Squared distance matrix of a weighted tree

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Abstract

Let $T$ be a tree with vertex set $\{1, \ldots, n\}$ such that each edge is assigned a nonzero weight. The squared distance matrix of $T$, denoted by $\Delta$, is the $n \times n$ matrix with $(i, j)$-element $d(i, j)^2$, where $d(i, j)$ is the sum of the weights of the edges on the $(ij)$-path. We obtain a formula for the determinant of $\Delta$. A formula for $\Delta^{-1}$ is also obtained, under certain conditions. The results generalize known formulas for the unweighted case.

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1 Introduction

Let $G$ be a connected graph with vertex set $V(G) = \{1, \ldots, n\}$. The distance between vertices $i, j \in V(G)$, denoted $d(i, j)$, is the minimum length (the number of edges) of a path from $i$ to $j$ (or an $ij$-path). We set $d(i, i) = 0$, $i = 1, \ldots, n$. The distance matrix $D(G)$, or simply $D$, is the $n \times n$ matrix with $(i, j)$-element $d_{ij} = d(i, j)$.

A classical result of Graham and Pollak [7] asserts that if $T$ is a tree with $n$ vertices, then the determinant of the distance matrix $D$ of $T$ is $(-1)^{n-1}(n-1)2^{n-2}$. Thus the determinant depends only on the number of vertices in the tree and not on the tree itself. A formula for the inverse of the distance matrix of a tree was given by Graham and Lovász [6]. Several extensions and generalizations of these results have been proved (see, for example [1], [2], [5], [8], [9] and the references contained therein).
Let $T$ be a tree with vertex set $\{1, \ldots, n\}$ and let $D$ be the distance matrix of $T$. The squared distance matrix $\Delta$ is defined to be the Hadamard product $D \circ D$, and thus has the $(i, j)$-element $d(i, j)^2$. A formula for the determinant of $\Delta$ was proved in [3], while the inverse and the inertia of $\Delta$ were considered in [4].

In this paper we consider weighted trees. Let $T$ be a tree with vertex set $V(T) = \{1, \ldots, n\}$ and edge set $E(T) = \{e_1, \ldots, e_{n-1}\}$. We assume that each edge is assigned a weight and let the weight assigned to $e_i$ be denoted $w_i$, which is a nonzero real number (not necessarily positive).

For $i, j \in V(T), i \neq j$, the distance $d(i, j)$ is defined to be the sum of the weights of the edges on the (unique) $ij$-path. We set $d(i, i) = 0$, $i = 1, \ldots, n$.

Let $D$ be the $n \times n$ distance matrix with $d_{ij} = d(i, j)$.

The Laplacian of $T$ is the $n \times n$ matrix defined as follows. The rows and the columns of $L$ are indexed by $V(T)$. For $i \neq j$, the $(i, j)$-element is 0 if $i$ and $j$ are not adjacent. If $i$ and $j$ are adjacent, and if the edge joining them is $e_k$, then the $(i, j)$-element of $L$ is set equal to $-1/w_k$. The diagonal elements of $L$ are defined so that $L$ has zero row (and column) sums.

The paper is organized as follows. In this section we review some basic properties of the distance matrix of a tree such as formulas for its determinant and inverse. Some preliminary results are obtained in Section 2. Sections 3 and 4 are devoted to the determinant and the inverse of $\Delta$, respectively.

Example Consider the tree

![Tree Diagram]

The Laplacian of the tree is given by

$$
\begin{bmatrix}
1/2 & 0 & -1/2 & 0 & 0 & 0 & 0 \\
0 & -1/3 & 1/3 & 0 & 0 & 0 & 0 \\
-1/2 & 1/3 & 7/6 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 19/20 & -1/5 & 1/2 & -1/4 \\
0 & 0 & 0 & -1/5 & 1/5 & 0 & 0 \\
0 & 0 & 0 & 1/2 & 0 & -1/2 & 0 \\
0 & 0 & 0 & -1/4 & 0 & 0 & 1/4
\end{bmatrix}
$$

We let $Q$ be the $n \times (n - 1)$ vertex-edge incidence matrix of the underlying unweighted tree, with an orientation assigned to each edge. Thus the rows and
the columns of $Q$ are indexed by $V(T)$ and $E(T)$ respectively. If $i \in V(T), e_j \in E(T)$, the $(i, j)$-element of $Q$ is 0 if $i$ and $e_j$ are not incident, it is $1(-1)$ if $i$ and $e_j$ are incident and $i$ is the initial (terminal) vertex of $e_j$. It is well-known [1] that $Q$ has rank $n - 1$ and any minor of $Q$ is either 0 or $\pm 1$ (thus $Q$ is totally unimodular).

Let $F$ be the $n \times n$ diagonal matrix with diagonal elements $w_1, \ldots, w_{n-1}$. It can be verified that $L = QF^{-1}Q'$.

**Lemma 1** The following assertions are true:

(i) $Q'DQ = -2F$.

(ii) $LDL = -2L$.

**Proof** (i). The result follows from the following observation which is easily verified: If $e_p = \{i, j\}$ and $e_q = \{k, \ell\}$ are edges of $T$, then

$$d(i, k) + d(j, \ell) - d(i, \ell) - d(j, k)$$

equals 0 if $e_p$ and $e_q$ are distinct, and equals $-2w_p$, if $e_p = e_q$.

(ii). We have

$$LDL = QF^{-1}Q'DQF^{-1}Q'$$

$$= QF^{-1}(-2F)F^{-1}Q'$$ by (i)

$$= -2QF^{-1}Q'$$

$$= -2L,$$

and the proof is complete. ■

Let $\delta_i$ denote the degree of the vertex $i, i = 1, \ldots, n$, and let $\delta$ be the $n \times 1$ vector with components $\delta_1, \ldots, \delta_n$. We set $\tau_i = 2 - \delta_i, i = 1, \ldots, n$, and let $\tau$ be the $n \times 1$ vector with components $\tau_1, \ldots, \tau_n$.

**Theorem 2** The following assertions are true:

(i) $\det D = (-1)^{n-1}2^{n-2}(\sum_i w_i)(\prod_i w_i)$.

(ii) If $\sum_i w_i \neq 0$, then $D$ is nonsingular and

$$D^{-1} = -\frac{1}{2}L + \frac{1}{2\sum_i w_i}\tau\tau'.$$

(iii) $D\tau = (\sum_i w_i)1$.

**Proof** Parts (i) and (ii) are well-known, see for example, [2]. To prove (iii), note that from (ii),

$$D^{-1}1 = \frac{1}{2\sum_i w_i}\tau\tau'1 = \frac{1}{\sum_i w_i}\tau,$$

since $1'\tau = 2$. It follows that $D\tau = (\sum_i w_i)1$ and the proof is complete. ■
2 Preliminary results

We now turn to the main results for the case of a weighted tree. Let $T$ be a tree with vertex set $V(T) = \{1, \ldots, n\}$ and edge set $E(T) = \{e_1, \ldots, e_{n-1}\}$. Let $w_1, \ldots, w_{n-1}$ be the edge-weights. Recall that $\delta_i$ is the degree of vertex $i$ and $\tau_i = 2 - \delta_i$. We write $j \sim i$ if vertex $j$ is adjacent to vertex $i$. We let $\hat{\delta}_i$ be the weighted degree of $i$, which is defined as

$$\hat{\delta}_i = \sum_{j: j \sim i} w(\{i, j\}), i = 1, \ldots, n.$$ 

Let $\hat{\delta}$ be the $n \times 1$ vector with components $\hat{\delta}_1, \ldots, \hat{\delta}_n$.

Let $\Delta$ be the squared distance matrix of $T$, which is the $n \times n$ matrix with its $(i, j)$-element equal to $d_{ij}^2$ or equivalently, $d(i, j)^2$. The next result was obtained in [4] for the unweighted case,

Lemma 3 $\Delta \tau = D \hat{\delta}$.

Proof Let $i \in \{1, \ldots, n\}$ be fixed. For $j \neq i$, let $\gamma(j)$ be the predecessor of $j$ on the $ij$-path (in the underlying unoriented tree). Let $e^j$ be the edge $\{\gamma(j), j\}$ and set $\theta^j = \hat{\delta}_j - w(e^j)$. We have

$$2 \sum_{j=1}^{n} d(i, j)^2 = \sum_{j=1}^{n} d(i, j)^2 + \sum_{j \neq i} (d(i, \gamma(j)) + w(e^j))^2$$

$$= \sum_{j=1}^{n} d(i, j)^2 + \sum_{j \neq i} d(i, \gamma(j))^2 + 2 \sum_{j \neq i} d(i, \gamma(j)) w(e^j) + \sum_{j \neq i} w(e^j)^2. \quad (1)$$

Note that

$$\sum_{j \neq i} d(i, \gamma(j))^2 = \sum_{j=1}^{n} (\delta_j - 1)d(i, j)^2, \quad (2)$$

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since vertex \( j \) serves as a predecessor of \( \delta_j - 1 \) vertices in paths from \( i \). Also note that
\[
\sum_{j \neq i} w(e_j)^2 = \sum_{k=1}^{n-1} w(e_k)^2. \tag{3}
\]

We have
\[
\sum_{j=1}^{n} d(i, j) \hat{\delta}_j = \sum_{j \neq i} (d(i, \gamma(j)) + w(e_j)) \sum_{j \neq i} w(e_j) + \sum_{j \neq i} (d(i, \gamma(j)) + w(e_j)) \theta^j. \tag{4}
\]

Observe that \( \theta^j \) is the sum of the weights of all the edges incident to \( j \), except the edge \( e_j \), which is on the \( ij \)-path. Thus \( (d(i, \gamma(j)) + w(e_j)) \theta^j \) equals \( \sum d(i, \gamma(\ell)) w(e_\ell) \), where the summation is over all vertices adjacent to \( j \), except \( i \). Therefore it follows that
\[
\sum_{j \neq i} d(i, \gamma(j)) w(e_j) = \sum_{j \neq i} (d(i, \gamma(j)) + w(e_j)) \theta^j. \tag{5}
\]

From (1)-(5) we get
\[
2 \sum_{i=1}^{n} d(i, j)^2 = \sum_{j=1}^{n} d(i, j)^2 \delta_j + \sum_{j=1}^{n} d(i, j) \hat{\delta}_j,
\]
which is equivalent to
\[
\sum_{i=1}^{n} d(i, j)^2 \tau_j = \sum_{j=1}^{n} d(i, j) \hat{\delta}_j,
\]
and the proof is complete.

Next we define the edge orientation matrix of \( T \). We assign an orientation to each edge of \( T \). Let \( e_i = (p, q); e_j = (r, s) \) be edges of \( T \). We say that \( e_i \) and \( e_j \) are similarly oriented, denoted by \( e_i \Rightarrow e_j \), if \( d(p, r) = d(q, s) \). Otherwise \( e_i \) and \( e_j \) are said to be oppositely oriented, denoted by \( e_i \Leftarrow e_j \). For example, in the following diagram \( e_i \) and \( e_j \) are similarly oriented.

\[\circ p \quad \circ q \quad \circ r \quad \circ s\]

The edge orientation matrix of \( T \) is the \((n - 1) \times (n - 1)\) matrix \( H \) having the rows and the columns indexed by the edges of \( T \). The \((i, j)\)-element of \( H \),
denoted by \( h(i, j) \) is defined to be \( 1(-1) \) if the corresponding edges \( e_i, e_j \) of \( T \) are similarly (oppositely) oriented. The diagonal elements of \( H \) are set to be 1. We assume that the same orientation is used while defining the matrix \( H \) and the incidence matrix \( Q \).

If the tree \( T \) has no vertex of degree 2, then we let \( \hat{\tau} \) be the diagonal matrix with diagonal elements \( 1/\tau_1, \ldots, 1/\tau_n \). We state some basic properties of \( H \) next, see [3].

**Theorem 4** Let \( T \) be a directed tree on \( n \) vertices, let \( H \) and \( Q \) be the edge orientation matrix and the vertex-edge incidence matrix of \( T \), respectively. Then

\[
\det H = 2^n - 2 \prod_{i=1}^{n} \tau_i.
\]

Furthermore, if \( T \) has no vertex of degree 2, then \( H \) is nonsingular and \( H^{-1} = \frac{1}{2} Q' \hat{\tau} Q \).

Let \( w_1, \ldots, w_{n-1} \) be the edge-weights. Recall that \( F \) be the diagonal matrix with diagonal elements \( w_1, \ldots, w_{n-1} \).

Also note that,

\[
(FHF)_{ij} = \begin{cases} w_i w_j & \text{if } e_i \Rightarrow e_j \\ -w_i w_j & \text{if } e_i \Leftarrow e_j \end{cases}
\]

**Lemma 5** \( Q' \Delta Q = -2HFH \).

**Proof** For \( i, j \in \{1, \ldots, n-1\} \), let the edge \( e_i \) be from \( p \) to \( q \) and the edge \( e_j \) be from \( r \) to \( s \). Then

\[
(Q' \Delta Q)_{ij} = \begin{cases} d(p, r)^2 + d(q, s)^2 - d(p, s)^2 - d(q, r)^2 & \text{if } e_i \Rightarrow e_j \\ d(p, s)^2 + d(q, r)^2 - d(p, r)^2 - d(q, s)^2 & \text{if } e_i \Leftarrow e_j \end{cases}
\]

Let \( d(r, s) = \alpha \). It follows from (6) that

\[
(Q' \Delta Q)_{ij} = \begin{cases} (w_i + \alpha)^2 + (w_j + \alpha)^2 - (w_i + w_j + \alpha)^2 - \alpha^2 = -2w_i w_j & \text{if } e_i \Rightarrow e_j \\ (w_i + w_j + \alpha)^2 + \alpha^2 - (w_i + \alpha)^2 - (w_j + \alpha)^2 = 2w_i w_j & \text{if } e_i \Leftarrow e_j \end{cases}
\]

and the proof is complete.

Let \( \bar{\tau} \) be the diagonal matrix with diagonal elements \( \tau_1, \ldots, \tau_n \).

**Lemma 6** \( \Delta L = 2D\bar{\tau} - 1\hat{\delta}' \).

**Proof** Let \( i, j \in \{1, \ldots, n\} \) be fixed. Let vertex \( j \) have degree \( p \). Suppose \( j \) is adjacent to vertices \( u_1, \ldots, u_p \) and let \( e_{\ell_1}, \ldots, e_{\ell_p} \) be the corresponding edges with weights \( w_{\ell_1}, \ldots, w_{\ell_p} \), respectively. We consider two cases.
Case (i). \( i = j \). We have

\[
(\Delta L)_{jj} = \sum_{k=1}^{n} d(j, k)^2 \ell_{kj} \\
= w_{\ell_1}^2 (-w_{\ell_1})^{-1} + \cdots + w_{\ell_p}^2 (-w_{\ell_p})^{-1} \\
= -(w_{\ell_1} + \cdots + w_{\ell_p}) \\
= -\hat{\delta}_j.
\]

Since the \((j, j)\)-element of \(2D \tilde{\tau} - 1\hat{\delta}'\) is \(-\hat{\delta}_j\), the proof is complete in this case.

Case (ii). \( i \neq j \). We assume, without loss of generality, that the \(ij\)-path passes through \(u_1\) (it is possible that \(i = u_1\)). Let \(d(i, u_1) = \alpha - w_{\ell_1}, d(i, u_2) = \alpha + w_{\ell_2}, \ldots, d(i, u_p) = \alpha + w_{\ell_p}\). We have

\[
(\Delta L)_{ij} = \sum_{k=1}^{n} d(i, k)^2 \ell_{kj} \\
= d(i, u_1)^2 (-w_{\ell_1})^{-1} + \cdots + d(i, u_p)^2 (-w_{\ell_p})^{-1} + d(i, j)^2 \ell_{jj} \\
= (\alpha - w_{\ell_1})^2 (-w_{\ell_1})^{-1} + (\alpha + w_{\ell_2})^2 (-w_{\ell_2})^{-1} + \cdots + (\alpha + w_{\ell_p})^2 (-w_{\ell_p})^{-1} \\
+ \alpha^2 ((w_{\ell_1})^{-1} + \cdots + (w_{\ell_p})^{-1}) \\
= (-2\alpha w_{\ell_1} + w_{\ell_1}^2) (-w_{\ell_1})^{-1} + (2\alpha w_{\ell_2} + w_{\ell_2}^2) (-w_{\ell_2})^{-1} + \cdots \\
+ (2\alpha w_{\ell_p} + w_{\ell_p}^2) (-w_{\ell_p})^{-1} \\
= 2\alpha - 2\alpha (p - 1) - (w_{\ell_1} + \cdots + w_{\ell_p}) \\
= 2\alpha \tau_j - (w_{\ell_1} + \cdots + w_{\ell_p}),
\]

which is the \((i, j)\)-element of \(2D \tilde{\tau} - 1\hat{\delta}'\) and the proof is complete.

\[\blacksquare\]

3 Determinant

Our next objective is to obtain a formula for the determinant of the squared distance matrix. We first consider the case when the tree has no vertex of degree 2.

**Theorem 7** Let \( T \) be a tree with vertex set \( V(T) = \{1, \ldots, n\} \), edge set \( E(T) = \{e_1, \ldots, e_{n-1}\} \), and edge weights \( w_1, \ldots, w_{n-1} \). Suppose \( T \) has no vertex of degree 2. Then

\[
\det \Delta = (-1)^{n-1} \frac{4^{n-2}}{2} \prod_{i=1}^{n} \tau_i \prod_{i=1}^{n-1} w_i^2 \sum_{i=1}^{n} \frac{\hat{\delta}_i^2}{\tau_i}.
\]  

(7)
Proof We assign an orientation to the edges of the tree and let $H$ and $Q$ be, respectively, edge orientation matrix and the vertex-edge incidence matrix of $T$.

Let $\Delta_i$ denote the $i$-th column of $\Delta$, and let $t_i$ be the column vector with 1 at the $i$-th place and zeros elsewhere, $i = 1, \ldots, n$. Then

$$
\begin{bmatrix}
Q' \\
Q' t'_1
\end{bmatrix}
\Delta
\begin{bmatrix}
Q & t_1
\end{bmatrix}
= 
\begin{bmatrix}
Q'\Delta Q & Q'\Delta_1 \\
\Delta_1^\top Q & 0
\end{bmatrix}.
\tag{8}
$$

Since $\det \begin{bmatrix}
Q' \\
Q' t'_1
\end{bmatrix} = \pm 1$, it follows from (8) that

$$
\det \Delta = 
\begin{bmatrix}
Q'\Delta Q & Q'\Delta_1 \\
\Delta_1^\top Q & 0
\end{bmatrix}
= 
\begin{bmatrix}
-2FHF & Q'\Delta_1 \\
\Delta_1^\top Q & 0
\end{bmatrix}
$$
by Lemma 4

$$
= (\det(-2FHF))(-\Delta_1^\top Q(-2FHF)^{-1}Q'\Delta_1)
$$

$$
= (-2)^{n-1}\prod_{i=1}^{n-1} w_i^2 (\det H) 2\Delta_1^\top Q F^{-1} H^{-1} F^{-1} Q'\Delta_1
$$

$$
= (-1)^{n-1}2^n \prod_{i=1}^{n-1} w_i^2 (\det H) \Delta_1^\top Q F^{-1} Q' \hat{\tau} Q F^{-1} Q' \Delta_1,
\tag{9}
$$
in view of Theorem 4.

By Lemma 5 we have

$$
\Delta_1^\top Q F^{-1} Q' \hat{\tau} Q F^{-1} Q' \Delta_1
= \sum_i (2d_{1i}\tau_i - \hat{\delta}_i)^2 \frac{1}{\tau_i}
$$

$$
= \sum_i (4d_{1i}^2\tau_i^2 + \hat{\delta}_i^2 - 4d_{1i}\tau_i\hat{\delta}_i) \frac{1}{\tau_i}
$$

$$
= \sum_i 4d_{1i}^2\tau_i + \sum_i \frac{\hat{\delta}_i^2}{\tau_i} - 4 \sum_i d_{1i}\hat{\delta}_i
\tag{10}
$$

It follows from (10) and Lemma 3 that

$$
\Delta_1^\top Q F^{-1} Q' \hat{\tau} Q F^{-1} Q' \Delta_1 = \sum_i \frac{\hat{\delta}_i^2}{\tau_i}
\tag{11}
$$

Also by Theorem 4,

$$
\det H = 2^{n-2}\prod_{i=1}^{n} \tau_i.
\tag{12}
$$

The proof is complete by substituting (11) and (12) in (9).
Corollary 8 [3] Let $T$ be an unweighted tree with vertex set $V(T) = \{1, \ldots, n\}$. Suppose $T$ has no vertex of degree 2. Then
\[
\det \Delta = (-1)^n 4^{n-2} \left( 2n - 1 - 2 \sum_i \frac{1}{\tau_i} \right) \prod_{i=1}^n \tau_i.
\] (13)

Proof We set $w_i = 1, i = 1, \ldots, n-1$ in Theorem 7. Then $\hat{\delta}_i = \delta_i = 2 - \tau_i, i = 1, \ldots, n$. We have
\[
\sum_i \frac{\hat{\delta}_i^2}{\tau_i} = \sum_i \frac{(2 - \tau_i)^2}{\tau_i}
\]
\[
= \sum_i \frac{4 + \tau_i^2 - 4\tau_i}{\tau_i}
\]
\[
= 4 \sum_i \frac{1}{\tau_i} + \sum_i \tau_i - 4n
\]
\[
= 4 \sum_i \frac{1}{\tau_i} + 2 - 4n
\]
\[
= -2 \left( 2n - 1 - 2 \sum_i \frac{1}{\tau_i} \right).
\] (14)

The proof is complete by substituting (14) in (7). \qed

We turn to the case when there is a vertex of degree 2.

Theorem 9 Let $T$ be a tree with vertex set $V(T) = \{1, \ldots, n\}$, edge set $E(T) = \{e_1, \ldots, e_{n-1}\}$, and edge weights $w_1, \ldots, w_{n-1}$. Let $q$ be a vertex of degree 2 and let $p$ and $r$ be neighbors of $q$. Let $e_i = (pq), e_j = (qr)$. Then
\[
\det \Delta = (-1)^{n-1} 2^{2n-5} (w_i + w_j)^2 \prod_{s=1}^{n-1} w_s^2 \prod_{k \neq q} \tau_k.
\] (15)

Proof We assume, without loss of generality, that $e_i$ is directed from $p$ to $q$ and $e_j$ is directed from $q$ to $r$.

Let $z_q$ be the $n \times 1$ unit vector with 1 at the $q$-th place and zeros elsewhere. Let $\Delta_q$ be the $q$-th column of $\Delta$. We have
\[
\begin{bmatrix}
Q'

\Delta

z_q
\end{bmatrix} = \begin{bmatrix}
Q' \Delta Q

\Delta_q Q

0
\end{bmatrix} = \begin{bmatrix}
-2FH F

\Delta_q \Delta_q

0
\end{bmatrix},
\] (16)
in view of Lemma 5. It follows from (16) that
\[
\begin{bmatrix}
F^{-1} & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
Q' \\
z_q
\end{bmatrix}
\Delta
\begin{bmatrix}
Q \\
z_q
\end{bmatrix}
\begin{bmatrix}
F^{-1} & 0 \\
0 & 1
\end{bmatrix}
= 
\begin{bmatrix}
-2H & F^{-1}Q'\Delta_q \\
\Delta'_qQF^{-1} & 0
\end{bmatrix}.
\tag{17}
\]

Taking determinants of matrices in (17) we get
\[
(det F^{-1})^2 det \Delta = det 
\begin{bmatrix}
-2H & F^{-1}Q'\Delta_q \\
\Delta'_qQF^{-1} & 0
\end{bmatrix}.
\tag{18}
\]

Note that the \(i\)-th and the \(j\)-th columns of \(H\) are identical.

Let \(H(j|j)\) denote the submatrix obtained by deleting row \(j\) and column \(j\) from \(H\). In \[
\begin{bmatrix}
-2H & F^{-1}Q'\Delta_q \\
\Delta'_qQF^{-1} & 0
\end{bmatrix},
\]
subtract column \(i\) from column \(j\), row \(i\) from row \(j\), and then expand the determinant along column \(j\). Then we get
\[
det 
\begin{bmatrix}
-2H & F^{-1}Q'\Delta_q \\
\Delta'_qQF^{-1} & 0
\end{bmatrix}
= -(det(\Delta'_qQF^{-1}))_j - (\Delta'_qQF^{-1})_j^2 det(-2H(j|j))
= \((-2)^{n-2} \det H(j|j)(-w_j - w_i)^2,
\tag{19}
\]

Note that \(H(j|j)\) is the edge orientation matrix of the tree obtained by deleting vertex \(q\) and replacing edges \(e_i\) and \(e_j\) by a single edge directed from \(p\) to \(r\) in the tree. Hence by Theorem 4,
\[
det H(j|j) = 2^{n-3} \prod_{k \neq q} \tau_k.
\tag{20}
\]
It follows from (17),(18) and (19) that
\[
det \Delta = -(det F)^2(-1)^n2^{n-2}2^{n-3}(\prod_{k \neq q} \tau_k)(w_i + w_j)^2
= \((-1)^{n-1}2^{n-5}(w_i + w_j)^2 \prod_{s=1}^{n-1} w_s^2 \prod_{k \neq q} \tau_k,
\tag{21}
\]
and the proof is complete.

\textbf{Corollary 10} Let \(T\) be a tree with vertex set \(V(T) = \{1,\ldots,n\}\), edge set \(E(T) = \{e_1,\ldots,e_{n-1}\}\), and edge weights \(w_1,\ldots,w_{n-1}\). Suppose \(T\) has at least two vertices of degree 2. Then \(\det \Delta = 0\).

\textbf{Proof} The result follows from Theorem 9 since \(\tau_i = 0\) for at least two values of \(i\). \[\blacksquare\]
4 Inverse

We now turn to the inverse of $\Delta$, when it exists. When the tree has no vertex of degree 2, we can give a concise formula for the inverse. We first prove some preliminary results.

**Lemma 11** Let the tree have no vertex of degree 2. Then

$$\Delta(2\tau - L\hat{\tau}\hat{\delta}) = (\hat{\delta}'\hat{\tau}\hat{\delta})\mathbf{1}. \quad (22)$$

**Proof** By Lemma 6, $\Delta L = 2D\hat{\tau} - \mathbf{1}\hat{\delta}'$. Hence

$$\Delta L\hat{\tau}\hat{\delta} = 2D\hat{\delta} - (\hat{\delta}'\hat{\tau}\hat{\delta})\mathbf{1}. \quad (23)$$

Since by Lemma 3, $\Delta\tau = D\hat{\delta}$, we obtain the result from (23). \[\Box\]

For a square matrix $A$, we denote by cof $A$, the sum of the cofactors of $A$.

**Lemma 12** Let $T$ be a tree with vertex set $V(T) = \{1, \ldots, n\}$, edge set $E(T) = \{e_1, \ldots, e_{n-1}\}$, and edge weights $w_1, \ldots, w_{n-1}$. Suppose $T$ has no vertex of degree 2. Then

$$\text{cof } \Delta = (-1)^{n-1}2^{2n-3} \prod_{k=1}^{n-1} w_k^2 \prod_{i=1}^{n} \tau_i. \quad (24)$$

**Proof** By Lemma 5, $Q'\Delta Q = -2FH^T$. Taking determinant of both sides and using Cauchy-Binet formula, we get

$$\text{cof } \Delta = (-2)^{n-1}(\det F)^2 \det H$$

$$= (-2)^{n-1} \prod_{k=1}^{n-1} w_k^2 2^{n-2} \prod_{i=1}^{n} \tau_i \text{ by Theorem 4}$$

$$= (-1)^{n-1}2^{2n-3} \prod_{k=1}^{n-1} w_k^2 \prod_{i=1}^{n} \tau_i, \quad (25)$$

and the proof is complete. \[\Box\]

**Corollary 13** Let the tree have no vertex of degree 2 and let $\beta = \hat{\delta}'\hat{\tau}\hat{\delta}$. If $\beta \neq 0$, then $\Delta$ is nonsingular and

$$\mathbf{1}'\Delta^{-1}\mathbf{1} = \frac{4}{\beta}. \quad (26)$$
Proof Observe that $\beta = \sum_{i=1}^{n} \frac{\hat{\delta}^2_i}{\tau_i}$. By Theorem 7,

$$\det \Delta = (-1)^{n-1} \frac{4^{n-2}}{2} \prod_{i=1}^{n} \tau_i \prod_{i=1}^{n-1} w_i^2 \sum_{i=1}^{n} \frac{\hat{\delta}^2_i}{\tau_i}. \tag{27}$$

If $\beta \neq 0$, then $\Delta$ is nonsingular by (27). Note that $1'\Delta^{-1}1 = \frac{\text{cof} \Delta}{\det \Delta}$. The proof is complete using Lemma 12 and (27).

**Theorem 14** Let the tree have no vertex of degree 2 and let $\beta = \hat{\delta}'\hat{\tau}\hat{\delta}$. Let $\eta = 2\tau - L\hat{\tau}\hat{\delta}$. If $\beta \neq 0$, then $\Delta$ is nonsingular and

$$\Delta^{-1} = -\frac{1}{4}L\hat{\tau}L + \frac{1}{4\beta} \eta\eta'. \tag{28}$$

**Proof** Let $X = -\frac{1}{4}L\hat{\tau}L + \frac{1}{4\beta} \eta\eta'$. Then

$$\Delta X = -\frac{1}{4} \Delta L\hat{\tau}L - \frac{1}{4\beta} \Delta \eta\eta'. \tag{29}$$

By Lemma 6, $\Delta L = 2D\tau - 1\hat{\delta}'$. Hence

$$\Delta L\hat{\tau}L = 2DL - 1\hat{\delta}'\hat{\tau}L. \tag{30}$$

Using Theorem 2, we can see that

$$DL = -2I + 1\tau'. \tag{31}$$

Finally, by Lemma 11, $\Delta \eta = \beta$. This fact and (29), (30) and (31) lead to

$$\Delta X = I - \frac{1}{2}1\tau' + \frac{1}{4} \hat{\delta}'\hat{\tau}L + \frac{1}{4\beta} 1\eta'. \tag{32}$$

Since $\eta = 2\tau - L\hat{\tau}\hat{\delta}$, it follows from (32) that $\Delta X = I$ and the proof is complete.

We conclude with an example to show that the condition $\beta \neq 0$ is necessary in Theorem 14.

**Example** Consider the tree

```
   o2
  /  |
 o3 ——— o1
     |   \
     1   |
     -—– ————
     |    |     |
     1   1     1
     ——– ————
    o4   o5
```


The distance matrix of the tree is given by

\[
D = \begin{bmatrix}
0 & 1 & 1 & \gamma & 1 \\
1 & 0 & 2 & 1+\gamma & 2 \\
1 & 2 & 0 & 1+\gamma & 2 \\
\gamma & 1+\gamma & 1+\gamma & 0 & 1+\gamma \\
1 & 2 & 2 & 1+\gamma & 0
\end{bmatrix}.
\]

It can be checked that \( \det \Delta = -32\gamma^2(\gamma^2 - 6\gamma - 3) \). Thus \( \Delta \) is singular if \( \gamma = 3 + 2\sqrt{3} \). Note that \( \tilde{\delta} = [\gamma + 3, 1, 1, \gamma, 1] \), \( \tau' = [-2, 1, 1, 1, 1] \) and hence, if \( \gamma = 3 + 2\sqrt{3} \), then \( \sum_{i=1}^{4} \frac{\tilde{\delta}_i^2}{\gamma_i} = 0 \).

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