Abstract. We construct an equivariant coherent, hence algebraic, cohomology class with values in the completion of the Poincaré bundle on an abelian scheme $A$. From this we obtain a cohomology class on the automorphism group of $A$ with values in some canonical bundles attached to the abelian scheme, which can be explicitly calculated in terms of Eisenstein-Kronecker series. As a consequence of this construction, we show that for an algebraic Hecke character $\chi$ of an arbitrary totally complex number field $L$ the (regularized) critical $L$-values $L(\chi, 0)$ divided by certain periods are algebraic integers. Moreover, using an infinitesimal trivialization of the Poincaré bundle, we construct a $p$-adic measure interpolating the critical $L$-values in the ordinary case. This generalizes previous results for CM fields by Damerell, Shimura and Katz.

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Critical values of Hecke $L$-functions. Let $\chi$ be an algebraic Hecke character of a number field $L$ and $L(\chi, s) = \sum_a \frac{\chi(a)}{N(a)^s}$ its $L$-function as defined by Hecke, where the sum runs over all integral ideals of $L$ coprime to the conductor of $\chi$. It is a classical problem to investigate the special values of $L(\chi, k)$ at integers $k$. As $L(\chi, k) = L(\chi \cdot N_{L/K}, 0)$ one can concentrate on the case $s = 0$.

Assume now that $s = 0$ is critical for $\chi$, which means that the $\Gamma$ factors occurring in the functional equation of $L(\chi, s)$ have a finite value at $s = 0$. It is known that this can happen only if $L$ is totally real or if $L$ contains a CM field $K$ (recall that a CM field is a totally imaginary quadratic extension of a totally real field).

In the totally real case one knows, thanks to work of Euler, Klingen and Siegel [Sie70], that the $L(\chi, 0)$ are algebraic numbers. This result was later refined by Barsky, Cassou-Nogues, Deligne and Ribet who showed that certain regularized values $L(\chi, 0)$ are even algebraic integers and they could construct a $p$-adic $L$-function.

In the case where $L = K$ is a CM field, work of Damerell [Dam70] (for $K/Q$ imaginary quadratic) and of Shimura [Shi75] in general, showed that $L(\chi, 0)$ divided by certain periods are algebraic numbers. Later Katz [Kat78] could even show that a certain regularization of these values are algebraic integers and he constructed a $p$-adic $L$-function in the ordinary case. Blasius [Bla86] could finally prove that for CM fields $K$ Deligne’s conjecture about critical $L$-values is true.

The remaining case where $L$ is an arbitrary extension of degree $n := [L : K]$ of a CM field $K$ was still open. Here an algebraic Hecke character has the form $\chi = \varrho(\chi_0 \circ N_{L/K})$ where $\chi_0$ is an algebraic Hecke character of $K$ and $\varrho$ is a Hecke character of finite order. Harder and Schappacher announced in [HS85] (see also [Har90]) that if $\chi$ has values in the number field $E$ one has that $\Delta(L/K)L(\varrho(\chi_0 \circ N_{L/K}), 0)$ divided by $L(\varrho|_{A_K^\times} x_0^n, 0)$ is in $E$, where $\Delta(L/K)$ is the quotient of two periods and $\varrho|_{A_K^\times}$ is $\varrho$ considered as a character of $K$. This together with Blasius theorem would then give a generalization of Shimura’s theorem. Unfortunately, no full details ever appeared on this result. We point out that even one would have a full proof of this statement, the construction proposed by Harder needs multiplicity one theorems as an input and is therefore not capable of giving good integrality results or even $p$-adic $L$-functions. As far as we know, the only other result previously obtained for extensions of CM fields is by Colmez [Col89], who could show the analogue of Shimura’s theorem for certain extensions $L$ of imaginary quadratic fields $K$. 

\section*{Introduction}
The main results. In this paper we generalize the results of Katz [Kat78] using an equivariant coherent Eisenstein-Kronecker class to the case of an arbitrary finite extension $L$ of an arbitrary CM field $K$. Our main result can be formulated in the case of a non-trivial conductor as follows (see [14.10] for the case of a trivial conductor):

**Theorem** (Integrality of critical values, see 4.10). Let $\chi$ be an algebraic Hecke character of $L$ of conductor $\mathfrak{f}$ and of critical infinity type $\beta - \alpha$ with respect to some CM type $\Sigma$ of $L$ (see section 1.1). Let $\mathcal{A}$ be an abelian scheme defined over a subring $R$ of an algebraic number field with CM by $\Theta_L$ and an analytic uniformization $\mathbb{C}^\Sigma/\Lambda_\mathfrak{f} \cong \mathcal{A}(\mathbb{C})$. Let $\mathfrak{c}$ be an integral ideal in $L$ coprime to $\mathfrak{f}$. Then if $\mathfrak{f} \neq \Theta_L$

\[
\frac{(\alpha - 1)!(2\pi i)^{\beta}}{\Omega^\alpha \Omega^{\beta}}(\chi(\mathfrak{c}) N \mathfrak{c} - 1)L(\chi, 0) \in R[L/(\mathfrak{f} \mathfrak{c})].
\]

Here $\Omega$ and $\Omega^\prime$ are periods of $\mathcal{A}$ and its dual $\mathcal{A}^\vee$. This generalizes the results of Katz in the case of CM fields to arbitrary totally complex fields. Such a result was anticipated by Katz in the introduction of [Kat78].

We understand that Bergeron-Charollois-Garcia can also prove the integrality of these $L$-values with a completely different method, but also relying on equivariant cohomology classes (personal communication, see also [BCG]).

It is a natural question if one could prove the full Deligne conjecture on critical values of Hecke $L$-functions starting from the above theorem and relying on the period relations shown by Blasius [Bl86]. Indeed, the functoriality of our construction makes it very easy to study the Galois action on the critical values. A little bit more difficult is to get the exact comparison of the periods occurring here and the ones used by Deligne in his conjecture. We hope to treat this in a future paper.

The main point of getting at integrality results for special values of $L$-functions is to prove $p$-adic interpolation as a consequence. Our method gives a geometric construction of $p$-adic measures in the ordinary case using the Poincaré bundle. We obtain the following generalization of the $p$-adic $L$-function constructed by Katz [Kat78]:

**Theorem** ($p$-adic interpolation, see 5.23). Suppose that the CM type $\Sigma$ is ordinary for the prime number $p$ (see 5.1). For every fractional ideal $\mathfrak{f}$ and every auxiliary fractional ideal $\mathfrak{c}$ co-prime to $p\mathfrak{f}$ there exists a $p$-adic measure $\mu_{\mathfrak{c}, \mathfrak{f}}$ on $\text{Gal}(L(p^{\infty})/L)$ with the following interpolation property: For every Hecke character $\chi$ of critical infinity type $\beta - \alpha$ and conductor dividing $p^{\infty}\mathfrak{f}$, we have:

\[
\frac{1}{\Omega^\alpha \Omega^{\beta}} \int_{\text{Gal}(L(p^{\infty})/L)} \chi(g)d\mu_{\mathfrak{c}, \mathfrak{f}}(g) = \text{Local}(\chi; \Sigma)[\mathcal{O}_L^\mathfrak{c} : \Gamma](N \mathfrak{c} - \chi(\mathfrak{c}^{-1})) \prod_{\mathfrak{p} \in \Sigma_p} \left(1 - \frac{\chi(\mathfrak{p})}{N \mathfrak{p}}\right)^\frac{(\alpha - 1)!(2\pi i)^{\beta}}{\Omega^\alpha \Omega^{\beta}} L_{\mathfrak{f}}(\chi, 0)
\]

Note that for extensions of imaginary quadratic fields, such an interpolation was proved by Colmez-Schneps [CS92] for the $L$-values of Hecke characters treated in [Col89].

The main ingredient in the proofs of these theorems above is an equivariant coherent cohomology class on an abelian scheme with values in $\mathcal{P}$, the completion of the Poincaré bundle $\mathcal{P}$ along $\mathcal{A} \times_S \mathbb{C}^\Sigma(S) \subset \mathcal{A} \times_S \mathcal{A}^\vee$. Let $d$ be the relative dimension and $\Gamma \subset \text{Aut}_S(\mathcal{A})$ a subgroup of automorphisms of $\mathcal{A}$ over $S$. Then for a finite subscheme $\mathcal{D} \subset \mathcal{A}$ consisting of torsion points and a $\Gamma$-invariant function $f : \mathcal{D} \to \mathcal{O}_S$ we construct a class

\[
EK_{\Gamma}(f) \in H^{d-1}(\mathcal{A} \setminus \mathcal{D}, \Gamma; \mathcal{P} \otimes \mathcal{O}_S^{\mathfrak{d}}),
\]
which we call the equivariant coherent Eisenstein-Kronecker class. It is absolutely essential for our applications to special values of Hecke $L$-functions to have classes in equivariant cohomology. The case we use is that $\Gamma \subset \mathcal{O}_L^\times$ for an abelian scheme with CM by $\mathcal{O}_L$, but there are many other interesting cases. For example, for an abelian scheme $\mathcal{A}/\mathcal{S}$ one can consider the $n$-fold product $\mathcal{A}^n$ of $\mathcal{A}$ over $\mathcal{S}$, which has an action of $GL_n(\mathbb{Z})$, or if $\mathcal{A}$ has already CM by $\mathcal{O}_L$, by $GL_n(\mathcal{O}_L)$. With this in mind the following theorem gives a construction of arithmetic cohomology classes with values in sections of certain algebraic bundles associated to $\mathcal{A}$.

**Theorem** (Eisenstein-Kronecker class, see 2.4). Let $\mathcal{A}$ be an abelian scheme over $\mathcal{R} := \text{Spec } \mathbb{R}$ of relative dimension $d$ and $\Gamma \subset \text{Aut}_{\mathcal{R}}(\mathcal{A})$. Then for any integers $a, b \geq 0$ there is an Eisenstein-Kronecker class $\text{EK}_{b,a}^{\mathcal{A}}(f,x) \in H^{d-1}(\Gamma, H^0(\mathcal{R}, \text{TSym}^a(\omega_{\mathcal{A}/\mathcal{R}}) \otimes \text{TSym}^b(\mathcal{H}) \otimes \omega_{\mathcal{A}/\mathcal{R}}^d))$, depending on a $\Gamma$-invariant function $f : \mathcal{D} \to \mathcal{O}_S$ and a torsion point $x \in (\mathcal{A} \setminus \mathcal{D})(\mathbb{R})$. Here $\omega_{\mathcal{A}/\mathcal{R}} := e^*\Omega_{\mathcal{A}/\mathcal{R}}, \mathcal{H} \cong H^1_{dR}(\mathcal{A}^\vee/\mathcal{R})$ and $\text{TSym}$ denote the tensor symmetric power algebra.

Equivariant cohomology classes receive a lot of attention lately, not only in the work of Bergeron-Charollois-Garcia [BCG] already mentioned, but also for example in the work of Graf [Gra16], of Sharifi-Venkatesh (in a different context) and in the work of Bannai-Hagihara-Yamada-Yamamoto [BHYY19] and in the work [BKL18]. The related Shintani-cocycles come up in the work of Charollois-Dasgupta [CD14].

Besides constructing such a group cohomology class, it is also important to be able to compute an explicit representative. This is accomplished in Section 3 of the paper. Using computations of Levin [Lev00] adapted to our equivariant case one can show that $\text{EK}_{b,a}^{\mathcal{A}}(f,x)$ can be represented in terms of generalized Eisenstein-Kronecker series, which was the reason for its name. Integrating theses series over the classifying space of $\Gamma$ then leads to partial $L$-values of $L(\chi,0)$ by a standard computation. The $p$-adic interpolation of these values relies on the ideas developed in [Spr19] who gives a conceptual approach of the results in Bannai-Kobayashi using the infinitesimal trivialization of the Poincaré bundle.

**The approach to the theorems.** We would like now to discuss our approach to the critical values of Hecke $L$-functions and why the strategy of Shimura or Katz has no straightforward generalization. Both, Shimura and Katz method relies on the existence of algebraic Eisenstein series on the Hilbert modular variety associated to the totally real subfield $F$ inside the CM field $K$. To get at the real-analytic Eisenstein series they use the Maaß-Shimura differential operators which are defined on the base. The evaluation of these Eisenstein series at a CM point is then essentially the $L$-value of the Hecke character in question. The algebraicity of the values is guaranteed by the algebraicity of the Eisenstein series on the Hilbert modular variety.

This approach of Shimura and Katz brakes down for arbitrary extensions $L$ of $K$ as the Eisenstein series in question no longer live in algebraic families. Their natural habitat is the locally symmetric space associated to the arithmetic group $GL_n(\mathcal{O}_K)$, which is not algebraic.

It turns out that one should not work on the base space of an abelian scheme, but on the abelian scheme with CM itself. That such an approach is possible was the pioneering insight by Bannai and Kobayashi [BK10] who showed that the results of Damerell and the $p$-adic interpolation by Katz for elliptic curves $\mathcal{E}$ with CM can be already achieved on $\mathcal{E} \times \mathcal{E}^\vee$. In fact, our work started by trying to understand their approach conceptually...
and to generalize it to higher dimensional abelian varieties. Working on a single abelian variety means that $q$-expansions to check algebraicity of Eisenstein series are no longer available. It is an important feature of our work that the algebraicity is built into our construction of the Eisenstein classes.

We think that it is an important insight that one should no longer work with sections of algebraic bundles as in the work of Shimura and Katz or of Bannai-Kobayashi, but with equivariant coherent cohomology classes. The equivariant approach in our paper has its source in the papers [BKL18] and [Gra16]. An equivariant motivic version of the polylog was advertised earlier by the first named author. Graf showed in fact, that one can recover the (algebraic) Eisenstein series used by Shimura and Katz with this method. This was for us the first indication that such an equivariant cohomological approach to the $L$-values should work.

The second input comes from [Spr19], which demonstrates that the differential operators used by Shimura and Katz can be realized using the canonical connection on the Poincaré bundle on the abelian variety times the universal vector extension of its dual. Underlying this last result is the fact that the completion of this Poincaré bundle with connection is an incarnation of the de Rham logarithm sheaf, which was first shown in the thesis of Scheider [Sch14]. With this insight the paper [Spr19] gives a conceptual construction of the $p$-adic Eisenstein measures of Katz in the elliptic case.

**Outline of the paper.** In the first section we fix our conventions about CM types and abelian varieties with CM by a number field $L$. Further we study the action of the units $\mathcal{O}_L^\times$ of $L$ on the co-Lie-algebra and the algebraic de Rham cohomology $H^1_{dR}(\mathcal{A})$.

In the second section we construct the equivariant coherent Eisenstein-Kronecker class. This is a cohomology class on the abelian scheme $\mathcal{A}$ where one removes some torsion points with values in the completion of the Poincaré bundle. We first develop the necessary properties of the completion of the Poincaré bundle. In fact, all the important properties of the completion are inherited from the Poincaré bundle itself. This Eisenstein-Kronecker class is still insufficient for our purposes as it would lead only to holomorphic Eisenstein series. To remedy this, we extend the class to the completion of the Poincaré bundle on the universal vector extension, where it acquires an integrable connection. We use this connection to construct derivatives of the Eisenstein-Kronecker class. This is in the spirit of Bannai-Kobayashi who used the usual derivative on the elliptic curves.

The third section is the technical heart of the paper. Here the equivariant Eisenstein-Kronecker class is explicitly computed in terms of generalized Eisenstein-Kronecker series. The computation of this class is facilitated by the relation of our class with polylogarithm on abelian schemes, which allows to follow the strategy developed by Levin [Lev00] to compute this class.

The fourth section contains a proof of the integrality statement of the Hecke $L$-function cited above. Here we also fix our complex periods, which is a little bit subtle as the abelian scheme $\mathcal{A}$ admits no $L$-linear polarization if $L$ is not a CM field. It turns out that one should use the periods of $\mathcal{A}$ and of $\mathcal{A}^\vee$ at the same time.

The final section is devoted to the $p$-adic interpolation of the critical $L$-values by a $p$-adic measure. The method used here relies on [Spr19], with some streamlining from [Kat81] and also follows partly [Kat78].

**Acknowledgements.** The attentive reader easily realizes our debt to the work of Katz [Kat78], Bannai-Kobayashi [BK10] and Beilinson-Levin [BL94]. We gratefully acknowledge the support of the SFB Higher Invariants - Interactions between Arithmetic Geometry and Global Analysis in Regensburg.
After the results of this work were completed we were kindly informed by Bergeron that he together with Charollois and Garcia obtained also integrality results for critical $L$-values of Hecke $L$-functions, but with a completely different method. The first named author thanks Bergeron for an invitation to Paris and interesting discussions with him and Charollois about the different approaches used.

1. Abelian schemes with CM

In this section we fix our notation concerning number fields and CM types, discuss abelian schemes with CM, prove some decomposition results and describe our set up.

1.1. CM types. We fix some notations concerning number fields and CM types. Let $L$ be an algebraic number field. We denote by $O_L$ its ring of integers and by $d_L$ its discriminant. We fix the algebraic closure $\overline{\mathbb{Q}} \subset \mathbb{C}$ of $\mathbb{Q}$ in $\mathbb{C}$.

For each set $\Xi$ we denote by $I_\Xi$ the free abelian group $\mathbb{Z}[\Xi]$ generated by $\Xi$. For $\mu = \sum_{\xi \in \Xi} \mu(\xi) \xi \in I_\Xi$ we let

$$|\mu| := \sum_{\xi \in \Xi} \mu(\xi) \in \mathbb{Z}$$

and we denote by

$$I^+_\Xi := \{ \mu = \sum_{\xi \in \Xi} \mu(\xi) \xi \in I_\Xi \mid \mu(\xi) \geq 0 \text{ for all } \xi \in \Xi \}$$

the elements of $I_\Xi$ with positive coefficients. Let $J_L := \text{Hom}_\mathbb{Q}(L, \overline{\mathbb{Q}})$ and $I_L := \mathbb{Z}[J_L]$ be the free abelian group generated by $J_L$. The Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $J_L$ by composition $\sigma \mapsto g\sigma$ and we extend this action to $I_L$ by

$$(1.1.1) \quad g\mu(\sigma) := \mu(g^{-1}\sigma) \text{ for } \mu \in I_L.$$ 

Let $L/K$ be an extension of number fields. We define $N_{L/K}^*: I_K \rightarrow I_L$ by

$$N_{L/K}^*(\mu)(\sigma) = \mu(\sigma|_K).$$

Recall that a CM field $K$ is a totally imaginary quadratic extension of a totally real field $F$. If a number field $L$ contains a CM field, it contains a maximal one and in this case we always denote the maximal CM subfield by $K \subset L$ and let

$$n := [L : K] \quad 2g := [K : \mathbb{Q}] \quad \text{and} \quad 2d := 2gn = [L : \mathbb{Q}].$$

Complex conjugation on $\overline{\mathbb{Q}} \subset \mathbb{C}$ induces an involution $\sigma \mapsto \overline{\sigma}$ on $J_L$. A CM type $\Sigma_K \subset J_K$ for a CM field $K$ is a subset of the embeddings of $K$ into $\overline{\mathbb{Q}}$ such that for $\Sigma_K := \{ \overline{\sigma} \mid \sigma \in \Sigma_K \}$ one has

$$\Sigma_K \cup \overline{\Sigma}_K = J_K \quad \text{and} \quad \Sigma_K \cap \overline{\Sigma}_K = \emptyset.$$ 

Given a CM type $\Sigma_K$ of $K$ we define the CM type $\Sigma$ of $L$ lifted from $\Sigma_K$ to be

$$(1.1.2) \quad \Sigma := \{ \sigma \in J_L \mid \sigma|_K \in \Sigma_K \} \quad \text{and} \quad \overline{\Sigma} := \{ \sigma \in J_L \mid \sigma|_K \in \overline{\Sigma}_K \}.$$ 

We point out that complex conjugation defines bijections $\Sigma_K \cong \overline{\Sigma}_K$ and $\Sigma \cong \overline{\Sigma}$.

**Notation 1.1.** The above decompositions allow to write

$$I_K = I_{\Sigma_K} \oplus I_{\Sigma_K} \quad \text{and} \quad I_L = I_{\overline{\Sigma}} \oplus I_{\overline{\Sigma}}$$

(note the order of $\overline{\Sigma}$ and $\Sigma$). We write accordingly $\mu_0 = \beta_0 + \alpha_0$ for $\mu_0 \in I_K$ and $\alpha_0 \in I_{\Sigma_K}$, $\beta_0 \in I_{\overline{\Sigma}}$ and similarly, $\mu = \beta + \alpha$ for $\mu \in I_L$. 

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We denote by
\[ T_L := \text{Res}_{L/k} \mathbb{G}_m \]
the restriction of \( \mathbb{G}_m \) to Spec \( \mathbb{Z} \). Let \( L^{\text{Gal}} \) be a Galois closure of \( L \). For each \( \mathcal{O}_{L^{\text{Gal}}}[1/d_L] \)-algebra \( R \) the base change of \( T_L \) to \( \mathcal{R} := \text{Spec } R \)
\[ (1.1.3) \]
\[ T_{L,R} := T_L \times_{\text{Spec } \mathbb{Z}} \mathcal{R} \]
is a split torus and the character group of \( T_{L,R} \) is identified with \( I_L \). More precisely, for such \( R \), \( R \otimes \mathcal{O}_L \) is a semi-simple algebra and one has an isomorphism
\[ R \otimes \mathcal{O}_L \cong \bigoplus_{\sigma \in I_L} R \]
\[ 1 \otimes \ell \mapsto (\sigma(\ell))_{\sigma \in I_L} \]
Observe that \( T_{L,R}(R) = (R \otimes \mathcal{O}_L)^\ast \).

An algebraic action of \( \mathcal{O}_L^\ast \) on a \( R \)-module \( M \) is a functorial way to give for each \( R \)-algebra \( R' \) a homomorphism
\[ (R' \otimes \mathcal{O}_L)^\ast \to \text{Aut}_R(R' \otimes_R M). \]
This is the same as a comodule structure under the group ring \( R[I_L] \). The following is well-known and easy to check:

**Theorem 1.2** *(Decomposition [GP11 Exposé I, Prop. 4.7.3]).* Let \( M \) be an algebraic \( T_{L,R} \)-module as above, then there is an isomorphism
\[ M \cong \bigoplus_{\mu \in I_L} M(\mu) \]
where \( T_{L,R} \) acts on \( M(\mu) \) via the character \( \mu : T \to \mathbb{G}_m \). The set of \( \mu \in I_L \) such that \( M(\mu) \neq 0 \) will be called the type of \( M \).

Suppose that \( R \) is contained in a number field \( k \). Let \( T_{L,k} \) be the base change to Spec \( k \). We are interested in the characters of \( T_{L,k} \) which are trivial on \( \mathcal{O}_L^\ast \subset (k \otimes \mathcal{O}_L)^\ast = T_{L,k}(k) \), which are the possible infinity types of algebraic Hecke characters. This question was answered by Serre in [Ser68]. There are two cases:

1) \( L \) does not contain a CM field. Then the only characters \( \mu : T_{L,k} \to \mathbb{G}_m \) trivial on \( \mathcal{O}_L^\ast \) are the powers of the \( N_{L/k} : T_{L,k} \to \mathbb{G}_{m,k} \), i.e. \( \mu \) is of the form
\[ \mu = w \sum_{\sigma \in I_L} \sigma. \]

2) \( L \) does contain a CM field. Let \( K \) be the maximal one. Then a character \( \mu \) which is trivial on \( \mathcal{O}_L^\ast \) has to be of the form \( \mu = \mu_0 \circ N_{L/K} \), where \( N_{L/K} : T_{L,k} \to T_{K,k} \) is induced by the norm from \( L \) to \( K \), and \( \mu_0 \) is a character of \( T_{K,k} \) which satisfies the following condition: There exists a CM type \( \Sigma_K \), elements \( \alpha_0 \in I_{\Sigma_K} \), \( \beta_0 \in I_{\Sigma_K} \) and \( w \in \mathbb{Z} \), such that \( \mu_0 = \beta_0 - \alpha_0 \) and
\[ (1.1.4) \]
\[ \beta_0(\sigma') - \alpha_0(\sigma') = w \text{ for all } \sigma' \in \Sigma_K. \]
In particular, \( \mu = \beta - \alpha \) with \( \alpha = N_{L/K}^\ast(\alpha_0) \) and \( \beta = N_{L/K}^\ast(\beta_0) \). The following definition is crucial for the whole paper.

**Definition 1.3.** Let \( L \) be a number field, which contains the (maximal) CM field \( K \), \( \Sigma_K \) a CM type for \( K \) and \( \Sigma \) the CM type of \( L \) lifted from \( \Sigma_K \). A \( \mu = \beta - \alpha \in I_L \) with \( \beta \in I_{\Sigma} \) and \( \alpha \in I_{\Sigma} \) is said to be of **Hecke character type**, if there are elements \( \beta_0 \in I_{\Sigma_K} \), \( \alpha_0 \in I_{\Sigma_K} \) and \( w \in \mathbb{Z} \) such that
\[ \alpha = N_{L/K}^\ast(\alpha_0) \]
\[ \beta = N_{L/K}^\ast(\beta_0) \]
and such that $\beta_0(\sigma') - \alpha_0(\sigma') = w$ for all $\sigma' \in \Sigma_K$. We write

$$H\text{Char}_L := \{ \mu = \beta - \alpha \in I_L \mid \mu \text{ of Hecke character type} \}$$

for the characters of Hecke character type. We call a $\mu = \beta - \alpha \in H\text{Char}_L$ critical, if $\beta(\sigma) \geq 0$ and $\alpha(\sigma) - 1 \geq 0$ for all $\sigma \in \Sigma$.

We write

$$\text{Crit}_L := \{ \mu = \beta - \alpha \in I_L \mid \mu \text{ of critical type} \}$$

**Remark 1.4.** The $\mu$ of Hecke character type are the possible infinity types of algebraic Hecke characters and the critical ones have a critical $L$-value at 0 (see 4.4 below).

We can now determine the $\Theta^*_L$ invariants of an algebraic $\mathbb{T}_{L,R}$-module $M$.

**Proposition 1.5.** Let $M$ be an algebraic $\mathbb{T}_{L,R}$-module and suppose that $R$ is contained in an algebraic number field $k$. Let $\Gamma \subset \Theta^*_L$ be a subgroup of finite index. Then the $\Gamma$-invariants of $M$ are

$$M^\Gamma = \bigoplus_{\mu \in \text{Char}_L} M(\mu).$$

**Proof.** On $M(\mu)$ the group $\Gamma \subset \Theta^*_L$ acts via the character $\mu : \Theta^*_L \rightarrow R^\times$. This is trivial if and only if $\mu$ is of Hecke character type. \qed

We end this section introducing a convenient notation.

**Definition 1.6.** Let $M$ be an algebraic $\mathbb{T}_{L,R}$-module such that $M = \bigoplus_{\sigma \in \Xi} M(\sigma)$ with $\Xi \subset J_L$. For $\alpha = \sum_{\sigma \in \Xi} \alpha(\sigma) \sigma \in I^L_\Xi$ a positive element we define

$$\text{TSym}^\alpha_R(M) := \bigotimes_{\sigma \in \Xi} \text{TSym}^{\alpha(\sigma)}(M(\sigma)).$$

Further, for an element $m \in M$, we denote by $m^{[\alpha]} \in \text{TSym}^\alpha_R(M)$ the element

$$m^{[\alpha]} = \bigotimes_{\sigma \in \Xi} m(\sigma)^{[\alpha(\sigma)]}$$

where $m(\sigma)^{[\alpha(\sigma)]} = m(\sigma) \otimes \cdots \otimes m(\sigma)$ $\alpha(\sigma)$-times.

Recall that $m(\sigma) \mapsto m(\sigma)^[k]$ is the divided power structure in $\text{TSym}_R(M(\sigma))$ and that $m(\sigma)^k = k! m(\sigma)^{[k]}$ where the left hand side is the $k$-th power calculated in $\text{TSym}_R M(\sigma)$.

With the above convention one has the formula

$$\text{TSym}^k_R(M) \cong \bigotimes_{\alpha \in I^L_\Xi, \alpha = [k]} \text{TSym}^\alpha_R(M)$$

1.2. **Abelian schemes with CM.** We fix some notations concerning abelian schemes. Let $S$ be a scheme and $\pi : A \rightarrow S$ an abelian scheme of relative dimension $d$. We write $e : S \rightarrow A$ for its unit section and $[N] : A \rightarrow A$ for the $N$-multiplication map. As usual, $A[N]$ denotes the kernel of $[N]$.

We denote by $\text{Lie}(A/S)$ the relative Lie algebra and let

$$\omega_{A/S} := e^* \Omega^1_{A/S} \cong \pi_* \Omega^1_{A/S}$$

be the sheaf of translation invariant differential forms. We define

$$\omega^i_{A/S} := \Lambda^i \omega_{A/S} \quad \omega^{-i}_{A/S} := \underline{\text{Hom}}_{O_S}(\omega^i_{A/S}, O_S) = \Lambda^i \text{Lie}(A/S).$$
Let \( \pi^\vee : \mathcal{O}^\vee \to S \) be the dual abelian scheme and write \( e^\vee : S \to A^\vee \) for its unit section. Then we have also the objects \( \text{Lie}(A^\vee /S) \) and \( \omega_{A^\vee /S}^\vee \). Denote by \( H^1_{dR}(A/S) \) the first relative de Rham cohomology of \( A/S \). Then there one has an exact sequence

\[
0 \to \omega_{A/S} \to H^1_{dR}(A/S) \to \text{Lie}(A^\vee /S) \to 0
\]

where one uses the isomorphism \( R^1\pi_* \mathcal{O}_A \cong \text{Lie}(A^\vee /S) \). We notice that there is a canonical pairing

\[
H^1_{dR}(A/S) \times H^1_{dR}(A^\vee /S) \to \mathcal{O}_S.
\]

**Definition 1.7.** We let

\[
\mathcal{H} := \mathcal{H}_A := \text{Hom}_{\mathcal{O}_S}(H^1_{dR}(A/S), \mathcal{O}_S) \cong H^1_{dR}(A^\vee /S).
\]

The isomorphism \( R^1\pi_* \mathcal{O}_A \cong \text{Lie}(A^\vee /S) \) also induces \( R^d\pi_* \mathcal{O}_A \cong \Lambda^d \text{Lie}(A^\vee /S) \), which by Grothendieck duality gives an isomorphism

\[
(1.2.1) \quad \omega_{A^\vee /S}^d \cong \pi_* \Omega^d_{A/S} \cong \omega_{A/S}^d.
\]

We need to review the Serre construction. Let \( T \) be a ring (not necessarily commutative) finite free over \( \mathbb{Z} \) and \( A/S \) an abelian scheme with an injection \( T \to \text{End}_S(A) \) and \( a \) be a projective finitely presented right \( T \)-module. Then we define as usual the abelian scheme \( a \otimes_T A \) (Serre construction), which for every \( S \)-scheme \( S' \) satisfies

\[
(1.2.2) \quad (a \otimes_T A)(S') = a \otimes_T A(S).
\]

Notice that \( \text{Lie}((a \otimes_T A)/S) \cong a \otimes_T \text{Lie}(A/S) \). The Serre construction behaves also well under isogenies: Each inclusion \( a \subset b \) of right \( R \)-modules gives rise to a homomorphism

\[
a \otimes_T A \to b \otimes_T A.
\]

Let \( L \) be a number field with ring of integers \( \mathcal{O}_L \). If \( T = \mathcal{O}_L \) and \( a \) and \( b \) are fractional ideals, then the homomorphism \( a \otimes_T A \to b \otimes_T A \) is an isogeny of degree \( [b : a] \). We introduce a special notation for this case.

**Definition 1.8.** Let \( c \subset L \) be an integral ideal, then the isogeny induced by \( \mathcal{O}_L \subset c^{-1} \) is denoted by

\[
(1.2.3) \quad [c] : A \to c^{-1} \otimes_{\mathcal{O}_L} A
\]

and we write \( A[c] \) for the kernel of \([c]\). In particular, \( \deg[c] = Nc \).

We are interested in considering abelian schemes \( AS \) together with the action of a group \( \Gamma \), i.e., a homomorphisms \( \Gamma \to \text{Aut}_S(A) \). An important source for such groups \( \Gamma \) comes from abelian schemes with complex multiplication.

**Definition 1.9.** Let \( L \) be an algebraic number field of degree \( 2d \) and with ring of integers \( \mathcal{O}_L \). An abelian scheme \( A/S \) of dimension \( d \) has complex multiplication (CM) by \( \mathcal{O}_L \), if there exists an injection

\[
i : \mathcal{O}_L \to \text{End}_S(A).
\]

In particular, one gets a homomorphism \( \mathcal{O}_L^\times \to \text{Aut}_S(A) \).

This is the situation which we study in this paper for the application to special values of Hecke \( L \)-functions. It is known that over a field \( S = \text{Spec} k \) the abelian scheme \( A \) is isogenous to a product of abelian varieties \( B \) with multiplication by the ring of integers \( \mathcal{O}_K \) of a CM field \( K \). In fact one should consider the following set up.

Let \( B/S \) be an abelian schemes with complex multiplication by \( \mathcal{O}_K \) for a CM field \( K \) and \( L/K \) an extension of degree \( n \), then

\[
\mathcal{O}_L \otimes_{\mathcal{O}_K} B
\]
has an action of $\text{End}_{\mathcal{O}_K}(\mathcal{O}_L)$ and hence of $\text{GL}_{\mathcal{O}_K}(\mathcal{O}_L)$. For our main result we will use only the action of $\mathcal{O}_L^\times \subset \text{GL}_{\mathcal{O}_K}(\mathcal{O}_L)$ but the cocycle we construct, lives on the bigger group $\text{GL}_{\mathcal{O}_K}(\mathcal{O}_L)$. Note that the sheaves $\text{Lie}(\mathcal{A}/\mathcal{S})$, $\omega_{\mathcal{A}/\mathcal{S}}$ and $\mathcal{H} \cong H^1_{dR}(\mathcal{A}/\mathcal{S})$ are all $\text{GL}_{\mathcal{O}_K}(\mathcal{O}_L)$-modules and in particular $\mathcal{O}_L^\times$-modules.

1.3. Decomposition results and set up. Let $\mathcal{A}$ be an abelian scheme with CM by $\mathcal{O}_L$. We want to study the decomposition of the sheaves $\text{Lie}(\mathcal{A}/\mathcal{S})$, $\omega_{\mathcal{A}/\mathcal{S}}$ and $\mathcal{H} \cong H^1_{dR}(\mathcal{A}/\mathcal{S})$ under the $\mathcal{O}_L^\times$-action.

We will consider the following situation. Let $L$ be a number field of degree 2, of discriminant $d_L$, which contains a CM field $K$ and let $n := [L : K]$, $2g := [K : \mathbb{Q}]$, so that $d = gn$. Let $R$ be a $\mathcal{O}_{L,\text{CM}}[1/d_L]$-algebra and write $\mathcal{R} = \text{Spec } R$. We let $\mathbb{T}_{L,R}$ be the torus defined in (1.1.3). Let $\mathcal{A}/\mathcal{R}$ be an abelian scheme with CM by $\mathcal{O}_L$. Note that the sheaves $\text{Lie}(\mathcal{A}/\mathcal{R})$, $\omega_{\mathcal{A}/\mathcal{R}}$ and $\mathcal{H} \cong H^1_{dR}(\mathcal{A}/\mathcal{R})$ are algebraic $\mathbb{T}_{L,R}$-modules and locally free as $R$-modules.

**Proposition 1.10.** Suppose that the $\mathcal{O}_{L,\text{CM}}[1/d_L]$-algebra $R$ is contained in $\mathbb{C}$. There exists a (lifted) CM type $\Sigma$ of $L$, called the CM type of $\mathcal{A}$, such that

$$\text{Lie}(\mathcal{A}/\mathcal{R}) \cong \bigoplus_{\sigma \in \Sigma} \text{Lie}(\mathcal{A}/\mathcal{R})(\sigma) \quad \text{and} \quad \omega_{\mathcal{A}/\mathcal{R}} \cong \bigoplus_{\sigma \in \Sigma} \omega_{\mathcal{A}/\mathcal{R}}(\sigma).$$

Moreover, the $\mathbb{T}_{L,R}$-module $\mathcal{H}$ splits into locally free $R$-modules of rank $d$

$$\mathcal{H} \cong \mathcal{H}(\Sigma) \oplus \mathcal{H}(\Sigma')$$

with

$$\mathcal{H}(\Sigma) := \bigoplus_{\sigma \in \Sigma} \mathcal{H}(\sigma) \quad \text{and} \quad \mathcal{H}(\Sigma') := \bigoplus_{\sigma \in \Sigma} \mathcal{H}(\sigma).$$

The above decompositions are compatible with the short exact sequence

$$0 \to \omega_{\mathcal{A}/\mathcal{R}} \to \mathcal{H} \to \text{Lie}(\mathcal{A}/\mathcal{R}) \to 0,$$

i.e. they provide a splitting of the Hodge filtration.

**Proof.** By the decomposition theorem it remains to determine the types of $\text{Lie}(\mathcal{A}/\mathcal{R})$, $\omega_{\mathcal{A}/\mathcal{R}}$ and $\mathcal{H}$. This can be checked after base change to $\mathbb{C}$. For $\mathcal{H} \otimes_R \mathbb{C}$ we get

$$\mathcal{H} \otimes_R \mathbb{C} \cong L \otimes_{\mathbb{Q}} \mathbb{C} \cong \bigoplus_{\sigma \in \text{Hom}_{\mathbb{Q}}(L, \mathbb{C})} \mathbb{C},$$

so every embedding appears exactly once in the rational representation. This determines the type of $\mathcal{H}$. The representation on $\text{Lie } \mathcal{A}/\mathcal{R} \otimes_R \mathbb{C}$ is the analytic representation. It is well known that the direct sum of the analytic representation with its complex conjugate gives the rational representation, cf. [BL04, Proposition 1.2.3]. This determines the type of $\text{Lie}(\mathcal{A}/\mathcal{R})$ and as the sequence

$$0 \to \omega_{\mathcal{A}/\mathcal{R}} \to \mathcal{H} \to \text{Lie}(\mathcal{A}/\mathcal{R}) \to 0.$$
Corollary 1.11. The $R \otimes \mathcal{O}_L$-module $\omega_{A/R}$ splits into

$$\omega_{A/R} \cong \bigoplus_{\sigma \in \Sigma} \omega_{A/R}(\sigma),$$

where $\gamma \in \Gamma = \mathcal{O}_L^\times$ acts on $\omega_{A/R}(\sigma)$ by multiplication with $\sigma(\gamma)^{-1}$.

The next decomposition will come up in our construction of coherent Eisenstein classes.

Corollary 1.12. Let $\Gamma \subseteq \mathcal{O}_L^\times$ be a subgroup of finite index, then

$$(\text{TSym}^\alpha(\omega_{A/R}) \otimes_R \text{TSym}^\beta(\mathcal{H}(\Sigma)) \otimes_R \omega_{A/R}^d)^\Gamma =
\begin{cases}
\text{TSym}^\alpha(\omega_{A/R}) \otimes_R \text{TSym}^\beta(\mathcal{H}(\Sigma)) \otimes_R \omega_{A/R}^d & \beta - \alpha - \frac{1}{2} \text{ critical,} \\
0 & \text{otherwise.}
\end{cases}$$

Proof. The action of $\gamma$ on $\text{TSym}^\alpha(\omega_{A/R}) \otimes_R \text{TSym}^\beta(\mathcal{H}(\Sigma)) \otimes_R \omega_{A/R}^d$ is by multiplication with

$$\prod_{\sigma \in \Sigma} \sigma(\gamma)^{-\alpha(\sigma)-1} \prod_{\sigma \in \Sigma} \sigma(\gamma)^{\beta(\sigma)}$$

which is equal to 1 if and only if $\mu = \beta - \alpha - \frac{1}{2}$ is of Hecke character type. As $\alpha(\sigma) \geq 0$ and $\beta(\sigma) \geq 0$ for all $\sigma \in \Sigma$, one sees that $\mu = \beta - \alpha - \frac{1}{2}$ has to be critical. □

Let $R$ be still contained in $\mathbb{C}$. We now consider the base change to $\mathbb{C}$. Fix a CM type $\Sigma$ of $L$ lifted from the CM type $\Sigma_K$ of a CM field $K \subset L$. Then $\Sigma$ induces an isomorphism

$$(1.3.1) \quad \Phi := \Phi_\Sigma : \mathbb{R} \otimes L \cong \mathbb{C}^\Sigma \quad \quad \quad l \otimes 1 \mapsto (\sigma(l))_{\sigma \in \Sigma}$$

and hence a complex structure on $\mathbb{R} \otimes L$. For a fractional ideal $a \subset L$ we let

$$\Lambda_a := \Phi(a) \subset \mathbb{C}^\Sigma$$

be the corresponding lattice in $\mathbb{C}^\Sigma$ and we consider the complex torus

$$(1.3.2) \quad X(a) := \mathbb{C}^\Sigma / \Lambda_a.$$ 

It is known ([Lan83, Theorem 4.1]) that $X(a)$ is in fact a complex abelian variety, which can be defined over some number field $k$ by the main theorem of complex multiplication. Further, for each complex abelian variety $A(\mathbb{C})$ with CM by $\mathcal{O}_L$ and type $\Sigma$, there exists a fractional ideal $a \subset L$ and an isomorphism

$$\theta : X(a) \cong A(\mathbb{C})$$

compatible with the $\mathcal{O}_L$-action. As usual, we call $A$ of type $(\mathcal{O}_L, \Sigma, a, \theta)$. If we start from an abelian scheme $B/\mathcal{O}$ with CM by $\mathcal{O}_K$ and type $\Sigma$, then one has a uniformization

$$\theta_0 : \mathbb{C}^\Sigma / \Lambda_b \cong B(\mathbb{C})$$

for some fractional ideal $b$ of $K$. In the case $b = \mathcal{O}_K$ one has a canonical isomorphism

$$a \otimes_{\mathcal{O}_K} \mathbb{C}^\Sigma / \Lambda_{\mathcal{O}_K} \cong \mathbb{C}^\Sigma / \Lambda_a$$

and hence the uniformization $\theta_0$ induces

$$(1.3.3) \quad \theta : \mathbb{C}^\Sigma / \Lambda_a \cong a \otimes_{\mathcal{O}_K} B.$$ 

We want to consider abelian schemes $A/\mathcal{O}$ with fixed $R \otimes \mathcal{O}_L$-bases of $\omega_{A/R}$ and of $\omega_{A'/\mathcal{O}}$. 

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**Definition 1.13.** Let $\mathcal{A}/\mathcal{R}$ be an abelian scheme over the $\mathcal{O}_{L,\text{Gal}}[1/d_L]$-algebra $R$ with CM by $\mathcal{O}_L$ and CM type $\Sigma$. A basis $\omega(\mathcal{A})$ of $\omega_{\mathcal{A}/\mathcal{R}}$ (resp. $\omega(\mathcal{A}^\vee)$ of $\omega_{\mathcal{A}^\vee/\mathcal{R}}$) is a collection of global sections

$$\omega(\sigma) \in H^0(\mathcal{R}, \omega_{\mathcal{A}/\mathcal{R}}(\sigma)) \quad \text{(resp. } \omega^\vee(\sigma) \in H^0(\mathcal{R}, \omega_{\mathcal{A}^\vee/\mathcal{R}}(\sigma)))$$

for $\sigma \in \Sigma$, which generate $\omega_{\mathcal{A}/\mathcal{R}}$ (resp. $\omega_{\mathcal{A}^\vee/\mathcal{R}}$) as $R \otimes_{\mathbb{Z}} \mathcal{O}_L$-module.

**Remark 1.14.** The $\omega_{\mathcal{A}/\mathcal{R}}(\sigma)$ and the $\omega_{\mathcal{A}^\vee/\mathcal{R}}(\sigma)$ are locally free $R$-modules of rank one. Hence in general there exists a basis only if one shrinks $\mathcal{R} = \text{Spec } R$ or one performs a base change to a bigger ring. Both possibilities influence the final integrality result.

By the duality between $\omega_{\mathcal{A}/\mathcal{R}}$ and Lie$(\mathcal{A}/\mathcal{R})$ one can either specify a basis of $\omega_{\mathcal{A}/\mathcal{R}}$ or of Lie$(\mathcal{A}/\mathcal{R})$. Hence a pair of bases $(\omega(\mathcal{A}), \omega(\mathcal{A}^\vee))$ of $\omega_{\mathcal{A}/\mathcal{R}}$ and of $\omega_{\mathcal{A}^\vee/\mathcal{R}}$ is the same as a basis of $\mathcal{H}$.

**Notation 1.15.** Let $K$ be a CM field with CM type $\Sigma_K$ and let $L/K$ be a finite extension of degree $n$. Let $R$ be a $\mathcal{O}_{L,\text{Gal}}[1/d_L]$-algebra and fix once and for all an embedding $\iota_{\infty} : \mathbb{R} \to \mathbb{C}$, which will be suppressed from the notation. Let $\mathcal{R} := \text{Spec } R$ and $\mathcal{B}/\mathcal{R}$ be an abelian scheme with CM by $\mathcal{O}_L$ such that there is an isomorphism $\theta_0 : \mathbb{C}^\times/\Lambda_0 \cong \mathcal{B}(\mathbb{C})$. Then for a fractional ideal $a$ of $L$ let

$$A := a \otimes_{\mathcal{O}_K} \mathcal{B}$$

which is an abelian scheme over $\mathcal{R}$ with CM by $\mathcal{O}_L$ and which has an uniformization $\theta : X(a) \cong \mathcal{A}(\mathbb{C})$ induced by $\theta_0$ as in (1.3.3), i.e. $\mathcal{A}$ is of type $(\mathcal{O}_L, \Sigma, a, \theta)$. We will write

$$(\mathcal{A}/\mathcal{R}, \Sigma, a, \theta, \omega(A), \omega(A^\vee), x)$$

to indicate that $\mathcal{A}/\mathcal{R} := a \otimes_{\mathcal{O}_K} \mathcal{B}$ is an abelian scheme over $\mathcal{R} = \text{Spec } R$ of type $(\mathcal{O}_L, \Sigma, a, \theta)$, $\omega(\mathcal{A})$ is a basis of $\omega_{\mathcal{A}/\mathcal{R}}$ and $\omega(\mathcal{A}^\vee)$ a basis of $\omega_{\mathcal{A}^\vee/\mathcal{R}}$ further $x : \mathcal{R} \to \ker \psi$ is a section of the kernel of an isogeny $\psi : \mathcal{A} \to \mathcal{A}'$ over $\mathcal{R}$, which has an étale dual $\psi^\vee : \mathcal{A}^\vee \to \mathcal{A}'$. Note that $\mathcal{A}/\mathcal{R}$ has an action

$$\mathcal{O}_L^\times \subset \text{GL}_{\mathcal{O}_K}(a) \to \text{Aut}_{\mathcal{R}}(\mathcal{A}).$$

If we write any subset of $(\mathcal{A}/\mathcal{R}, \Sigma, a, \theta, \omega(A), \omega(A^\vee), x)$ it is understood to have the same properties, for example $(\mathcal{A}/\mathcal{R}, \Sigma, a, \theta)$ is the abelian scheme $\mathcal{A}/\mathcal{R}$ with CM by $\mathcal{O}_L$ with uniformization $\theta : \mathbb{C}^\times/\Lambda_a \cong \mathcal{A}(\mathbb{C})$.

**Remark 1.16.** The theory of complex multiplication implies that one can choose the ring $R$ above to be a subring of an algebraic number field.

We finally remark that the $\mathcal{O}_K$-module $a$ is isomorphic to a direct sum $a = b_1 \oplus \cdots \oplus b_n$ with fractional ideals $b_i$ of $K$. Thus one has

$$\mathcal{A} \cong \bigoplus_{i=1}^n b_i \otimes_{\mathcal{O}_K} \mathcal{B}$$

and hence Lie$(\mathcal{A}/\mathcal{R}) \cong \bigoplus_{i=1}^n b_i \otimes_{\mathcal{O}_K} \text{Lie}(\mathcal{B}/\mathcal{S})$.

2. **Equivariant coherent Eisenstein classes**

In this section we present our construction of coherent Eisenstein classes. This construction uses the completion of the Poincaré bundle as main input and is modelled after the construction of the (de Rham) polylogarithm on abelian schemes.
2.1. Preliminaries on the completed Poincaré bundle. In this section we work with a general abelian scheme \( \pi : A \to S \) over a noetherian scheme \( S \) together with an action of a discrete group \( \Gamma \). The noetherian assumption can be weakened, see remark \( \ref{rem:noetherian} \).

Consider the universal vector extension \( A^\natural \) of \( A^\vee \), which sits in an exact sequence of commutative group schemes over \( S \)

\[
0 \to \omega_{A/S} \to A^\natural \xrightarrow{p} A^\vee \to 0
\]

and classifies isomorphism classes \((L, \nabla)\) of line bundles with integrable connection relative to \( S \) which satisfy the theorem of the square (see for example [Lau96] for further details).

We denote by \( \pi^\sharp : A^\natural \to S \) the structure map and by \( e^\sharp : S \to A^\natural \) the zero section. We let

\[
(2.1.2) \quad \text{pr}^\vee : A \times_S A^\vee \to A^\vee \quad \text{and} \quad \text{pr}^\natural : A \times_S A^\natural \to A^\natural
\]

be the projections. The relative Lie algebra of \( A^\natural \) identifies with the first de Rham cohomology \( \text{Lie}(A^\natural/S) \cong H^1_{dR}(A/S) \) and the exact sequence of relative Lie algebras

\[
0 \to \omega_{A/S} \to H^1_{dR}(A/S) \to \text{Lie}(A^\vee/S) \to 0
\]

is just the Hodge filtration of \( H^1_{dR}(A/S) \). We define

\[
(2.1.3) \quad \mathcal{H} := \text{Hom}_{\mathcal{O}_S}(H^1_{dR}(A/S), \mathcal{O}_S)
\]

equipped with the dual of the Gauss-Manin connection. The sheaf \( \mathcal{H} \) sits in an exact sequence

\[
0 \to \omega_{A^\vee/S} \to \mathcal{H} \to \text{Lie}(A/S) \to 0.
\]

**Definition 2.1.** Let \( \mathcal{P} \) be the Poincaré bundle on \( A \times_S A^\vee \). The universal line bundle with connection on \( A \times_S A^\natural \) is denoted by \((\mathcal{P}^\natural, \nabla_{\mathcal{P}^\natural})\).

The Poincaré bundle \( \mathcal{P} \) is rigidified along \( e(S) \times A^\vee \) and \( A \times e^\vee(S) \). As a line bundle \( \mathcal{P}^\natural \) is just the pull-back of \( \mathcal{P} \) via \( \text{id} \times p : A \times_S A^\natural \to A \times_S A^\vee \).

We now discuss the completions of \( \mathcal{P} \) and \( \mathcal{P}^\natural \) with respect to \( A \times_S e^\vee(S) \) resp. \( A \times_S e^\natural(S) \).

**Definition 2.2.** Let \( \mathcal{K} \subset \mathcal{O}_{A \times_S A^\vee} \) resp. \( \mathcal{K}^\natural \subset \mathcal{O}_{A \times_S A^\natural} \) be the ideal sheaves defining \( A \times_S e^\vee(S) \) resp. \( A \times_S e^\natural(S) \).

As \( e^\vee \) and \( e^\natural \) are regular immersions one has

\[
\mathcal{K}/\mathcal{K}^2 \cong \Omega^1_{A \times_S A^\vee} \cong \text{pr}^\vee \ast \Omega^1_{A^\vee/S} \cong \pi^\ast \omega_{A^\vee/S}
\]

and

\[
\mathcal{K}^\natural/\mathcal{K}^\natural^2 \cong \Omega^1_{A \times_S A^\natural} \cong \text{pr}^\natural \ast \Omega^1_{A^\natural/S} \cong \pi^\ast \omega_{A^\natural/S}.
\]

Moreover

\[
(2.1.4) \quad \mathcal{K}^n/\mathcal{K}^{n+1} \cong \text{Sym}^n(\omega_{A^\vee/S}) \quad \text{and} \quad \mathcal{K}^\natural^n/\mathcal{K}^\natural^{n+1} \cong \text{Sym}^n(\omega_{A^\natural/S}).
\]

Note that one has an isomorphism \( \omega_{A^\natural/S} \cong \mathcal{H} \) with \( \mathcal{H} \) defined in \((2.1.3)\), which implies

\[
\text{Sym}^n(\omega_{A^\natural/S}) \cong \text{Sym}^n(\mathcal{H}).
\]

To treat \( \mathcal{P} \) and \( \mathcal{P}^\natural \) at the same time, it is convenient to introduce the following notation:
**Notation 2.3.** We write $\mathcal{D}^{(2)}$ in all statements which hold for $\mathcal{D}$ and $\mathcal{D}^2$. Further let $\mathcal{O} := \mathcal{O}_{A \times S A^\vee}$ and $\mathcal{O}^2 := \mathcal{O}_{A \times S A^\wedge}$ and let

$$\mathcal{O}^{(n)} := \mathcal{O}_n \mathcal{O}^{n+1} \quad \text{and} \quad \mathcal{O}^2(n) := \mathcal{O}^2(n+1).$$

We consider these as $\mathcal{O}_A$-algebras and write

$$\hat{\mathcal{O}} := \lim_{\longleftarrow n} \mathcal{O}^{(n)} \quad \text{and} \quad \hat{\mathcal{O}}^2 := \lim_{\longleftarrow n} \mathcal{O}^2(n)$$

for the completions. The same notation will also be used for the co-Lie algebras

$$\hat{\omega} := \omega_{A^\vee / S} \quad \text{and} \quad \omega^2 := \omega_{A^\wedge / S}.$$

We remark that $\mathcal{O}^{(2)}(1)$ being an $\mathcal{O}_A$-algebra has a natural splitting $\mathcal{O}^{(2)}(1) \cong \mathcal{O}_A + \pi^* \mathcal{O}^{(2)}$. The group structure on $A^\vee$ and $A^\wedge$ induce a co-commutative comultiplication on $\hat{\mathcal{O}}$ and $\hat{\mathcal{O}}^2$ and hence maps

$$\mathcal{O}^{(2)}(n+m) \rightarrow \mathcal{O}^{(2)}(n) \otimes \mathcal{O}^{(2)}(m).$$

This gives rise to a map

$$\mathcal{O}^{(2)}(n) \rightarrow \text{TSym}^n \mathcal{O}^{(2)}(1) \cong \bigoplus_{b=0}^n \text{TSym}^b(\omega^{(2)}).$$

**Definition 2.4.** The map

$$\text{mom}^{(n)} : \mathcal{O}^{(2)}(n) \rightarrow \text{TSym}^n \mathcal{O}^{(2)}(1) \cong \bigoplus_{b=0}^n \text{TSym}^b(\omega^{(2)}).$$

induced by the comultiplication is called the moment map. Let $\text{TSym}^{(2)}$ be the completion of $\text{TSym}^{(2)}$ with respect to the augmentation ideal. Then in the limit one gets

$$\text{mom} : \hat{\mathcal{O}}^{(2)} \rightarrow \text{TSym}^{(2)}.$$

**Remark 2.5.** This map is the analogue of the moment map for completed group rings defined in [BKL18, 3.13]. Notice that in the case where $S$ is a scheme of characteristic zero mom is an isomorphism. This can be seen on the associated graded which is the canonical homomorphism $\text{Sym}(\omega^{(2)}) \rightarrow \text{TSym}^{(2)}$.

**Definition 2.6.** Let $\hat{\mathcal{D}}$ denote the completion of $\mathcal{D}$ along $A \times S e^\vee(S) \subset A \times S A^\vee$ and Denote by $\hat{\mathcal{D}}^2$ the completion of $\mathcal{D}^2$ along $A \times S e^\wedge(S) \subset A \times S A^\wedge$. We let $\nabla_{\hat{\mathcal{D}}}$ be the induced connection. We further define

$$\mathcal{D}^{(n)} := \mathcal{D} \otimes_{\hat{\mathcal{O}}^A} \mathcal{O}^{(n)} \quad \text{and} \quad \mathcal{D}^2(n) := \mathcal{D} \otimes_{\hat{\mathcal{O}}^A} \mathcal{O}^2(n).$$

considered as $\mathcal{O}^{(n)}$ and $\mathcal{O}^2(n)$-modules on $A$ respectively. The sheaves $\mathcal{D}^{(n)}$ also inherit a relative connection $\nabla_{\mathcal{D}^{(n)}}$.

Note that $\mathcal{D}^2(n) \cong \mathcal{D}^{(n)} \otimes_{\mathcal{O}^{(n)}} \mathcal{O}^2(n)$ and similarly for $\hat{\mathcal{D}}$ and $\hat{\mathcal{D}}^2$. Moreover, we have $\mathcal{D}^{(2)} = \lim_{\longleftarrow n} \mathcal{D}^{(2)(n)}$.

**Proposition 2.7.** There is an exact sequence of $\mathcal{O}_A$-modules

$$0 \rightarrow \pi^* \text{Sym}^n \omega^{(2)} \rightarrow \mathcal{D}^{(2)(n)} \rightarrow \mathcal{D}^{(2)(n-1)} \rightarrow 0.$$
Proof. This follows from the exact sequence
\[ 0 \to \mathcal{H}^{(2)}n / \mathcal{H}^{(2)}n-1 \to \mathcal{O}^{(2)}n \to \mathcal{O}^{(2)}n-1 \to 0 \]
and the isomorphism \((2.1.3)\). \qed

The rigidification \(\mathcal{O}_{A^\dag} \cong (e \times \text{id})^* \mathcal{P}\) and similarly for \(\mathcal{P}_B\) induce sections
\[ (2.1.5) \quad 1 : \mathcal{O}_S \to e^* \mathcal{P} \cong \mathcal{O} \quad \text{and} \quad 1 : \mathcal{O}_S \to e^* \mathcal{P}_B \cong \mathcal{O}_B \]
and the inclusion \(\mathcal{O} \subset \mathcal{O}_B\) induces an injection
\[ (2.1.6) \quad \mathcal{P} \subset \mathcal{P}_B \cong \mathcal{P} \otimes \mathcal{O}_B \]
compatible with the sections \(1\).

2.2. Properties of the completed Poincaré bundle. In this section we discuss the properties of the completed Poincaré bundle that are important for the construction of our coherent Eisenstein classes. The properties are similar to the one of the logarithm sheaf (see for example [HK]). In fact, one can show that \(\mathcal{P}_B\) is isomorphic to the logarithm sheaf if the base scheme \(S\) is of characteristic zero.

Let \(\varphi : A \to B\) be an isogeny and \(\varphi^\dag : B^\dag \to A^\dag\) its dual. The universal property of the Poincaré bundles gives rise to an isomorphism.
\[ (2.2.1) \quad (\varphi \times \text{id})^* \mathcal{P}_B \cong (\text{id} \times \varphi^\dag)^* \mathcal{P}_A. \]

Theorem 2.8 (Functoriality). Let \(\varphi : A \to B\) be an isogeny. Then one has a canonical map
\[ \varphi^\# : \mathcal{P}^{(1)}_{A^\dag} \to \varphi^* \mathcal{P}^{(1)}_B. \]
If \(\varphi^\dag\) is étale (for example if \(\text{deg} \varphi\) is invertible on \(S\)) \(\varphi^{\#(n)}\) is an isomorphism. The maps \(\varphi^{\#(n)}\) are compatible for different \(n\) and one obtains
\[ \varphi^{\#(n)} : \mathcal{P}^{(n)}_{A^\dag} \to \varphi^* \mathcal{P}^{(n)}_B. \]

Proof. We prove the statement for \(\mathcal{P}^{(n)}\). For \(\mathcal{P}^{(n)}_B\) it then follows by taking the tensor product with \(\mathcal{O}_B^{(n)}\).

Denote by \(A^{(n)}\) and \(B^{(n)}\) the \(n\)-th infinitesimal neighbourhood of \(e_A(S)\) resp. \(e_B(S)\). Let \(\varphi^{(n)} : B^{(n)} \to A^{(n)}\) be the map induced by \(\varphi^\dag\) and denote by \(\pi^{(n)}_A : A^{(n)} \to S\) and \(\pi^{(n)}_B : B^{(n)} \to S\) the structure maps. Then by definition \(\mathcal{P}^{(n)}_A \cong (\text{id} \times \pi^{(n)}_A)^* \mathcal{P}_A|_{A \times A^{(n)}}\) and similarly for \(\mathcal{P}^{(n)}_B\). From \((2.2.1)\) one has
\[ (\text{id}_A \times \varphi^{(n)})^* \mathcal{P}_A|_{A \times A^{(n)}} \cong (\varphi \times \text{id}_B^{(n)})^* \mathcal{P}_B|_{B \times B^{(n)}}. \]
The desired map \(\varphi^{\#(n)} : \mathcal{P}^{(n)}_A \to \varphi^* \mathcal{P}^{(n)}_B\) is now the composition
\[ \mathcal{P}^{(n)}_A = (\text{id}_A \times \pi^{(n)}_A)^* \mathcal{P}_A|_{A \times A^{(n)}} \to (\text{id}_A \times \pi^{(n)}_A)^* (\varphi \times \text{id}_B^{(n)})^* \mathcal{P}_B|_{B \times B^{(n)}} \]
\[ \cong (\text{id}_A \times \pi^{(n)}_A)^* (\varphi \times \text{id}_B^{(n)})^* \mathcal{P}_B|_{B \times B^{(n)}} \]
\[ \cong \varphi^* (\text{id}_B \times \pi^{(n)}_B)^* \mathcal{P}_B|_{B \times B^{(n)}} = \varphi^* \mathcal{P}^{(n)}_B \]
where the isomorphism between the last two lines comes from the base change
\[ A \times_S B^{(n)} \xrightarrow{\varphi \times \text{id}_B^{(n)}} B \times_S B^{(n)} \]
\[ \xrightarrow{\text{id}_A \times \pi^{(n)}_B} A \times_S B^{(n)} \]
If \( \varphi^v \) is étale the map \( \varphi^{v(n)} : B^{v(n)} \to A^{v(n)} \) is an isomorphism and hence the adjunction in (2.2.2) is an isomorphism.

The functoriality for isogenies leads to the important splitting principle.

**Corollary 2.9** (Splitting principle). Let \( \psi : A \to B \) an isogeny such that \( \psi^v \) is étale. Then one has an isomorphism

\[
\mathcal{D}(A)_{\ker \psi} \cong \mathcal{D}(B)_{\ker \psi}.
\]

Let \( t : S \to \ker \psi \) be a \( \psi \)-torsion section. Then there is a canonical isomorphism

\[
\varrho_t : t^* \mathcal{D}(A)_{\ker \psi} \cong \mathcal{D}(B)_{\ker \psi}.
\]

**Proof.** The first statement is clear. The isomorphism \( \varrho_t \) is the composition

\[
\varrho_t : t^* \mathcal{D}(A)_{\ker \psi} \xrightarrow{\psi^*} t^* \mathcal{D}(B)_{\ker \psi} = e^* \mathcal{D}(B)_{\ker \psi} \cong e^* \mathcal{D}(A)_{\ker \psi} \cong \mathcal{D}(B)_{\ker \psi}.
\]

The functoriality implies immediately that \( \mathcal{P} \) and \( \mathcal{P}^\vee \) are \( \Gamma \)-equivariant sheaves:

**Corollary 2.10** (\( \Gamma \)-equivariance). Let \( \Gamma \) be a discrete group acting (from the left) via automorphism on \( A/S \). For each \( \gamma \in \Gamma \) one has an isomorphism

\[
(\gamma^\#)^{-1} : \gamma^* \mathcal{P} \cong \mathcal{P}.
\]

**Proposition 2.11** (Comultiplication). For all \( m, n \) there are canonical homomorphisms

\[
\mathcal{P}(2(n+m)) \to \mathcal{P}(2(n)) \otimes_{\mathcal{O}_A} \mathcal{P}(2(m))
\]

and hence a homomorphism \( \mathcal{P}(2) \to \mathcal{P}(2) \otimes_{\mathcal{O}_A} \mathcal{P}(2) \) (completed tensor product) whose associated graded

\[
\pi^* \text{Sym}_{\mathcal{O}_S}(\mathcal{P}(2)) \to \pi^* \text{Sym}_{\mathcal{O}_S}(\mathcal{P}(2)) \otimes_{\mathcal{O}_A} \pi^* \text{Sym}_{\mathcal{O}_S}(\mathcal{P}(2))
\]

is the homomorphism induced by the diagonal \( \mathcal{P}(2) \to \mathcal{P}(2) \oplus \mathcal{P}(2) \). In particular, the comultiplication is co-commutative.

**Proof.** It is again enough to check this for \( \mathcal{P}(n) \) as the assertion then follows for \( \mathcal{P}(n) \) by taking the tensor product with \( \mathcal{O}^\vee(n) \).

Let \( \mu^{v(n,m)} : A^{v(n)} \times_A A^{v(m)} \to A^{v(n+m)} \) be induced by the group law. As \( \mathcal{P} \) satisfies the theorem of the square one gets

\[
(\text{id}_A \times \mu^{v(n,m)})^* \mathcal{P}|_{A \times A^{v(n+m)}} \cong (\text{id}_A \times \text{pr}_1)^* \mathcal{P}|_{A \times A^{v(n)}} \otimes (\text{id}_A \times \text{pr}_2)^* \mathcal{P}|_{A \times A^{v(m)}}.
\]

Using adjunction for \( (\text{id}_A \times \mu^{v(n,m)})^* \) and then push-forward with \( (\text{id}_A \times \pi^{v(n+m)})_* \) one gets a homomorphism (using \( \pi^{v(n+m)} \circ \mu^{v(n,m)} = \pi^{v(n)} \times \pi^{v(m)} \))

\[
\mathcal{P}(n+m) \to (\text{id}_A \times \pi^{v(n)})_* (\text{id}_A \times \text{pr}_1)^* \mathcal{P}|_{A \times A^{v(n)}} \otimes (\text{id}_A \times \text{pr}_2)^* \mathcal{P}|_{A \times A^{v(m)}}
\]

\[
\cong (\text{id}_A \times \pi^{v(n)})_* \mathcal{P}|_{A \times A^{v(n)}} \otimes (\text{id}_A \times \pi^{v(m)})_* \mathcal{P}|_{A \times A^{v(m)}}
\]

\[
= \mathcal{P}(n) \otimes_{\mathcal{O}_A} \mathcal{P}(m).
\]

The associated graded of \( \mathcal{P} \) is \( \pi^* \text{Sym}_{\mathcal{O}_S}(\mathcal{P}) \) which is the associated graded of \( e^* \mathcal{P} \cong \hat{\mathcal{O}} \).

Hence the associated graded of the map \( \hat{\mathcal{O}} \to \mathcal{P} \otimes \mathcal{P} \) coincides with the associated graded of the comultiplication \( \hat{\mathcal{O}} \to \hat{\mathcal{O}} \otimes_{\mathcal{O}_A} \hat{\mathcal{O}} \) of the formal group \( \varprojlim_n A^{v(n)} \), which is

\[
\text{Sym}_{\mathcal{O}_S}(\mathcal{P}) \to \text{Sym}_{\mathcal{O}_S}(\mathcal{P}) \otimes_{\mathcal{O}_S} \text{Sym}_{\mathcal{O}_S}(\mathcal{P})
\]

This map is induced from the dual of the addition \( \text{Lie}(A^v/S) \oplus \text{Lie}(A^v/S) \to \text{Lie}(A^v/S) \) which is the diagonal \( \mathcal{O} \to \mathcal{O} \oplus \mathcal{O} \).

□
Remark 2.12. The comultiplication structure for $\mathcal{R}(\mathfrak{P})$ in the proposition just reflects the fact that the torsor associated to $\mathcal{P}$ has a partial group law.

**Corollary 2.13.** One has a homomorphism

$$\mathcal{P}^{(n)} \to \text{TSym}_{\mathcal{O}_A}^n(\mathcal{P}^{(1)})$$

which is an isomorphism, if $n!$ is invertible on $S$.

**Proof.** The map $\mathcal{P}^{(n)} \to \mathcal{P}^{(1)} \otimes \cdots \otimes \mathcal{P}^{(1)}$ ($n$-factors) factors through $\text{TSym}_{\mathcal{O}_A}^n(\mathcal{P}^{(1)})$ and the map on the associated graded in degree $k \leq n$ is the canonical map

$$\text{Sym}_k^s(\omega^{(1)}) \to \text{TSym}_k^s(\omega^{(1)}),$$

which is an isomorphism if $k!$ is invertible. \qed

The known higher direct images of the Poincaré bundle under the projection $\text{pr}^\vee : A \times_S A^\vee \to A^\vee$ allow to compute the cohomology of $\mathcal{R}(\mathfrak{P})$.

**Theorem 2.14** (Vanishing of cohomology). There is a canonical isomorphism

$$R^i\pi_* (\mathcal{P} \otimes \pi^* \omega^d_{A/S}) \cong \begin{cases} \mathcal{O}_S & \text{if } i = d \\ 0 & \text{if } i \neq d. \end{cases}$$

Let $S$ be a scheme over a field of characteristic zero. Then

$$R^i\pi_* \Omega^*_{A/S}(\mathcal{P}) = \begin{cases} \mathcal{O}_S & i = 2d \\ 0 & i \neq 2d, \end{cases}$$

where $\Omega^*_{A/S}(\mathcal{P})$ is the de Rham complex of $(\mathcal{P}, \nabla)$.

**Proof.** We only prove the first statement. The second statement will not be used in this paper. A proof can be found in [Sch14, Theorem 1.2.1].

The projection formula gives

$$R^i\pi_* (\mathcal{P} \otimes \pi^* \omega^d_{A/S}) \cong (R^i\pi_* \mathcal{P}) \otimes \omega^d_{A/S}.$$ Let $\hat{\pi}^\vee : \hat{A}^\vee \to S$ be the structure map of the formal completion $\hat{A}^\vee$ at $e^\vee(S)$. Then one has

$$R^i\pi_*(\mathcal{P} \otimes \pi^* \omega^d_{A/S}) \cong \hat{\pi}^\vee_* \big( (R^i\pi_*(\mathcal{P})) \otimes \omega^d_{A/S} \big).$$

where $\hat{\pi}^\vee : A \times_S A^{\vee(n)} \to A^{\vee(n)}$ is the formal completion of $\text{pr}^\vee$. By [Gro61, Thm. 4.1.5] the cohomology of the formal completion is computed as

$$R^i\hat{\pi}^\vee_* \mathcal{P} \cong (R^i\pi^\vee_* \mathcal{P}) \otimes_{\mathcal{O}_{\hat{A}^\vee}} \mathcal{O}_{A \times \hat{A}^\vee}$$

Using the the computation of the cohomology of the Poincaré bundle

$$R \pi^\vee_* \mathcal{P} \cong e^-_{\omega_{\hat{A}^\vee/S}^-[-d]}$$

(see [Lau96, Lemme 1.2.5]) and the isomorphism $\omega_{\hat{A}^\vee/S}^- \cong e^{-d}_{\omega_{\hat{A}^\vee/S}}$ from (1.2.1) one obtains the theorem. \qed

**Corollary 2.15.** One has

$$H^i(A, \Gamma; \mathcal{P} \otimes \pi^* \omega^d_{A/S}) \cong \begin{cases} H^0(S, \mathcal{O}_S) & i = d \\ 0 & i < d. \end{cases}$$
Proof. From theorem 2.14 and with the Leray spectral sequence one gets
\[ H^i(A, \mathcal{P} \otimes \pi^* \omega^d_{A/S}) \cong H^i(S, R\pi_*(\mathcal{P} \otimes \pi^* \omega^d_{A/S})) \cong H^{i-d}(S, \mathcal{O}_S). \]
Hence, the spectral sequence for equivariant cohomology implies
\[ H^i(A, \Gamma; \mathcal{P} \otimes \pi^* \omega^d_{A/S}) = 0 \text{ for } i < d \]
and in degree \( d \) one gets \( H^0(S, \mathcal{O}_S) \Gamma = H^0(S, \mathcal{O}_S) \).

In characteristic zero the bundle \( \mathcal{P}^n \) is nothing but the de Rham logarithm sheaf, a fact which was first shown in [Sch14]. We recall the definition of the de Rham logarithm sheaf. For any scheme \( X \) in characteristic zero we denote by \( \mathcal{D}_{\mathcal{A}} \) the ring of differential operators. Recall that
\[ \mathcal{H} := \text{Hom}_{\mathcal{O}_S}(H^1_{\text{dR}}(A/S), \mathcal{O}_S) \]
equipped with the Gauss-Manin connection. There is an exact sequence [Sch14] (1.1.1) comming from the local to global spectral sequence for \( \text{Ext} \)
\[ 0 \to \text{Ext}^1_{\mathcal{O}_S}(\mathcal{O}_S, \mathcal{H}) \overset{\pi^*}{\longrightarrow} \text{Ext}^1_{\mathcal{O}_A}(\mathcal{O}_A, \pi^* \mathcal{H}) \to \text{Hom}_{\mathcal{O}_S}(\mathcal{H}, \mathcal{H}) \to 0 \]
which is split by \( e^* \). Then the first logarithm sheaf is an extension of \( \mathcal{D}_A \)-modules
\[ 0 \to \pi^* \mathcal{H} \to \mathcal{L}_{\log} \to \mathcal{O}_A \to 0 \]
which maps to \( \text{id} \in \text{Hom}_{\mathcal{O}_S}(\mathcal{H}, \mathcal{H}) \) and with a fixed splitting \( 1^{(1)} : e^* \mathcal{L}_{\log} \cong \mathcal{O}_S \oplus \mathcal{H} \).
The pair \( (\mathcal{L}_{\log}, 1^{(1)}) \) is unique up to unique isomorphism. One then defines
\[ \mathcal{L}_{\log}^{(n)} := \text{Sym}^n \mathcal{L}_{\log}^{(1)} \]
and \( \mathcal{L}_{\log} := \varinjlim_n \mathcal{L}_{\log}^{(n)} \). Recall that in characteristic zero there is also an isomorphism
\( \mathcal{P}^{(n)} \cong \text{TSpec}^n \mathcal{P}^{(1)} \) and that we have a section \( 1^{(1)} : \mathcal{O}_S \to \mathcal{P}^{(1)} \).

**Proposition 2.16** (Scheider, Theorem 2.3.1 [Sch14]). There is a canonical isomorphism \( (\mathcal{L}_{\log}^{(1)}, 1^{(1)}) \cong (\mathcal{P}^{(1)}, 1^{(1)}) \). In particular, one has an isomorphism \( \mathcal{L}_{\log} \cong \mathcal{P} \) respecting the sections along \( e : S \to A \).

### 2.3. The equivariant coherent Eisenstein-Kronecker class.
We consider the abelian scheme \( A \) over \( S \) with the action of \( \Gamma \). We want to consider subschemes of \( D \) consisting of torsion points as follows:

**Notation 2.17.** Let \( \varphi : A \to B \) be a \( \Gamma \)-equivariant isogeny with étale dual and \( D \subset \text{ker } \varphi \) be a non empty closed subscheme stable under the action of \( \Gamma \). We assume that the immersion \( i : D \to A \) is a locally complete intersection. Let \( \mathcal{U}_D := A \setminus D \) be the open complement of \( D \) and \( j : \mathcal{U}_D \to A \) the open immersion.

Under the condition of this notation \( \mathcal{U}_D \) is also stable under the action of \( \Gamma \) and one obtains the following \( \Gamma \)-equivariant diagram
\[ (2.3.1) \]
\[ \begin{array}{ccc}
D & \overset{e}{\longrightarrow} & A \\
\pi_D \downarrow & & \downarrow \pi \\
S & \overset{i}{\longrightarrow} & \mathcal{U}_D := A \setminus D \\
\end{array} \]

Note that \( \text{ker } \varphi \subset A \) is a locally complete intersection being the flat base change of the regular immersion \( e : S \to B \). Hence also the immersion of every open subscheme \( D \subset \text{ker } \varphi \) into \( A \) is a locally complete intersection.
With the vanishing result in Corollary 2.15 the localization sequence for the closed subscheme $D \subset A$ gives rise to a short exact sequence

$$(2.3.2) \quad 0 \to H^{d-1}(U_D, \Gamma; \mathcal{P} \otimes \pi^* \omega^d_{A/S}) \to H^d_D(A, \Gamma; \mathcal{P} \otimes \pi^* \omega^d_{A/S}) \to H^0(S, \mathcal{O}_S).$$

We introduce the following notation:

**Definition 2.18.** Let

$$\mathcal{O}_S[D]^{0, \Gamma} := \ker(H^0(S, \pi_{D*} \pi^*_D \mathcal{O}_S) \to H^0(S, \mathcal{O}_S))^\Gamma.$$ 

where the map $\pi_{D*} \pi^*_D \mathcal{O}_S \to \mathcal{O}_S$ is the trace map.

We remark that the group $\mathcal{O}_S[D]^{0, \Gamma}$ is not always trivial. In fact, $D$ as in the notation 2.17 can be decomposed into closed subschemes (possibly only into ker $\varphi \setminus \{e(S)\}$ and $e(S)$) each of which is $\Gamma$-stable. A function $f \in \mathcal{O}_S[D]^{0, \Gamma}$ is then a linear combination of characteristic functions on these $\Gamma$-stable subschemes, where the characteristic function takes the value one on the $\Gamma$-stable subscheme and zero everywhere else. In particular, there is always a non-trivial element $f_\varphi \in \mathcal{O}_S[D]^{0, \Gamma}$, which is minus the characteristic function on ker $\varphi \setminus \{e(S)\}$ and $\deg \varphi - 1$ times the characteristic function on $e(S)$.

$$(2.3.3) \quad f_\varphi = \begin{cases} 
\deg \varphi - 1 & \text{on } e(S) \\
-1 & \text{on ker } \varphi \setminus \{e(S)\}.
\end{cases}$$

**Theorem 2.19.** Assume $S$ noetherian and $D$ as in notation 2.17. There is a canonical inclusion

$$\mathcal{O}_S[D]^{0, \Gamma} \hookrightarrow \ker \left( H^d_D(A, \Gamma; \mathcal{P} \otimes \pi^* \omega^d_{A/S}) \to H^0(S, \mathcal{O}_S) \right).$$

The proof will be given in section 2.5.

From the theorem and the localization sequence (2.3.2) one gets a map

$$EK_\Gamma : \mathcal{O}_S[D]^{0, \Gamma} \to H^{d-1}(U_D, \Gamma; \mathcal{P} \otimes \pi^* \omega^d_{A/S}).$$

**Definition 2.20.** For $f \in \mathcal{O}_S[D]^{0, \Gamma}$ (and $S$ noetherian) we call

$$EK_\Gamma(f) \in H^{d-1}(U_D, \Gamma; \mathcal{P} \otimes \pi^* \omega^d_{A/S})$$

the equivariant coherent Eisenstein-Kronecker class associated to $f$. If it is necessary to indicate the dependence on $A$, we write $EK_{\Gamma,A}(f)$. Using the map $\mathcal{P} \to \mathcal{P}^z$ we denote by

$$EK_{\Gamma}^z(f) \in H^{d-1}(U_D, \Gamma; \mathcal{P}^z \otimes \pi^* \omega^d_{A/S})$$

the image of $EK_\Gamma(f)$ under the homomorphism

$$H^{d-1}(U_D, \Gamma; \mathcal{P} \otimes \pi^* \omega^d_{A/S}) \to H^{d-1}(U_D, \Gamma; \mathcal{P}^z \otimes \pi^* \omega^d_{A/S}).$$

**Proposition 2.21** (Compatibilty with base change). Let $u : T \to S$ be a flat morphism and write $A_T$, $D_T$ and $U_{DT}$ for the base change of $A$, $D$ and $U_D$ to $T$. Let $v : U_{DT} \to U_D$ and $u^* : \mathcal{O}_S[D]^{0, \Gamma} \to \mathcal{O}_T[D_T]^{0, \Gamma}$ be the induced maps, then

$$v^* EK_{\Gamma,A}(f) = EK_{\Gamma, A_T}(u^*(f)).$$

**Proof.** This follows from the commutative diagram

$$
\begin{array}{ccc}
\mathcal{O}_S[D]^{0, \Gamma} & \xrightarrow{EK_{\Gamma,A}} & H^{d-1}(U_D, \Gamma; \mathcal{P} \otimes \pi^* \omega^d_{A/S}) \\
\downarrow{u^*} & & \downarrow{v^*} \\
\mathcal{O}_T[D_T]^{0, \Gamma} & \xrightarrow{EK_{\Gamma, A_T}} & H^{d-1}(U_{DT}, \Gamma; \mathcal{P} \otimes \pi^* \omega^d_{A_T/T}).
\end{array}
$$
Remark 2.22. The noetherian assumption in the theorem and the definition can be weakened. We need it in the proof to compute the sections with support as a direct limit of Ext groups. As the polylogarithm is compatible with base change, one can usually assume that the polylogarithm comes from the universal abelian scheme over the moduli space. There the noetherian assumption is satisfied.

The class $\text{EK}_G^{(2)}(f)$ satisfies an important functoriality with respect to isogenies $\psi : \mathcal{A} \to \mathcal{A}'$. Suppose that $\Gamma$ acts on $\mathcal{A}'$ as well and that $\psi$ is compatible with the $\Gamma$-action. Let $\mathcal{D}' \subset \mathcal{A}'$ be a closed subscheme as in notation 2.17 and assume $\psi(\mathcal{D}) \subset \mathcal{D}'$. Then, if we let $\mathcal{V} := \mathcal{A} \setminus \psi^{-1}(\mathcal{D}')$, there is a cartesian diagram

$$
\begin{array}{ccc}
\psi^{-1}(\mathcal{D}') & \longrightarrow & \mathcal{A} \\
\downarrow \psi & & \downarrow \psi \\
\mathcal{D}' & \longrightarrow & \mathcal{A}' \longrightarrow \mathcal{U}_{\mathcal{D}'}.
\end{array}
$$

It follows that $\mathcal{U}_{\mathcal{D}} \subset \mathcal{V}$ and the composition

$$
\psi_* \mathcal{P}_A^{(2)} \xrightarrow{\psi_* (\psi)} \psi_* \psi^* \mathcal{P}_A^{(2)} \xrightarrow{\text{Tr}} \mathcal{P}_{A'}^{(2)}
$$

induces a trace map

$$
\text{Tr}_\psi : H^{d-1}(\mathcal{U}_{\mathcal{D}}, \Gamma; \mathcal{P}_A^{(2)} \otimes \pi^* \omega_{A/S}) \to H^{d-1}((\mathcal{V}, \Gamma; \mathcal{P}_A^{(2)} \otimes \pi^* \omega_{A/S}) \cong H^{d-1}(\mathcal{U}_{\mathcal{D}'}, \Gamma; \mathcal{P}_{A'}^{(2)} \otimes \pi^* \omega_{A'/S}).
$$

The trace map $\psi_* \psi^* \mathcal{O}_{\mathcal{D}'} \to \mathcal{O}_{\mathcal{D}'}$ induces also a homomorphism

$$
\text{Tr}_\psi : \mathcal{O}_S[\mathcal{D}]^{0, \Gamma} \to \mathcal{O}_S[\mathcal{D}]^{0, \Gamma}.
$$

Theorem 2.23 (Trace compatibility). Let $\psi : \mathcal{A} \to \mathcal{A}'$ be an isogeny and $\psi(\mathcal{D}) \subset \mathcal{D}'$ as above and $f \in \mathcal{O}_S[\mathcal{D}]^{0, \Gamma}$, then

$$
\text{Tr}_\psi(\text{EK}_G^{(2)}(f)) = \text{EK}_G^{(2)}(\text{Tr}_\psi(f)).
$$

Proof. By definition of $\text{EK}_G^{(2)}(f)$ it suffices to prove this result for $\text{EK}_G^{(2)}(f)$. By construction of the trace map one has a commutative diagram

$$
\begin{array}{ccc}
H^{d-1}(\mathcal{U}_{\mathcal{D}}, \Gamma; \mathcal{P}_A \otimes \pi^* \omega_{A/S}) & \longrightarrow & H^0_{\mathcal{D}}(\mathcal{A}, \Gamma; \mathcal{P}_A \otimes \pi^* \omega_{A/S}) \\
\downarrow \text{Tr}_\psi & & \downarrow \text{Tr}_\psi \\
H^{d-1}(\mathcal{U}_{\mathcal{D}'}, \Gamma; \mathcal{P}_{A'} \otimes \pi^* \omega_{A'/S}) & \longrightarrow & H^0_{\mathcal{D}'}(\mathcal{A}', \Gamma; \mathcal{P}_{A'} \otimes \pi^* \omega_{A'/S}).
\end{array}
$$

Further, by the construction of the inclusion

$$
\mathcal{O}_S[\mathcal{D}]^{0, \Gamma} \hookrightarrow \ker \left( H^d_{\mathcal{D}}(\mathcal{A}, \Gamma; \mathcal{P} \otimes \pi^* \omega_{A/S}) \to H^0_{\mathcal{D}}(\mathcal{A}, \mathcal{O}_S) \right),
$$

in Theorem 2.19 one has a commutative diagram

$$
\begin{array}{ccc}
\mathcal{O}_S[\mathcal{D}]^{0, \Gamma} & \longrightarrow & H^d_{\mathcal{D}}(\mathcal{A}, \Gamma; \mathcal{P}_A \otimes \pi^* \omega_{A/S}) \\
\downarrow \text{Tr}_\psi & & \downarrow \text{Tr}_\psi \\
\mathcal{O}_S[\mathcal{D}]^{0, \Gamma} & \longrightarrow & H^d_{\mathcal{D}'}(\mathcal{A}', \Gamma; \mathcal{P}_{A'} \otimes \pi^* \omega_{A'/S}).
\end{array}
$$
This gives the desired result by the definition of \( EK_{\Gamma, A'}(\text{Tr}_\psi(f)) \).

The class \( EK^2_1(f) \) is not yet sufficient to construct all Eisenstein classes we need. In fact, we will see in the explicit computation, that it gives only rise to the holomorphic Eisenstein series. For the real-analytic ones we use the connection on \( \mathcal{P}_1^x \). In this construction it is essential that we have a class in coherent cohomology. Consider the connection on \( \mathcal{P}_1^x \)

\[
\nabla := \nabla_{\mathcal{P}_1^x} : \mathcal{P}_1^x \to \Omega^1_{A/S} \otimes \mathcal{P}_1^x.
\]

Iterating this connection \( n \)-times gives a map

\[
\nabla^n : \mathcal{P}_1^x \to \text{TSym}^n(\Omega^1_{A/S}) \otimes \mathcal{P}_1^x.
\]

Applying this to \( EK^2_1(f) \) gives

\[
\nabla^n EK^2_1(f) \in H^{d-1}(\mathcal{U}_D, \Gamma; \text{TSym}^n(\Omega^1_{A/S}) \otimes \mathcal{P}_1^x \otimes \pi^* \omega^d_{A/S}).
\]

We note for later use, that this class is still compatible with flat base change.

**Proposition 2.24.** Using the notation of Proposition 2.23 one has

\[
v^* \nabla^n EK^2_1(f) = \nabla^n EK^2_1(\text{Tr}_\psi(f)).
\]

The derived classes satisfy the same functoriality as \( EK^2_1(f) \) because for an étale isogeny \( \psi : A \to A' \) the diagram

\[
\begin{array}{ccc}
\psi_* \mathcal{P}_1^x & \xrightarrow{\psi_*(\nabla)} & \Omega^1_{A'/S} \otimes \psi_* \mathcal{P}_1^x_A \\
\text{Tr}_\psi \downarrow & & \downarrow \text{id} \otimes \text{Tr}_\psi \\
\mathcal{P}_1^x_{A'} & \xrightarrow{\nabla'} & \Omega^1_{A'/S} \otimes \mathcal{P}_1^x_{A'}
\end{array}
\]

commutes. This gives:

**Corollary 2.25.** With the notations in Theorem 2.23 one has for an étale isogeny \( \psi_* \)

\[
\text{Tr}_\psi(\nabla^n EK^2_1(f)) = \nabla^n EK^2_1(\text{Tr}_\psi(f)).
\]

2.4. The Eisenstein-Kronecker class for CM abelian varieties. Let \( \mathcal{D} \) be as in 2.17 and \( x \in \mathcal{U}_D(R) \) a torsion section of \( \mathcal{U}_D \), i.e. it is contained in some \( A[N] \). Let \( \Gamma \) act on \( A/R \) and consider \( f \in R[\mathcal{D}]^{0,1} \). Assume that the torsion section \( x \) is fixed by \( \Gamma \). Then we can pull-back the class \( \nabla^n EK^2_1(f) \) from (2.3.4) along \( x \) and use the map

\[
\mathcal{P}_1^x \to \text{TSym}(\mathcal{P}_1^x(1))
\]

and the isomorphism \( x^* \mathcal{P}_1^x(1) \cong \mathcal{O}_R \oplus \mathcal{H} \) to get a class

\[
\varrho_x(x^* \nabla^n EK^2_1(f)) \in \prod_{b=0}^{\infty} H^{d-1}(\mathcal{R}, \Gamma; \text{TSym}^a(\omega_{A/R}) \otimes \text{TSym}^b(\mathcal{H}) \otimes \omega^d_{A/R}).
\]

As \( \mathcal{R} \) is affine the spectral sequence for equivariant cohomology collapses so that

\[
H^{d-1}(\mathcal{R}, \Gamma; \text{TSym}^a(\omega_{A/R}) \otimes \text{TSym}^b(\mathcal{H}) \otimes \omega^d_{A/R}) \cong H^{d-1}(\Gamma, \text{TSym}^a(\omega_{A/R}) \otimes \text{TSym}^b(\mathcal{H}) \otimes \omega^d_{A/R}),
\]

where we consider \( \text{TSym}^a(\omega_{A/R}) \otimes \text{TSym}^b(\mathcal{H}) \otimes \omega^d_{A/R} \) as \( R \)-Modul.
Definition 2.26. Let $\Gamma \subset \text{Aut}_R(A)$. Then the Eisenstein-Kronecker class

$$EK^{b,a}_\Gamma(f, x) \in H^{d-1}(\Gamma, \text{TSym}^a(\omega_{A/R}) \otimes \text{TSym}^b(\mathcal{H}) \otimes \omega^d_{A/R})$$

is the image of the $b$-component of $g_a(x^* \nabla^a EK^b_\Gamma(f))$ under the map in (2.4.1).

We now specialize to the case $(A/R, \Sigma, a, \theta, \omega(A), \omega(A^\vee), x)$ in [1.15] where $\mathcal{R} = \text{Spec } R$ and $A = \alpha \otimes_{\mathcal{O}_K} \mathcal{B}$ is an abelian scheme of type $(\mathcal{O}_L, \Sigma, \alpha)$ with $R \otimes \mathcal{O}_L$-bases $\omega(A), \omega(A^\vee)$ and a torsion section $x$. Note that in this case

$$EK^{b,a}_{\Gamma, \sigma_K}(\alpha)(f, x) \in H^{d-1}(\Gamma, \text{TSym}^a(\omega_{A/R}) \otimes \text{TSym}^b(\mathcal{H}) \otimes \omega^d_{A/R}).$$

For our main result we will only consider the restriction of $EK^{b,a}_{\Gamma, \sigma_K}(\alpha)(f, x)$ to a subgroup $\Gamma \subset \sigma_K \subset \text{GL}_{\sigma_K}(a)$, which is of finite index in $\sigma_K$.

For such a subgroup $\Gamma$, using the splitting of the Hodge filtration $\mathcal{H} \cong \mathcal{H}(\Sigma) \oplus \mathcal{H}(\Sigma)$, we can further project $EK^{b,a}_\Gamma(f, x)$ onto

$$H^{d-1}(\Gamma, \text{TSym}^a(\omega_{A/R}) \otimes \text{TSym}^b(\mathcal{H}(\Sigma)) \otimes \omega^d_{A/R}).$$

By Proposition [1.10] and Corollary [1.11] and using our conventions for multiindices from [1.9] one has

$$\text{TSym}^a(\omega_{A/R}) \cong \bigoplus_{|\alpha|=a} \text{TSym}^a(\omega_{A/R}) \quad \text{TSym}^b(\mathcal{H}(\Sigma)) \cong \bigoplus_{|\beta|=b} \text{TSym}^b(\mathcal{H}(\Sigma)).$$

By Corollary [1.12] the $\Gamma$-invariants are given by

$$(\text{TSym}^a(\omega_{A/R}) \otimes \text{TSym}^b(\mathcal{H}(\Sigma)) \otimes \omega^d_{A/R})^{\Gamma} \cong \bigoplus_{\beta - \alpha = -1}^{\beta^{\text{Crit}}_L} \text{TSym}^a(\omega_{A/R}) \otimes \text{TSym}^b(\mathcal{H}(\Sigma)) \otimes \omega^d_{A/R}.$$

We write

$$\text{pr}_{\Gamma}: \text{TSym}^a(\omega_{A/R}) \otimes \text{TSym}^b(\mathcal{H}(\Sigma)) \otimes \omega^d_{A/R} \to \bigoplus_{\beta - \alpha = -1}^{\beta^{\text{Crit}}_L} \text{TSym}^a(\omega_{A/R}) \otimes \text{TSym}^b(\mathcal{H}(\Sigma)) \otimes \omega^d_{A/R}$$

for the projection onto the $\Gamma$-invariants and obtain a class

$$\text{pr}_{\Gamma} EK^{b,a}_\Gamma(f, x) \in \bigoplus_{\beta - \alpha = -1}^{\beta^{\text{Crit}}_L} H^{d-1}(\Gamma, \mathbb{Z}) \otimes R \text{TSym}^a(\omega_{A/R}) \otimes R \text{TSym}^b(\mathcal{H}(\Sigma)) \otimes \omega^d_{A/R}.$$

Proposition 2.27. Let $\Gamma \subset \sigma_L$ be of finite index and $1 \in I_{\Sigma}$ the element with $1(\sigma) = 1$ for all $\sigma \in \Sigma$. Then there is a canonical homomorphism

$$H^{d-1}(\Gamma, \mathbb{Z}) \otimes R \text{TSym}^a(\omega_{A/R}) \otimes R \text{TSym}^b(\mathcal{H}(\Sigma)) \otimes \omega^d_{A/R} \to \text{TSym}^a(\omega_{A/R}) \otimes R \text{TSym}^b(\mathcal{H}(\Sigma)).$$

Proof. It is sufficient to define a canonical homomorphism $H^{d-1}(\Gamma, \omega^d_{A/R}) \to \text{TSym}^d(\omega_{A/R})$. Let $\Gamma' \subset \Gamma$ be of finite index. Then $H_{d-1}(\Gamma', \mathbb{Z})$ is non-canonically isomorphic to $\mathbb{Z}$ and for any generator $\xi' \in H_{d-1}(\Gamma', \mathbb{Z})$ we get an isomorphism

$$H^{d-1}(\Gamma', \omega^d_{A/R}) \xrightarrow{\xi'} \omega^d_{A/R}.$$ 

Choosing an ordering of $\Sigma$ allows to choose an orientation on $\mathbb{R} \otimes L$ and hence one on $L_{\mathbb{R}}^1 := \ker((\mathbb{R} \otimes L)^x \xrightarrow{NL_{\mathbb{R}}} \mathbb{R}^x)$. This orientation induces an isomorphism $\mathbb{Z} \cong H_{d-1}(\Gamma', L_{\mathbb{R}}^1, \mathbb{Z})$ \cong
Proposition 2.31. Let \( \omega_{A/R}^d \cong T \Sym^d \omega_{A/R} \) be the \( \omega \)-component of \( \pr \) \( \omega_{A/R} \). Then

\[
H^{d-1}(\Gamma', \omega_{A/R}^d) \cong T \Sym^d \omega_{A/R}
\]

independent of the choice of the generator \( \xi' \) and independent of the choice of the ordering.

Let \( \xi \in H_{d-1}(\Gamma, \mathbb{Z}) \) be an element with \( \text{res} \xi = \xi' \in H_{d-1}(\Gamma, \mathbb{Z}) \) (note that the restriction is surjective) and define using some ordering of \( \Sigma \)

\[
H^{d-1}(\Gamma, \omega_{A/R}^d) \xrightarrow{\text{res} \xi} H^{d-1}(\Gamma', \omega_{A/R}^d) \cong T \Sym^d \omega_{A/R}.
\]

To see that this is independent of the choice of \( \xi \) and the ordering, we note that the diagram

\[
\begin{array}{ccc}
H^{d-1}(\Gamma, \omega_{A/R}^d) & \xrightarrow{\text{res} \xi} & T \Sym^d \omega_{A/R} \\
\downarrow \quad \text{res} & & \downarrow \quad [\Gamma : \Gamma'] \\
H^{d-1}(\Gamma, \omega_{A/R}^d) & \xrightarrow{\text{res} \xi} & T \Sym^d \omega_{A/R}
\end{array}
\]

commutes, because for any \( \eta \in H^{d-1}(\Gamma, \omega_{A/R}^d) \) one has

\[
\text{res} \eta \cap \text{res} \xi = \text{cor}(\eta \cap \text{cor} \circ \text{res} \xi) = [\Gamma : \Gamma'] \eta \cap \xi.
\]

As \( R \subset \mathbb{C} \) is torsion free, the multiplicity with \( [\Gamma : \Gamma'] \) is injective. This gives the claim.

\[ \square \]

**Definition 2.28.** Let \( (A/R, \Sigma, a, \theta, \omega(A), \omega(A^\vee), x) \) be as in 2.15 and \( \Gamma \subset \mathcal{O}_L^\times \) a subgroup which is of finite index fixing \( x \). Let \( \mathcal{D} \) be \( \Gamma \)-stable as in 2.17 and assume that \( x \) is a section of \( \mathcal{U}_D \). Let \( f \in R[\mathcal{D}]^0, \mathcal{Z} \mathcal{H} \) and \( \mu = \beta - \alpha - 1 \in I_L \) critical. We define the class

\[
\text{EK}_\Gamma^{\beta, \alpha}(f, x) \in (T \Sym^{a+1}(\omega(A/R)) \otimes \text{TSym}^\beta(\mathcal{H}(\Sigma)))
\]

to be the \( \alpha, \beta \)-component of \( \text{pr}_\Gamma \text{EK}_\Gamma^{\beta, \alpha}(f, x) \) using the homomorphism from Proposition 2.27. Using the bases \( \omega(A), \omega(A^\vee) \) to trivialize \( T \Sym^{a+1}(\omega(A/R)) \otimes \text{TSym}^\beta(\mathcal{H}(\Sigma)) \) we obtain

\[
\text{EK}_\Gamma^{\beta, \alpha}(f, x)(\omega(A)^{a+1}, \omega(A^\vee)^{\beta}) \in R.
\]

**Remark 2.29.** The classes \( \text{EK}_\Gamma^{\beta, \alpha}(f, x) \) coincide in the case \( L = K, K \) a CM field, with the ones constructed by Katz [Kat78] as will follow from the explicit computation below (see 2.5).

We note for later use the behaviour of \( \text{EK}_\Gamma^{\beta, \alpha}(f, x) \) under change of \( \Gamma \).

**Corollary 2.30.** Let \( \Gamma' \subset \Gamma \subset \mathcal{O}_L^\times \) be two groups of finite index, then

\[
\text{EK}_\Gamma^{\beta, \alpha}(f, x) = [\Gamma : \Gamma'] \text{EK}_\Gamma^{\beta, \alpha}(f, x).
\]

**Proof.** This follows immediately from the commutative diagram 2.12. \[ \square \]

2.5. **Proof of theorem 2.19** We start with the computation of an Ext-group.

**Proposition 2.31.** Let \( \mathcal{I} \subset \mathcal{O}_A \) be the ideal sheaf defining \( \mathcal{D} \) and write \( \mathcal{O}_{A_n} := \mathcal{O}_A/I^{n+1} \). Then

\[
\text{Ext}^q_{\mathcal{O}_A}(\mathcal{I}^n/\mathcal{I}^{n+1}, \mathcal{P} \otimes \pi^* \omega_{A/S}^d) \cong \begin{cases} 0 & q < d \\ H^{q-d}(\mathcal{D}, \text{Hom}_{\mathcal{D}}(\mathcal{I}^n/\mathcal{I}^{n+1}, \pi^* \mathcal{P})) & q \geq d. \end{cases}
\]

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Proof. The local to global spectral sequence for Ext gives

$$H^p(\mathcal{A}, \text{Ext}^q_{\mathcal{O}_A}(\mathcal{I}^n/\mathcal{I}^{n+1}, \mathcal{P} \otimes \pi^* \omega_{A/S}^{d})) \Rightarrow \text{Ext}^{p+q}_{\mathcal{O}_A}(\mathcal{I}^n/\mathcal{I}^{n+1}, \mathcal{P} \otimes \pi^* \omega_{A/S}^{d}).$$

As $\mathcal{D}$ is locally a complete intersection, the local Ext can be computed by [Har66]. Note in addition that $\omega_{\mathcal{D}/\mathcal{A}}$ in loc. cit. is in our case equal to $\text{Hom}_{\mathcal{D}}(\mathcal{A}^d \mathcal{I}/\mathcal{I}^2, \mathcal{O}_D) \cong t^* \pi^* \omega_{A/S}^{d}$. Further, $\mathcal{I}^n/\mathcal{I}^{n+1}$ is locally free and if one writes

$$(\mathcal{I}^n/\mathcal{I}^{n+1})^v := \text{Hom}_{\mathcal{D}}(\mathcal{I}^n/\mathcal{I}^{n+1}, \mathcal{O}_D)$$

one gets

$$\text{Ext}^q_{\mathcal{O}_A}(\mathcal{I}^n/\mathcal{I}^{n+1}, \mathcal{P} \otimes \pi^* \omega_{A/S}^{d}) \cong \text{Ext}^q_{\mathcal{O}_A}(\mathcal{O}_D, \mathcal{P} \otimes \pi^* \omega_{A/S}^{d}) \otimes (\mathcal{I}^n/\mathcal{I}^{n+1})^v$$

$$\cong \begin{cases} 0 & q \neq d \\ \text{Hom}_{\mathcal{D}}(\mathcal{I}^n/\mathcal{I}^{n+1}, t^* \mathcal{P}) & q = d. \end{cases}$$

Inserting this result into the local to global spectral sequence gives the desired result. □

**Corollary 2.32.** For all $n \geq 1$ one has injections

$$\text{Ext}^d_{\mathcal{O}_A}(\mathcal{O}_{A_{n-1}}, \mathcal{P} \otimes \pi^* \omega_{A/S}^{d}) \subset \text{Ext}^d_{\mathcal{O}_A}(\mathcal{O}_{A_n}, \mathcal{P} \otimes \pi^* \omega_{A/S}^{d}).$$

In particular, one has

$$H^0(\mathcal{D}, t^* \mathcal{P}) \cong \text{Ext}^d_{\mathcal{O}_A}(\mathcal{O}_{A_0}, \mathcal{P} \otimes \pi^* \omega_{A/S}^{d}) \subset \lim_{n} \text{Ext}^d_{\mathcal{O}_A}(\mathcal{O}_{A_n}, \mathcal{P} \otimes \pi^* \omega_{A/S}^{d}).$$

**Proof.** This follows from the proposition and the long Ext-sequence associated to

$$0 \to \mathcal{I}^n/\mathcal{I}^{n+1} \to \mathcal{O}_{A_n} \to \mathcal{O}_{A_{n-1}} \to 0.$$

□

**Proof of Theorem 2.11.** If $\mathcal{S}$ is noetherian, then also $\mathcal{A}$ is noetherian. By [Gro68] one has then the following description of the cohomology with support

$$(2.5.1) \quad H^i_D(\mathcal{A}, \mathcal{P} \otimes \pi^* \omega_{A/S}^{d}) \cong \lim_{n} \text{Ext}^i_{\mathcal{O}_A}(\mathcal{O}_{A}/\mathcal{I}^{n+1}, \mathcal{P} \otimes \pi^* \omega_{A/S}^{d}).$$

Combining this with the corollary and using the splitting principle $t^* \mathcal{P} \cong \pi^*_D e^* \mathcal{P}$ of 2.9 gives

$$H^0(\mathcal{D}, \mathcal{O}_D) \subset H^0(\mathcal{D}, t^* \mathcal{P}) \subset H^d_D(\mathcal{A}, \mathcal{P} \otimes \pi^* \omega_{A/S}^{d})$$

which is the statement of Theorem 2.19. □

3. **Explicit computation of the equivariant coherent Eisenstein–Kronecker classes**

The aim of this chapter is to construct a suitable model to perform the explicit computation of the equivariant Eisenstein–Kronecker classes. The setup of this chapter is as in 1.15. We will choose an ordering of our CM type $\Sigma = \Sigma_L = \{\sigma_1, \ldots, \sigma_d\}$. For a family of objects $a = (a_\sigma)_{\sigma \in \Sigma}$ parametrized by $\Sigma$ we will write

$$a = (a_i)_{i=1}^d, \quad \text{with } a_i = a_{\sigma_i}.$$  

In particular, this gives the coordinates

$$z = (z_1, \ldots, z_d) := (z(\sigma_i))_{i=1}^d,$$
and \( I_L = I_{\Sigma} \oplus I_{\Sigma} \) gets identified with \( \mathbb{N}^d \times \mathbb{N}^d \). We fix a fractional ideal \( \mathfrak{a} \subseteq L \) and consider the complex abelian variety \( A(\mathbb{C}) := X(\mathfrak{a}) = \mathbb{C}^2/\Lambda_{\mathfrak{a}} \). The action of a subgroup \( \Gamma \subseteq \mathcal{O}_L^* \) of finite index is given by the formula

\[
\gamma \cdot z := \Phi_L(\gamma^{-1})z, \quad \gamma \in \mathcal{O}_L^*, \ z \in \mathbb{C}^2.
\]

Let us fix a \( \Gamma \)-stable lattice \( \Lambda' \) containing \( \Lambda_{\mathfrak{a}} \) and a \( \Gamma \)-invariant function

\[
f : \Lambda'/\Lambda_{\mathfrak{a}} \to \mathbb{R}
\]

satisfying \( \sum_{t \in \Lambda'/\Lambda_{\mathfrak{a}}} f(t) = 0 \). Notice that \( \Lambda' \) defines an isogeny \( \varphi : X(\mathfrak{a}) \to \mathbb{C}^2/\Lambda' \) and that \( \ker \varphi = \Lambda'/\Lambda_{\mathfrak{a}} \). Furthermore, let \( x \in \mathbb{Q} \otimes \Lambda_{\mathfrak{a}} \setminus \Lambda' \) such that \( x + \Lambda_{\mathfrak{a}} \) is \( \Gamma \)-stable. In the following we will compute the Eisenstein classes

\[
\text{EK}^{\beta,\alpha}(f, x)
\]

for a critical infinity type \( \beta - \alpha - 1 \). The computations use the ideas of Levin [Lev00].

3.1. Generalized Eisenstein-Kronecker series. In the following, we introduce the generalized Eisenstein–Kronecker series which appear in the explicit description of the equivariant coherent Eisenstein–Kronecker class.

**Definition 3.1.** Let \( \mu \in \mathbb{N}^d \), \( \mathfrak{a} \) a lattice in \( L \), \( z, w \in \mathbb{C}^2 \) and \( H = \text{diag}(h) \) be a Hermitian form which is in diagonal form in the standard basis of \( \mathbb{C}^2 \) with \( h \in \mathbb{R}_+^2 \) and \( \langle \cdot, \cdot \rangle := \text{Im} H(\cdot, \cdot) \) the associated alternating form. Then we define the Eisenstein-Kronecker series

\[
K^\mu(H, z, w, s, \Lambda_{\mathfrak{a}}) := \sum_{\lambda \in \Lambda_{\mathfrak{a}}} \sum' \left( \frac{z + \lambda}{|z + \lambda|_H^2} \right) e^{\pi i \langle \lambda, w \rangle}.
\]

Here \( ||z||_H^2 = H(z, z) := \sum_{i=1}^d h_i |z_i|^2 \) is the associated absolute value and \( \sum' \) means that we only sum over \( \lambda \in \Lambda_{\mathfrak{a}} \setminus \{ -z \} \).

**Remark 3.2.** These series are also a special case of the (generalization) of Epstein zeta functions considered by Siegel in [Sie80].

The series \( K^\mu(H, z, w, s, \Lambda_{\mathfrak{a}}) \) converges absolutely and uniformly for \( \text{Re} s > d + \frac{|\mu|}{2} \) and for \( z, w \) in a compact subset of \( \mathbb{C}^2 \).

**Definition 3.3.** For \( \mu \in \mathbb{N}^d \), \( t \in \mathbb{R}_{>0} \) and \( z, w \in \mathbb{C}^2 \) define the theta function

\[
\vartheta^\mu_t(H, z, w, \Lambda_{\mathfrak{a}}) := \sum_{\lambda \in \Lambda_{\mathfrak{a}}} (\tau + \lambda)^\mu e^{-\pi t |z + \lambda|^2} e^{2\pi i \langle \lambda, w \rangle}.
\]

The function \( \vartheta^\mu_t(H, z, w, \Lambda_{\mathfrak{a}}) \) is uniformly and absolutely convergent on each compact set in \( \mathbb{R}_{>0} \times \mathbb{C}^2 \times \mathbb{C}^2 \).

**Proposition 3.4.** The function \( \vartheta^\mu_t(H, z, w, \Lambda_{\mathfrak{a}}) \) satisfies the functional equation

\[
\vartheta^\mu_t(H, z, w, \Lambda_{\mathfrak{a}}) = t^{-d-|\mu|} \frac{2^d |\mathfrak{a}|}{N(\mathfrak{a})^{1/2} |d_L|^{1/2}} e^{2\pi i \langle w, z \rangle} \vartheta^\mu_{t-1}(H, w, z, \Lambda_{\mathfrak{a}}^*)
\]

where \( \Lambda_{\mathfrak{a}}^* \) is the dual of \( \Lambda_{\mathfrak{a}} \) with respect to \( \langle -,- \rangle \) and \( h^t = \prod_{i=1}^d h_i \).

**Proof.** The proof of the functional equation follows the usual lines (see [Sie80] Prop. 8) or [Neu92] Chapter VII,(3.6)]). \( \square \)
Proposition 3.5. For $\mu \in \mathbb{N}^d$ and $\Re s > 0$ one has the formula
\[
\pi^{-1} \Gamma(s) K^\mu(H, z, w, s, \Lambda_a) = \int_0^\infty (\delta_t^\mu H, z, w, \Lambda_a - \delta_{\mu z} e^{2\pi i (-z, w)}) t^s dt.
\]
where $\delta_{\mu z}$ is 1 if both $z \in \Lambda_a$ and $|\mu| = 0$, and zero in all other cases.

Proof. This follows immediately from the formula $\int_0^\infty e^{-tx} t^{-s} dt = \Gamma(s) a^{-s}$. \hfill \Box

As a corollary we get the analytic continuation and functional equation of $K^\mu(H, z, w, s, \Lambda_a)$.

Corollary 3.6. The Eisenstein-Kronecker series $K^\mu(H, z, w, s, \Lambda_a)$ has an analytic continuation to $\mathbb{C}$ with possible poles in $s = 0$ and $s = d$ of order one and residues $-\delta_{\mu z} e^{2\pi i (-z, w)}$ in $s = 0$ and residue $\delta_{\mu w} 2d N(a)^{-1} |d_L|^{-1/2}$ in $s = d$. Moreover, it satisfies the functional equation
\[
\frac{\Gamma(s) K^\mu(H, z, w, s, \Lambda_a)}{\pi^s} = \frac{2^d h_L e^{2\pi i (z, w)} \Gamma(d + |\mu| - s) K^\mu(H, w, z, d + |\mu| - s, \Lambda_a^*)}{\pi^d + |\mu| - s}.
\]

Proof. The proof follows in a standard way from the functional equation of the theta function, see for example [Sie80] Theorem 3]. \hfill \Box

3.2. The completed Poincaré bundle. Let us write $\hat{\mathcal{P}}$ and $\hat{\mathcal{P}}^i$ for the sheaves of $\mathcal{O}_{A(C)}$-modules on the complex manifold $A(C)$ induced by the sheaves $\mathcal{P}$ and $\mathcal{P}^i$ introduced in section 2.1. The connection $\nabla$ on $\hat{\mathcal{P}}$ induces a holomorphic connection $\nabla$ on $\hat{\mathcal{P}}^i$. We define a smooth connection $\nabla_{C_{\infty}}$ on $\hat{\mathcal{P}}_{C_{\infty}} := \hat{\mathcal{P}}^i \otimes_{\mathcal{O}_{A(C)}} \mathcal{C}_{A(C)}^\infty$ as follows. We have $\hat{\mathcal{P}}_{C_{\infty}} = (\hat{\mathcal{P}}^i)^{\nabla} \otimes_{C^\infty} \mathcal{C}_{A(C)}^\infty$, where $(\hat{\mathcal{P}}^i)^{\nabla}$ is the local system of $C^\infty$ vector spaces given by horizontal sections of $\hat{\mathcal{P}}^i$. Write $\mathcal{E}_{A(C)}$ for the differential graded algebra of sheaves of smooth differential forms. Then $\nabla_{C_{\infty}}$ is defined as:

\[
\nabla_{C_{\infty}} := \text{id} \otimes d: (\hat{\mathcal{P}}_{C_{\infty}})^{\nabla} \otimes_{C^\infty} \mathcal{E}_{A(C)} \to (\hat{\mathcal{P}}_{C_{\infty}})^{\nabla} \otimes_{C^\infty} \mathcal{E}_{A(C)}.
\]

The connection $\nabla_{C_{\infty}}$ decomposes into a holomorphic and an anti-holomorphic connection $\nabla_{C_{\infty}} = \nabla' + \nabla''$ where

\[
\nabla': \hat{\mathcal{P}}_{C_{\infty}} \to \hat{\mathcal{P}}_{C_{\infty}} \otimes \mathcal{E}_{A(C)}^{1,0}, \quad \nabla'': \hat{\mathcal{P}}_{C_{\infty}} \to \hat{\mathcal{P}}_{C_{\infty}} \otimes \mathcal{E}_{A(C)}^{0,1}.
\]

Let us observe, that the connection $\nabla''$ induces a Dolbeault resolution of the holomorphic sheaf $\hat{\mathcal{P}}^i$:

\[
(\hat{\mathcal{P}}^i)[0] \xrightarrow{\sim} \left( \mathcal{E}^{0,*} (\hat{\mathcal{P}}_{C_{\infty}}), \nabla'' \right).
\]

Our next aim is to describe $\hat{\mathcal{P}}_{C_{\infty}}$ explicitly. The splitting

\[
\mathcal{H}_C \cong \mathcal{H}_C(\Sigma) \oplus \mathcal{H}_C(\Sigma)
\]

is compatible with the dual of the splitting of the Hodge filtration

\[
\mathcal{H}_C \cong \mathcal{H}_C(\Sigma) \oplus \mathcal{H}_C(\Sigma) \cong \mathcal{E}_{A(C)}^{1,0} \oplus \mathcal{E}_{A(C)}^{0,1}.
\]

Definition 3.7. Let us define $\nu = \nu^{1,0} + \nu^{0,1} \in \mathcal{H}_C(\Sigma) \otimes \mathcal{C} (\omega_{A(C)} \oplus \omega_{A(C)})$ with

\[
\nu^{1,0} \in \mathcal{H}_C(\Sigma) \otimes \mathcal{C} \omega_{A(C)}, \quad \nu^{0,1} \in \mathcal{H}_C(\Sigma) \otimes \mathcal{C} \omega_{A(C)}
\]

respectively

\[
\mathcal{H}_C(\Sigma) \otimes \mathcal{C} \omega_{A(C)} \cong \omega_{A(C)}^{1,0} \otimes \mathcal{C} \mathcal{H}_C(\Sigma) = \text{Hom}_C(\omega_{A(C)}, \omega_{A(C)})
\]

\[
\mathcal{H}_C(\Sigma) \otimes \mathcal{C} \omega_{A(C)} \cong \omega_{A(C)}^{0,1} \otimes \mathcal{C} \mathcal{H}_C(\Sigma) = \text{Hom}_C(\omega_{A(C)}, \omega_{A(C)}).
\]
More explicitly, let us write $(\overline{u}_1, \ldots, \overline{u}_d, u_1, \ldots, u_d)$ for the basis of $\mathcal{H}_C$ corresponding to the basis
\[
\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_d}.\]
In this basis, we may write
\[
\nu^{0,1} = \sum_{i=1}^{d} \overline{u}_i dz_i, \quad \nu^{1,0} = \sum_{i=1}^{d} u_i dz_i.
\]

By abuse of notation, we will also write $\nu$, $\nu^{1,0}$ and $\nu^{0,1}$ for the corresponding sections $\nu \in \Gamma(A(\mathbb{C}), \mathcal{H}_C \otimes \mathcal{E}_A^1(\mathbb{C}))$, $\nu^{1,0} \in \Gamma(A(\mathbb{C}), \mathcal{H}_C \otimes \mathcal{E}_A^{1,0}(\mathbb{C}))$ and $\nu^{0,1} \in \Gamma(A(\mathbb{C}), \mathcal{H}_C \otimes \mathcal{E}_A^{0,1}(\mathbb{C}))$. With this notation, we have the following result:

**Theorem 3.8.** There is a horizontal isomorphism
\[
(\widehat{P}_C, \nabla_C) \cong \left( \prod_{b=0}^{\infty} \text{TSym}^b(\mathcal{H}_C \otimes \mathcal{C}_A^\infty), d + \nu \right).
\]
fitting into a commutative diagram
\[
\begin{array}{ccc}
\prod_{b=0}^{\infty} \text{TSym}^b(\mathcal{H}_C \otimes \mathcal{C}_A^\infty) & \cong & \prod_{b=0}^{\infty} \text{TSym}^b(\mathcal{H}_C \otimes \mathcal{C}_A^\infty) \\
\uparrow & & \uparrow \\
\widehat{P}_C & \cong & \text{TSym}^b(\mathcal{H}_C \otimes \mathcal{C}_A^\infty)
\end{array}
\]
where the vertical maps are the canonical inclusions. The pullback of the above isomorphism along $e$ is furthermore compatible with the moment maps on the formal group $\mathcal{A}$:
\[
\text{mom}_{\mathcal{A}}: e^* \widehat{P}_C \cong \mathcal{O}_A \cong \prod_{b=0}^{\infty} \text{TSym}^b \mathcal{H}_C.
\]

**Proof.** Let us first show that it is enough to prove that there is a horizontal isomorphism
\[
(\widehat{P}_C, \nabla_C) \cong \left( (\mathbb{C} \oplus \mathcal{H}_C) \otimes \mathcal{C}_A^\infty, d + \nu \right)
\]
such that the pullback of the above isomorphism along $e$ is the identity:
\[
e^* \widehat{P}_C = \mathbb{C} \oplus \mathcal{H}_C \cong \mathbb{C} \oplus \mathcal{H}_C.
\]
Indeed, the co-multiplication maps
\[
\mathcal{P}^{(n)}_C \to \text{TSym}^n \mathcal{P}^{(1)}_C
\]
are isomorphisms and we can define
\[
\mathcal{P}^{(1)}_C \cong (\mathbb{C} \oplus \mathcal{H}_C(\Sigma)) \otimes \mathcal{C}_A^\infty,
\]
as the restriction of (3.2.1) to $\mathcal{P}^{(1)}_C \subseteq \mathcal{P}^{(1)}_C$. The desired horizontal morphism
\[
(\widehat{P}_C, \nabla_C) \cong \left( \prod_{b=0}^{\infty} \text{TSym}^b(\mathcal{H}_C \otimes \mathcal{H}_C^\infty), d + \nu \right)
\]
comes now by taking the limit over $n$ of:
\[
\mathcal{P}^{(n)}_C \cong \text{TSym}^n \mathcal{P}^{(1)}_C \cong \text{TSym}^n(\mathbb{C} \oplus \mathcal{H}_C) \otimes \mathcal{C}_A^\infty \cong \prod_{b=0}^{n} \text{TSym}(\mathcal{H}_C \otimes \mathcal{H}_C^\infty).
\]
So it remains to construct the horizontal isomorphism \( \mathcal{G}_X(1) \). According to Scheider’s theorem there is a unique horizontal isomorphism
\[
\mathcal{G}_X(1) \sim \mathcal{L} \log(1)
\]
which is compatible with the trivialization
\[
\mathbb{C} \oplus \mathcal{H}_E \cong e^*\mathcal{P}_B(1) \sim e^*\mathcal{L} \log(1) = \mathbb{C} \oplus \mathcal{H}_C.
\]
It has been shown by Levin in [Lev00, Proposition 2.4.5] that there is a horizontal isomorphism
\[
(\mathcal{L} \log(1), \nabla) \sim ((\mathbb{C} \oplus \mathcal{H}_E) \otimes C^*_A, d + \nu)
\]
compatible with the splitting along \( \mathbb{R}^1 \) and the claim follows.

The anti-holomorphic part \( \nu^{0,1} \) of \( \nu \) is an anti-holomorphic differential form with values in \( \mathcal{H}_E(\Sigma) \subseteq \mathcal{H}_C(\Sigma) \):
\[
\nu^{0,1} \in \Gamma(A(\mathbb{C}), \mathcal{H}_E(\Sigma) \otimes \mathcal{E}^{0,1}_A) \subseteq \Gamma(A(\mathbb{C}), \mathcal{H}_E \otimes \mathcal{E}^{0,1}_A).
\]
Thus, it follows from the above theorem that the anti-holomorphic part \( \nabla'' \) of \( \nabla_E \) restricts to a connection on \( \mathcal{P}_E \) and we obtain a Dolbeault resolution for \( \mathcal{P} \):
\[
(\mathcal{P})[0] \sim (\mathcal{E}^{0,\bullet}(\mathcal{P}_E), \nabla'').
\]

Tensoring the left hand side with the sheaf of holomorphic \( p \)-forms gives the quasi-isomorphism
\[
(\mathcal{P} \otimes \Omega^p_A)[0] \sim (\mathcal{E}^{0,\bullet}(\mathcal{P}_E), \nabla'').
\]

Recall that the ultimate goal of this section is to compute the equivariant cohomology class and the associated Eisenstein classes explicitly. In order to compute the equivariant cohomology explicitly, we use the well-known fact that the equivariant sheaf cohomology
\[
H^i(X, \Gamma, \mathcal{F})
\]
of an equivariant sheaf \( \mathcal{F} \) on a \( \Gamma \)-manifold \( X \) can be computed by the sheaf cohomology \( H^i(\mathcal{E} \times \Gamma, X, \mathcal{F}) \). Here \( \mathcal{F} \) is the sheaf on \( \mathcal{E} \times \Gamma \) induced by the \( \Gamma \)-equivariant sheaf \( \mathcal{F} \) on \( \mathcal{E} \times X \) via pullback along \( \text{pr} \) \( \mathcal{E} \times X \rightarrow X \), see appendix A.

In order to make \( B\Gamma \) more explicit let us assume that \( \Gamma \) is torsion free. Then, we have the following explicit model for the classifying space \( B\Gamma \). Let us define
\[
L^1_{\mathbb{R}} := \left\{ (r_1, \ldots, r_d) \in \mathbb{R}^d_{>0} \mid \prod_{i=1}^d r_i = 1 \right\}.
\]
with an action of \( \Gamma \) given by
\[
\Gamma \times L^1_{\mathbb{R}} \rightarrow L^1_{\mathbb{R}}, \quad (\gamma, r) \mapsto (|\sigma_1(\gamma)|^2 r_1, \ldots, |\sigma_d(\gamma)|^2 r_d).
\]
This serves as an explicit model for the universal bundle \( \mathcal{E} \times \Gamma \) over the classifying space \( B\Gamma \):
\[
B\Gamma := L^1_{\mathbb{R}} \rightarrow B\Gamma := \Gamma \setminus \mathcal{E} \times \Gamma.
\]
The inclusion of \( L^1_{\mathbb{R}} \) into \( \mathbb{R}^d \) gives a function
\[
\gamma = (r_1, \ldots, r_d) : B\Gamma = L^1_{\mathbb{R}} \subseteq \mathbb{R}^d
\]
and we will consider \( (r_1, \ldots, r_{d-1}) \) as coordinates on \( B\Gamma \). In particular, we get the following explicit model for \( A(\mathbb{C}) \times \Gamma \times \mathcal{E} \):
\[
A(\mathbb{C}) \times \Gamma \times \mathcal{E} = \Gamma \setminus (A(\mathbb{C}) \times L^1_{\mathbb{R}})
\]
where $\gamma \in \Gamma$ acts on $(z, r) \in A(\mathbb{C}) \times L^1_\mathbb{R}$ by

$$\gamma.(z, r) = (\Phi_L(\gamma^{-1})z, (|\sigma_1(\gamma)|^2r_1, \ldots, |\sigma_d(\gamma)|^2r_d)).$$

Here, the topological space $ET$ carries the structure of a real manifold and $pr^{-1}_{A(\mathbb{C})} C^\infty_{A(\mathbb{C})}$ can be resolved by the complex of smooth relative differentials:

$$(3.2.4) \quad (pr^{-1}_{A(\mathbb{C})} C^\infty_{A(\mathbb{C})})[0] \sim \mathcal{E}^*_\Gamma \times_{A(\mathbb{C})/A(\mathbb{C})} C^\infty_{A(\mathbb{C})}.$$

It will be convenient to define

$$\mathcal{E}^{1,0}_{A(\mathbb{C}) \times ET} := pr^*_A(C^\infty_{A(\mathbb{C})}), \quad \mathcal{E}^{0,1}_{A(\mathbb{C}) \times ET} := pr^*_A(C^\infty_{A(\mathbb{C})}) \oplus pr^*_\Gamma \mathcal{E}^1_{ET}.$$

By summarizing the above discussion, we obtain:

**Lemma 3.9.** We have a quasi-isomorphism

$$(pr^{-1}_{A(\mathbb{C})}(\widehat{P} \otimes \Omega^d_{A(\mathbb{C})}) \sim (\mathcal{E}^{d,\bullet}_{A(\mathbb{C}) \times ET} \left( pr^*_A(\mathcal{P}_C^\infty), \nabla'' \right)).$$

In particular, the cohomology

$$H^i(U_D(\mathbb{C}), \Gamma, \widehat{P} \otimes \Omega^d_{A(\mathbb{C})})$$

can be described in terms of smooth $\Gamma$-invariant $\nabla''$-closed $(d, d-1)$-forms with values in $pr^*_A(\mathcal{P}_C^\infty)$.

**Proof.** In (3.2.3) we have shown that there is the resolution

$$(\widehat{P} \otimes \Omega^p_{A(\mathbb{C})})[0] \sim (\mathcal{E}^{p,\bullet}(\mathcal{P}_C^\infty), \nabla'').$$

The pullback of this along $pr^{-1}_{A(\mathbb{C})}: A(\mathbb{C}) \times ET \to A(\mathbb{C})$ tensored with $(3.2.4)$ over $\mathbb{C}$ gives after passing to the associated double complex the desired resolution:

$$pr^{-1}_{A(\mathbb{C})}(\widehat{P} \otimes \Omega^d_{A(\mathbb{C})}) \sim (\mathcal{E}^{d,\bullet}_{A(\mathbb{C}) \times ET} \left( pr^*_A(\mathcal{P}_C^\infty), \nabla'' \right)).$$

The equivariant cohomology $H^i(U_D(\mathbb{C}), \Gamma, \widehat{P} \otimes \Omega^d_{A(\mathbb{C})})$ coincides with

$$H^i(ET \times_{\Gamma} U_D(\mathbb{C}), (\widehat{P} \otimes \Omega^d_{A(\mathbb{C})})^\sim).$$

Since $\widehat{P} \otimes \Omega^d_{A(\mathbb{C})}$ is the sheaf on $ET \times_{\Gamma} U_D(\mathbb{C}) = (ET \times U_D(\mathbb{C}))/\Gamma$ induced by the $\Gamma$-equivariant sheaf $pr^{-1}_{A(\mathbb{C})}(\widehat{P} \otimes \Omega^d_{A(\mathbb{C})})$ this cohomology group coincides with the $i$-th cohomology of the complex

$$\left( \Gamma \left( A(\mathbb{C}) \times ET, \mathcal{E}^{d,\bullet}_{A(\mathbb{C}) \times ET} \left( pr^*_A(\mathcal{P}_C^\infty) \right) \right)^\Gamma, \nabla'' \right).$$

We define the sheaf of $(p, q)$-currents $\mathcal{D}^{pq}_{A(\mathbb{C}) \times ET} := \left( \mathcal{E}^{d-p,2d-1-q}_{A(\mathbb{C}) \times ET} \right)^*$ as the dual of smooth $(d-p,2d-1-q)$-forms with compact support. The map $\omega \mapsto (\eta \mapsto \int \eta \wedge \omega)$ gives a quasi-isomorphism of complexes

$$\mathcal{E}^{d,\bullet}_{A(\mathbb{C}) \times ET} \left( pr^*_A(\mathcal{P}_C^\infty) \right) \to \mathcal{D}^{d,\bullet}_{A(\mathbb{C}) \times ET} \left( pr^*_A(\mathcal{P}_C^\infty) \right).$$

In particular, we get:
Corollary 3.10. We have a quasi-isomorphism
\[ \text{pr}_{A(C)}^{-1}(\check{\mathcal{P}} \otimes \Omega^d_{A(C)}) \simeq \left( D^d_{A(C) \times ET}(\text{pr}^*_{A(C)} \check{\mathcal{P}}_{C^\infty}), \nabla'' \right) . \]
In particular, the cohomology
\[ H^i(U_D(C), \Gamma, \check{\mathcal{P}} \otimes \Omega^d_{A(C)}) \]
can be described in terms of smooth \( \Gamma \)-invariant \( \nabla'' \)-closed \( (d, d-1) \)-currents with values in \( \text{pr}^*_{A(C)} \check{\mathcal{P}}_{C^\infty} \).

3.3. The Eisenstein–Kronecker current. For the rest of the chapter we will fix the standard Hermitian form \( H = \text{diag}(1) \) on \( C^\Sigma \), whose imaginary part induces the alternating form
\[ \langle \cdot, \cdot \rangle : C^\Sigma \times C^\Sigma \to \mathbb{R} . \]

The dual lattice \( \Lambda^*_a \) is defined as follows
\[ \Lambda^*_a := \{ l \in C^\Sigma \mid \langle l, \Lambda_a \rangle \leq \mathbb{Z} \} . \]
Let us furthermore assume in this subsection that \( \Gamma \) is torsion-free. Let us define the volume form \( \text{vol} := \frac{(2\pi i)^d}{\text{vol}(A)} \wedge_{i=1}^d dz_i \wedge d\bar{z}_i \) with \( \text{vol}(A) = \int_A \wedge_{i=1}^d dz_i \wedge d\bar{z}_i \). Every element \( l \in \Lambda^*_a \) of the dual lattice gives us a character
\[ \chi_l : C^\Sigma / \Lambda_a \to \mathbb{C}^\times, \ w \mapsto \exp(2\pi i \langle l, w \rangle) . \]

Recall, that we write \( r : L_0 \to \mathbb{R}^d \) for the inclusion and \( \overline{\pi} = (\overline{\pi}_1, \ldots, \overline{\pi}_d) \) for the basis of \( \mathcal{H}_C(\Sigma) \). For \( l \in \Lambda^*_a \) let us define
\[ \tilde{l} := \frac{\overline{\pi}}{r} = \sum_{i=1}^d \overline{l}_i \frac{r_i}{r} \in \Gamma(A(C) \times ET, \text{pr}^*_{A(C)} \times ET(\mathcal{H}_C(\Sigma) \otimes C^\infty_{A(C)})) . \]

By identifying \( \omega_{A^\vee} \) with \( \overline{\mathcal{W}}_{A^\vee}(C) \) we may view \( \tilde{l} \) as an anti-holomorphic vector field
\[ \tilde{l} = \sum_{i=1}^d \frac{\overline{l}_i}{r_i} \frac{\partial}{\partial \overline{z}_i} \]
on \( A(C) \times ET \). We will write \( \iota_l \) for the contraction along \( \tilde{l} \). For \( t \in A(C) = C^\Sigma / \Lambda_a \) let us write \( \delta_t \) for the \( \delta \)-distribution concentrated in \( t \) and \( \delta_f := \sum_{t \in \Lambda_a / \Lambda_a} f(t) \delta_t \). Using the explicit description of \( \check{\mathcal{P}}_{C^\infty} \) from Theorem 3.8 we get:

Theorem 3.11. For non-negative integers \( b, j \) and \( s \in C \) with Re(\( s \)) > 0 define the following \( (d, d-1) \)-current with values in \( \text{TSym}^b \text{pr}^*_{A(C)}(\mathcal{H}_C(\Sigma) \otimes C^\infty_{A(C)}) \) on \( A(C) \times ET \):

\[ \phi^{(b,j)}_s := \frac{(-1)^j}{j!} \sum_{t \in \Lambda_a / \Lambda_a} f(t) \sum_{0 \neq \delta \in \Lambda_a} \int_{u \in \mathbb{R}^{>0}} \tilde{l}^{b|} \chi_{\delta} \left( u \left\| \frac{\tilde{l}}{\sqrt{r}} \right\|^2 u \right) u^{d-1} \frac{du}{\sqrt{r}} \chi_{\delta}(z - t) \iota_l(d\tilde{l})^j \text{vol} . \]

(1) The \( (d, d-1) \)-current
\[ \phi := \sum_{b \geq 0} \sum_{j=0}^{2d} \phi^{(b,j)}_s \in \Gamma \left( A(C) \times ET, D^{d,d-1}_{A(C) \times ET}(\text{pr}^*_{A(C)} \check{\mathcal{P}}_{C^\infty}) \right) \]
is \( \Gamma \)-equivariant.

(2) \( \phi \) solves the differential equations

\[ \nabla''(\phi) = \delta_f \text{vol} . \]
Proof. It is straightforward to check that \(3.3.1\) converges as a current.

(1): The \(\Gamma\)-invariance of \(\phi\) will follow from the more general formula \(\gamma^*\phi_{s}^{(b,j)} = \phi_{s}^{(b,j)}\). For \(\gamma \in \Gamma\) the function \(\gamma^* \langle l, \cdot \rangle\) is \(\mathbb{Z}\)-valued on \(\Lambda_s \subseteq \mathbb{C}\Sigma\), we deduce that \(\Phi_L(\gamma^{-1})l \in \Lambda^*_s \subseteq \mathbb{C}\Sigma\). Thus the formula

\[
\Gamma \times \Lambda^*_s \to \Lambda^*_s, \quad (\gamma, l) \mapsto \gamma.l := \Phi_L(\gamma^{-1})l
\]

is a well-defined action on \(\Lambda^*_s\). We have

\[
\gamma^* \left\| \frac{\gamma^*}{\sqrt{r}} \right\|^2_H = \gamma^* \left( \sum_{i=1}^{d} h_i \frac{|l_i|^2}{r_i} \right) = \sum_{i=1}^{d} h_i \frac{|l_i|^2}{\sigma_i(\gamma)\sigma_i(\gamma)r_i} = \left\| \frac{\gamma.l}{\sqrt{r}} \right\|^2_H
\]

and

\[
\gamma^* \circ t_l = t_{-\gamma^{-1}} \circ \gamma^*.
\]

Since \(f\) is \(\Gamma\)-invariant, we deduce

\[
\gamma^* \phi_{s}^{(b,j)} = \phi_{s}^{(b,j)}.
\]

(2): Our aim is to show that

\[
\phi := \sum_{b \geq 0} \sum_{j=0}^{2d} \phi_{s}^{(b,j)}
\]

satisfies the differential equation

\[
\nabla''(\phi) = \delta_f \text{vol}.
\]

Let us assume that there is a \((2d-1)\)-current \(\phi\) solving the above differential equation and consider its Fourier expansion

\[
\phi = \sum_{l \in \Lambda^*_s} \phi_l \chi_l.
\]

The differential equation

\[
\nabla''(\phi) = \delta_f \text{vol}
\]

can be restated as the following differential equation for the Fourier coefficients

\[
d\phi_l + (2\pi i d(l, \cdot) + \nu) \wedge \phi_l = \sum_{t \in \Lambda/\Lambda_s} f(t) \chi_l(-t).
\]

Our aim is to solve this equation coefficient-wise. For \(l = 0\) it is solved by \(\phi_0 = 0\). For \(l \in \Lambda^*_s\) define \(A_l := 2\pi i d(l, \cdot) + \nu \in \mathcal{E}_s^{(l)} \otimes \text{TSym} \mathcal{H}\). We get an operator

\[
C_l := d + A_l : \mathcal{E}_s^{(l)} \otimes \text{TSym} \mathcal{H} \to \mathcal{E}_s^{(l+1)} \otimes \text{TSym} \mathcal{H}.
\]

Claim: For \(l \neq 0\) the operator \(C_l \circ t_l + t_l \circ C_l\) is invertible and

\[
\phi_{l,t} := t_l \circ (C_l \circ t_l + t_l \circ C_l)(\exp(2\pi i (l, -t)) \text{vol})
\]

solves the equation

\[
d\phi_{l,t} + A_l \wedge \phi_{l,t} = \chi_l(-t) \text{vol}.
\]

Proof of the Claim: Let us first observe

\[
C_l \circ C_l = 0,
\]

\[
t_l \circ t_l = 0.
\]

Our next aim is to prove that the operator

\[
C_l \circ t_l + t_l \circ C_l = dt_l + A_l t_l + t_l A_l
\]
is invertible. The first term $\mathcal{L}_i := dt_i + \iota_i d$ is the Lie-derivative along the vector field $\tilde{t}$. It is nilpotent, more precisely $\mathcal{L}_i^{2d+1} = 0$. Let us compute the second term evaluated on a form $\omega$:

$$(\iota_i + \iota_i A_i)(\omega) = A_i \wedge \iota_i(\omega) + \iota_i (A_i \wedge \omega) = \iota_i (A_i) \wedge \omega.$$  

Thus the operator $A_i \iota_i + \iota_i A_i$ is just multiplication by

$$\iota_i (A_i) = 2\pi \iota_i d(l, \cdot) + \tilde{l} = \pi \left\| \frac{I}{\sqrt{r}} \right\|_H^2 + \tilde{l}.$$  

Since $\pi \left\| \frac{1}{\sqrt{r}} \right\|_H^2 \in \mathbb{C}^*$ and $\tilde{l} \in \text{TSym} \pi^* \mathcal{H}_C$, we conclude that $A_i \iota_i + \iota_i A_i$ is invertible in $\text{TSym} \pi^* \mathcal{H}_C$. We have already seen that $\mathcal{L}_i$ is nilpotent and deduce that $C_i \circ \iota_i + \iota_i \circ C_i$ is invertible.

Next, let us prove that $\phi_{l,t}$ solves the equation

$$d\phi_{l,t} + A_i \wedge \phi_{l,t} = \exp(2\pi i (l, -t)) \cdot \text{vol}.$$  

Let us define $\omega_{l,t} := \exp(2\pi i (l, -t)) \cdot \text{vol}$. We have

$$C_i (\omega_{l,t}) = d\omega_{l,t} + A_i \wedge \omega_{l,t} = 0$$  

and since $C_i$ commutes with $(C_i \iota_i + \iota_i C_i)^{-1}$ we deduce

$$\omega_{l,t} = (C_i \iota_i + \iota_i C_i)(C_i \iota_i + \iota_i C_i)^{-1} (\omega_{l,t}) = \iota_i (C_i \iota_i + \iota_i C_i)^{-1} C_i (\omega_{l,t}) + C_i \iota_i (C_i \iota_i + \iota_i C_i)^{-1} \omega_{l,t} = C_i \iota_i (C_i \iota_i + \iota_i C_i)^{-1} \omega_{l,t}.$$  

Let us recall the definitions $C_i := d + A_i \wedge l$ and $\omega_{l,t} := \exp(2\pi i (l, -t)) \cdot \text{vol}$. Thus $\phi_{l,t} = \iota_i (C_i \iota_i + \iota_i C_i)^{-1} \omega_{l,t}$ solves the equation

$$d\phi_{l,t} + A_i \wedge \phi_{l,t} = \chi_l (-t) \cdot \text{vol}$$  

as desired. This finishes the proof of the Claim.

In a next step, we compute $\phi_{l,t}$ explicitly.

$$\phi_{l,t} = \iota_i (C_i \iota_i + \iota_i C_i)^{-1} \omega_{l,t} = \iota_i (\mathcal{L}_i + \iota_i (A_i))^{-1} \omega_{l,t}$$  

$$= \sum_{j=0}^{2d} (-1)^j \iota_i (A_i)^{-(j+1)} \iota_i (\mathcal{L}_i)^j \omega_{l,t}$$  

$$= \sum_{j=0}^{2d} (-1)^j \iota_i (A_i)^{-(j+1)} \iota_i (dt_i)^j \omega_{l,t}$$  

$$= \sum_{j=0}^{2d} (-1)^j \frac{1}{\left( \pi \left\| \frac{1}{\sqrt{r}} \right\|_H^2 + \tilde{l} \right)^{j+1}} \iota_i (dt_i)^j \omega_{l,t}$$  

By the binomial series we have

$$\frac{1}{\left( \pi \left\| \frac{1}{\sqrt{r}} \right\|_H^2 + \tilde{l} \right)^{j+1}} = \sum_{b \geq 0} \frac{(j+b)!}{j!} \left( \pi \left\| \frac{1}{\sqrt{r}} \right\|_H^2 \right)^{-(b+j+1)} \tilde{l}^{|b|}$$  

and get the formula

$$\phi_{l,t} = \sum_{j=0}^{2d} \sum_{b \geq 0} (-1)^j \frac{(j+b)!}{j!} \left( \pi \left\| \frac{1}{\sqrt{r}} \right\|_H^2 \right)^{-(b+j+1)} \tilde{l}^{|b|} \iota_i (dt_i)^j \omega_{l,t}.$$
Combining everything and using
\[
\Gamma(s) \left( \frac{\pi}{\| l \sqrt{r} \|^2 H} \right)^s = \int_{R > 0} \exp \left( -\pi \left\| \frac{l}{\sqrt{r}} \right\|^2 u \right) u^s \frac{du}{u}
\]
gives
\[
\phi = \sum_{b \geq 0} \sum_{j = 0}^{2d} \phi^{(b, j)}_{b + j + 1}
\]
with
\[
\phi^{(b, j)}_{b + j + 1} = \frac{(-1)^j}{j!} \sum_{t \in \Lambda'/\Lambda} f(t) \sum_{0 \neq l \in \Lambda^*} \int_{u \in R > 0} \tilde{l}^{|l|} \exp \left( -\pi \left\| \frac{l}{\sqrt{r}} \right\|^2 u \right) u^s \frac{du}{u} \chi_l(z - t) \iota_l(d \iota_l)^j \text{vol}
\]

**Remark 3.12.** Using the inclusion \( \hat{P}_{C^\infty} \subseteq \hat{P}_{C^\infty} \) we may view \( \phi \) as a \((2d - 1)\) current in \( \hat{P}_{C^\infty} \) satisfying
\[
\nabla''(\phi) = \delta_f \text{vol}.
\]
Furthermore, we have \( \nabla'(\phi) = 0 \) for trivial reasons: \( \phi \) is already of top degree in the holomorphic differential forms. Thus, \( \phi \) is an equivariant current satisfying
\[
\nabla(\phi) = \delta_f \text{vol}.
\]
The sheaf \( \hat{P}^\natural \) is canonically isomorphic to the de Rham logarithm sheaf:
\[
\lim_{n \to 0} L^\log(n) \cong \hat{P}^\natural.
\]
Thus \([\phi]\) is a representative of the equivariant de Rham polylogarithm.

3.4. The equivariant coherent Eisenstein–Kronecker class. Our next aim is to show that \( \phi^{(b, j)} \) is represented by a smooth differential form. In this subsection we keep the assumption that \( \Gamma \) is torsion-free. Let us start with a more explicit expression for \( \iota_l(d \iota_l)^j \text{vol} \):

**Lemma 3.13.** Let \( j \) be a non-negative integer:

1. We have \( \iota_l(d \iota_l)^j \text{vol} = 0 \) for \( j \geq d \).
2. For \( 1 \leq j \leq d - 1 \) we have
\[
\iota_l(d \iota_l)^j \text{vol} = \frac{(-1)^j j!(2\pi i)^d}{\text{vol}(A)} \sum_{\substack{\xi \in \{0, 1\}^d \\ |\xi| = j + 1}} \left( \frac{l}{r} \right) \omega_\xi
\]
where
\[
\omega_\xi = l^2 \sum_{i=1}^d r_i \frac{\omega_i^\natural}{r_i} \left( \bigwedge_{i=1}^d \omega_i^j \right), \quad \omega_i^\natural = \begin{cases} dz_i \wedge \frac{dr_i}{r_i} & \varepsilon_i = 1 \\ dz_i \wedge d\bar{z}_i & \varepsilon_i = 0. \end{cases}
\]
3. For \( j = d - 1 \) we get:
\[
\iota_l(d \iota_l)^{d - 1} \text{vol} = \frac{(-1)^{d-1} d!(2\pi i)^d}{\text{vol}(A)} \left( \frac{l}{r} \right)^{d-1} \bigwedge_{i=1}^{d-1} \frac{dr_i}{r_i} \wedge \bigwedge_{i=1}^d dz_i.
\]
Here, we use the notation \( \xi = (1, \ldots, 1) \in \mathbb{N}^d \).
Proof. (1): This is clear since the degree of $d\bar{z}$ is $d$, so contracting more than $d$ times along $\tilde{l}$ gives 0.

(2): Recall that we have

$$\text{vol} = (2\pi i)^d \frac{d\text{vol}(A)}{d!}$$

Now the formula follows from

$$dt_\ell(dz_i \wedge d\bar{z}_i) = dz_i \wedge d\left(\frac{\tilde{l}}{r}_i\right) = -(\tilde{l})_i dz_i \wedge \frac{dr_i}{r_i}$$

and

$$t_\ell(dz_i \wedge d\bar{z}_i) = t^0 \sum_{i=1}^d r_i \frac{dr_i}{r_i} (dt_\ell(dz_i \wedge d\bar{z}_i)).$$

(3): In the case $j = d - 1$ the only $\epsilon$ in the above sum is $\epsilon = 1$. So the formula in (2) reduces to:

$$t_\ell(dt_\ell)^{d-1} \text{vol} = -\frac{(d-1)!}{d!} \frac{d\text{vol}(A)}{d!} \left(\frac{\tilde{l}}{r}\right) \wedge d\bar{z}_i$$

with

$$\left(\frac{dr}{r}\right)_i = \bigwedge_{i=1, i\neq j}^d \frac{dr_i}{r_i}.$$ 

The relation $1 = \prod_{i=1}^d r_i$ gives

$$\frac{dr_d}{r_d} = -\sum_{i=1}^{d-1} \frac{dr_i}{r_i}.$$ 

This shows for $j \neq d$ the equation

$$\left(\frac{dr}{r}\right)_j = \frac{dr_1}{r_1} \wedge \cdots \wedge \frac{dr_j}{r_j} \wedge \cdots \wedge \frac{dr_{d-1}}{r_{d-1}} \wedge \frac{dr_d}{r_d} =$$

$$= \frac{dr_1}{r_1} \wedge \cdots \wedge \frac{dr_j}{r_j} \wedge \cdots \wedge \frac{dr_{d-1}}{r_{d-1}} \wedge \left(\frac{dr_j}{r_j}\right) =$$

$$= (-1)^{d-j} \frac{dr_1}{r_1} \wedge \cdots \wedge \frac{dr_{d-1}}{r_{d-1}} = (-1)^{d-j} \left(\frac{dr}{r}\right)_j.$$ 

Putting this everything together gives

$$t_\ell(dt_\ell)^{d-1} \text{vol} = \frac{(-1)^{d-1}d!(2\pi i)^d}{d!} \frac{d\text{vol}(A)}{d!} \left(\frac{\tilde{l}}{r}\right) \bigwedge_{i=1}^{d-1} \frac{dr_i}{r_i} \wedge d\bar{z}_i .$$

□

For $r \in \mathbb{R}_{>0}$ we will use the notation $Hr$ for the Hermitian form

$$Hr = r \cdot \text{diag}(1) = \text{diag}(r, \ldots, r).$$

We will write $\Lambda_a^{Hr,*}$ for the lattice which is dual with respect to the alternating form

$$\langle \cdot, \cdot \rangle_{Hr} := \text{Im} Hr(\cdot, \cdot),$$

i.e.

$$\Lambda_a^{Hr,*} := \{ l \in \mathbb{C}^\Sigma | \langle l, \Lambda_a \rangle_{Hr} \subseteq \mathbb{Z} \}.$$ 

In a next step, let us prove that $\phi_s^{(k,j)}$ is represented by smooth differential forms:
Proposition 3.14. The current \( \phi^{(b,j)}_s \mid_{U_D(C) \times ET} \) is represented by a smooth \((d, d - 1)\)-form

\[
\phi^{(b,j)}_s \mid_{U_D(C) \times ET} \in \Gamma(U_D \times ET, \mathcal{E}^{d,d-1}_{A(C) \times ET}(pr^*_A \hat{P}_C))
\]

More precisely, we have the following explicit formula in terms of Eisenstein–Kronecker series:

\[
\phi^{(b,j)}_s \mid_{U_D(C) \times ET} = \frac{1}{\text{vol}(A)} \sum_{\beta \in \mathbb{N}^d \atop |\beta| = b} \sum_{t \in \Lambda'/\Lambda} f(t) \sum_{\varepsilon \in \{0,1\}^d \atop |\varepsilon| = j+1} \frac{\Gamma(s)K^{\beta+\varepsilon}(Hr, 0, w - t, s, \Lambda^{Hr,r}_a)}{\pi^s} \omega_{\varepsilon} \otimes \bar{u}^{[\beta]}
\]

(3.4.2)

with

\[
\omega_{\varepsilon} = \sum_{i=1}^d \epsilon_i \left( \frac{d}{\lambda_i} \right)^{\beta_i} \frac{1}{2} \bigg| \frac{d}{\lambda_i} \bigg| u_i \chi_i(w - t) = \Gamma(s)K^{\beta+\varepsilon}(Hr, 0, w - t, s, \Lambda^{Hr,r}_a)
\]

Proof. Both sides of (3.4.2) vary holomorphically in \( s \), so it is enough to prove the formula in the region where the defining series of the Eisenstein–Kronecker series converge, i.e. for \( \text{Re}(s) > 2d + b \). By lemma 9 and the equation

\[
\bar{l}_{[\beta]} = \sum_{\beta \in \mathbb{N}^d \atop |\beta| = b} \left( \frac{l}{r} \right)^{\beta} \frac{1}{2} \bigg| \frac{l}{r} \bigg| u_i \chi_i(w - t) = \Gamma(s)K^{\beta+\varepsilon}(Hr, 0, w - t, s, \Lambda^{Hr,r}_a)
\]

it is enough to show

Using \( \left| \frac{l}{r} \right|_H^2 = |l|_{Hr}^2 r \Lambda^{Hr,r}_a = \Lambda^{H,r}_a \) we get

\[
\sum_{l \in \Lambda^{Hr,r}_a \setminus \{0\}} \left( \frac{l}{r} \right)^{\beta+\varepsilon} \int_{u \in \mathbb{R}_{>0}} \exp \left( -\pi \left| \frac{l}{r} \right|_H^2 u \right) u \frac{du}{u} \chi_i(w - t) = \Gamma(s)K^{\beta+\varepsilon}(Hr, 0, w - t, s, \Lambda^{Hr,r}_a)
\]

In Lemma 33 we have seen that

\[
\text{pr}^{-1}_A(\hat{P}, \bigotimes_{A(C)} \mathbb{C}) \simeq \left( \mathcal{E}^{d,d-1}_{A(C) \times ET} \left( \text{pr}^*_A \hat{P}_C \right), \nabla'' \right)
\]

In particular, the cohomology

\[
H^i(U_D(C), \Gamma, \hat{P} \bigotimes \Omega^d_{A(C)})
\]

can be described in terms of smooth \( \Gamma \)-invariant \( \nabla'' \)-closed \((d, d - 1)\)-forms with values in \( \text{pr}^*_A \hat{P}_C \). As a corollary of the above Proposition and Theorem 3.11 we obtain:
Corollary 3.15. The $(d, d-1)$-form $\phi|_{U_D(C)}$ represents the equivariant coherent Eisenstein–Kronecker class

$$[\phi]_{coh} = EK_\Gamma(f) \in H^{d-1}(U_D(C), \Gamma, \tilde{\mathcal{P}} \otimes \Omega^d_{A(C)}).$$

Proof. The smooth $(d, d-1)$-form $\phi|_{U_D(C)}$ satisfies the differential equation

$$\nabla''(\phi|_{U_D(C)}) = 0,$$

so it is a $\Gamma$-equivariant $\nabla''$-closed $(d, d-1)$-form and gives a cohomology class in

$$H^{d-1}(U_D(C), \Gamma, \tilde{\mathcal{P}} \otimes \Omega^d_{A(C)}).$$

It remains to compute the residue map

$$H^{d-1}(U_D(C), \Gamma, \tilde{\mathcal{P}} \otimes \Omega^d_{A(C)}) \to H^d_{A|D}(A(C), \Gamma, \tilde{\mathcal{P}} \otimes \Omega^d_{A(C)}).$$

The residue map in Dolbeault cohomology is computed by extending $\phi|_{U_D(C)}$ to a current and applying $\partial$. By definition, the current $\phi$ extends the differential form $\phi|_{U_D(C)}$. It follows from the Dolbeault resolution (3.22) that the diagram

$$\begin{array}{ccc}
\tilde{\mathcal{P}} & \xrightarrow{\delta} & \tilde{\mathcal{P}} \otimes \mathcal{E}^{0,1} \\
\downarrow & & \downarrow \\
\tilde{\mathcal{P}}^\infty & \xrightarrow{\nabla''} & \tilde{\mathcal{P}}^\infty \otimes \mathcal{E}^{0,1}
\end{array}$$

commutes. Thus $[\nabla''(\phi)]$ represents the class $\text{res}([\phi|_{U_D(C)}])$. According to Theorem 3.11 we have $\nabla''(\phi) = \delta_f \text{vol}$, so the class $\delta_f \text{vol}$ represents the cohomology class $\sum t \in \Lambda'/\Lambda$ $f(t)|t|$ and the result follows.

Let us write $\omega$, $\mathcal{H}$ and $\mathcal{H}(\Sigma)$ for the local systems of $\mathbb{C}$-vector spaces on $B\Gamma$ associated to the $\mathbb{C}[\Gamma]$-modules $\omega_{A(C)}$, $\mathcal{H}_C$ and $\mathcal{H}_C(\Sigma)$. We have a canonical identification

$$H^{d-1}(\Gamma, \text{TSym}^\alpha \omega_{A(C)} \otimes \text{TSym}^\beta \mathcal{H}_C(\Sigma) \otimes \omega^d_{A(C)}) \cong H^{d-1}(B\Gamma, \text{TSym}^\alpha \omega \otimes \text{TSym}^\beta \mathcal{H}(\Sigma) \otimes \omega^d_{A(C)}).$$

The pullback of $\text{TSym}^\alpha \omega \otimes \text{TSym}^\beta \mathcal{H}(\Sigma) \otimes \omega^d_{A(C)}$ along the universal covering $p : E\Gamma \to B\Gamma$ gives a trivial local system $p^{-1}(\text{TSym}^\alpha \omega \otimes \text{TSym}^\beta \mathcal{H}(\Sigma) \otimes \omega^d_{A(C)})$ which can be resolved using the smooth de Rham complex on $E\Gamma$:

$$p^{-1}(\text{TSym}^\alpha \omega \otimes \text{TSym}^\beta \mathcal{H}(\Sigma) \otimes \omega^d_{A(C)})[0] \cong \mathcal{E}^{*,\Gamma}_{ET}(\text{TSym}^\alpha \omega_{A(C)} \otimes \text{TSym}^\beta \mathcal{H}_C(\Sigma) \otimes \omega^d_{A(C)}).$$

This allows us to describe

$$EK^{\beta,\alpha}_\Gamma(f, x) \in H^{d-1}(\Gamma, \text{TSym}^\alpha \omega_{A} \otimes \text{TSym}^\beta \mathcal{H}_C \otimes \omega^d_{A(C)})$$

in terms of $\Gamma$-invariant smooth $(d-1)$-forms on $E\Gamma$.

Proposition 3.16. Let $(\beta, \alpha) \in \mathbb{N}^d \times \mathbb{N}^d$ such that $(\beta, -\alpha - 1)$ is a critical infinity type and write $a = |\alpha|$ and $b = |\beta|$. The $\Gamma$-invariant differential form

$$\psi^{(\beta, \alpha)}(f, x) \in \Gamma(ET, \mathcal{E}^{d-1}_{ET}(\text{TSym}^\alpha \omega_{A(C)} \otimes \text{TSym}^\beta \mathcal{H}_C(\Sigma) \otimes \omega^d_{A(C)}))$$

with

$$\psi^{(\beta, \alpha)}(f, x) := d \sum_{t \in \Lambda'/\Lambda} f(t) \frac{\Gamma(a + d) K^{\beta + a + 1}(Hr, x - t, 0, a + d, \Lambda_a)}{\pi^{a+d}} \times (\pi r)^{d-1} \left(\bigotimes_{i=1}^d \left[\left(\int_{r_i} r_i dz^{[\alpha+1]} \otimes \bar{u}^{[\beta]} \otimes \bigotimes_{i=1}^d dz_i \right)\right] \right)$$

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represents the cohomology class
\[ \text{EK}_{\Gamma}^{\beta, \alpha}(f, x) = [\psi^{(\beta, \alpha)}(f, x)] \in H^{d-1}(\Gamma, \text{TSym}^a \omega_A \otimes \text{TSym}^b \mathcal{H}_C(\Sigma) \otimes \omega_A^d(\mathbb{C})). \]

**Proof.** Let us use the embedding \( \hat{\mathcal{P}} \subseteq \hat{\mathbb{P}}^2 \) to view \( \phi \) as a smooth \((d, d - 1)\)-form in \( \text{pr}^*_{A(\mathbb{C})} \hat{\mathbb{P}}^{\mathbb{C}}_{\mathbb{C} \infty} \). Then \( \text{EK}_{\Gamma}^{\beta, \alpha} \) is represented by the projection of
\[ \varrho_x(x^*(\nabla')^a \phi) \in \Gamma \left( ET, \mathcal{E}^{d-1}_E(\text{TSym}^a \omega_{A(\mathbb{C})} \otimes \mathcal{C} \prod_{b \geq 0} \text{TSym}^b \mathcal{H}_C(\Sigma) \otimes \omega_A^d(\mathbb{C})) \right) \]
to the direct summand \( \text{TSym}^a \omega_{A(\mathbb{C})} \otimes \text{TSym}^b \mathcal{H}_C(\Sigma) \otimes \omega_A^d(\mathbb{C}) \). The \( \beta \)-component \( \text{TSym}^b \mathcal{H}_C(\Sigma) \) is contained in the subspace of \( \text{TSym}^b \mathcal{H}_C(\Sigma) \) spanned by powers of \( \bar{u}_1, \ldots, \bar{u}_d \). Thus the holomorphic connection
\[ \nabla' = d^{1,0} + \nu^{1,0} = d^{1,0} + \sum_{i=1}^d dz_i \otimes u_i \]
acts just by the holomorphic exterior derivation \( d^{1,0} \) after projection to \( \text{TSym}^b \mathcal{H}_C(\Sigma) \). Furthermore, we have \( e^{*} \omega_{\phi} = 0 \) if \( \epsilon \neq (1, \ldots, 1) \). We deduce from Corollary 3.15 Proposition 3.14 and Lemma 3.13 (3) that the coherent equivariant Eisenstein–Kronecker class on \( B\Gamma \) is represented by the differential form:
\[ \frac{d}{\text{vol}(A)} \sum_{t \in A/\Delta_a} f(t) \frac{\Gamma(b + d)K^{\beta + 1}(H_r, 0, z - t, b + d, \Lambda_a^{|H_r \ast|})}{\pi^{b + d}} \bigg|_{z=x} \]
\[ \times \frac{d}{\text{vol}(A)} \sum_{t \in A/\Delta_a} f(t) \frac{\Gamma(b + d)K^{\beta + 1}(H_r, 0, z - t, b + d, \Lambda_a^{|H_r \ast|})}{\pi^{b + d}} \bigg|_{z=x} \]
\[ \times \frac{(\pi hr)^a}{r} \prod_{i=1}^{d} \frac{dz_i^{[\alpha + \frac{1}{2}]} \otimes \bar{u}_{[\beta]} \otimes \partial_i \otimes \frac{dz_i}{dz_i}}{\prod_{i=1}^{d} \frac{dz_i^{[\alpha + \frac{1}{2}]} \otimes \bar{u}_{[\beta]} \otimes \partial_i \otimes \frac{dz_i}{dz_i}}} \]
Combining this with
\[ \varrho_x^a K^{\beta + 1}(H_r, 0, z - t, b + d, \Lambda_a^{|H_r \ast|}) \bigg|_{z=x} = (\pi r)^a K^{\beta + \alpha + 1}(H_r, 0, z - t, b + d, \Lambda_a^{|H_r \ast|}) \bigg|_{z=x} \]
gives the representative
\[ \frac{d}{\text{vol}(A)} \sum_{t \in A/\Delta_a} f(t) \frac{\Gamma(b + d)K^{\beta + \alpha + 1}(H_r, 0, x - t, b + d, \Lambda_a^{|H_r \ast|})}{\pi^{b + d}} \]
\[ \times (\pi hr)^a \prod_{i=1}^{d} \frac{dz_i^{[\alpha + \frac{1}{2}]} \otimes \bar{u}_{[\beta]} \otimes \partial_i \otimes \frac{dz_i}{dz_i}}{\prod_{i=1}^{d} \frac{dz_i^{[\alpha + \frac{1}{2}]} \otimes \bar{u}_{[\beta]} \otimes \partial_i \otimes \frac{dz_i}{dz_i}}} \]
for \( \text{EK}_{\Gamma}^{\beta, \alpha}(f, x) \). Recall the formula \( \text{vol}(A) = \int_{A(\mathbb{C})} \prod_{i=1}^{d} dz_i \wedge d\bar{z}_i \). The measure \( \prod_{i=1}^{d} dz_i \wedge d\bar{z}_i \) differs from the measure \( \mu_{st} \) induced from the metric given by the standard scalar product by a factor \( (2i)^d \), so
\[ \text{vol}(A) = \frac{(2i)^d N(a)|d_L|^{1/2}}{2^d}. \]
Now, the desired result follows from the functional equation
\[ \frac{\Gamma(b + d)K^{\beta + \alpha + 1}(H_r, 0, x - t, b + d, \Lambda_a^{|H_r \ast|})}{\pi^{b + d} N(a)|d_L|^{1/2}} = \frac{r^d \Gamma(a + d)K^{\beta + \alpha + 1}(H_r, x - t, 0, a + d, \Lambda_a)}{\pi^{a + d}}. \]
\[ \square \]
3.5. **Eisenstein classes and fiber integration.** In the following \( \Gamma \subseteq \mathcal{O}_L \) is an arbitrary subgroup of finite index. The aim of this section is to compute the Eisenstein–Kronecker classes

\[
\text{EK}_{\Gamma}^{\beta,\alpha}(f, x) \in \text{TSym}^{\alpha+1} \omega_A \otimes \text{TSym}^\beta \mathcal{H}_\Sigma(\Sigma)
\]

for a critical infinity type \((\beta, -\alpha - 1)\). Recall that these classes are obtained by capping the classes \(\text{EK}_{\Gamma}^{\beta,\alpha}(f, x)\) with \(\xi \in H_{d-1}(\Gamma, \mathbb{Z})\), cf. Proposition 2.27.

\[H^{d-1}(\Gamma, \text{TSym}^\alpha \omega_A \otimes \text{TSym}^\beta \mathcal{H}_\Sigma(\Sigma) \otimes \omega_A(\Sigma)) \to \text{TSym}^{\alpha+1} \omega_A \otimes \text{TSym}^\beta \mathcal{H}_\Sigma(\Sigma)\].

**Definition 3.17.** Let \((\beta, \alpha) \in \mathbb{N}^d \times \mathbb{N}^d, s \in \mathbb{C}\) with \(\text{Re}(s) > \frac{b-a}{2} + d\) and \(O \subseteq \mathbb{Q} \otimes \Lambda_a\) a union of finitely many \(\Gamma\)-orbits then

\[
E^{\beta,\alpha}(O, s; a, \Gamma) := \sum_{l+t' \in \Gamma \setminus (\Lambda_a + O)} \frac{(l+t')^\beta}{(l+t')^\alpha N(l+t')^s}.
\]

For later reference, let us observe the following:

**Lemma 3.18.** For a subgroup of finite index \( \Gamma' \subset \Gamma \) and \((\beta, \alpha), s \) and \( O \) as above, we have:

\[
E^{\beta,\alpha}(O, s; a, \Gamma') = [\Gamma : \Gamma']E^{\beta,\alpha}(O, s; a, \Gamma).
\]

**Proof.** Since each fiber of the map

\[
\Gamma \setminus (\Lambda_a + O) \to \Gamma \setminus (\Lambda_a + O)
\]

has cardinality \([\Gamma : \Gamma']\), we get

\[
E^{\beta,\alpha}(O, s; a, \Gamma') = \sum_{l+t' \in \Gamma \setminus (\Lambda_a + O)} \frac{(l+t')^\beta}{(l+t')^\alpha N(l+t')^s} = [\Gamma : \Gamma'] \sum_{l+t' \in \Gamma \setminus (\Lambda_a + O)} \frac{(l+t')^\beta}{(l+t')^\alpha N(l+t')^s} = [\Gamma : \Gamma']E^{\beta,\alpha}(O, s; a, \Gamma).
\]

\[\square\]

**Lemma 3.19.** Let \((\beta, \alpha) \in \mathbb{N}^d \times \mathbb{N}^d\) and \(t \in \mathbb{Q} \otimes \Lambda_a\), then \(\Gamma(\alpha + s)E^{\beta,\alpha}(O, s; a, \Gamma)\) admits an analytic continuation to \( \mathbb{C} \). More precisely, we have

\[
\Gamma(\alpha + s)E^{\beta,\alpha}(O, s; a, \Gamma) =
\]

\[=d \sum_{t' \in (O + \Lambda_a) / \Lambda_a} \int_{r \in B_r} \frac{\Gamma(a + ds)K^{\beta+\alpha}(H_r, t', 0, a + ds; \Lambda_a)(\pi r)^{a+ds} \prod_{i=1}^{d-1} dr_i}{r_i}.
\]

Here, we write \(s = (s, \ldots, s)\) and \(\Gamma(\alpha + s) := \prod_{i=1}^{d} \Gamma(a_i + s)\).

**Proof.** The right hand side of the above equation is defined for \(s \in \mathbb{C}\), so it remains to compare it to the left hand side for \(\text{Re}(s) > \frac{b-a}{2} + d\):

\[
d \sum_{t' \in (O + \Lambda_a) / \Lambda_a} \int_{r \in B_r} \frac{\Gamma(a + ds)K^{\beta+\alpha}(H_r, t', 0, a + ds; \Lambda_a)(\pi r)^{a+ds} \prod_{i=1}^{d-1} dr_i}{r_i} =
\]

\[=d \sum_{t' \in (O + \Lambda_a) / \Lambda_a} \int_{r \in B_r} \frac{(l+t')^{\beta+\alpha}}{(l+t')^\alpha} \exp\left(-\pi \|l+t'\|^2\right) u^{a+ds} \prod_{i=1}^{d-1} \frac{dr_i}{r_i} =
\]

\[38\]
Let us now make the substitution $\tilde{r}_i := ur_i$ corresponding to the bijection

$$B\Gamma \times \mathbb{R}_{>0} = (\Gamma \backslash L^1_{\mathbb{R}}) \times \mathbb{R}_{>0} \cong \Gamma \backslash (\mathbb{R}_{>0})^d, \quad ((r_1, \ldots, r_d), u) \mapsto (ur_1, \ldots, ur_d).$$

This gives

$$= \sum_{l+t' \in \Gamma \backslash (\Lambda_a+O)} \frac{\Gamma(\alpha + \frac{1}{2})}{\Gamma(\alpha + \frac{1}{2})} \frac{(l+t')^{\beta+\alpha}}{(l+t)(l+t')} \pi^{\alpha+\lambda} = \Gamma(\alpha + \frac{1}{2}) E^{\beta,\alpha}(\Gamma t, s; a, \Gamma)$$

In the following we will compute $E_{\Gamma}^{\beta,\alpha}(f, x)$ explicitly:

**Theorem 3.20 (Explicit form of the Eisenstein-Kronecker class).** For $(\beta, \alpha) \in \mathbb{N}^d \times \mathbb{N}^d$ such that $(\beta, -\alpha - 1)$ is a critical infinity type, we have the following equality in $\mathrm{TSym}^{\alpha+1} \omega_{A(C)} \otimes \mathrm{TSym}^\beta \mathcal{H}_C(\Sigma)$:

$$E_{\Gamma}^{\beta,\alpha}(f, x) = \alpha! \sum_{\Gamma t \in \Gamma \backslash (\Lambda'/\Lambda_a)} f(-\Gamma t) E^{\beta,\alpha+1}(\Gamma(t + x), 0; a, \Gamma) \cdot dz^{(\alpha+1)} \otimes \bar{u}^{[\beta]}.$$

**Proof.** Let us first reduce the claim to a torsion-free subgroup. Let $\Gamma' \subseteq \Gamma$ be a torsion-free subgroup of finite index. Passing to $\Gamma'$ corresponds to multiplication with $[\Gamma : \Gamma']$ on both sides of the equation in the statement. For the right hand side this is Lemma 3.18 and for the left hand side this has been shown in Corollary 2.30. Thus it suffices to prove the formula for torsion-free groups.

Let us assume that $\Gamma$ is torsion-free. The cap product $H_{d-1}(B\Gamma, \mathbb{Z}) \otimes_\mathbb{Z} H^{d-1}(B\Gamma, \mathrm{TSym}^{\alpha+1} \omega \otimes \mathrm{TSym}^\beta \mathcal{H}_C(\Sigma)) \to H_0(B\Gamma, \mathrm{TSym}^{\alpha+1} \omega \otimes \mathrm{TSym}^\beta \mathcal{H}_C(\Sigma))$ is given by $([Z], [\omega]) \mapsto \int_Z \omega$ if $Z$ is a cycle in singular homology and $\omega$ a $(d-1)$-form representing a cohomology class in degree $d-1$. Using Proposition 3.10 and Lemma 3.19 we see that the image of $E_{\Gamma}^{\beta,\alpha}(f, x)$ under

$$H^{d-1}(B\Gamma, \mathrm{TSym}^\alpha \omega \otimes \mathrm{TSym}^\beta \mathcal{H}_C(\Sigma) \otimes \omega_{A(C)}^{d}) \cong \mathrm{TSym}^{\alpha+1} \omega_{A(C)} \otimes \mathrm{TSym}^\beta \mathcal{H}_C(\Sigma)$$

is given by the following fiber integral:

$$\int_{r \in B\Gamma} \psi^{(\beta,\alpha)}(f, x) = \Gamma(\alpha + 1) \sum_{\Gamma t \in \Gamma \backslash (\Lambda'/\Lambda_a)} f(-\Gamma t) E^{\beta,\alpha+1}(\Gamma(t + x), 0; a, \Gamma) \cdot dz^{(\alpha+1)} \otimes \bar{u}^{[\beta]}.$$

4. Application to special values of L-functions for Hecke characters

In this section we discuss the consequences of the computation 3.20 for the special values of $L$-functions for algebraic Hecke characters. We start with a discussion of the relevant periods.

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4.1. Complex periods. It is a special feature of our construction that canonically the periods of $A$ and of $A^\vee$ occur in the final formula. Using a polarization one can of course identify these periods but only with a loss of sharpness in the final integrality result.

Fix $(A/R, \Sigma, \alpha, \theta, \omega(A), \omega(A^\vee), x)$ as in \[\text{(4.1.1)}\]. Recall that $A = a \otimes_{\mathcal{O}_K} B$, where $B$ has CM by $\mathcal{O}_K$ and a uniformization $\theta_0 : C^\Sigma_1 / \Lambda_{\mathcal{O}_K} B(\Sigma)$. This induces $\theta : X(a) = C^\Sigma / \Lambda_a \cong A(\Sigma)$. Consider the coordinates

$$z(\sigma) : C^\Sigma = \bigoplus_{\sigma \in \Sigma} C \to C.$$ 

In $\omega_{A(\Sigma)} = \bigoplus_{\sigma \in \Sigma} \omega_{A(\Sigma)}(\sigma)$ we choose the $C \otimes \mathcal{O}_L$-basis

$$\omega_{an}(A) := (dz(\sigma))_{\sigma \in \Sigma}.$$ 

Using the transcendental Hodge decomposition

$$H^1_{dR}(A(\Sigma)) \cong \omega_{A(\Sigma)} \oplus \overline{\omega}_{A(\Sigma)}$$

we get a $C \otimes \mathcal{O}_L$-basis

$$(dz(\sigma), d\overline{\sigma}(\overline{\sigma}))_{\sigma \in \Sigma}$$

of $H^1_{dR}(A(\Sigma))$. We denote by $(\overline{u}(\sigma), u(\sigma))_{\sigma \in \Sigma}$ the dual basis in $\mathcal{H} \cong H^1_{dR}(A^\vee(\Sigma))$ and let $\omega_{an}(A^\vee(\Sigma))$ be the $C \otimes \mathcal{O}_L$-basis

$$\omega_{an}(A^\vee(\Sigma)) = (\overline{u}(\sigma))_{\sigma \in \Sigma}$$

of $\omega_{A^\vee(\Sigma)}$.

**Definition 4.1.** Denote by $\omega(A(\Sigma))$ and $\omega(A^\vee(\Sigma))$ the base change of the bases $\omega(A)$ and $\omega(A^\vee)$ to $C$. We define complex periods $\Omega(A) = (\Omega(A)(\sigma))_{\sigma \in \Sigma} \in C^\Sigma$ and $\Omega(A^\vee) = (\Omega(A^\vee)(\overline{\sigma}))_{\overline{\sigma} \in \Sigma} \in C^\Sigma$ by

$$\omega(A(\Sigma))(\sigma) = \Omega(A)(\sigma) dz(\sigma) \quad \text{and} \quad \omega(A^\vee(\Sigma))(\sigma) = (2\pi i)^{-1} \Omega(A^\vee)(\sigma) \overline{u}(\sigma).$$

For $\alpha \in I_{\Sigma}$ we use the notation

$$\Omega(A)^\alpha := \prod_{\sigma \in \Sigma} \Omega(A)(\sigma)^{\alpha(\sigma)}$$

and similarly for $\Omega(A^\vee)$.

The factor of $(2\pi i)^{-1}$ in the definition of $\Omega(A^\vee)$ is chosen, because the dual of the period pairing

$$H_1(A(\Sigma), \mathbb{Z}) \times H^1_{dR}(A(\Sigma)) \to \mathbb{C}$$

is the pairing

$$H_1(A^\vee(\Sigma), (2\pi i)\mathbb{Z}) \times H^1_{dR}(A^\vee(\Sigma)) \to \mathbb{C}$$

so that $\Omega(A)$ and $\Omega(A^\vee)$ are indeed the periods of $A(\Sigma)$ and $A^\vee(\Sigma)$ respectively.

For our applications it is also necessary to have a compatible choice of the bases $\omega(A(\Sigma))$ and $\omega(A^\vee(\Sigma))$ for isogenous abelian varieties arising from the Serre construction. The following strategy for normalization is due to Katz [Kat70].

First recall that for any fractional ideal $\mathfrak{a}$ one has

$$\text{Lie}((a \otimes_{\mathcal{O}_L} A)/R) \cong a \otimes_{\mathcal{O}_K} \text{Lie}(A/R).$$

In particular, if $R$ is flat over $\mathbb{Z}$ (for example contained in $\mathbb{C}$) and $Na$ is invertible in $R$, then one has an isomorphism $\text{Lie}((a \otimes_{\mathcal{O}_L} A)/R) \cong \text{Lie}(A/R)$. The $R \otimes_{\mathbb{Z}} \mathcal{O}_L$-dual of $a \otimes_{\mathcal{O}_L} \text{Lie}(A/R)$ is $a^{-1} \delta^{-1}_L \otimes \omega_{A/R}$, where $\delta^{-1}_L$ is the inverse different. As $d_L$ is also invertible in $R$ one gets an isomorphism

$$\omega(a \otimes_{A}/R) \cong a^{-1} \delta^{-1}_L \otimes \omega_{A/R} \cong \omega_{A/R}.$$
Moreover, we remark that
\[ \mathfrak{a} \otimes X(\mathcal{O}_L) \cong X(\mathfrak{a}). \]

**Definition 4.2.** Let \((\mathcal{A}/\mathcal{R}, \Sigma, \mathcal{O}_L, \theta, \omega(\mathcal{A}), \omega(\mathcal{A}^\vee))\) be as in [1.15]. Choose bases \(\omega(\mathcal{A})\) and \(\omega(\mathcal{A}^\vee)\) and define for a fractional ideal \(\mathfrak{a}\) the abelian scheme \(\mathcal{A}_\mathfrak{a} := \mathfrak{a} \otimes_{\mathcal{O}_L} \mathcal{A}\), so that \(\mathcal{A}_\mathfrak{a}\) is of type \((\mathcal{O}_L, \Sigma, \mathfrak{a})\). Assume that \(\mathcal{N} \mathfrak{a}\) is invertible in \(\mathcal{R}\). Then we define bases
\[ \omega(\mathcal{A}_\mathfrak{a}) = \omega(\mathcal{A}) \quad \text{and} \quad \omega(\mathcal{A}_\mathfrak{a}^\vee) = \omega(\mathcal{A}^\vee) \]
using the isomorphism \((4.1.1)\). Finally, we let
\[ \Omega := \Omega(\mathcal{A}) \quad \text{and} \quad \Omega^\vee := \Omega(\mathcal{A}^\vee). \]

From the above discussion one gets the following result:

**Proposition 4.3.** Let \((\mathcal{A}/\mathcal{R}, \Sigma, \mathcal{O}_L, \omega(\mathcal{A}), \omega(\mathcal{A}^\vee))\) be as in the definition and \(\mathfrak{a}\) a fractional ideal with \(\mathcal{N} \mathfrak{a}\) invertible in \(\mathcal{R}\). Then with the compatible choice of bases above one has
\[ \Omega(\mathcal{A}_\mathfrak{a}) = \Omega \quad \text{and} \quad \Omega((\mathcal{A}_\mathfrak{a})^\vee) = \Omega^\vee. \]

For comparison with the classical results of Damerell, Shimura and Katz, we briefly discuss the relation of the above periods \(\Omega\) and \(\Omega\) with the ones of \(\mathcal{B}\) and \(\mathcal{B}^\vee\):

**Remark 4.4.** Let us first observe, that the datum of algebraic bases \(\omega(\mathcal{A}), \omega(\mathcal{A}^\vee)\) is equivalent to give a generator of the free \(\mathcal{O}_L \otimes \mathcal{R}\)-module \(\mathcal{H}(\mathcal{A}/\mathcal{R}) = H^1_{dR}(\mathcal{A}^\vee/\mathcal{R})\). Let us assume that we have already chosen bases \(\omega(\mathcal{B}), \omega(\mathcal{B}^\vee)\) of algebraic differential forms for \(\mathcal{B}\) and an isomorphism of complex tori \(\theta_0: \mathcal{B}(\mathbb{C}) \cong \mathbb{C}^{\Sigma_\mathcal{K}}/\mathcal{O}_K\). Comparing the transcendental differential induced by \(\theta_0\) to the algebraic differential forms \(\omega(\mathcal{B})\) gives us periods of \(\mathcal{B}\), see above:
\[ \omega(\mathcal{B})(\sigma) = \Omega(\mathcal{B})(\sigma)dz(\sigma), \quad \text{and} \quad \omega(\mathcal{B}^\vee)(\overline{\sigma}) = (2\pi i)^{-1}\Omega(\mathcal{B}^\vee)(\overline{\sigma})\overline{\pi}(\overline{\sigma}), \quad \sigma \in \Sigma_\mathcal{K}, \overline{\sigma} \in \overline{\Sigma}_K. \]

Note, that these periods are indeed periods in the classical sense, since \(1 \in \mathcal{O}_K\) gives us a lattice point in \(\mathbb{C}^{\Sigma_\mathcal{K}}\) and hence an element in the singular homology \(H_1(\mathcal{B}(\mathbb{C}), \mathbb{Z})\). Integrating the algebraic differential form \(\omega(\mathcal{A})(\sigma)\) along this cycle gives \(\Omega(\mathcal{B})(\sigma)\) and similarly for \(\mathcal{B}^\vee\).

By the above remark, the given basis \(\omega(\mathcal{B}), \omega(\mathcal{B}^\vee)\) corresponds to an isomorphism
\[ \mathcal{H}(\mathcal{B}/\mathcal{R}) \cong \mathcal{O}_K \otimes \mathcal{R}. \]

This trivialization of \(\mathcal{H}(\mathcal{B}/\mathcal{R})\) yields a trivialization of \(\mathcal{A} = \mathcal{O}_L \otimes \mathcal{B}\):
\[ \mathcal{H}(\mathcal{A}/\mathcal{R}) = \mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{H}(\mathcal{B}/\mathcal{R}) \cong \mathcal{O}_L \otimes \mathcal{R} \]
and we get induced bases \(\omega(\mathcal{A}), \omega(\mathcal{A}^\vee)\). The complex uniformization \(\theta_0\) induces a uniformization \(\theta: \mathcal{A}(\mathbb{C}) \cong \mathbb{C}^{\Sigma}/\mathcal{O}_L\). By construction, we have
\[ \omega(\mathcal{A})(\sigma) = \Omega(\mathcal{B})(\sigma|_K)dz(\sigma), \quad \text{and} \quad \omega(\mathcal{A}^\vee)(\overline{\sigma}) = (2\pi i)^{-1}\Omega(\mathcal{B}^\vee)(\overline{\sigma|_K})\overline{\pi}(\overline{\sigma}), \quad \sigma \in \Sigma_L, \overline{\sigma} \in \overline{\Sigma}_L. \]

We can summarize the above discussion by saying that a careful choice of the algebraic bases \(\omega(\mathcal{A}), \omega(\mathcal{A}^\vee)\) allows us to identify \(\Omega(\mathcal{A})\) and \(\Omega(\mathcal{A}^\vee)\) with periods (in the classical sense) of the abelian varieties \(\mathcal{B}\) and \(\mathcal{B}^\vee\).
4.2. Special values of Eisenstein series. In this section we explain the consequences of our computation for the values of the Eisenstein series

\[ E^{\beta,\alpha}(O, s; a, \Gamma) := \sum_{t' \in \Gamma \cap (A_a + O)} (l + t')^\beta \frac{(l + t')^\alpha N(l + t')^s}{}, \]

from 3.17 at \( s = 0 \). Recall that for each integral ideal \( \mathfrak{c} \) of \( L \) one has an isogeny

\[ [\mathfrak{c}] : A_a \to A_{a^{-1}}. \]

**Theorem 4.5** (Values of Eisenstein series). Let \((A/\mathcal{R}, \Sigma, a, \omega(A), \omega(A^\vee), x)\) be as in \( I.13 \). Recall that \( R \subset \mathbb{C} \) is a \( \mathcal{O}_L \)-valued \( [1/d_L] \)-algebra. Let \( \Gamma \subset \mathcal{O}_L \) be a subgroup of finite index. Assume that \( x \in \mathcal{A}[\mathfrak{f}] \) is an \( \mathfrak{f} \)-torsion section fixed by \( \Gamma \), where \( \mathfrak{f} \) is an integral ideal in \( \mathcal{O}_L \) coprime to \( a \) and such that \( N \mathfrak{f} \) is invertible in \( R \). Let \( \mathfrak{c} \) be an integral ideal coprime to \( a \mathfrak{f} \) such that the isogeny \([\mathfrak{c}]^\vee \) is étale. (This is for example the case if \( N \mathfrak{c} \) is invertible in \( R \).) Let \( \mathcal{D} := \{x(S)\} \) and \( f \in R[\mathcal{D}]^{\Gamma} \), then

\[ \frac{(\alpha - 1)! (2\pi i)^\beta}{\Omega(A) \omega(A^\vee)^\beta} \sum_{\Gamma \in \Gamma \cap (\mathcal{D}(\mathbb{C}))} f(-\Gamma t) E^{\beta,\alpha}(\Gamma(t + x), 0; a, \Gamma) \in R[\frac{1}{N \mathfrak{f}}]. \]

**Proof.** One has by 2.28

\[ (\mathfrak{c} + \mathfrak{f})^{\alpha - 1}(f, x) \in \text{TSym}^\alpha(\omega_{A/\mathcal{R}}) \otimes \text{TSym}^\beta(\mathcal{H}(\Sigma)). \]

This class was explicitly computed in \( 3.20 \) and is given over the complex numbers by

\[ (\mathfrak{c} + \mathfrak{f})^{\alpha - 1}(f, x) = (\alpha - 1)! \sum_{\Gamma \in \Gamma \cap (\mathcal{D}(\mathbb{C}))} f(-\Gamma t) E^{\beta,\alpha}(t + x, 0; a, \Gamma) \cdot dz^{[\alpha]} \otimes \bar{u}^{[\beta]}, \]

where \( \Gamma t \) denotes the \( \Gamma \) orbit of \( t \). Finally, by definition

\[ \omega(\mathcal{A}(\mathbb{C}))^{[\alpha]} = \Omega(A)^\alpha dz^{[\alpha]} \quad \text{and} \quad \omega(\mathcal{A}^\vee(\mathbb{C}))^{[\beta]} = (2\pi i)^{-[\beta]} \Omega(A^\vee)^\beta \bar{u}^{[\beta]}. \]

Evaluation gives the theorem. \( \square \)

If \( N \mathfrak{a} \) is invertible in \( R \) one can identify the periods as in Proposition \( 4.3 \). This gives:

**Corollary 4.6.** Assume in addition that \( N \mathfrak{a} \) is invertible in \( R \) and that the conditions in the theorem are satisfied. Then

\[ \frac{(\alpha - 1)! (2\pi i)^\beta}{\Omega^\alpha \Omega^\vee,\beta} \sum_{\Gamma \in \Gamma \cap (\mathcal{D}(\mathbb{C}))} f(-\Gamma t) E^{\beta,\alpha}(\Gamma(t + x), 0; a, \Gamma) \in R[\frac{1}{N \mathfrak{f} a}]. \]

**Remark 4.7.** From the above result one can recover the result of Katz \( [\text{Kat78}, \text{Theorem } 3.5.2] \) in the case where \( L = K \). For this one puts \( x = 0 \), so that \( \mathfrak{f} = \mathcal{O}_L \) and uses \([\mathfrak{c}]^\vee \) which corresponds to the isogeny of dividing out the \( \Gamma_0(p^\infty) \)-structure in loc. cit. To recover the exact form of Katz’ result one needs to use the functional equation of the Eisenstein-Kronecker series \( 3.6 \) and to identify the periods of \( \mathcal{A}^\vee \) with respect to a \( \mathfrak{c} \)-polarization as in Katz. We leave it to the reader to verify the details.

4.3. Algebraic Hecke characters. Let \( L \) be a number field and \( \mathfrak{f} \subset \mathcal{O}_L \) an integral ideal, \( \mathcal{I}(\mathfrak{f}) \) the group of fractional ideals prime to \( \mathfrak{f} \). We let \( \mathcal{P}(\mathfrak{f}) \subset \mathcal{I}(\mathfrak{f}) \) be the subgroup of the principal ideals generated by \( \lambda \in L^\times \) with \( \lambda \equiv 1 \mod \mathfrak{f} \) and totally positive at the real places.

As before, let \( \overline{\mathbb{Q}} \subset \mathbb{C} \) be the algebraic closure of \( \mathbb{Q} \) in \( \mathbb{C} \). Recall that we let \( J_L := \text{Hom}_{\mathbb{Q}}(L, \overline{\mathbb{Q}}) \) and \( I_L \) the free abelian group generated by \( J_L \). We fix a number field \( E \). The Galois group \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) acts on \( J_L \times J_E \). We view elements in \( I_L \times J_E \) as integer valued functions on \( J_L \times J_E \).
Definition 4.8. An algebraic Hecke character $\chi$ of conductor dividing $f$ and infinity type $\mu \in I_L \times J_E$ is a homomorphism

$$\chi: I(f) \to E^\times$$

such that for all $\lambda \in P_f$ and $\tau \in J_E$ one has

$$\tau \circ \chi((\lambda)) = \prod_{\sigma \in J_L} \sigma(\lambda)^{\mu(\sigma, \tau)},$$

where $\mu$ has to satisfy $\mu(g\sigma, g\tau) = \mu(\sigma, \tau)$ for all $g \in \text{Gal}(\overline{Q}/Q)$.

If $f | f'$ one has $I(f') \subset I(f)$ and one identifies the Hecke character of conductor dividing $f$ with the ones of conductor dividing $f'$ obtained by restriction. The smallest $f$ where $\chi$ can be defined, is called the conductor of $\chi$.

It is known [Ser68] that not all $\mu \in I_L \times J_E$ can occur as the infinite type of an algebraic Hecke character. These are only the $\mu$ such that for fixed $\tau \in J_E$ the $\mu_\tau$ is of Hecke character type in the sense of Definition 1.3. One obtains from this the following classification:

1. Case: $L$ does not contain a CM field. Then the algebraic Hecke character $\chi$ is of the form

$$\chi = \varrho \cdot N_{L/Q}^w,$$

for some integer $w$ and where $\varrho$ is a character of $I(f)/P_f$ (a generalized Dirichlet character).

2. Case: $L$ does contain a CM field. Then it contains a maximal one, say $K$ and $L$ is necessarily totally complex. An algebraic Hecke character is then of the form

$$\chi = \varrho \cdot (\chi_0 \circ N_{L/K})$$

where $\chi_0$ is an algebraic Hecke character of $K$ and $\varrho$ a generalized Dirichlet character of $L$. Further the infinity type of $\chi_0$ has to be of Hecke character type in the sense of Definition 1.3.

4.4. Special values of $L$-series for algebraic Hecke characters. Each embedding $\tau: E \to \overline{Q} \subset \mathbb{C}$ allows to consider an algebraic Hecke character $\chi$ as a homomorphism $I(f) \to \mathbb{C}^\times$. In this section we fix such an embedding $\tau: E \to \mathbb{C}$ and suppress it from the notation. Thus we consider $\chi$ as a homomorphism

$$\chi: I(f) \to \mathbb{C}^\times$$

with values in $\tau(E) \subset \mathbb{C}$. For $\text{Re } s >> 0$ one defines the $L$-function of $\chi$ by

$$L_f(\chi, s) := \sum_{a \in I(f), a \subset P_L} \frac{\chi(a)}{N a^s},$$

where the sum is extended over all integral ideals $a$ coprime to $f$. The $L$-series has an analytic continuation to $\mathbb{C}$ and satisfies a functional equation. The value at $s = 0$ of $L_f(\chi, s)$ is called critical, if none of the $\Gamma$-factors occurring on each side of the functional equation have a pole at $s = 0$.

It is known that critical values can occur only if either $L$ is totally real or if $L$ contains a CM field (see [Del79]). In the latter case, one has that $s = 0$ is critical for $L_f(\chi, s)$ if and only if the infinity type $\mu$ of $\chi$ is critical in the sense of Definition 1.3. Thus $\chi$ with infinity type $\mu$ is critical at $s = 0$ if there is a CM type $\Sigma \subset J_L$ of $L$ such that

$$\mu = \beta - \alpha.$$
with $\beta \in I_{L}^+$ and $\alpha - 1 \in I_{L}^+$. For an integral ideal $b$ of $O_L$ with ideal class $[b] \in I(f)/P_f$ we introduce the partial $L$-function

$$L_f(\chi, s, [b]) := \sum_{a \in [b]} \frac{\chi(a)}{N a^s}$$

so that

$$L_f(\chi, s) = \sum_{[b] \in I(f)/P_f} L_f(\chi, s, [b]).$$

Let $O^x_L$ be the subgroup of units of $O_L$, which are congruent to 1 mod $f$. The partial $L$-functions can be expressed through the series $E^{\beta, \alpha}(O, s; fb^{-1}, O^x_L)$ defined in 3.17 as follows: There is a bijection

$$O^x_L \setminus (1 + fb^{-1}) \cong \{ a \in [b] \mid a \text{ integral} \} \quad \lambda \mapsto \lambda b$$

if the conductor $f$ is not trivial and for each $\lambda \in 1 + fb^{-1}$ one has $\chi((\lambda)) = \prod_{\sigma \in S} \overline{\chi(\lambda)^\beta}$. For the trivial conductor $f = O_L$ the integral ideals in the class $[b]$ are parametrized by $O^x_L \setminus (b^{-1} \setminus \{0\})$ and then $1 \in b^{-1}$. Thus, one gets for arbitrary conductor $f$

$$L_f(\chi, s, [b]) = \chi(b) N b^{-s} \sum_{\lambda \in O^x_L \setminus (1 + fb^{-1})} \frac{\chi((\lambda))}{N \lambda^s} = \chi(b) N b^{-s} E^{\beta, \alpha}_{O^x_L}(1, s, fb^{-1}).$$

If $\chi$ is critical at 0, i.e. $\beta - \alpha$ is critical in the sense of definition 1.3, we can evaluate both sides at $s = 0$ and get

$$L_f(\chi, 0, [b]) = \chi(b) E^{\beta, \alpha}_{O^x_L}(1, 0, fb^{-1}).$$

Note that $1 + fb^{-1}$ defines an $f$-torsion point in $X(fb^{-1})$, which is fixed by $O^x_L$.

Let $(A/R, \Sigma, O_L, \theta, \omega(A), \omega(A^{\vee}))$ be as in Definition 4.2 and $A_{fb^{-1}} := fb^{-1} \otimes_{O_L} A$. The uniformization $\theta : X(fb^{-1}) \cong A_{fb^{-1}}$ allows to consider 1 as $[f]$-torsion point on $A_{fb^{-1}}$, which is fixed by $O^x_L$.

Choose an auxiliary integral ideal $c \subset O_L$ coprime to $fb$. We identify the $[c]$-torsion points of $(A_{fb^{-1}}(C)$ with $f(bc)^{-1}/fb^{-1}$. Let $f|c| = N c(0) - \sum t \in A|c|(t) \in R[A|c]|^{\text{NP}}$ be the function defined in (2.3.3). Then we get from Corollary 4.6 that

$$\sum_{\sigma^x \in \sigma^x \setminus A(c)(C)} \frac{(\alpha - 1)! (2\pi i)^{\beta}}{\Omega^\alpha \Omega^{x, \beta}} f|c| \left(-\sigma^x t \right) E^{\beta, \alpha}(\sigma^x t + 1, 0; fb^{-1}, \sigma^x) \in R\left[\frac{1}{N(fb|c|)}\right],$$

where $R$ is the ring of definition of $(A/R, \Sigma, O_L, \theta, \omega(A), \omega(A^{\vee}), 1)$. To rewrite this, we use a distribution relation.

**Proposition 4.9.** One has the distribution relation

$$\sum_{\sigma^x \in \sigma^x \setminus A(c)(C)} E^{\beta, \alpha}(\sigma^x t + 1, 0; fb^{-1}, \sigma^x) = E^{\beta, \alpha}(1, 0; fb^{-1} c^{-1}, \sigma^x).$$

In particular,

$$\sum_{\sigma^x \in \sigma^x \setminus A(c)(C)} f|c| \left(-\sigma^x t \right) E^{\beta, \alpha}(\sigma^x t + 1, 0; fb^{-1}, \sigma^x) =$$

$$= N c E^{\beta, \alpha}(1, 0; fb^{-1}, \sigma^x) - E^{\beta, \alpha}(1, 0; fb^{-1} c^{-1}, \sigma^x).$$
Proof. Identify \( \mathcal{A}[\chi] \cong fb^{-1}c^{-1}/fb^{-1} \), then one has a disjoint decomposition

\[
fb^{-1}c^{-1} + 1 = \bigcup_{\beta \in \mathcal{A}[\chi](\mathbb{C})} (fb^{-1} + \mathcal{O}_f^\chi t + 1),
\]

which implies

\[
\mathcal{O}_f^\chi \setminus (fb^{-1}c^{-1} + 1) = \bigcup_{\beta \in \mathcal{A}[\chi](\mathbb{C})} \mathcal{O}_f^\chi \setminus (fb^{-1} + \mathcal{O}_f^\chi t + 1).
\]

Using the definition of \( E^{\beta,\alpha}(\mathcal{O}_f^\chi t + 1, 0; fb^{-1}, \mathcal{O}_f^\chi) \) in (4.17) the result follows. \( \square \)

Let \( b \) be an integral ideal in \( L \) and recall that the Hecke character \( \chi \) has values in the number field \( E \), i.e., \( \chi(b) \in E \). Then some power of \( b \) is a principal ideal say \( b^k = (\lambda) \) with \( \lambda \in \mathcal{O}_L \), so that \( \chi(b^k) = \prod_{\sigma \in \Sigma} \frac{\pi_i(\lambda)^{\beta(\sigma)}}{\pi_i(\lambda)^{\alpha(\sigma)}} \in \tau(\mathcal{O}_E[1/Nb]) \). This implies that \( \chi(b) \in \mathcal{O}_E[1/Nb] \). With this remark we get from the proposition and formulae (4.4.2) and (4.4.3)

\[
(4.4.4) \quad \frac{(\alpha - 1)!(2\pi i)^{\beta}}{\Omega^\alpha \Omega^\beta} \left( \chi(c) NcL_i(\chi, 0, [b]) - L_i(\chi, 0, [bc]) \right) \in \tau(\mathcal{O}_E)R_{\mathcal{C}t}\left[ \frac{1}{N(fb^\chi)} \right].
\]

Here \( \tau(\mathcal{O}_E)R \) denotes the subring of \( \mathbb{C} \) generated by \( \tau(\mathcal{O}_E) \) and \( R \). Recall that we can assume that \( R \subset \mathcal{O}_E \mathbb{C} \). The next theorem is one of the main results in this paper and generalizes the work of Katz [Katz83] in the case of CM fields.

Theorem 4.10 (Special values of Hecke \( L \)-functions). Let \( R \) be the ring of definition of \( (\mathcal{A}_f / R, \Sigma, \pi, \omega(\mathcal{A}_f), \omega(\mathcal{A}_f')) \), \( \chi \) a Hecke character with values in \( E \), conductor \( f \) and critical infinity type \( \beta - \alpha \) and \( \tau : E \to \mathbb{Q} \subset \mathbb{C} \) be an embedding, which allows to view \( \chi : \mathcal{I}(f) \to \mathbb{C}^\chi \). Then if \( f \neq \mathcal{O}_L \) one has for any \( c \) coprime to \( f \)

\[
\frac{(\alpha - 1)!(2\pi i)^{\beta} \Omega^\alpha \Omega^\beta}{N(c) L_i(\chi, 0, 1) - L_i(\chi, 0, [bc])} \in \tau(\mathcal{O}_E)R_{\mathcal{C}t}\left[ \frac{1}{N(fbc)} \right].
\]

If \( f = \mathcal{O}_L \) one has for any \( c, c' \) coprime to \( f \) and coprime to each other

\[
\frac{(\alpha - 1)!(2\pi i)^{\beta} \Omega^\alpha \Omega^\beta}{N(c' - 1) N(c - 1) L_i(\chi, 0, 1) - L_i(\chi, 0, [bc])} \in \tau(\mathcal{O}_E)R_{\mathcal{C}t}\left[ \frac{1}{N(fbc)} \right].
\]

In particular, the value

\[
\frac{L_i(\chi, 0)}{\Omega^\alpha \Omega^\beta} = \frac{(2\pi i)^{\beta}}{\Omega^\alpha \Omega^\beta} L_i(\chi, 0)
\]

is an algebraic number.

We understand that Bergeron-Charollois-Garcia can also prove the integrality of these \( L \)-values with a completely different method, but also relying on equivariant cohomology classes (personal communication, see also [BCG]).

Remark 4.11. With remark 4.4 we can reformulate the above theorem in more classical terms. Recall that \( \alpha = N_{L/K}^* \alpha_0 \) and \( \beta = N_{L/K}^* \beta_0 \). Then one has \( \Omega^\alpha = (\Omega(\mathcal{B})^{\alpha_0})^n \) and \( \Omega^\beta = (\Omega(\mathcal{B}^\beta))^n \). Thus one has

\[
\frac{(\alpha - 1)!(2\pi i)^{\beta} \Omega^\alpha \Omega^\beta}{N(\mathcal{B})^{\alpha_0} \Omega(\mathcal{B}^\beta)^{\beta_0}} = \left( \frac{(\alpha_0 - 1)!(2\pi i)^{\beta_0} \Omega(\mathcal{B})^{\alpha_0} \Omega(\mathcal{B}^\beta)^{\beta_0}}{N(\mathcal{B})^{\alpha_0} \Omega(\mathcal{B}^\beta)^{\beta_0}} \right)^n
\]

and hence \( \left( \frac{(2\pi i)^{\beta_0}}{N(\mathcal{B})^{\alpha_0} \Omega(\mathcal{B}^\beta)^{\beta_0}} \right)^n L_i(\chi, 0) \) is an algebraic number.
Proof. Let us first assume that \( f \neq \mathcal{O}_L \). Choose representatives \( \beta_1, \ldots, \beta_m \) for the ideal classes in \( \mathcal{I}(f)/\mathcal{P}_f \). Then we get from (14.4.2) by summing over all classes in \( \mathcal{I}(f)/\mathcal{P}_f \):

\[
\left( \alpha - 1 \right)! \left( 2\pi i \right)^3(\chi(\alpha \xi - 1) L_i(\chi, 0) \in \tau(\mathcal{O}_E) R[1/N(f \beta_1 \cdot \cdots \cdot \beta_m \xi)]].
\]

Choosing other representatives \( \tilde{\beta}_1, \ldots, \tilde{\beta}_m \) which are coprime to \( f \beta_1 \cdot \cdots \cdot \beta_m \xi \) one gets

\[
\left( \alpha - 1 \right)! \left( 2\pi i \right)^3(\chi(\alpha \xi - 1) L_i(\chi, 0) \in \tau(\mathcal{O}_E) R[1/N(f \beta_1 \cdot \cdots \cdot \beta_m \xi)]].
\]

The result follows, because the intersection of both rings is \( R[1/\mathcal{P}_f] \).

In the case where \( f = \mathcal{O}_L \), choose on \( (\mathcal{A}[\xi'] \setminus \{ e(S) \}) \rightarrow \mathcal{A}[\xi] \) the function \( \tilde{f}_i \) which is the pull-back of the function \( f_i \) under the projection to the second factor. Then the distribution relation gives

\[
\sum_{\mathcal{O} \in \mathcal{O} \setminus \mathcal{A}[\xi]} \tilde{f}_i(-\mathcal{O}^x t) E_{\mathcal{O}^x}^x(\mathcal{O}^x t + 1, 0; \mathcal{b}^{-1}, \mathcal{O}^x) = (N \mathcal{c}' - 1) (N \mathcal{c} E_{\mathcal{O}^x}^x(1, 0; \mathcal{b}^{-1}, \mathcal{O}^x)) - E_{\mathcal{O}^x}^x(1, 0; \mathcal{b}^{-1}(\mathcal{c} \xi')^{-1}, \mathcal{O}^x)).
\]

The rest of the argument is as in the case \( f \neq \mathcal{O}_L \). \( \square \)

5. \( p \)-adic interpolation

Let \( L \) be a totally imaginary field, \( p \) a prime co-prime to \( d_L \) splitting completely in the maximal CM subfield \( K \) of \( L \), and \( f \subseteq L \) a prime-to-\( p \) fractional ideal. In this section we will construct a \( p \)-adic measure interpolating \( p \)-adically all critical Hecke \( L \)-values of conductor dividing \( p^\infty f \).

5.1. The geometric setup. In the following \( \mathbb{C}_p \) denotes the completion of a fixed algebraic closure of \( \mathbb{Q}_p \) and \( t_p \) denotes a fixed embedding \( t_p : \overline{\mathbb{Q}} \rightarrow \mathbb{C}_p \). Let us denote by \( \Gamma \) a torsion-free subgroup of the units \( \mathcal{O}_L^* \) and let \( \xi \) be a generator of the free-abelian group \( H_{d-1}(\mathbb{G}, \mathbb{Z}) = \xi \mathbb{Z} \).

Let \( (\mathcal{A}/\mathcal{R}, \Sigma, \mathcal{O}_L, \theta, \omega(\mathcal{A}), \omega(\mathcal{A}^v), x) \) be an abelian variety over \( \mathcal{R} = \text{Spec} R \) with CM by \( \mathcal{O}_L \) as in Notation [14.15]. Let us assume that \( p \notin R^x \). We will say that the CM type \( \Sigma \) is ordinary at \( t_p \) or \( p \)-ordinary if the following condition holds:

\( \text{ORD-}p \) whenever \( \sigma \in \Sigma \) and \( \lambda \in \overline{\Sigma} \) the \( p \)-adic valuations induced by \( t_p \circ \lambda \) and \( t_p \circ \sigma \) are inequivalent.

A CM type which is ordinary at \( t_p \) induces a disjoint union

\( \{ \mathcal{P} \subseteq \mathcal{O}_L : \mathcal{P} \mid p \} = \Sigma_p \cup \overline{\Sigma}_p \)

of the primes dividing \( p \) with

\( \Sigma_p := \{ \mathcal{P} \text{ induced by the } p \text{-adic embeddings } t_p \circ \sigma \text{ with } \sigma \in \Sigma \} \)

\( \overline{\Sigma}_p := \{ \mathcal{P} \text{ induced by the } p \text{-adic embeddings } t_p \circ \lambda \text{ with } \lambda \in \overline{\Sigma} \} \).

In particular, a \( p \)-ordinary CM type \( \Sigma \) induces the decomposition

\( \mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{Z}_p \ni \mathcal{O}_L(\Sigma_p) \oplus \mathcal{O}_L(\overline{\Sigma}_p) \), \( \mathcal{O}_L(\Sigma_p) = \bigoplus_{\mathcal{P} \in \Sigma_p} \mathcal{O}_L(\mathcal{P}) \), \( \mathcal{O}_L(\overline{\Sigma}_p) = \bigoplus_{\mathcal{P} \in \overline{\Sigma}_p} \mathcal{O}_L(\mathcal{P}) \).

By abuse of notation, we will denote the base change of \( \mathcal{A} \) to \( \mathcal{O}_{\mathbb{C}_p} \) again by \( \mathcal{A} \).
Definition 5.1. A $\Gamma_0(p^\infty)$-structure is a pair of $O_L$-equivariant isomorphisms

$$
\theta_p: \mu_{p^\infty} \otimes \mathbb{Z} O_L(\Sigma_p) \sim \hat{A}[p^\infty],
$$

$$
\theta'_p: \mu_{p^\infty} \otimes \mathbb{Z} O_L(\Sigma_p) \sim \hat{A'}[p^\infty].
$$

Notation 5.2. For a prime $p$ let $(A/R, \Sigma, O_L, \omega(A), \omega(A'), x, \theta_p, \theta'_p)$ be a datum as above, i.e.:

- $R = \text{Spec } R$ with $\text{Quot}(R) = R^x$, $p \notin R^x$, $d_L \in R^x$ and $L^\text{Gal} \subseteq k$,
- $(A/R, \Sigma, O_L, \omega(A), \omega(A'), x)$ an abelian variety over $\mathcal{R}$ as in Notation 1.12,
- $\Sigma$ satisfies (ORD-$p$) with respect to the embedding $i_p$,
- $(\theta_p, \theta'_p)$ is a $\Gamma_0(p^\infty)$-structure.

From now on let $(A/R, \Sigma, O_L, \omega(A), \omega(A'), x, \theta_p, \theta'_p)$ be a tuple as in Notation 5.2

Lemma 5.3. We have canonical isomorphisms of formal groups over $O_{C_p}$

$$
\hat{A} \sim \text{Hom}_{\mathbb{Z}_p}(T_p\hat{A}^t, \hat{G}_m)
$$

and

$$
\hat{A}' \sim \text{Hom}_{\mathbb{Z}_p}(T_p\hat{A}'^t, \hat{G}_m).
$$

Here, $T_p\hat{A}^t$ denotes the $p$-adic Tate module of the Cartier dual $\hat{A}^t$ of the $p$-divisible group $\hat{A}$.

Proof. By Cartier theory we have an isomorphism

$$
T_p\hat{A}^t \cong \text{Hom}_{\text{Gr}, O_{C_p}}(\hat{A}, \hat{G}_m),
$$

where the right hand side denotes the $\mathbb{Z}_p$-module of homomorphisms of formal group from $\hat{A}$ to $\hat{G}_m$ over $O_{C_p}$. The image of the identity section under

$$
\text{Hom}_{\mathbb{Z}_p}(T_p\hat{A}^t, T_p\hat{A}^t) \cong T_p\hat{A}^t \otimes (T_p\hat{A}^t)^\vee \cong \text{Hom}_{\text{Gr}, O_{C_p}}(\hat{A}, \hat{G}_m \otimes (T_p\hat{A}^t)^\vee)
$$

gives us a canonical isomorphism

$$
(5.1.1) \quad \hat{A} \sim \hat{G}_m \otimes (T_p\hat{A}^t)^\vee.
$$

□

Since we assumed that $p$ does not divide $d_L$, the trace induces a canonical isomorphism

$$
O_L(\Sigma_p) \sim O_L(\Sigma_p)^\vee, \quad O_L(\Sigma_p) \sim O_L(\Sigma_p)^\vee.
$$

Lemma 5.4. The $\Gamma_0(p^\infty)$-structures on $A$ and $A'$ induce canonical $O_L$-equivariant isomorphisms

$$
T_p\hat{A}^t \cong O_L(\Sigma_p)^\vee \cong O_L(\Sigma_p)
$$

$$
T_p(\hat{A'})^t \cong O_L(\Sigma_p)^\vee \cong O_L(\Sigma_p).
$$

In particular, we obtain isomorphisms:

$$
(5.1.2) \quad \hat{A} \sim \hat{G}_m \otimes_{\mathbb{Z}_p} O_L(\Sigma_p)
$$

$$
(5.1.3) \quad \hat{A}' \sim \hat{G}_m \otimes_{\mathbb{Z}_p} O_L(\Sigma_p).
$$
Proof. The $\Gamma_{00}(p^\infty)$-structure on $\mathcal{A}$ induces an isomorphism of Tate modules

$$Z_p(1) \otimes_{\mathbb{Z}_p} O_L(\Sigma_p) \xrightarrow{\sim} T_p\hat{\mathcal{A}},$$

which corresponds to an isomorphism $T_p\hat{\mathcal{A}}(-1) \cong O_L(\Sigma_p)$ of $O_L(\Sigma_p)$-modules. The first claim follows from the perfect pairings

$$T_p\hat{\mathcal{A}} \times T_p\hat{\mathcal{A}}^t \to Z_p(1), \quad T_p\hat{\mathcal{A}}^t \times T_p(\hat{\mathcal{A}}^t)^t \to Z_p(1).$$

The isomorphisms (5.1.2) and (5.1.3) follow from Lemma 5.3. □

The $O_L^\times$-action on $\hat{\mathcal{A}}$ induces a decomposition

$$\omega_{\hat{\mathcal{A}}} \cong \bigoplus_{\sigma \in \Sigma} \omega_{\hat{\mathcal{A}}}(\sigma).$$

The isomorphism $\hat{\mathcal{A}} \cong \hat{G}_m \otimes O_L(\Sigma_p)$ allows us to be more concrete. We have

$$\text{Lie}(\hat{\mathcal{A}}) \cong \text{Lie}(\hat{G}_m) \otimes_{\mathbb{Z}_p} O_L(\Sigma_p) \cong O_{\mathbb{C}_p} \otimes_{\mathbb{Z}_p} O_L(\Sigma_p) \cong \prod_{\sigma \in \Sigma} O_{\mathbb{C}_p}.$$

Here, we have used the canonical isomorphism $\text{Lie}(\hat{G}_m) \cong O_{\mathbb{C}_p}$ coming from the canonical invariant derivation $(1 + T)\frac{\partial}{\partial T}$ on $\hat{G}_m$. Passing to the dual gives us a canonical generator

$$\omega_{\text{can}}(\mathcal{A})(\sigma) \in \omega_{\hat{\mathcal{A}}}(\sigma) = \omega_{\mathcal{A}}.$$

Similarly, we obtain

$$\omega_{\text{can}}(\mathcal{A}^\vee)(\sigma) \in \omega_{\hat{\mathcal{A}}^\vee}(\sigma) = \omega_{\mathcal{A}^\vee}.$$

The comparison of $\omega_{\text{can}}(\mathcal{A})$ to the algebraic basis $\omega(\mathcal{A})$ gives us the $p$-adic periods:

**Definition 5.5.** The $p$-adic periods $\Omega_p(\sigma) \in O_{\mathbb{C}_p}$ of $\mathcal{A}$ are defined by the equation:

$$\omega(\mathcal{A})(\sigma) = \Omega_p(\sigma)\omega_{\text{can}}(\mathcal{A})(\sigma),$$

$$\omega(\mathcal{A}^\vee)(\sigma) = \Omega^\vee_p(\sigma)\omega_{\text{can}}(\mathcal{A}^\vee)(\sigma).$$

We do not stress the dependence of $\Omega_p$ on $\mathcal{A}$ since the $p$-adic periods are independent under prime-to-$p$ isogenies in the following sense:

**Lemma 5.6.** Let $\mathcal{A} \to \mathcal{A}'$ be an isogeny of prime-to-$p$ degree and let $\omega(\mathcal{A}')(\sigma)$ be the basis of $\omega_{\mathcal{A}'}(\sigma)$ induced from the isomorphism:

$$\omega_{\mathcal{A}'}(\sigma) \cong \omega_{\mathcal{A}}(\sigma).$$

The $\Gamma_{00}(p^\infty)$-structure on $\mathcal{A}$ induces a $\Gamma_{00}(p^\infty)$-structure on $\mathcal{A}'$. In particular, we get by the same procedure as above a canonical basis

$$\omega_{\text{can}}(\mathcal{A}')(\sigma) \in \omega_{\hat{\mathcal{A}}'}(\sigma) = \omega_{\mathcal{A}'}(\sigma).$$

and we have the comparison

$$\omega(\mathcal{A}')(\sigma) = \Omega_p(\sigma)\omega_{\text{can}}(\mathcal{A}')(\sigma).$$

Proof. The equation

$$\omega(\mathcal{A}')(\sigma) = \Omega_p(\sigma)\omega_{\text{can}}(\mathcal{A}')(\sigma).$$

follows from the fact that both $\omega(\mathcal{A}')(\sigma)$ and $\omega_{\text{can}}(\mathcal{A}')(\sigma)$ are pullbacks of the corresponding differential forms on $\mathcal{A}$. □
The inclusion $\mathbb{Z}_p \subseteq \mathcal{O}_{\mathbb{C}_p}$ gives a map
\[
\mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{Z}_p \subseteq \mathcal{O}_L \otimes_{\mathbb{Z}} \mathcal{O}_{\mathbb{C}_p} \cong \mathcal{O}_{\mathbb{C}_p}^L.
\]
Since $\Sigma$ is assumed to be $p$-ordinary, we have $\mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{Z}_p = \mathcal{O}_L(\Sigma_p) \oplus \mathcal{O}_L(\Sigma_p)$ and the above map decomposes into
\[
t : \mathcal{O}_L(\Sigma_p) \rightarrow \mathcal{O}_{\mathbb{C}_p}^\Sigma
\]
and
\[
s : \mathcal{O}_L(\Sigma_p) \rightarrow \mathcal{O}_{\mathbb{C}_p}^\Sigma.
\]
For $\alpha \in I_{\Sigma}$, $\beta \in I_{\Sigma}$ we will write $t^\alpha s^\beta$ for the map
\[
\mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{Z}_p \rightarrow \mathcal{O}_{\mathbb{C}_p}, \ (z \otimes x) \mapsto \prod_{\sigma \in \Sigma} (x\sigma(z))^{\alpha} \cdot \prod_{\overline{\sigma} \in \Sigma} (x\overline{\sigma}(z))^{\beta}.
\]

5.2. **Infinitesimal trivialization of the Poincaré bundle.** Let us keep the notation of the previous section and denote by $\hat{\mathcal{P}}$ the $\mathcal{O}_A$-module obtained from the completion of the Poincaré bundle along $A \times e$, see [Spr19, II, §6]. Let us furthermore write $A$ for the base change of $A$ to $\mathbb{C}_p$ and $P$ for the associated Poincaré bundle. Motivated by Norman’s construction of $p$-adic theta functions, we obtain:

**Proposition 5.7.** There are canonical isomorphisms
\[
\hat{\mathcal{P}}|_A \sim \mathcal{O}_{\hat{A} \times \hat{A}^\vee}, \ \hat{P}|_A \sim \mathcal{O}_{\hat{A} \times \hat{A}^\vee}.
\]

**Proof.** The construction is the same as in the case of elliptic curves, see [Spr19, II, §6]. Let us first prove the claim for $A$. For the convenience of the reader, let us sketch the argument. Let us define $\mathfrak{p}_\Sigma := \prod_{p \in \Sigma} \mathfrak{p}_L$. The infinitesimal part of the $p^n$-torsion is given by the subgroupscheme
\[
C_n := A[\mathfrak{p}_L^n] = \ker[\mathfrak{p}_L^n].
\]
After the base change $A \times_S C_n \rightarrow C_n$ of $A$ to $C_n$ we have a canonical $C_n$-valued section
\[
\Delta : C_n \rightarrow A \times_S C_n
\]
given by the diagonal. The completed Poincaré bundle on $A \times_S C_n$ is given by $\text{pr}_A^* \hat{\mathcal{P}}$, where $\text{pr}_A : A \times_S C_n \rightarrow A$ is the projection to the first component. Since the dual isogeny $[\mathfrak{p}_L^n]^\vee$ is étale we can apply Corollary 2.9 and obtain an isomorphism
\[
\Delta^* \text{pr}_A^* \hat{\mathcal{P}} \cong e^* \text{pr}_A^* \hat{\mathcal{P}}.
\]
This gives an $\mathcal{O}_{C_n}$-linear isomorphism
\[
\hat{\mathcal{P}}|_{C_n} \cong \Delta^* \text{pr}_A^* \hat{\mathcal{P}} \cong e^* \text{pr}_A^* \hat{\mathcal{P}} \cong \mathcal{O}_{C_n} \otimes_{\mathcal{O}_S} \mathcal{O}_{\hat{A}^\vee}
\]
Passing to the limit over $n$ and observing $\lim_{n} C_n = \hat{A}$ proves the claim for $A$.

Restricting the integral isomorphism to the $n$-th respectively $m$-th infinitesimal neighborhoods of the zero section $A^{(n)} \times A^{(m)}$ gives
\[
\hat{\mathcal{P}}^{(m)}|_{A^{(n)}} \sim \mathcal{O}_{A^{(n)} \times A^{(m)}}.
\]
Passing to the limit over $(m, n)$ after base change to $\mathbb{C}_p$ proves the claim for $A$. \qed

In particular, there is an isomorphism of $\mathcal{O}_A$-modules
\[
(5.2.1) \quad \mathcal{P}^{(1)}|_{\hat{A}} \cong \mathcal{O}_{\hat{A}} \otimes_{\mathcal{O}_S} (\mathcal{O}_S \oplus \omega_{A^\vee}).
\]
Since $\mathcal{P}^{(1)}$ is the pushout of
\[
0 \rightarrow \pi^* \omega_{A^\vee} \rightarrow \mathcal{P}^{(1)} \rightarrow \mathcal{O}_A \rightarrow 0
\]
along $\pi^*\omega_{/V} \to \pi^*\mathcal{H}$ we also get a splitting

\[(5.2.2) \quad \mathcal{P}_{\hat{A}} \mid_{\hat{A}} \cong \mathcal{O}_{\hat{A}} \otimes \mathcal{O}_S (\mathcal{O}_S \oplus \mathcal{H}).\]

Applying $\text{TSym}^n$ to (5.2.1) and (5.2.2), using the co-multiplication maps and passing to the limit over $n$ gives

\[(5.2.3) \quad \mathcal{P}_{\hat{A}} \mid_{\hat{A}} \rightarrow \mathcal{O}_{\hat{A}} \otimes \text{TSym} \mathcal{H},\]

\[(5.2.4) \quad \mathcal{P}_{\hat{A}} \mid_{\hat{A}} \rightarrow \mathcal{O}_{\hat{A}} \otimes \text{TSym} \omega_{AV}.\]

These maps are injective since $n!$ is a non-zero divisor on $\mathcal{S}$ for all $n \in \mathbb{N}$. Recall, that the $\mathcal{O}_L$-action allows us to split the Hodge filtration $\omega_{AV} \subseteq \mathcal{H}$:

$$\mathcal{H} \twoheadrightarrow \omega_{AV}.$$  

**Lemma 5.8.** (1) There is a unique $\mathcal{O}_{\hat{A}}$-linear retraction $p: \mathcal{P}_{\hat{A}} \mid_{\hat{A}} \rightarrow \mathcal{P}_{\hat{A}}$ of the canonical injection $i: \mathcal{P}_{\hat{A}} \mid_{\hat{A}} \rightarrow \mathcal{P}_{\hat{A}}$ making the diagram

\[
\begin{array}{ccc}
\mathcal{P}_{\hat{A}} \mid_{\hat{A}} & \xrightarrow{i} & \mathcal{P}_{\hat{A}} \\
\downarrow & & \downarrow \\
\mathcal{P}_{\hat{A}} & \xrightarrow{\mathcal{P}_{\hat{A}} \mid_{\hat{A}}} & \mathcal{P}_{\hat{A}} \mid_{\hat{A}} \otimes \Omega^1_{\hat{A}/S} \\
\mathcal{O}_{\hat{A}} \otimes \mathcal{O}_S \text{TSym} \omega_{AV} & \cong & \mathcal{O}_{\hat{A}} \otimes \mathcal{O}_S \text{TSym} \omega_{AV} \\
\end{array}
\]

commutative. Here, the right vertical map is the canonical map splitting the Hodge filtration.

(2) The diagram

\[
\begin{array}{ccc}
\mathcal{P}_{\hat{A}} \mid_{\hat{A}} & \xrightarrow{\triangledown} & \mathcal{P}_{\hat{A}} \mid_{\hat{A}} \otimes \Omega^1_{\hat{A}/S} \\
i & & \downarrow p \\
\mathcal{P}_{\hat{A}} & \xrightarrow{\cong} & \mathcal{P}_{\hat{A}} \mid_{\hat{A}} \otimes \Omega^1_{\hat{A}/S} \\
\mathcal{O}_{\hat{A} \times \hat{A}} & \cong & \mathcal{O}_{\hat{A} \times \hat{A}} \otimes \Omega^1_{\hat{A}/S} \\
\end{array}
\]

commutes.

**Proof.** (1) : The inclusion $i: \mathcal{P}_{\hat{A}} \mid_{\hat{A}} \rightarrow \mathcal{P}_{\hat{A}}$ sits in a Cartesian diagram:

\[(5.2.5) \quad \mathcal{P}_{\hat{A}} \xrightarrow{\cong} \mathcal{O}_{\hat{A}} \otimes \mathcal{O}_S \text{TSym} \omega_{AV} \]

where the right vertical map is induced by the inclusion $\omega_{AV} \subseteq \mathcal{H}$. The splitting of the Hodge filtration induces a retraction

$$\text{TSym} \mathcal{H} \twoheadrightarrow \text{TSym} \omega_{AV}$$

of the right vertical map in (5.2.5). Since (5.2.5) is Cartesian, we obtain the desired splitting.
(2) By applying $\overline{\text{TSym}}$ it suffices to prove the commutativity of:

$$
\begin{array}{c}
\mathfrak{p}^\sharp_{\mathfrak{A}}(1) \mid_{\mathfrak{A}} \xrightarrow{\nabla_{\mathfrak{P}^{\sharp}(1)}} \mathfrak{p}^\sharp_{\mathfrak{A}}(1) \otimes \Omega^1_{\mathfrak{A}/S} \\
\mathfrak{P}(1) \mid_{\mathfrak{A}} \xrightarrow{\nabla_{\mathfrak{P}(1)}} \mathfrak{P}(1) \otimes \Omega^1_{\mathfrak{A}/S} \\
\mathfrak{O}_{\mathfrak{A} \times \mathfrak{A}^\vee, (1)} \xrightarrow{\cong} \mathfrak{O}_{\mathfrak{A} \times \mathfrak{A}^\vee, (1)} \otimes \Omega^1_{\mathfrak{A}/S}
\end{array}
$$

Let us recall that $\mathfrak{p}^\sharp_{\mathfrak{A}}(1)$ sits in a short exact sequence

$$
0 \to \pi^* \mathcal{H} \to \mathfrak{p}^\sharp_{\mathfrak{A}}(1) \to \mathfrak{O}_{\mathfrak{A}} \to 0
$$

which is horizontal if we equip $\pi^* \mathcal{H}$ and $\mathfrak{O}_{\mathfrak{A}}$ with the canonical $\mathcal{O}_S$-linear derivation. Since this sequence splits over $\mathfrak{A}$, the connection $\nabla_{\mathfrak{P}^{\sharp}(1)}$ is uniquely determined by $\nabla_{\mathfrak{P}(1)}(e^{(1)}) \in \pi^* \mathcal{H} \otimes \Omega^1_{\mathfrak{A}/S}$, where $e^{(1)}$ is the image of $1 \otimes (1, 0)$ under $\mathfrak{O}_{\mathfrak{A}^\vee, (1)} \cong \mathfrak{O}_{\mathfrak{A}^\vee, (1)} \otimes \Omega^1_{\mathfrak{A}^\vee}/S$.

The commutativity of the above diagram is now equivalent to the formula

$$
(5.2.6) \quad \nabla_{\mathfrak{P}^{\sharp}(1)}(e^{(1)}) \in \pi^* \mathcal{H}(\Sigma) \otimes \Omega^1_{\mathfrak{A}/S} = \ker(\pi^* \mathcal{H} \otimes \Omega^1_{\mathfrak{A}/S} \to \pi^* \omega_{\mathfrak{A}^\vee/S} \otimes \Omega^1_{\mathfrak{A}/S}).
$$

Before we verify this formula, let us recall that the canonical map $[p]^* \mathcal{H} \to \mathcal{H}$ acts by multiplication by $p$ on $\mathcal{H}(\Sigma)$ and is invertible on the 'unit root' space $\mathcal{H}(\Sigma)$. Let us write $\eta_{\Sigma}$ for the component of $\nabla_{\mathfrak{P}^{\sharp}(1)}(e^{(1)})$ in $\pi^* \mathcal{H}(\Sigma) \otimes \Omega^1_{\mathfrak{A}/S}$. Our aim is to show $\eta_{\Sigma} = 0$. The horizontality of the invariance under isogenies map $[p]^* \mathfrak{p}^\sharp_{\mathfrak{A}}(1) \to \mathfrak{p}^\sharp_{\mathfrak{A}}(1)$ and the fact that $[p]^* : \Gamma(\mathfrak{A}, \Omega^1_{\mathfrak{A}/S}) \to \Gamma(\mathfrak{A}, \Omega^1_{\mathfrak{A}/S})$ and $[p]^* \mathcal{H}(\Sigma) \to \mathcal{H}(\Sigma)$ are both multiplication by $p$ implies the formula

$$p^2 \eta_{\Sigma} = p \eta_{\Sigma}.
$$

Since $\mathcal{S}$ is $p$-torsion free this implies $\eta_{\Sigma} = 0$. \hfill $\square$

5.3. **Construction of $p$-adic theta functions.** Let us remind the reader that we have fixed a tuple

$$(\mathfrak{A}/\mathcal{R}, \Sigma, \Omega_{\mathfrak{L}}, \theta, \omega(\mathfrak{A}), \omega(\mathfrak{A}^\vee), x, \theta_p, \theta_p^\vee)$$

as in Notation 5.2. Additionally, let $\mathfrak{c} \subseteq \mathfrak{O}_{\mathfrak{L}}$ be a fractional ideal which is co-prime to $p$ and satisfies $x \notin \mathfrak{A}[\mathfrak{c}]$. We consider the $\Gamma$-invariant function

$$f = f_\mathfrak{c} = (N\mathfrak{c})(0) - \sum_{t \in \mathfrak{A}[\mathfrak{c}]} (t) \in R[\ker[\mathfrak{c}]]^0, \Gamma$$

which has been defined in Section 2.1. We keep the notation $\mathfrak{A}$ for the base change to $\mathbb{C}_p$ and $\mathcal{P}$ for the Poincaré bundle over $\mathfrak{A} \times \mathfrak{A}^\vee$. In the following, we will associate a $p$-adic theta function

$$\vartheta_\Gamma(f, x, \xi) \in H^0(\mathfrak{A} \times \mathfrak{A}^\vee, \mathfrak{O}_{\mathfrak{A} \times \mathfrak{A}^\vee})^\Gamma$$
to the equivariant coherent Eisenstein–Kronecker class $EK_{\mathcal{P}^\sharp}(f) \in H^{d-1}(A \setminus A[c], \Gamma, \hat{\mathcal{P}})$.

Let us write $\hat{A}_x$ for the completion of $A$ at $x$. The restriction of the equivariant coherent Eisenstein–Kronecker $EK_{\mathcal{P}^\sharp}(f)$ to $\hat{A}_x$ gives:

$$EK_{\mathcal{P}^\sharp}(f)|_{\hat{A}_x} \in H^{d-1}(\hat{A}_x, \Gamma, \hat{\mathcal{P}}|_{\hat{A}_x} \otimes \omega_A^d).$$

We will use the basis $\omega_{can}(A)$ to trivialize $\omega_A^d \cong \mathbb{C}_\nu$. Since $\hat{A}_x$ is an affine formal scheme, we have

$$H^{d-1}(\hat{A}_x, \Gamma, \hat{\mathcal{P}}|_{\hat{A}_x}) = H^{d-1}(\Gamma, H^0(\hat{A}_x, \hat{\mathcal{P}}|_{\hat{A}_x})).$$

Let us write $T_x$ for the translation by $x$ and let $\varphi$ an isogeny with $x \in \ker \varphi$. Pullback along $T_x$ gives an isomorphism

$$T_x^* : H^0(\hat{A}_x, \hat{\mathcal{P}}|_{\hat{A}_x}) \cong H^0(\hat{A}, (T_x^* \hat{\mathcal{P}})|_{\hat{A}})$$

By invariance under isogenies we get

$$T_x^* \hat{\mathcal{P}} \cong T_x^* \varphi^* \hat{\mathcal{P}} = \varphi^* \hat{\mathcal{P}} \cong \hat{\mathcal{P}}. \tag{5.3.1}$$

By Proposition [5.7] we have an isomorphism of $\mathcal{O}_{\hat{A}}$-modules:

$$\hat{\mathcal{P}}|_{\hat{A}} \cong \mathcal{O}_{\hat{A} \times \hat{A}^\vee} \tag{5.3.2}$$

and combining (5.3.1) and (5.3.2) gives:

$$\hat{T}_x : T_x^* \hat{\mathcal{P}}|_{\hat{A}} \cong \hat{\mathcal{P}}|_{\hat{A}} \cong \mathcal{O}_{\hat{A} \times \hat{A}^\vee}. \tag{5.3.3}$$

Let us recall that the moment map $\text{mom}_F$ of a formal group $\hat{F}$ is the map

$$\Gamma(\hat{F}, \mathcal{O}_F) \to \text{TSym}_\omega \hat{F}$$

induced by the co-multiplication. For later reference, let us observe the following basic properties of $\hat{T}_x$:

**Lemma 5.9.** We have the following identifications of maps:

1. $\text{mom}_{\hat{A}^\vee} \circ (e^* \hat{T}_x) = \hat{T}_x$ as maps $e^* \hat{\mathcal{P}} \to \text{TSym}_\omega \hat{A}^\vee$.
2. $\hat{T}_x \circ \nabla \circ i = d_{\hat{A}} \circ \hat{T}_x$ as maps $T_x^* \hat{\mathcal{P}}|_{\hat{A}} \to \mathcal{O}_{\hat{A} \times \hat{A}^\vee} \otimes_{\mathbb{C}_\nu} \omega_{\hat{A}}$. Here, $i : \hat{\mathcal{P}} \subseteq \hat{\mathcal{P}}^\sharp$ is the canonical inclusion into the completed Poincaré bundle with connection $(\hat{\mathcal{P}}^\sharp, \nabla)$.
3. For $f \in \hat{A}[p^n]$ we have a commutative diagram

$$\begin{array}{ccc}
H^0(\hat{A}, \hat{\mathcal{P}}) & \xrightarrow{(\cdot)_{\hat{A}^\vee} \circ T_x^*} & H^0(\hat{A}, T_x^* \hat{\mathcal{P}}) \\
\downarrow T_x^* & & \downarrow T_x^* \\
H^0(\hat{A}, T_{x+t}^* \hat{\mathcal{P}}) & \xrightarrow{(\cdot)_{\hat{A}^\vee} \circ T_x^*} & H^0(\hat{A}, T_{x+t}^* \hat{\mathcal{P}}) \\
\end{array}$$

**Proof.** The first claim follows immediately from the definitions of $\hat{T}_x$ and $\hat{T}_x$. The second claim follows from Lemma [5.8] using that the invariance under isogenies map

$$\varphi^* \hat{\mathcal{P}} \cong \hat{\mathcal{P}}^\sharp$$

is horizontal with respect to $\nabla$. The last claim reduces to the commutativity of the translation isomorphisms

$$\begin{array}{ccc}
T_{x+t}^* \mathcal{P} & \xrightarrow{T_x^*} & T_x^* \mathcal{P} \\
\downarrow T_{x+t}^* & & \downarrow T_x^* \\
T_x^* \mathcal{P} & \xrightarrow{\cdot} & \mathcal{P}.
\end{array}$$
Lemma 5.10. The inclusion $H^0(A \times \hat{A}, O_{A \times \hat{A}}) \subseteq H^0(A \times A^\vee, O_{A \times A^\vee})$ induces an isomorphism

$$H^{d-1}(\Gamma, H^0(A \times \hat{A}, O_{A \times \hat{A}})) = H^{d-1}(\Gamma, H^0(A \times A^\vee, O_{A \times A^\vee})).$$

In particular, we have by cap-product with $\xi$: $H_{d-1}(\Gamma, \mathbb{Z}) \cong \mathbb{Z}$ an isomorphism

$$H^{d-1}(\Gamma, H^0(A \times \hat{A}, O_{A \times \hat{A}})) \cong H^0(A \times A^\vee, O_{A \times A^\vee}).$$

Proof. On finite level each of the $\mathbb{C}_p[\Gamma]$-modules

$$H^0(A^{(n)} \times A^\vee(m), O_{A^{(n)} \times A^\vee(m)}).$$

decomposes into a product of one-dimensional representations and it follows that the induced map

$$H^{d-1}(\Gamma, \mathbb{C}_p) \otimes_{\mathbb{C}_p} H^0(\Gamma, H^0(A^{(n)} \times A^\vee(m), O_{A^{(n)} \times A^\vee(m)})) \xrightarrow{\cup} H^{d-1}(\Gamma, H^0(A^{(n)} \times A^\vee(m), O_{A^{(n)} \times A^\vee(m)}))$$

is an isomorphism, cf. [Gra16]. We deduce that the inclusion is an isomorphism:

$$H^{d-1}(\Gamma, H^0(A^{(n)} \times A^\vee(m), O_{A^{(n)} \times A^\vee(m)})) = H^{d-1}(\Gamma, H^0(A^{(n)} \times A^\vee(m), O_{A^{(n)} \times A^\vee(m)})).$$

Since all transition maps are surjective, we obtain the claimed isomorphism in the statement by passing to the limit over $n$ and $m$. □

Definition 5.11. The $p$-adic theta function $\vartheta_\Gamma(f, x, \xi) \in \Gamma(A \times \hat{A}, O_{A \times \hat{A}})^\Gamma$ associated with $EK_{\overline{p}}(f)$ at $x$ is defined as

$$\vartheta_\Gamma(f, x, \xi) := \vartheta_x T_x^{\ast}(EK_{\overline{p}}(f)|_{\hat{A}}).$$

For later reference, let us observe that $\vartheta_\Gamma(f, x, \xi)$ can be defined integrally as long as $x \in \ker \varphi$ with $\varphi^\vee$ étale:

Lemma 5.12. If $x \in \ker \varphi$ for an isogeny $\varphi : A \to A'$ with $\varphi^\vee$ étale then $\vartheta_\Gamma(f, x, \xi)$ is contained in the integral subring

$$\vartheta_\Gamma(f, x, \xi) \subseteq \Gamma(A \times \hat{A}, O_{A \times \hat{A}})^\Gamma \subseteq \Gamma(A \times A^\vee, O_{A \times A^\vee})^\Gamma.$$

Proof. If $x \in \ker \varphi$ for an isogeny $\varphi : A \to A'$ with $\varphi^\vee$ étale then

$$\varphi^* \mathcal{P} \cong \mathcal{P}$$

and the above construction works integrally as well. □

Finally, let us observe the behaviour of $\vartheta_\Gamma(f, x, \xi)$ under varying the subgroup $\Gamma \subseteq O_{\ell}^\vee$:

Lemma 5.13. If $\Gamma' \subseteq \Gamma$ is a subgroup of finite index then

$$\vartheta_{\Gamma'}(f, x) = [\Gamma : \Gamma'] \vartheta_\Gamma(f, x).$$

Proof. This follows exactly as in the proof of Proposition 2.27 from the fact that the restriction map

$$\mathbb{C}_p \cong H_{d-1}(\Gamma, \mathbb{C}_p) \otimes H^{d-1}(\Gamma, \mathbb{C}_p) \to H_{d-1}(\Gamma', \mathbb{C}_p) \otimes H^{d-1}(\Gamma', \mathbb{C}_p) \cong \mathbb{C}_p$$

is multiplication with $[\Gamma : \Gamma']$. □
5.4. Construction of the $p$-adic Eisenstein measure. We keep the setup from Section 5.1. Furthermore, we assume that $x \in \ker \varphi$ for an isogeny $\varphi: \mathcal{A} \to \mathcal{A}'$ with $\varphi^\vee$ étale.

**Definition 5.14.** For $\alpha \in I^+_p$, $\beta \in I^+_p$ let us define differential operators

$$\partial(\mathcal{A})^\alpha \partial(\mathcal{A}^\vee)^\beta: \mathcal{O}_{\hat{A} \times \hat{A}^\vee} \to \mathcal{O}_{\mathcal{C}_p}$$

by the formula

$$\text{mom}_{\hat{A} \times \hat{A}^\vee}(f) = \left( (\partial(\mathcal{A})^\alpha \partial(\mathcal{A}^\vee)^\beta f) \cdot \omega_{\text{can}}(\mathcal{A})^{[\alpha]} \omega_{\text{can}}(\mathcal{A}^\vee)^{[\beta]} \right)_{\alpha, \beta}. $$

Here $\text{mom}_{\hat{A} \times \hat{A}^\vee}$ denotes the moment map of the formal group $\hat{A} \times \hat{A}^\vee$.

According to Katz, we have the following isomorphism between functions on formal toroidal groups and $p$-adic measures:

**Proposition 5.15.** There is a $\Gamma$-equivariant isomorphism of rings

$$\Gamma(\hat{A} \times \hat{A}^\vee, \mathcal{O}_{\hat{A} \times \hat{A}^\vee}) \sim \text{Meas}(\mathcal{O}_L \otimes \mathbb{Z}_p, \mathcal{O}_{\mathcal{C}_p}), \ g \mapsto \mu_g$$

which is uniquely determined by the integration formula

$$\int_{\mathcal{O}_L \otimes \mathbb{Z}_p} t^\alpha s^\beta d\mu_g = \partial(\mathcal{A})^\alpha \partial(\mathcal{A}^\vee)^\beta g.$$

**Proof.** According to Lemma 5.3 we have

$$\hat{A} \times \hat{A}^\vee = \text{Hom}_{\mathbb{Z}_p}(\mathcal{O}_L \otimes \mathbb{Z}_p, \hat{G}_m).$$

Now we can apply [Kat81, Theorem 1]. The $\Gamma$-equivariance follows from the functoriality of Katz' construction of $p$-adic measures.

**Definition 5.16.** The unique $\Gamma$-invariant $p$-adic measure corresponding to $\hat{\varrho}_x(EK_{\varphi^\vee}(f)|_{\tilde{\mathcal{A}}})$ under

$$H^0(\hat{A} \times \hat{A}^\vee, \mathcal{O}_{\hat{A} \times \hat{A}^\vee})^\Gamma \cong \text{Meas}(\mathcal{O}_L \otimes \mathbb{Z}_p, \mathcal{O}_{\mathcal{C}_p})^\Gamma$$

is denoted by $\mu_{Eis}(f, x)$ and is called the $p$-adic Eisenstein measure associated to the tuple $(\mathcal{A}/\mathcal{R}, \Sigma, \mathcal{O}_L, \omega(\mathcal{A}), \omega(\mathcal{A}^\vee), x, \theta_p, \theta_p^\vee)$.

**Theorem 5.17** ($p$-adic Eisenstein measure). The $p$-adic Eisenstein measure has the following interpolation property: For $\alpha \in I^+_p$ and $\beta \in I^+_p$ we have

$$\frac{1}{\Omega_p^\alpha \Gamma_p^\beta} \int_{\mathcal{O}_L \otimes \mathbb{Z}_p} t^\alpha s^\beta d\mu_{Eis}(f, x) = EK^\beta_{\Gamma}(f, x)(\omega(\mathcal{A})^{[\alpha]} \cdot \omega(\mathcal{A}^\vee)^{[\beta]}).$$

**Proof.** Let us write $\text{mom}^{\alpha, \beta}_{\hat{A} \times \hat{A}^\vee}$ respectively $\text{mom}^\beta_{\hat{A}^\vee}$ for the $(\alpha, \beta)$ resp. $\beta$-component of the moment map. With this notation, we have by the defining property of $\mu_{Eis}(f, x)$ the integration formula

$$\int_{\mathcal{O}_L \otimes \mathbb{Z}_p} t^\alpha s^\beta d\mu_{Eis}(f, x) \omega_{\text{can}}(\mathcal{A})^{[\alpha]} \omega_{\text{can}}(\mathcal{A}^\vee)^{[\beta]} = \text{mom}^{\alpha, \beta}_{\hat{A} \times \hat{A}^\vee}(\hat{\varrho}_x(EK_{\varphi^\vee}(f)|_{\tilde{\mathcal{A}}}))$$. Using Lemma 5.9 and the definition of $EK^\beta_{\Gamma}(f, x)$ we get:

$$\text{mom}^{\alpha, \beta}_{\hat{A} \times \hat{A}^\vee}(\hat{\varrho}_x(EK_{\varphi^\vee}(f)|_{\tilde{\mathcal{A}}})) = \text{mom}^{\beta}_{\hat{A}^\vee}(\partial(\mathcal{A})^\alpha \hat{\varrho}_x(EK_{\varphi^\vee}(f)|_{\tilde{\mathcal{A}}}) \cdot \omega_{\text{can}}(\mathcal{A})^{[\alpha]} = \text{mom}^{\beta}_{\hat{A}^\vee}(\hat{\varrho}_x(\nabla^{[\alpha]} EK_{\varphi^\vee}(f)|_{\tilde{\mathcal{A}}})) = EK^\beta_{\Gamma}(f, x).$$

The formulas

$$\omega(\mathcal{A})^{[\alpha]} = \Omega_p^\alpha \omega_{\text{can}}(\mathcal{A})^{[\alpha]}, \ \omega(\mathcal{A}^\vee)^{[\beta]} = \Omega_p^\beta \omega_{\text{can}}(\mathcal{A}^\vee)^{[\beta]}$$
allow us to re-write $E^K_{\Gamma}(f, x)$ in terms of the basis $\omega_{\text{can}}(A), \omega_{\text{can}}(A^\vee)$

$$E^K_{\Gamma}(f, x) = E^K_{\Gamma}(f, x)(\omega(A)^{[\alpha]}, \omega(A^\vee)^{[\beta]}) \cdot \omega(A)^{[\alpha]} \otimes \omega(A^\vee)^{[\beta]} =$$

$$= E^K_{\Gamma}(f, x)(\omega(A)^{[\alpha]}, \omega(A^\vee)^{[\beta]}) \cdot \Omega_p^\alpha \Omega_p^\beta \omega_{\text{can}}(A)^{[\alpha]} \otimes \omega(A^\vee)^{[\beta]}.$$

Combining the above computations shows the Theorem. \hfill \Box

5.5. **Translation operators.** The aim of this section is to compute the value of certain translations of $\vartheta_{f, x}$ by infinitesimal torsion sections. More precisely, we will prove the following result:

**Theorem 5.18** (Translation). Let $(f, t) \in A[p^n] \times A^\vee[p^n]$ infinitesimal $\Gamma$-invariant torsion sections and assume that the order of $x$ is co-prime to $p$, then

$$\text{translation operators}.\text{Lemma 5.9 (3):}$$

$$\text{Proof of Theorem 5.18.} \text{The adjunction morphism} \text{shows a posteriori the integrality of the right hand side.}$$

**Remark 5.19.** A priori the right hand side of the equation in the above Theorem can only be defined rationally. Indeed, the dual of $\left[\mathcal{E}(x)\right]$ is not étale, hence $\vartheta_{\Gamma}(f, x + s + \hat{t})$ is not defined integrally for $s \in \ker[\mathcal{E}(x)]$. Nevertheless, $[\mathcal{E}(x)]$ becomes étale over the generic fiber $\mathbb{C}_p$ of $\mathcal{O}_{\mathbb{C}_p}$. Let us now assume that $x \in \ker \varphi$ with $\varphi^\vee$ étale. Then $(T_{\Gamma} \times T_{\Gamma})^* \vartheta_{\Gamma}(f, x)$ is defined integrally, i.e.

$$(T_{\Gamma} \times T_{\Gamma})^* \vartheta_{\Gamma}(f, x) \in \Gamma(\mathcal{A} \times \mathcal{A}^\vee, \mathcal{O}_{\mathcal{A} \times \mathcal{A}^\vee}).$$

In particular, the equation

$$(T_{\Gamma} \times T_{\Gamma})^* \vartheta_{\Gamma}(f, x) = \sum_{s \in A[p^n]} \langle s, t \rangle_{[p^n]} \cdot \vartheta_{\Gamma}(f, x + s + \hat{t})$$

shows a posteriori the integrality of the right hand side.

**Proof of Theorem 5.18.** It suffices to prove the statement for $(T_{\Gamma} \times \text{id})^*$ and $(\text{id} \times T_{\Gamma})^*$ separately. The translation by $\hat{t}$ is much easier, indeed this is just the statement of Lemma 5.9 (3):

$$(T_{\Gamma} \times \text{id})^* \vartheta_{\Gamma}(f, x, \xi) \overset{\text{Def.}}{=} T_{\Gamma}^* \hat{t}_x T_{\Gamma}^* (\mathcal{E}K_{\varphi^\vee}(f)|_{\mathcal{A}^\vee}) = \hat{t}_x + \hat{t} T_{\Gamma}^* (\mathcal{E}K_{\varphi^\vee}(f)|_{\mathcal{A}^\vee}) = \vartheta_{\Gamma}(f, x + \hat{t}).$$

Let us turn our attention to the translation along $t \in \mathcal{A}^\vee[p^n]$. First, we have to define translation operators on $\mathcal{P}$ corresponding to $(\text{id} \times T_{\Gamma})^*$ on the formal group. This works completely analogue to the construction of $T_{\Gamma}^* \mathcal{P} \cong \mathcal{P}$: The translation $T_{\Gamma}^* \mathcal{P} \rightarrow \mathcal{A}^\vee$ restricts to the formal group $T_{\Gamma}^* \mathcal{A}^\vee \rightarrow \mathcal{A}^\vee$. Let us define

$$\hat{\mathcal{P}}^t := \text{pr}_{A^\vee} \left((\text{id} \times T_{\Gamma})^* (\mathcal{P}|_{A^\times \mathcal{A}^\vee})\right).$$

The adjunction morphism

$$\mathcal{P} \rightarrow (\text{id} \times T_{\Gamma})^* (\text{id} \times T_{\Gamma})^* \mathcal{P}$$

induces a morphism

$$(T_{\Gamma} \times T_{\Gamma})^* \mathcal{P} \rightarrow \hat{\mathcal{P}}^t.$$
Let us define $A' := A/\mathcal{A}$$^n$ and consider the isogeny $[\mathcal{F}^\dagger_5]$: $A' \to A$ and its dual $\mathcal{A}$$^\dagger$ : $A^\dagger \to A'$$^\dagger$. The universal property of the Poincaré bundle gives us a chain of isomorphisms

\[
[\mathcal{F}^\dagger_5]|_{\hat{\mathcal{P}}} = \text{pr}_*((([\mathcal{F}^\dagger_5] \times T)\mathcal{P})|_{A \times \hat{A}^\dagger}) \cong \text{pr}_*(([\mathcal{id}_A \times \mathcal{A}$$^\dagger$])\mathcal{P}|_{A' \times \hat{A}^\dagger}) \cong \text{pr}_*(([\mathcal{F}^\dagger_5] \times \mathcal{id}_A)\mathcal{P}|_{A' \times \hat{A}^\dagger}) = [\mathcal{F}^\dagger_5]|_{\hat{\mathcal{P}}}
\]

(5.5.3)

Since $c$ is prime-to-$p$ there is a unique $c$-torsion point $x' \in A'[c](\mathbb{C}_p)$ mapping to $x$. Restricting this isomorphism to $\hat{A}^v$, and observing that $[\mathcal{F}^\dagger_5]$ gives an isomorphism $\hat{A}^v \cong \hat{A}_x$ gives the following translation map on the completed Poincaré bundle:

\[
\hat{U}^n_t : H^0(\hat{A}_x, \hat{P}) \xrightarrow{(5.5.2)} H^0(\hat{A}_x, \hat{\mathcal{P}}) \xrightarrow{(5.5.3)} H^0(\hat{A}_x, \hat{P}).
\]

Unwinding of definitions shows that this isomorphism is compatible with translation on $\hat{A}^\dagger$, i.e. the following diagram commutes:

\[
\begin{array}{ccc}
H^0(\hat{A}_x, \hat{P}) & \xrightarrow{\hat{U}^n_t} & H^0(\hat{A}_x, \hat{P}) \\
\phi_x \downarrow & & \phi_x \downarrow \\
H^0(\hat{P} \times \hat{A}^\dagger, \mathcal{O}_{\hat{A} \times \hat{A}^\dagger}) & = & H^0(\hat{P} \times \hat{A}^\dagger, \mathcal{O}_{\hat{A} \times \hat{A}^\dagger}).
\end{array}
\]

Thus, by the definition of $\phi_t(f, x, \xi)$ it suffices to prove, under the identification $\hat{A}_x \cong \hat{A}_x^v$, the formula

(5.5.4)

\[
\hat{U}^n_t(EK_{\mathcal{P}^n} f)|\hat{A}_x = EK_{A' : A^{\dagger}}(\hat{f})|\hat{A}_x^v
\]

with

\[
f : A'[\mathcal{F}^\dagger_5] \cong A'[c] \times A'[\mathcal{F}^\dagger_5] \to \mathbb{C}_p, \quad \hat{f}(c, s') := f([\mathcal{F}^\dagger_5]c) \cdot \langle s', t \rangle|_{\mathcal{F}^\dagger_5}.
\]

Here, observe that the isogeny $[\mathcal{F}^\dagger_5] : A \to A'$ induces an isomorphism $A|\mathcal{F}^\dagger_5 \cong A'|\mathcal{F}^\dagger_5$ and we have

\[
\langle [\mathcal{F}^\dagger_5]s, t \rangle|_{\mathcal{F}^\dagger_5} = \langle s, t \rangle|_{\mathcal{F}^\dagger_5}.
\]

The proof of equation (5.5.4) is not as formal as the proof for translations along $f$. Our strategy is to extend the translation operators $\hat{U}^n_t$ globally (on $A$) defined translation operators and prove the equality by a residue computation: The global translation operators are defined as follows:

\[
\hat{U}^n_t : H^{d-1}(A \setminus A[c], \Gamma, \hat{P}) \xrightarrow{(5.5.4)} H^{d-1}(A'[\mathcal{F}^\dagger_5], \Gamma, [\mathcal{F}^\dagger_5]|_{\hat{\mathcal{P}}}.
\]

By construction $\hat{U}^n_t$ fits into a commutative diagram:

\[
\begin{array}{ccc}
H^{d-1}(A \setminus A[c], \Gamma, \hat{P}) & \xrightarrow{\hat{U}^n_t} & H^{d-1}(A' \setminus A'[\mathcal{F}^\dagger_5], \Gamma, [\mathcal{F}^\dagger_5]|_{\hat{\mathcal{P}}}) \\
\downarrow & & \downarrow \\
H^{d-1}(\Gamma, H^0(\hat{A}, \hat{P})) & \xrightarrow{\hat{U}^n_t} & H^{d-1}(\Gamma, H^0(\hat{A}, \hat{P}))
\end{array}
\]

where the vertical maps come from restriction to $\hat{A}_x$ under the usual identifications. Thus, it remains to show

\[
\hat{U}^n_t(EK_{\mathcal{P}^n} f) = EK_{A' : A^{\dagger}}(\hat{f}).
\]
The translation morphisms $U^n_{\Gamma}$ fit into a commutative diagram

$$
\begin{array}{c}
\begin{align*}
H^{d-1}(A \setminus A[c], \Gamma, \tilde{\mathcal{P}}) \\ H^d_{\Gamma}(A, \Gamma, \tilde{\mathcal{P}}) \\ H^0(A[c], \tilde{\mathcal{P}}|_{A[c]} \Gamma)
\end{align*}
\end{array}
\begin{array}{c}
\xrightarrow{U^n_{\Gamma}}
\xrightarrow{\text{res}}
\xrightarrow{\text{incl}}
\end{array}
\begin{array}{c}
\begin{align*}
H^{d-1}(A' \setminus A'[\Sigma], \Gamma, [\Sigma]^* \tilde{\mathcal{P}}) \\ H^d_{A'[\Sigma]}(A', \Gamma, [\Sigma]^* \tilde{\mathcal{P}}) \\ H^0(A'[\Sigma], [\Sigma]^*(\tilde{\mathcal{P}}|_{A[c]})) \Gamma
\end{align*}
\end{array}
\end{array}
$$

where all horizontal maps are induced by $\circ [\Sigma]^* \circ (5.5.2)$ and the vertical maps are the residue maps, respectively the inclusions defined in section 2.5. Since $EK_{A', \varphi}(\hat{f})$ is uniquely determined by its image under the residue map, it suffices to show

$$U^n_{\Gamma}(i(f)) = i'(\hat{f}).$$

where $i$ resp. $i'$ denotes the inclusion $H^0(A[c], \mathcal{O}|_{A[c]}) \to H^0(A[c], \tilde{\mathcal{P}}|_{A[c]})$ respectively $H^0(A'[\Sigma], \mathcal{O}|_{A'[\Sigma]}) \to H^0(A'[\Sigma], \tilde{\mathcal{P}}|_{A'[\Sigma]})$. The inclusions $i$ and $i'$ have canonical retractions $r$ and $r'$ induced by

$$r: H^0(A[c], \tilde{\mathcal{P}}|_{A[c]}) = H^0(A[c] \times \overset{\sim}{A'}, \mathcal{P}|_{A[c] \times \overset{\sim}{A'}}) \xrightarrow{(\text{id} \times x)^*} H^0(A[c], \mathcal{O}|_{A[c]})$$

respectively

$$r': H^0(A'[\Sigma], [\Sigma]^*(\tilde{\mathcal{P}}|_{A[c]})) \xrightarrow{(\text{id} \times x)^*} H^0(A'[\Sigma], \mathcal{O}|_{A'[\Sigma]})$$

and it suffices to show

$$r'(U^n_{\Gamma}(i(f))) = \hat{f}.$$ 

The commutative diagram

$$
\begin{array}{c}
\begin{align*}
H^0(A[c], \tilde{\mathcal{P}}|_{A[c]} \Gamma) \\ H^0(A[c], \mathcal{O}|_{A[c]} \Gamma)
\end{align*}
\end{array}
\begin{array}{c}
\xrightarrow{(\cdot)|_{A'[\Sigma]} \circ U^n_{\Gamma}}
\xrightarrow{i}
\xrightarrow{r'}
\end{array}
\begin{array}{c}
\begin{align*}
H^0(A'[\Sigma], [\Sigma]^* \tilde{\mathcal{P}}|_{A'[\Sigma]} \Gamma) \\ H^0(A'[\Sigma], \mathcal{O}|_{A'[\Sigma]} \Gamma)
\end{align*}
\end{array}
\end{array}
$$

shows the formula

$$r'(U^n_{\Gamma}(i(f)))(c, 0) = f([\Sigma]^* c) = \hat{f}(c, 0).$$

The general formula will follow from the following claim:

**Claim:** For $s \in A'[\Sigma]$ we have $T^s_{\Gamma} r'(U^n_{\Gamma}(i(f)))(s, t|[\Sigma]) r'(U^n_{\Gamma}(i(f)))$. 

**Proof of the Claim:** Let us first recall the definition of Oda’s pairing: Let $\varphi: A' \to A$ be an isogeny and $\mathcal{L}$ a line bundle defining a torsion point $[\mathcal{L}] \in \ker \varphi^\vee$ in $A^\vee$. In particular there is an isomorphism $\alpha: \mathcal{O}_{A'} \cong \varphi^* \mathcal{L}$. For given $s \in \ker \varphi$ let us consider the $\mathcal{O}_{A'}$-linear isomorphism

$$\mathcal{O}_{A'} \xrightarrow{\alpha} \varphi^* \mathcal{L} = T^s_{\Gamma} \varphi^* \mathcal{L} \xrightarrow{T^s_{\alpha^{-1}}} \mathcal{O}_{A'}.$$
It’s deg $\varphi$-th power is the identity, so it is given by a deg $\varphi$-th root of unity. This root of unity is $(s, [\mathcal{L}])_{\varphi}$. If we apply this to $\mathcal{L} = (\text{id} \times \tau)^* \mathcal{P}$ we get a commutative diagram

$$
\begin{array}{ccc}
([\mathcal{P}]_\Sigma) \times \tau)^*(\mathcal{P}|_{A[\mathcal{L}] \times A[\mathcal{L}^\vee]}) & \to & \mathcal{O}_{A'[\mathcal{P}]_{\Sigma}} \\
|_{\text{id}} & & |_{(t, \tau)^*}\mathcal{P}_\Sigma \\
T_t^*(([\mathcal{P}]_\Sigma) \times \tau)^*(\mathcal{P}|_{A[\mathcal{L}] \times A[\mathcal{L}^\vee]}) & \to & \mathcal{O}_{A'[\mathcal{P}]_{\Sigma}}.
\end{array}
$$

Unwinding the definitions, we see that while 

$$H^0(A'[\mathcal{P}]_{\Sigma}), ([\mathcal{P}]_\Sigma) \times \tau)^*(\mathcal{P}|_{A[\mathcal{L}] \times A[\mathcal{L}^\vee]}) \to H^0(A'[\mathcal{P}]_{\Sigma}), \mathcal{O}_{A'[\mathcal{P}]_{\Sigma}})
$$

while $T_t^*(U_{t}^{\text{res}}(i(f)))$ is the image of $([\mathcal{P}]_\Sigma) \times \tau)^*(\mathcal{P}|_{A[\mathcal{L}] \times A[\mathcal{L}^\vee]})$ under 

$$H^0(A'[\mathcal{P}]_{\Sigma}), ([\mathcal{P}]_\Sigma) \times \tau)^*(\mathcal{P}|_{A[\mathcal{L}] \times A[\mathcal{L}^\vee]}) \to H^0(A'[\mathcal{P}]_{\Sigma}), \mathcal{O}_{A'[\mathcal{P}]_{\Sigma}}).$$

The Claim follows now from the commutativity of (5.5.6).

We conclude the proof by deriving the desired formula $r'(U_{t}^{\text{res}}(i(f))) = \hat{f}$ from the above claim:

$$r'(U_{t}^{\text{res}}(i(f)))(c, s) = T_t^* r'(U_{t}^{\text{res}}(i(f)))(c, 0) = (s, t)_{\mathcal{P}_\Sigma} r'(U_{t}^{\text{res}}(i(f)))(c, 0) =
$$

$$= (s, t)_{\mathcal{P}_\Sigma} f([\mathcal{P}]_\Sigma) = \hat{f}(c, s).$$

\[\square\]

**Definition 5.20.** For $\rho: \mathcal{O}_L/p^n \mathcal{O}_L \to \overline{\mathbb{Q}}$ let us define the partial Fourier transform $P\rho: A[p^n] \to \overline{\mathbb{Q}}$ by 

$$(P\rho)(s, \hat{f}) := \frac{1}{p^{dn}} \sum_{t \in A[\mathcal{P}]_{\Sigma}} (t, \hat{f})_{[p^n]}^{-1} \rho(t, s).$$

Here, we write $\rho(t, s)$ with $(t, s) \in A[\mathcal{P}]_{\Sigma} \times A[\mathcal{P}]_{\Sigma}$ and use the identification 

$$\mathcal{O}_L/p^n \mathcal{O}_L \cong A[\mathcal{P}]_{\Sigma} \times A[\mathcal{P}]_{\Sigma}$$

induced by the $\Gamma_0(p^\infty)$-structure.

As an immediate Corollary of the above Proposition, we deduce the following formula for integration of locally constant functions on the Tate module:

**Corollary 5.21.** Assume that $x \in \ker \varphi$ with $\varphi^\vee$ étale. For a function $\rho: A[\mathcal{P}]_{\Sigma} \times A[\mathcal{P}]_{\Sigma} \to \mathcal{O}_{\mathbb{C}_p}$ we have the integration formula

$$\frac{1}{\Omega_p^{\alpha+1} \Omega_q^{\beta}} \int_{\mathcal{O}_L \otimes \mathbb{Z}_p} t^a s^b \rho(t, s) d\mu_E(x) = \sum_{s' \in A[p^n]} (P\rho)(s') EK^\alpha_{\mathbb{R}}(f, x+s')(\omega(A)^{[\alpha+1]}, \omega(A)^{[\beta]}).$$

**Proof.** This follows immediately by combining Proposition 5.18 and Theorem 5.17 \[\square\]

5.6. **$p$-adic Hecke characters and the local factor.** Let $\hat{f}$ be a fractional ideal which is prime to $p$. Let $\chi: \mathcal{I}(\hat{f}) \to \overline{\mathbb{Q}}^\times$ be an algebraic Hecke character of conductor dividing $\hat{f}$ and of infinity type $\mu = \beta - \alpha \in I_L$ with $\alpha \in I_{\mathbb{C}}$ and $\beta \in I_{\mathbb{R}}$, i.e.

$$\chi((\lambda)) = \prod_{\sigma \in J_L} \sigma(\lambda)^{\mu(\lambda)}, \quad \lambda \in \mathcal{P}_f. $$

The fixed embedding $\overline{\mathbb{Q}} \to \mathbb{C}_p$ allows us to view $\chi$ as a character 

$$\chi: \mathcal{I}(\hat{f}) \to \mathbb{C}_p.$$
We deduce from (5.6.1) that \( \chi(I(pf)) \subseteq O_{C_p} \) and \( \chi(P_{pf}^n) \subseteq 1 + p^nO_{C_p} \). Passing to the inverse limit allows us to view \( \chi \) as a character
\[
\chi : \lim_n I(pf)/P_{pf}^n \rightarrow O_{C_p}^\times.
\]
By class field theory, we obtain an isomorphism \( \lim_n I(pf)/P_{pf}^n \cong \text{Gal}(L(p^\infty f)/L) \) and we may view \( \chi \) as a character of the Galois group
\[
\chi : \text{Gal}(L(p^\infty f)/L) \rightarrow O_{C_p}^\times.
\]
Similar as in [Kat78] we will associate a local term \( \text{Local}(\chi, \Sigma) \in \overline{Q} \) to the Hecke character \( \chi \) and the CM type \( \Sigma \). Above, we have defined the partial Fourier transform
\[
P : \text{Hom}(O_L/p^mO_L, \overline{Q}) \rightarrow \text{Hom}(A[p^m], \overline{Q}).
\]
The \( \Gamma_{00}(p^\infty) \)-structure gives us an isomorphism
\[
A[p^m] \cong O_L(\Sigma) \otimes \mathbb{Z}_p (\mu_{p^m} \times \mathbb{Z}/p^m\mathbb{Z}).
\]
Using our fixed embedding \( \overline{Q} \subseteq \mathbb{C} \) we have an isomorphism
\[
\mathbb{Z}/p^m\mathbb{Z} \overset{\sim}{\rightarrow} \mu_{p^m}, \ x \mapsto \exp\left(\frac{2\pi i x}{p^m}\right).
\]
Combining the above isomorphisms allows us to identify
(5.6.2)
\[
O_L/p^m \overset{\sim}{\rightarrow} A[p^m]
\]
and we get
\[
\tilde{P} : \text{Hom}(O_L/p^mO_L, \overline{Q}) \rightarrow \text{Hom}(O_L/p^mO_L, \overline{Q}), \quad \rho \mapsto \tilde{P}\rho := (P\rho) \circ (5.6.2).
\]
Let us now construct the local factors of the Hecke character \( \chi \) and the CM type \( \Sigma \). We have where \( \alpha = N_{L/K}\alpha_0, \beta = N_{L/K}\beta_0 \) for \( \alpha_0 \in I_{\Sigma_K} \) and \( \beta_0 \in I_{\Sigma_K} \). Let us denote by \( \chi_{fin} \) the unique locally constant character
\[
\chi_{fin} : (O_L \otimes \mathbb{Z}_p)^\times \rightarrow \overline{Q}^\times
\]
which satisfies
\[
\chi((\lambda)) = \chi_{fin}(\lambda) \frac{N_{L/K}(\lambda)^{\alpha_0}}{N_{L/K}(\lambda)^{\alpha_0}}
\]
for \( \lambda \in P_{pf} \). Let us write
\[
F : (O_L \otimes \mathbb{Z}_p)^\times = (O_L(\Sigma) \oplus O_L(\Sigma))^\times \rightarrow \overline{Q}^\times
\]
for the locally constant function given by
\[
F(x, y) := \chi_{fin}(x^{-1}, y).
\]
For \( p \in \Sigma_p \) and \( \overline{p} \in \Sigma_p \) let us write \( a_p \) and \( b_{\overline{p}} \) for the exact power of \( p \) respectively \( \overline{p} \) in \( \text{Cond}(\chi) \). Let us write
\[
\prod_{p \in \Sigma_p} p^{a_p} \prod_{\overline{p} \in \Sigma_p} \overline{p}^{b_{\overline{p}}} = (c)b
\]
with \( b \) a prime-to-\( p \) fractional ideal and \( c \in P_f \).

**Definition 5.22.** The local factor of \( \chi \) and \( \Sigma \) is defined as
\[
\text{Local}(\chi, \Sigma) := \frac{(N_{L/K}(c)^{\alpha_0}(PF)(c^{-1})(N_{L/K}(c)^{\beta_0}\chi(b))}{N_{L/K}(c)^{\beta_0}}.
\]
5.7. \textit{$p$-adic interpolation of Hecke $L$-values.} In the following, we would like to construct a $p$-adic measure on $\text{Gal}(L(p^{\infty})/L)$ interpolating all critical Hecke $L$-values for Hecke characters of conductor dividing $p^{\infty}$ where $\mathfrak{f}$ is a prime-to-$p$ fractional ideal of $L$. From here on, the exposition is quite similar to the one given in \cite{Kat78}. The following Theorem provides a $p$-adic measure $\mu_{i,\mathfrak{f}}$ on $\text{Gal}(L(p^{\infty})/L)$ which interpolates $p$-adically the critical $L$-values for Hecke characters of conductor dividing $p^{\infty}$\mathfrak{f}. This is a generalization of results of Katz in case of a CM field \cite{Kat78}.

\textbf{Theorem 5.23.} For every fractional ideal $\mathfrak{f}$ and every auxiliary fractional ideal $\mathfrak{c}$ co-prime to $\mathfrak{p}\mathfrak{f}$ there exists a $p$-adic measure $\mu_{i,\mathfrak{f}}$ on $\text{Gal}(L(p^{\infty})/L)$ with the following interpolation property: For every Hecke character $\chi$ of critical infinity type $\beta - \alpha$ and conductor dividing $p^{\infty}$\mathfrak{f}, we have:

\[
\frac{1}{\Omega_{p}^{\alpha}}\Omega_{p}^{\beta} \int_{\text{Gal}(L(p^{\infty})/L)} \chi(g) d\mu_{i,\mathfrak{f}}(g) = \text{Local}(\chi, \Sigma)(\mathcal{O}_{L}^{\infty}: \Gamma)(N\mathfrak{c} - \chi(e^{-1})) \prod_{p \in \Sigma_{p}} \left(1 - \frac{\chi(p)}{Np}\right) \frac{(\alpha - \beta)!(2\pi i)^{\beta}}{\Omega_{p}^{\alpha} \Omega_{p}^{\beta}} L_{\mathfrak{f}}(\chi, 0)
\]

Note that in the case of $L$ an extension of an imaginary quadratic field, such an interpolation was obtained by Colmez-Schneps \cite{CS92} for the $L$-values of Hecke characters treated in \cite{Col89}.

\textbf{Proof.} By class field theory, we have a short exact sequence

\[
0 \rightarrow G \rightarrow \text{Gal}(L(p^{\infty})/L) \rightarrow \text{Gal}(L(\mathfrak{f})/L) \rightarrow 0.
\]

with $G = (\mathcal{O}_{L} \otimes \mathbb{Z}_{p})^{\infty}/E(\mathcal{O}_{L}^{\infty})$ where $E(\mathcal{O}_{L}^{\infty})$ denotes the closure of $\mathcal{O}_{L}^{\infty}$ in $(\mathcal{O}_{L} \otimes \mathbb{Z}_{p})^{\infty}$. By abuse of notation, let us write $a \in \text{Gal}(L(p^{\infty})/L)$ for the image of a fractional ideal under the Artin map. Thus, if we choose prime-to-$p$ fractional ideals $\mathfrak{a}_{i}$ which are representatives of $\text{Gal}(L(\mathfrak{f})/L)$ we get an explicit co-set decomposition

\[
\text{Gal}(L(p^{\infty})/L) = \bigcup_{i=1}^{h_{\mathfrak{f}}} \mathfrak{a}_{i}G.
\]

By means of this decomposition it is equivalent to give $h_{\mathfrak{f}}$ distinct measures $\mu(\mathfrak{a}_{i})$ on $G \cong \mathfrak{a}_{i}G$:

\[
\int_{\text{Gal}(L(p^{\infty})/L)} \rho d\mu_{i,\mathfrak{f}} = \sum_{i=1}^{h_{\mathfrak{f}}} \int_{G} f(a, g) d\mu(\mathfrak{a}_{i})(g).
\]

Let us now define the measures $\mu(\mathfrak{a}_{i})$: Let $(\mathcal{A}/\mathcal{R}, \Sigma, \mathcal{O}_{L}, \theta, \omega(\mathcal{A}), \omega(\mathcal{A}^{\vee}), \theta_{p}, \theta_{p}^{\vee})$ be a tuple as in \textbf{5.2}. By \textbf{5.6} the Serre construction gives for every prime-to-$p$ fractional ideal $\mathfrak{a}$ a new tuple

\[
(A_{\mathfrak{a}}, \omega(A_{\mathfrak{a}}), \omega(A_{\mathfrak{a}}^{\vee}), \theta_{p, \mathfrak{a}}, \theta_{p, \mathfrak{a}}^{\vee})
\]

and an isomorphism $\theta'_{\mathfrak{a}}: A_{\mathfrak{a}} \cong X(\mathfrak{a})$. For each fractional ideal $\mathfrak{a}_{i}$ let $x_{i} \in A_{\mathfrak{a}_{i}}(\overline{\mathbb{Q}})$ be a torsion point corresponding to $1 \in \mathbb{C}^{X}$ under $\theta$. Define

\[
\mu'(\mathfrak{a}_{i}) := \mu_{\text{Eis}}(f_{i}, x_{i})|_{(\mathcal{O}_{L} \otimes \mathbb{Z}_{p})^{\times}} \in \text{Meas}((\mathcal{O}_{L} \otimes \mathbb{Z}_{p})^{\times}, \mathcal{O}_{\mathbb{C}_{p}})^{\Gamma} = \text{Meas}((\mathcal{O}_{L} \otimes \mathbb{Z}_{p})^{\times}/\Gamma, \mathcal{O}_{\mathbb{C}_{p}})
\]

as the restriction of the $p$-adic Eisenstein measure associated with

\[
(A_{\mathfrak{a}_{i}}, \omega(A_{\mathfrak{a}_{i}}), \omega(A_{\mathfrak{a}_{i}}^{\vee}), x_{i}, \theta_{p, \mathfrak{a}_{i}}, \theta_{p, \mathfrak{a}_{i}}^{\vee})
\]
for the group \( \Gamma := \mathcal{O}_L^\times \). Note that the measures \( \mu'(a_i) \) are measures on \( (\mathcal{O}_L \otimes \mathbb{Z}_p)^\times / E(\Gamma) \), where \( E(\Gamma) \) denotes the closure of \( \Gamma \) in \( \mathcal{O}_L \otimes \mathbb{Z}_p \). Let us write

\[
q: (\mathcal{O}_L \otimes \mathbb{Z}_p)^\times / E(\Gamma) \to (\mathcal{O}_L \otimes \mathbb{Z}_p)^\times / E(\mathcal{O}_L^\times)
\]

for the projection. The desired measure \( \mu(a_i) \) is obtained as follows:

\[
(5.7.1) \quad \int_{(\mathcal{O}_L \otimes \mathbb{Z}_p)^\times / \mathcal{O}_L^\times} \rho d\mu(a_i) := \int_{(\mathcal{O}_L \otimes \mathbb{Z}_p)^\times / E(\Gamma)} (q \circ \rho)(t^{-1}, s)t^{-1}d\mu'(a_i)(t, s).
\]

We define the measure \( \mu_{f, \epsilon} \) using the formula

\[
(5.7.2) \quad \int_{\text{Gal}(L(p^{\infty}f)/L)} \rho d\mu_{f, \epsilon} = \sum_{i=1}^{b_i} \int_G f(a_i g) d\mu(a_i)(g).
\]

It remains to check that this measure satisfies the desired interpolation property. From here on, we can follow Katz’ argument quite closely. Let \( \chi \) be a Hecke character of critical infinity type \( \beta - \alpha \). Without loss of generality, we may assume that \( f \) is the prime-to-\( p \) part of the conductor of \( \chi \). Let us denote by \( \chi_{fin} \) the unique locally constant character

\[
\chi_{fin}: (\mathcal{O}_L \otimes \mathbb{Z}_p)^\times \to \overline{\mathbb{Q}}^\times
\]

which satisfies

\[
\chi((\lambda)) = \frac{\chi_{fin}(\lambda) \overline{N_{L/K}(\lambda)}^{\frac{a_0}{b}}}{N_{L/K}(\lambda)^{a_0+1}},
\]

for \( \lambda \in \mathcal{P} \) co-prime to \( p \). Let us write

\[
F: (\mathcal{O}_L \otimes \mathbb{Z}_p)^\times = (\mathcal{O}_L(\Sigma) \oplus \mathcal{O}_L(\Sigma))^\times \to \overline{\mathbb{Q}}^\times
\]

for the locally constant function given by

\[
F(x, y) := \chi_{fin}(x^{-1}, y).
\]

The partial Fourier transform \( PF \) is supported in \( \mathcal{A}[p^a \overline{p}^b] \), where \( a \) and \( b \) are the multi-indices of exact orders of \( p \) and \( \overline{p} \) in the conductor of \( \chi \). By \( 5.21 \) we get

\[
(5.7.3) \quad \frac{1}{\Omega_p^{\nu'b}_p} \int_{\text{Gal}(L(p^{\infty}f)/L)} \chi(g) d\mu(g) = [\mathcal{O}_L^\times : \Gamma] \times \sum_{i=1}^{b_i} \chi(a_i) \sum_{s' \in A_{a_i}[p^{\overline{p}}]} (PF)(s') E_{1, \epsilon}(s', x_i + s') (\omega(A_{a_i})^{[a]}, \omega(A_{a_i}^{\vee})^{[\beta]}).
\]

Here, the factor \([\mathcal{O}_L^\times : \Gamma]\) appears since \( \mu(a_i) \) is defined as the pullback along \( q \), c.f. \( 5.7.1 \). Let us first assume that \( a_i \geq 1 \) for all \( i \in \Sigma_p \). The isomorphism \( A_{a_i}(\mathbb{C}) \cong \mathbb{C}^{\Sigma} / \Lambda_{a_i}^{-1} \) allows us to write

\[
A_{a_i}[p^a \overline{p}^b] \cong p^{-a} \overline{p}^{-b} a_i^{-1} / f a_i^{-1} = (c^{-1}) b^{-1} f a_i^{-1} / f a_i^{-1},
\]

where \( c \) and \( b \) are defined in section \( 5.16 \). For \( \lambda \in b^{-1} f a_i^{-1} \) a direct computation shows

\[
(PF)(c^{-1} \lambda) = (PF)(c^{-1}) \cdot \chi_{fin}(\lambda)
\]
and together with Theorem 3.20 we obtain the formula:

\[
\left(\frac{(\alpha - 1)![(2\pi i)|\beta|]}{\Omega(A^\alpha\Omega(A^\vee)^\beta)}\right)^{-1} \sum_{s' \in A_{\alpha_s}[p^s\mathbb{F}]} (PF)(s')EK_{\Gamma}^{\beta, \alpha - 1}(f_{c, x_i} + s')(\omega(\mathcal{A}_{a_i})^a, \omega(\mathcal{A}_{a_i}^\vee)^\beta) = \\
= \frac{N_{L/K}(c)^{\alpha} (\tilde{F}) (c^{-1})}{N_{L/K}(c)^\beta} \left( N\xi \sum_{\chi \in \Gamma \setminus (1 + b^{-1}f_{a_i}^{-1})} \frac{\chi_{fin}(\lambda)\lambda^\beta}{\lambda^\alpha N(\lambda)^s} - \sum_{\chi \in \Gamma \setminus (1 + c^{-1}b^{-1}f_{a_i}^{-1})} \frac{\chi_{fin}(\lambda)\lambda^\beta}{\lambda^\alpha N(\lambda)^s} \right) \bigg|_{s=0} = \\
= \frac{N_{L/K}(c)^{\alpha} (\tilde{F}) (c^{-1})}{N_{L/K}(c)^\beta} \chi(a_i^{-1}) \left( NcL_{pf}(\chi, 0, [ba_i]) - \chi(c^{-1})L_{pf}(\chi, 0, [bca_i]) \right) = \\
= \text{Local}(\chi, \Sigma) \chi(a_i^{-1}) \left( NcL_{pf}(\chi, 0, [ba_i]) - \chi(c^{-1})L_{pf}(\chi, 0, [bca_i]) \right).
\]

Summing over \( a_i \) gives

\[
\frac{1}{\Omega_p^\alpha \Omega_p^\beta} \int_{\text{Gal}(L(p^{\infty}f)/L)} \chi(g) d\mu_{i,c}(g) = \\
= a![(2\pi i)|\beta|] \Omega_p^\alpha \Omega_p^\beta \text{Local}(\chi, \Sigma)[O_L^\times : \Gamma](N\xi - \chi(c^{-1}))L_{pf}(\chi, 0).
\]

Since \( \chi(p) = 0 \) the claim follows in the case \( a_i \geq 1 \).

The general case follows essentially the same argument. We refer the reader to the computations in [Kat78].

\[\square\]

**Appendix A. Equivariant cohomology**

**A.1. Equivariant coherent cohomology.** Consider pairs \((X, \mathcal{F})\) consisting of a scheme \(X\) and a sheaf \(\mathcal{F}\) on it. A morphism \((f, f^\#) : (X, \mathcal{F}) \to (Y, \mathcal{G})\) consists of a map of schemes \(f : X \to Y\) and a sheaf homomorphism \(f^\# : \mathcal{G} \to f_* \mathcal{F}\) (or equivalently \(f^* \mathcal{G} \to \mathcal{F}\)).

**Definition A.1.** Let \(\Gamma\) be a group acting on \(X\) (from the left). A \(\Gamma\)-equivariant sheaf \(\mathcal{F}\) on \(X\) is a group homomorphism

\[
\Gamma \to \text{Aut}(X, \mathcal{F})
\]

such that the induced action on \(X\) coincides with the given one.

Let \(\pi : X \to S\) be morphism of schemes and \(\Gamma\) be a group acting on \(X/S\), i.e., each \(\gamma \in G\) gives an automorphism \(\gamma : X \to X\) over \(S\). The higher direct images \(\pi_* \mathcal{F}\) of a \(\Gamma\)-equivariant sheaf \(\mathcal{F}\) inherit a \(\Gamma\)-action and one can consider the subsheaf of \(\Gamma\)-invariant sections \(\pi_*^\Gamma \mathcal{F}\).

**Definition A.2.** The derived functors of \(\mathcal{F} \mapsto H^0(X, \mathcal{F})^\Gamma\) are the equivariant coherent cohomology groups which we denote by

\[
H^i(X, \Gamma ; \mathcal{F}).
\]

Immediate from this definition is the spectral sequence

\[
(A.1.1) \quad E_2^{p,q} = H^p(\Gamma, H^q(X, \mathcal{F})) \Rightarrow H^{p+q}(X, \Gamma ; \mathcal{F}).
\]
A.2. Borel-Equivariant sheaf cohomology. In this section we show that the coherent equivariant cohomology can be computed from a topological model.

We briefly recall the comparison between equivariant sheaf cohomology as derived cohomology of the invariant sections and Borel’s construction on the classifying space. We will only concentrate on the case of finitely generated free abelian groups. Let $\Gamma$ be a finitely generated free abelian group. We use $E = \Gamma \otimes \mathbb{Z} \mathbb{R} \to B = (\Gamma \otimes \mathbb{Z} \mathbb{R})/\Gamma$ as a model for the universal $\Gamma$-bundle $E$ over the classifying space $B$. Let $X$ be a topological $\Gamma$-space with a $\Gamma$-equivariant sheaf $F$ (for example $X$ a scheme or a manifold). By the $\Gamma$-equivariance of $F$ the pullback $pr^{-1}_X F$ of $F$ along $E \times X \to X$ descends to a sheaf $\tilde{F}$ on the topological space $E \times \Gamma X$. Let us define

$$H^i_{\Gamma}(X, F) := H^i(E \times \Gamma X, \tilde{F}).$$

Let us quickly recall the proof that $H^i_{\Gamma}(X, F)$ coincides with the derived functor of $\Gamma(X, F)^{\Gamma}$. We have $H^0_{\Gamma}(X, F) = \Gamma(X, F)^{\Gamma}$, so let us show that $H^0_{\Gamma}(X, \mathcal{I}) = 0$ for an injective sheaf $\mathcal{I}$. More precisely, we show that $\tilde{I}$ is a flabby sheaf on $E \times \Gamma X$: Since $\mathcal{I}$ is injective in the category of $\Gamma$-equivariant sheaves on $X$, the restriction homomorphism $\Gamma(X, \mathcal{I})^{\Gamma} = \text{Hom}_{\Gamma-Shv} (\mathbb{Z}, \mathcal{I}) \to \Gamma(U, \mathcal{I})^{\Gamma} = \text{Hom}_{\Gamma-Shv} (j_* \mathbb{Z}, \mathcal{I})$ is surjective for an open $\Gamma$-subspace $j : U \subseteq X$. On the other hand, we have $\Gamma(X, \mathcal{I})^{\Gamma} = \Gamma(E \times \Gamma X, \tilde{F})$ and $\Gamma(U, \mathcal{I})^{\Gamma} = \Gamma(E \times \Gamma U, \tilde{F})$.

By choosing an injective resolution of a bounded complex of equivariant sheaves $F^\bullet \to I^\bullet$ we see in particular, that

$$\mathbb{H}^i(X, \Gamma, F^\bullet) \to \mathbb{H}^i(E \times \Gamma X, \tilde{F}^\bullet)$$

computes the equivariant hypercohomology of $F^\bullet$. Applying this to the de Rham complex allows us to compute the equivariant de Rham cohomology of a scheme (or manifold) $X$ on $E \times \Gamma X$.

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Fakultät für Mathematik, Universität Regensburg, 93040 Regensburg, Germany