QUANTITATIVE LOCAL SENSITIVITY ESTIMATES FOR THE
RANDOM KINETIC CUCKER-SMALE MODEL WITH
CHEMOTACTIC MOVEMENT

SEUNG-YEAL HA

Department of Mathematical Sciences and Research Institute of Mathematics
Seoul National University, Seoul 08826, and
Korea Institute for Advanced Study, Hoegiro 85, Seoul, 02455, Republic of Korea

BORA MOON*

Department of Mathematical Sciences
Seoul National University
Seoul 08826, Republic of Korea

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ABSTRACT. In this paper, we present quantitative local sensitivity estimates for the kinetic chemotaxis Cucker-Smale (CCS) equation with random inputs. In the absence of random inputs, the kinetic CCS model exhibits velocity alignment under suitable structural assumptions on the turning kernel and reaction term despite of the random effect due to a turning operator. We provide a global existence of a regular solution with slow velocity alignment for the random kinetic CCS model within the proposed framework. Moreover, we investigate the propagation of regularity and stability of infinitesimal variations in random space.

1. Introduction. The purpose of this paper is to continue systematic studies begun in a series of works [1, 4, 10, 13, 14, 15, 16, 17, 18, 20] for the interplay of uncertainty quantification (UQ) and collective behaviors such as flocking and synchronization. In particular, we are interested in the velocity alignment in the ensemble of bacteria exhibiting an abrupt change of velocities in their motions. As reported in [22, 28, 35], collective behaviors in biological groups often occur through mutual communications. In 2007, Cucker and Smale introduced a simple analytical model for the flocking behavior of biological complex systems that generalizes Vicsek’s model [34]. After Cucker-Smale’s seminal work, most works on the modeling of flocking adapted their simple velocity alignment mechanism, which results in the smooth change of individual velocities. However, as we often see in the school of fish in aquarium, fish change their velocity abruptly, i.e., they have several components such as free swimming, tumbling and swarming etc. Recently, to model such...
an abrupt change of velocity in mesoscopic level, the kinetic chemotaxis Cucker-Smale (CCS) model was introduced in [8] by adopting a turning operator commonly used in the kinetic Keller-Segel type models [5, 6, 7, 24, 25, 33, 32]. We also refer to [19] for the application of flocking mechanism in phototaxis and [3, 29] for the collective modeling of chemotactic bacteria via a suitable modeling of the kernel in turning operator.

To incorporate uncertain effects to the kinetic CCS model, we introduce random vector $z$ defined on the sample space $\Omega \subset \mathbb{R}^d$. We assume that each component of $z$ is i.i.d., and let $\pi = \pi(z)$ be a probability density function for $z$. For the modeling of abrupt velocity change with random effects, we adopt the turning operator [8] involved with the randomness coefficients $\alpha = \alpha(z)$, $\kappa = \kappa(z)$: for a given $t > 0, x \in \mathbb{R}^d, z \in \Omega$, we set the turning kernel $T[S](t, x, v, v', z)$ to denote the rate of jumps from $v$ to $v'$, and define $T^+[S](t, x, v, v', z) := T[S](t, x, v', z)$ to denote the rate of jumps from $v'$ to $v$. Let $S = S(t, x, z)$ be the concentration of the chemical attractant whose dynamics is governed by the reaction-diffusion equation:

$$\partial_t S - \Delta S = -\kappa(z)\rho S, \quad \rho := \int_{\mathbb{R}^d} f dv, \quad x \in \mathbb{R}^d, \quad z \in \Omega, \quad t > 0, \quad (1)$$

where $\kappa(z)\rho S$ is a reaction term representing random chemical interactions between the Cucker-Smale particle and chemical substance with random reaction rate $\kappa$.

Note that this special ansatz for the reaction term is designed to decay to zero, as time goes on so that the effect of turning operator decays. In this way, the effect of abrupt velocity change with random effects is captured by the turning operator in the mesoscopic level. We refer to [2, 11, 12, 26, 27] for the contributions of the rate of change in a favorable direction with randomness:

$$T[S](f)(t, x, v, z) = \alpha(z) (T^+[S](f) - T^-[S](f)) := \alpha(z)\lambda[S](f), \quad (2)$$

where $\alpha$ is the desire of change in a favorable direction with randomness:

$$T^+[S](f)(t, x, v, z) := \int_{\mathbb{R}^d} T[S](t, x, v, v', z)f(t, x, v', z)dv',$$

$$T^-[S](f)(t, x, v, z) := \int_{\mathbb{R}^d} T^*[S](t, x, v, v', z)f(t, x, v, z)dv'$$

$$= \int_{\mathbb{R}^d} T[S](t, x, v', v, z)dv'f(t, x, v, z)$$

$$=: \lambda[S](t, x, v, z)f(t, x, v, z),$$

where the quantity $\lambda[S]$ denotes the turning frequency. Under this setting, we obtain the random kinetic CCS model by combining (1), (2) and the kinetic Cucker-Smale model: for $d \geq 2$,

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (F_a[f] f) = T[S](f), \quad x, v \in \mathbb{R}^d, \quad z \in \Omega, \quad t > 0,$$

$$\partial_t S - \Delta S = -\kappa(z)\rho S, \quad \rho := \int_{\mathbb{R}^d} f dv. \quad (3)$$

Here the alignment force $F_a[f]$ is given as follows:

$$F_a[f](t, x, v, z) := -\int_{\mathbb{R}^d} \psi(|x - x_*|, z)(v - v_*)f(t, x_*, v_*, z)dv_*dx_* \quad (4)$$
Finally, the dynamics of \( (f, S) \) is governed by (3) with the initial data:

\[
f(0, x, v, z) = f_0(x, v, z), \quad S(0, x, z) = S_0(x, z), \quad x, v \in \mathbb{R}^d, z \in \Omega.
\]

Note that the random variable \( z \) registers the uncertain effects in the communication weight function, parameters \( \alpha, \kappa \) and the initial data. If all randomness in the model are quenched, system (3)–(5) becomes the deterministic kinetic CCS model, which has been proposed to describe the dynamics of Cucker-Smale particles with chemotactic movements with velocity jumps and attraction toward chemotactic substances in [8].

In this paper, we are mainly interested in random effects on the flocking dynamics and regularity of the random kinetic equation via the local sensitivity analysis. Note that the kinetic density function \( f(t, x, v, z + dz) \) and chemotactic density function \( S(t, x, z + dz) \) can be expanded in \( z \)-variable via Taylor’s expansion:

\[
f(t, x, v, z + dz) = f(t, x, v, z) + \sum_{1 \leq j \leq m} \frac{\partial f}{\partial z_j} dz_j + \frac{1}{2!} \sum_{1 \leq j, k \leq m} \frac{\partial^2 f}{\partial z_j \partial z_k} dz_j dz_k + \cdots,
\]

\[
S(t, x, z + dz) = S(t, x, z) + \sum_{1 \leq j \leq m} \frac{\partial S}{\partial z_j} dz_j + \frac{1}{2!} \sum_{1 \leq j, k \leq m} \frac{\partial^2 S}{\partial z_j \partial z_k} dz_j dz_k + \cdots.
\]

Then, we can define the sensitivity matrices consisting of coefficients in the above expansion. The local sensitivity analysis deals with regularity and stability estimates for the sensitivity matrices [30, 31]. Such estimates are needed not only for analytical interest, but also for numerical methods such as stochastic Galerkin or collocation methods [23]. Note that for \( n \) random effects (i.e., \( z \in \Omega \subset \mathbb{R}^n \)), we can also get the same result which is presented in this paper, but it is rather lengthy and technical. So, we will assume that \( z \in \Omega \subset \mathbb{R}^1 \) for the simplicity of presentation. Next, we briefly discuss our three main results.

First, we present a framework on the initial data, the turning kernel and the communication weight leading to a regular global solution. Within our proposed framework in Section 3, our first result is concerned with pathwise \( W_{x,v}^{k,\infty} \times W_{x,v}^{k+1,\infty} \) estimate and flocking estimate using the Lyapunov functional approach. More precisely, we find a global unique regular solution \( (f, S) \) to (3)–(5) satisfying

\[
\int_{\mathbb{R}^{4d}} |v - v_s|^2 f(t, x, v_s, z) f(t, x, v, z) dv_s dv dx dv dx \leq \Phi(z)(1 + t)^{-d}, \quad z \in \Omega, \quad t \geq 0,
\]

where \( \Phi(z) \) is a nonnegative random variable (see Theorem 3.1 and Theorem 3.2).

Second, we provide the propagation of pathwise regularity, that is, for \( m \leq k \), every \( z \)-variations \( (\partial_z^m f, \partial_z^m S) \) is \( W_{x,v}^{k-m} \times W_{x,v}^{k+1-m} \)-regular if \( (f, S) \) is \( W_{x,v}^{k} \times W_{x,v}^{k+1} \) regular (see Theorem 4.4 and Theorem 4.5). We also show that \( (\partial_z^m f, \partial_z^m S) \) lies in the space \( L_x^2(\Omega; L_{x,v}^\infty(\mathbb{R}^{2d})) \times L_x^2(\Omega; L_{x,v}^\infty(\mathbb{R}^{2d})) \) (see Theorem 4.9).

Third, we present the pathwise stability estimates. Let \( (f, S) \) and \( (\tilde{f}, \tilde{S}) \) be global solutions corresponding to initial data \( (f_0, S_0) \) and \( (\tilde{f}_0, \tilde{S}_0) \), respectively. Then, for \( m \leq k \), there exists a nonnegative random variable \( C(T, z) \) for each \( z \in \Omega \), \( t \in (0, T) \) such that

\[
\sum_{0 \leq l \leq m} \left( \| \partial_z^l (f - \tilde{f})(t, z) \|_{W_{x,v}^{k-l-1,\infty}} + \| \partial_z^l (S - \tilde{S})(t, z) \|_{W_{x,v}^{k+1-l-1,\infty}} \right)
\]

\[
\leq C(T, z) \sum_{0 \leq l \leq m} \left( \| \partial_z^l (f_0 - \tilde{f}_0)(z) \|_{W_{x,v}^{k-l-1,\infty}} + \| \partial_z^l (S_0 - \tilde{S}_0)(z) \|_{W_{x,v}^{k+1-l-1,\infty}} \right),
\]
The rest of this paper is organized as follows. In Section 2, we propose structural assumptions on the turning kernel and the communication weight leading to the pathwise well-posedness estimates and flocking estimates, and provide a priori estimates for the random kinetic CCS equation. In Section 3, we provide pathwise $W_{x,v}^{k,\infty} \times W_{x,v}^{k+1,\infty}$ estimates and flocking estimates within the proposed framework. We also investigate the propagation of pathwise regularity and the pathwise stability for the $z$-variations of solutions in Section 4. Finally, Section 5 is devoted to a brief summary of our main results.

**Gallery of notation:** For simplicity, we use the following handy notation.

$$\sigma = (x,v), \quad \sigma_* = (x_*,v_*), \quad \|\psi\|_{\infty} := \|\psi\|_{L^\infty(\Omega; W^{k,\infty}(\mathbb{R}^d))},$$

$$\|\alpha\|_{k,\infty} := \|\alpha\|_{W^{k,\infty}(\Omega)}, \quad \|\kappa\|_{k,\infty} := \|\kappa\|_{W^{k,\infty}(\Omega)}, \quad k \geq 0.$$  

Let $\pi : \Omega \to \mathbb{R}_+ \cup \{0\}$ be a nonnegative probability density function, and we define a weighted $L^2$-space:

$$L^2_\pi(\Omega) := \{y : \Omega \to \mathbb{R} : \int_{\Omega} |g(z)|^2 \pi(z)dz < \infty\},$$

with an inner product and the corresponding $L^2$-norm:

$$\langle g_1, g_2 \rangle := \int_{\Omega} g_1(z)g_2(z)\pi(z)dz, \quad \|g\|_{L^2_\pi(\Omega)} := \left( \int_{\Omega} |g(z)|^2 \pi(z)dz \right)^{\frac{1}{2}}.$$  

For $m \in \mathbb{N} \cap \{0\}$, we set

$$\|g\|_{H^m_\pi(\Omega)} := \left( \sum_{l=0}^{m} \|\partial_z^l g\|_{L^2_\pi(\Omega)}^2 \right)^{\frac{1}{2}}.$$  

For $h = h(x,v,z) : \mathbb{R}^{2d} \times \Omega \to \mathbb{R}$, we define a mixed norm $H^m_\pi(\Omega; L_{x,v}^\infty)$ as follows:

$$\|h\|_{H^m_\pi(\Omega; L_{x,v}^\infty)} := \left( \sum_{l=0}^{m} \|\partial_z^l h\|_{L^2_\pi(\Omega; L_{x,v}^\infty(\mathbb{R}^{2d}))}^2 \right)^{\frac{1}{2}}.$$  

2. Preliminaries. In this section, we discuss structural assumptions on the turning kernel and communication weight function, and study a priori estimate for the random kinetic CCS model.

2.1. A framework. In this subsection, we discuss structural assumptions on the turning kernel $T$ and the communication weight $\psi$ leading to the pathwise well-posedness theory and the flocking estimate. Our proposed framework ($\mathcal{A}$) is formulated in terms of the turning kernel and communication weight function as follows.

($\mathcal{A}_1$) (Compactness and regularities of the turning kernel):

(i) The nonnegative function $T[S](t, x, v, v', z)$ has compact supports in $v$ and $v'$:

$$T[S](t, x, v, v', z) = 0, \quad v \text{ or } v' \notin V = B_R(0), \quad \text{for } x \in \mathbb{R}^d, \ z \in \Omega, \ t \geq 0.$$
(ii) If $S(t, x, z)$ is $W^{k+1}$-regular with respect to $x$, then $T[S](t, x, v, v', z)$ satisfies regularity conditions:

for $0 \leq |\mu| + |\nu| \leq k$,

$$|\partial_{\mu}^{\nu} T[S](t, x, v, v', z)| \leq |\partial_{x}^{\mu+v}[S(t, x - v', z)S(t, x + v, z)]|;$$

for $0 \leq |\mu| + |\nu| \leq k - 1$,

$$|\partial_{\mu}^{\nu} T[S](t, x, v, v', z) - T[\tilde{S}](t, x, v, v', z)|$$

$$\leq |\partial_{x}^{\nu}[S(t, x - v', z)S(t, x + v, z) - \tilde{S}(t, x - v', z)\tilde{S}(t, x + v, z)]|.$$  

(iii) If $\partial_{x} S(t, x, z)$ is $W^{k+1-l}$-regular with respect to $x$, then $T[S](t, x, v, v', z)$ satisfies the regularity conditions:

for $0 \leq |\mu| + |\nu| \leq k - l$,

$$|\partial_{\mu}^{\nu} \partial_{x} T[S](t, x, v, v', z)| \leq |\partial_{x}^{\mu+v}[S(t, x - v', z)S(t, x + v, z)]|,$$

for $0 \leq |\mu| + |\nu| \leq k - l - 1$,

$$|\partial_{\mu}^{\nu} \partial_{x} T[S](t, x, v, v', z) - T[\tilde{S}](t, x, v, v', z)|$$

$$\leq |\partial_{x}^{\nu}[S(t, x - v', z)S(t, x + v, z) - \tilde{S}(t, x - v', z)\tilde{S}(t, x + v, z)]|.$$  

$(A_2)$ (Positivity and boundedness of the communication weight function): there exist positive constants $\psi_m$ and $\psi_M$ such that

$$0 < \psi_m \leq \psi(|x|, z) \leq \psi_M < \infty, \quad x \in \mathbb{R}^d, \quad z \in \Omega.$$

**Remark 1.** In the sequel, we provide several comments on the above framework.

1. The assumptions (i) and (ii) in $(A_1)$ are the natural extensions of $[8]$ for a global well-posedness, and additional assumption (iii) in $(A_1)$ is needed for local sensitivity analysis. We also note that the difference $T[S] - T[\tilde{S}]$ in (ii) and (iii) in $(A_1)$ requires one-less regularity compared to the regularity of each turning kernel.

2. As in $[21]$, we can admit a relaxed communication weight $\psi$ satisfying

$$\frac{1}{(1 + |x - y|^2)^\beta} \leq \psi(|x - y|, z), \quad \beta \in \left(0, \frac{1}{2}\right), \quad z \in \Omega.$$  

(6)

to derive a well-posedness of strong solution in $W^{k,\infty}$-space and asymptotic flocking analysis pathwise. In fact, following the method of bi-characteristics, we only need to assume the uniform boundedness of $\psi$ in spatial variable for the boundedness of velocity support and second velocity moment (see Lemma 3.1 of $[8]$): for $T \in (0, \infty)$,

$$\sup_{0 \leq t < T} \sup_{x \in \mathbb{R}^d} \frac{1}{\psi(t, x, \cdot, \cdot)} + \sup_{0 \leq t < T} \int_{\mathbb{R}^d} |v|^2 f(t, x, v, z) dv dx \leq C(T), \quad z \in \Omega.$$

In $[8]$, the existence of a positive lower bound for $\psi$ as in $(A_2)$ was used for the simplicity of presentation and we also adopted the same condition for the same reason. Thus, one can use the assumption (6) to derive the well-posedness of strong solution and asymptotic flocking estimate pathwise with more technical estimates.
2.2. A priori estimates. In this subsection, we present a priori estimates for the random kinetic CCS model. First, we set the heat kernel on $\mathbb{R}^d \times \mathbb{R}^d$ as
\[
e^\Delta (x, y) := (4\pi t)^{-\frac{d}{2}} \exp \left(-\frac{|x-y|^2}{4t}\right),
\]
and we set $T \in (0, \infty)$.

Lemma 2.1. Suppose that the framework $(A_1)$-$(A_2)$ holds, and random parameters and the initial data $(f_0, S_0)$ satisfy
\[
\|\alpha\|_\infty < \infty, \|\kappa\|_\infty < \infty, f_0 \in L^\infty(\Omega; (L^1_+ \cap L^\infty)(\mathbb{R}^{2d})), S_0 \in L^\infty(\Omega; (L^1_+ \cap L^\infty)(\mathbb{R}^d)),
\]
and let $(f, S)$ be a regular solution process to (3). Then, for $d \geq 2$, $z \in \Omega$ and $t \in [0, T)$, we have
\[
\begin{align*}
\|S(t, \cdot, z)\|_{L^1} &\leq \|S_0(\cdot, z)\|_{L^1}, \quad \|S(t, \cdot, z)\|_{L^\infty} \leq \|S_0(\cdot, z)\|_{L^\infty}, \\
\|S(t, \cdot, z)\|_{L^\infty} &\leq (4\pi t)^{-\frac{d}{2}} \|S_0(\cdot, z)\|_{L^1}, \\
\|S(t, \cdot, z)\|_{L^p} &\leq C, \quad \|f(t, \cdot, z)\|_{L^\infty} \leq C(T)\|f_0(\cdot, z)\|_{L^\infty},
\end{align*}
\]
where $C$ and $C(T)$ are positive constants.

Proof. The estimates in (7) can be derived from the integral equation:
\[
S(t, x, z) = \int_{\mathbb{R}^d} e^{t \Delta} (x, y) S_0(y) dy - \kappa(z) \int_0^t \int_{\mathbb{R}^d} e^{(t-s) \Delta} (x, y) S(s, y, z) \rho(s, y, z) dy ds,
\]
which comes from Duhamel’s principle for the inhomogeneous heat equation. We refer to [8] for details.

Next, we discuss the propagation of velocity moments in $f$. We define the first three velocity moments, $M_0, M_1$ and $M_2$ which represent the total mass, momentum, and twice the value of energy, respectively: for $(t, z) \in \mathbb{R}_+ \times \mathbb{R}^{2d}$,
\[
M_0[f] := \int_{\mathbb{R}^{2d}} f(t, \sigma, z) d\sigma, \quad M_1[f] := \int_{\mathbb{R}^{2d}} v f(t, \sigma, z) d\sigma, \quad M_2[f] := \int_{\mathbb{R}^{2d}} |v|^2 f(t, \sigma, z) d\sigma.
\]
For notational simplicity, we drop $f$ from the moments of (3):
\[
M_i(t, z) := M_i[f](t, z), \quad i = 0, 1, 2.
\]

In the following two propositions, we study the temporal evolution of velocity moments.

Proposition 1. Let $(f, S) := (f(t, x, v, z), S(t, x, z))$ be a regular solution process to (3), and $f$ decays fast enough at infinity in phase space $\mathbb{R}^{2d}$. Then, for $d \geq 2$, $z \in \Omega$ and $t \geq 0$,
\[
(i) \partial_t M_0(t, z) = 0, \quad \partial_t M_1(t, z) = \int_{\mathbb{R}^{2d}} v T[S](f)(t, \sigma, z) d\sigma,
\]
\[
(ii) \partial_t M_2(t, z) = - \int_{\mathbb{R}^d} \psi(|x-y|, z) |v-v_\alpha|^2 f(t, \sigma, z) f(t, \sigma, z) d\sigma d\sigma
\]
\[
+ \alpha(z) \int_{\mathbb{R}^d} |v|^2 T(t, x, v, v', z) f(t, x, v', z) - |v|^2 T(t, x, v', v, z) f(t, x, v, z) dv' d\sigma.
\]

Proof. The proof is essentially the same as the deterministic counterpart. See [8] for details.
Proposition 2. Suppose that the framework (A1)-(A2) holds, and random parameters and the initial data \((f_0, S_0)\) satisfy
\[ \|\alpha\|_{\infty}, \|\kappa\|_{\infty} < \infty, \quad f_0 \in L^\infty(\Omega; (L_1^2 \cap L^\infty)(\mathbb{R}^d)), \quad S_0 \in L^\infty(\Omega; (L_1^1 \cap L^\infty)(\mathbb{R}^d)), \]
and let \((f, S)\) be a regular solution process to (3). Then, for \(d \geq 2, \ z \in \Omega\) and \(t \geq 0\),
\[ |M_1(t, z)| \leq |M_1(0, z)| + D_1(z), \quad M_2(t, z) \leq D_2(z)(M_2(0, z) + 1), \]
where
\[
D_1(z) := 2\|\alpha\|_{\infty}R|B_R(0)|M_0(0, z)\left(\|S_0(z)\|_{L^\infty}^2 + \frac{1}{(d - 1)(4\pi)^d}\|S_0(z)\|_{L_x^1}^2\right),
\]
\[
D_2(z) := \max\left\{1; \frac{D(z)}{2\psi_mM_0(0, z)}\right\},
\]
\[
D(z) := 2\psi_M(|M_1(0, z)| + D_1(z))^2 + \|\alpha\|_{\infty}R^2|B_R(0)||S_0(z)||_{L^\infty}^2M_0(0, z).
\]

Proof. (i) We use Proposition 1 to derive
\[
|M_1(t, z)| = |M_1(0, z)| + \int_0^t \int_{\mathbb{R}^d} vT[S]f(s, \sigma, z)\, d\sigma ds
\leq |M_1(0, z)| + \int_0^t (I_{11} + I_{12}) ds,
\]
where
\[
I_{11} := \left|\int_{\mathbb{R}^d} vT(s, x, v, v', z)f(s, x, v', z) dv' d\sigma\right|
\leq \|\alpha\|_{\infty} \int_{\mathbb{R}^d} \int_{B_R} \|v||S(s, x - v', z)S(s, x + v, z)|f(s, x, v', z) dv' d\sigma
\leq \|\alpha\|_{\infty}R|B_R(0)||M_0(0, z)||S(s, z)||_{L_x^\infty}^2,
\]
\[
I_{12} := \left|\int_{\mathbb{R}^d} vT(s, x, v, v', z)f(s, x, v, z) dv' d\sigma\right|
\leq \|\alpha\|_{\infty} \int_{\mathbb{R}^d} \int_{B_R} \|v||S(s, x - v, z)S(s, x + v', z)|f(s, x, v, z) dv' d\sigma
\leq \|\alpha\|_{\infty}R|B_R(0)||M_0(0, z)||S(s, z)||_{L_x^\infty}^2.
\]
By Lemma 2.1, we have
\[
\int_0^t \|S(s, z)||_{L_x^\infty}^2 ds \leq \int_0^1 \|S(s, z)||_{L_x^\infty}^2 ds + \int_{1}^{\infty} \|S(s, z)||_{L_x^\infty}^2 ds
\leq \|S_0(z)||_{L_x^\infty}^2 + \int_1^{\infty} (4\pi s)^{-d} \|S_0(z)||_{L_x^\infty}^2 ds \leq \|S_0(z)||_{L_x^\infty}^2 + \frac{1}{(d - 1)(4\pi)^d}\|S_0(z)||_{L_x^1}^2.
\]
Thus, we have
\[ |M_1(t, z)| \leq |M_1(0, z)| + D_1(z), \quad t \geq 0 \quad \text{and} \quad z \in \Omega. \]
(ii) Again, we use Proposition 1 to obtain

\[
\frac{\partial}{\partial t} M_2(t, z) \leq - \int_{\mathbb{R}^d} \psi(|x - x_s|, z) |v - v_s|^2 p(t, \sigma, z) f(t, \sigma, z) d\sigma_d \sigma d\sigma
\]

\[
+ \alpha(z) \int_{\mathbb{R}^d} |v|^2 T(t, x, v, v', z) f(t, x, v', z) dv' d\sigma
\]

\[
\leq -2\psi_m M_0(0, z) M_2(t, z) + 2\psi_M |M_1(t, z)|^2
\]

\[
+ \|\alpha\|_\infty \int_{\mathbb{R}^d} \int_{B_R} |v|^2 |S(t, x - v', z) S(t, x + v, z)| f(t, x, v', z) dv' d\sigma
\]

\[
\leq -2\psi_m M_0(0, z) M_2(t, z) + 2\psi_M |M_1(t, z)|^2
\]

\[
+ \|\alpha\|_\infty R^2 |B_R(0)| \|S_0(z)\|_{L^\infty}^2 M_0(0, z)
\]

\[
\leq -2\psi_m M_0(0, z) M_2(t, z) + D(z).
\]

Then Grönwall’s lemma yields

\[
M_2(t, z) \leq e^{-2\psi_m M_0(0, z)t} M_2(0, z) + \frac{D(z)}{2\psi_m M_0(0, z)}, \quad t \geq 0 \quad \text{and} \quad z \in \Omega.
\]

\[ \square \]

Now, we present a global well-posedness of a strong solution to (3).

**Theorem 2.2.** Suppose that the framework \((A_1)-(A_2)\) holds, and random parameters, the initial data satisfy

\[
\|\alpha\|_\infty, \|\kappa\|_\infty < \infty, f_0 \in L^\infty(\Omega; (L^1_1 \cap W^{1,\infty})(\mathbb{R}^d)), S_0 \in L^\infty(\Omega; (L^1_1 \cap W^{2,\infty})(\mathbb{R}^d)),
\]

\[
|M_1(0, z)| + M_2(0, z) < \infty, \quad \text{supp}_{\mathbb{C}} f_0(x, \cdot, z) \subset \mathcal{V}_0 \subset \subset \mathbb{R}^d, \quad \text{for} \ d \geq 2, \ x \in \mathbb{R}^d, \ z \in \Omega.
\]

Then, there is a unique global solution process \((f, S)\) to (3) such that for any \(T > 0\),

\[
f \in L^\infty((0, T); L^\infty(\Omega; L^1_1 \cap W^{1,\infty}(\mathbb{R}^d))), \ S \in L^\infty((0, T); L^\infty(\Omega; L^1_1 \cap W^{2,\infty}(\mathbb{R}^d))),
\]

\[
\text{supp}_{\mathbb{C}} f(t, x, \cdot, z) \subset \mathcal{V} \subset \subset \mathbb{R}^d, \quad \text{for} \ x \in \mathbb{R}^d, \ z \in \Omega, \ t \in [0, T),
\]

where \(\mathcal{V}_0\) is a compact set containing \(B_R(0)\) and \(\mathcal{V}\) is another compact set containing \(\mathcal{V}_0\).

**Proof.** The proof is basically the same as that of Theorem 3.1 in [8]. \[ \square \]

**Remark 2.** (1) The local well-posedness of a solution in Theorem 2.2 is shown in [8] by constructing of approximate solutions \((f^n, S^n)\) using the standard successive approximation. In other words, the proof is based on the method of characteristics for (3). For \(x, v \in \mathbb{R}^d\) and \(z \in \Omega\), we set

\[
(x^n(t, z), v^n(t, x), f^n(t, z)) := (x^n(t; 0, \sigma, z), v^n(t; 0, \sigma, z), f^n(t, x^n(t; 0, \sigma, z), v^n(t; 0, \sigma, z), z)),
\]
as a solution to the following ODE system: for $t > 0$ and $z \in \Omega$,
\[
\frac{\partial}{\partial t} x^n(t, z) = v^n(t, z),
\]
\[
\frac{\partial}{\partial t} v^n(t, z) = -\nu^n(t, z) \int_{\mathbb{R}^d} \psi(|x^n(t, z) - x|) f^{n-1}(t, \sigma, z) d\sigma \nu^n(t, z) d\sigma,
\]
\[
\frac{\partial}{\partial t} f^n(t, z) = d \left[ \int_{\mathbb{R}^d} \psi(|x^n(t, z) - x|) f^{n-1}(t, \sigma, z) d\sigma \right] f^n(t, z) - \alpha(z) \Lambda[S^{n-1}](t, x^n(t, z), v^n(t, z), z) f^n(t, z)
\]
\[
+ \alpha(z) T^+[S^{n-1}](f^{n-1})(t, x^n(t, z), v^n(t, z), z) := h(t, x^n(t, z), v^n(t, z), z) f^n(t, z)
\]
\[
+ \alpha(z) T^+[S^{n-1}](f^{n-1})(t, x^n(t, z), v^n(t, z), z),
\]
and Duhamel’s principle for (3) \text{2}:
\[
\partial_t S^n(t, x, z) = \int_{\mathbb{R}^d} e^{\Delta(x,y)} S_0(y) dy - \kappa \int_0^t \int_{\mathbb{R}^d} e^{(t-s)\Delta(x,y)} S^n \rho^{n-1}(s, y, z) dy ds.
\]
For a global well-posedness, the authors in [8] extended a local solution to a global one by showing the corresponding norm of solution does not blow-up in finite time.

(2) If we denote the diameter of the velocity support and the position support by $\Gamma$ for (3). For the sake of simplicity, we briefly denote the random turning operator by $\mathcal{T}[S](f)(t, x, v, z)$
\[
\mathcal{T}[S](f)(t, x, v, z) = \alpha(z) \int_{\mathbb{R}^d} T[S](t, x, v, v', z) f(t, x, v', z) - T^*[S](t, x, v, v', z) f(t, x, v, z) dv'
\]
\[
= \alpha(z) \int_{\mathbb{R}^d} (T f' - T f^*) dv' := \alpha(z) \Lambda[S](f)(t, x, v, z).
\]

3. Pathwise well-posedness and flocking estimate. In this section, we present the pathwise well-posedness and velocity alignment estimate for the smooth solution to (3). For the sake of simplicity, we briefly denote the random turning operator by $\mathcal{T}[S](f)(t, x, v, z)$
\[
\mathcal{T}[S](f)(t, x, v, z) = \alpha(z) \int_{\mathbb{R}^d} T[S](t, x, v, v', z) f(t, x, v', z) - T^*[S](t, x, v, v', z) f(t, x, v, z) dv'
\]
\[
= \alpha(z) \int_{\mathbb{R}^d} (T f' - T f^*) dv' := \alpha(z) \Lambda[S](f)(t, x, v, z).
\]

3.1. Pathwise well-posedness. In this subsection, we present a global existence of unique regular solution $(f, S) \in W^{k, \infty} \times W^{k+1, \infty}$ to (3). Although we considered the global well-posedness of a strong solution in $W^{1, \infty} \times W^{2, \infty}$-space in Section 2.2, we need to extend its analysis to regular solutions with high regularity for a later use.

Theorem 3.1. Suppose that the framework $(A_1)-(A_2)$ holds, and in addition, the following assumptions hold: for $d \geq 2$,

(1) Initial datum $f_0$ is compactly supported with respect to $v$ variable, and has finite moments: there exists a positive bounded function $V(z)$ such that
\[
V(0, z) \leq V(z), \quad \text{and} \quad M_0(0, z) + M_2(0, z) < \infty.
\]
Then, there exists a unique global solution process \((f, S)\) to (3) with the following regularity: for \(z \in \Omega\) and \(T \in (0, \infty)\), there exists a random variable \(C(T, z)\) such that
\[
\sup_{0 \leq t < T} \left( \sum_{0 \leq |\mu| + |\nu| \leq k} \|\partial_x^\mu \partial_v^\nu f(t, z)\|_{L^\infty_{x,v}} + \sum_{0 \leq |\mu| \leq k+1} \|\partial_x^\mu S(t, z)\|_{L^\infty_T}\right) \leq C(T, z). \tag{10}
\]

Proof. The proof is basically the same as that of Theorem 2.2. So, we will provide a priori estimate of (10) using induction argument on \(k \geq k' \geq 1\). To do this, we first introduce two functionals:

\[
A_k(t, z) := \sum_{0 \leq |\mu| + |\nu| \leq k} \|\partial_x^\mu \partial_v^\nu f(t, z)\|_{L^\infty_{x,v}} + \sum_{0 \leq |\mu| \leq k+1} \|\partial_x^\mu S(t, z)\|_{L^\infty_T},
\]

\[
B_{k+1}(t, z) := \sum_{|\mu| + |\nu| = k} \|\partial_x^\mu \partial_v^\nu f(t, z)\|_{L^\infty_{x,v}}, \quad C_{k+1}(t, z) := \sum_{|\mu| = k+1} \|\partial_x^\mu S(t, z)\|_{L^\infty_T}.
\]

• Initial step \((k' = 1)\): It is obviously true due to Theorem 2.2.

• Inductive step \((k' \geq 2)\): In this step, suppose that for \(z \in \Omega\),
\[
\sup_{0 \leq t < T} A_k(t, z) \leq c_1(T, z) < \infty.
\]

Next, we will show that for \(z \in \Omega\),
\[
\sup_{0 \leq t < T} B_{k+1}(t, z) \leq c_2(T, z), \quad \sup_{0 \leq t < T} C_{k+1}(t, z) \leq c_3(T, z),
\]

respectively. As a result we can conclude
\[
\sup_{0 \leq t < T} A_{k' + 1}(t, z) \leq C(T, z), \quad \text{for } z \in \Omega, \quad k \geq k' \geq 1.
\]

\(\diamond\) Step A. (finiteness of \(B_{k'+1}(t, z)\)): For this, we consider the cases: \(\nu = 0\) and \(\nu \geq 1\), separately.

(i) \((\nu = 0)\) case: For each \(\mu\) with \(|\mu| = k'\), we take \(\partial_x^\mu\) to (3) \(_1\), to obtain

\[
\partial_t (\partial_x^\mu f) + v \cdot \nabla_x (\partial_x^\mu f) + F_a[f] \cdot \nabla_v (\partial_x^\mu f) = -\nabla_v \cdot F_a[f] \partial_x^\mu f
\]

\[
- \sum_{0 < \mu' \leq \mu} \left( \mu \mu' \right) \left( \partial_x^{\mu'} F_a[f] \cdot \nabla_v (\partial_x^{\mu'-\mu} f) + \nabla_v \cdot (\partial_x^{\mu'} F_a[f]) \partial_x^{\mu'-\mu} f \right) + \alpha(z) \int_{\mathbb{R}^d} \sum_{0 \leq \mu' \leq \mu} \left( \mu \mu' \right) \left( \partial_x^{\mu'} T \partial_x^{\mu'-\mu} f - \partial_x^{\mu'} T' \partial_x^{\mu'-\mu'} f \right) dv'
\]

\[
= -\mathcal{I}_{21} - \mathcal{I}_{22} + \mathcal{I}_{23}.
\]

We integrate the equation (11) along the particle trajectory to obtain
\[
|\partial_x^\mu f(t, x, v, z)| \leq |\partial_x^\mu f_0(x, v, z)| + \int_0^t \mathcal{I}_{21} + \mathcal{I}_{22} + |\mathcal{I}_{23}| \, ds. \tag{12}
\]
For the term $F_u[f]$ in $I_{21}$ and $I_{22}$, we observe that
\[
|\nabla \cdot F_u[f]| \leq d\|\psi\|_\infty M_0(z), \quad |\partial \mu F_u[f]| \leq 2V(z)\|\psi\|_\infty M_0(z),
\]
\[
|\nabla \cdot (\partial \mu^T F_u[f])| \leq d\|\psi\|_\infty M_0(z).
\]
For the terms $T, T^*$ in $I_{22}$, we use the assumption $(A_1)$ to have
\[
|\partial^\mu T| \leq \sum_{0 \leq \mu'' \leq \mu'} |\partial^\mu'' S(t, x - v', z)\partial^\mu'',\nu'' S(t, x + v, z)|
\]
\[
\leq \sum_{0 \leq \mu'' \leq \mu'} \|\partial^\mu'' S(t, z)\|_{L^\infty} \|\partial^\mu'',\nu'' S(t, z)\|_{L^\infty}.
\]
Thus, we substitute these estimates into (12) to obtain
\[
\|\partial^\mu f(t, z)\|_{L^\infty, v} \leq \|\partial^\mu f_0(z)\|_{L^\infty, v} + \int_0^t \|\nabla \cdot M_0(z)\|_{L^\infty, v} + \int_0^t 2V(z)\|\psi\|_\infty M_0(z) \sum |t, z| = 1 \|\partial^\mu T f(t, z)\|_{L^\infty, v} ds
\]
\[
+ \int_0^t 2\|\alpha\|_\infty (A_{k'}(s, z))^2 (\|\partial^\mu f(s, z)\|_{L^\infty, v} + A_{k'}(s, z)) ds.
\]
This yields
\[
\|\partial^\mu f(t, z)\|_{L^\infty, v} \leq \|\partial^\mu f_0(z)\|_{L^\infty, v} + \int_0^t c(s, z) \left( \|\partial^\mu f(s, z)\|_{L^\infty, v} + \sum |t, z| = 1 \|\partial^\mu T f(t, z)\|_{L^\infty, v} + 1 \right) ds.
\]
\[
(13)
\]
(ii) $(|\nu| \geq 1)$ case: For $\mu$ and $\nu$ with $|\mu| + |\nu| = k'$, we take $\partial^\nu \partial^\mu$ to (3) to obtain
\[
\partial_t (\partial^\nu \partial^\mu f) + v \cdot \nabla (\partial^\nu \partial^\mu f) + F_u[f] \cdot \nabla (\partial^\nu \partial^\mu f) = -\sum |t, z| = 1 \left( \nu \right) \partial^\nu \partial^\mu f + \sum_{\mu', \mu''} \left( \mu'' \mu' \right) \nabla \partial^\nu \partial^\mu T \partial^\mu \partial^\nu f' - \sum_{\nu', \nu''} \left( \nu \nu' \right) \partial^\nu \partial^\mu T^* \partial^\nu \partial^\nu \partial^\nu f' dv'.
\]
\[
(14)
\]
We use the assumption (A1) in Section 2.1 for the terms $T$ and $T^*$:
\[
|\partial^\nu \partial^\mu T| \leq \sum_{\mu', \mu''} \left( \nu \nu' \right) \left( \mu' \mu'' \right) \partial^\nu \partial^\mu S(t, x - v, z) \partial^\nu \partial^\mu T \partial^\nu \partial^\nu \partial^\nu S(t, x + v, z)
\]
\[
\leq \sum_{\mu', \mu''} \left( \nu \nu' \right) \left( \mu' \mu'' \right) \|\partial^\nu \partial^\mu S(t, z)\|_{L^\infty} \|\partial^\nu \partial^\nu S(t, z)\|_{L^\infty}.
\]
Similar, the estimate for $T^\ast$ can be made as follows.

$$
\| \partial_x^{\mu} \partial_x^{\nu} f(t, z) \|_{L^\infty_{t, \nu}} \leq \| \partial_x^{\mu} \partial_x^{\nu} f_0(z) \|_{L^\infty_{x, \nu}} + \int_0^t c(s, z) \left( \| \partial_x^{\mu} \partial_x^{\nu} f(s, z) \|_{L^\infty_{x, \nu}} \sum_{|\mu|+|\nu|=k} \| \partial_x^{\mu'} \partial_x^{\nu'} f(s, z) \|_{L^\infty_{x, \nu}} + 1 \right) ds. \quad (15)
$$

Now, we combine (13) and (15) to derive

$$
\sum_{|\mu|+|\nu|=k} \| \partial_x^{\mu} \partial_x^{\nu} f(t, z) \|_{L^\infty_{x, \nu}} \leq \sum_{|\mu|+|\nu|=k} \| \partial_x^{\mu} \partial_x^{\nu} f_0(z) \|_{L^\infty_{x, \nu}} + \int_0^t c(s, z) \left( \sum_{|\mu|+|\nu|=k} \| \partial_x^{\mu} \partial_x^{\nu} f(s, z) \|_{L^\infty_{x, \nu}} + 1 \right) ds. \quad (16)
$$

Finally, we apply Grönwall’s lemma to (16) to get the finiteness of $B_{k' + 1}(t, z)$ for $0 \leq t < T$:

$$
\sup_{0 \leq t < T} B_{k' + 1}(t, z) \leq c_2(T, z).
$$

◊ Step B. (finiteness of $C_{k' + 1}(t, z)$): Consider $|\mu| = k' + 1$. Then, it follows from (3) that

$$
S(t, x, z) = \int_{\mathbb{R}^d} e^{t \Delta} (x, y) S_0(y) dy - \kappa(z) \int_0^t \int_{\mathbb{R}^d} e^{(t-s) \Delta} (x, y) S(s, y, z) \rho(s, y, z) dy ds.
$$

Next, we choose $j$ such that the $j$-th component of $\mu$ is not zero, and $e_j$ denotes the unit vector with $j$-th component 1. Then, we apply $\partial_x^{\mu}$ to $S(t, x, z)$ to obtain

$$
\partial_x^{\mu} S(t, x, z) = \int_{\mathbb{R}^d} e^{t \Delta} (x, y) \partial_x^{\mu} S_0(y, z) dy - \kappa(z) \int_0^t \int_{\mathbb{R}^d} [e^{(t-s) \Delta} (x, y)]_{x_j} \times \sum_{\mu' \leq \mu - e_j} \left( \begin{array}{c} \mu - e_j \\ \mu' \end{array} \right) \partial_x^{\mu} S(s, y, z) \partial_x^{\mu' - e_j} \rho(s, y, z) dy ds. \quad (17)
$$

Since $\int_{-\infty}^{\infty} e^{-r^2} dr = \sqrt{\pi}$, direct calculation shows

$$
\int_{\mathbb{R}^d} [e^{(t-s) \Delta} (x, y)]_{x_j} dy = \frac{1}{\sqrt{\pi(t-s)}}. \quad (18)
$$

We insert (18) into (17) to see

$$
\sup_{0 \leq t < T} \| \partial_x^{\mu} S(t, z) \|_{L^\infty} \leq \| \partial_x^{\mu} S_0(z) \|_{L^\infty} + \| \kappa \|_\infty c_1(T, z) c_2(T, z) \int_0^t \frac{1}{\sqrt{\pi(t-s)}} ds.
$$

This concludes

$$
\sup_{0 \leq t < T} C_{k' + 1}(t, z) \leq c_3(T, z).
$$
3.2. **Velocity alignment.** In this subsection, we present the velocity alignment estimate by using the Lyapunov functional approach in [8]. First we define Lyapunov functional \( \mathcal{L} \) as follows.

\[
\mathcal{L}[f](t, z) := \int_{\mathbb{R}^{d+1}} |v - v_s|^2 f(t, \sigma, z) f(t, \sigma, z) d\sigma d\sigma, \quad z \in \Omega, \ t \geq 0,
\]

where \( \sigma = (x, v) \) and \( \sigma_0 = (x_0, v_0) \). Then, it is easy to check that the functional can be rewritten as

\[
\mathcal{L}[f](t, z) = 2 \left( M_0(0, z) M_2(t, z) - |M_1(t, z)|^2 \right), \quad z \in \Omega, \ t \geq 0.
\]

The following theorem provides that the velocity alignment emerge algebraically fast under the same assumptions as in Theorem 3.1.

**Theorem 3.2.** Let \((f, S)\) be a global regular solution to (3) under the same assumptions in Theorem 3.1. Then, the Lyapunov functional \( \mathcal{L}(t, z) \) decays to zero:

\[
\mathcal{L}[f](t, z) \leq \Phi(z)(1 + t)^{-d}, \quad t \geq 0, \ z \in \Omega.
\]

**Proof.** We use Proposition 1 and 2 to find the time-variation of \( \mathcal{L}[f](t, z) \) as follows:

\[
\begin{align*}
\partial_t \mathcal{L}[f](t, z) &= 2M_0(0, z) \partial_t M_2(t, z) - 4M_1(t, z) \partial_t M_1(t, z) \\
&= -2M_0(0, z) \int_{\mathbb{R}^{d+1}} \psi(|x - y|, |z - v_s|^2 f(t, \sigma, z) f(t, \sigma, z) d\sigma d\sigma \\
&+ 2M_0(0, z) \alpha(z) \int_{\mathbb{R}^{d+1}} |v|^2 \Lambda[S](f)(t, \sigma, z) d\sigma \\
&- 4M_1(t, z) \cdot \alpha(z) \int_{\mathbb{R}^{d+1}} v \Lambda[S](f)(t, \sigma, z) d\sigma \\
&\leq -2M_0(0, z) \psi_m \mathcal{L}[f](t, z) + 2M_0(0, z) \int_{\mathbb{R}^{d+1}} |v|^2 \Lambda[S](f)(t, \sigma, z) d\sigma \\
&+ 4\|\alpha\|_\infty \int_{\mathbb{R}^{d+1}} |v| \Lambda[S](f)(t, \sigma, z) d\sigma \\
&\leq -2M_0(0, z) \psi_m \mathcal{L}[f](t, z) + 2M_0(0, z) V(z)^2 \int_{\mathbb{R}^{d+1}} \Lambda[S](f)(t, \sigma, z) d\sigma \\
&+ 4\|\alpha\|_\infty \int_{\mathbb{R}^{d+1}} \Lambda[S](f)(t, \sigma, z) d\sigma \\
&\leq -2M_0(0, z) \psi_m \mathcal{L}[f](t, z) \\
&+ 2\|\alpha\|_\infty \left( M_0(0, z) V(z)^2 + 2V(z) \sqrt{M_0(0, z) M_2(t, z)} \right) \int_{\mathbb{R}^{d+1}} \Lambda[S](f)(t, \sigma, z) d\sigma \\
&\leq -2M_0(0, z) \psi_m \mathcal{L}[f](t, z) + 2\|\alpha\|_\infty M_0(0, z) V(z)^2 \int_{\mathbb{R}^{d+1}} f(t, \sigma, z) d\sigma \|S\|^2_{L^\infty} \\
&+ 4V(z) \sqrt{M_0(0, z) D_2(z)(M_2(0, z) + 1) |B_R(0)|} \int_{\mathbb{R}^{d+1}} f(t, \sigma, z) d\sigma \|S\|^2_{L^\infty} \\
&\leq -2M_0 \psi_m \mathcal{L}[f](t, z) + \mathcal{O}(1)(1 + t)^{-d}.
\end{align*}
\]

The last inequality is due to the decay estimate of \( \|S\|_{L^\infty} \) in Lemma 2.1. Then, Grönwall type lemma yields

\[
\mathcal{L}[f](t, z) \leq \mathcal{O}(1)(1 + t)^{-d}.
\]

\( \square \)
4. Local sensitivity analysis: Higher-order estimates. In this section, we study two quantitative local sensitivity estimates such as the propagation of well-posedness and stability for the infinitesimal in a random space. For the simplicity of presentation, we assume that the random space $\Omega$ is one-dimensional, i.e. $z \in \mathbb{R}$. For $m \in \mathbb{N}$, we first apply the operator $\partial_z^m$ to equation (3) to get

$$
\partial_t(\partial_z^m f) + v \cdot \nabla_x(\partial_z^m f) + F_a[f] \cdot \nabla_v(\partial_z^m f)
= -\nabla_v \cdot F_a[f](\partial_z^m f) - \sum_{1 \leq l \leq m} \binom{m}{l} (\partial_z^l F_a[f] \cdot \nabla_v(\partial_z^{m-l} f) + \partial_z^l \nabla_v \cdot F_a[f] \partial_z^{m-l} f)
+ \sum_{0 \leq l \leq m} \binom{m}{l} \alpha^{(m-l)}(z) \partial_z^l \Lambda[S] f,
$$

$$
\partial_t(\partial_z^m S) - \Delta(\partial_z^m S) = - \sum_{0 \leq l \leq m} \binom{m}{l} \left( \frac{l}{l'} \right)(m-l)(z) \partial_z^{l-l'} S \partial_z^{l'} p,
$$

$$
(\partial_z^m f(0, x, v, z), \partial_z^m S(0, x, z)) = (\partial_z^m f_0(x, v, z), \partial_z^m S_0(x, z)).
$$

We can rewrite (19) as

$$
\partial_t(\partial_z^m f) + v \cdot \nabla_x(\partial_z^m f) + F_a[f] \cdot \nabla_v(\partial_z^m f)
= -(\nabla_v \cdot F_a[f]) + \alpha(z) \lambda[S](\partial_z^m f)
- \sum_{1 \leq l \leq m} \binom{m}{l} \left( \frac{l}{l'} \right)(m-l)(z) \partial_z^{l-l'} \nabla_v \cdot F_a[f] \partial_z^{m-l} f
+ \sum_{1 \leq l \leq m} \binom{m}{l} \alpha^{(l)}(z) \partial_z^{m-l} \Lambda[S](f) - \alpha(z) \partial_z^l \Lambda[S] \partial_z^{m-l} f
+ \alpha(z) \partial_z^m \mathcal{T}^+[S](f),
$$

to see that $f$ and its $z$-variations $\partial_z^m f$ have the same transport structure as in (8). Thus the global unique solvability of $(\partial_z^m f, \partial_z^m S)$ is fundamentally based on the same argument as in the proof of Theorem 2.2 and Theorem 3.1. Moreover, it is easy to check that the velocity and spatial supports of $z$-variations $\partial_z^m f$ are subsets of those of $f$. To be precise, for $m \in \mathbb{N}$, we denote the diameter of the velocity support and the position support for $\partial_z^m f$ by $V_m(t, z)$ and $X_m(t, z)$, respectively. If the initial datum $f_0$ is compactly supported in $x, v$, then it follows from (9) that

$$
V_m(t, z) \leq V(t, z) \leq V(0, z), \quad X_m(t, z) \leq X(t, z) \leq X(0, z) + V(0, z)t,
$$

for $z \in \Omega, t > 0$.

4.1. Propagation of pathwise regularities. In this subsection, we study the global existence of $z$-variations $\partial_z^m f, \partial_z^m S$ in $W^{k-m, \infty}_{x,v} \times W^{k+1-m, \infty}_z$-space.

In the sequel, we will consider the two cases:

$$
m = 1 \quad \text{and} \quad m \geq 2.
$$

Before we present our main results, we provide a series of a priori estimates.

**Lemma 4.1.** Let $(f, S) = (f(t, x, v, z), S(t, x, z))$ be a regular solution process to (3) whose initial datum $f_0$ is compactly supported in $x, v$. Then, for $d \geq 2$, there
exist $C_1(t, z)$ and $C_2(t, z)$ such that

$$(i) \left| \partial_z F_a[f] \right| + \left| \partial_z^l \partial_z F_a[f] \right| \leq C_1(t, z) \sum_{0 \leq l' \leq l} \left\| \partial_z^l f(t, z) \right\|_{L^\infty_{z,v}}.$$

$$(ii) \left| \nabla_v \cdot \partial_z F_a[f] \right| + \left| \nabla_v \cdot \partial_z^l \partial_z F_a[f] \right| \leq C_2(t, z) \sum_{0 \leq l' \leq l} \left\| \partial_z^l f(t, z) \right\|_{L^\infty_{z,v}},$$

where $1 \leq |\mu| \leq k$, $1 \leq l \leq m$.

**Proof.** (i) It follows from (4) that

$$\left| \partial_z F_a[f] \right| = \int_{\mathbb{R}^{d+1}_{t, \sigma, z}} \left( \frac{1}{\nu} \right) \partial_z^l \psi(|x - x_*|, z)(v - v_*) \partial_z^l f(t, \sigma, z) d\sigma_z$$

$$\leq 2 \|\psi\|_\infty \sum_{0 \leq l' \leq l} V(t, z) \int_{\mathbb{R}^{d+1}_{t, \sigma, z}} \left| \partial_z^l f(t, \sigma, z) \right| d\sigma_z$$

$$\leq 2^{d+1} \|\psi\|_\infty V(t, z) \sum_{0 \leq l' \leq l} \left( \frac{1}{\nu} \right) (X(t, z)V(t, z))^{\frac{d}{\nu}} \|\partial_z^l f(t, z)\|_{L^\infty_{z,v}}$$

$$\leq C_1(t, z) \sum_{0 \leq l' \leq l} \left\| \partial_z^l f(t, z) \right\|_{L^\infty_{z,v}},$$

where the random function $C_1$ is given as follows

$$C_1(t, z) := 2^{d+1} \|\psi\|_{L^\infty} V(t, z)^{d+1} X(t, z)^{\frac{d}{\nu}} \sum_{0 \leq l' \leq l} \left( \frac{1}{\nu} \right).$$

Similarly, we can derive

$$\left| \partial_z^l \partial_z F_a[f] \right| = \int_{\mathbb{R}^{d+1}_{t, \sigma, z}} \left( \frac{1}{\nu} \right) \partial_z^l \partial_z^l \psi(|x - x_*|, z)(v - v_*) \partial_z^l f(t, \sigma, z) d\sigma_z$$

$$\leq C_1(t, z) \sum_{0 \leq l' \leq l} \left\| \partial_z^l f(t, z) \right\|_{L^\infty_{z,v}}.$$

(ii) By direct calculations, one has

$$\left| \nabla_v \cdot \partial_z F_a[f] \right| = d \int_{\mathbb{R}^{d+1}_{t, \sigma, z}} \left( \frac{1}{\nu} \right) \partial_z^l \psi(|x - x_*|, z) \partial_z^l f(t, \sigma, z) d\sigma_z$$

$$\leq 2^d \|\psi\|_{L^\infty} \sum_{0 \leq l' \leq l} \left( \frac{1}{\nu} \right) (X(t, z)V(t, z))^{\frac{d}{\nu}} \|\partial_z^l f(t, z)\|_{L^\infty_{z,v}}$$

$$\leq C_2(t, z) \sum_{0 \leq l' \leq l} \left\| \partial_z^l f(t, z) \right\|_{L^\infty_{z,v}},$$

where the random function $C_2$ is given as follows

$$C_2(t, z) = 2^d \|\psi\|_{L^\infty} (V(t, z)X(t, z))^{\frac{d}{\nu}} \sum_{0 \leq l' \leq l} \left( \frac{1}{\nu} \right).$$

Similarly, we have

$$\left| \nabla_v \cdot \partial_z^l \partial_z F_a[f] \right| \leq C_2(t, z) \sum_{0 \leq l' \leq l} \left\| \partial_z^l f(z) \right\|_{L^\infty_{z,v}}.$$
Remark 3. Note that the relations (20) imply that the constants $C_1$ and $C_2$ in Lemma 4.1 under the assumption ($A_2$) satisfy the following estimates: for each $z \in \Omega$,
\[
C_1(z) = O(t^d), \quad C_2(z) = O(t^d).
\]

Lemma 4.2. Let $(f, S) = (f(t, x, v, z), S(t, x, z))$ be a regular solution process to (3) whose initial datum $f_0$ is compactly supported in $x, v$. Then, for $d \geq 2, z \in \Omega$ and $t > 0$,

(i) $\|\partial_z^2 S(t, z)\|_{L^\infty_v} \leq \|\partial_z^2 S_0(z)\|_{L^\infty_v}$
\[+
\int_0^t C_3(s, z) \left( \|\partial_z^2 S(s, z)\|_{L^\infty_v} \|f(s, z)\|_{L^\infty_v} + \|\partial^2 f(s, z)\|_{L^\infty_v} \|S(s, z)\|_{L^\infty_v} \right) ds
\]
\[+
\int_0^t C_3(s, z) \sum_{0 \leq l' \leq l-1} \|\partial_z^{l'} S(s, z)\|_{L^\infty_v} \sum_{0 \leq l'' \leq l-1} \|\partial_z^{l''} f(s, z)\|_{L^\infty_v} ds,
\]

(ii) $\|\partial^\mu \partial_z^2 S(t, z)\|_{L^\infty_v} \leq \|\partial^\mu \partial_z^2 S_0(z)\|_{L^\infty_v}$
\[+
\int_0^t \left[ \frac{C_3(s, z)}{\sqrt{\pi(t-s)}} \sum_{0 \leq \mu \leq l-1} \|\partial_z^{\mu'} \partial_z^{\mu''} S(s, z)\|_{L^\infty_v} \sum_{0 \leq l' \leq l} \|\partial_z^{l'} \partial_z^{l''} f(s, z)\|_{L^\infty_v} \right] ds,
\]

where $0 < |\mu| \leq k$, $0 < l \leq m$, $|\mu| + l < k + 1$.

Proof. (i) We use Duhamel’s principle to rewrite (19) as follows.
\[
\partial_z^2 S(t, x, z) = \int_{\mathbb{R}^d} e^{t\Delta}(x, y) \partial_z^2 S_0(y, z) dy - \int_0^t \int_{\mathbb{R}^d} e^{(t-s)\Delta}(x, y) \times G_1(s, y, z) dy ds,
\]
where
\[
G_1(s, y, z) := \sum_{0 \leq l' \leq l} \left( \frac{1}{l!} \right) \left( \frac{l'}{l''} \right) \kappa^{l''-l'}(z) \partial_z^{l''} S(s, y, z) \partial_z^{l'} \rho(s, y, z).
\]

Since
\[
\|G_1(s, z)\|_{L^\infty_v} \leq C \|\kappa\|_{k, \infty} \sum_{0 \leq l' \leq l} \|\partial_z^{l''-l'} S(s, z)\|_{L^\infty_v} \|\partial_z^{l'} \rho(s, z)\|_{L^\infty_v}
\]
\[\leq C \|\kappa\|_{k, \infty} (2V(s, z))^d \sum_{0 \leq l' \leq l} \|\partial_z^{l''-l'} S(s, z)\|_{L^\infty_v} \|\partial_z^{l'} f(s, z)\|_{L^\infty_v}
\]
\[\leq C_3(s, z) \left( \|\partial_z^2 S(s, z)\|_{L^\infty_v} \|f(s, z)\|_{L^\infty_v} + \|\partial_z^2 f(s, z)\|_{L^\infty_v} \|S(s, z)\|_{L^\infty_v} \right)
\]
\[+
C_3(s, z) \sum_{0 \leq l' \leq l-1} \|\partial_z^{l'} S(s, z)\|_{L^\infty_v} \sum_{0 \leq l'' \leq l-1} \|\partial_z^{l''} f(s, z)\|_{L^\infty_v},
\]
we obtain
\[
\|\partial_z^2 S(t, z)\|_{L^\infty_v} \leq \|\partial_z^2 S_0(z)\|_{L^\infty_v}
\]
\[+
\int_0^t C_3(s, z) \left( \|\partial_z^2 S(s, z)\|_{L^\infty_v} \|f(s, z)\|_{L^\infty_v} + \|\partial_z^2 f(s, z)\|_{L^\infty_v} \|S(s, z)\|_{L^\infty_v} \right) ds
\]
\[+
\int_0^t C_3(s, z) \sum_{0 \leq l' \leq l-1} \|\partial_z^{l'} S(s, z)\|_{L^\infty_v} \sum_{0 \leq l'' \leq l-1} \|\partial_z^{l''} f(s, z)\|_{L^\infty_v} ds.
\]
Here, we used the property:
\[
\int_{\mathbb{R}^d} e^{t\Delta}(x, y) dy = 1.
\]
Again we apply Duhamel’s principle to rewrite (19) as

$$
\partial_x^\mu \partial_z^\ell S(t, x, z) = \int_{\mathbb{R}^d} e^{t \Delta}(x, y) \partial_x^\mu \partial_z^\ell S_0(y, z) dy - \int_0^t \int_{\mathbb{R}^d} \left[ e^{(t-s) \Delta}(x, y) \right] \partial_x^\mu \partial_z^\ell S_0(y, z) dy \, ds,
$$

where

$$
G_2(s, y, z) := \sum_{0 \leq l' \leq l \leq l'' \leq \mu \leq \mu - e_j} \kappa(l-l') \rho_s(l', y, z) \partial_x^{\mu'-\mu} \partial_z^{l''} \rho(s, y, z).
$$

Note that we select the index $j$ such that the $j$th-component of $\mu$ is not zero and denote the unit vector with $j$th-component 1 by $e_j$. Again, it follows from (21) that

$$
\|G_2(s, z)\|_{L^\infty} \leq C \|\kappa\|_{k, \infty} (2V(s, z))^2 \sum_{0 \leq l' \leq l \leq l'' \leq \mu \leq \mu - e_j} \|\partial_x^{\mu'-\mu} \partial_z^{l''} S(s, z)\|_{L^\infty} \|\partial_x^{\mu'-\mu} \partial_z^{l''} f(s, z)\|_{L^\infty}.
$$

This yields

$$
\left\| \partial_x^\mu \partial_z^\ell S(t, z) \right\|_{L^\infty} \leq \left\| \partial_x^\mu \partial_z^\ell S_0(z) \right\|_{L^\infty} + \int_0^t \left[ \frac{C_3(s, z)}{\sqrt{\pi(t-s)}} \sum_{0 \leq l' \leq l \leq l'' \leq \mu \leq \mu - e_j} \|\partial_x^{\mu'-\mu} \partial_z^{l''} S(s, z)\|_{L^\infty} \sum_{0 \leq l' \leq l \leq l'' \leq \mu \leq \mu - e_j} \|\partial_x^{\mu'-\mu} \partial_z^{l''} f(s, z)\|_{L^\infty} \right] ds.
$$

The following lemma deals with a priori estimate for the turning operator

$$
\mathcal{T}[S](f)(t, x, v, z) = \alpha(z) \Lambda[S](f)(t, x, v, z).
$$

**Lemma 4.3.** Let $(f, S) = (f(t, x, v, z), S(t, x, z))$ be a regular solution process to (3) whose initial datum $f_0$ is compactly supported in $x, v$. Then, for $d \geq 2$, $z \in \Omega$
and \( t > 0 \),

(i) \(|\partial^l_x [\alpha(z) \Lambda[S](f)]| \leq C_3(t, z) (|\partial^l_x S(t, z)||_{L^\infty} + |\partial^l_x f(t, z)||_{L^\infty}\|S(t, z)\|_{L^\infty}^2)
+ C_3(t, z) \left( \sum_{0 \leq l' \leq l-1} |\partial^{l'}_x S(t, z)||_{L^\infty} \right)^2 \sum_{0 \leq l' \leq l-1} |\partial^{l''}_x f(t, v)||_{L^\infty},
\]

(ii) \(|\partial^l_x \partial^{\mu}_v \partial^{l}_x [\alpha(z) \Lambda[S](f)]| \leq C_3(t, z) \|\partial^{l''+\nu}_x \partial^{l}_x S(t, z)||_{L^\infty}\|S(t, z)\|_{L^\infty} + \|\partial^{l}_x f(t, z)||_{L^\infty}^2 \sum_{0 \leq l' \leq l-1} \sum_{0 \leq |\mu|+|\nu'|\leq l \leq |\mu|+|\nu|-1} |\partial^{l''}_x \partial^{l'}_x \partial^{l'}_x S(t, z)||_{L^\infty}
\]

where \( 0 < |\mu| + |\nu| \leq k, \ 0 < l \leq m, \ |\mu| + |\nu| + l < k + 1 \).

**Proof.** (i) We apply \( \partial^l_x \) for the turning operator to get

\[ |\partial^l_x [\alpha(z) \Lambda[S](f)]| = \left| \sum_{0 \leq l' \leq l} \binom{l}{l'} \alpha^{(l-l')}(z) \partial^{l'}_x \Lambda[S](f) \right|
\]

\[ \leq \sum_{0 \leq l' \leq l} \binom{l}{l'} \left( \sum_{0 \leq |\mu'| \leq l'} \alpha^{(l-l')} \right)^2 \left| \partial^{l''}_x T[S] \right| \left| \partial^{l'}_x - t'' f ||_{L^\infty}
\]

\[ \leq C_3(t, z) \left( \|\partial^l_x T[S]\||_{L^\infty} + \left| \partial^l_x f(t, z)||_{L^\infty} \|T[S]\| \right)\right)\]

\[ + C_3(t, z) \left( \sum_{0 \leq l' \leq l-1} \|\partial^{l'}_x S(t, z)||_{L^\infty} \right)^2 \sum_{0 \leq l' \leq l-1} \|\partial^{l'}_x f(t, v)||_{L^\infty}.\]

Note that

\[ |\partial^l_x T[S]| \leq \sum_{0 \leq l' \leq l} \binom{l}{l'} \|\partial^{l-l'}_x S(t, z)||_{L^\infty} \|\partial^l_x S(t, z)||_{L^\infty}
\]

\[ \leq 2\|\partial^l_x S(t, z)||_{L^\infty} \|S(t, z)||_{L^\infty} + \left( \sum_{0 \leq l' \leq l-1} \|\partial^{l'}_x S(t, z)||_{L^\infty} \right)^2.\]

This leads to

\[ |\partial^l_x [\alpha(z) \Lambda[S](f)]| \leq C_3(t, z) \left( \left( \sum_{0 \leq l' \leq l-1} \|\partial^{l'}_x S(t, z)||_{L^\infty} \right)^2 \sum_{0 \leq l' \leq l-1} \|\partial^{l'}_x f(t, v)||_{L^\infty}.\right)\]
(ii) Similarly, we take $\partial_x^\mu_0 \partial_z^\mu \partial_z^m$ to the turning operator to obtain
\[
|\partial_x^\mu_0 \partial_z^\mu \partial_z^m[\alpha(z)\Lambda[S](f)]| = \sum_{0 \leq l' \leq l} \binom{l}{l'} \binom{\mu}{\mu'} \binom{\nu}{\nu'} 2(2V(t,z))^d \alpha(k,\mu,\nu) |\partial_x^\mu_0 \partial_z^\mu \partial_z^m T[S]| \|\partial_x^\nu \partial_z^\nu \partial_z^{-l'-l'} f\|_{L_x^\infty}
\]
\[
\leq C_3(t,z) \left( \sum_{0 \leq l' \leq l} \sum_{0 \leq \mu' \leq \mu} \binom{l}{l'} \binom{\mu}{\mu'} \binom{\nu}{\nu'} |\partial_x^\mu_0 \partial_z^\mu \partial_z^m T[S]| \sum_{0 \leq \nu' \leq \nu} |\partial_x^\nu \partial_z^\nu \partial_z^{-l'-l'} f\|_{L_x^\infty} \right)
\]
Note that
\[
|\partial_x^\mu_0 \partial_z^\mu \partial_z^m T[S]| \leq \sum_{0 \leq l' \leq l} \binom{l}{l'} \binom{\mu}{\mu'} \binom{\nu}{\nu'} |\partial_x^\mu_0 \partial_z^\mu \partial_z^m S(t,z)|_{L_x^\infty} |\partial_x^\nu \partial_z^\nu \partial_z^{-l'-l'} S(t,z)|_{L_x^\infty}
\]
\[
\leq 2|\partial_x^\mu_0 \partial_z^\mu \partial_z^m S(t,z)|_{L_x^\infty} |\partial_x^\nu \partial_z^\nu \partial_z^{-l'-l'} S(t,z)|_{L_x^\infty} + \sum_{0 \leq l' \leq l} \binom{l}{l'} \binom{\mu}{\mu'} \binom{\nu}{\nu'} |\partial_x^\mu_0 \partial_z^\mu \partial_z^m S(t,z)|_{L_x^\infty}^2.
\]
Then, we have
\[
|\partial_x^\mu_0 \partial_z^\mu \partial_z^m[\alpha(z)\Lambda[S](f)]| \leq C_3(t,z) \|\partial_x^\mu_0 \partial_z^\mu \partial_z^m S(t,z)\|_{L_x^\infty} \|\partial_x^\nu \partial_z^\nu \partial_z^{-l'-l'} S(t,z)\|_{L_x^\infty} \|f(t,z)\|_{L_x^\infty}
\]
\[
+ C_3(t,z) \|\partial_x^\mu_0 \partial_z^\mu \partial_z^m f(t,z)\|_{L_x^\infty} \|\partial_x^\nu \partial_z^\nu \partial_z^{-l'-l'} S(t,z)\|_{L_x^\infty}^2 \sum_{0 \leq l' \leq l} \binom{l}{l'} \binom{\mu}{\mu'} \binom{\nu}{\nu'} |\partial_x^\mu_0 \partial_z^\mu \partial_z^m f(t,z)|_{L_x^\infty}.
\]

**Remark 4.** In Lemma 4.2 and 4.3, the constant $C_3(t,z)$ depends only on $V(t,z), m,k,d, \|\alpha\|_{k,\infty}$ and $\|k\|_{k,\infty}$.

Now, we are ready to provide a global well-posedness of $\delta$-variations $(\partial_x^m f, \partial_x^m S)$ to the equation (19) under the assumptions $(A_1)\sim(A_2)$, for each $z \in \Omega$. First we deal with the case $m = 1$.

**Theorem 4.4.** For $k \geq 1$ and for each $z \in \Omega$, suppose that the initial data $(f_0, S_0) = (f_0(x,v,z), S_0(x,z))$ satisfy the following conditions: for $d \geq 2$,

1. Initial datum $f_0$ is compactly supported with respect to $x,v$, and has finite moments:

there exist positive bounded functions $X(z), V(z)$ such that

\[
X(0,z) \leq X(z), \quad V(0,z) \leq V(z), \quad M_0(0,z) + M_2(0,z) < \infty.
\]
2. Initial data \((f_0(z), \partial_z f_0(z))\) is \(W^{k,\infty}_{x,v} \times W^{k-1,\infty}_{x,v}\)-regular:

\[
\sum_{0 \leq \mu_1 \leq 1} |\partial^\mu_x \partial^\nu_z \partial^\lambda_t f_0(z)|_{L^\infty_{x,v}} < \infty, \quad \text{for each} \quad z \in \Omega.
\]

3. Initial data \((S_0(z), \partial_z S_0(z))\) is \(W^{k+1,\infty}_{x} \times W^{k,\infty}_{z}\)-regular:

\[
\sum_{0 \leq \mu_1 \leq 1} |\partial^\mu_x \partial^\nu_z S_0(z)|_{L^\infty_z} < \infty \quad \text{for each} \quad z \in \Omega.
\]

4. \(\psi(|x|, z), \alpha(z)\) and \(\kappa(z)\) are sufficiently regular:

\[
\psi \in W^{1,\infty}(\Omega; W^{k,\infty}(\mathbb{R}^d)), \quad \alpha, \kappa \in W^{1,\infty}(\Omega).
\]

Then, for any \(T \in (0, \infty)\), there exists a global unique solution process satisfying

\[
\sup_{0 \leq \mu \leq k+1} \left( \sum_{0 \leq \mu_1 \leq 1} |\partial^\mu_x \partial^\nu_z \partial^\lambda_t f(t, z)|_{L^\infty_{x,v}} + \sum_{0 \leq \mu_1 \leq 1} |\partial^\mu_x \partial^\nu_z S(t, z)|_{L^\infty_z} \right) < \infty,
\]

for each \(z \in \Omega\).

**Proof.** The proof is almost the same as that of Theorem 3.1. So, we provide a priori \(W^{k,\infty}_{x,v}\)-estimate for \(\partial_z f\) and \(W^{k,\infty}_{x}\)-estimate for \(\partial_z S\). Throughout this proof, we set

\[
\mathcal{F}(t, z) := \sum_{0 \leq \mu_1 \leq 1} |\partial^\mu_x \partial^\nu_z f(t, z)|_{L^\infty_{x,v}}, \quad \mathcal{S}(t, z) := \sum_{0 \leq \mu_1 \leq k+1} |\partial^\mu_x \partial^\nu_z S(t, z)|_{L^\infty_z}.
\]

Then, by Theorem 3.1,

\[
\sup_{0 \leq t < T} \left( \mathcal{F}(t, z) + \mathcal{S}(t, z) \right) \leq C(T, z), \quad \text{for} \quad z \in \Omega.
\]

- **Case A \((k = 1)\):** Consider the equation (19) with \(m = 1\) to get

\[
\partial_t (\partial_z f) + v \cdot \nabla_x (\partial_z f) + F_a[f] \cdot \nabla_v (\partial_z f) = -\nabla_v \cdot [\partial_z F_a[f] f] - \nabla_v \cdot F_a[f] \partial_z f + \alpha'(z) \Lambda[S](f) + \alpha(z) \Lambda[S](f).
\]

We integrate the above equation along the particle trajectory to obtain

\[
|\partial_z f(t, x, v, z)| \leq |\partial_z f_0(x, v, z)| + \int_0^t \left[ |\nabla_v \cdot [\partial_z F_a[f] f]| + |\nabla_v \cdot F_a[f] |\partial_z f| \right] \, ds
\]

\[
+ \int_0^t |\alpha'(z)||\Lambda[S](f)| + |\alpha(z)||\partial_z \Lambda[S](f)| \, ds
\]

\[
\leq |\partial_z f_0(x, v, z)| + \int_0^t \mathcal{I}_{31} \, ds + \int_0^t \mathcal{I}_{32} \, ds.
\]

By Lemma 4.1 and 4.3, we have

\[
|\mathcal{I}_{31}| \leq |\nabla_v \cdot [\partial_z F_a[f] f]| + |\partial_z F_a[f]| |\nabla_v f| + |\nabla_v \cdot F_a[f]| |\partial_z f|
\]

\[
\leq \mathcal{F}(t, z) ||\partial_z f(t, z)||_{L^\infty_{x,v}},
\]

\[
|\mathcal{I}_{32}| \leq |\alpha'(z)||\Lambda[S](f)| + |\alpha(z)||\partial_z \Lambda[S](f)|
\]

\[
\leq 2\|\alpha\|_{1,\infty} C(t, z) \left( 1 + ||\partial_z S(t, z)||_{L^\infty} + ||\partial_z f(t, z)||_{L^\infty_z} \right).
\]
Thus, we have
\[
\|\partial_z f(t, z)\|_{L^\infty_{z,v}} \leq \|\partial_z f_0(z)\|_{L^\infty_{z,v}} + \int_0^t C(s, z) \left(\|\partial_z f(s, z)\|_{L^\infty_{z,v}} + \|\partial_z S(s, z)\|_{L^\infty_{z,v}} + 1\right) ds. \tag{23}
\]

On the other hand, one has
\[
|\partial_z S(t, x, z)| \leq \int_{R^d} e^{\Delta(x, y)}|\partial_z S_0(y, z)| dy + \kappa'(z) \int_0^t \int_{R^d} e^{(t-s)\Delta(x, y)}|S(s, y, z)||\rho(s, y, z)| dyds + \kappa(z) \int_0^t \int_{R^d} e^{(t-s)\Delta(x, y)}|\partial_z S(s, y, z)||\rho(s, y, z)| dyds + |\partial_z \rho(s, y, z)| dyds \leq \|\partial_z S_0(z)\|_{L^\infty_z} + \|\kappa\|_{1,\infty} \int_0^t (2V(s, z))^d S(s, z)F(s, z) ds \leq \|S_0(z)\|_{L^\infty_z} + \int_0^t C(s, z) \left(\|\partial_z f(s, z)\|_{L^\infty_{z,v}} + \|\partial_z S(s, z)\|_{L^\infty_{z,v}} + 1\right) ds.
\]

This yields
\[
\|\partial_z S(t, z)\|_{L^\infty_z} \leq \|S_0(z)\|_{L^\infty_z} + \int_0^t C(s, z) \left(\|\partial_z f(s, z)\|_{L^\infty_{z,v}} + \|\partial_z S(s, z)\|_{L^\infty_{z,v}} + 1\right) ds. \tag{24}
\]

Finally, we apply Gronwall’s lemma for (23) and (24) to get
\[
\sup_{0 \leq t < T} \left(\|\partial_z f(t, z)\|_{L^\infty_{z,v}} + \|\partial_z S(t, z)\|_{L^\infty_z}\right) \leq C(T, z).
\]

Moreover, the estimate (ii) in Lemma 4.2 gives
\[
\sup_{0 \leq t < T} \sum_{|\mu|=1} \|\partial^\mu_x \partial_z S(t, z)\|_{L^\infty_{z,v}} \leq C(T, z).
\]

- Case B \((k \geq 2)\): We will verify (22) by using an induction argument on \(k \geq k' \geq 2\). Suppose that the following estimate holds.
\[
\sup_{0 \leq t < T} \left(\sum_{0 \leq |\mu| + |\nu| \leq k'-1} \|\partial^\mu_x \partial^\nu_z f(t, z)\|_{L^\infty_{z,v}} + \sum_{0 \leq |\mu| \leq k'} \|\partial^\mu_x \partial_z f(t, z)\|_{L^\infty_{z,v}} + \sum_{0 \leq |\mu| \leq k'} \|\partial^\mu_x \partial_z S(t, z)\|_{L^\infty_z}\right) \leq C(T, z).
\]

Then, we have
\[
\sup_{0 \leq t < T} \left(\sum_{|\mu| + |\nu| = k'} \|\partial^\mu_x \partial^\nu_z f(t, z)\|_{L^\infty_{z,v}} + \sum_{|\mu| = k'+1} \|\partial^\mu_x \partial_z S(t, z)\|_{L^\infty_z}\right) \leq C(T, z).
\]
4.1, the estimate (ii) in Lemma 4.3 and the induction hypothesis to derive

\[
\|\partial_x^\nu \partial_z f(t, z)\|_{L^\infty_{t,z}} \leq \|\partial_x^\nu \partial_z f_0(z)\|_{L^\infty_z}
\]

\[
+ \int_0^t C(s, z) \left( \sum_{|i|=1} \|\partial_x^{\nu^i} \partial_z f(s, z)\|_{L^\infty} + \sum_{|i|=1, |j|=1} \|\partial_x^{\nu^i+j} \partial_z f(s, z)\|_{L^\infty} \right) ds
\]

\[
+ \int_0^t C(s, z) \left( \sum_{0 \leq |\mu| + |\nu| \leq k'-1} \|\partial_x^\mu \partial_z f(t, z)\|_{L^\infty_{t,z}} \right) ds
\]

\[
+ \int_0^t \|\alpha\|_{1, \infty} C(s, z) \left( 1 + \|\partial_x^{\nu^i} \partial_z S\|_{L^\infty} + \|\partial_x^\nu \partial_z f\|_{L^\infty_{t,z}} \right) ds.
\]  

(25)

\(\diamond\) Case B.2 (Estimate of \(\|\partial_x^\nu \partial_z f(t, z)\|_{L^\infty_{t,z}}\) with \(|\mu| + |\nu| = k'\): We take \(\partial_x^\nu\partial_z\) to (19)_1 with \(m = 1\) to obtain

\[
\partial_t (\partial_x^\nu \partial_z f) + v \cdot \nabla_x (\partial_x^\nu \partial_z f) + F_a[f] \cdot \nabla_v (\partial_x^\nu \partial_z f)
\]

\[
= - \sum_{|i|=1} \left( \mu^i \right) \partial_x^{\nu^i} \partial_z f - \sum_{0 < \mu' \leq \mu} \left( \mu \right) \partial_x^\mu F_a[f] \cdot \nabla_v (\partial_x^{\mu^i} \partial_z f)
\]

\[
- \sum_{|i|=1} \left( \mu^i \right) \partial_v \partial_x^\mu \partial_z f
\]

\[
- \sum_{\mu' \leq \mu} \left( \mu \right) \partial_x^\mu \partial_z F_a[f] \cdot \nabla_v (\partial_x^{\mu^i} \partial_z f + \alpha'(z) \partial_x^\nu \partial_z^1 f + \alpha(z) \partial_x^\nu \partial_z \Lambda.
\]

This leads

\[
\|\partial_x^\nu \partial_z f(t, z)\|_{L^\infty_{t,z}} \leq \|\partial_x^\nu \partial_z f_0(z)\|_{L^\infty_z}
\]

\[
+ \int_0^t C(s, z) \left( \sum_{|i|=1, |j|=1} \|\partial_x^{\nu^i+j} \partial_z f(s, z)\|_{L^\infty_z} + \sum_{|\mu| + |\nu| = k'} \|\partial_x^\mu \partial_z f(t, z)\|_{L^\infty_z} \right) ds
\]

\[
+ \int_0^t C(s, z) \left( 1 + \|\partial_x^\nu \partial_z S\|_{L^\infty} + \|\partial_x^\nu \partial_z f\|_{L^\infty_{t,z}} \right) ds.
\]  

(26)
We collect (25), (26) and use the induction hypothesis to find
\[
\sum_{|\mu|+|\nu|=k'} \|\partial_x^\mu \partial_z^\nu \partial_t f(t, z)\|_{L^\infty_{x,z}} 
\leq \sum_{|\mu|+|\nu|=k'} \|\partial_x^\mu \partial_z^\nu \partial_z f_0(z)\|_{L^\infty_{x,z}} + \int_0^t C(s, z) \left( \sum_{|\mu|+|\nu|=k'} \|\partial_x^\mu \partial_z^\nu \partial_z f(s, z)\|_{L^\infty_{x,z}} + 1 \right) ds.
\]

We apply Grönwall’s lemma to the above integral estimate to obtain
\[
\sup_{0 \leq t < T} \left( \sum_{|\mu| = k'+1} \|\partial_x^\mu \partial_z S(t, z)\|_{L^\infty_{x,z}} \right) \leq C(T, z).
\]

Finally, we substitute this estimate into the estimate (ii) in Lemma 4.2 to conclude
\[
\sup_{0 \leq t < T} \left( \sum_{|\mu| = k'+1} \|\partial_x^\mu \partial_z S(t, z)\|_{L^\infty_{x,z}} \right) \leq C(T, z).
\]

Now, we extend the result of Theorem 4.4 to the case $m \geq 2$.

**Theorem 4.5.** For $k \geq 2$ and $z \in \Omega$, suppose that the initial data $(f_0, S_0)$ satisfy the following conditions: for $d \geq 2$,

1. Initial datum $f_0$ is compactly supported with respect to $x, v$ and has finite moments:
   there exist positive bounded functions $X(z)$ and $V(z)$ such that
   \[
   X(0, z) \leq X(z), \quad V(0, z) \leq V(z), \quad M_0(0, z) + M_2(0, z) < \infty.
   \]

2. Initial datum $\partial_z^m f_0(z)$ is $W^{k-m, \infty}$-regular for each $0 \leq m \leq k$:
   \[
   \sum_{0 \leq m \leq k} \sum_{0 \leq |\mu|+|\nu| \leq k-m} \|\partial_x^\mu \partial_z^\nu \partial_z^m f_0(z)\|_{L^\infty_{x,z}} < \infty, \quad \text{for each} \quad z \in \Omega.
   \]

3. Initial datum $\partial_z^m S_0(z)$ is $W^{k+1-m, \infty}$-regular for each $0 \leq m \leq k$:
   \[
   \sum_{0 \leq m \leq k} \sum_{0 \leq |\mu| \leq k+1-m} \|\partial_x^\mu \partial_z^m S_0(z)\|_{L^\infty_{x,z}} < \infty, \quad \text{for each} \quad z \in \Omega.
   \]

4. $\psi(|x|, z), \alpha(z)$ and $\kappa(z)$ are sufficiently regular:
   \[
   \psi \in W^{k, \infty}(\mathbb{R}^d \times \Omega), \quad \alpha, \kappa \in W^{k, \infty}(\Omega).
   \]

Then, for any $T \in (0, \infty)$ and $z \in \Omega$, there exists a global unique solution process satisfying
\[
\sup_{0 \leq t < T} \left( \sum_{0 \leq \ell \leq k} \sum_{0 \leq |\mu|+|\nu| \leq k-\ell} \|\partial_x^\mu \partial_z^\nu \partial_t f(t, z)\|_{L^\infty_{x,z}} + \sum_{0 \leq \ell \leq k} \sum_{0 \leq |\mu| \leq k+1-\ell} \|\partial_x^\mu \partial_z^\ell S(t, z)\|_{L^\infty_{x,z}} \right) < \infty.
\]

**Proof.** Since the proof is rather lengthy, we leave its proof in Appendix A. \qed
4.2. Stability analysis. In this subsection, we derive $L^\infty$-stability estimates for the $z$-variations of solution processes. For this, we first present \textit{a priori} estimates.

Lemma 4.6. Let $(f(t, x, v, z), S(t, x, z))$ and $(\tilde{f}(t, x, v, z), \tilde{S}(t, x, v, z))$ be two $W^{k+1, \infty}_{x,v} \times W^{k+2, \infty}_x$-regular solutions to (3) whose initial data $f_0$ and $\tilde{f}_0$ are compactly supported in $x$ and $v$. Then, for $d \geq 2$ and $l \leq m$, $0 \leq |\mu| \leq k - l$, we have

(i) \quad $|\partial_x^\mu \partial_z^a (F_a[f] - F_a[\tilde{f}])| \leq C(t, z) \sum_{l'=0}^l \|\partial_x^l (f - \tilde{f})(t, z)\|_{L^\infty_x}$.

(ii) \quad $|\nabla_v \cdot \partial_x^\mu \partial_z^a (F_a[f] - F_a[\tilde{f}])| \leq C(t, z) \sum_{l'=0}^l \|\partial_x^l (f - \tilde{f})(t, z)\|_{L^\infty_x}$.

Proof. Since the proof for (ii) is similar to that of (i), we only derive the estimate (i):

$|\partial_x^\mu \partial_z^a (F_a[f] - F_a[\tilde{f}])|$

$\leq \sum_{0 \leq l' \leq l} \left( \frac{1}{\nu} \right) \int_{\mathbb{R}^{2d}} |v - v_\ast| \left| \partial_x^\mu \partial_z^{l'} \psi(|x - x_\ast|, z) \partial_z^a \left( f(t, \sigma, z) - \tilde{f}(t, \sigma_\ast, z) \right) \right| d\sigma_\ast$

$\leq 2^{d+1} \|\psi\|_\infty \|\nabla(t, z)\|^{d+1} \|\tilde{X}(t, z)\|^{d} \sum_{0 \leq l' \leq l} \|\partial_x^l (f - \tilde{f})(t, z)\|_{L^\infty_x}$

$\leq C(t, z) \sum_{0 \leq l' \leq l} \|\partial_x^l (f - \tilde{f})(t, z)\|_{L^\infty_x},$

where

$\tilde{V}(t, z) := \max\{V(t, z), \tilde{V}(t, z)\}$, \quad $\tilde{X}(t, z) := \max\{X(t, x), \tilde{X}(t, z)\}$,

and $\tilde{V}(t, z), \tilde{X}(t, z)$ are the diameters of the velocity and position supports of a solution $\tilde{f}$.

Next, we consider the $L^\infty$-stability estimates of global unique solution processes $(f, S)$.

Theorem 4.7. For each $z \in \Omega$, suppose that the following assumptions hold: for $d \geq 2$,

1. Initial data $f_0(z)$ and $\tilde{f}_0(z)$ are compactly supported with respect to $x,v$, and have finite moments: there exist positive $\overline{X}(z), \overline{V}(z)$ such that

$X(0, z) + \overline{X}(0, z) < \overline{X}(z), \quad V(0, z) + \overline{V}(0, z) < \overline{V}(z), \quad M_0[f](0, z) + M_0[\tilde{f}](0, z) < \infty.$

2. Initial data $f_0(z)$ and $\tilde{f}_0(z)$ are $W^{k+1, \infty}_{x,v}$-regular:

$\sum_{0 \leq |\mu| + |\nu| \leq k+1} \|\partial_x^\nu \partial_z^\mu f_0(z)\|_{L^\infty_x} + \sum_{0 \leq |\mu| + |\nu| \leq k+1} \|\partial_x^\nu \partial_z^\mu \tilde{f}_0(z)\|_{L^\infty_x} < \infty.$

3. Initial data $S_0(z)$ and $\tilde{S}_0(z)$ are $W^{k+2, \infty}_x$-regular:

$\sum_{0 \leq |\mu| \leq k+2} \|\partial_x^\mu S_0(z)\|_{L^\infty_x} + \sum_{0 \leq |\mu| \leq k+2} \|\partial_x^\mu \tilde{S}_0(z)\|_{L^\infty_x} < \infty.$
We leave its detailed proof in Appendix B.

Proof. We leave its detailed proof in Appendix B.

Now, we present a local sensitivity analysis on the infinitesimal variations of \( f \) and \( S \).

**Theorem 4.8.** For each \( z \in \Omega \) and for \( k \geq m \geq 1 \), suppose that the following assumptions hold: for \( d \geq 2 \),

1. Initial data \( f_0(z) \) and \( \tilde{f}_0(z) \) are compactly supported with respect to \( x,v \), and have finite moments: there exist positive constants \( A \) and \( S \) such that

\[
\|\tilde{f}(0, z)\|_{W^{k,\infty}_{v,x}} + \|\tilde{S}(0, z)\|_{W^{k+1,\infty}_x} \\
\leq C(T, z) \left( \|f_0 - \tilde{f}_0(z)\|_{W^{k,\infty}_{v,x}} + \|S_0 - \tilde{S}_0(z)\|_{W^{k+1,\infty}_x} \right),
\]

where \( (f, S) \) and \( (\tilde{f}, \tilde{S}) \) are global unique solution processes to (3) under the assumptions \((A_1)-(A_2)\) corresponding to initial processes \((f_0, S_0), (\tilde{f}_0, \tilde{S}_0)\), respectively.

Proof. Since the proof is rather lengthy, we leave its proof in Appendix C.

(4) \( \psi([x], z), \alpha(z) \) and \( \kappa(z) \) are sufficiently regular:

\[
\psi \in L^\infty(\Omega; W^{k,\infty}(\mathbb{R}^d)), \quad \alpha, \kappa \in L^\infty(\Omega).
\]

Then, for any \( T \in (0, \infty) \) and \( z \in \Omega \), there exists a nonnegative random variable \( C(T, z) \) such that for \( t \in (0, T) \),

\[
\sum_{0 \leq l \leq m} \left( \| \partial_t^l (f - \tilde{f})(t, z) \|_{W^{k,\infty}_{x,v}} + \| \partial_t^l (S - \tilde{S})(t, z) \|_{W^{k+1,\infty}_v} \right) \\
\leq C(T, z) \sum_{0 \leq l \leq m} \left( \| \partial_t^l (f_0 - \tilde{f}_0)(z) \|_{W^{k,\infty}_{x,v}} + \| \partial_t^l (S_0 - \tilde{S}_0)(z) \|_{W^{k+1,\infty}_v} \right),
\]

where \( (f, S), (\tilde{f}, \tilde{S}) \) are global unique solution process to (3) under the assumptions \((A_1)-(A_2)\) corresponding to initial data \((f_0, S_0), (\tilde{f}_0, \tilde{S}_0)\), respectively.

Proof. Since the proof is rather lengthy, we leave its proof in Appendix C.
4.3. Uniform bound in $H^m_x(L^\infty_v)$-norm. In this subsection, we provide $L^2$-estimates by using $L^\infty_w, L^\infty_v$-estimates for $\partial_x^m f, \partial_x^m S$. That is, we estimate $(\partial_x^m f, \partial_x^m S)$ in a space $L^2_x(\Omega; L^\infty_w(\mathbb{R}^{2d})) \times L^2_x(\Omega; L^\infty_v(\mathbb{R}^{d}))$.

**Theorem 4.9.** For $T \in (0, \infty)$, suppose that the following assumptions hold: for $d \geq 2$,

(1) Initial datum $f_0(z)$ has finite moments, compact velocity and position supports which are uniform in $z$:

$$\sup_{z \in \Omega} (X(0,z) + V(0,z)) < \infty, \quad \sup_{z \in \Omega} (M_0(0,z) + M_2(0,z)) < \infty.$$  

(2) For each $0 \leq l \leq m$, $z$-derivatives of initial datum $\partial_x^l f_0(z)$ is $W^{k-l, \infty}$-regular, for each $z \in \Omega$ and belongs to $(L^2_x \cap L^\infty)(\Omega; W^{k-l, \infty}(\mathbb{R}^{2d}))$:

$$\sum_{0 \leq l \leq m} \|\partial_x^l f_0(z)\|_{(L^2_x \cap L^\infty)(\Omega; W^{k-l, \infty}(\mathbb{R}^{2d}))} < \infty.$$  

(3) For each $0 \leq l \leq m$, $z$-derivatives of initial datum $\partial_x^l S_0(z)$ is $W^{k+1-l, \infty}$-regular, for each $z \in \Omega$ and belongs to $(L^2_x \cap L^\infty)(\Omega; W^{k+1-l, \infty}(\mathbb{R}^d))$:

$$\sum_{0 \leq l \leq m} \|\partial_x^l S_0(z)\|_{(L^2_x \cap L^\infty)(\Omega; W^{k+1-l, \infty}(\mathbb{R}^d))} < \infty.$$  

(4) $\psi(|x|, \alpha(z)$ and $\kappa(z)$ are sufficiently regular:

$$\psi \in W^{m, \infty}(\mathbb{R}^d \times \Omega), \quad \alpha, \kappa \in W^{m, \infty}(\Omega),$$

and let $(f, S)$ be a regular solution process to (3) under the assumptions (A$_1$)-(A$_2$). Then, one has

$$\|f(t)\|_{H^m_x(L^\infty_w)} + \|S(t)\|_{H^m_v(L^\infty_v)} \leq C(T) \left(\|f_0\|_{H^m_x(L^\infty_w)} + \|S_0\|_{H^m_v(L^\infty_v)}\right), \quad t \in (0, T).$$  

**Proof.** We set

$$F^l(t, z) := \sum_{0 \leq |\mu| + |\nu| \leq m-l} \|\partial_x^\mu \partial_t^\nu \partial_x^l f(t, z)\|_{L^\infty_x}, \quad S^l(t, z) \sum_{0 \leq |\mu| \leq m+1-l} \|\partial_x^\mu \partial_z^l S(t, z)\|_{L^\infty_x}.$$  

It follows from Theorem 4.5 that $F^l(t, z), S^l(t, z)$ are bounded by $C(t, z)$. Since $z$-dependency of $C(t, z)$ comes from $X(0, z), V(0, z), M_1(0, z), F^l(t, 0, z)$ and $S^l(t, 0, z)$ for $0 \leq l' \leq l$, we can remove $z$-dependency from $C(t, z)$ by using assumptions on uniform boundedness, (1)-(3). Similarly, we can get rid of $z$-dependency from constants $C_1(t, z), C_2(t, z), C_3(t, z)$ in Lemma 4.1 - Lemma 4.3. Here, we replace various bounds by a constant $C(t)$.

For $0 \leq l \leq m$, we recall (i) of Lemma 4.2 which can be estimated as

$$\|\partial_x^l S(t, z)\|_{L^\infty_x} \leq \|\partial_x^l S_0(z)\|_{L^\infty_x} + C(T) \int_0^t \sum_{0 \leq l' \leq l} \|\partial_x^l S(s, z)\|_{L^\infty_x} \sum_{0 \leq l' \leq l} \|\partial_x^l f(s, z)\|_{L^\infty_v} ds.$$  

(27)
The second inequality is due to Theorem 4.5. We multiply (27) by $\sqrt{\pi(z)}$ and take $L^2$-integration on $\Omega$ with the help of Minkowski’s inequality to obtain
\[
\|\partial_z^l s(t)\|_{L^2_\gamma(L^\infty)} \leq \|\partial_z^l s_0\|_{L^2_\gamma(L^\infty)} + C(T) \sum_{l'=0}^l \int_0^t \|\partial_z^{l'} S(s)\|_{L^2_\gamma(L^\infty)} ds.
\] (28)

We take a sum the square of (28) over all $0 \leq l \leq m$, and use Cauchy-Schwarz inequality to have
\[
\|S(t)\|^2_{H^m_\gamma(L^\infty)} \leq C(T) \left( \|S_0\|^2_{H^m_\gamma(L^\infty)} + \int_0^t \|S(s)\|^2_{H^m_\gamma(L^\infty)} ds \right).
\] (29)

Next, we integrate (19) along the characteristic curve $(x_l(t, z), v_l(t, z))$ of $\partial_z^l f$ to get
\[
\partial_z^l f(t, x, v, z) = \partial_z^l f_0(x_l(0), v_l(0), z) H(0, t)
\]
\[+ \sum_{1 \leq l' \leq l} \left( \int_0^t \nabla_v \cdot \left( \partial_z^{l'} F_a[f]\partial_z^{l''} f \right) (s, x_l(s), v_l(s), z) H(s, t) ds \right)
\]
\[- \sum_{0 \leq l' \leq l-1} \left( \int_0^t \alpha(l-l') \partial_z^{l''} \lambda[S](f)(s, x_l(s), v_l(s), z) H(s, t) ds \right)
\]
\[+ \int_0^t \alpha(z) \partial_z^l T^+[S](f)(s, x_l(s), v_l(s), z) H(s, t) ds,
\] (30)

where
\[
H(s, t) := \exp \left( \frac{1}{2} \int_0^s \nabla_v \cdot F_a[f](\tau, x_l(\tau), v_l(\tau), z) + \alpha(z) \lambda[S](\tau, x_l(\tau), v_l(\tau), z) d\tau \right).
\]

By Lemma 4.1 and Theorem 4.5, for $1 \leq l' \leq l$, one has
\[
|\partial_z^{l'} F_a[f]| \leq C(t) \sum_{0 \leq l'' \leq l'} \|\partial_z^{l''} f(t, z)\|_{L^\infty}, \quad \|\nabla_v \cdot \partial_z^{l''} f(t, z)\|_{L^\infty} \leq C(T),
\]
\[
|\nabla_v \cdot \partial_z^l F_a[f]| \leq C(t) \sum_{0 \leq l'' \leq l'} \|\partial_z^{l''} f(t, z)\|_{L^\infty} < C(T).
\] (31)

By (i) in Lemma 4.3 and Theorem 4.5, for $0 \leq l' \leq l$,
\[
|\partial_z^{l'} [\alpha(z) \lambda[S](f)](x)| \leq C(t) \left( \sum_{0 \leq l'' \leq l'} \|\partial_z^{l''} S(t, z)\|_{L^\infty} \right)^2 \sum_{0 \leq l'' \leq l'} \|\partial_z^{l''} f(t, z)\|_{L^\infty}
\]
\[\leq C(T) \sum_{0 \leq l'' \leq l'} \|\partial_z^{l''} f(t, z)\|_{L^\infty}.
\] (32)

We substitute these estimates (31),(32) into (30) to obtain
\[
\|\partial_z^l f(t, z)\|_{L^\infty} \leq \|\partial_z^l f_0(z)\|_{L^\infty} + C(T) \sum_{0 \leq l' \leq l} \int_0^t \|\partial_z^{l'} f(s, z)\|_{L^\infty} ds.
\] (33)

We apply the same argument to (33) to get
\[
\|f(t)\|^2_{H^m_\gamma(L^\infty)} \leq C(T) \left( \|f_0\|^2_{H^m_\gamma(L^\infty)} + \int_0^t \|f(s)\|^2_{H^m_\gamma(L^\infty)} ds \right).
\] (34)
Finally, we apply Gronwall’s lemma to (29) and (34) to conclude the desired result.

5. Conclusion. In this paper, we provided several quantitative local sensitivity estimates to the kinetic CCS model with random inputs. In the first author’s recent work [14] on kinetic CS model with random inputs, they suggested a systematic local sensitivity analysis for the CS flocking model, and found the structural assumptions on the communication weight results in the robustness of the flocking estimates. Thus, a natural extension of the previous work is to see whether similar local sensitivity analysis can be performed for kinetic CS model combined with turning operation effect [8]. In this work, we proposed some structural assumptions on the turning operator and communication weight with the supplement of random inputs, and showed the global existence of a regular solution having the robustness of a velocity alignment. Moreover, we provided the regularity and stability estimates for a global solution in random space. Recall that we used a kinetic approach to model the abrupt change of velocities in an interacting particle system. Of course, a more natural approach is to begin with a suitable second-order particle model capturing the abrupt change of velocities. However, as far as the authors know, this is an open problem in flocking community. We leave this interesting issue as a future work.

Appendix A. Proof of theorem 4.5. Note that we have already verified the initial step \((k \geq 1, m = 1)\) case in Theorem 4.4. To prove this, we will use induction on \(m\):

For fixed \(k \geq 2\) and \(1 \leq m \leq k\), under the assumptions in this theorem, if there exists a unique global solution \(\partial^m_z f(t, z), \partial^m_z S(t, z) \in L^\infty((0,T); W^{k-l,\infty}(\mathbb{R}^d) \times W^{k+1-l,\infty}(\mathbb{R}^d))\) for all \(l < m\), then there also exists a unique global solution \(\partial^m_z f(t, z), \partial^m_z S(t, z) \in L^\infty((0,T); W^{k-m,\infty}(\mathbb{R}^d) \times W^{k+1-m,\infty}(\mathbb{R}^d))\). That is, if we set

\[
F^l(t, z) := \sum_{0 \leq |\mu| + |\nu| \leq k-l} \|\partial^{\mu}_z \partial^{\nu}_v \partial^l_z f(t, z)\|_{L^\infty_x},
\]

\[
S^l(t, z) := \sum_{0 \leq |\mu| \leq k+1-l} \|\partial^{\mu}_z \partial^l_z S(t, z)\|_{L^\infty_x},
\]

the induction hypothesis implies

\[
\sum_{0 \leq l \leq m-1} F^l(t, z) + \sum_{0 \leq l \leq m-1} S^l(t, z) \leq C(t, z).
\]

We need to show that

\[
\sup_{0 \leq t \leq T} (F^m(t, z) + S^m(t, z)) \leq C(T, z).
\]

• (Zeroth order estimate): First, we apply Lemma 4.2 to (19) with the induction hypothesis to get

\[
\|\partial^m_z S(t, z)\|_{L^\infty_x} \leq \|\partial^m_z S_0(z)\|_{L^\infty_x} + \int_0^t C(s, z) \left[ \|\partial^m_z S(s, z)\|_{L^\infty_x} + \|\partial^m_z f(s, z)\|_{L^\infty_x} + 1 \right] ds. \tag{35}
\]
Next, we integrate (19) along the particle trajectory to obtain

\[
|\partial_z^m f(t, x, v, z)| \leq |\partial_z^m f_0(x, v, z)| + \int_0^t \sum_{1 \leq l \leq m} \left( \frac{m}{l} \right) |\partial_z^l F_a[f] \cdot \nabla_v (\partial_z^{m-l} f)| \, ds \\
+ \int_0^t \sum_{0 \leq l \leq m} \left( \frac{m}{l} \right) |\partial_z^l \nabla_v \cdot F_a[f] \partial_z^{m-l} f| \, ds \\
+ \int_0^t \sum_{0 \leq l \leq m} \left( \frac{m}{l} \right) \alpha^{(m-l)}(z) |\partial_z^l \Lambda[S](f)| \, ds \\
\leq |\partial_z f_0(x, v, z)| + \int_0^t (\mathcal{I}_{41} + \mathcal{I}_{42}) \, ds,
\]

where

\[
\mathcal{I}_{41} := \sum_{1 \leq l \leq m} \left( \frac{m}{l} \right) |\partial_z^l F_a[f] \cdot \nabla_v (\partial_z^{m-l} f)| + \sum_{0 \leq l \leq m} \left( \frac{m}{l} \right) |\partial_z^l \nabla_v \cdot F_a[f] \partial_z^{m-l} f|,
\]

\[
\mathcal{I}_{42} := \sum_{0 \leq l \leq m} \left( \frac{m}{l} \right) \alpha^{(m-l)}(z) |\partial_z^l \Lambda[S](f)|.
\]

By Lemma 4.1, Lemma 4.3 and the induction assumption, we have

\[
\mathcal{I}_{41} \leq \sum_{1 \leq l \leq m} \left( \frac{m}{l} \right) |\partial_z^l F_a[f]| \|\nabla_v (\partial_z^{m-l} f)| + \sum_{0 \leq l \leq m} \left( \frac{m}{l} \right) |\partial_z^l \nabla_v \cdot F_a[f] | \| \partial_z^{m-l} f| \\
\leq C(s, z) \left( 1 + \|\partial_z^m f(s, z)\|_{L_x^\infty} \right),
\]

\[
\mathcal{I}_{42} \leq \sum_{0 \leq l \leq m} \left( \frac{m}{l} \right) \alpha^{(m-l)}(z) |\partial_z^l \Lambda[S](f)| \\
\leq C(s, z) \sum_{0 \leq l \leq m} \left( 1 + \|\partial_z^l S(s, z)\|_{L_x^\infty} + \|\partial_z^l f(s, z)\|_{L_x^\infty} \right) \\
\leq C(s, z) \left( 1 + \|\partial_z^m S(s, z)\|_{L_x^\infty} + \|\partial_z^m f(s, z)\|_{L_x^\infty} \right).
\]

Thus, we have

\[
\|\partial_z^m f(t, z)\|_{L_x^\infty} \leq \|\partial_z^m f_0(z)\|_{L_x^\infty} \\
+ \int_0^t C(s, z) \left[ \|\partial_z^m S(s, z)\|_{L_x^\infty} + \|\partial_z^m f(s, z)\|_{L_x^\infty} + 1 \right] \, ds. \tag{36}
\]

Finally, we apply Gronwall lemma on (35) and (36) to conclude

\[
\sup_{0 \leq t < T} \left( \|\partial_z^m f(t, z)\|_{L_x^\infty} + \|\partial_z^m S(t, z)\|_{L_x^\infty} \right) \leq C(T, z).
\]

Moreover, it follows from Lemma 4.2 that

\[
\sup_{0 \leq t < T} \sum_{|\mu|=1} \left( \|\partial_z^\mu \partial_z^m S(t, z)\|_{L_x^\infty} \right) \leq C(T, z).
\]

• (Higher-order estimate): We consider two cases:

  either \( |\nu| \geq 1, |\mu| + |\nu| \leq k - m \) or \( \nu = 0, |\mu| \leq k - m \).
(i) \(|\nu| \geq 1, |\mu| + |\nu| \leq k - m\) case: We take \(\partial^\nu \partial^\mu \) on (19) and integrate along the particle trajectory to have

\[
|\partial^\nu \partial^\mu \partial^m f(t, x, v, z)| \leq |\partial^\nu \partial^\mu \partial^m f_0(x, v, z)| + \int_0^t \mathcal{I}_{43} + \mathcal{I}_{44} \, ds,
\]

where

\[
\mathcal{I}_{43} := - \sum_{|i|=1} \binom{\nu}{i} \partial^{\nu-i} \partial^m \partial^m f
\]

\[
- \sum_{0 \leq i \leq m} \sum_{0 \leq \mu \leq \mu'} \sum_{0 \leq |i| \leq 1} \binom{\mu'}{i} \binom{\mu}{\mu'} \binom{m}{i} \partial^\nu \partial^\mu \partial^l \partial^l F_a[f] \cdot \nabla_v (\partial^{\nu-i} \partial^m \partial^m l f)
\]

\[
- \sum_{0 \leq i \leq m} \binom{\mu}{\mu'} \binom{m}{i} \partial^\nu \cdot (\partial^\mu \partial^l F_a[f]) \partial^\nu \partial^\mu \partial^m \partial^m l f,
\]

\[
\mathcal{I}_{44} := \sum_{0 \leq l \leq m} \binom{m}{l} a^{(m-l)}(z) \partial^\nu \partial^\mu \partial^l \partial^l [S](f).
\]

For the estimate of \(\mathcal{I}_{43}\), we use Lemma 4.1 with the induction hypothesis to derive

\[
\mathcal{I}_{43} \leq \sum_{|i|=1} \binom{\nu}{i} |\partial^{\nu-i} \partial^m \partial^m f|
\]

\[
+ \sum_{0 \leq l \leq m} \sum_{0 \leq \mu \leq \mu'} \sum_{0 \leq |i| \leq 1} \binom{\mu'}{i} \binom{\mu}{\mu'} \binom{m}{i} |\partial^\nu \partial^\mu \partial^l \partial^l F_a[f]||\nabla_v (\partial^{\nu-i} \partial^m \partial^m l f)|
\]

\[
+ \sum_{0 \leq l \leq m} \binom{\mu}{\mu'} \binom{m}{i} \partial^\nu \cdot (\partial^\mu \partial^l F_a[f]) \left| \partial^\nu \partial^\mu \partial^m \partial^m l f \right|
\]

\[
\leq \sum_{|\nu| + |\nu'| = |\mu| + |\nu|} \|\partial^{\nu'} \partial^{\mu'} \partial^m \partial^m f\|_{L^\infty_{t,v}} + C(t, z) \left( \sum_{0 \leq l \leq m-1} \mathcal{F}^l(t, z) \right)^2
\]

\[
+ \mathcal{F}^0(t, z) \left( \sum_{|\nu'| + |\nu'| = |\mu'| + |\nu'|} \|\partial^{\nu'} \partial^{\mu'} \partial^m \partial^m f\|_{L^\infty_{t,v}} + \sum_{0 \leq |\nu'| + |\nu'| \leq |\mu| + |\nu| - 1} \|\partial^{\nu'} \partial^{\mu'} \partial^m \partial^m f\|_{L^\infty_{t,v}} \right).
\]
For the estimate of \( \mathcal{I}_{44} \), we apply Lemma 4.3 to obtain

\[
\mathcal{I}_{44} \leq \sum_{l=0}^{m} \binom{m}{l} |\alpha^{(m-l)}(z)||\partial^\nu_x \partial^\nu_z \partial^l_x A[S](f)|
\]

\[
\leq C(t, z) \left( \|\partial^\nu_x \partial^m_z S(t, z)\|_{L^\infty_x} S^0(t, z) F^0(t, z) + \|\partial^\nu_x \partial^m_x f(t, z)\|_{L^\infty_x} (S^0(t, z))^2 \right)
\]

\[
+ \left( \sum_{0 \leq |\nu'| + |\nu'| \leq |\nu| + |\mu|} \|\partial^\nu_x \partial^m_z \partial^\nu_x \partial^\nu_z S(t, z)\|_{L^\infty_x} \right)^2 \sum_{0 \leq |\nu'| + |\mu'| \leq |\nu| + |\mu|} \|\partial^\nu_x \partial^m_x f(t, z)\|_{L^\infty_x}
\]

\[
+ \left( \sum_{0 \leq |\nu'| \leq m-1} S^l(t, z) \right)^2 \sum_{0 \leq |\nu'| \leq m-1} F^l(t, z).
\]

(ii) \( |\nu| = 0, |\mu| \leq k - m \) case: As in (i), it follows from (19) \( \text{I} \) that

\[
|\partial^\nu_x \partial^m_z f(t, x, v, z)| \leq |\partial^\nu_x \partial^m_z f_0(x, v, z)| + \int_0^t \mathcal{I}_{45} + \mathcal{J}_{46} \, ds,
\]

where

\[
\mathcal{I}_{45} := - \sum_{0 \leq |\nu'| \leq m, \mu' \leq \mu, l + |\mu| > 0} \binom{m}{\mu'} \binom{l}{\mu} \partial^\nu_x \partial^m_z F_a[f] \cdot \nabla_v (\partial^\nu_x \partial^\nu_z \partial^l_x \partial^{m-l} f)
\]

\[
- \sum_{0 \leq |\nu'| \leq m, \mu' \leq \mu} \binom{m}{\mu'} \binom{l}{\mu} \nabla_v (\partial^\nu_x \partial^\nu_z \partial^l_x \partial^{m-l} f) \partial^\mu_x \partial^\mu_z \partial^l_x \partial^{m-l} f,
\]

\[
\mathcal{I}_{46} := \sum_{0 \leq |\nu'| \leq m} \binom{m}{\nu'} \alpha^{(m-|\nu'|)}(z) \partial^\nu_x \partial^\nu_z A[S](f).
\]

Lemma 4.1 and Lemma 4.3 with the induction hypothesis imply

\[
\mathcal{I}_{45} \leq \left( \sum_{0 \leq |\nu'| \leq m-1} F^l(t, z) \right)^2 + F^0(t, z) \|\partial^\nu_x \partial^m_x f(t, z)\|_{L^\infty_x},
\]

\[
\mathcal{I}_{46} \leq C(t, z) \left( \|\partial^\nu_x \partial^m_z S(t, z)\|_{L^\infty_x} S^0(t, z) F^0(t, z) + \|\partial^\nu_x \partial^m_x f(t, z)\|_{L^\infty_x} (S^0(t, z))^2 \right)
\]

\[
+ \left( \sum_{0 \leq |\nu'| \leq m-1} \|\partial^\nu_x \partial^m_z S(t, z)\|_{L^\infty_x} \right)^2 \sum_{0 \leq |\nu'| \leq m-1} \|\partial^\nu_x \partial^m_x f(t, z)\|_{L^\infty_x}
\]

\[
+ \left( \sum_{0 \leq |\nu'| \leq m-1} S^l(t, z) \right)^2 \sum_{0 \leq |\nu'| \leq m-1} F^l(t, z).
\]

Finally, for a fixed \( m \), we use induction principle on \( k' \) from 2 to \( k - m \), where \( k' := |\mu| + |\nu| \) in (i) and \( k' := |\mu| \) in (ii). Under the induction hypothesis on \( m \) and \( k' \), Lemma 4.2 shows

\[
\sup_{0 \leq t < T} \sum_{|\nu'| = k'} \|\partial^\nu_x \partial^m_z S(t, z)\|_{\infty} \leq C(T, z).
\]
We collect all the estimates for $\mathcal{I}_{43} - \mathcal{I}_{46}$ to derive
\[
\sum_{|\nu'| + |\mu'| = k'} \| \partial^\nu_{\nu'} \partial^\mu_{\mu'} \partial_z^m f(t, z) \|_{L_{x,v}^0} \leq \sum_{|\nu'| + |\mu'| = k'} \| \partial^\nu_{\nu'} \partial^\mu_{\mu'} \partial_z^m f_0(z) \|_{L_{x,v}^0}
+ \int_0^t C(s, z) \left( \sum_{|\nu'| + |\mu'| = k'} (\| \partial^\nu_{\nu'} \partial^\mu_{\mu'} \partial_z^m f(s, z) \|_{L_{x,v}^\infty} + \| \partial^\nu_{\nu'} + \partial^\mu_{\mu'} \partial_z^m S(s, z) \|_{L_{x,v}^\infty}) + 1 \right) ds.
\]

We apply Gronwall’s lemma to the above estimate with the estimate (37) to conclude
\[
\sup_{0 \leq t < T} \sum_{|\nu'| + |\mu'| = k'} \| \partial^\nu_{\nu'} \partial^\mu_{\mu'} \partial_z^m f(t, z) \|_{L_{x,v}^\infty} \leq C(T, z).
\]

Again, we use Lemma 4.2 to derive
\[
\sup_{0 \leq t < T} \sum_{|\nu'| = k' + 1} \| \partial^\nu_{\nu'} \partial_z^m S(t, z) \|_{\infty} \leq C(T, z).
\]

**Appendix B. Proof of theorem 4.7.** From Theorem 3.1, for each $z \in \Omega$, there exist unique solutions $(f, S)$ and $(\tilde{f}, \tilde{S})$ corresponding to initial data $(f_0, S_0)$ and $(\tilde{f}_0, \tilde{S}_0)$ respectively:
\[
F^0(t, z) + S^0(t, z) := \sum_{0 \leq |\nu| + |\mu|} \| \partial^\nu \partial^\mu f(t, z) \|_{L_{x,v}^\infty} + \sum_{0 \leq |\mu| \leq k + 2} \| \partial^\mu S(t, z) \|_{L_{x,v}^\infty} < \infty,
\]
\[
\tilde{F}^0(t, z) + \tilde{S}^0(t, z) := \sum_{0 \leq |\nu| + |\mu|} \| \partial^\nu \partial^\mu \tilde{f}(t, z) \|_{L_{x,v}^\infty} + \sum_{0 \leq |\mu| \leq k + 2} \| \partial^\mu \tilde{S}(t, z) \|_{L_{x,v}^\infty} < \infty.
\]

For notational simplicity, we use $C(t, z)$ to denote various functions satisfying $C(T, z) := \|C(\cdot, z)\|_{L^\infty}$ is finite for each $z \in \Omega$. For example, for $0 < t < T$,
\[
F^0(t, z) + S^0(t, z) < C(T, z).
\]

• **(Zeroth-order estimate):** From (3), difference of solutions satisfy
\[
\partial_t (f - \tilde{f}) + v \cdot \nabla_x (f - \tilde{f}) + \nabla_v \cdot \left[ F_a[f] \right] (f - \tilde{f}) + \nabla_v \cdot \left[ (F_a[f] - F_a[\tilde{f}]) \tilde{f} \right] = \alpha(z) \int_0^t (T - \tilde{T}) \tilde{f}^\prime + T (f' - \tilde{f}') - (T^\prime - \tilde{T}^\prime) \tilde{f} - T^\prime (f - \tilde{f}) \, dv,
\]
\[
(\partial_t - \Delta)(S - \tilde{S}) = -\kappa(z) \rho(S - \tilde{S}) - \kappa(z)(\rho - \tilde{\rho}) \tilde{S}.
\]

Here we used handy notation:
\[
\tilde{T} := T[\tilde{S}](t, x, v', z), \quad \tilde{T}^\prime := T[\tilde{S}](t, x, v', z).
\]

We first apply Duhamel’s principle to (38) to get
\[
(S - \tilde{S})(t, x, z) = \int_{\mathbb{R}^d} e^{t\Delta}(x, y) (S_0 - \tilde{S}_0)(y) dy
+ \kappa(z) \int_0^t \int_{\mathbb{R}^d} e^{(t-s)\Delta}(x, y) \left[ \rho(S - \tilde{S}) + (\rho - \tilde{\rho}) \tilde{S} \right] (s, y, z) dy ds.
\]
This gives
\[
\| (S - \tilde{S})(t, z) \|_{L^\infty_x} \\
\leq \| (S_0 - \tilde{S}_0)(z) \|_{L^\infty_x} + C(T, z) \int_0^t \| (S - \tilde{S})(s, z) \|_{L^\infty_x} + \| (f - \tilde{f})(s, z) \|_{L^\infty_{x,z}} ds.
\] (39)

On the other hands, we use the assumption \((A_1)\) to get
\[
|T[S] - T[\tilde{S}]| \\
\leq |S(t, x - v', z)S(t, x + v, z) - \tilde{S}(t, x - v', z)\tilde{S}(t, x + v, z)| \\
\leq \|S(t, z)\|_{L^\infty_x} \| (S - \tilde{S})(t, x - v', z) \| + \| \tilde{S}(t, z) \|_{L^\infty_x} \| (S - \tilde{S})(t, x + v, z) \| \\
\leq C(t, z) \| (S - \tilde{S})(t, z) \|_{L^\infty_x}.
\] (40)

As in the proof of Theorem 3.1, we integrate the equation (38), and use Lemma 4.3 and (40) to get
\[
\| (f - \tilde{f})(t, z) \|_{L^\infty_x} \\
\leq \| (f_0 - \tilde{f}_0)(z) \|_{L^\infty_x} + C(T, z) \int_0^t \| (S - \tilde{S})(s, z) \|_{L^\infty_x} + \| (f - \tilde{f})(s, z) \|_{L^\infty_{x,z}} ds.
\] (41)

We collect (39) and (41) to derive the desired zeroth-order estimate.

- (Higher-order estimate): We first suppose that the following induction hypothesis holds: for \(1 \leq k' \leq k\), there exists a nonnegative random variable \(C(T, z)\) such that
\[
\| (f - \tilde{f})(t, z) \|_{W^{k'-1}_x} + \| (S - \tilde{S})(t, z) \|_{W^{k'-1}_x} \\
\leq C(T, z) \left( \left\| (f_0 - \tilde{f}_0)(z) \right\|_{W^{k'-1}_x} + \| (S_0 - \tilde{S}_0)(z) \|_{W^{k'-1}_x} \right),
\]
for each \(z \in \Omega, t \in (0, T)\).

- (Estimate of \(\partial^\mu_x (S - \tilde{S})\) with \(|\mu| = k'\)): We use (17) to derive the equation for \(\partial^\mu_x (S - \tilde{S})\) as follows:
\[
\partial^\mu_x (S - \tilde{S})(t, x, z) \\
= \int_{\mathbb{R}^d} e^{t\Delta}(x, y) \partial^\mu_x (S_0 - \tilde{S}_0(y, z)) dy - \kappa(z) \int_0^t \int_{\mathbb{R}^d} [e^{(t-s)\Delta}(x, y)]_x H_1(s, x, z) dy ds,
\]
where
\[
H_1(s, y, z) : = \sum_{\mu' \leq \mu - \varepsilon_j} \binom{\mu - \varepsilon_j}{\mu'} \left( \partial^{\mu'}_x (S - \tilde{S}) \partial^{\mu' - \varepsilon_j} \rho + \partial^{\mu'}_x \tilde{S} \partial^{\mu' - \varepsilon_j} (\rho - \tilde{\rho}) \right) (s, y, z).
\]
Direct calculations show

\[
|H_1(s, y, z)| \leq C \sum_{|\mu| \leq |\mu|-1} (|\partial_2^\mu \rho| + |\partial_2^\mu \tilde{S}|) \sum_{|\mu'| \leq |\mu|-1} \left( |\partial_2^{\mu'} (S - \tilde{S})| + |\partial_2^{\mu'} (\rho - \tilde{\rho})| \right)
\]

\[
\leq C(s, z) \sum_{|\mu'| \leq |\mu|-1} \left( \|\partial_2^{\mu'} f(s, z)\|_{L^p_{x,v}} + \|\partial_2^{\mu'} \tilde{S}(s, z)\|_{L^p_{x,v}} \right)
\times \sum_{|\mu'| \leq |\mu|-1} \left( \|\partial_2^{\mu'} (S - \tilde{S})(s, z)\|_{L^p_{x,v}} + \|\partial_2^{\mu'} (f - \tilde{f})(s, z)\|_{L^p_{x,v}} \right)
\leq C(s, z) \sum_{|\mu'| \leq |\mu|-1} \left( \|\partial_2^{\mu'} (S - \tilde{S})(s, z)\|_{L^p_{x,v}} + \|\partial_2^{\mu'} (f - \tilde{f})(s, z)\|_{L^p_{x,v}} \right).
\]

This gives us

\[
\|\partial_2^{\mu'} (S - \tilde{S})(t, z)\|_{L^p_{x}} \leq \|\partial_2^{\mu'} (S_0 - \tilde{S}_0)(z)\|_{L^p_{x}}
+ \|\kappa\|_{\infty} \int_0^t \sum_{|\mu'| \leq |\mu|-1} \frac{C(s, z)}{\sqrt{\pi(t-s)}} \left( \|\partial_2^{\mu'} (S - \tilde{S})(s, z)\|_{L^p_{x,v}} + \|\partial_2^{\mu'} (f - \tilde{f})(s, z)\|_{L^p_{x,v}} \right) ds.
\]

By the induction hypothesis on \(k' - 1\), one has

\[
\sum_{|\mu| = k'} \|\partial_2^{\mu'} (S - \tilde{S})(t, z)\|_{L^p_{x,v}}
\leq C(T, z) \left( \sum_{|\mu'| \leq k'} \|\partial_2^{\mu'} (S_0 - \tilde{S}_0)\|_{L^p_{x,v}} + \sum_{|\mu'| \leq k' - 1} \|\partial_2^{\mu'} (f_0 - \tilde{f}_0)\|_{L^p_{x,v}} \right). \tag{42}
\]

\(\diamond\) (Estimate of \(\partial_2^{\mu'} \partial_2^{\mu'} (f - \tilde{f})\) with \(|\nu| \geq 1, |\mu| + |\nu| = k'\): We use (14) to derive the difference equation for \(\partial_2^{\mu'} \partial_2^{\mu'} (f - \tilde{f})\):

\[
\partial_t (\partial_2^{\mu'} \partial_2^{\mu'} (f - \tilde{f})) + v \cdot \nabla_x (\partial_2^{\mu'} \partial_2^{\mu'} (f - \tilde{f})) + F_a[f] \cdot \nabla_v (\partial_2^{\mu'} \partial_2^{\mu'} (f - \tilde{f})) = - \sum_{1 \leq \gamma \leq 0} I_{5,j}, \tag{43}
\]

where

\[
\begin{align*}
I_{51} &= (F_a[f] - F_a[f]) \cdot \nabla (\partial_2^{\mu'} \partial_2^{\mu'} f), \\
I_{52} &= \sum_{|\mu| \leq \mu'} \left( \sum_{|\nu| = 1} \right) \partial_2^{\nu} \partial_2^{\nu'} (f - \tilde{f}), \\
I_{53} &= \sum_{|\mu| \leq \mu'} \left( \sum_{|\nu| = 1} \right) \partial_2^{\nu} \partial_2^{\nu'} (F_a[f] - F_a[f]) \cdot \nabla (\partial_2^{\nu} \partial_2^{\nu'} (f - \tilde{f})), \\
I_{54} &= \sum_{|\mu| \leq \mu'} \left( \sum_{|\nu| = 1} \right) \partial_2^{\nu} \partial_2^{\nu'} (F_a[f] - F_a[f]) \cdot \nabla (\partial_2^{\nu} \partial_2^{\nu'} (f - \tilde{f})), \\
I_{55} &= \alpha(z) \int_{\partial dla} \sum_{|\mu| \leq \mu'} \left( \sum_{|\nu| = 1} \right) \partial_2^{\nu} \partial_2^{\nu'} \partial_2^{\nu} \partial_2^{\nu'} (f - \tilde{f}) - \sum_{\mu' \leq \mu} \left( \sum_{|\nu| = 1} \right) \partial_2^{\nu} \partial_2^{\nu'} \partial_2^{\nu} \partial_2^{\nu'} (f - \tilde{f}) dv', \\
I_{56} &= \alpha(z) \int_{\partial dla} \sum_{|\mu| \leq \mu'} \left( \sum_{|\nu| = 1} \right) \partial_2^{\nu} \partial_2^{\nu'} \partial_2^{\nu} \partial_2^{\nu'} (T - \tilde{T}) - \sum_{\nu \leq \nu} \left( \sum_{|\mu| \leq \mu'} \right) \partial_2^{\nu} \partial_2^{\nu'} \partial_2^{\nu} \partial_2^{\nu'} (T - \tilde{T}) dv'.
\end{align*}
\]
We use Theorem 3.1, Lemma 4.2 and Lemma 4.6 to find

\[ I_{51} \leq \sum_{|i|=1} \| \partial_x^{\nu_i} L_{v_i} f \|_{L_v} \| f - \tilde{f} \|_{L_v z} \leq C(T, z) \| f - \tilde{f} \|_{L_v z}, \]

\[ I_{52} \leq C(T, z) \sum_{|\mu'| + |\nu'| = k'} \| \partial_x^{\mu'} \partial_v^{\nu'} (f - \tilde{f}) \|_{L_v z}, \]

\[ I_{53} \leq C(T, z) \left( \sum_{\mu' \leq \mu} \| \partial_x^{\mu'} \partial_x^{\nu'} (f - \tilde{f}) \|_{L_v z} + \sum_{|\mu'| + |\nu'| \leq k'} \| \partial_x^{\mu'} \partial_v^{\nu'} (f - \tilde{f}) \|_{L_v z} \right) \]

\[ \leq C(T, z) \sum_{|\mu'| + |\nu'| \leq k'} \| \partial_x^{\mu'} \partial_v^{\nu'} (f - \tilde{f}) \|_{L_v z}, \]

\[ I_{54} \leq C(T, z) \left( \sum_{\mu' \leq \mu} \| \nabla_x \cdot \partial_x^{\mu'} (F_\nu[f] - F_\nu[\tilde{f}]) \|_{L_v z} + \sum_{|\mu'| + |\nu'| \leq k'} \| \nabla_x \cdot \partial_x^{\mu'} (F_\nu[f] - F_\nu[\tilde{f}]) \|_{L_v z} \right) \]

\[ \leq C(T, z) \| f - \tilde{f} \|_{L_v z}, \]

\[ I_{55} \leq C(T, z) \left( \sum_{\mu' \leq \mu} \| \partial_x^{\mu'} (f' - \tilde{f}) \|_{L_v z} + \sum_{|\mu'| + |\nu'| \leq k'} \| \partial_x^{\mu'} \partial_v^{\nu'} (f - \tilde{f}) \|_{L_v z} \right). \]

We rewrite the assumption \((A_1)\) for the difference \(\partial_x^{\mu'} \partial_v^{\nu'} (T[S] - T[\tilde{S}])\) as

\[ \| \partial_x^{\mu'} \partial_v^{\nu'} (T[S] - T[\tilde{S}]) \|_{L_v z} \leq C \sum_{\mu' \leq \mu} \| \partial_x^{\mu' + \nu'} S \|_{L_v z} \| \partial_x^{\mu + \nu' - \mu'} (S - \tilde{S}) (t, z) \|_{L_v z} \]

\[ + C \sum_{\mu' \leq \mu} \| \partial_x^{\mu + \nu' - \mu'} \tilde{S} (t, z) \|_{L_v z} \| \partial_x^{\mu + \nu'} (S - \tilde{S}) (t, z) \|_{L_v z} \]

\[ \leq C(T, z) \sum_{|\mu'| + |\nu'| \leq k'} \| \partial_x^{\mu'} (S - \tilde{S}) (t, z) \|_{L_v z}. \]

Thus, we have

\[ I_{56} \leq C(T, z) \left( \sum_{\mu' \leq \mu} \| \partial_x^{\mu'} \partial_x^{\nu'} (T - \tilde{T}) \|_{L_v z} + \sum_{|\mu'| + |\nu'| \leq k'} \| \partial_x^{\mu'} \partial_x^{\nu'} (T^* - \tilde{T}^*) \|_{L_v z} \right) \]

\[ \leq C(T, z) \sum_{|\mu'| + |\nu'| \leq k'} \| \partial_x^{\mu'} (S - \tilde{S}) (t, z) \|_{L_v z}. \]

Finally, we integrate (43) and use estimates for \(I_{4j}\) to have

\[ \| \partial_x^{\mu'} \partial_v^{\nu'} (f - \tilde{f})(t, z) \|_{L_v z} \leq \| \partial_x^{\mu'} \partial_v^{\nu'} (f_0 - \tilde{f}_0) (z) \|_{L_v z} \]

\[ + C(T, z) \int_0^1 \sum_{|\mu'| + |\nu'| \leq k'} \left( \| \partial_x^{\mu'} \partial_v^{\nu'} (f - \tilde{f}) (s, z) \|_{L_v z} + \| \partial_x^{\mu' + \nu'} (S - \tilde{S}) (s, z) \|_{L_v z} \right) ds. \]

(44)
\( \diamond \) (Estimate of \( \partial_{x}^{\mu}(f - \tilde{f}) \) with \( |\mu| = 0, k' = |\mu| \)): Note that \( \partial_{x}^{\mu}(f - \tilde{f}) \) satisfies
\[
\begin{align*}
\partial_{t}(\partial_{x}^{\mu}(f - \tilde{f})) + v \cdot \nabla_{x}(\partial_{x}^{\mu}(f - \tilde{f})) + F_{0}[f] \cdot \nabla_{x}(\partial_{x}^{\mu}(f - \tilde{f}))
&= -\nabla_{x}(\partial_{x}^{\mu}(f - \tilde{f})) \cdot (F_{0}[f] - \tilde{F}_{0}(f)) \\
&- \sum_{0 < \mu' \leq \mu} \left( \mu \right)^{\mu'} \partial_{x}^{\mu'} F_{0}[f] \cdot \nabla_{x}(\partial_{x}^{\mu'}(f - \tilde{f})) + \sum_{\mu' \leq \mu} \left( \mu \right)^{\mu'} \nabla_{x} \cdot (\partial_{x}^{\mu'} F_{0}[f]) \partial_{x}^{\mu'}(f - \tilde{f}) \\
&- \sum_{0 < \mu' \leq \mu} \left( \mu \right)^{\mu'} \nabla_{x} \cdot (\partial_{x}^{\mu'}(f - \tilde{f})) \cdot \partial_{x}^{\mu'}(f - \tilde{f}) - \sum_{\mu' \leq \mu} \left( \mu \right)^{\mu'} \partial_{x}^{\mu'} f \partial_{x}^{\mu'}(f - \tilde{f}) \\
&+ \alpha(z) \int_{0}^{t} \sum_{\mu' \leq \mu} \left( \mu \right)^{\mu'} \left( \partial_{x}^{\mu'} T \partial_{x}^{\mu'}(f - \tilde{f}) - \partial_{x}^{\mu'} T \partial_{x}^{\mu'}(f - \tilde{f}) \right) \, ds'
\end{align*}
\]
where the functionals
\[
\begin{align*}
F_{0}(f - \tilde{f}) &= F_{0}(f) - F_{0}(\tilde{f}) \\
S_{0}(f - \tilde{f}) &= S_{0}(f) - S_{0}(\tilde{f})
\end{align*}
\]
By Gröwall’s lemma, one has
\[
\|(f - \tilde{f})(t, z)\|_{W_{x,v}^{k,\infty}} \leq C(T, z) \left( \|(f_{0} - \tilde{f}_{0})(z)\|_{W_{x,v}^{k,\infty}} + \|(S_{0} - \tilde{S}_{0})(z)\|_{W_{x,v}^{k,\infty}} \right).
\]
Finally, we return to (42) and consider the case \( |\mu| = k' + 1 \). Then we have the desired result.

**Appendix C. Proof of Theorem 4.8.** It follows from Theorem 4.5 that for each \( z \in \Omega \), there exist unique solutions process \((f, S), (\tilde{f}, \tilde{S})\) to (3) satisfy
\[
\sum_{0 \leq l \leq m} (F_{l}(t, z) + S_{l}(t, z) + \tilde{F}_{l}(t, z) + \tilde{S}_{l}(t, z)) < \infty,
\]
where the functionals \( F_{l} \) and \( S_{l} \) are defined as follows:
\[
\begin{align*}
F_{l}(t, z) &:= \sum_{0 \leq |\mu| + |\mu'| \leq k + 1 - l} \| \partial_{x}^{\mu} \partial_{x}^{\mu'} f(t, z) \|_{L_{x,v}^{\infty}}, \\
S_{l}(t, z) &:= \sum_{0 \leq |\mu| + |\mu'| \leq k + 2 - l} \| \partial_{x}^{\mu} \partial_{x}^{\mu'} S(t, z) \|_{L_{x,v}^{\infty}}.
\end{align*}
\]
Since we have already proved the case when \( m = 0 \), we consider the case \( m \geq 1 \). Note that the case with \( l = 0 \) is covered in Theorem 4.3, we just consider the estimates when \( 1 \leq l \leq m \). We will present the zeroth-order estimate and higher-order
estimate for $\partial^2_t (f - \bar{f})$ separately. The proof is fundamentally based on induction argument used in the proof of Theorem 4.5.

- **(Zeroth-order estimates):** Here we use an induction argument on $l$. First, we use (19) to obtain the following equations for the difference of solutions:

$$
\partial_t (\partial^2_t (f - \bar{f})) + \nu \cdot \nabla_x (\partial^2_t (f - \bar{f})) + F_\nu[f] \cdot \nabla_x (\partial^2_t (f - \bar{f}))
$$

$$
= -(F_\nu[f] - F_\nu[\bar{f}]) \cdot \nabla_x \partial^2_t \tilde{f} - \sum_{1 \leq \nu \leq l} \left( \left( 1 \nu \right) \partial^2_t (F_\nu[f] - F_\nu[\bar{f}]) \cdot \nabla_x \partial^2_t \tilde{f}ight)
$$

$$
- \sum_{0 \leq \nu' \leq l} \left( \left( 1 \nu' \right) \nabla_x \cdot \partial^2_t (F_\nu[f] - F_\nu[\bar{f}]) \partial^2_t \tilde{f}ight)
$$

$$
- \sum_{0 \leq \nu' \leq l} \left( \left( 1 \nu' \right) \nabla_x \cdot \partial^2_t (F_\nu[f] - F_\nu[\bar{f}]) \partial^2_t \tilde{f}ight) + \sum_{0 \leq \nu' \leq l} \left( \left( 1 \nu' \right) \partial^2_t \tilde{f} \partial^2_t (S - \bar{S})ight)
$$

$$
\partial_t \partial^2_t (S - \bar{S}) - \Delta \partial^2_t (S - \bar{S})
$$

$$
= - \sum_{0 \leq \nu' \leq l} \left( \left( 1 \nu' \right) \kappa^{(0-l')}(z) \left( \partial^2_t \tilde{f} \partial^2_t (S - \bar{S}) + \partial^2_t \tilde{f} \partial^2_t (\rho - \bar{\rho})(S - \bar{S}) \right) \right),
$$

(i) We rewrite (46)2 by using Duhamel’s principle as

$$
\partial^2_t (S - \bar{S})(t, x, z)
$$

$$
= \int_{\mathbb{R}^d} e^{\Delta (x, y)} \partial^2_t (S_0 - \bar{S}_0)(y, z) dy - \int_0^t \int_{\mathbb{R}^d} e^{(t-s)\Delta} (x, y) H_2(s, y, z) dy ds,
$$

where

$$
H_2(s, y, z)
$$

$$
:= \sum_{0 \leq \nu' \leq l} \left( \left( 1 \nu' \right) \kappa^{(0-l')}(z) \left( \partial^2_t \tilde{f} \partial^2_t (S - \bar{S}) + \partial^2_t \tilde{f} \partial^2_t (\rho - \bar{\rho})(S - \bar{S}) \right) \right). (s, y, z).
$$

By Theorem 4.5, we have

$$
\| H_2(s, z) \|_{L^\infty} \leq C \| \kappa \|_{\infty} \sum_{0 \leq \nu' \leq l} \left( \| \partial^2_t \bar{f} \|_{L^\infty} + \| \partial^2_t \bar{S} \|_{L^\infty} \right),
$$

$$
\leq C \| \kappa \|_{\infty} \| \nabla (s, z) \|^2 \sum_{0 \leq \nu' \leq l} \left( \| \bar{f} \|_{L^\infty} + \| \bar{S} \|_{L^\infty} \right),
$$

$$
\times \sum_{0 \leq \nu' \leq l} \left( \| \partial^2_t (S - \bar{S}) \|_{L^\infty} + \| \partial^2_t (f - \bar{f}) \|_{L^\infty} \right),
$$

$$
\leq C(s, z) \sum_{0 \leq \nu' \leq l} \left( \| \partial^2_t (S - \bar{S}) \|_{L^\infty} + \| \partial^2_t (f - \bar{f}) \|_{L^\infty} \right).
$$
Under the induction hypotheses on \( l' \leq l - 1 \), we have
\[
\| \partial_x (S - \tilde{S})(t, z) \|_{L^p} \leq \| \partial_x (S_0 - \tilde{S}_0)(z) \|_{L^p} + C(t, z) \sum_{0 \leq l' \leq l - 1} \left( \left\| \partial_x (S_0 - \tilde{S}_0)(z) \right\|_{L^p} + \left\| \partial_x (f_0 - \tilde{f}_0)(z) \right\|_{L^p} \right)
\]
\[
+ \int_0^t C(s, z) \left( \left\| \partial_x (S - \tilde{S})(s, z) \right\|_{L^p} + \left\| \partial_x (f - \tilde{f})(s, z) \right\|_{L^p} \right) ds.
\]

(ii) We integrate (46) to find
\[
\| \partial_x (f - \tilde{f})(t, z) \|_{L^p} \leq \| \partial_x (f_0 - \tilde{f}_0)(z) \|_{L^p} + \int_0^t (\mathcal{I}_{61} + \mathcal{I}_{62} + \mathcal{I}_{63}) ds,
\]
where the term \( \mathcal{I}_{63} \) will be defined below.

\[
\mathcal{I}_{61} := - (F_a[f] - F_a[\tilde{f}]) \cdot \nabla_v \partial_x \tilde{f} - \sum_{1 \leq l' \leq l} \left( \frac{l}{l'} \right) \partial_x (F_a[f] - F_a[\tilde{f}]) \cdot \nabla_v \partial_x \tilde{f},
\]

\[
\mathcal{I}_{62} := - \sum_{1 \leq l' \leq l} \left( \frac{l}{l'} \right) \partial_x F_a [f] \cdot \nabla_v \partial_x \tilde{f} - \sum_{0 \leq l' \leq l} \left( \frac{l}{l'} \right) \nabla_v \cdot \partial_x F_a [f] \partial_x \tilde{f} - \sum_{0 \leq l' \leq l} \left( \frac{l}{l'} \right) \nabla_v \cdot \partial_x F_a [f] \partial_x \tilde{f},
\]

\[
\mathcal{I}_{63} := \sum_{0 \leq l' \leq l} \left( \frac{l}{l'} \right) \partial_x \tilde{f} \cdot \alpha \partial_x (\Lambda[S](f) - \Lambda[\tilde{S}](\tilde{f})).
\]

- (Estimate of \( \mathcal{I}_{61} \)): We apply Lemma 4.3 and Theorem 4.5 to have
\[
\mathcal{I}_{61} \leq \sum_{0 \leq l' \leq l} \left( \left\| \nabla_v \partial_x \tilde{f} (s, z) \right\|_{L^p} + \left\| f(s, z) \right\|_{L^p} \right) \sum_{0 \leq l' \leq l} \left\| \partial_x \tilde{f} (f - \tilde{f})(s, z) \right\|_{L^p},
\]
\[
\leq C(s, z) \sum_{0 \leq l' \leq l} \left\| \partial_x \tilde{f} (f - \tilde{f})(s, z) \right\|_{L^p}.
\]

- (Estimate of \( \mathcal{I}_{62} \)): We apply Lemma 4.1 and Theorem 4.5 to get
\[
\mathcal{I}_{62} \leq C(s, z) \left( \sum_{0 \leq l' \leq l - 1} \left\| \nabla_v \partial_x \tilde{f} (f - \tilde{f})(s, z) \right\|_{L^p} + \sum_{0 \leq l' \leq l} \left\| \partial_x \tilde{f} (f - \tilde{f})(s, z) \right\|_{L^p} \right).
\]

- (Estimate of \( \mathcal{I}_{63} \)): We use the assumption (A_1) and Theorem 4.5 to have
\[
\partial_x (\Lambda[S](f) - \Lambda[\tilde{S}](\tilde{f}))
\]
\[
= \sum_{0 \leq l' \leq l} \left( \frac{l}{l'} \right) \int_{\mathbb{R}^d} \partial_x T \partial_x \tilde{f} (f' - \tilde{f}') + \partial_x (T - \tilde{T}) \partial_x \tilde{f} dv' 
\]
\[
- \sum_{0 \leq l' \leq l} \left( \frac{l}{l'} \right) \int_{\mathbb{R}^d} \partial_x T \partial_x \tilde{f} (f' - \tilde{f}') + \partial_x (T - \tilde{T}) \partial_x \tilde{f} dv' 
\]
\[
\leq C(s, z) \sum_{0 \leq l' \leq l} \left( \left\| \partial_x \tilde{f} (f - \tilde{f})(s, z) \right\|_{L^p} + \left\| \partial_x (T - \tilde{T})(s, z) \right\|_{L^p} \right).
The last inequality is due to the similar observation for $\partial_z^\alpha (T - \tilde{T})$ as in (40). Thus, we have

$$I_{63} \leq C(s, z) \sum_{0 \leq \nu' \leq l} \left( \| \partial_z^{\nu'} (f - \tilde{f})(s, z) \|_{L_{t,x}^{\infty}} + \| \partial_z^{\nu'} (S - \tilde{S})(s, z) \|_{L_{t,x}^{\infty}} \right).$$

We collect these estimates and adopt the induction hypothesis $l' \leq l - 1$ to derive

$$\| \partial_z^{\nu} (f - \tilde{f})(t, z) \|_{L_{t,x}^{\infty}} \leq \| \partial_z^{\nu} (f_0 - \tilde{f}_0)(z) \|_{L_{t,x}^{\infty}} + C(t, z) \sum_{0 \leq \nu' \leq l'} \left( \| \partial_z^{\nu'} (S_0 - \tilde{S}_0)(z) \|_{L_{x,v}^{\infty}} + \| \partial_z^{\nu'} (f_0 - \tilde{f}_0)(z) \|_{L_{x,v}^{\infty}} \right) + \int_0^t C(s, z) \left( \| \partial_z^{\nu} (S - \tilde{S})(s, z) \|_{L_{t,x}^{\infty}} + \| \partial_z^{\nu} (f - \tilde{f})(s, z) \|_{L_{t,x}^{\infty}} \right) ds,$$

Finally, we add (47) and (48) and apply Gronwall’s inequality to derive the desired result for $l$.

- (Higher-order estimate): For a fixed $l$, we will use induction principle on $k'$.

  (i) For $|\mu| = k'$, we take $\partial_z^{\nu} \partial_l^k(S - \tilde{S})(t, x, z)$ to have

  $$\partial_z^\mu \partial_l^k (S - \tilde{S})(t, x, z) = \int_{\mathbb{R}^d} e^{t\Delta}(x, y) \partial_z^\mu \partial_l^k (S_0 - \tilde{S}_0)(y, z) dy - \int_0^t \int_{\mathbb{R}^d} \left[ e^{(t-s)\Delta}(x, y) \right] x_j H_3(s, y, z) dy ds,$$

  where

  $$H_3(s, y, z) := \sum_{\mu' \leq \mu - e_j} \sum_{0 \leq \nu' \leq l'} \left( \frac{\mu - e_j}{\mu'} \right) \left( \frac{l'}{l} \right) \frac{\mu'}{\nu'} \kappa^{l-l'}(z) \times \left( \partial_z^{\mu'-e_j} \partial_z^{l-l'} \rho \partial_z^\nu \partial_z^{\nu'} (S - \tilde{S}) + \partial_z^{\mu'-e_j} \partial_z^{l-l'} (\rho - \tilde{\rho})(\partial_z^\nu \partial_z^{\nu'} \tilde{S}) \right)(s, y, z).$$

  Similar to $\|H_2(s, z)\|_{L_{t,x}^{\infty}}$, one has

  $$\|H_3(s, z)\|_{L_{t,x}^{\infty}} \leq C \|\kappa\|_{\infty} \sum_{0 \leq \nu' \leq l} \left( \|\partial_z^{\mu'-e_j} \partial_z^{l-l'} \rho \|\partial_z^{\nu'} (S - \tilde{S})\| + \|\partial_z^{\mu'} \tilde{S}\| \|\partial_z^{\mu'-e_j} \partial_z^{l-l'} (\rho - \tilde{\rho})\| \right)(s, y, z) \leq C(s, z) \sum_{0 \leq \nu' \leq l} \left( \|\partial_z^\nu \partial_z^{\nu'} (S - \tilde{S})(s, z)\|_{L_{t,x}^{\infty}} + \|\partial_z^\nu \partial_z^{\nu'} (f - \tilde{f})(s, z)\|_{L_{t,x}^{\infty}} \right).$$

  Under the induction hypothesis on $|\mu'| \leq k' - 1$, we can derive

  $$\|\partial_z^\mu \partial_l^k (S - \tilde{S})(t, z)\|_{L_{t,x}^{\infty}} \leq \|\partial_z^\mu \partial_l^k (S_0 - \tilde{S}_0)(z)\|_{L_{t,x}^{\infty}} + \int_0^t \frac{C(s, z)}{\sqrt{t-s}} \sum_{0 \leq \nu' \leq l} \left( \|\partial_z^\nu \partial_l^k (S - \tilde{S})(s, z)\|_{L_{t,x}^{\infty}} + \|\partial_z^\nu \partial_l^k (f - \tilde{f})(s, z)\|_{L_{t,x}^{\infty}} \right) ds \leq C(t, z) \sum_{0 \leq \nu' \leq l} \left( \|\partial_z^\nu \partial_l^k (S_0 - \tilde{S}_0)(t, z)\|_{L_{t,x}^{\infty}} + \|\partial_z^\nu \partial_l^k (f_0 - \tilde{f}_0)(t, z)\|_{L_{t,x}^{\infty}} \right),$$

  (49)
for all $|\mu| = k'$.

(ii) For $|\mu| + |\nu| = k', |\nu| \geq 1$, we take $\partial_\nu^\mu \partial_x^\nu$ to (46) to get

$$
\partial_\nu (\partial_\nu^\mu \partial_x^\nu \partial_z^i (f - \bar{f})) + v \cdot \nabla_v (\partial_\nu^\mu \partial_x^\nu \partial_z^i (f - \bar{f})) + F_a[f] \cdot \nabla_v (\partial_\nu^\mu \partial_x^\nu \partial_z^i (f - \bar{f}))
= -I_{71} - I_{72} - I_{73} - I_{74} + I_{75},
$$

where the term $I_{7i}$ will be estimated below.

\*(Estimates for $I_{71}$ and $I_{72}$): We apply Lemma 4.3 and Theorem 4.5 to see

$$
I_{71} := (F_a[f] - F_a[\bar{f}]) \cdot \nabla_v (\partial_\nu^\mu \partial_x^\nu \partial_z^i (f - \bar{f})) \leq C(t, z) \| \partial_\nu^\mu \partial_z^i (f - \bar{f})(t, z) \|_{L_{\infty, v}^a},
$$

$$
I_{72} := \sum_{|i| = 1} \left( \begin{array}{c} \nu \\ i \end{array} \right) \partial_\nu^\mu - i \partial_x^\mu \partial_z^i (f - \bar{f}) \leq C \sum_{|\mu'| + |\nu'| = k'} \| \partial_\nu^\mu \partial_x^\nu \partial_z^i (f - \bar{f})(t, z) \|_{L_{\infty, v}^a}.
$$

\*(Estimates for $I_{73}$): We apply Lemma 4.1 and Theorem 4.5 to obtain

$$
I_{73} := \sum_{0 \leq \nu' \leq l} \left( \begin{array}{c} \nu \\ \nu' + \mu' \end{array} \right) \left( \begin{array}{c} \mu \\ 0 \end{array} \right) \partial_\nu^\mu \partial_{\nu'}^\nu \partial_z^{i'} (F_a[f] \cdot \nabla_v (\partial_\nu^\mu \partial_x^\nu \partial_z^{i'} (f - \bar{f})))
\leq \sum_{0 \leq \nu' \leq l} \| \partial_\nu^\mu \partial_{\nu'}^\nu \partial_z^{i'} (f - \bar{f}) \|_{L_{\infty, v}^a}.
$$

\*(Estimates for $I_{74}$): We use Lemma 4.3 and Theorem 4.5 to see

$$
I_{74} := \sum_{0 \leq \nu' \leq l} \left( \begin{array}{c} \nu \\ \nu' + \mu' \end{array} \right) \left( \begin{array}{c} \mu \\ 0 \end{array} \right) \partial_\nu^\mu \partial_{\nu'}^\nu \partial_z^{i'} (F_a[f] - F_a[\bar{f}]) \cdot \nabla_v (\partial_\nu^\mu \partial_x^\nu \partial_z^{i'} (f - \bar{f}))
\leq C \sum_{0 \leq \nu' \leq l} \| \partial_\nu^\mu \partial_{\nu'}^\nu \partial_z^{i'} (f - \bar{f})(t, z) \|_{L_{\infty, v}^a}.
$$
\( \diamond \) (Estimates for \( I_{75} \)): Similarly, one has

\[
I_{75} := \sum_{0 \leq |\nu'| \leq l, 0 \leq |\mu'| \leq k' - 1} \left( \frac{l}{l!} \right) \partial_x^{l-|\nu'|} \partial_z^{|\nu'|} \partial_t^{|\mu'|} (\Lambda[S](f) - \Lambda[S][\tilde{f}])
\]

\[
\leq C(t, z) \left( \| \partial_x^{\mu' + |\nu'|} \partial_z^{|\nu'|} (S - \tilde{S})(t, x) \|_{L^\infty} + \| \partial_x^\nu \partial_z^{\mu'} (f - \tilde{f}) \|_{L^\infty} \right)
\]

\[
+ C(t, z) \sum_{0 \leq |\nu'| \leq l, 0 \leq |\mu'| \leq k' - 1} \| \partial_x^{\mu' + |\nu'|} \partial_z^{|\nu'|} (S - \tilde{S})(t, z) \|_{L^\infty} + \| \partial_x^\nu \partial_z^{\mu'} (f - \tilde{f})(t, z) \|_{L^\infty} .
\]

We collect these estimates and adopt the induction hypothesis on \( |\mu'| + |\nu'| \leq k' - 1 \) to derive

\[
\| \partial_x^\mu \partial_z^\nu (f - \tilde{f})(t, z) \|_{L^\infty} \leq \| \partial_x^\mu \partial_z^\nu (f_0 - \tilde{f}_0)(z) \|_{L^\infty}
\]

\[
+ C(t, z) \sum_{0 \leq |\nu'| \leq l, 0 \leq |\mu'| \leq k' - 1} \left( \| \partial_x^\mu \partial_z^\nu \partial_{\nu'}^l (f_0 - \tilde{f}_0)(z) \|_{L^\infty} + \| \partial_x^\nu \partial_z^{\mu'} S_{x,z}^l (S_0 - \tilde{S}_0)(z) \|_{L^\infty} \right)
\]

\[
+ \int_0^t C(s, z) \sum_{0 \leq |\nu'| \leq l, 0 \leq |\mu'| \leq k'} \left( \| \partial_x^\mu \partial_z^\nu \partial_{\nu'}^l (f - \tilde{f})(s, z) \|_{L^\infty} + \| \partial_x^\nu \partial_z^{\mu'} S_{x,z}^l (S - \tilde{S})(s, z) \|_{L^\infty} \right) ds.
\]

(50)

For the case \( \nu = 0 \), we derive

\[
\| \partial_x^\mu \partial_z^\nu (f - \tilde{f})(t, z) \|_{L^\infty} \leq \| \partial_x^\mu \partial_z^\nu (f_0 - \tilde{f}_0)(z) \|_{L^\infty}
\]

\[
+ C(t, z) \left( \sum_{0 \leq |\nu'| \leq l, 0 \leq |\mu'| \leq k' - 1} \| \partial_x^\mu \partial_z^\nu \partial_{\nu'}^l (f_0 - \tilde{f}_0)(z) \|_{L^\infty} + \sum_{0 \leq |\nu'| \leq l, 0 \leq |\mu'| \leq k' - 1} \| \partial_x^\nu \partial_z^{\mu'} S_{x,z}^l (S_0 - \tilde{S}_0)(z) \|_{L^\infty} \right)
\]

\[
+ \int_0^t C(s, z) \sum_{0 \leq |\nu'| \leq l, 0 \leq |\mu'| \leq k'} \left( \| \partial_x^\mu \partial_z^{\mu'} \partial_{\nu'}^l (f - \tilde{f})(s, z) \|_{L^\infty} + \| \partial_x^\nu \partial_z^{\mu'} S_{x,z}^l (S - \tilde{S})(s, z) \|_{L^\infty} \right) ds.
\]

(51)

We substitute (49) into (50) and (51) and apply Grönwall’s lemma to get

\[
\| \partial_x^\mu (f - \tilde{f})(t, z) \|_{W^{k', \infty}} + \| \partial_x^\mu (S - \tilde{S})(t, z) \|_{W^{k', \infty}}
\]

\[
\leq C(T, z) \left( \| \partial_x^\mu (f_0 - \tilde{f}_0)(z) \|_{W^{k', \infty}} + \| \partial_x^\mu (S_0 - \tilde{S}_0)(z) \|_{W^{k', \infty}} \right);
\]

for \( 1 \leq k' \leq k - l \). This yields

\[
\sum_{0 \leq |\mu'| \leq k' - l} \left( \| \partial_x^\mu (f - \tilde{f})(t, z) \|_{W^{k'-1, \infty}} + \| \partial_x^\mu (S - \tilde{S})(t, z) \|_{W^{k'-1, \infty}} \right)
\]

\[
\leq C(T, z) \sum_{0 \leq |\mu'| \leq k' - l} \left( \| \partial_x^\mu (f_0 - \tilde{f}_0)(z) \|_{W^{k'-1, \infty}} + \| \partial_x^\mu (S_0 - \tilde{S}_0)(z) \|_{W^{k'-1, \infty}} \right),
\]

(52)

by induction on \( k' \) and \( l \).

Finally, we return to (49) with \( |\mu'| = k + 1 \). Then the estimate (52) yields

\[
\sum_{0 \leq |\mu'| \leq k + 1} \| \partial_x^\mu (S - \tilde{S})(t, z) \|_{W^{k+1-1, \infty}}
\]

\[
\leq C(T, z) \sum_{0 \leq |\mu'| \leq k + 1} \left( \| \partial_x^\mu (f_0 - \tilde{f}_0)(z) \|_{W^{k-1, \infty}} + \| \partial_x^\mu (S_0 - \tilde{S}_0)(z) \|_{W^{k+1-1, \infty}} \right).
\]

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E-mail address: syha@snu.ac.kr
E-mail address: boramoon@snu.ac.kr