Numerical Radius Inequalities Involving Commutators of $G_1$ Operators

Mojtaba Bakherad$^1$ · Fuad Kittaneh$^2$

Received: 30 December 2016 / Accepted: 6 September 2017 / Published online: 13 September 2017
© Springer International Publishing AG 2017

Abstract We prove numerical radius inequalities involving commutators of $G_1$ operators and certain analytic functions. Among other inequalities, it is shown that if $A$ and $X$ are bounded linear operators on a complex Hilbert space, then

$$w( f(A)X + X \bar{f}(A)) \leq \frac{2}{d_A^2} w(X - AXA^*),$$

where $A$ is a $G_1$ operator with $\sigma(A) \subset \mathbb{D}$ and $f$ is analytic on the unit disk $\mathbb{D}$ such that $\text{Re}(f) > 0$ and $f(0) = 1$.

Keywords $G_1$ operator · Numerical radius · Commutator · Analytic function

Mathematics Subject Classification Primary 47A12; Secondary 15A60 · 30E20 · 47A30 · 47B15 · 47B20

Communicated by Daniel Aron Alpay.

Fuad Kittaneh
fkitt@ju.edu.jo

Mojtaba Bakherad
mojtaba.bakherad@yahoo.com; bakherad@member.ams.org

1 Department of Mathematics, Faculty of Mathematics, University of Sistan and Baluchestan, Zahedan, Islamic Republic of Iran
2 Department of Mathematics, The University of Jordan, Amman, Jordan
1 Introduction

Let $\mathcal{H}$ be a complex Hilbert space and $\mathbb{B}(\mathcal{H})$ denote the $\mathbb{C}^\ast$-algebra of all bounded linear operators on $\mathcal{H}$ with the identity $I$. In the case when $\dim \mathcal{H} = n$, we identify $\mathbb{B}(\mathcal{H})$ with the matrix algebra $\mathbb{M}_n$ of all $n \times n$ matrices having entries in the complex field. The numerical radius of $A \in \mathbb{B}(\mathcal{H})$ is defined by

$$w(A) := \sup \{ |\langle Ax, x \rangle| : x \in \mathcal{H}, \|x\| = 1 \}.$$  

It is well known that $w(\cdot)$ defines a norm on $\mathbb{B}(\mathcal{H})$, which is equivalent to the usual operator norm $\| \cdot \|$. In fact, for any $A \in \mathbb{B}(\mathcal{H})$, $\frac{1}{2} \|A\| \leq w(A) \leq \|A\|$ (see [9, p. 9]). If $A^2 = 0$, then equality holds in the first inequality, and if $A$ is normal, then equality holds in the second inequality. For further information about numerical radius inequalities, we refer the reader to [1–3,12,16,17] and references therein.

An operator $A \in \mathbb{B}(\mathcal{H})$ is called a $G_1$ operator if the growth condition

$$\| (z - A)^{-1} \| = \frac{1}{\text{dist}(z, \sigma(A))}$$

holds for all $z$ not in the spectrum $\sigma(A)$ of $A$, where $\text{dist}(z, \sigma(A))$ denotes the distance between $z$ and $\sigma(A)$. For simplicity, if $z$ is a complex number, we write $z$ instead of $zI$. It is known that hyponormal (in particular, normal) operators are $G_1$ operators (see, e.g., [15]). Let $A \in \mathbb{B}(\mathcal{H})$ and $f$ be a function which is analytic on an open neighborhood $\Omega$ of $\sigma(A)$ in the complex plane. Then $f(A)$ denotes the operator defined on $\mathcal{H}$ by the Riesz–Dunford integral as

$$f(A) = \frac{1}{2\pi i} \int_C f(z)(z - A)^{-1} dz,$$

where $C$ is a positively oriented simple closed rectifiable contour surrounding $\sigma(A)$ in $\Omega$ (see e.g., [8, p. 568]). The spectral mapping theorem asserts that $\sigma(f(A)) = f(\sigma(A))$. Throughout this note, $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$ denotes the unit disk, $\partial \mathbb{D}$ stands for the boundary of $\mathbb{D}$ and $d_A = \text{dist}(\partial \mathbb{D}, \sigma(A))$. In addition, we denote

$$\mathfrak{A} = \{ f : \mathbb{D} \to \mathbb{C} : f \text{ is analytic, } \Re(f) > 0 \text{ and } f(0) = 1 \}.$$  

The Sylvester type equations $AXB \pm X = C$ have been investigated in matrix theory (see [4]). Several perturbation bounds for the norms of sums or differences of operators have been presented in the literature by employing some integral representations of certain functions. See [5,13,14] and references therein.

In this paper, we present some upper bounds for the numerical radii of the commutators and elementary operators of the form $f(A)X \pm X f(A)$, $(A)X f(B) - f(B)X f(A)$ and $f(A)X f(B) + 2X + f(B)X f(A)$, where $A, B, X \in \mathbb{B}(\mathcal{H})$ and $f \in \mathfrak{A}$. 
2 Main Results

To prove our first result, the following lemma concerning numerical radius inequalities and an equality is required.

Lemma 2.1 [10, 11] Let $A, B, X, Y \in \mathbb{B}(\mathcal{H})$. Then

(a) $w(A^*XA) \leq \|A\|^2 w(X)$.

(b) $w(AX \pm XA^*) \leq 2\|A\| w(X)$.

(c) $w(A^*XB \pm B^*YA) \leq 2\|A\|\|B\| w\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right)$.

(d) $w\left(\begin{bmatrix} 0 & AXB^* \\ BYA^* & 0 \end{bmatrix}\right) \leq \max\{|\|A\|^2, |\|B\|^2\} w\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right)$.

(e) $w\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) \leq \frac{w(X+Y)+w(X-Y)}{2}$.

(f) $w\left(\begin{bmatrix} 0 & X \\ e^{i\theta}X & 0 \end{bmatrix}\right) = w(X)$ for $\theta \in \mathbb{R}$.

Proof: Since all parts, except part (d), have been shown in [10, 11], we prove only part (d). If we take $C = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ and $S = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}$, then $CSC^* = \begin{bmatrix} 0 & AXB^* \\ BYA^* & 0 \end{bmatrix}$. Now, using part (a), we have

$$w\left(\begin{bmatrix} 0 & AXB^* \\ BYA^* & 0 \end{bmatrix}\right) = w(CSC^*)$$

$$\leq \|C\|^2 w(S)$$

$$= \max\{|\|A\|^2, |\|B\|^2\} w\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right),$$

as required. \qed

Now, we are in position to demonstrate the main results of this section by using some ideas from [13, 14].

Theorem 2.2 Let $A \in \mathbb{B}(\mathcal{H})$ be a $G_1$ operator with $\sigma(A) \subset D$ and $f \in \mathfrak{A}$. Then for every $X \in \mathbb{B}(\mathcal{H})$, we have

$$w(f(A)X + X\tilde{f}(A)) \leq \frac{2}{d_A^2} w(X - AXA^*)$$

and

$$w(f(A)X - X\tilde{f}(A)) \leq \frac{4}{d_A^2} \|A\| w(X).$$
Proof Using the Herglotz representation theorem (see e.g., [7, p.21]), we have
\[
    f(z) = \int_0^{2\pi} \frac{e^{i\alpha} + z}{e^{i\alpha} - z} d\mu(\alpha) + i \text{Im} f(0) = \int_0^{2\pi} \frac{e^{i\alpha} + z}{e^{i\alpha} - z} d\mu(\alpha),
\]
where \( \mu \) is a positive Borel measure on the interval \([0, 2\pi]\) with finite total mass \( \int_0^{2\pi} d\mu(\alpha) = f(0) = 1 \). Hence,
\[
    \bar{f}(z) = \int_0^{2\pi} \frac{e^{i\alpha} + z}{e^{i\alpha} - z} d\mu(\alpha) = \int_0^{2\pi} \frac{e^{-i\alpha} + \bar{z}}{e^{-i\alpha} - \bar{z}} d\mu(\alpha),
\]
where \( \bar{f} \) is the conjugate function of \( f \). So,
\[
    f(A)X + X\bar{f}(A) = \int_0^{2\pi} \left[ \left( e^{i\alpha} + A \right) \left( e^{i\alpha} - A \right)^{-1} X \\
    + X \left( e^{-i\alpha} + A^* \right) \left( e^{-i\alpha} - A^* \right)^{-1} \right] d\mu(\alpha)
    = \int_0^{2\pi} \left( e^{i\alpha} - A \right)^{-1} \left[ \left( e^{i\alpha} + A \right) X \left( e^{-i\alpha} - A^* \right) \\
    + \left( e^{i\alpha} - A \right) X \left( e^{-i\alpha} + A^* \right) \right] \left( e^{-i\alpha} - A^* \right)^{-1} d\mu(\alpha)
    = 2 \int_0^{2\pi} \left( e^{i\alpha} - A \right)^{-1} \left( X - AXA^* \right) \left( e^{-i\alpha} - A^* \right)^{-1} d\mu(\alpha).
\]
Hence,
\[
    w(f(A)X + X\bar{f}(A))
    = w \left( \int_0^{2\pi} \left[ \left( e^{i\alpha} + A \right) \left( e^{i\alpha} - A \right)^{-1} X \\
    + X \left( e^{-i\alpha} + A^* \right) \left( e^{-i\alpha} - A^* \right)^{-1} \right] d\mu(\alpha) \right)
    = 2 w \left( \int_0^{2\pi} \left( e^{i\alpha} - A \right)^{-1} \left( X - AXA^* \right) \left( e^{-i\alpha} - A^* \right)^{-1} d\mu(\alpha) \right)
    \leq 2 \int_0^{2\pi} w \left( \left( e^{i\alpha} - A \right)^{-1} \left( X - AXA^* \right) \left( e^{-i\alpha} - A^* \right)^{-1} \right) d\mu(\alpha)
    \text{(since } w(\cdot) \text{ is a norm)}
    \leq 2 \int_0^{2\pi} \left\| \left( e^{i\alpha} - A \right)^{-1} \right\|^2 w \left( X - AXA^* \right) d\mu(\alpha)
    \text{(by Lemma 2.1(a)).}
Since $A$ is a $G_1$ operator, it follows that

$$\left\| \left( e^{i\alpha} - A \right)^{-1} \right\| = \frac{1}{\text{dist}(e^{i\alpha}, \sigma(A))} \leq \frac{1}{\text{dist}(\partial \mathbb{D}, \sigma(A))} = \frac{1}{d_A},$$

and so

$$w \left( f(A)X + X \tilde{f}(A) \right) \leq \left( \frac{2}{d_A^2} \int_0^{2\pi} d\mu(\alpha) \right) w(X - AXA^*)$$

$$= \left( \frac{2}{d_A^2} f(0) \right) w(X - AXA^*)$$

$$= \frac{2}{d_A^2} w(X - AXA^*).$$

This proves the first inequality.

Similarly, it follows from the equations

$$f(A)X - X \tilde{f}(A) = \int_0^{2\pi} \left[ \left( e^{i\alpha} + A \right) \left( e^{i\alpha} - A \right)^{-1} X 
- X \left( e^{-i\alpha} + A^* \right) \left( e^{-i\alpha} - A^* \right)^{-1} \right] d\mu(\alpha)$$

$$= \int_0^{2\pi} \left( e^{i\alpha} - A \right)^{-1} \left[ \left( e^{i\alpha} + A \right) X \left( e^{-i\alpha} - A^* \right) 
- \left( e^{i\alpha} - A \right) X \left( e^{-i\alpha} + A^* \right) \left( e^{-i\alpha} - A^* \right)^{-1} \right] d\mu(\alpha)$$

$$= 2 \int_0^{2\pi} \left( e^{i\alpha} - A \right)^{-1} \left( e^{-i\alpha} AX - e^{i\alpha}XA^* \right) 
\times \left( e^{-i\alpha} - A^* \right)^{-1} d\mu(\alpha)$$

$$= 2 \int_0^{2\pi} \left( e^{i\alpha} - A \right)^{-1} \left( \left( e^{-i\alpha} A \right) X 
- X \left( e^{-i\alpha} A^* \right) \left( e^{i\alpha} - A^* \right)^{-1} \right) d\mu(\alpha)$$

that

$$w(f(A)X - X \tilde{f}(A))$$

$$= 2w \left( \int_0^{2\pi} \left( e^{i\alpha} - A \right)^{-1} \left( \left( e^{-i\alpha} A \right) X - X \left( e^{-i\alpha} A^* \right) \left( e^{-i\alpha} - A^* \right)^{-1} \right) d\mu(\alpha) \right)$$

$$\leq 2 \int_0^{2\pi} w \left( \left( e^{i\alpha} - A \right)^{-1} \left( \left( e^{-i\alpha} A \right) X - X \left( e^{-i\alpha} A^* \right) \left( e^{-i\alpha} - A^* \right)^{-1} \right) d\mu(\alpha) \right)$$

(since $w(\cdot)$ is a norm)
\begin{equation}
\leq 2 \int_{0}^{2\pi} \left\| (e^{i\alpha} - A)^{-1} \right\|^2 w \left( \left( e^{-i\alpha} A \right) X - X \left( e^{-i\alpha} A \right)^{*} \right) d\mu(\alpha)
\end{equation}
(by Lemma 2.1 (a))

\begin{equation}
\leq 4 \int_{0}^{2\pi} \left\| (e^{i\alpha} - A)^{-1} \right\|^2 \| e^{-i\alpha} A \| w(X) d\mu(\alpha)
\end{equation}
(by Lemma 2.1 (b))

\begin{equation}
\leq \frac{4}{d_A^2} \| A \| w(X) \int_{0}^{2\pi} d\mu(\alpha)
\end{equation}

\begin{equation}
\leq \frac{4}{d_A^2} \| A \| w(X).
\end{equation}

This proves the second inequality and completes the proof of the theorem. \qed

If we take \( X = I \) in Theorem 2.2, we get the following result. Observe that \( \bar{f}(A) = (f(A))^* \).

**Corollary 2.3** Let \( A \in \mathcal{B}(\mathcal{H}) \) be a \( G_1 \) operator with \( \sigma(A) \subset \mathbb{D} \) and \( f \in \mathfrak{A} \). Then

\begin{equation}
\| \text{Re}(f(A)) \| \leq \frac{1}{d_A^2} \| I - AA^* \|
\end{equation}

and

\begin{equation}
\| \text{Im}(f(A)) \| \leq \frac{2}{d_A^2} \| A \|.
\end{equation}

**Theorem 2.4** Let \( A, B \in \mathcal{B}(\mathcal{H}) \) be \( G_1 \) operators with \( \sigma(A) \cup \sigma(B) \subset \mathbb{D} \) and \( f \in \mathfrak{A} \). Then for every \( X \in \mathcal{B}(\mathcal{H}) \), we have

\begin{equation}
w(f(A)X \bar{f}(B) - f(B)X \bar{f}(A))
\leq \frac{2}{d_A d_B} \left[ 2w(X) + w(AXB^* + BXA^*) + w(AXB^* - BXA^*) \right]
\end{equation}

and

\begin{equation}
w(f(A)X \bar{f}(B) + 2X + f(B)X \bar{f}(A))
\leq \frac{2}{d_A d_B} \left[ 2w(X) + w(AXB^* + BXA^*) + w(AXB^* - BXA^*) \right].
\end{equation}

**Proof** We have

\begin{equation}
f(A)X \bar{f}(B) - f(B)X \bar{f}(A)
= \int_{0}^{2\pi} \int_{0}^{2\pi} \left[ (e^{i\alpha} - A)^{-1} (e^{i\alpha} + A) X (e^{-i\beta} + B^*) (e^{-i\beta} - B^*)^{-1}
- (e^{i\beta} - B)^{-1} (e^{i\beta} + B) X (e^{-i\alpha} + A^*) (e^{-i\alpha} - A^*)^{-1} \right] d\mu(\alpha) d\mu(\beta).
\end{equation}
Using the equations

\[
\left( e^{i\alpha} - A \right)^{-1} (e^{i\alpha} + A) X (e^{-i\beta} + B^*) (e^{-i\beta} - B^*)^{-1} \\
- (e^{i\beta} - B)^{-1} (e^{i\beta} + B) X (e^{-i\alpha} + A^*) (e^{-i\alpha} - A^*)^{-1} \\
= \left( e^{i\alpha} - A \right)^{-1} (e^{i\alpha} + A) X (e^{-i\beta} + B^*) (e^{-i\beta} - B^*)^{-1} + X \\
- X - (e^{i\beta} - B)^{-1} (e^{i\beta} + B) X (e^{-i\beta} + A^*) (e^{-i\beta} - A^*)^{-1} \\
= \left( e^{i\alpha} - A \right)^{-1} \left[ (e^{i\alpha} + A) X (e^{-i\beta} + B^*) \right. \\
+ (e^{i\alpha} - A) X (e^{-i\beta} - B^*) \left( e^{-i\beta} - B^* \right)^{-1} \\
- (e^{i\beta} - B)^{-1} \left[ (e^{i\beta} - B) X (e^{-i\alpha} - A^*) \right. \\
+ (e^{i\beta} + B) X (e^{-i\alpha} + A^*) \left( e^{-i\alpha} - A^* \right)^{-1} \\
= 2(e^{i\alpha} - A)^{-1} (e^{i\alpha} e^{-i\beta} X + AXB^*) (e^{-i\beta} - B^*)^{-1} \\
- 2(e^{i\beta} - B)^{-1} (e^{-i\alpha} e^{i\beta} X + AXA^*) (e^{-i\alpha} - A^*)^{-1},
\]

we have

\[
w(f(A)Xf(B) - f(B)Xf(A)) \\
= 2w \left( \int_0^{2\pi} \int_0^{2\pi} (e^{i\alpha} - A)^{-1} (e^{i\alpha} e^{-i\beta} X + AXB^*) (e^{-i\beta} - B^*)^{-1} \\
- (e^{i\beta} - B)^{-1} (e^{-i\alpha} e^{i\beta} X + BXA^*) (e^{-i\alpha} - A^*)^{-1} d\mu(\alpha) d\mu(\beta) \right) \\
\leq 2 \int_0^{2\pi} \int_0^{2\pi} w \left( (e^{i\alpha} - A)^{-1} (e^{i\alpha} e^{-i\beta} X + AXB^*) (e^{-i\beta} - B^*)^{-1} \\
- (e^{i\beta} - B)^{-1} (e^{-i\alpha} e^{i\beta} X + BXA^*) (e^{-i\alpha} - A^*)^{-1} \right) d\mu(\alpha) d\mu(\beta) \\
\text{(since } w(\cdot) \text{ is a norm)} \\
\leq 4 \int_0^{2\pi} \int_0^{2\pi} \| (e^{i\alpha} - A)^{-1} \| \| (e^{i\beta} - B)^{-1} \| \\
\times w \left( \begin{bmatrix} 0 & e^{i\alpha} e^{-i\beta} X + AXB^* \\ e^{-i\alpha} e^{i\beta} X + BXA^* & 0 \end{bmatrix} \right) d\mu(\alpha) d\mu(\beta) \\
\text{(by Lemma 2.1 (c))} \\
\leq \frac{4}{d_A d_B} \int_0^{2\pi} \int_0^{2\pi} \left[ w \left( \begin{bmatrix} 0 & e^{i\alpha} e^{-i\beta} X \\ -e^{-i\alpha} e^{i\beta} X & 0 \end{bmatrix} \right) \\
+ w \left( \begin{bmatrix} 0 & AXB^* \\ BXA^* & 0 \end{bmatrix} \right) \right] d\mu(\alpha) d\mu(\beta)
\]
we have

Similarly, we have

This proves the first inequality. Similarly, we have

Using the equations

we have

(by Lemma 2.1 (e) and (f)).
\[
\begin{align*}
&\leq 2 \int_0^{2\pi} \int_0^{2\pi} w \left( (e^{i\alpha} - A)^{-1}(e^{i\alpha} e^{-i\beta} X + AXB^*) (e^{-i\beta} - B^*)^{-1} \\
&\quad + (e^{i\beta} - B)^{-1}(e^{-i\alpha} e^{i\beta} X + BXA^*) (e^{-i\alpha} - A^*)^{-1} \right) d\mu(\alpha) d\mu(\beta) \\
&\leq 4 \int_0^{2\pi} \int_0^{2\pi} \left\| (e^{i\alpha} - A)^{-1} \right\| \left\| (e^{i\beta} - B)^{-1} \right\| \\
&\quad \times w \left( \begin{bmatrix} 0 & e^{i\alpha} e^{-i\beta} X + AXB^* \\ e^{-i\alpha} e^{i\beta} X + BXA^* & 0 \end{bmatrix} \right) d\mu(\alpha) d\mu(\beta) \\
&\leq \frac{4}{d_A d_B} \int_0^{2\pi} \int_0^{2\pi} \left[ w \left( \begin{bmatrix} 0 & e^{i\alpha} e^{-i\beta} X + AXB^* \\ e^{-i\alpha} e^{i\beta} X + BXA^* & 0 \end{bmatrix} \right) \right] d\mu(\alpha) d\mu(\beta) \\
&\quad + \frac{4}{d_A d_B} \int_0^{2\pi} \int_0^{2\pi} \left[ w \left( \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \right) + w \left( \begin{bmatrix} 0 & AXB^* \\ BXA^* & 0 \end{bmatrix} \right) \right] d\mu(\alpha) d\mu(\beta) \\
&\leq \frac{2}{d_A d_B} \left[ 2w(X) + w \left( AXB^* + BXA^* \right) + w \left( AXB^* - BXA^* \right) \right]
\end{align*}
\]

(by Lemma 2.1 (c))

This proves the second inequality and completes the proof of the theorem. \(\Box\)

**Remark 2.5** Under the assumptions of Theorem 2.4 and the hypothesis that \(X\) is self-adjoint, we have

\[
\| f(A)X \tilde{f}(B) - f(B)X \tilde{f}(A) \| \\
\leq \frac{4}{d_A d_B} \max\{ \|X\| + \|AXB^*\|, \|X\| + \|BXA^*\| \}
\]

and

\[
\| f(A)X \tilde{f}(B) + 2X + f(B)X \tilde{f}(A) \| \\
\leq \frac{4}{d_A d_B} \max\{ \|X\| + \|AXB^*\|, \|X\| + \|BXA^*\| \}.
\]

To see this, first note that if \(X\) is self-adjoint, then the operator matrix

\[
T = \begin{bmatrix}
0 & e^{i\alpha} e^{-i\beta} X + AXB^* \\
e^{-i\alpha} e^{i\beta} X + BXA^* & 0
\end{bmatrix}
\]

is self-adjoint, hence \(w(T) = \|T\|\). Moreover, \(T = M + N\), where

\[
M = \begin{bmatrix}
0 & e^{i\alpha} e^{-i\beta} X \\
e^{-i\alpha} e^{i\beta} X & 0
\end{bmatrix}, \quad N = \begin{bmatrix}
0 & AXB^* \\
BXA^* & 0
\end{bmatrix}
\]
are self-adjoint operators. Using the fact that \( \| C + D \| \leq \| |C| + |D| \| \) for any normal operators \( C \) and \( D \) (see [6]), we have

\[
\begin{align*}
\omega(T) &= \| M + N \| \leq \| |M| + |N| \| \\
&= \max\{ \| |X| \| + \| |AXB^*| \|, \| |X| \| + \| |BXA^*| \| \}.
\end{align*}
\]

Hence, we get the required inequalities by the same arguments as in the proof of Theorem 2.4.

If we take \( X = I \) in Theorem 2.4, we get the following result.

**Corollary 2.6** Let \( A, B \in \mathcal{B}(\mathcal{H}) \) be \( G_1 \) operators with \( \sigma(A) \cup \sigma(B) \subset \mathbb{D} \) and \( f \in \mathfrak{A} \). Then

\[
\| \text{Im}(f(A) \overline{f}(B)) \| \leq \frac{2}{d_A d_B} \left( 1 + \| A B^* \| \right)
\]

and

\[
\| \text{Re}(f(A) \overline{f}(B)) + I \| \leq \frac{2}{d_A d_B} \left( 1 + \| A B^* \| \right).
\]

**Remark 2.7** If instead of applying Lemma 2.1 (c) we use Lemma 2.1 (d) and (f) in the proof Theorem 2.4, we obtain the related inequalities

\[
\begin{align*}
\omega(f(A) X \overline{f}(B) - f(B) X \overline{f}(A)) &\leq \frac{4}{d_A d_B} \left[ 1 + \max\{ \| A \|^2, \| B \|^2 \} \right] \omega(X) \\
\omega(f(A) X \overline{f}(B) + 2X + f(B) X \overline{f}(A)) &\leq \frac{4}{d_A d_B} \left[ 1 + \max\{ \| A \|^2, \| B \|^2 \} \right] \omega(X).
\end{align*}
\]

**Acknowledgements** The first author would like to thank the Tusi Mathematical Research Group (TMRG).

**References**

1. Abu-Omar, A., Kittaneh, F.: Estimates for the numerical radius and the spectral radius of the Frobenius companion matrix and bounds for the zeros of polynomials. Ann. Func. Anal. 5(1), 56–62 (2014)
2. Abu-Omar, A., Kittaneh, F.: Numerical radius inequalities for products of Hilbert space operators. J. Oper. Theory 72(2), 521–527 (2014)
3. Abu-Omar, A., Kittaneh, F.: Notes on some spectral radius and numerical radius inequalities. Stud. Math. 227(2), 97–109 (2015)
4. Bao, L., Lin, Y., Wei, Y.: Krylov subspace methods for the generalized Sylvester equation. Appl. Math. Comput. 175(1), 557–573 (2006)
5. Bhatia, R., Sinha, K.B.: Derivations, derivatives and chain rules. Linear Algebra Appl. 302/303, 231–244 (1999)
6. Bourin, J.C.: Matrix subadditivity inequalities and block-matrices. Int. J. Math. 20(6), 679–691 (2009)
7. Donoghue, W.F.: Monotone Matrix Functions and Analytic Continuation. Springer, New York (1974)
8. Dunford, N., Schwartz, J.: Linear Operators I. Interscience, New York (1958)
9. Gustafson, K.E., Rao, D.K.M.: Numerical Range, The Field of Values of Linear Operators and Matrices. Springer, New York (1997)
10. Hirzallah, O., Kittaneh, F., Shebrawi, Kh: Numerical radius inequalities for commutators of Hilbert space operators. Numer. Funct. Anal. Optim. 32(7), 739–749 (2011)
11. Hirzallah, O., Kittaneh, F., Shebrawi, Kh: Numerical radius inequalities for certain $2 \times 2$ operator matrices. Integral Equ. Oper. Theory 71(1), 129–147 (2011)
12. Kittaneh, F.: A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix. Stud. Math. 158(1), 11–17 (2003)
13. Kittaneh, F.: Norm inequalities for commutators of $G_1$ operators. Complex Anal. Oper. Theory 10(1), 109–114 (2016)
14. Kittaneh, F., Moslehian, M.S., Sababheh, M.: Unitarily invariant norm inequalities for elementary operators involving $G_1$ operators. Linear Algebra Appl. 513, 84–95 (2017)
15. Putnam, C.R.: Operators satisfying a $G_1$ condition. Pac. J. Math. 84, 413–426 (1979)
16. Sheikholeslami, A., Moslehian, M.S., Shebrawi, K.: Inequalities for generalized Euclidean operator radius via Young’s inequality. J. Math. Anal. Appl. 445(2), 1516–1529 (2017)
17. Yamazaki, T.: On upper and lower bounds of the numerical radius and an equality condition. Stud. Math. 178, 83–89 (2007)