Equilibrium Points of a Singular Cooperative System with Free Boundary

J. Andersson, H. Shahgholian, N.N. Uraltseva and G.S. Weiss

Preprint 2015-08
Abstract. In this paper we initiate the study of maps minimising the energy
\[ \int_D (|\nabla u|^2 + 2u) \, dx, \]
which, due to Lipschitz character of the integrand, gives rise to the singular Euler equations
\[ \Delta u = \frac{u}{u^2 + v^2}, \quad u = (u_1, \ldots, u_m). \]

Our primary goal in this paper is to set up a road map for future developments of the theory related to such energy minimising maps.

Our results here concern regularity of the solution as well as that of the free boundary. They are achieved by using monotonicity formulas and epiperimetric inequalities, in combination with geometric analysis.

1. Introduction

1.1. Background. In this paper we shall study the singular system
\[ \Delta u = \frac{u}{u^2 + v^2}, \quad u = (u_1, \ldots, u_m), \]
where \( u : \mathbb{R}^n \supset D \to \mathbb{R}^m \), \( n \geq 2 \), \( m \geq 1 \), and \( |\cdot| \) is the Euclidean norm on the respective spaces. System (1) is a particular example of the equilibrium state of a cooperative system: the corresponding reaction-diffusion system
\[ u_t - \Delta u = -\frac{u}{\sqrt{u^2 + v^2}}, \quad v_t - \Delta v = -\frac{v}{\sqrt{u^2 + v^2}} \]
would mean that, considering the concentrations \( u \) and \( v \) of two species/reactants, each species/reactant slows down the extinction/reaction of the other species. The special choice of our reaction kinetics would assure a constant decay/reaction rate in the case that \( u \) and \( v \) are of comparable size.

System (1) may also be seen as one of the simplest extensions of the classical obstacle problem to the vector-valued case: Solutions of the classical obstacle problem are minimisers of the energy \( \int_D (\frac{1}{2} |\nabla u|^2 + \max(u, 0)) \, dx \), where \( u : \mathbb{R}^n \supset D \to \mathbb{R} \). Solutions of (1)
are minimisers of the energy

\begin{equation}
\int_D (|\nabla u|^2 + 2|u|) \, dx.
\end{equation}

It is noteworthy that in the scalar case, i.e. when $m = 1$, one recovers the two phase free boundary problem

$$\Delta u = \chi_{\{u>0\}} - \chi_{\{u<0\}},$$

contained in the analysis of [14]. While [14] as well as the two-phase result [6] relied essentially on the use of the monotonicity formula by Alt-Caffarelli-Friedman [4], a corresponding formula seems to be unavailable in our vector-valued problem.

There are several results concerning the obstacle problem for systems of various types: Optimal switching, multi-membranes, control of systems, constrained weakly elliptic systems, vector-valued obstacle problems, and probably many others. Although not directly relevant to our work, we refer to some papers that might be of interest for the readers [1], [2], [7], [8], [9], [10], [11].

1.2. Main Result and Plan of the paper. In this paper we are interested in qualitative behavior of the minimisers $u$ of the functional (2) as well as of the free boundary $\partial \{ x : |u(x)| > 0 \};$ here $u = (u_1, \ldots, u_m)$ and $m \geq 1$. Note that the part of the free boundary where the gradient $\nabla u \neq 0$ is, by the implicit function theorem, locally a $C^{1,\beta}$-surface, so that we are more concerned with the part where the gradient vanishes.

The main results of this paper (presented in Theorem 5) states that the set of "regular" free boundary points of the minimisers $u$ to the functional (2) is locally a $C^{1,\beta}$ surface.

In proving this result we need an array of technical tools including monotonicity formulas (Lemma 1 in Section 4), quadratic growth of solutions (Theorem 2), and an epiperimetric inequality (Theorem 1), for the balanced energy functional (3).

An epiperimetric inequality has been proved in [19] by one of the authors for the scalar obstacle problem. See also [12] for a related approach to the scalar obstacle problem with Dini continuous coefficients.

1.3. Notation. Throughout this paper $\mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^{nm}$ etc. will be equipped with the Euclidean inner product $\cdot \cdot$ and the induced norm $|x|$. $B_r(x^0)$ will denote the open $n$-dimensional ball of center $x^0$, radius $r$ and volume $r^n \omega_n$. $B_r^k(x^0):= \{ x \in B_r(x^0) : x_n = (x^0)_n \}, B_r^j(x^0):= \{ x \in B_r(x^0) : x_n > (x^0)_n \}$ and $e^i$ the $i$-th unit vector in $\mathbb{R}^n$. If the center $x^0$ is not specified, then it is assumed to be the origin. Given a set $A \subset \mathbb{R}^n$, we denote its interior by $A^\circ$ and its characteristic function by $\chi_A$. In the text we use the $n$-dimensional Lebesgue-measure $|A|$ of a set $A$ and the $k$-dimensional Hausdorff-measure $\mathcal{H}^k$. When considering the boundary of a given set, $\nu$ will typically denote the topological outward normal to the boundary and $\nabla_{\nu} f := \nabla f - \nabla f \cdot \nu \nu$ the surface derivative of a given function $f$. Finally, we shall often use abbreviations for inverse images like $\{ u > 0 \} := \{ x \in D : u(x) > 0 \}, \{ x_n > 0 \} := \{ x \in \mathbb{R}^n : x_n > 0 \}$ etc. and occasionally we employ the decomposition $x = (x', x_n)$ of a vector $x \in \mathbb{R}^n$. Last, let $\Gamma(u) := D \cap \partial \{ x \in D : |u(x)| > 0 \}$ and $\Gamma_0(u) := \Gamma(u) \cap \{ x : \nabla u(x) = 0 \}$.

2. The Epiperimetric Inequality

Following [19], we prove in this section an epiperimetric inequality, which tells us that close to half-plane solutions, the minimal energy achieved is lower than that of 2-homogeneous functions, and the energy difference can be estimated. This will imply in later sections a certain non-degeneracy of the energy close to half-plane solutions, and ultimately lead to regularity of the free boundary. Since the epiperimetric inequality is rather
an abstract property of the energy, and represents the core of our result, we put this section at the beginning. Although the proof follows partly the proof in [19], the PDE resulting from the “linearization” carried out in the proof is different from that in [19] and introduces new difficulties.

Let

\[ M(v) := \int_{B_1} (|\nabla v|^2 + 2|v|) - 2 \int_{\partial B_1} |v|^2 \, d\mathcal{H}^{n-1}, \]

and let

\[ \mathbb{H} := \{ \frac{\max(x \cdot v, 0)^2}{2} \rightarrow e : v \text{ is a unit vector in } \mathbb{R}^n \text{ and } e \text{ is a unit vector in } \mathbb{R}^m \}. \]

We define

\[ \frac{\alpha_n}{2} := M \left( \frac{\max(x \cdot v, 0)^2}{2} e \right). \]

**Theorem 1.** There exists \( \kappa \in (0, 1) \) and \( \delta > 0 \) such that if \( c \) is a homogeneous function of degree 2 satisfying \( \|c - h\|_{W^{1,2}(B_1; \mathbb{R}^m)} + \|c - h\|_{L^\infty(B_1; \mathbb{R}^m)} \leq \delta \) for some \( h \in \mathbb{H} \), then there is a \( v \in W^{1,2}(B_1; \mathbb{R}^m) \) such that \( v = c \) on \( \partial B_1 \) and

\[ M(v) \leq (1 - \kappa)M(c) + \kappa \frac{\alpha_n}{2}. \]

**Remark:** Note that the closeness in \( L^\infty \) is not really necessary and is assumed only in order to avoid capacity arguments in the proof.

**Proof of the Theorem.** Suppose towards a contradiction that there are sequences \( \kappa_k \rightarrow 0, \delta_k \rightarrow 0, c_k \in W^{1,2}(B_1; \mathbb{R}^m), \) and \( h_k \in \mathbb{H} \) such that \( c_k \) is a homogeneous function of degree 2 and satisfies

\[ \|c_k - h_k\|_{W^{1,2}(B_1; \mathbb{R}^m)} = \delta_k, \quad \|c_k - h_k\|_{L^\infty(B_1; \mathbb{R}^m)} \rightarrow_k \rightarrow 0 \]

and that

\[ M(v) > (1 - \kappa_k)M(c_k) + \kappa_k \frac{\alpha_n}{2} \quad \text{for all } v \in c_k + W^{1,2}_0(B_1; \mathbb{R}^m). \]

Rotating in \( \mathbb{R}^n \) and in \( \mathbb{R}^m \) if necessary we may assume that

\[ h_k(x) = \frac{\max(x_n, 0)^2}{2} e^l := h \]

where \( e^l = (1, 0, \ldots, 0) \in \mathbb{R}^n \). Subtracting from (7) \( M(h) = \frac{\alpha_n}{2} \), we obtain

\[ (1 - \kappa_k)(M(c_k) - M(h)) < M(v) - M(h) \quad \text{for every } v \in W^{1,2}(B_1; \mathbb{R}^m) \text{ such that } v = c_k \text{ on } \partial B_1. \]

Observe now that for each \( \phi \in W^{1,2}(B_1) \) and \( h := \frac{\max(x_n, 0)^2}{2} \)

\[ 2 \int_{B_1} (\nabla h \cdot \nabla \phi + \chi_{\{x_n > 0\}} \phi) - 2 \int_{\partial B_1} 2 h \phi \, d\mathcal{H}^{n-1} = 2 \int_{\partial B_1} (\nabla h \cdot v - 2h) \phi \, d\mathcal{H}^{n-1} = 0, \]

and therefore

\[ I := 2 \int_{B_1} (\nabla h \cdot \nabla (c_k - h) + \chi_{\{x_n > 0\}} e^l \cdot (c_k - h)) - 2 \int_{\partial B_1} 2 h \cdot (c_k - h) \, d\mathcal{H}^{n-1} = 0. \]
Subtracting $(1 - \kappa_k)I$ from the left-hand side of (8) and subtracting $I$ with $c_k$ replaced by $v$ from the right-hand side of (8), we obtain thus

\[
(1 - \kappa_k) \left[ \int_{B_1} (|\nabla c_k|^2 + 2|c_k|) - 2 \int_{\partial B_1} |c_k|^2 \, dH^{n-1} \right] - \int_{B_1} (\nabla h \cdot \nabla (c_k - h) + \chi(x, n \geq 0) e^1 \cdot (c_k - h)) + 2 \int_{\partial B_1} h \cdot (c_k - h) \, dH^{n-1}
\]

\[
< \int_{B_1} (|\nabla v|^2 + 2|v| - 2 \int_{\partial B_1} |v|^2 \, dH^{n-1}) - \int_{B_1} (|\nabla h|^2 + 2|h|) + 2 \int_{\partial B_1} h \, dH^{n-1}
\]

\[
- 2 \int_{B_1} \nabla h \cdot \nabla (v - h) + \chi(x, n \geq 0) e^1 \cdot (v - h)) + 2 \int_{\partial B_1} h \cdot (v - h) \, dH^{n-1}.
\]

Rearranging terms yields

\[
(1 - \kappa_k) \int_{B_1} (|\nabla c_k|^2 - 2 \int_{\partial B_1} |c_k| + 2 \int_{\partial B_1} |c_k| - e^1 \cdot c_k)
\]

\[
< \int_{B_1} (|\nabla (v - h)|^2 - 2 \int_{\partial B_1} |v - h|^2 \, dH^{n-1}) + 2 \int_{\partial B_1} (|v - e^1 \cdot v|)
\]

Definition now the sequence of functions $w_k := (c_k - h)/\delta_k$. Then $\|w_k\|_{W^{1,2}(B_1; \mathbb{R}^m)} = 1$ and, passing to a subsequence if necessary, $w_k \to w$ weakly in $W^{1,2}(B_1; \mathbb{R}^m)$. In order to obtain a contradiction, we are going to prove that $w_k \to w$ strongly in $W^{1,2}(B_1; \mathbb{R}^m)$ and that $w \equiv 0$ in $B_1(0)$.

**Step 1:** $w \equiv 0$ in $B_1$, and $\int_{B_1} (|c_k| - e^1 \cdot c_k) \leq C \delta_k^2$

Plug in $v := (1 - \zeta)c_k + \zeta h$ in (9), where $\zeta \in W^{1,2}_0(B_1)$ is radial symmetric and satisfies $0 \leq \zeta \leq 1$. Since $(v - h)/\delta_k = (1 - \zeta)w_k$, we obtain

\[
2 (1 - \kappa_k) \left[ \int_{B_1^*} \frac{|c_k|}{\delta_k^2} + \int_{B_1^*} \frac{|c_k| - e^1 \cdot c_k}{\delta_k^2} \right]
\]

\[
< C_1 + 2 \int_{B_1^*} (1 - \zeta) \frac{|c_k|}{\delta_k^2} + 2 \int_{B_1^*} (1 - \zeta) \frac{|c_k| - e^1 \cdot c_k}{\delta_k^2}
\]

and

\[
\int_{B_1^*} (\zeta - \kappa_k) \frac{|c_k|}{\delta_k^2} + \int_{B_1^*} (\zeta - \kappa_k) \frac{|c_k| - e^1 \cdot c_k}{\delta_k^2} \leq C_1.
\]

Using the homogeneity of $c_k$ we see that for large $k$,

\[
\int_{B_1^*} (\zeta - \kappa_k) |c_k| = \int_0^1 (\zeta(p) - \kappa_k)p^{n+1} \, dp \int_{|x, n| < \rho B_1} |c_k| \, dH^{n-1} \geq c_0 \int_{|x, n| < \rho B_1} |c_k| \, dH^{n-1},
\]

where $c_0 > 0$ depends only on $\zeta$ and $n$. We also get the corresponding estimate in $B_1^*$. It follows that

\[
\int_{B_1^*} |c_k| \leq C_2 \delta_k^2 \text{ and that } \int_{B_1^*} (|c_k| - e^1 \cdot c_k) \leq C_2 \delta_k^2.
\]

In particular,

\[
\int_{B_1^*} |w_k| \leq C_2 \delta_k,
\]

implying the statement of Step 1.

**Step 2:** $\Delta(e^1 \cdot w) = 0$ in $B_1^*(0)$, $e^1 \cdot w = d_j h$ in $B_1^*(0)$ for each $j > 1$, and some constant $d_j$. 

Fix a ball $B \subset B_1^+$ and plug $v := (1 - \zeta)c_k + \zeta(h + \delta_k g)$ into (9), where $\zeta \in C^0_0(B_1^+)$ and $g \in W^{1,2}(B_1; \mathbb{R}^m)$ such that $\zeta \equiv 1$ in $B$, $\zeta \equiv 0$ in $B_1^-$ and $g$ is a bounded $W^{1,2}(B_1; \mathbb{R}^m)$-function. Observing that

$$\frac{v - h}{\delta_k} = (1 - \zeta)\frac{c_k - h}{\delta_k} + \zeta g,$$

we obtain — using (10) as well as the fact that $\text{supp } \zeta \subset B_1^+$ — that

$$\int_{B_1^+} (2\zeta^2 - \zeta^2)\|\nabla w_k\|^2 + \frac{2}{\delta_k^2} \int_{B_1^+} \zeta|c_k - e^1 \cdot c_k|^2$$

$$\leq o(1) + \int_{B_1^+} \zeta^2\|\nabla g\|^2 + \frac{2}{\delta_k^2} \int_{B_1^+ \setminus B} \left(\nabla \zeta^2\|g - w_k\|^2 + (2\zeta - 2\zeta^2)\nabla w_k \cdot \nabla g\right)$$

$$+ 2 \int_{B_1^+} ((1 - \zeta)\nabla \zeta \cdot \nabla w_k (g - w_k) + \zeta \nabla \zeta \cdot \nabla g (g - w_k))$$

$$+ \frac{2}{\delta_k^2} \int_{B_1^+} \zeta \left(h + \delta_k g - e^1 \cdot (h + \delta_k g)\right).$$

Note that $\delta_k w_k \to 0$ uniformly in $B_1$. Therefore we have on $\text{supp } \zeta$

$$|c_k - e^1 \cdot c_k| = (h + \delta_k e^1 \cdot w_k) \left(\sqrt{1 + \frac{\|w_k\|^2 - (e^1 \cdot w_k)^2}{(h + \delta_k e^1 \cdot w_k)^2}} - 1\right) = o(\delta_k^2) + \frac{\delta_k^2 |w_k|^2 - (e^1 \cdot w_k)^2}{2h + \delta_k e^1 \cdot g}$$

and similarly

$$|h + \delta_k g| - e^1 \cdot (h + \delta_k g) = o(\delta_k^2) + \frac{\delta_k^2 |g|^2 - (e^1 \cdot g)^2}{2h + \delta_k e^1 \cdot g}.$$

Letting $k \to \infty$ we may then drop the assumption that $g$ is bounded. In particular, for $g$ such that $g = w$ in $B_1 \setminus B$, we arrive at the inequality

$$\int_B |\nabla w|^2 + \int_B \frac{|w|^2 - (e^1 \cdot w)^2}{h} \leq \int_B |\nabla g|^2 + \int_B \frac{|g|^2 - (e^1 \cdot g)^2}{h}$$

for all $g \in W^{1,2}(B_1; \mathbb{R}^m)$ coinciding with $w$ on $\partial B$.

Calculation of the first variation yields that

$$\Delta(e^1 \cdot w) = 0 \quad \text{in } B,$$

$$\Delta(e^j \cdot w) = \frac{e^j \cdot w}{h} \quad \text{in } B \quad \text{for } j > 1.$$

By Lemma 4 as well as the homogeneity of $w$ and the fact that $w \equiv 0$ in $B_1^-$ we obtain

$$e^j \cdot w(x) = d_j h(x) \quad \text{for each } j > 1,$$

where $d_j$ is a constant real number.

**Step 3:** $w := e^1 \cdot w = 0$ in $B_1^+$.

As $w$ is harmonic in $B_1^+$, homogeneous of degree 2 and satisfies $w = 0$ in $B_1^-$ we obtain (using for example odd reflection and the Liouville theorem) that $w(x) = \sum_{j=1}^{n-1} d_j x_j x_n$ in $B_1^+$. Remember that we have chosen $h$ as the minimiser of $\inf_{h \in H} \|c_k - h\|_{W^{1,2}(B_1; \mathbb{R}^m)}$. It follows that for $h_v := e^1 \max(x \cdot \nu, 0)^2/2$,

$$\frac{(w_k, h_v - h)_{W^{1,2}(B_1; \mathbb{R}^m)}}{|\nu - e^1|} \leq \frac{1}{2\delta_k} \frac{|h_v - h|_{W^{1,2}(B_1; \mathbb{R}^m)}^2}{|\nu - e^1|} \to 0 \quad \text{as } \nu \to e^1.$$
Therefore
\[
\alpha(1) \geq \int_{B_1} w_k \cdot e^1 \left[ \chi_{|x_n| > 0} \left( \frac{(x \cdot \nu)^2 - (x \cdot e^0)^2}{|\nu - e^0|} \right) - \chi_{|x_n| < 0} \left( \frac{(x \cdot e^0)^2}{|\nu - e^0|} \right) + \chi_{|x_n| \leq 0} \left( \frac{2(x \cdot \nu)(\nu - e^0)}{|\nu - e^0|} \right) \right] + \int_{B_1} x \cdot (\nu + e^0)(\nu - e^0) + x \cdot (\nu - e^0)(\nu + e^0) \frac{2x \cdot \nu}{|\nu - e^0|} \cdot \nabla w_k \cdot e^1.
\]

Setting \( \xi := \lim_{\nu \to e^0} \frac{x - e^0}{|\nu - e^0|} \), we see that for \( \nu \to e^0 \)
\[
\frac{(x \cdot \nu)^2 - (x \cdot e^0)^2}{|\nu - e^0|} \to 2x_n \cdot \xi, \quad \frac{x \cdot (\nu + e^0)(\nu - e^0)}{|\nu - e^0|} \to 2x_n \xi, \quad \frac{2x \cdot \nu}{|\nu - e^0|} \to 2x \cdot \xi e^0.
\]

On the other hand, on the set \( (\{ x_n > 0 \} \cap \{ x \cdot \nu < 0 \}) \cup (\{ x_n < 0 \} \cap \{ x \cdot \nu > 0 \}) \) and \( |x \cdot e^0| = O(|\nu - e^0|) \) we conclude that
\[
\alpha(1) \geq 2 \int_{B_1} [w_k \cdot e^1 \max(x_n, 0) + (\max(x_n, 0) \xi + \chi_{|x_n| > 0} x \cdot \xi e^0) \cdot \nabla w_k \cdot e^1].
\]

Passing next to the limit \( k \to \infty \), and taking into account that \( \xi_n = 0 \) and that
\[
\nabla w = \left( \sum_{j=1}^{n-1} a_{nj} x_j \right),
\]
we obtain that
\[
0 \geq \sum_{j=1}^{n-1} a_{nj} \int_{B_1} [\max(x_n, 0)^2 x \cdot \xi x_j + \max(x_n, 0)^2 \xi_j + \chi_{|x_n| > 0} x \cdot \xi x_j].
\]

Since also
\[
\int_{B_1} x_j x_i = 0 \quad \text{for } i \neq j,
\]
we deduce from (11) that
\[
0 \geq \sum_{j=1}^{n-1} \int_{B_1} (x_n^2 x_j^2 + x_n^2 + x_j^2) \quad \text{for every } \xi = (\xi_1, \ldots, \xi_{n-1}, 0).
\]
Thus \( a_{nj} = 0 \) for \( j = 1, \ldots, n - 1 \), that is \( w \equiv 0 \) in \( B_1^+ \).

\textbf{Step 4:} \( d_j = 0 \) for each \( j \geq 2 \).

From Step 2-3 we know that \( w_k = d h + z_k \), where \( d \cdot e^1 = 0 \) and \( z_k \to 0 \) weakly in \( W^{1,2}(B_1; \mathbb{R}^n) \) as \( k \to \infty \). It follows that \( c_k = h(e^1 + \delta_k d) + \delta_k z_k \). By assumption,
\[
1 = ||d h + z_k||^2_{W^{1,2}(B_1; \mathbb{R}^n)} = ||d||^2_{W^{1,2}(B_1; \mathbb{R}^n)} + 2 \langle d h, z_k \rangle_{W^{1,2}(B_1; \mathbb{R}^n)} + ||z_k||^2_{W^{1,2}(B_1; \mathbb{R}^n)}.
\]
Remember that we have chosen $h$ as the minimiser of $\inf_{f \in \mathbb{H}} \|c_k - f\|_{W^{1,2}(B_1; \mathbb{R}^m)}$. It follows that for $f := h(1 + \delta_k d) / \sqrt{1 + \delta_k^2 |d|^2} \in \mathbb{H}$,

$$\delta_k = \|c_k - h(1 + \delta_k d)\|_{W^{1,2}(B_1; \mathbb{R}^m)} \leq \|c_k - f\|_{W^{1,2}(B_1; \mathbb{R}^m)} = \left\| h(1 + \delta_k d) + \delta_k z_k - \frac{h(1 + \delta_k d)}{\sqrt{1 + \delta_k^2 |d|^2}} \right\|_{W^{1,2}(B_1; \mathbb{R}^m)}$$

$$= \left\| \delta_k z_k + \frac{h(1 + \delta_k d)}{\sqrt{1 + \delta_k^2 |d|^2}} (\sqrt{1 + \delta_k^2 |d|^2} - 1) \right\|_{W^{1,2}(B_1; \mathbb{R}^m)} \leq \delta_k \|z_k\|_{W^{1,2}(B_1; \mathbb{R}^m)} + C_3 \delta_k^2 |d|^2. \tag{14}$$

Hence, $1 \leq \|z_k\|_{W^{1,2}(B_1; \mathbb{R}^m)} + C_3 \delta_k^2 |d|^2$.

Combining (13) and (14), we obtain that

$$|d|^2 \|h\|_{W^{1,2}(B_1; \mathbb{R}^m)}^2 + 2(d \cdot z_k)_{W^{1,2}(B_1; \mathbb{R}^m)} + \|z_k\|_{W^{1,2}(B_1; \mathbb{R}^m)}^2 \leq \|z_k\|_{W^{1,2}(B_1; \mathbb{R}^m)}^2 + O(\delta_k).$$

Letting $k \to \infty$, we conclude that $|d|^2 \|h\|_{W^{1,2}(B_1; \mathbb{R}^m)}^2 = 0$ and that $|d| = 0$.

**Step 5: $w_k \to w$ strongly in $W^{1,2}(B_1; \mathbb{R}^m)$.**

Plug in $v := (1 - \zeta) c_k + \zeta h$ in (9), where $\zeta(x) = \min(2 \max(1 - |x|, 0), 1)$. Then

$$v - h = (1 - \zeta) w_k,$$

and we obtain that

$$\left(1 - \kappa_h\right) \int_{B_1} |\nabla w_k|^2 - 2 \int_{\partial B_1} |w_k|^2 d\mathcal{H}^{n-1} + 2 \int_{B_1} \left| c_k - f_1 \cdot c_k \right|^2 + 2 \int_{\partial B_1} \frac{|c_k - f_1 \cdot c_k|^2}{\delta_k^2}$$

$$\leq \int_{B_1} |\nabla((1 - \zeta) w_k)|^2 - 2 \int_{\partial B_1} |(1 - \zeta) w_k|^2 d\mathcal{H}^{n-1} + 2 \int_{\partial B_1} \frac{(1 - \zeta) |c_k|}{\delta_k^2}$$

$$+ 2 \int_{\partial B_1} \frac{|(1 - \zeta) c_k + \zeta h|}{\delta_k^2} - 2 \int_{\partial B_1} \frac{(1 - \zeta) c_k \cdot f_1 + \zeta h}{\delta_k^2}.$$

Using the definition of $\zeta$, it follows that

$$\int_{B_1/2} |\nabla w_k|^2 \leq C_4 \kappa_h + \int_{B_1} (|\nabla \zeta|^2 |w_k|^2 - 2(1 - \zeta) \nabla \zeta \cdot \nabla w_k w_k).$$

The integral on the left-hand side equals by homogeneity of $w_k$

$$2^{-n-2} \int_{B_1} |\nabla w_k|^2,$$

so that

$$\int_{B_1} |\nabla w_k|^2 \leq 2^{n+2} \left( C_4 \kappa_h + \int_{B_1} (|\nabla \zeta|^2 |w_k|^2 - 2(1 - \zeta) \nabla \zeta \cdot \nabla w_k w_k) \right) \to 0 \text{ as } k \to \infty.$$

Altogether we obtain a contradiction from $w \equiv 0$, the strong convergence of $w_k$ as well as the fact that $\|w_k\|_{W^{1,2}(B_1; \mathbb{R}^m)} = 1$. \qed
3. Introduction to the problem and technical tools

Let $D$ be a bounded Lipschitz domain in $\mathbb{R}^n$ and let $u = (u_1, \ldots, u_m)$ be a minimiser of

$$E(u) := \int_D (|\nabla u|^2 + 2|u|)$$

in the affine subspace $\{ v \in W^{1,2}(D; \mathbb{R}^m) : v = u_D \text{ on } \partial D \}$. Note that non-negativity, convexity and lower semicontinuity with respect to weak convergence imply existence of a minimiser for each $u_D \in W^{1,2}(D; \mathbb{R}^m)$.

In order to compute the first variation of the energy, we compute for $\phi \in W^{1,2}_0(D; \mathbb{R}^m)$

$$0 \leq \epsilon \int_D 2\nabla u \cdot \nabla \phi + \epsilon^2 \int_D |\nabla \phi|^2 + 2 \int_D (|u + \epsilon \phi| - |u|)$$

$$\leq \epsilon \int_D 2\nabla u \cdot \nabla \phi + \epsilon^2 \int_D |\nabla \phi|^2 + 2|\epsilon| \int_D |\phi|.$$  

Dividing by $\epsilon$ and letting $\epsilon \to 0$, it follows that

$$\left| \int_D \nabla u \cdot \nabla \phi \right| \leq ||\phi||_{L^2(D; \mathbb{R}^m)},$$

so that $\Delta u \in L^\infty(D; \mathbb{R}^m)$. Applying standard $L^p$- and $C^\alpha$-theory, we obtain that $u \in W^{2,p}_\text{loc}(D; \mathbb{R}^m) \cap C^1_{\text{loc}}(D; \mathbb{R}^m)$ for each $p \in [1, +\infty)$ and each $\alpha \in (0, 1)$. We see that $\Delta u = 0$ a.e. in $\{ u > \delta > 0 \}$. Moreover, in the open set $\{ |u| > \delta > 0 \}$, passing to the limit in (15) yields

$$\Delta u = \frac{u}{|u|} \chi_{\{ |u| > \delta > 0 \}}$$

Altogether we obtain that $u$ is a strong solution of the equation

$$\Delta u = \frac{u}{|u|} \chi_{\{ |u| = 0 \}}$$

in $D$.

Note that any other solution $v \in W^{1,2}(D; \mathbb{R}^m)$ with the same boundary data $u_D$ and satisfying the weak equation

$$\int_D (\nabla v \cdot \nabla \phi + \phi v - v \chi_{\{ |v| = 0 \}}) = 0 \text{ for every } \phi \in W^{1,2}_0(D; \mathbb{R}^m)$$

must coincide with $u$: subtracting the equation for $u$ and plugging in $\phi := v - u$ yields

$$\int_D |\nabla (u - v)|^2 \leq -\int_D \left( \frac{u}{|u|} \chi_{\{ |u| > 0 \}} - \frac{v}{|v|} \chi_{\{ |v| > 0 \}} \right) \cdot (u - v) \leq 0.$$  

Thus the weak solution is unique and equals the minimiser of the problem, so that it is sufficient to consider minimisers.

Note that in contrast to the classical (scalar) obstacle problem, it is an open problem whether $u \in W^{2,\infty}_{\text{loc}}(D; \mathbb{R}^m)$.

**Remark 1.** Using standard elliptic theory combined with the estimate $|\Delta u| \leq 1$ we obtain that

$$\sup_{B_{1/4}} |u| + \sup_{B_{1/4}} |\nabla u| \leq C_1(n, m) \left( ||u||_{L^2(B_{1/3}; \mathbb{R}^m)} + 1 \right).$$

**Remark 2.** If a sequence of solutions of our system $u_k$ converges weakly in $W^{1,2}(D; \mathbb{R}^m)$ to $u$, then Rellich’s theorem together with the fact that $D^2 u = 0$ a.e. in $\{ u = 0 \}$, implies that $u$ is a solution, too.
Proposition 1 (Non-Degeneracy). Let $u$ be a solution of (1) in $D$. If $x^0 \in |\{u| > 0\}$ and $B_r(x^0) \subset D$, then

$$\sup_{B_r(x^0)} |u| \geq \frac{1}{2n} r^2.$$ \hspace{1cm} (17)

Proof. It is sufficient to prove a uniform estimate for $x^0 \in |\{u| > 0\}$. Let $U(x) := |u(x)|$. Then

$$\Delta U = 1 + \frac{A}{U}$$ \hspace{1cm} in $|\{u| > 0\}$, where $A = |\nabla u|^2 - |\nabla U|^2 \geq 0.$

Assuming $\sup_{B_r(x^0)} |u| \leq \frac{1}{2n} r^2$, we obtain that the function

$$v(x) := U(x) - U(x^0) - \frac{1}{2n} |x - x^0|^2$$

is subharmonic in the connected component of $B_r(x^0) \cap |\{u| > 0\}$ containing $x^0$, that $v < 0$ on the boundary of that component and that $v(x^0) = 0$, contradiction. \hfill $\square$

Proposition 2. Let $u$ be a solution of (1) in $B_1(0)$ such that $||u - h||_{L^1(B_1(\mathbb{R}^n))} \leq \epsilon < 1$, where $h := \frac{\max_{B_1(0)} u}{2} e^1$. Then

$$B_{1/2}(0) \cap \text{supp } u \subset \{ x_n > -C\epsilon \frac{1}{\sqrt{m}} \}$$

with a constant $C = C(n, m)$.

Proof. Suppose that $B_{1/2} \cap |\{u| > 0\} \ni x^0$ and that $x^0_n = -\rho < 0$. It follows that

$$||u||_{L^1(B_{1/2}(\mathbb{R}^n))} \leq ||u - h||_{L^1(B_1(\mathbb{R}^n))} \leq \epsilon.$$

By the non-degeneracy property Proposition 1 we know that

$$|u(y)| = \sup_{B_{1/2}(x^0)} |u| \geq \frac{1}{8n} \rho^2$$

for some $y \in B_{1/2}(x^0)$. From Remark 1 we infer that

$$\inf_{B_{1/2}(y)} |u| \geq \frac{1}{8n} \rho^2 - 2C_1(n, m)\sigma \rho^2 \geq \frac{1}{16n} \rho^2,$$

provided that $\sigma$ has been chosen small enough, depending only on $n$ and $m$. Combining our estimates, we obtain that

$$\epsilon \geq ||u||_{L^1(B_{1/2}(y), \mathbb{R}^n)} \geq \left( \frac{1}{16n} \rho^2 \right) |B_1| \left( \sigma \rho^2 \right)^n,$$

a contradiction, if $\epsilon < C_2(n, m)\rho^{2n+2}$. It follows that $|u(x)| = 0$ for $x_n \leq -C(n, m) \epsilon \frac{1}{\sqrt{m}}$. \hfill $\square$

4. Monotonicity Formula and Consequences

Lemma 1. Let $u$ be a solution of (1) in $B_{r_0}(x^0)$ and let

$$W(u, x^0, r) = \frac{1}{r^{n+2}} \int_{B_r(x^0)} (|\nabla u|^2 + 2|u|) - \frac{2}{r^{n+3}} \int_{\partial B_r(x^0)} |u|^2 \, d\mathcal{H}^{n-1}.$$

For $0 < r < r_0$,

$$\frac{dW(u, x^0, r)}{dr} = 2 \int_{\partial B_r(0)} r \left| \frac{d}{dr} u \right|^2 \, d\mathcal{H}^{n-1},$$

where $u_r(x) = u(rx + x^0)/r^2$. 
Lemma 2. 1. The function \( r \mapsto W(u, x^0, r) \) has a right limit \( W(u, x^0, 0^+) \) in \([-\infty, +\infty) \) and in the case \( D = \mathbb{R}^n \) it has also a limit \( W(u, x^0, +\infty) \in (-\infty, +\infty] \).

2. Let \( 0 < r_k \to 0 \) be a sequence such that the blow-up sequence

\[
 u_k(x) := \frac{u(x^0 + r_k x)}{r_k^2}
\]

converges weakly in \( W^{1,2}_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^m) \) to \( u_0 \). Then \( u_0 \) is a homogeneous function of degree 2. Moreover

\[
 W(u, x^0, 0^+) = \int_{B_1(0)} |u_0| \geq 0,
\]

and \( W(u, x^0, 0^+) = 0 \) implies that \( u \equiv 0 \) in \( B_\delta(x^0) \) for some \( \delta > 0 \).

3. The function \( x \mapsto W(u, x, 0^+) \) is upper-semicontinuous.

Proof. 1. follows directly from the monotonicity formula.

2. By the assumption of convergence \((u_k)_{k \in \mathbb{N}}\) is bounded in \( W^{1,2}_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^m) \) and the limit \( W(u, x^0, 0^+) \) is finite. From the monotonicity formula we obtain for all \( 0 < \rho < \sigma < +\infty \)

\[
 \int_{\rho}^{\sigma} \frac{1}{r^{n+1}} \int_{B_r(0)} |x \cdot \nabla u_k(x) - 2u_k(x)|^2 \, d\mathcal{H}^{n-1} \, dr \to 0, \quad k \to \infty,
\]

proving the homogeneity of \( u_0 \).
We calculate, using the homogeneity of \( u_0 \) as well as Remark 2,

\[
W(u, x^0, 0+) = \int_{B_1(0)} (|\nabla u_0|^2 + 2|u_0|) - 2 \int_{\partial B_1(0)} |u_0|^2 \, dH^{n-1}
= \int_{B_1(0)} (-u_0 \cdot \Delta u_0 + 2u_0) + \int_{\partial B_1(0)} (x \cdot \nabla u_0 \cdot u_0 - 2|u_0|^2) \, dH^{n-1} = \int_{B_1(0)} |u_0| \geq 0.
\]

In the case \( W(u, x^0, 0+) = 0 \) we obtain a contradiction to the non-degeneracy Lemma 1 unless \( u \equiv 0 \) in some ball \( B_\delta(x^0) \).

3. For \( \varepsilon > 0, M < +\infty \) and \( x \in D \) we obtain from the monotonicity formula that

\[
W(u, x, 0+) \leq W(u, x, \rho) \leq \varepsilon + W(u, x^0, \rho) \leq \left\{ \begin{array}{ll}
\varepsilon + W(u, x^0, 0+), & W(u, x^0, 0+) > -\infty, \\
-M, & W(u, x^0, 0+) = -\infty,
\end{array} \right.
\]

if we choose first \( \rho \) and then \( |x - x^0| \) small enough. \( \Box \)

5. A QUADRATIC GROWTH ESTIMATE

**Theorem 2.** Any solution \( u \) to the system (1) in \( B_1(0) \) satisfies

\[
|u(x)| \leq C \text{dist}^2(x, \Gamma_0(u)) \text{ and } |\nabla u(x)| \leq C \text{dist}(x, \Gamma_0(u)) \text{ for every } x \in B_{1/2}(0),
\]

where the \text{dist} denotes the Euclidean distance in \( \mathbb{R}^n \) and constant \( C \) depends only on \( n \) and

\[
E(u, 0, 1) := \int_{B_1(0)} (|\nabla u|^2 + 2|u|).
\]

**Proof.** The statement of the theorem is equivalent to

\[
\sup_{B_r(x^0)} |u| \leq C_1 r^2 \text{ and } \sup_{B_r(x^0)} |\nabla u| \leq C_1 r \text{ for every } x^0 \in \Gamma_0(u) \cap B_{1/2}(0) \text{ and every } r \in (0, 1/4),
\]

which in turn can be readily derived by standard elliptic theory from

\[
(19) \quad \frac{1}{r^n} \int_{B_r(x^0)} |u| \leq C_2 r^2 \text{ for all } x^0 \text{ and } r \text{ as above.}
\]

Thus, our goal here is to show that (19) holds. To that end, notice first that by the monotonicity formula,

\[
W(u, x^0, r) \leq W(u, x^0, 1/2) \leq 2^{n+2} E(u, 0, 1) \text{ for every } x^0 \in B_{1/2}(0) \cap \Gamma_0(u) \text{ and } r \leq 1/2.
\]

Therefore

\[
\frac{2}{r^{n+2}} \int_{B_r(x^0)} |u| = W(u, x^0, r) - \frac{1}{r^{n+2}} \int_{B_r(x^0)} |\nabla u|^2 + \frac{2}{r^{n+3}} \int_{\partial B_r(x^0)} |u|^2 \, dH^{n-1}
= W(u, x^0, r) - \frac{1}{r^{n+2}} \int_{B_r(x^0)} |\nabla (u - S_x v p)|^2 + \frac{2}{r^{n+3}} \int_{\partial B_r(x^0)} |u - S_x v p|^2 \, dH^{n-1}
\leq 2^{n+2} E(u, 0, 1) + \frac{2}{r^{n+3}} \int_{\partial B_r(x^0)} |u - S_x v p|^2 \, dH^{n-1}
\]

for each \( p = (p_1, \ldots, p_m) \in \mathcal{H} \); here the set \( \mathcal{H} \) is the set of all \( p = (p_1, \ldots, p_m) \) such that each component \( p_j \) is a homogeneous harmonic polynomial of second order, \( S_x v f(x) := f(x - x^0) \).

Let \( x^0 \in \Gamma_0 \) and \( p_{v_0} \) be the minimiser of \( \int_{\partial B_r(x^0)} |u - S_x v p|^2 \, dH^{n-1} \) in \( \mathcal{H} \). It follows that

\[
0 = \int_{\partial B_r(x^0)} (u - S_x v p_{v_0}) \cdot S_x q \, dH^{n-1} \text{ for every } q \in \mathcal{H}.
\]
We maintain that there is a constant $C_1$ depending only on the dimension $n$ as well as $E(u, 0, 1)$ such that for each $x^0 \in B_{1/2}(0) \cap \Gamma_0$ and $r \leq 1/4$,
\[ \frac{1}{r^{n+3}} \int_{\partial B_r(x^0)} |u - S_n \cdot p_{u, n, r}|^2 \, d\mathcal{H}^{n-1} \leq C_1. \]

Suppose towards a contradiction that there is a sequence of solutions $u_k$ (to equation (1) in $B_t(0)$) and a sequence of points $x^k \in B_{1/2}(0) \cap \Gamma_0(u_k)$ as well as $r_k \to 0$ such that $I(u_k, 0, 1)$ are uniformly bounded,
\[ M_k := \frac{1}{r_k^{n+3}} \int_{\partial B_{r_k}(x^k)} |u_k - S_n \cdot p_{u, n, r_k}|^2 \, d\mathcal{H}^{n-1} \to \infty. \]

For $v_k(x) := u_k(x^k + r_k \cdot x) / \sqrt{r_k}$, and $w_k(x) := (v_k - p_{v, n, r_k}) / M_k$, we have $|w_k|_{L^2(\partial B_{r_k}(0) ; \mathbb{R}^n)} = 1$ and
\[
\int_{B_{r_k}(0)} |\nabla w_k|^2 = 2 \int_{\partial B_{r_k}(0)} |w_k|^2 = M_k^{-2} \left( \int_{B_{r_k}(0)} |\nabla(v_k - p_{v, n, r_k})|^2 - 2 \int_{\partial B_{r_k}(0)} |v_k - p_{v, n, r_k}|^2 \, d\mathcal{H}^{n-1} \right)
\leq M_k^{-2} \left( \int_{B_{r_k}(0)} |\nabla v_k|^2 - 2 \int_{\partial B_{r_k}(0)} |v_k|^2 \, d\mathcal{H}^{n-1} \right)
\leq M_k^{-2} W(u_k, x^k) + M_k^{-2} C_2 \to 0, \quad k \to \infty.
\]

It follows that $(w_k)_{k \in \mathbb{N}}$ is bounded in $W^{1,2}(B_1; \mathbb{R}^m)$ such that —passing to a subsequence if necessary— $w_k$ converges weakly in $W^{1,2}(B_1; \mathbb{R}^m)$ to $w_0 \in W^{1,2}(B_1; \mathbb{R}^m)$. By Rellich’s theorem, $w_k$ converges strongly in $L^2(\partial B_1(0) ; \mathbb{R}^m)$, $\|w_0\|_{L^2(\partial B_1(0) ; \mathbb{R}^m)} = 1$ and (by (20)) $0 = \int_{B_1(0)} w_0 \cdot q \, d\mathcal{H}^{n-1}$ for every $q \in \mathcal{H}$. Hence we obtain that
\[ \int_{B_{r_k}(0)} |\nabla w_0|^2 = 2 \int_{\partial B_{r_k}(0)} |w_0|^2 \, d\mathcal{H}^{n-1} = 2. \]

Moreover,
\[ |\Delta w_k| \leq \frac{C_3}{M_k} \to 0, \quad k \to \infty, \]

such that $w_0$ is harmonic in $B_1(0)$ and (by $C^{1,\alpha}$-estimates) $w_0(0) = \nabla w_0(0) = 0$.

Now by [17, Lemma 4.1], each component $z_j$ of $w_0$ must satisfy
\[ 2 \int_{\partial B_1(0)} z_j^2 \, d\mathcal{H}^{n-1} \leq \int_{B_1(0)} |\nabla z_j|^2. \]

Summing over $j$ we obtain
\[ 2 \int_{\partial B_1(0)} |w_0|^2 \, d\mathcal{H}^{n-1} \leq \int_{B_1(0)} |\nabla w_0|^2, \]

implying by (21) that
\[ 2 \int_{\partial B_1(0)} |w_0|^2 \, d\mathcal{H}^{n-1} = \int_{B_1(0)} |\nabla w_0|^2. \]

Thus
\[ 2 \int_{\partial B_1(0)} z_j^2 \, d\mathcal{H}^{n-1} = \int_{B_1(0)} |\nabla z_j|^2 \]

for each $j$, implying by [17, Lemma 4.1] that $z_j$ is a homogeneous harmonic polynomial of second order. But then $0 = \int_{B_1(0)} w_0 \cdot q \, d\mathcal{H}^{n-1}$ for every $q \in \mathcal{H}$ implies that $w_0 = 0$ on $\partial B_1(0)$, contradicting $\|w_0\|_{L^2(\partial B_1(0) ; \mathbb{R}^n)} = 1$.

The next section follows closely the procedure in [19] and [16].
6. An energy decay estimate and uniqueness of blow-up limits

In this section we show that an epiperimetric inequality always implies an energy decay estimate and uniqueness of blow-up limits. More precisely:

**Theorem 3** (Energy decay and uniqueness of blow-up limits). Let \( x^0 \in \Gamma_0(u) \), and suppose that the epiperimetric inequality holds with \( \kappa \in (0, 1) \) for each

\[ c_r(x) := |x|^2 u_r \left( \frac{x}{|x|} \right) = \frac{|x|^2}{|x|^2} u(x^0 + \frac{r}{|x|} x) \]

and for all \( r \leq r_0 < 1 \). Finally let \( \mathbf{u}_0 \) denote an arbitrary blow-up limit of \( u \) at \( x^0 \). Then

\[ |W(u, x^0, r) - W(u, x^0, 0+)| \leq \left| W(u, x^0, r_0) - W(u, x^0, 0+) \right| \left( \frac{r_0}{r} \right)^{\frac{n+2}{4}} \]

for \( r \in (0, r_0) \), and there exists a constant \( C \) depending only on \( n \) and \( \kappa \) such that

\[ \int_{\partial B_r(0)} \left| \frac{u(x^0 + rx)}{r^2} - \mathbf{u}_0(x) \right| dH^{n-1} \leq C \left| W(u, x^0, r_0) - W(u, x^0, 0+) \right|^{\frac{1}{2}} \left( \frac{r_0}{r} \right)^{\frac{n+2}{4}} \]

for \( r \in (0, \frac{r_0}{2}) \), and \( \mathbf{u}_0 \) is the unique blow-up limit of \( u \) at \( x^0 \).

**Proof.** We define

\[ e(r) := r^{n-2} \int_{\partial B_r(e)} (|\nabla u|^2 + 2u) - 2r^{n-3} \int_{\partial B_r(e)} |u|^2 dH^{n-1} - W(u, x^0, 0+) \]

Up to a constant \( e(r) \) is the function of the monotonicity identity, so that we have already computed \( e'(r) \). Here however, we need a different formula for \( e'(r) \):

\[ e'(r) = \left[ - \frac{n+2}{r} e(r) - \frac{n+2}{r} W(u, x^0, 0+) + \frac{2}{r} r^{n-3} \int_{\partial B_r(e)} |u|^2 dH^{n-1} \right. \\
- 2r^{n-3} \int_{\partial B_r(e)} 2v \cdot \nabla u \cdot u dH^{n-1} - \frac{2(n-1)}{r} r^{n-3} \int_{\partial B_r(e)} |u|^2 dH^{n-1} \\
+ \left. r^{n-2} \int_{\partial B_r(e)} (|\nabla u|^2 + 2u) dH^{n-1} \right] = r^{-1} \left[ \int_{\partial B_r(0)} (|\nabla u|^2 + 2u) dH^{n-1} \right. \\
+ 2|u| - 4v \cdot \nabla u \cdot u + 4|u|^2 + 4|\nabla u|^2 - 2(n+2)|u|^2 - 2(n+2)|u|^2 dH^{n-1} - (n+2)W(u, x^0, 0+) \right] \\
- \frac{n+2}{r} e(r) \geq r^{-1} \left[ \int_{\partial B_r(0)} (|\nabla u|^2 + 2|u| + 4|u|^2 - 2(n+2)|u|^2) dH^{n-1} \\
- (n+2)W(u, x^0, 0+) \right] - \frac{n+2}{r} e(r) \\
= \frac{n+2}{r} \left[ M(c_r) - W(u, x^0, 0+) - e(r) \right] .

Here we employ the minimal regularity assumption that the epiperimetric inequality \( M(v) \leq (1 - \kappa)M(c_r) + \kappa W(u, x^0, 0+) \) holds for some \( v \in W^{1,2}(B_1; \mathbb{R}^n) \) with \( c_r \)-boundary values and we obtain for \( r \in (0, r_0) \) the estimate

\[ e'(r) \geq \frac{n+2}{r} \left( M(u_r) - W(u, x^0, 0+) \right) - \frac{n+2}{r} e(r) \]
By the monotonicity formula Lemma 1, \( e(r) \geq 0 \), and we conclude in the non-trivial case \( e > 0 \) in \((r_1, r_0)\) that

\[
(\log(e(s)))' \geq \frac{(n + 2)\kappa 1}{1 - \kappa} \frac{1}{s} \text{ for } s \in (r_1, r_0).
\]

Integrating from \( r \) to \( r_0 \) we obtain that

\[
\log \left( \frac{e(r_0)}{e(r)} \right) \geq \frac{(n + 2)\kappa 1}{1 - \kappa} \log \left( \frac{r_0}{r} \right) \quad \text{and} \quad e(r_0) \geq e(r) \left( \frac{r_0}{r} \right)^{\frac{(n + 2)\kappa 1}{1 - \kappa}} \quad \text{for } r \in (r_1, r_0)
\]

and that \( e(r) \leq e(r_0) \left( \frac{r}{r_0} \right)^{\frac{(n + 2)\kappa 1}{1 - \kappa}} \) for \( r \in (0, r_0) \) which proves our first statement.

Using once more the monotonicity formula (Lemma 1) we get for \( 0 < \rho < \sigma \leq r_0 \) an estimate of the form

\[
\int_{\partial B(0)} \left| \frac{\partial u}{\partial r} \right|^2 \ dr \ d\mathcal{H}^{n-1} \leq \int_{\rho}^{r} r^{-1-n} \int_{\partial B(x)} \left| \nabla u - 2 \frac{u}{r} \right|^2 \ d\mathcal{H}^{n-1} \ dr
\]

\[
\leq \sqrt{n \omega_n} \int_{\rho}^{r} r^{-1-n} r^{\frac{n-2}{2}} \left( r^{-n-2} \int_{\partial B(x)} \left| \nabla u - 2 \frac{u}{r} \right|^2 \ d\mathcal{H}^{n-1} \right)^{\frac{1}{2}} \ dr
\]

\[
= \sqrt{n \omega_n} \int_{\rho}^{r} r^{\frac{2}{2-n}} \sqrt{e'(r)} \ dr \leq \sqrt{n \omega_n} \int_{\rho}^{r} \left( \log(\sigma) - \log(\rho) \right)^{\frac{1}{2}} \left( e(\sigma) - e(\rho) \right)^{\frac{1}{2}} \ dr
\]

Considering now \( 0 < 2\rho < 2\rho \leq r_0 \) and intervals \([2^{k-1}, 2^k) \ni \rho \) and \([2^{k-1}, 2^k) \ni \sigma \) the already proved part of the theorem yields that

\[
\int_{\partial B(0)} \left| \frac{u(x^0 + \rho \lambda)}{\rho^2} - \frac{u(x^0 + \sigma \lambda)}{\sigma^2} \right|^2 \ d\mathcal{H}^{n-1} \leq \frac{1}{2} \sum_{i=1}^{k} \int_{\partial B(0)} \left| \frac{u_i}{\partial r} \right|^2 \ dr \ d\mathcal{H}^{n-1}
\]

\[
\leq C_1(n) \sum_{i=1}^{k} \left( \log(2^{k}) - \log(2^{k-1}) \right)^{\frac{1}{2}} \left( e(2^{k}) - e(2^{k-1}) \right)^{\frac{1}{2}} = C_2(n) \sum_{i=1}^{k} \left( e(2^{k}) - e(2^{k-1}) \right)^{\frac{1}{2}}
\]

\[
\leq C_3(n, \kappa) \left| W(u, x^0, r_0) - W(u, x^0, 0) \right|^2 \sum_{j=1}^{k} \left( r_0 \right)^{\frac{(n + 2)\kappa 1}{1 - \kappa}}
\]

\[
\leq C_4(n, \kappa) \left| W(u, x^0, r_0) - W(u, x^0, 0) \right|^2 \left( \frac{c^f}{1 - c} \right) r_0^{\frac{(n + 2)\kappa 1}{1 - \kappa}}
\]

where \( c = 2^{\frac{(n + 2)\kappa 1}{1 - \kappa}} \in (0, 1) \). Thus

\[
\int_{\partial B(0)} \left| \frac{u(x^0 + \rho \lambda)}{\rho^2} - \frac{u(x^0 + \sigma \lambda)}{\sigma^2} \right|^2 \ d\mathcal{H}^{n-1}
\]

\[
\leq C_5(n, \kappa) \left| W(u, x^0, r_0) - W(u, x^0, 0) \right|^2 \left( \frac{r}{r_0} \right)^{\frac{(n + 2)\kappa 1}{1 - \kappa}}
\]

and letting \( \frac{u(x^0 + \rho \lambda)}{\rho^2} \to u_0 \) as a certain sequence \( \rho_j \to 0 \) finishes our proof. \( \square \)
7. Homogeneous solutions

In this section we consider homogeneous solutions $u \in W^{1,2}(B_1; \mathbb{R}^m)$, meaning that

$$u(\lambda x) = \lambda^2 u(x) \quad \text{for all } \lambda > 0 \text{ and } x \in B_1(0).$$

Obviously $u$ may be extended to a homogeneous solution on $\mathbb{R}^n$.

Moreover, if $\text{dist}_{L^1(B_1;\mathbb{R}^n)}(u, \mathbb{H}) \leq 1$ (see (4)) then due to Remark 1 we have

$$(22) \quad \sup_{h_i} |u| \leq C(n, m) \text{ and } \sup_{h_i} |\nabla u| \leq C(n, m).$$

**Proposition 3.** If $B_1 \cap \text{supp } u \subset \{x_n > -\delta(n, \sup_{B_1(0)} |u|)\}$, then $u \in \mathbb{H}$.

**Proof.** Observe first that each component $u_i$ is a solution of

$$\mathcal{L} u_i := -\Delta' u_i + u_i / |u| = 2nu_i$$

in every connected component $\Omega'$ of $\partial B_1 \cap \{|u| > 0\}$, where $\Delta'$ is the Laplace-Beltrami operator on the unit sphere in $\mathbb{R}^n$. In Lemma 4 of the Appendix we prove that $u_i = a_i f_{\Omega'}$ for a real number $a_i$ and a function $f_{\Omega'}$ depending only on $\Omega'$ which is positive on $\Omega'$ and vanishes on the boundary of $\Omega'$. It follows that for each connected component $\Omega$ of $B_1 \cap \{|u| > 0\}$ there exists a unit vector $a = (a_1, \ldots, a_m)$ such that $u(x) = a_i u(x)$ and $\Delta |u| = 1$ in $\Omega$.

Now, if $|\nabla u| = 0$ on $\partial \Omega$, then we may extend $u$ by $0$ outside $\Omega$, that is $|u|$ can be extended to a 2-homogeneous non-negative solution of the classical obstacle problem in $\mathbb{R}^n$. These solutions have been completely classified (see [5], cf. also [13]), and if $\text{supp } u \subset \{x_n > -\delta|x|\}$ (where $\delta = \delta(n, m, \sup_{B_1(0)} |u|)$) would in this case imply that up to rotation, $|u| = h$, and $u = ah$, where $h$ is a half-space solution for scalar problem.

If, on the other hand, there is a point $x' \in \partial \Omega \cap \{|\nabla u| \neq 0\}$, then the fact that $u$ is continuously differentiable, implies that a equals the vector of the adjoining connected component of $|u| > 0$ up to the sign. In this case we obtain, taking the maximal union of all such connected components, that each $u_i$ is a 2-homogeneous solution of the scalar two-phase obstacle problem

$$\Delta v = c(x_{(v > 0)} - x_{(v < 0)}) \in \mathbb{R}^n$$

with $c > 0$, satisfying $v = 0$ in $\{x_n \leq -\delta\}$. However, according to [14, Theorem 4.3], no such solution exists.

**Lemma 3.** The half-plane solutions are (in the $L^1(B_1(0);\mathbb{R}^m)$-topology) isolated within the class of homogeneous solutions of degree 2.

**Proof.** Let $\|u - h\|_{L^1(B_1;\mathbb{R}^m)} \leq \epsilon$, where rotating in $\mathbb{R}^n$ and in $\mathbb{R}^m$ if necessary we may assume that

$$h(x) = \frac{\max(x_n, 0)^2}{2} e^1.$$ 

From (22) as well as Propositions 2 and 3 we infer that $u \in \mathbb{H}$ if $\epsilon$ has been chosen small enough, depending only on $n$ and $m$.

We defined in (5) the constant $\alpha_n = 2M(h)$ where $h \in \mathbb{H}$. Now we are going to estimate the value of $M(u)$ for an arbitrary homogeneous solution $u$ of degree 2.

**Proposition 4.**

$$(23) \quad \alpha_n = \frac{\mathcal{H}^{n-1}(\partial B_1)}{2n(n + 2)}.$$
Let $u$ be a homogeneous solution of degree 2. Then

$$M(u) \geq \alpha_n \frac{\mathcal{H}^{n-1}(\partial B_1 \cap \{|u| > 0\})}{\mathcal{H}^{n-1}(\partial B_1)}.$$  

(24)

In particular,

$$M(u) \geq \alpha_n \text{ if } |u| > 0 \text{ a.e.}$$

(25)

Proof. Let $U := |u|$, and recall (17):

$$\Delta U = 1 + \frac{A}{U} \text{ in } |u| > 0, \text{ where } A = |\nabla u|^2 - |\nabla U|^2 \geq 0.$$  

It follows that —using the homogeneity of $u$—

$$\int_{B_1 \cap |u| > 0} \left(1 + \frac{A}{U}\right) = \int_{\partial B_1 \cap |u| > 0} \nabla U \cdot \nu \, d\mathcal{H}^{n-1}$$

$$= 2 \int_{\partial B_1 \cap |u| > 0} U \, d\mathcal{H}^{n-1} - 2 \int_{B_1 \cap |u| > 0} \mathcal{H}^{n-1} = 2(n + 2) \int_{B_1} U - 2 \int_{\partial B_1 \cap |u| > 0} \nabla U \, d\mathcal{H}^{n-1}.$$

On the other hand, using once more the homogeneity of $u$,

$$M(u) = \int_{B_1} (|\nabla u|^2 + 2|u|) - 2 \int_{\partial B_1} |u|^2 \, d\mathcal{H}^{n-1} = \int_{B_1} |u| = \int_{B_1} U.$$

(26)

In order to verify (23), observe that for $e \in \partial B_1 \subset \mathbb{R}^m$ and $h(x) = e \max(x_n, 0)^2/2$,

$$\frac{\alpha_n}{2} = M(h) = \frac{1}{2} \int_{B_1} \max(x_n, 0)^2 = \frac{1}{4} \int_{B_1} x_n^2 = \frac{1}{4n} \int_{B_1} |x|^2 = \frac{\mathcal{H}^{n-1}(\partial B_1)}{2n(n + 2)}.$$

Using the above estimates we conclude that

$$M(u) \geq \frac{1}{2(n + 2)} |B_1 \cap |u| > 0| = \frac{\mathcal{H}^{n-1}(\partial B_1 \cap |u| > 0))}{2n(n + 2)} = \frac{\alpha_n \mathcal{H}^{n-1}(\partial B_1 \cap |u| > 0))}{\mathcal{H}^{n-1}(\partial B_1)}.$$

(27)

Corollary 1. Let $u$ be a homogeneous solution of degree 2. Then

$$M(u) \geq \alpha_n \max \left(\frac{1}{2}, \frac{\mathcal{H}^{n-1}(\partial B_1 \cap |u| > 0))}{\mathcal{H}^{n-1}(\partial B_1)}\right) \geq \alpha_n/2,$$

and $M(u) = \alpha_n/2$ implies that $u \in \mathbb{H}$. Moreover, $\alpha_n/2 < \bar{\alpha}_n := \inf \{M(v) : v \text{ is a homogeneous solution of degree 2, but } v \notin \mathbb{H}\}$.

Proof. If $\mathcal{H}^{n-1}(\partial B_1 \cap |u| = 0)) = 0$, then (27) follows from (25). Otherwise $|u| = 0$ contains by the non-degeneracy property Lemma 1 an open ball $B_r(y)$, and we may choose it in such a way that there is a point $z \in \partial B_r(y) \cap \partial |u| > 0$. Let $u_0$ be a blow up of $u$ at $z$. Since supp $u_0$ is contained in a half-space it follows from Proposition 3 that $u_0 \in \mathbb{H}$. Note that by homogeneity, $|u(x)| \leq C|x|^2$ and $|\nabla u(x)| \leq C|x|$ for every $x \in \mathbb{R}^n$, implying that the limit $W(u, x^0, +\infty)$ does not depend on the choice of $x^0$.

From (23) we obtain therefore that

$$\frac{\alpha_n}{2} = M(u_0) = W(u, z, 0^+) \leq W(u, z, +\infty) = W(u, 0, +\infty) = M(u).$$

(28)

Now we have to prove that $M(u) = \alpha_n/2$ implies $u \in \mathbb{H}$. Consider a ball $B_r(y)$ and a point $z$ as above, that is $y = z + \rho e$ with a unit vector $e$. It follows from homogeneity of $u$ that $e$ is orthogonal to $z$. We consider two cases.
Case a) If \( z = 0 \) then, again due to homogeneity of \( u \), we have \(|u(x)| = 0 \) in a half-space \((x \cdot e) > 0\). Hence \( u \in \mathbb{H} \) by Proposition 3.

Case b) If \(|z| > 0\) then, since \( W(u, z, r) \) does not depend on \( r \) (by (28)), we conclude that \( u \) is homogeneous with homogeneity center \( z \). More exactly, we have \( u(z + kx) = k^2 u(z + x) \) for any \( k > 0, x \in \mathbb{R}^n \). Since also \( u(z + kx) = k^2 u(z/k + x) \) we obtain \( u(z + x) = u(z/k + x) \) for any \( k > 0, x \in \mathbb{R}^n \). It means that \( u \) is constant in direction of vector \( z \). In particular, \(|u| = 0 \) in the ball \( B_{r}(\rho e) \) touching the origin and we are again at the case a).

Last, we have to prove \( \tilde{a}_n > \alpha_n/2 \). If it is not true then there is a sequence of homogeneous global solutions \( \{u_k\} \) such that

\[
M(u_k) \searrow \frac{\alpha_n}{2} \quad \text{as} \quad k \to \infty.
\]

In particular it implies by (26) uniform boundedness of \( u_k \) in \( L^1(B_1(0)) \) and therefore, by (16) and by elliptic theory, uniform boundedness of solutions \( u_k \) in \( W^{2,q}_{\text{loc}}(\mathbb{R}^n) \) for any \( q < \infty \). Then there exists a limit \( \tilde{u} \), by subsequence, such that \( \tilde{u} \) is a homogeneous solution, \( \tilde{u} \not\in \mathbb{H} \) (by Lemma 3) and \( M(\tilde{u}) = \alpha_n/2 \). From the first part of the proof we infer that \( \tilde{u} \in \mathbb{H} \), and a contradiction arises.

Definition 1. A point \( x \) is a regular free boundary point for \( u \) if:

\[
x \in \Gamma_0(u) \quad \text{and} \quad \lim_{r \to 0} W(u, x, r) = \frac{\alpha_n}{2}.
\]

We denote by \( \mathcal{R}_a \) the set of all regular free boundary points of \( u \) in \( B_1 \).

Corollary 2. The set of regular free boundary points \( \mathcal{R}_a \) is open relative to \( \Gamma_0(u) \).

Proof. This is an immediate consequence of Corollary 1 and the upper semicontinuity Lemma 2.

8. Regularity

In this last section we prove that the set of regular free boundary points \( \mathcal{R}_a \) is locally in \( D \) a \( C^{1,\beta} \)-surface and we derive a macroscopic criterion for regularity: suppose that \( W(u, x, r) \) drops for some (not necessarily small) \( r \) below the critical value \( \tilde{a}_n \) : then \( \partial \{|u| > 0\} \) must be a \( C^{1,\beta} \)-surface in an open neighborhood of \( x \).

Theorem 4. Let \( C_h \) be a compact set of points \( x^0 \in \Gamma_0(u) \) with the following property: at least one blow-up limit \( u_0 \) of \( u \) at \( x^0 \) is a half-plane solution, say \( u_0(x) = \frac{1}{2} \max(x \cdot v(x^0), 0)^2 \) for some \( v(x^0) \in \partial B_1(0) \subset \mathbb{R}^n \) and \( e(x^0) \in \partial B_1(0) \subset \mathbb{R}^n \). Then there exist \( r_0 > 0 \) and \( C < \infty \) such that

\[
\int_{\partial B_1(0)} \frac{|u(x^0 + r x)|}{r^2} - \frac{1}{2} e(x^0) \max(x \cdot v(x^0), 0)^2 \, d\mathcal{H}^n \leq C r^{\alpha_n/2n}.
\]

for every \( x^0 \in C_h \) and every \( r \leq r_0 \).

Proof. Step 1: Due to Dini’s theorem, \( W(u, x, r) \leq \epsilon + \alpha_n/2 \) for all \( r \in (0, r_0) \) and all \( x \in C_h \).

Step 2: If \( \rho_j \to 0, x^j \in C_h \) and \( u_j := u(x^j + \rho_j) \) \( \rho_j^2 \to v \) in \( W^{1,2}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^m) \) as \( j \to \infty \), then \( v \in \mathbb{H} \): According to Remark 2, \( v \) is a global solution of (1). Moreover, by Step 1

\[
W(v, 0, \rho) = \lim_{j \to \infty} W(u_j, 0, \rho) = \lim_{j \to \infty} W(u, x^j, \rho_j) = \frac{\alpha_n}{2}.
\]

But then Lemma 1 and Corollary 1 imply that \( v \in \mathbb{H} \).
Step 3: For small $\rho$, $u(x+\rho y)/\rho^2$ is uniformly (in $x \in C_h$) close to $\mathbb{H}$ in the $W^{1,2}_0(\mathbb{R}^n; \mathbb{R}^m) \cap L^\infty_0(\mathbb{R}^n; \mathbb{R}^m)$-topology; assuming towards a contradiction that this is not the case, we obtain $\rho_j \to 0$ and $x^j \in C_h$ such that the distance in the same topology is $\geq \delta > 0$. As $\mathbf{u}_j := u(x^j + \rho_j y)/\rho_j^2$ is by Theorem 2 and $W^{2,p}$-theory bounded in $W^{2,p}_0(\mathbb{R}^n; \mathbb{R}^m)$ for each $q \in [1, +\infty)$, passing to a subsequence if necessary $\mathbf{u}_j \to \mathbf{w}$ in $W^{1,2}_0(\mathbb{R}^n; \mathbb{R}^m) \cap C_0^\infty(\mathbb{R}^n; \mathbb{R}^m)$, implying by Step 2 that $\mathbf{w} \in \mathbb{H}$, contradicting the distance being $\geq \delta > 0$.

\[ \text{Theorem 5 (regularity). The free boundary } \partial |u| > 0 \text{ is in an open neighborhood of the set } R_u \text{ locally a } C^{1,\beta} \text{-surface; here } \beta = \left( \frac{(n+2)k}{2(1-\alpha)} \right)^{-1}. \]

Proof. Let us consider a point $x^0 \in R_u$. By Theorem 4 there exists $\delta_0 > 0$ such that $B_{\delta_0}(x^0) \subset D$ and

\[
\int_{\partial B_r(0)} \left| \frac{\mathbf{u}(x^1 + rx) - \frac{1}{2}\mathbf{e}(x^1) \max(x \cdot \nu(x^1), 0)^2}{r^2} \right| dH^{n-1} \leq C r^{\frac{\alpha+2k}{2(1-\alpha)}},
\]

for every $x^1 \in R_u \cap B_{\delta_0}(x^0)$ and for every $r \leq \min(\delta_0, r_0)$. We now observe that $x^1 \mapsto \nu(x^1)$ and $x^1 \mapsto \mathbf{e}(x^1)$ are Hölder-continuous with exponent $\beta$ on $R_u \cap B_{\delta_1}(x^0)$ for some $\delta_1 \in (0, \delta_0)$:

\[
\frac{1}{2} \int_{\partial B_1(0)} \left| \mathbf{e}(x^1) \max(x \cdot \nu(x^1), 0)^2 - \mathbf{e}(x^2) \max(x \cdot \nu(x^2), 0)^2 \right| dH^{n-1} \leq 2C r^{\frac{\alpha+2k}{2(1-\alpha)}},
\]

\[
+ \int_{\partial B_1(0)} \int_0^1 \left| \nabla \mathbf{u}(x^1 + rx + t(x^2 - x^1)) \right| |x^1 - x^2| dt dH^{n-1} \leq 2C r^{\frac{\alpha+2k}{2(1-\alpha)}} + C_1 \max(r, |x^1 - x^2|) |x^1 - x^2|^{\frac{\alpha+2k}{2(1-\alpha)}}
\]

if we choose $\gamma := \left( 1 + \frac{(n+2)k}{2(1-\alpha)} \right)^{-1}$ and $r := |x^2 - x^1|^{\gamma} \leq \min(\delta_0, r_0)$, and the left-hand side

\[
\frac{1}{2} \int_{\partial B_r(0)} \left| \mathbf{e}(x^1) \max(x \cdot \nu(x^1), 0)^2 - \mathbf{e}(x^2) \max(x \cdot \nu(x^2), 0)^2 \right| dH^{n-1} \geq c(n) \left( |\nu(x^1) - \nu(x^2)| + |\mathbf{e}(x^1) - \mathbf{e}(x^2)| \right)
\]

which can be seen as follows: Suppose first that for $j \to \infty$, $\nu_j \to \nu^1$, $\nu_j \to \nu^2$, $\mathbf{e}_j \to \mathbf{e}^1$, $\mathbf{e}_j \to \mathbf{e}^2$ such that

\[
0 \leftarrow \int_{\partial B_1(0)} \left| \mathbf{e}_j^1 \max(x \cdot \nu_j, 0)^2 - \mathbf{e}_j^2 \max(x \cdot \nu_j, 0)^2 \right| dH^{n-1}.
\]

We obtain

\[
0 = \int_{\partial B_1(0)} \left| \mathbf{e}^1 \max(x \cdot \nu^1, 0)^2 - \mathbf{e}^2 \max(x \cdot \nu^2, 0)^2 \right| dH^{n-1},
\]

implying immediately that $\mathbf{e}^1 = \mathbf{e}^2$ and that $\nu^1 = \nu^2$. Next, suppose towards a contradiction that, setting $c_j := |\nu_j - \nu]| + |\mathbf{e}_j - \mathbf{e}|$, $(\nu_j - \nu)^2/c_j \to \eta$, $(\mathbf{e}_j - \mathbf{e})/c_j \to \xi$ and

\[
0 \leftarrow \frac{1}{c_j} \int_{\partial B_1(0)} \left| \mathbf{e}_j \max(x \cdot \nu_j, 0)^2 - \mathbf{e}_j \max(x \cdot \nu_j, 0)^2 \right| dH^{n-1} =: T_j.
\]
We obtain
\[
0 \leftarrow T_j \geq \frac{1}{e_j} \int_{\delta B_1(0) \cap \{x \cdot v > 0\} \cap \{x \cdot v'' > 0\}} \left| (e_j' - e_j^{''}) (x \cdot v') + e_j^{''} x \cdot (v'' + v') x \cdot (v' - v'') \right| \, dH^{n-1}
\]
\[
\rightarrow \int_{\delta B_1(0) \cap \{x \cdot v'' > 0\}} \left| \xi (x \cdot v')^2 + 2e_1 x \cdot v' x \cdot \eta \right| \, dH^{n-1}
\]
as \(j \to \infty\). We obtain \(\xi = -2v' \cdot \eta e_1\), contradicting the fact that \(0 = (\|e_j||^2 - |e_j^{''}|^2)/e_j = ((e_j')^2 - (e_j^{'})^2)/e_j \to 2e_1 \cdot \xi\) and thus proving (30).

Next, (29) as well as the regularity and non-degeneracy of \(u\) imply that for \(\epsilon > 0\) there exists \(\delta_2 \in (0, \delta_1)\) such that for \(x \in \mathcal{R}_u \cap B_{\delta_1}(x^0)\)

\[
(31) \quad u(y) = 0 \text{ for } y \in B_{\delta_2}(x^0) \text{ satisfying } (y - x^1) \cdot \nu(x^1) < -\epsilon \|y - x^1\| \text{ and } |u(y)| > 0 \text{ for } y \in B_{\delta_2}(x^0) \text{ satisfying } (y - x^1) \cdot \nu(x^1) > \epsilon \|y - x^1\|.
\]
assuming that (31) does not hold, we obtain a sequence \(\mathcal{R}_u \cap B_{\delta_1}(x^0) \ni x^m \to \bar{x}\) and a sequence \(y^m - x^m \to 0\) as \(m \to \infty\) such that

\[
(32) \quad \text{either } |u(y^m)| > 0 \text{ and } (y^m - x^m) \cdot \nu(x^m) < -\epsilon \|y^m - x^m\| \text{ or } u(y^0) = 0 \text{ and } (y^0 - x^m) \cdot \nu(x^0) > \epsilon \|y^0 - x^m\|.
\]
On the other hand we know from (29) as well as from the regularity and non-degeneracy of the solution \(u\), that the sequence \(u_j(x) := \frac{u(x + y - x^1)}{|y - x^1|^2}\) converges in \(C_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^m)\) to \(\frac{1}{2}\eta(\bar{x}) \max(x \cdot \nu(\bar{x}), 0)^2\) as \(j \to \infty\) and that \(u_j = 0\) on each compact subset \(C \ni \{x \cdot \nu(x) < 0\}\) provided that \(j \geq j(C)\). This, however, contradicts (32) for large \(j\).

Last, we use (31) in order to show that \(\partial |u| > 0\) is for some \(\delta_3 \in (0, \delta_2)\) in \(B_{\delta_3}(x^0)\) the graph of a differentiable function: applying two rotations we may assume that \(\nu(x^1) = \epsilon^n\) and \(e(x^1) = e_1\). Choosing now \(\delta_2\) with respect to \(\epsilon = \frac{1}{2}\) and defining functions \(g^+, g^- : B_\frac{\epsilon}{2}(0) \to [-\infty, \infty], g^+(x') := \sup_{x_n \in x^0 + (x', x_n) \in \partial |u| > 0} [x_n + x^0 + (x', x_n) \in \partial |u| > 0]\) and \(g^-(x') := \inf_{x_n \in x^0 + (x', x_n) \in \partial |u| > 0} [x_n + x^0 + (x', x_n) \in \partial |u| > 0]\), we first note that (applying (31) at \(x^0\)), that \(g^+ < -\infty\) and \(g^- > -\infty\) on \(B_\frac{\epsilon}{2}(0)\). As \(\nabla u(x^0 + (x', g^-(x'))) = 0\) for every \(x' \in B_{\frac{\epsilon}{2}}(0)\), we infer from Corollary 2 that \(x^0 + (x', g^-(x')) \in \mathcal{R}_u\). It follows that (31) is applicable at \(x^0 + (x', g^-(x'))\) for every \(x' \in B_{\frac{\epsilon}{2}}(0)\), yielding by the Hölder continuity of \(x^1 \mapsto \nu(x^1)\) that for sufficiently small \(\delta_3\) the functions \(g^+\) and \(g^-\) satisfy \(g^+ = g^-\) Lipschitz-continuous on \(B_{\delta_3}(0)\). In particular it follows that all free boundary points close to \(x^0\) belong to \(\mathcal{R}_u\), and there are no other free boundary points (for example free boundary points with non-vanishing gradient) in a neighborhood of \(x^0\).

Applying (31) once more with respect to arbitrary \(\epsilon\) we see that \(g^+\) is Fréchet-differentiable in \(B_{\delta_3}(0)\), which finishes our proof in view of the already derived Hölder-continuity of the normal \(\nu(x)\).

\[\square\]

**Corollary 3** (Macroscopic criterion for regularity). Let \(\bar{\alpha}_n\) be the constant defined in Corollary 1. Then \(B_{\delta_2}(x^0) \subset D, x^0 \in D \cap \partial |u| > 0\) and \(W(u, x^0, r) < \bar{\alpha}_n\) imply that \(\partial |u| > 0\) is in an open neighborhood of \(x^0\) a \(C^{1,\beta}\)-surface.

**Proof.** By \(C^{1,\beta}\)-regularity of \(u\) and Theorem 5, it suffices to show that for \(W(u, x^0, r) < \bar{\alpha}_n\) either (i) \(\nabla u(x^0) \neq 0\) or (ii) \(x^0 \in \mathcal{R}_u\). If both (i) and (ii) fail then by Lemma 1, Lemma 2 and Corollary 1, \(W(u, x^0, r) \geq \lim_{r \to 0} W(u, x^0, r) > \bar{\alpha}_n\), contradicting the assumption. \[\square\]
9. Appendix

Lemma 4. Let $\Delta'$ be the Laplace-Beltrami operator on the unit sphere in $\mathbb{R}^n$, let the domain $\Omega' \subset \partial B_1(0) \subset \mathbb{R}^3$, let $\mathcal{L} := -\Delta' + q$ where $q \in C^0(\Omega')$ such that $q \geq q_0 > 0$ in $\Omega'$, and let $\lambda_k(\mathcal{L}, \Omega')$ denote the $k$-th eigenvalue with respect to the eigenvalue problem

$$
\mathcal{L}v = \lambda v \text{ in } \Omega',
$$

$$
v = 0 \text{ on } \partial \Omega';
$$

here $\partial \Omega'$ denotes the boundary of $\Omega'$ relative to $\partial B_1$.

1. If $\Omega' \subset \Omega$ then $\lambda_k(\mathcal{L}, \Omega) \geq \lambda_k(\mathcal{L}, \Omega')$ for every $k \in \mathbb{N}$. For $k = 1$ the inequality is strict.

2. $\lambda_k(\mathcal{L}, \Omega) \geq q_0 + \lambda_k(-\Delta', \Omega')$ for every $k \in \mathbb{N}$; in case $q \neq q_0$ the inequality becomes a strict inequality.

3. $q = 1/h$ and $\Omega' \subset \partial B_1 \cap \{x_n > 0\}$ and $v \in W^{1,2}(\partial B_1)$ being an eigenfunction with respect to $\Omega'$ and $\lambda = 2n$ imply $v = ah$ for some real number $a \neq 0$. Here $h(x) = \frac{1}{4} \max (x_n, 0)^2$.

4. $\Omega' \subset \partial B_1 \cap \{x_n > -\delta(n, q_0)\}$ and $v \in W^{1,2}(\partial B_1(0))$ being an eigenfunction of $\mathcal{L}$ with respect to $\lambda = 2n$ and a domain $\Omega'$ imply $v = af_{\Omega'}$ for a real number $a \neq 0$ and a function $f_{\Omega'}$ which is positive on $\Omega'$ and depends only on $\Omega'$.

Proof. 1. It suffices to remark that $v \in W^{1,2}(\Omega)$ implies $v \in W^{1,2}(\Omega')$, after extending $v$ by zero outside $\tilde{\Omega}'$.

2. Let $M_{k-1}$ be a subspace of $W^{1,2}(\Omega')$ of codimension $k - 1$,

$$
\mu(\mathcal{L}, \Omega', M_{k-1}) = \inf_{\|v\|_{L^2(\Omega')} = 1} \int_{\Omega'} (|\nabla v|^2 + qv^2).
$$

Due to the Courant minimax principle we have

$$
\lambda_k(\mathcal{L}, \Omega') = \sup \mu(\mathcal{L}, \Omega', M_{k-1})
$$

where sup is taken over the set of all possible $M_{k-1}$. Since

$$
\mu(\mathcal{L}, \Omega', M_{k-1}) \geq \inf_{\|v\|_{L^2(\Omega')} = 1} \int_{\Omega'} (|\nabla v|^2 + qv^2) \geq q_0 + \mu(-\Delta', \Omega', M_{k-1}),
$$

we may take $M_{k-1} := \{v \in W^{1,2}_0(\Omega') : \int_{\Omega'} v w_i = 0, i \leq k - 1\}$, where $w_i$ is an eigenfunction with respect to the $i$-th eigenvalue of $-\Delta'$ on $\Omega'$. For such $M_{k-1}$ we obtain $\mu(-\Delta', \Omega', M_{k-1}) = \lambda_k(-\Delta', \Omega')$, and 2. is proved.

3. In this case the eigenvalue problem on the sphere becomes

$$
-\Delta' v + \frac{2}{\cos^2 \theta} v = 2nv \text{ in } \Omega' \subset \{x_n > 0\},
$$

$$
v = 0 \text{ on } \partial \Omega'.
$$

Since $2n = \lambda_2(-\Delta', \partial B_1(0) \cap \{x_n > 0\})$, we obtain from 1. and 2. that $\lambda_2(\mathcal{L}, \Omega') > 2 + 2n$. But then $v$ must be an eigenfunction with respect to the $2$-nd eigenvalue $\lambda_2(\mathcal{L}, \Omega')$, and $\lambda_2(\mathcal{L}, \Omega') = 2n$. Observe now that $h$ is an eigenfunction with respect to the $2$-nd eigenvalue $\lambda_1(\mathcal{L}, \partial B_1(0) \cap \{x_n > 0\})$, so that

$$
\lambda_1(\mathcal{L}, \partial B_1(0) \cap \{x_n > 0\}) = 2n = \lambda_1(\mathcal{L}, \Omega').
$$

This is only possible if $\Omega' = \partial B_1(0) \cap \{x_n > 0\}$ and $v = ah$ for some $a \neq 0$.

4. Observe that the second eigenvalue of $-\Delta'$ in a half-sphere is $2n$, therefore due to 1. and continuity of $\lambda_2$ with respect to the size of a spherical cap, $\lambda_2(-\Delta', \Omega') \geq 2n - \omega(n, \delta)$ where $\omega(n, \delta) \to 0$ as $\delta \to 0$. From 2. it follows that $\lambda_2(\mathcal{L}, \Omega') \geq 2n - \omega(n, \delta) + q_0 > 2n$ if $\delta = \delta(n, q_0)$ is small. Thus, $v$ must be an eigenfunction with respect to the $1$-st eigenvalue
\( \lambda_1(\mathcal{L}, \Omega') \), and \( \lambda_1(\mathcal{L}, \Omega') = 2n \). Again, the eigenspace is a one-dimensional space such that \( v = a f_{\Omega'} \) for some real number \( a \neq 0 \) and a function \( f_{\Omega'} \) depending only on \( \Omega' \); We also know that first eigenfunctions do not change sign in the connected set \( \Omega' \). \( \Box 

\textbf{References}

[1] D. R. Adams, Weakly elliptic systems with obstacle constraints: part I: a 2 \( \times \) 2 model problem, 1–14, IMA Vol. Math. Appl., 42, Springer, New York, 1992.
[2] D. R. Adams, Weakly elliptic systems with obstacle constraints: part II–an \( N \times N \) model problem, J. Geom. Anal. 10 (2000), 375–412.
[3] D. R. Adams Weakly Elliptic Systems with Obstacle Constraints III. Complex Eigenvalues and Singular Systems Journal of Functional Analysis 178, 258–268 (2000).
[4] H. W. Alt, L. A. Caffarelli, and Avner Friedman. Variational problems with two phases and their free boundaries. \textit{Trans. Amer. Math. Soc.}, 282(2):431–461, 1984.
[5] L. A. Caffarelli, Compactness methods in free boundary problems. \textit{Comm. Partial Differential Equations} 5 (1980), no. 4, 427448.
[6] L. A. Caffarelli, L. Karp, and H. Shahgholian. Regularity of a free boundary with application to the Pompeiu problem. \textit{Ann. of Math.} (2), 151(1):269–292, 2000.
[7] M. Chipot, G. Vergara-Caffarelli, The N-membranes problem. \textit{Appl. Math. Optim.} 13 (1985), no. 3, 231249.
[8] L. C. Evans and A. Friedman. Optimal stochastic switching and the Dirichlet problem for the Bellman equation. \textit{Trans. Amer. Math. Soc.}, 253:365389, 1979.
[9] M. Fuchs, An elementary partial regularity proof for vector-valued obstacle problems. Math. Ann. 279 (1987), no. 2, 217226.
[10] F. Duzaar, M. Fuchs Optimal regularity theorems for variational problems with obstacles. \textit{Manuscripta Math.} 56 (1986), no. 2, 209234.
[11] J. L. Lions, Optimal control of systems governed by partial differential equations. Translated from the French by S. K. Mitter. Die Grundlehren der mathematischen Wissenschaften, Band 170 Springer-Verlag, New York-Berlin 1971
[12] R. Monneau, Pointwise Estimates for Laplace Equation. Applications to the Free Boundary of the Obstacle Problem with Dini Coefficients. J. Fourier Anal. Appl (2009) 15: 279-335.
[13] A. Petrosyan, H. Shahgholian, N. Uraltseva, Nina Regularity of free boundaries in obstacle-type problems. Graduate Studies in Mathematics, 136. American Mathematical Society, Providence, RI, 2012. x+221 pp.
[14] H. Shahgholian, N. Uraltseva, and G. S. Weiss. Global solutions of an obstacle-problem-like equation with two phases. \textit{Monatsh. Math.}, 142(1-2):27–34, 2004.
[15] H. Shahgholian, N. Uraltseva, and G. S. Weiss. The two-phase membrane problem regularity of the free boundaries in higher dimensions. Int. Math. Res. Not. IMRN 2007, no. 8, Art. ID rnm026, 16 pp.
[16] G. S. Weiss. The free boundary of a thermal wave in a strongly absorbing medium. \textit{J. Differential Equations}, 160(2):357–388, 2000.
[17] G. S. Weiss. An obstacle-problem-like equation with two phases: pointwise regularity of the solution and an estimate of the Hausdorff dimension of the free boundary. \textit{Interfaces Free Bound.}, 3(2):121–128, 2001.
[18] G. S. Weiss. Partial regularity for weak solutions of an elliptic free boundary problem. \textit{Comm. Partial Differential Equations}, 23(3-4):439–455, 1998.
[19] G. S. Weiss. A homogeneity improvement approach to the obstacle problem. \textit{Invent. Math.}, 138(1):23–50, 1999.

\textsc{Department of Mathematics, Royal Institute of Technology, 100 44 Stockholm, Sweden}  
\textit{E-mail address: johnan@kth.se}

\textsc{Department of Mathematics, Royal Institute of Technology, 100 44 Stockholm, Sweden}  
\textit{E-mail address: henriksh@math.kth.se}  
\textit{URL: http://www.math.kth.se/~henriksh/}

\textsc{St-Petersburg State University, Universitetskij pr. 28, Stary Petergof, 198504, Russia}  
\textit{E-mail address: uraltsev@pdmi.ras.ru}

\textsc{Faculty of Mathematics, University of Duisburg-Essen, 45117 Essen, Germany}  
\textit{E-mail address: georg.weiss@uni-due.de}