Pulsating Strings With Angular Momenta

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Abstract: We derive the energy of pulsating string, as function of oscillation number and angular momenta, which oscillates in $AdS_3$ with an extra angular momentum along $S^1$. We find similar solutions for the strings oscillating in $S^3$ in addition to extra angular momentum. Further we generalize the result of the oscillating strings in Anti de-Sitter space in the presence of both spin and angular momentum in $AdS^5 \times S^1$.

Keywords: AdS-CFT Correspondence, Pulsating Strings
1. Introduction

Proving the AdS/CFT correspondence [1]-[3] or more generally the gauge-gravity duality has been a major area of research in string theory in the last many years. This duality maps the anomalous dimensions of gauge invariant operators in the gauge theory to the energy spectrum of the string theory states. It has been observed that the exact state-operator mapping is extremely difficult, because of the infinite number of stringy states in the string theory side. Hence various limits have been considered on both sides of the duality, so that the conjecture can be checked to some accuracy. The semiclassical strings have played a key role in exploring various aspects of the correspondence [4]-[20]. This shows that the semiclassical results are reliable enough to use for the duality and compare with the dimensions of the operators in the Super Yang-Mills (SYM) side. The matrix of anomalous dimensions can be mapped to an integrable Bethe spin chain [21], and it has empowered our understanding of the duality conjecture. The semiclassical calculations in the string theory side has shown that the multi-spin rotating and pulsating string solutions beyond their BPS limit with large charges are in perfect agreement with the ones calculated in dual gauge theory. The agreement beyond the BPS limit relies on the fact that on the string theory side the quantum corrections of strings are suppressed by the large quantum number, while in the field theory side the anomalous dimension matrices of the dual composite operators are related to the Hamiltonian of integrable spin chain. Among the classes of the semiclassical string solutions, the pulsating string solutions have better stability than the non-pulsating ones [2]. However unlike rotating strings, pulsating string are less explored. Pulsating string solution was first introduced in [23] and further generalized in [24]-[26]. They have also been studied in $AdS_5 \times S^5$ [27]-[31], $AdS_4 \times CP^3$ [32]-[34] and other backgrounds [35]-[40]. Pulsating strings concept was first introduced in [27] where they were expected to correspond to certain highly excited sigma model operators. In [23] and [33], pulsating string solutions in $AdS_5$ and $S^5$ have been worked
out separately where as in [31], simultaneously rotating and oscillating strings in $AdS_5$ have been derived. Strings spinning in $S^5$ whereas as pulsating in $AdS$ and spinning in $AdS$ whereas as pulsating in $S^5$ have been described in [28]. Recently pulsating string solution in the less supersymmetric Lunin-Maldacena background has been studied in [40] and dispersion relation for the string oscillating in $S^5$ with a constant $\rho$ value in $AdS_5$ has been found out in [34]. As oscillation number is adiabatic invariant, the relation between energy and oscillation number is presented as the solutions to characterize the string dynamics. So here we wish to study a few oscillating string solutions with an extra angular momentum and see how the energy and oscillation number relation is affected when the string is oscillating in $S$ and $AdS$ separately.

The rest of the paper is organized as follows. In the section-2, we show the relationship of the oscillation number with an angular momentum for the string which is pulsating in the radial direction of $AdS$ with an angular momentum along $S^1$. In the section-3, we present a class of circular string solution which is oscillating in $S^3$ and at the same time has an extra angular momentum. Section-4 is devoted to the rotating string which is pulsating in the $AdS$ with two spin along mutually perpendicular directions of $AdS_5$ and have an angular momentum along $S^1$. Finally, in the section-5, we conclude with some remarks.

2. Pulsating string with angular momentum in $AdS_3 \times S^1$

In this section we study a semiclassical quantization of a string which expands and contracts in $AdS_3$ and at the same time rotates along sphere. We start with the full metric for $AdS_5 \times S^5$ background

$$ds^2 = - \cosh^2 \rho dt^2 + \rho^2 d\sigma^2 + \sinh^2 \rho (d\phi^2 + \cos^2 \phi d\phi_1^2 + \sin^2 \phi d\phi_2^2) + d\psi^2$$

$$+ \sin^2 \psi d\theta^2 + \cos^2 \psi d\xi^2 + \sin^2 \psi \cos^2 \theta d\xi_1^2 + \sin^2 \psi \sin^2 \theta d\xi_2^2.$$

(2.1)

By appropriate substitution of the co-ordinates in the above metric (2.1), we get the following metric for studying the string which oscillates in the radial ($\rho$) direction of $AdS$ space and at the same time has an angular momentum along $S^1$:

$$ds^2 = - \cosh^2 \rho dt^2 + \rho^2 d\sigma^2 + \sinh^2 \rho d\phi^2 + d\psi^2,$$

(2.2)

where $\rho \in [0, \infty], \phi \in [0, 2\pi]$ and $\psi$ specifies $S^1$ direction. The Polyakov action for the fundamental string in the above background is given as

$$I = \frac{\sqrt{\lambda}}{4\pi} \int dt d\sigma \left[- \cosh^2 \rho(t^2 - t'^2) + \rho^2 - \rho'^2 + \sinh^2 \rho(\dot{\phi}^2 - \phi'^2) + \psi^2 - \psi'^2\right].$$

(2.3)

where $\lambda$ is the 't Hooft coupling, 'dots' and 'primes' denote the derivatives with respect to world sheet time and space coordinates respectively. We take the following ansatz for studying the pulsating string solution

$$t = t(\tau), \quad \rho = \rho(\tau), \quad \phi = m\sigma, \quad \psi = \psi(\tau).$$

(2.4)
The equations of motion for $t$ and $\rho$ are given by

$$2 \cosh \rho \sinh \rho \dot{t} + \cosh^2 \dot{\rho} = 0,$$

$$\ddot{\rho} = -\sinh \rho \cosh \rho (m^2 + \ell^2).$$ \hspace{1cm} (2.5)

The Virasoro constraints gives us the following

$$\dot{\rho}^2 + \psi^2 + m^2 \sinh^2 \dot{\rho} - \cosh^2 \rho t^2 = 0.$$ \hspace{1cm} (2.6)

The conserved quantities are as follows

$$\mathcal{E} = \cosh^2 \dot{t}, \quad \mathcal{J} = \dot{\psi},$$ \hspace{1cm} (2.7)

where $\mathcal{E} = \frac{\mathcal{E}}{\sqrt{\lambda}}$ and $\mathcal{J} = \frac{\mathcal{J}}{\sqrt{\lambda}}$. Now putting the above equation (2.7) in the equation (2.6), we get

$$\dot{\rho}^2 = \frac{\mathcal{E}^2}{\cosh^2 \rho} - m^2 \sinh^2 \dot{\rho} - \mathcal{J}^2.$$ \hspace{1cm} (2.8)

Some comments are in order. In the above equation, we can see that $\dot{\rho}^2$ goes from $\mathcal{E}^2 - \mathcal{J}^2$ to infinity as $\rho$ goes from 0 to $\infty$. This means that the coordinate $\rho$ oscillates between a minimal value ($0$) and a maximal value ($\rho_{\max}$). Now the oscillation number can be written as

$$N = \frac{\sqrt{\lambda}}{2\pi} \int d\rho \dot{\rho} = \frac{1}{2\pi} \int_{0}^{\rho_{\max}} d\rho \sqrt{\frac{\mathcal{E}^2}{\cosh^2 \rho} - m^2 \sinh^2 \dot{\rho} - \mathcal{J}^2}.$$ \hspace{1cm} (2.9)

Putting $\sinh \rho = x$ in the above equation (2.9), we get

$$N = \frac{1}{2\pi} \int_{0}^{\sqrt{R}} \frac{dx}{1 + x^2} \sqrt{\mathcal{E}^2 - m^2 x^2(1 + x^2) - \mathcal{J}^2(1 + x^2)},$$ \hspace{1cm} (2.10)

where $\mathcal{N} \sqrt{\lambda} = N$ and $R = \frac{-(m^2 + \mathcal{J}^2) + \sqrt{(m^2 + \mathcal{J}^2)^2 + 4m^2(\mathcal{E}^2 - \mathcal{J}^2)}}{2m^2}$. Taking the partial derivative of the above equation with respect to $m$ we get

$$\frac{\partial N}{\partial m} = -\frac{m}{\pi} \int_{0}^{\sqrt{R}} \frac{x^2}{1 + x^2} \sqrt{\frac{\mathcal{E}^2 - m^2 x^2(1 + x^2) - \mathcal{J}^2(1 + x^2)}} dx.$$ \hspace{1cm} (2.11)

The above can be written in terms of the standard elliptical integrals

$$\frac{\partial N}{\partial m} = \frac{1}{\sqrt{2\pi m}} \sqrt{a_+} \left[ \mathcal{K} \left( \frac{a_-}{a_+} \right) - \mathcal{E} \left( \frac{a_-}{a_+} \right) \right],$$ \hspace{1cm} (2.12)

where $a_\pm = (m^2 + \mathcal{J}^2) \pm \sqrt{(m^2 + \mathcal{J}^2)^2 + 4m^2(\mathcal{E}^2 - \mathcal{J}^2)}$ and $\mathcal{K}$ and $\mathcal{E}$ are complete elliptical integral of first and second kind respectively. Now expanding the above equation for small oscillation number with small $\mathcal{E}$ and $\mathcal{J}$

$$\frac{\partial N}{\partial m} = \left[ \frac{\mathcal{J}^2}{4m^2} + \frac{3\mathcal{J}^4}{32m^4} + \mathcal{O}[\mathcal{J}^6] \right] + \left[ -\frac{1}{4m^2} - \frac{9\mathcal{J}^2}{16m^4} - \frac{225\mathcal{J}^4}{256m^6} + \mathcal{O}[\mathcal{J}^6] \right] \mathcal{E}^2$$
\[ + \left[ \frac{15}{32m^4} + \frac{525J^2}{256m^6} + \frac{11025J^4}{2048m^8} + \mathcal{O}[J^6] \right] \mathcal{E}^4 + \mathcal{O}[\mathcal{E}^6]. \quad (2.13) \]

Integrating the above equation \((2.13)\) with respect to \(m\) and reversing the series, we get

\[ \mathcal{E} = 2\sqrt{mM} \ K_1(\mathcal{J}) \left[ 1 + K_2(\mathcal{J}) \frac{5M}{4m} + \mathcal{O}[M^2] \right], \quad (2.14) \]

where \[
\mathcal{M} = \mathcal{N} + \frac{J^2}{4m} + \frac{J^4}{32m^3} + \mathcal{O}[J^6] \]

\[ K_1(\mathcal{J}) = \left[ 1 + \frac{3J^2}{4m^2} + \frac{45J^4}{64m^4} + \mathcal{O}[J^6] \right]^{-1/2} \]

and \[ K_2(\mathcal{J}) = \left[ 1 + \frac{21J^2}{8m^2} + \frac{315J^4}{64m^4} + \mathcal{O}[J^6] \right] K_1^4(\mathcal{J}). \quad (2.15) \]

The equation \((2.14)\) represents the classical energy for the short strings which are oscillating near the center of AdS3 with an angular momentum in \(S^1\). With \(J \to 0\), \(\mathcal{M} \to \mathcal{N}\) and \(K_{1,2} \to 1\). This gives us the energy for the strings oscillating in one plane for small energy limit as in the \([31]\). Now expanding equation \((2.12)\), for large \(\mathcal{E}\) but small \(\mathcal{J}\), we get

\[
\frac{\partial \mathcal{N}}{\partial m} = k_1 m^{-1/2} \mathcal{E}^{1/2} + k_2 m^{1/2} \mathcal{E}^{-1/2} \left( 1 + \frac{J^2}{m^2} \right) + k_3 m^{3/2} \mathcal{E}^{-3/2} \left( 1 + \frac{k_4 J^2}{k_3 m^2} + \frac{J^4}{m^4} \right) + \mathcal{O}[\mathcal{E}^{-5/2}], \quad (2.16) \]

where \[ k_1 = \frac{1}{\pi} \left( \sqrt{\pi} \left[ \frac{5}{[\frac{3}{4}]} \right] - \mathcal{E}(-1) \right) = -0.19069, \]

\[ k_2 = -\frac{\mathcal{E}(-1)}{4\pi} + \frac{1}{8\pi} \left[ \frac{3}{[\frac{3}{2}]} \right] + \frac{1}{4\pi} \left[ \frac{5}{[\frac{5}{2}]} \right] + \frac{1}{\pi} 2 F_1 \left( \frac{3}{2}, \frac{3}{2}, 2, -1 \right) = 0.104328, \]

\[ k_3 = -\frac{\mathcal{E}(-1)}{32\pi} - \frac{1}{32\pi} \left[ \frac{1}{[\frac{3}{2}]} \right] + \frac{1}{32\pi} \left[ \frac{5}{[\frac{5}{2}]} \right] - \frac{1}{32} 2 F_1 \left( \frac{3}{2}, \frac{3}{2}, 2, -1 \right) + \frac{3}{128} 2 F_1 \left( \frac{5}{2}, \frac{5}{2}, 3, -1 \right) + \frac{9}{128} 2 F_1 \left( \frac{7}{2}, \frac{7}{2}, 3, -1 \right) = -0.0178772, \]

and \(k_4 = 0.0119181\). Integrating the above equation \((2.16)\), we get

\[
\mathcal{N} = \mathcal{N}_0 + 2k_1 m^{1/2} \mathcal{E}^{1/2} + \frac{2}{3} k_2 m^{3/2} \mathcal{E}^{-1/2} \left( 1 - 3 \frac{J^2}{m^2} \right) + \frac{2}{5} k_3 m^{5/2} \mathcal{E}^{-3/2} \left( 1 + \frac{5 k_4 J^2}{k_3 m^2} - \frac{5 J^4}{3 m^4} \right) + \mathcal{O}[\mathcal{E}^{-5/2}]. \quad (2.17) \]

The integration constant \(\mathcal{N}_0\) can be determined from the integral \((2.10)\) for \(m = 0\)

\[ \mathcal{N}_0 = \frac{1}{\pi} \int_0^r \frac{dx}{1 + x^2} \sqrt{\mathcal{E}^2 - J^2 - J^2 x^2}, \quad (2.18) \]

where \(r = \sqrt{\mathcal{E}^2 - J^2} \). Changing the variable we get

\[ \mathcal{N}_0 = \mathcal{J}^2 \pi \int_0^1 \frac{\sqrt{1 - x^2}}{1 + r^2 x^2} dx = \frac{1}{2} (\mathcal{E} - \mathcal{J}). \quad (2.19) \]
Now
\[ N = \frac{1}{2}(\mathcal{E} - J) + 2k_1m^{1/2}\mathcal{E}^{1/2} + \frac{2}{3}k_2m^{3/2}\mathcal{E}^{-1/2}\left(1 - 3\frac{J^2}{m^2}\right) + \mathcal{O}[\mathcal{E}^{-3/2}]. \] (2.20)

Reversing the series we get
\[ \mathcal{E} = 2L + a_1m^{1/2}L^{1/2} + a_2m - A(J) a_3 m^{3/2}L^{-1/2} + \mathcal{O}[L^{-3/2}], \] (2.21)
where
\[ L = \mathcal{N} + \frac{J^2}{2}, \quad A(J) = 1 - 5\frac{J^2}{m^2}, \]
\[ a_1 = 1.07871, \quad a_2 = 0.290901 \quad \text{and} \quad a_3 = 0.0591372. \]
In the above (2.21) equation, from \( \mathcal{O}[L^{-1/2}] \) term onwards the coefficients will depend on angular momentum \( J \) raised to the even exponent explicitly with \( L \). The \( J \) dependent terms will run up to the next even exponent of \( J \) i.e. the \( J \) dependent term of \( \mathcal{O}[L^{-3/2}] \) will run up to \( \frac{J^6}{m^6} \), \( \mathcal{O}[L^{-5/2}] \) up to \( \frac{J^8}{m^8} \) and so on. The above equation (2.21) is the energy expression for the long strings with small \( J \) in \( AdS_3 \times S^2 \). This is the same expression as in [23] with \( J = 0 \). Expanding (2.12) for short strings with large \( J \) we get
\[ \mathcal{E} = 2N + J - \frac{m^2}{4}J^{-1} - \frac{3m^4}{64}J^{-3} + \mathcal{O}[J^{-5}]. \] (2.22)

3. Pulsating string in \( \mathbb{R} \times S^3 \)
Here we wish to study a class of string solutions which is pulsating in \( S^3 \) as well as with an extra angular momentum. The background metric is
\[ ds^2 = -dt^2 + d\psi^2 + \sin^2 \psi d\theta^2 + \cos^2 \psi d\xi^2. \] (3.1)
The Polyakov action for the string in the above background is given by
\[ I = \frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \left[ -(\dot{t}^2 - t'^2) + \dot{\psi}^2 - \psi'^2 + \sin^2 \psi (\dot{\theta}^2 - \theta'^2) + \cos^2 \psi (\dot{\xi}^2 - \xi'^2) \right]. \] (3.2)
We chose the following ansatz for studying the pulsating string with an extra angular momentum in the \( \mathbb{R} \times S^3 \) space
\[ t = t(\tau), \quad \psi = \psi(\tau), \quad \theta = m\sigma, \quad \xi = \xi(\tau). \] (3.3)
The equations of motion for \( \psi \) and \( \xi \) are given by
\[ \ddot{\psi} = -(m^2 + \xi^2) \sin \psi \cos \psi, \]
\[ \ddot{\xi} = 2\dot{\psi} \dot{\xi} \tan \psi. \] (3.4)
On the other hand, the Virasoro constraints give us
\[ \dot{\psi}^2 - \dot{\xi}^2 + m^2 \sin^2 \psi + \cos^2 \psi \dot{\xi}^2 = 0. \] (3.5)
The conserved charges are given as
\[ \mathcal{E} = \dot{t}, \quad J = \cos^2 \psi \dot{\xi}. \] (3.6)
Putting above equation (3.6) in the equation (3.5), we get
\[
\dot{\psi}^2 = \mathcal{E}^2 - m^2 \sin^2 \psi - \frac{J^2}{\cos^2 \psi}.
\] (3.7)

Now the oscillation number is written as
\[
N = \frac{1}{2\pi} \int d\psi \sqrt{\mathcal{E}^2 - m^2 \sin^2 \psi - \frac{J^2}{\cos^2 \psi}}.
\] (3.8)

Putting \(\sin \psi = x\) in the above equation (3.8), we get
\[
N = 2\pi \int \sqrt{\mathcal{R}} \frac{dx}{1 - x^2} \sqrt{\mathcal{E}^2(1 - x^2) - m^2x^2(1 - x^2) - J^2},
\] (3.9)

where
\[
\mathcal{R} = \frac{-(m^2 + \mathcal{E}^2) + \sqrt{(m^2 + \mathcal{E}^2)^2 - 4m^2(\mathcal{E}^2 - J^2)}}{-2m^2},
\]

and
\[
\frac{\partial N}{\partial m} = \frac{-2m}{\pi} \int_0^{\sqrt{\mathcal{R}}} \frac{x^2}{\sqrt{\mathcal{E}^2(1 - x^2) - m^2x^2(1 - x^2) - J^2}} dx.
\] (3.10)

where \(a_\pm = (m^2 + \mathcal{E}^2) \pm \sqrt{(m^2 + \mathcal{E}^2)^2 - 4m^2(\mathcal{E}^2 - J^2)}\) and a condition of \((m^2 - \mathcal{E}^2)^2 + 4m^2J^2 > 0\), which give an upper bound to the \(N\). \(\mathcal{E}\) and \(\mathcal{K}\) are the usual Elliptic integral of first and second kind respectively. Now expanding equation (3.10) in small \(\mathcal{E}\) and \(J\) for small \(N\), we get
\[
\frac{\partial N}{\partial m} = \left[ \frac{J^2}{2m^2} - \frac{15J^4}{16m^4} + O[J^6] \right] + \left[ -\frac{1}{2m^2} + \frac{9J^2}{8m^4} - \frac{525J^4}{128m^6} + O[J^6] \right] \mathcal{E}^2
\]
\[
+ \left[ -\frac{3}{16m^4} + \frac{225J^2}{128m^6} + \frac{11025J^4}{1024m^8} + O[J^6] \right] \mathcal{E}^4 + O[\mathcal{E}^6].
\] (3.11)

Integrating the above equation (3.11) with respect to \(m\) and reversing the series, we get
\[
\mathcal{E} = \sqrt{2m\mathcal{L}} \ K_3(J) \left[ 1 - K_4(J) \frac{\mathcal{L}}{8m} + O[\mathcal{L}^2] \right],
\] (3.12)

where
\[
\mathcal{L} = N + \frac{J^2}{2m} - \frac{5J^4}{16m^3} + O[J^6]
\]
\[
K_3(J) = \left[ 1 - \frac{3J^2}{4m^2} + \frac{105J^4}{64m^4} + O[J^6] \right]^{-1/2}
\]
and
\[
K_4(J) = \left[ 1 - \frac{45J^2}{8m^2} + \frac{1575J^4}{64m^4} + O[J^6] \right] K_4^2(J).
\] (3.13)

The equation (3.12) gives the short string oscillation energy in \(\mathbb{R} \times S^3\) with an extra angular momentum. This is the same result of [33], except we have \(J\) dependent factor with every term in the expression. The one loop correction to the energy of this type of pulsating string can be calculated following [33] and appropriate correspondence to those kind of configurations can be made.
4. Pulsating and rotating string with two spins in $\text{AdS}_5 \times S^1$

In this section we wish to study a class of long pulsating strings (in the large energy limit) which is pulsating as well as rotating in the $\text{AdS}_5$ space. It has two spin in $\text{AdS}$ which are mutually perpendicular and has an angular momentum along the $S^1 \subset S^5$. We get the metric for $\text{AdS}_5 \times S^1$ background by putting $\theta = \xi_1 = 0$ and $\phi = \frac{\pi}{4}$ in the equation (2.1)

$$ds^2 = -\cosh^2 \rho dt^2 + d\rho^2 + \frac{1}{2} \sinh^2 \rho (d\phi_1^2 + d\phi_2^2) + d\psi^2.$$  \hspace{1cm} (4.1)

Now the Polyakov action for the string in the above background is given by

$$I = \frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \left[ -\cosh^2 \rho (\dot{t}^2 - \dot{t}'^2) + \dot{\rho}^2 - \dot{\rho}'^2 + \frac{1}{2} \sinh^2 \rho (\dot{\phi}_1^2 + \dot{\phi}_2^2 - \dot{\phi}_1^2 - \dot{\phi}_2^2) + \dot{\psi}^2 - \dot{\psi}'^2 \right].$$  \hspace{1cm} (4.2)

We take the following ansatz for studying the pulsating string in the above background

$$t = t(\tau), \quad \rho = \rho(\tau), \quad \phi_1 = \phi(\tau) + m\sigma, \quad \phi_2 = \phi(\tau) - m\sigma, \quad \psi = \psi(\tau).$$  \hspace{1cm} (4.3)

The equations of motion for $t$, $\rho$ and $\phi_1$ and $\phi_2$ are

$$2 \cosh \rho \sinh \rho \dot{\rho} \ddot{t} + \cosh^2 \rho \ddot{t} = 0,$$

$$\ddot{\rho} + \sinh \rho \cosh \rho (m^2 + \ell^2 - \dot{\phi}_2^2) = 0,$$

$$2 \cosh \rho \sinh \rho \dot{\rho} \ddot{\phi} + \sinh^2 \rho \ddot{\phi} = 0.$$  \hspace{1cm} (4.4)

The Virasoro constraint gives us

$$\dot{\rho}^2 + \dot{\psi}^2 + m^2 \sinh^2 \rho + \dot{\phi}_2^2 \sinh^2 \rho - \dot{t}^2 \cosh^2 \rho = 0.$$  \hspace{1cm} (4.5)

Now the conserved quantities are as follows

$$\mathcal{E} = \dot{t} \cosh^2 \rho, \quad \mathcal{S}_1 = \mathcal{S}_2 = \mathcal{S} = \frac{1}{2} \dot{\phi}_2 \sinh^2 \rho \quad \mathcal{J} = \dot{\psi},$$  \hspace{1cm} (4.6)

where $\mathcal{S} = \frac{\mathcal{S}}{\chi}$. Putting the above equation (4.6) in the equation (4.5), we get

$$\dot{\rho}^2 = \frac{\mathcal{E}^2}{\cosh^2 \rho} - \frac{4\mathcal{S}^2}{\sinh^2 \rho} - m^2 \sinh^2 \rho - \mathcal{J}^2.$$  \hspace{1cm} (4.7)

If we consider the above equation as the equation of motion of a test particle moving in a potential, then the particle experiences infinite potential at both zero and infinity having a minimum in between. So the radial coordinate oscillates in between a minimum ($\rho_{\text{min}}$) and maximum value ($\rho_{\text{max}}$). Now the oscillation number

$$\mathcal{N} = \frac{1}{2\pi} \int d\dot{\rho} \dot{\rho} = \frac{1}{\pi} \int_{\rho_{\text{min}}}^{\rho_{\text{max}}} d\rho \sqrt{\frac{\mathcal{E}^2}{\cosh^2 \rho} - \frac{4\mathcal{S}^2}{\sinh^2 \rho} - m^2 \sinh^2 \rho - \mathcal{J}^2}. $$  \hspace{1cm} (4.8)
Putting sinh $\rho = x$ in the above equation (4.8), we get
\[
N = \frac{1}{\pi} \int_{\sqrt{R_1}}^{\sqrt{R_2}} \frac{dx}{1 + x^2} \sqrt{\mathcal{E}^2 - \frac{4S^2(1 + x^2)}{x^2}} - m^2x^2(1 + x^2) - J^2(1 + x^2), \tag{4.9}
\]
where $R_1$ and $R_2$ are two positive roots of the cubic polynomial
\[
f(z) = m^2z^3 + (m^2 + J^2)z^2 + (4S^2 + J^2 - \mathcal{E}^2)z + 4S^2, \quad z \equiv x^2, \tag{4.10}
\]
with the condition $\mathcal{E}^2 \geq 4S^2 + J^2$. Now taking the partial derivative of the equation (4.13) with respect to $m$ we get
\[
\frac{\partial N}{\partial m} = -\frac{m}{\pi} \int_{\sqrt{R_1}}^{\sqrt{R_2}} \frac{x^2}{\sqrt{\mathcal{E}^2 - \frac{4S^2(1 + x^2)}{x^2}} - m^2x^2(1 + x^2) - J^2(1 + x^2)} dx. \tag{4.11}
\]
This can be written, by using the usual elliptical integral of first and second kind, as
\[
\frac{\partial N}{\partial m} = \frac{1}{\pi} \int_{\sqrt{R_2} - R_3}^{\sqrt{R_1} - R_3} \left[ (R_3 - R_2) \mathcal{E} \left( \frac{R_2 - R_1}{R_2 - R_3} \right) - R_3 \mathcal{E} \left( \frac{R_2 - R_1}{R_2 - R_3} \right) \right], \tag{4.12}
\]
For large $\mathcal{E}$, but small $S$ and $J$, we get the roots as
\[
R_{2,3} = \pm \frac{\mathcal{E}}{m} \left( 1 - \frac{J^2}{2m^2} \right) \pm \frac{m^2 - 2J^2 - 16S^2}{8m \mathcal{E}} + O[\mathcal{E}^{-2}],
\]
\[
R_1 = \frac{4S^2}{\mathcal{E}^2} + O[\mathcal{E}^{-4}]. \tag{4.13}
\]
Now expanding equation (4.13) for large $\mathcal{E}$ we get
\[
\frac{\partial N}{\partial m} = \frac{c_1}{m} \mathcal{E}^{1/2} + \frac{c_2}{m^{3/2}} \mathcal{E}^{-1/2} \left( 1 + \frac{J^2}{m^2} \right)
+ m^{3/2} \mathcal{E}^{-3/2} \left( c_3 + \frac{c_4}{m^2} \mathcal{E} + \frac{c_5}{m^4} \mathcal{E}^3 + \frac{c_6}{m^6} \mathcal{E}^5 \right) + O[\mathcal{E}^{-5/2}], \tag{4.14}
\]
where $c_1 = 4\frac{\sqrt{2\pi}}{\sqrt{\mathcal{E}}} - \mathcal{E}^{(-1)} \approx -0.19069$, $c_2 = \frac{4\sqrt{2\pi}}{\sqrt{\mathcal{E}^3}} = 0.104328$, $c_3 = -\frac{3\sqrt{\mathcal{E}}}{\sqrt{\pi^2}} = -0.0178772$, $c_4 = \frac{\sqrt{\mathcal{E}}}{\pi^2} = 0.0119181$, $c_5 = -\frac{\sqrt{\mathcal{E}^3}}{\pi^2} = -0.0059596$ and $c_6 = \frac{\sqrt{\mathcal{E}^5}}{\pi^2} = 0.19069$. Integrating the above equation (4.14), we get
\[
N = N_0 + 2c_1 \mathcal{E}^{1/2} + \frac{2}{3} c_2 m^{3/2} \mathcal{E}^{-1/2} \left( 1 - \frac{3J^2}{m^2} \right)
+ m^{5/2} \mathcal{E}^{-3/2} \left( \frac{2}{5} c_3 + \frac{2c_6}{m^2} \mathcal{E} + \frac{2c_5}{m^4} \mathcal{E}^3 + \frac{2c_6}{m^6} \mathcal{E}^5 \right) + O[\mathcal{E}^{-5/2}]. \tag{4.15}
\]
The integration constant $N_0$ can be determined from the integral (4.9) for $m = 0$
\[
N_0 = \frac{1}{\pi} \int_{r_1}^{r_2} \frac{dx}{1 + x^2} \sqrt{\mathcal{E}^2 - \frac{4S^2(1 + x^2)}{x^2} - J^2(1 + x^2)}, \tag{4.16}
\]
where \( r_{1,2}^2 = \frac{E^2 - 4S^2 - J^2 \pm \sqrt{(E^2 - 4S^2 - J^2)^2 - 16S^2J^2}}{2J^2} \). Changing the variable we get
\[
N_0 = \frac{r_1}{\pi} \int_1^{r_2/r_1} \frac{dx}{1 + r_1^2 x^2} \sqrt{r_1^2 J^2 (1 - x^2) + \frac{4S^2}{r_1^2} (1 - \frac{1}{x^2})} = \frac{1}{2} (E - 2S - J). \tag{4.17}
\]
Now putting (4.17) in (4.18) and reversing the series we get
\[
E = 2L + a_1 m^{1/2} L^{1/2} + a_2 m - A(J) a_3 m^{3/2} L^{-1/2} + B(J, S) a_4 m^{5/2} L^{-3/2} + \mathcal{O}[L^{-5/2}], \tag{4.18}
\]
where \( L = N + S + \frac{J}{T} \), \( B(J, S) = 1 - 3.48 \frac{J^2}{m^2} - 0.35 \frac{J^4}{m^4} - 34.05 \frac{S^2}{m^2} \), and \( a_4 = 0.00791996 \). In the above (4.18) series \( \mathcal{O}[L^{-3/2}] \) term onwards the coefficients will depend on even power of angular momentum \( J \) and spin \( S \) explicitly along with \( L \).

The above equation (4.18) is the energy expression for the long strings with small \( J \) and \( S \) in \( AdS_5 \times S^2 \). This is the same expression as in [2] with \( S = J = 0 \). In the above equation if we put \( S = 0 \), we get back to equation (2.21).

5. Conclusion

In this paper, we have studied some examples of pulsating strings with extra angular momentum along various subspace of the \( AdS_5 \times S^5 \). First we have found the energy for the short and long strings with small \( J \) and for the short string with large \( J \) in \( AdS_5 \times S^1 \). Then we have found the short string solution with small angular momentum in the \( \mathbb{R} \times S^3 \) background. Further we have shown the long spin energy behavior in the small \( S \) and \( J \) in \( AdS_5 \times S^1 \). An interesting study will be to find out the nature of gauge theory operators in the dual CFT. Similar studies have been performed earlier in, e.g. [31]-[34]. The operator for a rotating string in \( AdS_5 \times S^5 \) with charge \((S, J)\) has been suggested to be \( \text{Tr} \, D^S Z^J \) [12], where \( D \) is the complex combination of covariant derivative and \( Z \) is the complex scalar. Similarly some generic operator, dual to spinning and rotating string has been shown in [11], where the exact operator is a linear combination of operators containing \( S \) insertions of \( D_i \) into \( \text{Tr} \, Z^J \) for specific \((S, J)\) charge conditions. Furthermore, in [31] for a rotating and pulsating string in \( AdS_5 \) with two equal spins in two orthogonal directions, the dual operator has been suggested to be of the form \( \text{Tr} \, D_X^S D_Y^S \) (where \( D_X = D_1 + i D_2 \) and \( D_Y = D_3 + i D_4 \)) which is self dual components of gauge field strength [13]. We expect the solution presented in the chapter-4 may be dual to the gauge field operator of the generic form \( \text{Tr} \, D_X^S D_Y^S Z^J \). The anomalous dimensions and the dual gauge theory operators for the string solutions presented here are under construction.

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References

[1] J. M. Maldacena, “The Large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. 2, 231 (1998) [Int. J. Theor. Phys. 38, 1113 (1999)] [hep-th/9711200].
[2] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from noncritical string theory,” Phys. Lett. B 428 (1998) 105 [hep-th/9802109].

[3] E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. 2 (1998) 253 [hep-th/9802150].

[4] I. Bena, J. Polchinski and R. Roiban, “Hidden symmetries of the AdS(5) x S**5 superstring,” Phys. Rev. D 69, 046002 (2004) [hep-th/0305116].

[5] V. A. Kazakov, A. Marshall, J. A. Minahan and K. Zarembo, “Classical/quantum integrability in AdS/CFT,” JHEP 0405, 024 (2004) [hep-th/0402207].

[6] G. Arutyunov, S. Frolov and M. Staudacher, “Bethe ansatz for quantum strings,” JHEP 0410, 016 (2004) [hep-th/0406256].

[7] M. Kruczenski, “Spiky strings and single trace operators in gauge theories,” JHEP 0508, 014 (2005) [hep-th/0410226].

[8] G. Arutyunov and S. Frolov, “Integrable Hamiltonian for classical strings on AdS(5) x S**5,” JHEP 0502, 059 (2005) [hep-th/0411089].

[9] A. K. Das, J. Maharana, A. Melikyan and M. Sato, “The Algebra of transition matrices for the AdS(5) x S**5 superstring,” JHEP 0412, 055 (2004) [hep-th/0411200].

[10] N. Beisert, V. A. Kazakov, K. Sakai and K. Zarembo, “The Algebraic curve of classical superstrings on AdS(5) x S**5,” Commun. Math. Phys. 263, 659 (2006) [hep-th/0502226].

[11] L. F. Alday, G. Arutyunov and A. A. Tseytlin, “On integrability of classical superstrings in AdS(5) x S**5,” JHEP 0507, 002 (2005) [hep-th/0502240].

[12] A. K. Das, A. Melikyan and M. Sato, “The Algebra of flat currents for the string on AdS(5) x S**5 in the light-cone gauge,” JHEP 0511, 015 (2005) [hep-th/0508183].

[13] N. Beisert, “The SU(2—2) dynamic S-matrix,” Adv. Theor. Math. Phys. 12, 945 (2008) [hep-th/0511082].

[14] S. Frolov, J. Plefka and M. Zamaklar, “The AdS(5) x S**5 superstring in light-cone gauge and its Bethe equations,” J. Phys. A 39, 13037 (2006) [hep-th/0603008].

[15] R. A. Janik, “The AdS(5) x S**5 superstring worldsheet S-matrix and crossing symmetry,” Phys. Rev. D 73, 086006 (2006) [hep-th/0603038].

[16] D. M. Hofman and J. M. Maldacena, “Giant Magnons,” J. Phys. A 39, 13095 (2006) [hep-th/0604135].

[17] M. Kruczenski, J. Russo and A. A. Tseytlin, “Spiky strings and giant magnons on S**5,” JHEP 0610, 002 (2006) [hep-th/0607044].

[18] N. Beisert, R. Hernandez and E. Lopez, “A Crossing-symmetric phase for AdS(5) x S**5 strings,” JHEP 0611, 070 (2006) [hep-th/0609044].

[19] G. Arutyunov, S. Frolov, J. Plefka and M. Zamaklar, “The Off-shell Symmetry Algebra of the Light-cone AdS(5) x S**5 Superstring,” J. Phys. A 40, 3583 (2007) [hep-th/0609157].

[20] N. Beisert, B. Eden and M. Staudacher, “Transcendentality and Crossing,” J. Stat. Mech. 0701, P01021 (2007) [hep-th/0610251].

[21] J. A. Minahan and K. Zarembo, “The Bethe ansatz for N=4 superYang-Mills,” JHEP 0303, 013 (2003) [hep-th/0212208].
[22] A. Khan and A. L. Larsen, “Improved stability for pulsating multi-spin string solitons,” Int. J. Mod. Phys. A 21, 133 (2006) [hep-th/0502063].

[23] J. A. Minahan, “Circular semiclassical string solutions on AdS(5) x S**5,” Nucl. Phys. B 648, 203 (2003) [arXiv:hep-th/0209047].

[24] J. Engquist, J. A. Minahan and K. Zarembo, “Yang-Mills duals for semiclassical strings on AdS(5) x S**5,” JHEP 0311, 063 (2003) [arXiv:hep-th/0310188].

[25] H. Dimov and R. C. Rashkov, “Generalized pulsating strings,” JHEP 0405, 068 (2004) [arXiv:hep-th/0404012].

[26] M. Smedback, “Pulsating strings on AdS(5) x S**5,” JHEP 0407, 004 (2004) [arXiv:hep-th/0405102].

[27] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “A semi-classical limit of the gauge/string correspondence,” Nucl. Phys. B 636, 99 (2002) [arXiv:hep-th/0204051].

[28] A. Khan, A. L. Larsen, “Spinning pulsating string solitons in AdS(5) x S**5,” Phys. Rev. D69, 026001 (2004). [hep-th/0310019].

[29] G. Arutyunov, J. Russo, A. A. Tseytlin, “Spinning strings in AdS (5) x S**5: New integrable system relations,” Phys. Rev. D69, 086009 (2004). [hep-th/0311004].

[30] M. Kruczenski and A. A. Tseytlin, “Semiclassical relativistic strings in S**5 and long coherent operators in N = 4 SYM theory,” JHEP 0409, 038 (2004) [arXiv:hep-th/0406189].

[31] I. Y. Park, A. Tirziu and A. A. Tseytlin, “Semiclassical circular strings in AdS(5) and 'long' gauge field strength operators,” Phys. Rev. D 71, 126008 (2005) [hep-th/0505130].

[32] H. J. de Vega, A. L. Larsen, N. G. Sanchez, “Semiclassical quantization of circular strings in de Sitter and anti-de Sitter space-times,” Phys. Rev. D51, 6917-6928 (1995). [hep-th/9410219].

[33] M. Beccaria, G. V. Dunne, G. Macorini, A. Tirziu and A. A. Tseytlin, “Exact computation of one-loop correction to energy of pulsating strings in AdS_5 x S^5,” J. Phys. A 44, 015404 (2011) [arXiv:1009.2318 [hep-th]].

[34] K. L. Panigrahi and P. M. Pradhan, “On Rotating and Oscillating Four-Spin Strings in AdS_5 x S^5,” JHEP 1211, 053 (2012) [arXiv:1206.4920 [hep-th]].

[35] B. Chen and J. B. Wu, “Semi-classical strings in AdS_4 x CP^3,” JHEP 0809, 096 (2008) [arXiv:0807.0802 [hep-th]].

[36] H. Dimov and R. C. Rashkov, “On the pulsating strings in AdS_4 x CP^3,” Adv. High Energy Phys. 2009, 953987 (2009) [arXiv:0908.2218 [hep-th]].

[37] N. P. Bobev, H. Dimov and R. C. Rashkov, “Pulsating strings in warped AdS(6) x S^4 geometry,” arXiv:hep-th/0410262.

[38] D. Arnaudov, H. Dimov and R. C. Rashkov, “On the pulsating strings in AdS_5 x T^{1,1},” J. Phys. A 44, 495401 (2011) [arXiv:1006.1539 [hep-th]].

[39] D. Arnaudov, H. Dimov and R. C. Rashkov, “On the pulsating strings in Sasaki-Einstein spaces,” AIP Conf. Proc. 1301, 51 (2010) [arXiv:1007.3364 [hep-th]].

[40] S. Giardino and V. O. Rivelles, “Pulsating Strings in Lumin-Maldacena Backgrounds,” JHEP 1107, 057 (2011) [arXiv:1105.1353 [hep-th]].
[41] J. G. Russo, “Anomalous dimensions in gauge theories from rotating strings in AdS(5) x S**5,” JHEP **0206**, 038 (2002) [hep-th/0205244].

[42] N. Beisert, S. Frolov, M. Staudacher and A. A. Tseytlin, “Precision spectroscopy of AdS / CFT,” JHEP **0310**, 037 (2003) [hep-th/0308117].

[43] G. Ferretti, R. Heise and K. Zarembo, “New integrable structures in large-N QCD,” Phys. Rev. D **70**, 074024 (2004) [hep-th/0404187].