CORRECTIONS OF ORDER \((Z\alpha)^6 \frac{m_e^2}{m_\mu}\) IN THE MUONIUM FINE STRUCTURE

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Abstract

In the framework of the quasipotential method we calculate the contributions of the kind \((Z\alpha)^6 \frac{m_e^2}{m_\mu}\) to the energy spectrum of the muonium \(n^3S_1\) states. Numerical value of obtained correction for \(2^3S_1 \div 1^3S_1\) muonium fine structure interval is equal to 0.19 MHz.

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The investigation of muonium and positronium fine structure represents one of the basic tests of quantum electrodynamics, which is sensitive to radiative corrections of higher order on $\alpha$ [1]. Many papers [2, 3, 4, 5] are devoted to the calculation of different contributions to the fine structure of hydrogen-like atom energy levels. The interest to this problem remains unchanged [6, 7, 8]. The progress, achieved in the last years during the calculation of logarithmic contributions of order $\alpha^6 \ln \alpha$ in the positronium fine structure intervals $(2^3S_1 \div 1^3S_1, 2^3S_1 \div 2^3P_J)$ [10, 11, 12], doesn’t abolish the necessity of calculation of higher order corrections $O(\alpha^6)$ [13]. As discussed in the paper [9], there is some difference in the calculation of the corrections ($Z\alpha^6 m_1^2/m_2$), obtained in [8, 7]. The development of experimental methods, based on Doppler-free two-photon spectroscopy, allows the ”large” structure intervals to be measured for the muonium and the positronium [14, 15, 16].

$$\Delta E_{Ps}^{exp.}(2^3S_1 \div 1^3S_1) = \begin{cases} 1233607218, 9 \pm 10, 7 \text{ MHz} \\ 1233607216, 4 \pm 3, 2 \text{ MHz} \end{cases}$$

$$\Delta E_{Mu}^{exp.}(2^3S_1 \div 1^3S_1) = 2455527936 \pm 120 \pm 140 \text{ MHz}.$$ (2)

The frequency of Doppler-free two-photon transition $1S \div 2S$ in the hydrogen atom as well as hyperfine splitting of the ground state of hydrogen atom represents the quantity, which was measured with high accuracy [13]. The increase of the experimental accuracy of the muonium fine structure interval measurements (just as for positronium), planned in the near future, makes very actual the calculation of radiative corrections of higher order on $\alpha$. In this paper we have performed studies of the recoil contributions of order $O(\alpha^6)$ in the muonium fine structure. The contribution of such order to the muonium hyperfine structure was obtained in [17]. There exist many approaches for description of relativistic energy spectrums of two-particle bound states in quantum electrodynamics [3, 4, 18, 19, 20]. They all differ in organization of the calculation: the bound state equation for two-particle system, the construction of the interaction operator, the degree of complication in the calculation of the definite order corrections in the energy levels. But all such methods give equivalent results in the fixed order of perturbation theory on the small parameter $\alpha$ and $m_1/m_2$. Our calculations are based on the Schrödinger-type local quasipotential equation [21]

$$\left( \frac{b^2}{2\mu_R} - \frac{\mathbf{p}^2}{2\mu_R} \right) \psi_M(\mathbf{p}) = \int \frac{d\mathbf{q}}{(2\pi)^3} V(\mathbf{p}, \mathbf{q}, M) \psi_M(\mathbf{q}),$$

where $b^2 = E_1^2 - m_1^2 = E_2^2 - m_2^2$, $\mu_R = E_1E_2/M$ is relativistic reduced mass, $M = E_1 + E_2$ is the bound state mass, $m_1, m_2$ are the masses of the electron and the muon. As an initial approximation of quasipotential $V(\mathbf{p}, \mathbf{q}, M)$ for the bound state system ($e^-\mu^+$) we choose the ordinary Coulomb potential. On the basis of equation (3) in [22] we have obtained some relativistic corrections $m\alpha^6$ in the positronium fine structure from the one-photon, two-photon interaction and the second order perturbation theory. First of all, the contribution of order $(Z\alpha)^6 m_1/m_2$

$$\Delta B_1 = \frac{5m_1^2(Z\alpha)^6}{2m_2n^6}$$ (4)
appears from the condition of quantization of the energy levels for Coulomb interaction
\[ \frac{b^2}{\mu_R^2} = -\frac{\alpha^2}{n^2}, \quad (5) \]
which we have transformed for binding energy B.

1 One-photon interaction contribution to the fine structure

The basic contribution to the energy spectrum of two-particle bound state \((\mu^+e^-)\) is determined by one-photon interaction. The construction of one-photon quasipotential in the system of two spinor particles was done in [22, 23]. It is useful to note, that procedure of quasipotential derivation may be essentially simplified in this case, just as for two-photon and three-photon exchange diagrams by introducing the relativistic operator of projection onto \(^3S_1\) muonium state:
\[ \hat{\Pi} = \frac{1}{2\sqrt{2}} \sqrt{\frac{2}{\varepsilon_1 + m_1 \sqrt{\varepsilon_2 + m_2}}} (\hat{\rho}_1 + m_1 \gamma_0) \hat{\varepsilon} (-\hat{p}_2 + m_2) \]
\[ (6) \]

(7) over Coulomb wave functions [24] and extracting necessary order contributions, we have obtained:
\[ \Delta B_2 = -\frac{3m_1^2(Z\alpha)^6}{4m_2n^5} \left( 5 + \frac{2}{n} \right). \quad (8) \]

The quasipotential of one-photon interaction contains also a number of other terms, which lead to the energy corrections of order \(O(Z\alpha)^6\). They may be found when constructing \(V_{1\gamma}\) with the accuracy to terms of fourth order over \(|\vec{p}|/m_{1,2}\), \(|\vec{q}|/m_{1,2}\) in the form
\[ \Delta V_2(\vec{p}, \vec{q}, M) = -\frac{4\pi Z\alpha}{\vec{k}^2} \left\{ \frac{b^4}{16m_1^2} \left( 3 - \frac{2m_1}{m_2} \right) + \frac{\vec{q}^4}{96m_1^4} \left( 3 + \frac{m_1}{m_2} \right) - \right\} \]
\[ (9) \]
\[-\frac{(\vec{p}^2 + \vec{q}^2)(\vec{p}\vec{q})}{96m_1^4} \left(6 + \frac{13m_1}{m_2}\right) - \frac{(\vec{p}^2 + \vec{q}^2)b^2}{96m_1^4} \left(3 - \frac{m_1}{m_2}\right) - \frac{(\vec{p}\vec{q})b^2}{48m_1^4} \left(3 + \frac{7m_1}{m_2}\right)\].

Let consider calculation of \(O(Z\alpha)^6\) corrections from \(\Delta V_2\). A series of \(\Delta V_2\) terms will cause the divergent integrals in the energy spectrum. The reason behind this divergence is that, in deriving (9), we expanded all quantities determining the quasipotential in the relative momenta, which are proportional to \(\alpha\). A typical divergent integral with respect to momenta has the form \(\int \frac{d\vec{p}}{(2\pi)^3} \frac{\vec{p}^2}{\mu^2} \psi_{nS}(\vec{p}) = -\frac{3 + 2(n-1)(n+1)}{n^2} \alpha^2 \psi_{nS}(\vec{r} = 0).\) (10)

Taking into account (9) under averaging \(\Delta V_2\) we have calculated the relativistic corrections of necessary order for the energy levels with arbitrary principal quantum number \(n\):

\[\Delta B_3 = \frac{m_2^2(Z\alpha)^6}{m_2} \left( -\frac{139}{72n^3} + \frac{17\ln 2}{12n^3} + \frac{73}{72n^5} + \frac{43}{96n^6} + \frac{17}{12}(-1)^n \frac{1}{n^3}[C + \psi(n) - 1] \right).\] (11)

where \(\psi(z) = d\ln \Gamma(z)/dz, C=0.5772156649...\) is the Euler constant.

2 Second order of perturbation theory

In the second order of perturbation theory, the correction to the muonium energy spectrum is determined in our approach by the sum of two terms [25]:

\[\Delta B^{(2)} = \psi_n^c |\Delta V_1| \psi_n^c \mu \frac{\partial \Delta V_1}{B} |\psi_n^c > + \sum_{k=1, k\neq n}^\infty \frac{\psi_n^c |\Delta V_1| \psi_k^c >}{B_n - B_k^c} \psi_k^c |\Delta V_1| \psi_k^c >.\] (12)

The quasipotential (9) explicitly depends on the bound state energy \(B\) (the factors \(b^2\) and \(\mu_R\)). Bearing in mind that to a precision adopted here, the relation \(\partial b^2/\partial B = 2\mu\) holds, we obtain:

\[\Delta B_4 = \psi_n^c |\Delta V_1| \psi_n^c \mu \frac{\partial \Delta V_1}{\partial B} |\psi_n^c > = \frac{m_2^2(Z\alpha)^6 1}{n^5}.\] (13)

The sum over Coulomb states in (12) is defined by the reduced nonrelativistic Green’s function [19 20 27 28 29 30 31], which has the following partial expansion:

\[\tilde{G}_n(\vec{r}, \vec{r}', B) = \sum_{l,m} \tilde{g}_{nl}(r, r', B)Y_{lm}(\vec{n})Y^*_{lm}(\vec{n}')\] (14)

The radial function \(\tilde{g}_{nl}(r, r', B_n)\) was obtained in [31] as an expansion over Laguerre’s polynomials. In the case of S-wave states we have for it the next expression:

\[\tilde{g}_{n0}(r, r', B_n) = \frac{4Z\alpha \mu^2}{n} \left[ e^{-\frac{r+r'}{2}} \sum_{m=1, m\neq n}^\infty \frac{L^1_{m-1}(x)L^1_{m-1}(x')}{m(m-n)} + \right.\] (15)
\[ + \frac{1}{n^2} \left( \frac{5}{2} + x \frac{\partial}{\partial x} + x' \frac{\partial}{\partial x'} \right) e^{-\frac{x+x'}{2}} L^1_{n-1}(x)L^1_{n-1}(x') \],

where \( x = 2\mu Z\alpha r/n \), \( L^m_n \) are the ordinary Laguerre’s polynomials, which may be found by means the following relation:

\[
L^m_n(x) = e^x x^{-m} \left( \frac{d}{dx} \right)^n (e^{-x} x^n). \tag{16}
\]

The reduced Coulomb Green’s function (RCGF) depends from two variables \( r \) and \( r' \). Due to the appearance of the \( \delta \)-like potential in (9) the RCGF is needed for \( \vec{r} = 0 \). The necessary expression for RCGF was obtained by means of Hostler’s representation for Coulomb Green’s function in \([32]\), after subtraction of pole-like term:

\[
\bar{G}_n(\vec{r}, 0, B_n) = -\frac{Z\alpha\mu^2}{n\pi x} e^{\frac{-x}{2}} \sum_{s=0}^{n-1} \frac{(-x)^{n-s}}{s! \left( (n-s)! \right)^2} \frac{n!}{(n-s)!} \left\{ (n-s)[\psi(n+1) - 2\psi(n-s+1)] - \frac{2(n-s) + 3 - x}{2n} + \ln x + 1 \right\}, \tag{17}
\]

where \( \psi(z) = d\ln \Gamma(z)/dz \). Contrary to the paper \([32]\) this formula doesn’t contain free two-particle Green’s function \( G^f(r) = -\mu Re^{-Z\alpha\mu r}/2\pi r \), which controls the iteration part of quasipotential. Its contribution to the energy spectrum will be find separately. Let consider, for example, calculation of energy correction in the second order of perturbation theory, depending from \( \delta \)-like part of the quasipotential and the term of \( \Delta V_1 \sim 1/r^2 \). This contribution can be represented in the form:

\[
\delta B = -\frac{\mu^5(Z\alpha)^6}{3m_1 m_2 n^4} \sum_{k=1}^{n} (-1)^k \frac{n!}{(n-k)!(k!)^2} \int_0^{\infty} x^{k-1} e^{-x} L^1_{n-1}(x)dx \left\{ k[\psi(n+1) - 2\psi(k+1)] - \frac{2k + 3 - x}{2n} + \ln x + 1 \right\}. \tag{18}
\]

The expression (18) contains the integrals of two types with the power and the logarithmic functions correspondingly. Calculation of the first integral over \( x \) variable leads to the result:

\[
I_1 = \int_0^{\infty} x^{k-1} e^{-x} L^1_{n-1}(x) = \frac{(k-1)!\Gamma(n+1-k)}{(n-1)!\Gamma(2-k)}. \tag{19}
\]

So, we see that the sum over \( k \) contains only one term with \( k = 1 \). Second integral of (18)

\[
I_2 = \int_0^{\infty} x^{k-1} \ln xe^{-x} L^1_{n-1}(x) = \frac{(2-k)n-1\Gamma(k)}{(n-1)!}[\psi(k) + \psi(2-k) - \psi(n+1-k)] \tag{20}
\]

gives rise the sum of the kind

\[
\sum_{k=1}^{n} (-1)^k \frac{\psi(2-k)(2-k)n-1}{(n-k)!k!} = \frac{n-1}{n} + C, \tag{21}
\]
where \( C = \lim_{n \to \infty} \left[ - \ln n + \sum_{m=1}^{n} \frac{1}{m} \right] = 0.57721566 \ldots \). Taking into account that

\[
\lim_{k \to n} \frac{\Gamma(n-k)}{\Gamma(1-k)} = (-1)^{n-1}(n-1)!,
\]

we have obtained conclusively respective correction (18):

\[
\delta B = -\frac{m_2^2}{6m_2} (Z\alpha)^6 \frac{1}{n^4} (4n + 9).
\]

**Table 1. Second order contributions of the perturbation theory, defined by the RCGF and quasipotential (9) in the units \( \frac{(Z\alpha)^6 m_2^2}{m_2} \)**

| Term | \( -\frac{Z\alpha^2}{m_1 m_2} \) | \( -\frac{(Z\alpha)^2 \mu R}{2m_1^2} (1 + \frac{2m_1}{m_2}) \) | \( -\frac{Z\alpha (\nabla)}{4m_1^2 r^4} (1 + \frac{4m_1}{m_2}) \) | \( -\frac{\pi Z\alpha}{3m_1 m_2} \delta(\vec{r}) \) |
|------|------------------|------------------|------------------|------------------|
| \( \Delta V_1 \) | \( 2 - \frac{n}{n^3} \) | \( \frac{2n^2 - 5n + 1}{4n^6} \) | \( \frac{n^2 + 3n - 1}{4n^6} \) | \( \frac{(4n + 9)}{6n^4} \) |
| \( \frac{(Z\alpha)^2 \mu R}{2m_1^2} \) | \( \frac{2n^2 - 5n + 1}{4n^6} \) | \( \frac{-n^2 + 3n - 1}{4n^6} \) | \( \frac{-n(n-1)(n+1)}{24n^6} \) | \( \frac{(n^2 - 6n + 8)}{12n^6} \) |
| \( -\frac{Z\alpha (\nabla)}{4m_1^2 r^4} \) | \( -\frac{n^2 - 3n + 1}{4n^6} \) | \( \frac{-n(n-1)(n+1)}{24n^6} \) | \( \frac{-n(n-1)(n+1)}{24n^6} \) | \( \frac{(n^2 - 6n + 8)}{12n^6} \) |
| \( \frac{\pi Z\alpha}{3m_1 m_2} \delta(\vec{r}) \) | \( -\frac{(4n + 9)}{6n^4} \) | \( \frac{-n^2 + 3n - 1}{4n^6} \) | \( \frac{-n(n-1)(n+1)}{24n^6} \) | \( \frac{(n^2 - 6n + 8)}{12n^6} \) |

Similar calculations may be carried out to find the contributions of the other terms of quasipotential (9) in the second order of perturbation theory. The respective results of such calculation represented in the Table 1. The full correction, connected with reduced Coulomb Green’s function without considering \( G_f \) in the energy spectrum is:

\[
\Delta B_5 = \left( -\frac{25}{24} - \frac{3}{n} - \frac{49}{24n^2} + \frac{3}{2n^3} \right) \frac{m_1^2 (Z\alpha)^6}{m_2 n^3}.
\]

Let consider now omitting in the RCGF contribution of free two-particle propagator to the correction \( \Delta B^{(2)} \). It is convenient to perform necessary calculations in the momentum representation. Taking in mind that

\[
G_f(\vec{p}, \vec{q}, M) = \frac{(2\pi)^3 \delta(\vec{p} - \vec{q})}{\frac{p^2}{2\mu_R} - \frac{q^2}{2\mu_R}}.
\]

and \( \delta \)-like term of quasipotential (9) already contains the muon mass in the denominator, we may represent the iteration-type correction as:

\[
\Delta B_6 = \frac{2\mu (Z\alpha)^2 \pi \psi_{nS}(0)}{3m_2 m_1^3} \int \frac{d\vec{p}}{(2\pi)^3} \psi_{nS}(\vec{p}) \left[ \frac{\vec{p}^2}{k^2} - \frac{\vec{p}^2 + W^2}{k^2} \right] \frac{d\vec{q}}{(2\pi)^3(\vec{q}^2 + W^2)^3}.
\]
Figure 1: **Direct and crossed two-photon diagrams of exchange interaction**

where $W^2 = \mu_R^2 (Z\alpha)^2/n^2$. The divergence of this integral identical to that for (10). Using Feynman’s parameterization and the substitution of the type (10) to calculate (26), we obtain:

$$\Delta B_6 = -\frac{m_2^2 (Z\alpha)^6}{12m_2} \left( \frac{1}{n^3} - \frac{2\ln 2}{n^4} - \frac{2}{n^3 (-1)} C + \psi(n) - 1 \right).$$  (27)

### 3 Two-photon exchange interaction

The amplitude of two-photon exchange interaction is represented on two diagrams of Fig.1. The appropriate quasipotentials are:

\[ V^{(a)}_2 (\vec{p}, \vec{q}) = \frac{i(Z\alpha)^2}{\pi^2} \int \frac{f_1(k, m_1, m_2) d^4k}{[(k - p)^2 + i\epsilon][(-k - q)^2 + i\epsilon]} D_e(k) D_\mu(-k). \]  (28)

\[ f_1(k, m_1, m_2) = m_2(4m_1 + 2k_0) - 2m_1 k_0 - 2k_0^2 + \frac{2}{3} \vec{k}^2, \]

\[ D_\mu(-k) = k^2 - 2E_2 k_0 + b^2 + i\epsilon \approx -2m_2 k_0 + i\epsilon. \]

\[ V^{(b)}_2 (\vec{p}, \vec{q}) = \frac{i(Z\alpha)^2}{\pi^2} \int \frac{f_2(k, m_1, m_2) d^4k}{[(k - p)^2 + i\epsilon][(k - q)^2 + i\epsilon]} D_e(k) D_\mu(p - q - k), \]  (29)

\[ f_2(k, m_1, m_2) = m_2(4m_1 + 2k_0^0) - 2m_1 k_0 - 6k_0^2 + \frac{10}{3} \vec{p} \vec{k} + \frac{10}{3} \vec{q} \vec{k} + \frac{10}{3} \vec{k}^2, \]

\[ D_\mu(p - q - k) = k^2 + 2E_2 k_0 + 2\vec{k}(\vec{p} + \vec{q}) - (\vec{p} + \vec{q})^2 + b^2 + i\epsilon \approx 2m_2 k_0 + i\epsilon. \]

The main contribution of $V^{(a)}_2$ in the energy spectrum is proportional to $\alpha^5$. The analysis of corrections $O(Z\alpha)^6$ show, that they may exist in the energy levels also, if we take into account, for example, the contribution of photon poles, when $k_0 \sim \alpha$, $|\vec{p}| \sim \alpha$, $|\vec{q}| \sim \alpha$. The appropriate quasipotentials are:
$|q| \sim \alpha, \ |\vec{k}| \sim \alpha$. In order to extract such terms from $V_{2\gamma}$, let transform the product of electron and muon denominators in the direct two-photon diagram in the following manner:

$$\frac{1}{D_e(k)D_\mu(-k)} = \frac{-2\pi i\delta(k_0)}{-2E(k^2 - b^2)} - \frac{1}{2E}\left[\frac{1}{(k_0 + i\epsilon)D_e(k)} + \frac{1}{(-k_0 + i\epsilon)D_\mu(-k)}\right], \quad (30)$$

where the addendum with $\delta(k_0)$ in the right part of (30) cancels with the iteration term of quasipotential. First addendum in the square brackets has the same structure in the leading order over $1/m_2$ as a crossed amplitude. Then the two-photon interaction quasipotential, which generates the correction $(Z\alpha)^6m_1^2/m_2$ is equal:

$$V_{2\gamma}(\vec{p}, \vec{q}) = \frac{2i(Z\alpha)^2}{3\pi^2} \int \frac{d^4k[4\vec{k}^2 + 5\vec{k}(\vec{p} + \vec{q}) - 6k_0^2]}{[(k - p)^2 + i\epsilon][4(k - q)^2 + i\epsilon]D_e(k)(2m_2k_0 + i\epsilon)}. \quad (31)$$

We have calculated the contribution of $V_{2\gamma}$ to the energy spectrum for $n=1$ and $n=2$ by means of system for analytical calculations "Mathematica" \[8\] (package feynpar.m). The corresponding results are:

$$\Delta B_7 = \begin{cases} -\frac{7m_1^2}{2m_2^2}(Z\alpha)^6, & n = 1 \\ -\frac{31m_1^2}{16m_2^2}(Z\alpha)^6, & n = 2 \end{cases} \quad (32)$$

4 Three-photon exchange interaction

There are six diagrams, shown on Fig.2, which determine the three-photon exchange interaction in the muonium.

Let consider the first diagram of Fig.2. The corresponding amplitude already has the factor $\alpha^6$, which appears due to electromagnetic vertices and Coulomb wave function. So, in the first stage of our calculations we have neglected by the electron and muon vector momentum of relative motion in the initial and final states, taking into account that the
necessary accuracy is already achieved. Then the first diagram amplitude (Fig.2) takes the form:

\[ T_{1}^{3γ} = -\frac{(Zα)^{3}}{4π^5} \int d^{4}p \int d^{4}p' \frac{< \gamma_{1}^{p}(q_{1} - p' + m_{1})\gamma_{1}^{p'}(p_{1} - p + m_{1})\gamma_{1}^{p} >}{(p^2 - w^2 + iε)(p'^2 - w^2 + iε)[(p - p')^2 + iε]} \]  \tag{33}

\[ \left< \frac{\gamma_{2}^{p}(p_{2} - p + m_{2})\gamma_{2}^{p'}(q_{2} + p' + m_{2})\gamma_{2}^{p'}}{D_{e}(p)D_{e}(p')D_{\mu}(-p)D_{\mu}(-p')} \right>, \]

where \( D_{e,\mu}(p) \) are the denominators of electron and muon propagators:

\[ D(±p) = p^{2} - w^{2} ± 2mp^{0} + iε, \quad w^{2} = -b^{2}; \] \tag{34}

and the corner brackets designate the averaging over Dirac bispinors; \( p_{1}, p_{2} \) are the four-momenta of particles in the initial state; \( q_{1}, q_{2} \) are the particle four-momentum in the final state. As usually is, the factor \( Zα \) emphasizes exchanging character of photon interaction between particles. The exchange photon propagators were taken in Feynman covariant gauge. As it famous, using of Coulomb gauge is the most natural for exchange photons, because the Coulomb interaction dominates in the system (\( e^{-\mu^{+}} \)). Nevertheless, the equivalence of Coulomb and Feynman gauges in the scattering approximation for the three-photon diagrams calculations was shown in [17]. To construct the quasipotential of the system (\( e^{-\mu^{+}} \)) with \( L=0 \) and \( J=1 \), that corresponds to \( T_{1}^{3γ} \), let introduce the projector operator for initial and final states of the kind (6), setting also \( \vec{p} = \vec{q} = 0 \). Projecting the particles on \( ^{3}S_{1} \)- state by means of (6), we avoid cumbersome matrix multiplication in the bispinor averages and immediately pass on to calculation of total trace in (33). As a result, the quasipotential of first diagram may be written in the form:

\[ V_{1}^{3γ} = -\frac{(Zα)^{3}}{π^5} \int d^{4}p \int d^{4}p' \frac{F_{1}(p,p')}{D_{\gamma}(p)D_{\gamma}(p')D_{\gamma}(p - p')D_{e}(p)D_{e}(p')D_{\mu}(-p)D_{\mu}(-p')}, \] \tag{35}

\[ D_{\gamma}(p) = p^{2} - w^{2} + iε, \]

\[ F_{1}(p,p') = f_{12}(p,p')m_{2}^{2} + \frac{1}{3}f_{11}m_{2}, \quad f_{12} = pp' - 4m_{1}^{2} - 2m_{1}p_{0} - 2m_{1}p_{0}' - 2p_{0}p_{0'}, \] \tag{36}

\[ f_{11}(p,p') = 2m_{1}p^{2} + p_{0}p^{2} + 10m_{1}pp' + 2p_{0}pp' + 2p_{0}'pp' + 2m_{1}p^{2} + p_{0}'p^{2} + 6m_{1}p_{0} + 6m_{1}p_{0}' + 4m_{1}p_{0}^{2} + 4m_{1}p_{0}'^{2} - 4m_{1}p_{0}p_{0}'. \]

We kept in (35) only the terms proportional to \( m_{1}^{2} \) and \( m_{2} \), taking in mind the determination of contribution to muonium fine structure in the leading order over parameter \( m_{1}/m_{2} \). As will soon become evident, we can’t restrict in \( F(p,p') \) only by terms \( \sim m_{2}^{2} \). The quasipotentials of the rest amplitudes of Fig.2 may be constructed in a similar way. They differ from each other due to momentum dependence in muonic denominators and to the kind of functions \( f_{i1} \) (i=1,...,6).

The parts of \( F_{i}(p,p') \), proportional to \( m_{2}^{2} \), coincide in all six amplitudes. Let remark, that when substitute \( \hat{ε} \rightarrow γ_{5} \) in projector operator (6) (\(^{1}S_{0} \) state), we obtain the same function \( f_{12}(p,p') \), as for the \(^{3}S_{1} \) muonium. This means, that the muonium hyperfine splitting appears as effect of higher order on \( m_{1}/m_{2} \). Functions \( f_{i1} \) are equal to:

\[ f_{21} = -10m_{1}p^{2} - 5p_{0}p^{2} + 10m_{1}pp' + 4p_{0}pp' - 4p_{0}'pp' + 2m_{1}p^{2} + 4p_{0}'p^{2} + 12p_{0}m_{1} - \] \tag{37}
\[ f_{31} = 2m_1p'^2 + 4p_0p'^2 + 10m_1pp' - 4p_0pp' + 4p_0p' - 10m_1p^2 - 5p_0p^2 - 6m_1^2p_0 + 12m_1^2p_0' - 8m_1p_0p_0' + 4m_1p_0'^2 + 4p_0p_0'^2, \]
\[ f_{41} = 2m_1p'^2 + p_0p'^2 - 2m_1pp' + 4p_0pp' + 2p_0p' - 10m_1p^2 - 8p_0p^2 - 12m_1^2p_0 + 6m_1^2p_0' + 4m_1p_0p_0' - 8p_0p_0'^2, \]
\[ f_{51} = -10m_1p'^2 - 8p_0p'^2 - 2m_1pp' + 4p_0pp' + 2m_1p^2 + p_0p'^2 + 6m_1^2p_0, \]
\[ f_{61} = -10m_1p'^2 - 5p_0p'^2 - 2m_1pp' - 4p_0pp' - 4p_0p' - 10m_1p^2 - 5p_0p^2 - 6m_1^2p_0 - 8m_1p_0p_0' - 4m_1p_0p_0'^2 - 8m_1p_0'^2. \]

Integral function in (35) has simple poles over loop energies \( p_0, p_0' \) in electron, muon and photon propagators. So, the most natural way of integration (35) consists in the calculation of integrals on \( p_0, p_0' \) at the first step, using the method of residues. But such an approach of calculation leads, nevertheless, to rather complicated intermediate expressions, what makes highly questionable its subsequent analytical integration on spatial momenta \( \vec{p}, \vec{p}' \). So, we have used different approach of integration in (4), connected with transformation of muonic denominators, accounting the necessary calculational accuracy on parameter \( m_1/m_2 \). Considering that the spatial momentum of muonic motion in the intermediate state \( |\vec{p}| < m_2 \), we obtain:

\[ D_\mu(p) = p^2 - w^2 + 2m_2p_0 \approx 2m_2 \left( p_0 - \frac{p^2 + w^2}{2m_2} + i\epsilon \right) \approx 2m_2(p_0 + i\epsilon), \]  

where the second approximate equality means that we neglect by the muon kinetic energy in the intermediate state. Doing so, we suppose that the integration contour on variable \( p_0 \) must be closed in the lower halfplane. Considering the terms, proportional to \( m_2^2 \) in the numerators of all six diagrams (function \( f_{12}(p, p') \)), we have arrived to the need of expression transformation, which includes the sum of muonic denominators (34). Using the second approximate equality from (42), we obtain:

\[ \frac{1}{D_\mu(-p)D_\mu(-p')} + \frac{1}{D_\mu(-p)D_\mu(p'-p)} + \frac{1}{D_\mu(-p')D_\mu(p-p')} + \frac{1}{D_\mu(p)D_\mu(p-p')} + \frac{1}{D_\mu(p')D_\mu(p'-p)} + \frac{1}{D_\mu(p')D_\mu(p)} \approx \frac{(-2\pi i)\delta(p_0)}{2m_2} \frac{(-2\pi i)\delta(p_0')}{2m_2}. \]  

(43)

In the energy spectrum the expression (43) will cause the corrections of order \( O(\alpha^4) \), which are canceled by the similar terms from the iteration part of the quasipotential. Consequently, to find the necessary contribution of order of \( \alpha^6 \), we must use first approximate equality in (42). Taking the difference

\[ \frac{1}{2m_2 \left( p_0 - \frac{p^2 + w^2}{2m_2} + i\epsilon \right)} - \frac{1}{2m_2(p_0 + i\epsilon)} \approx \frac{(p^2 + w^2)}{4m_2^2(p_0 + i\epsilon)^2}, \]  

(44)
let represent the quantity $1/D_\mu(p)$ in the form:

$$
\frac{1}{D_\mu(p)} \approx \frac{1}{2m_2(p_0 + i\epsilon)} + \frac{(\vec{p}^2 + w^2)}{4m_2^2(p_0 + i\epsilon)^2}.
$$

(45)

Second addendum of (45) is of higher order on $m_1/m_2$ in comparison with the first. But it leads to the necessary order correction on the other parameter $\alpha$. Using the splitting (45), in the sum (43), let extract the terms, which generate the correction $O(\alpha^6)$ and $O(\alpha^6 \ln \alpha)$ in the energy spectrum. We may write them in the following manner:

$$
\frac{(\vec{p}^2 + w^2)}{8m_2^3} \left[ \frac{2\pi i\delta(p'_0)}{(p_0 + i\epsilon)^2} - \frac{2\pi i\delta(p'_0 - p_0)}{(p_0 + i\epsilon)^2} - \frac{2\pi i\delta(p_0)}{(p_0 + i\epsilon)^2} \right] + \ (46)
$$

$$
\frac{(\vec{p}^2 + w^2)}{8m_2^3} \left[ \frac{-2\pi i\delta(p'_0)}{(p_0 + i\epsilon)^2} - \frac{2\pi i\delta(p'_0 - p_0)}{(p_0 + i\epsilon)^2} + \frac{2\pi i\delta(p_0)}{(p_0 + i\epsilon)^2} \right] + \ (46)
$$

$$
\frac{(\vec{p} - \vec{p}')^2 + w^2}{8m_2^3} \left[ \frac{-2\pi i\delta(p'_0)}{(p_0 + i\epsilon)^2} + \frac{2\pi i\delta(p'_0 - p_0)}{(p_0 + i\epsilon)^2} - \frac{2\pi i\delta(p_0)}{(p_0 + i\epsilon)^2} \right]
$$

It is evident from three-photon interaction amplitude of the type (33), that the parts of (46) give the necessary order corrections on $\alpha$ in the studied fine structure intervals. The same order corrections $O(m_1/m_2)$, as well as (18), will arise from the quasipotential terms containing the functions $f_{i_1}(p, p')$, when we use the second approximation of (42) for muonic denominators. To do definite conclusion about the order of appearing terms in energy spectrum, which are determined by these quasipotential addenda, let transform them for greater simplification. Let consider for definiteness massless terms in the function $f_{i_1}(p, p')$, proportional to $\sim \vec{p}^2, \vec{p}'^2, pp'$:

$$
3\vec{p}^2 \left[ \frac{1}{D_\mu(p)} + \frac{1}{D_\mu(-p)} - \frac{1}{D_\mu(p - p')} - \frac{1}{D_\mu(p' - p)} \right] + \ (47)
$$

$$
+ 3\vec{p}'^2 \left[ \frac{1}{D_\mu(-p')} + \frac{1}{D_\mu(p')} - \frac{1}{D_\mu(p' - p)} - \frac{1}{D_\mu(p - p')} \right] - \ (47)
$$

$$
-6pp' \left[ \frac{1}{D_\mu(-p')} + \frac{1}{D_\mu(p')} + \frac{1}{D_\mu(-p)} + \frac{1}{D_\mu(p)} - \frac{1}{D_\mu(p' - p)} - \frac{1}{D_\mu(p - p')} \right] \approx \ (47)
$$

$$
\approx \frac{3\vec{p}^2}{2m_2} \left[ -2\pi i\delta(p_0) + 2\pi i\delta(p_0 - p'_0) \right] + \frac{3\vec{p}'^2}{2m_2} \left[ -2\pi i\delta(p'_0) + 2\pi i\delta(p_0 - p'_0) \right] - \ (47)
$$

$$
- \frac{6pp'}{2m_2} \left[ -2\pi i\delta(p_0) - 2\pi i\delta(p'_0) + 2\pi i\delta(p_0 - p'_0) \right].
$$

The transformation of other terms in functions $f_{i_1}(p, p')$ may be carried out by analogy. The next period of calculation consists in the integration over four-momenta in expressions (46)-(47). The typical two-loop integral, that results on this way has the following structure [17]:

$$
K_i = (4\pi)^2 \int \frac{d^4pd^4p'}{-(2\pi)^8 (\vec{p}^2 - w^2 + i\epsilon)(\vec{p} - \vec{p}')^2 + i\epsilon}(\vec{p}^2 - w^2 + i\epsilon)D_e(p')D_e(p), \ (48)
$$
where $G_i(p_0', p_0, m_1)$ contains one $\delta$-function, and $P(\vec{p}', \vec{p}, w)$ is a polynomial. To calculate fundamental integrals \((48)\) we have used Feynman parameterization in order to combine the denominators of the particle propagators, and the symmetry properties of the integral with the replacement $p \leftrightarrow p'$. There is the next set of functions $G_i(p_0', p_0, m_1)$, which appear in this paper:

$$G_1 = -\frac{2\pi i\delta(p_0 - p_0')2m_1}{(p_0 + i\epsilon)^2}, \quad G_2 = -\frac{2\pi i\delta(p_0)2m_1}{(p_0' + i\epsilon)^2}, \quad G_3 = -2\pi i\delta(p_0)2m_1, \quad G_4 = -2\pi i\delta(p_0')2m_1, \quad G_5 = -2\pi i\delta(p_0 - p_0')2m_1. \quad (49)$$

The results of the integrations for $K_i$ \((48)\) are presented in the table [4].

### Table of the integrations $K_i$ \((48)\), appearing in the muonium fine structure calculations

| $K_i$ | $\vec{p}^2(\vec{p}'\vec{p}')$ | $(\vec{p}'\vec{p}')^2$ | $\vec{p}^2(\vec{p}'\vec{p}')$ | $w^2(\vec{p}'\vec{p}')$ |
|-------|-----------------|-----------------|-----------------|-----------------|
| $K_1$ | $2\ln 2 - \frac{1}{2}$ | $2\ln 2 - \frac{1}{2}$ | $2\ln 2 - \frac{1}{2}$ | 0               |
| $K_2$ | $\frac{1}{2}\ln \frac{m_1}{2w} - \frac{1}{2}$ | $\frac{1}{2}\ln \frac{m_1}{2w} - \frac{1}{2}$ | $\frac{3}{2}$ | $\frac{3}{2}$ |
| $K_3$ | $\frac{1}{2}\ln \frac{m_1}{2w} - \frac{1}{2}$ | $\frac{1}{2}\ln \frac{m_1}{2w} - \frac{1}{2}$ | $\frac{3}{2}$ | $\frac{3}{2}$ |
| $K_4$ | $\frac{1}{2}\ln \frac{m_1}{2w} - \frac{1}{2}$ | $\frac{1}{2}\ln \frac{m_1}{2w} - \frac{1}{2}$ | $\frac{3}{2}$ | $\frac{3}{2}$ |
| $K_5$ | $\ln 2$ | $\ln 2 - \frac{1}{2}$ | $\ln 2$ | 0               |

Then the contributions, defined by expressions \((37-41)\) and \((46)\) are correspondingly equal:

$$\Delta B_1 = -\frac{1}{2}(Z\alpha)^6\frac{m_1^2}{m_2}, \quad (50)$$

$$\Delta B_2 = (Z\alpha)^6\frac{m_1^2}{m_2}(6\ln 2 - \frac{11}{48}) \quad (51)$$

We have introduced in \((48)\) the photon mass $w$ to avoid ”infrared” singularities. The ”infrared” logarithms $\ln w$, containing this photon mass (see table of integrals $K_i$), and appearing at intermediate expressions, are mutually cancelled in the corrections $\Delta B_1$, $\Delta B_2$.

Let consider now the quasipotential addenda, containing the momenta of particle relative motion in the initial and final states. We denote them by $\vec{r}_1$ and $\vec{r}_2$ correspondingly. Their consideration leads to modification of $f_{i1}$, which acquire the following additional terms:

$$\Delta f_{21} = 10m_1p'r_2 + 5p_0p'r_2 + m_1pr_2 + 3p_0'pr_2, \quad (52)$$

$$\Delta f_{31} = -m_1p'r_1 - 3p_0p'r_1 - 10m_1pr_1 - 5p_0'pr_1, \quad (53)$$

$$\Delta f_{41} = m_1(7p'r_1 + 5p'r_2 + 10pr_1 + 5pr_2) + 6p_0p'r_1 + 5p_0p'r_2 + 8p_0'pr_1 + 5p_0'pr_2, \quad (54)$$

$$\Delta f_{51} = m_1(-5p'r_1 - 10p'r_2 - 5pr_1 - 7pr_2) - 5p_0p'r_1 - 8p_0p'r_2 - 5p_0'pr_1 - 6p_0'pr_2, \quad (55)$$

$$\Delta f_{61} = m_1(-11p'r_1 - 11p'r_2 - 11pr_1 - 11pr_2) - 8p_0p'r_1 - 8p_0p'r_2 - 8p_0'pr_1 - 8p_0'pr_2. \quad (56)$$
Using again the symmetry properties of appearing integrals under simultaneous variable replacement $p \leftrightarrow p'$, $r_1 \leftrightarrow r_2$, we obtain cancellation of all integrations in (52)-(56). So, the contribution of the particle relative motion in the fine structure with the accuracy $O(m_1/m_2)$ is equal to zero. Thus the full value of the calculated correction $(Z\alpha)^6 m_1^2/m_2$ for hydrogen-like system S-states is defined as a sum of expressions (50) and (51):

$$\Delta B_8 = (Z\alpha)^6 \frac{1}{n^3 m_1 m_2} \left( 6 \ln 2 - \frac{35}{48} \right).$$

(57)

5 Discussion of the results

In this paper we have calculated all possible corrections of order $(Z\alpha)^6 m_1^2/m_2$ in the fine structure of hydrogen-like system on the basis of diagrammatic approach. Our total result is determined by the sum of terms $\Delta B_i$ (4), (8), (11), (13), (24), (27), (32) and (57):

$$\Delta B_{tot.} = \left( \frac{91}{12} \ln 2 - \frac{545}{144} - \frac{17}{6n} - \frac{37}{36n^2} + \frac{187}{96n^3} \right) \frac{m_1^2(Z\alpha)^6}{m_2 n^3} + \varepsilon_n$$

(58)

$$\varepsilon_n = \begin{cases} - \frac{23m_1^2(Z\alpha)^6}{12m_2}, & n = 1, \\ - \frac{31m_1^2(Z\alpha)^6}{128m_2}, & n = 2 \end{cases}$$

Numerical value of obtained contribution (58) for the "large" muonium fine structure interval $2^3S_1 \div 1^3S_1$ is equal to 0,19 MHz. Recently, the calculation of the recoil corrections $O(Z\alpha)^6 m_1^2/m_2$ for the S-levels of hydrogen atom was done in [6, 7, 8]. The total contribution of the necessary order in the energy spectrum, which was obtained by using Braun formula, is equal [8]:

$$\Delta E_{tot.} = \left( 1 + \frac{3}{8n} - \frac{1}{n^2} + \frac{1}{2n^2} \right) \frac{(Z\alpha)^6 m_1^2}{m_2} + \left( 4 \ln 2 - \frac{7}{2} \right) \frac{(Z\alpha)^6 m_1^2}{m_2}.$$ 

(59)

Comparing results (58) and (59), we see that the analytical expression of our correction (58) differs from (59), because we have calculated the contribution of necessary order to the energy spectrum, coming both from spin-independent and spin-dependant parts of the quasipotential. Numerical values of the contributions (58) and (59) for the levels with $n=1$ and $n=2$ are equal correspondingly: $n=1$: -0,212 MHz, -0,065 MHz; $n=2$: -0,021 MHz, -0,006 MHz. The contribution of correction (58) to the fine structure interval $2S \div 1S$ of hydrogen atom is equal to 21,5 KHz, whereas the values of similar contributions, obtained in [6, 7, 8] are equal correspondingly: 6,6 KHz; 14,5 KHz.

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