Stress-strain state of the interfacial layer in a visco-composite composite with longitudinal shear

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Annotation. In this paper, we propose a three-phase method for determining the effective properties of reinforced by whiskerized fibers composite materials. The model takes into account the presence of a whiskerized interfacial layer between the fiber and the matrix. The calculation of the stress-strain state of the interfacial layer loaded with a longitudinal shear using the Eshelby self-consistent field method is given.

Key Words: Fiber-reinforced interfacial layer, fibrous composites, Eshelby self-consistent method, micromechanics

1. Introduction
Composite materials are widely used in aircraft and rocket structures, and also find wide application in other areas of technology, including transport, medicine, and others. Composite structures with various combinations of components in a structure have many advantages compared to traditional materials based on metal alloys. It is known that the mechanical properties of composites are controlled by the contact conditions between the fiber and the matrix (i.e. interface characteristics) in the composite \cite{1, 2}. The influence of the structure on the nature of the interaction of the phases in the region of the contact of the phases turns out to be particularly significant. Various techniques have been developed to improve the quality of the interface, and the most common are: modifying the fiber surface, improving chemical interactions, or adding a third phase (interfacial layer) between the fiber and the matrix \cite{3-5}. The ideas underlying these methods are to improve the interfacial adhesion properties and to increase the surface area of the fiber for more efficient transfer of loads between the fibers and the matrix and to further improve the properties of the composites.

Currently, technologies for the production of modern fiber composites are being actively developed, in which special microstructures containing nanofibers (whiskers) - nanowires \cite{4, 6, 7} are grown on the circular surface of carbon fibers to increase the shear properties of the composite. For composite materials with whiskerized fibers, a higher ultimate strength and shear stiffness is realized compared to standard composites that do not have an additional microstructure on the fiber surface. In addition,
the presence of these microstructures increases the transversal strength and rigidity, as well as the
damping characteristics and electrical conductivity of the composites [8, 9, 10].
The aim of the work is to substantiate and develop a model of functional fibrous non-uniform
composite materials based on an analytical solution and able to reliably predict the elastic mechanical
properties of composites on the properties and phase concentrations taking into account the
characteristics of micro- or nanostructures (whiskers) grown on the surface of the base fiber.

2. Approximate structural model of the whiskerized interfacial layer. Effective properties

Fig. 1 shows a structural model of a whiskerized interfacial layer, which is built to find the effective
properties of a whiskerized interfacial layer, taking into account various characteristics of whiskers
(i.e. length, density, diameter, properties of whiskers) in this layer.

Fig. 1. An approximate structural model of a whiskerized interfacial layer.

Now we determine the maximum allowable number of whiskers $M_b^{\text{max}}$, which can fit on the
circumference of the fiber by the formula

$$M_b^{\text{max}} = \frac{\pi D}{d_b},$$  \hspace{1cm} (1)

where $D$ is the diameter of the base fiber; $d_b$ - diameter of whiskers. The angle between the whiskers $\alpha$ is given as

$$\alpha = 2\pi M_b,$$  \hspace{1cm} (2)

where $M_b$ is the number of whiskers grown on the surface of the fiber. The distance $h_1$ between the
whiskers is given by the formula

$$h_1 = \alpha R,$$  \hspace{1cm} (3)

where $R = D/2$ is the radius of the base fiber. The distance between the whiskers on the unit length of
the fiber $h_2$ we write as:

$$h_2 = Ph_1,$$  \hspace{1cm} (4)

where $P$ is a constant that defines the relationship between $h_1$ and $h_2$. The volume of whiskers in the
whiskerized interfacial layer $V_b$ is determined by the formula

$$V_b = \pi M_b l_b d_b^2/4,$$  \hspace{1cm} (5)

where $l_b$ is the length of the whiskers, and the volume of the whiskerized interfacial layer is

$$V_{\text{wm}} = \pi \left(r_2^2 - r_1^2\right) h_2,$$  \hspace{1cm} (6)
where \( r_1 = R \) and \( r_2 = R + l_b \). So, using the relation (5) and (6), we determine the volume fraction of whiskers in this layer:

\[
c_b = \frac{M_b d_b^3}{4(l_b + D)h_2}.
\]

(7)

Since, for simplicity, the cell at the root of the whiskers is considered square, we have \( p = 1 \). Therefore, based on (2), (3) and (4), we can rewrite (7) in the form

\[
c_b = \frac{M_b^2 d_b^4}{4\pi(l_b + D)D}.
\]

(8)

The density of whiskers is related to the number of whiskers grown on a single fiber surface. However, for simplicity, we determine in the calculation that the density of whiskers is the ratio of the number of whiskers grown on the fiber to the maximum allowable number of whiskers that fit along the circumference of the fiber. We have

\[
\rho_b = \left( \frac{M_b}{M_b^{max}} \right) 100%.
\]

(9)

After obtaining an estimate of the concentration of whiskers, we can use the well-known formulas of the three-phase method given in [11] for a two-phase composite to determine the effective properties of the whisker interfacial layer (see Fig. 2).

Fig. 2. Cell of the whiskerized interfacial layer using the three phase method.

Young's longitudinal modulus \( E_{11}^{eff} \):

\[
E_{11}^{eff} = c_b E_b + (1 - c_b) E_m + \frac{4c_b(1 - c_b)(v_b - v_m)^3}{(1 - c_b)\mu_m/(k_b + \mu_b/3) + c_b\mu_m/(k_m + \mu_m/3) + 1},
\]

(10)

Poisson's ratio (along the fibers) \( \nu_{12}^{eff} \):

\[
\nu_{12}^{eff} = c_b v_b + (1 - c_b) v_m + \frac{c_b(1 - c_b)(v_b - v_m)(\mu_m/(k_m + \mu_m/3) - \mu_m/(k_b + \mu_b/3))}{(1 - c_b)(\mu_m/(k_b + \mu_b/3)) + c_b(\mu_m/(k_m + \mu_m/3) + 1)},
\]

(11)

Longitudinal shear modulus \( \mu_{12}^{eff} \):

\[
\mu_{12}^{eff} = \frac{\mu_b(1 + c_b) + \mu_m(1 - c_b)}{\mu_b(1 - c_b) + \mu_m(1 + c_b)},
\]

(12)

Volumetric modulus of flat deformation \( K_{23}^{eff} \):

\[
K_{23}^{eff} = k_m + \frac{\mu_m}{3} \left( \frac{1}{(k_b - k_m + \frac{1}{2}\Delta\mu_m)} + (1 - c_b)/(k_m + 4\mu_m/3) \right)
\]

(13)
Transverse shear modulus $\mu_{23}^{\text{eff}}$:

$$A \left( \frac{\mu_{23}^{\text{eff}}}{\mu_m} \right)^2 + 2B \left( \frac{\mu_{23}^{\text{eff}}}{\mu_m} \right) + C = 0,$$

where $\eta = 3 - 4\nu$ and

$$A = 3c_b (1 - c_b)^2 \left( \frac{\mu_b}{\mu_m} - 1 \right) \left( \frac{\mu_b}{\mu_m} - \eta_b \right) +$$

$$+ \left[ \frac{\mu_b}{\mu_m} \eta_m + \eta_b \right] c_b + \left[ \frac{\mu_b}{\mu_m} \eta_m - \eta_b \right] c_b^2 \right],$$

$$B = -3c_b (1 - c_b)^2 \left( \frac{\mu_b}{\mu_m} - 1 \right) \left( \frac{\mu_b}{\mu_m} + \eta_b \right) +$$

$$+ \frac{1}{2} \left[ \eta_m \left( \frac{\mu_b}{\mu_m} - \eta_b \right) \right] c_b + \left[ \frac{\mu_b}{\mu_m} \eta_m - \eta_b \right] c_b^2 \right],$$

$$C = 3c_b (1 - c_b)^2 \left( \frac{\mu_b}{\mu_m} - 1 \right) \left( \frac{\mu_b}{\mu_m} + \eta_b \right) +$$

$$+ \left[ \eta_m \left( \frac{\mu_b}{\mu_m} - \eta_b \right) \right] c_b + \left[ \frac{\mu_b}{\mu_m} \eta_m - \eta_b \right] c_b^2 \right].$$

2.1 The model of whiskerized fiber composites, based on the Eshelby self-consistency method (three-phase method).

Fig. 3. (a) Cell of the whiskerized fiber composite; (b) cell of an effective composite (homogenized composite).
In each case of loading, the general admissible displacement fields for the orthotropic phases of the composite are determined from the solution of the corresponding problem of the theory of elasticity. With the help of the Cauchy relations and the Hooke’s equations, specific deformation and stress fields are found for all phases in the composite. Further, these fields of displacements and stresses are substituted into the boundary conditions at the boundaries of the phase contacts: the conditions of continuity of displacement and stress, the non-singularity conditions at the center of the fiber and the conditions at infinity (external boundary condition). Based on these conditions, a system of algebraic equations is formulated, from which, using the Eshelby integral formula

$$\int_S (\sigma_{ij}^{N-i} u_i^{N-j} - \sigma_{ij}^{N-j} u_i^{N-i}) n_j dS = 0,$$

(15)

all unknown constants are found and, therefore, all effective elastic moduli, which were also included in the number of unknown constants.

The tensors of elastic moduli ($C_{ijkl}^{(i)}$) phases in the composite that will be used for the proposed model are shown here. Based on Hooke’s rule, the tensor of elastic moduli of the first phase ($i = 1$) or base fiber transversely isotropic medium with axis of symmetry directed along its fiber axis has the form

$$
\begin{cases}
\sigma_{rr}^{(1)} = \begin{bmatrix}
C_{11}^{(1)} & C_{12}^{(1)} & C_{13}^{(1)} & 0 & 0 & 0 \\
C_{12}^{(1)} & C_{11}^{(1)} & C_{13}^{(1)} & 0 & 0 & 0 \\
C_{13}^{(1)} & C_{13}^{(1)} & C_{33}^{(1)} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44}^{(1)} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{66}^{(1)} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{44}^{(1)} - C_{66}^{(1)}
\end{bmatrix},
\end{cases}
$$

(16)

where the indices of the components of the tensor of elastic moduli $r, \theta, z$ are replaced by indices $1, 2, 3$, respectively. For the second phase ($i = 2$) or a whiskerized interfacial layer, the tensor of elastic moduli takes the form

$$
\begin{cases}
\sigma_{rr}^{(2)} = \begin{bmatrix}
C_{11}^{(2)} & C_{12}^{(2)} & C_{13}^{(2)} & 0 & 0 & 0 \\
C_{12}^{(2)} & C_{12}^{(2)} & C_{23}^{(2)} & 0 & 0 & 0 \\
C_{13}^{(2)} & C_{23}^{(2)} & C_{22}^{(2)} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{66}^{(2)} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{66}^{(2)} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{66}^{(2)}
\end{bmatrix},
\end{cases}
$$

(17)
It can be seen that the axis of symmetry of this transversely isotropic layer differs from the axis of the isotropy plane of the fiber. The obtained effective physical elastic moduli in (10-14) can be converted to elastic moduli \( C^{(2)}_{ij} \) (17) using the following relations:
\[
\begin{align*}
C_{11} &= E_{11} + 4\nu_{12}^2 K_{23}, \\
C_{12} &= 2K_{23}\nu_{12}, \\
C_{22} &= \mu_{23} + K_{23}, \\
C_{23} &= -\mu_{23} + K_{23}, \\
C_{66} &= \mu_{12} = \mu_{13}.
\end{align*}
\]
(18)

Further, the tensor of elastic moduli of an isotropic matrix or the third phase \((i = N = 3)\) can be written as
\[
\begin{pmatrix}
\sigma_{rr}^{(3)} \\
\sigma_{\theta\theta}^{(3)} \\
\sigma_{zz}^{(3)} \\
\sigma_{rz}^{(3)} \\
\sigma_{\theta r}^{(3)} \\
\sigma_{\theta z}^{(3)}
\end{pmatrix} = \begin{pmatrix}
C_{11}^{(3)} & C_{12}^{(3)} & C_{13}^{(3)} & 0 & 0 & 0 \\
C_{12}^{(3)} & C_{11}^{(3)} & C_{13}^{(3)} & 0 & 0 & 0 \\
C_{13}^{(3)} & C_{12}^{(3)} & C_{11}^{(3)} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{11}^{(3)} - C_{12}^{(3)} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{11}^{(3)} - C_{12}^{(3)} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{11}^{(3)} - C_{12}^{(3)}
\end{pmatrix} \begin{pmatrix}
\varepsilon_{rr}^{(3)} \\
\varepsilon_{\theta\theta}^{(3)} \\
\varepsilon_{zz}^{(3)} \\
\varepsilon_{rz}^{(3)} \\
\varepsilon_{\theta r}^{(3)} \\
\varepsilon_{\theta z}^{(3)}
\end{pmatrix},
\]
(19)

The elastic moduli tensor for an equivalent homogeneous medium or external phase \((i = N + 1)\) is assumed to be equal to the effective elastic modulus of the effective homogenized material and is the desired quantity. In addition, this medium has transversely isotropic properties along the axis of symmetry, directed along the axis of the base fiber, and therefore we have
\[
\begin{pmatrix}
\sigma_{rr}^{N+1} \\
\sigma_{\theta\theta}^{N+1} \\
\sigma_{zz}^{N+1} \\
\sigma_{rz}^{N+1} \\
\sigma_{\theta r}^{N+1} \\
\sigma_{\theta z}^{N+1}
\end{pmatrix} = \begin{pmatrix}
C_{11}^{N+1} & C_{12}^{N+1} & C_{13}^{N+1} & 0 & 0 & 0 \\
C_{12}^{N+1} & C_{11}^{N+1} & C_{13}^{N+1} & 0 & 0 & 0 \\
C_{13}^{N+1} & C_{12}^{N+1} & C_{11}^{N+1} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44}^{N+1} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{44}^{N+1} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{44}^{N+1}
\end{pmatrix} \begin{pmatrix}
\varepsilon_{rr}^{N+1} \\
\varepsilon_{\theta\theta}^{N+1} \\
\varepsilon_{zz}^{N+1} \\
\varepsilon_{rz}^{N+1} \\
\varepsilon_{\theta r}^{N+1} \\
\varepsilon_{\theta z}^{N+1}
\end{pmatrix},
\]
(20)

Finally, the elastic modulus tensor for an effective composite (medium without inclusions or a homogenized composite that has equivalent transversely isotropic properties) has the same elastic moduli tensor of the equivalent homogeneous medium \((i = C_{ij}^{eff} = C_{ij}^{N+1})\), and we write this in the form:
\[
\begin{pmatrix}
\sigma_{rr}^{eff} \\
\sigma_{\theta\theta}^{eff} \\
\sigma_{zz}^{eff} \\
\sigma_{rz}^{eff} \\
\sigma_{\theta r}^{eff} \\
\sigma_{\theta z}^{eff}
\end{pmatrix} = \begin{pmatrix}
C_{11}^{eff} & C_{12}^{eff} & C_{13}^{eff} & 0 & 0 & 0 \\
C_{12}^{eff} & C_{11}^{eff} & C_{13}^{eff} & 0 & 0 & 0 \\
C_{13}^{eff} & C_{12}^{eff} & C_{11}^{eff} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44}^{eff} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{44}^{eff} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{44}^{eff}
\end{pmatrix} \begin{pmatrix}
\varepsilon_{rr}^{eff} \\
\varepsilon_{\theta\theta}^{eff} \\
\varepsilon_{zz}^{eff} \\
\varepsilon_{rz}^{eff} \\
\varepsilon_{\theta r}^{eff} \\
\varepsilon_{\theta z}^{eff}
\end{pmatrix},
\]
(21)

2.2 Definition of longitudinal shear modulus
2.2.1. Statement of the problem of pure shear in the direction along the fibers in cylindrical coordinates for an orthotropic multiphase medium.

It is assumed that the fiber axes are directed along the z axis (axis 3) in a cylindrical coordinate system. The effective longitudinal shear modulus is determined from the solution of the pure shear problem along the fibers (see Fig. 5). To begin with, it is necessary to find the general fields of displacement of orthotropic phases that satisfy the equilibrium equations of this problem. According to [12, 13, 14], the field of displacement along the fibers \( u_i(r, \theta) \) is specified on the outer border of the cell. In this case, there are only phase movements along the fibers, which depend on the radius and angle, and the angular and radial movements are equal to zero:

\[
  u_i^{(i)}(r, \theta) \neq 0, \quad u_i^{(i)} = u_i^{(i)} = 0, \quad \text{where } i \text{ is the phase of the composite.}
\]

For simplicity, we introduce the notation \( u_i(r, \theta) = u_i \), and by the Cauchy deformation relations:

\[
  \varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{1}{r} \left( \frac{\partial u_\theta}{\partial \theta} + u_r \right), \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z},
  \varepsilon_{rz} = \frac{1}{2} \left( \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right), \quad \varepsilon_{r\theta} = \frac{1}{2} \left( \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right),
\]

and (22) we get

\[
  \varepsilon_{rr} = \frac{\partial u_r}{\partial z}, \quad 2\varepsilon_{r\theta} = \frac{\partial u_r}{\partial \theta} + \frac{1}{r} \frac{\partial u_\theta}{\partial r} = \frac{1}{r} \frac{\partial u_\theta}{\partial \theta}, \quad 2\varepsilon_{rz} = \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}
  \varepsilon_{\theta\theta} = \varepsilon_{\theta\theta} = 0.
\]

With the help of Hooke's law

\[
  \begin{bmatrix}
    \sigma_{rr} \\
    \sigma_{r\theta} \\
    \sigma_{zr} \\
    \sigma_{\theta\theta} \\
    \sigma_{zz} \\
    \sigma_{r\theta}
  \end{bmatrix}
  =
  \begin{bmatrix}
    C_{rr} & C_{r\theta} & C_{rz} & 0 & 0 & 0 \\
    C_{r\theta} & C_{\theta\theta} & C_{\theta z} & 0 & 0 & 0 \\
    C_{rz} & C_{\theta z} & C_{zz} & 0 & 0 & 0 \\
    0 & 0 & 0 & G_{\theta z} & 0 & 0 \\
    0 & 0 & 0 & 0 & G_{zz} & 0 \\
    0 & 0 & 0 & 0 & 0 & G_{r\theta}
  \end{bmatrix}
  \begin{bmatrix}
    \varepsilon_{rr} \\
    \varepsilon_{r\theta} \\
    \varepsilon_{rz} \\
    \varepsilon_{\theta\theta} \\
    \varepsilon_{zz} \\
    \varepsilon_{r\theta}
  \end{bmatrix},
\]

together with relations (24) we have
\[ \sigma_{\theta z} = G_{\theta z} 2 \varepsilon_{\theta z} = G_{\theta z} \left( \frac{1}{r} \frac{\partial u_z}{\partial \theta} \right), \]
\[ \sigma_{rz} = G_{rz} 2 \varepsilon_{rz} = G_{rz} \left( \frac{\partial u_z}{\partial r} \right). \]

Substituting (26) into the equilibrium equation
\[ \frac{\partial \sigma^{(i)}_{\theta \theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma^{(i)}_{\theta \theta}}{\partial \theta} + \frac{\sigma^{(i)}_{\theta \theta} - \sigma^{(i)}_{\theta \theta}}{r} + \frac{\partial \sigma^{(i)}_{rz}}{\partial r} = 0, \]
\[ \frac{\partial \sigma^{(i)}_{r \theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma^{(i)}_{r \theta}}{\partial \theta} + 2 \frac{\sigma^{(i)}_{r \theta}}{r} + \frac{\partial \sigma^{(i)}_{rz}}{\partial r} = 0, \]
\[ \frac{\partial \sigma^{(i)}_{r z}}{\partial r} + \frac{1}{r} \frac{\partial \sigma^{(i)}_{r z}}{\partial \theta} + \frac{\sigma^{(i)}_{r z} + \sigma^{(i)}_{r z}}{r} = 0, \]

We get:
\[ \frac{\partial}{\partial r} \left[ G_{\theta z} \left( \frac{\partial u_z}{\partial r} \right) \right] + \frac{1}{r} \frac{\partial}{\partial \theta} \left( G_{\theta z} \left( \frac{1}{r} \frac{\partial u_z}{\partial \theta} \right) \right) + \frac{1}{r} \left[ G_{rz} \left( \frac{\partial u_z}{\partial r} \right) \right] = 0 \]
\[ \Rightarrow G_{\theta z} \frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r} G_{\theta z} \frac{\partial u_z}{\partial r} + \frac{1}{r^2} G_{rz} \frac{\partial^2 u_z}{\partial \theta^2} = 0 \]
\[ \Rightarrow r^2 \frac{\partial^2 u_z}{\partial r^2} + r \frac{\partial u_z}{\partial r} + \lambda^2 \frac{\partial^2 u_z}{\partial \theta^2} = 0 \]
\[ \Rightarrow r^2 u''_z(r) + ru'_z(r) + \lambda^2 u''_z(\theta) = 0, \tag{28} \]

where \( \lambda = \sqrt{\frac{G_{\theta z}}{G_{rz}}} \). Another form of differential equation (28) can be written as
\[ r^2 R^2(r)T(\theta) + r R(\theta)T'(\theta) + \lambda^2 R(r)T''(\theta) = 0 \]
\[ \Rightarrow \frac{r^2 R'(r) + r R'(r)}{-R(r)} = \lambda^2 \frac{T''(\theta)}{T(\theta)}, \tag{29} \]

where the general solution of this equation has the form
\[ u_z(r, \theta) = R(r)T(\theta) = u_z(r)u_z(\theta), \tag{30} \]

Now we find the solution form for \( u_z(r) \). Let the right side of equation (29) be given by some constant, i.e. \( T''(\theta)/T(\theta) = K \), and then equation (29) has the form
\[ \frac{r^2 R' + r R'}{-\lambda^2 R} = K, \tag{31} \]

Let \( K = -1 \), and therefore, equation (31) takes the form
\[ r^2 R'(r) + r R'(r) - \lambda^2 R(r) = r^2 u'_z(r) + ru'_z(r) - \lambda^2 u_z(r) = 0, \tag{32} \]

By making in (32) a variable change
\[ u_z = r^m, \quad u'_z = mr^{m-1}, \quad u''_z = m(m-1)r^{m-2}, \tag{33} \]

we arrive at a characteristic equation for \( m \):
\[ r^2 \left[ m(m-1)r^{m-2} \right] + r \left[ mr^{m-1} \right] - \lambda^2 r^m = 0 \]
\[ \Rightarrow m^2 - \lambda^2 = 0 \]
\[ \Rightarrow m_{1,2} = \pm \lambda. \tag{34} \]

Thus, the general solution of the differential equation (35) is
\[
R(r) = u_r(r) = A_1 r^d + A_2 r^{-d},
\]
(35)

Next, look for the general form of solutions for \( u_r(r) \). Similarly, let the left side of equation (32) be \(-1\), and therefore we have
\[
T'(\theta) + T(\theta) = 0,
\]
(36)
and the general form of solving the differential equation (36) can be written as
\[
T(\theta) = u_\epsilon(\theta) = B_1 \cos \theta + B_2 \sin \theta,
\]
(37)

However, based on the state of cell deformation under the action of a pure longitudinal shear, the constant \( B_2 \) must be equal to zero, and therefore the solution (37) is reduced to
\[
T(\theta) = u_\epsilon(\theta) = B_1 \cos \theta,
\]
(38)

Thus, on the basis of (35) and (38) we have
\[
(12(\lambda_1, \lambda_2, d) \cos \theta \sin \theta) = \left( A_1 r^d + A_2 r^{-d} \right) B_1 \cos \theta = \left( A_1 B_1 r^d + A_2 B_2 r^{-d} \right) \cos \theta.
\]
(39)

So, the general allowable fields of displacement of an orthotropic multiphase composite are:
\[
u_i^{(i)}(r, \theta) = D_i^{(i)}(r \lambda_{i\alpha} + D_2^{(i)} r^{-\lambda_{i\alpha}}) \cos \theta,
\]
(40)

where \( D_i^{(i)}, D_2^{(i)} \) - unknown constants, which are determined from the boundary conditions; and \( \lambda_{i\alpha} = \sqrt{G_{i\alpha}^{(i)} / G_{i\alpha}^{(i)} \cdot \text{material phase constant.}} \)

Consequently, using the Cauchy relations, one can write the admissible deformation fields of orthotropic phases in the following form
\[
\begin{align*}
2\varepsilon_{\theta\theta}^{(i)}(r, \theta) &= \frac{1}{r} \frac{\partial u_r}{\partial r} = \frac{1}{r} \frac{\partial}{\partial r} \left[ \left( D_i^{(i)} r^{\lambda_{i\alpha}} + D_2^{(i)} r^{-\lambda_{i\alpha}} \right) \cos \theta \right] = \\
&= \left( D_i^{(i)} r^{\lambda_{i\alpha} - 1} + D_2^{(i)} r^{-\lambda_{i\alpha} - 1} \right) (-\sin \theta),
\end{align*}
\]
(41)
and using Hooke's law for (41) and (42), the admissible stress fields of orthotropic phases take the form
\[
\begin{align*}
\sigma_{\theta\theta}^{(i)}(r, \theta) &= G_{\theta\theta}^{(i)} 2\varepsilon_{\theta\theta}^{(i)} = -G_{\theta\theta}^{(i)} \left( D_i^{(i)} r^{\lambda_{i\alpha} - 1} + D_2^{(i)} r^{-\lambda_{i\alpha} - 1} \right) \sin \theta = \\
&= -C_{44}^{(i)} \left( D_i^{(i)} r^{\lambda_{i\alpha} - 1} + D_2^{(i)} r^{-\lambda_{i\alpha} - 1} \right) \sin \theta,
\end{align*}
\]
(43)

\[
\begin{align*}
\sigma_{r\theta}^{(i)}(r, \theta) &= G_{r\theta}^{(i)} 2\varepsilon_{r\theta}^{(i)} = G_{r\theta}^{(i)} \left( D_i^{(i)} r^{\lambda_{i\alpha} - 1} + (-\lambda_{i\alpha}) D_2^{(i)} r^{-\lambda_{i\alpha} - 1} \right) \cos \theta = \\
&= C_{55}^{(i)} \left( D_i^{(i)} r^{\lambda_{i\alpha} - 1} + (-\lambda_{i\alpha}) D_2^{(i)} r^{-\lambda_{i\alpha} - 1} \right) \cos \theta.
\end{align*}
\]
(44)

It can be seen that in equations (43) and (44), the tensors of elastic moduli \( G_{\theta\theta}^{(i)} \) and \( G_{r\theta}^{(i)} \) are replaced by indices \( C_{55}^{(i)} \) and \( C_{44}^{(i)} \), respectively.

In the case of isotropic or transversely isotropic media with the axis of the isotropy plane directed along the fibers, the material phase constant is equal to
\[
\lambda_{i\alpha} = \sqrt{G_{\theta\theta}^{(i)} / G_{r\theta}^{(i)}} = \sqrt{C_{44}^{(i)} / C_{55}^{(i)}} = 1,
\]
(45)
because in this case there is equality

\[ G_{\theta z} = G_z \quad \text{or} \quad C_{44}^{(i)} = C_{55}^{(i)}, \quad (46) \]

In addition, the ratio

\[ \mu_{23} = \mu_{32} = C_{44} = C_{55} = \mu_{31}, \quad (47) \]

is valid. Thus, taking into account (45), (46) and (47), we can rewrite (40-44) as follows:

\[ u_{\theta z}^{(i)}(r, \theta) = \left( D_{0}^{(i)} r + D_{2}^{(i)} r^{-2} \right) \cos \theta, \quad (48) \]

\[ 2e_{\theta z}^{(i)}(r, \theta) = - \left( D_{0}^{(i)} + D_{2}^{(i)} r^{-2} \right) \sin \theta, \quad (49) \]

\[ 2e_{rz}^{(i)}(r, \theta) = \left( D_{0}^{(i)} - D_{2}^{(i)} r^{-2} \right) \cos \theta, \quad (50) \]

\[ \sigma_{\theta z}^{(i)}(r, \theta) = - \mu_{23} \left( D_{0}^{(i)} + D_{2}^{(i)} r^{-2} \right) \sin \theta, \quad (51) \]

\[ \sigma_{rz}^{(i)}(r, \theta) = \mu_{23} \left( D_{0}^{(i)} - D_{2}^{(i)} r^{-2} \right) \cos \theta, \quad (52) \]

2.2.2 Method of obtaining an effective longitudinal shear modulus.

Consider again the pure shear problem for a layered structure and assume that a displacement field

\[ u_{\theta}(r, \theta) = 2 \varepsilon_{\theta z} r \cos \theta \]

is specified in the direction along the cell at infinity. In that case only the allowable movement along the fibers occur, which have the following form

\[ u_{\theta z}^{(i)}(r, \theta) = \left( D_{0}^{(i)} r + D_{2}^{(i)} r^{-2} \right) \cos \theta, \quad (53) \]

Where \[ \lambda_{(i)} = \sqrt{\frac{C_{44}^{(i)}}{C_{55}^{(i)}}} \], (54) for isotropic and transversely isotropic phases with an axis of symmetry directed along the fiber axis, \( \lambda_{(i)} = 1 \). From equations (43) and (44), the corresponding stress fields of each phase take the form

\[ \sigma_{\theta z}^{(i)}(r, \theta) = - C_{44}^{(i)} \left( D_{0}^{(i)} r \lambda_{(i)}^{-1} + D_{2}^{(i)} r^{-2} \lambda_{(i)}^{-1} \right) \sin \theta, \]

\[ \sigma_{rz}^{(i)}(r, \theta) = C_{55}^{(i)} \left( D_{0}^{(i)} \lambda_{(i)}^{-1} D_{2}^{(i)} r^{-2} \lambda_{(i)}^{-1} - \lambda_{(i)} D_{2}^{(i)} r^{-2} \lambda_{(i)}^{-1} \right) \cos \theta. \quad (55) \]

Now we define the fields of displacements and stresses for the equivalent homogeneous medium, which correspond to media with transversely isotropic properties with an isotropy plane perpendicular to the fiber axis. From equations (48), (51) and (52), we have

\[ u_{\theta z}^{N+1} (r, \theta) = \left( D_{0}^{N+1} r + D_{2}^{N+1} r^{-2} \right) \cos \theta, \quad (56) \]

\[ \sigma_{\theta z}^{N+1} (r, \theta) = - \mu_{23}^{N+1} \left( D_{0}^{N+1} + D_{2}^{N+1} r^{-2} \right) \sin \theta, \]

\[ \sigma_{rz}^{N+1} (r, \theta) = \mu_{23}^{N+1} \left( D_{0}^{N+1} - D_{2}^{N+1} r^{-2} \right) \cos \theta. \quad (57) \]

Obviously, we have nine unknown constants consisting of eight unknown constants (\( D_{0}^{(i)}, D_{2}^{(i)}, D_{1}^{(i)}, D_{2}^{(i)}, D_{1}^{(i)}, D_{5}^{(i)} \), \( D_{1}^{N+1} \) and \( D_{2}^{N+1} \)) and one unknown module - the effective module of the longitudinal shear \( \mu_{23}^{N+1} \).

Next, to find all the constants, the condition of non-singularity in the center of the fiber (\( r = 0 \)) is set:

\[ D_{2}^{(i)} = 0. \quad (58) \]

continuity condition at the boundaries of the ideal contact of the phases:

\[ u_{\theta z}^{(i)}(r_{(i)}, \theta) = u_{\theta z}^{N+1}(r_{(i)}, \theta), \quad \sigma_{\theta z}^{(i)}(r_{(i)}, \theta) = \sigma_{\theta z}^{N+1}(r_{(i)}, \theta), \quad (i = 1, 2,...,N); \quad (59) \]

outer boundary condition at infinity (\( r_{(N+1)} \to \infty \)).
\[ u^N_{N+1}(r_{N+1}) = 2\varepsilon_0 u_{N+1} \cos \theta , \]  
(60)

where the condition of the energy interaction over the surface at the contact \((r = r_N)\):

\[
\int_2 \left( \sigma^N_{\epsilon z} u^N_{\epsilon z} - \sigma^N_{\epsilon z} u^N_{\epsilon z} \right)_{r=r_N} dS = 0 ,
\]

(61)

It is seen that the angular stress fields are not included in the Eshelby integral formula (60), because the angular displacements are equal to zero. First, we define the displacement and stress fields of an equivalent homogeneous medium:

\[
u_{\epsilon}^0 (r, \theta) = \left( D_{11}^0 + D_{22}^0 r^{-2} \right) \cos \theta ,
\]

(62)

\[
\sigma_{\epsilon z}^0 (r, \theta) = \left( D_{11}^0 + D_{22}^0 r^{-2} \right) \sin \theta ,
\]

(63)

Using equality (58), which is also applicable to an effective composite, where \( D_{22}^0 = 0 \), (62) and (63) lead to the following

\[
u_{\epsilon}^0 (r, \theta) = D_{11}^0 \cos \theta ,
\]

(64)

\[
\sigma_{\epsilon z}^0 (r, \theta) = -\mu_{23}^0 D_{11}^0 \sin \theta ,
\]

(65)

It can be seen that on the basis of (59), which is also applicable for an effective composite together with (64), the unknown constant \( D_{11}^0 \) takes the form

\[
D_{11}^0 = 2\varepsilon_0 ,
\]

(66)

Thus, based on (56), (57), (64), (65) and (66), formula (61) is reduced to

\[
D_{22}^{N+1} = 0 ,
\]

(67)

where the cell length is considered equal to \( 2L \). Further, based on (56), (60) and (67), we can write the value \( D_{11}^{N+1} \) in the form

\[
D_{11}^{N+1} = 2\varepsilon_0 .
\]

(68)

Now we have five unknown constants \( D_{11}^{(1)}, D_{12}^{(1)}, D_{21}^{(1)}, D_{22}^{(1)} \) and one desired effective longitudinal shear modulus \( \mu_{23}^{(1)} \) and six unknown constants can be obtained using the conditions of continuity of displacement and stress (59) on the radii \( r_1, r_2, r_3 = r_N \):

1) \( D_{11}^{(1)} r_{12}^{(1)} - D_{12}^{(1)} r_{12}^{(1)} - D_{22}^{(1)} r_{12}^{(1)} = 0 , \)
2) \( D_{12}^{(2)} r_2^{(2)} + D_{21}^{(2)} r_2^{(2)} - D_{12}^{(2)} r_2^{(2)} - D_{22}^{(2)} r_2^{(2)} = 0 , \)
3) \( D_{12}^{(3)} r_3^{(3)} + D_{21}^{(3)} r_3^{(3)} - 2\varepsilon_0 r_3 = 0 , \)
4) \( C_{12}^{(1)} r_{12}^{(1)} - C_{12}^{(2)} r_{12}^{(2)} - C_{12}^{(3)} r_{12}^{(3)} = 0 , \)
5) \( C_{12}^{(2)} r_{23}^{(2)} r_{23}^{(2)} - C_{12}^{(3)} r_{23}^{(3)} r_{23}^{(3)} - C_{23}^{(3)} r_{23}^{(3)} + C_{55}^{(3)} r_{55}^{(3)} = 0 , \)
6) \( C_{55}^{(3)} r_{55}^{(3)} - C_{33}^{(3)} r_{33}^{(3)} r_{33}^{(3)} - 2\varepsilon_0 \mu_{23}^{(3)} = 0 , \)

(69)

It can be seen that the effective volumetric modulus of longitudinal shear is determined from the sixth equation (69).
\[ \mu_{23}^{N+1} = \mu_{23}^{\text{eff}} = \frac{1}{2e_0} C_{55}^{(3)} \left[ \lambda_{13} D_1^{(3)} r^{-\lambda_{13} \eta_{13}} \right] \cos \theta, \]

(70)

Thus, if we generalize the result (70) for multiphase composites, then the effective volumetric modulus of longitudinal shear is

\[ \mu_{23}^{\text{eff}} = \frac{1}{2e_0} C_{55}^{(N)} \left[ \lambda_{13}^{(N)} D_1^{(N)} r^{-\lambda_{13}^{(N)} \eta_{13}} \right] \cos \theta, \]

(71)

Where

\[ \lambda_{13}^{(N)} = \sqrt{\frac{C_{44}^{(N)}}{C_{55}^{(N)}}}, \]

(72)

Table 1. Material parameters and configuration of CNT-whiskered fiber composites

| PHASE          | Base Fiber | Whiskers | Matrix |
|---------------|------------|----------|--------|
| Materials     | Carbon T-650 | CNT | Epoxy  |
| Dimensions    |            |          |        |
| - Diameter (\( \mu \text{m} \)) | 5 \( \mu \text{m} \) | 0.00051 – 0.00085 | —      |
| - Length (\( \mu \text{m} \))     | —          | 1 – 2    | —      |
| Properties    |            |          |        |
| - Young’s longitudinal modulus, \( E_L \) (GPa) | 241        | 1100     | 3      |
| - Young’s transverse modulus, \( E_T \) (GPa) | 14.5       | —        | —      |
| - Longitudinal shear modulus, \( \mu_L \) (GPa) | 22.8       | —        | —      |
| - Transverse shear modulus, \( \mu_T \) (GPa) | 4.8        | —        | —      |
| - Poisson’s ratio, \( \nu_{LT} \) | 0.27       | 0.14     | 0.3    |

3. Results and Discussion

In order to determine the advantage of a composite with a whiskered interfacial layer over the classical composite, it is necessary to determine the destructive stresses by the criterion of the matrix failure, the destructive deformations will be taken equal to 0.4%. The calculation is performed according to the two-phase method based on the Eshelby self-consistent field method.

The method of two phases:

Assume \( e_0 = \frac{P_t}{\mu_{23}^{\text{eff}}} \), where \( P_t = 6.35MPa \), \( \mu_{23}^{\text{eff}} = 1.5911GPa \). When \( \theta = \pi / 2 \) stress and strain are as follows:
When $\theta = 0$:

\[
\sigma r z = \begin{cases} 
3 \times 10^{-7} & \text{for } r = 0.000002 \\
2 \times 10^{-7} & \text{for } r = 0.000006 \\
1 \times 10^{-7} & \text{for } r = 0.000010 
\end{cases}
\]

\[
\varepsilon r z = \begin{cases} 
0.000002 & \text{for } r = 0.000002 \\
0.000006 & \text{for } r = 0.000006 \\
0.000010 & \text{for } r = 0.000010 
\end{cases}
\]

\[
\sigma \theta z = \begin{cases} 
-1 \times 10^{7} & \text{for } r = 0.000002 \\
-2 \times 10^{7} & \text{for } r = 0.000006 \\
-3 \times 10^{7} & \text{for } r = 0.000010 
\end{cases}
\]

\[
\varepsilon \theta z = \begin{cases} 
0.000002 & \text{for } r = 0.000002 \\
0.000006 & \text{for } r = 0.000006 \\
0.000010 & \text{for } r = 0.000010 
\end{cases}
\]

\[
\sigma z = \begin{cases} 
-3 \times 10^{-7} & \text{for } r = 0.000002 \\
-2 \times 10^{-7} & \text{for } r = 0.000006 \\
-1 \times 10^{-7} & \text{for } r = 0.000010 
\end{cases}
\]

\[
\varepsilon z = \begin{cases} 
0.000002 & \text{for } r = 0.000002 \\
0.000006 & \text{for } r = 0.000006 \\
0.000010 & \text{for } r = 0.000010 
\end{cases}
\]

\[
\sigma \theta = \begin{cases} 
-2 \times 10^{7} & \text{for } r = 0.000002 \\
-1 \times 10^{7} & \text{for } r = 0.000006 \\
0 & \text{for } r = 0.000010 
\end{cases}
\]

\[
\varepsilon \theta = \begin{cases} 
0.000002 & \text{for } r = 0.000002 \\
0.000006 & \text{for } r = 0.000006 \\
0.000010 & \text{for } r = 0.000010 
\end{cases}
\]

Fig. 6. Shear stresses and shear deformations using the two phase method.

Making the interphase layer matrix properties:

In the case when we accept the properties of the interfacial layer as equal to the properties of the matrix, the result is absolutely similar, which indicates the correctness of the model we have constructed.

Assume $\mu_0 = P_t / \mu_{23}^{eff}$, where $P_t = 6.35MPa$, $\mu_{23}^{eff} = 1.5911GPa$. When $\theta = \pi / 2$ stress and strain are as follows:
When $\theta = 0$:

Fig. 7. Shear stresses and shear deformations with imparting matrix properties to the interfacial layer.

The method of three phases:

As can be seen from the results, the destructive load is $P_{t_1} = 6.35 \text{MPa}$. We take this value as a starting point for finding the destructive load for a three-phase composite with a whiskerized interfacial layer. At $P_{t_1} = 6.35 \text{MPa}$ the deformations in the matrix of the effective composite equal $\epsilon = 0.4\%$. Due to the increased modulus of elasticity of the longitudinal shear of the composite with the interfacial layer compared to the classical, the next step we “load” the composite to the level of destructive deformations $\epsilon = 0.4\%$.

Assume $\epsilon_0 = P_{t_1} / \mu_{23}^{\text{eff}}$, where $P_{t_2} = 8.13 \text{MPa}$, $\mu_{23}^{\text{eff}} = 2.0358 \text{GPa}$. At $\theta = \pi / 2$ stress and strain are as follows:
At $\theta = 0$:

![Graph showing shear stresses and shear deformations](image)

**Fig. 8.** Shear stresses and shear deformations with imparting matrix properties to the interfacial layer.

### 4. Conclusions

As can be seen, the destruction of the composite occurs under load $P_t = 8.13\, MPa$, from which we conclude that the hardening coefficient of the composite by the presence of a whiskerized interfacial layer is $n_{ypl} = \frac{P_t}{P_{t_1}} = \frac{8.13}{6.35} = 1.28$ times. In addition, there is a noticeable decrease in the level of stresses in the fiber, since part of the load is perceived by the interfacial layer, unloading the fiber, which has a positive effect on the strength characteristics of such a composite.
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