Nonclassical Lagrangian Dynamics and Potential Maps

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Abstract

Section 1 refines the theory of harmonic and potential maps. Section 2 defines a generalized Lorentz world-force law and shows that any PDEs system of order one generates such a law in suitable geometrical structure. In other words, the solutions of any PDEs system of order one are harmonic or potential maps, if we use semi-Riemann-Lagrange structures. Section 3 formulates open problems regarding the geometry of semi-Riemann manifolds \((J^1(T, M), S_1)\), \((J^2(T, M), S_2)\), and shows that the Lorentz-Udriste world-force law is equivalent to covariant Hamilton PDEs on \((J^1(T, M), S_1)\).

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1 Harmonic Maps and Potential Maps

All maps throughout the paper are smooth, while manifolds are real, finite-dimensional, Hausdorff, second-countable and connected.
Let \((T, h)\) and \((M, g)\) be semi-Riemann manifolds of dimensions \(p\) and \(n\). Hereafter we shall assume that the manifold \(T\) is oriented. Greek (Latin) letters will be used for indexing the components of geometrical objects attached to the manifold \(T\) (manifold \(M\)). Local coordinates will be written \(t = (t^\alpha), \quad \alpha = 1, \ldots, p\) \(x = (x^i), \quad i = 1, \ldots, n,\) and the components of the corresponding metric tensor and Christoffel symbols will be denoted by \(h_{\alpha\beta}, g_{ij}, H_{\beta\gamma}^\alpha, G_{jk}^i.\) Indices of tensors or distinguished tensors will be raised and lowered in the usual fashion.

Let \(\varphi : T \to M, \varphi(t) = x, x^i = x^i(t^\alpha)\) be a \(C^\infty\) map (parametrized sheet). We set

\[
\begin{align*}
  x^i_\alpha &= \frac{\partial x^i}{\partial t^\alpha}, \\
  x^i_{\alpha\beta} &= \frac{\partial^2 x^i}{\partial t^\alpha \partial t^\beta} - H_{\alpha\beta}^\gamma x^i_\gamma + G_{jk}^i x^j_\alpha x^k_{\beta}. 
\end{align*}
\]

Then \(x^i_\alpha, x^i_{\alpha\beta}\) transform like tensors under coordinate transformations \(t \to \bar{t}, x \to \bar{x}\). In the sequel \(x^i_\alpha, x^i_{\alpha\beta}\) will be interpreted like distinguished tensors.

The canonical form of the energy density \(E(\varphi)\) of the map \(\varphi\) is defined by

\[
E_0(\varphi)(t) = \frac{1}{2} h^{\alpha\beta}(t) g_{ij}(x(t)) x^i_\alpha(t) x^j_\beta(t).
\]

For a relatively compact domain \(\Omega \subset T\), we define the energy

\[
E_0(\varphi, \Omega) = \int_\Omega E_0(\varphi)(t) dv_h,
\]

where \(dv_h = \sqrt{|h|} dt^1 \wedge \ldots \wedge dt^p\) denotes the volume element induced by the semi-Riemann metric \(h\). A map \(\varphi\) is called harmonic map if it is a critical point of the energy functional \(E_0\), i.e., an extremal of the Lagrangian

\[
L = E_0(\varphi)(t) \sqrt{|h|},
\]

for all compactly supported variations (It should be remarked that every \(C^2\) harmonic map is automatically \(C^\infty\)). The harmonic map equation is a system of nonlinear PDEs of second order (generalized Laplace equations) and is expressed in local coordinates as

\[
\tau(\varphi)^i = h^{\alpha\beta} x^i_{\alpha\beta} = 0.
\]
The quantity \( \tau(\varphi)^i \) defines a section of the pull-back bundle \( \varphi^{-1}TM \) of the tangent bundle \( TM \) of the manifold \( M \) along \( \varphi \), and is called the tension field of \( \varphi \).

The product manifold \( T \times M \) is coordinated by \( (t^\alpha, x^i) \). The 1-order jet manifold \( J^1(T, M) \), i.e., the configuration bundle, is endowed with the adapted coordinates \( (t^\alpha, x^i, x^i_\alpha) \). The distinguished tensors fields and other distinguished geometrical objects on \( T \times M \) are introduced [4] using the jet bundle \( J^1(T, M) \).

Let \( X^i_\alpha(t, x) \) be a given \( C^\infty \) distinguished tensor field on \( T \times M \) and \( c(t, x) \) be a given \( C^\infty \) real function on \( T \times M \). The general energy density \( E(\varphi) \) of the map \( \varphi \) is defined by

\[
E(\varphi(t)) = \frac{1}{2} h^{\alpha\beta}(t) g_{ij}(x(t)) x^i_\alpha(t) x^j_\beta(t) - h^{\alpha\beta}(t) g_{ij}(x(t)) x^i_\alpha(t) X^j_\beta(t, x(t)) + c(t, x).
\]

Of course \( E(\varphi) \) is a perfect square iff

\[
c = \frac{1}{2} h^{\alpha\beta}(t) g_{ij}(x(t)) X^i_\alpha(t, x(t)) X^j_\beta(t, x(t)).
\]

Similarly, for a relatively compact domain \( \Omega \subset T \), we define the energy

\[
E(\varphi; \Omega) = \int_{\Omega} E(\varphi)(t) dv_h.
\]

A map \( \varphi \) is called potential map if it is a critical point of the energy functional \( E \), i.e., an extremal of the Lagrangian

\[
L = E(\varphi)(t) \sqrt{|h|},
\]

for all compactly supported variations. The potential map equation is a system of nonlinear PDEs (generalized Poisson equations) and is expressed locally by

\[
(3) \quad \tau(\varphi)^i = h^{\alpha\beta} x^i_{\alpha\beta} = g^{ij} \frac{\partial c}{\partial x^j} + h^{\alpha\beta}(\nabla_k X^i_\beta - g_{kj} g^{il} \nabla_l X^j_\beta) x^k_\alpha + h^{\alpha\beta} D_\alpha X^i_\beta,
\]

where \( D \) is the Levy-Civita connection of \( (T, h) \) and \( \nabla \) is the Levy-Civita connection of \( (M, g) \). Explicitly, we have

\[
(4) \quad \nabla_j X^i_\alpha = \frac{\partial X^i_\alpha}{\partial x^j} + C^i_{jk} X^k_\alpha, \quad D_\beta X^i_\alpha = \frac{\partial X^i_\alpha}{\partial t^\beta} - H^i_{\beta\gamma} X^\gamma_\alpha
\]
F_{j}^{i} = \nabla_{j} X_{i}^{\alpha} - g_{hj} g^{ik} \nabla_{k} X_{i}^{h}, \tag{5}

\frac{\partial g_{ij}}{\partial x^{k}} = G_{kj}^{h} g_{hi} + G_{kj}^{h} g_{hi}, \quad \frac{\partial h^{\alpha \beta}}{\partial t^{\gamma}} = -H_{\gamma \lambda}^{\alpha \beta} h^{\lambda \beta} - H_{\gamma \lambda}^{\beta} h^{\alpha \lambda}. \tag{6}

2 Lorentz-Udrişte World-Force Law

In nonquantum relativity there are three basic laws for particles: the Lorentz World-Force Law and two conservation laws [5]. Now we shall generalize the Lorentz World-Force Law (see also [6], [11]).

**Definition.** Let $F_{\alpha} = (F_{j}^{i})_{\alpha}$ and $U_{\alpha \beta} = (U_{i}^{j})_{\alpha \beta}$ be $C^\infty$ distinguished tensors on $T \times M$, where $\omega_{ji\alpha} = g_{hi} F_{j}^{h} \alpha$ is skew-symmetric with respect to $j$ and $i$. Let $c(t, x)$ be a $C^\infty$ real function on $T \times M$. A $C^\infty$ map $\varphi : T \to M$ obeys the Lorentz-Udrişte World-Force Law with respect to $F_{\alpha}$, $U_{\alpha \beta}$, $c$ iff

$$\tau(\varphi)^{i} = g^{ij} \frac{\partial c}{\partial x^{j}} + h^{\alpha \beta} F_{j}^{i} \alpha x_{\beta} + h^{\alpha \beta} U_{i}^{\alpha \beta}. \tag{7}$$

Now we show that the solutions of a system of PDEs of order one are potential maps in a suitable geometrical structure. First we remark that a $C^\infty$ distinguished tensor field $X_{i}^{\alpha}(t, x)$ on $T \times M$ defines a family of $p$-dimensional sheets as solutions of the PDEs system of order one

$$x_{\alpha}^{i} = X_{\alpha}^{i}(t, x(t)), \tag{8}$$

if the complete integrability conditions

$$\frac{\partial X_{\alpha}^{i}}{\partial t^{\beta}} + \frac{\partial X_{\alpha}^{i}}{\partial x^{j}} X_{\beta}^{j} = \frac{\partial X_{\alpha}^{j}}{\partial t^{\alpha}} + \frac{\partial X_{\alpha}^{j}}{\partial x^{j}} X_{\beta}^{j}$$

are satisfied.

The distinguished tensor field $X_{i}^{\alpha}$ and semi-Riemann metrics $h$ and $g$ determine the potential energy

$$f : T \times M \to R, \quad f = \frac{1}{2} h^{\alpha \beta} g_{ij} X_{\alpha}^{i} X_{\beta}^{j}.$$ 

The distinguished tensor field (family of $p$-dimensional sheets) $X_{\alpha}^{i}$ on $(T \times M, h + g)$ is called:
1) timelike, if \( f < 0 \);
2) nonspacelike or causal, if \( f \leq 0 \);
3) null or lightlike, if \( f = 0 \);
4) spacelike, if \( f > 0 \).

Let \( X^i_\alpha \) be a distinguished tensor field of everywhere constant energy. If \( X^i_\alpha \) (the system (8)) has no critical point on \( M \), then upon rescaling, it may be supposed that \( f \in \left\{ -\frac{1}{2}, 0, \frac{1}{2} \right\} \). Generally, \( \mathcal{E} \subset \mathcal{M} \) is the set of critical points of the distinguished tensor field \( X^i_\alpha \), and this rescaling is possible only on \( T \times (M \setminus \mathcal{E}) \).

Using the operator (derivative along a solution of (8)),
\[
\delta \frac{\partial}{\partial t^\alpha} x^i_\beta = \frac{\partial^2 x^i_\beta}{\partial t^\alpha \partial t^\beta} - H^\beta_\alpha x^i_\beta + G^{ij}_k x^k_\alpha x^j_\beta
\]
we obtain the prolongation (system of PDEs of order two)
\[
(9) \quad x^i_\alpha \beta = D_\beta X^i_\alpha + (\nabla^j X^i_\alpha) x^j_\beta.
\]

The distinguished tensor field \( X^i_\alpha \), the metric \( g \), and the connection \( \nabla \) determine the external distinguished tensor field
\[
F^i_\beta = \nabla^j X^i_\alpha - g^{ih} g^j_k \nabla^h X^k_\alpha,
\]
which characterizes the helicity of the distinguished tensor field \( X^i_\alpha \).

First we write the PDEs system (9) in the equivalent form
\[
x^i_\alpha \beta = g^{ih} g^j_k (\nabla^h X^k_\alpha) x^j_\beta + F^i_\alpha x^j_\beta + D_\beta X^i_\alpha.
\]

Now we modify this PDEs system into
\[
(10) \quad x^i_\alpha \beta = g^{ih} g^j_k (\nabla^h X^k_\alpha) x^j_\beta + F^i_\alpha x^j_\beta + D_\beta X^i_\alpha.
\]
Of course, the PDEs system (10) is still a prolongation of the PDEs system (8).

Taking the trace of (10) with respect to \( h^{\alpha \beta} \) we obtain that any solution of PDEs system (8) is also a solution of the PDEs system
\[
(11) \quad h^{\alpha \beta} x^i_\alpha \beta = g^{ih} h^{\alpha \beta} g^j_k (\nabla^h X^k_\alpha) x^j_\beta + h^{\alpha \beta} F^i_\alpha x^j_\beta + h^{\alpha \beta} D_\beta X^i_\alpha.
\]
(generalized Poisson equations).

**Theorem.** The PDEs system (11) is a prolongation of the PDEs system (8).

If \( F_{j}^{i} \alpha = 0 \), then the PDEs system (11) reduces to

\[
(12) \quad h^{\alpha \beta} x_{\alpha \beta}^i = g^{i h} h^{\alpha \beta} g_{k j} (\nabla_{h} X_{\alpha}^{k}) X_{\beta}^{j} + h^{\alpha \beta} D_{\beta} X_{\alpha}^{i}.
\]

The first term in the second hand member of the PDEs systems (11) or (12) is \((\text{grad } f)^{i}\). Consequently, choosing the metrics \( h \) and \( g \) such that \( f \in \{-\frac{1}{2}, 0, \frac{1}{2}\} \), then the preceding PDEs systems reduce to

\[
(11') \quad h^{\alpha \beta} x_{\alpha \beta}^i = h^{\alpha \beta} F_{j}^{i} \alpha x_{\beta}^j + h^{\alpha \beta} D_{\beta} X_{\alpha}^{i}
\]

\[
(12') \quad h^{\alpha \beta} x_{\alpha \beta}^i = h^{\alpha \beta} D_{\beta} X_{\alpha}^{i}.
\]

**Theorem.** 1) The solutions of PDEs system (11) are the extremals of the Lagrangian

\[
L = \frac{1}{2} h^{\alpha \beta} g_{ij} (x_{\alpha}^i - X_{\alpha}^i)(x_{\beta}^j - X_{\beta}^j) \sqrt{|h|} = \left( \frac{1}{2} h^{\alpha \beta} g_{ij} x_{\alpha}^i x_{\beta}^j - h^{\alpha \beta} g_{ij} x_{\alpha}^i X_{\beta}^j + f \right) \sqrt{|h|}.
\]

2) The solution PDEs system (12) are the extremals of the Lagrangian

\[
L = \left( \frac{1}{2} h^{\alpha \beta} g_{ij} x_{\alpha}^i x_{\beta}^j + f \right) \sqrt{|h|}.
\]

3) If the Lagrangians \( L \) are independent of the variable \( t \), then the PDEs systems (11) or (12) are conservative, the energy-impulse tensor field being

\[
T_{\alpha}^{\beta} = x_{\beta}^i \frac{\partial L}{\partial x_{\alpha}^i} - L \delta_{\beta}^{\alpha}.
\]

4) Both Lagrangians produce the same Hamiltonian

\[
H = \left( \frac{1}{2} h^{\alpha \beta} g_{ij} x_{\alpha}^i x_{\beta}^j - f \right) \sqrt{|h|}.
\]
Proof. 1) and 2) If we write $L = E \sqrt{|h|}$, where $E$ is the energy density, then the Euler-Lagrange equations of extremals

$$\frac{\partial L}{\partial x^k} - \frac{\partial}{\partial t^\alpha} \frac{\partial L}{\partial x^k_\alpha} = 0$$

can be written

(13) $$\frac{\partial E}{\partial x^k} - \frac{\partial}{\partial t^\alpha} \frac{\partial E}{\partial x^k_\alpha} - H^\gamma_\alpha \frac{\partial E}{\partial x^k_\alpha} = 0.$$ 

We compute

$$\frac{\partial E}{\partial x^k} = \frac{1}{2} h^{\alpha \beta} \frac{\partial g_{ij} x^i_x^j}{\partial x^k} x^j_{\alpha} x^j_{\beta} - h^{\alpha \beta} \frac{\partial g_{ij}}{\partial x^k} x^i_{\alpha} x^j_{\beta} +$$

$$+ \frac{1}{2} h^{\alpha \beta} g_{ij} x^i_{\alpha} x^j_{\beta} - h^{\alpha \beta} g_{ij} x^i_{\alpha} \frac{\partial x^j_{\beta}}{\partial x^k} + h^{\alpha \beta} g_{ij} \frac{\partial X^i_\alpha}{\partial x^k} X^j_{\beta},$$

$$\frac{\partial E}{\partial x^k_\alpha} = h^{\alpha \beta} g_{kj} x^j_{\beta} - h^{\alpha \beta} g_{kj} X^j_{\beta},$$

$$- \frac{\partial}{\partial t^\alpha} \frac{\partial E}{\partial x^k_\alpha} = - h^{\alpha \beta} \frac{\partial g_{kj}}{\partial t^\alpha} x^j_{\beta} - h^{\alpha \beta} \frac{\partial g_{kj}}{\partial x^l} x^l_{\alpha} x^j_{\beta} - h^{\alpha \beta} g_{kj} \frac{\partial^2 x^j_{\beta}}{\partial t^\alpha \partial t^\beta} +$$

$$+ \frac{\partial h^{\alpha \beta}}{\partial t^\alpha} g_{kj} X^j_{\beta} + h^{\alpha \beta} \frac{\partial g_{kj}}{\partial x^l} x^l_{\alpha} X^j_{\beta} + h^{\alpha \beta} g_{kj} \left( \frac{\partial X^j_{\alpha}}{\partial t^\alpha} + \frac{\partial X^j_{\alpha}}{\partial x^l} x^l_{\alpha} \right).$$

We replace in (13) taking into account the formulas (1), (4) and (6). We find

$$h^{\alpha \beta} g_{kj} x^j_{\alpha \beta} = h^{\alpha \beta} g_{ij} (\nabla_k X^i_\alpha) X^j_{\beta} + h^{\alpha \beta} g_{kj} (\nabla_l X^j_\beta) x^l_{\alpha} -$$

$$- h^{\alpha \beta} g_{ij} x^i_{\alpha} \nabla_k X^j_\beta + h^{\alpha \beta} g_{kj} D_\alpha X^j_{\beta}.$$

Transvecting by $g^{hk}$ and using the formula (5), we obtain

$$h^{\alpha \beta} x^j_{\alpha \beta} = g^{ik} h^{\alpha \beta} g_{lj} (\nabla_k X^i_\alpha) X^j_{\beta} + h^{\alpha \beta} F^i_{\alpha \beta} + h^{\alpha \beta} D_\alpha X^i_{\beta}.$$

3) Taking into account the Euler-Lagrange equations, we have

$$\frac{\partial T^\alpha}{\partial t^\alpha} = \frac{\partial^2 x^j_{\alpha \beta}}{\partial t^\alpha \partial t^\beta} \frac{\partial L}{\partial x^j_{\alpha \beta}} + x^j_{\alpha \beta} \frac{\partial^2 L}{\partial t^\beta \partial x^j_{\alpha \beta}} + x^j_{\alpha \beta} \frac{\partial^2 L}{\partial x^l_{\alpha \beta} \partial x^j_{\alpha \beta}}.$$
\[ +x_i^j \frac{\partial^2 L}{\partial x_i^\alpha \partial x^j} \frac{\partial x^j}{\partial t^\alpha} - \frac{\partial L}{\partial t^\alpha} \delta^i_\beta - \frac{\partial L}{\partial x^i} x_i^\gamma \frac{\partial x^\gamma}{\partial t^\alpha} \delta^j_\beta = -\frac{\partial L}{\partial t^\alpha}. \]

**Open problem.** Determine the general expression of the energy-impulse tensor field as object on \( J^1(T, M) \), and compute its divergence.

4) We use the formula

\[ H = x_i^\alpha \frac{\partial L}{\partial x_i^\alpha} - L. \]

**Corollary.** Every PDE generates a Lagrangian of order one via the associated first order PDEs system and suitable metrics on the manifold of independent variables and on the manifold of functions. In this sense the solutions of the initial PDE are potential maps.

**Theorem (Lorentz-Udrişte World-Force Law).**

1) Every solution of the PDEs system (12) is a potential map on the semi-Riemann manifold \((T \times M, h + g)\).

2) Every solution of the PDEs system (11) is a horizontal potential map of the semi-Riemann-Lagrange manifold

\[ (T \times M, h + g, N^{(i)}_j = C^{i}_{jk}x^k_\alpha - F^i_\alpha, M^{(i)}_\beta = -H^\gamma_{\alpha\beta}x^i_\gamma). \]

### 3 Covariant Hamilton Field Theory

**(Covariant Hamilton equations)**

Let us show that the PDEs systems (11) and (12) induce on \( J^1(T, M) \) Hamilton PDEs systems.

Let \((T, h)\) be a semi-Riemann manifold with \( p \) dimensions, and \((M, g)\) be a semi-Riemann manifold with \( n \) dimensions. Then \((J^1(T, M), h + g + h^{-1} \ast g)\) is a semi-Riemann manifold with \( p + n + pn \) dimensions.

We denote by \( X^i_\alpha \) a \( C^\infty \) distinguished tensor field on \( T \times M \), and by \( \omega_{ij\alpha} \) the distinguished 2-form associated to the distinguished tensor field

\[ F^i_\alpha = \nabla_j X^i_\alpha - g^{ij} g_{kj} \nabla_h X^k_\alpha \]

via the metric \( g \), i.e., \( \omega = \frac{1}{2} g \circ F \). Of course \( X^i_\alpha, F^i_\alpha \) are distinguished objects on \( J^1(T, M) \) globally defined.
If \((t^\alpha, x^i, x^j_{\alpha})\) are the coordinates of a point in \(J^1(T, M)\), and \(H^\gamma_{\beta\gamma}, G^i_{jk}\) are the components of the connection induced by \(h\) and \(g\), respectively, then

\[
\left( \frac{\delta}{\delta t^\alpha} = \frac{\partial}{\partial t^\alpha} + H^\gamma_{\alpha\beta} x^i_{\gamma} \frac{\partial}{\partial x^i_{\beta}}, \quad \frac{\delta}{\delta x^i_{\alpha}} = \frac{\partial}{\partial x^i_{\alpha}} - G^h_{ik} x^k_{\alpha} \frac{\partial}{\partial x^h_{\alpha}} \right),
\]

\[
(dt^\beta, dx^j, \delta x^j_{\beta} = dx^j_{\beta} - H^\gamma_{\beta\lambda} x^j_{\gamma} dt^\lambda + G^j_{hk} x^h \delta x^k_{\beta})
\]

are dual frames on \(J^1(T, M)\), i.e.,

\[
\begin{align*}
    dt^\beta \left( \frac{\delta}{\delta t^\alpha} \right) &= \delta^\beta_{\alpha}, & dt^\beta \left( \frac{\delta}{\delta x^i_{\alpha}} \right) &= 0, & dt^\beta \left( \frac{\partial}{\partial x^i_{\alpha}} \right) &= 0 \\
    dx^j \left( \frac{\delta}{\delta t^\alpha} \right) &= 0, & dx^j \left( \frac{\delta}{\delta x^i_{\alpha}} \right) &= \delta^j_{i}, & dx^j \left( \frac{\partial}{\partial x^i_{\alpha}} \right) &= 0 \\
    \delta x^j_{\beta} \left( \frac{\delta}{\delta t^\alpha} \right) &= 0, & \delta x^j_{\beta} \left( \frac{\delta}{\delta x^i_{\alpha}} \right) &= 0, & \delta x^j_{\beta} \left( \frac{\partial}{\partial x^i_{\alpha}} \right) &= \delta^j_{i} \delta^\alpha_{\beta}.
\end{align*}
\]

Using these frames, the induced Sasaki-like metric on \(J^1(T, M)\) is

\[
S_1 = h_{\alpha\beta} dt^\alpha \otimes dt^\beta + g_{ij} dx^i \otimes dx^j + h^\alpha_{\beta\gamma} g_{ij} dx^i_{\alpha} \otimes \delta x^j_{\beta}.
\]

**Open problems.**

1) The geometry of the semi-Riemann manifold \((J^1(T, M), S_1)\), which is similar to the geometry of the tangent bundle endowed with Sasaki metric, is now in working by our research group [4]. As was shown here this geometry permits the interpretation of solutions of PDEs systems of order one (8) as potential maps. In this sense the solutions of every PDE of any order are extremals of a Lagrangian of order one.

2) Study the geometry of the dual space of \((J^1(T, M), S_1)\).

3) Find a Sasaki-like \(S_2\) metric on the jet bundle of order two and develop the geometry of the semi-Riemann manifold \((J^2(T, M), S_2)\). In this manifold, the PDEs of Mathematical Physics (of order two) appear like hypersurfaces. Most of them are in fact algebraic hypersurfaces.

4) Study the geometry of the dual space of \((J^2(T, M), S_2)\).

Recall that on a symplectic manifold \((Q, \Omega)\) of even dimension \(q\), the Hamiltonian vector field \(X_{f_1}\) of a function \(f_1 \in \mathcal{F}(Q)\) is defined by

\[
X_{f_1} \cdot \Omega = df_1,
\]
and the Poisson bracket of } f_1, f_2 \text{ is defined by } \{f_1, f_2\} = \Omega(X_{f_1}, X_{f_2}).

The polysymplectic analogue of a function is a } q \text{-form called } momentum observable. The Hamiltonian vector field } X_{f_1} \text{ of such a momentum observable } f_1 \text{ is defined by } X_{f_1} \Omega = df_1,

where } \Omega \text{ is the canonical } (q + 2)-\text{form on the appropriate dual of } J^1(T, M). Since } \Omega \text{ is nondegenerate, this uniquely defines } X_{f_1}. The Poisson bracket of two such } n \text{-forms } f_1, f_2 \text{ is the } n\text{-form defined by } \{f_1, f_2\} = X_{f_1}(X_{f_2} \Omega).

Of course } \{f_1, f_2\} \text{ is, up to the addition of exact terms, another momentum observable.

**Theorem.** The PDEs system

\[ h^\alpha_\beta x^i_\alpha \cdot g^{ij} h^\alpha_\beta g_{jk} X^j_\beta \nabla_h X^k_\alpha \]

transfers in } J^1(T, M) \text{ as a covariant Hamilton PDEs system with respect to the Hamiltonian (momentum observable)}

\[ H = \left( \frac{1}{2} h^\alpha_\beta g_{ij} x^i_\alpha x^j_\beta - f \right) dv_h \]

and the non-degenerate distinguished polysymplectic } (p + 2)-\text{form}

\[ \Omega = \Omega_\alpha \otimes dt^\alpha, \quad \Omega_\alpha = g_{ij} dx^i \wedge \delta x^j_\alpha \wedge dv_h. \]

**Proof.** Let

\[ \theta = \theta_\alpha \otimes dt^\alpha, \quad \theta_\alpha = g_{ij} x^i_\alpha dx^j \wedge dv_h \]

be the distinguished Liouville } (p + 1)-\text{form on } J^1(T, M). It follows

\[ \Omega_\alpha = -d\theta_\alpha. \]

We denote by

\[ X_H = X^\beta_\alpha \frac{\delta}{\delta t^\beta}, \quad X^\beta_\alpha = u^{\beta l} \frac{\delta}{\delta x^l} + \delta u^{\beta l} \frac{\partial}{\partial t^\alpha} \frac{\partial}{\partial x^l}. \]
the distinguished Hamiltonian object of the observable \( H \). Imposing

\[ X^\alpha_H \llcorner \Omega_\alpha = dH, \]

where \( dH = (h^{\alpha\beta} g_{ij} x^j_\beta \delta x^i_\alpha - h^{\alpha\beta} g_{ij} X^j_\beta \nabla_k X^i_\alpha dx^k) \wedge dv_h, \)

we find

\[ g_{ij} u^{\alpha i} \delta x^j_\alpha - g_{ij} \frac{\delta u^{\alpha j}}{\partial t^\alpha} dx^i = h^{\alpha\beta} g_{ij} x^j_\beta \delta x^i_\alpha - h^{\alpha\beta} g_{ij} X^j_\beta \nabla_k X^i_\alpha dx^k \quad \text{modulo} \quad dv_h. \]

Consequently, it appears the Hamilton PDEs system

\[
\begin{cases}
    u^{\alpha i} = h^{\alpha\beta} x^i_\beta \\
    \delta u^{\alpha j} / \partial t^\alpha = g^{hi} h^{\alpha\beta} g_{jk} X^j_\beta (\nabla_h X^k_\alpha)
\end{cases}
\]

(up to the addition of terms which are cancelled by the exterior multiplication with \( dv_h \)).

**Theorem.** The PDEs system

\[ h^{\alpha\beta} x^i_\alpha = g^{ih} h^{\alpha\beta} g_{kj} (\nabla_h X^k_\alpha) X^j_\beta + h^{\alpha\beta} F^i_\alpha x^j_\beta + h^{\alpha\beta} D_\beta X^i_\alpha \]

transfers in \( J^1(T, M) \) as a covariant Hamilton PDEs system with respect to the Hamiltonian (momentum observable)

\[ H = \left( \frac{1}{2} h^{\alpha\beta} g_{ij} x^i_\alpha x^j_\beta - f \right) dv_h \]

and the non-degenerate distinguished polysymplectic \((p+2)\)-form

\[ \Omega = \Omega_\alpha \otimes dt^\alpha, \quad \Omega_\alpha = (g_{ij} dx^i \wedge \delta x^j_\alpha + \omega_{ij} dx^i \wedge dx^j + g_{ij} (D_\beta X^i_\alpha) dt^\beta \wedge dx^j) \wedge dv_h. \]

**Proof.** Let

\[ \theta = \theta_\alpha \otimes dt^\alpha, \quad \theta_\alpha = (g_{ij} x^i_\alpha dx^j - g_{ij} X^i_\alpha dx^j) \wedge dv_h \]

be the distinguished Liouville \((p+1)\)-form on \( J^1(T, M) \). It follows

\[ \Omega_\alpha = -d\theta_\alpha \]
(of course the term containing $dt^\beta$ disappears by exterior multiplication with $dv_h$). We denote by

$$X_H^\alpha = X_H^\beta \frac{\delta}{\delta t^\beta}, \quad X_H^\beta = h^\beta_\gamma \frac{\delta}{\delta t^\gamma} + u^{\beta l} \frac{\delta}{\delta x_l} + \frac{\delta u^{\beta l}}{\partial t^\alpha} \frac{\partial}{\partial x_l^\alpha}$$

the distinguished Hamiltonian object of the observable $H$. Imposing

$$X_H^\alpha \lrcorner \Omega_\alpha = dH,$$

where

$$dH = (h^{\alpha \beta} g_{ij} x^j_\beta \delta x^i_\alpha - h^{\alpha \beta} g_{ij} X^j_\beta (\nabla_k X^i_\alpha) dx^k) \wedge dv_h,$$

we find

$$(g_{ij} u^{\alpha j} \delta x^i_\alpha - g_{ij} \frac{\delta u^{\alpha j}}{\partial t^\alpha} dx^i + 2\omega_{ij\alpha} u^{\alpha i} dx^j + h^{\alpha \beta} g_{ij} (D_{ij}^\beta X^i_\alpha)dx^j) \wedge dv_h = dH.$$

Consequently, it appears the Hamilton PDEs system

$$\left\{ \begin{array}{l} u^{\alpha i} = h^{\alpha \beta} x^i_\beta \\
\frac{\delta u^{\alpha i}}{\partial t^\alpha} = g^{hi} h^{\alpha \beta} g_{jk} X^j_\beta (\nabla_h X^k_\alpha) + 2g^{hi} \omega_{hj\alpha} u^{\alpha j} + h^{\alpha \beta} D_{ij}^\beta X^i_\alpha
\end{array} \right.$$

(up to the addition of terms which are cancelled by the exterior multiplication with $dv_h$).

**Example.** The $C^\infty$ vector fields $\xi_\alpha$ on the manifold $M$ and the 1-forms $A^\alpha$ on the manifold $T$ satisfying

$$[\xi_\alpha, \xi_\beta] = C^\gamma_{\alpha \beta} \xi_\gamma, \quad C^\gamma_{\alpha \beta} = \text{constants},$$

$$\frac{\partial A^\alpha_\beta}{\partial t^\gamma} - \frac{\partial A^\alpha_\gamma}{\partial t^\beta} = C^\gamma_{\lambda \delta} A^\lambda_\beta A^\delta_\gamma$$

determine a continuous group of transformations via the PDEs

$$x^i_\alpha = \xi^i_\beta (x(t)) A^\beta_\alpha (t).$$

Conversely, if $x^i = x^i(t^\alpha, y^j)$ are solutions of a completely integrable system of PDEs of the preceding form, where the $A^\alpha$s and $\xi^i$s satisfy the conditions stated above, such that for values $t^\alpha_0$ of $t^\alpha$s the determinant of the $A^\alpha$s is not zero and

$$x^i(t^\alpha_0, y^j) = y^j,$$
then $x^i = x^i(t^\alpha, y^j)$ define a continuous group of transformations.

Using a semi-Riemann metric $h$ on the manifold $T$, a semi-Riemann metric $g$ on the manifold $M$, then the maps determining a continuous group of transformations appear like extremals (potential maps) of the Lagrangian

$$L = \frac{1}{2} h^{\alpha\beta} g_{ij}(x^i_\alpha - \xi^j_\lambda A^\lambda_\alpha)(x^j_\beta - \xi^j_\mu A^\mu_\beta) \sqrt{|h|}.$$

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