On the description of surface operators in $\mathcal{N} = 2^*$ SYM

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Abstract

In Ref. [1], Alday and Tachikawa observed that the Nekrasov partition function of $\mathcal{N} = 2$ $SU(2)$ superconformal gauge theories in the presence of fundamental surface operators can be associated to conformal blocks of a 2D CFT with affine $sl(2)$ symmetry. This can be interpreted as the insertion of a fundamental surface operator changing the conformal symmetry from the Virasoro symmetry discovered in [2] to the affine Kac-Moody symmetry. A natural question arises as to how such a 2D CFT description can be extended to the case of non-fundamental surface operators. Motivated by this question, we review the results of Refs. [3] and [4] and put them together to suggest a way to address the problem: It follows from this analysis that the expectation value of a non-fundamental surface operator in the $SU(2) \mathcal{N} = 2^*$ super Yang-Mills theory would be in correspondence with the expectation value of a single vertex operator in a two-dimensional CFT with reduced affine symmetry and whose central charge is parameterized by the integer number that labels the type of singularity of the surface operator.
1 Introduction

Of fundamental importance in theoretical physics is the question about non-perturbative effects in Yang-Mills theory (YM). In the last two decades, there has been important progress in this area, mainly due to our current understanding of the supersymmetric extensions of the theory. In the last few years, one of the most promising advances in the direction of understanding non-perturbative effects of supersymmetric YM theories has been the observation, due to Alday, Gaiotto, and Tachikawa [2], that the Nekrasov partition functions [5] of certain class of $\mathcal{N} = 2$ superconformal $SU(2)$ quiver theories in four dimensions are given by the conformal blocks of Liouville field theory. According to this, the full partition function of such gauge theories, meaning the partition function including instanton corrections, would be abstrusely encoded in the building blocks of a relatively well understood two-dimensional conformal field theory (CFT).

Specifically, Alday-Gaiotto-Tachikawa conjecture (AGT) states that the $n$-point conformal blocks of Liouville field theory formulated on an $n$-punctured genus-$g$ Riemann surface $C_{g,n}$ give the Nekrasov partition function of the Gaiotto’s quiver theory $T_{g,n}$ that is constructed, as in [6], by compactifying the six-dimensional $(2,0)$ theory of the $A_1$ type on $C_{g,n}$. This provides a very interesting correspondence between 2D conformal field theories and 4D superconformal gauge theories.

Soon after the paper [2] appeared, the extension of the 2D/4D correspondence to the cases in which both loop and surface operators are incorporated on the gauge theory side was proposed [7, 8]. In this generalized picture, not only the partition function, but also the expectation values of defects in the 4D theory happen to be described by Liouville correlation functions. It turns out that, to the insertion of a defect on the 4D gauge theory side, it corresponds the insertion of a degenerate Liouville field in the 2D CFT side.

More recently, it has been observed that the 2D CFT description of expectation values of defects in the gauge theory is naturally realized in terms of CFTs with affine symmetry [11]. For the case of $SU(2)$ gauge theories, this involves CFT with $\hat{sl}(2)_k$ affine Kac-Moody symmetry, with $k = \varepsilon_1/\varepsilon_2 + 2$, with $\varepsilon_{1,2}$ being the Nekrasov’s deformation parameters [5]. In some sense, it is adequate to say that, while Liouville field theory stands as the convenient language to represent the $\mathcal{N} = 2$ gauge theory partition function, the expectation values of defects in such theories are more conveniently described by conformal blocks of 2D CFTs with affine symmetry;
at least it seems to be the case for the simplest defects. One of the motivations of this paper is to propose a way of extending such affine CFT realization to the case of non-fundamental surface operators.

2 Surface operators and CFT

Our current understanding of the problem is that non-fundamental surface operators whose vacua are labeled by an integer number \( m \geq 1 \) exist in these \( SU(2) \mathcal{N} = 2 \) gauge theories, and the expectation value of such a surface operator would admit a 2D CFT description in terms of a Liouville correlation function with the additional insertion of a degenerate field of conformal dimension \( h = -(m/2)(1 + b^{-2}(1 + m/2)) \), namely the vertex \( e^{-m\varphi(x)/b} \). Inconveniently, a purely gauge theory description of the surface operators that correspond to \( m > 1 \) is still missing, and without a complete description of defects from the gauge theory point of view it is worthwhile studying the problem from different perspectives. Here, with the aim of contributing to the study of non-fundamental surface operators in the 4D \( \mathcal{N} = 2 \) SCFTs, we will draw the attention to a yet unexplored CFTs tool developed in Ref. [4]. We will focus our attention on the particular case of \( SU(2) \mathcal{N} = 2^* \) SYM. Invoking the result of [4], or more precisely its genus-one generalization, we will argue that the expectation value of a non-fundamental surface operator (labeled by an integer \( m \)) in the \( \mathcal{N} = 2^* \) theory is given by the expectation value of a single vertex operator in a 2D CFT which has central charge \( c(b,m) = 3 + 6(b^{-1} + (1 - m)b)^2, \) with \( b^2 = \varepsilon_1/\varepsilon_2 \).

Our line of argument is the following: According to the results of Ref. [7], the expectation value of a surface operator in \( SU(2) \mathcal{N} = 2^* \) SYM is associated to the Liouville 2-point function \( \langle e^{2\alpha\varphi}e^{-m\varphi/b} \rangle \) on the torus. On the other hand, a genus-one extension of the result of [4] permits to express such Liouville 2-point function in terms of the expectation value of a single vertex operator \( \langle \Phi_h \rangle \) in a 2D CFT with central charge \( c_{(b,m)} \) given above. In the case \( m = 1 \), which corresponds to fundamental surface operators, the 2D CFT is identified with the \( \hat{sl}(2)_k \) affine theory with level \( k = 2 + 1/b^2 \), suggesting a possible connection with the analysis of [1]. In the case \( m > 1 \), in contrast, such 2D CFT exhibits only part of the affine symmetry, being this part generated by the Borel subalgebra of \( \hat{sl}(2)_{k'} \) with \( k' = 2 + m^2/b^2 \).

As said, the specific CFTs we will consider are those that were introduced by Ribault in
Ref. [4]. It was shown in [4] that the genus-0 \((2n - 2)\)-point Liouville correlation functions that involve \(n - 2\) degenerate fields \(e^{-m\varphi/b}\) with \(m \in \mathbb{Z}_{\geq 0}\) provide the \(n\)-point correlation functions of a non-rational CFT with central charge \(c = 3 + 6(b^{-1} + (1 - m)b)^2\). These CFTs, if they actually exist for generic \(m \in \mathbb{Z}_{>1}\), would coincide with the \(\hat{sl}(2)_k\) WZW theory for \(m = 1\), and with Liouville theory itself for \(m = 0\). The point we want to make here is that other members of this \(m\)-labeled family of CFTs may have application to describe observables in 4D gauge theories as well. Here we will be concerned with the \(\mathcal{N} = 2^*\) SYM theory, and then, according to [7], this corresponds to the 2D CFT being formulated on the torus. Then, we first need to solve a preliminary problem: we need to extend the construction of [4] to genus one, \(g = 1\). This is basically the result of this paper: in what follows we will show that, as probably expected, the torus Liouville \(2n\)-point functions that involve \(n\) degenerate fields \(e^{-\varphi_m/b}\) and \(n\) generic primary fields \(e^{-2\alpha_i\varphi}\) with \(\alpha_i \in \mathbb{C}\) actually coincide with the torus \(n\)-point function of the \(m^{th}\) member of the family of CFTs proposed in [4]. Proving so follows from straightforwardly adapting the path integral techniques developed in Ref. [3] to the \(m > 1\) case. From the CFT point of view, this result is interesting in its own right as it provides further evidence of the consistency of the theories proposed in [4].

We begin in Sections III and IV by reviewing the theories introduced in [4] formulated on the genus-0 surface. In Section V, we extend the result of [4] to genus-1 by using the techniques developed in [3]; this represents a trivial extension of the results therein. Section VI contains the conclusions.

3 A family of conformal field theories

Let us introduce the family of CFTs we will be concerned with. Each member of the family, each CFT, depends on two parameters, \(m\) and \(b\). Here we will consider \(m \in \mathbb{Z}_{\geq 0}\) and \(b \in \mathbb{R}_{>0}\). The action can be written as [4]

\[
S_{(m,b)}[\phi, \beta, \gamma] = \frac{1}{2\pi} \int d^2z \left( \partial\phi \bar{\partial}\phi + \beta \bar{\partial}\gamma + \bar{\beta}\partial\gamma + \frac{Q_{(m,b)}}{4} \mathcal{R}\phi + b^2 (-\beta \bar{\beta})^m e^{2b\phi} \right),
\]

where the background charge takes the value

\[
Q_{(m,b)} = b + \frac{1 - m}{b}.
\]
Let us call \( Y_{m,b} \) the theory defined by the action (1). In (1) \( \mathcal{R} \) represents the scalar curvature of the surface on which the theory is defined. From this action we observe that \( Y_{0,b} \) corresponds to Liouville field theory with central charge \( c = 1 + 6(b + 1/b)^2 \) coupled to a free \( \beta\gamma \) ghost system. On the other hand, theory \( Y_{1,b} \) corresponds to the \( \mathbb{H}_3^+ = SL(2, \mathbb{C})/SU(2) \) WZNW theory with level \( k = b^{-2} + 2 \) expressed in the Wakimoto free-field representation [9]. Theory \( Y_{b,1/b} \) also corresponds to the \( \mathbb{H}_3^+ \) WZNW theory with the Langlands dual level \( k^l = b^{+2} + 2 \) [10, 11].

The stress-tensor associated to the free theory defined by action (1) is given by

\[
T(z) = -\beta(z)\partial\gamma(z) - (\partial\phi(z))^2 + Q_{(m,b)}\partial^2\phi(z) \tag{3}
\]

and by its anti-holomorphic counterpart \( \overline{T}(\overline{z}) \). This gives the central charge of the theory

\[
c_{(m,b)} = 1 + 6Q_{(m,b)}^2. \tag{4}
\]

Last term in the (1) represents a marginal operator with respect to the stress-tensor (3) as it can easily be verified by using the free field propagators.

As mentioned, the theory with \( m = 1 \) corresponds to the WZNW model, which exhibits \( \hat{sl}(2)_k \times \hat{sl}(2)_k \) affine Kac-Moody symmetry. This symmetry is generated by the currents

\[
J^-(z) = \beta(z), \quad J^3(z) = \beta(z)\gamma(z) + b^{-1}\partial\phi(z), \tag{5}
\]

and

\[
J^+(z) = \beta(z)\gamma^2(z) + 2b^{-1}\gamma(z)\partial\phi(z) - \left(b^{-2} + 2\right)\partial\gamma(z), \tag{6}
\]

together with the anti-holomorphic counterparts \( \overline{J}^3(\overline{z}), \overline{J}^\pm(\overline{z}) \), where \( b^{-2} = k - 2 \). In contrast, for the theory with \( m \neq 1 \) only a sub-algebra of (5) - (6) survives and the theory exhibits a remaining symmetry under the Borel subalgebra of \( \hat{sl}(2)_{k'} \) with modified level \( k' = 2 + m^2/b^2 \) generated by the currents

\[
J^-(z) = \beta(z), \quad J^3(z) = \beta(z)\gamma(z) + \frac{m}{b}\partial\phi(z) \tag{7}
\]

and the anti-holomorphic analogues. Currents (7) obey the following operator product expansion

\[
J^-(z)J^3(w) \simeq \frac{J^-(w)}{(z-w)} + \ldots \quad J^3(z)J^3(w) \simeq -\frac{(1 + m^2b^{-2}/2)}{(z-w)^2} + \ldots \quad J^-(z)J^-(w) \simeq \ldots \tag{8}
\]
where the ellipses stand for regular terms that are omitted. This realizes the Lie brackets
\[
\begin{align*}
[J_p^-, J_q^3] &= J_{p+q,0}^-; \\
[J_p^3, J_q^3] &= \frac{k'}{2} \delta_{p+q,0}; \\
[J_p^-, J_q^-] &= 0,
\end{align*}
\]
for the modes defined as usual,
\[
J_p^3 = \frac{1}{2\pi i} \int dz \, z^{-p-1} J^3(z), \quad J_p^- = \frac{1}{2\pi i} \int dz \, z^{-p-1} J^-(z).
\] (9)

The spectrum of the theory would consist of fields that are primary with respect to currents (7). Vertex operators creating such states are of the form
\[
\Phi_h(\mu|z) = |\mu|^{2m(j+1)} e^{\mu \gamma(z) - \bar{\mu} \bar{\gamma}(\bar{z})} e^{2b(j+1)\phi(z, \bar{z})}
\] (10)
whose holomorphic and anti-holomorphic conformal dimensions are given by
\[
h_j = \bar{h}_j = (-b^2 j + 1 - m)(j + 1).\]
In (10), \(\mu\) is a complex variable and \(j\) is an index that can be thought of as the momentum of the field. The spectrum of normalizable states of the theory would be ultimately determined by the values that \(j\) takes. Fields \(\Phi_h(\mu|z)\) should also include an overall factor \(|\omega(z, \bar{z})|^{2h}\) with \(\omega(z, \bar{z})\) being the Weyl factor of the two-dimensional metric \(ds^2 = |\omega(z, \bar{z})|^2 dz d\bar{z}\) on the surface. Such dependence of the conformal factor is required for \(\Phi_j(\mu|z)\) to transform as a primary \((h_j, \bar{h}_j)\)-dimension operator. For short, we will set \(\omega(z, \bar{z}) = 1\) in formulae below.

4 Genus-zero correlation functions

Let us begin by defining the theory defined by (1) on the sphere topology. More precisely, let us calculate the genus-zero correlation functions
\[
\Omega_{(m,b)}^{\gamma=0, n}(\mu_\nu|z_\nu) \equiv \left\langle \prod_{\nu=1}^n \Phi_h(\mu_\nu|z_\nu) \right\rangle = \int \mathcal{D}\phi \mathcal{D}^2 \beta \mathcal{D}^2 \gamma e^{-S_{(m,b)}[\phi, \beta, \gamma]} \prod_{\nu=1}^n \Phi_h(\mu_\nu|z_\nu),
\] (11)
where the expectation value is understood as the correlation function of primary operators (11) in the theory \(Y_{m,b}\) formulated on the \(n\)-puncture sphere \(\mathcal{C}_{0,n}\).

Functional integration over the fields \(\gamma\) and \(\bar{\gamma}\) yields \(\delta\)-functions that fix the conditions
\[
\partial \beta(w) = 2\pi \sum_{\nu=1}^n \mu_\nu \delta^2(w - z_\nu), \quad \partial \bar{\beta}(\bar{w}) = -2\pi \sum_{\nu=1}^n \bar{\mu}_\nu \delta^2(\bar{w} - \bar{z}_\nu).
\] (12)
These equations have solution only if \( \sum_{\nu=1}^{n} \mu_{\nu} = 0 \). Having in mind that \( \bar{\partial} (1/z) = \partial (1/\bar{z}) = 2\pi \delta^2(z) \), we write the most general solution to (12) in the form \[3\]

\[
\beta(w) = \sum_{\nu=1}^{n} \mu_{\nu} (w - z_{\nu})^{-1} = u \prod_{i=1}^{n-2} (w - y_{i}) \prod_{i=1}^{n} (w - z_{\nu}),
\]

with the following relation

\[
\mu_{\nu} = u \prod_{i=1}^{n-2} (z_{\nu} - y_{i}) \prod_{i \neq \nu} (z_{\nu} - z_{\mu}) / \prod_{\nu=1}^{n} (w - z_{\nu}),
\]

where a new variable \( u \) and \( n - 1 \) variables \( y_{i} \) have been introduced. (14) are \( n - 1 \) equations that permit to express the \( n - 1 \) independent variables \( \mu_{\nu} \) in terms of the \( n - 2 \) variables \( y_{i} \) and the variable \( u \). This permits to integrate over \( \beta \) and \( \bar{\beta} \) and obtain

\[
\Omega^{g=0,n}_{(m,b)} (\mu_{\nu} | z_{\nu}) = \int D\phi \ e^{-S_{\text{eff}}[\phi, \mathcal{X}_0]} \prod_{\nu=1}^{n} |\mu_{\nu}|^{2m(j_{\nu}+1)} e^{2b(j_{\nu}+1)\phi(z_{\nu})}.
\]

with the effective action

\[
S_{\text{eff}}[\phi, \mathcal{X}_0] = \frac{1}{2\pi} \int d^{2}z \left( \partial \phi \bar{\partial} \phi + \frac{Q_{(m,b)}}{4} R \phi + b^{2} |u|^{2m} |\mathcal{X}_0|^{2m} e^{2b\phi} \right)
\]

with

\[
\mathcal{X}_0(y_{i}, z_{\nu}; w) \equiv \prod_{i=1}^{n-2} (w - y_{i}) / \prod_{\nu=1}^{n} (w - z_{\nu}),
\]

see [3] for details.

The next step is to massage expression (16) to bring it into its Liouville form. To achieve so, one first performs the shifting \( \phi(w, \bar{w}) \rightarrow \phi(w, \bar{w}) - (m/b) \log |u| \), and arrives to

\[
\Omega^{g=0,n}_{(m,b)} (\mu_{\nu} | z_{\nu}) = |u|^{2m(1+(1-m)/b)} \int D\phi \ e^{-S_{\text{eff}}[\phi, \mathcal{X}_1]} \prod_{\nu=1}^{n} |\mu_{\nu}|^{2m(j_{\nu}+1)} |u|^{-2m(j_{\nu}+1)} e^{2b(j_{\nu}+1)\phi(z_{\nu})}.
\]

Then, defining the new variable

\[
\phi \equiv \varphi - \frac{m}{2b} \left( \sum_{i=1}^{n-2} \log |w - y_{i}|^{2} - \sum_{\nu=1}^{n} \log |w - z_{\nu}|^{2} - \log |\omega(w,w)|^{2} \right),
\]

and taking into account the powers of the conformal factor \( |\omega(z,\bar{z})|^{2} \) generated in the regularized coincident limit \( w \rightarrow z_{\nu} \), one finds that the background charge changes as \( Q_{(m,b)} \rightarrow Q_{(0,b)} = b + 1/b \). The latter corresponds to the Liouville background charge.
Finally, setting the overall factor \( |u|^{2m(1+(1-m)/b^2)} \) to one, one finds the expression obtained in \([1]\); namely
\[
\Omega_{(m,b)}^{\mu_\nu} |z_{\nu}| = \left\langle \prod_{\nu=1}^{n} e^{i \frac{m}{b} X(z_{\nu})} \prod_{i=1}^{n-2} e^{-i \frac{m}{b} X(y_{i})} \right\rangle_{X} \left\langle \prod_{\nu=1}^{n} V_{\alpha_{\nu}}(z_{\nu}) \prod_{i=1}^{n-2} V_{\frac{-m}{b}}(y_{i}) \right\rangle_{L} \tag{19}
\]
which is subject to conditions (14), in particular to the condition \( \sum_{\nu=1}^{n} \mu_{\nu} = 0 \). On the right hand side of (19), the Liouville correlation functions are given by
\[
\left\langle \prod_{\nu=1}^{n} V_{\alpha_{\nu}}(z_{\nu}) \prod_{i=1}^{n-2} V_{\frac{-m}{b}}(y_{i}) \right\rangle_{L} = \int \mathcal{D} \varphi e^{-S_{L}[\varphi]} \prod_{\nu=1}^{n} e^{2 \alpha_{\nu} \varphi(z_{\nu})} \prod_{i=1}^{n-2} e^{-\varphi(y_{i})} \tag{20}
\]
with the Liouville action
\[
S_{L}[\varphi] = \frac{1}{2\pi} \int d^{2}z \left( \partial \varphi \bar{\partial} \varphi + \frac{1}{4} (b + b^{-1}) \sqrt{g} R \varphi + b^{2} e^{2b \varphi} \right)
\]
and with momenta \( \alpha_{\nu} = b(j_{\nu} + 1 + b^{-2}/2) \). The overall factor is of the form
\[
\left\langle \prod_{\nu=1}^{n} e^{i \frac{m}{b} X(z_{\nu})} \prod_{i=1}^{n-2} e^{-i \frac{m}{b} X(y_{i})} \right\rangle_{X} = \prod_{\mu < \nu} |z_{\mu} - z_{\nu}|^{m^{2}b^{-2}} \prod_{i<j} |y_{i} - y_{j}|^{m^{2}b^{-2}} \prod_{\mu=1}^{n} \prod_{i=1}^{n-2} |z_{\mu} - y_{i}|^{-m^{2}b^{-2}}, \tag{21}
\]
which can be interpreted as the correlation function of a free boson \( X(z) \) with non-trivial background charge \( \hat{Q} = im/b \). That is, equation (19) can be thought of as expressing the equivalence between \( n \)-point correlation functions of the theory defined by action (1) and \( (2n-2) \)-point correlation functions of a theory composed by Liouville theory times a CFT with central charge \( c = 1 - 6m^{2}/b^{2} \). Expression (19) generalizes the relation between between the \( \mathbb{H}_{3}^{+} \) WZNW theory and Liouville field theory derived by Stoyanovsky [12] and by Ribault and Teschner [13], which is reobtained by replacing \( m = 1 \) in the formulae above. Here we have reviewed the derivation of (19) given by Hikida and Schomerus in Ref. [3], which, as we will see in the next Section, can be generalized to genus-one.

5 Genus-one correlation functions

Now, let us consider the theories (1) on the genus-one surface. We will follow the analysis of Ref. [3], adapting it to the case \( m > 1 \).
As usual, the torus is represented by the complex plane with periodic conditions under translations \( w \to w + 1 \) and \( w \to w + \tau \). The complex variable \( \tau = \tau_1 + i\tau_2 \) is the modular parameter of the torus. To fully parameterize the consistent boundary conditions, it is also necessary to introduce an additional parameter \( \lambda \) which amounts to consider twisted periodicity conditions

\[
\beta(w + p + q\tau) = e^{2\pi i q \lambda} \beta(w), \quad \gamma(w + p + q\tau) = e^{-2\pi i q \lambda} \gamma(w),
\]

and

\[
\phi(w + p + q\tau, \bar{w} + p + q\bar{\tau}) = \phi(w, \bar{w}) + \frac{2\pi mq \text{Im} \lambda}{b},
\]

where \( \text{Im} \lambda = \lambda_2 \) stands for the imaginary part of the twist parameter \( \lambda = \lambda_1 + i\lambda_2 \), and \( p \) and \( q \) are two arbitrary integer numbers that parameterize the steps on the lattice. The possibility of choosing conditions (22)-(23), even when they yield multivalued fields for \( \lambda \neq 0 \), comes from the fact that action (1) does remain univalued. For \( \lambda = 0 \), untwisted boundary conditions are recovered. For \( m = 0 \) the field \( \phi \) must be periodic; it acquires more freedom in the case \( m \neq 0 \) and such is parameterized by \( \lambda \), which labels different twist sectors. For \( m = 1 \), \( \lambda \) is identified with the Benard parameter that appears in the Bernard-Knizhnik-Zamolodchikov modular differential equation; see [3] for a detailed discussion. For \( m > 1 \), as long as \( m \neq b^2 \), the theory does not exhibit the full \( \hat{sl}(2)_k \) affine symmetry; nevertheless, \( \lambda \) may still be introduced.

The next step, before integrating over \( \beta \) and \( \gamma \), is to decompose the field \( \phi \) into its solitonic zero-mode part

\[
\phi_c(w, \bar{w}) = \frac{2\pi m \text{Im}(\lambda) \text{Im}(w)}{\text{Im}(\tau)}.
\]

and the fluctuations \( \phi_f \); namely \( \phi(w, \bar{w}) = \phi_c(w, \bar{w}) + \phi_f(w, \bar{w}) \). Solitonic configuration (24), together with \( \beta = 0 \) and \( \gamma = 0 \), represents the only solution to the classical equations of motion coming from the action (1) that satisfies the required periodic boundary conditions. The piece \( \phi_f \) is periodic under \( w \to w + 1 \) and \( w \to w + \tau \). We are now on the torus, so only the fluctuations \( \phi_f \) couple to the scalar curvature in the linear dilaton term. Although one considers the flat metric on the genus-one surface, this term is ultimately important to keep track of the background charge contribution when expressing the final result in terms of the Liouville field theory analogue. It can be restored wherever needed by writing the action (1).
Then, we are ready to compute the correlation functions. These are defined by

\[ \Omega^{g=1, n}_{(m,b)} = \left\langle \prod_{\nu=1}^{n} \Phi_{h_{\nu}}(\mu_{\nu}|z_{\nu}) \right\rangle_{(\lambda, \tau)} = \frac{1}{Z^{(m,b)}} \int \mathcal{D} \phi \mathcal{D}^{2} \beta \mathcal{D}^{2} \gamma e^{-S^{(m,b)}[\phi, \beta, \gamma]} \prod_{\nu=1}^{n} \Phi_{h_{\nu}}(\mu_{\nu}|z_{\nu}) \]  

where the subscript \((\lambda, \tau)\) on the left hand side stands to remind of the functional measure \(\int \mathcal{D} \phi \mathcal{D}^{2} \beta \mathcal{D}^{2} \gamma\) depending on the modular and the twist parameters. The definition of correlation functions in (25) includes the genus-one partition function \(Z^{(m,b)}\), which we will discuss below.

As in the case of genus-zero calculation, the integration over the fields \(\gamma\) and \(\bar{\gamma}\) yield \(\delta\)-functions that fix the conditions

\[ \bar{\partial} \beta(w) = 2\pi \sum_{\nu=1}^{n} \mu_{\nu} \delta(w - z_{\nu}), \quad \partial \bar{\beta}(\bar{w}) = -2\pi \sum_{\nu=1}^{n} \bar{\mu}_{\nu} \delta(\bar{w} - \bar{z}_{\nu}). \]  

However, the solutions to (26) in this case is more complicated. To integrate these equations on the torus it is convenient to introduce the \(\theta\)-function

\[ \theta(z|\tau) = -\sum_{n \in \mathbb{Z}} e^{i\pi(n+1/2)^{2}\tau + 2\pi i(n+1/2)(z+1/2)}, \]  

which obeys the periodic condition \(\theta(z + p + q\tau|\tau) = (-1)^{p-q} e^{-i\pi q(2z + q\tau)} \theta(z|\tau)\), for \(p, q \in \mathbb{Z}\). This property permits to build up from \(\theta(z|\tau)\) a new function \(\sigma_{\lambda}(z|\tau)\) which happens to have a single pole and the same periodicity condition that we asked for \(\beta\). That is, one can define

\[ \sigma_{\lambda}(z, w|\tau) = \frac{\theta(\lambda + w - z|\tau)\theta(0|\tau)}{\theta(z - w|\tau)\theta(\lambda|\tau)}, \]  

which, in fact, satisfies \(\sigma_{\lambda}(z + p + q\tau, w|\tau) = e^{2\pi i q\lambda} \sigma_{\lambda}(z, w|\tau)\). Then, one can use these modular functions to integrate (26). The integration of these equations is unique as long as the twist parameter \(\lambda\) does not vanish. We have

\[ \beta(w) = \sum_{\nu=1}^{n} \mu_{\nu} \sigma_{\lambda}(w, z_{\nu}|\tau) = u \prod_{i=1}^{n} \theta(\mu_{\nu}|\tau) \prod_{\nu=1}^{n} \theta(w - y_{\nu}|\tau), \]  

where, again, a function \(u\) appears. Field \(\beta\) is a meromorphic differential and depends on \(n + 1\) parameter; \(n\) of these parameters are the variables \(\mu_{\nu}\) and the other parameter is \(\lambda\). We can parameterize \(\beta\) in terms of \(u\) and \(n\) parameters \(y_{\nu}\) by defining the following set of \(n + 1\) implicit equations

\[ \mu_{\nu} = u \prod_{i=1}^{n} \theta(z_{\nu} - y_{\nu}|\tau) \prod_{\nu \neq \mu, \nu=1}^{n} \theta(z_{\nu} - z_{\mu}|\tau), \quad \lambda = \sum_{\nu=1}^{n} (y_{\nu} - z_{\nu}), \]  

where
where \( \theta'(0|\tau) \) refers to the derivative of the \( \theta \)-function. Then, one has \( n+1 \) equations that relate variables \( \mu_i \) and \( \lambda \) with variables \( y_i \) and \( u \). This is exactly what has been done in Ref. \( [3] \) for the case \( m = 1 \). Equation (30) comes from the computation of the residue of the function \( \beta \) at the pole \( w = z_\nu \). Then, the integration over \( \gamma \) and \( \bar{\gamma} \) leads to the following \( \delta \)-function

\[
\delta^{(2)}(\bar{\theta}\beta(w) - 2\pi \sum_{\nu=1}^{n} \mu_\nu \delta^2(w - z_\nu)) = |\det \partial_\lambda|^{-2} \delta^{(2)}(\beta(w) - u\lambda_1(y_i, z_\nu; w)), \tag{31}
\]

where

\[
\lambda_1(y_i, z_\nu; w) = \prod_{i=1}^{n-1} \theta(w - y_i|\tau) \prod_{\nu=1}^{n} \theta(w - z_\nu|\tau),
\tag{32}
\]

and where the factor \( |\det \partial_\lambda|^{-2} \) is the Jacobian of the change of variables from \( \partial \beta \) to \( \beta \). Then, one can integrate over \( \beta \) and \( \bar{\beta} \) and finally find

\[
\left< \prod_{\nu=1}^{n} \Phi_{\mu_\nu}(\mu_\nu|z_\nu) \right>_{(\lambda, \tau)} = \frac{1}{Z_{(m,b)}} \frac{1}{|\det \partial_\lambda|^2} \int D\phi \ e^{-S_{\text{eff}}[\phi, \lambda_1]} \prod_{\nu=1}^{n} |\mu_\nu|^{2m(j_\nu+1)} e^{2b(j_\nu+1)\phi(z_\nu)} \tag{33}
\]

with the effective action

\[
S_{\text{eff}}[\phi, \lambda_1] = \frac{1}{2b} \int d^2w \ (\partial \phi \bar{\partial} \phi + b^2|u|^{2m})|\lambda_1|^{2m} e^{2b\phi}.
\]

Shifting \( \phi(w, \bar{w}) \to \phi(w, \bar{w}) - (m/b) \log |u| \) and using (30), one finds

\[
\left< \prod_{\nu=1}^{n} \Phi_{j_\nu}(\mu_\nu|z_\nu) \right>_{(\lambda, \tau)} = \frac{1}{Z_{(m,b)}} \frac{1}{|\det \partial_\lambda|^2} \int D\phi \ e^{-S_{\text{eff}}[\phi, \lambda_1]} \prod_{\nu=1}^{n} e^{2b(j_\nu+1)\phi(z_\nu)} \times
\]

\[
\times \prod_{\nu=1}^{n} \prod_{\nu \neq \nu'}^{n} \left| \frac{\theta(z_\nu - z_{\nu'}|\tau)}{\theta'(0|\tau)} \right|^{2m(j_\nu+1)} \prod_{\nu=1}^{n} \prod_{i=1}^{n-1} \left| \frac{\theta(z_\nu - y_i|\tau)}{\theta'(0|\tau)} \right|^{2m(j_\nu+1)}. \tag{34}
\]

It is now convenient to define a new variable as

\[
\varphi(w, \bar{w}) \equiv \phi(w, \bar{w}) + \frac{m}{2b} \sum_{\nu=1}^{n} \left( \log |\theta(w - y_\nu)|^2 - \log |\theta(w - z_\nu)|^2 \right), \tag{35}
\]

and introduce a new function \( F \) defined as follows

\[
F(z - w|\tau) = e^{-2\pi(\text{Im}(z-w))^2/\text{Im}(\tau)} \left| \frac{\theta(z - w|\tau)}{\theta'(0|\tau)} \right|^2. \tag{36}
\]
which satisfies \( F(z+p+q\tau-w|\tau) = F(z-w|\tau) \). This permits to rewrite the relation between \( \varphi \) and \( \phi \) in terms of single valued variables, the fluctuation field \( \phi = \phi + \phi_c \) and the function \( F \). Namely,

\[
\varphi(w) = \phi(w) + \frac{m}{2\beta} \sum_{\nu=1}^{n} \left( \log F(w-y_{\nu}|\tau) - \log F(w-z_{\nu}|\tau) \right) + \Delta \tag{37}
\]

where \( \Delta = (\pi m/\text{Im}(\tau)b) \sum_{\nu=1}^{n} (\text{Im}(y_{\nu})^2 - \text{Im}(z_{\nu})^2) \). After some manipulation, one finds

\[
\left( \prod_{\nu=1}^{n} \Phi_{h_{\nu}}(\mu_{\nu}|z_{\nu}) \right)_{(\tau,\lambda)} = \frac{e^{-\pi m^2\text{Im}(\lambda)^2/\text{Im}(\tau)b^2}}{Z_{(m,b)}|\det \partial|_{\lambda}^2} \prod_{\mu<\nu}^{n} F(z_{\mu} - z_{\nu}|\tau) \frac{\pi}{2b} \prod_{i<j}^{n} F(y_{i} - y_{j}|\tau) \frac{\pi}{2b} \prod_{\mu=1}^{n} \prod_{i=1}^{n} F(z_{\mu} - y_{i}|\tau) \frac{\pi}{2b} \times \int D\varphi e^{-S_L[\varphi]} \prod_{\nu=1}^{n} e^{2\alpha_{\nu}\varphi(z_{\nu})} \prod_{i=1}^{n} e^{-\frac{\pi b}{2}\varphi(y_{i})}.
\tag{38}
\]

On the left hand side of (38) we already see the Liouville correlation functions to appear. In order to normalize the correlation functions we have to consider the partition function \( Z_{(m,b)} \), which depends on \( \lambda \) and \( \tau \). This function differs from the one for \( m = 1 \) by a factor \( e^{\pi (1-m^2)/(\text{Im}(\lambda)^2/b^2)} \); c.f. [3]. The case \( Z_{(m=1,b)} \) corresponds to the partition function of \( \mathbb{H}^3 \) WZW model. The case \( Z_{(m=0,b)} \) is, of course, the partition function of Liouville theory , \( Z_L \), times the contribution of the free \( \gamma-\beta \) ghost system. Equation (39) imposes \( \beta \) to be a constant, which has be zero for \( \lambda \neq 0 \); in turn, the integration yields just \( |\det \partial|_{\lambda}^{-2} \). In the untwisted case \( \lambda = 0 \), \( \beta \) may take any value and the integration diverges.

Collecting all the ingredients above, one arrives to the genus-one generalization of the Ribault formula of [3]; namely

\[
\Omega_{(m,b)}^{g=1,n} = \left\langle \prod_{\nu=1}^{n} e^{i\frac{\pi b}{2} X(z_{\nu})} \prod_{i=1}^{n} e^{-i\frac{\pi b}{2} X(y_{i})} \right\rangle_{X} \left\langle \prod_{\nu=1}^{n} V_{\alpha_{\nu}}(z_{\nu}) \prod_{i=1}^{n} V_{-\frac{\pi b}{2}}(y_{i}) \right\rangle_{L},
\tag{39}
\]

with the Liouville correlation function

\[
\left\langle \prod_{\nu=1}^{n} V_{\alpha_{\nu}}(z_{\nu}) \prod_{i=1}^{n} V_{-\frac{\pi b}{2}}(y_{i}) \right\rangle_{L} = \frac{1}{Z_L} \int D\varphi e^{-S_L[\varphi]} \prod_{\nu=1}^{n} e^{2\alpha_{\nu}\varphi(z_{\nu})} \prod_{i=1}^{n} e^{-\frac{\pi b}{2}\varphi(y_{i})} \tag{40}
\]

with exactly the same relation between the indices \( j_{\nu} \) and the Liouville momenta \( \alpha_{\nu} = b(j_{\nu} + 1 + \frac{m}{2}b^{-2}) \), but with two additional degenerate fields inserted. The prefactor takes the form

\[
\left\langle \prod_{\nu=1}^{n} e^{i\frac{\pi b}{2} X(z_{\nu})} \prod_{i=1}^{n} e^{-i\frac{\pi b}{2} X(y_{i})} \right\rangle_{X} = \prod_{\mu<\nu}^{n} F(z_{\mu} - z_{\nu}|\tau) \frac{\pi}{2b} \prod_{i<j}^{n} F(y_{i} - y_{j}|\tau) \frac{\pi}{2b} \prod_{\mu,i=1}^{n} F(z_{\mu} - y_{i}|\tau) \frac{\pi}{2b},
\]
which, again, can be thought of as the expectation value of exponential vertex operators in a theory of a free boson $X(z)$, now on the torus.

Relation (39) is valid for all values $m \in \mathbb{Z}_{\geq 0}$. It generalizes the genus-zero results of [4] to genus-one, which has been accomplished by straightforwardly adopting the analysis of [3] to the generic case $m \in \mathbb{Z}_{\geq 1}$. As a consequence, now we have (39), which relates the torus $n$-point function of the theory defined by action (1) to Liouville $2n$-point functions for $m \in \mathbb{Z}_{\geq 1}$.

6 Conclusions

Motivated by the question about how to extend the analysis of [1] to the case of non-fundamental surface operators in $\mathcal{N} = 2$ theories, we reviewed the results of Ref. [3] and used the path integral techniques developed therein to generalize the result of Ref. [4] to genus-one. That is, we have shown that torus $n$-point correlation functions of the conformal field theories proposed in [4], whose Lagrangian representation is given by (1), are given by $2n$-point correlation functions of Liouville field theory times a free field factor. In particular, this implies that the expectation value of a surface operator of the $\mathcal{N} = 2^*$ $SU(2)$ super Yang-Mills theory whose type of singularity is labeled by an integer number $m$ is in correspondence with the expectation value of a primary operator of one of the CFTs proposed in [4], whose central charge is $c_{(b,m)} = 3 + 6(b^{-1} + (1 - m)b)^2$, with $b^2 = \varepsilon_1/\varepsilon_2$, and with $\varepsilon_{1,2}$ being the Nekrasov deformation parameters. In the case $m = 1$, which corresponds to the simplest surface operators studied in [1], the relation between correlation functions mentioned above reduces to the WZNW-Liouville correspondence studied in Refs. [12, 13], or, more precisely, to its genus-one generalization done in Ref. [3]. Our aim here was to point out that the theories that correspond to other values of $m \in \mathbb{Z}_{>0}$ could also have applications to gauge theories through the AGT conjecture and its generalizations. In particular, we have that the Liouville two-point function that involves one degenerate field of level $m$, is given by correlator (39) in the case $n = 1$, which in the Coulomb gas representation takes the form of the following multiple integral over the complex plane

$$\Omega_{(m,b)}^{g=1,n=1} \sim \Gamma(j + 1) e^{2b(j+1)\delta(z,w)} \int \prod_{k=1}^{j-1} \frac{d^2 \omega_k}{2\pi i} \prod_{k=1}^{j-1} F\left(y - \omega_k|\tau\right)^m + \prod_{l \neq k} F\left(\omega_k - \omega_l|\tau\right)^{2b^2},$$

(41)
where $\delta(z, y) = \phi_c(z) + m((\text{Im} y)^2 - (\text{Im} z)^2 + \text{Im}(z - y)^2)/\text{Im} \tau$, $y = z + \lambda$.

The possible connection between the 2D CFT description of surface operators in $\mathcal{N} = 2$ $SU(2)$ gauge theories and the WZNW-Liouville correspondence of [12, 13] had been already suggested in Refs. [14, 1]. A generalization to higher genus-$g$ worked out in [3], which can be easily extended to $m > 1$ as we did here for $g = 1$, shows that $n$-point correlation functions of theories with affine symmetry are given by $(2n + 2g - 2)$-point correlation functions in Liouville theory, and the former are in correspondence with gauge theory observables associated to having a surface operator for each trinion in the Riemann surface decomposition. The question remains open as to how to make the relation between the affine description of [1] and the WZNW-Liouville correspondence of [12, 13] precise; if extended to generic $m$, it could provide a useful tool to investigate non-fundamental surface operators.

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