EQUIVALENCE OF SHARP TRUDINGER-MOSER-ADAMS INEQUALITIES

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Abstract. Improved Trudinger-Moser-Adams type inequalities in the spirit of Lions were recently studied in [21]. The main purpose of this paper is to prove the equivalence of these versions of the Trudinger-Moser-Adams type inequalities and to set up the relations of these Trudinger-Moser-Adams best constants. Moreover, using these identities, we will investigate the existence and nonexistence of the optimizers for some Trudinger-Moser-Adams type inequalities.

1. Introduction.

1.1. Trudinger-Moser inequality. Motivated by the applications to the prescribed Gauss curvature problem on two dimensional sphere $S^2$, J. Moser proved in [29] an exponential type inequality on $S^2$ with an optimal constant. In the same paper of Moser [29], he sharpened an inequality on any bounded domain $\Omega$ in the Euclidean space $\mathbb{R}^N$ studied independently by Pohozaev [30], Trudinger [35] and Yudovich [36], namely the embedding $W^{1,N}_0(\Omega) \subset L^{\varphi}(\Omega)$, where $L^{\varphi}(\Omega)$ is the Orlicz space associated with the Young function $\varphi(t) = \exp\left(\alpha |t|^{N/(N-1)}\right) - 1$ for some $\alpha > 0$. More precisely, using the Schwarz rearrangement, Moser proved the following inequality in [29] (see also [3, 10]):

**Theorem A (Trudinger-Moser inequality for finite-volume domain).** Let $\Omega$ be a domain with finite measure in Euclidean $N$-space $\mathbb{R}^N$, $N \geq 2$ and $0 \leq \beta < N$. Then there exists a constant $\alpha_N = N^2 \omega_{N-1}^{N-1}$, where $\omega_{N-1}$ is the area of the surface of the unit $N$-ball, such that

$$
\sup_{u \in W^{1,N}_0(\Omega) : \int_{\Omega} |\nabla u|^N \, dx \leq 1} \frac{1}{\int_{\Omega} |x|^{-\beta} \, dx} \int_{\Omega} \exp\left(\alpha_N \left( \frac{1 - \beta}{N} \right) |x|^{N/(N-1)} \right) \, dx \, |x|^\beta < \infty. \quad (1)
$$

The constant $\alpha_N$ is optimal in the sense that if we replace $\alpha_N$ by any number $\alpha > \alpha_N$, then the above supremum is infinite.

Moser used the following symmetrization argument: every function $u$ is associated to a radially symmetric function $u^*$ such that the sublevel-sets of $u^*$ are balls...
with the same area as the corresponding sublevel-sets of $u$. Moreover, $u$ is a positive and non-increasing function defined on $B_R(0)$ where $|B_R(0)| = |\Omega|$. Hence, by the layer cake representation, we can have that
\[
\int_{\Omega} f(u) \, dx = \int_{B_R(0)} f(u^*) \, dx
\]
for any function $f$ that is the difference of two monotone functions. In particular, we obtain
\[
\|u\|_p = \|u^*\|_p;
\]
\[
\int_{\Omega} \exp \left( \alpha \frac{|u|^{N-1}}{N} \right) \, dx = \int_{B_R(0)} \exp \left( \alpha \frac{|u^*|^{N-1}}{N} \right) \, dx.
\]
Moreover, the well-known Pólya-Szegö inequality
\[
\int_{B_R(0)} |\nabla u^*|^{p} \, dx \leq \int_{\Omega} |\nabla u|^{p} \, dx
\]
plays a crucial role in the approach of J. Moser.

As far as the existence of extremal functions of Moser’s inequality, the first breakthrough was due to the celebrated work of Carleson and Chang [5] in which they proved that the supremum
\[
\sup_{u \in W^{1,N}_0(\Omega), \int_{\Omega} |\nabla u|^N \, dx \leq 1} \frac{1}{|\Omega|} \int_{\Omega} \exp \left( \alpha_N \frac{|u|^{N-1}}{N} \right) \, dx
\]
can be achieved when $\Omega$ is an Euclidean ball. This result came as a surprise because it has been known that the Sobolev inequality does not have extremal functions supported on any finite ball. Subsequently, existence of extremal functions has been established on arbitrary domains in [7, 8, 12, 27], and on Riemannian manifolds in [24], etc.

We note when the volume of $\Omega$ is infinite, the Trudinger-Moser inequality (1) becomes meaningless. Thus, it becomes interesting and nontrivial to extend such inequalities to domains with infinite measure. Here we state the following such results in the Euclidean spaces (see [1, 4, 9, 21, 26, 31]):

**Theorem B (Trudinger-Moser inequality for infinite-volume domain).** Let $0 \leq \beta < N$ and $M > 1$. Then we have
\[
STM_\beta(\alpha) = \sup_{\|\nabla u\|_N \leq 1} \frac{1}{\|u\|_N^{N-\beta}} \int_{\mathbb{R}^N} \phi_{N,1} \left( \alpha \left( 1 - \frac{\beta}{N} \right) \frac{|u|^{N-1}}{|x|^\beta} \right) \, dx < \infty \text{ for } \alpha < \alpha_N.
\]
\[
LTM_{\beta,M}(\alpha) = \sup_{\|\nabla u\|_N \leq 1} \frac{1}{\|u\|_N^{N-\beta}} \int_{\mathbb{R}^N} \phi_{N,1} \left( M \left( \frac{\alpha}{N} \left( 1 - \frac{\beta}{N} \right) \frac{|u|^{N-1}}{|x|^\beta} \right) \right) \, dx
\]
\[
< \infty \text{ for } \alpha \leq \alpha_N.
\]
Here
\[
\phi_{N,1}(t) = e^t - \sum_{j=0}^{N-2} \frac{t^j}{j!}.
\]
Moreover, the constant \( \alpha_N \) is the best possible. As a consequence, there exists a constant \( C = C(N, \beta) > 0 \) such that

\[
TM_\beta (\alpha) \leq ITM_{\beta, M} (\alpha) \leq C(N, \beta),
\]

for all \( 0 \leq \alpha \leq \alpha_N \), where

\[
ITM_{\beta, M} (\alpha) = \sup_{u \in W^{1, N}(\mathbb{R}^N), \|\nabla u\|_N + \|u\|_N \leq 1} \int_{\mathbb{R}^N} \phi_{N, 1} \left( \frac{M}{M - 1 + \|\nabla u\|_N} \right) \frac{\alpha (1 - \beta / N)}{\|u\|^N} \frac{\alpha N}{\alpha N - 1} \frac{\alpha N - 1}{\alpha N - 1} \frac{\|u\|^N}{\|u\|^N} \, dx.
\]

\[
TM_\beta (\alpha) = \sup_{u \in W^{1, N}(\mathbb{R}^N), \|\nabla u\|_N + \|u\|_N \leq 1} \int_{\mathbb{R}^N} \phi_{N, 1} \left( \frac{1 - \beta / N}{\|u\|^N} \right) \frac{\alpha N}{\alpha N - 1} \frac{\alpha N - 1}{\alpha N - 1} \frac{\|u\|^N}{\|u\|^N} \, dx.
\]

The constant \( \alpha_N \) in the above supremums is sharp.

In the recent papers [6, 22], it was showed that

**Theorem C.** Let \( 0 \leq \beta < N \). Then there exist positive constants \( c = c(N, \beta) \) and \( C = C(N, \beta) \) such that when \( \alpha \) is close enough to \( \alpha_N \):

\[
\frac{c(N, \beta)}{\left( 1 - \left( \frac{\alpha}{\alpha_N} \right)^{N-1} \right)^{(N-\beta)/N}} \leq STM_\beta (\alpha) \leq \frac{C(N, \beta)}{\left( 1 - \left( \frac{\alpha}{\alpha_N} \right)^{N-1} \right)^{(N-\beta)/N}}.
\]

Also, for \( a, b > 0 \), denote

\[
TM_{a, b, \beta} (\alpha) = \sup_{\|\nabla u\|_N + \|u\|_N \leq 1} \int_{\mathbb{R}^N} \phi_{N, 1} \left( \frac{1 - \beta / N}{\|u\|^N} \right) \frac{\alpha N}{\alpha N - 1} \frac{\alpha N - 1}{\alpha N - 1} \frac{\|u\|^N}{\|u\|^N} \, dx.
\]

Then \( TM_{a, b, \beta} (\alpha_N) < \infty \) if and only if \( b \leq N \). The constant \( \alpha_N \) is sharp. Moreover, we have the following identities:

\[
TM_{a, b, \beta} (\alpha) = \sup_{\alpha \in (0, \alpha_N)} \left( 1 - \left( \frac{s}{\alpha} \right)^{N-1} \right)^{N-\beta} \frac{N-\beta}{N-1} \frac{N-\beta}{N-1} \frac{N-\beta}{N-1} \frac{\alpha}{\alpha} \frac{\alpha}{\alpha} \frac{\alpha}{\alpha} \frac{STM_\beta (s)}{STM_\beta (s)},
\]

\[
TM_\beta (\alpha) = \sup_{s \in (0, \alpha)} \left( 1 - \left( \frac{s}{\alpha} \right)^{N-1} \right)^{N-\beta} \frac{N-\beta}{N-1} \frac{N-\beta}{N-1} \frac{N-\beta}{N-1} \frac{\alpha}{\alpha} \frac{\alpha}{\alpha} \frac{\alpha}{\alpha} \frac{STM_\beta (s)}{STM_\beta (s)}.
\]

Concerning the existence and nonexistence of the best constants of the Trudinger-Moser type inequalities, the following results were proved in [11, 15, 16, 17, 23, 26, 31]:

**Theorem D.** 1/ \( STM_\beta (\alpha) \) can be achieved for all \( 0 < \alpha < \alpha_N \).

2/ Let \( N \geq 3 \). Then \( TM_\alpha (\alpha) \) can be attained for all \( 0 < \alpha \leq \alpha_N \).

3/ Let \( N = 2 \). Then there exists \( \delta \in (0, 4\pi) \) such that \( TM_\delta (\alpha) \) can be attained for all \( \delta < \alpha \leq 4\pi \) and is not achieved for all \( 0 < \alpha < \delta \). Moreover, in the later case, \( TM_\delta (\alpha) = \alpha \) for all \( 0 < \alpha < \delta \).

4/ Let \( N \geq 2, 0 < \beta < N \). Then \( TM_\beta (\alpha) \) can be attained for all \( 0 < \alpha < \alpha_N \).
1.2. Adams inequalities. It is worth noting that symmetrization has been a very useful and efficient (and almost inevitable) method when dealing with the sharp geometric inequalities. Thus, it is very fascinating to investigate such sharp geometric inequalities, in particular, the Trudinger-Moser type inequalities, in the settings where the symmetrization is not available such as on the higher order Sobolev spaces, the Heisenberg groups, Riemannian manifolds, sub-Riemannian manifolds, etc. Indeed, in these settings, an inequality like (2) is not available. In these situations, the first breakthrough came from the work of D. Adams [2] when he attempted to set up the Trudinger-Moser inequality in the higher order setting in Euclidean spaces. In fact, using a new idea that one can write a smooth function as a convolution of a (Riesz) potential with its derivatives, and then one can use the symmetrization for this convolution, instead of the symmetrization of the higher order derivatives, Adams proved the following inequality with boundary Dirichlet condition [2] which was extended to the Navier boundary condition in [34] when $\beta = 0$, and then to the case $0 \leq \beta < N$ in [19]. The following is taken from [2, 19]:

**Theorem E.** Let $\Omega$ be an open and bounded set in $\mathbb{R}^N$ with $N \leq \Omega$. If $m$ is a positive integer less than $N$, $0 \leq \beta < N$, then there exists a constant $C_0 = C(N, m, \beta) > 0$ such that for any $u \in W^{m, \frac{N}{m}}_0(\Omega)$ and $||\nabla^m u||_{L^\frac{N}{N-\beta}(\Omega)} \leq 1$, then

$$\frac{1}{|\Omega|^{1-\frac{\beta}{N}}} \int_{\Omega} \exp(\alpha \left(1 - \frac{\beta}{N}\right) |u(x)|^{\frac{N}{N-\beta}}) \frac{dx}{|x|^\beta} \leq C_0$$

for all $\alpha \leq \alpha(N, m)$ where

$$\alpha(N, m) = \begin{cases} \frac{N}{m(N-1)} \left[\frac{\pi^{N/2} 2^m \Gamma(\frac{m}{2}+1)}{\Gamma(\frac{N}{2}+1)}\right]^{\frac{N}{N-\beta}} & \text{when } m \text{ is odd} \\ \frac{N}{m(N-1)} \left[\frac{\pi^{N/2} 2^m \Gamma(\frac{m}{2})}{\Gamma(\frac{N}{2})}\right]^{\frac{N}{N-\beta}} & \text{when } m \text{ is even} \end{cases}.$$

Furthermore, the constant $\alpha(N, m)$ is optimal in the sense that for any $\alpha > \alpha(N, m)$, the integral can be made as large as possible.

In Theorem E, we use the symbol $\nabla^m u$, $m$ is a positive integer, to denote the $m$–th order gradient for $u \in C^m$, the class of $m$–th order differentiable functions:

$$\nabla^m u = \begin{cases} \Delta^\frac{m}{2} u & \text{for } m \text{ even} \\ \nabla \Delta^\frac{m-2}{2} u & \text{for } m \text{ odd} \end{cases}.$$

where $\nabla$ is the usual gradient operator and $\Delta$ is the Laplacian. Also, $W^{m, \frac{N}{m}}_0(\Omega)$ is the Sobolev space with homogeneous Navier boundary conditions:

$$W^{m, \frac{N}{m}}_0(\Omega) := \left\{ u \in W^{m, \frac{N}{m}}(\Omega) : \Delta^j u = 0 \text{ on } \partial \Omega \text{ for } 0 \leq j \leq \left[\frac{m-1}{2}\right] \right\}.$$

Notice that $W^{m, \frac{N}{m}}_0(\Omega)$ contains the Sobolev space $W^{m, \frac{N}{m}}(\Omega)$ as a closed subspace.

Adams inequalities have been extended to compact Riemannian manifolds in [13]. The Adams inequalities with optimal constants for high order derivatives on domains of infinite volume were recently established by Ruf and Sani in [32] in the case of even order derivatives and by Lam and Lu for all order of derivatives including fractional orders [14, 20]. The idea of [32] is to use the comparison principle for polyharmonic equations (thus could deal with the case of even order of derivatives) and thus involves some difficult construction of auxiliary functions. The arguments
in [14, 20] uses the representation of the (Bessel) potentials and thus avoids dealing with such a comparison principle. In particular, the method developed in [20] adapts the idea of deriving the sharp Moser-Trudinger-Adams inequalities on domains of finite measure to the entire spaces using the level sets of the functions under consideration. Thus, the argument in [20] does not use the symmetrization method and thus also works for the sub-Riemannian setting such as the Heisenberg groups [18]. The following general version is taken from [20].

**Theorem F** Let $0 < \gamma < n$ be an arbitrary real positive number, $p = \frac{n}{\gamma}$, $p' = \frac{n}{p-\gamma}$, and $\tau > 0$. There holds

$$
\sup_{u \in W^{\gamma,p}(\mathbb{R}^n), \| (\tau I - \Delta)^{\frac{\gamma}{2}} u \|_p \leq 1} \int_{\mathbb{R}^n} \phi \left( \beta_0 (n, \gamma) |u|^{p'} \right) dx < \infty
$$

where

$$
\phi(t) = e^t - \sum_{j=0}^{j_p} \frac{t^j}{j!},
$$

$$
j_p = \min \{ j \in \mathbb{N} : j \geq p \} \geq p,
$$

$$
\beta_0 (n, \gamma) = \frac{n}{\omega_{n-1}} \left[ \frac{\pi^{\frac{n}{2}} 2^{\frac{n}{2}} \Gamma \left( \frac{\gamma}{2} \right)}{\Gamma \left( \frac{n-\gamma}{2} \right)} \right]^{p'}.
$$

Furthermore this inequality is sharp in the sense that if $\beta_0 (n, \gamma)$ is replaced by any $\beta > \beta_0 (n, \gamma)$, then the supremum is infinite.

Here $W^{\gamma,p}(\mathbb{R}^n)$ is the fractional Sobolev space defined by the Bessel potential (see [33]). It is well-known that when $\gamma \in \mathbb{N}$, $W^{\gamma,p}(\mathbb{R}^n)$ is equivalent to the scale of the regular Sobolev space defined in the distribution sense.

We also have the following versions of the Adams type inequalities in the spirit of Lions (see [21]):

**Theorem G** (Adams type inequalities for infinite-volume domain). Let $N \geq 3$, $0 \leq \beta < N$ and $M > 1$. Then we have

$$
LA_{\beta,M} (\alpha) = \sup_{\| \Delta u \|_{\frac{N}{2}} \leq 1} \frac{1 - \| \Delta u \|_{\frac{N}{2}}^{\frac{N}{2} - \frac{\beta}{N}}}{\| u \|_{\frac{N}{2}}^{\frac{N}{2} - \frac{\beta}{N}}} \int_{\mathbb{R}^N} \phi_{N,2} \left( \frac{M^{\frac{1}{M-1}} \alpha \left( 1 - \frac{\beta}{N} \right) |u|_{\frac{N}{2}}^{\frac{N}{2} - \frac{1}{M-1}}}{M - 1 + \| \Delta u \|_{\frac{N}{2}}^{\frac{N}{2} - \frac{1}{M-1}}} \right) \frac{dx}{|x|^\beta} < \infty
$$

for all $\alpha \leq \alpha (N, 2)$.

$$
AT A_{\beta} (\alpha) = \sup_{\| \Delta u \|_{\frac{N}{2}} \leq 1} \frac{1}{\| u \|_{\frac{N}{2}}^{\frac{N}{2} (1-\frac{\beta}{N})}} \int_{\mathbb{R}^N} \phi_{N,2} \left( \alpha \left( 1 - \frac{\beta}{N} \right) |u|_{\frac{N}{2}}^{\frac{N}{2} - \frac{1}{M-1}} \right) \frac{dx}{|x|^\beta} < \infty
$$

for all $\alpha < \alpha (N, 2)$.

Here

$$
\phi_{N,2} (t) = \sum_{j \in \mathbb{N}, j \geq \frac{N}{2} - \frac{1}{2}} \frac{t^j}{j!}.
$$
Moreover, the constant $\alpha (N, 2)$ is the best possible. As a consequence, we have that there exists a constant $C = C(N, \beta) > 0$ such that

$$A_\beta (\alpha) \leq IA_{\beta, M} (\alpha) \leq C(N, \beta),$$

for all $0 \leq \alpha \leq \alpha (N, 2)$, where

$$IA_{\beta, M} (\alpha) = \sup_{\|\Delta u \|_{\frac{N}{2}} + \|u\|_{\frac{N}{2}} \leq R_N} \int_{\mathbb{R}^N} \phi_{N, 2} \left( M - 1 + \|\Delta u \|_{\frac{N}{2}} + \|u\|_{\frac{N}{2}} \right) \left( 1 - \frac{\beta}{N} \right) |u|^{\frac{N - 2}{2}} \frac{N}{2} \frac{1}{|x|^\beta} \; dx.$$

$$A_\beta (\alpha) = \sup_{\|\Delta u \|_{\frac{N}{2}} + \|u\|_{\frac{N}{2}} \leq R_N} \int_{\mathbb{R}^N} \phi_{N, 2} \left( \alpha \left( 1 - \frac{\beta}{N} \right) |u|^{\frac{N - 2}{2}} \frac{N}{2} \frac{1}{|x|^\beta} \right) \; dx.$$

The constant $\alpha (N, 2)$ in the above supremums is sharp.

In [22], it was also showed that

**Theorem H.** Let $0 \leq \beta < N$. Then there exist positive constants $c = c(N, \beta)$ and $C = C(N, \beta)$ such that when $\alpha$ is close enough to $\alpha (N, 2)$:

$$\frac{c(N, \beta)}{\left[ 1 - \left( \frac{\alpha}{\alpha (N, 2)} \right)^{\frac{N-2}{2}} \right]^{1 - \frac{\beta}{N}}} \leq AT A_\beta (\alpha) \leq \frac{C(N, \beta)}{\left[ 1 - \left( \frac{\alpha}{\alpha (N, 2)} \right)^{\frac{N-2}{2}} \right]^{1 - \frac{\beta}{N}}}.$$

Also, for $a, b > 0$, denote

$$A_{a, b} (\alpha) = \sup_{\|\Delta u \|_{\frac{N}{2}} + \|u\|_{\frac{N}{2}} \leq R_N} \int_{\mathbb{R}^N} \phi_{N, 2} \left( \alpha \left( 1 - \frac{\beta}{N} \right) |u|^{\frac{N - 2}{2}} \frac{N}{2} \frac{1}{|x|^\beta} \right) \; dx.$$

Then $A_{a, b} (\alpha (N, 2)) < \infty$ if and only if $b \leq \frac{N}{2}$. The constant $\alpha (N, 2)$ is sharp. Moreover, we have the following identity: for all $0 < \alpha \leq \alpha (N, 2)$:

$$A_{a, b} (\alpha) = \sup_{s \in (0, \alpha)} \left( \frac{1 - \left( \frac{a}{\alpha} \right)^{\frac{N-2}{2}} b}{\left( \frac{a}{\alpha} \right)^{\frac{N-2}{2}} b} \right)^{\frac{N - 2}{N}} \; AT A_\beta (s).$$

$$A_\beta (\alpha) = \sup_{s \in (0, \alpha)} \left( \frac{1 - \left( \frac{a}{\alpha} \right)^{\frac{N-2}{2}} b}{\left( \frac{a}{\alpha} \right)^{\frac{N-2}{2}} b} \right)^{\frac{N - 2}{N}} \; AT A_\beta (s).$$

Very little is known for existence of extremals for Adams inequalities. Existence of extremal functions for the Adams inequality on bounded domains in Euclidean spaces has been established in [28] and compact Riemannian manifolds by [25] only when $N = 4$ and $m = 2$ and is still widely open in other cases.

### 1.3. Our main results.

The first main purpose of this article is to study relations of the sharp constants of the Trudinger-Moser type inequalities. We will prove that

**Theorem 1.1.** Let $N \geq 2$ and $0 \leq \beta < N$. Then for $0 < \alpha \leq \alpha_N$:

$$LTM_{\beta, M} (\alpha) = ITM_{\beta, M} (\alpha) = \left( \frac{M}{M - 1} \right)^{\frac{N - \beta}{N}} T M_\beta (\alpha).$$
Theorem 1.6. Let

\[ \frac{M}{M-1} \sup_{s \in (0, \alpha)} \left( 1 - \left( \frac{s}{\alpha} \right)^{N-1} \right)^{\frac{N-\beta}{N}} \text{STM}_\beta (s) . \]

Using this result, we will then investigate the existence and nonexistence of optimizers for these Trudinger-Moser best constants. We will prove that

**Theorem 1.7.** \( \text{LTM}_{\beta, M} (\alpha) \) and \( \text{ITM}_{\beta, M} (\alpha) \) can be attained if and only if \( \text{TM}_\beta (\alpha) \) is achieved.

As a consequence, from the results in [11, 15, 16, 23, 26, 31] (Theorem D), we get that

**Theorem 1.3.** Let \( N \geq 3 \) and \( M > 1 \). Then \( \text{LTM}_{0, M} (\alpha) \) and \( \text{ITM}_{0, M} (\alpha) \) can be attained for all \( 0 < \alpha \leq \alpha_N \).

**Theorem 1.4.** Let \( N = 2 \) and \( M > 1 \). Then there exists \( \delta \in (0, 4\pi) \) such that \( \text{LTM}_{0, M} (\alpha) \) and \( \text{ITM}_{0, M} (\alpha) \) can be attained for all \( \delta < \alpha \leq 4\pi \) and are not achieved for all \( 0 < \alpha < \delta \). Moreover, in the later case, \( \text{LTM}_{0, M} (\alpha) = \text{ITM}_{0, M} (\alpha) = \frac{M}{M-1} \alpha \) for all \( 0 < \alpha < \delta \).

**Theorem 1.5.** Let \( N \geq 2 \), \( 0 < \beta < N \) and \( M > 1 \). Then \( \text{LTM}_{\beta, M} (\alpha) \) and \( \text{ITM}_{\beta, M} (\alpha) \) can be attained for all \( 0 < \alpha < \alpha_N \).

We also concern the best constants of the Adams type inequalities on \( W^{2, \infty} (\mathbb{R}^N) \):

**Theorem 1.6.** Let \( N \geq 3 \) and \( 0 \leq \beta < N \). Then for any \( 0 < \alpha \leq \alpha (N, 2) \):

\[
\text{LA}_{\beta, M} (\alpha) = \text{IA}_{\beta, M} (\alpha) = \left( \frac{M}{M-1} \right)^{\frac{N-\beta}{N}} A_\beta (\alpha) = \left( \frac{M}{M-1} \right)^{\frac{N-\beta}{N}} \sup_{s \in (0, \alpha)} \left( 1 - \left( \frac{s}{\alpha} \right)^{N-1} \right)^{\frac{N-\beta}{N}} \text{ATA}_\beta (s) .
\]

We will also investigate the attainability/unattainability of the Adams best constants in the singular case with subcritical growth. More precisely, we will prove that

**Theorem 1.7.** Let \( N \geq 3 \), \( 0 < \beta < N \) and \( M > 1 \). Then \( \text{ATA}_\beta (\alpha), A_\beta (\alpha), \text{LA}_{\beta, M} (\alpha) \) and \( \text{IA}_{\beta, M} (\alpha) \) can be attained for all \( 0 < \alpha < \alpha (N, 2) \). Also, \( A_0 (\alpha), \text{LA}_{0, M} (\alpha) \) and \( \text{IA}_{0, M} (\alpha) \) cannot be achieved for \( \alpha \gtrsim 0 \) if \( N = 4 \).

2. Some lemmata. We first recall the following lemma that the proof can be found in [11, 15, 23]:

**Lemma 2.1.** For \( \alpha < \alpha_N \), \( \text{STM}_\beta (\alpha) \) is attained by a function \( u : \| \nabla u \|_N = \| u \|_N = 1 \). Hence

\[
\text{STM}_\beta (\alpha) = \sup_{\| \nabla u \|_N = 1} \int_{\mathbb{R}^N} \phi_N \left( \alpha \left( 1 - \frac{\beta}{N} \right) \| u \|_{\frac{N}{N-\beta}} \right) \frac{dx}{|x|^\beta} = \sup_{\| \nabla u \|_N = 1} \int_{\mathbb{R}^N} \phi_N \left( \alpha \left( 1 - \frac{\beta}{N} \right) \| u \|_{\frac{N}{N-\beta}} \right) \frac{dx}{|x|^\beta} = \sup_{\| \nabla u \|_N = 1} \int_{\mathbb{R}^N} \phi_N \left( \alpha \left( 1 - \frac{\beta}{N} \right) \| u \|_{\frac{N}{N-\beta}} \right) \frac{dx}{|x|^\beta}.
\]
Moreover, $STM_{\beta}(\cdot)$ is continuous on $(0, \alpha_N)$.

**Lemma 2.2.** We have for $0 < \alpha < \alpha_N$:

$$STM_{\beta}(\alpha) \leq \left( \frac{M - 1}{M} - \frac{\alpha}{\alpha_N} \right)^{N-1} \frac{\alpha}{\alpha_N} \ ITM_{\beta,M} (\alpha_N). \quad (5)$$

**Proof.** Let $u \in W^{1,N}(\mathbb{R}^N) : \|\nabla u\|_N = \|u\|_N = 1$. Set

$$v(x) = \eta u(\lambda x)$$

$$\eta = \left( \frac{(M - 1) \left( \frac{\alpha}{\alpha_N} \right)^{N-1}}{M - \left( \frac{\alpha}{\alpha_N} \right)^{N-1}} \right)^{1/N}$$

$$\lambda = \left( \frac{\eta^N}{1 - \eta^N} \right)^{1/N}.$$

Then

$$\|\nabla v\|_N^N = \eta^N \|\nabla u\|_N = \eta^N$$

$$\|v\|_N^N = \eta^N \frac{1}{\lambda^N} \|u\|_N^N = 1 - \eta^N.$$

Hence $\|\nabla v\|_N^N + \|v\|_N^N = 1$. By the definition of $ITM_{\beta,M}(\alpha_N)$, we have

$$\int_{\mathbb{R}^N} \phi_{N,1} \left( \alpha(1 + \beta) |u|^{N/(N-1)} \right) \frac{d\lambda x}{|\lambda x|^\beta}$$

$$= \int_{\mathbb{R}^N} \phi_{N,1} \left( \alpha(1 + \beta) |u(\lambda x)|^{N/(N-1)} \right) \frac{d\lambda x}{|\lambda x|^\beta}$$

$$= \lambda^{N-\beta} \int_{\mathbb{R}^N} \phi_{N,1} \left( \alpha |u(\lambda x)|^{N/(N-1)} \right) \frac{d\lambda x}{|\lambda x|^\beta}$$

$$= \lambda^{N-\beta} \int_{\mathbb{R}^N} \phi_{N,1} \left( \frac{M^{1/\beta} \alpha_N}{M - \|\nabla v\|_N^{N-1}} \right) \frac{d\lambda x}{|\lambda x|^\beta}$$

$$\leq \left( \frac{M - 1}{M} - \frac{\alpha}{\alpha_N} \right)^{N-1} \ ITM_{\beta,M} (\alpha_N).$$

**Lemma 2.3.** We have for $0 < \alpha < \alpha_N$:

$$STM_{\beta}(\alpha) \leq \left( \frac{M - 1}{M} - \frac{\alpha}{\alpha_N} \right)^{N-1} \frac{\alpha}{\alpha_N} \ ITM_{\beta,M} (\alpha_N). \quad (6)$$

**Proof.** Let $u \in W^{1,N}(\mathbb{R}^N) : \|\nabla u\|_N = \|u\|_N = 1$. Set

$$v(x) = \eta u(\lambda x)$$

$$\eta = \left( \frac{(M - 1) \left( \frac{\alpha}{\alpha_N} \right)^{N-1}}{M - \left( \frac{\alpha}{\alpha_N} \right)^{N-1}} \right)^{1/N}$$

$$\lambda = \left( \frac{\eta^N}{1 - \eta^N} \right)^{1/N}.$$
\[ \eta = \left( \frac{(M - 1) \left( \frac{\alpha}{\alpha_N} \right)^{N-1}}{M - \left( \frac{\alpha}{\alpha_N} \right)^{N-1}} \right)^{\frac{1}{N}} \]

\[ \lambda = \left( \frac{\eta^N}{1 - \eta^N} \right)^{1/N}. \]

then

\[ \| \nabla v \|^N_N = \eta^N \| \nabla u \|^N_N = \eta^N \]

\[ \| v \|^N_N = \eta^N \frac{1}{N \lambda} \| u \|^N_N = 1 - \eta^N. \]

Hence \( \| \nabla v \|_N < 1 \). By the definition of \( LTM_{\beta, M}(\alpha_N) \), we have

\[ \int_{\mathbb{R}^N} \phi_{N, 1} \left( \alpha \left( 1 - \frac{\beta}{N} \right) |u|^{N/(N-1)} \right) \frac{dx}{|x|^{\beta}} \]

\[ = \int_{\mathbb{R}^N} \phi_{N, 1} \left( \alpha \left( 1 - \frac{\beta}{N} \right) |u| \left( \lambda x \right)^{N/(N-1)} \right) \frac{d(\lambda x)}{|\lambda x|^{\beta}} \]

\[ = \lambda^{N-\beta} \int_{\mathbb{R}^N} \phi_{N, 1} \left( \frac{M^{\frac{1}{N-\beta}} \alpha_N}{(M - 1 + \| \nabla v \|^N_N)^{\frac{1}{N-\beta}}} \left( 1 - \frac{\beta}{N} \right) \left| u \right|^{N/(N-1)} \right) \frac{dx}{|x|^{\beta}} \]

\[ \leq \left( \frac{M - 1}{M} \left( \frac{\alpha}{\alpha_N} \right)^{N-1} \right)^{\frac{N-\beta}{N}} LTM_{\beta, M}(\alpha_N). \]

Lemma 2.4.

\[ AT\Lambda_\beta (\beta) = \sup_{\| \Delta u \|_N = \| u \|_N = 1} \int_{\mathbb{R}^N} \phi_{N, 2} \left( \alpha \left( 1 - \frac{\beta}{N} \right) \left| u \right|^{\frac{N}{N-\beta}} \right) \frac{dx}{|x|^{\beta}}. \]

Proof. Let \( u \in W^{2, \frac{N}{N-\beta}} (\mathbb{R}^N) : \| \Delta u \|_N = c \leq 1 \). Then it is easy to see that

\[ \frac{1}{\| u \|^\frac{N}{N-\beta} (1 - \frac{\beta}{N})} \int_{\mathbb{R}^N} \phi_{N, 2} \left( \alpha \left( 1 - \frac{\beta}{N} \right) \left| u \right|^{\frac{N}{N-\beta}} \right) \frac{dx}{|x|^{\beta}} \]

\[ \leq \frac{1}{\| u \|^\frac{N}{N-\beta} (1 - \frac{\beta}{N})} \int_{\mathbb{R}^N} \phi_{N, 2} \left( \alpha \left( 1 - \frac{\beta}{N} \right) \left| u \right|^{\frac{N}{N-\beta}} \right) \frac{dx}{|x|^{\beta}}. \]

Hence

\[ AT\Lambda_\beta (\beta) = \sup_{\| \Delta u \|_N = 1} \int_{\mathbb{R}^N} \phi_{N, 2} \left( \alpha \left( 1 - \frac{\beta}{N} \right) \left| u \right|^{\frac{N}{N-\beta}} \right) \frac{dx}{|x|^{\beta}}. \]

Now, let \( u \in W^{2, \frac{N}{N-\beta}} (\mathbb{R}^N) : \| \Delta u \|_N = 1 \) and set

\[ v (x) = u (\lambda x); \]
\[ \lambda = \| u \|_\frac{N}{2}^\frac{3}{2} \]

Then it is easy to check that
\[ \Delta v (x) = \lambda^2 \Delta u (\lambda x) \]
and
\[ \| \Delta v \|_\frac{N}{2} = \| \Delta u \|_\frac{N}{2} = 1; \]
\[ \| v \|_\frac{N}{2} = \int_{\mathbb{R}^N} |v(x)|^\frac{N}{2} \, dx = \int_{\mathbb{R}^N} |u(\lambda x)|^\frac{N}{2} \, dx = \frac{1}{\lambda^N} \int_{\mathbb{R}^N} |u(x)|^\frac{N}{2} \, dx = 1. \]

Moreover
\[ \int_{\mathbb{R}^N} \phi_{N,2} \left( \alpha \left(1 - \frac{\beta}{N}\right) \right) \frac{|x|^{\frac{N-2}{2}}}{|x|^\beta} \, dx = \int_{\mathbb{R}^N} \phi_{N,2} \left( \alpha \left(1 - \frac{\beta}{N}\right) \right) \frac{|u(\lambda x)|^{\frac{N-2}{2}}}{|x|^\beta} \, dx \]
\[ = \frac{1}{\lambda^{N-\beta}} \int_{\mathbb{R}^N} \phi_{N,2} \left( \alpha \left(1 - \frac{\beta}{N}\right) \right) \frac{|u(x)|^{\frac{N-2}{2}}}{|x|^\beta} \, dx \]
\[ = \frac{1}{\| u \|_\frac{N}{2} \left(1 - \frac{\beta}{N}\right)} \int_{\mathbb{R}^N} \phi_{N,2} \left( \alpha \left(1 - \frac{\beta}{N}\right) \right) \frac{|u(x)|^{\frac{N-2}{2}}}{|x|^\beta} \, dx. \]

Lemma 2.5.

\[ \text{ATA}_{\beta} (\alpha) \leq \left( \frac{\alpha \left(\frac{N}{2}\right)^{\frac{N-2}{2}}}{1 - \left(\frac{\alpha \left(\frac{N}{2}\right)^{\frac{N-2}{2}}}{\frac{N}{2}}\right)^{\frac{N-2}{2}}} \right)^{\frac{N-\beta}{N}} \text{IA}_{\beta} (\alpha) \left(\frac{N}{2}\right) \]  \[ (7) \]

Proof. Let \( u \in W^{2, \frac{N}{2}} (\mathbb{R}^N) : \| \Delta u \|_\frac{N}{2} = \| u \|_\frac{N}{2} = 1 \). We define
\[ v (x) = \eta u (\lambda x) \]
\[ \eta = \left( \frac{(M - 1) \left(\frac{\alpha \left(\frac{N}{2}\right)^{\frac{N-2}{2}}}{\frac{N}{2}}\right)^{\frac{N-2}{2}}}{M - \left(\frac{\alpha \left(\frac{N}{2}\right)^{\frac{N-2}{2}}}{\frac{N}{2}}\right)^{\frac{N-2}{2}}} \right)^{\frac{N}{2}} \]
\[ \lambda = \left( \frac{\eta^{\frac{N}{2}}}{1 - \eta^{\frac{N}{2}}} \right)^{\frac{1}{N}}. \]

then
\[ \| \Delta v \|_\frac{N}{2} = \eta^{\frac{N}{2}} \| \Delta u \|_\frac{N}{2} = \eta^{\frac{N}{2}} \]
\[ \| v \|_\frac{N}{2} = \eta^{\frac{N}{2}} \frac{1}{\lambda^N} \| u \|_\frac{N}{2} = 1 - \eta^{\frac{N}{2}}. \]

Hence \( \| \Delta v \|_\frac{N}{2} + \| v \|_\frac{N}{2} = 1 \). By the definition of \( \text{IA}_{\beta} (\alpha) \left(\frac{N}{2}\right) \), we have
\[ \int_{\mathbb{R}^N} \phi_{N,2} \left( \alpha \left(1 - \frac{\beta}{N}\right) \right) \frac{|u|^{\frac{N}{2}(N-2)}}{|x|^\beta} \, dx \]
\[
\int_{\mathbb{R}^N} \phi_{N,2} \left( \alpha \left( 1 - \frac{\beta}{N} \right) |u(\lambda x)|^{N/(N-2)} \right) \frac{d(\lambda x)}{|\lambda x|^{\beta}}
\]

\[
= \lambda^{N-\beta} \int_{\mathbb{R}^N} \phi_N \left( \frac{M^{\frac{1}{N-1}} \alpha (N,2) \left( 1 - \frac{\beta}{N} \right)}{M - 1 + \|\Delta v\|_{\frac{N}{2}}^{\frac{N}{2}}} |v|^{N/(N-2)} \right) \frac{d\lambda x}{|\lambda x|^{\beta}}
\]

\[
\leq \left( \frac{M - 1}{M} - \left( \frac{\alpha}{\alpha(N,2)} \right)^{\frac{N-2}{2}} \right) \frac{\Delta v}{\lambda x} I \Lambda_{\beta} (\alpha (N,2)) \).
\]

\[\Box\]

Lemma 2.6.

\[
AT \Lambda_{\beta} (\alpha) \leq \left( \frac{\alpha}{\alpha(N,2)} \right)^{\frac{N-2}{2}} L \Lambda_{\beta} (\alpha (N,2)). \tag{8}
\]

Proof. Let \( u \in W^{2,\frac{N}{2}} (\mathbb{R}^N) : \|\Delta u\|_{\frac{N}{2}} = \|u\|_{\frac{N}{2}} = 1 \). We define

\[
v(x) = \eta u(\lambda x)
\]

\[
\eta = \left( \frac{M - 1}{M - \left( \frac{\alpha}{\alpha(N,2)} \right)^{\frac{N-2}{2}}} \right) \frac{\lambda x}{1 - \eta \frac{N}{2}}.
\]

then

\[
\|\Delta u\|_{\frac{N}{2}} = \eta \frac{N}{2} \|\Delta u\|_{\frac{N}{2}} = \eta \frac{N}{2}
\]

\[
\|u\|_{\frac{N}{2}} = \eta \frac{N}{2} \frac{1}{\lambda^N} \|u\|_{\frac{N}{2}} = 1 - \eta \frac{N}{2}.
\]

Hence \( \|\Delta u\|_{\frac{N}{2}} < 1 \). By the definition of \( L \Lambda_{\beta} (\alpha (N,2)) \), we have

\[
\int_{\mathbb{R}^N} \phi_{N,2} \left( \alpha \left( 1 - \frac{\beta}{N} \right) |u|^{N/(N-2)} \right) \frac{d\lambda x}{|\lambda x|^{\beta}}
\]

\[
= \int_{\mathbb{R}^N} \phi_{N,2} \left( \alpha \left( 1 - \frac{\beta}{N} \right) |u(\lambda x)|^{N/(N-2)} \right) \frac{d(\lambda x)}{|\lambda x|^{\beta}}
\]

\[
= \lambda^{N-\beta} \int_{\mathbb{R}^N} \phi_N \left( \frac{M^{\frac{1}{N-1}} \alpha (N,2) \left( 1 - \frac{\beta}{N} \right)}{M - 1 + \|\Delta v\|_{\frac{N}{2}}^{\frac{N}{2}}} |v|^{N/(N-2)} \right) \frac{d\lambda x}{|\lambda x|^{\beta}}.
\]
\[
\left( \frac{M-1}{M} \left( \frac{\alpha}{\alpha(N,2)} \right)^{\frac{N-2}{2}} \right)^{\frac{N-\beta}{\alpha}} L_{\alpha} (\alpha (N,2)) .
\]

3. Best constants of the Trudinger-Moser type inequalities.

3.1. Equivalence of Trudinger-Moser type inequalities-Proof of Theorem 1.1.

Proof of Theorem 1.1. We will first show that

\[
ITM_{\beta,M} (\alpha N) = LTM_{\beta,M} (\alpha N) = \sup_{\alpha \in (0, \alpha N)} \left( \frac{M - 1}{M - 1 + \|u_n\|_N} \right)^{\frac{N-\beta}{\alpha}} STM_{\beta} (\alpha).
\]

Indeed, by (5) and (6), we have

\[
\sup_{\alpha \in (0, \alpha N)} \left( \frac{M - 1}{M - 1 + \|u_n\|_N} \right)^{\frac{N-\beta}{\alpha}} STM_{\beta} (\alpha) \leq ITM_{\alpha b, \beta, M} (\alpha N).
\]

Now, let \( (u_n) \) be the maximizing sequence of \( ITM_{\beta,M} (\alpha N) \), i.e., \( u_n \in W^{1,N} (\mathbb{R}^N) \setminus \{0\} \) : \( \|\nabla u_n\|_N + \|u_n\|_N \leq 1 \) and

\[
\int_{\mathbb{R}^N} \phi_{N,1} \left( \frac{M^{\frac{1}{N \alpha N}} a N \left( 1 - \frac{\beta}{N} \right)}{\left( M - 1 + \|\nabla u_n\|_N \right)^{\frac{1}{N \alpha N}}} |u_n|^{\frac{N}{N \alpha N}} \right) \frac{dx}{|x|^\beta} \to n \to \infty ITM_{\beta,M} (\alpha N).
\]

We define

\[
v_n (x) = \frac{u (\lambda_n x)}{\|\nabla u_n\|_N}
\]

\[
\lambda_n = \left( 1 - \frac{\|\nabla u_n\|_N}{\|u_n\|_N} \right)^{1/N} > 0.
\]

Hence

\[ \|\nabla v_n\|_N = 1 \text{ and } \|v_n\|_N \leq 1. \]

Also,

\[
\int_{\mathbb{R}^N} \phi_{N} \left( \frac{M^{\frac{1}{N \alpha N}} a N \left( 1 - \frac{\beta}{N} \right)}{\left( M - 1 + \|\nabla u_n\|_N \right)^{\frac{1}{N \alpha N}}} |u_n|^{\frac{N}{N \alpha N}} \right) \frac{dx}{|x|^\beta}
\]
\[
\lambda_n^{N-\beta} \int_{R^N} \phi_N \left( \frac{M^{\frac{1}{N-\tau}} \|
abla u_n\|_N^{\frac{N}{N-\tau}} \alpha_N (1 - \frac{\beta}{N}) |u_n|^{N/(N-1)}}{\left( M - 1 + \|
abla u_n\|_N \right)^{\frac{N}{N-\tau}}} \right) \, dx \\
\leq \lambda_n^{N-\beta} STM_\beta \left( \frac{M^{\frac{1}{N-\tau}} \|
abla u_n\|_N^{\frac{N}{N-\tau}} \alpha_N, \beta}{\left( M - 1 + \|
abla u_n\|_N \right)^{\frac{N}{N-\tau}}} \right) \\
\leq \sup_{\alpha \in (0, \alpha_N)} \left( \frac{M}{M - 1} \left( 1 - \frac{(\alpha/\alpha_N)^{\frac{N-1}{N-\beta}}}{} \right)^{\frac{N-\beta}{\beta}} \right) STM_\beta (\alpha).
\]

Now, let \((w_n)\) be the maximizing sequence of \(LTM_{\beta,M}(\alpha_N)\), i.e., \(w_n \in W^{1,N}(\mathbb{R}^N) \setminus \{0\} : \|
abla w_n\|_N < 1\) and

\[
\frac{1 - \|
abla w_n\|_N^{1/N}}{\|
abla w_n\|_N^{N-\beta}} \int_{R^N} \phi_{N,1} \left( \frac{M^{\frac{1}{N-\tau}} \alpha_N (1 - \frac{\beta}{N}) |w_n|^{\frac{N}{N-\tau}}}{\left( M - 1 + \|
abla u_n\|_N \right)^{\frac{N}{N-\tau}}} \right) \, dx \to LTM_{\beta,M}(\alpha_N).
\]

We define

\[
z_n(x) = \frac{w_n(\mu_n x)}{\|
abla w_n\|_N} \quad \mu_n = \left( \frac{1 - \|
abla w_n\|_N^{1/N}}{\|
abla w_n\|_N^{N-\beta}} \right)^{1/N} > 0.
\]

Hence \(\|
abla z_n\|_N = 1\) and \(\|z_n\|_N \leq 1\).

Also,

\[
\frac{1 - \|
abla w_n\|_N^{1/N}}{\|
abla w_n\|_N^{N-\beta}} \int_{R^N} \phi_N \left( \frac{M^{\frac{1}{N-\tau}} \alpha_N (1 - \frac{\beta}{N}) |w_n|^{\frac{N}{N-\tau}}}{\left( M - 1 + \|
abla u_n\|_N \right)^{\frac{N}{N-\tau}}} \right) \, dx \\
= \mu_n^{N-\beta} \int_{R^N} \phi_N \left( \frac{M^{\frac{1}{N-\tau}} \|
abla w_n\|_N^{\frac{N}{N-\tau}} \alpha_N (1 - \frac{\beta}{N}) |z_n|^{N/(N-1)}}{\left( M - 1 + \|
abla w_n\|_N \right)^{\frac{N}{N-\tau}}} \right) \, dx \\
\leq \mu_n^{N-\beta} STM_\beta \left( \frac{M^{\frac{1}{N-\tau}} \|
abla w_n\|_N^{\frac{N}{N-\tau}} \alpha_N (1 - \frac{\beta}{N})}{\left( M - 1 + \|
abla w_n\|_N \right)^{\frac{N}{N-\tau}}} \right) \\
\leq \sup_{\alpha \in (0, \alpha_N)} \left( \frac{M}{M - 1} \left( 1 - \frac{(\alpha/\alpha_N)^{\frac{N-1}{N-\beta}}}{} \right)^{\frac{N-\beta}{\beta}} \right) STM_\beta (\alpha).
\]
Hence, we receive

\[
ITM_{\beta,M} (\alpha_N) = LTM_{\beta,M} (\alpha_N) = \sup_{\alpha \in (0, \alpha_N)} \left( \frac{M}{M-1} \frac{1 - \left( \frac{\alpha}{\alpha_N} \right)^{N-1}}{\left( \frac{\alpha}{\alpha_N} \right)^{N-1}} \right)^{\frac{N-\beta}{N}} S \beta M (\alpha)
\]

\[
= \left( \frac{M}{M-1} \right)^{\frac{N-\beta}{N}} \sup_{\alpha \in (0, \alpha_N)} \left( \frac{1 - \left( \frac{\alpha}{\alpha_N} \right)^{N-1}}{\left( \frac{\alpha}{\alpha_N} \right)^{N-1}} \right)^{\frac{N-\beta}{N}} S \beta M (\alpha)
\]

\[
= \left( \frac{M}{M-1} \right)^{\frac{N-\beta}{N}} TM_{\beta,M} (\alpha_N).
\]

Similarly, we get for \(0 < \alpha \leq \alpha_N :\)

\[
LTM_{\beta,M} (\alpha) = ITM_{\beta,M} (\alpha) = \left( \frac{M}{M-1} \right)^{\frac{N-\beta}{N}} \sup_{s \in (0,\alpha)} \left( \frac{1 - \left( \frac{s}{\alpha} \right)^{N-1}}{\left( \frac{s}{\alpha} \right)^{N-1}} \right)^{\frac{N-\beta}{N}} S \beta M (s)
\]

\[
= \left( \frac{M}{M-1} \right)^{\frac{N-\beta}{N}} TM_{\beta} (\alpha).
\]

\[\square\]

3.2. Attainability and Unattainability of the Trudinger-Moser best constants.

**Proof of Theorem 1.2.** Let \(u\) be the optimizer of \(TM_{\beta} (\alpha): \|\nabla u\|^N_N = \theta \in (0,1); \|u\|^N_N = 1 - \theta.\) We define

\[
v (x) = \frac{u (\lambda x)}{\|\nabla u\|^N_N}
\]

\[
\lambda = \left( \frac{1 - \|\nabla u\|^N_N}{\|\nabla u\|^N_N} \right)^{1/N} = \left( \frac{1 - \theta}{\theta} \right)^{1/N} > 0.
\]

Hence

\[
\|\nabla v\|^N_N = 1 \text{ and } \|v\|^N_N = 1.
\]

Also,

\[
TM_{\beta} (\alpha) = \int_{\mathbb{R}^N} \phi_N \left( \alpha \left( 1 - \frac{\beta}{N} \right) |u|^N_N \right) \frac{dx}{|x|^\beta}
\]

\[
= \lambda^{N-\beta} \int_{\mathbb{R}^N} \phi_N \left( \alpha \left( 1 - \frac{\beta}{N} \right) \|\nabla u\|^N_N |u|^{N/(N-1)} \right) \frac{dx}{|x|^\beta}
\]

\[
= \left( \frac{1 - \theta}{\theta} \right)^{1-N/\beta} \int_{\mathbb{R}^N} \phi_N \left( \alpha \theta^{1/\beta} \left( 1 - \frac{\beta}{N} \right) |u|^{N/(N-1)} \right) \frac{dx}{|x|^\beta}.
\]

Now, set

\[
w (x) = \eta v (\mu x)
\]
\[ \eta = \left( \frac{(M - 1) \left( \frac{\alpha \theta^{\frac{1}{\alpha}}}{\alpha} \right)^{N-1}}{M - \left( \frac{\alpha \theta^{\frac{1}{\alpha}}}{\alpha} \right)^{N-1}} \right)^{\frac{1}{N}} = \left( \frac{(M - 1) \theta^{\frac{1}{\theta}}}{M - \theta} \right)^{\frac{1}{N}} \]

\[ \mu = \left( \frac{\eta^N}{1 - \eta^N} \right)^{1/N} = \left( \frac{(M - 1) \theta^{\frac{1}{\theta}}}{M(1 - \theta)} \right)^{1/N} \]

then

\[ \|\nabla w\|_N^N = \eta^N \|\nabla v\|_N^N = \eta^N \]

\[ \|w\|_N^N = \eta^N \frac{1}{\mu^N} \|v\|_N^N = 1 - \eta^N. \]

Hence \( \|\nabla w\|_N^N + \|w\|_N^N = 1. \) Also,

\[ \frac{1 - \|\nabla w\|_N^N}{\|w\|_N^N} \int_{\mathbb{R}^N} \phi_{N,1} \left( \frac{M \frac{\alpha}{\alpha} \left( 1 - \frac{\beta}{N} \right)}{M - 1 + \|\nabla w\|_N^N} \right)^{N-1} |w|^{N/(N-1)} \frac{dx}{|x|^{\beta}} \]

\[ = \frac{1}{\mu^{N-\beta}} \int_{\mathbb{R}^N} \phi_{N,1} \left( \frac{M \frac{\alpha}{\alpha} \left( 1 - \frac{\beta}{N} \right)}{M - 1 + \|\nabla w\|_N^N} \right)^{N-1} |v|^N |(\mu x)|^{N/(N-1)} \frac{d(\mu x)}{|\mu x|^{\beta}} \]

\[ = \frac{1}{\mu^{N-\beta}} \int_{\mathbb{R}^N} \phi_{N,1} \left( \frac{M \frac{\alpha}{\alpha} \left( 1 - \frac{\beta}{N} \right)}{M - 1 + \|\nabla w\|_N^N} \right)^{N-1} \left( \frac{\theta}{1 - \theta} \right)^{1 - \frac{\theta}{N}} TM_{\beta,\alpha} \]

\[ = \left( \frac{M (1 - \theta)}{(M - 1) \theta} \right)^{1 - \frac{\theta}{N}} TM_{\beta,\alpha} = ITM_{\beta,M,\alpha} = LTM_{\beta,M,\alpha}. \]

In other words, \( w \) is an optimizer for \( ITM_{\beta,M,\alpha} \) and \( LTM_{\beta,M,\alpha} \).

Moreover, the above process can be reversed. Hence, we can conclude that \( LTM_{\beta,M,\alpha} \) and \( ITM_{\beta,M,\alpha} \) can be attained if and only if \( TM_{\beta,\alpha} \) is achieved.

\[ \square \]

4. Equivalence of Adams type inequalities-Proof of Theorem 1.6.

Proof of Theorem 1.6. We will show that

\[ IA_{\beta} (\alpha (N, 2)) = \left( \frac{M}{M - 1} \right)^{\frac{N-\beta}{N}} \sup_{\alpha \in (0, \alpha (N, 2))} \left( \frac{1 - \left( \frac{\alpha}{\alpha (N, 2)} \right)^{\frac{N-2}{2}}}{\left( \frac{\alpha}{\alpha (N, 2)} \right)^{\frac{N-2}{2}}} \right)^{\frac{N-\beta}{N}} \frac{N-\beta}{N} \]

Indeed, by (7):

\[ \left( \frac{M}{M - 1} \right)^{\frac{N-\beta}{N}} \sup_{\alpha \in (0, \alpha (N, 2))} \left( \frac{1 - \left( \frac{\alpha}{\alpha (N, 2)} \right)^{\frac{N-2}{2}}}{\left( \frac{\alpha}{\alpha (N, 2)} \right)^{\frac{N-2}{2}}} \right)^{\frac{N-\beta}{N}} \frac{N-\beta}{N} \]

\[ \leq IA_{\beta} (\alpha (N, 2)). \]
Now, let \((u_n)\) be the maximizing sequence of \(IA_{\beta} (\alpha (N, 2))\), i.e., \(u_n \in W^{2, \frac{N}{2}} (\mathbb{R}^N) \setminus \{0\} : \|\Delta u_n\|_{\frac{N}{2}} \leq \frac{\|u_n\|_{\frac{N}{2}}}{2} \leq 1\) and

\[
\int_{\mathbb{R}^N} \phi_{N, 2} \left( \frac{M \frac{1}{N-1} \alpha (N, 2) \left(1 - \frac{\beta}{N}\right)}{M - 1 + \|\Delta u_n\|_{\frac{N}{2}}^{\frac{N}{N-1}}} \right) \left| u_n \right|_{\frac{N}{2}}^{N-2} \frac{dx}{|x|^\beta} \to n \to \infty IA_{\beta} (\alpha (N, 2)) .
\]

We define a new sequence:

\[
v_n (x) = \frac{u (\lambda_n x)}{\|\Delta u_n\|_{\frac{N}{2}}}, \lambda_n = \left( 1 - \frac{\|\Delta u_n\|_{\frac{N}{2}}}{\|\Delta u_n\|_{\frac{N}{2}}} \right) > 0 .
\]

Then

\[
\|\Delta v_n\|_{\frac{N}{2}} = 1 \text{ and } \|v_n\|_{\frac{N}{2}} \leq 1 .
\]

Also,

\[
\int_{\mathbb{R}^N} \phi_{N, 2} \left( \frac{M \frac{1}{N-1} \alpha (N, 2) \left(1 - \frac{\beta}{N}\right)}{M - 1 + \|\Delta u_n\|_{\frac{N}{2}}^{\frac{N}{N-1}}} \right) \left| u_n \right|_{\frac{N}{2}}^{N-2} \frac{dx}{|x|^\beta} \cdot \frac{\lambda_n^{N-\beta} \int_{\mathbb{R}^N} \phi_{N, 2} \left( \frac{M \frac{1}{N-1} \alpha (N, 2) \left(1 - \frac{\beta}{N}\right)}{M - 1 + \|\Delta u_n\|_{\frac{N}{2}}^{\frac{N}{N-1}}} \right) \left| v_n \right|_{N/(N-2)}^{N/(N-2)} dx}{\lambda_n^{N-\beta} \int_{\mathbb{R}^N} \phi_{N, 2} \left( \frac{M \frac{1}{N-1} \alpha (N, 2) \left(1 - \frac{\beta}{N}\right)}{M - 1 + \|\Delta u_n\|_{\frac{N}{2}}^{\frac{N}{N-1}}} \right) \left| v_n \right|_{N/(N-2)} dx} \leq \lambda_n^{N-\beta} \sup_{\alpha \in (0, \alpha (N, 2))} \left( 1 - \left( \frac{\alpha}{\alpha (N, 2)} \right)^{\frac{N-2}{2}} \right) \left( \frac{\alpha (N, 2)}{\alpha (N, 2)} \right)^{\frac{N-2}{2}} A_{\beta} (\alpha) .
\]

Hence, we can conclude that

\[
IA_{\beta} (\alpha (N, 2)) = \left( \frac{M}{M - 1} \right)^{\frac{N-\beta}{N}} \sup_{\alpha \in (0, \alpha (N, 2))} \left( 1 - \left( \frac{\alpha}{\alpha (N, 2)} \right)^{\frac{N-2}{2}} \right) \left( \frac{\alpha (N, 2)}{\alpha (N, 2)} \right)^{\frac{N-2}{2}} \left( \frac{\alpha}{\alpha (N, 2)} \right)^{\frac{N-2}{2}} A_{\beta} (\alpha) .
\]

By the similar arguments, we get that for any \(0 < \alpha \leq \alpha (N, 2)\):

\[
LA_{\beta, M} (\alpha) = IA_{\beta, M} (\alpha) = \left( \frac{M}{M - 1} \right)^{\frac{N-\beta}{N}} A_{\beta} (\alpha) .
\]
1 and 

Now, let \( \gamma \) and for some \( q \) Also, since \( WLOG, we can assume that \( u \to u \) a.e. on \( \mathbb{R}^N \) and \( \|\Delta u\|_{\frac{N}{q}} \leq 1, \|u\|_{\frac{N}{q}} \leq 1 \). Now, for \( R \gg 1 \), we have 

Also, since 

and for some \( q \gg 1 \): 

by the Egorov theorem, we have that 

\[
\int_{\{|x|<R\}} \phi_{N,2} \left( \alpha \left( 1 - \frac{\beta}{N} \right) |u_n|^{\frac{N}{q}} \right) dx \to \int_{\{|x|<R\}} \phi_{N,2} \left( \alpha \left( 1 - \frac{\beta}{N} \right) |u|^{\frac{N}{q}} \right) dx 
\]

\[
\leq \int_{\mathbb{R}^N} \phi_{N,2} \left( \alpha \left( 1 - \frac{\beta}{N} \right) |u|^{\frac{N}{q}} \right) dx .
\]
As a consequence,

\[
ATA_\beta (\alpha) \leq \int_{\mathbb{R}^N} \phi_{N,2} \left( \alpha \left( 1 - \frac{\beta}{N} \right) |u| \frac{N}{x^2} \right) \frac{dx}{|x|^\beta}.
\]

Hence, \( u \neq 0 \) and

\[
ATA_\beta (\alpha) \leq \frac{1}{\|u\|^{\frac{\beta}{2}(1 - \frac{\beta}{N})}} \int_{\mathbb{R}^N} \phi_{N,2} \left( \alpha \left( 1 - \frac{\beta}{N} \right) |u| \frac{N}{x^2} \right) \frac{dx}{|x|^\beta}.
\]

Hence, \( u \) is an optimizer for \( ATA_\beta (\alpha) \).

Next, let \( \varepsilon_n \downarrow 0 \) and \( u_n \) be the optimizer for \( ATA_\beta (\alpha + \varepsilon_n) \). We note here that by Lemma 2.4, we can assume that \( \|\Delta u_n\|_2 = \|u_n\|_2 \downarrow 1 \). Again, WLOG, we can assume that \( u_n \to u \) weakly in \( W^2_2 (\mathbb{R}^N) \). As a consequence, \( u_n \to u \) a.e. on \( \mathbb{R}^N \) and \( \|\Delta u\|_2 \leq 1, \|u\|_2 \leq 1 \). It is also clear that

\[
0 \leq ATA_\beta (\alpha + \varepsilon_n) - ATA_\beta (\alpha)
\]

\[
\leq \int_{|x| \geq R} \phi_{N,2} \left( \alpha \left( 1 - \frac{\beta}{N} \right) |u_n| \frac{N}{x^2} \right) \frac{dx}{|x|^\beta} - \int_{|x| \geq R} \phi_{N,2} \left( \alpha \left( 1 - \frac{\beta}{N} \right) |u_n| \frac{N}{x^2} \right) \frac{dx}{|x|^\beta}
\]

\[
= \int_{|x| \geq R} \left[ \phi_{N,2} \left( \alpha \left( 1 - \frac{\beta}{N} \right) |u_n| \frac{N}{x^2} \right) - \phi_{N,2} \left( \alpha \left( 1 - \frac{\beta}{N} \right) |u_n| \frac{N}{x^2} \right) \right] \frac{dx}{|x|^\beta}
\]

\[
\leq \frac{2ATA_0 (\alpha)}{R^\beta} \to 0 \text{ as } R \to \infty.
\]

Also, noting that in \( \{|x| < R\} \):

\[
\left[ \phi_{N,2} \left( \alpha \left( 1 - \frac{\beta}{N} \right) |u_n| \frac{N}{x^2} \right) - \phi_{N,2} \left( \alpha \left( 1 - \frac{\beta}{N} \right) |u_n| \frac{N}{x^2} \right) \right] \frac{dx}{|x|^\beta}
\]

\[
\to 0 \text{ a.e.,}
\]

and there exists \( q \geq 1 \) such that

\[
\int_{|x| < R} \left[ \phi_{N,2} \left( \alpha \left( 1 - \frac{\beta}{N} \right) |u_n| \frac{N}{x^2} \right) - \phi_{N,2} \left( \alpha \left( 1 - \frac{\beta}{N} \right) |u_n| \frac{N}{x^2} \right) \right]^q \frac{dx}{|x|^\beta}
\]

\[
\leq ATA_{\beta} \left( \alpha + \frac{\alpha (N, 2)}{2} \right),
\]

we have by the Vitali theorem that the quantity

\[
\int_{|x| < R} \left[ \phi_{N,2} \left( \alpha \left( 1 - \frac{\beta}{N} \right) |u_n| \frac{N}{x^2} \right) - \phi_{N,2} \left( \alpha \left( 1 - \frac{\beta}{N} \right) |u_n| \frac{N}{x^2} \right) \right] \frac{dx}{|x|^\beta}
\]

tends to 0 as \( n \to \infty \). Thus \( ATA_\beta (\alpha + \varepsilon_n) - ATA_\beta (\alpha) \downarrow 0 \) as \( n \to \infty \).

Similarly, we could also show that \( ATA_\beta (\alpha - \varepsilon_n) - ATA_\beta (\alpha) \uparrow 0 \) as \( n \to \infty \). Hence, we can now conclude that \( ATA_\beta (\cdot) \) is continuous on \( (0, \alpha (N, 2)) \). \( \square \)
Lemma 5.2. \( A_\beta (\alpha) \) is achieved if and only if \( LA_{\beta,M}(\alpha) \) and \( IA_{\beta,M}(\alpha) \) can be attained.

Proof. Let \( (u) \) be the maximizer of \( A_\beta (\alpha) \), i.e., \( u \in W^{2,\frac{N}{N-\beta}} (\mathbb{R}^N) \setminus \{0\} : \|\Delta u\|_{\frac{N}{N-\beta}}^\frac{N}{N-\beta} = \theta \in (0,1); \|u\|_{\frac{N}{N-\beta}} = 1 - \theta \) and

\[
\int_{\mathbb{R}^N} \phi_{N,2} \left( \alpha \left(1 - \frac{\beta}{N}\right) |u_n|^{\frac{N-\beta}{N-2}} \right) \frac{dx}{|x|^\beta} = A_\beta (\alpha).
\]

We define a new function:

\[
v(x) = \frac{u(\lambda x)}{\|\Delta u\|_{\frac{N}{N-\beta}}^\frac{N}{N-\beta}}
\]

\[
\lambda_n = \left( \frac{1 - \|\Delta u\|_{\frac{N}{N-\beta}}^\frac{N}{N-\beta}}{\|\Delta u\|_{\frac{N}{N-\beta}}^\frac{N}{N-\beta}} \right)^\frac{\beta}{N-\beta} = \left( \frac{1 - \theta}{\theta} \right)^\frac{\beta}{N-\beta} > 0.
\]

Then

\[
\|\Delta v\|_{\frac{N}{N-\beta}} = 1 \quad \text{and} \quad \|v\|_{\frac{N}{N-\beta}} = 1.
\]

Also,

\[
A_\beta (\alpha) = \int_{\mathbb{R}^N} \phi_{N,2} \left( \alpha \left(1 - \frac{\beta}{N}\right) |u|^{\frac{N-\beta}{N-2}} \right) \frac{dx}{|x|^\beta}
\]

\[
= \lambda^{N-\beta} \int_{\mathbb{R}^N} \phi_{N,2} \left( \alpha \left(1 - \frac{\beta}{N}\right) \theta^{\frac{\beta}{N-2}} |v|^{N/(N-2)} \right) \frac{dx}{|x|^\beta}
\]

\[
= \left( \frac{1 - \theta}{\theta} \right)^{\frac{N-\beta}{N-\beta}} \int_{\mathbb{R}^N} \phi_{N,2} \left( \alpha \left(1 - \frac{\beta}{N}\right) \theta^{\frac{\beta}{N-2}} |v|^{N/(N-2)} \right) \frac{dx}{|x|^\beta}.
\]

Next, we define

\[
w(x) = \eta v(\mu x)
\]

\[
\eta = \left( \frac{(M-1) \left(\frac{\alpha}{\alpha \theta^{\frac{N-\beta}{N-2}}} \right)^{\frac{N-\beta}{2}}}{M - \left(\frac{\alpha}{\alpha \theta^{\frac{N-\beta}{N-2}}} \right)^{\frac{N-\beta}{2}}} \right)^\frac{\beta}{N-\beta} = \left( \frac{(M-1) \theta}{M - \theta} \right)^\frac{\beta}{N-\beta}
\]

\[
\mu = \left( \frac{\eta^{\frac{\beta}{N-\beta}}}{1 - \eta^{\frac{\beta}{N-\beta}}} \right)^\frac{\beta}{N-\beta}.
\]

Then

\[
\|\Delta w\|_{\frac{N}{N-\beta}}^\frac{N}{N-\beta} = \eta^\frac{N}{N-\beta} \|\Delta v\|_{\frac{N}{N-\beta}}^\frac{N}{N-\beta} = \eta^\frac{N}{N-\beta}
\]

\[
\|w\|_{\frac{N}{N-\beta}}^\frac{N}{N-\beta} = \eta^\frac{N}{N-\beta} \mu^\frac{\beta}{N-\beta} \|v\|_{\frac{N}{N-\beta}}^\frac{N}{N-\beta} = 1 - \eta^\frac{N}{N-\beta}.
\]
Hence \( \| \Delta u \|_{\frac{N}{2}}^\frac{N}{2} + \| u \|_{\frac{N}{2}}^\frac{N}{2} = 1 \). We have

\[
\left( \frac{\theta}{1 - \theta} \right)^{\frac{N - \beta}{N}} \cdot A_\beta(\alpha)
= \frac{1}{\| w \|_{\frac{N}{2}}^\frac{N}{2}} \int_{\mathbb{R}^N} \phi_{N,2} \left( \alpha \theta^{\frac{N - 2}{2}} \left( 1 - \frac{\beta}{N} \right) |w|^{N/(N - 2)} \right) \frac{dx}{|x|^\beta},
\]

\[
= \int_{\mathbb{R}^N} \phi_{N,2} \left( \alpha \left( 1 - \frac{\beta}{N} \right) |w|^{N/(N - 2)} \right) \frac{d(\mu x)}{|\mu x|^\beta},
\]

\[
= \mu^{N - \beta} \int_{\mathbb{R}^N} \phi_N \left( \frac{M^{\frac{1}{N - 1}} \alpha \left( 1 - \frac{\beta}{N} \right) |w|^{N/(N - 2)}}{M - 1 + \| \Delta u \|_{\frac{N}{2}}^\frac{N}{2}} \frac{1}{|x|^\beta} \right) dx.
\]

Noting that \( LA_{\alpha, M}(\alpha) = IA_{\beta, M}(\alpha) = \left( \frac{M}{M - 1} \right)^{\frac{N - \beta}{N}} A_\beta(\alpha) \), we get that \( w \) is an optimizer for \( IA_{\beta}(\alpha) \) and \( LA_{\beta}(\alpha) \).

Moreover, the above process can be reversed. Hence, we can conclude that \( LA_{\alpha, M}(\alpha) \) and \( IA_{\beta, M}(\alpha) \) can be attained if and only if \( A_\beta(\alpha) \) is achieved.

**Lemma 5.3.** Let \( N \geq 3, 0 < \beta < N \). Then \( A_\beta(\alpha) \) can be attained for all \( 0 < \alpha < \alpha(N, 2) \).

**Proof.** We recall that

\[
A_\beta(\alpha) = \sup_{s \in (0, \alpha)} \left( 1 - \frac{(\frac{\alpha}{s})^{\frac{N - 2}{2}}}{(\frac{\alpha}{s})^{\frac{N - 2}{2}}} \right)^{\frac{N - \beta}{N}} \cdot ATA_\beta(s).
\]

Also, it is clear that

\[
\lim_{s \uparrow \alpha} \left( 1 - \frac{(\frac{\alpha}{s})^{\frac{N - 2}{2}}}{(\frac{\alpha}{s})^{\frac{N - 2}{2}}} \right)^{\frac{N - \beta}{N}} \cdot ATA_\beta(s) = 0.
\]

Now, we will also show that

\[
\lim_{s \downarrow 0} \left( 1 - \frac{(\frac{\alpha}{s})^{\frac{N - 2}{2}}}{(\frac{\alpha}{s})^{\frac{N - 2}{2}}} \right)^{\frac{N - \beta}{N}} \cdot ATA_\beta(s) = 0.
\]

Indeed, let \( s_n \downarrow 0 \) and \( u_n : \| \Delta u_n \|_{\frac{N}{2}} = \| u_n \|_{\frac{N}{2}} = 1 \) be the optimizers for \( ATA_\beta(s_n) \).

Then

\[
ATA_\beta(s_n) = \int_{\mathbb{R}^N} \phi_{N,2} \left( s_n \left( 1 - \frac{\beta}{N} \right) |u_n|^{N/(N - 2)} \right) \frac{dx}{|x|^\beta}.
\]
Then

\[ \ds \frac{1}{\gamma} = \frac{1}{\gamma} \]
We note here that $\|\Delta w_\alpha\|_r^r + \|w_\alpha\|_r^r = 1$. Also,

$$\text{ATA}_\beta (s_\alpha) = \int_{\mathbb{R}^N} \phi_{N,2} \left( s_\alpha (1 - \frac{\beta}{N}) |v_\alpha|^{N/(N-2)} \right) \frac{dx}{|x|^\beta}$$

$$= \int_{\mathbb{R}^N} \phi_{N,2} \left( s_\alpha (1 - \frac{\beta}{N}) |v_\alpha (\mu x)|^{N/(N-2)} \right) \frac{d(\mu x)}{|\mu x|^\beta}$$

$$= \mu^{N-\beta} \int_{\mathbb{R}^N} \phi_{N,2} \left( \alpha \left( 1 - \frac{\beta}{N} \right) |w_\alpha|^{N/(N-2)} \right) \frac{dx}{|x|^\beta}$$

$$= \left( \frac{s_\alpha}{\alpha} \right)^{\frac{N-2}{N-2}} \int_{\mathbb{R}^N} \phi_{N,2} \left( \alpha \left( 1 - \frac{\beta}{N} \right) |w_\alpha|^{N/(N-2)} \right) \frac{dx}{|x|^\beta}$$

Hence

$$\int_{\mathbb{R}^N} \phi_{N,2} \left( \alpha \left( 1 - \frac{\beta}{N} \right) |w_\alpha|^{N/(N-2)} \right) \frac{dx}{|x|^\beta} = \left( 1 - \frac{s_\alpha}{\alpha} \right)^{\frac{N-2}{N-2}} \left( \frac{\alpha}{\alpha} \right)^{\frac{N-2}{N-2}} = \alpha_{\beta} (\alpha).$$

That means that $w_\alpha$ is a maximizer for $A_\beta (\alpha)$.

**Lemma 5.4.** If $N \geq 4$ and $N$ is even, then $A_0 (\alpha) \geq \frac{\alpha^{N-2}}{(\frac{N}{2})!}$ for all $0 \leq \alpha \leq 2 (N/2)$.

**Proof.** We choose a smooth function $v$ such that $\|v\|_r^r = 1$, and set

$$v_t (x) = t^{2/N} v \left( \frac{x}{t^2/N} \right),$$

then it is easy to check that

$$\|\Delta v_t\|_r^r = t^{2/N} \|\Delta v\|_r^r;$$

$$\|v_t\|_p = t^{2-p/N} \|v\|_p.$$
**Lemma 5.5.** \( A_0(\alpha) \) cannot be achieved for \( \alpha \geq 0 \) if \( N = 4 \).

**Proof.** Indeed, it is enough to show that for \( \alpha \geq 0 \):

\[
\int_{\mathbb{R}^N} \Phi_{N,2} \left( \alpha \left| u \right|^{\frac{N-2}{2}} \right) \ dx < \frac{\alpha^{N-2}}{(N-2)!} \ \forall u \neq 0 : \ |\Delta u|^{\frac{N}{2}} + \ |u|^{\frac{N}{2}} = 1. 
\]

We first note that by definition: for any \( \gamma \):

\[
\int_{\mathbb{R}^N} \Phi_{N,2} \left( \gamma \left| u \right|^{\frac{N}{2}} \right) \ dx \leq \ A_0(\gamma) \ \frac{|\Delta u|^{\frac{N}{2}}}{|\Delta u|^{\frac{N}{2}}}.
\]

Hence

\[
\left| u \right|^{\frac{N}{2}} + \left| u \right|^{\frac{N}{2}} \leq \ A_0(\gamma) \ \frac{(N-2+k)!}{\gamma^{N-2+k}} \left| \Delta u \right|^{\frac{N}{2}} \left| u \right|^{\frac{N}{2}} \ \forall k \geq 0
\]

\[
\Rightarrow \ A_0(\gamma) \ \frac{(N-2+k)!}{\gamma^{N-2+k}} \left| \Delta u \right|^{\frac{N}{2}} \left| u \right|^{\frac{N}{2}} \ \forall k \geq 0
\]

where \( \theta = \frac{|\Delta u|^{\frac{N}{2}}}{\gamma} \in (0,1) \).

Hence

\[
\int_{\mathbb{R}^N} \Phi_{N,2} \left( \alpha \left| u \right|^{\frac{N}{2}} \right) \ dx
\]

\[
= \frac{\alpha^{N-2}}{(N-2)!} \left| u \right|^{\frac{N}{2}} + \sum_{k=1}^{\infty} \frac{\alpha^{N-2+k}}{(N-2+k)!} \left| u \right|^{\frac{N}{2} + \frac{N}{2}}
\]

\[
\leq \frac{\alpha^{N-2}}{(N-2)!} \left| u \right|^{\frac{N}{2}} + \ A_0(\gamma) \sum_{k=1}^{\infty} \frac{\alpha^{N-2+k}}{(N-2+k)!} \frac{(N-2+k)!}{\gamma^{N-2+k}} \left| \Delta u \right|^{\frac{N}{2}} \left| u \right|^{\frac{N}{2}} \left( 1 - \theta^\frac{N}{2} \right)
\]

\[
= \frac{\alpha^{N-2}}{(N-2)!} \left| u \right|^{\frac{N}{2}} + \ A_0(\gamma) \left| \Delta u \right|^{\frac{N}{2}} \left| u \right|^{\frac{N}{2}} \left( 1 - \theta^\frac{N}{2} \right) \left[ \sum_{k=1}^{\infty} \frac{\alpha^{k}}{\gamma^k} \theta^{\frac{N}{2} - k} \right]
\]

Hence it is enough to choose \( \alpha \) such that

\[
\ A_0(\gamma) \frac{\alpha^{N-2}}{\gamma^{\frac{N}{2}}} \left( 1 - \theta^\frac{N}{2} \right) \left[ \frac{\alpha \theta^{\frac{N}{2}}}{\gamma - \alpha \theta^\frac{N}{2}} \right] \leq \frac{(N-2)!}{(N-2)!} \theta^\frac{N}{2}
\]

i.e.

\[
\alpha < \frac{\gamma \theta^\frac{N}{2}}{(N-2)!} \left[ \frac{1}{\gamma^{\frac{N}{2}}} + \frac{\alpha \theta^\frac{N}{2}}{(N-2)!} \right] \ \forall \theta \in (0,1).
\]
Noting that by L’Hôpital’s rule and $N = 4$:

$$\lim_{\theta \to 1} \frac{\gamma \theta^\frac{N}{2}}{(N-2)!} \frac{1}{\gamma} \frac{1}{\theta_{\gamma}^{\frac{N}{2}} + \theta \frac{\gamma^{N/2}}{(N-2)!}} > 0,$$

and

$$\lim_{\theta \to 0} \frac{\gamma \theta^\frac{N}{2}}{(N-2)!} \frac{1}{\gamma} \frac{1}{(1-\theta)^{\frac{N}{2}} \theta^{\frac{N}{2}} + \theta \frac{\gamma^{N/2}}{(N-2)!}} > 0,$$

we have

$$\inf_{\theta \in (0, 1)} \frac{\gamma \theta^\frac{N}{2}}{(N-2)!} \frac{1}{\gamma} \frac{1}{(1-\theta)^{\frac{N}{2}} \theta^{\frac{N}{2}} + \theta \frac{\gamma^{N/2}}{(N-2)!}} > 0.$$

Hence we now can conclude that $A_0(\alpha)$ cannot be achieved for $\alpha > 0$ if $N = 4$. 

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