GEOMETRIC REDUCTION IN OPTIMAL CONTROL THEORY WITH SYMMETRIES

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Abstract

A general study of symmetries in optimal control theory is given, starting from the presymplectic description of this kind of system. Then, Noether’s theorem, as well as the corresponding reduction procedure (based on the application of the Marsden-Weinstein theorem adapted to the presymplectic case) are stated both in the regular and singular cases, which are previously described.

Key words: Symmetries, reduction, optimal control, presymplectic Hamiltonian systems.

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1 Introduction

The application of modern differential geometry to optimal control theory has meant a great advance for this field in recent years, beginning with R.W. Brocket’s pioneering work [6], [7], up to recent developments, as reported, for instance, by H.J. Sussmann [31], [32], [33]. This paper is devoted to studying optimal control problems with symmetry in a geometric framework.

Hence, our standpoint is the natural presymplectic description of optimal control problems arising from the Pontryagin maximum principle. If this presymplectic system has symmetries, then the Marsden-Weinstein reduction theorem [23], generalized to the presymplectic case, as in [11], allows the dynamics to be simplified, thus reducing the number of degrees of freedom and giving a simpler structure to the equations of motion.

Previous works on this subject have been carried out by A.J. Van der Schaft [29] and H.J. Sussmann [31]. The first considers symmetries of the Lagrangian and the differential equation in order to arrive at a Noether theorem (although no intrinsic geometric structures are used in it). In the second, similar results are given, but in a more general context (relaxing the differentiability conditions). Recently, A. Bloch and P. Crouch offered a presentation of optimal control systems on coadjoint orbits related to reduction problems and integrability [5].

In this work, we give a general description of symmetries in optimal control, classifying them as the so-called natural ones (which come from diffeomorphisms in the configuration manifold of the original problem), and other symmetries of the associated presymplectic system. Moreover, the reduction procedure is described both in the regular and the singular case. (A different point of view on this problem, using Dirac structures and implicit Hamiltonian systems is given in [3] and [4]. Other approaches can be found in [9], [10] and [24]).

More precisely, Section 2 is devoted to stating the problem and describing the presymplectic formulation. Since in this situation there is no global dynamics, we study the application of a presymplectic algorithm (see [13]) to obtain the (maximal) manifold where the dynamics exists and the equation of motion on this last manifold. The analysis of the procedure leads to a distinction between the regular situation, where only one step of the algorithm is needed, and the singular one. Later on, in Section 3, after reviewing some basic facts of actions of Lie groups on presymplectic manifolds, different notions of symmetries for autonomous control problems are defined, and the reduction procedure as well as Noether’s theorem are stated for both the regular and singular cases. In Section 4, the results are then extended to the case of non-autonomous control problems. Finally, in Section 5, two examples are given: In the first one the reduction of regular optimal control problems invariant by a vector fields is considered, following the study given in [26] for time-dependent Lagrangian mechanics. The second corresponds to an example analyzed by H.J. Sussman in [31], and consists in searching for the shortest paths with a bounded curvature.

All the manifolds are real, connected, second countable and $C^\infty$. The maps are assumed to be $C^\infty$ and the differential forms have constant rank. Sum over crossed repeated indices is understood.
2 Geometric description of optimal control theory

2.1 Optimal control problems

Let $W = U \times V \subset \mathbb{R}^n$ equipped with coordinates $\{q^i, u^a\}$ ($i = 1, \ldots, m$, $a = 1, \ldots, n - m$). $\{q^i\}$ are the coordinates in the configuration space $V \subset \mathbb{R}^m$, and $\{u^a\}$ are said to be the control variables or coordinates of the control space $U \subset \mathbb{R}^{n-m}$. An optimal control problem consists in finding $C^1$-piecewise smooth curves $\gamma(t) = (q(t), u(t))$ with fixed endpoints in configuration space, $q(t_1) = q_1$ and $q(t_2) = q_2$, such that they satisfy the control equation

$$\dot{q}^i(t) = F^i(q(t), u(t)),$$

(2.1)

and minimize the objective functional

$$S[\gamma] = \int_{t_1}^{t_2} L(q(t), u(t)) dt,$$

where $F^i, L \in C^\infty(W)$. Solutions to this problem are called optimal trajectories (relative to the points $q_1$ and $q_2$).

It is well-known (see [20]) that the solution to this problem is provided by Pontryagin’s maximum principle, which is stated in the following way:

First, consider the co-state space $T^*V$, whose coordinates are denoted by $\{q^i, p_i\}$ ($i = 1, \ldots, m$), and take $M \equiv U \times T^*V$, with coordinates $\{q^i, p_i, u^a\}$. Then consider a family $\{H(q, p, u)\} \subset C^\infty(M)$ of Hamiltonian functions, parametrized by the control variables, given by

$$H(q, p, u) = p_i F^i(q, u) - p_0 L(q, u).$$

(2.2)

where $p_0$ can be regarded as another parameter. For each control function $u(t)$, we can find the integral curves $(q(t), p(t))$ of the Hamiltonian vector field which are the solutions to the Hamilton equations

$$\dot{q}^i = \frac{\partial H(q, p, u)}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H(q, p, u)}{\partial q^i} \quad (i = 1, \ldots, m).$$

(2.3)

Secondly, the maximal Hamiltonian function is defined as:

$$H_{\text{max}}(q, p) = \max_u H(q, p, u),$$

(2.4)

then we have:

**Pontryagin’s maximum principle**: If a curve $\gamma(t) = (q(t), u(t))$ is an optimal trajectory between $q_1$ and $q_2$, then there exists a curve $p(t)$ such that:

1. $(q(t), p(t), u(t))$ is the solution to (2.3), and
2. $H(q(t), p(t), u(t)) = H_{\text{max}}(q(t), p(t))$.

If $p_0 = 0$, then the optimal solutions are called abnormal. In this paper we confine our attention to the case in which $p_0 \neq 0$, and in particular we take the typical value $p_0 = 1$. 
It is clear that a necessary condition for $H$ to reach the maximum (if the maximum of $H$ is not on the boundary of the control set) is

$$
\chi_a = \frac{\partial H}{\partial u^a} = 0, \quad (a = 1, \ldots, n - m).
$$

(2.5)

Hence, the trajectories solution to the optimal control problem lie in a subset $M_1$ of the total space $M$, which is defined by the constraints $\chi_a = 0$.

In most cases, the constraint functions $\chi_a = 0$, called first order constraints, define implicitly $n - m$ functions $\psi^a$ such that

$$
u^a = \psi^a(q, p)
$$

whenever the matrix defined by $W_{ab} = \frac{\partial \chi_a}{\partial u^b}$ is non-singular, i.e., $\det W_{ab} \neq 0$. Under these circumstances, the function $\psi \equiv \{\psi^a\}$, given by equation (2.6), is called an optimal feedback function. If such a condition is satisfied, we say that the optimal control problem is regular. If $\det W_{ab} = 0$ on $M_1$, we say that the optimal control problem is singular. In any case, we will assume that rank $W_{ab}$ is constant on the domain of our analysis.

Remark 1 One of the consequences of the Maximum Principle is that the optimal control problems can be studied as presymplectic Hamiltonian systems; that is, those where the 2-form is degenerate. Next we give a concise description on this topic, as well as the associated constraint algorithm. Furthermore, in Section 3, we study the existence of first integrals for optimal control problems with symmetries (see Theorem 1).

2.2 Optimal control problems as presymplectic Hamiltonian systems

Taking into account the above considerations, a problem of optimal control, from a geometric viewpoint, may be given by the following data: a configuration space which is a differentiable manifold $Q$, locally described by the state variables $q^i$ ($i = 1, \ldots, m$), a fibre bundle $\pi: E \rightarrow Q$ whose fibres are locally described by the control variables $u^a$ ($a = 1, \ldots, n - m$), a vector field $X$ along the projection of the bundle, $X: E \rightarrow TQ$ (i.e., $\tau_Q \circ X = \pi$, where $\tau_Q: TQ \rightarrow Q$ denotes the canonical projection), and a “Lagrangian function” $L: E \rightarrow \mathbb{R}$. Consider the family of paths $\gamma: I \rightarrow E$ such that $\pi \circ \gamma$ has fixed end-points, which are solutions to the differential equation

$$T\pi \circ \dot{\gamma} = (\pi \circ \gamma)' = X \circ \gamma
$$

(2.7)

that rules the evolution of the state variables, i.e., in local coordinates it is equation (2.1), with boundary conditions $q^i_1 = q^i(t_1)$ to $q^i_2 = q^i(t_2)$ (there are no boundary conditions on the control variables). The problem is to find a minimum of the action

$$\int_{\gamma} L(\gamma(t)) \, dt
$$

for this family of paths $\gamma$. So we have the diagram

$$
\begin{array}{ccc}
I & \xrightarrow{\gamma} & E \\
\downarrow{\tau_E} & \searrow{X} & \searrow{\tau_Q} \\
TE & \xrightarrow{T\pi} & TQ
\end{array}
$$
Therefore, an optimal control problem is characterized by the data \((E, \pi, Q, L, X)\).

**Remark 2** It is easy to show that this is indeed a vakonomic problem on the manifold \(E\) (see [2],[15] for this kind of problem), where the Lagrangian \(L\) is singular, since it does not depend on the velocities. The constraint submanifold \(C \subset T E\), given by the differential equation above, is

\[
C = \{ w \in T E \mid T \pi(w) = (X \circ \tau_E)(w) \}.
\]

In this way, a path \(\gamma\) is admissible if, and only if, it is a solution to the differential equation (2.7) or, equivalently, if it takes values in the affine subbundle \(C\) of \(TE\). Notice that, in coordinates, and from a variational viewpoint, the constraints defining \(C\) as a submanifold of \(TE\) (which are the set of first order differential equations (2.1)) are very particular: they give the velocities of the state variables in terms of the state and control variables.

Optimal control theory admits several geometric formulations and expressions of the equations of motion (2.3) and (2.5). One of the most interesting is the presymplectic description which can be constructed on the manifold \(\pi^*T^*Q = E \times_Q T^*Q\) (for a description on presymplectic dynamical systems, in general, see for instance [8], [13]). We denote by \(\pi_1: \pi^*T^*Q \rightarrow E\) and \(\pi_2: \pi^*T^*Q \rightarrow T^*Q\) the projections onto the first and second factors, respectively. Then, the Hamiltonian function (2.2) can be defined intrinsically as

\[
H = \hat{X} - L ,
\]

where we identify the Lagrangian function \(L \in C^\infty(E)\) with its pull-back through \(\pi_1: \pi^*T^*Q \rightarrow E\), and \(\hat{X}: \pi^*T^*Q \rightarrow \mathbb{R}\) is the function defined by \(\hat{X}((q,u),\alpha) = \alpha(X(q,u))\), for \((q,u) \in E\) and \(\alpha \in T^*_qE\).

Let \(\theta_0\) and \(\omega_0 = -d\theta_0\) be the canonical 1 and 2-forms in \(T^*Q\). We can take their pull-back through \(\pi_2\), obtaining \(\theta = \pi_2^*\theta_0\) and \(\omega = \pi_2^*\omega_0\) in \(\pi^*T^*Q\). In local coordinates, \(\theta = p_i dq^i\) and \(\omega = dq^i \wedge dp_i\). If \(\eta\) is a path on \(\pi^*T^*Q\), then to solve the presymplectic equation

\[
i_\eta \omega = dH \circ \eta
\]

is equivalent to solving the equations of motion (2.3) and (2.5). To show this, it is enough to write their local expressions. If \(\eta\) is an integral curve of a vector field \(\Gamma\) on \(\pi^*T^*Q\), we can write the presymplectic equation above as

\[
i_\Gamma \omega = dH .
\]

Summarizing, our geometrical interpretation of the problem of optimal control stated in the beginning is given by the presymplectic Hamiltonian system \((\pi^*T^*Q,\omega, H)\), which arises from the original data \((E, \pi, Q, L, X)\).

Observe that relation (2.9) gives the critical curves of the variational problem for the Lagrangian \(L\) with constraints (2.7), and it is a weaker condition, in general, than the maximum principle posed by equations (2.3), (2.4). Throughout this paper, we will restrict our attention to the analysis of solutions to (2.9), and the existence of true extremals will not be studied.

From now on, we will denote \(M = \pi^*T^*Q\). Relation (2.5) is the local expression of the compatibility condition for the equation (2.9) and it is assumed that it defines a closed submanifold.
$M_1$ in $M$. In the following section, all these features will be studied in detail, and we show that
the regularity of the optimal control problem is equivalent to the existence of a unique vector field
solution to (2.9) tangent to the first order constraint submanifold $M_1$. Otherwise, the optimal
control problem is singular, and a constraint algorithm is needed to solve the problem, in general.

2.3 Constraint algorithm for optimal control problems

Consider the presymplectic dynamical system given by $(M, \omega, H)$, with $M = \pi^* T^* Q$, $\omega = \pi^*_2 \omega_0$, and $H$ being defined by (2.8). The presymplectic dynamical equation is (2.9). Notice that $\omega$ is
degenerate, and $\ker \omega = \mathfrak{X}^{(\pi_2)}(M) = \{ X \in \mathfrak{X}(M) \mid \pi_2 X = 0 \}$. It is well known [13], [25] that
there are vector fields satisfying equation (2.9) only at the points of the subset

$$M_1 = \{ x \in M \mid (L_Z H)(x) = 0, \text{ for every } Z \in \ker \omega \}$$

We assume that $M_1$ is a closed submanifold of $M$, and the natural embedding is denoted by
$j_1: M_1 \hookrightarrow M$. Now, there are vector fields $\Gamma$ satisfying equation (2.9) on the points of $M_1$. These
vector fields are defined in principle only at the points of $M_1$, and they take values in $TM|_{M_1}$
(obviously, they can be extended to vector fields in $M$). However, in general, these vector fields $\Gamma$
are not tangent to $M_1$, that is, they do not take values in $TM_1$. If $\ker \omega \cap \mathfrak{X}(M_1) = \{ 0 \}$ (where
$\mathfrak{X}(M_1)$ denotes the vector fields of $M$ which are tangent to $M_1$), then $(M_1, j^*_1 \omega)$ is a symplectic
manifold, and there is a unique vector field $\Gamma$ defined at the points of $M_1$ verifying (2.9). Moreover
$\Gamma$ is tangent to $M_1$ because $j^*_1 \omega$ is a symplectic form. As locally $\ker \omega = \langle \partial \partial u^a \rangle$, then $M_1$ is a
submanifold transverse to $\ker \omega$ if, and only if, $\det W_{ab} \neq 0$. So, a regular optimal control problem
corresponds to the case where $(M_1, j^*_1 \omega)$ is a symplectic manifold.

However, if the system is singular, the vector fields solutions to the Hamiltonian equation (2.9)
on the submanifold $M_1$ are not necessarily tangent to $M_1$. Thus, their integral curves can leave
the submanifold where the extremal trajectories must lie. Thus, we must take the points of $M_1$
where vector fields solutions to (2.9) being tangent to $M_1$ exist. The subset $M_2 \subset M_1$ made by
those points is defined as

$$M_2 = \{ x \in M_1 \mid \Gamma(\chi_a)(x) = 0, \text{ for every } \Gamma \text{ solution to (2.9) on } M_1 \} .$$

We assume that the subset $M_2$ is a closed submanifold of $M_1$. We denote the functions defining
$M_2$ on $M_1$ by $\chi_b^{(2)}$. Repeating the argument, we obtain a family of subsets (assuming that all of
them are closed submanifolds) defined recursively by

$$M_k = \{ x \in M_{k-1} \mid \Gamma(\chi_b^{(k-1)})(q) = 0, \text{ for every } \Gamma \text{ solution to (2.9) on } M_1 \} , \quad k > 1 .$$

The recursion stops, and $M_r = M_{r+1} = M_{r+2} = \cdots$ for a certain $r$. In this way, we obtain a stable
submanifold

$$M_f = \cap_{k \geq 1} M_k$$

where the dynamical equation has tangent solutions, and the integral curves of the corresponding
vector fields are the critical curves of the singular optimal control problem. We denote by $j_f: M_f \hookrightarrow M$
the natural embedding. This is the constraint algorithm for optimal control problems, which
is similar in nonlinear control to the so-called zero-dynamics algorithm based on the notion of (locally) controlled invariant submanifold (see [16], [27]).

Another geometric description of the condition of regularity and the submanifold $M_1$ can be given. In fact, as we know that $M = \pi^* T^* Q = E \times_Q T^* Q$, we can consider the fibre bundle $\pi_2: E \times_Q T^* Q \rightarrow T^* Q$. Now, consider the Hamiltonian function $H: E \times_Q T^* Q \rightarrow \mathbb{R}$, and the vertical bundle $V(\pi_2)$. The function $H$ defines a map

$$F_H: E \times_Q T^* Q \rightarrow V^*(\pi_2)$$

$$(e, \alpha) \mapsto T_e \left( H|_{\pi(e), \alpha} \right): V(\pi_2) \rightarrow \mathbb{R}$$

which is called the fibre derivative of $H$ (see [14],[12] for more details). In local coordinates, $F_H(q, p, u) = (q, p, u, \partial H/\partial u)$. Then, the submanifold $M_1$ can be characterized as $M_1 = F_H^{-1}(0)$. The local regularity condition $\det W_{ab} \neq 0$ is equivalent to demanding that $F_H$ has maximal rank everywhere, and it is also equivalent to the existence of a (local) section $\sigma: U \rightarrow E \times_Q T^* Q$, for some neighbourhood $U$ of each point $(q, p) \in T^* Q$. If there exists a global section $\sigma$ of $\pi_2$, such that $F_H^{-1}(0) = \sigma(T^* Q)$, then the optimal control problem is said to be hyper-regular and, of course, $\det W_{ab} \neq 0$ (i.e., it is regular).

In the regular case, the 2-form $\omega_1 = j_1^* \omega$ is non-degenerate and the manifold $M_1$ is locally symplectomorphic to $T^* Q$. In the hyper-regular situation, $M_1$ and $T^* Q$ are globally symplectomorphic, and the symplectomorphism is constructed by means of the global section $\sigma$. As a final remark, if $\Gamma_1 \in \mathfrak{X}(M_1)$ is the unique vector field solution to the dynamical equation

$$i_{\Gamma_1} \omega_1 = dh_1 , \quad (2.10)$$

where $h_1 = j_1^* H$, then the vector field $\Gamma$ solution to (2.9) is obtained from $\Gamma_1$ by using the optimal feedback condition (2.6).

Notice that, in both the regular and singular cases, there is no vector field on $M$ satisfying the presymplectic dynamical equation in $M$, but only on a submanifold $M_1 \neq M$.

## 3 Symmetries and reduction of optimal control problems: the autonomous case

(See the appendix 5.2 for the notation, the terminology and the fundamental concepts and results about presymplectic reduction, which are used in this section).

### 3.1 The regular case: symmetries and reduction

#### 3.1.1 Symmetries and first integrals

One of the most important features in the study of dynamical systems with symmetry is the so-called reduction theory.
First, we establish the concept of group of symmetries for the non-compatible presymplectic dynamical system \((M, \omega, H)\), where \(M = \pi^*T^*Q\) and \(H\) is given by (2.8), with compatible symplectic dynamical system \((M_1, \omega_1, h_1)\), where \(\omega_1 = j_1^*\omega\) and \(h_1 = j_1^*H\); i.e., we assume that the optimal control problem is regular. Notice that, in this case, in the notation of the above section, \((P, \Omega) = (M_1, \omega_1)\). Moreover, \(\omega_1\) is symplectic and exact, because \(\omega_1 = j_1^*\omega = j_1^*(-d\theta) = -d(j_1^*\theta)\).

**Definition 1** Let \(G\) be a connected Lie group and \(\Phi: G \times M \to M\) an action of \(G\) on \(M\). Let \((M, \omega, H)\) be a regular optimal control problem. \(G\) is said to be a symmetry group of \((M_1, \omega_1, h_1)\) if

1. \(\Phi\) leaves \(M_1\) invariant; that is, it induces an action \(\Phi_1: G \times M_1 \to M_1\).
2. The induced action \(\Phi_1\) is a symplectic action on \((M_1, \omega_1)\) (which is assumed to be Poissonian, free and proper); that is, for every \(g \in G\), \((\Phi_1)_g^*\omega_1 = -\omega_1\).
3. For every \(g \in G\), \((\Phi_1)_g^*h_1 = h_1\).

This definition is justified since, if \(G\) is a symmetry group of \((M_1, \omega_1, h_1)\), then \(\Phi_g\) maps solutions in solutions. To show this, let \(\Gamma\) be the vector field in \(M\) tangent to \(M_1\) solution to the dynamical system (2.9) in the points of \(M_1\). Then there is a vector field \(\Gamma_1 \in \mathfrak{X}(M_1)\) such that \(j_1^*\Gamma_1 = \Gamma\rvert_{M_1}\) and verifying \(i_{\Gamma_1} \omega_1 = d h_1\). Therefore,

\[
0 = (\Phi_1)_g^* (i_{\Gamma_1} \omega_1 - d h_1) = i_{(\Phi_1)_g^* \Gamma_1} (\Phi_1)_g^* \omega_1 - (\Phi_1)_g^* d h_1 =
\]

\[
i_{(\Phi_1)_g^* \Gamma_1} \omega_1 - d((\Phi_1)_g^* h_1) = i_{(\Phi_1)_g^* \Gamma_1} \omega_1 - d h_1.
\]

This definition of symmetry group applies to every presymplectic dynamical system, simply by identifying \(M_1\) with the final constraint submanifold \(M_f\), in the sense that it works for the case when the 2-form \(\omega_1\) (or \(\omega_f\) in the general case) is degenerate (in that case, in Condition 2, the action \(\Phi_f\) is a presymplectic action on \((M_f, \omega_f)\)).

In Definition 1, we have considered actions on \(M\) which induce symmetries on the symplectic manifold \(M_1\) if the optimal control problem is regular (symmetries on the symplectic final constraint submanifold, in general, if the problem is singular). It is straightforward to show that if \(G\) is a symmetry group of \((M, \omega, H)\) (i.e., \(\Phi_g^*\omega = \Omega\) and \(\Phi_g^*H = H\), for all \(g \in G\)), then it is a symmetry group of the associated compatible presymplectic dynamical system. However, the group of symmetries of \((M, \omega, H)\) is smaller, in general, than the group of symmetries of \((M_f, \omega_f, h_f)\), in the sense that it gives fewer symmetries of the dynamics \(\Gamma_f\).

There is a more natural definition of transformation of symmetry for optimal control systems strongly related to the special characteristics of the problem. Let us recall that an optimal control problem may be given by the data \((E, \pi, Q, X, L)\), where \(Q\) is the configuration space describing the state variables, \(\pi: E \to Q\) is a fibre bundle whose fibres describe the control variables, \(X: E \to T Q\) is a vector field along \(\pi\) (i.e., \(\tau_Q \circ X = \pi\)), and \(L: E \to \mathbb{R}\) is a Lagrangian function.

**Definition 2** Let \(\Psi: E \to E\) be a bundle diffeomorphism, and \(\varphi: Q \to Q\) the induced diffeomorphism on the base manifold \((\pi \circ \Psi = \varphi \circ \pi)\). We say that \(\Psi\) is a transformation of symmetry of the regular optimal control problem described by the data \((E, \pi, Q, X, L)\) if
1. $\Psi^* L = L$ ($\Psi$ is a symmetry of the Lagrangian function).
2. $\Psi_* X = X$ ($\Psi$ is a symmetry of the vector field).

In the definition, the push-forward $\Psi_* X$ is defined by

$$\Psi_* X = \varphi_* \circ X \circ \Psi^{-1}. \quad (3.1)$$

Hence the inverse $\Psi^*$ of the push-forward is $\Psi^* X = \varphi_*^{-1} \circ X \circ \Psi$.

If the diffeomorphism $\Psi$ is understood as a change of coordinates in the state and control variables then, in nonlinear control, it is called as feedback transformation, and the push-forward $\Psi_* X$ is the differential equation $X$ transformed via the feedback transformation (see [16], [17], [27]). Moreover, $\Psi$ being a symmetry of $X$ means that the induced diffeomorphism $\varphi$ is a symmetry of $X$ in the sense of [18] and [28].

The meaning of this definition will become clear in Theorem 1, where we prove that, if $G$ is a connected Lie group such that $\Psi_g$ is a transformation of symmetry of the regular optimal control problem, for every $g \in G$, then $\Psi_g$ maps optimal trajectories into optimal trajectories (see also [10]).

Now, let $Z \in \mathfrak{x}(E)$ be a vector field on $E$, and let $X: E \to TQ$ be a vector field along the projection $\pi: E \to Q$. If $Z$ is $\pi$-projectable, then we can define the Lie derivative of $X$ along $Z$ as follows: if $\Psi_t$ denotes the flow of $Z$ and $\varphi_t$ denotes the flow of the vector field $Z_0 = \pi_* Z \in \mathfrak{x}(Q)$, then

$$L_Z X = \frac{d}{dt} \bigg|_{t=0} [\Psi^*_t X], \quad (3.2)$$

or, equivalently,

$$L_Z X = \lim_{t \to 0} \frac{\Psi^*_t X - X}{t}. \quad (3.3)$$

Notice that the push-forward is well-defined since from the projectability of $Z$ we deduce that $\Psi_t$ is a bundle mapping. It is clear that $L_Z X: E \to TQ$ is a vector field along $\pi$.

The following lemma gives the algebraic expression of this Lie derivative.

**Lemma 1** Let $X: E \to TQ$ be a vector field along the projection $\pi: E \to Q$, and let $Z \in \mathfrak{x}(E)$ be a $\pi$-projectable vector field on $E$, with $Z_0 = \pi_* Z \in \mathfrak{x}(Q)$. Then, for every $f \in C^\infty(Q)$, the Lie derivative $L_Z X(f) \in C^\infty(E)$ is

$$L_Z X(f) = L_Z (L_X(f)) - L_X (L_{Z_0}(f)), \quad \text{i.e., } L_Z X = L_Z \circ L_X - L_X \circ L_{Z_0}. \quad (\text{Proof})$$

From (3.3) we must evaluate $(\Psi^*_t X - X)(f)(m)$, for every $f \in C^\infty(Q)$ and $m \in E$. Using (3.1), we obtain

$$(\Psi^*_t X - X)(f)(m) = X_{\Psi_t(m)}(f \circ \varphi_{-t}) - X_m(f) = X_{\Psi_t(m)}(f \circ \varphi_{-t}) - X_{\Psi_t(m)}(f) + X_{\Psi_t(m)}(f) - X_m(f).$$
Hence,
\[
(LZ \ X)_m(f) = \lim_{t \to 0} \left( \frac{\Psi_t \ X - X}{t} \right)_m(f) = \lim_{t \to 0} \frac{X_{\Psi_t(m)}(f) - X_m(f)}{t} \\
+ \lim_{t \to 0} \frac{X_{\Psi_t(m)}(f \circ \varphi_{-t}) - X_{\Psi_t(m)}(f)}{t} = (LZ(LX(f)))_m - (LX(LZ_0(f)))_m.
\]

\[\square\]

**Definition 3** Let \( Z \in \mathcal{X}(E) \) be a \( \pi \)-projectable vector field. The vector field \( Z \) is called an infinitesimal symmetry of the regular optimal control problem \((L, \pi, Q, X, L)\) if

1. \( LZ L = 0 \) (\( Z \) is an infinitesimal symmetry of the Lagrangian function).
2. \( LZ X = 0 \) (\( Z \) is an infinitesimal symmetry of the vector field \( X \)).

We finish these definitions of symmetries with the idea of symmetry group of an optimal control problem.

**Definition 4** Let \( G \) be a connected Lie group, and \( \Psi: G \times E \to E \) an action of \( G \) on \( E \) such that, for each \( g \in G \), \( \Psi_g \) is a bundle mapping, with induced mapping \( \varphi_g: Q \to Q \). \( G \) is said to be a symmetry group of the regular optimal control problem described by \((E, \pi, Q, X, L)\) if every \( \Psi_g \), \( g \in G \), is a transformation of symmetry.

This concept of symmetry group of regular optimal control problems is related to the idea of symmetry group of the presymplectic dynamical system \((M, \omega, H)\) (or, equivalently, symmetry group of \((M_1, \omega_1, h_1)\), since the problem is regular) as follows: given the above action \( \Psi: G \times E \to E \) preserving the bundle structure \( \pi: E \to Q \), we can lift this action to an action \( \Phi: G \times \pi^* T^*Q \to \pi^* T^*Q \) in a natural way: for every \((q, u, p) \in \pi^* T^*Q \) (where \( u \in E_q \) and \( p \in T^*_q Q \)),

\[
\Phi_g(q, u, p) = \left( \Psi_g(q, u), T^*_{\varphi_g(q)} \varphi_{g}^{-1}(p) \right).
\]

**Theorem 1** If \( G \) is a symmetry group of the regular optimal control problem \((E, \pi, Q, X, L)\) (i.e., \( \Psi_g: E \to E \) is a transformation of symmetry, for every \( g \in G \)), then the action \( \Phi \) given by (3.4) is a symmetry group of the presymplectic dynamical system \((\pi^* T^*Q, \omega, H)\). Moreover, the action is exact and there exists a comomemtum map

\[
\mathcal{J}^* : g \to C^\infty(\pi^* T^*Q) \\
\xi \mapsto i_\xi \theta
\]

in such a way that the functions \( f_\xi = i_\xi \theta \) are constants of motion.

Conversely, if the lifted transformations \( \Phi_g \) are symmetries of the presymplectic dynamical system, then the fundamental vector fields \( \xi \) are infinitesimal symmetries of \((E, \pi, Q, X, L)\).
Finally, the existence of the comomentum map follows from the exactness of the action. 

Then, since the new action $\Phi$ is the canonical lift of $\Psi$ to an action in $\pi^*T^*Q$, $\Phi(\pi^*\theta) = \pi^*\theta$ (if $\Phi$ is exact), so we have $\Phi_{\pi^*}\omega = \omega$. On the other hand, $\Phi_{\pi^*}H = \Phi_{\pi^*}\tilde{X} - \Phi_{\pi^*}L = \tilde{X} - L = H$. Finally, the existence of the comomentum map follows from the exactness of the action.

Conversely, notice that if $\Psi: G \times E \to E$ is a bundle action such that $G$ is a symmetry group of the presymplectic system $(\pi^*T^*Q, \omega, H)$, then $\Phi^*(H) = H$ implies that $\hat{\xi}^c(H) = 0$, where $\hat{\xi}^c$ are the fundamental vector fields associated with the action $\Phi$. It is clear that $\hat{\xi}^c$ are the lifting to $\pi^*T^*Q$ of the fundamental vector fields $\xi$ associated with the action $\Psi$. The local expressions of such fundamental vector fields associated with actions $\Psi$ and $\Phi$ are

$$\hat{\xi} = \xi^i(q) \frac{\partial}{\partial q^i} + \zeta^a(q,u) \frac{\partial}{\partial u^a}$$

$$\hat{\xi}^c = \xi^i(q) \frac{\partial}{\partial q^i} - p_i \frac{\partial \xi^i(q)}{\partial q^i} + \zeta^a(q,u) \frac{\partial}{\partial u^a}.$$ 

Then,

$$\hat{\xi}^c(H) = \hat{\xi}^c(p_i X^i(q,u)) - \hat{\xi}^c(L(q,u)) =$$

$$= p_i \left( \xi^i \frac{\partial X^i}{\partial q^j} - Y^j \frac{\partial \xi^i}{\partial q^j} + \zeta^a \frac{\partial Y^i}{\partial u^a} \right) - \left( \xi^i \frac{\partial L}{\partial x^i} + \zeta^a \frac{\partial L}{\partial u^a} \right) = 0.$$

Therefore, as $p_i$ are free, we obtain

$$\xi^i \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial \xi^i}{\partial x^j} + \zeta^a \frac{\partial Y^i}{\partial u^a} = 0, \quad \text{and}$$

$$\xi^i \frac{\partial L}{\partial x^i} + \zeta^a \frac{\partial L}{\partial u^a} = 0.$$

But, from Lemma 1, equation (3.5) is the expression in local coordinates of Condition 2 in Definition 3, $L_{\hat{\xi}^c} X = 0$, and equation (3.6) means $\hat{\xi}(L) = 0$, which is is equivalent to Condition 1 in the same definition. Therefore, the fundamental vector fields $\hat{\xi}$ are infinitesimal symmetries of the optimal control problem $(E, \pi, Q, X, L)$.

We can compare this statement with other versions of Noether’s theorem in control theory. For instance, in [19], a particular optimal control problem on a Lie group is studied: the conserved quantity related to the Casimir of the Lie group is used to find the shortest path of a car moving under the suitable conditions. In [29], the author studies the reduction of the Hamiltonian system associated with an optimal control problem (by the Maximum Principle), by a local symmetry given by a vector field. In [31], given an optimal control problem, the author uses the Maximum Principle to construct a family of symplectic Hamiltonian problems parametrized by the controls (instead of using the presymplectic alternative). Then, constants of motion are related to the action of a Lie algebra on $M$. The techniques are symplectic but local, and reduction is not studied. Finally, the situation studied in Theorem 3 in [34] is the following: if an optimal control problem, formulated on $\mathbb{R}^n$, is invariant by a one-parameter family of $C^1$-maps, then a conservation law is obtained.

**Remark 3** As mentioned in Section 2, an optimal control problem can be understood as a vakonomic problem where the Lagrangian function $L: TE \to \mathbb{R}$ is a basic function and the constraint
submanifold $C$ is the affine subbundle locally described by the contraints $\dot{q}^i = X^i(q,u), i = 1, \ldots, n$. Following Arnold et al [2], a transformation of symmetry of the vakonomic system is a diffeomorphism $\Phi: TE \rightarrow TE$ such that $\Phi|_C(C) \subset C$ and $\Phi^*(L|_C) = L|_C$. But $L|_C = L$, since $L$ is a basic function, so this last condition can be written as $\Phi^*(L) = L$ in this case. It is easy to show that these two conditions are the conditions we have assumed above when $\Phi$ is a diffeomorphism adapted to the bundle structure.

### 3.1.2 Momentum map and geometric reduction

Now, if we have the compatible dynamical system $(M_1, \omega_1, h)$ and the action $\Phi_1$, we are interested in removing the symmetries by following a reduction procedure in order to get a symplectic dynamical system. We apply the results of the appendix 5.2, where now $P \equiv M_1$ and $\Omega \equiv \omega_1$ is symplectic and exact. In what follows we assume that the action is Poissonian, free and proper.

Let $J$ be the momentum map associated with this action, $\mu \in g^*$ a weakly regular value, $j_\mu: J^{-1}(\mu) \rightarrow M$ the natural imbedding, and $\omega_\mu = j^*_\mu \omega_1$ and $h_\mu = j^*_\mu h_1$. Therefore:

**Proposition 1** $(J^{-1}(\mu), \omega_\mu, h_\mu)$ is a compatible presymplectic Hamiltonian system.

(Proof) If we denote by $\tilde{g}_{M_1}$ the set of fundamental vector fields on $M_1$ with respect to the action $\Phi_1$, and $\Gamma_1$ is the Hamiltonian vector field associated with the Hamiltonian function $h_1$, then, for every constraint $\zeta$, with $d\zeta = i_{\tilde{\Gamma}_1} \omega_1, \tilde{\Gamma}_1 \in \tilde{g}_{M_1}$, defining $J^{-1}(\mu),$

$$j^*_\mu \Gamma_1(\zeta) = j^*_\mu (i_{\Gamma_1} d\zeta) = j^*_\mu (i_{\Gamma_1} i_{\tilde{\Gamma}_1} \omega) = -j^*_\mu (i_{\tilde{\Gamma}_1} i_{\Gamma_1} \omega) = -j^*_\mu (i_{\tilde{\Gamma}_1} dh_1) = 0 .$$

Therefore, $\Gamma_1$ is tangent to $J^{-1}(\mu)$. Moreover, if $\Gamma_\mu \in \mathfrak{X}(J^{-1}(\mu))$ is a vector field such that $j_\mu * \Gamma_\mu = \Gamma_1|_{J^{-1}(\mu)}$, then

$$i_{\Gamma_\mu} \omega_\mu - dh_\mu = j^*_\mu (i_{\Gamma_1} \omega_1 - dh_1) = 0 ,$$

so the dynamical equation

$$i_{\Gamma_\mu} \omega_\mu - dh_\mu = 0 \tag{3.7}$$

is compatible and its solutions are $\Gamma_\mu + \ker \omega_\mu$. □

The last step is to obtain the orbit space $(J^{-1}(\mu)/G_\mu, \dot{\omega})$ (see Theorem 2). Consider the presymplectic Hamiltonian system $(J^{-1}(\mu), \omega_\mu, h_\mu)$, and the canonical projection $\pi_\mu: J^{-1}(\mu) \rightarrow J^{-1}(\mu)/\ker \omega_\mu$. As $(M_1, \omega_1)$ is symplectic, $J^{-1}(\mu)/G_\mu = J^{-1}(\mu)/\ker \omega_\mu$. Moreover, by the Marsden-Weinstein theorem, a symplectic form $\dot{\omega} \in \Omega^2(\ker \omega_\mu)$ exists such that $\omega_\mu = \pi^*_\mu \dot{\omega}$. Then:

**Proposition 2** The function $h_\mu$ and the vector field $\Gamma_\mu \in \mathfrak{X}(J^{-1}(\mu))$ satisfying (3.7) are $\pi_\mu$-projectable, and $(J^{-1}(\mu)/\ker \omega_\mu, \dot{\omega})$ is a symplectic Hamiltonian system, where $\pi^*_\mu \dot{h} = h_\mu$.

(Proof) In fact, $L_{\tilde{\Gamma}_1} h_\mu = 0$, for every $\tilde{\Gamma}_1 \in \tilde{g}_{M_1} \subset \tilde{g}_{M_1}$, since $h_1$ is $G$-invariant and then $h_\mu$ is $G_\mu$-invariant. Furthermore, for every $\tilde{\Gamma}_1 \in \tilde{g}_{M_1}$, since $\omega_\mu$ and $h_\mu$ are $G_\mu$-invariant, we have

$$i_{\tilde{\Gamma}_1} \omega_\mu = L_{\tilde{\Gamma}_1} i_{\Gamma_\mu} \omega_\mu - i_{\Gamma_\mu} L_{\tilde{\Gamma}_1} \omega_\mu = L_{\tilde{\Gamma}_1} dh_\mu = 0 ,$$

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and then \([\xi^M, \Gamma] \in \ker \omega_\mu\). But, as all the elements of \(\ker \omega_\mu\) can be expressed as \(Z_\mu = f^i \xi_{\mu i}\), then we also have that \([Z_\mu, \Gamma] \in \ker \omega_\mu\), for every \(Z_\mu \in \ker \omega_\mu\). Therefore, \(\Gamma_\mu\) is \(\pi_\mu\)-projectable.

Finally, since \(\tilde{g}M_\mu = \ker \omega_\mu\), for every \(x \in \mathcal{J}^{-1}(\mu)\), then \(\mathcal{J}^{-1}(\mu)/G_\mu = \mathcal{J}^{-1}(\mu)/\ker \omega_\mu\). As a consequence, \((\mathcal{J}^{-1}(\mu)/\ker \omega_\mu, \tilde{\omega})\) is a symplectic manifold. Hence, from (3.7), \((\mathcal{J}^{-1}(\mu)/\ker \omega_\mu, \tilde{\omega}, \tilde{h})\) is a symplectic Hamiltonian system and

\[
i_\Gamma \tilde{\omega} - d\tilde{h} = 0
\]

where \(\pi_\mu \Gamma_\mu = \tilde{\Gamma}\).

\[\square\]

### 3.2 The singular case: symmetries and reduction

As already pointed out, the concept of group of symmetries can be extended in a natural way to the case of singular optimal control problems. Given the presymplectic dynamical system \((M, \omega, H)\), if the optimal control problem is singular, then when we apply the constraint algorithm described in Section 3, a submanifold \(M_1\) is obtained. However, the 1-form \(\omega_1 = j_f^* \omega\) is now presymplectic and in the best of cases there will exist a family of vector fields satisfying the dynamical equation (2.9) in the points of \(M_1\) and tangent to \(M_1\). Otherwise, we apply the next steps in the constraint algorithm in order to obtain a final constraint submanifold \(M_f\) in which there exist vector fields \(\Gamma \in \mathfrak{X}(M)\) tangent to \(M_f\) such that

\[(i_\Gamma \omega - dH)|_{M_f} = 0.\]

If \(j_f: M_f \rightarrow M\) denotes the embedding, let us consider the presymplectic dynamical system \((M_f, \omega_f, h_f)\), where \(\omega_f = j_f^* \omega\) and \(h_f = j_f^* H\). If the 2-form \(\omega_f\) is nondegenerate, then the study of symmetries and the geometric reduction of the singular problem proceeds in a similar way to the regular case, just by replacing, in Definition 1, \((M_1, \omega_1, h)\) with the data \((M_f, \omega_f, h_f)\). It remains to consider the case when the compatible dynamical system \((M_f, \omega_f, h_f)\) is presymplectic. In this case we need to apply the generalization of the Marsden-Weinstein reduction theory to the case of presymplectic manifolds (see the Appendix 5.2), and replacing once again \((M_1, \omega_1, h)\) by \((M_f, \omega_f, h_f)\) in Definition 1. This action is denoted \(\Phi_f\). In both cases the problem is given by a group \(G\) acting on \(M\), and leaving \(M_f\) invariant.

If \(\mathcal{J}_f\) is the momentum map associated with the presymplectic action \(\Phi_f\) and \(\mu \in \mathfrak{g}^*\) is a weakly regular value, then the submanifold \(\mathcal{J}_f^{-1}(\mu)\) of \(M_f\), the form \(\omega_\mu = j_\mu^* \omega_f\) and the function \(h_\mu = j_\mu^* h_f\) make a compatible presymplectic Hamiltonian system \((\mathcal{J}_f^{-1}(\mu), \omega_\mu, h_\mu)\). The proof is similar to the regular case.

Let us denote by \(\tilde{\mathfrak{g}}_{M_f}\) the set of fundamental vector fields on \(M_f\) with respect to the action \(\Phi_f\), and let \(\Gamma_f\) be a solution to the dynamical system \(i_{\Gamma_f} \omega_f - dh_f = 0\). For every constraint \(\zeta\) defining \(\mathcal{J}_f^{-1}(\mu)\), if \(d\zeta = i_\xi M_f \omega_f\), with \(\xi^M \in \tilde{\mathfrak{g}}_{M_f}\), then \(j_\mu^* \Gamma_f(\zeta) = 0\). That is, the vector fields \(\Gamma_f\) are tangent to \(\mathcal{J}_f^{-1}(\mu)\). Moreover, if \(\Gamma_\mu \in \mathfrak{X}(\mathcal{J}_f^{-1}(\mu))\) is a vector field such that \(j_\mu^* \Gamma_\mu = \Gamma_f|_{\mathcal{J}_f^{-1}(\mu)}\), then the dynamical equation \(i_{\Gamma_\mu} \omega_\mu - dh_\mu = 0\) is compatible and its solutions are \(\Gamma_\mu + \ker \omega_\mu\). Finally, the procedure to obtain the orbit space \((\mathcal{J}_f^{-1}(\mu)/G_\mu, \tilde{\omega})\) and the Hamiltonian function \(\tilde{h}\) follows the same pattern as in the previous section.
4 Symmetries and reduction of optimal control problems: the non-autonomous case

In this section, we extend the previous results to the case of non-autonomous optimal control problems. After a description of a geometric formulation of the problem, we analyze the geometric reduction in the regular case. The extension of these results to the singular context is similar to that of autonomous problems.

4.1 Geometric description

If the optimal control problem is non-autonomous, then the control equation (2.1) becomes

\[ \dot{q}^i(t) = X^i(t, q(t), u(t)) \]  

and the objective functional to minimize is

\[ S[\gamma] = \int_{t_1}^{t_2} L(t, q(t), u(t)) \, dt \]

where at least one of the “functions” \( X^i \) \((i = 1, \ldots, n)\) or \( L \) depends explicitly on the time. A necessary condition for the existence of an optimum is still given by the Pontryagin’s maximum principle (2.4-2.3), where the Hamiltonian function is (2.2) (if we include the time-dependence).

Now we provide a geometric description of such equations when the maximum of \( H \) is not obtained on the boundary of the control set. A non-autonomous optimal control problem may be given by the following data: a configuration space, which is the trivial bundle \( \rho^1: \mathbb{R} \times Q \to \mathbb{R} \) (elements of \( Q \) describe the state variables, and \( \mathbb{R} \) is the time); a fibre bundle \( Id \times \pi: \mathbb{R} \times E \to \mathbb{R} \times Q \) (which is the identity in the first factor) whose fibres describe the control variables; a “Lagrangian function” \( L: \mathbb{R} \times E \to \mathbb{R} \); and a vector field \( X \) along the projection \( \rho_2 \circ (Id \times \pi) \) (where \( \rho_2: \mathbb{R} \times Q \to Q \) denotes the projection onto the second factor), i.e., \( X: \mathbb{R} \times E \to TQ \) is such that \( \tau_Q \circ X = \rho_2 \circ (Id \times \pi) \).

For sections \( \sigma: I \to \mathbb{R} \times E \) \((I = [t_1, t_2])\) such that \((Id \times \pi) \circ \sigma \) have fixed end-points, the problem is to find a section minimizing the action

\[ \int_{t_1}^{t_2} L(\sigma(t)) \, dt \]

when \( \sigma \) satisfies the differential equation

\[ \tilde{\rho}_2 \circ (Id \times \pi) \circ j^1 \sigma = X \circ \sigma, \]

where \( \tilde{\rho}_2: \mathbb{R} \times TQ \to TQ \) denotes the projection onto the second factor. So we have the following commutative diagramme:
The analog of the presymplectic description of autonomous systems shown in Section 2 is the following: we take the fiber bundle $\mathbf{R} \times \pi^* T^* Q$, which has canonical projections $Id \times \pi_1: \mathbf{R} \times \pi^* T^* Q \to \mathbf{R} \times E$ and $Id \times \pi_2: \mathbf{R} \times \pi^* T^* Q \to \mathbf{R} \times T^* Q$. Then the Hamiltonian function (2.2) can be defined intrinsically as

$$H = \tilde{X} - L,$$

(4.3)

where we identify the Lagrangian function $L \in C^\infty(\mathbf{R} \times E)$ with its pull-back through $Id \times \pi_1$ and $\tilde{X}: \mathbf{R} \times \pi^* T^* Q \to \mathbf{R}$ is the function defined by $\tilde{X}((t,q,u),\alpha) = \alpha(X(t,q,u))$.

If $\Theta$ and $\Omega$ are the pull-backs to $\mathbf{R} \times \pi^* T^* Q$ of the canonical forms in $T^* Q$, let $\Theta_H = \Theta + H dt = p_i dq^i + H dt$. Then the solutions to the equations of motion (2.3) and (2.5) (which are obtained as necessary conditions from the maximum Pontryagin’s principle if the control variables are interior points) are obtained from the integral curves of a vector field $\Gamma \in \mathfrak{X}(\mathbf{R} \times \pi^* T^* Q)$ verifying

$$i_\Gamma \Omega_H = 0, \quad i_\Gamma dt = 1,$$

(4.4)

where $\Omega_H = -d\Theta_H$, when we restrict the equations to the maximal manifold where a solution exists.

Once again, this is a presymplectic system, and a constraint algorithm similar to the one developed in Section 3 should be applied. It is easy to prove that, if we denote by $\tilde{M} = \mathbf{R} \times \pi^* T^* Q$, the maximal submanifold $\tilde{M}_1$ where a solution to equations (4.4) exists is described by equation (2.5), i.e., $\tilde{M}_1$ is defined locally by $\{ \varphi_a = 0 \}$, where $\varphi_a = \partial H/\partial u^a$. In this section we will assume that \( \det \left( \frac{\partial^2 H}{\partial u^a \partial u^b} \right) \neq 0 \), in such a way that we can solve locally the control variables as functions of the other variables, $u^a = \Psi^a(t,q,p)$. Then, the algorithm finishes in the first step, and there exists a unique vector field $\Gamma$ tangent to $\tilde{M}_1$ satisfying (4.4) in the points of $\tilde{M}_1$ (i.e., the optimal control problem is regular). In this case, $\tilde{M}_1$ is locally diffeomorphic to $\mathbf{R} \times T^* Q$. If $\tilde{j}_1: \tilde{M}_1 \to \tilde{M}$ denotes the embedding, let $h_1 = H(t,q,p,\Psi(t,q,p))$ be the pull-back through $\tilde{j}_1$ of $H$. Then $\Omega_{h_1} = \tilde{j}_1^* (\Omega_H) = \Omega_1 + h_1 dt$, where $\Omega_1 = \tilde{j}_1^* \Omega$. If $\dim Q = n$, then $\dim \tilde{M}_1 = 2n + 1$ and $\Omega_{h_1}$ is of maximal rank $2n$, in such a way that the pair $(\Omega_{h_1}, dt)$ is a cosymplectic structure of $\tilde{M}_1$, since $\Omega_{h_1} \wedge dt \neq 0$. If $\Gamma_1 \in \tilde{M}_1$ is the unique vector field solution to the dynamical equations

$$i_{\Gamma_1} (\Omega_1 + h_1 dt) = 0, \quad i_{\Gamma_1} dt = 1,$$

then the vector field $\Gamma$ is obtained from $\Gamma_1$ by using the feedback condition $u^a = \Psi^a(t,q,p)$ and the boundary conditions.

### 4.2 Symmetries and reduction

Concerning the study of symmetries, time-dependent optimal control problems display some particular characteristics which are worth consideration. Following the ideas in [11], if $G$ is a Lie group, $(\tilde{M}, \Omega_H)$ is a non-autonomous optimal control system and $\Phi: G \times \tilde{M} \to \tilde{M}$ is an action of $G$ on $\tilde{M}$, $G$ is said to be a group of standard symmetries of this system if, for every $g \in G$,

1. $\Phi$ leaves $\tilde{M}_1$ invariant, i.e., it induces an action $\Phi_1: G \times \tilde{M}_1 \to \tilde{M}_1$;
2. \((\Phi_1)_g^*\) preserves the forms \(\Omega_1\) and \(dt\) (it is a cosymplectic action), that is, \((\Phi_1)_g^*\Omega_1 = \Omega_1\); \((\Phi_1)_g^*dt = dt\).

3. and \((\Phi_1)_g^*\) preserves the dynamical function \(h_1\): i.e., \((\Phi_1)_g^*h_1 = h_1\).

The diffeomorphisms \(\Phi_g\) are called standard symmetries of the system.

As an immediate consequence of this definition, if \(G\) is a group of standard symmetries of the non-autonomous system \((\tilde{M}, \Omega_H)\) then, for every \(g \in G\), \((\Phi_1)_g^*\) preserves the form \(\Omega_1\): \((\Phi_1)_g^*\Omega_1 = \Omega_1\). Moreover, \(G\) is a group of standard symmetries of the non-autonomous system above if, and only if, the following three conditions hold for every \(\xi \in g\):

\[
\begin{align*}
(1) & \quad L_\xi \Omega_1 = 0 \\
(2) & \quad L_\xi dt = 0 \\
(3) & \quad L_\xi h_1 = 0
\end{align*}
\]

At this point, reduction of regular non-autonomous optimal control problems with symmetry follows a similar pattern to the reduction made above of autonomous optimal control problems with symmetry. Actually, singular optimal control problems can be studied by using similar ideas.

**Remark 4** We would like to point out that the reduced Hamiltonian system does not describe, in general, an optimal control problem. To show this, it is enough to recall that every variational problem can be written as an optimal control problem by taking \(E = TQ\) in the control bundle \(R \times E \to R \times Q\), and considering as control equations \(\dot{q}^i = u^i\) (where \((q^i, u^i)\) denote the local coordinates in \(E = TQ\)). However, in [26] it is shown that, in general, the reduced Hamiltonian system is not a Lagrangian system. To study when the reduced Hamiltonian system describes an optimal control problem, an inverse problem should be solved.

5 **Examples**

5.1 **Reduction of regular optimal control problems invariant by a vector field**

In order to illustrate the above results, we study the case where the optimal control problem is invariant by a vector field, that is, there exists a vector field \(Z \in \mathfrak{X}(E)\) which is an infinitesimal symmetry of the regular optimal control problem \((L, \pi, Q, X, L)\) (see Definition 3). As \(Z\) is \(\pi\)-projectable, let \(Z_0 = \pi_*Z \in \mathfrak{X}(Q)\). In local coordinates,

\[
Z^i = f^i(q) \frac{\partial}{\partial q^i} + g^a(q, u) \frac{\partial}{\partial u^a} \quad \text{and} \quad Z_0^i = f^i(q) \frac{\partial}{\partial q^i}.
\]

We can lift \(Z \in \mathfrak{X}(E)\) to a new vector field \(Z^c \in \mathfrak{X}(M)\) (where \(M = \pi^*T^*Q\), as usual) whose local expression is

\[
Z^c = f^i(q) \frac{\partial}{\partial q^i} - p_j \frac{\partial f^j}{\partial q^i} \frac{\partial}{\partial p_i} + g^a(q, u) \frac{\partial}{\partial u^a}.
\]

Let \(\varphi_t\), \(\Psi_t\) and \(\Phi_t\) be the flows of vector fields \(Z_0 \in \mathfrak{X}(Q)\), \(Z \in \mathfrak{X}(E)\) and \(Z^c \in \mathfrak{X}(M)\), respectively. Let \(\Gamma \in \mathfrak{X}(M)\) be the vector field in \(M\) constructed by the extension of the vector field \(\Gamma_1 \in \mathfrak{X}(M_1)\).
solution to the optimal control problem in $M_1$, using feedback condition (2.6). Then $\Phi_t^*\Gamma$ is also a solution to the dynamical system in the sense that its restriction to $M_1$, is again a vector field tangent to $M_1$ verifying the dynamical equation restricted to $M_1$.

Let us assume that the vector field $Z$ is complete. Then $Z^c$ is also complete, and it induces an action
\[ \Phi: \mathbb{R} \times M \to M \]
\[ (t, (q, u, p)) \mapsto \Phi_t(q, p, u) \]
which restricts to a new action
\[ \Phi_1: \mathbb{R} \times M_1 \to M_1. \]
Since $Z$ is an infinitesimal symmetry of the optimal control problem, then $\mathbb{R}$ is a symmetry group of the presymplectic dynamical system $(M_1, \omega_1, h_1)$ under this action, and $f_Z = \langle \theta, Z^c \rangle$ is a constant of motion ($f_Z$ is the comomentum map). Let $J$ be the dual momentum map, and consider the level set
\[ J^{-1}(\mu) = \{(q, p) \in M_1 \mid p_t f^i(q) = \mu \}, \]
where $\mu \in \mathbb{R}$. If $Z_0$ is non-vanishing everywhere, then $J^{-1}(\mu)$ is a submanifold of $M_1$. Moreover, in this case, $T_{(q,p)}J$ is surjective and therefore every $\mu \in \mathbb{R}$ is a regular value. In general, for an arbitrary $\pi$-projectable and complete vector field $Z$, $\mu \in \mathbb{R}$ will be neither regular nor almost regular. If we assume that $\mu$ is at least almost regular, let us consider the dynamical system $(J^{-1}(\mu), \omega_\mu, h_\mu)$ where $j_\mu: J^{-1}(\mu) \to M_1$ denotes the embedding, $\omega_\mu = j_\mu^* \omega_1 = (j_\mu^* \circ j_1^*) \omega = (j_1 \circ j_\mu)^* \omega$ and $h_\mu = j_\mu^* h_1 = (j_\mu^* \circ j_1^*) H = (j_1 \circ j_\mu)^* H$. Since $\dim J^{-1}(\mu) = 2 \dim Q - 1$, then the dynamical system is presymplectic, and the reduction procedure finishes quotienting by $\ker \omega_\mu$ ($h_\mu$ is projectable under this distribution).

It is interesting to realize that the momentum map $J: M_1 \to \mathbb{R}$ can be extended to a map $J: M \to \mathbb{R}$ whose local expression coincides with the local expression of the momentum map $J$. The level sets are again
\[ J^{-1}(\mu) = \{(q, u, p) \in M \mid \langle \theta, Z^c \rangle = p_t f^i(q) = \mu \}. \]
Moreover, $J$ is a momentum map, since the action is strictly presymplectic.

Let $(J^{-1}(\mu), \bar{\omega}_\mu, \bar{H}_\mu)$ be the presymplectic dynamical system given by $\bar{\omega}_\mu = j_\mu^* \omega$ and $\bar{H}_\mu = j_\mu^* H$, where $j_\mu: J^{-1}(\mu) \to M$ denotes the embedding (again, we assume that $\mu$ is, at least, an almost regular value of the momentum map). The presymplectic dynamical system $(J^{-1}(\mu), \bar{\omega}_\mu, \bar{H}_\mu)$ has solution in the points of $J^{-1}(\mu)$.

### 5.2 Shortest paths with bounded curvature

The following example is a free version of a problem which is studied from a different point of view in [31] (see also the quoted references), and consists in characterizing the shortest $C^1$-curves that are parametrized by arc length satisfying a curvature bound, and going from a given initial position and velocity to a final one.
In this model the configuration space is $Q = \mathbb{R}^3 \times S^2$, with local coordinates $(x^i, y^i)$ $(i = 1, 2, 3)$, with $\sum_i (y^i)^2 = 1$. The control space is, in principle, the closed unit ball $B^3$ in $\mathbb{R}^3$, and hence the bundle of controls is $E = \mathbb{R}^3 \times S^2 \times B^3$, with coordinates $(x^i, y^i, u^i)$ $(i = 1, 2, 3)$. The differential equations are

$$\dot{x} = y; \quad \dot{y} = y \times u$$

where $x \equiv (x^1, x^2, x^3)$, $y \equiv (y^1, y^2, y^3)$, $u \equiv (u^1, u^2, u^3)$, and $y \times u$ denotes the cross product in $\mathbb{R}^3$. The Lagrangian function for this problem is $L = 1$.

The presymplectic Hamiltonian description of this system is made in the manifold $E \times_Q T^*Q$, where we have the coordinates $(x^i, y^i, u^i; p_i, q_i)$ ($p_i, q_i$ are the conjugate momenta associated with the position coordinates $x^i, y^i$). The canonical forms are then

$$\theta = p_i dx^i + q_i dy^i, \quad \omega = dx^i \wedge dp_i + dy^i \wedge dq_i$$

The Hamiltonian function is

$$H = \langle p, y \rangle + \langle q, y \times u \rangle + 1$$

(where $\langle , \rangle$ denotes the usual scalar product in $\mathbb{R}^3$, arising from the duality between $TQ$ and $T^*Q$). Observe that $H$ is linear on the controls, and so it is known that the optimal solutions for the controls are in the boundary of $B^3$; that is in $S^2$, unless $y \times u = 0$, when every value of the controls gives an optimal solution. Hence we can take $E = \mathbb{R}^3 \times S^2 \times S^2$.

This system exhibits symmetries which are:

- Rigid translations in $\mathbb{R}^3$, whose action on $E = Q \times S^2$ is as follows: for a given $v \in \mathbb{R}^3$, if $\tau_v: \mathbb{R}^3 \to \mathbb{R}^3$ is the translation $x \mapsto x + v$, we have that $\tau_v(x, y, u) = (x + v, y, u)$.

- Rotations on $\mathbb{R}^3$ which act on $E$ in the following way: for a given rotation $R \in SO(3)$, we have that $R(x, y, u) = (Rx, Ry, Ru)$.

That is, the group of symmetries is $G = \mathbb{R}^3 \times SO(3)$. The infinitesimal generators are the following vector fields in $E$

$$\xi_i = \frac{\partial}{\partial x^i} \quad (i = 1, 2, 3)$$

$$\xi_4 = x^1 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^1} + y^1 \frac{\partial}{\partial y^2} - y^2 \frac{\partial}{\partial y^1} + u^1 \frac{\partial}{\partial u^2} - u^2 \frac{\partial}{\partial u^1}$$

$$\xi_5 = x^2 \frac{\partial}{\partial x^3} - x^3 \frac{\partial}{\partial x^2} + y^2 \frac{\partial}{\partial y^3} - y^3 \frac{\partial}{\partial y^2} + u^2 \frac{\partial}{\partial u^3} - u^3 \frac{\partial}{\partial u^2}$$

$$\xi_6 = x^3 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^3} + y^3 \frac{\partial}{\partial y^1} - y^1 \frac{\partial}{\partial y^3} + u^3 \frac{\partial}{\partial u^1} - u^1 \frac{\partial}{\partial u^3}$$

whose canonical liftings to $E \times_Q T^*Q$ give the following fundamental vector fields

$$\xi^c_i = \frac{\partial}{\partial x^i} \quad (i = 1, 2, 3)$$

$$\xi^c_4 = x^1 \frac{\partial}{\partial p_2} - x^2 \frac{\partial}{\partial p_1} + y^1 \frac{\partial}{\partial q_2} - y^2 \frac{\partial}{\partial q_1} + u^1 \frac{\partial}{\partial u_2} - u^2 \frac{\partial}{\partial u_1} +$$

$$p^1 \frac{\partial}{\partial p^1} - p^2 \frac{\partial}{\partial p^2} + q^1 \frac{\partial}{\partial q^1} - q^2 \frac{\partial}{\partial q^2}$$
\[\hat{\xi}_i^c = x^2 \frac{\partial}{\partial x^3} - x^3 \frac{\partial}{\partial x^2} + y^2 \frac{\partial}{\partial y^3} - y^3 \frac{\partial}{\partial y^2} + u^2 \frac{\partial}{\partial u} - u^3 \frac{\partial}{\partial u^2} + \]
\[p^2 \frac{\partial}{\partial p^3} - p^3 \frac{\partial}{\partial p^2} + q^2 \frac{\partial}{\partial q^3} - q^3 \frac{\partial}{\partial q^2} \]
\[\hat{\xi}_6 = x^3 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^3} + y^3 \frac{\partial}{\partial y^1} - y^1 \frac{\partial}{\partial y^3} + u^3 \frac{\partial}{\partial u^1} - u^1 \frac{\partial}{\partial u^3} + \]
\[p^3 \frac{\partial}{\partial p^1} - p^1 \frac{\partial}{\partial p^3} + q^3 \frac{\partial}{\partial q^1} - q^1 \frac{\partial}{\partial q^3} \]

Thus \(\{\hat{\xi}_1^c, \hat{\xi}_2^c, \hat{\xi}_3^c, \hat{\xi}_4^c, \hat{\xi}_5^c, \hat{\xi}_6^c\}\) is a set of generators of \(\hat{g}\), but observe that \(\dim \hat{g} = 5\). The action considered is strongly presymplectic, since it is an exact action in relation to the 1-form \(\theta\). The presymplectic Hamiltonian functions \(f_{\xi_j} \in C^\infty(E \times_Q T^*Q)\) \((j = 1, \ldots, 6)\) associated with \(\hat{\xi}_j^c\) are
\[f_{\xi_i} = p_i \quad (i = 1, 2, 3)\]
\[f_{\xi_4} = x^1 p_2 - x^2 p_1 + y^1 q_2 - y^2 q_1\]
\[f_{\xi_5} = x^2 p_3 - x^3 p_2 + y^2 q_3 - y^3 q_2\]
\[f_{\xi_6} = x^3 p_1 - x^1 p_3 + y^3 q_1 - y^1 q_3\]

So a momentum map \(J\) can be defined for this action, and for every weakly regular value \(\mu \equiv (\mu_1, \ldots, \mu_6) \in g^*\), its level sets \(J^{-1}(\mu)\) foliate \(E \times_Q T^*Q\), and are defined as submanifolds of \(E \times_Q T^*Q\) by the constraints \(f_{\xi_j} = \mu_j\) \((j = 1, \ldots, 6)\); that is, they are made by the points where the vectors \(p\) and \(x \times p + y \times q\) are constant. Observe that, locally, only 5 of these constraints are functionally independent and, as \(\dim (E \times_Q T^*Q) = 12\), then the submanifolds \(J^{-1}(\mu)\) are 7-dimensional (and presymplectic). Locally, each one of them can be described by coordinates \((x^i, u^i, z^k)\) \(((i = 1, 2, 3; k = 1, 2)\), where \(z^k\) are coordinates which can be chosen from the set \((y^i, q_i)\).

Next, the final step of the reduction procedure consists in constructing the quotient manifolds \((J^{-1}(\mu)/G_\mu, \hat{\Omega}_\mu)\) (with natural projections \(\sigma_\mu: J^{-1}(\mu) \to J^{-1}(\mu)/G_\mu\)). First notice that all the fundamental vector fields \(\hat{\xi}_j\) are tangent to the submanifolds \(J^{-1}(\mu)\), hence the isotropy group is \(G_\mu = G\), and the quotient manifolds \(J^{-1}(\mu)/G_\mu\) are 2-dimensional. They are described locally by coordinates \((u^k)\) \((k = 1, 2)\), such that \(\sigma^*_\mu u^k\) are functions of the coordinates \((u^i, z^k)\).

As a particular case, we can analyze when \(\mu = 0\). Then the constraints defining the submanifold \(J^{-1}(0)\) are
\[p_i = 0 \quad (i = 1, 2, 3) \quad ; \quad y^1 q_2 - y^2 q_1 = 0 \quad , \quad y^2 q_3 - y^3 q_2 = 0 \quad , \quad y^3 q_1 - y^1 q_3 = 0\]
that is, \(p = 0\), and \(y \times q = 0\) (i.e.; \(y = \lambda q\), with \(\lambda \in \mathbb{R}\)). Observe that, if \(j_0: J^{-1}(0) \hookrightarrow E \times_Q T^*Q\) denotes the natural embedding, then
\[H_0 := j_0^* H = 1 \quad , \quad \omega_0 := j_0^* \omega = 0\]
Therefore, in the quotient manifolds \(J^{-1}(0)/G_0\) we have
\[\hat{H}_0 = 1 \quad , \quad \hat{\omega}_0 = 0\]
where \(\hat{H}_0 \in C^\infty(J^{-1}(0)/G_0)\) and \(\hat{\omega}_0 \in \Omega^2(J^{-1}(0)/G_0)\) are such that \(\sigma^*_\mu \hat{H}_0 = H_0\) and \(\sigma^*_\mu \hat{\omega}_0 = \omega_0\). Hence, the dynamical equation in \(J^{-1}(0)/G_0\) has as solutions all the vector fields \(\hat{X}_0 \in \mathfrak{X}(J^{-1}(0)/G_0)\), whose local expressions are
\[\hat{X}_0 = \hat{F}_k(w) \frac{\partial}{\partial w^k} \quad (\hat{F}_k \in C^\infty(J^{-1}(0)/G_0))\]
This result agrees with the analysis made in [31].

Appendix: Actions of Lie groups on presymplectic manifolds and reduction

(This appendix is a review of the results given in [11]).

Given the presymplectic manifold \((P, \Omega)\), a vector field \(Y \in \mathfrak{X}(P)\) is said to be a Hamiltonian vector field if \(i_Y \Omega = d f_Y\). We denote by \(\mathfrak{X}_h(P)\) the set of Hamiltonian vector fields in \(P\). A function \(f \in C^\infty(P)\) is a Hamiltonian function if there exists a vector field \(Y_f \in \mathfrak{X}(P)\) such that the above equation holds. We denote by \(C^\infty_h(P)\) the set of Hamiltonian functions in \(P\).

A function \(f \in C^\infty(P)\) is said to be a locally Hamiltonian vector field if \(i_Y \Omega\) is a closed 1-form. We denote by \(\mathfrak{X}_{lh}(P)\) the set of locally Hamiltonian vector fields in \(P\). Clearly, \(\mathfrak{X}_h(P) \subset \mathfrak{X}_{lh}(P)\). Furthermore, \(Y \in \mathfrak{X}_{lh}(P)\) if and only if \(L_Y \Omega = 0\). For every \(Y \in \mathfrak{X}_{lh}(P)\) and \(Z \in \ker \Omega\), we have that \([Y, Z] \in \ker \Omega\).

Now, let \(\Phi: P \to P\) be a diffeomorphism. \(\Phi\) is said to be a canonical transformation for the presymplectic manifold \((P, \Omega)\) if \(\Phi^* \Omega = \Omega\). In a similar way, if \(Y \in \mathfrak{X}(P)\) is a vector field such that its flow \(\Phi_t\) satisfies \(\Phi_t^* \Omega = \Omega\), then \(Y\) is said to be an infinitesimal canonical transformation of the presymplectic manifold. It is clear that \(\Phi_t^* \Omega = \Omega\) if, and only if, \(L_Y \Omega = 0\) and, hence, \(Y\) is an infinitesimal canonical transformation if, and only if, it is a locally Hamiltonian vector field.

Let \(G\) be a connected Lie group, \(\mathfrak{g}\) its Lie algebra and \(\Phi: G \times P \to P\) a presymplectic action of \(G\) on \((P, \Omega)\); that is, \(\Phi_g^* \Omega = \Omega\), for every \(g \in G\). As a consequence, the fundamental vector fields \(\xi \in \mathfrak{X}(P)\), associated with \(\xi \in \mathfrak{g}\) by \(\Phi\), are locally Hamiltonian vector fields, \(\xi \in \mathfrak{X}_{lh}(P)\) (conversely, if for every \(\xi \in \mathfrak{g}\) we have that \(\xi \in \mathfrak{X}_{lh}(P)\), then \(\Phi\) is a presymplectic action of \(G\) on \(P\)). Therefore, for every \(\xi \in \mathfrak{g}\), \(L_\xi \Omega = 0\). We denote by \(\mathfrak{g}\) the set of fundamental vector fields. When \(\mathfrak{g} \subseteq \mathfrak{X}_h(P)\), the action \(\Phi\) is said to be strongly presymplectic or Hamiltonian. Otherwise, \(\Phi\) is called weakly presymplectic or locally Hamiltonian. In particular, if \((P, \Omega)\) is an exact presymplectic manifold, \(\Omega = -d\Theta\), and the action \(\Phi\) is exact (that is, \(\Phi_g^* \Theta = \Theta\), for every \(g \in G\)), then \(\Phi\) is strongly presymplectic and the fundamental vector fields are Hamiltonian, with associated Hamiltonian functions \(f_\xi = i_\xi \Theta\).

Given a presymplectic action \(\Phi\) of a connected Lie group \(G\) on the presymplectic manifold \((P, \Omega)\), the comomentum map associated with \(\Phi\), [30], is a map (if it exists)

\[
\mathcal{J}^* : \mathfrak{g} \to C^\infty_h(P)
\]

\[
\xi \mapsto f_\xi
\]

where, if \(\xi \in \mathfrak{g}\), and \(\tilde{\xi}\) is its associated fundamental vector field, then \(f_\xi\) is the function such that \(i_{\tilde{\xi}} \Omega = df_\xi\). The momentum map associated with \(\Phi\) is the dual map of the comomentum map; in other words, it is a map \(\mathcal{J}: P \to \mathfrak{g}^*\) such that, for every \(\xi \in \mathfrak{g}\) and \(x \in P\),

\[
(\mathcal{J}(x))(\xi) := \mathcal{J}^*(\xi)(x) = f_\xi(x)
\]
From the definitions, it follows that both the comomentum and momentum maps exist if, and only if, the presymplectic action $\Phi$ on $(P, \Omega)$ is strongly presymplectic (in particular, if $\Phi$ is exact then both mappings exist). In general, a comomentum map is not a Lie algebra homomorphism. An action $\Phi$ is said to be *Poissonian* or strongly *Hamiltonian* if there exists a comomentum map which is a Lie algebra homomorphism. Once again, if the action $\Phi$ is exact, then $\Phi$ is Poissonian and the comomentum map is given by $J^*(\xi) = i_\xi \Theta$, for every $\xi \in \mathfrak{g}$.

Let us assume that $\Phi$ is strongly presymplectic. If $J$ is the associated momentum map, then an element $\mu \in \mathfrak{g}^*$ is a weakly regular value of $J$ if $J^{-1}(\mu)$ is a submanifold of $P$, and $T_x(J^{-1}(\mu)) = \ker \Omega_x$, for every $x \in J^{-1}(\mu)$. Moreover, if $T_xJ$ is surjective for every $x \in J^{-1}(\mu)$, then $\mu$ is said to be a regular value. In this paper, here, every action $\Phi$ is assumed to be Poissonian, free and proper, and $\mu \in \mathfrak{g}^*$ is a weakly regular value of $J$. We denote by $j_\mu : J^{-1}(\mu) \hookrightarrow P$ the corresponding immersion.

Next, we give a brief description of $J^{-1}(\mu)$, for every almost regular value $\mu \in \mathfrak{g}^*$. If $\{\xi_i\}$ is a basis of $\mathfrak{g}$ with dual basis $\{\alpha^i\}$ in $\mathfrak{g}^*$, by writing $\mu = \mu_i \alpha^i$, a simple computation shows that there exist Hamiltonian functions associated with the fundamental vector fields $\{\xi_i\}$ such that $J^{-1}(\mu) = \{x \in P \mid f_\xi(x) = \mu_i\}$. In particular, if $\xi \in \mathfrak{g}$ is such that $\tilde{\xi} \in \ker \Omega$, the Hamiltonian functions can be taken to be equal to zero and, in this case, $\langle \mu, \xi \rangle = 0$.

The connected components of the level sets of the momentum map $J$ can be also obtained as the connected maximal integral submanifolds of the Pfaff system $i_\xi \Omega = 0$, for $\xi \in \mathfrak{g}$. Therefore, if $x \in J^{-1}(\mu)$, then $T_xJ^{-1}(\mu) = \tilde{\mathfrak{g}}_x^\perp$. As a consequence, since $\ker \Omega_x \subset \tilde{\mathfrak{g}}_x^\perp$, then $\ker \Omega \subset \tilde{\mathfrak{g}}(J^{-1}(\mu))$ (where $\tilde{\mathfrak{g}}(J^{-1}(\mu))$ denotes the set of vector fields of $\tilde{\mathfrak{g}}(P)$ which are tangent to $J^{-1}(\mu)$). If the action is exact, then $f_\xi = -i_\xi \Theta$ and the Pfaff system $i_\xi \Omega = 0$ can be expressed as $d(i_\xi \Omega) = 0$.

Let $G_\mu$ be the isotropy group of $\mu$ for the coadjoint action of $G$ on $\mathfrak{g}^*$. Then $G_\mu$ is the maximal subgroup of $G$ which leaves $J^{-1}(\mu)$ invariant. So, the quotient $J^{-1}(\mu)/G_\mu$ is well defined and it is called the reduced phase space or the orbit space of $J^{-1}(\mu)$. The Lie algebra $\tilde{\mathfrak{g}}_\mu$ of $G_\mu$ is made of vector fields tangent to $J^{-1}(\mu)$, and we have that $\tilde{\mathfrak{g}}_\mu = \tilde{\mathfrak{g}} \cap \tilde{\mathfrak{g}}(J^{-1}(\mu))$.

At this point, we indicate two different possibilities. If $\tilde{\mathfrak{g}} \cap \ker \Omega = \{0\}$, then all the fundamental vector fields give constraints which are not constant functions, and $\dim J^{-1}(\mu) < \dim P$. On the other hand, if $\tilde{\mathfrak{g}} \cap \ker \Omega \neq \{0\}$, only the fundamental vector fields not belonging to $\ker \Omega$ give constraints which are not constant functions, and $\dim J^{-1}(\mu) \leq \dim P$. Anyway, $J^{-1}(\mu)$ inherits a presymplectic structure $\Omega_\mu := j_\mu^* \tilde{\Omega}$, whose characteristic distribution is $\ker \Omega_\mu = \tilde{\mathfrak{g}}_\mu + \ker \Omega_x$, for every $x \in J^{-1}(\mu)$.

Finally, the generalization of the Marsden-Weinstein reduction theorem [23] to presymplectic actions of Lie groups on presymplectic manifolds is:

**Theorem 2** The orbit space $J^{-1}(\mu)/G_\mu$ is a differentiable manifold. If $\sigma : J^{-1}(\mu) \to J^{-1}(\mu)/G_\mu$ denotes the canonical projection, then there is a closed 2-form $\tilde{\Omega} \in \Omega^2(J^{-1}(\mu)/G_\mu)$ such that $\Omega_\mu = \sigma^* \tilde{\Omega}$ (that is, $\Omega_\mu$ is $\sigma$-projectable), and:

- $\tilde{\Omega}$ is symplectic if, and only if, for every $x \in J^{-1}(\mu)$, $\tilde{\mathfrak{g}}_{\mu_x} = \ker \Omega_{\mu_x}$ or, what is equivalent,
\[ \ker \Omega_x \cap T_x J^{-1}(\mu) \subseteq \tilde{g}_{\mu_x}. \]

- Otherwise, \( \hat{\Omega} \) is presymplectic. In particular, for every \( x \in J^{-1}(\mu) \), if \( \ker \Omega_x \subset T_x J^{-1}(\mu) \) and \( \tilde{g}_x \cap \ker \Omega_x = \{0\} \), then \( \text{rank} \hat{\Omega} = \text{rank} \Omega \).

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