A weighted random walk approximation to fractional Brownian motion

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Abstract

We present a random walk approximation to fractional Brownian motion where the increments of the fractional random walk are defined as a weighted sum of the past increments of a Bernoulli random walk.

Keywords: Fractional Brownian motion, random walks, discrete approximations, weak convergence

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The purpose of this brief note is to describe a discrete approximation to fractional Brownian motion. The approximation works for all Hurst indices $H$, but take slightly different forms for $H \leq \frac{1}{2}$ and $H > \frac{1}{2}$. There are already several discrete approximations to fractional Brownian motion in the literature (see, e.g., [11], [1], [3], [10], [4], [2], [5], [8] for this and related topics), and the advantage of the present approach is that the increments of the fractional random walk is given as a weighted sum of past increments of an ordinary (Bernoulli) random walk. This gives an excellent understanding of the dynamics of the process and is a good starting point for stochastic calculus with respect to fractional Brownian motion. A similar idea is exploited in much greater generality by Konstantopoulos and Sakhanenko in [5], but they assume that $H > \frac{1}{2}$, while the present paper is mainly of interest when $H < \frac{1}{2}$.

The discrete approximation is based on Mandelbrot and Van Ness’ [6] moving frame representation of fractional Brownian motion:

$$x_t = c_H \int_{-\infty}^{t} \left( (t - r)^{H - \frac{1}{2}} - (-r)^{H - \frac{1}{2}} \right) db_r$$

where the scaling constant $c_H$ is given by

$$c_H = \left( \int_0^{\infty} \left( (1 + u)^{H - \frac{1}{2}} - u^{H - \frac{1}{2}} \right)^2 du + \frac{1}{2H} \right)^{-\frac{1}{2}} = \frac{\sqrt{\Gamma(2H + 1) \sin(\pi H)}}{\Gamma(H + \frac{1}{2})}$$

(see also [9]). This representation will be used to establish the convergence.

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1 The main theorem

To state the main result, we need some notation. For each natural number \( N \), let \( \Delta t_N = \frac{1}{N} \) and think of \( T_N = \{ k \Delta t_N | k \in \mathbb{Z} \} \) as a timeline. We let \( T_N^+ \) denote the nonnegative part of \( T \). It is convenient to use the following convention for sums over elements in \( T_N \):

\[
\sum_{r=s}^{t} f(r) = f(s) + f(s + \Delta t) + \cdots + f(t - \Delta t_N)
\]

Note that the lower limit \( s \) is included in the sum, but the upper limit \( t \) is not.

We shall also write \( \Delta f(t) = f(t + \Delta t_N) - f(t) \) for the forward increment of \( f \) at \( t \).

For all \( t \in T_N \), let \( \omega_N(t) \) be independent random variables taking values \( \pm 1 \) with probability \( \frac{1}{2} \). We shall write \( \Delta B_N(t) = \sqrt{\Delta t_N} \omega_N(t) \) and think of \( B_N \) as a Bernoulli random walk approximating Brownian motion. For \( 0 < H < 1 \) and \( N \in \mathbb{N} \), define a process \( X_{H,N} : \Omega N \times T_N^+ \to \mathbb{R} \) by

\[
X_{N,H}(0) = 0 \quad \text{and} \quad \Delta X_{H,N}(s) = K_H \Delta t_N^{H-\frac{1}{2}} \Delta B_N(s) + \sum_{r=-\infty}^{s} (H - \frac{1}{2})(s-r)^{H-\frac{3}{2}} \Delta t_N \Delta B_N(r)
\]

(assuming, e.g., Kolmogorov’s one series theorem, see [12], one easily checks that the sum converges a.s.) where the constant \( K_H \) is defined by

\[
K_H = \begin{cases} 
-(H - \frac{1}{2})\zeta(\frac{3}{2} - H) & \text{for } H < \frac{1}{2} \\
1 & \text{for } H \geq \frac{1}{2}
\end{cases}
\]

(as usual, \( \zeta(s) = \sum_{n=1}^{\infty} n^{-s} \) when \( s > 1 \)). Except for the Mandelbrot-Van Ness scaling factor \( c_H \), \( X_{H,N} \) will be our random walk approximation to fractional Brownian motion. For convergence purposes it will be convenient to think of \( X_{H,N} \) as a càdlàg process defined on \( [0, \infty) \), and we do this simply by assuming that \( X_{H,N} \) is constant between points in \( T_N \).

Remark: Note that the increment \( \Delta X_{H,N}(s) \) is a weighted sum of increments of the Bernoulli random walk \( B_N \) — it is a linear combination of the current coin toss \( \omega_N(s) \) and all previous coin tosses \( \omega_N(r), r < s \). Observe also that since \( \lim_{H \to \frac{1}{2}} -(H - \frac{1}{2})\zeta(\frac{3}{2} - H) = \lim_{s \to 1}(s - 1)\zeta(s) = 1 \), the two cases meet continuously at \( H = \frac{1}{2} \). For \( H > \frac{1}{2} \), we may actually choose \( K_H \) as we please since the term will vanish in the limit (see below), but \( K_H = 1 \) is the natural value and probably the one that gives best results in numerical work.

We are now ready to state the main result. Note that when \( H = \frac{1}{2} \), \( \Delta X_{\frac{1}{2},N}(t) = \Delta B_N(t) \) and the theorem just reduces to the classical convergence of a Bernoulli random walk to Brownian motion.
Theorem 1 (Main Theorem) For all real numbers $H$, $0 < H < 1$, the processes $c_{H}X_{H,N}$ converge weakly in $D([0, \infty))$ to fractional Brownian motion with Hurst index $H$.

Notation: In the rest of the paper, we drop the notational dependence on $N$ and $H$, and write simply $X$, $B$, $T$, $\Delta t$ for $X_{H,N}$, $B_{N}$, $T_{N}$, $\Delta t_{N}$ etc. when no confusion can arise.

As we are interested in understanding the dynamics of fractional Brownian motion, we have defined $X$ by specifying its increments $\Delta X(s)$. To prove the main theorem, we need an expression for $X(t)$. This is just a small calculation:

$$X(t) = \sum_{s=0}^{t} \Delta X(s) = \sum_{s=0}^{t} K_{H} \Delta t^{H-\frac{1}{2}} \Delta B_{s} + \sum_{s=0}^{t} \sum_{r=-\infty}^{t} (H - \frac{1}{2})(s-r)^{H-\frac{3}{2}} \Delta t \Delta B_{r}$$

Changing the order of summation, we have

$$X(t) = K_{H} \Delta t^{H-\frac{1}{2}} B_{t} + \sum_{r=0}^{t} \sum_{s=r+\Delta t}^{t} (H - \frac{1}{2})(s-r)^{H-\frac{3}{2}} \Delta t \Delta B_{r}$$

where $B_{t} = \sum_{r=0}^{t} \Delta B_{r}$ is a random walk converging to Brownian motion. Observe that when $H > \frac{1}{2}$, the first term $K_{H} \Delta t^{H-\frac{1}{2}} B_{t}$ vanishes when $N \to \infty$ (this is why the choice of $K_{H}$ is irrelevant in this case), but when $H < \frac{1}{2}$, the term explodes. In this case we have a delicate balance between two terms going to infinity, and a correct choice of $K_{H}$ is crucial.

The idea is now to simplify the expression for $X$ by replacing the sums $\sum_{s=r}^{t} (s-r)^{H-\frac{3}{2}} \Delta t$ by the corresponding integrals $\int (H - \frac{1}{2})(s-r)^{H-\frac{3}{2}} ds$, and then performing the integration. This works nicely for $H > \frac{1}{2}$, but when $H < \frac{1}{2}$, one of the integrals diverges, and we have to be more careful. Put crudely, it is the divergence of this integral that will cancel the divergence of the term $K_{H} \Delta t^{H-\frac{1}{2}} B_{t}$.

We are ready to prove the main theorem, and start with the simplest case.

2 The case $H > \frac{1}{2}$

We start from the expression

$$X(t) = \Delta t^{H-\frac{1}{2}} B_{t} + \sum_{r=0}^{t} \sum_{s=r+\Delta t}^{t} (H - \frac{1}{2})(s-r)^{H-\frac{3}{2}} \Delta t \Delta B_{r}$$
\begin{align*}
&\sum_{r=-\infty}^{t} \sum_{s=0}^{t} (H - \frac{1}{2})(s-r)^{H-\frac{3}{2}} \Delta t \Delta B_r,
\end{align*}

above (remember that \(K_H = 1\) in this case). Since \(H > \frac{1}{2}\), we have no problem with convergence, and if we let \(\epsilon_N(r,t)\) be the error term:

\[
\epsilon_N(r,t) := \sum_{s=r+\Delta t}^{t} (H - \frac{1}{2})(s-r)^{H-\frac{3}{2}} \Delta t - \int_{r+\Delta t}^{t} (H - \frac{1}{2})(s-r)^{H-\frac{3}{2}} ds,
\]

we get

\[
\sum_{s=r+\Delta t}^{t} (H - \frac{1}{2})(s-r)^{H-\frac{3}{2}} \Delta t = \int_{r+\Delta t}^{t} (H - \frac{1}{2})(s-r)^{H-\frac{3}{2}} ds + \epsilon_N(r,t) =
\]

\[
= (t-r)^{H-\frac{1}{2}} - \Delta t^{H-\frac{1}{2}} + \epsilon_N(r,t)
\]

Similarly, with

\[
\delta_N(r,t) := \sum_{s=0}^{t} (H - \frac{1}{2})(s-r)^{H-\frac{3}{2}} \Delta t - \int_{0}^{t} (H - \frac{1}{2})(s-r)^{H-\frac{3}{2}} ds,
\]

we get

\[
\sum_{s=0}^{t} (H - \frac{1}{2})(s-r)^{H-\frac{3}{2}} \Delta t = \int_{0}^{t} (H - \frac{1}{2})(s-r)^{H-\frac{3}{2}} ds + \delta_N(r,t) =
\]

\[
= (t-r)^{H-\frac{1}{2}} - (-r)^{H-\frac{1}{2}} + \delta_N(r,t)
\]

This means that

\[
X(t) = \sum_{r=0}^{t} \left((t-r)^{H-\frac{1}{2}} + \epsilon_N(r)\right) \Delta B_r +
\]

\[
+ \sum_{r=-\infty}^{0} \left((t-r)^{H-\frac{1}{2}} - (-r)^{H-\frac{1}{2}} + \delta_N(r)\right) \Delta B_r =
\]

\[
= \sum_{r=-\infty}^{t} \left((t-r)^{H-\frac{1}{2}} - (-r)^{H-\frac{1}{2}}\right) \Delta B_r + \sum_{r=0}^{t} \epsilon_N(r,t) \Delta B_r + \sum_{r=-\infty}^{0} \delta_N(r,t) \Delta B_r
\]

We want to prove that \(X\) converges weakly to fractional Brownian motion. According to Theorem 1 in [5], it suffices to show that \(E(\epsilon_H^2 X(t)^2) \to t^{2H}\). This follows immediately from the Mandelbrot-Van Ness representation and the following lemma.

**Lemma 2** For \(\frac{1}{2} < H < 1\):

\(i\) \( E \left( \sum_{r=0}^{t} \epsilon_N(r,t) \Delta B_r \right)^2 \leq (H - \frac{1}{2})^2 t \Delta t^{2H-1} \)
(ii) $E \left( \left( \sum_{r=-\infty}^{0} \delta_N(r, t) \Delta B_r \right)^2 \right) \leq (H - \frac{1}{2})^2 \zeta(3 - 2H) \Delta t^{2H}$

Proof: (i) We first observe that

$$\epsilon_N(r, t) = \sum_{s=r+\Delta t}^{t} (H - \frac{1}{2})(s-r)^{H-\frac{3}{2}} \Delta t - \int_{r+\Delta t}^{t} (H - \frac{1}{2})(s-r)^{H-\frac{3}{2}} \, ds > 0$$

since $\sum_{s=r+\Delta t}^{t} (H - \frac{1}{2})(s-r)^{H-\frac{3}{2}} \Delta t$ is an upper Riemann sum for the integral.

Since $\sum_{s=r+2\Delta t}^{t} (H - \frac{1}{2})(s-r)^{H-\frac{3}{2}} \Delta t$ is a lower Riemann sum, we also have

$$0 \leq \epsilon_N(r, t) \leq (H - \frac{1}{2}) \Delta t^{H-\frac{3}{2}} \Delta t = (H - \frac{1}{2}) \Delta t^{H-\frac{1}{2}}$$

Thus

$$E \left( \left( \sum_{r=0}^{t} \epsilon_N(r, t) \Delta B_r \right)^2 \right) = \sum_{r=0}^{t} \epsilon_N(r, t)^2 \Delta t \leq$$

$$\leq \sum_{r=0}^{t} (H - \frac{1}{2})^2 \Delta t^{2H-1} \Delta t \leq (H - \frac{1}{2})^2 \Delta t^{2H-1}$$

(ii) Using approximating Riemann sums as in part (i), we see that

$$0 \leq \delta_N(r, t) \leq (H - \frac{1}{2})(-r)^{H-\frac{3}{2}} \Delta t,$$

and thus

$$E \left( \left( \sum_{r=-\infty}^{0} \delta_N(r, t) \Delta B_r \right)^2 \right) = \sum_{r=-\infty}^{0} \delta_N(r, t)^2 \Delta t \leq \sum_{r=-\infty}^{0} (H - \frac{1}{2})^2 (-r)^{2H-3} \Delta t^3$$

Letting $r = -k \Delta t$, we get

$$E \left( \left( \sum_{r=-\infty}^{0} \delta_N(r, t) \Delta B_r \right)^2 \right) \leq \sum_{k=0}^{\infty} (H - \frac{1}{2})^2 k^{2H-3} \Delta t^{2H} =$$

$$= (H - \frac{1}{2})^2 \zeta(3 - 2H) \Delta t^{2H}$$

This completes the proof of the lemma (and also the proof of the Main Theorem for the case $H > \frac{1}{2}$).

\[\Box\]

3 The case $H < \frac{1}{2}$

Again we start from the expression

$$X(t) = K_H \Delta t^{H-\frac{1}{2}} B_t + \sum_{r=0}^{t} \sum_{s=r+\Delta t}^{t} (H - \frac{1}{2})(s-r)^{H-\frac{3}{2}} \Delta t \Delta B_r +$$

$$+ \sum_{r=-\infty}^{0} \delta_N(r, t) \Delta B_r +$$

$$+ \sum_{r=0}^{t} \epsilon_N(r, t) \Delta B_r$$

$$= K_H \Delta t^{H-\frac{1}{2}} B_t + \sum_{r=0}^{t} \sum_{s=r+\Delta t}^{t} (H - \frac{1}{2})(s-r)^{H-\frac{3}{2}} \Delta t \Delta B_r +$$

$$+ \sum_{r=-\infty}^{0} \delta_N(r, t) \Delta B_r +$$

$$+ \sum_{r=0}^{t} \epsilon_N(r, t) \Delta B_r$$
\[ + \sum_{r=-\infty}^{0} \sum_{s=0}^{t} (H - \frac{1}{2})(s-r)^{H-\frac{3}{2}} \Delta t \Delta B_r. \]

In this case, one of the integrals we worked with above diverges, and we have to be more careful. Let us start with a closer look at the term \( \sum_{s=r+\Delta t}^{t} (H - \frac{1}{2})(s-r)^{H-\frac{3}{2}} \Delta t \). We obviously have

\[ \sum_{s=r+\Delta t}^{t} (H - \frac{1}{2})(s-r)^{H-\frac{3}{2}} \Delta t = \sum_{s=r+\Delta t}^{\infty} (H - \frac{1}{2})(s-r)^{H-\frac{3}{2}} \Delta t - \sum_{s=t}^{\infty} (H - \frac{1}{2})(s-r)^{H-\frac{3}{2}} \Delta t \]

and if we let \( r = N \Delta t, s = k \Delta t \), we get

\[ \sum_{s=r+\Delta t}^{\infty} (H - \frac{1}{2})(s-r)^{H-\frac{3}{2}} \Delta t = \sum_{k=N+1}^{\infty} (H - \frac{1}{2})(k \Delta t - N \Delta t)^{H-\frac{3}{2}} \Delta t \]

\[ = (H - \frac{1}{2}) \Delta t^{H-\frac{3}{2}} \sum_{k=N+1}^{\infty} (k - N)^{H-\frac{3}{2}} = (H - \frac{1}{2}) \Delta t^{H-\frac{3}{2}} \sum_{n=1}^{\infty} n^{H-\frac{3}{2}} \]

\[ = (H - \frac{1}{2}) \Delta t^{H-\frac{1}{2}} \zeta(\frac{3}{2} - H) = -K_H \Delta t^{H-\frac{1}{2}} \]

Substituting this into the expression for \( X(t) \), we get

\[ X(t) = \sum_{r=0}^{t} \sum_{s=0}^{\infty} -(H - \frac{1}{2})(s-r)^{H-\frac{3}{2}} \Delta t \Delta B_r \]

\[ + \sum_{r=\infty}^{0} \sum_{s=0}^{t} (H - \frac{1}{2})(s-r)^{H-\frac{3}{2}} \Delta t \Delta B_r \]

The two sums in this expression have less dangerous limits than the one we just got rid of, and can be approximated by integrals. If we let

\[ \tilde{\epsilon}_N(r, t) := \sum_{s=t}^{\infty} -(H - \frac{1}{2})(s-r)^{H-\frac{3}{2}} \Delta t - \int_{t}^{\infty} -(H - \frac{1}{2})(s-r)^{H-\frac{3}{2}} \, ds, \]

we get (remember that \( H < \frac{1}{2} \)):

\[ \sum_{s=1}^{\infty} -(H - \frac{1}{2})(s-r)^{H-\frac{3}{2}} \Delta t = \int_{t}^{\infty} -(H - \frac{1}{2})(s-r)^{H-\frac{3}{2}} \, ds + \tilde{\epsilon}_N(r, t) \]

\[ = \left[-(s-r)^{H-\frac{3}{2}}\right]_{s=t}^{s=\infty} + \tilde{\epsilon}_N(r, t) = (t-r)^{H-\frac{3}{2}} + \tilde{\epsilon}_N(r, t) \]
Similarly, if we let
\[
\tilde{\delta}_N(r, t) := \sum_{s=0}^{t} -\frac{1}{2}(s-r)^{H-\frac{3}{2}} \Delta t - \int_{0}^{t} -\frac{1}{2}(s-r)^{H-\frac{3}{2}} \, ds,
\]
we get
\[
\sum_{s=0}^{t} (H - \frac{1}{2})(s-r)^{H-\frac{3}{2}} \Delta t = \int_{0}^{t} (H - \frac{1}{2})(s-r)^{H-\frac{3}{2}} \, ds - \tilde{\delta}_N(r, t)
\]
\[
= \left[(s-r)^{H-\frac{3}{2}} \right]_{s=0}^{s=t} - \tilde{\delta}_N(r, t) + (t-r)^{H-\frac{3}{2}} - (-r)^{H-\frac{3}{2}} - \tilde{\delta}_N(r, t)
\]
We thus have
\[
X(t) = \sum_{r=0}^{t} \left((t-r)^{H-\frac{3}{2}} + \tilde{\epsilon}_N(r, t)\right) \Delta B_r + \sum_{r=-\infty}^{0} \left((t-r)^{H-\frac{3}{2}} - (-r)^{H-\frac{3}{2}} - \tilde{\delta}_N(r, t)\right) \Delta B_r
\]
\[
= \sum_{r=-\infty}^{t} \left((t-r)^{H-\frac{3}{2}} - (-r)^{H-\frac{3}{2}}\right) \Delta B_r + \sum_{r=0}^{t} \tilde{\epsilon}_N(r, t) \Delta B_r - \sum_{r=-\infty}^{0} \tilde{\delta}_N(r, t) \Delta B_r
\]
To prove that \( c_H X \) converges weakly to fractional Brownian motion, we can now longer use Theorem 1 of [5] as in the previous case since this theorem requires that \( H > \frac{1}{2} \). However, the first term in the expression above obviously converges weakly to
\[
\int_{r=-\infty}^{t} \left((t-r)^{H-\frac{3}{2}} - (-r)^{H-\frac{3}{2}}\right) \, dr,
\]
and the next lemma shows that error terms go uniformly to zero. Using the Mandelbrot-Van Ness representation, we then get the Main Theorem for \( H < \frac{1}{2} \).

**Lemma 3** For each \( H, 0 < H < \frac{1}{2} \), there is a constant \( K_H \in \mathbb{R}_+ \) (independent of \( N \) and \( t \)) such that
\[
\left| X(t) - \sum_{r=\infty}^{t} \left((t-r)^{H-\frac{3}{2}} - (-r)^{H-\frac{3}{2}}\right) \Delta B_r \right| \leq K_H \Delta t^H
\]

**Proof:** It clearly suffices to show that there are constants \( C_H, D_H \in \mathbb{R}_+ \) (independent of \( N \) and \( t \)) such that
\[
\left| \sum_{r=0}^{t} \tilde{\epsilon}_N(r, t) \Delta B_r \right| \leq C_H \Delta t^H \quad \text{and} \quad \left| \sum_{r=-\infty}^{0} \tilde{\delta}_N(r) \Delta B_r \right| \leq D_H \Delta t^H
\]
We begin with the \( \tilde{\epsilon}_N \)-case. By definition

\[
\tilde{\epsilon}_N(r, t) = \sum_{s=t}^{\infty} -(H - \frac{1}{2})(s - r)^{H - \frac{3}{2}} \Delta t - \int_t^{\infty} -(H - \frac{1}{2})(s - r)^{H - \frac{3}{2}} \, ds
\]

Since \( \sum_{s=t}^{\infty} -(H - \frac{1}{2})(s - r)^{H - \frac{3}{2}} \Delta t \) is an upper Riemann sum for the integral \( \int_t^{\infty} -(H - \frac{1}{2})(s - r)^{H - \frac{3}{2}} \, ds \), and \( \sum_{s=t+\Delta t}^{\infty} -(H - \frac{1}{2})(s - r)^{H - \frac{3}{2}} \Delta t \) is a lower Riemann sum, we have

\[
0 \leq \tilde{\epsilon}_N(r, t) \leq -(H - \frac{1}{2})(t - r)^{H - \frac{3}{2}} \Delta t
\]

Hence (remember that \( |\Delta B_r| = \Delta t^{\frac{1}{2}} \))

\[
\sum_{r=0}^{t} \tilde{\epsilon}_N(r, t) \Delta B_r \leq \sum_{r=0}^{t} -(H - \frac{1}{2})(t - r)^{H - \frac{3}{2}} \Delta t^{\frac{1}{2}}
\]

If we let \( t = K \Delta t \), \( r = k \Delta t \), we can rewrite the last sum as

\[
\sum_{k=0}^{K-1} -(H - \frac{1}{2})(K - k)^{H - \frac{3}{2}} \Delta t^{H} \leq -(H - \frac{1}{2})\zeta(\frac{3}{2} - H) \Delta t^{H}
\]

This completes the \( \tilde{\epsilon}_N \)-part of the argument.

Turning to the term \( \sum_{r=-\infty}^{0} \tilde{\delta}_N(r) \Delta B_r \), we first observe that by definition

\[
\tilde{\delta}_N(r) = \sum_{s=0}^{t} -(H - \frac{1}{2})(s - r)^{H - \frac{3}{2}} \Delta t - \int_0^{t} -(H - \frac{1}{2})(s - r)^{H - \frac{3}{2}} \, ds
\]

Again, \( \sum_{s=0}^{t} -(H - \frac{1}{2})(s - r)^{H - \frac{3}{2}} \Delta t \) is an upper Riemann sum, and we easily see that

\[
0 \leq \tilde{\delta}_N(r) \leq -(H - \frac{1}{2})(-r)^{H - \frac{3}{2}} \Delta t
\]

Letting \( r = -k \Delta t \), we get

\[
E \left| \sum_{r=-\infty}^{0} \tilde{\delta}_N(r) \Delta B_r \right| \leq \sum_{r=-\infty}^{0} -(H - \frac{1}{2})(-r)^{H - \frac{3}{2}} \Delta t^{\frac{1}{2}} \leq -(H - \frac{1}{2})\Delta t^{H} \sum_{k=0}^{\infty} k^{H - \frac{3}{2}} = -(H - \frac{1}{2})\zeta(\frac{3}{2} - H) \Delta t^{H}
\]

This proves the lemma (and hence the Main Theorem for the remaining case \( H < \frac{1}{2} \)). \( \square \)
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