INFRA-RED FINITE CHARGE PROPAGATION

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Abstract The Coulomb gauge has a long history and many uses. It is especially useful in bound state applications. An important feature of this gauge is that the matter fields have an infra-red finite propagator in an on-shell renormalisation scheme. This is, however, only the case if the renormalisation point is chosen to be the static point on the mass-shell, $p = (m,0,0,0)$. In this letter we show how to extend this key property of the Coulomb gauge to an arbitrary relativistic renormalisation point. This is achieved through the introduction of a new class of gauges of which the Coulomb gauge is a limiting case. A physical explanation for this result is given.

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In this letter we will study the propagation of a charged particle, such as an electron, in a mass-shell renormalisation scheme. To motivate this study, let us first recall the details of this calculation in a Lorentz gauge with gauge parameter $\xi$. Two renormalisation constants must be introduced: a mass shift, $m \rightarrow m - \delta m$, and a fermion wave function renormalisation, $\psi \rightarrow \sqrt{Z_2} \psi$. The mass shift, which we find by requiring the presence of a pole at the physical mass, is found to be gauge parameter independent as one would expect since the electron mass is physical. Problems arise when one now tries to demand that the residue of the pole be unity: the $Z_2$ renormalisation constant depends on the unphysical gauge parameter $\xi$ and has in general an infra-red divergence, which obscures the physical content of the theory.

Another way to see that there is a problem is to consider the general form of the propagator around the mass shell. Renormalisation group arguments indicate that the one-loop renormalised propagator near the mass shell has the form

$$\frac{(p^2 - m^2)^\beta}{p_\mu \gamma^\mu - m}, \quad \text{with} \quad \beta = \frac{e^2(-\xi - 3)}{8\pi^2}, \quad (1)$$

and we see that a pole structure emerges only in the Yennie gauge. However, even for this gauge the resulting propagator cannot be identified with that describing charge propagation.

A gauge which appears not to have these problems is the Coulomb gauge (for various treatments of QED in this gauge see Ref.’s 6-9). Here one obtains the same mass shift as in covariant gauges and $Z_2$ is infra-red finite. There is, however, an often unappreciated subtlety here: infra-red finiteness only holds if we are at the static point on the mass shell, i.e., if we demand $p = (m, 0, 0, 0)$. Details of this calculation can be found in Sect. 6 of Ref. 10. The form of the propagator in Coulomb gauge near the mass shell is quoted in Ref. 5. Although this, like (1), is in general not a simple pole, in the static limit their formula indeed reduces to a pole.

Such kinematical configurations naturally arise in many bound state problems. Indeed, even in QCD, the utility of this gauge in the calculation of the static inter-quark potential is well known (see, for example Ref. 11). A generalisation of the Coulomb gauge that allowed us to perform an on-shell renormalisation of the electron propagator at an arbitrary, relativistic point on the mass shell, i.e., for $p = m\gamma(1, v)$ where $\gamma = 1/\sqrt{1 - v^2}$, would improve our understanding of these fundamental topics, help extend the feasibility of such
bound state calculations and could be of use in the heavy quark effective theory where the
heavy quark has a well-defined velocity. We will provide such a generalisation below.

Our class of gauges, adapted to motion in the \( x^1 \)-direction, is described by the condition

\[
\gamma^{-2} \partial_1 A_1 + \partial_2 A_2 + \partial_3 A_3 + v^1 [\partial_0 A_1 - \partial_1 A_0] = 0,
\]

(2)

where \( v = (0, v^1, 0, 0) \). We will explain the physical arguments that lead to this gauge
choice in the conclusion. It is clear that in the limit \( v \to 0 \) this condition reduces to the
Coulomb gauge. However, we note that this class of gauges may not be reached by a boost
from the Coulomb gauge. The vector boson propagator in this gauge is

\[
D_{\mu \nu}^v = \frac{1}{k^2} \left\{ - g_{\mu \nu} + \frac{(1 - \xi) k^2 - [k \cdot (\eta - v)]^2 \gamma^{-2}}{[k^2 - (k \cdot \eta)^2 + (k \cdot v)^2]^2} k_\mu k_\nu \\
- \frac{k \cdot (\eta - v)}{k^2 - (k \cdot \eta)^2 + (k \cdot v)^2} [k_\mu (\eta + v)_\nu + (\eta + v)_{\mu} k_\nu] \right\},
\]

(3)

where \( \xi \) is a (smearing) gauge parameter that we set to zero in what follows and \( \eta = (1, 0, 0, 0) \) is a unit temporal vector. We are not aware of work on this class of gauges
previous to our studies.

The electron propagator in this gauge may now be directly calculated. The self-energy
has the form

\[
-i \Sigma = e^2 \int \frac{d^2 \omega k}{(2\pi)^2} \left\{ \frac{1}{k^2} \left( \frac{1}{m^2 - (p - k)^2} - \frac{1}{m^2 - (p - k)^2} \right) \left[ 2(\omega - 1)\hat{p} - 2\omega m - 2(\omega - 1)\hat{k} \right] \\
+ \frac{1}{k^2} \left( \frac{1}{(p - k)^2 - m^2} - \frac{1}{m^2} \right) \left[ - 2(\hat{p} - m) \right] \\
+ \frac{1}{(p - k)^2 - m^2} \frac{1}{k^2 - (k \cdot \eta)^2 + (k \cdot v)^2} \left[ 2(\hat{p} - m) + (\hat{\eta} + \hat{\eta}) k \cdot (\eta - v) \right] \\
+ \frac{1}{(p - k)^2 - m^2} \frac{1}{k^2 - (k \cdot \eta)^2 + (k \cdot v)^2} \left[ \gamma^{-2} (k \cdot \eta - k \cdot v)^2 - k^2 \right] \\
+ \frac{1}{k^2} \left( \frac{1}{(p - k)^2 - m^2} - \frac{1}{k^2 - (k \cdot \eta)^2 + (k \cdot v)^2} \right) \left[ - (p^2 - m^2)(\hat{\eta} + \hat{\eta}) k \cdot (\eta - v) - 2\hat{k} \cdot k \cdot (\eta - v) p \cdot (\eta + v) \right] \\
+ \frac{1}{k^2} \left( \frac{1}{(p - k)^2 - m^2} - \frac{1}{k^2 - (k \cdot \eta)^2 + (k \cdot v)^2} \right) (p^2 - m^2) \hat{k} \left[ k^2 - \gamma^{-2} (k \cdot \eta - k \cdot v)^2 \right] \right\}.
\]

(4)
This must be renormalised. With a little effort it may be seen that a simple multiplicative renormalisation is inadequate to the task. In this non-covariant gauge, this is not surprising. We find, however, that a *matrix multiplication* renormalisation scheme is appropriate. We use

\[ \psi \rightarrow \sqrt{Z_2} \exp \left\{ -i \frac{Z'}{Z_2} \sigma_{\mu\nu} \eta^{\mu} \nu^{\nu} \right\} \psi, \]

which is reminiscent of a naive Lorentz boost upon the fermion. We find it surprising and gratifying that this scheme is capable of multiplicatively renormalising the propagator in this highly non-covariant gauge. Note that we now have three renormalisation constants: \( \delta m \), \( Z' \) and \( Z_2 \). The possible counterterms are now (with \( Z_2 = 1 + \delta Z_2 \))

\[ -i \Sigma_{\text{counter}} = i \delta Z_2 (\not{p} - m) + 2i Z' (p \cdot \eta \not{\psi} - p \cdot v \not{\psi}) + i \delta m. \]

and the ultra-violet divergences of the self-energy are found to have a similar form\(^1\)

\[ -i \Sigma_{\text{UV}} = i \frac{\alpha}{4\pi} \frac{1}{2 - \omega} \left\{ -3m + (\not{p} - m) \left[ -3 - 2\chi(v) \right] \right. \]

\[ + 2 \left( p \cdot v \not{\psi} - p \cdot \eta \not{\psi} \right) \left[ \frac{1}{v^2} + \frac{1 + v^2}{2v^2} \chi(v) \right] \right\}, \]

in \( 2\omega \) dimensions with \( \alpha = (m^2)\omega^{-2}e^2/4\pi \) and \( \chi(v) = |v|^{-1}\ln\{(1 - |v|)/(1 + |v|)\} \).

The renormalised self-energy (i.e., including both loops and counter terms) may be written as

\[ -i \Sigma = m\alpha + \not{p} \beta + p \cdot \eta \delta + m\psi \epsilon, \]

where \( \alpha, \ldots, \epsilon \) are functions depending on \( p^2, p \cdot \eta, p \cdot v \) and \( v^2 \). Insisting that there is a pole at the physical mass, \( m \), fixes the mass shift condition. We find that with \( p^2 = m^2 \) for *any* values of \( p \cdot \eta, p \cdot v \) and \( v^2 \) the condition

\[ \tilde{\alpha} + \tilde{\beta} + \frac{(p \cdot \eta)^2}{m^2} \tilde{\delta} + \frac{p \cdot v}{m} \tilde{\epsilon} = 0, \]

(the tildes signify that we put the momentum \( p^2 \) on shell in the self-energy: \( p^2 = m^2 \)) yields \( \delta m = \alpha (3/\tilde{\epsilon} + 4)m/(4\pi) \) which is just the standard result for an arbitrary Lorentz gauge. (Note that \( 1/\tilde{\epsilon} = 1/(2 - \omega) - \gamma_E + \ln 4\pi. \) This gauge invariant result provides a check on our calculations.

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\(^1\) The integration procedures and detailed results will be presented elsewhere\(^{[12]}\).
In the on-shell scheme we now must demand that the residue of this pole is unity. This we recall is where infra-red divergences may lurk. It appears, however, as though non-covariance will immediately lead to other problems. This is because requiring that the renormalised propagator has the form of the tree level propagator now yields three equations

\[ i\delta Z_2 - 2v^2 iZ' = \bar{\delta}_L - \bar{\beta}_L - 2m^2 \bar{\Delta}, \]
\[ i\delta Z_2 - 2iZ' = \gamma^{-1} \bar{\epsilon}_L - \bar{\beta}_L - 2m^2 \bar{\Delta}, \]
\[ i\delta Z_2 = -\bar{\alpha}_L - 2\bar{\beta}_L - 2m^2 \bar{\Delta}, \]

(10)

where

\[ \Delta(p \cdot \eta, p \cdot v, v^2) = \left( \frac{\partial \alpha}{\partial p^2} + \frac{\partial \beta}{\partial p^2} + \frac{(p \cdot \eta)^2}{m^2} \frac{\partial \delta}{\partial p^2} + \frac{p \cdot v}{m} \frac{\partial \epsilon}{\partial p^2} \right), \]

(11)

and the bars denote that the functions are evaluated at the point on the mass shell given by, \( p = m\gamma(\eta + v) \). Note further that the \( L \) subscript signifies that the functions \( \alpha_L \) etc. only contain loop and mass shift contributions. Since these are three equations and we now have only two unknowns, \( Z' \) and \( Z_2 \), we cannot in general proceed any further. However, at our chosen point on the mass shell we find that these equations reduce to two independent ones and our counterterms are now uniquely determined in terms of the loop contributions.

The result for \( Z' \) can be seen to be free of infra-red divergences, but that for \( Z_2 \) contains various infra-red divergent integrals. However, we find that the overall combination of such divergences is:

\[ m^2 \bar{\Delta}_{IR} = i \frac{\alpha}{4\pi} \int_0^1 du u^{2\omega - 5} \left\{ -2 + 2 \int_0^1 \frac{dx}{\sqrt{1-x} \sqrt{1-v^2 x}} \left[ 1 + v^2 - 2v^2 x \right] \right. \]
\[ \left. - (1 - v^2) \int_0^1 \frac{dx}{\sqrt{1-x} \sqrt{1-v^2 x}} \frac{3 + v^2 - 2v^2 x}{2(1-v^2 x)} \right\}, \]

(12)

where infra-red finite terms are ignored. The integrals over \( x \) now yield exactly +2 and so the infra-red divergences all cancel. We have thus performed a (matrix) multiplicative, infra-red finite, on-shell renormalisation of the electron propagator in this class of gauges. This concludes the first part of this letter.

We now want to consider scalar QED in these gauges. It is known\(^{[13]} \) that in covariant gauges the infrared structure of the propagator is independent of the spin of the field. Thus the scalar electron propagator is also infra-red finite in the covariant Yennie gauge. Here,
in a non-covariant formalism, things are not so clear. Furthermore for a scalar ‘electron’ we cannot have a multiplicative matrix renormalisation scheme such as Eq. (5): this further tightens the conditions our gauge must fulfill.

The procedure we have to follow should be clear: we use the same photon propagator but a very different (scalar) electron propagator and calculate the one-loop self-energy. (Note that the tadpole diagram vanishes.) We demand a pole at the physical mass and that its residue is unity. Our wave function renormalisation is now just \( \phi \rightarrow \sqrt{Z_2} \phi \).

The mass shift has again a gauge invariant value in this theory. The condition that the pole has unit residue now implies just one equation since we do not have any gamma matrix structure and we find

\[
\delta Z_2 = \frac{\alpha}{4\pi} \left\{ (6 + 2\chi(v)) \frac{1}{\epsilon} + 4 \left( 1 - \gamma^{-2}\chi(v) - \frac{1}{|v|} [L_2(|v|) - L_2(-|v|)] \right) \right\} ,
\]

with \( L_2 \) being the dilogarithm. Once again all the infra-red divergences have cancelled and an on-shell, multiplicative renormalisation has been achieved. This is an important check of our earlier calculation and shows the power of the gauge we have introduced. We finally note that any attempt to renormalise at a different point on the mass shell leads to the appearance of infra-red divergences. This indicates the sensitivity of the above cancellation of the infra-red singularities.

To conclude this letter let us offer an explanation of why these gauges (parameterized by \( v \)) possess such attractive properties. We have seen that in the gauge (2) the propagator for the matter field (either scalar or spinor) is infra-red finite at the appropriate (moving) point on the mass shell. We recall that the infra-red problem reflects the fact that physical charged particles are always dressed with an electromagnetic cloud which the Lagrangian fermion or scalar field does not properly reflect — unless, as we now argue, our gauge is used.

We remember that we have also noted that the Yennie gauge propagator shares this property for the whole of the mass shell. However, in that case a connection with the propagation of charged particles is not visible. For our class of gauges, though, such a connection can be made\(^\text{[10]}\).

Recall that the gauge transformation that takes an electron into Coulomb gauge is just

\[
\psi(x) \rightarrow \exp \left( -ie \partial_i A_i / \nabla^2 \right) \psi(x) .
\]

(14)
The canonical commutator of the electric field with this transformed object gives the Coulomb electric field expected of a static charge\textsuperscript{[14,15]}. The general form of this transformation is in accord with what is known from rigorous studies\textsuperscript{[16–21,10]} of charged particles in gauge theories; i.e., that a charge is accompanied by a non-local electromagnetic cloud whose non-covariant form reflects the known difficulties in reconciling Lorentz transformations and gauge transformations for non-local, non-observable quantities. What we have used in this letter is a generalisation of Eq. 14, i.e., the transformation into our gauge condition (2) is such that\textsuperscript{[10]} the canonical commutators of the electric or magnetic fields with the gauge transformed charged matter fields give the results expected of a charge moving with velocity $v$. In other words the charged fields in the gauge we have introduced are automatically accompanied by the dressing which surrounds a charged particle moving with velocity, $(v_1,0,0)$. The reason that the gauge ‘works’ for both fermions and scalars is that the magnetic fields associated with the fermion’s anomalous magnetic moment fall off more rapidly than $1/r$ and thus these interactions do not lead to infra-red problems\textsuperscript{[22]}.

Although the argument used to construct the gauge choice (2) is essentially classical, we have seen that our quantum calculations solidly back up the idea that such fields may play the role of good asymptotic fields in gauge theory as was predicted in Ref. 10.

**Note added:** After this paper was completed we received a letter from R. Jackiw drawing our attention to his paper with L. Soloviev\textsuperscript{[23]}. Using the exponentiation of the soft divergences property found there, one can convince oneself that the infra-red finiteness property demonstrated here will hold at all orders. We thank Prof. Jackiw for this reference. The next stage in this programme is to consider vertices: here we must associate a different (velocity dependent) dressing to each charged leg and no gauge choice can remove all the dressings. Since we wrote this letter we have seen in explicit one-loop calculations that the soft divergences in the vertex can be removed by incorporating dressings. This and all-orders arguments will be presented elsewhere.

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