PROJECTIVE CURVATURE TENSOR WITH RESPECT TO ZAMKOVVOY CONNECTION IN LORENTZIAN PARA-SASAKIAN MANIFOLDS

ABHIJIT MANDAL¹, ASHOKE DAS²

¹Raiganj Surendranath Mahavidyalaya, Raiganj, Uttar Dinajpur, West Bengal, India,
Email: abhijit4791@gmail.com
²Raiganj University, Raiganj, Uttar Dinajpur, West Bengal, India,
Email: ashoke.avik@gmail.com

Abstract. The purpose of the present paper is to study some properties of Projective curvature tensor with respect to Zamkovoy connection in Lorentzian Para Sasakian manifold (briefly, LP-Sasakian manifold). We obtain some results on Lorentzian Para-Sasakian manifold with the help of Zamkovoy connection and Projective curvature tensor. Moreover, we study the LP-Sasakian manifold satisfying $P^*(\xi, U)\circ W^*_0 = 0$ and $P^*(\xi, U)\circ W^*_2 = 0$, where $P^*$, $W^*_0$ and $W^*_2$ are Projective curvature tensor, $W_0$—curvature tensor and $W_2$—curvature tensor with respect to Zamkovoy connection respectively.

Key words and Phrases: LP-Sasakian manifolds, Zamkovoy Connection, Projective Curvature tensor

1. Introduction

In 1989, K. Matsumoto [7] first introduced the notion of Lorentzian Para-Sasakian manifolds. Also, in 1992, I. Mihai and R. Rosca [8] introduced independently the notion of Lorentzian Para Sasakian manifolds (briefly, LP-Sasakian Manifolds) in classical analysis. In an $n$—dimensional metric manifold the signature of the metric tensor is the number of positive and negative eigenvalues of the metric. If the metric has $s$ positive eigenvalues and $t$ negative eigenvalues then the signature of the metric is $(s, t)$. For a non-degerate metric tensor $s + t = n$. A Lorentzian manifold is a special case of a semi Riemannian manifold, in which...
the signature of the metric is $(1, n - 1)$ or $(n - 1, 1)$. And the metric $g$ is called here a Lorentzian metric, which is named after the physicist Hendrik Lorentz. The LP-Sasakian manifold was further studied by several authors. We cite ([3], [9]) and their references.

The notion of Projective curvature tensor was first introduced by K. Yano and S. Bochner [13] in 1953. This curvature tensor was further studied by U. C. De and J. Sengupta [4], S. Ghosh [5]. If there exists a one-one mapping between each co-ordinate neighbourhood of a manifold $M$ to a domain of $\mathbb{R}^n$ such that any geodesic of $M$ corresponds to a straight line in $\mathbb{R}^n$, then the manifold $M$ is said to be locally projectively flat. Due to [4], the Projective curvature tensor $P$ of rank four for an $n$-dimensional Riemannian Manifold $M$ is given by

$$P(X, Y, Z, V) = R(X, Y, Z, V) - \frac{1}{n - 1} \left[ S(Y, Z) g(X, V) - S(X, Z) g(Y, V) \right]$$

(1)

for all $X, Y, V \in \chi(M)$, set of all vector fields of the manifold $M$, where $P$ denotes the Projective curvature tensor of type $(0, 4)$ and $R$ denotes the Riemannian curvature tensor of type $(0, 4)$ defined by

$$P(X, Y, Z, V) = g(P(X, Y) Z, V)$$

(2)

$$R(X, Y, Z, V) = g(R(X, Y) Z, V)$$

(3)

where $R$ is the Riemannian curvature tensor of type $(0, 3)$, $P$ is the Projective curvature tensor of type $(0, 3)$ and $S$ denotes the Ricci tensor of type $(0, 2)$.

In 2008, the notion of Zamkovoy connection on para contact manifold was introduced by S. Zamkovoy [14]. Zamkovoy connection was defined as a canonical paracontact connection whose torsion is the obstruction of paracontact manifold to be a para sasakian manifold. This connection was further studied by many researcher. For instance, we see ([2], [1], [6]). For an $n$-dimensional almost contact metric manifold $M$ equipped with an almost contact metric structure $(\phi, \xi, \eta, g)$ consisting of a $(1, 1)$ tensor field $\phi$, a vector field $\xi$, a 1-form $\eta$ and a Riemannian metric $g$, the Zamkovoy connection $(\nabla^*)$ in terms of Levi-Civita connection $(\nabla)$ is given by

$$\nabla^*_X Y = \nabla_X Y + (\nabla_X \eta)(Y) \xi - \eta(Y) \nabla_X \xi + \eta(X) \phi Y$$

(4)

for all $X, Y \in \chi(M)$.

In a LP-Sasakian manifold $M$ of dimension $(n > 2)$, the Projective curvature tensor $P$, $W_0$ Curvature tensor [10], $W_2$—Curvature tensor [12] with respect to the Levi-Civita connection are given by

$$P(X, Y) Z = R(X, Y) Z - \frac{1}{n - 1} \left[ S(Y, Z) X - S(X, Z) Y \right]$$

(5)

$$W_0(X, Y) Z = R(X, Y) Z - \frac{1}{n - 1} \left[ S(Y, Z) X - g(X, Z) QY \right]$$

(6)

$$W_2(X, Y) Z = R(X, Y) Z - \frac{1}{n - 1} \left[ g(Y, Z) QX - g(X, Z) QY \right]$$

(7)
The Projective curvature tensor, $W_0$-Curvature tensor and $W_2$-Curvature tensor with respect to the Zamkovoy connection are given by,

$$P^\ast (X,Y) Z = R^\ast (X,Y) Z - \frac{1}{n-1} [S^\ast (Y,Z) X - S^\ast (X,Z) Y]$$ (8)

$$W_0^\ast (X,Y) Z = R^\ast (X,Y) Z - \frac{1}{n-1} [S^\ast (Y,Z) X - g(X,Z) Q^\ast Y]$$ (9)

$$W_2^\ast (X,Y) Z = R^\ast (X,Y) Z - \frac{1}{n-1} [g(Y,Z) Q^\ast X - g(X,Z) Q^\ast Y]$$ (10)

where $R^\ast$, $S^\ast$ and $Q^\ast$ are Riemannian curvature tensor, Ricci tensor and Ricci operator with respect to Zamkovoy connection $\nabla^\ast$ respectively.

**Definition 1.1.** An $n$-dimensional LP -Sasakian manifold $M$ is said to be generalized $\eta$–Einstein manifold if the Ricci tensor of type (0,2) is of the form

$$S(Y,Z) = k_1 g(Y,Z) + k_2 \eta(Y) \eta(Z) + k_3 \omega(Y,Z)$$ (11)

for all $Y,Z \in \chi(M)$, set of all vector fields of the manifold $M$ and $k_1$, $k_2$ and $k_3$ are scalars and $\omega$ is a 2-form.

**Definition 1.2.** An $n$-dimensional LP-Sasakian manifold $M$ is said to be Projectively flat if $P(X,Y) Z = 0$ for all $X,Y,Z \in \chi(M)$.

**Definition 1.3.** An $n$-dimensional LP-Sasakian manifold $M$ is said to be $\xi$– Projectively flat if $P(X,Y) \xi = 0$ for all $X,Y,Z \in \chi(M)$.

This paper is structured as follows: after introduction, a short description of LP-Sasakian manifold is given in section (2). In section (3), we have discussed LP-Sasakian manifold admitting Zamkovoy connection $\nabla^\ast$ and obtain curvature tensor $R^\ast$, Ricci tensor $S^\ast$, Scalar curvature tensor $r^\ast$, in LP-Sasakian manifold. Section (4) contains Projectively flat LP-Sasakian manifold with respect to the connection $\nabla^\ast$. In section (5) we have discussed Locally Projectively $\phi$–symmetric LP-Sasakian manifold $M$ with respect to $\nabla^\ast$. In section (6) we have discussed a LP-Sasakian manifold satisfying $P^\ast (\xi,U) \circ W_0^\ast = 0$. In section (7) we have discussed a LP-Sasakian manifold satisfying $P^\ast (\xi,U) \circ W_2^\ast = 0$.

2. Preliminaries

An $n$-dimensional differentiable manifold is called a LP-Sasakian manifold if it admits a $(1,1)$ tensor field $\phi$, a vector field $\xi$, a 1-form $\eta$ and a Lorentzian metric $g$ which satisfies

$$\phi Y = Y + \eta(Y) \xi, \eta(\xi) = -1, \eta(\phi X) = 0, \phi \xi = 0$$ (12)

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y)$$ (13)

$$g(X,\phi Y) = g(\phi X,Y), \eta(Y) = g(Y,\xi)$$ (14)

$$\nabla_X \xi = \phi X, \quad g(X,\xi) = \eta(X)$$ (15)

$$\langle \nabla_X \phi \rangle Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi$$ (16)

$\forall X,Y \in \chi(M)$
where \( \nabla \) denotes the operator of covariant differentiation with respect to the Lorentzian metric \( g \).

Let us introduced a symmetric \((0, 2)\) tensor field \( \omega \) such that \( \omega (X, Y) = g(X, \phi Y) \). Also, since the vector field \( \eta \) is closed in LP- Sasakian manifold, we have

\[
(\nabla_X \eta) Y = \omega (X, Y), \omega (X, \xi) = 0, \forall X, Y \in \chi (M)
\]  

(17)

In LP- Sasakian manifold, the following relations also hold:

\[
\begin{align*}
\eta (R(X, Y) Z) &= g(Y, Z) \eta (X) - g(X, Z) \eta (Y) \\
R(X, Y) \xi &= \eta (Y) X - \eta (X) Y \\
R(\xi, Y) Z &= g(Y, Z) \xi - \eta (Z) Y \\
R(\xi, Y) \xi &= \eta (Y) \xi + Y \\
S(X, \xi) &= (n - 1) \eta (X) \\
S(\phi X, \phi Y) &= S(X, Y) + (n - 1) \eta (X) \eta (Y)
\end{align*}
\]

(18)

(19)

(20)

(21)

(22)

(23)

\[ Q\xi = (n - 1) \xi, Q\phi = \phi Q, S(X, Y) = g(QX, Y), S^2(X, Y) = S(QX, Y) \]

(24)

3. Some Properties of LP-Sasakian Manifolds with Respect to Zamkovoy Connection

Using (15) and (17) in (4), we get

\[
\nabla^*_X Y = \nabla_X Y + g(X, \phi Y) \xi - \eta (Y) \phi X + \eta (X) \phi Y
\]

(25)

with torsion tensor

\[
T^* (X, Y) = 2 [\eta (X) \phi Y - \eta (Y) \phi X]
\]

(26)

In view of (4) and (17), we have

\[
(\nabla_X g)(Y, Z) = -2g(Y, \phi Z) \eta (X)
\]

(27)

Putting \( Y = \xi \) in (25)

\[
\nabla^*_X \xi = 2\phi X
\]

(28)

Using (14), (15) and (16) in (25), we obtain

\[
\begin{align*}
\nabla_X (\phi Y) &= \phi (\nabla_X Y) + 2g(X, Y) \xi + \eta (Y) X + \eta (X) \eta (Y) \xi \\
\nabla_X g(Y, Z) &= g(\nabla_X Y, Z) + (Y, \nabla_X Z)
\end{align*}
\]

(29)

(30)

\[
\begin{align*}
\nabla_X g(Y, \phi Z) &= g(\nabla_X Y, \phi Z) + g(Y, \phi \nabla_X Z) + g(X, Z) \eta (Y) + g(X, Y) \eta (Z) + 2\eta (X) \eta (Y) \eta (Z)
\end{align*}
\]

(31)
In view of (25), (29), (30) and (31), we have

\[
\nabla_X^* \nabla_Y^* Z = \nabla_X \nabla_Y Z + g(X, \phi \nabla_Y Z) \xi - \eta(\nabla_Y Z) \phi X + \eta(X) \phi \nabla_Y Z
\]

\[
+ g(\nabla_Y Z, \phi Z) \xi + g(Y, \phi \nabla_X Z) \xi + g(X, Z) \eta(Y) \xi
\]

\[
+ g(X, Y) \eta(Z) \xi + 2 \eta(X) \eta(Y) \eta(Z) \xi + 2g(Y, \phi Z) \phi X
\]

\[
- g(X, \phi Z) \phi Y - \eta(\nabla_X Z) \phi Y - \phi(\nabla_Y Z) \eta(Z) - 2g(X, Y) \eta(Z) \xi - \eta(Y) \eta(Z) \xi - \eta(X) \eta(Z) \xi - 4\eta(Y) \eta(Z) \xi
\]

\[
+ g(X, \phi Z) \phi Z + \eta(\nabla_Y Z) \phi Z + \phi(\nabla_Y Z) \eta(X) + 2g(X, Z) \eta(Y) \xi + \eta(Y) \eta(Z) \xi + \eta(X) \eta(Y) \eta(Z) \xi (32)
\]

Interchanging \(X\) and \(Y\)

\[
\nabla_Y^* \nabla_X^* Z = \nabla_Y \nabla_X Z + g(Y, \phi \nabla_X Z) \xi - \eta(\nabla_X Z) \phi Y + \eta(Y) \phi \nabla_X Z
\]

\[
+ g(\nabla_X Z, \phi Z) \xi + g(X, \phi \nabla_Y Z) \xi + g(Y, Z) \eta(X) \xi
\]

\[
+ g(X, Y) \eta(Z) \xi + 2 \eta(Y) \eta(X) \eta(Z) \xi + 2g(X, \phi Z) \phi Y
\]

\[
- g(Y, \phi Z) \phi X - \eta(\nabla_Y Z) \phi X - \phi(\nabla_X Z) \eta(Z) - 2g(Y, X) \eta(Z) \xi - \eta(X) \eta(Z) \xi - \eta(Y) \eta(Z) \xi - 4\eta(Y) \eta(Z) \xi
\]

\[
+ g(Y, \phi Z) \phi Z + \eta(\nabla_X Z) \phi Z + \phi(\nabla_X Z) \eta(X) + 2g(Y, Z) \eta(X) \xi + \eta(X) \eta(Y) \eta(Z) \xi + 4\eta(Y) \eta(X) \eta(Z) \xi (33)
\]

Also we have

\[
\nabla_{[X,Y]}^* Z = \nabla_{[X,Y]}^* Z + g(\nabla_X Y, \phi Z) \xi - g(\nabla_Y X, \phi Z) \xi - \eta(Z) \phi \nabla_X Y
\]

\[
+ \eta(Z) \phi \nabla_Y X + \eta(\nabla_Y X) \phi Z - \eta(\nabla_X Y) \phi Z (34)
\]

Let \(R^*\) be the Riemannian curvature tensor with respect to Zamkovoy connection and it is defined as

\[
R^*(X, Y) Z = \nabla_X^* \nabla_Y^* Z - \nabla_Y^* \nabla_X^* Z - \nabla_{[X,Y]}^* Z (35)
\]

Using (25), (32), (33) and (34) in (35), we get

\[
R^*(X, Y) Z = R(X, Y) Z + 3g(X, Z) \eta(Y) \xi - 3g(Y, Z) \eta(X) \xi + 3g(Y, \phi Z) \phi X
\]

\[
- 3g(X, \phi Z) \phi Y - \eta(X) \eta(Z) Y + \eta(Y) \eta(Z) X (36)
\]

Consequently one can easily bring out the followings:

\[
\text{S}^*(Y, Z) = S(Y, Z) + (n - 1) \eta(Y) \eta(Z) + 3\psi g(Y, \phi Z) (37)
\]

\[
\text{S}^*(\xi, Z) = S^*(Z, \xi) = 0 (38)
\]

\[
\text{Q}^* Y = QY + (n - 1) \eta(Y) \xi + 3\psi \phi Y (39)
\]

\[
\text{Q}^* \xi = 0 (40)
\]

\[
r^* = r - n + 1 + 3\psi^2 (41)
\]
\[ R^* (X,Y) \xi = 0 \] (42)
\[ R^* (\xi,Y) Z = 4g(\phi Y, \phi Z) \xi \] (43)
\[ R^* (X,\xi) Z = -4g(\phi X, \phi Z) \xi \] (44)

for all \( X,Y,Z \in \chi(M) \), where \( \psi = \text{trace} (\phi) \)

Thus we can state the followings:

**Proposition 3.1.** Let \( M \) be an \( n \)-dimensional LP-Sasakian manifold admitting Zamkovoy connection \( \nabla^* \), then

(i) The curvature tensor \( R^* \) of \( \nabla^* \) is given by (36)
(ii) The Ricci tensor \( S^* \) of \( \nabla^* \) is given by (37)
(iii) The scalar curvature \( r^* \) of \( \nabla^* \) is given by (41)
(iv) The Ricci tensor \( S^* \) of \( \nabla^* \) is symmetric.
(v) \( R^* \) satisfies: \( R^* (X,Y) Z + R^* (Y,Z) X + R^* (Z,X) Y = 0 \).

4. **Projectively flat LP-Sasakian manifold with respect to the Zamkovoy connection**

**Theorem 4.1.** If an \( n \)-dimensional LP-Sasakian manifold \( M \) is Projectively flat with respect to Zamkovoy connection, then it is a generalized \( \eta \)-Einstein manifold.

**Proof.** In view of (8), (36) and (37), the Projective curvature tensor \( P^* \) with respect to the Zamkovoy connection \( \nabla^* \) on a LP-Sasakian manifold \( M \) of dimension \( (n > 2) \) takes the form

\[
P^* (X,Y) Z = R (X,Y) Z + 3g(X,Z) \eta(Y) \xi - 3g(Y,Z) \eta(X) \xi + 3g(Y,\phi Z) \phi X
- 3g(X,\phi Z) \phi Y - \eta(X) \eta(Z) Y + \eta(Y) \eta(Z) X
- \frac{1}{n-1} [S(Y,Z) X + (n-1) \eta(Y) \eta(Z) X + 3\psi g(Y,\phi Z) X]
+ \frac{1}{n-1} [S(X,Z) Y + (n-1) \eta(X) \eta(Z) Y + 3\psi g(X,\phi Z) Y] \tag{45}
\]

Let \( M \) be projectively flat with respect to Zamkovoy connection, then from (45), we get

\[
R (X,Y) Z = -3g(X,Z) \eta(Y) \xi + 3g(Y,Z) \eta(X) \xi - 3g(Y,\phi Z) \phi X
+ 3g(X,\phi Z) \phi Y + \eta(X) \eta(Z) Y - \eta(Y) \eta(Z) X
+ \frac{1}{n-1} [S(Y,Z) X + (n-1) \eta(Y) \eta(Z) X + 3\psi g(Y,\phi Z) X]
- \frac{1}{n-1} [S(X,Z) Y + (n-1) \eta(X) \eta(Z) Y + 3\psi g(X,\phi Z) Y] \tag{46}
\]
Taking inner product of (46) with a vector field $V$, we have

$$
R(X, Y, Z, V) = -3g(X, Z) \eta(Y) \eta(V) + 3g(Y, Z) \eta(X) \eta(V) - 3g(Y, \phi Z) g(\phi X, V)
$$

$$
+ 3g(X, \phi Z) g(\phi Y, V) + \eta(X) \eta(Z) g(Y, V) - \eta(Y) \eta(Z) g(X, V)
$$

$$
+ \frac{1}{n-1} \left[ S(Y, Z) + (n-1) \eta(Y) \eta(Z) + 3\psi g(Y, \phi Z) \right] g(X, V)
$$

$$
- \frac{1}{n-1} \left[ S(X, Z) + (n-1) \eta(X) \eta(Z) + 3\psi g(X, \phi Z) \right] g(Y, V)
$$

(47)

Setting $X = V = \xi$ and using (12), (22) in (47), we get

$$
S(Y, Z) = 4(n-1) g(Y, Z) + 3(n-1) \eta(Y) \eta(Z) - 3\psi \omega(Y, Z)
$$

where $\omega(Y, Z) = g(\phi Y, Z)$.

which shows that $M$ is an $\eta$–Einstein manifold. Hence the theorem is proved.

□

Corollary 4.2. An $n$– dimensional LP-Sasakian manifold $M$ is $\xi$– Projectively flat with respect to Zamkovoy connection iff it is so with respect to Levi-Civita connection.

Proof. Using (5) in (45), we get

$$
P^\ast(X, Y) Z = P(X, Y) Z + 3g(X, Z) \eta(Y) \xi - 3g(Y, Z) \eta(X) \xi
$$

$$
+ 3g(Y, \phi Z) \phi X - 3g(X, \phi Z) \phi Y
$$

$$
- \frac{3\psi}{n-1} \left[ g(Y, \phi Z) X + g(X, \phi Z) Y \right]
$$

(48)

Setting $Z = \xi$ in (48), we get

$$
P^\ast(X, Y) \xi = P(X, Y) \xi
$$

Therefore, $M$ is $\xi$–Projectively flat with respect to Zamkovoy connection iff it is so with respect to Levi-Civita connection.

□

5. Locally Projectively $\phi$–symmetric LP-Sasakian manifolds with respect to Zamkovoy connection

In 1977, Takahashi [11] first studied the concept of locally $\phi$-symmetry on Sasakian manifold. In this section we consider a locally projectively $\phi$-symmetric LP-Sasakian manifolds with respect to the connection $\nabla^\ast$.

Definition 5.1. An $n$–dimensional LP-Sasakian manifold $M$ is said to be locally projectively $\phi$-symmetric with respect to Zamkovoy connection $\nabla^\ast$ if the projective curvature tensor $P^\ast$ with respect to the connection $\nabla^\ast$ satisfies

$$
\phi^2 (\nabla^\ast_W P^\ast)(X, Y) Z = 0
$$

(49)

where $X, Y, Z$ and $W$ are horizontal vector fields on $M$, i.e $X, Y, Z$ and $W$ are orthonormal to $\xi$ on the manifold $M$. 
Theorem 5.2. An \( n \)-dimensional LP-Sasakian manifold \( M \) \( (n > 3) \) is locally projectively \( \phi \)-symmetric with respect to Zamkovoy connection if and only if it is so with respect to the Levi-Civita connection, provided \( \text{trace} (\phi) = 0 \).

Proof. In view of (25), we have

\[
(\nabla^* \phi^* \nabla) (X, Y) Z = (\nabla^* \phi^* \nabla) (X, Y) Z + g(W, \phi \phi^* (X, Y) Z) \xi
- \eta (\phi^* (X, Y) Z) \phi W + \eta(W) \phi \phi^* (X, Y) Z
\] (50)

Taking covariant differentiation of (48) in the direction of \( W \) and considering \( \text{trace} (\phi) = 0 \), we obtain

\[
(\nabla^* \phi^* \nabla) (X, Y) Z = (\nabla \phi^* \nabla) (X, Y) Z
+ 3 [g(X, Z) g(W, \phi Y) - g(Y, Z) g(W, \phi X)] \xi
+ 3 [g(W, Z) \eta(Y) + g(Y, W) \eta(Z) + 2 \eta(W) \eta(Y) \eta(Z)] \phi X
+ 3 g(Y, \phi Z) [g(W, X) \xi + \eta(X) W + 2 \eta(W) \eta(X) \xi]
- 3 [g(W, Z) \eta(X) + g(X, W) \eta(Z) + 2 \eta(W) \eta(X) \eta(Z)] \phi Y
- 3 g(X, \phi Z) [g(W, Y) \xi + \eta(Y) W + 2 \eta(W) \eta(Y) \xi]
\] (51)

In view of (12), (18) and (45), we obtain

\[\eta (\phi^* (X, Y) Z) = g(Y, Z) \eta(X) - g(X, Z) \eta(Y) - 3 g(X, Z) \eta(Y) + 3 g(Y, Z) \eta(X)\]

\[- \frac{1}{n-1} [S(Y, Z) + (n-1) \eta(Y) \eta(Z) + 3 \psi g(Y, \phi Z)] \eta(X) \]

\[+ \frac{1}{n-1} [S(X, Z) + (n-1) \eta(X) \eta(Z) + 3 \psi g(X, \phi Z)] \eta(Y) \] (52)

Using (51) and (52) in (50), we get

\[
(\nabla^* \phi^* \nabla) (X, Y) Z = (\nabla W \phi^* \nabla) (X, Y) Z + 3 [g(X, Z) g(W, \phi Y) - g(Y, Z) g(W, \phi X)] \xi
+ 3 [g(W, Z) \eta(Y) + g(Y, W) \eta(Z) + 2 \eta(W) \eta(Y) \eta(Z)] \phi X
+ 3 g(Y, \phi Z) [g(W, X) \xi + \eta(X) W + 2 \eta(W) \eta(X) \xi]
- 3 [g(W, Z) \eta(X) + g(X, W) \eta(Z) + 2 \eta(W) \eta(X) \eta(Z)] \phi Y
- 3 g(X, \phi Z) [g(W, Y) \xi + \eta(Y) W + 2 \eta(W) \eta(Y) \xi]
+ g(W, \phi \phi^* (X, Y) Z) \xi - g(Y, Z) \eta(X) \phi W + g(X, Z) \eta(Y) \phi W
+ 3 g(Y, Z) \eta(Y) \phi W - 3 g(Y, Z) \eta(X) \phi W
+ \frac{1}{n-1} [S(Y, Z) + (n-1) \eta(Y) \eta(Z) + 3 \psi g(Y, \phi Z)] \eta(X) \phi W
- \frac{1}{n-1} [S(X, Z) + (n-1) \eta(X) \eta(Z) + 3 \psi g(X, \phi Z)] \eta(Y) \phi W
+ \phi P(X, Y) Z \eta(W) + 3 g(Y, \phi Z) X \eta(W) - 3 g(X, \phi Z) Y \eta(W)
+ 3 g(Y, \phi Z) \eta(X) \eta(W) \xi - 3 g(X, \phi Z) \eta(Y) \eta(W) \xi \] (53)

Applying \( \phi^2 \) on both sides of (53) and using (12), we obtain

\[
\phi^2 (\nabla^* \phi^* \nabla) (X, Y) Z = \phi^2 (\nabla \phi^* \nabla) (X, Y) Z
\] (54)
where \( X, Y, Z, W \) are horizontal vector fields and \( \text{trace}(\phi) = 0 \). Hence the theorem is proved. \( \square \)

6. LP-SASAKIAN MANIFOLD SATISFYING \( P^* (\xi, U) \circ W_0^* = 0 \)

**Theorem 6.1.** In an \( n \)-dimensional \((n > 3)\) LP-Sasakian manifold \( M \) admitting Zamkovoy connection \( \nabla^* \), if the condition \( P^* (\xi, U) \circ W_0^* = 0 \) holds, then the equation

\[
S^2 (X, Y) = 4(n - 1) S (X, Y) - 6\psi S (\phi X, Y) + 12(n - 1) \psi g (X, \phi Y) - 9\psi^2 g (X, Y) + 3(n - 1)^2 - 9\psi^2 \eta (X) \eta (Y)
\]

is satisfied on the manifold \( M \), for all \( X, Y \in \chi (M) \).

**Proof.** It can be easily seen from (8) and (9), that

\[
P^* (\xi, U) X = 4g (\phi U, \phi X) \xi - \frac{1}{n - 1} S^* (U, X) \xi \tag{55}
\]

\[
P^* (\xi, Y) \xi = 0, P^* (\xi, \xi) Y = 0, P^* (X, Y) \xi = 0 \tag{56}
\]

\[
W_0^* (X, Y) \xi = \frac{1}{n - 1} \eta (X) Q^* Y, W_0^* (\xi, Y) \xi = - \frac{1}{n - 1} Q^* Y \tag{57}
\]

Let us consider a LP-Sasakian manifold \( M \) satisfying the condition

\[
(P^* (\xi, U) \circ W_0^*) (X, Y) Z = 0.
\]

Then we have

\[
0 = P^* (\xi, U) W_0^* (X, Y) Z - W_0^* (P^* (\xi, U) X, Y) Z - W_0^* (X, P^* (\xi, U) Y) Z - W_0^* (X, Y) P^* (\xi, U) Z \tag{58}
\]

Replacing \( Z \) by \( \xi \) in (58), we get

\[
0 = P^* (\xi, U) W_0^* (X, Y) \xi - W_0^* (P^* (\xi, U) X, Y) \xi - W_0^* (X, P^* (\xi, U) Y) \xi - W_0^* (X, Y) P^* (\xi, U) \xi \tag{59}
\]

Using (55), (56) and (57) in (59), we have

\[
0 = 4S^* (\phi U, \phi Y) \eta (X) \xi - \frac{1}{n - 1} S^* (U, Q^* Y) \eta (X) \xi + 4g (\phi U, \phi X) Q^* Y - \frac{1}{n - 1} S^* (U, X) Q^* Y \tag{60}
\]

The inner product of the equation (60) with vector field \( V \) gives

\[
0 = 4S^* (\phi U, \phi Y) \eta (X) \eta (V) - \frac{1}{n - 1} S^* (U, Q^* Y) \eta (X) \eta (V) + 4g (\phi U, \phi X) S^* (Y, V) - \frac{1}{n - 1} S^* (U, X) S^* (Y, V) \tag{61}
\]

Let \( \{ e_i \} (1 \leq i \leq n) \) be an orthonormal basis of the tangent space at any point of the manifold \( M \). Setting \( U = V = e_i \) and taking summation over \( i (1 \leq i \leq n) \) and using (23), (24), (37), (38) and (39) in (61), we get
\[ S^2(X,Y) = 4(n-1)S(X,Y) - 6\psi S(\phi X,Y) + 12(n-1)\psi g(X,\phi Y) \\
-9\psi^2 g(X,Y) + \left[3(n-1)^2 - 9\psi^2\right] \eta(X) \eta(Y) \] (62)

Hence the theorem is proved. \(\square\)

7. LP-Sasakian manifold satisfying \(P^*(\xi, U) \circ W^*_2 = 0\)

**Theorem 7.1.** In an \(n\)-dimensional LP-Sasakain manifold \(M\) of dimension \((n > 3)\) if the condition \(P^*(\xi, U) \circ W^*_2 = 0\) holds, then the equation

\[ S^2(X,Z) = 4(n-1)S(X,Z) - 6\psi S(\phi X,\phi Z) + 12(n-1)\psi g(X,\phi Z) \\
-9\psi^2 g(X,Z) + \left[3(n-1)^2 - 9\psi^2\right] \eta(X) \eta(Z) \] (63)

is satisfied on \(M\), for all \(X, Z \in \chi(M)\).

**Proof.** Let us consider a LP- Sasakian manifold \(M\) satisfying the condition

\[ (P^*(\xi, U) \circ W^*_2) (X,Y) Z = 0 \] (64)

Then we have

\[
0 = P^*(\xi, U) W^*_2 (X,Y) Z - W^*_2 (P^*(\xi, U) X,Y) Z \\
- W^*_2 (X, P^*(\xi, U) Y) Z - W^*_2 (X,Y) P^*(\xi, U) Z
\] (65)

Replacing \(Y\) by \(\xi\) in (64), we get

\[
0 = P^*(\xi, U) W^*_2 (X,\xi) Z - W^*_2 (P^*(\xi, U) X,\xi) Z \\
- W^*_2 (X, P^*(\xi, U) \xi) Z - W^*_2 (X,\xi) P^*(\xi, U) Z
\] (66)

It is seen that

\[
W^*_2 (X,\xi) Z = -4g(\phi X,\phi Z) \xi - \frac{1}{n-1} \eta(Z) Q^*_X \] (66)

\[
W^*_2 (\xi,\xi) Z = 0, W^*_2 (X,\xi) \xi = \frac{1}{n-1} Q^*_X \] (67)

Using (55), (56), (66) and (67) in (65), we get

\[
0 = \eta(Z) 4g(\phi U,\phi Q^*_X) \xi - \frac{1}{n-1} \eta(Z) S^*(U,Q^*_X) \xi \\
4g(\phi U,\phi Z) Q^*_X - \frac{1}{n-1} S^*(U,Z) Q^*_X \] (68)

The inner product of the equation (68) with vector field \(V\) gives

\[
0 = \left[4S^*(\phi U,\phi X) - \frac{1}{n-1} S^*(U,Q^*_X) \right] \eta(Z) \eta(V) \\
+ \left[4g(\phi U,\phi Z) - \frac{1}{n-1} S^*(U,Z) \right] S^*(X,V) \] (69)
Let \( \{e_i\} (1 \leq i \leq n) \) be an orthonormal basis of the tangent space at any point of the manifold \( M \). Setting \( U = V = e_i \) and taking summation over \( i (1 \leq i \leq n) \) and using (23), (24), (37), (38) and (39) in (69), we get

\[
0 = 4(n-1)S(X,Z) - 6\psi S(X,\phi Z) + 12(n-1)\psi g(X,\phi Z) - 9\psi^2 g(X,Z) - S^2(X,Z) \tag{70}
\]

Using (36) in (70), we have

\[
S^2(X,Z) = 4(n-1)S(X,Z) - 6\psi S(X,\phi Z) + 12(n-1)\psi g(X,\phi Z) - 9\psi^2 g(X,Z) - S^2(X,Z) \tag{71}
\]

Hence the theorem is proved. \( \square \)

**Acknowledgement.** The authors would like to thank the referee for their valuable suggestions to improve the paper.

**REFERENCES**

[1] Biswas, A. and Baishya, K. K., "Study on generalized pseudo (Ricci) symmetric Sasakian manifold admitting general connection", Bulletin of the Transilvania University of Brasov, 12(2) (2020) 233-246.

[2] Blaga, A. M., "Canonical connection on Para Kenmotsu manifold", Novi Sad J. Math, Vol 45, No.2 (2015) 131-142.

[3] De, U. C. and Matsumoto, K. and Shaikh, A. A., "On Lorentzian para-Sasakian manifolds", Rendiconti del Seminario Matematico di Messina, Serie II, Suplemento al n. 3(1999) , 149-158.

[4] De, U. C. and Sengupta, J., "On a Type of SemiSymmetric Metric Connection on an almost-contact metric connection", Facta Universitatis Ser. Math. Inform. 16(2001) 87-96.

[5] Ghosh, S., "On a class of \( (k,\mu) \)-contact manifolds", Bull. Cal. Math. Soc. 102 (2010), 219-226.

[6] Mandal, A. and Das, A., "On M-Projective Curvature Tensor of Sasakian Manifolds admitting Zamkovoy Connection", Adv. Math. Sci. J, 9(2020), no.10, 8929-8940.

[7] Matsumoto, K., "On Lorentzian paracontact manifolds", Bull. of Yamagata Univ., Nat. Sci.12 (1989), p151-156.

[8] Mihai, I. and Rosca, R., "On Lorentzian P-Sasakian manifolds, Classical Analysis", World Scientific Publi. (1992), 155-169.

[9] Ozgur, C., "\( \varphi \)-Conformally flat Lorentzian para Saskian manifolds", Radovi Matematichki,Vol(12). (2003), p99-106.

[10] Pokhariyal, G. P. and Mishra, R. S., "Curvature tensors and their relativistic significance", Yokohama Math. J., 18(1970), 105-108.

[11] Takahashi, T., "Sasakian \( \phi \)-symmetric spaces", Tohoku Mathematical Journal, Second Series, vol. 29, no.1 (1977), pp. 91–113.

[12] Tripathi, M. M. and Gupta, P., "On curvature tensor in K-contact manifold and Sasakian manifold", International Electronic Journal of Mathematics, V-04, (2011), p32-47.

[13] Yano, K. and Bochner, S., "Curvature and Betti numbers", Annals of Mathematics Studies 32, Princeton University Press (1955).

[14] Zamkovoy, S., "Canonical connections on paracontact manifolds", Ann. Global Anal. Geom. 36(1)(2008), 37-60.