ON THE EXISTENCE PROBLEM OF EINSTEIN-MAXWELL KÄHLER METRICS

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ABSTRACT. In this expository paper we review on the existence problem of Einstein-Maxwell Kähler metrics, and make several remarks. Firstly, we consider a slightly more general set-up than Einstein-Maxwell Kähler metrics, and give extensions of volume minimization principle, the notion of toric K-stability and other related results to the general set-up. Secondly, we consider the toric case when the manifold is the one point blow-up of the complex project plane and the Kähler class $\Omega$ is chosen so that the area of the exceptional curve is sufficiently close to the area of the rational curve of self-intersection number 1. We observe by numerical analysis that there should be a Killing vector field $K$ which gives a toric K-stable pair $(\Omega, K)$ in the sense of Apostolov-Maschler.

1. INTRODUCTION

Let $(M, J)$ be a compact Kähler manifold of complex dimension $m$. A Hermitian metric $\tilde{g}$ of constant scalar curvature on $(M, J)$ is said to be a conformally Kähler, Einstein-Maxwell (cKEM for short) metric if there exists a positive smooth function $f$ on $M$ such that $g = f^2 \tilde{g}$ is Kähler and that the Hamiltonian vector field $K = J \text{grad}_g f$ of $f$ with respect to the Kähler form $\omega_g$ of $g$ is a Killing vector field for both $g$ and $\tilde{g}$. In this case we call the Kähler metric $g$ an Einstein-Maxwell Kähler (EMK for short) metric. Let $\omega_0$ be a Kähler form, and consider $\Omega = [\omega_0] \in H^2_{\text{DR}}(M, \mathbb{R})$ as a fixed Kähler class. We look for an Einstein-Maxwell Kähler metric $g$ such that the Kähler form $\omega_g$ belongs to $\Omega$.

Let $G$ be a maximal torus of the reduced automorphism group, and pick $K \in g := \text{Lie}(G)$. Then the problem is to find a $G$-invariant Kähler metric $g$ with its Kähler form $\omega_g \in \Omega$ such that
(i) $\tilde{g} = f^{-2} g$ is a cKEM metric,
(ii) $J \text{grad}_g f = K$.

The scalar curvature $s_{\tilde{g}}$ of $\tilde{g} = f^{-2} g$ is given by

\begin{equation}
1 \quad s_{\tilde{g}} = f^2 s_g - 2(2m - 1)f \Delta_g f - 2m(2m - 1) |df|^2_g
\end{equation}

where $s_g$ is the scalar curvature of $g$ and $\Delta_g$ is the Hodge Laplacian with respect to $g$.

Now, starting with a Kähler metric $g$ and a Killing potential $f$, for any real number $n \in \mathbb{R}$ with $n \neq 0, 1, 2$ and $k \in \mathbb{R}$ with $k \neq 0$ we define the

Date: March 17, 2018.
(g, f, k, n)-scalar curvature $s_{g,f,k,n}$ by

\[(2)\]

\[s_{g,f,k,n} = f^{-k} \left\{ s_g + k(n - 1) \frac{1}{f} \Delta_g f + \frac{k}{4} (n - 1)(4 + 2k - kn) \frac{1}{f^2} |df|^2_g \right\}.\]

The case $n = 2m$ is the scalar curvature $s_{\tilde{g}}$ of the conformal metric $\tilde{g} = f^k g$, and for other values of $n$ such a meaning is lost. However, the cases of general values of $n$ appear in natural contexts such as in [2] and [13]. Moreover, Lahdili proves in [10] and [11] results for cKEM metrics generalizing to constant $(g, f, -2, n)$-scalar curvature.

In this expository paper we give extensions of the volume minimization principle [8], [9], the notion of toric K-stability [3] for $k = -2$ and other related results for cKEM metrics to the general set-up of constant $(g, f, k, n)$-scalar curvature. We consider the toric case where the manifold is the one point blow-up of the complex project plane and the Kähler class $\Omega$ is chosen so that the area of the exceptional curve is sufficiently close to the area of the rational curve of self-intersection number 1. We observe by numerical analysis that there should be a Killing vector field $K$ which gives a toric K-stable pair $(\Omega, K)$ in the sense of Apostolov-Maschler. For this purpose we show in Theorem 5.3 that we have only to consider the simple test configurations to test toric K-stability, extending the earlier works of Donaldson [4], Wang and Zhou [14], [15].

The rest of this paper is organized as follows. In section 2 we extend the volume minimization for Einstein-Maxwell Kähler metrics, see Theorem 2.1. In section 3 we review the normalized Einstein-Hilbert functional, and study its relation to the volume functional and the Futaki invariant. In section 4 we consider the normalized Einstein-Hilbert functional on toric Kähler manifolds. In section 5 we review toric K-stability, and prove Theorem 5.3. We then review the result of our paper [8] on the one-point blow-up of $\mathbb{C}P^2$ and show the graphics of the results of the numerical analysis which indicate that this case should be K-stable and there should be a conformally Kähler, Einstein-Maxwell metric.

2. Volume minimization for Einstein-Maxwell Kähler metrics

In this section we review the results in [8] and extend them to constant $(g, f, k, n)$-scalar curvature. Let $M$ be a compact smooth manifold. We denote by $\text{Riem}(M)$ the set of all Riemannian metrics on $M$, by $s_g$ the scalar curvature of $g$, and by $dv_g$ the volume form of $g$. For any given positive smooth function $f$ and real numbers $n \in \mathbb{R}$ with $n \neq 0, 1, 2$ and $k \in \mathbb{R}$ with $k \neq 0$, we define $s_{g,f,k,n}$ by the same formula as (2). We put

\[(3)\]

\[S(g, f, k, n) := \int_M s_{g,f,k,n} f^{\frac{k}{n}} dv_g\]

\[\]
and call it the total \((g,f,k,n)\)-scalar curvature, and put
\[
\text{Vol}(g,f,k,n) := \int_M f^{\frac{nk}{2}} dv_g
\]
and call it the \((g,f,k,n)\)-volume.

Let \(f_t\) be a smooth family of positive functions such that \(f_0 = f, d/dt|_{t=0} f_t = \phi\). Then by straightforward computations we have
\[
\frac{d}{dt} \bigg|_{t=0} S(g,f_t,k,n) = \frac{k}{2} (n-2) \int_M s_{g,f,k,n} \phi f^{\frac{nk}{2}-1} dv_g
\]
and
\[
\frac{d}{dt} \bigg|_{t=0} \text{Vol}(g,f_t,k,n) = \frac{nk}{2} \int_M \phi f^{\frac{nk}{2}-1} dv_g.
\]

Now we consider a compact Kähler manifold \((M,J)\) of complex dimension \(m\). As in section 1, let \(G\) be a maximal torus of the reduced automorphism group, and take \(K \in \mathfrak{g} := \text{Lie}(G)\). Consider a fixed Kähler class \(\Omega\) on \((M,J)\), and denote by \(K^G_\Omega\) the space of \(G\)-invariant Kähler metrics \(\omega\) in \(\Omega\). For any \((K,a,g) \in \mathfrak{g} \times \mathbb{R} \times K^G_\Omega\), there exists a unique function \(f_{K,a,g} \in C^\infty(M,\mathbb{R})\) satisfying the following two conditions:
\[
\iota_K \omega = -df_{K,a,g}, \quad \int_M f_{K,a,g} \omega^m = a.
\]
By (7), it is easy to see that \(f_{K,a,g}\) has the following properties:
\[
f_{K+H,a+b,g} = f_{K,a,g} + f_{H,b,g}
\]
\[
f_{0,a,g} = \frac{a}{\text{Vol}(M,\omega)}
\]
\[
f_{CgK,a,g} = Cf_{K,a,g}
\]

Hereafter the Kähler metric \(g\) and its Kähler form \(\omega_g\) are often identified, and \(\omega_g\) is often denoted by \(\omega\). Noting that \(\min\{f_{K,a,g} \mid x \in M\}\) is independent of \(g \in K^G_\Omega\) (this follows from the convexity of moment map images and the fact that the vertices do not move even if we change the Kähler metric in the fixed Kähler class \(\Omega\)), we put
\[
\mathcal{P}^G_\Omega := \{(K,a) \in \mathfrak{g} \times \mathbb{R} \mid f_{K,a,g} > 0\}.
\]
Note that the right hand side of (11) is independent of \(g \in K^G_\Omega\) again since the moment polytope is independent of \(g \in K^G_\Omega\). Fixing \((K,a) \in \mathcal{P}^G_\Omega, n \in \mathbb{R}\) and \(k \in \mathbb{R}\), put
\[
c_{\Omega,K,a,k,n} := \frac{\int_M s_{g,f_{K,a,g},k,n} \frac{j^{\frac{kn}{2}-1}}{j^{\frac{kn}{2}}} \frac{\omega^m}{m!}}{\int_M f_{K,a,g}^{\frac{kn}{2}-1} \frac{\omega^m}{m!}}
\]
and

\[ d_{\Omega,K,a,k,n} := \frac{S(g,f_{K,a,g},k,n)}{\text{Vol}(g,f_{K,a,g},k,n)} = \int_M \frac{s_{g,f_{K,a,g},k,n} f_{K,a,g}^{\frac{kn}{m!}}}{f_{K,a,g}^{\frac{kn}{m!}}} \omega^m. \]

Then \( c_{\Omega,K,a,k,n} \) and \( d_{\Omega,K,a,k,n} \) are constants independent of the choice of \( g \in K^G_{\Omega} \) since the integrands of (12) and (13) are part of equivariant cohomology, see e.g. [6], [5], [7]. Since \( P^G_{\Omega} \) is a cone in \( g \times \mathbb{R} \) by (10), with \( n \) and \( k \) fixed we consider its slice

\[ \tilde{P}^G_{\Omega} := \left\{ (K,a) \in P^G_{\Omega} \mid d_{\Omega,K,a,k,n} = \gamma \right\} \]

where \( \gamma \) is chosen to be \(-1, 0 \) or \( 1 \) depending on the sign of \( d_{\Omega,K,a,k,n} \). Let \( (K(t),a(t)), t \in (-\varepsilon, \varepsilon) \) be a smooth curve in \( \tilde{P}^G_{\Omega} \) such that \( (K(0), a(0)) = (K,a) \), \( (K'(0), a'(0)) \) is a smooth curve in \( \tilde{P}^G_{\Omega} \). Then

\[ S(g,f_{K(t),a(t),g},k,n) = \gamma \text{Vol}(g,f_{K(t),a(t),g},k,n) \]

holds for any \( t \in (-\varepsilon, \varepsilon) \). By differentiating this equation at \( t = 0 \) and noting \( k \neq 0 \), we have

\[ (n-2) \int_M s_{g,f_{K,a,g},k,n} f_{H,b,g} f_{K,a,g}^{\frac{nk-1}{m!}} \omega^m = n\gamma \int_M f_{H,b,g} f_{K,a,g}^{\frac{nk-1}{m!}} \omega^m. \]

The linear function \( \text{Fut}^G_{\Omega,K,a,k,n} : g \to \mathbb{R} \) defined by

\[ \text{Fut}^G_{\Omega,K,a,k,n}(H) := \int_M (s_{g,K,a,k,n} - c_{\Omega,K,a,k,n}) f_{H,b,g} f_{K,a,g}^{\frac{nk-1}{m!}} \omega^m \]

is independent of the choice of Kähler metric \( g \in K^G_{\Omega} \) and \( b \in \mathbb{R} \) (33). If there exists a Kähler metric \( g \in K^G_{\Omega} \) such that \( \tilde{g} = f_{K,a,g}^k g \) is a constant \( (g,f,k,n) \)-scalar curvature metric, then \( \text{Fut}^G_{\Omega,K,a,k,n} \) vanishes identically.

For the path \( (K(t), a(t)), t \in (-\varepsilon, \varepsilon) \) in \( \tilde{P}^G_{\Omega} \) with \( (K(0), a(0)) = (K,a) \), \( (K'(0), a'(0)) = (H,b) \) we have from (15)

\[ \text{Fut}^G_{\Omega,K,a,k,n}(H) = \left( \frac{n\gamma}{n-2} - c_{\Omega,K,a,k,n} \right) \int_M f_{H,b,g} f_{K,a,g}^{\frac{nk-1}{m!}} \omega^m \]

\[ = \left( \frac{n\gamma}{n-2} - c_{\Omega,K,a,k,n} \right) \left( \frac{2}{nk} \frac{d}{dt} \right)_{t=0} \text{Vol}(g,f_{K(t),a(t),g},k,n). \]

If there exists a constant \( (g,f,k,n) \)-scalar curvature metric \( \tilde{g} = f_{K,a,g}^k g \) with \( g \in K^G_{\Omega} \), then

\[ c_{\Omega,K,a,k,n} = d_{\Omega,K,a,k,n} = \gamma \]

and

\[ \text{Fut}^G_{\Omega,K,a,k,n}(H) = 0. \]
Therefore for $\gamma = \pm 1$ we have

\begin{equation}
\left. \frac{d}{dt} \right|_{t=0} \text{Vol}(g, f_{K(t),a(t),g}, k, n) = 0.
\end{equation}

The case of $\gamma = 0$ can be treated separately, see [8].

We summarize the result as follows.

**Theorem 2.1.** Let $\Omega$ be a fixed Kähler class, and $n \neq 0, 1, 2$ and $k \neq 0$ be fixed real numbers. Suppose that the pair $(K, a)$ of Killing vector field $K$ and normalization constant $a$ belongs to $\tilde{\mathcal{P}}^G_{\Omega}$. If there exists a $G$-invariant Kähler metric $g$ in the Kähler class $\Omega$, i.e. $g \in \mathcal{K}^G_{\Omega}$, such that the $(g, f, k, n)$-scalar curvature is constant for the Killing Hamiltonian function $f = f_{K,a,g}$ then $(K, a)$ is a critical point of $\text{Vol}_{n,k} : \tilde{\mathcal{P}}^G_{\Omega} \to \mathbb{R}$ given by

$$\text{Vol}_{n,k}(K, a) := \text{Vol}(g, f_{K,a,g}, k, n)$$

$$= \int_M f^{\frac{n}{2}}_{K,a,g} dv_g$$

for $(K, a) \in \tilde{\mathcal{P}}^G_{\Omega}$. Further, $(K, a)$ is a critical point of $\text{Vol}_{n,k} : \tilde{\mathcal{P}}^G_{\Omega} \to \mathbb{R}$ if and only if $\text{Fut}_{G_{\Omega},K,a,k,n} \equiv 0$.

**Corollary 2.2.** Let $\Omega$ be a fixed Kähler class. Take $n = 2m$ and $k = -2$, and let $(K, a) \in \tilde{\mathcal{P}}^G_{\Omega}$. If there exists a conformally Kähler, Einstein-Maxwell metric $\tilde{g} = f^{-2}_{K,a,g} g$ with $g \in \mathcal{K}^G_{\Omega}$, then $(K, a)$ is a critical point of $\text{Vol} : \mathcal{P}^G_{\Omega} \to \mathbb{R}$ given by $\text{Vol}(K, a) := \text{Vol}(g, f_{K,a,g}, -2, 2m)$ for $(K, a) \in \tilde{\mathcal{P}}^G_{\Omega}$. Further, $(K, a)$ is a critical point of $\text{Vol} : \mathcal{P}^G_{\Omega} \to \mathbb{R}$ if and only if $\text{Fut}_{G_{\Omega},K,a,-2,2m} \equiv 0$.

For a given Kähler class $\Omega$ the critical points of $\text{Vol} : \mathcal{P}^G_{\Omega} \to \mathbb{R}$ are not unique in general as can be seen from LeBrun’s construction [12].

3. **The normalized Einstein-Hilbert functional**

In the previous section we confined ourselves to the view point from the volume functional. In the present section we see that, when restricted to $\mathcal{P}^G_{\Omega}$, considering the volume functional is essentially the same as considering the normalized Einstein-Hilbert functional. The normalized Einstein-Hilbert functional $\text{EH} : \text{Riem}(M) \to \mathbb{R}$ on an $n$-dimensional compact Riemannian manifold is the functional on $\text{Riem}(M)$ defined by

$$\text{EH}(g) := \frac{S(g)}{(\text{Vol}(g))^{\frac{n-2}{n}}}$$

where $S(g)$ and $\text{Vol}(g)$ are respectively the total scalar curvature and the volume of $g$. It is a standard fact that the critical points of $\text{EH}$ are Einstein metrics, and that, when restricted to a conformal class, the critical points are metrics of constant scalar curvature.
Let us see this in a slightly different setting. In the equation (2), let us replace \( s_g \) by a smooth function \( \varphi \), and put
\[
(19) \quad s_{g,f,k,n,\varphi} = f^{-k} \left\{ \varphi + k(n-1)\frac{1}{f} \Delta_g f + \frac{k}{4} (n-1)(4+2kn-1) \frac{1}{f^2} |df|^2_g \right\}.
\]
Accordingly, we may replace (3) by
\[
(20) \quad S(g,f,k,n,\varphi) := \int_M s_{g,f,k,n,\varphi} f^\frac{n-k}{2} dv_g,
\]
and replace the normalized Einstein-Hilbert functional by
\[
EH(g,f,k,n,\varphi) := \frac{S(g,f,k,n,\varphi)}{(Vol(g,f,k,n,\varphi))^{\frac{n-2}{n}}}.
\]
As before, let \( f_t \) be a smooth family of positive functions such that \( f_0 = f, d/dt|_{t=0} f_t = \varphi \). Then one can show
\[
(21) \quad \left. \frac{d}{dt} \right|_{t=0} EH(g,f_t,k,n,\varphi) = \frac{(n-2)k}{2} \Vol(g,f,k,n,\varphi)^{\frac{2-n}{n}} \int_M \left( s_{g,f,k,n,\varphi} - \frac{S(g,f,k,n,\varphi)}{Vol(g,f,k,n)} \right) \varphi f^\frac{n-k}{2} - 1 dv_g \}
\]
Thus we have shown

**Proposition 3.1.** The function \( s_{g,f,k,n,\varphi} \) satisfies
\[
s_{g,f,k,n,\varphi} = \text{constant}
\]
if and only if \( f \) is a critical point of the functional \( f \mapsto EH(g,f,k,n,\varphi) \).

Let us return to the situation of the previous section where we considered a compact Kähler manifold with a maximal torus \( G \) of the reduced automorphisms group, with a fixed Kähler class \( \Omega \). Taking \( \varphi \) to be the \((g,f,k,n)\)-scalar curvature, we consider the Einstein-Hilbert functional \( EH(g,f,k,n) := EH(g,f,k,n,s_{g,f,k,n}) \). By the same reasoning from equivariant cohomology again, for a fixed \((K,a)\), \( EH(g,f_{K,a},g,k,n) \) is independent of the choice of \( g \in K_C^G \Omega \). Set \( EH_{k,n}(K,a) := EH(g,f_{K,a},g,k,n) \). Then using (8), (9) and (10), we see
\[
(22) \quad \left. \frac{d}{dt} \right|_{t=0} EH_{k,n}(K + tH,a) = \frac{(n-2)k}{2} \Vol_n(K,a)^{\frac{n-2}{n}} \int_M \left( s_{g,f_{K,a},g,k,n} - d_{\Omega,K,a,k,n} \right) f^{\frac{n-k}{2} - 1}_{K,a,g} f_{H,0,g} \frac{\omega^n_{n}}{m!}
\]
and
\[ \frac{d}{dt} \bigg|_{t=0} EH_{k,n}(K, a + tb) = \frac{(n - 2)kb}{2 \, \text{Vol}_n(K, a)^{\frac{n-2}{n}+1}} \left( c_{\Omega, K, a, n} - d_{\Omega, K, a, n} \right) \int_M \frac{f_{K,a,g}^n}{m!}. \]

If there exist \( g \in K^G_{\Omega}, K \) and \( a \) such that \( s_{g,fK,a,g,k,n} \) is constant, then
\[ s_{g,fK,a,g,k,n} = c_{\Omega, K, a, k,n} = d_{\Omega, K, a, k,n}, \]
and thus the pair \((K, a)\) is a critical point of the function \( EH_{k,n} : \mathcal{P}^G_{\Omega} \to \mathbb{R} \) given by
\[ (K, a) \mapsto EH_{k,n}(K, a) := EH(g, f_{K,a,g}, k, n). \]

Conversely, suppose that \((K, a)\) is a critical point of \( EH_{k,n} : \mathcal{P}^G_{\Omega} \to \mathbb{R} \). Then one can see \((K, a)\) satisfies \( c_{\Omega, K, a, k,n} = d_{\Omega, K, a, k,n} \). Hence, by (16) and (22), \( \text{Fut}_{\Omega, K, a, k,n} \) vanishes. More direct relation between the volume functional and the Einstein-Hilbert functional can be seen as follows.

**Remark 3.2.** Since \( EH_{k,n} \) is homogeneous of degree 0 on \( \mathcal{P}^G_{\Omega} \) we may restrict \( EH_{k,n} \) to the slice
\[ \mathcal{P}^G_{\Omega,n} := \{(K, a) \in \mathcal{P}^G_{\Omega,n} \mid d_{\Omega, K, a, k,n} = \gamma \}. \]

Then
\[ EH_{k,n}(K, a) = \gamma \, \text{Vol}_{k,n}(K, a)^{\frac{2}{n}} \]
on \( \mathcal{P}^G_{\Omega,n} \). This shows that the volume minimization Theorem 2.1 is equivalent to finding a critical point of the Einstein-Hilbert functional.

4. **The normalized Einstein-Hilbert functional for toric Kähler manifolds.**

In this section, we give the explicit formula for the Futaki invariant and the normalized Einstein-Hilbert functional when \((M, J, \omega)\) is a compact toric Kähler manifold and \( k = -2 \).

Let \((M, \omega)\) be a 2m-dimensional compact toric manifold and \( \mu : M \to \mathbb{R}^m \) the moment map. It is well-known that the image of \( \mu \), \( \Delta := \text{Image} \mu \), is an \( m \)-dimensional Delzant polytope in \( \mathbb{R}^m \). A \( T^m \)-invariant, \( \omega \)-compatible complex structure \( J \) on \( M \) gives a convex function \( u \), called a symplectic potential, on \( \Delta \) as follows. For the action-angle coordinates \((\mu_1, \ldots, \mu_m, \theta_1, \ldots, \theta_m) \in \Delta \times T^m \), there exists a smooth convex function \( u \) on \( \Delta \) which satisfies
\[ J \frac{\partial}{\partial \mu_i} = \sum_{j=1}^m u_{ij} \frac{\partial}{\partial \theta_j}, \quad J \frac{\partial}{\partial \theta_i} = \sum_{j=1}^m H_{ij}^u \frac{\partial}{\partial \mu_j}, \]
where, for a smooth function \( \varphi \) of \( \mu = (\mu_1, \ldots, \mu_m) \), we denote by \( \varphi_i \) the partial derivative \( \partial \varphi / \partial \mu_i \) and by \( H^u = (H^u_{ij}) \) the inverse matrix of the Hessian \((u_{ij})\) of \( u \). Conversely, if we give a smooth convex function \( u \) on \( \Delta \)
satisfying some boundary condition, by the formula above, we can recover a $T^m$-invariant $\omega$-compatible complex structure on $M$, see [1] for more detail.

Let $u$ be a symplectic potential on $\Delta$. Then the toric Kähler metric $g_J = \omega(\cdot, J \cdot)$ is represented as

$$ g_J = \sum_{i,j=1}^{m} u_{ij} d\mu_i d\mu_j + \sum_{i,j=1}^{m} H^u_{ij} d\theta_i d\theta_j. \quad (28) $$

According to Abreu [1], the scalar curvature $s_J$ of $g_J$ is

$$ s_J = - \sum_{i,j=1}^{m} H^u_{ij}. \quad (29) $$

In this case, a Killing potential is an affine linear function positive on $\Delta$. Fix a Killing potential $f$. Then $(g_J, f, k, n)$-scalar curvature $s_{J,f,k,n}$ is given by

$$ s_{J,f,k,n} = f - k s_J + \frac{4(n-1)}{n-2} f^{-\frac{k(n+2)}{4}} \Delta_J f^{\frac{k(n-2)}{4}}, \quad (30) $$

where $\Delta_J = \Delta_{g_J}$. For a smooth function $\varphi$ of $\mu_1, \ldots, \mu_m$,

$$ \Delta_J \varphi = - \sum_{i,j=1}^{m} \{ \varphi_{ij} H^u_{ij} + \varphi_{i} H^u_{ij,j} \} \quad (31) $$

holds (see the equation (20) in [3]). Since $f$ is affine linear, we have

$$ \Delta_J f^{\frac{k(n-2)}{4}} \quad (31) $$

By (29), (30) and (31), the $(g_J, f, k, n)$-scalar curvature is

$$ s_{J,f,k,n} = - f^{-k} \sum_{i,j=1}^{m} \left\{ H^u_{ij,i} + \frac{k(n-1)}{f^2} H^u_{ij,j} + \frac{k(n-1)}{f^2} \left( \frac{k(n-2)}{4} - 1 \right) f_{i,j} H^u_{ij} \right\}. \quad (32) $$

On the other hand, for any $\alpha \in \mathbb{R}$,

$$ \sum_{i,j=1}^{m} (f^\alpha H^u_{ij})_{,ij} = f^\alpha \sum_{i,j=1}^{m} \left\{ H^u_{ij,i} + \frac{2\alpha}{f^2} f_{i,j} H^u_{ij} + \frac{\alpha(\alpha-1)}{f^2} f_{i,j} H^u \right\} \quad (33) $$

holds. We easily see that $2\alpha = k(n-1)$ and $\alpha(\alpha-1) = k(n-1)(k(n-2)/4-1)$ hold if and only if $k = -2$ and $\alpha = 1-n$. In this case, we have

$$ s_{J,f,-2,n} f^{-1-n} = - \sum_{i,j=1}^{m} (f^{1-n} H^u_{ij})_{,ij}. \quad (34) $$
By Lemma 2 in [3], for any smooth function \( \phi \) on \( \mathbb{R}^m \),
\[
\int_{\Delta} \phi \sum_{i,j=1}^{m} \left( f^{1-n} H_{ij}^n \right)_{ij} d\mu = \int_{\Delta} f^{1-n} \sum_{i,j=1}^{m} H_{ij}^n \phi_{ij} d\mu - 2 \int_{\partial \Delta} f^{1-n} \phi d\sigma.
\]
In particular, when \( \phi \) is an affine function
\[
\int_{\Delta} \phi \sum_{i,j=1}^{m} \left( f^{1-n} H_{ij}^n \right)_{ij} d\mu = -2 \int_{\partial \Delta} f^{1-n} \phi d\sigma
\]
holds. Hence, if we define the constant \( c_{\Delta,f,-2,n} \) as
\[
c_{\Delta,f,-2,n} = 2 \int_{\partial \Delta} f^{1-n} d\sigma - \int_{\Delta} f^{-1-n} d\mu,
\]
the Futaki invariant (16) is given by
\[
\text{Fut}_{\Delta,f,-2,n}(\phi) = 2 \int_{\partial \Delta} f^{1-n} \phi d\sigma - c_{\Delta,f,-2,n} \int_{\Delta} f^{-1-n} \phi d\mu
\]
for any linear function \( \phi \) on \( \mathbb{R}^m \).

By (34) and (36), \( EH(g_J, f, -2, n) \) is given by
\[
EH_{-2,n}(f) := EH(g_J, f, -2, n) = \text{Const.} \frac{\int_{\partial \Delta} f^{2-n} d\sigma}{\left( \int_{\Delta} f^{-n} d\mu \right)^{\frac{n-2}{n}}}.
\]
If there exists a symplectic potential \( u \) such that the \((g_J, f, -2, n)\)-scalar curvature is constant, then \( \text{Fut}_{\Delta,f,-2,n} \) vanishes identically and \( f \) is a critical point of \( EH_{-2,n} \).

5. Toric K-stability

Let \((M, \omega)\) be a 2m-dimensional compact toric manifold with the moment image \( \Delta \subset \mathbb{R}^m \). Following the argument by Donaldson in [4], we may define the Donaldson-Futaki invariant with respect to a positive affine function \( f \) on \( \Delta \) as
\[
\text{DF}_{\Delta,f,n}(\phi) = 2 \int_{\partial \Delta} f^{1-n} \phi d\sigma - c_{\Delta,f,-2,n} \int_{\Delta} f^{-1-n} \phi d\mu
\]
for a convex function \( \phi \) on \( \Delta \), see also [3]. For any affine function \( \phi \),
\[
\text{Fut}_{\Delta,f,-2,n}(\phi) = \text{DF}_{\Delta,f,n}(\phi).
\]
We can prove the following straightforward analogue of the results in [4]:
Theorem 5.1. Suppose that there exists a symplectic potential $u$ on $\Delta$ such that the $(g_J, f, -2, n)$-scalar curvature is a constant $c$. Then $c = c(\Delta, f, -2, n)$ and $DF_{\Delta, f, n}(\phi) \geq 0$ for any smooth convex function $\phi$ on $\Delta$. Equality holds if and only if $\phi$ is affine.

Proof. Suppose that $s_{\Delta, f, -2, n} = c$. Then

$$c \int_{\Delta} f^{-1-n} \, d\mu = -\int_{\Delta} \sum_{i,j=1}^{m} (f^{-1-n}H_{ij}^{n}),ij \, d\mu = 2 \int_{\partial \Delta} f^{-1-n} \, d\sigma$$

by (34) and (36). Hence $c = c(\Delta, f, -2, n)$. By (35),

$$DF_{\Delta, f, n}(\phi) = -\int_{\Delta} \left( c(\Delta, f, -2, n)f^{-1-n} + \sum_{i,j=1}^{m} (f^{-1-n}H_{ij}^{n}),ij \right) \phi \, d\mu$$

$$+ \int_{\Delta} f^{-1-n} \sum_{i,j=1}^{m} H_{ij}^{n} \phi_{ij} \, d\mu$$

$$= \int_{\Delta} f^{-1-n} \sum_{i,j=1}^{m} H_{ij}^{n} \phi_{ij} \, d\mu \geq 0.$$

(40)

□

Definition 5.2. Let $\Delta \subset \mathbb{R}^m$ be a Delzant polytope, $n \neq 0, 1, 2$ and $f$ a positive affine function on $\Delta$. $(\Delta, f, n)$ is K-semistable if $DF_{\Delta, f, n}(\phi) \geq 0$ for any piecewise linear convex function $\phi$ on $\Delta$. $(\Delta, f, n)$ is K-polystable if it is K-semistable and the equality $DF_{\Delta, f, n}(\phi) = 0$ is only possible for $\phi$ affine linear.

Since any piecewise linear convex function on $\Delta$ can be approximated by smooth convex functions on $\Delta$, the existence of a constant $(g_J, f, -2, n)$-scalar curvature metric implies the K-semistability of $(\Delta, f, n)$.

We next consider compact toric surfaces and prove that the positivity of Donaldson-Futaki invariant for simple piecewise linear functions implies K-polystability. This is a generalization of the result by Donaldson [4] and Wang-Zhou [14, 15]. The proof is similar to the one given in [15], but to make this paper as self-contained as possible, we give a proof here.

Let $P \subset \mathbb{R}^m$ be an $m$-dimensional open convex polytope, $P^*$ a union of $P$ and the facets of $P$. Denote

$$C_1 := \{u : P^* \to \mathbb{R}, \text{ convex} | \int_{\partial P} u \, d\sigma < \infty\}.$$

For positive bounded functions $\alpha, \beta$ on $\bar{P}$ and an affine function $A$ on $\mathbb{R}^m$, we define the linear functional $L$ on $C_1$ as

$$L(u) := \int_{\partial P} \alpha u \, d\sigma - \int_{P} A \beta u \, d\mu.$$

(41)

Theorem 5.3. Suppose that $L(f) = 0$ for any affine function $f$ on $\mathbb{R}^m$. When $m = 2$, the following two conditions are equivalent.
(1) \( \mathcal{L}(u) \geq 0 \) for any \( u \in \mathcal{C}_1 \) and the equality holds if and only if \( u \) is affine.

(2) \( \mathcal{L}(u) > 0 \) for any simple piecewise linear convex function \( u \) with nonempty crease.

Here a convex function \( u \) is simple piecewise linear, sPL for short, if \( u = \max\{L,0\} \) for a non-zero affine function \( L \). The crease of sPL convex function \( u \) is the intersection of \( P \) and \{\( L = 0 \)\}.

Proof. It is sufficient to prove that (2) implies (1). Suppose that \( \mathcal{L} \) is positive for any sPL convex function with nonempty crease. Moreover we assume the case (1) does not occur, that is, the one of the following holds:

\begin{itemize}
  \item There exists \( v \in \mathcal{C}_1 \) such that \( \mathcal{L}(v) < 0 \).
  \item For any \( u \in \mathcal{C}_1 \), \( \mathcal{L}(u) \geq 0 \) and there exists \( v \in \mathcal{C}_1 \setminus \{\mathrm{affine\ function}\} \) such that \( \mathcal{L}(v) = 0 \).
\end{itemize}

We fix \( p_0 \in P \) and denote

\[ \tilde{\mathcal{C}}_1 := \left\{ u \in \mathcal{C}_1 \mid \int_{\partial P} \alpha u \, d\sigma = 1, \inf_P u = u(p_0) = 0 \right\}. \]

Since \( \mathcal{L} \) vanishes on the set of affine functions and \( \mathcal{L}(cu) = c\mathcal{L}(u) \) for any \( c > 0 \) and \( u \in \mathcal{C}_1 \), we may assume \( v \) in the condition above is an element of \( \tilde{\mathcal{C}}_1 \).

Lemma 5.4. The functional \( \mathcal{L} : \tilde{\mathcal{C}}_1 \to \mathbb{R} \) is bounded from below.

Proof. By Lemma 5.1.3 in [4], there exists a constant \( C > 0 \) such that

\[ \int_P u \, d\mu \leq C \int_{\partial P} u \, d\sigma \]

for all \( u \in \tilde{\mathcal{C}}_1 \). Since \( \alpha, \beta \) are positive and bounded on \( \bar{P} \)

\[ \int_P \beta u \, d\mu \leq \sup_P \beta \int_P u \, d\mu \leq \frac{C \sup_P \beta}{\inf_P \alpha} =: C' \]

for \( u \in \tilde{\mathcal{C}}_1 \). Hence, on \( \tilde{\mathcal{C}}_1 \),

\[ \mathcal{L}(u) = 1 - \int_P A \beta u \, d\mu \geq 1 - \max_P |A| \int_P \beta u \, d\mu \geq 1 - \max_P |A|C'. \]

By assumption, \( \inf_{\tilde{\mathcal{C}}_1} \mathcal{L} \leq 0 \). Moreover we see that there exists \( u_0 \in \tilde{\mathcal{C}}_1 \) which attains the infimum of \( \mathcal{L} \) on \( \tilde{\mathcal{C}}_1 \) by the same argument with the proof of Lemma 4.2 in [15] as follows. Let \( \{u_k\} \) be a sequence in \( \tilde{\mathcal{C}}_1 \) with \( \lim_{k \to \infty} \mathcal{L}(u_k) = \inf_{\tilde{\mathcal{C}}_1} \mathcal{L} \). By Lemma 5.4 above and Corollary 5.2.5 in [4], there is the limit function \( u_0 \) convex on \( P^* \). More precisely,

\[ u_0(p) = \begin{cases} 
  \lim_{k \to \infty} u_k(p) & \text{if } p \in P \\
  \lim_{t \searrow 1} u_0((1-t)p_0 + tp) & \text{if } p \text{ is in a facet of } P.
\end{cases} \]
The limit function $u_0$ satisfies
\[
\int_P A\beta u_0 \, d\mu = \lim_{k \to \infty} \int_P A\beta u_k \, d\mu \quad \text{and} \quad \inf_P u_0 = u_0(p_0) = 0.
\]
By convexity, $\int_{\partial P} \alpha u_0 \, d\sigma \leq 1$. Suppose that $\int_{\partial P} \alpha u_0 \, d\sigma < 1$. Then
\[
\mathcal{L}(u_0) = \int_{\partial P} \alpha u_0 \, d\sigma - \int_P A\beta u_0 \, d\mu < 1 - \int_P A\beta u_0 \, d\mu
\]
\[
= \lim_{k \to \infty} \mathcal{L}(u_k) = \inf_{\tilde{C}_1} \mathcal{L} \leq 0.
\]
On the other hand, since $\tilde{u}_0 := \left(\int_{\partial P} \alpha u_0 \, d\sigma\right)^{-1} u_0 \in \tilde{C}_1$, we have
\[
\left(\int_{\partial P} \alpha u_0 \, d\sigma\right)^{-1} \mathcal{L}(u_0) = \mathcal{L}(\tilde{u}_0) \geq \inf_{\tilde{C}_1} \mathcal{L}.
\]
Hence $\mathcal{L}(u_0) < \left(\int_{\partial P} \alpha u_0 \, d\sigma\right)^{-1} \mathcal{L}(u_0)$. Since $\mathcal{L}(u_0) < 0$, $\int_{\partial P} \alpha u_0 \, d\sigma > 1$. It is a contradiction. Therefore $u_0 \in \tilde{C}_1$ and it attains the infimum of $\mathcal{L}$ on $\tilde{C}_1$.

By the same argument with the proof of Lemma 4.3 in [15], we see that $u_0$ is a generalized solution to the degenerate Monge-Ampère equation
\[
\det D^2 u = 0.
\]

By convexity, $\mathcal{T} = \{x \in P \mid u_0(x) = 0\}$ is convex. Moreover any extreme point of $\mathcal{T}$ is a boundary point of $P$ by Lemma 4.1 in [14]. Since $P$ is two dimensional, $\mathcal{T}$ is either a line segment through $p_0$ with both endpoints on $\partial P$ or a convex polygon with vertices on $\partial P$. Note here that if the dimension of $P$ is greater than two the convex set $\mathcal{T}$ may be more complicated. We set an affine function $L$ on $\mathbb{R}^2$ as follows. When $\mathcal{T}$ is a line segment,
\[
L(x) := \langle n, x - p_0 \rangle,
\]
where $n$ is a unit normal vector of $\mathcal{T}$. When $\mathcal{T}$ is a polygon,
\[
L(x) := \langle n, x - p_1 \rangle,
\]
where $p_1 \in \partial \mathcal{T} \setminus \partial P$ and $n$ is the outer unit normal vector of $\partial \mathcal{T}$ at $p_1$. In either case, $\psi = \max\{0, L\}$ is a sPL convex function with nonempty crease.

We next define a function $a$ as
\[
a(p) = \lim_{t \searrow 0} \frac{u_0(p + tn) - u_0(p)}{t}.
\]
Here $p \in \mathcal{T}$ when $\mathcal{T}$ is a line segment or $p$ is in the edge of $\mathcal{T}$ containing $p_1$ when $\mathcal{T}$ is a polygon. By convexity of $u_0$, the limit exists and is nonnegative for any $p$.

**Lemma 5.5.** $a_0 := \inf a = 0$. 

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Proof. We give a proof only when $T$ is a line segment since the case when $T$ is a polygon is similar. Suppose $a_0 > 0$. Denote $u' := u_0 - a_0 \psi$. Then \[ \int_{\partial P} \alpha u' d\sigma < 1. \] By the definition of $a_0$, $u'$ is convex on $P^*$ and \[ \inf_P u' = u'(p_0) = u_0(p_0) - a_0 \psi(p_0) = 0. \] Since $\mathcal{L}(\psi) > 0$ by assumption, \[ \mathcal{L}(u_0) = \mathcal{L}(u') + a_0 \mathcal{L}(\psi) > \mathcal{L}(u'). \] Hence, since $\tilde{u}' := \left( \int_{\partial P} \alpha u' d\sigma \right)^{-1} u' \in \tilde{C}_1$, \[ 0 \geq \mathcal{L}(u_0) > \mathcal{L}(u') > \mathcal{L}(\tilde{u}'). \] This is a contradiction. \[ \square \]

By the definition of $\mathcal{T}$ and $L$, $u_0$ is positive on $P \cap \{ L > 0 \}$. For any $\varepsilon > 0$, $G_\varepsilon := \{ x \in P \mid u_0(x) < \varepsilon \psi(x) \}$ is nonempty because $a_0 = 0$. Since $\mathcal{T} \subset \{ L \leq 0 \}$, there exists $\delta(\varepsilon) > 0$ such that $G_\varepsilon \subset \{ 0 \leq L < \delta(\varepsilon) \}$. Denote \[ u_1 := u_0 \chi_-, \quad u_2 := (u_0 - \varepsilon \psi) \chi_+, \quad \tilde{u}_2 := \max\{0, u_2\}, \] where \[ \chi_-(x) = \begin{cases} 1 & \text{when } L(x) < 0, \\ 0 & \text{otherwise}, \end{cases} \quad \chi_+ = 1 - \chi_- . \] It is easy to see that $u_1 + \tilde{u}_2 \geq 0$ is convex and $(u_1 + \tilde{u}_2)(p_0) = 0$. Denote $\tilde{u} := u_1 + \tilde{u}_2 + \varepsilon \psi$. Then we have \[ \tilde{u} - u_0 = \tilde{u}_2 - u_2 = \begin{cases} -u_2 = \varepsilon L - u_0 \leq \varepsilon \delta(\varepsilon) & \text{on } G_\varepsilon, \\ 0 & \text{on } G_\varepsilon^c. \end{cases} \] Hence there exists a positive constant $C$ such that \[ \mathcal{L}(\tilde{u} - u_0) = \int_{\partial P} \alpha(\tilde{u} - u_0) d\sigma - \int_P A\beta(\tilde{u} - u_0) d\mu < C\varepsilon \delta(\varepsilon). \] Therefore we have \[ \mathcal{L}(u_1 + \tilde{u}_2) = \mathcal{L}(\tilde{u}) - \varepsilon \mathcal{L}(\psi) < \mathcal{L}(u_0) + \varepsilon (C \delta(\varepsilon) - \mathcal{L}(\psi)) < \mathcal{L}(u_0). \] for any sufficiently small $\varepsilon > 0$. Denote $u_3 := \left( \int_{\partial P} \alpha(u_1 + \tilde{u}_2) d\sigma \right)^{-1} (u_1 + \tilde{u}_2) \in \tilde{C}_1$. Since $u_1 + \tilde{u}_2 \leq u_0$, \[ \int_{\partial P} \alpha(u_1 + \tilde{u}_2) d\sigma \leq 1. \] Therefore we obtain $\mathcal{L}(u_3) \leq \mathcal{L}(u_1 + \tilde{u}_2) < \mathcal{L}(u_0)$. This is a contradiction. This completes the proof of Theorem 5.3. \[ \square \]
Finally we observe by numerical analysis that there exists a Killing vector field which gives a toric K-stable pair in the sense of Apostolov-Maschler.

Let $\Delta_p$ be the convex hull of $(0,0)$, $(p,0)$, $(p,1-p)$ and $(0,1)$ for $0 < p < 1$. By Delzant construction, the Kähler class of a toric Kähler metric on the one point blow up of $\mathbb{C}P^2$ corresponds to $\Delta_p$ up to multiplication of a positive constant.

Denote

$$\mathcal{P} := \{(a, b, c) \in \mathbb{R}^3 | c > 0, ap + c > 0, ap + b(1 - p) + c > 0, b + c > 0\}.$$ 

An affine function $a\mu_1 + b\mu_2 + c$ is positive on $\Delta_p$ if and only if $(a,b,c) \in \mathcal{P}$.

By the argument in Section 3 and 4, Fut$_\Delta(a\mu_1 + b\mu_2 + c, -2, n$ vanishes if and only if $(a,b,c) \in \mathcal{P}$ is a critical point of

$$EH_n(a,b,c) := \int_{\partial \Delta_p} (a\mu_1 + b\mu_2 + c)^{2-n} d\sigma \left( \int_{\Delta_p} (a\mu_1 + b\mu_2 + c)^{-n} d\mu \right)^{\frac{n-2}{n}}.$$ 

For $n = 4$, the authors identified in [8] such critical points as follows:

(a) $C \left(1,0,\frac{p(1-\sqrt{1-p})}{2\sqrt{1-p}+p-2}\right), \ C > 0, \ 0 < p < 1$,

(b) $C \left(-1,0,\frac{p(3p \pm \sqrt{9p^2 - 8p})}{2(p \pm \sqrt{9p^2 - 8p})}\right), \ C > 0, \ \frac{8}{9} < p < 1$,

(c) $C \left(-p^2 + 4p - 2 \pm \sqrt{F(p)}, \pm 2\sqrt{F(p)}, -p^2 - 2p + 2 \mp \sqrt{F(p)}\right), \ C > 0, \ 0 < p < \alpha$,

where $\alpha \approx 0.386$ is a real root of

$$F(x) := x^4 - 4x^3 + 16x^2 - 16x + 4 = 0.$$ 

For the affine functions corresponding to (a) and (b), LeBrun gave concrete examples of cKEM metrics in [12]. Hence $(\Delta_p, ap_1 + b\mu_2 + c, 4)$ is K-polystable by Corollary 3 in [3]. On the other hand, in case (c), we do not know whether there exists cKEM metrics. Denote

$$f^\pm_p = (\pm 2 \mp \sqrt{F(p)})\mu_1 \pm 2\sqrt{F(p)}\mu_2 - p^2 - 2p + 2 \mp \sqrt{F(p)},$$

$$= a^\pm_p \mu_1 + b^\pm_p \mu_2 + c^\pm_p.$$ 

By Theorem 5.3, if $DF_{\Delta_p, f^\pm_p A}(\phi)$ is positive for any sPL convex function $\phi$, $(\Delta_p, f^\pm_p, 4)$ is K-polystable. According to the position of the boundary points $u, v$ of creases, we divide into the following six cases.

1. $u = (0, e), v = (p, f) \ (0 \leq e \leq 1, 0 \leq f \leq 1 - p)$ : In this case, the corresponding sPL convex function is $\phi = \max\{(f - e)\mu_1 - p\mu_2 +$
\[ pe, 0 \}. \text{ Then} \]
\[
\int_{\partial \Delta_p} \frac{\phi}{(f_p^\pm)^3} \, d\sigma = \int_0^p \frac{(f - e)\mu_1 + pe}{(a_p^\pm \mu_1 + c_p^\pm)^3} \, d\mu_1 + \int_f^1 \frac{p(f - \mu_2)}{(a_p^\pm p + b_p^\pm \mu_2 + c_p^\pm)^3} \, d\mu_2
\]
\[
+ \int_e^0 \frac{p(e - \mu_2)}{(b_p^\pm \mu_2 + c_p^\pm)^3} \, d\mu_2
\]
and
\[
\int_{\Delta_p} \frac{\phi}{(f_p^\pm)^5} \, d\mu = \int_0^p \mu_1 \int_0^{f/e} \frac{(f - e)\mu_1 + p(1 - f)e}{(f_p^\pm)^5} \, d\mu_2
\]

It is too long and complicated to give the full description of \( DF_{\Delta_p, f_p^\pm, q} (\phi) \). We put the graph of \( DF_{\Delta_0, f_0^1, 4} \), as a function of \((e, f)\), instead. All graphics in this article are drawn by Mathematica.

2. \( u = (e, 0), \ v = (f, 1 - f) \) \((0 \leq e \leq p, 0 \leq f \leq p)\): In this case, the corresponding sPL convex function is \( \phi = \max\{(f - 1)\mu_1 + (f - e)\mu_2 + (1 - f)e, 0\} \). Then
\[
\int_{\partial \Delta_p} \frac{\phi}{(f_p^\pm)^3} \, d\sigma = \int_0^e \frac{(1 - f)(e - \mu_1)}{(a_p^\pm \mu_1 + c_p^\pm)^3} \, d\mu_1 + \int_0^f \frac{(f - e)\mu_2 + (1 - f)e}{(b_p^\pm \mu_2 + c_p^\pm)^3} \, d\mu_2
\]
\[
+ \int_0^f \frac{(f - 1)\mu_1 + (f - e)(1 - \mu_1) + (1 - f)e}{(a_p^\pm \mu_1 + b_p^\pm (1 - \mu_1) + c_p^\pm)^3} \, d\mu_1
\]
and
\[
\int_{\Delta_p} \frac{\phi}{(f_p^\pm)^5} \, d\mu = \int_0^1 \mu_2 \int_0^{f/e} \frac{(f - 1)\mu_1 + (f - e)\mu_2 + (1 - f)e}{(f_p^\pm)^5} \, d\mu_1
\]
\[
+ \int_0^1 \mu_2 \int_0^{1 - \mu_2} \frac{(f - 1)\mu_1 + (f - e)\mu_2 + (1 - f)e}{(f_p^\pm)^5} \, d\mu_1
\]
The graph of \( DF_{\Delta_0, f_0^1, 4} \) is as follows.
3. \( u = (0, e), \ v = (f, 1 - f) \ (0 \leq e \leq 1, 0 \leq f \leq p) \): In this case, the corresponding sPL convex function is \( \phi = \max\{(f + e - 1)\mu_1 + f\mu_2 - fe, 0\} \). Then

\[
\int_{\partial \Delta_p (f_p^\pm)^3} \frac{\phi}{(f_p^\pm)^3} \, d\sigma = \int_e^1 \frac{f(\mu_2 - e)}{(b_p^+ \mu_2 + c_p^+)^3} \, d\mu_2 + \int_0^f \frac{(f + e - 1)\mu_1 + f(1 - \mu_1) - fe}{(a_p^+ \mu_1 + b_p^+ (1 - \mu_1) + c_p^+)^3} \, d\mu_2
\]

and

\[
\int_{\Delta_p (f_p^\pm)^5} \frac{\phi}{(f_p^\pm)^5} \, d\mu = \int_0^f d\mu_1 \int_0^{1-\mu_1} \frac{(f + e - 1)\mu_1 + f\mu_2 - fe}{(f_p^\pm)^5} \, d\mu_2
\]

The graph of DF\(_{\Delta_{0.1}, f_{0.1}}\) is as follows.

4. \( u = (0, e), \ v = (f, 0) \ (0 \leq e \leq 1, 0 \leq f \leq p) \): In this case, the corresponding sPL convex function is \( \phi = \max\{-e\mu_1 - f\mu_2 + fe, 0\} \). Then

\[
\int_{\partial \Delta_p (f_p^\pm)^3} \frac{\phi}{(f_p^\pm)^3} \, d\sigma = \int_0^f \frac{e(f - \mu_1)}{(a_p^+ \mu_1 + c_p^+)^3} \, d\mu_1 + \int_0^e \frac{f(e - \mu_2)}{(b_p^+ \mu_2 + c_p^+)^3} \, d\mu_2
\]

and

\[
\int_{\Delta_p (f_p^\pm)^5} \frac{\phi}{(f_p^\pm)^5} \, d\mu = \int_0^f d\mu_1 \int_0^{-\mu_1 + e} \frac{-e\mu_1 - f\mu_2 + fe}{(f_p^\pm)^5} \, d\mu_2
\]
The graph of $\Delta_{0.1.f_{0.1}}$ is as follows.

5. $u = (p, e), \ v = (f, 0) \ (0 \leq e \leq 1 - p, 0 \leq f \leq p)$ : In this case, the corresponding sPL convex function is $\phi = \max\{e\mu_1 + (f - p)\mu_2 - fe, 0\}$. Then

$$\int_{\partial \Delta_p} \frac{\phi}{(f_+^p)^3} \ d\sigma = \int_f^p \frac{e(\mu_1 - f)}{(a_p^+ \mu_1 + c_p^+)^3} \ d\mu_1 + \int_0^e \frac{(p - f)(e - \mu_2)}{(a_p^+ p + b_p^+ \mu_2 + c_p^+)^3} \ d\mu_2$$

and

$$\int_{\Delta_p} \frac{\phi}{(f_+^p)^5} \ d\mu = \int_f^p \mu_1 \int_0^{1-p} \frac{e\mu_1 + (f - p)\mu_2 - fe}{(f_+^p)^5} \ d\mu_2$$

The graph of $\Delta_{0.1.f_{0.1}}$ is as follows.

6. $u = (p, e), \ v = (f, 1 - f) \ (0 \leq e \leq 1 - p, 0 \leq f \leq p)$ : In this case, the corresponding sPL convex function is $\phi = \max\{(1 - e - f)(\mu_1 - p) + (p - f)\mu_2 + (f - p)e, 0\}$. Then

$$\int_{\partial \Delta_p} \frac{\phi}{(f_+^p)^3} \ d\sigma = \int_f^p \frac{(1 - e - f)(\mu_1 - p) + (p - f)(1 - \mu_1) + (f - p)e}{(a_p^+ \mu_1 + b_p^+ (1 - \mu_1) + c_p^+)^3} \ d\mu_1$$

$$+ \int_e^{1-p} \frac{(p - f)(\mu_2 - e)}{(a_p^+ p + b_p^+ \mu_2 + c_p^+)^3} \ d\mu_2$$

and

$$\int_{\Delta_p} \frac{\phi}{(f_+^p)^5} \ d\mu = \int_f^p \mu_1 \int_0^{1-\mu_1} \frac{(1 - e - f)(\mu_1 - p) + (p - f)\mu_2 + (f - p)e}{(f_+^p)^5} \ d\mu_2$$

The graph of $\Delta_{0.1.f_{0.1}}$ is as follows.
Looking at the graphs, $(\Delta p, f_p^\pm, 4)$ must be K-polystable. By Theorem 5 in [3], cKEM metrics with Killing potential $f_p^\pm$ ought to exist. We leave this problem to the interested readers.

References

[1] M. Abreu, Kähler geometry of toric varieties and extremal metrics. Int. J. Math., 9(1998), 641-651.
[2] V. Apostolov, D. M. J. Calderbank, P. Gauduchon and E. Legendre: Levi-Kähler re-
duction of CR structures, products of spheres, and toric geometry, arXiv:1708.05253
[3] V. Apostolov and G. Maschler, Conformally Kähler, Einstein-Maxwell geometry,
arXiv preprint [arXiv:1512.06391]
[4] S.K. Donaldson: Scalar curvature and stability of toric varieties, J. Differential
Geometry, 62(2002), 289-349.
[5] A. Futaki : Kähler-Einstein metrics and integral invariants, Lecture Notes in Math.,
vol.1314, Springer-Verlag, Berlin-Heidelberg-New York,(1988).
[6] A. Futaki and S. Morita : Invariant polynomials of the automorphism group of a
compact complex manifold, J. Diff. Geom., 21(1985), 135–142.
[7] A. Futaki and T. Mabuchi : Moment maps and symmetric multilinear forms associ-
ated with symplectic classes, Asian J. Math., 6(2002), 349–372.
[8] A. Futaki and H. Ono : Volume minimization and Conformally Kähler, Einstein-
Maxwell geometry. To appear in J. Math. Soc. Japan. preprint, [arXiv:1706.07953].
[9] A. Futaki and H. Ono : Volume minimization and obstructions to solving some
problems in Kähler geometry, preprint.
[10] A. Lahdili : Automorphisms and deformations of conformally Kähler, Einstein-
Maxwell metrics. [arXiv:1708.01507]
[11] A. Lahdili : Conformally Kähler, Einstein-Maxwell metrics and boundedness of the
modified Mabuchi-functional. [arXiv:1710.00235]
[12] C. LeBrun: The Einstein-Maxwell equations and conformally Kähler geometry, Com-
mun. Math. Phys., 344, 621–653 (2016).
[13] M. Lejmi and M. Upmeier : Integrability theorems and conformally constant Chern
scalar curvature metrics in almost Hermitian geometry, [arXiv:1703.01323]
[14] X.-J. Wang and B. Zhou: On the existence and nonexistence of extremal metrics on
toric Kähler surfaces. Adv. Math. 226 (2011), no. 5, 4429–4455.
[15] X.-J. Wang and B. Zhou : K-stability and canonical metrics on toric manifolds. Bull.
Inst. Math. Acad. Sin. (N.S.) 9(2014), no. 1, 85–110.
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