Scale invariance implies conformal invariance for the three-dimensional Ising model

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Using Wilson renormalization group, we show under commonly accepted assumptions that scale invariance implies conformal invariance in dimension three for the Ising and O(N) models.

Conformal symmetry plays a considerable role both in high energy and condensed matter physics. It has witnessed a renewed interest these last years in particular because of the AdS/CFT conjecture [1] and the successful use of conformal methods in three-dimensional critical physics [2–4]. The groundbreaking papers of the seventies/eighties [7–12] solved two fundamental issues in two dimensions: First, scale invariance implies conformal invariance under mild assumptions [11, 12] and, second, conformal symmetry enables to solve most of the scale invariant problems that is, to determine critical exponents and correlation functions [12].

An important ingredient for the exact solution of two-dimensional conformal models is the existence of an infinite number of generators of the conformal group and thus an infinite number of conserved quantities. In higher dimensions, the number of generators is finite and we could naively conclude that symmetry arguments alone are not sufficient to solve a model in the critical regime. However, it is well-known that scale-invariant theories are characterized by a larger number of conserved quantities/eighties [7–12] solved two fundamental issues in two dimensions assuming, among other things, conformal invariance. A break-through in this direction has been achieved these last years with the conformal bootstrap program [24–26] that has led to the exact (although numerical) computation of the critical exponents of the Ising model in three dimensions assuming, among other things, conformal invariance [29].

In parallel, a large activity has been devoted to understand the relation between scale and conformal invariance in – or close to – four dimensions. It has been proven to all orders of perturbation theory that scale invariance implies conformal invariance [10] in four-dimensional unitary and Poincaré invariant theories. Moreover, there are strong indications that a non-perturbative proof could be at reach in this dimension [17–19].

Despite decades of efforts, it is still an open question to know whether a typical statistical model is conformally invariant at criticality in three dimensions. The aim of this article is to show that, in any dimension, conformal invariance is a consequence of scale invariance for the euclidean $\mathbb{Z}_2$ and $O(N)$ models. Our proof can probably be extended to most scalar models because our main arguments do not rely on specificities of the $O(N)$ models.

The proof of conformal invariance in all dimensions presented below is intimately related to the deep structure of the Wilson RG [40] and scale invariance and we therefore start by recalling the main results obtained when scale transformations alone are considered. In the modern formulations of the Wilson RG [22–25] (see [26] for a review), the coarse-graining procedure at some scale $k$ is implemented by smoothly decoupling the long-wavelength modes $\varphi(|q| < k)$ of the system, also called the slow modes, by giving them a large mass, while keeping unchanged the dynamics of the short-wavelength/rapid ones $\varphi(|q| > k)$. Here, $\varphi$ denotes collectively all the fields of the model. This decoupling is conveniently implemented by modifying the action or the Hamiltonian of the model: $S[\varphi] \rightarrow S[\varphi] + \Delta S_k[\varphi]$ where $\Delta S_k[\varphi]$ is quadratic in the field and reads, in Fourier space, $\Delta S_k[\varphi] = 1/2 \int_q R_k(q^2)\varphi(q)\varphi(-q)$. The precise shape of $R_k(q^2)$ does not matter for what follows as long as it can be written as

$$R_k(q^2) = Z_k k^2 r(q^2/k^2)$$

(1)

where $Z_k$ is the field renormalization factor and $r$ is a function that, (i) falls off rapidly to 0 for $q^2 > k^2$ – the rapid modes $\varphi(|q| > k)$ are not affected by $\Delta S_k$ – and (ii) goes to a constant for $q^2 = 0$ – the slow modes $\varphi(|q| < k)$ acquire a mass of order $k$ and thus smoothly decouple. The partition function, which now depends on the RG parameter $k$, reads:

$$Z_k[J] = \int D\varphi \exp [-S[\varphi] - \Delta S_k[\varphi] + \int_x J,\varphi].$$

(2)

It is convenient to define the free energy $\mathcal{W}_k[J] = \ln Z_k[J]$ and its (slightly modified) Legendre transform by

$$\Gamma_k[\phi] + \mathcal{W}_k[J] = \int_x J,\phi - \frac{1}{2} \int_{x,y} R_k(|x-y|) \phi(x)\phi(y),$$

(3)
with \( \phi(x) = \langle \phi(x) \rangle \), \( R_k(|x - y|) \) is the inverse Fourier transform of \( R_0(q^2) \) and where the last term has been added for the following reason. When \( k = \Lambda \), where \( \Lambda \) is the ultra-violet (UV) cut-off of the model (typically, the inverse lattice spacing for a statistical model defined on a lattice), all modes are completely frozen by the \( \Delta S_k \) term because, for all \( q, R_\Lambda(q^2) \) is very large. Thus, \( Z_{k=\Lambda} \) can be computed by the saddle-point method and it is then straightforward to show that the presence of the last term in Eq. (3) leads to \( \Gamma_\Lambda[\phi] \approx S[\phi = \phi] \). On the contrary, when \( k = 0 \), the definition of \( R_k \) implies that \( R_{k=0}(q^2) \equiv 0 \) and the original model is recovered: \( Z_{k=0}[J] = Z[J] \) and \( \Gamma_{k=0}[\phi] = \Gamma[\phi] \), with \( \Gamma[\phi] \) the usual Gibbs free energy or generating functional of one-particle-irreducible correlation functions.

The exact RG equation for \( \Gamma_k \) is nothing but its evolution under a change of \( k \) and reads \cite{23}:

\[
\partial_t \Gamma_k[\phi] = \frac{1}{2} \text{Tr} \int_{xy} \partial_t R_k(|x - y|) G_{k,xy}[\phi],
\]

where \( t = \ln(k/\Lambda) \), Tr means the trace over the \( O(N) \) indices and \( G_{k,xy}[\phi] \) is the field-dependent propagator:

\[
G_k = (\Gamma_1^{(2)} + R_k)^{-1}, \quad (\Gamma_2^{(k)}[\phi])_{ij} = \frac{\partial^2 \Gamma_k[\phi]}{\partial \phi_i(x) \partial \phi_j(y)},
\]

where the inverse is taken in both the \( x \)-space and internal-space. Notice the following important remark: Integrating \cite{11} down to \( k = 0 \), starting from the initial condition \( \Gamma_{k=\Lambda}[\phi] = S[\phi = \phi] \), amounts to solving the model and not only to changing the overall scale of the momenta in correlation functions as in the Gell-Mann - Low RG.

In the language of field theory, \( \Delta S_k[\phi] \) is nothing but a particular infrared regulator: If the original model described by \( S[\phi] \) is at criticality, \( W[J] \) and \( \Gamma[\phi] \) show non-analyticities whereas this is not the case for the model corresponding to \( S[\phi] + \Delta S_k[\phi] \) because of the scale \( k \) appearing explicitly in \( R_k \). This is one of the major advantages of working with the Wilson RG and we extensively make use of it in the following: At finite \( k \), all \( \Gamma_n^{(k)}(p_1, \ldots, p_n) \) are regular even when the original model is at criticality. The price to pay for regularity is that the Ward identity of scale invariance is modified by the presence of \( \Delta S_k \) (as in other situations where the regulator breaks symmetries \cite{28,29})...
earized flow by substituting $\tilde{\Gamma}_k[\phi] \rightarrow \tilde{\Gamma}^*[\phi]+\epsilon \exp(\lambda t) \tilde{\gamma}[\phi]$ and retaining only the $O(\epsilon)$ terms. (With our definition of $t$, a relevant operator has a negative eigenvalue.) This leads to the eigenvalue problem

$$\lambda \tilde{\gamma}[\phi] = \int \left( D^2 + D_\phi \right) \phi(\bar{x}) \cdot \frac{\delta \tilde{\gamma}}{\delta \phi(\bar{x})} - \frac{1}{2} \text{Tr} \int \left[ (D^2 + D_R)(r(\bar{x} - \bar{y})) \right] \tilde{G}_{\bar{x} \bar{y}} \cdot \tilde{\gamma}^{(2)} \cdot \tilde{G}_{\bar{x} \bar{y}},$$

where $\bar{x} = \{ \bar{x}, \bar{y}, \bar{z}, \bar{w} \}$, $\tilde{G}^*[\phi]$ is the dimensionless renormalized propagator at the fixed point: $\tilde{G}^* = (\tilde{\Gamma}^* + r)^{-1}$ and $r(\bar{x})$ is the dimensionless inverse Fourier transform of $r(q^2/k^2)$ defined in Eq. (1). The main result of [35–37] is that the physically acceptable eigenperturbations of the RG flow around $\tilde{\Gamma}^*$ have a power-law behavior at large field and constitute a discrete spectrum of eigenoperators with which are associated the physically interesting discrete eigenvalues.

We conclude from the above discussion that the large-field behavior selects among all the fixed-point functionals $\tilde{\Gamma}^*[\phi]$ those that are physical, that is, that can be reached by an RG flow from a physical action $S$ and that have a discrete spectrum of eigenperturbations.

Let us now study conformal invariance by following the same method as above. We start by considering a conformally invariant model described by an action $S$ and we derive the modified Ward identity of conformal invariance in the presence of the regulator $R_k$ by performing in Eq. (2) the following change of variables: $\varphi(x) \rightarrow \varphi(\bar{x}) + \epsilon(x \bar{x} \partial_k - 2x \bar{x} \nu \partial_\nu + 2a_\nu x)(x \varphi(x)$ (again, following [29]). By considering general cutoff functions as in Eq. (6), we find that reads:

$$\int_{xy}(K_{x,y} - D_R x \mu - K_{y,x} - D_R y \nu) R(x, y) \frac{\delta \Gamma_k}{\delta R(x, y)} + \int_{x}(K_{x}^{*} - 2D_\phi x \mu) \phi(x) \cdot \frac{\delta \Gamma_k}{\delta \phi(x)} = 0,$$

with $K_{x}^{*} = x^2 \partial_k - 2x \nu \partial_\nu$. By specializing to functions $R_k$ of the form Eq. (11) and requiring again that $Z_k \propto k^{-n},$ Eq. (6) can be rewritten

$$0 = \Sigma_k^\mu[\phi] \equiv \int_{x}(K_{x}^{*} - 2D_\phi x \mu) \phi(x) \cdot \frac{\delta \Gamma_k}{\delta \phi(x)} - \frac{1}{2} \text{Tr} \int_{xy} \partial_t R_k(|x - y|)(x \mu + y \nu)G_{k,x,y}. \tag{10}$$

As in the case of scale invariance, this identity boils down to the usual Ward identity of conformal invariance in the limit $k \rightarrow 0$ where $R_k$ becomes negligible.

At any fixed point, the scaling dimension of $\Sigma^\mu_k[\phi]$ is fixed by Eq. (10) to be $-1$. We thus define the dimensionless analogue of $\Sigma^\mu_k[\phi]$ by $\Sigma_k[\phi] = k \Sigma^\mu_k[\phi]$. Its flow equation reads:

$$\partial_t \Sigma_k^\mu[\phi] - \Sigma_k^\mu[\phi] = \int_{x}(D^2 + D_\phi) \phi(\bar{x}) \cdot \frac{\delta \tilde{\Sigma}_k^\mu}{\delta \phi(\bar{x})} - \frac{1}{2} \text{Tr} \int_{xy} \partial_t R_k(|x - y|)(x \mu + y \nu)G_{k,x,y}, \tag{11}$$

where $\tilde{\Sigma}_k^{(2)}[\phi]$ is the second functional derivative of $\Sigma_k^\mu$. We now assume that the model we are studying is scale invariant on scales much smaller than $\Lambda$. Then, the dimensionless flow is attracted towards the fixed point and thus, at sufficiently small $k$, $\tilde{G}_k \approx \tilde{G}^*$. The key point of our proof is that the fixed-point equation satisfied by $\Sigma^\mu_k[\phi]$ is formally identical to [3] with $\tilde{\gamma}[\phi]$ a vector eigenperturbation with eigenvalue $-1$. To prove that scale invariance implies conformal invariance, we must prove that $\Sigma^\mu_k[\phi] = 0$. A sufficient condition for conformal invariance is therefore to show that there is no vector eigenperturbation of $\tilde{\Gamma}^*[\phi]$ of scaling dimension $-1$.

To understand how conformal invariance is related with the scaling dimension of the vector eigenperturbations, it proves useful to consider two simple examples. The first one is the $O(N)$ model in $d = 4$. In this case, the $O(N)$-invariant vector operators with lowest dimension are quartic in the fields and have three derivatives [12] for the Ising model, there is only one such operator that reads $\int_x \phi \partial_\mu \phi(\partial \phi)^2$). It has therefore dimension +3. In the absence of vector operator of dimension $\pm 1$, we retrieve the well-known property that this model is conformally invariant at criticality $\pm 1$. By continuity, this also proves conformal invariance in a finite neighborhood of $d = 4$. The second example involves a vector field $A_\mu(x)$ and is described by the (euclidean) action

$$S = \int_x \frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \frac{\alpha}{2} (\partial_\mu A_\mu)^2 \tag{12}.$$
FIG. 1: Possible behavior of the lowest eigenvalue $\lambda$ associated with a vector perturbation as a function of dimension. Left panel: (a) and (b) correspond to typical behavior, (c) to the exceptional case where $\lambda = -1$ right in $d = 3$. In the three cases, conformal invariance holds. Right panel: the shaded area represents a continuum of eigenvalues and the curve an eigenvalue $\lambda$ having a plateau at $-1$ around $d = 3$. In these cases, conformal invariance can be broken.

to prove that the fixed point solutions of these equations are continuous functions of $d$. This has been shown only within various approximation schemes [24, 27]. For the Ising and $O(N)$ models, the Wilson-Fisher fixed point has been found for all $2 \leq d \leq 4$ in all approximation schemes studied so far and all quantities attached to this fixed point are continuous functions of $d$. Although there is no mathematical proof, it makes no doubt that this holds true beyond these approximations.

Determining the dependence in $d$ of the eigenvalues associated with vector perturbations of $\Gamma^*$ is a difficult task [43]. However, we can state a necessary condition for the model not to be conformally invariant at criticality in three dimensions: One eigenvalue associated with a vector eigenperturbation must be $-1$ in this dimension, see Fig. 1. This would mean that the $d = 3$ model has an integer critical exponent, a property that is highly improbable. Let us anyhow suppose that one of these eigenvalues crosses $-1$ right in $d = 3$, as in curve (c) of Fig. 1. Then, for any dimension infinitesimally smaller or larger than three, there would exist no eigenperturbation of dimension $-1$. The critical system would exhibit conformal symmetry above and below $d = 3$. Since correlation functions of the critical theory are expected to be continuous functions of $d$, we conclude that, even in this highly improbable situation, the model would exhibit conformal invariance at criticality in $d = 3$. We are thus led to the more stringent necessary (but not sufficient) condition: for a critical model not to be conformally invariant, there must exist a vector perturbation of scaling dimension $-1$ in a finite interval containing $d = 3$. This could happen either because a discrete eigenvalue is independent of the dimension in some range of dimension around three or because there exists a continuum of eigenvalues, see Fig. 1. Such a behavior is, to say the least, not standard. To our knowledge, this has never been observed in any interacting model.

To conclude, assuming that the spectrum of vector eigenperturbations of the fixed point is discrete and that the associated eigenvalues behave smoothly with $d$ and do not show a plateau at $-1$ around $d = 3$, we prove that at criticality, the Ising and $O(N)$ models are conformally invariant in three dimensions. Actually, we even expect that the lowest such eigenvalue is positive since it is three in $d = 4$ and, as any eigenvalue, it should not change much between three and four dimensions. This expectation could be tested numerically by adding to the lattice Hamiltonian interactions that lead to vector perturbations in the continuum limit and verify their irrelevance in the critical domain.

Notice finally that, at first sight, our approach could seem similar to the one based on the energy-momentum tensor and more specifically on the analysis of the virial current. This is not the case although there is perhaps a relationship between the two. $\Sigma^\mu$ is a functional of $\phi$ and not of $\phi^*$; it is built from $\Gamma_k$ and not from $S$; what matters is that its density vanishes up to a surface term and not that it is conserved. Moreover, as we already explained, we only deal with a regularized theory which enables us to consider only the analytic candidates for $\Sigma^\mu_k$ contrary to what should be done in a nonregularized theory.

Let us now point out some directions of research for the future. It is clear that the methodology presented above can be generalized to other three-dimensional theories (involving scalar, fermionic or vector fields with different symmetry groups) and it would be interesting to conclude on the fate of conformal invariance in this wider class of models. A marked difference between the $O(N)$ model and many other statistical models is that the phase transition is not necessarily of second order in all dimensions. This is for instance the case of the three-state Potts model and the frustrated antiferromagnets. The conformal bootstrap approach has been applied to the latter model [13] and it would be interesting to analyze an extension of our proof in those cases. Another promising line of investigation consists in making use of the conformal invariance in the Wilson framework to perform actual calculations of universal quantities. On the one hand, and in the best case, this would lead to closed and numerically tractable equations for the critical exponents. On the other hand, the approximation schemes currently used for solving the Wilson RG flow equation being incompatible with exact conformal invariance, we can expect that constraining them to be conformally invariant would improve their accuracy.

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[39] The bootstrap program has also been applied, for example, to three-dimensional O(N) scalar models [5], to the Ising universality class in generic dimensions [1] at to frustrated magnetic systems [13].

[40] The history of the relation between conformal invariance and Wilson RG is long, see for example [20]. During the writing of this article, a preprint by O. J. Rosten about the relationship between conformal invariance and the Wilson RG has appeared [21].

[41] Of course, there are also trivial Gaussian fixed points associated with high and low temperature regimes and with multicriticality for d < 4 or with the criticality for d > 4.

[42] This result is a consequence of the fact that (i) all vector operators of lowest dimension are surface terms and, (ii) Σμν is the integral of a density that depends only on the xμ. Point (ii) can be proven independently of a given model by observing that the left-hand-sides of Eqs. (9) and (10) can be interpreted as the action of the generators of dilatations D and conformal transformations Kμ on Γ. Similar expressions can be obtained for the generators of translations Pμ and rotations Jμν which, as expected, satisfy the algebra of the conformal group. In particular, applying [Pμ, Kν] = 2δμνD + 2Jμν to a translation, rotation and dilatation invariant Γ yields PμΣμν = 0. Thus, Σμν is the integral of a density that depends only on the fields and their derivatives.

[43] For the O(N) model, there exists two independent vector eigenperturbations of scaling dimension 3 in d = 4. Computing the first correction to these eigenvalues in ε = 4 − d leads to 3 + O(ε2) and 3 − 6ε/(N + 8) + O(ε2). Only the first perturbation exists in the Ising model.