An effective spin-1 Heisenberg chain in coupled cavities

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Received 19 November 2008, in final form 11 February 2009
Published 9 March 2009
Online at stacks.iop.org/JPhysB/42/065502

Abstract
A coupled array of $N$ identical cavities, each of which contains a five-level atom, is investigated. The results show that the atoms, via the exchange of virtual photons, can be effectively equal to a spin-1 Heisenberg model under certain conditions. By tuning the laser fields, the parameters of the effective Hamiltonian can be controlled.

(Some figures in this article are in colour only in the electronic version)

1. Introduction
The spin chain has played an important role in the quantum information field as well as in condensed matter physics. The interaction between on-site spins can offer us entanglement in a solid and realistic way [1–7]. It has been found that a spin chain with an open boundary condition can be applied to quantum communication [8] where information is translated from one end to another with high fidelity. Perfect state transfer can be realized by changing the interaction between the qubits in spin networks [9, 10].

However, because of the microscopic properties of solid-state materials, it is very hard to address individual spins even though it is the prerequisite for quantum information processing. Single spin addressability can also be very helpful to obtain deeper and more detailed insight into condensed matter physics. In order to do this, it has been shown that the arrays of Josephson junctions [11], quantum dots [12] and optical lattices [13] can provide an effective spin-chain Hamiltonian where spin coupling constants can be controlled. Recently, an array of coupled cavities [14–23], which are ideally suited to addressing individual spins, has been under active investigation. Intense interest has arisen from the demonstration that a polaritonic Mott transition and a Bose–Hubbard interaction can be generated in these structures [15–17]. Hartmann et al [18] have shown that single atoms in interacting cavities can form a Heisenberg spin-$\frac{1}{2}$ Hamiltonian in which all parameters of the effective Hamiltonian can be tuned individually.

On the other hand, people are very much interested in the multilevel systems (corresponding to a high spin if it can be described in a spin model). A number of papers treat multilevel systems in different areas of physics such as in condensed matter physics [24, 25], statistical mechanics [26, 27] as well as in quantum information [28–30]. Now, multilevel systems can be considered as an important field. In this paper, we consider a coupled array of $N$ identical cavities, each of which contains a five-level atom, and generalize the effective spin-$\frac{1}{2}$ model [18] to spin-1. We show that under a large detuning case, the atoms via the exchange of virtual photons can be effectively equal to a spin-1 model. We use the atomic bare basis as the spin level. It should be easy to manipulate an individual atom when we need to project a measure on the individual qutrit.

2. Model and the effective Hamiltonian
We consider an array of cavities coupled via an exchange of photons. Each of the cavities contains a five-level atom. The atomic configuration is shown in figure 1. The three long-lived levels $|a\rangle$, $|b\rangle$ and $|c\rangle$ represent three spin states, respectively. The cavity mode couples to the transitions $|d\rangle \leftrightarrow |a\rangle$, $|d\rangle \leftrightarrow |b\rangle$, $|e\rangle \leftrightarrow |b\rangle$ and $|e\rangle \leftrightarrow |c\rangle$ while four lasers drive the atom with Rabi frequencies $\Omega_i (i = 1, 2, 3, 4)$, respectively. The same atomic configurations have been used in [31] for addressing individual atoms in optical lattices with standing-wave driving fields. In the interaction picture, the Hamiltonian reads as

$$H = \sum_{j=1}^{N} \left( (g_1 a_j |d\rangle \langle a_j| + \Omega_1 |d\rangle \langle b_j|) e^{i\Delta_1 t} + (g_2 a_j |d\rangle \langle b_j| + \Omega_2 |d\rangle \langle a_j|) e^{i\Delta_2 t} \right)$$
We assume

\[ \frac{g_1^2}{\Delta_1} = \frac{g_2^2}{\Delta_2} = \frac{g_3^2}{\Delta_3} = \frac{g_4^2}{\Delta_4}. \]  

(3)

The effective Hamiltonian can be classified as

\[ H = H_0 + H_1 \]  

(4)

with

\[ H_0 = -\sum_{j=1}^{N} \left[ \frac{g_1^2}{\Delta_1} a_j - J(a_j a_{j+1} + a_j^+ a_{j+1}^+) \right] \]

\[ \sum_{j=1}^{N} \left[ \frac{g_2^2}{\Delta_2} |a_j\rangle \langle a_j| + \frac{g_3^2}{\Delta_3} |a_j\rangle \langle a_j| + \frac{g_4^2}{\Delta_4} |a_j\rangle \langle a_j| + \nu_j b_j \right] \]

(5)

Noting that every term in $H_0$ commutes with each other, when we perform unitary transformation we can do it separately. Diagonalize the cavity-hopping terms by employing a Fourier-transformation as $a_j = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} F_{jk} b_k$, where $F_{jk} = \exp(-i2\pi j k / N)$ and $\sum_{j=1}^{N} F_{jk} F^*_{jl} = N \delta_{kl}$. The diagonalized form reads as $\sum_{j=1}^{N} (a_j a_{j+1} + a_j^+ a_{j+1}^+) = \sum_{j \neq k} \nu_j b_j b_k$ with $\nu_j = 2J \cos(2\pi j / N)$. Now, we go into a new frame rotating with $H_0$, under the conditions

\[ \frac{\Omega_1^2}{\Delta_1} + \frac{\Omega_2^2}{\Delta_2} = \frac{1}{2} \left( \frac{\Omega_3^2}{\Delta_3} + \frac{\Omega_4^2}{\Delta_4} \right), \quad \frac{\Omega_1 g_1}{\Delta_1} = \frac{\Omega_3 g_3}{\Delta_3}, \]

(6)

we have the Hamiltonian

\[ H = -\sum_{j,k=1}^{N} \left[ \frac{\Omega_1 g_1}{\Delta_1} (|a_j\rangle \langle a_j| + |c_j\rangle \langle c_j|) e^{i \mu_{\pm} t} \right] \]

\[ + \frac{\Omega_2 g_2}{\Delta_2} (|a_j\rangle \langle b_j| + |b_j\rangle \langle c_j|) e^{i \mu_{\pm} t} \]

\[ \times e^{-i \nu_{j} t} F_{jk} b_k + \text{h.c.} \]

(7)

with $\mu_{\pm} = i \frac{\Pi^1}{\Delta_1} \pm \frac{i}{2} \left( \frac{\Omega_3}{\Delta_3} - \frac{\Omega_4}{\Delta_4} \right)$. We define that the eigenstates of $S_r$ for $S = 1$ are atomic bare states $|a_j\rangle$, $|b_j\rangle$ and $|c_j\rangle$; therefore, $S_{j+} = S_{j+} + i S_{j-} = \sqrt{2} (|b_j\rangle \langle a_j| + |b_j\rangle \langle c_j|)$ and $S_{j-} = S_{j+} - i S_{j-} = \sqrt{2} (|a_j\rangle \langle b_j| + |c_j\rangle \langle b_j|)$. Using the spin operator, we rewrite the Hamiltonian (7) as

\[ H = -\sum_{j,k=1}^{N} \left[ \frac{\Omega_1 g_1}{\sqrt{2} \Delta_1} S^- e^{i \mu_{\pm} t} + \frac{\Omega_2 g_2}{\sqrt{2} \Delta_2} S^+ e^{i \mu_{\pm} t} \right] \]

\[ \times e^{-i \nu_{j} t} F_{jk} b_k + \text{h.c.} \]

(8)

(2) If the Rabi frequency $\left| \Omega_1 g_1 / \sqrt{2 \Delta_1} \right| \ll |\mu_+ - \nu_j|, |\mu_- - \nu_j|$, this allows us to make use of the adiabatic elimination once
more. Considering a subspace without real photons, we deduce the effective Hamiltonian as

\[ H = \sum_{j,k=1}^{N} \left[ \frac{\Omega_{e}^{2} g_{j}^{2}}{2\Delta_{j}^{2}} \frac{1}{\mu_{+} - v_{k}} S_{j}^+ S_{k}^- + \frac{\Omega_{e}^{2} g_{j}^{2}}{2\Delta_{j}^{2}} \frac{1}{\mu_{-} - v_{k}} S_{j}^- S_{k}^+ \right] F_{jk} F_{ik}^* + \sum_{j,k=1}^{N} \left[ \frac{\Omega_{e}^{2} g_{j}^{2}}{2\Delta_{j}^{2}} \frac{1}{\mu_{+} - v_{k}} S_{j}^+ S_{k}^- \right] F_{jk} F_{ik}^* . \]  

(9)

In view of \( v_k = J(F_{ik} + F_{ik}^*) \), we expand \( F_{jk} F_{ik}^* \) as

\[ F_{jk} F_{ik}^* = \frac{1}{\mu_{\pm} - v_{k}} \left( F_{jk} F_{ik}^* + \frac{J}{\mu_{\pm}} F_{j+1k} F_{i+1k}^* + \frac{J}{\mu_{\pm}} F_{jk} F_{i+1k}^* \right) . \]  

(10)

This expansion demands the condition \( J \ll \mu_{\pm} \). Substituting equation (10) into equation (9), we finally obtain the effective Hamiltonian:

\[ H_{xy} = \sum_{j=1}^{N} A[S_{ij}^+ S_{ij}^-] + BS_{ij} + C(S_{ij} S_{ij+1} + S_{ij} S_{ij+1}) \]

with

\[ A = \frac{\Omega_{e}^{2} g_{i}^{2}}{2\Delta_{i}^{2}} + \frac{\Omega_{e}^{2} g_{i}^{2}}{2\Delta_{i}^{2}}, \quad B = \frac{\Omega_{e}^{2} g_{i}^{2}}{2\Delta_{i}^{2}}, \quad C = \frac{\Omega_{e}^{2} g_{i}^{2} J}{\Delta_{i}^{2}} \frac{\Omega_{e}^{2} g_{i}^{2} J}{\Delta_{i}^{2}} . \]

(12)

We clearly see that it is a spin-1 XY antiferromagnetic Hamiltonian (\( C > 0 \)). Because it is spin-1, the term \( S_{ij}^+ S_{ij}^- \) cannot be omitted, which is of essential importance in high-spin cases. Although individual control of the coefficients \( A, B, C \) is limited owing to their mutual dependence, we can still change them; because \( \{g_{i}, \Delta_{i}, \Omega_{i}\} \) \( (i = 1, \ldots, 4) \) meet with conditions (3) and (6), we still have seven variable so as to adjust the coefficients \( A, B, C \).

To confirm the validity of our approximations, we numerically simulate the dynamics generated by Hamiltonian (2) and compare it to the dynamics generated by an effective spin-chain equation (11) (red solid line). (a) The initial atomic state is \( |b_{1}, c_{2} \rangle \). (b) The initial atomic state is \( \frac{1}{\sqrt{2}}(|a_{1}| + |b_{1}|) \otimes |c_{2} \rangle \). To confirm the validity of our approximations, we numerically simulate the dynamics generated by Hamiltonian (2) and compare it to the dynamics generated by an effective spin-chain equation (11). (a) The initial atomic state is \( |b_{1}, c_{2} \rangle \). (b) The initial atomic state is \( \frac{1}{\sqrt{2}}(|a_{1}| + |b_{1}|) \otimes |c_{2} \rangle \). Figure 2(b) plots the population \( p(c_2) \) for the initial state \( \frac{1}{\sqrt{2}}(|a_{1}| + |b_{1}|) \otimes |c_{2} \rangle \). Under this case, all the three levels participate in the interaction so that we can see the two-step oscillation (see the red line). But now, we have much discrepancy between the Hamiltonian (2) and the effective Hamiltonian (11). The discrepancies come from the higher order term in the adiabatic elimination. Due to the two participators, the excited level and the middle level, the discrepancy between the Hamiltonian (2) and the effective Hamiltonian (11) becomes larger.

Now, we will obtain an effective \( S_{2} S_{2} \) interaction. We still employ the atomic level configuration but now we only need the two laser beams working between \( |e_{j} \rangle \leftrightarrow |c_{j} \rangle \) and \( |d_{j} \rangle \leftrightarrow |a_{j} \rangle \) shown in figure 3, and \( \Omega_{2} (\Omega_{3}) \) is withdrawn. The Hamiltonian is

\[ H = \sum_{j=1}^{N} \left[ (g_{a} a_{j} + \Omega_{2}) |d_{j} \rangle \langle a_{j}| e^{i\Delta_{j} t} + (g_{a} a_{j} + \Omega_{3}) |e_{j} \rangle \langle c_{j}| e^{i\Delta_{j} t} + h.c. \right] + \sum_{j=1}^{N} J(a_{j}^{|j+1} a_{j} + a_{j}^{|j+1} a_{j}^{|j+1}) . \]  

(13)
The other two cavity fields denoted by \( g_2 \) and \( g_3 \) in equation (1) will only induce a Stark shift at level \(|b\rangle\) and will not affect the effective interaction; hence, we do not write it in equation (13). Under larger detuning \(|\Delta'_2|, |\Delta'_3| \gg |g_1, g_2, \Omega_2, \Omega_3, J|\), we adiabatically eliminate the atomic excited state and have

\[
H = -\sum_{j=1}^{N} \left( \frac{g_j^2}{\Delta'_1} |a_j\rangle \langle a_j| + |c_j\rangle \langle c_j| \right) a_j
- J \left( a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j \right)
- \sum_{j=1}^{N} \left( \frac{\Omega_2^2}{\Delta'_3} |c_j\rangle \langle c_j| \right)
+ \frac{\Omega_2^2}{\Delta'_1} |a_j\rangle \langle a_j| - \sum_{j=1}^{N} \left( \frac{\Omega_2 g_1}{\Delta'_1} (a_j S_{j+} + \text{h.c.}) \right),
\]

where we need the condition \( \frac{g_j^2}{\Delta'_1} = \frac{g_j^2}{\Delta'_2}, \frac{\Omega_2 g_1}{\Delta'_1} = -\frac{\Omega_2 g_1}{\Delta'_2} \). Switch into the interaction picture and then once more adiabatically eliminate the atomic excited state under the condition \( |\Delta'_1| \ll |\Delta'_2 - \nu_k| \). Finally, we have the effective Hamiltonian

\[
H_{\text{eff}} = \sum_{j=1}^{N} \Omega_2^2 g_1^2 \frac{S_{j+}^2}{\Delta'_2 u} + 2 J \Omega_2^2 g_1^2 \frac{S_j S_{j+1}}{\Delta'_2 u^2},
\]

where \( u = \frac{\Delta'_2}{\Delta'_1} \). Now, the effective Hamiltonian is the \( S_j S_j \) interaction.

During the process of deduction of the effective Hamiltonians (11) and (15), we have twice changed the working picture. But the atomic population probability does not change when the picture changes. So, we can compare the atomic population probability in a different picture, as shown in figure 2. However, the Hamiltonians (13) and (15) do not affect the atomic population probability. Therefore, we do not plot it again.

The two Hamiltonians (11) and (15) can be combined into one effective Hamiltonian if we employ the method proposed by Hartmann et al [18]. The lasers that generate the Hamiltonian (11) are turned on for a short time interval \( \text{dt} \langle |H_{\text{xj}}| \rangle \ll 1 \) followed by another time interval \( \text{dt} \langle |H_{\text{xj}}| \rangle \ll 1 \) with the lasers that generate the Hamiltonian (15) being turned on. We repeat this sequence until the total time range to be simulated is covered. The effective Hamiltonian \( H = H_{\text{xj}} + H_{\text{eff}} \) can be finally obtained.

The decoherence of the system mainly results from the decay mechanisms via the photons or the excited state \(|e_j\rangle\) (\(|d_{j}\rangle\)). To overcome the decoherence, the coefficients of the effective Hamiltonians \( A, B, \) and \( C \) should be much larger than the decay rates of photons and the excited state \(|e_j\rangle\) (\(|d_{j}\rangle\)). Using the group of the parameters of figure 2 (all the parameters are scaled in \( g_1 \)), we have \( A = -0.0128, B = 0.0210, C = 0.0113 \). For a strongly coupled single quantum dot-cavity system, \( \kappa/\gamma < \approx 1800 (\kappa/\gamma \approx 1/300) \), in which \( \kappa/\gamma \) means that the decay of the cavity (the excited state) has been achieved for off-resonance [32]. Therefore, \( \{|A|, B, C\} \gg \{\kappa, \gamma\} \) can be realized in experiment.

3. Conclusion

We consider a coupled array of \( N \) identical cavities, each of which contains a five-level atom. We show that under a large detuning case, the atoms via the exchange of virtual photons can be effectively equal to a spin-1 Heisenberg model. Although the coefficients are related to one another, we can still tune them by controlling the laser fields so that our system is a good simulator for a spin-1 Heisenberg model. When operated in a two-dimensional array of cavities, the device is able to simulate spin lattices.

Acknowledgments

The project was supported by NSFC under grant no. 10774020, and also supported by SRF for ROCS, SEM.

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