Termination of threefold flips in mixed characteristic

Liam Stigant

Department of Mathematics, Imperial College London, London, UK

Abstract

This note gives a short proof of termination of threefold flips over positive dimensional base and in particular demonstrates that the results hold in mixed characteristic. The work draws on recent developments in mixed and positive characteristic birational geometry as well as earlier ideas from characteristic zero.

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1 | INTRODUCTION

Recent work in [2] establishes the bulk of the Minimal Model Program (MMP) for klt threefold pairs over suitable bases in mixed and positive characteristic. In particular, it is shown that one can always run an MMP with scaling when the residue fields of closed points are of characteristic $p > 5$. When the pair is pseudo-effective more is known, it is shown that in fact every MMP terminates without the need for scaling [2, Theorem 9.8]. Subsequent work in [12] establishes analogous results in the $p = 5$ case. For the special case of semi-stable fibrations, [11] covers the case of semi-stable threefolds without constraint on the characteristic, while [4] and [13] show MMP’s exist over Dedekind domains with perfect residue fields of characteristic $p > 5$.

The problem of termination remains open for threefold pairs which are not pseudo-effective however, and with the Abundance Theorem proven in [1], it is the only outstanding part of the core MMP for klt pairs in this setting.

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In this note, we study termination for klt threefold pairs over positive dimensional bases. In this setting, we will show that every \((K_X + \Delta)\text{-MMP}\) terminates for a dlt pair \((X, \Delta)/T\). We rely heavily on Theorem 2.6. The key remaining argument is if \(X \to T\) is a klt pair, then there is an open set on which every contraction has a horizontal exceptional locus.

We prove this by reducing to the case that \((X, \Delta)\) is terminal. In mixed characteristic this then follows from the liftablity of \(-1\) curves by generalizing the arguments of [7]. This argument does not work in purely positive characteristic but provides motivation for our approach. Instead we adapt a termination argument for terminal pairs, largely due to Shokurov [9].

The main restrictions are that the residue fields of \(R\) should have characteristic \(p = 0\) or \(p \geq 5\). A full characterization of suitable base rings is given in Definition 2.2. For such an \(R\), we prove the following.

**Theorem 1.1 (Corollary 3.5).** Let \(X\) be an integral, normal threefold over \(R\) equipped with a projective morphism \(X \to T\), where \(T\) is quasi-projective over \(R\). If \((X, \Delta)\) is a threefold dlt pair over \(R\) and the image of \(X\) in \(T\) is positive dimensional, then any \((K_X + \Delta)\text{-MMP}\) terminates.

### 2 | PRELIMINARIES

We will say that \(f : X \to Y\) is a contraction if \(f_* \mathcal{O}_X = \mathcal{O}_Y\). If \(f : X \to Y\) is a birational contraction of \(T\) schemes and the exceptional locus dominates \(T\), then we call it horizontal. We will use \(\mathbb{K}\) to mean either of \(\mathbb{Q}\) or \(\mathbb{R}\). Similarly, we call any sub-scheme \(Z\) of \(X\) horizontal if it dominates \(T\).

**Definition 2.1.** A sub-log pair \((X, \Delta)\) with \(\mathbb{K}\) boundary is an excellent, Noetherian, integral, normal scheme \(X\) of finite dimension admitting a dualizing complex together with an \(\mathbb{K}\)-divisor \(\Delta\) such that \(K_X + \Delta\) is \(\mathbb{K}\)-Cartier. If \(\Delta\) is effective, we say \((X, \Delta)\) is a log pair. When \(\Delta = 0\), we just say \(X\) is a log pair.

We adopt the notation and definitions of [6, Section 1.3] for singularities of pairs. In particular, for \(E\) with centre on \(X\), we denote the discrepancy by \(a(E, X, \Delta)\). If \(\Delta = 0\), we write \(a(E, X)\) for brevity.

**Definition 2.2.** An \(R\)-pair \((X, \Delta)/T\) with \(\mathbb{K}\)-boundary will be the following data:

- an excellent, normal ring \(R\) whose residue fields are of characteristic 0 or \(p \geq 5\) and of finite Krull dimension which admits a dualizing complex and whose residue fields have characteristic \(p = 0\) or \(p \geq 5\);
- an integral, normal quasi-projective \(R\)-variety \(T\);
- a normal, integral scheme \(X\);
- a dominant projective contraction \(f : X \to T\); and
- a \(\mathbb{K}\)-divisor \(\Delta \geq 0\) with \(K_X + \Delta\) \(\mathbb{K}\)-Cartier, for \(\mathbb{K} = \mathbb{R}\) or \(\mathbb{Q}\).

The dimension of such a pair is the dimension of \(X\). Equally the pair is said to \(\mathbb{Q}\)-factorial if \(X\) is. If no \(\mathbb{K}\) is specified, we default to \(\mathbb{K} = \mathbb{R}\). As with log pairs, we drop \(\Delta\) from the notation when \(\Delta = 0\).

In this paper, we consider exclusively \(T\) with \(\dim T \geq 1\).
The assumptions that $X \to T$ is a contraction and $T$ is integral, normal can be dropped in applications since we can always Stein factorize an arbitrary projective $X \to T$. In this case, it is necessary to assume the image of $X$ has dimension at least 1 in place of $T$ having dimension at least 1.

**Theorem 2.3** ([Cone Theorem] [2, Theorem I] [13, Theorem 1.5]). Let $(X, \Delta)/T$ be a dlt $\mathbb{Q}$-factorial threefold $R$-pair. Then there is a countable collection of curves $\{C_i\}$ on $X$ such that:

1. \[
\bar{N}E(X/T) = \bar{N}E(X/T)_{K_Y + \Delta} + \sum_i \mathbb{R}[C_i].
\]

2. The rays $C_i$ do not accumulate in $(K_Y + \Delta)_{<0}$.

3. For each $i$, there is $d_{C_i}$ with

\[
0 < -(K_X + \Delta).C_i \leq 4d_{C_i}
\]

and $d_{C_i}$ divides $L \cdot C_i$ for every Cartier divisor $L$ on $X$.

**Theorem 2.4** (Basepoint Free Theorem [2, Theorem H] [13, Theorem 1.4]). Let $(X, \Delta)/T$ be a klt $R$-pair of dimension 3. Let $L$ be an $f$-nef divisor with $L - (K_X + \Delta)$ is $f$-big and nef. Then $L$ is semi-ample.

**Theorem 2.5** ([2, Theorem G] [13, Theorem 1.3]). Let $(X, \Delta)/T$ be a dlt $R$-pair of dimension 3. We can run a $(K_X + \Delta)$-MMP. If $K_X + \Delta$ is pseudo-effective, then this MMP terminates. Further an MMP with scaling by an ample divisor will terminate for any choice of pair.

Termination for pseudo-effective pairs in this setting is assured by the following theorem, together with non-vanishing on the generic fibre.

**Theorem 2.6** [2, Theorem 9.8]. Let $(X, \Delta) \to T$ be a $\mathbb{Q}$-factorial threefold dlt $R$-pair. Suppose

\[
(X, \Delta) = \to (X_0, \Delta_0) \to (X_1, \Delta_1) \to \ldots
\]

is a sequence of $K_X + \Delta$ flips. Then neither the flipped nor the flipping locus are contained the support of $\Delta_n$ for all sufficiently large $n$.

We will also need the following construction, essentially due to [8].

**Lemma 2.7.** Let $\pi : X \to Y$ be a projective contraction from a regular scheme to a normal scheme, both of dimension 2. Let $E_1, \ldots, E_n$ be the exceptional curves. Choose a divisor $D$ on $Y$ and write $D'$ for the strict transform of $D$. Then, there are unique $m_i \geq 0$ with $D' + \sum m_i E_i \equiv_Y 0$. If $D$ is $\mathbb{Q}$-Cartier, then we have $\pi^* D = D' + \sum m_i E_i$.

**Proof.** By [6, Theorem 10.1], the intersection matrix $[E_i, E_j]$ is negative definite. Hence there is a unique choice of $m_i$ with $D' + \sum m_i E_i \equiv_Y 0$. It remains to show $m_i \geq 0$. By [6, Lemma 10.2], there
is \( E = \sum r_i E_i \) effective on \( X \) with \( -E \) ample over \( Y \). Then, \( E.E_i < 0 \) for each \( i \) ensures that \( r_i > 0 \) for all \( i \).

Now suppose for contradiction that \( m_k < 0 \) for some \( k \). Then, we may suppose that \( m_k / r_k \) is minimal, otherwise if \( m_j / r_j \) is minimal we just replace \( k \) with \( j \) as we must still have \( m_j < 0 \). We must have, for every \( j \), that \( D'.E_j \geq 0 \) as it does not contain any \( E_j \) and thus as \( D' = -\sum m_i E_i \), we have

\[
0 \geq \left( \sum_i m_i E_i \right).E_j = \sum_i \frac{m_i}{r_i} (r_i E_i).E_j \geq \frac{m_k}{r_k} \sum_i (r_i E_i).E_j > 0.
\]

This is a contradiction and hence in fact \( m_i \geq 0 \) for each \( i \). That this agrees with the pullback when \( D \) is \( \mathbb{Q} \)-Cartier is immediate from uniqueness. \( \square \)

**Lemma 2.8.** Let \( X \) be a \( \mathbb{Q} \)-factorial scheme together with a projective morphism \( f : X \to Y \) with geometrically connected fibres to an excellent normal scheme of dimension 2. Suppose \( V \) is a closed subscheme of \( X \) with \( f(V) \) contained in a divisor \( D \). Then there is a divisor \( D' \) on \( X \) lying over \( D \), numerically trivial over \( Y \) and containing \( V \).

**Proof.** Let \( \pi : Y' \to Y \) be a resolution of \( Y \) and \( X' \) be the normalization of the dominant component of the fibre product \( X \times_Y Y' \). By the previous lemma, since \( Y' \) is regular, we can find \( F = D' + \sum m_i E_i \) with \( F \equiv_Y 0 \) where \( E_i \) are the exceptional curves of \( \pi \) and \( D' \) is the strict transform of \( D \). Now \( g^*F \) is numerically trivial over \( Y \), and hence over \( X \). Thus, as \( X \) is \( \mathbb{Q} \)-factorial there is \( D' \) with \( \phi^* D' = F \). We can take \( D' = \phi_* F \), then \( F - \phi_* D \equiv_X 0 \) is exceptional and hence \( F - \phi_* D \approx_X 0 \) by the negativity lemma [2, Lemma 2.16]. It is clear from the construction that \( f_* D' = \pi_* F = D \). Suppose that \( C \) is a curve lying over \( D \), then we must have \( D'.C = 0 \). If \( C \) is not contained in \( D' \), then since \( f \) has connected fibres we may suppose that \( D' \) meets \( C \), up to replacing \( C \) with another curve in the same fibre, but then \( D'.C > 0 \), a contradiction. Hence \( D' \) contains every curve, and hence every fibre, over \( D \). In particular it contains \( V \). \( \square \)

## 3 TERMINATION

In this section, we prove the termination result. We follow the general approach of threefold termination results in characteristic 0, first the result is established for terminal \( K_X \) MMPs by using some variant of Shokurov’s difficulty. Terminal pairs \((X, \Delta)\) can then be handled by carefully considering the restrictions to each component of \( \Delta \) and then the general case of klt pairs is inferred from this and the existence of terminalizations.

In general, this final step can be a very delicate procedure, see, for instance, [3, Section 3] for a recent characteristic 0 proof in this vein, and it is this that presents the barrier in mixed and positive characteristic. We take a slightly different approach, instead making the observation that in fact terminal pairs admit no flips over some open set of a positive dimensional base. Unlike the termination, this readily descends to klt pairs via a terminalization. We then apply Theorem 2.6 to conclude.

If \( X \to T \) is projective and \( U \subseteq T \) is an open set, we will write \( X_U = X \times_T U \) and \( \Delta_U = \Delta|_{X_U} \).
Definition 3.1. Let $X$ be a terminal threefold log pair quasi-projective over $R$. We define the difficulty

$$d(X) = \#\{E : a(E, X) < 1\}.$$ 

This is finite by [5, Proposition 2.36], since log resolutions exist by [2, Proposition 2.14].

Clearly if $Y \hookrightarrow X$ is an open immersion, then $d(Y) \leq d(X)$ since every valuation with centre on $Y$ is also a valuation with centre on $X$. If $X \twoheadrightarrow X'$ is a $K_X$ flip, then $d(X') \leq d(X)$ by [5, Lemma 3.38]. We claim in fact this inequality is strict.

Lemma 3.2. Let $X/T$ be a terminal threefold $R$-pair and $X \twoheadrightarrow X'$ a $K_X$ flip, then $d(X') < d(X)$.

Proof. It suffices to find a divisor $E$ with $a(E, X) < 1$ and $a(E, X') \geq 1$. Let $C'$ be an irreducible component of the flipped curve. Then $X'$ is terminal, so it is smooth at the generic point $P$ of $C$ by [6, Corollary 2.30]. Let $Y \rightarrow X$ be the blowup of $C'$ and $E$ the dominant component of the exceptional divisor. By localizing at $P$, we see that $a(E, X') = 1$, since this is the blowup of a smooth point on a surface.

Let $C$ be the centre of $E$ on $X$. Then $C$ is a component of the flipping curve and so we have $a(E, X) < a(E, X')$ by [5, Lemma 3.38] concluding the proof. □

Theorem 3.3. Let $(X, \Delta)/T$ be a terminal threefold $R$-pair. Then there is an open set $U \subseteq T$ such that every $K_{X_U} + \Delta_U$ negative contraction is a horizontal divisorial contraction.

Proof. Write $\Delta = \sum a_kD_k$, we argue by induction on $n$. Suppose first that $n = 0$ and for contradiction there is no such $U$. Thus, we have a sequence of non-empty open sets $U_i \subseteq U_{i-1}$ such that there is a $K_{X_{U_i}}$ negative extremal ray $L_i$ supported away from $U_{i+1}$. We write $X_i = X \times U_i$.

If $L_i$ induces a divisorial contraction $f_i : X_i \rightarrow X'_i$, then $\rho(X_{i+1}) \leq \rho(X'_i) < \rho(X_i)$ since $f_i$ is an isomorphism over $U_{i+1}$. Similarly if $L_i$ induces a flip $f_i : X_i \twoheadrightarrow X'_i$, then $d(X_{i+1}) < d(X'_i) < d(X_i)$. Since both are positive integers, there can be only finitely many such $U_i$, a contradiction.

Now suppose $n > 0$. Let $\Delta^{n-1} = \sum a_iD_i$ then by induction there is an open set $U \subseteq T$ such that every $K_{X_U} + \Delta_U^{n-1}$ negative contraction is a horizontal divisorial contraction. If $D_n$ is not horizontal, we can shrink $U$ so it does not meet the image of $D_n$ and the result follows immediately by induction. This gives the result if $\dim T = 3$.

Otherwise if $D_n$ is horizontal, let $S$ be the normalization of $D_n$. If $\dim T = 2$, then there is an open set $V$ of $T$ on which $S_V \rightarrow T$ is finite, in particular $S_V$ contains no curves. If $\dim T = 1$, then by [10, Lemma 2.13] there is an open set $V$ of $T$ such that $S_V$ has relative Picard rank 1.

In either case, replace $U$ with $U \cap V$, then $X, S$ with $X_U, S_U$ and $\Delta$ with $\Delta|_{X_U}$. It suffices to show that every extremal $K_X + \Delta$ negative contraction is a horizontal divisorial contraction. Suppose for contradiction $\xi$ is an extremal ray inducing one that is not. We must have $D_n.\xi < 0$ from our choice of $U$. Thus, induced contraction restricts to a non-trivial birational morphism $S \rightarrow S'$ say. However, $S$ has Picard rank at most 1, so the only possibility is this map contracts $S$ entirely. In particular, this defines a horizontal divisorial contraction, a contradiction. The claim follows. □

We can extend this immediately to klt pairs.
Theorem 3.4. Let \((X, \Delta)/T\) be a klt threefold R-pair. Then there is an open set \(U \subseteq T\) such that every \(K_{X_U} + \Delta_U\) negative contraction is a horizontal divisorial contraction.

Proof. Let \(\pi: (Y, \Delta_Y) \to (X, \Delta)\) be a terminalization, which exists by [2, Proposition 9.19]. Then by Theorem 3.3, there is an open set \(U \subseteq T\) over which every \(K_{Y_U} + \Delta_{Y_U}\) negative contraction is divisorial. We claim the same holds for \(K_{X_U} + \Delta_U\) negative contractions.

Indeed if \(f: X_U \to Z\) is any such contraction, then \(K_{Y_U} + \Delta_{Y_U}\) is not nef over \(Z\). In particular, we get a contraction \(g: Y_U \to Z',\) factoring \(Y_U \to Z\), which is necessarily a horizontal divisorial contraction. In particular, \(g\) is not an isomorphism over the generic point \(\nu\) of \(T\). However then, neither can \(f\) be, else \(K_{Y_Y} + \Delta_{Y_Y}\) would be nef over \(Z_{\nu}\), a contradiction. Thus, \(f\) is a horizontal divisorial contraction as claimed.

Corollary 3.5. Let \(f: (X, \Delta) \to T\) be a \(\mathbb{Q}\)-factorial threefold dlt pair over \(R\), then any \((K_X + \Delta)\)-MMP terminates.

Proof. It is enough to show there is no infinite sequence of flips. Note that Theorem 2.6 ensures that the flipping and flipped curves are eventually disjoint from \(\lfloor \Delta \rfloor\). Therefore, replacing \(\Delta\) with \(\Delta - \lfloor \Delta \rfloor\), we may assume \((X, \Delta)\) is klt.

By Theorem 3.4, there is always some divisor \(D\) on \(T\) such that all the flips take place over \(D\). If \(T\) is \(\mathbb{Q}\)-factorial, then \((X, \Delta' = \Delta + tf^*D)\) is klt for small \(t > 0\) and a \((K_X + \Delta)\)-MMP is also a \((K_X + \Delta')\)-MMP. Since all the flips are contained in the support of \(\Delta'\), the sequence must terminate by Theorem 2.6. Otherwise we must have \(\dim T = 2\) so we use Lemma 2.8 in place of pulling back \(D\) and conclude exactly as above.

Remark 3.6. These arguments hold independently of any other restrictions on the base ring, assuming the requisite MMP results are known. In particular, the Cone Theorem, the existence of log resolutions and the existence of terminalizations are all that are required.

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ORCID
Liam Stigant https://orcid.org/0000-0002-6661-4545

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