On first order definability of equationally noetherian graphs

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Abstract. In this paper we show that there exist two simple graphs such that they are elementary equivalent but one of them is equationally noetherian and the other one is not. Therefore the equational noethericity property of simple graphs is not first order definable.

1. Introduction

Generally speaking, universal algebraic geometry studies equations over various algebraic structures. There are many papers devoted to this direction of modern mathematics. The monography [1] contains a quite complete description of achievements in this area and a large list of papers related to the universal algebraic geometry. The paper also belongs to this direction of research. On the other hand, this work might be considered as a continuation of studying model-theoretic properties of infinite simple graphs that we started in [2, 3, 4].

The notion of equational noethericity plays central role in universal algebraic geometry. An algebraic structure $A$ is called equationally noetherian if for any system $S$ of equations over $A$ in variables $x_1, \ldots, x_n$ there exists a finite subsystem of $S$ which is equivalent to $S$ (see section 2 for formal definitions). Equationally noetherian structures have many nice properties (see [1, 5] for details):

- It suffices to consider only finite systems of equations over $A$;
- There is a common approach to study algebraic sets of systems of equations over $A$;
- Any algebraic set over $A$ can be presented as a finite union of irreducible algebraic sets.

The first order language of simple graphs is quite expressible. Many important properties of graphs may be defined with help of first order formulas (see [6]). For example, it is easy to show that an arbitrary finite graph may be completely defined by a first order formula. On the other side, the property of graphs “to be connected”
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cannot be expressed by first order formulas: there exist two graphs such that they are elementary equivalent to each other but the first graph is connected and the second one is not.

In this paper we show that there exist two simple graphs such that they are elementary equivalent but one of them is equationally noetherian and the other one is not. Therefore the equational noethericity property of simple graphs is not first order definable.

2. Preliminaries

An undirected graph is a pair of sets \((V,E)\), where \(V\) is a nonempty set of vertices, and \(E\) is a set of pairs of elements from \(V\), called edges. We will introduce the concepts of algebraic geometry over algebraic systems, following the monograph [1]. Despite these concepts are suitable for any algebraic structure, we will present them in an adaptation to the language of graph theory.

Let \(L = \{ E(x,y) \}\) be the language of graph theory, where \(E(x,y)\) is the adjacency predicate for vertices \(x\) and \(y\). Note that for any vertex \(x\) \(E(x,x)\) is always false. An arbitrary graph is an algebraic structure over the language \(L\). Let \(\Gamma\) be an undirected graph. An extension of the language \(L\) by the set of vertices of the graph \(\Gamma\): \(L_\Gamma = L \cup V(\Gamma)\), we call the language of graphs with constants from \(\Gamma\). Since we consider systems of equations in a finite set of variables and the language \(L\) does not contain constants, then in the language \(L\) all graphs are equationally noetherian due to the fact that there are no infinite systems of equations in this language. Thus, it makes sense to consider equational noethericity of graphs in the language \(L\) extended by a certain set of constants. Therefore, for an arbitrary graph \(\Gamma\), we will consider equations in the language \(L_\Gamma = L \cup \{ V(\Gamma) \}\). Equations given in a language with a set of constants extended by all elements of an algebraic structure are called Diophantine equations. In this article, we deal only with Diophantine equations over graphs.

The language \(L_\Gamma\) allows only six types of equations: \(E(x,y), x = y, E(x,c), x = c, E(c_1,c_2), c_1 = c_2\), where \(x, y\) are variables, \(c, c_1, c_2\) are constants. Any set of such equations is called a system of equations in the language \(L_\Gamma\).

A point \(\alpha \in \Gamma^n\) is called a solution to the equation \(s(X)\) of the language \(L_\Gamma\) in \(n\) variables \(X = \{x_1,\ldots,x_n\}\) over the graph \(\Gamma\) if \(\Gamma \models s(\alpha)\). A point \(\alpha \in \Gamma^n\) is called a solution to the system of equations \(S(X)\) over the graph \(\Gamma\) if \(\alpha\) is a solution to each equation in the system \(S(X)\). The set of all solutions of the system of equations \(S(X)\) is called an algebraic set over \(\Gamma\) and is denoted by \(V_\Gamma(S(X))\).

Two systems of equations \(S_1(X)\) and \(S_2(X)\) of a language \(L\) are called equivalent over the graph \(\Gamma\) if their sets of solutions coincide. A graph \(\Gamma\) is called equationally noetherian if, for any positive integer \(n\), any system of equations \(S(X)\) in \(n\) variables \(X\) is equivalent to its own finite subsystem \(S_0(X) \subseteq S(X)\).

Equations consisting only of constants: \(E(c_1,c_2), c_1 = c_2\), are either always false or always true. Systems of such equations can be replaced with one knowingly false or
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true equation of the form \( c_1 = c_2 \). Therefore, we will not consider systems of equations containing an infinite number of equations consisting only of constants.

The paper [7] contains a criterion for equational non-noethericity of algebraic structures. In the original formulation of the criterion, there is a restriction on the language of the algebraic structure: it must not contain predicate symbols. However, this condition is not strict, and the proof of the lemma is true without any changes for an arbitrary algebraic structure. Let us present the formulation of this lemma.

**Lemma 1** An algebraic structure \( A = \langle A, L \rangle \) is not equationally noetherian if and only if there is a sequence of elements \((\alpha_i)_{i \in \mathbb{N}}, \alpha_i \in A^n, \) and a sequence of equations \((s_i(X))_{i \in \mathbb{N}}, X = \{x_1, \ldots, x_n\}\) in the language \( L \) such that \( A \not\models s_i(\alpha_i) \) for all \( i \in \mathbb{N} \) and \( A \models s_j(\alpha_i) \) for all \( j < i \).

It is known that all locally finite graphs are equationally noetherian, that is, graphs whose degree of any vertex is finite. This fact can be easily checked using the lemma above. At the same time, among the non-locally finite graphs, there are many graphs that are noetherian. Moreover there are also graphs that are equationally noetherian, in which there are an infinite number of vertices of infinite degree. The countable clique \( K = \{k_i, i \in \mathbb{N}\} \) is not equationally noetherian. Denote by \( s_i(x) \) the equation \( E(x, k_i) \). It is not hard to see that the series of equations \((s_i)_{i \in \mathbb{N}}\) and elements \((k_i)_{i \in \mathbb{N}}\) satisfy lemma 1.

3. Noethericity by equations and first order properties

We skip several known model-theoretic definitions such as first order formula, elementary equivalence and so on. We refer a reader to [6] to recall these notions. Also we refer to [8, 6] for description of Ehrenfaucht-Fraisse game which serves as a powerful criteria to determine when two algebraic structures are elementary equivalent.

Denote by \( K \) the countable clique. Let \( K_i, i \in \mathbb{N} \) be a finite clique on \( i \) vertices. Denote by \( \Gamma \sqcup \Delta \) a disjoint union of graphs \( \Gamma \) and \( \Delta \). Let \( \Gamma_1 = \sqcup_{i \in \mathbb{N}} K_i \) and \( \Gamma_2 = \Gamma_1 \sqcup K \). Then the following propositions hold.

**Proposition 1** In notations above two graphs \( \Gamma_1 \) and \( \Gamma_2 \) are elementary equivalent.

**Proof.** Denote by \( K^j_r \subset \Gamma_j \) a \( r \)-th connected component of \( \Gamma_j \) isomorphic to \( K_r, j = 1, 2 \) and \( K \) be the infinite clique from \( \Gamma_2 \).

We will use the Ehrenfaucht-Fraisse game for the graphs \( \Gamma_1 \) and \( \Gamma_2 \) to prove the proposition. Let \( k \) be a natural number and we consider a \( k \)-step Ehrenfaucht-Fraisse game.

Let’s describe a winning strategy of Duplicator. Before the first step he chooses a natural \( d > k \) and constructs a correspondence between connected components of \( \Gamma_1 \) and \( \Gamma_2 \): \( K^j_1 - K^j_r, K^j_2 - K^j_r + 1, K^j_3 - K^j_r + 2, K^j_4 - K^j_r + 3, \ldots \)

At each step if spoiler chose some vertex \( x \) from some component \( K^j_r \) of \( \Gamma^j \) then Duplicator chooses a vertex corresponded to \( K^j_r \) in another graph. Finally we get two \( k \)-tuples \( a_1, \ldots, a_k \in \Gamma^1 \) and \( b_1, \ldots, b_k \in \Gamma^2 \).
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Note that $\Gamma_1$ as well as $\Gamma_2$ have the property: for a natural $r$ any connected induced subgraph of $\Gamma_1$ or $\Gamma_2$ on $r$ vertices is isomorphic to $K_r$. According to the strategy the number of chosen vertices in some component $K^1_r$ equals to the number of chosen vertices in the corresponded component in $\Gamma_2$ and vice versa. Therefore induced subgraphs on vertices $a_1, \ldots, a_k \in \Gamma^1$ and $b_1, \ldots, b_k \in \Gamma^2$ are isomorphic.

Q.E.D.

Proposition 2 In notations above

(i) $\Gamma_1$ is equationally noetherian in the language $L_{\Gamma_1}$;

(ii) $\Gamma_2$ is not equationally noetherian in the language $L_{\Gamma_2}$. 

Proof.

Let’s prove the first part of the theorem. In the beginning we need to prove the following technical lemma about equationally non-noetherian systems of equations over graphs.

Lemma 2 Let $\Gamma$ be a non-noetherian simple graph in the language of graphs with constants $L_{\Gamma} = \{E(x)^{(2)}\} \cup V(\Gamma)$. Then there is a sequence of equations $S = (E(x,b_i))_{i \in \mathbb{N}}$ in one variable $x$ such that the conditions of the lemma 1 are satisfied.

Proof. By lemma 1, there exists an infinite system of equations $S(X)$ and a sequence of elements $(a_i)_{i \in \mathbb{N}}$ satisfying the lemma 1. Let $X = \{x_1, \ldots, x_n\}$ and $S = S^c \cup S' \cup S^{a_1} \cup \ldots \cup S^{a_n}$, where subsystem $S^{a_i}$ contains equations in a variable $x_i$, $i = 1, \ldots, n$, $S'$ contains equations without constants, $S^c$ contains equations with constants only. Since the number of unknowns is finite, the $S'$ subsystem is finite. Therefore at least some of the subsystems $S^{a_i}$, $i = 1, \ldots, n$, is infinite.

Hence there exists an infinite subsystem of equations $S' \subseteq S$ in one variable $x$. It is not hard to see by lemma 1 the subsystem $S'$ is also non-noetherian. Then $S'(x) = \{E(x,b_i)|i \in I_1\} \cup \{x = c_i|i \in I_2\}$. Since the system $S'$ is non-noetherian, it cannot contain equations of the form $x = c$. The proof of the lemma is completed.

Let $\Gamma_1$ is not equationally noetherian then by lemma 2 there exists a non-noetherian sequence of equations $S(x) = (E(x,b_i))_{i \in \mathbb{N}}$ and vertices $(a_i)_{i \in \mathbb{N}}$ such that satisfy lemma 1. It also follows from lemma 1 that $a_i \neq a_j$ for $i \neq j$ and all $a_i$ and $a_j$ are connected via $b_i$ for $i, j > 1$. But $G_1$ does not contain infinite connected graphs. A contradiction.

Let’s prove the second part of the theorem. Since an infinite clique $K = \{k_1, \ldots\}$ is an induced subgraph of $G_2$ then the equations $E(x,k_i), i \in \mathbb{N}$ and elements $k_i, i \in \mathbb{N}$ satisfy lemma 1. Therefore the graph $\Gamma_2$ is not equationally noetherian. Q.E.D.

Definition 1 We call a graph property $P$ is first order definable in the first order theory of graphs if there exists a set of first order sentences $T_P$ such that all graphs from $P$ are models of $T_P$ and all graphs outside $P$ are not models of $T_P$.

The theorem bellow shows that the property “to be a noetherian graph” cannot be expressed by first order sentences of graph theory in the language $L = \{E(x,y)\}$.
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**Theorem 1** The equational noethericity property is not first order definable in the first order theory of simple graphs.

**Proof.** The theorem follows from proposition 1 and proposition 2. Q.E.D.

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