The angular momentum of a magnetically trapped atomic condensate

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For an atomic condensate in an axially symmetric magnetic trap, the sum of the axial components of the orbital angular momentum and the hyperfine spin is conserved. Inside an Ioffe-Pritchard trap (IPT), we concentrate on the conserved quantity

\[ L_z \]

studied earlier in Ref. [5]. While Ho and Shenoy mainly studied the orbital angular momentum component in an IPT [3], instead we concentrate on the conserved quantity, the sum \((J_z)\) or difference \((D_z)\) for a QT or a IPT.

This paper is organized as follows. We first consider a spin-1 condensate in a QT. Making use of an effective energy functional appropriate for the adiabatic approximation [5], we prove \(J_z \in [-1, 1]\) with the actual value determined by the angle between the z-axis and the direction of the B-field. We then generalize to the spin-F case. Finally, our result is extended to an IPT.

The Hamiltonian of a spin-1 atomic condensate with \(N\) atoms in a magnetic trap is \(H = H_S + H_T\) with the single atom part

\[ H_S = \int \frac{\psi^†(\vec{r}) \left[ -\frac{\hbar^2}{2M} + \mu_{B} g_{F} \vec{B} \cdot \vec{n}(\vec{r}) \right] \psi(\vec{r}) d\vec{r}, \]

and the atom-atom interaction Hamiltonian

\[ H_T = \sum_{m,n,p,q} \int \frac{\psi^†_{m}(\vec{r}) \psi^†_{n}(\vec{r}) V_{pq}(\vec{r},\vec{r}′) \psi_{p}(\vec{r}) \psi_{q}(\vec{r}′) d\vec{r} d\vec{r}′. \]

\(\psi(\vec{r}) = [\psi_{-1}(\vec{r}), \psi_0(\vec{r}), \psi_{+1}(\vec{r})]^{\text{T}}\) denotes the annihilation field operator for the z-quantized \(F_z\)-component of \(m, n, p, q \in \pm 1\). \(M\) is the atomic mass, and \(\mu_{B}\) is the Bohr magneton. \(B(\vec{r})\) and \(\vec{n}(\vec{r})\) denote the strength and direction of the local B-field. \(h = 1\) is assumed. The Landé g factor is \(g_F = 1 = -1/2\).

Within the mean field approximation, the field operator \(\psi(\vec{r})\) is replaced by its average \(\langle \psi(\vec{r}) \rangle\). To introduce the adiabatic approximation, we define a group of normalized scalar wave functions \(\varphi_u(\vec{r})\):

\[ \langle \psi(\vec{r}) \rangle = \sum_{b=0,\pm 1} \sqrt{N} \xi^B(b, \vec{r}) \varphi_b(\vec{r}), \]

where \(\xi^B(b, \vec{r})\) is the eigenstate of the \(B\)-quantized spin component \(\vec{F} \cdot \vec{n}(\vec{r})\) with eigenvalue \(b\), satisfying the relations \(\vec{F} \cdot \vec{n}(\vec{r})\xi^B(b, \vec{r}) = b \xi^B(b, \vec{r})\) and \(\xi^B(b, \vec{r})\xi^B(b′, \vec{r}) = \delta_{b, b′}\). In the \(z\)-quantized representation, it takes the form \(\xi^B(b, \vec{r}) = [\xi^B_z(b, \vec{r}), \xi^B_0(b, \vec{r}), \xi^B_z(b, \vec{r})]^{\text{T}}\). In this study, it is important to distinguish \(\xi^B(b, \vec{r})\) from the eigenstates \(\xi_z(0, \pm 1)\) of \(F_z\) with eigenvalues \(0, \pm 1\). In explicit
form, we have $\xi_1(-1) = [1, 0, 0]^T$, $\xi_2(0) = [0, 1, 0]^T$, and $\xi_3(1) = [0, 0, 1]^T$.

A magnetic dipole precesses around the direction of a B-field. Majorona transitions between different $\xi_{B}(\theta, \phi)$ states can be neglected when $B(\vec{r})$ is large enough. Thus the atomic hyperfine spin adiabatically freezes in the low-field seeking state $\xi_{B}(-1, \vec{r})$ during the trapped center of mass motion, and $\varphi_{-1}(\vec{r}) = \varphi(\vec{r})$ and $\varphi_{0+1}(\vec{r}) = 0$. Similarly the z-quantized mean field becomes

$$\hat{\psi}(\vec{r}) = \sqrt{N} \xi_{B}(-1, \vec{r}) \varphi(\vec{r}),$$

with $\varphi(\vec{r})$ a B-quantized scalar function.

Substituting Eq. (2) into the expression of $H$, we can obtain the expression of the condensate energy $E_{ad}$ as a functional of the scalar wave function $\varphi(\vec{r})$: $E_{ad}[\varphi] = \sum_{m=0}^{\infty} \langle \psi(z) | \varphi(\vec{r}) \rangle_{g\xi_z}(m)$ gives

$$\langle \psi(z) | \varphi(\vec{r}) \rangle_{g\xi_z}(m) = \sqrt{N} \xi_{B}(-1, \vec{r}) \varphi_{\beta}(\rho, z) \exp(is\phi),$$

which is an eigenstate of $J_z$ with an eigenvalue $s$, i.e.,

$$J_z \langle \psi(z) | \varphi(\vec{r}) \rangle_{g\xi_z}(m) = s \langle \psi(z) | \varphi(\vec{r}) \rangle_{g\xi_z}(m),$$

because of Eq. (4).

Equation (5) and the expansion $\langle \psi(z) | \varphi(\vec{r}) \rangle_{g\xi_z}(m)$ gives

$$\langle \psi(z) | \varphi(\vec{r}) \rangle_{g\xi_z}(m) = \sqrt{N} \xi_{B}(-1, \vec{r}) \varphi_{\beta}(\rho, z) \exp(is\phi)$$

with $b_m = \xi_{B}(-1, \vec{r}) \varphi_{\beta}(\rho, z) \exp(is\phi)$ with any integer $m$, the energy functional $E_{ad}$ satisfies

$$E_{ad}[\varphi_{\beta}(\rho, z) \exp(is\phi)] = E_{ad}[\varphi_{\beta}] + \Delta E_{m}[\varphi_{\beta}],$$

with $E_{ad}[\varphi_{\beta}] = \int d\vec{r} \varphi_{\beta}^* \varphi_{\beta} |(m^2 + 4m \cos \beta)/(2M^2)|$. In addition to the centrifugal term proportional to $m^2$, a term linear in $m$ appears due to the $A \cdot \nabla$ term in $E_{ad}$.

In the following we show that the above linear term is important for the value $s$, which we determine with a variational approach. Because $E_{ad}$ takes its minimal value in the state $\varphi_{\beta} \exp(is\phi)$, we have $E_{ad}[\varphi_{\beta} \exp(is\phi)] \leq E_{ad}[\varphi_{\beta} \exp(is\chi) \exp(is\phi)]$. Together with Eq. (8), we find the necessary condition satisfied by $s$: $\Delta E_{ad}[\varphi_{\beta}] \leq \Delta E_{m}[\varphi_{\beta}]$ or $|s + C| \leq 1/2$, where the coefficient $C$ is defined as

$$C = \frac{\int d\vec{r} \varphi_{\beta}^* \varphi_{\beta} |(m^2 + 4m \cos \beta)/(2M^2)|}{\int d\vec{r} \varphi_{\beta}^* \varphi_{\beta} |(1/\rho^2)|}.$$
We now generalize our result to atoms with an arbitrary $F$ and inside any axially symmetric B-fields. Analogously we can prove that the value $s$ of $J_z$ in the ground state satisfies the necessary condition
\[
|s - \eta_F F| \leq 1/2,
\] (9)
and $s \in [-F, F]$ with $\eta_F = \text{sign}(g_F)$. The result of $s \in [-F, F]$ and the conservation of $J_z$ is independent of the form of the atomic interaction potential. Although its strength $g_F$ does affect the wave function shape, thus can influence the value of $s$ through the factor $B$.

The condition Eq. (9) also allows for a rough estimate of $L_z$. A straightforward calculation gives $\langle L_z \rangle = s - \eta_F F \int d\vec{r} |\psi_0(\rho, z)|^2 \cos \beta(\rho, z)$ for the spinor mean field $\langle \psi(\vec{r}) \rangle$. Neglecting the correlation between $\rho$ and $\beta(\rho, z)$ as before, the value of $\langle L_z \rangle$ becomes approximately $s - \eta_F FC$, which lies always in the region $[-1/2, 1/2]$ according to Eq. (9). Therefore, the value $\langle L_z \rangle$, or the weighted average of the winding numbers, is generally a small number, despite the winding number $s - m$ itself, for the component $\langle \psi_m(\vec{r}) \rangle$, may take any integer in the region $[-2F, 2F]$. We find $\langle F_z \rangle = \eta_F F \int d\vec{r} \hat{\sigma}_z |\psi_0(\rho, z)|^2 \cos \beta(\rho, z)$ from the expression of $\langle L_z \rangle$, a qualitative reflection that atomic hyperfine spin is aligned ($g_F > 0$) or anti-aligned ($g_F < 0$) with respect to the local B-field.

Our result above allows for the direct creation of vortex states in a quadrupole trapped atomic condensate. For example, assume a spin-1 condensate in a QT plus an “optical plug” [8] satisfies $V_o(\rho, z) = V_o(\rho, -z)$, then we find $C = 0$ and $s = 0$ due to the spatial reflection symmetry about the $x-y$ plane. The ground state components $\langle \psi_{m}(\vec{r}) \rangle$ then automatically carry persistent currents with winding numbers $\pm 1$ according to Eq. (9). In addition, the low field seeking atoms are trapped near the $x$-$y$ plane at $z = 0$ because $|B(\vec{r})|$ is an increasing function of $z$. The populations for the three $z$-quantized states, determined by $E^B(\vec{r})$ and $\xi^z_\pm(1, -\vec{r})$, are of the same order of magnitudes. Therefore, when a ground state condensate in the “plugged” QT is created, its $\pm 1$ components $\langle \psi_{\pm 1}(\vec{r}) \rangle$ are single quantized vortex states and can be directly resolved with a Stern-Gerlach B-field as used in Ref. [8].

The qualitative example above is confirmed by the numerical solution for a condensate of $5 \times 10^6$ $^{23}$Na atoms in a quadrupole plus a plug trap. We take $B^f = 22$ Gauss/cm and $V_o = U_o \exp[-\rho^2/\sigma^2]$ with $U_o = (2\pi)8 \times 10^4$Hz and $\sigma = 7.4 \mu$m. The ground state distribution $p_1 = \int d\vec{r} |\psi_{1}(\vec{r})|^2$ is found to be $p_{\pm 1} = 27.2\%$ and $p_0 = 45.6\%$. The phase and density distributions for the three components $\langle \psi_{0, \pm 1}(\vec{r}) \rangle$ are shown in Fig. 1 (a, b).

We also can expand the ground state $\langle \psi(\vec{r}) \rangle_g$ in terms of the eigenstates $\xi_x(m)$ of $F_x$ with eigenvalues $m$: $\langle \psi(\vec{r}) \rangle_g = \sum_{m=0,\pm 1} \sqrt{N} \langle \psi_m(\vec{r}) \rangle \xi_x(m)$. We then immediately note that $\langle \psi^{(z)}(\vec{r}) \rangle_g$ is a superposition of vortex states with definite winding numbers $0$ or $\pm 1$, e.g., $\langle \psi^{(z)}(\vec{r}) \rangle_g = \langle \sqrt{N/\sqrt{2}} |0(\rho, z) e^{i\phi} - b_{1}(\rho, z) e^{-i\phi} | \rangle$. The density distribution of $\langle \psi^{(z)}(\vec{r}) \rangle_g$ as shown in Fig. 1 (c) clearly illustrates the interference pattern along the $\hat{e}_{\phi}$ direction. As is demonstrated in Fig. 1 (c), the middle panel for $|\langle \psi^{(z)}(\vec{r}) \rangle|^2$ clearly displays the double peak structure along the azimuthal direction, arising from the interference of the terms proportional to $e^{\pm \phi}$. Thus, if a Stern-Gerlach B-field is used to separate the $x$-quantized components $\langle \psi_m(\vec{r}) \rangle_g$, a superposition of vortices with different winding numbers would be obtained.

We now extend our result for an axially symmetric magnetic trap to the widely used IPT whose B-field possesses a different symmetry. In the region near the z-axis, $\vec{B}(\vec{r}) = B^i(\cos(2\beta) \hat{e}_\rho - \sin(2\beta) \hat{e}_\phi + h\hat{z})$, the angle $\beta(\rho, z)$ between the local B-field and the z-axis satisfies $\cos \beta(\rho, z) = h/\sqrt{p^2 + h^2}$. In this case $J_z$ is no longer conserved due to the lack of the SO(2) symmetry. However, we find that $D_z$ is now conserved because it commutes with $\hat{F}_x, \hat{B}(\vec{r})$. Therefore, we can select the low field seeking hyperfine spin state $\xi^F(\eta_F, \vec{r})$ as the eigenstate of $D_z$ with an eigenvalue $-\eta_F F$, the same spin state as used in Ref. [8], again defined through

\[\xi^F(\eta_F, \vec{r}) = \frac{|\langle \psi^{(z)}(\vec{r}) \rangle |}{\sqrt{\int d\vec{r} |\psi^{(z)}(\vec{r})|^2}}.\]
a rotation $\hat{\xi}^B(\eta_F, \vec{r}) = \exp[-i\vec{F} \cdot \hat{n}_\perp \beta(\xi^z(\eta_F, F))$. For $h > 0$, we find the induced gauge potential becomes

$$A(\vec{r}) = -\eta_F F(1 - \cos(\beta(\rho, z)))\hbar e/\rho.$$

Adopting the same notation as before we denote

$$\langle \hat{\psi}(\vec{r}) \rangle_g = \sqrt{N}\phi_0(\vec{r})\xi^B(\eta_F, \vec{r}).$$

Interestingly we find $E_{\rm ad}[\phi(\rho, \phi, z)] = E_{\rm ad}[\phi(\rho, \phi + \theta, z)]$ remains satisfied, and the ground state takes the form $\phi_0 = \phi_0(\rho, z)\exp(\pm i\phi)$. Therefore $\langle \hat{\psi}(\vec{r}) \rangle_g$ is the eigenstate of $D_z$ with an eigenvalue $d = u - \eta_F F$, and its components $\langle \hat{\psi}_m^{(z)}(\vec{r}) \rangle_g = \sqrt{N}b_m'(\rho, z)e^{imu \beta(\xi^z(\eta_F, F))}$ analogously carry a persistent current with a winding number $m + u - \eta_F F$. Here $b_m' = \xi^z_m(m)e^{-iFz\beta(\xi^z(\eta_F, F))}$. This result is consistent with the ground state vortex phase diagram for an $F = 1$ condensate found numerically in the $z = 0$ plane of an IPT. The conservation of $D_z$ as found by us, however, calls for a simpler labelling of each vortex phase because only one of three integers $(m_1, m_0, m_{-1})$ is independent, as with Eq. (15) of Ref. [10].

Following the same reasoning as before, we find

$$|d + \eta_F F(1 - C)| \leq 1/2,$$

for $d \neq \eta_F F$, and $d \in [-F, F]$ or the value of $D_z$ in the ground state lies in the region $[-F, F]$.

In an IPT, atoms are trapped near the $z$-axis where the B-field is essentially along the $z$-axis direction and $\xi^B(\eta_F, \vec{r})$ is approximately the eigenstate $\xi^z(\eta_F, F)$. $L_z$ then is essentially always zero corresponding to a ground state without a vortex. The angular momentum difference $D_z$ then becomes $d = -\eta_F F$.

Several previous proposals [11] and experiments [12] on creating vortex states unknowingly have
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