New Limits of Treewidth-based Tractability in Optimization

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Abstract
Sparse structures are frequently sought when pursuing tractability in optimization problems. They are exploited from both theoretical and computational perspectives to handle complex problems that become manageable when sparsity is present. An example of this type of structure is given by treewidth: a graph theoretical parameter that measures how “tree-like” a graph is. This parameter has been used for decades for analyzing the complexity of various optimization problems and for obtaining tractable algorithms for problems where this parameter is bounded. The goal of this work is to contribute to the understanding of the limits of the treewidth-based tractability in optimization. Our results are as follows. First, we prove that, in a certain sense, the already known positive results on extension complexity based on low treewidth are the best possible. Secondly, under mild assumptions, we prove that treewidth is the only graph-theoretical parameter that yields tractability a wide class of optimization problems, a fact well known in Graphical Models in Machine Learning and in Constraint Satisfaction Problems, which here we extend to an approximation setting in Optimization.

1 Introduction

Treewidth is a graph-theoretical parameter used to measure, roughly speaking, how far a graph is from being a tree. It was explicitly defined by Robertson and Seymour [50] (also see [51]), but there are many equivalent definitions. An earlier discussion is found in [38] and closely related concepts have been used by many authors under different names, e.g., the “running intersection” property, and the notion of “partial k-trees”. Here we make will use the following definition: recall that a chordal graph is a graph where every induced cycle has exactly three vertices.

Definition 1.1. An undirected graph $G = (V,E)$ has treewidth $\leq \omega$ if there exists a chordal graph $H = (V,E')$ with $E \subseteq E'$ and clique number $\leq \omega + 1$. We denote as $tw(G)$ the treewidth of $G$. 
Note that $H$ in the definition above is sometimes referred to as a chordal completion of $G$. It can be shown that a graph has treewidth 1 if and only if it is a forest. On the other extreme, a complete graph of $n$ vertices has treewidth $n - 1$. An important fact is that an $n$-vertex graph with treewidth $\leq \omega$ has $O(\omega n)$ edges, and thus low treewidth graphs are sparse, although the converse is not true. This sparsity is accompanied by a compact decomposition of low-treewidth graphs that allows to efficiently address various combinatorial problems.

Bounded treewidth has been long and widely recognized as a measure of complexity for all kinds of problems involving graphs and there is expansive literature concerning polynomial-time algorithms for combinatorial problems on graphs with bounded treewidth. One of the earliest references is [3]; see also [2, 4, 20, 10, 14, 11]. These algorithms typically rely on Dynamic Programming techniques that yield algorithms with a non-polynomial dependency on the treewidth. A similar paradigm has been presented in Inference Problems of Graphical Models (see, e.g., [44]), where it is well known that an underlying graph with bounded treewidth yields tractable inference problems; see [49, 31, 25, 57, 21, 58] and references therein.

In a more general optimization context, treewidth-based sparsity has been studied using the concept of the intersection graph, which provides a representation of the variable interactions in a system of constraints. The intersection graph of a system of constraints was originally introduced in [32] and has been used by many authors, sometimes using different terminology.

**Definition 1.2.** The intersection graph of a system of constraints is the undirected graph which has a vertex for each variable and an edge for each pair of variables that appear in any common constraint. If an optimization problem instance or its system of constraints is denoted $I$, we call its intersection graph $\Gamma[I]$.

As it has been observed before (see [12, 13, 43, 42, 59, 57]), the combination of intersection graph and treewidth makes it possible to define a notion of structured sparsity in an optimization context. One example of a research stream that has made use of treewidth-based sparsity via intersection graphs is that of constraint satisfaction problems (CSPs). One can obtain efficient algorithms for CSPs, whenever the intersection graph of the constraints exhibits low treewidth. Moreover, one can find compact linear extended formulations (i.e., linear formulations with a polynomial number of constraints) in such cases [40, 41]. In the Integer Programming context, extended formulations for binary problems whose constraints present a sparsity pattern with small treewidth have been developed as well; see [13, 57, 43]. A different use of treewidth in Integer Programming is given in [24]. An alternative perspective on structured sparsity in optimization problems, without relying on an intersection graph, is taken in [17].

**Contribution**

In this article we focus on two questions related to tractability induced by treewidth. While we provide a precise statements of these questions in each corresponding section, roughly speaking these questions and our contribution can be summarized as follows:

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1 Also called primal constraint graph or Gaifman graph.
1. In general, whenever an optimization problem exhibits an intersection graph with bounded treewidth, it can be solved (or approximated, depending on the nature of the problem) in polynomial time (see [12, 41, 43]). As such it is natural to ask the following question:

*Is there any other graph-theoretical structure that yields tractability?*

It is known that the answer is negative in general. Grohe [36] and Marx [47] proved that, in a sense, CSPs are only tractable when bounded treewidth is present. Chandrasekaran et al. [21] proved that a family of graphs with unbounded treewidth can yield intractable inference problems in Graphical Models, under the $\text{NP} \not\subseteq \text{P}/\text{poly}$ hypothesis. Moreover, it is believed that many treewidth-based algorithms are best possible [46].

We complement these results by proving that a family of graphs with unbounded treewidth can yield intractable optimization problems, even if the variable domain is bounded and small violations to the constraints are allowed. This provides a converse to a recent theorem by Bienstock and Muñoz [12]. We follow the overall strategy of Chandrasekaran et al., but we make use of the hypothesis $\text{NP} \not\subseteq \text{BPP}$ instead. Besides the different complexity theory assumptions, we highlight other differences of our approach and results compared to that of Chandrasekaran et al. [21] and Marx [47] in Sections 3.1 and 3.2.

2. For sets in $\{0, 1\}^n$ defined using a set of constraints whose intersection graph has treewidth $\omega$, it is known that there exists a linear programming reformulation of its convex hull of size $O(n^2\omega)$. This yields the following question:

*For any given treewidth $\omega$, is there any 0/1 set that (nearly) meets this bound?*

We provide a positive answer to this question. Furthermore, we prove that this bound is tight even if we allow semidefinite programming formulations. This establishes that there is little to be gained from semidefinite programs over linear programs in general when exploiting low treewidth. Our analysis is based on the result of Briët et al. [18], where the existence of 0/1 sets with exponential semidefinite extension complexity is proved. We also prove a similar result for the stable set polytope, making use of the treewidth of the underlying graph directly instead of relying on a particular formulation, and discuss related results.

It is worth mentioning that the extension complexity upper bound is obtained enumerating *locally* feasible vectors along with a gluing argument. Moreover, the upper bound is oblivious to any other structure present in the constraints besides its sparsity pattern. Our result shows that, surprisingly, one cannot do much better than this seemingly straight-forward approach, even if semidefinite formulations are allowed.

Typically, one can find treewidth-based *upper bounds* on the extension complexity of certain polytopes [41, 40, 17], or extension complexity *lower bounds* on specific families of problems [15, 17, 30, 7] parametrized using the problem size. To the
best of our knowledge, much less attention has been devoted to providing extension complexity lower bounds parameterized using other features of the problem. As a matter of fact, we are only aware of two other articles in this domain: the work of Gajarský et al. [33], where the authors analyze the extension complexity of the stable set polytope based on the expansion of the underlying graph, and the work of Aboulker et al. [1] which, independently of this work, provided extension complexity lower bounds of the correlation polytope parameterized using the treewidth of the underlying graph. Our work contributes to this line of work, showing the existence of polytopes whose extension complexity lower bound depends on the treewidth parameter and nearly meet the aforementioned bound. We discuss the main difference of our approach to that of Aboulker et al. in Section 4.2.

We believe that addressing these two questions provides new valuable insights into the limitations of exploiting treewidth and provides strong lower bounds that allow for assessing the performance of current approaches. In fact, complementing the results by Chandrasekaran et al. [21] and Marx [47], our results show that the existing approaches are, in some sense, the best possible and that further improvement is only possible if more structure is considered. We emphasize that the two questions studied in this paper, although both related to treewidth, are different and their answers need distinct approaches and tools. The extension complexity is a concept that does not necessarily depend on whether a problem is easy or hard from an algorithmic perspective, nor on the assumption of $P = NP$. For example, there are instances of the matching polytope with an exponential extension complexity [54], whereas finding a maximum weight matching can be done in polynomial time for any graph. An example in the other direction is given by the stable set problem. For each $\epsilon > 0$, an $n^{1-\epsilon}$-approximate solution cannot be attained in polynomial time [39, 62] unless $P = NP$, but there exists a formulation of polynomial size of the stable set polytope with the property that, for each objective function $c \geq 0$, its optimal solution is a factor $O(\sqrt{n})$ away from the maximum weight stable set ([8, 9], by building on results from [28]).

Outline

The rest of the article is organized as follows. In Section 2 we provide the basic notation used in this article. The main contributions are divided in two sections. In Section 3 we provide the answer to the first question above, i.e., we prove that unbounded treewidth can yield intractable optimization problems, even if constraint violations are allowed, and in Section 4 we provide the answer to the second question, i.e., we show the existence of sparse problems with high extension complexity. Both sections are organized similarly: we begin by providing the necessary background, along with the known positive treewidth-exploiting results, and then move to the respective proofs. Section 5 provides additional results to complement Section 4.
We mostly follow standard linear algebra and graph theory notation. For \( n \in \mathbb{N} \), we use \([n]\) to denote the set of integers \( \{1, \ldots, n\} \). Further, we denote by \( \mathbb{R}^n \) the \( n \)-dimensional vector space of the reals and by \( \mathbb{Z}^n \) the \( n \)-dimensional free \( \mathbb{Z} \)-module over the integers. If we restrict vectors to have non-negative entries, we use \( \mathbb{R}_+^n \) and \( \mathbb{Z}_+^n \). We call \( e_i \) with \( i \in [n] \) the canonical vectors in \( \mathbb{R}^n \), i.e., \((e_i)_j = 1 \) if and only if \( i = j \). The space of symmetric \( n \times n \) positive semidefinite matrices is denoted as \( S_n^+ \). The standard inner product between two vectors \( v, w \in \mathbb{R}^n \) is denoted by \( v^T w \). Given two matrices \( A, B \) (of compatible dimension), the Frobenius inner product is denoted by \( \langle A, B \rangle = \text{trace}(A^T B) \). Given two set \( S_1, S_2 \), we denote the cartesian product by \( S_1 \times S_2 \). The convex hull of a set \( S \subseteq \mathbb{R}^n \) is denoted as \( \text{conv}(S) \), and its affine hull by \( \text{aff}(S) \). For a graph \( G = (V, E) \), we use \( V(G) \) and \( E(G) \) to denote its vertices and edges respectively. For \( v \in V(G) \), we use \( N_G(v) \) to denote the set of neighbors of \( v \) in \( G \), that is \( N_G(v) = \{ u : \{u, v\} \in E(G) \} \). Given two graphs \( G_i = (V_i, E_i) \) with \( i \in \{1, 2\} \), we have that \( G_1 \) is a subgraph of \( G_2 \) if \( V_1 \subseteq V_2 \) and \( E_1 \subseteq E_2 \), and \( G_1 \) is a minor of \( G_2 \) if \( G_1 \) can be obtained from \( G_2 \) using vertex deletions, edge deletions, and edge contractions. Lastly, for a polynomial \( p(x) \), we denote by \( \|p\|_1 \) the sum of the absolute values of its coefficients, i.e., if \( p(x) = \sum_{\alpha \in I(p)} p_\alpha x^\alpha \) with \( x^\alpha = \prod_{j=1}^n x_j^{\alpha_j} \) for some \( \alpha \in \mathbb{Z}_+^n \), \( p_\alpha \) rational and \( I(p) \subseteq \mathbb{Z}_+^n \), then

\[
\|p\|_1 = \sum_{\alpha \in I(p)} |p_\alpha|.
\]

The degree of \( p \) is defined as \( \deg(p) = \max_{\alpha \in I(p)} \sum_j \alpha_j \).

### 3 Unbounded treewidth can yield intractability

Our first goal is to study the question of whether low treewidth is the only graph-theoretical structure that yields tractability when approximating optimization problems. Here we work with the general Polynomial Optimization framework, i.e., we consider problems of the form:

\[
\text{(PO): } \min c^T x \quad \text{s.t. } f_i(x) \geq 0 \quad i \in [m] \\
x_j \in \{0, 1\} \quad j \in [p], \\
x_j \in [0, 1] \quad j = p + 1, \ldots, n.
\]

where each \( f_i \) is a polynomial of degree at most \( \rho \). When \( \rho = 2 \) we also use the term QCQP (quadratically constrained quadratic problem) to refer to PO.

**Remark 3.1.** Any problem with polynomial objective and constraints, and defined over a compact set, can be cast as a PO. This can be done by appropriately rescaling variables and by using an epigraph formulation to move the non-linear terms of the objective to the constraints.

As mentioned above, it is known that tractability of an instance \( J \) of PO is implied by an intersection graph \( \Gamma[J] \) of low treewidth. In the pure binary case, an exact optimal
solution of $\mathcal{I}$ can be computed in polynomial time whenever $\Gamma[\mathcal{I}]$ has bounded treewidth (see [12, 41, 43]). However, if continuous variables are present, exact solutions might not be computable in finite time as shown by the following simple example.

$$\begin{align*}
\text{max} & \quad x \\
\text{s.t.} & \quad x^2 \leq \frac{1}{2} \\
& \quad x \in [0, 1]
\end{align*}$$

has an irrational optimal solution. As such approximation is unavoidable from a computational perspective, therefore we make use of the following definition:

**Definition 3.2.** Given an instance $\mathcal{I}$ of PO, we say $x^* \in \{0, 1\}^p \times [0, 1]^{n-p}$ is $\epsilon$-feasible if $x^* \in S_\epsilon$, where

$$S_\epsilon = \{x \in \{0, 1\}^p \times [0, 1]^{n-p} : f_i(x) \geq -\epsilon \|f_i\|_1, \; 1 \leq i \leq m\}.$$ 

Given an instance $\mathcal{I}$ of PO an LP formulation that takes advantage of low treewidth of $\Gamma[\mathcal{I}]$ was proposed by Bienstock and Muñoz [12] in order to approximate $\mathcal{I}$. More specifically:

**Theorem 3.3** (Bienstock and Muñoz [12]). Consider a feasible instance $\mathcal{I}$ of PO and $\epsilon > 0$. Assume each $f_i(x)$ has degree at most $\rho$. If $\Gamma[\mathcal{I}]$ has treewidth $\leq \omega$ then there is an LP formulation with $O\left((2\rho/\epsilon)^{\omega+1} n \log(\rho/\epsilon)\right)$ variables and constraints such that

(a) all feasible solutions to the LP are $\epsilon$-feasible for $\mathcal{I}$

(b) every optimal LP solution $\hat{x}$ satisfies

$$c^T \hat{x} \leq c^T x^* + \epsilon \|c_N\|_1$$

where $x^*$ is an optimal solution to $\mathcal{I}$ and $c_N$ is the sub-vector of $c$ corresponding to continuous variables $j = p+1, \ldots, n$.

Moreover, given a chordal completion of $\Gamma[\mathcal{I}]$ with clique number $\leq \omega + 1$ (which exists whenever the treewidth is at most $\omega$), the LP can be constructed in time

$$O((2\rho/\epsilon)^{\omega+1} \log(\rho/\epsilon) \cdot \text{poly}(\|\mathcal{I}\|)),$$

where $\|\mathcal{I}\|$ is the size of the representation of $\mathcal{I}$.

Here we phrased the theorem in a slightly different way compared to [12]: (a) we assume that $\mathcal{I}$ is feasible and (b) the result in [12] only considers continuous variables, whereas we allow for binary variables as well. This can be done while ensuring that the error term in (2) only involves coefficients associated with continuous variables; see [48] for details.

We would like to stress that the approximation provided by Theorem 3.3 is different from the traditional notion of approximation used in approximation algorithms: we allow for $\epsilon$-feasibility, i.e., we allow (slightly) infeasible solutions, which is usually not the case in approximation algorithms.

For $\rho = 2$ we obtain the following immediate corollary of Theorem 3.3.
**Corollary 3.4.** For every fixed $\epsilon > 0$, there is an algorithm $A$ such that, given a feasible instance $I$ of QCQP and a chordal completion of $\Gamma[I]$ with clique number $\leq \omega + 1$, it computes an $\epsilon$-feasible solution satisfying (2) in time $O(C^{\omega+1}\text{poly}(\|I\|))$, where $C$ is a constant.

We establish an (almost) matching lower bound for Theorem 3.3 by providing an (almost) matching lower bound for Corollary 3.4. For this we use the strategy of Chandrasekaran et al. [21] adapted to the general optimization setting. We make use of the following definition:

**Definition 3.5.** We say a countable family of graphs $\{G_k\}_{k=1}^\infty$ is polynomial-time enumerable if there is an algorithm such that, given $k$, it outputs a description of $G_k$ in time $\text{poly}(k)$.

Using this definition, we prove the following; we discuss the complexity theoretic assumption $NP \not\subseteq BPP$ in Section 3.3:

**Main Theorem 3.6.** Fix $\epsilon < 1/10$ and let $\{G_k\}_{k=1}^\infty$ be an arbitrary polynomial-time enumerable family of graphs indexed by treewidth. Let $A$ be an algorithm such that for all instances $I_k$ of QCQP such that $\Gamma[I_k] = G_k$, algorithm $A$ computes an $\epsilon$-feasible solution satisfying (2) in time $T(k) \cdot \text{poly}(\|I_k\|)$, then assuming $NP \not\subseteq BPP$ implies that $T(k)$ grows super-polynomially in $k$.

Note that assuming the family is polynomial-time enumerable implies that an encoding of $G_k$ of size polynomial in $k$ exists. This is indeed a desirable feature, since we will be dealing with a polynomial-time reduction, and thus we need to have at least an efficient access to the graph family. In fact, Chandrasekaran et al. [21] assume this implicitly, as they assume access to the graph family via a polynomial time “advice”.

### 3.1 Related intractability results in CSPs

Many treewidth-based intractability results have been obtained in the CSP community. Two crucial contributions are those of Grohe [36] and Marx [47] who proved that treewidth, in a sense, is the only tractable graph structure in a CSP. More specifically, assuming FPT$\neq$W[1], Grohe [36] proved that CSPs defined over a recursively enumerable family of graphs are polynomially solvable if and only if the family has bounded treewidth. Later on, Marx [47] proved the following result that, assuming stronger complexity theoretic assumptions, leads to sharper lower bounds.

**Theorem 3.7.** (Marx [47, Theorem 1.3]) If there is a class $G$ of graphs with unbounded treewidth, an algorithm $\mathcal{A}^M$, and a function $f$ such that $\mathcal{A}^M$ correctly decides every binary CSP instance and the running time is $f(G)\|I\|^{o(tw(G)/\log \tw(G))}$ for binary CSP($G$) instances $I$ with intersection graph $G \in G$, then the Exponential Time Hypothesis (ETH) fails.

Here binary CSP refers to CSP problems where each constraint involves at most two variables, and does not imply that the variables’ domain is $\{0, 1\}$. Note that $\mathcal{A}^M$ in Theorem 3.7 is assumed to be defined over all CSP instances, meaning, instances
with any intersection graph (although the running-time assumption is only made on the family $G$). However, Marx also provides an alternative result that requires $\mathcal{A}^M$ to be defined only on CSPs whose intersection graph belongs to the family $G$, under the assumption of $G$ being recursively enumerable.

From this, it is natural to ask whether Theorem 3.6 can be obtained from these already known results. We argue why this is not the case and that our result is rather complementary.

The first evident difference lies in the complexity-theoretic assumption. Grohe \cite{Grohe} assumes FPT$\neq$W[1], Marx \cite{Marx} assumes ETH, whereas we assume $NP \not\subseteq BPP$. $NP \not\subseteq BPP$, roughly speaking, asserts that certain problems in $NP$ cannot be solved in randomized polynomial time. Not much is known about the relationship of $NP$ and $BPP$, but it is widely believed that $P = BPP$, which would make $NP \not\subseteq BPP$ equivalent to $P \neq NP$. We describe $BPP$ more precisely in Section 3.3.

Secondly, the results obtained by Grohe and Marx are impossibility results for solving CSPs exactly, while our goal is to provide a converse to Theorem 3.3—an approximation-type of result. Thus, we must allow algorithm $\mathcal{A}$ to return potentially infeasible solutions. It is not clear to us, and seems a challenging task, whether the sequence of reductions from e.g. \cite{Marx} can be extended to prove bounds on the approximation guarantee.

Finally, and most importantly, Grohe and Marx deal with CSPs defined over unbounded domains. In this case, the treewidth-based algorithmic complexity upper bound is roughly $n^{O(\omega)}$, which is what the authors work with. In our case, algorithm $\mathcal{A}$ in Theorem 3.6 is only assumed to be defined over QCQP instances (which can be viewed as a subset of CSP instances) and whose variables’ domain is only $\{0, 1\}$ or $[0, 1]$. This causes the upper bound in Corollary 3.4 to be better than $n^{O(\omega)}$ and thus we need a different procedure to provide an intractability result in this case. We note that due to the same observation, the result by Chandrasekaran et al. \cite{Chandrasekaran} that we discuss below does not follow from Marx’s.

### 3.2 Intractability in the 0/1 case

In the 0/1 case, a similar result to Theorem 3.6 can be obtained as a direct consequence of the work of Chandrasekaran et al. \cite{Chandrasekaran} in the exact setting, i.e., when no approximation is allowed. This was done in the context of graphical models.

Given a graph $G = (V, E)$, a collection of (binary) random variables $x_v$ with $v \in V$, and for each $K \subseteq V$ forming a clique of $G$ a function $\psi_K$ which only involves variables $x_v$ with $v \in K$, then the inference problem involves computing the partition function $Z(\psi)$ defined as

$$Z(\psi) = \sum_{x \in \{0, 1\}^V} \prod_{K \in \mathcal{K}} \psi_K(x_K),$$

where $\mathcal{K}$ is the set of all cliques in $G$. It is known that if the underlying graph $G$ has bounded treewidth, then the inference problem can be solved in polynomial time (see \cite{Kolmogorov}). One of the main results in \cite{Chandrasekaran} provides a converse to this statement: given any family of graphs $\{G_k\}_{k=1}^{\omega}$ indexed by treewidth—under the complexity assumptions of
Theorem 3.6—there exist instances defined over that family of graphs such that inference requires time super-polynomial in $k$.

The proof can be directly adapted to state the same result regarding computing an optimal solution for a 0/1 PO problem. Hence, the result by Chandrasekaran et al. [21] can be viewed as the 0/1 version Theorem 3.6, which does not involve $\epsilon$-feasible solutions, as in such context exact solutions can be computed.

**Remark 3.8.** The original proof in [21] makes use of the $NP \not\subseteq P/poly$ hypothesis and the so called Grid-minor hypothesis. Since then, the latter was shown to be true by Chekuri and Chuzhoy [22], along with an algorithmic result allowing the use of the $NP \not\subseteq BPP$ instead of $NP \not\subseteq P/poly$.

Here we extend these results to include continuous variables and show that even approximately solving the problem remains intractable. The proof is along the lines of [21] and we follow their overall strategy. Our contribution here is to replace reductions between distributions and potential functions with reductions involving QCQPs and their approximations, as well as making use of randomized algorithms instead of non-uniform algorithms. To avoid confusion, we would like to stress that the notion of Approximate Inference presented in [21] is a different concept compared to finding an $\epsilon$-feasible solution to a PO problem.

### 3.3 Complexity-theoretic Assumptions and Graph-theoretic Tools

For a precise definition of $BPP$ and the commonly believed $NP \not\subseteq BPP$ hypothesis, we refer the reader to [5]. Simplifying here, $BPP$ is the class of languages $L$ for which a polynomial time probabilistic Turing machine exists which, given an input $x$, provides a wrong answer to the decision $x \in L$ with probability of at most $1/3$, whether in fact $x \in L$ or $x \notin L$. In our context here, it is sufficient to know that this complexity-theoretic assumption implies that MAX-2SAT in planar graphs (an NP-hard problem; see [37]) does not belong to $BPP$.

The second important tool we will use stems from work on the famous graph minor theorem. We briefly recall relevant results here, phrased to match the language in [21].

**Theorem 3.9 (Robertson et al. [52]).** There exist universal constants $c_3$ and $c_4$ such that the following holds. Let $G$ be a $g \times g$ grid. Then, (a) $G$ is a minor of all planar graphs with treewidth greater than $c_3 g$. Further, (b) all planar graphs of size (number of vertices) less than $c_4 g$ are minors of $G$.

The next theorem relaxes the planarity assumption however only for one of the directions of Theorem 3.9.

**Theorem 3.10 (Robertson et al. [52]).** Let $G$ be a $g \times g$ grid. There exists a finite $\kappa_{GM}(g)$ such that $G$ is a minor of all graphs with treewidth greater than $\kappa_{GM}(g)$. Further, $c_1 g^2 \log g \leq \kappa_{GM}(g) \leq 2^{c_2 g^5}$, where $c_1$ and $c_2$ are universal constants (i.e., they are independent of $g$).

The last theorem provides bounds on the magnitude of $\kappa_{GM}(g)$ in order for it to have the $g \times g$ grid as a minor. The constant $\kappa_{GM}(g)$ was conjectured to be polynomial in $g$, and was used as a complexity-theoretic assumption (under the name Grid-minor
hypothesis) in [21]. Since then, a recent breakthrough by Chekuri and Chuzhoy [22] resolved this in the positive.

**Theorem 3.11** (Chekuri and Chuzhoy [22]).

\[ \kappa_{GM}(g) \in O(g^{98}\text{poly log}(g)) \]

Moreover, there is a polynomial time randomized algorithm that, given a graph \( G \) with treewidth at least \( \kappa_{GM}(g) \), with high probability\(^2\) outputs the sequence of grid minor operations transforming \( G \) into the \( g \times g \) grid.

**Remark 3.12.** In [22], the output of the randomized algorithm is a model of the minor. Such model can be directly turned into a set of minor operations.

**Remark 3.13.** There has been some considerable progress recently regarding the exponent of the polynomial dependency in Theorem 3.11. We refer the reader to [23] for these improvements. Nonetheless, these newer results are non-algorithmic, which is undesirable for our purposes.

**Theorem 3.11**, together with Theorem 3.9 and Theorem 3.10, yields the following corollary:

**Corollary 3.14.** Let \( G = (V, E) \) be a planar graph of \( n \) nodes. There exists a polynomial \( \kappa(n) \) such that \( G \) is a minor of all graphs of treewidth at least \( \kappa(n) \).

The above in particular implies that \( G \) is a minor of \( G_k \) for all \( k \geq \kappa(n) \) for the sequence of graphs in Theorem 3.6.

### 3.4 Proof of Theorem 3.6

The outline of the proof of Main Theorem 3.6 is as follows. We start from a NP-hard instance \( I \) of QCQP, whose intersection graph \( \Gamma[I] \) is planar. Recall that we assume we are given an arbitrary family of graphs \( \{G_k\}_{k=1}^\infty \) indexed by treewidth. Due to Corollary 3.14, \( \Gamma[I] \) is a minor of \( G_k \) for some \( k \) large enough. We then construct an instance \( I_k \) of QCQP equivalent to \( I \) whose intersection graph is exactly \( G_k \). This makes it possible to use algorithm \( \mathcal{A} \) over \( I_k \), which yields the conclusion. The key ingredient is the following: having a family with unbounded treewidth allows us to embed the graph defining the NP-Hard problem into a graph of the given family, even if this family is arbitrary.

#### 3.4.1 Formulating MAX-2SAT as a special PO problem

Consider the NP-Hard problem of planar MAX-2SAT with underlying planar graph \( G = (V, E) \). Denote \( \{C_i\}_{i=1}^m \) the clauses and \( E = \{e_i\}_{i=1}^m \) the edges of \( G \). Let the variables be \( x_j \) with \( j \in [n] \). Then

\[ e_i = \{x_{i_1}, x_{i_2}\} \iff C_i = \{x_{i_1} \lor x_{i_2}\} \lor \{x_{i_1} \lor \overline{x_{i_2}}\} \lor \{x_{i_1} \lor \overline{x_{i_2}}\} \lor \{x_{i_1} \lor x_{i_2}\} \lor \{x_{i_1} \lor x_{i_2}\} \]

\(^2\)probability at least \( 1 - 1/|V(G)|^c \) for some constant \( c > 1 \)
We can formulate MAX-2SAT directly as a QCQP:

\[(\text{MAX-2SAT-1}): \quad \text{max} \sum_{i=1}^{m} y_i \quad \text{s.t.} \quad y_i - f_i(x_{i_1}, x_{i_2}) \leq 0 \quad i \in [m] \]

\[x_j^2 - x_j = 0 \quad j \in [n] \]

\[y_i \in \{0, 1\} \quad i \in [m] \]

\[x_j \in [0, 1] \quad j \in [n] \]

where

\[f_i(x_{i_1}, x_{i_2}) = \begin{cases} x_{i_1} + x_{i_2} & \text{if } C_i = \{x_{i_1} \lor x_{i_2}\} \\ x_{i_1} + (1 - x_{i_2}) & \text{if } C_i = \{x_{i_1} \lor \overline{x_{i_2}}\} \\ (1 - x_{i_1}) + x_{i_2} & \text{if } C_i = \{\overline{x_{i_1}} \lor x_{i_2}\} \\ (1 - x_{i_1}) + (1 - x_{i_2}) & \text{if } C_i = \{\overline{x_{i_1}} \lor \overline{x_{i_2}}\} \end{cases} \]

thus \(y_i = 1\) implies that clause \(C_i\) is satisfied. Let \(I\) be an instance of MAX-2SAT-1.

Notethatusingthisformulationthegraph \(G\)isasubgraphoftheintersectiongraph \(\Gamma[I]\).

It is also not hard to see that \(\Gamma[I]\) is planar, as we only need to add vertices \(y_i\), and each vertex \(y_i\) is connected to the endpoints of one particular edge of \(G\). We would like to emphasize that constraints \(x_j^2 - x_j = 0\) are equivalent to simply requiring \(x_j \in \{0, 1\}\), so we could formulate MAX-2SAT as a pure binary problem, however, since we are aiming for statements about the complexity of approximating PO problems, we deliberately chose a formulation using variables that can be continuous in nature; this will become clear soon.

The above formulation of MAX-2SAT is straight-forward, however for technical reasons we use the following equivalent alternative. The advantage of this formulation is that all constraints involve only 1 or 2 variables, simplifying the later analysis.

\[(\text{MAX-2SAT}): \quad \text{max} \sum_{i=1}^{m} (y_{i_1} + y_{i_2}) \quad \text{s.t.} \quad y_{i_1} - f_{i_1}(x_{i_i}) \leq 0 \quad i \in [m] \]

\[y_{i_2} - f_{i_2}(x_{i_2}) \leq 0 \quad i \in [m] \]

\[y_{i_1} + y_{i_2} \leq 1 \quad i \in [m] \]

\[x_j^2 - x_j = 0 \quad j \in [n] \]

\[y_i \in \{0, 1\} \quad i \in [m] \]

\[x_j \in [0, 1] \quad j \in [n] \]

where

\[f_{i_1}(x_{i_i}) = \begin{cases} x_{i_i} & \text{if } C_i = \{x_{i_1} \lor x_{i_2}\} \text{ or } C_i = \{x_{i_1} \lor \overline{x_{i_2}}\} \\ 1 - x_{i_i} & \text{if } C_i = \{\overline{x_{i_1}} \lor x_{i_2}\} \text{ or } C_i = \{\overline{x_{i_1}} \lor \overline{x_{i_2}}\} \end{cases} \]

and \(f_{i_2}(x_{i_2})\) is similarly defined.
3.4.2 Graph Minor Operations

Let $I$ be an instance of MAX-2SAT with planar intersection graph $\Gamma[I]$. Given a target graph $H$ which has $\Gamma[I]$ as a minor, in this section we show how to construct a QCQP instance $I_H$ equivalent to $I$. The complexity of this reduction is polynomial in the number of minor operations (vertex deletion, edge deletion, edge contraction), assuming that we know in advance which those operations should be. We will first show this for $H$ being contractable to $\Gamma[I]$ using a single minor operation and then argue that this is without loss of generality by repeating the argument. We distinguish the following cases:

(a) **Vertex Deletion.** If the minor operation is a vertex deletion of a vertex $u \in V(H)$, we define $I_H$ as $I$ plus a new variable $x_u \in [0, 1]$ with objective coefficient 0. Additionally, for all $v \in N_H(u)$ we add the redundant constraint $x_v + x_u \geq 0$.

(b) **Edge Deletion.** If the minor operation is an edge deletion of an edge $(u, v) \in E(H)$, we define $I_H$ as $I$ plus the redundant constraint $x_v + x_u \geq 0$.

(c) **Edge contraction.** If the minor operation is an edge contraction of $(u, v) \in E(H)$ to form $w \in V(\Gamma[I])$, then we proceed as follows.

Let $N_H(u)$ be the neighbors of $u$ in $H$. Note that in $I$ all constraints involve at most 2 variables, hence there is a one-to-one correspondence of edges in $\Gamma[I]$ and constraints involving 2 variables in $I$, and all these constraints are linear. Such constraints have the form

$$a_w, t z_w + b_w, t z_t \leq d_w, t \quad t \in N_{\Gamma[I]}(w),$$

where variables $z$ can be either variables $x$ or $y$ in MAX-2SAT, depending on node $w$. Using this, we define $I_H$ from $I$ by removing variable $z_w$, adding variables $z_u$ and $z_v$, and adding the following constraints

$$a_w, t z_u + b_w, t z_t \leq d_w, t \quad t \in N_H(u)$$

$$a_w, t z_v + b_w, t z_t \leq d_w, t \quad t \in N_H(v)$$

$$z_u = z_v$$

If the objective value of $z_w$ was 1, then we ensure $z_u$ to have objective value 1 and $z_v$ to have objective value 0. If $z_w$ was a continuous variable we add the constraints $z_u(1 - z_u) = 0$ and $z_v(1 - z_v) = 0$, and if $z_w$ was a binary variable we enforce $z_u$ and $z_v$ to be binary as well.

Clearly, in any case we obtain that $\Gamma[I_H] = H$, and $I_H$ is equivalent to $I$. Note that constraints in $I_H$ involve at most 2 variables and the ones with exactly 2 variables are linear. This invariant makes it possible to iterate this procedure using any sequence of minor operations.

Let $s = |V(\Gamma[I])|$. Corollary 3.14 implies that $\Gamma[I]$ is a minor of $\mathcal{G}_{\kappa(s)}$, thus assuming the sequence of minor operations is known, we can use the procedure above to construct an instance $I_{\kappa(s)}$ which is equivalent to $I$ and whose intersection graph is exactly $\mathcal{G}_{\kappa(s)}$. 

12
It is not hard to see that \( I_{\kappa(s)} \) has the following form (after relabeling variables):

\[
(I_{\kappa(s)}) \quad \max \sum_{i=1}^{m'} z_i
\]

s.t. \( a_{i,j}z_i + b_{i,j}z_j \leq d_{i,j} \quad (i,j) \in E_1 \) \hspace{1cm} (5a)

\( z_i = z_j \quad (i,j) \in E_2 \) \hspace{1cm} (5b)

\( z_i(1 - z_i) = 0 \quad i = m', \ldots, n' \) \hspace{1cm} (5c)

\( z_i \in \{0, 1\} \quad i \in [m'] \)

\( z_i \in [0, 1] \quad i = m', \ldots, n' \),

for some appropriately defined \( E_1, E_2 \), and where \( a_{i,j}, b_{i,j} \in \{-1, 0, 1\}, d_{i,j} \in \{0, 1\} \).

**Remark 3.15.** Each constraint (5a) is either a redundant constraint (introduced with the vertex or edge deletion operation) or it involves at least one integer variable. This will be important in the next section.

### 3.4.3 From approximations to exact solutions

We will now show how to construct a (truly) feasible solution from an \( \epsilon \)-feasible solution to \( I_{\kappa(s)} \). This will provide the link of the hardness of approximating \( I_{\kappa(s)} \) to the hardness of solving \( I_{\kappa(s)} \) exactly.

**Lemma 3.16.** Let \( z \in \{0, 1\}^{m'} \times \{0, 1\}^{n'-m'} \) be an \( \epsilon \)-feasible solution to \( I_{\kappa(s)} \) satisfying (2) for \( \epsilon < 1/10 \). Then, from \( z \), we can construct \( \hat{z} \) such that \( \hat{z} \in \{0, 1\}^{m'} \times \{0, 1\}^{n'-m'} \) is feasible and optimal for \( I_{\kappa(s)} \).

**Proof.** Since \( z \) is an \( \epsilon \)-feasible solution, we have

\[ |z_i^2 - z_i| \leq 2\epsilon \quad m' \leq i \leq n', \]

where the 2 arises as the 1-norm of the coefficients. Thus either \( 0 \leq z_i \leq 4\epsilon \) or \( |z_i - 1| \leq 4\epsilon \): \( h(x) = x^2 - x \) is decreasing in \( [0, 1/2] \), increasing in \( [1/2, 1] \), \( h(0) = 0 \), and

\[ h(4\epsilon) + 2\epsilon = 16\epsilon^2 - 4\epsilon + 2\epsilon = 16\epsilon^2 - 2\epsilon = 2\epsilon(8\epsilon - 1) < 0, \]

as \( \epsilon < 1/10 \). Thus \( h(4\epsilon) \leq -2\epsilon \). From here we conclude \( z_i \leq 4\epsilon \) if \( z_i \leq 1/2 \). The case for \( z_i > 1/2 \) is symmetric.

Now from \( z \) we construct \( \hat{z} \) by rounding each component to the nearest integer, and we argue the feasibility and optimality of \( \hat{z} \).

(a) Constraints (5c) are clearly satisfied as \( \hat{z} \) is a binary vector.

(b) For constraints (5b), \( z \) being \( \epsilon \)-feasible implies \( |z_i - z_j| \leq 2\epsilon \) for all \((i,j) \in E_2 \) and thus, using the above and that \( \hat{z}_i, \hat{z}_j \) are binary, we have

\[
|\hat{z}_i - \hat{z}_j| \leq |z_i - z_i| + |\hat{z}_i - z_j| + 2\epsilon \\
\leq 10\epsilon.
\]

The left-hand side is an integer and \( \epsilon < 1/10 \), from where we conclude \( \hat{z}_i = \hat{z}_j \).
(c) For constraints (5a) fix \((i, j) \in E_1\) such that the corresponding constraint is not redundant. By Remark 3.15 either \(z_i\) or \(z_j\) is integer. Without loss of generality assume \(z_i \in \{0, 1\}\), and thus \(\hat{z}_i = z_i\). To make the argument clear, we rewrite the inequality as

\[a_{i,j}z_i - d_{i,j} \leq -b_{i,j}\hat{z}_j.\]

The left hand side is an integer, therefore rounding \(z_j\) will keep the inequality valid where we use that \(b_{i,j} \in \{-1, 0, 1\}\):

\[a_{i,j}\hat{z}_i - d_{i,j} \leq -b_{i,j}\hat{z}_j.\]

This proves \(\hat{z}\) is feasible. On the other hand, \(z\) satisfies (2), and only integer variables have non-zero objective coefficient:

\[\sum_{i=1}^{m'} \hat{z}_i = \sum_{i=1}^{m'} z_i \geq \sum_{i=1}^{m'} z_i^*\]

therefore \(\hat{z}\) is optimal. \(\square\)

3.4.4 Bringing it all together

Main Theorem 3.6. Suppose we are given a sequence of graphs \(G_k\), each having treewidth \(k\). We show that, under the conditions of Theorem 3.6, the existence of an algorithm \(A\) as in Theorem 3.6 (i.e., that can approximately solve QCQP problems \(I_k\) with \(\Gamma[I_k] = G_k\)), with running time \(T(k) \cdot \text{poly}(\|I_k\|)\) with \(T(k)\) polynomial in \(k\) implies that planar MAX-2SAT belongs to BPP, contradicting the assumption \(NP \notin BPP\).

1. Consider an instance of planar MAX-2SAT. We construct an instance \(I\) of a QCQP as in Section 3.4.1, whose intersection \(\Gamma[I]\) graph is planar. We denote \(s\) its number of vertices.

2. From Corollary 3.14 we know that \(\Gamma[I]\) is a minor of \(G_{\kappa(s)}\). Moreover, \(\kappa(s) := \kappa_{GM}(s/c_4)\) and, from the discussion in Section 3.4.2, \(I\) is equivalent to a QCQP problem \(I_{\kappa(s)}\) with \(\Gamma[I_{\kappa(s)}] = G_{\kappa(s)}\).

3. The minor operations transforming \(G_{\kappa(s)}\) into \(\Gamma[I]\), which are needed to construct \(I_{\kappa(s)}\), can be obtained as follows:

   (a) Since \(\Gamma[I]\) is planar, it is a minor of the \(s/c_4 \times s/c_4\) grid. This sequence of minor operations can be found in linear time using the results in [55].

   (b) The \(s/c_4 \times s/c_4\) grid is a minor of \(G_{\kappa(s)}\). We can find the corresponding sequence of minor operations (with high probability) in polynomial time using the algorithm by Chekuri and Chuzhoy [22] mentioned in Theorem 3.11.

4. Using the point above, we can construct (with high probability) instance \(I_{\kappa(s)}\).
5. Using $\mathcal{A}$ and a fixed $\epsilon < 1/10$, find an $\epsilon$-feasible solution satisfying (2) for $I_{\kappa(s)}$ in time $T(\kappa(s)) \cdot \text{poly}(\|I_{\kappa(s)}\|)$.

6. Given an $\epsilon$-feasible solution of $I_{\kappa(s)}$, we construct an optimal solution for $I_{\kappa(s)}$ as in Section 3.4.3.

7. From the optimal solution to $I_{\kappa(s)}$, we can find an optimal solution to $\bar{I}$ using the minor operations described in Section 3.4.2 in polynomial time.

Using the optimal solution, we can solve the decision problem associated to planar MAX-2SAT directly. The only place where our algorithm can make a mistake is in the sequence of minor operations, which happens with low probability. Since clearly $\|I_{\kappa(s)}\|$ is polynomial, and by assumption we have access to $G_{\kappa(s)}$ in polynomial time, if $T(\kappa(s))$ is also polynomial, we obtain that planar MAX-2SAT $\in \text{BPP}$, a contradiction. $\square$

4 Treewidth-based Extension Complexity Lower Bounds

In this section we analyze the tightness of the linear extension complexity results that exploit treewidth. While we provide precise definitions in Section 4.1, the linear extension complexity of a problem is the smallest number of inequalities needed to represent a given problem as linear program. In fact our lower bounds will also hold for semidefinite programs, showing that there is little to be gained from semidefinite programs over linear programs in terms of exploiting low treewidth.

To this end, we consider a set defined as

$$S = \{x \in \{0,1\}^n : \phi_i(x) = 0 \text{ with } i \in [m]\}$$

where each $\phi_i : \{0,1\}^n \rightarrow \{0,1\}$ is a boolean function. Note that the intersection graph does not only depend on the set $S$, but also on how it is formulated; we denote the intersection graph of (6) as $\Gamma[S_\phi]$.

Remark 4.1. Given the generality of the $\phi_i$ functions defining the constraints in (6), one could formulate $S$ using a single membership oracle of $S$. However, such a formulation would consist of a single constraint involving all variables, which would yield a very dense formulation of $S$, so that we could not exploit low treewidth.

Any pure binary PO can be formulated as (6). We have already seen in the previous section that unbounded treewidth of the intersection graph can yield intractability in the algorithmic sense. In this section, in contrast, we focus on studying how hard a sparse problem can be, using extension complexity as the measure of complexity.

4.1 Background on Extended Formulations

We will now briefly recall basics concepts from Extended Formulations needed for our discussion. Extended formulations aim for finding a formulation of an optimization problem in extended space where auxiliary variables are utilized with the aim to find an overall smaller formulations compared to formulations in the original space, involving only the problem-inherent variables. Note that optimizing a linear objective over an
extended formulation is no harder than over the original formulation, which makes extended formulations appealing. For a more detailed discussion we refer the reader to [29, 27].

**Definition 4.2 (Linear Extended Formulation).** Given a polytope $P \subseteq \mathbb{R}^n$, a linear extended formulation of $P$ is a linear system

$$Ex + Fy = g, \quad y \geq 0$$

(7)

with the property that $x \in P$ if and only if there exists $y$ such that $(x, y)$ satisfies (7). The size of the linear extension is given by the number of inequalities in (7), and the linear extension complexity of $P$ is the minimum size of a linear extended formulation of $P$, which we denote by $xc(P)$.

**Remark 4.3.** In the previous definition, system (7) can be made more general. We can also consider

$$Ex + Fy = g^\leq, \quad E^{\leq}x + F^{\leq}y \leq g^{\leq}$$

and define the size the same way as before. However, this more general definition does not affect the extension complexity of a polytope; see e.g., [61].

In Yannakakis’ ground-breaking paper [61], it is proved that the linear extension complexity of a polytope is strongly related to the concepts of slack matrix and non-negative rank:

**Definition 4.4 (Slack Matrix).** Let $P$ be a polytope that can be formulated as

$$P = \{x \in \mathbb{R}^n : \alpha^T_i x \leq b_i, \; i \in [m]\}.$$

Consider a set of points $V = \{x_j : j \in J\}$ such that $P = \text{conv}(V)$. Then, the slack matrix $S$ of $P$ associated to $Ax \leq b$ and $V$ is given by

$$S_{ij} = b_i - \alpha^T_i x_j.$$

**Definition 4.5 (Non-negative Factorization).** Given a non-negative matrix $M$, a rank-$r$ non-negative factorization of $M$ is given by two non-negatives matrix $T$ (of $r$ columns) and $U$ (of $r$ rows) such that

$$M = TU.$$

The non-negative rank of $M$, denoted as $rk_+(M)$, is the minimum rank of a non-negative factorization of $M$.

**Theorem 4.6** (Yannakakis [61]). Let $P = \{x \in \mathbb{R}^n : Ax \leq b\} = \text{conv}(V)$ be a polytope with $\text{dim}(P) \geq 1$ and let $S$ be the slack matrix of $P$ associated to $Ax \leq b$ and $V$. Then

$$xc(P) = rk_+(S).$$

In the linear case the $y$ variables in the extended formulation are required to be in the cone given by the non-negative orthant, i.e., $y \geq 0$. This was generalized to other cones, allowing for more expressiveness in the extended space. Of particular interest to this work is the generalization to semidefinite extended formulations; see [29, 35] for details on the following concepts and results.
Definition 4.7 (Semidefinite Extended Formulations). Given a convex set \( K \subseteq \mathbb{R}^n \), a semidefinite extended formulation of \( K \) is a system
\[
a_i^T x + (U_i, Y) = b_i, \quad i \in I, \quad Y \in \mathbb{S}_r^n(8)
\]
where \( I \) is an index set, \( a_i \in \mathbb{R}^n \), \( U_i \in \mathbb{S}_r^n \), with the property that \( x \in K \) if and only if there exists \( Y \) such that \((x, Y)\) satisfies (8). The size of the semidefinite extension is given by the size \( r \) of matrices \( U_i \) in (8), and the semidefinite extension complexity of \( K \) is the minimum size of a semidefinite extended formulation of \( K \). It is denoted \( xc_{SDP}(K) \).

Definition 4.8 (Semidefinite Factorization). Given a non-negative \( n \times m \) matrix \( M \), a rank-\( r \) semidefinite factorization of \( M \) is given by a set of pairs \((U_i, V_j)\) \((i, j) \in [n] \times [m]\) \( \subseteq \mathbb{S}_r^n \times \mathbb{S}_r^m \) such that
\[
M_{i,j} = (U_i, V^j) \quad \forall i \in [n], j \in [m].
\]
The semidefinite rank of \( M \), denoted as \( rk_{PSD}(M) \), is the minimum rank of a semidefinite factorization of \( M \).

Theorem 4.9 (Yannakakis’ Factorization Theorem for SDPs, [35]). Let \( P = \{x \in \mathbb{R}^n : Ax \leq b\} = \text{conv}(V) \) be a polytope with \( \dim(P) \geq 1 \) and let \( S \) be the slack matrix of \( P \) associated to \( Ax \leq b \) and \( V \). Then
\[
xc_{SDP}(P) = rk_{PSD}(S).
\]

Note that every linear extended formulation is a semidefinite extended formulation using diagonal matrices so that \( xc_{SDP}(P) \leq xc(P) \).

4.2 Low treewidth implies small extension complexity

We will now state the known upper bound on the linear extension complexity of low-treewidth problems, which we prove to be nearly optimal. The following strong result is well known; see e.g., [12, 41, 43]:

Theorem 4.10. Let \( S \subseteq \{0, 1\}^n \) be a set that exhibits a formulation as
\[
S = \{x \in \{0, 1\}^n : \phi_i(x) = 0 \text{ with } i \in [m]\}. \tag{9}
\]
If \( \Gamma[S_S] \) has treewidth \( \omega \), then \( \text{conv}(P) \) has linear extension complexity
\[
O(n2^{2\omega}). \tag{10}
\]

We will construct sets \( S \) that (a) can be formulated using sparse constraints (given by some treewidth \( \omega \)) and which (b) exhibit high extension complexity essentially of (10). By building on recent lower bounds on semidefinite extension complexity [18], we show the existence of such 0/1 sets, whose semidefinite extension complexity (nearly) meets the bound (10) (see Main Theorem 4.24). In fact, for those hard instances, we show a stronger result. The extension complexity does not take into account techniques that are routinely adopted to solve integer programs, such as e.g., reformulations or parallelization of separable sets. These techniques can be used to modify the original
instance to an equivalent integer programming problem, which may be computationally more attractive. We show that the hard instances we construct cannot be reformulated to have lower extension complexity or being separable.

The careful reader might have noticed an important fact: the extension complexity bound in (10) does not depend on a particular formulation of the set $S$, as opposed to the treewidth. To overcome this disparity and for simplicity in the upcoming discussion we focus on the “best possible” treewidth of a formulation, which we refer to as the treewidth (or treewidth complexity) of $S$. This definition prevents the results from depending on a particular formulation, or the type of constraints (e.g., linear, boolean, or polynomial).

**Definition 4.11.** Given $S \subseteq \{0, 1\}^n$, we denote as $tw(S)$ the smallest treewidth of the intersection graph of any formulation of $S$ as in (9).

**Remark 4.12.** It came to our attention that, independently of this work, in Aboulker et al. [1] it was recently proven that for any minor-closed family of graphs there exists a constant $c$ such that the correlation polytope of each graph of $n$ vertices in the minor-closed family has linear extension complexity at least

$$2^{c(\omega + \log n)}$$ (11)

where $\omega$ is the treewidth of the graph. This provides families of polytopes where (10) is almost tight. While this result is in the same spirit as the result we prove in this section, we highlight a few key differences:

1. The results in [1] study the important question of the linear extension complexity of the correlation polytope for various graphs providing (almost) optimal bounds, while we give ourselves more freedom with the polytope family.

2. The constant $c$ in (11) is at most $1/2$, and the correlation polytope of a graph with treewidth $\omega$ has ambient dimension $N \in O(\omega n)$—the number of edges of the graph. If additionally $N \in \Theta(\omega n)$, the lower bound in (11) satisfies

$$2^{c(\omega + \log n)} \in O\left(\sqrt{\frac{N}{\omega}}2^{\omega/2}\right).$$

The polytopes we construct here have a lower bound with a leading term $N/\omega$ as compared to $\sqrt{\frac{N}{\omega}}$. This is due to the fact that we rely on the stronger existential counting arguments in [53, 18, 19] along with a polytope composition procedure.

3. Our employed technique is drastically different: rather than reducing to a face of the correlation polytope we provide a general technique to construct high-extension-complexity polytopes from any seed polytope (under appropriate assumptions).

4. Our results apply to both the semidefinite and the linear case. Moreover, we also specialize our construction to Stable Set polytopes where the gluing operation that we use has a natural representation in terms of graph-theoretic operations.
4.3 Binary optimization problems with high extension complexity

In this section we analyze how high semidefinite extension complexity can be used to derive characteristics of the formulation of sets and their treewidth. Consider a family of sets \( \{ S_n \}_{n \in \mathbb{N}} \) with \( S_n \subseteq \{0, 1\}^n \) such that

\[
x_{\text{CSDP}}(S_n) \in \Omega\left(2^{f_n}\right)
\]

for some \( f_n \). For technical reasons we further assume that \( f_n \) satisfies

\[
\liminf_{n \to \infty} \frac{\log n}{f_n} < 1.
\]

Remark 4.13. Every family of sets \( S_n \) such that \( x_{\text{CSDP}}(S_n) \in \Omega(n^k) \) for some \( k > 1 \) satisfies (13). In such case, \( f_n \geq k \log n \) asymptotically and (13) can be easily verified.

Assuming (13) only excludes sets with linear or sub-linear semidefinite extension complexity (w.r.t. \( n \)), which are of little interest here. Moreover, by [18], we know there exist 0/1 sets whose semidefinite extension complexity satisfies (13).

Lemma 4.14. Any formulation of \( S_n \) has intersection graph with treewidth \( \Omega(f_n) \) and at most \( n - 1 \). In particular, \( \text{tw}(S_n) \) is \( \Omega(f_n) \) and \( O(n) \).

Proof. The upper bound is immediate, since \( S_n \) has \( n \) variables. For the lower bound, we know from Theorem 4.10 there exists \( c_1 \) such that

\[
x_{\text{CSDP}}(S_n) \leq c_1 n^{2\omega_n},
\]

where \( \omega_n \) is the treewidth obtained from a formulation (9). And since

\[
x_{\text{CSDP}}(S_n) \geq c_2 2^{f_n}
\]

for some \( c_2 \), we obtain

\[
f_n \leq \log(c_1/c_2) + \omega_n + \log n.
\]

If \( \omega_n \in o(f_n) \) this implies

\[
1 \leq \liminf_{n \to \infty} \frac{\log n}{f_n},
\]

a contradiction with (13). We conclude \( \omega_n \in \Omega(f_n) \). \( \square \)

4.4 Composition of Polytopes

The techniques in this section allow us to manipulate the sets \( S_n \) in a convenient way. Here we drop the index \( n \) for ease of notation as all definitions and results apply for any 0/1 set. We use the notation \( \alpha S \) with \( \alpha \in \mathbb{R}_+ \), to denote the set \( \{ x \mid x = \alpha \cdot y \text{ with } y \in S \} \); in particular \( 0 \in 0S \) for all \( S \).

Definition 4.15. For \( S \subseteq \{0, 1\}^n \), we define \( S^+ \subseteq \{0, 1\}^{n+1} \) as

\[
S^+ = \{(x, x_{n+1}) \in \{0, 1\}^{n+1} \mid x \in (1 - x_{n+1})S\}.
\]
In particular, \((x, 0) \in S^+\) for all \(x \in S\) and \(e_{n+1} \in S^+\). We obtain the following lemma:

**Lemma 4.16.**

\[
\text{conv}(S^+) = \{(x, x_{n+1}) \in [0, 1]^{n+1} \mid x \in (1 - x_{n+1})\text{conv}(S)\}. \quad (14)
\]

**Proof.** Inclusion \(\subseteq\) is direct, as the right-hand set is convex, and the inclusion can be directly verified for the extreme points.

Now consider \((x, x_{n+1})\) an extreme point of the right-hand set in (14). We first claim \(x_{n+1} \in \{0, 1\}\). Otherwise, we can write

\[
(x, x_{n+1}) = (1 - x_{n+1})(x/(1 - x_{n+1}), 0) + x_{n+1}e_{n+1},
\]

where \(e_{n+1}\) is the \((n + 1)\)-th canonical vector. By assumption \(x/(1 - x_{n+1}) \in S\) thus \((x/(1 - x_{n+1}), 0) \in S^+\) and \(e_{n+1} \in S^+\). This contradicts \((x, x_{n+1})\) being an extreme point.

As such \(x_{n+1} \in \{0, 1\}\) and we can easily verify that \((x, x_{n+1}) \in S^+\) which proves the remaining inclusion. \(\square\)

**Definition 4.17.** A polytope \(Q \subseteq \mathbb{R}^n\) is called a pyramid with base \(B \subseteq \mathbb{R}^n\) and apex \(v \in \mathbb{R}^n\) if

\[
Q = \text{conv}(B \cup \{v\})
\]

and \(v\) is not contained in the affine hull of \(B\).

In Tiwary et al. [56] the extension complexity of the Cartesian product of polytopes is analyzed and it is shown:

**Theorem 4.18.** Let \(Q_1, Q_2\) be non-empty polytopes such that one of the two polytopes is a pyramid. Then

\[
xc(Q_1 \times Q_2) = xc(Q_1) + xc(Q_2)
\]

This result provides us with a tool to combine polytopes in a way that their extension complexity is added up. Unfortunately, the result is limited to linear extended formulations. We generalize this result to the SDP case here:

**Theorem 4.19.** Let \(Q_1, Q_2\) be non-empty polytopes such that one of them is a pyramid. Then

\[
xc_{SDP}(Q_1 \times Q_2) \geq xc_{SDP}(Q_1) + xc_{SDP}(Q_2) - 1
\]

**Proof.** This result follows directly from combining the analysis by Tiwary et al. [56] with a result from Fawzi et al. [27]. We assume w.l.o.g. that \(Q_2\) is a pyramid and thus we may assume the slack matrix \(T\) of \(Q_2\) has the form

\[
T = \begin{bmatrix}
T' & 0 \\
0 & 1
\end{bmatrix}
\]

with \(T'\) a slack matrix of the base \(Q'_2\) of \(Q_2\). This implies

\[
xc_{SDP}(Q_2) = xc_{SDP}(Q'_2) + 1, \quad (15)
\]

20
(see e.g., [27, Theorem 2.10]). On the other hand, it also implies that there is a slack matrix $A$ of $Q_1 \times Q_2$ of the following form (see [56]):

$$A = \begin{bmatrix}
S & \cdots & S \\
\tilde{t}_1' & \cdots & \tilde{t}_k' & S & S \\
0 & \cdots & 0 & \cdots & 0 & 1 & 1
\end{bmatrix},$$

where each $t_i'$ corresponds to a column of $T'$ and $S$ is a slack matrix of $Q_1$. Further, the following matrix is a sub-matrix of $A$:

$$A' = \begin{bmatrix}
S \\
T' \\
0
\end{bmatrix}.$$

Since this is a block-triangular matrix by [27, Theorem 2.10], we know that

$$\operatorname{rank}_{PSD}(A') \geq \operatorname{rank}_{PSD}(S) + \operatorname{rank}_{PSD}(T').$$

Using the factorization theorem for semidefinite extended formulations (Theorem 4.9) and (15) we obtain

$$x_{\operatorname{SDP}}(Q_1 \times Q_2) \geq x_{\operatorname{SDP}}(Q_1) + x_{\operatorname{SDP}}(Q_2) - 1.\]
Since \( S = \{x \in \{0,1\}^n \mid (x,0) \in S^+\} \), we obtain
\[
S = \{x \in \{0,1\}^n \mid \varphi_i(x,0) = 0, \; i \in [m]\}.
\] (18)

The treewidth associated with formulation (18) is at most the treewidth of formulation (17), as the intersection graph of the former is obtained by removing a vertex from the intersection graph of the latter. By assumption, the treewidth of formulation (17) is \( tw(S^+) \), thus
\[
tw(S) \leq tw(S^+).
\]

In what follows, we will need a short technical lemma.

**Lemma 4.21.** Let \( S \subseteq \{0,1\}^n \). Then
\[
tw(S \times S) = tw(S).
\]

**Proof.** Inequality \( \leq \) follows directly, since any formulation of \( S \) can be used to formulate \( S \times S \). Moreover, the intersection graph of such formulation consists of 2 identical copies of the intersection graph of the formulation of \( S \). From here the inequality follows.

For the other inequality, take any formulation for \( S \times S \):
\[
S \times S = \{(x,y) \in \{0,1\}^{2n} : \varphi_i(x,y) = 0, \; i \in [m]\}.
\] (19)

Let \( \hat{y} \in S \) be arbitrary. By definition we must have that \( x \in S \) if and only if \((x, \hat{y}) \in S \times S \), thus \( S = \{x \in \{0,1\}^n : \varphi_i(x,\hat{y}) = 0, \; i = 1, \ldots, m\} \) is a valid formulation for \( S \). The intersection graph of such formulation is a sub-graph of the intersection graph of formulation (19), thus its treewidth is at most as large. This proves \( tw(S) \leq tw(S \times S) \). \( \square \)

The results above shows the key fact that taking Cartesian product of certain polytopes adds up their extension complexity, but roughly maintains their treewidth. We summarize this in the following Lemma.

**Lemma 4.22.** Let \( S \subseteq \{0,1\}^n \) and define
\[
S^{\otimes k} = S^+ \times \cdots \times S^+
\] where the Cartesian product is taken \( k \) times. Then
\[
xc_{SDP}(S^{\otimes k}) \geq k \cdot xc_{SDP}(S)
\]
and
\[
tw(S) \leq tw(S^{\otimes k}) \leq tw(S) + 1.
\]

**Proof.** Since \( \text{conv}(S^+) \) is a pyramid (part (a) of Lemma 4.20) and \( \text{conv}(S^+ \times S^+) = \text{conv}(S^+) \times \text{conv}(S^+) \) we obtain
\[
xc_{SDP}(S^{\otimes k}) \geq xc_{SDP}(S^{\otimes(k-1)}) + xc_{SDP}(S^+) - 1 \quad \text{(by Theorem 4.19)}
\]
\[
= xc_{SDP}(S^{\otimes(k-1)}) + xc_{SDP}(S) \quad \text{(by (15))}
\]
Applying this inductively we obtain \( xc_{SDP}(S^{\otimes k}) \geq k \cdot xc_{SDP}(S) \). On the other hand, applying Lemma 4.21 iteratively we have
\[
tw(S^{\otimes k}) = tw(S^+)
\]
and thus the treewidth claim follows from part (b) of Lemma 4.20. \( \square \)
4.5 Composing polytopes of high semidefinite extension complexity

We now use the results in Sections 4.3 and 4.4 and a family \( \{ S_n \}_{n \in \mathbb{N}} \) of (assumed) high (semidefinite) extension complexity, to construct a family of polytopes having a (semidefinite) extension complexity lower bounded by treewidth.

**Theorem 4.23.** Let \( \{ S_n \}_{n \in \mathbb{N}} \) be a family of sets satisfying (12), i.e.

\[
xc_{SDP}(S_n) \in \Omega \left( \frac{2^h}{\log(n)} \right),
\]

and technical condition (13). Consider a sequence \( \{ \omega_n \}_{n \in \mathbb{N}} \) with \( \omega_n \leq n - 1 \) for all \( n \in \mathbb{N} \). Then there exists a family of sets \( \{ S'_n \}_{n \in \mathbb{N}} \) such that:

\[
\text{tw}(S'_n) \leq \omega_n + 1 \quad \text{and} \quad xc_{SDP}(S'_n) \in \Omega \left( \frac{n}{\omega_n + 1} 2^{f_{\omega_n}} \right).
\]

Moreover, \( \text{conv}(S'_n) \) is a pyramid and \( \text{tw}(S'_n) \in \Omega(f_{\omega_n}) \).

**Proof.** Fix \( n \in \mathbb{N} \) and consider set \( S_{\omega_n} \). This set has \( \omega_n \) variables and from Lemma 4.14 \( \text{tw}(S_{\omega_n}) \) is \( \Omega(f_{\omega_n}) \) and at most \( \omega_n - 1 \). Now let \( k \in \mathbb{N} \) and consider \( S^{\omega_k}_{\omega_n} \). By Lemma 4.22

\[
\text{tw}(S_{\omega_n}) \leq \text{tw}(S^{\omega_k}_{\omega_n}) \leq \text{tw}(S_{\omega_n}) + 1
\]

which implies \( \text{tw}(S^{\omega_k}_{\omega_n}) \) is at most \( \omega_n \). Additionally

\[
xc_{SDP}(S^{\omega_k}_{\omega_n}) \geq k \cdot xc(S_{\omega_n}) \in \Omega \left( k \cdot 2^{f_{\omega_n}} \right).
\]

As a last step, we define

\[
S'_n = \left( S^{\omega_k}_{\omega_n} \right)^+, \tag{20}
\]

which inherits the extension complexity bounds from \( S^{\omega_k}_{\omega_n} \) and increases the treewidth by at most 1. The last requirement we need is \( S'_n \) to have at most \( n \) variables, hence, we require

\[
k \cdot (\omega_n + 1) \leq n - 1.
\]

Choosing \( k = \lceil \frac{n - 1}{\omega_n + 1} \rceil \) concludes the result. \( \square \)

The reader might notice that the last step taken in (20) is not necessary to obtain the extension complexity result. However, this will prove useful next, when we further analyze how hard these instances are.

We are now ready to apply the techniques we developed to some known hard polytopes, thus showing that Theorem 4.10 is essentially tight.

**Main Theorem 4.24.** For every \( \{ \omega_n \}_{n \in \mathbb{N}} \) satisfying \( \omega_n \leq n - 1 \) for all \( n \in \mathbb{N} \), there exists a family of sets \( \{ S'_n \}_{n \in \mathbb{N}} \) each with at most \( n \) variables and such that:

\[
\text{tw}(S'_n) \leq \omega_n + 1 \quad \text{and} \quad xc_{SDP}(S'_n) \in \Omega \left( \frac{n}{\omega_n + 1} 2^{\omega_n(1-o(1))} \right)
\]

Moreover, \( \text{tw}(S'_n) \) is \( \Omega(\frac{\omega_n}{4} (1 - o(1))) \) and \( \text{conv}(S'_n) \) is a pyramid.
Proof. In [18] the existence of $n$-dimensional 0/1 polytopes with semidefinite extension complexity lower bounded by
$$2^{n(1-o(1))},$$
is shown. We simply use the vertices of these polytopes as $\{S_n\}_{n\in\mathbb{N}}$ in Theorem 4.23 and the result is obtained. $\square$

Note that Theorem 4.24 provides a nice additional insight: as $tw(S'_n)\in O(\omega_n)$ the instances we construct can be formulated sparsely, but there is no valid formulation that is considerably sparser than that as $tw(S'_n)$ is $\Omega(\omega_n \cdot \omega_n)$.

4.6 Reformulations

When solving optimization problems in general and integer programming problems in particular, reformulation techniques are often employed to modify the original instance, in order to obtain a more well-behaved one. For instance, the affine map $(x_1,x_2-1)$ can be applied to the set $\{(0,0),(0,1),(1,1),(2,1)\} \subseteq \mathbb{R}^2$ to obtain the set $\{(0,0),(1,0),(1,1)\}$. The convex hull of the latter set can then be “decomposed” as the Cartesian product of the line segments $[0,1]$ and $[0,1]$; the same is not true for the former.

In this section we show that the hard instances from Theorem 4.24 are robust with respect to common reformulations techniques. A very general notion of reformulation was introduced by Braun et al. [17], where the authors deal with any nonnegative problem and allow to customize which objective functions (called evaluation) the original problem and the reformulation have to agree on (see [17] for details). Here, we restrict the definition to reformulations that agree with the original problem for any nonnegative objective function.

Definition 4.25. Let $S \subseteq \{0,1\}^n$. We define a reformulation of $S$ as a triple $(S',f,d)$, where:

(a) $S \subseteq \mathbb{R}^m$ is an arbitrary set;
(b) $f : S \rightarrow S'$ is a bijection;
(c) $d$ is a collection of affine functions $\{d_c : \mathbb{R}^m \rightarrow \mathbb{R} : c \in C\}$ with the property that $d_c(f(x)) = c(x)$ for all $x \in S$ and $c \in C$, where $C$ contains all affine functions $c : \mathbb{R}^n \rightarrow \mathbb{R}$ with $c(x) \geq 0$ for all $x \in S$.

The motivation for this definition is that, for each affine function $c$ that is nonnegative over $S$, one could find the optimal solution to the instance
$$\max\{c(x) : x \in S\}$$
by finding an optimal solution $y^*$ to the instance
$$\max\{d_c(y) : y \in S'\} \quad (21)$$

24
and then outputting $f^{-1}(y^*)$. Typically one is interested in the case when (21) is an integer programming problem, thus $f$ can be viewed as in integer programming re-encoding of an optimization problem. However, in the following analysis we do not need to restrict $S'$ to be an integer set. Furthermore, we do not need to assume $S \subseteq \{0,1\}^n$ either, but we will phrase everything with this assumption as the sets we construct in this work are all 0/1 valued.

For a set $S \subseteq \{0,1\}^n$, let $xc^{ref}(S)$ (resp. $xc_{SDP}^{ref}(S)$) be the minimum linear (resp. semidefinite) extension complexity of a reformulation for $S$. Clearly $xc^{ref}(S) \leq xc(S)$ and $xc_{SDP}^{ref}(S) \leq xc_{SDP}(S)$, as the set itself can be viewed as a trivial reformulation.

The following results appeared in [17], and show that the extension complexity of a set cannot be significantly reduced using a reformulation.

**Theorem 4.26.** $xc^{ref}(S) \geq xc(S) - 1$ and $xc_{SDP}^{ref}(S) \geq xc_{SDP}(S) - 1$.

Thus, reformulating problems as in Definition 4.25 can decrease their extension complexity by at most 1. This shows the sets we construct are robust, in terms of their extension complexity, for reformulations. However, extension complexity is not the only parameter to measure how hard a problem is, and separability can be also used to achieve tractability. We define and analyze this next.

### 4.6.1 Reformulations and separability

Given that the extension complexity cannot be reduced significantly via reformulations, one could aim at reformulating an optimization problem in a way that the resulting set is a Cartesian product of lower-dimensional sets (see the example at the beginning of Section 4.6).

More formally, for a $d$-dimensional set $S \subseteq \mathbb{R}^n$, we say it is decomposable if we can write $S = S_1 \times S_2$, with $S_i \subseteq \mathbb{R}^{n_i}$ (with $i \in \{1,2\}$) of dimension $d_i$ so that $d_1 + d_2 = d$, and $d > d_1 \geq d_2$. We say that a reformulation $(S', f, d)$ of $S$ is decomposable if so is $S'$. In this section, we prove that the hard instances from Theorem 4.24 cannot be reformulated to be decomposable.

**Lemma 4.27.** Let $(S', f, d)$ be a reformulation of $S \subseteq \{0,1\}^n$. If $conv(S)$ is a pyramid, then $conv(S')$ is also a pyramid.

**Proof.** Since $conv(S)$ is a pyramid, there exist $B \subseteq S$ and a point $v \in S \setminus aff(B)$ such that

$$conv(S) = conv(B \cup \{v\}).$$

Hence, there exists an affine function $\hat{c} : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$0 \leq M = \hat{c}(x) \neq \hat{c}(v) \geq 0 \quad \forall x \in S \cap B.$$

We claim that $conv(S')$ is a pyramid with base $conv(\{f(x) : x \in S \cap B\})$ and apex $f(v)$. This follows from the fact that $d_\hat{c}(f(x)) = \hat{c}(x)$, thus $M = d_\hat{c}(f(x)) \neq d_\hat{c}(f(v))$ \quad $\forall x \in S \cap B$. \[\square\]

Next we show that pyramids are not decomposable.
Lemma 4.28. Let $\text{conv}(S) \subseteq \mathbb{R}^n$ be a pyramid. Then $S$ is not decomposable.

Proof. Without loss of generality let $S$ have dimension $d \geq 3$; the statement is trivial otherwise. Assume for contradiction that $S = S_1 \times S_2$ with $S_i \subseteq \mathbb{R}^{n_i}$ of dimension $d_i$ for $i \in \{1, 2\}$ and without loss of generality let $d_1 \geq d_2$. Let $(\bar{y}_1, \bar{y}_2)$ be the apex of $\text{conv}(S)$. Since $d_1 = d - d_1 \geq 1$, we deduce that there exist $\tilde{y}_1 \in \mathbb{R}^{n_1}$, $\tilde{y}_2 \in \mathbb{R}^{n_2}$ with $\tilde{y}_i \neq \bar{y}_i$ for $i \in \{1, 2\}$ such that $(\tilde{y}_1, \tilde{y}_2)$ is an extreme point of $\text{conv}(S)$. Moreover, $(\tilde{y}_1, \bar{y}_2), (\bar{y}_1, \tilde{y}_2), (\tilde{y}_1, \tilde{y}_2)$ are all extreme points, and since they are different from the apex, they must lie on the base of $\text{conv}(S)$. Nonetheless, the affine space generated by those three latter points contains $(\tilde{y}_1, \tilde{y}_2)$, a contradiction. □

Note that the hypothesis of the previous lemma cannot be relaxed to the weaker assumption that $\text{conv}(S)$ (only) contains a pyramid, as the example from the beginning of Section 4.6 shows.

Lemma 4.29. Fix $n \in \mathbb{N}$, let $S := P'_n \subseteq \{0, 1\}^n$ as in Theorem 4.24. Then $S$ does not admit a reformulation that is decomposable.

Proof. Let $(S', f, d)$ be a reformulation of $S$. By Lemma 4.27, $\text{conv}(S')$ is a pyramid and by Lemma 4.28, $S'$ is not decomposable. □

5 Related Results

We will now present several related results.

5.1 Specialization to Stable Set Polytopes

We proved the existence of certain 0/1 polytopes with high exponential semidefinite extension complexity, parametrized using the treewidth of a formulation of the set itself in Theorem 4.24. For this purpose, we used a 0/1 set that does not necessarily correspond to a polytope of a combinatorial problem, such as, e.g., the stable set problem or the matching problem. And even if we had used a family of combinatorial polytopes as a starting point, there is no guarantee that the resulting polytopes in Theorem 4.24 correspond to a combinatorial problem as well. In this section we show that the argument in Theorem 4.23 is compatible with the stable set polytope, and one can state a similar parametrized lower bound on the semidefinite extension complexity of a family of stable set polytopes. Due to the restriction of the class of polytopes considered here, the lower bound is worse than that of Theorem 4.24, but nonetheless it is exponential in the treewidth parameter.

Definition 5.1. Given a graph $G = (V, E)$ on $n$ nodes, where $V = [n]$. We define

$$\text{STAB}(G) = \{x \in \{0, 1\}^n \mid x_i + x_j \leq 1 \ \forall \{i, j\} \in E(G)\}. \quad (22)$$

We first note there is a correspondence between the treewidth of the graph $G$, and the treewidth of the set $\text{STAB}(G)$ as defined in Definition 4.11. While expected this is a non-trivial fact, since one could conceive the existence of a boolean-formula-based formulation of $\text{STAB}(G)$ that can be sparser that $G$ itself. We prove that this is not the case.
Lemma 5.2. Given a graph $G = (V, E)$. Then:

$$tw(G) = tw(STAB(G))$$

Proof. Note that $tw(G) \geq tw(STAB(G))$, since the formulation given in (22) has as intersection graph $G$ itself. For the $\leq$ inequality, we prove that the intersection graph of any formulation of $STAB(G)$ has $G$ as subgraph. For contradiction, suppose there exist $\phi_i$ with $i \in [m]$ such that

$$STAB(G) = \{x \in \{0, 1\}^n \mid \phi_i(x) = 1, \ i \in [m]\}.$$ 

and that $G$ is not a subgraph of $\Gamma[STAB(G)_\emptyset] = (V, E')$. This implies that there must exist $\{k, l\} \in E$ such that $\{k, l\} \notin E'$. Defining

$$I_j \doteq \{i : x_j \in \text{supp}(\phi_i)\}$$

with supp($\phi_i$) being the set of variables that appear explicitly in $\phi$, we obtain $I_k \cap I_l = \emptyset$. On the other hand, both $e_k$ and $e_l$—the $k$-th and $l$-th canonical vectors—are indicator vectors of valid stable sets, thus

$$\phi_i(e_k) = 1 \land \phi_i(e_l) = 1 \quad \forall i.$$ 

We conclude the proof by noting that, since $e_k$ and $e_l$ only differ in the $k$-th and $l$-th coordinates, and $I_k \cap I_l = \emptyset$,

$$\phi_i(e_k + e_l) = 1 \quad \forall i.$$ 

This is not possible, since both $k$ and $l$ cannot be part of a stable set of $G$ simultaneously.

Slightly abusing notation, we now define a $(\cdot)^+$ operator for graphs (which is based on the $(\cdot)^+$ operator for polytopes) that will justify why we can use the same procedure as in Theorem 4.23 within the stable set family.

Definition 5.3. Let $G = (V, E)$ be a graph on $n$ vertices with $V = [n]$. We define $G^+$ as

(i) $V(G^+) = [n + 1]$.

(ii) $E(G^+) = E(G) \cup \{(i, n + 1) : \forall i \in [n]\}$

We are ready to formulate the following key lemma:

Lemma 5.4. Let $G$ be a graph on $n$ vertices, and define $G^+$ as before. Then

$$\text{conv}(STAB(G^+))$$

is a pyramid with base $\text{conv}(STAB(G))$ and apex $e_{n+1}$. Moreover,

$$STAB(G^+) = STAB(G)^+.$$ 

Proof. This follows directly since the stable sets of $G^+$ are either stable sets of $G$ or $\{n + 1\}$ by construction. This is in correspondence to the definition of $STAB(G)^+$. □
Using the lemma from above, we can retrace the proof of Theorem 4.23 using as a
starting point a family \( \{S_n\}_{n \in \mathbb{N}} \) with \( S_n = \text{STAB}(G_n) \) for some graph \( G_n \) over \( n \) nodes with \( n \in \mathbb{N} \). Note that, in addition, if we consider \( G_1, G_2 \) copies of a graph \( G \) we have

\[
\text{STAB}(G_1 \cup G_2) = \text{STAB}(G_1) \times \text{STAB}(G_2).
\]

All in all we obtain the following corollary:

**Corollary 5.5.** Given any sequence \( \{w_n\}_{n=1}^{\infty} \) of natural numbers such that \( w_n \leq n - 1 \) for all \( n \), there exists a family of connected graphs \( \{G'_n\}_{n=1}^{\infty} \) such that \( \text{tw}(G'_n) \leq w_n \) and

\[
\text{xcsdp}(\text{STAB}(G'_n)) \in \Omega \left( \frac{n}{w_n + 1} 2^{\Omega(w_n^{1/13})} \right).
\]

**Proof.** The proof follows along the same lines as the proof of Theorem 4.23. Our
starting point is a result by Lee et al. [45], where it is proven that for any \( n \), there exists
graph \( G_n \) on \( n \) vertices, such that

\[
\text{xcsdp}(\text{STAB}(G_n)) \geq 2^{\Omega(n^{1/13})}.
\]

Employing Lemma 5.4, the operations \((-)^+ \) and \( \times \) over stable set polytopes correspond
to operations performed directly over graphs, thus the result follows; the connectedness
requirement follows from the separability argument stated before. \( \square \)

We can also restrict ourselves to linear extension complexity to obtain sharper lower
bounds. Following the exact same strategy (using Theorem 4.18 instead of Theorem
4.19) and using a result of Göös et al. [34] that shows that there exist graphs \( G \) on \( n \) variables such that

\[
\text{xcl}(\text{STAB}(G_n)) \geq 2^{\Omega(n/\log n)}
\]

we obtain the following corollary for the linear extension complexity case:

**Corollary 5.6.** Given any family \( \{w_n\}_{n=1}^{\infty} \) such that \( w_n \leq n - 1 \) for all \( n \), there exists a family of graphs \( \{G'_n\}_{n=1}^{\infty} \) such that \( \text{tw}(G'_n) \leq w_n \) and

\[
\text{xcl}(\text{STAB}(G'_n)) \in \Omega \left( \frac{n}{w_n + 1} 2^{\Omega(w_n/\log w_n)} \right).
\]

### 5.2 Average Extension Complexity of Stable Set Polytopes

In Theorem 4.24 (resp. Corollary 5.6) it is shown that the upper bounds on the extension
complexity in terms of treewidth discussed in the introduction are essentially tight when
we consider 0/1 polytopes (resp. stable set polytopes), i.e., there exist polytopes that
almost satisfy the bound. It is a natural question to ask whether this fact holds with high
probability: if we sample a “random” 0/1 (resp. stable set) polytope, is its extension
complexity close to the bound given by Theorem 4.24 (resp. Corollary 5.6) with high
probability?

We show that the answer to the previous question is negative for stable set polytopes
in the classical Erdös-Renyi random graph model. Here a graph \( G(n, p) \) is the outcome
of a random process that starts from a graph on \( n \) nodes without any edges and then adds each potential edge independently with probability \( p \). For a graph \( G \), we denote by \( \alpha(G) \) the maximum size of its stable set. The average extension complexity of stable set polytope has been studied in [16]. However, we will only need the following basic fact, that can be found in e.g. [26, Lemma 11.2.1].

**Lemma 5.7.** Let \( n, r \geq 2 \) and \( G = G(n, p) \). Then \( \mathbb{P}(\alpha(G) \geq r) \leq \left( \frac{ne^{-p(r-1)/2}}{e^r} \right)^r \).

For \( \alpha(G) \geq 2 \), standard enumeration arguments imply that \( xc(STAB(G)) \leq n^{\alpha(G)} \). Hence, for \( r \gg \log n/p \), using Lemma 5.7 one has

\[
\mathbb{P}(\text{xc}(\text{STAB}(G)) \geq 2^r) \leq \mathbb{P}(\alpha(G) \geq r) \leq n^r e^{-pr(r-1)/2} = 2^r \log n \left( \frac{ne^{-p(r-1)/2}}{e^r} \right)^r \xrightarrow{n \to +\infty} 0.
\]

In the regime \( p \gg c(n) \cdot \frac{\log^2 n}{n} \) with \( c(n) = \Omega(1) \), we have therefore

\[
\lim_{n \to \infty} \mathbb{P}(\text{xc}(\text{STAB}(G)) \geq 2^{\frac{n}{\log n}}) = 0.
\]

On the other hand, random graphs in the same regime of \( p \) have linear treewidth with high probability, as shown in [60, Theorem 2].

**Theorem 5.8** (Wang et al. [60]). Let \( p \) be as above and \( G = G(n, p) \). Then

\[
\lim_{n \to \infty} \mathbb{P}(\text{tw}(G) \geq (1-t)n) = 1
\]

for every constant \( 0 < t < 1 \).

Therefore, for any \( p \) in this regime, the corresponding bound for \( xc(STAB(G)) \) given by Corollary 5.6 is of the order \( \Omega \left( 2^{\frac{n}{\log n}} \right) \) with high probability. This means that stable set polytopes like the ones constructed in Corollary 5.6 happen with very low probability, and thus the corresponding treewidth-based upper bound is loose with high probability.

Note that this also prevents us from using counting arguments (similar to [53, 18]) to establish high extension complexity as replacement for the construction used to establish Corollary 5.6 for stable sets.

### 5.3 Lower bounds for a fixed intersection graph family

Theorem 4.24 holds for any valid “target” treewidth, and we assume complete freedom on the set that we can construct, as long as its treewidth meets the target. A natural question is whether the same can be said for an arbitrary family of intersection graphs. In this section we prove that a similar result can be obtained even if we fix the graph family, and require the constructed sets to have the fixed family as intersection graph. Since we are in a much more restricted setting, the result is weaker than Theorem 4.24, but it remains exponential in the treewidth parameter.

29
5.3.1 Planar 2-SAT polytope with exponential extension complexity

In Avis and Tiwary [6] it is shown that:

**Theorem 5.9** (Avis and Tiwary [6]). For every \( n \) there exists a 2-SAT formula \( \phi \) in \( n \) variables such that the satisfiability polytope of \( \phi \) has extension complexity at least \( 2^{\Omega(\sqrt{n})} \). Moreover, one can assume the graph induced by the 2-SAT formula is planar.

Here, the satisfiability polytope is simply the convex hull of the 0/1 points that satisfy the boolean formula \( \phi \). The result follows from a stable set instance with linear extension complexity \( 2^{\Omega(\sqrt{n})} \) from [29], which is then used in [7] to obtain a stable set instance on a planar graph with extension complexity \( 2^{\Omega(\sqrt{n})} \). The latter can be cast as the feasible set of a 2-SAT formula derived from the underlying graph.

One can follow the same strategy along with the results regarding semidefinite extension complexity of Lee et al. [45] to see that there exist a family of 2-SAT formulas \( \{ \phi_n \}_{n=1}^{\infty} \) on \( n \) variables defined over planar graphs, such that their respective satisfiability polytopes have semidefinite extension complexity at least

\[
2^{\Omega(n^{1/26})}.
\]

Using this observation, we can follow a similar strategy as in Section 3.4 to prove the following lower bound.

5.3.2 Lower bound result

**Theorem 5.10.** Let \( \{G_k\}_{k=1}^{\infty} \) be an arbitrary family of graphs indexed by treewidth. There exists a sequence \( \{(S_{n_k}, G_{n_k})\}_{k=1}^{\infty} \), such that

(i) \( S_{n_k} \) is a 0/1 set.

(ii) \( \{G_{n_k}\}_{k=1}^{\infty} \) is a subsequence of \( \{G_k\}_{k=1}^{\infty} \).

(iii) \( S_{n_k} \) admits a formulation with \( G_{n_k} \) as intersection graph, and thus \( \text{tw}(S_{n_k}) \leq n_k \).

(iv) \( \text{xc}_{\text{SDP}}(S_{n_k}) \geq 2^{\text{poly}(n_k^{1/\epsilon})} \) for a universal constant \( \epsilon \).

**Proof.** Consider \( \phi_n \) a 2-SAT formula on \( n \) variables over a planar graph such that

\[
\text{xc}_{\text{SDP}}(S'_n) \geq 2^{\text{poly}(n^{1/\epsilon})},
\]

where \( S'_n \) is the set of binary vectors that satisfy \( \phi_n \). Let \( H_n \) be the planar graph on \( n \) vertices associated to \( \phi_n \) and fix \( k \in \mathbb{N} \). By Corollary 3.14, we know there exists a polynomial \( \kappa(k) \) such that \( H_k \) is a minor of \( G_{\kappa(k)} \). Following the proof in Section 3.4, we can start from a formulation of \( S'_n \subseteq \{0,1\}^k \) and obtain an equivalent formulation in a lifted space, which has \( G_{\kappa(k)} \) as intersection graph. We call the set of feasible solutions of this lifted formulation \( S''_{\kappa(k)} \).

Since the procedure generates equivalent formulations in an extended space, one can easily see that

\[
\text{xc}_{\text{SDP}}(S'_n) \leq \text{xc}_{\text{SDP}}(S''_{\kappa(k)})
\]

30
consequently,

\[ x_{\text{SDP}}(S_{\kappa(k)}') \geq 2^{\Omega(k^{3/26})}. \]

Defining \( n_k = \kappa(k) \) the result follows. The fact that \( c \) is a universal constant is justified by the fact that \( \kappa \) depends only on the number of vertices of \( H_k \).

The reader might have noticed that one can state the result in Theorem 5.10 without the need of a subsequence \( n_k \), since one can augment the sequence by defining “intermediate” pairs

\[ (S_{\kappa(k)+1}, G_{\kappa(k)+1}), \ldots, (S_{\kappa(k+1)-1}, G_{\kappa(k+1)-1}) \]

since \( H_k \) is also a minor of \( G_{k'} \) for \( k' > \kappa(k) \). This would not change the exponential lower bound, since there is only a polynomial number of sets in (23).

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