ON THE BOAS-BELLMAN INEQUALITY IN INNER PRODUCT SPACES

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Abstract. New results related to the Boas-Bellman generalisation of Bessel’s inequality in inner product spaces are given.

1. Introduction

Let \((H; \langle \cdot, \cdot \rangle)\) be an inner product space over the real or complex number field \(\mathbb{K}\). If \((e_i)_{1 \leq i \leq n}\) are orthonormal vectors in the inner product space \(H\), i.e., \(\langle e_i, e_j \rangle = \delta_{ij}\) for all \(i, j \in \{1, \ldots, n\}\) where \(\delta_{ij}\) is the Kronecker delta, then the following inequality is well known in the literature as Bessel’s inequality (see for example [6, p. 391]):

\[
\sum_{i=1}^{n} |\langle x, e_i \rangle|^2 \leq \|x\|^2 \quad \text{for any} \quad x \in H.
\]

For other results related to Bessel’s inequality, see [3] – [5] and Chapter XV in the book [6].

In 1941, R.P. Boas [2] and in 1944, independently, R. Bellman [1] proved the following generalisation of Bessel’s inequality (see also [6, p. 392]).

Theorem 1. If \(x, y_1, \ldots, y_n\) are elements of an inner product space \((H; \langle \cdot, \cdot \rangle)\), then the following inequality:

\[
\sum_{i=1}^{n} |\langle x, y_i \rangle|^2 \leq \|x\|^2 \left[ \max_{1 \leq i \leq n} \|y_i\|^2 + \left( \sum_{1 \leq i \neq j \leq n} |\langle y_i, y_j \rangle|^2 \right)^{\frac{1}{2}} \right]
\]

holds.

A recent generalisation of the Boas-Bellman result was given in Mitrović-Pečarić-Fink [6, p. 392] where they proved the following.

Theorem 2. If \(x, y_1, \ldots, y_n\) are as in Theorem 1 and \(c_1, \ldots, c_n \in \mathbb{K}\), then one has the inequality:

\[
\left| \sum_{i=1}^{n} c_i \langle x, y_i \rangle \right|^2 \leq \|x\|^2 \sum_{i=1}^{n} |c_i|^2 \left[ \max_{1 \leq i \leq n} \|y_i\|^2 + \left( \sum_{1 \leq i \neq j \leq n} |\langle y_i, y_j \rangle|^2 \right)^{\frac{1}{2}} \right].
\]

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They also noted that if in (1.2) one chooses \( c_i = (x, y) \), then this inequality becomes (1.2).

For other results related to the Boas-Bellman inequality, see [4].

In this paper we point out some new results that may be related to both the Mitrinović-Pečarić-Fink and Boas-Bellman inequalities.

2. SOME PRELIMINARY RESULTS

We start with the following lemma which is also interesting in itself.

**Lemma 1.** Let \( z_1, \ldots, z_n \in H \) and \( \alpha_1, \ldots, \alpha_n \in \mathbb{K} \). Then one has the inequality:

\[
\left\| \sum_{i=1}^{n} \alpha_i z_i \right\|^2 \leq \begin{cases} 
\max_{1 \leq i \leq n} |\alpha_i|^2 \sum_{i=1}^{n} \|z_i\|^2; \\
\left( \sum_{i=1}^{n} |\alpha_i|^{2\alpha} \right)^{\frac{1}{\alpha}} \left( \sum_{i=1}^{n} \|z_i\|^{2\beta} \right)^{\frac{1}{\beta}}, \text{ where } \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\
\sum_{i=1}^{n} |\alpha_i|^2 \max_{1 \leq i \leq n} \|z_i\|^2, \\
\left[ \left( \sum_{i=1}^{n} |\alpha_i|^2 \right)^{\frac{1}{2}} - \left( \sum_{i=1}^{n} |\alpha_i|^{2\gamma} \right)^{\frac{1}{2}} \right] \left( \sum_{1 \leq i \neq j \leq n} |(z_i, z_j)|^{2\delta} \right)^{\frac{1}{\delta}}, \text{ where } \gamma > 1, \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\
\left[ \left( \sum_{i=1}^{n} |\alpha_i|^2 \right)^{\frac{1}{2}} - \left( \sum_{i=1}^{n} |\alpha_i|^2 \right)^{\frac{1}{2}} \right] \max_{1 \leq i \neq j \leq n} |(z_i, z_j)|. 
\end{cases}
\]

**Proof.** We observe that

\[
\left\| \sum_{i=1}^{n} \alpha_i z_i \right\|^2 = \left( \sum_{i=1}^{n} \alpha_i z_i, \sum_{j=1}^{n} \alpha_j z_j \right) = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \overline{\alpha_j} (z_i, z_j) = \left| \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \overline{\alpha_j} (z_i, z_j) \right| \leq \sum_{i=1}^{n} \sum_{j=1}^{n} |\alpha_i| |\alpha_j| |(z_i, z_j)| = \sum_{i=1}^{n} |\alpha_i|^2 \|z_i\|^2 + \sum_{1 \leq i \neq j \leq n} |\alpha_i| |\alpha_j| |(z_i, z_j)|.
\]
Using Hölder’s inequality, we may write that

\[(2.3) \quad \sum_{i=1}^{n} |\alpha_i|^2 \|z_i\|^2 \leq \max_{1 \leq i \leq n} |\alpha_i|^2 \sum_{i=1}^{n} \|z_i\|^2 ; \]

\[\leq \left\{ \left( \sum_{i=1}^{n} |\alpha_i|^{2\alpha} \right)^{\frac{1}{\alpha}} \left( \sum_{i=1}^{n} \|z_i\|^{2\beta} \right)^{\frac{1}{\beta}} \right\}, \text{ where } \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \]

\[\sum_{i=1}^{n} |\alpha_i|^2 \max_{1 \leq i \leq n} \|z_i\|^2 . \]

By Hölder’s inequality for double sums we also have

\[(2.4) \quad \sum_{1 \leq i \neq j \leq n} |\alpha_i| |\alpha_j| |(z_i, z_j)| \]

\[\leq \left\{ \max_{1 \leq i \neq j \leq n} |\alpha_i, \alpha_j| \sum_{1 \leq i \neq j \leq n} |(z_i, z_j)| ; \]

\[\left\{ \left( \sum_{1 \leq i \neq j \leq n} |\alpha_i|^\gamma |\alpha_j|^\gamma \right)^{\frac{1}{\gamma}} \left( \sum_{1 \leq i \neq j \leq n} |(z_i, z_j)|^\delta \right)^{\frac{1}{\delta}} \right\}, \text{ where } \gamma > 1, \frac{1}{\gamma} + \frac{1}{\delta} = 1; \]

\[\sum_{1 \leq i \neq j \leq n} |\alpha_i| |\alpha_j| \max_{1 \leq i \neq j \leq n} |(z_i, z_j)| , \]

\[\left\{ \max_{1 \leq i \neq j \leq n} \{|\alpha_i, \alpha_j|\} \sum_{1 \leq i \neq j \leq n} |(z_i, z_j)| ; \]

\[\left\{ \left[ \left( \sum_{i=1}^{n} |\alpha_i|^{2\gamma} \right)^{\frac{1}{2\gamma}} - \left( \sum_{i=1}^{n} |\alpha_i|^{2\gamma} \right) \right] \left( \sum_{1 \leq i \neq j \leq n} |(z_i, z_j)|^\delta \right)^{\frac{1}{\delta}} \right\}, \text{ where } \gamma > 1, \frac{1}{\gamma} + \frac{1}{\delta} = 1; \]

\[\left[ \left( \sum_{i=1}^{n} |\alpha_i|^{2} \right) - \sum_{i=1}^{n} |\alpha_i|^2 \right] \max_{1 \leq i \neq j \leq n} |(z_i, z_j)| . \]

Utilising (2.3) and (2.4) in (2.2), we may deduce the desired result (2.1).  

Remark 1. Inequality (2.1) contains in fact 9 different inequalities which may be obtained combining the first 3 ones with the last 3 ones.

A particular case that may be related to the Boas-Bellman result is embodied in the following inequality.

Corollary 1. With the assumptions in Lemma 1, we have

\[(2.5) \quad \left\| \sum_{i=1}^{n} \alpha_i z_i \right\|^2 \]
\begin{align*}
&\leq \sum_{i=1}^{n} |\alpha_i|^2 \left\{ \max_{1 \leq i \leq n} \|z_i\|^2 + \frac{\left( \sum_{i=1}^{n} |\alpha_i|^2 \right)^2 - \sum_{i=1}^{n} |\alpha_i|^4 \right}\frac{1}{\sum_{i=1}^{n} |\alpha_i|^2} \right\} + \frac{1}{\sum_{1 \leq i \neq j \leq n} |(z_i, z_j)|^2} \\
&\leq \sum_{i=1}^{n} |\alpha_i|^2 \left\{ \max_{1 \leq i \leq n} \|z_i\|^2 + \left( \sum_{1 \leq i \neq j \leq n} |(z_i, z_j)|^2 \right)^{\frac{1}{2}} \right\}.
\end{align*}

The first inequality follows by taking the third branch in the first curly bracket with the second branch in the second curly bracket for $\gamma = \delta = 2$.

The second inequality in (2.5) follows by the fact that
\[
\left[ \left( \sum_{i=1}^{n} |\alpha_i|^2 \right)^2 - \sum_{i=1}^{n} |\alpha_i|^4 \right]^{\frac{1}{2}} \leq \sum_{i=1}^{n} |\alpha_i|^2.
\]

Applying the following Cauchy-Bunyakovsky-Schwarz type inequality
\begin{equation}
(\sum_{i=1}^{n} a_i)^2 \leq n \sum_{i=1}^{n} a_i^2, \quad a_i \in \mathbb{R}_+, \quad 1 \leq i \leq n,
\end{equation}
we may write that
\begin{equation}
\left( \sum_{i=1}^{n} |\alpha_i|^\gamma \right)^2 - \sum_{i=1}^{n} |\alpha_i|^{2\gamma} \leq (n - 1) \sum_{i=1}^{n} |\alpha_i|^{2\gamma} \quad (n \geq 1)
\end{equation}
and
\begin{equation}
\left( \sum_{i=1}^{n} |\alpha_i| \right)^2 - \sum_{i=1}^{n} |\alpha_i|^2 \leq (n - 1) \sum_{i=1}^{n} |\alpha_i|^2 \quad (n \geq 1).
\end{equation}

Also, it is obvious that:
\begin{equation}
\max_{1 \leq i \neq j \leq n} \{|\alpha_i \alpha_j|\} \leq \max_{1 \leq i \leq n} |\alpha_i|^2.
\end{equation}

Consequently, we may state the following coarser upper bounds for $\|\sum_{i=1}^{n} \alpha_i z_i\|^2$ that may be useful in applications.
Corollary 2. With the assumptions in Lemma 1 we have the inequalities:

\[(2.10) \quad \left\| \sum_{i=1}^{n} \alpha_i z_i \right\|^2 \]

\[\leq \left\{ \max_{1 \leq i \leq n} |\alpha_i|^2 \sum_{i=1}^{n} \|z_i\|^2 ; \right.\]

\[+ \left. \left( n-1 \right)^{\frac{\beta}{\gamma}} \left( \sum_{i=1}^{n} |\alpha_i|^{2\beta} \right)^{\frac{1}{\beta}} \left( \sum_{1 \leq i \neq j \leq n} |(z_i, z_j)|^{\beta} \right)^{\frac{1}{\beta}} \right\}, \]

where \(\gamma > 1, \quad \frac{1}{\gamma} + \frac{1}{\beta} = 1;\)

\[\left( n-1 \right)^{\frac{1}{\beta}} \left( \sum_{i=1}^{n} |\alpha_i|^{2\beta} \right)^{\frac{1}{\beta}} \left( \sum_{1 \leq i \neq j \leq n} |(z_i, z_j)|^{\beta} \right)^{\frac{1}{\beta}}.
\]

The proof is obvious by Lemma 1 in applying the inequalities (2.7) – (2.9).

Remark 2. The following inequalities which are incorporated in (2.10) are of special interest:

\[(2.11) \quad \left\| \sum_{i=1}^{n} \alpha_i z_i \right\|^2 \leq \max_{1 \leq i \leq n} |\alpha_i|^2 \left[ \sum_{i=1}^{n} \|z_i\|^2 + \sum_{1 \leq i \neq j \leq n} |(z_i, z_j)| \right];\]

\[(2.12) \quad \left\| \sum_{i=1}^{n} \alpha_i z_i \right\|^2 \leq \left( \sum_{i=1}^{n} |\alpha_i|^{2\beta} \right)^{\frac{1}{\beta}} \left( \sum_{i=1}^{n} \|z_i\|^{2\beta} \right)^{\frac{1}{\beta}} + (n-1)^{\frac{1}{\beta}} \left( \sum_{1 \leq i \neq j \leq n} |(z_i, z_j)|^{\beta} \right)^{\frac{1}{\beta}} \right],\]

where \(n > 1, \quad \frac{1}{\beta} + \frac{1}{\gamma} = 1;\) and

\[(2.13) \quad \left\| \sum_{i=1}^{n} \alpha_i z_i \right\|^2 \leq \sum_{i=1}^{n} |\alpha_i|^2 \left[ \max_{1 \leq i \leq n} \|z_i\|^2 + (n-1) \max_{1 \leq i \neq j \leq n} |(z_i, z_j)| \right].\]

3. Some Mitrinović-Pečarić-Fink Type Inequalities

We are now able to point out the following result which complements the inequality (1.3) due to Mitrinović, Pečarić and Fink [2, p. 392].
Theorem 3. Let $x, y_1, \ldots, y_n$ be vectors of an inner product space $(H; \langle \cdot, \cdot \rangle)$ and $c_1, \ldots, c_n \in K$ ($K = \mathbb{C}, \mathbb{R}$). Then one has the inequalities:

\begin{equation}
\| \sum_{i=1}^{n} c_i (x, y_i) \|^2 \leq \| x \|^2 \times \left\{ \sum_{i=1}^{n} |c_i|^2 \| y_i \|^2 + \max_{1 \leq i \leq n} \sum_{1 \leq i \neq j \leq n} \frac{|c_i c_j|}{(\sum_{i=1}^{n} |c_i|^\gamma)^{\frac{1}{\gamma}}} \frac{1}{(\sum_{1 \leq i \neq j \leq n} |(y_i, y_j)|^{\delta})^{\frac{1}{\delta}}} \right\},
\end{equation}

where $\alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1$.

Proof. We note that

$$\sum_{i=1}^{n} c_i (x, y_i) = \left( x, \sum_{i=1}^{n} c_i y_i \right).$$

Using Schwarz’s inequality in inner product spaces, we have:

$$\left| \sum_{i=1}^{n} c_i (x, y_i) \right|^2 \leq \| x \|^2 \left\| \sum_{i=1}^{n} c_i y_i \right\|^2.$$

Now using Lemma 1 with $\alpha_i = c_i$, $z_i = y_i$ ($i = 1, \ldots, n$), we deduce the desired inequality 3.1. 

The following particular inequalities that may be obtained by the Corollaries 1 and 2 and Remark 2 hold.
Corollary 3. With the assumptions in Theorem 3, one has the inequalities:

\[
\left(\sum_{i=1}^{n} c_i (x, y_i)\right)^2 \leq \left(\sum_{i=1}^{n} |c_i|^2 \left(\max_{1 \leq i \leq n} \|y_i\|^2 + \left(\sum_{1 \leq i \neq j \leq n} |(y_i, y_j)|^2\right)^{\frac{1}{2}}\right)\right)^\frac{1}{2}.
\]

Remark 3. Note that the first inequality in (3.2) is the result obtained by Mitrinović-Pečarić-Fink in [6]. The other 3 provide similar bounds in terms of the p–norms of the vector \(\left(|c_1|^2, \ldots, |c_n|^2\right)\).

4. SOME BOAS-BELLMAN TYPE INEQUALITIES

If one chooses \(c_i = \overline{(x, y_i)}\) \((i = 1, \ldots, n)\) in (3.1), then it is possible to obtain 9 different inequalities between the Fourier coefficients \((x, y_i)\) and the norms and inner products of the vectors \(y_i\) \((i = 1, \ldots, n)\). We restrict ourselves only to those inequalities that may be obtained from (3.2).

As Mitrinović, Pečarić and Fink noted in [6, p. 392], the first inequality in (3.2) for the above selection of \(c_i\) will produce the Boas-Bellman inequality (1.2).

From the second inequality in (3.2) for \(c_i = (x, y_i)\) we get

\[
\left(\sum_{i=1}^{n} |(x, y_i)|^2\right)^2 \leq \|x\|^2 \left(\max_{1 \leq i \leq n} |(x, y_i)|^2 \left(\sum_{i=1}^{n} \|y_i\|^2 + \sum_{1 \leq i \neq j \leq n} |(y_i, y_j)|\right)\right)^{\frac{1}{2}}.
\]

Taking the square root in this inequality we obtain:

\[
\sum_{i=1}^{n} |(x, y_i)|^2 \leq \|x\| \left(\max_{1 \leq i \leq n} |(x, y_i)|^2 \left(\sum_{i=1}^{n} \|y_i\|^2 + \sum_{1 \leq i \neq j \leq n} |(y_i, y_j)|\right)\right)^{\frac{1}{2}},
\]

for any \(x, y_1, \ldots, y_n\) vectors in the inner product space \((H; (\cdot, \cdot))\).

If we assume that \((e_i)_{1 \leq i \leq n}\) is an orthonormal family in \(H\), then by (4.1) we have

\[
\sum_{i=1}^{n} |(x, e_i)|^2 \leq \sqrt{n} \|x\| \left(\max_{1 \leq i \leq n} |(x, e_i)|\right), \quad x \in H.
\]
From the third inequality in (3.2) for \( c_i = (x, y_i) \) we deduce
\[
\left( \sum_{i=1}^{n} |(x, y_i)|^2 \right)^2 \leq \|x\|^2 \left( \sum_{i=1}^{n} |(x, y_i)|^{2p} \right)^{\frac{1}{p}} \times \left\{ \left( \sum_{i=1}^{n} \|y_i\|^{2q} \right)^{\frac{1}{q}} + (n-1)^{\frac{1}{p}} \left( \sum_{1 \leq i \neq j \leq n} |(y_i, y_j)|^q \right)^{\frac{1}{q}} \right\},
\]
for \( p > 1, \frac{1}{p} + \frac{1}{q} = 1 \).

Taking the square root in this inequality we get
\[
\left( \sum_{i=1}^{n} |(x, y_i)|^2 \right)^{\frac{1}{2}} \leq \|x\| \left( \sum_{i=1}^{n} |(x, y_i)|^{2p} \right)^{\frac{1}{2p}} \times \left\{ \left( \sum_{i=1}^{n} \|y_i\|^{2q} \right)^{\frac{1}{2q}} + (n-1)^{\frac{1}{p}} \left( \sum_{1 \leq i \neq j \leq n} |(y_i, y_j)|^q \right)^{\frac{1}{q}} \right\},
\]
for any \( x, y_1, \ldots, y_n \in H, p > 1, \frac{1}{p} + \frac{1}{q} = 1 \).

The above inequality becomes, for an orthonormal family \((e_i)_{1 \leq i \leq n}\),
\[
\sum_{i=1}^{n} |(x, e_i)|^2 \leq n^{\frac{1}{q}} \|x\| \left( \sum_{i=1}^{n} |(x, e_i)|^{2p} \right)^{\frac{1}{2p}}, \quad x \in H.
\]
Finally, the choice \( c_i = (x, y_i) \) \((i = 1, \ldots, n)\) will produce in the last inequality in (3.2)
\[
\left( \sum_{i=1}^{n} |(x, y_i)|^2 \right)^{\frac{1}{2}} \leq \|x\| \left\{ \max_{1 \leq i \leq n} \|y_i\|^2 + (n-1) \max_{1 \leq i \neq j \leq n} |(y_i, y_j)| \right\}
\]
giving the following Boas-Bellman type inequality
\[
\sum_{i=1}^{n} |(x, y_i)|^2 \leq \|x\|^2 \left\{ \max_{1 \leq i \leq n} \|y_i\|^2 + (n-1) \max_{1 \leq i \neq j \leq n} |(y_i, y_j)| \right\},
\]
for any \( x, y_1, \ldots, y_n \in H \).

It is obvious that (4.4) will give for orthonormal families the well known Bessel inequality.

**Remark 4.** In order the compare the Boas-Bellman result with our result (4.3), it is enough to compare the quantities
\[
A := \left( \sum_{1 \leq i \neq j \leq n} |(y_i, y_j)|^2 \right)^{\frac{1}{2}}
\]
and
\[
B := (n-1) \max_{1 \leq i \neq j \leq n} |(y_i, y_j)|.
\]
Consider the inner product space $H = \mathbb{R}$ with $(x, y) = xy$, and choose $n = 3$, $y_1 = a > 0$, $y_2 = b > 0$, $y_3 = c > 0$. Then

$$A = \sqrt{2} \left( a^2 b^2 + b^2 c^2 + c^2 a^2 \right)^{\frac{1}{2}}, \quad B = 2 \max(ab, ac, bc).$$

Denote $ab = p$, $bc = q$, $ca = r$. Then

$$A = \sqrt{2} \left( p^2 + q^2 + r^2 \right)^{\frac{1}{2}}, \quad B = 2 \max(p, q, r).$$

Firstly, if we assume that $p = q = r$, then $A = \sqrt{6}p$, $B = 2p$ which shows that $A > B$.

Now choose $r = 1$ and $p, q = \frac{1}{2}$. Then $A = \sqrt{3}$ and $B = 2$ showing that $B > A$.

Consequently, in general, the Boas-Bellman inequality and our inequality (4.5) cannot be compared.