DIFF($S^1$), TEICHHMÜLLER SPACE AND PERIOD MATRICES: CANONICAL MAPPINGS VIA STRING THEORY

by

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1. Introduction

1a. $Diff(S^1)$ and the Universal Teichmüller space: The quantizable coadjoint orbits of the infinite dimensional Frechet Lie group $Diff(S^1)$ are uniquely determinable as the following two remarkable homogeneous spaces: $N = Diff(S^1)/Rot(S^1)$ and $M = Diff(S^1)/Mobius(S^1)$; these symplectic manifolds are fundamental in the loop-space approach to the theory of the closed string. Basic references are the papers of Bowick, Rajeev, Zumino et. al. (see the references listed in [NV]), and Witten [W]. Actually $N$ and $M$ are homogeneous complex analytic (Frechet) manifolds with a Kähler form that is the symplectic form of Kirillov-Kostant-Souriau on these coadjoint orbits. Therefore $N$ and $M$ (the former is simply a holomorphic fiber space over the latter with fiber a unit disc) play the role of phase spaces for string theory. From (Ricci) curvature computations on these orbit manifolds with respect to the above canonical Kähler metric, [this was necessary in order to implement the fundamental requirement of reparametrization-invariance for closed string theory] there appears a proof that the bosonic string must be propagating in spacetime of 26 dimensions (and the fermionic string in 10).

On the other hand, in the Polyakov path integral approach to the closed bosonic string, the basic partition function has to be computed as an integral over the space of all world-sheets swept out by propagating strings. Since the Polyakov action has conformal invariance, the functional integral one has to perform reduces to one over the moduli space of all conformal (=complex) structures on the world-sheets. Thus the Teichmüller spaces parametrizing all the complex structures on a given topological surface become fundamental in the Polyakov path integral approach to string theory. The 26 dimensionality of bosonic spacetime from was discovered first (by Polyakov) from this approach as a consequence of demanding that the string partition function integral should be well-defined independent of how one parametrises the Teichmüller space as a space of conformally distinct Riemannian metrics on the world-sheet. In a non-perturbative string scattering amplitude calculation one would be required to do the Polyakov integral over some "universal parameter space" of Riemann surfaces that would contain the Teichmüller spaces of all topological types simultaneously.

Now, the Universal Teichmüller Space $T(1)$, is such a universal parameter space for all Riemann surfaces. $T(1)$ is a (non-separable) complex Banach manifold that may be
defined as the (holomorphically) homogeneous space $QS(S^1)/\text{M"ob}(S^1)$. The Universal Teichm"uller space is the universal ambient space for the Teichm"uller spaces of arbitrary Fuchsian groups – thus it contains as properly embedded complex submanifolds (multiple copies of) each of the classical Teichm"uller spaces of arbitrary Riemann surfaces. Here $QS(S^1)$ denotes the group of all quasisymmetric homeomorphisms of the unit circle $S^1$, and $\text{M"ob}(S^1)$ is the three-parameter subgroup of M"obius transformations of the unit disc (restricted to its boundary circle $S^1$).

In attempting to understand the relationship between the string-reparametrization complex manifolds of the loop space approach and the Teichm"uller spaces arising in the sum-over-histories approach, we were able to uncover some new structures in the Universal Teichm"uller space which had hitherto not been studied. Indeed, in [NV] [N2], we showed that the ”phase space” $M = \text{Diff}(S^1)/\text{M"ob}(S^1)$ sits embedded naturally in $T(1)$ (since any smooth diffeomorphism is quasisymmetric) – and that this embedding is an equivariant, injective holomorphic and K"ahler isometric immersion. $M$ can be interpreted as the submanifold of ”smooth points” of the Universal Teichm"uller space; in fact, if $T(1)$ is considered as the space of all (M"obius classes of) oriented quasicircles on the Riemann sphere, then $M$ corresponds precisely to the $C^\infty$ smooth ones. One thus obtains a picture of the Universal Teichm"uller space as foliated by copies of $M$ and its translates under the universal modular group – each leaf being a complex analytic Frechet submanifold carrying a well-defined K"ahler structure induced from the universal Weil-Petersson pairing on $T(1)$.

The K"ahler structure on $T(1)$ given by the universal form of the Weil-Petersson pairing (see [NV] Part II) is formal. That pairing converges on precisely the Sobolev class $H^{3/2}$ vector fields on the circle – whereas the general tangent vector to $T(1)$ is represented by vector fields on $S^1$ of the Zygmund class $\Lambda$. On the smooth vector fields we showed that this pairing pulled back under the injection of $M$ in $T(1)$ to the Kirillov-Kostant pairing of $M$.

The complex analytic structure of Teichmüller space is canonical and can be seen as arising from many points of view (see [N4]) – it is induced from the complex structure of the ”$\mu$-space” of Beltrami coefficients, which comprise a ball in a complex Banach space. On the other hand, the complex structure, dictated by representation theory, on the coadjoint ”phase spaces” $M$ and $N$ is obtained at the tangent space level by conjugation of Fourier series (namely ”Hilbert transform” on functions on $S^1$). [Recall that tangent vectors to $M$ or $N$ can be thought of as certain vector fields on the circle. The almost complex structure sends a vector field to its Hilbert transform!] That the injection of $M$ into $T(1)$ is a holomorphic immersion amount to proving that Hilbert transform on vector field corresponds at the Beltrami level to mapping $\mu$ to $i\mu$. That was shown to be true in [NV] Part I.

The convergence/divergence analysis of the universal Weil-Petersson form gave us
moreover a proof that the classical Teichmüller spaces of infinite order Fuchsian groups $G$ always intersect transversely the submanifold $M$ of the "smooth points" of $T(1)$. That assertion was an infinitesimal version of the Mostow rigidity theorem on the line. In fact, as is well known from papers of Bowen, Sullivan et. al., the quasicircles that are the limit sets of non-trivial quasi-Fuchsian deformations of $G$ are in general expected to be very non-smooth (indeed "fractal") – and consequently the quasicircles that represent points of $T(G)$ cannot be in the submanifold $M$ which comprises the smooth ones! In this article we shall explain how to create a Teichmüller space $T(H_\infty)$ that appears embedded in $T(1)$ (again in multiple copies) as the natural completion of the union of copies of the Teichmüller spaces of closed Riemann surfaces of all genera. Once again we will see that $T(H_\infty)$ is embedded transverse to $M = Diff(S^1)/Mob(S^1)$. There also appears a genus-independent form of the Weil-Petersson pairing on $T(H_\infty)$ that induces the classical pairing on the Teichmüller spaces of finite genus Riemann surfaces. This latter material is rather new and appears in our paper [NS] joint with Dennis Sullivan.

To complete our introductory description of the structure of the universal Teichmüller space that has arisen from the above considerations we should mention that the submanifold $M$ is not even locally closed in the Banach manifold $T(1)$. The closure of $M$ in the universal Teichmüller space is however identifiable (see [Re], [GS]), and turns out to be the complex Banach submanifold $M^{cl}$ which is $Symm(S^1)/Mob(S^1)$. Symmetric homeomorphisms are those that have quasiconformal extensions to the unit disc that are asymptotically conformal as one approaches the boundary circle. These are exactly the limits of smooth diffeomorphisms of $S^1$ in the Teichmüller (quasiconformal or "µ") topology. Thus $M^{cl}$ is a Banach thickening of the complex Frechet submanifold $M$, and, of course, copies of $M^{cl}$ will also foliate $T(1)$ by holomorphic leaves under translation of $M^{cl}$ by the biholomorphic (right-translation) action of the universal modular group on $T(1)$. Unfortunately, however, the universal Weil-Petersson does not converge on all tangents to the closure of $M$ [indeed, the tangent space to $M^{cl}$ can be identified as the vector fields on the circle of the small $\lambda$ Zygmund class – and these are more general than the convergence class $H^{3/2}$]. We have in summary:

Tangent space at origin of:

$M = Diff(S^1)/Mobius$ is $C^\infty$ vector fields on $S^1$ [modulo $sl(2, \mathbb{R})$]

$M^{cl} = Symm(S^1)/Mobius$ is Zygmund $\lambda$ class vector fields on $S^1$ [modulo $sl(2, \mathbb{R})$]

$T(1) = QS(S^1)/Mobius$ is Zygmund $\Lambda$ class vector fields on $S^1$ [modulo $sl(2, \mathbb{R})$]

and the Kirillov-Kostant = Universal Weil-Petersson pairing converges on the Sobolev class $H^{3/2}$ vector fields on $S^1$ [modulo $sl(2, \mathbb{R})$].

We produce below a schematic diagram of the Universal Teichmüller Space incorporating some of its structure discussed above:
1b. Teichmüller Space and Period Matrices: In subsequent work ([N1] [N2]) we showed that infinite dimensional ”period matrices” can be naturally associated to the smooth points $M$ of $T(1)$ – matrices which generalise exactly the usual $g \times g$ matrices of period integrals (of Abel-Jacobi) associated to closed Riemann surfaces of genus $g$.

We were motivated by Segal’s construction [S] of a certain representation of $Diff(S^1)$ as infinite dimensional symplectomorphisms. One realizes – by invoking considerations that go away at a tangent from Segal’s representation theoretic aims – that one is led immediately to a natural universal period mapping. We will describe this in detail below. In [NS] we succeeded in completing the theory of the period mapping from the submanifold $M$ to the entire space $T(1)$.

Before launching off upon the description of this mathematics, it may be mentioned that such a consideration is again natural from string physics. In fact, one may wish to compute string scattering amplitude integrals over spaces of Riemann surfaces by transferring them to integrals over the space of their Jacobi varieties – namely integrate over the Schottky locus in the Siegel space of principally polarized Abelian varieties instead of on the moduli space of Riemann surfaces. As is well-known (Torelli’s theorem), that classical association of period matrices (or, equivalently, Jacobi varieties) to Riemann surfaces is a one-to-one mapping – so that one should lose nothing by such a passage. There were speculations in the string physics literature on this matter. Our period mapping now shows that there is a genus-independent non-perturbative way to carry out this transfer from the space of all Riemann surfaces to a space of associated Jacobians, and that the rough edges of the finite-dimensional theory get smoothened out in passing to the universal parameter space since the universal period map is actually an isometry with respect to the canonical Kähler metrics.

Remark: Interestingly then, whether carrying out a string non-perturbative functional integral over the coadjoint orbit $M$, or over the corresponding submanifold in Universal Teichmüller space $T(1)$, or over the image (i.e., ”Schottky locus”) of $M$ in Siegel space should produce the same results in view of the holomorphy and isometry results established in [NV] [N1] [N2] [NS] – and surveyed above. The infinite-dimensional differential geometry of the three types of complex analytic parameter spaces are therefore canonically related to each other!

As mentioned, basic to our construction is the faithful representation ([S]) of $Diff(S^1)$ on the Frechet space

$$V = C^\infty \text{Maps}(S^1, \mathbb{R}) / \mathbb{R}( \text{the constant maps})$$

(1)

Here $Diff(S^1)$ acts by pullback on the functions in $V$ as a group of toplinear automorphisms preserving a basic symplectic form $S$ that $V$ carries. $V$ can be interpreted as a
smooth version of the Hodge theoretic first cohomology space of the unit disc, and then
$S$ becomes identified with the cup-product pairing on cohomology. Among the diffeomor-
phisms one notices that only the three-parameter Möbius group acts unitarily on $V$. Hence
one gets a natural map:

$$\Pi : M \to Sp(V)/\text{Unitary}$$

The target space is a complex Banach manifold that is the infinite dimensional version of
the Siegel space of period matrices built by Segal in [S]. This is the mapping we interpreted
as a ”period mapping” that keeps track of the varying space of holomorphic 1-forms on
a Riemann surface (as a point of a Grassmann manifold) as the complex structure on
the surface is varied; that is P. Griffiths way of describing just what the classical period
mappings do. The fundamental naturality of the map was evident from the result we
proved that, like the injection of $M$ in $T(1)$, this mapping $\Pi$ is an equivariant, injective
(genus-independent Torelli theorem!), holomorphic and Kähler isometric immersion of the
smooth-points-submanifold of the Universal Teichmüller space to an infinite dimensional
version of the Siegel space of period matrices.

**Note:** The complex structure of the Siegel space arises, as in the finite dimensional the-
ory, by interpreting the points of the Siegel space as positive polarizing subspaces in the
complexification of a symplectic vector space (the first cohomology vector space of the ref-
ence Riemann surface). Clearly then, the Siegel space appears embedded in a complex
Grassmannian – hence the complex structure is induced from that of the Grassmannian.
As for the metric, the finite dimensional Siegel space

$$S_g = Sp(2g, R)/\text{Maximal compact subgroup } U(g)$$
carries a unique (up to scaling constant) homogeneous Kähler metric for which the full
group $Sp(2g, R)$ of holomorphic automorphisms acts as isometries. That metric simply
goes over to the infinite dimensional Siegel space – and that canonical metric, we showed
in [N2], pulls back via $\Pi$ to universal Weil-Petersson = Kirillov-Kostant on the space $M$.

As we promised, we will also describe more recent joint work with Dennis Sullivan
([NS]) where we have succeeded in finding the natural extension of the above period map-
ing to the entire Universal Teichmüller space by utilising the Sobolev (Hilbert) space $H^{1/2}$
on the circle to complete the $C^\infty$ space $V$. It is completely clear that in order to be able
to extend the infinite dimensional period map to the full space $T(1)$, it is necessary to
replace $V$ by a suitable “completed” space that is invariant under quasisymmetric pull-
backs. Moreover, the quasisymmetric homeomorphisms should continue to act as bounded
symplectic automorphisms of this extended space. The Hilbert space $H^{1/2} = \mathcal{H}$, (say),
appears indicated from (at least!) two points of view and fits the bill exactly.

First, the above Hilbert space, which turns out to be exactly the completion of the pre-
Hilbert space $V$, actually characterizes quasisymmetric (q.s). homeomorphisms (amongst
all homeomorphisms of the circle) in the sense that q.s. homoeomorphisms, and only those, act as bounded operators on \( \mathcal{H} \) by pull-back (namely pre-composition). That will obviously be important for our understanding of the universal period mapping. Harmonic analysis tells us that a general \( H^{1/2} \) function on the circle is defined off a set of (logarithmic) capacity zero. Notice that the fact that capacity zero sets are preserved by quasisymmetric transformations – whereas merely being measure zero is not a q.s.-invariant notion – goes to exemplify how deeply quasisymmetry is connected to the properties of \( \mathcal{H} \).

Furthermore, the space \( H^{1/2} \) can be canonically identified as the Hodge theoretic first cohomology space of the universal Riemann surface (the unit disc) – namely the above Hilbert space is the space of square-integrable real harmonic 1-forms for the disc. The basic reason again comes from the ”trace theorems” of harmonic analysis – the \( H^{1/2} \) functions are exactly the non-tangential boundary values (”traces”) of harmonic functions of finite Dirichlet energy on the unit disc. The symplectic=cup-product structure, \( S \), (seen on \( V \)) extends to \( \mathcal{H} \) and is preserved by the pullback action of quasisymmetric homeomorphisms. Therefore, the extension of \( \Pi \) from \( M \) to all of \( T(1) \) utilizing this completion \( \mathcal{H} \) of \( V \) can again be interpreted as being a period mapping in P. Griffiths’ sense. Not only does the cup-product get identified with the canonical symplectic form carried by \( \mathcal{H} \), one finds that the fiducial complex structure on \( \mathcal{H} \) obtained from the Hodge star on harmonic 1-forms is given precisely by the Hilbert transform (which again preserves the space \( H^{1/2} \)). The target space for the period map is the universal Siegel space of period matrices; that is the space of all the complex structures \( J \) on \( \mathcal{H} \) that are compatible with the canonical symplectic structure (or, alternatively, all positive polarizing subspaces in the complexification of this first cohomology Hilbert space). The various incarnations of the Siegel space will be spelled out below (Section 8).

In effect, an arbitrary point \( X \) of \( T(1) \) produces a decomposition

\[
\text{Complexified } H^{1/2} = H^{1,0}(X) \oplus H^{0,1}(X)
\]

and the universal period map associates to \( X \) the positive polarizing subspace \( H^{1,0}(X) \) as a point of the universal Siegel space. That is why \( \Pi \) is the universal period mapping. \( \Pi \) thus provides us with a new realization of the Universal Teichmüller space as a complex submanifold of the Universal Siegel space.

To establish the complete naturality of the construction of \( \Pi \), we have also proved in [NS] that the pairing \( S \) is the unique symplectic structure available which is invariant under even the tiny finite-dimensional subgroup \( \text{Möb}(S^1) \) (\( \subset QS(S^1) \)). That proof uses the discrete series representations of \( SL(2, \mathbb{R}) \) and a version of Schur’s lemma.

**1c.Universal period mapping and Quantum calculus:** One must remember first of all that the generic quasisymmetric homeomorphisms of the circle that arise in the Teichmüller theory of Riemann surfaces as boundary values of quasiconformal homeomorphisms of the disk have fractal graphs in general, and are consequently not so amenable
to usual analytical or calculus procedures. Therefore we made use of the remarkable fact this group $QS(S^1)$ acts by substitution (i.e., pre-composition) as a family of bounded symplectic operators on the Hilbert space $\mathcal{H} = \mathcal{H}^{1/2}$.

But Alain Connes’ has suggested a "quantum differential" $d_Q f = [J, f]$ – commutator of the multiplication operator with the complex structure operator (Hilbert transform) – to replace the ordinary (often non-existent) classical derivative for a function $f$ (say on the circle). The idea is that operator theoretic properties of the quantum differential will capture smoothness characteristics of $f$. Utilizing this idea, we obtain ([N3]) in lieu of the problematical classical calculus a quantum calculus for quasisymmetric homeomorphisms. Namely, one has operators $\{h, L\}, d \circ \{h, L\}, d \circ \{h, J\}$, corresponding to the non-linear classical objects $\log(h'), h''dx, \frac{1}{6}\text{Schwarzian}(h)dx^2$. The point is that these quantum operators make sense whenever $h$ is merely quasisymmetric, whereas the classical forms could be defined only when $h$ was appropriately smooth. Any one of these objects is a quantum measure of the conformal distortion of $h$ in analogy with the classical calculus’ Beltrami coefficient $\mu$ for a quasiconformal homeomorphism of the disk. Here $L$ is the smoothing operator on the line (or the circle) with kernel $\log|x - y|$, $J$ is the Hilbert transform (which is $d \circ L$ or $L \circ d$), and $\{h, A\}$ means $A$ conjugated by $h$ minus $A$.

The period mapping and the quantum calculus are related in several ways. For example, $f$ belongs to $\mathcal{H}$ if and only if the quantum differential is Hilbert-Schmidt. Also, the Schottky locus or image of the period mapping is describable by a quantum integrability condition $[d_Q J, J] = 0$ for the complex structure $J$ on $\mathcal{H}$.

We will present below a section where we discuss quantum calculus. The idea being firstly to demonstrate that the $H^{1/2}$ functions have such an interpretation. That then allows us to interpret the universal Siegel space that is the target space for the period mapping as "almost complex structures on the line" and the Teichmüller points (i.e., the Schottky locus) can be interpreted as comprising precisely the subfamily of those complex structures that are integrable.

1d. Universal Period Mapping restricted to the Teichmüller space of the universal compact Riemann surface: As mentioned before, inside the Universal Teichmüller space there resides the separable complex submanifold $T(H_\infty)$ – the Teichmüller space of the universal hyperbolic lamination – that is exactly the closure of the union of certain copies of all the genus $g$ classical Teichmüller spaces of closed Riemann surfaces in $T(1)$. Genus-independent constructions like the universal period mapping proceed naturally to live on this completed version of the classical Teichmüller spaces.

The polarizing subspace determined by $\Pi$ for a point of $T_g$ has an intimate relationship with the subspace of $L^2(S^1)$ determined by the Krichever map on data living on this compact Riemann surface. We will spell this out. In fact, the restriction of the universal period mapping $\Pi$ to the submanifold $T(H_\infty)$ of $T(1)$ should have deep properties that
relate to the classical period mappings $\pi_g$ which live on the Teichmüller spaces of genus $g$ surfaces (which appear embedded within $T(H_\infty)$). To start with, we showed in [N3] that $T(H_\infty)$ carries a natural convergent Weil-Petersson pairing. We hope that in future publications we will show how the understanding of the universal Schottky locus that we will describe can be used to determine the Schottky locus in the classical genus $g$ situation.

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2. **Teichmüller Theory and Universal Weil-Petersson:**

**Basic definitions:** The universal Teichmüller space $T(1)$ is a holomorphically homogeneous complex Banach manifold that serves as the universal ambient space where all the Teichmüller spaces (of arbitrary Fuchsian groups) lie holomorphically embedded. Let $\Delta$ denote the open unit disc, and $S^1 = \partial \Delta$. Two classic models of the Universal Teichmüller Space $T(1) = T(\Delta)$ will be described below:

(a) the "real-analytic model" containing all (Möbius-normalised) quasisymmetric homeomorphisms of the unit circle $S^1$ ;

(b) the "complex-analytic model" comprising all normalized schlicht (Riemann mapping) functions on the exterior of the disc which allow quasiconformal extension to the whole Riemann Sphere – i.e., which conformally map the exterior of the disc to (the exterior of) a quasidisc.

The Teichmüller space of any Fuchsian group $G$ comprises those (Möbius-normalized) quasidiscs within which a quasi-Fuchsian deformation of $G$ acts discontinuously. Alternatively, (picture (a)), $T(G)$ is the family of quasisymmetric homeomorphisms of $S^1$ that are compatible with $G$ (namely conjugate $G$ again into groups of Möbius transformations).

The connection between them is via the rather mysterious operation called “Conformal Welding” (see [KNS],[N]). That means that the quasidisc in model (b) that is the image
of the given schlicht function determines a quasisymmetric homeomorphism of the circle by comparing the boundary values of the Riemann mappings to its interior and exterior.

To recall matters, we set the stage by introducing the chief actor – namely the space of (proper) Beltrami coefficients $L^\infty(\Delta)_1$; it is the open unit ball in the complex Banach space of $L^\infty$ functions on the unit disc $\Delta$. The principal construction is to solve the Beltrami equation.

$$w_\mu = \mu w_z \quad \text{(Bel)}$$

for any $\mu \in L^\infty(\Delta)_1$. The two above-mentioned models of Teichmüller space correspond to discussing two pertinent solutions for (Bel):

(a) $w_\mu$ - theory: The quasiconformal homeomorphism of $\mathbb{C}$ which is $\mu$-conformal (i.e. solves (1)) in $\Delta$, fixes $\pm 1$ and $-i$, and keeps $\Delta$ and $\Delta^*$ (= exterior of $\Delta$) both invariant. This $w_\mu$ is obtained by applying the existence and uniqueness theorem of Ahlfors-Bers (for (Bel)) to the Beltrami coefficient which is $\mu$ on $\Delta$ and extended to $\Delta^*$ by reflection ($\tilde{\mu}(1/\bar{z}) = \mu(z)z^2/\bar{z}^2$ for $z \in \Delta$).

[Note: The extension of $\mu$ to the whole plane by reflection above is not complex-linear, – that is why model (a) is being called a ”real-analytic” model of Teichmüller space. In this model the (almost-)complex structure appears mysteriously via the Hilbert transform on vector fields.]

(b) $w^\mu$ - theory: The quasiconformal homeomorphism on $\mathbb{C}$, fixing $0, 1, \infty$, which is $\mu$-conformal on $\Delta$ and conformal on $\Delta^*$. $w^\mu$ is obtained by applying the Ahlfors-Bers theorem to the Beltrami coefficient which is $\mu$ on $\Delta$ and zero on $\Delta^*$.

The fact is that $w_\mu$ depends only real analytically on $\mu$, whereas $w^\mu$ depends complex-analytically on $\mu$. We therefore obtain two standard models ((a) and (b) below) of the universal Teichmüller space, $T(\Delta)$ as stated above.

Thus, we define the universal Teichmüller space:

$$T(1) = T(\Delta) = L^\infty(\Delta)_1/\sim.$$  

Here $\mu \sim \nu$ if and only if $w_\mu = w_\nu$ on $\partial\Delta = S^1$, and that happens if and only if the conformal mappings $w^\mu$ and $w^\nu$ coincide on $\Delta^* \cup S^1$. We let

$$\Phi : L^\infty(\Delta)_1 \to T(\Delta)$$

denote the quotient (“Bers”) projection. $T(\Delta)$ inherits its canonical structure as a complex Banach manifold from the complex structure of $L^\infty(\Delta)_1$; namely, $\Phi$ becomes a holomorphic submersion.

The derivative of $\Phi$ at $\mu = 0$:

$$d_0\Phi : L^\infty(\Delta) \to T_0T(\Delta)$$
is a complex-linear surjection whose kernel is the space $N$ of “infinitesimally trivial Beltrami coefficients”.

$$N = \{ \mu \in L^\infty(\Delta) : \int \int_{\Delta} \mu \phi = 0 \text{ for all } \phi \in A(\Delta) \}$$

where $A(\Delta)$ is the Banach space of integrable ($L^1$) holomorphic functions on the disc. Thus, the tangent space at the origin of $T(\Delta)$ is canonically $L^\infty(\Delta)/N$. See [A3], [Le] and [N4] for this material and for what follows.

It is now clear that to $\mu \in L^\infty(\Delta)_1$ we can associate the quasisymmetric homeomorphism

$$f_\mu = w_\mu$$

on the boundary circle of the hyperbolic disc as representing the Teichmüller point $[\mu]$ in version (a) of $T(\Delta)$. Indeed $T(\Delta)_{(a)}$ is the homogeneous space comprising Möbius classes of quasisymmetric homeomorphisms of the unit circle. Alternatively, it is identified with the group (“partial topological group”) of quasisymmetric homeomorphisms fixing $1, -1$ and $-i$.

In the (b) model, $[\mu]$ is represented by the univalent function

$$f^\mu = w^\mu$$

on $\Delta^*$. That is version (b) of $T(\Delta)$. Smooth diffeomorphisms (recall $M$) in model (a) correspond to (namely they are the weldings for) univalent functions that map $\Delta^*$ onto smooth Jordan regions.

It is worth remarking here that the criteria that a power series expansion represents an univalent function, and that it allows quasiconformal extension, can be written down solely in terms of the coefficients $c_k$, (using the Grunsky inequalities etc.). Thus $T(\Delta)_{(b)}$ can be thought of as a certain space of sequences $(c_2, c_3, \ldots)$, (representing the power series coefficients of Riemann mappings to quasidisc regions). The tangent space can be given a concomitant description (see [N3]).

**Universal Weil-Petersson:** As usual, let $\text{Diff}(S^1)$ denote the infinite dimensional Lie group of orientation preserving $C^\infty$ diffeomorphisms of $S^1$. The complex- analytic homogeneous phase space” of string theory is, as we explained:

$$M = \text{Diff}(S^1)/\text{Mob}(S^1)$$

$M$ injects holomorphically into $T(\Delta)_{(a)}$. This was proved in [NV, Part I]. The submanifold $M$ comprises the “smooth points” of $T(\Delta)$ ; in fact, in version (b), the points from $M$ are those quasidiscs $F^\mu(\Delta^*)$ whose boundary curves are $C^\infty$.

Now, $M$, together with its modular group translates, foliates $T(\Delta)$ – and the fundamental Kirillov-Kostant Kähler (symplectic) form exists on each leaf of the foliation. Up to
an overall scaling this homogeneous Kähler metric gives the following hermitian pairing \( g \) on the tangent space at the origin of \( M \):

\[
g(V, W) = Re \left[ \sum_{k=2}^{\infty} a_k \overline{b}_k (k^3 - k) \right]
\]

where

\[
V = \sum_{2}^{\infty} a_k e^{ik\theta} + \sum_{2}^{\infty} \overline{a}_k e^{-ik\theta},
\]

and similarly \( W \) (with Fourier coefficients \( b_k \)), represent two smooth real vector fields on \( S^1 \) (tangents to \( M \)).  \[\textbf{Note:} \text{ The } (k^3 - k) \text{ appearing in the formula for } g \text{ is, of course, closely related to that same term appearing in the Gelfand-Fuks cohomology theory of the Lie algebra of smooth vector fields on } S^1.\]

The metric \( g \) on \( M \) was proved by this author [NV, Part II] to be nothing other than the universal Weil-Petersson metric (WP) of universal Teichmüller space. Of course, the classical Weil-Petersson metric is defined on the Teichmüller spaces of finite dimension by pairing holomorphic quadratic differentials for a Fuchsian group \( G \) using the Petersson pairing of automorphic forms. The point is that when \( G \) is taken to be the universal Fuchsian group \( G = \{1\} \), the pairing becomes the formula written out above.

Let us take a moment to recall the formula for the Weil-Petersson pairing in terms of Beltrami differentials representing tangent directions to Teichmüller space. Let \( \mu \) and \( \nu \) be \( g \)-invariant Beltrami (-1,1) coefficients on the disc. Then:

\[
WP(\mu, \nu) = \int_{\Delta/G} \times \int_{\Delta} [\mu(z) \overline{\nu(\zeta)}] / (1 - z\overline{\zeta})^4 d\zeta \wedge d\overline{\zeta} dz \wedge d\overline{z}
\]

A word about the proof. Using the perturbation formula for the theory of the Beltrami equation, the author calculated the Fourier coefficients of the vector fields on the circle corresponding to the Beltrami directions above. It then followed that the WP hermitian pairing above, for \( G = \{1\} \), coincides with the Kirillov-Kostant \( g \).

Moreover, we showed in [NV] Part II how to recover the classical W-P pairings on tangent vectors to finite dimensional Teichmüller spaces \( T(G) \) by regulating the universal Weil-Petersson formula. (Universal W-P diverges in general when applied to such \( G \)-invariant tangent vectors). That proved a form of the Mostow rigidity theorem expressing the transversality of \( M \) with the Teichmüller spaces \( T(G) \), as we mentioned in the Introduction.

Now, although ”conformal welding” operation that relates the two models shewn of \( T(1) \) is difficult to analyse, nevertheless, at the infinitesimal level, the above models have an amazingly simple relationship that forms the basis for our paper[N3]. Indeed, the \( k^{th} \) Fourier coefficient of the vector field representing a tangent vector in model (a),
and the (first variation of) the $k^{th}$ power series coefficient representing the same tangent vector in model (b), turn out to be just ($\sqrt{-1}$ times) complex conjugates of each other. That relationship can also be formulated as a direct identity relating the vector field on the circle with the holomorphic function in the exterior of the disc describing the perturbation in the complex-analytic model. It allows us to express the universal Weil-Petersson metric, namely the Kirillov-Kostant pairing on $M$, in the following deceptively simple (and equivalent) form:

We express the pairing for $g = WP$ in terms of 1-parameter flows of schlicht functions (from [N3]) as follows: Let $F_t(\zeta)$ and $G_t(\zeta)$ denote two curves through origin in $T(\Delta)_{(b)}$ representing two tangent vectors, say $\dot{F}$ and $\dot{G}$. Then the Weil-Petersson pairing assigns:

$$WP(\dot{F}, \dot{G}) = -\left[ \sum_{k=2}^{\infty} c_k(0) d_k(0) (k^3 - k) \right],$$

Here $c_k(t)$ and $d_k(t)$ are the power series coefficients for the schlicht functions $F_t$ and $G_t$ respectively. The series above converges precisely when the corresponding Zygmund class functions are in the Sobolev class $H^{3/2}$.

**Holomorphy of the inclusion map $I$ of $M$ into $T(1)$:** Utilizing the identity proved in [N3] between Fourier coefficients of Zygmund class vector fields and the corresponding power series for the Riemann mappings onto curves of quasidisks, we get an immediate proof of the fascinating fact that the almost complex structure of $T(\Delta)$ transmutes to the operation of Hilbert transform at the level of Zygmund class vector fields on $S^1$. In other words, one sees that the vector field determined by Beltrami direction $\mu$ is related to the field obtained from the Beltrami direction $i\mu$ as a pair of conjugate Fourier series. A proof of this was important for our previous work, and appeared in [NV, Part I] – establishing that $M$ injects holomorphically into Universal Teichmüller space.

**Remark:** The result above gives an independent proof of the fact that conjugation of Fourier series preserves the Zygmund class $\Lambda(S^1)$. That was an old theorem of Zygmund [Z].

In the next few sections we develop the theory necessary to work out the universal period mapping $\Pi$ on $T(1)$. We shall explain in detail the occurence in various incarnations of the Hilbert space $H^{1/2}(S^1)$ and demonstrate its canonical isomorphism with the Hodge-theoretic first cohomology space of the disc.

### 3. The Hilbert space $H^{1/2}$ on the circle:

Let $\Delta$ denote the open unit disc and $U$ the upper half-plane in the plane ($\mathbb{C}$), and $S^1 = \partial \Delta$ be the unit circle. Intuitively speaking, the real Hilbert space under concern:

$$\mathcal{H} \equiv H^{1/2}(S^1, \mathbb{R}) / \mathbb{R}$$

(2)
is the subspace of $L^2(S^1)$ comprising real functions of mean-value zero on $S^1$ which have a half-order derivative also in $L^2(S^1)$. Harmonic analysis will tell us that these functions are actually defined off some set of capacity zero (i.e., "quasi-e verywhere") on the circle, and that they also appear as the boundary values of real harmonic functions of finite Dirichlet energy in $\Delta$. Our first way (of several) to make this precise is to identify $\mathcal{H}$ with the sequence space

$$\ell_{1/2}^2 = \{\text{complex sequences } u \equiv (u_1, u_2, u_3, \cdots) : \{\sqrt{n} \ u_n\} \text{ is square summable }\}.$$  

The identification between (2) and (3) is by showing that the Fourier series

$$f(e^{i\theta}) = \sum_{n=-\infty}^{\infty} u_n e^{in\theta}; \quad u_{-n} = \overline{u}_n,$$

converges quasi-everywhere and defines a real function of the required type. The norm on $\mathcal{H}$ and on $\ell_{1/2}^2$ is, of course, the $\ell_2$ norm of $\{\sqrt{n} \ u_n\}$, i.e.,

$$\|f\|_{\mathcal{H}}^2 = \|u\|_{\ell_{1/2}^2}^2 = 2 \sum_{n=1}^{\infty} n |u_n|^2.$$  

Naturally $\ell_{1/2}^2$ and $\mathcal{H}$ are isometrically isomorphic separable Hilbert spaces. Note that $\mathcal{H}$ is a subspace of $L^2(S^1)$ because $\{\sqrt{n} \ u_n\}$ in $\ell_2$ implies $\{u_n\}$ itself is in $\ell_2$.

At the very outset let us note the fundamental fact that the space $\mathcal{H}$ is evidently closed under Hilbert transform ("conjugation" of Fourier series):

$$(Jf)(e^{i\theta}) = -\sum_{n=-\infty}^{\infty} i \text{sgn}(n) u_n e^{in\theta}.$$  

In fact, $J : \mathcal{H} \to \mathcal{H}$ is an isometric isomorphism whose square is the negative identity, and thus $J$ defines a canonical complex structure for $\mathcal{H}$. When identified with the real cohomology space of the disc, this Hilbert transform will appear as the Hodge star complex structure. See below.

Remark: In our earlier articles we had made use of the fact that the Hilbert transform defines the almost-complex structure operator for the tangent space of the coadjoint orbit manifolds ($M$ and $N$), as well as for the universal Teichmüller space $T(1)$. This fact is closely related to the matter at hand.

Whenever convenient we will pass to a description of our Hilbert space $\mathcal{H}$ as functions on the real line, $\mathbb{R}$. This is done by simply using the Möbius transformation of the circle onto the line that is the boundary action of the Riemann mapping ("Cayley transform")
of $\Delta$ onto $U$. We thus get an isometrically isomorphic copy, called $H^{1/2}(\mathbb{R})$, of our Hilbert space $\mathcal{H}$ on the circle defined by taking $f \in \mathcal{H}$ to correspond to $g \in H^{1/2}(\mathbb{R})$ where $g = f \circ R, R(z) = \frac{z-i}{z+i}$ being the Riemann mapping. The Hilbert transform complex structure on $\mathcal{H}$ in this version is then described by the usual singular integral operator on the real line with the "Cauchy kernel" $(x-y)^{-1}$.

Fundamental for our set up is the dense subspace $V$ in $\mathcal{H}$ defined by equation (1) in the Introduction. $V$ will be the "smooth part" of the Hodge-theoretic cohomology of the disc, and it will carry the theory of the period mapping on the submanifold $M$ of smooth points of $T(1)$. At the level of Fourier series, $V$ corresponds to those sequence $\{u_n\}$ in $\ell^2$ which go to zero more rapidly than $n^{-k}$ for any $k > 0$. Since trigonometric polynomials are in $V$, it is obvious that $V$ is norm-dense in $\mathcal{H}$.

$V$ carries the basic symplectic form that we utilised crucially in [N1], [N2]:

$$S : V \times V \to \mathbb{R}$$

given by

$$S(f, g) = \frac{1}{2\pi} \int_{S^1} f \cdot dg.$$  

(8)

This is essentially the signed area of the $(f(e^{i\theta}), g(e^{i\theta}))$ curve in Euclidean plane. On Fourier coefficients this bilinear form becomes

$$S(f, g) = 2 \text{Im} \left( \sum_{n=1}^{\infty} n u_n v_n \right) = -i \sum_{n=-\infty}^{\infty} n u_n v_{-n}$$

(9)

where $\{u_n\}$ and $\{v_n\}$ are respectively the Fourier coefficients of the (real-valued) functions $f$ and $g$, as in (4). Let us note that the Cauchy-Schwarz inequality applied to (9) shows that this non-degenerate bilinear alternating form extends from $V$ to the full Hilbert space $\mathcal{H}$. We will call this extension also $S : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$. We shall see that this very symplectic form is the cup-product in cohomology when we interpret $\mathcal{H}$ as Hodge-theoretic cohomology space.

Notice that Cauchy-Schwarz asserts:

$$|S(f, g)| \leq \|f\| \cdot \|g\|.$$  

(10)

Thus $S$ is a jointly continuous, in fact analytic, map on $\mathcal{H} \times \mathcal{H}$.

The important interconnection between the inner product on $\mathcal{H}$, the Hilbert-transform complex structure $J$, and the form $S$ is encapsulated in the identity:

$$S(f, Jg) = \langle f, g \rangle, \text{ for all } f, g \in \mathcal{H}$$

(11)
The point is that intertwining the cup-product with the Hodge star operator as in (11) always produces the inner product in $L^2$ cohomology of a Riemann surface.

To sum up, therefore, $V$ itself was naturally a pre-Hilbert space with respect to the canonical inner product arising from its symplectic form and its Hilbert-transform complex structure, and we have just established that the completion of $V$ is nothing other than the Hilbert space $\mathcal{H}$. Whereas $V$ carried the $C^\infty$ theory, because it was diffeomorphism invariant, the completed Hilbert space $\mathcal{H}$ allows us to carry through our constructions for the full Universal Teichmüller Space because it indeed is quasisymmetrically invariant – as we shall show below.

**Complexification of $\mathcal{H}$:** It will be important for us to *complexify* our spaces since we need to deal with isotropic subspaces and polarizations. Thus we set

$$\mathbb{C} \otimes V \equiv V_\mathbb{C} = C^\infty \text{Maps} (S^1, \mathbb{C}) / \mathbb{C}$$

$$\mathbb{C} \otimes \mathcal{H} \equiv \mathcal{H}_\mathbb{C} = H^{1/2} (S^1, \mathbb{C}) / \mathbb{C}$$

(12)

$\mathcal{H}_\mathbb{C}$ is a complex Hilbert space isomorphic to $\ell^2_{1/2}(\mathbb{C})$ - the latter comprising the Fourier series

$$f(e^{i\theta}) = \sum_{n=-\infty}^{\infty} u_n e^{in\theta}, \quad u_0 = 0$$

(13)

with $\{\sqrt{|n|} u_n\}$ being square summable over $\mathbb{Z} - \{0\}$. Note that the Hermitian inner product on $\mathcal{H}_\mathbb{C}$ derived from (5) is given by

$$\langle f, g \rangle = \sum_{n=-\infty}^{\infty} |n| u_n \overline{v_n}.$$  

(14)

[This explains why we introduced the factor 2 in the formula (5).] The fundamental orthogonal decomposition of $\mathcal{H}_\mathbb{C}$ is given by

$$\mathcal{H}_\mathbb{C} = W_+ \oplus W_-$$

(15)

where

$$W_+ = \{ f \in \mathcal{H}_\mathbb{C} : \text{all negative index Fourier coefficients vanish} \}$$

and

$$W_- = \overline{W_+} = \{ f \in \mathcal{H}_\mathbb{C} : \text{all positive index Fourier coefficients vanish} \}.$$  

We have denoted by bar the complex anti-linear automorphism of $\mathcal{H}_\mathbb{C}$ given by conjugation of complex scalars.
Let us extend \( C \)-linearly the form \( S \) and the operator \( J \) to \( \mathcal{H}_C \) (and consequently also to \( V_C \)). The complexified \( S \) is still given by the right-most formula in (9). Notice that \( W_+ \) and \( W_- \) can be characterized as precisely the \( -i \) and \( +i \) eigenspaces (respectively) of the \( C \)-linear extension of \( J \), the Hilbert transform. Further, each of \( W_+ \) and \( W_- \) is isotropic for \( S \), i.e., \( S(f, g) = 0 \), whenever both \( f \) and \( g \) are from either \( W_+ \) or \( W_- \) (see formula (9)). Moreover, \( W_+ \) and \( W_- \) are positive isotropic subspaces in the sense that the following identities hold:

\[
\langle f_+, g_+ \rangle = iS(f_+, \bar{g}_+) \text{, for } f_+, g_+ \in W_+ \tag{16}
\]

and

\[
\langle f_-, g_- \rangle = -iS(f_-, \bar{g}_-) \text{, for } f_-, g_- \in W_- \tag{17}
\]

**Remark:** (16) and (17) show that we could have defined the inner product and norm on \( \mathcal{H}_C \) from the symplectic form \( S \), by using these relations to define the inner products on \( W_+ \) and \( W_- \), and declaring \( W_+ \) to be perpendicular to \( W_- \). Thus, for general \( f, g \in \mathcal{H}_C \) one has the fundamental identity

\[
\langle f, g \rangle = iS(f_+, \bar{g}_+) - iS(f_-, \bar{g}_-) \tag{18}
\]

We have thus described the Hilbert space structure of \( \mathcal{H} \) simply in terms of the canonical symplectic form it carries and the fundamental decomposition (15). [Here, and henceforth, we will let \( f_\pm \) denote the projection of \( f \) to \( W_\pm \), etc.].

**\( \mathcal{H} \) and the Dirichlet space on the disc:** In order to survey our work on the universal period map, we have to rely on interpreting the functions in \( H^{1/2} \) as boundary values (“traces”) of functions in the disc \( \Delta \) that have finite Dirichlet energy, (i.e. the first derivatives are in \( L^2(\Delta) \)).

Define the following “Dirichlet space” of harmonic functions:

\[
\mathcal{D} = \{ F : \Delta \to \mathbb{R} : F \text{ is harmonic , } F(0) = 0, \text{ and } E(F) < \infty \} \tag{19}
\]

where the energy \( E \) of any (complex-valued) \( C^1 \) map on \( \Delta \) is defined as the \( L^2(\Delta) \) norm of \( \text{grad}(F) \) :

\[
\| F \|_\mathcal{D}^2 = E(F) = \frac{1}{2\pi} \int \int _\Delta \left( \left| \frac{\partial F}{\partial x} \right|^2 + \left| \frac{\partial F}{\partial y} \right|^2 \right) dx dy \tag{20}
\]

\( \mathcal{D} \), and its complexification \( \mathcal{D}_C \), will be Hilbert spaces with respect to this energy norm.
We want to identify the space $D$ as precisely the space of harmonic functions in $\Delta$ solving the Dirichlet problem for functions in $H$. Indeed, the Poisson integral representation allows us to map $P : H \to \mathcal{D}$ so that $P$ is an isometric isomorphism of Hilbert spaces.

To see this let $f(e^{i\theta}) = \sum_{n=-\infty}^{\infty} u_n e^{in\theta}$ be an arbitrary member of $H_{\mathbb{C}}$. Then the Dirichlet extension of $f$ into the disc is:

$$F(z) = \sum_{n=-\infty}^{\infty} u_n r^{|n|} e^{in\theta} = \left(\sum_{n=1}^{\infty} u_n z^n\right) + \left(\sum_{m=1}^{\infty} u_{-m} \bar{z}^m\right)$$

(21)

where $z = re^{i\theta}$. From the above series one can directly compute the $L^2(\Delta)$ norms of $F$ and also of $\text{grad}(F) = (\partial F/\partial x, \partial F/\partial y)$. One obtains the following:

$$E(F) = \frac{1}{2\pi} \int \int_{\Delta} |\text{grad}(F)|^2 = \sum_{n=-\infty}^{\infty} |n||u_n|^2 \equiv \|f\|_H^2 < \infty$$

(22)

We will require crucially the well-known formula of Jesse Douglas expressing the above energy of $F$ as the double integral on $S^1$ of the square of the first differences of the boundary values $f$.

$$E(F) = \frac{1}{16\pi^2} \int_{S^1} \int_{S^1} \left[\frac{f(e^{i\theta}) - f(e^{i\phi})}{\sin((\theta - \phi)/2)}\right]^2 d\theta d\phi$$

(23)

Transferring to the real line by the Möbius transform identification of $H$ with $H^{1/2}(\mathbb{R})$ as explained before, the above identity becomes as simple as:

$$E(F) = \frac{1}{4\pi^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \left[\frac{f(x) - f(y)}{x-y}\right]^2 dx dy = \|f\|^2$$

(24)

Calculating from the series (21), the $L^2$-norm of $F$ itself is:

$$\frac{1}{2\pi} \int \int_{\Delta} |F|^2 dx dy = \sum_{n=-\infty}^{\infty} \frac{|u_n|^2}{(|n| + 1)} \leq E(F) < \infty$$

(25)

(22) shows that indeed Dirichlet extension is isometric from $H$ to $\mathcal{D}$, whereas (25) shows that the functions in $\mathcal{D}$ are themselves in $L^2$, so that the the inclusion of $\mathcal{D} \hookrightarrow L^2(\Delta)$ is continuous. (Bounding the $L^2$ norm of $F$ by the $L^2$ norm of its derivatives is a “Poincaré inequality”).

It is therefore clear that $\mathcal{D}$ is a subspace of the usual Sobolev space $H^1(\Delta)$ comprising those functions in $L^2(\Delta)$ whose first derivatness (in the sense of distributions) are also in $L^2(\Delta)$. The theory of function spaces implies (by the “trace theorems”) that $H^1$ functions
lose half a derivative in going to a boundary hyperplane. Thus it is known that the functions in \( D \) will indeed have boundary values in \( H^{1/2} \).

Moreover, the identity (24) shows that for any \( F \in D \), the Fourier expansion of the trace on the boundary circle is a Fourier series with \( \sum |n||u_n|^2 < \infty \). But Fourier expansions with coefficients in such a weighted \( \ell_2 \) space, as in our situation, are known to converge \textit{quasi-everywhere} (i.e. off a set of logarithmic capacity zero) on \( S^1 \). See Zygmund [Z, Vol 2, Chap. XIII]. The identification between \( D \) and \( \mathcal{H} \) (or \( D_C \) and \( \mathcal{H}_C \)) is now complete.

It will be necessary for us to identify the \( W_\pm \) polarization of \( \mathcal{H}_C \) at the level \( D_C \). In fact, let us decompose the harmonic function \( F \) of (21) into its holomorphic and anti-holomorphic parts; these are \( F_+ \) and \( F_- \), which are (respectively) the two sums bracketed separately on the right hand side of (21). Clearly \( F_+ \) is a holomorphic function extending \( f_+ \) (the \( W_+ \) part of \( f \)), and \( F_- \) is anti-holomorphic extending \( f_- \). We are thus led to introduce the following space of holomorphic functions whose derivatives are in \( L^2(\Delta) \):

\[
\text{Hol}_2(\Delta) = \{ H : \Delta \to \mathbb{C} : H \text{ is holomorphic, } H(0) = 0 \text{ and } \int \int_{\Delta} |H'(z)|^2 dx dy < \infty \}.
\]  

(26)

This is a complex Hilbert space with the norm

\[
\| H \|^2 = \frac{1}{2\pi} \int \int_{\Delta} |H'(z)|^2 dx dy.
\]  

(27)

If \( H(z) = \sum_{n=1}^{\infty} u_n z^n \), a computation in polar coordinates (as for (21), (25)) produces

\[
\| H \|^2 = \sum_{n=1}^{\infty} n |u_n|^2.
\]  

(28)

Equations (27) and (28) show that the norm-squared is the Euclidean area of the (possibly multi-sheeted) image of the map \( H \).

We let \( \overline{\text{Hol}_2}(\Delta) \) denote the Hilbert space of antiholomorphic functions conjugate to those in \( \text{Hol}_2(\Delta) \). The norm is defined by stipulating that the anti-linear isomorphism of \( \text{Hol}_2 \) on \( \overline{\text{Hol}_2} \) given by conjugation should be an isometry. The Cauchy-Riemann equation for \( F_+ \) and \( \overline{F}_- \) imply that

\[
|\text{grad}(F)|^2 = 2 \left\{ |F'_+|^2 - |F'_-|^2 \right\}
\]  

(29)

and hence

\[
\| F_+ \|^2 + \| F_- \|^2 = \| f \|^2_{\mathcal{H}_C}.
\]  

(30)
Now, the relation between $\mathcal{D}$ (harmonic functions in $H^1(\Delta)$) and $\text{Hol}_2(\Delta)$ is transparent, so that the holomorphic functions in $\text{Hol}_2$ will have non-tangential limits quasi-everywhere on $S^1$ - defining a function $W_+$. We thus collect together, for the record, the various representations of our basic Hilbert space:

**Theorem 3.1:** There are canonical isometric isomorphisms between the following complex Hilbert spaces:

1. $\mathcal{H}_C = H^{1/2}(S^1, \mathbb{C}) / \mathbb{C} = \mathbb{C} \otimes H^{1/2}(\mathbb{R}) = W_+ \oplus W_-;
2. The sequence space $\ell_2^{1/2}(\mathbb{C})$ (constituting the Fourier coefficients of the above quasi-everywhere defined functions);
3. $\mathcal{D}_C$, comprising normalized finite-energy harmonic functions (either on $\Delta$ or on the half-plane $U$); [the norm-squared being given by (20) or (22) or (23) or (24)];
4. $\text{Hol}_2(\Delta) \oplus \overline{\text{Hol}_2}(\Delta)$.

Under the canonical identifications, $W_+$ maps to $\text{Hol}_2(\Delta)$ and $W_-$ onto $\overline{\text{Hol}_2}(\Delta)$.

**Remark:** Since the isomorphisms of the Theorem are all isometric, and because the norm arose from the canonical symplectic structure, (formulas (16), (17), (18)), it is possible and instructive to work out the formulas for the symplectic form $S$ on $\mathcal{D}_C$ and on $\text{Hol}_2(\Delta)$.

### 4. Quasisymmetric invariance:

Quasiconformal (q.c.) self-homeomorphisms of the disc $\Delta$ (or the upper half-plane $U$) are known to extend continuously to the boundary. The action on the boundary circle (respectively, on the real line $\mathbb{R}$) is called a quasisymmetric (q.s.) homeomorphism. Now, $\varphi : \mathbb{R} \to \mathbb{R}$ is quasisymmetric if and only if, for all $x \in \mathbb{R}$ and all $t > 0$, there exists some $K > 0$ such that

$$\frac{1}{K} \leq \frac{\varphi(x + t) - \varphi(x)}{\varphi(x) - \varphi(x - t)} \leq K \quad (31)$$

On the circle this condition for $\varphi : S^1 \to S^1$ means that $|\varphi(2I)|/|\varphi(I)| \leq K$, where $I$ is any interval on $S^1$ of length less that $\pi$, $2I$ denotes the interval obtained by doubling $I$ keeping the same mid-point, and $|\bullet|$ denotes Lebesgue measure on $S^1$.

Given any orientation preserving homeomorphism $\varphi : S^1 \to S^1$, we use it to pullback functions in $\mathcal{H}$ by precomposition:

$$V_\varphi(f) = \varphi^*(f) = f \circ \varphi - \frac{1}{2\pi} \int_{S^1} (f \circ \varphi). \quad (32)$$

This is the basic symplectic operator on $\mathcal{H}$ associated to any quasisymmetric homeomorphism of $S^1$ that produces the universal period mapping on $T(\Delta)$. 

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THEOREM 4.1: $V_\varphi$ maps $\mathcal{H}$ to itself (i.e., the space $\mathcal{H} \circ \varphi$ is $\mathcal{H}$) if and only if $\varphi$ is quasisymmetric. The operator norm of $V_\varphi \leq \sqrt{K + K^{-1}}$, whenever $\varphi$ allows a $K$-quasiconformal extension into the disc.

COROLLARY 4.2: The group of all quasisymmetric homeomorphism on $S^1$, $QS(S^1)$, acts faithfully by bounded toplinear automorphisms on the Hilbert space $\mathcal{H}$ (and therefore also on $\mathcal{H}_\mathbb{C}$).

Proof of sufficiency: Assume $\varphi$ is q.s. on $S^1$, and let $\Phi : \Delta \to \Delta$ be any quasiconformal extension. Let $f \in \mathcal{H}$ and suppose $P(f) = F \in \mathcal{D}$ is its unique harmonic extension into $\Delta$. Clearly $G = F \circ \Phi$ has boundary values $f \circ \varphi$, the latter being (like $f$) also a continuous function on $S^1$ defined quasi-everywhere. [Here we recall that q.s. homeomorphisms carry capacity zero sets to again such sets, although measure zero sets can become positive measure.] To prove that $f \circ \varphi$ minus its mean value is in $\mathcal{H}$, it is enough to prove that the Poisson integral of $f \circ \varphi$ again has finite Dirichelet energy. Indeed we will show

$$E(\text{harmonic extension of } \varphi^*(f)) \leq 2 \left( \frac{1 + k^2}{1 - k^2} \right)^{1/2} E(F).$$  \hspace{1cm} (33)

Here $0 \leq k < 1$ is the q.c. constant for $\Phi$, i.e.,

$$|\Phi_\bar{\bar{z}}| \leq k |\Phi_z| \quad \text{a.e. in } \Delta.$$

The operator norm of $V_\varphi$ is thus no more that $2^{1/2} \left( \frac{1 + k^2}{1 - k^2} \right)^{1/2}$. The last expression is equal to the bound quoted in the Theorem, where, as usual, $K = (1 + k)/(1 - k)$.

Towards establishing (33) we prove that the inequality holds with the left side being the energy of the map $G = F \circ \Phi$. Since $G$ is therefore also a finite energy extension of $f \circ \varphi$ to $\Delta$, Dirichlet’s principle (namely that the minimal energy amongst all extensions is achieved by the harmonic (Poisson integral) extension) implies the required inequality.

Remark: Since the Dirichlet integral in two dimensions is invariant under conformal mappings, it is not too surprising that it is quasi-invariant under quasiconformal transformations. Such quasi-invariance has been noted and applied before. See [A1].

Proof of necessity: The idea of this proof is taken from the notes of M. Zinsmeister. We express our gratitude for his generosity.

Since two-dimensional Dirichlet integrals are conformally invariant, we will pass to the upper half-plane $U$ and its boundary line $\mathbb{R}$ to aid our presentation. As explained earlier, using the Cayley transform we transfer everything over to the half-plane; the traces on the boundary constitute the space of quasi-everywhere defined functions called $H^{1/2}(\mathbb{R})$.

From the Douglas identity, equation (24), we recall that an equivalent way of expressing the Hilbert space norm on $H^{1/2}(\mathbb{R})$ is
Equation (34) immediately shows that \( \| g \| = \| \tilde{g} \| \) where \( \tilde{g}(x) = g(ax + b) \) for any real \( a(\neq 0) \) and \( b \). This will be important.

Assume that \( \varphi : \mathbb{R} \to \mathbb{R} \) is an orientation preserving homeomorphism such that \( V_{\varphi^{-1}} : H^{1/2}(\mathbb{R}) \to H^{1/2}(\mathbb{R}) \) is a bounded automorphism. Let us say that the norm of this operator is \( M \).

Fix a bump function \( f \in C_c^{\infty}(\mathbb{R}) \) such that \( f \equiv 1 \) on \([-1, 1] \) and \( 0 \leq f \leq 1 \) everywhere. Choose any \( c \in \mathbb{R} \) and any positive \( t \). Denote \( I_1 = [x - t, x] \) and \( I_2 = [x, x + t] \). Set \( g(u) = f(au + b) \), choosing \( a \) and \( b \) so that \( g \) is identically 1 on \( I_1 \) and zero on \([x + t, \infty)\).

By assumption, \( g \circ \varphi^{-1} \) is in \( H^{1/2}(\mathbb{R}) \) and \( \| g \circ \varphi^{-1} \| \leq M \| g \| = M \| f \| \). We have

\[
M \| f \| \geq \int \int_{\mathbb{R}^2} \left[ \frac{g \circ \varphi^{-1}(u) - g \circ \varphi^{-1}(v)}{u - v} \right]^2 \, du \, dv
\]

\[
\geq \int_{v = \varphi(x)}^{v = \varphi(x - t)} \int_{u = \varphi(x + t)}^{u = \varphi(x)} \frac{1}{(u - v)^2} \, du \, dv
\]

\[
= \log \left( 1 + \frac{\varphi(x) - \varphi(x - t)}{\varphi(x + t) - \varphi(x)} \right), \quad (35)
\]

[We have utilised the elementary integration \( \int_{\gamma}^{\beta} \frac{1}{(u - v)^2} \, du = \log \left( 1 + \frac{\beta - \alpha}{\gamma - \beta} \right) \), for \( \alpha < \beta < \gamma \).] We thus obtain the result that

\[
\frac{\varphi(x + t) - \varphi(x)}{\varphi(x) - \varphi(x - t)} \geq \frac{1}{e^{M \| f \|} - 1}
\]

for arbitrary real \( x \) and positive \( t \). By utilising symmetry, namely by shifting the bump to be 1 over \( I_2 \) and 0 for \( u \leq x - t \), we get the opposite inequality:

\[
\frac{\varphi(x + t) - \varphi(x)}{\varphi(x) - \varphi(x - t)} \leq e^{M \| f \|} - 1.
\]

The Beurling-Ahlfors condition on \( \varphi \) is verified, and we are through. Both the theorem and its corollary are proved. \( \blacksquare \)

5. The invariant symplectic structure:

The quasisymmetric homeomorphism group, \( QS(S^1) \), acts on \( \mathcal{H} \) by precomposition (equation (32)) as bounded operators, preserving the canonical symplectic form \( S : \mathcal{H} \times \mathcal{H} \to \mathbb{R} \).
\[ H \to \mathbb{R}. \] This is the central fact which we will now expose; it is the crux on which the definition of the period mapping on all of \( T(1) \) hinges:

**PROPOSITION 5.1:** For every \( \varphi \in QS(S^1) \), and all \( f, g \in \mathcal{H} \),

\[
S(\varphi^*(f), \varphi^*(g)) = S(f, g). \tag{36}
\]

Considering the complex linear extension of the action to \( \mathcal{H}_C \), one can assert that the only quasisymmetrics which preserve the subspace \( W_+ = \text{Hol}_2(\Delta) \) are the Möbius transformations. Thus \( \text{Möb} (S^1) \) acts as unitary operators on \( W_+ \) (and \( W_- \)).

Before proving the proposition we would like to point out that this canonical symplectic form enjoys a far stronger invariance property. The proof is an exercise in calculus.

**LEMMA 5.2:** If \( \varphi : S^1 \to S^1 \) is any (say \( C^1 \)) map of winding number (= degree) \( k \), then

\[
S(f \circ \varphi, g \circ \varphi) = kS(f, g) \tag{37}
\]

for arbitrary choice of \( (C^1) \) functions \( f \) and \( g \) on the circle. In particular, \( S \) is invariant under pullback by all degree one mappings.

**Proof of Proposition:** The Lemma shows that (36) is true whenever the quasisymmetric homeomorphism \( \varphi \) is at least \( C^1 \). By standard facts of quasiconformal mapping theory we know that for arbitrary q.s \( \varphi \), there exist real analytic q.s. homeomorphisms \( \varphi_m \) (with the same quasisymmetry constant as \( \varphi \)) that converge uniformly to \( \varphi \). An approximation argument (see [NS]) then proves the required result.

If the action of \( \varphi \) on \( \mathcal{H}_C \) preserves \( W_+ \), it is easy to see that \( \varphi \) must be the boundary values of some holomorphic map \( \Phi : \Delta \to \Delta \). Since \( \varphi \) is a homeomorphism one can see that \( \Phi \) is a holomorphic homeomorphism (as explained in [N1]) – hence a Möbius transformation. Since every \( \varphi \) preserves \( S \), and since \( S \) induces the inner product on \( W_+ \) and \( W_- \) by (16) (17), we note that such a symplectic transformation preserving \( W_+ \) must necessarily act unitarily.

**Motivational remark:** Theorem 4.1 and Proposition 5.1 enable us to consider \( QS(S^1) \) as a subgroup of the bounded symplectic operators on \( \mathcal{H} \). Since the heart of the matter in extending the period mapping from Witten’s homogeneous space \( M \) (as in [N1], [N2]) to \( T(1) \) lies in the property of preserving this symplectic form on \( \mathcal{H} \), we must prove that \( S \) is indeed the unique symplectic form that is \( \text{Diff} (S^1) \) or \( QS (S^1) \) invariant. It is all the more surprising that the form \( S \) is canonically specified by requiring its invariance under simply the 3-parameter subgroup \( \text{Möb} (S^1) \) \( \hookrightarrow \text{Diff} (S^1) \hookrightarrow QS (S^1) \). We sketch a proof of this uniqueness result:

**THEOREM 5.3:** Let \( S \equiv S_1 \) be the canonical symplectic form on \( \mathcal{H} \). Suppose \( S_2 : \mathcal{H} \times \mathcal{H} \to \mathbb{R} \) is any other continuous bilinear form such that \( S_2(\varphi^*(f), \varphi^*(g)) = S_2(f, g) \), for all \( f, g \) in \( \mathcal{H} \) whenever \( \varphi \) is in \( \text{Möb} (S^1) \). Then \( S_2 \) is necessarily a real multiple of
Thus every form on $\mathcal{H}$ that is M"ob $(S^1) \equiv PSL(2,\mathbb{R})$ invariant is necessarily non-degenerate (if not identically zero) and remains invariant under the action of the whole of $QS(S^1)$. (Also, it automatically satisfies the even stronger invariance property (37)).

The proof requires some representation theory. We start with an easy lemma:

**Lemma 5.4:** The duality induced by canonical form $S_1$ is (the negative of) the Hilbert transform (equation (6)). Thus the map $\Sigma_1$ (induced by $S_1$) from $\mathcal{H}$ to $\mathcal{H}^*$ is an invertible isomorphism.

The basic tool in proving the uniqueness of the symplectic form is to consider the "intertwining operator":

$$M = \Sigma_1^{-1} \circ \Sigma_2 : \mathcal{H} \to \mathcal{H} \quad (38)$$

which is a bounded linear operator on $\mathcal{H}$ by the above Lemma.

**Lemma 5.5:** $M$ commutes with every invertible linear operator on $\mathcal{H}$ that preserves both the forms $S_1$ and $S_2$.

**Proof:** $M$ is defined by the identity $S_1(v, Mw) = S_2(v, w)$. If $T$ preserves boths forms then one has the string of equalities:

$$S_1(Tv, TMw) = S_1(v, Mw) = S_2(v, w) = S_2(Tv, Tw) = S_1(Tv, MTw)$$

Since $T$ is assumed invertible, this is the same as saying

$$S_1(v, TMw) = S_1(v, MTw), \text{ for all } v, w \in \mathcal{H} \quad (39)$$

But $S_1$ is non-degenerate, namely $\Sigma_1$ was an isomorphism. Therefore (39) implies that $TM \equiv MT$, as desired.

It is clear that to prove that $S_2$ is simply a real multiple of $S_1$ means that the intertwining operator $M$ has to be just multiplication by a scalar. This can now be deduced by looking at the complexified representation of M"ob $(S^1)$ on $\mathcal{H}_C$, which is unitary, and applying Schur’s Lemma. [I am indebted to Graeme Segal and Ofer Gabber for helpful conversations in this regard.]

**Lemma 5.6:** The unitary representation of $SL(2,\mathbb{R})$ on $\mathcal{H}_C$ decomposes into precisely two irreducible pieces - namely on $W_+$ and $W_-$. In fact these two representations correspond to the two lowest (conjugate) members in the discrete series for $SL(2,\mathbb{R})$.

**Proof:** We claim that the representation given by the operators $V_\varphi$ on $W_+$ (equation (32)), $\varphi \in $ M"ob $(S^1)$, can be indentified with the "$m = 2$" case of the discrete series of irreducible unitary representations of $SL(2,\mathbb{R})$. Note, M"ob$(S^1) \equiv PSU(1,1) \cong SL(2,\mathbb{R})/(\pm I)$. Recall from Theorem 3.1 that $W_+$ is identifiable as Hol$_2(\Delta)$. The action of $\varphi$ is given on Hol$_2$ by:

$$V_\varphi(F) = F \circ \varphi - F \circ \varphi(0), \quad F \in Hol_2(\Delta). \quad (39)$$
But Hol₂ consists of normalized \((F(0) = 0)\) holomorphic functions in \(\Delta\) whose derivative is in \(L^2(\Delta, \text{Euclidean measure})\). From (39), by the chain rule,

\[
\frac{d}{dz} V_\varphi(F) = \left( \frac{dF}{dz} \circ \varphi \right) \varphi'
\]  

(40)

So we can rewrite the representation on the derivatives of the functions in Hol₂ by the formula (40) - which coincides with the standard formula for the discrete series irreducible representation at lowest index.

It is clear that the representation on the conjugate space will correspond to the \(m = -2\) (highest weight vector of weight \(-2\)) case of the discrete series. In particular, the representations we obtain of Möb \((S^1)\) by unitary operators of \(W_+\) and \(W_-\) are both irreducible. The Lemma is proved.

**Proof of Theorem 5.3:** By Lemma 5.5, (the \(\mathbb{C}\)-linear extension of ) the intertwining operator \(M\) commutes with every one of the unitary operators \(V_\varphi : \mathcal{H}_C \rightarrow \mathcal{H}_C\) as \(\varphi\) varies over Möb \((S^1)\). Since \(W_+\) and \(W_-\) are the only two invariant subspaces for all the \(V_\varphi\), as proved above, it follows that \(M\) must map \(W_+\) either to \(W_+\) or to \(W_-\). Let us first assume the former case. Then \(M\) commutes with all the unitary operators \(V_\varphi\) on \(W_+\), which we know to be an irreducible representation. *Schur's Lemma* says that a unitary representation will be irreducible if and only if the only operators that commute with all the operators in the representation are simply the scalars. Since \(M\) was a real operator to start with, that scalar must be real. The alternative assumption that \(M\) maps \(W_+\) to \(W_-\) is easily verified to be untenable.

**Remark:** Our proof of absolute naturality of the symplectic form is complete. We will utilise Theorem 5.3 in understanding the \(H^{1/2}\) space as a *Hilbertian* space, – namely a space possessing a fixed symplectic structure but a large family of compatible complex structures. That is the nature of a cohomology space on a Riemann surface as the complex structure on the surface varies – but the topology – which determines the cup-product symplectic form, remains invariant.

**6. The \(H^{1/2}\) space as first cohomology:**

The Hilbert space \(H^{1/2}\), that is the hero of our tale, can be interpreted as the first cohomology space with real coefficients of the "universal Riemann surface" – namely the unit disc – in a Hodge-theoretic sense. That will be fundamental for us in explaining the properties of the period mapping on the universal Teichmüller space.

In fact, in the classical theory of the period mapping, the vector space \(H^1(X, \mathbb{R})\) plays a basic role, \(X\) being a closed orientable topological surface of genus \(g\) to start with. This real vector space comes equipped with a canonical symplectic structure given by the cup-product pairing, \(S\). Now, whenever \(X\) has a complex manifold structure, this real space \(H^1(X, \mathbb{R})\) of dimension \(2g\) gets endowed with a *complex structure* \(J\) *that is compatible*
with the cup-pairing $S$. This happens as follows: When $X$ is a Riemann surface, the cohomology space above is precisely the vector space of real harmonic 1-forms on $X$, by the Hodge theorem. Then the complex structure $J$ is the Hodge star operator on the harmonic 1-forms. The compatibility with the cup form is encoded in the relationships (41) and (42):

$$S(J\alpha, J\beta) = S(\alpha, \beta), \quad \text{for all } \alpha, \beta \in H^1(X, \mathbb{R})$$

and that, intertwining $S$ and $J$ exactly as in equation (11),

$$S(\alpha, J\beta) = \text{inner product}(\alpha, \beta)$$

should define a positive definite inner product on $H^1(X, \mathbb{R})$. [In fact, as we will further describe in Section 8, the Siegel disc of period matrices for genus $g$ is precisely the space of all the $S$-compatible complex structures $J$.] Consequently, the period mapping can be thought of as the variation of the Hodge-star complex structure on the topologically determined symplectic vector space $H^1(X, \mathbb{R})$.

**Remark:** Whenever $X$ has a complex structure, one gets an isomorphism between the real vector space $H^1(X, \mathbb{R})$ and the $g$ dimensional complex vector space $H^1(X, \mathcal{O})$, where $\mathcal{O}$ denotes the sheaf of germs of holomorphic functions. That is so because $\mathbb{R}$ can be considered as a subsheaf of $\mathcal{O}$ and hence there is an induced map on cohomology. It is interesting to check that this natural map is an isomorphism, and that the complex structure so induced on $H^1(X, \mathbb{R})$ is the same as that given above by the Hodge star.

For our purposes it therefore becomes relevant to consider, for an arbitrary Riemann Surface $X$, the Hodge-theoretic first cohomology vector space as the space of $L^2$ (square-integrable) real harmonic 1-forms on $X$. This real Hilbert space will be denoted $\mathcal{H}(X)$. Once again, in complete generality, this Hilbert space has a non-degenerate symplectic form $S$ given by the cup (= wedge) product:

$$S(\phi_1, \phi_2) = \int_X \phi_1 \wedge \phi_2$$

and the Hodge star is the complex structure $J$ of $\mathcal{H}(X)$ which is again compatible with $S$ as per (41) and (42). In fact, one verifies that the $L^2$ inner product on $\mathcal{H}(X)$ is given by the triality relationship (42) – which is the same as (11).

Since in the universal Teichmüller theory we deal with the ”universal Riemann surface” – namely the unit disc $\Delta$ – (being the universal cover of all Riemann surfaces), we require the

**Proposition 6.1:** For the disc $\Delta$, the Hilbert space $\mathcal{H}(\Delta)$ is isometrically isomorphic to the real Hilbert space $\mathcal{H}$ of Section 3. Under the canonical identification the cup-wedge pairing is the canonical symplectic form $S$ and the Hodge star becomes the Hilbert-transform on $\mathcal{H}$.
Proof: For every $\phi \in \mathcal{H}(\Delta)$ there exists a unique real harmonic function $F$ on the disc with $F(0) = 0$ and $dF = \phi$. Clearly then, $\mathcal{H}(\Delta)$ is isometrically isomorphic to the Dirichlet space $\mathcal{D}$ of normalized real harmonic functions having finite energy. But in Section 3 we saw that this space is isometrically isomorphic to $\mathcal{H}$ by passing to the boundary values of $F$ on $S^1$.

If $\phi_1 = dF_1$ and $\phi_2 = dF_2$, then integrating $\phi_1 \wedge \phi_2$ on the disc amounts to, by Stokes’ theorem,

$$\int \int_{\Delta} dF_1 \wedge dF_2 = \int_{S^1} F_1dF_2 = S(F_1, F_2)$$

as desired.

Finally, let $\phi = udx + vdy$ be a $L^2$ harmonic 1-form with $\phi = dF$. Suppose $G$ is the harmonic conjugate of $F$ with $G(0) = 0$. Then $dF + idG$ is a holomorphic 1-form on $\Delta$ with real part $\phi$. It follows that the Hodge star maps $\phi$ to $dG$; hence, under the above canonical identification of $\mathcal{H}(\Delta)$ with $\mathcal{H}$, we see that the star operator becomes the Hilbert transform.

7. Quantum calculus and $H^{1/2}$:

A. Connes has proposed (see, for example, [CS] and Connes’ book ”Geometrie Non-Commutatif”) a ”quantum calculus” that associates to a function $f$ an operator that should be considered its quantum derivative – so that the operator theoretic properties of this $d^Q(f)$ capture the smoothness properties of the function. One advantage is that this operator can undergo all the operations of the functional calculus. The fundamental definition in one real dimension is

$$d^Q(f) = [J, M_f]$$

(44)

where $J$ is the Hilbert transform in one dimension explained in Section 3, and $M_f$ stands for (the generally unbounded) operator given by multiplication by $f$. One can think of the quantum derivative as operating (possibly unboundedly) on the Hilbert space $L^2(S^1)$ or on other appropriate function spaces.

Note: We will also allow quantum derivatives to be taken with respect to other Hilbert-transform like operators; in particular, the Hilbert transform can be replaced by some conjugate of itself by a suitable automorphism of the Hilbert space under concern. In that case we will make explicit the $J$ by writing $d^Q_J(f)$ for the quantum derivative. See below for applications.

As sample results relating the properties of the quantum derivative with the nature of $f$, we quote: $d^Q(f)$ is a bounded operator on $L^2(S^1)$ if and only if the function $f$ is of bounded mean oscillation. In fact, the operator norm of the quantum derivative is equivalent to the BMO norm of $f$. Again, $d^Q(f)$ is a compact operator on $L^2(S^1)$ if and
only if $f$ is in $L^\infty(S^1)$ and has vanishing mean oscillation. Also, if $f$ is Hölder, (namely in some Hölder class), then the quantum derivative acts as a compact operator on Hölder. See [CS], [CRW]. Similarly, the requirement that $f$ is a member of certain Besov spaces can be encoded in properties of the quantum derivative.

Our Hilbert space $H^{1/2}(\mathbb{R})$ has a very simple interpretation in these terms:

**PROPOSITION 7.1:** $f \in H^{1/2}(\mathbb{R})$ if and only if the operator $d^Q(f)$ is Hilbert-Schmidt on $L^2(\mathbb{R})$ [or on $H^{1/2}(\mathbb{R})$]. The Hilbert-Schmidt norm of the quantum derivative coincides with the $H^{1/2}$ norm of $f$.

**Proof:** Recall that the Hilbert transform on the real line is given as a singular integral operator with integration kernel $(x-y)^{-1}$. A formal calculation therefore shows that

$$(d^Q(f))(g)(x) = \int_{\mathbb{R}} \frac{f(x) - f(y)}{x-y} g(y) dy$$

But the above is an integral operator with kernel $K(x,y) = (f(x) - f(y))/(x-y)$, and such an operator is Hilbert-Schmidt if and only if the kernel is square-integrable over $\mathbb{R}^2$. Utilising now the Douglas identity – equation (24) – we are through.

Since the Hilbert transform, $J$, is the standard complex structure on the $H^{1/2}$ Hilbert space, and since this last space was shown to allow an action by the quasisymmetric group, $QS(\mathbb{R})$, some further considerations become relevant. Introduce the operator $L$ on 1-forms on the line to functions on the line by:

$$(L\varphi)(x) = \int_{\mathbb{R}} [\log|x-y|] \varphi(y) dy$$

One may think of the Hilbert transform $J$ as operating on either the space of functions or on the space of 1-forms (by integrating against the kernel $dx/(x-y)$). Let $d$ as usual denote total derivative (from functions to 1-forms). Then notice that $L$ above is an operator that is essentially a smoothing inverse of the exterior derivative. In fact, $L$ and $d$ are connected to $J$ via the relationships:

$$d \circ L = J_{\text{1-forms}}; \quad L \circ d = J_{\text{functions}}$$

**The Quasisymmetrically deformed operators:** Given any q.s. homeomorphism $h \in QS(\mathbb{R})$ we think of it as producing a q.s. change of structure on the line, and hence we define the corresponding transformed operators, $L^h$ and $J^h$ by $L^h = h \circ L \circ h^{-1}$ and $J^h = h \circ J \circ h^{-1}$. ($J$ is being considered on functions in $\mathcal{H} = H^{1/2}(\mathbb{R})$, as usual.) The q.s homeomorphism (assumed to be say $C^1$ for the deformation on $L$), operates standardly on functions and forms by pullback. Therefore, $J^h$ simply stands for the Hilbert transform conjugated by the symplectomorphism $T_h$ of $\mathcal{H}$ achieved by pre-composing by the q.s. homeomorphism $h$. $J^h$ is thus a new complex structure on $\mathcal{H}$.
Note: The complex structures on $\mathcal{H}$ of type $J^h$ are exactly those that constitute the image of $T(1)$ by the universal period mapping. (See Section 8.) The target manifold, the universal Siegel space, can be thought of as a space of $S$-compatible complex structures on $\mathcal{H}$.

Let us write the perturbation achieved by $h$ on these operators as the "quantum brackets":

$$\{h, L\} = L^h - L; \quad \{h, J\} = J^h - J.$$  \hspace{1cm} (48)

Now, for instance, the operator $d \circ \{h, J\}$ is represented by the kernel $(h \times h)^*m - m$ where $m = dxdy/(x - y)^2$. For $h$ suitably smooth this is simply $d_yd_x(\log((h(x) - h(y))/(x - y))).$

It is well known that $(h \times h)^*m = m$ when $h$ is a Möbius transformation. Interestingly, therefore, on the diagonal ($x = y$), this kernel becomes $(1/6$ times) the Schwarzian derivative of $h$ (as a quadratic differential on the line). For the other operators in the table below the kernel computations are even easier.

Set $K(x, y) = \log((h(x) - h(y))/(x - y))$ for convenience. We have the following table of quantum calculus formulas:

| Operator | Kernel | On diagonal | Cocycle on $QS(\mathbb{R})$ |
|----------|--------|-------------|-----------------------------|
| $\{h, L\}$ | $K(x, y)$ | $\log(h')$ | function – valued |
| $d \circ \{h, L\}$ | $d_xK(x, y)$ | $\frac{h''}{h'}dx$ | $1 - form$ – valued |
| $d \circ \{h, J\}$ | $d_yd_xK(x, y)$ | $\frac{1}{6}Schwarzian(h)dx^2$ | quadratic – form – valued |

The point here is that these operators make sense when $h$ is merely quasisymmetric. If $h$ happens to be appropriately smooth, we can restrict the kernels to the diagonal to obtain the respective nonlinear classical derivatives (affine Schwarzian, Schwarzian, etc.) as listed in the table above.

8. The universal period mapping on $T(1)$:

Having now all the necessary background results behind us, we are finally set to move into the theory of the universal period (or polarisations) map itself.

As we said at the very start, the Frechet Lie group, $Diff(S^1)$ operating by pullback (= pre-composition) on smooth functions, has a faithful representation by bounded symplectic operators on the symplectic vector space $V$ of equation (1). That induced the natural map $\Pi$ of the homogeneous space $M = Diff(S^1)/Möb(S^1)$ into Segal’s version of the Siegel space of period matrices. In [N1] [N2] we had shown that this map:

$$\Pi : Diff(S^1)/Möb(S^1) \to Sp_0(V)/U$$  \hspace{1cm} (50)

is equivariant, holomorphic, Kähler isometric immersion, and moreover that it qualifies as a generalised period matrix map. Remember that the domain is a complex submanifold of
the universal space of Riemann surfaces $T(1)$. Here $Sp_0$ denotes (see [S]) the symplectic automorphisms of Hilbert-Schmidt type. The canonical Siegel symplectic-invariant metric exists on the target space in (50).

From the results of Sections 2, 3, and 4, 5, we know that the full quasisymmetric group, $QS(S^1)$ operates as bounded symplectic operators on the Hilbert space $\mathcal{H}$ that is the completion of the pre-Hilbert space $V$. We also demonstrated that the subgroup of $QS$ acting unitarily is the Möbius subgroup. Clearly then we have obtained the extension of $\Pi$ (also called $\Pi$ to save on nomenclature) to the entire universal Teichmüller space:

$$\Pi : T(1) \to Sp(\mathcal{H})/U$$

(50)

**Universal Siegel (period matrix) space:** Let us first exhibit the nature of the complex Banach manifold that is the target space of the period map (71). This space, which is the universal Siegel period matrix space, denoted $S_\infty$, has several interesting descriptions:

(a): $S_\infty$ = the space of positive polarizations of the symplectic Hilbert space $\mathcal{H}$. Recall that a positive polarization signifies the choice of a closed complex subspace $W$ in $\mathcal{H}_C$ such that (i) $\mathcal{H}_C = W \oplus \overline{W}$; (ii) $W$ is $S$-isotropic, namely $S$ vanishes on arbitrary pairs from $W$; and (iii) $iS(w, \overline{w})$ defines the square of a norm on $w \in W$. [In the classical genus $g$ situation (ii) and (iii) are the bilinear relations of Riemann.]

(b): $S_\infty$ = the space of $S$-compatible complex structure operators on $\mathcal{H}$. That consists of bounded invertible operators $J$ of $\mathcal{H}$ onto itself whose square is the negative identity and $J$ is compatible with $S$ in the sense that requirements (41) and (42) are valid. Alternatively, these are the complex structure operators $J$ on $\mathcal{H}$ such that $H(f, g) = S(f, Jg) + iS(f, g)$ is a positive definite Hermitian form having $S$ as its imaginary part.

(c): $S_\infty$ = the space of bounded operators $Z$ from $W_+$ to $W_-$ that satisfy the condition of $S$-symmetry: $S(Z\alpha, \beta) = S(Z\beta, \alpha)$ and are in the unit disc in the sense that $(I - ZZ^*)$ is positive definite. The matrix for $Z$ is the "period matrix" of the classical theory.

(d): $S_\infty$ = the homogeneous space $Sp(\mathcal{H})/U$; here $Sp(\mathcal{H})$ denotes all bounded symplectic automorphisms of $\mathcal{H}$, and $U$ is the unitary subgroup defined as those symplectomorphisms that keep the subspace $W_+$ (setwise) invariant.

Introduce the Grassmannian $Gr(W_+, \mathcal{H}_C)$ of subspaces of type $W_+$ in $\mathcal{H}_C$, which is obviously a complex Banach manifold modelled on the Banach space of all bounded operators from $W_+$ to $W_-$. Clearly, $S_\infty$ is embedded in $Gr$ as a complex submanifold. The connections between the above descriptions of the Siegel universal space are transparent:

(a:b) the positive polarizing subspace $W$ is the $-i$-eigenspace of the complex structure operator $J$ (extended to $\mathcal{H}_C$ by complex linearity).
(a:c) the positive polarizing subspace $W$ is the graph of the operator $Z$.

(a:d) $Sp(H)$ acts transitively on the set of positive polarizing subspaces. $W_+$ is a polarizing subspace, and the isotropy (stabilizer) subgroup thereat is exactly $U$.

**$H$ as a Hilbertian space:** Note that the method (b) above describes the universal Siegel space as a space of Hilbert space structures on the fixed underlying symplectic vector space $H$. By the result of Section 4 we know that the symplectic structure on $H$ is completely canonical, whereas each choice of $J$ above gives a Hilbert space inner product on the space by intertwining $S$ and $J$ by the fundamental relationship (11) (or (42)). Thus $H$ is a "Hilbertian space", which signifies a complete topological vector space with a canonical symplectic structure but lots of compatible inner products turning it into a Hilbert space in many ways.

We come to one of our Main Theorems:

**THEOREM 8.1:** The universal period mapping $\Pi$ is an injective, equivariant, holomorphic immersion between complex Banach manifolds. Restricted to $M$ it is also an isometry between the canonical metrics.

**Proof:** From our earlier papers we know these facts for $\Pi$ restricted to $M$. The proof of equivariance is the same (and simple). The map is an injection because if we know the subspace $W_+$ pulled back by $w_\mu$, then we can recover the q.s. homeomorphism $w_\mu$. In fact, inside the given subspace look at those functions which map $S^1$ homeomorphically on itself. One sees easily that these must be precisely the M"obius transformations of the circle pre-composed by $w_\mu$. The injectivity (global Torelli theorem) follows.

Let us write down the matrix for the symplectomorphism $T$ on $H_C$ obtained by pre-composition by $w_\mu$. We will write in the standard orthonormal basis $e^{ik\theta}/k^{1/2}$, $k = 1, 2, 3..$ for $W_+$, and the complex conjugates as o.n. basis for $W_-$. In $H_C = W_+ \oplus W_-$ block form, the matrix for operator $T$ is given by maps $A : W_+ \to W_+$, and $B : W_- \to W_+$. The conjugates of $A$ and $B$ map $W_-$ to $W_-$, and $W_+$ to $W_-$, respectively. The matrix entries for $A = ((a_{pq}))$ and $B = ((b_{rs}))$ turn out to be:

$$a_{pq} = (2\pi)^{-1}p^{1/2}q^{-1/2} \int_0^{2\pi} (w_\mu(e^{i\theta}))^q e^{-ip\theta} d\theta, \quad p, q \geq 1$$

$$b_{rs} = (2\pi)^{-1}r^{1/2}s^{-1/2} \int_0^{2\pi} (w_\mu(e^{i\theta}))^{-s} e^{-ir\theta} d\theta, \quad r, s \geq 1$$

Recalling the standard action of symplectomorphisms on the Siegel disc (model (c) above), we see that the corresponding operator $[=\text{period matrix}] Z$ appearing from the Teichmüller point $[\mu]$ is given by:

$$\Pi[\mu] = BA^{-1}$$
The usual proof of finite dimensions shows that for any symplectomorphism $A$ must be invertible – hence the above explicit formula makes sense.

Since the Fourier coefficients appearing in $A$ and $B$ vary only real-analytically with $\mu$, it may be somewhat surprising that $\Pi$ is actually holomorphic. In fact, a computation of the first variation of $\Pi$ at the origin of $T(1)$ (i.e., the derivative map) in the Beltrami direction $\nu$ shows that the following $Rauch variational formula$ subsists:

\[
(d\Pi(\nu))_{rs} = \pi^{-1}(rs)^{1/2} \int \int_{\Delta} \nu(z) z^{r + s - 2} dx dy
\]  

The proof of this formula is as shown for $\Pi$ on the smooth points submanifold $M$ in our earlier papers. The manifest complex linearity of the derivative, i.e., the validity of the Cauchy-Riemann equations, combined with equivariance, demonstrates that $\Pi$ is complex analytic on $T(1)$, as desired.

Utilizing the derivative (51) one shows that the Siegel symplectic Kähler form pulls back on $M$ to the Weil-Petersson Kähler form.

**Interpretation of $\Pi$ as period map:** Why does the map $\Pi$ qualify as a universal version of the classical genus $g$ period maps? As we had explained in our previous papers, in the light of P.Griffiths’ ideas, the classical period map may be thought of as associating to a Teichmüller point a positive polarizing subspace of the first cohomology $H^1(X, \mathbb{R})$. The point is that when $X$ has a complex structure, then the complexified first cohomology decomposes as:

\[
H^1(X, \mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X)
\]  

The period map associates the subspace $H^{1,0}(X)$ – which is positive polarizing with respect to the cup-product symplectic form – to the given complex structure on $X$. Of course, $H^{1,0}(X)$ represents the holomorphic 1-forms on $X$, and that is why this is nothing but the usual period mapping.

*But that is precisely what $\Pi$ is doing in the universal Teichmüller space.* Indeed, by the results of Section 5, $\mathcal{H}$ is the Hodge-theoretic real first cohomology of the disc, with $S$ being the cup-product.

The standard complex structure on the unit disc has holomorphic 1-forms that are of the form $dF$ where $F$ is a holomorphic function on $\Delta$ with $F(0) = 0$. Thus the boundary values of $F$ will have only positive index Fourier modes – corresponding therefore to the polarizing subspace $W_+$. Now, an arbitrary point of $T(1)$ is described by the choice of a Beltrami differential $\mu$ on $\Delta$ perturbing the complex structure. We are now asking for the holomorphic 1-forms on $\Delta_\mu$. Solving the Beltrami equation on $\Delta$ provides us with the $\mu$-conformal quasiconformal self-homeomorphism $w_\mu$ of the disc. This $w_\mu$ is a holomorphic uniformising coordinate for the disc with the $\mu$ complex structure. The holomorphic 1-forms subspace, $H^{1,0}(\Delta_\mu)$, should therefore comprise those functions on $S^1$ that are the
functions \( w_+ \) precomposed with the boundary values of the q.c. map \( w_\mu \). That is exactly the action of \( \Pi \) on the Teichmüller class of \( \mu \). This explains in some detail why \( \Pi \) behaves as an infinite dimensional period mapping.

**Remark:** On Segal’s \( C^\infty \) version of the Siegel space – constructed using Hilbert-Schmidt operators \( Z \), there existed the universal Siegel symplectic metric, which we studied in [N1] [N2] and showed to be the same as the Kirillov-Kostant (= Weil-Petersson) metric on \( Diff(S^1)/Mob(S^1) \). For the bigger Banach manifold \( S_\infty \) above, that pairing fails to converge on arbitrary pairs of tangent vectors because the relevant operators are not any more trace-class in general. The difficulties associated with this matter will be addressed in Section 9 below, and in further work that is in progress.

9. The universal Schottky locus and quantum calculus:

Our object is to exhibit the image of \( \Pi \) in \( S_\infty \). The result (equation (53)) can be recognized to be a quantum “integrability condition” for complex structures on the circle or the line.

**PROPOSITION 9.1:** If a positive polarizing subspace \( W \) is in the ”universal Schottky locus”, namely if \( W \) is in the image of \( T(1) \) under the universal period mapping \( \Pi \), then \( W \) possesses a dense subspace which is multiplication-closed (i.e., an “algebra” under pointwise multiplication modulo subtraction of mean-value.) In quantum calculus terminology, this means that

\[
[d_J^q, J] = 0
\]

where \( J \) denotes the \( S \)-compatible complex structure of \( H \) whose \(-i\)-eigenspace is \( W \).

(Recall the various descriptions of \( S_\infty \) spelled out in the last section.)

**Multiplication-closed polarizing subspace:** The notion of being multiplication-closed is well-defined for the relevant subspaces in \( H_C \). Let us note that the original polarizing subspace \( W_+ \) contains the dense subspace of holomorphic trigonometric polynomials (with mean zero) which constitute an algebra. Indeed, the identity map of \( S^1 \) is a member of \( W_+ \), call it \( j \), and positive integral powers of \( j \) clearly generate \( W_+ \) – since polynomials in \( j \) form a dense subspace therein. Now if \( W \) is any other positive polarizing subspace, we know that it is the image of \( W_+ \) under some \( T \in Sp(H) \). Thus, \( W \) will be multiplication-closed precisely when the image of \( j \) by \( T \) generates \( W_+ \), in the sense that its positive integral powers (minus the mean values) also lie in \( W \) (and hence span a dense subspace of \( W \)).

In other words, we are considering \( W \in S_\infty \) (description (a)) to be multiplication-closed provided that the pointwise products of functions from \( W \) (minus their mean values) that happen to be \( H^{1/2} \) functions actually land up in the subspace \( W \) again. Multiplying \( f \) and \( g \) modulo arbitrary additive constants demonstrates that this notion is well-defined when applied to a subspace.

**Quantum calculus and equation (53):** We suggest a quantum version of complex structures in one real dimension, and note that the integrable ones correspond to the
universal Schottky locus under study.

In the spirit of algebraic geometry one takes the real Hilbert space of functions $\mathcal{H} = H^{1/2}(\mathbb{R})$ as the “coordinate ring” of the real line. Consequently, a complex structure on $\mathbb{R}$ will be considered to be a complex structure on this Hilbert space. Since $S_\infty$ was a space of (symplectically-compatible) complex structures on $\mathcal{H}$, we are interpreting $S_\infty$ as a space of quantum complex structures on the line (or circle).

Amongst the points of the universal Siegel space, those that can be interpreted as the holomorphic function algebra for some complex structure on the circle qualify as the “integrable” ones. But $T(1)$ parametrises all the quasisymmetrically related circles, and for each one, the map $\Pi$ associates to that structure the holomorphic function algebra corresponding to it; see the interpretation we provided for $\Pi$ in the last section. It is clear therefore that $\Pi(T(1))$ should be the integrable complex structures. The point is that taking the standard circle as having integrable complex structure, all the other integrable complex structures arise from this one by a $QS$ change of coordinates on the underlying circle. These are the complex structures $J^h$ introduced in Section 6 on quantum calculus. The $-i$-eigenspace for $J^h$ is interpreted as the algebra of analytic functions on the quantum real line with the $h$-structure. We will see in the proof that (53) encodes just this condition.

**Proof of Proposition 9.1:** For a point of $T(1)$ represented by a q.s. homeomorphism $\phi$, the period map sends it to the polarizing subspace $W_\phi = W_+ \circ \phi$. But $W_+$ was a multiplication-closed subspace, generated by just the identity map $j$ on $S^1$, to start with. Clearly then, $\Pi(\phi) = W_\phi$ is also multiplication-closed in the sense explained, and is generated by the image of the generator of $W_+$ – namely by the q.s homeomorphism $\phi$ (as a member of $\mathcal{H}_\mathbb{C}$).

We suspect that the converse is also true: that the $T(W_+)$ is such an ”algebra” subspace for a symplectomorphism $T$ in $Sp(\mathcal{H})$ only when $T$ arises as pullback by a quasisymmetric homeomorphism of the circle. This converse assertion is reminiscent of standard theorems in Banach algebras where one proves, for example, that every (conjugation-preserving) algebra automorphism of the algebra $C(X)$ (comprising continuous functions on a compact Hausdorff space $X$) arises from homeomorphisms of $X$. [Remark of Ambar Sengupta.] Owing to the technical hitch that $H^{1/2}$ functions are not in general everywhere defined on the circle, we are as yet unable to find a rigorous proof of this converse.

Here is the sketch of an idea for proving the converse. Thus, suppose we are given a subspace $E$ that is multiplication-closed in the sense explained. Now, $Sp(\mathcal{H})$ acts transitively on the set of positive polarizing subspaces. We consider a $T \in Sp(\mathcal{H})$ that maps $W_+$ to $E$ preserving the algebra structure (modulo subtracting off mean values as usual). Denote by $j$ the identity function on $S^1$ and let $T(j) = w$ be its image in $E$.

Since $j$ is a homeomorphism and $T$ is an invertible real symplectomorphism, one expects that $w$ is also a homeomorphism on $S^1$. (Recall the signed area interpretation of
the canonical form (8).) It then follows that the $T$ is nothing other that precomposition by this $w$. That is because:

$$T(j^m) = T(j)^m - \text{mean value} = (w(e^{i\theta}))^m - \text{mean value} = j^m \circ w - \text{mean value}.$$ 

Knowing $T$ to be so on powers of $j$ is sufficient, as polynomials in $j$ are dense in $W_+$. 

Again, since $T$ is the complexification of a real symplectomorphism, seeing the action of $T$ on $W_+$ tells us $T$ on all of $\mathcal{H}_C$; namely, $T$ is everywhere precomposition by that homeomorphism $w$ of $S^1$. By the necessity part of Theorem 4.1 we see that $w$ must be quasi-isometric, and hence that the given subspace $E$ is the image under $\Pi$ of the Teichmüller point determined by $w$ (i.e., the coset of $w$ in $QS(S^1)/\text{Möb}(S^1)$).

**Proof of equation (53):** Let $J$ be any $S$-compatible complex structure on $\mathcal{H}$, namely $J$ is an arbitrary point of $S_\infty$ (description (b) of Section 8). Let $J_0$ denote the Hilbert transform itself, which is the reference point in the universal Siegel space; therefore $J = TJ_0T^{-1}$ for some symplectomorphism $T$ in $Sp(\mathcal{H})$. The $-i$-eigenspace for $J_0$ is, of course, the reference polarizing subspace $W_+$, and the subspace $W$ corresponding to $J$ consists of the functions $(f + i(Jf))$ for all $f$ in $\mathcal{H}$. Now, the pointwise product of two such typical elements of $W$ gives:

$$(f + i(Jf))(g + i(Jg)) = [fg - (Jf)(Jg)] + i[f(Jg) + g(Jf)]$$

In order for $W$ to be multiplication closed the function on the right hand side must also be of the form $(h + i(Jh))$. Namely, for all relevant $f$ and $g$ in the real Hilbert space $\mathcal{H}$ we must have:

$$J[fg - (Jf)(Jg)] = [f(Jg) + g(Jf)]$$

(54)

Now recall from the concepts introduced in Section 8 that one can associate to functions $f$ their quantum derivative operators $d^Q_J(f)$ which is the commutator of $J$ with the multiplication operator $M_f$ defined by $f$. The quantum derivative is being taken with respect to any Hilbert-transform-like operator $J$ as explained above. But now a short computation demonstrates that equation (54) is the same as saying that:

$$J \circ d^Q_J(f) = -d^Q_J(f)$$

operating by $J$ on both sides shows that this is the same as (53). That is as desired. 

**Remark:** For the classical period mapping on the Teichmüller spaces $T_g$ there is a way of understanding the Schottky locus in terms of Jacobian theta functions satisfying the nonlinear K-P equations. In a subsequent paper we hope to relate the finite dimensional Schottky solution with the universal solution given above.
Remark: For the extended period-polarizations mapping $\Pi$, the Rauch variational formula that was exhibited in [N1], [N2], [N3], and also here in the proof of Theorem 8.1, continues to hold.

10. The Teichmüller space of the universal compact lamination:

The Universal Teichmüller Space, $T(1)=T(\Delta)$, is a non-separable complex Banach manifold that contains, as properly embedded complex submanifolds, all the Teichmüller spaces, $T_g$, of the classical compact Riemann surfaces of every genus $g \geq 2$. $T_g$ is $3g - 3$ dimensional and appears (in multiple copies) within $T(\Delta)$ as the Teichmüller space $T(G)$ of the Fuchsian group $G$ whenever $\Delta/G$ is of genus $g$. The closure of the union of a family of these embedded $T_g$ in $T(\Delta)$ turns out to be a separable complex submanifold of $T(\Delta)$ (modelled on a separable complex Banach space). That submanifold can be identified as being itself the Teichmüller space of the "universal hyperbolic lamination" $H_\infty$. We will show that $T(H_\infty)$ carries a canonical, genus-independent version of the Weil-Petersson metric, thus bringing back into play the Kähler structure-preserving aspect of the period mapping theory.

The universal laminated surfaces: Let us proceed to explain the nature of the (two possible) "universal laminations" and the complex structures on these. Starting from any closed topological surface, $X$, equipped with a base point, consider the inverse (directed) system of all finite sheeted unbranched covering spaces of $X$ by other closed pointed surfaces. The covering projections are all required to be base point preserving, and isomorphic covering spaces are identified. The inverse limit space of such an inverse system is the "lamination" – which is the focus of our interest.

The lamination $E_\infty$: Thus, if $X$ has genus one, then, of course, all coverings are also tori, and one obtains as the inverse limit of the tower a certain compact topological space – every path component of which (the laminating leaves) – is identifiable with the complex plane. This space $E_\infty$ (to be thought of as the "universal Euclidean lamination") is therefore a fiber space over the original torus $X$ with the fiber being a Cantor set. The Cantor set corresponds to all the possible backward strings in the tower with the initial element being the base point of $X$. The total space is compact since it is a closed subset of the product of all the compact objects appearing in the tower.

The lamination $H_\infty$: Starting with an arbitrary $X$ of higher genus clearly produces the same inverse limit space, denoted $H_\infty$, independent of the initial genus. That is because given any two surfaces of genus greater than one, there is always a common covering surface of higher genus. $H_\infty$ is our universal hyperbolic lamination, whose Teichmüller theory we will consider in this section. For the same reasons as in the case of $E_\infty$, this new lamination is also a compact topological space fibering over the base surface $X$ with fiber again a Cantor set. (It is easy to see that in either case the space of backward strings
starting from any point in \( X \) is an uncountable, compact, perfect, totally-disconnected space – hence homeomorphic to the Cantor set.) The fibration restricted to each individual leaf (i.e., path component of the lamination) is a universal covering projection. Indeed, notice that the leaves of \( H_\infty \) (as well as of \( E_\infty \)) must all be simply connected – since any non-trivial loop on a surface can be unwrapped in a finite cover. [That corresponds to the residual finiteness of the fundamental group of a closed surface.] Indeed, group-theoretically speaking, covering spaces correspond to the subgroups of the fundamental group. Utilising only normal subgroups (namely the regular coverings) would give a cofinal inverse system and therefore the inverse limit would still continue to be the \( H_\infty \) lamination. This way of interpreting things allows us to see that the transverse Cantor-set fiber actually has a group structure. In fact it is the pro-finite group that is the inverse limit of all the deck-transformation groups corresponding to these normal coverings.

**Complex structures**: Let us concentrate on the universal hyperbolic lamination \( H_\infty \) from now on. For any complex structure on \( X \) there is clearly a complex structure induced by pullback on each surface of the inverse system, and therefore \( H_\infty \) itself inherits a complex structure on each leaf, so that now biholomorphically each leaf is the Poincare hyperbolic plane. If we think of a reference complex structure on \( X \), then any new complex structure is recorded by a Beltrami coefficient on \( X \), and one obtains by pullback a complex structure on the inverse limit in the sense that each leaf now has a complex structure and the Beltrami coefficients vary continuously from leaf to leaf in the Cantor-set direction. Indeed, the complex structures obtained in the above fashion by pulling back to the inverse limit from a complex structure on any closed surface in the inverse tower, have the special property that the Beltrami coefficients on the leaves are locally constant in the transverse (Cantor) direction. These ”locally constant” families of Beltrami coefficients on \( H_\infty \) comprise the *transversely locally constant* (written “TLC”) complex structures on the lamination. The generic complex structure on \( H_\infty \), where all continuously varying Beltrami coefficients in the Cantor-fiber direction are admissible, will be a limit of the TLC subfamily of complex structures.

To be precise, a *complex structure* on a lamination \( L \) is a covering of \( L \) by lamination charts (disc) \( \times \) (transversal) so that the overlap homeomorphisms are complex analytic on the disc direction. Two complex structures are *Teichmüller equivalent* whenever they are related to each other by a homeomorphism that is homotopic to the identity through leaf-preserving continuous mappings of \( L \). For us \( L \) is, of course, \( H_\infty \). Thus we have defined the set \( T(H_\infty) \).

Note that there is a distinguished leaf in our lamination, namely the path component of the point which is the string of all the base points. Call this leaf \( l \). Note that all leaves are dense in \( H_\infty \), in particular \( l \) is dense. With respect to the base complex structure the leaf \( l \) gets a canonical identification with the hyperbolic unit disc \( \Delta \). Hence we have the natural
"restriction to $l$" mapping of the Teichmüller space of $H_\infty$ into the Universal Teichmüller space $T(l) = T(1)$. Since the leaf is dense, the complex structure on it records the entire complex structure of the lamination. The above restriction map is therefore actually injective (see [Sul]), exhibiting $T(H_\infty)$ as an embedded complex analytic submanifold in $T(1)$.

Indeed, as we will explain in detail below, $T(H_\infty)$ embeds as precisely the closure in $T(1)$ of the union of the Teichmüller spaces $T(G)$ as $G$ varies over all finite-index subgroups of a fixed cocompact Fuchsian group. These finite dimensional classical Teichmüller spaces lying within the separable, infinite-dimensional $T(H_\infty)$, comprise the TLC points of $T(H_\infty)$.

Alternatively, one may understand the set-up at hand by looking at the direct system of maps between Teichmüller spaces that is obviously induced by our inverse system of covering maps. Indeed, each covering map provides an immersion of the Teichmüller space of the covered surface into the Teichmüller space of the covering surface induced by the standard pullback of complex structure. These immersions are Teichmüller metric preserving, and provide a direct system whose direct limit, when completed in the Teichmüller metric, produces again $T(H_\infty)$. The direct limit already contains the classical Teichmüller spaces of closed Riemann surfaces, and the completion corresponds to taking the closure in $T(1)$.

Let us elaborate somewhat more on these various possible embeddings of $T(H_\infty)$ [which is to be thought of as the universal Teichmüller space of compact Riemann surfaces] within the classical universal Teichmüller space $T(\Delta)$.

**Explicit realizations of $T(H_\infty)$ within the universal Teichmüller space:** Start with any cocompact (say torsion-free) Fuchsian group $G$ operating on the unit disc $\Delta$, such that the quotient is a Riemann surface $X$ of arbitrary genus $g$ greater than one. Considering the inverse limit of the directed system of all unbranched finite-sheeted pointed covering spaces over $X$ gives us a copy of the universal laminated space $H_\infty$ equipped with a complex structure induced from that on $X$. Every such choice of $G$ allows us to embed the separable Teichmüller space $T(H_\infty)$ holomorphically in the Bers universal Teichmüller space $T(\Delta)$.

To fix ideas, let us think of the universal Teichmüller space as in model (a) of Section 2: $T(\Delta) = T(1) = QS(S^1)/Mob(S^1)$ For any Fuchsian group $\Gamma$ define:

$$QS(\Gamma) = \{ w \in QS(S^1) : w\Gamma w^{-1} \text{ is again a Möbius group.} \}$$

We say that the quasisymmetric homeomorphisms in $QS(\Gamma)$ are those that are compatible with $\Gamma$. Then the Teichmüller space $T(\Gamma)$ is $QS(\Gamma)/Mob(S^1)$ clearly sits embedded within $T(1)$. [We always think of points of $T(1)$ as left-cosets of the form $Mob(S^1) \circ w = [w]$ for arbitrary quasisymmetric homeomorphism $w$ of the circle.]
Having fixed the cocompact Fuchsian group $G$, the Teichmüller space $T(H_{\infty})$ is now the closure in $T(1)$ of the direct limit of all the Teichmüller spaces $T(H)$ as $H$ runs over all the finite-index subgroups of the initial cocompact Fuchsian group $G$. Since each $T(H)$ is actually embedded injectively within the universal Teichmüller space, and since the connecting maps in the directed system are all inclusion maps, we see that the direct limit (which is in general a quotient of the disjoint union) in this situation is nothing other than just the set-theoretic union of all the embedded $T(H)$ as $H$ varies over all finite index subgroups of $G$. This union in $T(1)$ constitutes the dense “TLC” (transversely locally constant) subset of $T(H_{\infty})$. Therefore, the TLC subset of this embedded copy of $T(H_{\infty})$ comprises the Möb-classes of all those QS-homeomorphisms that are compatible with some finite index subgroup in $G$.

We may call the above realization of $T(H_{\infty})$ as “the $G$-tagged embedding” of $T(H_{\infty})$ in $T(1)$.

Remark: We see above, that just as the Teichmüller space of Riemann surfaces of any genus $p$ have lots of realizations within the universal Teichmüller space (corresponding to choices of reference cocompact Fuchsian groups of genus $p$), the Teichmüller space of the lamination $H_{\infty}$ also has many different realizations within $T(1)$.

Therefore, in the Bers embedding of $T(1)$, this realization of $T(H_{\infty})$ is the intersection of the domain $T(1)$ in the Bers-Nehari Banach space $B(1)$ with the separable Banach subspace that is the inductive (direct) limit of the subspaces $B(H)$ as $H$ varies over all finite index subgroups of the Fuchsian group $G$. (The inductive limit topology will give a complete (Banach) space; see, e.g., Bourbaki’s ”Topological Vector Spaces".) It is relevant to recall that $B(H)$ comprises the bounded holomorphic quadratic forms for the group $H$.

By Tukia’s results, the Teichmüller space of $H$ is exactly the intersection of the universal Teichmüller space with $B(H)$.

**Remark:** Indeed one expects that the various $G$-tagged embeddings of $T(H_{\infty})$ must be sitting in general discretely separated from each other in the Universal Teichmüller space. There is a result to this effect for the various copies of $T(\Gamma)$, as the base group is varied, due to K. Matsuzaki (preprint – to appear in Annales Acad. Scient. Fennicae). That should imply a similar discreteness for the family of embeddings of $T(H_{\infty})$ in $T(\Delta)$.

It is not hard now to see how many different copies of the Teichmüller space of genus $p$ Riemann surfaces appear embedded within the $G$-tagged embedding of $T(H_{\infty})$. That corresponds to non-conjugate (in $G$) subgroups of $G$ that are of index $(p-1)/(g-1)$ in $G$. This last is a purely topological question regarding the fundamental group of genus $g$ surfaces.

**Modular group:** One may look at those elements of the full universal modular group $Mod(1)$ [quasisymmetric homeomorphism acting by right translation (i.e., pre-composition) on $T(1)$] that preserve setwise the $G$-tagged embedding of $T(H_{\infty})$. Since the modular group
$\text{Mod}(\Gamma)$ on $T(\Gamma)$ is induced by right translations by those QS-homeomorphisms that are in the normaliser of $\Gamma$:

$$N_{qs}(\Gamma) = \{ t \in QS(\Gamma) : t\Gamma t^{-1} = \Gamma \}$$

it is not hard to see that only the elements of $\text{Mod}(G)$ itself will manage to preserve the $G$-tagged embedding of $T(H_\infty)$. [Query: Can one envisage some limit of the modular groups of the embedded Teichmüller spaces as acting on $T(H_\infty)$?]

**The Weil-Petersson pairing:** In [Sul], it has been shown that the tangent (and the cotangent) space at any point of $T(H_\infty)$ consist of certain holomorphic quadratic differentials on the universal lamination $H_\infty$. In fact, the Banach space $B(c)$ of tangent holomorphic quadratic differentials at the Teichmüller point represented by the complex structure $c$ on the lamination, consists of holomorphic quadratic differentials on the leaves that vary continuously in the transverse Cantor-fiber direction. Thus locally, in a chart, these objects look like $\varphi(z, \lambda)dz^2$ in self-evident notation; ($\lambda$ represents the fiber coordinate). The lamination $H_\infty$ also comes equipped with an invariant transverse measure on the Cantor-fibers (invariant with respect to the holonomy action of following the leaves). Call that measure (fixed up to a scale) $d\lambda$. [That measure appears as the limit of (normalized) measures on the fibers above the base point that assign (at each finite Galois covering stage) uniform weights to the points in the fiber.] From [Sul] we have directly therefore our present goal:

**PROPOSITION 10.1:** The Teichmüller space $T(H_\infty)$ is a separable complex Banach manifold in $T(1)$ containing the direct limit of the classical Teichmüller spaces as a dense subset. The Weil-Petersson metrics on the classical $T_g$, normalized by a factor depending on the genus, fit together and extend to a finite Weil-Petersson inner product on $T(H_\infty)$ that is defined by the formula:

$$\int_{H_\infty} \varphi_1 \varphi_2 (\text{Poin})^{-2}dz \wedge d\bar{z} d\lambda$$

where $(\text{Poin})$ denotes the Poincare conformal factor for the Poincare metric on the leaves (appearing as usual for all Weil-Petersson formulas).  

**Remark on Mostow rigidity for $T(H_\infty)$:** The quasisymmetric homeomorphism classes comprising this Teichmüller space are again very non-smooth, since they appear as limits of the fractal q.s. boundary homeomorphisms corresponding to deformations of co-compact Fuchsian groups. Thus, the transversality proved in [NV, Part II] of the finite dimensional Teichmüller spaces with the coadjoint orbit homogeneous space $M$ continues to hold for $T(H_\infty)$. As explained there, that transversality is a form of the Mostow rigidity phenomenon. The formal Weil-Petersson converged on $M$ and coincided with the Kirillov-
Kostant metric, but that formal metric fails to give a finite pairing on the tangent spaces to the finite dimensional $T_g$. Hence there is interest in the above Proposition.

11. The Universal Period mapping and the Krichever map:

We make some remarks on the relationship of $\Pi$ with the Krichever mapping on a certain family of Krichever data. This could be useful in developing infinite-dimensional theta functions that go hand-in-hand with our infinite dimensional period matrices.

The positive polarizing subspace, $T_\mu(W_+)$, that is assigned by the period mapping $\Pi$ to a point $[\mu]$ of the universal Teichmüller space has a close relationship with the Krichever subspace of $L^2(S^1)$ that is determined by the Krichever map on certain Krichever data, when $[\mu]$ varies in (say) the Teichmüller space of a compact Riemann surface with one puncture (distinguished point). I am grateful to Robert Penner for discussing this matter with me.

Recall that in the Krichever mapping one takes a compact Riemann surface $X$, a point $p \in X$, and a local holomorphic coordinate around $p$ to start with (i.e., a member of the “dressed moduli space”). One also chooses a holomorphic line bundle $L$ over $X$ and a particular trivialization of $L$ over the given $(z)$ coordinate patch around $p$. We assume that the $z$ coordinate contains the closed unit disc in the $z$-plane. To such data, the Krichever mapping associates the subspace of $L^2(S^1)$ [here $S^1$ is the unit circle in the $z$ coordinate] comprising functions which are restrictions to that circle of holomorphic sections of $L$ over the punctured surface $X - \{p\}$.

If we select to work in a Teichmüller space $T(g, 1)$ of pointed Riemann surfaces of genus $g$, then one may choose $z$ canonically as a certain horocyclic coordinate around the point $p$. Fix $L$ to be the canonical line bundle $T^*(X)$ over $X$ (the compact Riemann surface). This has a corresponding trivialization via “$dz$”. The Krichever image of this data can be considered as a subspace living on the unit horocycle around $p$. That horocycle can be mapped over to the boundary circle of the universal covering disc for $X - \{p\}$ by mapping out by the natural pencil of Poincaré geodesics having one endpoint at a parabolic cusp corresponding to $p$.

We may now see how to recover the Krichever subspace (for this restricted domain of Krichever data) from the subspace in $H^{1/2}_C(S^1)$ associated to $(X, p)$ by $\Pi$. Recall that the functions appearing in the $\Pi$ subspace are the boundary values of the Dirichlet-finite harmonic functions whose derivatives give the holomorphic Abelian differentials of the Riemann surface. Hence, to get Krichever from $\Pi$ one takes Poisson integrals of the functions in the $\Pi$ image, then takes their total derivative in the universal covering disc, and restricts these to the horocycle around $p$ that is sitting inside the universal cover (as a circle tangent to the boundary circle of the Poincaré disc).

Since Krichever data allows one to create the tau-functions of the $KP$-hierarchy by the well-known theory of the Sato school (and the Russian school), one may now use the
tau-function from the Krichever data to associate a tau (or theta) function to such points of our universal Schottky locus. The search for natural theta functions associated to points of the universal Siegel space $S$, and their possible use in clarifying the relationship between the universal and classical Schottky problems, is a matter of interest that we are working upon.

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