Critical exponents for $3D O(n)$–symmetric model with $n > 3$

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Critical exponents for the $3D O(n)$–symmetric model with $n > 3$ are estimated on the base of six–loop renormalisation–group (RG) expansions. Simple Padé–Borel technique is used for resummation of RG series and Padé approximants $[L/1]$ are shown to give rather good numerical results for all calculated quantities. For large $n$, the fixed point location $g_c$ and critical exponents are also determined directly from six–loop expansions, without addressing to resummation procedure. Analysis of numbers obtained shows that resummation becomes unnecessary when $n$ exceeds 28 provided an accuracy about 0.01 is adopted as satisfactory for $g_c$ and critical exponents. Further, results of the calculations performed are used to estimate numerical accuracy of the $\frac{1}{n}$–expansion. The same value, $n = 28$, is shown to play a role of lower boundary of the domain where this approximation provides high–precision estimates for critical exponents.
I. INTRODUCTION

Field–theoretical 3D $O(n)$–symmetric model with self–interaction of $\lambda \varphi^4$ type is known to describe critical behaviour of many basic physical systems such as Ising ($n = 1$) and Heisenberg ($n = 3$) ferromagnets, superfluid Bose–liquid ($n = 2$), polymers ($n = 0$), etc. In 70–th Nickel, Meiron, and Baker, Jr. calculated all 2–point and 4–point Feynman graphs for this model up to six–loop order [1] paving the way for obtaining perturbative expansions of unprecedentended length for $\beta$–function and critical exponents. These expansions were then explicitly found and used, being resummed in various manners, to estimate the stable fixed point coordinate and numerical values of critical exponents [2,3,4]. The values obtained are referred today as most accurate (canonical) numbers [5].

Explicit expressions for RG functions and numerical estimates were presented in Refs. [2,3,4] only for $n = 0, 1, 2, 3$. At the same time, it is desirable to have such results for $n > 3$. They are interesting from, at least, three points of view. First, there are numerous physical systems with many–component order parameters and these results may be relevant to their critical or effective critical behaviour (see, e.g. Refs. [6,7,8]). Second, such calculations would enable one to clear up where resummation procedures applied to RG series become unnecessary, i.e. how large are the values of $n$ for which the theory may be thought as possessing a small parameter. And third, high–precision numerical estimates of critical exponents for $n \gg 1$ when compared with their counterparts given by $\frac{1}{n}$–expansion would provide an information about the numerical accuracy of this familiar approximation scheme.

Below, six–loop perturbative expansions for $\beta$–function and critical exponents $\eta$ and $\gamma (\gamma^{-1})$ are calculated for arbitrary $n$. The fixed point coordinate $g_c$ and critical exponents are estimated on the base of Padé–Borel resummation procedure and a comparison of these numbers with those given by unresummed RG series and $\frac{1}{n}$–expansion is made. The outline of the paper is as follows. In Sec. [I] the renormalisation scheme is formulated,
RG expansions are written down, the resummation technique is described, and numerical results obtained are collected. In Sec. [III] they are discussed along with their analogues resulting from unresummed six–loop series and $\frac{1}{n}$–expansion and corresponding inferences are presented. Section [IV] contains conclusions.

II. RG SERIES AND NUMERICAL RESULTS

The Hamiltonian of the model to be studied reads:

$$H = \frac{1}{2} \int d^3 x \left[ (\nabla \varphi_\alpha)^2 + m^2_0 \varphi_\alpha^2 + \frac{2}{4!} \lambda (\varphi_\alpha^2)^2 \right], \quad (2.1)$$

where $\varphi_\alpha$ is a vector order parameter field, $\alpha = 1, \ldots, n$, a bare mass squared $m^2_0$ being proportional to the deviation from the mean–field transition point.

We calculate the $\beta$–function and critical exponents within a massive theory. The renormalized Green function $G_R(p, m, g)$ and four–point vertex function $\Gamma_R(p, m, g)$ are normalized at zero momenta in a conventional way:

$$G_{R}^{-1}(0, m, g) = m^2 ,$$

$$\left. \frac{\partial G_{R}^{-1}(p, m, g)}{\partial p^2} \right|_{p^2=0} = 1 ,$$

$$\Gamma_{R}(0, m, g) = mg ,$$

with one extra condition being imposed on the $\varphi^2$ insertion:

$$\left. \Gamma_{R}^{1,2}(p, q, m, g) \right|_{p=q=0} = 1 . \quad (2.3)$$

Since combinatorial factors and momentum integrals for 2–point and 4–point Feynman graphs are known [1] the calculation of the $\beta$–function and critical exponents (anomalous dimensions) within six–loop approximation is straightforward (see, e.g. [9]). The results are as follows:
\[
\beta(g) = g - g^2 + \frac{1}{(n+8)^2}(6.07407408n + 28.14814815)g^3 - \frac{1}{(n+8)^3}(1.34894276n^2 \\
+ 54.94037698n + 199.6404170)g^4 + \frac{1}{(n+8)^4}(-0.15564589n^3 + 35.82020378n^2 \\
+ 602.5212305n + 1832.206732)g^5 - \frac{1}{(n+8)^5}(0.05123618n^4 + 3.23787620n^3 \\
+ 668.5543368n^2 + 7819.564764n + 20770.17697)g^6 + \frac{1}{(n+8)^6}(-0.0234217n^5 \\
- 1.07179839n^4 + 265.8357032n^3 + 12669.22119n^2 + 114181.4357n \\
+ 271300.0372)g^7, \tag{2.4}
\]

\[
\eta(g) = \frac{1}{(n+8)^2}(0.2962962964n + 0.5925925928)g^2 + \frac{1}{(n+8)^3}(0.0246840014n^2 \\
+ 0.246840014n + 0.3949440224)g^3 + \frac{1}{(n+8)^4}(-0.0042985626n^3 \\
+ 0.6679859202n^2 + 4.609221057n + 6.512109933)g^4 - \frac{1}{(n+8)^5}(0.0065509222n^4 \\
- 0.1324510614n^3 + 1.891139282n^2 + 15.18809340n + 21.64720643)g^5 \\
+ \frac{1}{(n+8)^6}(-0.0055489202n^5 - 0.0203994485n^4 + 3.054030987n^3 \\
+ 64.07744656n^2 + 300.7208933n + 369.7130739)g^6, \tag{2.5}
\]

\[
\gamma^{-1}(g) = 1 - \frac{n+2}{2(n+8)}g + \frac{n+2}{(n+8)^2}g^2 - \frac{1}{(n+8)^3}(0.8795588926n^2 + 6.48547686n \\
+ 9.452718166)g^3 + \frac{1}{(n+8)^4}(-0.1283321043n^3 + 7.966740703n^2 \\
+ 51.84421298n + 70.79480631)g^4 - \frac{1}{(n+8)^5}(0.0490966058n^4 \\
+ 4.288152493n^3 + 108.3618219n^2 + 537.8136105n + 675.6996077)g^5 \\
+ \frac{1}{(n+8)^6}(-0.0259267945n^5 - 1.618627843n^4 + 85.54569746n^3 \\
+ 1538.818235n^2 + 6653.956526n + 7862.074086)g^6. \tag{2.6}
\]

These series are known to be divergent (asymptotic). To extract the physical information which they contain some resummation procedure should be employed. We use the Padé–Borel method, i.e. construct Padé approximants \([L/M]\) for Borel transforms which are related to functions to be found (“sum of series”) by the formula
\[ f(x) = \sum_{k=0}^{\infty} c_k x^k = \int_{0}^{\infty} e^{-t} F(xt) dt , \quad (2.7) \]

\[ F(y) = \sum_{k=0}^{\infty} \frac{c_k}{k!} y^k , \quad (2.8) \]

and then evaluate the integral (2.7) where series (2.8) possessing non-zero radii of convergence are replaced by corresponding Padé approximants.

Starting from six-loop expansions available it is possible to construct different sets of Padé approximants: \([L/1], [L - 1/2],\) etc., where \(L = 6\) for \(\beta\)-function and \(L = 5\) for critical exponents. As we found, approximants

\[ [L/1] = (1 + b_1 y)^{-1} \sum_{i=0}^{L} a_i y^i \quad (2.9) \]

which generate following expressions for sums of the series

\[ f(x) = z e^{-z} Ei(z) \sum_{i=0}^{L} a_i (-b_1)^{-i} - \sum_{i=1}^{L} a_i (-b_1)^{-i} \sum_{k=0}^{i-1} k! z^{-k} , \]

\[ z = -\frac{1}{b_1 x} , \quad Ei(z) = \int_{-\infty}^{z} e^{t} t^{-1} dt \quad (2.10) \]

give the best results. They are presented in Table I. The estimates for \(\gamma\) and \(\eta\) originate from series (2.5) and (2.6) while numerical values of critical exponents \(\nu, \alpha,\) and \(\beta\) were determined by means of well-known scaling relations. The exponent \(\gamma\) was calculated also via resummed RG expansion for the exponent \(\eta_2 = (1 - \gamma)(2 - \eta)/\gamma\) and numbers were obtained which differ from those resulting from (2.6) by no more than 0.003; corresponding averages stand in Table I. This table contains as well, for comparison, numerical results found earlier for \(n = 0, 1, 2, 3\) on the base of higher-order RG expansions in 3 and 4 \(-\epsilon\) dimensions using alternative resummation techniques \([2,3,10]\). It is worthy to discuss these results along with ours in more detail.

As we can see, there are small differences between our estimates and their counterparts obtained in Refs. \([2,3]\) from 3D RG expansions of the same length. They are caused by
use of different resummation procedures. Indeed, the authors of Refs. [2,3] employed the Borel–Leroy transformation

$$f(x) = \int_0^\infty t^B e^{-t} F(xt) dt$$

instead of Eq. (2.7) in their calculations. The parameter $B$ was chosen to meet the known large–order behaviour of coefficients $c_k$ in perturbative expansions [11,12]:

$$c_k \sim k! (-a)^k k^b, \quad k \to \infty,$$

where $a = 0.147774$ for the model (2.1) and $b$ is equal to $2 + \frac{n}{2}$, or $3 + \frac{n}{2}$, or $5 + \frac{n}{2}$ depending on the RG function expanded. We use much simpler method which ignores some part of information (2.12) but leads, nevertheless, to numerical results rather close to those given by more sophisticated techniques. It is not surprising since the main property of $c_k$ – their factorial growth, is taken into account in our analysis while the rest of information about $c_k$ being incorporated enables one to reduce the apparent errors of estimation keeping the location of fixed point and critical exponents practically unchanged (see, e.g. Ref. [3] for detail). Dealing with simple Padé approximants $[L/1]$, we avoid also, to a certain extent, the problem of poles. The point is that these approximants turn out to have no real and positive poles for $n < 38$ in the case of critical exponents and up to $n = 80$ for the $\beta$–function. That is why Table I ends at $n = 32$. Since for $n = 0, 1, 2, 3$ our procedure gives critical exponents values which are almost identical to known high–precision estimates [2,3,10], we believe that the rest of the results listed in this table are also very close to exact numbers.

### III. LARGE $N$ AND $\frac{1}{N}$–EXPANSION

How can we estimate $g_c$ and critical exponents for $n \gtrsim 30$? It is well known (and clearly seen from Eqs. (2.4)–(2.6)) that coefficients of RG expansions are decreasing when $n$ grows
up. Hence, for large enough \( n \) the theory should possess a true small parameter as, say, the quantum electrodynamics does. In such a case, all quantities of interest can be obtained directly from corresponding perturbative expansions, without addressing to resummation technique. To find the minimal value of \( n \) which may be referred to as “large enough” we have calculated \( g_c \) for \( 20 \leq n \leq 60 \) using original and Padé–Borel–resummed series (2.4). (It should be reminded that the approximant \([6/1]\) for the Borel transform of \( \beta \)-function has no dangerous poles within this segment.) The results are presented in Table [I]. Values of \( g_c \) given by these two approximations are seen to differ from each other by 0.9% for \( n = 28 \) and this difference diminishes rapidly with increasing \( n \). So, if the accuracy of order of 1% for \( g_c \) was adopted as satisfactory, the resummation of six–loop expansion for \( \beta \)-function becomes unnecessary when \( n \) exceeds 28. The same turns out to be truth for the critical exponent \( \gamma \) as is seen from Table [II] (the first and the second lines).

For large \( n \), another approximate method may be used to calculate critical exponents. We mean famous \( \frac{1}{n} \)-expansion. Within the second order in \( \frac{1}{n} \) exponents \( \gamma \) and \( \eta \) are known to be [13]:

\[
\gamma = 2 - \frac{24}{\pi^2} \frac{1}{n} + \frac{64}{\pi^4} \left( \frac{44}{9} - \pi^2 \right) \frac{1}{n^2},
\]

\[
\eta = \frac{8}{3\pi^2} \frac{1}{n} - \frac{512}{27\pi^4} \frac{1}{n^2}.
\]

Series for other critical exponents are easily obtained via scaling relations.

It is interesting to evaluate the accuracy of numerical results given by \( \frac{1}{n} \)-expansion. We can get such an information comparing numbers resulting from Eqs. (3.1) and (3.2) for various \( n \) with their counterparts obtained on the base of resummed \((n \leq 32)\) and unresummed \((n > 32)\) six–loop RG series. On the other hand, this comparison would help us to determine the accuracy of the employed approximation itself in the limit \( n \to \infty \) where \( \frac{1}{n} \)-expansion’s results are exact.
Corresponding estimates for exponent $\gamma$ are listed in Table III. These numbers show that numerical accuracy of Eq. (3.1) becomes better than 1% when $n$ exceeds 28. Values of $\eta$ given by six–loop RG series and Eq. (3.2) are very small and not presented here. They differ from each other by approximately 10% for $n > 28$. Moreover, this discrepancy persists up to largest values of $n$ studied. It is not surprise. The point is that, for extremely large $n$, only leading terms in $n$ contribute to $\eta$ in each order in $g$. Since $g_c = 1 + O(n^{-1})$, $g_c$ should be put equal to unity within this limit. Hence, corresponding total contribution in the case of six–loop RG series may be found by summing of coefficients of all leading terms in Eq. (2.5). Such a procedure gives $\eta = \frac{0.30458}{n}$, while the exact asymptotic expression resulting from Eq. (3.2) is $\eta = \frac{0.27019}{n}$. So, the approximate asymptotic estimate for $\eta$ differs from the exact one by 13%. This difference, however, practically doesn’t influence upon numerical values of other critical exponents calculable by scaling relations since for $n > 28$ the exponent $\eta < 0.01$.

We see that simple formulas (3.1) and (3.2) enable one to estimate all critical exponents for the model (2.1) with an accuracy of order of 0.01 provided $n \geq 28$. Moreover, for such $n$ second–order terms in these formulas may be, in fact, neglected since their contributions are very small.

IV. CONCLUSION

Critical exponents of the $3D O(n)$–symmetric model have been estimated from six–loop RG series for $n > 3$. RG expansions have been resummed by means of simple Padé–Borel technique and approximants $[6/1]$ ($\beta$–function) and $[5/1]$ (critical exponents) have been shown to provide rather good numerical results for all calculated quantities. It has been found that for $n \geq 28$ the theory may be thought as possessing a small parameter, i.e. the fixed point coordinate and critical exponents may be determined with errors about 0.01 or
less directly from higher-order RG series, without use of resummation procedure. Numerical accuracy of the $\frac{1}{n}$-expansion has been also estimated. The same value, $n = 28$, has been shown to play a role of a lower boundary of the region where this approximation provides high-precision results for critical exponents.

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TABLE I. The stable fixed point location and critical exponents obtained within six-loop approximation using the Padé–Borel resummation technique.

| n | $g_c$  | $\gamma$ | $\eta$ | $\nu$  | $\alpha$ | $\beta$ |
|---|--------|----------|--------|--------|----------|--------|
| 0 | 1.402  | 1.160    | 0.034  | 0.589  | 0.231    | 0.305  |
|   | 1.421† | 1.161†   | 0.026† | 0.588† | 0.236†   | 0.302† |
|   | 1.417‡ | 1.162‡   | 0.026‡ | 0.588‡ | 0.302‡   |        |
| 1 | 1.401  | 1.239    | 0.038  | 0.631  | 0.107    | 0.327  |
|   | 1.416† | 1.241†   | 0.031† | 0.630† | 0.110†   | 0.324† |
|   | 1.414‡ | 1.240‡   | 0.032‡ | 0.630‡ | 0.325‡   |        |
|   |        |          | 0.035* | 0.628* |          |        |
| 2 | 1.394  | 1.315    | 0.039  | 0.670  | - 0.010  | 0.348  |
|   | 1.406† | 1.316†   | 0.032† | 0.669† | - 0.007† | 0.346† |
|   | 1.405‡ | 1.316‡   | 0.034‡ | 0.669‡ | 0.346‡   |        |
|   |        |          | 0.037* | 0.665* |          |        |
| 3 | 1.383  | 1.386    | 0.038  | 0.706  | - 0.117  | 0.366  |
|   | 1.392† | 1.390†   | 0.031† | 0.705† | - 0.115† | 0.362† |
|   | 1.391‡ | 1.387‡   | 0.034‡ | 0.705‡ | 0.365‡   |        |
| $n$ | $g_c$ (DS) | 20     | 24     | 28     | 32     | 36     | 40     | 50     | 60     |
|-----|-----------|--------|--------|--------|--------|--------|--------|--------|--------|
| 4   | 1.2184    | 1.1725 | 1.1458 | 1.1273 | 1.1134 | 1.1025 | 1.0830 | 1.0699 |
| 5   | 1.1768    | 1.1538 | 1.1359 | 1.1216 | 1.1099 | 1.1003 | 1.0822 | 1.0696 |
| 6   | 1.319     | 1.599  | 0.029  | 0.811  | 0.434  | 0.417  |
| 7   | 1.303     | 1.637  | 0.027  | 0.830  | 0.489  | 0.426  |
| 8   | 1.288     | 1.669  | 0.025  | 0.845  | 0.536  | 0.433  |
| 9   | 1.274     | 1.697  | 0.024  | 0.859  | 0.576  | 0.440  |
| 10  | 1.248     | 1.743  | 0.021  | 0.881  | 0.643  | 0.450  |
| 12  | 1.226     | 1.779  | 0.019  | 0.898  | 0.693  | 0.457  |
| 14  | 1.207     | 1.807  | 0.017  | 0.911  | 0.732  | 0.463  |
| 16  | 1.191     | 1.829  | 0.015  | 0.921  | 0.764  | 0.468  |
| 18  | 1.177     | 1.847  | 0.014  | 0.930  | 0.789  | 0.471  |
| 20  | 1.154     | 1.874  | 0.012  | 0.942  | 0.827  | 0.477  |
| 24  | 1.136     | 1.893  | 0.010  | 0.951  | 0.854  | 0.481  |
| 28  | 1.122     | 1.908  | 0.009  | 0.958  | 0.875  | 0.483  |
| 32  | 1.119     | 1.893  | 0.005  | 0.953  | 0.879  | 0.482  |

† Quoted from Ref. [3].
‡ Quoted from Ref. [2].
* Quoted from Ref. [10].

**TABLE II.** Coordinates of the fixed point obtained from Eq. (2.4) with use of Padé–Borel resummation procedure (PB) and by direct summation (DS).
TABLE III. Values of the critical exponent $\gamma$ obtained by direct summation of the RG expansion (DS), by means of Padé–Borel resummation technique (PB) and from $\frac{1}{n}$–expansion (Eq. (3.1)).

| $n$ | 20  | 24  | 28  | 32  | 36  | 40  | 50  | 70  | 100 | 500 |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| DS  | 1.8990 | 1.8991 | 1.9075 | 1.9165 | 1.9245 | 1.9314 | 1.9447 | 1.9606 | 1.9725 | 1.9946 |
| PB  | 1.8466 | 1.8737 | 1.8932 | 1.9078 | 1.9222 |
| $\frac{1}{n}$ | 1.8702 | 1.8930 | 1.9090 | 1.9208 | 1.9299 | 1.9372 | 1.9501 | 1.9646 | 1.9754 | 1.9951 |