Multiscaling and Structure Functions in Turbulence: An Alternative Approach

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We propose an alternative formulation of structure functions for the velocity field in fully-developed turbulence. Instead of averaging moments of the velocity differences as a function of the distance, we suggest to average moments of the distances as a function of the velocity difference. This is like an “inverted” structure function, with a different statistics. On the basis of shell model calculations we obtain a new multiscaling spectrum.

The understanding of intermittency effects in fully-developed turbulence and the associated multiscaling spectrum of exponents, is probably the most fundamental open problem in turbulence research [1]. The traditional way of describing this is, as already suggested by Kolmogorov [2], to consider the velocity difference between two points of the turbulent state, raise this difference to the moment $q$, and then study the variation with respect to the distance between the two points. To improve the statistics, the moments are averaged in space (and maybe time) and one obtains the well known structure functions where the corresponding scaling exponents are called structure function exponents [1]. During the last decades it has become clear both from many experimental [2,3], numerical [4] and theoretical considerations [5], that this set of exponents is very non-trivial, defining an infinity of independent exponents leading to a “curved” variation of the scaling exponent with the moment. Notable is also the recent fundamental advances in obtaining the multiscaling spectrum analytically for temporally non-correlated, velocity fields, the so-called Kraichnan model [6,7].

We propose simply to “invert” the structure function equation, and consider instead averaged moments of the distance between two points, given a velocity difference between those points. This leads to an alternative way of describing and analyzing a turbulent velocity field (in particular when measured experimentally) and one obtains a new set of exponents that we have not yet been able to relate to the traditionally estimated exponents, though we suspect that there might be a relation. This inversion is inspired by studies in passive scalar advection where one often, say for pair particles, considers averages of the advection time versus the distance, instead of averages of the distance versus time [1,4]. To a start let us introduce the well known structure functions for the velocity field $u(x,t)$ of a fully developed turbulent state, obtained either from the Navier-Stokes equations or from measurements

\[ \Delta u_x(\ell) = u(x + r) - u(x) , \quad \ell = |r| \]  

(2)

The average in Eq.(1) is over space (and maybe time). We have assumed full isotropy of the velocity field. The set of exponents $\zeta_q$ forms a multiscaling spectrum [7].

Alternatively, we now consider the following quantities, which is denoted the distance structure functions

\[ < \ell(\Delta u_x)^q > \sim |\Delta u_x|^{\delta_q} \]  

(3)

where the difference $\Delta u_x$ is again defined as in Eq. (2) and $\ell(\Delta u_x)$ is understood as the minimal distance in $r$, measured from $x$, for which the velocity difference exceeds the value $\Delta u_x$. In other words, we fix a certain set of values of the velocity difference $\Delta u_x$. Starting out from the point $x$, we monitor the distances $\ell(\Delta u_x)$ where the velocity differences are equal to the prescribed values. Performing an average over space (and maybe time) the distance structure functions $\ell(\Delta u_x)^q$ are obtained. By assuming self-similarity of the small scale velocity differences, one expects a trivial set of exponents $\delta_q$ where the variation with the moment $q$ is determined by one exponent. Say, in the standard Kolmogorov theory we know that the velocity differences behave as $\Delta u \sim \ell^{-1/3}$, forgetting for a moment the averaging brackets. Inverting this equations, we of course obtain $\ell \sim \Delta u^3$ and would expect a trivial relation $\delta_q = 3q$. In case of an intermittent and singular velocity field without self-similarity of the small scale velocity differences (see [5]), this would be completely different and the averaging brackets will be crucial, relating to the statistics of the varying quantity that is averaged. We will show, based on shell model calculations, that in turbulence there exists a new spectrum $\delta_q$, that appears not to be trivially related to the spectrum $\zeta_q$. Let us for a moment reflect on the case $q = 1$. Using the standard value $\zeta_1 \sim 0.38 - 0.40$, the simple inversion gives $\delta_1 \sim 2.5$. Our calculations indicate that this value is not obtained in a turbulent model field.

In order to apply this scheme in a direct calculation we employ the Gledzer-Ohkitani-Yamada, GOY, shell model [8,9] which has be intensively studied over the last years [20,21]. This model is a rough approximation to the Navier-Stokes equations and is formulated on a discrete set of $k$-values, $k_n = r^n$. We use the standard value $r = 2$. In term of a complex Fourier $u_n$, of the velocity field the model reads

Alternatingly, we have assumed full isotropy of the velocity field. The set of exponents $\zeta_q$ forms a multiscaling spectrum [7].
The constraints still leave a free parameter $\epsilon$, if helicity conservation is also demanded \cite{23}. The set \{1\} of $N$ coupled ordinary differential equations can be numerically integrated by standard techniques. We have used standard parameters in this paper $N = 27, \nu = 10^{-9}, k_0 = 0.05, f = 5 \cdot 10^{-3}$.

The GOY model is defined in $k$-space but our formalism is written in direct space and we therefore apply a sort of inverse Fourier transform \cite{27}. Here we employ an idea proposed by Vulpiani \cite{28,29} and write the three-dimensional velocity field in the following way

$$u(x, t) = \sum_{n=1}^{N} c_n[|u_n(t)|e^{i|k_n\cdot x|} + c.c.]$$

The wavevectors are defined by

$$k_n = k_0 e_n$$

where $e_n$ is a unit vector in a random direction, for each shell $n$. Also $c_n$ are unit vectors in random directions. One can easily ensure that the velocity field is incompressible, $\text{div} \ u = 0$, by the following constraint \cite{28}

$$c_n \cdot e_n = 0 \quad \forall n$$

Note, that this condition could be relaxed to $\sum_{n=1}^{N} c_n \cdot e_n = 0$. In our numerical computations we consider the vectors $c_n$ and $e_n$ quenched in time but nevertheless average over many different realizations of these; i.e. one,
or several, specific measurements of the distance structure functions are performed with one realization of the vectors. After that a new realization of \( c_n, c_m \) is applied in order to perform a good statistical average.

FIG. 3. Probability distribution functions \( P(\ell(\Delta u_\bf{x})|\Delta u_\bf{x}) \) for (a) the velocity \( \Delta u_\bf{x} = 0.0027 \), which is close to the dissipative length scale (see Fig.1), and for (b) the velocity \( \Delta u_\bf{x} = 0.26 \), close to the velocity of the outer cut-off.

Equipped with a real space time dependent velocity field we start out with a test of this field by computing the standard velocity structure functions, given by Eq. \[30\]. Indeed, the field exhibits nice scaling invariance (ESS) where one moment of a given variable is varied against another moment. In the present case this means a graph of one distance structure function \( < \ell(\Delta u)^q > \) versus another \( < \ell(\Delta u)^q' > \) for two different moments \( q, q' \), and this results in ESS plots which spans over three times as long a regime as compared to traditional ESS plots where the quantities are \( < \Delta u_\bf{x}(\ell)^q > \) are applied (the large regime is of course due to the Kolmogorov exponent relation). Applying ESS we have obtained the exponents \( \delta_q \) in an independent way and the results agree well with the values listed in Table 1. This property of a much larger scaling regime of the ESS plots could be one of the advantages of the presented formalism. Details will be given in a forthcoming publication.

To obtain a better understanding of the obtained results we need to consider the statistics of \( \ell(\Delta u_\bf{x})^q \) in the following way

\[
< \ell(\Delta u_\bf{x})^q > \simeq \int \ell(\Delta u_\bf{x})^q P(\ell(\Delta u_\bf{x})|\Delta u_\bf{x}) \, d\ell \quad (8)
\]

where we have introduced the conditional probability distribution function \( P(\ell(\Delta u_\bf{x})|\Delta u_\bf{x}) \). This measures the probability of a distance \( \ell \) given the velocity difference. We show this PDF for two different values of the velocity difference in Fig. 3 on linear scales. In both cases, the distributions are clearly non-Gaussian with long exponential (or in fact stretched exponential) tails, as expected in intermittent systems. The surprising difference to the standard PDF’s for velocity differences is, that it does not tend towards a Gaussian for large scales. We would have expected that. We have not been able to relate this PDF, \( P(\ell(\Delta u_\bf{x})|\Delta u_\bf{x}) \), to the “usual” PDF, \( P(\Delta u_\bf{x}|\ell) \); these two PDF’s measure simply very different things.

In conclusion, we have introduced the distance structure functions defined for a velocity field in fully developed turbulence. The corresponding multiscale spectrum appears not to be related to the well known spectrum for velocity structure functions. The distance structure function could be very relevant for experimental velocity data measured in one point [13]. Here one typically applies the Taylor hypothesis in order to relate a tempo-
ral segment to a spatial segment. For this type of time series, the distance structure functions should be easily extracted.

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| $q$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----|---|---|---|---|---|---|---|---|
| $\zeta_q$ | 0.39 | 0.73 | 1.0 | 1.28 | 1.53 | 1.77 | 2.0 | 2.20 |
| $\delta_q$ | 2.04 | 3.70 | 5.4 | 7.0 | 8.5 | 10.0 | 11.7 | 12.9 |

TABLE I. Values of the scaling exponents for velocity structure functions $\zeta_q$ and distance structure functions $\delta_q$ with selected error bars.