On the mixing property for a class of states of relativistic quantum fields

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Let $\omega$ be a factor state on the quasi-local algebra $A$ of observables generated by a relativistic quantum field, which in addition satisfies certain regularity conditions (satisfied by ground states and the recently constructed thermal states of the $P(\phi)^2$ theory). We prove that there exist space and time translation invariant states, some of which are arbitrarily close to $\omega$ in the weak* topology, for which the time evolution is weakly asymptotically abelian.

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I. INTRODUCTION AND SUMMARY

Let $(A, \tau)$ be a $C^*$- or $W^*$-dynamical system (see, e.g. [1, 2]), where $A$ is a quasi-local algebra [1, Vol. 1, Sec. 2.6] and $\tau$ is a group of time-translation automorphisms of $A$. Let $\omega$ be a $\tau$-invariant state, assumed to be normal in the $W^*$-case. The GNS triple $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ associated to the pair $(A, \omega)$ consists [1, Theorem 2.3.16, Vol. 1] of a (separable) Hilbert space $\mathcal{H}_\omega$, a representation $\pi_\omega$ of $A$ on $\mathcal{H}_\omega$, and a vector $\Omega_\omega$, which is cyclic for $\pi_\omega(A)$. The representation $\pi_\omega$ maps the triple $(A, \tau, \omega)$ into a new triple $(\mathcal{R}_\omega, \tilde{\tau}, \tilde{\omega})$, a $W^*$-dynamical system on the enveloping von Neumann algebra $\mathcal{R}_\omega = \pi_\omega(A)'$, with a normal invariant state

$$\tilde{\omega}(A) = (\Omega_\omega, A\Omega_\omega), \quad A \in \mathcal{R}_\omega. \quad (1)$$

$(\ldots)$ denotes the scalar product in $\mathcal{H}_\omega$. Since $\omega$ is $\tau$-invariant, the $W^*$-dynamics $\tilde{\tau}$,

$$\tilde{\tau}(A) = U^t_\omega A(U^{-t}_\omega)^*, \quad (2)$$

is implemented by a one-parameter group $\{U^t_\omega \mid t \in \mathbb{R}\}$ of unitary operators

$$U^t_\omega = \exp(itL_\omega) \quad (3)$$

acting on $\mathcal{H}_\omega$. The self-adjoint operator $L_\omega$, known as the $\omega$-Liouvillean [2], has the property

$$L_\omega \Omega_\omega = 0. \quad (4)$$

Quantum versions of the ergodic theorems of classical dynamics [3] have been formulated and proved [2, 4]. The natural quantum counterpart of the definition of ergodicity in classical mechanics is the notion of pure state (5).

$$\lim_{t \to \infty} \frac{1}{T} \int_0^T dt \ [\omega(A\tau_t(B)) - \omega(A)\omega(B)] = 0 \quad (5)$$

for all $A, B \in A$. Again in close analogy to classical dynamics [3], a $C^*$-dynamical system $(A, \tau, \omega)$ is said to be mixing (see, e.g., [2]):

$$\lim_{t \to \infty} \omega(A\tau_t(B)) - \omega(A)\omega(B) = 0 \quad \forall A, B \in A. \quad (6)$$

In classical mechanics, it is well-known that the mixing property [4] has a much more dramatic effect than ergodicity [3]: it represents the first step in a ergodic hierarchy crowned by Bernoulli or K-systems, which display fully chaotic behaviour [3]. The quantum theory of the latter has been developed in Ref. [5]. A necessary and sufficient condition for mixing is [2]:

$$w - \lim_{t \to \infty} \exp(itL_\omega) = \Omega_\omega(\omega, \ldots). \quad (7)$$

A sufficient condition for mixing, which follows from the Riemann-Lebesgue lemma, is (see, again, [2]):

**Lemma I.1** If the spectrum of $L_\omega$ on $\Omega_\omega$ is purely absolutely continuous, then $(A, \tau, \omega)$ is mixing.

The fact that the condition stated in Lemma I.1 is not necessary is due to the existence of singular continuous measures whose Fourier transform decays at infinity—the so-called Rajchman measures [6]. In spite of the great
conceptual and practical importance of the mixing condition \( C \), it has been seldom studied in quantum field theory. One exception is \( S \), the other is \( S \) (see also \cite{8, 11}). Here one must distinguish the vacuum state, for which we need only consider a \( C^* \)-dynamical system and \( L_\omega \) should be identified with the physical Hamiltonian \( H_\omega \), and thermal states, which satisfy the KMS condition.

In a beautiful paper, Maison \cite{7} proved that, in a unitary representation of the Poincaré group \( \mathcal{P}_+ \), the infinitesimal space-time translations have a spectral measure without singular continuous part. By Lemma \[11\] this implies \( C \) for the ground state, under the assumption that the latter is invariant under the group of Poincaré automorphisms: this yields a unitary representation of the Poincaré group by a well-known argument, already used to establish \( C \) (see, e.g. \cite{1, Vol. 1, Corollary 2.3.17, pg. 56}).

In recent years there arose a special interest in thermal quantum field theory \cite{12}, which is expected to play an important role in cosmology (see the concluding remarks of Section IV). Thermal states of quantum fields are important role in cosmology (see also \cite{13–11}).

Two new elements of the present extension are: (a) the boost relates space-translations and time-translations; (b) for space-translations, the large-distance behaviour is under control for the class of models considered.

Two new elements of the present extension are:

(c) the introduction of a time-dependent scale in the boosts;

(d) the explicit use of local commutativity.

In Section IV we present our framework, consisting of Assumptions A1-A5. The known examples included in this framework are also briefly reviewed there. In Section III we prove \( C \) for a dense set (in the weak* topology) of time- and space-time translation invariant states of a (relativistic) quantum field theory satisfying the assumptions of Section III (Theorem 1, Theorem 2 and Theorem 3). Section IV is reserved to the conclusion, open problems and conjectures.

II. THE FRAMEWORK: ASSUMPTIONS AND EXAMPLES

We denote by \((x^\mu), \mu = 0, 1, \ldots, \nu\), the points of Minkowski space-time \( \mathbb{R}^{1+\nu} \). Thus \( \nu \) is the space dimension and \( x^0 = t, c = 1 \) denotes the time-variable. The transformations \( T(a) \) and \( L(v) \) of \( \mathbb{R}^{1+\nu} \), corresponding to space-time translation by \( a \in \mathbb{R}^{1+\nu} \) and velocity boost by \( u \in (-1, 1) \) along \( x^1 \)-axis, are defined, respectively, by

\[ T(a)x = x + a \]

and

\[ L(v)x = \begin{pmatrix} x^0 \cosh v - x^1 \sinh v \\ x^1 \cosh v - x^0 \sinh v \\ x^2 \\ \vdots \\ x^\nu \end{pmatrix} \]

where \( u = \tanh v \) and \( \cosh v = (1 - u^2)^{-1/2} \). The corresponding automorphisms of \( \mathcal{A} \), denoted by \( \xi(a) \equiv (\tau_a^o, \sigma_\bar{a}) \) and \( \lambda_v \), satisfy the relations

\[ \xi(t, x) = \tau_t \circ \sigma_x = \sigma_x \circ \tau_t \quad \forall t \in \mathbb{R}, \quad \forall x \in \mathbb{R}^\nu, \]

and

\[ \lambda_v \circ \xi(a) \circ \lambda_{-v} = \xi(L(v)a) \quad \forall a \in \mathbb{R}^{1+\nu}, \quad \forall v \in \mathbb{R}. \]

We shall assume that we are given a relativistic quantum field theory described in terms of a quasi-local algebra \( \mathcal{A} \) satisfying

A1 the Haag-Kastler axioms \cite{14};

A2 for all \( A \in \mathcal{A} \), \( \lim_{t \to 0} \| \tau_t(A) - A \| = 0 \) and \( \lim_{x \to 0} \| \sigma_x(A) - A \| = 0 \);

A3 for all \( A \in \mathcal{A} \), \( \lim_{x \to 0} \| \lambda_x(A) - A \| = 0 \);

A4 either a pure state or a factor state;

A5 invariant under the automorphism group of space-time translations \( \{ \xi(a) \mid a \in \mathbb{R}^{1+\nu} \} \) and extremal space translation invariant.

In the relativistic case space translations are asymptotically abelian in norm:

\[ \lim_{|x| \to \infty} \| [A, \sigma_x(B)] \| = 0. \]

As \( \omega \) is, according to A4-A5, an extremal space translation invariant factor state, it is clustering (\cite{1, Example 4.3.24]):

\[ \omega(\Lambda \sigma_x(B)) - \omega(\Lambda) \omega(B) \to 0 \quad \text{as} \quad |x| \to \infty. \]

Let \( \mathcal{O} \to \mathcal{A}(\mathcal{O}) \) be the net of local algebras in \cite{14}, denote the representation obtained by the GNS construction from the state \( \omega \) by \( \pi_\omega \), and consider the von Neumann rings

\[ \mathcal{R}(\mathcal{O}) = \pi_\omega(\mathcal{A}(\mathcal{O}))''. \]
where the bar denotes the $\sigma$-weak topology on $\mathcal{R}(\mathcal{O})$; then one can easily see that $T_f(A) \in \mathcal{R}(\mathcal{O})$. Note that $\alpha_g(T_f(A))$ lies in a larger (but nevertheless strictly) local algebra $\mathcal{R}(\hat{\mathcal{O}})$ and therefore is well-defined. Moreover,
\begin{equation}
\lim_{g \to 2 \mathbb{R}^+} \left\| \alpha_g(T_f(A)) - T_f(A) \right\| = 0.
\end{equation}
Thus $T_f(A)$ as defined in (15) is a smooth element with respect to the Poincaré group automorphisms. Next let us consider $f_1, f_2 \in L^1(P^+_1, \nu)$ and
\begin{equation}
\{ \alpha_g(A_i) \mid g \in \text{supp} f_i \} \subset \mathcal{R}(\mathcal{O}), \quad i = 1, 2.
\end{equation}
Then (see [17])
\begin{equation}
\left\| \alpha_g(T_{f_1}(A_1)T_{f_2}(A_2)) - T_{f_1}(A_1)T_{f_2}(A_2) \right\| \to 0, \quad (18)
\end{equation}
as $P^+_1 \ni g \to e$. Thus elements of the form (15) generate a *-subalgebra $\mathcal{A}_0(\mathcal{O})$ of $\mathcal{R}(\hat{\mathcal{O}})$. Then by the above consideration,
\begin{equation}
\| \alpha_g (B) - B \| \to 0, \quad P^+_1 \ni g \to e, \quad (19)
\end{equation}
for $B \in \mathcal{A}_0(\mathcal{O})$. Let $\mathcal{A}_S(\mathcal{O})$ be the $C^*$-norm closure of $\mathcal{A}_0(\mathcal{O})$. It is easily seen that $\mathcal{A}_S(\mathcal{O})$ is $\sigma$-weakly dense in $\mathcal{R}(\mathcal{O})$. Moreover, it consists of smooth elements: for $C \in \mathcal{A}(\mathcal{O}), \epsilon > 0, let B \in \mathcal{A}_0(\mathcal{O})$ be an element such that $\| C - B \| \leq \epsilon$; then
\begin{equation}
\| \alpha_g (C) - C \| \leq \| \alpha_g (C) - \alpha_g (B) \| + \| \alpha_g (B) - B \| + \| B - C \|, \quad (20)
\end{equation}
Hence $\lim_{g \to e} \| \alpha_g (C) - C \| \leq 2\epsilon$. Since $\epsilon$ is arbitrary, $\lim_{g \to e} \| \alpha_g (C) - C \| = 0$.
Therefore the quasi-local algebra defined as
\begin{equation}
\mathcal{A} = \bigcup_{\mathcal{O} \subset \mathbb{R}^{1+1}} \mathcal{A}_S(\mathcal{O}), \quad (21)
\end{equation}
where the bar denotes the $C^*$-inductive limit [18, Proposition 11.4.1], together with the automorphisms $\{ \alpha_g \in Aut(\mathcal{A}) \mid g \in P^+_1 \}$ forms a $C^*$-dynamical system $(\mathcal{A}, P^+_1, \omega)$. Note that the Weyl algebra of the canonical commutation relations (CCR) does not satisfy Assumptions A2-A3 — for the violation of A2, see [1], Vol. 2, Theorem 5.2.8.

Important examples included in the above framework are the renormalised vacuum state of the $P(\phi)^2$ theory [16, 18, 20, 22, 23, 24] and the temperature states of the same theory [16, 19–22, 23, 24] as we now explain. In order to do that, we need a brief exposition of the barest elements of the theory.
Let $\mathfrak{H} := \bigotimes_{n=0}^{\infty} \mathcal{O}_n(\mathcal{H})$ be the bosonic Fock space (the subscript $s$ indicates the symmetric tensor product of copies of $\mathcal{H}$) over the one-particle space $\mathcal{H}$ given by $L^2(\mathbb{R}, dk)$; as usual $\bigotimes^0 \mathcal{H} := \mathbb{C}$. By $\Omega \equiv (1, 0, \ldots) \in \mathfrak{H}$ we denote the vacuum vector. The free Hamiltonian
\begin{equation}
H_0 := d\Gamma(\omega) \quad (22)
\end{equation}
is the second quantisation of the one-particle energy $\omega(k) := \sqrt{k^2 + m^2}$ with $k \in \mathbb{R}$ and mass $m > 0$, considered as a multiplication operator on $\mathcal{H}$. The number operator on the Fock space is $N := d\Gamma(1)$. There is a representation of the CCR by creation and annihilation operators $a^*(f)$ and $a(f), f \in \mathcal{O}$ (see, e.g., [22, Section 3.2]). Understanding these objects as operator-valued distributions and writing symbolically
\begin{equation}
a(h) = \int dk \overline{h}(k)a(k), \quad a^*(h) = \int dk h(k)a^*(k), \quad (23)
\end{equation}
the free field is given by
\begin{equation}
\phi(x) = \int \frac{dk}{\omega(k)^{1/2}} e^{-ikx} \left( a^*(k) + a(-k) \right). \quad (24)
\end{equation}
This expression is again considered as an operator-valued distribution and as such the multiplication of these objects at the same point $x$ is not a well defined operation. We define powers of the fields by “point splitting”, e.g.,
\begin{equation}
:\phi^2(x) : \equiv \lim_{y \to x} \left[ \phi(y)\phi(x) - \langle \Omega, \phi(y)\phi(x)\Omega \rangle \right], \quad (25)
\end{equation}
and similarly for higher powers. Using the CCR, one sees that this leads to the prescription of Wick ordering: all creation operators stand to the left of all annihilation operators.
We fix a real, semi-bounded polynomial of degree $2n$
\begin{equation}
P(\lambda) = \sum_{j=0}^{2n} a_j \lambda^j \quad (26)
\end{equation}
and choose a function $g \in C^0_{\infty}(\mathbb{R})$ with $0 \leq g(x) \leq 1$. Define the interaction Hamiltonian localised in a compact space region by
\begin{equation}
V(g) = \int dx \; g(x) : P(\phi(x)) : \quad (27)
\end{equation}
The Wick ordering in the powers of field operators makes $V(g)$ a well-defined unbounded quadratic form.
By investigating the smoothness and symmetry properties of the scalar kernel of $V(g)$, it is seen that $V(g)$ is an unbounded, symmetric operator with domain contained in $D(N^n)$. By [22, Theorem 6.4]

$$H(g) = H_0 + V(g)$$

(28)
is essentially self-adjoint on $D(H_0) \cap D(V(g))$. Moreover, $H(g)$ is semibounded from below. Let

$$E_g := \inf \{ \text{Spectrum } H(g) \}.$$  

(29)
The corresponding eigenvalue is an isolated eigenvalue of $H(g)$ with multiplicity one [20], thus corresponding to an eigenvector $\Omega_g \in \mathfrak{F}$, $\|\Omega_g\| = 1$, such that $H(g)\Omega_g = E_g\Omega_g$. As $g(\cdot) \to 1$ the scalar product $(\Psi, \Omega_g) \to 0$ for all $\Psi \in \mathfrak{F}$. This clearly demonstrates that Hilbert space methods are insufficient, and thus justifies the operator algebraic framework, which allows us to obtain (see [20]) the vacuum state $\omega$ as the $w^*$-limit, as $g(\cdot) \to 1$, of the states

$$\omega_g(A) = (\Omega_g, A\Omega_g),$$

(30)
with $A$ in the $C^*$-closure $\mathcal{A}$ of the local von Neumann algebras $\mathcal{R}(\mathcal{O})$ generated by the Weyl operators

$$W(f) = \exp \left( i \int dx f(x)\phi(x) \right),$$

(31)
with

$$f \in \omega^{1/2}D_{\mathcal{R}}(\mathcal{O}) + i\omega^{-1/2}D_{\mathcal{R}}(\mathcal{O})^L_2(\mathbb{R}, dx).$$

(32)
For the thermal field theory, the free Liouvillean $L_{\omega}$ (see [3]) is the Araki-Woods Liouvillean $L_{Aw}$ (see [12] and [23]). Euclidean techniques can be used to define the operator sum

$$H_\beta(g) := L_{Aw} + \int dx \; g(x) : P(\phi^2(x)) :_{C_\beta},$$

(33)
where the Wick ordering is defined from [24] in terms of the thermal covariance function $C_\beta$ [12, 23], as an essentially self-adjoint operator [24]: let its closure be defined by the same symbol. The vector

$$\Omega_\beta(g) := \frac{e^{-\frac{\beta}{2}H_\beta(g)}\Omega_{Aw}}{\|e^{-\frac{\beta}{2}H_\beta(g)}\Omega_{Aw}\|},$$

(34)
where $\Omega_{Aw}$ is the cyclic GNS vector associated to the Araki-Woods state, induces a KMS state $\omega(\cdot)$ for the $W^*$-dynamical system $(\pi_{Aw}(\mathcal{A})^\prime, \tau^g)$, where $\pi_{Aw}$ is the Araki-Woods representation and $\tau^g$ is the time evolution

$$\tau^g(A) := e^{itH_\beta}Ae^{-itH_\beta}, \quad A \in \mathcal{A}.$$  

(35)
Note that $H_\beta$ and $H_\beta(g)$ induce the same group of automorphisms on $\mathcal{A}$, thus there is no $\beta$ dependence on the level of automorphisms. It was proved in [15] that the limit

$$\omega_\beta := \lim_{g(\cdot) \to 1} \omega_\beta(g)$$

(36)
exists and defines a state on

$$\mathcal{A} := \bigcup_{\mathcal{O} \in \mathbb{R}^{1+\nu}} \mathcal{R}(\mathcal{O}).$$

(37)
The $C^*$-algebra (37) is isomorphic to the $C^*$-inductive limit of the local von Neumann algebra $\mathcal{R}_{Aw}(\mathcal{O})$ generated by the Weyl operators in the Araki-Woods representation [12, 23, 24]. The thermal states $\omega_\beta$, $\beta > 0$, defined by (36) satisfy a relativistic generalisation of the KMS condition (see [12] and references given there).

We now turn to the question of whether assumptions A1-A5 are satisfied for the above-mentioned examples. For A4, A5 (except purity), see [19] and [20]. For the unicity of the vacuum (the purity in A4) see [21] and references given there. Property A5 for the thermal state were proved in [12]. The replacement of $\mathcal{R}(\mathcal{O})$ by a weakly dense subalgebra such as the one of the form (15) does not alter the validity of the above-mentioned results, see the remarks after the definition pg. 399 of [19]. Thus A5 may also be assumed to hold for these examples, in particular the replacement of $\mathcal{R}(\mathcal{O})$ by [21]. Finally, for the factoriality property in A4, we may decompose the thermal state into factor states through the primary decomposition [1, Vol. 2, Theorem 5.3.30, pg. 116], and pick any one of the latter as our state. For the thermal $P(\phi^2)$ theory it is expected that the KMS state is unique and thus a factor state, but a proof is still missing. It would be interesting to know whether there exist non-equilibrium stationary states (NESS) which satisfy the properties A1-A5, but so far we are not aware of any rigorously constructed NESS for interacting relativistic quantum field theories.

### III. THE MAIN THEOREM

We want to inherit clustering properties of the time translation from those of space translation with the help of a smearing effect. First we specify the properties of the smearing functions. Let $f, g$ be $C^\infty$-functions of compact support, such that

$$\int_{-\infty}^{\infty} dx \; f(x) = 1, \quad f(x) \geq 0,$$

(38)
and

$$\text{supp } f \subset [-a - \delta, a - \delta],$$

(39)
where $\delta < a$. Set $\alpha_t := t^{-1/2-\epsilon}$, with $0 < \epsilon < 1/2$, and define

$$g_t(v) := \frac{1}{\alpha_t} g \left( \frac{v}{\alpha_t} \right) \quad \forall v \in \mathbb{R}.$$  

(40)
Clearly \( g_t(v) \to \delta(v) \) as \( t \to \infty \), thus \( \{g_t\}_{t>0} \) is an approximation of the Dirac delta function. For later usage we define also an approximation \( \{\hat{f}_s\}_{s>0} \) of the Dirac delta function, whose convergence is less rapid:

\[
\hat{f}_s(v) := \frac{1}{\alpha_s} g\left(\frac{v}{\alpha_s}\right), \quad v \in \mathbb{R}, \tag{41}
\]

with \( \alpha_s := s^{-1/4-\epsilon/4}, \quad 0 < \epsilon < 1/4 \).

Our main theorem may now be stated:

**Theorem III.1** Let a relativistic quantum field theory satisfy the Assumptions A1-A3 of Section 2 and let \( \omega \) be a state satisfying the Assumptions A4-A5. Then

(i) \( \omega \equiv \omega \circ \nu \) is an extremal space translation invariant and time invariant factor state;

(ii) \( \omega_f := \int dv \, f(v) \omega_v \) is not a factor state. Its centre contains space- and time-translation invariant elements \( B_f \) satisfying the following three properties:

(iia) \( \omega_f(B_f A) = \omega_f(B_f A) \);

(ii.b) \( \lim_{x \to \infty} \omega_f(A\sigma_x(B - B_f)) \) = 0;

(ii.c) \( \lim_{t \to \infty} \omega_f(\sigma_t(B - B_f)) \) = 0.

**Proof**

(i) by assumption \( \omega \) is invariant under space-time translations:

\( \omega \circ \xi(t, x) = \omega \) \( \forall (t, x) \in \mathbb{R}^{1+4} \). \tag{42}

From (11) we conclude that

\( \lambda_v \circ \xi_v = \xi_{L(v)\nu} \circ \lambda_v \). \tag{43}

Now consider the state \( \omega_v \equiv \omega \circ \nu \). Clearly

\( \omega_v \circ \xi_v(t, x) = \omega \circ \lambda_v \circ \xi_v(t, x) \)

\( = \omega \circ \xi_{L(v)\nu}(t, x) \circ \lambda_v = \omega \circ \lambda_v \)

\( = \omega_v \) \( \forall (t, x) \in \mathbb{R}^{1+4} \).

This shows that \( \omega_v \) is also space translation invariant and time invariant. Now assume that \( \omega_v \) allows a decomposition into space-translation invariant states. Then we could use the group property to derive a decomposition of \( \omega \) into space-translation invariant states, which is not allowed. Thus \( \omega_v \) is extremal space translation invariant. As \( \lambda_v \) is an automorphism of \( \mathcal{A} \), the state \( \omega_v \) is a factor state, just like \( \omega \).

(ii) Smearing out the state \( \omega \) (i.e., forming a convex combination) will lead to a non-trivial centre, thus we expect that \( \omega_f \) fails to be a factor state. Indeed, given some \( B \in \mathcal{A} \) we can weakly define a non-trivial element \( B_f \), which lies in the centre:

\[
\omega_f(AB_f C) := \int dv \, f(v) \omega_v(AC)\omega_v(B) \tag{45}
\]

for all \( A, B, C \in \mathcal{A} \). Clearly \( \omega_f(AB_f C) = \omega_f(ACB_f) \) and \( \omega_f(B_f C) = \omega_f(CB_f) \). Next we show that (ii.b) holds:

\[
\lim_{x \to \infty} \omega_f(A\sigma_x(B - B_f)) = \lim_{x \to \infty} \int dv f(v)\omega_v(A\sigma_x(B - \omega_v(B)) \mathbb{I}). \tag{46}
\]

We have made use of the fact that \( \omega_v(\sigma_x(B)) = \omega_v(B) \) for all \( x \in \mathbb{R}^4 \). The r.h.s. in (46) vanishes, as \( \omega \)-lim \( x \to \infty \sigma_x(B) = \omega_v(B) \mathbb{I} \).

Next we prove (ii.c). We proceed in several steps.

(a) Different from [8] the initial state will, except for the ground state or for the tracial state (provided it exists) not be invariant under Lorentz boosts. In order to use (9) and shift it between the operators \( A \) and \( B \) we have to control that the effect of the state is negligible. We define

\[
h_t(v) := \int dv_1 f(v - v_1) g_t(v_1), \tag{47}
\]

with \( g_t(v) \) as described in (10). It follows from the fact that the width of the supp \( g_t \) is proportional to \( t^{-1/2-\epsilon} \) that

\[|h_t(v) - f(v)| \leq c \sup d f \cdot t^{-1/2-\epsilon}. \tag{48}\]

This allows to approximate (ii.c) by

\[
\int dv f(v) \int dv_1 g_t(v_1) \times \omega_v\left(\lambda_{v_1}(A)\left(\lambda_{v_1} \circ \tau_t(B) - \omega_v(B) \mathbb{I}\right)\right). \tag{49}
\]

Note that we have shifted the integration over \( v \) by \( v_1 \).

(b) Let us now concentrate for a moment on the term

\[\lambda_{v_1}(A)(\lambda_{v_1} \circ \tau_t(B) - \omega_v(B) \mathbb{I}). \tag{50}\]

From (9) we conclude that

\[\lambda_{v_1} \circ \tau_t(B) = \tau_t \cosh v_1 \circ \sigma_t \sinh v_1 \circ \lambda_{v_1}(B). \tag{51}\]

Thus (50) equals \( \lambda_{v_1}(A)(\tau_t \cosh v_1 \circ \sigma_t \sinh v_1 \circ \lambda_{v_1}(B) - \omega_v(B) \mathbb{I}) \). Since \( g_t \) has shrinking compact support for \( t \to \infty \), there exist constants \( c_A \) and \( c_B \), which may depend on \( A \) and \( B \), respectively, such that

\[
\|\lambda_{v_1}(A) - A\| \leq c_A t^{-\epsilon-1/2}, \tag{52}
\]

\[
\|\lambda_{v_1}(B) - B\| \leq c_B t^{-\epsilon-1/2}, \tag{53}
\]

for all \( v_1 \in \text{supp} g_t \). Thus (50) can be approximated in norm:
\[ \left\| \lambda_{v_1}(A) \left( \tau_t \cosh v_1 \circ \sigma_t \sinh v_1 \circ \lambda_{v_1}(B) - \omega_v(B) \right) \right\| \\
- A \left( \tau_t \cosh v_1 \circ \sigma_t \sinh v_1 \circ \lambda_{v_1}(B) - \omega_v(B) \right) \leq \left\| (\lambda_{v_1}(A) - A) \left( \tau_t \cosh v_1 \circ \sigma_t \sinh v_1 \circ \lambda_{v_1}(B) - \omega_v(B) \right) \right\| \\
+ \left\| A \left( \tau_t \cosh v_1 \circ \sigma_t \sinh v_1 \circ \lambda_{v_1}(B) - \tau_t \cosh v_1 \circ \sigma_t \sinh v_1 (B) \right) \right\| \\
\leq c_A t^{-\epsilon/2} \cdot 2\|B\| + \|A\| \cdot c_B t^{-\epsilon/2}. \] (54)

(c) Using the time invariance of \( \omega_v \), we find that
\[ \left| \omega_v \left( \lambda_{v_1}(A) \left( \lambda_{v_1} \circ \tau_t(B) - \omega_v(B) \right) \right) - \omega_v \left( \tau_{-t} \cosh v_1 (A) \left( \sigma_t \sinh v_1 (B) - \omega_v(B) \right) \right) \right| \\
\leq c_A t^{-\epsilon/2} \cdot 2\|B\| + \|A\| \cdot c_B t^{-\epsilon/2}. \] (55)

(d) So far the considerations have been the same for Galilei invariant time evolutions as for relativistic time evolutions. In order that the \( v_1 \) integral is only connected with the space translation we need an additional estimate:
\[ \sup_{v_1 \in \text{supp} g_t} \left\| \tau_t(A) - \tau_t \cosh v_1(A) \right\| \leq c_A \|t\|^{-2\epsilon}, \] (56)

with a constant \( c'_A \) which may depend on \( A \). This follows from expanding \( \cosh v_1 \) in a power series and taking the support properties of \( g_t \) into account. Thus for \( t > 0 \)
\[ \left| \omega_v \left( \lambda_{v_1}(A) \left( \lambda_{v_1} \circ \tau_t(B) - \omega_v(B) \right) \right) - \omega_v \left( \tau_{-t} \left( \sigma_t \sinh v_1 (B) - \omega_v(B) \right) \right) \right| \\
\leq c_A t^{-\epsilon/2} \cdot 2\|B\| + \|A\| \cdot c_B t^{-\epsilon/2} \cdot 2\|B\|. \] (57)

We can now replace (ii.c) by estimating
\[ \left| \int dv_1 g_t(v_1) \omega_v \left( \tau_{-t} \left( \sigma_t \sinh v_1 (B) - \omega_v(B) \right) \right) \right| \\
\leq \|A\| \left\{ \int dv_1 dv_2 g_t(v_1) g_t(v_2) \omega_v \left( \left( \sigma_t \sinh v_1 (B^*) - \omega_v(B^*) \right) \right) \left( \sigma_t \sinh v_2 (B) - \omega_v(B) \right) \right\}^{1/2}. \] (58)

(e) Returning to (ii.c) we have still to integrate over \( v \). As \( \omega_v \) is an extremal space translation invariant primary state, \( \omega_v \) is clustering (1, Example 4.3.24):
\[ \omega_v \left( A \sigma_{x f} (B) \right) - \omega_v(A) \omega_v(B) \to 0 \] (59)
as \( |x| \to \infty \). With (59), (50) the integral \( \int dv_1... \) in (10) has an upper bound \( \|B\|(\|A\| + 1) \) which is independent of \( t \) and \( v \). For all \( v \) the \( v_1 \) integral converges to zero as \( t \to \infty \). This follows from (58) and (59) and Lebesgue dominated convergence: one makes in the double integral on the r.h.s. of (58) the change to variables \( w_1 = v_1/\alpha_t \) and \( w_2 = v_2/\alpha_t \) suggested by (39). Using space translation invariance of \( \omega_v \) and the choice of \( \alpha_t \), the integrand in the new variables is seen to tend to zero by (59) for all \( (w_1, w_2) \) except along the diagonal \( w_1 = w_2 \) (thus a set of zero Lebesgue measure in \( \mathbb{R}^2 \)), and is, uniformly in \( t \), bounded by \( g(w_1)g(w_2)(2\|B\|)^2 \). Since \( f \) has compact support and (59) is satisfied, a second application of the Lebesgue dominated convergence theorem finally proves the result.

Collecting all estimates proves the theorem. \[ \blacksquare \]

We now want to draw conclusions for the state \( \omega \) we started with:

**Theorem III.2** In QFT a primary state that is extremal space invariant and also time invariant is also extremal time invariant.

**Proof** We start with such a state and smear it to obtain \( \omega_f \). By Theorem III.1 (ii.c) a time translated operator converges weakly to an element in the centre. The
decomposition into extremal time-invariant states corresponds to a maximal abelian subalgebra within the commutant, which contains all time invariant elements. This algebra is contained in the centre, and coincides with the algebra of space-translation invariant elements. This follows from Theorem III.1 (ii.b). Therefore also the decomposition into extremal time invariant states coincides with the decomposition into extremal space translation invariant states. 

Unfortunately Theorem III.2 is not strong enough to guarantee the mixing property (6). There is still the possibility that the time invariant operators in the commutant are not weak limits but only invariant means. Notice that in our proof we used however just limits.

Another possibility to interpret Theorem III.1 is by varying \( f(v) \), in a sense to be made precise below.

\[
\text{Theorem III.3} \quad \text{Let} \quad \omega \quad \text{be a state satisfying A1-A5. Then for any state} \quad \omega_f \quad \text{of the form (ii) of Theorem 3.1 (hence space and time translation invariant), among which there are some arbitrarily close to} \quad \omega \quad \text{in the weak* topology, the time evolution is weakly asymptotically abelian:}
\]

\[
\lim_{t \to \infty} \omega_f \left( A \left[ \tau_t(B), C \right] D \right) = 0 \quad (60)
\]

for all \( A, B, C, D \in A \).

\[ \text{Proof} \quad \text{Let} \quad \epsilon > 0 \quad \text{and} \quad A \in A_S \quad \text{be given. We may then choose} \quad \delta > 0, \quad \text{depending on} \quad A \quad \text{and} \quad \epsilon, \quad \text{such that} \quad |\lambda_v(A) - A| < \epsilon. \quad \text{Choose now} \quad f \quad \text{smooth with compact support in} \quad [-\delta, \delta] \quad \text{such that} \quad \int_{-\infty}^{\infty} dv f(v) = 1. \quad \text{Then}
\]

\[
\left| \int dv f(v) \omega_v(A) - \omega(A) \right| < \epsilon,
\]

and thus a proper choice of \( f \) makes \( \omega_f \) arbitrarily close to \( \omega \) in the weak* topology. In the representation \( \pi_{\omega_f} \), corresponding to any state of the form (ii) of Theorem III.1 (close to \( \omega \) or not), (iic) of Theorem III.1 asserts that \( \tau_t(B) \) converges to an element of the centre, for any \( B \in A_S \). Thus (60) holds.

If, however, we try to obtain a statement that is close to (6) for an individual state, we have the possibility to scale the smearing function \( f \) and consider the limit of the states

\[
\omega_{fs} := \int dv \hat{f}_s(v) \omega_v \quad (61)
\]

as \( s \to \infty \). Note that \( \hat{f}_s \) was defined in (11). Clearly

\[
\lim_{s \to \infty} \omega_{fs} (A) = \omega(A), \quad A \in A. \quad (62)
\]

We now would like to estimate

\[
\left| \omega_{fs} (A \tau_t(B)) - \omega_{fs}(A) \omega_{fs}(B) \right| \quad (63)
\]

for \( s \) and \( t \) large. Setting \( s = t \) (i.e., taking the limits \( t \to \infty \) and \( s \to \infty \) simultaneously) we find that (48) changes to

\[
|\hat{h}_t(v) - \hat{f}_t(v)| \leq c \sup_v \left| \frac{d}{dt} \hat{f}_t(v) \right| \cdot t^{-\epsilon + 1/2}
\]

\[
\leq c \sup_v |f(v)| \cdot t^{-\epsilon/4}, \quad (64)
\]

for \( 0 < \epsilon < 1/4 \). The remaining estimates remain unchanged. Thus

\[
\lim_{t \to \infty} \omega_{fs} (A \tau_t(B)) = \omega(A) \omega(B), \quad (65)
\]

for all \( f \in C^\infty (\mathbb{R}) \) which satisfy (63), i.e., \( \int_{-\infty}^{\infty} dx \ |f(x)| = 1, \ f(x) \geq 0 \) and \( \sup f \in [-a - \delta, a + \delta] \), where \( \delta < a \). One can see from (64) that we are unable to control the limit \( a \to 0 \) with our estimates.

We note that we could still play with the support of the initial function \( f \). As a last remark, if \( \omega \) is Lorentz-invariant, we replace \( \omega_v \) by \( \omega(A) \) in (45), which, together with (e), yields the mixing property (6): this is Maison’s result [6].

Remark III.1 The relation between boost and shift was used by D. Buchholz [23] to show that for a large class of states \( \{ \omega \} \) (which are normal w.r.t. the vacuum state) the weak* limit points of the nets \( \{ \omega \circ \tau_t \}_{t > 0} \) as \( t \to \infty \) are states, which are invariant under spatial translations (and thus vacuum states). It should be possible to extend this result to a class of states of the form

\[
\hat{\omega}(.) = \omega(A, D) \quad (66)
\]

with \( \omega \) satisfying A4-A5 and \( A, D \in A \) operators with compact energy support (as defined, e.g., in [23]). Thus if \( \omega \) is the only space-translation invariant state w.r.t. the representation \( \pi_{\omega} \), then the weak* limit of all the states in the class \( \{ \hat{\omega} \} \) would be \( \omega \) itself. Whether such a result actually holds and is related to our work has to be further investigated.

IV. CONCLUSION, OPEN PROBLEMS AND CONJECTURES

We have shown that, given a distinguished state of a quantum field \( \omega \), which satisfies the assumptions A4-A5 of Section II, there exist space-time translation invariant states, some of which are arbitrarily close to \( \omega \) in the weak* topology, for which the time evolution is weakly asymptotically abelian.

The proof depends on two features: the fact that the Lorentz boost relates space translations and time translations, and, secondly, locality (12), which implies the existence of good space-like cluster properties (13). These properties are valid for both ground states and primary (factor) thermal states.

Assumptions A1-A5 hold for the ground state and thermal state(s) of the \( P(\phi)_2 \) model, but they are expected to hold for any relativistic quantum field theory,
and in this sense we have strengthened the conjecture in \([8]\) and \([28]\) that the observables of a Poincaré-invariant theory are a mixing system.

It follows from our results that these states, provided the corresponding GNS vector is separating (such states are called modular states in the literature), possess the properties of return to equilibrium

$$\lim_{t \to -\infty} \omega(A^* \tau_t(B)A) = \omega(A^* A)\omega(B)$$  \hspace{1cm} (67)

and corresponding consequences which are usually rather hard to prove. This fact may be regarded as a bonus from quantum theory, but, more specifically, of quantum field theory. Indeed, for quantum spin systems, the property of weak asymptotic abelianness (Theorem III.3), is not generally valid \([29, 30]\). The above reference to quantum field theory includes Galilean-invariant theories \([8, 10]\). One basic difference between fully relativistic quantum field theories and the Galilean-invariant theories \([8, 10]\), is that only the former display vacuum polarization (presently of about 3K) in the hot big-bang model (see, e.g., \([32, pg. 187]\)) must be of special relevance, which is, yet, to be fully explored (see, however, \([33]\)).

The above reference to quantum field theory includes Galilean-invariant theories \([8, 10]\). One basic difference between fully relativistic quantum field theories and the latter is that only the former display vacuum polarization, which leads to non-Fock representations of the CCR because of Haag’s theorem \([16, pg. 55]\). There is, however, an advantage of Galilean invariant theories over Poincaré-invariant ones: at least for the class considered in \([8, 10]\), a stronger state-independent mixing property, namely

$$\lim_{t \to -\infty} \|A \tau_t(B)\| = \|A\|\|B\| \quad \forall A, B \in \mathcal{A},$$  \hspace{1cm} (68)

may be proven \([8, 9]\). Quantum mixing systems in the sense of \([68]\) may be shown to be indeterministic and undecidable in a quite precise sense \([28, Lemma L4]\), with an interesting application: unpredictability of the symmetry breaking in a phase transition such as the one which presumably occurred after the big-bang \([28]\). It is an open problem to prove \([68]\) for our class of systems.

On the other hand, relativistic quantum fields and their equilibrium states play an important role in applications in cosmology, in particular in the dark energy problem \([31]\). In cosmology, thermal quantum fields associated to the temperature of background radiation (see, e.g., \([32, pg. 187]\)) must be of special relevance, which is, yet, to be fully explored (see, however, \([33]\)).

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