On mappings in the Orlicz-Sobolev classes

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IN MEMORY OF ALBERTO CALDERON (1920–1998)

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Abstract

First of all, we prove that open mappings in Orlicz-Sobolev classes $W^{1,\varphi}_{\text{loc}}$ under the Calderon type condition on $\varphi$ have the total differential a.e. that is a generalization of the well-known theorems of Gehring-Lehto-Menchoff in the plane and of Väisälä in $\mathbb{R}^n$, $n \geq 3$. Under the same condition on $\varphi$, we show that continuous mappings $f$ in $W^{1,\varphi}_{\text{loc}}$, in particular, $f \in W^{1,p}_{\text{loc}}$ for $p > n - 1$ have the $(N)$-property by Lusin on a.e. hyperplane. Our examples demonstrate that the Calderon type condition is not only sufficient but also necessary for this and, in particular, there exist homeomorphisms in $W^{1,n-1}_{\text{loc}}$ which have not the $(N)$-property with respect to the $(n-1)$-dimensional Hausdorff measure on a.e. hyperplane. It is proved on this base that under this condition on $\varphi$ the homeomorphisms $f$ with finite distortion in $W^{1,\varphi}_{\text{loc}}$ and, in particular, $f \in W^{1,p}_{\text{loc}}$ for $p > n - 1$ are the so-called lower $Q$-homeomorphisms where $Q(x)$ is equal to its outer dilatation $K_f(x)$ as well as the so-called ring $Q^*$-homeomorphisms with $Q^*(x) = [K_f(x)]^{n-1}$. This makes possible to apply our theory of the local and boundary behavior of the lower and ring $Q$-homeomorphisms to homeomorphisms with finite distortion in the Orlicz-Sobolev classes.

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1 Introduction

Moduli provide us the main geometric tool in the mapping theory. The recent development of the moduli method in the connection with modern classes of mappings can be found in the monograph [115], see also recent books in the moduli and capacity theory [7], [27] and [176] as well as the following papers and monographs [5], [79], [101], [161], [163], [169], [185] and further references therein. In the present paper we show that the theories of the so-called lower and ring $Q$-homeomorphisms developed in [115] can be applied to a wide range of mappings with finite distortion in the Orlicz-Sobolev classes. Note that the plane case has been recently studied in [84] and [106].

Recall, it was established therein that each homeomorphism of finite distortion in the plane is a lower and ring $Q$-homeomorphism with $Q(x) = K_f(x)$.

In what follows, $D$ is a domain in a finite-dimensional Euclidean space. Following Orlicz, see [126], given a convex increasing function $\varphi : [0, \infty) \to [0, \infty), \varphi(0) = 0$, denote by $L_\varphi$ the space of all functions $f : D \to \mathbb{R}$ such that

$$\int_D \varphi \left( \frac{|f(x)|}{\lambda} \right) \, dm(x) < \infty \tag{1.1}$$

for some $\lambda > 0$ where $dm(x)$ corresponds to the Lebesgue measure in $D$, see also the monographs [86] and [181]. $L_\varphi$ is called the Orlicz space. If $\varphi(t) = t^p$, then we write $L_p$. In other words, $L_\varphi$ is the cone over the class of all functions $g : D \to \mathbb{R}$ such that

$$\int_D \varphi (|g(x)|) \, dm(x) < \infty \tag{1.2}$$
which is also called the **Orlicz class**, see [14].

The **Orlicz-Sobolev class** $W^{1,\varphi}_{\text{loc}}(D)$ is the class of all locally integrable functions $f$ given in $D$ with the first distributional derivatives whose gradient $\nabla f$ belongs locally in $D$ to the Orlicz class. Note that by definition $W^{1,\varphi}_{\text{loc}} \subseteq W^{1,1}_{\text{loc}}$. As usual, we write $f \in W^{1,p}_{\text{loc}}$ if $\varphi(t) = t^p$, $p \geq 1$. It is known that a continuous function $f$ belongs to $W^{1,p}_{\text{loc}}$ if and only if $f \in AC^1L^p$, i.e., if $f$ is locally absolutely continuous on a.e. straight line which is parallel to a coordinate axis and if the first partial derivatives of $f$ are locally integrable with the power $p$, see, e.g., 1.1.3 in [119]. The concept of the distributional derivative was introduced by Sobolev [162] in $\mathbb{R}^n$, $n \geq 2$, and now it is developed under wider settings, see, e.g., [3], [47], [50], [52], [54], [110], [115], [146], [171] and [172].

Later on, we also write $f \in W^{1,\varphi}_{\text{loc}}$ for a locally integrable vector-function $f = (f_1, \ldots, f_m)$ of $n$ real variables $x_1, \ldots, x_n$ if $f_i \in W^{1,1}_{\text{loc}}$ and

$$\int_D \varphi(|\nabla f(x)|) \, dm(x) < \infty$$

(1.3)

where $|\nabla f(x)| = \sqrt{\sum_{i,j} \left(\frac{\partial f_i}{\partial x_j}\right)^2}$. In the main part of the paper we use the notation $W^{1,\varphi}_{\text{loc}}$ for more general functions $\varphi$ than in the classical Orlicz classes giving up the condition on convexity of $\varphi$. In fact we need the convexity of $\varphi$ only in Section 13. Note that the Orlicz-Sobolev classes are intensively studied in various aspects, see, e.g., [2], [5], [13], [16], [18], [26], [41], [60], [64], [74], [85], [102], [103], [170] and [178].

Recall that a homeomorphism $f$ between domains $D$ and $D'$ in $\mathbb{R}^n$, $n \geq 2$, is called of **finite distortion** if $f \in W^{1,1}_{\text{loc}}$ and

$$\|f'(x)\|^n \leq K(x) \cdot J_f(x)$$

(1.4)

with a.e. finite function $K$ where $\|f'(x)\|$ denotes the matrix norm of the Jacobian matrix $f'$ of $f$ at $x \in D$, $||f'(x)|| = \sup_{h \in \mathbb{R}^n, |h|=1} |f'(x) \cdot h|$. 
and $J_f(x) = \det f'(x)$ is its Jacobian. Later on, we use the notation $K_f(x)$ for the minimal function $K(x) \geq 1$ in (1.4), i.e., we set $K_f(x) = \|f'(x)\|^n / J_f(x)$ if $J_f(x) \neq 0$, $K_f(x) = 1$ if $f'(x) = 0$ and $K_f(x) = \infty$ at the rest points.

First this notion was introduced on the plane for $f \in W^{1,2}_{1,2}$ in the work [66]. Later on, this condition was changed by $f \in W^{1,1}_{1,1}$ but with the additional condition $J_f \in L^1_{1,1}$ in the monograph [65]. The theory of the mappings with finite distortion had many successors, see, e.g., [6], [20], [21], [30], [35], [53–56], [59], [63], [64], [68], [71], [72], [73], [75–78], [108], [109], [123–125], [129] and [137–140]. They had as predecessors the mappings with bounded distortion, see [144] and [177], in other words, the quasiregular mappings, see, e.g., [51], [111] and [148]. They are also closely related to the earlier mappings with the bounded Dirichlet integral, see, e.g., [105] and [166–168], and the mappings quasiconformal in the mean which had a rich history, see, e.g., [1], [11], [12], [39], [40], [44], [46], [87–99], [130], [131], [132], [149–151], [155], [164], [165], [172], [183] and [184].

Note that the above additional condition $J_f \in L^1_{1,1}$ in the definition of the mappings with finite distortion can be omitted for homeomorphisms. Indeed, for each homeomorphism $f$ between domains $D$ and $D'$ in $\mathbb{R}^n$ with the first partial derivatives a.e. in $D$, there is a set $E$ of the Lebesgue measure zero such that $f$ satisfies $(N)$-property by Lusin on $D \setminus E$ and

$$\int_A J_f(x) \, dm(x) = |f(A)|$$  \hspace{1cm} (1.5)

for every Borel set $A \subset D \setminus E$, see, e.g., 3.1.4, 3.1.8 and 3.2.5 in [31]. On this base, it is easy by the Hölder inequality to verify, in particular, that if $f \in W^{1,1}_{1,1}$ is a homeomorphism and $K_f \in L^q_{1,1}$ for $q > n - 1$, then also $f \in W^{1,p}_{1,1}$ for $p > n - 1$, that we often use further to obtain corollaries.
In this paper \( H^k, k \geq 0 \), denotes the **k-dimensional Hausdorff measure** in \( \mathbb{R}^n, n \geq 1 \). More precisely, if \( A \) is a set in \( \mathbb{R}^n \), then

\[
H^k(A) = \sup_{\varepsilon > 0} H^k_\varepsilon(A),
\]

where the infimum in (1.7) is taken over all coverings of \( A \) by sets \( A_i \) with \( \text{diam} \ A_i < \varepsilon \), see, e.g., [118] in this connection. Note that \( H^k \) is an **outer measure in the sense of Caratheodory**, i.e.,

1. \( H^k(X) \leq H^k(Y) \) whenever \( X \subseteq Y \),
2. \( H^k(\bigcup X_i) \leq \sum_i H^k(X_i) \) for each sequence of sets \( X_i \),
3. \( H^k(X \cup Y) = H^k(X) + H^k(Y) \) whenever \( \text{dist}(X,Y) > 0 \).

A set \( E \subset \mathbb{R}^n \) is called **measurable** with respect to \( H^k \) if \( H^k(X) = H^k(X \cap E) + H^k(X \setminus E) \) for every set \( X \subset \mathbb{R}^n \). It is well known that every Borel set is measurable with respect to any outer measure in the sense of Caratheodory, see, e.g., Theorem II (7.4) in [156]. Moreover, \( H^k \) is Borel regular, i.e., for every set \( X \subset \mathbb{R}^n \), there is a Borel set \( B \subset \mathbb{R}^n \) such that \( X \subset B \) and \( H^k(X) = H^k(B) \), see, e.g., Theorem II (8.1) in [156] and Section 2.10.1 in [31]. The latter implies that, for every measurable set \( E \subset \mathbb{R}^n \), there exist Borel sets \( B_* \) and \( B^* \subset \mathbb{R}^n \) such that \( B_* \subset E \subset B^* \) and \( H^k(B^* \setminus B_*) = 0 \), see, e.g., Section 2.2.3 in [31]. In particular, \( H^k(B^*) = H^k(E) = H^k(B_*) \).

If \( H^{k_1}(A) < \infty \), then \( H^{k_2}(A) = 0 \) for every \( k_2 > k_1 \), see, e.g., VII.1.B in [61]. The quantity

\[
\dim_H A = \sup_{H^k(A) > 0} k
\]

is called the **Hausdorff dimension** of a set \( A \).

It is known that the outer Lebesgue measure \( m(A) = \Omega_n 2^{-n} H^n(A) \) for sets \( A \) in \( \mathbb{R}^n \) where \( \Omega_n \) denotes the volume of the unit ball in \( \mathbb{R}^n \), see [158].
It was shown in [38] that a set \( A \) with \( \dim_H A = p \) can be transformed into a set \( B = f(A) \) with \( \dim_H B = q \) for each pair of numbers \( p \) and \( q \in (0, n) \) under a quasiconformal mapping \( f \) of \( \mathbb{R}^n \) onto itself, cf. also [8] and [13].

2 Preliminaries

First of all, the following fine property of functions \( f \) in the Sobolev classes \( W_{1,p}^{1,1} \) was proved in the monograph [41], Theorem 5.5, and can be extended to the Orlicz-Sobolev classes. The statement follows directly from the Fubini theorem and the known characterization of functions in Sobolev’s class \( W_{1,1}^{1,1} \) in terms of ACL (absolute continuity on lines), see, e.g., Section 1.1.3 in [119], and comments in Introduction.

**Proposition 2.1.** Let \( U \) be an open set in \( \mathbb{R}^n \) and let \( f : U \to \mathbb{R}^m \), \( m = 1, 2, \ldots \), be a mapping in the Orlicz-Sobolev class \( W_{1,\varphi}^{1,1}(U) \) with an increasing function \( \varphi : [0, \infty) \to [0, \infty) \). Then, for every \( k \)-dimensional direction \( \Gamma \) for a.e. \( k \)-dimensional plane \( \mathcal{P} \in \Gamma \), \( k = 1, 2, \ldots, n - 1 \), the restriction of the function \( f \) on the set \( \mathcal{P} \cap U \) is a function in the class \( W_{1,\varphi}^{1,1}(\mathcal{P} \cap U) \).

Here the class \( W_{1,\varphi}^{1,1} \) is well defined on a.e. \( k \)-dimensional plane because partial derivatives are Borel functions and, moreover, Sobolev’s classes are invariant with respect to quasi-isometric transformations of systems of coordinates, in particular, with respect to rotations, see, e.g., 1.1.7 in [119]. Recall also that a \( k \)-dimensional direction \( \Gamma \) in \( \mathbb{R}^n \) is the class of equivalence of all \( k \)-dimensional planes in \( \mathbb{R}^n \) that can be obtained each from other by a parallel shift. Note that each \( (n - k) \)-dimensional plane \( \mathcal{T} \) which is quite orthogonal to a \( k \)-dimensional plane \( \mathcal{P} \) in \( \Gamma \) intersects \( \mathcal{P} \) at a single point \( X(\mathcal{P}) \). If \( E \) is a subset of \( \Gamma \), then \( X(E) \) denotes the collection of all point \( X(\mathcal{P}) \), \( \mathcal{P} \in E \). It is clear that \( (n - k) \)-dimensional measure of the set \( X(E) \) does not depend of the choice of the plane \( \mathcal{T} \) and it is denoted by \( \mu_{n-k}(E) \). They say that a property holds for almost every plane in...
if $\mu_{n-k}(E) = 0$ for a set $E$ of all planes $\mathcal{P}$ for which the property fails.

Recall also the little-known Fadell theorem in [29] that makes possible us to extend the well-known theorems of Gehring-Lehto-Menchoff in the plane and Väisälä in $\mathbb{R}^n$, $n \geq 3$, see, e.g., [36], [120] and [174], on differentiability a.e. of open mappings in Sobolev’s classes to the open mappings in Orlicz-Sobolev classes in $\mathbb{R}^n$, $n \geq 3$.

**Proposition 2.2.** Let $f : \Omega \to \mathbb{R}^n$ be a continuous open mapping on an open set $\Omega$ in $\mathbb{R}^n$, $n \geq 3$. If $f$ has a total differential a.e. on $\Omega$ with respect to $n - 1$ variables, then $f$ has a total differential a.e. on $\Omega$.

Now, let us give the following Calderon result in [16], p. 208, cf. Lemma 3.2 in [70].

**Proposition 2.3.** Let $\varphi : [0, \infty) \to [0, \infty)$ be an increasing function such that $\varphi(0) = 0$ and

$$A : = \int_0^\infty \left[ \frac{t}{\varphi(t)} \right]^{\frac{1}{k-1}} dt < \infty \quad (2.1)$$

for a natural number $k \geq 2$ and let $f : G \to \mathbb{R}$ be a function given in a domain $G \subset \mathbb{R}^k$ of the class $W^{1,\varphi}(G)$. Then

$$\text{diam } f(C) \leq \alpha_k A^{\frac{k-1}{k}} \left[ \int_C \varphi (|\nabla f|) \, dm(x) \right]^\frac{1}{k} \quad (2.2)$$

for every cube $C \subset G$ whose edges are oriented along coordinate axes where $\alpha_k$ is a constant depending only on $k$.

Perhaps, the Calderon work [16] had time to be forgotten because it was published long ago in a badly accessible journal.

**Remark 2.1.** It is clear that the behavior of the function $\varphi$ about 0 is not essential and (2.2) holds with the replacement of the constant
A by the constant

\[ A_* : = \left[ \frac{1}{\varphi(1)} \right]^{\frac{1}{k-1}} + \int_1^\infty \left[ \frac{t}{\varphi(t)} \right]^{\frac{1}{k-1}} dt < \infty \quad \text{(2.3)} \]

and \( \varphi(t) \) by \( \varphi_*(t) \equiv \varphi(1) \) for \( t \in (0, 1) \), \( \varphi_*(0) = 0 \) and \( \varphi_*(t) = \varphi(t) \) for \( t \geq 1 \). Indeed, applying Proposition 2.3 to the one parameter family of the functions \( \varphi_\lambda(t) = \varphi(t) + \lambda \cdot [\varphi_*(t) - \varphi(t)] \), \( \lambda \in [0, 1] \), we obtain (2.2) with the changes \( A \mapsto A_* \) and \( \varphi \mapsto \varphi_* \) as \( \lambda \to 1 \).

Finally, one more statement of Calderon in [16], p. 209, 211-212, will be also useful later on.

**Proposition 2.4.** Let \( \varphi : [0, \infty) \to [0, \infty) \) be a convex increasing function such that \( \varphi(0) = 0 \) and

\[ \int_1^\infty \left[ \frac{t}{\varphi(t)} \right]^{\frac{1}{k-1}} dt = \infty \quad \text{(2.4)} \]

for a natural number \( k \geq 2 \). Then there is a continuously differentiable decreasing function \( F : (0, \infty) \to [0, \infty) \) with a compact support such that \( F(t) \to \infty \) as \( t \to 0 \), \( F'(t) \) is non-decreasing, \( F'(t) \to -\infty \) as \( t \to 0 \) and \( F_*(x) = F(|x|) \), \( x \in \mathbb{R}^k \), belongs to the class \( W^{1,\varphi}(\mathbb{R}^k) \), i.e., \( f \in W^{1,1}(\mathbb{R}^k) \) and

\[ \int_{\mathbb{R}^k} \varphi(|\nabla F_*|) \, dm(x) \leq 1. \quad \text{(2.5)} \]

**Remark 2.2.** The function \( F \) from Proposition 2.4 can be described in a more constructive way. More precisely, set

\[ \Phi(t) = \int_1^t \left[ \frac{\tau}{\varphi(\tau)} \right]^{\frac{1}{k-1}} d\tau \quad \text{(2.6)} \]

and

\[ \Psi(t) = \frac{\Phi'(t)}{\Phi(t)} = \left[ \frac{t}{\varphi(t)} \right]^{\frac{1}{k-1}} \frac{1}{\Phi(t)}. \quad \text{(2.7)} \]
The function $\Psi$ is continuous and decreasing and tends to 0 as $t \to \infty$ and to $\infty$ as $t \to 1$. Thus, its inverse function $h(s)$ is well defined for all $s > 0$. It was proved by Calderon in [16] that

$$\int_0^1 h(s) \, ds = \infty, \quad \int_0^1 \varphi(h(s)) \, s^{k-1} \, ds < \infty. \quad (2.8)$$

Then we may set $F(t) \equiv 0$ for $t \geq 1$ and

$$F(t) = \int_t^1 [h(s) - h(1)] \, ds \quad \forall \ t \in [0, 1]. \quad (2.9)$$

On the base of Proposition 2.3, it was proved by Calderon that each continuous function $f : G \to \mathbb{R}$ in $W^{1,\varphi}(G)$ under the condition

$$\int_1^\infty \left[ \frac{t}{\varphi(t)} \right]^{k-1} \, dt < \infty \quad (2.10)$$

has a total differential a.e. Moreover, on the base of Proposition 2.4, under the condition (2.4) Calderon has constructed a continuous function $f : \mathbb{R}^k \to \mathbb{R}$ which has not a total differential a.e. We use Propositions 2.3 and 2.4 for other purposes.

3 The differentiability of open mappings

Let us start from the following statement which is due to Calderon [16] but we prefer in comparison with [16] to prove it on the base of the Stepanoff theorem.

**Lemma 3.1.** Let $\Omega$ be an open set in $\mathbb{R}^k$, $k \geq 2$, and let $f : \Omega \to \mathbb{R}^m$, $m \geq 1$, be a continuous mapping in the class $W^{1,\varphi}_{loc}(\Omega)$ with some increasing $\varphi : [0, \infty) \to [0, \infty)$ such that $\varphi(0) = 0$ and

$$A := \int_1^\infty \left[ \frac{t}{\varphi(t)} \right]^{k-1} \, dt < \infty. \quad (3.1)$$
Then $f$ has a total differential a.e. in $\Omega$.

Proof. Given $x \in \Omega$, we set

$$L(x, f) = \limsup_{y \to x} \frac{|f(y) - f(x)|}{|y - x|}.$$

By the Stepanoff theorem, see, e.g., Theorem 3.1.9 in [31], the proof is reduced to the proof of the fact $L(x, f) < \infty$ a.e. in $\Omega$.

Denote by $C(x, r)$ the oriented cube centered at $x$ such that the ball $B(x, r)$ is inscribed in $C(x, r)$ where $r = |x - y|$. Then

$$L(x, f) = \limsup_{y \to x} \frac{|f(x) - f(y)|}{|x - y|} \leq \limsup_{r \to 0} \frac{d(fB(x, r))}{r} \leq \limsup_{r \to 0} \frac{d(fC(x, r))}{r}$$

and by Proposition 2.3 and Remark 2.1 we get

$$L(x, f) \leq \gamma_{k,m} A^*_{\frac{k-1}{k}} \limsup_{r \to 0} \left[ \frac{1}{r^{k}} \int_{C(x,r)} \varphi_* (|\nabla f|) \, dm(x) \right]^{\frac{1}{k}} < \infty$$

for a.e. $x \in \Omega$ by the Lebesgue theorem on differentiability of indefinite integral, see, e.g., Theorem IV.5.4 in [156]. The proof is complete.

Combining Lemma 3.1 and Proposition 2.1, we obtain the following statement.

**Corollary 3.1.** Let $\Omega$ be an open set in $\mathbb{R}^n$, $n \geq 3$, and let $f : \Omega \to \mathbb{R}^m$, $m \geq 1$, be a continuous mapping in the class $W^{1,\varphi}_{loc}(\Omega)$ with an increasing function $\varphi : [0, \infty) \to [0, \infty)$ such that $\varphi(0) = 0$ and

$$\int_{1}^{\infty} \left[ \frac{t}{\varphi(t)} \right]^{\frac{1}{n-2}} \, dt < \infty. \quad (3.2)$$

Then $f : \Omega \to \mathbb{R}^m$ has a total differential a.e. on a.e. hyperplane which is parallel to a coordinate hyperplane.
Combing Corollary 3.1 and the Fadell result in [29], see Proposition 2.2 above, we obtain the main result of this section.

**Theorem 3.1.** Let $\Omega$ be an open set in $\mathbb{R}^n$, $n \geq 3$, and let $f : \Omega \to \mathbb{R}^n$ be a continuous open mapping in the class $W^{1,\varphi}_{\text{loc}}(\Omega)$ with an increasing $\varphi : [0, \infty) \to [0, \infty)$ such that $\varphi(0) = 0$ and (3.2) holds. Then $f$ has a total differential a.e. in $\Omega$.

**Corollary 3.2.** If $f : \Omega \to \mathbb{R}^n$ is a homeomorphism in $W^{1,1}_{\text{loc}}$ with $K_f \in L^p_{\text{loc}}$ for $p > n - 1$, then $f$ is differentiable a.e.

**Remark 3.1.** In particular, the conclusion is true if $f \in W^{1,p}_{\text{loc}}$ for some $p > n - 1$. The latter statement is the Väisälä result, see Lemma 3 in [174]. Theorem 3.1 is also an extension of the well-known Gegring-Lehto-Menchoff result in the plane to high dimensions, see, e.g., [36], [104] and [120].

Calderon has shown in [16] the preciseness of the condition of (3.1) for differentiability a.e. of continuous mappings $f$. Theorem 3.1 shows that we may use the weaker condition (3.2) to obtain differentiability a.e. of open mappings $f$.

The condition (3.2) is not only sufficient but also necessary for open continuous mappings $f : W^{1,\varphi}_{\text{loc}}$ from $\mathbb{R}^n$ into $\mathbb{R}^n$, $n \geq 3$, to have a total differential a.e. Furthermore, if a function $\varphi : [0, \infty) \to [0, \infty)$ is increasing, convex and such that

$$
\int_1^\infty \left[ \frac{t}{\varphi(t)} \right]^{\frac{1}{n-2}} dt = \infty,
$$

then there is a homeomorphism $g : \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 3$, in the class $W^{1,\varphi}_{\text{loc}}$ which has not a total differential a.e. Indeed, if $f : \mathbb{R}^{n-1} \to \mathbb{R}$ is a function in the Calderon construction for $k = n - 1$ and $\varphi_*(t) = \varphi(t + k) - \varphi(k)$, then

$$
\int_1^\infty \left[ \frac{t}{\varphi_*(t)} \right]^{\frac{1}{n-2}} dt = \infty
$$

(3.4)
and \( g(x, y) = (x, y + f(x)) \), \( x \in \mathbb{R}^{n-1}, y \in \mathbb{R} \), is the desired example because of \(|\nabla g| \leq k + |\nabla f|\) and monotonicity of the function \( \varphi \). Thus, the condition \((3.2)\) already cannot be weakened even for homeomorphisms.

4 The Lusin and Sard properties on surfaces

Theorem 4.1. Let \( \Omega \) be an open set in \( \mathbb{R}^k, k \geq 2 \), and let \( f : \Omega \to \mathbb{R}^m, m \geq 1 \), be a continuous mapping in the class \( W^{1,\varphi}(\Omega) \) with an increasing \( \varphi : [0, \infty) \to [0, \infty) \) such that \( \varphi(0) = 0 \) and

\[
A := \int_1^\infty \left[ \frac{t}{\varphi(t)} \right]^{\frac{1}{k-1}} dt < \infty. \tag{4.1}
\]

Then

\[
H^k(f(E)) \leq \gamma_{k,m} A_*^{k-1} \int_E \varphi_*(|\nabla f|) \, dm(x) \tag{4.2}
\]

for every measurable set \( E \subset \Omega \) and \( \gamma_{k,m} = (m \alpha_k)^k \) where \( \alpha_k \) is a constant from \((2.2)\) depending only on \( k \), \( A_* = A + 1/[\varphi(1)]^{1/(k-1)} \), \( \varphi_*(0) = 0 \), \( \varphi_*(t) \equiv \varphi(1) \) for \( t \in (0, 1) \) and \( \varphi_*(t) = \varphi(t) \) for \( t \geq 1 \).

Thus, we obtain the following conclusions on the Lusin property of mappings in the Orlicz-Sobolev classes.

Corollary 4.1. Under the hypotheses of Theorem 4.1 the mapping \( f \) has the \( (N) \)-property of Lusin (furthermore, \( f \) is absolutely continuous) with respect to the \( k \)-dimensional Hausdorff measure.

Remark 4.1. Note that \( H^k(\mathbb{R}^m) = 0 \) for \( m < k \) and hence \((4.2)\) is trivial in this case without the condition \((4.1)\). However, the examples in Section 13 show that the condition \((4.1)\) is not only sufficient but also necessary for the \( (N) \)-property if \( m \geq k \), see Lemma 13.1 and Remark 13.2.

We obtain also the following consequence of Theorem 4.1 of the Sard type for mappings in the Orlicz-Sobolev classes, see in addition Theorem VII.3 in \([61]\).
Corollary 4.2. Under the hypotheses of Theorem 4.1, we have that \( H^k(f(E)) = 0 \) whenever \( |\nabla f| = 0 \) on a measurable set \( E \subset \Omega \) and hence \( \dim_H f(E) \leq k \) and also \( \dim f(E) \leq k - 1 \).

First such a statement was established by Sard in [157] for the set of critical points of \( f \) where \( J_f(x) = 0 \) and then similar problems studied by many authors for critical points of ranks \( r \) where \( \text{rank} f'(x) \leq r \) and, in particular, for supercritical points where the Jacobian matrix \( f'(x) \) is null at all, see, e.g., [10], [22], [24], [25], [28], [43], [48], [69], [122], [134], [159], [160] and [180]. Usually they requested the corresponding conditions of smoothness for \( f \) without which such statements, generally speaking, are not true.

In this connection, we would like to stress here that our result on supercritical points, Corollary 4.2, holds without any assumptions on smoothness of \( f \). For instance, this result holds for all continuous mappings \( f \) in the class \( W^{1,p}_{\text{loc}} \) with \( p > k \), see a fine survey on Sard’s theorems, in particular, for Sobolev mappings in the paper [15].

The proof of Theorem 4.1 is based on the following lemma.

Lemma 4.1. Let \( \Omega \) be a domain in \( \mathbb{R}^k \), \( k \geq 2 \), and let \( f : \Omega \to \mathbb{R}^m \), \( m \geq 1 \), be a continuous mapping in the class \( W^{1,\varphi}(G) \) with an increasing \( \varphi : [0, \infty) \to [0, \infty) \) such that \( \varphi(0) = 0 \) and

\[
A := \int_1^\infty \left[ \frac{t}{\varphi(t)} \right]^{1-k-1} dt < \infty. \tag{4.3}
\]

Then

\[
\text{diam } f(C) \leq m\alpha_k A_*^{-k} \left[ \int_C \varphi_* (|\nabla f|) \, dm(x) \right]^{1/k}. \tag{4.4}
\]

for every cube \( C \subset \Omega \) whose edges are oriented along coordinate axes where \( \alpha_k \) is a constant from (2.2) depending only on \( k \) and \( A_* \) and \( \varphi_* \) have been defined in Theorem 4.1.
Proof of Lemma 4.1. Let us prove (4.4) by induction in \( m = 1, 2, \ldots \). Indeed, (4.4) holds by Proposition 2.3 and Remark 2.1 for \( m = 1 \) and \( \alpha_k \) from (2.2). Let us assume that (4.4) is valid for some \( m = l \) and prove it for \( m = l + 1 \). Consider an arbitrary vector \( \vec{V} = (v_1, v_2, \ldots, v_l, v_{l+1}) \) in \( \mathbb{R}^{l+1} \) and the vectors \( \vec{V}_1 = (v_1, v_2, \ldots, v_l, 0) \) and \( \vec{V}_2 = (0, \ldots, 0, v_{k+1}) \). Then \(|\vec{V}| = |\vec{V}_1 + \vec{V}_2| \leq |\vec{V}_1| + |\vec{V}_2|\). Thus, denoting by \( \text{Pr}_1 \vec{V} = \vec{V}_1 \) and \( \text{Pr}_2 \vec{V} = \vec{V}_2 \) the projections of vectors from \( \mathbb{R}^{l+1} \) onto the coordinate hyperplane \( y_{l+1} = 0 \) and on the \((l + 1)\)th axis in \( \mathbb{R}^{l+1} \), correspondingly, we obtain that \( \text{diam } f(C) \leq \text{diam } \text{Pr}_1 f(C) + \text{diam } \text{Pr}_2 f(C) \) and, applying (4.4) under \( m = l \) and \( m = 1 \), we come by monotonicity of \( \varphi \) to the inequality (4.4) under \( m = l + 1 \). The proof is complete.

Proof of Theorem 4.1. In view of countable additivity of integral and measure we may assume with no loss of generality that \( E \) is bounded and \( \overline{E} \subset G \), i.e., \( \overline{E} \) is a compactum in \( G \). For each \( \varepsilon > 0 \) there is an open set \( \Omega \subset G \) such that \( E \subset \Omega \) and \(|\Omega \setminus E| < \varepsilon\), see, e.g., Theorem III (6.6) in [156]. By the above remark we may assume that \( \overline{\Omega} \) is a compactum and, thus, the mapping \( f \) is uniformly continuous in \( \Omega \). Hence \( \Omega \) can be covered by a countable collection of closed oriented cubes \( C_i \) whose interiorities are mutually disjoint and such that \( \text{diam } f(C_i) < \delta \) for any prescribed \( \delta > 0 \) and \( \bigcup_{i=1}^{\infty} \partial C_i = 0 \).

Thus, by Lemma 4.1 we have that

\[
H^k_\delta (f(E)) \leq H^k_\delta (f(\Omega)) \leq \sum_{i=1}^{\infty} \left[ \text{diam } f(C_i) \right]^k \leq \gamma_{k,m} A^{k-1}_* \int_{\Omega} \varphi_* (|\nabla f|) \, dm(x).
\]

Finally, by absolute continuity of the indefinite integral and arbitrariness of \( \varepsilon \) and \( \delta > 0 \) we obtain (4.2).

Combining Proposition 2.1 and Corollary 4.1 we obtain the following statement.
Proposition 4.1. Let \( k = 2, \ldots, n - 1 \), \( U \) be an open set in \( \mathbb{R}^n \), \( n \geq 3 \), and let \( f : U \to \mathbb{R}^m \), \( m \geq 1 \), be a continuous mapping in the class \( W_{1,\text{loc}}^{1,\varphi}(U) \) for some increasing function \( \varphi : [0, \infty) \to [0, \infty) \), \( \varphi(0) = 0 \), such that (4.1) holds. Then, for every \( k \)-dimensional direction \( \Gamma \) for a.e. \( k \)-dimensional plane \( \mathcal{P} \in \Gamma \), the restriction of the function \( f \) on the set \( \mathcal{P} \cap U \) has the \((N)\)-property (furthermore, it is locally absolutely continuous) with respect to the \( k \)-dimensional Hausdorff measure. Moreover, \( H^k(f(E)) = 0 \) whenever \( \nabla_k f = 0 \) on \( E \subset \mathcal{P} \) for a.e. \( \mathcal{P} \in \Gamma \).

Here \( \nabla_k \) denotes the \( k \)-dimensional gradient of the restriction of the mapping \( f \) to the \( k \)-dimensional plane \( \mathcal{P} \). However, the most important particular case of Proposition 4.1 for us is the following statement.

Theorem 4.2. Let \( U \) be an open set in \( \mathbb{R}^n \), \( n \geq 3 \), and let \( \varphi : [0, \infty) \to [0, \infty) \) be an increasing function such that \( \varphi(0) = 0 \) and
\[
\int_1^\infty \left[ \frac{t}{\varphi(t)} \right]^{\frac{1}{n-2}} dt < \infty. \tag{4.5}
\]
Then each continuous mapping \( f : U \to \mathbb{R}^m \), \( m \geq 1 \), in the class \( W_{1,\text{loc}}^{1,\varphi} \) has the \((N)\)-property (furthermore, it is locally absolutely continuous) with respect to the \((n-1)\)-dimensional Hausdorff measure on a.e. hyperplane \( \mathcal{P} \) which is parallel to a fixed hyperplane \( \mathcal{P}_0 \). Moreover, \( H^{n-1}(f(E)) = 0 \) whenever \( |\nabla f| = 0 \) on \( E \subset \mathcal{P} \) for a.e. such \( \mathcal{P} \).

Note that, if the condition (4.5) holds for an increasing function \( \varphi \), then the function \( \varphi_* = \varphi(ct) \) for \( c > 0 \) also satisfies (4.5). Moreover, the Hausdorff measures are quasi-invariant under quasi-isometries. By the Lindelöf property of \( \mathbb{R}^n \), \( U \setminus \{x_0\} \) can be covered by a countable collection of open segments of spherical rings in \( U \setminus \{x_0\} \) centered at \( x_0 \) and each such segment can be mapped onto a rectangular oriented segment of \( \mathbb{R}^n \) by some quasi-isometry, see, e.g., I.5.XI in [100].
for the Lindelöf theorem. Thus, applying Theorem 4.2 piecewise, we obtain the following conclusion.

**Corollary 4.3.** Under (4.5) each \( f \in W_{loc}^{1,\varphi} \) has the (N)-property (furthermore, it is locally absolutely continuous) on a.e. sphere \( S \) centered at a prescribed point \( x_0 \in \mathbb{R}^n \). Moreover, \( H^{n-1}(f(E)) = 0 \) whenever \( |\nabla f| = 0 \) on \( E \subseteq S \) for a.e. such sphere \( S \).

**Remark 4.2.** In particular, (4.5) holds for the functions \( \varphi(t) = t^p \), \( p > n - 1 \), i.e., the given properties hold for \( f \in W_{loc}^{1,p} \), \( p > n - 1 \).

Note also that (4.5) does not imply the (N)-property of \( f : U \to \mathbb{R}^n \) in \( U \) with respect to the Lebesgue measure in \( \mathbb{R}^n \). The latter conclusion follow, in particular, from the Ponomarev examples of homeomorphisms \( f \in W_{loc}^{1,p}(\mathbb{R}^n) \) for all \( p < n \) without (N)-property of Lusin, see [133].

If \( m < n - 1 \), then \( H^{n-1}(\mathbb{R}^m) = 0 \) and the (N)–property on a.e. hyperplane for the mapping \( f \) in Theorem 4.2 is obvious without the condition (4.5). However, the examples in the final section show that the condition (4.5) are not only sufficient but also necessary for the (N)-property on a.e. hyperplane if \( m \geq n - 1 \), see Remark 13.2 and Theorem 13.1.

The connection of estimates of the Calderon type (2.2) with the (N)–property and differentiability was first found under the study of the so-called generalized Lipschitzians in the sense of Rado, see, e.g., [16] and V.3.6 in [135], and also the recent works [9], [70] and [141].

5 On BMO and FMO functions

The BMO space was introduced by John and Nirenberg in [67] and soon became one of the main concepts in harmonic analysis, complex analysis, partial differential equations and related areas, see, e.g., [51] and [143].

Let \( D \) be a domain in \( \mathbb{R}^n \), \( n \geq 1 \). Recall that a real valued function \( \varphi \in L_{loc}^1(D) \) is said to be of **bounded mean oscillation** in \( D \), abbr.
\( \varphi \in \text{BMO}(D) \) or simply \( \varphi \in \text{BMO} \), if

\[
\| \varphi \|_* = \sup_{B \subset D} \int_B |\varphi(z) - \varphi_B| \, dm(z) < \infty \tag{5.1}
\]

where the supremum is taken over all balls \( B \) in \( D \) and

\[
\varphi_B = \int_B \varphi(z) \, dm(z) = \frac{1}{|B|} \int_B \varphi(z) \, dm(z) \tag{5.2}
\]

is the mean value of the function \( \varphi \) over \( B \). Note that \( L^\infty(D) \subset \text{BMO}(D) \subset L^p_{\text{loc}}(D) \) for all \( 1 \leq p < \infty \), see, e.g., \[143\].

A function \( \varphi \) in BMO is said to have \textbf{vanishing mean oscillation}, abbr. \( \varphi \in \text{VMO} \), if the supremum in (5.1) taken over all balls \( B \) in \( D \) with \( |B| < \varepsilon \) converges to 0 as \( \varepsilon \to 0 \). VMO has been introduced by Sarason in \[157\]. There are a number of papers devoted to the study of partial differential equations with coefficients of the class VMO, see, e.g., \[19\], \[66\], \[116\], \[128\] and \[136\].

Following \[62\], we say that a function \( \varphi : D \to \mathbb{R} \) has \textbf{finite mean oscillation at a point} \( z_0 \in D \), write \( \varphi \in \text{FMO}(x_0) \), if

\[
\lim_{\varepsilon \to 0} \int_{B(z_0,\varepsilon)} |\varphi(z) - \tilde{\varphi}_\varepsilon(z_0)| \, dm(z) < \infty \tag{5.3}
\]

where

\[
\tilde{\varphi}_\varepsilon(z_0) = \int_{B(z_0,\varepsilon)} \varphi(z) \, dm(z) \tag{5.4}
\]

is the mean value of the function \( \varphi(z) \) over the ball \( B(z_0, \varepsilon) \). The condition (5.3) includes the assumption that \( \varphi \) is integrable in some neighborhood of the point \( z_0 \). We also say that a function \( \varphi \) is of \textbf{finite mean oscillation in the domain} \( D \), write \( \varphi \in \text{FMO}(D) \) or simply \( \varphi \in \text{FMO} \), if this property holds at every point \( x_0 \in D \).

\textbf{Proposition 5.1.} If for some collection of numbers \( \varphi_\varepsilon \in \mathbb{R}, \ \varepsilon \in (0, \varepsilon_0] \),

\[
\lim_{\varepsilon \to 0} \int_{B(z_0,\varepsilon)} |\varphi(z) - \varphi_\varepsilon| \, dm(z) < \infty, \tag{5.5}
\]
then $\varphi$ is of finite mean oscillation at $z_0$.

Indeed, by the triangle inequality
\[
\int_{B(x_0,\varepsilon)} |\varphi(x) - \varphi_\varepsilon| \, dm(x) \leq \int_{B(x_0,\varepsilon)} |\varphi(x) - \varphi| \, dm(x) + |\varphi - \varphi_\varepsilon| \leq 2 \int_{B(x_0,\varepsilon)} |\varphi(x) - \varphi_\varepsilon| \, dm(x) .
\]

Choosing in Proposition 5.1 $\varphi_\varepsilon \equiv 0$, $\varepsilon \in (0, \varepsilon_0]$, we obtain the following statement.

**Corollary 5.1.** If for a point $z_0 \in D$
\[
\lim_{\varepsilon \to 0} \int_{B(z_0,\varepsilon)} |\varphi(z)| \, dm(z) < \infty ,
\]
then $\varphi$ has finite mean oscillation at $z_0$.

Recall that a point $z_0 \in D$ is called a Lebesgue point of a function $\varphi : D \to \mathbb{R}$ if $\varphi$ is integrable in a neighborhood of $z_0$ and
\[
\lim_{\varepsilon \to 0} \int_{B(z_0,\varepsilon)} |\varphi(z) - \varphi(z_0)| \, dm(z) = 0 .
\]
It is known that almost every point in $D$ is a Lebesgue point for every function $\varphi \in L^1(D)$. We thus have the following corollary.

**Corollary 5.2.** Every function $\varphi : D \to \mathbb{R}$, which is locally integrable, has a finite mean oscillation at almost every point in $D$.

**Remark 5.1.** Note that the function $\varphi(z) = \log(1/|z|)$ belongs to BMO in the unit disk $\Delta$, see, e.g., [143], p. 5, and hence also to FMO. However, $\tilde{\varphi}_\varepsilon(0) \to \infty$ as $\varepsilon \to 0$, showing that the condition (5.6) is only sufficient but not necessary for a function $\varphi$ to be of finite mean oscillation at $z_0$.

Clearly that BMO $\subset$ FMO. By definition FMO $\subset L^1_{\text{loc}}$ but FMO is not a subset of $L^p_{\text{loc}}$ for any $p > 1$ in comparison with BMO$_{\text{loc}} \subset L^p_{\text{loc}}$ for all $p \in [1, \infty)$. Here BMO$_{\text{loc}}$ stands for the local version of
the class BMO. So, let us give examples showing that FMO is not BMO_{loc}.

**Example 1.** Set \( z_n = 2^{-n}, \ r_n = 2^{-pn^2}, \ p > 1, \ c_n = 2^{2n^2}, \ D_n = \{ z \in \mathbb{C} : |z - z_n| < r_n \}, \) and

\[
\varphi(z) = \sum_{n=1}^{\infty} c_n \chi(D_n).
\]

Evidently by Corollary 5.1 that \( \varphi \in FMO(\mathbb{C} \setminus \{0\}) \).

To prove that \( \varphi \in FMO(0) \), fix \( N \) such that \( (p - 1)N > 1 \), and set \( \varepsilon = \varepsilon_N = z_N + r_N \). Then \( \bigcup_{n \geq N} D_n \subset \mathbb{D}(\varepsilon) := \{ z \in \mathbb{C} : |z| < \varepsilon \} \) and

\[
\int_{\mathbb{D}(\varepsilon)} \varphi = \sum_{n \geq N} \int_{D_n} \varphi = \pi \sum_{n \geq N} c_n r_n^2
\]

\[
= \sum_{n \geq N} 2^{2(1-p)n^2} < \sum_{n \geq N} 2^{2(1-p)n}
\]

\[
< C \cdot [2^{(1-p)N}]^2 < 2 C \varepsilon^2.
\]

Hence \( \varphi \in FMO(0) \) and, consequently, \( \varphi \in FMO(\mathbb{C}) \).

On the other hand

\[
\int_{\mathbb{D}(\varepsilon)} \varphi^p = \pi \sum_{n > N} c_n^p \cdot r_n^2 = \sum_{n > N} 1 = \infty.
\]

Hence \( \varphi \notin L^p(\mathbb{D}(\varepsilon)) \) and therefore \( \varphi \notin BMO_{loc} \) because BMO_{loc} \( \subset L^p_{loc} \) for all \( p \in [1, \infty) \).

**Example 2.** We conclude this section by constructing functions \( \varphi : \mathbb{C} \to \mathbb{R} \) of the class \( C^\infty(\mathbb{C} \setminus \{0\}) \) which belongs to FMO but not to \( L^p_{loc} \) for any \( p > 1 \) and hence not to BMO_{loc}.

In this example, \( p = 1 + \delta \) with an arbitrarily small \( \delta > 0 \). Set

\[
\varphi_0(z) = \begin{cases} 
\frac{1}{|z|^2 - 1}, & \text{if } |z| < 1, \\
0, & \text{if } |z| \geq 1.
\end{cases}
\]  

(5.8)
Then \( \varphi_0 \) belongs to \( C_0^\infty \) and hence to \( BMO_{\text{loc}} \). Consider the function
\[
\varphi(z) = \begin{cases} 
\varphi_k(z), & \text{if } z \in B_k, 
0, & \text{if } z \in \mathbb{C} \setminus \bigcup B_k
\end{cases}
\] (5.9)
where \( B_k = B(z_k, r_k), \ z_k = 2^{-k}, \ r_k = 2^{-(1+\delta)k^2}, \ \delta > 0 \), and
\[
\varphi_k(z) = 2^{2k^2} \varphi_0 \left( \frac{z - z_k}{r_k} \right), \ z \in B_k, \ k = 2, 3, \ldots .
\] (5.10)
Then \( \varphi \) is smooth in \( \mathbb{C} \setminus \{0\} \) and, thus, belongs to \( BMO_{\text{loc}}(\mathbb{C} \setminus \{0\}) \), and hence to \( FMO(\mathbb{C} \setminus \{0\}) \).

Now,
\[
\int_{B_k} \varphi_k(z) \, dm(z) = 2^{-2\delta k^2} \int_{\mathbb{C}} \varphi_0(z) \, dm(z).
\] (5.11)
Setting
\[
K = K(\varepsilon) = \left\lfloor \log_2 \frac{1}{\varepsilon} \right\rfloor \leq \log_2 \frac{1}{\varepsilon},
\] (5.12)
where \( [A] \) denotes the integral part of the number \( A \), we have
\[
J = \int_{D(\varepsilon)} \varphi(z) \, dm(z) \leq I \cdot \sum_{k=K}^{\infty} 2^{-2\delta k^2} / \pi 2^{-2(K+1)},
\] (5.13)
where \( I = \int_{\mathbb{C}} \varphi(z) \, dm(z) \). If \( K\delta > 1 \), i.e. \( K > 1 / \delta \), then
\[
\sum_{k=K}^{\infty} 2^{-2\delta k^2} \leq \sum_{k=K}^{\infty} 2^{-2k} = 2^{-2K} \sum_{k=0}^{\infty} \left( \frac{1}{4} \right)^k = \frac{4}{3} \cdot 2^{-2K},
\] (5.14)
i.e., \( J \leq 16I/3\pi \). Hence
\[
\lim_{{\varepsilon \to 0}} \int_{B(\varepsilon)} \varphi(z) \, dm(z) < \infty.
\] (5.15)
Thus, \( \varphi \in FMO \) by Corollary 5.11.

On the other hand,
\[
\int_{B_k} \varphi_{1+\delta}^{k}(z) \, dm(z) = \int_{\mathbb{C}} \varphi_{0+\delta}^{1}(z) \, dm(z)
\] (5.16)
and hence \( \varphi \notin L^{1+\delta}(U) \) for any neighborhood \( U \) of 0.
6 On some integral conditions

For every non-decreasing function $\Phi : [0, \infty] \to [0, \infty]$, the inverse function $\Phi^{-1} : [0, \infty] \to [0, \infty]$ can be well defined by setting

$$\Phi^{-1}(\tau) = \inf_{\Phi(t) \geq \tau} t. \quad (6.1)$$

As usual, here inf is equal to $\infty$ if the set of $t \in [0, \infty]$ such that $\Phi(t) \geq \tau$ is empty. Note that the function $\Phi^{-1}$ is non-decreasing, too.

**Remark 6.1.** Immediately by the definition it is evident that

$$\Phi^{-1}(\Phi(t)) \leq t \quad \forall t \in [0, \infty] \quad (6.2)$$

with the equality in (6.2) except intervals of constancy of the function $\Phi(t)$.

Since the mapping $t \mapsto t^p$ for every positive $p$ is a sense–preserving homeomorphism $[0, \infty]$ onto $[0, \infty]$ we may write Theorem 2.1 from [154] in the following form which is more convenient for further applications. Here, in (6.4) and (6.5), we complete the definition of integrals by $\infty$ if $\Phi_p(t) = \infty$, correspondingly, $H_p(t) = \infty$, for all $t \geq T \in [0, \infty)$. The integral in (6.5) is understood as the Lebesgue-Stieltjes integral and the integrals in (6.4) and (6.6)–(6.9) as the ordinary Lebesgue integrals.

**Proposition 6.1.** Let $\Phi : [0, \infty] \to [0, \infty]$ be a non-decreasing function. Set

$$H_p(t) = \log \Phi_p(t), \quad \Phi_p(t) = \Phi(t^p), \quad p \in (0, \infty). \quad (6.3)$$

Then the equality

$$\int_{\delta}^{\infty} H'_p(t) \frac{dt}{t} = \infty \quad (6.4)$$

implies the equality

$$\int_{\delta}^{\infty} \frac{dH_p(t)}{t} = \infty \quad (6.5)$$
and (6.5) is equivalent to
\[ \int_\delta^\infty H_p(t) \frac{dt}{t^2} = \infty \]  
(6.6)
for some \( \delta > 0 \), and (6.6) is equivalent to every of the equalities:
\[ \int_0^\Delta H_p\left(\frac{1}{t}\right) dt = \infty \]  
(6.7)
for some \( \Delta > 0 \),
\[ \int_{\delta^*}^\infty \frac{d\eta}{H_p^{-1}(\eta)} = \infty \]  
(6.8)
for some \( \delta^* > H(+0) \),
\[ \int_{\delta^*}^\infty \frac{d\tau}{\tau\Phi_p^{-1}(\tau)} = \infty \]  
(6.9)
for some \( \delta^* > \Phi(+0) \).

Moreover, (6.4) is equivalent to (6.5) and hence (6.4)–(6.9) are equivalent each to other if \( \Phi \) is in addition absolutely continuous. In particular, all the conditions (6.4)–(6.9) are equivalent if \( \Phi \) is convex and non-decreasing.

It is easy to see that conditions (6.4)–(6.9) become weaker as \( p \) increases, see e.g. (6.6). It is necessary to give one more explanation. From the right hand sides in the conditions (6.4)–(6.9) we have in mind \(+\infty\). If \( \Phi_p(t) = 0 \) for \( t \in [0, t_*] \), then \( H_p(t) = -\infty \) for \( t \in [0, t_*] \) and we complete the definition \( H_p'(t) = 0 \) for \( t \in [0, t_*] \). Note, the conditions (6.5) and (6.6) exclude that \( t_* \) belongs to the interval of integrability because in the contrary case the left hand sides in (6.5) and (6.6) are either equal to \(-\infty\) or indeterminate. Hence we may assume in (6.4)–(6.7) that \( \delta > t_0 \), correspondingly, \( \Delta < 1/t_0 \) where \( t_0 := \sup_{\Phi_p(t)=0} t \), \( t_0 = 0 \) if \( \Phi_p(0) > 0 \).
Recall that a function $\Phi : [0, \infty) \to [0, \infty]$ is called convex if
\[
\Phi(\lambda t_1 + (1 - \lambda)t_2) \leq \lambda \Phi(t_1) + (1 - \lambda) \Phi(t_2)
\]
for all $t_1$ and $t_2 \in [0, \infty]$ and $\lambda \in [0, 1]$.

Lemma 3.1 from [154] can be written in the following form.

**Lemma 6.1.** Let $Q : \mathbb{B}^n \to [0, \infty]$ be a measurable function and let $\Phi : [0, \infty) \to (0, \infty]$ be a non-decreasing convex function. Then
\[
\int_0^1 \frac{dr}{rq^{\frac{1}{p}}(r)} \geq \frac{1}{n} \int_{eM} \frac{d\tau}{\tau [\Phi^{-1}(\tau)]^{\frac{1}{p}}} \quad \forall \ p \in (0, \infty) \quad (6.10)
\]
where $q(r)$ is the average of the function $Q(x)$ over the sphere $|x| = r$ and $M$ is the average of the function $\Phi \circ Q$ over the unit ball $\mathbb{B}^n$.

**Remark 6.2.** Note that (6.10) is equivalent for each $p \in (0, \infty)$ to the inequality
\[
\int_0^1 \frac{dr}{rq^{\frac{1}{p}}(r)} \geq \frac{1}{n} \int_{eM} \frac{d\tau}{\tau [\Phi^{-1}(\tau)]^{\frac{1}{p}}} , \quad \Phi_p(t) : = \Phi(t^p) . \quad (6.11)
\]

**Theorem 6.1.** Let $Q : \mathbb{B}^n \to [0, \infty]$ be a measurable function such that
\[
\int_{\mathbb{B}^n} \Phi(Q(x)) \, dm(x) < \infty \quad (6.12)
\]
where $\Phi : [0, \infty) \to [0, \infty]$ is a non-decreasing convex function such that
\[
\int_{\delta_0}^\infty \frac{d\tau}{\tau [\Phi^{-1}(\tau)]^{\frac{1}{p}}} = \infty , \quad p \in (0, \infty) , \quad (6.13)
\]
for some $\delta_0 > \Phi(0)$. Then
\[
\int_0^1 \frac{dr}{rq^{\frac{1}{p}}(r)} = \infty \quad (6.14)
\]
where \( q(r) \) is the average of \( Q(x) \) over the sphere \(|x| = r\).

**Remark 6.3.** In view of Proposition 6.1, if (6.12) holds, then each of the conditions (6.4)–(6.9) implies the condition (6.14).

### 7 Moduli of families of surfaces

Let \( \omega \) be an open set in \( \mathbb{R}^k \), \( k = 1, \ldots, n - 1 \). A (continuous) mapping \( S: \omega \to \mathbb{R}^n \) is called a \( k \)-dimensional surface \( S \) in \( \mathbb{R}^n \). Sometimes we call the image \( S(\omega) \subseteq \mathbb{R}^n \) the surface \( S \), too. The number of preimages

\[
N(S, y) = \text{card } S^{-1}(y) = \text{card } \{x \in \omega : S(x) = y\}, \ y \in \mathbb{R}^n \quad (7.1)
\]

is said to be a **multiplicity function** of the surface \( S \). In other words, \( N(S, y) \) denotes the multiplicity of covering of the point \( y \) by the surface \( S \). It is known that the multiplicity function is lower semicontinuous, i.e.,

\[
N(S, y) \geq \lim \inf_{m \to \infty} N(S, y_m)
\]

for every sequence \( y_m \in \mathbb{R}^n, m = 1, 2, \ldots, \) such that \( y_m \to y \in \mathbb{R}^n \) as \( m \to \infty \); see, e.g., [135], p. 160. Thus, the function \( N(S, y) \) is Borel measurable and hence measurable with respect to every Hausdorff measure \( H^k \); see, e.g., [156], p. 52.

Recall that a \( k \)-dimensional Hausdorff area in \( \mathbb{R}^n \) (or simply area) associated with a surface \( S: \omega \to \mathbb{R}^n \) is given by

\[
\mathcal{A}_S(B) = \mathcal{A}_S^k(B) := \int_B N(S, y) \, dH^k y \quad (7.2)
\]

for every Borel set \( B \subseteq \mathbb{R}^n \) and, more generally, for an arbitrary set that is measurable with respect to \( H^k \) in \( \mathbb{R}^n \), cf. 3.2.1 in [31]. The surface \( S \) is called **rectifiable** if \( \mathcal{A}_S(\mathbb{R}^n) < \infty \), see 9.2 in [115].

If \( \varphi: \mathbb{R}^n \to [0, \infty] \) is a Borel function, then its **integral over** \( S \) is defined by the equality

\[
\int_S \varphi \, d\mathcal{A} := \int_{\mathbb{R}^n} \varphi(y) N(S, y) \, dH^k y. \quad (7.3)
\]
Given a family $\Gamma$ of $k$-dimensional surfaces $S$, a Borel function $\varrho: \mathbb{R}^n \to [0, \infty]$ is called admissible for $\Gamma$, abbr. $\varrho \in \text{adm } \Gamma$, if
\[
\int_S \varrho^k \, dA \geq 1
\] (7.4)
for every $S \in \Gamma$. Given $p \in (0, \infty)$, the $p$-modulus of $\Gamma$ is the quantity
\[
M_p(\Gamma) = \inf_{\varrho \in \text{adm } \Gamma} \int_{\mathbb{R}^n} \varrho^p(x) \, dm(x).
\] (7.5)
We also set
\[
M(\Gamma) = M_n(\Gamma)
\] (7.6)
and call the quantity $M(\Gamma)$ the modulus of the family $\Gamma$. The modulus is itself an outer measure on the collection of all families $\Gamma$ of $k$-dimensional surfaces.

We say that $\Gamma_2$ is minorized by $\Gamma_1$ and write $\Gamma_2 > \Gamma_1$ if every $S \subset \Gamma_2$ has a subsurface that belongs to $\Gamma_1$. It is known that $M_p(\Gamma_1) \geq M_p(\Gamma_2)$, see [32], p. 176-178. We also say that a property $P$ holds for $p$-a.e. (almost every) $k$-dimensional surface $S$ in a family $\Gamma$ if a subfamily of all surfaces of $\Gamma$, for which $P$ fails, has the $p$-modulus zero. If $0 < q < p$, then $P$ also holds for $q$-a.e. $S$, see Theorem 3 in [32]. In the case $p = n$, we write simply a.e.

**Remark 7.1.** The definition of the modulus immediately implies that, for every $p \in (0, \infty)$ and $k = 1, \ldots, n - 1$

1. $p$-a.e. $k$-dimensional surface in $\mathbb{R}^n$ is rectifiable,
2. given a Borel set $B$ in $\mathbb{R}^n$ of (Lebesgue) measure zero,
\[
\mathcal{A}_S(B) = 0
\] (7.7)
for $p$-a.e. $k$-dimensional surface $S$ in $\mathbb{R}^n$.

The following lemma was first proved in [81], see also Lemma 9.1 in [115].

**Lemma 7.1.** Let $k = 1, \ldots, n - 1$, $p \in [k, \infty)$, and let $C$ be an open cube in $\mathbb{R}^n$, $n \geq 2$, whose edges are parallel to coordinate axis.
If a property $P$ holds for $p$-a.e. $k$-dimensional surface $S$ in $C$, then $P$ also holds for a.e. $k$-dimensional plane in $C$ that is parallel to a $k$-dimensional coordinate plane $H$.

The latter a.e. is related to the Lebesgue measure in the corresponding $(n-k)$-dimensional coordinate plane $H^\perp$ that is perpendicular to $H$.

The following statement, see Theorem 2.11 in [82] or Theorem 9.1 in [115], is an analogue of the Fubini theorem, cf., e.g., [156], p. 77. It extends Theorem 33.1 in [173], cf. also Theorem 3 in [32], Lemma 2.13 in [112], and Lemma 8.1 in [115].

**Theorem 7.1.** Let $k = 1, \ldots, n-1$, $p \in [k, \infty)$, and let $E$ be a subset in an open set $\Omega \subset \mathbb{R}^n$, $n \geq 2$. Then $E$ is measurable by Lebesgue in $\mathbb{R}^n$ if and only if $E$ is measurable with respect to area on $p$-a.e. $k$-dimensional surface $S$ in $\Omega$. Moreover, $|E| = 0$ if and only if

$$A_S(E) = 0$$

on $p$-a.e. $k$-dimensional surface $S$ in $\Omega$.

**Remark 7.2.** Say by the Lusin theorem, see e.g. Section 2.3.5 in [31], for every measurable function $\varrho : \mathbb{R}^n \to [0, \infty]$, there is a Borel function $\varrho^* : \mathbb{R}^n \to [0, \infty]$ such that $\varrho^* = \varrho$ a.e. in $\mathbb{R}^n$. Thus, by Theorem 7.1, $\varrho$ is measurable on $p$-a.e. $k$-dimensional surface $S$ in $\mathbb{R}^n$ for every $p \in (0, \infty)$ and $k = 1, \ldots, n-1$.

We say that a Lebesgue measurable function $\varrho : \mathbb{R}^n \to [0, \infty]$ is $p$-**extensively admissible** for a family $\Gamma$ of $k$-dimensional surfaces $S$ in $\mathbb{R}^n$, abbr. $\varrho \in \text{ext}_p \text{adm } \Gamma$, if

$$\int_S \varrho^k \ dA \geq 1 \quad (7.9)$$

for $p$-a.e. $S \in \Gamma$. The $p$-**extensive modulus** $M_p(\Gamma)$ of $\Gamma$ is the quantity

$$M_p(\Gamma) = \inf_{\mathbb{R}^n} \int \varrho^p(x) \ dm(x) \quad (7.10)$$
where the infimum is taken over all $\varrho \in \text{ext}_p \text{adm} \Gamma$. In the case $p = n$, we use the notations $\overline{M}(\Gamma)$ and $\varrho \in \text{ext adm} \Gamma$, respectively. For every $p \in (0, \infty)$, $k = 1, \ldots, n - 1$, and every family $\Gamma$ of $k$-dimensional surfaces in $\mathbb{R}^n$,

$$
\overline{M}_p(\Gamma) = M_p(\Gamma). 
$$

(7.11)

8 Lower and ring $Q$-homeomorphisms

The following concept is motivated by Gehring’s ring definition of quasiconformality in [34].

Given domains $D$ and $D'$ in $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$, $n \geq 2$, $x_0 \in \overline{D} \setminus \{\infty\}$, and a measurable function $Q : D \to (0, \infty)$, we say that a homeomorphism $f : D \to D'$ is a lower $Q$-homeomorphism at the point $x_0$ if

$$
M(f\Sigma_\varepsilon) \geq \inf_{\varrho \in \text{ext adm} \Sigma_\varepsilon} \int_{D \cap R_\varepsilon} \frac{\varrho^n(x)}{Q(x)} \, dm(x) 
$$

(8.1)

for every ring

$$
R_\varepsilon = \{x \in \mathbb{R}^n : \varepsilon < |x - x_0| < \varepsilon_0\}, \quad \varepsilon \in (0, \varepsilon_0), \varepsilon_0 \in (0, d_0),
$$

where

$$
d_0 = \sup_{x \in D} |x - x_0|,
$$

(8.2)

and $\Sigma_\varepsilon$ denotes the family of all intersections of the spheres

$$
S(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| = r\}, \quad r \in (\varepsilon, \varepsilon_0),
$$

with $D$. As usual, the notion can be extended to the case $x_0 = \infty \in \overline{D}$ by applying the inversion $T$ with respect to the unit sphere in $\overline{\mathbb{R}^n}$, $T(x) = x/|x|^2$, $T(\infty) = 0$, $T(0) = \infty$. Namely, a homeomorphism $f : D \to D'$ is a lower $Q$-homeomorphism at $\infty \in \overline{D}$ if $F = f \circ T$ is a lower $Q_\ast$-homeomorphism with $Q_\ast = Q \circ T$ at 0.

We also say that a homeomorphism $f : D \to \overline{\mathbb{R}^n}$ is a lower $Q$-homeomorphism in $D$ if $f$ is a lower $Q$-homeomorphism at every point $x_0 \in \overline{D}$. 


Recall the criterion for homeomorphisms in $\mathbb{R}^n$ to be lower $Q$-homeomorphisms, see Theorem 2.1 in [81] or Theorem 9.2 in [115].

**Proposition 8.1.** Let $D$ and $D'$ be domains in $\mathbb{R}^n$, $n \geq 2$, let $x_0 \in \overline{D} \setminus \{\infty\}$, and let $Q : D \to (0, \infty)$ a measurable function. A homeomorphism $f : D \to D'$ is a lower $Q$-homeomorphism at $x_0$ if and only if

$$M(f\Sigma_\varepsilon) \geq \int_\varepsilon^{\varepsilon_0} \frac{dr}{||Q||_{n-1}(r)} \quad \forall \varepsilon \in (0, \varepsilon_0), \varepsilon_0 \in (0, d_0), \quad (8.3)$$

where

$$||Q||_{n-1}(r) = \left( \int_{D(x_0,r)} Q^{n-1}(x) \, dA \right)^{\frac{1}{n-1}} \quad (8.4)$$

is the $L_{n-1}$-norm of $Q$ over $D(x_0,r) = \{x \in D : |x - x_0| = r\} = D \cap S(x_0,r)$.

Note that the infimum of expression from the right-hand side in (8.1) is attained for the function

$$\varrho_0(x) = \frac{Q(x)}{||Q||_{n-1}(|x|)}.$$

Now, given a domain $D$ and two sets $E$ and $F$ in $\mathbb{R}^n$, $n \geq 2$, $\Delta(E, F, D)$ denotes the family of all paths $\gamma : [a, b] \to \mathbb{R}^n$ that join $E$ and $F$ in $D$, i.e., $\gamma(a) \in E$, $\gamma(b) \in F$, and $\gamma(t) \in D$ for $a < t < b$. Set

$$A(r_1, r_2, x_0) = \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\}, \quad (8.5)$$

$$S(x_0, r_i) = \{x \in \mathbb{R}^n : |x - x_0| = r_i\}, \quad i = 1, 2. \quad (8.6)$$

Given domains $D$ in $\mathbb{R}^n$ and $D'$ in $\mathbb{R}^n$, $n \geq 2$, and a measurable function $Q : D \to [0, \infty]$, they say that a homeomorphism $f : D \to D'$ is a ring $Q$-homeomorphism at a point $x_0 \in D$ if

$$M(\Delta(fS_1, fS_2, fD)) \leq \int_A Q(x) \cdot \eta^n(|x - x_0|) \, dm(x) \quad (8.7)$$
for every ring $A = A(r_1, r_2, x_0)$, $0 < r_1 < r_2 < d_0 = \text{dist}(x_0, \partial D)$, and for every measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that
\[
\int_{r_1}^{r_2} \eta(r) \, dr = 1. \tag{8.8}
\]

The notion was first introduced in the work \cite{154} in the connection with investigations of the Beltrami equations in the plane and then it was extended to the space in the work \cite{152}.

Let us recall the following criterion for ring $Q$-homeomorphisms, see Theorem 3.15 in \cite{152} or Theorem 7.2 in \cite{115}.

**Proposition 8.2.** Let $D$ be a domain in $\mathbb{R}^n$, $n \geq 2$, and $Q : D \to [0, \infty]$ a measurable function. A homeomorphism $f : D \to \mathbb{R}^n$ is a ring $Q$-homeomorphism at a point $x_0 \in D$ if and only if for every $0 < r_1 < r_2 < d_0 = \text{dist}(x_0, \partial D)$
\[
M(\Delta(fS_1, fS_2, fD)) \leq \frac{\omega_{n-1}}{I^{n-1}} \tag{8.9}
\]
where $\omega_{n-1}$ is the area of the unit sphere in $\mathbb{R}^n$, $S_j = \{x \in \mathbb{R}^n : |x - x_0| = r_j\}$, $j = 1, 2$, and
\[
I = I(r_1, r_2) = \int_{r_1}^{r_2} \frac{dr}{rq^{n-1}_{x_0}(r)}
\]
and $q_{x_0}(r)$ is the mean value of $Q(x)$ over the sphere $|x - x_0| = r$.

Note that the infimum from the right-hand side in \eqref{8.7} holds for the function
\[
\eta_0(r) = \frac{1}{Irq_{x_0}^{n-1}(r)}. \tag{8.10}
\]

**Remark 8.1.** By the Hesse and Ziemer equalities in \cite{58} and \cite{182}, see also the appendixes A3 and A6 in \cite{115}, we have
\[
M(\Delta(fS_1, fS_2, fD)) \leq \frac{1}{M^{n-1}(f\Sigma)} \tag{8.10}
\]
because $f\Sigma \subset \Sigma(fS_1, fS_2, fD)$ where $\Sigma$ is a collection of all spheres centered at $x_0$ between $S_1$ and $S_2$ and $\Sigma(fS_1, fS_2, fD)$ consists of all $(n - 1)$-dimensional surfaces in $fD$ that separate $fS_1$ and $fS_2$.

Thus, comparing the above criteria for lower and ring $Q$-homeomorphisms, we obtain the following conclusion at inner points.

**Corollary 8.1.** Each lower $Q$-homeomorphism $f : D \to D'$ in $\mathbb{R}^n$, $n \geq 2$, at a point $x_0 \in D$ is a ring $Q^*$-homeomorphism with $Q^* = Q^{n-1}$ at the point $x_0$.

**Corollary 8.2.** Each lower $Q$-homeomorphism $f : D \to D'$ in the plane at a point $x_0 \in D$ is a ring $Q$-homeomorphism at the point $x_0$.

It was proved in the work [84] that each homeomorphism $f$ of finite distortion in the plane is a lower $Q$-homeomorphism with $Q(x) = K_f(x)$. In the next section we show that the same is true for a homeomorphism $f$ of finite distortion in $\mathbb{R}^n$, $n \geq 3$, if, in addition, $f \in W^{1,\varphi}_{loc}$ where $\varphi$ satisfies the Calderon type condition (4.5).

# 9 Lower $Q$-homeomorphisms and Orlicz-Sobolev classes

The following statement is key for our further research.

**Theorem 9.1.** Let $D$ and $D'$ be domains in $\mathbb{R}^n$, $n \geq 3$, and let $\varphi : [0, \infty) \to [0, \infty)$ be an increasing function such that $\varphi(0) = 0$ and

$$\int_1^{\infty} \left[ \frac{t}{\varphi(t)} \right]^{\frac{1}{n-2}} \, dt < \infty. \quad (9.1)$$

Then each homeomorphism $f : D \to D'$ of finite distortion in the class $W^{1,\varphi}_{loc}$ is a lower $Q$-homeomorphism at every point $x_0 \in \overline{D}$ with $Q(x) = K_f(x)$.

**Proof.** Let $B$ be a (Borel) set of all points $x \in D$ where $f$ has a total differential $f'(x)$ and $J_f(x) \neq 0$. Then, applying Kirszbraun’s theorem and uniqueness of approximate differential, see, e.g., 2.10.43
and 3.1.2 in [31], we see that \( B \) is the union of a countable collection of Borel sets \( B_l, l = 1, 2, \ldots \), such that \( f_l = f|_{B_l} \) are bi-Lipschitz homeomorphisms, see, e.g., 3.2.2 as well as 3.1.4 and 3.1.8 in [31]. With no loss of generality, we may assume that the \( B_l \) are mutually disjoint. Denote also by \( B_* \) the rest of all points \( x \in D \) where \( f \) has the total differential but with \( f'(x) = 0 \).

By the construction the set \( B_0 := D \setminus (B \cup B_*) \) has Lebesgue measure zero, see Theorem 3.1. Hence by Theorem 4.4 \( \mathcal{A}_S(B_0) = 0 \) for a.e. hypersurface \( S \) in \( \mathbb{R}^n \) and, in particular, for a.e. sphere \( S_r := S(x_0, r) \) centered at a prescribed point \( x_0 \in \overline{D} \). Thus, by Corollary 4.3 \( \mathcal{A}_{S_r^*}(f(B_0)) = 0 \) as well as \( \mathcal{A}_{S_r^*}(f(B_*)) = 0 \) for a.e. \( S_r \) where \( S_r^* = f(S_r) \).

Let \( \Gamma \) be the family of all intersections of the spheres \( S_r, r \in (\varepsilon, \varepsilon_0), \varepsilon_0 < d_0 = \sup_{x \in D} |x - x_0| \), with the domain \( D \). Given \( \varrho_* \in \text{adm} f(\Gamma) \), \( \varrho_* \equiv 0 \) outside \( f(D) \), set \( \varrho \equiv 0 \) outside \( D \) and on \( B_0 \)

\[
\varrho(x) := \varrho_*(f(x)) \| f'(x) \| \quad \text{for } x \in D \setminus B_0.
\]

Arguing piecewise on \( B_l, l = 1, 2, \ldots \), we have by 1.7.6 and 3.2.2 in [31] that

\[
\int_{S_r} \varrho^{n-1} d\mathcal{A} \geq \int_{S_r^*} \varrho_*^{n-1} d\mathcal{A} \geq 1
\]

for a.e. \( S_r \) and, thus, \( \varrho \in \text{ext adm} \Gamma \).

The change of variables on each \( B_l, l = 1, 2, \ldots \), see, e.g., Theorem 3.2.5 in [31], and countable additivity of integrals give the estimate

\[
\int_D \frac{\varrho^n(x)}{K_f(x)} dm(x) \leq \int_{f(D)} \varrho_*^n(x) dm(x)
\]

and the proof is complete.

**Corollary 9.1.** Each homeomorphism \( f \) of finite distortion in \( \mathbb{R}^n \), \( n \geq 3 \), in the class \( W^{1, p}_{1, \text{loc}} \) for \( p > n - 1 \) is a lower \( Q \)-homeomorphism at every point \( x_0 \in \overline{D} \) with \( Q(x) = K_f(x) \).
Corollary 9.2. In particular, each homeomorphism $f$ of finite distortion in $\mathbb{R}^n$, $n \geq 3$, with $K_f \in L^p_{\text{loc}}$ for $p > n - 1$ is a lower $Q$-homeomorphism at every point $x_0 \in D$ with $Q(x) = K_f(x)$.

Corollary 9.3. Under the hypotheses of Theorem 9.1, each homeomorphism of finite distortion $f \in W^{1,\varphi}_{\text{loc}}$, in particular, $f \in W^{1,p}_{\text{loc}}$ for $p > n - 1$, is a ring $Q_\ast$-homeomorphism at every inner point $x_0 \in D$ with $Q_\ast(x) = [K_f(x)]^{n-1}$.

10 Equicontinuous and normal families

First of all, recall some general facts on normal families of mappings in metric spaces. Let $(X, d)$ and $(X', d')$ be metric spaces with distances $d$ and $d'$, respectively. A family $\mathcal{F}$ of continuous mappings $f : X \to X'$ is said to be normal if every sequence of mappings $f_m \in \mathcal{F}$ has a subsequence $f_{m_k}$ converging uniformly on each compact set $C \subset X$ to a continuous mapping. Normality is closely related to the following. A family $\mathcal{F}$ of mappings $f : X \to X'$ is said to be equicontinuous at a point $x_0 \in X$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that $d'(f(x), f(x_0)) < \varepsilon$ for all $f \in \mathcal{F}$ and $x \in X$ with $d(x, x_0) < \delta$. The family $\mathcal{F}$ is called equicontinuous if $\mathcal{F}$ is equicontinuous at every point $x_0 \in X$.

Given a domain $G$ in $\mathbb{R}^n$, $n \geq 2$, and an increasing function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$, $M \in [0, \infty)$ and $x_0 \in G$, denote by $\mathcal{F}^{\varphi} \mathcal{M}$ the collection of all continuous mappings $f : G \to \mathbb{R}^m$, $m \geq 1$, in the class $W^{1,1}_{\text{loc}}$ such that $f(x_0) = 0$ and

$$\int_G \varphi (|\nabla f|) \, dm(x) \leq M. \quad (10.1)$$

By Proposition 2.3 and Remark 2.1 and the Arzela-Ascoli theorem we obtain the following statement, cf., e.g., Theorem 8.1 in [64] and Theorem 4.3 in [45].
Corollary 10.1. If the function $\varphi$ satisfies the condition
\[ \int_{1}^{\infty} \left[ \frac{t}{\varphi(t)} \right]^{\frac{1}{n-1}} dt < \infty, \quad (10.2) \]
then the class $\mathcal{F}_M^\varphi$ is equicontinuous and hence normal. If in addition $\varphi$ is convex, then the class $\mathcal{F}_M^\varphi$ is also closed with respect to the locally uniform convergence.

Further we give the corresponding theorems for the classes of homeomorphic mappings under the condition (10.4) which is weaker than (10.2) and without (locally) uniform constraints of the type (10.1) in these classes.

In what follows, we use in $\mathbb{R}^n = \mathbb{R}^n \cup \{\infty\}$ the spherical (chordal) metric $h(x, y) = |\pi(x) - \pi(y)|$ where $\pi$ is the stereographic projection of $\mathbb{R}^n$ onto the sphere $S^n(\frac{1}{2}e_{n+1}, \frac{1}{2})$ in $\mathbb{R}^{n+1}$:
\[ h(x, y) = \frac{|x - y|}{\sqrt{1 + |x|^2} \sqrt{1 + |y|^2}}, \ x \neq \infty \neq y, \ h(x, \infty) = \frac{1}{\sqrt{1 + |x|^2}}. \]
Thus, by definition $h(x, y) \leq 1$ for all $x$ and $y \in \mathbb{R}^n$. The spherical (chordal) diameter of a set $E \subset \mathbb{R}^n$ is
\[ h(E) = \sup_{x,y \in E} h(x, y). \quad (10.3) \]
We use further the following statement of the Arzela-Ascoli type, see, e.g., Corollary 7.5. in [115].

Proposition 10.1. If $(X, d)$ is a separable metric space and $(X', d')$ is a compact metric space, then a family $\mathcal{F}$ of mappings $f : X \to X'$ is normal if and only if $\mathcal{F}$ is equicontinuous.

Combining Theorem 9.1 and Corollaries 9.1, 9.3 with the results of the work [152], see also Chapter 7 in [115], we have the following statements.

Theorem 10.1. Let $D$ and $D'$ be domains in $\mathbb{R}^n$, $n \geq 3$, and let $\varphi : [0, \infty) \to [0, \infty)$ be an increasing function such that $\varphi(0) = 0$
and
\[ \int_1^\infty \left[ \frac{t}{\varphi(t)} \right]^{\frac{1}{n-2}} dt < \infty. \]  

(10.4)

Let \( f : D \to D' \) be a homeomorphism of finite distortion in the Orlicz-Sobolev class \( W^{1,\varphi}_{\text{loc}} \) such that \( h(\mathbb{R}^n \setminus f(D)) \geq \Delta > 0 \). Then, for every \( x_0 \in D \) and \( x \in B(x_0, \varepsilon(x_0)) \), \( \varepsilon(x_0) < d(x_0) = \text{dist}(x_0, \partial D) \),

\[ h(f(x), f(x_0)) \leq \frac{\alpha_n}{\Delta} \exp \left\{ - \int_{|x-x_0|}^{\varepsilon(x_0)} dr \frac{1}{r k_{x_0}^{n-1}(r)} \right\} \]  

(10.5)

where \( \alpha_n \) is some constant depending only on \( n \) and \( k_{x_0}(r) \) is the average of \( [K_f(x)]^{n-1} \) over the sphere \( |x-x_0| = r \).

**Remark 10.1.** The estimate (10.5) can be written in the form

\[ h(f(x), f(x_0)) \leq \frac{\alpha_n}{\Delta} \exp \left\{ - \int_{|x-x_0|}^{\varepsilon(x_0)} dr \frac{1}{r \omega_{n-1}^{n-1}} \right\} \]  

(10.6)

where \( \|K_f\|_{n-1}(x_0, r) \) is the norm of \( K_f \) in the space \( L^{n-1} \) over the sphere \( |x-x_0| = r \) and \( \omega_{n-1} \) is the area of the unit sphere in \( \mathbb{R}^n \).

**Corollary 10.2.** The estimates (10.5) and (10.6) hold for homeomorphisms \( f \) of finite distortion in the Sobolev classes \( W^{1,p}_{\text{loc}} \), \( p > n-1 \). In particular, these estimates hold for homeomorphisms \( f \) of finite distortion with \( K_f \in L^q_{\text{loc}} \) for \( q > n-1 \).

**Corollary 10.3.** If

\[ k_{x_0}(r) \leq \left[ \log \left( \frac{1}{r} \right) \right]^{n-1} \]  

(10.7)

for \( r < \varepsilon(x_0) < \min \{1, d(x_0)\} \), then

\[ h(f(x), f(x_0)) \leq \frac{\alpha_n}{\Delta} \frac{1}{\log \frac{1}{\varepsilon(x_0)}} \]  

(10.8)

for all \( x \in B(x_0, \varepsilon(x_0)) \).
Corollary 10.4. If
\[ K_f(x) \leq \log \frac{1}{|x - x_0|}; \quad x \in B(x_0, \varepsilon(x_0)), \] (10.9)
then (10.8) holds in the ball \( B(x_0, \varepsilon(x_0)) \).

Corollary 10.5. Let \( n \geq 3 \), \( \varphi : [0, \infty) \to [0, \infty) \) be an increasing function, \( \varphi(0) = 0 \), satisfying (10.4). Let \( f : \mathbb{B}^n \to \mathbb{B}^n \), \( f(0) = 0 \), be a homeomorphism of finite distortion in the class \( W_{1,\varphi}^{1,\varphi} \) such that
\[ \int_{\varepsilon < |x| < 1} [K_f(x)]^{n-1} \frac{dm(x)}{|x|^n} \leq c \log \frac{1}{\varepsilon}, \quad \varepsilon \in (0, 1). \] (10.10)
Then
\[ |f(x)| \leq \gamma_n \cdot |x|^\beta_n \] (10.11)
where the constants \( \gamma_n \) and \( \beta_n \) depend only on \( n \).

Theorem 10.2. Let \( D \) and \( D' \) be domains in \( \mathbb{R}^n \), \( n \geq 3 \), and let \( \varphi : [0, \infty) \to [0, \infty) \) be an increasing function, \( \varphi(0) = 0 \), such that (10.4) holds. Suppose \( f : D \to D' \) is a homeomorphism of finite distortion in the class \( W_{1,\varphi}^{1,\varphi} \) such that \( h(\mathbb{R}^n \setminus f(D)) \geq \Delta > 0 \) and \( K_f(x) \leq Q(x) \) where \( Q^{n-1} \in \text{FMO}(x_0) \). Then
\[ h(f(x), f(x_0)) \leq \frac{\alpha_n}{\Delta} \left\{ \frac{\log \frac{1}{\varepsilon_0}}{\log \frac{1}{|x-x_0|}} \right\}^\beta \quad \forall \ x \in B(x_0, \varepsilon_0) \] (10.12)
where \( \varepsilon_0 < \text{dist}(x_0, \partial D) \) and \( \alpha_n \) depends only on \( n \) and \( \beta \) depends on the function \( Q \).

Corollary 10.6. In particular, the estimate (10.12) holds if
\[ \limsup_{\varepsilon \to 0} \int_{B(x_0, \varepsilon)} Q^{n-1}(x) \, dm(x) < \infty. \] (10.13)

Next, let \( D \) be a domain in \( \mathbb{R}^n \), \( n \geq 3 \), and let \( \varphi : [0, \infty) \to [0, \infty) \) be an increasing function, \( \varphi(0) = 0 \), \( Q : D \to [0, \infty] \) be a measurable function. Let \( \mathcal{O}_{Q,\Delta}^{\varphi} \) be the class of all homeomorphisms of finite distortion in the Orlicz-Sobolev class \( W_{1,\varphi}^{1,\varphi} \) such that \( h(\mathbb{R}^n \setminus f(D)) \)
\[\Delta > 0 \text{ and } K_f(x) \leq Q(x) \text{ a.e. Moreover, let } S_{Q,\Delta}^p, \ p \geq 1, \text{ denote the classes } O_{Q,\Delta}^\varphi \text{ with } \varphi(t) = t^p. \] Finally, let \( K_{Q,\Delta}^p \) be the class of all homeomorphisms with finite distortion such that \( K_f \in L_{\text{loc}}^p, \ p \geq 1, \ K_f(x) \leq Q(x) \text{ a.e. and } h(\mathbb{R}^n \setminus f(D)) \geq \Delta > 0. \)

By Proposition 10.1 the above estimates of distortion now yield:

**Theorem 10.3.** Let \( \varphi : [0, \infty) \to [0, \infty) \) be an increasing function such that \( \varphi(0) = 0 \) and (10.4) hold. If \( Q^{n-1} \in \text{FMO} \), then \( O_{\varphi Q, \Delta} \) is a normal family.

**Corollary 10.7.** Under (10.4) the class \( O_{\varphi Q, \Delta} \) is normal if

\[
\lim_{\varepsilon \to 0} \int_{B(x_0, \varepsilon)} Q^{n-1}(x) \, dm(x) < \infty \quad \forall \ x_0 \in D. \tag{10.14}
\]

**Corollary 10.8.** In particular, the classes \( S_{Q,\Delta}^p \) and \( K_{Q,\Delta}^p \) are normal under \( p > n - 1 \) if either \( Q^{n-1} \in \text{FMO} \) or (10.14) holds.

**Theorem 10.4.** Let \( \Delta > 0 \) and \( Q : D \to [0, \infty] \) be a measurable function such that

\[
\varepsilon(x_0) \int_0^{\varepsilon(x_0)} \frac{dr}{||Q||_{n-1}(x_0, r)} = \infty \quad \forall \ x_0 \in D \tag{10.15}
\]

where \( \varepsilon(x_0) < \text{dist}(x_0, \partial D) \) and \( ||Q||_{n-1}(x_0, r) \) denotes the norm of \( Q \) in \( L^{n-1} \) over the sphere \( |x - x_0| = r \). Then the classes \( O_{Q,\Delta}^\varphi, S_{Q,\Delta}^p, K_{Q,\Delta}^p \) form normal families if \( \varphi \) satisfies (10.4), correspondingly, \( p > n - 1 \).

**Corollary 10.9.** The classes \( O_{Q,\Delta}^\varphi, S_{Q,\Delta}^p, K_{Q,\Delta}^p \) form normal families if \( \varphi \) satisfies (10.4), correspondingly, \( p > n - 1 \) and \( Q(x) \) has singularities only of the logarithmic type.

Let \( D \) be a fixed domain in the extended space \( \overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}, \ n \geq 3, \ \varphi : [0, \infty) \to [0, \infty) \) be an increasing function, \( \varphi(0) = 0. \) Given a function \( \Phi : [0, \infty] \to [0, \infty], \ M > 0, \ \Delta > 0, \ O_{M,\Delta}^{\Phi,\varphi} \) denotes
the collection of all homeomorphisms of finite distortion in the Orlicz-Sobolev class \( W^1_{\text{loc}} \) such that \( h((\mathbb{R}^n \setminus f(D)) \geq \Delta > 0 \) and

\[
\int_D \Phi \left( K_f^{-1}(x) \right) \frac{dm(x)}{(1 + |x|^2)^n} \leq M. \tag{10.16}
\]

Similarly, \( S^{\Phi,\varphi}_{M,\Delta}, p \geq 1 \), denote the classes \( O^{\Phi,\varphi}_{M,\Delta} \) with \( \varphi(t) = t^p \).

Finally, let \( K^{\Phi,p}_{M,\Delta}, p \geq 1 \), be the class of all homeomorphisms with finite distortion such that \( K_f \in L^p_{\text{loc}}, p \geq 1 \), (10.16) holds for \( K_f \) and \( h((\mathbb{R}^n \setminus f(D)) \geq \Delta > 0 \).

Combining Theorem 9.1, Corollaries 9.1–9.3 and also Theorem 6.1 under \( p = n - 1 \), we have the following statements, cf. [153].

**Theorem 10.5.** Let \( \Phi : [0, \infty] \to [0, \infty] \) be a convex increasing function such that

\[
\int_{\delta_0}^{\infty} \frac{d\tau}{\tau [\Phi^{-1}(\tau)]^{n-1}} = \infty \tag{10.17}
\]

for some \( \delta_0 > \Phi(0) \). Then the classes \( O^{\Phi,\varphi}_{M,\Delta} \) under (10.4) and \( S^{\Phi,\varphi}_{M,\Delta} \) and \( K^{\Phi,p}_{M,\Delta} \) under \( p > n - 1 \) are equicontinuous and, consequently, form normal families of mappings for every \( M \in (0, \infty) \) and \( \Delta \in (0, 1) \).

**Remark 10.2.** As it follows from [153], the condition (10.17) is not only sufficient but also necessary for normality of the given classes. Moreover, by Proposition 6.1 we may use instead of (10.17) each of the equivalent conditions (6.4)–(6.9) under \( p = n - 1 \).

11 On domains with regular boundaries

Recall first of all the following topological notion. A domain \( D \subset \mathbb{R}^n, n \geq 2 \), is said to be **locally connected at a point** \( x_0 \in \partial D \) if, for every neighborhood \( U \) of the point \( x_0 \), there is a neighborhood \( V \subseteq U \) of \( x_0 \) such that \( V \cap D \) is connected. Note that every Jordan domain \( D \) in \( \mathbb{R}^n \) is locally connected at each point of \( \partial D \), see, e.g., [179], p. 66.
We say that $\partial D$ is **weakly flat at a point** $x_0 \in \partial D$ if, for every neighborhood $U$ of the point $x_0$ and every number $P > 0$, there is a neighborhood $V \subset U$ of $x_0$ such that

$$M(\Delta(E, F; D)) \geq P$$

(11.1)

for all continua $E$ and $F$ in $D$ intersecting $\partial U$ and $\partial V$. Here and later on, $\Delta(E, F; D)$ denotes the family of all paths $\gamma : [a, b] \to \mathbb{R}^n$ connecting $E$ and $F$ in $D$, i.e., $\gamma(a) \in E$, $\gamma(b) \in F$ and $\gamma(t) \in D$ for all $t \in (a, b)$. We say that the boundary $\partial D$ is **weakly flat** if it is weakly flat at every point in $\partial D$.

We also say that a point $x_0 \in \partial D$ is **strongly accessible** if, for every neighborhood $U$ of the point $x_0$, there exist a compactum $E$ in $D$, a neighborhood $V \subset U$ of $x_0$ and a number $\delta > 0$ such that

$$M(\Delta(E, F; D)) \geq \delta$$

(11.2)

for all continua $F$ in $D$ intersecting $\partial U$ and $\partial V$. We say that the boundary $\partial D$ is **strongly accessible** if every point $x_0 \in \partial D$ is strongly accessible.
Here, in the definitions of strongly accessible and weakly flat boundaries, one can take as neighborhoods $U$ and $V$ of a point $x_0$ only balls (closed or open) centered at $x_0$ or only neighborhoods of $x_0$ in another fundamental system of neighborhoods of $x_0$. These concepts can also be extended in a natural way to the case of $\mathbb{R}^n$ and $x_0 = \infty$. Then we must use the corresponding neighborhoods of $\infty$.

It is easy to see that if a domain $D$ in $\mathbb{R}^n$ is weakly flat at a point $x_0 \in \partial D$, then the point $x_0$ is strongly accessible from $D$. Moreover, it was proved by us that if a domain $D$ in $\mathbb{R}^n$ is weakly flat at a point $x_0 \in \partial D$, then $D$ is locally connected at $x_0$, see, e.g., Lemma 5.1 in [81] or Lemma 3.15 in [115].

The notions of strong accessibility and weak flatness at boundary points of a domain in $\mathbb{R}^n$ defined in [80] are localizations and generalizations of the corresponding notions introduced in [113]–[114], cf. with the properties $P_1$ and $P_2$ by Väisälä in [173] and also with the quasiconformal accessibility and the quasiconformal flatness by Näkki in [121]. Many theorems on a homeomorphic extension to the boundary of quasiconformal mappings and their generalizations are valid under the condition of weak flatness of boundaries. The condition of strong accessibility plays a similar role for a continuous extension of the mappings to the boundary. In particular, recently we have proved the following significant statements, see either Theorem 10.1 (Lemma 6.1) in [81] or Theorem 9.8 (Lemma 9.4) in [115].

**Proposition 11.1.** Let $D$ and $D'$ be bounded domains in $\mathbb{R}^n$, $n \geq 2$, $Q : D \rightarrow (0, \infty)$ a measurable function and $f : D \rightarrow D'$ a lower $Q$-homeomorphism on $\partial D$. Suppose that the domain $D$ is locally connected on $\partial D$ and that the domain $D'$ has a (strongly accessible) weakly flat boundary. If

$$
\delta(x_0) \int_0^\infty \frac{dr}{\|Q\|_{n-1}(x_0, r)} = \infty \quad \forall \ x_0 \in \partial D \quad (11.3)
$$
for some $\delta(x_0) \in (0, d(x_0))$ where $d(x_0) = \sup_{x \in D} |x - x_0|$ and

$$|\{Q\}_{n-1}(x_0, r) = \left( \int_{D \cap S(x_0, r)} Q^{n-1}(x) \, dA \right)^{\frac{1}{n-1}},$$

then $f$ has a (continuous) homeomorphic extension $\overline{f}$ to $\overline{D}$ that maps $\overline{D}$ (into) onto $\overline{D'}$.

Here as usual $S(x_0, r)$ denotes the sphere $|x - x_0| = r$ and the closure is understood in the sense of the extended space $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$.

A domain $D \subset \mathbb{R}^n$ is called a quasiextremal distance domain, abbr. QED-domain, see [37], if

$$M(\Delta(E, F; \mathbb{R}^n) \leq K \cdot M(\Delta(E, F; D)) \quad (11.4)$$

for some $K \geq 1$ and all pairs of nonintersecting continua $E$ and $F$ in $D$.

It is well known, see e.g. Theorem 10.12 in [173], that

$$M(\Delta(E, F; \mathbb{R}^n)) \geq c_n \log \frac{R}{r} \quad (11.5)$$

for any sets $E$ and $F$ in $\mathbb{R}^n$, $n \geq 2$, intersecting all the circles $S(x_0, \rho)$, $\rho \in (r, R)$. Hence a QED-domain has a weakly flat boundary. One example in [115], Section 3.8, shows that the inverse conclusion is not true even among simply connected plane domains.

A domain $D \subset \mathbb{R}^n$, $n \geq 2$, is called a uniform domain if each pair of points $x_1$ and $x_2 \in D$ can be joined with a rectifiable curve $\gamma$ in $D$ such that

$$s(\gamma) \leq a \cdot |x_1 - x_2| \quad (11.6)$$

and

$$\min_{i=1,2} s(\gamma(x_i, x)) \leq b \cdot d(x, \partial D) \quad (11.7)$$

for all $x \in \gamma$ where $\gamma(x_i, x)$ is the portion of $\gamma$ bounded by $x_i$ and $x$, see [117]. It is known that every uniform domain is a QED-domain.
but there exist QED-domains that are not uniform, see [37]. Bounded convex domains and bounded domains with smooth boundaries are simple examples of uniform domains and, consequently, QED-domains as well as domains with weakly flat boundaries.

A closed set $X \subset \mathbb{R}^n$, $n \geq 2$, is called a null-set for extremal distances, abbr. NED-set, if
\[ M(\Delta(E, F; \mathbb{R}^n)) = M(\Delta(E, F; \mathbb{R}^n \setminus X)) \]  
for any two nonintersecting continua $E$ and $F \subset \mathbb{R}^n \setminus X$.

**Remark 11.1.** It is known that if $X \subset \mathbb{R}^n$, $n \geq 2$, is a NED-set, then
\[ |X| = 0 \]  
and $X$ does not locally disconnect $\mathbb{R}^n$, i.e., see [61],
\[ \dim X \leq n - 2, \]  
and, conversely, if a set $X \subset \mathbb{R}^n$ is closed and
\[ H^{n-1}(X) = 0, \]  
then $X$ is a NED-set, see [175]. Note also that the complement of a NED-set in $\mathbb{R}^n$ is a very particular case of a QED-domain.

Further we denote by $C(X, f)$ the cluster set of the mapping $f : D \to \overline{\mathbb{R}^n}$ for a set $X \subset \overline{D}$,
\[ C(X, f) := \left\{ y \in \mathbb{R}^n : y = \lim_{k \to \infty} f(x_k), \; x_k \to x_0 \in X, \; x_k \in D \right\}. \]  
(11.12)

Note that the inclusion $C(\partial D, f) \subseteq \partial D'$ holds for every homeomorphism $f : D \to D'$, see, e.g., Proposition 13.5 in [115].

**12 The boundary behavior**

In this section we assume that $\varphi : [0, \infty) \to [0, \infty)$ is an increasing function with $\varphi(0) = 0$ such that
\[ \int_1^\infty \left[ \frac{t}{\varphi(t)} \right]^{\frac{1}{n-2}} dt < \infty. \]  
(12.1)
In view of Theorem 9.1, we have by Proposition 11.1 the following statement.

**Theorem 12.1.** Let $D$ and $D'$ be bounded domains in $\mathbb{R}^n$, $n \geq 3$, and let $f : D \to D'$ be a homeomorphism of finite distortion in $W^{1,\varphi}_{\text{loc}}$ with the condition (12.1). Suppose that the domain $D$ is locally connected on $\partial D$ and that the domain $D'$ has a (strongly accessible) weakly flat boundary. If

$$
\delta(x_0) \int_0^\delta \frac{dr}{|| K_f ||_{n-1}(x_0, r)} = \infty \quad \forall \ x_0 \in \partial D
$$

(12.2)

for some $\delta(x_0) \in (0, d(x_0))$ where $d(x_0) = \sup_{x \in D} |x - x_0|$ and

$$
|| K_f ||_{n-1}(x_0, r) = \left( \int_{D \cap S(x_0, r)} K_f^{n-1}(x) \, d\mathcal{A} \right)^{\frac{1}{n-1}},
$$

then $f$ has a (continuous) homeomorphic extension $\overline{f}$ to $\overline{D}$ that maps $\overline{D}$ (into) onto $\overline{D'}$.

In particular, as a consequence of Theorem 12.1 we obtain the following generalization of the well-known Gehring-Martio theorem on a homeomorphic extension to the boundary of quasiconformal mappings between QED domains, cf. [37].

**Corollary 12.1.** Let $D$ and $D'$ be bounded domains with weakly flat boundaries in $\mathbb{R}^n$, $n \geq 3$, and let $f : D \to D'$ be a homeomorphism of finite distortion in $D$ in the class $W^{1,p}_{\text{loc}}$, $p > n - 1$, in particular, $K_f \in L^q_{\text{loc}}$, $q > n - 1$. If the condition (12.2) holds at every point $x_0 \in \partial D$, then $f$ has a homeomorphic extension to $\overline{D}$.

The continuous extension to the boundary of the inverse mappings has a simpler criterion. Namely, in view of Theorem 9.1, we have by Theorem 9.1 in [81] or Theorem 9.6 in [115] the next statement.
Theorem 12.2. Let $D$ and $D'$ be domains in $\mathbb{R}^n$, $n \geq 3$, $D$ be locally connected on $\partial D$ and $\partial D'$ be weakly flat. If $f$ is a homeomorphism of finite distortion of $D$ onto $D'$ in the class $W^{1,\varphi}_{\text{loc}}$ with the condition (12.1) and $K_f \in L^{n-1}(D)$, then $f^{-1}$ has an extension to $\overline{D'}$ by continuity in $\mathbb{R}^n$.

However, as it follows from the example in Proposition 6.3 from [115], any degree of integrability $K_f \in L^q(D)$, $q \in [1, \infty)$, cannot guarantee the extension by continuity to the boundary of the direct mappings.

Similarly, in view of Theorem [9.1], we have by Theorem 8.1 in [81] or Theorem 9.5 in [115] the next result.

Theorem 12.3. Let $D$ be a domain in $\mathbb{R}^n$, $n \geq 3$, $X \subset D$, and let $f$ be a homeomorphism with finite distortion of $D \setminus X$ into $\mathbb{R}^n$ in $W^{1,\varphi}_{\text{loc}}$ with the condition (12.1). Suppose that $X$ and $C(X, f)$ are NED sets. If
\[
\int_0^{\varepsilon(x_0)} \frac{dr}{\| K_f \|_{n-1}(x_0, r)} = \infty \quad \forall \ x_0 \in \partial D \quad (12.3)
\]
where $0 < \varepsilon_0 < d_0 = \text{dist} (x_0, \partial D)$ and
\[
\| K_f \|_{n-1}(x_0, r) = \left( \int_{|x-x_0|=r} K_f^{n-1}(x) \, dA \right)^{\frac{1}{n-1}}, \quad (12.4)
\]
then $f$ is extended by continuity in $\overline{\mathbb{R}^n}$ to $D$.

Remark 12.1. In particular, the conclusion of Theorem [12.3] is valid if $X$ is a closed set with
\[
H^{n-1}(X) = 0 = H^{n-1}(C(X, f)). \quad (12.5)
\]

Finally, in view of Theorem [9.1], by Theorem [12.1] as well as by Theorem [6.1] under $p = n - 1$ we obtain the following result.
Theorem 12.4. Let $D$ and $D'$ be bounded domains in $\mathbb{R}^n$, $n \geq 3$, $D$ be locally connected on $\partial D$ and $D'$ have (strongly accessible) weakly flat boundary. Suppose $f : D \to D'$ is a homeomorphism of finite distortion in $D$ in the class $W^{1,\varphi}_{\text{loc}}$ with the condition \((12.1)\) such that

$$\int_D \Phi(K_f^{n-1}(x)) \, dm(x) < \infty$$ \hspace{1cm} (12.6)

for a convex increasing function $\Phi : [0, \infty] \to [0, \infty]$. If

$$\int_{\delta_0}^{\infty} \frac{d\tau}{\tau \left[ \Phi^{-1}(\tau) \right]^{\frac{1}{n-1}}} = \infty$$ \hspace{1cm} (12.7)

for some $\delta_0 > \Phi(0)$, then $f$ has a (continuous) homeomorphic extension $\overline{f}$ to $\overline{D}$ that maps $\overline{D}$ (into) onto $\overline{D'}$.

Remark 12.2. Note that by Theorem 5.1 and Remark 5.1 in [83] the conditions \((12.7)\) are not only sufficient but also necessary for continuous extension to the boundary of $f$ with the integral constraints \((12.6)\).

Recall that by Proposition 6.1 the condition \((12.7)\) is equivalent to each of the conditions \((6.4)\)–\((6.9)\) under $p = n - 1$ and, in particular, to the following condition

$$\int_{\delta}^{\infty} \log \Phi(t) \frac{dt}{tn'} = +\infty$$ \hspace{1cm} (12.8)

for some $\delta > 0$ where $\frac{1}{n'} + \frac{1}{n} = 1$, i.e., $n' = 2$ for $n = 2$, $n'$ is strictly decreasing in $n$ and $n' = n/(n - 1) \to 1$ as $n \to \infty$.

Finally note that all the results in this section hold, in particular, if $f \in W^{1,p}_{\text{loc}}$, $p > n - 1$ and, in particular, if $K_f \in L^q_{\text{loc}}$, $q > n - 1$, and, in particular, if $D$ and $D'$ are either bounded convex domains or bounded domains with smooth boundaries.
13 Some examples

The following lemma is a base for demonstrating preciseness of the Calderon type conditions in the above results.

**Lemma 13.1.** Let \( \varphi : [0, \infty) \to [0, \infty) \) be a convex increasing function such that

\[
\int_1^\infty \left[ t \varphi(t) \right]^{1/(k-1)} dt = \infty
\]

for a natural number \( k \geq 2 \). Then there is an embedding \( g \) of \( \mathbb{R}^k \) into \( \mathbb{R}^{k+1} \) of the form \( g(x) = (x, f(x)) \), such that \( g \in W^{1,\varphi}_{\text{loc}} \) but \( g \) has not \((N)\)-property with respect to \( k \)-dimensional Hausdorff measure.

**Corollary 13.1.** For every \( k \geq 2 \), there is an embedding \( g \) of \( \mathbb{R}^k \) into \( \mathbb{R}^{k+1} \) of the form \( g(x) = (x, f(x)) \) in the class \( W^{1,k}_{\text{loc}} \) that has not \((N)\)-property with respect to \( k \)-dimensional Hausdorff measure.

**Remark 13.1.** The corresponding examples of embeddings \( g \) in the class \( W^{1,k}_{\text{loc}} \) for \( k = 2 \) from \( \mathbb{R}^2 \) into \( \mathbb{R}^3 \) based on the theory of conformal mappings are known long ago, see, e.g., [142] and [145]. However, they have not the form \( g(x) = (x, f(x)) \) and cannot be applied for constructing examples of homeomorphisms in the class \( W^{1,2}_{\text{loc}} \) from \( \mathbb{R}^3 \) into \( \mathbb{R}^3 \) as in Theorem 13.1 and Corollary 13.2 further.

**Proof of Lemma 13.1.** We apply further for our purposes a little modified construction of Calderon in [16], p. 210-211. Let \( F \) and \( F_* \) be functions from Proposition 2.4 corresponding to the function \( \varphi_*(t) = \varphi(t + k) - \varphi(k) \). It is clear that \( \varphi_* \) satisfies (13.1), too.

Let us give 3 decreasing sequences of positive numbers \( r_l, \varrho_l \) and \( \varrho^*_l, l = 1, 2, \ldots \), by induction in the following way. Set \( r_1 \) is equal to the maximal number \( r > 0 \) such that \( r \leq 2^{-2} \) and

\[
\int_{|x| \leq r} \varphi_* (|\nabla F_*|) \, dm(x) \leq 2^{-k}.
\]
The numbers $\varrho_1$ and $\varrho_1^*$ are defined from the equalities $F(\varrho_1) = F(r_1) + 1$ and $F(\varrho_1^*) = F(r_1) + 3/4$, correspondingly. If the numbers $r_1, \ldots, r_{l-1}, \varrho_1, \ldots, \varrho_{l-1}$ and $\varrho_1^*, \ldots, \varrho_{l-1}^*$ have been given, then we set $r_l$ is equal to the maximal number $r > 0$ such that

$$
\int_{|x| \leq r} \varphi_\ast (|\nabla F|) \, dm(x) \leq 2^{-lk}
$$

and, moreover,

$$
r \leq \min \{ \varrho_{l-1}, 2^{-2l}(\varrho_{l-1}^* - \varrho_{l-1}), 1/[2^{l+2}(l - 1)|F'(\varrho_{l-1})|] \} .
$$

(13.3)

Then we define $\varrho_l$ and $\varrho_l^*$ from the equalities $F(\varrho_l) = F(r_l) + 1$ and $F(\varrho_l^*) = F(r_l) + 3/4$, respectively. Note that by monotonicity of the derivative $|F'(\varrho_{l-1})| = \max \{|F'(t)| : t \geq \varrho_{l-1} \}$. It is also clear by the construction that $\varrho_l < \varrho_l^* < r_l$ and that the sequence $\varrho_l^* - \varrho_l < r_l$ is decreasing because the function $F'(t)$ is non-decreasing.

Setting $F_l(r) = \min [1, F(r) - F(r_l)]$ for $r \in [0, r_l]$ and $F_l(r) \equiv 0$ for $r > r_l$, we see that $F_l$ satisfies a uniform Lipschitz condition, that $F_l(0) = 1$ and by (13.2)

$$
\int_{\mathbb{R}^k} \varphi_\ast (|\nabla F_l^\ast|) \, dm(x) \leq 2^{-lk}
$$

(13.4)

where $F_l^\ast(x) = F_l(|x|)$, $x \in \mathbb{R}^k$, $l = 1, 2, \ldots$.

Now, denote by $x^l_{j_1, \ldots, j_k}$, $l = 1, 2, \ldots, j_1, \ldots, j_k = 0, \pm 1, \pm 2, \ldots$, the points in $\mathbb{R}^k$ whose coordinates are integral multiples of $2^{-l}$ with the natural order in $j_1, \ldots, j_k$ along the corresponding coordinate axes. Let $B^l_{j_1, \ldots, j_k}$ be the closed balls centered at $x^l_{j_1, \ldots, j_k}$ with the radii $r_l$. Note that by the second condition in (13.3) $r_l \leq 2^{-2l}$ and the given balls are disjoint each to other for every fixed $l = 2, \ldots$. Next, define

$$
f_l(x) = \sum_{j_1, \ldots, j_k} F_l \left(|x - x^l_{j_1, \ldots, j_k}|\right),
$$

$$
f^\ast_p(x) = \sum_{l=1}^{p} 2^{-l} f_l(x)
$$
and
\[ f(x) = \sum_{l=1}^{\infty} 2^{-l} f_l(x) = \lim_{p \to \infty} f_p^*(x). \]

By the construction, the nonnegative functions \( f_l(x), f_p^*(x) \) and \( f(x) \leq 1 \) are continuous. Moreover, it is easy to estimate their oscillations on the balls \( B_{j_1, \ldots, j_k}^p \). In particular,

\[ \text{osc}_{B_{j_1, \ldots, j_k}^p} f_{p-1}^* \leq 2^{-p} \sum_{l=1}^{p-1} 2^{-l} \text{osc}_{B_{j_1, \ldots, j_k}^p} f_l \leq r_p \sum_{l=1}^{p-1} |F'(q_l)| \leq r_p (p-1) |F'(q_{p-1})| < 2^{-(p+2)} < 2^{-(p-1)}. \quad (13.5) \]

Indeed, by the triangle inequality and the monotonicity of \( F' \),

\[ \text{osc}_{B_{j_1, \ldots, j_k}^p} f_{p-1}^* \leq \sum_{l=1}^{p-1} 2^{-l} \text{osc}_{B_{j_1, \ldots, j_k}^p} f_l \leq r_p \sum_{l=1}^{p-1} |F'(q_l)| \leq r_p (p-1) |F'(q_{p-1})| \]

and, thus, applying the third condition in (13.3), we come to the (13.5).

Let us show that the mapping \( g(x) = (x, f(x)) \) belongs to the class \( W^{1,\varphi}_{1,\varphi} \). To this end, consider an arbitrary closed oriented unit cube \( C \) in \( \mathbb{R}^k \) whose vertices have irrational coordinates. Note that the cube \( C \) contains exactly \( 2^{lk} \) points \( x^l_{j_1, \ldots, j_k} \). Thus, by periodicity of the picture and the condition (13.4) we have that

\[ \int_C \varphi_* (|\nabla f_l|) \, dm(x) \leq 1 \quad (13.6) \]

and, applying the (discrete) Jensen inequality, see, e.g., Theorem 86 in [49], we obtain that

\[ \int_C \varphi_* (|\nabla f|) \, dm(x) \leq \int_C \varphi_* \left( \frac{\sum_{l=1}^{\infty} 2^{-l} |\nabla f_l|}{\sum_{l=1}^{\infty} 2^{-l}} \right) \, dm(x) \leq \sum_{l=1}^{\infty} 2^{-l} \int_C \varphi_* (|\nabla f_l|) \, dm(x) \leq 1, \]
Finally, since $|\nabla g| = \sqrt{k + |\nabla f|^2} \leq k + |\nabla f|$, we have that
\[ \int_C \varphi(|\nabla g|) \, dm(x) \leq 1 + \varphi(k). \quad (13.7) \]

Next, let us fix a closed oriented unit cube $C_0$ in $\mathbb{R}^k$ whose center has irrational coordinates and let $E_l$, $l = 1, 2, \ldots$, be the union of all balls $B_{j_1, \ldots, j_k}^l$ centered at points $x_{j_1, \ldots, j_k}^l$ in the cube $C_0$. By the second condition in (13.3) we have that $|E_l| \leq 2^{lk} \cdot \Omega_k \cdot 2^{-2lk} = \Omega_k 2^{-lk}$ where $\Omega_k$ is the volume of the unit ball in $\mathbb{R}^k$. Setting $E_m = \bigcup_{l=m}^\infty E_l$, $m = 1, 2 \ldots$, we see that $|E_m| \leq \frac{\Omega_k}{2^{k-1}} 2^{-k(m-1)} \to 0$ as $m \to \infty$, i.e., the set $E = \bigcap_{m=1}^\infty E_m$ is of the Lebesgue measure zero in $\mathbb{R}^k$. Similarly, $\mu_{k-1}(pr_i E_l) \leq \Omega_{k-1} 2^{-l(k-1)}$ and $\mu_{k-1}(pr_i E_m) \leq \frac{\Omega_{k-1}}{2^{k-1-1}} 2^{-(k-1)(m-1)} \to 0$ as $m \to \infty$, i.e., $\mu_{k-1}(pr_i E) = 0$ where $pr_i$ denotes the projection into the coordinate hyperplane $P_i$ which is perpendicular to the $i$-th coordinate axis, $i = 1, 2, \ldots, k$ in $\mathbb{R}^k$ and $\mu_{k-1}$ is the $(k - 1)$-dimensional Lebesgue measure on $P_i$.

Let us prove that every straight line segment $L$ in the cube $C$ which is parallel to a coordinate axis, say to the axis $Ox_1$, and does not intersect the set $E$ has only a finite number of joint points with the spheres $\partial B_{j_1, \ldots, j_k}^l$. Indeed, assume that such a segment $L$ intersects an infinite number of the closed balls $B_{j_1, \ldots, j_k}^l$. Recall that the cube $C$ intersects only a finite number of such balls under each fixed $l = 1, 2, \ldots$. Hence there exists an infinite sequence of balls $B_{j_1, \ldots, j_k}^{l_m}$ such that $L \cap B_{l_m} \neq \emptyset$, $m = 1, 2, \ldots$ and $l_{m_1} \neq l_{m_2}$ for $m_1 \neq m_2$, i.e., $l_m \to \infty$ as $m \to \infty$. Note that the end points of the segment $L$ can belong only to a finite number of the balls $B_{l_m}$ because in the contrary case it would be $L \cap E \neq \emptyset$. Thus, we may assume that length $L \cap B_{l_m} > 0$ for all $m = 1, 2, \ldots$. Remark that the distance between the centers $x_{j_1, \ldots, j_k}^{l_m}$ of the balls $B_{j_1, \ldots, j_k}^{l_m}$, $j_1 = 0, \pm 1, \pm 2, \ldots$, as well as between their projections on the straight line of $L$, is equal to $2^{-l_m} \to 0$ as $m \to \infty$. Hence we may assume without loss of generality that the sequence of the segments $L \cap B_{l_m}$
is monotone decreasing. However, then by the Cantor theorem, see e.g. 4.41.I (2') in [100], we obtain \( \bigcap_{m=1}^{\infty} B_{t_m} \cap L \neq \emptyset \) that contradicts the condition \( E \cap L = \emptyset \).

Thus, \( g \) is piecewise monotone and smooth and hence it is absolutely continuous on a.e. segment \( L \) in the cube \( C \) which is parallel to a coordinate axis. Consequently, \( g \) is ACL and by (13.7) \( g \in W^{1, \varphi}_{loc} \).

Let us show that the set \( E = g(E) \) in \( \mathbb{R}^{k+1} \) is not of \( k \)-dimensional Hausdorff measure zero.

Given a closed oriented unit cube \( C_* \) in \( \mathbb{R}^k \) whose center has irrational coordinates and whose edge length \( L = 2^{-m} \) for some positive integer \( m \), \( l \geq m \), we have that

\[
\sum [d(f(B_{j_1, \ldots, j_k}^l))]^k \leq 2^{3k} L^k
\]  

(13.8)

where the sum is taken over all balls \( B_{j_1, \ldots, j_k}^l \) whose centers \( x_{j_1, \ldots, j_k}^l \) belong to the cube \( C_* \). Indeed, \( C_* \) contains exactly \( 2^{(l-m)k} \) points \( x_{j_1, \ldots, j_k}^l \). Moreover, every set \( f(B_{j_1, \ldots, j_k}^l) \) is contained in a cylinder whose base radius is less or equal to \( 2^{-2l} \), see (13.3), and whose height is less or equal to \( 2^{-(l-1)} + 2^{-l} + \ldots = 2^{-(l-2)} \), see (13.5). Hence

\[
d(f(B_{j_1, \ldots, j_k}^l)) \leq \sqrt{2^{-2(l-2)} + 2^{2(-2l+1)}} = 2^{-(l-2)} \cdot \sqrt{1 + 2^{-2(l+1)}} \leq 2^{-(l-3)} = 8 \cdot 2^{-l}
\]

that implies (13.8).

Now, let us prove the following lower estimate of the diameters of the images of the balls \( B_{j_1, \ldots, j_k}^p \):

\[
d\left(f(B_{j_1, \ldots, j_k}^p)\right) \geq 2^{-(p+1)}.
\]  

(13.9)

It is sufficient for this purpose to show that

\[
\text{osc}_{B_{j_1, \ldots, j_k}^p} f \geq 2^{-(p+1)}
\]
and, in turn, it suffices to demonstrate that
\[
\text{osc}_{L^p_{j_1,\ldots,j_k}} f \geq 2^{-(p+1)}
\]
where \(L^p_{j_1,\ldots,j_k}\) is the intersection of the ball \(B^p_{j_1,\ldots,j_k}\) with the straight line \(L\) passing through its center \(x^p_{j_1,\ldots,j_k}\) parallelly to one of the coordinate axes. Indeed, by the condition (13.3), the length of the intersection of the line \(L\) with the set \(E_{p+1}\) can be easily estimated:
\[
\text{length}(L \cap E_{p+1}) \leq \sum_{l=p+1}^{\infty} 2r_l \cdot 2^l \leq 2 \sum_{l=p+1}^{\infty} 2^{-2l}(q^*_{l-1} - q_{l-1}) \cdot 2^l \leq 2(q^*_p - q_p) \sum_{l=p+1}^{\infty} 2^{-l} \leq 2^{-(p-1)}(q^*_p - q_p) \leq q^*_p - q_p.
\]
Hence by the choice of the number \(q^*_p\)
\[
\text{osc}_{\Delta^p_{j_1,\ldots,j_k}} f \geq \frac{3}{4} \text{osc}_{B^p_{j_1,\ldots,j_k}} f_p
\]
where \(\Delta^p_{j_1,\ldots,j_k} = L^p_{j_1,\ldots,j_k} \setminus E_{p+1}\). Thus, by the condition (13.5) and the triangle inequality
\[
\text{osc}_{\Delta^p_{j_1,\ldots,j_k}} f = \text{osc}_{\Delta^p_{j_1,\ldots,j_k}} f^*_p \geq \text{osc}_{\Delta^p_{j_1,\ldots,j_k}} f_p - \text{osc}_{\Delta^p_{j_1,\ldots,j_k}} f^*_{p-1} \geq \frac{1}{2} \cdot 2^{-p} \text{osc}_{B^p_{j_1,\ldots,j_k}} f_p = 2^{-(p+1)}
\]
and the lower estimate (13.9) follows.

Finally, let \(\varepsilon > 0\) and let \(\{A_j\}\) be a cover of \(E\) such that \(d(A_j) < \varepsilon\), \(j = 1, 2, \ldots\). Note that for each \(A_j\) there is a closed oriented cube \(C_j\) such that \(A_j \subseteq C_j\) and whose edge length \(L_j\) is less or equal to \(d(A_j)\). However, it is more convenient to use closed oriented cubes with \(L_j = 2^{-m_j}\) for some positive integer \(m_j\) such that \(L_j \leq 2d(A_j)\). Let \(\mathbb{N}\) be the collection of all positive integers. Set for arbitrary \(l \in \mathbb{N}\)
\[
S_l = \{(j_1, \ldots, j_k) : x^l_{j_1,\ldots,j_k} \in C_0\}, \quad J_l = \{j \in \mathbb{N} : m_j \leq l\},
\]
and

\[ S_l^* = \left\{ (j_1, \ldots, j_k) \in S_l : x_{j_1, \ldots, j_k}^l \in \bigcup_{j \in J_l} \text{pr } C_j \right\} . \]

Here \( \text{pr} \) denotes the natural projection from \( \mathbb{R}^{k+1} \) into \( \mathbb{R}^k \). Thus, we have by (13.8) that for every \( l \in \mathbb{N} \)

\[ 2^k \sum_{j=1}^{\infty} [d(A_j)]^k \geq \sum_{j=1}^{\infty} L_j^k \geq 2^{-3k} \sum_{(j_1, \ldots, j_k) \in S_l^*} [d \left( f(B_{j_1, \ldots, j_k}^l) \right)]^k . \]

Denote by \( N_l \) and \( N_l^* \) the numbers of indexes \((j_1, \ldots, j_k)\) in \( S_l \) and \( S_l^* \), correspondingly. Note that by the construction the ratio \( N_l^*/N_l \) is non-decreasing and it converges to 1 as \( l \to \infty \) because \( \{C_j\} \) covers \( E \), consequently, \( \{\text{pr } C_j\} \) covers \( E \) and hence \( \bigcup_{j=1}^{\infty} \text{pr } C_j \) includes all points \( x_{j_1, \ldots, j_k}^l \) with \((j_1, \ldots, j_k) \in S_l \). However, \( S_l \) contains exactly \( 2^{lk} \) indexes \((j_1, \ldots, j_k)\) and by (13.9) \( d \left( f(B_{j_1, \ldots, j_k}^l) \right) \geq 2^{-(l+1)} \). Consequently,

\[ \sum_{j=1}^{\infty} [d(A_j)]^k \geq 2^{-5k} \]

and, thus, \( H^k(E) \geq 2^{-5k} > 0 \) in view of arbitrariness of \( \varepsilon > 0 \). The proof is complete.

**Remark 13.2.** It is known that each homeomorphism of \( \mathbb{R}^k \) onto itself in the class \( W_{\text{loc}}^{1,k} \) has the \((N)\)-property, see Lemma III.6.1 in [104] for \( k = 2 \) and [147] for \( k > 2 \). The same is valid also for open mappings, see [107]. On the other hand, there exist examples of homeomorphisms \( W_{\text{loc}}^{1,p} \) for all \( p < k \) that have not the \((N)\)-property, see [133]. Moreover, Cezari in [17] proved that continuous plane mappings \( f : D \to \mathbb{R}^2 \) in the class \( \text{ACL}^p \), \( p > 2 \), has the \((N)\)-property and that there exist examples of such mappings in \( \text{ACL}^2 \) that have not the \((N)\)-property.

Applying the oblique projection \( h(x) = (x_1, \ldots, x_{k-1}, x_k + f(x)/4) \) of the surface \( g(x) = (x, f(x)) \), \( x \in \mathbb{R}^k \), onto \( \mathbb{R}^k \), \( k \geq 2 \), from
Lemma 13.1, we obtain the corresponding examples of the continuous mappings $h^\varphi_k$ of $\mathbb{R}^k$ onto itself in the class $W^{1,\varphi}_{\text{loc}}$ that have not the $(N)$-property for convex increasing functions $\varphi$ satisfying (13.1). In particular, we obtain in this way the example of a continuous mapping $h_k : \mathbb{R}^k \to \mathbb{R}^k$ in the class $W^{1,k}_{\text{loc}}$ for each integer $k \geq 2$ without the $(N)$-property.

Setting $H(x, y) = h_{n-1}^\varphi(x)$, $x \in \mathbb{R}^{n-1}$, $y \in \mathbb{R}$, $n \geq 3$, we obtain examples of continuous mappings $H : \mathbb{R}^n \to \mathbb{R}^{n-1}$ in the class $W^{1,\varphi}_{\text{loc}}$ without the $(N)$-property with respect to the $(n-1)$-dimensional Hausdorff measure on a.e. hyperplane for each convex increasing function $\varphi : [0, \infty) \to [0, \infty)$ satisfying the condition (13.10) further.

**Theorem 13.1.** Let $\varphi : [0, \infty) \to [0, \infty)$ be a convex increasing function such that

$$\int_1^\infty \left[ \frac{t}{\varphi(t)} \right]^{\frac{1}{n-2}} \, dt = \infty \quad (13.10)$$

for a natural number $n \geq 3$. Then there is a homeomorphism $H$ of $\mathbb{R}^n$ onto $\mathbb{R}^n$ of the form $H(x, y) = (x, y + f(x))$, $x \in \mathbb{R}^{n-1}$, $y \in \mathbb{R}$, such that $H \in W^{1,\varphi}_{\text{loc}}$ but $H$ has not $(N)$-property with respect to $(n-1)$-dimensional Hausdorff measure on any hyperplane $y = \text{const}$.

**Proof of Theorem 13.1.** Indeed, the function $\varphi_*(t) := \varphi(t + 1)$ satisfies (13.10). Set $H(x, y) = g(x) + (0, \ldots, 0, y) = (x, y + f(x))$, $x \in \mathbb{R}^{n-1}$, $y \in \mathbb{R}$, where $g(x) = (x, f(x))$ is the mapping in Lemma 13.1 under $k = n-1$ corresponding to the function $\varphi_*$. Then $|\nabla H| \leq 1 + |\nabla g|$ and by monotonicity of $\varphi$ we have that $H \in W^{1,\varphi}_{\text{loc}}$ because $g \in W^{1,\varphi_*}_{\text{loc}}$.

**Corollary 13.2.** For every $n \geq 3$, there is a homeomorphism of $\mathbb{R}^n$ onto $\mathbb{R}^n$ of the class $W^{1,n-1}_{\text{loc}}$ of the form $H(x, y) = (x, y + f(x))$, $x \in \mathbb{R}^{n-1}$, $y \in \mathbb{R}$, without $(N)$-property with respect to $(n-1)$-dimensional Hausdorff measure on any hyperplane $y = \text{const}$.

**Remark 13.3.** Note that $\mathbb{R}^n$ can be in the natural way embedded into $\mathbb{R}^m$ for each $m > n$. Thus, by Theorems 4.2 and 13.1...
and Remark 13.2, the Calderon type condition (4.5) is not only sufficient but also necessary for continuous mappings \( f : \mathbb{R}^n \to \mathbb{R}^m, n \geq 3, m \geq n - 1 \), in the Orlicz-Sobolev classes \( W^{1,\varphi}_{\text{loc}} \) to have the \((N)\)-property with respect to \((n - 1)\)-dimensional Hausdorff measure on a.e. hyperplane. Furthermore, Theorem 13.1 shows that the necessity of the condition (4.5) is valid for \( m = n \) even for homeomorphisms \( f \). In this connection note also that Corollaries 13.2 disproves Theorem 1.3 from the preprint [23].

References

[1] Ahlfors L.: On quasiconformal mappings. J. Anal. Math. 3, 1–58 (1953/54).

[2] Alberico A. and Cianchi A.: Differentiability properties of Orlicz-Sobolev functions. Ark. Mat. 43, 1–28 (2005).

[3] Ambrosio L.: Metric space valued functions of bounded variation. Ann. Scuola. Norm. Sup. Pisa Cl. Sci. (4) 17, no. 3, 439–478 (1990).

[4] Aseev V.V.: Moduli of families of locally quasisymmetric surfaces. Sib. Math. J. 30, no. 3, 353–358 (1989).

[5] Astala P., Iwaniec T. and Martin G.: Elliptic differential equations and quasiconformal mappings in the plane. Princeton Math. Ser., vol. 48. Princeton University Press, Princeton (2009).

[6] Astala K., Iwaniec T., Koskela P. and Martin G.: Mappings of BMO-bounded distortion. Math. Ann. 317, 703–726 (2000).

[7] Bahtin A.K., Bahtina G.P. and Zelinskii Yu.B.: Topologic-algebraic structures and geometric methods in complex analysis. Kiev, Inst. Mat. NAHU (2008).
[8] Balogh Z.M.: Hausdorff dimension distribution of quasiconformal mappings on the Heisenberg group. J. d’Anal. Math. 83, 289–312 (2001).

[9] Balogh Z.M., Monti R. and Tyson J.T.: Frequency of Sobolev and quasiconformal dimension distortion. Research Report 2010-11, 22.07.2010, 1–36 (2010).

[10] Bates S.M.: On the image size of singular maps. Proc. AMS 114, 699–705 (1992).

[11] Belinskii P.P.: General properties of quasiconformal mappings. Izdat. "Nauka" Sibirsk. Otdel., Novosibirsk (1974) [in Russian].

[12] Biluta P.A.: Extremal problems for mappings quasiconformal in the mean. Sib. Mat. Zh. 6, 717–726 (1965) [in Russian].

[13] Bishop C.J.: Quasiconformal mappings which increase dimension. Ann. Acad. Sci. Fenn. 24, 397–407 (1999).

[14] Birnbaum Z. and Orlicz W.: Über die Verallgemeinerungen des Begriffes der zueinander konjugierten Potenzen. Studia. Math. 3, 1–67 (1931).

[15] Bojarski B., Hajlasz P. and Strzelecki P.: Sard’s theorem for mappings in Hölder and Sobolev spaces. Manuscripta Math. 118, 383–397 (2005).

[16] Calderon A.P.: On the differentiability of absolutely continuous functions. Riv. Math. Univ. Parma 2, 203–213 (1951).

[17] Cesari L.: Sulle transformazioni continue. Annali di Mat. Pura ed Appl. IV 21, 157–188 (1942).

[18] Cianchi A.: A sharp embedding theorem for Orlicz-Sobolev spaces. Indiana Univ. Math. J. 45 (1), 39–65 (1996).

[19] Chiarenza F., Frasca M. and Longo P.: $W^{2,p}$-solvability of the Dirichlet problem for nondivergence elliptic equations with VMO coefficients. Trans. Amer. Math. Soc. 336, no. 2, 841–853 (1993).
[20] Cristea M.: Open discret mappings having local ACL$^n$ inverses. Preprint, Inst. Math., Romanian Acad., Bucuresti 6, 1–29 (2008).

[21] Cristea M.: Local homeomorphisms having local ACL$^n$ inverses. Complex Var. Elliptic Equ. 53 (1), 77–99 (2008).

[22] Cristea M.: Some properties of the maps in $\mathbb{R}^n$ with applications to the distortion and singularities of quasiregular mappings. Rev. Roumaine Math. Pures Appl. 36, 355–368 (1991).

[23] Csörnyei M., Hencl S. and Maly J.: Homeomorphisms in the Sobolev space $W^{1,n-1}$. Preprint MATH-KMA-2007/252, Prague Charles Univ., 1–15 (2007).

[24] Church P. T. and Timourian J. G.: Differentiable Maps with Small Critical Set or Critical Set Image. Indiana Univ. Math. J. 27, 953–971 (1978).

[25] Church P. T. and Timourian J. G.: Maps having 0–dimensional critical set image. Indiana Univ. Math. J. , 27, 813–832 (1978).

[26] Donaldson T.: Nonlinear elliptic boundary-value problems in Orlicz-Sobolev spaces. J. Diff. Eq. 10, 507–528 (1971).

[27] Dubinin V.N.: Capacities of condencers and symmetrization in geometric function theory of complex variable. Vladivostok, Dal’nauka (2009).

[28] Dubovickii A.Yu.: On the structure of level sets of differentiable mappings of an $n$–dimensional cube into a $k$–dimensional cube. Izv. Akad. Nauk SSSR. Ser. Mat. 21, 371–408 (1957).

[29] Fadell A.G.: A note on a theorem of Gehring and Lehto. Proc. Amer. Math. Soc. 49, 195–198 (1975).

[30] Faraco D., Koskela P. and Zhong X.: Mappings of finite distortion: The degree of regularity. Adv. Math. 190 (2), 300–318 (2005).
[31] Federer H.: Geometric Measure Theory. Springer-Verlag, Berlin (1969).

[32] Fuglede B.: Extremal length and functional completion. Acta Math. 98, 171–219 (1957).

[33] Gehring F.W.: Symmetrization of rings in space. Trans. Amer. Math. Soc. 101, 499–519 (1961).

[34] Gehring F.W.: Rings and quasiconformal mappings in space. Trans. Amer. Math. Soc. 103, 353–393 (1962).

[35] Gehring F.W. and Iwaniec T.: The limit of mappings with finite distortion, Ann. Acad. Sci. Fenn. Math. 24, 253–264 (1999).

[36] Gehring F.W. and Lehto O.: On the total differentiability of functions of a complex variable. Ann. Acad. Sci. Fenn. Ser. A1. Math. 272, 3–8 (1959).

[37] Gehring F.W. and Martio O.: Quasiextremal distance domains and extension of quasiconformal mappings. J. Anal. Math. 45, 181–206 (1985).

[38] Gehring F.W. and Väisälä J.: Hausdorff dimension and quasiconformal mappings. J. London Math. Soc. (2) 6, 504–512 (1973).

[39] Golberg A.: Homeomorphisms with finite mean dilatations. Contemporary Math. 382, 177–186 (2005).

[40] Golberg A. and Kud’yavin V.S.: Mean coefficients of quasiconformality of pair of domains. Ukrain. Mat. Zh. 43 (12), 1709–1712 (1991) [in Russian]; translation in Ukrain. Math. J. 43, 1594–1597 (1991).

[41] Gol’dshtein V. M. and Reshetyuk Yu.G.: Introduction to the theory of functions with distributional derivatives and quasiconformal mappings: Nauka, Moscow (1983); English transl., Quasiconformal mappings and Sobolev spaces. Kluwer, Dordrecht (1990).
[42] Gossez J.-P. and Mustonen V.: Variational inequalities in Orlicz-Sobolev spaces. Nonlinear Anal. Theory Meth. Appl. 11, 379–392 (1987).

[43] Grinberg E.L.: On the smoothness hypothesis in Sard’s theorem. Amer. Math. Monthly 92, no. 10, 733–734 (1985).

[44] Gutlyanskii V., Martio O., Sugawa T. and Vuorinen M.: On the degenerate Beltrami equation. Trans. Amer. Math. Soc. 357, 875–900 (2005).

[45] Gutlyanskii V.Ya., Martio O., Ryazanov V.I. and Vuorinen M.: On convergence theorems for space quasiregular mappings. Forum Math. 10, 353–375 (1998).

[46] Gutlyanskii V., Ryazanov V., Srebro U. and Yakubov E.: On recent advances in the Beltrami equations. Ukrainian Math. Bull. 7, no. 4, 875–900 (2005).

[47] Hajlasz P.: Sobolev spaces on an arbitrary metric space. Potential Anal. 5, 403–415 (1996).

[48] Hajlasz P.: Whitney’s example by way of Assouad’s embedding. Proc. Amer. Math. Soc. 131, 3463–3467 (2003).

[49] Hardy G.H., Littlewood J.E. and Polia G: Inequalities. Cambridge University Press, Cambridge (1934).

[50] Heinonen J.: Lectures of Analysis on Metric Spaces. Springer, New York etc. (2000).

[51] Heinonen J., Kilpelainen T. and Martio O.: Nonlinear Potential Theory of Degenerate Elliptic Equations. Oxford Mathematical Monographs. Clarendon Press, Oxford-New York-Tokio (1993).

[52] Heinonen J. and Koskela P.: Quasiconformal maps in metric spaces with controlled geometry Acta Math. 181, 1–41 (1998).

[53] Heinonen J. and Koskela P.: Sobolev mappings with integrable dilatations. Arch. Rational Mech. Anal. 125, 81–97 (1993).
[54] Heinonen J., Koskela P., Shanmugalingam N. and Tyson J.T.: Sobolev spaces of Banach space–valued functions and quasiconformal mappings. J. Anal. Math. **85**, 87–139 (2001).

[55] Hencl S., Koskela P. and Onninen J.: A note on extremal mappings of finite distortion. Math. Res. Lett. **12** (2-3), 231–237 (2005).

[56] Hencl S. and Maly J.: Mappings with finite distortion: Hausdorff measure of zero sets. Math. Ann. **324** (3), 451–464 (2002).

[57] Herron D.A. and Koskela P.: Locally uniform domains and quasiconformal mappings. Ann. Acad. Sci. Fenn. Ser. A1. Math. **20**, 187–206 (1995).

[58] Hesse J.: A $p$-extremal length and $p$-capacity equality. Ark. Mat. **13**, 131–144 (1975).

[59] Holopainen I. and Pankka P.: Mappings of finite distortion: Global homeomorphism theorem. Ann. Acad. Sci. Fenn. Math. **29**(1), 59–80 (2004).

[60] Hsini M.: Existence of solutions to a semilinear elliptic system through generalized Orlicz-Sobolev spaces. J. Partial Differ. Equ. **23** (2), 168–193 (2010).

[61] Hurewicz W. and Wallman H.: Dimension theory. Princeton Univ. Press, Princeton, NJ (1948).

[62] Ignat’ev A. and Ryazanov V.: Finite mean oscillation in the mapping theory. Ukrainian Math. Bull. **2** (3), 403–424 (2005).

[63] Iwaniec T., Koskela P. and Onninen J.: Mappings of finite distortion: Monotonicity and continuity. Invent. Math. **144** (3), 507–531 (2001).

[64] Iwaniec T., Koskela P. and Onninen J.: Mappings of finite distortion: Compactness. Ann. Acad. Sci. Fenn. Math. **27** (2), 391–417 (2002).
[65] Iwaniec T. and Martin G.: Geometric function theory and non-linear analysis. Oxford Math. Monogr., Oxford Univ. Press, Oxford (2001).

[66] Iwaniec T. and Sverak V.: On mappings with integrable dilatation. Proc. Amer. Math. Soc. 118, 181–188 (1993).

[67] John F. and Nirenberg L.: On functions of bounded mean oscillation. Comm. Pure Appl. Math. 14, 415–426 (1961).

[68] Kallunki S.: Mappings of finite distortion: The metric definition. Dissertation, Univ. Jyvaskyla, Jyvaskyla. Ann. Acad. Sci. Fenn. Math. Diss. 131, pp. 33 (2002).

[69] Kaufman R.: A singular map of a cube onto a square. J. Diff. Geom. 14, 593–594 (1979).

[70] Kauhanen J., Koskela P. and Maly J.: On functions with derivatives in a Lorentz space. Manuscripta math. 10, 87–101 (1999).

[71] Kauhanen J., Koskela P. and Maly J.: Mappings of finite distortion: Discreteness and openness. Arch. Rat. Mech. Anal. 160, 135–151 (2001).

[72] Kauhanen J., Koskela P. and Maly J.: Mappings of finite distortion: Condition N. Michigan Math. J. 49, 169–181 (2001).

[73] Kauhanen J., Koskela P., Maly J., Onninen J. and Zhong X.: Mappings of finite distortion: Sharp Orlicz-conditions. Rev. Mat. Iberoamericana 19 (3), 857–87 (2003).

[74] Khruslov E.Ya. and Pankratov L.S.: Homogenization of the Dirichlet variational problems in Sobolev-Orlicz spaces. Operator theory and its applications (Winnipeg, MB, 1998), 345-366, Fields Inst. Commun., 25, Amer. Math. Soc., Providence, RI, 2000.

[75] Koskela P. and Onninen J.: Mappings on finite distortion: The sharp modulus of continuity. Trans. Amer. Math. Soc. 355, 1905–1920 (2003).
[76] Koskela P., Onninen J.: Mappings of finite distortion: Capacity and modulus inequalities. J. Reine Agnew. Math. 599, 1–26 (2006).

[77] Koskela P., Onninen J. and Rajala K.: Mappings of finite distortion: Injectivity radius of a local homeomorphism. Future trends in geometrical function theory, 169–174, Rep. Univ. Jyvaskyla Dep. Math. Stat., 92, Univ. Jyvaskyla, Jyvaskyla (2003).

[78] Koskela P. and Rajala K.: Mappings of finite distortion: Removable singularities. Israel J. Math. 136, 269–283 (2003).

[79] Kovalev L.V.: Monotonicity of generalized reduced modulus. Zapiski Nauch. Sem. POMI 276, 219–236 (2001).

[80] Kovtonyuk D. and Ryazanov V.: On boundaries of space domains. Proc. Inst. Appl. Math. & Mech. NAS of Ukraine 13, 110–120 (2006) [in Russian].

[81] Kovtonyuk D. and Ryazanov V.: To the theory of lower $Q$-homeomorphisms. Ukrainian Math. Bull. 5 (2), 157–181 (2008).

[82] Kovtonyuk D. and Ryazanov V.: On the theory of mappings with finite area distortion. J. d’Anal. Math. 104, 291–306 (2008).

[83] Kovtonyuk D. and Ryazanov V.: On the boundary behavior of generalized quasi–isometries. ArXiv: 1005.0247 [math.CV], 20 p. (2010).

[84] Kovtonyuk D., Petkov I. and Ryazanov V.: On homeomorphisms with finite distortion in the plane. ArXiv: 1011.3310v2 [math.CV], 1-16 (2010).

[85] Koronel J.D.: Continuity and $k$-th order differentiability in Orlicz-Sobolev spaces: $W^k L_A$. Israel J. Math. 24 (2), 119–138 (1976).

[86] Krasnosel’skii M.A. and Rutitskii Ya.B.: Convex functions and Orlicz spaces. Noordhoff (1961) (Translated from Russian).
[87] Kruglikov V.I.: The existence and uniqueness of mappings that are quasiconformal in the mean. In: Metric Questions of the Theory of Functions and Mappings, pp. 123–147. Naukova Dumka, Kiev (1973) [in Russian].

[88] Kruglikov V.I. Capacities of condensors and quasiconformal in the mean mappings in space. Mat. Sb. 130 (2), 185–206 (1986) [in Russian].

[89] Kruglikov V.I. and Paikov V.I.: Capacities and prime ends of an $n$-dimensional domain. Dokl. Akad. Nauk Ukrain. SSR Ser. A. 5, 10–13, 84 (1987) [in Russian].

[90] Krushkal’ S.L.: On mappings quasiconformal in the mean. Dokl. Akad. Nauk SSSR 157 (3), 517–519 (1964) [in Russian].

[91] Krushkal’ S.L.: On the absolute integrability and differentiability of some classes of mappings of many-dimensional domains. Sib. Mat. Zh. 6 (3), 692–696 (1965) [in Russian].

[92] Krushkal’ S.L. and Kühnau R.: Quasiconformal mappings, new methods and applications. Nauka, Novosibirsk (1984) [in Russian].

[93] Kud’yavin V.S.: A characteristic property of a class of $n$-dimensional homeomorphisms. Dokl. Akad. Nauk Ukrain. SSR ser. A. 3, 7–9 (1990) [in Russian].

[94] Kud’yavin V.S.: Quasiconformal mappings and $\alpha$-moduli of families of curves. Dokl. Akad. Nauk Ukrain., 7, 11–13 (1992) [in Russian].

[95] Kud’yavin V.S.: Estimation of the distortion of distances under mappings quasiconformal in the mean. Dinam. Sploshn. Sred., 52, 168–171 (1981) [in Russian].

[96] Kud’yavin V.S.: Local boundary properties of mappings quasiconformal in the mean. In: Collection of Scientific Works, Institute of Mathematics, Siberian Branch of the Academy of Sciences of the USSR, pp. 168–171. Novosibirsk (1981) [in Russian].
Kud’yavin V.S.: Behavior of a class of mappings quasiconformal in the mean at an isolated singular point. Dokl. Akad. Nauk SSSR 277 (5), 1056–1058 (1984) [in Russian].

Kühnau R.: Über Extremalprobleme bei im Mittel quasikonformen Abbildungen. Lecture Notes in Math. 1013, 113–124 (1983) [in German].

Kühnau R.: Canonical conformal and quasiconformal mappings. Identities. Kernel functions. In: Handbook of Complex Analysis, Geometry Function Theory, vol. 2, pp. 131–163. Elsevier, Amsterdam (2005).

Kuratowski K.: Topology, vol. 1. Acad. Press, NY (1968).

Kuz’mina G.V.: Moduli of the curve families and quadratic differentials. Trudy Mat. Inst. AN SSSR 139, 1–240 (1980).

Landes R. and Mustonen V.: Pseudo-monotone mappings in Sobolev-Orlicz spaces and nonlinear boundary value problems on unbounded domains. J. Math. Anal. Appl. 88, 25–36 (1982).

Lappalainen V. and Lehtonen A.: Embedding of Orlicz-Sobolev spaces in Hölder spaces. Ann Acad. Sci. Fenn. Ser. AI Math. 14 (1), 41–46 (1989).

Lehto O. and Virtanen K.: Quasiconformal Mappings in the Plane. Springer-Verlag, New York (1973).

Lelong-Ferrand J.: Representation conforme et transformations à intégrale de Dirichlet bornée. Gauthier-Villars, Paris (1955).

Lomako T., Salimov R. and Sevost’yanov E.: On the local behaviour of homeomorphisms with finite distortion in the plane. ArXiv: 1012.4590v1 [math.CV], 1–18 (2010).

Maly J. and Martio O.: Lusin’s condition (N) and mappings of the class $W^{1,n}$. J. Reine Angew. Math. 485, 19–36 (1995).
[108] Manfredi J.J. and Villamor E.: Mappings with integrable di-
latation in higher dimensions. Bull. Amer. Math. Soc. 32 (2),
235–240 (1995).

[109] Manfredi J.J. and Villamor E.: An extension of Reshetnyak’s
theorem. Indiana Univ. Math. J. 47 (3), 1131–1145 (1998).

[110] Martio O.: Modern tools in the theory of quasiconformal maps.
Texts in Math. Ser. B, 27. Univ. Coimbra, Dept. Mat., Coimbra.
1–43 (2000).

[111] Martio O., Rickman S. and Vaisala J.: Definitions for quasireg-
ular mappings. Ann. Acad. Sci. Fenn. Ser. A1. Math. 448, 1–40
(1969).

[112] Martio O., Ryazanov V., Srebro U. and Yakubov E.: Mapp-
ings with finite length distortion. J. d’Anal. Math. 93, 215–236
(2004).

[113] Martio O., Ryazanov V., Srebro U. and Yakubov E.: Q-
homeomorphisms. Contemporary Math. 364, 193–203 (2004).

[114] Martio O., Ryazanov V., Srebro U., Yakubov E.: On Q-
homeomorphisms. Ann. Acad. Sci. Fenn. 30, 49–69 (2005).

[115] Martio O., Ryazanov V., Srebro U. and Yakubov E.: Moduli in
Modern Mapping Theory. Springer Monographs in Mathematics.
Springer, New York (2009).

[116] Martio O., Ryazanov V. and Vuorinen M.: BMO and Injectiv-
ity of Space Quasiregular Mappings. Math. Nachr. 205, 149–161
(1999).

[117] Martio O. and Sarvas J.: Injectivity theorems in plane
and space. Ann. Acad. Sci. Fenn. Ser. A1 Math. 4, 384–401
(1978/1979).

[118] Mattila P. Geometry of sets and measures in Euclidean spaces.
Cambridge University Press, Cambridge (1995).

[119] Maz’ya V.: Sobolev Spaces. Springer-Verlag, Berlin (1985).
[120] Menchoff D.: Sur les differencelles totales des fonctions univalentes. Math. Ann. 105, 75–85 (1931).

[121] Nakki R.: Boundary behavior of quasiconformal mappings in $n$-space. Ann. Acad. Sci. Fenn. Ser. A1. Math. 484, 1–50 (1970).

[122] Norton A.: A critical set with nonnull image has large Hausdorff dimension. Trans. Amer. Math. Soc. 296, no. 1, 367–376 (1986).

[123] Onninen J.: Mappings of finite distortion: future directions and problems. The $p$-harmonic equation and recent advances in analysis, Amer. Math. Soc., Contemp. Math., 370, 199–207, Providence, RI (2005).

[124] Onninen J.: Mappings of finite distortion: Minors of the differential matrix. Calc. Var. Partial Differ. Eq. 21 (4), 335–348 (2004).

[125] Onninen J.: Mappings of finite distortion: Continuity. Dissertation, Univ. Jyvaskyla, Jyvaskyla, 24 (2002).

[126] Orlicz W.: Über eine gewisse Klasse von Räumen vom Typus B. Bull. Intern. de l’Acad. Pol. Serie A, Cracovie (1932).

[127] Orlicz W.: Über Räume ($L^M$). Bull. Intern. de l’Acad. Pol. Serie A, Cracovie (1936).

[128] Palagachev D.K.: Quasilinear elliptic equations with VMO coefficients. Trans. Amer. Math. Soc. 347, no. 7, 2481–2493 (1995).

[129] Pankka P.: Mappings of finite distortion and weighted parabolicity. Future trends in geometrical function theory, 175–182, Rep. Univ. Jyvaskyla Dep. Math. Stat., 92, Univ. Jyvaskyla, Jyvaskyla (2003).

[130] Perovich M.: Isolated singularity of the mean quasiconformal mappings. Lect. Notes in Math. 743, 212–214 (1979).
[131] Perovich M.: Global homeomorphism of mappings quasiconformal in the mean. Dokl. Akad. Nauk SSSR 230 (4), 781–784 (1976). [in Russian].

[132] Pesin I.N.: Mappings quasiconformal in the mean. Dokl. Akad. Nauk SSSR 187 (4), 740–742 (1969) [in Russian].

[133] Ponomarev S.P.: On the $(N)$-property of homeomorphisms of the class $W_{1}^{p}$. Sibirsk. Mat. Zh. 28 (2), 140–148 (1987).

[134] Quinn F. and Sard A.: Hausdorff conullity of critical images of Fredholm maps. Amer. J. Math. 94, 1101–1110 (1972).

[135] Rado T., and Reichelderfer P.V.: Continuous Transformations in Analysis. Springer-Verlag, Berlin (1955).

[136] Ragusa M.A.: Elliptic boundary value problem in vanishing mean oscillation hypothesis. Comment. Math. Univ. Carolin. 40, no. 4, 651–663 (1999).

[137] Rajala K.: Mappings of finite distortion: the Rickman–Picard theorem for mappings of finite lower order. J. Anal. Math. 94, 235–248 (2004).

[138] Rajala K.: Mappings of finite distortion: removability of Cantor sets. Ann. Acad. Sci. Fenn. Math. 29 (2), 269–281 (2004).

[139] Rajala K.: Mappings of finite distortion: removable singularities for locally homeomorphic mappings. Proc. Amer. Math. Soc. 132 (11), 3251–3258 (2004) (electronic).

[140] Rajala K.: Mappings of finite distortion: Removable singularities. Dissertation, Univ. Jyvaskyla, Jyvaskyla, pp. 74 (2003).

[141] Rajala K., Zapadinskaya A. and Zürcher T.: Generalized Hausdorff dimension distortion in euclidean spaces under Sobolev mappings. arXiv:1007.2091v1 [math.CA], 1–13 (2010).

[142] Reimann H.M.: On the absolute continuity of surface representation. Comment. Math. Helv. 46, 44–47 (1971).
[143] Reimann H.M. and Rychener T.: Funktionen Beschränkter Mittlerer Oscillation. Lecture Notes in Math. 487 (1975).

[144] Reshetnyak Yu.G.: Space mappings with bounded distortion. Nauka, Novosibirsk (1982); English transl., Translations of Mathematical Monographs, vol. 73, Amer. Math. Soc., Providence, RI (1988).

[145] Reshetnyak Yu.G.: The condition (N) for $W^{1,n}_{loc}$ space mappings. Sibirsk. Math. Zh. 28, 149–153 (1987) [in Russian].

[146] Reshetnyak Yu.G.: Sobolev classes of functions with values in a metric space. Sibirsk. Mat. Zh. 38, 657–675 (1997).

[147] Reshetnyak Yu.G.: Some geometric properties of functions and mappings with generalized derivatives. Sibirsk. Mat. Zh. 7, 886–919 (1966).

[148] Rickman S.: Quasiregular Mappings. Springer, Berlin etc. (1993).

[149] Ryazanov V.I.: On compactification of classes with integral restrictions on the Lavrent’ev characteristics. Sibirsk. Mat. Zh. 33 (1), 87–104 (1992) [in Russian]; translation in Siberian Math. J. 33 (1), 70–86 (1992).

[150] Ryazanov V.I.: On quasiconformal mappings with measure restrictions. Ukrain. Mat. Zh. 45 (7), 1009–1019 (1993) [in Russian]; translation in Ukrain. Math. J. 45 (7), 1121–1133 (1993).

[151] Ryazanov V.I.: On mappings that are quasiconformal in the mean. Sibirsk. Mat. Zh. 37(2), 378–388 (1996) [in Russian]; translation in Siberian Math. J. 37 (2), 325–334 (1996).

[152] Ryazanov V. and Sevost’yanov E.: Toward the theory of ring $Q$-homeomorphisms. Israel J. Math. 168, 101–118 (2008).

[153] Ryazanov V. and Sevost’yanov E.: Equicontinuity of mappings quasiconformal in the mean. ArXiv: 1003.1199v4 [math.CV], 1-16 (2010).
[154] Ryazanov V., Srebro U. and Yakubov E.: On ring solutions of Beltrami equation. J. d’Anal. Math. 96, 117–150 (2005).

[155] Ryazanov V., Srebro U. and Yakubov E.: Integral conditions in the theory of the Beltrami equations. Complex Variables and Elliptic Equations (to appear).

[156] Saks S.: Theory of the Integral. Dover, New York (1964).

[157] Sard A.: The measure of the critical values of differentiable maps. Bull. Amer. Math. Soc. 48, 883–890 (1942).

[158] Sard A.: The equivalence of n-measure and Lebesgue measure in $E_n$. Bull. Amer. Math. Soc. 49, 758–759 (1943).

[159] Sard A.: Images of critical sets. Ann. Math. 68, no. 2, 247–259 (1958).

[160] Sard A.: Hausdorff measure of critical images on Banach manifolds. Amer. J. math. 87, 158–174 (1965).

[161] Shlyk V.A.: The equality between $p$-capacity and $p$-modulus. Sib. Math. J. 34 (6), 1196–1200 (1993).

[162] Sobolev S.L.: Applications of functional analysis in mathematical physics. Izdat. Gos. Univ., Leningrad (1950); English transl. Amer. Math. Soc., Providence, R.I. (1963).

[163] Solynin A.Yu.: Moduli and extremal-metric problems. Algebra and Analysis 11 (1), 3–86 (1999).

[164] Strugov Y.F.: Compactness of the classes of mappings quasiconformal in the mean. Dokl. Akad. Nauk SSSR 243 (4), 859–861 (1978) [in Russian].

[165] Strugov Y.F. and Sychov A.V.: On different classes of mappings quasiconformal in the mean. Vest. PANI, 7, 14–19 (2002) [in Russian].

[166] Suvorov G.D.: Generalized principle of the length and area in the mapping theory. Naukova Dumka, Kiev (1985) [in Russian].
[167] Suvorov G.D.: The metric theory of prime ends and boundary properties of plane mappings with bounded Dirichlet integrals. Naukova Dumka, Kiev (1981) [in Russian].

[168] Suvorov G.D.: Families of plane topological mappings. AN SSSR, Novosibirsk (1965) [in Russian].

[169] Sychev A.V.: Moduli and spatial quasiconformal mappings. Novosibirsk, Nauka (1983).

[170] Tuominen H.: Characterization of Orlicz-Sobolev space. Ark. Mat. 45 (1), 123–139 (2007).

[171] Ukhlov A. and Vodop’yanov S.: Sobolev spaces and \((P, Q)\)-quasiconformal mappings of Carnot groups. Siberian Math. J. 39, 665–682 (1998).

[172] Ukhlov A. and Vodop’yanov S.: Mappings associated with weighted Sobolev Spaces. Complex Anal. Dynam. Sys. III. Contemp. Math. 455, 363–382 (2008).

[173] Väisälä J.: Lectures on \(n\)-dimensional quasiconformal mappings. Lecture Notes in Math. 229, Springer-Verlag, Berlin (1971).

[174] Väisälä J.: On quasiconformal mappings in space. Ann. Acad. Sci. Fenn. Ser. A1. Math. 298, 1–36 (1961).

[175] Väisälä J.: On the null-sets for extremal distances. Ann. Acad. Sci. Fenn. Ser. A1. Math. 322, 1–12 (1962).

[176] Vasil’ev A.: Moduli of families of curves for conformal and quasiconformal mappings. Lecture Notes in Math. (1788), Springer-Verlag, Berlin–New York (2002).

[177] Vodop’yanov S.: Mappings with bounded distortion and with finite distortion on Carnot groups. Sibirsk. Mat. Zh. 40 (4), 764–804 (1999); transl. in Siberian Math. J. 40 (4), 644–677 (1999).
[178] Vuillermot P.A.: Hölder-regularity for the solutions of strongly nonlinear eigenvalue problems on Orlicz-Sobolev spaces. Houston J. Math. 13, 281–287 (1987).

[179] Wilder R.L.: Topology of Manifolds. AMS, New York (1949).

[180] Whitney H.: A function not constant on a connected set of critical points. Duke Math. J. 1, 514–517 (1935).

[181] Zaanen A.C.: Linear analysis. Noordhoff (1953).

[182] Ziemer W.P.: Extremal length and conformal capacity. Trans. Amer. Math. Soc. 126 (3), 460–473 (1967).

[183] Zorich V.A.: Admissible order of growth of the characteristic of quasiconformality in the Lavrent’ev theory. Dokl. Akad. Nauk SSSR 181 (1968) [in Russian].

[184] Zorich V.A.: Isolated singularities of mappings with bounded distortion. Mat. Sb. 81, 634–638 (1970) [in Russian].

[185] Zorich V.A.: Quasiconformal mappings and the asymptotic geometry of manifolds. Russ. Math. Surv. 57 (3), 437–462 (2002); transl. from Usp. Mat. Nauk 57 (3), 3–28 (2002).

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