The nature of gravitational singularities

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Abstract

The nature of gravitational singularities, long mysterious, has now become clear through a combination of mathematical and numerical analysis. As the singularity is approached, the time derivative terms in the field equations dominate, and the singularity behaves locally like a homogeneous oscillatory spacetime.

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A longstanding problem in general relativity has been to determine the nature of the singularities that form in gravitational collapse. Powerful theorems due to Hawking, Penrose and others[1] show that singularities form under very general circumstances. However, these theorems give almost no information about the nature of singularities, saying only that the worldline of some observer or light ray fails to be complete. There is also a longstanding conjecture due to Belinskii, Khalatnikov and Lifschitz (BKL)[2] on the general nature of singularities. The BKL conjecture is that as a general singularity is approached, the dynamics becomes local and oscillatory. The analysis of reference[2] was heuristic, so what was needed were two things: (1) a more precise way of stating the BKL conjecture and (2) a way of checking whether it is true.

It was realized by Berger and Moncrief[3] that numerical simulations provide a way of checking the BKL conjecture: simulate the evolution of the spacetime as the singularity is approached and see whether the behavior is as conjectured in reference[2]. What resulted from this insight was a program of research[4] that simulated the approach to the singularity in spacetimes with symmetry. The imposition of symmetry made the equations simpler and allowed the simulations (which were constrained by limits of computer memory) to be done with high spatial resolution. Both the simulations and the analysis of the results were done by casting the Einstein field equations as a Hamiltonian system. The results supported the BKL conjecture: in all cases the dynamics became local, i.e. spatial derivatives in the field equations became negligible compared to time derivatives. In some cases the dynamics was oscillatory and in some cases not. However, one could plausibly argue that the cases where
the dynamics was not oscillatory were not sufficiently general and that the general case would be expected to be oscillatory.

One limitation of this research program was the imposition of symmetry. As long as spacetimes with symmetry were treated, one could never be sure that the results reflected the behavior of the general spacetime without symmetry. Another difficulty came from the use of Hamiltonian variables. These were sufficiently different from the variables used in reference[2] that it was often difficult to compare the results of the simulations to the expected BKL behavior.

These difficulties came to be resolved with the use of scale invariant variables in the work of Uggla et al [5]. Here the key insight comes from the scale invariance of the vacuum Einstein equations, that is the property that a solution of these equations remains a solution if the overall length scale is changed. In the homogeneous, isotropic spacetimes of big bang cosmology a scale (or more precisely a scale at a given time) is given by the value of the Hubble constant. This is the rate of the expansion of space. The notion of the Hubble constant depends specifically on the homogeneity and isotropy of space; however it can be generalized to the case with no symmetry. In cosmology, the Hubble constant is one third of the divergence of the normal to the surfaces of homogeneity, which form the cosmological surfaces of constant time. In a general spacetime, given a choice of time slicing one can define the Hubble parameter $H$ at a given spacetime point to be simply one third of the divergence of the normal to the constant time surface. In physical terms, one can think of space as having three different rates of expansion (or contraction) in each of three orthogonal
directions. The Hubble parameter is then defined to be the average rate of expansion. Given this scale, one can then divide all other variables by (appropriate powers of) $H$ to make them scale invariant. One important variable is the shear $\sigma_{\alpha\beta}$ whose eigenvalues give the differences in the rates of expansion of the three orthogonal directions. In the equations, one uses the related scale invariant variable $\Sigma_{\alpha\beta} \equiv \sigma_{\alpha\beta}/H$. Another important variable is $n_{\alpha\beta}$ which measures the failure of derivatives along orthogonal spatial directions to commute and is therefore related to the curvature of space. The related scale invariant variable is $N_{\alpha\beta} \equiv n_{\alpha\beta}/H$. As the singularity is approached, $H$ diverges. However, the scale invariant quantities remain finite.

This set of variables also gives rise to a natural prescription for decomposing spacetime into space and time: pick an initial time slice and an orthonormal spatial frame on this slice. Choose a time orientation so that the singularity is to the past and choose time evolution towards the singularity to be motion by the amount $H^{-1}$ along the direction normal to the slice. Finally Fermi-Walker transport the spatial frame along the time evolution. This gives rise to a time coordinate that goes to minus infinity as the singularity is approached. Behavior near the singularity then becomes the $t \to -\infty$ behavior of solutions of the scale invariant system. This prescription then gives a rigorous way of stating the BKL conjecture: as $t \to -\infty$ the spatial derivatives of the scale invariant variables become negligible and the behavior at each spatial point becomes that of an oscillatory homogeneous spacetime.

What remained was to perform numerical simulations of the system of reference [5] for the general case of no symmetry, to see whether the conjecture was correct. Such simulations
Figure 1: components of $\Sigma_{\alpha\beta}$ vs time, in the asymptotic frame: $\Sigma_1$ (solid line), $\Sigma_2$ (dotted line) and $\Sigma_3$ (dot-dashed line) were performed by the author. The results support the BKL conjecture. As the singularity is approached, spatial derivatives become negligible. Each spatial point then behaves like a homogeneous universe. But what is the behavior of a homogeneous universe? And which type of homogeneous universe corresponds to the general behavior of singularities? Homogeneous universes can be classified by their symmetry groups. BKL conjectured that the general behavior of singularities is locally like that of a Mixmaster universe: a homogeneous universe whose symmetry group is $SU(2)$.

To see whether this conjecture is true, we must look at the dynamics of the scale invariant variables at a single point. Figures 1 and 2 show respectively the behavior of the variables $\Sigma_{\alpha\beta}$ and $N_{\alpha\beta}$ at a single spatial point in the numerical simulation of reference 6. Here what
Figure 2: components of $N_{\alpha\beta}$ vs time, in the asymptotic frame: $N_1$ (solid line), $N_2$ (dotted line) and $N_3$ (dot-dashed line)

is plotted are the diagonal components of $\Sigma_{\alpha\beta}$ and $N_{\alpha\beta}$ in the “asymptotic frame” which is the frame of eigenvectors of $\Sigma_{\alpha\beta}$ in the limit as the singularity is approached. In the times between 0 and $-20$ the spatial derivatives are not negligible and the behavior is complicated. However, for $t < -20$ the spatial derivatives are negligible and the behavior is simple: it consists of time intervals (called Kasner epochs) in which the components of $\Sigma_{\alpha\beta}$ are constant while those of $N_{\alpha\beta}$ are negligible. The Kasner epochs are punctuated by short “bounces” where during each bounce the components of $\Sigma_{\alpha\beta}$ change rapidly while one component of $N_{\alpha\beta}$ rapidly grows and then decays. This is exactly the behavior of a Mixmaster universe.

What determines how the components of $\Sigma_{\alpha\beta}$ change from one Kasner epoch to the next? To answer this, we must first characterize the Kasner epochs. The tensor $\Sigma_{\alpha\beta}$ is traceless,
and during a Kasner epoch its square is equal to 6. Thus the three eigenvalues of $\Sigma_{\alpha\beta}$ satisfy two relations and can therefore be characterized by one parameter. For each eigenvalue $\Sigma_i$ of $\Sigma_{\alpha\beta}$ introduce the number $p_i$ by $\Sigma_i = 3p_i - 1$. Then the properties of $\Sigma_{\alpha\beta}$ imply that during a Kasner epoch the sum of the $p_i$ and the sum of their squares is 1. Now define $u$ to be the ratio of the largest $p_i$ to the second largest one. Then $u \geq 1$ and since there are two relations satisfied by $3p_i$ it follows that the $p_i$ are completely characterized by $u$. The question of how $\Sigma_{\alpha\beta}$ changes from one Kasner epoch to the next then becomes the question of how $u$ changes from one epoch to the next. For Mixmaster spacetimes the answer was found in reference[2]. If $u \geq 2$ in one epoch, then it changes to $u - 1$ in the next; while if $u \leq 2$ in one epoch, then it changes to $1/(u - 1)$ in the next. This rule is called the $u$ map. One can compare the sequence of values of $u$ found in the simulations to the rule of the $u$ map. The result is that the general singularity satisfies the $u$ map. Thus the general singularity is local and oscillatory, with the oscillations having the Mixmaster form.

Note that the part of the $u$ map of the form $u \rightarrow 1/(u - 1)$ depends sensitively on initial conditions. Thus the $u$ map and therefore the general singularity are chaotic. Nearby spatial points begin with nearby values of $u$; but after a certain number of bounces they have very different values of $u$.

Finally note that the treatment of this essay has used classical general relativity; but the approach to the singularity necessarily involves growth of curvature up to the Planck scale where classical general relativity is no longer valid. Though the $u$ map involves an infinite sequence of bounces; the Planck scale will be reached in a finite number of bounces. The
results of this paper should therefore be read as describing the approach to the Planck scale in gravitational collapse. After the Planck scale is reached, a description of gravitational collapse requires the calculations of a quantum theory of gravity.

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