ELLIPITIC SOLUTIONS OF THE TODA LATTICE HIERARCHY
AND THE ELLIPTIC RUIJSENAARS–SCHNEIDER MODEL

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We consider solutions of the 2D Toda lattice hierarchy that are elliptic functions of the “zeroth” time $t_0 = x$. It is known that their poles as functions of $t_1$ move as particles of the elliptic Ruijsenaars–Schneider model. The goal of this paper is to extend this correspondence to the level of hierarchies. We show that the Hamiltonians that govern the dynamics of poles with respect to the $m$th hierarchical times $t_m$ and $\tilde{t}_m$ of the 2D Toda lattice hierarchy are obtained from the expansion of the spectral curve for the Lax matrix of the Ruijsenaars–Schneider model at the marked points.

Keywords: Toda lattice hierarchy, Ruijsenaars–Schneider model, elliptic solutions

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1. Introduction

1.1. Motivation and result. The 2D Toda lattice (2DTL) hierarchy [1] is an infinite set of compatible nonlinear differential–difference equations involving infinitely many time variables $t = \{t_1, t_2, t_3, \ldots\}$ (“positive” times), $\tilde{t} = \{\tilde{t}_1, \tilde{t}_2, \tilde{t}_3, \ldots\}$ (“negative” times), in which the equations are differential, and the “zeroth” time $t_0 = x$, in which the equations are difference. Among all solutions of these equations, of special interest are solutions that have a finite number of poles in the variable $x$ in a fundamental domain of the complex plane. In particular, one can consider solutions that are elliptic (double-periodic in the complex plane) functions of $x$ with poles depending on the times.

The investigation of the dynamics of poles of singular solutions of nonlinear integrable equations was initiated in the seminal paper [2], where elliptic and rational solutions of the Korteweg–de Vries and Boussinesq equations were studied. It was shown that the poles move as particles of the integrable Calogero–Moser many-body system [3]–[6] with some restrictions in the phase space. As was proved in [7], [8], this connection becomes most natural for the more general Kadomtsev–Petviashvili (KP) equation, in which case there are no restrictions in the phase space for the Calogero–Moser dynamics of poles. The method suggested by Krichever [9] for elliptic solutions of the KP equation consists in substituting the solution not in the KP equation itself but in the auxiliary linear problem for it (this implies a suitable pole ansatz for the wave function). This method allows obtaining the equations of motion together with their Lax representation.

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Later, Shiota showed [10] that the correspondence between rational solutions of the KP equation and the Calogero–Moser system with a rational potential can be extended to the level of hierarchies. The evolution of poles with respect to the higher times $t_k$ of the KP hierarchy was shown to be governed by the higher Hamiltonians $H_k = \text{tr} L^k$ of the integrable Calogero–Moser system, where $L$ is the Lax matrix. More recently, this correspondence was generalized to trigonometric and elliptic solutions of the KP hierarchy (see [11], [12] for the trigonometric solutions and [13] for the elliptic solutions).

The dynamics of poles of elliptic solutions of the 2DTL and modified KP (mKP) equations was studied in [14] (also see [15]). It was proved that the poles move as particles of the integrable Ruijsenaars–Schneider many-body system [16], [17], which is a relativistic generalization of the Calogero–Moser system. The extension to the level of hierarchies for rational solutions of the mKP equation was achieved in [18] (also see [19]): again, the evolution of poles with respect to the higher times $t_k$ of the mKP hierarchy is governed by the higher Hamiltonians $\text{tr} L^k$ of the Ruijsenaars–Schneider system. Recently, this result was generalized to trigonometric solutions [20]. However, the corresponding result for more general elliptic solutions was missing in the literature.

In this paper, we study the correspondence of the 2DTL hierarchy and the Ruijsenaars–Schneider hierarchy for elliptic solutions of the former. Our method consists in solving the auxiliary linear problems for the wave function and its adjoint using a suitable pole ansatz. The tau function of the 2DTL hierarchy for elliptic solutions [20]. However, the corresponding result for more general elliptic solutions was missing in the literature.

In this paper, we study the correspondence of the 2DTL hierarchy and the Ruijsenaars–Schneider hierarchy for elliptic solutions of the former. Our method consists in solving the auxiliary linear problems for the wave function and its adjoint using a suitable pole ansatz. The tau function of the 2DTL hierarchy for elliptic solutions has the form

$$
\tau(x, t, \bar{t}) = \exp \left( - \sum_{k \geq 1} k t_k \bar{t}_k \right) \prod_{i=1}^{N} \sigma(x - x_i(t, \bar{t})),
$$

(1.1)

where $\sigma(x)$ is the Weierstrass $\sigma$-function with quasiperiods $2\omega, 2\omega'$ such that $\text{Im}(\omega'/\omega) > 0$ (the definition is given in Sec. 3 below). The zeros $x_i$ of the tau function are poles of the solution. They are assumed to be all distinct.

We show that the dynamics of poles in the times $t, \bar{t}$ is Hamiltonian; we then identify the corresponding Hamiltonians, which turn out to be higher Hamiltonians of the elliptic Ruijsenaars–Schneider system. The generating function of the Hamiltonians is $\lambda(z)$, where the spectral parameters $\lambda$ and $z$ are connected by the equation of the spectral curve

$$
\det_{N \times N}(z e^{\zeta(\lambda)} I - L(\lambda)) = 0,
$$

(1.2)

where $\zeta(\lambda)$ is the Weierstrass $\zeta$ function, $I$ is the unity matrix, and $L(\lambda)$ is the Lax matrix. Any point of the spectral curve is $P = (z, \lambda)$, where $z, \lambda$ are connected by Eq. (1.2). There are to distinguished points on the spectral curve: $P_\infty = (\infty, 0)$ and $P_0 = (0, N \eta)$. The Hamiltonians corresponding to the positive-time flows $t$ are the coefficients of the expansion of the function $\lambda(z)$ in negative powers of $z$ around the point $P_\infty$, while the Hamiltonians corresponding to the negative-time flows $\bar{t}$ are coefficients of the expansion of the function $\lambda(z)$ in positive powers of $z$ around the point $P_0$. This is the main result of this paper.

1.2. Elliptic Ruijsenaars–Schneider model. Here, we collect the main facts on the elliptic Ruijsenaars–Schneider system following [17].

The $N$-particle elliptic Ruijsenaars–Schneider system is a completely integrable model. It can be regarded as a relativistic extension of the Calogero–Moser system. The dynamical variables are the coordinates $x_i$ and momenta $p_i$ with canonical Poisson brackets $\{x_i, p_j\} = \delta_{ij}$. The integrals of motion in involution have the form

$$
I_k = \sum_{I \subseteq \{1, \ldots, N\}, |I| = k} \exp \left( \sum_{i \in I} p_i \right) \prod_{i \in I, j \notin I} \frac{\sigma(x_i - x_j + \eta)}{\sigma(x_i - x_j)}, \quad k = 1, \ldots, N,
$$

(1.3)
where $\eta$ is a parameter (the inverse speed of light). In particular,

$$I_1 = \sum_i e^{p_i} \prod_{j \neq i} \frac{\sigma(x_i - x_j + \eta)}{\sigma(x_i - x_j)}, \quad I_N = \exp \left( \sum_{i=1}^N p_i \right).$$

Comparing to [17], our formulas differ by the canonical transformation

$$e^{p_i} \to e^{p_i} \prod_{j \neq i} \left( \frac{\sigma(x_i - x_j + \eta)}{\sigma(x_i - x_j - \eta)} \right)^{1/2}, \quad x_i \to x_i,$$

which allows eliminating square roots in [17]. The Hamiltonian of the model is $H_1 = I_1$.

The velocities of the particles are

$$\dot{x}_i = \frac{\partial H_1}{\partial p_i} = e^{p_i} \prod_{j \neq i} \frac{\sigma(x_i - x_j + \eta)}{\sigma(x_i - x_j)}.$$

The Hamiltonian equations $\dot{p}_i = -\partial H_1 / \partial x_i$ are equivalent to the equations of motion

$$\ddot{x}_i = -\sum_{k \neq i} \dot{x}_i \dot{x}_k (\zeta(x_i - x_k + \eta) + \zeta(x_i - x_k - \eta) - 2\zeta(x_i - x_k)) =$$

$$= \sum_{k \neq i} \dot{x}_i \dot{x}_k \frac{\wp'(x_i - x_k)}{\wp(\eta) - \wp(x_i - x_k)},$$

where $\wp(x)$ is the Weierstrass $\wp$ functions.

We can also introduce integrals of motion $I_{-k}$ as

$$I_{-k} = I_N^{-1} I_{N-k} = \sum_{I \subseteq \{1, \ldots, N\}, |I| = k} \exp \left( -\sum_{i \in I} p_i \right) \prod_{i \in I, j \not\in I} \frac{\sigma(x_i - x_j - \eta)}{\sigma(x_i - x_j)}.$$  (1.7)

In particular,

$$I_{-1} = \sum_i e^{-p_i} \prod_{j \neq i} \frac{\sigma(x_i - x_j - \eta)}{\sigma(x_i - x_j)}.$$  (1.8)

It is natural to set $I_0 = 1$. It can be easily verified that the equations of motion in the time $\tilde{t}_1$ corresponding to the Hamiltonian $\tilde{H}_1 = I_{-1}$ are the same equations (1.6).

This paper is organized as follows. In Sec. 2 we recall the main facts about the 2DTL hierarchy. We recall the Lax formulation in terms of pseudodifference operators, the bilinear identity for the tau function, and auxiliary linear problems for the wave function. Section 3 is devoted to solutions that are elliptic functions of $x = t_0$. We introduce double-Bloch solutions of the auxiliary linear problem and express them as a linear combination of elementary double-Bloch functions having just one simple pole in the fundamental domain. In Sec. 4, we obtain equations of motion for the poles as functions of the time $t_1$ together with their Lax representation. The properties of the spectral curve are discussed in Sec. 5. In Sec. 6, we consider the dynamics of poles with respect to the higher times and derive the corresponding Hamiltonian equations. Rational and trigonometric limits are addressed in Sec. 7. Explicit examples of the Hamiltonians are given in Sec. 8.
2. The 2D Toda lattice hierarchy

Here we very briefly review the 2DTL hierarchy (see [1]). We consider the pseudodifference Lax operators

\[ \mathcal{L} = e^{\eta \partial_x} + \sum_{k \geq 0} U_k(x) e^{-k\eta \partial_x}, \quad \overline{\mathcal{L}} = c(x) e^{-\eta \partial_x} + \sum_{k \geq 0} \overline{U}_k(x) e^{k\eta \partial_x}, \]  

(2.1)

where \( e^{\eta \partial_x} \) is the shift operator acting as \( e^{k\eta \partial_x} f(x) = f(x + \eta) \) and the coefficient functions \( U_k, \overline{U}_k \) are functions of \( x, t, \) and \( \overline{t} \). The equations of the hierarchy are differential–difference equations for the functions \( c, U_k, \) and \( \overline{U}_k \). They are encoded in the Lax equations

\[ \partial_{t_m} \mathcal{L} = [\mathcal{B}_m, \mathcal{L}], \quad \partial_{t_m} \overline{\mathcal{L}} = [\mathcal{B}_m, \overline{\mathcal{L}}], \quad \mathcal{B}_m = (L^m)_{\geq 0}, \]
\[ \partial_{\overline{t}_m} \mathcal{L} = [\overline{\mathcal{B}}_m, \mathcal{L}], \quad \partial_{\overline{t}_m} \overline{\mathcal{L}} = [\overline{\mathcal{B}}_m, \overline{\mathcal{L}}], \quad \overline{\mathcal{B}}_m = (\overline{L}^m)_{< 0}, \]  

(2.2)

where

\[ \left( \sum_{k \in \mathbb{Z}} U_k e^{k\eta \partial_x} \right)_{\geq 0} = \sum_{k \geq 0} U_k e^{k\eta \partial_x}, \quad \left( \sum_{k \in \mathbb{Z}} \overline{U}_k e^{k\eta \partial_x} \right)_{< 0} = \sum_{k < 0} \overline{U}_k e^{k\eta \partial_x}. \]

For example, \( \mathcal{B}_1 = e^{\eta \partial_x} + U_0(x), \overline{\mathcal{B}}_1 = c(x) e^{-\eta \partial_x}. \)

An equivalent formulation is provided by the zero-curvature (Zakharov–Shabat) equations

\[ \partial_{t_m} \mathcal{B}_m - \partial_{t_m} \mathcal{B}_n + [\mathcal{B}_m, \mathcal{B}_n] = 0, \]
\[ \partial_{\overline{t}_m} \mathcal{B}_m - \partial_{\overline{t}_m} \mathcal{B}_n + [\mathcal{B}_m, \overline{\mathcal{B}}_n] = 0, \]
\[ \partial_{t_m} \overline{\mathcal{B}}_m - \partial_{t_m} \overline{\mathcal{B}}_n + [\overline{\mathcal{B}}_m, \mathcal{B}_n] = 0. \]  

(2.3, 2.4, 2.5)

For example, from (2.4) with \( m = n = 1 \), we have

\[ \partial_{t_1} \ln c(x) = v(x) - v(x - \eta), \]
\[ \partial_{t_1} v(x) = c(x) - c(x + \eta), \]

where \( v = U_0 \). Eliminating \( v(x) \), we obtain a second-order differential–difference equation for \( c(x) \),

\[ \partial_{t_1} \partial_{t_1} \ln c(x) = 2v(x) - c(x + \eta) - c(x - \eta), \]

which is one of the forms of the 2D Toda equation. After the change of variables \( c(x) = e^{\varphi(x) - \varphi(x - \eta)} \), it acquires the most familiar form

\[ \partial_{t_1} \partial_{t_1} \varphi(x) = e^{\varphi(x) - \varphi(x - \eta)} - e^{\varphi(x + \eta) - \varphi(x)}. \]  

(2.6)

The zero-curvature equations are compatibility conditions for the auxiliary linear problems

\[ \partial_{t_m} \psi = \mathcal{B}_m(x) \psi, \quad \partial_{\overline{t}_m} \psi = \overline{\mathcal{B}}_m(x) \psi, \]  

(2.7)

where the wave function \( \psi \) depends on a spectral parameter \( z \): \( \psi = \psi(z; t, \overline{t}) \). The wave function has the following expansion in powers of \( z \):

\[ \psi = z^{x/\eta} e^{(t, \overline{t})} \left( 1 + \frac{\xi_1(x, t, \overline{t})}{z} + \frac{\xi_2(x, t, \overline{t})}{z^2} + \cdots \right), \]  

(2.8)
where
\[ \xi(t, z) = \sum_{k \geq 1} t_k z^k. \]

The wave operator is the pseudodifference operator of the form
\[ \mathcal{W}(x) = 1 + \xi_1(x)e^{-\eta \partial_x} + \xi_2(x)e^{-2\eta \partial_x} + \cdots \] (2.9)
with the same coefficient functions \( \xi_k \) as in (2.8). The wave function can then be written as
\[ \psi = \mathcal{W}(x)z^{x/\eta}e^{\xi(t, z)}. \] (2.10)

The adjoint wave function \( \psi^\dagger \) is defined by the formula
\[ \psi^\dagger = (\mathcal{W}^\dagger(x - \eta))^{-1}z^{-x/\eta}e^{-\xi(t, z)} \] (2.11)
(see, e.g., [21]), where the adjoint difference operator is defined according to the rule \((f(x) \circ e^{n\eta \partial_x})^\dagger = e^{-n\eta \partial_x} \circ f(x)\). The auxiliary linear problems for the adjoint wave function have the form
\[ -\partial_{-\eta} \psi^\dagger = \mathcal{B}_0(x - \eta)\psi^\dagger. \] (2.12)

In particular, we have
\[ \partial_1 \psi(x) = \psi(x + \eta) + v(x)\psi(x), \]
\[ -\partial_1 \psi^\dagger(x) = \psi^\dagger(x - \eta) + v(x - \eta)\psi^\dagger(x), \]
\[ \partial_1 \psi(x) = c(x)\psi(x - \eta). \]

The general solution of the 2DTL hierarchy is provided by the tau function \( \tau = \tau(x, t, \bar{t}) \) [22], [23]. The tau function satisfies the bilinear relation
\[ \oint_{\Gamma(x', \bar{t}')} z^{(x-x')/\eta - 1}e^{\xi(t, z) - \xi(t', z)}\tau(x, t - [z^{-1}], \bar{t})\tau(x' + \eta, t' + [z^{-1}], \bar{t})dz = \]
\[ = \oint_{\Gamma(x', \bar{t}')} z^{(x-x')/\eta - 1}e^{\xi(t, z) - \xi(t', \bar{t})}\tau(x + \eta, t, \bar{t} - [z])\tau(x', t', \bar{t}' + [z])dz, \] (2.14)
valid for all \( x, x', t, t', \bar{t}, \bar{t}' \), where
\[ t \pm [z] = \left\{ t_1 \pm z, t_2 \pm \frac{1}{2}z^2, t_3 \pm \frac{1}{3}z^3, \ldots \right\}. \]

The integration contour in the left-hand side of (2.14) is a big circle around infinity separating the singularities coming from the exponential factor from those coming from the tau functions. The integration contour in the right-hand side of (2.14) is a small circle around zero separating the singularities coming from the exponential factor from those coming from the tau functions.

The coefficient functions of the Lax operators can be expressed through the tau function. In particular,
\[ U_0(x) = v(x) = \partial_1 \ln \frac{\tau(x + \eta)}{\tau(x)}, \quad c(x) = \frac{\tau(x + \eta)\tau(x - \eta)}{\tau^2(x)}. \] (2.15)
In terms of the tau function, Toda equation (2.6) becomes

$$\partial_t \partial_t \ln \tau(x) = -\frac{\tau(x + \eta)\tau(x - \eta)}{\tau^2(x)}. \quad (2.16)$$

The wave function and its adjoint are expressed through the tau function as

$$\psi = z^{x/\eta} e^{\xi(t,z)} \frac{\tau(x, t - [z^{-1}], \bar{t})}{\tau(x, t, \bar{t})},$$

$$\psi^\dagger = z^{-x/\eta} e^{-\xi(t,z)} \frac{\tau(x, t + [z^{-1}], \bar{t})}{\tau(x, t, \bar{t})}. \quad (2.17)$$

We can also introduce the complementary wave functions $\bar{\psi}$, $\bar{\psi}^\dagger$ by the formulas

$$\bar{\psi} = z^{x/\eta} e^{\xi(t,z^{-1})} \frac{\tau(x + \eta, t, \bar{t} - [z])}{\tau(x, t, \bar{t})},$$

$$\bar{\psi}^\dagger = z^{-x/\eta} e^{-\xi(t,z^{-1})} \frac{\tau(x - \eta, t, \bar{t} + [z])}{\tau(x, t, \bar{t})}. \quad (2.19)$$

They satisfy the same auxiliary linear problems as the wave functions $\psi$ and $\psi^\dagger$. It is more convenient for us to work with the renormalized wave functions

$$\phi(x) = \frac{\tau(x)}{\tau(x + \eta)} \bar{\psi}(x) = z^{x/\eta} e^{\xi(t,z^{-1})} \frac{\tau(x + \eta, t, \bar{t} - [z])}{\tau(x, t, \bar{t})},$$

$$\phi^\dagger(x) = \frac{\tau(x)}{\tau(x - \eta)} \bar{\psi}^\dagger(x) = z^{-x/\eta} e^{-\xi(t,z^{-1})} \frac{\tau(x - \eta, t, \bar{t} + [z])}{\tau(x, t, \bar{t})}. \quad (2.20)$$

They satisfy the linear equations

$$\partial_t \phi(x) = \phi(x - \eta) - \bar{v}(x)\phi(x), \quad -\partial_t \phi^\dagger(x) = \phi^\dagger(x + \eta) - \bar{v}(x - \eta)\phi^\dagger(x), \quad (2.21)$$

where $\bar{v}(x) = \partial_t \ln(\tau(x + \eta)/\tau(x))$.

Finally, we note the useful corollaries of bilinear relation (2.14). Differentiating it with respect to $t_m$ and setting $x = x'$, $t = t'$ and $\bar{t} = \bar{t}'$ after that, we obtain

$$\frac{1}{2\pi i} \int_\infty^{-\infty} z^{m-1} \tau(x, t - [z^{-1}])\tau(x + \eta, t + [z^{-1}]) \, dz =$$

$$= \partial_{t_m} \tau(x + \eta, t)\tau(x, t) - \partial_{t_m} \tau(x, t)\tau(x + \eta, t) \quad (2.22)$$

or

$$\text{res}(z^m \psi(x)\psi^\dagger(x + \eta)) = \partial_{t_m} \ln \frac{\tau(x + \eta)}{\tau(x)}, \quad (2.23)$$

where $\text{res}$ is defined in accordance with the convention $\text{res}(z^{-n}) = \delta_{n1}$. Equivalently, Eq. (2.23) can be written in the form

$$\psi(x)\psi^\dagger(x + \eta) = 1 + \sum_{m \geq 1} z^{-m-1} \partial_{t_m} \ln \frac{\tau(x + \eta)}{\tau(x)}. \quad (2.24)$$

In a similar way, differentiating bilinear relation (2.14) with respect to $t_m$ and setting $x = x'$, $t = t'$ and $\bar{t} = \bar{t}'$ after that, we obtain the relation

$$\text{res}(z^{-m} \phi(x)\phi^\dagger(x + \eta)) = -\partial_{t_m} \ln \frac{\tau(x + \eta)}{\tau(x)}. \quad (2.25)$$

Here, $\text{res}$ is defined according to the convention $\text{res}(z^n) = \delta_{n1}$.
### 3. Elliptic solutions of the Toda equation and the double-Bloch wave functions

The ansatz for the tau function of elliptic (double-periodic in the complex plane) solutions of the 2DTL hierarchy is given by (1.1), where

\[
\sigma(x) = \sigma(x \mid \omega, \omega') = x \prod_{s \neq 0} \left( 1 - \frac{x}{s} \right) e^{x/s + x^2/2s^2}, \quad s = 2\omega m + 2\omega' m' \text{ with integer } m, m',
\]

is the Weierstrass \(\sigma\) function with quasiperiods \(2\omega\) and \(2\omega'\) such that \(\text{Im}(\omega'/\omega) > 0\). It is connected with the Weierstrass \(\zeta\) and \(\wp\) functions by the formulas \(\zeta(x) = \sigma'(x)/\sigma(x)\), \(\wp(x) = -\zeta'(x) = -\partial_x^2 \ln \sigma(x)\). The monodromy properties of \(\sigma(x)\) are

\[
\sigma(x + 2\omega) = -e^{2\zeta(\omega)(x + \omega)} \sigma(x), \quad \sigma(x + 2\omega') = -e^{2\zeta(\omega')(x + \omega')} \sigma(x),
\]

where the constants \(\zeta(\omega), \zeta(\omega')\) are related by \(\zeta(\omega)\omega' - \zeta(\omega')\omega = \pi i/2\).

For elliptic solutions,

\[
v(x) = \sum_i \dot{x}_i (\zeta(x - x_i) - \zeta(x - x_i + \eta))
\]

is an elliptic function, and we can therefore find double-Bloch solutions of the linear problem (2.13). The double-Bloch function satisfies the monodromy properties \(\psi(x + 2\omega) = B\psi(x)\) and \(\psi(x + 2\omega') = B'\psi(x)\) with some Bloch multipliers \(B\) and \(B'\). Any nontrivial (i.e., not exponential) double-Bloch function must have poles in \(x\) in the fundamental domain. The Bloch multipliers of wave function (2.17) are

\[
B = z^{2\omega/\eta} \exp\left( -2\zeta(\omega) \sum_i (e^{-D(z)} - 1)x_i \right),
\]

\[
B' = z^{2\omega'/\eta} \exp\left( -2\zeta(\omega') \sum_i (e^{-D(z)} - 1)x_i \right),
\]

where we define the differential operator

\[
D(z) = \sum_{k \geq 1} \frac{z^{-k}}{k} \partial_{x_k}.
\]

It follows from Eq. (2.24) that the Bloch multipliers of the function \(\psi^\dagger\) are \(1/B\) and \(1/B'\). Indeed, the right-hand side of (2.24) is an elliptic function of \(x\), and therefore the left-hand side must be also an elliptic function.

We introduce the elementary double-Bloch function \(\Phi(x, \lambda)\) defined as

\[
\Phi(x, \lambda) = \frac{\sigma(x + \lambda)}{\sigma(\lambda)\sigma(x)} e^{-\zeta(\lambda)x}.
\]

The monodromy properties of \(\Phi\) are

\[
\Phi(x + 2\omega, \lambda) = e^{2(\zeta(\omega)\lambda - \zeta(\lambda)\omega)} \Phi(x, \lambda),
\]

\[
\Phi(x + 2\omega', \lambda) = e^{2(\zeta(\omega')\lambda - \zeta(\lambda)\omega')} \Phi(x, \lambda),
\]

and therefore it is indeed a double-Bloch function. The function \(\Phi\) has a simple pole at \(x = 0\) with residue 1,

\[
\Phi(x, \lambda) = \frac{1}{x} + \alpha_1 x + \cdots, \quad x \to 0,
\]

where \(\alpha_1 = -\zeta(\lambda)/2\). When this does not lead to misunderstanding, we suppress the second argument of \(\Phi\), writing simply \(\Phi(x) = \Phi(x, \lambda)\). We also need the \(x\)-derivative \(\Phi'(x, \lambda) = \partial_x \Phi(x, \lambda)\).
Below, we need the following identities satisfied by $\Phi$:

$$\Phi(x,-\lambda) = -\Phi(-x,\lambda), \quad (3.6)$$

$$\Phi'(x) = \Phi(x)(\zeta(x + \lambda) - \zeta(\lambda) - \zeta(x)), \quad (3.7)$$

$$\Phi(x)\Phi(y) = \Phi(x + y)(\zeta(x) + \zeta(y) + \zeta(\lambda) - \zeta(x + y + \lambda)). \quad (3.8)$$

Equations (2.17) and (2.18) imply that the wave functions $\psi$ and $\psi^{\dagger}$ have simple poles at the points $x_i$. We can expand the wave functions using the elementary double-Bloch functions as

$$\psi = k^x/\eta e^{\xi(t,x)} \sum_i c_i \Phi(x - x_i, \lambda), \quad (3.9)$$

$$\psi^{\dagger} = k^{-x}/\eta e^{-\xi(t,x)} \sum_i c_i^* \Phi(x - x_i, -\lambda), \quad (3.10)$$

where $c_i, c_i^*$ are expansion coefficients, which do not depend on $x$, and $k$ is an additional spectral parameter. We note that the normalization of functions (2.17) and (2.18) implies that $c_i$ and $c_i^*$ are $O(\lambda)$ as $\lambda \to 0$. We can see that (3.9) is a double-Bloch function with the Bloch multipliers

$$B = e^{(2\omega/\eta)(\ln k - \eta\zeta(\lambda)) + 2\zeta(\omega)\lambda}, \quad B' = e^{(2\omega'/\eta)(\ln k - \eta\zeta(\lambda)) + 2\zeta'(\omega')\lambda}, \quad (3.11)$$

and (3.10) has the Bloch multipliers $B^{-1}$ and $B'^{-1}$. These Bloch multipliers should coincide with (3.3). Therefore, comparing Eqs. (3.3) and (3.11), we have

$$\frac{2\omega}{\eta} \left( \ln \frac{k}{z} - \eta\zeta(\lambda) \right) + 2\zeta(\omega) \left( \lambda + (e^{-D(z)} - 1) \sum_i x_i \right) = 2\pi in,$$

$$\frac{2\omega'}{\eta} \left( \ln \frac{k}{z} - \eta\zeta(\lambda) \right) + 2\zeta'(\omega') \left( \lambda + (e^{-D(z)} - 1) \sum_i x_i \right) = 2\pi in'$$

with some integers $n$ and $n'$. Regarding these equations as a linear system, we obtain the solution

$$\ln \frac{k}{z} - \eta\zeta(\lambda) = 2n'\eta\zeta(\omega) - 2n\eta\zeta(\omega'),$$

$$\lambda + (e^{-D(z)} - 1) \sum_i x_i = 2n\omega' - 2n'\omega.$$ 

Shifting $\lambda$ by a suitable vector of the lattice spanned by $2\omega, 2\omega'$, we obtain zeros in the right-hand sides of these equalities, and we can therefore represent the connection between the spectral parameters $k, z,$ and $\lambda$ in the form

$$k = ze^{\eta\zeta(\lambda)}, \quad \lambda = (1 - e^{-D(z)}) \sum_i x_i. \quad (3.12)$$

These two equations for three spectral parameters $k, z, \lambda$ define the spectral curve. Another description of the same spectral curve is obtained below as the spectral curve of the Ruijsenaars–Schneider system (it is given by the characteristic polynomial of the Lax matrix $L(\lambda)$ for the Ruijsenaars–Schneider system). It appears in the form $R(k, \lambda) = 0$, where $R(k, \lambda)$ is a polynomial in $k$ whose coefficients are elliptic functions of $\lambda$ (see Sec. 5 below). These coefficients are integrals of motion in involution. The spectral curve in the form $R(k, \lambda) = 0$ appears if we eliminate $z$ from Eqs. (3.12). Equivalently, we can represent the spectral curve as a relation connecting $z$ and $\lambda$:

$$R(ze^{\eta\zeta(\lambda)}, \lambda) = 0. \quad (3.13)$$
The second equation in (3.12) can be written as an expansion in powers of $z$:

$$\lambda = \lambda(z) = -\sum_{m \geq 1} z^{-m} \hat{h}_m \mathcal{X}, \quad \mathcal{X} := \sum_i x_i, \quad (3.14)$$

Here, $\hat{h}_k$ are differential operators of the form

$$\hat{h}_m = -\frac{1}{m} \partial_{t_m} + \text{higher order operators in } \partial_{t_1}, \partial_{t_2}, \ldots, \partial_{t_{m-1}}; \quad (3.15)$$

The first few are

$$\hat{h}_1 = -\partial_{t_1}, \quad \hat{h}_2 = \frac{1}{2} (\partial_{t_1}^2 - \partial_{t_2}), \quad \hat{h}_3 = \frac{1}{6} (-\partial_{t_1}^3 + 3 \partial_{t_1} \partial_{t_2} - 2 \partial_{t_3}).$$

As is mentioned above, the coefficients in expansion (3.14) are integrals of motion, i.e., $\partial_{t_j} \hat{h}_m \mathcal{X} = 0$ for all $j$ and $m$. It then follows from the explicit form of the operators $\hat{h}_m$ that $\partial_{t_j} \partial_{t_m} \mathcal{X} = 0$. A simple inductive argument then shows that $\partial_{t_j} \partial_{t_m} \mathcal{X} = 0$ for all $j, m$. This means that $-\hat{h}_m \mathcal{X} = \frac{1}{m} \partial_{t_m} \mathcal{X}$ and hence $\mathcal{X}$ is a linear function of the times,

$$\mathcal{X} = \sum_i x_i = \lambda_0 + \sum_{m \geq 1} V_m t_m \quad (3.16)$$

with some constants $V_m$ (which are velocities of the “center of masses” of the points $x_i$ with respect to the times $t_m$ multiplied by $N$). Therefore, the second equation in (3.12) can be written as

$$\lambda = D(z) \sum_i x_i = \sum_{j \geq 1} \frac{z^{-j}}{j} V_j, \quad (3.17)$$

We show in what follows that $H_m = V_m/m$ are Hamiltonians for the dynamics of poles in $t_m$, with $H_1$ being the standard Ruijsenaars–Schneider Hamiltonian.

4. Dynamics of poles with respect to $t_1$

The next procedure has become standard after paper [9]. We substitute $v(x)$ in form (3.2) and $\psi$ in form (3.9) into the left-hand side of the linear problem

$$\partial_{t_1} \psi(x) - \psi(x + \eta) - v(x)\psi(x) = 0$$

and cancel the poles at the points $x = x_i$ and $x = x_i - \eta$. The highest poles are of the second order, but it is easy to see that they cancel identically. A simple calculation shows that the cancellation of first-order poles leads to the conditions

$$zc_i + \dot{c}_i = \dot{x}_i \sum_{j \neq i} c_j \Phi(x_i - x_j) + c_i \sum_{j \neq i} \dot{x}_j \zeta(x_i - x_j) - c_i \sum_{j} \dot{x}_j \zeta(x_i - x_j + \eta),$$

$$kc_i - \dot{x}_i \sum_{j} c_j \Phi(x_i - x_j - \eta) = 0.$$

Here and below, the dot means the $t_1$-derivative. These conditions can be written in the matrix form as a system of linear equations for the vector $\mathbf{c} = (c_1, \ldots, c_N)^T$:

$$L(\lambda) \mathbf{c} = k \mathbf{c},$$

$$\dot{\mathbf{c}} = M(\lambda) \mathbf{c}, \quad (4.1)$$
where \((N \times N)\) matrices \(L\) and \(M\) are

\[
L_{ij} = \dot{x}_i \Phi(x_i - x_j - \eta), \tag{4.2}
\]

\[
M_{ij} = \delta_{ij} \left( k - z + \sum_{l \neq i} \dot{x}_l \zeta(x_i - x_l) - \sum_l \dot{x}_l \zeta(x_i - x_l + \eta) \right) + (1 - \delta_{ij}) \dot{x}_j \Phi(x_i - x_j - \eta). \tag{4.3}
\]

They depend on the spectral parameter \(\lambda\).

Differentiating the first equation in (4.1) with respect to \(t_1\) and substituting the second equation, we obtain the compatibility condition for linear problems (4.1):

\[
(\dot{L} + [L, M]) c = 0. \tag{4.4}
\]

The Lax equation \(\dot{L} + [L, M] = 0\) is equivalent to the equations of motion of the elliptic Ruijsenaars–Schneider system (see \([15]\) for the details of the calculation). We note that our matrix \(M\) differs from the standard one by the term \(\delta_{ij}(k - z)\), but this does not affect the compatibility condition because it is proportional to the unity matrix. It follows from the Lax representation that the time evolution is an isospectral transformation of the Lax matrix \(L\), and hence all traces \(\text{tr} L^m\) and the characteristic polynomial \(\det(kI - L)\), where \(I\) is the unit matrix, are integrals of motion. We note that the Lax matrix is written in terms of the momenta \(p_i\) as

\[
L_{ij} = \Phi(x_i - x_j - \eta) e^{p_i} \prod_{l \neq i} \frac{\sigma(x_i - x_l + \eta)}{\sigma(x_i - x_l)}. \tag{4.5}
\]

A similar calculation shows that the adjoint linear problem for function (3.10) leads to the equations

\[
\begin{align*}
\dot{c}^T \dot{X}^{-1} L(\lambda) \dot{X} &= k c^T, \\
\dot{c}^T &= -c^T M^*(\lambda),
\end{align*} \tag{4.6}
\]

where \(\dot{X} = \text{diag}(\dot{x}_1, \ldots, \dot{x}_N)\) and

\[
M_{ij}^* = \delta_{ij} \left( k - z - \sum_{l \neq i} \dot{x}_l \zeta(x_i - x_l) + \sum_l \dot{x}_l \zeta(x_i - x_l - \eta) \right) + (1 - \delta_{ij}) \dot{x}_j \Phi(x_i - x_j - \eta). \tag{4.7}
\]

5. The spectral curve

The first equation in (4.1) defines a connection between the spectral parameters \(k\) and \(\lambda\), which is the equation of the spectral curve:

\[
R(k, \lambda) = \det(kI - L(\lambda)) = 0. \tag{5.1}
\]

As was mentioned already, the spectral curve is an integral of motion. The matrix \(L = L(\lambda)\), which has an essential singularity at \(\lambda = 0\), can be represented in the form \(L(\lambda) = e^{N(\lambda)V \tilde{L}(\lambda)V^{-1}}\), where matrix elements of \(\tilde{L}(\lambda)\) do not have essential singularities and \(V\) is the diagonal matrix \(V_{ij} = \delta_{ij} e^{-\zeta(\lambda)x_i} \). The matrix \(\tilde{L}(\lambda)\) is given by

\[
\tilde{L}_{ij}(\lambda) = \frac{\sigma(x_i - x_j - \eta + \lambda)}{\sigma(\lambda) \sigma(x_i - x_j - \eta)} e^{p_i} \prod_{l \neq i} \frac{\sigma(x_i - x_l + \eta)}{\sigma(x_i - x_l)}. \tag{5.2}
\]
Using the connection between $k$ and $z$ in Eq. (3.12), we represent the spectral curve in the form
\begin{equation}
\det(zI - \tilde{L}(\lambda)) = 0. \quad (5.3)
\end{equation}
The spectral curve is an $N$-sheet covering over the $\lambda$-plane. Any point of the curve is $P = (z, \lambda)$, where $z$ and $\lambda$ are connected by Eq. (5.3) and there are $N$ points above each $\lambda$.

To represent the spectral curve in explicit form, we recall the identity
\begin{equation}
\det_{1 \leq i, j \leq N} \left( \frac{\sigma(x_i - y_j + \lambda)}{\sigma(\lambda)\sigma(x_i - y_j)} \right) = \frac{1}{\sigma(\lambda)} \sigma \left( \lambda + \sum_{i=1}^{N} (x_i - y_i) \right) \prod_{i<j} \sigma(x_i - x_j)\sigma(y_j - y_i) / \prod_{i,j} \sigma(x_i - y_j) \quad (5.4)
\end{equation}
for the determinant of the elliptic Cauchy matrix. Using this identity, we can represent (5.3) as
\begin{equation}
\sum_{n=0}^{N} \varphi_n(\lambda) I_n z^{N-n} = 0, \quad (5.5)
\end{equation}
where $I_n$ are the Ruijsenaars–Schneider integrals of motion (1.3) and
\begin{equation}
\varphi_n(\lambda) = \frac{\sigma(\lambda - n\eta)}{\sigma(\lambda)\sigma^n(\eta)} \quad (5.6)
\end{equation}

We now fix two distinguished points on the spectral curve. As $\lambda \to 0$, we have
\begin{equation}
\tilde{L}(\lambda) = \dot{X}E\lambda^{-1} + O(1),
\end{equation}
where $E$ is the rank-1 matrix with the entries $E_{ij} = 1$ for all $i$ and $j$. Therefore, using the formula for the determinant of the matrix $I + Y$, where $Y$ is a rank-1 matrix, we can write
\begin{equation}
\det(zI - \tilde{L}(\lambda)) = z^N - z^{N-1}\lambda^{-1}I_1 + O(z^{N-1}),
\end{equation}
or
\begin{equation}
\det(zI - \tilde{L}(\lambda)) = (z - h_N(\lambda)\lambda^{-1}) \prod_{j=1}^{N-1} (z - h_j(\lambda)),
\end{equation}
where the functions $h_j$ are regular functions at $\lambda = 0$ and $h_N(0) = I_1 = H_1$. We see that among the $N$ points above $\lambda = 0$, one is distinguished: it is the point $P_\infty = (\infty, 0)$. Another distinguished point is $P_0 = (0, N\eta)$. Because $\varphi_N(N\eta) = 0$, we see that this point indeed belongs to the curve.

6. Dynamics in higher times

6.1. Positive times. To study the dynamics of poles in the higher positive times, we take advantage of relation (2.23), which, after the substitution of the wave functions for elliptic solutions, acquires the form
\begin{equation}
\sum_{i,j} \text{res}_{z=\infty} (z^m k^{-1} c_i c_j^* \Phi(x - x_i, \lambda)\Phi(x - x_j + \eta, -\lambda)) = \sum_{n} \partial_t x_n(\zeta(x - x_n) - \zeta(x - x_n + \eta)). \quad (6.1)
\end{equation}
Equating the residues at $x = x_i - \eta$, we obtain
\begin{equation}
\partial_t x_i = -\sum_{j} \text{res}_{z=\infty} (z^m k^{-1} c_i c_j^* \Phi(x_i - x_j - \eta, \lambda)). \quad (6.2)
\end{equation}
Next, with (6.5), we can continue (6.3) as the following chain of equalities:

\[
\frac{\partial t_{m}}{x_{i}} = - \operatorname{res}(z^{m}k^{-1}c^{\ast T}X^{-1}(\partial_{p_{i}}, L)c) = \frac{\partial t_{m}}{x_{i}} = - \operatorname{res}(z^{m}k^{-1}c^{\ast T}X^{-1}Lc) + \operatorname{res}(z^{m}k^{-1}X^{-1}c) \quad \text{(6.3)}
\]

Summing (6.3) over \( i \) and using (3.17), we can write

\[
\operatorname{res}(z^{m}k^{-1}c^{\ast T}X^{-1}Lc) = \lambda(z),
\]

where we took into account that \( \sum_{i} \partial_{p_{i}} L = L \). Using the fact that \( c = O(1/z) \), we conclude that

\[
k^{-1}c^{\ast T}X^{-1}Lc = \lambda(z),
\]

or

\[
c^{\ast T}X^{-1}c = \lambda(z). \quad \text{(6.5)}
\]

Next, with (6.5), we can continue (6.3) as the following chain of equalities:

\[
0 = \frac{d\ln z}{dp_{i}} = \partial_{p_{i}} \ln z|_{\lambda = \text{const}} + \partial_{\lambda} \ln z|_{\lambda = \text{const}} \partial_{p_{i}} \lambda \quad \text{(6.6)}
\]

Therefore,

\[
\frac{\partial t_{m}}{x_{i}} = \operatorname{res}(z^{m-1} \partial_{p_{i}} \lambda(z)) = \frac{\partial H_{m}}{\partial p_{i}},
\]

where the Hamiltonian \( H_{m} \) is given by

\[
H_{m} = \operatorname{res}(z^{m-1} \lambda(z)). \quad \text{(6.8)}
\]

This is the first half of the Hamiltonian equations.

The calculation leading to the second half of the Hamiltonian equations is rather involved. We present the main steps of it. First of all, we note that

\[
X^{-1} \partial_{x_{i}} L = [E_{i}, B^{-}] + (\vec{D}^{+} - \vec{D}^{0})E_{i}A^{-} - \sum_{i} E_{i}(\zeta(x_{i} - x_{i} + \eta)A^{-} + \sum_{i \neq i} E_{i}(\zeta(x_{i} - x_{i})A^{-} := Y_{i}, \quad \text{(6.9)}
\]

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where we introduce the matrices $A^-$ and $B^-$ and diagonal matrices $E_i$, $D^0$, and $\bar{D}^\pm$ defined by their matrix elements as follows:

$$A_{jk}^+ = \Phi(x_j - x_k - \eta), \quad B_{jk}^- = \Phi'(x_j - x_k - \eta),$$

$$(E_i)_{jk} = \delta_{ij}\delta_{ik}, \quad \bar{D}^0_{jk} = \delta_{jk}\sum_{l \neq j} \zeta(x_j - x_l), \quad \bar{D}^\pm_{jk} = \delta_{jk}\sum_{l} \zeta(x_j - x_l \pm \eta).$$

Now, in order to prove the second half of the Hamiltonian equations, it is enough to obtain the equations

$$\partial_{t_m} p_i = \text{res}(z^n k^{-1} e^{*T} Y_i c),$$

(6.10)

where $Y_i$ is the right-hand side of (6.9). Indeed, repeating the argument given after Eq. (6.5) with the change $\partial_{p_i} \rightarrow \partial_{x_i}$, we obtain the Hamiltonian equations

$$\partial_{t_m} p_i = \frac{\partial H_m}{\partial x_i}$$

(6.11)

with the same Hamiltonian given by (6.8).

It is clear that

$$p_i = \ln \dot{x}_i - \sum_{k \neq i} \ln \frac{\sigma(x_i - x_k + \eta)}{\sigma(x_i - x_k)}.$$  

(6.12)

The $t_m$-derivative of this equality yields

$$\partial_{t_m} p_i = \dot{x}_i^{-1} \partial_{t_m} \dot{x}_i - \sum_{k \neq i} (\partial_{t_m} x_i - \partial_{t_m} x_k)(\zeta(x_i - x_k + \eta) - \zeta(x_i - x_k)).$$

(6.13)

We find the first term in the right-hand side. For this, we differentiate Eq. (6.3), which we write here in the form

$$\partial_{t_m} x_i = \text{res}(z^m k^{-1} e^{*T} E_i A^- c),$$

with respect to $t_1$ and use Eqs. (4.1), (4.6). In this way we arrive at

$$\partial_{t_m} \dot{x}_i = \text{res}(z^m k^{-1} e^{*T} (M^* E_i A^- - E_i A^- M - E_i \dot{A}^-) c),$$

(6.14)

where the matrices $M$ and $M^*$ are

$$M = (k - z)I + D^0 - D^+ + \dot{X} A - \dot{X} A^-,$$

$$M^* = (k - z)I - D^0 + D^- + A \dot{X} - A^- \dot{X},$$

(6.15)

and we define the off-diagonal matrix $A$ and diagonal matrices $D^0$ by specifying their matrix elements

$$A_{jk} = (1 - \delta_{jk})\Phi(x_j - x_k),$$

$$D^0_{jk} = \delta_{jk}\sum_{l \neq j} \dot{x}_l \zeta(x_j - x_l), \quad D^\pm_{jk} = \delta_{jk}\sum_{l} \dot{x}_l \zeta(x_j - x_l \pm \eta).$$

Taking into account that $e^{*T} A^- = k e^{*T} \dot{X}^{-1}$, $A^- c = k \dot{X}^{-1} c$, we have

$$e^{*T} (M^* E_i A^- - E_i A^- M - E_i \dot{A}^-) c =$$

$$= e^{*T} (k (D^- - D^0) E_i \dot{X}^{-1} + k AE_i + E_i A^- (D^+ - D^0) - E_i (A^- \dot{X} A - A \dot{X} A^-)) - k E_i A - E_i \dot{X} B^- + E_i B^- \dot{X}) c.$$
Using identities (3.7) and (3.8), can see after some calculations that the result is

\[ c^*T(M^* A^- E_i A^- M - E_i \dot{A}^-)c = kc^*T[A, E_i]c. \]  

(6.16)

It is easy to see that

\[ \dot{x}_i^{-1} c^*T [A, E_i]c = c^*T (AE_i A^- - A^- E_i A)c. \]

Identities (3.7), (3.8) allow us to prove the relation

\[
AE_i A^- - A^- E_i A = [E_i, B^-] + \sum_{l \neq i} \xi(x_l - x_i) E_l A^- - \sum_l \xi(x_l - x_i + \eta) A^- E_l + \\
+ \sum_{l \neq i} \xi(x_l - x_i) A^+ E_l - \sum_l \xi(x_l - x_i - \eta) E_l A^-.
\]

Therefore, we have

\[
\dot{x}_i^{-1} \partial_m \dot{x}_i = \text{res}_{\infty} \left( z^m k^{-1} c^*T \left[ E_i, B^- \right] + 2 \sum_{l \neq i} \xi(x_l - x_i) E_l A^- - \sum_l \xi(x_l - x_i + \eta) A^- E_l - \sum_l \xi(x_l - x_i - \eta) A^- E_l \right) c.
\]

(6.17)

We next transform the remaining part of (6.13):

\[
- \sum_{k \neq i} (\partial_m x_i - \partial_m x_k) (\xi(x_i - x_k + \eta) - \xi(x_i - x_k)) = \\
= \text{res}_{\infty} \left( z^m k^{-1} c^*T \left( A^- (\bar{D}^+ - \bar{D}^0) E_i - \sum_l \xi(x_l - x_i + \eta) A^- E_l + \sum_{l \neq i} \xi(x_l - x_i) A^- E_l \right) c \right) = \\
= \text{res}_{\infty} \left( z^m c^*T \left( \dot{X}^{-1} (\bar{D}^+ - \bar{D}^0) E_i - \dot{X}^{-1} \sum_l \xi(x_l - x_i + \eta) E_l + \dot{X}^{-1} \sum_{l \neq i} \xi(x_l - x_i) E_l \right) c \right).
\]

Collecting everything together, we finally find

\[
\partial_m p_i = \text{res}_{\infty} \left( z^m k^{-1} c^*T \left( k \dot{X}^{-1} (\bar{D}^+ - \bar{D}^0) E_i - k \dot{X}^{-1} \sum_l \xi(x_l - x_i + \eta) E_l + k \dot{X}^{-1} \sum_{l \neq i} \xi(x_l - x_i) E_l + \left[ E_i, B^- \right] + 2k \dot{X}^{-1} \sum_{l \neq i} E_l \xi(x_l - x_i) - k \dot{X}^{-1} \sum_l E_l \xi(x_l - x_i + \eta) - k \dot{X}^{-1} \sum_l E_l \xi(x_l - x_i - \eta) \right) c \right) = \\
= \text{res}_{\infty} \left( z^m k^{-1} c^*T \left[ E_i, B^- \right] + k \dot{X}^{-1} (\bar{D}^+ - \bar{D}^0) E_i + k \dot{X}^{-1} \sum_{l \neq i} \xi(x_l - x_i) E_l - k \dot{X}^{-1} \sum_l \xi(x_l - x_i) E_l \right) c \right).
\]

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where \( Y_i \) is the right-hand side of (6.9). Therefore, the second half of the Hamiltonian equations is proved.

We comment on how the result is to be modified if we consider a more general tau function of the form

\[
\tau(x, t) = e^{Q(x, t)} \prod_{i=1}^{N} \sigma(x - x_i(t)),
\]

where

\[
Q(x, t) = cx^2 + x \sum_{j=1}^{\infty} a_j t_j + b(t)
\]

with some constants \( c, a_j \). (Here, we set \( T = 0 \) for simplicity.) Repeating the arguments leading to (3.12), we can see that the first equation in (3.12) is modified as

\[
\dot{k} = ze^{-\eta c(\lambda) - \alpha(z)} \quad \alpha(z) = \sum_{j=1}^{\infty} \frac{a_j}{j} z^{-j}.
\]

Instead of (6.6), we then have

\[
\partial_{\mu_i} \ln k = -\frac{\partial_{\mu_i} \lambda(z)}{z \lambda'(z)} (1 - z \alpha'(z)),
\]

and hence the Hamiltonian for the \( m \)th flow is a linear combination of \( H_m \) and the \( H_j \) with \( 1 \leq j < m \).

### 6.2. Negative times.

We first obtain the relation between the velocities \( \dot{x}_i = \partial_\mu_i x_i \) and \( \dot{x}_i' = \partial_{\mu_i} x_i \).

The relation follows from Toda equation (2.16), where we substitute the tau function (1.1) for elliptic solutions:

\[
- \sum_i \dot{x}_i' \zeta(x - x_i) - \sum_i \dot{x}_i x_i' \phi(x - x_i) = 1 - \prod_i \frac{\sigma(x - x_i + \eta) \sigma(x - x_i - \eta)}{\sigma^2(x - x_i)}.
\]

Equating the coefficients in front of the second-order poles, we obtain

\[
\dot{x}_i x_i' = -\sigma^2(\eta) \prod_{j \neq i} \frac{\sigma(x_i - x_j + \eta) \sigma(x_i - x_j - \eta)}{\sigma^2(x_i - x_j)}.
\]

Our strategy is to solve linear problems (2.21) for the complementary wave functions \( \phi \) and \( \phi^\dagger \) represented as linear combinations of the elementary double-Bloch functions,

\[
\phi(x) = \tilde{k} x/\eta e^{\xi(x, z^{-1})} \sum_i b_i \Phi(x - x_i + \eta, -\mu),
\]

\[
\phi^\dagger(x) = \tilde{k}^{-1} x/\eta e^{-\xi(x, z^{-1})} \sum_i b_i^* \Phi(x - x_i - \eta, \mu),
\]

where \( \mu \) and \( \tilde{k} \) are new spectral parameters to be connected with \( \lambda \) and \( k \) later, and \( b_i \) and \( b_i^* \) are independent of \( x \). Identifying the monodromy properties of functions (6.23) and (2.20), we obtain the relations between \( \tilde{k}, z, \mu \) in the same way as relations (3.12) and (3.17) were obtained:

\[
\tilde{k} = ze^{-\eta c(\mu)}, \quad \mu = -\sum_{m \geq 1} z^m m \partial_{tm} \sum_i x_i.
\]
It then follows that
\[ \tilde{k} = ke^{-\eta \zeta(\lambda) - \eta \zeta(\mu)}. \] (6.25)

We note that the spectral parameters \( \tilde{k} \) and \( z \) here have a different meaning than in (3.12) and (3.17): they are local parameters on the spectral curve in the vicinity of the point \( P_0 \), while in (3.12) and (3.17) \( k^{-1} \) and \( z^{-1} \) are local parameters in the vicinity of \( P_\infty \).

The substitution of (6.23) in (2.21) with
\[ \bar{v}(x) = \sum_i x'_i (\zeta(x - x_i) - \zeta(x - x_i + \eta)) \]
gives, after the cancelation of poles at \( x = x_i \), the following conditions:
\[ x'_i \sum_j b_j \Phi(x_i - x_j + \eta, -\mu) = \tilde{k}^{-1} b_i, \]
\[ x'_i \sum_j b^*_j \Phi(x_i - x_j - \eta, \mu) = -\tilde{k}^{-1} b^*_i. \] (6.26)

Using identity (3.6), we can write them in matrix form as follows:
\[ b^T X'^{-1} \bar{L}^T(\mu) X' = -\tilde{k}^{-1} b^T, \quad \bar{L}(\mu)b = -\tilde{k}^{-1} b^*. \] (6.27)

Here, the matrix \( \bar{L}(\mu) \) is
\[ \bar{L}_{ij}(\mu) = x'_i \Phi(x_i - x_j - \eta, \mu) = -\sigma^2(\eta)e^{-p_i} \Phi(x_i - x_j - \eta, \mu) \prod_{l \neq i} \frac{\sigma(x_i - x_l - \eta)}{\sigma(x_i - x_l)}, \] (6.28)

where we use relation (6.22).

Equations (6.27) allow writing the equation of the spectral curve in the form
\[ \det(\tilde{k}^{-1} I + \bar{L}(\mu)) = 0. \] (6.29)

We show that it is the same spectral curve as the one given by (5.1) and discussed in Sec. 5. To show this, we find the inverse of the Lax matrix \( L(\lambda) \) given by (4.2). We have \( (L^{-1}(\lambda))_{kl} = (-1)^{k+l}/\det L(\lambda) \) and use the fact that both the numerator and the denominator here are determinants of the elliptic Cauchy matrices given explicitly by (5.4).

In particular,
\[ \det L(\lambda) = e^{N\eta \zeta(\lambda)} \left( \prod_i \hat{x}_i \right) \frac{\sigma(\lambda - N\eta)}{\sigma(\lambda)} \prod_{i \neq j} \frac{\sigma(x_i - x_j)}{\sigma(x_i - x_j - \eta)}. \] (6.30)

After some simple transformations, we obtain
\[ (L^{-1}(\lambda))_{kl} = -e^{(x_l - x_k - \eta) \zeta(\lambda)} \frac{\sigma(N\eta - \lambda + x_l - x_k + \eta)}{\sigma(N\eta - \lambda) \sigma(x_l - x_k + \eta)} \times \prod_{i \neq l} \frac{\sigma(x_l - x_i)}{\sigma(x_l - x_i + \eta)} \prod_{j \neq k} \frac{\sigma(x_k - x_j + \eta)}{\sigma(x_k - x_j)}. \] (6.31)

Hence,
\[ e^{\eta \zeta(\lambda)} (L^{T}(\lambda))^{-1} \cong -e^{-\eta \zeta(\mu)} \bar{L}(\mu), \quad \mu = N\eta - \lambda, \] (6.32)
where \( \cong \) means equality up to a similarity transformation. Taking Eqs. (6.32) and (6.25) into account, we write spectral curve (6.29) in the form

\[
\det(k^{-1}I - L^{-1}(\lambda)) = 0,
\]

which is the same as (5.1). However, we expand the spectral curve in the vicinity of different points in the previous sections and here: it was the point \( P_\infty = (\infty, 0) \) previously and is \( P_0 = (0, N\eta) \) now.

Using Eq. (2.25) and repeating the calculations leading to (6.3) and (6.5), we obtain the relations

\[
\partial_{\kappa_m} x_i = \text{res}_{0}(z^{-m-1}b^T X'^{-1} \partial_{p_i} \bar{L}(\mu)b^*)
\]

and

\[
b^T X'^{-1} b^* = \bar{k}^2 \mu'(z).
\]

A similar chain of equalities as the one after (6.35) leads to

\[
\partial_{\kappa_m} x_i = \text{res}_{0}(z^{-m-1} \partial_{p_i} \mu(z)) = \frac{\partial H_m}{\partial p_i},
\]

where

\[
H_m = \text{res}_{0}(z^{-m-1} \mu(z)) = -\text{res}_{0}(z^{-m-1} \lambda(z)).
\]

We see that the Hamiltonians for the negative-time flows are obtained from the expansion of \( \lambda(z) \) as \( z \to 0 \):

\[
\lambda(z) = N\eta - \sum_{m \geq 1} H_m z^m.
\]

The other half of the Hamiltonian equations,

\[
\partial_{\kappa_m} p_i = -\frac{\partial H_m}{\partial x_i},
\]

can be proved in the same way as Eqs. (6.11) were proved.

7. Degenerations of elliptic solutions

7.1. Rational limit. In the rational limit \( \omega, \omega' \to \infty, \sigma(x) = x \) and

\[
L_{ij}(\lambda) = \left( \frac{\dot{x}_i}{x_i - x_j - \eta} + \frac{\dot{x}_j}{\lambda} \right) e^{-(x_i-x_j)/\lambda+\eta/\lambda},
\]

where

\[
\dot{x}_i = e^{p_i} \prod_{l \neq i} \frac{x_i - x_l + \eta}{x_i - x_l}.
\]

The equation of the spectral curve is

\[
\det(k I - e^{\eta/\lambda}(L_{\text{rat}} + \lambda^{-1} \dot{X} E)) = 0,
\]

where \( L_{\text{rat}} \) is the Lax matrix of the rational Ruijsenaars–Schneider system with matrix elements

\[
(L_{\text{rat}})_{ij} = \frac{\dot{x}_i}{x_i - x_j - \eta}.
\]
Recalling the connection between the spectral parameters $k$, $z$, and $\lambda$, $k = ze^{\eta/\lambda}$, we can write the equation of the spectral curve in the form

$$\det(zI - L_{\text{rat}} - \lambda^{-1}\dot{XE}) = 0. \quad (7.3)$$

Because $E$ is a rank-1 matrix, we have

$$\det\left(I - \lambda^{-1}\dot{XE} \frac{1}{zI - L_{\text{rat}}}\right) = 1 - \lambda^{-1} \text{tr}\left(\dot{XE} \frac{1}{zI - L_{\text{rat}}}\right) = 0,$$

whence

$$\lambda = \text{tr}\left(\dot{XE} \frac{1}{zI - L_{\text{rat}}}\right). \quad (7.4)$$

As $z \to \infty$, we expand this as

$$\lambda(z) = \sum_{m \geq 1} z^{-m} \text{tr}(\dot{XE}^{-m').} \quad (7.5)$$

It is easy to verify the commutation relation

$$XL_{\text{rat}} - L_{\text{rat}} X = \dot{XE} + \eta L_{\text{rat}}. \quad (7.6)$$

Using it, we have

$$\text{tr}(\dot{XE}^{-m'}) = \text{tr}(XL_{\text{rat}}^{-m} - L_{\text{rat}}^{-m} X) = -\eta \text{tr} L_{\text{rat}}^{-m}.$$ 

We hence conclude that

$$\lambda(z) = -\eta \sum_{m \geq 1} z^{-m} \text{tr} L_{\text{rat}}^{m}, \quad z \to \infty, \quad (7.7)$$

and thus the Hamiltonians for positive-time flows are

$$H_{m} = -\eta \text{tr} L_{\text{rat}}^{m}. \quad (7.8)$$

This agrees with the result in [18], [19].

To find the Hamiltonians for negative-time flows, we expand (7.4) as $z \to 0$:

$$\lambda(z) = -\sum_{m \geq 0} z^{m} \text{tr}(\dot{XE}^{-m'}) = N\eta + \eta \sum_{m \geq 1} z^{m} \text{tr} L_{\text{rat}}^{-m}. \quad (7.9)$$

Therefore,

$$H_{m} = -\eta \text{tr} L_{\text{rat}}^{-m}. \quad (7.10)$$

This agrees with the result in [20].

### 7.2. Trigonometric limit.

We now pass to the trigonometric limit. Let $\pi i/\gamma$ be the period of the trigonometric (or hyperbolic) functions, with the second period tending to infinity. The Weierstrass functions in this limit are

$$\sigma(x) = \gamma^{-1}e^{-\gamma x^2/6} \sinh(\gamma x), \quad \zeta(x) = \gamma \coth(\gamma x) - \frac{1}{3}\gamma^2 x.$$ 

The tau function for trigonometric solutions is

$$\tau = \prod_{i=1}^{N}(e^{2\gamma x_i} - e^{-2\gamma x_i}), \quad (7.11)$$
and hence we should consider
\[ \tau = \prod_{i=1}^{N} \sigma(x - x_i)e^{\gamma(x-x_i)^2/6 + \gamma(x+x_i)}, \] (7.12)

where the exponential factor is inserted to ensure the correct trigonometric limit (7.11) of the elliptic tau function. Similarly to the KP case [13], Eq. (3.12) with this choice acquires the form
\[ \ln k = \ln z + \eta \gamma \coth(\gamma \lambda). \] (7.13)

The trigonometric limit of the function \( \Phi(x, \lambda) \) is
\[ \Phi(x, \lambda) = \gamma (\coth(\gamma x) + \coth(\gamma \lambda))e^{-\gamma x \coth(\gamma \lambda)}, \]
and \( L(\lambda) \) takes the form
\[ L_{ij}(\lambda) = \gamma e^{\eta \gamma \coth(\gamma \lambda)}e^{-\gamma \coth(\gamma \lambda)(x_i - x_j)}(\dot{x}_i \coth(\gamma (x_i - x_j) - \eta)) + \dot{x}_i \coth(\gamma \lambda)). \]

For further calculations, it is convenient to change the variables as
\[ w_i = e^{2\gamma x_i}, \quad q = e^{2\gamma \eta} \] (7.14)
and introduce the diagonal matrix \( W = \text{diag}(w_1, w_2, \ldots, w_N) \). In this notation, the equation of the spectral curve acquires the form
\[ \det(zI - q^{-1/2}W^{1/2}L_{\text{trig}}W^{-1/2} - \gamma (\coth(\gamma \lambda) - 1)\dot{X}E) = 0, \] (7.15)
where \( L_{\text{trig}} \) is the Lax matrix of the trigonometric Ruijsenaars–Schneider model:
\[ (L_{\text{trig}})_{ij} = 2\gamma q^{1/2} \frac{\dot{x}_i w_i^{-1/2} w_j^{1/2}}{w_i - qw_j} \] (7.16)
(see [20]). Again, because \( E \) is a rank-1 matrix, we can use (7.15) to obtain
\[ (\coth(\gamma \lambda) - 1)^{-1} = \gamma \text{tr} \left( \frac{1}{zI - q^{-1/2}W^{1/2}L_{\text{trig}}W^{-1/2}} \right) \]
or
\[ \lambda = \frac{1}{2\gamma} \ln \left( 1 + 2\gamma \text{tr} \left( \frac{1}{zI - q^{-1/2}W^{1/2}L_{\text{trig}}W^{-1/2}} \right) \right). \]

Applying the formula \( \det(I + Z) = 1 + \text{tr} Z \) for any matrix \( Z \) of rank 1 in the opposite direction, we have
\[ \lambda = \frac{1}{2\gamma} \ln \det \left( (zI - q^{-1/2}W^{1/2}L_{\text{trig}}W^{-1/2} + 2\gamma \dot{X}E) \times \frac{1}{zI - q^{-1/2}W^{1/2}L_{\text{trig}}W^{-1/2}} \right). \] (7.17)

Next, we use the trigonometric analogue of relation (7.6),
\[ 2\gamma \dot{X}E = q^{-1/2}W^{1/2}L_{\text{trig}}W^{-1/2} - q^{1/2}W^{-1/2}L_{\text{trig}}W^{1/2}, \] (7.18)
which can be verified easily. Using this relation, we deduce from (7.17) that
\[
\lambda = \frac{1}{2\gamma} \ln \det (zI - q^{1/2}W^{-1/2}L_{\text{trig}}W^{1/2}) - \frac{1}{2\gamma} \ln \det (zI - q^{-1/2}W^{1/2}L_{\text{trig}}W^{-1/2}) = 1 - z^{-1}q^{1/2}L_{\text{trig}}^{-1} = - \sum_{m \geq 1} z^{-m} q^{m/2} - q^{-m/2} \frac{2\gamma}{2\gamma} \ln \det L_{\text{trig}}^{-1}.
\]

Therefore, we finally obtain
\[
\lambda(z) = - \sum_{m \geq 1} z^{-m} \frac{\sinh(\gamma\eta m)}{\gamma m} \ln \det L_{\text{trig}}^{-1}, \tag{7.19}
\]
and hence the Hamiltonians are
\[
H_m = - \frac{\sinh(\gamma\eta m)}{\gamma m} \ln \det L_{\text{trig}}^{-1}. \tag{7.20}
\]
This agrees with the result in [20].

8. Examples of the Hamiltonians

We return to the general elliptic case and introduce renormalized integrals of motion
\[
J_m = \frac{\sigma(m\eta)}{\sigma(m\eta)} I_m, \quad m = \pm 1, \pm 2, \ldots, \pm N, \tag{8.1}
\]
where \(I_m\) are the integrals of motion in (1.3) and (1.7). The equation of the spectral curve, Eq. (5.5), is
\[
z^N + \sum_{n=1}^{N} \phi_n(\lambda) J_n z^{N-n} = 0, \tag{8.2}
\]
where
\[
\phi_n(\lambda) = \frac{\sigma(\lambda - n\eta)}{\sigma(\lambda)\sigma(n\eta)}. \tag{8.3}
\]
It can be expanded as \(\lambda \to 0\) (\(z \to \infty\)) as follows:
\[
\phi_n(\lambda) = - \frac{1}{\sigma(\lambda)} + \zeta(n\eta) \frac{\lambda}{\sigma(\lambda)} - (\zeta^2(n\eta) - \varphi(n\eta)) \frac{\lambda^2}{2\sigma(\lambda)} + O(z^{-2}).
\]
Expanding the equation of the spectral curve, we have
\[
1 + \zeta(\eta) J_1 z^{-1} - \frac{1}{2}(\zeta^2(\eta) - \varphi(\eta)) J_1^2 z^{-2} + \zeta(2\eta) J_2 z^{-2} = \frac{1}{\lambda} (J_1 J_2 z^{-1} + J_3 z^{-2}) + O(z^{-3})
\]
or
\[
\lambda(z) = \frac{z^{-1} (J_1 J_2 z^{-1} + J_3 z^{-2} + O(z^{-3}))}{1 + \zeta(\eta) J_1 z^{-1} + \zeta(2\eta) J_2 z^{-2} - (1/2)(\zeta^2(\eta) - \varphi(\eta)) J_1^2 z^{-2} + O(z^{-3})}
\]
Expanding this in a series, \(\lambda(z) = \sum_{m \geq 1} H_m z^{-m}\), we find the first three Hamiltonians:
\[
H_1 = J_1,
H_2 = J_2 - \zeta(\eta) J_1^2,
H_3 = J_3 - (\zeta(\eta) + \zeta(2\eta)) J_1 J_2 + \left(\frac{3}{2} \zeta^2(\eta) - \frac{1}{2} \varphi(\eta)\right) J_1^3. \tag{8.4}
\]
In the rational limit, we can obtain a general formula for $H_m$. In this limit,

$$\phi_n(\lambda) = \frac{1}{n\eta} - \frac{1}{\lambda}$$

and the equation of the spectral curve becomes

$$z^N + \frac{1}{\eta} \sum_{n=1}^{N} J_n z^{N-n} = \frac{1}{\lambda} \sum_{n=1}^{N} J_n z^{N-n},$$

(8.5)

or, recalling that $J_n = n\eta^{1-n}I_n$ in the rational limit,

$$\lambda(z) = \eta \sum_{n=1}^{N} nI_n(\eta z)^{-n} \left[ 1 + \sum_{n=1}^{N} I_n(\eta z)^{-n} \right].$$

(8.6)

Expanding the right-hand side in powers of $z^{-1}$, we obtain

$$H_1 = I_1,$$
$$H_2 = \eta^{-1}(2I_2 - I_1^2),$$
$$H_3 = \eta^{-2}(3I_3 - 3I_1I_2 + I_1^3).$$

(8.7)

In general, it follows from (8.6) that

$$H_m = (-\eta)^{1-m} \sum_{\nu=m} C^m_{\nu} I_{\nu_1} \cdots I_{\nu_\ell},$$

(8.8)

where the sum is taken over all Young diagrams $\nu$ with $|\nu| = m$ boxes having nonzero rows $\nu_1, \ldots, \nu_\ell$, and $C^m_{\nu}$ is the matrix of transition from the basis of elementary symmetric polynomials $e_j$ to the power sums $p_k$. Indeed, it is easy to see that the equality

$$\sum_{n=1}^{N} (-1)^{n-1} p_n z^{-n} = \sum_{m=1}^{N} me_m z^{N-m} \left[ \sum_{r=0}^{N} e_r z^{N-r} \right]$$

is equivalent to the well-known Newton identity

$$me_m = \sum_{n=1}^{m} (-1)^{n-1} p_n e_{m-n}$$

for the symmetric functions. The explicit formula is

$$H_m = -m\eta^{1-m} \sum_{r_1+2r_2+\cdots+mr_m=m} \frac{(r_1 + \cdots + r_m - 1)!}{r_1! \cdots r_m!} \prod_{i=1}^{m} (-I_i)^{r_i}.$$

(8.9)

We now consider Hamiltonians for the negative-time flows. The equation of the spectral curve (5.5) can be rewritten in the form

$$\varphi_N(\lambda) + \sum_{n=1}^{N} \varphi_{N-n}(\lambda) I_{-n} z^n = 0,$$

(8.10)
or, equivalently,

\[ \sigma(\mu) + \sum_{n=1}^{N} \frac{\sigma(n\eta - \mu)}{\sigma(n\eta)} J_{-n} z^n = 0, \quad \mu \equiv N\eta - \lambda. \]  \tag{8.11}

Expanding \( \mu \to 0 \) in powers of \( z \) as \( \mu(z) = \sum_{m \geq 1} H_m z^m \), we obtain

\[ H_1 = -J_{-1}, \]
\[ H_2 = -J_{-2} - \zeta(\eta) J_{-1}^2, \]
\[ H_3 = -J_{-3} - (\zeta(\eta) + \zeta(2\eta)) J_{-1} J_{-2} - \left( \frac{3}{2} \zeta^2(\eta) \right) J_{-1}^3. \]  \tag{8.12}

In the rational limit, we have

\[ H_1 = \eta^2 I_{-1}, \]
\[ H_2 = \eta^3 (2I_{-2} - I_{-1}^2), \]
\[ H_3 = \eta^4 (3I_{-3} - 3I_{-1} I_{-2} + I_{-1}^3). \]  \tag{8.13}

We can see that the properly arranged limit \( \eta \to 0 \) in (8.7) of the Hamiltonians for positive-time flows yields Hamiltonians of the Calogero–Moser model, while Hamiltonians (8.13) for negative-time flows vanish in this limit.

9. Conclusion

The main result in this paper is the precise correspondence between elliptic solutions of the 2D Toda lattice hierarchy and the hierarchy of the Hamiltonian equations for the integrable elliptic Ruijsenaars–Schneider model with higher Hamiltonians. The zeros of the tau function move as particles of the Ruijsenaars–Schneider model. We have shown that the \( m \)th-time flow \( t_m \) of the 2DTL hierarchy gives rise to the flow with the Hamiltonian \( H_m \) of the Ruijsenaars–Schneider model, obtained as the \( m \)th coefficient of the expansion of the spectral curve \( \lambda(z) \) in negative powers of \( z \) as \( z \to \infty \). Moreover, the \( m \)th-time flow \( \bar{t}_m \) of the 2DTL hierarchy corresponds to the flow with the Hamiltonian \( \bar{H}_m \) of the Ruijsenaars–Schneider model. obtained as the \( m \)th coefficient of the expansion of the spectral curve \( \lambda(z) \) in positive powers of \( z \) as \( z \to 0 \). The first few Hamiltonians were found explicitly.

For rational and trigonometric degenerations of elliptic solutions, the previous results in [18]–[20], obtained there by a different method, are reproduced here: the \( t_m \) time flow of the 2DTL hierarchy gives rise to the flow with the Hamiltonian \( H_m \) proportional to \( \text{tr} L^m \), where \( L \) is the Lax matrix, while the \( \bar{t}_m \) time flow gives rise to the Hamiltonian flow with the Hamiltonian \( \bar{H}_m \) proportional to \( \text{tr} L^{-m} \).

Conflict of interest. The authors declare no conflicts of interest.

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