Numerical computation of fractional Fredholm integro-differential equation of order $2\beta$ arising in natural sciences

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Abstract. This article investigates the approximate solutions to a class of fractional linear Fredholm integro-differential equations arising in physical phenomena based upon the use of an effective treatment technique, called the residual power series (RPS) technique. This approach relies on the generalized Taylor formula as well as the residual error function under the sense of Caputo, with the aim of deriving a supportive analytical solution in the form of a rapidly convergent fractional power series with easily computable components. The RPS algorithm is easy to implement without linearization, limitations or restrictions on the problem’s nature and the number of grid points. To justify the efficiency and accuracy of the proposed technique, an illustrative example is given. The results obtained indicate that the algorithm is simple, accurate, and powerful to solve such equations.

1. Introduction

During last years, differential equations and integro-differential equations of fractional order have been considerably referred to because of their capabilities to model several processes as well as their nice applications in engineering, physics and chemistry such as rheology, viscoelasticity, electromagnetic theory, electrochemistry and diffusion processes, fluid flow and electrical networks, etc. [1-3]. Consequently, many mathematical problems that arise in such phenomena and other sciences can be described very successfully by models using fractional integro-differential equations (FIDEs). Besides, most problems cannot be solved analytically, and hence finding good approximate solutions, using numerical methods will be very valuable.

On other hand, many researchers have presented numerical methods to solve the fractional integro-differential equations (FIDEs). To mention a few, the authors in [4] have applied a variational iteration method to investigate the solution of fractional integro-differential equations together with the nonlocal boundary conditions. The method of adomian decomposition is employed to solve the fractional integro-differential equations [5]. In [6] spectral-collocation method was used to solve the fractional Fredholm integro-differential equations. The method of homotopy perturbation is applied to find out the solution for the nonlinear fractional Fredholm integro-differential equations [7]. Generalized Taylor matrix method is presented in [8] to solve linear Voltera integro-fractional differential equations. Further research papers regarding numerical techniques for integro-differential equations, for more details, (see [9]-[14]).
The basic objective of the present article is to expand the applications of the residual power series (RPS) technique to investigate the fractional power series solution for a class of fractional Fredholm integro-differential equations (FFIDEs) of order $2\beta$, subject to certain initial conditions in the following form:

$$D_t^{2\beta} \varphi(t) + \int_{0}^{t} f(t, \tau) \varphi(\tau) d\tau = \omega(t), \quad \beta \in (0,1], \tag{1.1}$$

under the crisp initial conditions

$$\varphi(0) = c_0 \text{ and } D_t^{\beta} \varphi(0) = c_1 \tag{1.2}$$

where $\omega$ is a continuous function of $t: a \leq t \leq b$, $f(t, \tau)$ is a crisp kernel function, over the square $a \leq t, \tau \leq b$, $c_0, c_1 \in \mathbb{R}$ and the operator $D_t^{(\cdot)}$ indicated to the Caputo derivative of fractional order in crisp sense. Here $\varphi(t)$ is an unknown function which needs to be determined.

The residual power series (RPS) technique that developed in [15] consider an easy and applicable tool to determine power series solution for strongly linear and nonlinear equations without being linearized, discretized, or exposed to perturbation [15-21]. This technique is featured by the following characteristics: firstly, the method provides the solutions in Taylor expansions; therefore, the exact solutions will be available when the solutions are polynomials. Secondly, the solutions along with their derivatives can be applied for each arbitrary point in the given interval. Thirdly, the method does not require modifications while converting from the lower to the higher order. Consequently, this method can be easily and directly applied to the given system by selecting an appropriate value for the initial guesses approximations. Fourthly, the RPS technique needs minor computational requirements with less time and more accuracy.

The remainder of the current article is structured as follows. In Section 2, some essential definitions in addition to primary results relating fractional calculus and fractional power series are revisited. The main idea of the RPS algorithm is presented to derive the fractional power series solution for FFIDEs (1.1) and (1.2) in Section 3. In Section 4, a numerical example is given to show simplicity, capability, and potentiality of the technique. Finally, concluding remarks are given in Section 5.

2. Preliminaries

In the current section, some essential definitions and the basic properties of fractional calculus and fractional power series are introduced which are used in the current paper.

Definition 1. (See [2]). The Riemann-Liouville fractional integral operator of order $\beta$ is given by

$$I_t^\beta \varphi(t) = \frac{1}{\Gamma(\beta)} \int_{0}^{t} (t - \tau)^{\beta-1} \varphi(\tau) d\tau, \quad 0 < \tau < t, \beta > 0,$$

where $\varphi(t)$ is a continuous function of $t: a \leq t \leq b$. Then, the Riemann-Liouville fractional derivative of order $\beta$

$$D_t^\beta \varphi(t) = \frac{d}{dt} I_t^{(\beta)} \varphi(t), \quad 0 < \beta < 1,$$

is called a crisp kernel function, over the square $a \leq t, \tau \leq b$.

Definition 2. (See [2]). Let $\beta > 0, t, \beta \in \mathbb{R}$.

For the Caputo fractional derivative of order $\beta$

$$D_t^\beta \varphi(t) = \frac{d^n}{dt^n} I_t^{(\beta)} \varphi(t), \quad \beta = n,$$

For the Caputo fractional derivative of order $\beta$, we have the following facts:

1) $D_t^{\beta} c = 0$, for any constant $c \in \mathbb{R}$.

2) $D_t^{\beta} t^b = \begin{cases} 1 & \text{if } b < \beta, \cr \frac{b+1}{(b+1-\beta)} t^{\beta-b}, & \text{if } n-1 < \beta < n, \cr 0, & \text{otherwise,} \end{cases}$

3) If $\beta \geq 0, \varphi \in C[a,b]$, then $D_t^\beta \int_{0}^{t} \varphi(\tau) d\tau = \varphi(t)$.

4) For $\varphi \in C^n[a,b]$ and $n-1 < \beta \leq n$, with $n \in \mathbb{N}, b \in \mathbb{R}$

$$\int_{0}^{t} D_t^\beta \varphi(t) = \varphi(t) - \sum_{j=0}^{n-1} \varphi^{(j)}(\tau^+) \frac{(t-\tau)\beta}{\Gamma(j+1)}, \quad 0 \leq \tau < t,$$
Definition 3. (See [21]). A fractional power series (FPS) representation at \( t_0 \) has the following form
\[
\sum_{n=0}^{\infty} a_n (t - t_0)^n = a_0 + a_1 (t - t_0) + a_2 (t - t_0)^2 + \cdots,
\]
where \( 0 \leq m - 1 < \beta \leq m \) and \( t \geq t_0 \), and \( a_n \)'s are constants called the coefficients of the series.

Theorem 1. (See [21]). Suppose that \( f \) has the following FPS representation at \( t_0 \)
\[
f(t) = \sum_{n=0}^{\infty} f_n (t - t_0)^n.
\]
If \( f(t) \in C[t_0, t_0 + R] \), and \( D_t^n f(t) \in C[t_0, t_0 + R] \), for \( n = 0, 1, 2, \ldots \), then coefficients \( f_n \) will be in the form \( f_n = D_t^n f(t_0) \), where \( D_t^n = D_t^\beta D_t^\beta \cdots D_t^\beta \) \((n\text{-times})\).

3. Description of the RPS method

In this section, the RPS technique is introduced to construct and obtain an approximate solution of FFIDEs (1.1) and (1.2) through substituting the expansion of fractional power series (FPS) among the truncated residual function. In order to do so, we assume firstly the proposed solution according to RPS algorithm [16,17,21], for FFIDEs (1.1) and (1.2) at \( t_0 \) has the following form:
\[
\varphi(t) = \sum_{n=0}^{\infty} c_n \frac{t^n}{\Gamma(n\beta + 1)}.
\]

On the other hand, we can obtain the approximate solution of equation (3.1), by considering the following \( k\text{-truncated series:} \)
\[
\varphi_k(t) = \sum_{n=0}^{k} c_n \frac{t^n}{\Gamma(n\beta + 1)}.
\]

Obviously, by using the initial condition (1.2), the \( 1\text{st FPS approximate solution of } \varphi(t) \) will be \( \varphi_1(t) = c_0 + c_1 \frac{t}{\Gamma(1)} \). So, we can rewrite the equation (3.2) as follows:
\[
\varphi_k(t) = c_0 + c_1 \frac{t}{\Gamma(1)} + \sum_{n=2}^{k} c_n \frac{t^n}{\Gamma(n\beta + 1)}.
\]

Define the so-called kth-residual function, \( R_{\varphi}^k \) for \( k = 1, 2, 3, \ldots \) as follow:
\[
R_{\varphi}^k(t) = D_t^\beta \varphi_k(t) + \int_{t_0}^{t} f(t, \tau) \varphi_k(\tau) d\tau - r(t).
\]

and the following residual function
\[
R_{\varphi}^0(t) = \lim_{k \to \infty} R_{\varphi}^k(t) = D_t^\beta \varphi(t) + \int_{t_0}^{t} f(t, \tau) \varphi(\tau) d\tau - \omega(t).
\]

Clear that \( R_{\varphi}^0(t) = 0 \) for each \( t > 0 \). In [16,21], it has been proved that \( D_t^{(i-1)} R_{\varphi}^0(t) = 0 \), \( t \geq 0 \) for each \( i = 1, 2, 3, \ldots, k \), and \( D_t^{(i-1)} R_{\varphi}(0) = D_t^{(i-1)} R_{\varphi}(0) \), \( i = 1, 2, 3, \ldots, k \). However, \( D_t^{(k-2)} R_{\varphi}(0) = 0 \) holds for \( k = 2, 3, 4, \ldots \).

In view of that to determine the \( 2\text{nd unknown coefficient } c_2 \), substitute the \( 2\text{nd approximation } \varphi_2(t) = c_0 + c_1 \frac{t}{\Gamma(1)} + c_2 \frac{t^2}{\Gamma(2)} \), into the \( 2\text{nd residual function } R_{\varphi}^2(t) \) of equation (3.4) as:
\[
R_{\varphi}^2(t) = D_t^\beta \varphi_2(t) + \int_{t_0}^{t} f(t, \tau) \varphi_2(\tau) d\tau - \omega(t),
\]
\[
= c_2 + \int_{t_0}^{t} \frac{2^\beta}{\Gamma(2)} f(t, \tau) d\tau - \omega(t),
\]
and by using the fact \( R_{\varphi}^2(0) = 0 \) we have,
\[
c_2 = -\frac{\omega(0) - \int_{t_0}^{t} f(0, \tau) d\tau}{\int_{t_0}^{t} f(t, \tau) d\tau - \omega(t)}.
\]

Likewise, for the \( 3\text{rd unknown coefficient } c_3 \), substitute the \( 3\text{rd approximation } \varphi_3(t) = c_0 + c_1 \frac{t}{\Gamma(1)} + c_2 \frac{t^2}{\Gamma(2)} + c_3 \frac{t^3}{\Gamma(3)} \), into the \( 3\text{rd residual function } R_{\varphi}^3(t) \) of equation (3.4) and then by computing \( D_t^\beta R_{\varphi}^3(t) \), in addition using the fact \( D_t^\beta R_{\varphi}^3(0) = 0 \), yields
Using similar argument, the 4th unknown coefficient \( c_4 \), will be given utilizing the facts \( D_t^{2\beta} \text{Res}_\varphi^4(0) = 0 \). The same manner can be repeated until we obtain on the coefficients’ arbitrary order of the FPS solution for the FFIDEs (1.1) and (1.2). Furthermore, to achieve more accuracy, we select large \( k \) in the expansion (3.3) i.e, by calculating more components of the solution.

4. Numerical example

This section aims to verify the efficiency and applicability of the proposed algorithm by applying the RPS method to a numerical example. Here, all necessary calculations and analysis are implemented using Mathematica 10.

Example 1. Consider the following Fredholm fractional integro-differential equation

\[
D_t^{2\beta} \varphi(t) = (t^\beta - 2) + 60 \int_0^1 (t - \tau) \varphi(\tau) d\tau, \quad \beta \in (0,1],
\]

with the initial conditions

\[
\varphi(0) = 0 \quad \text{and} \quad D_t^{\beta} \varphi(0) = 1.
\]

which has the exact solution \( \varphi(t) = t(t - 1)^2 \) for \( \beta = 1 \).

Using the last description of RPS scheme, taking into account \( \varphi(0) = c_0 = 0 \) and \( D_t^{\beta} \varphi(0) = c_1 = 1 \), suppose that the \( k \)-th truncated series solution of \( \varphi(t) \) for FFIDEs (4.1) and (4.2) as a FPS about \( t_0 = 0 \) has the following form

\[
\varphi_k(t) = \frac{t^\beta}{\Gamma(\beta + 1)} + \sum_{n=2}^{k} c_n \frac{t^{n\beta}}{\Gamma(n\beta + 1)}.
\]

Consequently, we define the \( k \)-th residual defined function, \( \text{Res}_\varphi^k(t) \), as

\[
\text{Res}_\varphi^k(t) = D_t^{\beta} \varphi_k(t) - (t^\beta - 2) - 60 \int_0^1 (t - \tau) \varphi_k(\tau) d\tau,
\]

\[
= D_t^{\beta} \left( \frac{t^\beta}{\Gamma(\beta + 1)} + \sum_{n=2}^{k} c_n \frac{t^{n\beta}}{\Gamma(n\beta + 1)} \right) - (t^\beta - 2) - 60 \int_0^1 (t - \tau) \left( \frac{t^\beta}{\Gamma(\beta + 1)} \right) + \sum_{n=2}^{k} c_n \frac{t^{n\beta}}{\Gamma(n\beta + 1)} d\tau.
\]

To obtain the value of the coefficients \( c_n, n = 2,3,...k \), in equation (4.3), solve the algebraic system in \( c_n \) that obtained by the fact \( D_t^{(k-2)\beta} \text{Res}_\varphi^k(0) = 0, \enspace 0 < \beta \leq 1, \enspace k = 2,3,4,... \).

Following the procedure of RPS algorithm, we get

\[
c_2 = -\frac{2\Gamma(3+2\beta)\Gamma(2\beta+1)(30(1+3\beta)(1+3\beta))\Gamma(3+3\beta)}{\Gamma(3+\beta)\Gamma(3+3\beta)(60+120\beta+\Gamma(3+2\beta))},
\]

\[
c_3 = \beta \Gamma(\beta), \quad \text{and} \quad c_n = 0 \quad \text{for} \quad n \geq 4
\]

For particular case of \( \beta = 1 \), the solution of FFIDEs (4.1) and (4.2) can be written by the form \( \varphi(t) = \lim_{k \to \infty} \varphi_k(t) = t^2 - 2t^2 + t \), which coincides precisely with the exact solution of the second order Fredholm differential equation (4.1).

To demonstrate the accuracy of the RPS technique, numerical results of Example 1 at \( \beta = 1 \) for selected grid points with step size 0.1 on the interval [0,1] are given in Table 1. The behavior solution of the 3rd-RPS approximated solution at different values of fractional order \( \beta \) is plotted in Figure 1.
Table 1. Numerical results for Example 1 at $\beta = 1$.

| $t$ | Exact solution | Approximate solution | Absolute error |
|-----|----------------|----------------------|----------------|
| 0.1 | 0.08100000000000002 | 0.081000000000000002 | $1.387778780781 \times 10^{-17}$ |
| 0.2 | 0.12800000000000003 | 0.128000000000000000 | $2.77557561562 \times 10^{-17}$ |
| 0.3 | 0.147000000000000000 | 0.147000000000000002 | $2.77557561562 \times 10^{-17}$ |
| 0.4 | 0.144000000000000000 | 0.143999999999999996 | $2.77557561562 \times 10^{-17}$ |
| 0.5 | 0.125000000000000000 | 0.125000000000000000 | 0 |
| 0.6 | 0.09599999999999997 | 0.095999999999999997 | $1.387778780781 \times 10^{-17}$ |
| 0.7 | 0.062999999999999997 | 0.062999999999999999 | $2.77557561562 \times 10^{-17}$ |
| 0.8 | 0.031999999999999999 | 0.031999999999999999 | $6.938893903907 \times 10^{-17}$ |
| 0.9 | 0.008999999999999996 | 0.008999999999999996 | $1.214306433183 \times 10^{-17}$ |

Figure 1. The solution behavior of the 3rd-RPS approximated solution, $\varphi_3(t)$, of Example 1, at different values of $\beta$, where $\beta \in \{0.1,0.2,\ldots,0.9\}$ for all $t \in [0,1]$.

5. Concluding remarks
In this article, a recent efficient and accurate technique is applied to solve a class of fractional Fredholm integro-differential equations subjected to certain initial conditions based on RPS scheme. The proposed technique is used directly to obtain the approximate solution in quick convergent fractional power series with readily calculable components utilizing symbolic calculation software. The obtained numerical results show that the RPS algorithm is simple, straightforward and a powerful tool that can be applied to solve integro-differential equations of fractional order.

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