Tensor products of recurrent hypercyclic semigroups

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Abstract

We study tensor products of strongly continuous semigroups on Banach spaces that satisfy the hypercyclicity criterion, the recurrent hypercyclicity criterion or are chaotic.

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1 Introduction and preliminaries

In this note we study tensor products $T(t) \otimes S(t)$ of strongly continuous semigroups $T(t)$ and $S(t)$ acting on Banach spaces $X$ and $Y$. If $\alpha$ denotes a uniform crossnorm on the (algebraic) tensor product $X \otimes Y$ we denote by $X_{\tilde{\otimes}} \alpha Y$ the completion of the normed space $(X \otimes Y, \alpha)$. Our main purpose is to show that for strongly continuous semigroups $T(t), S(t)$ satisfying the recurrent hypercyclicity criterion and a uniform crossnorm $\alpha$ on $X \otimes Y$ the semigroup $T(t) \otimes S(t)$ acting on $X_{\tilde{\otimes}} \alpha Y$ satisfies the recurrent hypercyclicity criterion, too. An important ingredient in the proof of this result is the work by Desch and Schappacher in [4]. Our result is of particular interest when one is working with $L^p$ spaces of the form $L^p(M_1 \times M_2, \mu_1 \otimes \mu_2), p \geq 1$, for measure spaces $(M_i, \mu_i), i = 1, 2$, as there is a uniform crossnorm $\alpha$ such that $L^p(M_1 \times M_2, \mu_1 \otimes \mu_2) = L^p(M_1, \mu_1)_{\tilde{\otimes}} \alpha L^p(M_2, \mu_2)$, cf. [3]. Applications of our results to $L^p$ heat semigroups on certain Riemannian manifolds are contained in [7].

Similar results for tensor products of semigroups or operators can be found in [1, 9].

1.1 Hypercyclic and recurrent hypercyclic semigroups

A strongly continuous semigroup $T(t)$ on a Banach space $X$ is called hypercyclic if there exists an $x \in X$ such that its orbit $\{T(t)x : t \geq 0\}$ is dense in $X$.

If additionally the set of periodic points $\{x \in X : \exists t > 0 \text{ such that } T(t)x = x\}$ is dense.

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in $X$, the semigroup $T(t)$ is called *chaotic*. 

It is well known that a strongly continuous semigroup $T(t)$ on a separable Banach space $X$ is hypercyclic if and only if it is *topological transitive*, i.e. for any pair of non-empty open subsets $U, V \subset X$ there exists some $t > 0$ with $T(t)U \cap V \neq \emptyset$, cf. [5]. 

A sufficient condition for hypercyclicity is given by the so-called *hypercyclicity criterion*, cf. [6] for this variant:

**Definition 1.1.** (Hypercyclicity Criterion). A strongly continuous semigroup $T(t)$ on a separable Banach space $X$ satisfies the hypercyclicity criterion if for all non-empty open subsets $U, V, W \subset X$ with $0 \in W$ there exists a $t > 0$ such that

$$T(t)U \cap W \neq \emptyset \quad \text{and} \quad T(t)W \cap V \neq \emptyset.$$ 

(Note that the same $t$ is used in both cases.)

It should be remarked that a strongly continuous semigroup $T(t)$ on $X$ satisfies the hypercyclicity criterion if and only if the semigroup

$$T(t) \times T(t) := \begin{pmatrix} T(t) & 0 \\ 0 & T(t) \end{pmatrix}$$

is hypercyclic on $X \times X$, cf. [6, Theorem 2.5]. This result easily generalizes (with the same proof) to:

**Proposition 1.2.** Let $T(t)$ denote a strongly continuous semigroup on a Banach Space $X$ that satisfies the hypercyclicity criterion. Then the diagonal semigroup $T^n(t) := T(t) \times \cdots \times T(t)$ is hypercyclic on $X^n := X \times \cdots \times X$ for any natural number $n \geq 1$.

The discrete version of the next corollary can be found in [8, Corollary 6].

**Corollary 1.3.** Let $T(t)$ denote a strongly continuous semigroup on a Banach Space $X$ that satisfies the hypercyclicity criterion. Then the diagonal semigroup $T^n(t)$ satisfies the hypercyclicity criterion on $X^n$, $n \geq 1$, too.

**Proof.** It follows from Proposition 1.2 that the semigroup $T^n(t) \times T^n(t) = T^{2n}(t)$ is hypercyclic, and hence, by [6, Theorem 2.5], the semigroup $T^n(t)$ satisfies the hypercyclicity criterion. 

As in [4] we say that a strongly continuous semigroup satisfies the recurrent hypercyclicity criterion if for open subsets as in Definition 1.1 the set of all times $t > 0$ with $T(t)U \cap W \neq \emptyset$ and $T(t)W \cap V \neq \emptyset$ does not have arbitrarily large holes:

**Definition 1.4.** A strongly continuous semigroup $T(t)$ on a separable Banach space $X$ satisfies the *recurrent hypercyclicity criterion* if for all non-empty open subsets $U, V, W \subset X$ with $0 \in W$ there exists a constant $L \geq 0$ such that each interval $[t, t + L]$ contains an $s$ with

$$T(s)U \cap W \neq \emptyset \quad \text{and} \quad T(s)W \cap V \neq \emptyset.$$ 

Of course, any semigroup that satisfies the recurrent hypercyclicity criterion is hypercyclic as it satisfies the hypercyclicity criterion.
1.2 Tensor products

For Banach spaces $X$ and $Y$ we denote by $X \otimes Y$ their (algebraic) tensor product. Furthermore, let $\alpha$ be a tensor norm (or uniform crossnorm) on $X \otimes Y$ (for a definition see [3, 12.1] or [10, 6.1]). Then $\alpha$ is in particular a reasonable crossnorm on $X \otimes Y$ which implies that for $x \in X$ and $y \in Y$ we have

$$\alpha(x \otimes y) = ||x||_X \cdot ||y||_Y.$$ 

If we define for $z \in X \otimes Y$

$$\pi(z) = \inf \left\{ \sum_{i=1}^n ||x_i||_X \cdot ||y_i||_Y : z = \sum_{i=1}^n x_i \otimes y_i \right\}$$

this yields a tensor norm and is called projective norm. Actually, this norm is the greatest reasonable crossnorm on $X \otimes Y$, i.e. if $\alpha$ is another reasonable crossnorm it follows $\alpha \leq \pi$ (cf. [2, p. 64] or [10, Proposition 6.1]). For any norm $\alpha$ on $X \otimes Y$ we denote by $X \tilde{\otimes}_\alpha Y$ the completion of the normed space $(X \otimes Y, \alpha)$.

For bounded operators $T : X \rightarrow X$, $S : Y \rightarrow Y$, and any uniform crossnorm $\alpha$ the tensor product $T \otimes S$ is a bounded operator on $(X \otimes Y, \alpha)$ by definition of a uniform crossnorm. The unique extension of $T \otimes S$ to $X \tilde{\otimes}_\alpha Y$ is, for simplicity, also denoted by $T \otimes S$.

Similarly, if $T(t) : X \rightarrow X$ and $S(t) : Y \rightarrow Y$ are strongly continuous semigroups their tensor product $T(t) \otimes S(t)$ is a strongly continuous semigroup on $(X \otimes Y, \alpha)$ for any uniform crossnorm $\alpha$. To see this, let $z \in X \otimes Y$. Then we have

$$\alpha(T(t) \otimes S(t)z - z) \leq \alpha(T(t) \otimes S(t)z - T(t) \otimes Iz) + \alpha(T(t) \otimes Iz - z) \leq \pi(T(t) \otimes S(t)z - T(t) \otimes Iz) + \pi(T(t) \otimes Iz - z).$$

For the first term on the right hand side it follows if $z = \sum_i x_i \otimes y_i$ is any representation of $z$

$$\pi(T(t) \otimes S(t)z - T(t) \otimes Iz) \leq \sum_i ||T(t)x_i||_X \cdot ||S(t)y_i - y_i||_Y \rightarrow 0 \quad (t \rightarrow 0^+).$$

As an analogous argument shows that the second term goes to zero if $t \rightarrow 0^+$, it follows that $T(t) \otimes S(t)$ is strongly continuous.

2 Main results

Theorem 2.1. Let $T(t), S(t)$ denote strongly continuous semigroups on Banach spaces $X$ and $Y$, respectively, and assume that $T(t)$ satisfies the recurrent hypercyclicity criterion. Furthermore, $\alpha$ denotes a uniform crossnorm on $X \otimes Y$.

(a) If $S(t)$ satisfies the hypercyclicity criterion, the semigroup $T(t) \otimes S(t)$ on $X \tilde{\otimes}_\alpha Y$ satisfies the hypercyclicity criterion, too.
(b) If \( S(t) \) satisfies the recurrent hypercyclicity criterion, the semigroup \( T(t) \odot S(t) \) on \( X \tilde{\otimes}_\alpha Y \) satisfies the recurrent hypercyclicity criterion, too.

**Proof.** In the following, we use \( \| (x, y) \| = \sup \{ \| x \|_X, \| y \|_Y \} \) as norm on the product \( X \times Y \). Note, that the topology induced by this norm coincides with the usual product topology. As \( \alpha \) is a reasonable crossnorm, the canonical bilinear map

\[
\psi : (X \times Y, \| \cdot \|) \rightarrow (X \otimes Y, \alpha), (x, y) \mapsto x \otimes y
\]

is continuous and has norm \( \leq 1 \) (cf. [2, p. 64]). Hence, for any \( n \geq 1 \), the mapping

\[
\psi_n : \left\{ \begin{array}{c}
X^n \times Y^n \\
(x_1, \ldots, x_n, y_1, \ldots, y_n)
\end{array} \right\} \rightarrow X \otimes Y
\]

\[
\sum_{k=1}^n \psi(x_k, y_k)
\]

is continuous for the norm \( \| (x_1, \ldots, x_n, y_1, \ldots, y_n) \| = \sup \{ \| x_k \|_X, \| y_k \|_Y : k = 1, \ldots, n \} \) on \( X^n \times Y^n \).

For the proof of part (a) we proceed as follows: Let \( U, V, W \) be non-empty open subsets of \( X \tilde{\otimes}_\alpha Y \) with \( 0 \in W \). As \( X \otimes Y = \text{span} (\psi(X \times Y)) \) is dense in \( X \tilde{\otimes}_\alpha Y \), we find elements

\[
\sum_{k=1}^m x_k \otimes y_k \in U
\]

and

\[
\sum_{k=1}^n p_k \otimes q_k \in V.
\]

Extending one of the sums by zero summands if necessary we may assume \( m = n \). Then \( \psi_n^{-1}(U) \) and \( \psi_n^{-1}(V) \) are non-empty open subsets of \( X^n \times Y^n \). Furthermore, \( 0 \in \psi_n^{-1}(W) \).

As \( T(t) \) satisfies the recurrent hypercyclicity criterion and \( S(t) \) satisfies the hypercyclicity criterion, it follows from [4, Theorem 5.1] that the semigroup

\[
\left( \begin{array}{cc}
T(t) & 0 \\
0 & S(t)
\end{array} \right) : X \times Y \rightarrow X \times Y
\]

satisfies the hypercyclicity criterion and hence, by Corollary 1.3, the semigroup \( T^n(t) \times S^n(t) \) satisfies the hypercyclicity criterion, too. Therefore, there exists \( t > 0 \) such that

\[
\left( T^n(t) \times S^n(t) \psi_n^{-1}(U) \right) \cap \psi_n^{-1}(W) \neq \emptyset
\]

and

\[
\left( T^n(t) \times S^n(t) \psi_n^{-1}(V) \right) \cap \psi_n^{-1}(W) \neq \emptyset.
\]

As \( \psi_n(T^n(t) \times S^n(t) \psi_n^{-1}(U)) \subset T(t) \odot S(t) U \) and \( \psi_n(\psi_n^{-1}(V)) \subset V \) the proof of part (a) is complete.

For the proof of part (b) let \( U, V, W \subset X \tilde{\otimes}_\alpha Y \) be non-empty open subsets with \( 0 \in W \). As in part (a) we find an \( n \in \mathbb{N} \) such that the sets \( \psi_n^{-1}(U), \psi_n^{-1}(V), \) and \( \psi_n^{-1}(W) \) are non-empty open subsets of \( X^n \times Y^n \) with \( 0 \in \psi_n^{-1}(W) \). Since both semigroups \( T(t) \) and
S(t) satisfy the recurrent hypercyclicity criterion, it follows from [4, Corollary 5.6] that the semigroup $T^n(t) \times S^n(t)$ satisfies the recurrent hypercyclicity criterion, too. One can now conclude as in the proof of part (a) that the semigroup $T(t) \otimes S(t)$ satisfies the recurrent hypercyclicity criterion.

**Corollary 2.2.** Let $T(t), S(t)$ denote chaotic semigroups on Banach spaces $X$ and $Y$. If $\alpha$ denotes a uniform cross norm on the algebraic tensor product $X \otimes Y$ the semigroup $T(t) \otimes S(t)$ on $X \widehat{\otimes}_\alpha Y$ satisfies the recurrent hypercyclicity criterion.

**Proof.** This follows directly from Theorem 2.1 since any chaotic semigroup satisfies the recurrent hypercyclicity criterion, cf. [4, Corollary 6.2].

In order to state the next corollary, we need some preparation. Let $T$ denote a bounded operator on a Banach space $X$. $T$ is called chaotic, if – similar to the case of semigroups – there is an $x \in X$ whose orbit $\{T^n x : n \in \mathbb{N}\}$ is dense in $X$ and if the set of periodic points $\{x \in X : \exists n \in \mathbb{N} \text{ such that } T^n x = x\}$ is dense in $X$ as well.

**Corollary 2.3.** Let $T(t), S(t)$ denote strongly continuous semigroups on Banach spaces $X$ and $Y$ and $\alpha$ a uniform crossnorm.

(a) If there is a $t_0 > 0$ such that $T(t_0)$ is a chaotic operator and $S(t_0)$ has a dense set of periodic points, the semigroup $T(t) \otimes S(t)$ is chaotic.

(b) If there are $p_1, p_2, q_1, q_2 \in \mathbb{N}$ such that $T(p_1/q_1)$ and $S(p_2/q_2)$ are chaotic the tensor product $T(t) \otimes S(t)$ is chaotic.

**Proof.** We first prove (a). From [9, Corollary 1.12] it follows that the operator $T(t_0) \otimes S(t_0)$ is chaotic and hence, the semigroup $T(t) \otimes S(t)$ is chaotic. To show (b) we first remark that both semigroups $T(t)$ and $S(t)$ are chaotic and hence their tensor product satisfies the recurrent hypercyclicity criterion. It remains to show the density of the periodic points. Let $P_1$ (resp. $P_2$) be the set of periodic points of the operator $T(p_1/q_1)$ (resp. $S(p_2/q_2)$). These are dense linear spaces and hence, $P_1 \otimes P_2$ is dense in $X \widehat{\otimes}_\alpha Y$. Furthermore, if $x \otimes y \in P_1 \otimes P_2$ there exist $n, m \in \mathbb{N}$ with $T(p_1/q_1)^n x = T(p_1/q_1)^m y = T(p_2/q_2)^m y = T(p_2/q_2)^n y = y$, and $np_1/q_1 = mp_2/q_2 =: t$. Then we have

$$T(t) \otimes S(t)(x \otimes y) = T(t)x \otimes S(t)y = x \otimes y$$

and $x \otimes y$ is therefore a periodic point of $T(t) \otimes S(t)$. Since

$$P_1 \otimes P_2 = \text{span} \left\{ x_k \otimes y_k : x_k \in P_1, y_k \in P_2 \right\}$$

and as with $x_k \otimes y_k, k = 1, \ldots, n$, also $\sum_{k=1}^n x_k \otimes y_k$ is a periodic point, $P_1 \otimes P_2$ consists only of periodic points.

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