Note on the Painlevé V tau-functions

Yu. P. Bibilo, R. R. Gontsov

Abstract
We study some properties of tau-functions of an isomonodromic deformation leading to the fifth Painlevé equation. In particular, here is given an elementary proof of Miwa’s formula for the logarithmic differential of a tau-function.

1 Introduction
This work is an addition to the article [1], where we studied some properties of the Malgrange isomonodromic deformation of a linear differential $(2 \times 2)$-system defined on the Riemann sphere $\mathbb{C}$ and having at most two irregular singularities of Poincaré rank one. Here we consider a particular case of such a system:

$$\frac{dy}{dz} = \left( \frac{B_0}{z} + \frac{B_1}{z - t_0} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) y,$$

(1)

where $y(z) \in \mathbb{C}^2$ and $B_0, B_1$ are $(2 \times 2)$-matrices. This system has two Fuchsian singular points $z = 0, z = t_0 \in \mathbb{C} \setminus \{0\}$ and one non-resonant irregular singularity $z = \infty$ of Poincaré rank one.

As known (see [10, §§10,11] or [3, §21]), in a neighbourhood of (non-resonant) irregular singularity $z = \infty$ the system (1) is formally equivalent to a system

$$\frac{d\tilde{y}}{dz} = \left( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{A}{z} \right) \tilde{y},$$

where $A$ is a diagonal $(2 \times 2)$-matrix. This means that there is an invertible matrix formal Taylor series $\hat{F}(z)$ in $1/z$ beginning with the identity matrix such that these two systems are connected by means of the transformation $y = \hat{F}(z)\tilde{y}$ (and such a series $\hat{F}$ is unique). Thus, the system (1) possesses a uniquely determined formal fundamental matrix $\hat{Y}(z)$ of the form $\hat{Y}(z) = \hat{F}(z) e^{\text{diag}(z,0)}$.

According to Sibuya’s sectorial normalization theorem (see [3, Th. 21.13, Prop. 21.17]), a punctured neighbourhood of the point $z = \infty$ is covered by two sectors

$$S_1 = \left\{ \frac{\pi}{2} - \varepsilon < \arg z < \frac{3\pi}{2} + \varepsilon, \quad |z| > R \right\}, \quad S_2 = \left\{ -\frac{\pi}{2} - \varepsilon < \arg z < -\frac{\pi}{2} + \varepsilon, \quad |z| > R \right\},$$

with a sufficiently small $\varepsilon > 0$ and sufficiently large $R > 0$, such that in each $S_i$ there exists a unique actual fundamental matrix

$$Y_i(z) = F_i(z) e^{\text{diag}(z,0)}$$

of the system (1) whose factor $F_i(z)$ has the asymptotic expansion $\hat{F}(z)$ in $S_i$. 

The intersection $S_1 \cap S_2$ is a union of two sectors $\Sigma_1, \Sigma_2$,

$$
\Sigma_1 = \left\{ \frac{\pi}{2} - \varepsilon < \arg z < \frac{\pi}{2} + \varepsilon, \quad |z| > R \right\}, \quad \Sigma_2 = \left\{ \frac{3\pi}{2} - \varepsilon < \arg z < \frac{3\pi}{2} + \varepsilon, \quad |z| > R \right\}.
$$

In the sector $\Sigma_1$ the fundamental matrices $Y_1$ and $Y_2$ necessarily differ by a constant invertible matrix:

$$
Y_2(z) = Y_1(z)C_1, \quad C_1 \in \text{GL}(2, \mathbb{C}), \quad z \in \Sigma_1.
$$

In the sector $\Sigma_2$, one similarly has

$$
Y_1(z) = Y_2(z)C_2, \quad C_2 \in \text{GL}(2, \mathbb{C}), \quad z \in \Sigma_2.
$$

The matrices $C_1, C_2$ are called (Sibuya’s) Stokes matrices of the system (1) at the non-resonant irregular singular point $z = \infty$ (more precisely, the second Stokes matrix is $C_2 e^{-2\pi i A}$).

Further we will focus on isomonodromic deformations of the system (1) which are closely related to the fifth Painlevé equation. According to M. Jimbo [4] such an isomonodromic deformation is a family

$$
\frac{dy}{dz} = \left( \frac{B_0(t)}{z} + \frac{B_1(t)}{z-t} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) y, \quad B_{0,1}(t) = B_{0,1}^0,
$$

(2)

of differential systems holomorphically depending on the parameter $t \in D(t_0)$, where $D(t_0) \subset \mathbb{C}$ is a neighbourhood of the point $t_0$ and the matrix functions $B_0(t), B_1(t)$ are determined by the integrability condition $d\Omega = \Omega \wedge \Omega$ for the matrix meromorphic differential 1-form

$$
\Omega = \left( \frac{B_0(t)}{z} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) dz + \frac{B_1(t)}{z-t} d(z-t)
$$

on the space $\mathbb{C} \times D(t_0)$. The isomonodromy of the family (2) means that a part of its monodromy data is independent of $t$. This part consists of the monodromy matrices corresponding to some fundamental matrix $Y(z, t)$ of (2), Stokes matrices at the infinity and the connection matrix between $Y$ and $Y_1$.

Assuming the eigenvalues of $B_0(t) = (b_{0j}^i(t)), B_1(t) = (b_{1j}^i(t))$ to be $\pm \frac{1}{2} \theta_0, \pm \frac{1}{2} \theta_1 \not\in \frac{1}{2} \mathbb{Z}$ (they do not depend on $t$) and $B_0(t) + B_1(t)$ equivalent to diag($-\frac{1}{2} \theta_0, \frac{1}{2} \theta_0$), one has the function

$$
u(t) = \frac{b_{12}^1(t)(b_{01}^1(t) + \frac{1}{2} \theta_0)}{b_{02}^1(t)(b_{11}^1(t) + \frac{1}{2} \theta_1)}
$$

to satisfy the fifth Painlevé equation

$$
\frac{d^2u}{dt^2} = \left( \frac{1}{2u} + \frac{1}{u-1} \right) \left( \frac{du}{dt} \right)^2 - \frac{1}{t} \frac{du}{dt} + \frac{(u-1)^2}{t^2} \left( \alpha u + \beta \right) + \frac{\gamma u}{t} + \frac{\delta u(u+1)}{u-1},
$$

where

$$
\alpha = \frac{1}{8}(\theta_0 - \theta_1 + \theta_\infty)^2, \quad \beta = -\frac{1}{8}(\theta_0 - \theta_1 - \theta_\infty)^2, \quad \gamma = 1 - \theta_0 - \theta_1, \quad \delta = \frac{1}{2}.
$$

There is also a geometric approach of B. Malgrange [6] to the isomonodromic deformation of the system (1) we referred to in [1], but here we do not immerse in details of that approach in view of the sufficiency of an analytic language for the present note.
According to the Miwa–Malgrange–Helminck–Palmer theorem (see [7], [6, §3], [2] or [9, §3]) the matrix functions $B_0(t), B_1(t)$ holomorphic in $D(t_0)$ can be extended meromorphically to the universal cover $D \cong \mathbb{C}$ of the set $\mathbb{C} \setminus \{0\}$ of locations of the pole $t_0$. The set $\Theta \subset D$ of the poles of the extended matrix functions $B_0, B_1$ (which may be empty) is usually called the Malgrange $\Theta$-divisor of the family (2). For $t^* \in \Theta$, a local $\tau$-function of (2) is any holomorphic in $D(t^*)$ function $\tau^*$ such that $\tau^*(t^*) = 0$. There exists a function $\tau$ (called a global $\tau$-function of the isomonodromic deformation (2) or of the fifth Painlevé equation) holomorphic on the whole space $D$ whose zero set coincides with $\Theta$. Thus, the set $\Theta$ has no limit points in $D$. As follows from the results of [1], all the zeros of this $\tau$-function are simple (at least in the case of the irreducible monodromy of (2)). Here we give an elementary proof of Palmer’s theorem [9] concerning a $\tau$-function of the isomonodromic deformation (2).

**Theorem 1.** A $\tau$-function of the fifth Painlevé equation satisfies the equality

$$d \ln \tau(t) = \frac{1}{2} \text{res}_{z=t} \left( B(z,t) \right)^2 dt,$$

where $B(z,t)$ denotes the coefficient matrix of the family (2).

The above formula is also referred to as a definition of a $\tau$-function (see [5], [4]). Having this definition T. Miwa [7] has proved the analyticity of such a $\tau$-function on $D$ (then its zeros are a priori included in $\Theta$).

Another definition of a $\tau$-function of the Painlevé V (and of the other Painlevé equations) comes from a Hamiltonian form of this equation. As K. Okamoto has shown [8], there is a function $\tau$ holomorphic on $D$ whose logarithmic derivative is equal to a Hamiltonian along a solution (this $\tau$-function depends on a solution, its zeros are included in the set of poles of a solution).

2 Proof of Theorem 1

Consider a point $t^* \in \Theta$ of the Malgrange $\Theta$-divisor of (2). Assuming that $t^* \neq -1$ we make the transformation $\xi = 1/(z + 1)$ of the independent variable and come from (2) to the isomonodromic family

$$\frac{d\tilde{y}}{d\xi} = \left( \frac{B_0(t(s))}{\xi - 1} + \frac{B_1(t(s))}{\xi - s} - \frac{1}{\xi^2} \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) - \frac{B_0(t(s)) + B_1(t(s))}{\xi} \right) \tilde{y}, \quad \tilde{y}(\xi) = y(z(\xi)), \quad (3)$$

depending on the parameter

$$s = \frac{1}{t+1} \quad \left( \Rightarrow t = \frac{1-s}{s} \right),$$

with the Fuchsian singular points $\xi = 1, \xi = s$ and non-resonant irregular singularity $\xi = 0$ of Poincaré rank 1. The infinity is a non-singular point of (3), therefore this family is a particular case of those considered in the paper [11]. Using some properties of such families obtained in that paper we will prove Miwa’s formula for a local $\tau$-function of (3) in a neighbourhood of the point $s^* = 1/(t^* + 1) \in s(\Theta)$. Namely, the following statement holds whose proof is presented in the next section.

**Lemma 1.** There is a local $\tau$-function $\tilde{\tau}(s)$ of the family (3) near $s^*$ such that

$$d \ln \tilde{\tau}(s) = \frac{1}{2} \text{res}_{\xi=s} \left( \tilde{B}(\xi,s) \right)^2 ds,$$
where $\bar{B}(\xi, s)$ is the coefficient matrix of this family.

Having Lemma 1 we conclude for the local $\tau$-function $\tilde{\tau}(s(t))$ of (2) near $t^*$ that

$$d \ln \tilde{\tau}(s(t)) = \frac{1}{2} \text{res}_{z=t} \left( B(\xi, s) \right)^2 (-s^2) dt.$$

To connect $\text{res}_{z=t} \left( B(\xi, s) \right)^2 (-s^2)$ and $\text{res}_{z=t} \left( B(z, t) \right)^2$, let us compute the both. Since

$$\text{res}_{z=t} \left( B(z, t) \right)^2 = \frac{B_0 B_1 + B_1 B_0}{s} + B_1 \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) + \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) B_1,$$

one has

$$\frac{1}{2} \text{res}_{z=t} \left( B(z, t) \right)^2 = \text{tr} \left( \frac{B_0 B_1}{t} + B_1 \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) \right).$$

In a similar way,

$$\text{res}_{z=s} \left( B(\xi, s) \right)^2 = \frac{B_0 B_1 + B_1 B_0}{s - 1} - \frac{B_0 B_1 + B_1 B_0 + 2B_1^2}{s} - \frac{1}{s^2} \left( B_1 \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) + \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) B_1 \right),$$

and

$$\frac{1}{2} \text{res}_{z=s} \left( B(\xi, s) \right)^2 (-s^2) = \frac{s^2}{1 - s} \text{tr}(B_0 B_1) + s \text{tr}(B_0 B_1) + s \text{tr} B_1^2 + \text{tr} \left( B_1 \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) \right) = \frac{s}{1 - s} \text{tr}(B_0 B_1) + s \text{tr} B_1^2 + \text{tr} \left( B_1 \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) \right) = \text{tr} \left( \frac{B_0 B_1}{t} + B_1 \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) \right) + \frac{B_1^2}{t + 1}.$$

Therefore,

$$d \ln \tilde{\tau}(s(t)) = \frac{1}{2} \text{res}_{z=t} \left( B(z, t) \right)^2 dt + \text{tr} \frac{B_1^2}{t + 1} dt.$$

Denoting now $B_1(t) = (b_{ij}(t))$ we have

$$\text{tr} B_1^2 = (b_{11})^2 + 2b_{12}b_{21} + (b_{22})^2 = (b_{11} + b_{22})^2 - 2(b_{11}b_{22} - b_{12}b_{21}) = (\text{tr} B_1)^2 - 2 \det B_1 = \theta^2/2 = \text{const}$$

(recall that the eigenvalues of the matrix $B_1(t)$ are $\pm \theta_1/2$ not depending on $t$). Hence,

$$d \ln \tilde{\tau}(s(t)) = \frac{1}{2} \text{res}_{z=t} \left( B(z, t) \right)^2 dt.$$

Thus, near a point $t^* \in \Theta$ we have the formula

$$d \ln \tau^*(t) = \frac{1}{2} \text{res}_{z=t} \left( B(z, t) \right)^2 dt$$

for the local $\tau$-function $\tau^*(t) = \tilde{\tau}(s(t))/(t + 1)^{\theta^2/2}$ of (2).

Further we use standard reasonings to come from the local $\tau$-functions to a global one and finish the proof of the theorem. Consider a covering $\{U_\alpha\}$ of the deformation space $D$ such that
for every $U_\alpha$ there is a holomorphic function $\tau_\alpha(t)$ satisfying the equality (4) and non-vanishing in $U_\alpha$ if $U_\alpha \cap \Theta = \emptyset$. In every non-empty intersection $U_\alpha \cap U_\beta$ one has

$$\tau_\alpha(t) = c_{\alpha\beta} \tau_\beta(t), \quad (5)$$

where $c_{\alpha\beta} = \text{const} \neq 0$, since $d \ln c_{\alpha\beta} = d \ln \tau_\alpha - d \ln \tau_\beta = 0$. The equalities (5) imply

$$c_{\alpha\beta} c_{\beta\gamma} = c_{\alpha\gamma} \quad (6)$$

for non-empty intersections $U_\alpha \cap U_\beta \cap U_\gamma$.

Let us fix logarithms $l_{\alpha\beta} = \ln c_{\alpha\beta}$ in such a way that $l_{\alpha\beta} = -l_{\beta\alpha}$. Then, as follows from (6),

$$l_{\alpha\beta} - l_{\alpha\gamma} + l_{\beta\gamma} = 2\pi i l'_{\alpha\beta},$$

where the set of numbers $l_{\alpha\beta\gamma} \in \mathbb{Z}$ defines an element of the Čech cohomology group $H^2(D, \mathbb{Z})$. Since $D \cong \mathbb{C}$, one has $H^2(D, \mathbb{Z}) = 0$, hence there is a set $\{l'_{\alpha\beta}\} \subset \mathbb{Z}$ such that

$$l_{\alpha\beta\gamma} = l'_{\alpha\beta} - l'_{\alpha\gamma} + l'_{\beta\gamma} \quad \text{for} \quad U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset.$$ 

Therefore

$$(l_{\alpha\beta} - 2\pi i l'_{\alpha\beta}) - (l_{\alpha\gamma} - 2\pi i l'_{\alpha\gamma}) + (l_{\beta\gamma} - 2\pi i l'_{\beta\gamma}) = 0,$$

and the set of numbers $\lambda_{\alpha\beta} = l_{\alpha\beta} - 2\pi i l'_{\alpha\beta} \in \mathbb{C}$ defines an element of the Čech cohomology group $H^1(D, \mathbb{C})$. Since the latter is trivial, there is a set $\{\lambda_{\alpha}\} \subset \mathbb{C}$ such that

$$\lambda_{\alpha\beta} = \lambda_{\alpha} - \lambda_{\beta} \quad \text{for} \quad U_\alpha \cap U_\beta \neq \emptyset.$$ 

Thus, a set $\{c_{\alpha} = e^{\lambda_{\alpha}}\}$ of non-zero constants is such that

$$c_{\alpha\beta} = e^{l_{\alpha\beta}} = e^{\lambda_{\alpha\beta}} = c_{\alpha} c_{\beta}^{-1}$$

in every non-empty intersection $U_\alpha \cap U_\beta$. Therefore we have a global function $\tau(t)$ holomorphic on $D$ and equal to $c_{\alpha}^{-1} \tau_\alpha(t)$ in every $U_\alpha$, hence satisfying (4).

### 3 Proof of Lemma 1

Consider a point $s^* \in s(\Theta)$. Though the family (3) is not defined for this value of the parameter, one can construct an auxiliary linear meromorphic ($2 \times 2$)-system

$$\frac{dw}{d\xi} = A^*(\xi) w, \quad A^*(\xi) = \frac{A_0^*}{\xi} + \frac{A_0^*}{\xi^2} + \frac{A_1^*}{\xi - 1} + \frac{A_2^*}{\xi - s^*}, \quad (7)$$

with irregular non-resonant singular point $\xi = 0$ of Poincaré rank 1 and Fuchsian singular points 1, $s^*$. In a neighbourhood of zero this system is holomorphically equivalent to the image of the initial system (1) under the change of variable $\xi = 1/(z + 1)$, has the same monodromy matrices as the latter, but it has an apparent Fuchsian singularity at the infinity (i.e., the monodromy at this point is trivial).

We will use the following facts explained in [1]:

5
the auxiliary system (7) is included into a (Malgrange) isomonodromic family
\[
\frac{dw}{d\xi} = \left( \frac{A_{01}(s)}{\xi} + \frac{A_{02}(s)}{\xi^2} + \frac{A_1(s)}{\xi - 1} + \frac{A_2(s)}{\xi - s} \right) w
\]
whose isomonodromic fundamental matrix \( W(\xi, s) \) near the point \( \xi = \infty \) has the form
\[
W(\xi, s) = U(\xi, s)\xi^K, \quad U(\xi, s) = I + U_1(s)\frac{1}{\xi} + U_2(s)\frac{1}{\xi^2} + \ldots,
\]
where \( K = \text{diag} \left(-1, 1\right) \) and \( \frac{dU_1(s)}{ds} = -A_2(s) \);

- the upper right element \( u_1(s) \) of the matrix \( U_1(s) \) vanishes at the point \( s = s^* \) and is not equal to zero identically, hence it is a local \( \tau \)-function of the family (8) (and \( \frac{du_1(s)}{ds}(s^*) \neq 0 \) if the monodromy of (8) is irreducible);

- for \( s \neq s^* \) the family (8) is meromorphically equivalent to (3) via a gauge transformation \( \tilde{y} = \Gamma_1(\xi, s)w \), where
\[
\Gamma_1(\xi, s) = \left( \begin{array}{cc} f(s) + \xi & -u_1(s) \\ u_1(s) & 0 \end{array} \right),
\]
for some function \( f \) holomorphic at the point \( s = s^* \).

The coefficient matrices \( \tilde{B}(\xi, s) \) and \( A(\xi, s) \) of the families (3) and (8) respectively are connected by the equality
\[
\tilde{B}(\xi, s) = \frac{\partial\Gamma_1}{\partial\xi} \Gamma_1^{-1} + \Gamma_1 A(\xi, s) \Gamma_1^{-1},
\]
therefore
\[
\left( \tilde{B}(\xi, s) \right)^2 = \left( \frac{\partial\Gamma_1}{\partial\xi} \Gamma_1^{-1} \right)^2 + \frac{\partial\Gamma_1}{\partial\xi} A(\xi, s) \Gamma_1^{-1} + \Gamma_1 A(\xi, s) \Gamma_1^{-1} \left( \frac{\partial\Gamma_1}{\partial\xi} \Gamma_1^{-1} \right)^2 \Gamma_1^{-1}.
\]
As follows from the form of the matrix \( \Gamma_1(\xi, s) \), the product \( \left( \frac{\partial\Gamma_1}{\partial\xi} \right) \Gamma_1^{-1} \) is of the form
\[
\left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) \left( \begin{array}{c} 0 \\ -\frac{1}{u_1} \end{array} \right) = \left( \begin{array}{cc} 0 & u_1 \\ 0 & 0 \end{array} \right),
\]
hence its square is the zero matrix and
\[
\text{tr}\left( \tilde{B}(\xi, s) \right)^2 = 2 \text{tr}\left( \Gamma_1^{-1} \frac{\partial\Gamma_1}{\partial\xi} A(\xi, s) \right) + \text{tr}\left( A(\xi, s) \right)^2.
\]
Since
\[
\Gamma_1^{-1} \frac{\partial\Gamma_1}{\partial\xi} = \left( \begin{array}{cc} 0 & 0 \\ -\frac{1}{u_1} & 0 \end{array} \right),
\]
one has
\[
\text{tr}\left( \tilde{B}(\xi, s) \right)^2 = -2 \frac{a(\xi, s)}{u_1(s)} + \text{tr}\left( A(\xi, s) \right)^2,
\]
where \( a(\xi, s) \) is the upper right element of the matrix \( A(\xi, s) \). Thus,
\[
\text{res}_{\xi=s} \text{tr}\left( \tilde{B}(\xi, s) \right)^2 ds = -2 \frac{\text{res}_{\xi=s} a(\xi, s)}{u_1(s)} ds + \text{res}_{\xi=s} \text{tr}\left( A(\xi, s) \right)^2 ds.
\]
As \( \text{res}_{\xi=s}a(\xi, s) \) is equal to the upper right element of \( A_2(s) \), which is \(-\frac{du_1}{ds}\), one has

\[
\text{res}_{\xi=s}\text{tr}\left(\tilde{B}(\xi, s)\right)^2 ds = 2 d\ln u_1(s) + \text{res}_{\xi=s}\text{tr}\left(A(\xi, s)\right)^2 ds.
\]

The differential 1-form \( \text{res}_{\xi=s}\text{tr}\left(A(\xi, s)\right)^2 ds \) is closed and holomorphic in a neighbourhood \( D(s^*) \) of the point \( s = s^* \), hence there is a function \( f^* \) holomorphic and non-vanishing in \( D(s^*) \) such that \( d\ln f^*(s) = \frac{1}{2}\text{res}_{\xi=s}\text{tr}\left(A(\xi, s)\right)^2 ds \). Therefore,

\[
\text{res}_{\xi=s}\text{tr}\left(\tilde{B}(\xi, s)\right)^2 ds = 2 d\ln(f^*(s)u_1(s)),
\]

which finishes the proof of the lemma, with \( \tilde{\tau}(s) = f^*(s)u_1(s) \).

Acknowledgements. This work is supported by the Russian Foundation for Basic Research (grant no. RFBR-14-01-31145 mol_a) and RF President program for young scientists (grant no. MK-4594.2013.1).

References

[1] Yu. P. Bibilo, R. R. Gontsov, Some properties of Malgrange isomonodromic deformations of linear \( 2 \times 2 \) systems, Proc. Steklov Inst. Math. 277 (2012), 16–26.

[2] G. Helminck, Deformations of connections, the Riemann–Hilbert problem and \( \tau \)-functions, Computational and combinatorial methods in system theory (Stockholm 1985), North-Holland, Amsterdam, 1986, 75–89.

[3] Yu. S. Ilyashenko, S. Yakovenko, "Lectures on analytic differential equations", AMS Publ. Graduate Studies in Math. 86, 2008.

[4] M. Jimbo, Monodromy problem and the boundary condition for some Painlevé equations, Publ. RIMS, Kyoto Univ. 18 (1982), 1137–1161.

[5] M. Jimbo, T. Miwa, K. Ueno Monodromy preserving deformation of linear differential equations with rational coefficients. I, Physica D 2D:2 (1981), 306–352.

[6] B. Malgrange, Sur les déformations isomonodromiques. II. Singularités irrégulières, Progr. Math. 37 (1983), 427–438.

[7] T. Miwa, Painlevé property of monodromy preserving deformation equations and the analyticity of \( \tau \)-functions, Publ. RIMS, Kyoto Univ. 17 (1981), 703–721.

[8] K. Okamoto, On the \( \tau \)-function of the Painlevé equations, Physica D 2D:3 (1981), 525–535.

[9] J. Palmer, Zeros of the Jimbo, Miwa, Ueno tau function, J. Math. Phys. 40:12 (1999), 6638–6681.

[10] W. Wasow, "Asymptotic expansions for ordinary differential equations", John Wiley & Sons, New-York–London–Sydney, 1965.