Simple Finite Non-Abelian Flavor Groups

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Abstract

The recently measured unexpected neutrino mixing patterns have caused a resurgence of interest in the study of finite flavor groups with two- and three-dimensional irreducible representations. This paper details the mathematics of the two finite simple groups with such representations, the Icosahedral group $A_5$, a subgroup of $SO(3)$, and $PSL_2(7)$, a subgroup of $SU(3)$.

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1 Introduction

The gauge interactions of the Standard Model are associated with symmetries which naturally generalize to a Grand-Unified structure. Yet there is no corresponding recognizable symmetry in the Yukawa couplings of the three chiral families. I. I. Rabi’s old question about the muon, “Who ordered it?”, remains unanswered. Today, in spite of the large number of measured masses and mixing angles, the origin of the chirality-breaking Yukawa interactions remain shrouded in mystery.

A natural suggestion that the three chiral families assemble in three- or two-dimensional representations of a continuous group, single out SU(3) \([1]\), or SU(2) \([2]\), respectively, as natural flavor groups, which have to be broken at high energies to avoid flavor-changing neutral processes. Unfortunately, such hypotheses did not prove particularly fruitful.

However, the recently measured MNSP lepton-mixing matrix displays a strange pattern with two large and one small rotation angle. The near zero in one of its entries suggests a non-Abelian finite symmetry \([3-8]\). Indeed a matrix which approximates the measured mixing appears naturally in the smallest non-Abelian finite group, \(S_3\).

Since there are three chiral families, it is natural to concentrate on those finite group which have two and three dimensional irreducible representations: the possible finite flavor groups must be finite subgroups of continuous SU(2), SO(3), and SU(3). These finite subgroups were all identified early on by mathematicians, and are listed in a ninety years old textbook \([9]\).

One finds only two simple groups in their list. One is the Icosahedral group, a subgroup of SO(3); the other is PSL\(_2\)(7), a subgroup of SU(3). There are no finite simple subgroups of SU(2). The purpose of this paper is to study in some detail these groups, in particular PSL\(_2\)(7). For their application in quiver theories, see \([10]\).

2 Finite Groups with Two-dimensional Irreps

The finite groups with two-dimensional representations were identified as early as 1876 by Felix Klein, who considered the fractional linear transformations on two complex variables

\[
Z' = \begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = M Z = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix},
\]

where \(M\) is a unitary matrix with unit determinant. He then constructed the three quadratic forms, shown here in modern notation, (vector analysis had not yet been invented)

\[
\tilde{r} = Z^\dagger \tilde{\sigma} Z,
\]

where \(\tilde{\sigma}\) are known today as the Pauli (not Klein) matrices. He clearly understood how to go from SU(2) to SO(3), as he realized that the transformations
acted on the vector \( \vec{r} \) as rotations about the origin. These are five types of rotations,

- rotations of order \( n \) about one axis; they generate the finite Abelian group \( \mathbb{Z}_n \).
- rotations with two rotation axes which generate the Dihedral group \( D_n \) with \( 2n \) elements.
- rotations which map the tetrahedron into itself form \( T = A_4 \), the Tetrahedral group with 12 elements.
- rotations which map the octahedron into itself form \( O = S_4 \), the Octahedral group with 24 elements.
- rotations which map the icosahedron (dodecahedron) into itself form \( I = A_5 \), the Icosahedral group of order 60, the only simple finite rotation group.

The Dihedral groups, the symmetry groups of plane polygons, have only singlet and real doublet representations \((n > 2)\). The last three, \( T, O \) and \( I \), are symmetry groups of the Platonic solids. As \( SO(3) \) subgroups, all three have real triplet representations, but no two-dimensional spinor-like representations. Each can be thought of as the quotient group of its binary-equivalent group with at least one two-dimensional “spinor” representation and twice the number of elements (double cover). Below we give in some detail the construction of the Icosahedral group and of its binary form.

**Icosahedral Group \( A_5 \)**

We begin with a description of \( A_5 = I \), the only simple finite \( SO(3) \) subgroup. Its sixty group elements are generated by two elements, \( A \) and \( B \), with two equivalent presentations

\[
\langle A, B \mid A^2 = B^3 = (AB)^5 = 1 \rangle ,
\]

\[
\langle A, B, C \mid A^2 = B^3 = C^5 = (ABC) = 1 \rangle .
\] (3)

It is isomorphic to \( PSL_2(5) \), the group of projectively defined \((2 \times 2)\) matrices of unit determinant over \( \mathbb{F}_5 \), the finite Galois field with five elements. It is the symmetry group of the icosahedron, with 12 rotations of order 5 (by \( 72^\circ \)), 12 rotations of order 5 (by \( 144^\circ \)), 20 rotations of order 3 (by \( 60^\circ \)), and 15 rotations of order 2 (by \( 180^\circ \)). These correspond to its conjugacy classes, displayed in its character table (the prefactor and the square bracket denote the number and the order\(^1\) of the elements in the class, respectively).

\(^1\)The order of an element of a finite group is the smallest power needed to obtain the identity element.
with

\[ b_5 = \frac{1}{2}(1 + \sqrt{5}), \quad b'_5 = \frac{1}{2}(1 - \sqrt{5}). \]  

(4)

It has five real irreducible representations, two of which are triplets. We compute their Kronecker products, using the following general method: given two irreps \( r \) and \( s \), with Kronecker product

\[ r \otimes s = \sum_t d(r, s, t) t, \]  

(5)

the positive integers \( d(r, s, t) \) are given by the formula

\[ d(r, s, t) = \frac{1}{N} \sum_i n_i \chi_i^r \chi_i^s \chi_i^t. \]  

(6)

Here, \( N \) is the order of the group, the sum is over all classes and \( n_i \) denotes the corresponding number of elements in the class. Applied to \( A_5 \), we obtain the following table, where we indicate the symmetric and the antisymmetric products of \( r \otimes r \) by the subscripts \( s \) and \( a \), respectively.

| \( A_5 \) Kronecker Products |
|-----------------------------|
| \( 3_1 \otimes 3_1 = 3_1_1 + (1 + 5)_s \) |
| \( 3_2 \otimes 3_2 = 3_2_1 + (1 + 5)_s \) |
| \( 3_1 \otimes 3_2 = 4 + 5 \) |
| \( 3_1 \otimes 4 = 3_2 + 4 + 5 \) |
| \( 3_2 \otimes 4 = 3_1 + 4 + 5 \) |
| \( 3_1 \otimes 5 = 3_1 + 3_2 + 4 + 5 \) |
| \( 3_2 \otimes 5 = 3_1 + 3_2 + 4 + 5 \) |
| \( 4 \otimes 4 = (3_1 + 3_2)_a + (1 + 4 + 5)_s \) |
| \( 5 \otimes 5 = (3_1 + 3_2 + 4)_a + (1 + 4 + 5 + 5)_s \) |
| \( 4 \otimes 5 = 3_1 + 3_2 + 4 + 5 + 5 \) |
Binary Icosahedral Group $\mathcal{S}\mathcal{L}_2(5)$

Starting from $\mathcal{A}_5 = \mathcal{P}\mathcal{S}\mathcal{L}_2(5)$, we can construct a closely related group by dropping the projective restriction on the matrices. This yields $\mathcal{S}\mathcal{L}_2(5)$, a group with twice as many elements, the binary form of $\mathcal{A}_5$ (see e.g. [11]). This group is no longer simple since it has a normal subgroup of order two.

Its presentation is of the same form as for $\mathcal{A}_5$

$$< A, B, C \mid A^2 = B^3 = C^5 = (ABC) > ,$$  \hspace{1em} (7)

but here the element $A^2$ is not the identity, but in some sense the negative of the identity, so that its square is the identity element. Hence it has only one element of order two, and $\mathcal{A}_5$ is not a subgroup, but its largest quotient group.

In terms of representations, it corresponds to adding (at least) one two-dimensional spinor representation to those of $\mathcal{A}_5$. In that representation, the generators are given by

$$C = \begin{pmatrix} -\chi^2 & 0 \\ 0 & -\chi^3 \end{pmatrix} , \quad B = \frac{1}{\sqrt{5}} \begin{pmatrix} \chi^4 - \chi^2 & \chi - 1 \\ 1 - \chi^4 & \chi - \chi^3 \end{pmatrix} , \quad A = BC ,$$  \hspace{1em} (8)

where

$$\chi^5 = 1 .$$  \hspace{1em} (9)

$\mathcal{S}\mathcal{L}_2(5)$ has in fact four spinor-like representations, $2_s$, $2_s'$, $4_s$, and $6_s$. The character table is given by

| $\mathcal{S}\mathcal{L}_2(5)$ | $C_2$ | $12C_3^{[2]}$ | $12C_3^{[5]}$ | $12C_4^{[0]}$ | $12C_5^{[10]}$ | $20C_4^{[3]}$ | $20C_8^{[6]}$ | $30C_9^{[4]}$ |
|-----------------|-----|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| $\chi^{[1]}$    | 1   | 1              | 1              | 1              | 1              | 1              | 1              | 1              |
| $\chi^{[31]}$   | 3   | 3              | $b_3$          | $b_3'$         | $b_3$          | 0              | 0              | -1             |
| $\chi^{[32]}$   | 3   | 3              | $b_3$          | $b_3'$         | $b_3$          | 0              | 0              | -1             |
| $\chi^{[4]}$    | 4   | 4              | -1             | -1             | -1             | 1              | 1              | 1              |
| $\chi^{[5]}$    | 5   | 5              | 0              | 0              | 0              | -1             | -1             | 1              |
| $\chi^{[2s]}$   | 2   | -2             | $-b_5$         | $-b_5'$        | $b_5$          | 0              | 0              | 0              |
| $\chi^{[2s']}$  | 2   | -2             | $-b_5'$        | $-b_5$         | $b_5'$         | -1             | -1             | 0              |
| $\chi^{[4s]}$   | 4   | -4             | -1             | -1             | 1              | 1              | 1              | -1             |
| $\chi^{[6s]}$   | 6   | -6             | 1              | 1              | -1             | -1             | 0              | 0              |
We can easily find the Kronecker products of its irreps, by using the usual technique, but for the finite subgroups of $SU(2)$, there is another method which relies on the *McKay Correspondence* [12]. This theorem states that Kronecker products are related to extended Dynkin diagrams. In the following we illustrate this relation for $SL_2(5)$.

Any finite subgroup of $SU(2)$ will have (at least) one spinor doublet $2_s$, and of course one singlet irrep. We are going to generate its nine irreducible representations by taking repeated products with the spinor doublets, $2_s$. Start with the singlet

\[ 1 \otimes 2_s = 2_s \]

which we represent graphically as follows: assign a dot to the $1$ and one dot the $2_s$ which appears on the right-hand-side. We connect these two dots with a line, to indicate that they are obtained from one another by taking the product with $2_s$. We repeat the process with the right-most dot

\[ 2_s \otimes 2_s = 1 + 3_1 \]

which yields a third dot for the $3_1$, connected to the $2_s$ dot by a line. So far we have a linear diagram of three dots connected by a line. Next we consider the product $2_s \otimes 3_1$, which has to be a sum of spinorial representations. They are uniquely determined by the McKay Correspondence, which says that the diagram obtained in this way is the $E_8$ extended Dynkin diagram!

\[
\begin{array}{cccccc}
1 & 2_s & 3_1 & & & \\
\end{array}
\]

Hence there must be a $4_s$ at the fourth dot, that is

\[ 2_s \otimes 3_1 = 2_s + 4_s \]

Similarly,

\[ 2_s \otimes 4_s = 3_1 + 5 \]

yields a sum of tensor irreps which continues the linear chain. The product

\[ 2_s \otimes 5 = 4_s + 6_s \]

adds one more rung to the linear chain. Next, since the right-hand side can only contain tensor irreps, with a unique solution
we see that the linear chain breaks into two branches. The dot on the first branch is $3_2$, and it terminates, indicating that

$$2_s \otimes 3_2 = 6_s.$$ 

The second branch keeps going, requiring that

$$2_s \otimes 4 = 6_s + 2_s'.$$

The last dot associated with $2_s'$ terminates in order to satisfy the McKay Correspondence, that is

$$2_s \otimes 2_s' = 4.$$

This completes the process, with one irrep associated with each dot:

Similar considerations can be applied to all finite $SU(2)$ subgroups, all of which are associated with a distinctive extended Dynkin diagram. For the binary forms of the groups $Z_n$, $D_n$, $T$, $O$, and $I$ we have the extended Dynkin diagrams of $A_n$, $D_n$, $E_6$, $E_7$, and $E_8$, respectively.

3 $\mathcal{P}SL_2(7)$

Finite groups with complex three-dimensional representations are subgroups of continuous $SU(3)$. We refer the reader to the previously mentioned textbook [9] for the complete list, which contains only one simple group, $\mathcal{P}SL_2(7)$². In the remainder of this paper we work out its representations and discuss some of its properties.

$\mathcal{P}SL_2(7)$ is the projective special linear group of $(2 \times 2)$ matrices over $\mathbb{F}_7$, the finite Galois field of seven elements. It contains 168 elements, and has one complex three-dimensional irrep and its conjugate. It is isomorphic to $\mathcal{G}L_3(2)$, the group of non-singular $(3 \times 3)$ matrices with entries in $\mathbb{F}_2$. It is therefore an interesting candidate group for explaining the three chiral families.

²Labeled $\Sigma(168)$ in Refs. [9] [13].
As we have seen, it is economical to describe finite groups in terms of their presentation. The $\text{PSL}_2(7)$ presentation we use is given in terms of two generators $A$ and $B$, as

\[ < A, B | A^2 = B^3 = (AB)^7 = [A, B]^4 = 1 >, \]  

(10)

where 1 is the identity element, and $[A, B] = A^{-1}B^{-1}AB$. Written as $(2 \times 2)$ matrices over $\mathbb{F}_7$, the generators of $\text{PSL}_2(7)$ can be taken to be

\[ A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \]  

(11)

so that

\[ AB = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad [A, B] = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}. \]  

(12)

The group has six classes

\[ C_1^{[i]}(1) \quad \text{with one element}, \]
\[ C_2^{[i]}(A) \quad \text{with 21 elements}, \]
\[ C_3^{[i]}(B) \quad \text{with 56 elements}, \]
\[ C_4^{[i]}([A, B]) \quad \text{with 42 elements}, \]
\[ C_5^{[i]}(AB) \quad \text{with 24 elements}, \]
\[ C_6^{[i]}(AB^2) \quad \text{with 24 elements}, \]

where we have indicated the simplest element of each class in parentheses. It has therefore six irreps, whose properties are summarized in the character table:

\[
\begin{array}{ccccccc}
\text{PSL}_2(7) & C_1^{[i]} & \text{21C}_2^{[i]}(A) & 56C_3^{[i]}(B) & 42C_4^{[i]}([A, B]) & 24C_5^{[i]}(AB) & 24C_6^{[i]}(AB^2) \\
\chi^{[1]} & 1 & 1 & 1 & 1 & 1 & 1 \\
\chi^{[3]} & 3 & -1 & 0 & 1 & b_7 & \overline{b}_7 \\
\chi^{[3]} & 3 & -1 & 0 & 1 & \overline{b}_7 & b_7 \\
\chi^{[6]} & 6 & 2 & 0 & 0 & -1 & -1 \\
\chi^{[7]} & 7 & -1 & 1 & -1 & 0 & 0 \\
\chi^{[8]} & 8 & 0 & -1 & 0 & 1 & 1 \\
\end{array}
\]
where we have used the “Atlas of Finite Groups” notation

\[ b_7 = \frac{1}{2}(-1 + i\sqrt{7}), \quad \overline{b}_7 = \frac{1}{2}(-1 - i\sqrt{7}) . \]  

From the character table, we can directly compute the Kronecker products of the irreducible representations. The result is:

\[
\begin{array}{|c|c|}
\hline
\text{PSL}_2(7) & \text{Kronecker Products} \\
\hline
3 \otimes 3 &= \overline{3}_a + 6_s \\
3 \otimes \overline{3} &= 1 + 8 \\
3 \otimes 6 &= \overline{3} + 7 + 8 \\
\overline{3} \otimes 6 &= 3 + 7 + 8 \\
3 \otimes 7 &= 6 + 7 + 8 \\
\overline{3} \otimes 7 &= 6 + 7 + 8 \\
3 \otimes 8 &= 3 + 6 + 7 + 8 \\
\overline{3} \otimes 8 &= 3 + 6 + 7 + 8 \\
6 \otimes 6 &= (1 + 6 + 6 + 8)_s + (7 + 8)_a \\
6 \otimes 7 &= 3 + \overline{3} + 6 + 7 + 7 + 8 + 8 \\
6 \otimes 8 &= 3 + \overline{3} + 6 + 7 + 7 + 8 + 8 + 8 \\
7 \otimes 7 &= (1 + 6 + 6 + 7 + 8)_s + (3 + \overline{3} + 7 + 8)_a \\
7 \otimes 8 &= 3 + \overline{3} + 6 + 6 + 7 + 7 + 8 + 8 + 8 + 8 + 8 + 8 + 8 \\
8 \otimes 8 &= (1 + 6 + 6 + 7 + 8 + 8)_s + (3 + \overline{3} + 7 + 7 + 8)_a \\
\hline
\end{array}
\]

**SU(3) Subgroup**

Since \( \text{PSL}_2(7) \) is a subgroup of \( SU(3) \), linear combinations of its six irreducible representations must add up to \( SU(3) \) representations. Since the lowest non-trivial representation of both groups is the triplet, they must be the same. It immediately follows that the antitriplets also correspond to each other (identifying the triplet of one with the antitriplet of the other is a matter of convention).

Consider the product of two \( SU(3) \) triplets, which yields \( \overline{3} + 6 \). Comparison with the Kronecker product of \( \text{PSL}_2(7) \) shows that the \( 6 \) of \( SU(3) \) and the \( \overline{3} \) of \( \text{PSL}_2(7) \) correspond to each other. Similarly, the product of a triplet and an antitriplet shows that the \( 8 \) of both groups match. This method can be applied to obtain the decompositions of the smallest irreps of \( SU(3) \), as shown in the table below:
Although the smallest irreps of the two groups look the same, the differences begin at the level of the sextet. In SU(3) the sextet is a complex representation, while in PSL₂(7) it is real (see later). In SU(3), there is a unique invariant made up of three sextets, 666; in PSL₂(7) the Kronecker product shows two separate cubic invariants. A second and more obvious difference is that SU(3) has no seven-dimensional representation.

4 PSL₂(7) Irreducible Representations

The purpose of this section is to give explicit matrix realizations for the six irreducible representations of PSL₂(7). It is sufficient to derive the matrix expression for its two generators, $A$ and $B$; in the process we will also derive some of the Clebsch-Gordan coefficients.

4.1 The Triplet Representation

From the table of Kronecker products, we see that all irreducible representations can be generated from the 3; once we compute the PSL₂(7) generators in the triplet representation, we can deduce all the others by taking multiple products of the triplet representation.

We use the presentation and the character table to build the triplet representation. Let $\eta$ be a seventh root of unity, that is

$$\eta^7 = 1. \quad (14)$$

It is well known (in some circles), that

$$b_7 = \frac{1}{2}(-1 + i\sqrt{7}) = \eta + \eta^2 + \eta^4, \quad (15)$$
so that, in the triplet representation, the trace of the elements of order seven is a sum of three powers of $\eta$. Folding in the fact that we are interested in an irrep of $(3 \times 3)$ unitary matrices with unit determinant, we choose a basis where $AB$ is diagonal, and set

$$AB = \begin{pmatrix} \eta & 0 & 0 \\ 0 & \eta^2 & 0 \\ 0 & 0 & \eta^4 \end{pmatrix}.$$  \hspace{1cm} (16)

Since $B$ is an element of order 3, it follows that

$$A = \begin{pmatrix} \eta & 0 & 0 \\ 0 & \eta^2 & 0 \\ 0 & 0 & \eta^4 \end{pmatrix} B^2 ,$$

but $B$ is a unitary matrix, that is $B^2 = B^\dagger$, so that

$$A = \begin{pmatrix} \eta & 0 & 0 \\ 0 & \eta^2 & 0 \\ 0 & 0 & \eta^4 \end{pmatrix} B^\dagger .$$

In addition, $A^2 = 1$ infers that

$$B^2 \begin{pmatrix} \eta & 0 & 0 \\ 0 & \eta^2 & 0 \\ 0 & 0 & \eta^4 \end{pmatrix} = \begin{pmatrix} \eta^2 & 0 & 0 \\ 0 & \eta^2 & 0 \\ 0 & 0 & \eta^4 \end{pmatrix} B ,$$

which means that the matrix

$$\begin{pmatrix} \eta^2 & 0 & 0 \\ 0 & \eta^2 & 0 \\ 0 & 0 & \eta^4 \end{pmatrix} B ,$$

is its own inverse, and therefore hermitian. This restriction allows us to write

$$B = \begin{pmatrix} x\eta & a\eta^n & b\eta^m \\ a\eta^{3-n} & y\eta^2 & c\eta^p \\ b\eta^{5-m} & c\eta^{6-p} & z\eta^4 \end{pmatrix} ,$$  \hspace{1cm} (17)

where $a, b, c, x, y, z$ are real, and $m, n, p$ are integers to be determined. The character table for the triplet yields

$$-1 = \text{Tr} A = x + y + z ,$$

$$0 = \text{Tr} B = x\eta + y\eta^2 + z\eta^4 ,$$

from which it is straightforward to determine $x, y, z$

$$x = \frac{i}{\sqrt{7}}(\eta^2 - \eta^5) , \quad y = \frac{i}{\sqrt{7}}(\eta^4 - \eta^3) , \quad z = \frac{i}{\sqrt{7}}(\eta - \eta^6) .$$  \hspace{1cm} (18)
It follows that (absorbing $\sqrt{7}$ in $a, b, c$

\[
B = \frac{i}{\sqrt{7}} \begin{pmatrix}
\eta^3 - \eta^6 & -i\eta^n & -ib\eta^m \\
-ia\eta^{3-n} & \eta^6 - \eta^5 & -ic\eta^p \\
-ib\eta^{5-m} & -i\eta^6 - p & \eta^5 - \eta^3
\end{pmatrix}.
\]

(19)

Now

\[
B^\dagger = -\frac{i}{\sqrt{7}} \begin{pmatrix}
\eta^4 - \eta & ia\eta^{n-3} & ib\eta^{m-5} \\
ia\eta^{-n} & \eta - \eta^2 & ic\eta^{p-6} \\
ib\eta^{-m} & i\eta^{p-5} & \eta^2 - \eta^4
\end{pmatrix}.
\]

Since $B$ is unitary, $B^\dagger B = 1$, we find

\[
a(1 - \eta + \eta^2 - \eta^4) - ibc\eta^{m-p-n} = 0,
\]

\[
b(1 + \eta - \eta^3 - \eta^4) + iac\eta^{n+p-m} = 0,
\]

\[
c(\eta^2 - \eta - \eta^4 + \eta^6) - iab\eta^{m-p-n} = 0,
\]

for the off-diagonal elements, and

\[
2 - \eta^3 - \eta^4 + a^2 + b^2 = 7,
\]

\[
2 - \eta - \eta^6 + a^2 + c^2 = 7,
\]

\[
2 - \eta^2 - \eta^5 + b^2 + c^2 = 7,
\]

for the diagonal elements. The vanishing of the off-diagonal elements fixes

\[
n - m + p = 2 \mod 7,
\]

and

\[
bc = 2a \left( \sin \frac{4\pi}{7} - \sin \frac{6\pi}{7} \right),
\]

\[
ac = 2b \left( \sin \frac{4\pi}{7} + \sin \frac{2\pi}{7} \right),
\]

\[
ab = 2c \left( \sin \frac{2\pi}{7} - \sin \frac{6\pi}{7} \right).
\]

These have a sign ambiguity as we can change the sign of any two variables without affecting the equations. We choose the solutions

\[
a = i(\eta - \eta^6), \quad b = i(\eta^4 - \eta^3), \quad c = i(\eta^2 - \eta^5),
\]

(20)

as well as

\[
n = 2, \quad m = p = 4.
\]

(21)
Assembling these results, we find the explicit form of the generators in the triplet representation

\[
B^{[3]} = \frac{i}{\sqrt{7}} \begin{pmatrix} \eta^3 - \eta^6 & \eta^3 - \eta & \eta - 1 \\
\eta^2 - 1 & \eta^6 - \eta^5 & \eta^6 - \eta^2 \\
\eta^5 - \eta^3 & \eta^4 - 1 & \eta^5 - \eta^3 \end{pmatrix},
\]

(22)

\[
A^{[3]} = \frac{i}{\sqrt{7}} \begin{pmatrix} \eta^2 - \eta^5 & \eta - \eta^6 & \eta^4 - \eta^3 \\
\eta - \eta^6 & \eta^4 - \eta^3 & \eta^2 - \eta^5 \\
\eta^5 - \eta^3 & \eta^2 - \eta^5 & \eta - \eta^6 \end{pmatrix}.
\]

(23)

The matrix \(A\) is manifestly real and symmetric, reproducing Klein’s involution for \(A\), which can be found in Klein’s original 1878 paper [15].

The generators in the antitriplet representation, \(A^{[3]}\) and \(B^{[3]}\), are simply the complex conjugates of the generators in the triplet. To see this, we note that \(A = A^T\) is an involution \(AA = A^T A^T = (AA)^T = 1\). Also, the matrix \(B = (BB)^T\) obeys \(BB = B^T\), and is of order three as \(BBB = B^T B = B^3 = 1\). Hence

\[
A^{[3]} = \overline{A}^{[3]}, \quad B^{[3]} = \overline{B}^{[3]}.
\]

(24)

4.2 The Sextet Representation

The explicit construction of the sextet from the symmetric product of two triplets allows us to derive at the same time the Clebsch-Gordan coefficients.

Consider two independent irreps, \(3\) and \(3'\), spanned by \(|i\rangle\) and \(|i'\rangle\), \(i, i' = 1, 2, 3\). We denote the six symmetric states of the sextet \(6\) with a special ket notation as \(|\alpha\rangle\), \(\alpha = 1, 2, ..., 6\):

\[
|\alpha\rangle = \sum_{i,j} K^{ij}_{\alpha} |i\rangle |j'\rangle,
\]

(25)

where \(K^{ij}_{\alpha}\) are the Clebsch-Gordan coefficients. Their values are determined by setting

\[
|1\rangle = |1\rangle |1'\rangle, \quad |2\rangle = |2\rangle |2'\rangle, \quad |3\rangle = |3\rangle |3'\rangle, \quad |4\rangle = \frac{1}{\sqrt{2}}(|1\rangle |2'\rangle + |2\rangle |1'\rangle), \quad |5\rangle = \frac{1}{\sqrt{2}}(|3\rangle |1'\rangle + |1\rangle |3'\rangle), \quad |6\rangle = \frac{1}{\sqrt{2}}(|2\rangle |3'\rangle + |3\rangle |2'\rangle).
\]

(26)

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\(^3\) The generators \(A(\frac{2\pi}{7}, \frac{4\pi}{7}), E(0,0), Z(\xi)\) of Ref. [13] are related to our matrices \(A\) and \(B\) by \(A(\frac{2\pi}{7}, \frac{4\pi}{7}) = AB, \quad E(0,0) = [(AB)^4(AB^2)^2A] \cdot B \cdot [(AB)^4(AB^2)^2A]^{-1}, \quad Z(\xi = e^{\frac{4\pi}{7}}) = A.\)
We display the same information in the following Clebsch-Gordan table

\[
\begin{array}{c|cccc|cccc}
| i > | j' > & | 1 \rangle & | 2 \rangle & | 3 \rangle & | 4 \rangle & | 5 \rangle & | 6 \rangle \\
| 1 > | 1' > & 1 & 0 & 0 & 0 & 0 & 0 \\
| 1 > | 2' > & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\
| 1 > | 3' > & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\
| 2 > | 1' > & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\
| 2 > | 2' > & 0 & 1 & 0 & 0 & 0 & 0 \\
| 2 > | 3' > & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\
| 3 > | 1' > & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\
| 3 > | 2' > & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\
| 3 > | 3' > & 0 & 0 & 1 & 0 & 0 & 0 \\
\end{array}
\]

In order to get the explicit representations for the generators, we set

\[
A^{[6]} = A^{[3]} A^{[3]'} ,
\]

where \( A^{[3]} \) acts on \( | i > \), and \( A^{[3]'} \) on \( | i' > \), that is

\[
A^{[6]} | i > | j' > = A^{[3]} | i > A^{[3]'} | j' > = \sum_{m,n} a_{i,m} a_{j,n} | m > | n' > ,
\]

where \( a_{i,m} \) are the matrix elements of \( A \) in the triplet that we have just derived.

A tedious but straightforward calculation yields the \((6 \times 6)\) matrix

\[
A^{[6]} = -\frac{2\sqrt{2}}{7} \begin{pmatrix}
\frac{1}{\sqrt{2}}(c_3 - 1) & \frac{1}{\sqrt{2}}(c_2 - 1) & \frac{1}{\sqrt{2}}(c_1 - 1) & c_3 - c_1 & c_1 - c_2 & c_2 - c_3 \\
\frac{1}{\sqrt{2}}(c_2 - 1) & \frac{1}{\sqrt{2}}(c_3 - 1) & \frac{1}{\sqrt{2}}(c_1 - 1) & c_2 - c_3 & c_3 - c_1 & c_1 - c_2 \\
\frac{1}{\sqrt{2}}(c_1 - 1) & \frac{1}{\sqrt{2}}(c_2 - 1) & \frac{1}{\sqrt{2}}(c_3 - 1) & c_1 - c_2 & c_2 - c_3 & c_3 - c_1 \\
c_3 - c_1 & c_2 - c_3 & c_1 - c_2 & \frac{1}{\sqrt{2}}(c_1 - 1) & \frac{1}{\sqrt{2}}(c_2 - 1) & \frac{1}{\sqrt{2}}(c_3 - 1) \\
c_1 - c_2 & c_3 - c_1 & c_2 - c_3 & \frac{1}{\sqrt{2}}(c_2 - 1) & \frac{1}{\sqrt{2}}(c_3 - 1) & \frac{1}{\sqrt{2}}(c_1 - 1) \\
c_2 - c_3 & c_1 - c_2 & c_3 - c_1 & \frac{1}{\sqrt{2}}(c_3 - 1) & \frac{1}{\sqrt{2}}(c_1 - 1) & \frac{1}{\sqrt{2}}(c_2 - 1)
\end{pmatrix} ,
\]

written in terms of

\[
c_n = \cos \left( \frac{2n\pi}{7} \right) , \quad n = 1, 2, 3 ,
\]

which satisfy

\[
c_1 + c_2 + c_3 = -\frac{1}{2} .
\]
This enables us to check that the trace

\[ \text{Tr } A^{[6]} = -\frac{2}{7} (-6 + 2(c_1 + c_2 + c_3)) = 2 , \]

is consistent with the character table. We note also that

\[ s_n = \sin \left( \frac{2n\pi}{7} \right), \quad s_1 + s_2 - s_3 = \frac{\sqrt{7}}{2}. \tag{32} \]

In order to derive the form of \( B^{[6]} \), we first note that the order-seven combination of the generators is the diagonal matrix

\[ A^{[6]}B^{[6]} = \text{diag} (\eta^2, \eta^4, \eta, \eta^3, \eta^5, \eta^6) . \tag{33} \]

Since

\[ B^{[6]} = A^{[6]} \left( A^{[6]}B^{[6]} \right), \tag{34} \]

we easily find the second generator

\[
B^{[6]} = -\frac{2\sqrt{7}}{7} \times \\
\begin{pmatrix}
\frac{\eta^2}{\sqrt{2}}(c_3 - 1) & \frac{\eta^4}{\sqrt{2}}(c_2 - 1) & \frac{\eta^6}{\sqrt{2}}(c_1 - 1) & \eta^3(c_3 - c_1) & \eta^3(c_1 - c_2) & \eta^6(c_2 - c_3) \\
\frac{\eta^2}{\sqrt{2}}(c_2 - 1) & \frac{\eta^4}{\sqrt{2}}(c_1 - 1) & \frac{\eta^6}{\sqrt{2}}(c_3 - 1) & \eta^3(c_2 - c_3) & \eta^3(c_3 - c_1) & \eta^6(c_1 - c_2) \\
\frac{\eta^2}{\sqrt{2}}(c_1 - 1) & \frac{\eta^4}{\sqrt{2}}(c_3 - 1) & \frac{\eta^6}{\sqrt{2}}(c_2 - 1) & \eta^3(c_1 - c_2) & \eta^3(c_2 - c_3) & \eta^6(c_3 - c_1) \\
\eta^2(c_3 - c_1) & \eta^4(c_2 - c_3) & \eta(c_1 - c_2) & \frac{\eta^3}{\sqrt{2}}(c_1 - 1) & \frac{\eta^3}{\sqrt{2}}(c_2 - 1) & \frac{\eta^6}{\sqrt{2}}(c_3 - 1) \\
\eta^2(c_1 - c_2) & \eta^4(c_3 - c_1) & \eta(c_2 - c_4) & \frac{\eta^3}{\sqrt{2}}(c_3 - 1) & \frac{\eta^3}{\sqrt{2}}(c_2 - 1) & \frac{\eta^6}{\sqrt{2}}(c_1 - 1) \\
\eta^2(c_2 - c_3) & \eta^4(c_1 - c_2) & \eta(c_3 - c_4) & \frac{\eta^3}{\sqrt{2}}(c_2 - 1) & \frac{\eta^3}{\sqrt{2}}(c_1 - 1) & \frac{\eta^6}{\sqrt{2}}(c_2 - 1)
\end{pmatrix}. \tag{35}\]

This matrix is no longer hermitian, but we easily check that it is traceless, as required by the character table.

Although \( B^{[6]} \) and \( A^{[6]}B^{[6]} \) are complex matrices, their traces are real. It means that \( 6 \) and \( \bar{6} \) are equivalent, in the sense that their generators are related by a similarity transformation that is itself a group element (inner automorphism).

Our basis has the advantage that we can easily read-off the Clebsch-Gordan coefficients, but the quadratic invariant made out of two sextets involves a special matrix. To find it, we start from

\[ I^{[2]} = \sum_{\alpha,\beta} C^{\alpha\beta} | \alpha \rangle \langle \beta | , \tag{36} \]

where \( C \) is to be determined. Invariance under transformations generated by \( A \) and \( B \) requires, in matrix form,

\[ C = (A^{[6]})^T CA^{[6]} = (B^{[6]})^T CB^{[6]} . \tag{37} \]
From

\[ \mathbf{C} = (A^{[6]}B^{[6]})^T C A^{[6]}B^{[6]}, \]

we deduce that the non-zero elements of the symmetric \( \mathbf{C} \) are \( C_{15} = C_{51}, C_{24} = C_{42}, \) and \( C_{36} = C_{63} \). The unknown coefficients are easily determined by using

\[ A^{[6]}C = CA^{[6]}, \]

yielding (up to an overall normalization)

\[
\mathbf{C} = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}.
\] (38)

Since \( \mathbf{C} \) is a symmetric real matrix, we can find a basis where the matrices \( A^{[6]} \) and \( B^{[6]} \) are real. This is achieved by the similarity transformation generated by the matrix

\[ \mathbf{S} = \mathbf{VP}, \quad A^{[6]}_{\text{real}} = \mathbf{S} A^{[6]} \mathbf{S}^{-1}, \quad B^{[6]}_{\text{real}} = \mathbf{S} B^{[6]} \mathbf{S}^{-1}, \] (39)

where \( \mathbf{P} \) is a diagonal matrix

\[ \mathbf{P} = \text{diag}(i\eta^3, \eta^6, \eta^6, -\eta^4, i, -\eta^4), \]

and \( \mathbf{V} \) is a symmetric matrix with equal diagonal elements equal to \( 1/\sqrt{2} \), and with off-diagonal non-zero matrix elements

\[ V_{15} = V_{24} = V_{36} = \frac{i}{\sqrt{2}}. \] (41)

In this basis, \( \mathbf{C} \) becomes proportional to the unit matrix, and the quadratic invariant is trivial.

### 4.3 The Octet Representation

From the Kronecker product

\[ \mathbf{3} \otimes \mathbf{3} = \mathbf{1} + \mathbf{8}, \]

it is straightforward to construct the octet representation. The singlet is given by the combination

\[ \frac{1}{\sqrt{3}} (|1\rangle + |1\rangle + |2\rangle + |2\rangle + |3\rangle + |3\rangle). \] (42)

The eight states are arranged with the familiar \( SU(3) \) labeling, \( |A\rangle, \ A = 1, 2, ..., 8, \) with
\(|1\rangle = \frac{1}{\sqrt{2}} (|1\rangle |2\rangle + |2\rangle |\bar{1}\rangle)\), \(|2\rangle = \frac{-i}{\sqrt{2}} (|1\rangle |2\rangle - |2\rangle |\bar{1}\rangle)\),

\(|3\rangle = \frac{1}{\sqrt{2}} (|1\rangle |\bar{1}\rangle - |2\rangle |\bar{2}\rangle)\),

\(|4\rangle = \frac{1}{\sqrt{2}} (|1\rangle |\bar{3}\rangle + |3\rangle |\bar{1}\rangle)\), \(|5\rangle = \frac{-i}{\sqrt{2}} (|1\rangle |\bar{3}\rangle - |3\rangle |\bar{1}\rangle)\),

\(|6\rangle = \frac{1}{\sqrt{2}} (|2\rangle |\bar{3}\rangle + |3\rangle |\bar{2}\rangle)\), \(|7\rangle = \frac{-i}{\sqrt{2}} (|2\rangle |\bar{3}\rangle - |3\rangle |\bar{2}\rangle)\),

\(|8\rangle = \frac{1}{\sqrt{6}} (|1\rangle |\bar{1}\rangle + |2\rangle |\bar{2}\rangle - 2 |3\rangle |\bar{3}\rangle)\). \hspace{1cm} (43)

This corresponds to expressing the octet states, with the Gell-Mann matrices

acting as Clebsch-Gordan coefficients

\(|A\rangle = \frac{1}{\sqrt{2}} \sum_{i,j} (\lambda A)_{ij} |i\rangle |j\rangle\). \hspace{1cm} (44)

Using the form of \(AB\) in the triplet and antitriplet representations, it is relatively easy to work out the order-seven element, with the result

\[
\langle A^{[8]} B^{[8]} \rangle = \begin{pmatrix}
    c_1 & s_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    -s_1 & c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & c_{27} & s_3 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & -s_3 & c_3 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & c_2 & s_2 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & -s_2 & c_2 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix} . \hspace{1cm} (45)
\]

It is straightforward, albeit tedious, to work out \(A\) in the octet. We find

\[A^{[8]} = \frac{1}{7} \times \hspace{1cm} (46)\]

\[
\begin{pmatrix}
    2 - 2c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    2c_1 - 2c_2 - 4c_3 & 0 & 2c_1 + 2c_2 + 2c_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
    2 - 2c_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    2 - 2c_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    2c_1 - 2c_2 - 4c_3 & 0 & 2c_1 + 2c_2 + 2c_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
    2c_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    2 - 2c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    2c_1 + 2c_2 - 4c_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    2 - 2c_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

This real and symmetric matrix satisfies \((A^{[8]})^2 = 1\). Having determined \(A^{[8]}\) as well as \((AB)^{[8]}\), it is easy to obtain the order-three generator \(B^{[8]}\), by using

\[B^{[8]} = A^{[8]} (A^{[8]} B^{[8]}) . \hspace{1cm} (47)\]
The quadratic invariant for two octet fields is given by

$$I^{[2]} = \sum_{A,B} \delta_{AB} |A\rangle |B\rangle .$$  \hspace{1cm} (48)

Note that we do not distinguish upper and lower indices for this representation since this invariant is diagonal.

### 4.4 The Septet Representation

In order to work out the seven-dimensional representation, we consider the Kronecker product

$$3 \otimes 6 = \bar{3} + 7 + 8 .$$

To avoid confusion, we label the states of the 7 by $|a\rangle$, $a = 1, 2, ..., 7$. Since we are in a basis where $AB$ is diagonal in both the 3 and the 6, these eighteen states can be split into linear combinations of eigenstates of the form $|i\rangle |\alpha\rangle$.

For instance those with unit eigenvalue are

$$\mu_1 |1\rangle |6\rangle + \mu_2 |2\rangle |5\rangle + \mu_3 |3\rangle |4\rangle ,$$  \hspace{1cm} (49)

with $\mu_i$ to be determined by the orthogonality with respect to the two states in the octet and the one in the septet.

This enables us to construct the linear combinations that transform according to irreps of $PSL_2(7)$, and derive in the process the Clebsch-Gordan coefficients.

For the 3, we set

$$|\bar{j}\rangle = \sum_{i,\alpha} \tilde{K}^{ij\alpha} |i\rangle |\alpha\rangle ,$$  \hspace{1cm} (50)

and notice that these linear combinations have already been determined, since

$$\tilde{K}^{ij\alpha} = \sum_{\beta} K^{ij\beta} C^{\beta\alpha} .$$  \hspace{1cm} (51)

We express the states of the 8 as the linear combinations

$$|A\rangle = \sum_{i,\alpha} M^{i\alpha} |i\rangle |\alpha\rangle ,$$  \hspace{1cm} (52)

and find the Clebsch-Gordan coefficients. They are to be found in Table A-1.

For the 7, we set

$$|a\rangle = \sum_{i,\alpha} L^{i\alpha} |i\rangle |\alpha\rangle .$$  \hspace{1cm} (53)

These Clebsch-Gordan coefficients can be found in Table A-2.
Although in this basis the phases of the seven states have been chosen as to yield a real \( A^{[7]} \), \( (AB)^{[7]} \) is not real
\[
(A^{[7]}B^{[7]}) = \text{diag}(1, \eta, \eta^2, \eta^3, \eta^4, \eta^5, \eta^6),
\]
corresponding to our ordering of the states. One consequence is that the quadratic invariant is not diagonal (in this basis). Indeed, if we write it as
\[
I^{[2]} = \sum_{a,b} D^{ab} |a \rightarrow b>,
\]
we find the non-zero elements of the symmetric matrix \( D \) to be
\[
D^{11} = D^{27} = D^{36} = D^{45} = D^{54} = D^{63} = D^{72} = 1.
\]
We obtain for the generator \( A^{[7]} \)
\[
A^{[7]} = \frac{2}{\sqrt{i}} \times
\]
\[
\begin{pmatrix}
\sqrt{\pi} & \sqrt{\pi} & \sqrt{\pi} & \sqrt{\pi} & \sqrt{\pi} & \sqrt{\pi} & \sqrt{\pi} \\
\sqrt{\pi} & \pi & \sqrt{\pi} & \sqrt{\pi} & \sqrt{\pi} & \sqrt{\pi} & \sqrt{\pi} \\
2s_1 - 2s_2 - 4s_3 & 2s_1 - 2s_2 - 4s_3 & 2s_1 - 2s_2 - 4s_3 & 2s_1 - 2s_2 - 4s_3 & 2s_1 - 2s_2 - 4s_3 & 2s_1 - 2s_2 - 4s_3 & 2s_1 - 2s_2 - 4s_3
\end{pmatrix}
\]
Again \( B^{[7]} \) is obtained by multiplying \( A^{[7]} \) with \( (AB)^{[7]} \), since \( A^{[7]} \) is an involution.

As for the sextet, one can use a similarity transformation

\[
U = \frac{1}{\sqrt{2}} \begin{pmatrix}
\sqrt{2} e^{-\frac{\pi}{4}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -i & 0 \\
0 & 0 & 1 & 0 & 0 & -i & 0 \\
0 & 0 & 0 & 1 & -i & 0 & 0 \\
0 & 0 & 0 & -i & 1 & 0 & 0 \\
0 & 0 & -i & 0 & 0 & 1 & 0 \\
0 & -i & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
\]

to make these matrices real: \( A^{[7]} \) does not change, that is \( A^{[7]} \equiv UA^{[7]}U^\dagger \), while \( A^{[7]}B^{[7]} \) becomes

\[
U(AB)^{[7]}U^\dagger = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & c_1 & 0 & 0 & 0 & 0 & -s_1 \\
0 & 0 & c_2 & 0 & 0 & -s_2 & 0 \\
0 & 0 & 0 & c_3 & -s_3 & 0 & 0 \\
0 & 0 & 0 & s_3 & c_3 & 0 & 0 \\
0 & 0 & 0 & s_2 & 0 & c_2 & 0 \\
0 & s_1 & 0 & 0 & 0 & 0 & c_1
\end{pmatrix}.
\]

In this basis, the quadratic invariant will be trivial, since \( D \) is brought to the unit matrix (up to a normalization factor).
5 \( \mathcal{PSL}_2(7) \) Subgroups

\( \mathcal{PSL}_2(7) \) has two maximal subgroups, \( S_4 \), the order 24 permutation group on four letters, and the order 21 Frobenius group \( Z_7 \times Z_3 \), the semi-direct product of \( Z_7 \) and \( Z_3 \).

- \( \mathcal{PSL}_2(7) \supset S_4 \)

This 24 element group represents permutations on four objects. Its presentation is \( \langle a, b \mid a^4 = 1, b^2 = 1, (ba)^3 = 1 \rangle \), and its class structure with representative elements is \( \Gamma_2^{[2]}(b), \Gamma_3^{[3]}(a^2), \Gamma_4^{[3]}(ba), \Gamma_5^{[4]}(a) \), with character table

\[
\begin{array}{c|cccccc}
\mathcal{S}_4 & \Gamma_1^{[1]} & 6\Gamma_2^{[2]} & 3\Gamma_3^{[3]} & 8\Gamma_4^{[4]} & 6\Gamma_5^{[4]} \\
\hline
\chi^{[1]} & 1 & 1 & 1 & 1 & 1 \\
\chi^{[1]'} & 1 & -1 & 1 & 1 & -1 \\
\chi^{[2]} & 2 & 0 & 2 & -1 & 0 \\
\chi^{[3][1]} & 3 & 1 & -1 & 0 & -1 \\
\chi^{[3][2]} & 3 & -1 & -1 & 0 & 1 \\
\end{array}
\]

Its generators can be expressed as \((2 \times 2)\) matrices over \( \mathbb{F}_7 \); for instance

\[
a = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 4 & 4 \\ 1 & 3 \end{pmatrix},
\]

which can be expressed in terms of the \( \mathcal{PSL}_2(7) \) generators as,

\[
a = [A, B], \quad b = A(AB^2)^3(AB)^3.
\]

Its irreducible representations obey the following Kronecker products, see [5] for details and the explicit Clebsch-Gordan coefficients.
It has the presentation \( < c, d | c^7 = d^3 = 1, d^{-1}cd = c^4 > \), with classes \( \Theta_2^3(d), \Theta_3^3(d^2), \Theta_4^7(c), \Theta_5^7(c^3) \). Its character table reads

\[
\begin{array}{c|cccccc}
\mathbb{Z}_7 \times \mathbb{Z}_3 & \Theta_1^1 & 7\Theta_2^3 & 7\Theta_3^3 & 3\Theta_4^7 & 3\Theta_5^7 \\
\hline
\chi^1 & 1 & 1 & 1 & 1 & 1 \\
\chi^{1'} & 1 & e^{2\pi i/3} & e^{4\pi i/3} & 1 & 1 \\
\chi^{1''} & 1 & e^{4\pi i/3} & e^{2\pi i/3} & 1 & 1 \\
\chi^3 & 3 & 0 & 0 & b_7 & \overline{b}_7 \\
\chi^{3'} & 3 & 0 & 0 & \overline{b}_7 & b_7 \\
\end{array}
\]

Its generators can be expressed as \((2 \times 2)\) matrices over \( \mathbb{F}_7 \),

\[
c = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad d = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix},
\]

(62)

projectively defined. In terms of the \( \mathcal{P}\mathcal{S}\mathcal{L}_2(7) \) generators, they are

\[
c = AB, \quad d = AB(AB^2)^2(AB)^2AB^2.
\]

(63)
We note here for completeness the Kronecker products of its irreducible representations. First

\[ 3 \otimes 3 = 3_s + 3_a + 3, \]

with the Clebsch-Gordan decompositions

\[
(3 \otimes 3)_s \rightarrow 3 : \begin{cases} 
|3 > |3' > \\
|1 > |1' > \\
|2 > |2' > 
\end{cases} \quad \overline{3} : \begin{cases} 
\frac{1}{\sqrt{2}} (|3 > |2' > + |2 > |3' >) \\
\frac{1}{\sqrt{2}} (|1 > |3' > + |3 > |1' >) \\
\frac{1}{\sqrt{2}} (|2 > |1' > + |1 > |2' >) 
\end{cases} \\
(3 \otimes 3)_a \rightarrow \overline{3} : \begin{cases} 
\frac{1}{\sqrt{2}} (|3 > |2' > - |2 > |3' >) \\
\frac{1}{\sqrt{2}} (|1 > |3' > - |3 > |1' >) \\
\frac{1}{\sqrt{2}} (|2 > |1' > - |1 > |2' >) 
\end{cases} \quad (64)
\]

The further product

\[ 3 \otimes \overline{3} = 1 + 1' + \overline{1'} + 3 + \overline{3}, \]

yields

\[
3 \otimes \overline{3} \rightarrow 3 : \begin{cases} 
|2 > |1' > \\
|3 > |2' > \\
|1 > |3' > 
\end{cases} \quad \overline{3} : \begin{cases} 
|1 > |2' > \\
|2 > |3' > \\
|3 > |1' > 
\end{cases} \quad (65)
\]
as well as the one-dimensional representations

\[
1 : \frac{1}{\sqrt{3}} (|1 > |1' > + |2 > |2' > + |3 > |3' >) , \\
1' : \frac{1}{\sqrt{3}} (|1 > |1' > + \omega |2 > |2' > + \omega |3 > |3' >) , \\
\overline{1'} : \frac{1}{\sqrt{3}} (|1 > |1' > + \omega |2 > |2' > + \omega^2 |3 > |3' >) ,
\]

where \( \omega = \exp(2i\pi/3) \). Finally, we note that

\[ 3 \otimes 1' = 3 , \quad 3 \otimes \overline{1'} = 3 , \quad 1' \otimes 1' = \overline{1'} , \quad 1' \otimes \overline{1'} = 1 . \]
The relevant information is summarized in the following table.
\[
\begin{array}{|c|}
\hline
Z_7 \times Z_3 \text{ Kronecker Products} \\
\hline
1' \otimes 1' = 1' \\
1' \otimes \bar{1}' = 1 \\
3 \otimes 1' = 3 \\
3 \otimes \bar{1}' = 3 \\
3 \otimes 3 = (3 + \bar{3})_s + \bar{3}_a \\
3 \otimes \bar{3} = 1 + 1' + \bar{1}' + 3 + \bar{3} \\
\hline
\end{array}
\]

**Embeddings**

Invariance under group operations can be “spontaneously” broken down to subinvariances by assigning ”vacuum values” to components of fields belonging to irreps of \( PSL_2(7) \) which contain the singlet representation of the unbroken subgroup. This requires the decompositions of the irreducible representations of \( PSL_2(7) \) in terms of those of its subgroups, notably \( Z_7 \times Z_3 \) and \( S_4 \). We first discuss a general technique which relies on the way its classes contain elements of the subgroup.

Let \( G \) be a group of order \( n \), with irreps \( R[\alpha] \), of dimension \( D_\alpha \) and characters \( \Theta[\alpha] \). Let \( K \) be one of its subgroups, of order \( k \), with irreps \( T[a] \), of dimensions \( d_a \) and characters \( \chi[a] \). We set

\[
R[\alpha] = \sum_a f[\alpha]_a T[a]. 
\]

The embedding coefficients \( f[\alpha]_a \) are positive integers, subject to the dimension constraints,

\[
D_\alpha = \sum_a f[\alpha]_a d_a. 
\]

The embedding coefficients are simply determined from the way which \( K \) classes fit in the \( G \) classes:

- If the class \( C_i \) of \( G \) contains elements of \( K \) which live in its different classes \( \Gamma_{i_m} \), then
  \[
  \sum_\alpha f[\alpha]_b \Xi[\alpha]_i = \left( \frac{n}{k} \right) \sum_{i_m} k_{i_m} \chi[\alpha]_i, 
  \]
  where the sum is over the classes of \( K \) which are contained in the \( C_i \) class of \( G \).

- If the class \( C_\perp \) with character \( \Xi[\perp]_\perp \) has no element in \( K \), a similar reasoning leads to
  \[
  \sum_\alpha f[\alpha]_b \Xi[\alpha]_\perp = 0. 
  \]
We now apply these formulæ to the two maximal subgroups.

\( \bullet \mathcal{PSL}_2(7) \supset S_4 \)

The class structure is

\[
\begin{align*}
C_2^{[2]} &\supset \Gamma_2^{[2]} + \Gamma_3^{[2]}, & C_3^{[3]} &\supset \Gamma_4^{[3]}, & C_4^{[4]} &\supset \Gamma_5^{[4]}, \\
C_5^{[7]} &\supset \Gamma_5^{[7]} + \Gamma_6^{[7]}, & C_6^{[7]} &\supset \Gamma_7^{[7]},
\end{align*}
\tag{71}
\]

while \( C_5^{[7]} \) and \( C_6^{[7]} \) contain no \( S_4 \) classes. Hence we find

\[
\begin{pmatrix}
7\chi^{[a]}_1 \\
2\chi^{[a]}_2 + \chi^{[a]}_3 \\
\chi^{[a]}_4 \\
\chi^{[a]}_5 \\
0 \\
0
\end{pmatrix}
= 
\begin{pmatrix}
1 & 3 & 3 & 6 & 7 & 8 \\
1 & -1 & -1 & 2 & -1 & 0 \\
1 & 0 & 0 & 0 & 1 & -1 \\
1 & 1 & 1 & 0 & -1 & 0 \\
1 & b_7 & \bar{b}_7 & -1 & 0 & 1 \\
1 & \bar{b}_7 & b_7 & -1 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
f^{[1]}_a \\
f^{[3]}_a \\
f^{[6]}_a \\
f^{[7]}_a \\
f^{[8]}_a
\end{pmatrix},
\tag{72}
\]

valid for each \([a]\) irrep of \( S_4 \). The embedding coefficients are obtained by inverting this matrix and inserting the characters of the subgroup. We find

| \( \mathcal{PSL}_2(7) \supset S_4 \) |
|---|
| 3 = 3_2 |
| 3 = 3_2 |
| 6 = 1 + 2 + 3_1 |
| 7 = 1' + 3_1 + 3_2 |
| 8 = 2 + 3_1 + 3_2 |

We see that spontaneous breakdown to \( S_4 \) can be achieved with the sextet representation. Adopting our basis for the sextet representation, the singlet vev is \((\sqrt{2}, \sqrt{2}\eta, \sqrt{2}\eta^3, b_7\eta^4, b_7\eta^5, b_7\eta^6)\).

Once broken, \( S_4 \) can break to \( A_4 \), its maximal subgroup. We note here for convenience its character table

---

23
The Kronecker products and the decompositions of the $S_4$ representations into those of $A_4$ are given as follows. Note that a vev for the $1'$ of $S_4$ breaks this group down to $A_4$.

\[
\begin{array}{c|cccc}
A_4 & C_1^{[1]} & 4C_2^{[3]} & 4C_3^{[3]} & 3C_4^{[2]} \\
\hline
\chi^{[1]} & 1 & 1 & 1 & 1 \\
\chi^{[1']}& 1 & e^{2\pi i/3} & e^{4\pi i/3} & 1 \\
\chi^{[\overline{1}]}& 1 & e^{4\pi i/3} & e^{2\pi i/3} & 1 \\
\chi^{[3]} & 3 & 0 & 0 & -1
\end{array}
\]

The Frobenius group has only elements of order three and seven. Hence $C_2^{[2]}$ and $C_4^{[4]}$, with elements of order two and four respectively, contain no elements of $Z_7 \times Z_3$. On the other hand, we find

\[
C_3^{[3]} \supset \Theta_2^{[3]} + \Theta_3^{[3]} , \quad C_5^{[7]} \supset \Theta_4^{[7]} , \quad C_6^{[7]} \supset \Theta_5^{[7]} .
\]  

(73)

Applying our formulæ to these considerations yields the matrix equation

\[
\left( \begin{array}{cccc}
8 & 3 & 3 & 6 & 7 & 8 \\
1 & -1 & -1 & 2 & -1 & 0 \\
1 & 0 & 0 & 0 & 1 & -1 \\
1 & 1 & 1 & 0 & -1 & 0 \\
1 & b_7 & b_7 & -1 & 0 & 1 \\
1 & b^{-1}_7 & b^{-1}_7 & -1 & 0 & 1
\end{array} \right) = \left( \begin{array}{cccc}
\frac{f_a^{[1]}}{f_a} & \frac{f_a^{[3]}}{f_a} \\
\frac{f_a^{[2]}}{f_a} & \frac{f_a^{[4]}}{f_a} \\
\frac{f_a^{[6]}}{f_a} & \frac{f_a^{[7]}}{f_a} \\
\frac{f_a^{[8]}}{f_a}
\end{array} \right) ,
\]

(74)

(75)
valid for each irrep of $Z_7 \times Z_3$. This determines the embedding coefficients, with the following result.

\[
\begin{array}{c|c}
\mathcal{PSL}_2(7) & Z_7 \times Z_3 \\
\hline
3 & 3 \\
\bar{3} & \bar{3} \\
6 & 3 + \bar{3} \\
7 & 1 + 3 + \bar{3} \\
8 & 1' + \bar{1}' + 3 + \bar{3}'
\end{array}
\]

The singlet of the subgroup appears only in the septet, so that only a septet field can spontaneously break to this subgroup. In our representation, the septet vev is simply $(1, 0, 0, 0, 0, 0, 0)$. The only subgroups of the Frobenius group are $Z_7$ and $Z_3$, so that this breaking chain is rather simple.

### 6 Invariants

Since we have in mind applications to field theory, we represent the different irreducible representations in terms of fields:

- Triplet fields: $\varphi_i, \varphi'_j, \ldots$; antitriplets fields: $\bar{\varphi}^i, \bar{\varphi'}^j, \ldots, i, j = 1, 2, 3$;
- Sextet fields: $\chi_\alpha, \chi'_\beta, \ldots$, with $\alpha, \beta = 1, 2, \ldots, 6$;
- Septet fields: $\psi_a, \psi'_b, \ldots$, with $a, b = 1, 2, \ldots, 7$;
- Octet fields: $\Sigma_A, \Sigma'_B, \ldots, A, B = 1, 2, \ldots, 8$.

**Quadratic Invariants**

We have already built all possible quadratic invariants

\[
\varphi_i \varphi'^i; \quad \chi_\alpha \chi'_\beta C^{\alpha\beta}; \quad \psi_a \psi'_b D^{ab}; \quad \Sigma_A \Sigma'_B \delta_{AB}.
\]

Note that in our bases, the $C$ and $D$ matrices are used to raise and lower indices in the 6 and 7, respectively.\footnote{We could have chosen bases where $C$ and $D$ are the unit matrices, but the Clebsch-Gordan coefficients would have been more complicated.} Note that from now on, we always sum over repeated indices.
Cubic Invariants

Cubic invariants which contain triplets and antitriplets are given by

\[(3 \otimes 3)_a = \mathbf{3} \quad \rightarrow \quad \epsilon^{ijk} \varphi_i \varphi'_j \varphi''_k,\]
\[(3 \otimes 3)_s = 6 \quad \rightarrow \quad K^{ij}_\alpha C^{\alpha\beta} \varphi_i \varphi'_j \chi_\beta,\]
\[3 \otimes 6 = 7 \quad \rightarrow \quad L^{ia}_{ab} D^{ab} \varphi_i \chi_\alpha \psi'_a,\]
\[3 \otimes 6 = 8 \quad \rightarrow \quad M^{ia}_{AB} \varphi_i \chi_\alpha \Sigma_A,\]
\[3 \otimes \overline{3} = 8 \quad \rightarrow \quad (\lambda^i_A)^j \varphi_i \chi^j \Sigma_A,\]
\[3 \otimes 7 = 7 \quad \rightarrow \quad N^{ia}_{AB} D^{ab} \varphi_i \psi_a \psi'_b,\]
\[3 \otimes 7 = 8 \quad \rightarrow \quad P^{ia}_{AB} \varphi_i \psi_a \Sigma_A,\]
\[3 \otimes 8 = 8 \quad \rightarrow \quad Q^{i}_{AB} \varphi_i \psi_a \Sigma_A \Sigma_B.\]  

(75)

*\(N^{ia}_{AB}, P^{ia}_{AB}, \text{ and } Q^{i}_{AB}\) are the Clebsch-Gordan coefficients for the products

\[3 \otimes 7 = 7, \quad 3 \otimes 7 = 8, \quad \text{and} \quad 3 \otimes 8 = 8, \]

respectively. They are listed in the tables of Appendix A. Applying our rules for raising and lowering indices, one can easily derive other Clebsch-Gordan coefficients, e.g.

\[K^{ij}_\alpha C^{\alpha\beta} \varphi_i \chi_\beta,\]  

(76)

has a free upper index \(j\), and thus transforms as an antitriplet \(\mathbf{\overline{3}}\).

If we include the other representations, we can build many more cubic invariants, as we can infer from the table of Kronecker products. As an example, we see that there are two cubic invariants made out of one octet, and three if we consider several octets; as we mentioned earlier, there are two cubic invariants made out of one sextet, etc... 

The list of possible invariants one can construct out of several irreps grows very quickly. In the following we restrict ourselves to discuss the invariants which can be constructed out of triplets and antitriplets only.

Klein’s Quartic Invariant

With only one triplet, there is a unique quartic invariant

\[I^{[4]} = \frac{1}{2\sqrt{2}} (\varphi_i \varphi'_j K^{ij}_\alpha) C^{\alpha\beta} (\varphi_k \varphi'_l K^{kl}_\beta).\]  

(77)

In terms of components,

\[I^{[4]} = (\varphi_1)^3 \varphi_3 + (\varphi_3)^3 \varphi_2 + (\varphi_2)^3 \varphi_1.\]  

(78)

This quartic invariant was found by Felix Klein, and the equation

\[I^{[4]} = 0,\]  

(79)

defines Klein’s Quartic Curve. Obviously, \(\mathcal{PSL}_2(7)\) acts as the group of conformal transformations of the quartic into itself. Klein’s quartic can be parametrized in terms of one complex variable \(w\), with
\[ \varphi_1 = A^3B, \quad \varphi_2 = -2^{1/7}wB^3, \quad \varphi_3 = 2^{3/7}w^3A, \quad (80) \]

\[ A(w) = (1 + w^7)^{1/7}; \quad B(w) = (1 - w^7)^{1/7}. \quad (81) \]

It can be shown to be a Riemann surface of genus 3 of constant negative curvature. It is mathematically special, since it has the maximum number of symmetries allowed by its genus.

**Higher Order Invariants**

By taking the determinant of the \((3 \times 3)\) matrix

\[ M_{ij}^{[3]} = \frac{\text{det} I^{[4]}}{\partial \varphi_i \partial \varphi_j}, \quad (82) \]

Klein found the sixth-order invariant (the Hessian)

\[ I^{[6]} = -\frac{1}{54} \text{det} M^{[3]}. \quad (83) \]

He then formed the \((4 \times 4)\) matrix

\[
M^{[4]} = \begin{pmatrix}
\frac{\partial^2 I^{[4]}}{\partial \varphi_1^2} & \frac{\partial^2 I^{[4]}}{\partial \varphi_1 \partial \varphi_2} & \frac{\partial^2 I^{[4]}}{\partial \varphi_1 \partial \varphi_3} & \frac{\partial I^{[6]}}{\partial \varphi_1} \\
\frac{\partial^2 I^{[4]}}{\partial \varphi_2 \partial \varphi_1} & \frac{\partial^2 I^{[4]}}{\partial \varphi_2^2} & \frac{\partial^2 I^{[4]}}{\partial \varphi_2 \partial \varphi_3} & \frac{\partial I^{[6]}}{\partial \varphi_2} \\
\frac{\partial^2 I^{[4]}}{\partial \varphi_3 \partial \varphi_1} & \frac{\partial^2 I^{[4]}}{\partial \varphi_3 \partial \varphi_2} & \frac{\partial^2 I^{[4]}}{\partial \varphi_3^2} & \frac{\partial I^{[6]}}{\partial \varphi_3} \\
\frac{\partial I^{[6]}}{\partial \varphi_1} & \frac{\partial I^{[6]}}{\partial \varphi_2} & \frac{\partial I^{[6]}}{\partial \varphi_3} & 0
\end{pmatrix}, \quad (84)
\]

to find the invariant of order 14,

\[ I^{[14]} = \frac{1}{9} \text{det} M^{[4]}. \quad (85) \]

Finally, there is the invariant of order 21

\[ I^{[21]} = -\frac{1}{14} \text{det} K, \quad (86) \]

where

\[
K = \begin{pmatrix}
\frac{\partial I^{[4]}}{\partial \varphi_1} & \frac{\partial I^{[4]}}{\partial \varphi_2} & \frac{\partial I^{[4]}}{\partial \varphi_3} \\
\frac{\partial I^{[6]}}{\partial \varphi_1} & \frac{\partial I^{[6]}}{\partial \varphi_2} & \frac{\partial I^{[6]}}{\partial \varphi_3} \\
\frac{\partial I^{[14]}}{\partial \varphi_1} & \frac{\partial I^{[14]}}{\partial \varphi_2} & \frac{\partial I^{[14]}}{\partial \varphi_3}
\end{pmatrix}. \quad (87)\]
Extreme Values of Klein’s Invariant

In order to find the configurations which extremize Klein’s invariant, consider the following PSL\(_2(7)\)-invariant potential (\(m^2\) and \(\lambda\) are positive)

\[
V = -m^2 \sum_i \varphi_i^\dagger \varphi_i + \lambda \left( \sum_i \varphi_i^\dagger \varphi_i \right)^2 + \kappa \cdot (\varphi_1^3 \varphi_3 + \varphi_3^3 \varphi_2 + \varphi_2^3 \varphi_1 + \text{c.c.}) .
\] (88)

Setting

\[
\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} = r \cdot \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = r \cdot \phi , \quad r \in \mathbb{R} ,
\]

(89)

with the normalization \(\sum_i \phi_i^\dagger \phi_i = 1\), the value of the factor \(r\) that minimizes \(V\) is given by

\[
r^2 = \frac{m^2}{2(\lambda + \kappa E)} ,
\]

(90)

where

\[
E \equiv \begin{cases} 
\text{Max} (\phi_1^3 \phi_3 + \phi_3^3 \phi_2 + \phi_2^3 \phi_1 + \text{c.c.}) & \kappa < 0 , \\
\text{Min} (\phi_1^3 \phi_3 + \phi_3^3 \phi_2 + \phi_2^3 \phi_1 + \text{c.c.}) & \kappa > 0 .
\end{cases}
\]

(91)

Let us first discuss the case \(\kappa < 0\). We are looking for a maximum of Klein’s invariant with fixed normalization for the triplet \(\phi\). Adopting the ansatz

\[
\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \frac{1}{\sqrt{3}} \cdot \begin{pmatrix} e^{i\theta_1} \\ e^{i\theta_2} \\ e^{i\theta_3} \end{pmatrix} ,
\]

(92)

we readily find

\[
(\phi_1^3 \phi_3 + \phi_3^3 \phi_2 + \phi_2^3 \phi_1 + \text{c.c.}) = \frac{2}{9} \left[ \cos (3\theta_1 + \theta_3) + \cos (3\theta_3 + \theta_2) + \cos (3\theta_2 + \theta_1) \right] .
\]

This takes its maximum if, with \(n_i \in \mathbb{N}\),

\[
3\theta_1 + \theta_3 = 2\pi n_1 , \quad 3\theta_2 + \theta_1 = 2\pi n_2 , \quad 3\theta_3 + \theta_2 = 2\pi n_3 ,
\]

(93)

which we can rewrite as

\[
\begin{align*}
\theta_1 &= \frac{\pi}{14} (n_2 - 3n_3 + 9n_1) , \\
\theta_2 &= \frac{\pi}{14} (n_3 - 3n_1 + 9n_2) , \\
\theta_3 &= \frac{\pi}{14} (n_1 - 3n_2 + 9n_3) .
\end{align*}
\]

We now reparametrize this solution. We start with \(\theta_1\) which, due to the parameter \(n_2\), can take any multiple of \(\pi/14\). Let us therefore define the new integer parameter

\[
t \equiv n_2 - 3n_3 + 9n_1 ,
\]

(94)
and replace \( n_2 \) in the above equations by

\[
    n_2 = t + 3n_3 - 9n_1 .
\]

We obtain

\[
\begin{aligned}
\theta_1 &= t \frac{\pi}{14}, \\
\theta_2 &= 9t \frac{\pi}{14} - 6\pi n_1 + 2\pi n_3 ,
\theta_3 &= -3t \frac{\pi}{14} + 2\pi n_1 .
\end{aligned}
\]

As \( 2\pi \) rotations of the angles \( \theta_i \) are meaningless, the most general solution for our ansatz is

\[
(\theta_1 , \theta_2 , \theta_3) = \frac{\pi}{14} (t , 9t , -3t) , \quad t = 0 , \ldots , 27 .
\]  

These are 28 different solutions. The case with \( \kappa > 0 \) can be easily traced back to this result by observing that Klein’s quartic changes its sign under the phase transformation \( \phi_i \rightarrow e^{i\pi/4} \phi_i \).

Defining the phase factor

\[
\epsilon \equiv e^{i\pi/14} ,
\]  

the 28 vacuum configurations can be rewritten as

\[
\left\{ \frac{1}{\sqrt{3}} \begin{pmatrix} \epsilon^t \\ \epsilon^{9t} \\ \epsilon^{-3t} \end{pmatrix} \middle| t = 0 , \ldots , 27 \right\} = \frac{i^k}{\sqrt{3}} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} , \begin{pmatrix} \epsilon^{\pm 1} \\ \epsilon^{\pm 9} \\ \epsilon^{\mp 3} \end{pmatrix} , \begin{pmatrix} \epsilon^{\pm 3} \\ \epsilon^{\mp 1} \\ \epsilon^{\pm 9} \end{pmatrix} , \begin{pmatrix} \epsilon^{\pm 9} \\ \epsilon^{\mp 1} \\ \epsilon^{\pm 3} \end{pmatrix} \right\} ,
\]  

with \( k = 0 , \ldots , 3 \). Acting with the group generators on these vectors, we obtain new alignments. A tedious calculation yields the following set

\[
i^k \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} , \begin{pmatrix} i\epsilon^{\pm 1} \\ i\epsilon^{\pm 9} \\ i\epsilon^{\mp 3} \end{pmatrix} , \begin{pmatrix} a_1 \epsilon^{\pm 1} \\ a_1 \epsilon^{\pm 9} \\ a_1 \epsilon^{\mp 3} \end{pmatrix} , \begin{pmatrix} a_2 \epsilon^{\pm 1} \\ a_2 \epsilon^{\pm 9} \\ a_2 \epsilon^{\mp 3} \end{pmatrix} , \begin{pmatrix} a_3 \epsilon^{\pm 1} \\ a_3 \epsilon^{\pm 9} \\ a_3 \epsilon^{\mp 3} \end{pmatrix} , \text{cycl. perm.} \right\} ,
\]  

where normalization factors have been neglected and

\[
a_n \equiv 1 - \cos \left( \frac{2\pi n}{7} \right) .
\]  

Thus we end up with \( 4 \cdot (1 + 3 \cdot 9) = 112 \) different vacuum alignments that minimize the potential.

The question arises whether the degeneracy of these solutions is lifted if we go to higher order invariants. We therefore insert the above vacuum configurations
for $\kappa < 0$ into the invariants and find

$$\left. (I^{[6]} + \bar{I}^{[6]}) \right|_{\text{vac}} = \frac{4}{27} \begin{cases} 
-1 & \text{k even}, \\
+1 & \text{k odd},
\end{cases}$$

(100)

$$\left. (I^{[14]} + \bar{I}^{[14]}) \right|_{\text{vac}} = \frac{32}{729} \begin{cases} 
-1 & \text{k even}, \\
+1 & \text{k odd},
\end{cases}$$

(101)

$$\left. (I^{[21]} + \bar{I}^{[21]}) \right|_{\text{vac}} = 0.$$  

(102)

So with $\kappa < 0$, the 112-fold degeneracy transforms into a 56-fold one. On the other hand, with $\kappa > 0$ we get

$$\left. (I^{[6]} + \bar{I}^{[6]}) \right|_{\text{vac}} = (I^{[14]} + \bar{I}^{[14]}) \right|_{\text{vac}} = (I^{[21]} + \bar{I}^{[21]}) \right|_{\text{vac}} = 0.$$  

(103)

Therefore, the 112-fold degeneracy persists in this case when including the higher order invariants.

**Other Quartic Invariants**

As an illustration, we now consider quartic invariants made out of two triplets and two sextets. They are easy to form, starting from the cubic invariants. We have

$$\left[ L^{i\alpha}_{\quad a} \varphi_i \chi_\alpha \right] D^{ab} \left[ L^{j\beta}_{\quad b} \varphi_j \chi^\beta \right] ,$$

(104)

as well as

$$\left[ M^{i\alpha}_{\quad A} \varphi_i \chi_\alpha \right] \delta_{AB} \left[ M^{j\beta}_{\quad B} \varphi_j \chi^\beta \right] ,$$

(105)

both of which contain the same fields. An explicit calculation shows that both invariants are independent of each other and totally symmetric. We have

$$\begin{align*}
\frac{1}{3} & \left( \varphi_1 \varphi'_1 \chi_6 \chi'_6 + \varphi_2 \varphi'_2 \chi_5 \chi'_5 + \varphi_3 \varphi'_3 \chi_4 \chi'_4 \right) \\
+ \frac{1}{3} & \left( \varphi_1 \varphi'_2 \chi_5 \chi'_6 + \varphi_1 \varphi'_3 \chi_4 \chi'_6 + \varphi_2 \varphi'_3 \chi_4 \chi'_5 \right) + \text{sym.} \\
+ \frac{\sqrt{2}}{6} & \left( \varphi_1 \varphi'_2 \chi_3 \chi'_4 + \varphi_1 \varphi'_3 \chi_2 \chi'_5 + \varphi_2 \varphi'_3 \chi_1 \chi'_6 \right) + \text{sym.} \\
+ \frac{1}{6} & \left( \varphi_1 \varphi'_1 \chi_2 \chi'_3 + \varphi_2 \varphi'_2 \chi_1 \chi'_3 + \varphi_3 \varphi'_3 \chi_1 \chi'_2 \right) + \text{sym.} \\
- \frac{1}{\sqrt{2}} & \left( \varphi_1 \varphi'_1 \chi_1 \chi'_4 + \varphi_2 \varphi'_2 \chi_2 \chi'_6 + \varphi_3 \varphi'_3 \chi_3 \chi'_5 \right) + \text{sym.} \\
- \frac{1}{2} & \left( \varphi_1 \varphi'_2 \chi_1 \chi'_1 + \varphi_1 \varphi'_3 \chi_3 \chi'_3 + \varphi_2 \varphi'_3 \chi_2 \chi'_2 \right) + \text{sym.}
\end{align*}$$

(106)

and

$$\begin{align*}
\frac{2}{3} & \left( \varphi_1 \varphi'_1 \chi_6 \chi'_6 + \varphi_2 \varphi'_2 \chi_5 \chi'_5 + \varphi_3 \varphi'_3 \chi_4 \chi'_4 \right) \\
- \frac{1}{4} & \left( \varphi_1 \varphi'_2 \chi_5 \chi'_6 + \varphi_1 \varphi'_3 \chi_4 \chi'_6 + \varphi_2 \varphi'_3 \chi_4 \chi'_5 \right) + \text{sym.} \\
+ \frac{\sqrt{2}}{6} & \left( \varphi_1 \varphi'_2 \chi_3 \chi'_4 + \varphi_1 \varphi'_3 \chi_2 \chi'_5 + \varphi_2 \varphi'_3 \chi_1 \chi'_6 \right) + \text{sym.} \\
- \frac{2}{3} & \left( \varphi_1 \varphi'_1 \chi_2 \chi'_3 + \varphi_2 \varphi'_2 \chi_1 \chi'_3 + \varphi_3 \varphi'_3 \chi_1 \chi'_2 \right) + \text{sym.}
\end{align*}$$

(107)
where “sym.” indicates symmetrization in the triplet and the sextet indices (if they are not already symmetric). Explicitly, for \( i \neq j \) and \( \alpha \neq \beta \),

\[
\varphi_i \varphi_j' \chi_\alpha \chi_\beta' + \text{sym.} = \varphi_i \varphi_j' \chi_\alpha \chi_\beta' + \varphi_j \varphi_i' \chi_\alpha \chi_\beta' + \varphi_j \varphi_i' \chi_\beta \chi_\alpha' ,
\]

and, for \( i = j \) and \( \alpha \neq \beta \),

\[
\varphi_i \varphi_i' \chi_\alpha \chi_\beta' \text{ sym.} = \varphi_i \varphi_i' \chi_\alpha \chi_\beta' + \varphi_i \varphi_i' \chi_\beta \chi_\alpha' .
\]

We can understand the fact that these two quartic invariants are symmetric in the triplets and the antitriplets by looking at a rearranged version of products of representations. Using the Kronecker products, we obtain

\[
(3 \otimes 6) \otimes (3 \otimes 6) = \left( 3 + 6 \right) \otimes \left( 3 + 6 \right)
\]

\[
= \left( 3_a + 6_s \right) \otimes \left( 1 + 6 + 6 + 8 \right) + (7 + 8)_a .
\]

From this we can construct invariants only in two ways, namely through the symmetric sextets.

Another example is the quartic invariant

\[
\left[ (\lambda_A)^i_j \varphi_i' \varphi_j' \right] \delta_{AB} \left[ (\lambda_B)^k_l \varphi_k' \varphi_l' \right] .
\]

Using the identity

\[
(\lambda_A)^i_j (\lambda_A)^k_l = 2 \delta^i_j \delta^k_l - \frac{2}{3} \delta^i_j \delta^k_l ,
\]

this invariant reduces to a product of quadratic invariants.

The following table shows the number of independent quartic invariants which include at least two triplets (for the chiral fermions?), i.e. which are of type \( 3 \otimes 3 \otimes r \otimes s \).

| (r, s) | 3 | 3 | 6 | 7 | 8 |
|-------|---|---|---|---|---|
| 3     | 1 | 0 | 0 | 1 | 2 |
| 3     | 2 | 1 | 1 | 1 |   |
| 6     | 2 | 2 | 3 |   |   |
| 7     |   | 3 | 3 |   |   |
| 8     |   |   | 3 |   |   |

31
7 The Fano Plane Representation

We have seen how $\mathcal{P}SL_2(7)$ acts on Klein’s quartic curve through its three-dimensional representation. Its six-dimensional representation also has a remarkable geometrical meaning: it acts on the Fano projective plane, the simplest finite projective plane with seven points and seven lines.

In order to see this, consider seven points, imaginatively labeled as 1, 2, ..., 7. We wish to construct the group formed by those permutations that permute the seven points and the seven columns

\[
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
2 & 3 & 4 & 5 & 6 & 7 & 1 \\
4 & 5 & 6 & 7 & 1 & 2 & 3
\end{bmatrix}
\]

A convenient description of these arrangements is the Fano Plane where each triplet of points forms a line

This picture is often used to identify the octonion structure functions. It is also useful in identifying the subgroup that leaves the line [1 2 4] invariant. We can permute the remaining points without changing the picture: the three transpositions (6 7), (5 7), and (3 7) do not alter the picture but relabel the points while keeping the base of the triangle invariant. For example, (6 7) simply interchanges the two outer edges of the triangle. All three generate the permutations on the four points 3, 5, 6, 7: the subgroup that leaves one line invariant is the permutation group on four objects, $S_4$.

Now consider a transformation that maps the columns into one another, such as the order-seven permutation

\[ c = (1234567), \quad c^7 = 1, \]

(110)
which maps the first column into the second one. Similarly, $c^2$ maps the first column into the third one, and so on. This produces the group we are seeking through the coset decomposition

$$
\mathcal{G} = \mathcal{H} + \mathcal{H}c + \cdots + \mathcal{H}c^6.
$$

This yields our group of order $7 \cdot 24 = 168$. Geometrically, we can think of the Fano plane as describing a geometry with seven points and seven lines, such that two points are only on one line, and two lines cross at only one point. It is the smallest example of a finite projective plane.

The action of $\mathcal{PSL}_2(7)$ on the Fano plane yields a real representation in terms of permutations

$$
AB = (1234567), \quad B = (124)(3)(567),
$$

$$
A = (AB)B^2 = (1)(5)(6)(23)(47).
$$

Since

$$
A_{\text{Fano}} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 
\end{pmatrix}, \quad B_{\text{Fano}} = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 
\end{pmatrix}.
$$

These two matrices clearly satisfy the $\mathcal{PSL}_2(7)$ presentation. We can form its order four element

$$
[A,B]_{\text{Fano}} = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 
\end{pmatrix}.
$$

Their traces

$$
\text{Tr } A_{\text{Fano}} = 3, \quad \text{Tr } B_{\text{Fano}} = 1, \quad \text{Tr } [A,B]_{\text{Fano}} = 1,
$$

do not conform to the characters of the septet: the Fano plane representation is reducible, with

$$
7_{\text{Fano}} = 1 + 6.
$$

The singlet corresponds to the overall labeling of the seven points, and has no physical content. The same occurs in the Tetrahedral group, $A_4$ which acts on the four vertices of the tetrahedron, yielding a reducible representation of its triplet and singlet representations.
8 Outlook

Mathematicians have long recognized in the simple group $\text{PSL}_2(7)$ a remark-
ably rich structure with important applications in algebra, geometry and num-
ber theory. However, its beauty has not yet penetrated the consciousness of
physicists. We hope that our presentation will serve as a physicists’ gateway
to its further studies. Its maximal subgroups have already been used more or
less successfully to describe the family structure of quarks and leptons. It is
therefore tempting to study how $\text{PSL}_2(7)$ could be applied as a flavor group.
A first step in this direction has been taken in this article by determining the
extreme values of Klein’s quartic invariant, for which potentially CP-violating
phases naturally appear. Its geometrical incarnation as a Riemann surface of
genus three might also open new paths for string compactification.

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\(^{5}\text{Klein's Quartic Curve is immortalized in a sculpture at the entrance of the Mathematical}
\text{Sciences Institute in Berkeley.}\)
Appendix A

Tables of Clebsch-Gordan Coefficients

In this appendix, we tabulate the Clebsch-Gordan coefficients of $\mathcal{P}SL_2(7)$ which involve triplets and antitriplets. First, we summarize our notation

$3 \otimes 3 = 6$ : \[ \left\{ \alpha \right\} = \left\langle i > \left| j > \right. \right\rangle, \]
$3 \otimes 6 = 7$ : \[ \left\langle a > \right| = \left| \alpha \right\rangle, \]
$3 \otimes 6 = 8$ : \[ \left\{ A \right\} = \left| \alpha \right\rangle, \]
$3 \otimes 7 = 7$ : \[ \left\langle a > \right| = \left| b > \right. \right\rangle, \]
$3 \otimes 7 = 8$ : \[ \left\{ A \right\} = \left| b > \right. \right\rangle, \]
$3 \otimes 8 = 8$ : \[ \left\{ A \right\} = \left| B > \right. \right\rangle. \]

The numerical values for $K_{ij}^\alpha$ can be found in Section 4.2, while those for $M_{i\alpha}^A$, $L_{i\alpha}^a$, $N_{ib}^a$, $P_{ia}^A$, $Q_{iBA}$ are listed in the following tables. From Table A-1 we obtain, for example,

$|8\rangle = M_{i\alpha}^8 |i \rangle |\alpha \rangle = \frac{1}{\sqrt{2}} |2\rangle |5\rangle - \frac{1}{\sqrt{2}} |1\rangle |6\rangle$. \]

Notice that we do not give the Clebsch-Gordan coefficients for products like $3 \otimes 7 = 6$ since they are directly related to the previously derived decomposition $3 \otimes 6 = 7$. This can be seen easily by rewriting the invariant

$(L_{i\alpha}^a \phi_i \chi_{\alpha}) D_{ab} \psi_b = (L_{i\alpha}^a \phi_i \psi_a) C_{\alpha\beta} \chi_{\beta}^\prime$, \]

where we have defined $L_{i\alpha}^a = C_{\alpha\beta} D_{ab} L_{i\beta}^b$. \]

Similarly, the Clebsch-Gordan coefficients for products like $3 \otimes 6 = 7$ can be obtained from $3 \otimes 6 = 7$. Starting with the invariant

$(L_{i\alpha}^a \phi_i \chi_{\alpha}) \psi_a$, \]

its complex conjugate is also an invariant:

$(L_{i\alpha}^a \phi_i^* \chi_{\alpha}^*) \psi_a^*$. \]

Since $\phi_i^*$ transforms like a general $3$ field $\phi_i^*$, $\chi_{\alpha}^*$ like $C_{\alpha\beta} \chi_{\beta}^\prime \equiv \chi_{\alpha}^\prime$, and $\psi_a^*$ like $D_{ab} \psi^b \equiv \psi_a^\prime$, we get the invariant

$(L_{i\alpha}^a \phi_i^* \chi_{\alpha}^*) \psi_a^\prime$, \]

suggesting the definition

$L_{i\alpha}^a \equiv L_{i\alpha}^a$. \]

Thus we find

$(L_{i\beta}^b \phi_i^* C_{\alpha\beta} \chi_{\alpha}) D_{ab} \psi_a^\prime = (C_{\alpha\beta} D_{ab} L_{i\beta}^b \phi_i^* \chi_{\alpha}) \psi_a^\prime = (L_{i\alpha}^a \phi_i^* \chi_{\alpha}) \psi_a^\prime$. \]

35
$$3 \otimes 6$$

| $i > | \alpha$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|----------------|----|----|----|----|----|----|----|----|
| $1 > 1 \rangle$ | 0 0 0 0 0 0 0 0 |
| $1 > 2 \rangle$ | 0 0 0 0 0 $\sqrt{\frac{1}{3}}$ $-\sqrt{\frac{1}{3}}i$ 0 |
| $1 > 3 \rangle$ | 0 0 0 0 0 $-\sqrt{\frac{1}{3}}$ $-\sqrt{\frac{1}{3}}i$ 0 |
| $1 > 4 \rangle$ | 0 0 0 $\sqrt{\frac{1}{6}}$ $-\sqrt{\frac{1}{6}}i$ 0 0 0 |
| $1 > 5 \rangle$ | $-\sqrt{\frac{1}{3}}$ $\sqrt{\frac{1}{3}}i$ 0 0 0 0 0 |
| $1 > 6 \rangle$ | 0 0 $\sqrt{\frac{1}{6}}$ 0 0 0 0 0 $-\sqrt{\frac{1}{2}}$ |
| $2 > 1 \rangle$ | 0 0 0 $-\sqrt{\frac{1}{3}}$ $\sqrt{\frac{1}{3}}i$ 0 0 0 |
| $2 > 2 \rangle$ | 0 0 0 0 0 0 0 0 |
| $2 > 3 \rangle$ | 0 0 0 $\sqrt{\frac{1}{3}}$ $\sqrt{\frac{1}{3}}i$ 0 0 0 |
| $2 > 4 \rangle$ | 0 0 0 0 0 $-\sqrt{\frac{1}{6}}$ $\sqrt{\frac{1}{6}}i$ 0 |
| $2 > 5 \rangle$ | 0 0 $\sqrt{\frac{1}{6}}$ 0 0 0 0 0 $\sqrt{\frac{1}{2}}$ |
| $2 > 6 \rangle$ | $\sqrt{\frac{1}{3}}$ $\sqrt{\frac{1}{3}}i$ 0 0 0 0 0 |
| $3 > 1 \rangle$ | $\sqrt{\frac{1}{3}}$ $-\sqrt{\frac{1}{3}}i$ 0 0 0 0 0 |
| $3 > 2 \rangle$ | $-\sqrt{\frac{1}{3}}$ $-\sqrt{\frac{1}{3}}i$ 0 0 0 0 0 |
| $3 > 3 \rangle$ | 0 0 0 0 0 0 0 0 |
| $3 > 4 \rangle$ | 0 0 $-\sqrt{\frac{1}{3}}$ 0 0 0 0 0 |
| $3 > 5 \rangle$ | 0 0 0 0 0 $\sqrt{\frac{1}{6}}$ $\sqrt{\frac{1}{6}}i$ 0 |
| $3 > 6 \rangle$ | 0 0 0 $-\sqrt{\frac{1}{6}}$ $-\sqrt{\frac{1}{6}}i$ 0 0 0 |

Table A-1. The Clebsch-Gordan coefficients $M^{i\alpha}_A$ for $3 \otimes 6 = 8$.  

36
\[
\begin{array}{c|ccccccc}
3 \otimes 6 & 1 \!>\! & 2 \!>\! & 3 \!>\! & 4 \!>\! & 5 \!>\! & 6 \!>\! & 7 \!> \\
\hline
| i \!>\! & | \alpha \!>\! |
\end{array}
\begin{array}{cccccc}
& 1 \!\!>\! & 2 \!\!>\! & 3 \!\!>\! & 4 \!\!>\! & 5 \!\!>\! & 6 \!\!>\! & 7 \!\!> \\
\hline
| 1 \!>\! & | 1 \!>\! | & 0 & 0 & 0 & -\sqrt{\frac{1}{3}} & 0 & 0 & 0 \\
| 1 \!>\! & | 2 \!>\! | & 0 & 0 & 0 & 0 & \sqrt{\frac{1}{2}} & 0 & 0 \\
| 1 \!>\! & | 3 \!>\! | & 0 & 0 & \sqrt{\frac{1}{3}} & 0 & 0 & 0 & 0 \\
| 1 \!>\! & | 4 \!>\! | & 0 & 0 & 0 & 0 & \sqrt{\frac{2}{3}} & 0 & 0 \\
| 1 \!>\! & | 5 \!>\! | & 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{1}{3}} & 0 \\
| 1 \!>\! & | 6 \!>\! | & -\sqrt{\frac{1}{3}} & 0 & 0 & 0 & 0 & 0 \\
| 2 \!>\! & | 1 \!>\! | & 0 & 0 & 0 & 0 & \sqrt{\frac{1}{3}} & 0 & 0 \\
| 2 \!>\! & | 2 \!>\! | & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{1}{3}} \\
| 2 \!>\! & | 3 \!>\! | & 0 & 0 & 0 & \sqrt{\frac{1}{2}} & 0 & 0 & 0 \\
| 2 \!>\! & | 4 \!>\! | & 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{1}{3}} & 0 \\
| 2 \!>\! & | 5 \!>\! | & -\sqrt{\frac{1}{3}} & 0 & 0 & 0 & 0 & 0 \\
| 2 \!>\! & | 6 \!>\! | & 0 & \sqrt{\frac{2}{3}} & 0 & 0 & 0 & 0 \\
| 3 \!>\! & | 1 \!>\! | & 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{1}{2}} \\
| 3 \!>\! & | 2 \!>\! | & 0 & \sqrt{\frac{1}{3}} & 0 & 0 & 0 & 0 \\
| 3 \!>\! & | 3 \!>\! | & 0 & 0 & 0 & 0 & \sqrt{\frac{1}{3}} & 0 & 0 \\
| 3 \!>\! & | 4 \!>\! | & -\sqrt{\frac{1}{3}} & 0 & 0 & 0 & 0 & 0 \\
| 3 \!>\! & | 5 \!>\! | & 0 & 0 & \sqrt{\frac{2}{3}} & 0 & 0 & 0 & 0 \\
| 3 \!>\! & | 6 \!>\! | & 0 & 0 & 0 & \sqrt{\frac{1}{3}} & 0 & 0 & 0 \\
\end{array}
\]

Table A-2. The Clebsch-Gordan coefficients \( L^{i\alpha} \) for \( 3 \otimes 6 = 7 \).
Table A-3. The Clebsch-Gordan coefficients $N^{ia}_b$ for $3 \otimes 7 = 7$. 

| $i >$ | $a >$ | $1 >$ | $2 >$ | $3 >$ | $4 >$ | $5 >$ | $6 >$ | $7 >$ |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $|1 >|$ $|1 >|$    | 0     | $-\sqrt{\frac{1}{3}}$ | 0     | 0     | 0     | 0     |
| $|1 >|$ $|2 >|$    | 0     | 0     | $-\sqrt{\frac{1}{3}}$ | 0     | 0     | 0     |
| $|1 >|$ $|3 >|$    | 0     | 0     | 0     | $-\sqrt{\frac{1}{6}}$ | 0     | 0     |
| $|1 >|$ $|4 >|$    | 0     | 0     | 0     | 0     | $-\sqrt{\frac{1}{3}}$ | 0     |
| $|1 >|$ $|5 >|$    | 0     | 0     | 0     | 0     | 0     | $\sqrt{\frac{1}{6}}$ | 0     |
| $|1 >|$ $|6 >|$    | 0     | 0     | 0     | 0     | 0     | 0     | $\sqrt{\frac{1}{6}}$ |
| $|1 >|$ $|7 >|$    | $\sqrt{\frac{1}{3}}$ | 0     | 0     | 0     | 0     | 0     |
| $|2 >|$ $|1 >|$    | 0     | 0     | $-\sqrt{\frac{1}{3}}$ | 0     | 0     | 0     |
| $|2 >|$ $|2 >|$    | 0     | 0     | 0     | $\sqrt{\frac{1}{6}}$ | 0     | 0     |
| $|2 >|$ $|3 >|$    | 0     | 0     | 0     | 0     | $\sqrt{\frac{1}{3}}$ | 0     |
| $|2 >|$ $|4 >|$    | 0     | 0     | 0     | 0     | 0     | $\sqrt{\frac{1}{6}}$ |
| $|2 >|$ $|5 >|$    | 0     | 0     | 0     | 0     | 0     | 0     | $\sqrt{\frac{1}{6}}$ |
| $|2 >|$ $|6 >|$    | $\sqrt{\frac{1}{3}}$ | 0     | 0     | 0     | 0     | 0     |
| $|2 >|$ $|7 >|$    | 0     | 0     | 0     | 0     | 0     | 0     | 0     |
| $|3 >|$ $|1 >|$    | 0     | 0     | 0     | 0     | $-\sqrt{\frac{1}{6}}$ | 0     |
| $|3 >|$ $|2 >|$    | 0     | 0     | 0     | 0     | 0     | $-\sqrt{\frac{1}{3}}$ | 0     |
| $|3 >|$ $|3 >|$    | 0     | 0     | 0     | 0     | 0     | 0     | $\sqrt{\frac{1}{6}}$ |
| $|3 >|$ $|4 >|$    | $\sqrt{\frac{1}{3}}$ | 0     | 0     | 0     | 0     | 0     |
| $|3 >|$ $|5 >|$    | 0     | $-\sqrt{\frac{2}{3}}$ | 0     | 0     | 0     | 0     |
| $|3 >|$ $|6 >|$    | 0     | 0     | 0     | 0     | 0     | 0     |
| $|3 >|$ $|7 >|$    | 0     | 0     | 0     | 0     | 0     | 0     | 0     |
| \( i > \) | \( a > \) | \( 1 \) | \( 2 \) | \( 3 \) | \( 4 \) | \( 5 \) | \( 6 \) | \( 7 \) | \( 8 \) |
|---|---|---|---|---|---|---|---|---|---|
| \( 1 > 1 > \) | \( \sqrt{\frac{3}{21}} \) | \( \sqrt{\frac{3}{21}} i \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( 1 > 2 > \) | 0 | 0 | 0 | -\( \sqrt{\frac{8}{21}} \) | -\( \sqrt{\frac{8}{21}} i \) | 0 | 0 | 0 | 0 |
| \( 1 > 3 > \) | 0 | 0 | 0 | 0 | \( \sqrt{\frac{3}{21}} \) | -\( \sqrt{\frac{3}{21}} i \) | 0 | 0 | 0 |
| \( 1 > 4 > \) | 0 | 0 | 0 | 0 | 0 | \( \sqrt{\frac{3}{21}} \) | -\( \sqrt{\frac{3}{21}} i \) | 0 | 0 |
| \( 1 > 5 > \) | 0 | 0 | 0 | 0 | 0 | 0 | \( -\sqrt{\frac{2}{21}} \) | \( \frac{2}{21} i \) | 0 |
| \( 1 > 6 > \) | -\( \sqrt{\frac{2}{21}} \) | \( \sqrt{\frac{2}{21}} i \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( 1 > 7 > \) | 0 | 0 | -\( \frac{\sqrt{2}}{\sqrt{2}} \) | 0 | 0 | 0 | 0 | 0 | \( \frac{1}{\sqrt{2}} \) |
| \( 2 > 1 > \) | 0 | 0 | 0 | 0 | 0 | \( \sqrt{\frac{2}{21}} \) | \( \sqrt{\frac{2}{21}} i \) | 0 | 0 |
| \( 2 > 2 > \) | 0 | 0 | 0 | -\( \sqrt{\frac{2}{21}} \) | -\( \sqrt{\frac{2}{21}} i \) | 0 | 0 | 0 | 0 |
| \( 2 > 3 > \) | 0 | 0 | 0 | -\( \sqrt{\frac{2}{21}} \) | \( \sqrt{\frac{2}{21}} i \) | 0 | 0 | 0 | 0 |
| \( 2 > 4 > \) | 0 | 0 | 0 | 0 | 0 | -\( \sqrt{\frac{2}{21}} \) | \( \sqrt{\frac{2}{21}} i \) | 0 | 0 |
| \( 2 > 5 > \) | -\( \sqrt{\frac{2}{21}} \) | \( \sqrt{\frac{2}{21}} i \) | 0 | 0 | 0 | 0 | 0 | 0 | \( \sqrt{\frac{2}{21}} \) |
| \( 2 > 6 > \) | 0 | 0 | \( \sqrt{\frac{2}{21}} \) | 0 | 0 | 0 | 0 | 0 | \( \sqrt{\frac{2}{21}} i \) |
| \( 2 > 7 > \) | \( \sqrt{\frac{2}{21}} \) | \( \sqrt{\frac{2}{21}} i \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( 3 > 1 > \) | 0 | 0 | 0 | \( \frac{\sqrt{2}}{\sqrt{2}} \) | -\( \frac{\sqrt{2}}{\sqrt{2}} i \) | 0 | 0 | 0 | 0 |
| \( 3 > 2 > \) | 0 | 0 | 0 | 0 | 0 | -\( \sqrt{\frac{2}{21}} \) | \( \sqrt{\frac{2}{21}} i \) | 0 | 0 |
| \( 3 > 3 > \) | -\( \frac{\sqrt{2}}{\sqrt{2}} \) | \( \sqrt{\frac{2}{21}} i \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( 3 > 4 > \) | 0 | 0 | 0 | 0 | \( \frac{\sqrt{2}}{\sqrt{2}} \) | 0 | 0 | 0 | 0 |
| \( 3 > 5 > \) | -\( \sqrt{\frac{2}{21}} \) | -\( \frac{\sqrt{2}}{\sqrt{2}} i \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( 3 > 6 > \) | 0 | 0 | 0 | 0 | 0 | \( \frac{\sqrt{2}}{\sqrt{2}} \) | \( \sqrt{\frac{2}{21}} \) | \( \sqrt{\frac{2}{21}} i \) | 0 |
| \( 3 > 7 > \) | 0 | 0 | 0 | -\( \sqrt{\frac{2}{21}} \) | -\( \frac{\sqrt{2}}{\sqrt{2}} i \) | 0 | 0 | 0 | 0 |

Table A-4. The Clebsch-Gordan coefficients \( P_{ia}^{\alpha} \) for \( 3 \otimes 7 = 8 \).
| 3 ⊗ 8 | 1 > | 4 | 2 > | 5 | 3 > | 6 | 4 > | 7 | 5 > | 8 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 > 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 > 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 > 3 | $\sqrt{12}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 > 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 > 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 > 6 | $\sqrt{12}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 > 7 | $\sqrt{12}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 > 8 | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 > 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 > 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 > 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 > 4 | $\sqrt{12}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 > 5 | $\sqrt{12}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 > 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 > 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 > 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 > 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 > 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 > 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 > 4 | $\sqrt{12}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 > 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 > 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 > 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 > 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table A-5. The Clebsch-Gordan coefficients $Q^i_{AB}$ for $3 ⊗ 8 = 8$. 
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