LOCAL WELL-POSEDNESS FOR THE HIGHER-ORDER GENERALIZED KDV TYPE EQUATION WITH LOW-DEGREE OF NONLINEARITY

HAYATO MIYAZAKI

ABSTRACT. This paper is concerned with the local well-posedness for the higher-order generalized KdV type equation with low-degree of nonlinearity. The equation arises as a non-integrable and lower nonlinearity version of the higher-order KdV equation. As for the lower nonlinearity model of the KdV equation, Linares, the author and Ponce [10] prove the local well-posedness under a non-degenerate condition introduced by Cazenave and Naumkin [1]. In this paper, it turns out that the well-posedness result can be extended into the higher-order equation. We also give a lower bound for the lifespan of the solution. The lifespan depends on two quantities determined by the initial data.

1. Introduction

In this paper we consider the Cauchy problem for the higher-order generalized Korteweg-de Vries (KdV) type equation

\begin{equation}
\tag{HK}
\partial_t u + \partial_x^{2j+1}u \pm |u|^\alpha \partial_x^{2j-1}u = 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R},
\end{equation}

where $u = u(x, t)$ is a complex-valued unknown function, $\alpha \in (0, 1)$ and $j \in \mathbb{Z}^+$. The equation (HK) appears as a non-integrable and lower nonlinearity version of the higher order KdV equation

\begin{equation}
\partial_t u + \partial_x^{2j+1}u + c_j u \partial_x^{2j-1}u + P(u, \ldots, \partial_x^{2j-2}u) = 0
\end{equation}

corresponding to the KdV hierarchy introduced by Lax [9], where $c_j \in \mathbb{R} \setminus \{0\}$, $j \in \mathbb{Z}^+$ and $P$ is a certain polynomial (see also [8]). For instance, the KdV equation and the fifth order KdV equation can be described as

\begin{equation}
\partial_t u + \partial_x^3u = 0,
\end{equation}

\begin{equation}
\partial_t u + \partial_x^5u - 10u\partial_x^3u + 30u^2\partial_xu - 20\partial_xu\partial_x^2u = 0,
\end{equation}

respectively. These equation arise as various physical phenomena such as long wave propagating in a channel and the interaction effects between short and long waves. The KdV equation (1.2) on the line and the torus has been studied in huge mathematical and physical literatures. We refer to chapter 7-8 in [11] where we can summarize a lot of results for sharp local and global well-posedness, stability of special solutions, existence of blow-up solutions, and so on. In the fifth order KdV equation (1.3), Ponce [15] prove the local well-posedness in $L^2$-based Sobolev space $H^s(\mathbb{R})$, $s \geq 4$. Later on, Guo,
Then there exists \( (1.7) \) and \( (1.9) \). We emphasize that it is difficult to prove the local well-posedness for \( (1.4) \) with \( (1.6) \). Furthermore, Kenig and Pilod [3] introduced by Cazenave and Naumkin [1] (see (1.7)). Our purpose of this paper is to extend their result into the case of (HK). The main result is the following:

**Theorem 1.1 (Local well-posedness).** Denote \( m = \left[ \frac{3}{2} \right] + 1 \). Let \( s \in \mathbb{Z}^+ \) satisfy \( s - j + 1 \geq 2jm + 2j + 2 \). Assume that

\[
(1.5) \quad u_0 \in H^s(\mathbb{R}), \quad \langle x \rangle^m u_0 \in L^\infty(\mathbb{R}),
\]

\[
\langle x \rangle^m \partial_x^j u_0 \in L^2(\mathbb{R}), \quad \gamma = 1, \ldots, 2j + 2
\]

with

\[
(1.6) \quad \|u_0\|_{H^s} + \|\langle x \rangle^m u_0\|_{L^\infty} + \sum_{\gamma = 1}^{2j+2} \|\langle x \rangle^m \partial_x^\gamma u_0\|_{L^2} =: \delta,
\]

and

\[
(1.7) \quad \inf_{x \in \mathbb{R}} \langle x \rangle^m |u_0(x)| =: \lambda > 0.
\]

Then there exists \( T = T(\delta, \lambda, \alpha, s, j) > 0 \) such that (HK) has a unique local solution

\[
(1.8) \quad u \in C([0, T], H^s(\mathbb{R})), \quad \langle x \rangle^m u \in C([0, T], L^\infty(\mathbb{R})),
\]

with

\[
(1.9) \quad \langle x \rangle^m \partial_x^\gamma u \in C([0, T], L^2(\mathbb{R})), \quad \gamma = 1, \ldots, 2j + 2,
\]

\[
\partial_x^{s+j-l} u \in L^\infty(\mathbb{R}, L^2([0, T])), \quad l = 0, 1, \ldots, j - 1,
\]

and

\[
\sup_{0 \leq t \leq T} \|\langle x \rangle^m (u(t) - u_0)\|_{L^\infty} \leq \frac{\lambda}{2}.
\]

Moreover, the map \( u_0 \mapsto u(t) \) is continuous in the following sense: For any compact \( I \subset [0, T] \), there exists a neighborhood \( V \) of \( u_0 \) satisfying (1.5) and (1.7) such that the map is Lipschitz continuous from \( V \) into the class defined by (1.8) and (1.9).
Remark 1.2. The non-degenerate condition (1.7) comes from that \(|u|^\alpha\) is not regular enough (only \(C^\alpha\)). The condition (1.7) enables us to carry out the contraction principle. This approach is firstly introduced to a nonlinear Schrödinger equation by [1]. Later on, it is applied to the derivative nonlinear Schrödinger equation with low-degree of nonlinearity by Linares, Ponce and Santos [12, 13], as well as [10] for (1.4).

Remark 1.3. We employ the Kato smoothing effect (Lemma 2.1 below) to remove derivative loss of the nonlinearity. This provides us the additional regularity \(\partial_x^{s+j-l} u \in L^\infty(\mathbb{R}, L^2([0,T]))\) for any \(0 \leq l \leq j - 1\) in (1.9).

Remark 1.4. The regularity condition \(s - j + 1 \geq 2jm + 2j + 2\) arises from the estimate of the following norm:

\[
\| \langle x \rangle^m \partial_x^{2j+2} (|u|^\alpha \partial_x^{2j-1} u) \|_{L_t^q L_x^r}
\]

using the relation \(e^{i\omega_1^{j+1} x} e^{-i\omega_1^{2j+1}} = (x + (2j+1)t\omega_1^j)^m\) exhibited in Section 2. In detail, see the above estimate of (4.5).

We here define the lifespan of the solution to (HK) by

\[
T_{\delta, \lambda} := \sup \{T \in (0, \infty); \text{there exists a unique solution to (HK)} \text{ in the class given by Theorem 1.1},
\]

where \(\delta\) and \(\lambda\) are defined by (1.6) and (1.7). Once the local well-posedness is established by the contraction principle, we have a lower bound estimate for the lifespan of the solution.

Corollary 1.5 (Lower bound for the lifespan). Let \(\delta\) be as in (1.6). Define \(\lambda\) by (1.7). Under the same assumptions as in Theorem 1.1, there exists a constant \(C \in (0, 1)\) such that

\[
T_{\delta, \lambda} \geq \frac{C \lambda}{\delta \left(1 + \lambda (\frac{1}{1+\lambda})^{s-j+2-\alpha}\right)},
\]

where \(\kappa = s - j + 2\) if \(\delta \geq 1\), otherwise \(\kappa = \alpha\).

Remark 1.6. Note that \(\delta > \lambda\). In the case \(j = 1\), Corollary 1.5 was proven in [14]. However it is necessary that the lower bound in [14] is corrected slightly, because it is required to take an estimate as in (4.20) into account.

The rest of the paper is organized as follows: In Section 2, we collect some estimates for the linear evolution operator and an interpolation inequality. Section 3 is devoted to some nonlinear estimates playing a crucial role in Section 4. We finally prove Theorem 1.1 and Corollary 1.5 in Section 4.

We here introduce several notations used throughout this paper.

Notations: We set \(\langle x \rangle = (1 + |x|^2)^{\frac{1}{4}}\) and \([x]\) denotes the greatest integer less than or equal to \(x\) for any \(x \in \mathbb{R}\). For any \(q, r \geq 1\), We denote \(\|F\|_{L^q_x L^r_t} = \|F\|_{L^q_x L^r_t(\mathbb{R})}\) and \(\|F\|_{L^q_t L^r_x} = \|F\|_{L^q_t L^r_x(\mathbb{R})}\). We also define \(\|F\|_{L^q_x L^r_t} = \|F\|_{L^q_x L^r_t(\mathbb{R})}\) and \(\|F\|_{L^q_t L^r_x} = \|F\|_{L^q_t L^r_x(\mathbb{R})}\). \(\mathcal{F}\) stands for the usual Fourier transform on \(\mathbb{R}\). Let \(U_j(t) = e^{-i\omega_1^{j+1} t}\) be the linear evolution operator defined by \(U_j(t) = \mathcal{F}^{-1} e^{-t(|\xi|^{j+1}) \mathcal{F}}\) for any \(j \in \mathbb{Z}^+\).
2. Preliminary

We start this section presenting some linear estimates. The first one is concerning the sharp Kato smoothing effect found in [5, 6].

**Lemma 2.1 ([5, 7]).** Let \( j \in \mathbb{Z}^+ \). The following estimates hold:

\[
\left\| \partial_x^j U_j(t) \phi \right\|_{L^2_x L^2_t} \leq C \| \phi \|_{L^2_t} ,
\]

\[
\left\| \partial_x^j \int_\mathbb{R} U_j(t-s)f(s)ds \right\|_{L^2_x L^2_t} \leq CT^\alpha \| f \|_{L^p_x L^2_t} ,
\]

\[
\left\| \partial_x^{j+\sigma} \int_\mathbb{R} U_j(t-s)f(s)ds \right\|_{L^2_x L^2_t} \leq CT^\alpha \| f \|_{L^p_x L^2_t} ,
\]

for any \( \sigma = 0, \ldots, j \), where \( \alpha = (j - \sigma)/2j \) and \( p = 2j/(j + \sigma) \).

The next one is an interpolation inequality used throughout this paper.

**Lemma 2.2 ([10]).** Let \( \mu > 0 \) and \( r \in \mathbb{Z}^+ \). Denote \( \theta \in [0, 1] \) with \( (1 - \theta)r \in \mathbb{Z}^+ \). Then it holds that

\[
\left\| \langle x \rangle^{\mu} \partial_x^{(1-\theta)r} f \right\|_{L^2} \leq C \left\| \langle x \rangle^{\mu} f \right\|_{L^2} \left\| \partial_x^r f \right\|_{L^2}^{1-\theta} + L.O.T. ,
\]

where the lower order terms L.O.T. are bounded by

\[
\sum_{0 \leq \beta \leq 1, (1-\beta)(r-1) \in \mathbb{Z}^+} \left\| \langle x \rangle^{\beta(\mu-1)} \partial_x^{(1-\beta)(r-1)} f \right\|_{L^2} .
\]

**Proof.** We shall give a proof for self-containedness. Let us only show the case \( r \) is even \( (r = 2N) \), because the odd case is similar. We also only consider real-valued functions for simplicity. Set

\[
A_l = \left\| \langle x \rangle^{-l-r} \partial_x^l f \right\|_{L^2}
\]

for any \( 0 \leq l \leq r \). As for \( A_N \), combining \( A_0 \) with \( A_{2N} \), it follows from the integration by parts that

\[
\int_\mathbb{R} \langle x \rangle^\mu f \partial_x^{2N} f dx
\]

\[
\sim \int_\mathbb{R} \langle x \rangle^\mu \left( \partial_x^{N} f \right)^2 dx + \sum_{k=1}^N \int_\mathbb{R} \langle x \rangle^{\mu-k} \partial_x^{N-k} f \partial_x^{N} f dx .
\]

Here we write \( X \sim Y + Z \) to indicate \( X = \pm Y + cZ \) for some constant \( c \). For the last term of R.H.S in the above, we estimate

\[
\int_\mathbb{R} \langle x \rangle^{\mu-k} \partial_x^{N-k} f \partial_x^{N} f dx
\]

\[
\leq \left\| \langle x \rangle^{(\mu-1)\beta_0} \partial_x^{(r-1)(1-\beta_0)} f \right\|_{L^2} \left\| \langle x \rangle^{(\mu-1)\theta_0} \partial_x^{(r-1)(1-\theta_0)} f \right\|_{L^2} ,
\]

since it is possible to take \( \beta_0 = \beta_0(k), \theta_0 \in [0, 1] \) such that

\[
\begin{cases} 
\mu - k \leq (\mu - 1)(\beta_0 + \theta_0), \\
N - k = (r - 1)(1 - \beta_0), \quad N = (r - 1)(1 - \theta_0). 
\end{cases}
\]
Hence one sees from (2.1) and the Hölder inequality that
\[
A_N^2 \leq A_0 A_r
\]
\[
+ C \sum_{k=1}^{N} \left\| x^{(\mu-1)\beta_0(k)} \partial_x^{(r-1)(1-\beta_0(k))} f \right\|_{L^2}^2 
\times \left\| x^{(\mu-1)\theta_0} \partial_x^{(r-1)(1-\theta_0)} f \right\|_{L^2}^2.
\]
Let us next estimate \(A_l\) for all \(1 \leq l \leq N - 1\). Unifying \(A_0\) and \(A_{2l}\), by the integration by parts, we deduce that
\[
\int_{\mathbb{R}} (x)^l f (x) \frac{2(\mu-l)}{l} \partial_x^l f \, dx \sim \int_{\mathbb{R}} (x)^{2(\mu-1)} \left( \partial_x^l f \right)^2 \, dx + \sum_{k=1}^{l} L_{l,k},
\]
where
\[
L_{l,k} = \int_{\mathbb{R}} (x)^{2(\mu-1)-l} \partial_x^{l-k} f \partial_x^k f \, dx.
\]
A use of integration by parts gives us
\[
L_{l,1} = \frac{1}{2} \int_{\mathbb{R}} (x)^{\frac{2(\mu-l)}{l} \partial_x^l f} \, dx \leq C \left\| x^{\frac{2(\mu-1)}{l} \partial_x^l f} \right\|_{L^2}.
\]
We further obtain
\[
L_{l,k} \leq \left\| (x)^{(\mu-1)\beta_1} \partial_x^{(r-1)(1-\beta_1)} f \right\|_{L^2} \left\| (x)^{(\mu-1)\theta_1} \partial_x^{(r-1)(1-\theta_1)} f \right\|_{L^2},
\]
for any \(k \geq 2\), since there exist \(\beta_1 = \beta_1(k), \theta_1 \in [0, 1]\) such that
\[
\left\{ \begin{array}{l}
\frac{2(\mu-l)}{l} \mu - k \leq (\mu-1)(\beta_1 + \theta_1), \\
\mu - (l-k) = (r-1)(1-\beta_1), \quad l = (r-1)(1-\theta_1).
\end{array} \right.
\]
This implies
\[
A_l^2 \leq A_0 A_{2l}
\]
\[
+ C \sum_{k=1}^{l} \left\| x^{(\mu-1)\beta_1(k)} \partial_x^{(r-1)(1-\beta_1(k))} f \right\|_{L^2}^2 
\times \left\| x^{(\mu-1)\theta_1} \partial_x^{(r-1)(1-\theta_1)} f \right\|_{L^2}^2
\]
for any \(0 \leq l \leq N - 1\). Finally we shall consider \(A_l\) for all \(N + 1 \leq l \leq r - 1\). Combining \(A_{2l-r}\) with \(A_r\), using the integration by parts, we reach to
\[
\int_{\mathbb{R}} (x)^{\frac{2(\mu-r)}{r} \partial_x^{2l-r} f \partial_x^r f} \, dx \sim \int_{\mathbb{R}} (x)^{\frac{2(\mu-r)}{r} \partial_x^r f} \, dx + \sum_{k=1}^{r-l} L_{l,k}.
\]
Just arguing as in (2.3), one has
\[
A_l^2 \leq A_{2l-r} A_r
\]
\[
+ C \sum_{k=1}^{r-l} \left\| x^{(\mu-1)\beta_1(k)} \partial_x^{(r-1)(1-\beta_1(k))} f \right\|_{L^2}^2 
\times \left\| x^{(\mu-1)\theta_1} \partial_x^{(r-1)(1-\theta_1)} f \right\|_{L^2}^2
\]
for all \( N + 1 \leq l \leq r - 1 \). Collecting (2.2), (2.3) and (2.4), we have the desired estimate. \( \square \)

We finish this section by stating properties of \( U_j(t) \). Combining the fact
\[
U_j(-t)xU_j(t) = x + (2j + 1)t\partial_x^{2j}
\]
with Lemma 2.2, the following is valid:

**Lemma 2.3.** Let \( j \in \mathbb{Z}^+ \). It holds that
\[
\left\| \langle x \rangle^\beta U_j(t)f \right\|_{L^2} \leq C \langle t \rangle^\beta \left( \left\| \langle x \rangle^\beta f \right\|_{L^2} + \left\| \partial_x^{2j}f \right\|_{L^2} \right)
\]
for any \( \beta \in \mathbb{Z}^+ \).

**Proof.** By using (2.5), we have
\[
U_j(-t)x^\beta U_j(t)f = \left( x + (2j + 1)t\partial_x^{2j} \right)^\beta f
\]
for all \( \beta \in \mathbb{Z}^+ \). This implies that
\[
\left\| \langle x \rangle^\beta U_j(t)f \right\|_{L^2} \leq C \left( \left\| U_j(t)f \right\|_{L^2} + \left\| x^\beta U_j(t)f \right\|_{L^2} \right)
\]
\[
\leq C \left\| f \right\|_{L^2} + C \left\| \left( x + (2j + 1)t\partial_x^{2j} \right)^\beta f \right\|_{L^2}.
\]

Set \( \gamma = 2j \). A direct calculation gives us
\[
(x + (2j + 1)t\partial_x^j)^\beta f
\]
\[
= x^\beta f + t \sum_{k=0}^{\min(\gamma, \beta - 1)} c_{1,k}x^{\beta-1-k}\partial_x^{-k}f + t^2 \sum_{k=0}^{\min(2\gamma, \beta - 2)} c_{2,k}x^{\gamma-2-k}\partial_x^{2\gamma-k}f
\]
\[
\cdots + t^{\gamma-1} \left( c_{\beta-1,0}\partial_x^{(\beta-1)\gamma}f + c_{\beta-1,1}\partial_x^{(\beta-1)1}f \right) + t^\gamma \partial_x^{\beta\gamma}f,
\]
where \( c_{1,k}, \cdots, c_{\beta-1,k} \in \mathbb{Z}^+ \) depending on \( j \). Applying Lemma 2.2 to \( L^2 \) norm of each terms repeatedly, we obtain the desired estimate. \( \square \)

### 3. Nonlinear Estimates

In this section, we collect nonlinear estimates. To this end, we introduce the key norm corresponding to the solution space in Theorem 1.1:
\[
\|u\|_{X_T} := \|u\|_{L_T^\infty H^s} + \|\langle x \rangle^m u\|_{L_T^\infty L_x^\infty}
\]
\[
+ \sum_{\gamma=1}^{2j+2} \|\langle x \rangle^m \partial_x^\gamma u\|_{L_T^\infty L_x^{\max}} + \sum_{l=0}^{j-1} \|\partial_x^{s+j-l}u\|_{L_T^\infty L_x^2}
\]
for all \( s, j, m \in \mathbb{Z}^+ \) and any \( T > 0 \).

**Lemma 3.1.** Let \( m = \left[ \frac{1}{2} \right] + 1 \). Fix \( s, j \in \mathbb{Z}^+ \) with \( s - j + 1 \in \mathbb{Z}^+ \). Put \( q \in (1, 2] \) and \( \gamma \in \mathbb{Z}^+ \) with \( \gamma \leq 2j + 2 \). Then it holds that
\[
\|u^\alpha \partial_x^{2j-1}u\|_{L_T^2 L_x^2} \leq CT \|u\|_{X_T}^{\alpha+1},
\]
\[
\|\partial_x^{s+j-1}(u^\alpha \partial_x^{2j-1}u)\|_{L_T^2 L_x^2} \leq C \|u\|_{X_T}^{\alpha+1} + CT^{1/2}\lambda^{\alpha-1} \|u\|_{X_T}^2 + CT^{1/2} \lambda^{\alpha-(s-j+1)} \|u\|_{X_T}^{s-j+2}
\]
\[
\|\langle x \rangle^m \partial_x^\gamma (u^\alpha \partial_x^{2j-1}u)\|_{L_T^2 L_x^2} \leq CT \lambda^{\alpha-1} \|u\|_{X_T}^2 + CT\lambda^{\gamma-\gamma} \|u\|_{X_T}^{\gamma+1}
\]
for any $T > 0$ as long as $\langle x \rangle^m |u(t,x)| \geq \frac{1}{2}$ for any $(t,x) \in [0,T] \times \mathbb{R}$.

Proof. To simplify the exposition, we shall consider real-valued functions. (3.1) is immediate from Sobolev embedding. Let us prove (3.2). A use of the Leibniz rule gives us

$$
\| \partial_x^{s-j+1}(|u|^\alpha \partial_x^{j-1} u) \|_{L_x^3 L_T^2} \leq \sum_{k=0}^{s-j+1} \| \partial_x^k (|u|^\alpha \partial_x^{s-j} u) \|_{L_x^3 L_T^2} =: A_0 + \sum_{k=1}^{j-1} A_k + \sum_{k=0}^{s-j+1} A_k.
$$

By the Hölder inequality, we see that

$$
A_0 \leq C \| \partial_x^{s+j} u \|_{L_x^\infty L_T^2} \| \langle x \rangle |u|^\alpha \|_{L_x^p L_T^q} \| \langle x \rangle^{-1} \|_{L_T^\infty}
\leq C \| \partial_x^{s+j} u \|_{L_x^\infty L_T^2} \| \langle x \rangle^m |u|^\alpha \|_{L_x^\infty L_T^2}.
$$

In terms of $A_k$ for $1 \leq k \leq s-j+1$, the elements can be written as

$$
\partial_x^k (|u|^\alpha \partial_x^{s-j} u) = \sum_{n=1}^{k} \sum_{\beta_1 + \cdots + \beta_n = k} C_{\beta} |u|^{\alpha-2n} \partial_x^{\beta_1} u \cdots \partial_x^{\beta_n} u = \sum_{n=1}^{k} \sum_{\beta_1 + \cdots + \beta_n = k} F_{\beta},
$$

where $\beta_0 = s + j - k$. When $1 \leq k \leq j - 1$, in light of $\beta_0 \geq s + 1$, using $|x|^m |u(t,x)| \geq \frac{1}{2}$, we deduce from the Hölder and the Gagliardo-Nirenberg inequality that

$$
\| F_{\beta_1, \ldots, \beta_n} \|_{L_x^3 L_T^2} \leq C \chi^{\alpha-n} \| \langle x \rangle^{m(n-\alpha)} \partial_x^{\beta_1} u \cdots \partial_x^{\beta_n} u \partial_x^{\beta_0} u \|_{L_x^3 L_T^2} \leq C \chi^{\alpha-n} \| \langle x \rangle^{-1} \|_{L_x^2 L_T^2} \prod_{i=1}^{n} \| \langle x \rangle^{m} \partial_x^{\beta_i} u \|_{L_x^\infty L_T^2} \leq C \chi^{\alpha-n} \prod_{i=1}^{n} \left( \| \langle x \rangle^{m} \partial_x^{\beta_i} u \|_{L_x^\infty L_T^2} + \| \langle x \rangle^{m} \partial_x^{\beta_i+1} u \|_{L_x^\infty L_T^2} \right) \times \| \partial_x^{\beta_0} u \|_{L_x^\infty L_T^2} \leq C \chi^{\alpha-n} \| u \|_{X_T}^{n+1}.
$$

Note that $m \alpha > 1$. In the case $j \leq k \leq s-j+1$, in view of $\beta_i \leq s$, the similar calculation shows

$$
\| F_{\beta_1, \ldots, \beta_n} \|_{L_x^3 L_T^2} \leq C \chi^{\alpha-n} \| \langle x \rangle^{m(n-\alpha)} \partial_x^{\beta_1} u \cdots \partial_x^{\beta_n} u \partial_x^{\beta_0} u \|_{L_x^3 L_T^2}.
$$
iteratively, we deduce that for any \(0 \leq n \leq 1\) satisfying
\[
A_0 = \sum_{i=0}^{\infty} A_i. \quad \text{(3.4)}
\]
By using Lemma 2.2 iteratively, we deduce that for any \(0 \leq i \leq n\) and \(\sigma = 0, 1\),
\[
\begin{align*}
&\left\| \langle x \rangle^{m \theta_i} \frac{\partial^{\beta_i}}{\partial x^{\beta_i}} \partial^{(1-\theta_i)\ell s + \theta_i \gamma + \sigma} u \right\|_{L^\infty_T L^2_x} \\
&\leq C \left\| \langle x \rangle^m \frac{\partial^{\sigma}}{\partial x^{\beta_i}} \right\|_{L^\infty_T L^2_x}^{\theta_i} \left\| \partial^{n} u \right\|_{L^\infty_T L^2_x}^{1-\theta_i} + C \sum_{0 \leq n \leq 1, (1-n_i)(s-1)+n_i \gamma + \sigma \in \mathbb{Z}^+} \left\| \langle x \rangle^{(m-1)n_i} \frac{\partial^{n_i}}{\partial x^{n_i}} \partial^{(1-n_i)(s-1)+n_i \gamma + \sigma} u \right\|_{L^\infty_T L^2_x} \\
&\leq C \| u \|_{X_T}.
\end{align*}
\]
Combining the above, (3.2) is valid. To complete the proof, it remains to verify
\[
\theta_0 + \theta_1 + \cdots + \theta_n \geq n - \alpha.
\]
Collecting (3.4), together with \(\beta_1 + \cdots + \beta_n = k\) and \(\beta_0 = s + j - k\), we see
\[
\begin{align*}
j &= \left( n - (\theta_0 + \theta_1 + \cdots + \theta_n) \right) s + \theta_0 r_0 + \theta_1 r_1 + \cdots + \theta_n r_n.
\end{align*}
\]
By means of \(r_0, r_i \in \{1, \cdots, 2j + 1\}\), we obtain
\[
\theta_0 + \theta_1 + \cdots + \theta_n = n + \frac{\theta_0 r_0 + \theta_1 r_1 + \cdots + \theta_1 r_1 + \cdots + \theta_n r_n - j}{s} \geq n + \frac{\theta_0 + \theta_1 + \cdots + \theta_n - j}{s},
\]
Thanks to \(s - 1 \geq 2j + 3j\), it is concluded that
\[
\theta_0 + \theta_1 + \cdots + \theta_n \geq \frac{ns - j}{s - 1} > n - \frac{1}{2} > n - \alpha.
\]
Finally we shall show (3.3). By the Leibniz rule, it holds that
\[
\left\| \langle x \rangle^m \frac{\partial^\gamma}{\partial x} (|u|^n \partial^{2j-1} u) \right\|_{L^2_T L^2_x} \leq T \sum_{k=0}^{\gamma} \left\| \langle x \rangle^m \partial^k (|u|^n) \partial^{2j-1+\gamma-k} u \right\|_{L^2_T L^2_x} = T \left( \tilde{A}_0 + \sum_{k=1}^{\gamma} \tilde{A}_k \right)
\]
for any $\gamma \leq 2j + 2$. We then deduce from Lemma 2.2 and $\langle x \rangle^m |u(t,x)| \geq \frac{\lambda}{2}$ that

$$\tilde{A}_0 \leq C\lambda^{-1} \langle x \rangle^m u \|L^p_T L^\infty_x \| \left\| \langle x \rangle^{m-1} \partial_x^{2j+1+\gamma} u \right\|_{L^p_T L^2_x}$$

$$\leq C\lambda^{-1} \langle x \rangle^m u \|L^p_T L^\infty_x \| \times \left( \left\| \langle x \rangle^m \partial_x^{2j+1} u \right\|_{L^p_T L^2_x} \left\| \partial_x^{2m_j+\gamma-1} u \right\|_{L^p_T L^2_x} + L.O.T. \right)$$

$$\leq C\lambda^{-1} \langle x \rangle^m u \|L^p_T L^\infty_x \| \times \left( \left\| \langle x \rangle^m \partial_x^{2j+1} u \right\|_{L^p_T L^2_x} + \left\| \partial_x^{2m_j+\gamma-1} u \right\|_{L^p_T L^2_x} + L.O.T. \right).$$

Here, by means of a successive use of Lemma 2.2, one has

$$L.O.T. \leq C \sum_{0 \leq n \leq 1} \left\| \langle x \rangle^{(m-1)\eta} \partial_x^{(1-\eta)(2mj-1)+(\gamma-1)} u \right\|_{L^p_T L^2_x}$$

$$\leq C \left( \left\| \langle x \rangle^{m-1} \partial_x^{2j} u \right\|_{L^p_T L^2_x} + \|u\|_{L^p_T H^{\gamma-1}} \right.$$

$$\left. + \|u\|_{L^p_T H^{2mj-1+\gamma-1}} + \|u\|_{L^p_T H^{2mj-\gamma+1}} \right)$$

$$\leq C \left( \left\| \langle x \rangle^{m-1} \partial_x^{2j} u \right\|_{L^p_T L^2_x} + \|u\|_{L^p_T H^{2mj-1+\gamma-1}} \right),$$

which yields

$$\tilde{A}_0 \leq C\lambda^{-1} \|u\|_{X_T}^2.$$

As for $1 \leq k \leq \gamma$, the elements in $\tilde{A}_k$ can be written as

$$\langle x \rangle^m \partial_x^k (|u|^\alpha) \partial_x^{2j-1+\gamma-k} u$$

$$= \sum_{n=1}^{k} \sum_{1 \leq 1 \leq \beta_k \leq k} C_{\beta} \langle x \rangle^m |u|^{\alpha-2n} u^{\beta_1} \partial_x^{\beta_1} u \cdots \partial_x^{\beta_n} u \partial_x^{\beta_0} u$$

$$=: \sum_{n=1}^{k} \sum_{1 \leq 1 \leq \beta_k \leq k} \tilde{F}_k,$$
In the similar way, the other terms are estimated as follows:

\[ C^{\alpha-n} \| u \|_{X_T}^{n+1}. \]

One easily verifies the same estimate in the other cases \( \gamma - 3 \leq k \) or \( \gamma \leq 4 \), because of \( \beta_i \leq 2j + 2 \) for any \( 0 \leq i \leq n \). Indeed, if \( k = \gamma \), we then calculate that

\[
\left\| \langle x \rangle^{m(2-a)} | \partial_x^2 u | \partial_x^{2j-1} u \right\|_{L_T^\infty L_x^2} \leq C \left\| \langle x \rangle^{m(2-a)} | \partial_x^2 u | \partial_x^{2j-1} u \right\|_{L_T^\infty L_x^2} \\
+ C \lambda^{\alpha-n} \sum_{n=2}^{\gamma} \sum_{\beta_1 + \cdots + \beta_n = \gamma \atop 1 \leq \beta_1, \ldots, \beta_n \leq \gamma-1} \left\| \langle x \rangle^{m(n+1-a)} | \partial_x^{\beta_1} u \cdots \partial_x^{\beta_n} u | \partial_x^{2j-1} u \right\|_{L_T^\infty L_x^2}.
\]

For the first term, it follows from Sobolev embedding that

\[
\left\| \langle x \rangle^{m(2-a)} | \partial_x^2 u | \partial_x^{2j-1} u \right\|_{L_T^\infty L_x^2} \leq C \left( \left\| \langle x \rangle^{m} | \partial_x^{2j-1} u \right\|_{L_T^\infty L_x^2} + \left\| \langle x \rangle^{m} | \partial_x^2 u \right\|_{L_T^\infty L_x^2} \right) \left\| \langle x \rangle^{\alpha} \partial_x^2 u \right\|_{L_T^\infty L_x^2} \leq C \| u \|_{X_T}^2.
\]

In the similar way, the other terms are estimated as follows:

\[
\left\| \langle x \rangle^{m(n+1-a)} | \partial_x^{\beta_1} u \cdots \partial_x^{\beta_n} u | \partial_x^{2j-1} u \right\|_{L_T^\infty L_x^2} \leq C \| u \|_{X_T}^{n+1}.
\]

Combining these estimates, we find

\[
\tilde{A}_k \leq C \sum_{n=1}^{k} \lambda^{\alpha-n} \| u \|_{X_T}^{n+1} \leq C \lambda^{\alpha-1} \| u \|_{X_T}^2 + C \lambda^{\alpha-k} \| u \|_{X_T}^{k+1}
\]

for any \( 1 \leq k \leq \gamma \). These yield (3.3). This completes the proof.

**Lemma 3.2.** Let \( m = \left\lceil \frac{k}{2} \right\rceil + 1 \). Denote \( s, j \in \mathbb{Z}^+ \) with \( s-j+1 \in \mathbb{Z}^+ \). Put \( q \in (1, 2) \) and \( \gamma \in \mathbb{Z}^+ \) with \( \gamma \leq 2j + 2 \). Then it holds that

\[
\begin{align*}
\| u \|_{X_T}^n \| u \|_{X_T}^{n+1} & \leq C T \| u \|_{X_T}^n \| u \|_{X_T}^{n+1} + C \lambda^{\alpha-1} \| u \|_{X_T} \| u \|_{X_T} \\
& \leq C \| u \|_{X_T}^n \| u \|_{X_T}^{n+1} + C \lambda^{\alpha-1} \| u \|_{X_T} \| u \|_{X_T} + C T^{1/2} \left( \lambda^{\alpha-s-j+2} + \lambda^{\alpha-s-j+1} \right) \times \left( \| u \|_{X_T}^{s-j+2} + \| u \|_{X_T}^{s-j+1} \right) \| u \|_{X_T} \| u \|_{X_T} \\
& + C T^{1/2} \left( \lambda^{\alpha-2} + \lambda^{\alpha-1} \right) \left( \| u \|_{X_T}^2 + \| u \|_{X_T} \right) \| u \|_{X_T} \| u \|_{X_T}.
\end{align*}
\]
\[
\| (x)^m \partial_x^\gamma (|u|^\alpha \partial_x^{2j-1} u - |v|^\alpha \partial_x^{2j-1} v) \|_{L_x^1 L_t^2}
\leq C T \alpha^{-1} \| u \|_{X_T} \| u - v \|_{X_T}
+ C T (\lambda^{\alpha-\gamma-1} + \lambda^{\alpha-\gamma}) \left( \| u \|_{X_T}^{\gamma+1} + \| u \|_{X_T} \right) \| u - v \|_{X_T}
+ C T \alpha^{-2} \| u \|_{X_T}^2 \| u - v \|_{X_T},
\]
(3.7)

as long as \( \langle x \rangle^m \min (|u(t,x)|, |v(t,x)|) \geq \frac{1}{2} \) for any \( (t,x) \in [0,T] \times \mathbb{R} \).

**Proof.** For simplicity, we shall only consider real-valued functions. The proof is similar to that of Lemma 3.1 except for employing

\[
\| u|^{\alpha-2k} u^k - |v|^{\alpha-2k} v^k \| \leq C \left( \| u|^{\alpha-k-1} + |v|^{\alpha-k-1} \right) |u - v|
\]
for any \( k \in \mathbb{Z}_{\geq 0} \), so we shall only sketch the proof of (3.6). By the Leibniz rule, one has

\[
\sum_{k=0}^{s-j+1} \| \partial_x^{s-j+k} (|u|^\alpha \partial_x^{2j-1} u - |v|^\alpha \partial_x^{2j-1} v) \|_{L_x^s L_t^q}
\leq \sum_{k=0}^{s-j+1} \| \partial_x^{s-j+k} (|u|^\alpha \partial_x^{2j-1} u - |v|^\alpha \partial_x^{2j-1} v) \|_{L_x^s L_t^q}.
\]

Combining the Hölder inequality with \( \langle x \rangle^m \min (|u|, |v|) \geq \frac{1}{2} \) and (3.8), it is deduced that

\[
B_0 \leq C \| u|^\alpha \partial_x^{s+j} (u - v) \|_{L_x^1 L_t^2} + C \left( \| u|^{\alpha-1} + |v|^{\alpha-1} \right) |u - v| \partial_x^{s+j} v \|_{L_x^1 L_t^2}
\leq C \| \langle x \rangle^m u \|_{L_x^\infty L_t^\infty} \| \partial_x^{s+j} (u - v) \|_{L_x^\infty L_t^2}
+ C \lambda^{\alpha-1} \| \langle x \rangle^m (u - v) \|_{L_x^\infty L_t^\infty} \| \partial_x^{s+j} v \|_{L_x^\infty L_t^2}.
\]

The estimation of \( B_k \) for \( 1 \leq k \leq s-j+1 \) can be obtained in the similar way to the proof of (3.2). Hence we here only handle \( B_{s-j+1} \). Thanks to the Hölder inequality, one sees that

\[
B_{s-j+1} \leq C T^{1/2} \left( \| u|^{\alpha-2(s-j+1)} u^{s-j+1} (\partial_x u)^{s-j+1} \partial_x^{2j-1} u
- |v|^{\alpha-2(s-j+1)} v^{s-j+1} (\partial_x v)^{s-j+1} \partial_x^{2j-1} v \|_{L_x^\infty L_t^2} + \cdots
\right.
\left. + \| u|^{\alpha-2} (\partial_x^{s-j+1} u) \partial_x^{2j-1} u - |v|^{\alpha-2} (\partial_x^{s-j+1} v) \partial_x^{2j-1} v \|_{L_x^\infty L_t^2} \right)
\leq C T^{1/2} (B_{s-j+1,1} + \cdots + B_{s-j+1,s-j+1}).
\]

As in the proof of (3.2), the estimate of \( B_{s-j+1,k} \) for \( 2 \leq k \leq s-j+1 \) can be carried out, so we only give a proof of \( B_{s-j+1,1} \) and \( B_{s-j+1,s-j+1} \). By using the Gagliardo-Nirenberg inequality, (3.8) and \( \langle x \rangle^m \min (|u|, |v|) \geq \frac{1}{2} \), one has

\[
B_{s-j+1,1}
\leq \| \left( |u|^{\alpha-2(s-j+1)} u^{s-j+1} - |v|^{\alpha-2(s-j+1)} v^{s-j+1} \right) (\partial_x u)^{s-j+1} \partial_x^{2j-1} u \|_{L_x^\infty L_t^2}
+ \| |v|^{\alpha-(s-j+1)} \left( (\partial_x u)^{s-j+1} - (\partial_x v)^{s-j+1} \right) \partial_x^{2j-1} u \|_{L_x^\infty L_t^2}
+ \| |v|^{\alpha-(s-j+1)} (\partial_x v)^{s-j+1} \partial_x^{2j-1} (u - v) \|_{L_x^\infty L_t^2}
\]
Similarly to the above,

\[ \text{Combining these estimates, we reach to (3.6).} \]

\[ \Box \]

4. PROOF OF THE MAIN RESULTS.

Proof of Theorem 1.1. To simplify the exposition, we shall only consider real-valued functions. Let us introduce the complete metric space

\[ X_{T,M} = \{ u \in C([0,T], H^s(\mathbb{R})); \| u \|_{X_T} := \| u \|_{L_T^\infty L_x^2} + \| \langle x \rangle^m u \|_{L_T^\infty L_x^2} \] 

\[ + \sum_{\gamma=1}^{2j+2} \| \langle x \rangle^m \partial_x^\gamma u \|_{L_T^\infty L_x^2} + \sum_{l=0}^{j-1} \| \partial_x^{\gamma+j-l} u \|_{L_T^\infty L_x^2} \] 

\[ \leq M, \]

\[ \sup_{0 \leq t \leq T} \| \langle x \rangle^m (u(t) - u_0) \|_{L_\infty} \leq \frac{\lambda}{2} \] 

equipped with the distance function \( d_{X_T}(u,v) = \| u - v \|_{X_T} \). Here the constant \( M \) will be chosen later. Notice that

\[ \frac{\lambda}{2} \leq \langle x \rangle^m |u(x,t)| \leq \langle x \rangle^m |u_0(x)| + \frac{\lambda}{2} \] 

for any \( (x,t) \in \mathbb{R} \times [0,T] \) as long as \( u \in X_{T,M} \). Set

\[ \Phi(u(t)) = U_j(t)u_0 + \int_0^t U_j(t-s)(\langle x \rangle^m \partial_x^{2j-1} u)(s)ds. \]
We will prove that \( \Phi \) is a contraction map in \( X_{T,M} \). Let us first show that \( \Phi \) maps from \( X_{T,M} \) to itself. By Lemma 2.1, it holds that

\[
\| \partial_x^s \Phi(u) \|_{L^2 \times L_x^2} + \sum_{l=0}^{j-1} \| \partial_x^{s+j-l} \Phi(u) \|_{L^2 \times L_x^2} \\
\leq C \sum_{l=0}^{j-1} \| \partial_x^{s-l} u_0 \|_{L^2} + C \sum_{l=0}^{j-1} T \alpha_j \| \partial_x^{s-j+1} (|u|^\alpha \partial_x^{2j-1} u) \|_{L^2 \times L_x^2} \\
\leq C \| u_0 \|_{H^s} + C (T \alpha_j + T \alpha_{j-1}) \\
\times \left( \| \partial_x^{s-j+1} (|u|^\alpha \partial_x^{2j-1} u) \|_{L^2 \times L_x^2} + \| \partial_x^{s-j+1} (|u|^\alpha \partial_x^{2j-1} u) \|_{L^2 \times L_x^{p,j-1}} \right),
\]

where \( \alpha_j, \alpha_{j-1} = \frac{j+1}{2} \) and \( p_{j,t} = \frac{2j-1}{2} \). By employing (3.2) with \( q = p_{j,0} \) and \( p_{j,j-1} \), we obtain

\[
\| \partial_x^s \Phi(u) \|_{L^2 \times L_x^2} + \sum_{l=0}^{j-1} \| \partial_x^{s+j-l} \Phi(u) \|_{L^2 \times L_x^2} \\
\leq C \| u_0 \|_{H^s} + C T \frac{1}{T} \left( 1 + \lambda^{\alpha-(s-j+1)} \right) (M^{\alpha+1} + M^{s-j+2})
\]
as long as \( T \leq 1 \). One also sees from (3.1) that

\[
\| \Phi(u) \|_{L^2 \times L_x^2} \leq \| u_0 \|_{L^2} + C T M^{\alpha+1}.
\]

Let us next consider \( \| \langle x \rangle^m \partial_x^2 \Phi(u) \|_{L^2 \times L_x^2} \) for any \( 1 \leq \gamma \leq 2j+2 \). It follows from Lemma 2.3 that

\[
\| \langle x \rangle^m \partial_x^2 \Phi(u) \|_{L^2 \times L_x^2} \\
\leq C \| \langle x \rangle^m \partial_x^2 u_0 \|_{L^2} \\
+ C \langle T \rangle^m \left( \| \partial_x^2 u_0 \|_{L^2} + \| \partial_x^{2j+m-\gamma} u_0 \|_{L^2} \right) \\
+ C \| \langle x \rangle^m \partial_x^2 (|u|^\alpha \partial_x^{2j-1} u) \|_{L^2 \times L_x^2} \\
+ C \langle T \rangle^m \left( \| \partial_x^2 (|u|^\alpha \partial_x^{2j-1} u) \|_{L^2 \times L_x^2} + \| \partial_x^{2j+m+\gamma} (|u|^\alpha \partial_x^{2j-1} u) \|_{L^2 \times L_x^2} \right).
\]

Since \( s - j + 1 \geq 2jm + 2j + 2 \), we deduce from \( H^{s-j+1} \leftrightarrow H^\sigma \) for \( \sigma = \gamma, 2jm + \gamma \) that

\[
\| \partial_x^2 (|u|^\alpha \partial_x^{2j-1} u) \|_{L^2 \times L_x^2} + \| \partial_x^{2j+m+\gamma} (|u|^\alpha \partial_x^{2j-1} u) \|_{L^2 \times L_x^2} \\
\leq C \| u_0^\alpha \partial_x^{2j-1} u \|_{L^2 \times L_x^2} + C T^{1/2} \| \partial_x^{s-j+1} (|u|^\alpha \partial_x^{2j-1} u) \|_{L^2 \times L_x^2}.
\]

Applying (3.1) and (3.2) with \( q = 2 \), one has

\[
\| \partial_x^2 (|u|^\alpha \partial_x^{2j-1} u) \|_{L^2 \times L_x^2} + \| \partial_x^{2j+m+\gamma} (|u|^\alpha \partial_x^{2j-1} u) \|_{L^2 \times L_x^2} \\
\leq C T^{1/2} \left( 1 + \lambda^{\alpha-(s-j+1)} \right) (M^{\alpha+1} + M^{s-j+2})
\]
for any \( T \leq 1 \). Further, it comes from (3.3) that
\[
\| \langle x \rangle^m \partial_x^2 (|u|^{\alpha} \partial_x^{2j-1} u) \|_{L^1_T L^2_x} \leq C T \lambda^{-1} M^2 + C T \lambda^{-\gamma} M^{\gamma+1}.
\]
Hence, it is concluded from (4.4), (4.5) and (4.6) that
\[
\| \langle x \rangle^m \partial_x^2 \Phi(u) \|_{L^\infty_T L^2_x}
\leq C \left( \| \langle x \rangle^m \partial_x^2 u_0 \|_{L^2} + \| \partial_x^2 u_0 \|_{L^2} + \| \partial_x^{2j+\gamma} u_0 \|_{L^2} \right)
+ C T^{1/2} \left( 1 + \lambda^{-(s-j+1)} \right) \left( M^{\alpha+1} + M^{s-j+2} \right)
\]
as long as \( T \leq 1 \) for any \( 1 \leq \gamma \leq 2j + 2 \).
Let us next handle \( \| \langle x \rangle^m \Phi(u) \|_{L^\infty_T L^2_x} \). Thanks to
\[
\frac{d}{dt} U_j(t) u_0 = -\partial_x^{2j+1} U_j(t) u_0,
\]
combining Sobolev embedding with Lemma 2.3, we obtain
\[
\| \langle x \rangle^m (U_j(t) u_0 - u_0) \|_{L^\infty_T L^2_x}
\leq C T \left( \| \langle x \rangle^m \partial_x^{2j+1} u_0 \|_{L^2} + \| \langle x \rangle^m \partial_x^{2j+2} u_0 \|_{L^2} + \| u_0 \|_{H^{2j+2}} \right)
\]
whenever \( T \leq 1 \). Similarly to (4.8), one has
\[
\| \langle x \rangle^m \int_0^t U_j(t - s) \left( |u|^{\alpha} \partial_x^{2j-1} u \right) (s) ds \|_{L^\infty_T L^2_x}
\leq C (T)^m \left( \| \langle x \rangle^m |u|^{\alpha} \partial_x^{2j-1} u \|_{L^1_T L^2_x} + \| \langle x \rangle^m \partial_x^2 \left( |u|^{\alpha} \partial_x^{2j-1} u \right) \|_{L^1_T L^2_x} \right)
+ C (T)^m \left( \| |u|^{\alpha} \partial_x^{2j-1} u \|_{L^1_T L^2_x} + \| \partial_x^{2j+1} \left( |u|^{\alpha} \partial_x^{2j-1} u \right) \|_{L^1_T L^2_x} \right).
\]
By means of (4.1), we here compute
\[
\| \langle x \rangle^m |u|^{\alpha} \partial_x^{2j-1} u \|_{L^1_T L^2_x} \leq T \| \langle x \rangle^m u \|_{L^\infty_T L^2_x} \| |u|^{\alpha-1} \partial_x^{2j-1} u \|_{L^\infty_T L^2_x}
\leq C T \lambda^{-1} \| \langle x \rangle^m u \|_{L^\infty_T L^2_x} \| \langle x \rangle^m \partial_x^{2j-1} u \|_{L^\infty_T L^2_x}
\leq C T \lambda^{-\gamma} M^2.
\]
By (3.3) with \( \gamma = 1 \), it is observed that
\[
\| \langle x \rangle^m \partial_x (|u|^{\alpha} \partial_x^{2j-1} u) \|_{L^1_T L^2_x} \leq C T \lambda^{-1} M^2.
\]
Further, arguing as in (4.5), we deduce that
\[
\| |u|^{\alpha} \partial_x^{2j-1} u \|_{L^1_T L^2_x} + \| \partial_x^{2j+1} \left( |u|^{\alpha} \partial_x^{2j-1} u \right) \|_{L^1_T L^2_x}
\leq C T^{1/2} \left( 1 + \lambda^{-(s-j+1)} \right) \left( M^{\alpha+1} + M^{s-j+2} \right)
\]
for any \( T \leq 1 \). Therefore it follows from these estimates that
\[
\| \langle x \rangle^m \int_0^t U(t - s) \left( |u|^{\alpha} \partial_x^{2j-1} u \right) (s) ds \|_{L^\infty_T L^2_x}
\leq C T^{1/2} \left( 1 + \lambda^{-(s-j+1)} \right) \left( M^{\alpha+1} + M^{s-j+2} \right).
\]
as long as $T \leq 1$. Combining (4.8) with (4.9), we see that
\begin{align}
\|\langle x \rangle^m \Phi(u)\|_{L_x^\infty L_t^\infty} & \\
\leq & \|\langle x \rangle^m u_0\|_{L_x^\infty} \\
+ & CT \left( \|\langle x \rangle^m \partial_x^{2j+1} u_0\|_{L_x^2} + \|\langle x \rangle^m \partial_x^{2j+2} u_0\|_{L_x^2} + \|u_0\|_{H^{2m+2j+2}}(T)\right) \\
+ & CT^{1/2} \left( 1 + \lambda^{\alpha-(s-j+1)} \right) \left( M^{\alpha+1} + M^{s-j+2} \right)
\end{align}
(4.10)
whenever $T \leq 1$. By collecting (4.2), (4.3), (4.7), and (4.10), it is established that
\begin{align}
\|u\|_{X_T} & \leq C_0 (1 + T) \delta + CT^{1/2} \left( 1 + \lambda^{\alpha-(s-j+1)} \right) \left( M^{\alpha+1} + M^{s-j+2} \right) \\
& \leq C_1 \delta + CT^{1/2} \left( \frac{1 + \lambda}{\lambda} \right)^{s-j+1-\alpha} \left( M^{\alpha+1} + M^{s-j+2} \right)
\end{align}
(4.11)
for any $T \leq 1$, where $C_1 = 2C_0$. Therefore, taking $M = 2C_1 \delta$, we have $\|u\|_{X_T} \leq M$ whenever $T = T(\delta, \lambda; \alpha, s, j)$ satisfies
\begin{align}
CT^{1/2} \left( \frac{1 + \lambda}{\lambda} \right)^{s-j+1-\alpha} \left( \delta^\alpha + \delta^{s-j+1} \right) \leq 1.
\end{align}
Note that in view of $\delta \geq \lambda$, $T \leq 1$ holds as long as $T$ satisfies (4.11). Finally, combining (4.8) with (4.9), when $T \leq 1$, we easily have
\begin{align}
\|\langle x \rangle^m (\Phi(u(t)) - u_0)\|_{L_x^\infty} & \\
& \leq CT^{1/2} \delta + CT^{1/2} \left( \frac{1 + \lambda}{\lambda} \right)^{s-j+1-\alpha} \left( M^{\alpha+1} + M^{s-j+2} \right),
\end{align}
which implies
\begin{align}
\sup_{0 \leq t \leq T} \|\langle x \rangle^m (\Phi(u(t)) - u_0)\|_{L_x^\infty} \leq \frac{\lambda}{2}
\end{align}
as long as $T = T(\delta, \lambda; \alpha, s)$ satisfies
\begin{align}
CT^{1/2} \delta + CT^{1/2} \left( \frac{1 + \lambda}{\lambda} \right)^{s-j+1-\alpha} \left( \delta^\alpha + \delta^{s-j+1} \right) \leq \frac{\lambda}{2}
\end{align}
(4.12) since $M = 2C_1 \delta$. Thus taking $T = T(\delta, \lambda; \alpha, s, j)$ satisfying (4.11) and (4.12), $\Phi(u) \in X_{T,M}$ is valid.

Let us show $\Phi$ is a contraction map in $X_{T,M}$. The proof is very similar to the above proof, so we sketch the proof. Arguing as in the proof of (4.2), we deduce from (3.6) that
\begin{align}
\|\partial_x^j (\Phi(u) - \Phi(v))\|_{L_x^\infty L_t^2} & \\
& \leq CT^{1/2} \left( 1 + \lambda^{\alpha-(s-j+2)} \right) \left( M^\alpha + M^{s-j+2} \right) d_{X_T}(u, v)
\end{align}
(4.13)
as long as $T \leq 1$. Also, (3.5) gives us
\begin{align}
\|\Phi(u) - \Phi(v)\|_{L_x^\infty L_t^2} & \leq CT(M^\alpha + M) d_{X_T}(u, v).
\end{align}
(4.14) We will turn to treat
\begin{align}
\|\langle x \rangle^m \partial_x^j (\Phi(u) - \Phi(v))\|_{L_x^\infty L_t^2}
\end{align}
for any $1 \leq \gamma \leq 2j + 2$. It follows from Lemma 2.3 that
\[
\|(x)^m \partial_x^\gamma (\Phi(u) - \Phi(v))\|_{L_T^\infty L_x^2} \\
\leq C (T)^m \|(x)^m \partial_x^\gamma (|u|^\alpha \partial_x^{2j-1} u - |v|^\alpha \partial_x^{2j-1} v)\|_{L_T^1 L_x^2} \\
+ C (T)^m \left( \|\partial_x^\gamma (|u|^\alpha \partial_x^{2j-1} u - |v|^\alpha \partial_x^{2j-1} v)\|_{L_T^1 L_x^2} \\n\qquad + \|\partial_x^{2j+m+\gamma} (|u|^\alpha \partial_x^{2j-1} u - |v|^\alpha \partial_x^{2j-1} v)\|_{L_T^1 L_x^2} \right).
\] (4.15)
As for the first term of R.H.S in (4.15), it follows from (3.7) that
\[
\|(x)^m \partial_x^\gamma (|u|^\alpha \partial_x u - |v|^\alpha \partial_x v)\|_{L_T^1 L_x^2} \\
\leq CT \left( \lambda^{\alpha-2} + \lambda^{\alpha-1} \right) (M + M^2) d_{X_T}(u, v) \\
+ CT \left( \lambda^{\alpha-\gamma} + \lambda^{\alpha-(\gamma+1)} \right) (M^\gamma + M^{\gamma+1}) d_{X_T}(u, v).
\] (4.16)
Applying (3.5) and (3.6) with $q = 2$, together with $H^{s-j+1} \hookrightarrow H^\sigma$ for $\sigma = \gamma$, $2jm + \gamma$, the last term can be estimated as follows:
\[
\|\partial_x^\gamma (|u|^\alpha \partial_x^{2j-1} u - |v|^\alpha \partial_x^{2j-1} v)\|_{L_T^1 L_x^2} \\
+ \|\partial_x^{2j+m+\gamma} (|u|^\alpha \partial_x^{2j-1} u - |v|^\alpha \partial_x^{2j-1} v)\|_{L_T^1 L_x^2} \\
\leq CT^{1/2} \left( 1 + \lambda^{\alpha-(s-j+2)} \right) (M^\alpha + M^{s-j+2}) d_{X_T}(u, v)
\] (4.17)
for any $T \leq 1$. Hence, we conclude from (4.15), (4.16) and (4.17) that
\[
\|(x)^m \partial_x^\gamma (\Phi(u) - \Phi(v))\|_{L_T^\infty L_x^2} \\
\leq CT^{1/2} \left( 1 + \lambda^{\alpha-(s-j+2)} \right) (M^\alpha + M^{s-j+2}) d_{X_T}(u, v)
\] (4.18)
as long as $T \leq 1$ for any $1 \leq \gamma \leq 2j + 2$. Let us next estimate
\[
\|(x)^m (\Phi(u) - \Phi(v))\|_{L_T^\infty L_x^\infty}.
\]
Arguing as in (4.9), by Sobolev embedding and Lemma 2.3, we are led to
\[
\|(x)^m (\Phi(u) - \Phi(v))\|_{L_T^\infty L_x^\infty} \\
\leq CT \langle T \rangle^m \|(x)^m (|u|^\alpha \partial_x^{2j-1} u - |v|^\alpha \partial_x^{2j-1} v)\|_{L_T^1 L_x^2} \\
+ CT \langle T \rangle^m \|(x)^m \partial_x (|u|^\alpha \partial_x^{2j-1} u - |v|^\alpha \partial_x^{2j-1} v)\|_{L_T^1 L_x^2} \\
+ C \langle T \rangle^m \|(x)^m |u|^\alpha \partial_x^{2j-1} u - |v|^\alpha \partial_x^{2j-1} v\|_{L_T^1 L_x^2} \\
+ C \langle T \rangle^m \|(x)^m \partial_x^{2j+m+1} (|u|^\alpha \partial_x^{2j-1} u - |v|^\alpha \partial_x^{2j-1} v)\|_{L_T^1 L_x^2}.
\]
Thanks to (3.8) and (4.1), The first term of R.H.S in the above can be estimated as follows:
\[
\|(x)^m (|u|^\alpha \partial_x^{2j-1} u - |v|^\alpha \partial_x^{2j-1} v)\|_{L_T^1 L_x^2} \\
\leq C T^{\alpha-1} \|(x)^m (u - v)\|_{L_T^\infty L_x^\infty} \|(x)^m \partial_x^{2j-1} u\|_{L_T^\infty L_x^2} \\
+ C T^{\alpha-1} \|(x)^m \partial_x (u - v)\|_{L_T^\infty L_x^\infty} \|(x)^m \partial_x^{2j-1} (u - v)\|_{L_T^\infty L_x^2} \\
\leq C T^{\alpha-1} M d_{X_T}(u, v).
\]
We then see from (3.7) with $\gamma = 1$ from that
\[
\| (x)^m \partial_x \left( |u|^\alpha \partial_x^{2j-1}u - |v|^\alpha \partial_x^{2j-1}v \right) \|_{L_x^2 L_t^2} \\
\leq CT \left( \lambda^{\alpha-1} + \lambda^{\alpha-2} \right) (M + M^2) d_{X_T}(u, v).
\]
As in the proof of (4.17), one reaches to
\[
\| u^\alpha \partial_x^{2j-1}u - |v|^\alpha \partial_x^{2j-1}v \|_{L^1_x L^2_t} \\
+ \| \partial_x^{2jm+1} (u^\alpha \partial_x^{2j-1}u - |v|^\alpha \partial_x^{2j-1}v) \|_{L^1_x L^2_t} \\
\leq CT^{1/2} \left( 1 + \lambda^{\alpha-(s-j+2)} \right) (M^\alpha + M^{s-j+2}) d_{X_T}(u, v)
\]
for any $T \leq 1$. Therefore it is established from these estimates that
\[
\| (x)^m (\Phi(u) - \Phi(v)) \|_{L_x^2 L_t^\infty} \\
\leq CT^{1/2} \left( 1 + \lambda^{\alpha-(s-j+2)} \right) (M^\alpha + M^{s-j+2}) d_{X_T}(u, v)
\]
as long as $T \leq 1$. In conclusion, combining (4.13) with (4.14), (4.18) and (4.19), we see that
\[
d_{X_T}(\Phi(u), \Phi(v)) \leq CT \frac{1}{2} \left( 1 + \lambda^{\alpha-(s-j+2)} \right) (M^\alpha + M^{s-j+2}) d_{X_T}(u, v)
\]
for any $T \leq 1$. Therefore, recalling $M = 2C_1 \delta$, we have
\[
d_{X_T}(\Phi(u), \Phi(v)) \leq \frac{1}{2} d_{X_T}(u, v)
\]
whenever $T = T(\delta, \lambda; \alpha, s, j)$ satisfies
\[
CT \frac{1}{2} \left( \frac{1 + \lambda}{\lambda} \right)^{s-j+2-\alpha} (\delta^\alpha + \delta^{s-j+2}) \leq \frac{1}{2}.
\]
Note that $T \leq 1$ holds when $T$ satisfies (4.20). This implies that $\Phi$ is a contraction map in $X_{T_{1,\delta}}$, that is, (HK) has a unique local solution in $X_{T_{1,\delta}}$. The remainder of the proof is standard, so we omit the detail. \hfill \square

Proof of Corollary 1.5. Set
\[
T_1^\frac{1}{\delta} = \frac{C \lambda}{\delta \left( 1 + (\delta^\alpha + \delta^{s-j+2}) \left( \frac{1+\lambda}{\lambda} \right)^{s-j+2-\alpha} \right)}
\]
for some $C > 0$. Thanks to $\delta^\alpha + \delta^{s-j+1} \leq 2(\delta^\alpha + \delta^{s-j+2})$, since there exists $C \in (0, 1)$ such that $T_1$ satisfies (4.11), (4.12) and (4.20), it follows from Theorem 1.1 that (HK) has a unique solution in $X_{T_{1,\delta}}$. By the definition of $T_{\delta,\lambda}$, we obtain $T_{\delta,\lambda} \geq T_1$. This completes the proof. \hfill \square

Acknowledgments. This work was supported by the Research Institute for Mathematical Sciences, a Joint Usage/Research Center located in Kyoto University. The author is grateful to the referee for reading our manuscript carefully and giving useful suggestions.


References

[1] Thierry Cazenave and Ivan Naumkin, Local existence, global existence, and scattering for the nonlinear Schrödinger equation, Commun. Contemp. Math. 19 (2017), no. 2, 1650038, 20. MR3611666

[2] Zihua Guo, Chulkwang Kwak, and Soonsik Kwon, Rough solutions of the fifth-order KdV equations, J. Funct. Anal. 265 (2013), no. 11, 2791–2829. MR3096990

[3] Carlos E. Kenig and Didier Pilod, Well-posedness for the fifth-order KdV equation in the energy space, Trans. Amer. Math. Soc. 367 (2015), no. 4, 2551–2612. MR3301874

[4] Local well-posedness for the KdV hierarchy at high regularity, Adv. Differential Equations 21 (2016), no. 9-10, 801–836. MR3513119

[5] Carlos E. Kenig, Gustavo Ponce, and Luis Vega, Oscillatory integrals and regularity of dispersive equations, Indiana Univ. Math. J. 40 (1991), no. 1, 33–69. MR1101221

[6] Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle, Comm. Pure Appl. Math. 46 (1993), no. 4, 527–620. MR1211741

[7] Higher-order nonlinear dispersive equations, Proc. Amer. Math. Soc. 122 (1994), no. 1, 157–166. MR1195480

[8] On the hierarchy of the generalized KdV equations, Singular limits of dispersive waves (Lyon, 1991), 1994, pp. 347–356. MR1321214

[9] Peter D. Lax, Integrals of nonlinear evolution equations, Comm. Pure Appl. Math. 21 (1968), 467–490. MR0235310

[10] Felipe Linares, Hayato Miyazaki, and Gustavo Ponce, On a class of solutions to the generalized KdV type equation, Commun. Contemp. Math. 21 (2019), no. 7, 1850056, 21. MR4017781

[11] Felipe Linares and Gustavo Ponce, Introduction to nonlinear dispersive equations, Second, Universitext, Springer, New York, 2015. MR3308874

[12] Felipe Linares, Gustavo Ponce, and Gleison N. Santos, On a class of solutions to the generalized derivative Schrödinger equations, Acta Math. Sin. (Engl. Ser.) 35 (2019), no. 6, 1057–1073. MR3952703

[13] On a class of solutions to the generalized derivative Schrödinger equations II, J. Differential Equations 267 (2019), no. 1, 97–118. MR3944267

[14] Hayato Miyazaki, Lower bound for the lifespan of solutions to the generalized KdV equation with low degree of nonlinearity, Advanced Studies in Pure Mathematics 85 (2020), 303–313.

[15] Gustavo Ponce, Lax pairs and higher order models for water waves, J. Differential Equations 102 (1993), no. 2, 300–381. MR1216734

Teacher Training Courses, Faculty of Education, Kagawa University, Takamatsu, Kagawa 760-8522, Japan

Email address: miyazaki.hayato@kagawa-u.ac.jp