Ideals with Smital properties

Marcin Michalski¹ · Robert Rałowski¹ · Szymon Żeberski¹

Received: 5 November 2021 / Accepted: 17 January 2023 / Published online: 14 February 2023
© The Author(s) 2023

Abstract
A $\sigma$-ideal $\mathcal{I}$ on a Polish group $(X, +)$ has the Smital Property if for every dense set $D$ and a Borel $\mathcal{I}$-positive set $B$ the algebraic sum $D + B$ is a complement of a set from $\mathcal{I}$. We consider several variants of this property and study their connections with the countable chain condition, maximality and how well they are preserved via Fubini products. In particular we show that there are $\mathfrak{c}$ many maximal invariant $\sigma$-ideals with Borel bases on the Cantor space $2^{\omega}$.

Keywords Smital property · Steinhaus property · Countable chain condition · Fubini product · Maximal invariant ideal · Orthogonal ideals

Mathematics Subject Classification Primary 03E75 · 28A05; Secondary 03E17 · 54H05

1 Introduction

We adopt the usual set-theoretical notation. We say that $X$ is a Polish space if it is a separable and completely metrisable topological space. $\text{Bor}(X)$ denotes the family of Borel subsets of $X$. $\mathcal{M}(X)$ and $\mathcal{N}(X)$ are the families of meager and null subsets of $X$. The work has been partially financed by grant 8211204601, MPK: 9130730000 from the Faculty of Pure and Applied Mathematics, Wrocław University of Science and Technology. We would like to thank the referee for careful revision which improved the presentation of the paper.

Marcin Michalski
marcin.k.michalski@pwr.edu.pl

Robert Rałowski
robert.ralowski@pwr.edu.pl

Szymon Żeberski
szymon.zeberski@pwr.edu.pl

¹ Faculty of Pure and Applied Mathematics, Wrocław University of Science and Technology, Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland
X respectively. Sometimes we will write briefly \( \mathcal{M} \) and \( \mathcal{N} \) if the underlying space is clear from the context.

Let \( \mathcal{A} \) be a \( (\sigma-) \) algebra and \( \mathcal{I} \) be a \( (\sigma-) \) ideal on an Abelian Polish group \( (X, +) \). Throughout the paper we assume that \( \mathcal{I} \) contains all singletons and

\[
(\forall I \in \mathcal{I})(\exists A \in \mathcal{A} \cap I)(I \subseteq A).
\]

If \( \mathcal{A} \) is not explicitly stated we assume \( \mathcal{A} = \text{Bor}(X) \). In such a case \( \mathcal{I} \) has a Borel base.

We say that a set \( A \) is \( \mathcal{I} \)-positive, if \( A \not\in \mathcal{I} \). A is called \( \mathcal{I} \)-residual if \( A^c \in \mathcal{I} \), we denote this fact by \( A \in \mathcal{I}^* \).

For any sets \( A, B \subseteq X \) we denote the algebraic sum of these sets by \( A + B \), i.e.

\[
A + B = \{a + b : a \in A, \ b \in B\}.
\]

Let us now recall the classical notion of the Steinhaus property and probably less famous notions of Smital properties which were studied in [2].

**Definition 1** We say that a pair \( (A, \mathcal{I}) \) has

(i) the Steinhaus Property if for any \( A, B \in \mathcal{A} \setminus \mathcal{I} \) the set \( A - B \) has a nonempty interior;

(ii) the Smital Property, briefly SP , if for every dense set \( D \) and every \( A \in \mathcal{A} \setminus \mathcal{I} \) the set \( A + D \) is \( \mathcal{I} \)-residual;

(iii) the Weaker Smital Property, briefly WSP , if there exists a countable and dense set \( D \) such that for every \( A \in \mathcal{A} \setminus \mathcal{I} \) the set \( A + D \) is \( \mathcal{I} \)-residual;

(iv) the Very Weak Smital Property, briefly VWSP , if for every \( A \in \mathcal{A} \setminus \mathcal{I} \) there is a countable set \( D \) such that the set \( A + D \) is \( \mathcal{I} \)-residual.

Note that \( \mathcal{M} \) and \( \mathcal{N} \) have all of these properties.

The following Proposition seems to be folklore but we could not find the proof of the second part in the literature.

**Proposition 2** The Steinhaus Property is equivalent to SP.

**Proof** The Steinhaus Property implies SP. Let \( A \in \mathcal{A} \setminus \mathcal{I} \) and \( D \) be countable and dense. We may assume that \( D \) is a subgroup. Suppose that \( (A + D) \not\in \mathcal{I}^* \). Then \( (A + D)^c \not\in \mathcal{I} \). By the Steinhaus Property \( (A + D) - (A + D)^c \) contains an open neighborhood of 0. A contradiction since \( 0 \not\in (A + D) - (A + D)^c \).

SP implies the Steinhaus Property. Assume that \( A - B \) has an empty interior for \( A, B \in \mathcal{A} \setminus \mathcal{I} \). Then there is a countable dense set \( D \subseteq (A - B)^c \). It follows from SP that \( (B + D) \cap A \neq \emptyset \), a contradiction. \( \square \)

Let \( F \subseteq X \times Y \). Then for \( x \in X \)

\[
F_x = \{y \in Y : (x, y) \in F\}
\]

is the vertical section of \( F \) at \( x \). Similarly, for \( y \in Y \)

\[
F^y = \{x \in X : (x, y) \in F\}
\]
is the horizontal section of $F$ at $y$.

For $(\sigma)$-algebras $\mathcal{A} \subseteq P(X)$ and $\mathcal{B} \subseteq P(Y)$ let $\mathcal{A} \otimes \mathcal{B} \subseteq P(X \times Y)$ denote the $(\sigma)$-algebra generated by the rectangles of the form $A \times B$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

**Definition 3** Let $(\mathcal{A}, \mathcal{I})$ and $(\mathcal{B}, \mathcal{J})$ be pairs of $(\sigma)$-algebra-$(\sigma)$-ideal on Polish spaces $X$ and $Y$ respectively. Then we define the Fubini product of $\mathcal{I}$ and $\mathcal{J}$ as follows:

$$K \in \mathcal{I} \otimes \mathcal{J} \iff (\exists C \in \mathcal{A} \otimes \mathcal{B})(K \subseteq C \land \{x \in X : C_x \notin \mathcal{J}\} \in \mathcal{I}).$$

Notice that in the case of ideals possessing Borel bases, i.e. $\mathcal{A} = \text{Bor}(X)$ and $\mathcal{B} = \text{Bor}(Y)$ the above definition ensures the existence of a Borel base for $\mathcal{I} \otimes \mathcal{J}$.

**Proposition 4** If $(\mathcal{A} \otimes \mathcal{B}, \mathcal{I} \otimes \mathcal{J})$ has SP (WSP, VWSP), then $(\mathcal{A}, \mathcal{I})$ and $(\mathcal{B}, \mathcal{J})$ also have it.

**Proof** Let us consider the case of WSP for $\mathcal{I}$. Let $D$ be a witness that $\mathcal{I} \otimes \mathcal{J}$ has WSP. Let $A \in \mathcal{A} \setminus \mathcal{I}$. The set $R = D + (A \times Y)$ is $\mathcal{I} \otimes \mathcal{J}$-residual, therefore

$$\tilde{R} = \{x \in X : R_x \text{ is } \mathcal{J} \text{ - residual}\}$$

is $\mathcal{I}$-residual. Clearly, $A + \pi_X(D) = \tilde{R}$, hence we are done. $\Box$

The following definition is a variation of the ones found in [7, Definition 18.5], [9] and agrees with the notation given in [3].

**Definition 5** Let $X$ and $Y$ be Polish spaces and let $\mathcal{F} \subseteq P(X)$, $\mathcal{G} \subseteq P(Y)$, $\mathcal{H} \subseteq P(X \times Y)$ be families of sets. Then we say that $\mathcal{G}$ is $\mathcal{H}$-on-$\mathcal{F}$ if for each set $H \in \mathcal{H}$

$$\{x \in X : H_x \in \mathcal{G}\} \in \mathcal{F}.$$ 

Mainly we will be interested in the case where $\mathcal{G} = \mathcal{J} \subseteq P(Y)$ is a $\sigma$-ideal, $\mathcal{F} \in \{\text{Bor}(X), \sigma(\text{Bor}(X) \cup \mathcal{I})\}$ and $\mathcal{H} \in \{\text{Bor}(X \times Y), \sigma(\text{Bor}(X \times Y) \cup \mathcal{I} \otimes \mathcal{J})\}$. Here $\sigma(\mathcal{D})$ is the $\sigma$-algebra generated by the family $\mathcal{D}$. We will write, for example, that $\mathcal{J}$ is Borel-on-measurable instead of $\text{Bor}(X \times Y)$-on-$\sigma$ (Bor$(X)$ $\cup$ $\mathcal{I}$) if the context is clear. Notice that both $\mathcal{M}$ and $\mathcal{N}$ are Borel-on-Borel (see [7, Exercise 22.22, 22.25]).

**Example 6** Measurable-on-measurable not necessarily implies Borel-on-Borel.

**Proof** Let $\mathcal{J} = \{\emptyset\}$ and take a Borel set $B$ projection of which is analytic and not Borel. $\Box$

**Proposition 7** Borel-on-measurable implies measurable-on-measurable.

**Proof** Assume that $\mathcal{J}$ is Borel on measurable. Let $C \subseteq X \times Y$ be measurable with respect to $\mathcal{I} \otimes \mathcal{J}$. Then $C = (B \setminus A_1) \cup A_2$, where $B$ is Borel and $A_1, A_2 \in \mathcal{I} \otimes \mathcal{J}$. Clearly

$$\{x \in X : A_2x \notin \mathcal{J}\} \in \mathcal{I},$$
hence it is measurable. See that

\[ \{ x \in X : (B \setminus A_1)_x \notin \mathcal{J} \} = \{ x \in X : B_x \setminus A_{1x} \notin \mathcal{J} \} \]

\[ = \{ x \in X : B_x \notin \mathcal{J}, A_{1x} \in \mathcal{J} \} \cup \{ x \in X : B_x \notin \mathcal{J}, A_{1x} \notin \mathcal{J}, B_x \setminus A_{1x} \notin \mathcal{J} \}. \]

Since \( \{ x \in X : B_x \notin \mathcal{J} \} \) is measurable, \( \{ x \in X : A_{1x} \notin \mathcal{J} \} \) \( \in \mathcal{I}^* \) and

\[ \{ x \in X : B_x \notin \mathcal{J}, A_{1x} \notin \mathcal{J}, B_x \setminus A_{1x} \notin \mathcal{J} \} \subseteq \{ x \in X : A_{1x} \notin \mathcal{J} \} \in \mathcal{I}, \]

the set \( \{ x \in X : C_x \notin \mathcal{J} \} \) is measurable. \( \square \)

\section{Smital and ccc}

Let \( \mathcal{I} \) be a \( \sigma \)-ideal in a Polish space \( X \) and assume that \( \mathcal{I} \) has a Borel base. We say that \( \mathcal{I} \) satisfies the countable chain condition (briefly: ccc) if every family of pairwise disjoint Borel \( \mathcal{I} \)-positive sets is countable. Let us also recall the following cardinal coefficient

\[ \text{cov}(\mathcal{I}) = \min \left\{ |A| : A \subseteq \mathcal{I}, \bigcup A = X \right\}. \]

**Theorem 8** Let \( \mathcal{I} \) be a \( \sigma \)-ideal possessing WSP. Then \( \mathcal{I} \) satisfies ccc or \( \text{cov}(\mathcal{I}) = \omega_1 \).

**Proof** Let \( \mathcal{I} \) be \( \sigma \)-ideal with WSP and let \( D \) witness it. Let \( \{ B_\alpha : \alpha < \omega_1 \} \) be a family of pairwise disjoint Borel \( \mathcal{I} \)-positive sets. WSP implies that for each \( \alpha < \omega_1 \) a set \( D + B_\alpha \) is \( \mathcal{I} \)-residual. If \( \bigcap_{\alpha < \omega_1} (D + B_\alpha) = \emptyset \), then \( \text{cov}(\mathcal{I}) = \omega_1 \). On the other hand, if \( \bigcap_{\alpha < \omega_1} (D + B_\alpha) \neq \emptyset \) then for \( x \in \bigcap_{\alpha < \omega_1} (D + B_\alpha) \) we have

\[ (\forall \alpha \in \omega_1)(\exists d \in D)(x \in d + B_\alpha). \]

\( D \) is countable, hence there exist a set \( W \subseteq \omega_1 \) of cardinality \( \omega_1 \) and \( d \in D \) such that

\[ (\forall \alpha \in W)(x \in d + B_\alpha), \]

which gives \( x - d \in \bigcap_{\alpha \in W} B_\alpha \), a contradiction. \( \square \)

The following remark improves the result obtained in [4].

**Remark 9** Let \( \mathcal{I} \) be a \( \sigma \)-ideal possessing WSP. Then the following statements are equivalent:

(i) For each family of sets \( \{ B_\alpha : \alpha < \omega_1 \} \subseteq B \setminus \mathcal{I} \) there exists a set \( W \subseteq \omega_1 \) of cardinality \( \omega_1 \) such that \( \bigcap_{\alpha \in W} B_\alpha \neq \emptyset \);

(ii) \( \text{cov}(\mathcal{I}) > \omega_1 \).

\( \square \) Springer
Proof (ii) ⇒ (i) is a part of Theorem 8. To prove (i) ⇒ (ii) let us suppose that $\text{cov}(\mathcal{I}) = \omega_1$. Then there is a family of sets

$$\{A_\alpha : \alpha < \omega_1\} \subseteq B \cap \mathcal{I}$$

for which $\bigcup_{\alpha < \omega_1} A_\alpha = X$. Set $\tilde{A}_\alpha = \bigcup_{\beta \leq \alpha} A_\alpha$ for each $\alpha < \omega_1$. The family $\{\tilde{A}_\alpha : \alpha < \omega_1\}$ is ascending and covers $X$. Hence $\{\tilde{A}_\alpha^c : \alpha < \omega_1\}$ is descending family of $\mathcal{I}$-residual sets. Moreover, for every $W \subseteq \omega_1$ of cardinality $\omega_1$ we have $\bigcap_{\alpha \in W} \tilde{A}_\alpha^c = \emptyset$, which contradicts (i).

\[\Box\]

3 Preserving Smital properties via products

In [2] the authors present some results on various Smital properties in product spaces. Their setup is, in their words, as general as possible, concerned with algebras and ideals. It is not clear if they intended their results to hold for $\sigma$-algebras and $\sigma$-ideals or algebras and ideals only. The formulation of [2, Theorem 4.2] suggests the former since it is concerned with the Borel algebra and the families of meager and null sets. In their proof they rely implicitly on the following property.

Definition 10 Let $A \subseteq P(X \times Y)$ be a ($\sigma$-)algebra and let $I \subseteq P(X \times Y)$ be a ($\sigma$-)ideal. A pair $(A, I)$ has the positive rectangle property (PRP) if for every $I$-positive set $A \in A$ there is an $I$-positive rectangle $R$ satisfying $R \subseteq A \cup I$ for some $I \in I$.

In this section we show explicitly that PRP holds for pairs algebra-ideal. However, PRP does not hold for pairs $\sigma$-algebra - $\sigma$-ideal in general, including the relevant here pair of Borel $\sigma$-algebra and the family of null sets.

Example 11 The pair $(\text{Bor}(\mathbb{R}^2), [\mathbb{R}]^{\leq \omega} \otimes [\mathbb{R}]^{\leq \omega})$ does not have PRP.

Proof Let $P \subseteq \mathbb{R}$ be a perfect set such that $P \cap (P + x)$ is at most 1-point for $x \neq 0$ (see [8]). Let us set

$$B = \{(x, y) : x \in P \land y \in P - x\}\setminus(\mathbb{R} \times \{0\}).$$

$B$ is Borel and $B \notin [\mathbb{R}]^{\leq \omega} \otimes [\mathbb{R}]^{\leq \omega}$. If $x \in B^y$, then $x \in P$ and $x \in P - y$, therefore $B^y$ is at most 1-point. Let us suppose that there are sets $A_1, A_2 \in \text{Bor}(\mathbb{R}^2)\setminus[\mathbb{R}]^{\leq \omega}$ and a set $K \in [\mathbb{R}]^{\leq \omega} \otimes [\mathbb{R}]^{\leq \omega}$ such that $(A_1 \times A_2) \setminus K \subseteq B$. Let $T = \{x \in \mathbb{R} : |K_x| > \omega\}$ and notice that $|T| \leq \omega$. Pick $x_0 \in A_1 \setminus T$. Then $K_{x_0}$ is countable. By the definition of $B$ for each $y \in A_2 \setminus K_{x_0}$ we have $K^y \supseteq A_1 \setminus \{(x_0) \cup T\}$. So $A_2 \setminus K_{x_0} \subseteq K_{x_1}$ for $x_1 \in A_1 \setminus (\{(x_0) \cup T\})$, a contradiction.

It is clear that the pair $(\text{Bor}(\mathbb{R}^2), \mathcal{M})$ has PRP, since every nonmeager set possessing the property of Baire is nonempty and open, modulo a set of the first category. What about $(\text{Bor}(\mathbb{R}^2), \mathcal{N})$? As a warm up let us recall the following folklore result.
Proposition 12 Every set $E \subseteq [0, 1]^2$ of positive measure contains a subset of the same measure which does not contain a rectangle of positive measure.

Proof Let $E \subseteq [0, 1]^2$ have positive measure. Consider $E' = \{(x, y) \in E : x - y \in \mathbb{Q}^{c}\}$. To see that $\lambda(E') = \lambda(E)$ let us observe that $E'_x = E_x \cap (\mathbb{Q}^{c} + x)$ for every $x \in [0, 1]$ and $\mathbb{Q}^{c}$ is co-null. Now, if $A \times B \subseteq E'$, $A$ and $B$ of positive measure, then $A - B$ should contain a nonempty open set (Steinhaus Theorem), but clearly $\mathbb{Q} \cap (A - B) = \emptyset$. A contradiction. This completes the proof. \(\square\)

This result may be improved with the following Lemma.

Lemma 13 There exists a set $F \subseteq \mathbb{R}$ such that $\lambda(F \cap U) > 0$ and $\lambda(F^{c} \cap U) > 0$ for every nonempty open set $U$.

Proof Let $(B_n : n \in \omega)$ be an enumeration of the basis of $\mathbb{R}$. At the step 0 let $C^1_0, C^2_0 \subseteq B_0$ be two disjoint Cantor sets of positive measure. At the step $n + 1$ let assume that we have two sequences of pairwise disjoint Cantor sets $(C^1_k : k \leq n)$ and $(C^2_k : k \leq n)$ which for all $i \in \{0, 1\}$ and $k \leq n$ satisfy $\lambda(C^i_k \cap B_k) > 0$. The set

$$B_{n+1} \setminus \bigcup_{k \leq n}(C^1_k \cup C^2_k)$$

is nonempty and open, hence it contains two disjoint Cantor sets of positive measure. Denote them by $C^1_{n+1}$ and $C^2_{n+1}$. This completes the construction and $F = \bigcup_{n \in \omega} C^1_k$ is the desired set. \(\square\)

The above Lemma will serve as a tool to prove the result from [6].

Example 14 (Erdös, Oxtoby). There is a set $E \subseteq \mathbb{R}^2$ such that $E \cap (A \times B)$ and $E^{c} \cap (A \times B)$ have positive measure for each $A, B \subseteq \mathbb{R}$ of positive measure.

Proof Let $F$ be as in the formulation of Lemma 13 and set $E = \{(x, y) \in \mathbb{R}^2 : x - y \in F\}$. Let $A, B \subseteq \mathbb{R}$ have a positive measure. Then

$$\lambda(E \cap A \times B) = \int \int \chi_F(x - y)\chi_A(x)\chi_B(y)dxdy$$

$$= \int \int \chi_F(x)\chi_A(x + y)\chi_B(y)dxdy$$

$$= \int_{F} \lambda((A - x) \cap B)dx.$$

$\lambda((A - x) \cap B)$ is a continuous non-negative function. Furthermore, it is positive on some interval, since $\int_{\mathbb{R}} \lambda((A - x) \cap B)dx = \lambda(A \times B)$, so $\int_{F} \lambda((A - x) \cap B)dx > 0$. \(\square\)

Corollary 15 $(\text{Bor}(\mathbb{R}^2), \mathcal{N})$ does not have PRP.

Now we will focus on PRP for products of algebras.
Lemma 16 Let $\mathcal{A} \subseteq P(X)$ and $\mathcal{B} \subseteq P(Y)$ be algebras. Then $\mathcal{A} \otimes \mathcal{B} \subseteq P(X \times Y)$ consists of finite unions of rectangles.

Proof First let us observe that complements of rectangles are finite unions of rectangles:

$$(A \times B)^c = (A \times B^c) \cup (A^c \times B) \cup (A^c \times B^c).$$

Next, see that finite intersection of a finite union of rectangles is again a finite union of rectangles. Let $T_1, T_2 \subseteq \omega$ be finite. Let $A^n_k, B^n_k$ be rectangles from $\mathcal{A}$ and $\mathcal{B}$ respectively for every $n \in T_1$ and $k \in T_2$. Then

$$\bigcap_{n \in T_1} \bigcup_{k \in T_2} A^n_k \times B^n_k = \bigcup_{f \in T_2^{T_1}} \bigcap_{n \in T_1} A^n_{f(n)} \times B^n_{f(n)}.$$

$T_2^{T_1}$ is finite and finite intersections of rectangles are also rectangles, hence the proof is complete. $\square$

For algebra $\mathcal{A}$ and ideal $\mathcal{I}$ let $\mathcal{A}[\mathcal{I}]$ denote the algebra generated by $\mathcal{A} \cup \mathcal{I}$.

Proposition 17 Let $\mathcal{A} \subseteq P(X)$ and $\mathcal{B} \subseteq P(Y)$ be algebras and let $\mathcal{I}$ be an ideal in $X \times Y$. Then $((\mathcal{A} \otimes \mathcal{B})[\mathcal{I}], \mathcal{I})$ has PRP.

Proof Notice that $(\mathcal{A} \otimes \mathcal{B})[\mathcal{I}] = \{C \triangle I : C \in \mathcal{A} \otimes \mathcal{B}, I \in \mathcal{I}\}$. PRP follows from the previous Lemma. $\square$

From now on let $\mathcal{A} \subseteq P(X), \mathcal{B} \subseteq P(Y)$ be $\sigma$-algebras, and $\mathcal{I} \subseteq P(X), \mathcal{J} \subseteq P(Y)$ $\sigma$-ideals.

Theorem 18 Let $\mathcal{I}$ and $\mathcal{J}$ possess WSP and assume one of the following properties

(i) $\mathcal{J}$ is Borel-on-Borel;
(ii) $\mathcal{J}$ measurable-on-measurable;
(iii) $(\text{Bor}(X \times Y), \mathcal{I} \otimes \mathcal{J})$ has PRP.

Then $\mathcal{I} \otimes \mathcal{J}$ also has WSP.

Proof Let $D_1$ and $D_2$ witness WSP for $\mathcal{I}$ and $\mathcal{J}$ respectively. Let $B \in \text{Bor}(X \times Y) \setminus \mathcal{I} \otimes \mathcal{J}$. If any of the properties (i)-(iii) holds then a set $\tilde{B} = \{x \in X : B_x \notin \mathcal{J}\}$ contains a Borel, $\mathcal{I}$-positive set and $D_1 + \tilde{B}$ is $\mathcal{I}$-residual. Let us observe that

$$(D_1 \times D_2) + B \supseteq \bigcup_{d_1 \in D_1} \bigcup_{x \in \tilde{B}} ([d_1 + x] \times (D_2 + B_x)),$$

therefore for every $x \in D_1 + \tilde{B}$ the set $((D_1 \times D_2) + B)_x$ is $\mathcal{J}$-residual. Since $D_1 + \tilde{B}$ is $\mathcal{I}$-residual, the proof is complete. $\square$

In [1] the authors showed that $\mathcal{M} \otimes \mathcal{N}$ and $\mathcal{N} \otimes \mathcal{M}$ have SP and thus WSP. The following corollary extends this result regarding WSP.
Corollary 19  Let $n \in \omega$ and $I_k \in \{\mathcal{M}, \mathcal{N}\}$ for any $k \leq n$. Then $I_0 \otimes I_1 \otimes \ldots \otimes I_n$ has WSP.

Proof  By [5, Lemma 3.1] the ideal $I_0 \otimes I_1 \otimes \ldots \otimes I_n$ is Borel-on-Borel for any $n \in \omega$ and $I_k \in \{\mathcal{M}, \mathcal{N}\}$, $k \leq n$.

In [2, Theorem 4.3] the authors showed that if $B = J \cup J^*$ and $(A, I)$ has SP then $(A \otimes B, I \otimes J)$ also has SP. We will generalize this result (Theorem 24). Let us start with two technical definitions.

Definition 20  We say that a pair $(A \otimes B, I \otimes J)$ has the Tall Rectangle Hull Property (TRHP) if for every set $C \in A \otimes B$

$$(\exists \tilde{C} \in A, I \in I, J \in J)((\tilde{C} \setminus I) \times (Y \setminus J) \subseteq C \subseteq (\tilde{C} \times Y) \cup (I \times Y) \cup (X \times J))$$

If a set $C$ fulfills the above condition we will say that it has TRHP witnessed by the triple $(\tilde{C}, I, J)$.

Analogously we define Wide Rectangle Hull Property (WRHP):

$$(\exists \tilde{C} \in B, I \in I, J \in J)((X \setminus I) \times (\tilde{C} \setminus J) \subseteq C \subseteq (X \times \tilde{C}) \cup (I \times Y) \cup (X \times J))$$

Proposition 21  If a pair $(A \otimes B, I \otimes J)$ have TRHP or WRHP then it has PRP.

Proof  Let $C \in A \otimes B$ has TRHP witnessed by $(\tilde{C}, I, J)$:

$$(\tilde{C} \setminus I) \times (Y \setminus J) \subseteq C \subseteq (\tilde{C} \times Y) \cup (I \times Y) \cup (X \times J))$$

and assume that $C \notin I \otimes J$. Then $\tilde{C} \times Y$ is the desired rectangle. Clearly $\tilde{C} \times Y \subseteq C$ modulo a set from $I \otimes J$. It is also $I \otimes J$-positive, otherwise

$$(\tilde{C} \times Y) \cup (I \times Y) \cup (X \times J)) \in I \otimes J$$

and also $C \in I \otimes J$.

The proof of WRHP case is almost identical.

Lemma 22  The family of sets possessing TRHP is closed under countable unions and complements. The same is true for the family of sets possessing WRHP.

Proof  Proofs for both cases follow the same pattern, so without loss of generality let us focus on the case of TRHP.

Let $C = \bigcup_{n \in \omega} C_n$ and $(\tilde{C}_n, I_n, J_n)$ witness TRHP for $C_n, n \in \omega$. Then for each $n \in \omega$

$$(\tilde{C}_n \setminus I_n) \times (Y \setminus J_n) \subseteq C_n \subseteq (\tilde{C}_n \times Y) \cup (I_n \times Y) \cup (X \times J_n).$$

Springer
Then
\[
\left( \bigcup_{n \in \omega} \tilde{C}_n \setminus \bigcup_{n \in \omega} I_n \right) \times \left( \bigcup_{n \in \omega} (\tilde{C}_n \setminus I_n) \times (Y \setminus J_n) \right) \subseteq \bigcup_{n \in \omega} (\tilde{C}_n \times Y) \cup (I_n \times Y) \cup (X \times J_n) \\
\subseteq \left( \bigcup_{n \in \omega} C_n \right) \times Y \cup \left( \bigcup_{n \in \omega} I_n \right) \times Y \\
\cup \left( X \times \left( \bigcup_{n \in \omega} J_n \right) \right).
\]

Hence, setting \( \tilde{C} = \bigcup_{n \in \omega} \tilde{C}_n, I = \bigcup_{n \in \omega} I_n, J = \bigcup_{n \in \omega} J_n \) completes this part of the proof.

Now let \( C = D^c \) for \( D \) witnessing TRHP with \((\tilde{D}, I, J)\). We have
\[
(\tilde{D} \setminus I) \times (Y \setminus J) \subseteq D \subseteq (\tilde{D} \times Y) \cup (I \times Y) \cup (X \times J).
\]

Through complementation
\[
((\tilde{D} \times Y) \cup (I \times Y) \cup (X \times J))^c \subseteq C \subseteq ((\tilde{D} \setminus I) \times (Y \setminus J))^c.
\]

Let us focus on the right-hand side
\[
((\tilde{D} \setminus I) \times (Y \setminus J))^c \subseteq ((\tilde{D} \times Y) \cup (I \times Y) \cup (X \times J))^c.
\]

Now the left-hand side. It is an intersection of the following sets
\[
(\tilde{D} \times Y)^c = (\tilde{D})^c \times Y, \quad (I \times Y)^c = (X \setminus I) \times Y, \quad (X \times J)^c = X \times (Y \setminus J),
\]

which is equal to
\[
((\tilde{D})^c \setminus I) \times (Y \setminus J).
\]

In summary
\[
((\tilde{D})^c \setminus I) \times (Y \setminus J) \subseteq C \subseteq ((\tilde{D})^c \times Y) \cup (X \times J) \cup (I \times Y).
\]

Then \(((\tilde{D})^c, I, J)\) witnesses TRHP for \( C \). The proof is complete. \( \square \)
Theorem 23 Let $C = A \otimes B$. Then

(i) if $A = I \cup I^*$ then $(C, I \otimes J)$ has WRHP.
(ii) if $B = J \cup J^*$ then $(C, I \otimes J)$ has TRHP.

Proof Let us prove (ii) (the proof of (i) is similar). Let us notice that rectangles from $C$ have TRHP. Indeed, if $C = A \times B$ then set $\tilde{C} = A$, $I = \emptyset$, $J = B^c$ if $B \in J^*$.

The rest of the proof relies on Lemma 22 which allows us to perform an induction over the hierarchy of sets making up the $\sigma$-algebra $C$. $\square$

Theorem 24 Let $C = A \otimes B$ and assume that

(i) $(C, I \otimes J)$ has TRHP and $(A, I)$ has SP, or
(ii) $(C, I \otimes J)$ has WRHP and $(B, J)$ has SP.

Then $(C, I \otimes J)$ has SP.

Proof Assume (i). Let $D \subseteq X \times Y$ be dense, set $D_1 = \pi_1(D)$ and let $B \in C$ be $I \otimes J$-positive. Then there are $\tilde{B} \in A \setminus J$ and $J \in J$ such that $\tilde{B} \times (Y \setminus J) \subseteq B$. It follows that

$$D + B \supseteq D + (\tilde{B} \times (Y \setminus J)) \supseteq \bigcup_{d_1 \in D_1} \bigcup_{d_2 \in D_1} (d_1 + \tilde{B}) \times (d_2 + Y \setminus J)).$$

Therefore for every $x \in D_1 + \tilde{B}$ the set $(D + B)_x$ contains a translation of $Y \setminus J$. By SP $D_1 + \tilde{B} \in I^*$ thus $D + B$ is $I \otimes J$-residual.

The proof assuming (ii) is analogous. $\square$

4 Maximal invariant $\sigma$-ideals with Borel bases

There is a surprising connection between maximal invariant $\sigma$-ideals with Borel bases and Smital properties.

Proposition 25 The following are equivalent:

(i) $I$ has VWSP;
(ii) $I$ is maximal among invariant proper $\sigma$-ideals with Borel base.

Proof (i) $\Rightarrow$ (ii) : Let us suppose that $J \supseteq I$ is such an ideal. Let $A$ be a Borel set from $J \setminus I$. Then there exists a countable set $D$ such that $D + A$ is $I$-residual, therefore $J$-residual, hence $J$ is not proper.

(ii) $\Rightarrow$ (i) : Let us suppose that there is a set $B \in Bor \setminus I$ for which $B + D$ is not $I$-residual for every countable set $D$. Consider the family

$$I' = \{ A \cup C : A \in I \land (\exists D)(C \subseteq D + B \land |D| \leq \omega) \}.$$
Ideals with Smítal properties

It is an invariant proper $\sigma$-ideal satisfying $I \subset I'$, which leads to a contradiction. 

Proposition 26 Let $\{I_n : n \in \omega\}$ be a countable family of pairwise distinct maximal invariant $\sigma$-ideals on $X$ with Borel bases. Then for each $n \in \omega$ the $\sigma$-ideal $I_n$ is orthogonal to $\bigcap_{k \in \omega \setminus \{n\}} I_k$.

Proof Fix $n \in \omega$. There are sets $A_k \in (\text{Bor}(X) \cap I_n) \setminus I_k$, $k \in \omega \setminus \{n\}$. By Proposition 25 for each of them there is a countable set $C_k$ such that $A_k + C_k \in I_n \cap I_k^*$. Therefore

$$\bigcup_{k \in \omega \setminus \{n\}} (A_k + C_k) \in I_n \cap \bigcap_{k \in \omega \setminus \{n\}} I_k^*.$$

Corollary 27 Let $I$ be a maximal invariant $\sigma$-ideal with a Borel base different from $M$ and $N$. Then there is $A \in I \cap (M \cap N)^*$.

Let us now focus on $X = 2^\omega$. The following result incorporates techniques similar to those used in [10, Theorem 3.1].

Theorem 28 There are $c$ many maximal invariant $\sigma$-ideals on $2^\omega$.

Proof Let $\{A_\alpha : \alpha < c\}$ be an AD family on $\omega$, i.e. for all distinct $\alpha, \beta < c$ the set $A_\alpha \cap A_\beta$ is finite. For every $\alpha < c$ set

$$I_\alpha = \{A \subseteq 2^\omega : (\exists B \in \text{Bor}(2^\omega)A \subseteq B \land (\exists M \in M(2^{A_\alpha}))(\forall x \in 2^{A_\alpha})
(x \notin M \rightarrow B^a_x \in N(2^\omega \setminus A_\alpha))\},$$

where $B^a_x = \{y \upharpoonright \omega \setminus A_\alpha : y \in B, \ y \upharpoonright A_\alpha = x\}$.

Let us show that $I_\alpha \neq I_\beta$ if $\alpha \neq \beta$. Set

$$M \in M(2^{A_\alpha \setminus A_\beta}) \setminus N(2^{A_\alpha \setminus A_\beta}),$$

$$C = \{x \in 2^\omega : x \upharpoonright (A_\alpha \setminus A_\beta) \in M\}.$$

Notice that $C \in I_\alpha$. The set

$$M_\alpha = \bigcup_{t \in 2^{A_\alpha \setminus A_\beta}} \{y \in 2^{A_\alpha} : y \upharpoonright (A_\alpha \cap A_\beta) = t \land y \upharpoonright (A_\alpha \setminus A_\beta) \in M\}$$

is a finite union of meager sets, hence meager. For each $x \in 2^{A_\alpha \setminus M_\alpha}$ the set $C^a_x$ is empty.

On the other hand for each $x \in 2^{A_\beta}$

$$C^\beta_x = \{y \in 2^{\omega \setminus A_\beta} : y \upharpoonright (A_\alpha \setminus A_\beta) \in M\}.$$
The above set may be considered as a product $M \times 2^{\omega \setminus (A_{\alpha} \cup A_{\beta})}$ of non-null set and the whole space, which is not null.

Now we will show that every $I_\alpha$ is maximal among invariant $\sigma$-ideals on $2^{\omega}$ with Borel bases. Each $I_\alpha$ is essentially $\mathcal{M}(2^{A_{\alpha}}) \otimes \mathcal{N}(2^{\omega \setminus A_{\alpha}})$. It follows that $I_\alpha$ has WSP, thus by Proposition 25 the proof is complete. \hfill $\square$

The reasoning in the above Theorem does not translate to the case of $\mathbb{R}$. We may ask the following question.

**Question 29** Are $\mathcal{M}$ and $\mathcal{N}$ the only maximal invariant $\sigma$-ideals with Borel bases in $\mathbb{R}$?

**Question 30** Is it true that for every set $G \in (\mathcal{M} \cap \mathcal{N})^*$ there is a countable set $C$ such that $C + G = \mathbb{R}$?

A positive answer to Question 30 would also answer positively Question 29.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

References

1. Balcerzak, M., Kotlicka, E.: Steinhaus property for products of ideals. Publicationes Mathematicae Debrecen 63(1–2), 235–248 (2003)
2. Bartoszewicz, A., Filipczak, M., Natkaniec, T.: On smital properties. Topol. Appl. 158, 2066–2075 (2011)
3. Borodulin-Nadzieja, P., Głąb, SZ: Ideals with bases of unbounded Borel complexity. Math. Log. Quart. 57(6), 582–590 (2011)
4. Cichoń, J., Szymański, A., Węglorz, B.: On intersections of sets of positive Lebesgue measure. Colloq. Math. 52(2), 173–174 (1987)
5. Cieślak, A., Michalski, M.: Bulletin of the polish academy of sciences. Mathematics 66, 157–166 (2018)
6. Erdős, P., Oxtoby, J.C.: Partitions of the plane into sets having positive measure in every non-null measurable product set. Trans. Am. Math. Soc. 79(1), 91–102 (1955)
7. Kechris, A.S.: Classical descriptive set theory, Graduate Texts in Mathematics 156. Springer, New York (1995)
8. Michalski M., Żeberski Sz: Some properties of $\mathcal{I}$-Luzin sets. Topol. Appl. 189, 122–135 (2015)
9. Srivastava, S.M.: A course on Borel sets, Graduate Texts in Mathematics 180. Springer, New York (1998)
10. Zakrzewski, P.: On invariant ccc $\sigma$-ideals on $2^{\mathbb{N}}$. Acta Math. Hungar. 143, 367–377 (2014)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.