The rotation curve of a point particle in stringy gravity

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Abstract. Double Field Theory suggests to view the whole massless sector of closed strings as the gravitational unity. The fundamental symmetries therein, including the $O(D, D)$ covariance, can determine unambiguously how the Standard Model as well as a relativistic point particle should couple to the closed string massless sector. The theory also refines the notion of singularity. We consider the most general, spherically symmetric, asymptotically flat, static vacuum solution to $D = 4$ Double Field Theory, which contains three free parameters and consequently generalizes the Schwarzschild geometry. Analyzing the circular geodesic of a point particle in string frame, we obtain the orbital velocity as a function of $R/(M_\infty G)$ which is the dimensionless radial variable normalized by mass. The rotation curve generically features a maximum and thus non-Keplerian over a finite range, while becoming asymptotically Keplerian at infinity, $R/(M_\infty G) \rightarrow \infty$. The adoption of the string frame rather than Einstein frame is the consequence of the fundamental symmetry principle. Our result opens up a new scheme to solve the dark matter/energy problems by modifying General Relativity at ‘short’ range of $R/(M_\infty G)$.

Keywords: dark matter theory, modified gravity, dark energy theory

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1 Introduction

The galaxy rotation curve is a plot of the orbital velocities of visible stars versus their radial distance from the galactic center, see figure 1. While Einstein gravity, i.e. General Relativity, predicts the Keplerian (inverse square root) monotonic fall-off of the velocities, observations however show rather ‘flat’ (100 ∼ 200 km/s) curves after a fairly rapid rise [1]. The resolution of the discrepancy might call for dark matter or modifications of the law of gravity [2], or perhaps both, e.g. [3]. However — despite remarkable improvements of experimental sensitivity — there has been no direct evidence of detecting any dark matter candidate. This failure might well motivate to explore various possibilities of modifying gravity, General Relativity.

In General Relativity the metric is the only geometric object. All other fields are viewed as matter or radiation; they source the gravity. On the other hand, string theory puts a two-form gauge potential and a scalar dilaton on an equal footing along with the metric, since the three of them, conventionally denoted by $g_{\mu\nu}$, $B_{\mu\nu}$, $\phi$, correspond to the massless sector of closed strings and form a multiplet of T-duality. This may indicate the existence of an alternative gravitational theory where the whole closed string massless sector becomes geometric as the gravitational unity. Such an idea, or Stringy Gravity, has been materialized in recent years through the developments of Double Field Theory (DFT).

The primary goal of DFT [6–8] was to reformulate supergravity with doubled coordinates, $x^A = (\tilde{x}_\mu, x^\nu)$, in a way that realizes T-duality as a manifest symmetry of the action and unifies diffeomorphisms and $B$-field gauge symmetry into ‘doubled diffeomorphisms’ [9–11]. The closed string massless sector should be then better represented by T-duality or $O(D, D)$ covariant field variables, namely the DFT-metric, $\mathcal{H}_{AB}$, and the DFT-dilaton, $d$. The underlying differential geometry has been subsequently explored in various manners [7, 12–19] which all suggested to generalize the Riemannian geometry, often making contact with the ‘Generalized Geometry’ a la Hitchin [20], e.g. [21] (we refer to review papers [22–24] on various aspects of DFT).
Figure 1. Observed galaxy rotation curve from ref. [1] (figure 7 therein). The curve shows a fairly rapid velocity rise and a slower rise (or flat) thereafter. For more figures, we refer to [4, 5], or Google Search.

In particular, in [14] based on [12], the stringy extension of the Christoffel connection, \( \Gamma_A \), was derived (2.20) which is made up of the whole closed string massless sector now given by \( H_{AB} \) and \( d \). Subsequently, it constitutes the two-indexed (Ricci-type) as well as zero-indexed (scalar) covariant curvatures of DFT (2.23), and hence furnishes the theory with geometrical interpretations. Further, (from the covariant constancy of the DFT-vielbeins), the connection, \( \Gamma_A \), determines a pair of spin connections, \( \Phi_A \) \& \( \bar{\Phi}_A \), for the doubled local Lorentz symmetries, \( \text{Spin}(1, D-1) \times \text{Spin}(D-1, 1) \) [15]. This twofold spin group reflects the existing two separate locally inertial frames for each left and right closed string mode [25]. Crucially, combining all the connections, a master derivative is at our disposal (see [26] for a concise review),

\[
D_A = \partial_A + \Gamma_A + \Phi_A + \bar{\Phi}_A, \tag{1.1}
\]

which takes care of the fundamental symmetries of the stringy gravity, i.e. DFT:

\begin{itemize}
  \item \( \text{O}(D, D) \) T-duality,
  \item Doubled diffeomorphisms,
  \item Two-fold local Lorentz symmetries.
\end{itemize}

The master derivative has been successfully utilized to complete the full order supersymmetrizations of DFT [28–30], making each term in every formula completely covariant under the fundamental symmetries (cf. [21, 31]).

Besides the direct applications to string theory, the master derivative naturally provides the minimal coupling of the closed string massless sector to the Standard Model [26]. Each fermion therein couples to the closed string massless sector as [15, 21, 26],

\[
e^{-2d} \bar{\psi} \gamma^A D_A \psi = e^{-2d} \bar{\psi} \gamma^A \left( \partial_A \psi + \frac{1}{4} \Phi_{A pq} \gamma^{pq} \psi \right)
\]

\[
\equiv \frac{1}{\sqrt{2}} g e^{-2\phi} \bar{\psi} \gamma^A \left( \partial_A \psi + \frac{1}{4} \omega_{\mu pq} \gamma^{pq} \psi + \frac{1}{24} H_{\mu pq} \gamma^{pq} \psi - \partial_\mu \phi \psi \right) \tag{1.2}
\]

\[
\equiv \sqrt{-g} \bar{\chi} \gamma^A \left( \partial_{\mu} \chi + \frac{1}{4} \omega_{\mu pq} \gamma^{pq} \chi + \frac{1}{24} H_{\mu pq} \gamma^{pq} \chi \right) ,
\]

\( ^1 \)Consequently, each fermion sources the two-indexed DFT-curvature, \( S_{\mu \nu} \), by \( \bar{\psi} \gamma_\nu D_\mu \psi \) [28, 29].
where the $O(4, 4)$ covariant DFT-field variables on the top line have been parametrized, \(\equiv\), in terms of the conventional (undoubled) spin connection, $\omega_{\mu pq}$, the $H$-flux and the scalar dilaton, $\phi$. Further, especially for the last expression, the field redefinition of the fermion, $\chi \equiv 2^{-\frac{1}{4}} e^{-\phi}\psi$, has been performed to remove the scalar dilaton completely. The result of (1.2) shows that, (not only the fundamental string but also) the Standard Model fermions can source the $H$-flux! It indicates the stringy nature, if not origin, of the fermion, $\chi$. Similarly, gauge bosons in the Standard Model couple to the closed string massless sector as \[\text{and hence they can source } \phi, \text{ the scalar dilaton. Surely, all the fields in the Standard Model source the (string framed) metric, } g_{\mu\nu}.\]

This line of development suggests that DFT is not a mere reformulation of supergravity; it gives rise to the stringy extension of General Relativity as a (theoretically) possible alternative theory of gravity.

It is the purpose of the present paper to push this idea further, and to derive novel theoretical predictions which differ from that of Einstein gravity and can be, in principle, tested against observations. Yet, we intend neither to make any provocative claim that DFT should replace General Relativity as the correct theory of gravity, nor to falsify DFT comparing with precise observational data. Rather, we merely aspire to postulate DFT as a ‘theoretically’ plausible gravitational theory and to explore its various physical aspects, especially the implications of the fundamental symmetries of $O(D, D)$ T-duality and doubled diffeomorphisms.

In the sense that DFT does not take only the metric, $g_{\mu\nu}$, as the gravitational fields, it is somewhat similar to the Brans-Dicke theory where the gravitational interaction is mediated by a scalar field as well as the metric. The Brans-Dicke theory contains a tunable dimensionless parameter, $\omega$, while DFT does not admit any free parameter as strongly constrained by the fundamental symmetries. Observations of the light deflection in solar system — in particular derived from the Cassini-Huygens experiment — currently set the lower bound, $\omega > 40,000$. From the “principle” of Occam’s razor, the Brans-Dicke theory appears then less favored in comparison to General Relativity. This might motivate a haste tendency to rule out any gravitational theory with a massless scalar field, such as DFT. However, because of the enlarged fundamental symmetries, DFT includes $H$-flux in addition to a scalar dilaton, $\phi$, and consequently it enriches the possible spherically symmetric vacuum geometry — see (3.12) — and renders a priori more room to meet the observational constraints, e.g. light deflection. Furthermore, after field redefinition, $\phi \rightarrow \Phi := e^{-\phi}$, the scalar, $\Phi$, acquires an effective mass given by the scalar curvature in the Lagrangian (2.28), as $4\partial_{\mu}\Phi\partial^{\mu}\Phi + R\Phi^2$. Thus, on a curved background, the scalar $\Phi$ is effectively massive and can modify the short distance gravity, which is indeed the case as we show in the present work. Note that, for this, it is crucial to adopt not the Einstein frame but the string frame. We shall justify this choice of the frame from the symmetry principle.

In this work, motivated by the Standard Model coupling to the closed string massless sector \[D = 4\text{ DFT,}\] we firstly look for spherically symmetric vacuum solutions to which are in analogy to the Schwarzschild solution to Einstein gravity. We address the notion of spherical symmetry in DFT, in terms of a priori not the conventional field variables, \{\(g_{\mu\nu}, B_{\mu\nu}, \phi\)\}, but the \(O(4, 4)\) covariant DFT-metric and DFT-dilaton, \{\(H_{AB}, d\)\}: we spell three DFT-Killing vectors (3.6) which form an \(\text{so}(3)\) algebra through C-bracket (3.4).
By solving directly the DFT-Killing equations and the Euler-Lagrangian equations of motion, we identify the most general form of the spherically symmetric, asymptotically flat, static vacuum solutions to $D = 4$ DFT, (3.10), which turns out to possess three free parameters, including one for the electric $H$-flux.

Though the backbone of the present work is the fundamental symmetry principle of DFT, in practice, with the spherically symmetric ansatz (3.5), we are solving the full Euler-Lagrangian equations of the rather familiar gravity action of the closed string massless sector (2.28). They are equivalent to the vanishing of both the two-indexed Ricci and the zero-indexed scalar DFT-curvatures (2.23), and thus the solution can be identified as the spherically symmetric $DFT$-vacuum. Essentially, our result of the solution is a re-derivation of the known one by Burgess, Myers and Quevedo (BMQ) [32]. Historically, Fischer in 1948 [33], and Janis, Newman and Winicour later in 1968 [34] (F-JNW) obtained the most general spherical solution to the Einstein gravity coupled to a massless scalar field. Then adding one more scalar, or an axion, and making use of the $\text{SL}(2,\mathbb{R})$ S-duality, BMQ managed to generate a three-parameter family of spherical solutions. The axion is dual to the $H$-flux and our solution fully agrees with the BMQ solution. Yet, since they focused on a time-independent dual scalar, the possibility of having magnetic $H$-flux was excluded from the beginning in their analysis. Our spherically symmetric ansatz allows both electric and magnetic $H$-fluxes. Nonetheless we show that, the magnetic $H$-flux is inconsistent with the asymptotic flatness, see (B.14) and (B.17).

Having more than one parameters, the BMQ solution is generically ‘hairy’; the center would correspond to a “naked singularity”. Only if the scalar dilaton, $\phi$, and the $H$-flux (axion) are trivial, the solution is free of a naked singularity and reduces identically to the Schwarzschild metric. However, strictly speaking, within the framework of DFT, the notion of singularity should be addressed in terms of its own covariant curvatures. Since we are solving for the $DFT$-vacuum with the vanishing DFT-curvatures (both two-index and zero-index), while there seems no fully covariant Riemann-type four-index curvature in DFT [14, 16], we shall rather not be concerned with the issue of singularity (cf. [35] for an intriguing analysis on the photon sphere of the F-JNW geometry).

Once the spherically symmetric vacuum solution to $D = 4$ DFT is fully identified, we shall proceed to analyze the geodesic of a point particle on the vacuum geometry and derive the corresponding rotation curve, with the intention of making a comparison with the galaxy observations [figures 1 and 2]. In contrast to the null geodesic of a massless photon, the massive particle geodesic depends sensitively on the choice of the frame, i.e. string (Jordan) versus Einstein. Whilst this ambiguity cannot be resolved to full satisfaction in the conventional theories based on the Riemannian geometry (cf. [36]), we show that the fundamental symmetries of DFT do the job: the symmetries of $\text{O}(D,D)$ T-duality and doubled diffeomorphisms dictate that the point particle should follow the geodesic defined not in the Einstein frame but in the string frame. Specifically, we spell an $\text{O}(D,D)$ covariant (doubled) action for a relativistic point particle coupled to the DFT-metric in (2.11) which can reduce consistently to the conventional (undoubled) particle action coupled to the string frame metric, (2.17).

The rest of the present paper is organized as follows. In section 2, we first review the concept of ‘doubled-yet-gauged spacetime’ [37], which provides a geometric meaning to the doubled coordinates and the associated section condition. These two are characteristics of DFT, i.e. the stringy extension of Einstein gravity. We spell an action for a relativistic point particle which propagates in the doubled-yet-gauged spacetime and couples to the closed string massless sector in an $\text{O}(D,D)$ covariant manner, (2.11). We also review briefly the
geometric formulation of DFT and its Euler-Lagrangian equations of motion. Section 3 contains most of our main results. We write the three DFT-Killing vectors which form the \( \text{so}(3) \) \( C \)-bracket relation. We identify the most general form of the DFT-vacuum solutions which are spherically symmetric, static and asymptotically flat (cf. BMQ \[32\]). We then focus on the circular geodesic motion of a relativistic point particle propagating around the spherically symmetric DFT-vacuum. We compute the orbital velocity and depict the rotation curves as a function of radius in various limits of the three free parameters. In contract to the Keplerian (inverse square root) monotonic fall-off on the Schwarzschild geometry, the radial curve around a generic, spherically symmetric DFT-vacuum features a maximum. Yet, eventually at spatial infinity, it becomes asymptotically Keplerian. We conclude with comments in section 4. In particular, we point out that, observations of galaxies far away might well reveal the short-distance nature of the gravitational law (cf. ‘Cosmic Uroboros’). Appendix contains more rotation curves for various choices of the free parameters of the DFT-vacua as well as some technical derivations of the main results.

2 Point particle and stringy gravity in doubled-yet-gauged spacetime

2.1 \( O(D, D) \) covariant action for a point particle coupled to the DFT-metric

In order to describe the phenomenologically apparent, four-dimensional spacetime, we employ the eight-dimensional, doubled-yet-gauged coordinate system \[37\], where \( \mathcal{J} \) an \( O(4, 4) \) group is postulated with the \( 8 \times 8 \) invariant metric put in the off-block diagonal form,

\[
\mathcal{J}_{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

(2.1)

which along with its inverse, \( \mathcal{J}^{AB} \), can freely lower or raise the doubled vector indices, \( A = 1, 2, \cdots, 8 \), and \( ii) \) the doubled coordinates, \( x^A = (\bar{x}_i, x^\nu) \), are gauged through an equivalence relation, namely the coordinate gauge symmetry,

\[
x^A \sim x^A + \mathcal{J}^{AB} \Phi_i(x) \partial_B \Phi_j(x).
\]

(2.2)

Here and henceforth, \( \Phi_i, \Phi_j \) denote the arbitrary fields and their arbitrary derivatives which should ‘belong’ to the theory, i.e. DFT. The equivalence relation (2.2) is realized in DFT — as for a target spacetime perspective — simply by requiring that all the functions are invariant under the coordinate gauge symmetry shift,

\[
\Phi_i(x) = \Phi_i(x + \Delta), \quad \Delta^A = \Phi_j(x) \partial^A \Phi_k(x).
\]

(2.3)

In fact, as can be seen easily from the power series expansion \[37, 38\], the invariance is equivalent to the so-called ‘section condition’ \[7\]:

\[
\partial_A \Phi_i \partial^A \Phi_j = 0, \quad \partial_A \partial^A \Phi_k = 0.
\]

(2.4)

Upon the section condition, the generalized Lie derivatives \[7, 9\],

\[
\hat{L}_V T_{M_1 \cdots M_n} := V^N \partial_N T_{M_1 \cdots M_n} + \omega \partial_N V^N T_{M_1 \cdots M_n} + \sum_{i=1}^n (\partial_M V_N - \partial_N V_M) T_{M_1 \cdots M_{i-1} N M_{i+1} \cdots M_n},
\]

(2.5)

In (2.4), the former (strong) constraint implies the latter (weak) one, since \( \partial_A \partial_B \Phi \partial^B \partial^C \Phi = 0 \) means that \( \partial_A \partial^B \Phi \) is a nilpotent matrix and hence is traceless. Yet, replacing \( \Phi_k \) by the product, \( \Phi_i \Phi_j \), the latter may give the former.
are closed under commutations through so-called the C-bracket,
\[
\left[ \hat{\mathcal{L}}_U, \hat{\mathcal{L}}_V \right] = \hat{\mathcal{L}}_{[U,V]}^C, \quad [U,V]^C := U^N \partial_N V^M - V^N \partial_N U^M + \frac{1}{2} V^N \partial^M U_N - \frac{1}{2} U^N \partial^M V_N .
\]
(2.6)
Thus, it generates the diffeomorphisms in the doubled-yet-gauged spacetime.

On the other hand, on a particle worldline, or on a string worldsheet, the doubled coordinates are dynamical fields and need to be gauged explicitly with the introduction of a relevant gauge potential [38],
\[
D^A_x := d^A_x - A^A_A .
\]
(2.7)
As in any gauge theory the gauge potential, $A^A_A$, should meet precisely the same property as the gauge generator which is in the present case, ‘derivative-index-valued’ $\Delta^A$ in (2.3), such that
\[
\Delta^A \partial_A = 0, \quad A^A_A A_A = 0 .
\]
(2.8)
It is crucial to note that $D^A_x$ is a covariant vector of DFT and also is invariant under the coordinate gauge symmetry, but the ordinary infinitesimal one-form, $d^A_x$, is anomalous [38]. Under infinitesimal diffeomorphisms, $\delta_V x^A = V^A(x)$, as well as the coordinate gauge symmetry, $\delta_\Delta x^A = \Delta^A(x)$, provided the the potential transforms properly, respecting (2.8), as
\[
\delta_V A^A = -\partial^A V_B A^B + \partial^A V_B dx^B, \quad \delta_\Delta A^A = d\Delta^A,
\]
(2.9)
we have the covariance as well as the invariance,
\[
\delta_V (\partial_A) = (\partial^B V_A - \partial_A V^B)\partial_B, \quad \delta_\Delta (\partial_A) = 0 ,
\]
\[
\delta_V (D^A_x) = (\partial^B V_A - \partial_A V^B)D^B_x, \quad \delta_\Delta (D^A_x) = 0 .
\]
(2.10)
The fundamental symmetries, together with the coordinate gauge symmetry, then uniquely fix the relativistic point particle action on a generic closed string background:
\[
S_{\text{particle}} = \int d\tau e^{-1} D^A_x D^A_x H_{AB}(x) - \frac{1}{4} m^2 e ,
\]
(2.11)
where $e$ is an einbein, $m$ is the mass of the particle and $H_{AB}$ is the DFT-metric which is, by definition, a symmetric $O(D,D)$ element:
\[
H_{AB} = H_{BA}, \quad H^C_A H^D_B J_{CD} = J_{AB} .
\]
(2.12)
In general, up to $O(D,D)$ rotations, the section conditions of (2.4) and (2.8) are solved by letting
\[
\partial_A = (\tilde{\partial}^\mu, \partial_\nu) \equiv (0, \partial_\nu), \quad A_A \equiv (0, A_\nu) .
\]
(2.13)
Consequently, only the dual tilde-coordinates are gauged:
\[
D_x^A \equiv (\tilde{x}^\mu - A_\mu, \tilde{x}^\nu) .
\]
(2.14)
Also, the DFT-metric and the DFT-dilaton can be conventionally parametrized by the string frame metric, the $B$-field and the scalar dilaton:\footnote{In fact, eq. (2.15) gives the generic parametrization of the DFT-metric whose upper left $D \times D$ block is non-degenerate. If it is degenerate, the DFT-metric should be parametrized differently. Such a background was explicitly obtained through a T-duality rotation along temporal directions [38], and was shown in [41] to realize a ‘non-relativistic’ string background.}
\[
H_{AB} \equiv \begin{pmatrix} g^{-1} & -g^{-1} B \\ B g^{-1} & g - B g^{-1} B \end{pmatrix}, \quad e^{-2d} \equiv \sqrt{-g} e^{-2\phi} .
\]
(2.15)
An instructive relation follows
\[ D_\tau x^A D_\tau x^B \mathcal{H}_{AB} \equiv \dot{x}^\mu \dot{x}^\nu g_{\mu\nu} + \left( \dot{x}_\mu - A_\mu + \dot{x}^\rho B_{\rho\mu} \right) \left( \dot{x}_\nu - A_\nu + \dot{x}^\sigma B_{\sigma\nu} \right) g^{\mu\nu}. \tag{2.16} \]

Now, integrating out the auxiliary gauge potential, \( A_\mu \), the fully symmetric action (2.11) reduces to the standard action for a relativistic point particle coupled only to the string frame metric:
\[ S_{\text{particle}} \equiv \int d\tau \, e^{-1} \dot{x}^\mu \dot{x}_\mu - e^{-1} m^2. \tag{2.17} \]

This implies that the particle follows the geodesic path defined in the string frame. We stress that this preferred choice of the frame is due to the fundamental symmetry principle of DFT.

### 2.2 Pure DFT and its Euler-Lagrangian equations of motion

In DFT, the massless sector of closed strings is represented by the DFT-dilaton, \( d \), and the DFT-metric, \( \mathcal{H}_{AB} \), satisfying the defining property (2.12). With the \( O(D,D) \) invariant metric, \( J_{AB} \), the latter defines a pair of projectors,
\[ P_{AB} = \frac{1}{2}(J_{AB} + \mathcal{H}_{AB}), \quad P_A^B P_B^C = P_A^C, \quad \bar{P}_{AB} = \frac{1}{2}(J_{AB} - \mathcal{H}_{AB}), \quad \bar{P}_A^B \bar{P}_B^C = \bar{P}_A^C, \tag{2.18} \]
which are symmetric, orthogonal and complete,
\[ P_{AB} = P_{BA}, \quad \bar{P}_{AB} = \bar{P}_{BA}, \quad P_A^B \bar{P}_B^C = 0, \quad P_{AB} + \bar{P}_{AB} = J_{AB}. \tag{2.19} \]
The stringy or DFT extension of the Christoffel connection is, from [14],
\[ \Gamma_{CAB} = 2 \left( P \partial_{\partial C} P \right)_{[AB]} + 2 \left( P_A^D \bar{P}_B^E - P_A^D P_B^E \right) \partial_D P_{EC} \]
\[ - \frac{4}{D-1} \left( \bar{P}_{C[A} P_{B]}^D + P_{C[A} P_{B]}^D \right) (\partial_{DD} + (P \partial E P \tilde{P})_{[ED]}). \tag{2.20} \]

Further, if we set
\[ R_{CDAB} := \partial_A \Gamma_{BCD} - \partial_B \Gamma_{ACD} + \Gamma_{AC}^E \Gamma_{BED} - \Gamma_{BC}^E \Gamma_{AED}, \tag{2.21} \]
we may define so-called the ‘semi-covariant’ four-indexed Riemann-type DFT-curvature,
\[ S_{ABCD} := \frac{1}{2}(R_{ABCD} + R_{CDAB} - \Gamma_{AC}^E \Gamma_{BDE} - \Gamma_{BC}^E \Gamma_{AED}), \tag{2.22} \]
which in turn sets the completely covariant, zero-indexed scalar and two-indexed Ricci-type DFT-curvatures,
\[ (P_A^B \bar{P}_B^D - \bar{P}_A^B \bar{P}_B^D) S_{ACBD}, \quad P_A^C \bar{P}_B^D S_{CED}. \tag{2.23} \]
The scalar curvature defines the pure DFT Lagrangian, multiplied by the weightful DFT-dilaton factor,
\[ \mathcal{L}_{\text{DFT}} = e^{-2d}(P_A^B \bar{P}_B^D - \bar{P}_A^B \bar{P}_B^D) S_{ACBD}. \tag{2.24} \]
The full Euler-Lagrangian equations of motion are nothing but the vanishing of the two DFT-curvatures (2.23).
With the ‘conventional’ parametrization of the DFT-dilaton and the DFT-metric (2.15), all the Euler-Lagrangian equations of the pure DFT reduce to

\[ R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \phi - \frac{1}{4} H_{\mu\rho\sigma} H_{\nu}{}^{\rho\sigma} = 0, \]  
(2.25)

\[ \nabla^\lambda H_{\lambda\mu\nu} - 2(\nabla^\lambda \phi) H_{\lambda\mu\nu} = 0, \]  
(2.26)

\[ R + 4\Box \phi - 4\partial_\mu \phi \partial^\mu \phi - \frac{1}{12} H_{\mu\nu\rho} H_{\mu\nu\rho} = 0. \]  
(2.27)

Basically, the first two equations correspond to the symmetric and the anti-symmetric parts of the two-indexed DFT-curvature (after pulled back by DFT-vielbeins, \( S^p \bar{q} \)), while the last is precisely the scalar DFT-curvature. These formulae can be also derived as the equations of motion of the conventional (undoubled) action for the closed string massless sector,

\[ \int d^x e^{-2\phi} \left( R + 4\partial_\mu \phi \partial^\mu \phi - \frac{1}{12} H_{\mu\nu\rho} H_{\mu\nu\rho} \right). \]  
(2.28)

Eq. (2.26) can be rewritten in terms of the form notation,

\[ d \star \left( e^{-2\phi} H_{(3)} \right) = 0. \]  
(2.29)

Combining (2.27) with the trace of (2.25), we have

\[ \Box \phi - 2\partial_\mu \phi \partial^\mu \phi + \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} = 0. \]  
(2.30)

After all, the Euler-Lagrangian equations of motion boil down to (2.25), (2.29) and (2.30).

It is worth while to note that the equations of motion ensure the conservation of the Noether current,

\[ \nabla_\mu J^\mu = 0, \quad J^\mu = e^{-2\phi} \left( \partial^\mu \phi + \frac{1}{4} H_{\mu\nu\rho} B_{\nu\rho} \right), \]  
(3.1)

which corresponds to the global scale symmetry present in the action (2.28),

\[ \phi \rightarrow \phi + (D - 2) \lambda, \quad g_{\mu\nu} \rightarrow e^{4\lambda} g_{\mu\nu}, \quad B_{\mu\nu} \rightarrow e^{4\lambda} B_{\mu\nu}. \]  
(3.2)

3 Circular geodesic around the spherical vacuum in \( D = 4 \) DFT

In this section we spell our main results. Appendix contains detailed derivations and more rotation curves.

- **O(\( D, D \)) covariant action for a point particle in doubled-yet-gauged spacetime.**

We recall (2.11) that, requiring \( O(\( D, D \)) \) T-duality, doubled diffeomorphisms and the coordinate gauge symmetry, the action for a point particle in the doubled-yet-gauged spacetime is uniquely determined:

\[ S_{\text{particle}} = \int d\tau e^{-1} D_\tau x^A D_\tau x^B H_{AB}(x) - \frac{1}{4} m^2 e, \]  
(3.1)

which reduces to the conventional (undoubled) action (2.17) for a relativistic point particle coupled to the string frame metric:

\[ S_{\text{particle}} \equiv \int d\tau e^{-1} \dot{x}^\mu \dot{x}^\nu g_{\mu\nu} - \frac{1}{4} m^2 e. \]  
(3.2)

Thus, the geodesic motion of the particle should be analyzed in the *string frame.*
• Spherically symmetric ansatz for DFT.
  We prescribe that any spherically symmetric DFT configuration should admit three
  doubled Killing vectors, \( V^A_a, a = 1, 2, 3 \), which satisfy the DFT-Killing equations in
  terms of the generalized Lie derivative [42],
  \[
  \hat{L}_{V^A_a} H_{AB} = 0, \quad \hat{L}_{V^A_a} (e^{-2d}) = 0, \tag{3.3}
  \]
  and form an \( \text{so}(3) \) algebra through the \( C \)-bracket,
  \[
  [V^A_a, V^B_b]_C = \sum_c \epsilon^{abc} V^C_c. \tag{3.4}
  \]
  With (2.15), such a spherically symmetric and static closed string background assumes
  the generic form:
  \[
  ds^2 = e^{2\phi(r)} \left[ -A(r)dt^2 + A^{-1}(r)dr^2 + A^{-1}(r)C(r) d\Omega^2 \right], \tag{3.5}
  \]
  \[
  B^{(2)} = B(r) \cos \vartheta \, dr \wedge d\varphi + h \cos \vartheta \, dt \wedge d\varphi, \tag{3.6}
  \]
  which contains four unknown radial functions, \( A(r), B(r), C(r) \) and the scalar dilaton,
  \( \phi(r) \). We also set \( d\Omega^2 = d\vartheta^2 + \sin^2 \vartheta \, d\varphi^2 \) and put the \( B \)-field into a two-form, \( B^{(2)} = \frac{1}{2} B_{\mu \nu} dx^\mu \wedge dx^\nu \). The \( H \)-flux then takes the most general spherically symmetric form,
  \[
  H^{(3)} = dB^{(2)} = B(r) \sin \vartheta \, dr \wedge d\vartheta \wedge d\varphi + h \sin \vartheta \, dt \wedge d\vartheta \wedge d\varphi,
  \tag{3.7}
  \]
  which is closed for constant \( h \).
  Writing \( V^A_a = (\lambda_{a\mu}, \xi^a_\nu) \), the doubled \( \text{so}(3) \) Killing vectors are given concretely by
  \[
  \lambda_1 = \frac{\cos \varphi}{\sin \vartheta} \left[ hdt + B(r) dr \right], \quad \xi_1 = \sin \varphi \partial_\varphi + \cot \vartheta \cos \varphi \partial_\vartheta, \tag{3.8}
  \]
  \[
  \lambda_2 = \frac{\sin \varphi}{\sin \vartheta} \left[ hdt + B(r) dr \right], \quad \xi_2 = -\cos \varphi \partial_\varphi + \cot \vartheta \sin \varphi \partial_\vartheta, \tag{3.9}
  \]
  \[
  \lambda_3 = 0, \quad \xi_3 = -\partial_\varphi.
  \tag{3.10}
  \]
  They meet, with the ordinary (undoubled) Lie derivative,
  \[
  \mathcal{L}_{\xi_a} g_{\mu \nu} = 0, \quad \mathcal{L}_{\xi_a} \phi = \xi^a_\mu \partial_\mu \phi = 0, \quad \mathcal{L}_{\xi_a} B^{(2)} = -d\lambda_a, \tag{3.11}
  \]
  and hence, as expected for the \( H \)-flux,
  \[
  \mathcal{L}_{\xi_a} H^{(3)} = 0. \tag{3.12}
  \]

• Spherically symmetric, static and asymptotically flat vacuum solution to
  \( D = 4 \) DFT.
  We insert the spherically symmetric static ansatz (3.5) into the Euler-Lagrangian equations of motion, especially (2.25), (2.29) and (2.30). We impose the boundary condition of the asymptotic flatness and obtain the most general form of such solutions. Appendix B contains the details of our direct derivation of the solution. The asymptotic flatness turns out to be inconsistent with the magnetic \( H \)-flux, and hence we put \( B(r) = 0 \) and
  \[
  H^{(3)} = h \sin \vartheta \, dt \wedge d\vartheta \wedge d\varphi. \tag{3.13}
  \]
The most general, spherically symmetric, asymptotically flat, static vacuum solution to \( D = 4 \) DFT is, with the ansatz (3.5), given by

\[
A(r) = \left( \frac{r - \alpha}{r + \beta} \right) \frac{1}{\sqrt{a^2 + b^2}}, \quad C(r) = (r - \alpha)(r + \beta),
\]

\[
B(2) = h \cos \vartheta \, dt \wedge d\varphi,
\]

\[
e^{2\phi} = \gamma_+ \left( \frac{r - \alpha}{r + \beta} \right) \frac{b}{\sqrt{a^2 + b^2}} + \gamma_- \left( \frac{r - \alpha}{r + \beta} \right) \frac{-b}{\sqrt{a^2 + b^2}},
\]

where \( a, b, h \) are three real constants satisfying \( b^2 \geq h^2 \), and \( \alpha, \beta, \gamma_{\pm} \) are associated shorthand,

\[
\alpha := \frac{a}{a + b} \sqrt{a^2 + b^2}, \quad \beta := \frac{b}{a + b} \sqrt{a^2 + b^2}, \quad \gamma_{\pm} := \frac{1}{2} (1 \pm \sqrt{1 - h^2/b^2}).
\]

(3.11)

This result is a re-derivation of the BMQ solution which was generated by the \( SL(2, \mathbb{R}) \) S-duality [32]. After shifting the radius, \( r \rightarrow r - \beta \), we may rewrite the solution as

\[
e^{2\phi} = \gamma_+ \left( 1 - \frac{\sqrt{a^2 + b^2}}{r} \right) \frac{b}{\sqrt{a^2 + b^2}} + \gamma_- \left( 1 - \frac{\sqrt{a^2 + b^2}}{r} \right) \frac{-b}{\sqrt{a^2 + b^2}}, \quad B(2) = h \cos \vartheta \, dt \wedge d\varphi,
\]

\[
ds^2 = e^{2\phi} \left[ - \left( 1 - \frac{\sqrt{a^2 + b^2}}{r} \right) \frac{b}{\sqrt{a^2 + b^2}} \, dt^2 + \left( 1 - \frac{\sqrt{a^2 + b^2}}{r} \right) \frac{-b}{\sqrt{a^2 + b^2}} \left( dr^2 + r - \sqrt{a^2 + b^2} d\Omega^2 \right) \right],
\]

(3.12)

where the radial origin, \( r = 0 \), corresponds to the coordinate singularity.

It is worth while to note the expression of the DFT integral measure,

\[
e^{-2\ell} = e^{2\phi} CA^{-1} \sin \vartheta = g_{\vartheta \vartheta}(r) \sin \vartheta.
\]

(3.13)

**Circular geodesic and orbital velocity.**

We proceed to analyze the circular geodesic, with both \( r \) and \( \vartheta \equiv \frac{\pi}{2} \) fixed. We introduce the ‘proper’ radius,

\[
R := \sqrt{g_{\vartheta \vartheta}(r)} = \sqrt{C(r)/A(r)} e^{\phi(r)},
\]

(3.14)

which would convert the metric into a canonical form where the angular part is ‘properly’ normalized (hence comparable to observations, e.g. galaxy rotation curves):

\[
ds^2 = g_{tt} dt^2 + g_{rr} dr^2 + R^2 d\Omega^2 = -e^{2\phi} A \, dt^2 + e^{2\phi} A^{-1} \left( \frac{dR}{dr} \right)^2 \, dR^2 + R^2 d\Omega^2.
\]

(3.15)

The radial component of the geodesic equation determines the angular velocity as a function of \( r \), or the proper radius, \( R \),

\[
\left( \frac{d\varphi}{dt} \right)^2 = \frac{d}{dr} (A e^{2\phi}) \left/ \frac{d}{dr} (CA^{-1} e^{2\phi}) \right. = -\frac{1}{2} R^{-1} \frac{dg_{\vartheta \vartheta}}{dR}.
\]

(3.16)
The orbital velocity is given by the proper radius times the angular velocity, computable from (3.10), (3.14), (3.16),
\[
V_{\text{orbit}} = \left| R \frac{d\varphi}{dt} \right| = \left| -\frac{1}{2} R \frac{dg_\mu}{dR} \right|^\frac{1}{2}. \tag{3.17}
\]
Clearly, from (3.16) and (3.17), the gravitational force can be repulsive if and only if the orbital velocity is pure imaginary, i.e. \( \frac{dg_\mu}{dR} > 0 \).
Explicitly we have for the solution (3.10),
\[
R = \left( r-\alpha \right) \left( r+\beta \right) \left( \frac{r-0}{r+\beta} \right) \left( \frac{r-0}{r+\beta} \right) \left( \frac{r-0}{r+\beta} \right)^2 \tag{3.18}
\]
\[
\left( \frac{d\varphi}{dt} \right)^2 = \frac{1}{\left( r-\alpha \right) \left( r+\beta \right) \left( r-\alpha \right) \left( r+\beta \right)} \left( \frac{r-0}{r+\beta} \right) \left( \frac{r-0}{r+\beta} \right) \left( \frac{r-0}{r+\beta} \right)^2 \tag{3.19}
\]
\[
V_{\text{orbit}} = \left[ \left( \frac{r-0}{r+\beta} \right) \left( \frac{r-0}{r+\beta} \right) \left( \frac{r-0}{r+\beta} \right) \left( \frac{r-0}{r+\beta} \right) \right]^{\frac{1}{2}} \tag{3.20}
\]
We refer readers to appendix C for details.

- **Physical observables of the spherically symmetric DFT-vacuum.**
  There are three physical observables,\(^4\) on account of the three free parameters, \(a, b, h\) \(\left( b^2 \geq h^2 \right)\),
\[
M_\infty G := \lim_{R \rightarrow \infty} \left( RV_{\text{orbit}}^2 \right) = \frac{1}{2} \left( a + b \sqrt{1 - h^2/b^2} \right), \tag{3.21}
\]
\[
R_{\text{photon}} = R(r_{\text{photon}}) , \quad r_{\text{photon}} = a + \frac{1}{2} \left( \frac{a - b}{a + b} \right) \sqrt{a^2 + b^2}, \tag{3.22}
\]
\[
Q[\partial_t] = \frac{1}{4} \left[ a + \left( \frac{a - b}{a + b} \right) \sqrt{a^2 + b^2} \right]. \tag{3.23}
\]
The first defines the asymptotic mass, \(M_\infty\), from the Keplerian fall-off of the orbital speed which, from (3.18), (3.20), eventually takes place at spatial infinity,
\[
ds^2|_{R \rightarrow \infty} \rightarrow - \left( 1 - \frac{2M_\infty G}{R} \right) dt^2 + \left( 1 + \frac{a - b \sqrt{1 - h^2/b^2}}{R} \right) dR^2 + R^2 d\Omega^2. \tag{3.24}
\]
Hence, the rotation curve can be non-Keplerian only over a finite range. The second, with (3.18), gives the radius of a photon sphere (if positive). The last is the conserved

\(^4\)Alternative to (3.23), by analyzing the asymptotic behaviors of the scalar dilaton \(\phi\) and the \(H\)-flux, one may obtain the dilaton and the \(H\)-flux “charges” [32]. But, since they are merely parametrization-dependent components of the DFT-metric and the DFT-dilaton, the “charges” cannot quite be qualified as \(O(D, D)\) covariant quantities, nor physical observables.
global charge of the time translational symmetry, computable straightforwardly following the prescription of [42] which generalized the Wald formalism [44–46] to DFT and contains the $O(D,D)$ covariant extensions of both Noether potential and boundary two-form (cf. [47]). Appendix D contains details.

- Rotation curves around the spherically symmetric DFT-vacuum.
  Various choices of the parameters, $a, b, h$, are informative.

  - If we set $h = 0$, our solution reduces to that of F-JNW [33, 34], and further with $b = 0$ to the Schwarzschild metric,
    \[ ds^2 = -(1 - a/R)dt^2 + \frac{dR^2}{1 - a/R} + R^2d\Omega^2, \quad V_{\text{orbit}} = \sqrt{\frac{a}{2R}}. \quad (3.25) \]
  - If $a = h = 0$, we reproduce the renowned orbital velocity formula proposed by Hernquist [43],
    \[ ds^2 = -dt^2 + \frac{R^2}{1 + b/R} + R^2d\Omega^2, \quad e^{2\phi} = \frac{1}{1 + b/R}, \quad V_{\text{orbit}} = \sqrt{\frac{bR}{2(R + b)^2}}. \quad (3.26) \]

    Remarkably, the orbital velocity is not monotonic; it assumes its maximum value, about 35% of the speed of light, at $R = b$,
    \[ \max[V_{\text{orbit}}] = \frac{1}{2\sqrt{2}} \approx 0.35. \quad (3.27) \]
  - If $h = 0$ and $a = b$, we obtain with $\alpha = \sqrt{\frac{1}{2}|a|} \geq 0$,
    \[ ds^2 = -\left(\frac{\sqrt{R^2 + \alpha^2 - \alpha}}{\sqrt{R^2 + \alpha^2} + \alpha}\right)^{\frac{\sqrt{2}}{2}} dt^2 + \frac{R^2}{R^2 + \alpha^2} dR^2 + R^2d\Omega^2, \quad e^{2\phi} = \frac{\sqrt{R^2 + \alpha^2 - \alpha}}{\sqrt{R^2 + \alpha^2 + \alpha}}, \quad (3.28) \]
    and
    \[ V_{\text{orbit}} = \left(\frac{2\alpha^2}{R^2 + \alpha^2}\right)^{\frac{1}{4}} \left(\frac{\sqrt{R^2 + \alpha^2 - \alpha}}{R}\right)^{\frac{\sqrt{2}}{2}}. \quad (3.29) \]
    The orbital velocity is maximal at $R = \left(4 + 2\sqrt{6}\right)^{\frac{1}{2}} \alpha$ as
    \[ \max[V_{\text{orbit}}] = \left(\frac{2}{5 + 2\sqrt{6}}\right)^{\frac{1}{4}} \left(\frac{\sqrt{5 + 2\sqrt{6}} - 1}{\sqrt{4 + 2\sqrt{6}}}\right)^{\frac{\sqrt{2}}{2}} \approx 0.42. \quad (3.30) \]
  - A yet more interesting limit is the case of $a/b \to 0^+$ with nontrivial $H$-flux, $h \neq 0$. Especially when $a = 0$, we have
    \[ ds^2 = e^{2\phi} \left(-dt^2 + \frac{R^2}{R^2 - h^2/4} dR^2\right) + R^2d\Omega^2, \quad (3.31) \]
    \[ e^{2\phi} = \frac{R_h^2}{R_h^2 - 1/2 + \tan v \sqrt{R_h^2 - 1/4}}, \quad B_{(2)} = h \cos \vartheta dt \wedge d\varphi, \quad (3.32) \]
    \[ V_{\text{orbit}} = \frac{R_h}{R_h^2 - 1/2 + \tan v \sqrt{R_h^2 - 1/4}} \left[\frac{1}{2}\tan v \left(\frac{R_h^2 - 1/2}{\sqrt{R_h^2 - 1/4}} - \frac{1}{2}\right)\right]^{\frac{1}{2}}, \quad (3.33) \]
Figure 2. Rotation curves (dimensionless, nonexhaustive). The curves with \(a/b \sim 0^+\) and \(h/b \sim 1^-\) feature a maximum of the orbital velocity after a fairly rapid rise. It is roughly about 150 km/s\(^{-1}\) which is comparable to observations [1]. Further, if we let \(R\) and \(M_\infty\) assume the radius and the mass of the visible matter in the Milky Way, i.e. approximately 15 kpc and \(2 \times 10^{11} M_\odot\) respectively, we have as an order of magnitude, \(R/(M_\infty G) \approx 1.5 \times 10^6\). This number fits our scale of the horizontal axis above, and is thousand times smaller compared with the Earth, \(R_\oplus/(M_\oplus G) \approx 1.4 \times 10^9\), cf. ‘Cosmic Uroboros’. For small enough \(R/(M_\infty G)\), the gravity becomes repulsive and \(V_{\text{orbit}}\) is not defined (or pure imaginary).

where we set two dimensionless shorthand variables,

\[
R_h := \frac{R}{|h|} \geq 1/2, \quad \tan \varphi := b\sqrt{h^2 - b^2}. \tag{3.34}
\]

By tuning the variable as \(\varphi \to 0^+\) (\(h/b \to 1^-\)), it is possible to make the maximal velocity, \(\max[|V_{\text{orbit}}|]\), arbitrarily small. Hence it may be comparable to observations; it may simulate the galaxy rotation curve, see figure 2.

4 Discussion

In this work of theoretical interest, we have aspired to assume DFT as the stringy extension of General Relativity. From the fundamental symmetry principle, such as \(O(D, D)\) covariance and doubled diffeomorphisms, we have unambiguously determined the action for a point particle coupled to the closed string massless sector. We have showed that the particle follows the geodesic set in not the Einstein but the string frame. We have analyzed the circular geodesic motion around the most general, spherically symmetric, asymptotically flat, static \(D = 4\) DFT-vacuum. Crucially, the resulting rotation curve features generically a maximum and thus non-Keplerian over a finite range (short-distance), while becoming asymptotically Keplerian at infinity (long-distance), all measured in terms of the dimensionless radial variable, \(R/(M_\infty G)\), which is normalized by the mass in natural units. Furthermore, the gravitational force can be even repulsive quite close to the origin (far-short-distance) [see (3.33) and figure 2]. By tuning the three free parameters of the spherically symmetric DFT-vacuum, such as \(a/b \sim 0^+\) and \(h/b \sim 1^-\), we have attempted to simulate quantitatively, fitting order of magnitude the scales of both vertical and horizontal axes, the flat or slowly rising galaxy rotation curves observed for finite regions outside the visible matter [figure 2].

While the proper radius, \(R\), is the dimensionful physical radius, the normalized radius, \(R/(M_\infty G)\), is the mathematically natural dimensionless variable which essentially probes
Table 1. ‘Uroboros’ spectrum of the dimensionless radial variable normalized by mass in natural units. The orbital speed is also dimensionless, and depends on the single variable, \( R/(M_\infty G) \).

| \( R/(M_\infty G) \) | 0 \( ^+ \) | 7.1 \( \times 10^{15} \) | 2.0 \( \times 10^{13} \) | 2.4 \( \times 10^{26} \) | 1.4 \( \times 10^{7} \) | 1.0 \( \times 10^{3} \) | 1.5 \( \times 10^{6} \) | 0 \( ^+ \) |
|------------------|-----------|----------------|----------------|----------------|----------------|----------------|----------------|-----------|

the theoretical nature of the gravitational force, not exclusively, in Double Field Theory. Intriguingly, the normalized dimensionless radius is thousand times smaller for the Milky Way compared to the Earth at each surface (of the visible matter): 1.5 \( \times 10^6 \) versus 1.4 \( \times 10^9 \). Note also 1AU/(\( M_\odot G \)) \( \simeq 1.0 \times 10^8 \) for the solar system.

Generically, if the mass density is constant, the dimensionless radial variable, \( R/(M_\infty G) \), becomes smaller as the physical radius, \( R \), grows. This seems to imply that, the observations of stars and galaxies far away, or the dark matter and the dark energy problems, are actually revealing the short-distance nature of gravity, as they are essentially based on small \( R/(M_\infty G) \) observations (long distance divided by far heavier mass). Perhaps, the repulsive gravitational force at very short-distance, \( R/(M_\infty G) \rightarrow 0^+ \), might be responsible for the inflation or the accelerating expansion of the Universe. We believe this speculation of solving the dark matter/energy problems by modifying short gravity deserves further explorations, even not necessarily restricted to the framework of Double Field Theory.

From the coupling of the closed string massless sector to the Standard Model (1.2), (1.3), the \( B \)-field (or the axion) couples to the fermions only: it does not interact with any gauge bosons, and hence transparent, or dark, to electromagnetic radiation. In contrast, the scalar dilaton, \( \phi \), couples to the bosons only but not to any fermion, \( \chi \) in (1.2). As the scalar dilaton, \( \phi \), and the \( B \)-field are “massless”, they tend to spread over larger space and get diluted, but not completely, as our asymptotically flat solution is anyhow non-Keplerian up to finite range, \( R/(M_\infty G) \ll \infty \). It is worth while to note that, in the string frame the scalar field, \( \Phi \equiv e^{-\phi} \), acquires an effective mass given by the scalar curvature as \( 4\partial_\mu \Phi \partial^\mu \Phi + R \Phi^2 \) in the Lagrangian. While DFT modifies the law of Einstein gravity, from the conventional GR point of view, the scalar dilaton and the \( B \)-field may well be then regarded as extra ‘dark matter’ (cf. axion [48–51]), or ‘dark gravity’ (as part of stringy gravity). This identification appears consistent with the ‘bullet cluster’ observation [54] which often rules out theories of modified gravity. Furthermore, with the identification of the asymptotic mass, \( M_\infty \) defined in (3.23), as the (baryonic) mass of the visible matter, it is worth while to note that even if \( M_\infty \) vanishes, there exists a class of nontrivial DFT-vacuum solutions. This might also explain the observed gravitational lensing without visible matter.

Certainly, the phenomenological validity of DFT, as an alternative to GR, is still questionable, requires and deserves further thorough verifications. Compared to other theories of modified gravity, DFT is singled out as the string theory extension of Einstein gravity guided entirely by the symmetry principle: the fundamental symmetries of \( O(D,D) \) T-duality, doubled diffeomorphisms and twofold local Lorentz symmetries rigidly fix the theory, including the couplings to the Standard Model and to a point-like particle. Thus, while testing DFT against more high-precision data of observations in future, it should be taken into account that i) a relativistic point particle follows the geodesic motion not in the Einstein frame but in the string frame, and ii) the scalar dilaton and the \( B \)-field are transparent or ‘dark’ to the Standard Model fermions and the gauge bosons respectively. Deeper understanding of
the three parameters, perhaps as the intrinsic properties of matter or an elementary particle, would be desirable. For this, once again, the minimal coupling between DFT and the Standard Model (1.2), (1.3), could be a starting point.

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A More rotation curves around various spherically symmetric DFT-vacua

Here, for various choices of the free parameters, \{a, b, h\} (3.10), we depict the corresponding rotation curve as a plot of the two \textit{dimensionless} quantities, namely the orbital velocity, \(V_{\text{orbit}}\), versus the scaled proper radius, \(R/(M_\infty G)\), where the ‘asymptotic’ mass, \(M_\infty\), is defined in (3.23).

- If \(b = h = 0\) and \(a = 2M_\infty G > 0\), we recover the Schwarzschild metric, as in (3.25),

\[
\text{d}s^2 = -\left(1 - \frac{2M_\infty G}{R}\right)\text{d}t^2 + \left(1 - \frac{2M_\infty G}{R}\right)^{-1}\text{d}R^2 + R^2\text{d}\Omega^2, \quad \phi = 0, \quad B_{\mu\nu} = 0.
\]  

(A.1)

The corresponding Keplerian rotation curve is depicted in figure 3.

If \(b = h = 0\) and \(a = 2M_\infty G < 0\), we have

\[
\text{d}s^2 = -\left(1 + \frac{2M_\infty G}{r}\right)^{-1}\text{d}t^2 + \left(1 + \frac{2M_\infty G}{r}\right)\text{d}r^2 + (r + 2M_\infty G)^2\text{d}\Omega^2, \quad \phi = 0, \quad B_{\mu\nu} = 0,
\]  

(A.2)

which, after the radial coordinate redefinition, \(r \to R - 2M_\infty G\), reduces to the Schwarzschild metric (A.1), yet with the negative mass.
Figure 4. Rotation curve of $V_{\text{orbit}} = \sqrt{\frac{bR}{2(R+b)^2}}$: Hernquist Model [43], $a = h = 0$. The orbital velocity assumes its maximum value, $\text{max}[V_{\text{orbit}}] = \frac{1}{2\sqrt{2}} \approx 0.353553$ (about 35% of the speed of light) at $R = b = 2M_\infty G$.

- If $a = h = 0$, regardless of the sign of $b$ (up to possible radial coordinate shift), we get
  \[
  \text{d}s^2 = -\frac{R}{R+b}\text{d}t^2 + \frac{R}{R+b}\text{d}R^2 + R^2\text{d}\Omega^2, \quad e^{2\phi} = \frac{R}{R+b}, \quad B_{\mu\nu} = 0. \quad (A.3)
  \]
  The corresponding orbital velocity coincides with that of the Hernquist model [43] up to an overall constant factor [see figure 4],
  \[
  V_{\text{orbit}} = \sqrt{\frac{bR}{2(R+b)^2}}, \quad (A.4)
  \]
  which is valid for positive $b$. Otherwise the gravity is repulsive. The orbital velocity would have been trivial if we had computed it in the Einstein frame where the temporal component of the Einstein frame metric is constant, $'g_{tt}^E = 1'$. 

- If $h = 0$, we recover the F-JNW solution [33, 34] which is the most general static spherical solution of the Einstein gravity coupled to the scalar dilaton,
  \[
  \text{d}s^2 = -\left(1 - \frac{\sqrt{a^2+b^2}}{r}\right) \frac{a+b}{\sqrt{a^2+b^2}} \text{d}t^2 + \left(1 - \frac{\sqrt{a^2+b^2}}{r}\right) \frac{-a+b}{\sqrt{a^2+b^2}} \left[\text{d}r^2 + r \left(r - \sqrt{a^2+b^2}\right) \text{d}\Omega^2\right],
  \]
  \[
  e^{2\phi} = \left(1 - \frac{\sqrt{a^2+b^2}}{r}\right) \frac{b}{\sqrt{a^2+b^2}}, \quad B_{\mu\nu} = 0. \quad (A.5)
  \]
  In this case of $h = 0$, the T-duality over the temporal direction, $t \leftrightarrow \tilde{t}$ (cf. [39, 40]) preserves the form of the F-JNW solution given in the string frame, and it results in exchanging the two parameters, $(a, b) \leftrightarrow (-b, -a)$.

The solution can be also rewritten, after the shift, $r \rightarrow r + \frac{b}{a+b} \sqrt{a^2+b^2}$, as
  \[
  \text{d}s^2 = -\left(\frac{r - \alpha}{r + \beta}\right) \frac{a+b}{\sqrt{a^2+b^2}} \text{d}t^2 + \left(\frac{r - \alpha}{r + \beta}\right) \frac{-a+b}{\sqrt{a^2+b^2}} \left[\text{d}r^2 + \left(r - \alpha\right) \left(r + \beta\right) \text{d}\Omega^2\right],
  \]
  \[
  e^{2\phi} = \left(\frac{r - \alpha}{r + \beta}\right) \frac{b}{\sqrt{a^2+b^2}}, \quad B_{\mu\nu} = 0. \quad (A.6)
  \]
  which manifestly interpolates (A.1) and (A.3), in a unifying manner.
Figures 5 and 6 show the orbital velocities for the choices of the parameters, \(a/b = 0.5\) and \(a/b = 2\), commonly with \(h = 0\).

- If \(h = 0\) and \(a = b\), with the proper radius, \(R \equiv \sqrt{r^2 - \alpha^2}\), and a positive number, \(\alpha \equiv \frac{1}{\sqrt{2}}|a| > 0\), the above solution reduces to

\[
\begin{align*}
\text{ds}^2 &= -\left(\frac{\sqrt{R^2 + \alpha^2} - \alpha}{\sqrt{R^2 + \alpha^2} + \alpha}\right)^{\sqrt{2}} \text{d}t^2 + \frac{R^2}{R^2 + \alpha^2} \text{d}r^2 + R^2 \text{d}\Omega^2, \\
e^{2\phi} &= \left(\frac{\sqrt{R^2 + \alpha^2} - \alpha}{\sqrt{R^2 + \alpha^2} + \alpha}\right)^{\sqrt{2}}, \\
B_{\mu\nu} &= 0.
\end{align*}
\]

The orbital velocity is [see figure 7],

\[
V_{\text{orbit}} = \left(\frac{2\alpha^2}{R^2 + \alpha^2}\right)^{\frac{\sqrt{2}}{4}} \left(\frac{\sqrt{R^2 + \alpha^2} - \alpha}{R}\right)^{\sqrt{2}}, \quad (A.8)
\]

which assumes its maximum value, about 42% of the speed of light, at \(R = (4 + 2\sqrt{6})\frac{1}{2}\alpha\):

\[
\text{max}[V_{\text{orbit}}] = \left(\frac{2}{5 + 2\sqrt{6}}\right)^{\frac{1}{4}} \left(\frac{\sqrt{5 + 2\sqrt{6}} - 1}{\sqrt{4 + 2\sqrt{6}}}\right)^{\sqrt{2}} \simeq 0.420868. \quad (A.9)
\]

- If \(a = 0\) with \(b^2 \geq h^2\), up to some alternative radial coordinate shift, we get

\[
\begin{align*}
\text{ds}^2 &= e^{2\phi} \left(\text{d}t^2 + \text{d}r^2\right) + \left(r^2 + \frac{1}{4}h^2\right) \text{d}\Omega^2, \\
e^{2\phi} &= \frac{4r^2 + h^2}{4r^2 \pm 4r\sqrt{b^2 - h^2} - h^2 - h^2}, \\
H_{(3)} &= h \sin \vartheta \text{d}t \wedge \text{d}\vartheta \wedge \text{d}\varphi,
\end{align*}
\]

where the free sign, \(\pm\), coincides with that of \(b\). This solution can be rewritten in terms
of the proper radius satisfying $R^2 = r^2 + \frac{1}{4}h^2$, to take the form:

$$
\frac{\dd s^2}{e^{2\phi}} = e^{2\phi} \left( -\dd t^2 + \frac{R^2}{R^2 - h^2/4} \dd R^2 \right) + R^2 \dd \Omega^2,
$$

$$
e^{2\phi} = \frac{R^2}{R^2 - h^2/2 \pm \sqrt{b^2 - h^2 \sqrt{R^2 - h^2/4}}}, \quad H_{(3)} = h \sin \vartheta \, \dd t \wedge \dd \vartheta \wedge \dd \varphi. \tag{A.11}
$$

Requiring the reality, we need to constrain the range of the proper radius, at least,

$$
R \geq \frac{1}{2} |h|. \tag{A.12}
$$

The orbital velocity is [see figures 8 and 9],

$$
V_{\text{orbit}} = \frac{R}{R^2 - \frac{1}{2}h^2 \pm \sqrt{b^2 - h^2 \sqrt{R^2 - \frac{1}{4}h^2}}} \sqrt{\pm \frac{b^2 - h^2}{4R^2 - h^2} \left( R^2 - \frac{1}{2}h^2 \right) - \frac{1}{2}h^2}, \tag{A.13}
$$

which reduces to (A.4) when $h = 0$ for consistency. If $h \neq 0$, it can be rewritten as

$$
V_{\text{orbit}} = \frac{R_h}{\left| R_h^2 - \frac{1}{2} + \tan v \sqrt{R_h^2 - \frac{1}{4}} \right|} \left[ \frac{1}{2} \tan v \left( \frac{R_h^2 - \frac{1}{2}}{\sqrt{R_h^2 - \frac{1}{4}}} \right) - \frac{1}{2} \right]^{1/2}, \tag{A.14}
$$

for which we set dimensionless variables,

$$
R_h := R/|h|, \quad \cos v := |h|/b, \quad \sin v := \sqrt{1 - h^2/b^2} \geq 0. \tag{A.15}
$$

As for figures 2 and 9, it is worth while to note the dimensionless unit, $100\text{ km/s} \, c^{-1} \simeq 3.336 \times 10^{-4}$. 

---

Figure 7. Rotation curve: $a/b = 1, \ h = 0$. 

---
In particular, if $a = 0$ and $b^2 = h^2$ saturated, the above solution (A.11) reduces to

$$
\begin{align*}
\frac{ds^2}{R^2} &= -\frac{R^2}{R^2 - \frac{1}{2}h^2}dt^2 + \frac{R^4}{(R^2 - \frac{1}{2}h^2)(R^2 - \frac{1}{4}h^2)}dR^2 + R^2dt^2, \\
e^{2\varphi} &= \frac{R^2}{R^2 - \frac{1}{2}h^2}, \\
H_{(3)} &= h \sin \vartheta \, dt \wedge d\vartheta \wedge d\varphi. 
\end{align*}
$$

(A.16)

It is no longer necessary to impose the constraint (A.12). Yet, the gravity is now repulsive and the orbital velocity becomes imaginary making no physical sense,

$$V_{\text{orbit}} = \sqrt{-\frac{1}{2} \times \frac{hR}{|R^2 - \frac{1}{2}h^2|}}. 
$$

(A.17)

For a generic case with all non-vanishing parameters, $a, b, h$, we may plot the corresponding rotation curve numerically, based on the exact expressions of $R(r)$ and $V_{\text{orbit}}(r)$, (3.18) and (3.20) respectively, see figures 2 and 10.
B Derivation of the spherically symmetric vacuum solution to $D = 4$ DFT

Without loss of generality, utilizing the radial diffeomorphisms, we assume the following static, spherically symmetric ansatz for the string frame metric,

\[ ds^2 = e^{2\phi(r)} \left[ -A(r)dt^2 + A^{-1}(r)dr^2 + A^{-1}(r)C(r)d\Omega^2 \right], \tag{B.1} \]

where we put as shorthand notation,

\[ d\Omega^2 = d\vartheta^2 + \sin^2 \vartheta \, d\phi^2. \tag{B.2} \]

It is worth while to note that our string frame metric ansatz takes the product form of the dilaton factor, $e^{2\phi}$, times the Einstein frame metric.

If the spacetime is asymptotically 'flat', our metric ansatz (B.1) should meet boundary conditions,

\[ \lim_{r \to \infty} A(r) = 1, \quad \lim_{r \to \infty} r^{-2} C(r) = 1, \quad \lim_{r \to \infty} \phi(r) = 0, \tag{B.3} \]

and also from the asymptotic ‘smoothness’,

\[ \lim_{r \to \infty} A'(r) = \lim_{r \to \infty} A''(r) = 0, \quad \lim_{r \to \infty} r^{-1} C'(r) = \lim_{r \to \infty} C''(r) = 2, \quad \lim_{r \to \infty} \phi'(r) = \lim_{r \to \infty} \phi''(r) = 0. \tag{B.4} \]

We often write the $B$-field using the form notation,

\[ B_{(2)} = \frac{1}{2} B_{\mu\nu} dx^\mu \wedge dx^\nu = B(r) \cos \vartheta \, dr \wedge d\varphi + h \cos \vartheta \, dt \wedge d\varphi, \tag{B.5} \]

such that its field strength, or the $H$-flux, takes the most general spherically symmetric form,

\[ H_{(3)} = \frac{1}{3!} H_{\lambda\mu\nu} dx^\lambda \wedge dx^\mu \wedge dx^\nu = B(r) \sin \vartheta \, dr \wedge d\vartheta \wedge d\varphi + h \sin \vartheta \, dt \wedge d\vartheta \wedge d\varphi, \tag{B.6} \]

which is closed for constant $h$. It is spherically symmetric as it admits the $\mathfrak{so}(3)$ Killing vectors given by the usual angular momentum differential operators,

\[ \mathcal{L}_{\xi_a} H_{(3)} = d \left( i_{\xi_a} H_{(3)} \right) + i_{\xi_a} \left( dH_{(3)} \right) = 0, \]

\[ \xi_1 = \sin \varphi \partial_\vartheta + \cot \vartheta \cos \varphi \partial_\varphi, \quad \xi_2 = -\cos \varphi \partial_\vartheta + \cot \vartheta \sin \varphi \partial_\varphi, \quad \xi_3 = -\partial_\varphi, \]

\[ [\xi_a, \xi_b] = \sum_c \epsilon_{abc} \xi_c, \tag{B.7} \]

where $i_{\xi_a}$ denotes the inner product. Since, in general, the exterior derivative and the Lie derivative commute, i.e. $[d, \mathcal{L}_{\xi_a}] = 0$, eq. (3.6) implies that there must be a one-form, $\lambda_a$, for each Killing vector, $\xi_a$, satisfying

\[ \mathcal{L}_{\xi_a} B_{(2)} = -d\lambda_a. \tag{B.8} \]

Explicitly, we have

\[ \lambda_1 = \frac{\cos \varphi}{\sin \vartheta} \left[ h dt + B(r) dr \right], \quad \lambda_2 = \frac{\sin \varphi}{\sin \vartheta} \left[ h dt + B(r) dr \right], \quad \lambda_3 = 0. \tag{B.9} \]
It follows that the DFT-Killing equations hold: both the DFT-metric, $\mathcal{H}_{AB}$, and the DFT-dilaton are annihilated by the generalized Lie derivative, cf. [42],

$$\hat{\mathcal{L}}_{V_a} \mathcal{H}_{AB} = 0, \quad \hat{\mathcal{L}}_{V_a} (e^{-2d}) = 0,$$

where the $O(4,4)$ vectorial parameter is given by the $\text{so}(3)$ angular momenta (3.6) and the one-forms (B.9),

$$V_a^A = (\lambda_{\mu}, \xi_a^\nu), \quad [V_a, V_b]_C = \sum_c \epsilon_{abc} V_c^c.$$

The nontrivial Christoffel symbols for the metric ansatz (B.1) are exhaustively,

$$\Gamma^t_r = \Gamma^t_{tr} = \frac{1}{2} A' A^{-1} + \phi', \quad \Gamma^r_t = \frac{1}{2} A' A + \phi' A^2,$$

$$\Gamma^r_r = -\frac{1}{2} A' A^{-1} + \phi', \quad \Gamma^s_{\phi\theta} = \frac{1}{2} A' A^{-1} C - \frac{1}{2} C' - C \phi',$$

$$\Gamma^\phi_r = \frac{1}{2} A' A^{-1}, \quad \Gamma^\phi_\theta = -\frac{1}{2} A' A^{-1} + 1 + \frac{1}{2} C' C^{-1} + \phi', \quad \Gamma^\phi_\phi = \Gamma^\phi_{\phi\theta} = \cot \theta,$$

where the prime denotes the radial derivative.

The Ricci curvature, $R_{\mu\nu}$, and the second derivative, $\nabla_\mu \nabla_\nu \phi$, are automatically diagonal, such that the equation of motion of the string frame metric, i.e. (2.25) is almost diagonal,

$$R_{tt} + 2 \nabla_t \partial_t \phi - \frac{1}{4} H_{t\rho\sigma} H_{t}^{\rho\sigma} = \frac{1}{2} A' A \frac{d}{dr} \ln(A' A^{-1} C) + \phi' A^2 \frac{d}{dr} \ln(\phi' C) - \frac{1}{2} h^2 A^2 C^{-2} e^{-4\phi},$$

$$R_{rr} + 2 \nabla_r \partial_r \phi - \frac{1}{4} H_{t\rho\sigma} H_{t}^{\rho\sigma} = \frac{1}{2} A' A^{-1} \frac{d}{dr} \ln(A' A^{-1} C) - \frac{1}{2} A^2 A^{-2} - C' C^{-1} + \frac{1}{2} C C^{-2}$$

$$- 2 \phi'^2 - \phi' \frac{d}{dr} \ln(\phi' C) - \frac{1}{2} A^2 B^2 C^{-2} e^{-4\phi},$$

$$R_{\phi\theta} + 2 \nabla_\phi \partial_\theta \phi - \frac{1}{4} H_{t\rho\sigma} H_{t}^{\rho\sigma} = 1 + \frac{1}{2} C A' A^{-1} C^{-1} - \frac{1}{2} A^2 A^{-2} C + \frac{1}{2} A' A^{-1} C'$$

$$- \phi' C \frac{d}{dr} \ln(\phi' C) - \frac{1}{2} (A^2 B^2 - h^2) C^{-1} C e^{-4\phi},$$

$$R_{\phi\phi} + 2 \nabla_\phi \partial_\phi \phi - \frac{1}{4} H_{t\rho\sigma} H_{t}^{\rho\sigma} = \sin^2 \theta \left( R_{\phi\phi} + 2 \nabla_\phi \partial_\phi \phi - \frac{1}{4} H_{t\rho\sigma} H_{t}^{\rho\sigma} \right).$$

The only exception is the following off-diagonal component,

$$R_{tr} + 2 \nabla_t \partial_r \phi - \frac{1}{4} H_{t\rho\sigma} H_{t}^{\rho\sigma} = -\frac{1}{4} H_{t\rho\sigma} H_{t}^{\rho\sigma} = -\frac{1}{2} h B e^{-4\phi} A^2 C^{-2}. \quad \text{(B.14)}$$

This implies either $h = 0$ or $B(r) = 0$. The remaining equations of motion (2.29), (2.30) become

$$d \star \left( e^{-2\phi} H_{(3)} \right) = d \left( e^{-4\phi} A^2 B C^{-1} dt + e^{-4\phi} h C^{-1} dr \right) = \frac{d}{dr} \left( e^{-4\phi} A^2 B C^{-1} \right) dr \wedge dt = 0,$$

(B.15)
and
\[ \Box \phi - 2 \partial_\mu \phi \partial^\mu \phi + \frac{1}{12} H_{\mu \nu \rho} H^{\mu \nu \rho} = e^{-2\phi} \phi' A \frac{d}{dr} \ln(\phi') + \frac{1}{2} e^{-6\phi} (A^3 B^2 C^{-2} - h^2 A C^{-2}) = 0. \] (B.16)

In this way, all the equations of motion boil down to the following six relations:
\[ e^{-4\phi} A^2 B C^{-1} = q, \]
\[ h q = 0, \]
\[ C'' - 2 = q^2 A^{-2} C e^{4\phi}, \]
\[ A' A \frac{d}{dr} \ln(A'^{-1} C) = q^2 e^{4\phi}, \]
\[ \phi' \frac{d}{dr} \ln(\phi') = \frac{1}{2} h^2 C^{-2} e^{-4\phi} - \frac{1}{2} q^2 A^{-2} e^{4\phi}, \]
\[ 4\phi'^2 + A'^2 A^{-2} + C'' C^{-1} - C'^2 C^{-2} + 2 C^{-1} + h^2 C^{-2} e^{-4\phi} = 0, \] (B.17)

where \( h, q \) are constants, associated with the electric and magnetic \( H \)-fluxes, see (B.6). Apparently, either \( h \) or \( q \) must be trivial. If \( h = 0, q \neq 0 \) and hence the \( H \)-flux were magnetic, it is easy to see that the above relations are inconsistent with the asymptotically flat smoothness boundary conditions, (B.3) and (B.4). Henceforth we set \( q \equiv 0, B(r) \equiv 0, \) and focus on electric \( H \)-flux solutions. The differential equations above reduce to
\[ C'' = 2, \] (B.18)
\[ \frac{d}{dr} (A'^{-1} C) = 0, \] (B.19)
\[ \phi'' + \phi' C^{-1} = \frac{1}{2} h^2 C^{-2} e^{-4\phi}, \] (B.20)
\[ 4\phi'^2 + (B'^{-1} C)^2 + 4 C - C'^2 + h^2 e^{-4\phi} = 0. \] (B.21)

From (B.18) and (B.19), imposing the boundary condition of (B.3), \( A(r) \) and \( C(r) \) are straightforwardly determined,
\[ A(r) = \left( \frac{r - c_+}{r - c_-} \right)^{-\frac{a}{a^2 + h^2 e^{-4\phi}}}, \quad C(r) = (r - c_+) (r - c_-), \] (B.22)

where \( a \) is the constant of the integral from (B.19), and \( c_+, c_- \) are two roots of the quadratic real polynomial, \( C(r) \). Taking the radial derivative of (B.21) gives nothing but the second order differential equation (B.20). Thus, we only need to solve (B.21) which becomes with the substitution of (B.22),
\[ 4 \left[ (r - c_+) (r - c_-) \phi' \right]^2 + a^2 + h^2 e^{-4\phi} = (c_+ - c_-)^2. \] (B.23)

This result implies that — as the left hand side of the equality is positive — the two roots, \( c_+, c_- \), must be real, and further that eq. (B.23) can be rewritten as an integral relation,
\[ \pm \int \frac{\sqrt{c_+ - c_-})^2 - a^2 - h^2 e^{-4\phi}}{(r - c_+)(r - c_-)} = \int \frac{dr}{r - c_+}(r - c_-). \] (B.24)
The left hand side integral gives
\[ \pm \int \frac{2d\phi}{\sqrt{(c_+ - c_-)^2 - a^2 - b^2 e^{-4\phi}}} = b^{-1} \ln \left( e^{2\phi} + \sqrt{e^{4\phi} - h^2 b^{-2}} \right) + \text{constant}, \]  
(B.25)
where we have absorbed the sign factor, \( \pm \), into the newly introduced integration constant, \( b \), satisfying
\[ a^2 + b^2 = (c_+ - c_-)^2. \]  
(B.26)
From (B.23), we note that \( (c_+ - c_-)^2 - a^2 \) is positive and hence \( b \) is a real number too. The right hand side of (B.24) gives
\[ \int \frac{dr}{(r - c_+)(r - c_-)} = \frac{1}{c_+ - c_-} \ln \left( \frac{r - c_+}{r - c_-} \right) + \text{constant}'. \]  
(B.27)
Combining (B.25) with (B.27) and fixing the integration constant from the boundary condition (B.3), we obtain
\[ e^{2\phi} = \frac{1}{2} \left( 1 + \sqrt{1 - h^2 b^{-2}} \right) \left( \frac{r - c_+}{r - c_-} \right)^{\frac{b}{\sqrt{a^2 + b^2}}} + \frac{1}{2} \left( 1 - \sqrt{1 - h^2 b^{-2}} \right) \left( \frac{r - c_+}{r - c_-} \right)^{-\frac{b}{\sqrt{a^2 + b^2}}}. \]  
(B.28)
Thus, with four real constants, \( a, b, c, h \), and
\[ c_+ = c + \frac{1}{2} \sqrt{a^2 + b^2}, \quad c_- = c - \frac{1}{2} \sqrt{a^2 + b^2}, \quad \gamma_\pm = \frac{1}{2} \left( 1 \pm \sqrt{1 - h^2/b^2} \right), \]  
(B.29)
the string frame metric takes the form,
\[ ds^2 = - \left[ \gamma_+ \left( \frac{r - c_+}{r - c_-} \right)^{\frac{a+b}{\sqrt{a^2 + b^2}}} + \gamma_- \left( \frac{r - c_+}{r - c_-} \right)^{-\frac{a-b}{\sqrt{a^2 + b^2}}} \right] dt^2 \]
\[ + \left[ \gamma_+ \left( \frac{r - c_+}{r - c_-} \right)^{\frac{a-b}{\sqrt{a^2 + b^2}}} + \gamma_- \left( \frac{r - c_+}{r - c_-} \right)^{-\frac{a+b}{\sqrt{a^2 + b^2}}} \right] \left[ dr^2 + (r - c_+)(r - c_-)d\Omega^2 \right], \]  
(B.30)
and the \( B \)-field as well as the \( H \)-flux are
\[ B_{(2)} = h \cos \vartheta \, dt \wedge d\varphi, \quad H_{(3)} = h \sin \vartheta \, dt \wedge d\vartheta \wedge d\varphi. \]  
(B.31)
For the metric to be real valued, we must require
\[ b^2 \geq h^2. \]  
(B.32)
Namely, the presence of the electric \( H \)-flux \( (h \neq 0) \) induces the nontrivial string dilaton, \( (b \neq 0) \), see (B.28). The results of (B.28), (B.30) and (B.31) provide the most general form of the static, asymptotically flat and spherically symmetric vacuum solutions to \( D = 4 \) Double Field Theory.

Up to the radial diffeomorphisms, we may set the free parameter, \( c \), arbitrarily. It is worth while then to rewrite the general solution in slightly different styles.
Further, if the orbital motion is circular, we have
\[ \frac{d}{dr} \left( r \right) = 0 \]
for (B.29), the solution can be rewritten as (3.10) which we recall here:

\[
e^{2\phi} = \gamma_+ \left( 1 - \frac{\sqrt{a^2 + b^2}}{r} \right)^{b} + \gamma_- \left( 1 - \frac{\sqrt{a^2 + b^2}}{r} \right)^{-b}, \quad B_{(2)} = h \cos \vartheta \, dt \wedge d\varphi,
\]

\[
ds^2 = e^{2\phi} \left[ - \left( 1 - \frac{\sqrt{a^2 + b^2}}{r} \right)^{b} \, dt^2 + \left( 1 - \frac{\sqrt{a^2 + b^2}}{r} \right)^{-b} \left( \frac{dr^2}{r^2} + \frac{\sqrt{a^2 + b^2}}{r^2} \, d\varphi^2 \right) \right],
\]

where the radial origin, \( r = 0 \), corresponds to the coordinate singularity.

Alternatively, if we choose \( c \equiv \frac{1}{2} \left( \frac{a-b}{a+b} \right) \sqrt{a^2 + b^2} \) for (B.29), the solution can be rewritten as (3.10) which we recall here:

\[
e^{2\phi} = \gamma_+ \left( \frac{r - \alpha}{r + \beta} \right)^{\frac{b}{\sqrt{a^2 + b^2}}} + \gamma_- \left( \frac{r - \alpha}{r + \beta} \right)^{-\frac{b}{\sqrt{a^2 + b^2}}}, \quad B_{(2)} = h \cos \vartheta \, dt \wedge d\varphi,
\]

\[
ds^2 = e^{2\phi} \left[ - \left( \frac{r - \alpha}{r + \beta} \right)^{\frac{a}{\sqrt{a^2 + b^2}}} \, dt^2 + \left( \frac{r - \alpha}{r + \beta} \right)^{-\frac{a}{\sqrt{a^2 + b^2}}} \left( \frac{dr^2}{r^2} + (r - \alpha)(r + \beta) d\varphi^2 \right) \right],
\]

where

\[
\alpha = \frac{a}{a+b} \sqrt{a^2 + b^2}, \quad \beta = \frac{b}{a+b} \sqrt{a^2 + b^2}, \quad \gamma_\pm = \frac{1}{2} \left( 1 \pm \sqrt{1 - h^2/b^2} \right).
\]

After all, there are three real parameters left, \( a, b, h \), which satisfy \( b^2 \geq h^2 \) and have the dimension of length.

One side remark is that, from (B.20), we may get the dual ‘axion’, \( \Phi \), which has been a dark-matter candidate, cf. [48–53],

\[
\ast \left( e^{-2\phi} H_{(3)} \right) = h e^{-4\phi} C^{-1} \, d\varphi = 2 h^{-1} d(\varphi C) = d\Phi, \quad \Phi = 2 h^{-1} \varphi C.
\]

We refer readers to [55] for the discussion on the related S-duality.

C Derivation of the orbital velocity, eq. (3.20)

For a planar orbital motion where the \( \vartheta \) angle is fixed at \( \vartheta \equiv \frac{\pi}{2} \), the geodesic equation for the other angle, \( \varphi \), reads

\[
\frac{d^2 \varphi}{d\tau^2} + 2 \Gamma^r_{\varphi r} \frac{d\varphi}{d\tau} \frac{dr}{d\tau} = \frac{d^2 \varphi}{d\tau^2} + \frac{d\varphi}{d\tau} \frac{d}{d\tau} \ln \left( e^{2\phi} A^{-1} C \right) = 0.
\]

This gives the conservation of the angular momentum,

\[
L_{\varphi} := e^{2\phi} A^{-1} \, C \frac{d\varphi}{d\tau}, \quad \frac{dL_{\varphi}}{d\tau} = 0.
\]

Further, if the orbital motion is circular, we have \( \frac{d}{d\tau} \equiv 0 \), and the geodesic formula for the radial coordinate,

\[
\frac{d^2 r}{d\tau^2} + \Gamma^r_{tt} \left( \frac{dt}{d\tau} \right)^2 + \Gamma^r_{\varphi r} \left( \frac{d\varphi}{d\tau} \right)^2 = 0.
\]
the angular part of the metric is properly normalized,
\[
\left( \frac{d\varphi}{dt} \right)^2 = - \left( \frac{dg_{tt}}{dr} \right) \left( \frac{dg_{\varphi\varphi}}{dr} \right)^{-1} = \frac{d}{dr} \left( A e^{2\varphi} \right). \tag{C.4}
\]

Especially for a massless particle or photon to be captured in a circular orbit, and thus to form a ‘photon sphere’, we further require the ‘null’ condition: directly from the metric ansatz (B.1),
\[
\left( \frac{d\varphi}{dt} \right)^2 = A^2 C^{-1} = - \frac{g_{tt}}{g_{\vartheta\vartheta}}. \tag{C.5}
\]

This relation must match with (C.4). Straightforward computation shows that, the photon sphere is located at a radius, \( r = r_{\text{photon}} \), extremizing the angular velocity,
\[
\frac{d}{dr} \left( A^2 C^{-1} \right) \bigg|_{r=r_{\text{photon}}} = 0. \tag{C.6}
\]

As it should be, the dilaton factor has disappeared above, thanks to the null property of the photon.

With the the ‘proper’ radius,
\[
R := \sqrt{g_{\vartheta\vartheta}(r)} = \sqrt{C(r)/A(r)} e^{\varphi(r)}, \tag{C.7}
\]

the angular part of the metric is properly normalized,
\[
ds^2 = g_{tt} dt^2 + g_{\vartheta\vartheta} dR^2 + R^2 d\Omega^2 = -e^{2\varphi} A dt^2 + e^{2\varphi} A^{-1} \left( \frac{dR}{dr} \right)^{-2} dR^2 + R^2 d\Omega^2, \tag{C.8}
\]

and in particular, the circumference of a circle is \( 2\pi R \).

Explicitly for the most general spherically symmetric solution (B.34), the radius of the photon sphere is
\[
r_{\text{photon}} = a + \frac{1}{2} \left( \frac{a-b}{a+b} \right) \sqrt{a^2 + b^2}; \tag{C.9}
\]

the angular velocity is
\[
\left( \frac{d\varphi}{dt} \right) = \frac{1}{(r-a)(r+\beta)} \left[ \gamma_+ \left( a+b \right) \left( \frac{r-a}{r+\beta} \frac{\sqrt{a^2+b^2}}{2} \right) + \gamma_- \left( a-b \right) \right]; \tag{C.10}
\]

the proper radius is
\[
R = \left[ (r-a)(r+\beta) \right] \left[ \gamma_+ \left( \frac{r-a}{r+\beta} \frac{\sqrt{a^2+b^2}}{2} \right) + \gamma_- \left( \frac{r-a}{r+\beta} \frac{\sqrt{a^2+b^2}}{2} \right) \right]^{\frac{1}{2}}; \tag{C.11}
\]

and the orbital velocity is
\[
V_{\text{orbit}} = \left| R \frac{d\varphi}{dt} \right| = \left[ -\frac{1}{2} R \frac{dg_{tt}}{dR} \right]^{\frac{1}{2}} \left[ \gamma_+ \left( \frac{r-a}{r+\beta} \frac{\sqrt{a^2+b^2}}{2} \right) + \gamma_- \left( \frac{r-a}{r+\beta} \frac{\sqrt{a^2+b^2}}{2} \right) \right]^{\frac{1}{2}} \left[ \gamma_+ \left( 2r-a+\beta-a+b \right) \left( \frac{r-a}{r+\beta} \frac{\sqrt{a^2+b^2}}{2} \right) + \gamma_- \left( 2r-a+\beta-a-b \right) \right]^{\frac{1}{2}}. \tag{C.12}
\]
D Derivation of the global Noether charge, eq. (3.23)

In [42], the general formula of the conserved global charge in DFT has been worked out, including also the Yang-Mills sector [27]. For pure DFT, the conserved global charge reads

$$Q[X] = \oint_{\partial M} d^2 x_{AB} \ e^{-2d} \left( K^{AB} + 2X^{[A}B^{]B} \right),$$

(D.1)

where $\partial M$ denotes the spatial infinity; $K^{AB}$ is a skew-symmetric Noether potential for DFT,

$$K^{AB} = 4(\bar{P}\nabla)^{[A}(P\nabla)^{B]} - 4(P\nabla)^{[A}(\bar{P}\nabla)^{B]},$$

(D.2)

and the second term, $2X^{[A}B^{]B}$, corresponds to the DFT extension of the counter two-form *à la* Wald [44–46]. Explicitly, $B^A$ is given by

$$B^A = 2(P^{AC}P^{BD} - \bar{P}^{AC}\bar{P}^{BD})\Gamma_{BCD} = 4(P - \bar{P})^{AB} \partial_B d - 2\partial_B P^{AB}.$$  

(D.3)

The global charge (D.1) is conserved if $X^A$ meets

$$\partial_A \partial_B X_C = 0.$$  

(D.4)

For example, a constant vector, corresponding to a rigid translational symmetry, satisfies this condition.

In terms of the conventional field variables, $\{g_{\mu\nu}, B_{\mu\nu}, \phi\}$, of the closed string massless sector, the conserved global charge above reduces to

$$Q[X] = \int_{\partial M} d^{D-2} x_{\mu\nu} \sqrt{-g} \ e^{-2\phi} \left( K^{\mu\nu}[X] + 2X^{[\mu}B^{\nu]} \right),$$

(D.5)

where now, with $X^A = (\zeta^\mu + B_{\mu\rho}\xi^\rho, \xi^\nu)$,

$$K^{\mu\nu}[X] = 2\xi^{[\mu\nu]} - H^{\mu\nu\rho} \xi^\rho, \quad B^\mu = 2g^{\mu\nu} \left( 2\partial_\nu \phi - \partial_\nu \ln \sqrt{-g} \right) - \partial_\nu g^{\mu\nu}. $$

(D.6)

Specifically for the general solution (B.34), it is straightforward to compute the conserved global charge for the time translational symmetry:

$$Q[\partial_t] = \frac{1}{4} \left[ a + \left( \frac{a-b}{a+b} \right) \sqrt{a^2 + b^2} \right].$$

(D.7)

As known for Jordan (i.e. string) frame, e.g. [56], this time translational global charge is not necessarily positive definite. We speculate that only $M_\infty$ in (3.23) ought to be non-negative.

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