Research Article

Iterative Learning Consensus Control for Nonlinear Partial Difference Multiagent Systems with Time Delay

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This paper considers the consensus control problem of nonlinear spatial-temporal hyperbolic partial difference multiagent systems and parabolic partial difference multiagent systems with time delay. Based on the system’s own fixed topology and the method of generating the desired trajectory by introducing virtual leader, using the consensus tracking error between the agent and the virtual leader agent and neighbor agents in the last iteration, an iterative learning algorithm is proposed. The sufficient condition for the system consensus error to converge along the iterative axis is given. When the iterative learning number approaches infinity, the consensus error in the sense of the $L^2$ norm between all agents in the system will converge to zero. Furthermore, simulation results illustrate the effectiveness of the algorithm.

1. Introduction

As sensor technology advances, communication and network technology, when solving complex practical problems, the ability of communication and coordination between various agents in a multiagent system (MAS) is used to solve the problem that a single agent cannot complete complex task [1, 2]. In recent years, MASs’s cooperative control has been widely used in the industry, such as formation control for drones [3], satellite attitude control [4], and congestion control of communication networks [5]. At present, the main problems are collaborative control of MASs cluster control [6], formation control [7], and consensus control [8]. Among them, the consensus problem of MASs is its most fundamental problem. The research on the MASs consensus problem is mainly through designing appropriate controllers for agents, so that all agents must be consensus in some given state or output [9].

In recent years, the consensus of MASs has been extensively studied from different perspectives [10]. In [11], Vicsek et al. first designed a classic consensus model and verified its effectiveness through simulation examples. In [12, 13], Ren studied the consensus problem of discrete and continuous models of MASs. In [14], Olfati-Saber et al. handled the progressive consensus of the leaderless MASs. Subsequently, based on the characteristics of the MASs, scholars designed more control algorithms to achieve consensus control of the MASs. In practical applications, there are many tasks that need to be executed repeatedly, and coordination should be carried out among several independent individuals [15]. For this reason, we need to study MASs. For example, in the coordinated control of a multimanipulator system with repeated operations on a production line, it is impossible to fully realize the control of the tracking desired trajectory within a given finite-time control [16]. For example, the periodic operation of multisatellite systems cannot ignore the influence of changes in the internal parameters of the system and external disturbances, and the application of the sliding mode control strategy cannot guarantee the stability of such systems [17]. Due to the limitation of information obtained by each agent, the expected information (or trajectory) of the leader specified by the topology diagram is only available to some agents. For each agent, this requires that the designed distributed algorithm only allows the use of local information obtained by certain agents to track the desired trajectory as perfectly as
possible. Iterative learning control (ILC) has great advantages in dealing with MASs that continuously repeat tasks within the finite time interval [18].

Other control methods [19–21] (such as feedback control, adaptive control, and robust control) are compared with ILC. Many formerers consider a continuous process system; that is, the operation of the system will continue to run over time. ILC is to complete a given task in a period of time and repeat it continuously. Therefore, compared with other control methods, in designing ILC algorithms, not only the time axis but also the iteration axis must be considered. The main control goal is to achieve convergence along the iteration axis, which is different from other control methods that only consider the time axis. In other words, ILC is a typical 2D process, and this characteristic also causes the theoretical analysis method of ILC algorithm to be different from other control methods. The controller of the ILC algorithm has a simple structure, and the ILC algorithm is effective for complex dynamic systems (such as uncertain, nonlinear, difficult to model, and missing information). It was proposed by Arimoto et al. in 1984 and after more than 30 years of development. There are many results in the research on ILC [22–25]. Because ILC has the above advantages, it provides an effective and feasible method for solving the consensus control problem of networked MASs in the finite-time interval. For example, in [26], the ILC method is used to solve the consensus problem that the leader with packet loss follows the nonlinear MASs. In [27], in order to solve the consensus tracking problem of the heterogeneous MASs, a corresponding ILC protocol was designed. In [28], P-type and PI-type ILC update algorithms are designed to solve the consensus tracking problem of a class of nonlinear fractional MASs. In [29], ILC technology is used to solve the consensus tracking problem in singular MASs. In [30], the problem of iterative learning consensus control in a continuous-time MAS is studied, and, during the learning process, the test length of each iteration is allowed to change randomly.

With the rapid development of digital sensors and digital controllers, in practical applications, although the system itself is a continuous process, since the channel bandwidth is limited in the communication system, the controller can only apply the sampled data obtained at discrete times [31]. Therefore, related research on discrete systems is necessary. Many scholars have begun to study the issues related to discrete-time MASs [32–36]. In [34], a discrete form of ILC algorithm was designed for the discrete MASs consensus tracking problem with random noise. In [35], the consensus tracking problem of a class of nonlinear discrete MASs is studied when the iteration length changes randomly. In [36], with the help of ILC algorithm, the consensus tracking problem of discrete-time MASs with cooperative competition network is solved.

Analysis of the above literatures reveals that the current results of using ILC algorithms to solve the consensus problems of MASs are mostly studies on ordinary differential MASs, which study the convergence of all agents in the system in the time domain. However, in practice, MASs have spatial dynamic behaviors, such as biological systems, chemical reactions, food networks, and other processes that depend on time and space [37]. The distributed parameter MASs consider the convergence of both the time domain and space domain, which has been of certain practical significance for the research of MASs related problems. Recently, there have been reports on related research on distributed parameter MASs. For example, in [38], the consensus of the continuous distributed parameter model MASs was achieved based on the Lyapunov function method. In [39], an ILC algorithm was proposed for continuous distributed parameter model MASs. In [40], the consensus problem of continuous distributed parameter MASs with time delay is studied. In [41], a second-order ILC algorithm is proposed for a class of consensus problems with continuous distributed parameter MASs with time delay.

Considering these factors, in this paper, a class of nonlinear first-order hyperbolic partial difference MASs and parabolic partial difference MASs with time delay are studied in the finite-time interval. Based on the fixed topology for the MASs, an ILC algorithm is proposed with a method for the virtual leader agent to generate the desired trajectory. This algorithm uses the tracking error between any agent with the virtual leader agent and neighbor agents in the last iteration and combines the characteristics of the MASs communication topology to continuously modify the previous control law to obtain the ideal control law. When the iterative learning index $k$ approaches infinity, in the sense of the $L^2$ norm, the consensus error between all agents in the MASs will converge to zero.

The main contributions of this paper are summarized as follows:

1. It can be noted that the works in [37–41] present a type of consensus problem in the study of continuous distributed parameter MASs. However, due to the limited channel bandwidth in the communication system, the controller can only apply the sampling data obtained at discrete moments. Therefore, it can be considered that the consensus control of discrete distributed parameter MASs is more practical.

2. In the works in [38–41], the consensus control problem of parabolic distributed parameter MASs is studied. Since hyperbolic system is an important system type, a kind of consensus control of hyperbolic distributed parameter MASs has been studied. When the leader agent number is unknown, complete consensus control of all agents in the system is achieved within a finite time.

3. From the analysis of the works in [38–42], in reality, the MASs have relatively complex nonlinearities and time delay. Therefore, an ILC algorithm is designed, which contains consensus errors between any two agents. After the system has been iteratively learned enough times, the consensus problem of nonlinear hyperbolic partial difference MASs and parabolic partial difference MASs with time delay is better solved.

The remainder of the article is organized as follows. In Section 2, we consider the iterative learning consensus
control for nonlinear first-order hyperbolic partial difference MASs with time delay. In Section 3, we discuss iterative learning consensus control of the nonlinear parabolic partial difference MASs with time delay. In Section 4, two examples illustrate the results. Section 5 gives the conclusion.

Notations. This article uses the following symbols: $I_N$ is the $N \times N$ dimension unit matrix, $1_N$ is a column vector of $N$ order, and each element is 1. $\otimes$ denotes the Kronecker product. For a function $f(\zeta)$, $\|f\|$ is expressed as $\|f\| = \left(\sum_{i=0}^{Z} f^2(\zeta)\right)^{1/2}$, where $0 \leq \zeta \leq Z$, and $\zeta = 0, 1, 2, \ldots, Z$. Let $\lambda (0 < \lambda < 1)$ be a positive constant, and, for the binary function $\psi (\zeta, h)$, $[0, Z] \times [0, H] \rightarrow \mathbb{R}$, $\zeta = 0, 1, 2, \ldots, Z$, $h = 0, 1, 2, \ldots, H$, its $\lambda$ norm is expressed as $\|\psi\|_\lambda^1 = \sup_{h \in [0, H]} \{\|\psi(\zeta, h)\|\}$. Using the knowledge of graph theory to describe the interrelationship between any two agents, the corresponding graph theory knowledge is as follows: let $G' = (V', E', A')$ represent a weighted directed graph, where $V' = \{1, 2, \ldots, N\}$ is a collection of nodes, and $E' \subseteq V' \times V'$ is the set of edges. $A' = \{a_{ij}\}$ represents the adjacency matrix of graph $G$; if node $j$ to node $i$ has an edge, then $a_{ij} > 0$, and this indicates that agent $j$ can accept the information of agent $i$; otherwise, $a_{ij} = 0$. Define the Laplacian matrix of the graph $G$ as $L_{G}^{\max} = \hat{D}' - A'$, degree matrix $D' = \text{diag}(d_{1}^{in}, d_{2}^{in}, \ldots, d_{N}^{in})$, $d_{j}^{in} = \sum_{i=1}^{N} a_{ij}$ be node $j$.

2. Iterative Learning Consensus Control for Nonlinear Hyperbolic Partial Difference Mass with Time Delay

Consider a class of nonlinear hyperbolic partial difference MASs composed of $N$ homogeneous discrete multiple agents, which run repeatedly in the finite-time interval. The dynamic model of the $i$-th agent is

\begin{align}
\Delta \tau w_{i,k}(\zeta, h) &= a(h)\Delta \tau w_{i,k}(\zeta + 1, h) + f(w_{i,k}(\zeta, h - \tau), u_{i,k}(\zeta, h), h), \\
y_{i,k}(\zeta, h) &= b(w_{i,k}(\zeta, h)) + c(h)u_{i,k}(\zeta, h),
\end{align}

where $\zeta, h$ are discrete variables of space and time, respectively. $Z$ and $H$ are given integers, and $0 \leq \zeta \leq Z$, $0 \leq h \leq H$. $k$ is the number of iterations, and $i \in \{1, 2, \ldots, N\}$ indicates the $i$-th agent. $\{a(h), \{c(h)\}$ are uncertain bound real sequences, and $\{a(h)\} > 0$.

Assumption 1. Graph $G$ contains the spanning tree, and the virtual leader agent is the root of the spanning tree.

Assumption 2. For all $k$, the initial boundary condition of the $i$-th agent in system (1) is given as

\begin{align}
w_{i,k}(0, h) = w_{i,k}(Z, h), \quad i = 1, 2, \ldots, N, \quad -\tau \leq h \leq H, \\
w_{i,k}(\zeta, h) = \varphi_i(\zeta, h), \quad i = 1, 2, \ldots, N, \quad -\tau \leq h \leq 0, 0 \leq \zeta \leq Z.
\end{align}

Assumption 3. The dynamic characteristics of system (1) are reversible, ensuring that there is a single desired control input such that the state and output of system (1) are expected.

Assumption 4. The nonlinear functions $f, b$, respectively, satisfy the following uniform global Lipschitz condition:

\begin{align}
\|f(w_{i}(\zeta, h - \tau), u_{i}(\zeta, h), h) - f(w_{i}(\zeta, h - \tau), u_{j}(\zeta, h), h))\| &\leq l_f(\|w_{i}(\zeta, h - \tau) - w_{j}(\zeta, h - \tau)\| + \|u_{i}(\zeta, h) - u_{j}(\zeta, h)\|), \\
\|b(w_{i}(\zeta, h), h) - b(w_{j}(\zeta, h), h))\| &\leq l_b(\|w_{i}(\zeta, h) - w_{j}(\zeta, h)\|),
\end{align}

where $l_f$ and $l_b$ are constants.

Remark 1. Agents can only get information from virtual leader and neighbor agents, while virtual leaders cannot get information from other agents.

Lemma 1 (see [43]). Let $\{A(v)\}$, $\{B(v)\}$, and $\{C(v)\}$ be real sequences defined for $v \geq 0$ satisfying

\begin{align}
A(v + 1) &\leq B(v)A(v) + C(v), \quad B(v) \geq 0, v \geq 0.
\end{align}

Then,

\begin{align}
A(h) &\leq \prod_{v=0}^{h-1} B(v)A(0) + \sum_{v=1}^{h-1} C(v) \prod_{s=v+1}^{h-1} B(s), \quad h \geq 0.
\end{align}

In this paper, the trajectory of the virtual leader agent $y_{d}(\zeta, h)$ is the desired trajectory of the discrete distributed parameter MAS (see (1)) $y_{d}(\zeta, h)$. Because MAS has a distributed structure, only a few agents in the system are allowed to directly obtain the required trajectory.
where $y_i, k$ is the output of the $i$-th agent in the system (1). This equation means that after the MAS learns with an ILC algorithm, in system (1), the consensus error between any two agents is able to converge to zero.

Based on the characteristics of information transfer between agents, the following ILC algorithm is designed:

$$u_{i,k+1}(\zeta, h) = u_{i,k}(\zeta, h) + \gamma(h)$$

$$+ \sum_{j=1}^{N} a_{ij}[(e_{i,j}(\zeta, h) - e_{j,k}(\zeta, h)) + s_i e_{i,k}(\zeta, h)]$$

where $\gamma(h)$ is the learning gain, $k$ is the number of iterations. $e_{i,j}(\zeta, h) = y_i(\zeta, h) - y_{i,j}(\zeta, h)$ is the tracking error of the $i$-th agent, and $e_{j,k}(\zeta, h) = y_j(\zeta, h) - y_{j,k}(\zeta, h)$ is the tracking error of the $j$-th agent. $s_i$ reflects the communication relationship between the $i$-th agent and virtual leader agent. $s_i > 0$ indicates that the $i$-th agent can communicate with virtual leader agent, and $s_i = 0$ is the opposite.

For the simplicity of the subsequent convergence analysis, the compact form of system (1) is as follows:

$$\begin{cases}
\Delta_2 w_k(\zeta, h) = a(h) \Delta_1 w_k(\zeta + 1, h) + f(w_k(\zeta, h), h), \\
y_{\epsilon}(\zeta, h) = b(w_k(\zeta, h), h) + c(h) u_k(\zeta, h),
\end{cases}$$

where $\Delta_2 w_k(\zeta, h) = \begin{bmatrix} w_{T_{12}}^T(\zeta, h), w_{T_{22}}^T(\zeta, h), \ldots, w_{T_{N,2}}^T(\zeta, h) \end{bmatrix}^T \in \mathbb{R}^N$, $u_k(\zeta, h) = \begin{bmatrix} u_{T_{12}}^T(\zeta, h), u_{T_{22}}^T(\zeta, h), \ldots, u_{T_{N,2}}^T(\zeta, h) \end{bmatrix}^T \in \mathbb{R}^N$, $y_{\epsilon}(\zeta, h) = \begin{bmatrix} y_{T_{12}}^T(\zeta, h), y_{T_{22}}^T(\zeta, h), \ldots, y_{T_{N,2}}^T(\zeta, h) \end{bmatrix}^T \in \mathbb{R}^N$.

Define the consensus error of system (10) as

$$e_k(\zeta, h) = 1_N y_{\epsilon}(\zeta, h) - y_k(\zeta, h), \quad 0 \leq h \leq H,$$

where $y_{\epsilon}(\zeta, h)$ is the trajectory of virtual leader agent and $e_k(\zeta, h) = [e_{T_{12}}^T(\zeta, h), e_{T_{22}}^T(\zeta, h), \ldots, e_{T_{N,2}}^T(\zeta, h)]^T \in \mathbb{R}^N$.

Further, the compact form corresponding to ILC law (see (9)) is can be written as

$$u_{k+1}(\zeta, h) = u_k(\zeta, h) + [\gamma(h)(L + S)1_N] e^T_k(\zeta, h), \quad (13)$$

where $L \in \mathbb{R}^{N \times N}$ is the Laplacian matrix of graph $G$ and $S \in \mathbb{R}^{N \times N}$. After system (10) iterates $k$ times, the corresponding difference form is expressed as

$$\Delta_2 w_k(\zeta, h) = w_k(\zeta + 1, h) - w_k(\zeta, h), \quad (14)$$

$$\Delta_1 w_k(\zeta + 1, h) = w_k(\zeta + 1, h) - w_k(\zeta, h). \quad (15)$$

**Theorem 1.** If $\gamma(h)$ in the ILC law (see (13)) satisfies $|1 - \gamma(h)(L + S)1_N| < 0.5, 0 \leq h \leq H$ and satisfies Assumptions 1–4 at the same time, and when the number of iterations $k \rightarrow \infty$, the consensus error of system (10) can converge to zero in the following sense:

$$\lim_{k \rightarrow \infty} \|y_{i,k}(h) - y_{j,k}(h)\|^2 = 0,$$

$$i, j \in \{1, 2, \ldots, N\}, 0 \leq h \leq H.$$

**Proof.** From equations (12) and (13), after the $k$-th iteration, the consensus error of system (10) can be written as

$$e_{k+1}(\zeta, h) = 1_N y_{\epsilon}(\zeta, h) - y_{k+1}(\zeta, h),$$

$$= e_k(\zeta, h) - [b(w_{k+1}(\zeta, h), h) - b(w_k(\zeta, h), h)]$$

$$- c(t)(u_{k+1}(\zeta, h) - u_k(\zeta, h)),$$

$$= e_k(\zeta, h) - [b(w_{k+1}(\zeta, h), h) - b(w_k(\zeta, h), h)]$$

$$- c(t)\gamma(h)[(L + S)1_N] e_k(\zeta, h),$$

$$= [1 - c(h)l(h)] e_k(\zeta, h) - [b_{k+1} - b_k].$$

$$e_{k+1}^2(\zeta, h) = [1 - c(h)l(h)] e_k^2(\zeta, h) - [b_{k+1} - b_k]^2,$$

$$\leq 2[1 - c(h)l(h)] e_k^2(\zeta, h) + 2[1 - c(h)l(h)] e_k^2(\zeta, h).$$

Summing from $\zeta = 0$ to $Z$ on both sides of inequality (19), one obtains
\[
\sum_{\zeta=0}^{Z} e_k^2(\zeta, h) \leq \sum_{\zeta=0}^{Z} 2\lambda_{ti} \frac{1}{h} \|e_k(\zeta, h)\|^2 + 2\lambda_{ti} \|w_k(\zeta, h)\|^2,
\]

where
\[
\lambda_{ti} = \sup_{0 \leq t \leq T} \frac{1}{h} \|e_k(\zeta, h)\|^2.
\]

It can be seen from inequality (20) that, to prove the convergence of consensus error \(\|e_k(\zeta, h)\|^2\) of system (10), it is necessary to estimate \(\bar{w}_k(\zeta, h)\). Thus, by the first equation in system (10) and equation (14), one can derive
\[
w_k(\zeta, h + 1) = a(h)\Delta_1 w_k(\zeta + 1, h) + w_k(\zeta, h) + f(w_k(\zeta, h - \tau), u_k(\zeta, h)).
\]

Further, during the \(k + 1\)-th iteration, the agents state can also be written as
\[
w_{k+1}(\zeta, h + 1) = a(h)\Delta_1 w_{k+1}(\zeta + 1, h) + w_{k+1}(\zeta, h) + f(w_{k+1}(\zeta, h - \tau), u_{k+1}(\zeta, h)).
\]

Next, after subtracting equation (22) from equation (23) and combining with equation (15), we have the following results:
\[
\bar{w}_k(\zeta, h + 1) = a(h)\left[\bar{w}_k(\zeta + 1, h) - \bar{w}_k(\zeta, h)\right] + \bar{w}_k(\zeta, h) + f(w_{k+1}(\zeta, h - \tau), u_{k+1}(\zeta, h), h) - f(w_k(h - \tau), u_k(\zeta, h), h).
\]

The two sides of equation (24) are squared at the same time. Since \((a + b + c) \leq 3a^2 + 3b^2 + 3c^2\), equation (24) can be rewritten as
\[
[\bar{w}_k(\zeta, h + 1)]^2 = \left[a(h)[\bar{w}_k(\zeta + 1, h) - \bar{w}_k(\zeta, h)] + \bar{w}_k(\zeta, h) + f(w_{k+1}(\zeta, h - \tau), u_{k+1}(\zeta, h), h) - f(w_k(h - \tau), u_k(\zeta, h), h)]^2
\]

Summing from \(\zeta = 0\) to \(Z\) on both sides of inequality (25) and using Lipschitz condition (5), one obtains
\[
\sum_{\zeta=0}^{Z} [\bar{w}_k(\zeta, h + 1)]^2 \leq 3a^2(h) \sum_{\zeta=0}^{Z} [\bar{w}_k(\zeta + 1, h) - \bar{w}_k(\zeta, h)]^2 + 3 \sum_{\zeta=0}^{Z} \bar{w}_k^2(\zeta, h)
\]

where \(\Sigma_1 = 3a^2(h) \sum_{\zeta=0}^{Z} [\bar{w}_k(\zeta + 1, h) - \bar{w}_k(\zeta, h)]^2 + 3\|\bar{w}_k(\zeta, h)\|^2\), \(\Sigma_2 = 3f\left[\|\bar{w}_k(h, h - \tau)\|^2 + \|\bar{u}_k(h, h)\|^2\right]\). Combining the boundary condition (3), \(\Sigma_1\) in inequality (26) can be written as
\[ \Sigma_1 = 3a^2(h) \sum_{\zeta=0}^Z \left[ \bar{w}_k(\zeta + 1, h) - w_h(\zeta, h) \right]^2 + 3\| \bar{w}_k(\cdot, h) \|^2, \]

\[ \leq 3a^2(h) \sum_{\zeta=0}^Z \left\{ 2\bar{w}_k^2(\zeta + 1, h) + 2\bar{w}_k^2(\zeta, h) \right\} + 3\| \bar{w}_k(\cdot, h) \|^2 \]

\[ \leq 12a^2(h) \sum_{\zeta=0}^Z \bar{w}_k^2(\zeta, h) - 6a^2(h)\bar{w}_k(1, h) + 3\| \bar{w}_k(\cdot, h) \|^2 \]

\[ \leq \left( 12a^2(h) + 3 \right)\| \bar{w}_k(\cdot, h) \|^2. \]

(27)

Substituting (27) into (26), one has

\[ \| \bar{w}_k(\cdot, h + 1) \|^2 \leq \left( 12a^2(h) + 3 \right)\| \bar{w}_k(\cdot, h) \|^2 \]

\[ + 3l_s \left\{ \| \bar{w}_k(\cdot, h - \tau) \|^2 + \| \bar{w}_k(\cdot, h) \|^2 \right\} \]

\[ + m_1\| \bar{w}_k(\cdot, h) \|^2 \]

\[ + m_2 \left\{ \| \bar{w}_k(\cdot, h - \tau) \|^2 + \| \bar{w}_k(\cdot, h) \|^2 \right\}, \]

(28)

where \( m_1 = \sup_{h \in [0, H]} \left\{ 12a^2(h) + 3 \right\}, m_2 = 3l_s \). Combining Lemma 1 and the initial value condition (4), we have the following results:

\[ \sum_{\zeta=0}^Z \bar{w}_k^2(\zeta, h) \leq \sum_{\zeta=0}^{h-1} m_1 \left\{ \sum_{\zeta=0}^{h-1} \bar{w}_k^2(\zeta, s - \tau) + \sum_{\zeta=0}^{h-1} \bar{w}_k^2(\zeta, s) \right\} m_1^{(h-s-2)}. \]

(29)

So, we can see the relationship between \( \bar{u}_k(\zeta, h) \) and \( \bar{u}_k(\zeta, h) \). Next, choosing ILC algorithm (13), one obtains

\[ \{ \bar{u}_k(\zeta, h) \}^2 = \{ u_{k+1}(\zeta, h) - u_k(\zeta, h) \}^2 \leq l^2 e_k^2(\zeta, h), \]

(30)

where \( l = \sup_{h \in [0, H]} \gamma(h) (L + S)_1 \).

In addition, by inequality (30), inequality (29) can be further written as

\[ \sum_{\zeta=0}^Z \bar{w}_k^2(\zeta, h) \leq \sum_{\zeta=0}^{h-1} m_1 \left\{ \sum_{\zeta=0}^{h-1} \bar{w}_k^2(\zeta, s - \tau) + \sum_{\zeta=0}^{h-1} \bar{w}_k^2(\zeta, s) \right\} m_1^{(h-s-2)}. \]

(31)

Multiplying both sides of inequality (31) by \( \lambda^h (0 < \lambda < 1) \), (31) can be rewritten as follows:

\[ \sum_{\zeta=0}^Z \bar{w}_k^2(\zeta, h) \lambda^h \leq \sum_{\zeta=0}^{h-1} m_1 \left\{ \sum_{\zeta=0}^{h-1} \bar{w}_k^2(\zeta, s - \tau) \lambda^h + \sum_{\zeta=0}^{h-1} \bar{w}_k^2(\zeta, s) \lambda^h \right\} m_1^{(h-s-2)}. \]

(32)

Estimating state \( \bar{w}_k(\zeta, s - \tau) \) with time delay in inequality (32) and letting \( \Sigma_3 = m_2 \sum_{h=0}^{h-r-1} \sum_{\zeta=0}^Z \bar{w}_k^2(\zeta, s - \tau) \lambda^h m_1^{(h-s-3)} \), from the known conditions, when \( h \in [-r, 0] \), there is \( w_k(\zeta, h) = \phi_k(\zeta, h) \), and one has

\[ \Sigma_3 = m_2 \lambda^{(r+2)} \sum_{h=0}^{h-r-1} \lambda m_1^{(h-m-r-2)} \| \bar{w}_{\phi}(\cdot, \tau) \|^2 \]

\[ + m_2 \lambda^{(r+2)} \sum_{m=0}^{h-r-1} \lambda m_1^{(h-n-m-r-2)} \| \bar{w}_k(\cdot, \tau) \|^2. \]

(33)

By the initial value condition \( \| \bar{w}_{\phi}(\cdot, \tau) \|^2 = 0 \), inequality (33) can be written as

\[ \Sigma_3 = m_2 \lambda^{(r+2)} \sum_{h=0}^{h-r-1} \lambda m_1^{(h-m-r-2)} \| \bar{w}_k(\cdot, \tau) \|^2. \]

(34)

Substituting inequality (32) into inequality (34), one has

\[ \| \bar{w}_k(\cdot, \tau) \|^2 \leq \frac{m_2}{1 - \lambda m_1} \| \bar{w}_k(\cdot, \tau) \|^2 + m_2 \lambda^{(r+2)} \| \bar{w}_k(\cdot, \tau) \|^2 \]

\[ = \frac{h_2 \lambda^{(r+2)}}{1 - \lambda m_1} \| \bar{w}_k(\cdot, \tau) \|^2 + m_2 \lambda^{(r+2)} \| \bar{w}_k(\cdot, \tau) \|^2. \]

(35)

Since \( \lambda (0 < \lambda < 1) \) is sufficiently small, \( 0 < 1 - \lambda m_1 < 1 \) can be obtained. Inequality (35) can be written as

\[ \| \bar{w}_k(\cdot, \tau) \|^2 \leq \frac{m_2}{1 - \lambda m_1} \| \bar{w}_k(\cdot, \tau) \|^2 \leq \| \bar{w}_k(\cdot, \tau) \|^2. \]

(36)

Multiplying both sides of inequality (20) by \( \lambda^{h} (0 < \lambda < 1) \), one obtains

\[ \| e_{k-1} \|^2 \leq 2\lambda \| e_k \|^2 + 2l \| \bar{w}_k \|^2. \]

(37)

Substituting inequality (37) into inequality (36), one obtains

\[ \| e_{k-1} \|^2 \leq (2\lambda + 2l) \| e_k \|^2. \]

(38)

Letting \( \rho = 2\lambda + 2l, \lambda = \sup_{h \in [0, H]} \{ 1 - \gamma(h)(L + S)_1 \} \), \( 2\lambda < 1 \) can be obtained from the condition \( |1 - \gamma(h)(L + S)_1| \leq 0.5, 0 \leq h \leq H \), of Theorem 1. Therefore, \( \rho < 1 \). After recursion by inequality (38), one gets

\[ \| e_{k-1} \|^2 \leq \rho^k \| e_0 \|^2 \leq \ldots \leq \rho^k \| e_0 \|^2. \]

(39)

Then, by the contraction mapping principle, one obtains

\[ \lim_{k \to \infty} \| e_k \|^2 = 0. \]

(40)

Further, we have the following results:

\[ 0 \leq \| e_k(\cdot, h) \|^2 = \lambda^{h} \| e_k(\cdot, h) \|^2, \]

\[ \leq \sup_{0 \leq h \leq H} \{ \| e_k(\cdot, h) \|^2 \} \lambda^{-H} \]

\[ = \lambda^{-H} \| e_0 \|^2. \]

(41)

Thus, one has

\[ \lim_{k \to \infty} \| e_k(\cdot, h) \|^2 = 0, 0 \leq h \leq H. \]

(42)

Next, combining equation (22), the \( i \)-th agent consensus tracking error can be rewritten as follows:
Further, due to \( \| \varepsilon_{i,k} (\cdot, h) \|^2 = \| y_r (\cdot) - y_{i,k} (\cdot, h) \|^2 \), one gets
\[
\| y_{i,k} (\cdot, h) - y_{j,k} (\cdot, h) \|^2 = \| y_r (\cdot, h) - y_{i,k} (\cdot, h) \|^2 - \| y_r (\cdot, h) - y_{j,k} (\cdot, h) \|^2
\]
\[
\leq \| e_{i,k} (\cdot, h) \|^2 + \| e_{j,k} (\cdot, h) \|^2.
\]

Finally, based on equation (42) and inequality (43), one obtains
\[
\lim_{k \to \infty} \| y_{i,k} (\cdot, h) - y_{j,k} (\cdot, h) \|^2 = 0,
\]
\( i, j \in \{1, 2, \ldots, N\}, 0 \leq h \leq H. \)  

The proof of Theorem 1 is completed.

\( \square \)

3. Iterative Learning Consensus Control for Nonlinear Parabolic Partial Difference Mass with Time Delay

In this section, consider a nonlinear parabolic partial difference MAS (45) composed of \( N \) homogeneous discrete multiple agents, which run repeatedly within the finite-time interval, where the \( i \)-th agent’s dynamic equation is
\[
\Delta_2 w_{i,k} (\zeta, h) = a(h) \Delta_2^2 w_{i,k} (\zeta - 1, h) + f (w_k (\zeta, h - \tau), u_k (\zeta, h), h),
\]
\[
y_{i,k} (\zeta, h) = b (w_k (\zeta, h), h) + c (h) u_{i,k} (\zeta, h),
\]
where \( \zeta, h \) are discrete variables of space and time, respectively. \( Z \) and \( H \) are given integers, and \( 0 \leq \zeta \leq Z, 0 \leq h \leq H \). \( k \) is the number of iterations, and \( i \in \{1, 2, \ldots, N\} \) indicates the \( i \)-th agent. \( a(h), c(h) \) are uncertain bounded real sequences, and \( a(h) > 0 \). \( w_{i,k} (\zeta, h), u_{i,k} (\zeta, h), y_{i,k} (\zeta, h) \in \mathbb{R} \), respectively, denote the state of the \( i \)-th agent in system (41), control input, and control output. \( \tau \) is the time delay; let the initial state function be \( \varphi (\cdot) \); when \( h \in [-\tau, 0] \), \( w_{i,k} (\zeta, h) = \varphi_{i,k} (\zeta, h) \). \( f : \mathbb{R} \times \mathbb{R} \times [0, H] \longrightarrow \mathbb{R}, b : \mathbb{R} \times [0, H] \longrightarrow \mathbb{R} \) are both nonlinear functions. The differential form of the first expression in system (45) is defined as
\[
\Delta_2 w_{i,k} (\zeta, h) = w_{i,k} (\zeta, h + 1) - w_{i,k} (\zeta, h),
\]
\[
\Delta_2^2 w_{i,k} (\zeta - 1, h) = w_{i,k} (\zeta + 1, h) - 2 w_{i,k} (\zeta, h) + w_{i,k} (\zeta - 1, h).
\]

Here, it is still necessary to achieve consensus error between any two agents in system (45) to satisfy \( \lim_{k \to \infty} \| y_{i,k} (\cdot, h) - y_{j,k} (\cdot, h) \|^2 = 0 \). Under the conditions of Assumptions 1–4, we use ILC algorithm (13) to achieve this goal.

For the simplicity of the subsequent convergence analysis, the compact form of system (45) is as follows:
\[
\begin{align*}
\Delta_2 w_k (\zeta, h) = & \ a(h) \Delta_2^2 w_k (\zeta - 1, h) + f (w_k (\zeta, h - \tau), u_k (\zeta, h), h), \\
y_k (\zeta, h) = & \ b (w_k (\zeta, h), h) + c (h) u_k (\zeta, h),
\end{align*}
\]
where
\[
\begin{align*}
w_k (\zeta, h) = & \begin{bmatrix} u_{1,k}^T (\zeta, h), u_{2,k}^T (\zeta, h), \ldots, u_{N,k}^T (\zeta, h) \end{bmatrix}^T \in \mathbb{R}^N, \\
u_k (\zeta, h) = & \begin{bmatrix} u_{1,k}^T (\zeta, h), u_{2,k}^T (\zeta, h), \ldots, u_{N,k}^T (\zeta, h) \end{bmatrix}^T \in \mathbb{R}^N, \\
y_k (\zeta, h) = & \begin{bmatrix} y_{1,k}^T (\zeta, h), y_{2,k}^T (\zeta, h), \ldots, y_{N,k}^T (\zeta, h) \end{bmatrix}^T \in \mathbb{R}^N.
\end{align*}
\]

After system (45) iterates \( k \) times, the corresponding difference form is expressed as
\[
\begin{align*}
\Delta_2 w_k (\zeta, h) = & \ w_k (\zeta, h + 1) - w_k (\zeta, h), \\
\Delta_2^2 w_k (\zeta - 1, h) = & \ w_k (\zeta + 1, h) - 2 w_k (\zeta, h) + w_k (\zeta - 1, h).
\end{align*}
\]

Theorem 2. If \( \gamma(h) \) is in use, ILC algorithm (13) satisfies \( |1 - \gamma(h) (L + S) |^2 \leq 0.5, 0 \leq h \leq H \) and satisfies Assumptions 1–4 at the same time; then when the iteration index \( k \longrightarrow \infty \), in the following sense, consensus error for the nonlinear parabolic partial difference system (47) can converge to zero; that is,
\[
\lim_{k \to \infty} \| y_{i,k} (\cdot, h) - y_{j,k} (\cdot, h) \|^2 = 0,
\]
\( i, j \in \{1, 2, \ldots, N\}, 0 \leq h \leq H. \)

Proof. To prove the convergence for system (47) consensus error \( \| e_k (\cdot, h) \|^2 \), we need to estimate \( \bar{w}_k (\cdot, h) \). By equation (49) and the first equation in system (47), one has
\[
\begin{align*}
w_k (\zeta, h + 1) = & \ a(h) \Delta_2 w_k (\zeta - 1, h) + w_k (\zeta, h) \\
& \ + f (w_k (\zeta, h - \tau), u_k (\zeta, h), h),
\end{align*}
\]
Similar to equation (52), during the \( k + 1 \)-th iteration, the agent’s state can also be written as
\[
\begin{align*}
w_{k+1} (\zeta, h + 1) = & \ a(h) \Delta_2^2 w_{k+1} (\zeta - 1, h) + w_{k+1} (\zeta, h) \\
& \ + f (w_{k+1} (\zeta, h - \tau), u_{k+1} (\zeta, h), h).
\end{align*}
\]

Next, subtracting equation (52) from equation (53) and combining with equation (50), one has
\[
\begin{align*}
\bar{w}_h (\zeta, h + 1) = & \ a(h) \bigg[ \Delta_2^2 w_{k+1} (\zeta - 1, h) - \Delta_2 w_k (\zeta - 1, h) \bigg] \\
& \ + \bar{w}_h (\zeta, h) + f (w_{k+1} (\zeta, h - \tau), u_{k+1} (\zeta, h), h), \\
& \ = a(h) \big[ \bar{w}_h (\zeta + 1, h - 2 \bar{w}_h (\zeta, h) + \bar{w}_h (\zeta, h) \big] \\
& \ + \bar{w}_h (\zeta, h) + f (w_{k+1} (\zeta, h - \tau), u_{k+1} (\zeta, h), h), \\
& \ - f (w_k (\zeta, h - \tau), u_k (\zeta, h), h),
\end{align*}
\]
Here, squaring both sides of equation (54) and by \( (a + b + c) \leq 3a^2 + 3b^2 + 3c^2 \), one obtains

\[
\begin{align*}
\{ \tilde{w}_k (\zeta, h + 1) \}^2 &= [a (h) [\tilde{w}_k (\zeta + 1, h) - 2\tilde{w}_k (\zeta, h) + \tilde{w}_k (\zeta - 1, h)] + \tilde{w}_k (\zeta, h) + \sum_{k=0}^{\infty} f (w_{k+1} (\zeta, h - \tau), u_{k+1} (\zeta, h), h) - f (w_k (\zeta, h - \tau), u_k (\zeta, h), h)]^2, \\
&\leq 3a^2 (h) [\tilde{w}_k (\zeta + 1, h) - 2\tilde{w}_k (\zeta, h) + \tilde{w}_k (\zeta - 1, h)]^2 + 3\{ f (w_{k+1} (\zeta, h - \tau), u_{k+1} (\zeta, h), h) - f (w_k (\zeta, h - \tau), u_k (\zeta, h), h)]^2.
\end{align*}
\]

(55)

By the boundary condition (4), the following inequality holds:

\[
\begin{align*}
\sum_{\zeta=0}^{Z} \{ \tilde{w}_k (\zeta, h + 1) \}^2 &\leq \sum_{\zeta=0}^{Z} 3a^2 (h) [\tilde{w}_k (\zeta + 1, h) - 2\tilde{w}_k (\zeta, h) + \tilde{w}_k (\zeta - 0, h)]^2 \\
&\quad + \sum_{\zeta=0}^{Z} 3\{ f (w_{k+1} (\zeta, h - \tau), u_{k+1} (\zeta, h), h) - f (w_k (\zeta, h - \tau), u_k (\zeta, h), h)]^2 + \sum_{\zeta=0}^{Z} 3\tilde{w}_k^2 (\zeta, h), \\
&\leq \sum_{\zeta=0}^{Z} 3a^2 (h) [\tilde{w}_k (\zeta + 1, h) - 2\tilde{w}_k (\zeta, h) + \tilde{w}_k (\zeta - 0, h)]^2 + 3\{ \tilde{w}_k (\cdot, h) \}^2 + 3l_j \left( \tilde{w}_k (\cdot, h - \tau) \right)^2 + \tilde{u}_k (\cdot, h) \right)^2 \\
&\triangleq \Sigma_1 + \Sigma_2.
\end{align*}
\]

(56)

where

\[
\Sigma_1 = 3a^2 (h) \sum_{\zeta=0}^{Z} \{ \tilde{w}_k (\zeta + 1, h) - 2\tilde{w}_k (\zeta, h) + \tilde{w}_k (\zeta - 1, h)]^2 \\
+ 3\{ \tilde{w}_k (\cdot, h) \}^2,
\]

\[
\Sigma_2 = 3l_j \left( \tilde{w}_k (\cdot, h - \tau) \right)^2 + \tilde{u}_k (\cdot, h) \right)^2.
\]

(57)

and

\[
\begin{align*}
\Sigma_1' &= 3a^2 (h) \sum_{\zeta=0}^{Z} \{ \tilde{w}_k (\zeta + 1, h) - 2\tilde{w}_k (\zeta, h) + \tilde{w}_k (\zeta - 1, h)]^2 \\
&\quad + 3\{ \tilde{w}_k (\cdot, h) \}^2 \\
&\leq 3a^2 (h) \sum_{\zeta=0}^{Z} \{ 3\tilde{w}_k^2 (\zeta + 1, h) + 12\tilde{w}_k^2 (\cdot, h) + 3\tilde{w}_k^2 (\zeta - 1, h)] + 3\{ \tilde{w}_k (\cdot, h) \}^2 \\
&\leq 54a^2 (h) \sum_{\zeta=0}^{Z} \tilde{w}_k^2 (\zeta, h - \tau) \tilde{w}_k^2 (\zeta, h) - 9a^2 (h) \tilde{w}_k^2 (\zeta, h + 3\{ \tilde{w}_k (\cdot, h) \}^2 \\
&\leq (54a^2 (h) + 3)\{ \tilde{w}_k (\cdot, h) \}^2.
\end{align*}
\]

(58)
Substituting (58) into (56), one has
\[
\|\tilde{w}_k(\cdot, h + 1)\|^2 \leq (54a^2(h) + 3)\|\tilde{w}_k(\cdot, h)\|^2 \\
+ 3M_l\left(\|\tilde{w}_k(\cdot, h - r)\|^2 + \|\tilde{u}_k(\cdot, h)\|^2\right),
\]
\[
\leq M_1\|\tilde{w}_k(\cdot, h)\|^2 \\
+ M_2\left(\|\tilde{w}_k(\cdot, h - r)\|^2 + \|\tilde{u}_k(\cdot, h)\|^2\right),
\]
(59)

where \( M_1 = \sup_{h \in \mathbb{R}_+} \{54a^2(h) + 3\} \), \( M_2 = 3M_l \). Combining Lemma 1 and the initial condition (4), one has
\[
Z \sum_{\zeta = 0}^h \tilde{w}_k^2(\zeta, h) \leq \sum_{\zeta = 0}^{h-1} M_2 \left( Z \sum_{\zeta = 0}^\infty \tilde{w}_k^2(\zeta, s - r) + Z \sum_{\zeta = 0}^\infty \tilde{u}_k^2(\zeta, s) \right) M_1^{(h-r-2)}.
\]
(60)

Here, we found the relationship between \( \tilde{w}_k(\cdot, h)^2 \) and \( \tilde{u}_k(\cdot, h)^2 \). Following on, by the ILC law (see (13)), one gets
\[
\|\tilde{w}_k(\cdot, h)\|^2 = \|u_{k+1}(\cdot, h) - u_k(\cdot, h)\|^2 \\
\leq \lambda^2 \tilde{e}_k^2(\cdot, h),
\]
(61)

where \( l = \sup \left\{ \gamma(h) | L + S \right\} \). It is observed that the output equation of system (45) is the same as that of system (1), which gives the following results:
\[
Z \sum_{\zeta = 0}^h \tilde{w}_k^2(\zeta, h) \leq \sum_{\zeta = 0}^{h-1} 2\left[1 - c(h)l(h)\right]\tilde{e}_k^2(\zeta, h) + \sum_{\zeta = 0}^\infty \left[b_{k+1} - b_k\right]^2 \\
= 2\left[1 - c(h)l(h)\right]\|\tilde{e}_k(\cdot, h)\|^2 \\
+ 2L_b\|\tilde{u}_{k+1}(\cdot, h) - \tilde{u}_k(\cdot, h)\|^2 \\
\leq 2L_b\|\tilde{e}_k(\cdot, h)\|^2 + 2L_b\|\tilde{u}_k(\cdot, h)\|^2,
\]
(62)

where
\[
\lambda_{cl} = \sup_{0 \leq h \leq H} \left\{1 - c(h)l(h)\right\},
\]
(63)
\[
\tilde{w}_k(\cdot, h) = \tilde{w}_{k+1}(\cdot, h) - \tilde{u}_k(\cdot, h).
\]

In addition, by inequality (61), inequality (60) can be further written as
\[
Z \sum_{\zeta = 0}^h \tilde{w}_k^2(\zeta, h) \leq \sum_{\zeta = 0}^{h-1} M_2 \left( Z \sum_{\zeta = 0}^\infty \tilde{w}_k^2(\zeta, s - r) + Z \sum_{\zeta = 0}^\infty \tilde{e}_k^2(\zeta, s) \right) M_1^{(h-r-2)}.
\]
(64)

Multiplying both sides of inequality (64) by \( \lambda^h(0 < \lambda < 1) \), one obtains
\[
Z \sum_{\zeta = 0}^h \tilde{w}_k(\zeta, s - r) \lambda^h \leq \sum_{\zeta = 0}^{h-1} M_1 \left( Z \sum_{\zeta = 0}^\infty \tilde{w}_k(\zeta, s - r) \lambda^h + Z \sum_{\zeta = 0}^\infty \tilde{e}_k(\zeta, s) \lambda^h \right) M_1^{(h-r-2)}.
\]
(65)

Estimating state \( \tilde{w}_k(\cdot, s - r) \) with time delay in inequality (65) and letting \( \Sigma_1 = M_2 \sum_{\zeta = 0}^{h-1} \left[ \sum_{s = 0}^\infty \tilde{w}_k^2(\zeta, s - r) \lambda^h \right] \), \( M_1^{(h-r-2)} \), from the known conditions, when \( h \in [-\tau, 0] \), there is \( \tilde{w}_k(\cdot, h) = \tilde{q}_k(\cdot, h) \), and one has
\[
\Sigma_1' = M_2 \lambda^{(r+2)} \sum_{m = 0}^{h-r-1} M_1^{(h-m-r-2)} \|\tilde{q}_k\|^2 \\
+ M_2 \lambda^{(r+2)} \sum_{m = 0}^{h-r-1} M_1^{(h-m-r-2)} \|\tilde{w}_k\|^2
\]
(66)

By the initial value condition \( \|\tilde{q}_k\|^2 = 0 \), inequality (66) can be written as
\[
\Sigma_1' = M_2 \lambda^{(r+2)} \sum_{m = 0}^{h-r-1} M_1^{(h-m-r-2)} \|\tilde{w}_k\|^2.
\]
(67)

Substituting inequality (65) into inequality (67), one has
\[
\|\tilde{w}_k\|^2 \leq \frac{M_2 \lambda^{(r+2)}}{1 - M_1 \lambda} \|\tilde{w}_k\|^2 + M_2 \lambda^2 \sum_{m = 0}^{h-r-1} M_1 \lambda (h-r-2) \|e_k\|^2
\]
\[
= \frac{M_2 \lambda^{(r+2)}}{1 - M_1 \lambda} \|\tilde{w}_k\|^2 + M_2 \lambda^2 \frac{\|e_k\|^2}{1 - M_1 \lambda}.
\]
(68)

Since \( \lambda(0 < \lambda < 1) \) is sufficiently small, \( 0 < 1 - M_1 \lambda < 1 \) can be obtained. Inequality (68) can be written as
\[
\|\tilde{w}_k\|^2 \leq \frac{M_2 \lambda^{(r+2)}}{1 - M_1 \lambda} \|e_k\|^2 \leq \|e_k\|^2.
\]
(69)

Because the output equation of system (45) is the same as that of system (1), one gets
\[
\|e_{k+1}\|^2 \leq 2\lambda_{cl} \|e_k\|^2 + 2L_b \|\tilde{w}_k\|^2.
\]
(70)

Substituting inequality (70) into inequality (69), we have the following results:
\[
\|e_{k+1}\|^2 \leq (2\lambda_{cl} + 2L_b) \|e_k\|^2.
\]
(71)

Let \( \rho^* = 2\lambda_{cl} + 2L_b \), and \( 2\lambda_{cl} < 1 \) can be obtained from the condition of Theorem 2. Therefore, \( \rho^* < 1 \). After recursion by inequality (71), one has
\[
\|e_{k+1}\|^2 \leq \rho^* \|e_k\|^2 \leq \cdots \leq \rho^{kh} \|e_1\|^2.
\]
(72)

Finally, using proof method similar to that in formulas (40)–(43) in Section 2, one obtains
\[
\lim_{k \to +\infty} \|y_{ik}(\cdot, h) - y_{jk}(\cdot, h)\|^2 = 0,
\]
where
\[
i, j \in \{1, 2, \ldots, N\}, 0 \leq h \leq H.
\]
(73)

This completes the proof of Theorem 2. \( \square \)

4. Simulations

In order to verify the effectiveness of the ILC scheme proposed in this paper for the nonlinear partial difference MASs with time delay, two specific numerical examples are given as follows.

Take the corresponding Laplacian matrix:

\[
\begin{bmatrix}
0 & 1 & 0 \\
1 & 2 & 1 \\
0 & 1 & 0
\end{bmatrix}
\]

and the initial state vectors:
\[
\begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}
\]

The desired output is given by:
\[
y_d(\cdot, h) = \begin{cases}
1 & \text{if } h = 0 \\
0 & \text{otherwise}
\end{cases}
\]

The simulation results are shown in the following figures.
Example 1. The dynamic model of the $i$-th agent in the nonlinear hyperbolic partial difference MAS with time delay is

$$u_{k+1}(\zeta, h) = u_k(\zeta, h) + [y(h)(L + S)1_N]e_k(\zeta, h).$$

Example 2. The dynamic model of the $i$-th agent in the nonlinear parabolic partial difference MASs is

$$\begin{align*}
\triangle_2 w_{i,k}(\zeta, h) &= a(h)\triangle_1^2 w_{i,k}(\zeta + 1, h) \\
+ f(w_{i,k}(\zeta, h - \tau), u_{i,k}(\zeta, h), h), \\
y_{i,k}(\zeta, h) &= b(w_{i,k}(\zeta, h), h) + c(h)u_{i,k}(\zeta, h),
\end{align*}$$

where $i = 1, 2, 3$, $(\zeta, h) \in [1, 15] \times [0, 250]$, $Z = 15, H = 250$, while $Z$ and $H$ are given integers, and $a(h) = 0.21 - (1/(15(h + 1)))$, $c(h) = 1$. Let the desired output of system (77) be (the virtual leader’s motion trajectory)

$$y_{r1}(\zeta, h) = 4(\sin (h - 1))^2(\zeta - 1)(1 - e^{-2.5(Z - \zeta)}).$$

The simulation diagram and results of Example 1 can be obtained, as shown in Figures 1–3.
Figure 2: Error surface between the three agents and the virtual leader agent in nonlinear hyperbolic partial MASs. (a) Agent1 error surface ($k = 30$). (b) Agent2 error surface ($k = 30$). (c) Agent3 error surface ($k = 30$).

Figure 3: Changes in the consensus error between the three agents and the virtual leader agent in nonlinear hyperbolic partial MASs.
Figure 4: Actual output of three agents and the virtual leader agent in nonlinear parabolic partial MASs. (a) Virtual \( y_{\text{virtual}}(\zeta, h) \). (b) Agent 1 \( y_1(\zeta, h)(k = 30) \). (c) Agent 2 \( y_2(\zeta, h)(k = 30) \). (d) Agent 3 \( y_3(\zeta, h)(k = 30) \).

Figure 5: Continued.
where \( i = 1, 2, 3 \), \((\zeta, h) \in [1, 15] \times [0, 250], Z = 15, H = 250\), while \( Z \) and \( H \) are given integers, and 
\( a(h) = 0.2 - (1/(20(h + 1))) \), \( c(h) = 1 \). Let the desired output of this system be (the virtual leader's motion trajectory) 

\[ y_{r2}^i(\zeta, h) = 2 \left( \sin \frac{\zeta}{5} \right) \left( 1 - e^{-2(Z-0)} \right) \sin h. \]  

The simulation diagram and results of Example 2 can be obtained, as shown in Figures 4–6.

Finally, the simulation results are shown in Figures 1–6. Next, we analyze the simulation results.

Figure 1 depicts the actual output changes of the three agents and virtual leaders \((y, 1(\zeta, h))\) in hyperbolic partial MASs when iterating 30 times \((k = 30)\). Figure 4 depicts the actual output changes of the three agents and virtual leaders \((y_2(\zeta, h))\) in nonlinear parabolic partial MASs when iterating 30 times \((k = 30)\). It can be seen that system (77) and the three other agents in system (79) track the output changes of the virtual leader. It can be seen from Figures 1–3 that the control goal has been basically achieved.

Figures 2 and 5 show error surfaces between the three agents and virtual leader agent in nonlinear hyperbolic partial MASs and nonlinear parabolic partial MASs when iterating 30 times \((k = 30)\), respectively. In Figures 2 and 4, it can be seen that the absolute value for largest consensus error surface is less than \(3 \times 10^{-10}\). Therefore, from Figures 2 and 4, it can be seen that the three agents in system (77) and system (79), respectively, track the motion trajectories of the respective virtual leader agent.

Figures 3 and 6 show the variation of the consensus error of three agents in nonlinear hyperbolic partial MASs and nonlinear parabolic partial MASs in the sense of \(L^2\) norm. From Figures 3 and 6, it can be seen that the consensus error of the three agents in system (77) and system (79) along the iterative axis gradually converges to zero in the sense of \(L^2\) norm.

From Figures 1–6, within the finite-time interval \([0, T]\), when the designed ILC algorithm is used in nonlinear hyperbolic partial MASs and parabolic partial MASs, respectively, in the sense of \(L^2\) norm, the consensus error along the iteration axis between any two agents in system (77) and system (79) can gradually converge to zero. Therefore, in this paper, simulation results illustrate the effectiveness the ILC algorithm designed in nonlinear hyperbolic partial MASs and parabolic partial MASs with time delay, respectively.

5. Conclusions

The consensus control problem for nonlinear hyperbolic partial MASs and nonlinear parabolic partial MASs with time delay is studied in this paper, taking into consideration the space-time discrete problem of agents. Based on the fixed topology of the MASs, an ILC algorithm is proposed by
applying a method for the virtual leader agent to generate the desired trajectory. This algorithm uses the tracking error between any agent and the virtual leader and neighbor agents in the last iteration and combines the characteristics of the system communication topology to continuously modify the previous control law to obtain the ideal control law. When the iterative learning index $k$ approaches infinity, in the sense of the $L^2$ norm, the consensus error between all agents in the system can converge to zero. Furthermore, simulation results illustrate the effectiveness of the algorithm. In the future, the iterative learning consensus control problem of discrete linear MASs with random noise and measurement saturation can be further carried out, and, for the topological structure between multiple agents, the topology of the iterative axis can be studied.

Data Availability

The simulations program code used to support the findings of this study is available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors’ Contributions

This work was carried out in collaboration between all authors. Xisheng Dai raised these interesting problems in this research. Cun Wang (gxutwc@163.com) proved the theorems, interpreted the results, and wrote the article. The numerical example is given by Cun Wang (gxutwc@163.com), Kene Li (likene@163.com), and Zupeng Zhou (zupengzhou@163.com).

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