A fresh look on three-loop sum-integrals

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Abstract: In order to prepare the ground for evaluating classes of three-loop sum-integrals that are presently needed for thermodynamic observables, we take a fresh and systematic look on the few known cases, and review their evaluation in a unified way using coherent notation. We do this for three important cases of massless bosonic three-loop vacuum sum-integrals that have been frequently used in the literature, and aim for a streamlined exposition as compared to the original evaluations. In passing, we speculate on options for generalization of the computational techniques that have been employed.
1 Introduction

Our knowledge about sum-integrals, needed for evaluating phenomenologically relevant equilibrium observables in thermal field theories (some examples being [1–3]), is by far not as developed as the knowledge about continuum integrals [4] needed for standard high-energy (but zero-temperature) phenomenology, as e.g. reviewed recently in [5]. One of the reasons seems to be the impenetrable structure of the multiple infinite sums that are involved.

Even considering the simplest class of sum-integrals, dimensionally regularized massless bosonic vacuum sum-integrals (which constitute zero-scale problems since their only dimensionful scale – the temperature $T$ – scales out trivially) that we will call hot tadpoles in the
following, only a very limited number of cases are known in practice (for a review, see [6]), let alone the number of technical tools that have been developed for handling such sum-integrals.

All such presently known hot tadpoles contain one-loop two-point sub-integrals \( \Pi(P) \), whose structure is heavily exploited in the process of evaluation. While the 1-loop tadpoles can be computed exactly, i.e. as functions of the dimension \( d \), all 2-loop sum-integrals can be factorized into 1-loop ones e.g. by systematic use of integration-by-parts (IBP) methods [5, 7], the first non-trivial cases occur at the 3-loop level.

It turns out that the few hot 3-loop tadpoles that have been evaluated in the literature have been dealt with on a case-by-case basis by hand, often in some painstaking process, involving inspired tensor transformations, elaborate (UV- and IR-) subtractions, skillful integration tricks, mixed momentum and coordinate space techniques, numerical integration and the such [8–10].

At present, however, there are not only pressing open three-loop questions [11], but even interesting open problems at four-loop order [12] that involve a (large) number of yet unknown sum-integrals. While exactly one type of 4-loop tadpole has been evaluated so far [9], it is clear that a more systematic treatment is urgently needed. The purpose of the present note is to re-analyze the known non-trivial 3-loop cases and to streamline their derivations in terms of a unified notation [13], in order to prepare the ground for tackling further 3-loop and 4-loop tadpoles in an efficient way.

The relevance of hot tadpoles can be appreciated from the following: In modern treatments of equilibrium thermodynamics, despite the known problem of infrared (IR) divergences in massless thermal gauge theories [14], an effective field theory (EFT) approach [15] allows for clean separation of IR and ultraviolet scales; the sum-integrals treated here constitute, in the jargon of those EFT’s, the hard contributions to the corresponding observables, and hence need no IR regulator (other than being dimensionally regularized) [16].

The three key examples of known non-trivial massless 3-loop vacuum sum-integrals that exhibit a range of useful techniques are of ‘spectacles-’, ‘basketball-’ and ‘tensor-spectacles-type’, and can be represented as special instances of a general class as defined by (cf. Eqs. (A.23),(A.30) of Ref. [2]),

\[
\begin{align*}
M_{i,j} &\equiv \sum_{PQR} \left[ \frac{(Q-R)^2}{[P^2]^j} \right] \frac{1}{Q^2 (Q+P)^2} \frac{1}{R^2 (R+P)^2},
\end{align*}
\]

Our Euclidean notation is such that we use bosonic four-momenta \( P \) with \( P^2 = p_0^2 + p^2 = (2\pi n_p T)^2 + p^2 \), and where the sum-integral symbol is a shorthand for

\[
\sum_P^f \equiv T \sum_{p_0} \int \frac{d^d p}{(2\pi)^d},
\]

with \( d = 3-2\varepsilon \) and the sum taken over all integers \( n_p \in \mathbb{Z} \). In the notation of Eq. (1.1), \( M_{0,0} \) and \( M_{1,0} \) are the basketball- and spectacles-type 3-loop tadpoles, respectively, while we refer to \( M_{2,-2} \) as the tensor-spectacles-case, owing to its numerator structure.
In the remainder of the paper, we will discuss the cases $M_{1,0}$, $M_{0,0}$ and $M_{2,-2}$ of Eq. (1.1) in turn, in Sections 2–4. Sec. 5 contains conclusions, while a few technical details are relegated to the appendices.

Before turning to the specific cases, let us note that (by splitting $4QR = (Q + Q)(R + R)$ and exploiting the denominator’s invariance by shifting $Q → -Q - P$ or $R → -R - P$ in the second instance only), the sub-class

\[ M_{N,-1} = \sum \int P \int Q \frac{Q^2 + R^2 - 4QR/2}{Q^2(Q + P)^2 R^2(R + P)^2} = 2I_1 \sum \int P \frac{\Pi(P)}{[P^2]^N} - \frac{1}{2} \sum \int P \frac{[\Pi(P)]^2}{[P^2]^{N-1}} \]  

(1.3)

with $\Pi(P) \equiv \int Q \frac{1}{Q^2(P + Q)^2}$

(1.4)

is seen to involve scalar 1-loop sub-integrals $\Pi$ only. It can hence be treated with the scalar methods employed for $M_{1,0}$ and $M_{0,0}$ see Sec. 2 and Sec. 3, respectively. As an example, from Eq. (1.3) we immediately get $M_{1,-1} = -M_{0,0}/2$, where we have used that the 2-loop sunset sum-integral

\[ \bigodot \equiv S \equiv \sum \int P Q \frac{1}{P^2 Q^2(Q + P)^2} = \sum \int P \frac{1}{P^2} \Pi(P) = 0 \]  

(1.5)

vanishes identically in dimensional regularization as can be shown via integration-by-parts (IBP) techniques [5, 6].

2 The 3-loop spectacles

In the notation of Eq. (1.1), the 3-loop spectacles-type sum-integral $M_{1,0}$ is defined in terms of the 1-loop 2-point function $\Pi$ as

\[ \bigtriangledown \equiv M_{1,0} \equiv \sum \int P \frac{1}{P^2} [\Pi(P)]^2 . \]  

(2.1)

We shall translate the original computation of $M_{1,0}$ from Ref. [10] (relying on the methods pioneered by [8]) to our notation and systematics, using less than their seven pages, in a transparent way, and in a notation that is generalizable to other cases.

2.1 Decomposition of $M_{1,0}$

The spectacles can be identically re-written as

\[ M_{1,0} = \sum \int P \frac{1}{P^2} \left\{ 2\Pi_D \Pi \right\} + \sum \int P \frac{\delta_{PD}}{P^2} \left\{ [\Pi - \Pi_A]^2 + 2[\Pi_A - \Pi_D] \Pi - [\Pi_A]^2 \right\} + \]  

\[ + \sum \int P \frac{1}{P^2} \left\{ [\Pi - \Pi_B]^2 + 2[\Pi_B - \Pi_D] [\Pi - \Pi_C] - [\Pi_B]^2 + 2[\Pi_B - \Pi_D] \Pi_C \right\} , \]  

(2.2)

\[ ^{1}\text{To unclutter the text, we have collected various definitions in the Appendix (cf. Eq. (A.2) for $I_1$).} \]
where $\delta_{p_0}$ picks out the Matsubara zero-mode, the primed sum excludes the zero-mode and we have suppressed the argument ($P$) of all functions in curly brackets. This re-written form becomes useful if $\Pi_{A,B,C}$ have the structure $\sum_i f_i(T, \epsilon) / (P^2)^{\gamma_i(\epsilon)}$, and $\Pi_D = f(T, \epsilon)$ does not depend on the momentum $P$. For then, the first term of Eq. (2.2) is proportional to the 2-loop function $S$, which vanishes as already discussed above; all terms that do not involve $\Pi$ are trivial 1-loop tadpoles $I$, which are known analytically in $d = 3 - 2\epsilon$ dimensions (see Eq. (A.2)); the zero-mode term involving $[\Pi_A - \Pi_D]\Pi$ is known analytically in terms of the 2-loop function $A$ introduced in Eq. (21) of [13] (see Eq. (A.3)); and in the three remaining terms that involve $\Pi$, the three subtraction terms $\Pi_{A,B,C}$ can be independently chosen such as to facilitate their evaluation.

The strategy of Ref. [10] amounts to choosing (for a motivation of this choice, see below)

$$\Pi_A = \frac{\beta}{(P^2)^\epsilon} + \frac{T G(1,1,d)}{(P^2)^{\epsilon+\epsilon}}, \quad \Pi_B = \frac{\beta}{(P^2)^\epsilon}, \quad \Pi_C = \frac{\beta}{(P^2)^\epsilon} + 2 \frac{I_1}{P^2}, \quad \Pi_D = \frac{\beta}{(\alpha T^2)^\epsilon}, \quad (2.3)$$

where $\beta \equiv G(1,1,d+1)$ with $G$ given in Eq. (A.1) and $\alpha$ is a constant to be fixed later. Eq. (2.2) then reduces to

$$M_{1,0} = 0 + A + 2 \beta A (1+\epsilon, 1, 1) + 2 T G(1,1,d) A (3/2 + \epsilon, 1, 1) - 0 +$$

$$+ B + 2 C - \beta^2 I_{1+2\epsilon} + 2 \beta^2 I_{\epsilon+\epsilon} + 4 \beta I_1 I_2, \quad (2.4)$$

where we have introduced the shorthand $f(x) = f(x+\epsilon) - f(x)/(\alpha T^2)^\epsilon$ for convenience. The first zero in Eq. (2.4) is the 2-loop sunset discussed above and the second zero comes from the fact that $\sum \delta_{p_0} [\Pi_A]_f^2 / P^2$ is scale-free and hence vanishes in dimensional regularization. We have introduced the notation $A, B, C$ for the three non-trivial 3-loop sum-integrals\footnote{We take a slightly different $\Pi_D$ here, whose leading term at $\epsilon \to 0$ equals the choice $1/(16\pi^2\epsilon)$ of [10].}

$$A \equiv \int_P \frac{\delta_{p_0}}{P^2} [\Pi - \Pi_A]_f^2, \quad B \equiv \int_P \frac{\delta_{p_0}}{P^2} [\Pi - \Pi_B]_f^2, \quad C \equiv \int_P \frac{\delta_{p_0}}{P^2} [\Pi_B - \Pi_D] [\Pi - \Pi_C] \quad (2.5)$$

that involve $\Pi$, and whose evaluation we shall discuss in the next subsection. In fact, the subtraction terms $\Pi_{A,B,C,D}$ defined above were chosen such that $A, B, C$ are finite in $d = 3$, and can be evaluated numerically after simplification by e.g. the spatial Fourier transform method of [8]. More concretely, $\Pi_A$ subtracts the leading UV- and IR-divergences in $A$; $\Pi_B$ subtracts the leading UV-divergence in $B$; $\Pi_D$ was chosen such that $[\Pi_B - \Pi_D]$ is finite as $d \to 3$; and $\Pi_C$ subtracts the leading and sub-leading UV-divergences in $C$.

### 2.2 Evaluation of $A, B, C$

Let us now bring the 3-loop sum-integrals $A, B, C$ into a form suitable for numerical evaluation. The (inverse) 3d spatial Fourier transforms that will be used below read [9]

\[
\left\{ \Pi - \Pi_B, \frac{2 I_1}{P^2} \right\} = \frac{T}{(4\pi)^2} \int \frac{d^3 r}{r^2} e^{i P r} e^{-|p_0|r} \left\{ \coth (2\pi T r) - \frac{1}{2\pi T r} \frac{2\pi T r}{3} \right\} + O(\epsilon). \quad (2.6)
\]
For $A$, re-writing $\delta_{p_0} [\Pi - \Pi_A] = \delta_{p_0} [\Pi - \Pi_B] - \frac{1}{8} \frac{T}{p} \times \int \frac{d^3r}{r} e^{ipr} \frac{8p}{(4\pi)^2} + \mathcal{O}(\epsilon)$ (where $\frac{1}{8} = G(1, 1, 3)$, while the extra integral is unity and introduced here for notational simplicity); using the 3d spatial Fourier transform of $[\Pi - \Pi_B] (at p_0 = 0)$; integrating over angles via $\frac{1}{r} \int_1^r du e^{ipru} = \frac{\sin(pru)}{pr}$; and letting $|r| = x/(2\pi T)$, $|r'| = y/(2\pi T)$, $|p| = 2\pi T z$:

$$
A = \frac{T^2}{(4\pi)^4} A + \mathcal{O}(\epsilon),
$$

(2.7)

$$
A = \int_0^\infty \frac{dx}{x} \left( \coth(x) - \frac{1}{x} - 1 \right) \int_0^\infty \frac{dy}{y} \left( \coth(y) - \frac{1}{y} - 1 \right) \frac{4}{\pi} \int_0^\infty \frac{dz}{z^2} \sin(zx) \sin(zy)
$$

$$
= \int_0^\infty \frac{dx}{x} \left( \coth(x) - \frac{1}{x} - 1 \right) \int_0^\infty \frac{dy}{y} \left( \coth(y) - \frac{1}{y} - 1 \right) \left( |x + y| - |x - y| \right)
$$

$$
= 2 \int_0^\infty \frac{dx}{x} \left( \coth(x) - \frac{1}{x} - 1 \right) \int_0^x \frac{dy}{y} \left( \coth(y) - \frac{1}{y} - 1 \right) (2y)
$$

$$
= 4 \int_0^\infty \frac{dx}{x} \left( \coth(x) - \frac{1}{x} - 1 \right) \left[ \ln \left( \frac{\sinh(x)}{x} \right) - x \right]
$$

$$
= 2 \int_0^\infty \frac{dx}{x^2} \left[ \ln \left( \frac{\sinh(x)}{x} \right) - x \right]^2 \approx 9.5763057898 . . .
$$

(2.8)

For $B$, using the Fourier transform of $[\Pi - \Pi_B]$, recognizing $\int_p e^{ipr} \frac{1}{p^2 + p_0^2} = \frac{e^{-|p_0|r}}{4\pi r}$; summing over $p_0$ via geometric series; scaling $|r| = x/(2\pi T)$, $|r'| = y/(2\pi T)$; and integrating over angles:

$$
B = \frac{T^2}{(4\pi)^4} B + \mathcal{O}(\epsilon),
$$

(2.9)

$$
B = 4 \int_0^\infty \frac{dx}{x} \left( \coth(x) - \frac{1}{x} \right) \int_0^x \frac{dy}{y} \left( \coth(y) - \frac{1}{y} \right) \left[ \ln \left( \frac{\sinh(x+y)}{\sinh(x)} \right) - y \right]
$$

(2.10)

$$
\approx 0.058739245719 . . .
$$

(2.11)

For $C$, using the Fourier transform of $[\Pi - \Pi_C]$, expanding $[\Pi_B - \Pi_B] = \frac{1}{(4\pi)^2} \ln \frac{\alpha T^2}{p^2} + \mathcal{O}(\epsilon)$; integrating over angles; letting $|p| = |p_0| |y|$, $|r| = x/(2\pi T)$; and using the exponential-integral $\text{Ei}(x) \equiv - \int_x^\infty \frac{dt}{t} e^{-t}$ for $\frac{2}{\pi} e^{iz} \int_0^\infty \frac{dy}{y^2 + 1} \ln \frac{\alpha}{y^2 + 1} = e^{2|z|} \text{Ei}(-2|z|) + \gamma_E + \ln \frac{1}{2}$:

$$
C = -\frac{1}{3} \frac{T^2}{(4\pi)^4} C + \mathcal{O}(\epsilon),
$$

(2.12)

$$
C = -6 \int_0^\infty \frac{dx}{x} \left( \coth(x) - \frac{1}{x} - \frac{x}{3} \right) \sum_{n=1}^\infty \left[ \text{Ei}(-2nx) + e^{-2nx} \ln \left( \frac{2x \alpha e^{-\gamma_E}}{n 16\pi^2} \right) \right]
$$

(2.13)

$$
\approx 0.003496 . . .
$$

(2.14)

where the numerical value is given for $\alpha = 16\pi^2/e^{\gamma_E}$ and corresponds to Eq. (D.27) of [10]. For an discussion of the numerical evaluation, we refer to App. B.

Note that for this choice of $\alpha$, we avoid the computation of $\xi$ in Eqs. (D.20-25) in [10].
2.3 Result

Expanding Eq. (2.4) around \( d = 3 - 2e \) (for \( \alpha = 16\pi^2/e\gamma_E \)), we finally obtain

\[
\mathcal{M}_{1,0} = -\frac{1}{4} \frac{T^2}{(4\pi)^4} \frac{(4\pi e\gamma_E T^2)^{-3e}}{e^2} \left[ 1 + v_1 \epsilon + v_2 \epsilon^2 + \mathcal{O}(\epsilon^3) \right],
\]

(2.15)

\[
v_1 = \frac{4}{3} + 4\gamma_E + 2\frac{\zeta'(1)}{\zeta(-1)},
\]

(2.16)

\[
v_2 = \frac{1}{3} \left[ 46 - 16\gamma_E^2 + \frac{45\pi^2}{4} + 24\ln^2(2\pi) - 104\gamma_1 - 8\gamma_E - 24\gamma_E \ln(2\pi) + 16\gamma_E \frac{\zeta'(1)}{\zeta(-1)} + 24 \right] - 38.5309 \ldots,
\]

(2.17)

which coincides with Eq. (D.51) of [10]. The numerical value in Eq. (2.17) is \(-4\left(A + B - \frac{2}{3}C\right)\).

3 The 3-loop basketball

In the notation of Eq. (1.1), the basketball-type sum-integral \( \mathcal{M}_{0,0} \) is defined in terms of the 1-loop 2-point function \( \Pi \) as

\[
\begin{array}{c}
\begin{array}{c}
\infty \\
\sum \\
\int \\
\left\{ \Pi(P) \right\}^2
\end{array}
\end{array}
\equiv \mathcal{M}_{0,0} \equiv \sum_p \int [\Pi(P)]^2.
\]

(3.1)

Historically, the evaluation of \( \mathcal{M}_{0,0} \) was performed by in Ref. [8], where many of the techniques that were later generalized to other cases of sum-integrals, such as basketball-type tadpoles with different powers on the propagators and/or factors in the numerator [11, 13], were introduced. Here, we translate this pioneering computation of \( \mathcal{M}_{0,0} \) to our notation and systematics.

3.1 Decomposition of \( \mathcal{M}_{0,0} \)

The basketball can be identically re-written as

\[
\mathcal{M}_{0,0} = \sum_p \left\{ 2 \Pi_D \Pi \right\} + \sum_p \delta_{p_0} \left\{ \left[ \Pi - \Pi_B \right]^2 + 2 \left[ \Pi_B - \Pi_D \right] \Pi - \left[ \Pi_B \right]^2 \right\} + \sum_p \left\{ 2 \left[ \Pi_C - \Pi_B \right] \Pi \right\} + \sum_p \left\{ \left[ \Pi - \Pi_C \right]^2 + 2 \left[ \Pi_B - \Pi_D \right] \left[ \Pi - \Pi_E \right] - \left[ \Pi_C \right]^2 + 2 \left[ \Pi_B - \Pi_D \right] \Pi_E \right\},
\]

(3.2)

where, in full analogy to Eq. (2.2), \( \delta_{p_0} \) picks out the Matsubara zero-mode, the primed sum excludes the zero-mode and we have suppressed the argument \( P \) of all functions in curly brackets.

Let us slightly refine the strategy of Ref. [8] by choosing

\[
\Pi_B = \frac{\beta}{(P^2)^e}, \quad \Pi_C = \Pi_B + \frac{2I_1}{P^2}, \quad \Pi_D = \frac{\beta}{(\alpha T^2)^e}, \quad \Pi_E = \Pi_C + 8 \frac{T^4 J_1}{[P^2]^2} \frac{P^2 - (d+1)p_0^2}{d P^2},
\]

(3.3)
where \( \beta \equiv G(1,1,d+1) \) as above, \( \alpha \) is a constant to be fixed later and \( J_n = \left( \frac{4\pi}{T^2} \right)^\epsilon \frac{\Gamma(d+n)(d+n)}{4\pi^{d/2}\Gamma(d/2)} \) as in Eq. (B5) of [8]. With this choice, the first term of Eq. (3.2) can be shifted to the square of a trivial 1-loop tadpole \( I \); for the last term in the first line of Eq. (3.2), note that after \( \sum' \to \sum - \sum \delta_{p_0} \) the full sum is proportional to the 2-loop sunset sum-integral \( S \) and hence vanishes; the rest as well as the other zero modes involving \([\Pi_B - \Pi_D]\) and \([\Pi_B - \Pi_C]\) are known analytically in terms of the 2-loop function \( A \) of Eq. (A.3); all terms that do not involve \( \Pi \) are trivial 1-loop tadpoles \( I \); and in the three remaining terms that involve \( \Pi \), the subtraction terms \( \Pi_B, C,D,E \) have been chosen such as to subtract UV divergences in order to render the sum-integrals finite.

Eq. (3.2) then reduces to

\[
\mathcal{M}_{0,0} = 2 \beta I_1 I_1/(\alpha T^2)^\epsilon + D + 2 \beta A(\bar{0},1,1) - 0 + 0 - 4 I_1 A(1,1,1) + \frac{16 \beta J_1}{dT-4} \left[ I_2 - (d+1)I_3^2 \right],
\]

where we have again used the shorthand \( f(\bar{x}) = f(x+\epsilon) - f(x)/(\alpha T^2)^\epsilon \). The first zero in Eq. (3.4) comes from the fact that \( \sum \delta_{p_0} |[\Pi_B]|^2 \) is scale-free and hence vanishes in dimensional regularization and the second zero is the 2-loop sunset discussed above. We have introduced the notation \( D, E, F \) for the three non-trivial 3-loop sum-integrals\(^5\)

\[
D \equiv \sum_P \delta_{p_0} |[\Pi - \Pi_B]|^2, \quad E \equiv \sum_P' |[\Pi - \Pi_C]|^2, \quad F \equiv \sum_P' (\Pi_B - \Pi_D) |[\Pi - \Pi_E]|
\]

that involve \( \Pi \), and whose evaluation we shall discuss in the next subsection.

### 3.2 Evaluation of \( D, E, F \)

The 3-loop sum-integrals \( D, E, F \) are finite as \( d \to 3 \), such that the 3d spatial Fourier transform method of [8] proves fruitful. The (inverse) transforms needed below are [8, 9]

\[
\{ \Pi, \Pi_B, \Pi_C - \Pi_B, \Pi_E - \Pi_C, \Pi_B - \Pi_D \} = \frac{T}{(4\pi)^2} \frac{d^3r}{r^2} e^{ipr} e^{-|p_0|r} \left\{ \text{coth}(\tilde{r}) + \frac{|p_0|}{2\pi T}, \frac{|p_0|}{2\pi T} + \frac{1}{\tilde{r}}, \frac{\tilde{r}}{4}, \frac{\tilde{r}^3}{45}, \frac{1 + |p_0|r}{r} \right\} + \mathcal{O}(\epsilon),
\]

where \( \tilde{r} = 2\pi Tr \), and for the last term we have expanded \( [\Pi_B - \Pi_D] = -\frac{1}{(4\pi)^2} \ln \frac{\alpha T^2}{2\pi} + \mathcal{O}(\epsilon) \) and used the inverse Fourier transform of the logarithm as derived e.g. in Eqs. (D.11),(D.12) of [8]\(^6\).

For \( D \), using the 3d spatial Fourier transform of \( [\Pi - \Pi_B] \) (at \( p_0 = 0 \)); integrating over angles; and letting \( |r| = x/(2\pi T) \):

\[
D = \frac{T^4}{(4\pi)^2} D + \mathcal{O}(\epsilon),
\]

\(^5\)In the notation of Ref. [8], \( D = \text{Eq. (2.34)}; E \sim \text{Eq. (2.31, 32)}; F \sim I_4 \) as treated in Eqs. (D9-D14) of [8].

\(^6\)Note that it is defined up to a Delta function \( \delta(r) \) which however vanishes in \( F \) below.
\[ D = \frac{1}{2} \int_0^\infty \frac{dx}{x^2} \left( \coth(x) - \frac{1}{x} \right)^2 = \frac{2 \zeta(3)}{\pi^2}, \quad (3.8) \]

where the analytic value was obtained via the recursion of App. C.

For \( E \), using the Fourier transform of \([\Pi - \Pi_C]\); integrating over angles; summing over \( p_0 \) via geometric series and re-writing \( 2/(e^{2x} - 1) = \coth(x) - 1 \); and scaling \( |r| = x/(2\pi T) \):

\[ E = \frac{T^4}{(4\pi)^2} E + \mathcal{O}(\epsilon), \quad (3.9) \]

\[ E = \frac{1}{2} \int_0^\infty \frac{dx}{x^2} \left( \coth(x) - \frac{1}{x} - \frac{x}{3} \right)^2 (\coth(x) - 1) \]
\[ = \frac{1}{18} \left[ \frac{67}{30} + \gamma_E - 6 \ln(2\pi) - \frac{36\zeta(3)}{\pi^2} - 2 \zeta'(-3) \zeta(-3) + 7 \zeta'(-1) \right], \quad (3.10) \]

where the analytic value was again obtained via the recursion of App. C.

For \( F \), using the Fourier transforms of \([\Pi - \Pi_E]\) and \([\Pi_B - \Pi_D]\); integrating over angles; letting \( p_0 = 2\pi T n, \ |r| = x/(2\pi T) \); and summing over \( n \) via geometric series and re-writing \( 2/(e^{2x} - 1) = \coth(x) - 1 \):

\[ F = \frac{T^4}{(4\pi)^2} F + \mathcal{O}(\epsilon), \quad (3.12) \]

\[ F = \frac{1}{2} \int_0^\infty \frac{dx}{x^3} \left( \coth(x) - \frac{1}{x} - \frac{x}{3} + \frac{x^3}{45} \right) \left( 1 - \frac{x}{2} \partial_x \right) (\coth(x) - 1) \]
\[ = \frac{1}{180} \left[ -\frac{23}{6} + 3\gamma_E + \frac{90\zeta(3)}{\pi^2} - 2 \zeta'(-3) \zeta(-3) - 5 \zeta'(-1) \zeta(-1) \right], \quad (3.13) \]

where the analytic value was obtained by first introducing the regulator \( x^\delta \) as in App. C; re-writing \( 2 \coth(x) \partial_x \coth(x) = \partial_x \coth^2(x) \); integrating by parts all terms involving \( \partial_x \) while dropping all boundary terms, which vanish due to the regulator; and using the recursion of App. C, letting \( \delta \to 0 \) in the end.

### 3.3 Result

Expanding Eq. (3.4) around \( d = 3 - 2\epsilon \) (\( \alpha \) does not contribute yet), we finally obtain

\[ \mathcal{M}_{0,0} = \frac{T^4}{(4\pi)^2} \left( \frac{4\pi e^{\gamma_E} T^2}{24 \epsilon} \right)^{-3\epsilon} \left[ 1 + b_{11} \epsilon + \mathcal{O}(\epsilon^2) \right], \quad (3.15) \]

\[ b_{11} = \frac{37}{9} - \frac{32\gamma_E}{15} + 8 \ln(2\pi) - \frac{24\zeta(3)}{\pi^2} + \frac{2}{15} \zeta'(-3) + 24(D + E + 2F) \]
\[ = \frac{91}{15} - \frac{2}{15} \zeta'(-3) + 8 \zeta'(-1) \zeta(-1), \quad (3.16) \]

which coincides with Eq. (2.36) of [8].
4 The 3-loop tensor spectacles

A first non-trivial representative of the class Eq. (4.1) involving numerator structure is $M_{2,-2}$. In this case, we do not have an easy way of dealing with the scalar products in the numerator as was still the case for $M_{1,-1}$, see Eq. (1.3). Here, we wish to first relate the computation of $M_{2,-2}$ to an auxiliary one [2], and then evaluate the latter, for historical reasons denoted $I_{\text{sqed}}$ [8], using our notation and systematics. Let us note that $M_{2,-2}$ is quite another category compared the previous two, as it needs tensor methods. Let us re-write\footnote{Note that in our conventions $d = 3 - 2\epsilon$ and hence the trace of the metric tensor is $g_{\mu\nu} = d + 1$.}

$$4M_{2,-2} = 4(5 - d)I_2I_1I_1 + M_{0,0} + \sum_p \frac{1}{|P|^2} [\bar{\Pi}_{\mu\nu}(P)]^2 \quad (4.1)$$

where $\bar{\Pi}_{\mu\nu}(P)$ was chosen transverse, $P_{\mu} \bar{\Pi}_{\mu\nu}(P) = 0$, consequences of which will be exploited next.

4.1 Relating $M_{2,-2}$ to $I_{\text{sqed}}$

Transversality $P_{\mu} \bar{\Pi}_{\mu\nu}(P) = 0$ constrains the structure of the symmetric tensor $\bar{\Pi}_{\mu\nu}$ to

$$\bar{\Pi}_{\mu\nu}(P) = A_{\mu\nu} \bar{\Pi}_A(P) + B_{\mu\nu} \bar{\Pi}_B(P), \quad (4.3)$$

where $A_{\mu\nu} = A_{\nu\mu}$, $B_{\mu\nu} = B_{\nu\mu}$ are projectors $A A = A$, $B B = B$, $A B = 0$ (with traces $\text{tr} A = d - 1$, $\text{tr} B = 1$) which are orthogonal to the external momentum $P A = 0 = P B$. Concretely, trading the 4-vector $U = (1, 0)$ for the linear combination $V \equiv P^2 U - (P U)P$ that satisfies $P V = 0$, we have $A = g - \frac{P P}{P^2} - \frac{V V}{V^2}$ (for which in fact also $V A = 0 = U A$) and $B = \frac{V V}{V^2}$. The scalar coefficients $\bar{\Pi}_{A,B}$ of Eq. (4.3) can hence be obtained via projections\footnote{Clearly, $\bar{\Pi}_B$ will be the “hard” case whenever it occurs, involving $1/|P|^2$ and $q_0$ etc.}

$$\bar{\Pi}_A(P) = \frac{\text{tr} A \bar{\Pi}(P)}{\text{tr} A A} = \frac{1}{d - 1} \left( g_{\mu\nu} - \frac{|P|^2}{V^2} U_{\mu} U_{\nu} \right) \bar{\Pi}_{\mu\nu}(P) = \frac{\bar{\Pi}_{\mu\nu}(P) - \bar{\Pi}_B(P)}{d - 1}, \quad (4.4)$$

$$\bar{\Pi}_B(P) = \frac{\text{tr} B \bar{\Pi}(P)}{\text{tr} B B} = \frac{|P|^2}{V^2} U_{\mu} U_{\nu} \bar{\Pi}_{\mu\nu}(P) = \frac{|P|^2}{P^2} \bar{\Pi}_{00}(P). \quad (4.5)$$

For the specific case at hand, we can read off $\bar{\Pi}_{\mu\nu}$ and $\bar{\Pi}_{00}$ from Eq. (4.2) to obtain

$$\bar{\Pi}_{\mu\nu} = P^2 \Pi(P) + 2I_1(d - 1), \quad \bar{\Pi}_{00}(P) = \frac{P^2}{P^2} \left( 2I_1 - \sum_Q \frac{(2q_0 + p_0)^2}{Q^2 (Q + P)^2} \right). \quad (4.6)$$

Returning to Eq. (4.1), $M_{0,0}$ is the 3-loop basketball sum-integral, while the last term is related to $I_{\text{sqed}}$ of Appendix H in [8] by an IR subtraction in the $p_0 = 0$ mode defined by

$$\bar{\Pi}_{\mu\nu}^{\text{IR}} = A_{\mu\nu} \bar{\Pi}_{A}^{\text{IR}} + B_{\mu\nu} \bar{\Pi}_{B}^{\text{IR}} \quad (4.7)$$
when choosing\(^9\)  \(\Pi^\text{IR}_A = \frac{2(d-2)L + 4L^2}{d-1}\) and  \(\Pi^\text{IR}_B = 2I_1 - 4I_2^2\) as momentum-independent:

\[
\int_P \frac{1}{[P^2]^2} \left[ (\Pi_{\mu\nu}(P))^2 - \delta_{\mu\nu}\Pi_{\mu\nu}^\text{IR}\right]^2 = \int_P \frac{1}{[P^2]^2} \left[ (\Pi_{\mu\nu}(P) - \delta_{\mu\nu}\Pi_{\mu\nu}^\text{IR})^2 + \Pi_{\mu\nu}^\text{UV} \left(2\Pi_{\mu\nu} - \Pi_{\mu\nu}^\text{UV} - 2\delta_{\mu\nu}\Pi_{\mu\nu}^\text{IR}\right)\right]
\]

\[
= \mathcal{I}_{\text{sqed}} + 2\mathcal{I}_{\text{A}} \int_P \frac{\delta_{\mu\nu}}{[P^2]^2} \Pi_{\mu\nu} + 2 \left( \Pi_{\mu\nu}^\text{IR} - \Pi_{\mu\nu}^\text{IR}_A \right) \int_P \frac{\delta_{\mu\nu}}{[P^2]^2} \Pi_{\mu\nu} + 0_{\text{scale-free}}
\]

\[
= \mathcal{I}_{\text{sqed}} + 2\mathcal{I}_{\text{A}} A(1,1,1;0) + 8 \left( \Pi_{\mu\nu}^\text{IR} - \Pi_{\mu\nu}^\text{IR}_B \right) A(2,1,1;2) + 0 .
\] (4.8)

Here, the 1st line is a trivial re-writing; the 2nd line follows via Eqs. (4.3),(4.7), using the properties of the projectors, plugging in Eq. (4.4), noting that \(\Pi^\text{IR}_{(A,B)}\) are momentum-independent and dropping scale-free integrals that vanish in dimensional regularization; and in the last line we have used Eq. (4.6), again dropped scale-free integrals and expressed the remaining sum-integrals in terms of the 2-loop tadpole \(A\) from Eq. (A.4).

### 4.2 Evaluation of \(\mathcal{I}_{\text{sqed}}\)

Owing to Eqs. (4.1) and (4.8), instead of \(\mathcal{M}_{2,-2}\) the authors of Ref. [8] choose to compute

\[
\mathcal{I}_{\text{sqed}} \equiv \int_P \frac{1}{[P^2]^2} \left[ (\Pi_{\mu\nu}(P) - \delta_{\mu\nu}\Pi_{\mu\nu}^\text{IR})^2 \right]
\] (4.9)

\[
= \int_P \frac{1}{[P^2]^2} \left\{ \left( \Pi_{\mu\nu} - \Pi_{\mu\nu}^\text{UV} - \delta_{\mu\nu}\Pi_{\mu\nu}^\text{IR} \right)^2 + \Pi_{\mu\nu}^\text{UV} \left(2\Pi_{\mu\nu} - \Pi_{\mu\nu}^\text{UV} - 2\delta_{\mu\nu}\Pi_{\mu\nu}^\text{IR}\right) \right\}
\] (4.10)

\[
= \int_P \left\{ \left( \tilde{\Pi}_A - \tilde{\Pi}_B \right)^2 \left[ \frac{\delta_{\mu\nu}}{d-1} \right]^2 + \frac{\Pi_{\mu\nu}^\text{UV}}{dP^2} \left(2\Pi_{\mu\nu} - P^2\Pi_{\mu\nu}^\text{UV} - 2\delta_{\mu\nu}\left[(d-1)\Pi_{\mu\nu}^\text{IR} + \Pi_{\mu\nu}^\text{IR}_B\right]\right) \right\}
\]

\[
= \frac{1}{d-1} \int_P \left\{ \tilde{\Pi}_B \left[ d\tilde{\Pi}_B - 2\tilde{\Pi}_A \right] + \left[ \tilde{\Pi}_A \right]^2 \right\} + \frac{1}{d} \int_P \frac{\Pi_{\mu\nu}^\text{UV}}{P^2} \left(2\Pi_{\mu\nu} - \Pi_{\mu\nu}^\text{UV}\right) + 0 ,
\] (4.11)

where in the 2nd line a UV subtraction \(\Pi_{\mu\nu}^\text{UV} = (A_{\mu\nu} + B_{\mu\nu}) \frac{P^2}{d}\Pi_{\mu\nu}^\text{UV}\) was introduced\(^{10}\); for the 3rd line we have used projector properties as well as Eq. (4.4), and defined

\[
\tilde{\Pi}_A = \frac{1}{P^2} \left( \Pi_{\mu\nu} - P^2\Pi_{\mu\nu}^\text{UV} - \delta_{\mu\nu}\left[ (d-1)\Pi_{\mu\nu}^\text{IR} + \Pi_{\mu\nu}^\text{IR}_B\right]\right) ,
\] (4.12)

\[
\tilde{\Pi}_B = \frac{1}{P^2} \left( \Pi_{\mu\nu} - P^2\Pi_{\mu\nu}^\text{UV} / (d - \delta_{\mu\nu}\Pi_{\mu\nu}^\text{IR}_B) \right) ;
\] (4.13)

and for the 4th line we have assumed \(\Pi_{\mu\nu}^\text{UV} = \sum_i c_i / [P^2]^{n_i}\) and dropped scale-free integrals.

The essence of the computation of Ref. [8] is now the treatment of the terms involving \(\tilde{\Pi}_B\) in Eq. (4.11) (note the similarity to Eq. (H.14) of [8]). Choosing different UV subtractions for the zero- and non-zero modes via \(\Pi_{\mu\nu}^\text{UV} = \Pi_{\mu\nu} - (1 - \delta_{\mu\nu})dP_{1}/P^2\) these terms are finite and can be treated in \(d = 3\) by spatial Fourier transform methods. Ref. [8] states the simple result

\[
\frac{1}{d-2} \int_P \tilde{\Pi}_B \left[ d\tilde{\Pi}_B - 2\tilde{\Pi}_A \right] = \int_P \left[ \tilde{\Pi}_A \right]^2 + \mathcal{O}(\epsilon) ,
\] (4.14)

\(^{9}\)This particular choice in fact reflects \(\Pi_{\mu\nu}^\text{IR}_{(A,B)} = \Pi_{\mu\nu}^\text{IR}_{(A,B)}(0, p \to 0)\).

\(^{10}\)Note that its tensor structure is \((A_{\mu\nu} + B_{\mu\nu}) = (g_{\mu\nu} - P_{\mu}P_{\nu}/P^2)\), as expected at zero temperature.
which follows from a rather lengthy calculation, involves an “amazing cancellation” and is explained in our Appendix D. As a result, the computation of \( \mathcal{I}_{\text{squad}} \) (up to the constant term) is reduced to elements that already appear in the basketball case \( \mathcal{M}_{0,0} \).

In detail, with Eq. (4.6) and the choices of \( \Pi^R \) and \( \Pi^\text{UV} \) given above,

\[
\Pi_A = \Pi - \Pi_B - (1 - \delta_{p_0}) 2I_1/P^2,
\]

\[
\Pi_B = \frac{1}{P^2} (\Pi_B - P^2 \Pi_B/d - 2I_1 + \delta_{p_0} A I_2^2).
\]  

Plugging Eq. (4.14) into Eq. (4.11) then results in

\[
\mathcal{I}_{\text{squad}} = \int f^* \left[ \Pi_A \right]^2 + \frac{1}{d} \int f^* \Pi^\text{UV} \left( \frac{2 \Pi_{\mu\nu}}{P^2} - \Pi^\text{UV} \right) + \mathcal{O}(\epsilon)
\]

\[
= \int f^* \left\{ [\Pi]^2 + \frac{d-1}{d} \left( (4I_1/P^2 + \Pi_B - 2\Pi) \Pi_B + (1 - \delta_{p_0}) d [2I_1/P^2]^2 \right) \right\} + \mathcal{O}(\epsilon).
\]  

Recognizing in Eq. (4.18) the term quadratic in \( \Pi \) as \( \mathcal{M}_{0,0} \), re-writing the linear term as

\[
\int f^* \left\{ \Pi \Pi_B \right\} = \int f^* \left\{ [\Pi - \Pi_E] [\Pi_B - \Pi_D] \right\} + \left\{ \Pi_E [\Pi_B - \Pi_D] \right\} + \left\{ \Pi [\Pi_B - \Pi_D] \right\} + \left\{ \Pi_D \right\}
\]

where the first term on the right-hand side (rhs) is \( \mathcal{F} \) (of the \( \mathcal{M}_{0,0} \) calculation of Sec. 3, cf. Eqs. (3.5), (3.14)) and the others are elementary (as are the remaining terms in Eq. (4.18)), \( \mathcal{I}_{\text{squad}} \) evaluates to (using again \( f(\bar{x}) = f(x+\epsilon) - f(x)/(\alpha T^2)^\epsilon \) as well as the functions collected in Eq. (3.3) and in App. A)

\[
\mathcal{I}_{\text{squad}} = \mathcal{M}_{0,0} + \frac{d-1}{d} \left( \beta^2 I_{2e} + 4 \beta I_1 I_{1+\epsilon} - 2 \left\{ \mathcal{F} + \beta^2 I_\epsilon + 2 \beta I_1 I_1 + \frac{8 \beta I_1}{d T-4} \right\} I_2 - (d+1) I_3^2 \right) + \beta A(0, 1, 1; 0) + \beta I_1 I_1/(\alpha T^2)^\epsilon + 4 d I_2 I_1 I_1 + O_{\text{scale-free}} + \mathcal{O}(\epsilon).
\]  

The expansion around \( d = 3 - 2\epsilon \) coincides with Eq. (H.30) of [8]:

\[
\mathcal{I}_{\text{squad}} = \frac{T^4}{(4\pi)^2} \left( \frac{4\pi e^\gamma T^2}{2\epsilon} \right)^{-3\epsilon} \left[ \frac{23}{10} + \frac{517}{10} + 12\gamma_E - 11 \zeta'(-3) / \zeta(-3) + 68 \zeta'(-1) / \zeta(-1) \right] + \mathcal{O}(\epsilon).
\]  

4.3 Result

Putting together Eqs. (4.1), (4.8), (4.20) and expanding around \( d = 3 - 2\epsilon \), we finally obtain

\[
\mathcal{M}_{2-2} = \frac{T^4}{(4\pi)^2} \left( \frac{4\pi e^\gamma T^2}{2\epsilon} \right)^{-3\epsilon} \left[ \frac{11}{\epsilon} + \frac{73}{2} + 12\gamma_E - 10 \zeta'(-3) / \zeta(-3) + 64 \zeta'(-1) / \zeta(-1) \right] + \mathcal{O}(\epsilon),
\]

which coincides with Eq. (A.30) of [2].
5 Conclusions

We have re-examined the three most prominent cases of massless bosonic three-loop vacuum sum-integrals, in order to simplify their derivation and translate the original calculations to a language that is amenable to generalizations.

First, we have re-derived the result for the spectacles-type 3-loop vacuum sum-integral given first by Andersen and Kyllingstad in \[10\], streamlining the computation quite a bit by using our notation from \[13\]. As an improvement over \[10\], we give a one-dimensional integral representation of \(A\) (which was given as a triple integral there). Further effort would be welcome in order to derive a high-precision result for the numerical coefficient \(C\), involving an infinite sum and a one-dimensional integral, leading to extremely slow convergence behavior. It would be interesting to study generalizations of the computation outlined in Sec. 2, such as \(1/P^2 \rightarrow 1/(P^2)^N\) as was done for the 3-loop basketball topology in \[13\], or including factors of \(p_0\) or other scalar products in the numerator, in order to derive some of the integrals needed in our 3-loop computations.

Second, we have re-derived the result for the basic basketball-type 3-loop vacuum sum-integral given first by Arnold and Zhai in \[8\], streamlining the computation quite a bit by using our notation from \[13\].

Third, we have re-derived the result for the first non-trivial 3-loop vacuum sum-integral involving scalar products in the numerator given first by Arnold and Zhai in \[8\], somewhat streamlining the computation. Here is a summary of this computation in a nutshell:

\[M_{2,-2} \sim (QR)^2 = Q_\mu Q_\nu R_\mu R_\nu \mapsto [\Pi_{\mu\nu}]^2 \mapsto [\Pi_{00}, \Pi_{\mu\mu}]^2 \mapsto 3d\text{FT} \mapsto [\Pi_{\mu\mu}]^2 + \mathcal{O}(\epsilon) \sim M_{0,0} + \mathcal{O}(\epsilon)\]

One wonders whether there is a simpler way to compute \(M_{2,-2}\). Note that the projection method, acting on the level of sub-integrals, seems to over-complicate the computation by triggering factors of \(1/p^2\) (stemming from \(1/V^2 = 1/P^2 p^2\)), which leads outside the class of integrals Eq. (1.1) we started with. It even leads outside the natural generalization of this class as suggested by IBP methods (which allows for factors of \(q_0\) etc in the numerators \[5\]). One idea to avoid this change of structure could be to explore applicability of the generic tensor method of Ref. [17] to the case of finite-temperature sum-integrals as discussed here. However, this is clearly beyond the scope of the present paper but should be explored in the future.

In closing, we hope that our unified exposure of known techniques for sum-integral evaluation leads to a program of generalizing them to other cases – be it with irreducibles in the numerator or with different powers of the denominators – as needed for example for determining matching coefficients in effective field theories \[11\], or for advancing to the next loop order \[9\]. In the short term, it seems that the class of hot bosonic tadpoles \(M_{N,-2}\) is a suitable candidate deserving further study. Finally, an extension to fermionic cases (ultimately involving masses as well as chemical potentials) would be another possible line of future work.
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A Standard integrals

For convenience, we collect here the functions used above, as defined in [13]. They are the 1-loop massless propagator at zero temperature

\[ G(s_1, s_2, d) \equiv (p^2)^{s_{12} - d \over 2} \int \frac{d^d q}{(2\pi)^d} \frac{1}{|q|^2 |s_1| (p + q)^2 |s_2|} = \frac{\Gamma(d/2 - s_1)\Gamma(d/2 - s_2)\Gamma(s_{12} - d/2)}{(4\pi)^{d/2}\Gamma(s_1)\Gamma(s_2)\Gamma(d - s_{12})}, \quad (A.1) \]

the 1-loop bosonic tadpoles

\[ I^a_s \equiv \oint \frac{|q_0|^2}{|Q|^2} = \frac{2T \zeta(2s - a - d)}{(2\pi T)^{2s - a - d}} \frac{\Gamma(s - d/2)}{(4\pi)^{d/2}\Gamma(s)}, \quad I^s_s \equiv \oint \frac{1}{|Q|^2} = I^0_s, \quad (A.2) \]

and a specific 2-loop tadpole

\[ A(s_1, s_2, s_3) \equiv A(s_1, s_2, s_3; 0) \]

\[ A(s_1, s_2, s_3; s_4) \equiv \oint \frac{\delta_{q_0}|q_0|^{s_4}}{|P|^{s_1} |Q|^{s_2} |(P + Q)|^{s_3}} = \frac{2T^2 \zeta(2s_{123} - 2d - s_4)}{(2\pi T)^{2s_{123} - 2d - s_4}} \frac{\Gamma(s_{13} - d/2)\Gamma(s_{12} - d/2)\Gamma(s_{123} - d)}{(4\pi)^d\Gamma(s_2)\Gamma(s_3)\Gamma(d/2)\Gamma(s_{1123} - d)}, \quad (A.4) \]

where \( s_{abc...} \equiv s_c + s_b + s_c + ... \).

B Numerical evaluation of \( A, B, C \)

The integrals \( A \) and \( B \) are easily evaluated numerically e.g. with Mathematica[19],

\[ A \approx 9.5763057898113125, \quad \text{(B.1)} \]

\[ B \approx 0.058739245719225247, \quad \text{(B.2)} \]

while \( C \) is tougher to get with high precision. Maybe it is easier to handle it in pieces:

\[ C_1 = -6 \int_1^\infty \frac{dx}{x} \left( \coth(x) - \frac{1}{x} - \frac{x}{3} \right) \sum_{n=1}^\infty \left[ \text{Ei}(-2nx) + e^{-2nx} \ln \left( \frac{2x}{n} \right) \right] \]

\[ \approx +0.016232689597, \]

\[ C_2 = -6 \int_0^1 \frac{dx}{x} \left( \coth(x) - \frac{1}{x} - \frac{x}{3} \right) \sum_{n=1}^\infty \left[ \text{Ei}(-2nx) - e^{-2nx} \ln (2nx e^{\gamma_E}) \right] \]

\[ \approx -0.022965150204, \]
\[
C_3 = -6 \int_0^1 \frac{dx}{x} \left[ \left( \coth(x) - \frac{1}{x} - \frac{x}{3} \right) \frac{1}{e^{2x} - 1} + \frac{x^2}{90} \right] \ln \left( 4x^2 e^\gamma \right) \\
\approx -0.021888498587,
\]

\[
C_4 = +6 \int_0^1 \frac{dx x^2}{90} \ln \left( 4x^2 e^\gamma \right) = \frac{\gamma E + 2 \ln(2) - 1}{30} \\
\approx +0.03217000867,
\]

\[
C = C_1 + C_2 + C_3 + C_4 \\
\approx +0.003496041673. \quad (B.3)
\]

The sum in \( C_1 \) converges reasonably fast, the one in \( C_2 \) rather slowly. The following piece of Mathematica code was used to obtain the approximate numerical results given above:

wp = 60; ag = 30; max = 10000;

(*here the sum converges reasonably fast*)
Clear[cc1];
cc1[n_] :=
  cc1[n] = NIntegrate[-6/
    x (Coth[x] - 1/x - x/3) (ExpIntegralEi[-2 n x] +
      Exp[-2 n x] Log[2 x/n]), {x, 1, Infinity},
      WorkingPrecision -> wp, AccuracyGoal -> ag];
c1 = Sum[cc1[n], {n, 1, max}]

(*this is the part in which the sum converges extremely slowly*)
Clear[cc2];
cc2[n_] :=
  cc2[n] = NIntegrate[-6/
    x (Coth[x] - 1/x - x/3) (ExpIntegralEi[-2 n x] -
      Exp[-2 n x] Log[2 n x Exp[EulerGamma]]), {x, 0, 1},
      WorkingPrecision -> wp, AccuracyGoal -> ag];
c2 = Sum[cc2[n], {n, 1, max}]

(*these are not problematic at all*)
c3 =
  NIntegrate[-6/
    x ((Coth[x] - 1/x - x/3)/(Exp[2 x] - 1) + x^2/90) (2 Log[2 x] +
      EulerGamma), {x, 0, 1}, WorkingPrecision -> wp,
      AccuracyGoal -> ag]
c4 = Integrate[x/15 (2 Log[2 x] + EulerGamma), {x, 0, 1}]

(*sum up to get C*)
\{c_1, c_2, c_3, N[c_4, wp]\}
Total[%]

C     Integrals needed for \(D, E, F\)

Following Appendix C of Ref. [8], let us introduce – in a manner similar to dimensional regularization – a convergence factor \(x^\delta\) into convergent integrals of the form Eqs. (3.8), (3.10) and (3.13), enabling us to integrate term by term and letting \(\delta \to 0\) in the end. The basic relations needed are

\[
H(a, b) \equiv \int_0^\infty dx x^{a+\delta} \coth^b(x),
\]

\[
H(a, 0) = \int_0^\infty dx x^{a+\delta} = 0,
\]

\[
H(a, 1) = \int_0^\infty dx x^{a+\delta} \left[ 1 + 2 \sum_{n=1}^\infty e^{-2nx} \right] = H(a, 0) + \frac{\Gamma(1+a+\delta) \zeta(1+a+\delta)}{2^a + \delta}
\]

allowing for a recursive solution of integrals \(\int_0^\infty dx x^b \coth^b(x)\) for \(b \in \mathbb{Z}\).

D     Derivation of Eq. (4.14)

In order to derive Eq. (4.14), we closely follow Eqs. (H.15-27) of Ref. [8]. The (inverse) 3d spatial Fourier transforms that we need derive from Eq. (22) of [13],

\[
\hat{\Gamma}(s) = \frac{T}{(4\pi)^2} \int \frac{d^3r}{r^2} e^{ipr} e^{-|p_0|r} \left\{ \frac{1}{2} \frac{(4\pi)^2}{(2\pi T)^2} \sum_{n=0}^{s-1} (s+n)! |p_0|^{1-s-n} \frac{r^{s-n}}{2^n n!} \right\} + \mathcal{O}(\epsilon)
\]

where \(s = |3/2 - s| - 1/2\), and read (using \(\beta/\Gamma(\epsilon) = 1/(4\pi)^2 + \mathcal{O}(\epsilon)\) and \(I_1 = T^2/12 + \mathcal{O}(\epsilon))\)

\[
\left\{ \frac{1}{P^2} \right\}_{P^2, P^2, P^2} = \frac{1}{Q^2} \int \frac{d^3r}{r^2} e^{ipr} e^{-|p_0|r} \times
\]

\[
\left\{ \frac{2\bar{r}}{T^2}, \frac{\bar{r}}{3}, |\bar{p}|, \frac{1}{\bar{r}}, -2(2\pi T)^2 \left( \frac{p_0^2}{\bar{r}} + \frac{3|p_0|}{\bar{r}^2} + \frac{3}{\bar{r}^3} \right), e^{-((|p_0|+|p_0+|p_0|-|p_0|)r)} \right\} + \mathcal{O}(\epsilon).
\]

From Eqs. (4.16) and (4.6) we can therefore compute the transform of Eq. (4.16):

\[
p^2 \hat{\Pi}_B = -\frac{\int_{\hat{Q}} d^3Q}{Q^2 (Q + P)^2} \frac{4|p_0^2 - \frac{p_0^2}{\bar{r}}|}{2^2 \Pi_B \frac{d}{d} + p_0^2 \left[ \Pi_B \frac{d}{d} + \frac{2I_1}{P^2} \right]} + \delta_{p_0} 4I_2
\]

\[
= \frac{T}{(4\pi)^2} \int \frac{d^3r}{r^2} e^{ipr} e^{-|p_0|r} (2\pi T)^2 \left\{ - \left[ p_0 \frac{2 + \frac{p_0^2}{3}}{\bar{r}} + p_0^2 \coth(\bar{r}) + 2 \frac{\coth(\bar{r}) + |p_0|}{\sinh^2(\bar{r})} \right] + \right.
\]

\[
+ \left. \frac{2}{3} \left[ \frac{p_0^2}{\bar{r}} + \frac{3|p_0|}{\bar{r}^2} + \frac{3}{\bar{r}^3} \right] + p_0^2 \left[ \frac{1}{3} \left( \frac{p_0}{\bar{r}} + \frac{1}{\bar{r}} \right) + \frac{\bar{r}}{3} \right] - \delta_{p_0} \frac{T^2}{6} + \mathcal{O}(\epsilon) \right) - \frac{\delta_{p_0} 4I_2}{\hat{Q}^2}.
\]
\[-\frac{T}{(4\pi)^2} \int \frac{d^3r}{r^2} \ e^{i\vec{p}\cdot\vec{r}} \partial_r^2 e^{-|p_0|r} \left( \coth(\tilde{r}) - \frac{1}{\tilde{r}} - \frac{\tilde{r}}{3} \right) - \delta_{p_0} \frac{T^2}{6} + \mathcal{O}(\epsilon) \]

\[-\frac{T}{4\pi} \int_0^\infty dr \ e^{-|p_0|r} \left( \coth(\tilde{r}) - \frac{1}{\tilde{r}} - (1 - \delta_{p_0})\frac{\tilde{r}}{3} \right) \partial_r^2 \frac{\sin(pr)}{pr} + \mathcal{O}(\epsilon) , \quad (D.3)\]

where in the first line we have transformed the numerator of the sum-integral in Eq. (4.6) as 
\[(2q_0 + p_0)^2 = 4q_0^2 + p_0^2 + 2(q_0 + q_0)p_0 \rightarrow 4q_0^2 - p_0^2 \] (shifting half of the term linear in \(q_0\) as \(Q \rightarrow -Q - P\) using the denominator’s symmetry) and re-arranged terms; in the second line we have applied Eq. (D.2), solved the Matsubara sum\(^\text{11}\) and used \(I_2^2 = -T^2/24 + \mathcal{O}(\epsilon)\); as a third step we have done an identical re-writing in terms of derivatives; and in the last line we have integrated over angles via \(\int \frac{d^3r}{r^4} \ e^{i\vec{p}\cdot\vec{r}} = 4\pi \int_0^\infty dr \ \sin(pr) \) and integrated by parts (twice; surface terms cancel against \(\delta_{p_0} T^2/6\)). Analogously, for Eq. (4.15) one easily gets

\[
\tilde{\Pi}_A = \frac{T}{(4\pi)^2} \int \frac{d^3r}{r^2} \ e^{i\vec{p}\cdot\vec{r}} \ e^{-|p_0|r} \left( \coth(\tilde{r}) - \frac{1}{\tilde{r}} - (1 - \delta_{p_0})\frac{\tilde{r}}{3} \right) + \mathcal{O}(\epsilon) \quad (D.4)
\]

\[
\tilde{\Pi}_A = \frac{T}{4\pi} \int_0^\infty dr \ e^{-|p_0|r} \left( \coth(\tilde{r}) - \frac{1}{\tilde{r}} - (1 - \delta_{p_0})\frac{\tilde{r}}{3} \right) \frac{\sin(pr)}{pr} + \mathcal{O}(\epsilon) . \quad (D.5)
\]

To prove Eq. (4.14), let us now use Eq. (D.4) on its rhs and integrate over angles:

\[
\sum_{p_0} \left( \frac{T^2}{(4\pi)^2} \int \frac{d^3r}{r^4} \ e^{-|p_0|r} \left( \coth(\tilde{r}) - \frac{1}{\tilde{r}} - (1 - \delta_{p_0})\frac{\tilde{r}}{3} \right) \right)^2 + \mathcal{O}(\epsilon) \quad (D.6)
\]

\[
= \frac{T}{4\pi} \sum_{p_0} \int_0^\infty dr \ e^{-|p_0|r} \left( \coth(\tilde{r}) - \frac{1}{\tilde{r}} - (1 - \delta_{p_0})\frac{\tilde{r}}{3} \right)^2 + \mathcal{O}(\epsilon) . \quad (D.7)
\]

On the other hand, plugging Eqs. (D.3) and (D.5) into the lhs of Eq. (4.14) results in

\[
\frac{1}{d-2} \sum_{p_0} T^3 \left( \int_0^\infty dr \int_0^\infty ds \ e^{-|p_0|(r+s)} \left( \coth(\tilde{r}) - \frac{1}{\tilde{r}} - (1 - \delta_{p_0})\frac{\tilde{r}}{3} \right) \left( \coth(\tilde{s}) - \frac{1}{\tilde{s}} - (1 - \delta_{p_0})\frac{\tilde{s}}{3} \right) \right) \times
\]

\[
\int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\mathbf{p}^2} \partial_r^2 \left( \frac{\sin(ps)}{ps} + \mathcal{O}(\epsilon) \right) . \quad (D.8)
\]

Using Eqs. (H.21),(H.22) of [8], the integral in the last line of Eq. (D.8) is

\[
\frac{1}{2\pi} \left( -\frac{d}{3r^3} + \frac{\theta(r-s) - r}{r^3} + (d-2) \frac{\delta(r-s)}{4\pi rs} \right) + (d-2) \frac{\delta(r-s)}{4\pi rs} = \left( \frac{3}{d} - \frac{1}{3r^3} \right) + \frac{1}{2\pi} \delta(r-s) . \quad (D.9)
\]

where \(r_\rightarrow = r \theta(r-s)+s \theta(s-r)\) and in the last step we used that the first line of Eq. (D.8) is symmetric under \(r \leftrightarrow s\). Up to terms of \(\mathcal{O}(\epsilon)\), the second line of Eq. (D.8) thus reduces\(^\text{12}\) to a delta function, transforming Eq. (D.8) into Eq. (D.7), which completes the proof.

\(^{11}\)From Eq. (19) of [13], it immediately follows that \(\sum_{p_0} e^{-(|p_0|+|q_0+p_0|-|p_0|)r}\) = \(\coth(\tilde{r}) + |\tilde{p}_0|\) and that \(\sum_{p_0} 2q_0^2 e^{-(|q_0|+|q_0+p_0|-|p_0|)r}\) = \(|\tilde{p}_0|/(1 + 2q_0^2/3 + |\tilde{p}_0|)\coth(\tilde{r}) + (\coth(\tilde{r}) + |\tilde{p}_0|)/\sinh^2(\tilde{r})\).

\(^{12}\)This is the “amazing cancellation” the authors of Ref. [8] refer to in their Appendix H.
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