On connected degree sequences

Jonathan McLaughlin

Department of Mathematics, St. Patrick’s College, Dublin City University, Dublin 9, Ireland

Abstract
This note gives necessary and sufficient conditions for a sequence of non-negative integers to be the degree sequence of a connected simple graph. This result is implicit in a paper of Hakimi. A new alternative characterisation of these necessary and sufficient conditions is also given.

Keywords: connected graph, degree sequence
2010 MSC: 05C40

1. Introduction
A finite sequence of non-negative integers is called a graphic sequence if it is the degree sequence of some finite simple graph. Erdős and Gallai [4] first found necessary and sufficient conditions for a sequence of non-negative integers to be graphic and these conditions have since been refined by Hakimi [6] (stated in the sequel as Theorem 3.2) as well as (independently) by Havel [7]. Alternative characterisations and generalisations are due to Choudum [2], Sierksma & Hoogeveen [8] and Tripathi et al. [9], [10], [11]. This note states a result which is implicit in Hakimi [6], before giving an alternative characterisation of these necessary and sufficient conditions for a finite sequence of non-negative integers to be the degree sequence of a connected simple graph.

2. Preliminaries
Let \( G = (V_G, E_G) \) be a graph where \( V_G \) denotes the vertex set of \( G \) and \( E_G \subseteq [V_G]^2 \) denotes the edge set of \( G \) (given that \([V_G]^2\) is the set of all 2-element subsets of \( V_G \)). An edge \( \{a, b\} \) is denoted \( ab \) in the sequel. A graph is finite when \( |V_G| < \infty \) and \( |E_G| < \infty \), where \( |X| \) denotes the cardinality of the set \( X \). A graph is simple if it contain no loops (i.e. \( a \neq b \) for all \( ab \in E_G \)) or parallel/multiple edges (i.e. \( E_G \) is not a multiset). The degree of a vertex \( v \) in a graph \( G \), denoted \( deg(v) \), is the number of edges in \( G \) which contain \( v \). A path is a graph with \( n \) vertices in which two vertices, known as the endpoints, have
degree 1 and \( n - 2 \) vertices have degree 2. A graph is connected if there exists at least one path between every pair of vertices in the graph. A tree is a connected graph with \( n \) vertices and \( n - 1 \) edges. \( K_n \) denotes the complete graph on \( n \) vertices. All basic graph theoretic definitions can be found in standard texts such as [1], [3] or [5]. All graphs in this note are undirected and finite.

3. Degree sequences and graphs

A finite sequence \( s = \{s_1, ..., s_n\} \) of non-negative integers is called realisable if there exists a finite graph with vertex set \( \{v_1, ..., v_n\} \) such that \( \deg(v_i) = s_i \) for all \( i = 1, ..., n \). A sequence \( s \) which is realisable as a simple graph is called graphic. Given a graph \( G \) then the degree sequence of \( G \), denoted \( d(G) \), is the monotonic non-increasing sequence of degrees of the vertices in \( V_G \). This means that every realisable (resp. graphic) sequence \( s \) is equal to the degree sequence \( d(G) \) of some graph (resp. simple graph) \( G \) (subject to possible rearrangement of the terms in \( s \)). The maximum degree of a vertex in \( G \) is denoted \( \Delta_G \) and the minimum degree of a vertex in \( G \) is denoted \( \delta_G \). In this note all sequences will have positive terms as the only connected graph which has a degree sequence containing a zero is \( (\{v\}, \{\} ) \).

The following theorem states necessary and sufficient conditions for a sequence to be realisable (though not necessarily graphic).

**Theorem 3.1 (Hakimi)** Given a sequence \( s = \{s_1, ..., s_n\} \) of positive integers such that \( s_i \geq s_{i+1} \) for \( i = 1, ..., n - 1 \) then \( s \) is realisable if and only if \( \sum_{i=1}^{n} s_i \) is even and \( \sum_{i=2}^{n} s_i \geq s_1 \).

To address the issue of when a sequence is graphic, Hakimi describes in [6] a process he called a reduction cycle and uses it to state the following result.

**Theorem 3.2 (Havel, Hakimi)** Given a sequence \( \{s_1, ..., s_n\} \) of positive integers such that \( s_i \geq s_{i+1} \) for \( i = 1, ..., n - 1 \) then the sequence \( \{s_1, ..., s_n\} \) is graphic if and only if the sequence \( \{s_2 - 1, s_3 - 1, ..., s_{s_1 + 1} - 1, s_{s_1 + 2}, ..., s_n\} \) is graphic.

4. Degree sequences and connected graphs

**Definition 4.1** A finite sequence \( s = \{s_1, ..., s_n\} \) of positive integers is called connected (resp. connected and graphic) if \( s \) is realisable as a connected graph (resp. connected simple graph) with vertex set \( \{v_1, ..., v_n\} \) such that \( v_i \) has degree \( s_i \) for all \( i = 1, ..., n \).
Of course disconnected realisations of connected and graphic degree sequences exist, for example, \((2, 2, 2, 2, 2, 2)\) can be realised as a 6-cycle or as two disjoint 3-cycles.

As a graph is connected if and only if it contains a spanning tree, then a simple induction argument on the number of edges shows that every spanning tree of a graph \(G\), with \(|V_G| = n\), has exactly \(n - 1\) edges. Hence, a necessary condition for a graph \(G\), with \(|V_G| = n\), to be connected is that \(|E_G| \geq n - 1\).

The following theorem states necessary and sufficient conditions for a sequence to be connected but not necessarily simple.

**Theorem 4.2 (Hakimi)** Given a sequence \(s = \{s_1, \ldots, s_n\}\) of positive integers such that \(s_i \geq s_{i+1}\) for \(i = 1, \ldots, n - 1\) then \(s\) is connected if and only if \(s\) is realisable and \(\sum_{i=1}^{n} s_i \geq 2(n - 1)\).

The two main tools used in the proof of Theorem 4.2 are \(d\)-invariant operations (which leave degree sequences unchanged) and Lemma 4.3 (which appears in [6] as Lemma 1).

Consider a graph \(G\) with \(d(G) = (d_1, \ldots, d_n)\). Given any two edges \(ab, cd \in E_G\), where \(a, b, c\) and \(d\) are all distinct, then \(G\) is transformed by a \(d\)-invariant operation into \(G'\) when either

- \(V_{G'} = V_G\) and \(E_{G'} = (E_G \setminus \{ab, cd\}) \cup \{ac, bd\}\), or
- \(V_{G'} = V_G\) and \(E_{G'} = (E_G \setminus \{ab, cd\}) \cup \{ad, bc\}\).

Figure 1 shows both \(d\)-invariant operations.

![Figure 1](image1.png)

Figure 1: The two possible \(d\)-invariant operations on \(G\) resulting in \(G'\) such that \(d(G) = d(G')\)

These \(d\)-invariant operations are used to prove the following important result.

**Lemma 4.3** Let \(G_1, G_2, \ldots, G_r\), (with \(r > 1\)), be maximally connected subgraphs of \(G\) such that not all of the \(G_i\) are acyclic, then there exists a graph \(G'\) with \(r - 1\) maximally connected subgraphs such that \(d(G') = d(G)\).

The essence of Lemma 4.3 is presented in Figure 2. Note that worst-case-scenarios are assumed i.e. \(G_1\) is a cycle and \(G_2\) is acyclic with the graph \((V_{G_2}, (E_{G_2} \setminus \{cd\}))\) being disconnected.

3
5. Results

The first result, Theorem 5.1, is an explicit statement of a result implicit in [6]. The second result, Theorem 5.2, is a new alternative characterisation of Theorem 5.1 and has a similar flavour to that of Theorem 3.2.

Theorem 5.1 Given a sequence \( s = \{s_1, ..., s_n\} \) of positive integers such that \( s_i \geq s_{i+1} \) for \( i = 1, ..., n - 1 \) then \( s \) is connected and graphic if and only if the sequence \( s' = \{s'_1, ..., s'_{n-1}\} = \{s_2-1, s_3-1, ..., s_{s_1+1}-1, s_{s_1+2}, ..., s_n\} \) is graphic and \( \sum_{i=1}^{n} s_i \geq 2(n-1) \).

Proof (\( \Rightarrow \)) Suppose that \( s \) is connected and graphic. It is required to show that \( s' \) is graphic and that \( \sum_{i=1}^{n} s_i \geq 2(n-1) \).

As \( s = d(G) \) for some simple (connected) graph \( G \) then \( s_i \leq n - 1 \) for all \( i = 1, ..., n \) and there exists a graph \( G' \) with vertex set \( V_{G'} = V_G \setminus \{v_1\} \) and edge set \( E_{G'} = E_G \setminus \{v_1v_i \mid v_1v_i \in E_G\} \) such that \( d(G') = s' \). As \( G \) is a simple graph then it follows that \( G' \) is also a simple graph, hence \( s' \) is graphic. As \( s = d(G) \) for some (simple) connected graph \( G \), where \( |V_G| = n \), then as \( G \) is connected \( |E_G| \geq n - 1 \), hence \( \sum_{i=1}^{n} s_i \geq 2(n-1) \).

(\( \Leftarrow \)) Suppose that \( s' \) is graphic and that \( \sum_{i=1}^{n} s_i \geq 2(n-1) \). It is required to show that \( s \) is both graphic and connected.

As \( s' \) is graphic then \( s' \leq n-2 \) for all \( i \in \{1, ..., n-1\} \). Adding a term \( s_1 \) (which is necessarily \( \leq n-1 \)) results in \( s \) also being graphic as in the worst case scenario i.e. where \( s_1 = s_n = n - 1 \), then \( s = \{s_1, ..., s_n\} = \{n-1, ..., n-1\} \) which is the degree sequence of the simple graph \( K_n \). Suppose that \( \sum_{i=1}^{n} s_i = 2(n-1) \), then \( s \) is the degree sequence of a graph \( G \) with \( n - 1 \) edges and \( n \) vertices.
which means that $G$ is a tree, hence $s$ is connected. If $\sum_{i=1}^{n} s_i > 2(n-1)$ then either $s = d(G)$ for some connected graph $G$ or it is possible to apply Lemma 4.3 repeatedly until a graph $G$ is found such that $G$ is connected and $d(G) = s$. $\square$

The following result is what can be thought of as a connected version of Theorem 3.2. However, note that it is not possible to simply add the word connected to the statement of Theorem 3.2 as $s = (2, 2, 1)$ is connected but $s' = (1, 0, 1)$ is not connected.

**Theorem 5.2** Given a sequence $s = \{s_1, ..., s_n\}$ of positive integers such that $s_i \geq s_{i+1}$ for $i = 1, ..., n-1$ then $s$ is connected and graphic if and only if the sequence $s' = \{s'_1, ..., s'_{n-1}\} = \{s_1 - 1, s_2 - 1, ..., s_n - 1, s_{n+1}, ..., s_{n-1}\}$ is connected and graphic.

**Proof** ($\Rightarrow$) Suppose that $s = \{s_1, ..., s_n\}$ is connected. It is required to show that $s'$ is both graphic and connected.

To show that $s'$ is graphic it is required to show that $\sum_{i=1}^{n-1} s'_i$ is even and that all vertices have degree less than or equal to $n - 2$. Observe that

$$\sum_{i=1}^{n-1} s'_i = \sum_{i=1}^{n-1} (s_i - s_n) = \sum_{i=1}^{n} s_i - 2s_n.$$ 

As $s$ is graphic then $\sum_{i=1}^{n} s_i$ is even and so $\sum_{i=1}^{n} s_i - 2s_n$ is also even. As $s$ is graphic then all vertices $v_i \in V_G$ with $i \in \{1, ..., n\}$ must satisfy $\text{deg}(v_i) \leq n - 1$. All vertices with degree $n - 1$ in $G$ are necessarily connected to $v_n$ whereas vertices with degree less than $n - 1$ may or may not be connected to $v_n$. It follows that after deleting $v_n$ and all edges containing $v_n$ that the maximum degree which any vertex can have in any $G'$ is $n - 2$ (where $d(G') = s'$).

To show that $s'$ is connected it is required to show that $\sum_{i=1}^{n-1} s'_i \geq 2(n-2)$ i.e. there exists a graph $G'$ with $d(G') = s'$ and $|V_{G'}| = n - 1$ such that $|E_{G'}| \geq n - 2$.

As $s$ is graphic then $1 \leq s_i \leq n - 1$.

Let $s_n = 1$: As $s_n = 1 = \delta_G$ then $s' = (s_1 - 1, s_2, ..., s_{n-1})$. Not all $s_i = 1$ except in the case where $s = (1, 1)$ resulting in $s' = (0)$ which is a connected degree sequence. As $s$ is connected and $\text{deg}(v_n) = 1$ then $v_n$ is a leaf of a connected graph $G$ and so deleting $v_n$ cannot result in a disconnected graph $G'$, hence $s'$ is connected when $s_n = 1$.

Let $s_n = k$ where $2 \leq k \leq n - 1$: As $s_n = k = \delta_G$ then

$$s' = (s_1 - 1, s_2 - 1, ..., s_{s_n - 1}, s_{s_n + 1}, ..., s_{n-1}).$$

5
Assuming the worst case scenario i.e. $\Delta_G = \delta_G = k$, then this gives

$$s' = (k - 1, ..., k - 1, k, ..., k)$$

which means that

$$\sum_{i=1}^{n-1} s_i' \geq k(k - 1) + k(n - k - 1) = k(n - 2) \geq 2(n - 2)$$

whenever $2 \leq k \leq n - 1$. Hence, $s'$ is connected when $s_n = k$ where $2 \leq k \leq n - 1$.

$(\Leftarrow)$ Suppose that $s' = \{s'_1, ..., s'_{n-1}\} = \{s_1 - 1, s_2 - 1, ..., s_{n-1} - 1, s_{s_n+1}, ..., s_{n-1}\}$ is connected. It is required to show that $s$ is both graphic and connected.

As $s'$ is connected then there exists some $G'$ with $|V_{G'}| = n - 1$ and $d(G') = s'$ where the degree of all vertices in $V_{G'}$ is less than or equal to $n - 2$. As $V_G = V_{G'} \cup \{v_n\}$ (and all $s_i$ are necessarily $\leq n - 1$) then all vertices in $V_G$ will have degree at most $n - 1$. Observe that

$$\sum_{i=1}^{n} s_i = \sum_{i=1}^{n-1} s_i' + 2s_n.$$

As $s'$ is graphic then $\sum_{i=1}^{n-1} s_i'$ is even and so $\sum_{i=1}^{n-1} s_i' + 2s_n$ is also even.

As $s'$ is connected then there exists some $G'$ with $d(G') = s'$ where $|E_{G'}| \geq n - 2$ as $|V_{G'}| = n - 1$. As $s_n \geq 1$ then this means that there is at least one edge in $G$ which has $v_n$ as an endpoint and some $v_i$ with $i \in \{1, ..., n - 1\}$ as the other endpoint.

This observation along with the fact that $|E_G| \geq |E_{G'}| + 1 > n - 2$, where $|V_G| = n$, means that $s$ is connected.

**Example 5.3** An example, using Theorem 5.2, of what Hakimi would term a “set of successive reduction cycles” is shown in Figure 4.
6. Comments

Theorem 3.2 is used in [6] to check algorithmically when a given sequence \( s \) is graphic. In a similar manner, Theorem 5.2 suggests an algorithm which can be used to determine if a given sequence \( s \) is connected and graphic.

References

[1] J. A. Bondy and U. S. R. Murty. *Graph theory*, volume 244 of *Graduate Texts in Mathematics*. Springer, New York, 2008.

[2] S. A. Choudum. A simple proof of the Erdős-Gallai theorem on graph sequences. *Bull. Austral. Math. Soc.*, 33(1):67–70, 1986.

[3] R. Diestel. *Graph theory*, volume 173 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 2000.

[4] P. Erdős and T. Gallai. Graphs with prescribed degrees of vertices. *Mat. Lapok*, 11:264–274 (in Hungarian), 1960.

[5] R. Gould. *Graph theory*. The Benjamin/Cummings Publishing Co. Inc., Menlo Park, CA, 1988.

[6] S. L. Hakimi. On realizability of a set of integers as degrees of the vertices of a linear graph. I. *J. Soc. Indust. Appl. Math.*, 10:496–506, 1962.

[7] V. Havel. A remark on the existence of finite graphs. *Časopis Pěst. Mat.*, 80:477–480 (in Czech), 1955.

[8] G Sierksma and H Hoogeveen. Seven criteria for integer sequences being graphic. *J. Graph Theory*, 15(2):223–231, 1991.

[9] A. Tripathi, S. Venugopalan, and D. B. West. A short constructive proof of the Erdős and Gallai characterization of graphic lists. *Discrete Math.*, 310(4):843 – 844, 2010.

[10] A. Tripathi and S. Vijay. A note on a theorem of Erdős and Gallai. *Discrete Math.*, 265:417 – 420, 2003.

[11] A. Tripathi and S. Vijay. A short proof of a theorem on degree sets of graphs. *Discrete Appl. Math.*, 155(5):670 – 671, 2007.