\textbf{Abstract}

We compute the multipliers (2-cocycles) of the free nilpotent groups of class 2 and rank $n$ and give conditions for simplicity of the associated twisted group $C^*$-algebras. The free nilpotent groups of class 2 and rank $n$ can also be considered as a family of generalized Heisenberg groups with higher-dimensional center and their group $C^*$-algebras are in a natural way isomorphic to continuous fields over $\mathbb{T}^{\frac{1}{2}(n-1)}$ with the noncommutative $n$-tori as fibers. In this way, the twisted group $C^*$-algebras associated with the free nilpotent groups of class 2 and rank $n$ may be thought of as “second order” noncommutative $n$-tori.

\textbf{Introduction}

The discrete Heisenberg group may be described as the group generated by three elements $u_1, u_2, v_{12}$ satisfying the commutation relations

$$[u_1, v_{12}] = [u_2, v_{12}] = 1, \quad [u_1, u_2] = v_{12}.\]$$

The group has received much attention in the literature, partly because it is one of the easiest examples of a nonabelian torsion-free group. Moreover, the continuous Heisenberg group (see below) is a connected nilpotent Lie group that arises in certain quantum mechanical systems.

As a natural consequence of this attention, several classes of generalized Heisenberg groups have been investigated. For example, in \cite{MilnesWalters, MilnesWalters2} Milnes and Walters describe all the four and five-dimensional nilpotent groups, and in \cite{LeePacker, LeePacker2}, Lee and Packer study the finitely generated torsion-free two-step nilpotent groups with one-dimensional center.

In this paper, on the other hand, we will consider a family of generalized Heisenberg groups, denoted by $G(n)$ for $n \geq 2$, with larger center. The groups $G(n)$ are the so-called free nilpotent groups of class 2 and rank $n$ and will be defined properly in Section \ref{sec:free-nilpotent-groups}. Here we also provide further motivation for our
investigation of these groups. Inspired by the work of Packer [13], we compute
the second cohomology group $H^2(G(n), T)$ of $G(n)$ and study the structure
of the twisted group $C^*(G(n), \sigma)$ associated with multipliers $\sigma$ of
$G(n)$.

Section 2 is devoted to the multiplier calculations, where we decompose
$G(n)$ into a semidirect product and apply techniques introduced by Mackey [9]. In
particular, we will see that

$$H^2(G(n), T) \cong T^{\frac{1}{2}(n+1)n(n-1)},$$

and in Theorem 2.7 we give explicit formulas for the multipli ers of
$G(n)$ up to
similarity.

Next, in Section 3 we describe $C^*(G(n), \sigma)$ as a universal
$C^*$-algebra of
a set of generators and relations. Then we construct the algebra that in a
natural way appear as a continuous field over the compact spac e
$H^2(G(n), T)$ with $C^*(G(n), \sigma)$ as fibers. We also conjecture that this algebra is the group
$C^*$-algebra of the free nilpotent group of class 3 and rank $n$, which is indeed the
case for $n = 2$.

In Section 4 we investigate the center of $C^*(G(n), \sigma)$ and give conditions for
simplicity of these twisted group $C^*$-algebras.

Finally, in Section 5 we study the automorphism group of $G(n)$ and discuss
isomorphism invariants of $C^*(G(n), \sigma)$ coming from $\text{Aut} G(n)$.

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comments throughout this work.

1 The free nilpotent groups $G(n)$

For each natural number $n \geq 2$, let $G(n)$ be the group generated by elements
$\{u_i\}_{1 \leq i \leq n}$ and $\{v_{jk}\}_{1 \leq j < k \leq n}$ subject to the relations

$$[v_{jk}, v_{lm}] = [u_i, v_{jk}] = 1, \quad [u_j, u_k] = v_{jk} \quad (1)$$

for $1 \leq i \leq n$, $1 \leq j < k \leq n$, and $1 \leq l < m \leq n$. Clearly, $G(2)$ is the usual
(discrete) Heisenberg group. For some purposes, it can be useful to set $G(1) = \langle u_1 \rangle = \mathbb{Z}$. Remark that $G(n)$ is generated by $n + \frac{1}{2}n(n-1) = \frac{1}{2}n(n+1)$ ele-
ments.

The group $G(n)$ is called the free nilpotent group of class 2 and rank $n$. Indeed, $G(n)$ is a free object on $n$ generators in the category of nilpotent groups
of step at most two. To see this, note first that $G(n)$ is the group generated by
$\{u_i\}_{i=1}^n$ subject to the relations that all commutators of order greater than two
involving the generators are trivial. Let $G'(n)$ be any other nilpotent group of
step at most two and let $\{u'_i\}_{i=1}^n$ be any set of $n$ elements in $G'(n)$. Then there is a unique homomorphism from $G(n)$ to $G'(n)$ that maps $u_i$ to $u'_i$ for $1 \leq i \leq n$.

Of course, every free object on $n$ generators in this category is isomorphic with
$G(n)$. For a more extensive treatment of free nilpotent groups, see the post on Terence Tao’s webpage [17] (see also 2. in the list below).
Furthermore, we will need the following concrete realization, say $\tilde{G}(n)$, of $G(n)$. For $n \geq 2$, we denote the elements of $\tilde{G}(n)$ by

$$r = (r_1, \ldots, r_n, r_{12}, r_{13}, \ldots, r_{n-1,n})$$

where all entries are integers, and define multiplication by

$$r \cdot s = (r_1 + s_1, \ldots, r_n + s_n,\]

$$r_{12} + r_1 s_2, r_{13} + r_1 s_3, \ldots, r_{n-1,n} + s_{n-1,n} + r_{n-1,n}s_n).$$

By letting $u_i$ have 1 in the $i$'th spot and 0 else and $v_{jk}$ have 1 in the $jk$'th spot and 0 else, the relations are satisfied for these elements. Next, we define the map

$$\tilde{G}(n) \rightarrow G(n), \quad r \mapsto v_{12} \cdots v_{n-1,n} u_{n} \cdots u_{1},$$

and then it is not difficult to see that $\tilde{G}(n)$ is isomorphic to $G(n)$. Henceforth, we will not distinguish between $G(n)$ and the realization $\tilde{G}(n)$ just described, but this should cause no confusion.

Denote by $V(n)$ the subgroup of $G(n)$ generated by the $v_{jk}$'s. Then $V(n)$ coincides with the center $Z(G(n))$ of $G(n)$ and

$$V(n) = Z(G(n)) \cong \mathbb{Z}^{n(n-1)}.$$}

Moreover, consider the subgroups $G(n-1)$ and $H(n)$ of $G(n)$ defined by

$$G(n-1) = \langle u_i, v_{jk} : 1 \leq i \leq n-1, 1 \leq j < k \leq n-1 \rangle,$$

$$H(n) = \langle u_n, v_{jn} : 1 \leq j < n \rangle.$$}

Note that $G(n-1)$ sits inside $G(n)$ as a subgroup and that $H(n) \cong \mathbb{Z}^n$ is a normal subgroup of $G(n)$. Clearly, we have that $G(n)/V(n) \cong \mathbb{Z}^n$ and $G(n)/H(n) \cong G(n-1)$. Therefore, there are short exact sequences

$$1 \rightarrow V(n) \rightarrow G(n) \rightarrow \mathbb{Z}^n \rightarrow 1$$

and

$$1 \rightarrow H(n) \rightarrow G(n) \rightarrow G(n-1) \rightarrow 1$$

where the second one splits and the first does not. In particular, $G(n)$ is a central extension of $\mathbb{Z}^n$ by $\mathbb{Z}^{\frac{n(n-1)}{2}}$ and consequently, $G(n)$ is a two-step nilpotent group.

To motivate our investigation of $G(n)$, we present a few aspects about these groups and some appearances in the literature.

1. Consider in the first place the continuous Heisenberg group. We will represent this group in two different ways, $G_{\text{matrix}}$ and $G_{\text{wedge}}$, both with elements $(x, x') = (x_1, x_2, x') \in \mathbb{R}^3$, i.e. $x = (x_1, x_2) \in \mathbb{R}^2$, and with multiplication as follows. For $G_{\text{matrix}}$ we define

$$(x_1, x_2, x')(y_1, y_2, y') = (x_1 + y_1, x_2 + y_2, x' + y' + x_1 y_2),$$

\footnote{To be absolutely precise, the entries with double index are colexicographically ordered, that is, $(i, j) < (k, l)$ if $j < l$ or if $j = l$ and $i < k.$}
and for $G_{\text{wedge}}$ we set

$$(x_1, x_2, x')(y_1, y_2, y') = (x_1 + y_1, x_2 + y_2, x' + y' + \frac{1}{2}(x_1y_2 - x_2y_1)).$$

One can deduce that $G_{\text{matrix}} \cong G_{\text{wedge}}$. To motivate the notation, note that $G_{\text{matrix}}$ can be represented as matrix multiplication in $M_2(\mathbb{R})$ if one identifies

$$(x_1, x_2, x') \mapsto \begin{pmatrix} 1 & x_1 & x' \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix},$$

and that the multiplication in $G_{\text{wedge}}$ may be written as

$$(x, x')(y, y') = (x + y, x' + y' + \frac{1}{2}(x \wedge y)).$$

In general, the wedge product on $\mathbb{R}^n$ is defined as a certain bilinear map (see e.g. [16], p. 79)

$$\mathbb{R}^n \times \mathbb{R}^n \to \bigwedge^2(\mathbb{R}^n),$$

where $\bigwedge^2(\mathbb{R}^n)$ is a $\frac{1}{2}n(n-1)$-dimensional real vector space. The elements of $\bigwedge^2(\mathbb{R}^n)$ are called bivectors and if $\{e_i\}_{i=1}^n$ is a basis for $\mathbb{R}^n$, then $\{e_i \wedge e_j\}_{i<j}$ is a basis for $\bigwedge^2(\mathbb{R}^n)$. For $n \geq 2$, define the group $\tilde{G}(n, \mathbb{R})$ with elements

$$(x, x') \in \mathbb{R}^n \oplus \bigwedge^2(\mathbb{R}^n), \text{ where } x = (x_1, \ldots, x_n), x' = (x'_1, x'_2, \ldots, x'_{n-1,n}),$$

and where multiplication is given by

$$(x, x')(y, y') = (x + y, x' + y' + \frac{1}{2}(x \wedge y)).$$

This group is of dimension $n + \frac{1}{2}n(n-1) = \frac{1}{2}n(n+1)$. Remark especially that if $n = 3$ the wedge product can be identified with the vector cross product on $\mathbb{R}^3$. That is, the product in $\tilde{G}(3, \mathbb{R})$ is given by

$$(x, x')(y, y') = (x + y, x' + y' + \frac{1}{2}(x \times y)).$$

It is not hard to see that the groups $\tilde{G}(n, \mathbb{R})$ are isomorphic with groups consisting of the same elements, but with multiplication given by

$$(x, x')(y, y') = (x + y, x' + y' + (x_1 y_2, x_1 y_3, \ldots, x_{n-1,n})). \quad (2)$$

For $n \geq 2$, let $G(n, \mathbb{R})$ denote the group defined by (2). Then $G(n)$ is the integer version of $G(n, \mathbb{R})$.

2. One may define the free nilpotent group $G(m, n)$ of class $m$ and rank $n$ for every $m \geq 1$. Indeed, $G(m, n)$ is the group generated by $\{u_i\}_{i=1}^n$ subject to the relations that all commutators of order greater than $m$ involving the generators are trivial. More precisely, for $m = 1, 2, 3$ and $n \geq 2$, the group $G(m, n)$ can be described as the groups with presentation

$$G(1, n) = \langle \{u_i\}_{i=1}^n : [u_i, u_j] = 1 \rangle \cong \mathbb{Z}^n,$$

$$G(2, n) = \langle \{u_i\}_{i=1}^n : [[u_i, u_j], u_k] = 1 \rangle = G(n),$$

$$G(3, n) = \langle \{u_i\}_{i=1}^n : [[[u_i, u_j], u_k], u_l] = 1 \rangle. \quad (3)$$

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and it should now be clear how to define $G(m, n)$ for all $m \geq 1, n \geq 2$.
Finally, we set $G(m, 1) = \langle u_1 \rangle \cong \mathbb{Z}$ for each $m \geq 1$. Moreover, for all $m, n \geq 1$, the group $G(m, n)$ is the free object on $n$ generators in the category of nilpotent groups of step at most $m$. In particular, notice that $G(m, n)$ is $m$-step nilpotent and that

$$G(m, n) \cong G(m + 1, n)/Z(G(m + 1, n)).$$

Again, we refer to [17] for additional details.

In [11] Section 4, Milnes and Walters describe the simple quotients of the $C^*$-algebra associated with a five-dimensional group denoted by $H_{5, 4}$. One can check that $H_{5, 4}$ is isomorphic with the group $G(3, 2)$ (see Remark 5.3 for more about this group).

The homology of free nilpotent groups of class 2 has been calculated in [6] (see Remark 2.10 below).

3. The group $G(3)$ is briefly discussed by Baggett and Packer [3] Example 4.3]. The purpose of that paper is to describe the primitive ideal space of group $C^*$-algebras of some two-step nilpotent groups. However, $G(3)$ only serves as an example of a group the authors could not handle.

Moreover, as remarked in [3], the (ordinary) irreducible representation theory of $G(3)$ coincides with the projective irreducible representation theory of $\mathbb{Z}^3$.

4. Fix $n \geq 2$. It follows from [3] Section 1 that the group $C^*$-algebra $A = C^*(G(n))$ may be described as the universal $C^*$-algebra generated by unitaries $\{U_i\}_{1 \leq i \leq n}$ and $\{V_{jk}\}_{1 \leq j < k \leq n}$ satisfying the relations

$$[V_{jk}, V_{lm}] = [U_i, V_{jk}] = I, \quad [U_j, U_k] = V_{jk}$$

for all $1 \leq i \leq n, 1 \leq j < k \leq n$, and $1 \leq l < m \leq n$.

For $\lambda = (\lambda_{12}, \lambda_{13}, \ldots, \lambda_{n-1,n}) \in \mathbb{T}^{n(n-1)}$, let $A_\lambda$ be the noncommutative $n$-torus. It is the universal $C^*$-algebra generated by unitaries $\{W_i\}_{i=1}^n$ and relations $[W_i, W_j] = \lambda_{ij}I$ for $1 \leq i < j \leq n$. The universal property of $A$ gives that for each $\lambda$ in $\mathbb{T}^{n(n-1)}$ there is a surjective $*$-homomorphism

$$\pi_\lambda : A \to A_\lambda$$

satisfying $\pi_\lambda(U_i) = W_i$ for $1 \leq i \leq n$ and $\pi_\lambda(V_{jk}) = \lambda_{jk}I$ for $1 \leq j < k \leq n$.

Furthermore, $A$ has center $Z(A) = C^*(V(n))$. Indeed, this is the case since $G(n)$ is amenable and its finite conjugacy classes are precisely the one-point sets of central elements (see Lemma 4.1). Therefore, we set

$$T = \text{Prim} \ Z(A) \cong \widehat{Z(A)} = \mathbb{T}^{n(n-1)}.$$
such that $\rho(W_i) = U_i + I_\lambda$ for $1 \leq i \leq n$. Hence, $\rho \circ \pi_\lambda$ coincides with the quotient map $A \to A/I_\lambda$ and consequently, $\ker \pi_\lambda \subset I_\lambda$. Therefore, $A_\lambda \cong A/I_\lambda$ and $\pi_\lambda$ may be regarded as the quotient map $A \to A/I_\lambda$.

For $a \in A$, let $\tilde{a}$ be the section $T \to \bigsqcup_{\lambda} A_\lambda$ given by $\tilde{a}(\lambda) = \pi_\lambda(a)$ and let $\tilde{A} = \{ \tilde{a} \mid a \in A \}$ be the set of sections. Then the following can be deduced from the Dauns-Hofmann Theorem [5].

**Theorem 1.1.** The triple $(T, \{A_\lambda\}, \tilde{A})$ consisting of the base space $T$, $C^*$-algebras $A_\lambda$ for each $\lambda$ in $T$, and the set of sections $\tilde{A}$, is a full continuous field of $C^*$-algebras. Moreover, the $C^*$-algebra associated with this continuous field is naturally isomorphic to $A$.

This result may also be obtained as a corollary to [15, Theorem 1.2] by taking $G = G(n)$ and $\sigma = 1$ in that theorem, but our proof is more direct, in the spirit of [14, Theorem 1.1] which covers the case where $n = 2$.

## 2 The multipliers of $G(n)$

Let $G$ be any discrete group with identity $e$. A function $\sigma : G \times G \to \mathbb{T}$ satisfying

$$
\sigma(r, s)\sigma(rs, t) = \sigma(r, st)\sigma(s, t)
$$

$$
\sigma(r, e) = \sigma(e, r) = 1
$$

for all elements $r, s, t \in G$ is called a multiplier of $G$ or a 2-cocycle on $G$ with values in $\mathbb{T}$. Moreover, two multipliers $\sigma$ and $\tau$ are said to be similar if

$$
\tau(r, s) = \beta(r)\beta(s)\beta(rs)\sigma(r, s)
$$

for all $r, s \in G$ and some $\beta : G \to \mathbb{T}$. The set of similarity classes of multipliers of $G$ is an abelian group under pointwise multiplication. This group is the second cohomology group $H^2(G, \mathbb{T})$.

To compute the multipliers of $G(n)$ up to similarity, we will proceed in the following way. Consider $G(n)$ as the split extension of $G(n - 1)$ by $H(n)$ as described in Section 4. We will identify the elements

$$
a = (0, \ldots, 0, a_n, 0, \ldots, 0, a_{1n}, \ldots, a_{n-1,n}),
$$

$$
b = (b_1, \ldots, b_{n-1}, 0, b_{12}, \ldots, b_{n-2,n-1}, 0, \ldots, 0),
$$

of $H(n)$ and $G(n - 1)$, respectively, with ones of the form

$$
a \leftrightarrow (a_n, a_{1n}, \ldots, a_{n-1,n}),
$$

$$
b \leftrightarrow (b_1, \ldots, b_{n-1}, b_{12}, \ldots, b_{n-2,n-1}).
$$

By properties of the semidirect product, the elements of $G(n)$ can be uniquely written as a product $ab$, where $a$ belongs to $H(n)$ and $b$ belongs to $G(n - 1)$. Define the action $\alpha$ of $G(n - 1)$ on $H(n)$ by

$$
\alpha_b(a) = bab^{-1} = (a_n, a_{1n} + b_1a_n, \ldots, a_{n-1,n} + b_{n-1}a_n).
$$
Hence, alternatively, one may write \( G(n) = H(n) \times H(n - 1) \), but to simplify the notation, we still denote the elements of \( G(n) \) by \( ab \) instead of \( (a, b) \), and write the group product in \( G(n) \) as \((ab)(a' b') = a \alpha_b(a') b b'\) for \( a, a' \in H(n) \) and \( b, b' \in G(n - 1) \). Hopefully, the reader is familiar with semidirect products so that this does not cause any confusion.

Next, we may apply MacKey’s theorem [7] Theorem 9.4] and obtain:

**Remark 2.1.** For notational reasons, to minimize the number of primes in the computations, we are switching \( a \) and \( a' \).

**Theorem 2.2.** Every multiplier of \( G(n) \) is similar to a multiplier \( \sigma_n \) of \( G(n) \) of the form

\[
\sigma_n(a'b, b') = \sigma_{H(n)}(a', \alpha_b(a))g_n(a, b)\sigma_{n-1}(b, b'),
\]

where \( \sigma_{H(n)} \) and \( \sigma_{n-1} \) are multipliers of \( H(n) \) and \( G(n-1) \), respectively, \( g_n \) is a function \( H(n) \times G(n-1) \rightarrow \mathbb{T} \) such that \( g_n(a, e) = g_n(e, b) = 1 \) for all \( a \in H(n), b \in G(n-1) \), and \( \sigma_{H(n)} \) and \( g_n \) satisfy

\[
\begin{align*}
g_n(a + a', b) & = \sigma_{H(n)}(\alpha_b(a), \alpha_b(a'))\sigma_{H(n)}(a, a') \cdot g_n(a, b)g_n(a', b), \\
g_n(ab, bb') & = g_n(\alpha_v(a), b)g_n(a, b').
\end{align*}
\]

Moreover, for every choice of \( \sigma_{H(n)}, g_n \) and \( \sigma_{n-1} \) satisfying the conditions above, \( \sigma_n \) is a multiplier of \( G(n) \).

**Proposition 2.3.** Let \((\sigma_{H(n)}, g_n, \sigma_{n-1})\) and \((\sigma'_{H(n)}, g'_n, \sigma'_{n-1})\) be triples satisfying the conditions of Theorem 2.2 and let \( \sigma_n \) and \( \sigma'_n \) be the corresponding multipliers of \( G(n) \). Then \( \sigma_n \sim \sigma'_n \) if and only if the following conditions hold:

(i) \( \sigma_{n-1} \sim \sigma'_{n-1} \),

(ii) There exists \( \beta : H(n) \rightarrow \mathbb{T} \) such that

\[
\begin{align*}
\sigma'_{H(n)}(a, a') & = \beta(a)\beta(a')\beta(a + a')\sigma_{H(n)}(a, a'), \\
g'_n(a, b) & = \beta(\alpha_b(a)); \beta(a)g_n(a, b).
\end{align*}
\]

**Remark 2.4.** If (ii) holds, then \( \sigma_{H(n)} \sim \sigma'_{H(n)} \). If \( \sigma_{H(n)} \sim \sigma'_{H(n)} \) and \( \beta \) and \( \beta' \) are two functions implementing the similarity, then \( \beta' = f \cdot \beta \) for some homomorphism \( f : H(n) \rightarrow \mathbb{T} \).

**Proof.** Suppose \( \sigma_n \sim \sigma'_n \), then there exists some \( \gamma : G(n) \rightarrow \mathbb{T} \) such that

\[
\sigma_n(a'b, ab') = \gamma(a'b)\gamma(ab')\gamma(a' bab')\sigma'_n(a'b, ab')
\]

for all \( a, a' \in H(n) \) and \( b, b' \in G(n - 1) \). In particular, if \( a = a' = 0 \), then

\[
\sigma_{n-1}(b, b') = \gamma(b)\gamma(b')\gamma(bb')\sigma'_{n-1}(b, b')
\]

for all \( b, b' \in G(n - 1) \), so \( \sigma_{n-1} \sim \sigma'_{n-1} \). Moreover, the formula (5) from Theorem 2.2 with \( a = 0 \) and \( b = e \) gives that

\[
\sigma_n(a', b') = 1 = \sigma'_n(a', b')
\]
for all \(a' \in H(n)\) and \(b' \in G(n-1)\). Applying this fact to (7) shows that 
\[ \gamma(a'b') = \gamma(a')\gamma(b') \text{ for all } a' \in H(n) \text{ and } b' \in G(n-1) \]
Define \(\beta\) on \(H(n)\) by \(\beta(a) = \gamma(a)\). Then, by letting \(b = b' = e\) in (5) and (7), we get 
\[ \sigma'_{H(n)}(a', a) = \beta(a')\beta(a)\beta(a') + a\sigma_{H(n)}(a', a) \]
for all \(a', a \in H(n)\). Furthermore, by letting \(a' = 0\) and \(b' = e\) in (5) and (7), we get that 
\[ g_n(a, b) = \gamma(b)\gamma(a)\gamma(b)g'_n(a, b) \]
\[ = \gamma(b)\gamma(a)(\alpha_b(a)b)\gamma(b)g'_n(a, b) \]
\[ = \gamma(a)(\alpha_b(a))\gamma(b)g'_n(a, b) \]
\[ = \gamma(a)b\alpha_b(a)g'_n(a, b) \]
for all \(a \in H(n)\) and \(b \in G(n-1)\).

Assume next that \(\beta\) is such that (ii) holds, and that (i) holds through \(\delta\), that is, 
\[ \sigma_{n-1}(a, b') = \delta(a)\delta(b)\delta(b')\sigma_{n-1}(a, b') \]
Define \(\gamma\) on \(G(n)\) by \(\gamma(ab) = \beta(a)\delta(b)\). Then 
\[ \sigma_n(a'b', ab') = \sigma_{H(n)}(a', \alpha_b(a))g_n(a, b)\sigma_{n-1}(a, b') \]
\[ = \beta(a')\beta(\alpha_b(a))\beta(a' + \alpha_b(a))\sigma'_{H(n)}(a', \alpha_b(a)) \]
\[ \cdot \delta(a)\delta(b)\delta(b')\delta(b')\sigma_{n-1}(a, b') \]
\[ = \beta(a')\delta(a)\delta(b')\cdot \beta(a' + \alpha_b(a))\delta(b')\sigma_{n-1}(a'b', ab') \]
\[ = \gamma(a'b')\gamma(ab')\gamma(a'bab')\sigma_{n-1}(a'b', ab') \]

\[ \square \]

**Remark 2.5.** Clearly, a similar result may be shown to hold for any semidirect product.

The result can be deduced from [15, Appendix 2], but in any case it may be useful to give a proof by a direct computation.

Let \(\tau_n\) be a multiplier of \(G(n)\) coming from a pair \((\sigma_{H(n)}, g_n)\), that is, 
\[ \tau_n(a'b', ab') = \sigma_{H(n)}(a', \alpha_b(a))g_n(a, b), \] (8)
where \((\sigma_{H(n)}, g_n)\) satisfies (4). By Theorem 2.2 and Proposition 2.3, every multiplier of \(G(n)\) that is trivial on \(G(n-1)\) is similar to one of this form. Denote the abelian group of similarity classes of multipliers of this type by \(\check{H}^2(G(n), \mathbb{T})\).

**Corollary 2.6.** For all \(n \geq 2\), the second cohomology group of \(G(n)\) may be decomposed as 
\[ H^2(G(n), \mathbb{T}) = \check{H}^2(G(n), \mathbb{T}) \oplus H^2(G(n-1), \mathbb{T}) = \bigoplus_{k=2}^n \check{H}^2(G(k), \mathbb{T}). \]
Proof. It follows from Theorem 2.2 and Proposition 2.3 (see our comment above) that
\[ H^2(G(n), \mathbb{T}) = \tilde{H}^2(G(n), \mathbb{T}) \oplus H^2(G(n-1), \mathbb{T}). \]
Thus, the second inequality is proven by induction after noticing that
\[ \{1\} = H^2(\mathbb{Z}, \mathbb{T}) = H^2(G(1), \mathbb{T}) = \tilde{H}^2(G(1), \mathbb{T}). \]

\[ \square \]

**Theorem 2.7.** We have that
\[ H^2(G(n), \mathbb{T}) \cong \mathbb{T}^+(n+1)n(n-1), \]
and for each set of \( \frac{1}{3}(n+1)n(n-1) \) parameters
\[ \{\lambda_{i,j,k} : 1 \leq i \leq k, 1 \leq j < k \leq n\} \subset \mathbb{T}, \]
the associated \([\sigma] \in H^2(G(n), \mathbb{T})\) may be represented by
\[
\sigma(r, s) = \prod_{i<j<k} \lambda_{i,j,k}^{s_i r_i + s_j r_j + s_k (r_i - r_j)} \cdot \prod_{j<k} \lambda_{j,k}^{s_j r_j + \frac{1}{2} s_k (r_j - 1)} \cdot \lambda_{k,j}^{r_k + r_j + s_k (s_k - 1)}.
\]

The proof of this theorem will be given in Section 2.4.

**Remark 2.8.** Note that \( \lambda_{i,j,k} \) for \( i > k \) is not involved in the expression above. See Remark 5.2 and [7] for comments regarding this fact. This is also a consequence of \([18]\) in the proof below.

**Example 2.9.** For \( G(1) = \mathbb{Z} \) there are no nontrivial multipliers. The multipliers of the usual Heisenberg group \( G(2) \) are, up to similarity, given by two parameters (as computed in \([13]\) Proposition 1.1]):
\[
\sigma(r, s) = \lambda_{1,12}^{s_1 r_1} \lambda_{2,12}^{s_2 r_1 (r_1 - 1)} \lambda_{2,13}^{s_2 r_2} \lambda_{3,12}^{s_3 r_2} \lambda_{3,13}^{s_3 r_3}
\]

The multipliers of \( G(3) \) are, up to similarity, given by eight parameters:
\[
\sigma(r, s) = \lambda_{1,23}^{s_1 r_1} \lambda_{2,12}^{s_2 r_1 (r_1 - 1)} \lambda_{2,13}^{s_2 r_2} \lambda_{3,12}^{s_3 r_2} \lambda_{3,13}^{s_3 r_3} \lambda_{1,23}^{s_1 r_2} \lambda_{2,13}^{s_2 r_2 (r_2 - 1)} \lambda_{3,13}^{s_3 r_3 (r_2) (r_3 - 1)} \lambda_{1,23}^{s_1 r_3} \lambda_{2,13}^{s_2 r_3} \lambda_{3,13}^{s_3 r_3 (r_3 - 1)} \lambda_{1,23}^{s_1 r_3} \lambda_{2,13}^{s_2 r_3 (r_3 - 1)} \lambda_{3,13}^{s_3 r_3 (r_3 - 1)} \lambda_{1,23}^{s_1 r_3} \lambda_{2,13}^{s_2 r_3 (r_3 - 1)} \lambda_{3,13}^{s_3 r_3 (r_3 - 1)}
\]

**Remark 2.10.** One may associate a Lyndon-Hochschild-Serre spectral sequence with the extension (see exs. \([18] 6.8.2\]):
\[
1 \rightarrow V(n) \rightarrow G(n) \rightarrow \mathbb{Z}^n \rightarrow 1
\]

By applying [6] Theorem 4 to this sequence, one can then compute the second homology group of \( G(n) \) and deduce that
\[ H_2(G(n), \mathbb{Z}) \cong \mathbb{Z}^+(n+1)n(n-1), \]
which gives that \( H^2(G(n), \mathbb{T}) \cong \mathbb{T}^+(n+1)n(n-1) \) after dualizing. However, this does not give an explicit description of \( H^2(G(n), \mathbb{T}) \).
2.1 Proof of Theorem 2.7

Fix \( n \geq 2 \). We will in the proof compute \( H^2(G(n), \mathbb{T}) \) through several lemmas.

**Lemma 2.1.1.** Every element of \( H^2(G(n), \mathbb{T}) \) may be represented by a pair \((\sigma_{H(n)}, g_n)\), where \( \sigma_{H(n)} \) is a multiplier of \( H(n) \) given by

\[
\sigma_{H(n)}(a', a) = \prod_{i=1}^{n-1} \lambda_i \alpha_i^{a_n a_i} \tag{11}
\]

for some \( \lambda_1, \ldots, \lambda_{n-1} \in \mathbb{T} \), and \( g_n \) satisfies

\[
g_n(a + a', b) = \left( \prod_{i=1}^{n-1} \lambda_i \alpha_i^{a_n a_i} \right) g_n(a, b) g_n(a', b) \tag{12}
\]

for all \( a, a' \in H(n) \) and \( b \in G(n - 1) \).

**Proof.** Every element of \( H^2(G(n), \mathbb{T}) \) may be represented by a multiplier of the form \( \sigma \), that is, by a pair \( (\sigma_{H(n)}, g_n) \) satisfying \( \sigma \).

Moreover, it is well-known (see e.g. [2]) that every multiplier of \( H(n) \cong \mathbb{Z}^n \) is similar to one of the form

\[
\sigma_{H(n)}(a', a) = \prod_{1 \leq i \leq n-1} \lambda_i^{a_n a_i} \prod_{1 \leq j < k \leq n-1} \mu_{jk}^{a_j a_k}
\]

for some sets of scalars \( \{\lambda_i\}_{1 \leq i \leq n-1}, \{\mu_{jk}\}_{1 \leq j < k \leq n-1} \subset \mathbb{T} \). Since \( H(n) \) is abelian, \( \sigma \) gives that

\[
\sigma_{H(n)}(\alpha_b(a), \alpha_b(a')) \sigma_{H(n)}(a, a') = g_n(a + a', b) g_n(a, b) g_n(a', b) = g_n(a' + a, b) g_n(a', b) g_n(a, b) = \sigma_{H(n)}(\alpha_b(a'), \alpha_b(a)) \sigma_{H(n)}(a', a)
\]

for all \( a, a' \in H(n) \) and \( b \in G(n - 1) \). Furthermore, we have

\[
\sigma_{H}((\alpha_b(a), \alpha_b(a')) \sigma_{H}(a, a') = \prod_{1 \leq i \leq n-1} \lambda_i^{a_n(a'_n + b_n a'_n) - a_n a'_n} \prod_{1 \leq j < k \leq n-1} \mu_{jk}^{(a_j a_n + b_j a_n)(a'_n a'_k - a_n a'_k)}
\]

This is equal to \( \sigma_{H}(\alpha_b(a'), \alpha_b(a)) \sigma_{H}(a', a) \) for all \( a, a' \in H(n) \) and \( b \in G(n - 1) \) if and only if the expression remains unchanged under the substitution \( a \leftrightarrow a' \), that is, if and only if all the \( \mu_{jk} \)'s are 1.

**Lemma 2.1.2.** For every element of \( H^2(G(n), \mathbb{T}) \) there is a unique associated pair \( (\sigma_{H(n)}, g_n) \) satisfying the conditions of Lemma 2.1.1 such that

\[
g_n(u_n, u_i) = 1 \text{ for all } 1 \leq i \leq n - 1. \tag{13}
\]
Proof. Suppose that \((\sigma_{H(n)}, g_n)\) satisfies \((\ref{11})\) and \((\ref{12})\). Let \(f : H(n) \to \mathbb{T}\) be the homomorphism determined by \(f(u_n) = 1\) and \(f(v_i) = g_n(u_n, u_i)\) for all \(1 \leq i \leq n-1\), and define \(g'_n\) by \(g'_n(a, b) = f(\alpha b(a))f(a)g_n(a, b)\). Then, \(g'_n(u_n, u_i) = 1\) for all \(1 \leq i \leq n-1\) and by Proposition \((\ref{23})\) \((\sigma_{H(n)}, g'_n)\) determines a multiplier on \(H(n)\) in the same similarity class as the one coming from \((\sigma_{H(n)}, g_n)\).

Suppose now that there are two pairs \((\sigma'_{H(n)}, g_n')\) and \((\sigma''_{H(n)}, g_n'')\) both satisfying the conditions of Lemma \((\ref{11})\). Then \(\sigma'_{H(n)} = \sigma''_{H(n)}\), so by Proposition \((\ref{23})\) and the succeeding remark, there is a homomorphism \(f : H(n) \to \mathbb{T}\) such that

\[
g'_n(a, b) = f(\alpha b(a))f(a)g_n(a, b) = \left(\prod_{i=1}^{n-1} f(v_i)^{a_{\alpha b(i)}}\right)g_n(a, b)
\]

for all \(a \in H(n), b \in G(n-1)\). In particular,

\[
g'_n(u_n, u_i) = f(v_i)g_n(u_n, u_i) \quad \text{for all} \quad 1 \leq i \leq n-1,
\]

so that \(g'_n = g_n\) if \(g'_n(u_n, u_i) = g_n(u_n, u_i)\) for all \(1 \leq i \leq n-1\).

In the forthcoming lemmas we fix an element of \(\tilde{H}^2(G(n), \mathbb{T})\), and let \((\sigma_{H(n)}, g)\) be the unique associated pair satisfying \((\ref{11}), (\ref{12})\) and \((\ref{13})\) for some set of scalars \((\lambda_i)_{i=1}^{n-1} \subset \mathbb{T}\).

For computational reasons, we now introduce the following notation. For \(a = (a_n, a_{1n}, \ldots, a_{n-1,n}) \in H(n)\), we write \(a = w(a) + z(a)\), where \(w(a) = (a_n, 0, \ldots, 0)\), and \(z(a)\) is the “central part”, i.e. \(z(a) = (0, a_{1n}, \ldots, a_{n-1,n})\). Similarly, for \(b = (b_1, \ldots, b_{n-1}, b_{12}, \ldots, b_{n-2,n-1}) \in G(n-1)\), we write \(b = w(b)z(b)\), where \(w(b) = (b_1, \ldots, b_{n-1}, 0, \ldots, 0)\) and \(z(b) = (0, \ldots, 0, b_{12}, \ldots, b_{n-2,n-1})\). Remark that \(\alpha_{\lambda b}(a) = a\) if either \(w(a)\) or \(w(b)\) is trivial, i.e. if either \(a\) or \(b\) is central.

Lemma \((\ref{11})\). For all \(a \in H(n)\) and \(b \in G(n-1)\) we have

\[
g(a, b) = g(w(a), w(b))g(w(a), z(b))g(z(a), w(b)).
\]

Proof. It follows immediately from Lemma \((\ref{11})\) that if \(a, a' \in H(n)\) and \(w(a)\) or \(w(a')\) is 0, then

\[
g(a + a', b) = g(a, b)g(a', b),
\]

hence,

\[
g(a, b) = g(w(a) + z(a), b) = g(w(a), b)g(z(a), b)
\]

for all \(a \in H(n)\) and \(b \in G(n-1)\). If \(b' \in G(n-1)\) and \(w(b') = e\), then \(b'\) is central and \(\alpha_{\lambda b'}(a) = a\) for all \(a \in H(n)\). Therefore,

\[
g(a, b)g(a, b') = g(a, bb') = g(a, b') = g(\alpha_{\lambda b}(a), b')g(a, b)
\]

for all \(a \in H(n), b \in G(n-1)\). By \((\ref{13})\), we then get

\[
1 = g(\alpha_{\lambda b}(a), b')g(a, b') = g(a + (0, b_1a_n, \ldots, b_{n-1}a_n), b')g(a, b') = g(a, b')g((0, b_1a_n, \ldots, b_{n-1}a_n), b')g(a, b') = g((0, b_1a_n, \ldots, b_{n-1}a_n), b')
\]

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for all $a \in H(n)$ and $b \in G(n-1)$. Consequently, since this holds for all $a \in H(n)$ and $b \in G(n-1)$, and central $b' \in G(n-1)$, we get that if $a$ and $b$ are any elements in $H(n)$ and $G(n-1)$, respectively, then $g(z(a), z(b)) = 1$. Moreover, (13) also imply that if $b, b' \in G(n-1)$ and either $w(b)$ or $w(b')$ is equal to $e$, that is, either $b$ or $b'$ is central, then

$$g(a, bb') = g(a, b)g(a, b').$$

Hence, by (16) and (14),

$$g(a, b) = g(a, w(b)z(b)) = g(a, w(b))g(a, z(b)) = g(w(a), w(b))g(z(a), w(b))g(w(a), z(b)) \cdot 1$$

for all $a \in H(n)$ and $b \in G(n-1)$. □

**Lemma 2.1.4.** For all $a \in H(n)$ and $b, b' \in G(n-1)$ we have

$$g(z(a), w(b)) = \prod_{i,j=1}^{n-1} g(v_{i,n}, u_j)^{a_{n,b_j}},$$

$$g(w(a), z(b)) = \prod_{1 \leq i < j \leq n} g(u_n, v_{ij})^{a_{n,b_{ij}}} = \prod_{1 \leq i < j \leq n} \left( g(v_{ij}, u_j)g(v_{ij}, u_i) \right)^{a_{n,b_{ij}}},$$

and

$$g(a, bb') = \left( \prod_{i,j=1}^{n-1} g(v_{i,n}, u_j)^{b'_{n,a_{n}}} \right) g(a, b)g(a, b').$$

Proof. Let $z(H(n)) = \{ z(a) \mid a \in H(n) \}$ and $z(G(n-1)) = \{ z(b) \mid b \in G(n-1) \}$. Then note that $g$ is a bihomomorphism when restricted to $z(H(n)) \times G(n-1)$ or $H(n) \times z(G(n-1))$. Therefore, the first two identities hold. Indeed, this follows directly from (9) after noticing that since $z(a)$ and $z(b)$ are central,

$$\alpha_{w(b)}(z(a)) = z(a) \text{ and } \alpha_{z(b)}(w(a)) = w(a).$$

Moreover, for $i < j$ we have that $u_iu_j = v_{ij}u_iu_j$. By (9) and the previous lemma,

$$g(u_n, u_iu_j) = g(\alpha_{u_i}(u_n), u_i)g(u_n, u_j) = g(u_nv_{jn}, u_i)g(u_n, u_j) = g(u_n, u_i)g(v_{jn}, u_i)g(u_n, u_j)$$

and

$$g(u_n, v_{ij}u_ju_i) = g(u_n, v_{ij})g(u_n, u_ju_i) = g(u_n, v_{ij})g(\alpha_{u_i}(u_n), v_{ij})g(u_n, u_i) = g(u_n, v_{ij})g(u_nv_{jn}, u_i)g(u_n, u_i) = g(u_n, v_{ij})g(u_n, v_{ijn})g(u_n, u_i)g(u_n, u_i),$$

so that

$$g(v_{jn}, u_i) = g(u_n, v_{ij})g(v_{jn}, u_j).$$

(18)
which gives the last identity in the second line of the statement. Finally,
\[ g(a, bb') = g(\alpha \psi(a), b)g(a, b') = g(a + (0, b'_1a_n, \ldots, b'_{n-1}a_n), b)g(a, b') \]
\[ = (\prod_{i=1}^{n-1} g(v_{in}, w(b))b'_{ia_n})g(a, b)g(a, b') \]
\[ = (\prod_{i=1}^{n-1} \prod_{j=1}^{n-1} g(v_{in}, u_{ij})b'_{ia_n})g(a, b)g(a, b'). \]

\[ \square \]

Lemma 2.1.5. For all \( a \in H(n) \) and \( b \in G(n-1) \) we have
\[ g(w(a), w(b)) = \left( \prod_{i=1}^{n-1} \lambda_i^{b_{ia_n}(a_{n-1})} g(v_{in}, u_i)^{b_{ia_n}(b_{i-1})} \right) \cdot \prod_{1 \leq i < j \leq n-1} g(v_{in}, u_j)^{b_{ia_n}b_{ja_n}}. \]

Proof. First we see from (17) that if \( b_j \geq 1 \), then
\[ g(u_n, u_j^{b_j}) = g(u_n, u_j^{b_j-1}u_j) = g(v_{jn}, u_j)^{b_j-1}g(u_n, u_j^{b_j-1})g(u_n, u_j) = \cdots = g(v_{jn}, u_j)^{b_j(b_j-1)}g(u_n, u_j^{b_j}) \]
and it is not hard to see that
\[ g(u_n, u_j^{b_j}) = g(v_{jn}, u_j)^{b_j(b_j-1)}g(u_n, u_j^{b_j}) \]
for negative \( b_j \) as well, for example by applying (17) again.
Moreover, note that \( w(b) = u_n^{b_{n-1}} \cdots u_1^{b_1} \), so that by (17),
\[ g(u_n, w(b)) = g(u_n, u_n^{b_{n-1}} \cdots u_1^{b_1}) \]
\[ = (\prod_{j=2}^{n-1} g(v_{jn}, u_j)^{b_j^2}g(u_n, u_n^{b_{n-1}} \cdots u_2^{b_2})g(u_n, u_1^{b_1})) \]
\[ = \cdots = (\prod_{1 \leq i < j \leq n-1} g(v_{in}, u_j)^{b_{ia_n}b_{ja_n}})(\prod_{j=1}^{n-1} g(u_n, u_j^{b_j})). \]
Again, let $G$ be any discrete group, $\sigma$ a multiplier of $G$ and $H$ a nontrivial Hilbert space. A map $U$ from $G$ into the unitary group of $H$ satisfying

\[ U(r)U(s) = \sigma(r, s)U(rs) \]

for all $r, s \in G$ is called a $\sigma$-projective unitary representation of $G$ on $H$.

3 The twisted group $C^*$-algebras of $G(n)$

Again, let $G$ be any discrete group, $\sigma$ a multiplier of $G$ and $H$ a nontrivial Hilbert space. A map $U$ from $G$ into the unitary group of $H$ satisfying

\[ U(r)U(s) = \sigma(r, s)U(rs) \]

for all $r, s \in G$ is called a $\sigma$-projective unitary representation of $G$ on $H$. 
We recall the following facts about twisted group $C^*$-algebras and refer to Zeller-Meier [19] for further details of the construction.

To each pair $(G, \sigma)$, we may associate the full twisted group $C^*$-algebra $C^*(G, \sigma)$. Denote the canonical injection of $G$ into $C^*(G, \sigma)$ by $i_{\tau}$. Then $C^*(G, \sigma)$ satisfies the following universal property. Every $\sigma$-projective unitary representation of $G$ on some Hilbert space $\mathcal{H}$ (or in some $C^*$-algebra $A$) factors uniquely through $i_{\tau}$.

The reduced twisted group $C^*$-algebra $C^*_r(G, \sigma)$ is generated by the left regular $\sigma$-projective unitary representation $\lambda_{\sigma}$ of $G$ on $B(F(G))$. Consequently, $\lambda_{\sigma}$ extends to a $^*$-homomorphism of $C^*(G, \sigma)$ onto $C^*_r(G, \sigma)$. If $G$ is amenable, then $\lambda_{\sigma}$ is faithful. Note especially that every nilpotent group is amenable, so that $C^*(G(n), \sigma) \cong C^*_r(G(n), \sigma)$ through $\lambda_{\sigma}$ for every $n \geq 1$ and all multipliers $\sigma$ of $G(n)$.

Finally, remark that if $\tau \sim \sigma$ through some $\beta : G \to T$, then the assignment $i_{\tau}(r) \mapsto \beta(r)i_{\sigma}(r)$ induces an isomorphism $C^*(G, \tau) \to C^*(G, \sigma)$.

**Theorem 3.1.** Fix $n \geq 2$ and let $\sigma$ be a multiplier of $G(n)$ of the form (I), that is, determined by the $\frac{1}{2}(n+1)n(n-1)$ parameters

$$\{\lambda_{i,j,k} : 1 \leq i \leq k, 1 \leq j < k \leq n\} \subset \mathbb{T}.$$  

Moreover, set

$$\lambda_{k,ij} = \overline{\lambda_{i,j,k}}$$  \hspace{1cm} (19)

when $1 \leq i < j < k \leq n$.

Then the twisted group $C^*$-algebra $C^*(G(n), \sigma)$ is the universal $C^*$-algebra generated by unitaries $\{U_i\}_{1 \leq i \leq n}$ and $\{V_{jk}\}_{1 \leq j < k \leq n}$ satisfying the relations

$$[V_{jk}, V_{lm}] = I, \quad [U_i, V_{jk}] = \lambda_{i,j,k}I, \quad [U_j, U_k] = V_{jk}$$  \hspace{1cm} (20)

for $1 \leq i \leq n$, $1 \leq j < k \leq n$, and $1 \leq l < m \leq n$.

**Proof.** Set $U_i = i_{\sigma}(u_i)$ and $V_{jk} = i_{\sigma}(v_{jk})$ and note that (19) gives that $\sigma(u_i, v_{jk}) = \lambda_{i,j,k}$ and $\sigma(v_{jk}, u_i) = 1$ for all $1 \leq i \leq n$ and $1 \leq j < k \leq n$. Thus,

$$[U_i, V_{jk}] = \sigma(u_i, v_{jk})\sigma(v_{jk}, u_i)I = \lambda_{i,j,k}I$$

for all $1 \leq i \leq n$, $1 \leq j < k \leq n$.

Moreover, note that $\sigma(u_i, u_j) = 1$ for all $1 \leq i, j \leq n$ and $\sigma(v_{jk}, v_{lm}) = 1$ for all $1 \leq j < k \leq n$ and $1 \leq l < m \leq n$. Hence, it is clear that $C^*(G(n), \sigma)$ is generated as a $C^*$-algebra by unitaries satisfying (20).

Next, suppose that $A$ is any $C^*$-algebra generated by a set of unitaries satisfying the relations (20). For each $r$ in $G(n)$ we define the unitary $W_r$ in $A$

by

$$W_r = V_{12}^{r_{12}} \cdots V_{n-1,n}^{r_{n-1,n}} \cdot U_{n}^{r_{n}} \cdots U_{1}^{r_{1}}.$$  

Then a computation using (20) repeatedly gives that

$$W_r W_s = \tau(r, s) W_{rs},$$

where $\tau(r, s)$ is a scalar in $\mathbb{T}$ for all $r, s \in G(n)$. Now, the associativity of $A$ immediately implies that $\tau$ is a multiplier of $G(n)$, so that $W$ is a $\tau$-projective unitary representation of $G(n)$ in $A$. Furthermore, note that $\tau$ satisfies

$$\tau(u_i, v_{jk})\tau(v_{jk}, u_i) = \lambda_{i,j,k}$$

for $1 \leq i \leq n$, $1 \leq j < k \leq n$.

---

2In general, it will require much work to compute the formula for $\tau$ and it is not needed for this argument. However, for $n = 2$, the expression for $\tau$ is precisely of the form (16).
By the universal property of the full twisted group $C^*$-algebra, there exists a unique $^*$-homomorphism $\varphi$ of $C^*(G(n), \tau)$ onto $A$ such that $\varphi(i_r(r)) = W(r)$ for all $r \in G(n)$.

Therefore, it is sufficient to show that $\tau \sim \sigma$, because then $C^*(G(n), \tau)$ is canonically isomorphic with $C^*(G(n), \sigma)$. By Theorem 2.7 there is some $\beta : G(n) \to \mathbb{T}$ such that $\sigma'$, given by

$$\sigma'(r, s) = \beta(r)\beta(s)\beta(rs)\tau(r, s),$$

is of the form (9). We calculate that

$$\sigma'(u_i, v_{jk})\sigma'(v_{jk}, u_i) = \beta(u_i)\beta(v_{jk})\beta(u_i, v_{jk})\beta(v_{jk})\beta(u_i, v_{jk})\beta(v_{jk}, u_i)\tau(v_{jk}, u_i)$$

$$= \tau(u_i, v_{jk})\tau(v_{jk}, u_i) = \lambda_{i, jk}$$

for all $1 \leq i \leq n$ and $1 \leq j < k \leq n$. Hence, $\sigma' = \sigma$, so $\tau \sim \sigma$. \hfill $\square$

**Remark 3.2.** To explain the relation (19), consider the three-dimensional case. Let $U_1, U_2, U_3$ and $V_{12}, V_{13}, V_{23}$ be unitaries in a $C^*$-algebra $B$ satisfying

$$[V_{jk}, V_{lm}] = I, \quad [U_i, V_{jk}] = \mu_{i, jk}I, \quad [U_j, U_k] = V_{jk}$$

for $1 \leq i \leq 3, 1 \leq j < k \leq 3$, and $1 \leq l < m \leq 3$ where $\{\mu_{i, jk}\}$ is any set of nine scalars in $T$. Then we can compute that

$$U_1U_2U_3 = V_{12}U_2U_1U_3 = \cdots = \mu_{2,13}V_{12}V_{13}V_{23}U_3U_2U_1,$$

$$U_1U_2U_3 = U_1V_{23}U_2U_3 = \cdots = \mu_{1,23}\mu_{1,23}V_{13}V_{23}U_3U_2U_1,$$

that is, we must have $\mu_{2,13} = \mu_{1,23}\mu_{1,23}$.

For dimensions $n > 3$, any choice of a triple of unitaries from the family $\{U\}_{i=1}^n$ gives a similar dependence. In the $n \cdot \frac{1}{3} n(n-1)$ commutation relations, these $\binom{n}{3}$ dependencies are the only possible ones since

$$n \cdot \frac{1}{3} n(n-1) - \binom{n}{3} = \frac{1}{3} n(n-1)(n-1) - \frac{1}{3} (n-1) = \frac{1}{3} (n+1)n(n-1).$$

**Remark 3.3.** Let $\omega$ be the dual 2-cocycle of $G(n)$, that is,

$$\omega : G(n) \times G(n) \to H^2(G(n), \mathbb{T}) \cong \mathbb{Z}_{\frac{1}{3}}^{(n+1)n(n-1)}$$

is determined by $\omega(r, s)(\sigma) = \sigma(r, s)$ for a multiplier $\sigma$ of $G(n)$. Let the group $K(n)$ be defined as the set $\mathbb{Z}_{\frac{1}{3}}^{(n+1)n(n-1)} \times G(n)$ with product

$$(j, r)(k, s) = (j + k + \omega(r, s), rs).$$

It is not entirely obvious that $\omega$ and $K(n)$ are well-defined and we refer to [15] Corollary 1.3 for details. Moreover, according to [15] Corollary 1.3, we may construct a continuous field $A$ over $H^2(G(n), \mathbb{T})$ with fibers $A_\lambda \cong C^*(G(n), \sigma_\lambda)$ for each $\lambda \in H^2(G(n), \mathbb{T})$. Then the $C^*$-algebra associated with this continuous field will be natural isomorphic to the group $C^*$-algebra of the group $K(n)$.

Next, we briefly consider the group $G(3, 2)$ generated by $u_1, u_2, v_{12}, w_1, w_2$ satisfying

$$[u_1, u_2] = v_{12}, \quad [u_1, v_{12}] = w_1, \quad [u_2, v_{12}] = w_2, \quad w_1, w_2 \text{ central}.$$
Then we have that $Z(G(3, 2)) \cong \mathbb{Z}^2$ and $Z(C^*(G(3, 2))) \cong C(T^2)$.

Let $i$ denote the canonical injection of $G(3, 2)$ into $C^*(G(3, 2))$. For each $\lambda = (\lambda_1, \lambda_2) \in \mathbb{T}^2$, let $C^*(G(2), \sigma_\lambda)$ be generated by unitaries satisfying (20). By a similar argument as in Theorem 1.1, there is a surjective $\ast$-homomorphism

$$\pi_\lambda : C^*(G(3, 2)) \to C^*(G(2), \sigma_\lambda)$$

such that $i(u_i) = U_i$, $i(v_{12}) = V_{12}$, and $i(w_i) = \lambda_i I$ for $i = 1, 2$. Moreover, the kernel of $\pi_\lambda$ coincides with the ideal of $C^*(G(3, 2))$ generated by

$$\lambda \in \text{Prim} \, Z(C^*(G(3, 2))) \cong Z(C^*(\widehat{G(3, 2)})) = \mathbb{T}^2 \cong H^2(G(2), \mathbb{T}).$$

Again, similarly as in Theorem 4.1, we define a set of sections and apply the Dauns-Hofmann Theorem. In this way, the triple

$$\left( H^2(G(2), \mathbb{T}), \{C^*(G(2), \sigma_\lambda)\}_\lambda, C^*(\widehat{G(3, 2)}) \right)$$

is a full continuous field of $C^*$-algebras, and the $C^*$-algebra associated with this continuous field is naturally isomorphic to $C^*(G(3, 2))$.

It is not difficult to see that $K(2)$ is isomorphic with $G(3, 2)$ (and with $H_{n, 4}$). We conjecture that $K(n) \cong G(3, n)$ also for $n \geq 3$, where $G(3, n)$ is the free nilpotent group of class 3 and rank $n$ as described in [3], so that $A$ is isomorphic with $C^*(G(3, n))$. For $n \geq 3$, the complicated part is to construct an isomorphism $Z_{\lambda}^{G(3,n)} \cong Z(G(3, n))$ and produce a commuting diagram (recall (4)):

\[
\begin{array}{cccccc}
1 & \overset{i}{\longrightarrow} & Z_{\lambda}^{G(3,n)} & \cong & K(n) & \longrightarrow & G(n) & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & Z(G(3, n)) & \overset{i}{\longrightarrow} & G(3, n) & \longrightarrow & G(n) & \longrightarrow & 1 \\
\end{array}
\]

4 Simplicity of $C^*(G(n), \sigma)$

Let $\sigma$ be a multiplier of any group $G$. An element $r$ of $G$ is called $\sigma$-regular if $\sigma(r, s) = \sigma(s, r)$ whenever $s$ in $G$ commutes with $r$. If $r$ is $\sigma$-regular, then every conjugate of $r$ is also $\sigma$-regular. Therefore, we say that a conjugacy class of $G$ is $\sigma$-regular if it contains a $\sigma$-regular element.

The conjugacy class $C_r$ of $r \in G(n)$ is infinite if $r \notin Z(G(n))$. Indeed, for any $s \in G(n)$ we have that

$$\left( srs^{-1} \right)_i = r_i \quad \text{and} \quad \left( srs^{-1} \right)_{jk} = r_{jk} + s_j r_k - r_j s_k. \quad (21)$$

Hence, $|C_r| = \infty$ if $r_i \neq 0$ for some $i$. Of course, $C_r = \{ r \}$ if $r \in Z(G(n))$.

Now, we fix a multiplier $\sigma$ of $G(n)$ of the form (4).

Lemma 4.1. Let $S(G(n))$ be the set of $\sigma$-regular central elements of $G(n)$, that is,

$$S(G(n)) = \{ r \in Z(G(n)) \mid \sigma(r, s) = \sigma(s, r) \text{ for all } s \in G(n) \}.$$ 

Then $S(G(n))$ is a subgroup of $G(n)$ and $Z(C^*_r(G(n)), \sigma)) \cong C(S(\widehat{G(n)})).$
This means that altogether we get Theorem 4.4. The following are equivalent:

(i) $\lambda_\sigma(s) = 0$ for some $\sigma$. By [14, Theorem 1.7], if $\sigma$ is a nontrivial $C^*$-subalgebra of $C^*_r(G(n), \sigma)$ generated by $\{\lambda_\sigma(s) \mid s \in S(G(n))\}$. Since $G(n)$ is amenable, it follows from [13] last paragraph of 4.26 that $T$ belongs to $C^*_r(S(G(n), \sigma)$.

This means that $Z(C^*_r(G(n), \sigma) \cong C^*_r(S(G(n), \sigma))$. As the reverse inclusion obviously holds, we have $Z(C^*_r(G(n), \sigma)) = C^*_r(S(G(n), \sigma))$.

Now, it is not difficult to see that $C^*_r(S(G(n))) \cong C^*_r(S(G(n)))$. Indeed, as $s \mapsto \lambda_\sigma(s)$ is a unitary representation of $S(G(n))$ into $C^*_r(S(G(n), \sigma)$ and the canonical tracial state $\tau$ on $C^*_r(S(G(n), \sigma)$ is faithful and satisfies $\tau(\lambda_\sigma(s)) = 0$ for each nonzero $s \in S(G(n))$, this is just a consequence of [19] Théorème 4.22. Altogether, we get

$$Z(C^*_r(G(n), \sigma)) = C^*_r(S(G(n), \sigma) \cong C^*_r(S(G(n))) \cong C^*_r(S(G(n))).$$

\[ \square \]

Remark 4.2. If $S(G(n))$ is nontrivial, we can describe $C^*(G(n), \sigma)$ as a continuous field of $C^*$-algebras over the base space $S(G(n))$. The fibers will be isomorphic to $C^*(G(n)/S(G(n)), \omega)$ for some multiplier $\omega$ of $G(n)/S(G(n))$ (see [15] Theorem 1.2 for further details).

Example 4.3 ([17] Lemma 3.8 and Theorem 3.9]). Fix a multiplier $\sigma$ of $G(2)$ of the form [10] such that both $\lambda_{1, 12}$ and $\lambda_{2, 12}$ are torsion elements. Let $p$ and $q$ be the smallest natural numbers such that $\lambda_{1, 12}^p = \lambda_{2, 12}^q = 1$ and set $k = \text{lcm}(p, q)$. Clearly, $Z(G(2)) = \mathbb{Z}$ and $S(G(2)) = k\mathbb{Z}$. Moreover, $G(2)/S(G(2))$ can be identified with the group with product

$$(r_1, r_2, r_{12})(s_1, s_2, s_{12}) = (r_1 + s_1, r_2 + s_2, r_{12} + s_{12} + r_1 s_2 \text{ mod } k\mathbb{Z})$$

for $r_1, r_2, s_1, s_2 \in \mathbb{Z}$ and $r_{12}, s_{12} \in \{0, 1, \ldots, k - 1\}$. Then $C^*(G(2), \sigma)$ is a continuous field of $C^*$-algebras over the base space $\text{S}(G(2)) \cong \mathbb{T}$. The fibers will be isomorphic to $C^*(G(n)/S(G(n), \omega_\lambda)$, where $\lambda \in \mathbb{T}$ and

$$\omega_\lambda(r, s) = \sigma(r, s)\mu^{-s_2}$$

for some $\mu \in \mathbb{T}$ with $\mu^k = \lambda$.

Theorem 4.4. The following are equivalent:

(i) $C^*(G(n), \sigma)$ is simple.

(ii) $C^*(G(n), \sigma)$ has trivial center.

(iii) There are no nontrivial central $\sigma$-regular elements in $G(n)$.

Proof. By [14] Theorem 1.7, $C^*(G(n), \sigma)$ is simple if and only if every nontrivial $\sigma$-regular conjugacy class of $G(n)$ is infinite. Since every finite conjugacy class of $G(n)$ is a one-point set of a central element, then (i) is equivalent with (iii).

Moreover, (iii) is the same as saying that $S(G(n))$ is trivial, so therefore, (ii) is equivalent with (iii) by Lemma [11].

\[ \square \]
Lemma 4.5. A central element \( s = (0, \ldots, 0,s_{12}, s_{13}, \ldots, s_{n-1,n}) \) is \( \sigma \)-regular if and only if
\[
\prod_{1 \leq j < k \leq n} \lambda_{i,jk}^{s_{jk}} = 1
\]
for all \( 1 \leq i \leq n \).

Proof. A central element \( s = (0, \ldots, 0,s_{12}, s_{13}, \ldots, s_{n-1,n}) \) is \( \sigma \)-regular if and only if \( \sigma(s, r) = \sigma(r, s) \) for all \( r \in G(n) \). By a direct calculation from the multiplier formula (9), we get that
\[
\sigma(r, s) \sigma(s, r) = \left( \prod_{i<j<k} \lambda_{i,jk}^{s_{jk} r_i} \lambda_{j,ik}^{s_{ik} r_j} \lambda_{k,ij}^{s_{ij} r_k} \right) \left( \prod_{1 \leq j<k \leq n} \lambda_{i,jk}^{s_{jk} r_i} \lambda_{j,ik}^{s_{ik} r_j} \right)
\]
is equal to 1 for all \( r \in G(n) \) if and only if the inner parenthesis is 1 for each \( 1 \leq i \leq n \).

Corollary 4.6. \( C^*(G(n), \sigma) \) is simple if and only if for each nontrivial central element \( s = (0, \ldots, 0,s_{12}, s_{13}, \ldots, s_{n-1,n}) \) there is some \( 1 \leq i \leq n \) such that
\[
\prod_{1 \leq j < k \leq n} \lambda_{i,jk}^{s_{jk}} \neq 1.
\]

Example 4.7. \( C^*(G(3), \sigma) \) is simple if and only if for each nontrivial central element \( s = (0,0,0,s_{12}, s_{13}, s_{23}) \) at least one of the following hold:
\[
\lambda_{1,12}^{s_{12}} \lambda_{1,13}^{s_{13}} \lambda_{1,23}^{s_{23}} \neq 1,
\lambda_{2,12}^{s_{12}} \lambda_{2,13}^{s_{13}} \lambda_{2,23}^{s_{23}} \neq 1,
\lambda_{3,12}^{s_{12}} \lambda_{3,13}^{s_{13}} \lambda_{3,23}^{s_{23}} \neq 1.
\]

Set \( \lambda_{i,jk} = e^{2\pi i t_{i,jk}} \) for \( t_{i,jk} \in [0,1) \) and consider the \( n \times \frac{1}{2} n(n-1) \)-matrix \( T \) with entries \( t_{i,jk} \) in the corresponding spots. Then \( T \) induces a linear map
\[
\mathbb{R}^{\frac{1}{2} n(n-1)} \to \mathbb{R}^n.
\]

Corollary 4.8. Let \( T \) be the matrix described above. Then following are equivalent:

(i) \( C^*(G(n), \sigma) \) is simple
(ii) \( T^{-1}(\mathbb{Z}^n) \cap \mathbb{Z}^{\frac{1}{2} n(n-1)} = \{0\} \)
(iii) \( T(\mathbb{Z}^{\frac{1}{2} n(n-1)} \setminus \{0\}) \cap \mathbb{Z}^n = \emptyset \)

Remark 4.9. Clearly, the condition (ii) above is equivalent to that \( T \) restricts to an injective map
\[
\mathbb{Z}^{\frac{1}{2} n(n-1)} \to \mathbb{R}^n / \mathbb{Z}^n \cong T^n.
\]

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Furthermore, for $1 \leq j < k \leq n$, define
\[ \Lambda_{jk} = \{ t_{i,j,k} \in [0,1); 1 \leq i \leq n \mid e^{2\pi i t_{i,j,k}} = \lambda_{i,j,k} \} \]
and for $1 \leq i \leq n$, define
\[ \Lambda_i = \{ t_{i,j,k} \in [0,1); 1 \leq j < k \leq n \mid e^{2\pi i t_{i,j,k}} = \lambda_{i,j,k} \}. \]

**Proposition 4.10.** If there exists $i$ such that all the elements of $\Lambda_i$ are irrational and linearly independent over $\mathbb{Q}$, then $C^*(G(n), \sigma)$ is simple.

**Proof.** It follows immediately from Lemma 4.5 that “equation i” cannot be satisfied unless $s = 0$. Hence, no nontrivial $\sigma$-regular central elements exists. \( \square \)

**Proposition 4.11.** If there exists $j < k$ such that $\Lambda_{jk}$ consists of only rational elements, then $C^*(G(n), \sigma)$ is not simple.

**Proof.** Let $q$ be the least common multiplier of the denominators of the elements of $\Lambda_{jk}$. Then $qv_{jk}$ is central and $\sigma$-regular. Indeed,
\[ \sigma(r, qv_{jk})\sigma(qv_{jk}, r) = \prod_{i=1}^{n-1} \lambda_{i,j,k}^{qr_i} = 1 \]
for all $r \in G(n)$. \( \square \)

## 5 On isomorphisms of $C^*(G(n), \sigma)$

Fix $n \geq 2$ and let $\sigma$ be a multiplier of $G(n)$. If $\varphi$ is an automorphism of $G(n)$, define the multiplier $\sigma_\varphi$ of $G(n)$ by
\[ \sigma_\varphi(r, s) = \sigma(\varphi(r), \varphi(s)). \quad (22) \]

Then the associated twisted group $C^*$-algebras $C^*(G(n), \sigma)$ and $C^*(G(n), \sigma_\varphi)$ are isomorphic. Indeed, the map
\[ i(G, \sigma)(r) \mapsto i(G, \sigma_\varphi)(\varphi^{-1}(r)) \]
extends to an isomorphism $C^*(G(n), \sigma) \rightarrow C^*(G(n), \sigma_\varphi)$. Moreover, for any automorphism $\varphi$ of $G(n)$, it is easily seen that $\sigma \sim \tau$ if and only if $\sigma_\varphi \sim \tau_\varphi$. Hence, there is a well-defined group action of the automorphism group $\operatorname{Aut} G(n)$ on $H^1(G(n), T)$ defined by $\varphi \cdot [\sigma] = [\sigma_\varphi]$.

Furthermore, we have that the inner automorphism group of $G(n)$ can be described as
\[ \operatorname{Inn} G(n) \cong G(n)/V(n) \cong \mathbb{Z}^n. \]

Indeed, this is the case since the central part of an element does not contribute in a conjugation (and can also be seen from (21)). The outer isomorphism group $\operatorname{Out} G(n) = \operatorname{Aut} G(n)/\operatorname{Inn} G(n)$ turns out to be harder to describe directly. Therefore, we introduce another subgroup of $\operatorname{Aut} G(n)$, consisting of the automorphisms $G(n) \rightarrow G(n)$ of the form $u_i \mapsto z_i u_i$ for $1 \leq i \leq n$ and central elements $z_i \in V(n)$. In particular, these automorphisms leave all the $v_{jk}$’s fixed. Clearly, this subgroup of $\operatorname{Aut} G(n)$ is isomorphic with $V(n)^n$ and contains $\operatorname{Inn} G(n)$. In fact, in the case $n = 2$, we have $V(2)^2 = \operatorname{Inn} G(2)$.  

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Theorem 5.1. There is a split short exact sequence:

\[
1 \longrightarrow V(n)^n \longrightarrow \text{Aut}(n) \longrightarrow GL(n, \mathbb{Z}) \longrightarrow 1
\]

Proof. Assume that \( \varphi \) is any endomorphism \( G(n) \rightarrow G(n) \). Clearly, the image of any central element under \( \varphi \) must be central, so \( \varphi \) restricts to an endomorphism \( \varphi_1 : V(n) \rightarrow V(n) \). Therefore, \( \varphi \) also induces an endomorphism \( \varphi_2 : G(n)/V(n) \rightarrow G(n)/V(n) \), determined by \( \varphi_2(q(r)) = q(\varphi(r)) \). Consider now the following commutative diagram:

\[
\begin{array}{c}
1 & \rightarrow & V(n) & \rightarrow & G(n) & \rightarrow & \mathbb{Z}^n & \rightarrow & 1 \\
& & \varphi_1 & & \varphi & & q & & \\
1 & \rightarrow & V(n) & \rightarrow & G(n) & \rightarrow & \mathbb{Z}^n & \rightarrow & 1
\end{array}
\]

Assume that \( \varphi_2 \) is an automorphism. First, since \( \varphi_2 \) is surjective, then for all \( u_i \), there is some \( s_i \in G(n) \) such that \( \varphi(s_i) = z_i u_i \) for some \( z_i \in V(n) \). Hence, for all \( j < k \), we have \( \varphi(s_j s_k s_j^{-1} s_k^{-1}) = v_{jk} \) and therefore, \( \varphi_1 \) is surjective. Every surjective endomorphism of \( \mathbb{Z}^n \) is also injective, so \( \varphi_1 \) is an automorphism as well. Thus, by the “short five-lemma”, \( \varphi \) is an automorphism.

The converse obviously holds, and hence, \( \varphi \) is an automorphism if and only if \( \varphi_2 \) is an automorphism.

Furthermore, the construction of \( G(n) \) in terms of generators and relations means that every endomorphism \( G(n) \rightarrow G(n) \) is uniquely determined by its values at \( \{u_i\}_{i=1}^n \). In particular, we let \( \varphi : G(n) \rightarrow G(n) \) be determined by the pair of matrices given by its entries

\[
(\varphi(u_i)_j), (\varphi(u_i)_j)_k) \in M_n(\mathbb{Z}) \times M_n(\mathbb{Z})
\]

so that the induced endomorphism \( \varphi_2 \) is coming from a matrix in \( M_n(\mathbb{Z}) \).

By the above argument, the following map between endomorphism groups

\[
\text{End } G(n) \rightarrow \text{End } \mathbb{Z}^n, \quad (\varphi(u_i)_j), (\varphi(u_i)_j)_k) \mapsto (\varphi(u_i)_j)
\]

restricts to a surjective map \( \text{Aut } G(n) \rightarrow \text{Aut } \mathbb{Z}^n = GL(n, \mathbb{Z}) \).

To conclude the argument, we need the following.

Lemma 5.2. If \( \varphi \) and \( \varphi' \) are two endomorphisms of \( G(n) \), then

\[
(\varphi \circ \varphi')(u_i)_j = \sum_{k=1}^n \varphi'(u_i)_k \varphi(u_k)_j.
\]

If \( \varphi \) and \( \varphi' \) are two endomorphisms of \( G(n) \) that both induce the trivial map on \( G(n)/V(n) \), then

\[
(\varphi \circ \varphi')(u_i)_j = \varphi'(u_i)_j + \varphi(u_i)_j.
\]

Proof. For the moment, set \( \varphi(u_i)_j = r_{ij} \) and \( \varphi'(u_i)_j = s_{ij} \). Then

\[
(\varphi \circ \varphi')(u_i) = \varphi(u_n^{s_{1n}} \cdots u_1^{s_{1n}} z) = (u_n^{r_{1n}} \cdots u_1^{r_{1n}})^{s_{1n}} \cdots (u_n^{r_{nn}} \cdots u_1^{r_{nn}})^{s_{1n}} z',
\]
for some elements $z, z' \in V(n)$. Moreover, we can change the order of the $u_i$'s in the expression just by replacing $z'$ by another central element $z''$ and thus,

$$(\varphi \circ \varphi')(u_i)_j = r_{nj}s_m + r_{n-1}js_i, n-1 + \cdots + r_{1j}s_1 = \sum_{k=1}^{n} s_{ik}r_{kj}.$$ 

If both $\varphi$ and $\varphi'$ are trivial, then $\varphi(u_i) = z_iu_i$ and $\varphi'(u_i) = z'_i u_i$ for all $1 \leq i \leq n$ and some elements $z_i, z'_i \in V(n)$. Hence, $\varphi(v_{jk}) = \varphi'(v_{jk}) = v_{jk}$ for all $j < k$ and thus,

$$(\varphi \circ \varphi')(u_i) = \varphi(z'_i u_i) = z'_i z_i u_i.$$ 

Therefore, $(\varphi)$ restricts to a surjective homomorphism $\text{Aut} \ G(n) \to \text{GL}(n, \mathbb{Z})$ with kernel isomorphic to the group $M_n, \pm(n-1)(\mathbb{Z})$ under addition, that is, to $V(n)^n$.

Moreover, it should also be clear that $\text{GL}(n, \mathbb{Z})$ sits inside $\text{Aut} \ G(n)$ as a subgroup so that the sequence splits. In fact, we can calculate the action of $\text{GL}(n, \mathbb{Z})$ on $V(n)^n$ and see that $A \in \text{GL}(n, \mathbb{Z})$ acts on $(s_{i,j}) \in M_n, \pm(n-1)(\mathbb{Z})$ by the natural action on each column.

**Proposition 5.3.** If $\varphi \in V(n)^n$, then $\sigma \sim \sigma_{\varphi}$. Thus, the action of $V(n)^n$ on $H^2(G(n), \mathbb{T})$ given by $(\varphi)$ is trivial.

**Proof.** It is not hard to see that

$$\sigma(u_i, v_{jk})\sigma(v_{jk}, u_i) = \sigma_{\varphi}(u_i, v_{jk})\sigma_{\varphi}(v_{jk}, u_i),$$

that is,

$$[i_{(G,\sigma)}(u_i), i_{(G,\sigma)}(v_{jk})] = [i_{(G,\sigma_{\varphi})}(u_i), i_{(G,\sigma_{\varphi})}(v_{jk})]$$

for all $1 \leq i \leq n$ and $1 \leq j < k \leq n$. Hence, by the proof of Theorem 8.1, $\sigma_{\varphi}$ is similar to $\sigma$. 

**Remark 5.4.** To describe the $\text{GL}(n, \mathbb{Z})$-action on $H^2(G(n), \mathbb{T})$ requires more work. In particular, for $A \in \text{GL}(n, \mathbb{Z})$, we will need to define another square matrix $\tilde{A}$ of dimension $\frac{1}{2}n(n-1)$, with entries coming from the determinant of all $2 \times 2$-matrices inside $A$. More precisely, if $A = (a_{ij})$, $\tilde{A}$ is given by entries $\tilde{a}_{ij,kl}$ for $i < j, k < l$ such that $\tilde{a}_{ij,kl} = a_{ik}a_{jl} - a_{il}a_{jk}$. Then $A$ acts on the matrix $T$ of Corollary 1.8 by $A \cdot T = A\tilde{T}$. 

Further description of this action and an attempt of determining the isomorphism classes of $C^*(G(n), \sigma)$, or maybe of matrix algebras $M_k(C^*(G(n), \sigma))$, will hopefully be included in a future work.

Finally, we remark that for $n = 2$, it is shown by Packer [13, Theorem 2.9] that $C^*(G(2), \sigma)$ and $C^*(G(2), \sigma')$, where $\sigma$ and $\sigma'$ are of the form $[\sigma]$, are isomorphic if and only if there is a $\text{GL}(2, \mathbb{Z})$-matrix $A$ taking $\sigma$ to $\sigma'$. Note in this case that $A = \det A = \pm 1$. 

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