Langlands correspondence for isocrystals and existence of crystalline companion for curves

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Abstract

In this paper, we show the Langlands correspondence for isocrystals on curves. This shows the existence of crystalline companion in the curve case. For the proof, we construct the theory of arithmetic $\mathcal{D}$-modules for algebraic stacks whose diagonal morphisms are finite. Finally, combining with methods of Deligne and Drinfeld, we show existence of “$\ell$-adic companion” for any isocrystals on smooth scheme of any dimension.

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Introduction

The Weil conjecture was proven by Deligne in 80’s and culminated in the theory of weights for $\ell$-adic cohomology in his celebrated paper \cite{De1}. In the paper, he conjectured existence of “compatible system”. The following is an excerpt of a part of the conjecture:

Conjecture (\cite{De1} 1.2.10). — Soient $X$ normal connexe de type fini sur $\mathbb{F}_p$, et $\mathcal{F}$ un faisceau lisse irréductible dont le déterminant est défini par un caractère d’ordre fini du groupe fondamental.
When $X$ is a curve, the conjecture except for (vi) are consequences of the Langlands correspondence, which was proven by V. Drinfeld in the rank 2 case and by L. Lafforgue unconditionally. Using the Langlands correspondence, Deligne and Drinfeld showed the conjecture, except for (vi), in the case $X$ is smooth. In this paper, we prove (vi) in the case $X$ is a curve. In fact, we show even a stronger result: a correspondence between irreducible overconvergent $F$-isocrystals with finite determinant on an open dense subscheme of $X$ and cuspidal automorphic representations of the function field of $X$ with finite central character (see Theorem 4.13). Finally, in Theorem 4.13.3 we show the converse of Deligne’s conjecture when $X$ is smooth using the techniques of Deligne and Drinfeld in [EK] and [Dr]: for any overconvergent $F$-isocrystal over a smooth scheme, there exists an $\ell$-adic companion for any $\ell \neq p$.

For a proof, we follow the strategy of the $\ell$-adic case. First, we prove the “product formula for epsilon factor”, which had already been carried out in $p$-adic setting by the author together with A. Marmora in [AM]. By using Deligne’s “principe de récurrence”, construction of the correspondence had been reduced to producing an isocrystal out of a cuspidal automorphic representation in [A2]. Now, we use the method of Drinfeld using the moduli space of “shtukas”, which was further refined by L. Lafforgue in his establishment of the Langlands correspondence for function fields. Since we already have “motives” to construct the isocrystals, a main thing we need to do left in the $p$-adic case is to build a suitable cohomology theory for certain algebraic stacks.

Before explaining our construction, let us review the history of attempts to construct a six functor formalism of $p$-adic cohomology theory. For more general history, we suggest to refer to [I], [Ke2]. The first $p$-adic cohomology defined for arbitrary separated scheme of finite type over a perfect field of characteristic $p$ was proposed by P. Berthelot around 80’s, which is named the rigid cohomology. He also defined a coefficient theory, called overconvergent $F$-isocrystals, which can be seen as a $p$-adic analogue of vector bundles with integrable connection. As we can see from this analogy, it is not reasonable to expect for six functor formalism of A. Grothendieck in the framework of overconvergent $F$-isocrystals. To remedy this, keeping the analogy with complex situation, Berthelot introduced the theory of arithmetic $\mathcal{D}$-modules. See [Ber2] for a beautiful survey by the founder himself. The theory is very involving since we need to deal with differential operators of infinite order, and several fundamental properties had been left as conjectures. Among which, a crucial one was the finiteness. A big step toward this problem was introduction of overholonomic modules by D. Caro, which potentially bypasses Berthelot’s original strategy to construct a formalism. His work was successful in proving stability for most of the standard cohomological operations, but the finiteness of overconvergent $F$-isocrystals had still been a difficult problem. A breakthrough was achieved by K. S. Kedlaya in his resolution of Shiho’s conjecture, or the proof of semistable reduction theorem [Ke4]. This extremely powerful theorem enabled us to answer to many tough questions in arithmetic $\mathcal{D}$-modules: a finiteness result by Caro and N. Tsuzuki (cf. [CT]), an analogue of Weil II by Caro and the author (cf. [AC]). Even though we do not explicitly use the theorem in the proof of the Langlands correspondence, the theorem can be seen as another application of it. Thanks to the theorem, before our work, the theory of arithmetic $\mathcal{D}$-modules had been established for “realizable schemes”. See [1.1] for the detailed review. In particular, quasi-projective schemes were included in the framework. However, to construct the isocrystals corresponding to cuspidal automorphic representations, the category of quasi-projective schemes is too restricting. A large part of this paper is devoted to construct such a six functor formalism for “admissible stacks”, with which, in particular, we end our seeking, since Monsky and Washnitzer, for a six functor formalism of $p$-adic cohomology for separated schemes of finite type over a perfect field.

Our construction of six functor formalism is more or less formal: making full use of the existence of the formalism in local situation, we glue. Even though we do not axiomize, the construction can be carried out for any cohomology theory over a field admitting reasonable six functor formalism locally. First, let us explain the construction in the scheme case. As we have already mentioned, for “realizable schemes” (e.g. quasi-projective schemes) we already have the formalism thanks to works of Caro. For a realizable scheme $X$, we denote by $D^b_{\text{hol}}(X)$ the associated triangulated category with $t$-structure, and by $\text{Hol}(X)$ its heart, the category corresponding to that of perverse sheaves in the philosophy of Riemann-Hilbert correspondence. When $X$ is a scheme of finite type over $k$, we are able to take a finite open covering $\{U_i\}$
by realizable schemes. By standard gluing, we define $\text{Hol}(X)$ out of $\{\text{Hol}(U_i)\}$. The first difficulty is to define the derived category. A starting point of our construction is an analogy of Beilinson’s equivalence

$$D^b(\text{Hol}(X)) \xrightarrow{\sim} D^b_{\text{hol}}(X)$$

where $X$ is a realizable scheme, which is proven in [AC2]. This equivalence suggests to define the derived category naively by $D^b(\text{Hol}(X))$ for general scheme $X$. The next problem is to construct the cohomological functors. Let us explain this when $f: X \to Y$ is a finite morphism between realizable schemes, the easiest case. The push-forward $f_+$ is an exact functor, so we can define $f_+: \text{Hol}(X) \to \text{Hol}(Y)$ for finite morphism between general schemes by gluing, and we derive this functor to get the functor between derived categories. The definition of $f^!$ is more technical. When the morphism $f$ is between realizable schemes, by using some general non-sense, we are able to show that $f^!$ is the right derived functor of $\mathcal{H}^0 f_!$ using the fact that $(f_+, f^!)$ is an adjoint pair and $f_+$ is exact. Thus, for a general finite morphism $f$, we define $\mathcal{H}^0 f_!$ by gluing, and define the functor between derived categories by taking the right derived functor. Since the category $\text{Hol}(X)$ does not possess enough injectives, we need techniques of Ind-categories to overcome this deficit. Even though the construction is more involving, we can define the cohomological functors for projections $X \times Y \to Y$ using similar idea. Since any morphism between separated schemes of finite type can be factorized into a closed immersion and a projection, we may define the cohomological operations by composition.

For an algebraic stack $\mathfrak{X}$, we use a simplicial technique to construct the derived category $D^b_{\text{hol}}(\mathfrak{X})$: take a presentation $X \to \mathfrak{X}$, and we consider the simplicial algebraic space $X_\bullet := \cosk_0(X \to \mathfrak{X})$. The derived category of $\mathfrak{X}$ should coincide with that of $X_\bullet$ with suitable condition on the cohomology. Since $\mathcal{D}$-modules behave like perverse sheaves, there are differences with the construction of [LO], but mostly parallel. Now, we want to construct the cohomological operations for algebraic stacks similarly to those for schemes. However, in general, morphisms between algebraic stacks cannot be written as composition of finite morphisms and projections. Because of this obstacle, we give up constructing the formalism in full generality we can expect, and restrict our attention to admissible stacks, algebraic stacks whose diagonal morphisms are finite.

This formalism is especially used to show the $\ell$-independence of the trace of the action of the correspondence on a cohomology groups, which we use to calculate the trace of the action of Hecke algebra on the cohomology groups of the moduli spaces of shtukas. See [LO] for more explanation of the proof of the main theorem.

Let us overview the organization of this paper. We begin with collecting known results concerning arithmetic $\mathcal{D}$-modules in [1.1] and the subsection contains few new facts. In [1.2] we show some elementary properties of Ind categories. In [1.3] we introduce a $t$-structure corresponding to constructible sheaves in the spirit of Riemann-Hilbert correspondences. This $t$-structure is useful when we construct various types of trace maps. In the arithmetic $\mathcal{D}$-module theory, the coefficient categories are $K$-additive where $K$ is a complete discrete valuation field whose residue field is $k$. However, for the Langlands correspondence it is convenient to work with $\mathbb{Q}_p$-coefficients. Passing from $K$-coefficients to $\mathbb{Q}_p$-coefficients is rather formal, and some generality is developed in [1.4]. In [1.5] we conclude the first section by constructing the trace maps for flat morphisms in the style of SGA 4. This foundational property had been lacking in the theory of arithmetic $\mathcal{D}$-modules, and it plays an important role in the proof of $\ell$-independence type theorem, which is the main theme in [2].

In [2] we develop a theory for algebraic stacks. Most of the properties used in this section are formal in the six functor formalism, and almost no knowledge of arithmetic $\mathcal{D}$-modules is required. In [2.1] we define the triangulated category of holonomic complexes for algebraic stacks. Some cohomological operations for algebraic stacks are introduced in [2.2]. In [2.3] we restrict our attention to so called “admissible stacks”. Any morphism between admissible stacks can be factorized into morphisms which has already been treated in [2.2] and our construction of six functor formalism for this type of stacks completes. We show basic properties of the operations in this subsection. The final subsection [2.4] is complementary, in which we collect some facts which is needed in the proof of the Langlands correspondence.

In [3] we show an $\ell$-independence type theorem of the trace of the action of a correspondence on cohomology groups. With the trace formalism developed in [1.4] even though there are some differences since we are dealing with algebraic stacks, our task is to translate the proof of [KS] in our language.

In the final section [4] we show the Langlands correspondence. We state the main theorem and explain the idea of the proof in the first subsection. The actual proof is written in the second subsection, and we conclude the paper with some well-known applications.
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Convention and notation

0.0.1. In this paper, we usually use Roman fonts (e.g. $X$) for schemes, script fonts (e.g. $\mathcal{P}$) for formal schemes, and Gothic fonts (e.g. $\mathfrak{X}$) for algebraic stacks. When we write $(-)^{(0)}$ it means “$(-)$ (resp. $(-')$”). Throughout this paper, we fix a prime number $p$. When a discrete valuation field $K$ is fixed, we fix an algebraic closure of $K$, and denote it by $\overline{\mathbb{Q}}_p$. In the whole paper, we fix a universe $U$.

0.0.2. For the terminology of algebraic stacks, we follow [LM]. Especially, any scheme, algebraic space, or algebraic stack is assumed to be quasi-separated. For an algebraic stack $\mathfrak{X}$, we denote by $X_{sm}$ the category of affine schemes over $\mathfrak{X}$ such that the structural morphism $X \to \mathfrak{X}$ is smooth. Recall that a presentation of $\mathfrak{X}$ is a smooth surjective morphism $\mathcal{X} \to \mathfrak{X}$ from an algebraic space $\mathcal{X}$. Finite morphism or universal homeomorphism between algebraic stacks is always assumed to be representable.

0.0.3. Let $P: \mathcal{X} \to \mathfrak{X}$ be a smooth morphism from an algebraic space to an algebraic stack. Then the continuous function $\dim(P): \mathcal{X} \to \mathbb{N}$ is defined in [LM] (11.14). This function is called the relative dimension of $P$, and sometimes denoted by $d_{\mathcal{X}/\mathfrak{X}}$. When $d$ is an integer, smooth morphisms between algebraic stacks of relative dimension $d$ are assumed to be equidimensional.

0.0.4. Let $X$ be a topological space, and let $\pi_0(X)$ be the set of connected components of $X$. Let $d: \pi_0(X) \to \mathbb{Z}$ be a map. For any connected component $Y$ of $X$, assume that a category $C_Y$ with an auto-functor $T_Y: C_Y \to C_Y$ is attached, and $C_X \cong \prod_{Y \in \pi_0(X)} (C_Y)$ via which $T_X$ is identified with $(T_Y)_{Y \in \pi_0(X)}$. For $M \in C_X$, we define $T^d(M)$ as follows: let $M = (M_Y)_{Y \in \pi_0(X)} \in \prod C_Y$. Then $T^d(M) := (T^d(Y)M_Y)_{Y}$. This notation is used for shifts and Tate twists $[d]$, $(d)$, and when $T = [1]$ (resp. $T = (1)$), then $T^d$ is denoted by $[d]$ (resp. $(d)$).

1. Preliminaries

1.1. Review of arithmetic $\mathcal{P}$-modules

Let us recall the status of the theory of arithmetic $\mathcal{P}$-modules briefly. Let $s$ be a positive integer, and put $q := p^s$. Let $R$ be a complete discrete valuation ring whose residue field, which is assumed to be perfect of characteristic $p$, is denoted by $k$. Put $K := \text{Frac}(R)$. We moreover assume that the $s$-th absolute Frobenius homomorphism $\sigma: k \xrightarrow{\sim} k$ sending $x$ to $x^q$ lifts to an automorphism $R \xrightarrow{\sim} R$ denoted by $\sigma$.

1.1.1 Definition ([ACL 1.1.3]). — A scheme over $k$ is said to be realizable if it can be embedded into a proper smooth formal scheme over $\text{Spf}(R)$. We denote by $\text{Real}(k/R)$ the full subcategory of $\text{Sch}(k)$ consisting of realizable schemes.

For a realizable scheme $X$, the triangulated category with t-structure of holonomic complexes $\text{D}^b_{\text{hol}}(X/K)$ is defined. Let us recall the construction. Let $\mathcal{P}$ be a proper smooth formal scheme over $\text{Spf}(R)$. Then the category of overholonomic $\mathcal{P}^!_{\mathcal{P},q}$-modules (without Frobenius structure) is defined in [Ca3]. We denote by $\text{Hol}(\mathcal{P})$ its thick full subcategory generated by overholonomic $\mathcal{P}^!_{\mathcal{P},q}$-modules which can be endowed with $s'$-th Frobenius structure for some positive integer $s'$ divisible by $s$ (but we do not consider Frobenius structure). We call objects of $\text{Hol}(\mathcal{P})$ holonomic modules. By definition, $\text{D}^b_{\text{hol}}(\mathcal{P}^!_{\mathcal{P},q})$ is the
full subcategory of $D^b(\mathcal{P}^b_{\mathcal{X}^b})$ whose cohomology complexes are holonomic. Of course this subcategory is triangulated by [KSc 13.2.7].

Let $X \hookrightarrow \mathcal{P}$ be an embedding into a proper smooth formal scheme. Then $D^b_{\text{hol}}(X/K)$ is the subcategory of $D^b_{\text{hol}}(\mathcal{P})$ which is supported on $X$. This category does not depend on the choice of the embedding, and well-defined. Moreover, the t-structure is compatible with this equivalence (cf. [AC1 1.2.8]). The heart of the triangulated category is denoted by $\text{Hol}(X/K)$. For further details of this category, one can refer to [AC1 §1.1, §1.2] and [AC2 §1]. In [AC2], $\text{Hol}(X/K)$ and $D^b_{\text{hol}}(X/K)$ are denoted by $\text{Hol}_F(X/K)$ and $D^b_{\text{hol},F}(X/K)$ respectively.

1.1.2 Remark. — Lifting $R \xrightarrow{\sim} R$ of the Frobenius automorphism of $k$ is not unique in general. Let $\sigma': R \xrightarrow{\sim} R$ be another lifting. Let $\mathcal{X}$ be a smooth formal scheme over $R$. We denote by $\mathcal{X}^{\sigma'} := \mathcal{X} \otimes_{R, \sigma'} R$. Locally on $\mathcal{X}$, we have the following commutative diagram over $\text{Spf}(R)$:

$$
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{F} & \mathcal{X}^{\sigma} \\
\downarrow & & \downarrow G \\
\mathcal{X}^{\sigma'} & \xrightarrow{F'} & 
\end{array}
$$

where $F$ and $F'$ denote liftings of the relative $s$-th Frobenius. For a $\mathcal{P}^b_{\mathcal{X}^b}$-module $\mathcal{M}$, we have

$$
F^*(\mathcal{M}^{\sigma}) \cong F^*(G^*.\mathcal{M}^{\sigma'}) \cong F^*(\mathcal{M}^{\sigma'}),
$$

where $\mathcal{M}^{\sigma'}$ denotes the base change by the isomorphism $\mathcal{X}^{\sigma'} \xrightarrow{\sim} \mathcal{X}$. This shows that $\mathcal{M}$ being able to endow with a Frobenius structure with respect to $\sigma$ is equivalent to endowing that with respect to $\sigma'$. Thus, our category $\text{Hol}(X/K)$ does not depend on the choice of $\sigma$. However, the category of modules with Frobenius structure (cf. [AC1 1.4.4]) does depend on the choice. For example, assume $k$ is algebraically closed and consider $\sigma$ and $\sigma'$. Put

$$
K_0^{(\ell)} := \{ x \in K \mid \sigma^{(\ell)}(x) = x \}.
$$

We assume further that $K_0$ and $K_0'$ are not the same. Consider the unit object $K$ in $\text{Hol}($Spf$(R))$. Endow it with the trivial Frobenius structure $\Phi^{(\ell)}$ with respect to $\sigma^{(\ell)}$. Then

$$
\text{Hom}_{F^{(\ell)}, \text{Hol}(X/K)}((K, \Phi^{(\ell)}), (K, \Phi^{(\ell)})) \cong K_0^{(\ell)}.
$$

Thus, we are not able to expect an equivalence of categories between $F^*\text{-Hol}(X/K)$ and $F'^*\text{-Hol}(X/K)$ compatible with the forgetful functors to $\text{Hol}(X/K)$.

1.1.3. Six functors are defined for realizable schemes. For details one can refer to [AC1], [AC2]. For the convenience of the reader, we collect known results. Let $f: X \to Y$ be a morphism in $\text{Real}(k/R)$. Then we have the triangulated functors

$$
f_!, f_+ : D^b_{\text{hol}}(X/K) \to D^b_{\text{hol}}(Y/K), \quad f^!, f^+ : D^b_{\text{hol}}(Y/K) \to D^b_{\text{hol}}(X/K).
$$

These functors satisfy the following fundamental properties of six functor formalism:

1. $D^b_{\text{hol}}(X/K)$ is a closed symmetric monoidal category, namely it is equipped with tensor product $\otimes$ and the unit object $K_0^F$ forming a symmetric monoidal category (cf. [KSc 4.2.17]), and $\otimes$ has the left adjoint functor $\text{Hom}$ called the internal hom. (cf. [AC1 1.1.6, Appendix])

2. $f^+$ is monoidal, namely it commutes with $\otimes$ and preserves the unit object. Moreover, given composable morphisms $f$ and $g$, there exists a canonical transition isomorphism $(f \circ g)^+ \cong g^+ \circ f^+$. With this pull-back and transition, we associate $D^b_{\text{hol}}(X/K)$ to $X \in \text{Real}(k/R)$, and we have the fibered category over $\text{Real}(k/R)$. (cf. [AC1 1.3.14], checking of the category being fibered readily follows from the construction of the functor.)

3. $(f^+, f_+)$ and $(f^!, f_!)$ are adjoint pairs. (cf. [AC1 1.3.14 (viii)])
4. We have a morphism of functors $f_i \to f_+$ compatible with transition isomorphisms of composition. This morphism is an isomorphism when $f$ is proper. (cf. \textbf{AC1} 1.3.7, 1.3.14 (vi))

5. When $j$ is an open immersion, there exists the isomorphism $j^+ \cong j'$ compatible with transition isomorphism of the composition of two open immersions. (cf. \textbf{V2} I.3.5)

6. Let $\text{Vec}_K$ be the abelian category of $K$-vector spaces, and denote by $D^b_{\text{fin}}(\text{Vec}_K)$ the derived category consisting of bounded complexes whose cohomologies are finite dimensional. There exists a canonical equivalence of monoidal categories $\mathbb{R}\Gamma : D^b_{\text{hol}}(\text{Spec}(k)/K) \xrightarrow{\sim} D^b_{\text{fin}}(\text{Vec}_K)$. For $X \to \text{Spec}(k)$ in $\text{Real}(k/R)$, we put $\mathbb{R}\text{Hom}(-,-) := \mathbb{R}\Gamma \circ f_+ \circ \text{Hom}(-,-)$.

7. Consider the following cartesian diagram of schemes:

$$
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow f' & & \downarrow f \\
Y' & \xrightarrow{g} & Y.
\end{array}
$$

Assume schemes to be realizable. Then we have a canonical isomorphism $g^+ f_i \cong f'_i g'^+$ compatible with compositions. When $f$ is proper (resp. open immersion), this isomorphism is the base change homomorphism defined by the adjointness of $(f^+, f_+) \text{ (resp. } (f, f'))$ via the isomorphism $[\mathbf{4}]$ (resp. \textbf{9}). (cf. \textbf{AC1} 1.3.14 (vii))

8. We have canonical isomorphisms (cf. \textbf{AC1} Appendix)

$$
f_i \mathcal{F} \otimes \mathcal{G} \cong f_i(\mathcal{F} \otimes f^+ \mathcal{G}), \quad \text{Hom}(f_i \mathcal{F}, \mathcal{G}) \cong f_+ \text{Hom}(\mathcal{F}, f^! \mathcal{G}), \quad f^! \text{Hom}(\mathcal{F}, \mathcal{G}) \cong \text{Hom}(f^+ \mathcal{F}, f^! \mathcal{G}).
$$

Before recalling several more properties, let us show the following lemma:

**Lemma.** — Let $\iota : X \to X'$ be a universal homeomorphism in $\text{Real}(k/R)$. Then the adjoint pair $(\iota^+, \iota')$ induces an equivalence between $D^b_{\text{hol}}(X/K)$ and $D^b_{\text{hol}}(X'/K)$, and we have a canonical isomorphism $\iota^+ \cong \iota'$. Moreover assume the following commutative diagram where $\iota$ and $\iota'$ are universal homeomorphisms:

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow \iota & & \downarrow \iota' \\
X' & \xrightarrow{f'} & Y'.
\end{array}
$$

Then $f^!(\iota), f^!(\iota'), f^{!+},$ and $f^{!\dagger}$ commute canonically with $\iota^{!+} \cong \iota'^{\dagger}$.

**Proof.** The first equivalence is nothing but \textbf{AC1} 1.3.12. Since $\iota_1 \simeq \iota_+$ by \textbf{4} above, we have $\iota^+ \cong (\iota_1)^{-1} \cong (\iota_+)^{-1} \cong \iota'^{\dagger}$. Commutation results follows by transitivity of push-forwards or pull-backs.

This result can be applied in particular when $f$ is the relative Frobenius morphism (see remark below). We need a few more properties, which may not be regarded as standard properties of six functor formalism:

9. For $X$ in $\text{Real}(k/R)$, let $X^\sigma := X \otimes_{k, \sigma} k$. Then we have a pull-back $\sigma^* : D^b_{\text{hol}}(X/K) \xrightarrow{\sim} D^b_{\text{hol}}(X^\sigma/K)$, which is exact, and all the cohomological functors commute canonically with this pull-back. (This follows easily from the definition of the cohomological functors. See also \textbf{Ber3} 4.5.)

Now, for a separated scheme of finite type over $k$, we denote by $\text{Isoc}^!(X/K)$ the thick full subcategory of the category of overconvergent isocrystals on $X$ generated by those which can be endowed with $s'$-th Frobenius structure for some $s|s'$. Caution that the notation is slightly different from the standard one in \textbf{Ber1} 2.3.6.
10. Let $X$ be a realizable scheme such that $X_{\text{red}}$ is a smooth realizable scheme of dimension $d$: $\pi_0(X) \rightarrow \mathbb{N}$ (cf. [0.0.3]). Then there exists a fully faithful functor $\text{sp}_+: \text{Isoc}^\downarrow(X/K) \rightarrow \text{Hol}(X/K)$ called the specialization functor. We denote the essential image of $\text{sp}_+$ shifted by $-d$ by $\text{Sm}(X/K) \subset \text{Hol}(X/K)[-d] \subset D^b_{\text{hol}}(X/K)$. (cf. [Ca2])

**Remark.** — Let $F: X \rightarrow Y$ be the $s$-$t$ absolute Frobenius endomorphism. Combining [7] and the lemma above applied to the $s$-$t$ relative Frobenius morphism $F_{X/k}: X \rightarrow X^s$, we get an equivalence of categories

$$F^* := F^s_{X/k} \circ \sigma^*: D^b_{\text{hol}}(X/K) \rightarrow D^b_{\text{hol}}(X/K).$$

This pull-back is nothing but the one used in [Ber3, Definition 4.5]. For a cohomological functor $C: D^b_{\text{hol}}(X/K) \rightarrow D^b_{\text{hol}}(Y/K)$, we say that $C$ commutes with Frobenius pull-back if there exists a “canonical” isomorphism $C \circ F^* \cong F^* \circ C$.

It is convenient to introduce the following additional cohomological functors: Let $p: X \rightarrow \text{Spec}(k)$ be the structural morphism of a realizable scheme. We put $K_X := p^*(K)$ and call it the dualizing complex. We put $\mathbb{D}_X := \mathcal{H}om(-, K_X^\vee)$, and call it the dual functor. Then it is known that the canonical homomorphism $\text{id} \rightarrow \mathbb{D}_X \otimes \mathcal{O}_X$ is an isomorphism (cf. [V2 II.3.5]). Finally, for realizable schemes $X_1$, $X_2$ and $\mathcal{M}_1 \in D^b_{\text{hol}}(X_1)$, $\mathcal{M}_2 \in D^b_{\text{hol}}(X_2)$, we put $\mathcal{M}_1 \boxtimes \mathcal{M}_2 := p_1^*(\mathcal{M}_1) \boxtimes p_2^*(\mathcal{M}_2)$ where $p_1: X_1 \times X_2 \rightarrow X_1$ denotes the $i$-th projection.

In the current formalism, cycle class map is missing. We shall construct trace maps and a cycle class formalism in the coming subsection, which are important to show the $\ell$-independence type result.

1.1.4. Consider the cartesian diagram (1.1.3.1). We assume that the schemes are realizable. Then we define the base change homomorphism $g^+ \circ f^1 \rightarrow f^d \circ g^+$ to be the adjunction of the following composition:

$$f^!_1 \circ g^+ \circ f^1 \xrightarrow{\sim} g^+ \circ f^!_1 \circ f^1 \xrightarrow{\text{adj}_{f^!}} g^+.$$ 

By definition, the following diagram is commutative, which we will use later:

$$g^+ f^! f^1 \xrightarrow{\sim} f^!_1 g^+ f^1 \xrightarrow{\text{adj}_{f^!}} f^!_1 f^+ g^+.$$ 

1.1.5. Let $f^{(i)}: X^{(i)} \rightarrow Y^{(i)}$ be a morphism of realizable schemes, and $\mathcal{M}^{(i)}$ in $D^b_{\text{hol}}(X^{(i)}/K)$. We have the canonical isomorphism $(f \times f')^+((-) \boxtimes (-)) \cong f^+(-) \boxtimes f'^+(-)$ since $f^+$ and $f'^+$ are monoidal. By taking the adjoint, we have a homomorphism

$$f_+(\mathcal{M}) \boxtimes f'^+_+(\mathcal{M}') \rightarrow (f \times f')_+(\mathcal{M} \boxtimes \mathcal{M}').$$

**Proposition.** — This homomorphism is an isomorphism.

**Proof.** When $f$ and $f'$ are immersions, the proposition is essentially contained in the proof of [AC1] 1.3.3 (i). Thus, we may assume $f$ and $f'$ to be smooth and $X$, $X'$ can be lifted to proper smooth formal schemes. In this situation, we have the canonical isomorphism $f^!(-) \boxtimes f'^!(-) \cong (f \times f')^!((-) \boxtimes (-))$, and by taking the adjunction of [V1], we have the homomorphism $\rho: (f \times f')_+(\mathcal{M} \boxtimes \mathcal{M}') \rightarrow f_+(\mathcal{M}) \boxtimes f'^+_+(\mathcal{M}')$.

The homomorphism in the statement is the dual of this homomorphism. Thus it suffices to show that $\rho$ is an isomorphism. Let $f^{(i)}_n: X^{(i)}_n \rightarrow Y^{(i)}_n$ be a smooth morphism of relative dimension $d^{(i)}_n$ between proper smooth schemes over $R/\pi^{n+1}$ where $\pi$ is a uniformizer of $R$. For bounded coherent $\mathcal{O}^{(m)}$-modules (thus perfect) on $X^{(i)}_n$ and $Y^{(i)}_n$, have the homomorphism $\rho_n: (f_n \times f'_n)_+((-) \boxtimes (-)) \rightarrow f_{n+}(-) \boxtimes f'_{n+}(-)$ by similar construction to $\rho$, and it suffices to show that the latter homomorphism is an isomorphism. By [H VII.4.1], the following diagram commutes:

$$R^d f_*(\omega_{X_n/Y_n}) \boxtimes R^d f'_*(\omega_{X'_n/Y'_n}) \rightarrow \mathcal{O}_{Y_n} \boxtimes \mathcal{O}_{Y'_n}$$

$$R^{d+d'} (f_n \times f'_n)_*(\omega_{X_n \times X'_n/Y_n \times Y'_n}) \rightarrow \mathcal{O}_{Y_n \times Y'_n}$$

(1) In [Ca2], the essential image is denoted by $\text{Isoc}^\uparrow(X/K)$. 


where the vertical homomorphisms are trace maps, and $\omega$ denotes the canonical bundle sheaf. The commutativity shows that $\rho_n$ is nothing but the homomorphism induced by the isomorphism

$$\mathcal{D}_{Y_n \times Y_n \leftarrow X_n \times X_n}^{(m)} \cong \mathcal{D}_{Y_n \leftarrow X_n}^{(m)} \boxtimes \mathcal{D}_{Y_n \leftarrow X_n}^{(m)}$$

(cf. [A1 Lemma 4.5 (ii)]), and we get the proposition by using the Künneth formula for quasi-coherent sheaves. ■

**1.1.6.** Finally, we recall the following result:

**Theorem ([AC2]).** — Let $X$ be a realizable scheme. Then the canonical functor $D^b(\text{Hol}(X/K)) \to D^b_{\text{hol}}(X/K)$ induces an equivalence of triangulated categories.

### 1.2. Ind-categories

**1.2.1 Lemma.** — Let $A$, $B$ be abelian categories, and assume $A$ has enough injective objects. Let $F: A \to B$ be a left exact functor, and assume that we have an adjoint pair $(G, F)$ such that $G$ is exact. Then for $M \in D^+(A)$ and $N \in D(B)$, we have

$$\text{Hom}_{D^b(A)}(G(N), M) \cong \text{Hom}_{D^b(B)}(N, RF(M)).$$

**Proof.** By the exactness of $G$, $F$ sends injective objects to injective objects. Thus, for $I^* \in C^+(A)$ consisting of injective objects and $N^* \in C(B)$, it suffices to show that

$$\text{Hom}_{K^b(A)}(G(N^*), I^*) \cong \text{Hom}_{K^b(B)}(N^*, F(I^*)).$$

By the adjointness, we have $\text{Hom}^*(G(N^*), I^*) \cong \text{Hom}^*(N^*, F(I^*))$ in $C(\text{Ab})$ where $\text{Hom}^*$ is the functor defined in [H I.§6]. Since $\text{Hom}_{K^b(A)} = \mathcal{R}\text{Hom}^*$, we get the isomorphism. ■

**Remark.** — The proof shows that if moreover $B$ has enough injectives, we have

$$\mathcal{R}\text{Hom}_{D^b(A)}(G(N), M) \cong \mathcal{R}\text{Hom}_{D^b(B)}(N, RF(M)).$$

**1.2.2.** Let us collect some facts on Ind-categories. Let $A$ be a category. Let $A^\wedge$ be the category of presheaves on $A$, and $h_A: A \to A^\wedge$ be the canonical embedding. Then $\text{Ind}(A)$ is the full subcategory of $A^\wedge$ consisting of objects which can be written as filtrant small inductive limit of the image of $h_A$. By definition, $h_A$ induces a functor $\iota_A: A \to \text{Ind}(A)$. We sometimes abbreviate this as $\iota$. Since $h_A$ is fully faithful by the Yoneda lemma [KS 1.4.4], $\iota_A$ is fully faithful as well. For the detail see [KS §6].

Now, we assume that $A$ is an abelian category. We have the following properties:

1. The category $\text{Ind}(A)$ is abelian, and the functor $\iota_A$ is exact. Moreover, $\text{Ind}(A)$ admits small inductive limits, and small filtrant inductive limits are exact. (cf. [KS 8.6.5])

2. Assume $A$ to be essentially small. Then $\text{Ind}(A)$ is a Grothendieck category, and in particular, it possesses enough injectives, and admits small projective limits. (cf. [KS 8.6.5, 9.6.2, 8.3.27])

3. The category $A$ is a thick subcategory of $\text{Ind}(A)$ by [KS 8.6.11]. This in particular shows that any direct factor of objects of $A$ is in $A$, since direct factor is the kernel of a projector.

4. Let $X_i: I \to A$ be an inductive system. Since $\iota_A$ is fully faithful, if $\lim \iota_A(X_i)$ is in the essential image of $\iota_A$, then $\lim X_i$ exists in $A$ and $\iota_A(\lim X_i) \cong \lim \iota_A(X_i)$.

Now, let $F: A \to B$ be an additive functor between abelian categories. Then it extends uniquely to an additive functor $IF: \text{Ind}(A) \to \text{Ind}(B)$ such that $IF$ commutes with arbitrary small filtrant inductive limits by [KS 6.1.9]. Since small direct sum can be written as a filtrant inductive limit of finite sums, $IF$ commutes with small direct sums as well. We have the following more properties:

5. If $F$ is left (resp. right) exact, so is $IF$. (cf. [KS 8.6.8])

6. Let $G: B \to C$ be another additive functor between abelian categories. Then $IG \circ IF \cong I(G \circ F)$. (cf. [KS 6.1.11])

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If there is nothing to be confused, by abuse of notation, we denote $IF$ simply by $F$.

**Remark.** — In general $i_A$ does not commute with inductive limits (cf. [KSc 6.1.20]), and in [KSc], inductive limits in $\text{Ind}(A)$ are distinguished by using “$\text{lim}$”. In this paper, we simply denote this limit by $\text{lim}$ if no confusion can arise, and when we use inductive limits, it is understood to be taken in $\text{Ind}(A)$, not in $A$, unless otherwise stated.

1.2.3 Lemma. — Let $A, B$ be abelian categories, and assume that $B$ admits small filtrant inductive limits. Then the restriction functor yields an equivalence $\text{Fct}^{\text{add}}(\text{Ind}(A), B) \rightarrow \text{Fct}^{\text{add}}(A, B)$ where the target (resp. source) is the category of additive functors (resp. which commute with small filtrant inductive limits).

Proof. This is a reorganization of [KSc 6.3.2], or see [SGA 4, I, 8.7.3]. The quasi-inverse is the functor sending $F$ to $\sigma_B \circ IF$ where $\sigma_B: \text{Ind}(B) \rightarrow B$ is the functor taking the inductive limit (cf. [KSc 6.3.1]).

1.2.4. Let $A, B$ be abelian categories, and assume moreover that $A$ is a noetherian category (i.e. the category is essentially small and all the objects are noetherian cf. [Ga III.4]). Let $f_*: A \rightarrow B$ be a left exact functor. Recall that $\text{Ind}(A)$ has enough injectives (cf. [1.2.2]). Thus, $I_{f_*}$ can be derived to get $\mathbb{R}f_*: D^+(\text{Ind}(A)) \rightarrow D^+(\text{Ind}(B))$, by abuse of notation. We recall that the canonical functor $\omega_A: D^b(A) \rightarrow D^B_+(\text{Ind}(A))$ gives an equivalence by [KSc 15.3.1], and the same for $B$.

Lemma. — Let $f_*: A \rightarrow B$ be a left exact functor as above. Then for any integer $i \geq 0$, $\mathbb{R}^if_*: \text{Ind}(A) \rightarrow \text{Ind}(B)$ commutes with arbitrary small filtrant inductive limit.

Proof. Since we are assuming $A$ to be noetherian, $\text{Ind}(A)$ is locally noetherian category (cf. [Ga III.4]), and by [Ga I.4, Cor 1 of Thm 1], small filtrant inductive limits of injective objects in $\text{Ind}(A)$ remain to be injective. Thus, we may apply [KSc 15.3.3] to conclude the proof.

1.2.5. Recall that a $\delta$-functor $\{f^i\}$ between abelian categories is called the right satellite of $f^0$ if $f^i = 0$ for $i < 0$, and it is universal among them (cf. [Ga 2.2]).

Lemma. — The composition functor $\{\mathbb{R}^if_* \circ i\}_*: A \rightarrow \text{Ind}(A) \rightarrow \text{Ind}(B)$ is the right satellite functor of $I_{f_*} \circ i_A \cong i_B \circ f_*$. 

Proof. Since $\{\mathbb{R}^if_* \circ i\}_*$ is a right $\delta$-functor, it remains to show that it is universal. Let $\{G^i\}_*: A \rightarrow \text{Ind}(B)$ be a right $\delta$-functor with a morphism of functors $i_B \circ f_* \rightarrow G^0$. By Lemma [1.2.3], this extends uniquely to a collection of functors $\{G^i\}_*: \text{Ind}(A) \rightarrow \text{Ind}(B)$. Then $\{G^i\}_*$ is a right $\delta$-functor as well by [KSc 8.6.6], with a morphism $f_* \rightarrow G^0$. By the universal property of $\{\mathbb{R}^if_*\}_*$, we get a morphism $\{\mathbb{R}^if_* \circ i\}_* \rightarrow \{G^i\}_*$. This induces the morphism $\varphi: \{\mathbb{R}^if_* \circ i\}_* \rightarrow \{G^i\}_*$. Now, any morphism $\mathbb{R}^if_* \circ i \rightarrow G^i$ extends uniquely to $\mathbb{R}^if_* \rightarrow G^i$ by Lemma [1.2.4] and [1.2.3]. Thus the uniqueness of $\varphi$ follows, and we conclude that the $\delta$-functor is universal.

1.2.6 Lemma. — Let $f^*: B \rightarrow A$ be an exact functor such that $(f^*, f_*)$ is an adjoint pair. Assume given a functor $f_+: D^b(A) \rightarrow D^b(B)$ such that $(f^+, f_*)$ is an adjoint pair. Then $f_+ \cong \mathbb{R}f_* \circ i \circ D^b(A)$.

Proof. First, let us show that for $X \in A$, $\mathcal{H}^i f_+(X) = 0$ for $i < 0$. If $f_+(X) \neq 0$, by boundedness condition, there exists an integer $d$ such that $\mathcal{H}^d f_+(X) \neq 0$ and $\mathcal{H}^i f_+(X) = 0$ for $i < d$. Assume $d < 0$. Then for any $Y \in B$, we have

$$\text{Hom}_B(Y, \mathcal{H}^d f_+(X)) \cong \text{Hom}_{D(B)}(Y, f_+(X)[d]) \cong \text{Hom}_{D(A)}((f^*)^*(Y), X[d]) = 0$$

where the last equality holds since $\mathcal{H}^i(X[d]) = 0$ for $i \leq 0$. This contradicts with the assumption, and thus, $\mathcal{H}^i f_+(X) = 0$ for $i < 0$. In the same way, we get that $(f^+, \mathcal{H}^0 f_+)$ is an adjoint pair, and in particular, $\mathcal{H}^0 f_+ \cong f_+$.

This shows that the collection of functors $\{\mathcal{H}^i f_+\}$ is a (right) $\delta$-functor. Since $\{\mathcal{H}^i f_+\}$ is a $\delta$-functor from $A$ to $B$ with the isomorphism $f_* \cong \mathcal{H}^0 f_+$, Lemma [1.2.5] yields a homomorphism $\{\mathbb{R}^if_* \circ i\}_* \rightarrow \{\mathcal{H}^i f_+\}_*$ of $\delta$-functors. For $X \in A$, we have

$$\text{Hom}_{D(\text{Ind}(B))}(f_+(X), \mathbb{R}^if_* f_+(X)) \cong \text{Hom}_{D(\text{Ind}(A))}(f^* f_+(X), f_+(X)) \cong \text{Hom}_{D(A)}((f^*)^*(f_+(X)), f_+(X)) \cong \text{Hom}_{D(A)}(f^*(f_+(X)), f_+(X)) \cong \text{Hom}_{D(\text{Ind}(B))}(f_+(X), f_+(X))$$
where we used the canonical equivalence $D^b(A) \xrightarrow{\sim} D^b_\A$ (Ind($A$)) recalled in [1.2.4]. Thus, the identity of $f_+(X)$ defines a homomorphism $\rho: f_+(X) \to Rf_+(X)$, which induces the isomorphism on $\mathcal{H}^0$. By the universal property of satellite functor, the composition $\{R^if_+(X)\} \to \{\mathcal{H}^if_+(X)\}$ is the identity, which shows that $R^if_+(X)$ is a direct factor of $\mathcal{H}^if_+(X)$. This shows that $R^if_+(X)$ is in $\mathcal{B}$ by [1.2.2.1] and $R^if_+(X) = 0$ for $i \gg 0$, which means that $Rf_+(X)$ is in $D^b_\B(\text{Ind}(\mathcal{B})) \xleftarrow{\sim} D^b(\mathcal{B})$. Thus $Rf_+$ induces a functor from $D^b(A)$ to $D^b(\mathcal{B})$. For any $Y \in D^b(\mathcal{B})$, we have

$$\text{Hom}_{D^b(\mathcal{B})}(Y, Rf_+(X)) \cong \text{Hom}_{D^b(A)}(f^+(Y), X) \cong \text{Hom}_{D^b(\mathcal{B})}(Y, f_+(X)).$$

Thus $Rf_+(X) \xrightarrow{\sim} f_+(X)$ as required.

**1.2.7.** Now, let us apply the preceding general results to the theory of arithmetic $\mathcal{D}$-modules. First, we need:

**Lemma.** — For a realizable scheme $X$, the category $\text{Hol}(X/K)$ is a noetherian and artinian.

**Proof.** Let $X$ be a realizable variety. Then the category $\text{Hol}(X/K)$ is essentially small. Indeed, to check this, it suffices to show that for a smooth formal scheme $\mathcal{X}$, the category of coherent $\mathcal{D}_{\mathcal{X}, \mathcal{O}}^-$-modules is essentially small. The verification is standard. Now, any object of $\text{Hol}(X/K)$ is noetherian and artinian by \[AC2\] 1.3.

**Definition.** — For a realizable scheme $X$, we put $M(X/K) := \text{Ind}(\text{Hol}(X/K))$. This is a Grothendieck category by [1.2.2.2] and the lemma above.

**1.2.8.** Let $\phi: X \to Y$ be a smooth morphism of relative dimension $d$ between realizable schemes. We have the following functors via the equivalence of Theorem [1.1.6] compatible with Frobenius pull-backs:

$$\phi_+[-d]: D^b(\text{Hol}(X/K)) \xrightarrow{\sim} D^b(\text{Hol}(Y/K)) : \phi^+[d].$$

**Lemma.** — We have an adjoint pair $(\phi^+[d], \phi_+[-d])$, and $\phi^+[d]$ is exact. The adjunction map is compatible with Frobenius pull-backs.

**Proof.** Since $(\phi^+, \phi_+)$ is an adjoint pair, the adjointness follows. The exactness is by [AC1] 1.3.2 (i).

Now, we put $\phi_* := \mathcal{H}^0(\phi_+[-d])$, $\phi_* := \mathcal{H}^0(\phi^+[d])$. We have the right derived functor $R\phi_* : D^+(M(X/K)) \to D^+(M(Y/K))$. By Lemma [1.2.6] together with Lemma [1.2.7], $\phi_+[-d]$ is the right derived functor of $\phi_*$, namely, $\phi_+[-d] \cong R\phi_*$ on $D^b(\text{Hol}(X/K))$, which is a full subcategory of $D^+(M(X/K))$.

Let $f: X \to Y$ be a smooth morphism which may not be equidimensional. Then there exists a decomposition $X = \bigsqcup X_i$ where $X_i$ is an open subscheme of $X$ such that the induced morphism $f_i: X_i \to Y$ is equidimensional. We put $f_* := \sum f_{i*}$ and $f^* := \sum f^*_i$. Note that when $\phi$ is an open immersion, then we have $R\phi_* \cong \phi_*$.}

**1.2.9.** Let $f: X \to Y$ be a finite morphism between realizable schemes. Consider the functors

$$f_* \cong f_+ : D^b(\text{Hol}(X/K)) \xrightarrow{\sim} D^b(\text{Hol}(Y/K)) : f^!.$$

We have the following:

**Lemma.** — Then $f_+ \cong f_!$ are exact and $(f_+, f^!)$ is an adjoint pair compatible with Frobenius pull-backs.

**Proof.** The exactness is by [AC1] 1.3.13, and the other claims follows by [1.2.6].

Now, we have the right derived functor $R(\mathcal{H}^0f^!): D^+(M(X/K)) \to D^+(M(Y/K))$. By Lemma [1.2.6] together with Lemma [1.2.7] we have $f^! \cong R(\mathcal{H}^0f^!)$ on $D^b(\text{Hol}(Y/K))$. 

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1.2.10. Let $X$, $Y$ be realizable schemes, and consider the projections $p: X \times Y \to Y$, $q: X \times Y \to X$. Let $\mathcal{A}$ be an object in $\text{Hol}(X/K)$. We have functors:

$$p_{\mathcal{A}^+}(-) := p_+ \text{Hom}(q^+ \mathcal{A}, -): D^b(\text{Hol}(X \times Y/K)) \to D^b(\text{Hol}(Y/K)): \mathcal{A} \boxtimes (-) =: p^+_\mathcal{A}. $$

Now, assume that $\mathcal{A}$ is endowed with Frobenius structure $\mathcal{A} \overset{\sim}{\longrightarrow} F^* \mathcal{A}$. Then we have an isomorphism of functors $F^* \circ p_{\mathcal{A}^+} \cong p_{\mathcal{A}^+} \circ F^*$, and $F^* \circ p_+ \mathcal{A} \cong p^+_\mathcal{A} \circ F^*$. Thus, $p_{\mathcal{A}^+}$ and $p^+_\mathcal{A}$ are compatible with Frobenius pull-backs. We have:

**Lemma.** — The functor $p^+_\mathcal{A}$ is exact, and $(p_{\mathcal{A}^+}, p^+_\mathcal{A})$ is an adjoint pair. Moreover, if $\mathcal{A}$ is endowed with Frobenius structure, the pair is compatible with Frobenius pull-backs.

**Proof.** The exactness of $p^+_\mathcal{A}$ follows from [AC1, 1.3.3 (ii)]. By definition [AC1, 1.1.8 (i)], we have $q^+ \mathcal{A} \otimes p^+(-) \cong \mathcal{A} \boxtimes (-)$. Thus, we get

$$\text{Hom}_{X \times Y}(p^+_\mathcal{A}(-), -) \cong \text{Hom}_{X \times Y}(q^+ \mathcal{A} \otimes p^+(-), -) \cong \text{Hom}_{X \times Y}(p^+(-), \text{Hom}(q^+ \mathcal{A}, -))$$

$$\cong \text{Hom}_{Y}(-, p_+ \text{Hom}(q^+ \mathcal{A}, -)),$$

where the second and the last isomorphism holds by the adjunction properties (cf. 1.13).

We put $p^*_\mathcal{A} := \mathcal{H}^0 p_{\mathcal{A}^+}$, $p^+_\mathcal{A} := \mathcal{H}^0 p^+_\mathcal{A}$. Once again, we get $p_{\mathcal{A}^+} \cong \mathbb{R}p^*_\mathcal{A}$ on $D^b(\text{Hol}(X \times Y/K))$.

1.2.11 Lemma. — Let $X$ be a realizable scheme, $j: U \to X$ be an open immersion, and $i: Z \to X$ be its complement. For an injective object $\mathcal{I}$ in $\text{M}(X/K)$, we have an exact sequence

$$0 \to \mathcal{I}^0 i^+_+(\mathcal{I}) \to \mathcal{I} \to \mathcal{I}^0 j^+_+(\mathcal{I}) \to 0.$$ 

**Proof.** Let us put

$$F := \text{Coker}(\text{id} \to \mathcal{H}^0(j^+_+(\mathcal{I})): \text{Hol}(X/K) \to \text{Hol}(Y/K)).$$

We claim that $\mathbb{R}^1(\mathcal{H}^0 i^+_+(\mathcal{I}))(\mathcal{M}) \cong IF(\mathcal{M})$, since $\mathcal{H}^0 i^+_+(\mathcal{I})$ is left exact and $i^+_+$ is exact, we have $\mathbb{R}^1(\mathcal{H}^0 i^+_+(\mathcal{I})) \cong i^+_+ \mathbb{R}^1(\mathcal{H}^0 i^+_+(\mathcal{I}))$. For $\mathcal{M} \in \text{Hol}(X/K)$, we have the isomorphism by Lemma 1.2.3 and the localization triangle. Lemma 1.2.7 and Lemma 1.2.8 shows that the functor $\mathbb{R}^1(i^+_+\mathcal{I}^i)$ commutes with small filtrant inductive limits. Thus, by Lemma 1.2.3, this isomorphism uniquely extends to the isomorphism we want. This shows that for an injective object $\mathcal{I}$, $IF(\mathcal{I}) = 0$, and we get the short exact sequence in the statement of the lemma.

1.3. Constructible t-structures

We need to introduce a t-structure on $D^b_{\text{hol}}(X/K)$ whose heart corresponds to the category of “constructible sheaves” in the philosophy of Riemann-Hilbert correspondences.

1.3.1. Let $X$ be a realizable scheme. For $\mathcal{M} \in \text{Hol}(X/K)$ we define the support, denoted by $\text{Supp}(\mathcal{M})$, to be the smallest closed subset $Z \subset X$ such that $\mathcal{M}$ is 0 if we pull-back to $X \setminus Z$. When $X_{\text{red}}$ is smooth of dimension $d$, we say that a complex $\mathcal{M} \in D^b_{\text{hol}}(X/K)$ is smooth if $\mathcal{H}^i(\mathcal{M})[-d]$ is in $\text{Sm}(X/K)$ for any $i$ (cf. [1.1.3 (10]).

Now, we define the following two full subcategories of $D^b_{\text{hol}}(X/K)$:

- $^cD^{\geq 0}$ consists of complexes $\mathcal{M}$ such that $\dim(\text{Supp}(\mathcal{H}^n(\mathcal{M}))) \leq n$ for any $n \geq 0$.
- $^cD^{\leq 0}$ consists of complexes $\mathcal{M}$ such that $\mathcal{H}^k i^+_W(\mathcal{M}) = 0$ for any closed subscheme $i_W: W \to X$ and $k > \dim(W)$.

We note that the extension property holds, namely, for a triangle $\mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \xrightarrow{+1} $, if $\mathcal{M}'$ and $\mathcal{M}''$ are in $^cD^{\ast}(X)$ ($\ast \in \{0, \leq 0\}$) then so is $\mathcal{M}$.

**Example.** — Let $X$ be a smooth curve. Then $^cD^{\geq 0}$ consists of complexes $\mathcal{M}$ such that $\mathcal{H}^i(\mathcal{M}) = 0$ for $i < 0$, and $\mathcal{H}^0(\mathcal{M})$ is supported on finite union of points. The category $^cD^{\leq 0}$ consists of complexes $\mathcal{N}$ such that $\mathcal{H}^i(\mathcal{N}) = 0$ for $i > 1$, and $\mathcal{H}^0 i^+_x(\mathcal{H}^1(\mathcal{M})) = 0$ for any closed point $x$. For example, $i^+_x(K)$ and $K_X(\geq sp_+(O_X,q)[-1])$ are in both $^cD^{\geq 0}$ and $^cD^{\leq 0}$. For a smooth realizable scheme $X$, any object of $\text{Sm}(X/K)$ (cf. [1.1.3 (10)] is in both $D^{\geq 0}$ and $D^{\leq 0}$. This can be checked by the right exactness of $i^+$ (cf. [AC1, 1.3.2 (ii)]).
1.3.2 Lemma. — Let $i : Z \hookrightarrow X$ be a closed immersion, and $j : U \hookrightarrow X$ be its complement. Then $i^+,$ $j_+,$ $j^+$ all preserve both $\mathcal{D}^{\geq 0}$ and $\mathcal{D}^{\leq 0}.$

Proof. Since $i^\star \cong i_+$ and $j^+$ are exact by [AC1, 1.3.2], the verification is easy. Let us show the preservation for $i^+.$ Since the verification is Zariski local with respect to $X,$ we may assume that $X$ is affine. Then the verification is reduced to the case where $Z$ is defined by a function $f \in \mathcal{O}_X.$ In this case, we know that for any $\mathcal{M} \in \mathcal{H}(X),$ $\mathcal{H}^k i^+ \mathcal{M} = 0$ for $i \neq 0,$ $-1.$

Since $i^+_W$ is right exact by [AC1, 1.3.2 (ii)], the preservation for $\mathcal{D}^{\leq 0}$ is easy. Let us show the preservation for $\mathcal{D}^{\geq 0}.$ By the extension property, it suffices to check for $\mathcal{M}$ of the form $\mathcal{M} = \mathcal{N}[-n]$ such that $\mathcal{N} \in \mathcal{H}(X)$ and $\dim(\text{Supp}(\mathcal{N})) \leq n.$ By using the extension property again, we are reduced even to the case where $\mathcal{N}$ is irreducible. In particular, we may assume that the support of $\mathcal{N}$ is irreducible. In this case, we have two possibilities: $\text{Supp}(\mathcal{N}) \subset Z$ or $\text{Supp}(\mathcal{N}) \not\subset Z.$ When $\text{Supp}(\mathcal{N}) \subset Z,$ we get $\mathcal{H}^{-1} i^+ (\mathcal{M}) = 0,$ and the other case follows since $\dim(\text{Supp}(\mathcal{N}) \cap Z) < \dim(\text{Supp}(\mathcal{N})).$

Let us show the lemma for $j^+$ by using the induction on the dimension of $X.$ When $j$ is affine, the claim follows easily since $j^+$ is exact by [AC1, 1.3.13]. In general, take $\mathcal{M} \in \mathcal{D}^+(U).$ Let $j^+ : V \hookrightarrow U$ be an affine open dense subscheme, and $j'$ be the closed immersion into $U$ defined by the complement. Consider the triangle $\xymatrix{ j^+_j j^+ \mathcal{M} \ar[r] & \mathcal{M} \ar[r] & i^+_c \mathcal{M} \ar[r] & }.$ Since $j \circ j'$ is affine, $j^+_j j^+ \mathcal{M}$ is in $\mathcal{D}^+(X),$ and $j^+_c \mathcal{M}$ is in $\mathcal{D}^+(X)$ as well by the induction hypothesis and the lemma for $i^+$ we have already treated. Using the extension property, we conclude.

1.3.3 Proposition. — The categories $\mathcal{D}^{\geq 0}$ and $\mathcal{D}^{\leq 0}$ define a t-structure on $\mathcal{D}^b_{\mathcal{H}}(X/K).$

Proof. We put $D(X) := D^b_{\mathcal{H}}(X/K).$ Let $U$ be an open subset of $X$ and $Z$ be its complement. Put $i : Z \hookrightarrow X$ and $j : U \hookrightarrow X.$ For $* \in \{0, \leq, \geq\},$ $\mathcal{M}$ is in $\mathcal{D}^*(X)$ if and only if $i^* (\mathcal{M})$ and $j^* (\mathcal{M})$ are in $\mathcal{D}^*(Z)$ and $\mathcal{D}^*(U)$ respectively. This follows by the extension property, and Lemma [KW, p.143].

Now, we proceed as [KW, p.143]. We use the induction on the dimension of $X.$ We may assume $X$ to be reduced by Lemma [1.3.2]. It suffices to check, for a smooth open affine subscheme $j : U \hookrightarrow X$ equidimensional of dimension $U,$ the restriction of $\mathcal{D}^{\geq 0}$ and $\mathcal{D}^{\leq 0}$ to the subcategory $T(X,U) := \{ \mathcal{E} \in D(X) \mid \mathcal{H}^i (j^+ \mathcal{E}) \text{ is smooth on } U \text{ for any } i \}$ defines a t-structure, since $\bigcup \mathcal{H}^i (T(X,U)) = D(X).$ Let $i : Z \hookrightarrow X$ be the complement of $U.$ By the observation above, $\mathcal{M} \in T(X,U)$ is in $\mathcal{D}^*(X)$ if and only if $j^* (\mathcal{M})$ and $i^* (\mathcal{M})$ are in $\mathcal{D}^*(U)$ and $\mathcal{D}^*(Z)$ respectively. We note that $\mathcal{D}^*(Z)$ defines a t-structure by induction hypothesis.

Let us check the axioms of t-structure [BBD, 1.3.1]. Axiom (ii) is obvious, and axiom (iii) can be shown by a similar argument to [KW, p.140, 141] using the t-structures of $D(Z).$ Let us check (i). By dévissage using the localization triangle $j^+ \to j^\star \to j^+_1 \to$ two times and by induction hypothesis, we are reduced to showing that $\text{Hom}(i_+ \mathcal{B}, j^+ \mathcal{E}) = 0$ when $\mathcal{B} \in \mathcal{D}^{\leq 0}(Z)$ and $\mathcal{E} \in \mathcal{D}^{\geq 0}(U)$ such that $\mathcal{H}^k (\mathcal{E})$ is smooth for any $i.$ Then, $\mathcal{H}^{k-1} (i_+ \mathcal{B}) = 0$ for $k > \dim(Z).$ On the other hand, $j^+ \mathcal{B}$ is exact by [AC1, 1.3.13] since $j$ is affine, and thus $\mathcal{H}^k (j^+ \mathcal{E}) = 0$ for $k < \dim(U),$ so the claim follows.

Definition. — The t-structure on $\mathcal{D}^b_{\mathcal{H}}(X/K)$ is called the constructible t-structure, and shortly, $c$-t-structure. The heart of the $t$-structure is denoted by $\mathcal{D}(X/K),$ and called the category of constructible modules. The cohomology functor for this t-structure is denoted by $\mathcal{H}^*.$

Remark. — Our constructible t-structure can be regarded as a generalization of “pervasive t-structure” introduced in [Le], and also as a p-adic analogue of the t-structure defined in [Ka].

1.3.4 Lemma. — Let $f : X \to Y$ be a morphism between realizable schemes. Then the following holds.

(i) The functor $f^+$ is $c$-t-exact, and $f_+$ is left $c$-t-exact. Moreover, we have an adjoint pair $(\mathcal{H}^0 f^+, \mathcal{H}^0 f_+).$

(ii) When $f =: i$ is a closed immersion, $i_+$ is $c$-t-exact and $\mathcal{H}^0 i_+$ is left $c$-t-exact. Moreover, the pair $(i_+, \mathcal{H}^0 i_+)$ is adjoint.

(iii) When $f =: j$ is an open immersion, $j^+$ is $c$-t-exact, and the pair $(j^+, j^+_1)$ is adjoint.

Proof. Claims (ii) and (iii) are nothing but Lemma [1.3.2] and we reproduced these for the reminder.

Let us show (i). We only need to show the exactness of $f^+.$ When $f$ is a closed immersion this is Lemma [1.3.2]. The verification is Zariski local, so we may assume $X$ and $Y$ to be affine. By (ii), which has already been verified, it suffices to show the case where $Y$ is smooth and $f$ is smooth. In this case, we can check the $c$-t-exactness of $f^+$ easily using [AC1, 1.3.2 (i)].
1.3.5 Lemma. — Let $X$ be an irreducible realizable scheme. Let $\mathcal{M}$ be a constructible module on $X$ such that $\text{Supp}(\mathcal{M}) = X$. Then there exists an open dense subscheme $j: U \to X$ such that $j^+\mathcal{M}$ is smooth. The rank of $j^+\mathcal{M}$ is called the generic rank of $\mathcal{M}$.

Proof. For any complex in $D_{\text{hol}}^b(X)$, there exists an open dense subscheme $j: U \to X$ such that the cohomology modules of $j^+\mathcal{M}$ is smooth.

1.3.6 Lemma. — Let $X$ be a realizable scheme. Then the category $\text{Con}(X/K)$ is noetherian.

Proof. Since $\text{Hol}(X/K)$ is essentially small by Lemma 1.2.7, so is $D^b(\text{Hol}(X))$. Since $\text{Con}(X)$ is a full subcategory, it is also essentially small.

Let us show that the category is noetherian. It suffices to show it over each irreducible component of $X$, so we may assume $X$ to be irreducible. Assume $X$ is smooth, and let $\mathcal{M}$ be a smooth module on $X$. We claim that any submodule $\mathcal{N}$ of $\mathcal{M}$, there exists an open dense subscheme $U$ such that $\mathcal{N}$ is non-zero smooth on $U$. Assume contrary. Then there exists a nowhere dense closed subset $i: Z \to X$ such that we have non-zero homomorphism $i_*\mathcal{N} \to \mathcal{M}$. Shrinking $X$ if necessary, we may assume that both $X$ and $Z$ are smooth and $\mathcal{N}$ is smooth on $Z$. Taking the adjoint, we get a non-zero homomorphism $\mathcal{N} \to i^+\mathcal{M}$. By [A1] 5.6, we have $i^+\mathcal{M} \cong i^+\mathcal{M}(-d)[-2d]$ where $d$ is the codimension of $Z$ in $X$, which is impossible.

We use the noetherian induction on the support of $\mathcal{M}$. We may assume $X$ is reduced. Moreover, we may assume $\text{Supp}(\mathcal{M}) = X$. Otherwise, we can conclude by the induction hypothesis. Let $\mathcal{M}$ be a constructible module, and let $\{\mathcal{M}_i\}_{i \in \mathbb{N}}$ be an ascending chain of submodules of $\mathcal{M}$. There exists $N$ such that the generic rank of $\mathcal{M}$ is the same for any $i \geq N$. Since it suffices to show that the ascending chain $\{\mathcal{M}_i/\mathcal{M}_N\}_{i \geq N}$ is stationary in $\mathcal{M}/\mathcal{M}_N$, we may assume that $\text{Supp}(\mathcal{M}_i) \subset \text{Supp}(\mathcal{M})$ is nowhere dense. Let $U$ be an open dense smooth subscheme of $X$ such that $\mathcal{M}$ is smooth on $U$. By what we have shown, $\mathcal{M}_i$ is 0 on $U$. Let $i: Z \to X$ be the complement. Then by the $c$-exactness of $i_+$ and $i^+$, and induction hypothesis, $\mathcal{M}_i \cong i_+i^+\mathcal{M}_i$ is stationary in $i_+i^+\mathcal{M}$ as required.

Remark. — Obviously, $\text{Con}(X/K)$ is not artinian, contrary to $\text{Hol}(X/K)$.

1.3.7 Lemma. — Let $X$ be a realizable scheme. For a closed point $x \in X$, denote by $i_x: \{x\} \to X$ the closed immersion.

(i) For $\mathcal{F} \in \text{Con}(X)$, $\mathcal{F} = 0$ if and only if $i_x^+(\mathcal{F}) = 0$ for any closed point $x$. In particular, a homomorphism $\phi$ in $\text{Con}(X)$ is 0 if and only if $i_x^+(\phi) = 0$ for any closed point $x$.

(ii) Let $f: s' \to s$ be a morphism of points (e.g. connected schemes of dimension 0 of finite type over $k$). Then $f^+$ is faithful and conservative.

Proof. For (i), use [AC1] 1.3.11, and (ii) is left to the reader.

1.3.8 Lemma. — Let $f: X \to Y$ be a morphism of realizable schemes such that for any $y \in Y$, the dimension of the fiber $f^{-1}(y)$ is $\leq d$. Then for any $\mathcal{M} \in \text{Con}(X)$, $\mathcal{H}^i f_!(\mathcal{M}) = 0$ for $i > 2d$.

Proof. By Lemma 1.3.7, it suffices to show that for any closed point $y \in Y$, $i_y^+\mathcal{H}^i f_!(\mathcal{M}) = 0$ for $i > 2d$. By the $c$-exactness of $i_y^+$ and base change, it is reduced to showing that $\mathcal{H}^i f_!(\mathcal{M}) = 0$ for $i > 2d$ where $f_y: X \times_Y \{y\} \to \{y\}$. Since over a point, $c$-structure and the usual t-structure coincide, it remains to show that if $X$ is a realizable scheme of dimension $d$, and $\mathcal{M} \in \text{Con}(X)$, then $\mathcal{H}^i f_!\mathcal{M} = 0$ for $i > 2d$. We use the induction on the dimension of $X$. By Lemma 1.1.3, we may assume that $X$ is reduced. By the $c$-exactness $1.3.4$, we may shrink $X$ by its open dense subscheme, and may assume that $X$ is smooth liftable and $\mathcal{M} = \mathcal{N}[-d]$ where $\mathcal{N} \in \text{Hol}(X)$ by Lemma 1.3.5. Then the lemma follows by the definition of $f_!$.

1.3.9 Lemma. — Let $\mathcal{M} \in \text{Con}(X)$, and let $\{u_i: U_i \to X\}$ be a finite open covering of $X$. Let $u_{ij}: U_i \cap U_j \to X$ be the immersion. Then the following sequence is exact in $\text{Con}(X)$:

$$\bigoplus_{i,j} u_{ij}^!u_{ij}^+\mathcal{M} \to \bigoplus_i u_i^!u_i^+\mathcal{M} \to \mathcal{M} \to 0.$$  

Proof. To check the exactness, it suffices to check it after taking $i_x^+$ for each closed point $x \in X$ by Lemma 1.3.7. By the commutativity of $i_x^+$ and $u_{ij}$, the verification is standard.
1.3.10. We use the category \( \text{Ind}(\text{Con}(X)) \) later. Let us prepare some properties of this category. Let \( f : X \to Y \) be a morphism between realizable schemes. Since \( f^+ \) is c-t-exact, we have a functor \( f^+: \text{Ind}(\text{Con}(Y)) \to \text{Ind}(\text{Con}(X)) \).

**Lemma.** — We use the notation of Lemma 1.3.7.

(i) Let \( \mathcal{F} \in \text{Ind}(\text{Con}(X)) \). Assume that \( i_x^+ \mathcal{F} = 0 \) for any closed point \( x \). Then \( \mathcal{F} = 0 \).

(ii) Let \( g : s' \to s \) be a morphism of points. Then for \( \mathcal{F} \in \text{Ind}(\text{Con}(s)) \), \( \mathcal{F} = 0 \) if and only if \( g^+(\mathcal{F}) = 0 \). In particular, a homomorphism \( \phi \) in \( \text{Ind}(\text{Con}(s)) \) is an isomorphism if and only if so is \( g^+(\phi) \).

**Proof.** Let us show (i). Write \( \mathcal{F} = \lim_{\to j} \mathcal{F}_j \) such that \( I \) is a filtrant category and \( \mathcal{F}_i \in \text{Con}(X) \). Fix \( i \in I \), and let \( E_i := \text{Ker}(\mathcal{F}_i \to \mathcal{F}_j) \). By assumption, we have \( \lim_{\to j} i_x^+ (\mathcal{F}_i/E_i) = 0 \), so \( \lim_{\to j} E_i = \mathcal{F}_i \). Since \( \text{Con}(X) \) is noetherian by Lemma 1.3.7, there exists \( j_0 \in I \) such that \( E_{j_0} = \mathcal{F}_i \). This shows that the homomorphism \( \mathcal{F}_i \to \mathcal{F}_{j_0} \) is 0, and the claim follows.

Let us show (ii). Let \( \mathcal{F} = \lim_{\to j} \mathcal{F}_i \), where \( I \) is filtrant and \( \mathcal{F}_i \in \text{Con}(s) \). For each \( i \in I \), there exists \( j \in I \) such that \( g^+ \mathcal{F}_i \to g^+ \mathcal{F}_j \) is 0. Thus, by Lemma 1.3.7 we get that \( \mathcal{F}_i \to \mathcal{F}_j \) is 0 as well.

1.3.11 Question. — Unify the rigid cohomology theory into the framework of arithmetic \( \mathcal{D} \)-modules. Namely, let \( X \) be a realizable scheme. Define the category of smooth objects \( \text{Sm}(X/K) \) in \( \text{Con}(X) \), and establish an equivalence of categories \( \text{Isoc}^0(X/K) \to \text{Sm}(X/K) \). This equivalence should be the same as the one in 1.1.3.10 when \( X \) is smooth. Finally, compare the rigid cohomology and the push-forward in the sense of \( \mathcal{D} \)-modules. We note that when \( X \) is smooth and realizable, then [Ca2] gives us an answer to these questions (cf. [11.11]). We ask the same question when \( X \) is a separated scheme of finite type over \( k \) using the theory we are going to develop in the next section.

1.4. Extension of scalars and Frobenius structures

So far, the coefficient categories we have treated (e.g. \( \text{Hol}(X/K) \) and \( D^b_{\text{hol}}(X/K) \)) are \( K \)-additive. For the Langlands correspondence, we need to consider \( L \)-coefficients with Frobenius structure where \( L \) is a finite field extension of \( K \). We introduce such category in this subsection.

**Extension of scalars**

1.4.1. Let \( A \) be a \( K \)-additive category. We take a finite field extension \( L \) of \( K \). We define the category \( A_L \) as follows. An object is a pair \( (X, \rho) \) such that \( X \in \text{Ob}(A) \), and a \( K \)-algebra homomorphism \( \rho : L \to \text{End}(X) \), called the \( L \)-structure (cf. [DM after Remark 3.10]). Morphisms are morphisms in \( A \) compatible with \( L \)-structures.

We have the forgetful functor \( f_L : A_L \to A \). Let \( X \in A \). Then we define \( X \otimes_K L \in A_L \) as follows: Take a basis \( x_1, \ldots, x_d \) of \( L \) over \( K \). Then \( X \otimes_K L := \left( \bigoplus_{i=1}^d X \otimes x_i, \rho' \right) \) such that for \( x \in L \), write \( x : x_l = \sum a_i x_i \) with \( a_i \in K \), and \( \rho'(x)|_{X \otimes x_k} := \sum \rho(a_i) \otimes x_i \). We can check easily that this does not depend on the choice of the basis up to canonical isomorphism. We denote by \( \iota_L := (-) \otimes_K L : A \to A_L \).

For \( X \in A \) and \( Y \in A_L \), we have

\[ \text{Hom}_A(X, f_L(Y)) \cong \text{Hom}_{A_L}(\iota_L(X), Y), \]

in other words, we have an adjoint pair \( (\iota_L, f_L) \). Thus, if \( A \) is an enough injective abelian category, the functor \( f_L \) sends injective objects in \( A_L \) to injective objects in \( A \).

Let \( f : A \to B \) be a \( K \)-additive functor between \( K \)-additive categories. Then there exists a unique functor \( f_L : A_L \to B_L \), which is compatible with both \( f \) and \( f_L \).

1.4.2. Let \( A \) be a \( K \)-additive category. Let \( X, Y \in A_L \). On \( \text{Hom}(f_L(X), f_L(Y)) \), we endow with \( L \otimes_K L \)-module structure as follows: we define the left \( L \)-structure by \( (a \cdot \phi)(x) := a(\phi(x)) \), and the right \( L \)-structure by \( (\phi \cdot a)(x) := \phi(ax) \). For \( a \in K \), both \( L \)-structures are compatible, and we get the \( L \otimes_K L \)-module structure. We denote by \( \text{Hom}_K(X, Y) \) for this \( L \otimes_K L \)-module. By definition, we have

\[ \text{Hom}(X, Y) \cong \{ \phi \in \text{Hom}_K(X, Y) \mid (a \otimes 1)\phi = (1 \otimes a)\phi = 0 \text{ for any } a \in L \}. \]

Note that if \( L/K \) is a separable extension, then \( L \otimes_K L \) is a product of fields, and any \( L \otimes_K L \)-module is flat.
Lemma. — Let \( L/K \) be a separable extension, and \( M \) be an \( L \otimes_K \) \( L \)-module. Let \( I := \text{Ker}(L \otimes_K L \to L) \). Then we have a canonical isomorphism

\[
M_0 := \{ m \in M \mid a \cdot m = 0 \text{ for any } a \in I \} \xrightarrow{\sim} M/IM.
\]

Proof. Left to the reader. \( \square \)

Corollary. — Let \( L/K \) be a separable extension. Then, for \( X, Y \in \mathcal{A}_L \), we have a canonical isomorphism

\[
\text{Hom}(X, Y) \xrightarrow{\sim} L \otimes L \otimes_{K, L} \text{Hom}_K(X, Y).
\]

1.4.3. Now, let \( \mathcal{A} \) be a \( K \)-abelian category. Assume \( L/K \) is a separable extension. Now, we consider the derived category \( D(\mathcal{A}_L) \). We have the following functor

\[
\text{Hom}_K^*: C(\mathcal{A}_L)^\circ \times C(\mathcal{A}_L) \to C(L \otimes_K L); \quad (X^\bullet, Y^\bullet) \mapsto \prod_{i \in \mathbb{Z}} \text{Hom}_K(X^i, Y^{i+\bullet}),
\]

and the differential is defined as in \([H, I.6]\). Since for \( L \) sends injective objects to injective objects, we can derive this functor and get \( \mathbb{R}\text{Hom}_K^*(\cdot, \cdot) \). We have the following:

Lemma. — Let \( X \in D(\mathcal{A}_L), Y \in D^+(\mathcal{A}_L) \). Then, we have an isomorphism

\[
\mathbb{R}\text{Hom}(X, Y) \xrightarrow{\sim} L \otimes L \otimes_{K, L} \mathbb{R}\text{Hom}_K^*(X, Y).
\]

Proof. Use Corollary 1.4.2. \( \square \)

Corollary. — The functor \( D^b(\mathcal{A}_L) \to D^b(\mathcal{A})_L \) is fully faithful.

Proof. For \( X \in D(\mathcal{A}_L) \), let us denote by \( X' \) the image in \( D(\mathcal{A})_L \). We have

\[
\mathbb{R}^i\text{Hom}_K^*(X, Y) \cong \text{Hom}_K(X', Y'[i])
\]

as \( L \otimes K \) \( L \)-modules. This shows that the functor is fully faithful by the lemma above and Corollary 1.4.2. \( \square \)

Remark. — This corollary shows that if \( X, Y \in D^b(\mathcal{A}_L) \) are isomorphic in \( D^b(\mathcal{A})_L \), then they are isomorphic in \( D^b(\mathcal{A}_L) \). For example, assume given two functors \( F, G: D^b(\mathcal{A}) \to D^b(\mathcal{B}) \) and a morphism \( \alpha: F \to G \) of functors. Further, assume that these functors have liftings \( \tilde{F}, \tilde{G}: D^b(\mathcal{A}_L) \to D^b(\mathcal{B}_L) \). Then \( \alpha \) can be lifted to \( \tilde{\alpha}: \tilde{F} \to \tilde{G} \). If \( \alpha \) is an isomorphism so is \( \tilde{\alpha} \).

1.4.4 Lemma. — Assume \( \mathcal{A} \) be a noetherian category. Then we have a canonical equivalence \( \text{Ind}(\mathcal{A}_L) \xrightarrow{\sim} \text{Ind}(\mathcal{A})_L \).

Proof. It is easy to check that it is fully faithful. Let \((X, \rho)\) be an object in \( \text{Ind}(\mathcal{A})_L \). We may write \( X = \lim_{\rightarrow \downarrow \in I} X_i \) where \( X_i \in \mathcal{A} \) and \( X_i \subset X \) by \([\text{De}2, 4.2.1 (ii)]\). Let \( x \in L \) such that \( K[x] = L \), and \([L : K] := d\). Put \( X'_i := \sum_{j=0}^{d-1} \rho(x^j)(X_i) \) where the sum is taken in \( X \). Then \( X'_i \) is stable under the action of \( L \), and defines an object in \( \mathcal{A}_L \). The limit \( \lim_{\rightarrow \downarrow \in I} X'_i, \rho \) is sent to \((X, \rho)\). \( \square \)

1.4.5. Let \( F: \mathcal{A} \to \mathcal{B} \) be a \( K \)-additive functor between \( K \)-abelian categories. Assume that \( F \) is left exact and \( \mathcal{A}_L \), then \( \mathcal{A} \) as well, has enough injectives. Then \( RF \) and \( \iota_L \) commute. Indeed, since \( F \) and \( \iota_L \) commute, it suffices to show that for an injective object \( I \) in \( \mathcal{A} \), \( RF(\iota_L(I)) = 0 \) for \( i > 0 \). For this, it suffices to show that for \( L \circ (R^i F) \circ \iota_L(I) = 0 \). We have

\[
\text{for } L \circ (R^i F) \circ \iota_L(I) = (R^i F) \circ \text{for } L \circ \iota_L(I) = 0,
\]

where the first equality holds since \( for_L \) preserves injective objects and commutes with \( F \), and the second by the fact that \( for_L \circ \iota_L(I) \) is a finite direct sum of copies of \( I \) and thus injective.

Frobenius structure
1.4.6. To consider “Frobenius structure”, we need to fix a base as in \(\mathfrak{F}\). A base tuple is a 5-tuple \((k, R, K, s, \sigma)\) where \(R\) is a complete discrete valuation ring with residue field \(k\) which is assumed to be perfect, \(K = \text{Frac}(k)\), \(s\) is a positive integer, and \(\sigma: K \xrightarrow{\sim} K\) is a lifting of the \(s\)-th absolute Frobenius automorphism of \(K\). We fix a base tuple \(\mathfrak{F}\).

1.4.7. Now, let us consider the “Frobenius structure”. Let \(\mathcal{A}\) be a \(K\)-additive category, and \(F^*: \mathcal{A} \rightarrow \mathcal{A}\) be a \(\sigma\)-semilinear functor, namely for \(X, Y \in \mathcal{A}\) the homomorphism \(\text{Hom}(X, Y') \rightarrow \text{Hom}(F^*X, F^*Y)\) is \(\sigma\)-semilinear. We put \(K_0 := K^{s=1}\). We define the category \(F\mathcal{A}\) to be the category of pairs \((X', \Phi)\) such that \(X' \in \text{Ob}(\mathcal{A})\), and an isomorphism \(\Phi: X' \xrightarrow{\sim} F^*X'\) called the Frobenius structure. Morphisms in \(F\mathcal{A}\) are morphisms in \(\mathcal{A}\) respecting \(\Phi\). Then the category \(F\mathcal{A}\) is \(K_0\)-additive category. Let \(X := (X', \Phi)\) be an object of \(F\mathcal{A}\). For an integer \(n\), we define \(X(n) := (X', p^{sn} \cdot \Phi)\), and call it the \(n\)-th Tate twist \(\mathcal{1}(2)\) of \(X\).

There exists the forgetful functor
\[
\text{for}_F: F\mathcal{A} \rightarrow \mathcal{A}; \quad (X', \Phi) \mapsto X'.
\]

This functor is faithful. For \(X, Y \in F\mathcal{A}\), we have a \(K_0\)-linear endomorphism
\[
(1.4.7.1) \quad F: \text{Hom}(\text{for}_F(X), \text{for}_F(Y)) \rightarrow \text{Hom}(\text{for}_F(F^*X), \text{for}_F(F^*Y)) \cong \text{Hom}(\text{for}_F(X), \text{for}_F(Y)),
\]
where the last isomorphism is induced by the Frobenius structures of \(X\) and \(Y\).

Now, assume that \(\mathcal{A}\) is abelian. Then \(F\mathcal{A}\) is abelian as well. Indeed, assume given a morphism \(f: X \rightarrow Y\) in \(F\mathcal{A}\). Then the Frobenius structure on \(X\) induces a Frobenius structure on \(\text{Ker}(\text{for}_F(f))\), which is the kernel of \(f\). This construction shows that \(\text{for}_F(\text{Ker}(f)) \cong \text{for}_F(\text{Ker}(f))\). Replacing \(\text{Ker}\) by \(\text{Coker}\), we get the same result. Thus, we get the claim.

The construction shows that \(\text{for}_F\) is an exact functor, and the following diagram is commutative:
\[
\begin{array}{ccc}
D(F\mathcal{A}) & \xrightarrow{\Phi^1} & F\mathcal{A} \\
\downarrow \text{for}_F & & \downarrow \text{for}_F \\
D(\mathcal{A}) & \xrightarrow{\Phi^1} & \mathcal{A}.
\end{array}
\]

Moreover, if \(\text{for}_F(X) = 0\) then \(X = 0\). This implies that a sequence in \(F\mathcal{A}\) is exact if and only if so is the sequence after taking \(\text{for}_F\).

Finally, let \((\mathcal{A}, F)\) and \((\mathcal{B}, G)\) be \(K\)-additive categories with automorphism. Assume given a functor \(f: \mathcal{A} \rightarrow \mathcal{B}\) and an equivalence \(f \circ F \cong G \circ f\). Then we have a canonical functor \(\overline{f}: F\mathcal{A} \rightarrow G\mathcal{B}\) such that \(f \circ \text{for}_F \cong \text{for}_G \circ \overline{f}\).

1.4.8. Now, assume that \(F^*\) is an equivalence of categories, and \(\mathcal{A}\) is a Grothendieck category. We define
\[
(-)_F: \mathcal{A} \rightarrow F\mathcal{A}; \quad X \mapsto X_F := \bigoplus_{n \in \mathbb{Z}} (F^*)_n X.
\]

Then it can be checked easily that \((-)_F, \text{for}_F\) is an adjoint pair. Furthermore, \((-)_F\) is exact, since the functor \(\text{for}_F \circ (-)_F\) is exact. Thus \(\text{for}_F\) sends injective object to injective object. The filtrant inductive limit is representable in \(F\mathcal{A}\), and commutes with \(\text{for}_F\). Let \(G\) be a generator of \(\mathcal{A}\). Then \(G_F\) is a generator of \(F\mathcal{A}\). Indeed, assume given two morphisms \(f, g: X \rightarrow Y\) in \(F\mathcal{A}\). Then there exists \(\phi: G \rightarrow \text{for}_F(X)\) such that \(\phi \circ \text{for}_F(f) \neq \phi \circ \text{for}_F(g)\). By taking adjoint, we have \(\phi_F: G^\mathcal{A} \rightarrow X\). Then \(\phi_F \circ f \neq \phi_F \circ g\) as required. This shows that \(F\mathcal{A}\) is a Grothendieck category as well.

In the following, we often assume:
\[
(*) \quad F^* \text{ is an equivalence, and } \mathcal{A} \text{ is a noetherian category.}
\]

This assumption implies that \(\text{Ind}(\mathcal{A})\) is a Grothendieck category endowed with semilinear auto-functor \(F^*\). Thus by the result we get above, the category \(F\text{-Ind}(\mathcal{A})\) is a Grothendieck category.

\(\mathcal{2}\) In many texts, Frobenius structure is defined to be an isomorphism with the opposite direction \(\Psi: F^*X' \xrightarrow{\sim} X'\) and \(n\)-th Tate twist to be \(p^{-sn} \cdot \Psi\). See footnote (1) in 2.7 of \([\mathcal{A}1]\).
1.4.9. We retain the assumption ($\dagger$) in 1.4.8. Take $X, Y$ in $F\text{-Ind}(\mathcal{A})$. Then the homomorphism $F$ in (1.4.7) is an isomorphism, and $\text{Hom}(\text{for}_F(X),\text{for}_F(Y))$ is a $K_0[F^{\pm 1}]$-module. Here the $K_0[F^{\pm 1}]$-module structure is defined so that $F \cdot \varphi := F \circ \varphi \circ F^{-1}$ for $\varphi: \text{for}_F(X) \to \text{for}_F(Y)$. This module is denoted by $\text{Hom}_\rho(X, Y)$. On the other hand, for a $K_0[F^{\pm}]$-module $M$ and $X \in F\text{-Ind}(\mathcal{A})$, we define $M \otimes_{K_0} X$ as follows: Write $M = \varinjlim M_i$ as $K_0$-vector spaces such that $M_i$ is finite dimensional. As an object in $\text{Ind}(\mathcal{A})$, it is $\varinjlim (M_i \otimes_{K_0} X)$. The Frobenius structure is defined by $F \otimes F$, namely sending $m \otimes x$ to $(F(m) \otimes F(x)$ if we use the description by taking elements. The functor $M \otimes_{K_0}$ is exact. Now, for any $K_0[F^{\pm 1}]$-module $M$, we have

$$\text{Hom}(M \otimes_{K_0} X, Y) \xrightarrow{\sim} \text{Hom}_{K_0[F^{\pm 1}]}(M, \text{Hom}_\rho(X, Y)).$$

This shows that if $Y$ is an injective object, $\text{Hom}_\rho(X, Y)$ is an injective $K_0[F^{\pm 1}]$-module.

As in [H 1.6], for $X, Y \in C(F\text{-Ind}(\mathcal{A}))$, we define a complex of $K_0[F^{\pm 1}]$-modules $\text{Hom}_\rho^\bullet(X, Y)$. Then this functor can be derived, and get

$$\mathbb{R}\text{Hom}_\rho^\bullet: D(F\text{-Ind}(\mathcal{A})) \times D^+(F\text{-Ind}(\mathcal{A})) \to D^+(K_0[F^{\pm 1}]).$$

**Lemma.** We regard $K_0$ as a $K_0[F^{\pm}]$-module such that $F$ acts trivially. For $X, Y \in D(F\text{-Ind}(\mathcal{A}))$ such that $X \in D^-, Y \in D^+$, we have

$$\mathbb{R}\text{Hom}_{K_0[F^{\pm 1}]}(K_0, \mathbb{R}\text{Hom}_\rho^\bullet(X, Y)) \cong \mathbb{R}\text{Hom}(X, Y).$$

**Proof.** We have a canonical isomorphism

$$\text{Hom}_{K_0[F^{\pm 1}]}(K_0, \text{Hom}_\rho(X, Y)) \cong \text{Hom}(X, Y).$$

Since the functor $\text{Hom}_\rho(X, -)$ preserves injective objects, we get the lemma. ■

For a $K_0[F^{\pm}]$-module $M$, we put

$$M^F := \text{Hom}_{K_0[F^{\pm}]}(K_0, M), \quad M_F := \text{Ext}^1_{K_0[F^{\pm}]}(K_0, M).$$

**Corollary.** For $Z \in D(F\text{-Ind}(\mathcal{A}))$, we denote by $Z' \in F^{-}\text{D}(\text{Ind}(\mathcal{A}))$ the image of $Z$. Let $X, Y \in D(F\text{-Ind}(\mathcal{A}))$ such that $X \in D^-$ and $Y \in D^+$. Then there exists the short exact sequence:

$$0 \to \text{Hom}_\rho(X, Y'[\{-1\}]_F) \to \text{Hom}(X, Y) \to \text{Hom}_\rho(X', Y')^F \to 0.$$

**Proof.** When $X$ and $Y$ are in $\text{Ind}(\mathcal{A})$, we have $\text{Hom}_\rho(X', Y') \xrightarrow{\sim} \mathbb{R}^0\text{Hom}_\rho(X, Y)$. ■

1.4.10. Let $\mathcal{A}$ be a $K$-additive category, and $F^\ast: \mathcal{A} \to \mathcal{B}$ be a $\sigma$-semilinear functor. We fix a finite field extension $L$ and an isomorphism $\sigma_L: L \to L$ compatible with $\sigma$. Put $L_0 := L^{\sigma=1}$. We define $F_\lambda^\ast: \mathcal{A}_L \to \mathcal{A}_L$ as follows: Let $\rho: L \to \text{End}(X)$ be an object of $\mathcal{A}_L$. We have a $\sigma$-semilinear homomorphism $F^\ast(\rho): L \xrightarrow{\sim} \text{End}(X) \to \text{End}(F^\ast(X))$. We put

$$F_\lambda^\ast(\rho) := F^\ast(\rho) \circ \sigma_L^{-1}: L \to \text{End}(F^\ast X).$$

This is a homomorphism of $K$-algebras. We define $F_\lambda^\ast: \mathcal{A}_L \to \mathcal{A}_L$ by sending $(X, \rho)$ to $(F^\ast X, F_\lambda^\ast(\rho))$. The $L_0$-additive category $F_\lambda^\ast\mathcal{A}_L$ is sometimes denoted by $F\mathcal{A}_L$. The $K$-additive functors $\iota_L: \mathcal{A} \to \mathcal{A}_L$ and $\text{for}_L: \mathcal{A}_L \to \mathcal{A}$ induce the functors $F\mathcal{A} \to F_\lambda^\ast\mathcal{A}_L$ and $F_\lambda^\ast\mathcal{A}_L \to F\mathcal{A}$ denoted abusively by $\iota_L$ and $\text{for}_L$, respectively. We can check that $(\iota_L, \text{for}_L)$ is an adjoint pair, and $\iota_L$ is exact. In particular, $\text{for}_L$ preserves injective objects.

Let $\mathcal{B}$ be another $K$-additive category endowed with a $\sigma$-semilinear endo-functor $G^\ast$. Let $f: \mathcal{A} \to \mathcal{B}$ be a $K$-additive functor between $K$-abelian categories compatible with $F^\ast$ and $G^\ast$. Then there exists a unique functor $f_\lambda: F\mathcal{A}_L \to G\mathcal{B}_L$ compatible with $\text{for}_L$ and $\iota_L$.

**Application of the theory**
1.4.11. Fix a base tuple $\Sigma_K$. We take $\star \in \{\emptyset, 0\}$. Depending on $\star$, we consider the following situations:

$\star = \emptyset$ case: We fix a finite field extension $L$ of $K$.

$\star = 0$ case: We fix a finite field extension $L$ and an automorphism $\sigma_L : L \to L$ extending $\sigma$.

We put $L_0 := L^{\sigma = 1}$.

**Definition.** — Let $X$ be a realizable scheme over $k$. The category $\text{Hol}(X/K)$ (cf. 1.1.3) is endowed with $(s\text{-th})$ Frobenius pull-back $F^s$ (cf. Remark 1.1.3). Moreover, $F^s$ induces an auto-equivalence of $\text{Hol}(X/K)$. Thus, we can apply the general results developed in the preceding paragraphs. We put:

$\star = \emptyset$ case: $\text{Hol}_{\Sigma_K}(X/L) := \text{Hol}(X/K)_L$, $\text{Isoc}^i_{\Sigma_K}(X/L) := \text{Isoc}^i(X/K)_L$, $M(X/L) := \text{Ind}(\text{Hol}(X/L))$, $D(X/L) := D(\text{Hol}(X/L))$.

$\star = 0$ case: $\text{Hol}_{\Sigma_K}(X_0/L) := F^0\text{-Hol}(X/K)_L$, $\text{Isoc}^i_{\Sigma_K}(X_0/L) := F^0\text{-Isoc}^i(X/K)_L$, $M(X_0/L) := F_\ell\text{-Ind}(\text{Hol}(X/L))$, $D(X_0/L) := D(M(X_0/L))$.

If there is nothing to be confused, we often omit the subscripts $(\cdot)_{\Sigma_K}$.

Let us introduce a convenient convention. Let $C(k)$ be a category of objects “over $k$” (e.g. category of schemes over $k$, category of algebraic stacks over $k$, etc.). The category $C(k_0)$ of $k_0$-objects is just a copy of $C(k)$. There exists a functor $s : C(k_0) \to C(k)$ taking the underlying object. An object or morphism in $C(k_0)$ is said to have property $P$ if it satisfies $P$ after applying $s$. For a scheme $X$ over $k_0$, we put

$$\text{Hol}(X/L) := \text{Hol}(s(X)_0/L),$$

and similarly for $\text{Isoc}^i(X/L)$, $M(X/L)$, $D(X/L)$.

**Remark.** — (i) For a realizable scheme over $k$, recall that $\text{Isoc}^i(X/K)$ is slightly smaller than the category of overconvergent isocrystals (cf. 1.1.3). However, $\text{Isoc}^i(X_0/K)$ coincides with $F^0\text{-Isoc}^i(X/K)$ using the notation of [Ber1, 2.3.7].

(ii) For a scheme $X$ over a field, let us denote by $D^b_c(X)$ the category of constructible $\mathbb{Q}_p$-complexes. Let $k$ be a finite field, $\overline{k}$ is its algebraic closure, and $X$ be a scheme of finite type over $k$. Put $\overline{X} := X \otimes_k \overline{k}$. Under the philosophy of Riemann-Hilbert correspondences, $D^b_c(X)$ (resp. $D^b_{\text{hol}}(X_0)$) plays a role of $D^b_c(\overline{X})$ (resp. $D^b_{\text{hol}}(X)$) in $p$-adic cohomology theory.

Now, note that $D^b_c(X)$ does not depend on the base field $k$. On the other hand, a priori, $D^b_{\text{hol}}(X_0)$ does depend on $\Sigma_K$. However, we show in Corollary 1.4.12 that the category, in fact, does not depend on the choice of the base tuple under some conditions, which justifies further the analogue.

1.4.12 Lemma. — Let $\Sigma' = (k', R', K', s', \sigma')$ be a base tuple over $\Sigma$, namely $K'/K$ is a finite extension, $s' - s := a \geq 0$, and $\sigma'$ is an extension of $\sigma_K$.

(i) There exists a canonical equivalence $\text{Hol}_{\Sigma'}(X \otimes_k k'/K') \cong \text{Hol}_{\Sigma_K}(X/K')$.

(ii) Assume further that $s = s'$. Then $\text{Hol}_{\Sigma'}(X_0 \otimes_k k'/K') \cong \text{Hol}_{\Sigma_K}(X_0/K')$.

**Proof.** We may reduce to the case where $X$ can be lifted to a smooth formal scheme $\mathscr{X}$ over $R$. Let $\mathscr{X}' := \mathscr{X} \otimes_R R'$. There exists the functor

$$M(\mathscr{D}^i_{\mathscr{X}'//R', \mathbb{Q}}) \to M(\mathscr{D}^i_{\mathscr{X}//R, \mathbb{Q}})_{K'},$$

where $M(\mathscr{A})$ denotes the category of $\mathscr{A}$-modules. It is straightforward to show that this functor induces an equivalence of categories. By Remark 1.4.12 we get (i). Now, the following diagram is commutative:

$$\begin{array}{ccc}
M(\mathscr{D}^i_{\mathscr{X}'//R', \mathbb{Q}}) & \xrightarrow{F^*} & M(\mathscr{D}^i_{\mathscr{X}//R, \mathbb{Q}})_{K'} \\
\downarrow & & \downarrow F^*
\end{array}$$

This diagram implies (ii).
Corollary. — Assume $k$ is a finite field of $q = p^s$-elements, and $\sigma = \text{id}$. Let $K'$ be a finite extension of $K$, and put $\Sigma_K := (k', R', K'), s' := [k' : k] + s, \text{id})$. Let $X$ be a scheme over $k'$. Then, we have an equivalence of categories

$$\text{Hol}_{\Sigma_K}(X_0/K') \cong \text{Hol}_{\Sigma_K}(X_0/K').$$

Proof. When the extension $K'/K$ is totally ramified, the claim follows from (ii) of the lemma. Thus, we may assume that the extension is unramified. In this case, the verification is essentially the same as in [Del1 1.1.10], so we only sketch. Since $K'/K$ is assumed to be unramified, we have $K' \cong K \otimes_{W(k)} W(k')$. Let $\sigma' := \text{id} \otimes F$ where $F$ denotes the $(s$-th absolute) Frobenius automorphism on $W(k')$. As a scheme over $k$, we have a canonical isomorphism $X \otimes_k k' \cong \bigoplus_{\sigma \in \text{Gal}(k'/k)} X^\sigma$ where each $X^\sigma$ is canonically isomorphic to $X$, and the Galois action on $k'$ is compatible with an obvious sense. Put $\Sigma' := (k', R', K', s, \sigma')$. Then by the lemma, we get $\text{Hol}_{\Sigma'}(X_0 \otimes_k k'/K') \cong \text{Hol}_{\Sigma_K}(X_0/K')$. There exists $\varphi \in \text{Gal}(k'/k)$ such that, by $F$, each $X^\sigma$ is sent to $X^\sigma \varphi$. Assume given $(\mathcal{M}, \Phi) \in \text{Hol}_{\Sigma_K}(X_0/K')$. For $0 \leq i \leq [k' : k]$, we put $(F^*)^i(\mathcal{M})$ on $X^{\varphi^i}$, which defines $\mathcal{N}$ in $\text{Hol}(X' \otimes_k k'/K')$. The Frobenius structure $\Phi$ defines an $s'$-th Frobenius structure on $\mathcal{N}$. It is easy to check that this correspondence yields the equivalence of categories. ■

1.4.13 Remark. — In the proof of the Langlands correspondence, it is convenient to work with $\mathbb{Q}_p$-coefficient. For this, we use 2-inductive limit method as in [Del1 1.1.4] to construct the theory. The detail is explained in 2.4.4.4.

1.4.14. Let $L := \iota_L(K)$ in $\text{Hol}(\text{Spec}(k)_* / L)$. We have the left exact functor

$$\Gamma: \mathcal{M}(\text{Spec}(k)_* / L) \rightarrow \text{Vec}_L; \quad \mathcal{M} \mapsto \text{Hom}(L, \mathcal{M}).$$

This functor can be derived, and get a functor $\mathbb{R}\Gamma: D^+(\text{Spec}(k)_* / L) \rightarrow D^+(\text{Vec}_L)$.

1.5. Trace map

In order to establish the formalism of cycle classes, we need the trace map axiomatically, following the construction of SGA 4. We need to remark that, in [A1 5.5], we showed the isomorphism $f^! \cong f^+ (d)[2d]$ for a smooth morphism $f$ of relative dimension $d$. However, the construction of this homomorphism is ad hoc., and does not seem to be easy to check the properties that the trace map should satisfy, for example, transitivity, and we moreover need the trace maps for flat morphisms to define the cycle class map.

1.5.1. We fix a base tuple $(k, R, K, s, \sigma)$ (cf. 1.4.6) and $\Delta \in \{0, 0\}$ in this subsection. Let $X$ be a realizable scheme over $k_{\Delta}$. When $\Delta = 0$, we denote $D^b(X/K)$ simply by $D^b(X)$, in which case, the Tate twist $(n)$ is the identity functor. When $\Delta = 0$, we denote $F \cdot D^b(X/K)$ by $D^b(X)$. Caution that when $\Delta = 0$, the notation is not compatible with later sections where we use $D^b(X_0/K)$. The main result of this subsection is the following theorem on the existence of the trace map:

Theorem. — Let $f: X \rightarrow Y$ be a morphism between realizable schemes over $k_{\Delta}$. Let $\mathfrak{M}_d$ be the following set of morphisms of realizable schemes:

- there exists an open subscheme $U \subset Y$ such that $X \times_Y U \rightarrow U$ is flat of relative dimension $d$, and for each $x \in Y \setminus U$, the dimension of $f^{-1}(x)$ is $< d$.

Then there exists a unique homomorphism $\text{Tr}_f: f_* f^+ \mathcal{F}(d)[2d] \rightarrow \mathcal{F}$ for any $\mathcal{F}$ in $D^b_{\text{hol}}(Y)$, called the trace map, satisfying the following conditions.

- (Var 1) $\text{Tr}_f$ is functorial with respect to $\mathcal{F}$.
- (Var 2) Consider the cartesian diagram (1.1.3.1) of realizable schemes. Assume $f \in \mathfrak{M}_d$. Then the following diagram is commutative:

$$
\begin{array}{ccc}
g^* f_* f^+ (d)[2d] & \xrightarrow{\sim} & f'_* g^+ f^+(d)[2d] \\
g^+ \text{Tr}_f & & f'_* g^+ \text{Tr}_f \\
g^+ & \xrightarrow{\sim} & g^+ \\
\end{array}
$$

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(Var 3) Let \( X \to Y \to Z \) be morphisms of realizable schemes such that \( f \in \mathcal{M}_d \) and \( g \in \mathcal{M}_e \). Then the following diagram is commutative:

\[
\begin{array}{c}
\begin{tikzcd}
\text{Tr}_g & f_*g^*(d+e)(2d+e) \\
(f \circ g)_!(f \circ g)^*(d+e)(2d+e) \\
\end{tikzcd}
\end{array}
\]

(Var 4-I) Let \( f \in \mathcal{M} \) to be a locally free morphism of rank \( n \). Then the composition

\[
\mathcal{F} \to f_+ f^+ \mathcal{F} \xleftarrow{\sim} f_! f^+ \mathcal{F} \xrightarrow{\text{Tr}_f} \mathcal{F}
\]

is the multiplication by \( n \).

(Var 4-II) When \( X \) and \( Y \) can be lifted to a proper smooth formal scheme, and \( f \) can be lifted to a smooth morphism of relative dimension 1 between them, then the trace map is the one defined in 1.5.11 below.

(Var 5) The following diagram is commutative

\[
\begin{array}{c}
\begin{tikzcd}
\mathcal{F} \otimes \mathcal{G} & (f_! f^+ \mathcal{F})(d)[2d] \\
(f_! f^+ \mathcal{F}(d)[2d]) \otimes \mathcal{G} \\
\end{tikzcd}
\end{array}
\]

where the upper horizontal homomorphism is the projection formula.

1.5.2. Even though there are many technical differences, the idea of construction of trace map is essentially the same as that in [SGA 4, XVIII]. Let us start to construct the trace map. First, we list up direct consequences from the requested properties.

1. By (Var 5), it suffices to construct the trace homomorphism for \( \mathcal{F} = K_Y \).

2. By assumption, for \( \mathcal{F} \in \text{Con}(X) \), we have \( c_* \mathcal{H}^i f_! f^+ \mathcal{F} = 0 \) for \( i > 2d \) by Lemma 1.3.8. Thus, we have \( \text{Hom}(f_! f^+ \mathcal{F}(d)[2d], \mathcal{F}) \cong \text{Hom}(c_* \mathcal{H}^{2d} f_! f^+ \mathcal{F}(d), \mathcal{F}) \).

3. Assume that we have already constructed trace morphism when \( Y \) is a point. Then by 1.2 Lemma 1.3.7 (i), and the base change property shows that extension of this trace map to the general situation is unique, if they exist.

4. When \( f = f \circ f' : X' \coprod X'' \to Y \), and assume \( \text{Tr}_f \) and \( \text{Tr}_{f'} \) have already been constructed. Then, by the same argument as [SGA 4, XVII 6.2.3.1], \( \text{Tr}_f = \text{Tr}_{f'} + \text{Tr}_{f''} \).

5. If \( f \) is a universal homeomorphism, then the canonical homomorphism

\[
\alpha : \mathcal{F} \to f_+ f^+ \mathcal{F} \xleftarrow{\sim} f_! f^+ \mathcal{F}
\]

is an isomorphism. By (Var 4-I), we have \( \text{Tr}_f := \deg(f) \cdot \alpha^{-1} \).

6. Consider the cartesian diagram of realizable schemes. Then the compatibility (Var 2) is equivalent to the commutativity of one of the following diagrams:

\[
\begin{array}{c}
\begin{tikzcd}
\mathcal{F} & f^+ f^+ \mathcal{F} \\
& f_! f^+ \mathcal{F} \\
\end{tikzcd}
\end{array}
\]

where the vertical arrows are the base change homomorphisms. The verification is standard using 1.1.4.
1.5.3 Lemma. — Let \( f : X \to Y \) be a morphism of realizable schemes of relative dimension \( \leq d \). Let \( \{ U_i \} \) be a finite open covering of \( X \), and \( U_{ij} := U_i \cap U_j \). For \( * \in \{ i, ij \} \), we put \( u_* : U_* \to X \), and \( f_* : U_* \to X \to Y \). Then for \( \mathcal{F} \in \text{Con}(X) \), the following sequence is exact:

\[
\bigoplus_{i,j} \mathcal{H}^{2d} f_{ij}u_{ij}^* \mathcal{F} \to \bigoplus_{i} \mathcal{H}^{2d} f_i u_i^* \mathcal{F} \to \mathcal{H}^{2d} f_* \mathcal{F} \to 0.
\]

Proof. By Lemma 1.3.8, \( \mathcal{H}^{2d} f_i \) is right exact, and the claim of the lemma follows by applying this functor to the exact sequence in Lemma 1.3.9. □

1.5.4. First, let \( \mathcal{M}_{et} \) be the set of étale morphisms between realizable schemes. We show the theorem for \( \mathcal{M}_{et} \) instead of \( \mathcal{M} \). By 1.5.2.9 combining with 1.5.2.10 and Lemma 1.3.7 (ii), if the trace maps exist for morphisms in \( \mathcal{M}_{et} \), then they are unique. We show the following lemma:

Lemma. — For \( f \in \mathcal{M}_{et} \), there exists a unique trace map \( f'_+ \mathcal{M} \to \mathcal{M} \) for \( \mathcal{M} \in D_{hol}^b(Y) \) satisfying the properties (Var 1,2,3,4-1,5) if we replace \( \mathcal{M} \) by \( \mathcal{M}_{et} \). Moreover, the homomorphism \( f'_+(\mathcal{M}) \to f'_i(\mathcal{M}) \) defined by taking the adjoint is an isomorphism.

The proof is divided into several parts, and it is given in 1.5.7.

1.5.5 Lemma (Smooth base change for open immersion). — Consider the following cartesian diagram

\[
\begin{array}{ccc}
U' & \rightarrow & X' \\
\downarrow & & \downarrow \\
U & \rightarrow & X
\end{array}
\]

Assume \( g \) is smooth, \( j \) is an open immersion, and \( i \) is the complement. Then the base change homomorphisms \( g^+ j_+(\mathcal{M}) \to j'_+ g'^+(\mathcal{M}) \) and \( g''^+ i'_+(\mathcal{M}) \to i'^+ g^+(\mathcal{M}) \) are isomorphisms for any \( \mathcal{M} \in D_{hol}^b(U) \).

Proof. By the localization triangle \( i'_+i^! \to \text{id} \to j_+ j^+ \to +1 \), it suffices to show only the first isomorphism. Obviously, if we restrict the base change homomorphism to \( U' \), the homomorphism is an isomorphism. Thus, by the localization triangle, it suffices to show that \( i'^! g^+ j_+ \mathcal{M} = 0 \). Since the verification is Zariski local with respect to \( X' \), we may assume that \( g \) factors through étale morphism followed by the projection \( \mathbb{A}^n_X \to X \). We can treat étale and projection cases separately. Thus, using [EGA VI, 18.4.6], we may assume that there is a smooth morphism \( \mathcal{P}' \to \mathcal{P} \) of smooth formal schemes and a closed embedding \( X \hookrightarrow \mathcal{P} \) such that \( X' \cong X \times_{\mathcal{P}} \mathcal{P}' \). Then we may use [Ber2 4.3.12] and [A1 Theorem 5.5] to conclude. □

Corollary (Smooth base change). — Consider the diagram of realizable schemes \( 1,1,3,4 \). Assume that \( g \) is smooth. Then the base change homomorphism \( g^+ \circ f'_+ \to f'_+ \circ g'^+ \) is an isomorphism.

Proof. We may factor \( f \) as \( X \xrightarrow{j} X \xrightarrow{p} Y \) where \( j \) is an open immersion and \( p \) is proper. The base change for \( p \) is the proper base change theorem (cf. 1.1.3.7), and that for \( j \) is the lemma above. □

1.5.6. First, suppose that \( Y \) is smooth liftable purely of dimension \( d \), and \( f \) is affine. In this case let us construct an isomorphism \( f^+ K_Y \cong f^+ K_Y \). By taking the dual, it is equivalent to constructing \( f^+ K_Y^\vee \cong f^+ K_Y^\vee \) (cf. 1.1.3 for \( K_Y^\vee \)). Since the étale site of \( X \) and \( X \) are equivalent, there exists the following cartesian diagram where \( X \) and \( \mathcal{P} \) are smooth formal schemes and \( X \) and \( Y \) are special fibers:

\[
\begin{array}{ccc}
X & \rightarrow & X \vee \\
\downarrow & & \downarrow \\
Y & \rightarrow & Y
\end{array}
\]

By [Ca4 4.1.8, 4.1.9, 4.3.5] and [A1 3.12], we have canonical isomorphisms

\[
D_{X} \circ f^3 \circ D_{\mathcal{P}}(\text{sp}_+(\mathcal{O}_{\mathcal{P}_K})) \cong \text{sp}_+(f^\vee(O_{\mathcal{P}_K})(-d)) \cong \text{sp}_+(O_{X_K}).
\]
This gives us a canonical isomorphism

\[ \rho_f: \tilde{f}^+O_{Y,Q} \sim \rightarrow O_{X,Q} \cong \tilde{f}^1O_{Y,Q}. \]

By Kedlaya’s fully faithfulness [Ke1], \( \rho_f \) extends to the desired isomorphism. Taking the adjoint, we get a homomorphism \( K_Y^r \rightarrow f_+f^!K_Y^r \).

**Remark.** — We remark that for an isocrystal \( M \), the following diagram is commutative:

\[
\begin{array}{ccc}
\text{sp}_+(M) & \xrightarrow{\sim} & \text{sp}_+((M^\vee)^\vee) \\
\downarrow \cong & & \downarrow \\
\mathcal{D} \circ \mathcal{D}(\text{sp}_+(M)). & & \\
\end{array}
\]

To check this, by definition (cf. [Ca1, 2.2.12]), it suffices to check the commutativity for the middle square immediately follows by definition. To check the commutativity for the middle square, we do not make any difference between \( \tilde{f} \) and \( f \). In this case, the right vertical homomorphism is, in fact, isomorphic. Let \( \mathcal{M} \) be a holonomic module on \( Y \). By definition, there exists a smooth proper formal scheme \( \mathcal{P} \) such that \( \mathcal{M} \) can be realized as a \( D_{\mathcal{P},Q} \)-module \( \mathcal{M} \mathcal{P} \). There exists an immersion (not necessarily closed) \( i: Y \rightarrow \mathcal{P} \). Then the \( D_{\mathcal{P},Q} \)-module \( i^!(\mathcal{M} \mathcal{P}) \) (which is canonically isomorphic to \( i^+\mathcal{M} \mathcal{P} \)) is denoted by \( \mathcal{M} \mathcal{P} \| Y \) for a moment. This module does not depend on the auxiliary choice up to canonical equivalence.

**Lemma.** — The following diagram is commutative

\[
\begin{array}{ccc}
K_Y^r \| Y & \xrightarrow{\sim} & f_+f^!K_Y^r \| Y \\
\downarrow \cong & & \downarrow \\
O_{Y,Q} & \xrightarrow{\text{adj}_f} & \tilde{f}_f^*O_{Y,Q}. \\
\end{array}
\]

**Proof.** First, let us show in the case where \( f \) is finite étale of rank \( n \). Since \( \tilde{f} \) is finite étale, we can identify \( \tilde{f}_+ \) and \( \tilde{f}^! \) by \( f_+ \) and \( f^! \) if we consider the underlying \( O_{Y,Q} \)-module structure. In the following, for simplicity, we do not make any difference between \( \tilde{f} \) and \( f \). In this case, the right vertical homomorphism is, in fact, isomorphic. Let \( \mathcal{F} \) be an \( O_{Y,Q} \)-module, and \( \iota: f_+(f^!*\mathcal{F})^\vee \rightarrow (f_+f^!\mathcal{F})^\vee \) be the homomorphism sending \( \varphi \) to \( \text{Tr}_f \circ \varphi \), where \( \text{Tr}_f: f_+f^*O_{Y,Q} \rightarrow O_{Y,Q} \) is the classical trace map. If \( \mathcal{F} \) is a locally free \( O_{Y,Q} \)-module, \( \iota \) is an isomorphism. We have the following diagram, where we omit \( \text{sp}_+ \) and the subscripts \( Q \):

\[
\begin{array}{ccc}
O_{Y,Q} & \xrightarrow{\sim} & f_+f^+O_{Y,Q} & \xrightarrow{\rho_f} & f_+f^1O_{Y,Q} \\
\downarrow & & \downarrow & & \downarrow \\
(O_{Y,Q}^\vee)_{(\text{Tr}_f)^\vee} & \xrightarrow{\sim} & (f_+f^*O_{Y,Q}^\vee)_{(\iota_1)^\vee} & \xrightarrow{\sim} & f_+f^*O_{Y,Q} \\
\end{array}
\]

Here the vertical morphisms are isomorphic. This diagram is commutative. The commutativity of the left and right square immediately follows by definition. To check the commutativity for the middle one, we need to go back to the definition, which is [VI IV.1.3]. We note that since \( f \) is finite étale, the trace morphism \( f_+O_{X,Q} \rightarrow O_{Y,Q} \) defined in [VI III.5.1] is equal to \( \text{Tr}_f \) via the identification \( f_+O_{X,Q} \cong f_!O_{X,Q} \). Since the commutativity is long and tedious routine work, we leave the detail to the reader. Now, the verification of the lemma in the finite étale case is reduced to showing the composition of the lower row is the adjunction homomorphism. This is easy.

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In general, there exists an open dense formal subscheme \( j : \mathcal{U} \subset \mathcal{Y} \) such that \( f' : \mathcal{X}' := \mathcal{X} \times_{\mathcal{Y}} \mathcal{U} \to \mathcal{Y} \) is finite étale. Put \( j' : \mathcal{X}' \hookrightarrow \mathcal{X} \). Then by [Ber2, 4.3.10], we have an injection \( K_X^\sim \hookrightarrow j_{+}^! K_{\mathcal{X}}^\sim \), where \( X' \) is the special fiber of \( \mathcal{X}' \). Since \( f \) is affine, \( f_+ \) is exact, and the homomorphism \( f_+^! K_X^\sim \to j_{+} f_+^! K_{\mathcal{U}}^\sim \) is injective. Consider the following diagram where we omit \( ||_{\mathcal{Y}} \):

\[
\begin{array}{ccc}
K_Y^\sim & \xrightarrow{j_{+}^!} & f_+^! K_Y^\sim \\
\sim & & \sim \\
O_{\mathcal{U}, Q} & \xrightarrow{j_+ f_+^* O_{\mathcal{U}, Q}} & f_+ f_+^* O_{\mathcal{U}, Q}.
\end{array}
\]

The diagram is known to be commutative except for the forehead square diagram. By the injectivity of \( \ast \), the desired commutativity follows by the commutativity of other surfaces.

The trace homomorphism satisfies the base change property: namely considering the cartesian diagram of (1.1.3.1) such that \( Y \) and \( Y' \) are smooth liftable, then the diagram of (Var 2) is commutative. To check this, it suffices to check the (dual of) base change property for the dual of trace map \( K_Y^\sim \to f_+ f_+^! K_Y^\sim \). By using [Ber2, 4.3.10], it suffices to show the base change property after taking \( ||_{\mathcal{Y}} \). The lemma above reduces the verification to the base change property for the adjunction homomorphism \( O_{\mathcal{U}, Q} \to f_+ f_+^* O_{\mathcal{U}, Q} \) which is easy. The transitivity can also be checked by a similar argument.

**1.5.7. Proof of Lemma 1.5.4**

Let us construct the trace map for general étale morphism. Assume that we have the following cartesian diagram

(1.5.7.1)

\[
\begin{array}{ccc}
X & \overset{j}{\leftarrow} & \tilde{X} \\
\downarrow f & \square & \downarrow g \\
Y & \overset{i}{\leftarrow} & \tilde{Y}
\end{array}
\]

where \( \tilde{Y} \) is smooth liftable, \( g \) is affine étale, and the horizontal morphisms are closed immersions. We have the following uncompleted diagram of solid arrows:

\[
\begin{array}{ccc}
i_{+}^! g_{+}^! K_{\tilde{Y}} & \xrightarrow{\text{Tr}_{\tilde{g}}} & i_{+}^! g_{+}^! K_{\tilde{Y}} \\
\sim & & \sim \\
f_+ i_{+}^! K_{\tilde{Y}} & \sim & f_+ i_{+}^! K_{\tilde{Y}}.
\end{array}
\]

The left vertical homomorphism is an isomorphism by the transitivity, and the dotted homomorphism is defined so that the diagram commutes, which is \( f_+^! K_X \sim \xrightarrow{\sim} f_+^! K_X \). This induces a homomorphism \( f_+ f_+^! K_Y \to K_Y \). Since the base change property for liftable schemes have already been checked, this homomorphism is compatible with base change by a closed immersion from a point \( s \hookrightarrow Y \). The compatibility with composition follows as well. By the uniqueness mentioned at the beginning of 1.5.4, the map does not depend on the choice of the diagram (1.5.7.1).

In general, we can take the diagram (1.5.7.1) locally on \( X \) and \( Y \). By using Lemma 1.5.3, we can glue, and get the trace map \( f_+ f_+^! K_Y \to K_Y \) in general. The verification is the same as [SGA 4, XVIII 2.9 c)], and the details are left to the reader.

Now, let \( \mathcal{F} \in D^b_{\text{hol}}(Y) \). By (Var-5), the trace map for \( \mathcal{F} \) should be

\[
f_+ f_+^! \mathcal{F} \simeq f_+ f_+^! K_Y \otimes \mathcal{F} \xrightarrow{\text{Tr}_f} \mathcal{F}.
\]
This map satisfies (Var 1) to (Var 4) since they hold for \( \mathcal{F} = K_Y \). Taking the adjunction, we have a homomorphism \( f^+(\mathcal{F}) \to f^!(\mathcal{F}) \). Let us show that this is an isomorphism by induction. When \( Y \) is a point, this is easy. Let \( s \in Y \) be a closed point, and consider the following diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{i^*} & X \\
f_s \downarrow & \xrightarrow{\square} & f \\
\tau \downarrow & & \tau \\
Y & \xrightarrow{i_s} & Y
\end{array}
\]

By the compatibility of trace map by base change, the following diagram is commutative

\[
\begin{array}{ccc}
i^+_s f^+(\mathcal{F}) & \xrightarrow{\sim} & i^+_s f^!\mathcal{F} \\
\downarrow & & \downarrow \\
f^+_s i^+_s (\mathcal{F}) & \xrightarrow{\sim} & f^+_s i^+_s (\mathcal{F}).
\end{array}
\]

By (the dual of) smooth base change [1.5.5], the right vertical homomorphism is an isomorphism, and by the point case, the bottom horizontal homomorphism is an isomorphism as well. Thus by [AC1] 1.3.11, the claim follows.

**1.5.8.** Let us construct the trace map for quasi-finite flat morphism. We follow the construction of [SGA 4, XVII 6.2]. Let \( f : X \to Y \) be a quasi-finite flat morphism. For an étale morphism \( U \to Y \), we consider the category \( \Psi_f(U) \) defined as follows: it consists of collections \( (V_i)_{i \in I} \) of pointed set \( 0 \in I \), and a decomposition \( X \times_Y U = \coprod_{i \in I} V_i \) such that \( V_i \to U \) is finite for \( i \neq 0 \). We denote by \( I^0 := I \setminus \{0\} \).

A morphism from \( \varphi = (V_i)_{i \in I} \) to \( \varphi' = (V'_i)_{i \in I'} \) is a morphism \( \sigma : I \to I' \) such that \( \sigma(0) = 0 \) and \( V_i = \bigcup_{j \in \sigma^{-1}(i)} V'_j \). For a morphism \( U \to V \) in \( Y_{et} \), there exists the obvious functor \( \Psi_f(V) \to \Psi_f(U) \), and \( \Psi_f(U) \) is a fibered category over \( Y_{et} \). This category is denoted by \( \Psi_f \), and an object of the fiber over \( U \in Y_{et} \) is denoted by \( \{U; (V_i)_{i \in I}\} \). We refer to [ibid.] for some details.

**Lemma.** — Let \( \mathcal{F} \in \text{Con}(Y) \). Then there exists the canonical isomorphism

\[
\tau_f : \lim_{(U; \varphi) \in \Psi_f} j_U^! j_U^+ \mathcal{F}^* \xrightarrow{\sim} f^! f^+ \mathcal{F},
\]

where \( j_U : U \to Y \) is the étale morphism.

**Remark.** — Before proving the lemma, we remark that the inductive limit is not filtrant.

**Proof.** The verification is essentially the same as [ibid.]. Let us construct the homomorphism. Take \( \varphi = \{U; (V_i)_{i \in I}\} \). Let \( j_i : V_i \to X \) be the étale morphism. Since \( f_i : V_i \to U \) is assumed to be finite for \( i \in I^0 \), we have the left homomorphism below:

\[
\mathcal{F} \to f_i^+ f^+_i (\mathcal{F}) \xleftarrow{\sim} f_i j_i^+_i (\mathcal{F}), \quad j_U j_U^! j_U^+ (\mathcal{F}) \to j_U j_U^! f_i^+ j_i^+_i (\mathcal{F}) \cong f j_i^+ (\mathcal{F}) \xrightarrow{\text{Tr}^\gamma} f_i^+ (\mathcal{F}).
\]

By using the trace map in Lemma [1.5.3] we get the right homomorphism above, which induces the homomorphism in the statement.

Now, by [1.2.2.1] it suffices to show the isomorphism in \( \text{Ind}(\text{Con}(X)) \). When \( f \) is a universal homeomorphism, the canonical homomorphism \( \mathcal{F} \to f^! f^+ \mathcal{F} \) is an isomorphism by Lemma [1.3.3]. Assume \( Y := s \) is a point. There is a separable extension \( s' \to s \) such that \( X \times s' \to s' \) is disjoint union of universal homeomorphisms. Thus, the lemma follows by Lemma [1.3.10] (ii).

Let \( s \) be a closed point of \( Y \). Put \( i_s : s \to Y \) to be closed immersion. Since \( i_s^+ \) is an exact functor and commutes with direct sum, it commutes with arbitrary inductive limits. Thus, we have

\[
i_s^+ (\lim_{\Psi_f} j_U^! \mathcal{F}^*) \cong \lim_{\Psi_f} i_s^+ j_U^! \mathcal{F}^*.\]

Let \( f_s : X \times_Y s \to s \). There exists a functor \( \Psi_f \to \Psi_f \). This functor is cofinal by [EGA IV, 18.12.1]. Then by Lemma [1.5.3] \( i_s^+ \tau_f \cong \tau_{f_s} \), and by the proven case where \( Y \) is a point, \( i_s^+ \tau_f \) is an isomorphism. By Lemma [1.3.10] (i), this implies that \( \tau_f \) is an isomorphism as required. ■
1.5.9. Let \( f : X \to Y \) be a quasi-finite flat morphism between realizable schemes. Let us construct the unique trace map \( f_! f^+ K_Y \to K_Y \) satisfying (Var 1.2.3.4.1). When \( f \) is étale, we remark that this trace map coincides with that of Lemma [1.5.4] by the uniqueness. The construction is the same as \([SGA 4, XVII 6.2]\), so we only sketch.

Let \( \Psi_f \) be the full subcategory of \( \Psi_f \) consisting of \( \{ U; (V_i)_{i \in I} \} \) such that \( V_i \) is locally free of constant rank over \( U \) for any \( i \neq 0 \). This category is cofinal in \( \Psi_f \). For each \( \{ U; (V_i)_{i \in I} \} \in \Psi_f \), we have a homomorphism

\[
\sum_{i \in I} \deg(V_i/U) \cdot \text{Tr}_{j_U} : j_{U!} j_U^+ (K_Y^P) \to K_Y.
\]

Since the compatibility follows by that of Lemma 1.5.4, we get a homomorphism

\[
f_! f^+ K_Y \cong \lim_{\varphi \in \Psi_f} j_{U!} j_U^+ (K_Y^P) \to K_Y.
\]

It is easy to check that this is what we are looking for.

1.5.10 Lemma. — Let \( f : X \to Y \) be the special fiber of a finite flat morphism between smooth formal curves \( \bar{f} : \mathcal{X} \to \mathcal{Y} \). By taking the dual of the trace map, we get \( K_Y^P \to f_+ K_Y^P \). When we restrict this homomorphism to \( \mathcal{Y} \) (i.e. taking \( \|\|_\mathcal{Y} \) of [1.5.4]), this dual of trace map is nothing but the homomorphism induced by the adjunction homomorphism \( \phi_f \) : \( \mathcal{O}_{\mathcal{X}, \mathcal{Q}} \to f_* \mathcal{O}_{\mathcal{X}, \mathcal{Q}} \) composed with the canonical homomorphism \( \bar{f} \mathcal{O}_{\mathcal{X}, \mathcal{Q}} \to \bar{f}_* \mathcal{O}_{\mathcal{X}, \mathcal{Q}} \).

Proof. We may assume \( \mathcal{X} \) and \( \mathcal{Y} \) to be connected. Let \( L_{\mathcal{X}}, L_{\mathcal{Y}} \) be the largest finite extension of \( K \) in \( \mathcal{X}, \mathcal{Y} \). If \( L_{\mathcal{X}} \neq L_{\mathcal{Y}} \), then \( f \) factors as

\[
\mathcal{X} \cong \mathcal{Y} \otimes_{R_{\mathcal{Y}}} R_{\mathcal{X}} \to \mathcal{Y},
\]

where \( R_* \) denotes the ring of integers of \( L_* \), and it suffices to show the lemma for \( \alpha \) and \( \beta \) separately. The verification for \( \beta \) is easy. Let us show that for \( \alpha \). In this case, we let \( L_{\mathcal{X}} = L_{\mathcal{Y}} = L \).

Since the formal schemes are curves, by Kedlaya’s faithfulness theorem [Kel], the homomorphism \( \mathcal{O}_{\mathcal{Y}, \mathcal{Q}} \to \bar{f}_* \mathcal{O}_{\mathcal{X}, \mathcal{Q}} \) induced by \( \phi_f \) extends uniquely to the homomorphism \( \phi : K_Y^P \to f_+ K_Y^P \). We have

\[
\text{Hom}(K_Y^P, f_+ f^! K_Y^P) \cong \text{Hom}(f_+ K_Y^P, f^! K_Y^P) \sim \text{Hom}(K_Y^P, K_Y^P) \cong L
\]

where \( \sim \) is the isomorphism induced by [A1] Theorem 5.5 for \( \mathcal{X} \) and \( \mathcal{Y} \) are smooth. Thus, there exists \( c \in L \) such that \( c \cdot \phi = D(Tr_f) \). It remains to show that \( c = 1 \). By definition, the composition

\[
K_Y \to f_! f^+ K_Y \to K_Y
\]

is the multiplication by \( n := \deg(f) \). Take the dual of this homomorphism, and we get

\[
n : K_Y^P \xrightarrow{D(Tr_f)} f_+ f^! K_Y^P \xrightarrow{\text{Tr}^\text{Vir}_f} K_Y^P,
\]

where the second homomorphism is the trace homomorphism of [V1]. On the other hand, by property of \( \text{Tr}^\text{Vir}_f \) (cf. [V1] III.5.4), we get \( \text{Tr}^\text{Vir}_f \circ \phi = n \). Thus \( c = 1 \) since \( \text{Hom}(K_Y^P, K_Y^P) \cong L \).

1.5.11. Let \( f : \mathcal{X} \to \mathcal{Y} \) be a proper smooth morphism of relative dimension 1 between smooth proper formal schemes. We define the homomorphism

\[
1.5.11(1) \quad \mathcal{O}_{\mathcal{Y}, \mathcal{Q}} \to \mathbb{R} f_* [0 \to \mathcal{O}_{\mathcal{X}, \mathcal{Q}} \to \Omega_{\mathcal{X}/\mathcal{Y}, \mathcal{Q}} \to 0]
\]

in \( D(\mathcal{O}_{\mathcal{Y}, \mathcal{Q}}) \), by sending \( 1 \in \mathcal{O}_{\mathcal{Y}, \mathcal{Q}} \) to \( 1 f_* \mathcal{O}_{\mathcal{X}, \mathcal{Q}} \). Since the target of the homomorphism is canonically isomorphic to \( f_* f^! \mathcal{O}_{\mathcal{X}, \mathcal{Q}} [-1] \), we have a homomorphism \( \alpha_f : \mathcal{O}_{\mathcal{Y}, \mathcal{Q}}(1)[2] \to f_* f^! \mathcal{O}_{\mathcal{Y}, \mathcal{Q}} \). By [A1] 3.14, this homomorphism is compatible with Frobenius structure when \( \alpha = 0 \). This trace morphism only depends on the special fibers because unit element is sent to unit element by ring homomorphism. Thus, we have a homomorphism \( \mathcal{O}_{\mathcal{Y}, \mathcal{Q}}(1)[2] \to f_* f^! \mathcal{O}_{\mathcal{Y}, \mathcal{Q}} \). By construction, this homomorphism is compatible with base change, namely, given a morphism of proper smooth formal schemes \( g : \mathcal{Y}' \to \mathcal{Y} \) such that \( d := \dim(\mathcal{Y'}) - \dim(\mathcal{Y}) \), let \( f' : \mathcal{X}' := \mathcal{X} \times_\mathcal{Y} \mathcal{Y}' \). Then the following diagram is commutative:

\[
\begin{array}{c}
\mathcal{O}_{\mathcal{Y}, \mathcal{Q}}(1)[2] \xrightarrow{\sim} g^! \mathcal{O}_{\mathcal{Y}, \mathcal{Q}}(1)[2 - d] \\
\downarrow \alpha_f \downarrow \alpha_f \downarrow \alpha_f \\
f_* f^! \mathcal{O}_{\mathcal{Y}, \mathcal{Q}} \xrightarrow{\sim} g^! f_* f^! \mathcal{O}_{\mathcal{Y}, \mathcal{Q}}[-d]
\end{array}
\]
where the horizontal homomorphisms are canonical homomorphisms. By taking the duality, we get the trace map \( \text{Tr}_f: f_! f^+ K_Y(1)[2] \to K_Y \). This trace morphism is compatible with pull-back \( g^+ \) where \( g \) is a morphism between liftable proper smooth schemes.

Now, let us consider the case where \( \mathcal{P} \) is a point:

\[
\begin{array}{ccc}
\mathcal{U} & \xrightarrow{g} & \mathcal{X} \\
\downarrow f & & \downarrow \text{Spf}(R) \\
\end{array}
\]

where \( \mathcal{U} \) is dense open in \( \mathcal{X} \). Put \( Z := X \setminus U \) where \( X, U \) are the special fibers of \( \mathcal{X}, \mathcal{U} \), and assume \( Z \) is a divisor of \( X \). Then we have an injection

\[
\mathcal{H}^{-2} g_+ g^1 K \cong \mathcal{H}^{-1} f_+ O_{\mathcal{X}, \mathcal{U}}(1) Z \hookrightarrow \mathcal{H}^{-1} g_+ O_{\mathcal{U}, \mathcal{Q}}.
\]

1.5.12. Proof of Theorem 1.5.1

Now, we construct the trace map. We need several steps for the construction.

(i) Absolute curve case: Let \( f: X \to \text{Spec}(k) \) be a realizable variety. We put \( H^1_c(X) := \text{Hom}(K, f_!(K_X)[i]) \). Now, assume \( f \) to be a curve. Let us construct the trace map. We denote by \( \iota: X_{\text{red}} \hookrightarrow X \) the closed immersion. Let \( U \subset X \) be an open dense subscheme. Then there exists an isomorphism \( H^2_c(U) \xrightarrow{\sim} H^2_c(X) \).

Thus we may assume that \( f^! : X_{\text{red}} \to \text{Spec}(k) \) is proper smooth. We define

\[
H^2_c(X)(1) \xrightarrow{\text{Tr}^!_{U}} H^2_c(X_{\text{red}})(1) \xrightarrow{\text{Tr}^!} K.
\]

Lemma. — Let \( X \xrightarrow{f} Y \to \text{Spec}(k) \) be a morphism of realizable schemes such that \( f \) is quasi-finite flat morphism and \( g \) is of relative dimension 1. Then we have

\[
\text{Tr}_{g \circ f} = \text{Tr}_g \circ g_!(\text{Tr}_f) : (g \circ f)_! K_X(1)[2] \to K.
\]

Proof. Arguing as [SGA 4, XVIII, 1.1.5], we may assume that \( X \) and \( Y \) are connected smooth affine. Shrink \( X \) and \( Y \) further, we may assume that there exist smooth liftings \( \mathcal{X} \xrightarrow{\tilde{f}} \mathcal{Y} \to \text{Spf}(R) \) such that \( \tilde{f} \) is finite flat. In this case, it suffices to check the transitivity after removing the boundary by the injectivity of \([1.5.11.2] \), and the lemma follows by Lemma [1.5.10] and the definition of \([1.5.11.1] \).

(ii) Relative affine space case: Let \( Y \) be a realizable scheme, and consider the projection \( f: X := \mathbb{P}^1_Y \to Y \). There exists a proper smooth formal scheme \( \mathcal{P} \) such that \( Y \to \mathcal{P} \). Then \( f \) can be lifted to the following cartesian diagram:

\[
\begin{array}{ccc}
\mathbb{P}^1_Y & \xrightarrow{f} & \mathbb{P}^1_{\mathcal{P}} \\
\downarrow i & & \downarrow i \\
Y & \xrightarrow{i} & \mathcal{P}.
\end{array}
\]

We define the trace map

\[
\text{Tr}_f: f_! f^+ K_Y(1)[2] \cong i^+ i_! f_! f^+ K_{\mathcal{P}}(1)[2] \xrightarrow{i^+ \text{Tr}^!_f} i^+ K_{\mathcal{P}} \cong K_Y,
\]

where \( \text{Tr}_f \) is the one defined in [1.5.11]. This map does not depend on the choice of \( \mathcal{P} \) by the base change property of \( \text{Tr}_f \).

When \( f: \mathbb{A}^1_X \to X \) is the projection, then we have the factorization \( \mathbb{A}^1_X \xrightarrow{i} \mathbb{P}^1_X \xrightarrow{p} X \), and the trace map \( \text{Tr}_f \) is defined to be the composition \( \text{Tr}_{\mathbb{P}} \circ p_!(\text{Tr}_i) \). The base change property can be checked by the base change property for \( j \) and \( p \). Now, let \( f: X := \mathbb{A}^1_Y \to Y \). In this case, we define by iteration as in [SGA 4, XVIII, 2.8].

(iii) Factorization case: Let \( f: X \to Y \) be a morphism which possesses a factorization \( X \xrightarrow{u} \mathbb{A}^d_Y \xrightarrow{a} Y \) such that \( u \) is a quasi-finite flat morphism. Then we define \( t(f, u) := \text{Tr}_{a_!} \circ a_!(\text{Tr}_u) \). We need to check
that the definition does not depend on the choice of the factorization. By using the lemma in (i), the
verification is the same as [SGA 4, XVIII, 2.9 b]).

(iv) General case: The construction is the same as [ibid.]. We sketch the construction. When \( f \) is
Cohen-Macaulay morphism, then there exists a finite covering of \( \{ U_i \} \) of \( X \) such that the compositions
\( U_i \to X \to Y \) possess factorization considered in case (iii). By gluing lemma 1.5.3, we have the trace
map in this case. In general, we shrink \( X \) suitably, so that \( f \) is Cohen-Macaulay. Thus the trace map is
constructed, and we conclude the proof of Theorem 1.5.1.■

1.5.13 Theorem (Poincaré duality). — Let \( X \to Y \) be a smooth morphism of relative dimension \( d \)
between realizable schemes. Then for \( F \in D^b_{hol}(Y) \), the adjoint of the trace map \( \phi_F : f^+ F(d)[2d] \to f^! F \)
is an isomorphism.

Proof. Since the verification is local on \( X \), it suffices to treat the case where \( f \) is étale and the projection
\( \bar{\mathcal{V}} \to Y \) separately. The étale case has already been treated in Lemma 1.5.4.

Let us treat the projection case. We may shrink \( Y \). Then we can embed \( Y \) into a proper smooth
formal scheme. By using [A1] Theorem 5.5, we have an isomorphism \( f^+ F(d)[2d] \sim f^! F \), which may not
be the same as the one defined by the trace map. It suffices to show the theorem for \( F \in \text{Hol}(Y) \). For
this, we may assume \( F \) to be irreducible. Let \( k' \) be a finite extension of \( k \). It suffices to show that \( \phi_F \)
is an isomorphism after pulling-back to \( X \otimes_k k' \). Thus, we may assume moreover that \( F \) is irreducible
also on \( X \otimes_k k' \) for any extension \( k' \) of \( k \). For a closed point \( a \in \mathbb{A}^1 \), we denote \( i_a : Y \otimes_k k(a) \to \bar{\mathcal{V}} \)
the closed immersion defined by \( a \). We claim that \( f^+(F) \) is irreducible. Indeed, first, let us assume \( Y \)
is smooth and \( F \) is smooth. Assume \( f^+(F) \) were not irreducible. Then there would exist a smooth object
\( N \subset f^+(F) \) and a closed point \( a \) of \( \mathbb{A}^1(F) \) such that \( i_a^+ N \) and \( i_a^+(f^+(F)/N) \) are non-zero. This is a
contradiction. In general, \( F \) can be written as \( j_+ \) of a smooth irreducible object by [AC] 1.4.9 where
\( j \) is an open immersion of \( Y \). Since \( f \) is smooth, \( f^+ \) and \( j_+ \) commute, and \( f^+(F) \) is irreducible.

Now, we know that
\[
\text{Hom}(f^+ F(d)[2d], f^! F) \sim \text{Hom}(f^+ F(d)[2d], f^+ F(d)[2d]).
\]
Since \( f^+ F \) is irreducible, the Hom group is a division algebra, and it remains to show that \( \phi_F \) is not
0. For this, it suffices to check that the trace homomorphism \( f^+ f^+ F(d)[2d] \to F \) is non-zero. By base
change property of trace map, we may assume \( Y \) to be a point, in which case, the trace map is non-zero
by construction.■

Corollary. — Let \( f : X \to Y \) be a flat morphism of relative dimension \( d \) between smooth realizable
schemes. Then the adjoint of trace map \( f^+ KY(d)[2d] \to f^! KY \) is an isomorphism.

Proof. This follows by the transitivity of trace map and the Poincaré duality.■

1.5.14. Let \( i : Z \to X \) be a closed immersion of codimension \( c \) between smooth realizable schemes. By
using the Poincaré duality, we have a canonical isomorphism \( i^+ K_X^c \to i^! i^+ K_X^c \). Let us denote
\( \cdot \otimes (-) := \mathbb{D}(\cdot \otimes \mathbb{D}(\cdot)) \). The projection formula yields the homomorphism \( i^+ (N \otimes M) \to
i^+ (N) \otimes i^! (M) \) for \( M, N \) in \( D^b_{hol}(X/K) \). Using this homomorphism, we get a homomorphism
\[
i^+ (M)(-c)[-2c] \cong i^+ (K_X^c \otimes N)(-c)[-2c] \to i^+ (K_X^c)(-c)[-2c] \otimes i^! (M)
\]
\[
\cong i^! (M).
\]
(1.5.14.1)

Theorem. — If \( M \) is smooth, then the canonical homomorphism (1.5.14.1) is an isomorphism.

Proof. It suffices to show that when \( M \) is a smooth holonomic module, the canonical homomorphism
\( i^+ (N \otimes M) \to i^! (N) \otimes i^! (M) \) is an isomorphism for any \( N \in D^b_{hol}(X/K) \). Since \( i_+ \) is conservative, it
suffices to show that the homomorphism
\[
\rho : i_+ i^+(N \otimes M) \to i_+ (i^+(N) \otimes i^+(M)) \cong i_+ i^+(N) \otimes i_+ M
\]

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is an isomorphism. By definition, this is the unique homomorphism which makes the following diagram commutative:

\[
\begin{array}{ccc}
N \otimes M & \xrightarrow{i_+ i^+} (N \otimes M) \\
\alpha & \downarrow \rho & \downarrow \beta \\
i_+ i^+ (N \otimes M) & \end{array}
\]

where \( \alpha := \text{adj}_j \) and \( \beta := \text{adj}_j \otimes \text{id} \). Now, since the verification is local, we may assume that \( Z \) and \( X \) can be lifted to smooth formal schemes \( \mathcal{Z} \) and \( \mathcal{X} \). It suffices to show the claim after removing the boundaries by [Ber2, 4.3.10]. In this situation, recall that \((-) \otimes (-) \cong (-) \otimes^\mathbb{L} \omega_{\mathcal{X}}(-)[- \text{dim}(\mathcal{X})] \) (cf. [AC1, 1.1.6]). Since \( M \) is a coherent \( \mathcal{O}_\mathcal{X} \)-module, we have a canonical isomorphism

\[
\mathbb{R} \text{Hom}_{\mathcal{O}_\mathcal{X}}(N \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{M}, \mathcal{D}_\mathcal{X} \otimes \omega_{\mathcal{X}}^{-1}) \cong \mathbb{R} \text{Hom}_{\mathcal{O}_\mathcal{X}}(N, \mathcal{D}_\mathcal{X} \otimes \omega_{\mathcal{X}}^{-1} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{M}^\vee),
\]

where \( \mathcal{D}_\mathcal{X} \) denotes \( \mathcal{D}^+_\mathcal{X}, \mathbb{Q} \). This yields an isomorphism \( \mathcal{D}(N \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{M}) \cong \mathcal{D}(N) \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{M}^\vee \). Consider the following diagram:

\[
\begin{array}{ccc}
\mathcal{D}(i_+ i^+ (N \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{M})) & \xrightarrow{\sim} i_+ i^+ (\mathcal{D}(N \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{M})) & \xrightarrow{\sim} \mathcal{D}(N) \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{M}^\vee \\
\mathcal{D}(i_+ i^+ (N \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{M})) & \xrightarrow{\sim} i_+ i^+ (\mathcal{D}(N) \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{M}^\vee) & \xrightarrow{\sim} \mathcal{D}(N) \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{M}^\vee
\end{array}
\]

where the right horizontal homomorphisms are adjunction homomorphisms, and we define the isomorphism * so that the diagram is commutative. The right square is commutative by [Ca5, 2.2.7]. The composition of upper (resp. lower) horizontal homomorphisms is the dual of \( \alpha \) (resp. \( \beta \)). By the characterization of \( \rho \), the isomorphism * is the dual of \( \rho \), which implies that \( \rho \) is an isomorphism. \[\square\]

2. Arithmetic \( \mathcal{D} \)-modules for algebraic stacks

This section is devoted to construct a \( p \)-adic cohomology theory for algebraic stacks. In fact, even though we do not try to axiomize, the results of this section hold also for any non-torsion cohomology theories with standard six functors formalism without essential changes. (e.g. algebraic \( \mathcal{D} \)-module theory, étale cohomology theory with perverse t-structures over a separably closed field with non-torsion coefficients, etc.)

2.0.1. In this section, \( \Delta \in \{0, \emptyset\} \) is fixed. Throughout this section, we fix a base tuple \((k, R, K, s, \sigma)\), and a finite extension \( L \) of \( K \). When \( \Delta = 0 \), we moreover fix an extension \( \sigma_L : L \sim L \sigma \) of \( \sigma \) (cf. [1.4.1]1). When \( \Delta = \emptyset \), Tate twists \( (n) \) are understood to be the identity functors.

2.0.2. To construct a theory for general algebraic stacks, we first need to construct a theory for algebraic spaces, and making use of the theory, we may construct that for algebraic stacks. Since the process for the construction from realizable schemes to algebraic spaces and that from algebraic spaces to algebraic stacks are parallel, we try to present these processes at once. Thus to get the theory for general algebraic spaces, one needs to read in the following way: First, read \( \S 2.1 \ S 2.2 \) by replacing the indicated terminologies in these sections according to the following table. Then reread \( \S 2.1 \ S 2.2 \) This time, replace the terminologies by the “second read” following the table below.

| 1st read | 2nd read |
|----------|----------|
| space    | good stack | good presentation |
| quasi-projective scheme | algebraic stack whose diagonal morphism is quasi-projective | smooth surjective morphism from a quasi-projective scheme |
| separated algebraic space of finite type | algebraic stack of finite type | smooth surjective morphism from a separated quasi-compact algebra space |

Note that since good stacks are of finite type, they are in particular quasi-compact. See paragraph \( \S 2.2.22 \ S 2.2.23 \) for additional explanation. Finally, we remark that admissible stacks we define in \( \S 2.3 \) are good stack in the sense of first read, and second read is not needed if the reader is only interested in the proof of the Langlands correspondence.
2.1. Definition of the derived category of $\mathcal{D}^i$-modules for stacks

2.1.1. We first need basic cohomological operations for spaces. Let $X$ be a space over $k$. In the “first read case”, we have already defined $M(X/L)$ and $D(X/L)$ in 1.4.11 and in the “second read case”, see 2.2.22.

**Smooth morphism:** Let $f : X \to Y$ be a smooth morphism between spaces over $k$. The exact functor $f^* : \text{Hol}(Y/K) \to \text{Hol}(X/K)$ (cf. 1.2.8 or 2.2.5) can be extended canonically to $M(Y/L) \to M(X/L)$ by 1.2.2 and 1.4.10 which remains to be exact. The derived functor is also denoted by $f^*$. Similarly, we have the left exact functor $f_* : M(X/L) \to M(Y/L)$, and we can take the derived functor $\mathbb{R}f_* : D^+(X/L) \to D^+(Y/L)$. By 1.2.1, $(f^*, \mathbb{R}f_*)$ is an adjoint pair. Since for $L$ and for $F$ commutes with $f^*$ and $\mathbb{R}f_*$ by 1.4.1 and 1.4.7, $f^*$ and $\mathbb{R}f_*$ preserve holonomicity, and induce functors between $D^+_{\text{hol}}(X/L)$ and $D^+_{\text{hol}}(Y/L)$.

**Finite morphism:** Let $f : X \to Y$ be a finite morphism between spaces over $k$. Just as in the smooth morphism case, we can define functors

$$f_+: D^+(X/L) \cong D^+(Y/L) : f^!$$

where $* = \emptyset$ for $f_+$ and $+ = f^!$. The pair $(f_+, f^!)$ is adjoint. These functors commute with $f_+$ and $f_+^!$, and preserve boundedness and holonomicity.

**External tensor product:** Let $X, Y$ be spaces over $k$. The bifunctor $\boxtimes : M(X/L) \times M(Y/L) \to M(X \times Y/L)$, extending the external tensor product functor, is exact. Thus, we can take the derived functor. This derived functor preserves boundedness and holonomicity as well.

**Dual functor:** Let $X$ be a space over $k$. The dual functor extends canonically to $\mathbb{D}_X : \text{Hol}(X/L)^\circ \to \text{Hol}(X/L)^\circ$. This induces the functor

$$\mathbb{D}_X : D^b(\text{Hol}(X/L))^\circ \to D^b(\text{Hol}(X/L))^\circ.$$ 

**Lemma.** (i) Consider the cartesian diagram (1.3.7) of spaces such that $f$ is smooth and $g$ is finite. Then the base change homomorphisms $f^* \circ \mathcal{H}^0(g^! ) \to \mathcal{H}^0(g^! ) \circ f^* : M(Y/L) \to M(X/L)$ and $f^* \circ \mathcal{H}^0(g^! ) \to \mathcal{H}^0(g^! ) : M(X/L) \to M(Y/L)$ are isomorphisms. Moreover, we have $\mathcal{H}^0(g^! ) \circ f_* \cong f'_* \circ \mathcal{H}^0(g^! ) : M(X/L) \to M(Y'/L)$.

(ii) For smooth morphisms $f : X \to Y$ of spaces of relative dimension $d$, we have the canonical isomorphism

$$(f^* \circ \mathbb{D}_Y)(d) \cong \mathbb{D}_X \circ f^* : \text{Hol}(Y)^\circ \to \text{Hol}(X).$$

**Remark.** These equalities also holds on the level of $D^b_{\text{hol}}(-/L)$, which we show later.

**Proof.** In the second read case, we can easily reduce to the first read case, so we assume we are in the first read case. When $\Delta = \emptyset$ and $L = K$, we only need to show the equality for $\text{Hol}(-/K)$, and these are consequences of Corollary 1.5.5, Theorem 1.5.13, 1.1.3.7. Since the equality is on the level of modules, these results can be extended automatically to general $\Delta$ and $L$.

2.1.2. Now, let us fix some terminologies on simplicial spaces.

**Definition.** We denote by $[i]$ the totally ordered finite set $[0, i]$. Let $\Delta^+$ be the category having $[i]$ ($i \geq 0$) as objects, and morphisms are increasing injective maps between $[i]$.

(i) An admissible simplicial space is a contravariant functor $\Delta^+ \to \text{Sp}^{sm}(k)$ where $\text{Sp}^{sm}(k)$ denotes the category of spaces over $k$ whose morphisms are smooth. Let $X_\bullet$ be an admissible simplicial space. For $i \geq 0$, we often denote $X_\bullet(i)$ by $X_i$. This can be described as follows:

$$X_\bullet : \left[ X_0 \rightleftarrows X_1 \rightarrow X_2 \rightarrow \cdots \right].$$

For a space $S$, let $S_\bullet$ be the constant admissible simplicial space. A morphism of an admissible simplicial space to a space $X_\bullet \to S$ is a morphism $X_\bullet \to S$ is the morphism is said to be smooth if $X_0 \to S$ is.
(ii) A morphism of simplicial spaces \( f : X \to Y \) is said to be cartesian if for any \( \phi : [i] \to [j] \), the following diagram is cartesian:

\[
\begin{array}{c}
X_j \xrightarrow{f_j} Y_j \\
\downarrow X(\phi) \quad \square \quad \downarrow Y(\phi) \\
X_i \xrightarrow{f_i} Y_i,
\end{array}
\]

(iii) An admissible double simplicial space \( X_{\bullet \bullet} \) is a functor \( ((\Delta^+)^2)^o \to \text{Sp}^{sm}(k) \). A morphism from an admissible double simplicial space to an admissible simplicial space \( X_{\bullet \bullet} \to S_{\bullet} \) is defined similarly.

**Remark.** — Note that we do not consider “degeneracy maps”, and only “face maps”. This type of objects are sometimes called strictly simplicial scheme (e.g. [LO]).

**2.1.3 Definition.** — Let \( X \) be an algebraic stack, and \( \mathcal{X} \to X \) be a presentation. Put \( \mathcal{X}_n := \text{cosk}_0(\mathcal{X} \to X) \) (i.e. the simplicial space such that \( \mathcal{X}_n := X \times X \times \cdots \times X \mathcal{X} \) and the face morphisms are projections).

We say such a simplicial algebraic space a simplicial presentation algebraic space of \( X \). Let \( \mathcal{P} \) be a set of algebraic spaces. A simplicial presentation \( \mathcal{P} \) is a simplicial presentation algebraic space \( X_{\bullet} \), consisting of algebraic spaces belonging to \( \mathcal{P} \) (e.g. simplicial presentation realizable schemes,......).

Now, let \( X \) be a good stack, and \( X \to \mathcal{X} \) be a good presentation. Since \( \mathcal{X} \) is a good stack, \( \text{cosk}_0(X \to \mathcal{X}) \) is an admissible simplicial space. In particular, for any good stack, we may take a simplicial presentation space.

**2.1.4 Definition.** — Let \( X_{\bullet} \) be an admissible simplicial space. For a morphism \( \phi : [i] \to [j] \), since \( X(\phi) \) is smooth, the pull-back \( X(\phi)^* : M(X_i/L) \to M(X_j/L) \), which is exact, is defined (cf. [2.1.4]). This defines a cofibered category \( M(X_{\bullet}/L)_{\bullet} \) over \( \Delta^+ \).

(i) We put \( M(X_{\bullet}/L) := \text{sec}_1(M(X_{\bullet}/L)) \) (see [2.1.4] for the notation). We often denote \( M(X_{\bullet}/L) \) by \( M(X_{\bullet}) \). For \( \mathcal{M} \) or \( \mathcal{M}^o \) in \( M(X_{\bullet}/L) \), the fiber over \([i]\) is denoted by \( \mathcal{M}_i \). For \( \phi : [i] \to [j] \), the homomorphism \( X(\phi)^* \mathcal{M}_i \to \mathcal{M}_j \) is called the gluing homomorphism.

(ii) We denote by \( \text{Hol}(X_{\bullet}/L) \) or simply by \( \text{Hol}(X_{\bullet}) \) the full subcategory of \( M(X_{\bullet}/L)_{\bullet} \) consisting of \( \mathcal{M}_i \) such that \( \mathcal{M}_i \in \text{Hol}(X_{\bullet}/L) \) for any \( i \geq 0 \).

(iii) We denote by \( D^+_\text{hol}(X_{\bullet}/L) \) or \( D^+\text{hol}(X_{\bullet}) \) the full subcategory of \( D^+(M(X_{\bullet}/L)) \) whose cohomology sheaves are in \( \text{Hol}(X_{\bullet}/L) \). We denote \( D^+_\text{hol}(M(X_{\bullet}/L)_{\bullet}) \) by \( D^+\text{hol}(X_{\bullet}/L) \) or \( D^+\text{hol}(X_{\bullet}) \).

**2.1.5.** Let \( \mathcal{M} \) and \( \mathcal{N} \) be in \( D^+_\text{hol}(X_{\bullet}/L) \). Then by (A1.2.1), we have the following spectral sequence:

\[
E^{p,q}_1 := \text{Ext}^q_{D^+(X_{\bullet})}(\mathcal{M}_p, \mathcal{N}_q) \Rightarrow \text{Ext}^{p+q}_{D^+(X_{\bullet})}(\mathcal{M}, \mathcal{N}).
\]

We have kernel, cokernel, and inductive limit on \( M(X_{\bullet}) \), and they can be calculated termwise. This is because, for any \( \phi : [i] \to [j] \), the functors \( \text{Ker} \), \( \text{Coker} \), \( \text{lim} \) commute with \( X(\phi)^* \). In particular, \( M(X_{\bullet}) \) is an abelian category. Moreover, projective limits are representable in \( M(X_{\bullet}) \) and they can be calculated termwise as well, by using the canonical homomorphism \( X(\phi)^* \circ \text{lim} \to \text{lim} \circ X(\phi)^* \).

**2.1.6.** Let us define three basic functors. Take \( i \geq 0 \). We define a functor \( \rho_i^* : M(X_{\bullet}) \to M(X_i) \) by sending \( \mathcal{M} \in M(X_{\bullet}) \) to \( \mathcal{M}_i \). Obviously, this is an exact functor. Now, take \( \mathcal{N} \in M(X_i) \). We define

\[
\rho_\bullet^*(\mathcal{N}) := \left\{ \bigotimes_{\phi : [k] \to [i]} X(\phi)^*_s(\mathcal{N}) \right\}_k,
\rho_\Sigma(\mathcal{N}) := \left\{ \bigoplus_{\phi : [i] \to [k]} X(\phi)^*_a(\mathcal{N}) \right\}_k,
\]

and the gluing homomorphisms are define as follows: for \( \psi : [k] \to [k'], X(\psi)^*\rho_\Sigma(\mathcal{N})_k \to \rho_\Sigma(\mathcal{N})_{k'} \) (resp. \( X(\psi)^*\rho_\Sigma(\mathcal{N})_k \to \rho_\Sigma(\mathcal{N})_{k'} \)) is the product (resp. direct sum) of the adjunction (resp. canonical) homomorphisms

\[
X(\psi)^*X(\phi)^*_s(\mathcal{N}) \to X(\phi')^*_s(\mathcal{N}), \quad (\text{resp. } X(\psi)^*X(\phi')^*_a(\mathcal{N}) \to X(\phi)^*_a(\mathcal{N})).
\]

where \( \phi : [k] \xrightarrow{\phi} [k'] \xrightarrow{\phi'} [i] \) and 0 if \( \phi \) cannot be factored through \( \psi \) (resp. \( \phi' : [i] \xrightarrow{\phi'} [k] \xrightarrow{\phi} [k'] \)). These data define functors \( \rho_i^*, \rho_\Sigma : M(X_i) \to M(X_{\bullet}) \).
Lemma. — (i) We have adjoint pairs \((\rho_1^*, \rho_*^i)\) and \((\rho_0, \rho_1^*)\).
(ii) The functors \(\rho_*^i\) and \(\rho_1\) are exact. In particular \(\rho_*^i\) and \(\rho_*\) preserves injective objects.
(iii) The category \(M(X_\bullet)\) is a Grothendieck category.

Proof. Since the verification of (i) is standard, we leave the details to the reader. The first claim of (ii) follows from the exactness of \(X(\phi)^*\). Let us check (iii). We have arbitrary inductive limit as observed in \(#\) and filtrant inductive limits are exact. We need to show that it has a generator. Let \(\mathcal{G}_k\) be a generator of \(M(X_k)\). Then \(\{\rho_0(\mathcal{G}_k)\}_k\) is a set of generators. Indeed, let \(\mathcal{M} \in M(X_\bullet)\), and assume \(\text{Hom}(\rho_0, \mathcal{M}) = 0\) for any \(i \geq 0\). Then by (i), we have \(\text{Hom}(\mathcal{G}_k, \rho_1^*(\mathcal{M})) = 0\), and thus \(\mathcal{M} = 0\). Thus by definition, \(\mathcal{M} = 0\).

\[\mathcal{M}_k := \lim_{\phi \in D_k} X(\phi)^*(\mathcal{M}_i).\]

Then \(\{\mathcal{M}_i\}_i \leq k\) with obvious gluing homomorphism defines the desired object. The functor is denoted by \(\sigma_k^l\). We can check easily that the pair \((\sigma_k^l, \sigma_k^r)\) is adjoint.

2.1.8 Remark. — The functors defined in the last two paragraphs have natural interpretation in terms of the language of topos. Let \(X_\bullet\) be a (strictly) simplicial topos (cf. \([SGA 4, \text{V}bis]\)). Since sheaves \(\mathcal{F}\) on \(X_\bullet\) can be described as data \(\{\mathcal{F}_i, \phi_{ij}\}\) where \(\mathcal{F}_i\) is the sheaf on \(X_i\) and \(\phi_{ij}\) is the gluing homomorphism. We have a functor sending a sheaf \(\mathcal{F}\) on \(X_\bullet\) to the sheaf \(\mathcal{F}_i\) on \(X_i\). This functor defines a morphism of topos \(\epsilon_i : X_i \rightarrow X_\bullet\). The functors \(\rho_0^l, \rho_1^*, \rho_*\) are nothing but analogue of the functors \(\epsilon_i, \epsilon^*\), \(\epsilon_*\) (cf. \([\text{ibid., 1.2.8–1.2.12}]\)). The interpretation of \(\sigma_\delta, \sigma^\delta\) is similar.

2.1.9 Lemma. — Let us denote by \(\text{Hol}(X_\bullet)\) the cofibered category over \(\Delta^+\) such that the fiber over \([i]\) is \(\text{Hol}(X_i)\). Let \(D_{\text{tot}}^b(\text{Hol}(X_\bullet))\) be the derived category defined in \(\text{[A.1]}\). The canonical functor \(D_{\text{hol}}^b(\text{Hol}(X_\bullet)) \rightarrow D_{\text{tot}}^b(\text{Hol}(X_\bullet))\) is an equivalence of categories.

Proof. We can argue as \([\text{KSc 15.3.1}]\). It suffices to show that the functor

\[D_{\text{tot}}^b(\text{Hol}(X_\bullet)) \rightarrow D_{\text{tot}}^b(X_\bullet)\]

is fully faithful. By \([\text{KSc 13.2.8}]\), it suffices to show the following: given a surjection \(\mathcal{A} \rightarrow \mathcal{M}\) such that \(\mathcal{M} \in \text{Hol}(X_\bullet)\) and \(\mathcal{A} \in M(X_\bullet)\), there exists a homomorphism \(\mathcal{N} \rightarrow \mathcal{A}\) such that \(\mathcal{N} \in \text{Hol}(X_\bullet)\) and the composition \(\mathcal{N} \rightarrow \mathcal{M}\) is surjective. To check this, it suffices to construct, for each \(k \geq 0\), the following \(\mathcal{N}_k\) in \(M(\text{sk}_k(X_\bullet))\): 1. \(\sigma_k^*(\mathcal{N}_k) \cong \mathcal{N}_k\); 2. we have a homomorphism \(\mathcal{N}_k \rightarrow \sigma_k^*(\mathcal{A})\) such that the composition \(\mathcal{N}_k \rightarrow \sigma_k^*(\mathcal{A}) \rightarrow \sigma_k^*(\mathcal{M})\) is surjective.

We use the induction on \(k\). For \(\mathcal{N}_0\), take one as in \([\text{ibid., 15.3.1}]\). Assume we have constructed \(\mathcal{N}_k\). Take \(\mathcal{N}' \rightarrow \mathcal{A}_k\) such that \(\mathcal{N}' \in \text{Hol}(\mathcal{X}_k)\) and the composition with \(\mathcal{A}_k \rightarrow \mathcal{M}_k\) is surjective, as in \([\text{ibid., 15.3.1}]\). Given \(\phi : [i] \rightarrow [k]\), we have the following diagram:

\[
\begin{array}{ccc}
X(\phi)^*\mathcal{N}_i & \rightarrow & X(\phi)^*\mathcal{A}_i \\
\downarrow & & \downarrow \\
\mathcal{N}' & \rightarrow & \mathcal{A}_k \\
\downarrow & & \downarrow \\
\mathcal{M}_k & \rightarrow & \mathcal{M}_k.
\end{array}
\]

We need to construct the dotted homomorphism making the diagram commutative, for which we modify \(\mathcal{N}'\). We put

\[
(\mathcal{N}_k)_i := \begin{cases} 
(\mathcal{N}_k-1)_i & \text{for } i < k \\
\mathcal{N}' \oplus (\sigma_k(\mathcal{N}_k-1))_k & \text{for } i = k,
\end{cases}
\]

and with the obvious gluing homomorphisms, \(\mathcal{N}_k\) becomes an object in \(M(\text{sk}_k(X_\bullet))\), which is what we are looking for. ■
2.1.10. Let \( f_\bullet : X_\bullet \to Y_\bullet \) be a cartesian morphism of admissible simplicial spaces such that \( f_i \) is finite for any \( i \geq 0 \). For a morphism \( \phi : [j] \to [k] \), we have the following cartesian diagram

\[
\begin{array}{ccc}
X_k & \xrightarrow{f_k} & Y_k \\
\downarrow \phi & & \downarrow \phi \\
X_j & \xrightarrow{f_j} & Y_j
\end{array}
\]

where \( f_j \) and \( f_k \) are finite. Let \( \mathcal{M} \) be an object in \( M(Y_\bullet) \). For a finite morphism \( g \), we denote \( \mathcal{M}^g \) by \( g^\circ \), which is left exact by \([2.4.3]\) We have a canonical homomorphism

\[ X(\phi)^* f_j^g(\mathcal{M}_j) \cong f_k^g Y(\phi)^* (\mathcal{M}_j) \to f_k^g (\mathcal{M}_k) \]

by Lemma \([2.1.1]\). Using this homomorphism, \( \{ f_k^g(\mathcal{M}_k) \} \) defines an object in \( M(X_\bullet) \), and defines a functor \( f^\circ : M(Y_\bullet) \to M(X_\bullet) \). Since \( f_k^g \) is left exact, \( f^\circ \) is left exact as well. We can derive this functor to get

\[ f^! := \mathbb{R}f^\circ : D^+(M(Y_\bullet)) \to D^+(M(X_\bullet)) \]

In turn, the functor \( f_{k+} \) is exact. For \( \mathcal{N} \in M(X_\bullet) \), using Lemma \([2.1.1]\) we have a homomorphism

\[ Y(\phi)^* f_{j+}(\mathcal{N}_j) \cong f_{k+} X(\phi)^*(\mathcal{N}_j) \to f_{k+}(\mathcal{N}_k) \]

which defines an object \( \{ f_{k+}(\mathcal{N}_k) \} \) in \( M(Y_\bullet) \). The functor is denoted by \( f_+ \). Since the functor is exact, we can derive to get a functor

\[ f_+ : D(M(X_\bullet)) \to D(M(Y_\bullet)) \]

Lemma. — Let \( X_\bullet \xrightarrow{f} Y_\bullet \xrightarrow{g} Z_\bullet \) be cartesian finite morphisms of admissible simplicial spaces.

(i) We have a canonical isomorphism \( \rho_k^* \circ f^! \cong f_k^* \circ \rho_k^g \). In particular, \( f^! \) sends \( D^+_{hol}(Y_\bullet) \) into \( D^+_{hol}(X_\bullet) \) for \( * \in \{ +, b \} \).

(ii) We have an adjoint pair \( (f_+, f^!) \), and \( f_+ \) is exact. In particular, \( f^\circ \) preserves injective objects.

(iii) We have a canonical isomorphism \( f^! \circ g^! \cong (g \circ f)^! \).

Proof. Let us show (i). Since \( \rho_k^* \) is exact and preserves injective objects by Lemma \([2.1.6]\), the first isomorphism follows by definition. Let us check the second one. The functor \( f^! \) preserves total complexes by Lemma \([2.1.1]\)(i). It remains to show that it preserves holonomicity, which is an immediate consequence of (i).

The verification of (ii) is easy. To show (iii), by (ii), it suffices to show that \( f^\circ \circ g^\circ \cong (g \circ f)^\circ \). This follows by definition, and the corresponding statement for spaces. ■

2.1.11. Now, we use the notation of \([2.1.1]\). Let \( f : X_\bullet \to S \) be a smooth morphism from an admissible simplicial space to a space. In this situation, let us define an adjoint pair of functors \( (f^*, \mathbb{R}f_*) \). Let \( f_j : X_j \to S \) be the induced morphism. The pull-back is easy to define: Let \( \mathcal{N} \in M(S) \). We put \( \mathcal{N}_i := f_i^*(\mathcal{N}) \) which is defined in \( M(X_i) \). Let \( \phi : [i] \to [j] \) be a map. Then we define a homomorphism, which is in fact an isomorphism, \( X(\phi)^* \mathcal{N}_i \to \mathcal{N}_j \) to be the transition homomorphism. The object we have in \( M(X_\bullet) \) is denoted by \( f^*(\mathcal{N}) \). Thus, we have a functor

\[ f^* : M(S) \to M(X_\bullet) \]

The functor \( f^* \) is exact since each \( f_i^* \) is.

Let us define its right adjoint. Take \( \mathcal{M} \) in \( M(X_\bullet) \). For \( \phi : [i] \to [j] \), we have the homomorphism

\[ \alpha_\phi : f_i^*(\mathcal{M}_i) \to f_j^*(\mathcal{M}_j) \]

We put

\[ f_* (\mathcal{M}) := \ker (f_{0*} (\mathcal{M}_0) \Rightarrow f_{1*} (\mathcal{M}_1)) \]

Since \( f_{0*} \) and \( f_{1*} \) are left exact, the functor \( f_* \) is left exact as well. Thus we may derive this functor to get

\[ \mathbb{R}f_* : D^+(X_\bullet) \to D^+(S) \]

This construction can be generalized to a smooth morphism \( X_\bullet \to S_\bullet \) from a double simplicial space to a simplicial space, and get the derived functor \( \mathbb{R}f_* : D^+(X_\bullet) \to D^+(S_\bullet) \).
Lemma. — We have an adjoint pair \((f^*, f_*)\). Thus, the pair \((f^*, \mathbb{R}f_*)\) as well, and the obvious analogue holds for the double simplicial case.

Proof. The adjointness of \((f^*, \mathbb{R}f_*)\) follows by the first one using Lemma 2.1.1. Let \(S_\bullet\) be the constant simplicial space. We have the morphism \(f_\bullet: X_\bullet \to S_\bullet\), and the adjoint pair \((f^*_\bullet, f_{*\bullet})\) defines a pair of functors \((f^*_\bullet, f_{*\bullet})\) between \(M(X_\bullet)\) and \(M(S_\bullet)\). It is straightforward to check that this is an adjoint pair. Thus, it suffices to check the lemma for the morphism \(S_\bullet \to S\). This follows from the following general fact: Let \(A\) be an abelian category, and \(\Delta^+A\) be the category of cosimplicial objects, namely the abelian category of functors \(\Delta^+A \to A\). Let \(\rho^*: A \to \Delta^+A\) be funtor assigning the constant object, and \(\rho_*: \Delta^+A \to A\) to be the functor associating Ker\((M_0 \rightrightarrows M_1)\) to \(\{M_i\}\). Then \((\rho^*, \rho_*)\) is an adjoint pair. The verification is straightforward. The double simplicial case is similar. ■

2.1.12. The following spectral sequence is one of the keys to show the cohomological descent.

Lemma. — Let \(\mathcal{M} \in D^+(X_\bullet)\). Then we have the following spectral sequence:

\[ E^{p,q}_0 := \mathbb{R}^q f_{p\bullet}(\mathcal{M}_p) \Rightarrow \mathbb{R}^{p+q} f_\bullet(\mathcal{M}). \]

Proof. The construction is essentially the same as [O3 Corollary 2.7]. Since we use the similar argument again later, we sketch the proof. Let \(\mathcal{N} \in M(X_\bullet)\). We have a homomorphism \(f_{p\bullet}(\rho_{p\bullet}(\mathcal{N}_0)) \cong \prod_{[i] \to [i]} f_\bullet(\mathcal{N}) \to f_\bullet(\mathcal{N})\) where the second homomorphism is the projection to the component of the map \(\alpha: [0] \to [i]\) such that the image is \(0 \in [i]\). This induces a homomorphism from the Čech type complex:

\[ C^\bullet_\alpha(\mathcal{N}) := \left[ 0 \to f_{\alpha\bullet}(\rho_{\alpha\bullet}(\mathcal{N})) \to f_{\bullet}(\rho_{\bullet}(\mathcal{N})) \to f_{2\bullet}(\rho_{2\bullet}(\mathcal{N})) \to \cdots \right] \]

to \(f_\bullet(\mathcal{N})\). This homomorphism is in fact a homotopy equivalence. Indeed, the cohomologies of the complex \(\overline{T} := \left[ 0 \to \prod_{[i] \to [i]} L \to \prod_{[1]} L \to \cdots \right]\), which is isomorphic to that of the \(i\)-simplex, vanish except for degree 0, and since \(K(\text{Vec}_L) \cong D(\text{Vec}_L)\), the homomorphism \(\overline{T} \to L\) of the projection to \(\alpha\)-component is a homotopy equivalence. Since \(C^\bullet_\alpha(\mathcal{N}) \cong \overline{T} \otimes L f_\bullet(\mathcal{N})\), we get the claim.

Now, for any \(\mathcal{N} \in M(X_\bullet)\), there exists an embedding \(\mathcal{N} \hookrightarrow \mathcal{F}\) into an injective object in \(M(X_\bullet)\) such that the complex

\[ (2.1.12.1) \quad 0 \to f_{\alpha\bullet}(\rho_{\alpha\bullet}(\mathcal{F})) \to f_{\bullet}(\rho_{\bullet}(\mathcal{F})) \to f_{2\bullet}(\rho_{2\bullet}(\mathcal{F})) \to \cdots \]

is exact away from degree 0 part. For this, take an embedding \(\rho^\bullet: \mathcal{N} \hookrightarrow \mathcal{F}\) into an injective object in \(M(X_\bullet)\), and put \(\mathcal{F} := \prod_i \rho_{\bullet}(\mathcal{F}(i))\). Note that products can be calculated termwise by 2.1.5. Moreover, small products and \(f_{\bullet}\) commute by [KS2 2.1.10] since \(f_{\bullet}\) admits a left adjoint \(f^{-}\). This implies that \eqref{2.1.12.1} is isomorphic to \(\prod_i C^\bullet_\alpha(\mathcal{F}(i))\). Now, each \(C^\bullet_\alpha(\mathcal{F}(i))\) is homotopic to \(f_{\bullet}(\mathcal{F}(i))\) by the observation above. Since homotopy equivalence is preserved even after taking product, we get that \eqref{2.1.12.1} is homotopic to \(\prod_i f_{\bullet}(\mathcal{F}(i))\), in particular, the complex is exact away from 0. Since \(\rho_{\bullet}\) preserves injectivity object by Lemma 2.1.6 and the product of injective objects remains to be injective, \(\mathcal{F}\) is a desired object.

Finally, let \(\mathcal{M} \to \mathcal{F}\) be an resolution of \(\mathcal{M}\) consisting of complex as above. We consider the double complex \(\{f_{p\bullet}(\rho_{p\bullet}(\mathcal{F}))\}_{p \geq 0}\). The acyclicity of \eqref{2.1.12.1} except for degree 0 shows that the total complex is \(\mathbb{R}f_{\bullet}(\mathcal{M})\), and thus the associated spectral sequence is the one we want. ■

Recall that by Lemma 1.2.8 or by 2.2.5 in the second read case, \(\mathbb{R}f_{\bullet}\) preserves holonomicity. Thus, we have the following corollary:

Corollary. — The functor \(\mathbb{R}f_{\bullet}\) preserves holonomicity, and induces a functor \(D_{\text{hol}}^+(X_\bullet) \to D_{\text{hol}}^+(S)\).

2.1.13 Proposition. — Let \(X \to S\) be a smooth morphism between spaces, and put \(f: X_\bullet := \cosk_0(X/S) \to S\). The adjoint pair of functors \((f^*, \mathbb{R}f_*)\) induces an equivalence between \(D^*_\text{tot}(X_\bullet)\) and \(D^+(S)\) for \(x \in \{+, b\}\). Moreover, it induces an equivalence between \(D_{\text{hol}}^+(X_\bullet)\) and \(D_{\text{hol}}(S)\).

Proof. The second claim follows by the first one since \(f^*\) and \(\mathbb{R}f_*\) preserve the holonomicity by Corollary 2.1.12. Thus, it suffices to show that the canonical homomorphisms \(\text{id} \to \mathbb{R}f_*f^*\) and \(f^*\mathbb{R}f_* \to \text{id}\) are isomorphisms. For the first one, it suffices to show the equalities after taking \(i^*_s\) where \(s\) is a closed point of \(S\) and \(i^*_s: \{s\} \to S\). By taking the fiber product, it induces a morphism of simplicial spaces \(i_{X,s}: X_{s\bullet} \hookrightarrow X_\bullet\). Using Lemma 2.1.1, we have \(\mathbb{R}f_{*s} \circ i^*_s \cong i^*_s \circ \mathbb{R}f_*\). Thus, we
can reduce to the situation where we have a section \( S \to X_0 \). In this case, the verification for the first homomorphism is the same as [C3, Thm 7.2].

Let us show the second one. It suffices to show that the homomorphism of functors before derivation \( f^* f_* \to \text{id} \) is an isomorphism. Indeed, if this is shown, we get that for \( \mathcal{M} \in M(X_\bullet) \), we have

\[
\mathbb{R}f_*(\mathcal{M}) \xleftarrow{\sim} \mathbb{R}f_*(f^* f_*(\mathcal{M})) \xleftarrow{\sim} f_*(\mathcal{M})
\]

where the second quasi-isomorphism follows by the the first part of the proof. Thus we have \( \mathbb{R}^i f_*(\mathcal{M}) = 0 \) for \( i \neq 0 \). Finally, let us show \( f^* f_*(\mathcal{M}) \to \mathcal{M} \) is an isomorphism. As for the first isomorphism, it suffices to show the claim when \( S \) is a point, and in particular there is a section \( S \to X \). In this case, it suffices to show that there exists \( \mathcal{N} \in M(S) \) such that \( f^* (\mathcal{N}) \cong \mathcal{M} \), namely, \( \mathcal{M} \) is “effective descent”. Because of the existence of the section, this is automatic (for example, see [C3, right after 6.15]).

2.1.14. Let \( f : X_\bullet \to S \) be a smooth morphism from an admissible simplicial space to a space, and \( g : S' \to S \) be a smooth morphism between spaces. Consider the following commutative diagram:

\[
\begin{array}{ccc}
X'_\bullet & \xrightarrow{g} & X_\bullet \\
\downarrow{f'} & & \downarrow{f} \\
S' & \xrightarrow{g} & S \\
\end{array}
\]

For \( \mathcal{M} \in M(X_\bullet) \), the system \( \{g_i^*(\mathcal{M}_i)\}_i \) defines an object of \( M(X'_\bullet) \), denoted by \( g^*(\mathcal{M}) \). This functor is exact, and preserves holonomicity. We can derive to get the functor \( g^* : D^b_{\text{hol}}(X_\bullet) \to D^b_{\text{hol}}(X'_\bullet) \). Similarly, for \( \mathcal{N} \in M(X'_\bullet) \), the system \( \{g'_{j!}(\mathcal{N})\}_j \) defines an object of \( M(X_\bullet) \) by Corollary 1.5.5 or by the following lemma in the second read, and the assumption that \( g' \) is cartesian. This functor is right exact, and we have the right derivation \( \mathbb{R}g'_{\ast} \). The couple \( (g^*, \mathbb{R}g'_{\ast}) \) is adjoint.


**Lemma.** — The canonical homomorphism \( g^* \circ \mathbb{R}f_* \to \mathbb{R}f'_* \circ g^* \) is an isomorphism.

**Proof.** By using the spectral sequence of Lemma 2.1.12, the verification is reduced to Corollary 1.5.5 or to the present lemma of the first read case in the second read. □

2.1.15 Proposition. — Let \( \mathfrak{X} \) be a good stack, and \( X_\bullet \to \mathfrak{X} \) be a simplicial presentation space. Then for \( * \in \{b, +\} \), the category \( D^*_{\text{tot}}(X_\bullet) \) and \( D^*_{\text{hol}}(X_\bullet) \) does not depend on the choice of the presentation up to canonical equivalence, and the t-structure as well.

**Proof.** Let \( X_\bullet \to \mathfrak{X} \) and \( X'_\bullet \to \mathfrak{X} \) be two presentations. Let \( Z_{n,n'} := X_n \times_{\mathfrak{X}} X'_{n'} \). This defines a double simplicial space \( Z_{\bullet, \bullet} \) with morphisms \( Z_{\bullet, 
\bullet} \to X_\bullet \) and \( Z_{\bullet, 
\bullet} \to X'_{\bullet} \). Thus, it suffices to show the following: given a smooth morphism \( f : Z_{\bullet, \bullet} \to X_\bullet \) such that \( (Z_{\bullet, \bullet} \to X_\bullet) = \text{cosk}_0(Z_{00} \to X_0) \), the functors \( \mathbb{R}f_* \) and \( f^* \) induce an equivalence of categories. First, the functors \( \mathbb{R}f_* \) and \( f^* \) preserve holonomicity. Indeed, the holonomicity for \( f^* \) is easy, and for \( \mathbb{R}f_* \), use (double simplicial analogue of) Lemma 2.1.12 (ii), Lemma 2.1.14 and Corollary 2.1.12 If \( f^* \), \( \mathbb{R}f_* \) yield an equivalence between \( D^*(X_\bullet) \) and \( D^*(Z_{\bullet, \bullet}) \), since they preserve holonomicity, they induce the equivalence of \( D^*_{\text{hol}}(X_\bullet) \) and \( D^*_{\text{hol}}(Z_{\bullet, \bullet}) \) and the proposition follows.

Now, it remains to show that for \( \mathcal{N} \in M(X_\bullet) \) and \( \mathcal{M} \in M(Z_{\bullet, \bullet}) \), the homomorphisms

\[
f^* \mathbb{R}f_*(\mathcal{M}) \to \mathcal{M}, \quad \mathcal{N} \to \mathbb{R}f_* f^*(\mathcal{N})
\]

are isomorphisms. Since it suffices to show the isomorphism for each \( X_i \), this follows from Proposition 2.1.13 □

2.1.16 Definition. — Let \( \mathfrak{X} \) be a good stack. Take a simplicial presentation space \( X_\bullet \to \mathfrak{X} \). By the proposition above, for \( * \in \{b, +\} \), the categories \( D^*_{\text{tot}}(X_\bullet/\mathfrak{X}) = D^*_{\text{hol}}(X_\bullet/\mathfrak{X}) \) does not depend on the choice of the presentation. We denote these categories by \( D^*(\mathfrak{X}/\mathfrak{L}) \) and \( D^*_{\text{hol}}(\mathfrak{X}/\mathfrak{L}) \). These categories are endowed with t-structure, and their hearts are denoted by \( M(\mathfrak{X}/\mathfrak{L}) \) and \( \text{Hol}(\mathfrak{X}/\mathfrak{L}) \) respectively. Objects of \( \text{Hol}(\mathfrak{X}/\mathfrak{L}) \) are called holonomic modules on \( \mathfrak{X} \). As usual, we often omit “/\mathfrak{L}” from the notation of categories.

**Remark.** — When \( \mathfrak{X} \) is a realizable scheme \( D^b_{\text{hol}}(\mathfrak{X}) \) is equivalent to the one defined in 1.1.1 by Proposition 2.1.13.
2.1.17 Definition. — (i) Let \( \mathcal{X} \) be a good stack. Assume further that the associated reduced algebraic stack \( \mathcal{X}_{\text{red}} \) is smooth. Let \( X_\bullet \to \mathcal{X} \) be a simplicial presentation space. The category of smooth objects denoted by \( \text{Sm}(\mathcal{X}/L) \) is the full subcategory of \( D^b_{\text{hol}}(\mathcal{X}/L) \) consisting of \( \mathcal{M} \) such that, for any \( i \), \( f_i \in D^b_{\text{hol}}(X_i/K) \) is in \( \text{Sm}(X_i/K)/d_i \) where \( d_i \) denotes the relative dimension function (cf. [0.0.3]) of \( X \to \mathcal{X} \), and see [1.1.10] for the notation of \( \text{Sm}(X/K) \). It is straightforward to check that the category does not depend on the choice of the presentation.

(ii) Let \( \mathcal{X} \) be a good stack, and \( \mathcal{M} \in \text{Hol}(\mathcal{X}) \). The support of \( \mathcal{M} \) is the minimum closed subset \( Z \subset \mathcal{X} \) such that the restriction of \( \mathcal{M} \) to \( \mathcal{X} \setminus Z \) is 0. The support is denoted by \( \text{Supp}(\mathcal{M}) \). For \( \mathcal{M} \in D^b_{\text{hol}}(\mathcal{X}/L) \), we put \( \text{Supp}(\mathcal{M}) := \bigcup_i \text{Supp}(\mathcal{M}^i) \).

2.1.18. Let \( X_\bullet \) be an admissible simplicial space. Since \( M(X_\bullet) \) is enough injectives, we have the bifunctor \( \mathbb{R}\text{Hom}_{D(X_\bullet)}(-, -) : D(X_\bullet)^{\circ} \times D^+(X_\bullet) \to D(\text{Vec}_L) \) (cf. [I, §6]). This induces the bifunctor

\[
\mathbb{R}\text{Hom}_{D(X_\bullet)}(-, -) : D^+(\mathcal{X}) \times D^+(\mathcal{X}) \to D^+(\text{Vec}_L).
\]

Indeed, \( \mathbb{R}\text{Hom}_{D(X_\bullet)} \) does not depend on the choice of simplicial presentation space \( X_\bullet \to \mathcal{X} \). To check this, let \( f : Z_\bullet \to X_\bullet \) be as in the proof of Proposition 2.1.15. Then we have a canonical homomorphism

\[
\mathbb{R}\text{Hom}_{D(X_\bullet)}(\mathcal{M}, \mathcal{N}) \to \mathbb{R}\text{Hom}_{D(Z_\bullet)}(f^*(-\mathcal{M}), f^*(-\mathcal{N}))
\]

for \( \mathcal{M} \in D(X_\bullet), \mathcal{N} \in D^+(X_\bullet) \). It suffices to show that this homomorphism is a quasi-isomorphism when \( \mathcal{M}, \mathcal{N} \in D^+_\text{hol}(X_\bullet) \). This follows since the pair \( (f^*, \mathbb{R}f_*) \) is an equivalence of categories.

In the following, for simplicity, we use particularly \( D^+_\text{hol}(\mathcal{X}/L) \) even when we can generalize statements or constructions to \( D^<(\mathcal{X}/L) \) easily.

2.1.19. We use the following lemma later, whose proof is similarly to the proof of Proposition 2.1.15.

Lemma. — Let \( \mathcal{X} \) and \( \mathcal{Y} \) be good stacks, and \( X_\bullet \) and \( Y_\bullet \) be simplicial presentation space. Let \( (X \times Y)_{n,n'} := X_n \times Y_{n'} \), which forms a double simplicial spaces denoted by \( (X \times Y)_{\bullet\bullet} \). Then we have a canonical equivalence

\[
D^+_\text{hol}(\mathcal{X} \times \mathcal{Y}) \sim D^+_\text{hol}((X \times Y)_{\bullet\bullet}).
\]

2.1.20. Now, we translate the functor constructed in 2.1.19 in the language of algebraic stacks. Let \( f : \mathcal{X} \to \mathcal{Y} \) be a finite morphism between good stacks. Let us define \( f^i \) and \( f_+ \). Take a simplicial presentation space \( Y_\bullet \to \mathcal{Y} \). By pulling-back, we get a simplicial presentation space \( X_\bullet \to \mathcal{X} \). Let \( f_+ : X_\bullet \to Y_\bullet \) be the finite cartesian morphism. Let \( * \in \{b, +\} \). We define

\[
f_+ : D^*_\text{hol}(\mathcal{X}) \cong D^*_\text{hol}(X_\bullet) \cong D^*_\text{hol}(Y_\bullet) \cong D^*_\text{hol}(\mathcal{Y}) : f^i.
\]

We need to check the well-definedness, namely independence of the presentation. By the adjointness property, it suffices to show the independence for \( f_+ \). As in the proof of Proposition 2.1.15 it suffices to show the following: Consider the cartesian diagram

\[
\begin{array}{ccc}
Z_{\bullet\bullet} & \overset{g}{\longrightarrow} & W_{\bullet\bullet} \\
p \downarrow & \square & \downarrow q \\
X_\bullet & \overset{f}{\longrightarrow} & Y_\bullet
\end{array}
\]

Then \( g^* \circ f_+ = g_+ \circ p^* : M(X_\bullet) \to M(W_{\bullet\bullet}) \). The verification is straightforward and left to the reader. We have the pair \((f_+^*, f^i)\) of adjoint functors between \( D^*_\text{hol}(\mathcal{X}) \) and \( D^*_\text{hol}(\mathcal{Y}) \).

Now, let \( \mathcal{X} \overset{f}{\to} \mathcal{Y} \overset{g}{\to} \mathcal{Z} \) be finite morphisms of good stacks. We have canonical isomorphisms

\[
c_{(g,f)} : f^i \circ g^i \sim (g \circ f)^i, \quad c_{(g,f)}^+ : (g \circ f)_+ \sim g_+ \circ f_+.
\]

These isomorphisms are subject to the following two conditions: 1. we have \( c_{(f,\text{id})} = c_{(\text{id},f)} = \text{id} \); 2. given homomorphisms \( \mathcal{X} \overset{f}{\to} \mathcal{Y} \overset{g}{\to} \mathcal{Z} \overset{h}{\to} \mathcal{W} \), we have

\[
c_{(h, g) \circ f} \circ c_{(g,f)}(h^i) = c_{(h, g) \circ f} \circ f^i(c_{(h, g)}) , \quad h_+ c_{(g,f)}^+ \circ c_{(h, g) \circ f} = c_{(h, g)}^+(f_+) \circ c_{(h, g) \circ f}.
\]
These results can be rephrased by using the language of (co)fibered categories (cf. [SGA 1, Exp. VII, end of 7]) as follows. Let $\text{St}^{\text{fin}}(k_\bullet)$ the be the category of good stacks (we do not consider the 2-morphisms) over $k_\bullet$ such that the morphisms are finite morphisms between good stacks. To a good stack $X$, we associate the triangulated category $D^b_{\text{hol}}(X)$. For a finite morphism $X \to Y$, we consider the functor $f_!$, and $c_{(f,g)}$. Then these data form a fibered category $\mathcal{F}^! \to \text{St}^{\text{fin}}(k_\bullet)$. Considering $f_+$ and $c(f,g)$, we get a cofibered category $\mathcal{F}^+ \to \text{St}^{\text{fin}}(k_\bullet)$.

We conclude this subsection with the following lemma:

2.1.21 Lemma. — Let $f : \mathcal{C} \to \mathcal{C}'$ be a morphism in $\text{St}^{\text{fin}}(k_\bullet)$.

(i) Assume $f$ is finite surjective radicial morphism. Then $f_+$ and $f^+$ defines an equivalence of categories between $D^b_{\text{hol}}(\mathcal{C}')$ and $D^b_{\text{hol}}(\mathcal{C})$, and $\text{Hol}(\mathcal{C}')$ and $\text{Hol}(\mathcal{C})$.

(ii) Assume $f$ is a finite étale morphism. Then for any $\mathcal{M} \in \text{Hol}(\mathcal{C})$, it is a direct factor of $f_+f^+(\mathcal{M})$.

(iii) If $f$ is a finite flat morphism, then for any $\mathcal{M} \in D^b_{\text{hol}}(\mathcal{C}')$, $\mathcal{M}$ is a direct factor of $f_+f^+(\mathcal{M})$.

Proof. For (i), it suffices to show the claim when $\mathcal{C}$ and $\mathcal{C}'$ are schemes. This is nothing but Lemma 2.1.9 in the first read case, and the second read can be reduced to the first one immediately. For (iii), we can define homomorphisms $f_+f^+(\mathcal{M}) \to \mathcal{M}$ using the trace map of realizable schemes, and the claim follows easily.

For (ii), let $C'_i \to \mathcal{C}'$ be a simplicial presentation. Put $\mathcal{M}_i := \rho_i^*(\mathcal{M})$, and $f_i : \mathcal{C} \times \mathcal{C}' C'_i \to C'_i$. We have the morphisms $\mathcal{M}_i \to f_i^+f_+\mathcal{M}_i \to \mathcal{M}_i$. Here the first morphism is defined by the trace map in the first read case and the homomorphism defined in the first read in the second read. It is standard that the composition is the identity. Moreover, these homomorphisms are compatible with gluing homomorphisms, so they define homomorphisms $\mathcal{M} \to f_+f^+\mathcal{M} \to \mathcal{M}$ whose composition is an isomorphism, thus the claim follows.

2.2. Cohomological functors

In this subsection, we define some cohomological functors for algebraic stacks. Even though the six functor formalism is expected for algebraic stacks, unfortunately, at this moment, we can obtain full formalism only for certain stacks, which is enough for our purpose. In this subsection, we define functors that we can define for general algebraic stacks.

2.2.1. Dual functors

Let $f : X \to Y$ be a smooth morphism of relative dimension $d$ between spaces. Then we have a canonical isomorphism $(f^* \circ D_Y)(d) \tilde{\to} D_X \circ f^*$ by Lemma 2.1.1. Now, let $X_\bullet$ be an admissible simplicial space, and assume given a smooth morphism $X_\bullet \to X$ to an algebraic stack. We have the dual auto-functor $D_X : \text{Hol}(X_\bullet) \tilde{\to} \text{Hol}(X_\bullet)$. We modify this functor by putting $D_i := (DX_\bullet/X) \circ D_X$, where $DX_\bullet/X$ denotes the relative dimension function (cf. 0.1.3).

Now, we use the notation of Lemma 2.1.9 and 2.1.1. Let $\mathcal{M}_i \in \text{sec}_+(\text{Hol}(X_\bullet))$. For a morphism $\phi : [i] \to [j]$, let $\alpha_\phi : X(\phi)^*\mathcal{M}_i \to \mathcal{M}_j$ be the transition homomorphism. Let

$$\beta_\phi : D_j(\mathcal{M}_j) \to D_j(X(\phi)^*(\mathcal{M}_i)) \hookrightarrow X(\phi)^*D_i(\mathcal{M}_i).$$

The data $\{D_i(\mathcal{M}_i), \beta_\phi\}$ defines an object in $\text{sec}_-(\text{Hol}(X_\bullet))$ and defines a functor

$$D_X/\text{sec}_- : \text{sec}_+(\text{Hol}(X_\bullet)) \to \text{sec}_-(\text{Hol}(X_\bullet)).$$

Similarly, we can define the functor $D_{X_\bullet/X} : \text{sec}_-(\text{Hol}(X_\bullet)) \to \text{sec}_+(\text{Hol}(X_\bullet))$, and we have canonical isomorphisms $c_\pm \circ D_{X_\bullet/X} \cong \text{sec}_\pm$. These functors are exact since $D_i$ are. Then we have

$$D_{\text{tot}(\text{sec}_\pm\text{Hol}(X_\bullet))} \sim D_{\text{tot}(\text{Hol}(X_\bullet))} \sim D_{\text{hol}(X_\bullet)}.$$

where the right horizontal isomorphisms follow by Lemma 2.1.9. We define the dotted functor so that the square is commutative. The dotted functor is called the dual functor on $D(X_\bullet)$. By construction, the functor is exact. Moreover, we have a canonical isomorphism $D'_X \circ D_X \cong \text{id}$. 

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Let $X$ be a good stack. Take a simplicial presentation space $X_\bullet \to X$. We can check that $D_{X_\bullet/X}$ does not depend on the choice of presentation. Thus, we get a functor
\[
D'_X: D^{\bullet}_{\text{hol}}(X)^\circ \to D^{\bullet}_{\text{hol}}(X).
\]

2.2.2 Lemma. — Let $f: X \to \mathbb{Y}$ be a finite morphism between good stacks. Then there exists a canonical isomorphism
\[
D'_{\mathbb{Y}} \circ f_+ \cong f_+ \circ D'_X: D^{\bullet}_{\text{hol}}(X)^\circ \to D^{\bullet}_{\text{hol}}(\mathbb{Y}).
\]
Proof. Take a simplicial presentation space $Y \to \mathbb{Y}$, and let $X \to \mathbb{Y}$ be the pull-back. Denote by $f_X: X_k \to Y_k$ the finite morphism induced by $f$. Since $f_X(\mathcal{M})$ is in $\text{Hol}(Y_k)$ when $\mathcal{M} \in \text{Hol}(X_k)$ and $f_X$ and $X(\phi)^*$ commute canonically, we can define the push-forward functors $f_{+\pm}: \text{sec}_\pm(\text{Hol}(X_\bullet)_\bullet) \to \text{sec}_\pm(\text{Hol}(Y_\bullet)_\bullet)$ by sending $\{\mathcal{M}_i\}$ to $\{f_X(\mathcal{M}_i)\}$ with obvious gluing homomorphism. By the definition of the functors $c_{\pm}$, the following diagrams commute:
\[
\begin{array}{ccc}
\text{sec}_+(\text{Hol}(X_\bullet)_\bullet) & \xrightarrow{f_{+\pm}} & \text{sec}_+(\text{Hol}(Y_\bullet)_\bullet) \\
M(X_\bullet) & \xrightarrow{f_+} & M(Y_\bullet),
\end{array}
\]
\[
\begin{array}{ccc}
\text{sec}_-(\text{Hol}(X_\bullet)_\bullet) & \xrightarrow{f_{-\pm}} & \text{sec}_-(\text{Hol}(Y_\bullet)_\bullet) \\
C(M(X_\bullet)) & \xrightarrow{f_+} & C(M(Y_\bullet)).
\end{array}
\]
Thus, it is reduced to constructing an isomorphism $D_{Y_\bullet/\mathbb{Y}} \circ f_{+\pm} \cong f_{-\pm} \circ D_{X_\bullet/X}$. Since all the functors appearing are exact, the verification is easy.

Definition. — The lemma shows that $f_+$ has a left adjoint functor
\[
D'_X \circ f^! \circ D'_{\mathbb{Y}}: D^{\bullet}_{\text{hol}}(\mathbb{Y}) \to D^{\bullet}_{\text{hol}}(X).
\]
This right adjoint functor is denoted by $f^*$. Since $f^!$ is left exact, $f^*$ is right exact. Summing up, when $f$ is finite, we have two pairs of adjoint functors $(f^+, f_+)$ and $(f^+, f^!)$.

2.2.3. Exterior tensor product
Let us define the exterior tensor product. Let $X_\bullet$ and $Y_\bullet$ be admissible simplicial spaces. Given $\mathcal{M}$ and $\mathcal{N}$ in $M(X_\bullet)$ and $M(Y_\bullet)$ respectively, the collection $\{\mathcal{M} \boxtimes \mathcal{N}\}_i$ defines an object in $M(X_\bullet \times Y_\bullet)$. This is denoted by $\mathcal{M} \boxtimes \mathcal{N}$. The functor is exact, and we can derive to get a functor
\[
(-) \boxtimes (-): D(M(X_\bullet)) \times D(M(Y_\bullet)) \to D(M(X_\bullet \times Y_\bullet)).
\]
We can check easily that this preserves holonomicity and boundedness.

Let $X$ and $\mathbb{Y}$ be good stacks, and take simplicial presentation spaces $X_\bullet \to X$ and $Y_\bullet \to \mathbb{Y}$. Then $X_\bullet \times Y_\bullet$ is a simplicial presentation space of the good stack $X \times \mathbb{Y}$. Thus, we get the exterior tensor product on good stacks.

2.2.4 Lemma. — Let $f: X \to \mathbb{Y}$, $g: X' \to \mathbb{Y}'$ be finite morphisms between good stacks. Then we have canonical isomorphisms $f_\pm(-) \boxtimes g_\pm(-) \cong (f \times g)_\pm((-) \boxtimes (-))$, $f^*\pm(-) \boxtimes g^*(\pm(-)) \cong (f \times g)^*\pm((-) \boxtimes (-))$ where $\pm \in \{+!,\}$, and $D'(\pm(-) \boxtimes (-)) \cong D'(\pm(-) \boxtimes D'(-)$.

Proof. For the commutation of external tensor product and dual functors, see [AC1, 1.3.3], and for the push-forward, use Proposition [1.1.5]. Since the proofs are straightforward, we leave the details.

2.2.5. Smooth morphism case
Let $f: X \to \mathbb{Y}$ be a smooth morphism between good stacks. Take simplicial presentation spaces $Y_\bullet \to \mathbb{Y}$ and $X_\bullet \to X$. Let $X_{n,m} := X_n \times_{\mathbb{Y}} Y_m$, which defines a double simplicial space $X_{\bullet\bullet}$ with obvious face morphisms, and morphisms $f: X_{\bullet\bullet} \to Y_\bullet$ and $g: X_{\bullet\bullet} \to X_\bullet$. By (double simplicial analogue of) Proposition 24.4.8, $g^*$ and $R^\infty g_*$ induces an equivalence between $D^+(X_{\bullet\bullet})$ and $D^+(X_{\bullet\bullet}) \cong D^+(X)$. Thus, we have functors
\[
Rf_*: D^+(X) \cong D^+(X_{\bullet\bullet}) \cong D^+(Y_\bullet) \cong D^+(\mathbb{Y}): f^*
\]
These functors preserve holonomicity by Corollary 24.4.12. We need to check that these functors do not depend on the choice of the presentations. By adjointness property, it suffices to check it for $f^*$. The verification is easy and left to the reader.
**Lemma.** — Assume $f$ is of relative dimension $d$. We have a canonical isomorphism $\mathbb{D}'_X \circ f^* \cong (d) \circ f^* \circ \mathbb{D}'_\mathcal{Y}$.

*Proof.* The proof is similar to Lemma 2.2.2 using Lemma 2.1.1 (ii). □

**Projection case**

We define the push-forward functor for a projection $X \times \mathcal{Y} \to \mathcal{Y}$. The method here is close to the definition of $Rf^!$ in [SGA4, XVIII 3.1].

**2.2.6 Lemma.** — Let $X_\bullet$ and $Y_\bullet$ be admissible simplicial spaces, and $\mathcal{A}$ be an object in $M(X_\bullet)$. Let

$$p^*_\mathcal{A} := \mathcal{A} \boxtimes (-) : M(Y_\bullet) \to M((X \times Y)_\bullet),$$

where we use the notation of Lemma 2.1.19. Then there exists the right adjoint denoted by $p^*_\mathcal{A}$. □

*Proof.* Since the functor $p^*_\mathcal{A}$ is exact and commutes with direct sums by definition, it commutes with arbitrary inductive limits by [KSc, 2.2.9]. Since $M(Y_\bullet)$ is a Grothendieck category and $p^*_\mathcal{A}$ commutes with inductive limits, the existence follows from [KSc, 8.3.27 (iii)]. □

Given a homomorphism $\mathcal{A} \to \mathcal{B}$ in $M(X_\bullet)$, we have a natural transform of functors

$$p^*_\mathcal{A} p^*_\mathcal{B} \to p^*_\mathcal{B} p^*_\mathcal{A} \to \text{id}$$

where the last homomorphism is the adjunction. Taking the adjoint, we get a homomorphism $p^*_\mathcal{B} \to p^*_\mathcal{A}$. Obviously, if the homomorphism $\mathcal{A} \to \mathcal{B}$ is 0, the induced morphism of functors is 0 as well. Thus, for a complex $\mathcal{A}^* \in C(M(X_\bullet))$, we have a complex of functors

$$p^*_\mathcal{A}^* : [\cdots \to p^*_\mathcal{A}^{i+1} \to p^*_\mathcal{A}^i \to \cdots]$$

where $p^*_\mathcal{A}^*$ is placed at degree $-i$ part.

**2.2.7 Lemma.** — Let $\mathcal{I}$ be an injective object in $M((X \times Y)_\bullet)$. Then the contravariant functor

$$p^-_\mathcal{I}(\mathcal{I}) : M(X_\bullet)^\circ \to M(Y_\bullet)$$

sending $\mathcal{I}$ to $p^-_\mathcal{I}(\mathcal{I})$ is exact.

*Proof.* Let $0 \to \mathcal{A}' \to \mathcal{A} \to \mathcal{A}'' \to 0$ be a short exact sequence. It suffices to show that, for any $\mathcal{N}$ in $M(Y_\bullet)$, the complex

$$0 \to \text{Hom}(\mathcal{N}, p^*_\mathcal{A}^\circ(\mathcal{I})) \to \text{Hom}(\mathcal{N}, p^*_\mathcal{A}^\circ(\mathcal{I})) \to \text{Hom}(\mathcal{N}, p^*_\mathcal{A}^\circ(\mathcal{I})) \to 0$$

is exact, which implies in fact the sequence $0 \to p^*_\mathcal{A}^\circ(\mathcal{I}) \to p^*_\mathcal{A}^\circ(\mathcal{I}) \to p^*_\mathcal{A}^\circ(\mathcal{I}) \to 0$ to be split exact. This follows by definition. □

**2.2.8 Corollary.** — Let $\mathcal{C}$ be in $C(M(X_\bullet))$, and $\mathcal{M} \in C((X \times Y)_\bullet)$. We have the spectral sequence

$$E_2^{p,q} := \mathbb{R}^p p^*_\mathcal{M}(\mathcal{C}) \Rightarrow \mathbb{R}^{p+q} p^*_\mathcal{M}(\mathcal{M}).$$

*Proof.* The lemma shows that if $\mathcal{I}$ is an injective object in $M((X \times Y)_\bullet)$, we have

$$\mathcal{M}^{-1}(\cdots \to p^*_\mathcal{A}^{i+1}(\mathcal{I}) \to p^*_\mathcal{A}^i(\mathcal{I}) \to p^*_\mathcal{A}^{i-1}(\mathcal{I}) \to \cdots) \cong p^*_\mathcal{A}^{-1}(\mathcal{I}).$$

Let $\mathcal{I}^*$ be an injective resolution of $\mathcal{M}$. Then the spectral sequence associated with the double complex $p^*_\mathcal{A}^*(\mathcal{I}^*)$ is the desired one. □

**2.2.9 Corollary.** — If a homomorphism of complexes $\mathcal{A}^* \to \mathcal{B}^*$ is a quasi-iso morphism, the induced homomorphism of derived functors $\mathbb{R}p^*_\mathcal{A}^* \to \mathbb{R}p^*_\mathcal{B}^*$ is a quasi-iso morphism as well.
Proof. The homomorphism of functors \( \mathbb{R}p_{\mathscr{E}_{*}} \to \mathbb{R}p_{\mathscr{E}_{*}} \) induces the homomorphism of spectral sequences

\[
\begin{array}{c}
\begin{array}{c}
E_2^{p,q} = \mathbb{R}p_{\mathscr{E}^{-q}(\mathscr{E})_*} \\
E_2^{'p,q} = \mathbb{R}p_{\mathscr{E}^{-q}(\mathscr{E}')_*}
\end{array}
\end{array}
\]

Since the left vertical homomorphism is an isomorphism, so is the right.

2.2.10 Definition. — Let \( * \in \{0,b,+,\} \), and \( \mathscr{C} \in C^*(M(X_*)) \). We can derive the functor \( p_\mathscr{C}_{*} \) to get

\[
p_\mathscr{C}_{*} := \mathbb{R}p_\mathscr{C}_{*} : D^*((X \times Y)_{**}) \to D(Y_{*}), \quad p_\mathscr{C}_{*}^{\bullet} := \mathbb{R}p_\mathscr{C}_{*}^{\bullet} : D^*(Y_{*}) \to D^*((X \times Y)_{**}).
\]

By Corollary 2.2.9 we may even take \( \mathscr{C} \in D(X_{*}) \). By definition, the pair \( (p_\mathscr{C}_{*}^{\bullet}, p_\mathscr{C}_{*}^{\bullet}) \) is adjoint.

Remark. — Assume \( \mathscr{C} \in M(X_{*}) \). Then by definition, for \( \mathscr{M} \in M((X \times Y)_{**}) \), \( H^0 p_\mathscr{C}_{*}^{\bullet}(\mathscr{M}) = 0 \) for \( i < 0 \), and in particular, it sends \( D^+ \) to \( D^+ \). Now, let \( \mathscr{C} \in D(X_{*}) \) such that there exists an integer \( a \) with \( H^a \mathscr{C} = 0 \) for \( a < i \). Then by Corollary 2.2.9 for \( \mathscr{M} \in M((X \times Y)_{**}) \), we have \( H^a p_\mathscr{C}_{*}^{\bullet}(\mathscr{M}) = 0 \) for \( i < -a \). In particular, the functor sends \( D^+ \) to \( D^+ \) as well.

2.2.11. Let us compute \( p_{\mathscr{E}_{*}} \) more concretely when \( \mathscr{E} \in \text{Hol}(X_{*}) \). Let \( X_{*} \) be an admissible simplicial space, and \( Y \) be a space. Let \( p : X_{*} \times Y \to Y \) be the projection. Let \( \mathscr{E}_{*} \in \text{Hol}(X_{*}) \). Take \( \mathscr{M}_{*} \in M(X_{*} \times Y) \). For \( \phi : [i] \to [j] \), we have the commutative diagram

\[
\begin{array}{c}
\begin{array}{c}
X_{*} \times Y \\
\downarrow \quad (X \times Y)(\phi) \\
X_{*} \times Y
\end{array}
\end{array}
\]

Recall 2.2.10 Since \( (X \times Y)(\phi)^* \circ p_{\mathscr{E}_{*}}^{\bullet} \circ p_{\mathscr{E}_{*}}^{\bullet} \circ (X \times Y)(\phi)_{*} \) is an adjoint pair and we have the canonical isomorphism \( (X \times Y)(\phi)^* \circ p_{\mathscr{E}_{*}}^{\bullet} \cong p_{\mathscr{E}_{*}}^{\bullet} \), we have

\[
p_{\mathscr{E}_{*}}^{\bullet} \circ (X \times Y)(\phi)_{*} \cong p_{\mathscr{E}_{*}}^{\bullet}.
\]

The gluing homomorphism of \( \mathscr{M}_{*} \) induces a canonical homomorphism \( \alpha_{\phi} : p_{\mathscr{E}_{*}}^{\bullet}(\mathscr{M}_{i}) \to p_{\mathscr{E}_{*}}^{\bullet}(\mathscr{M}_{j}) \). With this homomorphism, we define

\[
p_{\mathscr{E}_{*}}^{\bullet}(\mathscr{M}_{*}) := \text{Ker}(p_{0,\mathscr{E}_{*}}^{\bullet}(\mathscr{M}_{0}) \to p_{1,\mathscr{E}_{*}}^{\bullet}(\mathscr{M}_{1})).
\]

Now, let \( Y_{*} \) be an admissible simplicial space, and \( \mathscr{M}_{**} \) be in \( M((X \times Y)_{**}) \). Given a morphism \( \psi : [k] \to [l] \), consider the following diagram:

\[
\begin{array}{c}
\begin{array}{c}
X_{*} \times Y_{*} \\
\downarrow \quad (X \times Y)(\psi) \\
X_{*} \times Y_{*}
\end{array}
\end{array}
\]

Then we have a canonical homomorphisms in \( M(Y_{*}) \)

\[
Y(\psi)^* p_{\mathscr{E}_{*}}^{\bullet} (((X \times Y)(\psi)^* \circ (X \times Y)(\psi)) \to p_{\mathscr{E}_{*}}^{\bullet}((X \times Y)(\psi)).
\]

With this transition homomorphism, we get a left exact functor

\[
p_{\mathscr{E}_{*}}^{\bullet} : p_{\mathscr{E}_{*}}^{\bullet} : M((X \times Y)_{**}) \to M(Y_{*}).
\]

Lemma. — The pair \( (p_{\mathscr{E}_{*}}^{\bullet}, p_{\mathscr{E}_{*}}^{\bullet}) \) is an adjoint pair. Thus, we have an isomorphism \( p_{\mathscr{E}_{*}}^{\bullet} \cong p_{\mathscr{E}_{*}}^{\bullet} \).

Proof. The proof is essentially the same as that of Lemma 2.1.1 and we do not repeat here. ■
2.2.12 Lemma. — Assume \( \mathcal{A} \in \mathrm{Hol}(X_\bullet) \). Then, there exists the following spectral sequence
\[
E_1^{p,q} = \mathbb{R}^q p_{\mathcal{A} \times \bullet}(\mathcal{M}) \Rightarrow \mathcal{H}^{p+q} p_{\mathcal{A} \bullet}(\mathcal{M})
\]
where \( p_p : X_p \times Y_p \to Y_p \) is the projection.

Proof. By Lemma 2.2.11 it suffices to construct the spectral sequence for \( p_{\mathcal{A} \times \bullet} \). The construction is the same as that of Lemma 2.1.12 following [O3 Corollary 2.7].

2.2.13 Proposition. — Let \( \mathcal{A} \) be an object of \( D^b_{\mathcal{H}ol}(X_\bullet) \).

(i) The functors \( p_{\mathcal{A} \times \bullet} \) and \( p_{\mathcal{A} \bullet}^+ \) yield functors between \( D^b_{\mathcal{H}ol}((X \times Y)_\bullet) \) and \( D^b_{\mathcal{H}ol}(Y_\bullet) \).

(ii) We have an adjoint pair \( (p_{\mathcal{A} \times \bullet}^+, p_{\mathcal{A} \bullet}^+) \) between \( D^b_{\mathcal{H}ol}(Y_\bullet) \) and \( D^b_{\mathcal{H}ol}((X \times Y)_\bullet) \).

Proof. Let us show (i). First, let us check that for \( \mathcal{M} \in D^b_{\mathcal{H}ol}((X \times Y)_\bullet) \), \( \mathcal{H}^n p_{\mathcal{A} \times \bullet}(\mathcal{M}) \) is in \( \mathrm{Hol}(Y_i) \). By combining the spectral sequences of Corollary 2.2.5 and Lemma 2.2.12 it suffices to check the holonomicity of \( \mathbb{R}^q p_{\mathcal{A} \times \bullet}(\mathcal{M}) \) for any integers \( p, q, i \). This follows from 2.1.11 or 2.2.5. It remains to show that \( \mathcal{H}^n p_{\mathcal{A} \times \bullet}(\mathcal{M}) \) is a total complex, namely for \( \psi : [k] \to [l] \), the homomorphism
\[
Y(\psi)^* p_{\mathcal{A} \times \bullet}^+ (\mathcal{M}_{\bullet k}) \to p_{\mathcal{A} \times \bullet}^+ (\mathcal{M}_{\bullet l})
\]
is a quasi-isomorphism. We may assume \( \mathcal{A} \in \mathrm{Hol}(X_\bullet) \). In this case, this follows by smooth base change (cf. Corollary 1.5.3) and Lemma 2.2.12. Using (i), (ii) follows immediately by construction.

Remark. — Unfortunately, the boundedness is not preserved as we can see from the standard example [LM 18.3.3]. Thus, to get a six functor formalism for algebraic stacks, dealing with unbounded derived category is essential as in [LO]. However, we only construct the formalism for “admissible stacks” (cf. Definition 2.3.4), under which situation the boundedness is preserved, we do not use unbounded category.

2.2.14. As one can expect, these functors define functors for good stacks. Let \( \mathfrak{X} \) and \( \mathfrak{Y} \) be good stacks, and \( \mathcal{A} \in D^b_{\mathcal{H}ol}(\mathfrak{X}) \). Take simplicial presentation spaces \( X_\bullet \to \mathfrak{X} \) and \( Y_\bullet \to \mathfrak{Y} \). Then we have a functor
\[
D^b_{\mathcal{H}ol}(X_\bullet \times \mathfrak{Y}) \cong D^b_{\mathcal{H}ol}((X \times Y)_\bullet) \xrightarrow{p_{\mathcal{A} \times \bullet}} D^b_{\mathcal{H}ol}(Y_\bullet) \cong D^b_{\mathcal{H}ol}(\mathfrak{Y}),
\]
and the same for \( p_{\mathcal{A} \bullet}^+ : D^b_{\mathcal{H}ol}(\mathfrak{Y}) \to D^b_{\mathcal{H}ol}(X_\bullet \times \mathfrak{Y}) \). The pair \( (p_{\mathcal{A} \times \bullet}^+, p_{\mathcal{A} \bullet}^+) \) is adjoint. Now, we have:

Lemma. — The functors does not depend on the choice of presentation.

Proof. By the adjointness property, it suffices to show the lemma for \( p_{\mathcal{A} \bullet}^+ \), in which case the verification is easy.

Remark. — When \( \mathfrak{X} \) and \( \mathfrak{Y} \) are realizable schemes, then \( p_{\mathcal{A} \times \bullet} \) coincide with the functor defined in 1.2.10 which justifies the notation. This follows since both functors are right adjoint to \( p_{\mathcal{A} \bullet}^+ \).

2.2.15 Proposition. — Let \( p : \mathfrak{X} \to \text{Spec}(k) \) be the structural morphism of a good stack. Let \( \mathcal{A} \) be in \( D^b_{\mathcal{H}ol}(\mathfrak{X}) \). For any \( \mathcal{M} \) in \( D^+_{\mathcal{H}ol}(\mathfrak{X}) \), we have a canonical isomorphism (recall 1.4.14 and 2.1.18 for the notation)
\[
\mathbb{R}\Gamma \circ p_{\mathcal{A} \bullet}(\mathcal{M}) \cong \mathbb{R}\mathrm{Hom}_{D(\mathfrak{X})}(\mathcal{A}, \mathcal{M}).
\]

Proof. Take a simplicial presentation space \( X_\bullet \to \mathfrak{X} \). For \( \mathcal{A} \in C^b(X_\bullet) \) and \( \mathcal{M} \in M(X_\bullet) \), we have
\[
\Gamma \circ p_{\mathcal{A} \bullet}(\mathcal{M}) \cong \mathrm{Hom}_{M(\text{Spec}(k))}(L, p_{\mathcal{A} \bullet}(\mathcal{M})) \cong \mathrm{Hom}_{M(X_\bullet)}(\mathcal{A}, \mathcal{M}).
\]
Now, \( p_{\mathcal{A} \bullet}^+ \) preserves injective objects since the left adjoint functor \( p_{\mathcal{A} \bullet}^+ \) is exact. This shows that \( \mathbb{R}((\Gamma \circ p_{\mathcal{A} \bullet}) \cong \mathbb{R} \circ p_{\mathcal{A} \bullet} \). Thus by the definition of \( p_{\mathcal{A} \bullet} \), the proposition follows.
2.2.16. We have defined a pair of adjoint functors $(p^+_\mathcal{A}, p_{\mathcal{A}+})$, which depends on the choice of the complex $\mathcal{A}$. For the construction of normal push-forward and pull-back, we need a “canonical choice” of $\mathcal{A}$, which is nothing but the unit object $L_X$ when $X$ is a smooth realizable scheme. To construct this complex for good stacks, we need the following theorem of [BBD], as in the construction of $[\mathcal{O}]$.

**Theorem ([BBD 3.2.4]).** — Let $X_\bullet$ be an admissible simplicial spaces. Assume given a data $\{\mathcal{E}_i, \alpha_\phi\}$ where $\mathcal{E}_i \in D^b(X_\bullet)$ and for $\phi: [i] \to [j]$, $\alpha_\phi: X(\phi)^\ast \mathcal{E}_i \to \mathcal{E}_j$ satisfying the cocycle conditions. Assume moreover that

$$R^i \text{Hom}_{D(X)}(\mathcal{E}_i, \mathcal{E}_i) = 0$$

for any $i < 0$. Then there exists a unique $\mathcal{E} \in D^b(X_\bullet)$ such that $\rho_+^i(\mathcal{E}) \cong \mathcal{E}_i$ (cf. [2.1.10]) and the gluing isomorphism is equal to $\alpha_\phi$ via this isomorphism.

**Proof.** For the uniqueness, use the spectral sequence [2.1.5(1)]. The existence is more difficult. We use a construction of Beilinson and Drinfeld. In [BD 7.4.10], they define an abelian category $\text{hot}_+(M(X_\bullet))$. This category is nothing but $\text{tot}(A^+)$ in the notation of [BBD 3.2.7] by taking $A(n)$ to be $\text{tot}(X_n)$. In [BD], they construct an equivalence of categories $s_+ : D\text{sec}_+(M(X_\bullet)) \to D\text{hot}_+(M(X_\bullet))$ and characterize $D\text{tot}(X_\bullet)$ in terms of $D\text{hot}$. Even though the appearance is slightly different, this is the statement corresponding to [BBD 3.2.17]. Our task is, thus, to construct an object in $K(\text{hot}_+(M(X_\bullet)))$. For this, we can copy the argument of [BBD 3.2.9]. ■

2.2.17 **Lemma.** — Let $p: X \to \text{Spec}(k)$ be a realizable space. Let $L_X := p^+(L)$. Then we have

$$R^i \text{Hom}_{D(X)}(L_X, L_X) = 0 \text{ for } i < 0.$$ 

**Proof.** Consider the “first read” case, namely the case where $X$ is a realizable scheme. Since for $L$ is conservative, we may assume that $L = K$. We have isomorphisms

$$R^i \text{Hom}_{D(X)}(L_X, L_X) \cong R^i \Gamma p^+ \text{Hom}(L_X, L_X) \cong R^i \Gamma p^+ p^+(L)$$

where we used Proposition 2.2.15 for the first isomorphism, and the second one follows by 1.1.3.8. Now the lemma follows by the left c-t-exactness of $p_+$ and $R^i \Gamma$. For the “second read” case, we can reduce to the first read case by using 2.1.5(1). ■

2.2.18. Let $X$ be a good stack, and $X_\bullet \to X$ be a simplicial presentation space. Let us construct the unit complex on $X_\bullet$. The unit complex $L_{X_\bullet}$ in $D^b_{\text{hol}}(X_\bullet)$ has already been defined. Let $\phi: [i] \to [j]$. Recall the notation (2.1.3) and (2.1.4). We have a canonical isomorphism

$$X(\phi)^\ast (L_{X_i}[d_{X_i/X_i}]) \cong L_{X_j}[d_{X_j/X_j}].$$

By Lemma 2.2.16, the conditions in Theorem 2.2.10 are satisfied. Thus $\{L_{X_i}[d_{X_i/X_i}]\}_i$ can be glued to get $L_{X_\bullet/X}$ in $D^b_{\text{hol}}(X_\bullet)$.

**Lemma.** — The object $L_{X_\bullet/X}$ does not depend on the choice of simplicial presentation up to canonical isomorphism.

**Proof.** The proof is straightforward. ■

**Definition.** — (i) We define the constant complex $L_X$ to be $L_{X_\bullet/X}$ in $D^b_{\text{hol}}(X)$.

(ii) We define the dualizing complex $L_X^\omega$ to be $D_X(L_X)$.

2.2.19 **Lemma.** — (i) Let $X$ and $Y$ be good stacks. Then we have $L_{X \times Y} \cong L_X \boxtimes L_Y$, and $L_{X \times Y}^\omega \cong L_X^\omega \boxtimes L_Y^\omega$.

(ii) Let $f: X \to Y$ be a finite morphism between good stacks. Then we have an isomorphism $t_f: f^+(L_Y) \cong L_X$ such that given another finite morphism $g: Y \to Z$, the isomorphism is compatible with composition: the composition $(g \circ f)^+(L_Z) \cong g^+(f^+(L_Z)) \xrightarrow{t_g \circ t_f} L_Y \xrightarrow{t_f} L_X$ is equal to $t_{g \circ f}$.

**Proof.** For (i), the first claim follows from the corresponding statement for spaces, and the second by Lemma 2.2.4. Verification of (ii) is left to the reader. ■
2.2.20 Definition. — Let $\mathcal{X}$ and $\mathcal{Y}$ be good stacks. We put

$$p_+ := p_{LX^+} : D^+_{\text{hol}}(\mathcal{X} \times \mathcal{Y}) \to D^+_{\text{hol}}(\mathcal{Y}), \quad p^+ := p^+_{LX} : D^+_{\text{hol}}(\mathcal{Y}) \to D^+_{\text{hol}}(\mathcal{X} \times \mathcal{Y}).$$

Let $\mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \xrightarrow{f} \mathcal{Y} \times \mathcal{Z} \xrightarrow{g} \mathcal{Z}$ be projections. By using the canonical isomorphism in Lemma 2.2.19, we have a canonical isomorphisms

$$(2.2.20.1) \quad f^+ \circ g^+ \cong L_X \boxtimes (L_{\mathcal{Y}} \boxtimes (-)) \cong L_{\mathcal{X} \times \mathcal{Y}} \boxtimes (-) \cong (g \circ f)^+.$$

By taking the adjoint, we also get a canonical isomorphism $(g \circ f)_+ \cong g_+ \circ f_+.$

2.2.21. Let $\mathcal{X}'(t), \mathcal{Y}'(t)$ be good stacks, and take $\mathcal{A}'(t) \in D^b_{\text{hol}}(\mathcal{X}'(t)).$ Let $p'(t) : \mathcal{X}'(t) \times \mathcal{Y}'(t) \to \mathcal{Y}'(t).$ Let $q : (\mathcal{X} \times \mathcal{X}') \times (\mathcal{Y} \times \mathcal{Y}') \to \mathcal{Y} \times \mathcal{Y}'$ be the projection. By definition, we have a canonical isomorphism

$$(2.2.21.1) \quad p^+_{\mathcal{A}'(t)}(-) \boxtimes p^+_{\mathcal{A}'(t)}(-) \cong q^+_{\mathcal{A} \boxtimes \mathcal{A}'}(- \boxtimes -).$$

Now, we have

$$(2.2.21.2) \quad q^+_{\mathcal{A} \boxtimes \mathcal{A}'}(p_{\mathcal{A}(t)}(-) \boxtimes p_{\mathcal{A}'(t)}(-)) \cong p^+_{\mathcal{A}(t)}p_{\mathcal{A}(t)}(-) \boxtimes p^+_{\mathcal{A}'(t)}p_{\mathcal{A}'(t)}(-) \to (-) \boxtimes (-)$$

where we used (2.2.21.1) at the first isomorphism.

Lemma. — The homomorphism $p_{\mathcal{A}(t)}(-) \boxtimes p_{\mathcal{A}'(t)}(-) \to q_{\mathcal{A} \boxtimes \mathcal{A}'}(- \boxtimes (-))$ defined by taking adjoint to (2.2.21.1) is an isomorphism.

Proof. We can easily reduce to the case where $\mathcal{X}(t)$ and $\mathcal{Y}(t)$ are spaces using the spectral sequences of Corollary 2.2.8 and Lemma 2.2.12. Using the same spectral sequences, we may assume further that $\mathcal{X}(t)$ and $\mathcal{Y}(t)$ are realizable schemes in the “second read” case. Now, for realizable schemes $X$ and $X'$, and $\mathcal{A}(t), \mathcal{A}'(t)$ in $D^b_{\text{hol}}(\mathcal{X}(t)),$ we have

$$\text{Hom}(\mathcal{A} \boxtimes \mathcal{A}', \mathcal{A}' \boxtimes \mathcal{A}') \cong \text{Hom}(\mathcal{A}, \mathcal{A}') \boxtimes \text{Hom}(\mathcal{A}', \mathcal{A'}).$$

This follows by the isomorphism $\text{Hom}(\mathcal{A}, \mathcal{A}') \cong \mathbb{D}(\mathcal{A} \boxtimes \mathcal{A}'.)$ (cf. [AC1 A.1]) and the commutativity of $\mathbb{D}$ and $\boxtimes$ (cf. [AC1 1.3.3 (i)]). Using the Künneth formula for realizable schemes 1.1.4, the lemma follows.

For the second read

2.2.22. Before moving on to the “second read” (cf. 2.0.2) to establish the theory for more general algebraic stacks, we need the following proposition.

Proposition. — Let $\mathcal{X}$ be a Deligne-Mumford stack of finite type. Moreover, assume that $\mathcal{X}$ is separated. Then the canonical functor $D^b(\text{Hol}(\mathcal{X})) \to D^b_{\text{hol}}(\mathcal{X})$ is an equivalence.

Proof. First, we note that since the Deligne-Mumford stack is separated and of finite type, the diagonal morphism is finite, and in particular, schematic. Fully faithfulness is the only problem, so it suffices to show that the canonical functor $D^+(\text{Ind}(\text{Hol}(\mathcal{X}))) \to D^+(\mathcal{X})$ is equivalent, since the canonical functor $D^b(\text{Hol}(\mathcal{X})) \to D^b_{\text{hol}}(\text{Ind}(\text{Hol}(\mathcal{X}))) \to D^+_{\text{hol}}(\text{Ind}(\text{Hol}(\mathcal{X})))$ is fully faithful. Let $f : X \to Y$ be an affine étale morphism of realizable schemes. Then the pair $(f_!, f^*)$ of functors between $M(X/L)$ and $M(Y/L)$ is an adjoint pair, and $f_!$ is exact by [AC1 1.3.13]. In particular, $f^*$ sends injective objects to injective objects. Now, let $f : X \to \mathcal{X}$ be a smooth morphism from an affine scheme. The functors in 2.2.19 define a pair of adjoint functors $(f^*, f_*)$ between $M(X/L)$ and $\text{Ind}(\text{Hol}(\mathcal{X}/L))$. Then we can define functors

$$Rf_* : D^+(X/L) \to D^+(\text{Ind}(\text{Hol}(\mathcal{X}/L)) : f^*$$

as in the scheme case. Consider the following cartesian diagram:

\[
\begin{array}{ccc}
Y' & \xrightarrow{f'} & Y \\
\downarrow{g'} & & \downarrow{g} \\
X & \xrightarrow{f} & \mathcal{X}
\end{array}
\]
where $X$, $Y$ are affine schemes, and $g$ is an étale morphism. We can easily check that the canonical homomorphism $g^* \circ f_* \to f'_* \circ g^*$ is an isomorphism. Since $X$ is assumed to be separated, $g'$ is affine étale morphism, and thus $g^*$ preserves injective objects. This implies that the canonical homomorphism

\[(*) \quad g^* \circ \mathbb{R}f_* \to \mathbb{R}f'_* \circ g^* : D^+(X/L) \to D^+(Y/L).
\]

is an isomorphism.

Now, let $f : X \to \mathfrak{X}$ be a simplicial presentation realizable scheme. Then we can define pair of adjoint functors

\[
\mathbb{R}f_* : D^+(X/\mathfrak{X}) \rightleftarrows D^+(\text{Ind}(\text{Hol}(\mathfrak{X}/L))): f^*
\]

in the similar way to \[2.1.11\]. To conclude the proof, we need to show an analogue of Proposition \[2.1.13\] in this context. For this, use the base change \[\ref{43}\] above, and we are reduced to the realizable scheme situation we have already treated.

\[\boxrulebox\]

2.2.23. For an algebraic stack of finite type over $k$, we may take a presentation $X \to \mathfrak{X}$ such that $X$ is a separated algebraic space of finite type (or even affine scheme of finite type). Then $\text{cosk}_0(X \to \mathfrak{X})$ consists of separated algebraic spaces of finite type, since $\mathfrak{X}$ is assumed to be quasi-separated. In the “first read”, the starting point of the construction was Theorem \[1.1.9\]. In the “second read”, Proposition \[2.2.22\] plays the role of the theorem. Replacing the definitions of the terminologies accordingly to the table of \[2.0.2\] we can construct the theory for quasi-separated algebraic stacks of finite type over $k$.

2.2.24 Remark. — (i) As one can see from the construction of the cohomology theory explained in \[2.2.23\] quasi-separatedness is important. Otherwise, non-separated algebraic spaces appear in the simplicial space $\text{cosk}_0(X \to \mathfrak{X})$, and we are not able to apply Proposition \[2.2.22\]. Non quasi-separated stacks naturally appear in the work of \[\text{[Ol]}\].

(ii) We may treat the non quasi-compact case with much effort. For a separated scheme $X$ locally of finite type over $k$, we take an affine covering $\{U_i\}$, and define $M(X/L)$ by gluing $M(U_i/L)$. Note that this category is not equivalent to $\text{Ind}(\text{Hol}(X/L))$ in general. Even though we need to care about finiteness and so on, the constructions in \[2.1\] and \[2.2\] can be carried out almost in parallel. Since we do not treat such stacks, we do not go into further details.

2.2.25 Question. — Let $X$ be a separated scheme of finite type over $k$. We may construct a functor $D^b(\text{Con}(X/L)) \to D^b_{\text{hol}}(X/L)$ as in \[\text{[B]}\]. We ask if this is an equivalence of categories.

Remark. — (i) If we have a positive answer for this problem, we can define the six functor formalism exactly as in \[\text{[LQ]}\], or more precisely, we may define the push-forward to be the derived functor of $\mathcal{H}^0f_*$. If the proof of the question is “motivic”, then suitable six functor formalism can be extended to that for algebraic stacks automatically.

(ii) The problem is solved by Nori \[\text{[N]}\] when $k = \mathbb{C}$ and $\text{Hol}(X/L)$ is replaced by the category of perverse sheaves.

Theory of weights

2.2.26. Let $k$ be a finite field with $q = p^r$ elements, and consider the situation where the base tuple (cf. \[1.4.6\]) is $(r, K, s, \sigma = \text{id})$, and put $\sigma_L := \text{id}$. We fix an isomorphism $\iota : \overline{\mathbb{Q}_p} \cong \mathbb{C}$. Let $X$ be a realizable scheme over $k$. We say that $\mathcal{M} \in D^b_{\text{hol}}(X_0/L)$ is $\iota$-mixed (resp. $\iota$-mixed of weight $\leq w$, $\iota$-mixed of weight $\geq w$) if $\text{for}_L(\mathcal{M}) \in D^b_{\text{hol}}(X_0/K)$ is. The results \[\text{[AC1]}\ 4.1.3, 4.2.3\] are automatically true also for $D^b_{\text{hol}}(-/L)$ since cohomological operators commutes with for$L$ by \[1.4.10\] The same for \[\text{[AC1]}\ 4.3\].

Remark. — For $(V, \varphi) \in F\text{-Vec}_L$, if $f_\varphi(x)$ is the characteristic polynomial of $\varphi$ and $\{\alpha_i\}$ is the set of eigenvalues, then the set of eigenvalues for $\text{for}_L((V, \varphi)) \in F\text{-Vec}_K$ is $\{\sigma(f_\varphi)(x) = 0\}_{\sigma \in \text{Hom}(L, \overline{\mathbb{Q}}_p)}$. This is equal to the finite set $\{\sigma(\alpha_i)\}_{\sigma \in \text{Gal}(\overline{\mathbb{Q}}_p/K)}$ (without multiplicity).

2.2.27. Let $\mathfrak{X}$ be an algebraic stack over $k$. We say that $\mathfrak{X} \in \text{Hol}(X_0/L)$ is $\iota$-pure of weight $w$ (resp. $\iota$-mixed, $\iota$-mixed of weight $\leq w$, $\iota$-mixed of weight $\geq w$) if for any $f : X \to \mathfrak{X}$ in $\mathfrak{X}_{\text{sm}}$ of relative dimension $d$, $f^*(\mathfrak{M})$ is $\iota$-pure of weight $w+d$ (resp. $\iota$-mixed, $\iota$-mixed of weight $\leq w+d$, $\iota$-mixed of weight $\geq w+d$).

By the existence of weight filtration \[\text{[AC1]}\ 4.3.4\], if $\mathfrak{M}$ is $\iota$-mixed, then there exists an increasing filtration $W$ such that $\text{gr}^W_i(\mathfrak{M})$ is $\iota$-pure of weight $i$. A complex $\mathcal{C} \in D^b_{\text{hol}}(X_0/L)$ is said to be $\iota$-mixed complex
of weight \( \star \in \{ \leq w, \geq w, \emptyset \} \), if \( \mathcal{H}^i \mathcal{C} \) is \( \iota \)-mixed of weight \( \star + i \). We say that the complex \( \mathcal{C} \) is \( \iota \)-pure of weight \( w \) if it is \( \iota \)-mixed of weight both \( \leq w \) and \( \geq w \).

2.2.28. Let \( f : \mathcal{X}_0 \to \mathcal{Y}_0 \) be a morphism of algebraic stacks of finite type over \( k_0 \). We have the following properties:

1. For any algebraic stack \( \mathcal{X}_0 \), the functor \( \mathcal{D}_{\mathcal{X}_0} \) preserves \( \iota \)-mixed complexes, and exchanges \( \iota \)-mixed of weight \( \leq w \) and \( \geq -w \).

2. Assume \( f \) is finite. Then \( f_+ \), \( f^! \) preserves \( \iota \)-mixedness. Moreover, \( f_+ \) preserves weight, and \( f^! \) preserves complexes of weight \( \geq w \).

3. Let \( \mathcal{M} \) and \( \mathcal{N} \) be \( \iota \)-mixed complexes in \( D^b_{\text{hol}}(\mathcal{X}_0/L) \) and \( D^b_{\text{hol}}(\mathcal{Y}_0/L) \) respectively. Then \( \mathcal{M} \boxtimes \mathcal{N} \) is \( \iota \)-mixed as well. Moreover if \( \mathcal{M} \) and \( \mathcal{N} \) are of weight \( \geq w \) and \( \geq w' \) (resp. \( \leq w \) and \( \leq w' \)), then \( \mathcal{M} \boxtimes \mathcal{N} \) is of weight \( \geq w + w' \) (resp. \( \leq w + w' \)).

4. The complex \( L_r^{\mathcal{X}} \) (resp. \( L_x \)) is \( \iota \)-mixed of weight \( \geq 0 \) (resp. \( \leq 0 \)).

5. Assume \( f \) is a projection \( \mathcal{X}_0 \times \mathcal{Y}_0 \to \mathcal{Y}_0 \), and \( \mathcal{A} \) be an \( \iota \)-mixed complex of weight \( \leq w' \) on \( \mathcal{X}_0 \). Then \( f_{\mathcal{A}+} \) sends \( \iota \)-mixed complexes of weight \( \geq w \) to that of weight \( \geq w - w' \).

We think only the last property needs some explanation. Let \( \mathcal{M} \in \text{Hol}(\mathcal{X}_0 \times \mathcal{Y}_0) \) be \( \iota \)-mixed of weight \( \geq w \). We may assume \( \mathcal{A} \in \text{Hol}(\mathcal{X}_0) \). Let us use the notation of 2.2.11. We denote by \( d_p \) (resp. \( d'_q \)) the relative dimension of \( X_p \to X \) (resp. \( Y_q \to \mathcal{Y} \)). By definition, \( \mathcal{M}_{p,q} \) is \( \iota \)-mixed of weight \( \geq w + d_p + d'_q \), and \( \mathcal{A}_p \) is of weight \( \leq w' + d_p \). Recall that

\[
\mathbb{R}p_{p}^{*}(\mathcal{M}_{p,q}) \cong p_+ \mathcal{H}\text{om}(q_{p,0}^* \mathcal{A}_p, \mathcal{M}_{p,q}).
\]

where \( q_p : X_p \times Y_q \to Y_q \) is the projection. Using [AC1, 4.1.3], \( \mathbb{R}p_{p}^{*}(\mathcal{M}_{p,q}) \) is of weight \( \geq (w + d_p + d'_q) - (d_p + w') = w - w' + d'_q \). Now, by the spectral sequence of Lemma 2.2.12 the claim follows.

2.3. Six functor formalism for admissible stacks

In this subsection, we construct the six functor formalism for admissible stacks, namely algebraic stacks with finite diagonal morphisms.

2.3.1 Definition. — A morphism \( f : \mathcal{X} \to \mathcal{Y} \) between algebraic stacks is said to be admissible if it is of finite type, and the diagonal morphism \( \Delta_f : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \) is finite. An algebraic stack \( \mathcal{X} \) over \( k \) is said to be admissible if the structural morphism is admissible.

Remark. — (i) An admissible morphism is quasi-compact and separated by definition.

(ii) For an admissible stack \( \mathcal{X} \), there exists a finite covering \( \{ \mathcal{U}_i \} \) such that \( \mathcal{U}_i \) possess a quasi-finite flat morphism \( V_i \to \mathcal{U}_i \) from a scheme. This is possible by [SGA 3, V, 7.2]. In particular, there is a dense open substack \( \mathcal{U} \) of \( \mathcal{X} \) such that there exists a finite locally free morphism \( V \to \mathcal{U} \) from a scheme.

2.3.2 Lemma. — Let \( \mathcal{X} \xrightarrow{f} \mathcal{Y} \xrightarrow{g} \mathcal{Z} \) be morphisms between algebraic stacks.

(i) If \( f \) and \( g \) are admissible, so is \( g \circ f \).

(ii) Let \( \mathcal{Y}' \to \mathcal{Y} \) be a morphism between algebraic stacks. If \( f \) is admissible, then the base change \( \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \to \mathcal{Y}' \) is admissible as well.

(iii) If \( g \circ f \) is admissible, so is \( f \).

(iv) Separated representable morphisms of finite type between algebraic stacks are admissible. In particular, immersions are admissible.

(v) Any morphism \( \mathcal{X} \to \mathcal{Y} \) from a scheme to an admissible stack is schematic.

Proof. The proof for (i) and (ii) are the same as [EGA I, 5.5.1]. Let us show (iii). Consider the factorization of \( \Delta_{g \circ f} \) into representable morphisms

\[
\mathcal{X} \xrightarrow{\Delta_{g \circ f}} \mathcal{X} \times \mathcal{Y} \xrightarrow{p} \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}.
\]
We need to show that $\Delta_f$ is finite. By definition of algebraic stacks, $\mathcal{Y} \rightarrow \mathcal{Y} \times \mathcal{Y}$ is representable separated. Thus, by [EGA I, 5.5.1 (v)], $\Delta_y$ is separated as well. This implies that $p$ is separated since it is the base change of $\Delta_y$. Since the composition $p \circ \Delta_f$ is assumed to be finite and the morphisms are representable, we conclude that $\Delta_f$ is finite by [EGA II, 6.1.5 (v)].

For (iv), assume $f$ is representable and separated. Then [LM 8.1.3] shows that the diagonal $\mathfrak{X} \rightarrow \mathfrak{X} \times_\mathcal{Y} \mathfrak{X}$ is a monomorphism, and since $f$ is assumed to be separated, it is a closed immersion. For the latter assertion, use [EGA I, 5.5.1 (i)].

For (v), factorize the morphism as $X \rightarrow X \times \mathcal{Y} \rightarrow \mathcal{Y}$. The first morphism is finite since $\mathcal{Y}$ is admissible, so it is schematic. The second one is schematic as well since $X$ is a scheme.

2.3.3. The following variant of Chow’s lemma for admissible stack is important for showing foundational properties of cohomological operations:

**Proposition.** Let $\mathfrak{X}$ be an admissible stack. Then there exists a morphism $p: X \rightarrow \mathfrak{X}$ such that $X$ is a scheme and $p$ is a surjective generically finite proper morphism.

**Proof.** We modify slightly the proof of [O2 (1.1)]. From [ibid., 2.1 to 2.4], the argument is the same. In [ibid., 2.5], he replaces $\mathfrak{X}$ by the closure of $U :\rightarrow \mathfrak{X} \times \mathbb{P}(V)$. This replacement is not finite, but birational over $\mathfrak{X}$, so this replacement is harmless in our situation. In [ibid., 2.6], it suffices to take $P'$ such that $\text{dim}(P') = \text{dim}(\mathfrak{X})$ in addition to the conditions there. If we have a surjective morphism $a: P' \rightarrow \mathfrak{X}$ then this is generically finite. Indeed, let $q: Q \rightarrow \mathfrak{X}$ be a smooth presentation, and $P'_Q := P' \times_{\mathfrak{X}} Q$. By generically finiteness and some standard limit argument, there exists an open dense subscheme $V \subset Q$ such that $P'_Q \times_Q V \rightarrow \mathfrak{X}$ is finite. By fpqc descent, $a$ is finite over $q(V) \subset \mathfrak{X}$.

We do not need [ibid., 2.7, 2.8]. Take a quasi-finite flat covering $\{V_i \rightarrow \mathfrak{X}\}$ as in Remark 2.3.1 (ii). Put $P' = \mathbb{P}_P$, where $r := \text{dim}(\mathfrak{X}) - \text{dim}(P')$. By the copying the argument of [ibid., 2.9] (taking $P_1$ to be $P'$), we can shrink $V_i$ and may assume that there exist morphisms $V_i \rightarrow P'$ factoring $V_i \rightarrow \mathfrak{X} \rightarrow P$. Indeed, in [ibid., 2.9], he uses only the fact that $V_i \rightarrow P$ has equidimensional fibers. In our case, since $V_i \rightarrow \mathfrak{X}$ and $\mathfrak{X} \rightarrow P$ are flat, the equidimensionality holds. For [ibid., 2.10 to 2.13], we just copy word by word. Since he only takes normalizations and blow-ups of $P'$, the dimension does not change, and we get the desired morphism.

**Remark.** We are not able to take $p$ to be generically étale in general. Indeed, let $k$ be an algebraically closed field of characteristic $p$, and let $G$ be a connected finite flat group scheme of dimension $0$ over $k$ which is not étale (e.g. $\alpha_p := \text{Spec}(k[T]/(T^p))$). Consider the admissible stack $\mathfrak{X} := BG := [\text{Spec}(k)/G]$. Assume there exists a generically finite étale proper surjective morphism $p: X \rightarrow BG$. Then since $\text{dim}(BG) = 0$, the dimension of $X$ is $0$ as well. By taking a connected component, we may assume that $p: \text{Spec}(k) \rightarrow BG$, since $k$ is assumed to be algebraically closed. Since any $G$-torsor on an $\text{Spec}(k)$ splits, the category $BG(\text{Spec}(k))$ is a singleton, and $p$ is nothing but the universal torsor. The morphism $p$ cannot be étale, since if it were, $G$ would be étale.

2.3.4 Corollary. Let $\mathfrak{X}$ be an admissible stack. Then there exists a generically finite proper surjective morphism $f: X \rightarrow \mathfrak{X}$ such that $f$ is generically finite and $X$ is smooth quasi-projective.

**Proof.** Use Chow’s lemma above, and then de Jong’s alteration theorem.

2.3.5. Let $f: \mathfrak{X} \rightarrow \mathcal{Y}$ be a morphism of admissible stacks. When $f$ is a finite morphism (resp. projection), we denote by $f_\oplus$ and $f^\oplus$ the functors $f_*$ and $f^+$ defined in [2.1.20] (resp. [2.2.20]). We denote the fibered and cofibered categories defined in [2.4.20] by $\mathcal{F}_\oplus$ and $\mathcal{F}^{\oplus}$ respectively.

Consider the canonical factorization $\mathfrak{X} \rightarrow \mathfrak{X} \times \mathcal{Y} \rightarrow \mathcal{Y}$. Since $\mathcal{Y}$ is admissible, $i$ is finite, and the following definition makes sense:

$$f_+ := p_\oplus \circ i_\oplus : D^+_{\text{hol}}(\mathfrak{X}) \rightarrow D^+_{\text{hol}}(\mathcal{Y}), \quad f^+ := i^{\oplus} \circ p^{\oplus} : D^b_{\text{hol}}(\mathcal{Y}) \rightarrow D^b_{\text{hol}}(\mathfrak{X}).$$

We have the adjoint pair $(f^+, f_+)$.  

2.3.6 Lemma. Let $f: \mathfrak{X} \rightarrow \mathcal{Y}$ be a finite morphism between admissible stacks. Then there is a canonical isomorphism $f_\oplus \cong f_+$, and $f^{\oplus} \cong f^+$. 

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Proof. Let \( \mathfrak{X} \overset{i}{\to} \mathfrak{X} \times \mathfrak{Y} \overset{p}{\to} \mathfrak{Y} \) be the standard factorization. By adjointness property, it suffices to construct the isomorphism for the pull-back. We construct the isomorphism for their duals, namely \( f^! \) and \( i^! \) where \( p^! := L_X^\mathfrak{Y} \mathcal{E} (-) \) (cf. Definition 2.2.18). Let \( Y \rightarrow \mathfrak{Y} \) be a simplicial presentation realizable scheme, and \( X_\bullet \to \mathfrak{X} \) be its pull-back. We may assume \( X_0 = \mathfrak{X} \) and \( Y_0 = \mathfrak{Y} \) are equidimensional. For \( \mathcal{M} \in M(Y_\bullet) \), let us construct an isomorphism \( \alpha : \mathcal{H}^0 f^!(\mathcal{M}) \simeq \mathcal{H}^0 i^!(L_X^\mathfrak{Y} \mathcal{E} \mathcal{M}) \). Put \( Y = Y_i \) for \( i \geq 0 \), and \( X := \mathfrak{X} \times \mathfrak{Y} Y \). We have the following commutative diagram

![Diagram](https://via.placeholder.com/150)

Let \( d := d_\beta - d_{\beta'} \) where \( d_* \) denotes the relative dimension of \( * \). We have canonical isomorphisms

\[ g^* f^!(\mathcal{M}) \simeq (p^! \circ i^!) g^* (\mathcal{M}) \simeq i^! (L_X^\mathfrak{Y} \mathcal{E} g^* \mathcal{M}) \simeq i^! g^* (L_X^\mathfrak{Y} \mathcal{E} \mathcal{M})(d)[d] \simeq g^* i^!(L_X^\mathfrak{Y} \mathcal{E} \mathcal{M}). \]

Apply \( \mathcal{H}^i \) to this isomorphism, and since it satisfies the cocycle condition, we have the desired isomorphism \( \alpha \). We moreover have

\[ (\ast) \quad \mathcal{H}^i (i^!(L_X^\mathfrak{Y} \mathcal{E} \mathcal{M})) = 0 \quad \text{for } i \neq 0 \]

if \( \mathcal{M} \) is an injective object in \( M(Y_\bullet) \). Now, for \( \mathcal{M} \in D^+(X_\bullet) \), take an injective resolution \( \mathcal{M} \to \mathcal{I}^\bullet \). We denote by \( \mathcal{H}^0 (i^!(L_X^\mathfrak{Y} \mathcal{E} \mathcal{I}^\bullet)) \) the complex whose degree \( i \) part is \( \mathcal{H}^0 (i^!(L_X^\mathfrak{Y} \mathcal{E} \mathcal{I}^i)) \). We have quasi-isomorphisms

\[ f^!(\mathcal{M}) \simeq (\mathcal{H}^0 f^!)(\mathcal{I}^\bullet) \simeq \mathcal{H}^0 (i^!(L_X^\mathfrak{Y} \mathcal{E} \mathcal{I}^\bullet)) \simeq (\mathcal{H}^0 (i^!(L_X^\mathfrak{Y} \mathcal{E} \mathcal{I}^\bullet))) \]

where the first isomorphism is induced by \( \alpha \), and the second by the vanishing \( \ast \). Thus, the lemma follows.

2.3.7 Lemma. — Let \( \mathfrak{X} \overset{f}{\to} \mathfrak{Y} \overset{g}{\to} \mathfrak{Z} \) be morphisms of admissible stacks. Then we have canonical homomorphisms of functors

\[ \alpha : \text{id}^+ \simeq \text{id}, \quad \beta : \text{id} \simeq \text{id}_+, \quad c_{g,f} : f^+ \circ g^+ \simeq (g \circ f)^+, \quad d^{\beta,f} : (g \circ f)_+ \simeq g_+ \circ f_+. \]

These homomorphisms are subject to the following conditions: 1. We have identities \( c_{f,\text{id}} = \alpha(f^+) \), \( c_{\text{id},f} = f^+ \alpha \). 2. Assume given another morphism of admissible stacks \( h : \mathfrak{Z} \to \mathfrak{W} \). Then we have an equality

\[ c_{h,g \circ f} \circ c_{g,f}(h^+) = c_{h \circ g,f} \circ f^+(c_{h,g}). \]

We have the similar equalities for \( \beta \) and \( d^{\beta,f} \).

Proof. By the adjointness property, it suffices to show the lemma for the pull-back. First, we define \( \alpha \) to be the isomorphism of Lemma 2.3.6. Consider the following diagram:

![Diagram](https://via.placeholder.com/150)

The transition isomorphism for (1), namely the isomorphism \( a^\bullet \circ b^\bullet \simeq c^\bullet \circ d^\bullet \), is defined by 2.2.20. That for (3), use Lemma 2.2.19 and for (5), use Lemma 2.2.4. Finally, for the transition isomorphism for (2), it suffices to construct \( \Gamma^+(L_{X \times Y}) \simeq L_X \) where \( \Gamma : \mathfrak{X} \to \mathfrak{X} \times \mathfrak{Y} \) is the graph morphism. This follows by Lemma 2.2.19. The verification of the compatibility conditions is straightforward, so we leave it as an exercise. ■
Let $\text{St}^{\text{adm}}(k_\bullet)$ be the full subcategory of the category of algebraic stacks (we do not consider the 2-morphisms) consisting of admissible stacks. For an admissible stack $X$, we associate the triangulated category $D^b_{\text{hol}}(X)$ (resp. $D^b_{\text{hol}}(\mathcal{X})$). By the data of the lemma above, we have a cofibered category $\mathcal{F}_+ \to \text{St}^{\text{adm}}$ by considering $f_+$ (resp. a fibered category $\mathcal{F}_+ \to \text{St}^{\text{adm}}$ by considering $f^+$). The isomorphism of Lemma 2.3.6 yields an isomorphism of fibered and cofibered categories $\mathcal{F}_+ \cong \mathcal{F}^+$ and $\mathcal{F}_+ \cong \mathcal{F}_+$ over the category of admissible stack with finite morphism $\text{St}^{\text{adm}}$.

2.3.10 Proposition. — Let $\rho: X \to \mathcal{X}$ be a smooth scheme over an admissible stack $\mathcal{X}$. Let $d$ be the relative dimension of $\rho$. Then $\rho^* \cong \rho^+[d]$, and $\mathbb{R}\rho_* \cong \rho_+[d]$ (cf. 2.2.3).

Proof. By adjointness, it suffices to prove $\rho^* \cong \rho^+[d]$. Since the proof is similar to that of Lemma 2.3.6, we only sketch. Let $Y_\bullet \to \mathcal{X}$ be a simplicial presentation scheme, and $Y := Y_\bullet$. Consider the following cartesian diagram:

$$
\begin{array}{ccc}
X \times_{\mathcal{X}} Y & \to & X \times Y \\
\alpha \downarrow & & \downarrow \\
X & \to & X \times \mathcal{X}.
\end{array}
$$

For $\mathcal{M} \in M(Y_\bullet)$, we have $\alpha^*\rho^*(\mathcal{M}) = \alpha^*\rho^+[d](\mathcal{M})$. This follows by using the fact that if $f$ is a smooth morphism of relative dimension $d_f$ between realizable schemes, then $f^* \cong f^+[d_f]$ by definition. We finish the proof by descent argument. □

For a smooth morphism $\rho: X \to \mathcal{X}$ from a realizable scheme to an admissible scheme and $\mathcal{M} \in D^b_{\text{hol}}(X)$, we often denote $\rho^*(\mathcal{M})$ by $\mathcal{M}_X$.

2.3.9 Lemma. — Let $f: \mathcal{X} \to \mathcal{Y}$ be a smooth morphism of relative dimension $d$ between admissible stacks. Then $f^* \cong f^+[d]$ and $\mathbb{R}f_* \cong f_*[-d]$. In particular, $f^+[d]$ is an exact functor.

Proof. Let us construct the functor. Let $\mathcal{M} \in \text{Hol}(\mathcal{Y})$. By Lemma 2.3.8 for any $X \in X_{\text{sm}}$, we have the canonical morphism $f^*(\mathcal{M})_X \to (\mathcal{M}_{\text{hol}}d^f(\mathcal{M}))_X$. Since it satisfies the cocycle condition, it yields a homomorphism $f^* \to \mathcal{M}_{\text{hol}}d^f$. Since $f^*$ is exact, by the universal property, it induces the desired homomorphism on the derived category. Once we have the homomorphism, it is easy to check that it is an isomorphism: Since the verification is local, it suffices to show that for any smooth presentation $\rho: X \to \mathcal{X}$, we have $\rho^* \circ f^* \cong \rho^* \circ f^+[d]$. By Lemma 2.3.8 and transitivity of $(\cdot)^*$ and $(\cdot)^+$, the lemma follows. □

2.3.10 Proposition (Smooth base change). — Consider the following cartesian diagram of admissible stacks:

$$
\begin{array}{ccc}
\mathcal{X}' & \xrightarrow{\text{g}} & \mathcal{X} \\
\text{f'} \downarrow & & \text{f} \\
\mathcal{Y} & \xrightarrow{\text{g}} & \mathcal{Y}
\end{array}
$$

where $g$ is smooth. Then the base change homomorphism $g^+f_+ \to f'_+g'^+: D^b_{\text{hol}}(\mathcal{X}) \to D^b_{\text{hol}}(\mathcal{Y}')$ is an isomorphism.

Proof. In the verification, it suffices to replace $g^+$ and $g'^+$ by $g^*$ and $g'^*$ by Lemma 2.3.9. We use the standard factorization of $f$ into a finite morphism and a projection, and the verification is reduced to these cases separately. In both cases, the verification is straightforward from the definition, so we leave the details to the reader. □

2.3.11. Let $j: \mathcal{U} \hookrightarrow \mathcal{X}$ be an open immersion, and $i: \mathcal{Z} \hookrightarrow \mathcal{X}$ be its complement. Then Lemma 2.3.9 shows that $j^+$ is an exact functor. We define the functor $j_!$ so that the pair $(j_!, j^+)$ is adjoint. Such a functor exists since by Lemma 2.3.9 and Lemma 2.2.5 $D^+_{\mathcal{X}} \circ j_+ \circ D^+_{\mathcal{U}}$ is left adjoint to $j^+$. Thus, we have pairs of adjoint functors $(j^+, j_+)$ and $(j_!, j^+)$. We have the following exact triangle:

$$i_! i^! \to \text{id} \to j_! j^+ \xrightarrow{\pm 1}, \quad j_! j^+ \to \text{id} \to i^! i_+ \xrightarrow{\pm 1}.$$
Let us check the left one. Let $X_i \to X$ be a simplicial presentation realizable scheme, and let $\mathcal{F}^\bullet$ be a complex of injective objects in $C(X_i)$. By abuse of notation, we denote the open and closed immersions $U \times_X X_i \to X_i$ and $Y \times_X X_i \to X_i$ by $j$ and $i$ respectively. By construction and Remark 2.3.10 $i_+^!$ and $i^!$ commute with $\rho_k^\ast$. By Lemma 2.3.11, $j^!$ commutes also with $\rho_k^\ast$. Moreover, $j^!$ commutes with $\rho_k^\ast$ as well, which implies that their right adjoint functors $j_!$ and $j_+$ commute. Thus, it suffices to show that the sequence

$$0 \to i_+^! (\rho_k^\ast(\mathcal{F}^l)) \to \rho_k^\ast(\mathcal{F}^l) \to j_+ j^!(\rho_k^\ast(\mathcal{F}^l)) \to 0$$

is exact. This follows by Lemma 2.3.11. The right triangle is exact by duality.

2.3.12 Proposition. — Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism between admissible stacks. Then $f_+$ induces a functor $D^{b}_{\text{hol}}(\mathcal{X}) \to D^{b}_{\text{hol}}(\mathcal{Y})$.

Proof. We may assume $\mathcal{X}$ to be reduced. It suffices to show that $f_+(\mathcal{M})$ is bounded for $\mathcal{M} \in \text{Hol}(\mathcal{X})$. Let us show by the induction on the dimension of the support of $\mathcal{M}$. We may assume that the support of $\mathcal{M}$ is equal to $\mathcal{X}$. Let $g : Y \to \mathcal{Y}$ be a smooth presentation from a scheme. Since the functor $g^\ast$ is conservative, we may replace $\mathcal{Y}$ by $Y$, and assume that $Y$ is a scheme. Now, we may shrink $\mathcal{X}$. Indeed, consider the localization triangle $i_* i^! \to \text{id} \to j_+ j^! \to$ of (2.3.11). We know that $i^!$ and $i_+$ preserves boundedness. This implies that $j_+ j^!$ preserves boundedness as well. By induction hypothesis, we are reduced to showing the proposition for $j_+ j^!$, and the claim follows. Since $\mathcal{X}$ is an admissible stack, we may assume that there exists a finite surjective morphism $h : V \to \mathcal{X}$ from a scheme $V$. Shrinking $\mathcal{X}$ further, we may assume that $h$ is finite locally free since $\mathcal{X}$ is assumed to be reduced. In this situation, since $\mathcal{M}$ is a direct factor of $h_+ h^!(\mathcal{M})$ by Lemma 2.1.21 we reduce the verification to the scheme case. 

2.3.13. Let $\mathcal{X}$ be an admissible stack. Then the diagonal morphism $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is finite. For $i = 1, 2$, let $p_i : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ be the $i$-th projection. Let $p : \mathcal{X} \to \text{Spec}(k)$ be the structural morphism. We define the internal Hom functor by

$$\text{Hom}(\mathcal{M}, \mathcal{N}) := p_{1,\mathcal{M}}(\Delta_!(\mathcal{N})) : D^{b}_{\text{hol}}(\mathcal{X}) \to D^{b}_{\text{hol}}(\mathcal{X}).$$

Let $\mathcal{L} \in D^{b}_{\text{hol}}(\mathcal{X})$. Since $(p_{1,\mathcal{M}}^+, p^+)_{1,\mathcal{M}}$ is an adjoint pair, with Remark 1.2.1, we have

$$\mathcal{R}\text{Hom}(\mathcal{L}, \text{Hom}(\mathcal{M}, \mathcal{N})) \cong \mathcal{R}\text{Hom}(\mathcal{L} \otimes_{\mathcal{M}} \Delta_+(\mathcal{N})).$$

We can check easily that when $\mathcal{X}$ is a realizable scheme, $\text{Hom}$ coincides with that in (1.15). Now, let $f : \mathcal{X} \to \mathcal{Y}$, $g : \mathcal{X}' \to \mathcal{Y}'$ be morphisms of admissible stacks. Then, we have an isomorphism $(f \times g)^{\pm}(\Delta_{\mathcal{X}' \times \mathcal{Y}'}) \cong f^{\pm}(\Delta_{\mathcal{X} \times \mathcal{Y}}) \otimes g^{\mp}(\Delta_{\mathcal{X}' \times \mathcal{Y}'})$. This follows by combining Lemma 2.2.1 and 2.2.21. Using this, we have

$$\text{Hom}(\mathcal{L}, f_+ \text{Hom}(\mathcal{M}, \mathcal{N})) \cong \text{Hom}(f^{\pm}(\mathcal{L}) \otimes f^{\pm}(\mathcal{M}), \Delta_{\mathcal{X} \times \mathcal{Y}}(\mathcal{N})) \cong \text{Hom}(f^{\mp}(\mathcal{L} \otimes \mathcal{M}), \mathcal{N}) \cong \text{Hom}(\mathcal{L}, \text{Hom}(\mathcal{M}, f_+ \mathcal{N})).$$

Thus, we get an isomorphism

$$f_+ \text{Hom}(\mathcal{M}, \mathcal{N}) \cong \text{Hom}(\mathcal{M}, f_+ \mathcal{N}).$$

2.3.14 Lemma. — Let $\mathcal{M}$ and $\mathcal{N}$ be in $D^{b}_{\text{hol}}(\mathcal{X})$. For a presentation $\rho : X \to \mathcal{X}$ from a realizable scheme, there is a canonical isomorphism

$$\rho^\ast \text{Hom}(\mathcal{M}, \mathcal{N}) \cong \text{Hom}(\rho^\ast(\mathcal{M}), \rho^\ast(\mathcal{N}))[d],$$

and $d$ denotes the relative dimension function of $\rho$.

Proof. When $\mathcal{X}$ is a realizable scheme, it can be checked easily that $\text{Hom}$ coincides with the internal Hom recalled in (1.15). We denote $\rho^\ast(\mathcal{N})$ by $\mathcal{N}'$. We have the following diagram

$$\begin{array}{ccc}
X \times X & \xrightarrow{\rho \times \text{id}} & \mathcal{X} \times \mathcal{X} \\
q \downarrow & & \downarrow g \\
X & & X
\end{array}$$

where $g$ is the second projection. By the definition of the functor $p_{\mathcal{X} \times \mathcal{X}}$, we have

$$(*) \quad \rho^\ast \text{Hom}(\mathcal{M}, \mathcal{N}) \cong q_{\mathcal{X} \times \mathcal{X}}(\Delta'_{\mathcal{X} \times \mathcal{X}}(\mathcal{N}'))$$

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where $\Delta' := (\rho, \text{id}): X \to X \times X$. Let $\delta$ be the diagonal morphism of $X$.

\[
\text{Hom}(\rho^+(\mathcal{M}), \mathcal{N}') \cong q_+(\rho \times \text{id})_+ \text{Hom}(\rho^+(\mathcal{M}) \boxtimes L_X, \Delta_+(\mathcal{N}'))
\cong q_+ \text{Hom}(\mathcal{M} \boxtimes L_X, \Delta^+_+(\mathcal{N}'))
\cong q_+ \text{Hom}(\mathcal{M}, \Delta^+_+(\mathcal{N}')).
\]

Thus, combining with the isomorphism $\rho^+ \cong \rho^*[-d]$ and (2.3.14), the lemma follows.

2.3.15. Recalling Definition 2.2.18, we define the dual functor to be

\[
\mathbb{D}_X(\mathcal{M}) := \mathcal{H}om(\mathcal{M}, L_X^\omega): D^b_{\text{hol}}(\mathfrak{X}) \to D^b_{\text{hol}}(\mathfrak{X}).
\]

If no confusion may arise, we often omit the subscript $\mathfrak{X}$ from $\mathbb{D}_X$.

**Proposition** (Biduality). — There exists a canonical isomorphism of functors

\[
\text{id} \xrightarrow{\sim} \mathbb{D}_X \circ \mathbb{D}_X: D^b_{\text{hol}}(\mathfrak{X}) \to D^b_{\text{hol}}(\mathfrak{X}).
\]

**Proof.** We have isomorphisms

\[
(2.3.15.1) \quad \text{Hom}(\mathbb{D}(\mathcal{M}), \mathbb{D}(\mathcal{N})) \cong \text{Hom}(\mathcal{M} \boxtimes \mathcal{N}, \Delta_+(L_X^\omega))
\cong \text{Hom}(\mathcal{M} \boxtimes \mathbb{D}(\mathcal{N}), \Delta_+(L_X^\omega)) \cong \text{Hom}(\mathcal{M}, \mathbb{D}(\mathbb{D}(\mathcal{N})),
\]

where the second isomorphism is induced by the morphism $\mathfrak{X} \times \mathfrak{X} \xrightarrow{\sim} \mathfrak{X} \times \mathfrak{X}$ exchanging the first and second factor. The image of the identity morphism induces a homomorphism $\mathcal{M} \to \mathbb{D}(\mathcal{N})$. By Lemma 2.3.14 this homomorphism is in fact an isomorphism since so is the realizable scheme case.

**Remark.** — (i) We have a canonical isomorphism $\mathcal{H}^i \mathbb{D} \cong \mathcal{H}^i \mathbb{D}'$ for any $i$. However, even though there is no doubt to believe that $\mathbb{D}_X$ and $\mathbb{D}_X'$ coincide, we do not know the proof.

(ii) We have a canonical isomorphism $L_X^\omega \cong \mathbb{D}_X(L_X)$. This follows since for any $X \in \mathfrak{X}_{\text{sm}}$, we have $(L^\omega)_X \cong \mathbb{D}(L)_X$, and the uniqueness of Theorem 2.2.16.

(iii) When $j$ is an open immersion, we have $j^+ \circ \mathbb{D} \cong \mathbb{D} \circ j^+$. Thus, $j_! \cong \mathbb{D} \circ j_+ \circ \mathbb{D}$.

(iv) When $f$ is a finite morphism, we shall prove in Lemma 2.3.20 that we have an isomorphism $f^+ \cong \mathbb{D} \circ f^! \circ \mathbb{D}$.

2.3.16. We define the tensor product by

\[
(\_ \times \, ) := \Delta^+(\, \boxtimes \, ) : D^b_{\text{hol}}(\mathfrak{X}) \times D^b_{\text{hol}}(\mathfrak{X}) \to D^b_{\text{hol}}(\mathfrak{X}).
\]

Let $\mathcal{M}, \mathcal{N}, \mathcal{L}$ be objects in $D^b_{\text{hol}}(\mathfrak{X})$. We have

\[
R\text{Hom}(\mathcal{M}, R\text{Hom}(\mathcal{N}, \mathcal{L})) \cong R\text{Hom}(\mathcal{M}, p_{1,\mathcal{N}^+}(\Delta_+(\mathcal{L}))) \cong R\text{Hom}(\mathcal{M} \boxtimes \mathcal{N}, \Delta_+(\mathcal{L}))
\cong R\text{Hom}(\Delta_+^+(\mathcal{M} \boxtimes \mathcal{N}), \mathcal{L}) = R\text{Hom}(\mathcal{M} \otimes \mathcal{N}, \mathcal{L}).
\]

The identity homomorphism of $\text{Hom}(\mathcal{M}, \mathcal{N})$ induces the evaluation homomorphism

\[
\text{Hom}(\mathcal{M}, \mathcal{N}) \otimes \mathcal{M} \to \mathcal{N}.
\]

Now, since $p_2 \circ \Delta \cong \text{id}$, we have

\[
L_X \otimes \mathcal{M} = \Delta^+(L_X \boxtimes \mathcal{M}) \cong \Delta^+(p_2^+(\mathcal{M})) \cong (p_2 \circ \Delta)^+(\mathcal{M}) \cong \mathcal{M}
\]

where the second isomorphism holds by the definition of $p_2^+$ (cf. Definition 2.2.10). Using these, we have

\[
R\Gamma \circ p_+ \text{Hom}(\mathcal{M}, \mathcal{N}) \cong R\text{Hom}_{R(\mathfrak{X})}(L_X, \text{Hom}(\mathcal{M}, \mathcal{N})) \cong R\text{Hom}_{R(\mathfrak{X})}(\mathcal{M}, \mathcal{N})
\]

where the first isomorphism holds by Proposition 2.2.15, and the second by what we have just proven.

2.3.17 Proposition. — Let $f: \mathfrak{X} \to \mathfrak{X}', g: \mathfrak{Y} \to \mathfrak{Y}'$ be morphisms of admissible stacks.

1. We have an isomorphism $(f \times g)^+((-) \boxtimes (-)) \cong f^+(-) \boxtimes g^+(-)$.
2. We have $\mathbb{D}((-) \boxtimes (-)) \cong \mathbb{D}(-) \boxtimes \mathbb{D}(-)$.

3. We have $f^+((-) \otimes (-)) \cong f^+(-) \otimes f^+(-)$.

4. We have $\text{Hom}(\mathcal{M} \otimes \mathcal{N}, \mathcal{L}) \cong \text{Hom}(\mathcal{M}, \text{Hom}(\mathcal{N}, \mathcal{L}))$.

5. We have $\text{Hom}(\mathbb{D}(\mathcal{M}), \mathbb{D}(\mathcal{N})) \cong \text{Hom}(\mathcal{N}, \mathcal{M})$.

6. We have $\text{Hom}(\mathcal{M}, \mathcal{N}) \cong \mathbb{D}(\mathcal{M} \otimes \mathbb{D}(\mathcal{N}))$.

**Proof.** The first one is just a reproduction from 2.3.13. The second claim follows by combining Lemma 2.2.2 (the commutativity of $f_+$ and $\boxtimes$), Lemma 2.2.21 and Lemma 2.2.19 (i). Let us show 5. Let $\mathcal{M}'$, $\mathcal{N}'$ be objects in $D^b_{\text{hol}}(\mathfrak{X}')$. Consider the following commutative diagram:

\[
\begin{array}{ccc}
\mathfrak{X} & \xrightarrow{\Delta} & \mathfrak{X} \times \mathfrak{X} \\
\downarrow{f} & & \downarrow{f \times f} \\
\mathfrak{Y} & \xrightarrow{\Delta_{\mathfrak{Y}}} & \mathfrak{Y} \times \mathfrak{Y}.
\end{array}
\]

Using this diagram, we have

\[f^+(\mathcal{M}' \otimes \mathcal{N}') \cong f^+\Delta^+_{\mathfrak{Y}}(\mathcal{M}' \boxtimes \mathcal{N}') \cong \Delta^+_{\mathfrak{X}}(f \times f)^+(\mathcal{M}' \boxtimes \mathcal{N}') \cong f^+(\mathcal{M}') \otimes f^+(\mathcal{N}').\]

Let us show 4. For any $\mathcal{L} \in D^b(\mathfrak{X})$, we have a canonical isomorphism

\[\text{Hom}(\mathcal{L}, \text{Hom}(\mathcal{M} \otimes \mathcal{N}, \mathcal{L})) \cong \text{Hom}(\mathcal{L}, \mathcal{M}, \text{Hom}(\mathcal{N}, \mathcal{L}))\]

by using 2.3.10 (i) two times, thus the claim follows. For 5, the isomorphisms 2.3.10 (i) still hold if we replace Hom by $\text{Hom}$ with suitable push-forwards. For the last claim 6, it suffices to construct a canonical isomorphism $\text{Hom}(\mathcal{L}, \text{Hom}(\mathcal{M}, \mathcal{N})) \cong \text{Hom}(\mathcal{L}, \mathbb{D}(\mathcal{M} \otimes \mathbb{D}(\mathcal{N})))$. This can be shown by using 4 and 5.

**2.3.18.** Let $f: \mathfrak{X} \to \mathfrak{Y}$ be a morphism of admissible stacks, and let $\mathcal{M}$, $\mathcal{N}$ be objects in $D^b_{\text{hol}}(\mathfrak{X})$. The adjunction homomorphisms $f^+f_+ \to \text{id}$ induces a homomorphism

\[f^+(f_+(\mathcal{M}) \otimes f_+(\mathcal{N})) \cong f^+f_+(\mathcal{M}) \otimes f^+f_+(\mathcal{N}) \to \mathcal{M} \otimes \mathcal{N},\]

where the first isomorphism follows by Proposition 2.3.17. This induces the homomorphism

\[f_+(\mathcal{M}) \otimes f_+(\mathcal{N}) \to f_+(\mathcal{M} \otimes \mathcal{N}).\]

Using this, we have a homomorphism

\[f_+\text{Hom}(\mathcal{M}, \mathcal{N}) \otimes f_+(\mathcal{M}) \to f_+(\text{Hom}(\mathcal{M}, \mathcal{N}) \otimes \mathcal{M}) \to f_+(\mathcal{N})\]

where the second homomorphism is induced by the evaluation homomorphism. Taking the dual, we get a canonical homomorphism

\[(2.3.18.1) \quad f_+\text{Hom}(\mathcal{M}, \mathcal{N}) \to \text{Hom}(f_+\mathcal{M}, f_+\mathcal{N}).\]

**Duality results**

**2.3.19.** Recall the trace map defined in Theorem 1.5.1. This trace map is defined in the category $D^b_{\text{hol}}(S/K)$ or $F-D^b_{\text{hol}}(S/K)$. Now, we have functors $f_+$, $f^+$ on the level of $D^b_{\text{hol}}(S_{\text{hol}}/L)$, and in fact, the theorem holds even if we replace $D^b_{\text{hol}}(S)$ by $D^b_{\text{hol}}(S_{\text{hol}}/L)$. When $\Delta = \emptyset$, this follows by the commutativity of $\text{for}_L$ and the cohomological functors $f_+$, $f^+$, and the fully faithfulness of Corollary 1.4.3. When $\Delta = 0$, we moreover use the commutativity of $\text{for}_F$ and $f_+$, $f^+$. 1.3.2.2 and Corollary 1.4.9.
2.3.20. First, let us construct the trace morphism for projective morphisms. Let \( f : X \to \mathcal{Y} \) be a projective morphism between admissible stacks. Let \( Y_\bullet \to \mathcal{Y} \) be a simplicial presentation quasi-projective scheme, and since \( f \) is assumed to be projective, the cartesian product \( X_\bullet := X \times_{\mathcal{Y}} Y_\bullet \) is an admissible simplicial quasi-projective scheme as well. Since \( f_+L^i_{X_\bullet} \) are in \( D_{\text{hol}}^b(Y_\bullet) \), we have a spectral sequence

\[
E_1^{p,q} = \text{Ext}^q_{D(Y_\bullet)}(f_+L^p_{X_\bullet}, L^q_{Y_\bullet}) \Rightarrow \text{Ext}^{p+q}_{D(Y_\bullet)}(f_+L^i_{X_\bullet}, L^j_{Y_\bullet})
\]

by (2.3.20.1). For each \( p \), the usual trace morphism defines \( \text{Tr}_{f_+} : \text{Hom}(\cdot, L^p_{X_\bullet}) \to \text{Hom}(\cdot, f_+L^p_{X_\bullet}) \). Moreover, consider the following cartesian diagram of admissible stacks:

\[
\begin{array}{ccc}
X & \xrightarrow{g'} & X \\
\downarrow f' & \Box & \downarrow f \\
\mathcal{Y} & \xrightarrow{g} & \mathcal{Y} \\
\end{array}
\]

Assume \( f \) is projective. Then the canonical homomorphism \( g^+f_+ \to f'_+g'^+ \) is an isomorphism. If, moreover, \( g \) is an open immersion, we have the canonical isomorphism \( g \circ f'_+ \cong f_+ \circ g' \).

2.3.21 Proposition (Proper base change). — Consider the cartesian diagram of admissible stacks (2.3.20.1). Assume that \( f \) is proper. Then the canonical homomorphism \( g^+f_+ \to f'_+g'^+ \) is an isomorphism.

Proof. It suffices to show that \( g^+f_+(\mathcal{M}) \cong f'_+g'^+(\mathcal{M}) \) for \( \mathcal{M} \in \text{Hol}(X) \). We may assume \( X \) to be reduced by Lemma [2.1.21] and Lemma [2.3.20]. By smooth base change theorem [2.3.11], we may replace \( \mathcal{Y} \) by its smooth presentation. In particular, we may assume \( \mathcal{Y} =: Y \) to be a realizable scheme. Now, we use the induction on the dimension of the support of \( \mathcal{M} \). We may assume \( \text{Supp}(\mathcal{M}) =: \mathcal{X} \). By induction hypothesis, it suffices to show the equality for the equality for \( \mathcal{M} = j_!(\mathcal{N}) \) where \( j : \mathcal{U} \to \mathcal{X} \) which is open dense. By Corollary [2.3.3], there is a smooth quasi-projective scheme \( X \) projective and generically finite over \( \mathcal{X} \), and projective over \( Y \). Since \( h : X \to \mathcal{X} \) is projective, we already know the base change by Lemma [2.3.20].

We may shrink \( \mathcal{U} \) so that \( h \) is finite flat over \( \mathcal{U} \) since \( \mathcal{X} \) is assumed to be reduced. We denote \( h^{-1}(\mathcal{U}) \to \mathcal{U} \) by \( h \) abusing the notation. Then we have \( h_+j_!(h^+(\mathcal{N})) \cong j_h_+h^+(\mathcal{N}) \), where \( h' : h^{-1}(\mathcal{U}) \to X \), and this contains \( j_!(\mathcal{N}) \) as a direct factor by Lemma [2.1.21]. Thus, the verification is reduced to the case \( \mathcal{M} = j_!(h^+(\mathcal{N})) \). Indeed, let \( \mathcal{F} \in D_{\text{hol}}^b(X) \), and \( \mathcal{E} \) be a direct factor of \( \mathcal{F} \). For any integer \( i \), we have the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{H}^i g^+f_+\mathcal{E} & \longrightarrow & \mathcal{H}^i g^+f_+\mathcal{F} \\
\downarrow & & \downarrow \\
\mathcal{H}^i f'_+g'^+\mathcal{E} & \longrightarrow & \mathcal{H}^i f'_+g'^+\mathcal{F}
\end{array}
\]

where the composition of the horizontal homomorphisms are identities. If the homomorphism \( \ast \) is an injection (resp. surjection), so is the left (resp. right) vertical homomorphism, so it suffices to show that \( \ast \) is an isomorphism.

Thus we may replace \( \mathcal{X} \) by \( X \). In this case, the verification is local with respect to \( Y \), and may assume it to be an affine scheme. In this situation, \( X \) is realizable as well since \( X \) is projective over \( Y \), and the proper base change theorem has already been known.

\[
\square
\]
2.3.22 Definition. — A morphism of admissible stacks \( f: \mathfrak{X} \to \mathfrak{Y} \) is said to be compactifiable if it can be factorized as

\[
\mathfrak{X} \xrightarrow{j} \overline{\mathfrak{X}} \xrightarrow{f'} \mathfrak{Y}
\]

where \( \overline{\mathfrak{X}} \) is admissible, \( j \) is an open immersion, and \( f' \) is proper. We say that \( \mathfrak{X} \to \overline{\mathfrak{X}} \) is a compactification of \( f \). An admissible stack \( \mathfrak{X} \) is said to be compactifiable if the structural morphism is compactifiable. For short, we abbreviate compactifiable admissible stack as \( c\text{-admissible} \).

2.3.23. In this subsection, we fix a subcategory \( \mathcal{S}_{\text{adm}} \) of the category of admissible stacks satisfying the following conditions: 1. open immersion and proper morphisms are morphisms in \( \mathcal{S}_{\text{adm}} \), 2. any morphism \( f: \mathfrak{X} \to \mathfrak{Y} \) in \( \mathcal{S}_{\text{adm}} \) is compactifiable by an object in \( \mathcal{S}_{\text{adm}} \), and 3. for a proper morphism \( \mathfrak{X} \to \mathfrak{Y} \) and any morphism \( \mathfrak{Y}' \to \mathfrak{Y} \) in \( \mathcal{S}_{\text{adm}} \), the fiber product \( \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Y}' \to \mathfrak{Y}' \) is in \( \mathcal{S}_{\text{adm}} \). An example of such category is the following:

Lemma. — The category of c-admissible stacks satisfies the conditions.

Proof. We need to show that any morphism between c-admissible stacks is compactifiable. Let \( f: \mathfrak{X} \to \mathfrak{Y} \) be a morphism between c-admissible stacks, and let \( \overline{\mathfrak{X}} \) be a compactification of the structural morphism of \( \mathfrak{X} \). Then \( f \) is factorized as \( \mathfrak{X} \xrightarrow{f} \overline{\mathfrak{X}} \times \mathfrak{Y} \xrightarrow{p} \mathfrak{Y} \) where \( \Gamma \) is the graph morphism and \( p \) is the projection. Since \( \overline{\mathfrak{X}} \) is assumed to be proper, \( p \) is proper. Thus, it suffices to show that \( \Gamma \) is compactifiable. Since \( \mathfrak{Y} \) is admissible, \( \Gamma \) is a quasi-finite morphism. By [LM, 16.5], any quasi-finite morphism between admissible stacks is compactifiable, and the claim follows.

2.3.24. An advantage of considering the category \( \mathcal{S}_{\text{adm}} \) is that it satisfies the conditions of [SGA 4, XVII 3.2.4] if we take \( (S) \) to be \( \mathcal{S}_{\text{adm}} \), \( (S,i) \) to be the subcategory consisting of open immersions, and \( (S,p) \) to be the subcategory consisting of proper morphisms.

Now, for \( \mathfrak{X} \in \mathcal{S}_{\text{adm}} \), we associate the category \( D^b_{\text{hol}}(\mathfrak{X}) \). We shall further endow with data which satisfies the conditions of [SGA 4, XVII 3.3.1]. For a proper morphism \( p \), we consider the push-forward \( p_+ \) and the transition isomorphism for composition. For an open immersion \( j \), we consider \( j_! \) with transition isomorphism. These are the data of \([\text{ibid.}, (i), (i'), (ii), (ii')]\). These functors are subject to the conditions \([\text{ibid.}, (a), (a'), (b), (b')]\). Finally, for \([\text{ibid.}, (iii)]\), we use the proper base change [2.3.21] and localization exact triangle [2.3.11]. This isomorphism is subject to the conditions \([\text{ibid.}, (c), (c')]\). Thus, we may apply [ibid., Proposition 3.3.2]. Summing up, we come to get the following definition.

Definition. — Let \( f: \mathfrak{X} \to \mathfrak{Y} \) be a morphism in \( \mathcal{S}_{\text{adm}} \). Take a compactification \( j: \mathfrak{X} \to \overline{\mathfrak{X}} \), and \( g: \overline{\mathfrak{X}} \to \mathfrak{Y} \) be the proper morphism. Then the functor \( g_+ \circ j_! \) does not depend on the choice of the factorization up to canonical equivalence. This functor is denoted by \( f_! \). Given composables morphisms \( f \) and \( g \) in \( \mathcal{S}_{\text{adm}} \), we have a canonical equivalence \( (f \circ g)_! \cong f_! \circ g_! \).

2.3.25 Lemma. — Consider the cartesian diagram \( \xymatrix{ \mathfrak{X} \ar[r]^j \ar[d]^f & \overline{\mathfrak{X}} \ar[d]_{f'} \ar[r]^g & \mathfrak{Y} \ar[d]^{g'} \ar@{.>}[ld]_j \ar@{.>}[ld]_g } \) in \( \mathcal{S}_{\text{adm}} \) (where we do not assume \( f \) to be projective). Then there exists a canonical isomorphism \( g^+ \circ f_! \cong f'_! \circ g'^+ \).

Proof. By definition of \( f_! \), it suffices to treat the case where \( f \) is proper and an open immersion separately. When \( f \) is proper, this is nothing but proper base change theorem [2.3.21] When \( f = j \) is an open immersion, we have the canonical homomorphism \( j'! \circ g'^+ \to g^+ \circ j_! \). By definition, this homomorphism is an isomorphism if we take \( j' \). Thus by the localization triangle (cf. [2.3.11]), we get the isomorphism. Finally, we need to show that the resulting isomorphism does not depend on the choice of the factorization. Since the verification is standard, we leave it to the reader.

2.3.26. Let us construct a trace map. For this, we need to introduce a new t-structure.

Definition. — Let \( X \) be a realizable scheme. For \( * \in \{\geq 0, \leq 0\} \), let \( cD^* \) be the full subcategory of \( D^b_{\text{hol}}(X/L) \) consisting of \( \mathfrak{V} \) such that for \( L \) for \( \mathfrak{V} \in cD^* \) \( \mathfrak{V} \in D^b_{\text{hol}}(X/K) \). The pair \( (cD^+, cD^\geq) \) defines a t-structure on \( D^b_{\text{hol}}(X/L) \) called the constructible t-structure. We define \( dcD^\leq := D(cD^\leq) \) and \( dcD^\geq := D(cD^\geq) \). Then \( (dcD^\leq, dcD^\geq) \) defines a t-structure on \( D^b_{\text{hol}} \). This is called the dual constructible t-structure.

2.3.27 Definition. — Let \( \mathfrak{X} \) be an algebraic stack. Let \( \mathcal{M} \to \mathfrak{X} \) be a simplicial presentation realizable scheme, and let \( M = D^b_{\text{hol}}(\mathfrak{X}) \). Put \( d_! := d_{\mathfrak{X}^\circ} \) (cf. [1.1.3]).
1. A complex \( \mathcal{M} \in D^b_{hol}(X) \) is in \( cD^* (\ast \in \mathbb{Z}) \) if and only if \( \rho_!^*(\mathcal{M}) \in cD^{*-d} \).

2. A complex \( \mathcal{N} \in D^b_{hol}(X) \) is in \( dcD^* \) \((\ast \in \mathbb{Z})\) if and only if \( \rho_!^*(\mathcal{N}) \in dcD^{*-d} \).

We leave the reader to check that \( cD^\leq \) (resp. \( dcD^\leq \)) define t-structures, and do not depend on the choice of the simplicial schemes. These t-structures are called the constructible t-structure and dual constructible t-structure, and abbreviate as \( c-t \)-structure and \( dc-t \)-structure respectively. We denote the cohomology functor for the \( c \)-t-structure (resp. \( dc-t \)-structure) by \( c\mathcal{H}^* \) (resp. \( dc\mathcal{H}^* \)), and objects in the heart are called \( c \)-modules (resp. \( dc \)-modules).

**2.3.28 Lemma.** — (i) We have \( dc\mathcal{H}^i \cong \mathbb{D} \circ c\mathcal{H}^{-i} \circ \mathbb{D} \). In particular, \( \mathcal{M} \in D^b_{hol}(X) \) is a \( c \)-module if and only if \( \mathbb{D}(\mathcal{M}) \) is a dc-module, and a homomorphism \( f : \mathcal{M} \to \mathcal{N} \) of \( c \)-modules is \( c \)-injective (resp. \( c \)-surjective) if and only if \( \mathbb{D}(f) \) is dc-injective (resp. dc-surjective).

(ii) Let \( f : X \to Y \) be a smooth morphism. Then the functor \( f^+ \) is \( c \)-t-exact.

**Proof.** For (ii), see Lemma \[1.3.3\]. The detailed proof is left to the reader. \[\square\]

**2.3.29 Lemma.** — (i) For an admissible stack \( X \), \( L_X^c \) is a dc-module.

(ii) Let \( f : X \to Y \) be a morphism in \( \mathcal{S}_{adm} \). Then \( f_* \) is right dc-t-exact.

**Proof.** To check (i), it suffices to show that \( \mathbb{D}(L_X^c) \cong L_X^c \) is a \( c \)-module. This follows from Lemma \[1.3.4\](i). Let us check (ii). First we may assume \( Y = Y' / X \) to be a realizable scheme. For a dc-module \( \mathcal{M} \) on \( X \), we need to show that \( dc\mathcal{H}^i f_*(\mathcal{M}) = 0 \) for \( i > 0 \). We use the induction on the support of \( \mathcal{M} \). We may assume \( \text{Supp}(\mathcal{M}) = X \). For an open dense substack \( j : U \to X \), it suffices to check that \( dc\mathcal{H}^i f_*(j_! j^+ \mathcal{M}) = 0 \) for \( i > 0 \). Indeed, consider the following exact sequence of dc-modules:

\[
0 \to \mathcal{E} \to j^+ j^+ (\mathcal{M}) \to \mathcal{M} \to \mathcal{E}' 
\]

Since \( \mathcal{E}' \) is supported on the complement of \( U \), we know that \( dc\mathcal{H}^i j_! (\mathcal{E}') = 0 \) for \( i > 0 \). Thus, if we know the vanishing for \( j^+ j^+ (\mathcal{M}) \), so do for \( \mathcal{M} \). By shrinking \( X \), we can take a finite flat morphism \( h : X \to X \) from a realizable scheme. By Lemma \[2.1.2\], \( \mathcal{M} \) is a direct factor of \( \mathbb{D} h^{-1} h^+ \mathbb{D}(\mathcal{M}) \cong h^+(\mathcal{M}) \) (cf. Lemma \[2.3.20\]). Since \( h^+ \) is dc-t-exact, it remains to prove the right dc-t-exactness of \( f \circ h : X \to Y \).

Since \( \mathbb{D} f \circ h \cong f^+ \) is left c-t-exact by Lemma \[1.3.4\] we get the result. \[\square\]

**2.3.30 Lemma.** — Let \( X_* \to X \) be an admissible simplicial scheme. Assume given a data \( \{ \mathcal{M}_i, \alpha_\phi \} \) where \( \mathcal{M}_i \) is a dc-module on \( X_i \), and for \( \phi : [i] \to [j] \), \( \alpha_\phi : X(\phi)^*(\mathcal{M}_i) \cong \mathcal{M}_j \) satisfying the cocycle condition. Then there exists a dc-module \( \mathcal{M} \) on \( X \), the descent, such that \( p_!^*(\mathcal{M}) \cong \mathcal{M}_i \). Moreover, given another data \( \{ \mathcal{N}_i, \beta_\phi \} \) and its descent \( \mathcal{N} \), homomorphism \( \mathcal{M} \to \mathcal{N} \) corresponds bijectively to a system of homomorphisms \( \mathcal{M}_i \to \mathcal{N}_i \) compatible in the obvious sense. We also have the similar results for c-modules.

**Proof.** To check this, it suffices to show that \( \mathbb{R}^k \text{Hom}(\mathcal{M}_i, \mathcal{M}_j) = 0 \) for \( k < 0 \) by Theorem \[2.2.16\]. By the definition of dc-t-structure and duality (cf. Proposition \[2.3.15\]), we may assume that \( \mathcal{M}_i \) is a dc-module. In this case, since \( \mathbb{R}^k \text{Hom}(\cdot, -) \) is left c-t-exact, the claim follows. \[\square\]

**2.3.31 Lemma.** — Let \( f : X \to Y \) be a smooth morphism of realizable schemes. Let \( \mathcal{M}' \) be a complex in \( D^b_{hol}(Y) \), and put \( \mathcal{M} := f^+ (\mathcal{M}') \). Then there exists an open dense subscheme \( V \subset Y \) such that \( \text{Supp} (\mathcal{M}') \cap V \) is dense in \( \text{Supp} (\mathcal{M}') \), and for any closed immersion from a point \( g : \{ y \} \to Y \), the base change homomorphism \( g^+ f_+ (\mathcal{M}) \to f'_+ g^+ (\mathcal{M}) \) is an isomorphism, where \( f' : X' := X \times_Y \{ y \} \to \{ y \} \) and \( g' : X' \to X \) are the base change of \( f \) and \( g \).

**Proof.** We may assume \( \mathcal{M} \in \text{Hol}(X/L) \). By replacing \( Y \) by the support of \( \mathcal{M} \), we may assume that the support of \( \mathcal{M} \) is equal to \( Y \). We may assume that \( Y \) is smooth and \( \mathcal{M}' \) is smooth on \( Y \). We may take \( V \) such that each cohomology of \( f_+ (\mathcal{M}) \) is smooth. Let \( c \) be the codimension of \( \{ y \} \) in \( Y \). In this case, we have \( g^+ f_+ (\mathcal{M}) \cong g' f'_+ (\mathcal{M}) (c)[2c] \) and \( f'_+ g^+ (\mathcal{M}) \cong f'_+ g'^+ (\mathcal{M}) (c)[2c] \) by Theorem \[1.5.14\] and the claim follows by \[1.1.3\] \[\square\]

**2.3.32 Lemma.** — Let \( X_* \to X \) be a simplicial presentation realizable scheme of a c-admissible stack \( X \). Let \( p_! : X_I \to X \) be the induced morphism. We put \( p_!^1 := c\mathcal{H}^0 p_! \) (resp. \( p_!^0 := dc\mathcal{H}^0 p_! \)), and
similarly for $p^0_i$. For a $c$-module $\mathcal{M}$ (resp. $dc$-module $\mathcal{N}$), we denote by $\mathcal{M}_i$ (resp. $\mathcal{N}_i$) the module on $X_i$. We have the following exact sequences of $c$-modules (resp. $dc$-modules):

$$0 \to \mathcal{M} \to p^0_+(\mathcal{M}_0[\delta_0]) \to p^0_+(\mathcal{M}_1[\delta_1]), \quad (\text{resp. } 0 \to \mathcal{N} \to p^0_+(\mathcal{N}_0[\delta_0]) \to p^0_+(\mathcal{N}_1[\delta_1]))$$

$$p^0_+(\mathcal{M}_1[d](\delta_1)) \to p^0_0(\mathcal{M}_0[\delta_0](d)) \to \mathcal{M} \to 0, \quad (\text{resp. } p^0_+(\mathcal{N}_1[d](\delta_1)) \to p^0_0(\mathcal{N}_0[\delta_0](d)) \to \mathcal{N} \to 0).$$

**Proof.** Let us prove the upper left sequence. The sequence is defined by the adjunction. We only need to show that it is exact. Thus, we may assume $X = X$ to be a scheme. First, let us show that there exists an open subscheme $j : U \hookrightarrow X$ which is dense in the support of $\mathcal{M}$ such that $j_+\mathcal{M}|_U$ satisfies the exactness property. We may take $U$ such that $p_0$ and $p_1$ possess the base change property by Lemma 1.3.4. Indeed, it suffices to show the exactness after restricting to $U$ since $j_+$ is left $c$-exact by Lemma 2.3.31. In this case, we are reduced to the case where $X$ is a point by the base change. Then $c$-t-structure coincides with the usual t-structure, and the exactness follows by Proposition 2.1.13.

We may take the "stalk", and reduce to the case where $\mathcal{M}$ is exact. Thus, we may assume $\mathcal{M}$ such that the sequence is exact for $j_+\mathcal{M}|_U$ where $j : U \hookrightarrow X$. Consider the following diagram of $c$-modules where we omit shifts and twists:

$$\begin{array}{cccccc}
0 & \to & \mathcal{C} & \to & \mathcal{M} & \to & 0 \\
0 & \to & p^0_0 + \mathcal{M}_0 & \to & p^0_0 + \mathcal{M}_0 & \to & 0 \\
0 & \to & p^0_1 + \mathcal{M}_1 & \to & p^0_1 + \mathcal{M}_1 & \to & 0 \\
\end{array}$$

The horizontal sequences are complexes and sequences with solid arrows are exact. By induction hypothesis, the vertical sequences are known to be exact except for the one starting from $\mathcal{M}$. Then by diagram chasing, we get that the vertical sequence starting from $\mathcal{M}$ is exact as well, and we get the lemma.

For the exactness of lower right sequence we just argue dually. For the lower left sequence, the argument is similar, and even simpler: We may assume $\mathcal{M}$ to be a scheme. We can check the exactness by taking the "stalk", and reduce to the case where $X$ is a point, in which case we get the exactness by Proposition 2.1.13. We can show dually for the upper right sequence, and may finish the proof. ■

2.3.33 Theorem. — Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism in $\mathcal{S}_\text{adm}$. Then there exists a unique homomorphism $Tr^p_{\mathcal{Y}} : f_*L^c_{\mathcal{X}} \to L^c_{\mathcal{Y}}$ satisfying the following conditions:

1. Transitivity: given $\mathcal{X} \xrightarrow{f} \mathcal{Y} \xrightarrow{g} \mathcal{Z}$ in $\mathcal{S}_\text{adm}$, the composition of the following homomorphisms is equal to $Tr^p_{\mathcal{Z}}$

   $$(g \circ f)_*L^c_{\mathcal{X}} \cong g_*(f_*L^c_{\mathcal{X}}) \xrightarrow{g_{Tr}^p} g_*L^c_{\mathcal{Y}} \xrightarrow{Tr^p_{\mathcal{Y}}} K^c_{\mathcal{Z}}.$$ 

2. When $\mathcal{X} = X$ and $\mathcal{Y} = Y$ are realizable schemes and $L = K$, then $Tr^p_{\mathcal{Y}}$ is the adjunction homomorphism $f^\dag K^c_{\mathcal{X}} \cong f^\dag f^!K^c_{\mathcal{Y}} \to K^c_{\mathcal{Y}}$. Moreover, $Tr^p_{\mathcal{Y}}$ commutes with $f_{!L}$.

3. The trace map is compatible with smooth pull-back on $\mathcal{Y}$. Namely, consider 2.3.20.1) in $\mathcal{S}_\text{adm}$ such that $g$ is smooth of relative dimension $d$. Then the composition

   $$f^*L^c_{\mathcal{X}}(d) \cong f^!g^*L^c_{\mathcal{X}}(d) \xrightarrow{g_{Tr}^p f^*} g^*L^c_{\mathcal{Y}}(d) \cong L^c_{\mathcal{Y}},$$

   where the second map is the base change homomorphism, coincides with $Tr^p_{\mathcal{Y}}$.

**Proof.** We put the $dc$-t-structure on $D^b_{\text{hol}}(\mathcal{X})$ and $D^b_{\text{hol}}(\mathcal{Y})$. Since $f_1$ is right $dc$-t-exact by Lemma 2.3.29, it suffices to construct a morphism of $dc$-modules $f_!^0L^c_{\mathcal{X}} \cong L^c_{\mathcal{Y}}$ where $f_!^0 := dc\mathcal{H}^0f_1$. When $\mathcal{X}$ and $\mathcal{Y}$ are realizable schemes, the trace map for $L = K$ extends uniquely to general $L$ by (II). Let us construct in the case where $\mathcal{Y} = Y$ is a realizable scheme. Let $X_0 \to \mathcal{X}$ be a presentation from a quasi-projective scheme, $X_1 := X \times X, p_0, p_1 : X_1 \to X_0$ be the first and second projection, and put $f_i : X_i \to \mathcal{X} \xrightarrow{f} \mathcal{Y}$. 

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By the property of adjunction homomorphism, we have the following commutative diagram:

\[
\begin{array}{ccc}
  f_1^! L_{X_1}^\omega & \xrightarrow{\text{Tr}_j^p} & L_Y^\omega \\
  \downarrow p_1 & & \downarrow \text{Tr}_{j_0} \\
  f_0^! L_{X_0}^\omega & \xrightarrow{\text{Tr}_j^p} & L_Y^\omega
\end{array}
\]

Thus by the lower right exact sequence in Lemma 2.3.32, we have a homomorphism \( \text{Tr}_j^p : f_1^! L_{X_1}^\omega \to L_Y^\omega \) as required. By condition (I), this map is uniquely determined. It is straightforward to check that this map does not depend on the choice of the smooth presentation and satisfies (II).

Finally consider the case where \( \mathcal{Y} \) is not a realizable scheme. Take an simplicial presentation realizable scheme \( Y_\bullet \to \mathcal{Y} \). By Lemma 2.3.30 it suffices to construct a homomorphism \((f_1^! L_{X_1}^\omega)_{Y_\bullet} \to L_Y^\omega\) with compatibility conditions. By condition (III), this map should be the one we have already constructed, and we conclude the proof. \( \blacksquare \)

2.3.34. Let \( f : \mathcal{X} \to \mathcal{Y} \) be a proper morphism between admissible stacks. Then we have the homomorphism \( f_+ L_X^\omega \overset{\sim}{\to} f_+ L_Y^\omega \). This homomorphism induces \( f_+ \circ \mathbb{D}_X \to \mathbb{D}_Y \circ f_+ \) as in 2.3.20. Now, let \( f \) be a morphism in \( \mathcal{S}_{\text{adm}} \). Let \( \mathcal{X} \to \mathcal{X} \to \mathcal{Y} \) be a compactification of \( f \) in \( \mathcal{S}_{\text{adm}} \). We have the homomorphism

\[
(f_+ \circ \mathbb{D}_X \cong f_+ \circ j_+ \circ \mathbb{D}_X \overset{\sim}{\to} f_+ \circ \mathbb{D}_X) \circ f_1 \cong \mathbb{D}_Y \circ f_1
\]

where the second isomorphism follows by Remark 2.3.15 (iii). We may check that this homomorphism does not depend on the choice of the factorization up to canonical equivalence.

**Theorem** (Poincaré duality). — For any morphism \( f \) in \( \mathcal{S}_{\text{adm}} \), the homomorphism \((\mathcal{R})\) is, in fact, an isomorphism.

**Proof.** First, we may assume \( \mathcal{Y} \) is scheme by (III) of Theorem 2.3.33. Let us show the isomorphism for \( \mathcal{M} \in \text{Hol}(\mathcal{X}) \). We use the induction on \( \dim \text{Supp}(\mathcal{M}) \). Assume the theorem holds for \( \dim \text{Supp}(\mathcal{M}) < k \).

Let \( \mathfrak{J} \) be the support of \( \mathcal{M} \). We may shrink \( \mathcal{X} \) so that \( \mathfrak{J} \) is shrunk by its open dense subscheme. Indeed, let \( j : \mathfrak{U} \to \mathcal{X} \) be an open immersion such that \( \mathfrak{J} \cap \mathfrak{U} \) is dense in \( \mathfrak{J} \), and \( i : \mathfrak{W} \to \mathcal{X} \) is its complement. The proposition holds for \( i_+ \circ \mathcal{T}(\mathcal{M}) \) by the induction hypothesis. Thus, it suffices to show for \( \mathcal{M} = j_+ \circ \mathcal{T}(\mathcal{M}) \).

Since the theorem holds for \( f = j \), we may replace \( \mathcal{X} \) by \( \mathfrak{U} \).

Shrinking \( \mathcal{X} \) by its open dense substack, we may assume that there exists a finite flat morphism \( g : X \to \mathcal{X} \) from a realizable scheme. Since \( \mathcal{M} \) is a direct factor of \( g_+ g^+ \mathcal{M} \), by arguing as in the proof of Proposition 2.3.21, it suffices to show that the homomorphism \( f_+ \circ \mathbb{D}_X(g_+ g^+ \mathcal{M}) \to \mathbb{D}_Y \circ f_1(g_+ g^+ \mathcal{M}) \) is an isomorphism. Thus, we are reduced to the realizable scheme case. \( \blacksquare \)

2.3.35 Definition. — Let \( f : \mathcal{X} \to \mathcal{Y} \) be a morphism in \( \mathcal{S}_{\text{adm}} \). We define \( f^! : = \mathbb{D}_X \circ f^+ \circ \mathbb{D}_Y \). The couple \((f_!, f^!)\) is adjoint. Transitivity holds since it holds for \( f^+ \).

2.3.36 Lemma. — Let \( f : \mathcal{X} \to \text{Spec}(k)_\bullet \) be the structural morphism of a c-admissible stack of dimension \( d \). Then for \( \mathcal{M} \in \text{Con}(\mathcal{X}) \), the cohomology \( \mathcal{H}^i f_!(\mathcal{M}) = 0 \) for \( i > 2d \).

**Proof.** We may use the induction on the dimension of \( \mathcal{M} \). By standard dévissage using the induction hypothesis, we may shrink \( \mathcal{X} \) by its open sense substack. Shrinking \( \mathcal{X} \) and taking a finite flat morphism from a realizable scheme, we may assume that \( X \) is a realizable scheme. Then the proposition is reduced to Lemma 1.3.8. \( \blacksquare \)

2.3.37 Theorem (Relative Poincaré duality). — Let \( \mathcal{M}_d^\text{st} \) be the set of morphisms \( f : \mathcal{X} \to \mathcal{Y} \) of \( \mathcal{S}_{\text{adm}} \) such that there exists an open substack \( \mathfrak{U} \subset \mathcal{Y} \) such that \( \mathfrak{X} \times_{\mathcal{Y}} \mathfrak{U} \to \mathfrak{U} \) is flat of relative dimension \( d \), and the dimension of any fiber of \( \mathcal{Y} \setminus \mathfrak{U} \) is \( < d \). Then for \( f \in \mathcal{M}_d^\text{st} \) there is a unique trace map \( \text{Tr}_{f}^p : f_! f^+(d)[2d] \to \text{id} \) satisfying the following:

(I) When \( \mathfrak{X} \) and \( \mathcal{Y} \) are realizable schemes and \( L = K \), then it coincides with the trace map in Theorem 1.5.7. Moreover, it commutes with \( \text{for}_i \).

(II) It commutes with base change in the sense of (Var 2) of 1.5.1 if we replace the diagram of realizable schemes by that in \( \mathcal{S}_{\text{adm}} \) and \( f \in \mathcal{M}_d^\text{st} \).
(III) It is transitive with respect to the composition of morphisms in \( D^b \) in the sense of (Var 3) of \( \text{(1.5.1)} \).

Taking the adjoint, we have a homomorphism \( f^+(d)[2d] \to f' \), which is an isomorphism when \( f \) is smooth.

**Proof.** First, we need to construct the trace map \( \text{Tr}^m_\text{hol} : f_! f^+ L^2 f (d) [2d] \to L^2 f \). Put c-t-structure on \( D^\text{hol}_\text{hol}(\mathcal{X}) \). By Lemma 2.3.36, it suffices to construct a homomorphism \( \mathcal{H}^2 f_! f^+ L^2 f (d) \to L^2 f \) of c-modules. When \( \mathcal{X} \) and \( \mathcal{Y} \) are schemes, this is the trace map of Theorem 1.5.11 when \( L = K \), and in general defined extending the scalar. For careful reader, we remark that, when \( \mathcal{A} = 0 \), in \textit{ibid.}, we used the category \( F^2 D^\text{hol}_\text{hol}(s(Y)/K) \) (cf. see 1.4.11 for \( s(Y) \)) to define the trace map. However, the isomorphism defining the Frobenius structure in \( D^\text{hol}_\text{hol}(s(Y)/K) \) induces an isomorphism in \( \text{Con}(s(Y)/K) \), which defines an object in \( \text{Con}(Y/K) \), so \textit{ibid.} is enough to get a trace map in \( D^b(Y/L) \).

For the construction of trace map in the general case, let \( Y_\bullet \to \mathcal{Y} \) be an admissible simplicial scheme. By Lemma 2.3.30, it suffices to construct the trace map for \( \mathcal{X} \times_{\mathcal{Y}} Y_i \to Y_i \) for each \( i \) compatible with each other. The construction is similar to that of Theorem 2.3.33 using Lemma 2.3.32 so we leave the details to the reader.

The trace map defines a morphism \( f^+(d)[2d] \to f' \). Let us show that this is an isomorphism when \( f \) is smooth. By base change property, we may assume \( \mathcal{Y} \) to be a scheme. Moreover, it suffices to show the identity after pulling back to schemes which are smooth over \( \mathcal{X} \). Then we are reduced to the scheme case we have already treated in Theorem 1.5.13 \( \square \).

### 2.3.38. Finally, we have the projection formula, whose proof is similar to the proper base change theorem, and left to the reader:

**Proposition.** — Let \( f : \mathcal{X} \to \mathcal{Y} \) be a morphism in \( \mathfrak{S}_\text{adm} \). Then for \( \mathcal{M} \in D^b_\text{hol}(\mathcal{X}) \) and \( \mathcal{N} \in D^b_\text{hol}(\mathcal{Y}) \), we have a canonical isomorphism:

\[ f_!(\mathcal{M} \otimes \mathcal{N}) \cong f_!(\mathcal{M} \otimes f^+ \mathcal{N}) \cdot \]

**Künneth formula**

### 2.3.39 Proposition. — Consider morphisms of admissible stacks \( f : \mathcal{X} \to \mathcal{X}' \) and \( g : \mathcal{Y} \to \mathcal{Y}' \). Let \( \mathcal{M} \in D^b_\text{hol}(\mathcal{X}) \) and \( \mathcal{N} \in D^b_\text{hol}(\mathcal{Y}) \). Then there exists a canonical isomorphism

\[ f_!(\mathcal{M} \boxtimes g_+(\mathcal{N})) \cong f_!(f \times g)_+ (\mathcal{M} \boxtimes \mathcal{N}). \]

Moreover, if \( f \) and \( g \) are in \( \mathfrak{S}_\text{adm} \), we get an isomorphism

\[ f_!(\mathcal{M} \boxtimes g_!(\mathcal{N})) \cong (f \times g)_!(\mathcal{M} \boxtimes \mathcal{N}). \]

**Proof.** Let us construct the first homomorphism. We have the following homomorphism

\[ (f \times g)^+ (f_+ (\mathcal{M} \boxtimes g_+(\mathcal{N}))) \cong f^+ f_+ (\mathcal{M} \boxtimes g^+ g_+(\mathcal{N})) \to \mathcal{M} \boxtimes \mathcal{N} \]

where the first isomorphism follows by Proposition 2.3.17 and the second homomorphism is by adjunction. By taking the adjunction, we get the homomorphism we are looking for. To check that this homomorphism is an isomorphism, it suffices to treat the finite morphism case and projection case separately. The finite morphism case follows by Proposition 2.2.4 and projection case follows by Lemma 2.2.21 and Lemma 2.2.19(i). By Theorem 2.3.34, we get that \( f_! \cong \Delta_{\mathcal{X}} \circ f^+ \circ \Delta_{\mathcal{X}} \). Thus, the second isomorphism holds by the first one and the commutativity of \( \otimes \) and \( \boxtimes \) by Proposition 2.3.17 \( \square \).

### 2.3.40. Let \( f : \mathcal{X} \to \mathcal{Y} \) and \( g : \mathcal{X}' \to \mathcal{Y} \) be morphisms between admissible stacks. Consider the following cartesian diagram:

\[
\begin{array}{ccc}
\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}' & \to & \mathcal{X} \times \mathcal{X}' \\
\downarrow h & & \downarrow (f \times g) \\
\mathcal{Y} & \to & \mathcal{Y} \times \mathcal{Y}
\end{array}
\]

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For $\mathcal{M} \in D^b_{\text{hol}}(\mathfrak{X})$, $\mathcal{N} \in D^b_{\text{hol}}(\mathfrak{X'})$, we put

$$\mathcal{M} \boxtimes_{\mathcal{N}} := i^+(\mathcal{M} \boxtimes \mathcal{N}).$$

When $f$ and $g$ are the identities, $(-) \boxtimes_{\mathcal{N}} (-)$ is nothing but $(-) \otimes (-)$.

**Corollary.** Assume $f$ and $g$ are $\mathcal{S}_{\text{adm}}$. Then, we have a canonical isomorphism

$$f_!(\mathcal{M}) \otimes g_!(\mathcal{N}) \cong h_!(\mathcal{M} \boxtimes_{\mathcal{N}}).$$

**Proof.** Use Lemma 2.3.26.

---

**Theory of weights revisited**

**2.3.41.** The theory of six functors for c-admissible stacks fits perfectly well with the theory of weights. Consider the situation as in 2.2.20. The following is a direct consequence of 2.2.28.

**Theorem.** Let $f : \mathfrak{X}_0 \to \mathfrak{Y}_0$ be a morphism between admissible stacks. Then the functors $f_+$, $f^+$, $\mathbb{D}$, $\otimes$ preserves mixed complexes. Moreover, $f_+$ (resp. $f^+$) preserves complexes of weight $\geq w$ (resp. $\leq w$), $\mathbb{D}$ exchanges complexes of weight $\leq w$ and $\geq -w$, and $\otimes$ sends complexes of weight $(\leq w, \leq w')$ to $(\leq w + w')$.

In particular, by using the duality 2.3.34, if $f$ is proper, $f_+$ sends a complex pure of weight $w$ to a pure complex of weight $w$.

**2.3.42.** Another important functor is the intermediate extension functor. Let $j : \mathfrak{U} \to \mathfrak{X}$ be an immersion of admissible stacks. For $\mathcal{M} \in \text{Hol}(\mathfrak{U})$, we define the intermediate extension of $\mathcal{M}$ to be

$$j_!(\mathcal{M}) := \text{Im}(\mathcal{H}^0(j_!(\mathcal{M})) \to \mathcal{H}^0(j_!(\mathcal{M}))),$$

which is defined in $\text{Hol}(\mathfrak{X})$. For $f : \mathfrak{X} \to \mathfrak{Y}$ in $\mathfrak{X}_{\text{sm}}$, let $j_X : U := \mathfrak{U} \times_{\mathfrak{X}} \mathfrak{X} \to \mathfrak{X}$ be the immersion. We can check easily that $f^*(j_!(\mathcal{M})) \cong j_{X!}f^*(\mathcal{M})$. Now, if $j$ is defined on $k_0$, we have the following by [ACI] 4.2.4:

**Theorem.** If $\mathcal{M} \in \text{Hol}(\mathfrak{U}_0)$ is pure of weight $w$, so is $j_!(\mathcal{M})$.

---

**2.4.** Miscellaneous on the cohomology theory for algebraic stacks

Before moving on to the next section, we pose a little and collect some miscellaneous results which are used in the proof of the Langlands correspondences.

**2.4.1.** We denote by $\text{St}^\mathbb{R}(k_\mathbb{A})$ be the category of algebraic stacks of finite type over $k_\mathbb{A}$. Let $\mathfrak{X}$ be in $\text{St}^\mathbb{R}(k_\mathbb{A})$. For $X \in \mathfrak{X}_{\text{sm}}$, we associate the category $\text{Hol}(X/L)$. For $f : X \to Y$ in $\mathfrak{X}_{\text{sm}}$, we have the pull-back functor $f^* : \text{Hol}(Y/L) \to \text{Hol}(X/L)$, with obvious transition isomorphism. These data form a fibered category $\text{Hol}_X \to \mathfrak{X}_{\text{sm}}$. When $X$ is quasi-compact, the category of holonomic modules $\text{Hol}(X/L)$ is equivalent to the category of cartesian object of the fibered category $\text{Hol}_X$. For an algebraic stack locally of finite type, we define $\text{Hol}(X/L)$ to be the category of cartesian objects of $\text{Hol}_X$. For a smooth morphism $f : \mathfrak{X} \to \mathfrak{Y}$ in $\text{St}^\mathbb{R}(k_\mathbb{A})$ and $\mathcal{M} \in D^b_{\text{hol}}(\mathfrak{Y})$, we denote $f^*(\mathcal{M}) \in D^b_{\text{hol}}(\mathfrak{X})$ by $\mathcal{M}_X$.

**2.4.2.** Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a representable morphism between algebraic stacks of finite type. Let us construct $f_+^i$ and $f_+^i$. To construct these, let $Y \in \mathfrak{Y}_{\text{sm}}$. This defines a morphism $f_Y : \mathfrak{X}_Y := \mathfrak{X} \times_{\mathfrak{Y}} Y \to Y$. Note that $\mathfrak{X}_Y$ is an algebraic space by assumption. Now, for $\mathcal{M} \in \text{Hol}(\mathfrak{Y})$, put $(f_+^i \mathcal{M})_Y := \mathcal{H}^i f_{Y+}(\mathcal{M})_Y$. For a smooth morphism $\phi : Y' \to Y$ in $\mathfrak{Y}_{\text{sm}}$, the base change theorem 2.3.10 gives us the isomorphism

$$\phi^*(f_+^i \mathcal{M})_Y \cong (f_+^i \mathcal{M})_Y.$$ We can check the transitivity of this isomorphism easily, and defines an object $\{(f_+^i \mathcal{M})_Y\}_{Y \in \mathfrak{Y}_{\text{sm}}}$ in $\text{Hol}(\mathfrak{Y})$. We define $f_+^i := D_{\mathfrak{Y}} \circ f_+^i \circ D_{\mathfrak{X}}$.

**2.4.3 Lemma.** (i) If $f : \mathfrak{X} \to \mathfrak{Y}$ is a morphism in $\mathfrak{S}_{\text{adm}}$, then $\mathcal{H}^i f_+ \cong f_+^i$. 

(ii) Consider the following cartesian diagram of algebraic stacks

\[
\begin{array}{ccc}
\mathcal{X}' & \xrightarrow{g'} & \mathcal{X} \\
\downarrow f' & \square & \downarrow f \\
\mathcal{E}' & \xrightarrow{g} & \mathcal{E}.
\end{array}
\]

Assume that \( f \) is quasi-projective, \( g \) is smooth, and \( \mathcal{E}' \) (thus \( \mathcal{X}' \) as well) is admissible. Then there exists a canonical isomorphism \( g^* \circ f' \cong f'^* \circ g^* \).

\textbf{Proof.} The first claim follows from the definition, and the verification for (ii) is easy from the base change. We note that quasi-projectiveness assumption is made only to assure that, for \( C \in \mathcal{E}_\text{sm} \), the morphism \( \mathcal{X} \times_C C \to C \) is compactifiable. \( \square \)

\textbf{2.4.4.} Let \( \text{St}^{\text{ft,sm}}(k_\bullet) \) be the subcategory of \( \text{St}^{\text{ft}}(k_\bullet) \) consisting of smooth morphisms. By associating \( \text{Hol}(\mathcal{X}) \) to \( \mathcal{X} \in \text{St}^{\text{ft,sm}}(k_\bullet) \), and considering the pull-back \( f^* \) for smooth morphism, we have the fibered category \( \text{Hol} \to \text{St}^{\text{ft,sm}}(k_\bullet) \). Then we have the following descent result:

\textbf{Lemma.} — Smooth surjective representable morphisms are universal effective descent morphisms in the fibered category \( \text{Hol} \to \text{St}^{\text{ft,sm}}(k_\bullet) \).

\textbf{Proof.} Let \( \text{St}^{\text{ft,sm,rep}}(k_\bullet) \) be the subcategory of \( \text{St}^{\text{ft,sm}}(k_\bullet) \) consisting of representable morphisms. We may consider \( \text{Hol} \) as a fibered category over \( \text{St}^{\text{ft,sm,rep}}(k_\bullet) \). By Proposition \( 2.1.13 \), a smooth surjective morphism from an algebraic space is an effective descent morphism, thus it is a universal effective descent morphism in \( \text{St}^{\text{ft,sm,rep}}(k_\bullet) \). Since universal effective descent morphisms form a topology by \([\text{Gir}, 6.23]\), we get the lemma. \( \square \)

\textbf{2.4.5.} We may extend the \( i \)-th pull-back functor to arbitrary morphism between algebraic stacks. Let \( f: \mathcal{X} \to \mathcal{Y} \) be a morphism in \( \text{St}^{\text{ft}}(k_\bullet) \), and take \( Y \in \mathcal{Y}_\text{sm} \). Put \( f': \mathcal{X}' := \mathcal{X} \times_{\mathcal{Y}} Y \to Y \). Now, let \( X' \in \mathcal{X}'_\text{sm} \). Then we have the morphism \( f_{X'}^*: X' \to Y \), and

\[
f^{i+1}_X(M_Y)_{X'} := \mathcal{H}^i f^+_X(M_Y),
\]

where \( d \) denoted the relative dimension (function) of \( X' \to \mathcal{X}' \), is defined. We can check easily that the collection of these modules satisfies the compatibility condition, and defines a cartesian section of the fibered category \( \text{Hol}_X \), which is the module \( f^{i+1}_X(M_Y) \) in \( \text{Hol}(\mathcal{X}) \). These modules yield a descent data with respect to the representable smooth morphism \( \mathcal{X}' \to \mathcal{X} \). By using the Lemma \( 2.4.4 \) we get a holonomic module \( f^{i+1}(M) \). We put \( f^i_\text{sm} := f^{i+1}_X \circ D_\mathcal{Y} \).

\textbf{2.4.6 Lemma.} — Let \( \mathcal{X} \) be an algebraic stack in \( \text{St}^{\text{ft}}(k_\bullet) \), \( i: \mathcal{Z} \hookrightarrow \mathcal{X} \) be a closed substack, \( j: \mathcal{U} \hookrightarrow \mathcal{X} \) be its complement. Then, for \( M \in \text{Hol}(\mathcal{X}) \), there exists the following long exact sequence:

\[
0 \to i^* j^{i+1}(M) \to j^0 j^+(M) \to M \to i^* j^{i+0}(M) \to j^1 j^+(M) \to 0.
\]

\textbf{Proof.} By definition of the functors, we can easily reduce to the scheme case. \( \square \)

\textbf{2.4.7 Lemma.} — Let \( \mathcal{X} \) be a algebraic space of finite type, and \( G \) be a flat algebraic group space over \( \mathcal{X} \). We assume that \( G \) is moreover finite radical surjective over \( \mathcal{X} \), and let \( \rho: \mathcal{X} := [\mathcal{X}/G] \to \mathcal{X} \). Then \( \rho^{+0} \) induces the equivalence of categories between \( \text{Hol}(\mathcal{X}) \) and \( \text{Hol}(\mathcal{X}) \).

\textbf{Proof.} By assumption we have the following commutative diagram

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{u} & [\mathcal{X}/G] \\
\downarrow \text{id} & \square & \downarrow \rho \\
\mathcal{X} & \xrightarrow{\text{id}} & \mathcal{X}
\end{array}
\]

where \( u \) is the universal \( G \)-torsor, which is a universal homeomorphism by assumption. Thus we can use Lemma \( 2.1.21 \) (i) to conclude. \( \square \)
Corollary. — Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a gerb-like morphism (cf. [Beh 5.1.3]) whose structural group is flat finite and radicial. Then there exists a canonical equivalence of categories $\text{Hol}(\mathfrak{X}) \cong \text{Hol}(\mathfrak{Y})$.

Proof. Since the structural group is flat, there exists a smooth surjective morphism from an algebraic space $P : \mathcal{Y}_0 \to \mathfrak{Y}$ such that $f_0 : \mathfrak{X}_0 := \mathfrak{X} \times_{\mathfrak{Y}} \mathcal{Y}_0 \to \mathcal{Y}_0$ is a neutral gerb by Lemma A.2.1. Let $\mathcal{Y}_1 := \mathcal{Y}_0 \times_{\mathfrak{Y}} \mathcal{Y}_0$, $\mathcal{Y}_2 := \mathcal{Y}_0 \times_{\mathfrak{Y}} \mathcal{Y}_0 \times_{\mathfrak{Y}} \mathcal{Y}_0$, and $f_i : \mathfrak{X}_i := \mathfrak{X} \times_{\mathfrak{Y}} \mathcal{Y}_i \to \mathcal{Y}_i$ be the projection. We have the following diagram:

$$
\begin{array}{c}
\text{Hol}(\mathfrak{Y}) \\
\downarrow f_0^* \\
\text{Hol}(\mathfrak{X}) \\
\downarrow f_1^* \\
\text{Hol}(\mathfrak{X}_0) \\
\downarrow f_2^* \\
\text{Hol}(\mathfrak{X}_1) \\
\downarrow f_2^* \\
\text{Hol}(\mathfrak{X}_2)
\end{array}
$$

By the assumption on the structural group, $f_0^*, f_1^*, f_2^*$ are equivalence of categories by Lemma 2.4.4. Since $P$ is a presentation, we may use Lemma 2.4.4 to conclude.

2.4.8 Lemma. — Let $\mathcal{X}$ be an algebraic space of finite type over $k$ such that $\mathcal{X}_{\text{red}}$ is smooth, and $G$ be a smooth fiberwise connected algebraic group over $\mathcal{X}$. Then we have a canonical equivalence of categories $\text{Sm}(\mathcal{X}) \to \text{Sm}(BG)$.

Proof. We may replace $\mathcal{X}$ by $\mathcal{X}_{\text{red}}$ since the derived category do not change, and we may assume that $\mathcal{X}$ is smooth. The canonical morphism $p : \mathcal{X} \to BG$ is a smooth presentation and $\mathcal{X} \times BG \cong G$ such that the $i$-th projection $p_i : \mathcal{X} \times BG \to BG$ is the structural morphism $p : G \to \mathcal{X}$ by [LM 4.6.1]. By Lemma 2.4.4, taking an object of $\text{Sm}(BG)$ is equivalent to taking $E \in \text{Sm}(\mathcal{X})$ endowed with an isomorphism $\alpha : p^* E \to p^* F$ satisfying the cocycle condition. Let $\mathcal{K} := \text{Ker}(\alpha - \text{id} : p^* E \to p^* F)$. Since $p^* E \in \text{Hol}(G)$ is smooth, $\mathcal{K}$ is smooth as well. Let $e : \mathcal{K} \to G$ be the unit morphism. Since $p^* E$ is smooth, we get that the sequence $0 \to \mathcal{H}^{-d+e}(\mathcal{K}) \to \mathcal{H}^{-d+e+p^* E} \to \mathcal{H}^{-d+e+p^* F}$, where $d$ denotes the dimension of $G$, is exact. By the cocycle condition, $e^+ (\alpha - \text{id})$ is 0, and thus the rank of $\mathcal{K}$ is equal to that of $p^* E$ since $G$ is connected. Thus $\alpha$ is the identity, and we get the lemma.

Remark. — The assumption that $\mathcal{X}_{\text{red}}$ is smooth is made only for the simplicity. In fact, the lemma remains to be true even if we replace $\text{Sm}$ by $\text{Hol}$. For this, we need an analogue of [BBD 4.2.6.2] for the morphism $p$.

2.4.9 Lemma. — Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a diagonally connected gerb-like morphism (cf. [Beh 5.1.3]). Shrinking $\mathfrak{Y}$ by its open dense substack if necessary, the functor $f^* : \text{Sm}(\mathfrak{Y}) \to \text{Sm}(\mathfrak{X})$ induces an equivalence.

Proof. Take a presentation $P : Y \to \mathfrak{Y}$ from a scheme. There exists an open dense subscheme $U \subset Y$ such that $U_{\text{red}}$ is smooth. By replacing $\mathfrak{Y}$ by $P(U) \subset \mathfrak{Y}$, we may assume that $\mathfrak{Y}_{\text{red}}$ is smooth. Now, by using smooth descent, we may assume that $\mathfrak{Y} := Y$ is a scheme and $\mathfrak{X} = BG$ with a connected flat algebraic group space $G$ over $Y$. Since the category is stable under universal homomorphism, we may replace $Y$ by $Y_{\text{red}}$, and assume that $BG$ and $Y$ are smooth.

When $G$ is smooth, the lemma follows from Lemma 2.4.8. In the general case, we use the argument of [Beh 5.1.17]. Take the relative Frobenius $G \to G' \to G^{(p)}$. By Corollary 2.4.7, $\text{Sm}(BG') \cong \text{Sm}(BG)$, so we may replace $G$ by $G'$. Repeating this, we come down to the case where $G$ is smooth over a dense open subscheme of $Y$ by [SGA 3, VIIA, 8.3]. Thus, by shrinking $Y$, we are reduced to the case where $G$ is smooth.

2.4.10. — In the proof of the Langlands correspondence, we need smoothness of certain holonomic modules. For this, we need to use the functors of Beilinson. Let $k'$ be a finite extension of $k$, and put $A' := \text{Spec}(k'[2])$. Let $f : \mathfrak{X} \to A'_k$ be a morphism from a c-admissible stack. We put $f_* : \mathfrak{Y} := f^* \mathfrak{Y} := f^{-1}(0) \to \mathfrak{X}$, and $j_f : \mathfrak{Y}_f := \mathfrak{X} \setminus \mathfrak{Y} \hookrightarrow \mathfrak{X}$. Then for any $\mathcal{M} \in \text{Hol}(\mathfrak{Y}_f)$ and integers $a \leq b$, holonomic modules $\Pi^{a,b}_f(\mathcal{M}), \Phi^{a,b}_f(\mathcal{M}) \in \text{Hol}(\mathfrak{Y}_f)$ are defined in [AC2 §2] using a technique of Beilinson. To clarify $f$, we denote it by $\Pi^{a,b}_f(\mathcal{M})$. Put $\Pi^{b,0}_f := \Psi^{a,0}_f$, the unipotent nearby cycle functor. Explicitly, we can compute using the notation of [AC2 2.5] that

$$
\Psi^{a,0}_f(\mathcal{M}) \cong \lim_{s} \ker (j_{f+}(\mathcal{M}^{-s,0}) \to j_{f+}(\mathcal{M}^{-s,0})) \cong \lim_{s} \mathcal{H}^{-1} j_{f+}(\mathcal{M}^{-s,0}).
$$
2.4.11. For a Galois extension $L/k'(\langle x \rangle)$, we define $\Psi_{L,f}(\mathcal{M})$ as follows: Let us denote by $\mathcal{L}_L$ the differential module on the Robba ring of $k'(\langle x \rangle)$ defined by taking the push-forward along the extension $L/k'(\langle x \rangle)$ of the trivial differential module on the Robba ring of $L$. Let $\mathcal{L}_L$ be the canonical extension of $\mathcal{L}_L$ on $\mathbb{G}_m,k'$ (cf. [AM §2.1]). Then we put $\Psi_{L,f}(\mathcal{M}) := \Phi_{f,\mathcal{M} \circ f^+L_L}$. We remark that $\mathcal{M} \otimes f^+\mathcal{L}_L$ is in $\text{Hol}(\mathcal{X})$. Indeed, it suffices to check this when $\mathcal{X} := X$ is a realizable scheme. In this case, we may take a closed immersion $i: X \to P$ to a smooth scheme. By shrinking $X$, we may assume that there exists $g: P \to \mathbb{A}^1$ such that $g \circ i = f$. Now, by projection formula $i_+(\mathcal{M} \otimes f^+\mathcal{L}_L) \cong i_+(\mathcal{M}) \otimes g^+\mathcal{L}_L$, and the latter is in $\text{Hol}(P)$. Since we have the action of $\text{Gal}(L/k'(\langle x \rangle))$ on $\mathcal{L}_L$, it induces the Galois action on $\Psi_{L,f}$.

Lemma. — Let $f: C \to \mathbb{A}^1_k$ be an étale morphism from a curve such that $f^{-1}(0) = \{s\}$ and $k(s) = k'$. Let $\mathcal{M} \in \text{Hol}(C)$. Assume that $\mathcal{M}$ is smooth outside of $f^{-1}(0)$. If $\Phi_{f,\mathcal{M}} = 0$ and the action of $\text{Gal}(L/k'(\langle x \rangle))$ and monodromy operator on $\Psi_{L,f}(\mathcal{M})$ is trivial for any $L$, then $\mathcal{M}$ is smooth.

Proof. Since $\Phi_{f,\mathcal{M}} = 0$, we get that $i_+(\mathcal{M})[-1] \cong \Phi_{f,\mathcal{M}}$. It suffices to show that the rank of $\Phi_{f,\mathcal{M}}$ is equal to that of $\mathcal{M}$. By [2.4.10.1] and [AC1 1.5.9 (iii)], $\Psi_{L,f}$ depends only on the differential module on the Robba ring around $s$ defined by restricting $\mathcal{M}$, and we can compute $\Psi_{L,f}(\mathcal{M})$ by using the local monodromy theorem. Since the argument is standard, we leave the details to the reader. ■

2.4.12 Lemma. — Let $f: \mathcal{X} \to \mathcal{Y} \to \mathbb{A}^1$ be morphisms between c-admissible stacks. Assume that $h$ is proper. Then we have

$$\prod^a_h(\mathcal{H}^1h_{+\mathcal{M}}) \cong \mathcal{H}^1h_{+\prod^a_h(\mathcal{M})}.$$  

The same isomorphism holds if we replace $\prod^a_h$ by $\Psi_{L,*}$ or $\Phi_{*,*}^a$.

Proof. Since $h$ is proper, we have $h \circ j_+ \cong j_+h_+ \circ f^+f_+$ for $f^+f_+ \in \{!,+\}$. Thus, by projection formula, we have $\prod^a_h(\mathcal{H}^1h_{+\mathcal{M}})^{**} \cong \lim \mathcal{H}^1h_{+\prod^a_h(\mathcal{M})}^{**}$ where $f^+f_+ \in \{!,+\}$. Thus, by construction, we get the commutativity for $\prod^a_h$. Now, let us define $K_f(\mathcal{M}) := \ker(\mathcal{E}_f(\mathcal{M}) \otimes \mathcal{M} \to j_f(\mathcal{M}))$, where $\mathcal{E}_f := \prod^a_f$. Since $\mathcal{E}_f$ and $j_f$ commute with $\mathcal{H}^1h_{+\mathcal{M}}$, we get that $K_f(\mathcal{H}^1h_{+\mathcal{M}}) \cong \mathcal{H}^1h_{+\mathcal{M}}$. Similarly, $\text{Im}(j_f(\mathcal{M}) \to \mathcal{E}_f(\mathcal{M}) \otimes \mathcal{M}) \cong j_f(\mathcal{M})$ commutes with $\mathcal{H}^1h_{+\mathcal{M}}$ as well, and the lemma for $\Phi_{*,*}^a$ follows by definition. ■

2.4.13 Lemma ([La2 A.9 (i)]). — Let $p_X: \mathcal{X} \to S$ be a proper morphism from a c-admissible stack to a smooth scheme, and $\text{Res}: \mathcal{X} \to \mathcal{E}$ be a morphism to an algebraic stack. Assume that $(p_X, \text{Res}): \mathcal{X} \to S \times \mathcal{E}$ is smooth. Then for any $\mathcal{M} \in \text{Hol}(\mathcal{E})$, the complex $p_X^*\text{Res}^*(\mathcal{M})$ is smooth.

Proof. We put $\mathcal{H}^*_\mathcal{X} := \mathcal{H}^0p_X^*\text{Res}^*(\mathcal{M})$. Assume given a smooth morphism $f: S \to \mathbb{A}^1_k$. Put $g := f \circ p_X$. Since $f$ and $S$ are smooth, we get that $\Phi_{f,\text{Res}^*(\mathcal{M})} = 0$ and the Galois and monodromy action on $\Psi_{L,g}(\text{Res}^*(\mathcal{M}))$ is trivial. Now, since $p_X$ is assumed to be proper, $\Phi_{f,\text{Res}^*(\mathcal{M})} = 0$ and the Galois and monodromy action on $\Psi_{L,f}(\mathcal{H}^*_\mathcal{X})$ is trivial by Lemma 2.4.12. This, in particular, implies that $\mathcal{H}^1f_{+\mathcal{H}^*_\mathcal{X}}(\mathcal{M}) \cong \mathcal{H}^1f_{+\mathcal{M}}$. Moreover, when $S$ is a curve, the lemma holds. Indeed, take an open subscheme $U \subset S$ such that $\mathcal{H}^1\mathcal{X}_U$ is smooth on $U$. We may replace $S$ by $S \otimes_k k'$, and may assume that $S \setminus U$ is $k'$-rational. Since the verification is local, we may assume that we are given an étale morphism $f: S \to \mathbb{A}^1_{k'}$ such that $f^{-1} = S \setminus U$ consists of one point, and then apply Lemma 2.4.11.

Let us treat the general case. Let $c: C \to S$ be an immersion from a smooth curve $C$. Let $U \subset S$ be an open dense subscheme on which $p_X^*\text{Res}^*(\mathcal{M})$ is smooth. Then by base change and purity, we have $c^*(\mathcal{H}^1\mathcal{X}_U) \cong \mathcal{H}^1c^*\mathcal{X}_U \otimes C^r$ where $r$ is the codimension of $C$ is $S$. By the curve case we have already treated, this is smooth. By Shiho’s cut by curve theorem [S Thm 0.1] and [Kc3 5.2.1], the smooth module $\mathcal{H}^1\mathcal{X}_U$ on $U$ can be extended to a smooth module on $S$. Since $j_{f_{+\mathcal{H}^1\mathcal{X}}}(\mathcal{M}) \cong \mathcal{H}^1\mathcal{X}$ for any $f$, we get the lemma.

(4) In [ibid.], there is an assumption that $k$ is uncountable. However, Shiho pointed out to the author that this assumption is not needed if $\mathcal{E}$ in [ibid., Thm 0.1] is endowed with Frobenius structure. Indeed, let us use the notation of the proof of the theorem in [ibid., §2.3]. By using [ibid., Thm 2.5] instead of Thm 2.10, $E_{\mathcal{X},L}$ is slope 0. Since we have a Frobenius structure, the exponents are in $\mathbb{Q}$, and we can use [ibid., Prop 1.20] to conclude.
2.4.14. Finally, we make a few comments on the $\mathcal{O}_p$-coefficient cohomology theory. Consider an algebraic extension $L$ of $K$, which may not be finite. In the case $\Delta = 0$:

(*) take an isomorphism $\sigma_L: L \to L$ which extends $\sigma$, and there exists sequence of finite extensions $M_n$ of $K$ in $L$ such that $\sigma_L(M_n) \subset M_n$ and $\bigcup_n M_n = L$.

We use the 2-inductive limit method of Deligne [De1 1.1.3] to construct the $L$-theory. For an algebraic stack of finite type over $k$, we define

$$D^b(\mathcal{X}/L) := \lim_{M \supset K} D^b(\mathcal{X}/M), \quad \text{Sm}(\mathcal{X}/L) := \lim_{M \supset K} \text{Sm}(\mathcal{X}/M)$$

where $M = M_n$ in the case of $\Delta = 0$. By taking limits, the results we get in this paper can be generalized automatically to this category since cohomological operators we have defined so far commute with $\iota_L$ by 1.4.9. Let $f$ be the structural morphism of $\mathcal{X}$. We denote by $L_\mathcal{X} := f^+(L)$ as usual. Further detail is left to the reader.

3. Cycle classes, correspondences, and $\ell$-independence

The aim of this section is to prove an $\ell$-independence result. This is a key tool to compute the trace of the action of Hecke algebra on the cohomology of certain moduli spaces. In this section, we fix $\Delta \in \{0, \emptyset\}$ and a base tuple as usual. We take an algebraic extension $L/K$, which can be infinite. When $\Delta = 0$, we moreover assume the condition (*) in 2.4.14. For simplicity, smooth admissible stacks over $k$ are assumed to be equidimensional.

3.1. Generalized cycles and correspondences

3.1.1. Let $p: \mathcal{X} \to \text{Spec}(k)$ be the structural morphism of a c-admissible stack $\mathcal{X}$. If no confusion may arise, we denote $L_\mathcal{X}$ by $L$. For $\mathcal{M} \in D^b_{\text{hol}}(\mathcal{X})$, we put

$$H^i_+(\mathcal{X}, \mathcal{M}) := \text{Hom}_{D^b(\text{Spec}(k))}(L, p_+(\mathcal{M})[i]), \quad H^i_+(\mathcal{X}, \mathcal{M}) := \text{Hom}_{D^b(\text{Spec}(k))}(L, p_+(\mathcal{M})[i]).$$

For a morphism $i: \mathcal{Z} \to \mathcal{X}$, we define the local cohomology to be

$$H^i_3(\mathcal{X}, \mathcal{M}) := \text{Hom}_{D^b(\text{Spec}(k))}(L, p_{i+i_1^*}(\mathcal{M})[i]).$$

Furthermore, we put $H^c_3(\mathcal{X}) := H^c_3(\mathcal{X}, L_\mathcal{X})$ where $\emptyset \in \{0, c, 3\}$.

Now, given a proper morphism of c-admissible stacks $f: \mathcal{X} \to \mathcal{Y}$, we have a homomorphism $H^c_+(\mathcal{Y}) \to H^c_+(\mathcal{X})$ induced by the adjunction homomorphism. Consider the following commutative diagram of c-admissible stacks:

$$\begin{array}{ccc}
\mathcal{Z} & \xrightarrow{f'} & \mathcal{Y} \\
\mathcal{X} & \xrightarrow{f} & \mathcal{Y} \\
\end{array}$$

If the diagram is cartesian, then we have the base change isomorphism $i^* f_+ \cong f'_+ i'^*$ by (the dual of) Lemma 2.3.25, which induces the homomorphism $f^* : H_{3\mathcal{Y}}^c(\mathcal{Y}) \to H_{3\mathcal{X}}^c(\mathcal{X})$.

The most important case to consider the local cohomology is when $i$ is proper. If $f$ is the identity and $f'$ is proper, the adjunction $f'_* f'^! \to \text{id}$ induces the push-forward homomorphism $H^c_3(\mathcal{X}) \to H^c_{\mathcal{Y}}(\mathcal{X})$.

3.1.2. Let $f: \mathcal{X} \to \mathcal{S}$ be a morphism of c-admissible stacks, and $p$ be the structural morphism of $\mathcal{X}$. Let $\mathcal{M}$ and $\mathcal{N}$ be objects in $D^b_{\text{hol}}(\mathcal{X}/L)$. We have a canonical homomorphism

$$\cup: p_!(\mathcal{M}) \otimes f_! \mathcal{N} \cong (p \times f)_! (\mathcal{M} \boxtimes \mathcal{N}) \xrightarrow{\Delta^*} f_! (\mathcal{M} \boxtimes \mathcal{N})$$

where the first isomorphism is induced by Proposition 2.3.32 and $\Delta: \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is the diagonal morphism. This is called the cup product. Note that we have a homomorphism $H^c_+(\mathcal{X}, \mathcal{M}) \to \mathcal{M} \boxtimes p_!(\mathcal{M})$ as vector spaces over $L$. 61
Now, let $j: \mathfrak{X} \rightarrow \mathfrak{X}'$ be an open immersion. Assume given a homomorphism $\mathcal{M} \otimes \mathcal{N} \rightarrow \mathcal{L}$ in $D^b_{\text{hol}}(\mathfrak{X}/L)$. This induces a homomorphism

$$j_+ (\mathcal{M}) \otimes j_! (\mathcal{N}) \cong j_! (\mathcal{M} \otimes \mathcal{N}) \rightarrow j_! (\mathcal{L}).$$

On the other hand, let $i: \mathfrak{Z} \rightarrow \mathfrak{X}$ be a morphism. Then we have the homomorphism $i_!(i^! \mathcal{M} \otimes i^+ \mathcal{N}) \cong i^! \mathcal{M} \otimes i^+ \mathcal{N} \rightarrow i^! \mathcal{L}$. Taking the adjoint, we get a homomorphism

$$i^! \mathcal{M} \otimes i^+ \mathcal{N} \rightarrow i^! \mathcal{L}.$$

3.1.3 Definition. — (i) Let $\mathfrak{X}$ be a $c$-admissible stack of dimension $d$. A generalized cycle of codimension $c$ is a proper morphism $g: \Gamma \rightarrow \mathfrak{X}$ between $c$-admissible stacks such that $\dim(\Gamma) = \dim(\mathfrak{X}) = c$.

(ii) Let $S$ be a scheme of finite type over $k$, $\varphi$ be a proper endomorphism of $S$, and $f^{(i)}: \mathfrak{X}^{(i)} \rightarrow S$ be $c$-admissible $S$-stacks. Let $c_{\Gamma}: \Gamma \rightarrow \mathfrak{X} \times_{\varphi,S} \mathfrak{X}'$ be a morphism between $c$-admissible stacks. For $i = 1, 2$, put $p_i := \pi_i \circ c_{\Gamma}$ where $\pi_i$ denotes the $i$-th projection. The morphism $c_{\Gamma}$ is called a correspondence over $\varphi$ if $\Gamma$ is of dimension $\dim(\mathfrak{X})$, and $p_2$ is proper. We sometimes denote the correspondence by $\Gamma: \mathfrak{X} \leadsto \mathfrak{X}'$. Note that $c_{\Gamma}$ is a generalized cycle of codimension $\dim(\mathfrak{X}')$ of $\mathfrak{X} \times_S \mathfrak{X}'$.

3.1.4. In this section, we denote $\text{Tr}^{\alpha\alpha}$ in Theorem 2.3.37 simply by $\text{Tr}_f$. Let $\alpha: \mathfrak{X} \rightarrow \text{Spec}(k)^\wedge$ be the structural morphism of a $c$-admissible stack $\mathfrak{X}$. We often denote $\text{Tr}_\alpha$ by $\text{Tr}_\mathfrak{X}$. Now, let $f: \mathfrak{Z} \rightarrow \mathfrak{X}$ be a morphism of $c$-admissible stacks, and assume that $\mathfrak{X}$ is smooth. Then $\text{Tr}_f$ defines a homomorphism $f^\# : \text{Hom}(\mathfrak{Z}, \mathfrak{X}) \rightarrow \text{Hom}(\mathfrak{Z}, \mathfrak{X})$ where $d := \dim(\mathfrak{Z}) - \dim(\mathfrak{X})$, called the fake trace map of $f$, as follows. Let $p: \mathfrak{Y} \rightarrow \text{Spec}(k)$ be the structural morphism. We have the isomorphism

$$\text{Hom}(f^* f^+ L_X(d)[2d], L_X) \cong \text{Hom}(f^+ L_X(d)[2d], f^! L_X) \cong \text{Hom}(p^+ L(d_Y)[2d_Y], p^! L)$$

where $d_Y := \dim(\mathfrak{Y})$, and we used the Poincaré duality [2.3.37] for the second isomorphism. The trace map $\text{Tr}_f$ is an element on the right hand side of the equality. Sending this to the left side, we get the desired homomorphism $f^\#$. This homomorphism induces a homomorphism

$$(3.1.4.1) \quad f^* : H^{c+2d}(\mathfrak{Z})(d) \rightarrow H^c_{\mathfrak{X}}(\mathfrak{X}).$$

(i) Let $i: \mathfrak{Z} \rightarrow \mathfrak{X}$ be a generalized cycle of codimension $c$ on a smooth $c$-admissible stack $\mathfrak{X}$. Then by taking the adjoint to $f^\#$, we have

$$c_{\mathfrak{Z}}: i^+ L_X \rightarrow i^! L_X(c)[2c].$$

(ii) Let us construct a similar homomorphism when we are given a correspondence. We use the notation of Definition 3.1.3(ii). We assume further that $\mathfrak{X}$ is smooth. Then we have $f^\# p_{i_1} : p_{i_1} i^+ L_X \rightarrow L_X$, where we used the assumption that $\dim(\Gamma) = \dim(\mathfrak{X})$. Thus, we have

$$i_{\Gamma} : p_{i_2} L_X^* \cong p_{i_1} L_X \rightarrow p_{i_1} L_X$$

where we used the adjoint of $f^\# p_{i_1}$ for the second homomorphism.

3.1.5. Let us characterize the fake trace map in the style of [SGA 4_1/2, Cycle]. Let $\mathfrak{S} := \text{Spec}(k)^\wedge$. Let us consider the following diagram of $c$-admissible stacks

$$\begin{array}{ccc}
\mathfrak{Z} & \xrightarrow{i} & \mathfrak{X} \\
\downarrow f & & \downarrow g \\
\mathfrak{S} & \xrightarrow{\varphi} & \mathfrak{X}
\end{array}$$

where $\mathfrak{X}$ is smooth, and $\mathfrak{X}$ and $\mathfrak{Z}$ are of dimension $N$ and $d$ respectively. Put $c := N - d$. The cup product (cf. 3.1.2) defines a coupling

$$\cup: H^d_{\mathfrak{Z}}(\mathfrak{X})(c) \otimes f_! L_{\mathfrak{Z}}(d)[2d] \rightarrow g_! L_X(N)[2N]$$

Now, by taking the adjoint, we may regard $f^\# i_{\mathfrak{Z}}$ as an element in $H^d_{\mathfrak{Z}}(\mathfrak{X})(c)$. We have the following characterization of the fake trace map.
**Lemma.** — The class \( f_{\mathrm{Tr}} \in H^{2c}_{\mathcal{X}}(X)(c) \) is the unique element such that

\[
\mathrm{Tr}_f(u) = \mathrm{Tr}_y(f_{\mathrm{Tr}} \cup u).
\]

**Proof.** As in [SGA 4\(\frac{1}{2}\), Cycle, 2.3], we have the following commutative diagram:

\[
\begin{array}{ccc}
H^{2c}_{\mathcal{X}}(X) & \xrightarrow{\text{Hom}(f_{L\mathcal{Y}}(d)[2d], f_{iL\mathcal{Y}}(N)[2N])} & \text{Hom}(f_{L\mathcal{Y}}(d)[2d], L_{\mathcal{X}}) \\
\downarrow & & \downarrow_{\text{adj}} \\
\text{Hom}(g_{iL\mathcal{Y}}(d)[2d], g_{iL\mathcal{Y}}(N)[2N]) & \xrightarrow{\text{adj}_i} & \text{Hom}(g_{iL\mathcal{Y}}(d)[2d], L_{\mathcal{X}}).
\end{array}
\]

Here, \( \text{adj}_i \) is induced by the adjunction using the assumption of \( X \) being smooth. The homomorphism \( \text{adj}_i : i^! \rightarrow \text{id} \) is the adjunction. The homomorphism \( \text{adj} \) is induced by \( \cup \) defined above. By the definition of fake trace map, the upper horizontal homomorphism maps \( f_{\mathrm{Tr}} \) to \( \mathrm{Tr}_f \). Moreover, the composition of the upper horizontal maps is an isomorphism. Thus we can conclude the proof. \( \square \)

**Remark.** — The assumption that \( \mathcal{S} = \text{Spec}(k) \) is used only for the existence of the fake trace over \( \mathcal{S} \). It is not hard to generalize the definition of the fake trace to relative situations, and we are able to prove the lemma in this generality, although we are not sure if it is meaningful in our situation.

**Corollary.** — Assume that \( f : \mathcal{X} \rightarrow \mathcal{Y} \) be a flat morphism of \( c \)-admissible stacks such that \( \mathcal{Y} \) is smooth. Then \( \mathrm{Tr}_f = f_{\mathrm{Tr}} \).

**Proof.** This follows readily from the characterization lemma of \( f_{\mathrm{Tr}} \) above. \( \square \)

**3.1.6.** (i) Consider the situation as in [3.1.4 (i)]. We have an isomorphism

\[
\text{Hom}(i^+ L_X, i^! L_X(c)[2c]) \cong \text{Hom}(L_X, i_* i^! L_X(c)[2c]) =: H^{2c}_{\mathcal{X}}(X)(c).
\]

The image of \( c \) is denoted by \( c_{\mathcal{X}}(3) \), and called the cycle class of \( 3 \). Since the homomorphism \( 3 \rightarrow \mathcal{X} \) is proper, we have the homomorphism \( H^{2c}_{\mathcal{X}}(X) \rightarrow H^*(X) \). The image of \( c_{\mathcal{X}}(3) \) in \( H^{2c}(X)(d) \) is also called the cycle class. Note that if the morphism \( 3 \rightarrow g(3) \) is not generically finite, then \( H^{2c}_{g(3)}(X) = 0 \), and in particular the cycle class in \( H^{2c}(X)(c) \) is 0.

(ii) Consider the situation as in [3.1.4 (ii)]. Recall that \( \varphi \) is assumed to be proper. We have the action of correspondence on the cohomology, which is the composition of the homomorphisms

\[
\Gamma^* : f^! L_{\mathcal{Y}} \rightarrow f^! p_{2+}^! L_{\mathcal{Y}} \xrightarrow{\text{cl}} (\varphi \circ f \circ p_1)_! p_{2+}^! L_{\mathcal{X}} \xrightarrow{\text{adj}} \varphi + f_! p_{1+}^! L_{\mathcal{X}} \rightarrow \varphi + f_! L_{\mathcal{X}}.
\]

When \( S \) is a point, this is nothing but the composition using [3.1.4 (i)]:

\[
H^*(\mathcal{X}') \xrightarrow{p_2^*} H^*(\Gamma) \xrightarrow{p_1^*} H^*(X).
\]

**3.1.7 Lemma.** — Consider the following diagram on the left:

\[
\begin{array}{ccc}
\mathcal{Y}' & \xrightarrow{g'} & \mathcal{Y} \\
\downarrow f' & & \downarrow f \\
\mathcal{X}' & \xrightarrow{g} & \mathcal{X},
\end{array}
\]

where \( \mathcal{X} \) and \( \mathcal{X}' \) are smooth. Assume moreover that there exists an open dense substack \( \mathcal{U} \subset \mathcal{Y} \) such that the morphism \( \mathcal{U} \rightarrow \mathcal{X} \) is flat of relative dimension \( d \), and \( g^{-1}(\mathcal{U}) \subset \mathcal{Y}' \) is dense. Then the above diagram on the right is commutative. In particular, if \( g \) is proper, we have an equality

\[
g^* f_* = f'_* g^* : H^*_c(\mathcal{Y}) \rightarrow H^*_{c-2d}(\mathcal{X})(-d).
\]
Proof. Since the commutativity of the diagram on the right can be interpreted as coincidence of two elements in $H^{2d}(\mathcal{Y})(a)^\vee$ where $a$ denotes the dimension of $\mathcal{Y}$, we may shrink $\mathcal{Y}$ by its open dense substack by Lemma 2.3.37. Thus we may replace $\mathcal{Y}$ by $\mathfrak{Y}$, and may assume that $f$ is flat of relative dimension $d$. Now the lemma follows by Corollary 3.1.5 and the base change property of the trace map (cf. Theorem 2.3.37 (II)).

Corollary (Projection formula). — Let $f : \mathcal{X} \to \mathcal{Y}$ be a proper morphism of c-admissible stacks and $\mathcal{Y}$ is smooth. Assume that there exists an open dense substack $\mathbb{U} \subset \mathcal{X}$ such that the morphism $\mathbb{U} \to \mathcal{Y}$ is flat of relative dimension $d$. Then for $\alpha \in H^2_c(\mathcal{X})$ and $\beta \in H^2_c(\mathcal{Y})$, we have the following equality in $H^{c+j-2d}(\mathcal{Y})(-d)$.

$$f_*(\alpha \cup f^* \beta) = f_* \alpha \cup \beta.$$  

Proof. Consider the following commutative diagram of proper morphisms:

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{id} & \mathcal{X} \\
\downarrow & & \downarrow \\
\mathcal{X} \times \mathcal{Y} & \xrightarrow{f} & \mathcal{Y}
\end{array}
\]

By the hypothesis on $f$, we can apply the lemma above for the right cartesian diagram (take $f$, $g$ in the lemma to be $id \times f$ and $\Delta$ respectively). We have

$$f_*(\alpha \cup \beta) = \Delta^*((f \times id)_*(\alpha \boxtimes \beta)) = f_* \Delta^* (\alpha \boxtimes \beta)$$

where we used the lemma at the second equality. The diagram above and the transitivity of the pull-back show that $\Delta^* (\alpha \boxtimes \beta) = \alpha \cup \star^f \beta$, and the corollary follows.

3.1.8 Lemma. — Let $\rho : \mathcal{X} \to \mathcal{X}'$ be a proper morphism over $\varphi$ such that $\mathcal{X}$ is smooth. Let $\Gamma_{\rho}$ denote the graph of $\rho$ and regard it as a correspondence $\mathcal{X} \rightsquigarrow \mathcal{X}'$. Then $\rho_* = \Gamma_{\rho}^*.$

Proof. We have $\mathcal{X} \xrightarrow{\rho^!!} \Gamma_{\rho} \xrightarrow{\rho^*} \mathcal{X}'$. The via the identification $p_{1!}K_{\Gamma^*} \sim \rightarrow K_X$, the fake trace $f_{Tr_{p_1}} : p_{1!}p_{1}^* K_X \to K_X$ is the identity. Thus the lemma follows.

3.1.9. Let $\mathcal{X}$, $\mathcal{Y}$ are c-admissible stacks of dimension $d$. Let $g : \mathcal{Y} \to \text{Spec}(k)$, $h : \mathcal{X} \to \mathcal{Y}$ and $f := g \circ h$. Assume $h$ to be proper. Then we have the canonical homomorphism $(h^*)^Y : (H^{2d}_c(\mathcal{X})(d))^Y \to (H_c^{2d}(\mathcal{Y})(d))^Y$ where $(\cdot)^Y$ denotes $\text{Hom}(\cdot, L)$. Note that $\text{Tr}_X \in H^{2d}_c(\mathcal{X})(d)^Y$.

Lemma. — Assume that $h$ is generically locally free morphism of constant degree. Then $(h^*)^Y$ sends the trace map $\text{Tr}_f$ to $\deg(h) \cdot \text{Tr}_g$.

Proof. Since the trace maps are determined generically, we may shrink $\mathcal{Y}$ by its open dense subscheme. Shrinking $\mathcal{Y}$, we may assume that $h$ is finite flat. We have the following commutative diagram:

$$\begin{array}{ccc}
g(h)h^+g^+(L)(d)[2d] & \xrightarrow{\text{Tr}_h} & g(g^+(L)(d)[2d] & \xrightarrow{\text{Tr}_g} & L \\
\downarrow_{\text{adj}_h} & & \downarrow_{\deg(h)} & & \\
gg^+L(d)[2d]. & & & & \\
\end{array}$$

The composition of the first row is $\text{Tr}_f$. By definition, $(h^*)^Y(\text{Tr}_f) = \text{adj}_h \circ \text{Tr}_f$. Thus the corollary follows.

Corollary. — (i) Let $\mathcal{X}$ be a c-admissible stack, and $\mathfrak{Z}$, $\mathfrak{Z}'$ be generalized cycles of codimension $c$. Assume given a generically locally free morphism of constant degree $\rho : \mathfrak{Z}' \to \mathfrak{Z}$ over $\mathcal{X}$. Then the push-forward homomorphism $\rho_* : H^{2c}_c(\mathcal{X})(c) \to H^{2c}_c(\mathcal{X})(c)$, $c_1(\mathfrak{Z})$ is sent to $\deg(\rho) \cdot c_1(\mathfrak{Z})$.

(ii) Let $\Gamma, \Gamma' : \mathcal{X} \rightsquigarrow \mathcal{X}'$ be correspondences. Assume that there exists a morphism $\rho : \Gamma \to \Gamma'$ such that $c_\Gamma \circ \rho = c_{\Gamma'}$. Assume that $\rho$ is generically locally free of constant degree. Then $\Gamma'^* \cong \deg(\rho)^{-1} \cdot \Gamma^*$.  

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Proof. They are straightforward from the lemma.  

3.1.10 Definition. — We denote by Corr_φ(X, X') the Q-vector space generated by the set \{Γ: X ∼→ X' | correspondence over φ\}. We have a homomorphism

\[\text{Corr}_φ(X, X') \rightarrow \text{Hom}_S(φ^+ f^*_1 L_{X'}, f^*_1 L_X)\]

by sending Γ to Γ*. Let I be the Q-vector space generated by \((Γ' − \text{deg}(ρ))^{-1} Γ\) where we have a generically locally free morphism of constant degree ρ: Γ → Γ'. By Corollary 3.1.9 (ii), the homomorphism above factors through Corr_φ(X, X')/I.

Let Corr_φ(X, X') for * = et (resp. fin) be the Q-vector subspace of Corr_φ(X, X') generated by integral correspondences Γ (i.e. Γ is integral) such that the first projection Γ → X is étale (resp. finite). There exists a map

\[o: \text{Corr}^{et}_φ(X, X') \times \text{Corr}^{et}_ψ(X', X'') \rightarrow \text{Corr}^{et}_ψ(\phi, \phi')\]

defined by sending \((Γ, Γ')\) to \(Γ \times \phi'\).

Lemma. — Let Γ: X ∼→ X', Γ': X' ∼→ X'' be correspondences over id. Then we have \((Γ' ∘ Γ)^* \cong Γ^* ∘ Γ''\).

Proof. The verification is standard. See, for example, [La2 A.7].

3.1.11 Lemma. — Let S be a scheme of finite type over k, X be a smooth c-admissible stack, and f: X → S be a morphism. Let Γ be a correspondence on X over φ. Assume there is a dense open substack Γ' ⊂ Γ such that the first projection Γ' → S is flat. For a closed point \(i_s: s \mapsto S\), we denote by \(X_s\), \(Γ_s\), \(Γ'_s\), \(φ_s\) the fibers over s. If \(Γ'_s ⊂ Γ_s\) is dense and non-empty, \(Γ_s\) is a correspondence on \(X_s\) over \(φ_s\), and the the following diagram is commutative:

\[
\begin{array}{ccc}
i^+_s f^*_1 L_X & \xrightarrow{Γ'} & i^+_s f^*_1 L_X \\
\downarrow & & \downarrow \\
f^*_s L_{X_s} & \xrightarrow{Γ'_s} & f^*_s L_{X_s}.
\end{array}
\]

Here the vertical homomorphisms are the base change maps.

Proof. Since Γ' is flat over S, \(Γ_s\) is a correspondence for any s ∈ S if it is non-empty. Now, to show the commutativity, it suffices to show the commutativity of the following diagrams:

\[
\begin{array}{ccc}
i^+_s p^+_1 L_X & \xrightarrow{i^+_s (g^+_s)} & p^+_1 L_{X_s} \\
\downarrow & & \downarrow \\
i^+_s p^+_1 L_X & \xrightarrow{Γ'_s} & p^+_1 L_{X_s},
\end{array} \quad \begin{array}{ccc}
i^+_s L_X & \xrightarrow{~} & L_{X_s} \\
\downarrow & & \downarrow \\
i^+_s p^+_2 L_X & \xrightarrow{~} & p^+_2 L_{X_s}.
\end{array}
\]

The commutativity of the left diagram follows by Lemma 3.1.7. The commutativity of the right one follows by the fact that \((g^+, g^+)_s\) is an adjoint pair when g is a morphism of admissible stacks.

Remark. — In general, if the fiber Γ over the generic point of S is non-empty, there exists an open dense subscheme \(U ⊂ S\) such that the condition of the lemma holds for \(Γ_s\) (s ∈ S).

3.1.12 Given a correspondence, the results in this subsection hold in exactly the same manner for ℓ-adic étale cohomology. However, in [La2 §A], he uses slightly different definition of the actions of correspondences on the cohomology, and we need to compare these.

For an admissible stack X, we denote by \(X^{\text{gr}}\) the associated coarse moduli algebraic space of Keel and Mori (cf. [La2 A.2]). By [Behi 5.1.12, 13, 14], by shrinking \(X^{\text{gr}}\), the morphism X → \(X^{\text{gr}}\) is a gerb. Now, let Γ: X ∼→ X' be an integral correspondences such that the first projection Γ → X is generically finite. Then the morphism Γ → \(Γ^{\text{gr}} := \underbrace{X^{\text{gr}} \times_{X^{\text{gr}}} X^{\text{gr}}}\) is generically finite locally free. The degree of this morphism is denoted by d_Γ. We can check easily that for composable correspondences Γ, Γ', we have d_Γ ∘ Γ' = d_Γ + d_Γ'. We define a ring homomorphism

\[\text{norm}: \text{Corr}^{\text{fin,et}}_S(X) \rightarrow \text{Corr}^{\text{fin,et}}_S(X); \ Γ \mapsto (d_Γ)^{-1} \cdot Γ.\]
Now, let \( \mathcal{Y} \) be a c-admissible stack, and \( q_{\mathcal{Y}} : \mathcal{Y} \to \mathcal{Y}'^e \) be the canonical morphism, which is known to be proper (cf. [Co2, 1.1]). By [La2, A.3], the adjunction homomorphism \( \mathcal{Q}_\ell \to \mathbb{R}q_{\mathcal{Y}}^*(\mathcal{Q}_\ell) \) is an isomorphism. For \( \Gamma \in \text{Corr}_S(\mathcal{X}, \mathcal{X}') \), we define

\[
\Gamma^* \text{Laf} : \mathcal{H}_c^*(\mathcal{X}') \xrightarrow{\sim} \mathcal{H}_c^*(\mathcal{X}'^e) \xrightarrow{\Gamma^*} \mathcal{H}_c^*(\mathcal{X})
\]

where, for a c-admissible stack \( \mathcal{Y} \) denotes \( \mathcal{H}^* f_! q_{\mathcal{Y}}^* \). This is nothing but the action of correspondence defined in [La2, A.6].

**Lemma.** — Let \( \Gamma \in \text{Corr}_S(\mathcal{X}, \mathcal{X}') \) such that the first projection \( \Gamma \to \mathcal{X} \) is generically finite. Then we have \( \Gamma^* \text{Laf} = (\text{norm} (\Gamma))^* \).

**Proof.** Consider the following commutative diagram on the left:

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\Gamma} & \mathcal{X}' \\
\downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quad
3.2.2. We do not assume \( k \) to be finite here, and we fix \( \star \in \{0, \varnothing\} \).

Lemma ([SGA 4\_1, Cycle, 2.3.8 (ii)]). — Consider the following cartesian diagram of \( c \)-admissible stacks over \( k_\star \)

\[
\begin{array}{ccc}
\Gamma' & \xrightarrow{g'} & X' & \xrightarrow{h'} & S' \\
\downarrow f' & & \downarrow f & & \downarrow g & \partial \\
\Gamma & \xrightarrow{g} & X & \xrightarrow{h} & S
\end{array}
\]

where \( X' \) is smooth (over \( k_\star \)), \( S' \) is a scheme, \( g \) is a generalized cycle of codimension \( c \), and \( h \) and \( h \circ g \) are flat equidimensional. Then \( g' \) is a generalized cycle of codimension \( c \) as well, and we have \( f^* \cl(\Gamma') = \cl(\Gamma) \in H^2_{\cl}(X')(c) \).

Proof. We have the following commutative diagram:

\[
\begin{array}{ccc}
\Hom(g^+L_X, g'L_X) & \xrightarrow{\sim} & H^2_{\cl}(X)(c) \\
\downarrow f^* & & \downarrow f^* \\
\Hom(g'^+L_{X'}, g'^{!}L_{X'}) & \xrightarrow{\sim} & H^2_{\cl}(X')(c)
\end{array}
\]

where the dotted arrow can be described as follows: Let \( \phi \in \Hom(g^+L_X, g'L_X) \). Then the image of \( \phi \) is the unique dotted homomorphism, in the following diagram on the left, which makes the diagram commutative:

\[
\begin{array}{ccc}
g'^+L_{X'} & \sim & g'^{!}L_{X}(c)[2c] \\
\downarrow f'^+g'^+L_{X'} & \sim & f'^+g'^{!}L_{X}(c)[2c],
\end{array}
\]

Here the right vertical homomorphism is the base change homomorphism. Thus, by taking \( \phi = \Tr_\Gamma \), the problem is reduced to showing the commutativity of the right diagram above. Now, \( \Tr_\Gamma \) and \( \Tr_\Gamma' \) can be regarded as \( \Tr_{\log} \) and \( \Tr_{h' \circ g'} \) by the transitivity of trace. By the base change property of trace, we get the lemma.

3.2.3 Lemma. — Let \( X \) be a smooth scheme of dimension \( d \) over \( k_\star \), and let \( Z \to X \) be a generalized cycle. Then the cycle class map induces a homomorphism \( \cl_X : \CH_i(Z) \to H^2_{\cl}(Z)(d-i) \).

Proof. Let \( W \) be a closed integral subscheme of dimension \( i+1 \) of \( Z \) and \( W \to \mathbb{P}^1 \) be a dominant morphism (hence flat by [EGA, IV.2.8.2]). Let \( W_i \) be the fiber of \( i \in \mathbb{P}^1 \). By [Fu Ch.I, Prop 1.6], it suffices to show that \( \cl_X(W_0) = \cl_X(W_1) \). Let \( W^0 \) be the pull-back of \( \mathbb{A}^1 \subset \mathbb{P}^1 \) by the morphism \( W \to \mathbb{P}^1 \). Note that \( W^0 \to Z \times \mathbb{A}^1 \) is a closed immersion. Consider the following commutative diagram on the left:

\[
\begin{array}{ccc}
W_0 & \sim & W^0 & \sim & W_1 \\
\downarrow & & \downarrow & & \downarrow \\
Z & \sim & Z \times \mathbb{A}^1 & \sim & Z \\
\downarrow & & \downarrow & & \downarrow \\
\{0\} & \sim & \mathbb{A}^1 & \sim & \{1\},
\end{array}
\]

This induces the commutative diagram on the right where \( j := d - i \) and we omit Tate twists. The bottom arrows are isomorphisms since \( H^i(\mathbb{A}^1) = 0 \) for \( i \neq 0 \) and \( H^0(\mathbb{A}^1) \cong L \). By Lemma 3.2.2 and the flatness of \( X^\circ \to \mathbb{A}^1 \), \( \cl_{X \times \mathbb{A}^1}(W^0) \) on the top middle is sent to \( \cl_X(W_0) \) and \( \cl_X(W_1) \) via \( 0^* \) and \( 1^* \), so the lemma follows.

Remark. — When \( Z \) is a cycle in \( X \), we believe that the method of [GH] can be applied to construct the cycle class map. However, the author does not know how we define the Zariski sheaves \( \Gamma^*(i) \) in [ibid., 1.1].
3.2.4 Lemma ([KS 2.1.1]). — Let \( \mathcal{X} \) be a c-admissible stack\(^{(5)} \) \( \mathcal{U} \) be an open substack of \( \mathcal{X} \) which is smooth, and \( \Gamma \) be a generalized cycle of codimension \( d \) on \( \mathcal{U} \). Consider the following commutative diagram

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{i} & \mathcal{U} \\
\downarrow{j} & & \downarrow{j'} \\
\mathcal{X} & , & 
\end{array}
\]

where \( j \) is the open immersion, and \( i \) is the generalized cycle. Assume \( i' \) is proper. Recall the homomorphism \( \mathcal{X} \). Using this homomorphism, the composition

\[
c: H^2_c(\mathcal{X}, j^*L_\mathcal{U}) \xrightarrow{i'^*} H^2_c(\Gamma) \xrightarrow{\alpha} H^2_c(\mathcal{U})(d)
\]

is the cup product (cf. 3.1.2) with the image of \( cl \) by the canonical homomorphism

\[
H^2_c(\mathcal{U})(d) \cong H^2(\mathcal{X}, j^*i^*L_\mathcal{U})(d) \rightarrow H^2(\mathcal{X}, j^*L_\mathcal{U}(d))
\]

where the first isomorphism follows since \( i' \) is assumed to be proper as well.

Proof. We can copy the proof of [ibid.]. \( \blacksquare \)

3.2.5. Let \( f: X \to \mathcal{Y} \) be a representable l.c.i. morphism from a scheme\(^{(6)} \) of finite type purely of dimension \( n \) to a c-admissible stack purely of dimension \( m \) over \( k \). For example, a representable morphism between smooth c-admissible stacks is l.c.i.. We note that \( f \) is schematic when \( \mathcal{Y} \) is admissible. Let \( \Gamma \to \mathcal{Y} \) be a generalized cycle of codimension \( d \) which is a scheme, and put \( \Gamma' := \Gamma \times_{\mathcal{Y}} X \), which is a generalized cycle of \( X \) which is a scheme as well since \( f \) is schematic. Let us briefly recall the construction of \( f'([\Gamma]) \in CH_{n-d}(\Gamma') \) by Kresch [K].

Let \( f': X' \to \mathcal{Y}' \) be a representable separated morphism between algebraic stacks. In [K] 5.1, Kresch constructs \( o: M_{X'}^0(\mathcal{Y}') \to \mathbb{A}^1 \), whose fiber over 0 is called the normal cone denoted by \( C_{X'}(\mathcal{Y}') \to X' \), and the general fiber is just \( \mathcal{Y}' \). When \( f': X' \to Y' \) is a closed immersion of schemes, \( M_{X'}^0(\mathcal{Y}') \) is nothing but the one introduced in [Fu] Ch.5. We remark that by construction, there is a canonical morphism \( X' \times_{\mathbb{A}^1} \mathcal{Y}' \to M_{X'}^0(\mathcal{Y}') \) defined by the strict transform, and \( o \) is flat by using [Fu] B.6.7. When \( f' \) is l.c.i., \( C_{X'}(\mathcal{Y}') \) is known to be a vector bundle over \( X \), in which case we denote it by \( N_{f'} \).

Now, assume that \( X \) and \( \mathcal{Y} \) are smooth. In this situation, \( M_{X}^0(\mathcal{Y}) \) is smooth. We put \( \Gamma' := \Gamma \times_{\mathcal{Y}} X \). By definition, we have the diagram below on the left:

\[
\begin{array}{ccc}
M_{X}^0(\mathcal{U}) & \xrightarrow{\alpha} & M_{X}^0(\mathcal{Y}) \times_{\mathcal{Y}} \Gamma \\
\downarrow{\alpha} & & \downarrow{\alpha} \\
\Gamma & \xrightarrow{\square} & \mathcal{Y} \\
\downarrow{\square} & & \downarrow{\square} \\
\Gamma' & \xrightarrow{\square} & \mathcal{Y}' \\
\downarrow{\square} & & \downarrow{\square} \\
\Gamma' \times \mathbb{A}^1 & \xrightarrow{\square} & X \times \mathbb{A}^1 \\
\downarrow{\square} & & \downarrow{\square} \\
\Gamma' \times \mathbb{A}^1 & \xrightarrow{\square} & M_{X}^0(\mathcal{Y}) \times_{\mathcal{Y}} \Gamma \\
\downarrow{\square} & & \downarrow{\square} \\
\Gamma' \times \mathbb{A}^1 & \xrightarrow{\square} & M_{X}^0(\mathcal{Y}) \times_{\mathcal{Y}} \Gamma \\
\end{array}
\]

This diagram induces the cartesian diagram on the right.

Let us define \( f'([\Gamma]) \in CH_{n-d}(\Gamma') \). For the detail, see [ibid., 3.1.5.1]. By taking the pull-back by the morphism \( X \to \mathcal{Y} \) of the diagram above on the left, we have the following diagram of schemes:

\[
\begin{array}{ccc}
C_{\Gamma'} \times \mathbb{A}^1 & \xrightarrow{\square} & N_f \\
\downarrow{\square} & & \downarrow{\square} \\
\Gamma' & \xrightarrow{\square} & X \\
\end{array}
\]

The closed immersion on the upper left is induced by \( \alpha \) of \( \square \). By definition, \( f'([\Gamma]) \) is the image of \([C_{\Gamma'}] \) by the homomorphism

\[
CH_{m-d}(C_{\Gamma'} \times \Gamma) \to CH_{m-d}(N_f) \xrightarrow{\delta} CH_{n-d}(\Gamma')
\]

\(^{(5)}\) In the corresponding statement of [KS], they assume \( X \) to be smooth. We think that this assumption is redundant and not enough for their and our purposes. In fact, in the proof of [ibid., 2.3.2], they apply [ibid., 2.1.1] in the situation where \( X \) is not necessarily smooth.

\(^{(6)}\) This assumption is made for simplicity. In fact, with suitable changes, similar results can be obtained when \( X \) is a c-admissible stack.

\(^{(7)}\) In fact, he constructs over \( \mathbb{F}^1 \) instead of \( \mathbb{A}^1 \). However, for convenience, we restrict his construction over \( \mathbb{A}^1 \) and, by abusing the notation, we still denote by \( M_{X}^0(\mathcal{Y}) \).
where the last isomorphism follows by [75, Theorem 3.3].

3.2.6 Lemma ([KS, 2.1.2]). — We preserve the notation. Let \( f^* : H^{2d}_T(\mathcal{M}) \to H^{2d}_T(X) \) be the pull-back. Then the class \( \text{cl}(f^!(\Gamma)) \in H^{2d}_T(X) \) is equal to \( f^! \text{cl}(\Gamma) \).

**Proof.** The verification is essentially the same as [KS]. We have the following commutative diagram:

\[
\begin{array}{ccc}
H^{2d}_T(\mathcal{M})(d) & \xrightarrow{1^*} & H^{2d}_T(\mathcal{M}_\mathcal{X})(d) \xrightarrow{0^*} H^{2d}_T(N_f)(d) \\
\downarrow{f^*} & & \downarrow{g^*} \\
H^{2d}_T(X)(d) & \xrightarrow{1^*} & H^{2d}_T(X \times \mathcal{K})(d) \xrightarrow{0^*} H^{2d}_T(X)(d)
\end{array}
\]

where \( Z := M^2_T, C' := C_T \mathcal{X}, \) and the middle vertical homomorphism is induced by the morphism \( X \times \mathcal{K} \to M^2_T \) defined by the strict transform. In the upper row, the image of the cycle class \( \text{cl}(Z) \) is sent to \( \text{cl}(\Gamma) \) and \( \text{cl}(C') \) by \( 1^* \) and \( 0^* \) respectively, by Lemma 3.2.2 and the flatness of \( Z \to \mathcal{K}^1 \).

Recall the diagram of schemes (3.2.5.1). It is reduced to showing that \( \text{cl}(g^*(C')) = g^* \text{cl}(C') \). Using Lemma 3.2.3, this amounts to proving the commutativity of the following diagram on the left:

\[
\begin{array}{ccc}
\text{CH}_{m-d}(N') & \xrightarrow{g'^*} & H^{2d}_{\mathcal{N}'}(N_f)(d) \\
\downarrow{g'^*} & & \downarrow{p'^*} \\
\text{CH}_{m-d}(\Gamma') & \xrightarrow{p'^*} & H^{2d}_{\mathcal{N}'}(X)(d),
\end{array}
\]

Note that the stacks appearing in these diagrams are, in fact, schemes. Consider the diagram on the right above, where \( p \) denotes the projection \( N_f \to X \). Since \( g^* \circ p^* \) is the identity on \( H^{2d}_T(X)(d) \), and \( g^* \) is an isomorphism whose inverse is \( p^* \) on the Chow groups, the verification of the commutativity of the left diagram is reduced to that of the right one. There exists an open dense subscheme \( U \subset X \) such that \( U \cap \Gamma' \subset \Gamma' \) is dense, and \( N_f \times_X U \to U \) is a trivial bundle, and we can write \( N_f \cong X \times A^n \). Since \( H^{2d}_T(X)(d) \cong H^{2d}_{\mathcal{N}'}(U)(d) \), we may replace \( X \) by \( U \) and \( \Gamma' \) by \( \Gamma' \cap U \). Thus, the claim follows by Lemma 3.2.2. \( \blacksquare \)

3.2.7. Now, we only consider the case where \( \Delta = 0 \) (but \( k \) is still not necessarily finite). Let us recall the construction of [KS] after Lemma 2.3.1. Let \( \mathcal{U} \) be a smooth \( c \)-admissible stack of dimension \( d \), and let \( \Gamma \) be a correspondence on \( \mathcal{U} \) (over \( \text{Spec}(k) \)). Let \( j : \mathcal{U} \to \mathcal{X} \) be a compactification. Since the second projection \( p_2 : \Gamma \to \mathcal{U} \) is assumed to be proper, the morphism \( (j \circ p_1, p_2) : \Gamma \to \mathcal{X} \times \mathcal{U} \) is proper, and we have isomorphisms

\[
H^{2d}_T(\mathcal{X} \times \mathcal{U}, (j \times i\Delta)L(d)) \cong H^{2d}_T(\mathcal{X} \times \mathcal{U}, L(d)) \cong H^{2d}_T(\mathcal{U} \times \mathcal{U}, L(d)).
\]

Thus, the cycle class \( \text{cl}(\Gamma) \) defined in \( H^{2d}_T(\mathcal{U} \times \mathcal{U}, L(d)) \) induces an element in

\[
H^{2d}_T(\mathcal{U} \times \mathcal{U}, (j \times i\Delta)L(d)).
\]

This cohomology is isomorphic to \( \prod_n \text{End}(H^{2d}_T(\mathcal{U})) \) as in [ibid.] using Künneth formula (cf. Corollary 2.3.40) and the Poincaré duality (cf. Theorem 2.3.31).

**Lemma** ([KS, 2.3.2]). — The action \( \Gamma^* \) can be identified with the class \( \text{cl}(\Gamma) \) via this isomorphism.

**Proof.** Using Lemma 3.2.4, the proof of [ibid.] works in exactly the same manner. \( \blacksquare \)

3.2.8. **Proof of Theorem 3.2.1**

By Corollary 2.3.1, we can take a proper generically finite surjective morphism \( f : X \to \mathcal{X} \) such that \( X \) is a smooth scheme. Using the same corollary and Corollary 3.1.9, we may assume \( \Gamma \) to be a scheme. Let \( H^*(\mathcal{X}) \) be \( H^*(\mathcal{X} \times \mathcal{K}, \mathbb{Q}_l) \) or \( H^*(\mathcal{X}, L_X) \). Using Lemma 3.2.7 and Corollary 3.1.7, by arguing as [KS] 2.3.3, we have

\[
(*) \quad \text{Tr}(\Gamma^* : H^*(\mathcal{X})) = \text{deg}(f)^{-1} \cdot \text{Tr}((f \times f)^* \text{cl}_{X \times X}(\Gamma) : H^*_c(X)),
\]

(8) This isomorphism holds even when \( X \) is an admissible stack. Indeed, by [K], 3.5.7, an admissible stack admits “a stratification by global quotients”. Then by [ibid., 4.3.2], we have the required homotopy invariance property.
where we regard classes in $H^d_c(X \times X)(d := \dim(X))$ as endomorphisms of $H^d_c(X)$ using $\boxtimes$. Consider the following commutative diagram

$$
\begin{array}{ccc}
\Gamma' & \longrightarrow & X \times X \\
\downarrow & & \pi_2 \\
\Gamma & \longrightarrow & X \times X \\
\end{array}
$$

Since the composition of horizontal morphisms below is proper by assumption, the composition morphism from $\Gamma'$ to $X$ is proper. Thus the composition of the horizontal morphisms of the first row is proper. By Lemma 2.2.6 $(f \times f)^+\epsilon_{X \times X}(\Gamma)$ is equal to $\epsilon_{X \times X}((f \times f)^+(\Gamma))$. Let $\Gamma'$ be the correspondence defined by $(f \times f)^+(\Gamma)$. By applying Lemma 3.2.7 once again, the trace of the right hand side of $(\Gamma)$ is equal to $\operatorname{Tr}(\Gamma'; H^*_c(X))$. This implies that it suffices to show the theorem in the case where $X = X$ and $\Gamma$ are schemes. By Corollary 3.1.9 we may replace $\Gamma$ by its image in $X \times X$, and assume that $\Gamma \subset X \times X$. In this case, we just repeat the argument of [KS] Prop 2.3.6. Further detail is left to the reader.

**Remark.** — We assumed $k$ to be finite since some delicate arguments might be needed to reduce to the finite field case as in the proof of [KS] 2.3.6.1. However, the theorem should hold without the assumption on the base field.

4. Langlands correspondence

In this final section, we establish the Langlands correspondence, and in particular, prove the existence of *petits camarades cristallins* for curves.

4.1. Langlands correspondence

4.1.1. In this section, in order to make the atmosphere more standard, we prefer using $\text{Isoc}^+(X_0/\mathbb{Q}_p)$ to $\text{Sm}(X_0/\mathbb{Q}_p)$. Let us clarify the relation between these categories. Consider the situation of 2.0.1 or 2.4.14 if $L/K$ is not finite. Let $X$ be a smooth scheme separated of finite type over $k$. Recall the functor $\text{sp}_+$ in [14.10]. By extension of scalar and gluing, we define the following functor

$$
\tilde{\text{sp}}_+: = \text{sp}_+(-d)[-d] \cdot \text{Isoc}^+(X_0/L) \sim \text{Sm}(X_0/L) \subset D^b_{\text{hol}}(X_0/L).
$$

From now on, we identify $\text{Isoc}^+(X_0/L)$ and $\text{Sm}(X_0/L)$ via $\tilde{\text{sp}}_+$. For a morphism $f: X \to Y$ between smooth schemes separated of finite type, let $d := \dim(X) - \dim(Y)$. Then, we have a canonical equivalence $\text{sp}_+ \circ f^* \cong (f^![-d]) \circ \text{sp}_+$ compatible with the composition of morphisms between smooth schemes by [14.2] (2.3.8.1). Thus, by Theorem 1.5.13 and Theorem 1.5.14 we have a canonical equivalence $\tilde{\text{sp}}_+ \circ f^* \cong f^+ \circ \tilde{\text{sp}}_+$. Via this equivalence, we identify $f^*$ and $f^+$. We also know that $\tilde{\text{sp}}_+$ commutes with tensor products by [14.2] 2.3.14. Finally, let $f: X \to \text{Spec}(k)$ be the structural morphism of a smooth realizable scheme, and $M \in \text{Isoc}^+(X_0/L)$. By [14.2] 2.3.12(10) and [11] 5.9, we have canonical isomorphisms

$$
H^*_\text{rig}(X, M) \cong H^*_f(\text{sp}_+(M)), \quad H^*_\text{rig,c}(X, M) \cong H^*_f(\tilde{\text{sp}}_+(M))
$$

as objects in $F\text{-Vec}_L$.

4.1.2. In the rest of this section, unless otherwise stated, $k$ is assumed to be a finite field with $q = p^s$ elements. We fix a base tuple $\mathcal{T} := (k, R := W(k), K := \text{Frac}(R), s, \text{id})$, and take $\sigma_{\mathcal{T}_p} = \text{id}$ as an extension of $\sigma$. We mainly use the $\mathcal{Q}_p$-coefficient cohomology theory. Let $X$ be a smooth scheme over $k$, and let $i_x: x \to X$ be a closed point of $X$. Put $s' := [k(x) : k] + s$, and let $\mathcal{T}_x := (k(x), W(k(x)), K_x, s', \text{id})$. We have the following functor

$$
i_x: \text{Isoc}^+(X_0/\mathcal{Q}_p) \xrightarrow{\iota_x} \text{Isoc}^+(k(x)_0/\mathcal{Q}_p) \cong \text{Isoc}^+_x(k(x)_0/\mathcal{Q}_p)
$$

(9) If one needs more standard notation, $\text{Isoc}^+(X_0/\mathcal{Q}_p)$ should be denoted by $F\text{-Isoc}^+(X/\mathcal{Q}_p)$, but we think that this notation is a little heavy (cf. Definition and Remark of 1.4.1).

(10) There is a mistake in this theorem as pointed out in [11] 3.15.
where the last equivalence follows by Corollary 1.4.12. The last category is equivalent to the category of finite dimensional $\overline{\mathbb{Q}}_p$-vector spaces with automorphism. For $\mathcal{E} \in \text{Isoc}^\dagger(U_0/\overline{\mathbb{Q}}_p)$, the set of Frobenius eigenvalues at $x$ is the set of eigenvalues of the automorphism of $\iota_x(\mathcal{E})$.

4.1.3 Theorem (Langlands correspondence for isocrystals). — Let $X$ be a geometrically connected proper smooth curve over $k$. Denote by $F$ the function field of $X$, and let $\mathbb{A}_F$ be the ring of adeles. For an integer $r \geq 1$, consider the following two sets:

- $\mathcal{I}_r$: The set of isomorphism classes of irreducible isocrystal of rank $r$ in $\lim_{\to} \text{Isoc}^\dagger(U_0/\overline{\mathbb{Q}}_p)$, where the limit runs over open subschemes $U \subset X$, such that the determinant is of finite order.
- $\mathcal{A}_r$: The set of isomorphism classes of cuspidal automorphic representations $\pi$ of $\text{GL}_r(\mathbb{A}_F)$ such that the order of the central character of $\pi$ is finite.

1. There exist maps

$$\mathcal{E}_\bullet: \mathcal{A}_r \rightleftarrows \mathcal{I}_r: \pi_\bullet$$

which correspond in the sense of Langlands: for $\pi \in \mathcal{A}_r$ (resp. $\mathcal{E} \in \mathcal{I}_r$), the sets of unramified places of $\pi$ (resp. $\mathcal{E}$) and $\mathcal{E}_\pi$ (resp. $\pi_\mathcal{E}$) coincide, which we denote by $U$, and for any $x \in |U|$, the set of Frobenius eigenvalues of $\mathcal{E}_\pi$ (resp. $\pi_\mathcal{E}$) at $x$ and that of Hecke eigenvalues of $\pi$ (resp. $\pi_\mathcal{E}$) at $x$ coincide.

2. Assume that $\pi^{\prime}(\cdot) \in \mathcal{A}_r(\cdot)$ and $\mathcal{E}^{\prime}(\cdot) \in \mathcal{I}_r(\cdot)$ corresponds in the sense of Langlands. Then the local $L$-functions and local $\varepsilon$-factors of pairs $(\pi, \pi')$ and $(\mathcal{E}, \mathcal{E}')$ coincide for any point $x \in |X|$. (cf. [A2] and [La2, VI.9 (ii)])

Remark. — The correspondence is unique if it exists.

4.1.4. For the proof of the theorem, we follow the program of Drinfeld and Lafforgue. We shortly recall the outline of the proof to introduce some notation we use in the next subsection. The idea is explained in detail and clearly in the introduction of [La2], so we encourage the reader who are not familiar with Lafforgue’s proof to read through it before our proof.

In [A2], with the help of the product formula proven in [AM], we have the following theorem, which is nothing but the $p$-adic version of “principe de récurrence” by Deligne:

Theorem ([A2, §5]). — Let $r_0$ be a positive integer, and assume Theorem 4.1.3 is known for $r, r'< r_0$, then we have the map $\mathcal{I}_{r_0} \rightarrow \mathcal{A}_{r_0}$ in the sense of Langlands such that the corresponding cuspidal representation is unramified at the places where the isocrystal is. Moreover, if we have a map $\mathcal{A}_{r_0} \rightarrow \mathcal{I}_{r_0}$ in the sense of Langlands such that the corresponding isocrystal is unramified at the places where the cuspidal representation is, the theorem holds for $r', r \leq r_0$.

4.1.5. Thanks to the theorem above, our task is only to construct a map $\mathcal{A}_r \rightarrow \mathcal{I}_r$ such that the corresponding isocrystal is unramified at the places where the cuspidal representation is. Basic idea is to realize this as a relative cohomology of moduli spaces of “shtukas” à la Drinfeld.

In this paper, we use the following various types of moduli spaces of shtukas. Let $r$ be a positive integer. Let $N$ be a level, i.e. a closed subscheme $N = \text{Spec}(\mathcal{O}_N) \hookrightarrow X$ which is not equal to $X$, $p: [0, r] \rightarrow \mathbb{R}$ be a convex polygon, and $a \in \mathbb{A}_{r}$. Given these data, we have admissible stacks over the surface $(X-N) \times (X-N)$ as follows:

(11) In fact, they are moreover “serene stack” in the terminology of [La2, Appendix A].
In this table “smooth?” (resp. “proper?”) asks if the spaces are smooth (resp. proper) over \((X - N) \times (X - N)\) or not. The components of these spaces are indexed by \(1 \leq d \leq r\) called the\(^{1}\) degree. The corresponding components are denoted by \(\text{Cht}_{N}^{r,d,\mathcal{P} \leq p}/a^{\mathcal{Z}}\), \(\text{Cht}_{N}^{r,d,\mathcal{P} \leq p}/a^{\mathcal{Z}}\), and \(\text{Cht}_{N}^{r,d,\mathcal{P} \leq p}/a^{\mathcal{Z}}\).

\[\begin{array}{|c|c|c|c|}
\hline
\text{Situkas} & \text{smooth?} & \text{proper?} & \text{correspondence} \\
\hline
1 & \text{Cht}_{N}^{r,\mathcal{P} \leq p}/a^{\mathcal{Z}} & \circ & \times & \Delta^{1} \\
2 & \text{Cht}_{N}^{r,\mathcal{P} \leq p}/a^{\mathcal{Z}} & \times & \circ & - \\
3 & \text{Cht}_{N}^{r,\mathcal{P} \leq p}/a^{\mathcal{Z}} & \circ & \times & \Delta^{2} \\
\hline
\end{array}\]

\(^1\) We have ring action of Hecke correspondence only after taking the inductive limit over the convex polygon \(p\).

4.1.6. Let \(f: \mathcal{X} \to (X - N) \times (X - N)\) be one of the three moduli spaces of situkas. Then \(H_{N}^{\mathcal{E}} := f_{!}[\mathcal{E}]_{p},\mathcal{X}\) contains the isocrystals which correspond to cuspidal representations in the set \(\{\pi\}_{N}\) (cf. \([12,29]\)). However, it also contains a lot of “garbages” which have already appeared in the Langlands correspondence of lower ranks and we need to throw away. The garbage is called the “\(r\)-negligible part”, and the part we need is called the “essential part”. We first need to show that the essential part is concentrated at a certain degree of \(H_{N}^{\mathcal{E}}\). For this, we need to use the purity of intersection cohomology, and we need the compact space \(\mathcal{Y}\). Still, the essential part is a mixture of isocrystals corresponding to \(\{\pi\}_{N}\), and we need to extract the very isocrystal which corresponds to a fixed cuspidal representation \(\pi \in \{\pi\}_{N}\). For this, we need to define an action of the Hecke algebra \(H_{N}\). We have ring homomorphism from \(H_{N}\) to the ring of correspondences on the moduli space \(\mathcal{X}\) if we pass to the limit of \(p\). Since we are passing to the limit to define the action, the resulting stack is not finite type anymore. For the calculation of the trace of the action of correspondences, we use \(\mathcal{Y}\). We note that even though we have the correspondence associated to elements of the Hecke algebra on \(\mathcal{X}\), this map might not be a homomorphism of rings. Finally, we use the \(\ell\)-independence result to calculate the trace, and extract the exact information we need.

4.2. Proof of the theorem

4.2.1. In this paragraph, we fix a base tuple \(\mathcal{E} := (k, R, K, s, \sigma)\), and we do not need to assume \(k\) to be a finite field. Let \(Y\) be a smooth scheme of finite type over \(k\), which is assumed to be \(\text{geometrically connected}\). Take a closed point \(i_{y}: y \to Y\). Let \(K_{y}\) be the unramified extension of \(K\) induced by the finite extension \(k(y)/k\) of \(k\). Then we have the fiber functor

\[\omega_{y}: \text{Isoc}^{\dagger}(Y/K) \xrightarrow{i_{y}^{*}} \text{Isoc}^{\dagger}(k(y)/K) \cong \text{Vec}_{K_{y}}.\]

Since \(\text{End}(K_{Y}) \cong K\), by [DM 3.10.1], this induces the fiber functor

\[\omega_{y}: \text{Isoc}^{\dagger}(Y/L) \to \text{Vec}_{L}\]

for any finite extension \(L\) of \(K_{y}\), by sending \(E\) to \(i_{y}^{*}(E) \otimes i_{y}^{*}L\). By [DM 3.11], and taking 2-inductive limit, we have the following equivalence of tensor categories

\[\text{Isoc}^{\dagger}(Y/\overline{Q}_{p}) \xrightarrow{\sim} \text{Rep}_{\overline{Q}_{p}}(\pi_{1}(Y, y))\]

where \(\pi_{1}^{\text{isoc}}(Y, y) := \text{Aut}^{\otimes}(\omega_{y})\), the isocrystal fundamental group, which is an affine group scheme over \(\overline{Q}_{p}\).

Remark. — If \(Y \to \text{Spec}(k)\) is not geometrically connected, then \(\text{Isoc}^{\dagger}(Y/\overline{Q}_{p})\) is not a Tannakian category over \(\overline{Q}_{p}\), since \(\text{End}(\overline{Q}_{p}, Y) \cong (\overline{Q}_{p})^{\times C}\) where \(C\) is the number of connected components of \(Y \otimes_{k} \overline{F}_{k}\).

4.2.2. Let us go back to the case where \(k\) is a finite field. For the notation being compatible with [La2], we denote the relative \(s\)-th Frobenius endomorphism on \(X\) by \(\text{Frob}_{X}\) instead of \(F_{X}\). Let \(Y\) be a geometrically connected smooth scheme of finite type over \(k\). Take a closed point \(i_{y}: y \leftrightarrow Y\). Let
$$\mathcal{E} \in \text{Isoc}^1(Y/\overline{\mathbb{Q}}_p).$$ Since $K_y \cong W(k(y)) \otimes_{W(k)} K$, the $s$-th Frobenius automorphism on $W(k(y))$ induces an automorphism $\text{Frob}_y^s: K_y \to K_y$. The fiber $i_y^+\mathcal{E}$ can be seen as an $i_y^+\overline{\mathbb{Q}}_p,Y$-module, where the latter endomorphism ring is isomorphic to $K_y \otimes_K \overline{\mathbb{Q}}_p$ since $Y$ is geometrically connected. Thus, we have isomorphisms

$$\omega_Y(\text{Frob}_Y^+\mathcal{E}) \cong (K_y \otimes_{\text{Frob}_y^s} K_y) \otimes_{K_y \otimes_K \overline{\mathbb{Q}}_p} \overline{\mathbb{Q}}_p \cong i_y^+\mathcal{E} \otimes_{K_y \otimes_K \overline{\mathbb{Q}}_p} \overline{\mathbb{Q}}_p \cong \omega_Y(\mathcal{E}),$$

where the homomorphism $\alpha$ sends $e \otimes a$ to $1 \otimes e \otimes a$. Thus, we get the following 2-commutative diagram:

This diagram induces a homomorphism $\text{Frob}_Y^+: \pi_1^{\text{isoc}}(Y,y) \to \pi_1^{\text{isoc}}(Y,y)$. This homomorphism is in fact an isomorphism, since $\text{Frob}_Y^+$ gives an equivalence of categories by Remark 1.1.3. We define $\rho: \mathbb{Z} \to \text{Aut}(\pi_1^{\text{isoc}}(Y,y))$ to be the homomorphism sending 1 to $\text{Frob}_Y^+$. Using this homomorphism we put $W^{\text{isoc}}(Y,y) := \pi_1^{\text{isoc}}(Y,y) \times \mathbb{Z}$, which is called the isocrystal Weil group of $Y$. By construction, we have the equivalence of tensor categories

$$\text{Isoc}^1_2(Y/\overline{\mathbb{Q}}) \cong \text{Rep}_{\overline{\mathbb{Q}}_p}(W^{\text{isoc}}(Y,y)).$$

induced by $\omega_Y$.

In general, let $Y \to \text{Spec}(k)$ be a connected scheme of finite type, which may not be geometrically connected, and take a closed point $y$. The structural morphism factors as $Y \to \text{Spec}(k') \to \text{Spec}(k)$ where $k'$ is a finite field extension of $k$ of degree $d$ and $Y$ is geometrically connected over $k'$. Consider the base tuple $\mathcal{S} := (k', R', s + d, 1)$. Then we define $W^{\text{isoc}}(Y,y)$ to be the isocrystal Weil group of $Y$ over $\mathcal{S}$. Despite the base base tuple being changed, by Corollary 1.1.2 the equivalence remains to be true. We note that, by definition, $W^{\text{isoc}}(Y,y)$ does not depend on the choice of the base field $k$.

Assume $Y$ is geometrically connected, and $k'$ be a Galois extension of $k$. Take a closed point $y'$ of $Y'$, and let $y$ be the projection to $Y$. Then we have the following exact sequence:

$$1 \to W^{\text{isoc}}(Y \otimes_k k', y') \to W^{\text{isoc}}(Y,y) \to \text{Gal}(k'/k) \to 1.$$  

Remark. — Let $Y$ be a geometrically connected scheme of finite type over $k$. Assume moreover that we have a $k$-rational point $i_y: \text{Spec}(k) \to Y$ for simplicity. Since $\text{Isoc}^1_k(Y_0/\overline{\mathbb{Q}}_p)$ is a neutral Tannakian category over $\overline{\mathbb{Q}}_p$ by using the fiber functor $\omega := i_y^+$, we could have used $\text{Aut}^\omega(\omega)$ as the fundamental group. However, this algebraic group is complicated to handle with, and we used the simpler substitute $W^{\text{isoc}}$.

4.2.3. Let $Y', Y''$ be smooth schemes of finite type and geometrically connected over $k$. Put $Y := Y' \times Y''$, which is geometrically connected over $k$ as well. Let $U \subset Y$ be an open subscheme. The endomorphism $\text{Frob}_{Y'} \times \text{id}: Y' \times Y'' \to Y' \times Y''$ defines an endomorphism on $U$, also denoted by $\text{Frob}_{Y'} \times \text{id}$. Take a closed point $y \in U$. Arguing as in 4.2.2, the pull-back $(\text{Frob}_{Y'} \times \text{id})^*$ induces an automorphism of $W^{\text{isoc}}(U,y)$, and yields a homomorphism $\mathbb{Z} \to \text{Aut}(W^{\text{isoc}}(U,y))$ sending 1 to $(\text{Frob}_{Y'} \times \text{id})^*$. We put $ZW^{\text{isoc}}(Y,y) := W^{\text{isoc}}(U,y) \times \mathbb{Z}$.

Lemma [La 2 VI.13]. — Take closed points $y'$ and $y''$ of $Y'$ and $Y''$. Put $y := (y', y'')$. Then the canonical homomorphism $ZW^{\text{isoc}}(Y,y) \to W^{\text{isoc}}(Y', y') \times W^{\text{isoc}}(Y'', y'')$ is surjective.

Proof. Let $k'$ be a Galois extension of $k$, and put $G := \text{Gal}(k'/k)$. Consider the following diagram, where we omitted the base points of the Weil groups:

$$
\begin{array}{ccccccc}
1 & \to & W^{\text{isoc}}(Y' \otimes k') \times W^{\text{isoc}}(Y'' \otimes k') & \to & W^{\text{isoc}}(Y') \times W^{\text{isoc}}(Y'') & \to & G \times G & \to & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \to & ZW^{\text{isoc}}(Y \otimes k') & \to & ZW^{\text{isoc}}(Y) & \to & G \times G & \to & 1.
\end{array}
$$

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Thus, we may replace \( k \) by \( k' \), and may assume that \( y' \) and \( y'' \) are rational points of \( Y' \) and \( Y'' \). These rational points define morphisms \( Y' \to Y, Y'' \to Y \), and defines a homomorphism

\[
\pi_1^{\text{isoc}}(Y', y') \times \pi_1^{\text{isoc}}(Y'', y'') \to \pi_1^{\text{isoc}}(Y, y),
\]

which is a section of the canonical homomorphism \( \pi_1^{\text{isoc}}(Y, y) \to \pi_1^{\text{isoc}}(Y', y') \times \pi_1^{\text{isoc}}(Y'', y'') \), thus the latter homomorphism is surjective. By definition of the Weil groups, the lemma follows. ■

4.2.4 Lemma. — Let \( Y \) be a smooth connected scheme, and \( U \subset Y \) be an open subscheme. Take a closed point \( y \) in \( U \). Then the homomorphism \( W_{\text{isoc}}(U, y) \to W_{\text{isoc}}(Y, y) \) is surjective.

Proof. It suffices to show that the canonical homomorphism \( \pi_1^{\text{isoc}}(U, y) \to \pi_1^{\text{isoc}}(Y, y) \) is surjective. Let \( j: U \to Y \) be the open immersion. By [DM 2.21], this is equivalent to showing that \( j^+ \) is fully faithful and any subobject of \( j^+ \mathcal{E} \) for an overconvergent isocrystal \( \mathcal{E} \) on \( Y \) in the image of \( j^+ \). Fully faithfulness follows by purity (cf. Theorem 1.5.14). It remains to show that if \( \mathcal{E} \) is an irreducible overconvergent isocrystal on \( Y \), then \( \mathcal{E}|_U \) is irreducible. This is [AC1 1.4.6]. ■

4.2.5 Definition. — Let \( Y \) be a smooth scheme of finite type and geometrically connected over \( k \). We denote the category of overconvergent isocrystals on \( Y \) by \( \mathcal{I}(Y) \). Taking a closed point \( y \) of \( Y \), objects of \( \mathcal{I}(Y) \) correspond to representations of \( W_{\text{isoc}}(Y, y) \). Let \( U \subset Y \times Y \). We denote by \( W_{\text{isoc}}(U) \) the category of overconvergent isocrystals on \( U \) with Frobenius structure equipped with an isomorphism \( \text{Frob}_Y \times \text{id}^+ \mathcal{E} \cong \mathcal{E} \). This corresponds to a representation of \( W_{\text{isoc}}(U) \). Let \( F \) be the function field of \( Y \). We put

\[
\mathcal{I}(F) := 2 \cdot \lim_{U \subset Y} \mathcal{I}(U), \quad W_{\text{isoc}}(F^2) := 2 \cdot \lim_{U \subset Y \times Y} W_{\text{isoc}}(U).
\]

4.2.6. We preserve the notation, and \( q', q'' : Y \times Y \to Y \) be the first and second projection respectively.

Definition ([La2 VI.14]). — Let \( r \geq 1 \) be an integer. An object \( \mathcal{E} \in \mathcal{I}(F^2) \) is said to be \( r \)-negligible if any of its subquotient is a direct factor of an object of the form \( q'^+ \mathcal{E}' \otimes q'^+ \mathcal{E}'' \) where \( \mathcal{E}' \) and \( \mathcal{E}'' \) are objects of rank \( < r \) in \( \mathcal{I}(F) \). It is said to be essential if all the subquotients are not \( r \)-negligible. A semi-simple (resp. virtual) \( r \)-negligible object of \( \mathcal{I}(F^2) \) is said to be complete if it is a direct sum of objects of the form \( q'^+ \mathcal{E}' \otimes q'^+ \mathcal{E}'' \).

4.2.7 Corollary. — (i) Let \( \mathcal{E}', \mathcal{E}'' \) be irreducible objects in \( \mathcal{I}(F) \). Then \( q'^+ \mathcal{E}' \otimes q'^+ \mathcal{E}'' \) is irreducible as an object in \( W_{\text{isoc}}(F^2) \).

(ii) A semi-simple or virtual \( r \)-negligible object \( \mathcal{E} \) is complete if it is invariant by the action of \( (\text{Frob}_X \times \text{id})^+ \), namely if there exists an isomorphism \( (\text{Frob}_X \times \text{id})^+ (\mathcal{E}) \cong \mathcal{E} \).

Proof. Let us prove (i). We may assume \( \mathcal{E}' \) and \( \mathcal{E}'' \) are defined on \( U \subset X \). By Lemma 4.2.4, it suffices to show that \( q'^+ \mathcal{E}' \otimes q'^+ \mathcal{E}'' \) is irreducible as an object in \( W_{\text{isoc}}(U \times X) \). This follows by Lemma 4.2.3. Let us check (ii). There exists \( U \subset X \) such that \( \mathcal{E} \) is an isocrystal on \( U \times U \). Since \( \mathcal{E} \) is assumed to be negligible, there exists a complete \( r \)-negligible object \( \hat{\mathcal{E}} \in W_{\text{isoc}}(X \times X) \) such that \( \mathcal{E} \subset \hat{\mathcal{E}} \) by definition. We may assume that \( \mathcal{E} \) is invariant under the fixed isomorphism \( (\text{Frob}_X \times \text{id})^+ (\hat{\mathcal{E}}) \cong \hat{\mathcal{E}} \). Let \( \rho_E \) be the representations of \( W_{\text{isoc}}(U \times U) \), and \( \rho_E \) be the representation of the same group for \( \mathcal{E} \) defined since \( \mathcal{E} \) is invariant by the action of \( (\text{Frob}_X \times \text{id})^+ \). Then \( \rho_E \to \rho E \) is injective. Let \( K := \text{Ker}(W_{\text{isoc}}(U \times U) \to W_{\text{isoc}}(U) \times W_{\text{isoc}}(U)) \). Then since \( \hat{\mathcal{E}} \) is assumed to be complete, \( \rho_E(K) = \text{id} \). Thus, \( \rho_E(K) = \text{id} \), and defines a representation of \( W_{\text{isoc}}(U) \times W_{\text{isoc}}(U) \). ■

4.2.8. We start the proof of the theorem from here. From now on, we use the notation of [La1] freely. We prove slightly stronger statement than the theorem. For an integer \( n \geq 1 \), what we shall prove is the following:

1. The theorem is true for \( r, r' \leq n \). Thus, we have \( I_{n+1} \to A_{n+1} \) as well.
2. For \( r' < n \), \( H^i_{\text{c}}(\text{Ch}_{\mathbb{V}(p)}^\text{r,\text{c}}, a^r) \) is \( n \)-negligible for any \( i \), level \( N \); \( \text{Spec}(O_N) \to X, a \in A_F^\text{r} \) such that \( \text{deg}(a) = 1 \), and convex polygon \( p \) big enough with respect to \( X \) and \( N \).

Let us call these statements \( (S)_n \). We know that \( (S)_1 \) holds. Indeed, [La1] is nothing but the class field theory, and [La2] can be shown in the same way as [La2] remark of VI.15]. In the following, we fix \( r > 1 \), and assume that \( (S)_1 \) holds for \( i < r \). The proof concludes at the very end of this subsection.
4.2.9. In the following, we fix $a \in \mathbb{A}^r_F$, and a level $N = \text{Spec}(\mathcal{O}_N) \hookrightarrow X$. Let $p$ be an big enough convex function. For $\pi \in \mathcal{A}(F)$, we denote by $\chi_\pi$ the central character of $\pi$. We define a set by
\[
\{\pi\}_N := \{\pi \in \mathcal{A}(F) \mid \chi_\pi(a) = 1 \text{ and } \pi \cdot 1_N \neq 0\},
\]
where $1_N$ is the quotient of the characteristic function of $K_N := \text{Ker}(\text{GL}_r(\mathbb{A}_F) \to \text{GL}_r(\mathcal{O}_N))$ by its volume. It suffices to construct isocrystals corresponding to the cuspidal representations belonging to $\{\pi\}_N$. We put $S_N := (X - N) \times (X - N)$.

Let $q', q'': X \times X \to X$ be the first and second projection respectively. For a morphism of c-admissible stacks $f: \mathfrak{X}_0 \to S_{N,0}$, we denote the relative cohomology $(\mathcal{H}^{i+2}f_!\mathcal{O}_{\mathfrak{X}_0})[-2]$ by $\mathcal{H}^i_c(\mathfrak{X})$. This is an object in $\text{Hol}(S_{N,0})[-2]$. Assume $f$ is proper. There exists an open dense substack $j: \mathfrak{X}_0 \to \mathfrak{X}$ such that $\mathcal{O}_{\mathfrak{X},\mathfrak{X}_0}$ is pure of weight 0. Then we denote by $\mathcal{H}^i_c(\mathfrak{X}) := (\mathcal{H}^{i+2}f_!j_+\mathcal{O}_{\mathfrak{X},\mathfrak{X}_0})[-2]$, which is pure of weight $i$.

We also use $\ell$-adic cohomologies. We denote by $\mathcal{H}^i_c(\mathfrak{X},\mathfrak{X}_i) := \mathcal{H}^i_f(\mathfrak{X},\mathfrak{X}_i)$ and $\mathcal{H}^i_c(\mathfrak{X},\mathfrak{X}_i) := \mathcal{H}^i_f(\mathfrak{X}_i,\mathfrak{X}_i)$, where $\mathcal{H}^i$ denotes the standard (constructible) $\ell$-structure.

Finally, for an abelian category $\mathcal{A}$, we denote by $\text{Gr}(\mathcal{A})$ the Grothendieck group of $\mathcal{A}$, and $\mathcal{Q}\text{Gr}(\mathcal{A}) := \text{Gr}(\mathcal{A}) \otimes \mathbb{Q}$. For an object of $X$ of $\mathcal{A}$ of finite length, we denote by $X^{\text{ss}}$ the semi-simplification of $X$.

Remark. — In the definition of $\mathcal{H}^i_c(\mathfrak{X},\mathfrak{X}_i)$, we used the standard $t$-structure. Thus, if $\mathcal{H}^i_c(\mathfrak{X},\mathfrak{X}_i)$ is smooth, then we have $\mathcal{H}^i_c(\mathfrak{X},\mathfrak{X}_i) \cong (\mathcal{H}^{i+2}f_+j_+\mathcal{O}_{\mathfrak{X},\mathfrak{X}_i})[-2]$, where $\mathcal{H}^i$ denotes the perverse cohomology functor in the derived category of $\mathcal{O}_{\mathfrak{X}}$-sheaves.

4.2.10 Definition. — Let $d_0 \geq 1$ be an integer. We denote by $F_{d_0}$ and $F_{d_0}^2$ the function fields of $X \otimes_{\mathbb{F}_q} \mathbb{F}_{q_{d_0}}$ and $(X \times X) \otimes_{\mathbb{F}_q} \mathbb{F}_{q_{d_0}}$. An irreducible object in $\mathcal{H}(F)$ is said to be $r$-negligible if it is of rank $< r$. An irreducible object $\mathcal{E}$ in $\mathcal{H}(F_{d_0})$ (resp. $\mathcal{H}(F_{d_0}^2)$) is said to be $r$-negligible if there exists an irreducible $r$-negligible object $\mathcal{E}'$ in $\mathcal{H}(F)$ (resp. $\mathcal{H}(F^2)$) such that $\mathcal{E} \otimes_{\mathbb{F}_{q_{d_0}}} \mathbb{F}_{q_{d_0}}$ contains $\mathcal{E}$. Sums of $r$-negligible objects are said to be $r$-negligible as well.

4.2.11 Lemma ([La2 VI.16]). — Let $d_0 \geq 1$ be an integer. An irreducible object $\mathcal{E}$ in $\mathcal{H}(F^2)$ is $r$-negligible if $\mathcal{E} \otimes_{\mathbb{F}_{q_{d_0}}} \mathbb{F}_{q_{d_0}}$ contains an $r$-negligible object in $\mathcal{H}(F_{d_0}^2)$.

Proof. The proof is the same as [ibid.]

4.2.12. We need to show the following technical proposition. In the statement and the proof, the algebraic stack $\mathcal{C}^{r,N}$ and its variant $\mathcal{C}^{r,N}_{(12)}$ are used. Since the geometry of these stacks are rather complicated and we use these only to show the following proposition, we do not try to recall them. Since the proof is parallel, and the notation is the same as [La2 VI.17], refer to the corresponding part of [ibid.] for the details of those stacks.

Proposition ([La2 VI.17]). — Let $p$ be a convex polygon big enough with respect to $X$, $N$, and the degree $d \in \mathbb{Z}$. Let $\mathfrak{C}$ be an algebraic stack representable quasi-projective over $\mathcal{C}^{r,N}$, and consider the following cartesian diagram:

\[
\begin{array}{ccc}
\mathfrak{X} & \xrightarrow{\text{Res}} & \mathfrak{C} \\
\downarrow{g'} & & \downarrow{g} \\
\text{Ch}_{r,d,p} \times X \times X & \xrightarrow{\text{Res}} & \mathcal{C}^{r,N}.
\end{array}
\]

We denote by $p_X: \mathfrak{X} \to S_N$ the projection. Let $\mathfrak{M} \in \text{Hol}(\mathfrak{C})$ which is supported on the boundary (cf. [ibid.] of $\mathfrak{C}$. Then $\mathcal{H}^i g_!^r (\mathfrak{M})$ is smooth on $S_N$ and $r$-negligible as an object in $\mathcal{H}(F^2)$ for any $i$.

Proof. Recall that the morphism $\text{Res}$ is smooth. Using Lemma 2.4.3, we have
\[
\mathcal{H}^i g_!^r (\mathfrak{M}) \cong \text{Res}^+ g_!^r (\mathfrak{M}).
\]

Since we have $p_X g_!^r \cong p_{\mathfrak{X}_0} \circ g_!^r$, for $\mathfrak{M} \in D_{\text{hol}}^+(\mathfrak{X})$, $\mathcal{H}^k p_{\mathfrak{X}_0} (\mathfrak{M})$ can be expressed as extensions of subquotients of $\bigoplus_{i+j-k} \mathcal{H}^i \mathcal{P}_{2k} (\mathcal{H}^i g_!^r (\mathfrak{M})$. Thus it suffices to show the proposition for $\mathfrak{C} = \mathcal{C}^{r,N}$. Since the morphism

(12) In [La2], Lafforgue uses script fonts (e.g. $\mathcal{C}^{r,N}$).
In the following, we initialize the notation $\mathcal{C}$, and use it for other stacks.

Let $c_0 := \sup \{|\dim(L_x)| \mid x \in |\Res(\mathcal{X})|\}$ where $L_x$ is the inertia algebraic group space of $\mathcal{C}_r^N$ at $x$. The dimension of any locally closed substack of $\mathcal{C}_r^N$ in the image of $\Res$ can be bounded below by $-c_0$ by the quasi-compactness of $\mathcal{X}$. We use the induction on $k := c_0 + \dim(\mathcal{C}(\geq 0))$ where $\mathcal{C} := \Supp(\mathcal{M})$ (cf. (2.4.6) for the definition of support). Assume that the proposition holds for $k = k_0 - 1$. We will show the proposition for modules whose support for degree is of dimension $k = k_0 + 1$. Now, we have the following two reductions which we use frequently:

1. Let $j : \mathcal{U} \subset \mathcal{C}$ be an open dense (i.e. $\dim(\mathcal{C} \setminus \mathcal{U}) < \dim(\mathcal{C})$) substack. Let $\mathcal{M}' \in \Hol(\mathcal{C})$ such that $j^+\mathcal{M} \cong j^+\mathcal{M}'$. Then proving the proposition for $\mathcal{M}$ and $\mathcal{M}'$ are equivalent.

2. It suffices to show that $\mathcal{M}/\mathcal{X}!(\Res^+\mathcal{M}) \otimes \mathbb{F}_{q_0}$ is $r$-negligible in $\mathcal{I}(E_r^2)$ (cf. Definition 2.2.11).

The first one follows by using the localization exact sequence Lemma 2.4.6 and the second one by Lemma 2.2.11.

By Lemma 2.2.11, we may assume that $\mathcal{M}$ is the extension by zero of a smooth object on a locally closed substack $\mathcal{C}$ of dimension $k$ in the stratum $\mathcal{C}_r^N$ of $\mathcal{C}_r^N$ with non-trivial partition $r = (r_1, \ldots, r_k)$ of $r$. Considering the stratum $\mathcal{C}_r^N$ in $\mathcal{C}_r^N$, which contains $\mathcal{C}_{r_1}^N$ as an open substack, we have a morphism

$$\gamma : \mathcal{C}_{r_1}^N \to \mathcal{C}_r^N : = \mathcal{C}_{r_1}^N \times_{\mathcal{F}_{d_0}} \mathcal{F}_{d_0}^\times \times_{\mathcal{F}_{d_0}} \mathcal{F}_{d_0}^\times \cdots \times_{\mathcal{F}_{d_0}} \mathcal{F}_{d_0}^\times .$$

As in Lemma 2.2.11, consider the following commutative diagram:

$$\begin{array}{c}
\text{Cht}_{\alpha}^{r,d} \eta_{\mathcal{X}} \times X \times X S_N \to S_N \times \mathcal{C}_r^N \text{ is smooth by [La2 III, 3)-(a)]}, \text{we get the smoothness by Lemma 2.4.13.}
\end{array}$$

As shown in [La2], for any valued point of $\mathcal{C}_r^N$ which sits in the image of the upper horizontal morphism, the automorphism group of $\mathcal{C}_r^N$ over its image in $\mathcal{C}_r^N$ is flat and geometrically connected, and moreover, the dimension of the automorphism group is equal to the relative dimension of $\gamma$ at the given point. This implies that the preimage of a point of $\mathcal{C}_r^N$ by $\gamma$ which sits in the image of the upper horizontal morphism consists of finitely many points, and by Lemma 2.1.21 shrinking $\mathcal{C}$ by its dense open substack if necessarily, we may assume that $\gamma'$ is the composition of a gerb-like morphism whose structural group is flat and fiber-wise geometrically connected, a finite flat radical morphism, and a finite etale morphism. By using Lemma 2.4.9, Lemma 2.1.21 and changing $\mathcal{C}$ by its open dense substack, which is allowed by [La2], we may assume that $\mathcal{M}$ is a pull-back by $\gamma'$ of a smooth object on a locally closed substack of $\mathcal{C}_r^N$.

Now, we have a finite surjective radical morphism

$$\mathcal{C}_r^N \to \mathcal{C}_{r_1}^N \times_{\mathcal{F}_{d_0}} \mathcal{F}_{d_0}^\times \times_{\mathcal{F}_{d_0}} \mathcal{F}_{d_0}^\times \cdots \times_{\mathcal{F}_{d_0}} \mathcal{F}_{d_0}^\times .$$

In particular, we may identify the holonomic modules on these stacks by Lemma 2.1.21. It is explained in [La2] that the image of $\mathcal{C}$ by the morphism $\mathcal{C}_r^N \to \mathcal{C}_r^N \to \mathcal{C}_r^N$ ($* = 1$ or $k$) consists of finitely many points. Thus by shrinking $\mathcal{C}$ further, we may assume that the image of $\mathcal{C}$ in $\mathcal{C}_r^N$ consists of a locally closed point

$$\mathcal{B}_* \cong \left[ \text{Spec}(\mathbb{F}_{q_0})/\Aut(\mathcal{B}_*) \right]$$

where $\mathbb{F}_{q_0}$ is the field extension of $\mathbb{F}_q$ of degree $d_0$. We define the Galois coverings $\mathcal{B}_*'$ of $\mathcal{B}_*$ defined by the “discrete part” $\Aut(\mathcal{B}_*)/(\Aut(\mathcal{B}_*)^g \circ \Aut(\mathcal{B}_*))$. Let $\alpha : \mathcal{C} \to \mathcal{C}$ be the Galois covering induced by $\mathcal{B}_* \to \mathcal{B}_*$, $\mathcal{B}_* \to \mathcal{B}_*$. We may replace $\mathcal{M}$ by $\mathcal{M}' := \alpha_+ \alpha^+ \mathcal{M}$. By considering Lemma 2.4.13, we may assume that $\mathcal{M}' \cong \mathcal{M} \otimes_{\mathcal{F}_{d_0}} \mathcal{M}' \otimes_{\mathcal{F}_{d_0}} \mathcal{F}_{d_0}^\times \cdots \otimes_{\mathcal{F}_{d_0}} \mathcal{F}_{d_0}^\times$.

- $\mathcal{M}_* \ (* = 1, k)$ is the push-forward of the trivial smooth object on $\mathcal{B}_*'$ by the finite etale covering $\mathcal{B}_\eta \to \mathcal{B}_1$.  

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\( M'' \) is a smooth object on a locally closed substack of \( \mathcal{C}^{2, N}_{\mathbb{A}^N/G_{\mathbb{A}^N}^N, \mathbb{F}_p} \times \mathbb{A}^{N-1}/G_{\mathbb{A}^N}^N, \mathbb{F}_p \). Consider the following diagram

\[
\begin{array}{ccc}
\text{Ch}_{r-d}^\times \mathcal{C}^{p} & \xrightarrow{\beta} & \mathcal{C}^{r,N} \\
\downarrow g & & \downarrow \beta' \\
\text{Ch}_{r-d}^\times \mathcal{C}^{p} & \xrightarrow{\gamma} & \mathcal{C}^{r,N} \\
\end{array}
\]

Let us treat the case where \( B_1' \) has a zero (resp. “does not have zero”) (cf. \textit{ibid.}) for the terminologies, then the cohomology group \( H^i f_+ \mathcal{F}_p \) where

\[
f: \text{Ch}_{r_1^1, d_1, \mathcal{P}^\leq p_1 \times \mathcal{F}^{1, N}_{\mathbb{A}^N/G_{\mathbb{A}^N}^N} \to X \times X \times \mathbf{F}_p \to X \times X \times X, \text{Prob}(X \times (X - N)) \text{ over the generic point of } X \text{ (resp. } X \times X \text{) is } r\text{-negligible in } \mathcal{I}(F_{d_2}) \text{ (resp. } \mathcal{I}(F_{d_2}^2)\).}
\]

Similar claim for \( B_2' \) as in \text{[La2]}.

Indeed, if these hold, we can write \( q_* \text{Res}^+(\mathcal{M}) \) as a direct factor of \( \mathcal{E}_* \times \mathcal{F}_* \) such that \( \mathcal{E}_* \) and \( \mathcal{F}_* \) are irreducible of rank \( < r \) or supported on points. Using Künneth formula again, we get that arbitrary irreducible subquotient of \( H^i f_+ \mathcal{F}_p \) is a direct factor of

\[
(\mathcal{E}_* \times \mathcal{F}_k) \otimes H^i (X \times X, r^+ q_{11}^* \text{Res}^+(\mathcal{M}) \otimes (\mathcal{F}_1 \otimes \mathcal{E}_k)),
\]

and, by considering \text{[2]} the proposition follows.

Let us show the claim. We only consider \(* = 1\) case. When \( B_1' \) does not have a zero, we have

\[
\text{Ch}_{r_1^1, d_1, \mathcal{P}^\leq p_1 \times \mathcal{F}^{1, N}_{\mathbb{A}^N/G_{\mathbb{A}^N}^N} \to X \times X \times \mathbf{F}_p \to X \times X \times X, \text{Prob}(X \times (X - N)) \text{ over the generic point of } X \text{ (resp. } X \times X \text{) is } r\text{-negligible in } \mathcal{I}(F_{d_2}) \text{ (resp. } \mathcal{I}(F_{d_2}^2)\).}
\]

and by the induction hypothesis \text{[2,8,2]} we get the desired \( r\)-negligibility.

Let us treat the case where \( B_1' \) has a zero \( \{0\} \), which is a point in \( N \) defined over \( \mathbf{F}_{q_0} \). We put

\[
X^1_{d_1} := \text{Ch}_{r_1^1, d_1, \mathcal{P}^\leq p_1 \times \mathcal{F}^{1, N}_{\mathbb{A}^N/G_{\mathbb{A}^N}^N} \to X \times X \times \mathcal{F}_{1, N} \mathbf{F}_{q_0}, \}
\]

and by the induction hypothesis \text{[2,8,2]} we get the desired \( r\)-negligibility.

Let us treat the case where \( B_1' \) has a zero \( \{0\} \), which is a point in \( N \) defined over \( \mathbf{F}_{q_0} \). We put

\[
X^1_{d_1} := \text{Ch}_{r_1^1, d_1, \mathcal{P}^\leq p_1 \times X \times X \times \mathcal{F}_{1, N} \mathbf{F}_{q_0}, \}
\]

Let \( (p_1, \text{Res}_1): X^1_{d_1} \to (X \times N) \times \mathcal{C}^{1, N}_{\mathbb{A}^N/G_{\mathbb{A}^N}^N} \{0\}) \) be the canonical smooth morphism. We are now reduced to the following claim:
Claim ([La2 VI.18]). — For any holonomic module $\mathcal{M}_1$ on $E_{r,N} \times_{\mathbb{A}^N} \mathbb{C}_m \{0\}$, the cohomology $\mathcal{H}^i_{\mathcal{P}1! \text{Res}_1^+} \mathcal{M}_1$ is $r$-negligible for any $i$.

Proof. The proof is essentially the same as [ibid.]. We just remark a few things: In the proof, we need a purity result for intersection cohomology (i.e. the purity of the cohomology of $\mathcal{F}_1$ using the notation of Lafforgue). To do this, we use Theorem 2.3.41 [2.3.42]

Now, arguing as [ibid.] using the induction, we are reduced to the negligibility of the relative cohomology of $\text{Chr}^r_{\mathbb{A}^N, \mathbb{B}_N}$ over $X - N$. For this, we need to calculate the trace of the Frobenius action on the cohomology of the moduli space of shtukas. Since the moduli space is smooth over $(X - N) \otimes_{\mathbb{F}_q} \{0\}$ and admissible by [La2 II.4], we can use Theorem 3.2.1 and the Frobenius eigenvalues of each point of $(X - N) \otimes 0$ coincide with those of the $\ell$-adic cohomology. Thus, by the induction hypothesis 4.2.8.1, the relative cohomology is actually $r$-negligible.

4.2.13. We extract the “essential part” from the cohomology group $\mathcal{H}_c^r(\text{Chr}^r_{\mathbb{A}^N, \mathbb{B}_N}^*, \mathbb{F}_p^\infty / a^\mathbb{Z})$. The cohomology is smooth on $S_N$ by Proposition 4.2.12. Now, there exists a unique element $\mathcal{H}_{N, \text{ess}}^r$ in $\mathbb{Q}\text{Gr}(S_N)$ satisfying the following two properties:

1. The formal difference

$$\mathcal{H}_{N, \text{ess}}^r - \frac{1}{r!} \sum_{n=1}^{r!} (\text{Frob}_{n}^r \times \text{id}_X)^+ \mathcal{H}_c^r(\text{Chr}^r_{\mathbb{A}^N, \mathbb{B}_N}^*, \mathbb{F}_p^\infty / a^\mathbb{Z}),$$

considered as an element of $\mathbb{Q}\text{Gr}(F^2)$, is complete $r$-negligible.

2. Let $\mathcal{H}_{N, \text{ess}, \ell}^r$ be the $\ell$-adic essential part defined in [La2 VI.19] for some prime number $\ell \neq p$. Then for any closed point $x \in U$, using the notation of [3.2.41] we have

$$\text{Tr} \mathcal{H}_{N, \text{ess}}^r(\text{Frob}_{x}^r) = \text{Tr} \mathcal{H}_{N, \text{ess}, \ell}^r(\text{Frob}_{x}^r).$$

Indeed, there exists complete $r$-negligible $\ell$-adic sheaf $\sigma_i$ and rational constant $c_i$ such that

$$\mathcal{H}_{N, \text{ess}, \ell}^r - \frac{1}{r!} \sum_{n=1}^{r!} (\text{Frob}_{n}^r \times \text{id}_X)^+ \mathcal{H}_c^r(\text{Chr}^r_{\mathbb{A}^N, \mathbb{B}_N}^*, \mathbb{Q}_\ell) = \sum_i c_i \cdot \sigma_i.$$

Since $\sigma_i$ are assumed to be complete $r$-negligible, there exists a unique $\tilde{\sigma}_i$ in $\mathbb{Q}\text{Gr}(S_N)$ corresponding to $\sigma_i$ in the sense of Langlands by the induction hypothesis 4.2.8.1 By Theorem 3.2.1, the following element meets our need:

$$\mathcal{H}_{N, \text{ess}}^r := \frac{1}{r!} \sum_{n=1}^{r!} (\text{Frob}_{n}^r \times \text{id}_X)^+ \mathcal{H}_c^r(\text{Chr}^r_{\mathbb{A}^N, \mathbb{B}_N}^*, \mathbb{Z}) - \sum_i c_i \cdot \tilde{\sigma}_i.$$

Now we have

4.2.14 Proposition ([La2 VI.20]). — (i) None of the irreducible component of $\mathcal{H}_{N, \text{ess}}^r$ is $r$-negligible. All the components have positive multiplicity, and pure of weight $2r - 2$.

(ii) The following difference is $r$-negligible:

$$\mathcal{H}_{N, \text{ess}}^r - \sum_{n=1}^{r!} (\text{Frob}_{n}^r \times \text{id}_X)^+ \mathcal{H}_c^{2r-2}(\text{Chr}^r_{\mathbb{A}^N, \mathbb{B}_N}^*, \mathbb{Z}).$$

Proof. We can just copy the proof of [ibid.], in which Proposition 4.2.12 is used. Making use of the $\ell$-adic result, we can also argue as follows. By Proposition 4.2.12, the difference

$$\mathcal{H}_{N, \text{ess}}^r - \mathcal{H}^r(\text{Chr}^r_{\mathbb{A}^N, \mathbb{B}_N}^*, \mathbb{F}_p^\infty / a^\mathbb{Z})$$

is $r$-negligible. By the purity (cf. Theorem 2.3.42 and 2.3.41), and $\ell$-independence Theorem A.4.4, we know that $\mathcal{H}^r(\text{Chr}^r_{\mathbb{A}^N, \mathbb{B}_N}^*, \mathbb{F}_p^\infty / a^\mathbb{Z})$ corresponds in the sense of Langlands to $\mathcal{H}^r(\text{Chr}^r_{\mathbb{A}^N, \mathbb{B}_N}^*, \mathbb{F}_p^\infty / a^\mathbb{Z})$ for any $i$. Using the corresponding result for $\ell$-adic cohomology, our proposition follows.

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4.2.15 Corollary ([La2 VI.21]). — Let $p \leq q$ be big enough convex polygons. The kernel and cokernel of the homomorphisms

$$H^{2r-2}_c(Cht_{\mathbb{P}^2}^r/a^2) \to H^{2r-2}_c(Cht_{\mathbb{P}^2}^{r+p}/a^2), \quad H^{2r-2}_c(Cht_{\mathbb{P}^2}^{r+p}/a^2) \to H^{2r-2}_c(Cht_{\mathbb{P}^2}^{r+q}/a^2)$$

defined by the inclusions $Cht_{\mathbb{P}^2}^r/a^2 \hookrightarrow Cht_{\mathbb{P}^2}^{r+p}/a^2$ and $Cht_{\mathbb{P}^2}^{r+p}/a^2 \hookrightarrow Cht_{\mathbb{P}^2}^{r+q}/a^2$ are $r$-negligible.

Proof. Copy the proof of [ibid.]. We note that Proposition 4.2.12 is once again used. ■

4.2.16. Next thing we need to do is to define the Hecke action on $H^\ast_{N,\text{ess}}$. Lafforgue constructed a homomorphism of algebras (cf. 3.1.10)

$$\varrho_\ast: H^\ast_{N}/a^2 \to \text{Corr}^{\text{inv-et}}(Cht^r_{N}/a^2)$$

sending $f$ to $\Gamma_f$, which is a finite étale correspondence on a certain open dense subscheme of $X \times X$ (cf. [La2 V 2) a)] or [La3 14, Theorem 5]). To be compatible with [La2], we consider $\varrho_\ast := \text{norm} \circ \varrho$ for the action on the cohomology (cf. 3.1.12). Moreover, we have partial Frobenius endomorphisms $\text{Frob}_a$ and $\text{Frob}_b$ over $\text{Frob}_X \times \text{id}_X$ and $\text{id}_X \times \text{Frob}_Y$ respectively. For any convex polygon $p, f \in H^\ast_{N}/a^2, s, u \in \mathbb{N}$, there exists a convex polygon $q \geq p$ such that the correspondence $f \times \text{Frob}_s \times \text{Frob}_u$ sends $Cht^r_{N}/a^2$ to $Cht_{N}/a^2$. Thus,

$$H^{2r-2}_c(Cht_{N}/a^2) := \lim_{\to \leftarrow} H^{2r-2}_c(Cht_{N}^{r+p}/a^2)$$

can be seen as an element of $\text{Ind}((\mathbb{Z}I(F^2)))$ with the action of Hecke algebra $H^\ast_{N}/a^2$ via $\varrho_\ast$ commuting with actions of partial Frobenius endomorphisms (cf. Lemma 3.1.10).

Now, let us define a filtration on $H := H^{2r-2}_c(Cht_{N}/a^2)$ in $\text{Ind}((\mathbb{Z}I(F^2)))$ stable by the action of $H^\ast_{N}/a^2$ as in [La2 VI, 3(a)]. Put $F^0H = 0$. Assume $F^2H$ has already been constructed. We define $F^{2i+1}H$ to be the largest submodule such that $F^{2i+1}H/F^{2i}H$ is $r$-negligible. Then, we put $F^{2i+2}H$ to be the largest submodule such that $F^{2i+2}H/F^{2i+1}H$ is essential. For a convex polygon $p$, we define the filtration $F^\ast H^{2r-2}_c(Cht_{N}^{r+p}/a^2)$ by pulling-back the one using the canonical homomorphism $H^{2r-2}_c(Cht_{N}^{r+p}/a^2) \to H^{2r-2}_c(Cht_{N}^{r+p}/a^2)$. By Corollary 4.2.15 the homomorphism

$$\bigoplus_{i \geq 0} F^{2i+2}/F^{2i+1} \to \bigoplus_{i \geq 0} F^{2i+2}/F^{2i+1}$$

is an isomorphism in $\mathbb{Z}I(F^2)$ for big enough $p$, and these objects possesses the action of $H^\ast_{N}/a^2$. In particular, the coefficients of $H^\ast_{N,\text{ess}}$, which is $a$ priori rational numbers by construction, are positive integers. We denote by $H^\ast_{N,\text{ess}}$ the semi-simplification of the sum above as an object in $\mathbb{Z}I(F^2)$ with action of $H^\ast_{N}/a^2$. The action of $H^\ast_{N}/a^2$ on $H^\ast_{N,\text{ess}}$ as an object in $\mathbb{Z}I(F^2)$ can be extended uniquely to that as an object in $\mathbb{Z}I(S_N)$. For an abelian category $\mathcal{A}$, we denote by $H^\ast_{N}/a^2,\mathcal{A}$ the category of objects of $\mathcal{A}$ endowed with action of $H^\ast_{N}/a^2$.

4.2.17. We need to calculate the trace of the Hecke action on $H^\ast_{N,\text{ess}}$. Let $f \in H^\ast_{N}/a^2$. Even though $\Gamma_f$ does not stabilize $Cht_{N}^{r+p}/a^2$, we can take an open dense substack $\mathcal{U}$ of $Cht_{N}^{r+p}/a^2 \times Cht_{N}^{r+p}/a^2$ on which the restriction $\Gamma_f \mid \mathcal{U}$ of $\Gamma_f$ acts. Take the normalization of the of the morphism

$$\Gamma_f \rightarrow \mathcal{U} \subset Cht_{N}^{r+p}/a^2 \times Cht_{N}^{r+p}/a^2 \rightarrow Cht_{N}^{r+p}/a^2 \times Cht_{N}^{r+p}/a^2.$$

Then Lafforgue shows in [La2 V.14] that the correspondence stabilizes $Cht_{N}^{r+p}/a^2$, and defines a correspondence on it. Arguing similarly to 4.2.16 the action of the correspondence $\Gamma_f$ on $H^\ast_{N}/(Cht_{N}^{r+p}/a^2)$ induces an action on $H^\ast_{N,\text{ess}}$. We need to compare the trace of the action of correspondences on $H^\ast_{N,\text{ess}}$ defined using $\text{Cht}'$ and $\text{Cht}$.

4.2.18 Lemma ([La2 VI.23, 24]). — Let $f \in H^\ast_{N}/a^2$, and take a big enough convex polygon $p$. Let $\{E\}'$ be the set of irreducible objects in $\mathbb{Z}I(S_N)$ appearing in $H^\ast_{N,\text{ess}}$. Then there exists a unique set of scalars $\{c_E\}$ for $E \in \{E\}'$ such that, for any $x \in |S_N|$ and $n \in \mathbb{Z}$, we have

$$\text{Tr} \mid_H(c \times \text{Frob}_a^n) = \sum_{E \in \{E\}'} c_E \cdot \text{Tr}_{E}(\text{Frob}_a^n) \quad (=: \text{Tr}_{H^\ast_{N,\text{ess}}}(c \times \text{Frob}_a^n))$$
where, on the left side, the action of $f$ on $\mathcal{H}_{N,\text{ess}}'$ is the one induced by that of $\mathcal{H}_{c}^\nu(\text{Ch}_{N,F}^{z\leq P}/a^Z)$. Moreover, we have
\[
\text{Tr}_{\mathcal{H}_{N,\text{ess}}}(f \times \text{Frob}_x^n) = \text{Tr}_{\mathcal{H}_{N,\text{ess}}'}(f \times \text{Frob}_x^n)
\]
where the action on the left hand side is the one defined in 4.2.16.

Proof. The proof is exactly the same as [ibid.]. Since $\mathcal{H}_{c}^\nu(\text{Ch}_{N,F}^{z\leq P}/a^Z)$ is $\iota$-mixed by Theorem 2.3.41, the uniqueness follows by the Cebotarev density A.3.1. The existence of $\{c_\nu\}$ follows by commutativity of the action of $f$ and $\text{Frob}_x^n$. The details and the verification of the last equality is the same as [ibid.], so we leave it to the reader.

4.2.19. For an unramified irreducible admissible representation $\pi$ of $\text{GL}_r(F_\tau)$ at some place $x$ of $F$ and $t \in \mathbb{Z}$, we put
\[
z_t^i(\pi) := z_1(\pi)^t + \cdots + z_r(\pi)^t,
\]
where $z_i(\pi)$ denotes the Hecke eigenvalue of $\pi$. Take a closed point $x$ in $X \times X$. We denote the closed point on $X$ defined by the first (resp. second) projection by $\infty_x$ (resp. $0_x$). We note that $\text{deg}(x) = \text{lc}(\text{deg}(\infty_x), \text{deg}(0_x))$.

Lemma ([La2], VI.25). — Let $f \in \mathcal{H}_N'/a^2$. Then there exists an open dense subscheme $U_f \subset S_N$ such that for any $x \in U_f$, we have
\[
\text{Tr}_{\mathcal{H}_{N,\text{ess}}}(f \times \text{Frob}_x^{-s/\text{deg}(x)}) = q^{(r-1)s} \sum_{\pi \in \pi_N} \text{Tr}_{\pi}(f) \cdot z_{-\nu}^i(\pi_{\infty_x}) \cdot z_{\nu}^i(\pi_{0_x}),
\]
where $s = \text{deg}(\infty_x) = \text{deg}(0_x)u \in \mathbb{Z} \cdot \text{deg}(x)$.

Proof: For $x \in S_N$, we denote by $\text{Ch}_{N,F}^{z\leq P}/a^Z$ the fiber of $\text{Ch}_{N,F}^\nu(a^2)$ over $x$. By Corollary 3.2.1, there exists $U_f \subset S_N$ such that
\[
\text{Tr} \left( f \times \text{Frob}_x^{-s/\text{deg}(x)}; \mathcal{H}_{c}^\nu(\text{Ch}_{N,F}^{z\leq P}/a^Z) \right) = \text{Tr} \left( f \times \text{Frob}_x^{-s/\text{deg}(x)}; \mathcal{H}_{c}^\nu(\text{Ch}_{N,F}^{z\leq P}/a^Z) \right)
\]
for any $x \in U_f$. Let $\mathcal{H}_{N,\text{ess}}, \ell$ be $\mathcal{H}_{N,\text{ess}}$ defined in [La2] after VI.22]. Using the Langlands correspondence for $r$ and Lemma 4.2.18, we can write
\[
\text{Tr} \left( f \times \text{Frob}_x^{-s/\text{deg}(x)}; \mathcal{H}_{N,\text{ess}} \right) = \text{Tr} \left( f \times \text{Frob}_x^{-s/\text{deg}(x)}; \mathcal{H}_{N,\text{ess}}, \ell \right) + q^{(r-1)s} \sum_{\ell} c_{\ell} \lambda_{\ell}^s \text{Tr}_{q^{-\nu} \chi' \otimes q^{\nu} \chi''}(\text{Frob}_x^{-s/\text{deg}(x)}),
\]
where $\chi', \chi''$ are of rank $r$ and pure of weight 0. Let $\{\mathcal{H}\}$ be the set of irreducible objects in $ZI(S_N)$ which appears in $\mathcal{H}_{N,\text{ess}}$. By Lemma 4.2.18, there exists $c_{\mathcal{H}}$ such that
\[
\text{Tr} \left( f \times \text{Frob}_x^n; \mathcal{H}, \mathcal{H}_{N,\text{ess}} \right) = \sum_{\mathcal{H} \in \{\mathcal{H}\}} c_{\mathcal{H}} \cdot \text{Tr}_{\mathcal{H}}(\text{Frob}_x^n)
\]
for any $n \in \mathbb{Z}$ and closed point $x \in |S_N|$. By the calculation of in the $\ell$-adic situation [La2], VI.25], we get
\[
(4.2.19.1) \quad \sum_{\mathcal{H} \in \{\mathcal{H}\}} c_{\mathcal{H}} \frac{\text{Tr}_{\mathcal{H}}(\text{Frob}_x^{-s/\text{deg}(x)})}{q^{(r-1)s}} - \sum_{\pi \in \pi_N'} \text{Tr}_{\pi}(f) \cdot z_{-\nu}^i(\pi_{\infty_x}) \cdot z_{\nu}^i(\pi_{0_x})
\]
\[
= \sum_{\ell} c_{\ell} \lambda_{\ell}^s \text{Tr}_{q^{-\nu} \chi' \otimes q^{\nu} \chi''}(\text{Frob}_x^{-s/\text{deg}(x)}).
\]
Since $\mathcal{H}$ is pure of weight $2r - 2$, and we know that $|z_i(\pi_{\infty_x})|$ and $|z_j(\pi_{0_x})|$ are 1, we have $|\lambda_{\ell}| = 1$. We need to show that $c_{\ell} = 0$. The argument is essentially the same, but for the reader, we recall it. Assume otherwise. Then there would exist $\mathcal{E}'$, $\mathcal{E}''$ of rank $r$ such that the series
\[
\sum_{\ell} c_{\ell} \frac{d}{d\overline{Z}} \log L_{\mathcal{E}'}((\mathcal{E}' \otimes q^{\nu} \chi') \otimes (\mathcal{E}'' \otimes q^{\nu} \chi''), \lambda_{\ell}, \overline{Z})
\]

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has a pole at $|Z| = q^{-2}$ by a $p$-adic analogue of [ibid., VI.3]. On the other hand, the series

$$\sum_{\mathcal{H}} c_{\mathcal{H}} \frac{d}{dZ} \log L_{U_1}(\mathcal{H} \otimes (q^r \mathcal{E}^r \otimes q^{r''} \mathcal{E}^{r''}), q^{1-r} Z)$$

does not have poles at $|Z| = q^{-2}$ since $\mathcal{H}$ is essential. Now, let $\pi'$ and $\pi''$ be the automorphic cuspidal representations corresponding to $\mathcal{E}'$ and $\mathcal{E}''$. For a locally closed subscheme $Y$ of $(X - N) \times (X - N)$, let us denote by $\mathcal{E}_{s, Y}$ the subset of $(x, s', u') \in |Y| \times \mathbb{N} \times \mathbb{N}$ such that $s \cdot \deg(\infty_x)^{-1} = s' \in \mathbb{N}$ and $s \cdot \deg(0_x)^{-1} = u' \in \mathbb{N}$, and put

$$\text{Ser}_Y(Z) := \sum_{s \geq 1} Z^{s-1} \sum_{(x, s', u') \in \mathcal{E}_{s, Y}} (\deg(\infty_x) \cdot z_s^{s'}(\pi_{\infty_x}) \cdot z_s^{u'}(\pi_{0_x})) \cdot (\deg(0_x) \cdot z_s^{u'}(\pi_{0_x}) \cdot z_s^{u''}(\pi_{0_x})).$$

The series

$$\frac{d}{dZ} \log L_{X-N}(\pi \times \pi'', Z), \quad \frac{d}{dZ} \log L_{X-N}(\pi' \times \pi''', Z)$$

do not have poles at $|Z| \leq q^{-1}$ by [ibid., B.10]. Thus, the “product series” $\text{Ser}_S$ does not have poles at $|Z| \leq q^{-2}$. We claim that the series $\text{Ser}_{U_1}$ does not have poles at $|Z| \leq q^{-2}$ as well. Indeed, putting $W := S_N \setminus U_f$, we have $\text{Ser}_W = \text{Ser}_S - \text{Ser}_{U_1}$. Since $|z_1(\pi)| = |z_1(\pi')| = |z_1(\pi'')| = 1$, we have

$$|\text{Ser}_W(Z)| \leq \sum_{s} |Z|^{s-1} \sum_{(x, s', u') \in \mathcal{E}_W} \deg(\infty_x) \cdot \deg(0_x).$$

Since $W$ is of dimension 1, the latter series converges on $|Z| < q^{-1}$, and thus $\text{Ser}_W$ converges absolutely on the same area, which implies the claim. Combining these, if we put

$$\sum_{s} Z^{s-1} \sum_{(x, s', u') \in \mathcal{E}_{s, U_f}} \deg(\infty_x) \cdot \deg(0_x) \cdot z_s^{s'}(\pi_{\infty_x}) \cdot z_s^{u'}(\pi_{0_x})$$

at the head of the both sides of [4.2.19.1], the left side does not have poles at $|Z| = q^{-2}$ whereas the right side does at $|Z| = q^{-2}$, which is a contradiction. ■

4.2.20 Lemma ([La2, VI.26]). — As an object of $\mathcal{H}_N^r/a^2 \mathcal{I}(S_N)$, we can write $\mathcal{H}_{N, ess}$ as

$$\bigoplus_{\pi \in \{\pi\}_N} \pi \otimes \mathcal{H}_\pi(1 - r)$$

and there exists an open dense subscheme $U_\pi \subset S_N$ for any $\pi \in \{\pi\}_N$ such that $\mathcal{H}_\pi$ is pure of weight 0, and for any closed point $x \in U_\pi$ and $s = \deg(\infty_x) s' = \deg(0_x) u'$, we have

$$\text{Tr}_{\mathcal{H}_\pi}(\text{Frob}_x^{-s} \cdot \deg(x)) = z_s^{s'}(\pi_{\infty_x}) \cdot z_s^{u'}(\pi_{0_x}).$$

Proof. We can copy the proof of [ibid.]. ■

4.2.21 Theorem ([La2, VI.27]). — For any $\pi \in \{\pi\}_N$, we have

$$\mathcal{H}_\pi = q^r \mathcal{E}_\pi \otimes q^{r''} \mathcal{E}''_\pi$$

as objects in $\mathcal{I}(S_N)$, where $\mathcal{E}_\pi$ is an isocrystal of rank $r$ on $X - N$ pure of weight 0 corresponding to $\pi$ in the sense of Langlands.

Proof. Take a closed point $x \in U_\pi$, which places over $(\infty, 0) \in |X - N| \times X - N$. Let $X^0 := X \times X$ be the closed immersion, and let $(X - N)^0 \subset X^0$ be the pull-back of $S_N$ by the closed immersion. Let $\mathcal{E}^0$ be the semi-simplification in $\mathcal{I}((X - N)^0)$ of the pull-back of $\mathcal{H}_\pi$, which is pure of weight 0. Let $\mathcal{H}_0^\pi$ be the pull-back on $S_N \otimes k(0)$. Then there exist characters $\chi_1, \ldots, \chi_r$ (i.e. isocrystals on the point 0) such that two semi-simple objects in $\mathcal{I}(S_N \otimes k(0))$

$$q^r \mathcal{E}_\pi \otimes q^{r''} \mathcal{E}''_\pi = \bigoplus_{i=1}^r (\mathcal{H}_0^\pi \otimes \chi_i)$$

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have the same Frobenius eigenvalues for each closed point of $U_\pi \otimes k(0) \cap S_N \otimes k(0)$. Thus by the Čebotarev density [A3.1] these two objects coincide.

Thus, there exist $E'$ and $E''$ on $X - N \subset X$ which are pure of weight 0 such that $q^r E' \otimes q^{r'} E''$ and $H_\pi$ have the same irreducible subobject. We may assume that $E'$ and $E''$ are irreducible, and since $H_\pi$ is stable under the action of $(\text{Frob}_X \times \text{id})^*$, we may assume that $q^r E' \otimes q^{r'} E''$ is a subobject of $H_\pi$.

Let us show that $E'$ is of finite order. Otherwise, there exists an automorphic cuspidal representation $\pi'$ corresponding to $E'$. As in [ibid.], the series

$$\frac{d}{dZ} \log L_{X-N}(\pi \times \pi', Z), \quad \frac{d}{dZ} \log L_{X-N}(\pi' \times E''', Z)$$

converge absolutely on $|Z| < q^{-1+\epsilon}$ for some $\epsilon > 0$ and on $|Z| < q^{-1}$ respectively, and thus the “product series” converges absolutely on $|Z| < q^{-2+\epsilon}$. On the other hand, consider the series

\[(4.2.21.1) \quad \frac{d}{dZ} \log L_{U_\pi} (H_\pi \otimes (q^r \sigma' \otimes q^{r'} \sigma'''))\]

Let $C := V \times V \setminus U_\pi$. The difference with the product series is nothing but

\[(4.2.21.2) \quad \sum_{s \geq 1} Z^{s-1} \sum_{(x, x', w) \in \mathcal{C}_s} (\deg(\infty_x) \cdot z^{-s}(\pi_{\infty_x}) \cdot z^{s}(\pi'_{\infty_x})) \cdot (\deg(0_x) \cdot z^{s}(\pi_0) \cdot z^{s}(\pi''_0))\]

where $\mathcal{C}_s \subset |C| \times |N| \times |N|$ such that $s = \deg(\infty_x)^{-1} = s' \in \mathbb{N}$ and $s = \deg(0_x)^{-1} = u' \in \mathbb{N}$. Since $C$ is of dimension 1 and the complex absolute value of $z_{\infty}(\pi_x)$, $z_{\infty}(\pi'')$ are 0, we get that the series (4.2.21.1) converge in $|Z| < q^{-1}$. Thus, (4.2.21.1) converges on $|Z| < q^{-2+\epsilon}$. However, since by [ibid., VI.3], it should have a pole at $|Z| = q^{-2}$, which is a contradiction.

This shows that the rank of $E'$ is $\geq r$. By symmetry, the rank of $E''$ is $\geq r$ as well. Since the rank of $H_\pi$ is $r^2$, we get that the ranks of $E'$ and $E''$ are $r$, and $H_\pi \cong q^r E' \otimes q^{r'} E''$. By the induction hypothesis, we get that there exist cuspidal automorphic representations $\pi', \pi''$ of $\text{GL}_r(A)$ corresponding to $E'$ and $E''$. Now, since $\dim(V) = 1$ and $|z_{\infty}(\pi)| = |z_{\infty}(\pi')| = |z_{\infty}(\pi'')| = 1$, the rational functions

$$\frac{d}{dZ} \log L_{X-N}(\pi \times \pi', Z), \quad \frac{d}{dZ} \log L_{X-N}(\pi'' \times \pi''', Z)$$

do not have poles at $|Z| < q^{-1}$. This implies that they have poles at $Z = q^{-1-s}$ and $q^{-1+s}$ respectively for some $s \in \mathbb{C}$ such that $\text{Re}(s) = 0$. Otherwise, the series

$$\frac{d}{dZ} \log L_{U_\pi} (H_\pi \otimes (q^r \sigma' \otimes q^{r'} \sigma''')) = \frac{d}{dZ} \log L_{U_\pi} (H_\pi \otimes \pi')$$

would not have pole at $q^{-2}$. This shows that $E' \cong \pi'$, and $E_\pi = E'(s)$ as requested, and $H_\pi \cong q^r E'_\pi \otimes q^{r'} E''_\pi$. ■

\[4.2.22. \quad \text{Conclusion of the proof}\]

This theorem shows that (4.2.8) holds for $n = r + 1$. By using Theorem 4.1.3, (4.2.11) also holds, and (S)$_{r+1}$ is shown. Thus, we conclude the proof of Theorem 4.1.3. ■

\[4.3. \quad \text{A few applications}\]

To conclude this paper, let us collect some applications of the Langlands correspondence.

\[4.3.1 \quad \text{Theorem (}\text{[De1], 1.2.10 (vi)})]. \quad \text{Let } X \text{ be a smooth curve over a finite field } k \text{ of characteristic } p. \text{ Let } \ell \text{ be a prime number different from } p. \text{ Then for any irreducible smooth } \mathcal{O}_\ell\text{-sheaf whose determinant is of finite order, there exists a "petit camarade cristallin".}\]

\textbf{Proof.} Use Lafforgue’s result and Theorem 4.1.3. ■

\[4.3.2 \quad \text{Theorem.} \quad \text{Assume we are in the situation of } 4.1.3 \text{ Let } X \text{ be a scheme of finite type over } k. \text{ Then any complex in } D^b_{\text{hol}}(X_0) \text{ is mixed.}\]

\textbf{Proof.} See [A2, 6.3]. ■
4.3.3 Corollary. — The Čebotarev density theorem holds for overconvergent $F$-isocrystals.

Proof. Apply Proposition A.3.1 and Theorem 1.4.2. ■

4.3.4 Theorem. — Let $X$ be a smooth scheme of finite type over a finite field $k$. Let $E$ be an irreducible $\mathbb{Q}_p$-isocrystal on $X_0$ such that the determinant is of finite order.

(i) There exists a number field $E/\mathbb{Q}$ such that, for any $x \in |X|$, all the coefficients of the Frobenius eigenpolynomial of $E$ at $x$ is in $E$.

(ii) For any prime $\ell \neq p$, there exists an $\overline{\mathbb{Q}}_\ell$-adic smooth sheaf $F$ corresponding to $E$ such that the sets of Frobenius eigenvalues coincide for any closed point of $X$.

Proof. Let us show (i). We denote by $\mathcal{L}_r(X)$ the set of $\mathbb{Q}_p$-isocrystals of rank $r$ on $X$ up to isomorphism and semi-simplification. We use the notation of [EK] §2. We use the induction on the dimension of $X$. Let $\overline{X}$ be a normal compactification of $X$, and $\overline{X}\setminus X$ is a Cartier divisor. Then there exists a map $\mathcal{L}_r(X) \to \mathcal{V}_r(X)$ using the Langlands correspondence. This is injective by the Čebotarev density. We show the following: Let $E \in \mathcal{L}_r(X)$. Then there exists an dense open subsheaf $U \subset X$ and Cartier divisor $D$ of $\overline{X}$ contained in $\overline{X}\setminus U$ such that $E|_U \in \mathcal{V}_r(U,D)$ where $E|_U$ is the restriction of $E$ on $U$. Once this is shown, we get (i) of the theorem for $E|_U$ by [EK] 8.2, and by induction hypothesis, we conclude.

When $\overline{X}$ is proper smooth, $\overline{X}\setminus X$ is a simple normal crossing divisor, and $E$ is log-extendable to $\overline{X}$, then we may take $D = 0$. Indeed, for a smooth curve $U$, let $E$ be an $\overline{\mathbb{Q}}_p$-isocrystal whose determinant is finite and $\mathcal{F}$ be its $\ell$-adic companion. Then the ramification of $E$ and $\mathcal{F}$ at the boundary are the same by Theorem 1.1.32. In general, take a semi-stable reduction $\overline{V} \to \overline{X}$ of $E$ (cf. [Ke3]). We take $U \subset X$ so that $p: V := U \times_{\overline{X}} \overline{Y} \to U$ is finite étale. There exists a Cartier divisor $D$ such that the ramification of $p_*\overline{\mathbb{Q}}_\ell$ is in $\mathcal{V}_r(U,D)$ where $d = \deg(V/U)$. Then we can check easily that $E \in \mathcal{V}_r(X,D)$.

For (ii), copy the proof of Drinfeld [Dr] §2.3.

Remark. — We think that some further study on overconvergent $F$-isocrystals is needed to show the existence of crystalline companions on smooth schemes of finite type over $k$.

A. Appendix

A.1. Beilinson-Drinfeld’s gluing of derived category

In this section, we recall the construction and some results of Beilinson and Drinfeld [BD] 7.4 shortly, which is used in the main text.

A.1.1. Let $\mathcal{M} \to \Delta^+$ be a cofibered category such that its fiber $\mathcal{M}_i$ over $[i] \in \Delta^+$ is an abelian category, and for $\phi: [i] \to [j]$, the push-forward $\phi_*$ is exact. We denote by $\mathcal{M}_{\text{tot}}$ the abelian category of cartesian objects; the category of collections $\{\mathcal{M}_n, \alpha_\phi\}$ such that $\mathcal{M}_n \in \mathcal{M}_n$ and for $\phi: [i] \to [j]$, $\alpha_\phi: \mathcal{M}_i \to \mathcal{M}_j$ satisfying the cocycle condition. Now, we want to construct a suitable triangulated category associated to $\mathcal{M}$ whose heart is $\mathcal{M}_{\text{tot}}$. For this, we consider the category $\mathcal{S}(\mathcal{M})$. The objects consist of collections $\{\mathcal{M}_n, \alpha_\phi\}$ where $\mathcal{M}_n \in \mathcal{M}_n$, and for $\phi: [i] \to [j]$, $\alpha_\phi: \mathcal{M}_i \to \mathcal{M}_j$, satisfying the condition that $\alpha_{\phi_\psi} = \alpha_{\psi} \circ \phi_*(\alpha_\phi)$ for composable morphisms $\phi$ and $\psi$ in $\Delta^+$, and $\alpha_{id} = id$. We put $\mathcal{S}(\mathcal{M}) := (\mathcal{S}(\mathcal{M})^\circ)^\circ$. A profound observation of Beilinson and Drinfeld is that there are functors

$c_+: C(\mathcal{S}(\mathcal{M})) \to C(\mathcal{S}(\mathcal{M}))$,

$c_-: C(\mathcal{S}(\mathcal{M})) \to C(\mathcal{S}(\mathcal{M}))$

such that $(c_+, c_-)$ is an adjoint pair, and the adjunction homomorphisms $c_+c_- \to id$ and $id \to c_-c_+$ are quasi-isomorphisms (cf. [BD] 7.4.4). With these functors, we are able to identify $D(\mathcal{S}(\mathcal{M}))$ and $D(\mathcal{S}(\mathcal{M}))$. Now, let $C_{\text{tot}} \subset C(\mathcal{S}(\mathcal{M}))$ be the full subcategory consisting of complexes $M$ such that $\mathcal{S}(\mathcal{M})(M) \in \mathcal{M}_{\text{tot}}$. We denote by $K_{\text{tot}}(\mathcal{M})$ and $D_{\text{tot}}(\mathcal{M})$ the corresponding homotopy and derived categories. By means of $c_\pm$, we are able to identify $D_{\text{tot}}(\mathcal{M})$, and denote it by $D_{\text{tot}}(\mathcal{M})$. The functors $\mathcal{S}(\mathcal{M}): D_{\text{tot}}(\mathcal{M}) \to \mathcal{M}_{\text{tot}}$ induces $D_{\text{tot}}(\mathcal{M}) \to D_{\text{tot}}(\mathcal{M})$.

A.1.2. Now, an important aspect of the theory is the existence of a spectral sequence connecting $\{\mathcal{M}_n\}$ and $\mathcal{M}$. For $N \in D^-(\mathcal{S}(\mathcal{M}))$ and $M \in D^+(\mathcal{S}(\mathcal{M}))$, we have the following spectral sequence by [BD] 7.4.8:

(A.1.2.1) $E^{p,q}_1 = \text{Ext}^q_{\mathcal{M}_p}(N_p, M_p) \Rightarrow \text{Hom}_{D(\mathcal{M})}(c_+(N), M[p+q]) \cong \text{Hom}_{D(\mathcal{M})}(N, c_-(M)[p+q])$. 83
Remark. — Since the proof of [ibid.] is rather sketchy, it might be hard to follow their argument in some cases. Let us add a short explanation. When sec\(_+(\mathcal{M})\) is enough injectives, then the functor \(\text{Hom}(N, -): K^+(\text{sec}_+(\mathcal{M})) \to DF\), where \(DF\) denotes the derived category of filtered modules, can be derived in a usual way and we get the spectral sequence as written in [ibid.]. However, there might be a situation that the functor does not admit a right derived functor. Even in this case, we can define the derived functor \(\mathbb{R}\text{Hom}(N, -): D^+(\text{sec}_+(\mathcal{M})) \to \text{Ind}(DF)\) as in [SGA 4, XVII, 1.2]. We have a functor

\[
\mathcal{F}^i: \text{Ind}(DF) \xrightarrow{\mathcal{F}^i} \text{Ind}(F\text{Ab}) \xrightarrow{\lim} F\text{Ab},
\]

and with which, we have \(\mathcal{F}^i\text{gr}^n\mathbb{R}\text{Hom}(N, M) \cong \text{Hom}_{\text{DF}(\mathcal{M}_n)}(N_n, M_n[n - i])\). This follows from the fact that we have an isomorphism \(\text{gr}^n\text{Hom}(N, M) \cong \text{Hom}_{\mathcal{K}(\mathcal{M}_n)}(N_n, M_n)[-n]\) by construction of the functor \(\text{Hom}(N, -)\), and the functor \(M \mapsto M_n\) has an exact right adjoint as in [EGA I, 5.5.1 (vi)] which implies the existence of \(g\) and \(h\) in [ibid.]. Since the category of spectral sequences of abelian groups admits inductive limits, we get the desired result.

A.2. Some properties of algebraic stacks

A.2.1 Lemma. — Let \(f: \mathcal{X} \to \mathcal{Y}\) be a gerb-like morphism [Beh, 5.1.3] such that the structural group is flat. Then there exists a presentation \(Y \to \mathcal{Y}\) such that \(\mathcal{X} \times_{\mathcal{Y}} Y \to Y\) is a neutral gerb.

Proof. First, we note that \(f\) is smooth surjective. Indeed, since the verification is fpqc-local, we may assume that \(f\) is neutral, and thus \(\mathcal{Y} =: Y\) is a scheme and \(\mathcal{X} = B(G/Y)\). Since \(G\) is assumed to be flat, \(f\) is smooth by [Beh, 5.1.2].

Let \(P: Y \to \mathcal{X}\) be a presentation, and consider the smooth morphism \(Q := f \circ P: Y \to \mathcal{Y}\), which is a presentation of \(\mathcal{Y}\) since \(f\) is smooth surjective. We have the morphism \((P, \text{id})\): \(Y \to \mathcal{X} \times_{\mathcal{Y}} Y\). This defines a section of the second projection \(\mathcal{X} \times_{\mathcal{Y}} Y \to Y\). \qed

A.2.2 Lemma. — Let \(f: \mathcal{X} \to \mathcal{Y}\) be a representable morphism of algebraic stacks over an integral scheme \(S\). For the generic point \(\eta \in S\), if \(f_\eta\) is separated, then there exists an open subscheme \(U \subset \mathcal{S}\) such that \(f_U: \mathcal{X} \times_\mathcal{Y} U \to \mathcal{Y}\) is separated.

Proof. Let \(\mathcal{Y} \to \mathcal{Y}\) be a presentation, and \(\mathcal{X} \to \mathcal{X}\) be the induced presentation. Let \(f': \mathcal{X} \to \mathcal{Y}\) be the induced morphism. The morphism \(f\) being separated is equivalent to \(f'\) being separated, and thus by [EGA VI, 8.10.5], the lemma follows. \qed

A.2.3 Lemma. — Let \(f: \mathcal{X} \to \mathcal{Y}\) be a representable morphism of locally noetherian algebraic stacks. Then there exists an open dense substack \(U\) of \(\mathcal{X}\) such that \(f|_U\) is separated.

Proof. By [EGA I, 5.5.1 (vi)], we may assume that \(\mathcal{X}\) and \(\mathcal{Y}\) are reduced. By shrinking \(\mathcal{X}\) and \(\mathcal{Y}\), we may assume that \(\mathcal{X} \to \mathcal{X}\) and \(\mathcal{Y} \to \mathcal{Y}\) are gerbs over algebraic spaces by [LM, 11.5]. By shrinking \(\mathcal{X}\), we may assume that \(\mathcal{X}\) is a scheme and separated. Consider the following diagram:

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\alpha} & \mathcal{X}' \\
\downarrow & & \downarrow \\
\mathcal{Y} & & \mathcal{Y}. \\
\end{array}
\]

Note that \(\alpha\) is representable by [LM, 3.12 (c)]. Let \(\eta\) be a generic point of \(\mathcal{X}\). By Lemma A.2.2, it suffices to show that \(\alpha_\eta\) is separated. So the statement is reduced to the following special case of the lemma:

Claim. — Let \(\mathcal{X}\) and \(\mathcal{X}'\) be a gerbs over \(\text{Spec}(K)\), and let \(\alpha: \mathcal{X} \to \mathcal{X}'\) be a representable morphism. Then this is separated.

Proof. Since there exists a scheme \(X \to \text{Spec}(K)\) such that \(\mathcal{X}\) and \(\mathcal{X}'\) are neutral over \(X\), by taking a closed point of \(X\), \(\mathcal{X}\) and \(\mathcal{X}'\) are neutral over a finite extension of \(K\). Since the claim is stable under finite extension, that \(\mathcal{X}\) and \(\mathcal{X}'\) are neutral gerbs over \(\text{Spec}(K)\). Thus, by taking the automorphism groups.
G and G' of X(K) and X'(K), we have a homomorphism of K-group spaces ρ: G → G' over X, which induces α. We have the cartesian diagram

\[
\begin{array}{c}
G \\
\underline{\rho} \\
\downarrow \\
BG \\
\Delta \\
BG \times_{BG'} BG.
\end{array}
\]

Since the diagonal morphism Δ is quasi-compact by [LM 7.7], ρ is quasi-compact. By [EGA IV, 6.7], ρ decomposes as G → G/N → G'. This induces a morphism BG → B(G/N) → BG'. Since α is assumed to be representable, the morphism BG → B(G/N) is representable as well. This can only happen when N is trivial. Thus, ρ is a closed immersion by the same corollary of [EGA 3]. This shows that Δ is a closed immersion, and thus α is separated. □

A.2.4 Lemma. — Let f: X → Y be a morphism locally of finite type between reduced algebraic stacks such that

\[\dim(\text{Aut}_Y X) = \dim(Y) - \dim(X).\]

Then locally on X and Y, f can be factorized as X \xrightarrow{p} X' \xrightarrow{g} Z \xrightarrow{h} Y where p is a gerb-like morphism with the structure group space \text{Aut}_Y X, g is a representable universal homeomorphism, and h is a representable finite étale morphism.

**Proof.** Locally on X, f factors as X \xrightarrow{p} X' \xrightarrow{α} Y such that p is gerb-like and α is a representable morphism [Bell 5.1.13, 5.1.14]. By the assumption on the dimension, α is representable quasi-finite morphism. By shrinking X, by using Lemma A.2.2 we may assume that α is separated. By using Zariski's main theorem (cf. [LM 16.5]), by shrinking if necessary, we may assume that α is a finite morphism.

Let f: X → Y be a finite morphism between integral schemes such that X is normal. The finite extension K(X)/K(Y) of fields can be factorized canonically as K(X)/K(M) such that K(X)/M is purely inseparable and M/K(Y) is separable. Let Z be the normalization of Y in Spec(M). The morphism f factors as composition of finite morphisms X → Z → Y. This construction is compatible with smooth base change Y' → Y by [LM 16.2]. Thus, given a finite morphism X → Y of reduced algebraic stacks, by shrinking X and Y if necessarily, we have a factorization X → Z → Y such that the first morphism is generically purely inseparable and the second is generically finite étale by [LM 14.2.4].

Apply this factorization to α, and we get a factorization X' \xrightarrow{p} Z \xrightarrow{h} Y satisfying the condition above. Take a presentation Z → Z, and let X' → X' be the pull-back. Then by construction, \tilde{g}: X' → Z is generically purely inseparable. By [EGA IV, 1.8.7], by shrinking Z, we may assume that \tilde{g} is radicial surjective for any fiber of Z. Thus \tilde{g} is radicial surjective as well, and since moreover \tilde{g} is finite, it is a universal homeomorphism by [EGA IV, 2.4.5]. By replacing Z by the image of Z and X' by the pull-back of newly constructed Z, g can be made universally homeomorphic. By removing the ramification locus of h from Y, we may assume that h is finite étale. □

A.3. Čebotarev density

The Čebotarev density theorem for curves and mixed isocrystals is proven in [A2]. We need the Čebotarev density for surfaces and mixed isocrystals, which we show in this appendix. The following simple proof is due to N. Tsuzuki. We consider the situation of [L12] and let L/K be an algebraic extension with σ_L = id.

A.3.1 Proposition. — Let X be a smooth variety over a finite field k. Let ℰ and ℰ' be ϵ-mixed overconvergent F-isocrystals in Isoc^1(X_0/L) such that the sets of Frobenius eigenvalues are the same for any closed point of X. Then ℰ^ss = ℰ'^ss where the semi-simplification is taken in Isoc^1(X_0/L).

**Proof.** In this proof, we denote Isoc^1(X_0/L) simply by Isoc^1(X_0). Since we have weight filtration on ℰ and ℰ' by [AC1 4.3.4], we may assume that ℰ and ℰ' are ϵ-pure. It suffices to show that Hom_{Isoc^1(X_0)}(ℰ', ℰ) ≠ 0. Indeed, let φ: ℰ' → ℰ be a non-zero homomorphism. This implies that there exist submodules
Thus, $L$ implies that the characteristic polynomial of the Frobenius automorphism acting on $A$.4. Gabber-Fujiwara’s $\ell$-independence results (cf. [F]) for admissible stacks. For a category $\mathcal{C}$, we denote by $[\mathcal{C}]$ the set of isomorphism classes of $\mathcal{C}$.

A.4. Gabber-Fujiwara’s $\ell$-independence

We show Gabber-Fujiwara’s $\ell$-independence results (cf. [F]) for admissible stacks. For a category $\mathcal{C}$, we denote by $[\mathcal{C}]$ the set of isomorphism classes of $\mathcal{C}$.

A.4.1 Theorem (Trace formula). — Let $\mathcal{X}$ be a c-admissible stack over finite field $\mathbb{F}_q$. Let $\mathcal{M}$ be a complex in $D^{b}_{hol}(\mathcal{X}/\mathbb{Q}_p)$. For $x \in [\mathcal{X}(\mathbb{F}_q)]$, we denote by $i_x : \text{Spec}(\mathbb{F}_q) \to \mathcal{X}$ the corresponding morphism. Then, for any $n > 0$, we have

$$\sum_{i \in \mathbb{Z}} (-1)^i \cdot \text{Tr}(F^n; H^i_c(\mathcal{X}, \mathcal{M})) = \sum_{x \in [\mathcal{X}(\mathbb{F}_q)]} \frac{1}{\# \text{Aut}(x)} \cdot \text{Tr}(F^n; i^*_x(\mathcal{M})).$$

Remark. — Note that both sides of the equality are finite sums, since we are dealing with c-admissible stacks, contrary to the case of more general algebraic stacks. This proves the compatibility with the convergence issues as in [Beh], which makes it much easier to formulate and prove.

Proof. Since we can prove similarly to [La2] A.14 or [Beh] 6.4.10, we only sketch. Let us denote the right hand side of the equality by $L(\mathcal{X}, \mathcal{M})$. For a morphism $f : \mathcal{X} \to \mathcal{Y}$ of c-admissible stacks, it suffices to show the equality $L(\mathcal{X}, \mathcal{M}) = L(\mathcal{Y}, f_*(\mathcal{M}))$ since the theorem is the particular case where $\mathcal{Y} = \text{Spec}(\mathbb{F}_q)$. When $f$ is a morphism between flat schemes, then this equality is already known by using [EL]. Now, by localization triangle, the verification is local with respect to $\mathcal{X}$. By using [LM] 11.5 and some standard dévissage argument, it suffices to treat the case where $f$ is gerb-like. By definition of $L(-, -)$ combining with [Beh] 6.4.2, it is reduced to showing the theorem in the case $\mathcal{X} = BG$ with a finite flat group scheme $G$ over Spec($\mathbb{F}_q$) and $\mathcal{M}$ is $\mathbb{Q}_p$. Since the morphism $BG_{\text{red}} \to BG$ is a representable universal homeomorphism, we have $H^i(BG, \mathbb{Q}_p) \simeq H^i(BG_{\text{red}}, \mathbb{Q}_p)$, and we may assume $G$ to be smooth. By considering the universal torsor $\text{Spec}(\mathbb{F}_q) \to BG$, which is finite since $G$ is, we get $H^i(BG, \mathbb{Q}_p) = 0$ for $i \neq 0$. The calculation of $H^0$ is left to the reader.

A.4.2 Definition. — Let $\mathcal{X}$ be an algebraic stack over $\mathbb{F}_q$, and $\mathcal{E}$ (resp. $\mathcal{F}$) be an object in $D^b_{hol}(\mathcal{X}/\mathbb{Q}_p)$ (resp. $D^b_{hol}(\mathcal{X}/\mathbb{Q}_p)$). For $x \in [\mathcal{X}(\mathbb{F}_q)]$, we denote by $i_x : \text{Spec}(\mathbb{F}_q) \to \mathcal{X}$ and $\rho : \text{Spec}(\mathbb{F}_q) \to \text{Spec}(\mathbb{F}_q)$ the canonical morphisms. We say that $\mathcal{E}$ and $\mathcal{F}$ are compatible if for any point $x \in [\mathcal{X}(\mathbb{F}_q)]$, the Frobenius trace of $\rho_+ \circ i^*_x(\mathcal{E})$ and $\rho_+ \circ i^*_x(\mathcal{F})$ are equal.

A.4.3 Lemma. — The couple $(\mathcal{E}, \mathcal{F})$ are compatible if and only if for any $X \in \mathcal{X}_{\text{sm}}$, the pull-back $\mathcal{E}_X$ and $\mathcal{F}_X$ are compatible.

Proof. Use [LM] 6.3.
A.4.4 Theorem. — (i) Let \( f : \mathcal{X} \to \mathcal{Y} \) be a morphism between c-admissible stacks. Then \( f_+^*, f_!^*, f^!_+^* \), \( \mathcal{D} \), and \( \otimes \) preserves compatible system.

(ii) When \( j : \mathcal{U} \to \mathcal{X} \) is an immersion of c-admissible stacks, \( j_+^* \) preserves compatible system.

Proof. By Lemma [A.4.3] the theorem for \( f^!_+^*, \mathcal{D}, j_+^* \) follows from [AC1 4.3.11]. We only need to show the theorem for \( f_!^* \). For this, use the trace formula [A.4.1].

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