A critical review of the concept of ‘ignorability’ when drawing direct likelihood inferences

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January 29, 2019

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Abstract

With incomplete data, the standard argument for when the response mechanism can be ignored for modelling purposes requires that realised Missing at Random (MAR) holds for each density in the model and that distinctness of parameters holds for the model’s parameter space. We critically review these conditions and argue that a MAR assumption for the underlying response mechanism alone is the optimal criterion for ignorability. Under this assumption, we argue that ignoring the response mechanism is optimal when compared to modelling the response mechanism explicitly, regardless of whether or not distinctness of parameters holds for the hypothetical full model.

Key words and phrases: incomplete data, missing data, ignorable, ignorability, missing at random, distinctness of parameters, likelihood theory.

1 Introduction

Missing data is a common problem in empirical research, and particularly so in medical and epidemiological studies. A central feature of the corresponding statistical methods pertains to conditions under which the response mechanism need not be modelled; the so-called ‘ignorability’ conditions. Users of these methods require a clear and logical rationale for when ignoring the response mechanism is justified when analysing incomplete data. We argue that the standard pair of conditions, Missing at Random (MAR) and distinctness of parameters (see Section 2 for definitions) has drawbacks in this regard, and that these conditions should be replaced with the single condition that a suitable Missing at Random (MAR) assumption holds for the underlying response mechanism.

2 A review of Rubin’s ignorability conditions

Given a random vector $Y$ comprising data for all units jointly, Rubin considered a corresponding random vector of binary variables, $R$, and a full (joint) distribution for $(Y, R)$. Given a model

$$\mathcal{M}_g = \{ f_\theta(y) g_\psi(r|y) : (\theta, \psi) \in \Delta \}$$

of full (joint) densities for $(Y, R)$, Rubin formulated conditions under which the response mechanism $g_\psi(r|y)$ could be ignored for modelling purposes and valid inferences could be drawn based on a simpler model derived from the model

$$\mathcal{M}_s = \{ f_\theta(y) : \theta \in \Theta \}$$
of joint densities for $Y$ alone. For drawing direct likelihood inferences, Rubin identified a pair of conditions, *missing at random* and *distinctness of parameters*, under which the response mechanism can be ignored for modelling purposes. We consider these in turn.

Given a realisation $(y, r)$ of the random vector $(Y, R)$, let $y^{ob}(r)$ and $y^{mi}(r)$ denote the vectors of observed and missing components of $y$, respectively. Analysis of the observed data $(y^{ob}(r), r)$ then proceeds by restriction of the random vector $(Y, R)$ to the event:

\[
\{ (y_*, r_*) : r_*=r \text{ and } y_*^{ob}(r)=y^{ob}(r) \}
\]

comprising all datasets $y_*$ (together with $r$) which correspond to $(y, r)$ on the observed data values $y^{ob}(r)$ but may differ on the unobserved values $y^{mi}(r)$. A response mechanism $g_v(r|y)$ is called Missing at Random (MAR) with respect to $(y, r)$ if $g$ is a constant function on the set \([1]\). Rubin \([3]\) defined missing at random to be a property of the model $\mathcal{M}_g$ by requiring each density in the model to be MAR with respect to $(y, r)$. The terminology *realised* MAR was introduced in \([4]\) to distinguish this weaker form of MAR framed by Rubin from a stronger form needed for frequentist likelihood inference.

Note that MAR is a functional constraint and not a stochastic constraint on the response mechanism because $g$ does not determine probabilities of events that arise by (i) holding the response pattern $r$ fixed and (ii) allowing $y$ to vary. (There seems to be misconception about this point in the literature, presumably due to the use of notation of the form \(P(R|Y_{obs},Y_{mis}) = P(R|Y_{obs})\) to denote the MAR property.)

The second of Rubin’s conditions, distinctness of parameters, requires that the parameter space $\Delta$ of $\mathcal{M}_g$ be a direct product $\Delta = \Theta \times \Psi$ of parameter spaces $\Theta$ for the densities for $Y$ and $\Psi$ for the response mechanisms for $R$. This means that in the model $\mathcal{M}_g$ every response mechanism $g_v(r|y)$ is paired with every data density $f_\theta(y)$. We refer the reader to Rubin’s original paper \([3]\) for details of why these conditions are sufficient for ignorability of the response mechanism for direct likelihood inferences, and we refer the reader to \([4]\) for additional details.

Since the appearance of \([3]\), the pair of conditions, MAR and distinctness of parameters, together seem to have been adopted as the standard conditions upon which justification for not modelling the response mechanism is based (for drawing direct likelihood inferences about the distribution for $Y$; a third condition, *independence of of parameters*, applies in the case of Bayesian inferences) \([2]\). We note, however, that this is not necessarily universal. See \([1]\), for example.

### 3 Two drawbacks of the distinctness of parameters criterion

Rubin’s ignorability conditions are derived under an assumption that a full model $\mathcal{M}_g$ is the starting point of interest for an analysis of incomplete data, and the objective is to determine when the same inference about $\theta$ can be obtained from the simpler, related model $\mathcal{M}_s$. However, in most cases an analyst does not have a pre-conceived model $\mathcal{M}_g$ in mind, and simply wishes to analyse the incomplete data at hand. Under these conditions, the choice of a model $\mathcal{M}_g$ as the starting point for deriving conditions for ignorability has several drawbacks.

One drawback that tends to undermines the cogency of the resulting case for justifying ignoring the response mechanism is that the distinctness of parameters condition seems merely to replace one arbitrary choice with another. Hypothetical models $\mathcal{M}_g$ are partitioned into two sets: those for which distinctness of parameters holds and those for which it does not. The choice simply to go straight to the model $\mathcal{M}_s$ is then replaced with the choice to restrict consideration only to models $\mathcal{M}_g$ for which distinctness of parameters holds, and this latter choice would seem not to be any more principled than the former. To put it another way, Rubin’s conditions show that inferences based on a
model of the from $\mathcal{M}_g$ will coincide with inferences drawn from a model derived from $\mathcal{M}_s$ for some models and not for other models. So an analyst is no better informed about what to do because who is to say that the models in the first set are any better than the models in the second set.

A second, related drawback is that conditions under which something need not be modelled ought to be based on properties of the thing being modelled, and not on properties of the hypothetical models an analyst might consider. But distinctness of parameters is a model-level property only. Unlike MAR, which is a property of a given response mechanism, and only becomes a model-level property by assuming that it holds for all response mechanisms in the model, distinctness of parameters cannot be assumed to hold for the underlying data generation process.

4 Another argument against distinctness of parameters

One assumption underlying Rubin’s joint modelling of the data vector $Y$ and response pattern $R$ is that the data variables are sufficiently rich to capture the probabilities of response. That is, one cannot have two different subjects whose respective values on all of the $Y$ variables agree but for which their probabilities of response differ (otherwise it is impossible to have a correctly-specified model). This is equivalent to the assumption that there exists a well-defined function (underlying response mechanism) $g(r|y)$ which captures the probabilities of response for all subjects.

It would seem difficult to mount a case that one could do better than to use this (implicitly assumed) truth about the response mechanism to complete the model $\mathcal{M}_s$ for the data vector $Y$:

$$\mathcal{M}_t = \{ f_\theta(y) g(r|y) : \theta \in \Theta \},$$

and therefore we take $\mathcal{M}_t$ as the most appropriate starting point for an analysis of the incomplete data. The likelihood derived from $\mathcal{M}_t$ for the observable data is

$$L_t(\theta) = \int f_\theta(y) g(r|y) \, dy_{mi}(r). \quad (2)$$

In general, maximising (2) is not possible because a functional form for the response mechanism is not known, and there is no way to evaluate the function $g(r|y)$. However, if $g(r|y)$ is MAR with respect to the realised values $(y, r)$, then (2) factorises in the usual way

$$L_t(\theta) = g(r|y) \int f_\theta(y) \, dy_{mi}(r) \quad (3)$$

and (3) can be maximised without the need to evaluate the constant $g(r|y)$. Therefore, under this assumption about $g(r|y)$, direct likelihood inferences for $\theta$ can be drawn based on the proportional likelihood

$$L_s(\theta) = \int f_\theta(y) \, dy_{mi}(r) \quad (4)$$

derived from the corresponding simpler model $\mathcal{M}_s$. That is, under a MAR assumption about $g(r|y)$, the analyst can use the (implicitly assumed) true underlying response mechanism in the analysis because there is no need for the analyst to know how to evaluate it at the specific observed values $y$ and $r$.

A second argument against the distinctness of parameters criterion is that it may lead to a suboptimal course of action with respect to models $\mathcal{M}_g$ for which distinctness of parameters does not hold. These models fail to meet the standard conditions for ignorability, so an analyst may proceed to model the response mechanism explicitly. However, if the underlying response mechanism satisfies the appropriate MAR condition,
then for a given model $\mathcal{M}_s$, the analyst cannot improve on an estimate for $\theta$ obtained from (4), and risks making an inferior inference through mis-specification of the model for the response mechanism.

5 Discussion

In [4] the authors observed that one difference in statements of the Missing at Random condition in the literature is that some authors had left off the parameter $\psi$ for the response mechanism, whereas other authors, following Rubin [3], had stated the MAR condition in terms of a model of the form $\mathcal{M}_g$. We can only speculate as to the various authors’ reasons for doing this, but a reasonable hypothesis is that the authors who omitted the parameter $\psi$ from the model consider $\mathcal{M}_t$ as the natural starting point for determination of whether or not an analysis based on $\mathcal{M}_s$ can be justified.

The main motivation for writing this note is that we feel Rubin’s presentation starting from a general model of the form $\mathcal{M}_g$ is just as likely to confuse as much as inform readers of the literature on incomplete data methods. We have given several reasons for this, but all can be traced to fact that models of the form $\mathcal{M}_g$ contain information about the response mechanism that is unnecessary for deciding whether or not an analyst needs to consider a model for the response mechanism, and distinctness of parameters is a condition pertaining to this unnecessary information. The argument given in Section 4 more specifically explains why the distinctness of parameters condition is irrelevant for deciding whether or not to model the response mechanism. Specifically, if MAR holds for the underlying response mechanism $g(r|y)$, then ignoring the response mechanism is equivalent to using the true response mechanism in the analysis, and this analysis cannot be improved upon by explicitly modelling the response mechanism instead (regardless of whether or not distinctness of parameters holds for the model).

In the introduction we stated that users of incomplete data methods require a clear rationale for when ignoring the response mechanism is justified. We conclude that the standard conditions of distinctness of parameters and all response mechanisms in the model are (realised) MAR should be replaced with the single assumption that the underlying response mechanism is (realised) MAR with respect to the observed data. The rationale for ignoring the response mechanism then becomes that, under this assumption, direct likelihood inferences drawn by ignoring the response mechanism coincide with the inferences an analyst would draw using the true underlying response mechanism. This cannot be improved upon by positing an explicit model the response mechanism: one can do no better, and in fact, one actually could well do worse.

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