Faber polynomial coefficient estimates for a subclass of bi-univalent functions involving $q$-analogue of Ruscheweyh operator

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Abstract. In this article, we introduce a new subclass of analytic bi-univalent functions using the Ruscheweyh type $q$-analogue operator. In addition, we estimate upper bounds for general and early bounds of Taylor-Maclaurin coefficients in functions of the class which is considered by using Faber polynomial expansions.

1. Introduction
The function class is denoted by $\mathcal{A}$ which represented by the following form:

$$k(z) = z + \sum_{m=2}^{\infty} a_m z^m, \quad (z \in \Delta)$$

(1.1)

that are analytic in $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and satisfy the following normalization conditions:

$$k(0) = 0 \quad \text{and} \quad k'(0) = 1.$$ 

In addition, Let $S$ be the subfamily of $\mathcal{A}$ consisting of univalent functions. Further, Let $\mathcal{P}$ be the subclass of analytic functions in $\Delta$ and satisfy the inequality $Re(\Psi(z)) > 0$ in $\Delta$ of the form

$$\Psi(z) = 1 + \sum_{m=1}^{\infty} \Psi_m z^m,$$

(1.2)

where $|\Psi_m| < 2$, by Caratheodory’s Lemma (see, [1]).

For the two functions $k(z)$ and $h(z)$ analytic in $\Delta$, we say that $k(z)$ is subordinate to $h(z)$, usually denoted by $k(z) < h(z)$ ($z \in \Delta$), if there exists a Schwarz function $\varphi(z)$ within $\Delta$ with $\varphi(0) = 0$ and $|\varphi(z)| < 1$ ($z \in \Delta$), such that $k(z) = h(\varphi(z)),$ ($z \in \Delta$).

It is well known that $k(\Delta) \geq 1/4$ for any $k \in S$, by the Koebe’s theorem [1]. Also, every univalent function $k$ has an inverse $k^{-1}$, which is defined by

$$k^{-1}(k(z)) = z, \quad (z \in \Delta),$$
A function \( k \in \mathcal{A} \) is said to be bi-univalent in \( \Delta \) if both \( k \) and \( k^{-1} \) are univalent in \( \Delta \). Let \( \sigma \) denote the class of bi-univalent functions given by the Taylor-Maclaurin series expansion (1.1).

Previously, work by Lewin [2] investigated the bi-univalent function class, found the bound 1.51 for the modulus of \( |a_2| \). This was followed by hypotheses from Brannan and Clunie [3] that \( |a_2| \leq \sqrt{2} \) for \( k \in \sigma \). Subsequently, Netanyahu [4] demonstrated that, \( k \in \sigma \), \( \max |a_2| = \frac{5}{3} \). In addition, various subclasses within \( \sigma \) as the class of bi-univalent functions were found by Brannan and Taha [5], which have similarities with the well-known \( S^* (\beta) \) and \( K (\beta) \) subclasses respectively for convex and starlike functions of order \( 0 \leq \beta < 1 \) in \( \Delta \). Several researchers have recently examined boundaries of various subclasses of bi-univalent functions (as in e.g. [6], [7], [8], [9], [10] and [11]).

The Faber polynomials presented by Faber [12] play a vital part in different ranges of mathematical sciences, particularly in the theory of geometric functions. Grunsky [13] succeeded in defining a set of necessary and sufficient conditions for the univalence for a given function and in these conditions, the coefficients of the Faber polynomials play a significant role. Schiffer [14] used the Faber polynomials to give differential equations for univalent functions that solved such extreme problems with regard to coefficients of such functions.

Faber polynomial expansion of functions \( k \in \mathcal{A} \) of the form (1.1), can be used to represent the coefficients of \( g = k^{-1} \) as:

\[
g(\omega) = k^{-1}(\omega) = \omega + \sum_{m=2}^{\infty} \frac{1}{m!} K_{m-1}^{m}(a_2, a_3, \ldots) \omega^m,
\]

where

\[
K_{m-1}^{m} = \frac{(-m)!}{(2m + 1)!(m - 1)!} a_2^{m-1} + \frac{(-m)!}{[2(-m + 1)]!(m - 3)!} a_2^{m-3} a_3 + \frac{(-m)!}{(2m - 3)!} a_2^{m-4} a_4
\]

\[
+ \frac{(-m)!}{[2(-m + 2)]!(m - 5)!} a_2^{m-5} [a_5 + (-m + 2) a_3^2] + \frac{(-m)!}{(2m - 6)!} a_2^{m-6} [a_6 + (-2m + 5) a_3 a_4]
\]

\[
+ \sum_{i \geq 7} a_2^{m-i} V_i,
\]

and \( V_i \) with \( 7 \leq i \leq m \) is a homogeneous polynomial of degree \( i \) in the variables \( a_2, a_3, \ldots, a_m \), (all details can be found in [15]). Particularly, the first three terms of \( K_{m-1}^{m} \) are

\[
\frac{1}{2} K_1^{2} = -a_2,
\]

\[
\frac{1}{3} K_2^{3} = 2a_2^2 - a_3,
\]

\[
\frac{1}{4} K_3^{4} = -(5a_2^3 - 5a_2a_3 + a_4).
\]

In general, an expansion of \( K_m^{p} \) is given by (see, for details, [16]).

\[
K_m^{p} = p a_m + \frac{(p - 1)}{2} F_m^2 + \frac{p!}{(p - 3)!} F_m^3 + \cdots + \frac{p!}{(p - m)!} F_m^m, \quad (p \in \mathbb{Z}),
\]
where  
\[ Z = \{0, \pm 1, \ldots\} \text{ and } F^p_m = F^p_m(a_2, a_3, \ldots), \]
and, alternatively, by (see, [15]).  
\[ F^m_n(a_1, a_2, \ldots, a_m) = \sum_{n=1}^{\infty} \frac{n!(a_1)^{\sigma_1} \cdots (a_m)^{\sigma_m}}{\sigma_1! \cdots \sigma_m!}, \]
while \( a_1 = 1 \), and the sum is taken over all nonnegative integers \( \sigma_1, \ldots, \sigma_m \) satisfying
\[ \sigma_1 + \sigma_2 + \ldots + \sigma_m = n, \quad \sigma_1 + 2\sigma_2 + \ldots + m\sigma_m = m. \]
Evidently,
\[ F^m_n(a_1, a_2, \ldots, a_m) = a_1^n. \]

The application of \( q \)-calculus is very important in the theory in analytic functions. Jackson is known to be the first to succeed in developing \( q \)-integral and \( q \)-derivative in a systematic way (for more details, see [17, 18]). Purohit and Raina [19] for example examined the use of fractional \( q \)-calculus operators in defining a number of analytic function classes for \( \Delta \) as an open unit disk. Meanwhile, Mohammed and Darus [20] evaluated \( q \)-operator characteristics in terms of geometry and approximation with reference to particular analytic function subclasses within compact disks. In addition, fractional \( q \)-derivative and fractional \( q \)-integral operators, among other \( q \)-calculus operators, have been applied in constructing a number of analytic function subclasses, as in [21], [22], [23], [24], [25], [26], [27], [28], [29] and [19]. A more complete treatment of applied \( q \)-analysis within the theory of operators may be found in [30] and [31].

This work starts by defining key terms and detailed concepts within the \( q \)-calculus applied here. For the purposes of the report, the following assumption is made: 0 < \( q \) < 1. Firstly, fractional \( q \)-calculus operators for a function with complex values \( k(z) \) are defined below:

**Definition 1.1** Let 0 < \( q \) < 1 and define the \( q \)-number \([m]_q \) by
\[
[m]_q = \begin{cases} 
1-q^m, & (m \in \mathbb{C}), \\
1-q^{m-1} \sum_{n=0}^{m-1} q^n = 1 + q + q^2 + \ldots + q^{j-1}. & (m = j \in \mathbb{N}) 
\end{cases}
\]

**Definition 1.2** Let 0 < \( q \) < 1 and define the \( q \)-factorial \([m]_q! \) by
\[
[m]_q! = \begin{cases} 
[m]_q[m-1]_q \cdots [1]_q, & m = 1, 2, \ldots, \\
1, & m = 0.
\end{cases}
\]

**Definition 1.3** (see [17],[18]) The \( q \)-derivative operator \( \partial_q \) of a function \( k \) is determined by
\[
\partial_q k(z) = \begin{cases} 
\frac{k(qz) - k(z)}{(q-1)z}, & (z \neq 0) \\
k'(z). & (z = 0)
\end{cases}
\]

We note from Definition 1.3 that
\[
\lim_{q \to 1} (\partial_q k)(z) = \lim_{q \to 1} \frac{k(qz) - k(z)}{(q-1)z} = k'(z).
\]
From (1.1) and (1.6), we get

\[ \partial_q k(z) = 1 + \sum_{m=2}^{\infty} [m]_q a_m z^{m-1}. \]

In 2014, the authors in [21] defined the Ruscheweyh type \( q \)-analogue operator \( R_q^\delta \) by

\[ R_q^\delta k(z) = z + \sum_{m=2}^{\infty} \frac{\left[ m + \delta - 1 \right]_q}{[\delta]_q [m-1]_q} a_m z^m, \quad (1.7) \]

where \( \delta \geq 0 \) and \([m]_q\) defined by (1.5).

Also, as \( q \to 1^- \) we have

\[ \lim_{q \to 1^-} R_q^\delta k(z) = z + \sum_{m=2}^{\infty} \frac{(m + \delta - 1)!}{(\delta)! (m-1)!} a_m z^m \]

\[ = R^\delta k(z), \]

where \( R^\delta k(z) \) is Ruscheweyh differential operator described in [32] and studied by many researchers, for instance [33] and [34].

Now, using the differential operator \( R_q^\delta k(z) \) and the concept of subordination, we define a new subclass of \( \sigma \) as:

**Definition 1.4** Let \( \delta \geq 0 \) and \( \xi \geq 1 \). Then \( k \in \sigma \) is said to be in the class \( B\sigma^d (\delta, \xi; \Psi) \) if and only if

\[ (1 - \xi) \frac{R_q^\delta k(z)}{z} + \xi \partial_q (R_q^\delta k(z)) < \Psi(z), \]

and

\[ (1 - \xi) \frac{R_q^\delta g(\omega)}{\omega} + \xi \partial_q (R_q^\delta g(\omega)) < \Psi(\omega), \]

where \( g(\omega) = k^{-1}(\omega) \) and \( R_q^\delta \) are given by (1.3) and (1.7), respectively.

**Remark 1.1** It can be noted that the class \( B\sigma^d (\delta, \xi; \Psi) \) is reduced to various subclasses of bi-univalent functions by specializing the parameters, for instance:

- For \( q \to 1^- \) and \( \delta = 0 \), the class \( B\sigma^d (\delta, \xi; \Psi) \) reduces to the class \( B(\xi; \Psi) \) examined by Peng and Han [35].
- For \( \xi = 1, q \to 1^- \) and \( \delta = 0 \) the class \( B\sigma^d (\delta, \xi; \Psi) \) reduces to the class \( B(\Psi) \) examined by Peng and Han [35].

In the following main results, the Faber polynomial expansion is used to determine the upper bounds for general coefficient \( |a_n| \). In addition, we offer estimates for the initial coefficients for functions belong to the class \( B\sigma^d (\delta, \xi; \Psi) \).
2. Main Results

In the following theorem, we estimated the coefficients bound for such functions which belong to the class $B\sigma^q(\delta, \xi; \Psi)$.

**Theorem 2.1** For $\delta \geq 0$ and $\xi \geq 1$, let $k \in B\sigma^q(\delta, \xi; \Psi)$. If $a_k = 0; k = 2\ldots m - 1$, then

$$|a_m| \leq \frac{2(1 - q)}{|1 - q + \xi(q - q^m)|} \frac{[m + \delta - 1]q!}{[\delta]q![m - 1]q!}, m \geq 4. \quad (2.1)$$

**Proof.** Let $k$ be given by (1.1), we have

$$1 \leq (1 - \xi) - \frac{R^q_k(z)}{z} + \xi \partial_q(R^q_k(z)) = 1 + \sum_{m=2}^{\infty} [1 + \xi([m]_q - 1)] \times \frac{[m + \delta - 1]q!}{[\delta]q![m - 1]q!} a_m z^{m-1}, \quad (2.2)$$

and for $g = k^{-1}$, we have

$$1 \leq (1 - \xi) - \frac{R^q_k(\omega)}{\omega} + \xi \partial_q(R^q_k(\omega)) = 1 + \sum_{m=2}^{\infty} [1 + \xi([m]_q - 1)] \times \frac{[m + \delta - 1]q!}{[\delta]q![m - 1]q!} \omega z^{m-1}, \quad (2.3)$$

where $\mathcal{K}^{-m}_{m-1}$ as in (1.4).

On the other hand, since $k \in B\sigma^q(\delta, \xi; \Psi)$ and $k^{-1} \in B\sigma^q(\delta, \xi; \Psi)$, there exist two Schwartz functions

$$\nu(z) = \sum_{m=1}^{\infty} b_m z^m \quad \text{and} \quad \theta(\omega) = \sum_{m=1}^{\infty} d_m \omega^m,$$

such that

$$1 = (1 - \xi) - \frac{R^q_k(z)}{z} \sum_{m=1}^{\infty} b_m z^m = \Psi(\nu(z)), \quad (2.4)$$

and

$$1 = (1 - \xi) - \frac{R^q_k(\omega)}{\omega} \sum_{m=1}^{\infty} d_m \omega^m = \Psi(\theta(\omega)), \quad (2.5)$$

where

$$\Psi(\nu(z)) = 1 + \sum_{m=1}^{\infty} \sum_{n=1}^{m} \Psi_n F^n_m(b_1, \ldots, b_m) z^m, \quad (2.6)$$

and

$$\Psi(\theta(\omega)) = 1 + \sum_{m=1}^{\infty} \sum_{n=1}^{m} \Psi_n F^n_m(d_1, \ldots, d_m) \omega^m. \quad (2.7)$$

From (2.2), (2.4), and (2.6) we get

$$[1 + \xi([m]_q - 1)] \times \frac{[m + \delta - 1]q!}{[\delta]q![m - 1]q!} a_m = \sum_{n=1}^{m-1} \Psi_n F^n_{m-1}(b_1, \ldots, b_{m-1}), \quad (m \geq 2). \quad (2.8)$$
Similarly, form (2.3), (2.5), and (2.7) we have
\[
\left[ 1 + \xi([m]q - 1) \right] \times \frac{[m + \delta - 1]q!}{[\delta]q! [m-1]q} c_m = \sum_{n=1}^{m-1} \Psi_n F_{m-1}^n (d_1, \ldots, d_{m-1}), \ (m \geq 2).
\] (2.9)

Note that for \( a_k = 0; \ \kappa = 2, \ldots, m - 1 \), we have \( c_m = -a_m \) and so
\[
\left[ 1 + \xi([m]q - 1) \right] \times \frac{[m + \delta - 1]q!}{[\delta]q! [m-1]q}, a_m = \Psi_1 b_{m-1}, \quad - \left[ 1 + \xi([m]q - 1) \right] \times \frac{[m + \delta - 1]q!}{[\delta]q! [m-1]q}, a_m = \Psi_1 d_{m-1}.
\]

Taking the absolute value of the above two equations, we obtain
\[
|a_m| = \left| \Psi_1 b_{m-1} \right| = \left| \left[ 1 + \xi([m]q - 1) \right] \times \frac{[m + \delta - 1]q!}{[\delta]q! [m-1]q} \right| = \left| \left[ 1 + \xi([m]q - 1) \right] \times \frac{[m + \delta - 1]q!}{[\delta]q! [m-1]q} \right|.
\]
by applying the Caratheodory's Lemma, we have
\[
|a_m| \leq \frac{2(1 - q)}{|1 - q + \xi(q - q^m)| \frac{[m + \delta - 1]q!}{[\delta]q! [m-1]q}}.
\]

Which obviously completes the proof of Theorem 2.1.

By putting \( \delta = 0 \) in Theorem 2.1, we obtain the following corollary.

**Corollary 2.1** [36] For \( \xi \geq 1 \), let \( k \in \mathcal{B}\sigma^q(\xi; \Psi) \). If \( a_k = 0; \ \kappa = 2, \ldots, m - 1 \), then
\[
|a_m| \leq \frac{2(1 - q)}{1 - q + \xi(q - q^m)}, \ m \geq 4.
\]
Setting \( q \to 1^- \) and \( \delta = 0 \) in the above theorem, we obtain the following result.

**Corollary 2.2** For \( \xi \geq 1 \), let \( k \in \mathcal{B}(\xi; \Psi) \). If \( a_k = 0; \ \kappa = 2, \ldots, m - 1 \), then
\[
|a_m| \leq \frac{2}{1 + \xi(m - 1)}, \ m \geq 4.
\]

In next theorem, we estimate the following early coefficient bounds for functions belong to the class \( \mathcal{B}\sigma^q(\delta, \xi; \Psi) \).

**Theorem 2.2** Let \( \delta \geq 0 \) and \( \xi \geq 1 \). If \( k \in \mathcal{B}\sigma^q(\delta, \xi; \Psi) \) then we have the following consequence

1. \( |a_2| \leq \min \left\{ \frac{2}{(1 + \xi q)\delta + 1]q}, \frac{2\sqrt{1 + q}}{\sqrt{(1 + \xi q)(q^2)}\delta + 2]q} \right\} \).
2. \( |a_3| \leq \min \left\{ \frac{4}{(1 + \xi q)\delta + 1]q + 2]q}, \frac{2(1 + q)}{\sqrt{(1 + \xi q)(q^2)}\delta + 2]q}, \frac{6(1 + q)}{(1 + \xi q)(q^2)}\delta + 2]q \right\} \).
3. \( |a_3 - 2a_2^2| \leq \frac{4(1 + q)}{\sqrt{(1 + \xi q)(q^2)}\delta + 2]q} \).


Upon substituting the value of ... d_1, (2.12)

Next, in order to find the bounds of \( |a_2| \), we subtract (2.11) from (2.13), we get

\[
2 \left[ 1 + \xi([3]_q) - 1 \right] \times \frac{[\delta + 1]_q[\delta + 2]_q}{[2]_q} (a_2 - a_3) = \Psi_1(b_2 - d_2) + \Psi_2(b_1^2 - d_1^2),
\]

or, equivalently,

\[
|a_2| \leq \frac{2\sqrt{1 + q}}{\sqrt{[1 + \xi(q + q^2)][\delta + 1]_q[\delta + 2]_q}}.
\] (2.15)

Next, in order to find the bounds of \( |a_3| \), we substitute (2.13) from (2.11), we get

\[
2 \left[ 1 + \xi([3]_q) - 1 \right] \times \frac{[\delta + 1]_q[\delta + 2]_q}{[2]_q} (a_3 - a_2^2) = \Psi_1(b_2 - d_2) + \Psi_2(b_1^2 - d_1^2),
\]

then

\[
|a_3| \leq |a_2|^2 + \frac{[2]_q |\Psi_1(b_2 - d_2)|}{2 \left[ 1 + \xi([3]_q - 1) \right] [\delta + 1]_q[\delta + 2]_q}.
\] (2.16)

Upon substituting the value of \( a_2 \) from equation (2.14) and (2.15) into (2.16) to get

\[
|a_3| \leq \frac{4}{\left[ 1 + \xi(q + q^2) \right][\delta + 1]_q[\delta + 2]_q} + \frac{2(1 + q)}{\left[ 1 + \xi(q + q^2) \right][\delta + 1]_q[\delta + 2]_q},
\]

and

\[
|a_3| \leq \frac{6(1 + q)}{\left[ 1 + \xi(q + q^2) \right][\delta + 1]_q[\delta + 2]_q}.
\]

Finally, we rewrite (2.13) as

\[
\left[ 1 + \xi([3]_q - 1) \right] \times \frac{[\delta + 1]_q[\delta + 2]_q}{[2]_q} (a_3 - 2a_2^2) = -(\Psi_1d_2 + \Psi_2d_1^2).
\]
and therefore
\[ |a_3 - 2a_2^2| = \left| \frac{2|q(\Psi_1 d_2 + \Psi_2 d_2^2)}{[1 + \xi((3|q - 1) [\delta + 1]_q [\delta + 2]_q] \leq \frac{4(1 + q)}{[1 + \xi(q + q^2)] [\delta + 1]_q [\delta + 2]_q}. \]

This completes the proof of Theorem 2.2.

By putting \( \delta = 0 \) in Theorem 2.2, we obtain the following corollary.

**Corollary 2.3** [36] For \( \xi \geq 1 \), let \( k \in \mathcal{B}(\xi; \Psi) \). Then

1. \( |a_2| \leq \frac{2}{1+\xi(\delta)} \),
2. \( |a_3| \leq \frac{4}{(1+\xi(\delta)^2} + \frac{2}{1+\xi(\delta+2)} \),
3. \( |a_3 - 2a_2^2| \leq \frac{4}{1+\xi(\delta+2)} \).

Setting \( q \to 1^- \) and \( \delta = 0 \) in Theorem 2.2, we get the following result.

**Corollary 2.4** For \( \xi \geq 1 \), let \( k \in \mathcal{B}(\xi; \Psi) \), then

1. \( |a_2| \leq \frac{2}{1+\xi} \),
2. \( |a_3| \leq \frac{4}{(1+\xi)^2} + \frac{2}{1+2\xi} \).

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