli polynomial of $s_{m,j}$, $1 \leq j \leq \beta(m)$.

5 Application to the flag manifolds $\mathcal{G}/\mathcal{T}$

A calculus, or science of calculation, is one which has organized processes by which passage is made, mechanically, from one result to another. -De Morgan.

Among all the flag manifolds $\mathcal{G}/\mathcal{P}$ associated to a Lie group $\mathcal{G}$ it is the complete flag manifold $\mathcal{G}/\mathcal{T}$ that is of crucial importance, since the inclusion $\mathcal{T} \subseteq \mathcal{P} \subseteq \mathcal{G}$ of subgroups induces the fibration

$\mathcal{P}/\mathcal{T} \hookrightarrow \mathcal{G}/\mathcal{T} \longrightarrow \mathcal{G}/\mathcal{P}$

in which the induced map $\pi^*$ embeds the ring $H^*(\mathcal{G}/\mathcal{P})$ as a subring of $H^*(\mathcal{G}/\mathcal{T})$, see the proof of Theorem 4.3. Further, according to E. Cartan [54, p.674] all the 1-connected simple Lie groups consist of the three infinite families $SU(n)$, $Sp(n)$, $Spin(n)$ of the classical groups, as well as the five exceptional ones: $G_2, F_4, E_6, E_7, E_8$, while for any compact connected Lie group $\mathcal{G}$ with a maximal torus $\mathcal{T}$ one has a diffeomorphism

$\mathcal{G}/\mathcal{T} = G_1/T_1 \times \cdots \times G_k/T_k$

with each $G_i$ an 1-connected simple Lie group and $T_i \subseteq G_i$ a maximal torus.

Thus, the problem of finding Schubert presentations of flag manifolds may be reduced to the special cases $\mathcal{G}/\mathcal{T}$ where $\mathcal{G}$ is 1-connected and simple.

In this section we determine the Schubert presentation of the ring $H^*(\mathcal{G}/\mathcal{T})$ in accordance to $\mathcal{G}$ is classical or exceptional. Recall that for a simple Lie group $\mathcal{G}$ with rank $n$ the fundamental dominant weights $\{\omega_1, \ldots, \omega_n\}$ of $\mathcal{G}$ [4] is precisely the Schubert basis of $H^2(\mathcal{G}/\mathcal{T})$ [17, Lemma 2.4].
5.1 The ring $H^*(G/T)$ for a classical $G$

If $G = SU(n+1)$ or $Sp(n)$ Borel [3] has shown that

$$H^*(G/T) = \mathbb{Z}[\omega_1, \ldots, \omega_n] / \left\langle \mathbb{Z}[\omega_1, \ldots, \omega_n]^{+, W} \right\rangle,$$

where $\mathbb{Z}[\omega_1, \ldots, \omega_n]^{+, W}$ is the ring of the integral Weyl invariants of $G$ in positive degrees. It follows that if we let $c_r(G) \in H^{2r}(G/T)$ be respectively the $r^{th}$ element symmetric polynomial in the sets

$$\{\omega_1, \omega_k - \omega_{k-1}, -\omega_n \mid 2 \leq k \leq n\} \text{ or } \{\pm \omega_1, \pm (\omega_k - \omega_{k-1}) \mid 2 \leq k \leq n\},$$

then we have

**Theorem 5.1.** For $G = SU(n)$ or $Sp(n)$ Schubert presentation of $H^*(G/T)$ is

\begin{align}
(5.1) & \quad H^*(SU(n)/T) = \mathbb{Z}[\omega_1, \ldots, \omega_{n-1}] / \langle c_2, \ldots, c_n \rangle, \quad c_r = c_r(SU(n)); \\
(5.2) & \quad H^*(Sp(n)/T) = \mathbb{Z}[\omega_1, \ldots, \omega_n] / \langle c_2, \ldots, c_{2n} \rangle, \quad c_{2r} = c_{2r}(Sp(n)). \quad \square
\end{align}

Turning to the group $G = Spin(2n)$ let $y_k$ be the Schubert class on $Spin(2n)/T$ associated to the Weyl group element

$$w_k = \sigma[n - k, \ldots, n - 2, n - 1], \quad 2 \leq k \leq n - 1$$

(in the notation of Section 4.1). According to Marlin [46, Proposition 3]

\begin{align}
(5.3) & \quad H^*(Spin(2n)/T) = \mathbb{Z}[\omega_1, \ldots, \omega_n, y_2, \ldots, y_{n-1}] / \langle \delta_i, \xi_j, \mu_k \rangle
\end{align}

where

$$\delta_i := 2y_i - c_i(\omega_1, \ldots, \omega_n), \quad 1 \leq i \leq n - 1,$$

$$\xi_j := y_{2j} + (-1)^j y_j^2 + 2 \sum_{1 \leq r \leq j - 1} (-1)^r y_r y_{2j-r}, \quad 1 \leq j \leq \left\lfloor \frac{n-1}{2} \right\rfloor,$$

$$\mu_k := (-1)^k y_k^2 + 2 \sum_{2k-n+1 \leq r \leq k-1} (-1)^r y_r y_{2k-r}, \quad \left\lfloor \frac{k}{2} \right\rfloor \leq k \leq n - 1,$$

and where $c_i(\omega_1, \ldots, \omega_n)$ is the $i^{th}$ elementary symmetric function on set

$$\{\omega_n, \omega_i - \omega_{i-1}, \omega_{n-1} + \omega_n - \omega_{n-2}, \omega_{n-1} - \omega_n \mid 2 \leq i \leq n - 2\}.$$ 

Since each Schubert class $y_{2j}$ with $1 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor$ can be expressed as a polynomial in the $y_{2r+1}$’s by the relations of the type $\xi_k$, we obtain that

**Theorem 5.2.** The Schubert presentation of $H^*(Spin(2n)/T)$ is

\begin{align}
(5.4) & \quad H^*(Spin(2n)/T) = \mathbb{Z}[\omega_1, \ldots, \omega_n, y_3, y_5, \ldots, y_{2\left\lfloor \frac{n+1}{2} \right\rfloor - 1}] / \langle r_i, h_k \rangle,
\end{align}

where $r_i$ and $h_k$ are the polynomials obtained respectively from $\delta_i$ and $\mu_k$ by replacing the classes $y_{2r}$ with the polynomials (by the relation $\xi_r$)

$$(-1)^{r-1} y_r^2 + 2 \sum_{1 \leq k \leq r-1} (-1)^{k-1} y_k y_{2r-k}. \quad \square$$
Similarly, if $G = \text{Spin}(2n + 1)$ one can deduce a Schubert presentation of the ring $H^*(G/T)$ from Marlin’s formula \cite{Marlin} Proposition 2.

**Remark 5.3.** For the classical Lie groups $G$ the Giambelli polynomials (i.e. Schubert polynomials) of the Schubert classes on $G/T$ have been determined by Billey and Haiman \cite{Billey-Haiman}.

In comparison with Marlin’s formula (5.3) the presentation (5.4) is more concise for involving fewer generators and relations. A basic requirement of topology is to present the cohomology of a space $X$ by a minimal system of generators and relations, so that the (rational) minimal model and $\kappa$-invariants of the Postnikov tower of $X$ \cite{Postnikov} can be formulated accordingly. □

### 5.2 The ring $H^*(G/T)$ for an exceptional $G$

Having clarified the Schubert presentation of the ring $H^*(G/T)$ for the classical $G$ we proceed to the exceptional cases $G = F_4$, $E_6$ or $E_7$ (the result for the case $E_8$ comes from the same calculation, only the presentation \cite{Billey-Haiman} Theorem 5.1 is slightly lengthy). In these cases the dimension $s = \dim G/T$ and the number $t$ of the Schubert classes on $G/T$ are

$$(s, t) = (48, 1152), (72, 51840) \text{ or } (126, 2903040),$$

respectively. Instead of describing the ring $H^*(G/T)$ using the totality of $t^3$ Littlewood-Richardson coefficients, the idea of Schubert presentation brings us the following concise formulae of the ring $H^*(G/T)$.

**Theorem 5.4.** For $G = F_4, E_6$ and $E_7$ the Schubert presentations of the cohomologies $H^*(G/T)$ are

\begin{align*}
H^*(F_4/T) &= \mathbb{Z}[\omega_1, \omega_2, \omega_3, \omega_4, y_3, y_4]/\langle \rho_2, \rho_4, r_3, r_6, r_8, r_{12} \rangle, \quad \text{where} \\
\rho_2 &= c_2 - 4\omega_1^2; \\
\rho_4 &= 3y_4 + 2\omega_1y_3 - c_4; \\
r_3 &= 2y_3 - \omega_1^2; \\
r_6 &= y_3^2 + 2c_6 - 3\omega_1^2y_4; \\
r_8 &= 3y_4^2 - \omega_1^2c_6; \\
r_{12} &= y_3^3 - c_6^2. \\
\end{align*}

\begin{align*}
H^*(E_6/T) &= \mathbb{Z}[\omega_1, \ldots, \omega_6, y_3, y_4]/\langle \rho_2, \rho_3, \rho_4, \rho_5, r_6, r_8, r_9, r_{12} \rangle, \quad \text{where} \\
\rho_2 &= 4\omega_2^2 - c_2; \\
\rho_3 &= 2y_3 + 2\omega_2^3 - c_3; \\
\rho_4 &= 3y_4 + \omega_2^4 - c_4; \\
\rho_5 &= 2\omega_2^3y_3 - \omega_2c_4 + c_5; \\
r_6 &= y_3^3 - \omega_2c_5 + 2c_6; \\
r_8 &= 3y_4^2 - 2c_8y_3 - \omega_2^3c_6 + \omega_2^2c_8; \\
r_9 &= 2yc_6 - \omega_2^2c_6; \\
r_{12} &= y_3^3 - c_6^2. \\
\end{align*}
(5.7) \( H^*(E_7/T) = \mathbb{Z}[^\omega_1, \ldots, ^\omega_7, y_1, y_4, y_5, y_9]/\langle \rho_1, r_3 \rangle \), where
\[
\rho_2 = 4\omega_2^2 - c_2; \\
\rho_3 = 2y_3 + 2\omega_3 - c_3; \\
\rho_4 = 3y_4 + \omega_4^2 - c_4; \\
\rho_5 = 2y_5 - 2\omega_2y_3 + \omega_2c_4 - c_5; \\
r_6 = y_6^2 - \omega_2c_5 + 2c_6; \\
r_8 = 3y_8^2 + 2y_3y_5 - 2y_3c_5 + 2\omega_2c_7 - \omega_2^2c_6 + \omega_3c_5; \\
r_9 = 2y_9 + 2y_4y_5 - 2y_3c_6 - \omega_2^2c_7 + \omega_3^2c_6; \\
r_{10} = y_5^2 - 2y_3c_7 + \omega_3^2c_7; \\
r_{12} = y_6^2 - 4y_5y_7 - c_7^2 - 2y_3y_9 - 2y_3y_4y_5 + 2\omega_2y_5c_6 + 3\omega_2y_4c_7 + c_5c_7; \\
r_{14} = c_7^2 - 2y_5y_9 + 2y_3y_4c_7 - \omega_3^2y_4c_7; \\
r_{18} = y_8^2 + 2y_5c_6c_7 - y_4c_7^2 - 2y_4y_9y_7 + 2y_3y_5^2 - 5\omega_2y_5^2c_7,
\]
where the \( c_r \)'s are the polynomials \( c_r(P) \) in the weights \( \omega_1, \ldots, \omega_n \) defined in (5.17); and where the \( y_i \)'s are the Schubert classes on \( G/T \) associated to the Weyl group elements tabulated below:

| \( y_i \) | \( F_4/T ) ( F/P_{(4)} ) \) | \( E_6/T ) ( E_6/P_{(2)} ) \) | \( E_7/T ) ( E_7/P_{(2)} ) \) |
|---|---|---|---|
| \( y_3 \) | \( \sigma[3,2,1] \) | \( \sigma[5,4,2] \) | \( \sigma[5,4,2] \) |
| \( y_4 \) | \( \sigma[4,3,2,1] \) | \( \sigma[6,5,4,2] \) | \( \sigma[6,5,4,2] \) |
| \( y_5 \) | \( \sigma[3,2,4,3,2,1] \) | \( \sigma[1,3,6,5,4,2] \) | \( \sigma[7,6,5,4,2] \) |
| \( y_6 \) | \( \sigma[3,2,4,3,2,1] \) | \( \sigma[1,3,6,5,4,2] \) | \( \sigma[1,3,7,6,5,4,2] \) |
| \( y_7 \) | \( \sigma[3,2,4,3,2,1] \) | \( \sigma[1,3,6,5,4,2] \) | \( \sigma[1,5,4,3,7,6,5,4,2] \) |
| \( y_9 \) | \( \sigma[3,2,4,3,2,1] \) | \( \sigma[1,3,6,5,4,2] \) | \( \sigma[1,5,4,3,7,6,5,4,2] \) |

To reduce the computational complexity of showing Theorem 5.4 we choose for each \( G = F_4, E_6 \) or \( E_7 \) a parabolic subgroup \( P \) associated to a singleton \( K = \{ i \} \), where the index \( i \), as well as the isomorphism types of \( P \) and its simple part \( P^s \), is stated in the table below

| \( G \) | \( F_4 \) | \( E_6 \) | \( E_7 \) |
|---|---|---|---|
| \( i \) | \( 1 \) | \( 2 \) | \( 2 \) |
| \( P; P^s \) | \( \Sp(3) \cdot \Sp^1; \Sp(3) \) | \( SU(6) \cdot \Sp^1; SU(6) \) | \( SU(7) \cdot \Sp^1; SU(7) \) |

In view of the circle bundle associated to \( P \) (see in Section 4.3)

(5.10) \( S^1 \hookrightarrow G/P^s \xrightarrow{p} G/P \)

the calculation will be divided into three steps, in accordance to cohomologies of the three homogeneous spaces \( G/P^s, G/P \) and \( G/T \).

**Step 1. The cohomologies** \( H^*(G/P^s) \). By the formulae (4.8) and (4.9) the additive structure of \( H^*(G/P^s) \) is determined by the homomorphisms

\[
H^{2r-2}(G/P) \xrightarrow{\cup \omega} H^{2r}(G/P).
\]
Explicitly, with respect to the Schubert basis \( \{ s_{r,1}, \ldots, s_{r,\beta(r)} \} \) of \( H^{2r}(G/P) \),
\( \beta(r) = |W^r(P; G)| \), the expansions
\[
\omega \cup s_{r-1,i} = \sum a_{i,j} \cdot s_{r,j}
\]
give rise to a \( \beta(r-1) \times \beta(r) \) matrix \( A_r \) that satisfies the linear system
\[
\begin{pmatrix}
\omega \cup s_{r-1,1} \\
\omega \cup s_{r-1,2} \\
\vdots \\
\omega \cup s_{r-1,\beta(r-1)}
\end{pmatrix} = A_r
\begin{pmatrix}
s_{r,1} \\
s_{r,2} \\
\vdots \\
s_{r,\beta(r)}
\end{pmatrix}.
\]
Since \( \omega \cup s_{r-1,i} \) is a monomial in Schubert classes, the Characteristics is applicable to evaluate the entries of \( A_r \). Diagonalizing \( A_r \) by the integral row and column reductions [20 p.162-166] one obtains the non-trivial groups \( H^r(G/P^*) \), together with their basis, as that tabulated below, where

1) \( y_i := P^i(y_i) \) with \( y_i \) the Schubert classes in table (5.8);
2) For simplicity the non-trivial groups \( H^r(G/P^*) \) are printed only up to the stage where all the generators and relations of the ring \( H^{even}(G/P^*) \) emerge.

Table 1. Non-trivial cohomologies of \( F_4/P^* \):

| nontrivial \( H^k \) | basis elements | relations |
|----------------------|---------------|-----------|
| \( H^0 \cong \mathbb{Z}_2 \) | \( s_{3,1}(= \overline{y}_3) \) | \( 2\overline{y}_2 = 0 \) |
| \( H^8 \cong \mathbb{Z} \) | \( s_{4,2}(= \overline{y}_4) \) | \( 2\overline{y}_6 = \overline{y}_3 \) |
| \( H^{12} \cong \mathbb{Z}_4 \) | \( s_{6,2}(= \overline{y}_6) \) | \( -2\overline{y}_6 = \overline{y}_3 \) |
| \( H^{14} \cong \mathbb{Z}_2 \) | \( \overline{y}_4 \overline{y}_4 \) | \( 3\overline{y}_4 = 0 \) |
| \( H^{16} \cong \mathbb{Z}_4 \) | \( \overline{y}_4 \) | \( -2\overline{y}_6 = \overline{y}_3 \) |
| \( H^{18} \cong \mathbb{Z}_2 \) | \( \overline{y}_4 \overline{y}_6 \) | \( 2\overline{y}_6 = 0 \) |
| \( H^{20} \cong \mathbb{Z}_4 \) | \( \overline{y}_4 \overline{y}_6 \) | \( \overline{y}_4 \overline{y}_6 = 0 \) |
| \( H^{23} \cong \mathbb{Z} \) | \( d_{23} = \beta^{-1}(2s_{11,1} - s_{11,2}) \) | \( \overline{y}_4 \overline{y}_6 = 0 \) |

Table 2. Non-trivial cohomologies of \( E_6/P^* \):

| nontrivial \( H^k \) | basis elements | relations |
|----------------------|---------------|-----------|
| \( H^0 \cong \mathbb{Z} \) | \( s_{3,1}(= \overline{y}_3) \) | \( 2\overline{y}_2 = 0 \) |
| \( H^8 \cong \mathbb{Z}_2 \) | \( s_{4,2}(= \overline{y}_4) \) | \( 2\overline{y}_6 = \overline{y}_3 \) |
| \( H^{12} \cong \mathbb{Z}_4 \) | \( s_{6,1}(= \overline{y}_6) \) | \( -2\overline{y}_6 = \overline{y}_3 \) |
| \( H^{14} \cong \mathbb{Z}_4 \) | \( \overline{y}_4 \overline{y}_4 \) | \( 3\overline{y}_4 = 0 \) |
| \( H^{16} \cong \mathbb{Z}_4 \) | \( \overline{y}_4 \) | \( -2\overline{y}_6 = \overline{y}_3 \) |
| \( H^{18} \cong \mathbb{Z}_2 \) | \( \overline{y}_4 \overline{y}_6 \) | \( 2\overline{y}_6 = 0 \) |
| \( H^{20} \cong \mathbb{Z}_4 \) | \( \overline{y}_4 \overline{y}_6 \) | \( \overline{y}_4 \overline{y}_6 = 0 \) |
| \( H^{22} \cong \mathbb{Z}_4 \) | \( \overline{y}_4 \overline{y}_6 \) | \( \overline{y}_4 \overline{y}_6 = 0 \) |
| \( H^{24} \cong \mathbb{Z}_4 \) | \( \overline{y}_4 \overline{y}_6 \) | \( \overline{y}_4 \overline{y}_6 = 0 \) |
| \( H^{23} \cong \mathbb{Z}_4 \) | \( \overline{y}_4 \overline{y}_6 \) | \( \overline{y}_4 \overline{y}_6 = 0 \) |
| \( H^{24} \cong \mathbb{Z} \) | \( d_{23} = \beta^{-1}(s_{11,1} - s_{11,2} - s_{11,3} + s_{11,4} - s_{11,5} + s_{11,6}) \) | \( 2d_{29} = \pm \overline{y}_4 d_{23} \) |
| \( H^{23} \cong \mathbb{Z} \) | \( d_{29} = \beta^{-1}(-s_{14,1} + s_{14,2} + s_{14,4} - s_{14,5}) \) | \( 2d_{29} = \pm \overline{y}_4 d_{23} \) |
Table 3. Non-trivial cohomologies of $E_7/P^s$

| $H^K$          | basis elements |
|----------------|----------------|
| $H^4 \cong \mathbb{Z}$ | $s_{3,2} = y_3$ |
| $H^6 \cong \mathbb{Z}$ | $s_{4,3} = y_4$ |
| $H^{10} \cong \mathbb{Z}$ | $s_{5,4} = y_5$ |
| $H^{14} \cong \mathbb{Z}$ | $s_{6,5} = y_6 + y_6$ |
| $H^{18} \cong \mathbb{Z} \oplus \mathbb{Z}$ | $s_{7,6} = -y_7 - y_4 y_6$ |
| $H^{18} \cong \mathbb{Z}$ | $y_3 y_5 - 2 y_4^2$ |
| $H^{18} \cong \mathbb{Z} \oplus \mathbb{Z}$ | $y_3 y_7 - y_4^2 - y_3 y_7 + y_4 y_6$ |
| $H^{18} \cong \mathbb{Z} \oplus \mathbb{Z}$ | $y_3 y_7 - y_3 y_7 + y_4 y_7 - 2 y_4 y_7 + y_5 y_6$ |
| $H^{18} \cong \mathbb{Z} \oplus \mathbb{Z}$ | $y_3 y_6 + y_4 y_6 + y_4 y_5 - y_4 y_5 - y_4^2 + y_4 y_7 + y_5^2$ |
| $H^{18} \cong \mathbb{Z} \oplus \mathbb{Z}$ | $y_3 y_6 + y_4 y_6 + y_4 y_5 - y_4 y_5 - y_4^2 + y_4 y_7 + y_5^2$ |
| $H^{18} \cong \mathbb{Z} \oplus \mathbb{Z}$ | $y_3 y_6 + y_4 y_6 + y_4 y_5 - y_4 y_5 - y_4^2 + y_4 y_7 + y_5^2$ |
| $H^{18} \cong \mathbb{Z} \oplus \mathbb{Z}$ | $y_3 y_6 + y_4 y_6 + y_4 y_5 - y_4 y_5 - y_4^2 + y_4 y_7 + y_5^2$ |
| $H^{18} \cong \mathbb{Z} \oplus \mathbb{Z}$ | $y_3 y_6 + y_4 y_6 + y_4 y_5 - y_4 y_5 - y_4^2 + y_4 y_7 + y_5^2$ |
| $H^{18} \cong \mathbb{Z}$ | $y_3 y_6 + y_4 y_6 + y_4 y_5 - y_4 y_5 - y_4^2 + y_4 y_7 + y_5^2$ |
| $H^{18} \cong \mathbb{Z}$ | $y_3 y_6 + y_4 y_6 + y_4 y_5 - y_4 y_5 - y_4^2 + y_4 y_7 + y_5^2$ |

**Step 2. the cohomologies $H^*(G/P)$**. Summarizing the contents of Table 1, we find that

$$H^{even}(F_4/P^s) = \mathbb{Z}[g_3, g_4, g_6]/\langle p^*(h_3), p^*(h_6), p^*(h_8), p^*(h_{12}) \rangle,$$

where

$$h_3 = 2y_3, \quad h_6 = 2y_6 + y_4^2, \quad h_8 = 3y_8, \quad h_{12} = y_6^2 - y_4^2,$$

and that $H^{odd}(F_4/P^s)$ has the $H^{even}(F_4/P^s)$-module basis $\{d_{23}\}$. By Theorem 4.3 we obtain the partial presentation

$$H^*(F_4/P) = \mathbb{Z}[\omega_1, y_3, y_4, y_6]/\langle r_3, r_6, r_8, r_{12}, r'_{12} \rangle,$$

indicating that the inclusion $\{\omega_1, y_3, y_4, y_6\} \subset H^*(F_4/P)$ induces the surjective ring map $f: \mathbb{Z}[\omega_1, y_3, y_4, y_6] \to H^*(F_4/P)$. Further, according to Lemma 4.7, computing with the Null-space $N(f_m)$ in the order $m = 3, 6, 8$ and 12 suffices to decide the generators of the ideal $\ker f$ to yields the Schubert presentation...
(5.11) \( H^*(F_4/P) = \mathbb{Z}[\omega_1, y_3, y_4, y_6]/\langle r_3, r_6, r_8, r_{12} \rangle \), where
\[
\begin{align*}
    r_3 &= 2y_3 - \omega_1^2; \\
    r_6 &= 2y_6 + y_4^2 - 3\omega_1^2 y_4; \\
    r_8 &= 3y_4^2 - \omega_1^2 y_6; \\
    r_{12} &= y_6^2 - y_4^2.
\end{align*}
\]

Similarly, combining the Null-space with the contents of Tables 2 and 3 one gets the Schubert presentations of the ring \( H^*(G/P) \) for \( G = E_6 \) and \( E_7 \) as

(5.12) \( H^*(E_6/P) = \mathbb{Z}[\omega_2, y_3, y_4, y_6]/\langle r_6, r_9, r_{12} \rangle \), where
\[
\begin{align*}
    r_6 &= 2y_6 + y_4^2 - 3\omega_2^2 y_4 + 2\omega_2 y_3 - \omega_2^3; \\
    r_9 &= 2y_6 y_9 - \omega_2^2 y_6; \\
    r_{12} &= y_6^2 - y_4^2.
\end{align*}
\]

(5.13) \( H^*(E_7/P) = \mathbb{Z}[\omega_2, y_3, y_4, y_5, y_6, y_7, y_9]/\langle r_j \rangle_{j \in \Lambda} \), where \( \Lambda = \{6, 8, 9, 10, 12, 14, 18\} \),
\[
\begin{align*}
    r_6 &= 2y_6 + y_2^2 + 2\omega_2 y_5 - 3\omega_2^2 y_4 + 2\omega_2^3 y_3 - \omega_2^3; \\
    r_9 &= 2y_6 y_9 - \omega_2^2 y_6; \\
    r_{12} &= y_6^2 - y_4^2; \\
    r_{18} &= y_6^2 + 2y_5 y_9 - y_4^2 + 2y_3 y_9 + 2y_3 y_4 y_5 + 2\omega_2 y_5 y_6 - 6\omega_2 y_4 y_7 + \omega_2^2 y_5^2; \\
    r_{14} &= y_6^2 - 2y_5 y_7 + y_4 y_5^2; \\
    r_{16} &= y_6^2 + 2y_5 y_6 y_9 - y_4 y_7^2 - 2y_4 y_5 y_6 + 2y_3 y_6^2 - \omega_2 y_3^2 y_7.
\end{align*}
\]

\textbf{Step 3. Computing with the Weyl invariants.} In addition to (5.10) the parabolic subgroup \( P \) on \( G \) specified by table (5.9) induces also the fibration

(5.14) \( P/T \xrightarrow{\iota} G/T \xrightarrow{\pi} G/P \) (i.e. (4.3)),

where Schubert presentation of the cohomology of the base space \( G/P \) has been decided by (5.11), (5.12) and (5.13). On the other hand, with

\[
P/T = Sp(3)/T^3, \ SU(6)/T^5 \text{ or } SU(7)/T^6 \text{ for } G = F_4, E_6 \text{ or } E_7
\]

the cohomology of the fiber space \( P/T \) has been decided by Theorem 5.1 as

(5.15) \( H^*(P/T) = \begin{cases} 
    \mathbb{Z}[\omega_1, \ldots, \omega_n] & \text{if } G = F_4 \\
    \mathbb{Z}[\omega_1, \ldots, \omega_n] / (c_r, \ldots, c_{r+n}) & \text{if } G = E_n \text{ with } n = 6, 7.
\end{cases} \)

Thus, Theorem 4.3 may be applicable to fashion the ring \( H^*(G/T) \) in question from the known ones \( H^*(P/T) \) and \( H^*(G/P) \). To this end we need only to specify a system \( \{r_i\} \) satisfying the constraints (4.5). The invariant theory of Weyl groups helps to implement this final task.

Recall that the Weyl group \( W \) of \( G \) can be identified with the subgroup of \( \text{Aut}(H^2(G/T)) \) generated by the automorphisms \( \sigma_i, 1 \leq i \leq n \), whose action on the Schubert basis \( \{\omega_1, \ldots, \omega_n\} \) of \( H^2(G/T) \) is given by the Cartan matrix \( (a_{ij})_{n \times n} \) of \( G \) as
(5.16) \( \sigma_i(\omega_k) = \begin{cases} 
\omega_k & \text{if } k \neq i; \\
\omega_k - \sum_{1 \leq j \leq n} a_{ij} \omega_j & \text{if } k = i. 
\end{cases} \)

Introduce for each \( G = F_4, E_6 \) and \( E_7 \) the polynomials \( c_r(P) \) in \( \omega_1, \ldots, \omega_n \) by the formula

\[
(5.17) \quad c_r(P) := \begin{cases} 
\sigma_r(o(\omega_4, W(P))), 1 \leq r \leq 4 & \text{if } G = F_4; \\
\sigma_r(o(\omega_n, W(P))), 1 \leq r \leq n & \text{if } G = E_n, n = 6, 7. 
\end{cases}
\]

where \( o(\omega, W(P)) \subset H^2(G/T) \) denotes the \( W(P) \)-orbit through \( \omega \in H^2(G/T) \), and where \( e_r(o(\omega, W(P))) \in H^{2r}(G/T) \) is the \( r \)-th elementary symmetric function on the set \( o(\omega, W(P)) \). Explicitly, for \( G = F_4, E_6 \) we have by (5.16) that

\[
o(\omega_4, W(P)) = \{ \omega_4, \omega_3 - \omega_4, \omega_3 - \omega_1 - \omega_2 + \omega_3, \omega_1 - \omega_3 + \omega_4, \omega_1 - \omega_4 \};
\]

\[
o(\omega_6, W(P)) = \{ \omega_6, \omega_5 - \omega_6, \omega_4 - \omega_5, \omega_2 + \omega_3 - \omega_4, \omega_1 + \omega_2 - \omega_3, \omega_2 - \omega_1 \}.
\]

On the other hand, according to Bernstein-Gelfand-Gelfand \[5\] Proposition 5.1] the induced map \( \pi^* \) in (4.3) injects, and satisfies the relation

\[
\Im \pi^* = H^*(G/T)^W(P) = H^*(G/P),
\]

implying \( c_r(P) \in H^*(G/P) \). Since \( c_r(P) \) is an explicit polynomial in the Schubert classes \( \omega_i \), the Giambelli polynomials is functional to express it as a polynomial \( g_r \) in the special Schubert classes on \( H^*(G/P) \) given by table (5.8):

\[
(5.18)
\begin{array}{|c|c|c|c|}
\hline
G & F_4 & E_6 & E_7 \\
\hline
\hline
\quad g_2 \quad & 4\omega_2 & 4\omega_2 & 4\omega_2 \\
\quad g_3 \quad & 4y_3 + 2\omega_3 & 2y_3 + 2\omega_3 & 2y_3 + 2\omega_3 \\
\quad g_4 \quad & 3y_4 + 2\omega_4 + \omega_4 & y_4 + \omega_4 & 3y_4 + \omega_4 \\
\quad g_5 \quad & 3y_5 - 2\omega_5 y_3 + \omega_5 & 5y_5 + 3\omega_5 y_4 - 2\omega_5 y_3 + \omega_5 & 5y_5 + 3\omega_5 y_4 - 2\omega_5 y_3 + \omega_5 \\
\quad g_6 \quad & y_6 & y_6 & y_6 \\
\quad g_7 \quad & & & y_7 \\
\hline
\end{array}
\]

Up to now we have accumulated sufficient information to show Theorem 5.4.

**Proof of Theorem 5.4.** For each \( G = F_4, E_6 \) or \( E_7 \) Schubert presentations for the cohomologies of the base \( G/P \) and of the fiber \( P/T \) have been determined by (5.11)-(5.13) and (5.15), respectively, while a system \( \{ \rho_i \} \) satisfying the relation (4.5) is seen to be \( \rho_r := c_r(P) - g_r \). Therefore, Theorem 4.3 is directly applicable to formulate a presentation of the ring \( H^*(G/T) \). The results can be further simplified to yield the desired formul\( (5.5)-(5.7) \) by the following observations:

a) Certain Schubert classes \( y_i \) from the base space \( G/P \) can be eliminated against appropriate relations of the type \( \rho_y \), e.g. if \( G = E_7 \) the generators \( y_6, y_7 \) and the relations \( \rho_6, \rho_7 \) can be excluded by the formul\( e_i \) of \( g_6 \) and \( g_7 \), which implies that \( y_6 = \sigma_6 - 2\omega_2 y_5 \) and \( y_7 = c_7 \), respectively;

b) Without altering the ideal, higher degree relations of the type \( r_i \) may be simplified modulo the lower degree ones by the following fact. For two ordered sequences \( \{ f_i \}_{1 \leq i \leq n} \) and \( \{ h_i \}_{1 \leq i \leq n} \) of a graded polynomial ring with

\[
deg f_1 < \cdots < \deg f_n \text{ and } \deg h_1 < \cdots < \deg h_n
\]

write \( \{ h_i \}_{1 \leq i \leq n} \sim \{ f_i \}_{1 \leq i \leq n} \) to denote the statements that \( \deg h_i = \deg f_i \) and that \( (f_i - h_i) \in (f_j)_{1 \leq j < i} \). Then \( \{ f_i \}_{1 \leq i \leq n} \sim \{ h_i \}_{1 \leq i \leq n} \) implies that \( (h_1, \ldots, h_n) = (f_1, \ldots, f_n) \). \[\square\]
5.3 A type free characterization of the ring \(H^*(G/T)\)

For an 1-connected simple Lie group \(G\) with rank \(n\) denote by \(D(G) \subset H^*(G/T)\) the ideal of decomposable elements. Let \(h(G)\) be the cardinality of a basis of the quotient group \(H^*(G/T)/D(G)\) and set \(m = h(G) - n - 1\). The results of Theorems 5.1, 5.2 and 5.4 can be summarized into one formula, without referring to the types of the group \(G\) (see [21 Theorems 1.2 and 1.3]).

**Theorem 5.5.** For each simple Lie group \(G\) there exist a set \(\{y_1, \ldots, y_m\}\) of \(m\) Schubert classes on \(G/T\) with \(2 < \deg y_1 < \cdots < \deg y_m\), so that the inclusion \(\{\omega_1, \ldots, \omega_n, y_1, \ldots, y_m\} \in H^*(G/T)\) induces the Schubert presentation

\[
H^*(G/T) = \mathbb{Z}[\omega_1, \ldots, \omega_n, y_1, \ldots, y_m] / \langle e_i, f_j, g_j \rangle_{1 \leq i \leq k, 1 \leq j \leq m},
\]

where

i) \(k = n - m\) for all \(G \neq E_8\) but \(k = n - m + 2\) for \(G = E_8\);

ii) \(e_i \in \langle \omega_1, \ldots, \omega_n \rangle, 1 \leq i \leq k\);

iii) the pair \((f_j, g_j)\) of polynomials is related to the Schubert class \(y_j\) in the fashion

\[
f_j = p_j \cdot y_j + \alpha_j, \quad g_j = y_j^{k_j} + \beta_j, \quad 1 \leq j \leq m,
\]

where \(p_j \in \{2, 3, 5\}\) and \(\alpha_j, \beta_j \in \langle \omega_1, \ldots, \omega_n \rangle\);

iv) ignoring the ordering, the sequence \(\{\deg e_i, \deg g_j\}\) of integers agrees with the degree sequence of the basic Weyl invariants of the group \(G\) (over the field of rationals).

Concerning assertion iv) we remark that for each simple Lie group \(G\) the degree sequence, as well as explicit formulae, of the basic Weyl invariants \(P_1, \ldots, P_n\) of \(G\) has been determined by Chevalley and Mehta [12, 47].

6 Further remarks on the characteristics

6.1. Certain parameter spaces of the geometric figures concerned by Schubert [51 Chapter 4] may fail to be flag manifolds, but can be constructed by performing finite number of blow-ups on flag manifolds along the centers again in flag manifolds, see the examples in Fulton [27 Example 14.7.12], or in [18] for the construction of the parameter spaces of the complete conics and quadrics on the 3-space \(\mathbb{P}^3\). As results the relevant characteristics can be computed from those of flag manifolds via strict transformations (e.g. [18 Examples 5.11; 5.12]).

6.2. As the intersection multiplicities of certain Schubert varieties on \(G/P\), the characteristics \(a_{w_1, \ldots, w_k}\) are always non-negative by Van der Waerden [60]. Due to the importance of these numbers in geometry their effective computability (rather than positivity) had been the top priority in the classical approach [51] [52] [54], see also Fulton [27 14.7].

Motivated by the Littlewood-Richardson rule [45] for the structure constants of the Grassmannian \(G_{n,k}\) a remarkable development of Schubert calculus has taken place in algebraic combinatorics since 1970’s, where the main concern is to find combinatorial descriptions of characteristics by which the positivity become transparent. This idea has inspired beautiful results on the enumerations of
Yong tableaux, Mondrian tableaux, Chains in the Bruhat order, and puzzles by A. Buch, W.A. Graham, I. Coskun, A. Knutson and T.C. Tao \[7, 14, 28, 33, 44\], greatly enhanced the classical Schubert calculus.

6.3. According to Van der Waerden \[60\] and A. Weil \[61, p.331\] Hilbert’s 15th problem has been solved satisfactorily. In particular, in the context of modern intersection theory (e.g. \[27, 29\]) rigorous treatment of the major enumerative results of Schubert \[51\] had been completed independently by many authors (e.g. \[1, 41, 42, 49\])

\[; \text{granted with the basis theorem the characteristics of flag manifolds can be evaluated uniformly by the formula (2.2), while the Schubert presentations of the cohomology rings of flag manifolds have also been available (e.g. \[3, 6, 20, 21, 46\].}

However, Schubert calculus remains a vital and powerful tool in constructing the cohomologies of much broad spaces, such as the homogeneous spaces \(G/H\) associated to Lie groups \(G\). In contrast to the basis theorem (i.e. Theorem 1.2) the cohomologies of such spaces may be nontrivial in odd degrees, and may contain torsion elements. Nevertheless, inputting the formula (5.19) into the Koszul complex

\[E^*_2(G) = H^*(G/T) \otimes H^*(T)\]

associated to the fibration \(G \to G/T\) a unified construction of the integral cohomology rings of all the 1-connected simple Lie groups \(G\) has been completed by Duan and Zhao in \[23\]. In addition, the formula (2.2) of the characteristics have been extended to evaluate the Steenrod operators on the mod \(p\) cohomologies of flag manifolds \[24\], and of the exceptional Lie groups \[25\].

6.4. As is of today Schubert calculus has entered the intersection of several rapidly developing fields of mathematics, and has been extended to the studies of other generalized cohomology theories, such as equivariant, quantum cohomology, K-theory, and cobordism, all of them are different deformations of the ordinary cohomology \[31\]. In this regard the present paper is by no means a comprehensive survey on the contemporary Schubert calculus. Instead, it talks about an explicit passage from the Cartan matrices of Lie groups to the cohomology of homogeneous spaces, where Schubert’s characteristics play a central role.

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\(^2\)The enumerative results in Schubert \[51\] were mutually verifiable with the results of other geometers (e.g. Salmon, Clebsch, Chasles and Zeuthen) of the same period, hence were already known to be correct at that time.
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