On the singular spectrum of the Almost Mathieu operator. 
Arithmetics and Cantor spectra of integrable models.

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I review a recent progress towards solution of the Almost Mathieu equation (A.G. Abanov, J.C. Talstra, P.B. Wiegmann, Nucl. Phys. B 525, 571, 1998), known also as Harper’s equation or Azbel-Hofstadter problem. The spectrum of this equation is known to be a pure singular continuum with a rich hierarchical structure. Few years ago it has been found that the Almost Mathieu operator is integrable. An asymptotic solution of this operator became possible due analysis the Bethe Ansatz equations.

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1. Introduction In this lecture I review a recent progress towards decoding of one of the most puzzling strange set generated by a quasiperiodic Schrödinger operator:

\[ \psi_{n+1} + \psi_{n-1} + 2\lambda \cos(\theta + 2\pi n\eta)\psi_n = E\psi_n \]  

(1)

The history of this equation as well as its applications in different branches of physics and mathematics are rich. This equation, known as Harper’s equation, describes the electronic spectrum of one-dimensional quasicrystal (a particle in a quasiperiodic potential) and often used to study localization-delocalization transition (see e.g. [4,5]). It is a standard example of the operator, also known as Azbel-Hofstadter problem [6,7]. This list may be continued.

The spectrum of this equation is complex. In the commensurate case, when \( \eta \) is a rational number

\[ \eta = P/Q, \]

one may impose the Bloch condition:

\[ \psi_n = e^{ik}\psi_{n+Q} \]

Then, the spectrum consists of \( Q \) bands, separated by \( Q - 1 \) gaps. In the incommensurate limit, when \( \eta \) is an irrational number (\( P \to \infty, Q \to \infty \)) the spectrum becomes an infinite Cantor set with total bandwidth (Lebesgue measure of the spectrum) \( 4|\lambda - 1| \).

The most interesting “critical” case appears at \( |\lambda| = 1 \). Then the spectrum becomes a purely singular continuum.

\footnote{Although some properties of the spectrum depend on the type of irrational number \( \eta \), here we consider typically Diophantine numbers. These numbers have a full Lebesgue measure and thus sufficient for almost all physical applications.}

\footnote{\textsuperscript{1}closed, nowhere dense set with no isolated points}

\[ \textbf{[1]} \] In this case wave functions lost their extended character and not yet localized but exhibit a power law scaling. Moreover, there is numerical evidence and almost a consensus, that in this case (\( \lambda = 1 \)) the spectrum and wave functions are multifractal [11].

Multifractal sets exhibit a sort of conformal invariance and are expected to be described by methods of conformal field theory. This theory, however, is yet to be developed and scaling properties of sets generated by dynamical systems and by closely related quasiperiodic systems are far from being understood.

Since the empirical observations of Hofstadter [12], the evidence has been mounting that the spectrum of (1) (Hofstadter butterfly) as well as other quasiperiodic equations with a differential potentials are regular and universal rather than erratic or “chaotic”. Few years ago it has been shown [13,14] that despite the complexity of the spectrum, the Harper-Azbel-Hofstadter-almost Mathieu operator (\[ 1 \]) equation at any rational \( \eta = P/Q \) is integrable and can be “solved” by employing methods of integrable systems Ansatz (BA). This had opened the possibility of describing the complex behavior of an incommensurate system as a limit of a sequence of integrable models. I hope that this solution will help to apply conformal field theory to dynamical systems.

The symmetries of the problem which eventually lead to its integrability are the most transparent it its “magnetic” interpretation. Consider a particle on a two dimensional square lattice in a magnetic field with a flux \( \Phi = 2\pi\eta \) per plaquette. Its Hamiltonian is:

\[ H = T_x + T_x^{-1} + \lambda(T_y + T_y^{-1}), \]

where operators \( T_x \) and \( T_y \) describe translations of the particle in \( x \) and \( y \) direction by a lattice site. In magnetic field translations form a Weyl pair:

\[ T_x T_y = q^2 T_y T_x, \]

(2)

where

\[ q^2 = e^{i\Phi}. \]
The Harper’s equation, then appears as a result of representation of translation operators as a shift and multiplication:

\[ T_x \psi_n = \psi_{n+1}, \]
\[ T_y \psi_n = q^2 e^{ik} \psi_n \]  \hspace{1cm} (3)

2. Hierarchical tree. I begin by describing the scaling hypothesis for the spectrum of an incommensurate (quasiperiodic) operator with a purely singular continuum hypothesis for the spectrum of an incommensurate 

so that:

\[ |\eta^{(j)} - \eta| < c(Q_j)^{-2}, \]  \hspace{1cm} where \( c \) is a \( j \)-independent constant. A Harper equation taken for each \( \eta^{(j)} \) is generations of the hierarchy. Let us consider a graph (with no loop), which connects the \( k \)-th band of the generation \( \eta^{(j)} \) (the daughter generation) to a certain band \( k' \), respectively. We call it a hierarchical tree if energies \( E_j(J) \) belonging to any branch \( J \) of the tree form a sequence converging to the point \( E(J) \) of the spectrum in such a way that the sequence \( Q^{j-1} |E_j(J) - E(J)| \) is bounded but does not converge to zero.

The set of numbers \( \epsilon, J \) are anomalous exponents. In a multifractal spectrum, anomalous dimensions depend on the branch. They and the tree characterize ultrametric properties of the spectrum.

Let us stress that the very existence of the hierarchical tree is a hypothesis and the tree constructed below is the conjecture. We call it scaling hypothesis.

To construct the hierarchical tree it is necessary to find the sequence of generations and a rule to determine the parent generation and a parent band out of a given band of a given generation. In other words, the hierarchical tree is determined by a sequence of rational approximants \( \eta^{(j)} \rightarrow \eta \) and by a mapping

\[ (k, P, Q) \rightarrow (k', P', Q'). \]  \hspace{1cm} (4)

where \( k \) and \( k' \) are labels of a daughter’s band and the parent’s band of generations \( P/Q \) and \( P'/Q' \) respectively. To describe the hierarchical tree we will need a notion of a discrete spectral flow and its rate Hall conductance.

A heuristic definition of the spectral flow is as follows. Let us consider a spectrum of the problem with a given \( \eta \) and choose some (big) gap. We label it by \( k \). Now let us change \( \eta \) by a small \( \delta \eta \), such that newly appeared gaps in the vicinity of the edge of the big gap, will be smaller than the big gap \( k \). Then we can look on new levels appeared within the big gap, close to the bottom of the gap. The number of these levels, i.e., the number of levels \( \delta N_k \), crossing an energy \( E \), lying inside the gap, close to its lower edge, is a spectral flow. One can also treat the spectral flow as a number of levels appeared within a "big" band adjacent to the gap from below. The rate of the spectral flow is

\[ \sigma_k = \frac{\delta N_k}{\delta \eta} \]  \hspace{1cm} (5)

is known to be the Hall conductance of the gap [17] (Streda’s formula) of two dimensional particles in magnetic field. This number is an integer and depends on the gap. The difference between the Hall conductances of nearest gaps, i.e., the spectral flow into the \( k \)-th band

\[ \sigma(k) = \sigma_k - \sigma_{k+1} \]  \hspace{1cm} (6)

is the Hall conductance of the band.

The index theorem identifies the Hall conductance of the band with the number of zeros of the wave function \( \psi_n(k, k') \) within the Brillouin zone: \( 0 < k < 2\pi, 0 < k' < 2\pi/Q \), i.e., with the Chern class of a band:

\[ \sigma(k) = \frac{1}{2\pi} \oint_{\partial B} \vec{\nabla} \ln \psi_n(k, k') d\vec{k}, \]

where the integral goes over the boundary of the magnetic Brillouin zone of the \( k \) th band.

The Hall conductivity of the \( k \)-th gap varies the range \( -Q/2 < \sigma_k \leq Q/2 \) and obeys the Diophantine equation [18][19]

\[ P\sigma_k = k \pmod{Q}. \]  \hspace{1cm} (7)

In it turns the Hall conductance of a band \( \sigma(k) \) is allowed to have only two values. They can be found explicitly.

Let us consider a continued fraction expansion of

\[ \eta^{(j)} = \frac{1}{n_1 + \frac{1}{n_2 + \frac{1}{n_3 + \ldots}}} \equiv [n_1, n_2, n_3, \ldots, n_j] \]  \hspace{1cm} (8)

Then the Hall conductance of the gap \( k \) is

\[ \sigma_k = \frac{Q_j}{2} - Q_j \left\{ (-1)^j \frac{Q_j^{-1} + 1}{2} \right\}, \]  \hspace{1cm} (9)

while the two values of the Hall conductance of the band \( k \) may be

\[ \sigma(k) = (-1)^{j-1} Q_{j-1}, \quad \text{or} \quad (-1)^j (Q_j - Q_{j-1}) \]  \hspace{1cm} (10)

here \( \{x\} \) is fractional part of \( x \). Now let us turn to the hierarchical tree. We conjecture that the hierarchical tree is the spectral hierarchy - an integral version of the spectral flow. Let us consider two close rational \( P/Q \) and \( P'/Q' \) with \( Q' < Q \). The number
of states per lattice site in a band is $1/Q$ and $1/Q'$ respectively. If there is a band $k$ of the problem with a flux $P/Q$, such that its Hall conductance is a ratio between the difference of the number of states and the fluxes

$$\frac{1}{Q} - \frac{1}{Q'} = \sigma(k)\left(\frac{P}{Q} - \frac{P'}{Q'}\right) \quad (11)$$

then we say that the band $k$ of the generation $P/Q$ has a "parent" band in the generation $P'/Q'$. The absolute value of the Hall conductance is the difference between number of states in the parent and a daughter band:

$$Q - Q' = |\sigma(k)| \quad (12)$$

This formula may be viewed as an integrated Streda formula. It determines the flux $P'/Q'$ and and by virtue of an iterative procedure, generates a sequence of rational approximants (generations of the hierarchical tree), $\eta^{(j)}$ to $\eta$.

The integrated Streda formula is not enough to determine the tree. We complete it by the adiabatic principle, which states that the levels do not cross each other along a tree. This proposition may be put in symbols. Let us enumerate all states from the bottom of the spectrum and characterize them by a fraction $\nu$ of the form

$$\nu = \frac{P}{Q}$$

then we say that the band $k$ of the generation $P/Q$ has a "parent" band in the generation $P'/Q'$. The absolute value of the Hall conductance is the difference between number of states in the parent and a daughter band:

$$Q - Q' = |\sigma(k)| \quad (12)$$

Each path of the tree may also be characterized by a fraction $\nu^{(j)} = k/Q_j$ lying on the path and converged to a given irrational fraction $\nu$. According to eq. (11) the parent fraction is determined by the daughter one as

$$\nu^{(j-1)} = \frac{1}{Q_{j-1}} \left(\left\lfloor \frac{Q_{j-1}(\nu^{(j)} - \frac{1}{Q_j})} \right\rfloor + 1\right)$$

where $\sigma(j) = \nu^{(j)} - \nu$. The sequence $\nu^{(j)}$ converges to the irrational $\nu$ faster than $Q_j^{-1}$, i.e., $|\nu^{(j)} - \nu| < c Q_j^{-1}$. In terms of the fractions one may reformulate the scaling hypothesis as $|E_j - E| < c|\nu^{(j)} - \nu|^{1/2}$, defining the scaling exponent $\alpha(J)$.

The hierarchical tree, we just described, has been suggested by the Bethe Ansatz equations for the Harper’s equation. However, it seems plausible that the construction is universal and valid for a general quasiperiodic equation, regardless, whether it is integrable or not. A set of Hall conductances is the only input of the algorithm.

3. Integrability. The Harper’s equation is integrable as soon as $\eta$ is a rational. Here I adopt a restricted definition of the integrability of a linear equation: there is an isospectral transformation which turns all Bloch solutions of the Harper’s equation to discrete polynomials of degree $Q$. In symbols

$$\psi_n = e^{i k' n} \sum_{m=0}^{Q-1} c_{nm} \Psi_m, \quad (16)$$

where $c_{nm}$ is a unitary $Q \times Q$ matrix and $\Psi(z)$ is a polynomial of the degree $Q - 1$. In other words, there is a gauge (a choice of the gage potential), or a representation of the algebra of translations in a magnetic field (2), where all wave functions are polynomials.

In this sense the Harper’s equation appears to be integrable for any point of the Brillouin zone $0 \leq k' < 2\pi/Q$, $0 \leq k < 2\pi$ [12], although the Bethe Ansatz equations look especially simple at the so called rational points of the Brillouin zone. The latter correspond to the centers and edges of bands. The study of these points is sufficient for our purposes. In this case

$$\Psi_n = \sum_{j=0}^{Q-1} a_j (\rho q^n)^j$$

where $a_j$ do not depend on $n$ and $\rho$ is a constant.

It appears to be convenient to parameterized polynomials by its roots:
\[ \Psi(z) \equiv \sum_{j=0}^{Q-1} a_j z^j = \prod_{i=0}^{Q-1} (z - z_i). \] (17)

Below we sketch the results of the Bethe-Ansatz solution and skip all aspects of integrability related to cyclic representations of \( U_q(sl_2) \) \[2\].

Rational points form a zoo. To characterize them we introduce parameters \( \tau, \kappa, \mu = \pm 1 \). The choice \( \tau = 1 \) yields levels at the center of bands, while \( \tau = -1 \) corresponds to edges of bands. The Chambers relation

\[ \Lambda(k', k) \equiv \det H = 2 \cos Qk' + 2\lambda \cos Qk \] (18)

implies that the energy depends on \( k' \) and \( k \) via \( \Lambda(k', k) \). Therefore the edges of the energy bands are given by extremum of \( \Lambda \) which assumes a minimum/maximum given by \( \Lambda = \pm (2 + 2\lambda) \). The middle of bands corresponds to \( \Lambda = 0 \). If \( P \) is even \((Q \text{ is odd})\), the rational points are labeled by additional discrete parameters \( \kappa, \mu = \pm 1 \). The middle points of bands at \( \kappa = \pm 1 \) are given by the equation

\[ \cos \left( \frac{2\pi}{Q}(k + \frac{\kappa Q}{2}) \right) = \nu(-1)^{Q-1}. \] The edges \((\tau = -1)\) of bands \( k' = \frac{\pi}{Q} \frac{1-(-1)^\tau}{2} \) and \( k = \frac{\pi}{Q} \left( 1 - \frac{(-1)^\tau}{2} \right) + 2l \)

are distinguished by parameter \( \mu \). Being count from the bottom of the spectrum, these edges are ordered as bottom-top-bottom... if \( \mu = (-1)^{\frac{\pi}{2}\text{-odd}} \) and top-bottom-top... if \( \mu = (-1)^{\frac{\pi}{2}\text{-even}} \) (see [1] for details).

For the rational points the transformation \( (16) \) is given by “quantum dilogarithms"

\[ c_{nm} = \prod_{j=0}^{m-1} \left( e^{i k q^{2n+1/2} \lambda^{1/2} + \tau \kappa \rho^{-1} q^{j-1/2}} \right), \] (19)

where \( \rho = i \exp(i \frac{kQ-2P}{2}) \). Under this transformation the Harper’s equation becomes:

\[ i(q \left( \frac{z}{\lambda} \right) + \frac{i\tau \kappa}{\lambda} z) \left( \frac{z}{\lambda} - \frac{i}{q} \right) \Psi(qz) \]

\[ -i q^{-1} \left( \frac{z}{\lambda} - \frac{i\tau \kappa}{\lambda} \right) \left( \frac{z}{\lambda} + \frac{i}{q} \right) \Psi(zq^{-1}) = \mu \kappa \lambda^{-1/2} E \Psi(z), \] (20)

where one suppose to set \( z = \rho q^n \). However, there is a certain advantage to consider the difference equation for \( \Psi(z) \) in a complex plane \( z \).

The integrability now reads: all Bloch solutions of the difference equation \( (20) \) are polynomials.

I try to unmasked this transformation by a comment bellow, however it becomes meaningful in the \( U_q(sl_2) \) setup.

Let us represent translation operators \( (3) \) by another Weyl pair

\[ UV = qVU, \] (21)

and setting

\[ T_x = UVU + a \]

\[ T_y = VU^{-1}U + a \]

where

\[ a = -i \nu q^{-1/2} \lambda^{-1/2} \]

\[ b = -i\tau \nu q^{-1/2} \lambda^{-1/2} \] (22)

Equation \( (23) \) appears, by choosing a standard representation of \( U \) and \( V \):

\[ (U\Psi)_n = -\rho^{-1} q^{-n} \Psi_n, \quad (V\Psi)_n = -i \nu \mu \Psi_{n+1}. \] (23)

4. The Bethe Ansatz. Being sure that solutions of the Eq.\( (20) \) are polynomials, we may evaluate it at one of the roots of the polynomial \( z_i \). This gives the Bethe-Ansatz (BA) equations:

\[ q^Q \prod_{k=1}^{Q-1} \left( \frac{z_i - q^{-1} z_i}{z_i - q^i} \right) \equiv \left( \frac{z_i - i \kappa \lambda^{-1/2} q^{1/2}}{z_i + i \kappa \lambda^{1/2} q^{1/2}} \right) \]

\[ \prod_{i=1}^{Q-1} \left( \frac{z_i - \lambda^{-1/2}}{q^{1/2} - q^{-1/2}} \right). \] (25)

Solutions of the BA equations give the wave functions of the Harper equation at band’s edges and centers. Their energy is given by

\[ E = i \mu \lambda^{1/2} q^Q (q - q^{-1}) \left[ \sum_{i=1}^{Q-1} \frac{z_i - \lambda^{-1/2}}{q^{1/2} - q^{-1/2}} \right]. \] (26)

The latter is obtained by evaluating the leading terms at \( z \to \infty \) at eq.\( (23) \).

At first glance, the BA equations \( (25) \) look even more complicated than the original Harper equation. This is true as long as \( Q \) is not large. However, the BA equations \( (25) \) provide a better description of the problem in the most interesting, incommensurate, limit \( P, Q \to \infty, \eta \to \) irrational number.

At \( P \) odd the BA equations admit an exact zero mode solution at for \( E = 0 \) \[24\]. For a quasiclassical analysis of the BA equations at \( \eta \to 0 \), see \[24\].

Below, we consider the most interesting case \( \lambda = 1 \).

3. String hypothesis. Here we formulate the string hypothesis which allows us to obtain the solutions of the BA equations \((\lambda = 1)\) with an accuracy \( O(Q^{-2}) \). The hypothesis is based on the analysis of singularities of the BA, and is supported by extensive numerics \[4\]. Here we just formulate the string hypothesis and present some immediate consequences. To proceed, we need the notion of strings.

A string of spin \( l \), parity \( v = \pm 1 \) and center \( x_1 \) is a set of \( 2l + 1 \) complex numbers:

\[ z_m^{(l)} = v x_l q^m, \quad m = -l, -l + 1, \ldots, l. \] (27)

which have a common modulus \( x_1 > 0 \) (a center of the string), a parity \( v = \pm 1 \) and differ by multiples of \( q \).

Now we are ready to formulate the string hypothesis — a central concept of this analysis:
At large $Q$ each solution of the BA consists of strings.

Each solution can be labeled by spins $\{l_j, l_{j-1}, \ldots\}$ and parities $\{v_j, v_{j-1}, \ldots\}$ of strings, such that the total number of roots $\sum_{i=1}^{k}(2l_i + 1) = Q - 1$. We refer to the set of lengths and parities of strings constituting the solution for a given energy level as to a string content of this level. Not more than two strings with a given length and parity may be found in a string content of the solution.

The length of the longest string in a string content of a given energy level is the Hall conductance of the corresponding band: $2l + 1 = |\sigma(k)|$. The period of this string $q_l = e^{i\pi\eta_l}$ is uniquely determined by the requirement that $\eta_l = \frac{\theta_{q_l^{2l+1}}}{2l+1}$ is the best approximant for the period $\eta_l$, so that $q_l^{2l+1} = \pm 1$.

The parity of the longest string is $v_l = -i q_l^{l+1/2} = (-1)^l |q_l|^{1/2}$. The center of the longest string is $x_l = 1 + O(1/l)$.

The remaining roots of the state is a solution of the BA equation for the parent state of the parent generation.

The string hypothesis states that

$$\Psi_{\text{daughter}}(z) \approx \prod_{m=-l}^{l} (z - x_l v_l q_l^m) \Psi_{\text{parent}}(z).$$

and that $2l + 1 = |\sigma(k)|$ is the absolute value of the Hall conductance of the "daughter" band.

This simple hypothesis allows one to construct a complete set of wave functions by virtue of the iterative procedure:

Starting from an irrational $\eta$, we first generate a hierarchical tree. Let us choose a branch of the tree $J$. Find the Hall conductance of the bands belong to the branch down to the origin. This determines the lengths, periods and parities of the string content and therefore zeros of the wave function of the chosen branch.

The only unknowns are the centers of strings. They, however approach 1 with accuracy $O(1/l)$. The accuracy of the recursive eq. (28) is $O(l^{-2})$. The string content of a state (i.e., lengths and parities of strings) is a topological characteristics, while the centers of strings are not.

The strings hierarchy has been obtained by analysis of singularities of BA (see [1]). As it was expected a set of possible lengths of strings is a set of Takahashi-Suzuki numbers, known in the Bethe-Ansatz literature. Eq. (28) provides a relation between them and Hall conductances.

To illustrate the iterative procedure let us consider the bottom edges of the lowest band of the spectrum and choose $\eta = \frac{\sqrt{5}-1}{2}$ to be the golden mean. The sequence of rational approximants is given by ratios of subsequent Fibonacci numbers $\eta_l = \frac{F_{l+1}}{F_l}$, where the $F_l$ are Fibonacci numbers ($F_1 = F_{-2} + F_{-1}$ and $F_0 = F_1 = 1$). The set of Hall conductances = Takahashi-Suzuki numbers = allowed lengths of strings are again Fibonacci numbers: $Q_k - 1 = F_{k-1}$. The considered branch of hierarchical tree connects edges $(\tau = -1, k = 1, \mu = 1)$ of the lowest bands of generations $\eta_{3k} = F_{3k-1}/F_{3k}$. Their string content consists of pairs of strings with length $2l_n + 1 = F_{3n+1}$, $n = 0, 1, \ldots, k - 1$, parities $+1$ and inverted centers $x_k$ and $x_k^{-1}$. According to the string hypothesis the wave function of this state is

$$\Psi(z|\eta_{3k}) \approx \prod_{n=0}^{k-1} \prod_{j=-l_n}^{l_n} (z - x_{l_n} q_{l_n}^j) (z - x_{l_n}^{-1} q_{l_n}^j).$$

Centers of the strings $x_{l_n}$ are close to 1 but can not be obtained from the string hypothesis alone.

5. Gaps. A direct application of the string hypothesis, suggested by J. Bellissard, is the calculation of the gap distribution $\rho(D)$, i.e., the number of gaps with magnitude between $D$ and $D + dD$. The result is $\rho(D) \sim D^{-3/2}$, which essentially means that the width of the smallest gaps scales as $D_{\text{min}} \sim 1/Q^2$. This result confirms numerical analysis of Ref. [25].

6. Scaling hypothesis and finite size corrections: stating the problem. The string hypothesis solves the Bethe Ansatz equations with an accuracy $O(Q^{-2})$, i.e., is asymptotically exact in the incommensurate limit $Q \to \infty$. It alone gives the explicit asymptotically exact form of wave functions and provides the hierarchical tree and topology of the Cantor set spectrum. However, the most interesting quantitative characteristics of the spectrum are hidden in the finite size corrections of the order of $Q^{-2}$ to the bare value of strings. Among them are the anomalous dimensions of the spectrum $\epsilon_{ij}$. They depend on the branch and on arithmetics of $\eta$ (according to ref. [29]). Exponents $\epsilon_{ij}$ vary between 0.171 and −0.374 for the golden mean $\eta = \frac{\sqrt{5}-1}{2}$. Can anomalous dimensions be found analytically, by finding finite size corrections to the string solutions? This is a technically involved but a fascinating and important problem. Its solution would provide an ultimate information of the spectrum and most interesting physical properties of the system. It also may suggest the conformal bootstrap and operator algebra approach, which has been proven to be effective for finding the finite size corrections of integrable systems, without the actual solving the Bethe Ansatz.

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[1] A.G. Abanov, J.C. Talstra, P.B. Wiegmann, Phys. Rev. Lett. 78, idem 4103 (1997), Nucl. Phys. B 525, 571 (1998).
[2] S. Aubry, G. Adnré, Ann. Israel Phys. Soc. 3, 133 (1980).
[3] Ya.G. Sinai, J. Stat. Phys. 46, 861-909 (1987).
[4] M.Ya. Azbel, Zh. Eksp. Teor. Fiz. 46, 929 (1964)
[5] D.R. Hofstadter, Phys. Rev. B 14, 2239 (1976).
[6] J. Bellissard, B. Simon, J. Funct. Anal. 48, 408-419 (1982)
[7] Y. Last, Proc. XI Intern. Congress of Math. Phys. (Paris 1994); S. Jitomirskaya, ibid.,
[8] D.J. Thouless, Phys. Rev. B 28, 4272-4276 (1983)
[9] J. Avron, P. van Mouche, B. Simon, Commun. Math. Phys. 132, 103-118 (1990)
[10] C. Tang, M. Kohmoto, Phys. Rev. B 34, 2041 (1986)
[11] P.B. Wiegmann and A.Zabrodin, Nucl. Phys. B 422, 495 (1994); idem Phys. Rev. Lett. 72, 1890 (1994);
[12] L.D. Faddeev, R.M. Kashaev, Commun. Math. Phys., 169, 181 (1995).
[13] N. Kutz, Phys.Lett. A187,365 (1994)
[14] Ch. Kreft, R. Seiler, Sfb 288 Preprint No.209
Models of Hofstadter type.
[15] E. G. Floratos and S.Nicolis, J. Phys. A:Math. Gen. 31 3961 (1998)
[16] I.M. Vinogradov, An introduction to the theory of numbers., Pergamon press (1961)
[17] P. Streda, J. Phys. C: Solid State Phys., 15, L717 (1982).
[18] D.J. Thouless, M. Kohmoto, M.P. Nightingale, M. den Nijs, Phys. Rev. Lett. 49, 405 (1982).
[19] I. Dana, Y. Avron, J. Zak, J. Phys. C 18, L679 (1985).
[20] M. Takahashi, M. Suzuki, Progr. Theor. Phys. 48, 2187 (1972).
[21] L. Mezincescu, R. I. Nepomechie, Phys. Lett., B246, 412 (1990).
[22] For details about rational points see [1, 2].
[23] Y. Hatsugai, M. Kohmoto, Y.S. Wu, Phys. Rev. Lett. 73, 1134 (1994); idem Phys. Rev. B 53, 9697 (1996).
[24] I. V. Krasovsky, to appear in Phys. Rev. B.
[25] T. Geisel, R. Ketzmerick, G. Petschel, Phys. Rev. Lett. 66, 1651 (1991).
[26] H. Hiramoto, M. Kohmoto, Phys. Rev. B 40, 8225 (1989).