Performance Fuzzy Multinomial Control Chart

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Abstract. One of the most useful tools in Statistics Process Control is control chart. This technique has been used widely in industry and services. One of the most simple attribute control chart is p chart, when the item classified into two categories. In its development, if each item of quality characteristics classify in more than two categories multinomial control chart more appropriate. The classify such as Excellent, Good, Fair and Bad. However, if there is vagueness of the classification for each item, the fuzzy multinomial control chart is more appropriately used. By using triangular fuzzy number to calculate the representative value and simulation study, the control chart will be evaluated based on the value of average run length. For small shift of parameter $p_i$, the value of average run length when the process is in control closed to 370 and when the process out of control the value of average run length are decrease. Based on this this value, it shown that the Fuzzy Multinomial control chart is sensitive.

1. Introduction

Control chart is a technique that has been used widely in industries and services. One of the simplest control chart for attribute is p control charts. In this chart, each item is classified into two categories, conforming and nonconforming with the specifications related to quality characteristics. P chart constructed based on binomial distribution[1].

In practice, each item can be classify in more than two categories such as "bad", "fair", "good" and "excellent". Then to monitor the process, p multinomial control chart was used. In traditional control chart, a multinomial control chart is an extension of p control chart. Moreover, p multinomial control chart was used to monitor the process. Multinomial p control chart is a control that follows a random multinomial distribution with $(n, p)$ parameters, where $n$ is the sample size and $p$ is the probability vector [2,3]. This chart uses crisp data. However, if the classification is an element of vagueness, the fuzzy control chart is more appropriate.[4] In case, if there are more there 2 classification fuzzy Multinomial (FM) control chart is more appropriately to use [5,6]. The statistics of FM chart \( \bar{L} \) depends on the degrees of membership (the representative value) and the form of the fuzzy membership function. In fuzzy theory, there are several methods of transformation that can be used to determine the representative value (fuzzy mode, $\alpha$-level midrange, median and average) [7,8]. And the membership function there are several form (linear, triangular, trapezoid, S, etc)[9]. The control limit of FM control chart which constructed using multinomial distribution. By using triangular fuzzy number (TFN) and fuzzy mode transformation, the degrees of membership for the four categories could be obtained such as 0 for “excellent”, 0.25 for “good”, 0.5 for “medium” and 1 for “bad”. This chart compared with $p$ chart using data food process industry, packaging of frozen food. Based on the probability of type II error, the FM chart gives the better result than the $p$ chart [10].
triangular fuzzy number and fuzzy mode transformation was used to determine the representative values. Therefore, performance of FM control chart will be evaluated to know the sensitivity for the level shift of parameter \( p_i \). Simulation study based on Average Run Length (ARL) criteria was used to evaluate performance of FM chart.

2. Fuzzy Multinomial Control Chart

Fuzzy multinomial control chart can be applied in cases where there is a given language variable consisting of more than two categories \([10]\). Multinomial fuzzy control charts based on multinomial distribution. Multinomial fuzzy control charts will provide more information than the fuzzy \( p \) control charts.

Assume if \( L \) is the language variables that are categorized in the set \( \{l_1, l_2, \ldots, l_k\} \), Each \( l_i \) have weight \( (L(l_i)) \) stating the degree of membership in the set. Furthermore, in the set of fuzzy language variables can be expressed as \( L = \{(l_1, L(l_1)), (l_2, L(l_2)), \ldots, (l_k, L(l_k))\} \). For example \( p_i \) is the probability item category \( l_i \) with \( i = 1, 2, 3, \ldots, k \). If assume that a random sample of size \( n \) is taken and there is an item \( X_i \) \( (i = 1, 2, 3, \ldots, k) \) category \( l_i \), then \( (X_1, X_2, \ldots, X_k) \) have multinomial distribution with parameters \( n \) and \( p = (p_1, p_2, \ldots, p_k) \). Thus, each \( X_i \) has the binomial distribution with a mean \( np_i \) and variance \( np_i(1-p_i) \). Therefore, it can be defined statistically weighted average of \( (L(l_i)) \) is:

\[
\overline{L} = \sum_{i=1}^{k} \frac{n X_i(l_i)}{n} \tag{1}
\]

As for the mean and variance of \( L \) described in the following theorem:

**Theorem 1.**
If \( L = \{(l_1, L(l_1)), (l_2, L(l_2)), \ldots, (l_k, L(l_k))\} \), a variable language so that \( p_i \) is the probability of item categories \( l_i \), \( i = 1, 2, 3, \ldots, k \). Assume selected random sample \( n \). If \( X_i \), \( i = 1, 2, 3, \ldots, k \) is the number of items of the product which is a category \( l_i \), \( i = 1, 2, 3, \ldots, k \), then:

\[ i) E(L) = \sum_{i=1}^{k} p_i L(l_i) \]

\[ ii) \text{Var}(L) = \frac{\sum_{i=1}^{k} p_i (i-p_i) L(l_i) - 2 \sum_{i=1}^{k} \sum_{j=1}^{k} p_i p_j L(l_i) L(l_j)}{n} \tag{2} \]

**Corollary :**
If \( L = \{(l_1, 1), (l_2, 0)\} \) as language variables, the multinomial fuzzy control charts will become diagram \( p \) with \( p = \text{Pr} \) (which is an item).

Thus, obtained control limits for fuzzy diagram multinomial is

\[
LCL = E(L) - k \sqrt{\text{Var}(L)}
\]

\[
CL = E(L)
\]

\[
UCL = E(L) + k \sqrt{\text{Var}(L)} \tag{3}
\]

where the value of \( k \) is 3 associated with false alarms 0.0027.
3. Method
The steps to evaluate the performance of FM are:

a. Generate $m$ observations of $X \sim Mult(n, p_1, p_2, p_3, \ldots, p_k)$, $\sum_{i=1}^{k} p_i = 1$

b. Calculate the representative values for each classify. In this case the membership function using fuzzy triangle and fuzzy transformation using fuzzy Mode.

c. Calculate the weighted averages $\left( \tilde{T}_i \right)$

d. Calculate the mean and variance of fuzzy multinomial fuzzy, as in (2).

e. Calculate the control limits (UCL and LCL)

f. Determining the ARL value,
   i. Generating $m$ observation $X \sim Mult(n, p_1, p_2, p_3, \ldots, p_k)$, replicate $g$ times.
   ii. Determine the Fuzzy transformation to get a representative value.
   iii. Determine weighted average for each observation
   iv. In each replication, will stop until we find the first out of control.
   v. Calculate the average number out of control (ARL)
   vi. Repeat step (i)-(v) with the parameter of the $p_i$ shifted by $\Delta$.

The performance of control chart will be evaluated by the from or pattern from ARL. If we determine $\alpha = 0.0027$, the value of $ARL_0$ closed to 370 and the value of $ARL_1$ is decrease for the level shift of parameter.

4. Results and Discussion
Using simulation study, first, 300 observations will be generated from Multinomial distribution with parameter $(n=10, p_1=0.33, p_2=0.35, p_3=0.14, p_4=0.19)$. The values of parameter $p_i$ are generated by using 4 variables $(x_1, x_2, x_3, x_4)$ from Gamma $(4,2)$ with the probability $p_i$ is calculated as follows:

$$p_i = \frac{x_i}{\sum_{i=1}^{4} x_i}$$

This formula to guarantee the value of $\sum_{i=1}^{4} p_i = 1$.

Based on TFN and Mode transformation obtained the representative values of 0, 0.25, 0.5 and 1. By using equation (1), (2) and (3), statistic of FM and control limit can be calculated, and the FM chart as shown at Figure 1.

![Figure 1. FM Control chart](image)
Using this control limit, evaluation performance of FM chart will be simulated by ARL criteria. ARL is a tool to determine the performance of a control chart, which states the average number of samples (subgroups) which must be observed until it was out of control the first in a control chart. There are two kinds of ARL, the ARL in out of control conditions and ARL in the in control condition. ARL in a condition out of control called ARL$_1$, while the in control ARL called ARL$_0$.

Based on the simulation methodology in section 3, the value of ARL is shown at Table 1. According to Table 1, the first value is ARL$_0$ (373.72) which close to 370. It mean when the process is in control, the average observation until find the first out of control is 373. When the value of parameter $p_i$ shifted by 0.01, the value of ARL$_1$ will obtained. In this case, $p_1$ and $p_2$ shifted by 0.01 and $p_3$ and $p_4$ shifted by -0.01. When the probabilities are shifted to $p_1=0.33+0.01$, $p_2=0.35+0.01$, $p_3=0.14-0.01$ and $p_4=0.19-0.01$, the ARL$_1$ will be equal to 315.46, it mean when the process is out of control, the average observation until find the first out of control is 315. The ARL$_1$ is decrease when the probability ($p_i$) is shifted. It means that larger probability shift, the faster ability of fuzzy multinomial control chart to detect the actual shift. Based on Table 1, the ARL$_1$’s value is decrease for the small shift of $p_i$. Figure 2 also shows that the ARL$_1$ of fuzzy Multinomial sharply declined for the large shift of parameter $p_i$.

![Average Run Length](image)

**Figure 2. ARL’s value**

| No | ARL   | No | ARL   | No | ARL   |
|----|-------|----|-------|----|-------|
| 1  | 373.72| 11 | 98.39 | 21 | 53.75 |
| 2  | 315.46| 12 | 90.93 | 22 | 51.41 |
| 3  | 260.93| 13 | 84.49 | 23 | 49.27 |
| 4  | 218.42| 14 | 78.89 | 24 | 47.29 |
| 5  | 187.66| 15 | 73.96 | 25 | 45.47 |
| 6  | 163.68| 16 | 69.62 | 26 | 43.78 |
| 7  | 145.05| 17 | 65.75 | 27 | 42.22 |
| 8  | 129.95| 18 | 62.28 | 28 | 40.76 |
| 9  | 117.55| 19 | 59.15 | 29 | 39.40 |
| 10 | 107.10| 20 | 56.32 | 30 | 38.13 |

**5. Conclusion**

Fuzzy Multinomial control chart using TFN and Mode transformation, is quite sensitive for small parameter shifted. The value of ARL$0$ close to 370 and the values of ARL$1$ is drops sharply.
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