Pairs emission in a uniform background field: an algebraic approach

Roberto Soldati

Dipartimento di Fisica, Università di Bologna, Istituto Nazionale di Fisica Nucleare, Sezione di Bologna, Bologna, Italy

E-mail: roberto.soldati@bo.infn.it

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Abstract
A fully algebraic general approach is developed to treat the pairs emission and absorption in the presence of some uniform external background field. In particular, it is shown that the pairs production and annihilation operators, together with the pairs number operator, do actually fulfil the SU(2) functional Lie algebra. As an example of application, the celebrated Schwinger formula is consistently and nicely recovered, within this novel approach, for a Dirac spinor field in the presence of a constant and homogeneous electric field in four spacetime dimensions.

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1. Introduction

Soon after the discovery of the relativistic Dirac equation [1], it was quickly realized [2, 3] that electron positron pairs might have been produced out of the vacuum by, e.g., a constant homogeneous very strong electric field. Later on, Julian Schwinger succeeded in retrieving [4] the electron positron pairs creation rate per unit time and volume, by exploiting the proper time method and the analytic continuation from the four-dimensional Minkowski spacetime to the Euclidean space. Since then, on the one hand, a huge amount of works, papers and monographies on this subject has been put forward [5–7]. On the other hand, the experimental probe of the famous Schwinger formula for the electron–positron pairs creation rate is still lacking, because of the very large value of the corresponding electric field strength critical scale [8]. Nonetheless, it has recently been suggested [9] that in the pseudorelativistic planar QED effective model for graphene, the detectable emission of massless Dirac quasiparticles–antiquasiparticles pairs might occur.

Surprisingly enough, it was only after nearly 20 years, since the seminal Schwinger article, that the exact solutions of the Dirac equation in the presence of a uniform electric field was
obtained by Nikishov [10]. In this remarkable paper, as well as within some subsequent ones
[11], arguments have been put forward to reobtain the Schwinger formula for the electron–
positron pairs creation rate from the knowledge of the nonperturbative exact solutions of the
Dirac equation in a constant and homogeneous electric field. However, although physically
and heuristically well posed, those arguments do not appear, at least in my opinion, to be
completely developed and fully convincing from the modern field theoretical point of view. It
is the main target of this paper to fill this gap.

It is worthwhile to remark that those exact solutions of the Dirac equation are
nonstationary. This is because of the gauge choice, which renders the problem exactly
solvable, but is such that the ensuing Dirac Hamiltonian becomes time dependent. Moreover,
those exact solutions are truly nonperturbative in the sense that there is no asymptotic regime
in which they reduce to the free field plane wave solutions, just because a uniform electric
field is never negligible. As a consequence, it turns out that the asymptotic states and fields
can never be identified with the conventional ones, which are solutions of the free field theory
without external fields.

It is the aim of this paper to reformulate the pairs emission and absorption issue in a
purely algebraic context based upon the canonical quantization in the presence of background,
external, uniform fields. This algebraic approach is quite general for it does not rely on the
specific form of the Dirac equation and of the uniform field. For example, it might be used to
describe matter production in a constant gravitational field, or the Unruh effect [12] in a
Rindler spacetime.

The main feature of this approach lies in the fact that the pairs creation, the pairs destruction
and the pairs number operator do realize a representation of the functional SU(2) Lie algebra.
This idea is not new, since it dates back to the seminal paper by Marinov and Popov [13]
in which a group theoretical approach to the problem of pair creation in a time-dependent
electric field has been developed. However, it is my aim to fully develop this approach, in the
modern language of quantum field theory and in a rigorous way, taking the massive Dirac field
in a constant electric field as a paradigmatic example. This allows in turn to understand the
Bogoliubov transformations [14] as similarity transformations acting upon the Fock space of
the states, while the vacuum state will correspond to a coherent state in the manner of Dirac
[1] with an infinite sea of pairs. In so doing, the Schwinger formula will nicely appear to be
the vacuum expectation value of an operator which represents a complex functional rotation
around a fixed axis.

For the sake of clarity and to provide a paradigmatic example, in the first two sections
I will provide a short overview for the solutions of the Dirac equation in the presence of a
constant homogeneous electric field on the Minkowski spacetime, as well as the canonical
quantization of the Dirac field and the ensuing derivation of the Schwinger formula. In so
doing, I will establish my notations and conventions. Then, in section 4, the algebraic approach
will be developed and all the formulas will be checked in terms of the four-dimensional spinor
QED with a uniform electric field. Finally, a short discussion of the concluding remarks will
be attempted.

2. Exact solutions of the Dirac equation

In this section, I briefly review the structure of the nonstationary (time-dependent) exact
(nonperturbative) solutions of the Dirac equation, in the presence of a constant and

1 I am truly indebted to Gerald V Dunne for bringing this absolutely relevant paper to my attention.
2 In the literature, unfortunately, there are several authors that roughly and wrongly disregard this crucial feature and
come unavoidably to incorrect conclusions.
homogeneous electric field on the four-dimensional Minkowski spacetime. I will use the metric $g_{\mu\nu} = \text{diag} \{ +, -, -, - \}$, natural units $\hbar = c = 1$ and the ordinary or standard representation for the Dirac matrices [15]. The Dirac equation can be written either in the covariant form

$$ (i\partial + eA - M)\Psi(x) = (\gamma^\mu P_\mu - M)\Psi(x) = 0, $$

(1)

where $q = -e (e > 0)$ is the negative electron charge and $P_\mu = p_\mu - qA_\mu(x) = i\partial_\mu + eA_\mu(x)$ is the usual Abelian covariant derivative, or in the manner of Schrödinger, namely,

$$ \frac{i\partial}{\partial t} \Psi = H\Psi, \quad H = -eA_0 + \alpha^k P^k + \beta M $$

(2)

in which we have set as it is customary:

$$ \beta \equiv \gamma^0, \quad \gamma^0 \gamma^k \equiv \alpha^k, \quad P = -i\nabla + eA. $$

If we assume the electrostatic field towards the positive $OX$ axis, that means $F_{10} = F_{01} = E_0$, after setting $x_\mu \equiv (t, r) = (t, x, y, z)$, we get the time-dependent Hamiltonian in the Nikishov [10] temporal gauge\(^3\) $A_\mu = (0, -Et, 0, 0)$:

$$ H_t = -i(\partial_t + eEt)\alpha^1 - i\partial_y \alpha^2 - i\partial_z \alpha^3 + M\beta $$

(3)

which turns out to be a symmetric operator but does not allow for stationary states. It follows that the first-order Dirac operators read

$$ i\partial + eA \pm M = \begin{pmatrix} i\partial_t \pm M & 0 & i\partial_z & iD_x + \partial_y \\ 0 & i\partial_t \pm M & iD_x - \partial_y & -i\partial_z \\ -i\partial_z & -iD_x - \partial_y & -i\partial_t \pm M & 0 \\ -iD_x + \partial_y & i\partial_z & 0 & -i\partial_t \pm M \end{pmatrix}, $$

(4)

where $D_x \equiv \partial_x - ieEt$, whilst the related second order differential operator turns out to be

$$ (i\partial + eA + M)(i\partial + eA - M) = P^2 - M^2 + e\sigma^{\mu\nu}F_{\mu\nu}, $$

(5)

where $\sigma^{\mu\nu} \equiv (i/4) [\gamma^\mu, \gamma^\nu]$ so that the second order differential operator becomes

$$ (i\partial + eA + M)(i\partial + eA - M) \equiv P^2 - M^2 + e\sigma^{\mu\nu}F_{\mu\nu} $$

$$ = -\partial_t^2 - M^2 + (\partial_x - ieEt)^2 + \partial_y^2 + \partial_z^2 + iE\alpha^1. $$

(6)

It is convenient to obtain the solution of the Dirac equation from the second order equation

$$ (P^2 - M^2 + e\sigma^{\mu\nu}F_{\mu\nu}) f(x)\Upsilon = 0, $$

(7)

where $f(x)$ is a Lorentz invariant complex scalar function, while $\Upsilon$ is one of the constant eigenbispinors of the matrix

$$ e\sigma^{\mu\nu}F_{\mu\nu} = ieE\alpha^1 = ieE \begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix} = ieE \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. $$

(8)

Since the above matrix commutes with the matrix of the spin component along the direction of the electrostatic field, i.e.

$$ [\alpha^1, \Sigma_1] = 0 \quad \Sigma_1 = \frac{1}{2} i\gamma^2\gamma^3 = \frac{1}{2} \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix}, $$

(9)

\(^3\) Actually, the Nikishov temporal gauge is a particular case of the general Fock–Schwinger gauge choice $A_\mu = \frac{1}{2} F_{\mu\nu} x^\nu$. 
we can suitably introduce the four real bispinors
\[
\begin{align*}
\Upsilon^+_{↑} &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}, \\
\Upsilon^+_{↓} &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}, \\
\Upsilon^+_{↑} &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}, \\
\Upsilon^+_{↓} &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}.
\end{align*}
\]
\[ (10) \]

\[ e^\mu_{\nu} F_{\mu\nu} \Upsilon^r = \pm ie \epsilon \Upsilon^r, \quad \beta \Upsilon^r = \Upsilon^r \quad (r = \uparrow, \downarrow) \]
\[ (11) \]

which satisfy the orthonormality and completeness relations
\[
\alpha^r \Upsilon^r_\pm = \pm \Upsilon^r_\pm, \quad \tilde{\Upsilon}^r_\pm \beta \Upsilon^r_\pm = \delta^{rt} \quad (r, s = \uparrow \downarrow) \]
\[ (12) \]

\[ \Sigma^r \Upsilon^r_\pm = \Upsilon^r_\pm, \quad \Sigma \Upsilon^r_\pm = -\Upsilon^r_\pm, \quad \tilde{\Upsilon}^r_\pm \beta \Upsilon^r_\pm = 0 \quad (r, s = \uparrow \downarrow). \]
\[ (13) \]

On the one hand, the bispinors of the first couple \( \{ \Psi^r_+(x) \mid r = \uparrow \downarrow \} \), corresponding to the eigenvalue \( +ie \), are mutually orthogonal and can be determined from the relationship
\[
\Psi^r_+(x) = (i \phi + eA + M) f(x) \Upsilon^r_+ \quad (r = \uparrow \downarrow).
\]
\[ (14) \]

On the other hand, the bispinors of the second couple \( \{ \Psi^r_-(x) \mid r = \uparrow \downarrow \} \), related to the eigenvalue \( -ie \), are in turn mutually orthogonal and are constructed according to
\[
\Psi^r_-(x) = (i \phi + eA + M) f^*(x) \Upsilon^r_- = (i \phi + eA + M) \beta \Upsilon^r_+ f^*(x) \quad (r = \uparrow \downarrow). \]
\[ (15) \]

From equation (7), we obtain the second order differential equation for the eigenvalue \( +ie \) and the scalar functions \( f(x) \) which reads
\[
\left[ \partial^2_t + M^2 - (\partial_x - ieEt)^2 - \partial_y^2 - \partial_z^2 - ie \right] f(x) = 0
\]
\[ (16) \]

so that, after turning to the momentum space
\[
\tilde{f}(x) = f(t, \mathbf{r}) \equiv (2\pi)^{-3/2} \int \! dp \, \exp(i \mathbf{p} \cdot \mathbf{r}) \tilde{f}(t, \mathbf{p})
\]
\[ (17) \]

we eventually obtain, with \( (p_x, p_y, p_z) \equiv (p^1, p^2, p^3) \),
\[
\left[ \partial_t^2 + M^2 + (p_x - eEt)^2 + p_y^2 + p_z^2 - ie \right] \tilde{f}(t, \mathbf{p}) = 0.
\]
\[ (18) \]

It is convenient to introduce the dimensionless quantities
\[
\xi \equiv \frac{p_x - eEt}{\sqrt{eE}}, \quad \lambda \equiv \frac{p_y^2 + p_z^2 + M^2}{eE}
\]
\[ (19) \]

so that we can rewrite the above equation in the standard form
\[
\left( \partial^2_\xi + \xi^2 + \lambda - i \right) \tilde{f}(\xi, \lambda) = 0.
\]

It can be verified (see the appendix) that the forthcoming two couples of linearly independent solutions of the above equation actually occur and read
\[
\tilde{f}^{(1)}_{\lambda}(\pm z) = D_{\lambda/2}[\pm(1-i)\xi]
\]
\[ (20) \]
\[
\tilde{f}^{(2)}_{\lambda}(\pm z) = D_{-\lambda/2-1}[\pm(1+i)\xi]
\]
\[ (21) \]

\[ z_{\pm} \equiv (1 \pm i)\xi = z_{\mp}^* \]
\[ (22) \]

where \( D_{\nu}(z) \) are the parabolic cylinder functions.
In order to construct the true physically relevant solutions of the first order Dirac problem, we have to take carefully into account the physical content of the parabolic cylinder functions, according to their asymptotic behaviour [16]—see the appendix. Namely, to a given particle of momentum $p$ we will associate the moving wave fronts, i.e. the surfaces of stationary phases

$$\exp(i\phi_k(t, \mathbf{r})) \equiv \exp\left[i\mathbf{p} \cdot \mathbf{r} \pm \frac{1}{2}i\xi^2(t)\right] = \text{constant.}$$

Then, in so doing, the temporal evolution is provided by

$$i\partial_t \exp(i\phi_k(t, \mathbf{r})) = -\phi_k(t, \mathbf{r}) \exp[i\phi_k(t, \mathbf{r})] = \mp \xi(t) \sqrt{\mathcal{E}} \exp[i\phi_k(t, \mathbf{r})]$$

in such a manner that the phase factors $\phi_k(t, \mathbf{r})$ lead to positive frequency solutions when $t \rightarrow \mp \infty$ that will describe, according to the free field theory jargon, the particles of given momentum $p$ and negative charge ($-e$).

The normalized solutions of the original first order Dirac equation can be obtained either from equation (14) or from equation (15), the final achievement being the same. The result is

in order to construct the complete orthonormal sets of $in$-states: namely, $u_{p, \gamma}^- (t, \mathbf{r})$, describing incoming electrons of momentum $p$ and helicity $\gamma$ while $v_{\gamma, p}^+ (t, \mathbf{r})$ describing incoming positrons of momentum $p$ and helicity $\gamma$, respectively. Analogously, we can in turn construct the complete orthonormal sets of $out$-states: $u_{p, \gamma}^+ (t, \mathbf{r})$, describing outgoing electrons of momentum $p$ and helicity $\gamma$, and $v_{\gamma, p}^- (t, \mathbf{r})$ describing outgoing positrons of momentum $p$ and helicity $\gamma$, respectively. Taking all of that into account, we have

$$(\gamma^\mu P_\mu + M) \exp[i\mathbf{p} \cdot \mathbf{r}]$$

$$= \begin{bmatrix} -\sqrt{\mathcal{E}}\tilde{\mathcal{E}}\tilde{i}_\xi + M & 0 & -p_z & -\xi \sqrt{\mathcal{E}} + ip_y \\ 0 & -\sqrt{\mathcal{E}}\tilde{\mathcal{E}}\tilde{i}_\xi + M & -\xi \sqrt{\mathcal{E}} - ip_y & p_z \\ p_z & \xi \sqrt{\mathcal{E}} - ip_y & \sqrt{\mathcal{E}}\tilde{\mathcal{E}}\tilde{i}_\xi + M & 0 \\ \xi \sqrt{\mathcal{E}} + ip_y & -p_z & 0 & \sqrt{\mathcal{E}}\tilde{\mathcal{E}}\tilde{i}_\xi + M \end{bmatrix} e^{i\mathbf{p} \cdot \mathbf{r}}$$

after setting

$$i\partial_t = -\sqrt{\mathcal{E}}\tilde{\mathcal{E}}\tilde{i}_\xi = -\sqrt{\mathcal{E}}(1 \pm i) \frac{id}{dz_{\pm}}$$

and then we immediately obtain

$$(\gamma^\mu P_\mu + M) \Upsilon_+^\dagger (\sigma) = M\Upsilon_+^\dagger (\sigma) - \Upsilon_+^\dagger \sqrt{\mathcal{E}}(\xi + id_\xi)$$

$$\Upsilon_+^\dagger (\sigma) \equiv \Upsilon_+^\dagger + i\sigma \Upsilon_+^\dagger \quad \sigma \equiv (p_y + ip_z)/M$$

and from the recursive relations

$$(\xi + id_\xi) D_{\alpha,2/1}(\pm z_{\pm}) = \mp \frac{1}{2} \lambda (1 - i) D_{\alpha,2/1}(\pm z_{\pm})$$

we eventually find

$$(\gamma^\mu P_\mu + M) \Upsilon_+^\dagger D_{\alpha,2/1}(\pm z_{\pm}) = \Upsilon_+^\dagger (\sigma) MD_{\alpha,2/1}(\pm z_{\pm}) \pm \frac{1}{2} \lambda (1 - i) \sqrt{\mathcal{E}} D_{\alpha,2/1}(\pm z_{\pm}) \Upsilon_+^\dagger.$$
It follows therefrom that the bispinor solutions which describe an incoming electron can be written as \((r = \uparrow, \downarrow)\)

\[
u_{p,r}^{in}(t, \mathbf{r}) = \frac{\lambda \varepsilon E (2\pi)^3}{\sqrt{[\lambda E (2\pi)^3]^3 - i\lambda (1 - i) \sqrt{\varepsilon E D_{ijk}/2(-z_0)\Upsilon^s}}}
\]

whilst the bispinor solutions which describe an outgoing positron read

\[
u_{p,r}^{out}(t, \mathbf{r}) = \frac{\lambda \varepsilon E (2\pi)^3}{\sqrt{[\lambda E (2\pi)^3]^3 - i\lambda (1 - i) \sqrt{\varepsilon E D_{ijk}/2(-z_0)\Upsilon^s}}}
\]

Notice that we actually obtain the standard normalization \((r, s = \uparrow, \downarrow)\)

\[
\int \mathrm{d}\mathbf{r} \, \bar{\nu}_{p,r}^{in}(t, \mathbf{r})\beta v_{q,s}^{out}(t, \mathbf{r}) = \delta(\mathbf{p} - \mathbf{q})\delta_{rs} = \int \mathrm{d}\mathbf{r} \, \bar{\nu}_{p,r}^{out}(t, \mathbf{r})\beta v_{q,s}^{out}(t, \mathbf{r})
\]
\[ v_{\text{out}}^{p,s}(t, r) \sim \frac{M (2 \xi^2(t))^{3/4}}{\sqrt{\hbar c E(2\pi)^3}} \exp \left\{ i p \cdot r + \frac{1}{2} i \xi^2(t) \right\} Y_s^i(\sigma) \quad (t \to +\infty). \] (37)

The two sets of incoming and outgoing bispinor wavefunctions of definite momentum and polarization
\[ \{ u_{p,r}^{\text{in}}, v_{p,r}^{\text{in}} \}, \{ u_{q,s}^{\text{out}}, v_{q,s}^{\text{out}} \} \quad (p, q \in \mathbb{R}^3, r, s = \uparrow, \downarrow) \] (38)
turn out to be orthonormal and complete in the generalized sense, i.e. in the sense of the theory of the tempered distributions. Thus, they represent the bases in the Hilbert spaces \( \mathcal{H}_{\text{in}} \) and \( \mathcal{H}_{\text{out}} \) of the Dirac bispinors in the presence of an electrostatic background field. As a matter of fact, we can readily verify that the scalar products are time independent owing to the self-adjointness of the one-particle Dirac Hamiltonian
\[ (u_{q,s}^{\text{in}}, v_{p,r}^{\text{in}}) \equiv \int \! dr \bar{u}_{q,s}^{\text{in}}(t, r) \beta v_{p,r}^{\text{in}}(t, r) = \delta(p - q) \delta_{rs} = (v_{q,s}^{\text{out}}, u_{p,r}^{\text{out}}). \] (39)

Note that we can also immediately obtain that
\[ (u_{q,s}^{\text{in}}, v_{p,r}^{\text{in}}) \equiv \int \! dr \bar{u}_{q,s}^{\text{in}}(t, r) \beta v_{p,r}^{\text{in}}(t, r) = 0 \] (41)
\[ (v_{q,s}^{\text{out}}, u_{p,r}^{\text{out}}) \equiv \int \! dr \bar{v}_{q,s}^{\text{out}}(t, r) \beta u_{p,r}^{\text{out}}(t, r) = 0 \] (42)
which means that the positive and negative frequency solutions of the Dirac equations are mutually orthogonal. Moreover, it is straightforward although tedious to check by direct inspection the equal time closure relations
\[ \sum_{p,r} [u_{p,r}^{\text{as}} \otimes \bar{u}_{p,r}^{\text{as}} + v_{p,r}^{\text{as}} \otimes \bar{v}_{p,r}^{\text{as}}] = \beta, \] (43)
where we have set for the sake of brevity
\[ \sum_{p,r} \equiv \int \! dp \sum_{r=\uparrow, \downarrow} \] while ‘as’ stands for either ‘in’ or ‘out’. As a consequence, we can write in the very same way the most general operator solutions of the Dirac equation as follows: namely,
\[ \psi(x) = \sum_{p,r} \left[ a_{\text{as}}(p, r) u_{p,r}^{\text{as}}(x) + b_{\text{as}}^\dagger(\bar{p}, \bar{r}) \bar{u}_{p,r}^{\text{as}}(x) \right] \] (44)
\[ \bar{\psi}(x) = \sum_{p,r} \left[ a_{\text{as}}^\dagger(\bar{p}, \bar{r}) v_{p,r}^{\text{as}}(x) + b_{\text{as}}(p, r) \bar{v}_{p,r}^{\text{as}}(x) \right], \] (45)
where the creation and destruction operators satisfy the canonical anticommutation relations
\[ \{ a_{\text{as}}(p, r), a_{\text{as}}^\dagger(q, s) \} = \delta(p - q) \delta_{rs} = \{ b_{\text{as}}(p, r), b_{\text{as}}^\dagger(q, s) \} \]
all the other anticommutators being equal to zero. From the closure relations (43), we can easily derive the canonical equal time anticommutation relations
\[
\{ \psi(t, r'), \bar{\psi}(t, r) \} = \sum_{p,r} \left[ u^{as}_{p,r}(t, r') u^{as}_{p,r}(t, r) + v^{as}_{p,r}(t, r') \bar{v}^{as}_{p,r}(t, r) \right] = \beta \delta(r - r').
\]
Accordingly, the causal Green’s function, or Feynman propagator, for the Dirac spinor under the influence of a background electrostatic homogeneous field has been firstly reported in [10] in full detail.

This means that we have to introduce the Fock spaces \( F^\text{as} \) for the incoming and outgoing particles and antiparticles. For example, starting from the in-vacuum \( |0\text{ in}\rangle \) one can generate as usual the one-particle state of charge \(-e\) momentum \( p \) and polarization \( s = \uparrow \downarrow \) describing an incoming electron, namely
\[
a^\dagger_{\text{in}}(p, s)|0\text{ in}\rangle = \begin{vmatrix} -e \end{vmatrix} |p s \text{ in}\rangle,
\]
the corresponding wavefunction being \( u^\text{in}_{p,r}(t, r) \). All the other one-particle and many-particle states for incoming and outgoing electrons and positrons can be constructed in close analogy. It follows therefrom that the charge operator, the momentum operator and the helicity operator, i.e. the projection of the spin angular momentum along the electric field direction, will correspond to the customary expressions
\[
Q \equiv (-e) \int dr : \psi^\dagger(t, r)\psi(t, r) : = (-e) \sum_{p,r} \left[ a^\dagger(p, r)a(p, r) - b^\dagger(p, r)b(p, r) \right] \\
P \equiv (-i) \int dr : \psi^\dagger(t, r)\nabla\psi(t, r) : = \sum_{p,r} \mathbf{p} \left[ a^\dagger(p, r)a(p, r) - b^\dagger(p, r)b(p, r) \right] \\
h \equiv \frac{1}{2} \int_{-\infty}^{\infty} dx : \psi^\dagger(t, x)\Sigma_1\psi(t, x) : = \frac{1}{2} \int_{-\infty}^{\infty} dp \left[ a^\dagger(p, \uparrow)a(p, \uparrow) - a^\dagger(p, \downarrow)a(p, \downarrow) + b^\dagger(p, \uparrow)b(p, \uparrow) - b^\dagger(p, \downarrow)b(p, \downarrow) \right]
\]
where the suffix ‘in’ or ‘out’ is understood.

Let us now turn to the calculation of the invariant inner product between the wavefunctions of an incoming electron with quantum numbers \( p, r \) and of an outgoing positron of quantum numbers \( q, s \): we find
\[
\int dr \bar{v}^\text{out}_{q,s}(t, r)\beta u^\text{in}_{p,r}(t, r) = e^{-\pi\lambda/2}\delta(p - q)\delta_{rs}
\]
and analogously
\[
\int dr \bar{v}^\text{in}_{q,s}(t, r)\beta u^\text{out}_{p,r}(t, r) = e^{-\pi\lambda/2}\delta(p - q)\delta_{rs}
\]
so that we can write in a compact way
\[
(v^\text{in}_{q,s}, u^\text{out}_{p,r}) = e^{-\pi\lambda/2}\delta(p - q)\delta_{rs} = (u^\text{in}_{q,s}, v^\text{out}_{p,r}) \quad \forall \mathbf{p}, \mathbf{q} \in \mathbb{R}^3, \quad r, s = \uparrow \downarrow.
\]
The above equality has led Nikishov [10] to the attempt of understanding the real positive quantity

\[ w_\lambda \equiv \exp \left\{ -\pi \left( p_y^2 + p_z^2 + M^2 \right) / eE \right\} \]

as the probability of creating one electron positron pair out of the vacuum, in which the particle and the antiparticle have opposite good quantum numbers \((\mp e, p, r)\). This intriguing interpretation is supported by the fact that indeed we have

\[
\begin{align*}
(u_{\text{out}, q,s}, u_{\text{in}, p,r}) &= \delta_{rs} \delta(p-q) N_{\lambda}, \\
(v_{\text{out}, q,s}, v_{\text{in}, p,r}) &= N_{\lambda}^* \delta(p-q) \delta_{rs},
\end{align*}
\]

where we have set

\[
N_{\lambda} \equiv \exp \left\{ -\pi \lambda / 4 \right\} \sqrt{\frac{\lambda}{\pi}} \Gamma \left( \frac{i\lambda}{2} \right) \sinh \frac{\pi \lambda}{2}.
\]

Now, since we find

\[
N_{\lambda} N_{\lambda}^* = 1 - e^{-\pi \lambda} = 1 - w_\lambda,
\]

it is natural to understand the complex quantity \(N_{\lambda}\) as the probability amplitude that one pair, in which the particle and the antiparticle have opposite quantum numbers \((\mp e, p, r)\), is not created out of the vacuum by absorbing energy from the electrostatic external field. Actually, in fact, it is also straightforward to verify that for positrons we get

\[
\int \text{d}r \, \bar{v}_{\text{out}, q,s}(t, r) \beta v_{\text{in}, p,r}(t, r) = N_{\lambda}^* \delta(p-q) \delta_{rs},
\]

thus providing the full endorsement to the whole construction. By the way, it is apparent that the probability amplitudes for the absorption and nonabsorption of one pair \((\mp e, p, r)\) are still expressed by \(w_\lambda\) and \(1 - w_\lambda\), respectively, owing to obvious symmetry reasons.

3. Pairs production annihilation mechanism

Here, I shortly discuss how, according to the Nikishov’s original proposal [10], one can try to recover the Schwinger formula for the rate of pairs emission and absorption, per unit time and unit volume, from the knowledge of the exact solutions of the Dirac equation. As we shall see here, this attempt is not at all flawless, because of the presence of divergences and the consequent unavoidable introduction of some regularization method. To this concern, it is useful to perform the Bogoliubov transformations connecting the incoming and outgoing complete orthonormal sets of the exact nonperturbative solutions of the Dirac equation in the presence of the homogeneous electrostatic field.

The above four sets (38) of incoming and outgoing solutions are indeed related throughout a Bogoliubov transformation, that is,

\[
\begin{align*}
&u_{\text{in}, p,r} = c_{1\lambda} u_{\text{out}, p,r} + c_{2\lambda} v_{\text{out}, p,r}, \\
v_{\text{in}, p,r} = c_{1\lambda}^* u_{\text{out}, p,r} + c_{2\lambda}^* v_{\text{out}, p,r},
\end{align*}
\]

\[
|c_{1\lambda}|^2 + |c_{2\lambda}|^2 = 1.
\]

We have

\[
\begin{align*}
(u_{\text{out}, q,s}, u_{\text{in}, p,r}) &= N_{\lambda} \delta(p-q) \delta_{rs}, \\
(v_{\text{out}, q,s}, v_{\text{in}, p,r}) &= N_{\lambda}^* \delta(p-q) \delta_{rs}, \\
(u_{\text{out}, q,s}, v_{\text{in}, p,r}) &= \exp \left\{ -\frac{1}{2} \pi \lambda \right\} \delta(p-q) \delta_{rs} = (u_{\text{in}, p,r}, v_{\text{out}, q,s}).
\end{align*}
\]
Then, using the above-listed orthonormality relations, we immediately find
\[ c_{1\lambda} = N_\lambda, \quad c_{2\lambda} = \exp \left\{ -\frac{i}{4}\pi \lambda \right\} = c_{2\lambda}^* \] (61)
In turn, the relative probability of a pair production will be coherently given by
\[ w_\lambda = 1 - |N_\lambda|^2 = c_{2\lambda}^2 = \exp \left\{ -\pi \lambda \right\}. \]

The vacuum to vacuum persistence amplitude can be written in the form—see also the recent up-to-date review [7]:
\[
\langle \text{out} 0 | 0 \text{i n} \rangle = \exp \left\{ \frac{i}{\hbar} [\Re \Gamma_{\text{eff}}(E) + i \Im \Gamma_{\text{eff}}(E)] \right\}
\]
\[
|\langle \text{out} 0 | 0 \text{i n} \rangle|^2 = \exp \left\{ -2V(t_f - t_i) \Im \mathcal{L}_{\text{eff}}(E) \right\}
\]
where \( \mathcal{L}_{\text{eff}}(E) \) is referred to as the effective Lagrangian density in the presence of a background electrostatic field \( E \), whereas \( t_f - t_i \) does indicate the very long total time during which the pairs production or annihilation takes place, while \( V \) denotes the total volume of e.g. a very large cubic box of sides \( L_x, L_y, L_z \) in the three-dimensional space. Needless to say, the very large-dimensional quantities \( t_f - t_i \) and \( V \) are in fact infrared divergences. It turns out that \( \Gamma_{\text{eff}}(E) \) contains a real part that describes dispersive effects like the Faraday’s birefringence, as well as an imaginary part that concerns absorptive effects like vacuum pairs production. Now, according to the above-described interpretation, the absolute probability for no pairs creation will be thereof given by the formal, divergent expression
\[
|\langle \text{out} 0 | 0 \text{i n} \rangle|^2 = \prod_{k,r} N_k N_r^* \times \prod_k (1 - e^{-\pi \lambda})^2 \equiv \exp \left\{ 2 \sum_{k} \ln(1 - e^{-\pi \lambda}) \right\}, \quad (64)
\]
where the mandatory dimensionless spatial vector index \( \mathbf{k} = (p_x L_x/2\pi, p_y L_y/2\pi, p_z L_z/2\pi) \) has been introduced to vindicate the probabilistic interpretation with
\[
\sum_{k} \equiv V_{\text{a}}(2\pi)^{-3} \int dp.
\]
Note that when the electric field is switched off then \( w_\lambda \to 0 \) and consequently the vacuum to vacuum probability becomes trivially equal to 1. It is important to realize that the validity of the above relation (64) stems from the natural assumption that the production and the destruction of any two pairs with different quantum numbers are statistically independent events. Now we find
\[
\ln |\langle \text{out} 0 | 0 \text{i n} \rangle|^2 = 2L_x L_y L_z (2\pi)^{-3} \int_{-\infty}^{\infty} dp_x \times \int_{-\infty}^{\infty} dp_y \int_{-\infty}^{\infty} dp_z \ln \left\{ 1 - \exp \left\{ -\pi \left( p_x^2 + p_y^2 + M^2 \right)/eE \right\} \right\}
\]
which is, as it stands, an ill-defined expression owing to the infrared (large volume) and ultraviolet (large longitudinal momentum \( p_x \)) divergences. The latter one can be expressed in terms of the large total time during which the process of pairs emission or absorption takes place: namely,
\[
\int_{-\infty}^{\infty} dp_x \equiv eE(t_f - t_i)
\]
that corresponds to a suitable ultraviolet regularization because, in the presence of a uniform
field, this time interval is arbitrarily long. To sum up, we can express the total absolute
probability of pairs production out of the vacuum (or pairs absorption into the vacuum) as

\[ W_{\text{pairs}} \equiv 1 - |\langle \text{out} 0 | \text{in} 0 \rangle|^2 \]

\[ = 1 - \exp \left\{ -\frac{2}{\hbar} \Im \Gamma_{\text{eff}}(E) \right\} \]

\[ \approx \frac{2}{\hbar} \nu(t_f - t_i) \Im \mathcal{L}_{\text{eff}}(E) \]  \hspace{1cm} (66)

in which we have taken into account that the quantity \( \frac{\Im \Gamma_{\text{eff}}(E)}{\hbar} \) is typically very small
for massive particle–antiparticle pairs. Hence, the average probability of pairs production per
unit volume and unit time is approximately given by

\[ \Gamma_{\text{pairs}} \equiv W_{\text{pairs}}[\nu(t_f - t_i)]^{-1} \approx \frac{2}{\hbar} \Im \mathcal{L}_{\text{eff}}(E) \]

\[ = -\frac{eE}{4\pi^3} \int_{-\infty}^{\infty} dp_x \int_{-\infty}^{\infty} dp_y \ln (1 - \exp \{ -\pi \lambda \}) \]

\[ = \frac{e^2 E^2}{4\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^2} \exp \left\{ -n \frac{\pi M^2}{eE} \right\} \]  \hspace{1cm} (67)

the n-th term of the series being \textit{grosso modo} understood to be the probability of the emission
or absorption of \( n \) pairs. From the above equation, one can readily extract the value of the
critical electrostatic field, above which the probability of production of e.g. electron–positron
pairs becomes appreciable, namely

\[ E_{\text{cr}} = \frac{m^2 \pi^3}{\hbar e} \simeq 1.3 \times 10^{18} \text{ V m}^{-1} \]

which is far beyond the present experimental capabilities. Nonetheless, it is worthwhile to
observe to this concern, in accordance with [9], that for massless charged spinor particle,
the approximation \( \frac{1}{2} \Gamma_{\text{pairs}} \approx \Im \mathcal{L}_{\text{eff}}(E) \) does not certainly hold true. This opens the interesting
possibility of detecting the pairs production and destruction in a two-dimensional graphene
sample.

As a final remark, I recall that, as is well known, the nonperturbative complex effective
action can be rigorously derived from the Euclidean formulation and the zeta function
regularization—see appendix B.

4. The algebraic approach

In order to develop the most general algebraic approach to the pairs production and annihilation
processes in the presence of an external background uniform field, it is convenient to introduce
a multi-valued index \( i, j, \ell, \kappa, \ldots \) to label the whole set \( Q \) of discrete and continuous quantum
numbers that correspond to the conserved quantities of the system under consideration, but for
the quantum number that distinguishes particles from antiparticles. For example, in the case
of a spinor field in a constant homogeneous electric field on the four-dimensional Minkowski
spacetime we have \( i = (p, r) \) with \( p \in \mathbb{R}^3, \ r = 1, 2. \) From now on, I will indicate with
\( a_i, a_i^\dagger \) \( (i \in Q) \) the creation and destruction operators for particles of negative electric charge
\( q = -e \) \( (e > 0) \) while \( b_j, b_j^\dagger \) \( (j \in Q) \) the corresponding operators for the antiparticles of
positive electric charge \( e. \)
4.1. Pairs operators algebra

Consider the pairs annihilation and pairs production dimensionless operators
\[
\Pi(z) \equiv \sum_{j \in \Omega} z_j a_j b_j = \sum_{j \in \Omega} z_j \Pi_j,
\]
\[
\Pi^\dagger(\bar{z}) \equiv \sum_{i \in \Omega} \bar{z}_i a_i^\dagger b_i^\dagger = \sum_{i \in \Omega} \bar{z}_i \Pi_i^\dagger,
\]
where we use the short notation
\[
\sum_{i \in \Omega} = \int d\mathbf{p} \sum_{r \cdots}(\mathbf{p} \in \mathbb{R}^n, \quad n = 1, 2, 3),
\]
whereas \(z_i = z(\mathbf{p}, r, \ldots)\) are complex-valued dimensionless functions and such that
\[
\mathcal{V}_n(2\pi)^{-n} \sum_{i \in \Omega} z_i \bar{z}_i = \nu_0 \quad (n = 1, 2, 3)
\]
is a pure number that will be named the characteristic number of the given pairs distribution function \(z(\mathbf{p}, r, \ldots)\), while \(\mathcal{V}_n\) is the volume of a very large cubic box in the \(n\)th dimensional Euclidean space. The creation and destruction operators for particles \((a_i^\dagger, a_j)\) and antiparticles \((b_i^\dagger, b_j)\) satisfy the usual canonical anticommutation relations
\[
\{a_i^\dagger, a_j\} = \{b_i^\dagger, b_j\} = \delta_{ij} \quad (i, j \in \Omega)
\]
all the remaining ones being equal to zero. The Fock vacuum is defined as usual
\[
a_i |0\rangle = 0 = \langle 0 | a_i^\dagger \quad b_i |0\rangle = 0 = \langle 0 | b_i^\dagger \quad (\forall i \in \Omega).
\]
In a quite general manner, if we denote by \(Q_a\) \((a = 1, 2, \ldots, A)\) all the conserved charges of the system which are allowed by the background field configuration, i.e.
\[
Q_a = \sum_{j \in \Omega} q_a (a_j^\dagger a_j - b_j^\dagger b_j) \quad (a = 1, 2, \ldots, A),
\]
where e.g. \(q_\alpha\) \((\alpha = 1, 2, \ldots, A)\) are the particle charges while \(-q_\alpha\) the antiparticle charges, then for any one-pair state \(b_i^\dagger a_i^\dagger |0\rangle\) of definite quantum numbers \(i \in \Omega\) we evidently find
\[
Q_a |b_i^\dagger a_i^\dagger 0\rangle = 0 \quad (\forall i \in \Omega, \quad a = 1, 2, \ldots, A).
\]
More generally, from the commutation relations
\[
[Q_a, \Pi(z)] = \sum_{j \in \Omega} q_a \sum_{i \in \Omega} z_i [a_i^\dagger a_j - b_j^\dagger b_i, a_i, b_i] = (-q_a) \sum_{j \in \Omega} z_i (a_i b_i + b_i a_i) = 0
\]
which obviously also imply \([Q_a, \Pi^\dagger(\bar{z})] = 0 \quad (a = 1, 2, \ldots, A)\), it follows that if we set
\[
\Pi^\dagger(\bar{z}) |0\rangle = |\bar{z}\rangle
\]
than we find
\[
Q_a |\bar{z}\rangle = 0 \quad (a = 1, 2, \ldots, A)
\]
which means that the generic one-pair state of distribution function \(z(\mathbf{p}, r, \ldots)\) is a common null eigenstate of all the conserved charge operators. For example, in the case of Dirac spinors in the uniform electric field, the conserved charges are the electric charge, the three components of momentum and the helicity, so that \(A = 5\).
The pairs creation and annihilation operators satisfy the commutation relations

\[
\left[ \Pi(z), \Pi^\dagger(\bar{z}) \right] = \sum_{i \in \Omega} z_i \sum_{j \in \Omega} \bar{z}_j \left[ a_i b_j, b^\dagger_j a_i^\dagger \right]
\]

\[
= \sum_{i \in \Omega} z_i \sum_{j \in \Omega} \bar{z}_j \left( a_i [b_j, b^\dagger_j a_i^\dagger] + [a_i, b^\dagger_j a_i^\dagger] b_j \right)
\]

\[
= \sum_{i \in \Omega} z_i \sum_{j \in \Omega} \bar{z}_j \left( a_i a^\dagger_j - b^\dagger_j b_j \right) \delta_{ij}
\]

\[
= \sum_{i \in \Omega} z_i \bar{z}_i \left( a_i a^\dagger_i - b^\dagger_i b_i \right) \equiv -2N(\bar{z}z) \quad (75)
\]

in which

\[
N(\bar{z}z) \equiv \sum_{i \in \Omega} \frac{1}{2} z_i \bar{z}_i \left( b_i^\dagger b_i - a_i a_i^\dagger \right)
\]

\[
= \sum_{i \in \Omega} z_i \bar{z}_i n_i = N^\dagger(z\bar{z})
\]

\[
= \sum_{i \in \Omega} \frac{1}{2} z_i \bar{z}_i \left( a_i^\dagger a_i + b_i^\dagger b_i \right) - \frac{1}{2} v_i. \quad (76)
\]

Finally for \( v_i \in \mathbb{R} \quad (\forall i \in \Omega) \), we get

\[
\left[ N(v), \Pi(z) \right] = \sum_{j \in \Omega} \frac{1}{2} v_j \sum_{i \in \Omega} z_i [b_j b_j + a_i^\dagger a_j^\dagger, a_i, b_j]
\]

\[
= -\sum_{i \in \Omega} v_i z_i a_i b_i = -\Pi(zv) \quad (77)
\]

\[
\left[ N(v), \Pi^\dagger(\bar{z}) \right] = \sum_{i \in \Omega} v_i z_i b_i^\dagger a_i^\dagger = \Pi^\dagger(v\bar{z}). \quad (78)
\]

It follows therefrom that the above three operators do satisfy the well-known commutation relations

\[
\left[ N(v), \Pi^\dagger(\bar{z}) \right] = \Pi^\dagger(v\bar{z})
\]

\[
\left[ \Pi(z), N(v) \right] = \Pi(vz)
\]

\[
\left[ \Pi^\dagger(\bar{z}), \Pi(z) \right] = 2N(\bar{z}z) \quad (79)
\]

in which

\[
\Pi^\dagger(\bar{z}) = J_x(\bar{z}) = J_x(u) + iJ_y(v)
\]

\[
\Pi(z) = J_-(z) = J_x(u) - iJ_y(v)
\]

\[
N(v) = J(\mu v), \quad \text{(82)}
\]

where the threesome of operators

\[
J_x(u) \equiv \frac{1}{2} \left( \Pi(z) + \Pi^\dagger(\bar{z}) \right)
\]

\[
J_y(v) \equiv \frac{1}{2i} \left( \Pi^\dagger(\bar{z}) - \Pi(z) \right)
\]

\[
J_z(\mu v) = N(v) \quad \text{(83)}
\]

are a basis of Hermitian generators obeying the well known SU(2) Lie algebra

\[
\left[ J_a(u), J_b(v) \right] = i\hbar \varepsilon_{abc} J_c(\mu v) \quad (a, b, c = 1, 2, 3). \quad \text{(84)}
\]
The quantum state \(|\tilde{z}\rangle\) that represents a generic one-pair state with a momentum distribution function \(\bar{z}\) \((\bar{t} \in \Omega)\) does satisfy
\[
|\tilde{z}\rangle = \Pi^\dagger(\tilde{z})|0\rangle \quad \langle z| = \langle 0|\Pi(z)
\]
and has the norm
\[
\langle z|\tilde{z}\rangle = \langle 0|\Pi(z), \Pi^\dagger(\tilde{z})|0\rangle = -2\langle 0|N(\bar{z}z)|0\rangle = \nu_o.
\]

4.2. The Schwinger formula from the algebraic approach

The general feature that characterizes the pairs production and annihilation processes in the presence of external background uniform fields is the existence of a nonsingular Bogoliubov similarity transformation \(S\), the generator of which is acting on the Fock space according to
\[
A_\bar{t} = S^{-1}a_\bar{t}S \equiv c_{1\bar{t}}a_\bar{t} - c_{2\bar{t}}b_\bar{t}^\dagger
\]
\[
B_\bar{t}^\dagger = S^{-1}b_\bar{t}^\dagger S \equiv c_{1\bar{t}}^*b_\bar{t}^\dagger + c_{2\bar{t}}a_\bar{t}
\]
where
\[
|c_{1\bar{t}}|^2 + |c_{2\bar{t}}|^2 = 1 \quad (\forall \bar{t} \in \Omega)
\]
in such a manner that the canonical anticommutation relations
\[
\{A_\bar{t}, B_\bar{j}^\dagger\} = 0 = \{a_\bar{t}, b_\bar{j}^\dagger\}, \quad \text{etc}
\]
keep unchanged, thanks to the similarity nature of the invertible transformation \(S\). It follows that we come to the two Fock spaces \(\mathcal{F}_{\text{in}}\) and \(\mathcal{F}_{\text{out}}\) which are generated by the cyclic vacuum states normalized to 1 and defined by
\[
a_\bar{t}|0\text{ in}\rangle = b_\bar{t}|0\text{ in}\rangle = 0 \quad (\forall \bar{t} \in \Omega)
\]
\[
A_\bar{t}|0\text{ out}\rangle = B_\bar{j}|0\text{ out}\rangle = 0 \quad (\forall \bar{j} \in \Omega).
\]
Now we have for example
\[
A_\bar{t}a_\bar{t}^\dagger|0\text{ in}\rangle = c_{1\bar{t}}a_\bar{t}a_\bar{t}^\dagger|0\text{ in}\rangle - c_{2\bar{t}}b_\bar{t}^\dagger a_\bar{t}^\dagger|0\text{ in}\rangle
\]
\[
= \nu_n(2\pi)^{-n}c_{1\bar{t}}|0\text{ in}\rangle - c_{2\bar{t}}\Pi_\bar{t}|0\text{ in}\rangle
\]
so that
\[
\langle 0|A_\bar{t}a_\bar{t}^\dagger|0\text{ in}\rangle = \nu_n(2\pi)^{-n}c_{1\bar{t}} \quad (\forall \bar{t} \in \Omega)
\]
whence it follows that, as expected, the Bogolyubov coefficient \(c_{1\bar{t}}\) is nothing but the probability amplitude that a pair of quantum numbers \(\bar{t} \in \Omega\) is not created out of the vacuum or not absorbed into the vacuum, i.e. the relative vacuum persistence probability density. In a similar way, by taking the in vacuum expectation value
\[
(2\pi)^n\nu_n^{-1}\langle 0|\Pi_\bar{t}A_\bar{t}a_\bar{t}^\dagger|0\text{ in}\rangle = -c_{2\bar{t}}^* (2\pi)^n\nu_n^{-1}\langle 0|\Pi_\bar{t}\Pi_\bar{t}^\dagger|0\text{ in}\rangle
\]
\[
= -\nu_n(2\pi)^{-n}c_{2\bar{t}} \quad (\forall \bar{t} \in \Omega),
\]
it is also clear that we can understand the Bogoliubov coefficient \(c_{2\bar{t}}\) as the probability amplitude that a pair of quantum numbers \(\bar{t} \in \Omega\) is created out of the vacuum or absorbed into the vacuum. To proceed further on, let me define
\[
S(\theta, \tilde{\n}) \equiv \exp[-i\theta \cdot \mathbf{T}(z, \tilde{z}, v)]
\]
\[
\theta \cdot \mathbf{T}(z, \tilde{z}, v) \equiv \Pi^\dagger(\theta\tilde{z}) + \Pi(z\theta) + 2\mathbf{N}(\theta v)
\]
As a consequence, we actually obtain the most general Bogoliubov transformations in the form

\[ \hat{\mathbf{n}}^2 = z_i \bar{z}_i + v_i^2 = 1 \quad (\forall i \in \mathcal{Q}). \]

For example, a suitable functional parametrization is provided by a pair of polar angles, latitude \( \Theta_i \) and azimuth \( \phi_i \), in such a manner to set

\[ v_i = \cos \Theta_i \quad z_i = \sin \Theta_i \exp(-i\phi_i). \]

From the basic commutation relation

\[ [T(z, \bar{z}, v), a_i] = -\bar{z}_i b_i^\dagger - v_i a_i \quad [T(z, \bar{z}, v), b_i^\dagger] = -z_i a_i + v_i b_i^\dagger, \] (95)

we readily obtain

\[ [T, [T, a_i]] = a_i \quad [T, [T, b_i^\dagger]] = b_i^\dagger, \] (96)

As a consequence, we actually obtain the most general Bogoliubov transformations in the form

\[ A_i = a_i + i \theta_i [T, a_i] + \frac{1}{3} (i \theta_i)^2 [T, [T, a_i]] + \frac{1}{3} (i \theta_i)^3 [T, [T, [T, a_i]]] + \cdots \]
\[ = (\cos \theta_i - iv_i \sin \theta_i) a_i - ib_i^\dagger z_i \sin \theta_i \]
\[ = c_{1i} a_i - c_{2i} b_i^\dagger \quad (97) \]

\[ B_j^\dagger = b_j^\dagger + i \theta_j [T, b_j^\dagger] + \frac{1}{3} (i \theta_j)^2 [T, [T, b_j^\dagger]] + \frac{1}{3} (i \theta_j)^3 [T, [T, [T, b_j^\dagger]]] + \cdots \]
\[ = (\cos \theta_j + iv_j \sin \theta_j) b_j^\dagger - ia_j z_j \sin \theta_j \]
\[ = c_{1j}^* b_j^\dagger + c_{2j} a_j \] (98)

with

\[ c_{1j} \equiv \cos \theta_j - iv_j \sin \theta_j \quad c_{2j} \equiv i z_j \sin \theta_j \] (99)

\[ \hat{n}_j^2 = z_j \bar{z}_j + v_j^2 = 1 \quad |c_{1j}|^2 + |c_{2j}|^2 = 1. \]

It follows thereby that the functional unitary operator

\[ S(\theta, z, v) = S(\theta, \hat{\mathbf{n}}) = S(c_1, c_2) \quad S^{-1} = S^\dagger \] (100)

does generate the Bogoliubov similarity transformations which connect the extended in and out states and fields according to the suitable definitions

\[ \Psi_{\text{out}}(x) = S^{-1} \Psi(x) S \quad \Psi_{\text{in}}(x) = S \Psi(x) S^{-1} \]
\[ |\text{out}\rangle = S^{-1} |\text{in}\rangle \quad |\text{in}\rangle = S |\text{out}\rangle. \] (101)

Moreover, we obtain

\[ \Psi_{\text{out}}(x) = \sum_{j \in \Omega} S^{-1} [a_j, u_{j-}(x) + b_j^\dagger v_{j-}(x)] S \]
\[ = \sum_{j \in \Omega} [A_j, u_{j-}(x) + B_j^\dagger v_{j-}(x)] \] (102)

\[ \Psi_{\text{in}}(x) = \sum_{j \in \Omega} S^{-1} [a_j, u_{j+}(x) + b_j^\dagger v_{j+}(x)] S \]
\[ = \sum_{j \in \Omega} [a_j, u_{j+}(x) + b_j^\dagger v_{j+}(x)] \] (103)
where, in the case of the spinor QED in a uniform electric field in four spacetime dimensions, we have e.g.

$$u_{r\pm}(x) \equiv u_{p,r}^{(\pm)}(t, \mathbf{r}), \quad v_{j\pm}(x) \equiv v_{p,j}^{(\pm)}(t, \mathbf{r}).$$

(104)

Hence, the most general Bogoliubov transformation is nothing but a functional rotation in the Fock space with parameter functions $\langle \theta, z, v \rangle = (\theta, \tilde{\mathbf{n}}) = (c_1, c_2)$, the generators of which are the pairs emission $\Pi^\dagger(\theta z)$, the pairs absorption $\Pi(\bar{v} z)$ and the pairs number $N(\theta v)$ operators, which actually fulfill the functional commutation relations (79) arising from the SU(2) Lie algebra.

Suppose that at a very remote time $t_e \rightarrow -\infty$ the system is in a definite state, e.g. the vacuum $|0\text{ in}|$ for instance, so that it contains no physical fermion particles. The final state at a very future time $t_f \rightarrow +\infty$ has some calculable probability of containing zero, one, two, etc emitted pairs of fermion particles and antiparticles. For example, the probability amplitude to remain in the vacuum state, i.e. the probability amplitude of emitting no pairs, is given by

$$\langle 0\text{ out}|0\text{ in}\rangle = \langle 0\text{ out}|S|0\text{ in}\rangle = \langle 0\text{ out}|S|0\text{ out}\rangle.$$  

(105)

For this interpretation to make sense, one has to actually verify that the vacuum to vacuum probability

$$W_{0, f \rightarrow e} \equiv |\langle 0\text{ out}|0\text{ in}\rangle|^2$$

is not greater than 1. To this concern, consider the Hermitian operator

$$N_i \equiv \frac{1}{2}(A_i^\dagger A_i - B_i B_i) = S^{-1} n_i S$$

(106)

$$= \frac{1}{2}(c_1^j a_1^j - c_2^j b_2^j)(c_1^j a_1^j - c_2^j b_2^j)$$

$$- \frac{1}{2}(c_1^j b_1^j + c_2^j b_2^j)(c_1^j b_1^j + c_2^j b_2^j)$$

$$= (|c_1|^2 - |c_2|^2) n_i + c_1^j c_2^j \Pi_i^\dagger + c_1^j c_2^j \Pi_i,$$

where

$$4|c_1^j c_2^j|^2 + (|c_2|^2 - |c_1|^2)^2 = 1.$$  

It follows that if we set

$$\zeta_j \equiv 2c_1^j c_2^j, \quad w_j \equiv |c_2|^2 - |c_1|^2$$

(107)

we can write the following golden operator identity that is valid $\forall t, j, \ell, \ldots \in \Omega$:

$$2N_i = \zeta_i \Pi_i^\dagger + \bar{\zeta}_i \Pi_i^\dagger + 2w_i n_i, \quad \zeta_i + w_i^2 = 1.$$  

(108)

It is important to remark that the above equality (108) holds true, thanks to the unitarity property satisfied by the Bogoliubov coefficients $c_{1j}, c_{2j}$. Then, for any complex distribution function $\varphi(\mathbf{p}, r, \ldots) = \varphi_i$, we can write

$$2N(\varphi) \equiv \sum_{i \in \Omega} 2N_i \varphi_i = \sum_{i \in \Omega} (A_i^\dagger A_i - B_i B_i) \varphi_i$$

$$= \Pi^\dagger(\zeta \varphi) + \Pi(\bar{\zeta} \varphi) + 2J(\varphi w) = \varphi \cdot T(\zeta, \bar{\zeta}, w)$$

(109)

where, for example,

$$w_i = \cos \Theta_i, \quad \zeta_i = \sin \Theta_i, \exp(-i \phi_i) \quad (\forall \mathbf{r} \in \Omega).$$

Now, owing to

$$2N_i|0\text{ out}\rangle = [B_i^\dagger(), B_i^\dagger]|0\text{ out}\rangle = \delta_{ni}|0\text{ out}\rangle$$

$$= \delta_{ni}^n(0)|0\text{ out}\rangle \equiv V_i(2\pi)^{-n}|0\text{ out}\rangle,$$

(110)
where \( V_n \) denotes the total volume occupied by a large box in the \( n \)-th-dimensional space (\( n = 1, 2, 3 \)), then we eventually obtain

\[
\langle 0 \text{ out} | 0 \text{ in} \rangle = \langle 0 \text{ out} | \exp \{-2iN(\varphi)\} | 0 \text{ out} \rangle = \prod_{i \in \Omega} \exp \{-i\varphi_i V_n(2\pi)^{-n}\} = \exp \{-iV_n(2\pi)^{-n}\sum_{i \in \Omega} \varphi_i\}. \tag{111}
\]

However, according to the natural interpretation which arises from equation (91), it is mandatory for consistency to identify

\[
\varphi_j \equiv i \ln c^*_1 \equiv \text{Arg} c^*_1 + \frac{1}{2} \ln |c^*_1|^2 \quad (\forall j \in \Omega). \tag{112}
\]

As a matter of fact, according to Nikishov [10], the logarithm of vacuum to vacuum transition amplitude is provided by

\[
\ln \langle 0 \text{ out} | 0 \text{ in} \rangle = \sum_{i \in \Omega} \langle 0 \text{ out} | 0 \text{ in} \rangle = \sum_{i \in \Omega} c^*_i = \text{Tr} \ln c^*_1 = V_n(2\pi)^{-n}\sum_{i \in \Omega} \ln c^*_1. \tag{113}
\]

A close comparison evidently yields once again (112). As a consequence, it is possible to express the out vacuum in terms of the in operators in the explicit form:

\[
| 0 \text{ out} \rangle = S^{-1}_\varphi | 0 \text{ in} \rangle = \exp \{-2N(\ln c^*_1)\} | 0 \text{ in} \rangle = \prod_{i \in \Omega} B_i B^*_i \ln c^*_1 | 0 \text{ in} \rangle. \tag{114}
\]

Then, from the commutation relations

\[
[A^*_i A_j, A_k] = -A_k \delta_{ij}, \quad [B_i B^*_j, A_k] = 0, \tag{115}
\]

it immediately follows that, by the very construction,

\[
A_j | 0 \text{ out} \rangle = 0 \quad (\forall \lambda \in \Omega) \tag{116}
\]

and in a quite analogous way one can readily check the other relation

\[
B_j | 0 \text{ out} \rangle = 0 \quad (\forall j \in \Omega) \tag{117}
\]

and the corresponding ones under the exchange of in and out. Note that the invertible operator \( S_\varphi \) is not unitary, owing to the presence of an imaginary part in the distribution function \( \varphi = i \ln c^*_1 \). Furthermore, from the golden operator identity (108), it follows that we can write

\[
2N(\varphi) = \varphi \cdot T(\xi, \bar{\xi}, w) = \Pi^1(\varphi \bar{\xi}) + \Pi(\varphi \bar{\xi}) + 2N(\varphi w) \tag{118}
\]

in such a manner that the out vacuum state can be expressed in the manner of Dirac as an infinite sea of pairs, i.e. a coherent like state involving any number of pairs of any quantum numbers, namely

\[
| 0 \text{ out} \rangle = S^{-1}_\varphi | 0 \text{ in} \rangle = \exp \{-2N(\ln c^*_1)\} | 0 \text{ in} \rangle = \exp \{-\Pi^1(\xi \ln c^*_1) - \Pi(\xi \ln c^*_1) - 2N(w \ln c^*_1)\} | 0 \text{ in} \rangle. \tag{119}
\]
in which
\[ \zeta_j \equiv 2c_1e_{2j} \quad w_j \equiv |c_{2j}|^2 - |c_{1j}|^2 \quad (\forall j \in \Omega). \]

Finally, as an example, in the case of spinor QED in the presence of a uniform electric field on the four-dimensional Minkowski spacetime, i.e. \( n = 3 \), we have
\[ \langle 0 \text{ out} | 0 \text{ in} \rangle = \exp \left\{ -iV(t_f - t_i) \frac{eE}{8\pi^3} \int d^2p_\perp \varphi(\lambda) \right\} \]
with
\[ V_3 = V \quad p_\perp \equiv (p_y, p_z) \quad \lambda \equiv \left( p_\perp^2 + M^2 \right) / eE \]
in such a manner that we can write
\[ |\langle 0 \text{ in} | 0 \text{ out} \rangle|^2 = \exp \left\{ V(t_f - t_i) \frac{eE}{(2\pi)^3} \int d^2p_\perp \ln |c_1(\lambda)|^2 \right\} \]
\[ = \exp \left\{ V(t_f - t_i) \frac{eE}{8\pi^3} \int d^2p_\perp \ln \left( 1 - e^{-\pi \lambda} \right) \right\} \]
which is nothing but the Schwinger formula.

5. Discussion and conclusions

In this paper, I have shown that the processes of emission and absorption of charged fermion–antifermion pairs, in the presence of a background uniform electric field, can be fully described by means of an algebraic approach based upon the functional SU(2) Lie algebra, as obeyed by the pairs operators. Actually, the threesome of functional generators of SU(2) are nothing but the pair creation, the pair destruction and the pairs number operators, respectively. This result allows us to put the Schwinger pair production mechanism within a rigorous framework of quantum field theory in the presence of external, classical, background fields. In particular, the Bogoliubov transformations leading to the Bogoliubov coefficients nicely appear to be nothing but a functional rotation in the Fock space. The present derivation is strongly tailored to the spinor QED, but is not difficult to imagine that the algebraic approach can be generalized to other contexts, such as varying external fields [20], charged scalar or vector quantized fields, etc. Last but not least, even the Bogoliubov transformations relating inertial and noninertial observers and leading to the famous Unruh and Hawking effects could be revisited in the light of the algebraic approach. This might simplify and clarify the main formulae, once the suitable functional Lie groups have been identified.

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Appendix A. The parabolic cylinder functions

The parabolic cylinder functions, of the special form we are interested in the present context, can be defined, e.g., by the integral representation 9.241 1. p 1092 of [16]:
\[ D_{-\mu/2}[\pm(1 + i)\xi] = \frac{1}{\sqrt{\pi}} 2^{\mu/2 + 1/2} e^{-\pi \lambda/4} e^{\mu^2/2} \int_{-\infty}^{\infty} x^{-\mu/2} e^{-2x^2 - 2(1 + i)\xi} dx, \quad (A.1) \]
where $\lambda > 0$, $\xi \in \mathbb{R}$, $\arg x^{-i\xi/2} = \lambda/2$ for $x < 0$, so that

$$D_{-i\xi/2}\left[\pm(1+i)\xi\right] = \frac{1}{\sqrt{\pi}} 2^{i\xi/2+1/2} e^{-\pi\lambda/4} e^{-i\xi^2/2} \int_{-\infty}^{\infty} x^{i\xi/2} e^{-2x^2+i\pi x(1-i)\xi} \, dx.$$  
(A.2)

After the change of variable $x \mapsto -x$,

$$D_{-i\xi/2}\left[\pm(1+i)\xi\right] = \frac{1}{\sqrt{\pi}} 2^{i\xi/2+1/2} e^{\pi\lambda/4} e^{-i\xi^2/2} \int_{-\infty}^{\infty} x^{i\xi/2} e^{-2x^2+i\pi x(1-i)\xi} \, dx,$$  
(A.3)

we eventually come to the conjugation property

$$D_{-i\xi/2}\left[\pm(1+i)\xi\right] = D_{i\xi/2}\left[\pm(1-i)\xi\right],$$  
(A.4)

as naively expected. The following special values appear in our calculations:

$$D_{\pm i\xi/2}(0) = \pi^{-1/2} 2^{i\xi/4} \Gamma\left(\frac{1}{2} \pm \frac{i\lambda}{4}\right) \cosh \frac{\pi \lambda}{4},$$  
(A.5)

$$D_{\pm i\xi/2-1}(0) = \pm i\pi^{-1/2} 2^{i\xi/4} \Gamma\left(\frac{1}{2} \pm \frac{i\lambda}{4}\right) \sinh \frac{\pi \lambda}{4},$$  
(A.6)

$$\pm \frac{\lambda}{2} |D_{\pm i\xi/2-1}(0)|^2 = \pm \sinh \frac{\pi \lambda}{4},$$  
(A.7)

$$|D_{\pm i\xi/2}(0)|^2 = \cosh \frac{\pi \lambda}{4}.$$  
(A.8)

The parabolic cylinder functions fulfill the recursion formulas

$$\frac{d}{dz} D_{\xi}(z) = -\frac{1}{2} z D_{\xi}(z) + v D_{\xi-1}(z),$$  
(A.9)

$$\frac{d}{dz} D_{\xi}(z) = \frac{1}{2} z D_{\xi}(z) - D_{\xi+1}(z).$$  
(A.10)

Consider the combination

$$D_{\xi} \equiv D_{-i\xi/2}\left[(1+i)\xi\right] D_{i\xi/2}\left[(1-i)\xi\right] + \frac{\lambda}{2} D_{-i\xi/2-1}\left[(1+i)\xi\right] D_{i\xi/2-1}\left[(1-i)\xi\right].$$  
(A.11)

From the recursion formulae, we get

$$-2 \frac{d}{dz} D_{-i\xi/2}\left[(1+i)\xi\right] D_{i\xi/2}\left[(1-i)\xi\right] = \lambda(1+i) D_{-i\xi/2}\left[(1+i)\xi\right] D_{i\xi/2-1}\left[(1-i)\xi\right] + \text{c.c.}$$

$$\lambda \frac{d}{dz} D_{-i\xi/2-1}\left[(1+i)\xi\right] D_{i\xi/2-1}\left[(1-i)\xi\right] = \lambda(1+i) D_{-i\xi/2}\left[(1+i)\xi\right] D_{i\xi/2-1}\left[(1-i)\xi\right] + \text{c.c.},$$

so that the above combination $D_{\xi}$ does not depend upon $\xi$ and from the conjugation property (A.4) we can write

$$D_{\xi} = |D_{-i\xi/2}(0)|^2 + \frac{\lambda}{2} |D_{-i\xi/2-1}(0)|^2 = \exp[\pi \lambda/4] .$$  
(A.12)

Let us now consider the further combination

$$D_{\xi} \equiv D_{-i\xi/2}\left[(1+i)\xi\right] D_{i\xi/2}\left[-(1-i)\xi\right] - \frac{\lambda}{2} D_{-i\xi/2-1}\left[(1+i)\xi\right] D_{i\xi/2-1}\left[-(1-i)\xi\right].$$  
(A.13)
From the recursion formulae we get
\[
2 \frac{d}{d \xi} \{ D_{\nu/2}((i-1)\xi) D_{-\nu/2}((i+1)\xi) \} = \lambda (1-i) D_{\nu/2}((i-1)\xi) D_{-\nu/2-1}((i+1)\xi)
\]
\[+ \lambda (1+i) D_{-\nu/2}((i+1)\xi) D_{\nu/2-1}((i-1)\xi) \]
\[
\lambda \frac{d}{d \xi} \{ D_{-\nu/2-1}((i+1)\xi) D_{\nu/2-1}((i-1)\xi) \} = \lambda (1-i) D_{\nu/2}((i-1)\xi) D_{-\nu/2-1}((i+1)\xi)
\]
\[+ \lambda (1+i) D_{-\nu/2}((i+1)\xi) D_{\nu/2-1}((i-1)\xi) ,
\]
which leads to the conclusion that also the quantity \(D_-\) is independent of \(\xi\) and yields
\[
D_- = |D_{-\nu/2}(0)|^2 - \frac{\lambda}{2} |D_{-\nu/2-1}(0)|^2 = \exp[-\pi \lambda/4].
\] (A.15)

The above important properties of the parabolic cylinder functions can be summarized in the
remarkable formula
\[
\text{Two pairs of linearly independent solutions for the upper sign equation are}
\]
\[
f_+^{(1)}(\pm \xi, \lambda) = D_{-\nu/2}(\pm \xi \sqrt{2} e^{i \pi/4})
\] (A.18)
\[
f_+^{(2)}(\pm \xi, \lambda) = D_{\nu/2-1}(\pm \xi \sqrt{2} e^{-i \pi/4})
\] (A.19)
while two couples of linearly independent solutions for the lower sign equation are
\[
f_-^{(1)}(\pm \xi, \lambda) = D_{-\nu/2-1}(\pm \xi \sqrt{2} e^{i \pi/4})
\] (A.20)
\[
f_-^{(2)}(\pm \xi, \lambda) = D_{\nu/2}(\pm \xi \sqrt{2} e^{-i \pi/4}).
\] (A.21)

To the aim of verifying linear independence, we have to compute the Wronskian. Let us first
calculate derivatives by means of the recursion formulae (A.9) and (A.10) that yield
\[
\frac{d}{d \xi} D_{-\nu/2}(\pm \xi \sqrt{2} e^{i \pi/4}) = -i \xi D_{-\nu/2}(\pm \xi \sqrt{2} e^{i \pi/4}) \mp \frac{\lambda}{\sqrt{2}} e^{3 \pi i/4} D_{-\nu/2-1}(\pm \xi \sqrt{2} e^{i \pi/4})
\]
and thereby
\[
W[f_+^{(1)}(\xi, \lambda), f_+^{(1)}(-\xi, \lambda)] = \frac{1}{\sqrt{\pi}} \Gamma \left( -\frac{i\lambda}{2} \right) \sinh \left( \frac{\pi \lambda}{2} \right).
\] (A.22)

On the other side, we readily find
\[
W[f_+^{(1)}(\pm \xi, \lambda), f_+^{(2)}(\pm \xi, \lambda)] = \mp(1-i) \exp[-\pi \lambda/4]
\] (A.23)
and analogous relationships for the other solutions.

In order to understand the physical meaning of the solutions of the wave field equations,
we have to analyse the leading asymptotic behaviour of the parabolic cylinder functions. Then
from equation 9.246 1. p 1093 of [16], we have
\[
D_{-\nu/2}(\xi \sqrt{2} e^{i \pi/4}) \sim \left(2\xi^2\right)^{-\nu/4} e^{\pi \lambda/8} \exp \left\{ -i \xi^2/2 \right\}
\]
\[
D_{-\nu/2-1}(\xi \sqrt{2} e^{i \pi/4}) \sim O(\xi^{-1}) \quad (\xi \gg \lambda > 0).
\]
If instead \( \xi \ll -\lambda \), we have either \( \xi e^{\pi i/4} = |\xi| e^{\pi i/4} \) or else \( \xi e^{\pi i/4} = |\xi| e^{-3\pi i/4} \). Now, for \( \arg(\xi e^{\pi i/4}) = 5\pi i/4 \), no reliable asymptotic expansion is available, so that from equation 9.246, p 1094 of \[16\] we obtain the bona fide leading behaviour for \( \xi \ll -\lambda < 0 \), namely

\[
D_{-i\lambda/2-1}(\xi|\sqrt{2}e^{\pi i/4}) = D_{-i\lambda/2-1}(\xi|\sqrt{2}e^{-3\pi i/4}) \sim \frac{\sqrt{2\pi}}{\Gamma(1+i\lambda/2)} (2\xi^2)^{i\lambda/4} \exp \left\{ -\frac{\pi \lambda}{8} + \frac{i\xi^2}{2} \right\},
\]

\[
D_{-i\lambda/2}(\xi|\sqrt{2}e^{-3\pi i/4}) \sim (2\xi^2)^{-i\lambda/4} \exp \left\{ \frac{-3\pi \lambda}{8} - \frac{i\xi^2}{2} \right\}.
\]

(A.24)

Of course, the situation becomes exactly time-reversed for the two other linearly independent solutions: namely, for \( \xi \gg \lambda > 0 \) we find

\[
D_{-i\lambda/2-1}(\xi|\sqrt{2}e^{\pi i/4}) = D_{-i\lambda/2-1}(\xi|\sqrt{2}e^{-3\pi i/4}) \sim \frac{\sqrt{2\pi}}{\Gamma(1+i\lambda/2)} (2\xi^2)^{i\lambda/4} \exp \left\{ -\frac{\pi \lambda}{8} + \frac{i\xi^2}{2} \right\},
\]

\[
D_{-i\lambda/2}(\xi|\sqrt{2}e^{-3\pi i/4}) = D_{-i\lambda/2}(\xi|\sqrt{2}e^{\pi i/4}) \sim (2\xi^2)^{-i\lambda/4} \exp \left\{ \frac{-3\pi \lambda}{8} - \frac{i\xi^2}{2} \right\}.
\]

(A.25)

whereas for \( \xi \ll -\lambda < 0 \) we obtain

\[
D_{-i\lambda/2}(\xi|\sqrt{2}e^{\pi i/4}) \sim (2\xi^2)^{-i\lambda/4} e^{\pi \lambda/8} \exp \left\{ -i\xi^2/2 \right\},
\]

\[
D_{-i\lambda/2-1}(\xi|\sqrt{2}e^{\pi i/4}) \sim (2\xi^2)^{i\lambda/4} \exp \left\{ i\xi^2/2 \right\}.
\]

(A.26)

For a given particle momentum \( p = e \), we will associate the stationary asymptotic phase

\[
\frac{1}{2} \xi^2(t) = \frac{1}{2} e^2t^2 - pt + p^2/2eE \quad \frac{1}{2} \xi^2(t) = eEt - p \sim \omega t
\]

to the positive frequency solutions \( \omega t \) which describe either a particle, i.e. an electron of momentum \( p \) and charge \( -e \) when \( t \to +\infty \), or an antiparticle, i.e. a positron of momentum \( -p \) and charge \( e \) when \( t \to -\infty \). Conversely, the negative frequency solutions

\[
-\frac{1}{2} \xi^2(t) = -\frac{1}{2} e^2t^2 + p\omega t - p^2/2eE \quad -\frac{1}{2} \xi^2(t) = -eEt + p \sim -\omega t
\]

(A.27)

will instead describe either a particle of momentum \( p \) and charge \( -e \) or an antiparticle of momentum \( -p \) and charge \( e \) when \( t \to -\infty \).

### Appendix B. The Euclidean formulation

In order to make contact with the celebrated Schwinger’s formula (67) it is expedient to turn to the Euclidean formulation [17]. To this aim, consider the Euclidean Dirac operator for an electron

\[
\not{D} + M \equiv (\partial_\mu - ieA_\mu)\gamma_\mu + M
\]

(B.1)

where

\[
x_\mu = (\tau, x, y, z) = (it, x, y, z) \quad A_\mu = (0, i\mathcal{E} \tau, 0, 0)
\]

(B.2)

\[
\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu} \quad \gamma_0 = \gamma_0^\dagger \quad \gamma_\mu = (\gamma^0, -i\gamma^k)
\]

(B.3)

in which \( \mathcal{E} = -ieE \) is the Euclidean electric field. The Euclidean Dirac operator is a normal elliptic operator so that one can safely define its complex power and the corresponding
Euclidean effective action in the one-loop approximation by means of the zeta function regularization [18, 19], namely
\[ \mathcal{S}_{\text{eff}}^E[A;] = \mathcal{S}_{\text{eff}}^E[A;] - \frac{1}{2} \ln \det[(\nabla \psi + M^2)/\mu^2] \]
\[ = \frac{1}{2} \varepsilon^2 \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \text{Tr}[(\nabla \psi + M^2)/\mu^2]^{-\varepsilon} \]
\[ \mathcal{S}_{\text{eff}}^L[A;] = \int d^4x \, \bar{\psi}(x)(\gamma_{\mu} \partial_\mu - eA_\mu(x)] + M)\psi(x) \]  \hspace{1cm} (B.4)
in which \( \mu \) is a suitable reference mass scale. The Euclidean second order differential operator turns out to be elliptic and reads
\[ \vec{\nabla}^2 + M^2 = -\partial_x^2 + (p_x + e\varepsilon \tau)^2 + p_y^2 + p_z^2 + M^2 + e\varepsilon \alpha^1 \]  \hspace{1cm} (B.5)
where, of course,
\[ \mathbf{p} = -i\nabla \quad \alpha^1 = \begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix} . \]  \hspace{1cm} (B.6)
Furthermore, we have
\[ \left[-\partial_x^2 + (p_x + e\varepsilon \tau)^2 + p_y^2 + p_z^2 + M^2, e\varepsilon \alpha^1 \right] = 0, \]  \hspace{1cm} (B.7)
and we can easily find the spectrum and degeneracy of the second order differential scalar operator \(-\partial_x^2 + (p_x + e\varepsilon \tau)^2 + p_y^2 + p_z^2 + M^2\), namely
\[ \lambda_n, p_y, p_z = p_y^2 + p_z^2 + M^2 + e\varepsilon(2n + 1) \]
\[ p_y, p_z \in \mathbb{R} \quad n + 1 \in \mathbb{N} \quad \Delta = e\varepsilon/2\pi . \]  \hspace{1cm} (B.8)
It follows therefrom that we can write [16]
\[ \frac{d}{ds} \left. \text{Tr} \left[ (\nabla \psi + M^2)/\mu^2 \right]^{-s} \right|_{s=0} = \varepsilon^2 4\pi^2 (\text{vol}) \frac{d}{ds} \left. \left. \left( \frac{\mu^2}{e\varepsilon} \right)^{-s} \right|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty dy \, y^{s-2} e^{-ay} \coth y \right) \]
\[ = (\text{vol}) \frac{e\varepsilon/\pi}{2} \frac{d}{ds} \left. \left. \left( \frac{\mu^2}{2e\varepsilon} \right)^{-s} \right|_{s=0} \frac{1}{s-1} \left[ \zeta \left( s - 1, \frac{a}{2} \right) - \frac{a^2}{4} \left( \frac{a}{2} \right)^{-s} \right] \right) \]
with \( a = M^2/e\varepsilon \), \( \Re s > 2 \), where we can identify \( \text{vol} = V \mathcal{T} \) in which \( \mathcal{T} = i(t_f - t_i) \) is the total Euclidean time, so that, taking the simplest renormalization prescription \( \mu = M \) into account, we end up with
\[ \mathcal{L}_{\text{eff}}^E = \frac{1}{2} \varepsilon^2 - \frac{M}{2\pi} + \frac{e\varepsilon^2}{2\pi^2} \left[ 1 + \ln \frac{M^2}{2e\varepsilon} \right] \zeta \left( -1, \frac{M^2}{2e\varepsilon} \right) - \zeta' \left( -1, \frac{M^2}{2e\varepsilon} \right) \]  \hspace{1cm} (B.9)
On the other hand, from the representation as a series of the Riemann zeta function \( \zeta(z, q) \) for \( \Re z < 0 \), \( 0 < q \leq 1 \), we have
\[ \zeta(z, q) = \frac{2\Gamma(1 - z)}{(2\pi)^{1-\zeta}} \sum_{n=1}^\infty n^{z-1} \sin \left( 2\pi nq + \frac{z\pi}{2} \right) \]
\[ \zeta'(z, q) = \frac{2\Gamma(1 - z)}{(2\pi)^{1-\zeta}} \left[ \ln 2\pi - \psi(1 - z) \right] \sum_{n=1}^\infty n^{z-1} \sin \left( 2\pi nq + \frac{z\pi}{2} \right) \]
\[ + \frac{2\Gamma(1 - z)}{(2\pi)^{1-\zeta}} \sum_{n=1}^\infty n^{z-1} \ln(n) \sin \left( 2\pi nq + \frac{z\pi}{2} \right) \]
\[ + \frac{2\Gamma(1 - z)}{(2\pi)^{1-\zeta}} \sum_{n=1}^\infty n^{z-1} \pi \cos \left( 2\pi nq + \frac{z\pi}{2} \right) \]
and consequently
\[
\zeta \left( -1, \frac{M^2}{2eE} \right) = \frac{1}{2\pi^2} \sum_{n=1}^{\infty} n^{-2} \cos \left( \frac{n\pi M^2}{eE} \right)
\]
\[
\zeta' \left( -1, \frac{M^2}{2eE} \right) = \frac{1}{2\pi^2} \left[ 1 - C - \ln 2\pi \right] \sum_{n=1}^{\infty} n^{-2} \cos \left( \frac{n\pi M^2}{eE} \right) \]
\[
- \frac{1}{2\pi^2} \sum_{n=1}^{\infty} n^{-2} (\ln n) \cos \left( \frac{n\pi M^2}{eE} \right) - \frac{1}{4\pi} \sum_{n=1}^{\infty} n^{-2} \sin \left( \frac{n\pi M^2}{eE} \right),
\]
where C is the Euler–Mascheroni constant. It turns out that the effective Lagrangian density in the four-dimensional Minkowski spacetime is achieved under the inverse Wick rotation \( E \to -iE \) which yields
\[
L_{\text{eff}}^M(E) = L_{\text{eff}}^E(-iE) = -\frac{1}{2}E^2 - \left( \frac{M}{2\pi} \right)^2 - \frac{e^2 E^2}{2\pi^2} \left\{ 1 + \ln \left( \frac{iM^2}{2eE} \right) \right\}
\]
\[
\zeta \left( -1, \frac{iM^2}{2eE} \right)
\]
\[
\zeta' \left( -1, \frac{iM^2}{2eE} \right).
\]
It follows therefrom that we eventually find
\[
\Im L_{\text{eff}}^M(E) = \frac{e^2 E^2}{8\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^2} \exp \left\{ -\frac{n\pi M^2}{eE} \right\}
\]
\[
\Re L_{\text{eff}}^M(E) = -\frac{e^2 E^2}{4\pi^4} \left( C + \ln \frac{\pi M^2}{eE} \right) \sum_{n=1}^{\infty} \frac{1}{n^2} \cosh \left( \frac{n\pi M^2}{eE} \right)
\]
\[
-\frac{e^2 E^2}{4\pi^4} \sum_{n=1}^{\infty} \frac{\ln n}{n^2} \cosh \left( \frac{n\pi M^2}{eE} \right).
\]
As a consequence, the total rate of pairs production and/or annihilation in the whole space and during all the time will be given by the manifestly Lorentz invariant expression
\[
w \equiv 1 - \exp \left\{ -2VT \Im L_{\text{eff}}(\mathcal{F}) \right\}
\]
\[
= 1 - \exp \left\{ \frac{e^2 \mathcal{F}}{4\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^2} \exp \left\{ -\frac{n\pi M^2}{e\mathcal{F}^{1/2}} \right\} \right\}
\]
\[
\approx \frac{e^2 \mathcal{F}}{4\pi^3} \exp \left\{ -\frac{\pi M^2}{e\mathcal{F}^{1/2}} \right\}
\]
\[
\mathcal{F} \equiv -\frac{1}{2}F_{\mu\nu}F^{\mu\nu} = E^2 - B^2
\]
when \( G = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} = E \cdot B = 0 \). As a matter of fact, once a uniform magnetic field is switched on, orthogonal to the electrostatic field, we have to solve the corresponding Dirac equation. Instead of solving explicitly the above equation, we can take profit of being within the context of a relativistic theory, so that it is expedient to consider a new inertial frame \( K' \) moving along the negative \( OY \) axis with the velocity \( v_2 = -v, \quad v > 0 \) with respect to the previously considered inertial frame \( K \). Then, the crossed magnetostatic field does actually appear in \( K' \), i.e.
\[
E' = (E' = \gamma E, 0, 0, ) \quad B' = (0, 0, \gamma v E = B')
\]
so that \( v = (B'/E') \). Note, en passant, that by means of a Lorentz transformation and just owing to the Lorentz invariance, we will be restricted to the case in which \( E' \cdot B' = 0 \), namely electric and magnetic orthogonal fields, as well as in the first two sections \( E^2 > B^2 \) that means relatively weak magnetostatic field, namely \( 0 < v < 1 \). The spinor field in the \( K' \) reference frame can be obtained in turn from the transformation law \( \Psi'(x') = \Lambda(v) \Psi(x) \). It follows therefrom, taking equations (24), (27) and (30) suitably into account, that the spinor solutions which describe an incoming electron, in the presence of crossed constant electromagnetic fields, can be written as

\[
\left[ u^{(-)}_{\mu, r}(t, r) \right]' = \left[ 2e E \lambda (2\pi)^3 \right]^{-1/2} \exp \left\{ i p \cdot r - \pi \lambda /8 \right\} \left\{ \Upsilon'_r D_{\lambda/2}(z_-) \right\} + \tilde{\Upsilon}'_r (1 - i) \lambda E D_{\lambda/2-1}(z_-)
\]

with

\[
\Upsilon'_r = \Lambda(v) \Upsilon_r \quad \tilde{\Upsilon}'_r = \Lambda(v) \tilde{\Upsilon}_r \quad (r = 1, 2)
\]

and corresponding rather analogous expressions for all other solutions of the Dirac equation. From the Lorentz invariance of the effective Lagrangian, we immediately obtain

\[
\Im \mathcal{L}_{\text{eff}}^M (E', B') = \frac{e^2}{8\pi^2} (E^2 - B^2) \sum_{n=1}^{\infty} n^{-2} \exp \left\{ - \frac{n\pi M^2}{e E} \right\}
\]

in accordance with the result of [5].

If, instead, in addition to the electrostatic field there is also a uniform magnetic field not orthogonal to the electric field, then in a suitable Lorentz coordinate system we can always take \( E \) and \( B \) to be directed along the \( Ox \) axis. In such a circumstance, it turns out that the imaginary part of the effective Lagrangian is provided by the celebrated Schwinger’s formula [4, 10]

\[
\Im \mathcal{L}_{\text{eff}}^M (E, B) = \frac{e^2 B E}{8\pi^2} \sum_{n=1}^{\infty} \frac{1}{n} \exp \left\{ - \frac{n\pi M^2}{e E} \right\} \coth \left( \frac{n\pi B}{E} \right).
\]

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