Aurelian Cernea

ON AN EVOLUTION INCLUSION
IN NON-SEPARABLE BANACH SPACES

Abstract. We consider a Cauchy problem for a class of nonconvex evolution inclusions in non-separable Banach spaces under Filippov-type assumptions. We prove the existence of solutions.

Keywords: Lusin measurable multifunctions, selection, mild solution.

Mathematics Subject Classification: 34A60.

1. INTRODUCTION

In this paper we study differential inclusions of the form

\[ x'(t) \in A(t)x(t) + \int_0^t K(t,s)F(s,x(s))ds, \quad x(0) = x_0, \quad (1.1) \]

where \( F : [0,T] \times X \to P(X) \) is a set-valued map, Lipschitzian with respect to the second variable, \( X \) is a Banach space, \( A(t) \) is the infinitesimal generator of a strongly continuous evolution system of a two parameter family \( \{ G(t,\tau), t \geq 0, \tau \geq 0 \} \) of bounded linear operators of \( X \) into \( X, D = \{(t,s) \in [0,T] \times [0,T]; t \geq s\}, K(.,.) : D \to R \) is continuous and \( x_0 \in X \).

The existence and qualitative properties of mild solutions of problem (1.1) have been obtained in [1,2–7,13] etc.. Most of the existence results mentioned above are obtained using fixed point techniques. In [9] it is shown that Filippov’s ideas ([11]) can suitably be adapted in order to prove the existence of solutions to problem (1.1). All these approaches are have proved successful the Banach space \( X \) separable.

De Blasi and Pianigiani ([10]) established the existence of mild solutions for semilinear differential inclusions on an arbitrary, not necessarily separable, Banach space \( X \). Even if Filippov’s ideas are still present, the approach in [10] is fundamental different: it consists in the construction of the measurable selections of the multifunction.
This construction does not use classical selection theorems such as Kuratowski and Ryll-Nardzewski’s ([12]) or Bressan and Colombo’s ([8]).

The aim of this paper is to obtain an existence result for problem (1.1) similar to the one in [10]. We will prove the existence of solutions for problem (1.1) in an arbitrary space $X$ under Filippov-type assumptions on $F$.

The paper is organized as follows: in Section 2 we present the notations, definitions and preliminary results to be used in the sequel, and in Section 3 we prove the main result.

2. PRELIMINARIES

Consider $X$, an arbitrary real Banach space with norm $|.|$ and with the corresponding metric $d(.,.)$. Let $\mathcal{P}(X)$ be the space of all bounded nonempty subsets of $X$ endowed with the Hausdorff pseudometric

$$d_H(A,B) = \max\{d^*(A,B),d^*(B,A)\}, \quad d^*(A,B) = \sup_{a \in A} d(a,B),$$

where $d(x,A) = \inf_{a \in A} |x-a|$, $A \subset X, x \in X$.

Let $\mathcal{L}$ be the $\sigma$-algebra of the (Lebesgue) measurable subsets of $R$ and, for $A \in \mathcal{L}$, let $\mu(A)$ be the Lebesgue measure of $A$.

Let $X$ be a Banach space and $Y$ be a metric space. An open (resp., closed) ball in $Y$ with center $y$ and radius $r$ is denoted by $B_Y(y,r)$ (resp., $\overline{B_Y}(y,r)$). In what follows, $B = B_X(0,1)$.

A multifunction $F : Y \to \mathcal{P}(X)$ with closed bounded nonempty values is said to be $d_H$-continuous at $y_0 \in Y$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any $y \in B_Y(y_0,\delta)$ there is $d_H(F(y), F(y_0)) \leq \varepsilon$. $F$ is called $d_H$-continuous if it is so at each point $y_0 \in Y$.

Let $A \in \mathcal{L}$, with $\mu(A) < \infty$. A multifunction $F : Y \to \mathcal{P}(X)$ with closed bounded nonempty values is said to be Lusin measurable if for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset A$, with $\mu(A \setminus K_\varepsilon) < \varepsilon$ such that $F$ restricted to $K_\varepsilon$ is $d_H$-continuous.

It is clear that if $F,G : A \to \mathcal{P}(X)$ and $f : A \to X$ are Lusin measurable, then so are $F$ restricted to $B$ ($B \subset A$ measurable), $F + G$ and $t \to d(f(t),F(t))$.

Moreover, the uniform limit of a sequence of Lusin measurable multifunctions is Lusin measurable, too.

Let $I$ stand for the interval $[0,T]$, $T > 0$.

In what follows, $\{A(t); t \in I\}$ is the infinitesimal generator of a strongly continuous evolution system $G(t,s)$, $0 \leq s \leq t \leq T$ depending on two parameters is said to be a strongly continuous evolution system if the following conditions hold: $G(s,s) = I$, $G(t,r)G(r,s) = G(t,s)$ for $0 \leq s \leq r \leq t \leq T$ and $(t,s) \to G(t,s)$ is strongly continuous for $0 \leq s \leq t \leq T$, i.e., $\lim_{t \to s, t > s} G(t,s)x = x$ for all $x \in X$. 

Aurelian Cernea
In what follows, we are concerned with the evolution inclusion

\[ x'(t) \in A(t)x(t) + \int_0^t K(t,s)F(s,x(s))ds, \quad x(0) = x_0, \quad (2.1) \]

where \( F : I \times X \to \mathcal{P}(X) \) is a set-valued map, \( X \) is a Banach space, \( A(t) \) is the infinitesimal generator of a strongly continuous evolution system of a two parameter family \( \{G(t,\tau), t \geq 0, \tau \geq 0\} \) of bounded linear operators of \( X \) into \( X \), \( D = \{(t,s) \in I \times I; t \geq s\}, K(,,) : D \to \mathbb{R} \) is continuous and \( x_0 \in X \).

A continuous mapping \( x(,) \in C(I,X) \) is called a **mild solution** of problem (2.1) if there exists a (Bochner) integrable function \( f(,) \in L^1(I,X) \) such that

\[ f(t) \in F(t,x(t)) \quad a.e. (I), \quad (2.2) \]

\[ x(t) = G(t,0)x_0 + \int_0^t G(t,\tau) \int_0^\tau K(\tau,s)f(s)dsd\tau, \quad t \in I. \quad (2.3) \]

In this case, we shall call \((x(,),f(,))\) a **trajectory-selection pair** of (2.1).

We note that condition (2.3) can be rewritten as

\[ x(t) = G(t,0)x_0 + \int_0^t U(t,s)f(s)ds, \quad t \in I, \quad (2.4) \]

where \( U(t,s) = \int_s^t G(t,\tau)K(\tau,s)d\tau. \)

In what follows, we assume the following hypotheses.

**Hypothesis 2.1.**

(i) \( \{A(t); t \in I\} \) is the infinitesimal generator of the strongly continuous evolution system \( G(t,s), 0 \leq s \leq t \leq T \).

(ii) \( F(,,) : I \times X \to \mathcal{P}(X) \) has nonempty closed bounded values and, for any \( x \in X \), \( F(,,x) \) is Lusin measurable on \( I \).

(iii) There exists \( l(,) \in L^1(I,(0,\infty)) \) such that for each \( t \in I \):

\[ d_H(F(t,x_1),F(t,x_2)) \leq l(t)|x_1 - x_2|, \quad \forall x_1, x_2 \in X. \]

(iv) There exists \( q(,) \in L^1(I,(0,\infty)) \) such that for each \( t \in I \):

\[ F(t,0) \subset q(t)B. \]

(v) \( D = \{(t,s) \in I \times I; t \geq s\}, K(,,) : D \to \mathbb{R} \) is continuous.

Set \( n(t) = \int_0^t l(u)du, \quad t \in I, \quad M := \sup_{t,s \in I} |G(t,s)| \) and \( M_0 := \sup_{(t,s) \in D} |K(t,s)| \)

and note that \( |\hat{U}(t,s)| \leq MM_0(t - s) \leq MM_0T. \)

The technical results summarized in the following lemma are essential in the proof of our result. For the proof, we refer the reader to [10].
Aurelian Cernea

Lemma 2.2 ([10] i)). Let \( F_i : I \to \mathcal{P}(X), \ i=1,2, \) be two Lusin measurable multifunctions and let \( \varepsilon_i > 0, \ i=1,2 \) be such that
\[
H(t) := (F_1(t) + \varepsilon_1 B) \cap (F_2(t) + \varepsilon_2 B) \neq \emptyset, \quad \forall t \in I.
\]
Then the multifunction \( H : I \to \mathcal{P}(X) \) has a Lusin measurable selection \( h : I \to X. \)

ii) Assume that Hypothesis 2.1 is satisfied. Then for any continuous \( x(.) : I \to X, \)
\( u(.) : I \to X \) measurable and any \( \varepsilon > 0 \) there is:
\begin{enumerate}[(a)]
  \item the multifunction \( t \to F(t,x(t)) \) is Lusin measurable on \( I, \)
  \item the multifunction \( G : I \to \mathcal{P}(X) \) defined by
    \[
    G(t) := (F(t,x(t)) + \varepsilon B) \cap B_X(u(t), d(u(t), F(t,x(t))) + \varepsilon)
    \]
    has a Lusin measurable selection \( g : I \to X. \)
\end{enumerate}

3. THE MAIN RESULT

We are now ready to prove our main result.

Theorem 3.1. We assume that Hypothesis 2.1 is satisfied. Then, for every \( x_0 \in X, \)
Cauchy problem (1.1) has a mild solution \( x(.) \in C(I,X). \)

Proof. Let us first note that if \( z(.) : I \to X \) is continuous, then every Lusin measurable selection \( u : I \to X \) of the multifunction \( t \to F(t,z(t)) + B \) is Bochner integrable on \( I. \) More precisely, for any \( t \in I, \) there holds
\[
|u(t)| \leq d_H(F(t,z(t)) + B,0) \leq d_H(F(t,z(t)), F(t,0)) + d_H(F(t,0),0) + 1 \leq \ell(t)|z(t)| + q(t) + 1.
\]
Let \( 0 < \varepsilon < 1, \quad \varepsilon_n = \frac{\varepsilon}{\sqrt{n+1}}. \)
Consider \( f_0(.) : I \to X, \) an arbitrary Lusin measurable, Bochner integrable function, and define
\[
x_0(t) = G(t,0)x_0 + \int_0^t U(t,s)f_0(s)ds, \quad t \in I.
\]
Since \( x_0(.) \) is continuous, by Lemma 2.2 ii) there exists a Lusin measurable function \( f_1(.) : I \to X \) which, for \( t \in I, \) satisfies
\[
f_1(t) \in (F(t,x_0(t)) + \varepsilon_1 B) \cap B(f_0(t), d(f_0(t), F(t,x_0(t))) + \varepsilon_1).
\]
Obviously, \( f_1(.) \) is Bochner integrable on \( I. \) Define \( x_1(.) : I \to X \) by
\[
x_1(t) = G(t,0)x_0 + \int_0^t U(t,s)f_1(s)ds, \quad t \in I.
\]
On an evolution inclusion in non-separable Banach spaces

By induction, we construct a sequence \( x_n : I \rightarrow X, n \geq 2 \) given by

\[
x_n(t) = G(t, 0)x_0 + \int_0^t U(t, s)f_n(s)ds, \quad t \in I,
\]

where \( f_n : I \rightarrow X \) is a Lasin measurable function which, for \( t \in I \), satisfies:

\[
f_n(t) \in (F(t, x_{n-1}(t)) + \varepsilon_n B) \cap B(f_{n-1}(t), d(f_{n-1}(t), F(t, x_{n-1}(t))) + \varepsilon_n).
\]  \( (3.2) \)

At the same time, as we saw at the beginning of the proof, \( f_n(.) \) is also Bochner integrable.

From (3.2), for \( n \geq 2 \) and \( t \in I \), we obtain

\[
|f_n(t) - f_{n-1}(t)| \leq d(f_{n-1}(t), F(t, x_{n-1}(t))) + \varepsilon_n \leq d(f_{n-1}(t), F(t, x_{n-2}(t))) + +d_B(F(t, x_{n-2}(t)), F(t, x_{n-1}(t))) + \varepsilon_n \leq \varepsilon_n + l(t)|x_{n-1}(t) - x_{n-2}(t)| + \varepsilon_n.
\]  \( (3.3) \)

Since \( \varepsilon_n + \varepsilon_n < \varepsilon_n - 2 \), for \( n \geq 2 \), we deduce that

\[
|f_n(t) - f_{n-1}(t)| \leq \varepsilon_n - 2 + l(t)|x_{n-1}(t) - x_{n-2}(t)|,
\]  \( (3.3) \)

Denote \( p_0(t) := d(f_0(t), F(t, x_0(t))) \), \( t \in I \). We next prove by recurrence, that for \( n \geq 2 \) and \( t \in I \):

\[
|x_n(t) - x_{n-1}(t)| \leq \sum_{k=0}^{n-2} \int_0^t \varepsilon_{n-2-k} (MM_0T)^{k+1}(n(t) - n(u))^k k! du + + \varepsilon_0 \int_0^t (MM_0T)^n(n(t) - n(u))^{n-1} (n-1)! du + \int_0^t (MM_0T)^n(n(t) - n(u))^{n-1} (n-1)! p_0(u)du.
\]  \( (4.4) \)

We start with \( n = 2 \). In view of (3.1), (3.2) and (3.3), for \( t \in I \), there is

\[
|x_2(t) - x_1(t)| \leq \int_0^t |U(t, s)| \cdot |f_2(s) - f_1(s)| ds \leq \int_0^t MM_0T|\varepsilon_1 + l(s)|x_1(s) - x_0(s)||ds \leq
\]

\[
\leq \varepsilon_0 MM_0Tt + \int_0^t [MM_0Tl(s)] \int_0^s |U(s, r)| \cdot |f_1(r) - f_0(r)| dr ds \leq
\]

\[
\leq \varepsilon_0 MM_0Tt + \int_0^t [(MM_0T)^2l(s)] \int_0^s (p_0(u) + \varepsilon_1) du ds \leq
\]

\[
\leq \varepsilon_0 MM_0Tt + \int_0^t [(MM_0T)^2(p_0(u) + \varepsilon_1)] \int_u^t l(s) ds du =
\]

\[
= \varepsilon_0 MM_0Tt + \int_0^t (MM_0T)^2(n(t) - n(s)) [p_0(s) + \varepsilon_0] ds,
\]  \( (4.4) \)

i.e, (4.4) is verified for \( n = 2 \).
Using again (3.3) and (3.4), we conclude:

\[
|x_{n+1}(t) - x_n(t)| \leq \int_0^t |U(t,s)| \cdot |f_{n+1}(s) - f_n(s)| ds \leq \\
\leq \int_0^t M M_0 T |\varepsilon_{n-1} + I(s)| x_n(s) - x_{n-1}(s)| ds \leq \varepsilon_{n-1} M M_0 T t + \\
+ \int_0^t l(s) \sum_{k=0}^{n-2} \varepsilon_{n-2-k} \left( \frac{M M_0 T)^{k+2}(n(s) - n(u))^k}{k!} \right) du + \\
+ \int_0^s \frac{(M M_0 T)^{n+1}(n(s) - n(u))^{n-1}}{(n-1)!} (p_0(u) + \varepsilon_0) du ds = \\
= \varepsilon_{n-1} M M_0 T t + \sum_{k=0}^{n-2} \varepsilon_{n-2-k} \left( \int_0^t \frac{(M M_0 T)^{k+2}(n(s) - n(u))^k}{k!} l(s) du ds \right) \\
+ \int_0^t l(s) \left( \int_0^s \frac{(M M_0 T)^{n+1}(n(s) - n(u))^{n-1}}{(n-1)!} l(s) [p_0(u) + \varepsilon_0] du ds \right) = \\
= \varepsilon_{n-1} M M_0 T t + \sum_{k=0}^{n-2} \varepsilon_{n-2-k} \left( \int_0^t \frac{(M M_0 T)^{k+2}(n(s) - n(u))^k}{k!} l(s) ds du \right) \\
+ \int_0^t \left( \int_0^t \frac{(M M_0 T)^{n+1}(n(s) - n(u))^{n-1}}{(n-1)!} l(s) ds du \right) [p_0(u) + \varepsilon_0] du = \\
= \varepsilon_{n-1} M M_0 T t + \sum_{k=0}^{n-2} \varepsilon_{n-2-k} \left( \int_0^t \frac{(M M_0 T)^{k+2}(n(s) - n(u))^k}{(k+1)!} du \right) \\
+ \int_0^t \frac{(M M_0 T)^{n+1}(n(s) - n(u))^n}{n!} [p_0(u) + \varepsilon_0] du = \\
= \sum_{k=0}^{n-1} \varepsilon_{n-1-k} \left( \int_0^t \frac{(M M_0 T)^{k+1}(n(s) - n(u))^k}{k!} du \right) \\
+ \int_0^t \frac{(M M_0 T)^{n+1}(n(s) - n(u))^n}{n!} [p_0(u) + \varepsilon_0] du,
\]

and statement (3.4) it is true for \(n+1\).
From (3.4) it follows that for \( n \geq 2 \) and \( t \in I \):

\[
|x_n(t) - x_{n-1}(t)| \leq a_n,
\]

where

\[
a_n = \sum_{k=0}^{n-2} \varepsilon_{n-2-k} \frac{(MM_0T)^{k+1}n(T)^k}{k!} + \frac{(MM_0T)^n n(T)^{n-1}}{(n-1)!} \int_0^1 p_0(u)du + \varepsilon_0,
\]

Obviously, the series whose \( n \)-th term is \( a_n \) converges. So, from (3.5) we infer that \( x_n(.) \) converges to a continuous function, \( x(.) : I \to X \), uniformly on \( I \).

On the other hand, in view of (3.3) there is

\[
|f_n(t) - f_{n-1}(t)| \leq \varepsilon_{n-2} + l(t)a_{n-1}, \quad t \in I, n \geq 3
\]

which implies that the sequence \( f_n(.) \) converges to a Lusin measurable function \( f(.) : I \to X \).

Since \( x_n(.) \) is bounded and

\[
|f_n(t)| \leq l(t)|x_{n-1}(t)| + q(t) + 1,
\]

we infer that \( f(.) \) is also Bochner integrable.

Passing with \( n \to \infty \) in (3.1) and using the Lebesgue dominated convergence theorem, we obtain

\[
x(t) = G(t, 0)x_0 + \int_0^t U(t, s)f(s)ds, \quad t \in I.
\]

On the other hand, from (3.2) we get

\[
f_n(t) \in F(t, x_n(t)) + \varepsilon_n B, \quad t \in I, n \geq 1
\]

and letting \( n \to \infty \) we obtain

\[
f(t) \in F(t, x(t)), \quad t \in I,
\]

which completes the proof.

**Remark 3.2.** If \( A(t) \equiv A \) and \( A \) is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators \( \{G(t); t \geq 0\} \) from \( X \) to \( X \), then problem (1.1) reduces to the problem

\[
x'(t) \in Ax(t) + \int_0^t K(t, s)F(s, x(s))ds, \quad x(0) = x_0,
\]

well known ([1,2–7,13] etc.) as an integrodifferential inclusion.

Obviously, a result similar to that of Theorem 3.1 may be obtained for problem (3.6).
REFERENCES

[1] A. Anguraj, C. Murugesan, *Continuous selections of set of mild solutions of evolution inclusions*, Electronic J. Diff. Equations **2005** (2005) 21, 1–7.

[2] K. Balachandran, P. Balasubramanian, J.P. Dauer, *Controllability of nonlinear integrodifferential systems in Banach spaces*, J. Optim. Theory Appl. **74** (1995), 83–91.

[3] M. Benchohra, S.K. Ntouyas, *Existence results for neutral functional differential and integrodifferential inclusions in Banach spaces*, Electronic J. Diff. Equations **2000** (2000) 20, 1–15.

[4] M. Benchohra, S.K. Ntouyas, *Nonlocal Cauchy problems for neutral functional differential and integrodifferential inclusions in Banach spaces*, J. Math. Anal. Appl. **258** (2001), 573–590.

[5] M. Benchohra, S.K. Ntouyas, *Controllability of infinite time horizon for functional differential and integrodifferential inclusions in Banach spaces*, Commun. Applied Nonlin. Anal. **8** (2001), 63–78.

[6] M. Benchohra, S.K. Ntouyas, *Existence results for functional differential and integrodifferential inclusions in Banach spaces*, Indian J. Pure Applied Math. **32** (2001), 665–675.

[7] M. Benchohra, S.K. Ntouyas, *Controllability for functional and integrodifferential inclusions in Banach spaces*, J. Optim. Theory Appl. **113** (2002), 449–472.

[8] A. Bressan, G. Colombo, *Extensions and selections of maps with decomposable values*, Studia Math. **90** (1988), 69–86.

[9] A. Cernea, *On the mild solutions of a class of evolution inclusions*, Int. J. Evolution Equations **3** (2008), 157–167.

[10] F.S. De Blasi, G. Pianigiani, *Evolution inclusions in non separable Banach spaces*, Comment. Math. Univ. Carolinae **40** (1999), 227–250.

[11] A.F. Filippov, *Classical solutions of differential equations with multivalued right-hand side*, SIAM J. Control Optim. **5** (1967), 609–621.

[12] K. Kuratowski, C. Ryll-Nardzewski, *A general theorem on selectors*, Bull. Acad. Pol. Sci. Math. Astron. Phys. **13** (1965), 397–403.

[13] B. Liu, *Controllability of neutral functional differential and integrodifferential inclusions with infinite delay*, J. Optim. Theory Appl. **123** (2004), 573–593.

Aurelian Cernea
acernea@fmi.unibuc.ro

University of Bucharest
Faculty of Mathematics and Computer Science
Academiei 14, 010014 Buharest, Romania

Received: October 1, 2008.
Accepted: April 8, 2009.