Temporal correlations and its connection to coherence

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Time evolution is an indivisible part in any physics theory. It is now known that there are also correlation in evolutions of quantum systems. In this paper, we generalize the entangled-history theory to arbitrary quantum states and quantum channels. Seeing quantum channels as temporal correlations in quantum mechanics, we show how to describe this temporal correlation based on our generalization. Besides, we give a physical explanation of the entanglement of a quantum channel’s Choi matrix and show the relation between temporal correlation and coherence in quantum mechanics.

I. INTRODUCTION

Correlation is one core of quantum information theory. Since the famous work of J. S. Bell in 1964 [1], people began to know the extraordinary of quantum spatial correlations. On the other hand, the abnormal of temporal correlation is not known until [2]. But due to the strong action of measurement in quantum mechanics, these temporal can not reflect the information of evolution of the original state very well. It is until recent days, people began to realize that there also exits correlations in an evolution of a quantum system. To show this correlation, two-vector formalism [3–5] and entangled-history theory [6–9] make huge progress. Besides, in [10], they regard the whole evolution process as the temporal resource, using the process matrix method. In this paper, we embrace arbitrary quantum states and quantum channels into the entangled-history theory. With this generalization, we show that quantum channels can be seen as the temporal correlation in quantum mechanics. And we also shows how to measure the strength of this temporal correlation, which has a close connection with Choi matrix of quantum channels. Besides, the strength of this temporal correlation is also correlated with the decoherence power of quantum channels. This shows that the strength of this temporal correlation is closed connected with coherence, which is very different from the relation between spatial entanglement and coherence.

II. THE ENTANGLLED-HISTORY THEORY

In [9], they show that for a system evolves from one instant to the instant, it is possible to record the evolution of a system by using auxiliary systems. Through this recording, we can get information of the system’s evolution.

In the entangled-history theory, they prefer to use the signature ⊙ instead of ⊗ to represent the tensor product structure, emphasizing its temporal nature. Now let us a unitary evolution of a pure state. Suppose that we choose an orthonormal base {αi} for the Hilbert space H0 and an orthonormal base {βj} for the Hilbert space H1, then a unitary operator between the two instants t0 and t1 can be described by a matrix (uij), where

\[ u_{ij} = (β_j | U | α_i) \]

Then if the initial state of the system at the instant t0 is

\[ |φ⟩ = \sum_i a_i |α_i⟩ \]

and it goes under a unitary evolution U to the instant t1. By [9], this process can be described as

\[ \sum_{i,j} u_{ji} a_i |β_j⟩ ⊙ |α_i⟩ \]

Regard the set \{β_j ⊙ α_i\} as the basic evolution paths given by orthonormal basis \{α_i\} for H0 and \{β_j\} for H1, we see that it is the initial state of the system and the unitary operator together giving the superposition of these basic evolution paths. Usually this superposition will make the pure state entangled and using the monitor systems method in [9], we can set up corresponding experiments to detect this kind of entanglement. This is the reason that this theory is called entangled-history theory. What’s more, from (1), we can see it is the unitary operator that correlates the system at the two instants t0 and t1.

III. GENERALIZATION TO COMPLEX CASES

In [9], they show structures of (1) for pure initial states and unitary evolutions between two instants t0 and t1. For generalizations of this idea to complex cases, there are two basic directions. One is to embrace more instants keeping the initial state pure and evolutions between different instants unitary. The other way also focuses on the two-instant setting, but allows the initial state arbitrary and the evolution between t0 and t1 being any quantum channel. Then combine these two
together, we can get a generalization. But the first one is trivial, which resembles the tensor product of $n$ Hilbert spaces. So in the following, when we refer to generalization, we talk about the second type.

Suppose we have a system initialized in a state $\rho$ at the instant $t_0$, then it goes through some quantum channel $\Lambda$ and finally it arrives at the instant $t_1$. In the above setting, $\rho$ is a density matrix and $\Lambda$ is a quantum channel. Now, we show how to generalize the idea in [9] to describe this process. Firstly, we fix the orthonormal bases $\{|\alpha_i\rangle\}_{i}$ and $\{|\beta_j\rangle\}_{j}$ for Hilbert spaces $H_0$ and $H_1$ corresponding to instants $t_0$ and $t_1$, similarly as before. By the Stinespring theorem [11], we can assume that $\dim(H_0) = \dim(H_1)$. Then using

$$E_{ij} = |\alpha_i\rangle \langle \alpha_j|, \quad F_{kl} = |\beta_k\rangle \langle \beta_l|,$$

the initial mixed state $\rho$ is expressed as

$$\rho = \sum_{i,j} \rho_{ij} E_{ij}. \quad (1)$$

In the unitary case, when we fix orthonormal basis, we can give a matrix representation of the unitary operator then use this matrix representation to describe the whole process. Similarly, with $\{E_{ij}\}_{ij}$ and $\{F_{kl}\}_{kl}$, we can also give a matrix representation of a quantum channel, its Choi matrix [12], whose element is given by

$$\Lambda_{kl,ij} = \langle F_{kl}| \Lambda |E_{ij}\rangle = tr(F_{kl} \Lambda(E_{ij})). \quad (2)$$

With these, the above process is given as

$$\rho_{\Lambda} = \sum_{ij,kl} \Lambda_{kl,ij} \rho_{ij} F_{kl} \otimes E_{ij}. \quad (2)$$

In particular, when the initial state is pure and the quantum channel is unitary, like the example in the above section, our construction (2) will give $|\phi_{\Lambda}\rangle = \sum_{ij} \mu_{ij} |\alpha_i\rangle \langle \beta_j| \otimes |\alpha_i\rangle$. So our construction is actually a generalization of [9].

Next, we show that operators of the form $\rho_{\Lambda}$ are density operators. By calculation, we can directly get $tr(\rho_{\Lambda}) = 1$. To show its semi-positivity, note that

$$\rho_{\Lambda} = \sum_{ij,kl} \Lambda_{kl,ij} \rho_{ij} F_{kl} \otimes E_{ij} = \sum_{ij} \rho_{ij} \Lambda_{ij}(E_{ij}) \otimes E_{ij}$$

$$= SWAP \{ I \otimes \Lambda(\sum_{ij} \rho_{ij} E_{ij} \otimes E_{ij}) \}, \quad (3)$$

where $SWAP$ is the swap gate, a unitary channel. So we only need to verify the semi-positivity of $\sum_{ij} \rho_{ij} E_{ij} \otimes E_{ij}$. For any vector $|\mu\rangle = \sum_{ij} \mu_{ij} |\alpha_i\rangle |\alpha_j\rangle \in H_0 \otimes H_0$, we have that

$$\langle \mu | \sum_{ij} \rho_{ij} E_{ij} \otimes E_{ij} |\mu\rangle = \langle \mu | \rho |\mu\rangle \geq 0,$$

where $|\mu\rangle = \sum_{ij} \mu_{ij} |\alpha_i\rangle$. So $\rho_{\Lambda}$ is a density operator, for any initial state and any quantum channel.

Because $\rho_{\Lambda}$ is a density operator on a bipartite system, so we can measure how entangled or nonlocal it is. But the physics behind the entanglement of $\rho_{\Lambda}$ is not clear. Note that there is a big difference between spatial bipartite systems and temporal bipartite systems. For spatial bipartite systems, we can construct any global state we like. However, for temporal bipartite systems, we have to have an initial quantum state and let it evolves through some quantum channel to a later instant. It is the quantum channel that connect systems of different instants. In this way, temporal correlations in quantum mechanics are quantum channels. On the other hand, in terms of spatial correlations, for bipartite systems, we can classify them by their nonlocal properties or entanglement properties. In the following, we show that the entanglement of $\rho_{\Lambda}$ actually reflects the strength of temporal correlations represented by the quantum channel $\Lambda$. And it is also possible to classify quantum channels by the strength of their temporal correlations.

In the spatial case, when a quantum channel can map a product state, a trivial state, into an entangled state or a nonlocal state, a nontrivial state, then we say that this quantum channel is nontrivial. This is quite often in resource theory. Following this way, if we can find a trivial initial state $\rho$ and $\rho_{\Lambda}$ is entangled, then we may say that the quantum channel $\Lambda$ or its temporal correlation nontrivial. For the initial space $H_0$ with orthonormal basis $\{|\alpha_i\rangle\}_{i}$, the most trivial quantum state is $|\alpha_i\rangle \langle \alpha_i| \otimes |\alpha_i\rangle \forall i$, because it is pure and has no coherence. However, if $\rho = |\alpha_i\rangle \langle \alpha_i|$, then no matter what quantum channel $\Lambda$ is and what orthonormal basis we choose for $H_1$, the corresponding $\rho_{\Lambda}$ is always separable. So this idea can not work in the temporal setting. We have to take a different strategy.

From the above analysis, we see that to be able to classify quantum channels, the initial quantum state must be coherent. And note that unlike the usual way used in entanglement or resource theory, we can also to describe a quantum channel by how bad it is. That is, for entanglement, if a quantum channel can map a maximally entangled state into separable states, then this quantum channel is bad. Thinking this way and considering the necessity of initial states, we may choose $|\mu\rangle = \frac{1}{d} \sum_{i} |\alpha_i\rangle$ as the initial state, which is the most nontrivial state in terms of coherence under the orthonormal basis $\{|\alpha_i\rangle\}$. Let $\rho_{\mu} = |\mu\rangle \langle \mu|$, then by how separable the state $\rho_{\mu,\Lambda}$ is, we may be able to give the strength of temporal correlations represented by a quantum channel $\Lambda$ and furthermore classify quantum channels by their temporal correlations.

Take the initial state $\rho_{\mu}$ into the equation (3), with $\{|\alpha_i\rangle\}_{i}$ and $\{|\beta_j\rangle\}_{j}$ being orthonormal basis for $H_0$ and $H_1$. Then for a quantum channel $\Lambda$, we have that

$$\rho_{\mu,\Lambda} = \frac{1}{d} \sum_{i,j,k,l} \Lambda_{kl,ij} e^{i(\theta_j - \theta_k)} F_{kl} \otimes E_{ij}. \quad (4)$$

As the swap gate does not change the entanglement of a quantum state. So the separability of $\rho_{\mu,\Lambda}$ is the same as the state
SWAP(\rho_{\mu,\Lambda}). However,
\[
SWAP(\rho_{\mu,\Lambda}) = \frac{1}{d} \sum_{i,j,k,l} A_{k,l,i,j} e^{i(\theta_{k,l} - \theta_{i,j})} E_{ij} \otimes F_{kl} = \frac{1}{d} \sum_{i,j} E_{ij} \otimes \Lambda(E_{ij}) = I \otimes \Lambda(\frac{1}{d} \sum_{i,j} e^{i(\theta_{i,j})} E_{ij} \otimes E_{ij}).
\]
(5)

Note that \(\frac{1}{d} \sum_{i,j} e^{i(\theta_{i,j})} E_{ij} \otimes E_{ij}\) is a maximally entangled state on \(H_0 \otimes H_0\). So actually, SWAP(\rho_{\mu,\Lambda}) is a Choi matrix of the quantum channel. By the above, we see that the separability of \(\rho_{\mu,\Lambda}\) is actually the separability of the Choi matrix of the quantum channel \(\Lambda\) under the orthonormal basis \(|\alpha_i\rangle\) and \(|\beta_j\rangle\) of \(H_0\) and \(H_1\). So this tells us that the entanglement of the bipartite state \(\rho_{\mu,\Lambda}\) describes the strength of temporal correlations represented by some quantum channel \(\Lambda\) with some orthonormal basis of \(H_0\) on \(H_1\). As the entanglement of the bipartite state \(\rho_{\mu,\Lambda}\) is actually the entanglement of the Choi matrix of this channel, so of course, we can classify quantum channels by strength of their temporal correlations.

Now, let us see two extreme cases.

**Theorem 1** For an initial quantum state \(\rho\), if there exists a quantum channel \(\Lambda\), such that the operator \(\rho_{\Lambda} = \sum_{i,j,k} A_{k,l,i,j} \rho_{ij} F_{kl} \otimes E_{ij}\) is maximally entangled, then the quantum channel can be and can only be unitary channels and the initial state has to be maximally coherent under the orthonormal basis \(|\alpha_i\rangle\) of \(H_0\).

**Proof.**

Suppose that \(\rho_{\Lambda}\) is maximally entangled. It means that it is pure and its reductive state is \(\frac{1}{d} I\).

To be pure, it is equivalent to
\[
tr(\rho_{\Lambda}) = \sum_{i,j,k,l} \rho_{ij}^2 A_{k,l,i,j} A_{l,j,k,i} = \sum_{i,j} \rho_{ij}^2 (\sum_{k,l} A_{k,l,i,j})^2 = 1,
\]
where we use the fact \(A_{k,l,i,j} = tr(F_{ik}^\dagger \Lambda(E_{ji})) = tr(F_{ik}^\dagger \Lambda(E_{ji}))\) because \(\Lambda\) is a quantum channel.

On the other hand, because \(\Lambda\) is a quantum channel, so it has the Kraus operator representation
\[
\Lambda(M) = \sum_p A_p M A_p^\dagger, \forall M \in L(H_0)
\]
(6), and for every \(i\), \(\Lambda(E_{ii})\) is a quantum state. So we have
\[
tr(\Lambda(E_{ii})^\dagger \Lambda(E_{ii})) \leq 1, \forall i.
\]
Now let us define \(|a_{i,p}\rangle = A_p |\alpha_i\rangle\). Then by calculation,
we have that for every \(i\),
\[
tr(\Lambda(E_{ii})^\dagger \Lambda(E_{ii})) = tr\{\sum_p \rho_{ii} |a_{i,p}\rangle \langle a_{i,p}| \}^\dagger \sum_q \rho_{ii} |a_{i,q}\rangle \langle a_{i,q}| \} = \sum_{p,q} \rho_{ii} |a_{i,p}\rangle \langle a_{i,q}| \}^\dagger \sum_q \rho_{ii} |a_{i,q}\rangle \langle a_{i,q}| \} = \sum_{p,q} |a_{i,p}|^2 \leq 1.
\]
(7)

In the above calculation, the vector \(|\alpha_{i,p}\rangle\) is the vector whose coordinates are conjugate of coordinates of the vector \(|a_{i,p}\rangle\).

So for every \(i, j\), we have
\[
\sum_{k,l} |A_{k,l,i,j}|^2 = tr(\Lambda(E_{ij})^\dagger \Lambda(E_{ij})) = \sum_{k,l} \sum_{p,q} |\beta_k| |a_{i,p}||a_{i,q}| \langle a_{i,q}| \} = \sum_{p,q} |a_{i,p}|^2 \leq 1.
\]
(8)

The last inequality is based on (7) and the Cauchy-Schwarz inequality.

With (8), we know that
\[
tr(\rho_{\Lambda}) = \sum_{i,j} |\rho_{ij}|^2 (\sum_{k,l} |A_{k,l,i,j}|^2) \leq \sum_{i,j} |\rho_{ij}|^2.
\]

So for \(\rho_{\Lambda}\) to be pure, the initial state \(\rho\) has to be pure and for the quantum channel \(\Lambda\), we have \(\sum_{k,l} |A_{k,l,i,j}|^2 = 1, \forall i, j\). In particular, \(\Lambda(E_{ii})\) is a pure state, \(\forall i\).

Next, let us consider the second requirement of maximally entangled states. We should have
\[
tr_{H_1}(\rho_{\Lambda}) = \frac{1}{d} I, \quad tr_{H_0}(\rho_{\Lambda}) = \frac{1}{d} I
\]
. This is equivalent to
\[
\sum_k \rho_{ij} A_{k,k,i,j} = 0, \forall i \neq j, \sum_k \rho_{ii} A_{k,k,i,i} = \frac{1}{d}, \forall i.
\]
(9)

Note that for every \(i, \sum_k A_{k,k,i} = tr(\Lambda(E_{ii})) = 1\), so from the above, we must have \(\rho_{ii} = \frac{1}{d}, \forall i\). But we have verified that the initial state must be pure. So we have that \(\rho = |\alpha\rangle \langle \alpha|, \forall i.\)

where
\[
|\alpha\rangle = \sum_i \frac{1}{\sqrt{d}} e^{i\theta_i} |\alpha_i\rangle.
\]
With this and (1), we know that

$$SWAP(\bar{\rho}) = \frac{1}{d} \sum_{i,j,k,l} \Lambda_{kl,ij} e^{i(\theta_i - \theta_j)} E_{ij} \otimes F_{kl}$$

$$= \frac{1}{d} \sum_{i,j} e^{i(\theta_i - \theta_j)} E_{ij} \otimes \Lambda(E_{ij})$$

$$= I \otimes \Lambda(\frac{1}{d} \sum_{i,j} e^{i(\theta_i - \theta_j)} |\alpha_i\alpha_i\rangle \langle \alpha_j\alpha_j|) \quad (10)$$

As $\frac{1}{\sqrt{d}} \sum_{i} e^{i\theta_i} |\alpha_i\alpha_i\rangle$ is a maximally entangled state on $H_0 \otimes H_0$. So from the isomorphism between quantum channels and their Choi matrices, we know that for a quantum channel $\Lambda$, $I \otimes \Lambda$ maps some maximally entangled state into another maximally entangled state, if and only if this quantum channel is unitary. So above all, we have the following correspondence

$$\rho_\Lambda$$ is maximally entangled $\iff$ $SWAP(\rho_\Lambda)$ is maximally entangled $\Rightarrow$ $\Lambda$ is unitary and the initial state is maximally coherent.

For the other direction, we can choose the initial state to be $|\alpha\rangle = \frac{1}{\sqrt{d}} \sum_i |\alpha_i\rangle$. Then from (1), we only need to show that the Choi matrix of unitary channels are proportional to some maximally entangled state, which can be checked easily by direct calculation.

The above theorem shows the case when $\rho_{\mu,\Lambda}$ is the least separable, that is it is maximally entangled. And also it shows the necessity of using maximally coherent states as the initial state.

The other extreme is when $\rho_{\mu,\Lambda}$ is the most separable or classical, that is the matrix representation of $\rho_{\mu,\Lambda}$ is a diagonal matrix under orthonormal basis $\{|\alpha_i\rangle\}_i$ and $\{|\beta_j\rangle\}_j$. Then we have

$$\rho_{\mu,\Lambda} = \frac{1}{d} \sum_{i,j,k,l} \Lambda_{kl,ij} e^{i(\theta_i - \theta_j)} F_{kl} \otimes E_{ij} = \sum_{i,k} \lambda_k |\alpha_i\rangle \langle \alpha_i| \otimes |\beta_k\rangle \langle \beta_k|.$$  

So in this case, we have $\Lambda(E_{ij}) = 0$ and $\Lambda(E_{ii})$ is a diagonal matrix for every $i$. This just says that the quantum channel $\Lambda$ is a coherence destroying map [13], which is the most classical channel considering coherence.

From these two extreme cases, under the choice of orthonormal basis for $H_0$ and $H_1$, for an arbitrary quantum channel, its temporal correlation is between the above two cases. And taking $\{|\beta_j\rangle \otimes |\alpha_i\rangle\}_{i,j}$ as the basic evolution paths, based on the above analysis, physically, the initial state being maximally coherent gives us the strongest coherence in terms of $\{|\alpha_i\rangle\}_i$. Because unitary evolution keeps all information in quantum systems, so it also preserves coherence of the initial system, in other words, their power of decoherence is the weakest, so it makes $\rho_{\mu,\Lambda}$ the best superposition of basic evolution paths, which is maximally entangled. On the other hand, for coherence destroying maps, they destroy all ingredients of coherence, in other words, their power of decoherence is the strongest, so in this case, although the initial state is still maximally entangled, the evolution destroys all and makes $\rho_{\mu,\Lambda}$ most classical. For arbitrary quantum channels, their decoherence power is between the above two, so their $\rho_{\mu,\Lambda}$ is in the middle.

Our analysis shows that there exists a close relation between decoherence of quantum channels and the strength of quantum channels’ temporal correlations, which connects temporal correlation with coherence. Although, for spatial bipartite systems, the entanglement also relies on coherence, we have to classify coherence which is helpful for entanglement and which has no connection with entanglement to show this reliance.

IV. CONCLUSION

Superposition and linearity are the most important and fundamental features of quantum mechanics. From these two traits, people get a set of no-go theorems, such as the No-cloning theorem [14], the No-deleting theorem [15]. These make quantum mechanics depart quite far from the classic world. But usually, we refer superposition to quantum states at some fixed instant and linearity to the evolution way of quantum systems. In the entangled-history theory, they are combined together and through this combination we see the superposition of evolution paths of quantum systems and can get a better understanding of temporal correlations. In this paper, we generalize the entangled-history theory to embrace arbitrary quantum states and quantum channels. Using this generalization, we propose a method to describe the temporal correlation given by a quantum channel, which has a close connection with the Choi matrix of quantum channels. Besides, we also shows that the strength of temporal correlation given by some quantum channel is closely connected with the decoherence power of this channel.

ACKNOWLEDGMENTS

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