Cosmological Perturbations with Multiple Fluids and Fields

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October 27, 2018

Abstract

We consider the evolution of perturbed cosmological spacetime with multiple fluids and fields in Einstein gravity. Equations are presented in gauge-ready forms, and are presented in various forms using the curvature (\(\Phi\) or \(\varphi\chi\)) and isocurvature (\(S_{(ij)}\) or \(\delta\varphi_{(ij)}\)) perturbation variables in the general background with \(K\) and \(\Lambda\). We clarify the conditions for conserved curvature and isocurvature perturbations in the large-scale limit. Evolutions of curvature perturbations in many different gauge conditions are analysed extensively. In the multi-field system we present a general solution to the linear order in slow-roll parameters.

PACS numbers: 98.80.Cq, 98.80.Hw, 98.70.Vc, 04.62+v

1 Introduction

Cosmological models with multiple components of underlying energy-momentum content are not only general but also necessarily appear in many cosmological situations including the early and the late evolution stages of our world models. Assuming the gauge degrees of freedom are properly fixed, for the \(n\) component medium with fluids or fields in Einstein gravity we anticipate a set of \(n\) coupled second-order differential equations for the scalar-type perturbation, a set of \(n\) (in general) coupled first-order differential equations for the vector-type perturbation, and one second-order differential equation for the tensor-type perturbation. The situations of the vector- and tensor-type perturbations are rather trivial to handle, and, in fact, the situation is similar even in some classes of generalized versions of gravity theories [1, 2]. However, the scalar-type perturbation in such a multi-component system is naturally more complicated, and often requires numerical methods. Not only the equations are complicated for the scalar-type perturbation, the behavior and the way of handling the equations depend on our choice of the gauge condition which we need to make. In either analytic or numerical handling of the scalar-type perturbations, it is useful and is often necessary to examine the problems from the perspectives of different gauge conditions.

In the following, for the benefit of future studies, we would like to present various useful forms of the scalar-type perturbation equations considering arbitrary numbers of imperfect fluids and scalar fields with general mutual interactions among the components in Einstein gravity. Equations are presented in the gauge-ready forms which allow to take any of our favorite gauge conditions easily, and also allow easy translation of a solution derived in one gauge condition to the ones in other gauge conditions. We also present closed form equations in terms of the curvature and isocurvature perturbation variables in several useful forms, and derive general solutions if available. We recently have presented similar work on systems with multiple minimally coupled scalar fields [3]. In the present work we present a complementary part now including the multiple fluids as well as the interactions among fluids and fields; we also extend [3] by considering general \(K\) and \(\Lambda\) and the fluid formulation of the field perturbations in §4.

The manuscript is mainly concerned with the formal aspects of perturbations in the multi-component situations. There are some advantages of investigating perturbations in a formal way keeping their most general forms. For example, we emphasize that the basic set of equations for a single component case remains valid even in the multi-component situations with additional equations for the components. Also, by including most general energy-momentum tensor in the fluids system we will show that the same equations remain valid for the mixture of fluids and fields, and sometimes even for the generalized gravity theories. By comparing the fluid system and field system we can clarify the similarity and difference between them. Although, we intentionally have kept the presentation formal, many new results are presented together with the known ones; see §6 for a summary. Previous studies of
Perturbations in the multi-component fluid system in a similar spirit can be found in [3, 4, 5, 6, 7]. We start by introducing our notations, fundamental equations, and strategy in §1. We set \( c \equiv 1 \).

## 2 Perturbed world model and basic equations

We consider an arbitrary number of mutually interacting imperfect fluids in Einstein gravity. The gravitational field equation and the energy-momentum conservation equation are

\[
G_{ab} \equiv 8\pi GT_{ab} - \Lambda g_{ab}, \quad T^{b}_{a; b} = 0. \tag{1}
\]

The energy-momentum tensor can be decomposed into the sum of the individual one as

\[
T_{ab} = \sum_{i} T_{(i)ab}, \tag{2}
\]

and the energy-momentum conservation gives

\[
T^{b}_{(i)a; b} \equiv Q_{(i)a}, \quad \sum_{i} Q_{(i)a} = 0, \tag{3}
\]

where \((i)\) indicates the \( i \)-th component of \( n \) matters (including fluids and fields) with \( i, j, k, \ldots = 1, 2, \ldots, n \).

We consider the general perturbations in the FLRW world model. In the spatially homogeneous and isotropic background we are considering, to the linear order the three different types of perturbation decouple from each other and evolve independently. The presence of tensor-type anisotropic stress contributes as the source or sink of the gravitational wave of the perturbed spacetime. The rotational perturbation of the individual fluid is described by the angular-momentum conservation where the vector-type anisotropic stress and the mutual interaction terms among fluids can work as the source or sink of the rotation of individual component. The most general situations of vector- and tensor-type perturbations are presented in §1. In the following we will consider the scalar-type perturbation only.

A metric with the most general scalar-type perturbation in the FLRW background is

\[
d\xi^{2} = -a^{2} (1 + 2\alpha) \, d\eta^{2} - 2a^{2} \beta_{, a} dx^{a} + a^{2} \left[ g^{(3)}_{\alpha\beta} (1 + 2\varphi) + 2\gamma_{,\alpha\beta} \right] dx^{\alpha} dx^{\beta}, \tag{4}
\]

where \( \bar{\eta} \) is the cosmic scale factor. \( \alpha, \beta, \gamma \) and \( \varphi \) are spacetime dependent perturbed order variables. A vertical bar \( \vert \) indicates a covariant derivative based on \( g^{(3)}_{\alpha\beta} \). Considering the FLRW background and the scalar-type perturbation, the energy-momentum tensor can be decomposed into the collective fluid quantities as

\[
T_{0}^{0} = -(\bar{\mu} + \delta\mu), \quad T_{0}^{a} = -\frac{1}{k} (\mu + p) v_{, a}, \quad T^{a}_{\alpha} = (\bar{\rho} + \delta p) \delta^{a}_{\alpha} + \left( \frac{1}{k^{2}} \nabla^{\alpha} \nabla_{\beta} + \frac{1}{3} \delta^{a}_{\beta} \right) \pi^{(s)}, \tag{5}
\]

where \( \mu(\equiv \bar{\mu} + \delta\mu) \), \( p(\equiv \bar{p} + \delta p) \), \( v \), and \( \pi^{(s)} \) are the energy density, the isotropic pressure, the velocity or flux related variable, and the anisotropic stress, respectively; \( k \) is the comoving wavenumber. In terms of the individual matter’s fluid quantities we have

\[
\bar{\mu} = \sum_{i} \bar{\mu}_{(i)}, \quad \delta\mu = \sum_{i} \delta\mu_{(i)}, \tag{6}
\]

and similarly for \( \bar{\rho}, \delta p, (\mu + p)v \), and \( \pi^{(s)} \); an overbar indicates the background order quantity, and will be ignored unless necessary. The interaction terms among fluids introduced in eq. (3) are decomposed as

\[
Q_{(i)0} \equiv -\alpha \left[ \dot{Q}_{(i)} (1 + \alpha) + \delta Q_{(i)} \right], \quad Q_{(i)a} \equiv J_{(i)a}. \tag{7}
\]

The equations for the background are:

\[
H^{2} = \frac{8\pi G}{3} \bar{\mu} + \frac{\Lambda}{3} - \frac{K}{a^{2}}, \quad \dot{H} = -4\pi G (\mu + p) + \frac{K}{a^{2}}, \tag{8}
\]

\[
\dot{\mu}_{(i)} + 3H (\mu_{(i)} + p_{(i)}) = Q_{(i)}, \tag{9}
\]

where an overdot indicates the time derivative based on \( t \) defined as \( dt \equiv a d\eta \). \( K \) is the normalized spatial curvature, and \( H \equiv \dot{a}/a \). The two equations in eq. (8) follow from \( G_{0}^{0} \) and \( G_{a}^{a} - 3G_{0}^{0} \) components of the field equation in eq.
gauge condition, i.e., in a gauge-ready form [8, 1, 2]:

which also follows from eq. (8).

metric and fluid quantities change as [1, 2]:

In the following we briefly summarize the gauge-ready strategy suggested by Bardeen [8] and elaborated in [1]; see eqs. (46,47) in [3]. Equations (13,14) also remain valid in the generalized gravity where we can reinterpret the fluid quantities as the effective ones; see the Appendix B of [3].

By adding properly eqs. (15,16) over all components of the fluids, and using properties in eq. (6), we get

where \( \chi \equiv \alpha(\beta + a^2) \). \( \chi \), \( \varphi \) and \( \kappa \) correspond to the shear, the three-space curvature and the perturbed expansion of the normal-frame vector field, respectively, see [3, 2] for details. \( \nabla_\alpha \) and \( \Delta \) are the covariant derivative and the Laplacian based on \( \eta_{\alpha\beta}^{(3)} \). Equations (13,14) are: the definition of \( \kappa \), ADM energy constraint (\( G_\alpha^\alpha \) component), tracefree part of ADM propagation (\( G_\beta^\beta - \frac{1}{3}G_\gamma^\gamma G_\alpha^\alpha \) component), and Raychaudhuri equation (\( G_\gamma^\gamma - G_\gamma^\gamma \) component), respectively. Equations (13,14) follow from the energy conservation (\( T_{\alpha\beta}^{(b)} = Q_{(i)\alpha\beta} \)) and the momentum conservation (\( T_{(i)\alpha\beta}^{(b)} = Q_{(i)\alpha\beta} \)), respectively. We note that eqs. (13,14) in the same forms are valid even in a class of generalized gravity theories considered in [3]; see eqs. (46,47) in [3]. Equations (13,14) also remain valid in the generalized gravity where we can reinterpret the fluid quantities as the effective ones; see the Appendix B of [3].

By adding properly eqs. (15,16) over all components of the fluids, and using properties in eq. (6), we get equations for the collective fluid quantities as:

where \( e_{(i)} \) and \( e \) are the entropic perturbations. Equation (17) can be written in another form as

where \( Q_{(i)} \equiv 3H(\mu_{(i)}) + p_{(i)}q_{(i)} \), and we used \( \dot{w}_{(i)} = -3H(c_{(i)}^2 - w_{(i)})(1 + w_{(i)})(1 - q_{(i)}) ; \delta_{(i)} = \delta\mu_{(i)}/\mu_{(i)} \).

In the following we briefly summarize the gauge-ready strategy suggested by Bardeen [3] and elaborated in [1]; see [2] for a recent extension. Under the gauge transformation \( \tilde{x}^a = x^a + \xi^a \) with \( \xi^a \equiv a\xi^a \) (0 = \( \eta \)) the perturbed metric and fluid quantities change as [3, 2]:

\( \tilde{\alpha} = \alpha - \dot{\xi}^t, \quad \tilde{\varphi} = \varphi - H\xi^t, \quad \tilde{\kappa} = \kappa + \left( 3\dot{H} + \frac{\Delta a^2}{a^2} \right) \xi^t, \quad \tilde{\chi} = \chi - \xi^t, \quad \tilde{v} = v - \frac{k}{a} \xi^t, \quad \tilde{\dot{v}}(i) = v_{(i)} - \frac{k}{a} \xi^t, \)

\( \delta\tilde{\mu} = \delta\mu - \dot{\mu}\xi^t, \quad \delta\tilde{\mu}_{(i)} = \delta\mu_{(i)} - \dot{\mu}_{(i)}\xi^t, \quad \delta\tilde{p} = \delta p - \dot{p}\xi^t, \quad \delta\tilde{\rho}_{(i)} = \delta p_{(i)} - \dot{p}_{(i)}\xi^t, \quad \delta\tilde{Q}_{(i)} = \delta Q_{(i)} - \dot{Q}_{(i)}\xi^t, \quad \tilde{\dot{J}}_{(i)} = \dot{J}_{(i)} + Q_{(i)}\xi^t. \)
As the temporal gauge fixing condition we can impose one condition in any of these temporally gauge dependent variables: \( \alpha \equiv 0 \) (synchronous gauge), \( \varphi \equiv 0 \) (uniform-curvature gauge), \( \kappa \equiv 0 \) (uniform-expansion gauge), \( \chi \equiv 0 \) (zero-shear gauge), \( v/k \equiv 0 \) (comoving gauge), \( \delta \mu \equiv 0 \) (uniform-density gauge), \( \delta p \equiv 0 \), \( v/(i)/k \equiv 0 \), \( \delta \mu(i) \equiv 0 \), \( \delta \rho(i) \equiv 0 \), etc. By examining these we notice that, except for the synchronous gauge condition (which fixes \( \alpha = 0 \)), each of the gauge conditions fixes the temporal gauge mode completely. Thus, a variable in such a gauge condition uniquely corresponds to a gauge-invariant combination which combines the variable concerned and the variable used in the gauge condition. As examples, one can recognize the following combinations are gauge-invariant.

\[
\delta \mu_v \equiv \delta \mu - {a \over k} \dot{v}, \quad \varphi_v \equiv \varphi - {a H \over k} v, \quad \varphi_\chi \equiv \varphi - H \chi \equiv - \dot{H} \chi, \quad v_\chi \equiv v - {k \over a} \dot{\chi},
\]

etc. The original study of cosmological perturbation by Lifshitz [3] was made in the synchronous gauge. The zero-shear gauge and the comoving gauge were introduced by Harrison [10] and Nariai [11], respectively. The combination \( \varphi_v \) was first introduced by Lukash [12] and Bardeen [13]. The gauge-invariant combinations \( \varphi_\chi \) and \( v_\chi \) are equivalent to \( \varphi \) and \( v \) in the zero-shear gauge which takes \( \chi \equiv 0 \) as the gauge condition, and \( \delta \mu_v \) is the same as \( \delta \mu \) in the comoving gauge, etc. In this way, we can systematically construct various gauge-invariant combinations for a given variable. Since there exist many gauge-invariant combinations even for a given variable, say for \( \delta \mu \) we could have \( \delta \mu_v \), \( \delta \mu_\varphi \), \( \delta \mu_\chi \), etc.., using our notation we can easily recognize which gauge-invariant combination we are using. Generally, we do not know the suitable gauge condition a priori. The proposal made in [8, 1, 2] is that we write the set of equation without fixing the (temporal) gauge condition and arrange the equations so that we can implement easily various fundamental gauge conditions: eqs. (20-22) are arranged accordingly. We term it a gauge-ready approach.

### 3 Fluid system

#### 3.1 Newtonian analog

The following equations most closely resemble Newtonian hydrodynamic equations in the context of linear perturbation of the Friedmann world model: from eqs. (11-12), eqs. (17,16,12), eqs. (13,18), eqs. (10,12,13), and eq (12), respectively we have

\[
k^2 - 3K \over a^2 \varphi_\chi = 4 \pi G \delta \mu_v, \quad (23)
\]

\[
\delta \mu_v + 3H \delta \mu_v = - {k^2 - 3K \over k^2} \left[ {k \over a} (\mu + p) \varphi_\chi + 2H \dot{\pi}(s) \right], \quad (24)
\]

\[
\dot{\varphi}_\chi + H \varphi_\chi = {k \over a} \left( - \varphi_\chi + \delta \mu_v \right) - 8 \pi G a^2 K \over k \pi(s) - 2 \left( k^2 - 3K \right) \over 3 k^2 \over \mu + p \right), \quad (25)
\]

\[
\dot{\varphi}_\chi + H \varphi_\chi = - 4 \pi G a^2 K \over k (\mu + p) \varphi_\chi - 8 \pi G H a^2 \over k^2 \pi(s), \quad (26)
\]

\[
\alpha_\chi = - \varphi_\chi - 8 \pi G a^2 \over k^2 \pi(s). \quad (27)
\]

Equations (24-25) can be compared with the Poisson, the continuity, and the Euler (Navier-Stokes) equations in Newtonian theory, respectively. By setting \( p = 0 = \pi(s) \) and ignoring \( K = 0 \) we recover the well known equations in the Newtonian perturbation theory with \( - \varphi_\chi \), \( \delta \mu_v \) and \( \varphi_\chi \) corresponding to the Newtonian potential, density and velocity perturbations, respectively. Notice the appearance of pressure terms in rather unexpected places in eqs. (24-25) and none appearing in eq. (23). Although the zero-shear gauge and the comoving gauge were first studied by Harrison and Nariai, respectively [10, 11], these equations using mixed gauge-invariant combinations were first presented by Bardeen in eqs. (4.3-4.8) of [13]. Newtonian correspondence of these fully general relativistic equations and variables was investigated in [14]. In Fig. 1 we present the behaviors of \( \varphi_\chi \), \( \varphi_\chi \) and \( \varphi_\chi \) in the case of three component system with the photon, baryon and the cold dark matter (c). We also compared the behavior with \( \varphi, v \) ((v)\chi) and \( \delta \) (\delta (c)) in the uniform-expansion gauge.

In [14] we noted that in the sub-horizon scale in matter dominated era \( \phi, \delta \), and \( v \) in the uniform-expansion gauge all behave like \( \varphi_\chi \), \( \delta v \), and \( \varphi_\chi \) which most closely resemble the corresponding Newtonian behavior, see also §84 of [15]. We compared the behaviors in the uniform-expansion gauge with the variables with Newtonian behavior in Fig. 1.
Figure 1: In the first figure, evolutions of $\varphi_x$ (black, solid line), $\varphi_{(c)}$ (red, solid line), $v_x$ (blue, dot and long-dash line) are compared with $\varphi_\kappa$ (black, dotted line), $\varphi_{(c)}$ (red, dotted line), $v_\kappa$ (blue, dot and short-dash line). In the second figure, evolutions of $\varphi_x$ (black, solid line), $\varphi_{(c)}$ (red, solid line), $v_{(c)}x$ (blue, dot and long-dash line) are compared with $\varphi_\kappa$ (black, dotted line), $\varphi_{(c)}$ (red, dotted line), $v_{(c)}\kappa$ (blue, dot and short-dash line). The vertical scale indicates the relative amplitude; we normalized $\varphi_\kappa = 1$ in the early radiation era. We took $K = 0 = \Lambda$, $\Omega_{b0} = 0.06$, $H = 65 \text{ km/secMpc}$, and ignored the direct coupling between the baryon and the photon before recombination. Radiation-matter equality ($t_{eq}$) occurs around $\log(a_{eq}/a_0) \approx -4.2$. We took the adiabatic initial condition in the radiation dominated era. As the scale we considered $\lambda_0 \equiv 2 \pi a_0/k = 4 \pi \text{ Mpc}$. We indicate the horizon-crossing epochs of the given scale as $t_1$ where $k/aH = 1$, and $t_2$ where $\lambda/\lambda_H = 2 \pi aH/k = 1$.

From eqs. 14, 17, 18 we can derive
\begin{equation}
\frac{1 + w}{a^2 H} \left[ \frac{H^2}{a(\mu + p)} \left( \frac{a^3 \mu}{H} \delta_v \right) \right] + c_s^2 \frac{k^2}{a^2} \delta_v = - \frac{k^2 - 3K}{a^2} \left[ \frac{e}{\mu} + 2 \frac{1 + w}{H} \left( \frac{a^2 H^2}{k^2} \frac{\pi^{(s)}}{\mu + p} \right) - \frac{2 \pi^{(s)}}{3 \mu} \right]. \tag{28}
\end{equation}

The multi-component version of this equation will be presented later, see eq. (61). This equation is fully relativistic but most closely resembles the well known Newtonian density perturbation equation; only for pressureless fluid our comoving gauge coincides with the synchronous gauge. $\delta_v$ is directly related to the density gradient variable introduced in the covariant approach [10]; see IV and the note added in [17]. By using eq. (23), the equation for $\varphi_x$ can be derived simply from eq. (28), see eq. (53).

In the multi-component situation, although eqs. 24, 25 are still valid for the total fluid quantities, we can also derive the equations for individual component. From eqs. (15, 16, 12) and eqs. (16, 13), respectively, we have
\begin{equation}
\delta \dot{\mu}_{(i)v(i)} + 3H \delta \mu_{(i)v(i)} = - \frac{k^2 - 3K}{k^2} \left[ \frac{k}{a} (\mu_{(i)} + p_{(i)}) \dot{v}_{(i)x} + 2H \pi^{(s)}_{(i)} \right] + 12\pi G \frac{\alpha}{k} (\mu_{(i)} + p_{(i)}) (\mu_{(i)} + p_{(i)}) (v_X - v_{(i)x}) + \delta Q_{(i)v(i)} + Q_{(i)\alpha v(i)} - 3HJ_{(i)v(i)}, \tag{29}
\end{equation}
\begin{equation}
\dot{\nu}_{(i)x} + H \nu_{(i)x} + \frac{Q_{(i)}}{\mu_{(i)} + p_{(i)}} \nu_{(i)x} = \frac{k}{a} \left[ - \varphi_X + \frac{\delta p_{(i)v(i)}}{\mu_{(i)} + p_{(i)}} - 8\pi G \frac{a^2}{k^2} \frac{\pi^{(s)}}{\mu_{(i)} + p_{(i)}} - \frac{2 k^2 - 3K}{3 k^2} \frac{\pi^{(s)}}{\mu_{(i)} + p_{(i)}} - \frac{J_{(i)x}}{\mu_{(i)} + p_{(i)}} \right]. \tag{30}
\end{equation}

By adding eqs. 29, 30 over the components properly we have eqs. 24, 25. Later we will notice that the above equations are not only valid for multiple fluids but also for the system with additional multiple fields.
3.2 Curvature perturbations

We introduce \[18, 13, 17\]

\[
\Phi \equiv \varphi_v - \frac{K/a^2}{4\pi G (\mu + p)} \varphi_\chi,
\]

which becomes \(\varphi_v\), the Lukash variable, for vanishing \(K\). Using eqs. \([22, 24]\) we can show

\[
\Phi = \frac{H^2}{4\pi G (\mu + p) a} \left( \frac{a}{H} \varphi_\chi \right) + \frac{2H^2 a^2}{\mu + p} k^2 \pi(s).
\]

(32)

On the other hand, from eqs. \([28, 23, 32]\) or eqs. \([23, 26, 25]\) we have

\[
\Phi = -\frac{H c_s^2}{4\pi G (\mu + p)} \frac{k^2}{a^2} \varphi_\chi - \frac{H}{\mu + p} \left( e - \frac{2}{3} \pi(s) \right).
\]

(33)

Thus, in a pressureless case, if we could ignore \(e\) and \(\pi(s)\), \(\Phi(x, t) = C(x)\) is an exact solution valid in general scale \([18]\). Combining eqs. \([32, 33]\) we have closed form equations for \(\Phi\) and \(\varphi_\chi\):

\[
\frac{H^2 c_s^2}{(\mu + p) a^2} \left[ (\mu + p) a^3 \Phi \right] + c_s^2 k^2 a^2 \Phi = -\frac{H^2 c_s^2}{(\mu + p) a^3} \left[ \frac{a^3}{c_s^2 H} \left( e - \frac{2}{3} \pi(s) \right) \right] + 2H^2 \frac{\pi^2}{\mu + p} \pi(s),
\]

(34)

\[
\frac{\mu + p}{H} \left[ \frac{H^2}{a(\mu + p)} \left( \frac{a}{H} \varphi_\chi \right) \right] + c_s^2 k^2 a^2 \varphi_\chi = -4\pi G \left( e - \frac{2}{3} \pi(s) \right) - \frac{4\pi G (\mu + p)}{H} \left( \frac{2H^2 a^2}{\mu + p} \pi(s) \right).
\]

(35)

Equation \([34]\) is valid for \(c_s^2 \neq 0\); for \(c_s^2 = 0\) we have eq. \([33]\) instead.

In the case of multiple ideal fluids without direct interactions among them, later we will see that the differences in the sound velocities cause nonvanishing \(e\) term which mixes the curvature perturbation \(\Phi\) with the isocurvature modes, see eq. \([16]\). In \([19]\) we will see that a minimally coupled scalar field has a nonvanishing entropic perturbation, see eq. \([76]\). Thus, we can derive a similar equation for \(\Phi\) as in eq. \([34]\) which is the form valid even in the multi-component case.

Although introduced in a different way, a variable proportional to \(\Phi\) was first introduced by Field and Shepley in 1968 \([18]\), see also \([20]\). In the single component ideal fluid case, using \(v \equiv z \Phi\) with \(z \equiv a\sqrt{\mu + p}/(c_s H)\) and \(t \equiv \frac{\partial}{\partial \eta}\), eq. \([34]\) can be written as

\[
v'' + \left( c_s^2 k^2 - z''/z \right) v = 0,
\]

(36)

which is a well known equation in the cosmological perturbation, \([13, 19, 21, 22]\). This equation can be found in eq. \((45)\) of \([13]\) by Field and Shepley. The variable \(H\) in eq. \((43)\) of \([13]\) and \(\varphi\) in eq. \((3.3)\) of \([19]\) are proportional to \(v\), and \(\phi\) in eq. \((122)\) of \([7]\) is the same as \(-\Phi\).

The \(\zeta\) variable introduced in \([24, 8]\) is the same as \(\varphi_\delta \equiv \varphi + \frac{\delta\mu}{3\pi(\mu + p)}\). From eqs. \([13, 17]\) we can derive

\[
\varphi_\delta = -\frac{k}{3a} \varphi_\chi - \frac{He}{\mu + p}.
\]

(37)

Using eqs. \([13, 18]\) we have

\[
\varphi_\delta + (2 - 3c_s^2) H \varphi_\delta + c_s^2 k^2 a^2 \varphi_\delta = \left( \frac{1}{3} + c_s^2 \right) \frac{k^2}{a^2} \varphi_\chi - \left( \frac{He}{\mu + p} \right) \left[ (2 - 3c_s^2) H^2 + \frac{k^2}{3a^2} \right] \frac{e - 8\pi G}{3(\mu + p)} + \frac{2k^2 - 3K}{9} \frac{\pi(s)}{a^2 \mu + p}.
\]

(38)

We can also derive the closed form equation for \(\varphi_\delta\). From the definition of \(\varphi_\delta\), evaluating it in the comoving gauge, and using eqs. \([23, 31]\) we have

\[
\varphi_\delta(k, t) = \Phi + \frac{k^2}{a^2} \frac{1}{12\pi G (\mu + p)} \varphi_\chi,
\]

(39)

\footnote{Ignorant of this rich history, and even forgetting our own contribution unfortunately, we have rediscovered this important variable \(\Phi\) in \([23]\).}
where $\phi$ and $\phi_\chi$ are given above. For $\phi_\kappa$ from the definition of $\phi_\kappa$ and using eq. (11) and eq. (12) we have

$$
\phi_\kappa = \left(1 + \frac{k^2 - 3K}{12\pi G(\mu + p)a^2}\right)^{-1} \phi_\delta = \phi_\chi - \left(1 + \frac{k^2 - 3K}{12\pi G(\mu + p)a^2}\right)^{-1} \frac{aH}{k} \phi_\chi.
$$

(40)

Using the first expression in eq. (58) we can guess the equation for $\phi_\kappa$. All the equations above are generally valid in the multi-component situation including fluids and fields.

In the single component ideal fluid (later we will see that it also applies for multiple ideal fluids under an adiabatic condition, thus vanishing $\epsilon$) with ignorable $c^2 k^2$ term compared with $z''/z$ term in eq. (59), thus valid in the super-sound-horizon scale, we have general solutions (17), (18).

$$
\Phi(k, t) = C(k) - d(k) \frac{k^2}{4\pi G} \int^t \frac{c^2 H^2}{a^3(\mu + p)} dt,
$$

(41)

$$
\phi_\chi(k, t) = 4\pi G C(k) \frac{H}{a} \int^t \frac{a(\mu + p)}{H^2} dt + d(k) \frac{H}{a},
$$

(42)

where $C(k)$ and $d(k)$ are the coefficients of the growing and decaying solutions, respectively. The coefficients are matched using eqs. (32,33). In order to derive $\Phi$ from $\phi_\chi$ and vice versa to proper order, we need to consider the next leading order terms in the growing solution of $\Phi$ and the decaying solution of $\phi_\chi$; these follow trivially from eqs. (34,35) as

$$
\Phi = C(k) \left\{ \frac{1}{2} + k^2 \left[ \frac{1}{2} \int^t \frac{a(\mu + p)}{H^2} dt \right] \phi_\chi \right\} - d(k) \frac{k^2}{4\pi G} \int^t \frac{c^2 H^2}{a^3(\mu + p)} dt,
$$

(43)

$$
\phi_\chi = 4\pi G C(k) \frac{H}{a} \int^t \frac{a(\mu + p)}{H^2} dt + d(k) H \left\{ \frac{1}{2} + k^2 \left[ \frac{1}{2} \int^t \frac{a(\mu + p)}{H^2} dt \right] \phi_\chi \right\}.
$$

Equation (41) includes $c^2 = 0$ limit; thus for a pressureless fluid we simply have $\Phi = C$, (39). Solution for $\phi_\delta$ follows from eq. (11). Solution for $\phi_\kappa$ follows using the first expression in eq. (40). We emphasize that these solutions are valid in the super-sound-horizon (thus virtually in all scales in the matter dominated era).

Notice that the decaying solutions of $\Phi$, $\phi_\delta$, and $\phi_\kappa$ (ignoring $K$ in this case) are higher order in the large-scale expansion compared with the one of $\phi_\chi$. In the super-horizon scale we have

$$
\phi_\kappa = \phi_\delta = \phi_\kappa = C,
$$

(44)

thus time independent, whereas $\phi_\chi$ evolves in time according to eq. (12). As mentioned in (11), despite its complex behavior compared with the curvature variables in other gauges, $\phi_\chi$ most closely resembles the behavior of the perturbed Newtonian gravitational potential (14). In the $K = 0 = \Lambda$ background dominated by an ideal fluid with constant $w$, in the super-sound horizon, the growing mode behaves as

$$
\phi_\chi = \frac{3 + 3w}{5 + 3w} C,
$$

(45)

thus $\phi_\kappa = \frac{2}{3} \phi_\chi$ and $\frac{3}{5} \phi_\chi$ in the radiation and the matter dominated eras, respectively. We show the solutions in Figs. 2 and 3. The amplitudes of $\phi_\kappa$, $\phi_\kappa$ and $\phi_\chi$ drop as the scale comes inside the sound-horizon; compare Figs. 2 and 3.

In the sub-horizon scale in matter dominated era we have the following behaviors:

(i) $\phi_\kappa$ remains conserved.

(ii) $\phi_\delta$ approaches $\frac{3}{5} \phi_\chi$. This is because the second term in eq. (59) dominates the first, thus $\phi_\delta(\equiv \phi_\kappa + \frac{1}{3(1+w)} \phi_\kappa) \simeq \frac{1}{3} \phi_\kappa$ (in the sub-horizon $K$ term is not important); it also follows from eq. (58).

(iii) $\phi_\kappa$ approaches $\phi_\chi$. This follows from the second expression in eq. (40) and using eq. (23).

We compared these behaviors in Fig. 2. Oscillatory behaviors of $\phi_\kappa$ and others occur as the scale comes inside horizon during the radiation dominated era where the sound-horizon scale is comparable to the (visual) horizon scale. Since the above analytic solutions are valid for the super-sound-horizon we do not anticipate the above
behaviors in (ii) and (iii) to be valid inside horizon in the radiation dominated era. The numerical results confirm (ii) and (iii) even in such a situation: Case (iii) is more understandable by examining the second expression in eq. (40) and eq. (26) without using the large-scale solution.

Although many of the results in this section were presented in our own previous work in [14, 23], which were mainly concerned with the single component fluid case, we emphasize that by keeping the most general form of fluid quantities the equations are valid even in the context of the multiple fluids and fields as well. Figures 2 and 3 show how the conserved quantities in the single component case are affected by the multi-component nature of the more realistic situations.

### 3.3 Isocurvature perturbations

We emphasize that eqs. (23-40) are generally valid for the multi-component situation. In the multi-component case $e$ has additional contributions besides the entropic perturbation from the individual component. Following a pioneering study by Kodama and Sasaki, using eq. (19) we decompose $e$

$$e = e_{\text{rel}} + e_{\text{int}}, \quad e_{\text{int}} \equiv \sum_k c_i^2 \delta \mu(k),$$

$$e_{\text{rel}} \equiv \sum_k (c_i^2 - c_s^2) \delta \mu(k) = \frac{1}{2} \sum_{k,j} \left( \frac{\mu(k) + p(k)}{\mu + p} \right) (c_i^2 - c_s^2) S_{(kl)} + \sum_k \frac{\mu(k) + p(k)}{\mu + p} q(k) c_i^2 \delta \mu,$$

where we introduced the following combinations

$$S_{(ij)} \equiv \frac{\delta \mu(i)}{\mu(i) + p(i)} - \frac{\delta \mu(j)}{\mu(j) + p(j)}.$$

$S_{(ij)}$ is gauge-invariant for $q(i) = 0$, see eq. (21). Our definition differs from a gauge-invariant definition based on the comoving gauge ($v/k = 0$) originally introduced in [14], coinciding only for $q(i) = 0$; for another gauge-invariant definition see [24].

Thus, we notice that the RHSs of eqs. (34,35) vanish for the following situations:

(i) single ideal fluid ($e = 0 = \pi(i)$),

(ii) multiple pressureless ideal fluids, thus $c_i^2 = 0$,

(iii) multiple ideal fluids ($e(i) = 0 = \pi(i)$) with the $c_i^2 = c_j^2$, implying $c_i^2 = c_s^2$, thus $e_{\text{rel}} = 0.$

Figure 2: The first figure shows evolutions of $\varphi_\chi$ (black, solid line), $\varphi_\kappa$ (red, dotted line), $\varphi_\nu$ (blue, dot and short-dash line), $\varphi_{\nu(c)}$ (blue, dot and long-dash line). The same evolutions are reproduced in the second figure which also shows evolutions of $\varphi_\delta$ (red, solid line), $\varphi_{\delta(c)}$ (red, dotted line), $\frac{1}{3} \delta_{\nu(c)}$ (blue, dot and short-dash line), $\frac{1}{3} \delta_{\nu(c)}$ (blue, dot and long-dash line). The conditions used are the same as in Fig. 1.
(iv) multiple ideal fluids with adiabatic perturbation, i.e., $S_{(ij)} = 0$ for $q(i) = 0$; for $q(i) \neq 0$ it is not meaningful to set $S_{(ij)} = 0$ without imposing the gauge condition.

In these situations the large-scale asymptotic solutions in eqs. [41],[42] remain valid, thus $\Phi$ is generally conserved in the super-sound-horizon scale considering general $K$, $\Lambda$ and time-varying $p(\mu)$. The case (iv) shows that the strictly adiabatic mode ($S_{(ij)} = 0$) is conserved in the large-scale limit for general number of fluid components.

In terms of $S_{(ij)}$ and a gauge-invariant combination $v_{(ij)}(\equiv v(i) - v(j))$, from eqs. (20,16) we can derive [1]

$$
\dot{S}_{(ij)} + \frac{3}{2} H \left\{ q(i)(1 + c_{(ij)}^2) + q(j)(1 + c_{(ij)}^2) \right\} S_{(ij)}
$$

$$
+ \left[ q(i)(1 + c_{(ij)}^2) - q(j)(1 + c_{(ij)}^2) \right] \sum_k \frac{\mu(k) + p(k)}{\mu + p} (S_{(ik)} + S_{(jk)}) \right\} \equiv 0
$$

$$
\dot{v}_{(ij)} + H v_{(ij)} - \frac{3}{2} H \left\{ \left[ c_{(i)}^2 + c_{(j)}^2 - q(i)(1 + c_{(i)}^2) - q(j)(1 + c_{(j)}^2) \right] v_{(ij)}
$$

$$
+ \left[ c_{(i)}^2 - c_{(j)}^2 - q(i)(1 + c_{(i)}^2) + q(j)(1 + c_{(j)}^2) \right] \sum_k \frac{\mu(k) + p(k)}{\mu + p} (v_{(ik)} + v_{(jk)}) \right\} \equiv 0
$$

$$
= 3H \left[ c_{(i)}^2 - c_{(j)}^2 - q(i)(1 + c_{(i)}^2) + q(j)(1 + c_{(j)}^2) \right] v + \frac{k}{a} \left[ (c_{(i)}^2 - c_{(j)}^2) \delta \mu \right] + \frac{1}{2} (c_{(i)}^2 + c_{(j)}^2) S_{(ij)}
$$

$$
+ \frac{1}{2} (c_{(i)}^2 - c_{(j)}^2) \sum_k \frac{\mu(k) + p(k)}{\mu + p} (S_{(ik)} + S_{(jk)}) + e_{(ij)} - \frac{2 k^2 - 3 K}{3 k^2} \pi_{(ij)}^{(s)} - J_{(ij)}
$$

where $e_{(ij)}$, $\pi_{(ij)}^{(s)}$, $\delta Q_{(ij)}$, and $J_{(ij)}$ are defined similarly as $S_{(ij)}$:

$$
e_{(ij)} \equiv \frac{e(i)}{\mu(i) + p(i)} - \frac{e(j)}{\mu(j) + p(j)}
$$

$$
\pi_{(ij)}^{(s)} \equiv \frac{\pi_{(ij)}^{(s)}}{\mu(i) + p(i)} - \frac{\pi_{(ij)}^{(s)}}{\mu(j) + p(j)}
$$

$$
\delta Q_{(ij)} \equiv \frac{\delta Q(i)}{\mu(i) + p(i)} - \frac{\delta Q(j)}{\mu(j) + p(j)}
$$

$$
J_{(ij)} \equiv \frac{J(i)}{\mu(i) + p(i)} - \frac{J(j)}{\mu(j) + p(j)}.
$$

Although $S_{(ij)}$, $\delta Q_{(ij)}$ and $J_{(ij)}$ are not gauge-invariant for nonvanishing $q(i)$, eqs. (35,36) are written in a gauge-ready form; these equations were derived in eqs. (35,36) of [1]. Under the comoving gauge condition based on
\( v/k = 0 \), removing \( \alpha_v \) using eq. (18), we can show that eqs. (18,19) reduce to (the corrected) eqs. (5.53,5.57) in [1]; typographical errors in [1] were corrected in eqs. (A.37,A.38) of [3]. We emphasize that, since eqs. (13,16) are valid in the context of generalized gravity theories considered in [2], eqs. (18,43) in the same forms remain valid in the context of the generalized gravity theories as well.

In the case of multiple field system, we have nonvanishing entropic perturbation of the individual component. Later we will show that the relative entropic perturbation \( e(ij) \) causes a curious situation of making eq. (41) simple, and in fact making it identically satisfied, and eq. (18) alone gives the second order system; see eqs. (23,31).

For \( q(ij) = 0 \), \( S(ij) \) is gauge-invariant. In such a case eqs. (18,19,28,31) form a closed set of equations for \( S(ij) \) and \( \delta_\mu \). \( \delta_\mu \) and \( v \) term in eq. (41) combine to give \( \delta_\mu_v \). In this way, \( \delta_\mu_v \) and \( S(ij) \) may provide the curvature and isocurvature perturbations, respectively. Using eqs. (23,33) we can express \( \delta_\mu_v \) in terms \( \Phi \) and \( \varphi_\chi \) as

\[
\delta_\mu_v = \frac{k^2 - 3K}{4\pi G a^2} \varphi_\chi = -\frac{1 - 3K/k^2}{c_s^2} \left( \frac{\mu + p}{H} \Phi + e - \frac{2}{3} \pi(s) \right),
\]

where \( e \) is given in eq. (16), and eqs. (18,43) provide the equation for \( \Phi \) and \( \varphi_\chi \). In this way, \( \Phi \) (or \( \varphi_\chi \)) and \( S(ij) \) may provide the curvature and isocurvature perturbations. Our use of these nomenclature is somewhat misleading in the sense that, actually, the conditions \( S(ij) = 0 \) make the perturbation adiabatic, and the condition \( \Phi = 0 \) (or \( \varphi_\chi = 0 \)) makes the perturbations isocurvature. [In the literature, often \( \varphi_\chi \) is simply related to \( \delta_\mu_v \) as in eqs. (23), and it is related to \( \Phi \) as in eq. (22).] We consider \( \Phi \) (or \( \varphi_\chi \)) and \( S(ij) \) characterize the curvature (adiabatic) and isocurvature perturbations, respectively. For nonvanishing \( q(ij) \) we still have a freedom to impose a temporal gauge-condition depending on the situation, and we can use eqs. (10,14) to replace the metric and matter variables in terms of the relevant curvature and isocurvature perturbation variables. For \( n \)-component fluids we can derive \( n \)-set of coupled second-order differential equations.

In the case we could ignore the mutual interaction among the fluids for the background (thus, set \( q(ij) = 0 \)) eqs. (18,19) give

\[
\ddot{S}(ij) + \left[ 2 - \frac{3}{2}(c_i^2 + c_j^2) \right] H \dot{S}(ij) - \frac{3}{2} H(c_i^2 - c_j^2) \sum_k \frac{\mu(k) + p(k)}{\mu + p} \left( \dot{S}(ik) + \dot{S}(jk) \right)
\]

\[
+ \frac{k^2}{a^2} \left( c_i^2 + c_j^2 \right) \frac{\delta_\mu_v}{\mu + p} - 3 H e(ij) - \delta Q(ij) \right) - \left[ 2 - \frac{3}{2}(c_i^2 + c_j^2) \right] H \left( 3 H e(ij) - \delta Q(ij) \right)
\]

\[
+ \frac{3}{2} H(c_i^2 - c_j^2) \sum_k \frac{\mu(k) + p(k)}{\mu + p} \left[ 3 H (e(ik) + e(jk)) - \delta Q(ik) - \delta Q(jk) \right]
\]

\[
- \frac{k^2}{a^2} \left( e(ij) - \frac{2 k^2}{3} \frac{3K}{k^2} n(ij) - J(ij) \right),
\]

where \( \delta_\mu_v \) is given in eq. (51); equations for \( \delta_\mu_v \), \( \Phi \) and \( \varphi_\chi \) are given in eqs. (28,34,35). Assuming no mutual interaction among fluids (thus, \( e(ij) = 0 = J(ij) \)) and ideal fluids (thus, \( e(ij) = 0 = n(ij) \)), the term in the RHS of eq. (12) becomes

\[
- \frac{k^2}{a^2} (c_i^2 - c_j^2) \frac{\delta_\mu_v}{\mu + p},
\]

which vanishes for

(i) pressureless fluids,

(ii) fluids with \( c_i^2 = c_j^2 \),

(iii) \( \delta_\mu_v = 0 \),

(iv) most importantly, by comparing eqs. (52,54,28,40) or eqs. (52,51,54,40), we notice that, for \( K = 0 \), the RHS becomes negligible in the large-scale limit. Therefore, in the large-scale limit the curvature mode in the RHS of eq. (12) does not contribute to the isocurvature modes, thus, isocurvature modes decouple from the curvature one; this result was known by Kodama and Sasaki, see eq. (2.16) in [1]; for another argument in a two-component situation, see [28]. This should be compared with the scalar field system where we expect less strong decoupling in general; see below eq. (83) and [3].
In the cases of (i) and (ii), assuming ideal fluids \( e_{(ij)} = 0 = \pi^{(s)}_{(ij)} \) and without interactions among fluids \( (\delta Q_{(ij)} = 0 = J_{(ij)}) \), eq. \((53)\) becomes

\[
\ddot{S}_{(ij)} + (2 - 3c_z^2) \frac{H}{c_i} \dot{S}_{(ij)} + c_z^2 \frac{k^2}{a^2} S_{(ij)} = 0,
\]

where \( c_z^2 = c(ij) \). In such a case the RHS of eq. \((53)\) vanishes, thus we have a general large-scale solution for \( \delta v \)

\[
\delta v = \frac{1}{\Omega} \sum_k \Omega(k) \delta(k)v = (k^2 - 3K) \frac{H}{a^3 \mu} \left[ C(k) \int \frac{a(\mu + p)}{H^2} dt + \frac{1}{4\pi G} d(k) \right],
\]

where the coefficients are matched using eqs. \((23, 42)\). If \( w = w_{(i)} = \) constant we have \( c_z^2 = w \), in the large-scale limit the general solution of eq. \((54)\) becomes

\[
S_{(ij)} = C_{ij}(k) + D_{ij}(k) \int \frac{dt}{a^{2-3w}}.
\]

We note that although equations in terms of \( \delta v \) and \( S_{(ij)} \) are decoupled in this case, the equations directly in terms of \( \delta v \) are generally coupled even in the same situation, see eq. \((6)\) as an example; similar situations in the scalar field system can be found below eq. \((60)\).

In the two-component ideal fluid system without mutual interactions, using \( S \equiv S_{(12)} \), eq. \((53)\) becomes

\[
\ddot{S} + H (2 - 3c_z^2) \dot{S} + c_z^2 \frac{k^2}{a^2} S = - \frac{k^2}{a^2} (c_{(1)}^2 - c_{(2)}^2) \frac{\delta \mu_i}{\mu_i + p},
\]

\[
c_z^2 = \frac{c_{(2)}^2 (\mu_{(1)} + p_{(1)}) + c_{(1)}^2 (\mu_{(2)} + p_{(2)})}{\mu_i + p_i}.
\]

Various asymptotic solutions in the case of radiation (ideal fluid with \( p(r) = \frac{1}{3} \mu(r) \)) plus dust (pressureless ideal fluid with \( p_{(c)} = 0 \)) system were studied thoroughly by Kodama and Sasaki in \((3, 8)\).

The adiabatic and the various isocurvature initial conditions in a realistic four component system with the cold dark matter, the massless neutrinos and the tightly coupled baryon and photon in the radiation dominated limit are presented in \((2)\).

### 3.4 Component equations

In this subsection we assume no interaction among fluids, thus \( q_{(i)} = 0 \) and \( \delta Q_{(i)} = 0 = J_{(i)} \). We have

\[
\varphi \delta_{(i)} \equiv \varphi - \frac{H}{\mu_{(i)}} \delta \mu_{(i)} = \varphi + \frac{\delta \mu_{(i)}}{3(\mu_{(i)} + p_{(i)})},
\]

where in the second step we used \( q_{(i)} = 0 \). From eqs. \((10, 13)\) we can derive

\[
\dot{\varphi} \delta_{(i)} = - \frac{k}{3a} \dot{\phi}_{(i)} \chi - \frac{He_{(i)}}{\mu_{(i)} + p_{(i)}}.
\]

Using eqs. \((3, 10)\) we can derive

\[
\varphi \delta_{(i)} + H (2 - 3c_{(i)}^2) \dot{\phi}_{(i)} + c_{(i)}^2 \frac{k^2}{a^2} \varphi_{(i)} = \left( \frac{1}{3} + c_{(i)}^2 \right) \frac{k^2}{a^2} \varphi \chi
\]

\[
- \left( \frac{He_{(i)}}{\mu_{(i)} + p_{(i)}} \right) \left[ (2 - 3c_{(i)}^2)H^2 + \frac{k^2}{3a^2} \right] \frac{e_{(i)}}{\mu_{(i)} + p_{(i)}} + \frac{8\pi G}{3} \pi^{(s)} + \frac{2k^2 - 3K}{9a^2} \frac{\pi^{(s)}}{\mu_{(i)} + p_{(i)}}.
\]

Using \( \varphi \delta_{(c)} = \frac{\delta_{(c)} \varphi}{3(1 + w_{(c)})} \) eq. \((60)\) can be directly converted to a set of equations for \( \delta_{(c)} \varphi \), i.e., equations for \( \delta_{(i)} \) in the uniform-curvature gauge. We plot the evolution of \( \varphi \delta_{(c)} \) for the cold dark matter \( (c) \) in Figs. \((3, 8)\). During the super-horizon scale \( \varphi_{(c)} \) and \( \dot{\varphi}_{(c)} \) are conserved like \( \varphi_{(c)} \) and others. As they come inside the horizon they are dominated by the density parts; in the matter dominated era using eqs. \((60, 38)\) we can show \( \varphi \delta_{(c)} \) and \( \dot{\varphi}_{(c)} \) behave as \( \frac{1}{3} \delta_{(c)} \approx \frac{1}{3} \delta_{(c)} \approx \frac{1}{3} \delta_{(c)} v_{(c)} \).
It is also convenient to have equations for the individual component of density perturbation in a gauge-invariant combination with the velocity of the same component, i.e., in terms of $\delta_{(i)v_{(i)}} \equiv \delta_{(i)} + 3H(a/k)(1 + w_{(i)})v_{(i)}$. By evaluating eqs. (64,15,10) in the $v_{(i)} = 0$ gauge condition we can derive

$$
\ddot{\delta}_{(i)v_{(i)}} + \left[ 2 + 3c_{(i)}^2 - 2w_{(i)} \right] H\dot{\delta}_{(i)v_{(i)}} + \left[ c_{(i)}^2 \frac{k^2}{a^2} - 3\dot{H}w_{(i)} + c_{(i)}^2 + 3H^2(c_{(i)}^2 - 5w_{(i)}) \right] \delta_{(i)v_{(i)}}
- 4\pi G(1 + w_{(i)}) \sum_k \mu(k)(1 + 3c_{(k)}^2)\delta(k)v_{(k)}
= \frac{12\pi G}{\mu_{(i)}} \sum_k \left[ (\mu_{(i)} + p_{(i)})c_{(k)} - (\mu_{(i)} + p_{(i)}) e_{(i)} \right] - \frac{k^2 - 3K e_{(i)}}{a^2 \mu_{(i)}} \frac{1}{a^2} \sum_k \frac{a^2 H^2 \pi_{(i)}}{\mu_{(i)} + p_{(i)}} \cdot \left[ 2 - \frac{k^2 - 3K e_{(i)}}{a^2 \mu_{(i)}} \right],
$$

(61)

with

$$
4\pi G(1 + w_{(i)}) \sum_k \mu(k)(1 + 3c_{(k)}^2)\delta(k)v_{(k)} = 4\pi G(1 + w_{(i)}) \sum_k \mu(k)(1 + 3c_{(k)}^2)\delta(k)v_{(k)} + \frac{12\pi GH}{3H - k^2/a^2} \sum_k \mu_{(k)}(1 + 3c_{(k)}^2) \left[ (1 + w_{(i)}) \left( \delta_{(i)v_{(i)}} - 3Hw_{(i)}\delta_{(i)v_{(i)}} \right) - (1 + w_{(i)}) \left( \delta_{(i)v_{(i)}} - 3Hw_{(i)}\delta_{(i)v_{(i)}} \right) \right],
$$

(62)

where we used

$$
\left( 3\dot{H} - \frac{k^2}{a^2} \right) a \frac{v_{(i)}}{k} = \frac{1}{1 + w_{(i)}} \left( \delta_{(i)v_{(i)}} - 3Hw_{(i)}\delta_{(i)v_{(i)}} \right) - \frac{1}{1 + w_{(i)}} \left( \delta_{(i)v_{(i)}} - 3Hw_{(i)}\delta_{(i)v_{(i)}} \right),
$$

(63)

which follows from eq. (15) using $\kappa_{(i)v_{(i)}} = \kappa + (3\dot{H} - k^2/a^2)(a/k)v_{(i)}$. We have

$$
\delta_v = \frac{1}{\mu} \sum_k \mu(k)\delta(k)v_{(k)}, \quad S_{(ij)} = \frac{\delta_{(i)v_{(i)}}}{\mu_{(i)} + p_{(i)}} - \frac{\delta_{(j)v_{(j)}}}{\mu_{(j)} + p_{(j)}} - 3H a \frac{v_{(i)}}{k}.
$$

(64)

Equations (61,62) provide a closed set of second-order differential equations in terms of $\delta_{(i)v_{(i)}}$’s. This includes the case with one additional scalar field, see eq. (70) where $\delta\mu_{(i)v_{(i)}} = \delta\mu_{(i)}\delta\phi_{(i)} = -\delta\phi_{(i)}\alpha\delta\phi_{(i)}$; for multiple scalar fields we also have nonvanishing $\delta\mu_{(i)}\delta\phi_{(i)}$ in eq. (71). In the single component situation $\delta\mu_{(i)v_{(i)}}$ becomes $\delta_v$, which behaves similarly as in the Newtonian situation [3]. In the single component limit eq. (21) reduces to eq. (28). Using eqs. (1,13,18) one can also derive a closed form equations in terms of $\delta_{(i)v_{(i)}}$.

For pressureless ideal fluids ($w_{(i)} = 0$ and $e_{(i)} = 0 = \pi_{(i)}$) eq. (64) gives

$$
\ddot{\delta}_{(i)v_{(i)}} + 2H\dot{\delta}_{(i)v_{(i)}} + c_{(i)}^2 \frac{k^2}{a^2} \delta_{(i)v_{(i)}} - 4\pi G \sum_k \mu(k)\delta_{(k)v_{(k)}} = 0,
$$

(65)

under the $v_{(i)} = 0$ gauge condition; thus the complete set of equations is made under mixed gauge conditions. Using eqs. (1,13,18) we can show that eq. (65) is valid under the $v_{(j)} = 0$ or $\nu = 0$ gauge conditions as well. Thus we can regard eq. (65) is valid under any single comoving gauge condition of $v_{(j)} = 0$ or $\nu = 0$; the $v_{(c)} = 0$ gauge is the same as the synchronous gauge without remaining gauge mode. In fact, one can derive eq. (65) in the Newtonian context.

In the case of ideal fluids with one component a pressureless fluid ($w_{(c)} = 0$), eq. (65) for $i = c$ becomes

$$
\ddot{\delta}_{(c)v_{(c)}} + 2H\dot{\delta}_{(c)v_{(c)}} + c_{(c)}^2 \frac{k^2}{a^2} \delta_{(c)v_{(c)}} - 4\pi G \sum_k \mu(k)(1 + 3c_{(k)}^2)\delta_{(k)v_{(k)}} = 0.
$$

(66)

Solutions in the case of radiation and dust were studied in [24].

### 4 Scalar field system

Considering our previous study of the multiple scalar field system in [3] the following can be considered as a supplement to [3]. Besides considering general $K$ and $\Lambda$ the equations are presented in the context of fluid formulation
using fluid quantities; in fact, part of the fluid formulation was presented in Sec. 4.4.2 of [1]. By presenting the scalar field system together with the real fluid system we can compare the similarity and difference of the two systems more easily.

A system of minimally coupled scalar fields can be re-interpreted as a system of fluids with special fluid quantities. For an arbitrary number of minimally coupled scalar fields we have

\[
[T]_{ab} = \sum_k \left( \phi^{(k)\cdot a} \phi^{(k)\cdot b} - \frac{1}{2} g_{ab} \phi^{(k)\cdot c} \phi^{(k)\cdot c} \right) - V g_{ab},
\]

(67)

where \( \phi^{(i)} \) indicates \( i \)-th scalar field with \( i, j, k, . . . = 1, 2, . . . n; V = V(\phi^{(k)}) = V(\phi^{(1)}, \phi^{(2)}, . . . , \phi^{(n)}) \). Additionally we have equations of motion for the scalar fields as

\[
\phi^{(i)\cdot \alpha} - V^{(i)} = 0,
\]

(68)

where \( V^{(i)} \equiv \partial V/(\partial \phi^{(i)}) \).

We decompose the scalar fields as \( \phi^{(i)} = \phi^{(i)} + \delta \phi^{(i)} \). To the background order, from eqs. (67,68) we have

\[
\mu = \frac{1}{2} \sum_k \dot{\phi}^{(k)\cdot 2} + V, \quad p = \frac{1}{2} \sum_k \ddot{\phi}^{(k)} - V;
\]

(69)

\[
\ddot{\phi}^{(i)} + 3H \dot{\phi}^{(i)} + V^{(i)} = 0.
\]

(70)

Equations (68,69,70) provide a complete set for the background evolution. In order to match with eq. (9) it is convenient to introduce fluid quantities of the individual field as

\[
\mu^{(i)} + p^{(i)} \equiv \ddot{\phi}^{(i)}, \quad \dot{\mu}^{(i)} \equiv -3H \left( \mu^{(i)} + p^{(i)} \right).
\]

(71)

In this way we have \( Q^{(i)} = 0 \) by definition. From eq. (71) we have

\[
\ddot{x}^{(i)} + \frac{k^2}{a^2} \delta \phi^{(i)} + \sum_j V^{(i)(j)} \delta \phi^{(j)} = \dot{\phi}^{(i)} (\kappa + \dot{\alpha}) + \left( 2 \phi^{(i)} + 3H \dot{\phi}^{(i)} \right) \alpha.
\]

(73)

Equations (10-14,72,73) provide a complete set. Under the gauge transformation we have

\[
\delta \phi^{(i)} = \phi^{(i)} \xi^i.
\]

(74)

Thus, in a single component case, the uniform-field gauge \( \delta \phi \equiv 0 \) gives \( v = 0 \) which is the comoving gauge condition [24].

In order to match with eqs. (15,16) it is convenient to introduce fluid quantities of the individual field as

\[
\delta \mu^{(i)} \equiv \dot{\phi}^{(i)} \delta \phi^{(i)} - \dot{\phi}^{(i)\cdot 2} \alpha + V^{(i)} \delta \phi^{(i)}, \quad \delta p^{(i)} \equiv \dot{\phi}^{(i)} \delta \phi^{(i)} - \dot{\phi}^{(i)\cdot 2} \alpha - V^{(i)} \delta \phi^{(i)}, \quad \dot{v}^{(i)} \equiv \frac{k}{a} \frac{\delta \phi^{(i)}}{\delta \phi^{(i)}} \dot{\phi}^{(i)}, \quad \pi^{(s)} \equiv 0.(75)
\]

In this way, we can show \( e^{(i)} \) in eq. (13) becomes

\[
e^{(i)} = (1 - c^{(i)\cdot 2}) \delta \mu^{(i)} v^{(i)}.
\]

(76)

With these definitions of fluid quantities eqs. (15,16) are satisfied with the following results

\[
\delta Q^{(i)} = \sum_k V^{(i)(k)} \left( \phi^{(k)} \delta \phi^{(i)} - \phi^{(i)} \delta \phi^{(k)} \right), \quad J^{(i)} = 0,
\]

(77)
where we used eq. (73) to derive eq. (15). From eq. (76) we can derive
\[ e_{(ij)} = \frac{1}{2}(2 - c_{(i)}^2 - c_{(j)}^2) \left( S_{(ij)} + 3H \frac{a}{k} v_{(ij)} \right) \]
- \( (c_{(i)}^2 - c_{(j)}^2) \left\{ \frac{\delta \mu}{\mu + p} + 3H \frac{a}{k} v + \frac{1}{2} \sum_k \frac{\mu_{(k)} + p_{(k)}}{\mu + p} \left[ S_{(ik)} + S_{(jk)} + 3H \frac{a}{k} (v_{(ik)} + v_{(jk)}) \right] \right\}. \] (78)

Thus, eq. (49) simplifies to become \( \dot{v}_{(ij)} - 2H v_{(ij)} = (k/a) S_{(ij)} \) which is satisfied identically; these were found in [1]. Introducing a gauge-invariant combination
\[ \delta \phi_{(ij)} = \frac{\delta \phi_{(i)}}{\phi_{(i)}} - \frac{\delta \phi_{(j)}}{\phi_{(j)}}, \] (79)
we have \( v_{(ij)} = (k/a) \delta \phi_{(ij)} \), and eq. (48) leads to
\[
\delta \dot{\phi}_{(i)} - \frac{3}{2} H \left[ (c_{(i)}^2 + c_{(j)}^2) \delta \dot{\phi}_{(i)} + (c_{(i)}^2 - c_{(j)}^2) \sum_k \frac{\mu_{(k)} + p_{(k)}}{\mu + p} \left( \delta \dot{\phi}_{(ik)} + \delta \dot{\phi}_{(jk)} \right) \right] + \left( -3H + \frac{k^2}{a^2} \right) \delta \phi_{(ij)} - \delta Q_{(ij)}
\]
where, from eq. (72) we have
\[
\delta Q_{(ij)} = \sum_k \left( V_{(i)(k)} \frac{\dot{\phi}_{(k)}}{\phi_{(i)}} \delta \phi_{(ik)} - V_{(j)(k)} \frac{\dot{\phi}_{(k)}}{\phi_{(j)}} \delta \phi_{(jk)} \right).
\] (81)

Using \( c_{(i)}^2 = 1 + 2V_{(i)}/(3H\dot{\phi}_{(i)}) \) we have
\[
\frac{1}{a^3 \phi_{(i)} \phi_{(j)}} \left( a^3 \frac{\dot{\phi}_{(i)} \dot{\phi}_{(j)} \delta \phi_{(ij)}}{\phi_{(i)} \phi_{(j)}} \right) - \left( \frac{V_{(i)}}{\phi_{(i)}} - \frac{V_{(j)}}{\phi_{(j)}} \right) \sum_k \frac{\dot{\phi}_{(k)}}{\mu + p} \left( \delta \dot{\phi}_{(ik)} + \delta \dot{\phi}_{(jk)} \right)
\]
\[
+ \left( -3H + \frac{k^2}{a^2} \right) \delta \phi_{(ij)} - \sum_k \left( V_{(i)(k)} \frac{\dot{\phi}_{(k)}}{\phi_{(i)}} \delta \phi_{(ik)} - V_{(j)(k)} \frac{\dot{\phi}_{(k)}}{\phi_{(j)}} \delta \phi_{(jk)} \right)
\]
\[
= 2 \left( \frac{V_{(i)}}{\phi_{(i)}} - \frac{V_{(j)}}{\phi_{(j)}} \right) \frac{\delta \mu_v}{\mu + p}
\] (82)

Using eq. (21), assuming \( K = 0 \), eq. (22) becomes eq. (26) in [3]; in the present case it is valid for general \( K \). From eqs. (19, 22) we have \( e = (1 - c_{(i)}^2) \delta \mu_v - 2 \sum_k V_{(i)(k)} \delta \phi_{(k)v} \). Using eqs. (21, 41) we have
\[
e = (1 - c_{(i)}^2) \delta \mu_v - \frac{2}{\mu + p} \sum_{k,l} V_{(i)(k)} \frac{\dot{\phi}_{(k)}}{\phi_{(i)}} \delta \phi_{(kl)}.
\]
Thus, eq. (41) gives
\[
\delta \mu_v = \frac{k^2 - 3K}{4\pi Ga^2} \frac{\varphi_\chi}{4} = -\frac{1}{1 - 3(1 - c_{(i)}^2)K/k^2} \left( \frac{\mu + p}{H} \dot{\phi} - \frac{2}{\mu + p} \sum_{k,l} V_{(i)(k)} \delta \phi_{(i)\delta \phi_{(kl)}} \right).
\] (83)

The RHS of eq. (22) can be compared with the one of eq. (22), i.e., eq. (53). As mentioned below eq. (53), in the large-scale limit isocurvature modes decouple from the curvature one for the multiple fluid system; in terms of \( \varphi_\chi \) eq. (22) shows \( k^4/a^4 \) factor. However, it is different for the multiple field system; in terms of \( \varphi_\chi \) eq. (22) shows \( k^3/a^3 \) factor. Equation (53) remains valid with vanishing \( \pi_v \), thus
\[
\frac{H^2 c_A^2}{(\mu + p)a^3} \left[ \frac{(\mu + p)a^3}{H^2 c_A^2} \dot{\phi} \right] + c_A^2 \frac{k^2}{a^2} \Phi = \frac{2H^2 c_A^2}{(\mu + p)a^3} \left[ a^3 \left( \frac{a^3}{H^2} \right) \sum_{k,l} V_{(i)(k)} \delta \phi_{(i)} \delta \phi_{(kl)} \right],
\] (84)
\[
\text{where } c_A^2 = 1 - 3(1 - c_{(i)}^2)K/k^2. \text{ This equation can be compared with eq. (24). Since } k^2 \text{ does not imply } c_A^2 k^2 \text{ does not imply } 0, \text{ contrary to the ideal fluid situation, we have to be careful in examining the large-scale limit for } K \neq 0 \text{ case. From eq. (28) or from eq. (41), using eq. (28), we can derive}
\]
\[
\frac{1 + w}{a^2 H} \left[ \frac{H^2}{a(\mu + p)} \left( \frac{a^3}{H} \delta_v \right) \right] + c_A^2 \frac{k^2}{a^2} \delta_v = \frac{k^2 - 3K}{2(\mu + p)} \sum_{k,l} V_{(i)(k)} \delta \phi_{(i)} \delta \phi_{(kl)}.
\] (85)
Equations (82,84) provide a complete set in terms of the curvature mode $\Phi$ (or $\delta\nu$, or $\varphi_\chi$ using eq. (23)) and isocurvature modes $\delta\phi_{(ij)}$. Various analytic solutions for the curvature and the isocurvature modes in a system of scalar fields were studied in §3 assuming $K = 0 = \Lambda$. Here we have considered general $K$ and $\Lambda$ in the background FLRW world model and presented the fluid formulation of the system. Literatures on the subject can be found in §3; some selected ones are in [31], and for recent additions, see [32, 33, 34].

### 4.1 Slow-roll: linear-order solutions

We introduce the following slow-roll parameters (for different definitions, see [32, 33]):

$$\epsilon \equiv \frac{\dot{H}}{H^2}, \quad \epsilon_i \equiv \frac{\ddot{\phi}_i}{H\dot{\phi}_i}, \quad \epsilon_{ij} \equiv \frac{V_{(ij)(i)}}{3H^2}. \quad (86)$$

In §4 of §3 we have presented various solutions of eqs. (82,84) to the zero-th order slow-roll limit of $\epsilon_i$, i.e., ignoring $\epsilon_i$, thus $V_{(i)} = -3H\dot{\phi}_i$. In this case eqs. (82,84) decouple from each other: see eqs. (29,30) in §3. We consider $K = 0 = \Lambda$. The assisted inflation based on multiple fields each with an exponential potential [35, 36].

In addition to zero-th order in $\epsilon_i$, by further assuming zero-th order in $\epsilon$, we have

$$\frac{1}{a^3} \left(a^3\dot{\varphi}_v\right) + \frac{k^2}{a^2}\varphi_v = 0, \quad \frac{1}{a^3} \left(a^3\delta\phi_{(ij)}\right) + \frac{k^2}{a^2}\delta\phi_{(ij)} = 0, \quad (87)$$

where we used that $V_{(i)(k)}$ terms in eq. (82) are of the order $\epsilon$ and $\epsilon_i$. Thus, the curvature and the isocurvature modes are decoupled, and in the large-scale limit we have the general solutions $\varphi_v(k,t) = C(k) - \bar{D}(k) \int^t \frac{1}{a^3} dt, \quad \delta\phi_{(ij)}(k,t) = C_{ij}(k) - D_{ij}(k) \int^t \frac{1}{a^3} dt. \quad (88)$

Notice that the non-transient solutions remain constant in time. From eqs. (22,72) we have

$$\varphi_v = -\frac{H}{\mu + p} \sum_k \dot{\phi}_k \delta\phi_{(k)\nu}. \quad (89)$$

In the single component scalar field, equation for $\delta\phi_{\nu}$, which is $\delta\phi$ in the uniform-curvature gauge, resembles most closely the scalar field equation in the given (i.e., without accompanying metric fluctuations) curved background [37]. Equation (17) in §3 can be written using the slow-roll parameters as

$$\ddot{\delta\phi}_{(i)\nu} + 3H\dot{\delta\phi}_{(i)\nu} + \frac{k^2}{a^2}\delta\phi_{(i)\nu} = \sum_k \left[8\pi G\dot{\phi}_i \dot{\phi}_k \left(3 - \epsilon + \epsilon_k - 3H^2\epsilon_{ik}\right) \delta\phi_{(k)\nu}\right], \quad (90)$$

which is still exact. Thus, notice that although equations for $\varphi_v$ and $\delta\phi_{(ij)}$ are decoupled to the zero-th order in the slow-roll parameters, equations for $\delta\phi_{(ij)}$ are generally coupled to the same order.

Now, to the linear order in the slow-roll parameters, and considering the non-transient solutions of eq. (88), eqs. (83,32) give

$$\frac{1}{a^3} \left(a^3\dot{\varphi}^{(1)}_{(i)}\right) + \frac{k^2}{a^2}\varphi^{(1)}_{(i)} = -\frac{2H^2}{(\mu + p)^2} \sum_{j,k} \frac{1}{a^3} \left(a^3\epsilon_j\right) \left(\phi^{(2)}_{(j)k}\right) \delta\phi_{(jk)} = S^{(1)}, \quad (91)$$

$$\frac{1}{a^3} \left(a^3\dot{\delta\phi}^{(1)}_{(ij)}\right) + \frac{k^2}{a^2}\delta\phi^{(1)}_{(ij)} = 3H^2 \left[\epsilon\delta\phi_{(0)}^{(ij)} + \sum_k \left(\epsilon_{ik}\delta\phi_{(i)k} - \epsilon_{jk}\delta\phi_{(j)k}\right)\right] = S_{ij}^{(1)}, \quad (92)$$

where superscripts (0) and (1) indicate the variables to the zero-th order and the linear order in the slow-roll parameters, respectively. Apparently, we can extend this perturbative procedure to the higher order in the slow-roll parameters. Therefore, to the linear order in the slow-roll parameters the general solutions in the large-scale limit become

$$\varphi_v^{(1)} = -\int^t a^3 \left(\int^t \frac{1}{a^3} dt\right) S^{(1)} dt + \left(\int^t \frac{1}{a^3} dt\right) \int^t a^3 S^{(1)} dt, \quad (93)$$

$$\delta\phi^{(1)}_{(ij)} = -\int^t a^3 \left(\int^t \frac{1}{a^3} dt\right) S_{ij}^{(1)} dt + \left(\int^t \frac{1}{a^3} dt\right) \int^t a^3 S_{ij}^{(1)} dt. \quad (94)$$
We note that to the linear order in the slow-roll parameter $\epsilon_i$, and considering the non-transient solution in the large-scale limit, the RHS of eq. (2), using eq. (3), vanishes. Thus, in such a case the isocurvature modes decouple from the curvature one, whereas the curvature mode is coupled with the isocurvature ones, see eqs. (11,12). Studies in more general situation with nonlinear sigma type coupling are presented in [32] keeping covariant forms in field space.

5 Fluid and field system

The formulations presented in §4 and 5 are valid for a mixture of arbitrary numbers of fluids and scalar fields. Thus, we will not rewrite the complete set of equations again.

As an example, in this section we consider a two-component system of a fluid and a field with a general interaction term between them. As the Lagrangian we consider

$$\mathcal{L} = \sqrt{-g} \left( \frac{1}{16\pi G} R - \frac{1}{2} \dot{\phi}^2 - V(\phi) + L_f \right),$$

(95)

$L_f$ is the fluid part Lagrangian which also depends on the scalar field as $L_f = L_f(\text{fluid}, g_{ab}, \phi)$. The variation with respect to $g_{ab}$ leads to eq. (1) with an identification $T_{ab} = T_{(f)ab} + T_{(\phi)ab}$. The variation with respect to $\phi$ leads to the equation of motion for $\phi$

$$\phi^2 - V(\phi) = -L_{(f)\phi} \equiv \Gamma.$$

(96)

Thus, from eq. (3) we have $Q_{(f)a} = -Q_{(\phi)a}$ and

$$T_{(f)ab} = \frac{\delta Q}{\delta \phi} = L_{(f),\phi,a}$$

(97)

where we used $i,j, \ldots = 1,2 \equiv f, \phi$. As we decompose $\Gamma = \bar{\Gamma} + \delta \Gamma$, from eq. (5) we have

$$\bar{Q}_{(f)} = \bar{\Gamma} \bar{\phi}, \quad \delta Q_{(f)} = \delta \bar{\Gamma} \bar{\phi} + \Gamma (\delta \dot{\phi} - \dot{\phi} \delta \bar{\phi}), \quad J_{(f)} = -\bar{\Gamma} \delta \bar{\phi}.$$

(98)

We have $\bar{Q}_{(f)} = -\bar{Q}_{(f)}$ and similarly for $\delta Q_{(f)}$ and $J_{(f)}$.

By interpreting the fluid quantities properly, equations in §4 and 5 remain valid. For the background we set $\mu = \mu_f + \mu_{(\phi)}$ and similarly for $p$ with

$$\mu_{(\phi)} = \frac{1}{2} \dot{\phi}^2 + V, \quad p_{(\phi)} = \frac{1}{2} \dot{\phi}^2 - V.$$

(99)

Equation (8) remains valid. Equation (8) is valid for $(i) = (f)$, and eq. (9) leads to

$$\ddot{\phi} + 3H \dot{\phi} + V_{,\phi} = -\Gamma.$$

(100)

For the perturbations we set $\delta \mu = \delta \mu_f + \delta \mu_{(\phi)}$ and similarly for $\delta p$, $(\mu + p)v$, and $\pi(s)$, with

$$\delta \mu_{(\phi)} = \phi \delta \dot{\phi} - \dot{\phi}^2 \alpha + V_{,\phi} \delta \phi, \quad \delta p_{(\phi)} = \phi \delta \dot{\phi} - \dot{\phi}^2 \alpha - V_{,\phi} \delta \phi, \quad v_{(\phi)} = \frac{k \delta \phi}{a \phi}, \quad \pi_{(s)} = 0.$$

(101)

Equations (10), (11) are valid for collective fluid quantities. Equations (12,13) are valid for the fluid with $(i) = (f)$, and eq. (12) leads to

$$\ddot{\phi} + 3H \dot{\phi} + \frac{k^2}{a^2} \delta \phi + V_{,\phi,\phi} \delta \phi = \dot{\phi} (\kappa + \dot{\alpha}) + \left( 2 \dot{\phi} + 3H \phi \right) \alpha - \delta \Gamma.$$

(102)

After choosing the temporal gauge condition we can derive a fourth-order differential equation, or a set of coupled second-order differential equations. As an example, we derive one form of such a set of equations based on a gauge condition $v_{(f)}/k = 0$ which is the comoving gauge based on the fluid velocity; in the following we simply write $v_{(f)} = 0$ as the gauge condition. Under our gauge condition $v_{(f)} = 0$ the perturbation variables are gauge invariant, i.e., $\delta_{(f)}(\equiv \delta \mu_{(f)}/\mu_{(f)}) = \delta_{(f)}v_{(f)}$ and $\delta \phi = \delta \phi_{v_{(f)}}$, where, from eqs. (2,3,4), we have:

$$\delta_{(f)}v_{(f)} \equiv \delta_{(f)} - \frac{\dot{\mu}_{(f)}}{\mu_{(f)}} \frac{a}{k} v_{(f)} , \quad \delta \phi_{v_{(f)}} \equiv \delta \phi - \frac{\dot{\phi}}{k} v_{(f)}.$$

(103)
We assume an ideal fluid with \( w \equiv p_f/\rho_f = \text{constant} \). Using eq. (16) for the fluid component we can express \( \alpha \) in terms of \( \delta(f) \) and \( \delta \phi \). Using eq. (15) we can express \( \kappa \) in terms of \( \delta(f) \) and \( \delta \phi \). Thus, eqs. (14,102) lead to

\[
\ddot{\delta}(f) + (2 - 3w)H \dot{\delta}(f) + \left[ w \frac{k^2}{a^2} - 6w \left( H + H^2 \right) - 4\pi G \mu(f)(1 + w)(1 + 3w) \right] \delta(f) \\
= 8\pi G \left[ 2w \phi^2 \delta(f) + (1 + w) \left( 2 \dot{\delta}\phi - V,\phi \delta\phi \right) \right] \\
- \frac{1}{a^2} \left\{ \frac{\Gamma \delta(f)}{\mu(f)} - \frac{\phi \delta \Gamma}{\mu(f)} - \frac{\Gamma}{\mu(f)} \left( \dot{\delta}\phi + 3H \delta\phi \right) \right\} + \left( 3H + 16\pi G \phi^2 - \frac{k^2}{a^2} \right) \frac{\Gamma}{\mu(f)} \delta\phi, \\
\frac{\Gamma}{\mu(f)} \delta\phi + 3H \delta \phi + \frac{k^2}{a^2} + V,\phi \delta \phi \right\} \delta\phi = \frac{1}{1 + w} \left[ (1 - w) \dot{\delta}\phi_f - 2w \left( \ddot{\phi} + 3H \dot{\phi} \right) \delta(f) \right] + \frac{\Gamma}{\mu(f)} \delta\phi. \\
- \left[ 1 + \frac{\phi^2}{(1 + w)\mu(f)} \right] \delta\Gamma + \frac{1}{1 + w} \left[ - \frac{\Gamma}{\mu(f)} \delta\phi - \phi \left( \frac{\Gamma \delta\phi}{\mu(f)} \right) + 2 \left( \ddot{\phi} + 3H \dot{\phi} \right) \frac{\Gamma}{\mu(f)} \delta\phi \right].
\]

Although the RHS of eq. (104) contains \( \delta\phi \) term, it can be replaced by using eq. (105). Notice that eqs. (104,105) are valid for general \( K \) and \( \Lambda \). The LHS of eq. (104) is the familiar density perturbation in the comoving gauge derived in [11, 13], and the LHS of eq. (105) is also the familiar perturbed scalar field equation without the metric perturbation; compare with eq. (102). Thus, our \( \nu(f) = 0 \) gauge choice allows simple equations for the uncoupled parts of the system.

This set of equations or eqs. (61,76) will be useful to handle the following situations:
(i) Warm inflation scenario with nonvanishing interaction term, \( \Gamma \) and \( \delta\Gamma \), between the field and the radiation \( (w = \frac{1}{3}) \).
(ii) Time-varying cosmological constant simulated using the scalar field, often called a quintessence; in this case we may ignore the direct interaction between the fluid and the field \( \Gamma = 0 = \delta\Gamma \). An application of eqs. (104,105) to a system with an exponential type field potential is made in [10].

6 Summary

In this paper we have investigated aspects of scalar-type cosmological perturbation in the context of multiple numbers of mutually interacting imperfect fluids and minimally coupled scalar fields in Einstein gravity. Equations are presented using the curvature (\( \Phi \) or \( \varphi \)) and isocurvature (\( S_{(ij)} \) or \( \delta\phi_{(ij)} \)) perturbation variables. It looks there exists no clear consensus about the curvature/isocurvature decomposition of perturbation in the literature. Since the equations are all coupled, such a decomposition may have clearer meaning in setting up the initial conditions.

In \[8\] and \[9\] we have shown that the equations can be meaningfully classified into the two definitions of the curvature (\( \Phi \) or \( \varphi \)) and isocurvature (\( S_{(ij)} \) or \( \delta\phi_{(ij)} \)) modes; see below eq. (31). We have shown that either \( \Phi \) or \( \varphi \) can characterize the curvature mode depending on situations.

We have presented the equations in a gauge-ready form where we have a freedom to choose one temporal gauge condition from the variables in eqs. (21,23) or the linear combinations of them. In the general situation, eqs. (14,18) provide a complete set. In the case of scalar fields we also have eqs. (7,22). In terms of the curvature and isocurvature perturbations, eqs. (14,18,19,20,21) with eqs. (14,14) provide a complete set, and in the special case with multiple fields eqs. (23,24) or eqs. (61,70) provide complete sets. In \[8\] we have shown how to consider the case with mutual interaction among fluids and fields.

Some new results found in this paper are the following:
(i) Gauge-ready formulation of the fluid-field system in Einstein gravity, \[8\] and \[8\].
(ii) Several useful forms of the individual fluid equations, eqs. (23,23) and \[34\].
(iii) Analyses of various curvature perturbations in realistic situations of multi-component system, \[8\].
(iv) Equations (14,18) are valid even in a class of generalized gravity theories, see below eq. (14,14) and eq. (34).
(v) Extension of fluid formulation of the multiple field system, \[8\].
(vi) In the scalar field system the isocurvature modes are less decoupled from the curvature mode in the large-scale limit compared with the fluids system, see below eq. (33). The couplig term vanishes to the linear order in slow-roll expansion, see below eq. (14).
(vii) Solutions for \( c_s^2 = c_e^2 \) fluids, eqs. (14,56).
(viii) Solutions valid to the linear order in the slow-roll parameters, \[4,1\].
The sets of equations in §5 may be useful to analyze the evolution of structures in the world models with quintessence and in the warm inflation scenario. However, these sets of equations are the ones derived in certain gauge conditions, and when we encounter new problems we believe it is best to go back to the original set of equations in the gauge-ready form and see whether we could gather any new perspective from other gauge conditions as well: the set is in eqs. (10-18,73). Corresponding set of equations in a gauge-ready form which is applicable in a wide class of generalized gravity theories including contributions from the kinetic components based on Boltzmann equations is presented in [1, 2]. Specific applications will be made in future occasions.

Acknowledgments

We thank Winfried Zimdahl for suggesting the study and for his useful discussions and comments during the work. We also wish to thank Stefan Groot Nibbelink and David Wands for useful comments. HN was supported by grant No. 2000-0-113-001-3 from the Basic Research Program of the Korea Science and Engineering Foundation. JH was supported by the Korea Research Foundation Grants (KRF-2000-013-DA004 and 2000-015-DP0080).

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