Symmetries of field equations of axion electrodynamics

A.G. Nikitin

Institute of Mathematics, National Academy of Sciences of Ukraine,
3 Tereshchenkivs’ka Street, Kyiv-4, Ukraine, 01601

Oksana Kuriksha

Petro Mohyla Black Sea State University,
10, 68 Desantnukiv Street, 54003 Mukolaiv, Ukraine

(Dated: February 19, 2022)

Abstract

The group classification of models of axion electrodynamics with arbitrary self interaction of axionic field is carried out. It is shown that extensions of the basic Poincaré invariance of these models appear only for constant and exponential interactions. The related conservation laws are discussed. The maximal continuous symmetries of the 3d Chern-Simons electrodynamics and Carroll-Field-Jackiw electrodynamics are presented. Using the Inönü-Wigner contraction the nonrelativistic limit of equations of axion electrodynamics is found. Exact solutions for the electromagnetic and axion fields are discussed including those which describe propagation with group velocities faster than the speed of light. However these solutions are causal since the corresponding energy velocities are subluminal.

PACS numbers: 03.65.Pm, 03.65.Fd, 03.65.Ge, 03.50.Kk

*Electronic address: nikitin@imath.kiev.ua
†Electronic address: kuriksha@imath.kiev.ua
I. INTRODUCTION

To explain the absence of the CP symmetry violation in interquark interactions Peccei and Quinn [1] suggested that a new symmetry must be present. The breakdown of this gives rise to the axion field proposed later by Weinberg [2] and Wilczek [3]. And it was Wilczek who presented the first analysis of possible effects caused by axions in electrodynamics [4]. Notice that the idea to include an extra pseudoscalar field into electrodynamics was proposed by Ni [5] as early as 1974, and so this date can be treated as the birth year of the axion prototype.

Axions belong to the main candidates to form the dark matter, see, e.g. [6] and references cited therein. New arguments for the materiality of axion theories were created in solid states physics. Namely, it was found recently [7] that the axionic-type interaction terms appear in the theoretical description of a class of crystalline solids called topological insulators. Axion electrodynamics gains plausibility by results of Heht et al [8] who extract the existence of a pseudoscalar field from the experimental data concerning electric field-induced magnetization on Cr$_2$O$_3$ crystals or the magnetic field-induced polarization. In other words, although their existence is still not confirmed experimentally axions are stipulated at least in the three fundamental fields: QCD, cosmology and condensed matter physics.

There are many other interesting aspects of axion electrodynamics. In particular, its reduced version (corresponding to the external axion field linear in independent variables) was used by Carroll, Field and Jackiw (CFJ) [9] to examine the possibility of Lorentz and CPT violations in Maxwell’s electrodynamics. In addition, just the interaction Lagrangian of axion electrodynamics generalizes the Chern-Simons form $\varepsilon_{abc} A^a \nabla^b A^c$ [10] to the case of (1+3)-dimensional Minkowski space.

Let us present more arguments for materiality of axion electrodynamics which are very inspiring for us. Recently new exactly solvable models for neutral Dirac fermions had been discovered [11], [32]. These models involve the external electromagnetic fields which do not solve Maxwell equations with physically reasonable currents. However, these fields solve equations of axion electrodynamics. We had classified exactly solvable quantum mechanical models with matrix potentials [13], [14], and superintegrable models of cold neutrons [15]. Some of these systems also include external fields which solve equations of axion electrodynamics. In addition, these field equations appear to be a relativistic counterpart of Galilei invariant systems classified in [16]. Thus we have a particular interest to study equations of axion electrodynamics, and
we will do it using the tools of group theory.

Group theory, and especially the theory of Lie groups is one of the cornerstones of modern theoretical physics. Symmetries of Lagrangians and of the corresponding motion equations form a very essential constituent part of any physical theory. However, except the analysis of symmetries of the CFJ model presented in paper [17], we do not know any systematical investigation of symmetries of axion theories. Notice that such an investigation would generate group-theoretical backgrounds for axion models and enable to construct their exact solutions.

In the present paper we make the group classification of the field equations of axion electrodynamics with arbitrary self interaction of axion field. The considered model includes the standard axion electrodynamics as a particular case. We prove that an extension of the basic Poincaré invariance appears only for the exponential, constant and trivial interaction terms. These and other results of group classification are presented in Section 3 and Appendix A.

In addition, we carry out the group analysis of two other theories which are close to axion electrodynamics. Namely, we describe Lie symmetries of the field equations of classical electrodynamics modified by adding the Chern-Simons term, and symmetries of the CFJ model. As it is shown in Appendix B, the maximal continuous group of Chern-Simon electrodynamics is the 17-parametrical extended conformal group.

A special subject of our analysis are conservation laws which correspond to found symmetries. They are discussed in Section 4, where we present a simple proof that the interaction between the electromagnetic and axion fields does not affect the energy-momentum tensor.

In Section 5 we present selected invariant solutions of field equations of axion electrodynamics. Some of these solutions play the key role in formulation of exactly solvable problems of quantum mechanics in both relativistic [11], [12] and nonrelativistic [15], [18] approaches.

In Section 6 we analyze plane wave solutions which are smooth and bounded functions which generate positive definite and bounded energy density. We show that these solutions describe waves whose group velocity can be superluminal. Nevertheless, they are causal since the corresponding energy velocities are smaller than the velocity of light.

An important constituent of any relativistic model is its nonrelativistic limit. This is true also for models including massless fields. As it was shown long time ago [19], there exist a reasonable (and very important) nonrelativistic approximation for the Maxwell equations, which makes them invariant w.r.t. the Galilei group. Namely, in this approximation we obtain equations of Faraday electrodynamics. This result justifies Galilei invariance of quantum mechanical
systems including particles interacting with an external electromagnetic field.

A natural question arises whether it is possible to extend this result to the case of field equations of axion electrodynamics. Notice that the correct definition of the nonrelativistic limit of a physical model is by no means a simple problem in general and in the case of theories of massless fields in particular, see, for example, [20]. Such limit is not necessary unique, and simple passing the speed of light to infinity we can obtain a physically meaningless theory.

By definition, any relativistic system is invariant w.r.t. the Poincaré group P(1,3), and a correct nonrelativistic approximation of this system should be invariant w.r.t. the Galilei group G(1,3). Thus to obtain a well defined nonrelativistic limit it is necessary to take a care on the attending transformation P(1,3)→G(1,3). This idea had been proposed long time ago by Inönü and Wigner [21] who presented definitions and justifications for such transformation. It is a special limiting procedure called contraction, which is an important subject of modern group theory.

In Section 7 we find a nonrelativistic limit of equations of axion electrodynamics with using a generalized Inönü-Wigner (IW) contraction. As a result we prove that the Galilei-invariant wave equations for an abstract ten-component vector field, deduced in [16], are nothing but a contracted version of the field equations of axion electrodynamics.

Appendix A includes a rather detailed proof of the results formulated in Section 3. Finally, in Appendix B we present the results of group analysis of the field equations of classical electrodynamics modified by adding the Chern-Simons terms, and of the CFJ model.

II. FIELD EQUATIONS OF AXION ELECTRODYNAMICS

Let us start with the following model Lagrangian:

\[ L = \frac{1}{2} p_\mu p^\mu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\kappa}{4} \theta F_{\mu\nu} \tilde{F}^{\mu\nu} - V(\theta). \] (1)

Here \( F_{\mu\nu} \) is the strength tensor of electromagnetic field, \( \tilde{F}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \), \( p_\mu = \partial_\mu \theta \), \( \theta \) is the pseudoscalar axion field, \( V(\theta) \) is a function of \( \theta \), \( \kappa \) is a dimensionless constant, and the summation is imposed over the repeating indices over the values 0, 1, 2, 3. Moreover, the strength tensor can be expressed via four-potential \( A = (A^0, A^1, A^2, A^3) \) as:

\[ F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \] (2)
Setting in (1) \( \theta = 0 \) we obtain the Lagrangian for Maxwell field. Moreover, if \( \theta \) is a constant then (1) coincides with the Maxwell Lagrangian up to constant and four-divergence terms. Finally, the choice \( V(\theta) = \frac{1}{2} m^2 \theta^2 \) reduces \( L \) to the standard Lagrangian of axion electrodynamics.

We will investigate symmetries of the generalized Lagrangian (1) with arbitrary \( V(\theta) \). More exactly, we will make the group classification of the corresponding Euler-Lagrange equations:

\[
\partial_\nu F^{\mu\nu} = \kappa p_\nu \tilde{F}^{\mu\nu}, \tag{3}
\]

\[
\partial_\nu \partial^\nu \theta = -\frac{\kappa}{2} F^{\mu\nu} \tilde{F}_{\mu\nu} + F \tag{4}
\]

where \( F = -\frac{\partial V}{\partial \theta} \). In addition, in accordance with its definition, \( \tilde{F}^{\mu\nu} \) satisfies the Bianchi identity

\[
\partial_\nu \tilde{F}^{\mu\nu} = 0. \tag{5}
\]

Substituting (2) into (3) one obtains the second order equation for potential \( A_\mu \):

\[
\partial_\nu \partial^\nu A^\mu = -\kappa p_\nu \tilde{F}^{\mu\nu} \tag{6}
\]

provided \( A_\mu \) satisfies the Lorentz gauge condition:

\[
\partial_\mu A^\mu = 0. \tag{7}
\]

Just the system of equations (3)–(5) will be the main subject of group classification. In addition, we shall discuss symmetries of field equations of the Chern-Simons electrodynamics, i.e., of the system including equations (3) and (4). In this theory \( p_\mu \) is treated as an external field whose motion equation is not specified, i.e., equation (4) is ignored.

**III. GROUP CLASSIFICATION OF SYSTEMS (3)–(5)**

Equations (3)–(5) include an arbitrary function \( F(\theta) \) so we can expect that the variety of symmetries of this system depends on the explicit form of \( F \). The group classification of these equations presupposes finding their symmetry groups for arbitrary \( F \).

In this section we present the results of group classification while the related calculations details are given in Appendix A.

The maximal continuous symmetry of system (3)–(5) with arbitrary function \( F(\theta) \) is given by Poincaré group \( P(1,3) \). On the set of solutions of equations (3)–(5) written as a seven component vector

\[
F = \text{column}(F^{01}, F^{02}, F^{03}, F^{23}, F^{31}, F^{12}, \theta) \tag{8}
\]
infinitesimal generators of this group take the following form:

\[ P_\mu = \partial_\mu, \quad J_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu + S_{\mu\nu} \]  

(9)

where indices \( \mu \) and \( \nu \) independently take values 0, 1, 2, 3,

\[
S_{ab} = \varepsilon_{abc} \begin{pmatrix} S_c & \cdots & \cdots \\ \cdots & S_c & \cdots \\ \cdots & \cdots & 0 \end{pmatrix}, \quad S_{0c} = \begin{pmatrix} S_c & \cdots & \cdots \\ \cdots & S_c & \cdots \\ \cdots & \cdots & 0 \end{pmatrix}, \quad a, b, c \neq 0,
\]

\( S_c \) are \( 3 \times 3 \) matrices of spin 1, whose entries are \( (S_c)_{ab} = i\varepsilon_{cab} \), \( \varepsilon_{cab} \) is the Levi-Chivita symbol, and the dots denote the zero matrices of an appropriate dimension.

Alternatively, spin matrices \( S_{\mu\nu} \) can be represented as the first order differential operators:

\[
S_{\mu\nu} = F_{\mu\lambda} \partial_F^{\lambda\nu} - F_{\nu\lambda} \partial_F^{\lambda\mu}.
\]  

(10)

Such notation is both convenient and usual for group analysis of differential equations.

Operators (9) form a basis of the Lie algebra \( p(1,3) \) of the Poincaré group.

For some special functions \( F(\theta) \) symmetry of system (3)–(5) appears to be more extended. Namely, if \( F = 0 \), \( F = c \) or \( F = b \exp(a\theta) \) then the basis (9) of symmetry algebra of this system is extended by the following additional operators \( P_4 \), \( D \) and \( X \):

\[
P_4 = \partial_\theta, \quad D = x_0 \partial_0 + x_i \partial_i - \frac{1}{2} F_{\mu\nu} \partial_F^{\mu\nu} \quad \text{if} \quad F(\theta) = 0,
\]

\[
P_4 = \partial_\theta \quad \text{if} \quad F(\theta) = c,
\]

\[
X = aD - 2P_4 \quad \text{if} \quad F(\theta) = be^{a\theta}.
\]  

(11)

Operator \( P_4 \) generates shifts of dependent variable \( \theta \), \( D \) is the dilatation operator generating a consistent scaling of dependent and independent variables, and \( X \) generates the simultaneous shift and scaling. Note that arbitrary parameters \( a, b \) and \( c \) can be reduced to the fixed values \( a = \pm 1 \), \( b = \pm 1 \) and \( c = \pm 1 \) by scaling dependent and independent variables.

Thus the continues symmetries of system (3)–(5) where \( F(\theta) \) is an arbitrary function of \( \theta \) are exhausted by the Poincaré group. The same symmetry is accepted by the standard equations of axion electrodynamics which correspond to \( F(\theta) = -m^2\theta \). In the cases indicated in (11) we have the extended Poincaré groups.

Notice that symmetries of the equations (6), (7), (5) for potentials also are generated by infinitesimal operators of the form (9) where

\[ S_{\mu\nu} = A_\mu \partial_A^\nu - A_\nu \partial_A^\mu. \]  

(12)

Additional symmetries again are given by equations (11) where, however, \( D \rightarrow x^\mu \partial_\mu - A^\mu \partial_A^\mu \).
IV. CONSERVATION LAWS

An immediate consequence of symmetries presented above is the existence of conservation laws. Indeed, the system (3)–(5) admits a Lagrangian formulation. Thus, in accordance with the Noether theorem, symmetries of equations (3)–(5) which keep the shape of Lagrangian (1) up to four divergence terms should generate conservation laws. Let us present them explicitly.

First we represent generators (9), (12) and (11) written in terms of the variational variables $A^\mu$ and $A^4 = \theta$ in the following unified form:

$$Q = \xi^\mu \partial_\mu + \varphi^\tau \partial_{A^\tau}$$

(13)

where the summation is imposed over the values $\tau = 0, 1, 2, 3, 4$ and $\mu = 0, 1, 2, 3$.

Conserved current corresponding to symmetry (13) can be represented as [22]:

$$J_\sigma = \varphi^\tau \frac{\partial L}{\partial (\partial_\sigma A^\tau)} + \xi^\sigma L - \xi^\nu \partial_\nu A^\tau \frac{\partial L}{\partial (\partial_\sigma A^\tau)}.$$  

(14)

The basic conserved quantity is the energy-momentum tensor which corresponds to symmetries $P_\mu$ presented in (9). In this case

$$\varphi^\tau \equiv 0 \quad \text{and} \quad \xi_\mu = 1$$

(15)

where $\mu$ successively takes the values 0, 1, 2, 3. Substituting (1), (15) into (14) and using three dimensional notations

$$F_{0a} = E_a, \quad F_{ab} = \varepsilon_{abc}B_c.$$  

(16)

we find the components of the conserved energy-momentum tensor in the following form:

$$T^{00} = \frac{1}{2}(E^2 + B^2 + p^2_0 + p^2) + V(\theta), \quad T^{0a} = \varepsilon_{abc}E_bB_c + p^0p^a,$$

$$T^{ab} = -E^aE^b - B^aB^b + p^a p^b + \frac{1}{2} \delta^{ab}(E^2 + B^2 + p^2_0 - p^2 - 2V(\theta)).$$  

(17)

The tensor $T^{\mu\nu}$ is symmetric and satisfies the continuity equation $\partial_\nu T^{\mu\nu} = 0$. Its components $T^{00}$ and $T^{0a}$ are associated with the energy and momentum densities.

It is important to note that the energy-momentum tensor does not depend on parameter $\kappa$ and so is not affected by the term $\frac{\kappa}{4} \theta F_{\mu\nu} \tilde{F}^{\mu\nu}$ present in Lagrangian (1). In fact this tensor is nothing but a sum of energy momenta tensors for the free electromagnetic field and scalar field. Moreover, the interaction of these fields between themselves is not represented in (17).
The conservation of tensor (17) is caused by the symmetry of Lagrangian (1) w.r.t. shifts of independent variables $x_\mu$. The symmetries w.r.t. rotations and Lorentz transformations give rise to conserving of the following tensor:

$$G^{\alpha\nu\mu} = x^\alpha T^{\mu\nu} - x^\nu T^{\mu\alpha}$$

(18)

which satisfies the continuity equation w.r.t. the index $\mu$. In particular, for $\alpha, \nu = 1, 2, 3$ equation (18) represents the angular momentum tensor.

The tensors (17)–(18) exhaust the conserved quantities whose existence is caused by the Lie symmetries of equations (3)–(5) with arbitrary function $F(\theta)$.

The additional symmetries presented in (11) are neither variational nor divergent symmetries of Lagrangian (1), and so they do not generate conservation laws. However, we can indicate another conservation laws which have nothing to do with Lie symmetries.

First let us note that equation (3) in itself can be written in the divergence form $\partial_\nu F^{\mu\nu} = 0$, where

$$F^{\mu\nu} = F^{\mu\nu} - \kappa \theta \tilde{F}^{\mu\nu}$$

is the antisymmetric conserved tensor. In addition, this equation can be rewritten as $j_\mu = 0$ where

$$j_\mu = \partial_\nu F^{\mu\nu} - \kappa p_\nu \tilde{F}^{\mu\nu}$$

(19)

is a conserved current. Changing equation (3) by (19) with $j_\mu \neq 0$ we obtain the system which represents the field equations of axion electrodynamics with nontrivial currents.

Equation (4) in its turn can be represented as

$$\partial_\mu J^\mu = F(\theta)$$

where

$$J^\mu = p^\mu + \kappa \tilde{F}^{\mu\nu} A_\nu.$$

(20)

If $F = 0$ then current (20) satisfies the continuity equation.

In addition to (17)–(20) there exist the infinite number of (trivial) conserved currents corresponding to the gauge symmetries of Lagrangian (1). An example of such conserved current is:

$$N^\mu = F^{\mu\nu} p_\nu \varphi(\theta)$$

(21)

where $\varphi(\theta)$ is an arbitrary differentiable function of $\theta$. Vector $N^\mu$ satisfies the continuity equation $\partial_\mu N^\mu = 0$ provided equations (3) are satisfied (remember that $p_\nu = \partial_\nu \theta$).
V. SELECTED EXACT SOLUTIONS

The field equations of axion electrodynamics form a rather complicated system of nonlinear partial differential equations. However, this system admits an extended symmetry algebra which makes it possible to find a number of exact solutions. Here we present some of these solutions while the completed list of them can be found in [23].

The algorithm for construction of group solutions of partial differential equations goes back to Sophus Lie and is expounded in various monographs, see, e.g., [22]. Roughly speaking, to find such solutions we have to change the dependent and independent variables by invariants of the subgroups of our equations symmetry group. Solving equations (3)–(5) it is reasonable to restrict ourself to three-parametrical subgroups of P(1,3) which enables to reduce (3)–(5) to systems of ordinary differential equations. The complete list of these subgroups can be found in [24].

To make solutions of equations (3)–(5) more physically transparent, we write them in terms of electric field $E$ and magnetic field $B$ whose components are expressed via the strengths tensor $F_{\mu\nu}$ as shown in (16). In addition, we rescale the dependent variables such that $\kappa \to 1$.

A. Plane wave solutions

Let us present solutions of system (3)–(5) which are invariant w.r.t. subalgebras of p(1,3) whose basis elements have the following unified form: $\langle P_1, P_2, kP_0 + \varepsilon P_3 \rangle$ where $\varepsilon$ and $k$ are parameters satisfying $\varepsilon^2 \neq k^2$, while $P_1$, $P_2$, $P_3$ and $P_0$ are generators given in (1).

The invariants $\omega$ of the corresponding three-parametrical group should solve the equations

$$P_1 \omega = 0, \quad P_2 \omega = 0, \quad (kP_0 + \varepsilon P_3)\omega = 0. \quad (22)$$

Solutions of (22) include all dependent variables $E_a, B_a, \theta$ ($a = 1,2,3$) and the only independent variable $\omega = \varepsilon x_0 - kx_3$. Thus we can search for solutions which are functions of $\omega$ only. As a result we reduce equations (3)–(5), (16) to the system of ordinary differential equations whose solutions are

$$B_1 = -kc_1\theta, \quad B_2 = \varepsilon c_1 + kc_2, \quad B_3 = c_3,$$

$$E_1 = \varepsilon c_2 + kc_1, \quad E_2 = \varepsilon c_1\theta, \quad E_3 = c_3\theta - c_4(\varepsilon^2 - k^2). \quad (23)$$
where \( c_1, \ldots, c_4 \) are arbitrary real numbers. The corresponding bounded solutions of equation (4) with \( F = -m^2 \theta \) are:

\[
\theta = a_\mu \cos \mu \omega + r_\mu \sin \mu \omega + \frac{c_3 c_4}{\mu^2}
\]  

(24)

where \( a_\mu, r_\mu \) and \( \mu \) are arbitrary constants restricted by the following constraint:

\[
\mu^2 = \left( \frac{c_1}{\varepsilon^2} + \frac{c_3^2}{\varepsilon^2} + m^2 \right)
\]  

(25)

Notice that for the simplest nonlinear function \( F = \lambda \theta^2 \) equation (4) is reduced to Weierstrass one and admits a nice soliton-like solution

\[
\theta = \frac{c_3 c_4}{2} \tanh^2 (\omega + C)
\]  

(26)

where \( C \) is an integration constant. The related parameters \( \varepsilon, k \) and \( \lambda \) should satisfy the conditions

\[
\varepsilon^2 = k^2 + \frac{c_3^2}{8 - c_1^2}, \quad \lambda c_3 c_4 = 12.
\]  

(27)

The corresponding magnetic, electric and axion fields are localized waves moving along the third coordinate axis.

In analogous (but as a rule much more complicated) way we can find solutions corresponding to the other three dimensional subalgebras of the Poincaré algebra. One more and rather specific solution of equations (3)–(5) with \( \kappa = 1 \) and \( F = 0 \) (obtained with using the subalgebra spanned on basis elements \( \langle J_{12} + k P_0 + \varepsilon P_1, P_2, P_3 \rangle \) can be written as follows:

\[
E_1 = \varepsilon (c_k \sin(\omega) - d_k \cos(\omega)), \quad E_2 = \varepsilon (c_k \cos(\omega) + d_k \sin(\omega)), \quad E_3 = e,
\]

\[
B_1 = -\frac{k}{\varepsilon} E_2, \quad B_2 = -\frac{k}{\varepsilon} E_1, \quad B_3 = 0, \quad \theta = \alpha x_0 + \nu x_3 + \mu
\]  

(28)

where \( e, c_k, d_k, \varepsilon, k, \alpha, \nu, \mu \) are constants satisfying the following conditions:

\[
\varepsilon^2 - k^2 = \nu \varepsilon - \alpha k, \quad \varepsilon \neq 0.
\]  

(29)

Solutions (28) depend on two different plane wave variables, i.e., \( \omega = \varepsilon x_0 - k x_1 \) and \( \alpha x_0 + \nu x_1 \). They satisfy the superposition principle since a sum of solutions with different \( \varepsilon, k, c_k \) and \( d_k \) is also a solution of equations (3)–(5) with \( \kappa = 1 \) and \( F = 0 \). Thus it is possible to generate much more general solutions by summing up functions (28) over \( k \) and treating \( c_k \) and \( d_k \) as arbitrary functions of \( k \).
B. Radial and planar solutions

Consider solutions which include the Coulomb electric field. They can be obtained using invariants of the subalgebra spanned on \(\langle J_{12}, J_{23}, J_{31}\rangle\) and have the following form:

\[
B_a = \frac{c_1 x_a}{r^3}, \quad E_a = \frac{(c_1 \theta - c_2)x_a}{r^3}, \quad \theta = \frac{\varphi}{r}, \quad (30)
\]

where \(\varphi\) is a function of \(x_0\) and \(r = \sqrt{x_1^2 + x_2^2 + x_3^2}\) satisfying the following equation:

\[
\frac{\partial^2 \varphi}{\partial r^2} - \frac{\partial^2 \varphi}{\partial x_0^2} = \left( \frac{c_1^2}{r^4} + m^2 \right) \varphi - \frac{c_1 c_2}{r^3}. \quad (31)
\]

Setting in (30) \(c_1 = 0\) we come to the electric field of point charge which is well defined for \(r > 0\). A particular solution for (31) corresponding to \(c_1 = -q^2 < 0\) and \(c_2 = 0\) is \(\varphi = c_3 r \sin(mx_0) e^{-\frac{x^2}{r^2}}\) which gives rise to the following field components:

\[
B_a = -\frac{q^2 x_a}{r^3}, \quad E_a = -\frac{q^2 \theta x_a}{r^3}, \quad \theta = c_3 \sin(mx_0) e^{-\frac{x^2}{r^2}}. \quad (32)
\]

The components of magnetic field \(B_a\) are singular at \(r = 0\) while \(E_a\) and \(\theta\) are bounded for \(0 \leq r \leq \infty\).

Separating variables it is possible to find the general solution of equation (31), see [23].

One more solution of equations (3)-(5) for \(F = 0\) with a radial electric field is:

\[
E_a = \frac{x_a}{r^2}. \quad (33)
\]

The corresponding magnetic and axion fields take the following forms:

\[
B_1 = \frac{x_1 x_3}{r^2 x}, \quad B_2 = \frac{x_2 x_3}{r^2 x}, \quad B_3 = -\frac{x}{r^2}, \quad \theta = \arctan \left( \frac{x}{x_3} \right)
\]

where \(x = \sqrt{x_1^2 + x_2^2}\).

The electric field (33) is requested in the superintegrable model with Fock symmetry proposed in [12].

Let us present planar solutions which depend on spatial variables \(x_1\) and \(x_2\). Namely, the functions

\[
E_1 = x_1 \left( c_1 x^{c_3-2} + c_2 x^{-2-c_3} \right), \quad E_2 = x_2 \left( c_1 x^{c_3-2} + c_2 x^{-2-c_3} \right), \quad E_3 = 0, \quad (34)
\]

\[
B_1 = x_2 \left( c_1 x^{c_3-2} - c_2 x^{-2-c_3} \right), \quad B_2 = x_1 \left( c_2 x^{-2-c_3} - c_1 x^{c_3-2} \right), \quad B_3 = 0, \quad (35)
\]

\[
\theta = c_3 \arctan \frac{x_2}{x_1} + c_4
\]
where \( c_1, \ldots, c_4 \) are arbitrary parameters, solve equations (3)–(5) with \( \kappa = 1 \) and \( F = 0 \).

In particular, for \( c_2 = 0, \ c_3 = 1 \) and \( c_2 = 0, \ c_3 = -1 \) we have:

\[
E_1 = -B_2 = \frac{c_1 x_1}{x}, \quad B_1 = E_2 = \frac{c_1 x_2}{x}, \quad B_3 = E_3 = 0, \quad \theta = \arctan \frac{x_2}{x_1}
\]

(36)

and

\[
E_1 = -B_2 = \frac{c_1 x_1}{x^3}, \quad B_1 = E_2 = \frac{c_1 x_2}{x^3}, \quad B_3 = E_3 = 0, \quad \theta = \arctan \frac{x_2}{x_1}.
\]

(37)

Solutions (34), (35) can be found with using invariants of a subgroup of the extended Poincaré group whose Lie algebra is spanned on the basis \( \{ P_0, P_3, J_{12}+P_4 \} \), see equations (9), (11) for definitions.

Let us present an example of solutions describing fields in a constantly charged space. The related equation (3) for \( \mu = 0 \) should be changed to

\[
\nabla \cdot E = p \cdot B + j_0
\]

(compare with (46)) where \( j_0 \) represents a constant charge density. The remaining equations (3)–(5) are kept uncharged, and the considered system is solved by the following fields:

\[
E_1 = \frac{1}{2} j_0 x_1 \ln x, \quad E_2 = \frac{1}{2} j_0 x_2 \ln x, \quad E_3 = B_3 = 0, \quad B_1 = \frac{1}{2} j_0 x_2 \left( \ln x - \frac{1}{4} \right), \quad B_2 = -\frac{1}{2} j_0 x_1 \left( \ln x - \frac{1}{4} \right),
\]

\[
\theta = 2 \arctan \frac{x_2}{x_1}.
\]

(38)

The fields (36) and (38) appears in superintegrable models for particles with spin 1 and \( \frac{3}{2} \) correspondingly [18] while the fields (37) are requested in the exactly solvable system described by the Dirac equation [11].

VI. PHASE, GROUP AND ENERGY VELOCITIES

In this section we consider some of the found solutions in more detail and discuss the propagation velocities of the corresponding fields. There are various notions of field velocities, see, e.g., [25], [26], [27]. We shall discuss the phase, group and energy velocities.

Let us start with the plane wave solutions given by equations (23) and (24). They describe oscillating waves moving along the third coordinate axis. Setting for simplicity \( c_2 = c_3 = c_4 = \)}
$r_{\mu} = 0$ we obtain:

$$
B_1 = c_1 k \theta, \quad B_2 = -c_1 \varepsilon, \quad B_3 = 0, \\
E_1 = -c_1 k, \quad E_2 = -c_1 \varepsilon \theta, \quad E_3 = 0, \quad \theta = a_\mu \cos(\mu (\varepsilon x_0 - k x_3)).
$$

(39)

Here $\varepsilon$, $k$, and $a_\mu$ are arbitrary parameters which, in accordance with (25), should satisfy the following dispersion relations:

$$(\varepsilon^2 - k^2)(\mu^2 - c_1^2) = m^2.$$  

(40)

If $m \neq 0$ the version $\mu^2 = c_1^2$ is forbidden, and we have two qualitatively different possibilities: $\mu^2 > c_1^2$ and $\mu^2 < c_1^2$.

Let $\mu^2 > c_1^2$ then $(\varepsilon^2 - k^2) = \frac{m^2}{\mu^2 - c_1^2} > 0$. The corresponding group velocity $V_g$ is equal to the derivation of $\varepsilon$ w.r.t. $k$, i.e.,

$$V_g = \frac{\partial \varepsilon}{\partial k} = \frac{k}{\varepsilon}.$$ 

(41)

Since $\varepsilon > k$, the group velocity appears to be less than the velocity of light (remember that we use the Heaviside units in which the velocity of light is equal to 1).

On the other hand the phase velocity $V_p = \frac{\varepsilon}{k}$ is larger than the velocity of light, but this situation is rather typical in relativistic field theories.

In the case $\mu^2 < c_1^2$ the wave number $k$ is larger than $\varepsilon$. As a result the group velocity (41) exceeds the velocity of light, and we have a phenomenon of superluminal motion. To understand whether the considered solutions are causal let us calculate the energy velocity which is equal to the momentum density divided by the energy density:

$$V_e = \frac{T_0^3}{T_{00}}.$$ 

(42)

Substituting (39) into (17) we find the following expressions for $T_{00}$ and $T_0^3$:

$$T_{00} = \frac{1}{2} (\varepsilon^2 + k^2) \Phi + \frac{1}{2} m^2 \theta^2, \quad T_0^3 = \varepsilon k \Phi$$

where $\Phi = c_1^2 (\theta^2 + 1) + \mu^2 (a_\mu^2 - \theta^2)$. Thus

$$V_e = \frac{2 \varepsilon k \Phi}{(\varepsilon^2 + k^2) \Phi + \frac{1}{2} m^2 \theta^2} < \frac{2 \varepsilon k}{\varepsilon^2 + k^2} < 1,$$

and this relation is valid for $\varepsilon > k$ and for $\varepsilon < k$ as well.

We see that the energy velocity is less than the velocity of light. Thus solutions (39) can be treated as causal in spite of the fact that for $\mu^2 < c_1^2$ the group velocity is superluminal.
Analogously, analyzing dispersion relations (29) we conclude, that the electromagnetic fields (29) propagate along the third coordinate axis with the group velocity

\[ V_g = \frac{\partial \varepsilon}{\partial k} = \frac{1}{\sqrt{1 + \delta}} \]  

were

\[ \delta = 2 \frac{\nu^2 - \alpha^2}{(2k - \alpha)^2}. \]  

If \(-1 < \delta < 0\) the group velocity (44) is larger than the velocity of light. However, the corresponding energy velocity (42) which is equal to

\[ V_e = \frac{2\varepsilon k + 2\nu \alpha}{\varepsilon^2 + k^2 + \nu^2 + \alpha^2} \]  

cannot exceed the velocity of light since \(2\varepsilon k \leq \varepsilon^2 + k^2\) and \(2\nu \alpha \leq \nu^2 + \alpha^2\).

Thus solutions (28) also can propagate with a superluminal group velocity. However, these solutions are causal since their energy velocities (45) are subluminal. Analogous results can be proven for soliton-like solutions (26).

VII. NONRELATIVISTIC LIMIT

To find a nonrelativistic limit of the field equations of axion electrodynamics we shall use the generalized IW contraction [21] which guarantees Galilean symmetry of the limiting theory.

Let us consider more general field equations including currents (see (19)), but restrict ourselves to the limiting case of the zero axionic mass. More explicitly, we start with the following system:

\[ \nabla \cdot \mathbf{E} = \kappa \mathbf{p} \cdot \mathbf{B} + j_0, \]  

\[ \partial_\nu \mathbf{E} - \nabla \times \mathbf{B} = \kappa (p_0 \mathbf{B} + \mathbf{p} \times \mathbf{E}) + \mathbf{j}, \]  

\[ \partial_\nu \mathbf{B} + \nabla \times \mathbf{E} = 0, \]  

\[ \nabla \cdot \mathbf{B} = 0, \]  

\[ \partial_\nu p_0 - \nabla \cdot \mathbf{p} = -\kappa \mathbf{E} \cdot \mathbf{B} + j_4, \]  

\[ \partial_\nu \mathbf{p} - \nabla p_0 = 0, \]  

\[ \nabla \times \mathbf{p} = 0. \]
If \( j_0 = j_4 = 0 \) and \( j = 0 \) then the subsystem (46)–(49) reduces to equations (3), (5), while the subsystem (50)–(52) becomes equivalent to (5) (remember that \( p_0 = \partial_0 \theta \) and \( \mathbf{p} = \nabla \theta \)).

Like (3)–(5) the system with currents, i.e., (46)–(52) is Poincaré invariant. The related representation of the Lie algebra of Poincaré group can be obtained by the prolongation of the basis elements (9) to the first derivatives of \( \theta \) and adding analogous terms acting on components of the current four-vector \( j = (j_0, j_1, j_2, j_3) \):

\[
\hat{P}_0 = \partial_0, \quad \hat{P}_a = \partial_a,
\]
\[
\hat{J}_{ab} = x_a \partial_b - x_b \partial_a + B^a \partial_{E^b} - B^b \partial_{E^a} + E^a \partial_{B^b} - E^b \partial_{E^a} + p^a \partial_{p^b} - p^b \partial_{p^a} + j^a \partial_{j^b} - j^b \partial_{j^a}, \quad (53)
\]
\[
\hat{J}_{0a} = x_0 \partial_a + x_a \partial_0 + \varepsilon_{abc} \left( E^b \partial_{B^c} - B^b \partial_{E^c} \right) + p^0 \partial_{p^a} - p^a \partial_{p^0} + j^0 \partial_{j^a} - j^a \partial_{j^0}.
\]

Generators (53) do not include differentials w.r.t. \( j^4 \) since this current component is not changed under Lorentz transformations (i.e., it should be scalar).

Being applied to basis elements of algebra \( \mathfrak{p}(1,3) \) the IW contraction consists of the transformation to a new basis

\[
\hat{J}_{ab} \rightarrow J'_{ab} = \hat{J}_{ab}, \quad \hat{J}_{0a} \rightarrow J'_{0a} = \varepsilon \hat{J}_{0a}, \quad \hat{P}_0 \rightarrow P'_0 = \varepsilon^{-1} \hat{P}_0, \quad \hat{P}_a \rightarrow P'_a = \hat{P}_a \quad (54)
\]

where \( \varepsilon \) is a small parameter equal to the inverse speed of light. In fact we deal with the generalized IW contraction since \( P'_0 \) is proportional to the inverse power of the small parameter.

In addition, the dependent and independent variables in (53) undergo the invertible transformations \( E^a \rightarrow E'^a, B^a \rightarrow B'^a, p^\mu \rightarrow p'^\mu, x^\mu \rightarrow x'^\mu \) where the primed quantities are functions of all the unprimed ones and of \( \varepsilon \). Moreover, the transformed quantities should depend on the contracting parameter \( \varepsilon \) in a tricky way, such that all transformed generators \( P'_\mu, J'_{ab}, J'_{0a} \) are kept nontrivial and nonsingular when \( \varepsilon \rightarrow 0 \).

The contractions of relativistic bi-vector fields (like \( \mathbf{E}, \mathbf{B} \)) and four-vectors \( p^\mu \) has been described in papers [28] and [29]. However, we need to contract simultaneously two subjects, i.e., the basis elements of the Lorentz algebra given by equations (53) and the system of equations (46)–(52), which is a much more sophisticated problem.

In order that transformation (54) be nonsingular in \( \varepsilon \) it should to be attended by the following transformations of the dependent and independent variables:

\[
x'_0 = t = \varepsilon x_0, \quad x'_a = x_a,
\]
\[
p'_0 = p_0, \quad \mathbf{p}' = \frac{\varepsilon}{2} (\mathbf{E} + \mathbf{p}), \quad \mathbf{E}' = \varepsilon^{-1} (\mathbf{E} - \mathbf{p}), \quad \mathbf{B}' = \mathbf{B}, \quad (55)
\]
\[
j'_0 = \varepsilon^{-1} (j_0 + j_4), \quad j'_4 = \frac{\varepsilon}{2} (j_0 - j_4), \quad \mathbf{j}' = \mathbf{j}.
\]

15
Making changes (55) in equations (53), applying transformation (54) and tending $\varepsilon \rightarrow 0$ we obtain the following set of first order differential operators which for a basis of the Galilei algebra:

\begin{align*}
P'_{\alpha} &= \partial_{\alpha}, \quad P'_{\alpha} = \partial_{\alpha}, \\
J'_{ab} &= x_a \partial_b - x_b \partial_a + B'^a \partial_{B'^b} - B'^b \partial_{B'^a} + E'^a \partial_{E'^b} - E'^b \partial_{E'^a} \\
&+ p'^a \partial_{p'^b} - p'^b \partial_{p'^a} + j'^a \partial_{j'^b} - j'^b \partial_{j'^a}, \\
J'_{0a} &= t \partial_{a} + \varepsilon_{abc} (p'^b \partial_{B'^c} - B'^b \partial_{E'^c}) + p'^0 \partial_{E'^a} - p'^a \partial_{p'^0} - j'^a \partial_{j'^0}.
\end{align*}

(56)

To obtain a consistent system of equations, we have to make changes (55) not directly in equations (46)–(52), but in the equivalent system which includes equation (47), (49), (51) and sums and half divergences of pairs of equation (46), (50) and (48), (52). Then equating terms with lowest powers of $\varepsilon$ we obtain the following system:

\begin{align*}
&\partial_t p'_0 + \nabla \cdot E' + \kappa B' \cdot E' = j'_0, \\
&\partial_t p' - \nabla \times B' - \kappa (p'_0 B' + p' \times E') = j', \\
&\nabla \cdot p' - \kappa p' \cdot B' = j'_4, \\
&\nabla \cdot B' = 0, \\
&\partial_t B' + \nabla \times E' = 0, \\
&\partial_t p' - \nabla p'_0 = 0, \quad \nabla \times p' = 0
\end{align*}

(57)

and $p'^0 = \partial_t \theta'$, $p' = \nabla \theta'$.

Just equations (57) present the nonrelativistic limit of system (46)–(52). The Galilei invariance of system (57) can be proven directly using the following transformation laws which can be found by integrating the Lie equations for generators (56):

\begin{align*}
x &\rightarrow x + v t, \quad t \rightarrow t, \\
p'_0 &\rightarrow p'_0 - v \cdot p', \quad p' \rightarrow p', \quad B' = B' + v \times p', \\
E' &\rightarrow E' - v \times B' + v p'_0 + v (v \cdot p') - \frac{1}{2} v^2 p'.
\end{align*}

(58)

Thus we find the nonrelativistic limit for field equations of axion electrodynamics with zero axion mass and nontrivial currents. It is interesting to note that system (57) has been discovered in paper [16], starting with the requirement of Galilei invariance for abstract vector fields. Thus our contraction procedure presents a physical interpretation for the Galilei invariant system for the indecomposable ten component field deduced in [16], see equation (67) there. Namely, this Galilei invariant system is nothing but a nonrelativistic limit of the field equations of axion electrodynamics.
VIII. DISCUSSION

The aim of the present paper is multifold. First we make group classification of field equations of axion electrodynamics \((3)-(5)\) which include an arbitrary function \(F\) depending on \(\theta\), and find the conservation laws generated by these equations. Secondly, we find all continuous symmetries of Chern-Simon electrodynamics and CFJ model which are closely related to axion electrodynamics. At the third place, we discuss exact solutions for axion and e.m. fields which can be obtained using found symmetries. Finally, we define a correct nonrelativistic limit of the field equations of axion electrodynamics.

As a result of the group classification we prove that the Poincaré invariance is the maximal symmetry of the standard axion electrodynamics and indicate the special forms of \(F\) for which the theory admits more extended symmetries. In accordance with \((11)\) such an extension appears for trivial, constant and exponential interaction terms.

In spite of the transparent Poincaré invariance of the analyzed equations, these results are nontrivial, since possible extensions \((11)\) of the basic symmetries and the absence of other ones were not evident a priori. Our results form group-theoretical grounds for constructing of various axionic models. The detailed calculations of Lie symmetries of equations \((3)-(5)\) are presented in Appendix A.

In addition, in Appendix B we describe Lie symmetries of classical electrodynamics modified by Chern-Simons term (i.e., symmetries of the subsystem \((3), (5)\) where \(p_\mu\) is treated as an external field), and symmetries of CFJ electrodynamics. It is proven that the maximal continuous symmetry of the Chern-Simons electrodynamics is given by the 17-parametrical extended conformal group.

Conservation laws and the corresponding symmetries of the CFJ model have been already studied in paper \([17]\). However, the symmetries which generate conservation laws are not the half of all Lie symmetries \([22]\). Our research justifies and completes the results of \([17]\). In particular we prove the completeness of the list of space-time symmetries theory presented in paper \([17]\), and add this list by symmetry \(D_1\) given by equation \((B10)\).

Analyzing conservation laws caused by equations \((3)-(5)\) we prove that the energy-momentum tensor does not depend on interaction between the electromagnetic and axion fields and indicate that the additional symmetries \((11)\) do not generate conservation laws. The first statement generalizes the observation present in book \([30]\) to the case of arbitrary self interaction.
of axionic field.

Exact solutions of the field equations of axion electrodynamics are discussed in Sections 5 and 6. There is a sufficiently large amount of group solutions for these equations, but we restrict ourselves only to those which can be potentially important to physical applications.

Analyzing the plane wave solutions we recognize the possibility of the faster than light group velocities in axion electrodynamics. However the corresponding energy velocities are subluminal and do not lead to causality violation.

Let us note that functions (28) solve the field equations of CFJ electrodynamics which coincide with the system (3), (5) where $p_{\mu}$ are constants. The existence of faster than light solutions for these equations was indicated in paper [9], the enhanced discussion of the plane wave solutions for (3), (5) can be found in [31]. However, in CFJ electrodynamics the energy is not positive definite [9] while the energy density of axion electrodynamics, discussed in section 5, is positive. In addition, in the CFJ theory variables $p_{\mu}$ represent an external field while in our approach they are dynamical variables satisfying equations (4).

An important aspect of the presented exact solutions is that they are requested in some problems of nonrelativistic and relativistic quantum mechanics. In particular, as it was indicated in [11], just the vectors of the electric and magnetic fields described by relations (37) give rise to exactly solvable Dirac equation for a charged particle anomalously interacting with these fields. These fields are involved also into the superintegrable nonrelativistic system discussed in [15]. The electric field given by equation (33) was used in [12] to construct a new exactly solvable QM model with Fock symmetry. The fields (36) and (38) are requested in superintegrable models for particles with higher spins [18]. Using the formalism presented in [32] it is possible to construct also relativistic versions of these models.

Summarizing, the presented exact solutions are not nice mathematical toys only. In contrary, they have a nontrivial physical content. These solutions give rise to exactly solvable QM systems enumerated in the previous paragraph, which in principle can be used to predict physically verifiable effects. That opens ways for finding new arguments for the real existence of the pseudoscalar axion field.

Solutions discussed in Section 4 represent only a part of exact solutions which can be obtained using three dimensional subalgebras of algebra p(1,3). The complete list of them can be found in [23]. Notice that among these solutions there are rather general ones which include six arbitrary functions [23].
A special goal of this paper was to present a correct nonrelativistic limit of equations of axion electrodynamics. To achieve this goal we use the generalized IW contraction of the corresponding representation of the Poincaré group. As a result we prove that the limiting case of these equations is nothing but the Galilei-invariant system for the ten-component vector field obtained earlier in paper [16]. This result casts light on the physical content of the model discussed in [16]. In addition, the contraction procedure can be used for transforming solutions discussed in Sections 4, 5 (and other solutions found in [23]) to solutions of system (57).

Some results of this paper had been announced in conferences, see Proceedings [33].

ACKNOWLEDGEMENTS

This work was partially supported by research program Cosmomicrophysics of the National Academy of Sciences of Ukraine (state registration number 0109U003207).

Appendix A: LIE SYMMETRIES of equations (3)–(5)

The group analysis of differential equations is a nice field of mathematics whose fundamentals had been created by Sophus Lie as long ago as in the end of the eighteenth century. The foundations of the group analysis are expounded in many books, the most popular of which is monograph [22]. Nevertheless we will present a very short sketch of the Lie algorithm addressed to more physically oriented readers.

The basic idea of the classical Lie algorithm is to treat a differential equation (or a system of equations) as a manifold in the multidimensional Euclidean space whose basis elements include the dependent and independent variables and also the derivatives present in the equation. In our case we have four independent variables \(x^0, x^1, x^2, x^3\), seven dependent variables \(F^\mu{}_{\nu}, \theta\) and 38 differential variables \(F^\mu{}_{\lambda\nu} = \partial_\lambda F^\mu{}_{\nu}, \theta^\nu = \partial_\nu \theta\) and \(\theta^\mu = \partial_\mu \theta\). Equations (3)–(5) define a hypersurface \(\mathcal{F}\) in the Euclidean space \(\mathbb{R}^{49}\) with variables \(x^\mu, F^\mu{}_{\nu}, \theta, F^\mu{}_{\lambda\nu}, \theta^\nu, \theta^\mu\). Moreover, \(\mathcal{F}\) is a smooth manifold in \(\mathbb{R}^{49}\).

Local continuous symmetries of system (3)–(5) can be treated as continuous transformations in \(\mathbb{R}^{49}\) which keep this manifold invariant. To find these symmetries it is possible to imply the infinitesimal invariance criterium, i.e., consider transformations close to the identity one:

\[
x^\mu \to x'^\mu = x^\mu + \varepsilon \xi^\mu, \quad F^{\mu\nu} \to F'^{\mu\nu} = F^{\mu\nu} + \varepsilon \eta^{\mu\nu}, \quad \theta \to \theta' = \theta + \varepsilon \sigma
\]  

(A1)
where $\varepsilon$ is a small transformation parameter and $\xi^\mu$, $\eta^{\mu\nu}$, $\sigma$ are some functions of $x^\mu$, $F^{\mu\nu}$ and $\theta$, and ask for the form invariance of equations (3)–(5) w.r.t. the change of variables (A1)).

Transformations (A1) can be formally represented as

$$ f \to f' = (1 + \varepsilon Q)f $$

where $f$ is a vector whose components are independent variables $x^\mu$ and dependent variables $F^{\mu\nu}$, $\theta$, and $Q$ is the infinitesimal operator:

$$ Q = \xi^\mu \partial_\mu + \frac{1}{2} \eta^{\mu\nu} \partial_{F^{\mu\nu}} + \sigma \partial_\theta. \quad (A2) $$

Starting with (A1) we can find transformations for the differential variables:

$$ F^{\mu\nu} \to F'^{\mu\nu} = F^{\mu\nu} + \varepsilon \eta^{\mu\nu}, \quad \theta^\mu \to \theta'^\mu = \theta^\mu + \varepsilon \sigma^\mu, \quad \theta^{\mu\nu} \to \theta'^{\mu\nu} = \theta^{\mu\nu} + \varepsilon \sigma^{\mu\nu} \quad (A3) $$

where

$$ \eta^\mu_\sigma = D_\mu(\eta^{\nu\sigma}) - F^{\nu\sigma}_\lambda D_\mu(\xi^\lambda), \quad \sigma_\mu = D_\mu(\sigma) - \theta^\lambda D_\mu(\xi^\lambda), $$

$$ \sigma_{\nu\mu} = D_\nu(\sigma_\mu) - \theta^{\lambda\mu} D_\nu(\xi^\lambda) $$

and $D_\mu = \partial_\mu + \frac{1}{2} F^{\mu\sigma}_\nu \partial_{F^{\nu\sigma}} + \theta^{\nu\sigma} \partial_\theta + \theta^{\mu\sigma} \partial_{\theta^{\nu\sigma}}$.

Using (A3), the invariance condition for system (3)–(5) can be written in the following form:

$$ Q(2) F|_{F=0} = 0 \quad (A4) $$

where $Q(2)$ is the infinitesimal operator (A2) prolonged to the first and second derivatives:

$$ Q(2) = Q + \frac{1}{2} \eta^{\mu\sigma}_\nu \partial_{F^{\sigma\mu}} + \sigma_\mu \partial_{\theta^\mu} + \frac{1}{2} \sigma_{\mu\nu} \partial_{\theta^{\mu\nu}} \quad (A5) $$

and $F$ is the manifold defined by relations (3)–(5).

Acting by operator (A5) on differential forms (3)–(5) and equating coefficients for linearly independent functions $F^{\mu\nu}$, $\theta$ and their derivatives we obtain the following system of determining equations for the coefficients $\xi^\mu$, $\eta^{\mu\nu}$ and $\sigma$:

$$ \xi_{F^{\nu\lambda}}^\mu = 0, \quad \xi_\theta^\mu = 0, \quad \xi_\theta^{\mu\nu} + \xi^{\mu}_{\nu} = \frac{1}{2} \delta^{\mu\nu} \xi_{\sigma}, \quad (A6) $$

$$ \sigma_{F^{\mu\nu}} = 0, \quad \sigma_{\theta^\theta} = 0, \quad (A7) $$

$$ \partial_\mu \partial^{\mu} \sigma + \left( \sigma_\theta - \frac{1}{2} \xi^{\mu}_{\mu} \right) \left( F + \frac{\kappa}{2} F^{\mu\nu} \tilde{F}^{\mu\nu} \right) - \frac{\kappa}{2} \eta^{\mu\nu} \tilde{F}^{\mu\nu} - \sigma F_\theta = 0, \quad (A8) $$

20
\[ \eta_{\mu\nu} = \tilde{F}^{\mu\nu} \sigma_{\nu}, \quad \varepsilon_{\alpha\sigma} \eta_{\mu}^{\sigma} = 0, \quad 2\sigma_{\theta, \mu} = \partial_{\nu} \partial^{\nu} \xi_{\mu}, \]
\[ \eta_{\mu} - \frac{1}{2} \varepsilon_{\mu\sigma} \eta_{\theta}^{\sigma} + F_{\mu\nu} \sigma_{\theta} + F_{\nu\alpha} \xi_{\alpha}^{\mu} - F_{\mu\alpha} \xi_{\alpha} = 0, \quad (A9) \]
\[ \xi_{\mu} + \eta_{\lambda\mu} = 0, \quad \eta_{\lambda\nu} = \eta_{\lambda\mu}^{\mu}, \quad \eta_{\lambda\nu} = F_{\mu\nu} \eta_{\lambda\mu}^{\mu} \]

where the subscripts \( F_{\nu\lambda} \) and \( \theta \) denote the derivatives with respect to the corresponding variables: \( \xi_{\mu} = \frac{\partial}{\partial x_{\mu}} \), \( \eta_{\mu}^{\mu} = \frac{\partial}{\partial \theta_{\mu}} \), etc., \( \delta_{\mu}^{\nu} \) is the Kronecker symbol and there are no sums over the repeating indices in the last line of equation (A9).

In accordance with equations (A6) functions \( \xi_{\mu} \) do not depend on \( F_{\mu\nu} \) and \( \theta \) and, moreover, they are Killing vectors in the space of independent variables:

\[ \xi_{\mu} = 2x_{\mu} f_{\nu} x_{\nu} - f_{\mu} x_{\nu} x_{\nu} + \varepsilon_{\mu} x_{\nu} + \xi_{\mu} = 0 \quad (A10) \]

where \( f_{\mu}, d, e^{\mu} \) and \( c^{\mu\nu} = -c^{\nu\mu} \) are arbitrary constants.

It follows from (A7) that \( \sigma = \varphi_{1} \theta + \varphi_{2} \), where \( \varphi_{1} \) and \( \varphi_{2} \) are functions of \( x_{\mu} \). Substituting this expression into (A8) we obtain the following equation:

\[ \varphi_{1} \theta F_{0} + \varphi_{2} F_{0} + (2\varepsilon_{0} - \varphi_{1}) \left( F + \frac{\kappa}{2} F_{\mu\nu} \tilde{F}^{\mu\nu} \right) + \kappa \eta^{\mu\nu} \tilde{F}_{\mu\nu} - \theta \partial_{\nu} \partial^{\nu} \varphi_{1} - \partial_{\nu} \partial^{\nu} \varphi_{2} - 2p^{\mu} \partial_{\mu} \varphi_{1} = 0. \quad (A11) \]

Let the terms \( \theta F_{0}, F_{0}, F, \) and \( 1 \) be linearly independent. Then it follows from (A11) and (A10) that \( \varphi_{1} = \varphi_{2} = 0, f^{\nu} = 0 \) and \( \eta^{\mu\nu} \tilde{F}_{\mu\nu} = 0 \). Thus, using (A9) and (A10) we obtain:

\[ \xi_{\mu} = \varepsilon_{\mu} x_{\nu} + e^{\mu}, \quad \eta^{\mu\nu} = c^{\nu} F^{\alpha}_{\nu} - c^{\alpha} F^{\mu}_{\alpha}, \quad \sigma = 0. \quad (A12) \]

Substituting (A12) into (A2) we receive a linear combination of infinitesimal operators (9) which form a basis of the Lie algebra of Poincaré group P(1,3). Thus the group P(1,3) is the maximal continuous symmetry group of system (3)–(5) with arbitrary function \( F(\theta) \).

The possible extensions of this symmetry which appears for some particular functions \( F \) are enumerated in equations (11).

**Appendix B: Symmetries of Chern-Simons and Carrol-Field-Jackiw models**

Let us discuss symmetries of two models which are closely related to the axion electrodynamics. The first is the classical electrodynamics modified by adding the Chern-Simons terms. It is based on field equation (3), (5), but does not include the dynamical equation for \( p_{\mu} \) which is treated as an external field.
Let us suppose that vector $p_{\mu}$ can be presented as a four-gradient of some scalar function $\theta$. Then, repeating the procedure used above we again come to the determining equations \(A6\), \(A7\), \(A9\) while equation \(A7\) would be absent. Solving these determining equations we obtain the 17-dimensional symmetry algebra whose basis elements are given by equations \(9\) and by the following equations:

\[
\begin{align*}
D_1 &= F^{\mu\nu} \partial_{F^{\mu\nu}}, \\
D_2 &= x^{\mu} \partial_{\mu} - D_1, \\
K_{\mu} &= 2x_{\mu}D_2 - x_{\nu}x^{\nu} \partial_{\mu} + 2x^{\nu}S_{\mu\nu}, \\
P_4 &= \frac{\partial}{\partial \theta}
\end{align*}
\]

where $S_{\mu\nu}$ are operators defined by equation \((10)\).

Operators \(9\) and \(B1\) form a basis of the Lie algebra of conformal group, while operators $P_4$ and $D_2$ belong to the central extension of this algebra.

Thus the field equations of Chern-Simons electrodynamics are invariant w.r.t. the conformal group and admit two additional symmetries, i.e., scaling of the strength tensor of electromagnetic field and shifts of function $\theta$.

If vector field $p_{\mu}$ is not supposed a priori to be a four-gradient, we come to the 16-dimensional symmetry algebra of equations \(3\), \(5\). The explicit form of its basis elements can be obtained from \(9\) and \(B1\) by the change $S_{\mu\nu} \rightarrow \tilde{S}_{\mu\nu}$ and $D_2 \rightarrow \tilde{D}_2$ where

\[
\tilde{S}_{\mu\nu} = S_{\mu\nu} + p_{\mu} \partial_{\nu} - p_{\nu} \partial_{\mu}, \quad \tilde{D}_2 = D_2 - p_{\mu} \partial_{\mu}.
\]

The second model which we consider here is the Carrol-Field-Jackiw one \[9\]. This is a particular version of the Chern-Simons electrodynamics corresponding to a constant vector $p_{\mu}$. Considering equations \(3\), \(5\) with the additional condition

\[
\partial_{\nu}p_{\mu} = 0,
\]

we conclude that in this case the symmetry algebra is reduced to $\tilde{p}(1,3) \oplus A_1$ where $\tilde{p}(1,3)$ is the extended Poincaré algebra whose basis elements are operators \(9\) and operator $\tilde{D}_2$, while $A_1$ is the one-dimensional algebra spanned on $D_1$. The reason of this reduction is that equations \(B3\) are not invariant w.r.t. the conformal transformations generated by operators $K_{\mu}$.

Thus the system \(3\), \(5\), \(B3\) is not invariant w.r.t. the conformal transformations, but it is still invariant w.r.t. the extended Poincaré group $\tilde{P}(1,3)$ which includes the ordinary Poincaré group and accordant scalings of dependent and independent variables. Equations \(B3\) require
that \( p_\mu \) are (arbitrary) constants. If these constants are fixed, the symmetry group \( P(1,3) \) is reduced to its subgroup which keeps them invariant. Namely, for time-like, space-like and light-like vectors \( p = (p_0, p_1, p_2, p_3) \) we can set

\[
p = (\mu, 0, 0, 0), \quad (B4)
\]
\[
p = (0, 0, \mu, 0), \quad (B5)
\]

and

\[
p = (\mu, 0, 0, \mu) \quad (B6)
\]
correspondingly. Then the subgroups of \( P(1,3) \) which keep (B4), (B5) or (B6) invariant are the Euclid group \( E(3) \), the Poincaré group \( P(1,2) \) in \( (1+2) \)-dimensional space or the extended Galilei group \( \tilde{G}(1,2) \) respectively. The basis elements of the corresponding Lie algebras are

\[
\langle P_1, P_2, P_3, J_{12}, J_{23}, J_{31} \rangle, \quad (B7)
\]
\[
\langle P_1, P_2, P_3, J_{12}, J_{01}, J_{02} \rangle
\]
or

\[
\langle P_1, P_2, P_3, J_{12}, G_1 = J_{01} + J_{31}, G_2 = J_{02} + J_{32}, D_3 = \tilde{D}_2 + J_{03} \rangle \quad (B8)
\]

where \( P_\alpha, J_{\mu\nu} \) and \( D_2 \) are generators given in (9) and (B1). In addition, in all cases (B4)–(B6) there are additional symmetries

\[
P_0 = \frac{\partial}{\partial x_0} \quad (B9)
\]
and

\[
D_1 = F^{\mu\nu} \partial_{F^{\mu\nu}}. \quad (B10)
\]

The list of symmetries (B7) – (B8) had been found in paper [17] starting with conservation laws for equations (3) and (5). Our approach is more straightforward and guarantees finding of all Lie symmetries including nonvariational ones. An example of such nonvariational symmetry which cannot be found with approach used in [17] is the dilatation whose generator \( D_1 \) is given by equation (B10).

[1] R. D. Peccei and H. R. Quinn, Phys. Rev. Lett. 38, 1440 (1977).
[2] S. Weinberg, Phys. Rev. Lett. 40, 223 (1978).
[3] F. Wilczek, Phys. Rev. Lett. 40, 279 (1978).
[4] F. Wilczek, Phys. Rev. Lett. 58, 1799 (1987).
[5] W.-T. Ni, Bull. Am. Phys. Soc. 19, 655 (1974).
[6] G. G. Raffelt, Phys. Rep. 198, 1 (1990).
[7] X-L. Qi, T. L. Hughes, and S-C. Zhang, Phys. Rev. B 78, 195424 (2008).
[8] F. W. Hehl, Y.N. Obukhov J.-P. Rivera and H. Schmid, Eur. Phys. J. B 71, 321329 (2009)
[9] S. M. Carroll, G. B. Field and R. Jackiw, Phys. Rev D 41 1231 (1990).
[10] S.S. Chern and J. Simons, Annals Math. 99 48 (1974).
[11] E. Ferraro, A. Messina and A.G. Nikitin, Phys. Rev. A 81, 042108 (2010).
[12] A. G. Nikitin, arXiv:1205.3094 (2012).
[13] A. G. Nikitin and Y. Karadzhov, J. Phys. A: 44 (2011) 305204.
[14] A. G. Nikitin and Y. Karadzhov, J. Phys. A: 44 (2011) 445202.
[15] A.G. Nikitin, arXiv:1204.5902v2 (2012).
[16] J. Niederle and A.G. Nikitin, J. Phys. A: Math. Theor. 42 105207 (2009).
[17] A.J. Hariton and R. Lehnert, Phys. Lett. A 367 11 (2007).
[18] A. G. Nikitin, J. Phys. A: Math. Theor. 45 (2012) 225205.
[19] M. Le Bellac and J.-M. Lévy-Leblond, Nuovo Cimento B 14, 217 (1973).
[20] P. Holland and H.R. Brown, Studies in History and Philosophy of Science 34, 161, (2003).
[21] E. Inönü and E.P. Wigner, Proc. Nat. Acad. Sci. US 39 510 (1953).
[22] P. Olver, Application of Lie groups to Differential equations (Second Edition, Springer-Verlag, New York, 2000), electronic version: PJ Olver - 2000 - books.google.com
[23] Oksana Kuriksha and A.G. Nikitin, arXiv:1002.0064v4 (2012),
http://dx.doi.org/10.1016/j.cnsns.2012.04.009 (2012).
[24] J. Patera, P. Winternitz, and H. Zassenhaus, J. Math. Phys. 16, 1597 (1975).
[25] L. Brillouin, Wave Propagation and Group Velocity (Academic, New York, 1960).
[26] R. L. Smith, Am. J. Phys. 38, 978-983 (1970).
[27] S. C. Bloch, Am. J. Phys. 45, 538 (1977).
[28] M. de Montigny, J. Niederle and A.G. Nikitin, J. Phys. A: Math. Theor. 39 9365 (2006).
[29] J. Niederle and A.G. Nikitin, Czech. J. Phys. 56 1243 (2006).
[30] F.W. Hehl and Y.N. Obukhov, Foundations of Classical Electrodynamics – Charge, Flux and
Metric (Birkhauser 2003).

[31] Y. Itin, Phys. Rev. D 76 087505 (2007); Y. Itin. Gen. Rel. Grav. 40 1219 (2008).

[32] J. Niederle and A. G. Nikitin, Phys. Rev. D 64 125013 (2001).

[33] J. Niederle, A. G. Nikitin and O. Kuriksha, In: Proceedings of the Fifth International Workshop "Group Analysis of Differential Equations and Integrable Systems", June 6-10, 2010, Protaras, Cyprus, University of Cyprus, Nikosia, 2011, pp. 152-163;
J. Niederle, A. G. Nikitin and O. Kuriksha, Acta Polytechnica 50 96 (2010).