ON THE FREE FRACTIONAL WISHART PROCESS.

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ABSTRACT. We investigate the process of eigenvalues of a fractional Wishart process defined as \( N = B^* B \), where \( B \) is a matrix fractional Brownian motion recently studied in [20]. Using stochastic calculus with respect to the Young integral we show that the eigenvalues do not collide at any time with probability one. When the matrix process \( B \) has entries given by independent fractional Brownian motions with Hurst parameter \( H \in (1/2, 1) \) we derive a stochastic differential equation in a Malliavin calculus sense for the eigenvalues of the corresponding fractional Wishart process. Finally a functional limit theorem for the empirical measure-valued process of eigenvalues of a fractional Wishart process is obtained. The limit is characterized and referred to as the free fractional Wishart process which constitutes the family of fractional dilations of the free Poisson distribution.

Key words and phrases: Fractional Wishart matrix process, measure valued process, free probability, Young integral, fractional calculus.

1. INTRODUCTION.

In this paper, we make a systematic study of the dynamics and the limiting non-commutative distribution of the eigenvalue process of the fractional Wishart matrix process. More specifically, let \( H \in (0, 1) \), \( n, p \geq 1 \) and \( B = \{b_{ij}(t), t \geq 0\}, 1 \leq i \leq p, 1 \leq j \leq n \) be a set of \( p \times n \) independent one-dimensional fractional Brownian motions with the same Hurst parameter \( H \). That is, each \( b_{ij} \) is a zero mean Gaussian process with covariance

\[
E[b_{ij}(t)b_{ij}(s)] = \frac{1}{2} \left(t^{2H} + s^{2H} - |t-s|^{2H}\right).
\]

Similarly as in [20], we introduce \( (N(t), t \geq 0) \), the matrix fractional Brownian motion process with parameter \( H \) whose components satisfy \( N_{ij}(t) = b_{ij}(t), \) for \( t \geq 0 \).

The fractional Wishart process is the nonnegative definite \( n \times n \) matrix process defined by \( X = N^* N \), where \( N^* \) denotes the transpose of the matrix \( N \). Let \( (\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t), t \geq 0) \) be the \( n \)-dimensional stochastic process of eigenvalues of \( X \) and consider the empirical spectral process of the eigenvalues \( \lambda_1^{(n)}(t) \geq \lambda_2^{(n)}(t) \geq \cdots \geq \lambda_n^{(n)}(t) \geq 0 \) of \( X^{(n)} = n^{-1} X \), i.e.

\[
\mu_i^{(n)} = \frac{1}{n} \sum_{j=1}^{n} \delta_{\lambda_j^{(n)}(t)}, \quad t \geq 0.
\]
Different aspects of dynamics and asymptotics of the spectral process have been considered by several authors in the case $H = 1/2$ of the classical Wishart process. In this case, for $n \geq 1$ fixed, Bru [3] considered the dynamics and non-colliding phenomena of the eigenvalue process, proving that the spectral process is an $n$-dimensional diffusion given by the system of non-smooth diffusion equations

$$\lambda_i(t) = \lambda_i(0) + 2 \int_0^t \sqrt{\lambda_i(s)} \cdot d\nu_i(s) + \int_0^t \left( p + \sum_{i \neq j} \frac{\lambda_i(s) + \lambda_j(s)}{\lambda_i(s) - \lambda_j(s)} \right) ds,$$

where $\nu_i$ are independent Brownian motions for $i = 1, \ldots, n$, and “$\cdot$” denotes the Itô stochastic integral. Moreover, Bru [3] also showed that if $\lambda_1^{(n)}(0) \geq \cdots \geq \lambda_n^{(n)}(0)$, then the eigenvalues do not collide at any time a.s., in other words

$$\mathbb{P}(\lambda_1(t) > \cdots > \lambda_n(t), \forall t > 0) = 1.$$  

The main tool to prove (1.2) is Itô’s formula for matrix-valued semimartingales and for (1.3) a McKean type argument in classical stochastic calculus.

Still in the classical case $H = 1/2$, for fixed $t > 0$, the asymptotic distribution of $\mu_1^{(n)}$ is given by the classical Marchenko and Pastur pioneering work [14]. Namely, recall that the free Poisson distribution (or Marchenko-Pastur distribution) $\mu_{f,p}^c$, $c > 0$ is the probability measure on $\mathbb{R}_+$ defined by

$$\mu_{f,p}^c(dx) = \begin{cases} 
\nu_c(dx), & c \geq 1, \\
(1-c)\delta_0(dx) + \nu_c(dx), & c < 1,
\end{cases}$$

where

$$\nu_c(dx) = \frac{\sqrt{(x-a)(b-x)}}{2\pi x}1_{\{(a,b)\}}(x)dx,$$

$$a = (1 - \sqrt{c})^2, \quad b = (1 + \sqrt{c})^2.$$  

It was shown in [14] that $\mu_{f,p}^c$ is the asymptotic spectral distribution, when $t = 1$, of the empirical spectral measure $\mu_1^{(n)}$ when $\lim_{n \to \infty} \frac{p}{n} = c > 0$. For fixed general positive $t \neq 1$, the asymptotic spectral distribution of the empirical spectral measure $\mu_t^{(n)}$, when $\lim_{n \to \infty} \frac{p}{n} = c > 0$, is the family $(\mu_c(t), t > 0)$ of dilations of the free Poisson distribution which is given by $\mu_c(t) = \mu_{f,p}^c \circ h_t^{-1}$, where $h_t(x) = tx$ (see for instance Cavanal-Dubilard and Guionnet [4] and Perez-Abreu and Tudor [25]).

The asymptotic behavior of the empirical spectral measure-valued process of (1.2) falls into the framework of the study of limiting measure-valued process of interacting diffusion particles governed by Itô stochastic diffusion equations with strong interaction, as the eigenvalue process of matrix diffusions with the property that the particles never collide. The general aim in this frame is to show that the empirical spectral measure process converges weakly in the space of continuous probability measure valued processes to a deterministic law. This general direction of study was considered by Cepa and Lepingle [5], Chan [6] and Rogers and Shi [27], among others, in the case of some Gaussian matrix diffusions, turning
out to limiting non-commutative processes as the free Brownian motion. See also \cite{[13]}, \cite{[11]}, \cite{[12]}, \cite{[26]} and references therein.

In this trend, for the case $H = 1/2$ of the Wishart process, it was proved in \cite{[4]}, \cite{[25]} that the empirical spectral process $\{(\mu_t^{(n)}, t \geq 0); n \geq 1\}$ converges weakly in the space of continuous probability measure valued process to the family $(\mu_c(t), t \geq 0)$ of dilations of the free Poisson distribution. The proof of this result is mainly based on an appropriate Itô’s formula for matrix-valued semimartingales and large deviations or classical Itô calculus inequalities estimates.

The aim of this paper is to make a systematic study of the spectra of the fractional matrix Wishart process of Hurst parameter $H \in (1/2, 1)$ and understand several properties like the dynamics of its eigenvalue process, its noncolliding phenomena and the limiting family of the corresponding empirical spectral measure-valued processes. Since fractional Brownian motion is not a semimartingale, the main tools we use are based on Skorohod and Young stochastic calculus. Recently, Nualart and Perez-Abreu \cite{[20]} considered the dynamics and noncolliding property of the eigenvalues of a symmetric matrix fractional Brownian motion and the authors in \cite{[24]} derive the functional limit of the corresponding empirical spectral measure-valued processes, which is the non-commutative fractional Brownian motion considered by Nourdin and Taqqu \cite{[18]}. The final goal of the present work is to find the non-commutative limit process of the empirical spectral measure-valued processes $\{(\mu_t^{(n)}, t \geq 0); n \geq 1\}$ of the fractional Wishart matrix process of Hurst parameter $H \in (1/2, 1)$, as $n$ goes to infinity. Specifically, in this paper we introduce the free fractional Wishart process of Hurst parameter $H \in [1/2, 1)$ as the family $(\mu_{c,H}(t), t > 0)$ of fractional dilations of the free Poisson distribution given by $\mu_{c,H}(t) = \mu_{c,H}^{f \nu} \circ (h_t^H)^{-1}$, where $h_t^H = t^{2H}x$. That is,

$$
\mu_{c,H}(t)(dx) = \begin{cases} 
\nu_c(t)(dx), & c \geq 1, \\
(1-c)\delta_0(dx) + \nu_c(t)(dx), & c < 1,
\end{cases}
$$

(1.5)

with $\mu_{c,H}(0) = \delta_0$. Then, as our main results we prove the following functional limit theorem for the empirical spectral measure-valued processes $\{(\mu_t^{(n)}, t \geq 0); n \geq 1\}$. Let $Pr(\mathbb{R})$ be the space of probability measures on $\mathbb{R}$ endowed with the topology of weak convergence and let $C(\mathbb{R}_+, Pr(\mathbb{R}))$ be the space of continuous functions from $\mathbb{R}_+$ into $Pr(\mathbb{R})$, endowed with the topology of uniform convergence on compact intervals of $\mathbb{R}_+$.

**Theorem 1.1.** Let $H \in (1/2, 1)$ and $(\lambda_1^{(n)}(t) \geq \lambda_2^{(n)}(t) \geq \cdots \geq \lambda_n^{(n)}(t) \geq 0, t \geq 0)$ be the eigenvalue process of $X^{(n)}$, the scaled fractional Wishart process of Hurst parameter $H$. Assume that $\mu_0^{(n)}$ converges weakly to a point mass $\mu_0$ at an arbitrary element of $Pr(\mathbb{R})$, and that $\lim_{n \to \infty} \mathbb{L}_n = c > 0$. Then the family of empirical spectral measure-valued processes $\{(\mu_t^{(n)}, t \geq 0); n \geq 1\}$ converges weakly in $C(\mathbb{R}_+, Pr(\mathbb{R}))$ to the unique continuous probability-measure valued function satisfying, for each $t \geq 0$
\( f \in C^2_b(\mathbb{R}), \)

\[
\langle \mu_t, f \rangle = \langle \mu_0, f \rangle + H \int_0^t \int_{\mathbb{R}^2} (f'(x) - f'(y)) \frac{x + y}{x - y} s^{2H - 1} \mu_s(dx) \mu_s(dy) ds
+ 2Hc \int_0^t \int_{\mathbb{R}} f'(x) s^{2H - 1} \mu_s(dx). \tag{1.6}
\]

Moreover, the family \((\mu_t, t \geq 0)\) corresponds to the law of the free fractional Wishart process of Hurst parameter \(H\) (\(\mu_{c,H}(t), t > 0\)) described in (1.5).

The strategy to prove this theorem is as follows, including some results that are important on their own. We first consider the dynamics and noncolliding of the eigenvalues of a fractional Wishart process. The goal of Section 2 is to derive a stochastic differential equation of the eigenvalues of fractional Wishart process in the framework of Skorokhod integral with respect to the multivariate fractional Brownian motion. For preliminaries on the stochastic calculus with respect to fractional Brownian motion, we refer to [16], [17] and [19]. We start with results on the first and second derivative of the eigenvalues of a nonnegative definite matrix \(X = N^*N\) as functions of the entries of the matrix \(N\). A detailed consideration of derivatives of eigenvalues of an Hermitian matrix \(X\) but in terms of the elements of \(X\) is considered in Anderson et al [2] and Tao [28]. Then we consider in Theorem 2.1 a new Itô’s formula for the Skorokhod integral of functions related to the growth of the second derivative of the eigenvalues of fractional Wishart processes, here denoted by \(X\). In Section 3 we prove the noncolliding phenomena of the eigenvalues of \(X\) at any time. We follow closely the proof in the case of the fractional symmetric matrix Brownian motion in [20], using stochastic calculus with respect to Young’s integral as well as appropriate estimates for moments of the repulsion force of the eigenvalue processes and the joint distribution of the eigenvalues of the fractional Wishart matrix.

The functional asymptotics of the empirical spectral measure-valued process is considered in Section 4. We first apply our Itô’s formula to find appropriate expressions for the integrated processes

\[
\langle \mu_t^{(n)}, f \rangle = \frac{1}{n} \sum_{i=1}^{n} f(\lambda_i^{(n)}(t)), \quad t \geq 0.
\]

Then we prove tightness and the weak convergence of the family of measures \(\{(\mu_t^{(n)}, t \geq 0); n \geq 1\}\) in the space \(C(\mathbb{R}_+, \mathcal{P}(\mathbb{R}))\), for which estimates of some moments of Young integrals are first derived as well as the right order of convergence of \(q\)-moments involving the repulsion force (Lemma 4.3), as \(n\) goes to infinity. Finally, we prove a characterization of the family of laws \((\mu_{c,H}(t), t \geq 0)\) of the free fractional Wishart process of Hurst parameter \(H\) in terms of an initial valued problem for the corresponding Cauchy transform \(G_{c,H}\) of \(\mu_{c,H}\) (see Proposition 4.2).

2. Stochastic differential equation for the eigenvalues.

2.1. Matrix Calculus and notation. In this section, we recall some results on the eigenvalues of a nonnegative definite symmetric matrix that will be needed during the development of this work.
We denote by \( \mathcal{N}_{pn} \) to the collection of \( p \times n \) matrices. For a matrix \( N \in \mathcal{N}_{np} \), we use the coordinates \( N_{ij} \), with \( 1 \leq i \leq p, 1 \leq j \leq n \), to denote the element on the \( i \)-th row and on the \( j \)-th column of \( N \). For simplicity, we write \( N = (N_{ij}) \). Let \( N^* \) denote the transpose of \( N \). In order to work with Wishart matrices, we define \( X := N^* N \), which is clearly symmetric and therefore belongs to \( \mathcal{H}_n \), the space of symmetric \( n \)-dimensional matrices.

Let \( \mathcal{U}_{n}^{vg} \) be the set of orthogonal matrices \( U \) such that \( U_{ii} > 0 \) for all \( i \), \( U_{ij} \neq 0 \) for all \( i, j \) and all minors of \( U \) have non zero determinants. We denote by \( \mathcal{N}_{np}^{vg} \) the set of matrices \( N \in \mathcal{N}_{pn} \) such that there is a factorization

\[
X := N^* N = U \Lambda U^*,
\]

where \( \Lambda \) is a diagonal matrix with entries \( \lambda_i = \Lambda_{ii} \) such that \( \lambda_1 > \lambda_2 > \cdots > \lambda_n \) and \( U \in \mathcal{U}^{vg}_n \). We also denote by \( \mathcal{H}_{n}^{vg} \) for the space of symmetric \( n \)-dimensional matrices \( X \) such that \( X = N^* N \) and \( N \in \mathcal{N}_{np}^{vg} \). The matrices in the set \( \mathcal{N}_{np}^{vg} \) will be called very good matrices, and we identify \( \mathcal{N}_{np}^{vg} \) as an open subset of \( \mathbb{R}^{np} \). Moreover, the complement of \( \mathcal{N}_{np}^{vg} \) has zero Lebesgue measure.

Let \( S_n \) be the set

\[
S_n = \left\{ (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n : \lambda_1 > \lambda_2 > \cdots > \lambda_n \right\}.
\]

For any \( \lambda \in S_n \), let \( \Lambda^{\lambda} \) be the diagonal matrix such that \( \Lambda_{ii}^{\lambda} = \lambda_i \). We consider the mapping \( T : \mathcal{U}_{n}^{vg} \rightarrow \mathbb{R}^{n(n-1)/2} \) defined as follows

\[
T(U) := \left( \frac{U_{12}}{U_{11}}, \ldots, \frac{U_{1p}}{U_{11}}, \ldots, \frac{U_{n-1n}}{U_{n-1n-1}} \right).
\]

It is known that \( T \) is bijective and smooth. Next, we introduce the mapping \( \hat{T} : S_n \times T(\mathcal{U}_{n}^{vg}) \rightarrow \mathcal{H}_{n}^{vg} \) by \( \hat{T}(\lambda, Z) = T^{-1}(Z)\Lambda^{\lambda}T^{-1}(Z)^* \) which turns out to be a smooth bijection. We denote by \( \hat{\Phi} \) the inverse of \( \hat{T} \), i.e. \( \hat{\Phi}(X) := (\lambda, T(U)) \) and observe that it can be defined as a function of the associated very good matrix of \( X \), in other words we define a function \( \Phi : \mathcal{N}_{n}^{vg} \rightarrow S_n \times T(\mathcal{U}_{n}^{vg}) \) such that \( \Phi(N) := \hat{\Phi}(X) \). As a consequence of these facts, it is clear that \( \lambda(X) \) is a smooth function of \( N \in \mathcal{N}_{np}^{vg} \).

Next, we suppose that \( N \) is a smooth function of a parameter \( \theta \in \mathbb{R} \). Then, we know

\[
\frac{\partial \lambda_i}{\partial \theta} = \left( U^* \frac{\partial X}{\partial \theta} U \right)_{ii} \quad \text{and} \quad \frac{\partial^2 \lambda_i}{\partial \theta^2} = \left( U^* \frac{\partial^2 X}{\partial \theta^2} U \right)_{ii} + 2 \sum_{j \neq i} \left| \frac{\partial^2 X}{\partial \theta U_{ij} \partial \theta U_{ij}} \right|^2. \tag{2.1}
\]

On the one hand, if we compute the values of the eigenvalues of \( X \) in terms of the entries of the matrix \( N \), we observe

\[
\lambda_i = \sum_{s=1}^{p} \sum_{r=1}^{n} \sum_{l=1}^{n} U_{ri} N_{sr} N_{sl} U_{li} = \sum_{s=1}^{p} \left( \sum_{r=1}^{n} U_{ri} N_{sr} \right)^2. \tag{2.2}
\]

On the other hand, if we take \( \theta = N_{kh} \) we deduce

\[
\frac{\partial X_{rl}}{\partial N_{kh}} = \sum_{s=1}^{p} \left( \frac{\partial N_{sr}}{\partial N_{kh}} N_{sl} + \frac{\partial N_{sl}}{\partial N_{kh}} N_{sr} \right) = N_{kl} 1_{\{r=h\}} + N_{kr} 1_{\{l=h\}}. \tag{2.3}
\]
Therefore putting all the pieces together, we obtain

$$\frac{\partial \lambda_i}{\partial N_{kh}} = \sum_{r=1}^{n} \sum_{l=1}^{n} U_{rl} N_{kl} \mathbf{1}_{\{r=h\}} U_{li} + \sum_{r=1}^{n} \sum_{l=1}^{n} U_{ri} N_{kr} \mathbf{1}_{\{t=h\}} U_{li} = 2 U_{hi} \sum_{r=1}^{n} U_{ri} N_{kr}.$$ 

Now, we are interesting in computing the second derivative of the eigenvalues. We start with the first term of (2.1) with $\theta = N_{kh}$. Thus from (2.3), we deduce

$$\frac{\partial^2 X_{rl}}{\partial N_{kh}^2} = 2 \mathbf{1}_{\{r=l=h\}},$$

implying

$$\left( U^* \frac{\partial^2 X}{\partial N_{kh}^2} U \right)_{ii} = 2 \sum_{r=1}^{n} \sum_{l=1}^{n} U_{rl} U_{li} \mathbf{1}_{\{r=l=h\}} = 2 U_{hi}^2.$$ 

For the second term of (2.1), we use again (2.3) and observe

$$\left( U^* \frac{\partial X}{\partial N_{kh}} U \right)_{ij} = \sum_{l=1}^{n} U_{lj} U_{hi} N_{kl} + \sum_{r=1}^{n} U_{ri} U_{hj} N_{kr}.$$ 

Therefore

$$\frac{\partial^2 \lambda_i}{\partial N_{kh}^2} = 2 U_{hi}^2 + 2 \sum_{j \neq i} \frac{\left| \sum_{l=1}^{n} U_{lj} U_{hi} N_{kl} + \sum_{r=1}^{n} U_{ri} U_{hj} N_{kr} \right|^2}{\lambda_i - \lambda_j}.$$ 

Now, let us apply the above computations to the particular case of the fractional Wishart process which is defined below. Let us consider a family of independent fractional Brownian motions with Hurst parameter $H \in (1/2, 1)$, $B = \{ (b_{ij}(t), t \geq 0), 1 \leq i \leq p, 1 \leq j \leq n \}$. Similarly as in [20], we introduce $(N(t), t \geq 0)$, the matrix fractional Brownian motion process with parameter $H$ whose components satisfies $N_{ij}(t) = b_{ij}(t)$, for $t \geq 0$.

**Definition 2.1.** Let $(N(t), t \geq 0)$ be the matrix fractional Brownian motion with parameter $H$. We call fractional Wishart process of order $n$ with parameter $H$ to the process $(X(t), t \geq 0)$ satisfying $X(t) = N^*(t)N(t)$, for $t \geq 0$.

Following the previous discussion, for any $i \in \{1, \ldots, n\}$, we deduce that there exists a function $\Phi_i : \mathbb{R}^{pn} \to \mathbb{R}$, which is $C^\infty$ in an open subset $G \subset \mathbb{R}^{pn}$, with $G^c$ has Lebesgue measure equals 0, and such that $\lambda_i(t) = \Phi_i(N(t))$, for $t \geq 0$. Therefore using the fact that $N_{kh}(t) = b_{kh}(t)$, we have

$$\frac{\partial \Phi_i}{\partial b_{kh}} = 2 U_{hi} \sum_{r=1}^{n} U_{ri} b_{kr}, \quad (2.4)$$
and
\[
\frac{\partial^2 \Phi_i}{\partial b_{kh}^2} = 2U_{hi}^2 + 2 \sum_{i \neq j} \frac{\left| \sum_{l=1}^{n} U_{ij} b_{kl} + \sum_{r=1}^{n} U_{ri} b_{kr} \right|^2}{\lambda_i - \lambda_j} = 2U_{hi}^2 + 2 \sum_{i \neq j} \left( \frac{U_{hi}^2 (\sum_{l=1}^{n} U_{ij} b_{kl})^2 + U_{hj}^2 (\sum_{l=1}^{n} U_{il} b_{kl})^2 + 2U_{hi} U_{hj} \sum_{l=1}^{n} U_{ij} b_{kl} \sum_{l=1}^{n} U_{il} b_{kl}}{\lambda_i - \lambda_j} \right).
\]

(2.5)

On the other hand, we note
\[
\sum_{h=1}^{p} \sum_{h=1}^{n} \frac{\partial^2 \Phi_i}{\partial b_{kh}^2} = 2 \sum_{h=1}^{p} \sum_{h=1}^{n} U_{hi}^2 + 2 \sum_{i \neq j} \frac{\sum_{h=1}^{n} U_{hi}^2 \sum_{k=1}^{n} (\sum_{l=1}^{n} U_{ij} b_{kl})^2 + \sum_{h=1}^{n} U_{hj}^2 \sum_{k=1}^{n} (\sum_{l=1}^{n} U_{il} b_{kl})^2}{\lambda_i - \lambda_j} + 4 \sum_{i \neq j} \frac{1}{\lambda_i - \lambda_j} \sum_{h=1}^{n} U_{hi} U_{hj} \sum_{k=1}^{n} \left( \sum_{l=1}^{n} U_{ij} b_{kl} \sum_{l=1}^{n} U_{il} b_{kl} \right)
\]
\[
= 2 \left( p + \sum_{i \neq j} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j} \right),
\]

(2.6)

where in the last identity we have used the fact that $U$ is an orthogonal matrix and identity (2.2).

2.2. **Stochastic calculus for the fbm.** In order to describe the evolution of the eigenvalues of a matrix fractional Brownian motion we present a modification of Theorem 3.1 in [20], which is a multidimensional version of the Itô formula for the Skorokhod integral, in the case of functions that are smooth only on a dense open subset of the Euclidean space and satisfy some growth requirements. In specific the modification is related to the growth of the second derivative.

We refer to the monograph of Nualart [19] for the definition of the Skorokhod integral. For the definition of the space $L_{H,i}^{1,p}$, for $p > 1$ and $1 \leq i \leq n$, we refer to section 2 in [20].

**Theorem 2.1.** Suppose $B^H$ is a $n$-dimensional fractional Brownian motion with Hurst parameter $H > 1/2$. Consider a function $F : \mathbb{R}^n \to \mathbb{R}$ such that:

(i) There exists an open set $G \subset \mathbb{R}^n$ such that $G^c$ has zero Lebesgue measure and $F$ is twice continuously differentiable in $G$.

(ii) $|F(x)| + |\frac{\partial F}{\partial x_i}| \leq C(1 + |x|^M)$, for some constants $C > 0$ and $M > 0$ and for all $x \in G$ and $i = 1, \ldots, n$.

(iii) For each $i \in \{1, \ldots, n\}$ and for each $s > 0$ and $q \in [1, 2]$,
\[
\mathbb{E} \left[ \left| \frac{\partial^2 F}{\partial x_i^2} (B^H_s) \right|^q \right] \leq C,
\]

for some constant $C > 0$. 

Then, for each \( i = 1, \ldots, n \) and \( t \in [0, T] \), the process \( \{ \frac{\partial F}{\partial x_i}(B^H_s)1_{[0,t]}(s), s \in [0, T] \} \) belongs to the space \( L_{1,H,i}^{1/1} \) and

\[
F(B^H_t) = F(B^H_0) + \sum_{i=1}^{n} \int_0^t \frac{\partial F}{\partial x_i}(B^H_s) \delta B^H_{si} + H \sum_{i=1}^{n} \int_0^t \frac{\partial^2 F}{\partial x_i^2}(B^H_s) s^{2H-1} ds. \tag{2.7}
\]

**Proof.** We observe that the proof of this result follows similar arguments as those used in the proof of Theorem 3.1 in [20], with the exception of the argument that verifies that equation (2.7) is well defined.

To this end, it is enough to verify that the process \( u_i(s) = \frac{\partial F}{\partial x_i}(B^H_s)1_{[0,t]}(s) \) belongs to the space \( L_{1,H,i}^{1/1} \). Indeed, using conditions (ii) and (iii) we have

\[
E \left[ \int_0^T |u_i(s)|^{1/H} ds \right] \leq C^{1/H} E \left[ \int_0^T (1 + |B^H_s|^M)^{1/H} ds \right] < \infty,
\]

and

\[
E \left[ \int_0^T \int_0^T |D_r^{(i)} u_i(s)|^{1/H} dr ds \right] = E \left[ \int_0^T s \left| \frac{\partial^2 F}{\partial x_i^2}(B^H_s) \right|^{1/H} ds \right] \leq \frac{C}{2} T^2.
\]

On the other hand, taking \( q = 1 \) in condition (iii), we also have for each \( i \in \{ 1, \ldots, n \} \),

\[
E \left[ \int_0^t \left| \frac{\partial^2 F}{\partial x_i^2} \right|^{2H-1} ds \right] < \infty, \quad \text{for} \quad t > 0.
\]

given the fact that \( H > 1/2 \). As a consequence, all terms in (2.7) are well defined. \( \square \)

### 2.3. SDE followed by the eigenvalues of a fractional Wishart process.

In this section, we are interested in studying the dynamics of the eigenvalues of a fractional Wishart process as a stochastic differential equation that depends on the Skorokhod integral.

Recall from Theorem 7.1.2 in [8] that the joint density of the eigenvalues of \( X(t) \) on \( \mathcal{S}_n \), with respect to the Lebesgue measure, satisfies

\[
c_{n,p} \prod_{j=1}^{n} \left( \frac{\lambda_j^{(p-n-1)/2} s^{-pnH} \exp \left( -\frac{\lambda_j}{2s^{2H}} \right)}{\prod_{j<k} |\lambda_k - \lambda_j|} \right).
\]

**Theorem 2.2.** Let \( H \in (1/2, 1) \) and \( \{ N(t), t \geq 0 \} \) be a matrix fractional Brownian motion of parameter \( H \) defined as above. We also let \( N(0) \) be an arbitrary deterministic \( p \times n \) matrix. For each \( t \geq 0 \), let \( \lambda_1, \ldots, \lambda_n \) be the eigenvalues of the fractional Wishart process of order \( n \), \( X = N^* N \). Then, for any \( t > 0 \) and \( i = 1, \ldots, n \),

\[
\lambda_i(t) = \lambda_i(0) + \sum_{k=1}^{p} \sum_{h=1}^{n} \int_0^t \frac{\partial \Phi_i}{\partial b_{kh}}(N(s)) \delta b_{kh}(s) + 2H \int_0^t \left( p + \sum_{i \neq j} \frac{\lambda_i(s) + \lambda_j(s)}{\lambda_i(s) - \lambda_j(s)} \right) s^{2H-1} ds. \tag{2.9}
\]

Proof. Without loss of generality, we assume that $N(0) = 0$. Now let us check \( \{\lambda_i(t), i = 1, \ldots, n\} \) satisfies the conditions of Theorem 2.1. Using (2.2) and the Cauchy-Schwarz inequality, we observe

\[
\sum_{i=1}^{n} \Phi_i^2(N(t)) = \sum_{i=1}^{n} \left( \sum_{l=1}^{p} \left( \sum_{r=1}^{n} U_{ri}(t)b_{lr}(t) \right)^2 \right) \leq \sum_{i=1}^{n} \left( \sum_{l=1}^{p} \sum_{r=1}^{n} b_{lr}^2(t) \right) \leq n ||N(t)||_2^4,
\]

where \( || \cdot ||_2 \) denotes the Euclidian norm of columns of a matrix. In particular, for each \( 1 \leq i \leq n \), we have

\[
|\Phi_i(N(t))| \leq \sqrt{n} ||N(t)||_2.
\]

On the other hand using (2.4) and Cauchy-Schwarz inequality, we obtain

\[
\left| \frac{\partial \Phi_i}{\partial b_{kh}}(N(t)) \right|^2 = 4U_{hi}^2(t) \left( \sum_{r=1}^{n} U_{ri}(t)b_{kr}(t) \right)^2 \leq 4 ||N(t)||_2^2,
\]

implying

\[
\left| \frac{\partial \Phi_i}{\partial b_{kh}}(N(t)) \right| \leq 2 ||N(t)||_2.
\]

Therefore, for each \( 1 \leq i \leq n \), we have

\[
|\Phi_i(N(t))| + \left| \frac{\partial \Phi_i}{\partial b_{kh}}(N(t)) \right| \leq \sqrt{n} ||N(t)||_2^2 + 2 ||N(t)||_2.
\]

Now, observe that using (2.6) we verify

\[
\left| \frac{\partial^2 \Phi_i}{\partial b_{kh}^2}(N(t)) \right| \leq 2U_{hi}^2(t) + 2 \sum_{i \neq j} \left| \sum_{l=1}^{n} U_{ij}(t)U_{hi}(t)b_{kl}(t) + \sum_{r=1}^{n} U_{ri}(t)U_{hj}(t)b_{kr}(t) \right| \frac{1}{|\lambda_i(t) - \lambda_j(t)|}
\]

\[
\leq \sum_{k=1}^{p} \sum_{h=1}^{n} \left( 2U_{hi}^2(t) + 2 \sum_{i \neq j} \left| \sum_{l=1}^{n} U_{ij}(t)U_{hi}(t)b_{kl}(t) + \sum_{r=1}^{n} U_{ri}(t)U_{hj}(t)b_{kr}(t) \right| \frac{1}{|\lambda_i(t) - \lambda_j(t)|} \right)
\]

\[
= 2 \left( p + \sum_{i \neq j} \lambda_i(t) + \lambda_j(t) \right) \frac{1}{|\lambda_i(t) - \lambda_j(t)|}.
\]

Hence using the joint density of the eigenvalues given by (2.8), the previous equation and Jensen inequality, we obtain for \( q \in [1, 2] \)

\[
\mathbb{E} \left[ \left| \frac{\partial^2 \Phi_i}{\partial b_{kh}^2}(N(t)) \right| ^q \right] \leq 2^q \mathbb{E} \left[ \left( p + \sum_{i \neq j} \lambda_i(t) + \lambda_j(t) \right) \frac{1}{|\lambda_i(t) - \lambda_j(t)|} \right] ^q \leq 4^q p^q + K_{n,p,q} \sum_{i \neq j} \mathbb{E} \left[ \left| \frac{\lambda_i(t) + \lambda_j(t)}{|\lambda_i(t) - \lambda_j(t)|} \right| ^q \right]
\]

\[
= 4^q p^q + K_{n,p,q} \sum_{i \neq j} \int_{S^p} \prod_{j=1}^{n} \left( \lambda_j^{(p-n-1)/2} t^{-npH} \exp \left( -\frac{\lambda_j}{2t^{2H}} \right) \right) \prod_{j<k} \left| \lambda_k - \lambda_j \right| \frac{1}{|\lambda_i - \lambda_j|^q} \, d\gamma,
\]

where \( \gamma \) denotes the uniform distribution on the set of all \( \lambda \) such that \( \lambda_i > \lambda_j \) for \( i < j \).
where $\gamma$ denotes the Lebesgue measure and $K_{n,p,q}$ is a positive constant that only depends on $q,p,n$. Using the following change of variables $\lambda_i = t^H \mu_i$, we observe the last integral in the previous inequality is a constant that only depends on $q,p,n$, thus

$$\mathbb{E} \left[ \left| \frac{\partial^2 \Phi_i}{\partial b_{kh}^2}(N(t)) \right|^q \right] \leq \tilde{K}_{n,p,q}, \tag{2.10}$$

where $\tilde{K}_{n,p,q}$ is a positive constant that only depends on $q,p,n$. The result now follows using Theorem 2.1 and (2.6).

**Remark 2.1.** Let us consider the case $H = 1/2$, therefore the process $(N(t), t \geq 0)$ correspond to a $p \times n$ matrix of independent Brownian motions. So we have that equation (2.9) takes the following form

$$\lambda_i(t) = \lambda_i(0) + 2 \sum_{k=1}^{p} \sum_{h=1}^{n} \int_0^t \frac{\partial \Phi_i}{\partial b_{kh}}(N(s)) \delta b_{kh}(s) + 2H \int_0^t \left( p + \sum_{i \neq j} \lambda_i(s) + \lambda_j(s) \right) ds.$$

Using that the process $\frac{\partial \Phi_i}{\partial b_{kh}}(N(s))$ is adapted to the filtration generated by the process $N$, then the Skorokhod integral coincides with the Itô integral, implying that the stochastic integral in (2.9) satisfies

$$\sum_{k=1}^{p} \sum_{h=1}^{n} \int_0^t \frac{\partial \Phi_i}{\partial b_{kh}}(N(s)) \delta b_{kh}(s) = \sum_{k=1}^{p} \sum_{h=1}^{n} \int_0^t \frac{\partial \Phi_i}{\partial b_{kh}}(N(s)) \cdot \delta b_{kh}(s) = 2 \int_0^t \sqrt{\lambda_i(s)} \cdot dY^i(s),$$

where "·" denotes Itô integral. Using (2.4), the process $Y^i$ satisfies

$$Y^i_t = \sum_{k=1}^{p} \sum_{h=1}^{n} \int_0^t U_{hi}(s) \frac{\sum_{l=1}^{n} b_{kl}(s) U_{li}(s)}{\sqrt{\lambda_i(s)}} \cdot \delta b_{kh}(s).$$

Computing the quadratic variation of $Y^i$, we observe

$$\langle Y^i, Y^i \rangle_t = \sum_{k=1}^{p} \sum_{h=1}^{n} \int_0^t U_{hi}^2(s) \left( \sum_{l=1}^{n} b_{kl}(s) U_{li}(s) \right)^2 \delta b_{kh}(s) ds = t,$$

where in the last equality we used (2.2) and (2.4). Hence from Lévy’s Characterization Theorem, we deduce that $Y^i$ is a Brownian motion. Therefore equation (2.9) takes the following form

$$\lambda_i(t) = \lambda_i(0) + 2 \int_0^t \sqrt{\lambda_i(s)} \cdot d\nu^i(s) + \int_0^t \left( p + \sum_{i \neq j} \lambda_i(s) + \lambda_j(s) \right) ds,$$

where $\nu^i$ is a Brownian motion for $i = 1, \ldots, n$, which is the system of SDE’s obtained by Bru [3].

### 3. No Colliding of Eigenvalues.

In this section we show that the eigenvalues of the fractional Wishart process do not collide almost surely. Recall that the fractional Wishart process is defined as $X = N^*N$, where $N$ denotes the matrix fractional Brownian motion with Hurst parameter $H > 1/2$. Let us assume that $X(0)$ is a fixed deterministic symmetric matrix.
The following result follows closely the proof of Theorem 4.1 in [20], despite that the fractional Wishart process is not Gaussian. For the sake of completeness, we provide its proof.

**Theorem 3.1.** Denote by \( \{ (\lambda_i(t))_{i \geq 0}, 1 \leq i \leq n \} \) the eigenvalues of fractional Wishart process \( (X(t), t \geq 0) \). Assuming that \( \lambda_1(0) \geq \cdots \geq \lambda_n(0) \). Then,

\[
\mathbb{P}(\lambda_1(t) > \cdots > \lambda_n(t), \forall t > 0) = 1.
\]

**Proof.** We first assume that for fixed \( t_0 > 0 \), we have \( \lambda_1(t_0) > \cdots > \lambda_n(t_0) \). Since the matrix \( X(t) \) is symmetric then we can use the Hoffman-Weilandt inequality (see [9]) to deduce the following

\[
\sum_{i=1}^{n} (\lambda_i(t) - \lambda_i(s))^2 \leq \frac{1}{n} \sum_{i,j=1}^{n} \left( X_{ij}(t) - X_{ij}(s) \right)^2 = \frac{1}{n} \sum_{i,j=1}^{n} \left| \sum_{k=1}^{p} \left( b_{ki}(t)b_{kj}(t) - b_{ki}(s)b_{kj}(s) \right) \right|^2.
\]

On the one hand from the previous identity and applying Jensen inequality twice, we get for \( r \geq 2 \),

\[
\left| \lambda_i(t) - \lambda_i(s) \right|^r \leq \frac{1}{n^{r/2}} \left( \sum_{i,j=1}^{n} \sum_{k=1}^{p} \left| b_{ki}(t)b_{kj}(t) - b_{ki}(s)b_{kj}(s) \right|^r \right)^{r/2} \leq n^{r/2-2} \sum_{i,j=1}^{n} \left| \sum_{k=1}^{p} \left( b_{ki}(t)b_{kj}(t) - b_{ki}(s)b_{kj}(s) \right)^r \right|^r \leq n^{r/2-2} p^{r-1} \sum_{i,j=1}^{n} \sum_{k=1}^{p} \left| b_{ki}(t)b_{kj}(t) - b_{ki}(s)b_{kj}(s) \right|^r.
\]

On the other hand, observe

\[
(b_{ki}(t)b_{kj}(t) - b_{ki}(s)b_{kj}(s)) = b_{ki}(t)(b_{kj}(t) - b_{kj}(s)) + b_{kj}(s)(b_{ki}(t) - b_{ki}(s)).
\]

Hence using independence between \( b_{ki} \) and \( b_{kj} \) and the previous identity, we get

\[
\mathbb{E}\left[ \left| b_{ki}(t)b_{kj}(t) - b_{ki}(s)b_{kj}(s) \right|^r \right] \leq C_r |t - s|^{rH} (t^{rH} + s^{rH}), \tag{3.1}
\]

where \( C_r \) is a positive constant that only depends on \( r \). Therefore putting all the pieces together, we deduce that for any \( T > 0 \) and \( s, t \in [t_0, T] \) there exists a constant \( C_{n,p,r,T} \) depending on \( n, p, r, T \), such that

\[
\mathbb{E}\left[ \left| \lambda_i(t) - \lambda_i(s) \right|^r \right] \leq n^{r/2-2} p^{r-1} \sum_{i,j=1}^{n} \sum_{k=1}^{p} \mathbb{E}\left[ \left| b_{ki}(t)b_{kj}(t) - b_{ki}(s)b_{kj}(s) \right|^r \right] \leq C_{n,p,r,T} |t - s|^{rH}. \tag{3.2}
\]

Since \( rH > 1 \), we deduce that the paths of \( \lambda_i \) are Hölder continuous of order \( \beta \) for any \( \beta < H \).

We now follow the same arguments as those used in Theorem 4.1 in [20]. We consider the stopping time

\[
\tau := \inf \left\{ t > t_0 : \lambda_i(t) = \lambda_j(t) \text{ for some } i \neq j \right\}.
\]
Observe that \( \tau > t_0 \) a.s., and that on the random interval \([t_0, \tau)\) the function \( \log(\lambda_i(t) - \lambda_j(t)) \), where \( i \neq j \), is well defined. Since the paths of \( \lambda_i \) are Hölder continuous for each \( \beta < H \), we can apply stochastic calculus with respect to the Young’s integral and deduce that for any \( t < \tau \wedge T \),

\[
\log(\lambda_i(t) - \lambda_j(t)) = (\lambda_i(t_0) - \lambda_j(t_0)) + \int_{t_0}^{t} \frac{1}{\lambda_i(s) - \lambda_j(s)} d(\lambda_i(s) - \lambda_j(s)). \tag{3.3}
\]

Therefore for \( 1 - H < \alpha < \frac{1}{2} \), we obtain

\[
\int_{t_0}^{t} \frac{1}{\lambda_i(s) - \lambda_j(s)} d\lambda_i(s) = \int_{t_0}^{t} I_{i,j}(s) J_j(s) ds + \frac{\lambda_i(t) - \lambda_i(t_0)}{\lambda_i(t_0) - \lambda_j(t_0)},
\]

where

\[
I_{i,j}(s) := D_{t_0}^\alpha (\lambda_i - \lambda_j)_0^{-1}(s) = \frac{1}{\Gamma(1 - \alpha)} \left[ s^{-\alpha} \left( \frac{1}{\lambda_i(s) - \lambda_j(s)} - \frac{1}{\lambda_i(t_0) - \lambda_j(t_0)} \right) + \alpha \int_{t_0}^{s} (\lambda_i(s) - \lambda_j(s))^{-1} - (\lambda_i(y) - \lambda_j(y))^{-1} (s - y)^{\alpha + 1} dy \right],
\]

and

\[
J_j(s) := D_{t_0}^{1 - \alpha} \lambda_i t_j^{-}(s) = \frac{1}{\Gamma(\alpha)} \left( \frac{\lambda_i(s) - \lambda_i(t)}{(t - s)^{1 - \alpha}} + (1 - \alpha) \int_{s}^{t} \frac{\lambda_i(s) - \lambda_i(y)}{(y - s)^{2 - \alpha}} dy \right).
\]

We claim that

\[
\mathbb{P} \left( \int_{t_0}^{t} |I_{i,j}(s)||J_j(s)| ds < \infty, \text{ for all } t \geq t_0, i \neq j \right) = 1. \tag{3.4}
\]

Before we prove the above identity, we first observe that Hölder’s inequality with exponents \( \ell, q > 1 \) such that \( 1/\ell + 1/q = 1 \), imply

\[
\mathbb{E} \left[ |I_{i,j}(s)||J_j(s)| \right] \leq \mathbb{E} \left[ |I_{i,j}(s)|^\ell \right]^{1/\ell} \mathbb{E} \left[ |J_j(s)|^q \right]^{1/q}.
\]

From (3.2), we deduce that for any \( \beta \in (1 - \alpha, H) \), there exists a r.v. \( G \) with moments of all orders such that

\[
|\lambda_i(u) - \lambda_i(s)| \leq k_{n,p,T} G |s - u|^\beta, \quad \text{for all } i \in \{1, \ldots, n\}, \tag{3.5}
\]

for all \( s, u \in [t_0, t] \) with \( t \leq T \), and \( k_{n,p,T} \) is a positive constant that only depends on \( n, p \) and \( T \). The above leads to the estimate

\[
\mathbb{E} \left[ |J_j(s)|^q \right] \leq k_{n,p,T,q} \mathbb{E} \left[ G^q \right], \tag{3.6}
\]

for all \( q > 1 \) and for some constant \( k_{n,p,T,q} > 0 \). In order to estimate \( \mathbb{E}[|I_{i,j}(s)|^q] \), we consider the integral part in the definition of \( I_{i,j} \) and denote by

\[
K_{i,j}(s) := \int_{t_0}^{s} \frac{(\lambda_i(s) - \lambda_j(s))^{-1} - (\lambda_i(y) - \lambda_j(y))^{-1}}{(s - y)^{\alpha + 1}} dy.
\]


Thus using the same estimates as in the proof of Theorem 4.1 in [20], we obtain
\[
\|K_{i,j}(s)\|_\ell \leq 2^\alpha \|C^\alpha\|_{p_1} \int_{t_0}^\infty \left( \|\lambda_i(y) - \lambda_j(y)\|^{b-1}_{p_2} \|\lambda_i(s) - \lambda_j(s)\|^{-1}_{p_3} + \|\lambda_i(y) - \lambda_j(y)\|^{-1}_{p_3} \|\lambda_i(s) - \lambda_j(s)\|^{b-1}_{p_2} \right) (s - y)^{a_2 - a_1 - 1} dy, \tag{3.7}
\]
where \( \ell = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \), with \( p_i > 1 \) for \( i = 1, 2, 3 \). We choose \( a, p_1, p_2 \) and \( p_3 \) such that
\[
a > \frac{\alpha}{\beta}, \quad p_3 < 2, \quad p_2 < \frac{2\beta}{\alpha},
\]
which is possible by taking \( \ell \) and \( p_1 \) close to 1 and since \( \alpha < \frac{1}{2} < \beta \). In order to proof identity (3.4), we need to estimate
\[
\mathbb{E}\left[|\lambda_i(s) - \lambda_j(s)|^{-q}\right], \quad \text{for} \quad q < 2.
\]
Recall that the joint density of the eigenvalues \( \lambda_1(s) > \cdots > \lambda_n(s) \) is given by (2.8) and that \( \gamma \) denotes the Lebesgue measure. Then,
\[
\mathbb{E}\left[|\lambda_i(s) - \lambda_j(s)|^{-q}\right] = c_{n,p} \int_{S_n} \prod_{j=1}^n \left( \lambda_j^{(p-3)/2} |\lambda_i - \lambda_j|^{-q} s^{-np} \exp\left( -\frac{\lambda_j}{2s^2} \right) \right) \prod_{j<k} |\lambda_k - \lambda_j| d\gamma.
\]
By making the change of variable \( \lambda_i = \mu_i s^H \) and performing the integral, we observe
\[
\mathbb{E}\left[|\lambda_i(s) - \lambda_j(s)|^{-q}\right] \leq C_{p,n} s^{-2qH},
\]
for a positive constant that depends on \( p \) and \( n \). In other words \( \mathbb{E}[|\lambda_i(s) - \lambda_j(s)|^{-q}] \) is uniformly bounded on the interval \([t_0, T]\). Therefore for all \( i \neq j \), \( \int_{t_0}^T |I_{i,j}(s)||J_j(s)| ds < \infty \), and so the claim (3.4) follows.

Finally, the identity (3.4) implies that \( \mathbb{P}(T < \tau) = 1 \), otherwise we would get a contradiction since \( \log(\lambda_i(\tau) - \lambda_j(\tau)) = -\infty \). Therefore taking \( T \) goes to \( \infty \), we obtain \( \mathbb{P}(\tau = \infty) = 1 \). We obtain the desired result by letting \( t_0 \) goes to zero. \( \square \)

4. Functional limit for the Fractional Wishart Process.

For a probability measure \( \mu \) and a \( \mu \)-integrable function \( f \), we write \( \langle \mu, f \rangle = \int f(x)\mu(dx) \). Hence, since the empirical measure \( \mu_i^{(n)} \) is a point measure we have, for \( f \in C_b^2 \), that
\[
\langle \mu_i^{(n)}, f \rangle = \frac{1}{n} \sum_{i=1}^n f(\lambda_i^{(n)}(t)). \tag{4.1}
\]
Therefore, applying the chain rule to the last equation, we get
\[
\langle \mu_i^{(n)}, f \rangle = \langle \mu_0^{(n)}, f \rangle + \frac{1}{n} \sum_{i=1}^n \int_0^t f'(\lambda_i^{(n)}(s)) d\lambda_i^{(n)}(s). \tag{4.2}
\]
In order to consider the dynamics of the measure-valued process \((\mu_t^{(n)}, t \geq 0)\), we prove the following result. Recall that the fractional Wishart process \(X^{(n)}(t)\) is defined as follows

\[ X^{(n)}(t) = (N^{(n)}(t))^* N^{(n)}(t), \quad \text{for} \quad t \geq 0, \]

where \(N^{(n)}(t) = n^{-1/2} N(t)\) and \(N\) denotes the matrix fractional Brownian motion \((n \times p)\).

**Lemma 4.1.** Let \((\mu_t^{(n)}, t \geq 0)\) be the empirical measure-valued process of the eigenvalues of the fractional Wishart process \((X^{(n)}(t), t \geq 0)\). Then for \(f \in C^2_0(\mathbb{R})\), we have

\[
\langle \mu_t^{(n)}, f \rangle = \langle \mu_0^{(n)}, f \rangle + \frac{1}{n^{3/2}} \sum_{i=1}^{n} \sum_{k=1}^{p} \sum_{h=1}^{n} \int_0^t f'(\lambda_i^{(n)}(s)) \frac{\partial \Phi_i}{\partial b_{kh}}(N^{(n)}(s)) \delta b_{kh}(s) \]

\[+ H \int_0^t \int_{\mathbb{R}^2} (f'(x) - f'(y)) \frac{x + y}{x - y} s^{2H-1} \mu_s^{(n)}(dx) \mu_s^{(n)}(dy) ds \]

\[+ \frac{2H}{n} \int_0^t f'(x)s^{2H-1} \mu_s^{(n)}(dx) + \frac{2H}{n} \int_0^t \int_{\mathbb{R}} f''(x)x s^{2H-1} \mu_s^{(n)}(dx) ds. \quad (4.3)\]

**Proof.** We first observe from [20] that we can apply Itô’s formula with respect to the Young integral to the eigenvalues of the process \(X^{(n)}\) and obtain

\[\lambda_i^{(n)}(t) = \lambda_i^{(n)}(0) + \frac{1}{\sqrt{n}} \sum_{k=1}^{p} \sum_{h=1}^{n} \int_0^t \frac{\partial \Phi_i}{\partial b_{kh}}(N^{(n)}(s)) db_{kh}(s), \quad (4.4)\]

for any \(t \geq 0\) and \(i = \{1, 2, \ldots, n\}\). Hence using (4.2) and (4.4), we obtain

\[\langle \mu_t^{(n)}, f \rangle = \langle \mu_0^{(n)}, f \rangle + \frac{1}{n^{3/2}} \sum_{i=1}^{n} \sum_{k=1}^{p} \sum_{h=1}^{n} \int_0^t f'((\Phi_i(N^{(n)}(s)))) \frac{\partial \Phi_i}{\partial b_{kh}}(N^{(n)}(s)) \delta b_{kh}(s) \]

Now we will be interested in replacing the Young integrals by Skorohod integrals in the above expression. To this end, we prove that the condition of Proposition 3 in [1] is satisfied, i.e.

\[\int_0^t \int_0^t D_r \left( f'(\Phi_i(N^{(n)}(s))) \frac{\partial \Phi_i}{\partial b_{kh}}(N^{(n)}(s)) \right) |s - r|^{2H-2} dr ds < \infty, \quad \mathbb{P}\text{-a.s.}, \quad (4.5)\]

where \(D_r\) denotes the Malliavin derivative. In this direction, we first observe

\[\int_0^t \int_0^t D_r \left( f'(\Phi_i(N^{(n)}(s))) \frac{\partial \Phi_i}{\partial b_{kh}}(N^{(n)}(s)) \right) |s - r|^{2H-2} dr ds \]

\[= \frac{1}{n(2H-1)} \int_0^t f''(\Phi_i(N^{(n)}(s))) \left( \frac{\partial \Phi_i}{\partial b_{kh}}(N^{(n)}(s)) \right)^2 s^{2H-1} ds \]

\[+ \frac{1}{n(2H-1)} \int_0^t f'(\Phi_i(N^{(n)}(s))) \frac{\partial^2 \Phi_i}{\partial b_{kh}^2}(N^{(n)}(s)) s^{2H-1} ds. \quad (4.6)\]

Now, using (2.4) and Cauchy-Schwartz inequality

\[
\left| \frac{\partial \Phi_i}{\partial b_{kh}}(N^{(n)}(s)) \right|^2 = \frac{4}{n} \left( U_{hi}^{(n)}(s) \right)^2 \left( \sum_{r=1}^{n} U_{ri}^{(n)}(s) b_{kr}(s) \right)^2 \leq \frac{4}{n} \sum_{r=1}^{n} b_{kr}^2(s), \quad (4.7)\]
implying
\[
\mathbb{E} \left[ \left| \int_0^t f''(\Phi_i(N(n)(s))) \left( \frac{\partial \Phi_i}{\partial b_{kh}}(N(n)(s)) \right)^2 s^{2H-1} ds \right| \right] \leq \frac{4}{n} \|f''\|_\infty \left| \int_0^t \sum_{r=1}^n \mathbb{E} \left[ b_{kr}^2(s) \right] s^{2H-1} ds \right|
= \frac{t^{4H}}{H} \|f''\|_\infty < \infty.
\]

On the other hand using inequality (2.10), we obtain
\[
\mathbb{E} \left[ \left| \int_0^t f'(\Phi_i(N(n)(s))) \frac{\partial^2 \Phi_i}{\partial b_{kh}^2}(N(n)(s)) s^{2H-1} ds \right| \right] \leq \|f'\|_\infty \left| \int_0^t \mathbb{E} \left[ \left| \frac{\partial^2 \Phi_i}{\partial b_{kh}^2}(N(n)(s)) \right| \right] s^{2H-1} ds \right|
\leq \frac{K_{n,p,1}}{2H} t^{2H} < \infty,
\]
thus, we conclude
\[
\left| \int_0^t f'(\Phi_i(N(n)(s))) \frac{\partial^2 \Phi_i}{\partial b_{kh}^2}(N(n)(s)) s^{2H-1} ds \right| < \infty \quad \mathbb{P}\text{-a.s.}
\]
Putting the pieces together, we obtain that (4.5) holds.

Now, we apply Proposition 3 in [11] (see also Proposition 5.2.3 in [19]) in order to express the Young integrals that appear in (4.4) in terms of Skorohod integrals. Therefore
\[
\langle \mu_i^{(n)}, f \rangle = \langle \mu_0^{(n)}, f \rangle + \frac{1}{n^{3/2}} \sum_{i=1}^n \sum_{k=1}^p \sum_{h=1}^1 \int_0^t f'(\Phi_i(N(n)(s))) \frac{\partial \Phi_i}{\partial b_{kh}}(N(n)(s)) \delta b_{kh}(s)
+ \frac{H(2H-1)}{n^2} \sum_{i=1}^n \sum_{k=1}^p \sum_{h=1}^1 \int_0^t \int_0^t D_r \left( f'(\Phi_i(N(n)(s))) \frac{\partial \Phi_i}{\partial b_{kh}}(N(n)(s)) \right) \left| s - r \right|^{2H-2} dr ds.
\]

Next, we apply identity (4.6) and deduce
\[
\langle \mu_i^{(n)}, f \rangle = \langle \mu_0^{(n)}, f \rangle + \frac{1}{n^{3/2}} \sum_{i=1}^n \sum_{k=1}^p \sum_{h=1}^1 \int_0^t f'(\Phi_i(N(n)(s))) \frac{\partial \Phi_i}{\partial b_{kh}}(N(n)(s)) \delta b_{kh}(s)
+ \frac{H}{n^2} \sum_{i=1}^n \sum_{k=1}^p \sum_{h=1}^1 \int_0^t f''(\Phi_i(N(n)(s))) \left( \frac{\partial \Phi_i}{\partial b_{kh}}(N(n)(s)) \right)^2 s^{2H-1} ds
+ \frac{H}{n^2} \sum_{i=1}^n \sum_{k=1}^p \sum_{h=1}^1 \int_0^t f'(\Phi_i(N(n)(s))) \frac{\partial^2 \Phi_i}{\partial b_{kh}^2}(N(n)(s)) s^{2H-1} ds.
\]

Since $U^{(n)}$ is an orthogonal matrix, we observe from (4.7) and (2.2) that
\[
\sum_{k=1}^p \sum_{h=1}^1 \left( \frac{\partial \Phi_i}{\partial b_{kh}}(N(n)(s)) \right)^2 = 4 \sum_{k=1}^p \left( \sum_{r=1}^n U_{ri}^{(n)}(s) b_{kr}^{(n)}(s) \right)^2 = 4 \lambda_i^{(n)}(s).
\]
Therefore from the latter identity and applying (2.6), we deduce

\[
\langle \mu_i^{(n)}, f \rangle = \langle \mu_0^{(n)}, f \rangle + \frac{1}{n^{3/2}} \sum_{i=1}^{n} \sum_{k=1}^{p} \sum_{h=1}^{n} \int_0^t f'(\Phi_i(N^{(n)}(s))) \frac{\partial \Phi_i}{\partial b_{kh}}(N^{(n)}(s)) \delta b_{kh}(s)
\]
\[
+ \frac{2H}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} \int_0^t f'(\Phi_i(N^{(n)}(s))) \left( p + \frac{\lambda_i^{(n)}(s) + \lambda_j^{(n)}(s)}{\lambda_i^{(n)}(s) - \lambda_j^{(n)}(s)} \right) s^{2H-1} ds
\]
\[
+ \frac{4H}{n^2} \sum_{i=1}^{n} \int_0^t f''(\Phi_i(N^{(n)}(s))) \lambda_i^{(n)}(s) s^{2H-1} ds
\]
\[
= \langle \mu_0^{(n)}, f \rangle + \frac{1}{n^{3/2}} \sum_{i=1}^{n} \sum_{k=1}^{p} \sum_{h=1}^{n} \int_0^t f'(\Phi_i(N^{(n)}(s))) \frac{\partial \Phi_i}{\partial b_{kh}}(N^{(n)}(s)) \delta b_{kh}(s)
\]
\[
+ H \int_0^t \int_{\mathbb{R}^2} (f'(x) - f'(y)) \frac{x+y}{x-y} s^{2H-1} \mu_s^{(n)}(dx) \mu_s^{(n)}(dy) ds
\]
\[
+ \frac{2H}{n} \int_0^t \int_{\mathbb{R}} f'(x) s^{2H-1} \mu_s^{(n)}(dx) ds + \frac{2H}{n} \int_0^t \int_{\mathbb{R}} f''(x) x s^{2H-1} \mu_s^{(n)}(dx) ds,
\]

where in the last identity we used the definition of \( \mu^{(n)} \). This completes the proof. \( \square \)

4.1. Tightness. In this section, we prove the tightness of the family of measure \( \{ (\mu_i^{(n)}, t \geq 0) : n \geq 1 \} \). For this purpose, we prove first the following auxiliary result.

**Lemma 4.2.** Let \( (\lambda_i^{(n)}(t), \ldots, \lambda_n^{(n)}(t); t \geq 0) \) be the eigenvalues of the fractional Wishart process \( X^{(n)} \) with parameter \( H \in (1/2, 1) \). Then for all \( s, t \in [0, T] \), we have

\[
\mathbb{E} \left[ \left( \frac{1}{n} \sum_{i=1}^{n} |\lambda_i^{(n)}(t) - \lambda_i^{(n)}(s)| \right)^4 \right] \leq \left( \frac{p}{n} \right)^4 C|t - s|^{4H T^{4H}},
\]

(4.8)

where \( C \) is a positive constant.

**Proof.** As in the proof of Theorem 3.1 we can use the Hoffman-Weilandt inequality (see [9]) to deduce

\[
\sum_{i=1}^{n} \left( \lambda_i^{(n)}(t) - \lambda_i^{(n)}(s) \right)^2 \leq \frac{1}{n} \sum_{i=1}^{n} \left( \sum_{k=1}^{p} \left( b_{ki}^{(n)}(t)b_{kj}^{(n)}(t) - b_{ki}^{(n)}(s)b_{kj}^{(n)}(s) \right) \right)^2.
\]

From similar arguments as those used at the beginning of the proof of Theorem 3.1, we get that there exists a constant \( C > 0 \) such that

\[
\mathbb{E} \left[ \left( b_{ki}^{(n)}(t)b_{kj}^{(n)}(t) - b_{ki}^{(n)}(s)b_{kj}^{(n)}(s) \right)^4 \right] \leq \frac{C}{n^4} |t - s|^{4H T^{4H}}.
\]
Therefore using the Cauchy-Schwartz inequality, Jensen’s inequality twice and the previous inequality, we obtain
\[
\mathbb{E} \left[ \left( \frac{1}{n} \sum_{i=1}^{n} |\lambda_i^{(n)}(t) - \lambda_i^{(n)}(s)| \right)^4 \right] \leq \frac{1}{n^2} \mathbb{E} \left[ n^2 \left( \sum_{i=1}^{n} |\lambda_i^{(n)}(t) - \lambda_i^{(n)}(s)|^2 \right)^2 \right] 
\leq \frac{1}{n^2} \mathbb{E} \left[ \left( \frac{1}{n} \sum_{i,j=1}^{n} \left( \sum_{k=1}^{p} \left( b_{ki}^{(n)}(t)b_{kj}^{(n)}(t) - b_{ki}^{(n)}(s)b_{kj}^{(n)}(s) \right) \right) \right)^2 \right] 
\leq \frac{1}{n^2} \mathbb{E} \left[ \sum_{i,j=1}^{n} \left( \sum_{k=1}^{p} \left( b_{ki}^{(n)}(t)b_{kj}^{(n)}(t) - b_{ki}^{(n)}(s)b_{kj}^{(n)}(s) \right) \right)^4 \right] 
\leq \frac{p^2}{n^2} \sum_{i,j=1}^{n} \sum_{k=1}^{p} \mathbb{E} \left[ \left( b_{ki}^{(n)}(t)b_{kj}^{(n)}(t) - b_{ki}^{(n)}(s)b_{kj}^{(n)}(s) \right)^4 \right] 
\leq \left( \frac{p}{n} \right)^4 C |t - s|^{4H} T^{-4H}.
\]

This completes the proof.  

\[\square\]

**Theorem 4.1.** Assume that \( p := p(n) \) is such that \( p/n \to c \), as \( n \to \infty \). Then the family of measures \( \{\mu_i^{(n)}, t \geq 0; n \geq 1\} \) is tight.

**Proof.** Following the ideas in [26], it is easily seen using (4.1) that for every \( 0 \leq t_1 \leq t_2 \leq T \), \( n \geq 1 \) and \( f \in C^1_b \),
\[
|\langle \mu_t^{(n)}, f \rangle - \langle \mu_{t_1}^{(n)}, f \rangle | \leq \frac{1}{n} \sum_{i=1}^{n} \left| f(\lambda_i^{(n)}(t)) - f(\lambda_i^{(n)}(t_1)) \right|. 
\]
(4.9)

On the other hand by (3.5), we know that for each \( n \geq 1 \), the functions \( \lambda_i^{(n)} \) are Hölder continuous of order \( \beta < H \). Therefore, since \( f' \) is bounded and applying the Mean Value Theorem, we deduce
\[
\left| f'(\lambda_i^{(n)}(r)) - f'(\lambda_i^{(n)}(s)) \right| \leq \|f'\|_\infty \left| \lambda_i^{(n)}(r) - \lambda_i^{(n)}(s) \right|.
\]

Hence using the above estimate, identity (4.8), and Jensen’s inequality we obtain
\[
\mathbb{E} \left[ |\langle \mu_t^{(n)}, f \rangle - \langle \mu_{t_1}^{(n)}, f \rangle |^4 \right] \leq \mathbb{E} \left[ \left( \frac{1}{n} \sum_{i=1}^{n} \left| f(\lambda_i^{(n)}(t)) - f(\lambda_i^{(n)}(t_1)) \right| \right)^4 \right] 
\leq \|f'\|_\infty^4 \mathbb{E} \left[ \left( \frac{1}{n} \sum_{i=1}^{n} \left| \lambda_i^{(n)}(t) - \lambda_i^{(n)}(t_1) \right| \right)^4 \right] 
\leq C_{f,T}|t_1 - t_2|^{4H}, \quad (4.10)
\]

where in the last inequality we used our assumption and \( C_{f,T} \) is a constant that depends on \( f' \) and \( T \).
Therefore, by the well known criterion that appears in [7] (see Prop. 2.4), we have that the sequence of continuous real processes \( \{(\mu_t^{(n)}, f), t \geq 0; n \geq 1\} \) is tight and consequently the sequence of processes \( \{(\mu_t^{(n)}, t \geq 0); n \geq 1\} \) is tight on \( C(\mathbb{R}_+, \mathbb{P}(\mathbb{R})) \).

4.2. **Weak convergence of the empirical measure of eigenvalues.** In the previous section, we proved that the family of measures \( \{(\mu_t^{(n)}, t \geq 0); n \geq 1\} \) is tight on \( C(\mathbb{R}_+, \mathbb{P}(\mathbb{R})) \). Now we proceed to identify the limit of any subsequence of such family. To this end, we first prove an estimate for the \( q \)-th moment, for \( q \in (1, 2) \), of the repulsion force between the eigenvalues of the fractional Wishart process, as the dimension goes to infinity.

**Lemma 4.3.** For all \( q \in (1, 2), i \in \{1, \ldots, n-1\} \), and \( t \geq 0 \) we have

\[
\mathbb{E} \left[ \left| \frac{|\lambda_i^{(n)}(t)|^q}{|\lambda_i^{(n)}(1) - \lambda_{i+1}^{(n)}(1)|^q} \right| \right] = O(n^{-1}), \quad \text{as } n \to \infty.
\]

**Proof.** For fix \( t \geq 0 \), let us consider the eigenvalues \( \{\lambda_i^{(n)}(t)\}_{i=1}^n \) of the matrix \( X^{(n)}(t) \). We observe that \( X^{(n)}(t) \) and \( t^{2H} X^{(n)}(1) \) have the same distribution, thus \( \{\lambda_i^{(n)}(t)\}_{i=1}^n \) and \( t^{2H} \{\lambda_i^{(n)}(1)\}_{i=1}^n \) also have the same distribution. In other words, it is enough to prove our result for \( t = 1 \).

On the other hand from identities (7.2.30) and (19.2.22) in [15] and following that same notation as in [15], we have that the joint distribution of two consecutive eigenvalues satisfies

\[
\mathbb{P} \left( \lambda_i^{(n)}(1) \in dx_i, \lambda_{i+1}^{(n)}(1) \in dx_{i+1} \right) = C_H \frac{n^2}{(n-1)} \det[K_{n1}(nx, ny)]_{x,y=x_i,x_{i+1}} dx_i dx_{i+1},
\]

where \( C_H \) is a normalizing constant that only depends on \( H \). Hence,

\[
\mathbb{E} \left[ \left| \frac{|\lambda_i^{(n)}(1)|^q}{|\lambda_i^{(n)}(1) - \lambda_{i+1}^{(n)}(1)|^q} \right| \right] = C_{n,H} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|x_i|^q}{|x_i - x_j|^q} n^2 \det[K_{n1}(nx, ny)]_{x,y=x_i,x_{i+1}} dx_i dx_{i+1}
\]

\[
= C_{n,H} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|u_i|^q}{|u_i - u_j|^q} \det[K_{n1}(x, y)]_{x,y=u_i,u_{i+1}} du_i du_j,
\]

with \( C_{n,t,H} = C_H n^{-1} (n - 1)^{-1} \). On the other hand using the identity (7.2.41) in [15] we have

\[
\lim_{n \to \infty} \frac{1}{n} \det[K_{n1}(x, y)]_{x,y=u_i,u_{i+1}}^2 = CK(u_i, u_{i+1}),
\]

where

\[
K(u_i, u_{i+1}) = 1 - \left[ s^2(r) + \left( \int_0^r s(t)dt - \frac{1}{2} \text{sign}(r) \right) s'(r) \right], \quad s(r) = \frac{\sin(\pi r)}{\pi r},
\]

with \( r = (u_i - u_{i+1})u_i^{-1/2} \) and \( C \) is a constant. Hence using the estimate (7.2.44) in [15], we note that

\[
\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|u_i|^q}{|u_i - u_{i+1}|^q} K(u_i, u_{i+1}) du_i du_{i+1} < \infty,
\]
which in turn implies,
\[ \lim_{n \to \infty} n \mathbb{E} \left[ \frac{|\lambda_i^{(n)}(1)|^q}{|\lambda_i^{(n)}(1) - \lambda_{i+1}^{(n)}(1)|^q} \right] < \infty. \]

This completes the proof. \( \square \)

The previous lemma allow us to prove the following proposition which is related to the convergence of multidimensional Skorokhod integrals that appears in (4.3). This result turns out to be crucial for identifying the limit of any subsequence of \( \{(\mu_t^{(n)}, t \geq 0); n \geq 1\} \).

**Proposition 4.1.** For any \( T > 0 \) and \( f \in C_0^2(\mathbb{R}) \), we have
\[ \lim_{n \to \infty} \frac{1}{n^{3/2}} \sum_{i=1}^n \sum_{k=1}^p \sum_{h=1}^n \int_0^t f'((\Phi_i(N^{(n)}(s)))) \frac{\partial \Phi_i}{\partial \delta b_{kh}}(N^{(n)}(s)) \delta b_{kh}(s) \right] = 0, \quad t \in [0, T], \] (4.11)
in probability.

**Proof.** For simplicity, we use the following notation for the Skorokhod integral with respect to the matrix fractional Brownian motion \( \{N^{(n)}(t), t \geq 0\} \),
\[ \int_0^t g_{i,n}^i(N^{(n)}(s)) \delta B(s) := \sum_{k=1}^p \sum_{h=1}^n \int_0^t g_{kh}^i(N^{(n)}(s)) \delta b_{kh}(s) \]
where
\[ g_{kh}^i(N^{(n)}(s)) := f'((\Phi_i(N^{(n)}(s)))) \frac{\partial \Phi_i}{\partial \delta b_{kh}}(N^{(n)}(s)), \quad \text{for} \quad i = 1, \ldots, n. \]

First, we show an \( L^q \) estimate which follows from the multidimensional Meyer’s inequalities (see for instance the proof of Proposition 3.5 of [22]). Hence for \( q \in (1, 2) \) and \( i \in \{1, \ldots, n\} \), we have
\begin{align*}
\mathbb{E} \left[ \left( \int_0^T \frac{1}{n^{3/2}} \sum_{i=1}^n g_{i,n}^i(N^{(n)}(s)) \delta N^{(n)}(s) \right)^q \right] \\
\leq c_{p,T,H} \left( \mathbb{E} \left[ \frac{1}{n^{3/2}} \sum_{i=1}^n g_{i,n}^i(N^{(n)}) \right]^q \right) + \mathbb{E} \left[ \left( \frac{1}{n^{3/2}} \sum_{i=1}^n g_{i,n}^i(N^{(n)}) \right)^q \right] \\
\leq c_{p,T,H} \left( \int_0^T \mathbb{E} \left[ \frac{1}{n^{3/2}} \sum_{i=1}^n g_{i,n}^i(N^{(n)}(s)) \right]^q ds \right) \\
+ \mathbb{E} \left[ \int_0^T \left( \int_0^T \frac{1}{n^{3/2}} \sum_{i=1}^n D_s g_{i,n}^i(N^{(n)}(s)) \right)^q ds \right]^{qH} dr \right) \] (4.12)
where \( c_{p,T,H} \) is a positive constant depending on \( p, H, \) and \( T; | \cdot | \) denotes the Euclidian norm and in the last inequality we apply Jensen’s inequality.
Therefore, we get recalling the definition of $g^{i,n}$ and using identity (2.4), it is clear by Jensen’s inequality

$$
\left| \mathbb{E} \left[ \frac{1}{n^{3/2}} \sum_{i=1}^{n} g^{i,n}(N^{(n)}(s)) \right] \right|^q = \left( \sum_{k=1}^{n} \sum_{h=1}^{n} \left( \mathbb{E} \left[ \frac{1}{n^{3/2}} \sum_{i=1}^{n} f'(\Phi_i(N^{(n)}(s))) \frac{\partial \Phi_i}{\partial b_{kh}}(N^{(n)}(s)) \right] \right)^2 \right)^{q/2}
$$

\[ \leq \left\| f' \right\|_{\infty}^q \left( \sum_{i=1}^{n} \sum_{k=1}^{p} \sum_{h=1}^{n} \mathbb{E} \left[ \frac{\partial \Phi_i}{\partial b_{kh}}(N^{(n)}(s)) \right] \right)^2 \]

\[ = 2^q \left\| f' \right\|_{\infty}^q \left( 1 \right)^q \left( \sum_{i=1}^{n} \sum_{k=1}^{p} \sum_{h=1}^{n} \mathbb{E} \left[ U^{(n)}_{hi}(s) \sum_{r} U^{(n)}_{ri}(s) b^{(n)}_{kr}(s) \right] \right)^{2q/2}
\]

\[ = 2^q \left\| f' \right\|_{\infty}^q \left( 1 \right)^q \left( \sum_{i=1}^{n} \sum_{k=1}^{p} \sum_{r} \mathbb{E} \left[ (U^{(n)}_{ri}(s))^2 (b^{(n)}_{kr}(s))^2 \right] \right)^{q/2}
\]

\[ = 2^q \left\| f' \right\|_{\infty}^q \frac{1}{qH} s^{qH}.
\]

Therefore, we get

$$
\int_{0}^{T} \left| \mathbb{E} \left[ \frac{1}{n^{3/2}} \sum_{i=1}^{n} g^{i,n}(N^{(n)}(s)) \right] \right|^q ds \leq \frac{2^q}{qH + 1} n^{-q/2} \left\| f' \right\|_{\infty}^q T^{qH+1}.
$$

For the other term in the second integral in the right hand side of (4.12), we use (2.5) to upper bound for the norm of the Malliavin derivative of $g^{i,n}$,

$$
\left( \left| \frac{1}{n^{3/2}} \sum_{i=1}^{n} D_s g^{i,n}(N^{(n)}(s)) \right| \right)^2 \leq \left( \frac{4}{n^3} \sum_{i=1}^{n} \sum_{k=1}^{p} \sum_{h=1}^{n} \left( \left\| f'' \right\|_{\infty}^2 \left( \frac{\partial \Phi_i}{\partial b_{kh}}(N^{(n)}(s)) \right)^4 + \left\| f' \right\|_{\infty}^2 \left| \frac{\partial^2 \Phi_i}{\partial b_{kh}^2}(N^{(n)}(s)) \right|^2 \right) \right)^{q/2}
$$

\[ \leq \left( \frac{4}{n^3} \sum_{i=1}^{n} \sum_{k=1}^{p} \sum_{h=1}^{n} \left( \left\| f'' \right\|_{\infty}^2 \left( \frac{\partial \Phi_i}{\partial b_{kh}}(N^{(n)}(s)) \right)^4 + 16 \left\| f' \right\|_{\infty}^2 \left[ (U^{(n)}_{hi}(r))^4 + H^2_{i,k,h,n}(r) \right] \right) \right)^{q/2},
\]

where

$$
H_{i,k,h,n}(r) = \sum_{i \neq j} \left| \sum_{r=1}^{n} U^{(n)}_{ij}(r) U^{(n)}_{hi}(r) b^{(n)}_{ki}(r) + \sum_{r=1}^{n} U^{(n)}_{ri}(r) U^{(n)}_{hj}(r) b^{(n)}_{kr}(r) \right|^2 \left| \lambda^{(n)}_i(r) - \lambda^{(n)}_j(r) \right|.
$$
Therefore there exists a constant $K_q > 0$, such that

\[
\left( \left| \frac{1}{n^{3/2}} \sum_{i=1}^{n} D_s g^{i,n}(N^{(n)}(s)) \right| \right)^2 
\leq \left( \frac{4}{n^3} \sum_{i=1}^{n} \left( 16 p \|f''\|_\infty^2 + \sum_{k=1}^{p} \sum_{h=1}^{n} \left( \|f''\|_\infty^2 \left( \frac{\partial \Phi_i}{\partial b_{kh}}(N^{(n)}(s)) \right)^4 + 16 \|f''\|_\infty^2 H^2_{i,k,h,n}(r) \right) \right) \right)^{q/2} 
\leq K_q \left( \frac{2q}{n^{3/2q}} \left( 4^q (np)^{q/2} \|f''\|_\infty^q + \|f''\|_\infty^q \left( \sum_{i=1}^{p} \sum_{k=1}^{p} \sum_{h=1}^{n} \left( \frac{\partial \Phi_i}{\partial b_{kh}}(N^{(n)}(s)) \right)^4 \right)^{q/2} 
\right) \right) + 4^q \|f''\|_\infty^q \left( \sum_{i=1}^{p} \sum_{k=1}^{p} \sum_{h=1}^{n} H_{i,k,h,n}(r) \right)^q \right)^{q/2}.
\]

where $K_q$ is a positive constant depending on $q$. Hence from the previous and Jensen’s inequalities, we have

\[
E \left[ \int_0^T \left( \int_0^T \left| \frac{1}{n^{3/2}} \sum_{i=1}^{n} D_s g^{i,n}(N^{(n)}(s)) \right| \frac{1}{n} \, ds \right)^{qH} \, dr \right] (4.13)
\]

On the other hand from Cauchy-Schwartz inequality, we obtain

\[
\sum_{k=1}^{p} \sum_{h=1}^{n} \left( \frac{\partial \Phi_i}{\partial b_{kh}}(N^{(n)}(s)) \right)^4 = 2^4 \sum_{k=1}^{p} \sum_{h=1}^{n} (U_{hi}^{(n)}(r))^4 \left( \sum_{r} U_{ri}^{(n)}(r) b_{kr}^{(n)}(r) \right)^4 \leq 2^4 \sum_{k=1}^{p} \sum_{r} b_{kr}^4(r).
\]

Therefore by Jensen’s inequality, we get

\[
E \left[ \left( \sum_{i=1}^{p} \sum_{k=1}^{p} \sum_{h=1}^{n} \left( \frac{\partial \Phi_i}{\partial b_{kh}}(N^{(n)}(s)) \right)^4 \right)^{q/2} \right] = 2^{2q} / n^{q/2} \left[ \sum_{i=1}^{n} \sum_{k=1}^{p} \sum_{r} b_{kr}^4(r) \right]^{q/2} \leq \tilde{K}_q 2^{2q} (np)^{q/2} r^{2qH},
\]

where $\tilde{K}_q$ is a positive constant depending only on $q$. 

In order to complete proof, we use Jensen’s inequality and observe that for
\[
\sum_{i=1}^{n} \sum_{k=1}^{p} \sum_{h=1}^{n} H_{i,k,h,n}(r) = \sum_{i=1}^{n} \sum_{k=1}^{p} \sum_{h=1}^{n} \sum_{i \neq j} |\sum_{l=1}^{n} U_{ij}^{(n)}(r)U_{hi}^{(n)}(r)b_{kl}^{(n)}(r) + \sum_{r=1}^{n} U_{ri}^{(n)}(r)U_{hj}^{(n)}(r)b_{kr}^{(n)}(r)|^{2} |\lambda_{i}^{(n)}(r) - \lambda_{j}^{(n)}(r)|
\]
\[
= \sum_{i=1}^{n} \sum_{i \neq j} \left| \frac{\lambda_{i}^{(n)}(r) + \lambda_{j}^{(n)}(r)}{\lambda_{i}^{(n)}(r) - \lambda_{j}^{(n)}(r)} \right|.
\]

Again, Jensen’s inequality implies
\[
\mathbb{E} \left[ \left( \sum_{i=1}^{n} \sum_{i \neq j} \left| \frac{\lambda_{i}^{(n)}(r) + \lambda_{j}^{(n)}(r)}{\lambda_{i}^{(n)}(r) - \lambda_{j}^{(n)}(r)} \right| \right)^{q} \right] \leq 2^{q+1} n^{2q-1} \sum_{i=1}^{n} \left( \mathbb{E} \left[ \left| \frac{\lambda_{i}^{(n)}(r)}{\lambda_{i}^{(n)}(r) - \lambda_{i+1}^{(n)}(r)} \right|^{q} \right] + \mathbb{E} \left[ \left| \frac{\lambda_{i+1}^{(n)}(r)}{\lambda_{i}^{(n)}(r) - \lambda_{i+1}^{(n)}(r)} \right|^{q} \right] \right). \]

Now using Lemma 4.3, we can conclude that there exists a constant \( c_{q} \) such that for \( n \) sufficiently large, the following upper bound holds
\[
\mathbb{E} \left[ \left( \sum_{i=1}^{n} \sum_{k=1}^{p} \sum_{h=1}^{n} H_{i,k,h,n}(r) \right)^{q} \right] \leq C_{q} n^{2q-1}.
\]

Therefore for \( n \) sufficiently, we have
\[
\mathbb{E} \left[ \left( \sum_{i=1}^{n} \sum_{k=1}^{p} \sum_{h=1}^{n} H_{i,k,h,n}(r) \right)^{q} \right] \leq C_{q} n^{2q-1}. \tag{4.15}
\]

Using the estimates (4.14) and (4.15) in (4.13), it is clear that there exist two positive constants \( K_{f,q,T} \) and \( \tilde{K}_{f,q,T} \) that depend only on \( T, q, \) and \( f \), such that for \( n \) sufficiently large
\[
\mathbb{E} \left[ \int_{0}^{T} \left( \int_{0}^{T} \left| \frac{1}{n^{3/2}} \sum_{i=1}^{n} D_{s} g^{i,n}(N^{(n)}(s)) \frac{1}{n} ds \right|^{q} \right) dr \right] \leq K_{f,q,T} p^{q/2} n^{-q} + \tilde{K}_{f,q,T} n^{q/2-1}.
\]

In order to complete proof, we use Jensen’s inequality and observe that for \( n \) sufficiently large
\[
\mathbb{E} \left[ \left| \int_{0}^{T} \frac{1}{n^{3/2}} \sum_{i=1}^{n} g^{i,n}(N^{(n)}(s)) \delta N^{(n)}(s) \right|^{q} \right] \leq \hat{K}_{f,q,T} n^{-q/2} + K_{f,q,T} p^{q/2} n^{-q} + \tilde{K}_{f,q,T} n^{q/2-1},
\]
where \( \hat{K}_{f,q,T} \) is a positive constant that depends on \( T, q, \) and \( f \).

Finally for \( \varepsilon, T > 0 \) and \( q < 2 \), we deduce
\[
\lim_{n \to \infty} \mathbb{P} \left( \left| \int_{0}^{T} \frac{1}{n^{3/2}} \sum_{i=1}^{n} g^{i,n}(N^{(n)}(s)) \delta N^{(n)}(s) \right| > \varepsilon \right) \leq \lim_{n \to \infty} \frac{1}{\varepsilon^{2}} \left( \hat{K}_{f,q,T} n^{-q/2} + K_{f,q,T} p^{q/2} n^{-q} + \tilde{K}_{f,q,T} n^{q/2-1} \right) = 0,
\]
implying
\[
\left| \int_0^T \frac{1}{n^{3/2}} \sum_{i=1}^n g^{i,n}(N^{(n)}(s)) \delta N^{(n)}(s) \right| \to 0,
\]
in probability, as \( n \) goes to \(+\infty\).

Before we prove our main result, we first characterize the family of laws \( (\mu_{c,H}(t), t \geq 0) \) of fractional dilations of a free Poisson distribution in terms of an initial valued problem of its corresponding Cauchy transforms. We note that the case \( H = 1/2 \) was proved in Corollary 3.1 in [4].

**Proposition 4.2.** The family \( (\mu_{c,H}(t), t \geq 0) \) is characterized by the property that its Cauchy transform is the unique solution to the initial value problem
\[
\begin{cases}
\frac{\partial G_{c,H}}{\partial t}(t, z) = 2H \left[ G_{c,H}^2(t, z) + \left[ 1 - c + 2zG_{c,1/2}(t, z) \right] \frac{\partial G_{c,H}}{\partial z}(t, z) \right] t^{2H-1}, & t > 0, \\
G_{c,H}(0, z) = \int_{\mathbb{R}} \frac{\mu_{c,H}(0)(dx)}{x - z}, & z \in \mathbb{C}^+,
\end{cases}
\]

satisfying \( G_t(z) \in \mathbb{C}^+ \) for \( z \in \mathbb{C}^+ \) and
\[
\lim_{\eta \to \infty} \eta |G_t(i\eta)| < \infty, \quad \text{for each} \quad t > 0.
\]

**Proof.** Recall from Section 1 that the family of fractional dilations of a free Poisson distribution satisfies that for each \( t > 0, \mu_{c,H}(t) = \mu_c^{f,p} \circ (h_t^H)^{-1} \), where \( \mu_c^{f,p} \) is the free Poisson distribution and \( h_t^H(x) = t^{2H}x \).

Therefore, the Cauchy transform \( G_{c,H}(t, z) \) of the distribution \( \mu_{c,H}(t) \) satisfies
\[
G_{c,H}(t, z) = \int_{\mathbb{R}} \frac{\mu_{c,H}(t)(dx)}{x - z} = \int_{\mathbb{R}} \frac{\mu_c^{f,p} \circ (h_t^H)^{-1}(dx)}{x - z} = \int_{\mathbb{R}} \frac{\mu_c^{f,p}(dx)}{xt^{2H} - z} = \frac{1}{t^{2H}} \int_{\mathbb{R}} \frac{\mu_c^{f,p}(dx)}{x - zt^{-2H}},
\]
for all non-real \( z \) with \( \text{Im}(z) \neq 0 \). The above equality implies that
\[
G_{c,H}(t, z) = \frac{1}{t^{2H}} G_c^{f,p}(zt^{-2H}) = G_{c,1/2}(t^{2H}, z),
\]
where \( G_c^{f,p} \) is the Cauchy transform of the free Poisson distribution.

If we assume that \( \mu_{c,H}(0) = \mu_{c,1/2}(0) \), then by Corollary 3.1 in [4] (see also Proposition 2.1 in [25]) we know that \( G_{c,1/2} \) is the unique solution to the initial value problem
\[
\begin{cases}
\frac{\partial G_{c,1/2}}{\partial t}(t, z) = G_{c,1/2}^2(t, z) + \left[ 1 - c + 2zG_{c,1/2}(t, z) \right] \frac{\partial G_{c,1/2}}{\partial z}(t, z), & t > 0, \\
G_{c,1/2}(0, z) = \int_{\mathbb{R}} \frac{\mu_{c,H}(0)(dx)}{x - z}, & z \in \mathbb{C}^+.
\end{cases}
\]
Therefore
\[
\frac{\partial G_{c,H}(t, z)}{\partial t} = \frac{\partial G_{c,1/2}(t^{2H}, z)}{\partial t} = 2H t^{2H-1} \frac{\partial G_{c,1/2}(t, z)}{\partial t} \bigg|_{(t,z)=(t^{2H},z)}
\]
\[
= 2H \left[ G_{c,1/2}(t^{2H}, z) + \left[ 1 - c + 2z G_{c,1/2}(t^{2H}, z) \right] \frac{\partial G_{c,1/2}(t^{2H}, z)}{\partial z} \right] t^{2H-1}
\]
\[
= 2H \left[ G_{c,H}(t, z) + \left[ 1 - c + 2z G_{c,H}(t, z) \right] \frac{\partial G_{c,H}(t, z)}{\partial z} \right] t^{2H-1}.
\]

On the other hand, observe
\[
G_{c,H}(0, z) = \int_{\mathbb{R}} \frac{\mu_{c,H}(0)(dx)}{x-z}.
\]

Finally the uniqueness to (4.16) follows from Corollary 3.1 in [4].

Now, we are ready to prove our main result, i.e., that the weak limit of the sequence of the measure-valued processes \{((\mu^{(n)}_t, t \geq 0); n \geq 1\} satisfies (1.6).

**Proof of Theorem 1.1.** From Theorem 4.1 we know that the family \{((\mu^{(n)}_t, t \geq 0); n \geq 1\} is relative compact. Hence, for our purposes, we take a subsequence \{((\mu^{(n)}_t, t \geq 0); \ell \geq 1\} and we assume that it converges weakly to \((\mu_t, t \geq 0)\). Therefore from (4.3) and with \(p_{\ell} = p(n_{\ell})\), we have
\[
\langle \mu^{(n_{\ell})}_t, f \rangle = \langle \mu^{(0)}_t, f \rangle + \frac{1}{n_{\ell}^{3/2}} \sum_{i=1}^{n_{\ell}} \sum_{k=1}^{n_{\ell}} \sum_{h=1}^{n_{\ell}} \int_0^t f'(\lambda^{(n_{\ell})}(s)) \frac{\partial \Phi_i}{\partial b_{kh}} (N^{(n_{\ell})}(s)) \delta_{bkh}(s)
\]
\[
+ H \int_0^t \int_\mathbb{R} (f'(x) - f'(y)) \frac{x+y}{x-y} \int_0^s (2H-1) \mu^{(n_{\ell})}_s(dx) \mu^{(n_{\ell})}_s(dy) ds
\]
\[
+ \frac{2H p_{\ell}}{n_{\ell}} \int_0^t f'(x)s^{2H-1} \mu^{(n_{\ell})}_s(dx) + \frac{2H}{n_{\ell}} \int_0^t \int_\mathbb{R} f''(x)s^{2H-1} \mu^{(n_{\ell})}_s(dx) ds.
\]

We observe that for \(0 \leq t \leq T\) the following holds
\[
\lim_{\ell \to \infty} \mathbb{E} \left[ \frac{2H}{n_{\ell}} \int_0^t \int_\mathbb{R} f''(x)s^{2H-1} \mu^{(n_{\ell})}_s(dx) ds \right] = \lim_{\ell \to \infty} \mathbb{E} \left[ \frac{H}{n_{\ell}^2} \int_0^t \sum_{i=1}^{n_{\ell}} f''(\lambda^{(n_{\ell})}(s)) 2\lambda^{(n_{\ell})}(s) s^{2H-1} ds \right]
\]
\[
\leq \lim_{\ell \to \infty} \frac{2H}{n_{\ell}^2} \|f''\|_\infty \int_0^t \mathbb{E} \left[ \sum_{i=1}^{n_{\ell}} \lambda^{(n_{\ell})}(s) \right] s^{2H-1} ds
\]
\[
\leq \lim_{\ell \to \infty} \frac{H p_{\ell}}{n_{\ell}^2} C_T \|f''\|_\infty \int_0^t s^{3H-1} ds = 0,
\]
where the last inequality follows from the \(L_q\) embedding and the upper bound from (4.8) and \(C_T\) is a positive constant that only depends on \(T\).

On the other hand, from Proposition 4.1 it is clear
\[
\lim_{\ell \to \infty} \left| \frac{1}{n_{\ell}^{3/2}} \sum_{i=1}^{n_{\ell}} \sum_{k=1}^{n_{\ell}} \sum_{h=1}^{n_{\ell}} \int_0^t f'(\lambda^{(n_{\ell})}_i(s)) \frac{\partial \Phi_i}{\partial b_{kh}} (N^{(n_{\ell})}(s)) \delta_{b_{kh}} \right| = 0
\]
in probability. Therefore there exists a subsequence \((n_{k_\ell})_{\ell \geq 0}\) such that the limit in (4.18) holds \(P\) a.s. Therefore using (4.17), we can conclude
\[
\langle \mu_t, f \rangle - \langle \mu_0, f \rangle - H \int_0^t \int_{\mathbb{R}^2} (f'(x) - f'(y)) \frac{x + y}{x - y} s^{2H-1} \mu_s(dx) \mu_s(dy) ds
- 2Hc \int_0^t \int_{\mathbb{R}} f'(x) s^{2H-1} \mu_s(dx)
= \lim_{\ell \to \infty} \langle \mu_t^{(n_{k_\ell})}, f \rangle - \langle \mu_0^{(n_{k_\ell})}, f \rangle - H \int_0^t \int_{\mathbb{R}^2} (f'(x) - f'(y)) \frac{x + y}{x - y} s^{2H-1} \mu_s^{(n_{k_\ell})}(dx) \mu_s^{(n_{k_\ell})}(dy) ds
- \frac{2Hp}{n} \int_0^t \int_{\mathbb{R}} f'(x) s^{2H-1} \mu_s^{(n_{k_\ell})}(dx) = 0.
\]
In other words, any weak limit \((\mu_t, t \geq 0)\) of a subsequence \((\mu_t^{(n_{k_\ell})}, t \geq 0)\) should satisfy (4.6).

Next, applying (4.18) to the deterministic sequence of functions
\[f_j(x) = \frac{1}{x - z_j}, \quad z_j \in (\mathbb{Q} \times \mathbb{Q}) \cap \mathbb{C}^+,
\]
and using a continuity argument, we get that the Cauchy-Stieltjes transform \((G_t, t \geq 0)\) of \((\mu_t, t \geq 0)\) satisfies the integral equation
\[
G_t(z) = \int_{\mathbb{R}} \frac{\mu_0(dx)}{x - z} + H \int_0^t \int_{\mathbb{R}^2} \left( \frac{1}{(y - z)^2} - \frac{1}{(x - z)^2} \right) \frac{x + y}{x - y} s^{2H-1} \mu_s(dx) \mu_s(dy) ds
- 2Hc \int_0^t \int_{\mathbb{R}} \frac{1}{(x - z)^2} s^{2H-1} \mu_s(dx).
\]
(4.19)
Using the above identity and making some straightforward computations, it is easy to verify that \((G_t, t \geq 0)\) satisfies (4.16), and therefore the family \((\mu_t, t \geq 0)\) corresponds to the family of fractional dilations of a free Poisson distribution.

Therefore, we conclude that all limits of subsequences of \((\mu_t^{(n)}, t \geq 0)\) coincide with the family \((\mu_t, t \geq 0)\), with Cauchy–Stieltjes transform given as the solution to (4.19), and thus the sequence \(\{(\mu_t^{(n)}, t \geq 0) : n \geq 1\}\) converges weakly to \((\mu_t, t \geq 0)\).

As a corollary of Theorem 1.1., we have the following result for the \(p\)-moments of the family of fractional dilations of a free Poisson distribution. For the \(p\)-moment, we use the following notation
\[m^p(t) = \int_0^\infty x^p \mu_t(dx).
\]

Corollary 4.1. For each \(p \geq 1\), and \(t > 0\), we have
\[m^p(t) = m^p(0) + 2Hpc \int_0^t m^{p-1}(s)s^{2H-1} ds + 2Hp \sum_{i=0}^{p-2} \int_0^t m^{p-1-i}(s)m^i(s)s^{2H-1} ds.
\]

Proof. The proof follows from (4.6) with \(f(x) = x^p\).
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