Estimating the number of negative eigenvalues of a relativistic Hamiltonian with regular magnetic field

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Abstract
We prove the analog of the Cwikel-Lieb-Rosenblum estimation for the number of negative eigenvalues of a relativistic Hamiltonian with magnetic field \( B \in C^\infty_\text{pol}(\mathbb{R}^d) \) and an electric potential \( V \in L^1_{\text{loc}}(\mathbb{R}^d) \), \( V_- \in L^d(\mathbb{R}^d) \cap L^{d/2}(\mathbb{R}^d) \). Compared to the nonrelativistic case, this estimation involves both norms of \( V_- \) in \( L^{d/2}(\mathbb{R}^d) \) and in \( L^d(\mathbb{R}^d) \). A direct consequence is a Lieb-Thirring inequality for the sum of powers of the absolute values of the negative eigenvalues.

1 Introduction
For the Schrödinger operator \(-\Delta + V\) on \( L^2(\mathbb{R}^d) \) \((d \geq 3)\), one has the well-known CLR (Cwikel-Lieb-Rosenblum) estimation for \( N(V) \), the number of negative eigenvalues:

\[
N(V) \leq c(d) \int_{\mathbb{R}^d} dx |V_-(x)|^{d/2}.
\]

(1.1)

\( V \) is the multiplication operator with the function \( V \in L^1_{\text{loc}}(\mathbb{R}^d) \) and \( V_- := (|V| - V)/2 \in L^{d/2}(\mathbb{R}^d) \); the constant \( c(d) > 0 \) only depends on the dimension \( d \geq 3 \) (see [RS], Th. XII.12).

There exist at least four different proofs of this inequality. Rosenblum [R] uses "piece-wise polynomial approximation in Sobolev spaces". Lieb [L] relies on the Feynman-Kac formula. Cwikel [C] uses ideas from interpolation theory. Finally, Li and Yau [LY] make a heat kernel analysis.

The inequality (1.1) has been extended in [AHS] and [S1] to the case of operators with magnetic fields \((-i\nabla - A)^2 + V\), where the components of the vector potential \( A = (A_1, \ldots, A_d) \) belong to \( L^2_{\text{loc}}(\mathbb{R}^d) \). The basic ingredient of the proof is the Feynman-Kac-Itô formula. Melgaard and Rosenblum [MR]
generalizes this result (by a different method) to a class of differential operators of second order with variable coefficients. The idea for treating the relativistic Hamiltonian (without a magnetic field), by replacing Brownian motion with a Lévy process, appears in [D] and we follow it in our work giving all the technical details. Some similar results but for a different Hamiltonian and with different techniques have been obtained recently in [FLS].

Our aim in this paper is to obtain an estimation of the type (1.1) for an operator that is a good candidate for a relativistic Hamiltonian with magnetic field (for scalar particles); it is gauge covariant and obtained through a quantization procedure from the classical candidate. We shall make use of a "magnetic pseudodifferential calculus" that has been introduced and developed in some previous papers [M], [MP1], [KO1], [KO2], [MP2], [MP4], [IMP].

Let us denote by
\[ C^\infty_{\text{pol}}(\mathbb{R}^d) \]
the family of functions \( f \in C^\infty(\mathbb{R}^d) \) for which all the derivatives \( \partial^\alpha f \), \( \alpha \in \mathbb{N}^d \) have polynomial growth.

Let \( B \) be a magnetic field (a 2-form) with components \( B_{jk} \in C^\infty_{\text{pol}}(\mathbb{R}^d) \). It is known that it can be expressed as the differential \( B = dA \) of a vector potential \( (a 1\text{-form}) \)
\[ A = (A_1, \ldots, A_d) \]
with \( A_j \in C^\infty_{\text{pol}}(\mathbb{R}^d) \), \( j = 1, \ldots, d \); an example is the transversal gauge:
\[ A_j(x) = -\sum_{k=1}^n \int_0^1 ds B_{jk}(sx)\dot{s}x_k. \]

We denote by
\[ \Gamma^A(x,y) := \int_0^1 ds A((1-s)x+sy) = \int_{[x,y]} A, \quad x,y \in \mathbb{R}^d. \]
\[ (1.2) \]
the circulation of \( A \) along the segment \( [x,y] \), \( x,y \in \mathbb{R}^d \). If \( a \) is a symbol on \( \mathbb{R}^d \), one defines by an oscillatory integral the linear continuous operator \( \mathcal{O}p^A(a) : S(\mathbb{R}^d) \to S^*(\mathbb{R}^d) \) by
\[ \left[ \mathcal{O}p^A(a) \right](x) := (2\pi)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} dyd\xi e^{i(x-y)\cdot\xi} e^{-i\int_{[x,y]} A} a \left( \frac{x+y}{2}, \xi \right) u(y), \]
\[ (1.3) \]
The correspondence \( a \mapsto \mathcal{O}p^A(a) \) is meant to be a quantization and could be regarded as a functional calculus \( \mathcal{O}p^A(a) = a(Q, \Pi^A) \) for the family of non-commuting operators \( (Q_1, \ldots, Q_d; \Pi^A_1, \ldots, \Pi^A_d) \), where \( Q \) is the position operator, \( \Pi^A := D - A(Q) \) is the magnetic momentum, with \( D := -i\nabla \).

If \( a \) belongs to the Schwartz space \( S(\mathbb{R}^{2d}) \), then \( \mathcal{O}p^A(a) \) acts continuously in the spaces \( S(\mathbb{R}^d) \) and \( S^*(\mathbb{R}^d) \), respectively. It enjoys the important physical property of being gauge covariant: if \( \varphi \in C^\infty_{\text{pol}}(\mathbb{R}^d) \) is a real function, \( A \) and \( A' := A + d\varphi \) define the same magnetic field and one prove easily that \( \mathcal{O}p^{A'}(a) = e^{i\varphi} \mathcal{O}p^A(a)e^{-i\varphi} \). The property is not shared by the quantization \( a \mapsto \mathcal{O}p_A(a) := \mathcal{O}p(a \circ \nu_A) \), where \( \mathcal{O}p \) is the usual Weyl quantization and \( \nu_A : \mathbb{R}^d \to \mathbb{R}^d \)
\[ \nu_A(x,\xi) := (x,\xi - A(a)) \]
is an implementation of "the minimal coupling".
We mention that in the references quoted above, a symbolic calculus is developed for the magnetic pseudodifferential operators (1.3). In particular, a symbol composition \((a, b) \mapsto a^\sharp b\) is defined and studied, verifying \(\mathcal{D} p^A(a)\mathcal{D} p^A(b) = \mathcal{D} p^A(a^\sharp b)\). It depends only on the magnetic field \(B\), no choice of a gauge being needed. The formalism has a \(C^*\)-algebraic interpretation in terms of twisted crossed products, cf. [MP1], [MP3], [MPR1] and it has been used in [MPR2] for the spectral theory of quantum Hamiltonians with anisotropic potentials and magnetic fields.

We shall denote by \(H_A\) the unbounded operator in \(L^2(\mathbb{R}^d)\) defined on \(C^\infty_0(\mathbb{R}^d)\) by \(H_Au := \mathcal{D} p^A(h)u\), with \(h(x, \xi) \equiv h(\xi) := <\xi > - 1 = (1 + |\xi|^2)^{1/2} - 1\). One can express it as

\[
(H_Au)(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} dy \, d\xi \, e^{i(x-y)\cdot\xi} h(\xi - \Gamma^A(x, y)) \, u(y). \tag{1.4}
\]

\(H_A\) is a symmetric operator and, as seen below, essentially self-adjoint on \(C^\infty_0(\mathbb{R}^d)\). Also denoting its closure by \(H_A\), we will have \(H_A \geq 0\).

Ichinose and Tamura [IT1], [IT2], using the quantization \(a \mapsto (Op)_A(a)\), study another relativistic Hamiltonian with magnetic field defined by

\[
(H_A^u)(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} dy \, d\xi \, e^{i(x-y)\cdot\xi} h(\xi - A(x, y)) \, u(y), \tag{1.5}
\]

for which they prove many interesting properties. Unfortunately, \(H_A^u\) is not gauge covariant (cf. [IMP]). Many of the properties of \(H_A^u\) also hold for \(H_A\) (by replacing \(A \left(\frac{x+y}{2}\right)\) with \(\Gamma^A(x, y)\) in the statements and proofs) and this will be used in the sequel.

Aside the magnetic field \(B = dA\), we shall also consider an electric potential \(V \in L^1_{\text{loc}}(\mathbb{R}^d)\), real function expressed as \(V = V_+ - V_-\), \(V_\pm \geq 0\), such that \(V_- \in L^{d+k}(\mathbb{R}^d) \cap L^{d/2+k}(\mathbb{R}^d)\) for some \(k \geq 0\). We are interested in the operator \(H(A, V) := H_A + V\); it will be shown that it is well-defined in form sense as a self-adjoint operator in \(L^2(\mathbb{R}^d)\), with essential spectrum included into the positive real axis. Taking advantage of gauge covariance, we denote by \(N(B, V)\) the number of strictly negative eigenvalues of \(H(A, V)\) (multiplicity counted); it only depends on the potential \(V\) and the magnetic field \(B\).

The main result of the article is

**Theorem 1.1.** Let \(B = dA\) be a magnetic field with \(B_{jk} \in C^\infty_0(\mathbb{R}^d)\), \(A_j \in C^\infty_0(\mathbb{R}^d)\) and let \(V = V_+ - V_- \in L^1_{\text{loc}}(\mathbb{R}^d)\) be a real function with \(V_\pm \geq 0\) and \(V_- \in L^d(\mathbb{R}^d) \cap L^{d/2}(\mathbb{R}^d)\). Then there exists a constant \(C_d\), only depending on the dimension \(d \geq 3\), such that

\[
N(B, V) \leq C_d \left( \int_{\mathbb{R}^d} dx \, V_-(x)^d + \int_{\mathbb{R}^d} dx \, V_-(x)^{d/2} \right). \tag{1.6}
\]

A standard consequence is the next Lieb-Thirring-type estimation:
Corollary 1.2. We assume that the components of $B$ belong to $C^\infty_c(\mathbb{R}^d)$ and that $V = V_+ - V_- \in L^1_{loc}(\mathbb{R}^d)$ is a real function with $V_+ \geq 0$ and $V_- \in L^{d+k}(\mathbb{R}^d) \cap L^{d/2+k}(\mathbb{R}^d)$, $k > 0$. We denote by $\lambda_1 \leq \lambda_2 \leq \ldots$ the strictly negative eigenvalues of $H(A, V)$ (with multiplicity). For any $d \geq 2$ there exists a constant $C_d(k)$ such that

$$
\sum_j |\lambda_j|^k \leq C_d(k) \left( \int_{\mathbb{R}^d} dx V_+(x)^{d+k} + \int_{\mathbb{R}^d} dx V_-(x)^{d/2+k} \right). \quad (1.7)
$$

Sections 2, 3, 4 will contain essentially known facts (usually presented without proofs), needed for checking Theorem 1.1. So, in Section 2 we introduce the Feller semigroup ([IT2], [Ic2], [J]) associated to the operator $H_0 := \langle D \rangle - 1$. In the third section we define properly the operator $H(A, V)$ and study its basic properties. In Section 4 we recall some probabilistic results, as the Markov process associated to the semigroup defined by $H_0$ ([IW], [DvC], [J]) and the Feynman-Kac-Itô formula adapted to a Lévy process ([IT2]).

In Section 5 we prove Theorem 1.1 for $B = 0$, using some of Lieb’s ideas for the non-relativistic case (see [S1]) in the setting proposed in [D]. The last section contains the proof of Theorem 1.1 with magnetic field as well as Corollary 1.2. The main ingredient is the Feynman-Kac-Itô formula.

2 The Feller semigroup.

We consider the following symbol (interpreted as a classical relativistic Hamiltonian for $m = 1, c = 1$) $h: \mathbb{R}^d \to \mathbb{R}_{+}$ defined by $h(\xi) := \langle \xi \rangle + 1 \equiv \sqrt{1 + |\xi|^2} - 1$. Let us observe (as in [Ic2]) that it defines a conditional negative definite function (see [RS]) and thus has a Lévy-Khinchin decomposition (see Appendix 2 to Section XIII of [RS]). Computing $(\nabla h)(\xi)$ and $(\Delta h)(\xi)$ and using the general Lévy-Khinchin decomposition (see for example [RS]), one obtains that there exists a Lévy measure $n(dy)$, i.e. a non-negative, $\sigma$-finite measure on $\mathbb{R}^d$, for which $\min\{1, |y|^2\}$ is integrable on $\mathbb{R}^d$, such that

$$
h(\xi) = -\int_{\mathbb{R}^d} n(dy) \left\{ e^{iw \cdot \xi} - 1 - i(y \cdot \xi)I_{\{|x|<1\}}(y) \right\}, \quad (2.1)
$$

where $I_{\{|x|<1\}}$ is the characteristic function of the open unit ball in $\mathbb{R}^d$. One has the following explicit formula (see [Ic2]):

$$
n(dy) = 2(2\pi)^{-(d+1)/2}|y|^{-(d+1)/2}\kappa_{(d+1)/2}(|y|) dy, \quad (2.2)
$$

with $\kappa_{\nu}$ the modified Bessel function of third type and order $\nu$. We recall the following asymptotic behaviour of these functions:

$$
0 < \kappa_{\nu}(r) \leq C \max(r^{-\nu}, r^{-1/2})e^{-r}, \quad \forall r > 0, \quad \forall \nu > 0. \quad (2.3)
$$

We shall denote by $\mathcal{H}^s(\mathbb{R}^d)$ the usual Sobolev spaces of order $s \in \mathbb{R}$ on $\mathbb{R}^d$ and by $H_0$ the pseudodifferential operator $h(D) \equiv \mathcal{D}p(h)$ considered either as a continuous operator on $S(\mathbb{R}^d)$ and on $S^*(\mathbb{R}^d)$ or as a self-adjoint operator in
$L^2(\mathbb{R}^d)$ with domain $H^1(\mathbb{R}^d)$. The semigroup generated by $H_0$ is explicitly given by the convolution with the following function (for $t > 0$ and $x \in \mathbb{R}^d$):

$$
\hat{\varphi}_t(x) := (2\pi)^{-d} \frac{t}{\sqrt{|x|^2 + t^2}} \int_{\mathbb{R}^d} d\xi e^{(t-\sqrt{|\xi|^2+t^2})(|\xi|^2+1)} = 2^{-(d-1)/2} \pi^{-(d+1)/2} t e^{t(|x|^2+t^2)} K_{(d+1)/2}(\sqrt{|x|^2+t^2})
$$

(2.4)

(see [IT2], [CMS]). We have

$$
\hat{\varphi}_t(x) > 0 \quad \text{and} \quad \int_{\mathbb{R}^d} dx \hat{\varphi}_t(x) = 1.
$$

(2.5)

From (2.3) one easily can deduce the following estimation

$$
\exists C > 0 \quad \text{such that} \quad \hat{\varphi}_t(0) \leq C t^{-d} (1 + t^{d/2}), \quad \forall t > 0.
$$

(2.6)

Let us set

$$
C_\infty(\mathbb{R}^d) := \left\{ f \in C(\mathbb{R}^d) \mid \lim_{|x| \to \infty} f(x) = 0 \right\}
$$

(2.7)

and endow it with the Banach norm $\|f\|_\infty := \sup_{x \in \mathbb{R}^d} |f(x)|$. Using the above properties of the function $\hat{\varphi}_t$ we can extend $e^{-tH_0}$ to a well-defined bounded operator $P(t)$ acting in $C_\infty(\mathbb{R}^d)$.

**Remark 2.1.** One can easily verify that $\{P(t)\}_{t \geq 0}$ is a Feller semigroup, i.e.:

1. $P(t)$ is a contraction: $\|P(t)f\|_\infty \leq \|f\|_\infty$, $\forall f \in C_\infty(\mathbb{R}^d)$;
2. $\{P(t)\}_{t \geq 0}$ is a semigroup: $P(t+s) = P(t)P(s)$;
3. $P(t)$ preserves positivity: $P(t)f \geq 0$ for any $f \geq 0$ in $C_\infty(\mathbb{R}^d)$;
4. We have $\lim_{t \downarrow 0} \|P(t)f - f\|_\infty = 0$, $\forall f \in C_\infty(\mathbb{R}^d)$.

### 3 The perturbed Hamiltonian.

Suppose given a magnetic field of class $C^\infty(\mathbb{R}^d)_{\text{pol}}$ and let us choose a potential vector $A$, such that $B = dA$, with components also of class $C^\infty(\mathbb{R}^d)_{\text{pol}}$ (this is always possible, as said before). We shall denote by $H_A$ the operator $\mathcal{D}p^A(h)$, considered either as a continuous operator on $S(\mathbb{R}^d)$ and on $S^*(\mathbb{R}^d)$ (by duality) or as an unbounded operator on $L^2(\mathbb{R}^d)$ with domain $C_0^\infty(\mathbb{R}^d)$.

Using the Fourier transform one easily proves that for $u \in C_0^\infty(\mathbb{R}^d)$:

$$
[H_0 u](x) = -\int_{\mathbb{R}^d} n(dy) \left[ u(x+y) - u(x) - I_{|y|<1}(y) (y \cdot \partial_x u)(x) \right].
$$

(3.1)

5
Recalling the definition of $\mathbf{Op}^A(h)$, we remark that

$$[H_A u](x) = \left[ \mathbf{Op}^A(h) u \right](x) = \left[ \mathbf{Op}(h) \left( e^{i(x-x^-) \Gamma^A(x^-)} u \right) \right](x) =$$

$$= \left[ H_0 \left( e^{i(x-x^-) \Gamma^A(x^-)} u \right) \right](x).$$

Combining the above two equations one gets easily

$$[H_A u](x) = - \int_{\mathbb{R}^d} n(dy) \left[ e^{-iy \cdot \Gamma^A(x,x+y)} u(x+y) - u(x) - I_{|z|<1}(y) \left( y \cdot (\partial_x - iA(x))u \right)(x) \right].$$

Repeating the arguments in [Ic2] with $\Gamma^A(x,x+y)$ replacing $A((x+y)/2)$ one proves the following results similar to those in [Ic2].

**Proposition 3.1.** Considered as unbounded operator in $L^2(\mathbb{R}^d)$, $H_A$ is essential self-adjoint on $C_0^\infty(\mathbb{R}^d)$. Its closure, also denoted by $H_A$, is a positive operator.

**Proposition 3.2.** For any $u \in L^2(\mathbb{R}^d)$ such that $H_A u \in L^1_{\text{loc}}(\mathbb{R}^d)$

$$\Re \left[ \langle \text{sign} u \rangle H_A u \right] \geq H_0 |u|.$$ 

Using the method in [S2] we can prove the following result.

**Proposition 3.3.** For any $u \in L^2(\mathbb{R}^d)$ we have:

1. for any $\lambda > 0$ and for any $r > 0$

$$\left| (H_A + \lambda)^{-r} u \right| \leq (H_0 + \lambda)^{-r} |u|; \quad (3.4)$$

2. for any $t \geq 0$

$$|e^{-t H_A} u| \leq e^{-t H_0} |u|. \quad (3.5)$$

We associate to $H_A$ its sesquilinear form

$$\mathcal{D}(h_A) = \mathcal{D}(H_A^{1/2}),$$

$$h_A(u,v) := \langle H_A^{1/2} u, H_A^{1/2} v \rangle, \quad \forall (u,v) \in \mathcal{D}(h_A)^2. \quad (3.6)$$

Consider now a function $V \in L^1_{\text{loc}}(\mathbb{R}^d)$, $V \geq 0$ and associate to it the sesquilinear form

$$\mathcal{D}(q_V) := \{ u \in L^2(\mathbb{R}^d) \mid \sqrt{V} u \in L^2(\mathbb{R}^d) \},$$

$$q_V(u,v) := \int_{\mathbb{R}^d} dx \sqrt{V(x)} u(x) \overline{v(x)}, \quad \forall (u,v) \in \mathcal{D}(q_V)^2. \quad (3.7)$$

Both these sesquilinear forms are symmetric, closed and positive. We shall abbreviate $h_A(u) \equiv h_A(u,u)$ and $q_V(u) \equiv q_V(u,u).$
Proposition 3.4. Let \( V : \mathbb{R}^d \to \mathbb{R} \) be a measurable function that can be decomposed as \( V = V_+ - V_- \) with \( V_\pm \geq 0 \) and \( V_\pm \in L_{\text{loc}}^1(\mathbb{R}^d) \). Moreover let us suppose that the sesquilinear form \( q_{V_-} \) is small with respect to \( h_0 \) (i.e. it is \( h_0 \)-relatively bounded with bound strictly less than 1). Then the sesquilinear form \( h_A + q_{V_-} - q_{V_+} \), that is well defined on \( \mathcal{D}(h_A) \cap \mathcal{D}(q_{V_-}) \), is symmetric, closed and bounded from below, defining thus an inferior semibounded self-adjoint operator \( H(A;V) \equiv H := H_A + V \) (sum in sense of forms).

Proof. The sesquilinear form \( h_A + q_{V_+} \) (defined on the intersection of the form domains) is clearly positive, symmetric and closed. We shall prove now that the conclusion of the proposition follows by standard arguments.

Let us denote by \( H_+ := H_A + V_+ \) the unique positive self-adjoint operator associated to the sesquilinear form \( h_A + q_{V_+} \) by the representation theorem 2.6 in §VI.2 of [K]. As \( V_+ \in L_{\text{loc}}^1(\mathbb{R}^d) \), we have \( C_0^\infty(\mathbb{R}^d) \subset \mathcal{D}(h_A) \cap \mathcal{D}(q_{V_+}) \) and thus we can use the form version of the Kato-Trotter formula from [KM]:

\[
e^{-tH_+} = s - \lim_{n \to -\infty} \left( e^{-(t/n)H_A} e^{-(t/n)V_+} \right)^n, \quad \forall t \geq 0. \tag{3.8}
\]

Let us recall the formula \((r > 0 \text{ and } \lambda > 0)\)

\[
(H_+ + \lambda)^{-r} = \Gamma(r)^{-1} \int_0^\infty dt \ t^{r-1} e^{-t\lambda} e^{-tH_+}. \tag{3.9}
\]

Combining the above two equalities we obtain

\[
|((H_+ + \lambda)^{-r} f)| \leq \Gamma(r)^{-1} \int_0^\infty dt \ t^{r-1} e^{-t\lambda} |e^{-tH_+} f| \leq \Gamma(r)^{-1} \int_0^\infty dt \ t^{r-1} \left| s - \lim_{n \to -\infty} \left( e^{-(t/n)H_A} e^{-(t/n)V_+} \right)^n f \right| \leq (H_0 + \lambda)^{-r} |f|,
\]

by using the second point of Proposition 3.3.

Taking \( u = (H_0 + \lambda)^{-1/2} g \) with \( g \in L^2(\mathbb{R}^d) \) arbitrary and \( \lambda > 0 \) large enough and using the hypothesis on \( V_- \) we deduce that there exists \( a \in [0, 1), b \geq 0 \) and \( a^\prime \in [0, 1) \) such that

\[
q_{V_-}(u) \leq a\|H_0^{1/2} u\|^2 + b\|u\|^2 = a\|H_0^{1/2} (H_0 + \lambda)^{-1/2} g\|^2 + b\|(H_0 + \lambda)^{-1/2} g\|^2 \leq (a + b/\lambda)\|g\|^2 \leq a^\prime\|g\|^2. \tag{3.11}
\]

For any \( v \in \mathcal{D}(h_A) \cap \mathcal{D}(q_{V_-}) \) let \( f := (H_+ + \lambda)^{1/2} v \) and \( g := |f| \). Using now (3.10) with \( r = 1/2 \), (3.11) and the explicit form of \( q_{V_-} \) we conclude that

\[
q_{V_-}(v) = q_{V_-}((H_+ + \lambda)^{-1/2} f) \leq q_{V_-}((H_0 + \lambda)^{-1/2} g) \leq a^\prime\|g\|^2 = a^\prime \|(H_+ + \lambda)^{1/2} v\|^2 = a^\prime [h_A(v) + q_+(v) + \lambda\|v\|^2]. \tag{3.12}
\]

\(\Box\)
Definition 3.5. For a potential function $V$ satisfying the hypothesis of Proposition 3.4, we call the operator $H = H(A; V)$ introduced in the same proposition the relativistic Hamiltonian with potential $V$ and magnetic vector potential $A$.

The spectral properties of $H$ only depend on the magnetic field $B$, different choices of a gauge giving unitarily equivalent Hamiltonians, due to the gauge covariance of our quantization procedure.

Proposition 3.6. Let $B$ be a magnetic field with $C^\infty_{pol}(\mathbb{R}^d)$ components and $A$ a vector potential for $B$ also having $C^\infty_{pol}(\mathbb{R}^d)$ components. Assume that $V : \mathbb{R}^d \to \mathbb{R}$ is a measurable function that can be decomposed as $V = V_+ - V_-$ with $V_\pm \geq 0$, $V_+ \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $V_- \in L^p(\mathbb{R}^d)$ with $p \geq d$. Then

1. $q_-$ is a $\hbar_0$-bounded sesquilinear form with relative bound $\theta$;

2. the Hamiltonian $H$ defined in Definition 3.5 is bounded from below and we have $\sigma_{\text{ess}}(H) = \sigma_{\text{ess}}(H_A + V_+) \subset [0, \infty)$.

Proof. 1. Using Observation 3 in §2.8.1 from [T], we conclude that for $d > 1$, the Sobolev space $H^{1/2}(\mathbb{R}^d)$ (that is the domain of the sesquilinear form $h_0$) is continuously embedded in $L^r(\mathbb{R}^d)$ for $2 \leq r \leq 2d/(d-1) < \infty$. Also using Hölder inequality, we deduce that for $r = 2p/(p-1) \in [2, 2d/(d-1)]$, for $p \geq d$

$$\|V_+^{1/2}u\|_2^2 \leq \|V_-\|_p\|u\|_p^2 \leq c\|V_-\|_p\|u\|_{H^{1/2}(\mathbb{R}^d)}^2, \quad (3.13)$$

$\forall u \in H^{1/2}(\mathbb{R}^d) = \mathcal{D}(h_0)$. Thus $V_+^{1/2} \in \mathcal{B}(H^{1/2}(\mathbb{R}^d); L^2(\mathbb{R}^d))$; now let us prove that it is even compact. Let us observe that for $d \leq p < \infty$, $C^\infty_{0}(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$. Thus, for $d \leq p < \infty$ let $\{W_\epsilon\}_{\epsilon > 0} \subset C^\infty_{0}(\mathbb{R}^d)$ be an approximating family for $V_+^{1/2}$ in $L^{2p}(\mathbb{R}^d)$, i.e. $\|V_+^{1/2} - W_\epsilon\|_{2p} \leq \epsilon$. Moreover, for any sequence $\{u_j\} \subset H^{1/2}(\mathbb{R}^d)$ contained in the unit ball (i.e. $\|u_j\|_{H^{1/2}} \leq 1$) we may suppose that it converges to $u \in H^{1/2}(\mathbb{R}^d)$ for the weak topology on $H^{1/2}(\mathbb{R}^d)$ and thus $\|u\|_{H^{1/2}} \leq 1$. It follows that $W_\epsilon u_j$ converges to $W_\epsilon u$ in $L^2(\mathbb{R}^d)$ and due to (3.13) we have:

$$\|(V_+^{1/2} - W_\epsilon)(u - u_j)\| \leq C^{1/2}\|V_+^{1/2} - W_\epsilon\|_{L^{2p}}\|u - u_j\|_{H^{1/2}} \leq 2c^{1/2}\epsilon, \quad \forall j \geq 1.$$

We conclude that $V_+^{1/2} u_j$ converges in $L^2(\mathbb{R}^d)$ to $V_+^{1/2} u$ and using the duality we also get that $V_-$ is a compact operator from $H^{1/2}(\mathbb{R}^d)$ to $H^{-1/2}(\mathbb{R}^d)$. Using exercise 39 in ch. XIII of [RS] we deduce that $q_-$ has zero relative bound with respect to $h_0$.  

2. The conclusion of point 1 implies that the operator $V_+^{1/2}(H_0 + 1)^{-1/2} \in \mathcal{B}(L^2(\mathbb{R}^d))$ is compact. Using the first point of Proposition 3.3 with $\lambda = -1$ and $r = 1/2$, and Pitt Theorem in [P], we conclude that the operator $V_-^{1/2}(H_A + V_+ + 1)^{-1/2} \in \mathcal{B}(L^2(\mathbb{R}^d))$ is also compact. Thus $V_- : \mathcal{D}(h_A + q_-) \to \mathcal{D}(h_A + q_-)$ is compact and the conclusion (2) follows from exercise 39 in ch. XIII of [RS].
4 The Feynman-Kac-Itô formula.

In this section we gather some probabilistic notions and results needed in the proof of Theorem 1.1. The main idea is that we obtain a Feynman-Kac-Itô formula (following [IT2]) for the semigroup defined by $H(A, V)$ and this allows us to reduce the problem to the case $B = 0$. For this last one we repeat then the proof in [D] giving all the necessary details for the case of singular potentials $V$; here an essential point is an explicit formula for the integral kernel of the operator $e^{-tH(0, V)}$ in terms of a Lévy process.

Let $(\Omega, \mathfrak{F}, P)$ be a probability space, i.e. $\mathfrak{F}$ is a $\sigma$-algebra of subsets of $\Omega$ and $P$ is a non-negative $\sigma$-aditive function on $\mathfrak{F}$ with $P(\Omega) = 1$. For any integrable random variable $X : \Omega \to \mathbb{R}$ we denote its expectation value by

$$E(X) := \int_{\Omega} X(\omega)P(d\omega). \quad (4.1)$$

For any sub-$\sigma$-algebra $\mathfrak{G} \subset \mathfrak{F}$ we denote its associated conditional expectation by $E(X \mid \mathfrak{G})$; this is the unique $\mathfrak{G}$-measurable random variable $Y : \Omega \to \mathbb{R}$ satisfying

$$\int_B Y(\omega)P(d\omega) = \int_B X(\omega)P(d\omega), \quad \forall B \in \mathfrak{G}. \quad (4.2)$$

Let us recall the following properties of the conditional expectation (see for example [J]):

$$E(E(X \mid \mathfrak{G})) = E(X), \quad (4.3)$$

$$E(XZ \mid \mathfrak{G}) = ZE(X \mid \mathfrak{G}), \quad (4.4)$$

for any $\mathfrak{G}$-measurable random variable $Z : \Omega \to \mathbb{R}$, such that $ZX$ is integrable.

We also recall the Jensen inequality ([S1], [J]): for any convex function $\varphi : \mathbb{R} \to \mathbb{R}$, and for any lower bounded random variable $X : \Omega \to \mathbb{R}$ the following inequality is valid

$$\varphi(E(X)) \leq E(\varphi(X)). \quad (4.5)$$

Following [DvC], we can associate to our Feller semigroup $\{P(t)\}_{t \geq 0}$, defined in Section 2, a Markov process $\{(\Omega, \mathfrak{F}, P_x), \{X_t\}_{t \geq 0}, \{\theta_t\}_{t \geq 0}\}$; that we briefly recall here:

- $\Omega$ is the set of "cadlag" functions on $[0, \infty)$, i.e. functions $\omega : [0, \infty) \to \mathbb{R}^d$ (paths) that are continuous to the right and have a limit to the left in any point of $[0, \infty)$.

- $\mathfrak{F}$ is the smallest $\sigma$-algebra for which all the coordinate functions $\{X_t\}_{t \geq 0}$, with $X_t(\omega) := \omega(t)$, are measurable.
Proposition 4.1. Under the same conditions as in Definition 3.5, for any function \( u \in L^2(\mathbb{R}^d) \) we have

\[
    (e^{-tH}u)(x) = \mathbb{E}_x \left( (u \circ X_t) e^{-S(t,X)} \right), \quad t \geq 0, x \in \mathbb{R}^d
\]

where

\[
    S(t, X) := i \int_0^{t^+} \int_{\mathbb{R}^d} \hat{N}_X(ds dy) \left( \int_0^1 dr \left( A(X_{s-} + ry) \right), y \right) +
\]

\[
    \hat{N}_X(ds dy) := N_X(ds dy) - \hat{N}_X(ds dy), \quad \hat{N}_X(ds dy) := \mathbb{E}_x (N_X(ds dy)) = \langle x, y \rangle
\]

\[
    \# \{ s \in (t, t'] \mid X_s \neq X_{s-}, X_s X_{s-} \in B \}.
\]
\[
+ i \int_0^t \int_{\mathbb{R}^d} \mathcal{N}_X(ds \, dy) \left( \left( \int_0^1 dr \, A(X_s + ry) - A(X_s) \right), y \right) + \\
+ \int_0^t ds \, V(X_s).
\] (4.12)

In the sequel we shall take \( A = 0 \) and \( V \in C_0^\infty(\mathbb{R}^d) \). As it is proved in [DvC], the operator \( e^{-t(H_0 + V)} \) has an integral kernel that can be described in the following way. Let us denote by \( \mathcal{F}_t^- \) the sub-\( \sigma \)-algebra of \( \mathcal{F} \) generated by the random variables \( \{X_s\}_{0 \leq s < t} \). For any pair \((x, y) \in [\mathbb{R}^d]^2\) and any \( t > 0 \) we define a measure \( \mu_{t,x}^{t,y} \) on the Borel space \((\Omega, \mathcal{F}_t^-)\) by the equality

\[
\mu_{t,x}^{t,y}(M) := \mathbb{E}_x \left[ \chi_M \circ \varphi_{t-s}(X_s - y) \right],
\] (4.13)

for any \( M \in \mathcal{F}_s \) and \( 0 \leq s < t \), where \( \chi_M \) is the characteristic function of \( M \). This measure is concentrated on the family of ‘paths’ \( \{\omega \in \Omega \mid X_0(\omega) = x, X_t(\omega) = y\} \) and we have \( \mu_{t,x}^{t,y}(\Omega) = \circ \varphi_t(x - y) \).

**Proposition 4.2.** Let \( F : \Omega \to \mathbb{R} \) be a non-negative \( \mathcal{F}_t^- \)-measurable random variable and let \( f : \mathbb{R}^d \to \mathbb{R} \) be a positive borelian function. Then the following equality holds for any \( t > 0 \) and any \( x \in \mathbb{R}^d \):

\[
\int \mathbb{R}^d dy \left\{ \int_{\Omega} \mu_{t,x}^{t,y}(d\omega) F(\omega) e^{-\int_0^s ds \, V(X_s)} \right\} f(y) = \mathbb{E}_x \left( F e^{-\int_0^t ds \, V(X_s)} f(X_t) \right).
\] (4.14)

**Proof.** This is a direct consequence of relations (2.29) and (2.33) from [DvC]. \( \Box \)

Let us now take \( A = 0 \) in Proposition 4.1 and \( F = 1 \) in Proposition 4.2 in order to deduce that the operator \( e^{-t(H_0 + V)} \) is an integral operator with integral kernel given by the function

\[
\varphi_t(x, y) := \int_{\Omega} \mu_{t,x}^{t,y}(d\omega) e^{-\int_0^t ds \, V(X_s)}, \quad t > 0, \ (x, y) \in \mathbb{R}^d \times \mathbb{R}^d.
\] (4.15)

Proposition 3.3 from [DvC] implies that the function \([0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \ni (t, x, y) \mapsto \varphi_t(x, y) \in \mathbb{R}\) is non-negative, continuous and verifies \( \varphi_t(x, y) = \varphi_t(y, x) \). We shall also need the following result.

**Proposition 4.3.** For any \( t > 0 \), any \( x \in \mathbb{R}^d \) and any function \( g : \Omega \to \mathbb{R} \) that is integrable with respect to the measure \( \mu_{t,x}^{t,x} \) we have the equality:

\[
\int_{\Omega} \mu_{t,x}^{t,x}(d\omega) g(\omega) = \int_{\Omega} \mu_{t,0}^{t,0}(d\omega) g(x + \omega).
\] (4.16)
Proof. It is evidently sufficient to prove that for any \( s \in [0, t) \) and any \( M \in \mathfrak{F}_s \), we have

\[
\mu_{t,x}^{t,s}(M) = \left( \mu_{0,0}^{t,0} \circ S_x^{-1} \right)(M)
\]

where the map \( S_x : \Omega \to \Omega \) is defined by \((S_x(\omega))(t) := x + \omega(t)\). We noticed previously the identity \( P_x = P_0 \circ S_x^{-1} \); thus for any function \( F : \Omega \to \mathbb{R} \) integrable with respect to \( P_x \), we have \( \mathbb{E}_x(F) = \mathbb{E}_0(F \circ S_x) \). We remark that \( X_s(\omega + x) = \omega(s) + x = X_s(\omega) + x \), and using the definition of the measure \( \mu_{t,x}^{t,s} \) in (4.13), we obtain

\[
\mu_{t,x}^{t,s}(M) = \mathbb{E}_x \left[ \chi_M \mathcal{S}_{t-s}(X_s - x) \right] = \mathbb{E}_0 \left[ (\chi_M \circ S_x) \mathcal{S}_{t-s}(X_s) \right] = \mathbb{E}_0 \left[ (\chi_{S_x^{-1}(M)} \circ \mathcal{S}_{t-s})(X_s) \right] = \mu_{0,0}^{t,0}(S_x^{-1}(M)) = \left( \mu_{0,0}^{t,0} \circ S_x^{-1} \right)(M).
\]

\[ \square \]

5 Proof of the bound for \( N(0; V) \).

In this Section we will consider \( A = 0 \) and we shall work only with a potential \( V = V_+ - V_- \) satisfying the properties:

- \( V_\pm \geq 0 \),
- \( V_+ \in L^1_{\text{loc}}(\mathbb{R}^d) \),
- \( V_- \in L^d(\mathbb{R}^d) \cap L^{d/2}(\mathbb{R}^d) \).

We shall use the notations \( H := H_0 + V, H_+ := H_0 + V_+, H_- := H_0 + (-V_-) \) for the operators associated to the sesquilinear forms \( \mathfrak{h} = \mathfrak{h}_0 + q_V, \mathfrak{h}_+ = \mathfrak{h}_0 + q_{V_+}, \mathfrak{h}_- = \mathfrak{h}_0 - q_{V_-} \).

Due to the results of Proposition 3.6 we have \( \sigma_{\text{ess}}(H) = \sigma_{\text{ess}}(H_+) \subset \sigma(H_+) \subset (0, \infty) \) and \( \sigma_{\text{ess}}(H_-) = \sigma_{\text{ess}}(H_0) \subset \sigma(H_0) = [0, \infty) \).

For any potential function \( W \) verifying the same conditions as \( V \) above, we denote by \( N(W) \) the number of strictly negative eigenvalues (counted with their multiplicity) of the operator \( H_0 + W \). The following result reduces our study to the case \( V_+ = 0 \).

Lemma 5.1. The following inequality is true:

\[
N(V) \leq N(-V_-).
\]

In particular we have that \( N(V) = \infty \) implies that \( N(-V_-) = \infty \).

Proof. We apply the Min-Max principle (see Theorem XIII.2 in [RS]) noticing that \( \mathcal{D}(\mathfrak{h}_-) = \mathcal{D}(\mathfrak{h}_0) \supset \mathcal{D}(\mathfrak{h}) \) and \( \mathfrak{h}_- \leq \mathfrak{h} \) and we deduce that the operator \( H_- \) has at least \( N(V) \) strictly negative eigenvalues. \( \square \)

Thus we shall suppose from now on that \( V_+ = 0 \).
5.1 Reduction to smooth, compactly supported potentials

In this subsection we shall prove that we can suppose \( V_- \in C_0^\infty(\mathbb{R}^d) \). This will be done by approximation, using a result of the type of Theorem 4.1 from [S3].

**Lemma 5.2.** Let \( V \) and \( V_n \) \((n \geq 1)\) functions as in proposition 3.4. In addition, \( V_+ = V_{n,+} = 0 \) for all \( n \geq 1 \) and \( \lim_{n \to \infty} V_{n,-} = V_- \) in \( L^1_{\text{loc}}(\mathbb{R}^d) \) and \( V_{n,-} \) are uniformly \( h_0 \)-bounded with relative bound < 1. We set \( H_n := H_A + V_n \). Then \( H_n \to H \) when \( n \to \infty \) in strong resolvent sense.

**Proof.** We denote by \( h_n \) the quadratic form associated to \( H_n \), i.e. \( h_n = h_A - q_{n,-} \), where \( q_{n,-} \) is associated to \( V_{n,-} \) by (3.7). We have \( D(h_n) = D(h_A) \subset D(q_{n,-}) \), and according to Proposition 3.4 there exist \( \alpha \in (0,1) \) and \( \beta > 0 \) such that

\[
q_{n,-}(v) \leq \alpha h_A(v) + \beta \| v \|, \quad \forall v \in D(h_A), \; \forall n \geq 1. \tag{5.1}
\]

It follows that \( h_n \) are uniformly lower bounded and the norms defined on \( D(h_A) \) by \( h_A \) and \( h_n \) are equivalent, uniformly with respect to \( n \geq 1 \). Moreover, \( C_0^\infty(\mathbb{R}^d) \) is a core for \( H_A \), thus for \( h_A, h \) and \( h_n \) also.

Let \( f \in L^2(\mathbb{R}^d) \) and \( u_n := (H_n + i)^{-1} f \in D(H_n) \subset D(h_A), \; n \geq 1 \). We have clearly

\[
\| u_n \| \leq \| f \|, \quad |h_n(u_n)| = |(H_n u_n, u_n)| \leq \| f \|, \quad \forall n \geq 1. \tag{5.2}
\]

From (5.1), the subsequent comments and (5.2) it follows that the sequence \((u_n)_{n \geq 1}\) is bounded in \( D(h_A) \), while the sequence \((V_{n,-}^{1/2} u_n)_{n \geq 1}\) is bounded in \( L^2(\mathbb{R}^d) \). Let \( u \in L^2(\mathbb{R}^d) \) be a limit point of the sequence \((u_n)_{n \geq 1}\) with respect to the weak topology on \( L^2(\mathbb{R}^d) \). By restricting maybe to a subsequence, we may assume that there exist \( \psi, \eta \in L^2(\mathbb{R}^d) \) such that \( H_{1/2} A u_n \underset{n \to \infty}{\to} \psi \) and \( V_{n,-}^{1/2} u_n \underset{n \to \infty}{\to} \eta \) in the weak topology of \( L^2(\mathbb{R}^d) \). For \( g \in D\left(H_{1/2}^A\right) \) we have

\[
\left(H_A^{1/2} g, u\right) = \lim_{n \to \infty} \left(H_A^{1/2} g, u_n\right) = \lim_{n \to \infty} \left(g, H_A^{1/2} u_n\right) = (g, \psi),
\]

thus \( u \in D(H_{1/2}^A) \) and \( H_{1/2}^A u = \psi \). Then \( u \in D(q_-) \) and for any \( g \in C_0^\infty(\mathbb{R}^d) \)

\[
(h, g) = \lim_{n \to \infty} \left(V_{n,-}^{1/2} u_n, g\right) = \lim_{n \to \infty} \left(u_n, V_{n,-}^{1/2} g\right) = (u, V_{n,-}^{1/2} g) = (V_{n,-}^{1/2} u, g),
\]

implying \( V_{1/2}^{1/2} u = \eta \).

It follows that for every \( g \in C_0^\infty(\mathbb{R}^d) \) we have

\[
(g, f) = (g, (H_n + i) u_n) = h_n(g, u_n) - i(g, u_n) =
\]

\[
= \left(H_A^{1/2} g, H_A^{1/2} u_n\right) - \left(V_{n,-}^{1/2} g, V_{n,-}^{1/2} u_n\right) - i(g, u_n) \to h(g, u) - i(g, u).
\]
Consequently, \( u \in D(H) \) and \((H + i)u = f \). Thus the sequence \((u_n)_{n \geq 1}\) has the single limit point \( u = (H + i)^{-1}f \) for the weak topology of \( L^2(\mathbb{R}^d) \). It follows that \((H_n \pm i)^{-1}f \to (H \pm i)^{-1}f\) weakly in \( L^2(\mathbb{R}^d) \) for \( n \to \infty \).

By the resolvent identity we get

\[
\| (H_n + i)^{-1}f \|^2 = \frac{i}{2} ((f, (H_n - i)^{-1}f) - (f, (H_n + i)^{-1}f)) \to (H + i)^{-1}f \|^2,
\]

therefore \((H_n + i)^{-1}f \to (H + i)^{-1}f\) in \( L^2(\mathbb{R}^d) \).

A direct consequence of Lemma 5.2 and Theorem VIII.20 from [RS] is

**Corollary 5.3.** Under the hypothesis of Lemma 5.2, for any function \( f \) bounded and continuous on \( \mathbb{R} \) and any \( u \in L^2(\mathbb{R}^d) \), we have \( f(H_n)u \to f(H)u \).

Approximating \( V_- \) is done by the standard procedures: cutoffs and regularization. The first of the lemmas below is obvious.

**Lemma 5.4.** Let \( V_- \in L^1_{\text{loc}}(\mathbb{R}^d) \) with \( V_- \geq 0 \) and assume that its associated sesquilinear form is \( h_0 \)-bounded with relative bound strictly less then 1. Let \( \theta \in C^\infty_0([0, \infty)) \) satisfy the following: \( 0 \leq \theta \leq 1 \), \( \theta \) is a decreasing function, \( \theta(t) = 1 \) for \( t \in [0, 1] \) and \( \theta(t) = 0 \) for \( t \in [2, \infty) \).

If we denote by \( \theta^n(x) := \theta(|x|/n) \) and \( V^n_\eta = \theta^nV_- \), then \( V^n_\eta \to V_- \) in \( L^1_{\text{loc}}(\mathbb{R}^d) \), \( 0 \leq V^n_\eta \leq V^{n+1}_\eta \) and the sesquilinear forms associated to \( V^n_\eta \) are \( h_0 \)-bounded with relative bound strictly less then 1, uniformly in \( n \in \mathbb{N}^* \).

Moreover, if we denote by \( h^n \) the sesquilinear form associated to the operator \( H_A + (-V^n_\eta) \), we have \( h^{(n)} \geq h^{(n+1)} \geq h \) and \( h^{(n)}(u) \to h(u) \) for any \( u \in D(h_A) \).

If, in addition, \( V_- \in L^p(\mathbb{R}^d) , p \geq 1 \), then \( V^n_\eta \in L^p_{\text{comp}}(\mathbb{R}^d) \), \( \|V^n_\eta\|_{L^p} \leq \|V_\eta\|_{L^p} \) for any \( n \geq 1 \), and \( V^n_\eta \to V_- \) in \( L^p(\mathbb{R}^d) \).

**Lemma 5.5.** (a) Let \( V_- \in L^1_{\text{loc}}(\mathbb{R}^d) \), \( V_- \geq 0 \) and \( h_0 \)-bounded with relative bound \( < 1 \). Let \( \theta \in C^\infty(\mathbb{R}^d), \theta \geq 0 \) and \( \int_{\mathbb{R}^d} \theta = 1 \). We set \( \theta_n(x) := n^d\theta(nx) \), \( x \in \mathbb{R}^d, n \in \mathbb{N}^* \) and \( V_{n,-} := V_- \ast \theta_n \in C^\infty(\mathbb{R}^d) \). In particular, \( V_{n,-} \in C^\infty(\mathbb{R}^d) \) if \( V_- \in L^1_{\text{comp}}(\mathbb{R}^d) \).

Then \( V_{n,-} \to V_- \) in \( L^1_{\text{loc}}(\mathbb{R}^d) \) for \( n \to \infty \) and the functions \( V_{n,-} \) are non-negative and uniformly \( h_0 \)-bounded, with relative bound \( < 1 \). Moreover, \( \delta_n(u) \to \delta(u) \) for any \( u \in D(\delta_A) \), where \( \delta_n \) is the quadratic form associated to \( H_n := H_A + (-V_n) \).

(b) If, in addition, \( V_- \in L^p(\mathbb{R}^d) \) with \( p \geq 1 \), then \( V_{n,-} \in L^p(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d) \), \( \|V_{n,-}\|_{L^p} \leq \|V_-\|_{L^p} \), \( \forall n \geq 1 \) and \( V_{n,-} \to V_- \) in \( L^p(\mathbb{R}^d) \).

**Proof.** (a) We have for any \( x \in \mathbb{R}^d \)

\[
V_{n,-}(x) = \int_{\mathbb{R}^d} dy \theta_n(y)V_-(x - y) = \int_{\mathbb{R}^d} dy \theta(y)V_-(x - n^{-1}y). \tag{5.3}
\]

By the Dominated Convergence Theorem, for any compact \( K \subset \mathbb{R}^d \)

\[
\int_K dx |V_{n,-}(x) - V_-(x)| \leq \int_{\mathbb{R}^d} dx \theta(y) \int_K dx |V_-(x - n^{-1}y) - V_-(x)| \to 0,
\]
hence $V_{n,-}$ converges to $V_-$ in $L^1_{\text{loc}}(\mathbb{R}^d)$ when $n \to \infty$.

If $V_-$ is relatively small with respect to $h_0$, we use the fact that $H_0^{1/2}$ is a convolution operator (hence it commutes with translations) and using the comments after inequality (5.1), we deduce that for any $u \in C_0^\infty(\mathbb{R}^d)$ there exists $\alpha \in (0, 1)$ and $\beta \geq 0$ such that

$$
\int_{\mathbb{R}^d} dx V_{n,-}|u|^2 = \int_{\mathbb{R}^d} dy \theta_n(y) \int_{\mathbb{R}^d} dz V_-(z)|u(z + y)|^2 \leq
$$

$$
\leq \int_{\mathbb{R}^d} dy \theta_n(y) \left[ \alpha \| H_0^{1/2} u(z + y) \|^2 + \beta \| u(z + y) \|^2 \right] =
$$

$$
= \alpha \| H_0^{1/2} u \|^2 + \beta \| u \|^2 .
$$

(b) From (5.3) it follows that

$$
\| V_{n,-} \|_{L^p} \leq \int_{\mathbb{R}^d} dy \theta_n(y) \| V_-(\cdot - y) \|_{L^p} \leq \| V_- \|_{L^p} .
$$

Also, using the Dominated Convergence Theorem, we infer that

$$
\| V_{n,-} - V_- \|_{L^p} \leq \int_{\mathbb{R}^d} dy \theta(y) \| V_-(\cdot) - V_-(-n^{-1}y) \|_{L^p} \to 0.
$$

Thus Lemmas 5.4 and 5.5 imply, for a potential function $V_-$ satisfying the hypothesis of the Lemma, the existence of a sequence $(V_{n,-})_{n \geq 1} \subset C_0^\infty(\mathbb{R}^d)$ such that $V_{n,-} \geq 0$, $\| V_{n,-} \|_{L^p} \leq \| V_- \|_{L^p}$, $\forall n \geq 1$, $V_{n,-} \to V_-$ in $L^p(\mathbb{R}^d)$ (for $p = d$ and $p = d/2$) when $n \to \infty$ and the functions $V_{n,-}$ are uniformly $h_0$-bounded with relative bound $< 1$.

**Lemma 5.6.** Assume that there exists a constant $C > 0$, such that the inequality

$$
N(-V_{n,-}) \leq C \left( \int_{\mathbb{R}^d} dx |V_{n,-}(x)|^d + \int_{\mathbb{R}^d} dx |V_{n,-}(x)|^{d/2} \right)
$$

(5.4)

holds for any $n \geq 1$. Then one also has

$$
N(-V_-) \leq C \left( \int_{\mathbb{R}^d} dx |V_-(x)|^d + \int_{\mathbb{R}^d} dx |V_-(x)|^{d/2} \right) .
$$

(5.5)

**Proof.** We set $H_{n,-} := H_0 + (-V_{n,-})$; $(E_{n,-}(\lambda))_{\lambda \in \mathbb{R}}$ will be the spectral family of $H_{n,-}$ and $(E_-(\lambda))_{\lambda \in \mathbb{R}}$ the spectral family of $H_-$. For $\lambda < 0$, we denote by $N_\lambda(W)$ the number of eigenvalues of $H_0 + W$ which are strictly smaller than $\lambda$ (for any potential function $W$ satisfying the hypothesis at the beginning of this section). It suffices to show that for any $\lambda < 0$ not belonging to the spectrum of $H_-$, one has the inequality

$$
N_\lambda(-V_-) \leq C \left( \int_{\mathbb{R}^d} dx |V_-(x)|^d + \int_{\mathbb{R}^d} dx |V_-(x)|^{d/2} \right) .
$$

(5.6)
Since $V_{n,-}$ converges to $V_-$ in $L^1_{\text{loc}}(\mathbb{R}^d)$, cf. Lemma 5.2, $H_{n,-}$ will converge to $H_-$ in strong resolvent sense. By [K], Ch.VIII, Th.1.15, this implies the strong convergence of $E_{n,-}(\lambda)$ to $E_-(\lambda)$ for any $\lambda \notin \sigma(H_-)$. By Lemmas 1.23 and 1.24 from [K], Ch.VII, for $\lambda < 0$, $\lambda \notin \sigma(H_-)$, one also has $\|E_{n,-}(\lambda) - E_-(\lambda)\|_\infty \to 0$.

Let us suppose that there exists some $\lambda < 0$ not belonging to $\sigma(H_-)$ and such that for it the inequality (5.6) is not verified. Thus for the given $\lambda < 0$ we have $\forall n \geq 1$:

$$N_{\lambda}(V_{n,-}) \leq C \left( \int_{\mathbb{R}^d} dx |V_-(x)|^d + \int_{\mathbb{R}^d} dx |V_-(x)|^{d/2} \right) < N_{\lambda}(V_-).$$

But for $n$ large enough, one has $N_{\lambda}(V_-) = N_{\lambda}(V_{n,-})$ and thus

$$N_{\lambda}(V_-) = N_{\lambda}(V_{n,-}) \leq N(-V_{n,-}) \leq N_{\lambda}(V_-) \leq C \left( \int_{\mathbb{R}^d} dx |V_{n,-}(x)|^d + \int_{\mathbb{R}^d} dx |V_{n,-}(x)|^{d/2} \right) \leq C \left( \int_{\mathbb{R}^d} dx |V_-(x)|^d + \int_{\mathbb{R}^d} dx |V_-(x)|^{d/2} \right)$$

that is a contradiction with our initial hypothesis. 

\[\square\]

5.2 Proof of the Theorem 1.1 for $B = 0$

We shall assume from now on that $V_+ = 0$ and $0 \leq V_- \in C_0^\infty(\mathbb{R}^d)$. We check a Birman-Schwinger principle. For $\alpha > 0$ we set $K_\alpha := V_-^{1/2}(H_0 + \alpha)^{-1}V_-^{1/2}$; it is a positive compact operator on $L^2(\mathbb{R}^d)$.

**Lemma 5.7.**

$$N_{-\alpha}(V_-) \leq \# \{ \mu > 1 \mid \mu \text{ eigenvalue of } K_\alpha \}. \quad (5.7)$$

**Proof.** We introduce the sequence of functions $\mu_n : [0, \infty) \to (-\infty, 0]$, $n \geq 1$, where $\mu_n(\lambda)$ is the $n$th eigenvalue of $H_0 - \lambda V_-$ if this operator has at least $n$ strictly negative eigenvalues and $\mu_n(\lambda) = 0$ if not. Cf. [RS] §XIII.3, $\mu_n$ is continuous and decreasing (even strictly decreasing on intervals on which it is strictly negative). Obviously, we have $N_{-\alpha}(V_-) \leq \# \{ n \geq 1 \mid \mu_n(1) < -\alpha \}$.

Now fix some $n$ such that $\mu_n(1) < -\alpha$ and recall that $\mu_n(0) = 0$. The function $\mu_n$ is continuous and injective on the interval $[\epsilon_n, 1]$, where $\epsilon_n := \sup \{ \lambda \geq 0 \mid \mu_n(\lambda) = 0 \}$, therefore it exists a unique $\lambda \in (0, 1)$ such that $\mu_n(\lambda) = -\alpha$. Thus

$$N_{-\alpha}(V_-) = \# \{ \lambda \in (0, 1) \mid \exists n \geq 1 \text{ s.t. } \mu_n(\lambda) = -\alpha \} =$$

$$= \# \{ \lambda \in (0, 1) \mid \exists \varphi \in D(H_0) \setminus \{ 0 \} \text{ s.t. } (H_0 - \lambda V_-)\varphi = -\alpha \varphi \} \leq$$

$$\leq \# \{ \lambda \in (0, 1) \mid \exists \psi \in L^2(\mathbb{R}^d) \setminus \{ 0 \} \text{ s.t. } K_\alpha \psi = \lambda^{-1} \psi \},$$

where for the last inequality we set $\psi := V_-^{1/2} \varphi$, noticing that the equality $(H_0 + \alpha)\varphi = \lambda V_- \varphi$ implies $\psi \neq 0$. 

\[\square\]

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Lemma 5.8. Let \( F : [0, \infty) \to [0, \infty) \) be a strictly increasing continuous function with \( F(0) = 0 \). Then \( F(K_\alpha) \) is a positive compact operator and the next inequality holds:
\[
N_{-\alpha}(-V_-) \leq F(1)^{-1} \sum_{F(\mu) \in \sigma(F(K_\alpha)), F(\mu) > F(1)} F(\mu).
\]

Proof. The first part is obvious. Using (5.7) and \( F \)'s monotonity, we get
\[
N_{-\alpha}(-V_-) \leq \#\{\mu > 1 \mid \mu \in \sigma(K_\alpha)\} = \# \{F(\mu) \mid \mu > 1, F(\mu) \in \sigma(F(K_\alpha))\} = \sum_{\mu > 1, F(\mu) \in \sigma(F(K_\alpha))} \frac{F(\mu)}{F(1)} \leq F(1)^{-1} \sum_{\mu > 1, F(\mu) \in \sigma(F(K_\alpha))} F(\mu).
\]

So, we shall be interested in finding functions \( F \) having the properties in the statement above, such that \( F(K_\alpha) \in B_1 \) (the ideal of trace-class operators in \( L^2(\mathbb{R}^d) \)) and such that \( \text{Tr}[F(K_\alpha)] \) is conveniently estimated.

Using an idea from [S1], we are going to consider functions of the form
\[
F(t) := t \int_0^\infty ds e^{-s} g(ts), \quad t \geq 0,
\]
where \( g : [0, \infty) \to [0, \infty) \) is continuous, bounded and \( g \not\equiv 0 \). Plainly, \( F : [0, \infty) \to [0, \infty) \) is continuous, \( F(0) = 0 \), satisfies \( F(t) \leq Ct \) for some \( C > 0 \) and the identity
\[
F(t) = \int_0^\infty dr e^{-rt^{-1}} g(r)
\]
implies that \( F \) is strictly increasing. We shall use the notations \( F = \Phi(g) \), \( \tilde{g}(t) := tg(t) \).

In particular, \( g_\lambda(t) = e^{-\lambda t}, \lambda > 0 \) leads to \( F_\lambda(t) = t(1 + \lambda t)^{-1} \). In the sequel, relations valid for this particular case will be extended to the following case, that we shall be interested in:
\[
g_\infty : [0, \infty) \to [0, \infty), \quad g_\infty(t) = 0 \text{ if } 0 \leq t \leq 1, \quad g_\infty(t) = 1 - 1/t \text{ if } t > 1, \quad (5.8)
\]
by using an approximation that we now introduce. The first lemma is obvious.

Lemma 5.9. Let \( g_\infty \) be given by (5.8). For \( n \geq 1 \) we define \( g_n : [0, \infty) \to [0, 1] \), \( g_n(t) = g(t) \) for \( 0 \leq t \leq n \), \( g_n(t) = \frac{2^{n-1}}{2^{n-1}} - 1 \) for \( n \leq t \leq 2n - 1 \), \( g_n(t) = 0 \) for \( t \geq 2n - 1 \). Then \( g_n \in C_0([0, \infty)) \), \( 0 \leq g_n \leq g_{n+1} \leq g_\infty \), \( \forall n \) and \( g_n \to g_\infty \) when \( n \to \infty \) uniformly on any compact subset of \( [0, \infty) \).

Lemma 5.10. Let \( f \) be a nonnegative continuous function on \([0, \infty), \lim_{t \to \infty} f(t) = 0 \). There exists a sequence \((f_k)_{k \geq 1}\) of real functions on \([0, \infty)\) with the properties
(a) Every \( f_k \) is a finite linear combination of functions of the form \( g_\lambda, \lambda > 0 \).
(b) \( f_k \geq f^{k+1} \geq f \geq 0 \) on \([0, \infty), \forall k \geq 1,\)
(c) \( f_k \to f \) uniformly on \([0, \infty)\) when \( k \to \infty \).
Proof. We define the function \( h : [0, 1] \to [0, \infty) \), \( h(s) := f(-\ln s) \) for \( s \in (0, 1) \), \( h(0) := 0 \). It follows that \( h \in C([0, 1]) \). We can chose now two sequences of positive numbers \( \{\epsilon_k\}_{k \geq 1} \) and \( \{\delta_k\}_{k \geq 1} \) verifying the properties: \( \lim_{k \to \infty} (\epsilon_k + \delta_k) = 0 \) and \( \delta_k - \epsilon_k \geq \epsilon_{k+1} + \delta_{k+1} > 0, \forall k \geq 1 \) (for example we may take \( \delta_k = (k+2)^{-1} \) and \( \epsilon_k = (k+2)^{-3} \)). Using the Weierstrass Theorem we may find for any \( k \geq 1 \) a real polynomial \( P_k' \) such that \( \sup_{s \in [0,1]} |h(s) - P_k'(s)| \leq \epsilon_k \) and let us denote by \( P_k := P_k' + \delta_k \). We get:

\[
\sup_{s \in [0,1]} |h(s) - P_k(s)| \leq \epsilon_k + \delta_k \to 0,
\]

\[
h \leq h + \delta_{k+1} - \epsilon_{k+1} \leq P'_{k+1} + \delta_{k+1} = P_{k+1} \leq h + \delta_{k+1} + \epsilon_{k+1} \leq h + \delta_k - \epsilon_k \leq P'_{k} + \delta_k = P_k
\]
on \([0, 1]\). Thus \( f^k(t) := P_k(e^{-t}) \) defined on \([0, \infty)\) for \( k \geq 1 \) have the required properties.

Proposition 5.11. Let \( F_\infty := \Phi(g_\infty) \). The operator \( F_\infty(K_\alpha) \) is self-adjoint, positive and compact on \( L^2(\mathbb{R}^d) \). It admits an integral kernel of the form

\[
[F_\infty(K_\alpha)](x, y) = \int_0^\infty dt e^{-\alpha t} \int_{\Omega} \mu_{0,x}^\perp(d\omega)g_\infty \left( \int_0^t ds V_-(X_s) \right),
\]

which is continuous, symmetric, with \( [F_\infty(K_\alpha)](x, x) \geq 0 \).

Proof. The first part is clear. To establish (5.9), we treat first the operator \( B_\lambda := F_\lambda(K_\alpha), \lambda > 0 \). We have

\[
B_\lambda = K_\alpha(1 + \lambda K_\alpha)^{-1} \implies B_\lambda = K_\alpha - \lambda B_\lambda K_\alpha.
\]

The second resolvent identity gives

\[
(H_0 + \alpha)^{-1} - (H_0 + \lambda V_- + \alpha)^{-1} = \lambda(H_0 + \lambda V_- + \alpha)^{-1} V_- (H_0 + \alpha)^{-1}.
\]

Multiplying by \( V_-^{1/2} \) to the left and to the right and taking into account (5.10) and the definition of \( K_\alpha \), one gets

\[
B_\lambda = V_-^{1/2}(H_0 + \lambda V_- + \alpha)^{-1} V_-^{1/2} = V_-^{1/2} \left[ \int_0^\infty dt e^{-\alpha t} e^{-t(H_0 + \lambda V_-)} \right] V_-^{1/2}.
\]

By Proposition 4.2 and its consequences, for any \( u \in C_0(\mathbb{R}^d), u \geq 0 \), we have

\[
[F_\lambda(K_\alpha) u](x) = \int_0^\infty dt e^{-\alpha t} \int_{\mathbb{R}^d} dy \left[ \int_{\Omega} \mu_{0,x}^\perp(d\omega) g_\lambda \left( \int_0^t ds V_-(X_s) \right) \right] V_-^{1/2}(y) u(y).
\]
Since $\Phi$ maps monotonous convergent sequences into monotonous convergent sequences, by applying Lemmas 5.9 and 5.10 and the Monotonous Convergence Theorem (B. Levi), we get (5.11) for $\lambda = \infty$, for the couple $(g_{\infty}, F_{\infty})$.

We introduce the notation

$$G_{\lambda}(t; x, y) := \int_{0}^{t} \mu_{0,x}^{t,y}(d\omega) g_{\lambda} \left(\int_{0}^{t} ds V_{-}(X_s)\right), \quad t > 0, \ x, y \in \mathbb{R}^d, \ 0 < \lambda \leq \infty.$$  \hfill (5.12)

By the consequences of Proposition 4.2, for any $0 < \lambda < \infty$ the function $G_{\lambda}$ is continuous on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ and symmetric in $x, y$. To obtain the same properties for $\lambda = \infty$, we approximate $g_{\infty}$ by using once again Lemmas 5.9 and 5.10. So it exists a sequence $(f_n)_{n \geq 1}$ of real continuous functions on $[0, \infty)$, each one being a finite linear combination of functions of the form $g_{\lambda}$, such that $f_n$ converges to $g_{\infty}$ uniformly on any compact subset of $[0, \infty)$. On the other hand, if $M > 0$ is an upper bound for $V_{-}$, we have

$$0 \leq \int_{0}^{t} ds V_{-}(X_s) \leq Mt,$$

and $\mu_{0,x}^{t,y}(\Omega) = \varphi_t(x - y)$. It follows that $G_{\infty}$ is, uniformly on compact subsets of $[0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$, the limit of a sequence of continuous functions, which are symmetric in $x, y$. Thus $G_{\infty}$ has the same properties. Moreover, since $0 \leq g_{\infty} \leq 1$ and $g_{\infty}(t) = 0$ for $0 \leq t \leq 1$, we have $G_{\infty}(t; x, y) = 0$ for $t \leq 1/M$. Using (2.4) and (2.3), there is a constant $C > 0$ such that

$$0 \leq G_{\infty}(t; x, y) \leq C, \quad \forall t > 0, \ \forall x, y \in \mathbb{R}^d.$$ \hfill (5.13)

From (5.11) for $\lambda = \infty$, we infer that $F_{\infty}(K_{\alpha})$ has an integral kernel of the form

$$[F_{\infty}(K_{\alpha})](x, y) = V_{-}^{1/2}(x)V_{-}^{1/2}(y) \int_{0}^{\infty} dt e^{-\alpha t} G_{\infty}(t; x, y),$$ \hfill (5.14)

so (5.9) is verified. The continuity of $F_{\infty}(K_{\alpha})$ follows from the Dominated Convergence Theorem and from (5.13). The symmetry is obvious, and the last property of the statement follows from $F_{\infty}(K_{\alpha}) \geq 0$. \hfill \(\square\)

Remark 5.12. By a lemma from [RS], §XI.4, $F_{\infty}(K_{\alpha}) \in B_1$ if the function $\mathbb{R}^d \ni x \mapsto [F_{\infty}(K_{\alpha})](x, x)$ is integrable and one has

$$\text{Tr} [F_{\infty}(K_{\alpha})] = \int_{\mathbb{R}^d} dx \ [F_{\infty}(K_{\alpha})](x, x).$$ \hfill (5.15)

Setting $D_{\infty}(t; x) := V_{-}(x)G_{\infty}(t; x, x)$, $t > 0, x \in \mathbb{R}^d$, we have

$$[F_{\infty}(K_{\alpha})](x, x) = \int_{0}^{\infty} dt e^{-\alpha t} D_{\infty}(t; x).$$ \hfill (5.16)
To check the integrability of this function, one introduces
\[ \Psi_\infty : (0, \infty) \times \mathbb{R}^d \to \mathbb{R}_+, \]
\[ \Psi_\infty(t; x) := t^{-1} \int_\Omega \mu_{t,x}^{**}(d\omega) \tilde{g}_\infty \left( \int_0^t ds \, V_-(X_s) \right), \]
where \( \tilde{g}_\infty(t) := t g_\infty(t). \) The role of this function is stressed by

**Lemma 5.13.** For \( d \geq 3 \) consider the following constant depending only on \( d \):
\[ \overline{C}_d := C \left( \int_1^\infty ds \, s^{-d} g_\infty(s) \vee \int_1^\infty ds \, s^{-d/2} g_\infty(s) \right) = C \int_1^\infty ds \, s^{-d/2} g_\infty(s) \]
where \( C \) is the constant verifying (2.6). One has
\[ \int_0^\infty dt \, e^{-\alpha t} \int_{\mathbb{R}^d} dx \, \Psi_\infty(t; x) \leq \overline{C}_d \left( \int_{\mathbb{R}^d} dx \, V^d(x) + \int_{\mathbb{R}^d} dx \, V^{d/2}_-(x) \right). \] (5.17)

**Proof.** The function \( \tilde{g}_\infty \) is convex and \( \frac{ds}{t} \) is a probability on \((0,t)\); thus by the Jensen inequality we obtain
\[ \tilde{g}_\infty \left( \int_0^t ds \, V_-(X_s) \right) \leq \int_0^t \frac{ds}{t} \tilde{g}_\infty \left( t V_-(X_s) \right). \]

Let us also remark that for the constant \( \overline{C}_d \) to be finite we have to ask that \( d \geq 3 \) for the factor \( s^{-d/2} \) to be integrable at infinity, because the convexity condition on \( \tilde{g}_\infty \) rather implies that \( \tilde{g}_\infty \) cannot vanish at infinity.

Then
\[ \int_0^\infty dt \, e^{-\alpha t} \int_{\mathbb{R}^d} dx \, \Psi_\infty(t; x) \leq \int_0^\infty dt \, t^{-2} e^{-\alpha t} \int_{\mathbb{R}^d} dx \left[ \int_{\Omega} \mu_{t,x}^{**}(d\omega) \int_0^t ds \, \tilde{g}_\infty \left( t V_-(X_s) \right) \right]. \]

Using now Proposition 4.3, the last expression is equal to:
\[ \int_0^\infty dt \, t^{-2} e^{-\alpha t} \int_{\mathbb{R}^d} dx \left[ \int_{\Omega} \mu_{t,0}^{**}(d\omega) \int_0^t ds \, \tilde{g}_\infty \left( t V_-(X_s + \omega(s)) \right) \right] = \int_0^\infty dt \, t^{-2} e^{-\alpha t} \left[ \int_{\Omega} \mu_{t,0}^{**}(d\omega) \int_0^t ds \, \tilde{g}_\infty \left( t V_-(X_s) \right) \right] = \int_0^\infty dt \, t^{-1} e^{-\alpha t} \left[ \int_{\Omega} \mu_{t,0}^{**}(d\omega) \right] \int_{\mathbb{R}^d} dx \tilde{g}_\infty \left( t V_-(X) \right) = \int_0^\infty dt \, t^{-1} e^{-\alpha t} \int_{\mathbb{R}^d} dx \tilde{g}_\infty \left( t V_-(X) \right) \leq \overline{C}_d \left( \int_{\mathbb{R}^d} dx \, V^d_-(x) + \int_{\mathbb{R}^d} dx \, V^{d/2}_-(x) \right), \]
where we have used the fact that \( s < 1 \) implies \( g_\infty(s) = 0. \)
The next result gives the connection between $D_{\infty}$ and $\Psi_{\infty}$:

**Proposition 5.14.**

$$ \int_{\mathbb{R}^d} dx \, D_{\infty}(t, x) = \int_{\mathbb{R}^d} dx \, \Psi_{\infty}(t, x). $$

**Proof.** First let us verify the following identity for any $t > 0$:

$$ \int_{\mathbb{R}^d} dx \, D_{\lambda}(t, x) = \int_{\mathbb{R}^d} dx \, \Psi_{\lambda}(t, x), \quad \text{for } \lambda \in (0, \infty) \quad (5.18) $$

where $D_{\lambda}$ and $\Psi_{\lambda}$ are defined in terms of $g_{\lambda}$ in the same way that $D_{\infty}$ and $\Psi_{\infty}$ are defined in terms of $g_{\infty}$. Let us point out that both $D_{\lambda}$ and $\Psi_{\lambda}$ are positive measurable functions on $(0, \infty) \times \mathbb{R}^d$ but only the integral on the left hand side of (5.18) is evidently finite by what we have proven so far. For simplifying the writing we shall take $\lambda = 1$. For any $r \in [0, t]$ we denote by

$$ S_r := e^{r(H_0 + V_{-})} - e^{-(t-r)(H_0 + V_{-})}. $$

Following the remarks after Proposition 4.2 above, for $r \in (0, t)$, both exponentials appearing in the above right hand side are integral operators with non-negative continuous integral kernels; thus $S_r$ will also be an integral operator with non-negative continuous kernel that we shall denote by $K_r$, and we can compute it explicitely as follows. For a non-negative $u \in C_0(\mathbb{R}^d)$, using Proposition 4.1 with $A = 0$ gives

$$ (S_r u)(x) = \mathbb{E}_x \left\{ e^{-\int_0^r V_{-}(X_\sigma)d\sigma} V_{-}(X_r)e^{-\int_0^t V_{-}(X_\sigma)d\sigma} u(X_{t-r}) \right\} $$

and using the Markov property (4.8) we obtain

$$ \mathbb{E}_{X_r} \left[ e^{-\int_0^r V_{-}(X_\sigma)d\sigma} u(X_{t-r}) \right] = \mathbb{E}_x \left[ e^{-\int_0^t V_{-}(X_\sigma)d\sigma} u(X_t) \mid \mathcal{F}_r \right] = \mathbb{E}_x \left[ e^{-\int_0^t V_{-}(X_\sigma)d\sigma} u(X_t) \mid \mathcal{F}_r \right]. $$

As the function $e^{-\int_0^r V_{-}(X_r)d\sigma} V_{-}(X_r) : \Omega \to \mathbb{R}$ is evidently $\mathcal{F}_r$-measurable, we get (using the property (4.4) of conditional expectations)

$$ (S_r u)(x) = \mathbb{E}_x \left\{ \mathbb{E}_x \left[ V_{-}(X_r)e^{-\int_0^r V_{-}(X_\sigma)d\sigma} u(X_t) \mid \mathcal{F}_r \right] \right\}. $$

We use now the property (4.3) and Proposition 4.2 taking $F := V_{-}(X_r)$ in order to get

$$ (S_r u)(x) = \mathbb{E}_x \left\{ V_{-}(X_r)e^{-\int_0^r V_{-}(X_\sigma)d\sigma} u(X_t) \right\} = \int_{\mathbb{R}^d} dy \left\{ \int_{\Omega} \mu_{0,x}^{t,y}(d\omega)V_{-}(X_r)e^{-\int_0^r V_{-}(X_\sigma)d\sigma} u(\omega) \right\} u(y). $$

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In conclusion for any \((x, y) \in \mathbb{R}^d \times \mathbb{R}^d\) we have
\[
K_r(x, y) = \int_\Omega \mu^{t, x}_{0,\omega}(d\omega)V_-(X_r)e^{-\int_0^t V_-(X_\rho)d\sigma}.
\] (5.19)

Using Proposition 4.3 we obtain
\[
\int_{\mathbb{R}^d} dx K_r(x, x) \leq \int_{\mathbb{R}^d} dx \left[ \int_\Omega \mu^{t, x}_{0,\omega}(d\omega)V_-(\omega) \right] = \int_{\mathbb{R}^d} dx \left[ \int_\Omega \mu^{t, x}_{0,0}(d\omega)V_-(x + \omega) \right] = \varphi_c(0) \int_{\mathbb{R}^d} dx V_-(x) < \infty, \quad \forall t > 0.
\]

Thus, for any \(r \in [0, t]\) the operator \(S_r\) is trace class. Moreover, due to the properties of the trace we have \(\text{Tr} S_r = \text{Tr} S_0\), \(\forall r \in [0, t]\). We have:
\[
\text{Tr}(S_0) = \frac{1}{t} \int_0^t dt \left( \int_{\mathbb{R}^d} dx \left( \int_\Omega \mu^{t, x}_{0,\omega}(d\omega) \right) \right) = \int_{\mathbb{R}^d} dx \left[ \int_\Omega \mu^{t, x}_{0,0}(d\omega) \right] = \int_{\mathbb{R}^d} dx \Psi_1(t, x).
\]

In particular, for any \(t > 0, \Psi_1(t, \cdot)\) is integrable on \(\mathbb{R}^d\).

On the other hand
\[
\text{Tr}(S_0) = \int_{\mathbb{R}^d} K_0(x, x)dx = \int_{\mathbb{R}^d} dx V_-(x) \int_{\mathbb{R}^d} \mu^{t, x}_{0,\omega}(d\omega)e^{-\int_0^t V_-(X_\rho)d\sigma} = \int_{\mathbb{R}^d} dx D_1(t; x).
\]

One uses the approximation properties contained in Lemmas 5.9 and 5.10 as well as the Monotone Convergence Theorem.

Proof. of Theorem 1.1 for \(B = 0\).

We can assume \(V_+ = 0\) and \(V_- \in C_0^\infty(\mathbb{R}^d)\). Lemma 5.8 implies that for any \(\alpha > 0\) one has
\[
N_{-\alpha}(-V_-) \leq F_\infty(1)^{-1}\text{Tr}[F_\infty(K_\alpha)].
\]

Using (5.15), (5.16), we obtain
\[
\text{Tr}[F_\infty(K_\alpha)] = \int_0^\infty dt e^{-\alpha t} \int_{\mathbb{R}^d} dx D_\infty(t; x) = \int_0^\infty dt e^{-\alpha t} \int_{\mathbb{R}^d} dx \Psi_\infty(t; x).
\]

Inequality (1.6) for \(B = 0\) follows from (5.20) and Lemma 5.13. In addition \(C_d = F_\infty(1)^{-1}C_d\).
6 Proof of the bounds in the magnetic case.

Proof. of Theorem 1.1 for \( B \neq 0 \).

Analogously to Section 5, we can assume \( V_+ = 0 \) and \( V_- \in C_0^\infty(\mathbb{R}^d) \). For \( \alpha > 0 \) one sets \( K_\alpha(A) := V_-^{1/2}(H_A + \alpha)^{-1}V_-^{1/2} \). By inequality (3.4) for \( r = 1 \) and also using Pitt’s Theorem [P], \( K_\alpha(A) \) is a positive compact operator, and the same can be said about \( F_\infty[K_\alpha(A)] \). We show that \( F_\infty[K_\alpha(A)] \in B_1 \) and we estimate the trace-norm. As at the beginning of the proof of Proposition 5.11,

\[
F_\lambda[K_\alpha(A)] = V_-^{1/2} \int_0^\infty dt \, e^{-\alpha t} e^{-t(H_A + \lambda V_-)} V_-^{1/2}.
\]

(6.1)

By using Proposition 4.1, we get for any \( u \in C_0(\mathbb{R}^d) \), \( u \geq 0 \)

\[
[F_\lambda[K_\alpha(A)] u](x) =
\]

\[
= V_-^{1/2}(x) \int_0^\infty dt \, e^{-\alpha t} E_x \left[ u(X_t)V_-^{1/2}(X_t)e^{-iS_A(t,X)}g\lambda \left( \int_0^t ds V_-(X_s) \right) \right].
\]

Approximating \( g_\infty \) by means of Lemmas 5.9 and 5.10 and using the Monotone Convergence Theorem, we see that (6.2) also holds for the pair \((g_\infty, F_\infty)\). The next inequality follows:

\[
|F_\infty[K_\alpha(A)] u| \leq F_\infty(K_\alpha)|u|, \quad \forall u \in L^2(\mathbb{R}^d).
\]

(6.3)

By Lemma 15.11 from [S1], we have \( F_\infty[K_\alpha(A)] \in B_1 \) and

\[
\text{Tr} \left( F_\infty[K_\alpha(A)] \right) \leq \text{Tr}(F_\infty[K_\alpha]) .
\]

(6.4)

Denoting by \( N_{-\alpha}(B,-V_-) \) the number of eigenvalues of \( H_A - V_- \) strictly less than \( -\alpha \), analogously to Lemmas 5.7 and 5.8, we deduce that

\[
N_{-\alpha}(B,-V_-) \leq F_\infty(1)^{-1} \text{Tr}(F_\infty[K_\alpha]) .
\]

(6.5)

Inequality (1.6) follows from (6.5) by using the estimations at the end of Section 5. The constant \( C_d \) is the same as for the case \( B = 0 \).

\[
\square
\]

Proof. of Corollary 1.2. The idea of the proof is standard (cf. [S1] for instance), but one has to use parts of the arguments from the proof of Theorem 1.1 in the case \( B = 0 \).

1. We show that it is enough to treat the case \( V_+ = 0 \).

We denote by \( N \) (resp. \( N_- \)) the number of strictly negative eigenvalues of \( H_A + V \) (resp. \( H_A + (-V_-) \)). We have \( N, N_- \in [0, \infty] \) and the min-max principle shows that \( N \leq N_- \). In addition, if \( H_A + V \) has strictly negative eigenvalues \( \lambda_1 \leq \lambda_2 \leq \ldots \), then \( H_A + (-V_-) \) has strictly negative eigenvalues \( \lambda_1^- \leq \lambda_2^- \leq \ldots \) and \( \lambda_j^- \leq \lambda_j, \quad j \geq 1 \). Therefore, one has \( \sum_{j \geq 1} |\lambda_j| \leq \sum_{j \geq 1} |\lambda_j^-| \).

2. We show that treating compactly supported \( V_- \) is enough (remark that this property implies that \( V_- \in L^p(\mathbb{R}^d) \) for any \( p \in [1, d + k] \)).

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We take into account the approximation sequence defined in Lemma 5.4. The sequence of forms \((b^n)_{n \geq 1}\) satisfies the hypothesis of Theorem 3.11, Ch. VIII from [K]. If we denote by \(\lambda_1 \leq \lambda_2 \leq \ldots\) the strictly negative eigenvalues of \(H_A + V\) and by \(\lambda^{(n)}_1 \leq \lambda^{(n)}_2 \leq \ldots\) the strictly negative eigenvalues of \(H^{(n)} := H_A + V^{(n)}\), once again by Theorem 3.15, Ch. VIII from [K], we have \(\lambda^{(n)}_j \geq \lambda_j\), \(\forall j, n \in \mathbb{N}^*\) and \(\lambda^{(n)}_j\) converges to \(\lambda_j\). So it will be sufficient to prove \((1.6)\) for the operators \(H^{(n)}\).

3. We assume from now on that \(V = -V_-, V_- \in L^{d+k}(\mathbb{R}^d) (k > 0)\) and that \(\text{supp}(V_-)\) is compact. Let \(\beta_0 > 0\) and for \(\beta \in (0, \beta_0]\) let

\[
\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{N_{-\beta}} < -\beta
\]

be the eigenvalues of \(H = H_A + (-V_-)\) strictly smaller than \(-\beta\) and let

\[
\overline{\lambda}_1 \leq \overline{\lambda}_2 \leq \cdots \leq \overline{\lambda}_{M(\beta)} < -\beta
\]

be the distinct eigenvalues with \(m_j\) the multiplicity of \(\overline{\lambda}_j\), \(1 \leq j \leq M(\beta)\). We have \(N_{-\alpha} := N_{-\alpha}(B, -V_-)\). Using the definition of the Stieltjes integral and integration by parts, we get

\[
\sum_{j=1}^{N_{-\beta}} |\lambda_j|^k = \sum_{j=1}^{M(\beta)} m_j |\overline{\lambda}_j|^k = \sum_{j=1}^{M(\beta)} |\overline{\lambda}_j|^k \left(N_{\overline{\lambda}_{j+1}} - N_{\overline{\lambda}_j}\right) = \int_{\lambda_1}^{-\beta} |\lambda|^k dN_\lambda = |\beta|^k N_{-\beta} + k \int_{\lambda_1}^{-\beta} |\lambda|^{k-1} N_\lambda d\lambda. \tag{6.6}
\]

We denote by \(I\) the last integral and use \((6.5)\) and \((5.20)\) and the arguments in the proof of Lemma 5.13 to estimate \(I\):

\[
I = \int_{\beta}^{-\lambda_1} \alpha^{k-1} N_{-\alpha} d\alpha = [F_\infty(1)]^{-1} \int_{\beta}^{-\lambda_1} \alpha^{k-1} \text{Tr} F_\infty(K_\alpha) d\alpha =
\]

\[
= [F_\infty(1)]^{-1} \int_{\mathbb{R}^d} dx \int_0^\infty dt \Psi_\infty(t, x) \int_{\beta}^{-\lambda_1} d\alpha \alpha^{k-1} e^{-\alpha t} \leq
\]

\[
\leq [F_\infty(1)]^{-1} \int_{\mathbb{R}^d} dx \int_0^\infty dt \left(\frac{t^{-1}}{\varphi_\lambda(0)}\right) \tilde{g}_\infty(tV_-(x)) \int_{\beta}^{-\lambda_1} d\alpha \alpha^{k-1} e^{-\alpha t} \leq
\]

\[
\leq C [F_\infty(1)]^{-1} \int_{\mathbb{R}^d} dx \int_0^\infty dt \left(t^{-d-1} + t^{-d/2-1}\right) \tilde{g}_\infty(tV_-(x)) \int_{\beta}^{-\lambda_1} d\alpha \alpha^{k-1} e^{-\alpha t}
\]

The \(\alpha\) integral may be bounded by:

\[
\int_0^\infty d\alpha \alpha^{k-1} e^{-\alpha t} = t^{-k} \int_0^\infty ds s^{k-1} e^{-s} \leq Ct^{-k}.
\]
Recalling that $\tilde{g}_\infty(t) = 0$ for $t \leq 1$ and $\tilde{g}_\infty(t) = t - 1$ for $t > 1$, we get that $\tilde{g}_\infty(tV_-(x)) = 0$ for $V_-(x) = 0$ and for $V_-(x) > 0$

$$\int_0^\infty dt \, t^{-k} \left( t^{-d-1} + t^{-d/2-1} \right) \tilde{g}_\infty(tV_-(x)) =$$

$$= [V_-(x)]^{d+k} \int_1^\infty s^{-d-k-1}(s-1)ds + [V_-(x)]^{d/2+k} \int_1^\infty s^{-d/2-k-1}(s-1)ds,$$

the integrals being convergent for $d \geq 2$.

Using these estimations in (6.6) we conclude that

$$\sum_{j=1}^{N-\beta} (|\lambda_j|^k - |\beta|^k) \leq C \left\{ \int_{\mathbb{R}^d} [V_-(x)]^{d+k} dx + \int_{\mathbb{R}^d} [V_-(x)]^{d/2+k} dx \right\},$$

thus

$$\sum_{j=1}^{N-(\beta_0)} (|\lambda_j|^k - |\beta|^k) \leq C \left\{ \int_{\mathbb{R}^d} [V_-(x)]^{d+k} dx + \int_{\mathbb{R}^d} [V_-(x)]^{d/2+k} dx \right\},$$

with the constant $C$ not depending on $\beta$ or $\beta_0$. We end the proof by letting $\beta \searrow 0$. \(\square\)

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