The Optimal Faucet

H. Henry Chen and Michael P. Brenner

1Department of Physics, Harvard University, Cambridge, MA 02138
2Division of Engineering and Applied Sciences, Harvard University, Cambridge, MA 02138

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The production of small fluid droplets relies on an instability of solutions to the Young-Laplace equation. We ask whether smaller droplets can be produced by changing the shape of the nozzle. At a given critical pressure, the circular nozzle actually produces the largest droplet. The droplet volume can be decreased by up to 18% using a triangular nozzle with stretched corners.

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A standard protocol for producing small droplets is as follows: a pipette, of circular cross-section, is pressurized at one end, pushing out a small fluid droplet. If the nozzle is sufficiently small, force balance requires that the droplet has constant mean curvature. At a critical pressure, this equilibrium shape becomes unstable, ultimately leading to the droplet detaching from the nozzle.

The volume of fluid entrained during this process is set by the total fluid volume contained in the critical droplet. This volume scales like \( r^3 \), where \( r \) is the nozzle radius. On the other hand, the critical pressure for ejecting this droplet scales like \( \gamma/r \), where \( \gamma \) is the liquid surface tension. Thus, ejecting smaller droplets requires higher pressures. The smallest size droplet that can be ejected is thus determined by the highest pressure that can be reliably applied to the nozzle, without material failure, etc.

One strategy for creating smaller droplets than those dictated by the instability of a static droplet is to use a time varying forcing at the nozzle. This method has achieved an order of magnitude decrease in droplet volume \( \dagger \). However, typical nozzles use a circular cross section. It is not unreasonable to imagine that changing the shape of the cross section to be some other shape may decrease the ejected droplet volume, while maintaining the same applied pressure. For example, imagine that we have a circular nozzle with a pendant droplet just below the critical volume: by “squeezing” the shape of the nozzle cross section into an elliptical shape, one might cause the droplet to detach at a lower volume.

In this paper we address the question: what is the shape of a nozzle for which the ejected droplet volume is minimized, for a given applied pressure? We demonstrate that circular nozzles do not eject the smallest droplets; instead, the optimal nozzle more closely resembles an equilateral triangle, albeit with “stretched” corners. The best nozzle shape that we have found has an ejected droplet volume about twenty percent smaller than the circular nozzle with the same critical pressure. Our method is inspired by and extends J. Keller’s classic treatment of the Euler buckling problem with a beam of nonuniform cross section \( \mathbb{B} \). Recently, the method has been applied to the optimization of a bistable switch \( \mathbb{H} \). For a detailed mathematical treatment of capillary surfaces in general, see \( \mathbb{I} \).

This Letter is organized as follows. We first explain the origin of the pendant droplet instability. Then we describe our method for reducing droplet size. Lastly we provide numerical calculations implementing the method, and present the candidate optimal nozzle.

**Pendant Droplet Instability.** - The instability of a droplet protruding from a nozzle is due to a bifurcation, most easily seen in the case of a circular nozzle that is much smaller than the capillary length, which allows us to neglect gravity. The shape of the droplet is then determined by the Young-Laplace equation \( p = \gamma K \), where \( p \) is the pressure difference across the liquid/air interface, \( \gamma \) is the surface tension, and \( K \) is the mean curvature of the droplet. This equation describes a surface of constant mean curvature \( p/\gamma \) with the nozzle edge as its boundary. If the boundary is a circle, then the solution must be a section of the sphere with mean curvature \( p/\gamma \). From the familiar relation

\[
K_{\text{sphere}} = \frac{2}{\text{sphere radius}}
\]

we deduce that the radius of curvature of the droplet is \( 2\gamma/p \). For small \( p \), such that the sphere radius is much greater than the nozzle radius, the solution is a shallow spherical cap. But note that its complement, the rest of the sphere, is also a solution. As \( p \) is increased, these two solutions approach each other until both become a hemisphere with the nozzle at the equator. The pressure at which the two solutions meet is the critical pressure \( p^* \), and the corresponding degenerate solution is unstable. Note that the critical pressure is also the maximum pressure, for the nozzle cannot support a sphere smaller than itself.

For a noncircular nozzle, we no longer have such a simple geometric picture, however key features remain. The unstable solution is still characterized by a bifurcation at which two solutions meet, corresponding to the maximum pressure achievable for the given nozzle. The critical pressure for a general nozzle can be computed as fol-
lows: let the droplet surface be parameterized as a function $R(u,v)$ over a domain $D$ in the $uv$-plane, which takes value in three dimensional physical space. The boundary of the domain $\partial D$ corresponds to a closed curve $C$ which represents the nozzle. The curvature is a nonlinear functional of the surface and its derivatives up to second order, hence the equation for the droplet shape has the form

$$\gamma K[\vec{R}, \vec{\nabla} R, \vec{\nabla}^2 R] = p, \quad (2)$$

where $\vec{\nabla}$ is the gradient operator in the $uv$-plane.

Upon increasing the pressure $p \rightarrow p + \delta p$, the surface changes: $\vec{R} \rightarrow \vec{R} + \delta \vec{R}$. Equation (2) implies that the variation $\delta \vec{R}$ and $\delta p$ are related by

$$\gamma \dot{L} \delta \vec{R} = \delta p, \quad (3)$$

where $\dot{L} \delta \vec{R}$ is the change in mean curvature induced by the surface change. $\dot{L}$ is a differential operator acting on $\delta \vec{R}$.

At the critical solution, the pressure is at a maximum; therefore, there must be a solution $w = \delta \vec{R}$ to equation (3) with $\delta p = 0$. The solution $w$ satisfies

$$\dot{L} w = 0 \quad (4)$$

with boundary condition $w = 0$ at $\partial D$. Note that the pressure dependence in this formula arises because $\dot{L} = \dot{L}[\vec{R}]$ depends implicitly on the pressure $p$ through $\vec{R}$. Hence, the existence of a nonzero $w$ is a diagnostic for finding the critical solution to (2) and the corresponding critical pressure $p^*$.

Optimization Method. - Now, to find the optimal nozzle, we need to derive a relation between the change in critical pressure and change in nozzle shape. Since pressure and volume are conjugate variables, increasing critical pressure is tantamount to decreasing critical volume. By iteratively changing the nozzle shape to increase critical pressure, we will thus arrive at a nozzle which produces smaller droplets. We compare the critical volume of the deformed nozzle with that of the circular nozzle that corresponds to the same critical pressure, since pressure is the control variable in practical situations.

Suppose that a given nozzle shape $C$ has a critical pressure $p^*$, a critical droplet shape $R^*$, and a corresponding $w$. All of these quantities change when the nozzle shape $C \rightarrow C + \delta C$. The change in the droplet shape $\delta \vec{R}$ is linearly related to the pressure change $\delta p$ by equation (3) with the boundary condition $\delta \vec{R} = \delta C$ at $\partial D$. On the other hand, since the critical solution maximizes the critical pressure, $w$ does not change to leading order in $\delta C$.

The change in critical pressure induced by $\delta C$ can therefore be computed by taking the inner product of both sides of (3) with $w$:

$$\gamma \langle w, \dot{L} \delta \vec{R} \rangle = \gamma \langle \delta \vec{R}, \dot{L} w \rangle + \gamma \int b(\delta \vec{R}, w)$$

$$= 0 + \gamma \int b(\delta C, w) = \langle w, \delta p \rangle.$$

Therefore

$$\delta p = \frac{\gamma \int b(\delta C, w)}{\langle w, 1 \rangle}. \quad (5)$$

Here $b(\bullet, \bullet)$ denotes the boundary integrand from integrating by parts. The derivation also uses the self adjointness of $\dot{L}$, which is readily demonstrable by explicit computation. Equation (5) is an explicit relation between a change in the nozzle shape ($\delta C$) and the resulting change in critical pressure.

Explicit Formula for $\delta p$. - We choose the nozzle $C$ to lie in the $xy$-plane, enclosing the origin. Then the droplet surface may be given in spherical coordinates by the distance from the origin ($R$) as a function of the two angles $\theta \in [0, \pi/2]$ and $\phi \in (0, 2\pi]$. To avoid the coordinate singularity at the pole ($\theta = 0$) we use $u, v$ given by $u = \tan(\theta/2) \cos(\phi)$ and $v = \tan(\theta/2) \sin(\phi)$. Hence the surface is a scalar function $R(u,v)$; its domain $D$ is the unit disk in the $uv$-plane. We retain $\phi$ to denote the polar angle in the $uv$-plane.

An appealing feature of this coordinate system is that the line element remains diagonal:

$$ds^2 = dR^2 + \Gamma (du^2 + dv^2),$$

where $\Gamma = 4R^2/(1 + u^2 + v^2)^2$. It is then straightforward to compute the free energy $E = \int (\gamma dA - p dV)$ which yields, upon variation, the Young-Laplace equation

$$- \vec{\nabla} \cdot (C \vec{\nabla} R) + AR = F, \quad (6)$$

where $\vec{\nabla}$ is the usual gradient operator in the $uv$-plane. The coefficients are

$$C = \frac{1}{\sqrt{1 + (\frac{4\rho^2}{AR})^2 (\vec{\nabla} R)^2}}, \quad A = C \left( \frac{(\vec{\nabla} R)^2}{R^2} + \frac{8}{(1 + \rho^2)^2} \right), \quad F = \rho \frac{4R^2}{(1 + \rho^2)^2},$$

where $\rho^2 \equiv u^2 + v^2$ is the radial coordinate in the $uv$-plane. $\partial D$ corresponds to $\rho = 1$.

In our coordinate system, the pressure change is

$$\delta p = \frac{1}{\delta_w V} \int d\phi \delta C \frac{w_p R(R^2 + R_z^2)}{(R^2 + R_p^2 + R_z^2)^{3/2}}, \quad (7)$$

where $\delta_w V \equiv \int d^2 \rho \ w \frac{4\rho^2}{(1 + \rho^2)^2}$. Here and in the following we use subscripts to denote partial differentiation.

We can recast this expression into a form that is more geometric. First, the contact angle $\alpha$ between the
drop and the plane of the nozzle is given by \( \cot \alpha(\phi) = \frac{R_p}{(R^2 + R_\phi^2)^{1/2}} \) where the right hand side is evaluated at the boundary. Second, we define \( w_\perp = w_p/(R^2 + R_\phi^2)^{1/2} \) which can be understood as follows - note that \( w \) is the difference between the outer and inner solutions as the pressure approaches bifurcation. Using the contact angle given above, this expression is the difference between the slopes (with respect to the vertical) of the outer and inner solutions at the boundary. This is a coordinate independent quantity. Third, we observe that

\[
d\phi \delta c R = \left( d\phi \sqrt{R^2 + R_\phi^2} \right) \left( \frac{\delta c R}{\sqrt{R^2 + R_\phi^2}} \right) = dl \delta N,
\]

where \( dl \) is the line element, and \( \delta N \) is the change of the nozzle in the direction locally normal to the nozzle. Lastly, the denominator \( \delta_w V \) in \( \mathbf{7} \) is just the change in volume from changing the surface by \( w \). Putting these facts together, the pressure change is

\[
\delta p = \frac{1}{\delta_w V} \int dl \delta N w_\perp \sin^3 \alpha,
\]

which leads to the prescription for changing the nozzle

\[
\delta N \sim \frac{1}{\delta_w V} w_\perp \sin^3 \alpha. \tag{8}
\]

Clearly, for the circular nozzle, symmetry implies that \( \delta N \) should be constant. But this amounts to a mere reduction in the size of the nozzle; the shape remains a circle. So the circular nozzle is at an extremum, in fact a minimum of critical pressure for fixed nozzle area.

For a noncircular nozzle, the contact angle isn't constant, and hence the change according to the above formula cannot be constant. So one may change the critical pressure while fixing the nozzle area. Moreover, since the circular nozzle is the only one (except the infinite strip) with a constant contact angle, the process of deformation does not end.

We apply \( \mathbf{8} \) iteratively to a perturbed circular nozzle to see how the shape evolves away from the circle. Figure \( \mathbf{4} \) shows the result of iterations starting with a circle deformed by a perturbation with a three-fold symmetry. The perturbation grows with each iteration, and eventually the nozzle shape becomes concave. With each iteration, we have applied a rescaling in order to maintain the nozzle area. Without the area constraint, the nozzle would become arbitrarily small in accordance with \( \mathbf{8} \). We are interested in the shape of the nozzle, not its size. We also apply the Savitzky-Golay filter \( \mathbf{9} \) at each iteration to smooth out the mesh noise. The solutions to the Young-Laplace equations are obtained using the nonlinear PDE solver in the MATLAB® PDE Toolbox, which implements the finite element method for elliptic equations with variable coefficients, exactly of the form in \( \mathbf{10} \). For each nozzle shape, we start at a pressure below the bifurcation and by choosing different trial solutions obtain both solutions. Then we bring both solutions to just below the critical pressure by stepping up the pressure, using the solution at each step as the trial solution for the next step. We then use the average of the two solutions for our surface, and their difference for \( w \). The validity of this procedure can be rigorously shown for a circular nozzle, and we expect it to remain valid for noncircular nozzles as long as the pressure is brought close to critical.

In order to compare and select among nozzle shapes, we need a measure of optimality independent of size. For every nozzle, we rescale its critical volume by the critical volume corresponding to the circular nozzle with the same critical pressure. This dimensionless volume is given by

\[
\tilde{V} = \frac{V^*}{2\pi \left( \frac{2}{p} \right)^3}. \tag{9}
\]

Figure \( \mathbf{1} \) shows a particular sequence of critical properties obtained through our iteration procedure. We see that the critical pressure begins to increase rapidly about the fifth iteration, after which the decrease in \( \tilde{V} \) slows down, and the nozzle shape becomes stretched out (see Figure \( \mathbf{4} \)). This means that in order to decrease droplet size at a given pressure, one should use a nozzle shape that is roughly triangular, perhaps with somewhat stretched out corners; but further deformation does not lead to significant improvement. Moreover, gravitational instabilities will inevitably become relevant if the “arms” become too long \( \mathbf{8} \).

It should be emphasized that we have shown a particular example of an improved nozzle, generated by a choice

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d\phi \delta c R = \left( d\phi \sqrt{R^2 + R_\phi^2} \right) \left( \frac{\delta c R}{\sqrt{R^2 + R_\phi^2}} \right) = dl \delta N,
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It should be emphasized that we have shown a particular example of an improved nozzle, generated by a choice
of the initial perturbation. We have tried other perturbations, leading to shapes with, say, four-fold symmetry or without any symmetry, but the three-fold perturbation has yielded the biggest reduction in the normalized critical volume.

So far we have ignored the effects of gravity, but our formalism applies just as well to the problem with gravity. Including gravity means that the pressure would no longer be constant throughout the drop surface, but rather a linear function of height: \( p \rightarrow p - \rho_m gh(u,v) \), where \( \rho_m \) is the mass density of the liquid, \( g \) is the gravitational acceleration, \( h \) is the distance below the nozzle, and \( p \) now denotes the pressure at the nozzle \( (h = 0) \). Although (9) acquires a new term as a result, this term does not contain derivatives and thus does not contribute to the boundary integral. So our formula for the pressure change remains the same in the presence of gravity. To be sure, the nozzle evolution would differ because the contact angle and \( w_\perp \) will be affected by gravity. Moreover, if the nozzle is too large relative to the capillary length, then gravity destabilizes all solutions; it is not possible to suspend a water drop from a meter wide faucet. It would be interesting to examine the case of the intermediate sized nozzle, small enough to have stable solutions, yet large enough to be affected by gravity.

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FIG. 2: Sequence of iterations away from the circular nozzle with an initial three-fold perturbation. The normalized critical volume given by (9) is shown in the bottom graph. The arrows indicate the corresponding shapes in Figure 1.

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* Electronic address: chen@physics.harvard.edu
† Electronic address: brenner@deas.harvard.edu
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[8] In computing an inner product \( \langle f, Lg \rangle \equiv \int dudv \langle Lg \rvert f \rangle \), where \( f \) and \( g \) are arbitrary functions of \( u, v \), and the integration is over \( D \), we can undo the differentiation on \( g \), and instead let the adjoint operator \( L^\dagger \) act on \( f \). Self-adjointness \((L = L^\dagger)\) can be verified explicitly. In the derivation, this allows us to simply interchange \( f \) and \( g \), while introducing the boundary terms from integration by parts.