Brane Death via Born-Infeld String

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Abstract

I revisit the solution of Born-Infeld theory which corresponds to a 3-brane and anti-brane joined by a (fundamental) string. The global instability of this configuration makes possible the semiclassical tunneling into a wide, short tube which keeps expanding out, thus annihilating the brane. This tunneling is suppressed exponentially as \( \exp\left\{-\frac{S_{cl}}{\hbar}\right\} \). The attraction between the branes causes them to approach and at some point to tunnel, because the action of the bounce solution goes to zero. The energy of the solution at the top of the barrier, the sphaleron, goes like \( \sim D^3 \) for large separarions \( D \), while the energy of the string is proportional to its length \( D \).
1 Introduction

Recently, Callan and Maldacena [1] considered among other configurations the 3-brane and anti-brane joined by a (fundamental) string in the framework of Born-Infeld theory. The string is made of a 3-brane, wrapped around $S^2$ sphere. When looked from some distance, such an object does not appear to carry RR charge as a whole, but is rather like a RR dipole, and has energy per unit length proper to the fundamental string.

Such a configuration is only quasi-stable, since globally it is possible to lose energy by making the throat very wide: if it had radius $R$, the change in energy is mostly due to tension $\mathcal{T}$ and goes like $\sim R^2 \cdot D - R^3$ and is arbitrarily negative. However, there is a potential energy barrier and one needs to construct the bouncing euclidean solution in order to address the problem in a complete way. In [1] it was attempted to approach the problem by dropping the contributions due to the electric field, in that case the lagrangian is Lorentz invariant with respect to $r,t$ and it is possible to construct some approximation to the bouncing solution.

In this paper I will compute exactly the energy of the string and sphaleron solution (the unstable static solution at the top of the potential barrier). This will allow to conclude that the tunneling rate in fact goes to infinity when the branes approach each other but still are at a finite distance $\sim \sqrt{A}$.

2 The Two Static Configurations

I will review the construction of the relevant solution from [1]. Similar solutions were also considered by Gibbons in [3].

Consider the case when the worldbrane gauge field is purely electric and only one transverse coordinate $X$ is excited. The worldbrane action reduces to

$$L = -\frac{1}{g_p} \int d^4x \sqrt{(1 - \vec{E}^2)(1 + (\vec{\nabla}X)^2) + (\vec{E}\vec{\nabla}X)^2} - \dot{X}^2$$  \hspace{0.5cm} \text{(1)}

where $g_p = (2\pi)^3 g$, and $g$ is the string coupling ($\alpha' = 1$).

The canonical momentum associated with $\vec{A}$ is

$$g_p\vec{\Pi} = \frac{\vec{E}(1 + (\vec{\nabla}X)^2) - \vec{\nabla}X(\vec{E}\vec{\nabla}X)}{\sqrt{(1 - \vec{E}^2)(1 + (\vec{\nabla}X)^2) + (\vec{E}\vec{\nabla}X)^2}}$$.  \hspace{0.5cm} \text{(2)}

The constraint equation is $\vec{\nabla} \cdot \vec{\Pi} = 0$. The Hamiltonian is

$$H = \frac{1}{g_p} \sqrt{(1 + (g_p\vec{\Pi})^2)(1 + (\vec{\nabla}X)^2) - (g_p\vec{\Pi} \times \vec{\nabla}X)^2}$$  \hspace{0.5cm} \text{(3)}

\hspace{0.5cm} \text{The second term has the same origin as in [2]. The two branes act as a sort of capacitor to create a uniform bulk 3-form RR field to which the cylindrically wrapped brane couples. The only difference is dimensionality: in case of the 2-brane the potential goes as $-R^2$, and for the 3-brane it’s $\sim -R^3$.}
We are looking for the most general static, spherically symmetric solution. The equation for $X$, which follows from varying the energy, after setting $\dot{X} = 0$ is

$$
\nabla \frac{(1 - \vec{E}^2) \nabla X + \vec{E}(\vec{E} \nabla X)}{\sqrt{(1 - \vec{E}^2)(1 + (\vec{\nabla} X)^2) + (\vec{E} \nabla X)^2}} = 0 .
$$

(4)

From (2) follows that $g_p \Pi = A \vec{r} r^3$, and from (4) $\nabla X \sqrt{1 - \vec{E}^2 + (\nabla X)^2} = B \vec{r} r^3$. Here $A$ and $B$ are integration constants.

Here $g_p \Pi = \vec{E} \sqrt{1 - \vec{E}^2 + (\nabla X)^2}$.

The solution for the coordinate and electric field is

$$
\nabla X = \frac{B \vec{r}}{\sqrt{r^4 + A^2 - B^2}} , \quad \vec{E} = \frac{A \vec{r}}{\sqrt{r^4 + A^2 - B^2}} .
$$

(5)

One can view this solution as a way to minimally break supersymmetry, instead of $E = \nabla X$ we have $E = \frac{A}{B} \nabla X$. In principle, $A$ should be quantized as electric charge in units of $\pi g$. We will be interested in $B > A$, in this case the resulting configuration will be the 3-brane and anti-brane joined by a smooth throat. To see this one needs to explicitly exploit the geometry by finding $X$:

$$
X(r) = B \int_r^\infty \frac{ds}{\sqrt{s^4 - r_0^4}} .
$$

(6)

Here $r_0^4 = B^2 - A^2$. We have set $X(\infty) = 0$, e.g. far away the brane is flat and is at zero coordinate in the transverse direction. $X(r_0)$ is finite, but $X'(r_0)$ is infinite: the throat becomes vertical at that radius. This can be continued back out through $r_0$ to give the two branes. Branes possess orientation and continuing it through the throat we see that the new brane is of the opposite orientation: an antibrane.

The relations between $A,B$ and $r_0, X(r_0) = D/2$ can be reversed to express $r_0$ and $B$ in terms of $D$ and $A$:

$$
D/2 = c \sqrt{\frac{A^2}{r_0^2} + r_0^2} \quad \text{and} \quad B^2 = A^2 + r_0^4 , \quad \text{where} \quad c = \int_1^\infty \frac{dz}{\sqrt{z^4 - 1}} .
$$

(7)

In the limit of large $D$ the two possible radii at the throat are $r_0 \sim D/2c$, and $A 2c/D$. A remark is in place here that the minimal separation for which a real root exists is $D_{\text{min}} = 2c \sqrt{2A}$.

Knowing the energy function $H = \sqrt{(1 + \nabla X^2)(1 + g \Pi^2)}$ allows to compute the energies of the solutions exactly,

$$
E_{\text{tot}} = \frac{1}{g_p} \int_{r_0}^\infty \frac{1 + A^2 + r_0^4}{r^4 - r_0^4} \sqrt{1 + \frac{A^2}{r^4}} \ 4\pi r^2 \ dr .
$$

(8)

At this point the temptation to make the problem completely dimensionless becomes irresistible. Let me introduce the parameters $\mu = \frac{D}{2c \sqrt{A}}$, $r = z \sqrt{A}$. I am now measuring length in units of $\sqrt{A}$, energy in $A^{3/2}$, and also $r_0 = \xi \sqrt{A}$:
\[ E_{1,2} = \frac{1}{g_p} \int_{\xi_{1,2}}^{\infty} \sqrt{1 + \frac{1 + \xi^4}{z^4 - \xi^4}} \sqrt{1 + \frac{1}{z^4}} \, 4\pi z^2 \, dz . \] 

(9)

One can check that \( \xi_1 \cdot \xi_2 = 1 \), because (7) becomes \( \mu^2 = \xi^2 + 1/\xi^2 \). The minimum separation is \( \mu_{\text{min}} = \sqrt{2} \), at which point the two roots of the quadratic equation become degenerate: \( \xi_1 = \xi_2 = 1 \).

After some manipulations with the energy integral, and a variable change \( y = \xi/z \), I get

\[ \frac{g_p}{8\pi} E = \xi^3 \int_0^1 \frac{dy}{y^4 \sqrt{1 - y^4}} + \frac{1}{\xi} \int_0^1 \frac{dy}{\sqrt{1 - y^4}} . \] 

(10)

I have not dealt yet with the volume infinity which manifests itself in the fact that the first of these integrals is divergent at \( y \to 0 \). Regularize it by substracting \( 1/4^n \) from the integrand: if one were to go back, it is exactly equivalent to computing the energy with respect to the configuration when the two branes are flat, parallel and not joined by any string. Denote the integrals in (10) as \( u \) and \( v \),

\[ u = \int_0^1 \frac{dy}{y^4 \sqrt{1 - y^4}} - 1 - \int_1^\infty \frac{dy}{y^4} = -\frac{1}{3} + \sum_{n=1}^\infty \frac{1}{4n-3} \frac{(2n-1)!!}{n! 2^n} \]

\[ u = \frac{\sqrt{\pi} \Gamma(-3/4)}{4 \Gamma(-1/4)} = 0.43701 \ldots \]

\[ v = \int_0^1 \frac{dy}{\sqrt{1 - y^4}} = \frac{\sqrt{\pi} \Gamma(5/4)}{\Gamma(3/4)} = 1.31103 \ldots \] 

(11)

The answers are not surprising, since both integrals are generalized B functions, and the second one is in fact the quarter period of the Jacoby elliptic functions. One can also show, by using the functional properties of the Gamma function, that \( 3u = v \). Also, constant \( c \) from (6) is also related, in fact \( c = v \). The energies of the string and the sphaleron become, respectively

\[ E_{\text{string}} = u(\xi^{-3} + 3\xi) \quad E_{\text{sph}} = u(\xi^3 + 3\xi^{-1}) \]

Note, that I have now taken the larger of the two roots to be \( \xi \), the other root being \( 1/\xi \). One can even write a relation between the energies of the string and sphaleron solutions \( E_{\text{string}}(1/\xi) = E_{\text{sph}}(\xi) \). Even though this relation is formal, (each function is defined only for arguments larger than one) it might be of significance in the future. The energy of a long string after fully substituting the dimensionful units becomes

\[ E_{\text{str}} = \frac{4\pi}{g_p} DA = \frac{4\pi}{8\pi^3 g} D\pi g = \frac{1}{2\pi} D \]

. This reproduces the correct tension of the fundamental string. The slope of the energy function is zero at the minimum possible value of \( \xi = 1 \) (Fig 1).

One further interesting elaboration is possible. The energy surface in the parameter space of \( r_0 \) and \( D \) (or \( \mu, \xi \)) apparently has a kink well known from catastrophe theory. The point at which the kink first appears corresponds to \( \mu^2 = 2 \) and \( \xi = 1 \). For \( \mu^2 < 2 \) the energy is monotonous in \( \xi \), and there is no static solutions, otherwise it has one minimum and one maximum, corresponding to the string and the sphaleron respectively.
3 Annihilation by Tunneling.

Let us recall that two branes, a 3-brane and an anti-brane, are going to gravitationally attract, even though weakly, causing them to move closer and to eventually annihilate. The tunneling through the potential barrier, the sphaleron being the unstable solution at the top, will be exponentially weak \[ \mu \], but as the branes move closer \( D \) becomes smaller. So are \( \mu \) and \( \xi \). This process makes the tunneling easier in a twofold way: first by making the barrier narrower (distance between \( \xi \) and \( 1/\xi \) becomes smaller), and second, the energy at the top decreasing to become equal to the energy of the string. As \( \xi \to 1 \), the rate of tunneling rate increases indefinitely, and at \( \xi = 1 \) the metastable string state ceases to exist. This is an interesting example of physical continuity: before the state disappeared as we change the parameter \( (D \text{ or } \xi) \), its physical width due to tunneling had to become infinite.

Unfortunately, even though these conclusions are self consistent, they may still be physically incomplete due to the fact that as branes approach to planckian distance new nonperturbative phenomena of annihilation may kick in. One possible way out of this difficulty is to crank up the string coupling, this leads into unknown territory too but our quasi-classical tunneling may indeed be the dominant mode of annihilation in that regime.

4 Acknowledgements

This work was finished in December 1997, I decided that these calculationd might be interesting after similar formulas appeared in [4] and [5] in calculations of the quark-anti-quark potential. I am grateful to Professor C.Callan for introducing me to this area of research and in fact suggesting this problem. I would also like to thank W.Taylor, G.Savvidy and S.Lee for useful discussions.

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