The role of special conformal Killing tensors in general relativity

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Abstract. Special conformal Killing tensors have a number of interesting properties, an overview of which is given in the Riemannian case. Some of these properties remain valid in pseudo-Riemannian manifolds. Presently, we are investigating the latter problem, where, in order to obtain a better insight, we first examine some specific cases, such as special conformal Killing tensors in conformally flat or Petrov type D space-times.

1. Introduction
A symmetric type $(0, 2)$ tensor field $L$ is a special conformal Killing tensor of a metric $g$ if there exists a 1-form $\lambda$ such that

$$L_{ij|k} = \frac{1}{2} (g_{jk} \lambda_i + g_{ik} \lambda_j),$$

where the rule denotes covariant differentiation with respect to the Levi-Civita connection of $g$. These solutions satisfy

$$L_{ij|k} + L_{jk|i} + L_{ki|j} = \lambda_i g_{jk} + \lambda_j g_{ki} + \lambda_k g_{ij},$$

and are therefore conformal Killing tensors. Moreover, taking the trace of equation (1) on $i$ and $j$ shows that $\lambda = d(\text{tr} L)$, so $L$ is of gradient type, and $\lambda$ is in fact defined by the equation (1).

Special conformal Killing tensors have some very interesting properties. In the first place they can be used to construct Killing tensors. Furthermore the Nijenhuis torsion of any special conformal Killing tensor vanishes (actually we should refer here to the type $(1, 1)$ tensor, obtained by raising an index with the metric):

$$L^k_i \left( L^l_{i|j} - L^l_{j|i} \right) - L^l_j L^k_{i|l} + L^l_i L^k_{j|l} = 0.$$}

This has significant consequences for the eigenfunctions of $L$, when they are functionally independent. At each point $x$ of the manifold $M$, $L(x)$ is a linear transformation of the tangent space $T_x M$, which is symmetric with respect to the scalar product defined by the metric. There is thus an orthogonal basis of $T_x M$ with respect to which $L(x)$ is diagonal. The diagonal entries are then the eigenvalues of $L$. If $L$ has functionally independent eigenvalues, $n$ functions $u^1, u^2, \ldots, u^n$ exist, such that at each point $x$ of the manifold, $u^i(x)$ is an eigenvalue of $L(x)$ for each $i$, and
the Jacobian matrix \((\partial u^i/\partial x^j)\) is everywhere non-singular. The \(u^i\) may then be taken as local coordinates. As a consequence of the fact that its torsion vanishes, \(L\) takes the form

\[
L = \sum_{i=1}^{n} u^i \frac{\partial}{\partial u^i} \otimes d\bar{u}^i. \tag{4}
\]

As they are orthogonal coordinates, \(g_{ij} = 0\) for \(i \neq j\).

2. Constructing Killing tensors

Vector fields \(K^\mu\) generating isometries of a metric \(g\) are known as Killing vector fields. Killing tensors are a generalisation of Killing vectors. They are linked with integrals of geodesic motion and the theory of separation of variables. Killing tensors are symmetric type \((0,n)\) tensors which satisfy the Killing equation

\[
\nabla_{(\sigma} K_{\mu_1...\mu_n)} = 0. \tag{5}
\]

This equation is hard to solve, therefore methods are being used to construct Killing tensors from vectors and other tensors with special properties. We will only examine valence 2 Killing tensors.

Let us consider a (non-singular) special conformal Killing tensor \(L\) of metric \(g\) and take its cofactor tensor \(A\), defined by

\[
A_{ij} = (\det L) g_{il}. \tag{6}
\]

Then it is easy to see that \(A\) is a Killing tensor of \(g\). To prove this take the covariant derivative of the equation (6) above, and use the SCK-equation (1) and the fact that \(L\) has vanishing torsion. Then

\[
A_{ij[k]} = (\det L) \left( L_{ij} \bar{L}_{kl} - \frac{1}{2} \bar{L}_{ik} \bar{L}_{jl} - \frac{1}{2} \bar{L}_{il} \bar{L}_{jk} \right) \lambda^l, \tag{7}
\]

where \(\bar{L}\) is the inverse of \(L\). This tensor clearly satisfies the Killing equation

\[
A_{(ij|k)} = 0. \tag{8}
\]

Also note that this Killing tensor has the same eigenvectors as \(L\).

The special conformal Killing tensor equations (1) are linear, and obviously have the solution \(L_{ij} = kg_{ij}\) for any constant \(k\). So, the cofactor tensor of \(aL_1 + bL_2\) is a Killing tensor for every constant \(a, b\) (or at least those for which \(aL_1 + bL_2\) is non-singular). In particular, if \(L_{ij}\) is a special conformal Killing tensor, so is \(L_{ij} + kg_{ij}\). By taking the coefficients of powers of \(k\) in the cofactor tensor of this special conformal Killing tensor, we obtain \(n\) valence 2 Killing tensors, simultaneously diagonal with \(L_{ij}\), one of which is the metric itself. When the special conformal Killing tensor has simple eigenvalues, these Killing tensors are independent. Since the eigenvectors of \(aL + kg\) are the same as the eigenvectors of \(L\), these Killing tensors have the same eigenvectors.

It is important to note that the space of solutions of (1) is a finite-dimensional vector space. Therefore only a limited number of Killing tensors can be generated from special conformal Killing tensors. The maximal dimension of the space of solutions of (1), \(\frac{1}{2}(n+1)(n+2)\), is achieved if and only if the space is of constant curvature \([1]\).

3. Separation of variables

Related to the Killing tensor problem, is the question of separation of variables of classical differential equations. When a manifold admits a special conformal Killing tensor, the eigenfunctions of which are simple and functionally independent, these eigenfunctions are orthogonal separation coordinates for classical differential equations, such as the Hamilton-Jacobi equations for the geodesics of the manifold. To prove this, one has to use the fact
that the Nijenhuis torsion of the special conformal Killing tensor vanishes. As we said in the introduction, the eigenvalues $u^i$ of $L$ can be used, when simple and functionally independent, as local orthogonal coordinates. These coordinates then are the orthogonal separation coordinates for the Hamilton-Jacobi equation for the geodesics of the manifold.

4. **Projective equivalence**

Two metrics are said to be projectively equivalent if they have the same geodesics up to reparametrization. This occurs if and only if a certain metric formed out of the two metric tensors is a special conformal Killing tensor.

Let $h$ and $g$ be two projectively equivalent metrics, then

$$L_{ij} = \left( \frac{\det h}{\det g} \right)^{1/(n+1)} g_{ik}g_{jl}h^{kl}$$

(9)

defines a special conformal Killing tensor $L$ of metric $g$. Conversely, given $L$, a (non-singular) special conformal Killing tensor of $g$

$$h_{ij} = (\det L)^{-1} g_{ik}g_{jl}\tilde{L}^{kl}$$

(10)

defines a metric projectively equivalent to $g$.

As mentioned before, the special conformal Killing tensors form a linear space, so if $g$ admits an equivalent metric $h$, it admits a one-parameter family of equivalent metrics $h^k$. Indeed, starting with $L$, the special conformal Killing tensor generating $h$, we can consider all equivalent metrics determined by the pencil $L_k = L + k g$ (the constant $-k$ must not coincide with the eigenvalues of $L$).

Remark: if $n$ is even, $L$ in (9) is always well defined. If $n$ is odd and $\det h/\det g$ is negative, we can consider $-h$ instead of $h$. This changes the sign of $\det h$ and makes $L$ well defined.

For proofs and more information on special conformal Killing tensors in connection with projective equivalence, we refer the reader to [2]

5. **Example: aligned Petrov type D and future work**

We have examined the Petrov type D space-times, which admit an aligned special conformal Killing tensor, other than the trivial one (a constant times the metric). By aligned we mean that $L$ has only a contribution along the principal null-directions of the tetrad. We choose the tetrad such that $\Psi_i = 0$, $i \neq 2$. It can be shown that the only possible special conformal Killing tensor in this case is a constant Killing tensor. Furthermore, we find that $\Psi_2 = -\frac{R}{72}$, $R$ being the Ricci constant, $\Phi_{ij} = 0$ except $\Phi_{11}$. If $R$ is a constant, these solutions belong to the Bertotti-Robinson family.

Presently we are examining which Petrov type D metrics allow non-aligned special conformal Killing tensors. Also the cases of Petrov type N and conformally flat spaces are being looked at.

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[1] Crampin M 2007 *Differential geometric methods in mechanics and field theory* ed F Cantrijn et al (Ghent: Academia press) pp 57-70

[2] Bolsinov A V and Matveev V S 2003 *Jour. geometry and physics* 44 489