Rate-Optimal Denoising with Deep Neural Networks

Reinhard Heckel*  Wen Huang*  Paul Hand**  Vladislav Voroninski†

Department of Electrical and Computer Engineering*, Rice University
School of Mathematical Sciences*, Xiamen University, Xiamen, Fujian, P.R.China
Department of Mathematics and College of Computer and Information Science**, Northeastern University
Helm.ai†

April 9, 2019

Abstract

Deep neural networks provide state-of-the-art performance for image denoising, where the goal is to recover a near noise-free image from a noisy observation. The underlying principle is that neural networks trained on large datasets have empirically been shown to be able to generate natural images well from a low-dimensional latent representation of the image. Given such a generator network, a noisy image can be denoised by i) finding the closest image in the range of the generator or by ii) passing it through an encoder-generator architecture (known as an autoencoder). However, there is little theory to justify this success, let alone to predict the denoising performance as a function of the network parameters. In this paper we consider the problem of denoising an image from additive Gaussian noise using the two generator based approaches. In both cases, we assume the image is well described by a deep neural network with ReLU activations functions, mapping a $k$-dimensional code to an $n$-dimensional image. In the case of the autoencoder, we show that the feedforward network reduces noise energy by a factor of $O(k/n)$. In the case of optimizing over the range of a generative model, we state and analyze a simple gradient algorithm that minimizes a non-convex loss function, and provably reduces noise energy by a factor of $O(k/n)$. We also demonstrate in numerical experiments that this denoising performance is, indeed, achieved by generative priors learned from data.

1 Introduction

We consider the denoising problem, where the goal is to remove noise from an unknown image or signal. In more detail, our goal is to obtain an estimate of an image or signal $y_\ast \in \mathbb{R}^n$ from a noisy measurement

$$y = y_\ast + \eta.$$ 

Here, $\eta$ is unknown noise, which we model as a zero-mean white Gaussian random variable with covariance matrix $\sigma^2/nI$. Image denoising relies on generative or prior assumptions on the image $y_\ast$, such as self-similarity within images [Dab+07], sparsity in fixed [Don95] and learned bases [EA06], and most recently, by assuming the image can be generated by a pre-trained deep-neural network [Bur+12; Zha+17]. Deep-network based approaches, typically yield the best denoising performance. This success can be attributed to their ability to efficiently represent and learn realistic image priors, for example via auto-decoders [HS06] and generative adversarial models [Goo+14].

Motivated by this success story, we assume that the image $y_\ast$ lies in the range of an image-generating network. In this paper, we propose the first algorithm for solving denoising with deep generative priors that provably finds an approximation of the underlying image. As the influence of deep networks in denoising and inverse problems grows, it becomes increasingly important to understand their performance at a theoretical level. Given that most optimization approaches for deep learning are first-order gradient methods, a justification is needed for why they do not get stuck in local minima.
The most related work that establishes theoretical reasons for why gradient methods might succeed when using deep generative priors for solving inverse problems, is [HV18]. In it, the authors establish global favorability for optimization of a $\ell_2$-loss function under a random neural network model. Specifically, they show existence of a descent direction outside a ball around the global optimizer and a negative multiple of it in the latent space of the generative model. This work does not justify why the one spurious point is avoided by gradient descent, nor does it provide a specific algorithm which provably estimates the global minimizer, nor does it provide an analysis of the robustness of the problem with respect to noise. This work was subsequently extended to include the case of generative convolutional neural networks by [Ma+18], but that work too does not prove convergence of a specific algorithm.

Contributions: The goal of this paper is to analytically quantify the denoising performance of deep-prior based denoisers. Specifically, we characterize the performance of two simple and efficient algorithms for denoising based on a $d$-layer generative neural network $G: \mathbb{R}^k \to \mathbb{R}^n$, with $k < n$.

We first provide a simple result for an encoder-generator network $G(E(y))$ where $E: \mathbb{R}^n \to \mathbb{R}^k$ is an encoder network. We show that if we pass noise through an encoder-decoder network $G(E(y))$ that acts as the identity on a class of images of interest, then it reduces the random noise by $O(k/n)$. This result requires no assumptions on the weights of the network.

The second and main result of our paper pertains to denoising by optimizing over the latent code of a generator network with random weights. We propose a gradient method that attempts to minimize the least-squares loss $f(x) = \frac{1}{2} \| G(x) - y \|^2$ between the noisy image $y$ and an image in the range of the generator, $G(x)$. Even though $f$ is non-convex, we show that a gradient method yields an estimate $\hat{x}$ obeying

$$\| G(\hat{x}) - y_s \|^2 \leq \sigma^2 \frac{k}{n},$$

with high probability, where the notation $\leq$ absorbs a constant factor depending on the number of layers of the network, and its expansitivity, as discussed in more detail later. Our result shows that the denoising rate of a deep generator based denoiser is optimal in terms of the dimension of the latent representation. We also show in numerical experiments, that this rate—shown to be analytically achieved for random priors—is also experimentally achieved for priors learned from real imaging data.

Related work: We hasten to add that a close theoretical work to the question considered in this paper is the paper [Bor+17], which solves a noisy compressive sensing problem with generative priors by minimizing an $\ell_2$-loss. Under the assumption that the network is Lipschitz, they show that if the global optimizer can be found, which is in principle NP-hard, then a signal estimate is recovered to within the noise level. While the Lipschitzness assumption is quite mild, the resulting theory does not provide justification for why global optimality can be reached.

2 Background on denoising with classical and deep-learning based methods

As mentioned before, image denoising relies on modeling or prior assumptions on the image $y_s$. For example, suppose that the image $y_s$ lies in a $k$-dimensional subspace of $\mathbb{R}^n$ denoted by $\mathcal{Y}$. Then we can estimate the original image by finding the closest point in $\ell_2$-distance to the noisy observation
y on the subspace \( \mathcal{Y} \). The corresponding estimate, denoted by \( \hat{y} \), obeys

\[
\|\hat{y} - y_*\|^2 \lesssim \sigma^2 k/n,
\]

with high probability (throughout, \( \|\cdot\| \) denotes the \( \ell_2 \)-norm). Thus, the noise energy is reduced by a factor of \( k/n \) over the trivial estimate \( \hat{y} = y \) which does not use any prior knowledge of the signal. The denoising rate (1) shows that the more concise the image prior or image representation (i.e., the smaller \( k \)), the more noise can be removed. If on the other hand the image model (the subspace, in this example) does not include the original image \( y_* \), then the error bound (1) increases, as we would remove a significant part of the signal along with noise when projecting onto the range of the image prior. Thus a concise and accurate model is crucial for denoising.

Real world signals rarely lie in \textit{a priori} known subspaces, and the last few decades of image denoising research have developed sophisticated algorithms based on accurate image models. Examples include algorithms based on sparse representations in overcomplete dictionaries such as wavelets [Don95] and curvelets [Sta+02], and algorithms based on exploiting self-similarity within images [Dab+07]. A prominent example of the former class of algorithms is the (state-of-the-art) BM3D [Dab+07] algorithm. However, the nuances of real world images are difficult to describe with handcrafted models. Thus, starting with the paper [EA06] that proposes to learn sparse representation based on training data, it has become common to learn concise representation for denoising (and other inverse problems) from a set of training images.

Burger et al. [Bur+12] applied deep networks to the denoising problem, by training a plain neural network on a large set of images. Since then, deep learning based denoisers [Zha+17] have set the standard for denoising. The success of deep network priors can be attributed to their ability to efficiently represent and learn realistic image priors, for example via auto-decoders [HS06] and generative adversarial models [Goo+14]. Over the last few years, the quality of deep priors has significantly improved [Kar+17; Uly+18; HH19]. As this field matures, priors will be developed with even smaller latent code dimensionality and more accurate approximation of natural signal manifolds. Consequently, the representation error from deep priors will decrease, and thereby enable even more powerful denoisers.

3 Denoising with a neural network with an hourglass structure

Perhaps the most straight-forward and classical approach to using deep networks for denoising is to train a deep network with an autoencoder or hourglass structure end-to-end to perform denoising. An autoencoder compresses data from the input layer into a low-dimensional code, and then generates an output from that code. In this section, we analyze such networks from the perspective of denoising.

Specifically, we show mathematically that a simple model for neural networks with an hourglass structure achieves optimal denoising rates, as given by the dimensionality of the low-dimensional code. An autoencoder \( H(x) = G(E(x)) \) consists of an encoder network \( E : \mathbb{R}^n \to \mathbb{R}^k \) mapping an image to a low-dimensional latent representation, and a decoder or generator network \( G : \mathbb{R}^k \to \mathbb{R}^n \) mapping the latent representation to an image. To see that the size of the low-dimensional code, \( k \), determines the denoising rate, consider a simple one-layer encoder and multilayer decoder of the form

\[
E(y) = \text{relu}(W'y), \quad G(x) = \text{relu}(W_d \ldots \text{relu}(W_2 \text{relu}(W_1 x)) \ldots),
\]

where \( \text{relu}(x) = \max(x, 0) \) applies entrywise, \( W' \) are the weights of the encoder, \( W_i \in \mathbb{R}^{n_i \times n_{i-1}} \) are the weights in the \( i \)-th layer of the decoder, and \( n_i \) is the number of neurons in the \( i \)-th layer.
Typically, the autoencoder network \( H \) is trained such that \( H(y) \approx y \) for some class of signals of interest (say, natural images). The following proposition shows that the structure of the network alone guarantees that an autoencoder “filters out” most of the noise.

**Proposition 1.** Let \( H = G \circ E \) be an autoencoder of the form (2) and note that it is piecewise linear, i.e., \( H(y) = U y \) in a region around \( y \). Suppose that \( \|U\|^2 \leq 2 \) for all such regions. Let \( \eta \) be Gaussian noise with covariance matrix \( \sigma I \), \( \sigma > 0 \). Then, provided that \( k \cdot 32 \log(2n_1 n_2 \ldots n_d) \leq n \), we have with probability at least \( 1 - 2e^{-k \log(2n_1 n_2 \ldots n_d)} \), that

\[
\|H(\eta)\|_2^2 \leq 5 \frac{k}{n} \log(2n_1 n_2 \ldots n_d) \|\eta\|_2^2.
\]

Note that the assumption \( \|U\|^2 \leq 2 \) implies that \( \|H(y)\|_2^2/\|y\|_2^2 \leq 2 \) for all \( y \). This in turn guarantees that the autoencoder does not “amplify” a signal too much. Note that we envision an autoencoder that is trained such that it obeys \( H(y) \approx y \) for \( y \) in a class of images. The proposition would then justify why the feedforward network reduces noise by \( O(k/n) \).

The proof of Proposition 1, contained in the appendix, shows that \( H \) lies in the range of a union of \( k \)-dimensional subspaces and then uses a standard concentration argument showing that the union of those subspaces can represent no more than a fraction of \( O(k/n) \) of the noise.

In the remainder of the paper we shows that denoising via enforcing a generative prior gives us an analogous denoising rate.

## 4 Denoising via enforcing a generative model

We consider the problem of estimating a vector \( y_* \in \mathbb{R}^n \) from a noisy observation \( y = y_* + \eta \). We assume that the vector \( y_* \) belongs to the range of a \( d \)-layer generative neural network \( G: \mathbb{R}^k \to \mathbb{R}^n \), with \( k < n \). That is, \( y_* = G(x_*) \) for some \( x_* \in \mathbb{R}^k \). We consider a generative network of the form

\[
G(x) = \text{relu}(W_d \ldots \text{relu}(W_2 \text{relu}(W_1 x_*) \ldots)),
\]

where \( \text{relu}(x) = \max(x, 0) \) applies entrywise, \( W_i \in \mathbb{R}^{n_i \times n_{i-1}} \), are the weights in the \( i \)-th layer, \( n_i \) is the number of neurons in the \( i \)-th layer, and the network is expansive in the sense that \( k = n_0 < n_1 < \cdots < n_d = n \). The problem at hand is: Given the weights of the network \( W_1 \ldots W_d \) and a noisy observation \( y \), obtain an estimate \( \hat{y} \) of the original image \( y_* \) such that \( \|\hat{y} - y_*\| \) is small and \( \hat{y} \) is in the range of \( G \).

### 4.1 Enforcing a generative model

As a way to solve the above problem, we first obtain an estimate of \( x_* \), denoted by \( \hat{x} \), and then estimate \( y_* \) as \( G(\hat{x}) \). In order to estimate \( x_* \), we minimize the loss

\[
f(x) := \frac{1}{2} \|G(x) - y\|^2.
\]

Since this objective is nonconvex, there is no a priori guarantee of efficiently finding the global minimum. Approaches such as gradient methods could in principle get stuck in local minima, instead of finding a global minimizer that is close to \( x_* \).

However, as we show in this paper, under appropriate conditions, a gradient method —introduced next—finds a point that is very close to the original latent parameter \( x_* \), with the distance to the
parameter $x_\ast$ controlled by the noise. In order to state the algorithm, we first introduce a useful quantity. For analyzing which rows of a matrix $W$ are active when computing $\text{relu}(Wx)$, we let

$$W_{+,x} = \text{diag}(Wx > 0)W.$$  

For a fixed weight matrix $W$, the matrix $W_{+,x}$ zeros out the rows of $W$ that do not have a positive dot product with $x$. Alternatively put, $W_{+,x}$ contains weights from only the neurons that are active for the input $x$. We also define $W_{1,+} = (W_1)_{+,x} = \text{diag}(W_1x > 0)W_1$ and

$$W_{i,+} = \text{diag}(W_iW_{i-1,+} \cdots W_{2,+}W_{1,+}x > 0)W_i.$$  

The matrix $W_{i,+}$ consists only of the weights of the neurons in the $i$th layer that are active if the input to the first layer is $x$.

We are now ready to state our algorithm: a gradient method with a tweak informed by the loss surface of the function to be minimized. Given a noisy observation $y$, the algorithm starts with an arbitrary initial point $x_0 \neq 0$. At each iteration $i = 0, 1, \ldots$, the algorithm computes the step direction

$$\tilde{v}_{x_i} = (\Pi_{i=d}W_{i,+}x_i)^T(G(x_i) - y),$$

which is equal to the gradient of $f$ if $f$ is differentiable at $x_i$. It then takes a small step opposite to $\tilde{v}_{x_i}$. The tweak is that before each iteration, the algorithm checks whether $f(-x_i)$ is smaller than $f(x_i)$, and if so, negates the sign of the current iterate $x_i$.

This tweak is informed by the loss surface. To understand this step, it is instructive to examine the loss surface for the noiseless case in Figure 1. It can be seen that while the loss function has a global minimum at $x_\ast$, it is relatively flat close to $-x_\ast$. In expectation, there is a critical point that is a negative multiple of $x_\ast$ with the property that the curvature in the $\pm x_\ast$ direction is positive, and the curvature in the orthogonal directions is zero. Further, around approximately $-x_\ast$, the loss function is larger than around the optimum $x_\ast$. As a simple gradient descent method (without the tweak) could potentially get stuck in this region, the negation check provides a way to avoid converging to this region. Our algorithm is formally summarized as Algorithm 1 below.
Algorithm 1 Gradient method

Require: Weights of the network $W_i$, noisy observation $y$, and step size $\alpha > 0$

1: Choose an arbitrary initial point $x_0 \in \mathbb{R}^k \setminus \{0\}$
2: for $i = 0, 1, \ldots$ do
3: \hspace{1em} if $f(-x_i) < f(x_i)$ then
4: \hspace{2em} $\tilde{x}_i \leftarrow -x_i$
5: \hspace{1em} else
6: \hspace{2em} $\tilde{x}_i \leftarrow x_i$
7: \hspace{1em} end if
8: \hspace{1em} Compute $\nu_{\tilde{x}_i} \in \nabla f(\tilde{x}_i)$, in particular, if $G$ is differentiable at $\tilde{x}_i$, then set $\nu_{\tilde{x}_i} = \tilde{v}_{\tilde{x}_i}$, where
9: \hspace{2em} $\tilde{v}_{\tilde{x}_i} := (\prod_{i=0}^{l} W_{i+1, \tilde{x}_i})^t (G(\tilde{x}_i) - y)$;
10: $x_{i+1} \leftarrow \tilde{x}_i - \alpha \nu_{\tilde{x}_i}$;
end for

Other variations of the tweak are also possible. For example, the negation check in Step 3 could be performed after a convergence criterion is satisfied, and if a lower objective is achieved by negating the latent code, then the gradient descent can be continued again until a convergence criterion is again satisfied.

5 Main results

For our analysis, we consider a fully-connected generative network $G: \mathbb{R}^k \to \mathbb{R}^n$ with Gaussian weights and no bias terms. Specifically, we assume that the weights $W_i$ are independently and identically distributed as $\mathcal{N}(0, 2/n_i)$, but do not require them to be independent across layers. Moreover, we assume that the network is sufficiently expansive:

Expansivity condition. We say that the expansivity condition with constant $\epsilon > 0$ holds if

$$n_i \geq ce^{-2} \log(1/\epsilon) n_{i-1} \log n_{i-1}, \quad \text{for all } i,$$

where $c$ is a particular numerical constant.

In a real-world generative network the weights are learned from training data, and are not drawn from a Gaussian distribution. Nonetheless, the motivation for selecting Gaussian weights for our analysis is as follows:

1. The empirical distribution of weights from deep neural networks often have statistics consistent with Gaussians. AlexNet is a concrete example [Aro+15].

2. The field of theoretical analysis of recovery guarantees for deep learning is nascent, and Gaussian networks can permit theoretical results because of well-developed theories for random matrices.

3. It is not clear which non-Gaussian distribution for weights is superior from the joint perspective of realism and analytical tractability.
The network model we consider is fully connected. We anticipate that the analysis of this paper can be extended to the case of generative convolutional neural networks. This extension has already happened for theoretical results concerning the favorability of the optimization landscape for compressive sensing under generative priors [Ma+18], as mentioned previously.

We are now ready to state our main result.

**Theorem 2.** Consider a network with the weights in the $i$-th layer, $W_i \in \mathbb{R}^{n_i \times n_{i-1}}$, i.i.d. $\mathcal{N}(0, 2/n_i)$ distributed, and suppose that the network satisfies the expansivity condition for $\epsilon = K/d^{80}$. Also, suppose that the noise variance $\omega$, defined for notational convenience as

$$
\omega := \sqrt{\frac{18\sigma^2 k}{n} \log(n_1^d n_2^{d-1} \ldots n_d)},
$$

obeys

$$
\omega \leq \|x_*\| \frac{K_1}{d^{16}}.
$$

Consider the iterates of Algorithm 1 with stepsize $\alpha = K_4 \frac{1}{d^{72}}$. Then, there exists a number of steps $N$ upper bounded by

$$
N \leq K_2 d^{80} \frac{f(x_0)}{\|x_*\|}
$$

such that after $N$ steps, the iterates of Algorithm 1 obey

$$
\|x_i - x_*\| \leq \frac{K_5}{d^{96}} \|x_*\| + K_6 d^6 \omega, \quad \text{for all } i \geq N,
$$

with probability at least $1 - 2e^{-2km_0 n} - \sum_{i=2}^d 8n_i e^{-K_7 n_{i-2}} - 8n_1 e^{-K_7 \log d/d^{480}}$. In addition, for all $i > N$, we have

$$
\|x_i - x_*\| \leq (1 - \alpha/7)^{i-N}\|x_N - x_*\| + K_8 \omega \quad \text{and} \quad \|G(x_i) - G(x_*)\| \leq 1.2(1 - \alpha/7)^{i-N}\|x_N - x_*\| + 1.2K_8 \omega,
$$

where $\alpha < 1$ is the stepsize of the algorithm. Here, $K_1, K_2, \ldots$ are numerical constants, and $x_0$ is the initial point in the optimization.

Our result guarantees that after polynomially many iterations (with respect to $d$), the algorithm converges linearly to a region satisfying

$$
\|x_i - x_*\|^2 \lesssim \sigma^2 \frac{k}{n},
$$

where the notation $\lesssim$ absorbs a factor logarithmic in $n$ and polynomial in $d$. One can show that $G$ is Lipschitz in a region around $x_*^1$,

$$
\|G(x_i) - G(x_*)\|^2 \lesssim \sigma^2 \frac{k}{n}.
$$

Thus, the theorem guarantees that our algorithm yields the denoising rate of $\sigma^2 k/n$, and, as a consequence, denoising based on a generative deep prior provably reduces the energy of the noise in the original image by a factor of $k/n$.

In the case of $\sigma = 0$, the theorem guarantees convergence to the global minimizer $x_*$. We note that the intention of this paper is to show rate-optimality of recovery with respect to the noise

---

1The proof of Lipschitzness follows from applying the Weight Distribution Condition in Section 5.1.
power, the latent code dimensionality, and the signal dimensionality. As a result, no attempt was made to establish optimal bounds with respect to the scaling of constants or to powers of \( d \). The bounds provided in the theorem are highly conservative in the constants and dependency on the number of layers, \( d \), in order to keep the proof as simple as possible. Numerical experiments shown later reveal that the parameter range for successful denoising are much broader than the constants suggest. As this result is the first of its kind for rigorous analysis of denoising performance by deep generative networks, we anticipate the results can be improved in future research, as has happened for other problems, such as sparsity-based compressed sensing and phase retrieval.

Finally, we remark that Theorem 2 can be generalized to the case where \( y_s \) only approximately lies in the range of the generator, i.e., \( G(x_s) \approx y_s \). Specifically, if \( \|G(x_s) - y_s\| \) is sufficiently small, the error induced by this perturbation is proportional to \( \|G(x_s) - y_s\| \).

### 5.1 The Weight Distribution Condition (WDC)

To prove our main result, we make use of a deterministic condition on \( G \), called the Weight Distribution Condition (WDC), and then show that Gaussian \( W_i \), as given by the statement of Theorem 2 are such that \( W_i/\sqrt{2} \) satisfies the WDC with the appropriate probability for all \( i \), provided the expansivity condition holds. Our main result, Theorem 2, continues to hold for any weight matrices such that \( W_i/\sqrt{2} \) satisfy the WDC.

The condition is on the spatial arrangement of the network weights within each layer. We say that the matrix \( W \in \mathbb{R}^{n \times k} \) satisfies the **Weight Distribution Condition** with constant \( \epsilon \) if for all nonzero \( x, y \in \mathbb{R}^k \),

\[
\sum_{i=1}^{n} 1_{(w_i,x) > 0} 1_{(w_i,y) > 0} \cdot w_i w_i^T - Q_{x,y} \leq \epsilon, \quad \text{with} \quad Q_{x,y} = \frac{\pi - \theta_0}{2\pi} I_k + \frac{\sin \theta_0}{2\pi} M_{\hat{x} \rightarrow \hat{y}},
\]

where \( w_i \in \mathbb{R}^k \) is the \( i \)th row of \( W \); \( M_{\hat{x} \rightarrow \hat{y}} \in \mathbb{R}^{k \times k} \) is the matrix\(^2\) such that \( \hat{x} \mapsto \hat{y} \), \( \hat{y} \mapsto \hat{x} \), and \( z \mapsto 0 \) for all \( z \in \text{span}\{x, y\} \); \( \hat{x} = x/\|x\|_2 \) and \( \hat{y} = y/\|y\|_2 \); \( \theta_0 = \angle(x, y) \); and \( 1_S \) is the indicator function on \( S \). The norm in the left hand side of (6) is the spectral norm. Note that an elementary calculation\(^3\) gives that \( Q_{x,y} = \mathbb{E}[\sum_{i=1}^{n} 1_{(w_i,x) > 0} 1_{(w_i,y) > 0} \cdot w_i w_i^T] \) for \( w_i \sim \mathcal{N}(0, I_k/n) \). As the rows \( w_i \) correspond to the neural network weights of the \( i \)th neuron in a layer given by \( W \), the WDC provides a deterministic property under which the set of neuron weights within the layer given by \( W \) are distributed approximately like a Gaussian. The WDC could also be interpreted as a deterministic property under which the neuron weights are distributed approximately like a high dimensional vector chosen uniformly from a sphere of a particular radius. Note that if \( x = y \), then \( Q_{x,y} \) is an isometry up to a factor of 1/2.

### 5.2 Sketch of proof of Theorem 2

The proof relies on a characterization of the loss surface. We show that outside of two balls around \( x = x_s \) and \( x = -\rho_d x_s \), with \( \rho_d \) a constant defined in the proof, the direction chosen by the algorithm is a descent direction, with high probability.

\(^2\)A formula for \( M_{\hat{x} \rightarrow \hat{y}} \) is as follows. If \( \theta_0 = \angle(\hat{x}, \hat{y}) \in (0, \pi) \) and \( R \) is a rotation matrix such that \( \hat{x} \) and \( \hat{y} \) map to \( e_1 \) and \( \cos \theta_0 \cdot e_1 + \sin \theta_0 \cdot e_2 \) respectively, then \( M_{\hat{x} \rightarrow \hat{y}} = R^T \begin{pmatrix} \cos \theta_0 & \sin \theta_0 \\ \sin \theta_0 & -\cos \theta_0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} R \), where \( 0_{k-2} \) is a \( k-2 \times k-2 \) matrix of zeros. If \( \theta_0 = 0 \) or \( \pi \), then \( M_{\hat{x} \rightarrow \hat{y}} = \hat{x} \hat{x}^T \) or \( -\hat{x} \hat{x}^T \), respectively.

\(^3\)To do this calculation, take \( x = e_1 \) and \( y = \cos \theta_0 \cdot e_1 + \sin \theta_0 \cdot e_2 \) without loss of generality. Then each entry of the matrix can be determined analytically by an integral that factors in polar coordinates.
We show that the stepdirection $\tilde{v}_x$ concentrates around a particular $h_x \in \mathbb{R}^k$, that is a continuous function of nonzero $x, x_\star$ and is zero only at $x = x_\star, x = -\rho_d x_\star$, and 0, using a concentration argument similar to [HV18]. Around $x = x_\star$, the loss function has a global minimum, close to 0 it has a saddle point, and close to $x = -\rho_d x_\star$ potentially a local minimum. In a nutshell, we show that i) the algorithm moves away from the saddle point at 0, and ii) the algorithm escapes the local minimum close to $x = -\rho_d x_\star$ with the twist in Steps 3-5 of the algorithm. Finally, the iterates end up close to the $x_\star$.

The proof is organized as described next and as illustrated in Figure 2. The algorithm is initialized at an arbitrary point; for example close to 0. Algorithm 1 moves away from 0, at least till its iterates are outside the gray ring, as 0 is a local maximum; and once an iterate $x_i$ leaves the gray ring around 0, all subsequent iterates will never be in the white circle around 0 again (see Lemma 11 in the supplement). Then the algorithm might move towards $-\rho_d x_\star$, but once it enters the dashed ball around $-\rho_d x_\star$, it enters a region where the function value is strictly larger than that of the dashed ball around $x_\star$, by Lemma 13 in the supplement. Thus steps 3-5 of the algorithm will ensure that the next iterate $x_i$ is in the dashed ball around $x_\star$. From there, the iterates will move into the region $S^+_\beta$, since outside of $S^+_\beta \cup S^-_\beta$ the algorithm chooses a descent direction in each step (see the argument around equation (22) in the supplement). The region $S^+_\beta$ is covered by a ball of radius $r$, by Lemma 12 in the supplement, determined by the noise and $\epsilon$. This establishes the bound (3) in the theorem.

The proof proceeds be showing that within a ball around $x_\star$, the algorithm then converges linearly, which establishes equations (4) and (5).

6 Applications to Compressed Sensing

In this section we briefly discuss another important scenario to which our results apply to, namely regularizing inverse problems using deep generative priors. Approaches that regularize inverse problems using deep generative models [Bor+17] have empirically been shown to improve over sparsity-based approaches, see [Luc+18] for a review for applications in imaging, and [Mar+17] for an application in Magnetic Resonance Imaging showing a significant performance improvement over conventional methods.

Consider an inverse problem, where the goal is to reconstruct an unknown vector $y_\star \in \mathbb{R}^n$ from $m < n$ noisy linear measurements:

$$z = Ay_\star + \eta \in \mathbb{R}^m,$$

where $A \in \mathbb{R}^{m \times n}$ is called the measurement matrix and $\eta$ is zero mean Gaussian noise with covariance matrix $\sigma^2/nI$, as before. As before, assume that $y_\star$ lies in the range of a genera-
tive prior $G$, i.e., $y_* = G(x_*)$ for some $x_*$. As a way to recover $x_*$, consider minimizing the empirical risk objective $f(x) = \frac{1}{2} \| AG(x) - z \|$, using Algorithm 1, with Step 6 substituted by $	ilde{v}_{xi} = (A\Pi_{i=\tilde{d}}^{1}W_{i,+}x_{i})^t(AG(x_i) - y_i)$, to account for the fact that measurements were taken with the matrix $A$.

Suppose that $A$ is a random projection matrix, for concreteness assume that $A$ has i.i.d. Gaussian entries with variance $1/m$. One could prove an analogous result as Theorem 2, but with $\omega = \sqrt{18\sigma^2 k \log(n_1^n d_{i=\tilde{d}} - 1 \ldots n_d)}$, (note that $n$ has been replaced by $m$). This extension shows that, provided $\epsilon$ is chosen sufficiently small, that our algorithm yields an iterate $x_i$ obeying

$$\| G(x_i) - G(x_*) \|^2 \leq \sigma^2 \frac{k}{m},$$

where again $\leq$ absorbs factors logarithmic in the $n_i$’s, and polynomial in $d$. Proving this result would be analogous to the proof of Theorem 2, but with the additional assumption that the sensing matrix $A$ acts like an isometry on the union of the ranges of $\Pi_{i=\tilde{d}}^{1}W_{i,+}x_{i}$, analogous to the proof in [HV18]. This extension of our result shows that Algorithm 1 enables solving inverse problems under noise efficiently, and quantifies the effect of the noise.

We hasten to add that the paper [Bor+17] also derived an error bound for minimizing empirical loss. However, the corresponding result (for example Lemma 4.3) differs in two important aspects to our result. First, the result in [Bor+17] only makes a statement about the minimizer of the empirical loss and does not provide justification that an algorithm can efficiently find a point near the global minimizer. As the program is non-convex, and as non-convex optimization is NP-hard in general, the empirical loss could have local minima at which algorithms get stuck. In contrast, the present paper presents a specific practical algorithm and proves that it finds a solution near the global optimizer regardless of initialization. Second, the result in [Bor+17] considers arbitrary noise $\eta$ and thus can not assert denoising performance. In contrast, we consider a random model for the noise, and show the denoising behavior that the resulting error is no more than $O(k/n)$, as opposed to $\| \eta \|^2 \approx O(1)$, which is what we would get from direct application of the result in [Bor+17].

### 7 Experimental results

In this section we provide experimental evidence that corroborates our theoretical claims that denoising with deep priors achieves a denoising rate proportional to $\sigma^2 k/n$. We focus on denoising by enforcing a generative prior, and consider both a synthetic, random prior, as studied theoretically in the paper, as well as a prior learned from data. All our results are reproducible with the code provided in the supplement.

#### 7.1 Denoising with a synthetic prior

We start with a synthetic generative network prior with ReLU-activation functions, and draw its weights independently from a Gaussian distribution. We consider a two-layer network with $n = 1500$ neurons in the output layer, 500 in the middle layer, and vary the number of input neurons, $k$, and the noise level, $\sigma$. We next present simulations showing that if $k$ is sufficiently small, our algorithm achieves a denoising rate proportional to $\sigma k/n$ as guaranteed by our theory.

Towards this goal, we generate Gaussian inputs $x_*$ to the network and observe the noisy image $y = G(x_*) + \eta$, $\eta \sim \mathcal{N}(0, \sigma^2/nI)$. From the noisy image, we first obtain an estimate $\hat{x}$ of the latent representation by running Algorithm 1 until convergence, and second we obtain an estimate of the image as $\hat{y} = G(\hat{x})$. In the left and middle panel of Figure 4, we depict the normalized mean squared error of the latent representation, $\text{MSE}(\hat{x}, x_*)$, and the mean squared error in the
image domain, $\text{MSE}(G(\hat{x}), G(x_\ast))$, where we defined $\text{MSE}(z, z') = \|z - z'\|^2$. For the left panel, we fix the noise variance to $\sigma^2 = 0.25$, and vary $k$, and for the middle panel we fix $k = 50$ and vary the noise variance. The results show that, if the network is sufficiently expansive, guaranteed by $k$ being sufficiently small, then in the noiseless case ($\sigma^2 = 0$), the latent representation and image are perfectly recovered. In the noisy case, we achieve a MSE proportional to $\sigma^2 k/n$, both in the representation and image domains.

We also observed that for the problem instances considered here, the negation trick in step 3-4 of Algorithm 1 is often not necessary, in that even without that step the algorithm typically converges to the global minimum. Having said this, in general the negation step is necessary, since there exist problem instances that have a local minimum opposite of $x_\ast$.

### 7.2 Denoising with a learned prior

We next consider a prior learned from data. Technically, for such a prior our theory does not apply since we assume the weights to be chosen at random. However, the numerical results presented in this section show that even for the learned prior we achieve the rate predicted by our theory pertaining to a random prior. Towards this goal, we consider a fully-connected autoencoder parameterized by $k$, consisting of a decoder and encoder with ReLU activation functions and fully connected layers. We choose the number of neurons in the three layers of the encoder as 784, 400, $k$, and those of the decoder as $k$, 400, 784. We set $k = 10$ and $k = 20$ to obtain two different autoencoders. We train both autoencoders on the MNIST [Lec+98] training set.

We then take an image $y_\ast$ from the MNIST test set, add Gaussian noise to it, and denoise it using our method based on the learned decoder-network $G$ for $k = 10$ and $k = 20$. Specifically, we estimate the latent representation $\hat{x}$ by running Algorithm 1, and then set $\hat{y} = G(\hat{x})$. See Figure 3 for a few examples demonstrating the performance of our approach for different noise levels.

We next show that this achieves a mean squared error (MSE) proportional to $\sigma^2 k/n$, as suggested by our theory which applies for decoders with random weights. We add noise to the images with noise variance ranging from $\sigma^2 = 0$ to $\sigma^2 = 6$. In the right panel of Figure 4 we show the MSE in the image domain, $\text{MSE}(G(\hat{x}), G(x_\ast))$, averaged over a number of images for the learned decoders with $k = 10$ and $k = 20$. We observe an interesting tradeoff: The decoder with $k = 10$ has fewer parameters, and thus does not represent the digits as well, therefore the MSE is larger than that for $k = 20$ for the noiseless case (i.e., for $\sigma = 0$). On the other hand, the smaller number of parameters results in a better denoising rate (by about a factor of two), corresponding to the steeper slope of the MSE as a function of the noise variance, $\sigma^2$.

### Acknowledgements

RH is partially supported by a NSF Grant ISS-1816986 and PH is partially supported by a NSF CAREER Grant DMS-1848087 as well as NSF Grant DMS-1464525, and the authors would like to
Figure 4: Mean square error in the image domain, \( \text{MSE}(G(\hat{x}), x_*) \), and in the latent representation, \( \text{MSE}(\hat{x}, x_*) \), as a function of the dimension of the latent representation, \( k \), with \( \sigma^2 = 0.25 \) (left panel), and the noise variance, \( \sigma^2 \) with \( k = 50 \) (middle panel). As suggested by the theory pertaining to decoders with random weights, if \( k \) is sufficiently small, and thus the network is sufficiently expansive, the denoising rate is proportional to \( \sigma^2 k/n \). Right panel: Denoising of handwritten digits based on a learned decoder with \( k = 10 \) and \( k = 20 \), along with the least-squares fit as dotted lines. The learned decoder with \( k = 20 \) has more parameters and thus represents the images with a smaller error; therefore the MSE at \( \sigma = 0 \) is smaller. However, the denoising rate for the decoder with \( k = 20 \), which is the slope of the curve is larger as well, as suggested by our theory.

thank Tan Nguyen for helpful discussions.

References

[Aro+15] S. Arora, Y. Liang, and T. Ma. “Why are deep nets reversible: A simple theory, with implications for training”. In: arXiv:1511.05653 (2015).

[Bor+17] A. Bora, A. Jalal, E. Price, and A. G. Dimakis. “Compressed sensing using generative models”. In: arXiv:1703.03208 (2017).

[Bur+12] H. C. Burger, C. J. Schuler, and S. Harmeling. “Image denoising: Can plain neural networks compete with BM3D?” In: 2012 IEEE Conference on Computer Vision and Pattern Recognition. 2012, pp. 2392–2399.

[Cla17] C. Clason. “Nonsmooth Analysis and Optimization”. In: arXiv:1708.04180 (2017).

[Dab+07] K. Dabov, A. Foi, V. Katkovnik, and K. Egiazarian. “Image denoising by sparse 3-D transform-domain collaborative filtering”. In: IEEE Transactions on Image Processing 16.8 (2007), pp. 2080–2095.

[Don95] D. L. Donoho. “De-noising by soft-thresholding”. In: IEEE Transactions on Information Theory 41.3 (1995), pp. 613–627.

[EA06] M. Elad and M. Aharon. “Image denoising via sparse and redundant representations over learned dictionaries”. In: IEEE Transactions on Image Processing 15.12 (2006), pp. 3736–3745.

[Goo+14] I. Goodfellow, J. Pouget-Abadie, M. Mirza, B. Xu, D. Warde-Farley, S. Ozair, A Courville, and Y. Bengio. “Generative adversarial nets”. In: Advances in Neural Information Processing Systems 27. 2014, pp. 2672–2680.
A Proof of Proposition 1

We first show that $H(y)$ lies in the range of a union of $k$-dimensional subspaces, and upper-bound the number of the subspaces. Towards this goal, first note that the effect of the ReLU operation $\text{relu}(z)$ can be described with a diagonal matrix $D$ which contains a one on its diagonal if the respective entry of $z$ is larger than zero, and zero otherwise, so that $Dz = \text{relu}(z)$. With this notation, we can write

$$H(y) = D_dW_dD_{d-1}\ldots D_2W_2D_1W_1D'yW'y.$$

The matrix $U \in \mathbb{R}^{n \times n}$ has at most rank $k$, thus $H(y)$ lies in the range of a union of at most $k$-dimensional subspaces, where each subspace is determined by the matrices $D_d, \ldots, D_1, D'$. We
next bound the number of subspaces. First note that since \( W' \in \mathbb{R}^{n \times k} \), there are only \( 2^k \) many different choices for \( D' \), corresponding to all the sign patterns. Next note that, with the lemma below, we have that for a fixed \( D W' \), the number of different matrixes \( D_1 \) can be bounded by \( n_1^k \). Likewise, for fixed \( W_2 D_1 W_1 D' W' \), the number of different matrices \( D_2 \) can be bounded by \( n_2^k \) and so forth. Thus, the total number of different choices of the matrices \( D_d, \ldots, D_1, D' \) is upper bounded by
\[
2^k n_1^k \ldots n_d^k.
\]

**Lemma 3.** For any \( U \in \mathbb{R}^{n \times k} \) and \( k \geq 5 \),
\[
\|\{\text{diag}(U v > 0) U | v \in \mathbb{R}^k\}\| \leq n^k.
\]

Next note that by assumption we have that
\[
\|U \eta\|_2^2 / \|\eta\|_2^2 \leq 2,
\]
for all vectors \( \eta \) and for all \( U \) defined by the matrices \( D_d, \ldots, D_1, D' \). For fixed \( U \), let \( S \) be the span of the right singular vectors of \( U \), and note that \( S \) has dimension at most \( k \). Let \( P_S \) be the orthogonal projector onto a subspace \( S \). We have that
\[
\frac{\|U \eta\|_2^2}{\|\eta\|_2^2} = \frac{\|U P_S \eta\|_2^2}{\|\eta\|_2^2} \leq 2 \frac{\|P_S \eta\|_2^2}{\|\eta\|_2^2},
\]
again for all \( \eta \). Now, we make use of the following bound on the projection of the noise \( \eta \) onto a subspace, which follows from standard Gaussian concentration inequalities [LM00, Lem. 1].

**Lemma 4.** Let \( S \subset \mathbb{R}^n \) be a subspace with dimension \( k \). Let \( \eta \sim \mathcal{N}(0, I_n) \) and \( \beta \geq 1 \). Then,
\[
\mathbb{P}\left[ \frac{\|P_S \eta\|_2^2}{\|\eta\|_2^2} \leq \frac{10 \beta k}{n} \right] \geq 1 - e^{-\beta k} - e^{-n/16}.
\]

Taking a union bound over all subspaces, we obtain with the lemma above that
\[
\mathbb{P}\left[ \frac{\|H(\eta)\|_2^2}{\|\eta\|_2^2} \leq \frac{20 \beta k}{n} \right] \geq 1 - (2^k n_1^k \ldots n_d^k) (e^{-\beta k} + e^{-n/16}).
\]
Choosing \( \beta = 2 \log(2n_1 n_2 \ldots n_d) \) concludes the proof.

**B Proof of Theorem 2**

In this section we prove our main result, Theorem 2. Instead of proving Theorem 2 as stated, we will prove the following equivalent rescaled statement for when \( W_i \) have i.i.d. \( \mathcal{N}(0, 1/n_i) \) entries. Because of this rescaling, \( G(x) \) scales like \( 2^{-d/2} \|x\| \), the noise \( \omega \) is assumed to scale like \( 2^{-d/2} \), \( \nabla f \) scales like \( 2^d \), and \( \alpha \) scales like \( 2^d \). Theorem 2 is the \( \epsilon = K/d^{60} \) case of what follows.

**Theorem 5.** Consider a network with the weights in the \( i \)-th layer, \( W_i \in \mathbb{R}^{n_i \times n_{i-1}} \), i.i.d. \( \mathcal{N}(0, 1/n_i) \) distributed, and suppose that the network satisfies the expansivity condition for some \( \epsilon \leq K/d^{60} \). Also, suppose that the noise variance obeys
\[
\omega \leq \frac{\|x\| K_1 2^{-d/2}}{d^{16}}, \quad \omega := \sqrt{\frac{18 \sigma^2 k}{n} \log(n_1^{d-1} n_2 \ldots n_d)}.
\]

Consider the iterates of Algorithm 1 with stepsize \( \alpha = K_4 d^{4d} \).
A. Then, there exists a number of steps $N$ upper bounded by
\[ N \leq \frac{K_2 f(x_0)2^d}{d^2 \epsilon \|x_*\|^2} \]
such that after $N$ steps, the iterates of Algorithm 1 obey
\[ \|x_i - x_*\| \leq K_5 d^\theta \|x_*\| \sqrt{\epsilon} + K_6 d^{6/2} \omega, \quad \text{for all } i \geq N, \] (8)
with probability at least $1 - 2e^{-2k \log n} - \sum_{i=2}^{d} 8n_i e^{-K \tau n_i - 2} - 8n_1 e^{-K \tau^2 \log(1/\epsilon) k}$.

B. In addition, for all $i \geq N$, we have
\[ \|x_{i+1} - x_*\| \leq (1 - \alpha 7/8)^{i+1-N} \|x_N - x_*\| + K_8 d^{2/2} \omega \] and
\[ \|G(x_{i+1}) - G(x_*)\| \leq \frac{1.2}{2d^2} (1 - \alpha 7/8)^{i+1-N} \|x_N - x_*\| + 1.2 K_8 \omega, \] (9,10)
where $\alpha < 1$ is the stepsize of the algorithm. Here, $K_1, K_2, \ldots$ are numerical constants, and $x_0$ is the initial point in the optimization.

As mentioned in Section 5.1, our proof makes use of a deterministic condition, called the Weight Distribution Condition (WDC), formally defined in Section 5.1. The following proposition establishes that the expansivity condition ensures that the WDC holds:

**Lemma 6** (Lemma 9 in [HV18]). Fix $\epsilon \in (0,1)$. If the entries of $W_i \in \mathbb{R}^{n_i \times n_i}$ are i.i.d. $\mathcal{N}(0,1/n_i)$ and the expansivity condition
\[ n_i > c \epsilon^{-2} \log(1/\epsilon) n_i-1 \log n_i-1 \]
holds, then $W_i$ satisfies the WDC with constant $c$ with probability at least $1 - 8n_i e^{-K \epsilon^2 n_i - 1}$. Here, $c$ and $K$ are numerical constants.

It follows from Lemma 6, that the WDC holds for all $W_i$ simultaneously with probability at least $1 - \sum_{i=2}^{d} 8n_i e^{-K \tau n_i - 2} - 8n_1 e^{-K \tau^2 \log(1/\epsilon) k}$.

In the remainder of the proof we work on the event that the WDC holds for all $W_i$.

**B.1 Preliminaries**

Recall that the goal of our algorithm is to minimize the empirical risk objective
\[ f(x) = \frac{1}{2} \|G(x) - y\|^2, \]
where $y := G(x_*) + \eta$, with $\eta \sim \mathcal{N}(0, \sigma^2/nI)$.

Our results rely on the fact that outside of two balls around $x = x_*$ and $x = -\rho_d x_*$, with $\rho_d$ a constant defined below, the direction chosen by the algorithm is a descent direction, with high probability. In order to prove this, we use a concentration argument, similar to the arguments used in [HV18]. First, define
\[ \Lambda_x := \Pi_{i=1}^d W_{i,x}, \]
with $W_{i,x}$ defined in Section 4 for notational convenience, and note that the step direction of our algorithm can be written as
\[ \tilde{v}_x = \tilde{v}_x + \tilde{q}_x, \quad \text{with} \quad \tilde{v}_x := \Lambda_x^t \Lambda_x x - (\Lambda_x)^t (\Lambda_x x) x_*, \quad \text{and} \quad \tilde{q}_x := \Lambda_x^t \eta. \] (11)
Note that at points $x$ where $G$ (and hence $f$) is differentiable, we have that $\tilde{v}_x = \nabla f(x)$.

The proof is based on showing that $\tilde{v}_x$ concentrates around a particular $h_x \in \mathbb{R}^k$, defined below, that is a continuous function of nonzero $x, x_*$ and is zero only at $x = x_*$ and $x = -\rho q x_*$. The definition of $h_x$ depends on a function that is helpful for controlling how the operator $x \mapsto W_{+,x}x$ distorts angles, defined as:

$$g(\theta) := \cos^{-1}\left(\frac{(\pi - \theta) \cos \theta + \sin \theta}{\pi}\right).$$

(12)

With this notation, we define

$$h_x := -\frac{1}{2d} \left( \prod_{i=0}^{d-1} \frac{\pi - \bar{\theta}_i}{\pi} \right) x_* + \frac{1}{2d} \left[ x - \sum_{i=0}^{d-1} \sin \bar{\theta}_i \left( \prod_{j=i+1}^{d-1} \frac{\pi - \bar{\theta}_j}{\pi} \right) \frac{\|x_*\|_2}{\|x\|_2} \right].$$

where $\bar{\theta}_0 = \angle(x, x_*)$ and $\bar{\theta}_i = g(\bar{\theta}_{i-1})$. Note that $h_x$ is deterministic and only depends on $x, x_*$, and the number of layers, $d$.

In order to bound the deviation of $\tilde{v}_x$ from $h_x$ we use the following two lemmas, bounding the deviation controlled by the WDC and the deviation from the noise:

**Lemma 7** (Lemma 6 in [HV18]). Suppose that the WDC holds with $\epsilon < 1/(16\pi d^2)^2$. Then, for all nonzero $x, x_* \in \mathbb{R}^k$,

$$\|\pi - h_x\|_2 \leq K \frac{d^3 \sqrt{e}}{2d} \max(\|x\|_2, \|x_*\|_2),$$

(13)

$$\langle \Lambda x, \Lambda_{x_*} x_* \rangle \geq \frac{1}{4\pi} \frac{1}{2d^2} \|x\|_2 \|x_*\|_2,$$

(14)

$$\|\Lambda x\|^2 \leq \frac{1}{2d} (1 + 2\epsilon)^d \leq \frac{13}{12} 2^{-d}.$$  

(15)

**Proof.** Equation (13) and (14) are Lemma 6 in [HV18]. Regarding (15), note that the WDC implies that $\|W_{+,x}\|^2 \leq 1/2 + \epsilon$. It follows that

$$\|\Lambda x\|^2 = \left| \prod_{i=1}^d W_{+,x} \right|^2 \leq \frac{1}{2d} (1 + 2\epsilon)^d = \frac{1}{2d} \frac{e^{d \log(1+2\epsilon)}}{2d} \leq \frac{1 + 4\epsilon d}{2d} \leq \frac{13}{12} 2^{-d},$$

where the last inequalities follow by our assumption on $\epsilon$ (i.e., $\epsilon < 1/(16\pi d^2)^2$).

**Lemma 8.** Suppose the WDC holds with $\epsilon < 1/(16\pi d^2)^2$, that any subset of $n_{i-1}$ rows of $W_i$ are linearly independent for each $i$, and that $\eta \sim \mathcal{N}(0, \sigma^2/nI)$. Then the event

$$\mathcal{E}_{\text{noise}} := \left\{ \|\Lambda^T \eta\| \leq \frac{\omega}{2d^{2/3}}, \text{ for all } x \right\}, \quad \omega := \sqrt{16\sigma^{-k/2} \log(n_1 n_2^{d-1} \ldots n_d)}$$

holds with probability at least $1 - 2e^{-2k \log n}$.

As the cost function $f$ is not differentiable everywhere, we make use of the generalized subdifferential in order to reference the subgradients at nondifferentiable points. For a Lipschitz function $\check{f}$ defined from a Hilbert space $\mathcal{X}$ to $\mathbb{R}$, the Clarke generalized directional derivative of $\check{f}$ at the point $x \in \mathcal{X}$ in the direction $u$, denoted by $\check{f}^o(x;u)$, is defined by $\check{f}^o(x;u) = \limsup_{y \rightarrow x, t \downarrow 0} \frac{\check{f}(y+tu) - \check{f}(y)}{t}$, and the generalized subdifferential of $\check{f}$ at $x$, denoted by $\partial \check{f}(x)$, is defined by

$$\partial \check{f}(x) = \{ v \in \mathbb{R}^k \mid \langle v, u \rangle \leq \check{f}^o(x;u), \text{ for all } u \in \mathcal{X} \}.$$
Proof. By (17), \( \partial f(x) = \text{conv}(v_1, \ldots, v_t) \) for some finite \( t \), and thus \( \partial f(x) = a_1v_1 + \ldots + a_tv_t \) for some \( a_1, \ldots, a_t \geq 0 \), \( \sum_i a_i = 1 \). For each \( v_i \), there exists a \( w \) such that \( v_i = \lim_{t \to 0} \tilde{v}_x + tw \). On the event \( \mathcal{E}_{\text{noise}} \), we have that for any \( x \neq 0 \), for any \( \tilde{v}_x \in \partial f(x) \)

\[
\|v_x - h_x\| \leq \|\tilde{v}_x - h_x\| = \|\nabla x + q_x - h_x\| \\
\leq \|\nabla x - h_x\| + \|q_x\| \\
\leq K \frac{d^3 \sqrt{\epsilon}}{2^d} \max(\|x\|_2, \|x_*\|_2) + \frac{\omega}{2^{d/2}},
\]

where the last inequality follows from Lemmas 7 and 8 above. The proof is concluded by appealing to the continuity of \( h_x \) with respect to nonzero \( x \), and by noting that

\[
\|v_x - h_x\| \leq \sum_i a_i \|v_i - h_x\| \leq K \frac{d^3 \sqrt{\epsilon}}{2^d} \max(\|x\|_2, \|x_*\|_2) + \frac{\omega}{2^{d/2}},
\]

where we used the inequality above and that \( \sum_i a_i = 1 \).

We will also need an upper bound on the norm of the step direction of our algorithm:

**Lemma 9.** Under the assumption of Lemma 8, and assuming that \( \mathcal{E}_{\text{noise}} \) holds, we have that, for any \( x \neq 0 \) and any \( v_x \in \partial f(x) \),

\[
\|v_x\| \leq \frac{dK}{2^d} \max(\|x\|, \|x_*\|),
\]

where \( K \) is a numerical constant.

**Proof.** Define for convenience \( \zeta_j = \prod_{i=1}^{d-1} \frac{\pi - \theta_{j,x,x_*}}{\pi} \). We have

\[
\|v_x\| \leq \|h_x\| + \|h_x - v_x\| \\
\leq \frac{1}{2^d} x - \frac{1}{2^d} \zeta_0 x_* - \frac{1}{2^d} \sum_{i=1}^{d-1} \frac{\sin \theta_{i,x}}{\pi} \zeta_{i+1} \frac{\|x_*\|}{\|x\|} \|x_*\|_2 + K_1 \frac{d^3 \sqrt{\epsilon}}{2^d} \max(\|x\|_2, \|x_*\|_2) + \frac{\omega}{2^{d/2}} \\
\leq \frac{1}{2^d} \|x\| + \left( \frac{1}{2^d} + \frac{d}{\pi 2^d} \right) \|x_*\| + K_1 \frac{d^3 \sqrt{\epsilon}}{2^d} \max(\|x\|, \|x_*\|) + \frac{\omega}{2^{d/2}} \\
\leq \frac{dK}{2^d} \max(\|x\|, \|x_*\|),
\]

where the second inequality follows from the definition of \( h_x \) and Lemma 9, the third inequality uses \( |\zeta_j| \leq 1 \), and the last inequality uses the assumption \( \omega \leq \frac{2^{-d/2} \|x_*\|}{8\pi} \). \( \square \)
B.2 Proof of Theorem 5A

We are now ready to prove Theorem 5A. The logic of the proof is illustrated in Figure 2. Recall that \( x_i \) is the \( i \)th iterate of \( x \) as per Algorithm 1. We first ensure that we can assume throughout that \( x_i \) is bounded away from zero:

**Lemma 11.** Suppose that WDC holds with \( \epsilon < 1/(16\pi d^2) \) and that \( \mathcal{E}_{\text{noise}} \) holds with \( \omega \) in (16) obeying \( \omega \leq \frac{2^{-d/2}||x_\ast||}{8\pi} \). Moreover, suppose that the step size in Algorithm 1 satisfies \( 0 < \alpha < \frac{K^d}{d^2} \), where \( K \) is a numerical constant. Then, after at most \( N = \left( \frac{38\pi K_0^2d^2}{\alpha} \right)^2 \) steps, we have that for all \( i > N \) and all \( t \in [0,1] \) that \( t\tilde{x}_i + (1-t)x_{i+1} \notin B(0, \frac{1}{32\pi}||x_\ast||) \).

In particular, if \( \alpha = K^2d/d^2 \), then \( N \) is bounded by a constant times \( d^4 \).

We can therefore assume throughout this proof that \( x_i \notin B(0,K_0||x_\ast||) \), \( K_0 = \frac{1}{32\pi} \). We prove Theorem 5 by showing that if \( ||h_x|| \) is sufficiently large, i.e., if the iterate \( x_i \) is outside of set

\[
S_\beta = \left\{ x \in \mathbb{R}^k \mid ||h_x|| \leq \frac{1}{2d} \beta \max(||x||, ||x_\ast||) \right\},
\]

with

\[
\beta = 4Kd^3\sqrt{\epsilon} + 13\omega d^{4d/2}/||x_\ast||, \tag{19}
\]

then the algorithm makes progress in the sense that \( f(x_{i+1}) - f(x_i) \) is smaller than a certain negative value. The set \( S_\beta \) is contained in two balls around \( x_\ast \) and \( -\rho x_\ast \), whose radius is controlled by \( \beta \):

**Lemma 12.** For any \( \beta \leq \frac{1}{64^2d^2} \),

\[
S_\beta \subset B(x_\ast, 5000d^6\beta||x_\ast||_2) \cup B(-\rho_d x_\ast, 500d^{11}\sqrt{\beta}||x_\ast||_2). \tag{20}
\]

Here, \( \rho_d > 0 \) is defined in the proof and obeys \( \rho_d \to 1 \) as \( d \to \infty \).

Note that by the assumption \( \omega \leq \frac{||x_\ast||K_12^{-d/2}}{d^6} \) and \( Kd^{45}\sqrt{\epsilon} \leq 1 \), our choice of \( \beta \) in (19) obeys \( \beta \leq \frac{1}{64^2d^2} \) for sufficiently small \( K_1,K \), and thus Lemma 12 yields:

\[
S_\beta \subset B(x_\ast, r) \cup B(-\rho_d x_\ast, \sqrt{r||x_\ast||d^8}).
\]

were we define the radius \( r = K_2d^6\sqrt{\epsilon||x_\ast||} + K_3d^6\omega d^{4d/2} \), where \( K_2,K_3 \) are numerical constants. Note that hat the radius \( r \) is equal to the right hand side in the error bound (8) in our theorem. In order to guarantee that the algorithm converges to a ball around \( x_\ast \), and not to that around \( -\rho_d x_\ast \), we use the following lemma:

**Lemma 13.** Suppose that the WDC holds with \( \epsilon < 1/(16\pi d^2) \). Moreover suppose that \( \mathcal{E}_{\text{noise}} \) holds, and that \( \omega \) in the event \( \mathcal{E}_{\text{noise}} \) obeys \( \frac{\omega}{2^{-d/2}||x_\ast||_2} \leq K_0/d^2 \), where \( K_0 < 1 \) is a universal constant. Then for any \( \phi_d \in [\rho_d,1] \), it holds that

\[
f(x) < f(y) \tag{21}
\]

for all \( x \in B(\phi_d x_\ast, K_3d^{-10}||x_\ast||) \) and \( y \in B(-\phi_d x_\ast, K_3d^{-10}||x_\ast||) \), where \( K_3 < 1 \) is a universal constant.
In order to apply Lemma 13, define for convenience the two sets:

\[ S^+_\beta := S_\beta \cap B(x_*, r), \quad \text{and} \quad S^-_\beta := S_\beta \cap B(-\rho_d x_*, \sqrt{r\|x_*\|^2}). \]

By the assumption that \( K d^{15} \sqrt{\epsilon} \leq 1 \) and \( \omega \leq K_1 d^{-16} 2^{-d/2} \|x_*\| \), we have that for sufficiently small \( K_1, K \),

\[ S^+_\beta \subseteq B(x_*, K_3 d^{-10} \|x_*\|) \quad \text{and} \quad S^-_\beta \subseteq B(-\rho_d x_*, K_3 d^{-10} \|x_*\|). \]

Thus, the assumptions of Lemma 13 are met, and the lemma implies that for any \( x \in S^-_\beta \) and \( y \in S^+_\beta \), it holds that \( f(x) > f(y) \). We now show that the algorithm converges to a point in \( S^+_\beta \).

This fact and the negation step in our algorithm (line 3-5) establish that the algorithm converges to a point in \( S^+_\beta \) if we prove that the objective is nonincreasing with iteration number, which will form the remainder of this proof.

Consider \( i \) such that \( x_i \notin S_\beta \). By the mean value theorem \([Cla17, \text{Theorem 8.13}]\), there is a \( t \in [0, 1] \) such that for \( \hat{x}_i = x_i - t\alpha \tilde{v}_{x_i} \), there is a \( \tilde{v}_{x_i} \in \partial f(\hat{x}_i) \), where \( \partial f \) is the generalized subdifferential of \( f \), obeying

\[
f(x_i - \alpha \tilde{v}_{x_i}) - f(x_i) = \langle \tilde{v}_{x_i}, -\alpha \tilde{v}_{x_i} \rangle = \langle \tilde{v}_{x_i}, -\alpha \tilde{v}_{x_i} \rangle + \langle \tilde{v}_{x_i} - \tilde{v}_{x_i}, -\alpha \tilde{v}_{x_i} \rangle \leq -\alpha \|\tilde{v}_{x_i}\|^2 + \alpha \|\tilde{v}_{x_i} - \tilde{v}_{x_i}\| \|\tilde{v}_{x_i}\| = -\alpha \|\tilde{v}_{x_i}\| (\|\tilde{v}_{x_i}\| - \|\tilde{v}_{x_i} - \tilde{v}_{x_i}\|). \tag{22}\]

In the next subsection, we guarantee that for any \( t \in [0, 1], \tilde{v}_{x_i} \) with \( \hat{x}_i = x_i - t\alpha \tilde{v}_{x_i} \) is close to \( \tilde{v}_{x_i} \):

\[
\|\tilde{v}_{x_i} - \tilde{v}_{x_i}\| \leq \left( 5 + \alpha K_7 d^2 \right) \|\tilde{v}_{x_i}\|, \quad \text{for all } \tilde{v}_{x_i} \in \partial f(\hat{x}_i). \tag{23}\]

Applying (23) to (22) yields

\[
f(x_i - \alpha \tilde{v}_{x_i}) - f(x_i) \leq -\frac{1}{12} \alpha \|\tilde{v}_{x_i}\|^2,
\]

where we used that \( \alpha K_7 d^2 \leq \frac{1}{12} \), by our assumption on the stepsize \( \alpha \) being sufficiently small.

Thus, the maximum number of iterations for which \( x_i \notin S_\beta \) is \( f(x_0)12/(\alpha \min_i \|\tilde{v}_{x_i}\|^2) \). We next lower-bound \( \|\tilde{v}_{x_i}\| \). We have that on \( \mathcal{E}_{\text{noise}} \), for all \( x \notin S_\beta \), with \( \beta \) given by (19) that

\[
\|\tilde{v}_{x_i}\| \geq \|h_x\| - \|h_x - \tilde{v}_x\| \geq 2^{-d} \max(\|x\|, \|x_*\|) \left( \beta - K_4 d^{3} \sqrt{\epsilon} - \omega \frac{2^{d/2}}{\|x_*\|} \right) \geq 2^{-d} \max(\|x\|, \|x_*\|) \left( 3K d^{3} \sqrt{\epsilon} + 12 \omega \frac{2^{d/2}}{\|x_*\|} \right) \geq 2^{-d} \|x_*\| 3K d^{3} \sqrt{\epsilon}, \tag{24}\]

where the second inequality follows by the definition of \( S^+_{\beta} \) and Lemma 9, and the third inequality follows from our definition of \( \beta \) in equation (19). Thus,

\[
f(x_i - \alpha \tilde{v}_{x_i}) - f(x_i) \leq -\alpha K_5 2^{-d} d^6 \epsilon \|x_*\|^2 \leq -2^{-d} d^4 K_6 \epsilon \|x_*\|^2 \]

19
Lemma 14. Suppose the WDC holds with $x \notin S_\beta$. Hence, there can be at most $f(x_0)^{2d}/K_0d^2 \ell(x_0)^2$ iterations for which $x_i \notin S_\beta$.

In order to conclude our proof, we remark that once $x_i$ is inside a ball of radius $r$ around $x_*$, the iterates do not leave a ball of radius $2r$ around $x_*$. To see this, note that by the bound on $\|v_x\|$ given in equation (18) and our choice of stepsize,

$$\alpha \|v_x\| \leq \frac{K}{d} \max(\|x_i\|, \|x_*\|).$$

This concludes the proof of Theorem 5A.

### B.3 Proof of Theorem 5B

Theorem 5A establishes that after $N$ iterations the iterates $x_i$ are inside a ball of radius $2r$ around $x_*$. With the assumption that $\epsilon \leq K_1/d^{90}$ for sufficiently small $K_1$ and the definition of $r$, this implies that the iterates lie in a ball around $x_*$ of radius at most $K_3d^{-10}\|x_*\|$. In this proof of Theorem 5B, we prove convergence within this ball.

In this proof, we show that for any $i \geq N$, it holds that $x_i \in B(x_*, a_4d^{-10}\|x_*\|), \tilde{x}_i = x_i$, and

$$\|x_{i+1} - x_*\| \leq b_2^{i+1-N} \|x_N - x_*\| + b_4 2^{d/2} \omega.$$  

where $K_3$ is defined in Lemma 13, $b_2 = 1 - \frac{\alpha}{2d} \epsilon$ and $b_4$ is a universal constant.

We need Lemma 14 which guarantees that the search directions of the iterates afterward point only up to the noise $\omega$:

**Lemma 14.** Suppose the WDC holds with $200d \sqrt{d} \sqrt{\epsilon} < 1$ and $x \in B(x_*, d \sqrt{\epsilon} \|x_*\|)$. Then for all $x \neq 0$ and for all $v_x \in \partial f(x)$,

$$\|v_x - \frac{1}{2d}(x - x_*)\| \leq \frac{1}{2d} \|x - x_*\| + \frac{1}{2d/2} \omega.$$  

Suppose $\tilde{x}_i \in B(x_*, K_3d^{-10}\|x_*\|)$. By the assumption $\epsilon \leq K_1/d^{90}$ for sufficiently small $K_1$, the assumptions in Lemma 14 are met. Therefore,

$$\|x_{i+1} - x_*\| = \|\tilde{x}_i - \alpha v\tilde{x}_i - x_*\|$$

$$= \|\tilde{x}_i - x_*\| - \alpha v\tilde{x}_i - x_*\|$$

$$\leq \left(1 - \frac{\alpha}{2d}\right) \|\tilde{x}_i - x_*\| + \alpha \|v\tilde{x}_i - \frac{1}{2d}(\tilde{x}_i - x_*)\|$$

$$\leq \left(1 - \frac{\alpha}{2d}\right) \|\tilde{x}_i - x_*\| + \alpha \left(1 - \frac{1}{8} \frac{\alpha}{2d}\|\tilde{x}_i - x_*\| + \frac{1}{2d/2} \omega\right)$$

$$= \left(1 - \frac{\alpha}{2d}\right) \|\tilde{x}_i - x_*\| + \alpha \frac{1}{2d/2} \omega,$$  

(25)

where the second inequality holds by Lemma 14. By the assumptions $\tilde{x}_i \in B(x_*, K_3d^{-10}\|x_*\|), \omega \leq K_1d^{-10}/2d^{2d/2}$, and using (25), we have $x_{i+1} \in B(x_*, K_3d^{-10}\|x_*\|)$. In addition, using Lemma 13 yields that $\tilde{x}_{i+1} = x_{i+1}$. Repeat the above steps yields that $x_i \in B(x_*, K_3d^{-10}\|x_*\|)$ and $\tilde{x}_i = x_i$ for all $i \geq N$.

Using (25) and $\alpha = K_1d^{2d/2}$, we have

$$\|x_{i+1} - x_*\| \leq b_2 \|x_i - x_*\| + b_3 \frac{2d/2}{d^2} \omega,$$  

(26)
where \( b_2 = 1 - 7K_1/(8d^2) \) and \( b_3 \) is a universal constant. Repeatedly applying (26) yields

\[
\|x_{i+1} - x_*\| \leq b_2^i 1^{-N} \|x_N - x_*\| + (b_2^{-N} + b_2^{-N-1} + \cdots + 1) \frac{b_4^{2d^2/2}}{(1-b_2)d^2} \omega \leq b_2^i 1^{-N} \|x_N - x_*\| + b_4^{2d^2/2} \omega,
\]

where the last inequality follows from the definition of \( b_2 \), and \( b_4 \) is a universal constant. This finishes the proof for (9). Inequality (10) follows from Lemma 21.

This concludes the proof.

The remainder of the proof is devoted to prove the lemmas used in this section.

### B.4 Proof of Equation (23)

Our proof relies on \( h_x \) being Lipschitz, as formalized by the lemma below, which is proven in Section B.10:

**Lemma 15.** For any \( x, y \notin B(0, K_0\|x_*\|) \), where \( K_0 \) and \( K_4 \) are numerical constants,

\[
\|h_x - h_y\| \leq K_4d^2 \frac{2d}{\|x - y\|}.
\]

By Lemma 15, for all \( t \in [0, 1] \) and \( i > N \) (recall that by Lemma 11, after at most \( N \) steps, \( x_i \neq B(0, K_0\|x_*\|) \)):

\[
\|h_{\tilde{x}_i} - h_{x_i}\| \leq K_4d^2 \frac{2d}{\|\tilde{x}_i - x_i\|},
\]

where \( \tilde{x}_i = x_i - \alpha \tilde{v}_{x_i} \). Thus, we have that on \( E_{\text{noise}} \), for any \( v_{\tilde{x}_i} \in \partial f(\tilde{x}_i) \) by Lemma 9,

\[
\|v_{\tilde{x}_i} - \tilde{v}_{x_i}\| \leq \|v_{\tilde{x}_i} - h_{\tilde{x}_i}\| + \|h_{\tilde{x}_i} - h_{x_i}\| + \|h_{x_i} - \tilde{v}_{x_i}\| \leq K_1 \frac{d^3 \sqrt{\epsilon}}{2d} \max(\|x_i\|, \|x_*\|) + \frac{\omega}{2d} + \frac{K_4d^2}{2d} \|\tilde{x}_i - x_i\| + K_1 \frac{d^3 \sqrt{\epsilon}}{2d} \max(\|x_i\|, \|x_*\|) + \frac{\omega}{2d} \leq K_1 \frac{d^3 \sqrt{\epsilon}}{2d} \max(\|x_i\|, \|x_*\|) + \frac{K_4d^2}{2d} \|\tilde{v}_{x_i}\| + K_1 \frac{d^3 \sqrt{\epsilon}}{2d} \max(\|x_i\|, \|x_*\|) + \frac{\omega}{2d} \leq K_1 \frac{d^3 \sqrt{\epsilon}}{2d} \left( 2 + \alpha dK \right) \max(\|x_i\|, \|x_*\|) + \frac{K_4d^2}{2d} \|\tilde{v}_{x_i}\| + \frac{K_4d^2}{2d} \|x_*\| \leq K_4d^2 \frac{2d}{\|x_*\|} \leq K_9/d^2.
\]

Combining (28) and (24), we get that

\[
\|v_{\tilde{x}_i} - \tilde{v}_{x_i}\| \leq \left( \frac{5}{6} + \alpha K_7 \right) \|\tilde{v}_{x_i}\|,
\]

with the appropriate constants chosen sufficiently small. This concludes the proof of Equation (23).
B.5 Proof of Lemma 11

First suppose that \( x_i \in B(0, 2K_0\|x_*\|) \). We show that after a polynomial number of iterations \( N \), we have that \( x_{i+N} \notin B(0, 2K_0\|x_*\|) \). Below, we prove that

\[
\langle x, v_x \rangle < 0 \text{ and } \|v_x\| \geq \frac{1}{2^{d+16\pi}}\|x_*\| \text{ for all } x \in B(0, 2K_0\|x_*\|) \text{ and } v_x \in \partial f(x).
\]

(29)

It follows that for any \( \tilde{x}_i \in B(0, 2K_0\|x_*\|) \), \( \tilde{x}_i \) and the next iterate produced by the algorithm, \( x_{i+1} = \tilde{x}_i - \alpha v_{\tilde{x}_i} \), form an obtuse triangle. As a consequence,

\[
\|\tilde{x}_{i+1}\|^2 = \|x_{i+1}\|^2 \geq \|\tilde{x}_i\|^2 + \alpha^2 \|v_{\tilde{x}_i}\|^2
\]

\[
\geq \|\tilde{x}_i\|^2 + \alpha^2 \left( \frac{2^{d+16\pi}}{2^d} \right)^2 \|x_*\|^2,
\]

where the last inequality follows from (29). Thus, the norm of the iterates \( \tilde{x}_i \) will increase until after \( (2K_02^{d+16\pi})^2 \) iterations, we have \( \tilde{x}_{i+N} \notin B(0, 2K_0\|x_*\|) \).

Consider \( \tilde{x}_i \notin B(0, 2K_0\|x_*\|) \), and note that

\[
\alpha \|v_{\tilde{x}_i}\| \leq \alpha \frac{dK}{2^d} \max(\|\tilde{x}_i\|, \|x_*\|) \leq \alpha \frac{16\pi Kd}{2^d} \|\tilde{x}_i\| \leq \frac{1}{2} \|\tilde{x}_i\|,
\]

where the first inequality follows from (18), the second inequality from \( \|\tilde{x}_i\| \geq 2K_0\|x_*\| \), and finally the last inequality from our assumption on the sufficiently small step size \( \alpha \). Therefore, from \( x_{i+1} = \tilde{x}_i - \alpha v_{\tilde{x}_i} \), we have that \( t\tilde{x}_i + (1-t)x_{i+1} \notin B(0, K_0\|x_*\|) \) for all \( t \in [0, 1] \), which completes the proof.

**Proof of (29):** It remains to prove (29). We start with proving \( \langle x, v_x \rangle < 0 \). For brevity of notation, let \( \Lambda_z = \prod_{i=d}^1 w_{i,.+z} \). We have

\[
x^T v_x = \langle \Lambda_x^T \Lambda_x x - \Lambda_x^T \Lambda x_* x_* + \Lambda_x^T \eta, x \rangle
\]

\[
\leq \frac{13}{12} 2^d - \frac{1}{4\pi} \frac{1}{2^d} \|x\| \|x_*\| + \|x\| \frac{\omega}{2^d/2}
\]

\[
\leq \|x\| \left( \frac{13}{12} 2^d - \frac{1}{4\pi} \frac{1}{2^d} \|x_*\| - \frac{1}{16\pi} \frac{1}{2^d} \|x_*\| \right)
\]

\[
\leq \|x\| \frac{1}{2^d} \left( 2\|x\| - \frac{3}{16\pi} \|x_*\| \right).
\]

The first inequality follows from (14) and (15), and the second inequality follows from our assumption on \( \omega \). Therefore, for any \( x \in B(0, \frac{1}{16\pi} \|x_*\|) \), \( \langle x, v_x \rangle < -\frac{1}{16\pi 2^d} \|x\| \|x_*\| \leq 0 \), as desired.

If \( G(x) \) is differentiable at \( x \), then \( v_x = \tilde{v}_x \) and \( \langle x, v_x \rangle < 0 \). If \( G(x) \) is not differentiable at \( x \), by equation (17), we have

\[
x^T v_x = x^T \left( c_1 v_1 + c_2 v_2 + \cdots + c_d v_d \right) \leq (c_1 + c_2 + \cdots + c_d) \|x\| \frac{1}{2^d} \left( 2\|x\| - \frac{3}{16\pi} \|x_*\| \right)
\]

\[
= \|x\| \frac{1}{2^d} \left( 2\|x\| - \frac{3}{16\pi} \|x_*\| \right) < -\frac{1}{16\pi 2^d} \|x\| \|x_*\| \leq 0,
\]

(30)

for all \( v_x \in \partial f(x) \).

Using (30) yields

\[
\|v_x\| = \max_{\|u\|=1} \langle v_x, u \rangle \geq \langle v_1, x \|x\| \rangle = -\frac{x^T v_x}{\|x\|} > \frac{1}{2^d 16\pi} \|x_*\|,
\]

which concludes the proof of (29).
B.6 Proof of Lemma 8
Let \( \Lambda_x = \Pi_{i=d}^1 W_{i,x} \). We have that
\[
\| \tilde{q}_x \|^2 = \| \Lambda_x' \eta \|^2 \leq \| \Lambda_x \|^2 \| P_{\Lambda_x} \eta \|^2,
\]
where \( P_{\Lambda_x} \) is a projector onto the span of \( \Lambda_x \). As a consequence, \( \| P_{\Lambda_x} \eta \|^2 \) is \( \chi^2 \)-distributed random variable with \( k \)-degrees of freedom scaled by \( \sigma/n \). A standard tail bound (see [Lug+13, p. 43]) yields that, for any \( \beta \geq k \),
\[
P \left[ \| P_{\Lambda_x} \eta \|^2 \geq 4\beta \right] \leq 2e^{-\beta}.
\]
Next, we note that by applying Lemmas 13-14 from [HV18, Proof of Lem. 15])\(^4\), with probability one, that the number of different matrices \( \Lambda_x \) can be bounded as
\[
| \{ \Lambda_x | x \neq 0 \} | = | \{ \Pi_{i=d}^1 W_{i,x} | x \neq 0 \} | \leq 10^{d^2} (n_1^d n_2^{d-1} \ldots n_d)^k \leq (n_1^d n_2^{d-1} \ldots n_d)^{2k},
\]
where the second inequality holds for \( \log(10) \leq k/4 \log(n_1) \). To see this, note that \( (n_1^d n_2^{d-1} \ldots n_d)^k \geq 10^{d^2} \) is implied by \( k(d \log(n_1) + (d-1) \log(n_2) + \ldots \log(n_d)) \geq kd^2/4 \log(n_1) \geq d^2 \log(10) \). Thus, by the union bound,
\[
P \left[ \| P_{\Lambda_x} \eta \|^2 \leq 16k \log(n_1^d n_2^{d-1} \ldots n_d), \text{ for all } x \right] \geq 1 - 2e^{-2k \log(n)},
\]
where \( n = n_d \). Recall from (15) that \( \| \Lambda_x \| \leq \frac{13}{12} \). Combining this inequality with \( \| \tilde{q}_x \|^2 \leq \| \Lambda_x \|^2 \| P_{\Lambda_x} \eta \|^2 \) concludes the proof.

B.7 Proof of Lemma 12
We now show that \( h_x \) is away from zero outside of a neighborhood of \( x_* \) and \( -\rho_d x_* \). We prove Lemma 12 by establishing the following:

**Lemma 16.** Suppose \( 64d^6 \sqrt{\beta} \leq 1 \). Define
\[
\rho_d := \sum_{i=0}^{d-1} \frac{\sin \tilde{\theta}_i}{\pi} \left( \prod_{j=i+1}^{d-1} \frac{\pi - \tilde{\theta}_j}{\pi} \right),
\]
where \( \tilde{\theta}_0 = \pi \) and \( \tilde{\theta}_i = g(\tilde{\theta}_i-1) \). If \( x \in S_\beta \), then we have that either
\[
| \tilde{\theta}_0 | \leq 32d^4 \beta \quad \text{and} \quad \| x \|_2 - \| x_* \|_2 \leq 132d^6 \beta \| x_* \|_2
\]
or
\[
| \tilde{\theta}_0 - \pi | \leq 8\pi d^4 \sqrt{\beta} \quad \text{and} \quad \| x \|_2 - \| x_* \|_2 \rho_d \| \leq 200d^7 \sqrt{\beta} \| x_* \|_2.
\]
In particular, we have
\[
S_\beta \subset B(x_*, 5000d^6 \beta \| x_* \|_2) \cup B(-\rho_d x_*, 500d^{11} \sqrt{\beta} \| x_* \|_2).
\]
(31)
Additionally, \( \rho_d \to 1 \) as \( d \to \infty \).

\(^4\)The proof in that argument only uses the assumption of independence of subsets of rows of the weight matrices.
Proof. Without loss of generality, let \( \|x_\ast\| = 1 \), \( x_\ast = e_1 \) and \( \hat{x} = r \cos \bar{\theta}_0 \cdot e_1 + r \sin \bar{\theta}_0 \cdot e_2 \) for \( \bar{\theta}_0 \in [0, \pi] \). Let \( x \in S_\beta \).

First we introduce some notation for convenience. Let
\[
\xi = \prod_{i=0}^{d-1} \frac{\pi - \bar{\theta}_i}{\pi}, \quad \zeta = \sum_{i=0}^{d-1} \sin \bar{\theta}_i \prod_{j=i+1}^{d-1} \frac{\pi - \bar{\theta}_j}{\pi}, \quad r = \|x\|_2, \quad M = \max(r, 1).
\]

Thus, \( h_x = -\frac{1}{2\pi} \xi \hat{x}_0 + \frac{1}{2\pi} (r - \zeta) \hat{x} \). By inspecting the components of \( h_x \), we have that \( x \in S_\beta \) implies
\[
|\xi + \cos \bar{\theta}_0 (r - \zeta)| \leq \beta M
\]
\[
|\sin \bar{\theta}_0 (r - \zeta)| \leq \beta M
\]

Now, we record several properties. We have:
\[
\bar{\theta}_i \in [0, \pi/2] \text{ for } i \geq 1
\]
\[
\bar{\theta}_1 \leq \bar{\theta}_{i-1} \text{ for } i \geq 1
\]
|\xi| \leq 1
|\xi| \leq \frac{d}{\pi} \sin \theta_0
\[
\bar{\theta}_i \leq \frac{3\pi}{i+3} \text{ for } i \geq 0
\]
\[
\bar{\theta}_i \geq \frac{\pi}{i+1} \text{ for } i \geq 0
\]
\[
\xi = \prod_{i=0}^{d-1} \frac{\pi - \bar{\theta}_i}{\pi} \geq \frac{\pi - \bar{\theta}_0}{\pi} d^{-3}
\]
\[
\bar{\theta}_0 = \pi + O_1(\delta) = \bar{\theta}_i + O_1(i\delta)
\]
\[
\bar{\theta}_0 = \pi + O_1(\delta) \Rightarrow |\xi| \leq \frac{\delta}{\pi}
\]
\[
\bar{\theta}_0 = \pi + O_1(\delta) \Rightarrow \zeta = \rho_d + O_1(3d^3\delta) \text{ if } \frac{d^2\delta}{\pi} \leq 1
\]

We now establish (36). Observe \( 0 < g(\theta) \leq (\frac{1}{3\pi} + \frac{1}{\pi})^{-1} =: \tilde{g}(\theta) \) for \( \theta \in (0, \pi] \). As \( g \) and \( \tilde{g} \) are monotonic increasing, we have \( \tilde{\theta}_i = g^{-1}(\bar{\theta}_0) = g^{-1}(\pi) \leq \tilde{g}^{-1}(\pi) = (\frac{i}{3\pi} + \frac{1}{\pi})^{-1} = \frac{3\pi}{i+3} \). Similarly, \( g(\theta) \geq (\frac{1}{3} + \frac{1}{\pi})^{-1} \) implies that \( \tilde{\theta}_i \geq \frac{\pi}{i+1} \), establishing (37).

We now establish (38). Using (36) and \( \bar{\theta}_i \leq \tilde{\theta}_i \), we have
\[
\prod_{i=1}^{d-1} \left(1 - \frac{\bar{\theta}_i}{\pi}\right) \geq \prod_{i=1}^{d-1} \left(1 - \frac{3}{i+3}\right) \geq d^{-3},
\]
where the last inequality can be established by showing that the ratio of consecutive terms with respect to \( d \) is greater for the product in the middle expression than for \( d^{-3} \).

We establish (39) by using the fact that \( |g'(\theta)| \leq 1 \) for all \( \theta \in [0, \pi] \) and using the same logic as for [HV18, Eq. 17].

We now establish (41). As \( \bar{\theta}_0 = \pi + O_1(\delta) \), we have \( \bar{\theta}_i = \tilde{\theta}_i + O_1(i\delta) \). Thus, if \( \frac{d^2\delta}{\pi} \leq 1 \),
\[
\prod_{j=i+1}^{d-1} \frac{\pi - \bar{\theta}_j}{\pi} = \prod_{j=i+1}^{d-1} \left(\frac{\pi - \bar{\theta}_j}{\pi} + O_1(\frac{i\delta}{2\pi})\right) = \left(\prod_{j=i+1}^{d-1} \frac{\pi - \bar{\theta}_j}{\pi}\right) + O_1(d^2\delta)
\]

24
Thus (41) holds.

Next, we establish that \( x \in S_\beta \Rightarrow r \leq 4d \), and thus \( M \leq 4d \). Suppose \( r > 1 \). At least one of the following holds: \( |\sin \theta_0| \geq 1/\sqrt{2} \) or \( |\cos \theta_0| \geq 1/\sqrt{2} \). If \( |\sin \theta_0| \geq 1/\sqrt{2} \) then (33) implies that \( |r - \xi| \leq \sqrt{2} \beta r \). Using (35), we get \( r \leq \frac{\pi}{1 - \sqrt{2} \beta} \leq d/2 \) if \( \beta < 1/4 \). If \( |\cos \theta_0| \geq 1/\sqrt{2} \), then (32) implies that \( |r - \xi| \leq \sqrt{2}(\beta r + |\xi|) \). Using (34), (35), and \( \beta < 1/4 \), we get \( r \leq \frac{\sqrt{2} \beta |\xi| + |\xi|}{1 - \sqrt{2} \beta} \leq \frac{d + \sqrt{2}}{1 - \sqrt{2} \beta} \leq 4d \). Thus, we have \( x \in S_\beta \Rightarrow r \leq 4d \Rightarrow M \leq 4d \).

Next, we establish that we only need to consider the small angle case \( (\theta_0 \approx 0) \) and the large angle case \( (\theta_0 \approx \pi) \), by considering the following three cases:

(Case I) \( \sin \theta_0 \leq 16d^4 \beta \): We have \( \bar{\theta}_0 = O_1(32d^4 \beta) \) or \( \bar{\theta}_0 = \pi + O_1(32d^4 \beta) \), as \( 32d^4 \beta < 1 \).

(Case II) \( |r - \xi| < \sqrt{3}M \): Applying case II to inequality (32) yields \( |\xi| \leq 2\sqrt{3}M \). Using (38), we get \( \bar{\theta}_0 = \pi + O_1(2\pi d^3 \sqrt{3}M) \).

(Case III) \( \sin \theta_0 > 16d^4 \beta \) and \( |r - \xi| \geq \sqrt{3}M \): Finally, consider Case III. By (33), we have \( |r - \xi| \leq \frac{\beta M}{\sin \theta_0} \). Using this inequality in (32), we have \( |\xi| \leq \beta M + \frac{\beta M}{\sin \theta_0} \leq \frac{2\beta M}{\sin \theta_0} \leq \frac{1}{2} d^{-3} \), where the second to last inequality uses \( \sin \theta_0 > 16d^4 \beta \) and the last inequality uses \( M \leq 4d \). By (38), we have \( \frac{\pi - \theta_0}{\pi} d^{-3} \leq \xi \leq \frac{1}{2} d^{-3} \), which implies that \( \theta_0 > \pi/2 \). Now, as \( |r - \xi| \geq \sqrt{3}M \), then by (33), we have \( |\sin \theta_0| \leq \sqrt{3} \). Hence, \( \bar{\theta}_0 = \pi + O_1(2\sqrt{3}) \), as \( \bar{\theta}_0 \approx \pi/2 \) and \( \beta < 1 \).

At least one of the Cases I, II, or III hold. Thus, we see that it suffices to consider the small angle case \( \bar{\theta}_0 = O_1(32d^4 \beta) \) or the large angle case \( \bar{\theta}_0 \approx \pi + O_1(8\pi d^4 \sqrt{3}) \).

**Small Angle Case.** Assume \( \bar{\theta}_0 = O_1(\delta) \) with \( \delta = 32d^4 \beta \). As \( \delta_i \leq \bar{\theta}_0 \leq \delta \) for all \( i \), we have \( 1 \geq \xi \geq (1 - \frac{\delta}{\pi}) d = 1 + O_1(\frac{2\delta d}{\pi}) \) provided \( \delta d/\pi \leq 1/2 \) (which holds by our choice \( \delta = 32d^4 \beta \) by assumption \( 64d^6 \sqrt{3} \beta \leq 1 \)). By (35), we also have \( \zeta = O_1(\frac{4d}{\pi} \delta) \). By (32), we have

\[
| - \xi + \cos \bar{\theta}_0 (r - \xi) | \leq \beta M.
\]

Thus, as \( \cos \bar{\theta}_0 = 1 + O_1(\bar{\theta}_0^2/2) = 1 + O_1(\delta^2/2) \),

\[
- \left( 1 + O_1(\frac{2\delta d}{\pi}) \right) + (1 + O_1(\frac{2\delta d}{\pi}))(r + O_1(\frac{\delta d}{\pi})) = O_1(4d \beta),
\]

and \( r \leq M \leq 4d \) (shown above) provides,

\[
r - 1 = O_1(4d \beta + \frac{2\delta d}{\pi} + \frac{\delta d}{\pi} + \frac{2\delta d}{\pi} + \frac{2\delta d^2}{\pi^2})
\]

\[
= O_1(4\beta d + 4\delta d^2).
\]

By plugging in that \( \delta = 32d^4 \beta \), we have that \( r - 1 = O_1(132d^6 \beta) \), where we have used that \( \frac{32d^6 \beta}{\pi} \leq 1/2 \).
Large Angle Case. Assume \( \theta_0 = \pi + O_1(\delta) \) where \( \delta = 8\pi d^4 \sqrt{3} \). By (40) and (41), we have \( \xi = O_1(\delta/\pi) \), and we have \( \zeta = \rho_d + O_1(3d^3 \delta) \) if \( 8d^6 \sqrt{3} \leq 1 \). By (32), we have
\[
| - \xi + \cos \theta_0 (r - \zeta) | \leq \beta M,
\]
so, as \( \cos \theta_0 = 1 - O_1(\theta_0^2/2) \),
\[
O_1(\delta/\pi) + (1 + O_1(\delta^2/2))(r - \rho_d + O_1(3d^3 \delta)) = O_1(\beta M),
\]
and thus, using \( r \leq 4d, \rho_d \leq d, \) and \( \delta = 8\pi d^4 \sqrt{3} \leq 1 \),
\[
r - \rho_d = O_1(\beta M + \delta/\pi + 3d^3 \delta + \tfrac{5}{2} \delta^2 d + \tfrac{3}{2} d^3 \delta^3)
= O_1(4 \beta d + \delta(\tfrac{1}{\pi} + 3d^3 + \tfrac{5}{2} d + \tfrac{3}{2} d^3))
= O_1(200d^7 \sqrt{3})
\tag{47}
\]
\[
(48)
\]
To conclude the proof of (31), we use the fact that
\[
\| x - x_\ast \| \leq \|x\|_2 - \|x_\ast\|_2 + (\|x_\ast\|_2 + \|x\|_2 - \|x_\ast\|_2) \overline{\theta}_0.
\]
This fact simply says that if a 2d point is known to have magnitude within \( \Delta r \) of some \( r \) and is known to be within angle \( \Delta \theta \) from 0, then its Euclidean distance to the point of polar coordinates \((r, 0)\) is no more than \( \Delta r + (r + \Delta r) \Delta \theta \).

Finally, we establish that \( \rho_d \to 1 \) as \( d \to \infty \). Note that \( \rho_{d+1} = (1 - \frac{\tilde{\theta}_d}{\pi}) \rho_d + \frac{\sin \tilde{\theta}_d}{\pi} \) and \( \rho_0 = 0 \). It suffices to show \( \tilde{\rho_d} \to 0 \), where \( \tilde{\rho_d} := 1 - \rho_d \). The following recurrence relation holds:
\[
\tilde{\rho_d} = (1 - \frac{\theta_{d-1}}{\pi}) \tilde{\rho}_{d-1} \leq \tilde{\theta}_{d-1} - \sin \frac{\tilde{\theta}_{d-1}}{\pi}, \text{ with } \tilde{\rho}_0 = 1.
\]
Using the recurrence formula [HV18, Eq. (15)] and the fact that \( \tilde{\theta}_0 = \pi \), we get that
\[
\tilde{\rho}_d = \sum_{i=1}^d \tilde{\theta}_{i-1} - \sin \frac{\tilde{\theta}_{i-1}}{\pi} \prod_{j=i+1}^d \left( 1 - \frac{\tilde{\theta}_{j-1}}{\pi} \right)
\tag{50}
\]
using (37), we have that
\[
\prod_{j=i+1}^d \left( 1 - \frac{\tilde{\theta}_{j-1}}{\pi} \right) \leq \prod_{j=i+1}^d \left( 1 - \frac{1}{j} \right) = \exp \left( -\sum_{j=i+1}^d \frac{1}{j} \right) \leq \exp \left( -\int_{i+1}^{d+1} \frac{1}{s} \, ds \right) = \frac{i+1}{d+1}
\]
Using (36) and the fact that \( \tilde{\theta}_{i-1} \leq \tilde{\theta}_{i-1}^3/6 \), we have that \( \tilde{\rho}_d \leq \sum_{i=1}^d \frac{\tilde{\theta}_{i-1}^3}{6\pi} \cdot \frac{i+1}{d+1} \to 0 \) as \( d \to \infty \).

\[\square\]

B.8 Proof of Lemma 13

Consider the function
\[
f_\eta(x) = f_0(x) - \langle G(x) - G(x_\ast), \eta \rangle,
\]
and note that \( f(x) = f_\eta(x) + \| \eta \|^2 \). Consider \( x \in \mathcal{B}(\phi_d x_\ast, \varphi \| x_\ast \|) \), for a \( \varphi \) that will be specified later. Note that
\[
| \langle G(x) - G(x_\ast), \eta \rangle | \leq | \langle \Pi_{i=d}^1 W_{i,+} x, \eta \rangle | + | \langle \Pi_{i=d}^1 W_{i,+} x_\ast, \eta \rangle | = | \langle x, (\Pi_{i=d}^1 W_{i,+}) x_\ast \rangle | + | \langle x_\ast, (\Pi_{i=d}^1 W_{i,+}) x \rangle | \leq (\| x \| + \| x_\ast \|) \frac{\omega}{2d^{7/2}} \leq (\varphi \| x_\ast \| + \| x_\ast \|) \frac{\omega}{2d^{7/2}},
\]

26
where the second inequality holds on the event $\mathcal{E}_{\text{noise}}$, by Lemma 8, and the last inequality holds by our assumption on $x$. Thus, for $x \in \mathcal{B}(\phi_d x_*, \varphi \| x_* \|)$

\[
    f_\eta(x) \leq \mathbb{E} f_0(x) + |f_0(x) - \mathbb{E} f_0(x)| + |\langle G(x) - G(x_*), \eta \rangle| \\
    \leq \frac{1}{2d+1} \left( \phi_d^2 - 2\phi_d + \frac{10}{K_3^2} d\varphi \right) \| x_* \|^2 + \frac{1}{2d+1} \| x_* \|^2 \\
    + \frac{\epsilon (1 + 4ed)}{2d} \| x \|^2 + \frac{\epsilon (1 + 4ed) + 48d^3 \sqrt{\epsilon}}{2d+1} \| x \| \| x_* \| + \frac{\epsilon (1 + 4ed)}{2d} \| x_* \|^2 \\
    + (\varphi \| x_* \| + \| x_* \|) \frac{\omega}{2d^2} \\
    \leq \frac{1}{2d+1} \left( \phi_d^2 - 2\phi_d + \frac{10}{K_3^2} d\varphi \right) \| x_* \|^2 + \frac{1}{2d+1} \| x_* \|^2 \\
    + \frac{\epsilon (1 + 4ed)}{2d} \left( \phi_d + \varphi \right)^2 \| x_* \|^2 + \frac{\epsilon (1 + 4ed) + 48d^3 \sqrt{\epsilon}}{2d+1} (\phi_d + \varphi) \| x_* \|^2 + \frac{\epsilon (1 + 4ed)}{2d} \| x_* \|^2 \\
    + (\varphi \| x_* \| + \| x_* \|) \frac{\omega}{2d^2} \\
    \leq \frac{\| x_* \|^2}{2d+1} \left( 1 + \phi_d^2 - 2\phi_d + \frac{10}{K_3^2} d\varphi + 68d^2 \sqrt{\epsilon} \right) + (\varphi \| x_* \| + \| x_* \|) \frac{\omega}{2d^2} \tag{51}
\]

where the last inequality follows from $\epsilon < \sqrt{\epsilon}$, $\rho_d \leq 1$, $4ed < 1$, $\varphi < 1$ and assuming $\varphi = \epsilon$.

Similarly, we have that for any $y \in \mathcal{B}(-\phi_d x_*, \varphi \| x_* \|)$

\[
    f_\eta(y) \geq \mathbb{E} [f(y)] - |f(y) - \mathbb{E} [f(y)]| - |\langle G(x) - G(x_*), \eta \rangle| \\
    \geq \frac{1}{2d+1} \left( \phi_d^2 - 2\phi_d \rho_d - 10d^3 \varphi \right) \| x_* \|^2 + \frac{1}{2d+1} \| x_* \|^2 \\
    - \left( \frac{\epsilon (1 + 4ed)}{2d} \| y \|^2 + \frac{\epsilon (1 + 4ed) + 48d^3 \sqrt{\epsilon}}{2d+1} \| y \| \| x_* \| + \frac{\epsilon (1 + 4ed)}{2d} \| x_* \|^2 \right) \\
    - (\varphi \| x_* \| + \| x_* \|) \frac{\omega}{2d^2} \\
    \geq \frac{\| x_* \|^2}{2d+1} \left( 1 + \phi_d^2 - 2\phi_d \rho_d - 10d^3 \varphi - 68d^2 \sqrt{\epsilon} \right) - (\varphi \| x_* \| + \| x_* \|) \frac{\omega}{2d^2} \tag{52}
\]

Using $\epsilon < \sqrt{\epsilon}$, $\rho_d \leq 1$, $4ed < 1$, $\varphi < 1$ and assuming $\varphi = \epsilon$, the right side of (51) is smaller than the right side of (52) if

\[
    \varphi = \epsilon \leq \left( \frac{\phi_d - \rho_d \theta - 13 \| \eta \|_2}{(125 + \frac{5}{K_3^2}) d^3} \right)^2. \tag{53}
\]

We can establish that:

**Lemma 17.** For all $d \geq 2$, that

\[
    1/(K_1 (d + 2)^2) \leq 1 - \rho_d \leq 250/(d + 1).
\]

Thus, it suffices to have $\varphi = \epsilon = \frac{K_3}{d^3}$ and $13 \| \eta \|_2 \leq \frac{K_2}{d^3} \leq \frac{1}{2} \frac{K_2}{K_1 (d+2)^2}$ for an appropriate universal constant $K_9$, and for an appropriate universal constant $K_3$. 

27
B.9 Proof of Lemma 17

It holds that

\[ \|x - y\| \geq 2 \sin(\theta_{x,y}/2) \min(\|x\|, \|y\|), \quad \forall x, y \]  
\[ \sin(\theta/2) \geq \theta/4, \quad \forall \theta \in [0, \pi] \]  
\[ \frac{d}{d\theta} g(\theta) \in [0, 1] \quad \forall \theta \in [0, \pi] \]  
\[ \log(1 + x) \leq x \quad \forall x \in [-0.5, 1] \]  
\[ \log(1 - x) \geq -2x \quad \forall x \in [0, 0.75] \]  

where \( \theta_{x,y} = \angle(x, y) \). We recall the results (36), (37), and (50) in [HV18]:

\[ \tilde{\theta}_i \leq \frac{3\pi}{i + 3} \quad \text{and} \quad \tilde{\theta}_i \geq \frac{\pi}{i + 1} \quad \forall i \geq 0 \]

\[ 1 - \rho_d = \prod_{j=1}^{d-1} \left( 1 - \frac{\tilde{\theta}_j}{\pi} \right) + \sum_{i=1}^{d-1} \frac{\tilde{\theta}_i - \sin \tilde{\theta}_i}{\pi} \prod_{j=i+1}^{d-1} \left( 1 - \frac{\tilde{\theta}_j}{\pi} \right). \]

Therefore, we have for all \( 0 \leq i \leq d - 2 \),

\[ \prod_{j=i+1}^{d-1} \left( 1 - \frac{\tilde{\theta}_j}{\pi} \right) \leq \prod_{j=i+1}^{d-1} \left( 1 - \frac{1}{j + 1} \right) = e^{\sum_{j=i+1}^{d-1} \log \left( 1 - \frac{1}{j + 1} \right)} \leq e^{-\sum_{j=i+1}^{d-1} \frac{1}{j + 1} ds} = \frac{i + 2}{d + 1}, \]
\[ \prod_{j=i+1}^{d-1} \left( 1 - \frac{\tilde{\theta}_j}{\pi} \right) \leq \prod_{j=i+1}^{d-1} \left( 1 - \frac{3}{j + 3} \right) = e^{\sum_{j=i+1}^{d-1} \log \left( 1 - \frac{3}{j + 3} \right)} \leq e^{-\sum_{j=i+1}^{d-1} \frac{6}{j + 3} ds} = \left( \frac{i + 3}{d + 2} \right)^6, \]

where the second and the fifth inequalities follow from (57) and (58) respectively. Since \( \pi^3/(12(i + 1)^3) \leq \tilde{\theta}_i/12 \leq \tilde{\theta}_i - \sin \tilde{\theta}_i \leq \tilde{\theta}_i^3/6 \leq 27\pi^3/(6(i + 3)^3) \), we have that for all \( d \geq 3 \)

\[ 1 - \rho_d \leq \frac{2}{d + 1} + \frac{27\pi^3}{6(i + 3)^3 d + 1} + \frac{2}{d + 1} + \frac{3\pi^3}{4(d + 1)} \leq \frac{250}{d + 1}, \]

and

\[ 1 - \rho_d \geq \left( \frac{3}{d + 2} \right)^6 + \sum_{i=1}^{d-1} \frac{\pi^3}{12(i + 3)^3} \left( \frac{i + 3}{d + 2} \right)^6 \geq \frac{1}{K_1(d + 2)^3}, \]

where we use \( \sum_{i=4}^{\infty} \frac{1}{i^2} \leq \frac{\pi^2}{6} \) and \( \sum_{i=1}^{n} i^3 = O(n^4) \).

B.10 Proof of Lemma 15

To establish Lemma 15, we prove the following:

Lemma 18. For all \( x, y \neq 0 \),

\[ \|h_x - h_y\| \leq \left( \frac{1}{2^d} + \frac{6d + 4d^2}{\pi 2^d} \sin \left( \frac{1}{\|x\|, \|y\|} \right) \|x\| + \|y\| \right) \|x - y\| \]

Lemma 15 follows by noting that if \( x, y \notin B(0, r\|x\|) \), then \( \|h_x - h_y\| \leq \left( \frac{1}{2^d} + \frac{6d + 4d^2}{\pi r 2^d} \right) \|x - y\| . \)
Proof of Lemma 18. For brevity of notation, let \( \zeta_{j,z} = \prod_{i=j}^{d-1} \frac{\pi - \theta_{i,z}}{\pi} \). Combining (54) and (55) gives
\[
|\theta_{0,x} - \bar{\theta}_{0,y}| \leq 4 \max \left( \frac{1}{\|x\|}, \frac{1}{\|y\|} \right) \|x - y\|. \tag{56}
\]
Inequality (56) implies \( |\theta_{i,x} - \bar{\theta}_{i,y}| \leq |\theta_{j,x} - \bar{\theta}_{j,y}| \) for all \( i \geq j \).

It follows that
\[
\|h_x - h_y\| \leq \frac{1}{2d} \|x - y\| + \frac{1}{2d} \|\zeta_{0,x} - \zeta_{0,y}\| x_* \|_T \tag{59}
\]
\[
+ \frac{1}{2d} \left| \sum_{i=0}^{d-1} \frac{\sin \bar{\theta}_{i,x}}{\pi} \zeta_{i+1,x} \hat{x} - \sum_{i=0}^{d-1} \frac{\sin \bar{\theta}_{i,y}}{\pi} \zeta_{i+1,y} \hat{y} \right| x_* \|_T.
\]

By Lemma 19, we have
\[
T_1 \leq \frac{d}{\pi} |\hat{\theta}_{0,x} - \hat{\theta}_{0,y}| \leq \frac{4d}{\pi} \max \left( \frac{1}{\|x\|}, \frac{1}{\|y\|} \right) \|x - y\|. \tag{60}
\]

Additionally, it holds that
\[
T_2 = \left| \sum_{i=0}^{d-1} \frac{\sin \bar{\theta}_{i,x}}{\pi} \zeta_{i+1,x} \hat{x} - \sum_{i=0}^{d-1} \frac{\sin \bar{\theta}_{i,y}}{\pi} \zeta_{i+1,y} \hat{y} \right| \leq \frac{d}{\pi} \|\hat{x} - \hat{y}\| + \left| \sum_{i=0}^{d-1} \frac{\sin \bar{\theta}_{i,x}}{\pi} \zeta_{i+1,x} \right| - \left| \sum_{i=0}^{d-1} \frac{\sin \bar{\theta}_{i,y}}{\pi} \zeta_{i+1,y} \right| \tag{61}
\]

We have
\[
T_3 \leq \sum_{i=0}^{d-1} \left[ \left| \frac{\sin \bar{\theta}_{i,x}}{\pi} \zeta_{i+1,x} \right| - \left| \frac{\sin \bar{\theta}_{i,y}}{\pi} \zeta_{i+1,y} \right| \right] + \left| \frac{1}{\pi} \left( \frac{d - i - 1}{\pi} |\hat{\theta}_{i-1,x} - \hat{\bar{\theta}}_{i-1,y}| \right) + \frac{1}{\pi} |\sin \bar{\theta}_{i,x} - \sin \bar{\theta}_{i,y}| \right| \right| \leq \frac{d^2}{\pi} |\bar{\theta}_{0,x} - \bar{\theta}_{0,y}| \leq \frac{4d^2}{\pi} \max \left( \frac{1}{\|x\|}, \frac{1}{\|y\|} \right) \|x - y\|. \tag{62}
\]

Using (54) and (55) and noting \( \|\hat{x} - \hat{y}\| \leq \theta_{x,y} \) yield
\[
\|\hat{x} - \hat{y}\| \leq \theta_{x,y} \leq 2 \max \left( \frac{1}{\|x\|}, \frac{1}{\|y\|} \right) \|x - y\|. \tag{63}
\]

Finally, combining (59), (60), (61), (62) and (63) yields the result. \( \square \)

**Lemma 19.** Suppose \( a_i, b_i \in [0, \pi] \) for \( i = 1, \ldots, k \), and \( |a_i - b_i| \leq |a_j - b_j|, \forall i \geq j \). Then it holds that
\[
\left| \prod_{i=1}^{k} \frac{\pi - a_i}{\pi} - \prod_{i=1}^{k} \frac{\pi - b_i}{\pi} \right| \leq \frac{k}{\pi} |a_1 - b_1|. \]
Proof. Prove by induction. It is easy to verify that the inequality holds if \( k = 1 \). Suppose the inequality holds with \( k = t - 1 \). Then

\[
\left| \prod_{i=1}^{t} \frac{\pi - a_i}{\pi} - \prod_{i=1}^{t} \frac{\pi - b_i}{\pi} \right| \leq \left| \prod_{i=1}^{t} \frac{\pi - a_i}{\pi} - \prod_{i=1}^{t-1} \frac{\pi - b_i}{\pi} \right| + \left| \prod_{i=1}^{t} \frac{\pi - a_i}{\pi} - \prod_{i=1}^{t-1} \frac{\pi - b_i}{\pi} \right| \\
\leq \frac{t-1}{\pi} |a_1 - b_1| + \frac{1}{\pi} |a_t - b_t| \leq \frac{t}{\pi} |a_1 - b_1|.
\]

\[\square\]

### B.11 Proof of Lemma 14

We first need Lemmas 20, 21 and 22.

**Lemma 20.** Suppose \( W \in \mathbb{R}^{n \times k} \) satisfies the WDC with constant \( \epsilon \). Then for any \( x, y \in \mathbb{R}^k \), it holds that

\[
\|W_{+,x}x - W_{+,y}y\| \leq \left( \sqrt{\frac{1}{2} + \epsilon} + \sqrt{2(\epsilon + \theta)} \right) \|x - y\|,
\]

where \( \theta = \angle(x, y) \).

**Proof.** We have

\[
\|W_{+,x}x - W_{+,y}y\| \leq \|W_{+,x}x - W_{+,x}y\| + \|W_{+,x}y - W_{+,y}y\| \\
= \|W_{+,x}(x - y)\| + \|(W_{+,x} - W_{+,y})y\| \leq \|W_{+,x}\|\|x - y\| + \|(W_{+,x} - W_{+,y})y\|. \tag{64}
\]

By WDC assumption, we have

\[
\|W_{+,x}^T(W_{+,x} - W_{+,y})\| \leq \|W_{+,x}^T - W_{+,y}^T(x - I/2) + \|W_{+,x}^T(W_{+,y} - Q_{x,y}) + \|Q_{x,y} - I/2\| \\
\leq 2\epsilon + \theta. \tag{65}
\]

We also have

\[
\|W_{+,x} - W_{+,y}\|^2 = \sum_{i=1}^{n} (1_{w_i > 0} - 1_{w_i < 0})^2 (w_i \cdot y)^2 \\
\leq \sum_{i=1}^{n} (1_{w_i > 0} - 1_{w_i < 0})^2 ((w_i \cdot x)^2 + (w_i \cdot y)^2 - 2(w_i \cdot x)(w_i \cdot y)) \\
= \sum_{i=1}^{n} (1_{w_i > 0} - 1_{w_i < 0})^2 (w_i \cdot (x - y))^2 \\
= \sum_{i=1}^{n} 1_{w_i > 0} (w_i \cdot (x - y))^2 + \sum_{i=1}^{n} 1_{w_i < 0} (w_i \cdot (x - y))^2 \\
= (x - y)^TW_{+,x}^T(W_{+,x} - W_{+,y})(x - y) + (x - y)^TW_{+,y}^TW_{+,y}(x - y) \\
\leq 2(2\epsilon + \theta)\|x - y\|^2. \quad \text{(by (65))} \tag{66}
\]

Combining (64), (66), and \( \|W_{+,x}\|^2 \leq 1/2 + \epsilon \) given in [HV18, (10)] yields the result. \[\square\]
Lemma 21. Suppose \( x \in \mathcal{B}(x_*, d\sqrt{\epsilon}\|x_*\|) \), and the WDC holds with \( \epsilon < 1/(200)^4/d^6 \). Then it holds that

\[
\left\| \prod_{i=j}^1 W_{i,+,x} - \prod_{i=j}^1 W_{i,+,x_*} \right\| \leq \frac{1.2}{2^\frac{j}{2}} \| x - x_* \|.
\]

Proof. In this proof, we denote \( \theta_{i,x,x_*} \) and \( \bar{\theta}_{i,x,x_*} \) by \( \theta_i \) and \( \bar{\theta}_i \) respectively. Since \( x \in \mathcal{B}(x_*, d\sqrt{\epsilon}\|x_*\|) \), we have

\[
\bar{\theta}_i \leq \theta_0 \leq 2d\sqrt{\epsilon}.
\]

By [HV18, (14)], we also have \( |\theta_i - \bar{\theta}_i| \leq 4i\sqrt{\epsilon} \leq 4d\sqrt{\epsilon} \). It follows that

\[
2\sqrt{\theta_i + 2\epsilon} \leq 2\sqrt{\bar{\theta}_i + 4d\sqrt{\epsilon} + 2\epsilon} \leq 2\sqrt{2d\sqrt{\epsilon} + 4d\sqrt{\epsilon} + 2\epsilon} \\
\leq 2\sqrt{8d\sqrt{\epsilon}} \leq \frac{1}{30d}\cdot \text{by the assumption on } \epsilon
\]

Note that \( \sqrt{1 + 2\epsilon} \leq 1 + \epsilon \leq 1 + \sqrt{d\sqrt{\epsilon}} \). We have

\[
\prod_{i=d-1}^0 \left( \sqrt{1 + 2\epsilon} + 2\sqrt{\theta_i + 2\epsilon} \right) \leq \left( 1 + 7\sqrt{d\sqrt{\epsilon}} \right)^d \leq 1 + 14d\sqrt{d\sqrt{\epsilon}} \leq \frac{107}{100} < 1.2,
\]

where the second inequality is from that \((1 + x)^d \leq 1 + 2dx \) if \( 0 < x \leq 1 \). Combining the above inequality with Lemma 20 yields

\[
\left\| \prod_{i=j}^1 W_{i,+,x} - \prod_{i=j}^1 W_{i,+,x_*} \right\| \leq \prod_{i=d-1}^0 \left( \sqrt{\frac{1}{2} + \epsilon} + \sqrt{2\sqrt{\theta_i + 2\epsilon}} \right) \| x - x_* \| \leq \frac{1.2}{2^\frac{j}{2}} \| x - x_* \|.
\]

\[\square\]

Lemma 22. Suppose \( x \in \mathcal{B}(x_*, d\sqrt{\epsilon}\|x_*\|) \), and the WDC holds with \( \epsilon < 1/(200)^4/d^6 \). Then it holds that

\[
\left( \prod_{i=d}^1 W_{i,+,x} \right)^T \left[ \left( \prod_{i=d}^1 W_{i,+,x} \right) x - \left( \prod_{i=d}^1 W_{i,+,x} \right) x_* \right] = \frac{1}{2d} (x - x_*) + \frac{1}{2d} \frac{1}{16} \| x - x_* \| O_1(1).
\]

Proof. For brevity of notation, let \( \Lambda_{d,i,x} = \prod_{i=j}^k W_{i,+}. \) We have

\[
\Lambda_{d,1,x}^T (\Lambda_{d,1,x} x - \Lambda_{d,1,x_*} x_*) \\
= \Lambda_{d,1,x}^T \left[ \Lambda_{d,1,x} x - \sum_{j=1}^d (\Lambda_{d,j,x} \Lambda_{j-1,1,x_*} x_*) + \sum_{j=1}^d (\Lambda_{d,j,x} \Lambda_{j-1,1,x_*} x_*) - \Lambda_{d,1,x_*} x_* \right] \\
= \Lambda_{d,1,x}^T \Lambda_{d,1,x} (x - x_*) + \Lambda_{d,1,x}^T \sum_{j=1}^d (\Lambda_{j-1,1,x_*} x_*) (W_{j,+,x} - W_{j,+,x_*}) \Lambda_{j-1,1,x_*} x_*.
\]

For \( T_1 \), we have

\[
T_1 = \frac{1}{2d} (x - x_*) + \frac{4d}{2^d} \| x - x_* \| O_1(\epsilon). \quad \text{[HV18, (10)]}
\]

\[\square\]
For $T_2$, we have

$$T_2 = O_1(1) \sum_{j=1}^{d} \left( \frac{1}{2^{d-j}} + \frac{(4d - 2j)e}{2^{d-j}} \right) \| (W_{j,+} - W_{j,+}x_*) \Lambda_{j-1,1,x_*} \|
$$

$$= O_1(1) \sum_{j=1}^{d} \left( \frac{1}{2^{d-j}} + \frac{(4d - 2j)e}{2^{d-j}} \right) \| (\Lambda_{j-1,1,x}x - \Lambda_{j-1,1,x_*}x) \| \sqrt{2(\theta_{i,x,x_*} + 2e)}
$$

$$= O_1(1) \frac{1.2}{2^j} \| x - x_* \| \frac{1}{30 \sqrt{2d}}
$$

$$= \frac{1}{16} \frac{1}{2^d} \| x - x_* \| O_1(1). \tag{71}$$

where the first equation is by [HV18, (10)]; the second equation is by (66); the third equation is by Lemma 21 and (68). The result follows from (69), (70) and (71).

Now, we are ready to prove Lemma (14). For brevity of notation, let $\Lambda_{j,z} = \prod_{i=j}^{1} W_{i,+z}$. Using Lemma 22 yields

$$\| \vec{v}_x - \frac{1}{2^d}(x - x_*) \| \leq 1 \frac{1}{2^d} \frac{1}{16} \| x - x_* \| .$$

It follows that

$$\| \vec{v}_x - \frac{1}{2^d}(x - x_*) \| = \| \vec{v}_x + \vec{q}_x - \frac{1}{2^d}(x - x_*) \| \approx \frac{1}{2^d} \frac{1}{16} \| x - x_* \| + \frac{1}{2d} \omega.$$

For any $x \neq 0$ and for any $v \in \mathcal{C}(x)$, by (17), there exist $c_1, c_2, \ldots, c_t \geq 0$ such that $c_1 + c_2 + \ldots + c_t = 1$ and $v = c_1 v_1 + c_2 v_2 + \ldots + c_t v_t$. It follows that $\| v - \frac{1}{2^d}(x - x_*) \| \leq \sum_{j=1}^{t} c_j \| v_j - \frac{1}{2^d}(x - x_*) \| \leq \frac{1}{2^d} \frac{1}{16} \| x - x_* \| + \frac{1}{2d} \omega.$