ON THE SEMICIRCULAR LAW OF LARGE DIMENSIONAL RANDOM QUATERNION MATRICES

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Abstract. It is well known that Gaussian symplectic ensemble (GSE) is defined on the space of $n \times n$ quaternion self-dual Hermitian matrices with Gaussian random elements. There is a huge body of literature regarding this kind of matrices (see [7, 8, 12, 15, 16, 9]). As a natural idea we want to get more universal results by removing the Gaussian condition. For the first step, in this paper we prove that the empirical spectral distribution of the common quaternion self-dual Hermitian matrices tends to semicircular law. The main tool to establish the universal result is given as a lemma in this paper as well.

1. Introduction and Main Results

Suppose $H_n$ is an $n \times n$ Hermitian matrix with eigenvalues $s_i, i = 1, 2, \cdots n$. The empirical spectral distribution (ESD) of $H_n$ is defined as:

$$F_{H_n}(x) = \frac{1}{n} \sum_{i=1}^{n} I(s_i \leq x)$$

where $I(\cdot)$ is indicator function. It is shown that if the entries on and above the diagonal of $H_n$ (known as Wigner matrix) are independent random variables with zero-mean and variance $n^{-1}\sigma^2$, then $F_{H_n}$ converges almost surely (a.s.) to a non-random distribution $F$ which has the density function

$$(1.1) \quad f(x) = \frac{1}{2\pi\sigma} \sqrt{4\sigma^2 - x^2}, \quad x \in [-2, 2].$$

This is also known as the semicircular law (see [18]). Winger matrices and semicircular law play important roles in physics and pure mathematics. Thus in recent years, there are a lot of subsequent work which were trying to obtain a better understanding of Winger matrices and semicircular law. Much more details can be found in [14, 17, 2, 11, 6] and references therein.

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In mathematics, the quaternions were first described by Irish mathematician William Rowan Hamilton in 1843 [11], and applied to mechanics in three-dimensional space. In recent years, quaternions are found uses in both theoretical and applied mathematics, such as in three-dimensional computer graphics and computer vision, the theory of peg-top using, navigation, enginery and organ, robot technology and artificial satellite attitude control, and so on. However, random quaternion matrices have not been studied as substantially as those in real and complex fields, except for Gaussian Symplectic Ensemble (GSE).

The GSE is defined on the space of $n \times n$ quaternion self-dual Hermitian matrices. For a matrix $X = (x_{jk})_{n \times n}$ drawn from the GSE, where

$$x_{jk} = a_{jk} + b_{jk} \cdot i_1 + c_{jk} \cdot i_2 + d_{jk} \cdot i_3$$

is a random quaternion. Here $\{i_1, i_2, i_3\}$ denotes the standard quaternion basis with

\[
\begin{align*}
  i_1^2 &= i_2^2 = i_3^2 = -1, & i_1 &= i_2 i_3 = -i_3 i_2, \\
  i_2 &= i_1 i_3 = -i_3 i_1, & i_3 &= i_1 i_2 = -i_2 i_1. 
\end{align*}
\]

The four coefficients $\{a_{jk}, b_{jk}, c_{jk}, d_{jk}\}$ are independent real Gaussian random variables with zero mean. For $j > k$,

$$Ea_{jk}^2 = Eb_{jk}^2 = Ec_{jk}^2 = Ed_{jk}^2 = 1/4,$$

and for the diagonal elements $Ex_{jj}^2 = Ea_{jj}^2 = 1$. It was shown that the ESD of GSE tends to semicircular law almost surely, and there are also many local and bulk results about GSE. Details can be found in [7, 8, 12, 15, 16, 9, 3].

Therefore it motives us to investigate the universal results of the quaternion self-dual Hermitian matrices. Before giving the main theorem, we introduce some notation and basic properties of quaternion. Let

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

where $i = \sqrt{-1}$. It’s easy to verify that

$$i^2 = j^2 = k^2 = -e, \quad i = jk = -kj,$$

$$j = ik = -ki, \quad k = ij = -ji.$$

Thus a quaternion

$$x = a + bi_1 + ci_2 + di_3$$

can be represented as

$$x = a \cdot e + b \cdot i + c \cdot j + d \cdot k = \begin{pmatrix} \lambda & \omega \\ -\omega & \lambda \end{pmatrix},$$
where $a, b, c, d$ are real and $\lambda = a + bi$, $\omega = c + di$ are complex. Here $i$ denotes the usual imaginary unit. The quaternion conjugate of $x$ is defined by

$$\bar{x} = a \cdot e - b \cdot i - c \cdot j - d \cdot k = \begin{pmatrix} \lambda & -\omega \\ \omega & \lambda \end{pmatrix},$$

and its norm is defined by

$$\|x\| = \sqrt{a^2 + b^2 + c^2 + d^2} = \sqrt{|\lambda|^2 + |\omega|^2}.$$

**Remark 1.1.** Apparently, an $n \times n$ Hermitian quaternion matrix $X = (x_{jk})_{n \times n}$ can be represented as a $2n \times 2n$ Hermitian matrix (see Section 2.4 in [14]). That is, we represent the entries of $W_n$ as

$$x^{(n)}_{jk} = \begin{pmatrix} a^{(n)}_{jk} + b^{(n)}_{jk}i & c^{(n)}_{jk} + d^{(n)}_{jk}i \\ -c^{(n)}_{jk} & d^{(n)}_{jk} \end{pmatrix} = \begin{pmatrix} \lambda^{(n)}_{jk} & \omega^{(n)}_{jk} \\ -\bar{\omega}^{(n)}_{jk} & \bar{\lambda}^{(n)}_{jk} \end{pmatrix}, 1 \leq j < k \leq n,$$

and $x^{(n)}_{jj} = \begin{pmatrix} a^{(n)}_{jj} & 0 \\ 0 & a^{(n)}_{jj} \end{pmatrix}$. Then, $W_n$ is represented as a $2n \times 2n$ Hermitian complex matrix, denoted by $W_n$. It is well known (see [19]) that the multiplicities of all the eigenvalues of $W_n^R$ are even and at least 2. Taking one from each of the $n$ pairs of eigenvalues $W_n^R$, the $n$ values are defined as the eigenvalues of $W_n$. Throughout the rest of this paper, we still use $W_n$ to denote the represented one and omit the superscript $(n)$ from the notations for brevity.

The theorem can be described as following:

**Theorem 1.2.** Suppose that $W_n := \frac{1}{\sqrt{n}}X_n = \frac{1}{\sqrt{n}}\left(x^{(n)}_{jk}\right)_{n \times n}$, where $x^{(n)}_{jk} = a^{(n)}_{jk} + b^{(n)}_{jk}i_1 + c^{(n)}_{jk}i_2 + d^{(n)}_{jk}i_3$, is a quaternion self-dual Hermitian matrix whose entries above and on the diagonal are independent and satisfy:

(i) $E x^{(n)}_{jk} = 0$, for all $1 \leq j < k \leq n$.

(ii) $E\|x^{(n)}_{jk}\|^2 < \infty$, $E\|x^{(n)}_{jk}\|^2 = 1$, for all $1 \leq j < k \leq n$.

(iii) For any constant $\eta > 0$,

$$\lim_{n \to \infty} \frac{1}{n^2} \sum_{jk} E \left\|x^{(n)}_{jk}\right\|^2 I\left(\left\|x^{(n)}_{jk}\right\| \geq \eta \sqrt{n}\right) = 0.$$

Then we have as $n \to \infty$, the ESD of $W_n$ converges to semicircular law almost surely.

**Remark 1.3.** Actually, the Hermitian quaternion matrix $W_n$ can be viewed as a $2n \times 2n$ Wigner complex matrix with dependent entries. For recent progress in this direction, we refer to [10] and references therein.

**Remark 1.4.** By the density function (1,1) of semicircular law, we can find that the condition (iii) is also necessary for Theorem 1.2.
Remark 1.5. Note that condition (1.2) is equivalent to: for any \( \eta > 0 \),

\[
\lim_{n \to \infty} \frac{1}{\eta^2 n^2} \sum_{jk} \mathbb{E}[|x_{jk}^{(n)}|^2] I(\|x_{jk}^{(n)}\| \geq \eta \sqrt{n}) = 0.
\]

Thus we can select a sequence \( \eta_n \downarrow 0 \) such that (1.3) remains true when \( \eta \) is replaced by \( \eta_n \).

The remainder of this paper is organized as follows. A main mathematical tool of proving the theorem is established in Section 2. Theorem 1.2 is proved in Section 3 and some technical lemmas are given in Section 4.

2. The main tool

In this section, we will give a lemma which is the key tool to prove Theorem 1.2. Before that, we introduce some definitions firstly.

Definition 2.1. A matrix is called Type-T matrix if it has the following structure:

\[
\begin{pmatrix}
  t & 0 & a_{12} & b_{12} & \cdots & a_{1n} & b_{1n} \\
  0 & t & c_{12} & d_{12} & \cdots & c_{1n} & d_{1n} \\
  d_{12} & -b_{12} & t_2 & 0 & \cdots & a_{2n} & b_{2n} \\
  -c_{12} & a_{12} & 0 & t_2 & \cdots & c_{2n} & d_{2n} \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  d_{1n} & -b_{1n} & d_{2n} & -b_{2n} & \cdots & t_n & 0 \\
  -c_{1n} & a_{1n} & -c_{2n} & a_{2n} & \cdots & 0 & t_n
\end{pmatrix}.
\]

Definition 2.2. A matrix is called Type-I matrix if it has the following structure:

\[
\begin{pmatrix}
  t_1 & 0 & a_{12} & b_{12} & \cdots & a_{1n} & b_{1n} \\
  0 & t_1 & c_{12} & d_{12} & \cdots & c_{1n} & d_{1n} \\
  d_{12} & -b_{12} & t_2 & 0 & \cdots & a_{2n} & b_{2n} \\
  -c_{12} & a_{12} & 0 & t_2 & \cdots & c_{2n} & d_{2n} \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  d_{1n} & -b_{1n} & d_{2n} & -b_{2n} & \cdots & t_n & 0 \\
  -c_{1n} & a_{1n} & -c_{2n} & a_{2n} & \cdots & 0 & t_n
\end{pmatrix}.
\]

Definition 2.3. A matrix is called Type-II matrix if it has the following structure:

\[
\begin{pmatrix}
  t_1 & 0 & a_{12} + c_{12} i & b_{12} + d_{12} i & \cdots & a_{1n} + c_{1n} i & b_{1n} + d_{1n} i \\
  0 & t_1 & b_{12} - d_{12} i & a_{12} + c_{12} i & \cdots & b_{1n} - d_{1n} i & a_{1n} + c_{1n} i \\
  a_{12} + c_{12} i & -b_{12} - d_{12} i & t_2 & 0 & \cdots & a_{2n} + c_{2n} i & b_{2n} + d_{2n} i \\
  b_{12} + d_{12} i & a_{12} + c_{12} i & 0 & t_2 & \cdots & -b_{2n} - d_{2n} i & a_{2n} + c_{2n} i \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  a_{1n} + c_{1n} i & -b_{1n} - d_{1n} i & a_{2n} + c_{2n} i & -b_{2n} - d_{2n} i & \cdots & t_n & 0 \\
  b_{1n} + d_{1n} i & a_{1n} + c_{1n} i & b_{2n} + d_{2n} i & a_{2n} + c_{2n} i & \cdots & 0 & t_n
\end{pmatrix}.
\]

Here \( i = \sqrt{-1} \) denotes the usual imaginary unit and all the other notations denote complex numbers.
Definition 2.4. Let $A_{k1}$ and $A_{k2}$ ($1 \leq k \leq n$) be $2 \times 2$ complex matrices, and have the structure that $A_{k1} = \begin{pmatrix} a_{k1} & a_{k2} \\ a_{k3} & a_{k4} \end{pmatrix}_{2 \times 2}$ and $A_{k2} = \begin{pmatrix} a_{k4} & -a_{k2} \\ -a_{k3} & a_{k1} \end{pmatrix}_{2 \times 2}$.

Then we denote $A_1 \leftrightarrow A_2$, if

$$A_1 = (A_{11}, \cdots, A_{n1})_{2 \times 2n}, \quad A_2 = (A'_{11}, \cdots, A'_{n1})_{2 \times 2n}.$$ 

Let $B_k$ and $C_k$ be any $2 \times 2$ complex matrices, $1 \leq k \leq n$. Then we denote $D_1 \leftrightarrow D_2$, if

$$D_1 = (B_1 + C_1 \cdot i, \cdots, B_n + C_n \cdot i)_{2 \times 2n},$$

and

$$D_2 = (B_1' + C_1' \cdot i, \cdots, B_n' + C_n' \cdot i)'_{2 \times 2n}.$$ 

Here superscript $'$ and $*$ stand for the transpose and complex conjugate transpose of a matrix respectively.

Now we are in position to present the following important lemma which is the main tool to prove Theorem 1.2.

Lemma 2.5. For all $n \geq 1$, if $\Omega_n$ is an invertible complex matrix of Type-II matrix, then $\Omega_n^{-1}$ is of Type-I.

Proof. We prove this lemma by the method of induction. First of all, we can easily verify that the conclusion is correct when $n = 1$ and 2. Now, suppose the conclusion is true when $n = m$ ($m > 2$). Then let $n = m + 1$ and suppose $t_1 \neq 0$. Actually, the condition $t_1 \neq 0$ is equivalent to there exists a $t_j \neq 0$, for some $1 \leq j \leq n$. That is because if $t_1 = 0$ and $t_j \neq 0$, We can exchange $t_1$ and $t_j$ by using elementary transformation of matrices without changing the structures of Type-I and Type-II matrices.

Write: $\Omega_{m+1} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$, where $\Sigma_{11} = \begin{pmatrix} t_1 & 0 \\ 0 & t_1 \end{pmatrix}$,

$$\Sigma_{12} = \begin{pmatrix} a_{12} + c_{12}i & b_{12} + d_{12}i & \cdots & a_{1,m+1} + c_{1,m+1}i & b_{1,m+1} + d_{1,m+1}i \\ -\bar{b}_{12} - \bar{d}_{12}i & \bar{a}_{12} + \bar{c}_{12}i & \cdots & -\bar{b}_{1,m+1} - \bar{d}_{1,m+1}i & \bar{a}_{1,m+1} + \bar{c}_{1,m+1}i \end{pmatrix},$$

$$\Sigma_{21} = \begin{pmatrix} \bar{a}_{12} + \bar{c}_{12}i & \bar{b}_{12} + \bar{d}_{12}i & \cdots & \bar{a}_{1,m+1} + \bar{c}_{1,m+1}i & \bar{b}_{1,m+1} + \bar{d}_{1,m+1}i \\ -\bar{b}_{12} - \bar{d}_{12}i & \bar{a}_{12} + \bar{c}_{12}i & \cdots & -\bar{b}_{1,k+1} - \bar{d}_{1,m+1}i & \bar{a}_{1,m+1} + \bar{c}_{1,m+1}i \end{pmatrix}' ,$$

$$\Sigma_{22} = \begin{pmatrix} t_2 & 0 & a_{23} + c_{23}i & b_{23} + d_{23}i & \cdots \\ 0 & t_2 & \bar{b}_{23} - \bar{d}_{23}i & \bar{a}_{23} + \bar{c}_{23}i & \cdots \\ \bar{a}_{23} + \bar{c}_{23}i & -\bar{b}_{23} - \bar{d}_{23}i & t_3 & 0 & \cdots \\ \bar{b}_{23} + \bar{d}_{23}i & a_{23} + c_{23}i & 0 & t_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}_{2m \times 2m}.$$

According to Lemma 1.3 to complete the proof it is sufficient to show that:

(1): $\Sigma_{22} - \Sigma_{21} \Sigma_{11}^{-1} \Sigma_{12}$ is a Type-II matrix.
(2): $\Sigma_{11}^{-1} + \Sigma_{11}^{-1} \Sigma_{22}^{-1} \Sigma_{21} \Sigma_{11}^{-1}$ is a Type-T matrix.

(3): $-\Sigma_{11}^{-1} \Sigma_{12} \Sigma_{22}^{-1} d \leftrightarrow -\Sigma_{22}^{-1} \Sigma_{21} \Sigma_{11}^{-1}$.

We now proceed in our proof by taking these three steps. Let $A_{jk} = \begin{pmatrix} a_{jk} & b_{jk} \\ -\overline{b_{jk}} & \overline{a_{jk}} \end{pmatrix}$, $B_{jk} = \begin{pmatrix} c_{jk} & d_{jk} \\ -\overline{d_{jk}} & \overline{c_{jk}} \end{pmatrix}_{2 \times 2}$ for $1 \leq j < k \leq m + 1$.

**Step 1: Proof of (1).** Apparently, $\Sigma_{22}$ is a Type-II matrix and $\Sigma_{11}^{-1}$ is a scalar complex matrix. What’s more, we can easily verify that if two $2 \times 2$ matrices $A \leftrightarrow B$, then for any $t \in \mathbb{C}$, $tA \leftrightarrow tB$. Therefore, it’s sufficient to show that $\Sigma_{21} \Sigma_{12}$ is a Type-II matrix. Rewrite

$$
\Sigma_{21} \Sigma_{12} = \begin{pmatrix}
A_{12} & B_{12}^* \\
\vdots & \\
A_{1,m+1} & B_{1,m+1}^*
\end{pmatrix}
$$

Write $\Gamma = \Sigma_{21} \Sigma_{12} = (\gamma_{jk})_{j,k=2,\ldots,m+1}$, where $\gamma_{jk}$ is a $2 \times 2$ complex matrix.

For $j = 2, \ldots, m + 1$, the diagonal $2 \times 2$ block entries $\gamma_{jj}$ are

$$
(A_{1j}^* + B_{1j}^*) \cdot (A_{1j} + B_{1j}i) = (A_{1j}^* A_{1j} - B_{1j}^* B_{1j}) + (A_{1j}^* B_{1j} + B_{1j}^* A_{1j})i
$$

(2.1) $= (|a_{1j}|^2 + |b_{1j}|^2 - |c_{1j}|^2 - |d_{1j}|^2 + 2i \Re(\overline{a_{1j}} c_{1j} + b_{1j} d_{1j})) I_2.$

Thus $\gamma_{jj}$ are all Type-T matrices.

**Note:** In the equation (2.1), if we consider the $2 \times 2$ matrices $A_{1j}$ and $B_{1j}$ as quaternions and their algebraic operations as those in quaternions, then on the right hand side of the first line of (2.1), the expressions $||A_{1j}||^2 - ||B_{1j}||^2$ and $A_{1j}^* B_{1j} + B_{1j}^* A_{1j} = 2 \Re(A_{1j}^* B_{1j})$ are also quaternions. For simplicity of expressions, we shall use either of the double interpretations of the matrices $A_{jk}$ and $B_{jk}$ in accordance with convenience in the following arguments.

Next, for $j \neq k$ we have

$$
(A_{1j}^* + B_{1j}^*) \cdot (A_{1k} + B_{1k}i) = (A_{1j}^* A_{1k} - B_{1j}^* B_{1k}) + (A_{1j}^* B_{1k} + B_{1j}^* A_{1k})i
$$

and

$$
(A_{1k}^* + B_{1k}^*) \cdot (A_{1j} + B_{1j}i) = (A_{1k}^* A_{1j} - B_{1k}^* B_{1j}) + (A_{1k}^* B_{1j} + B_{1k}^* A_{1j})i
$$

$$
= (A_{1j}^* A_{1k} - B_{1j}^* B_{1k})^* + (A_{1j}^* B_{1k} + B_{1j}^* A_{1k})^* i,
$$

which implies, for any $2 \leq j < k \leq m + 1$, $\gamma_{jk} \leftrightarrow \gamma_{kj}$. Thus, the proof of (1) is complete.
Step 2: Proof of (2). Since $\Sigma_{11}^{-1}$ is a Type-T matrix, we only need to prove that $\Sigma_{12}\Sigma_{22,1}^{-1}\Sigma_{21}$ is also a Type-T matrix. Write

$$
\Sigma_{22,1}^{-1} = \begin{pmatrix}
P_{11} & P_{12} & \cdots & P_{1,m} \\
P_{21} & P_{22} & \cdots & P_{2,m} \\
\vdots & \vdots & \ddots & \vdots \\
P_{m,1} & P_{m,2} & \cdots & P_{m,m}
\end{pmatrix}
$$

where $P_{jk}$ are $2 \times 2$ matrices. Then we have,

$$
\Sigma_{12}\Sigma_{22,1}^{-1}\Sigma_{21} = \sum_{jk} (A_{1,j+1} + B_{1,j+1}i)P_{jk}(A_{1,k+1}^* + B_{1,k+1}^*i).
$$

From the induction hypothesis, we know $P_{jj}$ is a Type-T matrix and $P_{jk} \overset{d}{\leftrightarrow} P_{kj}$.

If $j = k$, since $P_{jj}$ is a Type-T matrix and

$$
(A_{1,j+1} + B_{1,j+1}i) \cdot (A_{1,j+1}^* + B_{1,j+1}^*i)
= (A_{1,j+1}A_{1,j+1}^* - B_{1,j+1}B_{1,j+1}^*) + (A_{1,j+1}B_{1,j+1}^* + B_{1,j+1}A_{1,j+1}^*)i
= (\|A_{1,j+1}\|^2 - \|B_{1,j+1}\|^2)I_2 + (A_{1,j+1}B_{1,j+1}^* + B_{1,j+1}A_{1,j+1}^*)i,
$$

we get that $(A_{1,j+1} + B_{1,j+1}i)P_{jj}(A_{1,j+1}^* + B_{1,j+1}^*i)$ is a Type-T matrix.

If $j \neq k$, then we have

$$
(A_{1,j+1} + B_{1,j+1}i)P_{jk}(A_{1,k+1}^* + B_{1,k+1}^*i) + (A_{1,k+1} + B_{1,k+1}i)P_{kj}(A_{1,j+1}^* + B_{1,j+1}^*i)
= (A_{1,j+1}P_{jk}A_{1,k+1}^* + A_{1,k+1}P_{kj}A_{1,j+1}^*) - (B_{1,j+1}P_{jk}B_{1,k+1}^* + B_{1,k+1}P_{kj}B_{1,j+1}^*)
+ (A_{1,j+1}P_{jk}B_{1,k+1}^* + B_{1,k+1}P_{kj}A_{1,j+1}^*)i + (B_{1,j+1}P_{jk}A_{1,k+1}^* + A_{1,k+1}P_{kj}B_{1,j+1}^*)i.
$$

Since $P_{jk} \overset{d}{\leftrightarrow} P_{kj}$, thus we can assume

$$
P_{jk} = \begin{pmatrix} e_{jk} & g_{jk} \\
h_{jk} & f_{jk} \end{pmatrix}, \quad P_{kj} = \begin{pmatrix} f_{jk} & -g_{jk} \\
-h_{jk} & e_{jk} \end{pmatrix}.
$$

It is easy to verify that

$$
(A_{1,j+1}P_{jk}A_{1,k+1}^* + A_{1,k+1}P_{kj}A_{1,j+1}^*)
= \begin{pmatrix} a_{1,j+1} & b_{1,j+1} \\
-b_{1,j+1} & a_{1,j+1} \end{pmatrix}
\begin{pmatrix} e_{jk} & g_{jk} \\
h_{jk} & f_{jk} \end{pmatrix}
\begin{pmatrix} \tau_{1,k+1} & -\tau_{1,k+1} \\
\tau_{1,k+1} & \tau_{1,k+1} \end{pmatrix}
\begin{pmatrix} -b_{1,k+1} & a_{1,k+1} \\
a_{1,k+1} & -b_{1,k+1} \end{pmatrix}
\begin{pmatrix} a_{1,k+1} & b_{1,k+1} \\
-b_{1,k+1} & a_{1,k+1} \end{pmatrix}
\begin{pmatrix} \tau_{1,j+1} & -\tau_{1,j+1} \\
\tau_{1,j+1} & \tau_{1,j+1} \end{pmatrix}
\begin{pmatrix} e_{jk} & g_{jk} \\
h_{jk} & f_{jk} \end{pmatrix}
\begin{pmatrix} -b_{1,j+1} & a_{1,j+1} \\
a_{1,j+1} & -b_{1,j+1} \end{pmatrix}
= \begin{pmatrix} 2x & 0 \\
0 & 2x \end{pmatrix},
$$

where $x$ is a constant.
where
\[ x = (\overline{a}_{1,j+1}a_{1,k+1} + b_{1,j+1}\overline{b}_{1,k+1})f_{jk} + (a_{1,j+1}\overline{a}_{1,k+1} + \overline{b}_{1,j+1}b_{1,k+1})e_{jk} \\
+ (a_{1,j+1}\overline{b}_{1,k+1} - \overline{b}_{1,j+1}a_{1,k+1})g_{jk} + (b_{1,j+1}\overline{a}_{1,k+1} - \overline{a}_{1,j+1}b_{1,k+1})h_{jk}. \]

Hence, we conclude that \((A_{1,j+1}P_{jk}A_{1,k+1}^* + A_{1,k+1}P_{kj}A_{1,j+1}^*)\) is a Type-T matrix. Similarly, we can verify that \((B_{1,j+1}P_{jk}B_{1,k+1}^* + B_{1,k+1}P_{kj}B_{1,j+1}^*)\), \((A_{1,j+1}P_{jk}B_{1,k+1}^* + B_{1,k+1}P_{kj}A_{1,j+1}^*)\), and \((B_{1,j+1}P_{jk}A_{1,k+1}^* + A_{1,k+1}P_{kj}B_{1,j+1}^*)\) are all Type-T matrices, which complete the proof.

**Step 3: Proof of (3).** Since \(\Sigma_{11}^{-1}\) is a diagonal matrix, thus we only need to prove that \(\Sigma_{12}\Sigma_{22,1}^{-1}\) \(\not\rightarrow\) \(\Sigma_{22,1}^{-1}\Sigma_{21}\). Write \(Q = \Sigma_{12}\Sigma_{22,1}^{-1} = (Q_1, Q_2, \cdots, Q_m)\), \(V = \Sigma_{22,1}^{-1}\Sigma_{21} = (V_1', V_2', \cdots, V_m')\) and

\[
\Sigma_{22,1}^{-1} = \begin{pmatrix}
P_{11} & P_{12} & \cdots & P_{1,m} \\
P_{21} & P_{22} & \cdots & P_{2,m} \\
\vdots & \vdots & \ddots & \vdots \\
P_{m,1} & P_{m,2} & \cdots & P_{m,m}
\end{pmatrix},
\]

Then for any \(k\), we have

\[ Q_k = \sum_j (A_{1,j+1} + B_{1,j+1}i)P_{jk}, \quad V_k = \sum_j P_{kj}(A_{1,j+1}^* + B_{1,j+1}^*i). \]

To complete the proof, it is sufficient to show that for any \(k\), \(Q_k \not\rightarrow V_k\).

From the induction hypothesis, we assume that \(P_{jk} = \begin{pmatrix} e_{jk} & f_{jk} \\ g_{jk} & h_{jk} \end{pmatrix}\), \(P_{kj} = \begin{pmatrix} h_{jk} & -f_{jk} \\ -g_{jk} & e_{jk} \end{pmatrix}\).

Then we have

\[
(A_{1,j+1} + B_{1,j+1}i) \cdot \begin{pmatrix} e_{jk} & f_{jk} \\ g_{jk} & h_{jk} \end{pmatrix} \\
= A_{1,j+1} \begin{pmatrix} e_{jk} & f_{jk} \\ g_{jk} & h_{jk} \end{pmatrix} + B_{1,j+1} \begin{pmatrix} e_{jk} & f_{jk} \\ g_{jk} & h_{jk} \end{pmatrix}i \\
= \begin{pmatrix} e_{jk}a_{1,j+1} + g_{jk}b_{1,j+1} & f_{jk}a_{1,j+1} + h_{jk}b_{1,j+1} \\ -e_{jk}\overline{b}_{1,j+1} + g_{jk}\overline{a}_{1,j+1} & -f_{jk}\overline{b}_{1,j+1} + h_{jk}\overline{a}_{1,j+1} \end{pmatrix} \\
+ \begin{pmatrix} e_{jk}c_{1,j+1} + g_{jk}d_{1,j+1} & f_{jk}c_{1,j+1} + h_{jk}d_{1,j+1} \\ -e_{jk}\overline{d}_{1,j+1} + g_{jk}\overline{c}_{1,j+1} & -f_{jk}\overline{d}_{1,j+1} + h_{jk}\overline{c}_{1,j+1} \end{pmatrix}i.
\]
and
\[
\begin{pmatrix}
  h_{jk} & -f_{jk} \\
  -g_{jk} & e_{jk}
\end{pmatrix}
(\hat{A}^*_{1,j+1} + B^*_{1,j+1}i) \\
= \begin{pmatrix}
  h_{jk} & -f_{jk} \\
  -g_{jk} & e_{jk}
\end{pmatrix} A^*_{1,j+1} + \begin{pmatrix}
  h_{jk} & -f_{jk} \\
  -g_{jk} & e_{jk}
\end{pmatrix} B^*_{1,j+1}i \\
= \left( -f_{jk}b_{1,j+1} + h_{jk}\bar{a}_{1,j+1} - f_{jk}a_{1,j+1} - h_{jk}b_{1,j+1} \right) \\
+ \left( -f_{jk}d_{1,j+1} + h_{jk}\bar{c}_{1,j+1} - f_{jk}c_{1,j+1} - h_{jk}d_{1,j+1} \right) i,
\]
which together with (2.2) complete the proof of (3). Therefore, we get Lemma 2.5 with \( t_1 \neq 0 \).

Since all the entries of \( \Omega_m^{-1} \) are in fact a polynomial of \( t_1 \), thus by the continuity of polynomial, we have the conclusion of Lemma 2.5 is true even \( t_1 = 0 \). Then we complete the proof. □

3. Proof of Theorem 1.2

In this section we give the proof of Theorem 1.2. The tools we use here are Stieltjes transform and Burholder inequality for the martingale difference sequence, which was first introduced in [13]. The proof is following the same steps as Section 2 in [4].

3.1. Truncation, centralization and rescale. Define
\[
\hat{W}_n = n^{-1/2}(\hat{x}_{ij})_{j,k=1}^{n} = \frac{1}{\sqrt{n}}(x_{jk}I(\|x_{jk}\| \leq \eta_n\sqrt{n}))_{j,k=1}^{n}.
\]
Then by lemma 4.2, we obtain that
\[
(3.1) \quad \| F^W_n - F^{\hat{W}_n} \| \leq \frac{1}{2n} \text{rank} \left( W_n - \hat{W}_n \right) \leq \frac{1}{n} \sum_{1 \leq j \leq k \leq n} I(\|x_{jk}\| \geq \eta_n\sqrt{n}).
\]
By condition (1.3), we have
\[
E \left( \frac{1}{n} \sum_{1 \leq j \leq k \leq n} I(\|x_{jk}\| \geq \eta_n\sqrt{n}) \right) \leq \frac{1}{\eta_n^2 n^2} \sum_{j,k} E\|x_{jk}\|^2 I(\|x_{jk}\| \geq \eta_n\sqrt{n}) = o(1),
\]
and
\[
\text{Var} \left( \frac{1}{n} \sum_{1 \leq j \leq k \leq n} I(\|x_{jk}\| \geq \eta_n\sqrt{n}) \right) \leq \frac{1}{\eta_n^2 n^3} \sum_{j,k} E\|x_{jk}\|^2 I(\|x_{jk}\| \geq \eta_n\sqrt{n}) = o(\frac{1}{n}).
\]
Then by Bernstein’s inequality, for all small $\varepsilon > 0$ and large $n$, we have

$$P\left(\frac{1}{n} \sum_{1 \leq j \leq k \leq n} I(\|x_{jk}\| \geq \eta n \sqrt{n}) \geq \varepsilon\right) \leq 2e^{-\varepsilon n},$$

which is summable. Thus combining (3.1), (3.2) and Borel-Cantelli lemma, we obtain

$$\left\|FW_n - F\tilde{W}_n\right\| \to 0, \ a.s.$$

Next we will remove the diagonal elements. Let $\hat{\tilde{W}}_n$ be the matrix obtained from $\tilde{W}_n$ by replacing the diagonal elements with 0. Then using lemma 4.1, we have:

$$L^3(F\hat{\tilde{W}}_n, F\hat{W}_n) \leq \frac{1}{2n} \text{tr} \left(\tilde{W}_n - \hat{\tilde{W}}_n\right)\left(\tilde{W}_n - \hat{\tilde{W}}_n\right)^* \leq \frac{1}{n^2} \sum_{j=1}^{n} \|x_{jj}\|^2 I(\|x_{jj}\| < \eta_n \sqrt{n}) \leq \eta_n^2 \to 0.$$

In addition, by lemma 4.1 and Remark 1.5 we have:

$$L^3(F\hat{\tilde{W}}_n, F\hat{\tilde{W}}_n - E\hat{\tilde{W}}_n) \leq \frac{1}{n^2} \sum_{j \neq k} \|E(x_{jk} I(\|x_{jk}\| \leq \eta_n \sqrt{n}))\|^2 \leq \frac{1}{\eta_n^2 n^2} \sum_{j \neq k} E\|x_{jk}\|^2 I(\|x_{jk}\| \geq \eta_n \sqrt{n}) \to 0.$$

Now the remaining work is rescaling. Let $\tilde{x}_{jk} = \text{Var}(\tilde{x}_{jk})$. If $j < k$ and $\sigma_{jk}^2 < 1/2$, then we replace $\tilde{x}_{jk} - E\tilde{x}_{jk}$ by a bounded quaternion random variable $\tilde{x}_{jk}$ with mean 0, variance 1 and being independent of the other entries. If $j < k$ and $\sigma_{jk}^2 \geq 1/2$, we denote $\tilde{x}_{jk} = \tilde{x}_{jk} - E\tilde{x}_{jk}$. And if $j \geq k$, we denote $\tilde{x}_{jk} = \tilde{x}_{kj}$. Let $E_n$ be the set of pairs $\{(j, k): \sigma_{jk}^2 < \frac{1}{2}\}$ and $N_n$ be the cardinal number of $E_n$. Because $\frac{1}{n^2} \sum_{j \neq k} \sigma_{jk}^2 \to 1$, we can conclude that $N_n = o(n^2)$. Write $\tilde{q}_{jk} = \tilde{x}_{jk} - \tilde{x}_{jk} + E\tilde{x}_{jk}$ and write $\tilde{W}_n = \frac{1}{\sqrt{n}}(\tilde{x}_{jk})$. By lemma 4.1 we get that

$$L^3(F\tilde{W}_n, F\tilde{W}_n - E\tilde{W}_n) \leq \frac{1}{n^2} \sum_{(j, k) \in E_n} \|\tilde{q}_{jk}\|^2.$$

Rewrite the right hand side above as

$$\frac{1}{n^2} \sum_{(j, k) \in E_n} \|\tilde{q}_{jk}\|^2 := \frac{2}{n^2} \sum_{k=1}^{K} u_k,$$
where $K = \frac{1}{2}N_n$. Then, select $m = \lfloor \log n \rfloor$ and for any fixed $t > 0$, we have:

$$
E\left(\frac{2}{n^2} \sum_{k=1}^{K} u_k\right)^m = \frac{2^m}{n^{2m}} \sum_{m_1 + \ldots + m_k = m} \frac{m!}{m_1! \ldots m_k!} E u_{m_1}^1 \ldots E u_{m_k}^m
$$

$$
\leq \frac{2^m}{n^{2m}} \sum_{l=1}^{m} \sum_{m_1 + \ldots + m_l = m} \frac{m!}{l! m_1! \ldots m_l!} \prod_{l=1}^{K} (\sum_{k=1}^{K} E u_k^{m_l})
$$

$$
\leq C \sum_{l=1}^{m} 2^m n^{-2m} \frac{l^m}{l!} (2\eta^2 n)^{m-l} 2^l K^l
$$

$$
\leq C \sum_{l=1}^{m} \left(\frac{12K}{n^2}\right)^l \left(\frac{4\eta^2 n}{n}\right)^{m-l}
$$

$$
\leq C \left(\frac{12K}{n^2} + \frac{4\eta^2 n}{n}\right)^m = o(n^{-t}),
$$

where we have used the fact that for all $l \geq 1$, $l! \geq (l/3)^l$ and the last inequality follows from facts that the two terms in the parentheses tend to 0 and $m = \lfloor \log n \rfloor$. From the inequality above with $t = 2$ and (3.4) we conclude that:

$$
L(F \tilde{W}_n, F \hat{W}_n - E \tilde{W}_n) \rightarrow 0, a.s.
$$

Write

$$
\tilde{W}_n = \sqrt{n} \left(\bar{\sigma}_{jk} - 1 \bar{x}_{jk} (1 - \delta_{jk})\right),
$$

where $\bar{\sigma}_{jk}^2 = E \|\bar{x}_{jk}\|^2$ and $\delta_{jk}$ is the Kronecker delta, i.e. equal to 1 when $j = k$ and 0 otherwise. By Lemma 4.1 it follows that:

$$
L^3(F \tilde{W}_n, F \hat{W}_n) \leq \frac{1}{n^2} \sum_{i \neq j} (1 - \bar{\sigma}_{ij}^{-1})^2 \|\bar{x}_{ij}\|^2.
$$

Note that

$$
E\left(\frac{1}{n^2} \sum_{j \neq k} (1 - \bar{\sigma}_{jk}^{-1})^2 \|\bar{x}_{jk}\|^2\right)
$$

$$
= \frac{1}{n^2} \sum_{j} (1 - \bar{\sigma}_{jk})^2 \leq \frac{1}{n^2} \sum_{j} (1 - \bar{\sigma}_{jk}^2)
$$

$$
\leq \frac{1}{n^2} \sum_{(j,k) \notin E_n} \left[ E \|x_{jk}\|^2 I(\|x_{jk}\| \geq \eta_n \sqrt{n}) + E^2 \|x_{jk}\|^2 I(\|x_{jk}\| \geq \eta_n \sqrt{n})\right] \rightarrow 0.
$$
Also, applying Lemma 4.8, we have:

\[
E \left| \frac{1}{n^2} \sum_{(j,k) \notin E_n} (1 - \tilde{\sigma}_{jk}^{-1})^2 \left( \|\tilde{x}_{jk}\|^2 - E(\bar{x}_{jk})^2 \right) \right|^4 \leq \frac{C}{n^8} \left[ \sum_{j \neq k} E\|x_{jk}\|^2 I(\|x_{jk}\| \leq \eta_n \sqrt{n}) + \left( \sum E\|x_{jk}\|^4 I(\|x_{jk}\| \leq \eta_n \sqrt{n}) \right)^2 \right]
\leq C n^{-2} [n^{-1} \eta_n^6 + \eta_n^4]
\]

which is summable. From the two estimates above, we conclude that

\[
L(F_{\tilde{W}_n}, F_{W_n}) \to 0, \text{ a.s.}
\]

Therefore, we conclude that:

\[
L^3(F_{W_n} - F_{\tilde{W}_n}) \to 0, \text{ a.s.}
\]

Thus in the proof of the theorem, we may assume that:

1. The variables \(x_{jk}, 1 \leq j < k \leq n\) are independent and \(x_{jj} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\).
2. \(E x_{jk} = 0, \text{Var}(x_{jk}) = 1, 1 \leq j < k \leq n\).
3. \(\|x_{jk}\| \leq \eta_n \sqrt{n}, 1 \leq j < k \leq n\).

For brevity, we still use \(x_{ij}\) to denote the truncated and normalized variables in the sequel.

3.2. Proof of Theorem 1.2. The main mathematical tool of the proof of Theorem 1.2 is Stieltjes transform, which is defined as: For any function of bounded variation \(G\) on the real line, its Stieltjes transform is defined by

\[
s_G(z) = \int \frac{1}{y - z} dG(y), \ z \in \mathbb{C}^+ \equiv \{ z \in \mathbb{C} : \Re z > 0 \}.
\]

From Theorems B.8-B.10 in [4], we conclude that we just need to proceed in our proof by the following three steps:

1 : For any fixed \(z \in \mathbb{C}^+\), \(s_n(z) - E s_n(z) \to 0, \text{ a.s.}\).
2 : For any fixed \(z \in \mathbb{C}^+\), \(E s_n(z) \to s(z)\).
3 : Outside a null set, \(s_n(z) \to s(z)\) for every \(z \in \mathbb{C}^+\).

Here \(z = u + vi\) with \(v > 0, s_n(z) = s_{F_{W_n}}(z)\) and \(s(z)\) is the Stieltjes transform of the semicircular law (see lemma 4.7). Similar to the Step 3 in Section 2.3 of [4], the last step is implied by the first two steps and thus its proof is omitted. We now proceed with the first two steps.
Step 1: Let $E_k(\cdot)$ denote the conditional expectation given $\{x_{j,l}, j,l > k\}$, then we have

$$s_n(z) - E_s_n(z) = \frac{1}{2n} \sum_{k=1}^{n} [E_{k-1} \text{tr}(W_n - zI_{2n})^{-1} - E_k \text{tr}(W_n - zI_{2n})^{-1}]$$

(3.5)

where

$$\gamma_k = E_{k-1} \text{tr}(W_n - zI_{2n})^{-1} - E_k \text{tr}(W_n - zI_{2n})^{-1}$$

(3.6)

and $W_n^{(k)}$ is the matrix obtained from $W_n$ with the $k$-th quaternions row and quaternions column removed. Notice that here we use the fact that

$$s_n(z) = \int \frac{1}{\lambda - z} dF_{W_n}(\lambda) = \frac{1}{2n} \text{tr}(W_n - zI_{2n})^{-1}.$$

By lemma 4.3 we have:

$$|\text{tr}(W_n - zI_{2n})^{-1} - \text{tr}(W_n^{(k)} - zI_{2n-2})^{-1}| \leq \frac{2}{v}.$$  

(3.7)

Note that here we use the fact that $W_n^{(k)}$ has two rows and two columns fewer than $W_n$. Check that $\{\gamma_k\}$ forms a sequence of bounded martingale differences, thus by lemma 4.7, we obtain

$$E|s_n(z) - E_s_n(z)|^4 \leq \frac{K_4}{(2n)^4} E \left( \sum_{k=1}^{n} |\gamma_k|^2 \right)^2 \leq \frac{16K_4}{n^2 v^4} = O(n^{-2}).$$

which together with Borel-Cantelli lemma implies that, for each fixed $z \in \mathbb{C}^+$,

$$s_n(z) - E_s_n(z) \to 0 \text{ a.s.}$$

Step 2: Denote $Q_k = (x_{1k}', \ldots, x_{(k-1)k}', x_{(k+1)k}', \ldots, x_{nk}')'$, $R_k(j,l) = \left( \begin{array}{c} e_{jl}(k) \\ h_{jl}(k) \\ f_{jl}(k) \end{array} \right)$, $R_k = (W_n^{(k)} - zI_{2n-2})^{-1} = (R_k(j,l))_{j,l \neq k}$, $R(j,l) = \left( \begin{array}{c} e_{jl} \\ h_{jl} \\ f_{jl} \end{array} \right)$ and $R = (W_n - zI_{2n})^{-1} = (R(j,l))_{j,l \neq k}$.

By lemma 4.4 we have

$$s_n(z) = \frac{1}{2n} \text{tr}(W_n - zI_{2n})^{-1} = \frac{1}{2n} \sum_{k=1}^{n} \text{tr}(-zI_2 - Q_k^*(W_n^{(k)} - zI_{2n-2})^{-1}Q_k)^{-1}.$$
Let $\varepsilon_k = \mathcal{E}_{s_n}(z) I_2 - \frac{1}{n} Q_k^* R_k Q_k$, then we have
\[
\mathcal{E}_{s_n}(z) = -\frac{1}{z + \mathcal{E}_{s_n}(z)} + \delta_n
\]
where
\[
\delta_n = \frac{1}{2n} \sum_{k=1}^{n} \text{Etr}\left( \frac{1}{z + \mathcal{E}_{s_n}(z)} \varepsilon_k ((-z - \mathcal{E}_{s_n}(z)) I_2 + \varepsilon_k)^{-1} \right).
\]
Solving for $\mathcal{E}_{s_n}(z)$ from the equation above and according to the analysis in Page 36 of [4], it is suffices to show that
\[
\delta_n \to 0.
\]
Now, rewrite
\[
\delta_n = -\frac{1}{2n(z + \mathcal{E}_{s_n}(z))^2} \sum_{k=1}^{n} \text{Etr}\varepsilon_k
\]
\[
+ \frac{1}{2n(z + \mathcal{E}_{s_n}(z))^2} \sum_{k=1}^{n} \text{Etr}(\varepsilon_k^2 (\varepsilon_k - z I_2 - \mathcal{E}_{s_n}(z) I_2)^{-1})
\]
\[
= J_1 + J_2. \tag{3.8}
\]
By (3.7), we have
\[
|\text{Etr}\varepsilon_k| = |\text{tr}(\mathcal{E}_{s_n}(z) I_2 - \frac{1}{n} \mathcal{E}Q_k^*(W_n^{(k)} - z I_{2n-2})^{-1} Q_k)|
\]
\[
= |2\mathcal{E}_{s_n}(z) - \frac{1}{n} \text{Etr}(W_n^{(k)} - z I_{2n-2})^{-1}|
\]
\[
= \frac{1}{n} |\text{Etr}(W_n - z I_{2n})^{-1} - \text{Etr}(W_n^{(k)} - z I_{2n-2})^{-1}|
\]
\[
\leq \frac{2}{nu}. \tag{3.9}
\]
By (3.9) and the fact that
\[
|z + s_n(z)| \geq \Im(z + s_n(z)) = v + \Im(s_n(z)) \geq v,
\]
we obtain, for any fixed $z \in \mathbb{C}^+$,
\[
|J_1| \leq \frac{1}{nu^3} \to 0. \tag{3.11}
\]
Now we begin to prove $J_2 \to 0$. Let $\alpha_k$ denote the first column of $Q_k$, and $\beta_k$ denote the second column of $Q_k$. Write:
\[
(\varepsilon_k - z I_2 - \mathcal{E}_{s_n}(z) I_2)
\]
\[
= \begin{pmatrix}
-\frac{1}{n} \alpha_k^* R_k \alpha_k & -\frac{1}{n} \alpha_k^* R_k \beta_k \\
-\frac{1}{n} \beta_k^* R_k \alpha_k & -\frac{1}{n} \beta_k^* R_k \beta_k
\end{pmatrix}.
\]
By Lemma 2.5 we obtain

\[
(\varepsilon_k - zI_2 - E_{s_n}(z)I_2)^{-1} = \left( \begin{array}{cc} -z - \frac{1}{n} \alpha_k^* R_k \alpha_k & 0 \\ 0 & -z - \frac{1}{n} \beta_k^* R_k \beta_k \end{array} \right)^{-1}
\]

(3.12)

and

\[
-z - \frac{1}{n} \alpha_k^* R_k \alpha_k = -z - \frac{1}{n} \beta_k^* R_k \beta_k.
\]

(3.13)

Using (3.12) and (3.13), we have

\[
E|\text{tr}\varepsilon_k^2| = E|E_{s_n}(z) - \frac{1}{n} \alpha_k^* R_k \alpha_k|^2 + |E_{s_n}(z) - \frac{1}{n} \beta_k^* R_k \beta_k|^2|
\]

\[
= 2E|E_{s_n}(z) - \frac{1}{n} \alpha_k^* R_k \alpha_k|^2
\]

\[
= 2E|E_{s_n}(z) - \frac{1}{2n}(\alpha_k^* R_k \alpha_k + \beta_k^* R_k \beta_k)|^2
\]

\[
= 2E|E_{s_n}(z) - \frac{1}{2n}\text{tr}Q_k^* R_k Q_k|^2
\]

(3.14)

\[
\leq 8E\left\{|\frac{1}{2n}\text{tr}Q_k^* R_k Q_k - \text{tr}R_k|^2 + \frac{1}{2n}|\text{tr}R - \text{tr}R_k|^2 + |s_n(z) - E_{s_n}(z)|^2\right\}.
\]

What is more, from (3.7), we have

\[
E\left|\frac{1}{2n}\text{tr}R - \text{tr}R_k\right|^2 \leq \frac{1}{n^2 \nu^2}.
\]

(3.15)

By (3.2), (3.6), (3.7) and applying the fact that the martingale difference $\gamma_k$ are uncorrelated, for $k = 1, \cdots, n$, we obtain

\[
E|s_n(z) - E_{s_n}(z)|^2 = \frac{1}{4n^2} \sum_{k=1}^n E|\gamma_k|^2
\]

(3.16)

\[
\leq \frac{4}{nv^2}.
\]
Now considering the first term of (3.14), we have
\[
E[|\text{tr}Q^*_k R_k Q_k - \text{tr}R_k|^2]
= E\left| \sum_{j \neq k} \text{tr}x_{lk}^* x_{jk} R_k(j, l) - \sum_{j \neq k} \text{tr}R_k(j, j) \right|^2
\]
\[
= E\left| \sum_{j \neq k, j \neq l, l \neq k} \text{tr}x_{lk}^* x_{jk} R_k(j, l) + \sum_{j \neq k} \text{tr}(x_{jk} x_{jk}^* - I_2) R_k(j, j) \right|^2
\]
\[
\leq 2 \sum_{j \neq k} \sum_{j \neq l, l \neq k} E[|\text{tr}x_{lk}^* x_{jk} R_k(j, l)|^2 + \sum_{j \neq k} E[|\text{tr}(x_{jk} x_{jk}^* - I_2) R_k(j, j)|^2].
\]
Since for \( l \neq j, \)
\[
E[|\text{tr}x_{lk}^* x_{jk} R_k(j, l)|^2] = E[|\text{tr}\left( \frac{\lambda_{lk}}{\omega_{lk}} \frac{\omega_{lk}}{\lambda_{lk}} - \frac{\lambda_{jk}}{\omega_{jk}} \frac{\omega_{jk}}{\lambda_{jk}} \right) (e_{jl}(k) g_{jl}(k))|^2
= E[(\lambda_{lk}\bar{\lambda}_{jk} + \omega_{lk}\bar{\omega}_{jk})e_{jl}(k) + (\omega_{lk}\lambda_{jk} - \lambda_{lk}\omega_{jk})h_{jl}(k)
+ (\bar{\lambda}_{lk}\lambda_{jk} - \bar{\lambda}_{jk}\lambda_{lk})f_{jl}(k)]^2
\leq 4E|\lambda_{lk}\bar{\lambda}_{jk} + \omega_{lk}\bar{\omega}_{jk}|e_{jl}(k)|^2 + |\omega_{lk}\lambda_{jk} - \lambda_{lk}\omega_{jk}|h_{jl}(k)|^2
\]
\[
\leq 4E|e_{jl}(k)|^2 + |f_{jl}(k)|^2 + |h_{jl}(k)|^2 + |g_{jl}(k)|^2
\]
and for \( j = l, \)
\[
E[|\text{tr}(x_{jk} x_{jk}^* - I_2) R_k(j, j)|^2
= E[(|\lambda_{jk}|^2 + |\omega_{jk}|^2 - 1)|\text{tr}\left( \frac{f_{jj}(k)}{0} \frac{0}{f_{jj}(k)} \right)|^2
= E[(|\lambda_{jk}|^2 + |\omega_{jk}|^2 - 1)(2f_{jj}(k))]^2
\]
\[
\leq 4\eta_n^2 n E(|e_{jj}(k)|^2 + |f_{jj}(k)|^2 + |h_{jj}(k)|^2 + |g_{jj}(k)|^2).
\]
Therefore, for all large \( n, \) we have
\[
E[|\text{tr}Q_k^* R_k Q_k - \text{tr}R_k|^2 \leq 8 \sum_{j \neq k} \sum_{j \neq l, l \neq k} E(|e_{jl}(k)|^2 + |f_{jl}(k)|^2 + |h_{jl}(k)|^2 + |g_{jl}(k)|^2)
+ 4\eta_n^2 n \sum_{j \neq k} E(|e_{jj}(k)|^2 + |f_{jj}(k)|^2 + |h_{jj}(k)|^2 + |g_{jj}(k)|^2)
\leq 4\eta_n^2 n \text{Etr}(R_k R_k^*).
\]
By the fact
\[
\text{Etr}(R_k R_k^*) = 2(n - 1) \int_{-\infty}^{+\infty} \frac{1}{|x - z|^2} dE F^{(k)}(x) \leq 2n \nu^{-2},
\]
we have

\[
(3.17) \quad \frac{1}{4n^2} E|\text{tr}Q_k^* R_k Q_k - \text{tr}R_k|^2 \leq \frac{2\eta_n^2}{\nu^2} \to 0.
\]

Combining (3.14), (3.15), (3.16), and (3.17), we obtain, for all large \( n \),

\[
(3.18) \quad E|\text{tr}\varepsilon_k^2| \to 0.
\]

By (3.12), we have

\[
(\varepsilon_k - zI_2 - E s_n(z) I_2)^{-1} = \left( \begin{array}{cc} -z - \frac{1}{n} \alpha_k^* R_k \alpha_k & 0 \\ 0 & -z - \frac{1}{n} \beta_k^* R_k \beta_k \end{array} \right)^{-1}
\]

\[
(3.19)
\]

Thus from (3.18) and (3.19) we have

\[
E|\text{tr}(\varepsilon_k^2 \ast (\varepsilon_k - zI_2 - s_n(z) I_2)^{-1})| = E|\frac{\text{tr}\varepsilon_k^2}{z - \frac{1}{n} \alpha_k^* R_k \alpha_k} - \frac{\text{tr}\varepsilon_k^2}{\nu} | \to 0.
\]

Here we use the fact that

\[
\Im(-z - \frac{1}{n} \alpha_k^* R_k \alpha_k)
\]

\[
(3.21)
\]

\[
= -\nu \left( 1 + \frac{1}{n} \alpha_k^* R_k R_k^* \alpha_k \right) < -\nu.
\]

Therefore, by (3.8), (3.10), (3.11), and (3.20), we conclude that, for all large \( n \),

\[
(3.22) \quad |\delta_n| \to 0,
\]

which completes the proof of the mean convergence \( E s_n(z) \to s(z) \). And the proof of Theorem 1.2 is complete.

4. Appendix

Let us make a list of lemmas that were used in the process of the proofs of Lemma 2.5 and Theorem 1.2

**Lemma 4.1** (Corollary A.41 in [4]). Let \( A \) and \( B \) be two \( n \times n \) normal matrices with their ESDs \( F^A \) and \( F^B \). Then,

\[
L^2(F^A, F^B) \leq \frac{1}{n}\text{tr}[(A - B)(A - B)^*].
\]

**Lemma 4.2** (Theorem A.43 in [4]). Let \( A \) and \( B \) be two \( p \times n \) Hermitian matrices. Then,

\[
\|F^A - F^B\| \leq \frac{1}{n}\text{rank}(A - B).
\]
Lemma 4.3 (See appendix A.1.4 in [4]). Suppose that the matrix $\Sigma$ is non-singular and has the partition as given by 
\[
\begin{pmatrix}
\Sigma_{11} & \Sigma_{12} \\
\Sigma_{21} & \Sigma_{22}
\end{pmatrix},
\]
then, if $\Sigma_{11}$ is non-singular, the inverse of $\Sigma$ has the form 
\[
\left( \begin{array}{cc}
\Sigma_{11}^{-1} + \Sigma_{11}^{-1}\Sigma_{12}\Sigma_{21}\Sigma_{11}^{-1} & -\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22,1}^{-1} \\
-\Sigma_{22,1}^{-1}\Sigma_{21}\Sigma_{11}^{-1} & \Sigma_{22,1}^{-1}
\end{array} \right)
\]
where $\Sigma_{22,1}^{-1} = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$.

Lemma 4.4 (Theorem A.4 in [4]). For an $n \times n$ Hermitian $A$, define $A_k$, called a major submatrix of order $n - 1$, to be the matrix resulting from the $k$-th row and column from $A$. If both $A$ and $A_k$, $k = 1, \cdots, n$, are nonsingular, and if we write $A^{-1} = [a^{kl}]$, then 
\[
a^{kk} = \frac{1}{a_{kk} - \alpha_k A_k^{-1} \beta_k}
\]
and hence
\[
\text{tr}(A^{-1}) = \sum_{k=1}^{n} \frac{1}{a_{kk} - \alpha_k A_k^{-1} \beta_k},
\]
where $a_{kk}$ is the $k$-th diagonal entry of $A$, $\alpha_k'$ is the vector obtained from the $k$-th row of $A$ by deleting the $k$-th entry, and $\beta_k$ is the vector from the $k$-th column by deleting the $k$-th entry.

Lemma 4.5 (See appendix A.1.5 in [4]). Let $z = u + iv, v > 0$, and let $A$ be an $n \times n$ Hermitian matrix. $A_k$ be the $k$-th major submatrix of $A$ of order $(n - 1)$, to be the matrix resulting from the $k$-th row and column from $A$. Then 
\[
|\text{tr}(A - zI_n)^{-1} - \text{tr}(A_k - zI_{n-1})^{-1}| \leq \frac{1}{v}.
\]

Lemma 4.6 (Lemma 2.11 in [4]). Let $z = u + iv, v > 0$, $s(z)$ be the Stieltjes transform of the semicircular law. Then, we have $s(z) = -\frac{1}{2}(z - \sqrt{z^2 - 4})$.

Lemma 4.7 (Lemma 2.12 in [4]). Let $\{\tau_k\}$ be a complex martingale difference sequence with respect to the increasing $\sigma$-field. Then, for $p > 1$, $E|\sum \tau_k|^p \leq K_p E(\sum |\tau_k|^2)^{p/2}$.

Lemma 4.8 (Page 29 in [4]). Let $\tau_j$ are independent with zero means, then we have, for some constant $C_k$, 
\[
E|\sum \tau_j|^{2k} \leq C_k(\sum E|\tau_j|^{2k} + (\sum E|\tau_j|^2)^k).
\]
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