The polytropic hydrodynamic vortex describes an effective (2+1)-dimensional acoustic spacetime with an inner reflecting boundary at $r = r_c$. This physical system, like the spinning Kerr black hole, possesses an ergoregion of radius $r_e$ and an inner non-pointlike curvature singularity of radius $r_s$. Interestingly, the fundamental ratio $r_e/r_s$ which characterizes the effective geometry is determined solely by the dimensionless polytropic index $N_p$ of the circulating fluid. It has recently been proved that, in the $N_p = 0$ case, the effective acoustic spacetime is characterized by an infinite countable set of reflecting surface radii, $\{r_c(N_p,n)\}_{n=1}^{\infty}$, that can support static (marginally-stable) sound modes. In the present paper we use analytical techniques in order to explore the physical properties of the polytropic hydrodynamic vortex in the $N_p > 0$ regime. In particular, we prove that in this physical regime, the effective acoustic spacetime is characterized by a finite discrete set of reflecting surface radii, $\{r_c(N_p,m;n)\}_{n=1}^{N_{max}}$, that can support the marginally-stable static sound modes (here $m$ is the azimuthal harmonic index of the acoustic perturbation field). Interestingly, it is proved analytically that the dimensionless outermost supporting radius $r_{c,\text{max}}/r_e$, which marks the onset of superradiant instabilities in the polytropic hydrodynamic vortex, increases monotonically with increasing values of the integer harmonic index $m$ and decreasing values of the dimensionless polytropic index $N_p$.

I. INTRODUCTION

One of the most intriguing phenomena in general relativity is the dragging of inertial frames by rotating bodies [1]. This fundamental physical effect is demonstrated most dramatically in the spinning Kerr spacetime [2]. In particular, this black-hole solution of the classical Einstein field equations is characterized by an ergoregion [1], a spacetime region located between the horizon and the static surface [3] in which physical observers cannot appear static with respect to inertial observers at asymptotic spatial infinity.

Interestingly, it has been demonstrated more than four decades ago [4–6] that energy and angular momentum can be extracted from the ergoregion of a spinning black hole by a co-rotating cloud of bosonic (integer-spin) fields. Nevertheless, the Kerr spacetime solution of the Einstein field equations is known to be stable to linearized perturbations of massless fields [6–8]. This important physical feature of the spinning black-hole spacetime is attributed to the fact that a classical black-hole horizon acts as an absorbing one-way membrane. Thus, massless perturbation fields are eventually swallowed by the black hole or radiated away to infinity [6–12].

As opposed to the asymptotically flat Kerr black-hole spacetime, whose stability to massless bosonic perturbations is guaranteed by the absorbing nature of its horizon [4–8], horizonless spinning spacetimes with reflecting boundaries and ergoregions may develop characteristic superradiant instabilities to co-rotating bosonic fields [7, 13–15]. In particular, it has recently been demonstrated [7, 16, 17] that exponentially growing superradiant instabilities may develop in the effective ergoregions of rotating fluid systems [18–21].

The physically interesting polytropic hydrodynamic vortex studied in [16] describes an horizonless (2 + 1)-dimensional purely circulating fluid with an effective acoustic ergoregion of radius $r_e$ and an inner reflecting boundary of radius $r_c$. As explicitly proved in [7, 16, 17], the presence of an ergoregion together with the absence of an absorbing horizon, guarantee that the effective acoustic spacetime described by the rotating fluid is superradiantly unstable to linearized sound perturbation modes.

Intriguingly, for a given value of the dimensionless polytropic index $N_p$ which characterizes the circulating fluid [see Eq. (1) below], the horizonless acoustic spacetime of the hydrodynamic vortex is characterized by static (marginally-stable) resonances that mark the boundary between stable and superradiantly unstable spinning fluid configurations [7, 16, 17]. The main goal of the present paper is to study analytically the physical and mathematical properties of these characteristic marginally-stable acoustic spacetime resonances.
II. DESCRIPTION OF THE SYSTEM

The polytropic hydrodynamic vortex describes a \((2+1)\)-dimensional purely circulating fluid which is characterized by the compact pressure-density functional relation \[P(\rho) = k_p \rho^{1+1/N_p},\] (1)
where the coefficient \(k_p\) is the polytropic constant and the dimensionless constant \(N_p\) is the polytropic index of the fluid \[22\]. This physical system is described by the effective acoustic geometry \[16\]
\[ds^2 = \frac{\rho}{c_s} \left[ -c_s^2 dt^2 + \left( r d\theta - v_\theta dt \right)^2 + dr^2 + dz^2 \right],\] (2)
where the radially dependent physical parameters \(\rho = \rho(r)\) and \(v_\theta = v_\theta(r)\) are respectively the mass density and the angular velocity of the fluid, and \(c_s\) is the propagation speed of sound perturbation modes in the fluid.

Interestingly, as explicitly shown in \[16\], the effective acoustic spacetime of the polytropic hydrodynamic vortex, like the spinning Kerr black-hole spacetime, is characterized by a non-pointlike singularity of radius \(r_s\) \[23\]. At this inner radius, the mass density \(\rho\) of the fluid and the speed \(c_s\) of sound modes go to zero, signaling that the scalar curvature of the corresponding acoustic geometry becomes singular \[16\]. In addition, the spinning acoustic spacetime \[2\], which describes the polytropic hydrodynamic vortex, is characterized, like the familiar Kerr black-hole spacetime, by an ergoregion of radius \(r_e\) \[16, 24\].

As explicitly proved in \[16\], the dimensionless ratio \(r_e/r_s\) between the radius of the ergoregion and the radius of the inner singularity which characterizes the polytropic hydrodynamic vortex is determined solely by the polytropic index \(N_p\) of the fluid \[16\]:
\[\frac{r_e}{r_s} = \sqrt{1 + 2N_p}.\] (3)

Below we shall focus on the physical regime \(N_p > 0\), in which case one finds from \(3\) the inequality \(r_s < r_e\), a relation which also characterizes the rotating Kerr black-hole spacetime.

The spatial and temporal properties of linearized perturbations modes in the effective acoustic spacetimes \[2\] are governed by the Klein-Gordon equation \[7, 16, 19, 25\]
\[\nabla^\nu \nabla_\nu \Psi = \frac{1}{\sqrt{|g|}} \partial_\mu \left( \sqrt{|g|} g^{\mu \nu} \partial_\nu \Psi \right) = 0.\] (4)
We shall henceforth focus on the static perturbation modes of the polytropic hydrodynamic vortex which, as nicely demonstrated numerically in \[16\], mark the onset of rotational instabilities in the acoustic curved spacetime \[2\]. Substituting the metric components of the curved line element \[2\] into \[4\], defining the dimensionless radial coordinate
\[x \equiv \frac{r}{r_s},\] (5)
and using the mathematical decomposition \[26\]
\[\Psi(x, \phi, z) = \frac{1}{\sqrt{x}} \sum_{m=-\infty}^{\infty} \psi_m(x) e^{im\phi},\] (6)
one obtains the radial differential equation \[16\]
\[\left[ x^2 (x^2 - 1) \frac{d^2}{dx^2} + 2N_p x \frac{d}{dx} + (2m^2 - 1)N_p - (x^2 - 1)(m^2 - \frac{1}{4}) \right] \psi_m(x) = 0\] (7)
which determines the spatial behavior of the static (marginally-stable) linearized perturbation modes that characterize the polytropic hydrodynamic vortex. Note that the radial wave equation \[7\] for the static resonant modes of the polytropic hydrodynamic vortex is invariant under the reflection transformation \(m \rightarrow -m\). Hence, we shall henceforth assume
\[m > 0\] (8)
without loss of generality.
Interestingly, the ordinary differential equation (7) can be solved analytically in terms of the familiar hypergeometric function [27], yielding the physically acceptable radial eigenfunction [16, 27, 28]

$$\psi_m(x; N_p) = A \cdot x^{\frac{1}{2} - m} {}_2F_1 \left[ \frac{N_p + m - \sqrt{N_p^2 + m^2(1 + 2N_p)}}{2}, \frac{N_p + m + \sqrt{N_p^2 + m^2(1 + 2N_p)}}{2}; 1 + m; x^{-2} \right]$$

(9)

for the static (marginally-stable) resonances of the polytropic hydrodynamic vortex, where $\psi (r = r_c) = 0$. In particular, we shall explicitly prove below that, in the $N_p > 0$ case, the acoustic spacetime is characterized by an infinite countable set of reflecting cylinder radii, $\{r_c(N_p = 0, m; n)\}_{n=1}^{\infty}$, that can support the static (marginally-stable) resonant acoustic modes. In the present paper we shall use analytical techniques in order to study the characteristic resonance condition (13) of the polytropic hydrodynamic vortex in the physical regime $N_p > 0$. In particular, we shall explicitly prove below that, in the $N_p > 0$ regime of the circulating fluid, the corresponding acoustic spacetime is characterized by a finite discrete set of reflecting cylinder radii, $\{r_c(N_p, m; n)\}_{n=1}^{N_{max}}$, that can support the marginally-stable static acoustic modes.

III. ONSET OF THE SUPERRADIANT INSTABILITIES IN THE POLYTROPIC HYDRODYNAMIC VORTEX

A necessary (though not a sufficient) condition for the existence of exponentially growing (superradiantly amplified) resonant modes in the polytropic hydrodynamic vortex is provided by the compact inequality [10]

$$r_c < r_s$$

(14)

The characteristic inequality (14) simply reflects the physical requirement that the acoustic ergoregion of the effective (2 + 1)-dimensional rotating spacetime be part of the fluid system.

Intriguingly, however, one finds that not every circulating fluid system with inner reflecting boundary conditions at $r_c < r_s$ is superradiantly unstable to acoustic perturbation modes [2, 10, 17]. In particular, for a given set of the dimensionless physical parameters $\{N_p, m\}$ that characterize the polytropic hydrodynamic vortex, there exists a critical cylinder radius

$$x_c^* = \max_n \{x_c(N_p, m; n)\}_{n=1}^{N_{max}}$$

(15)

which marks the boundary between stable and superradiantly unstable circulating fluid configurations. Specifically, circulating fluid systems with supporting radii in the physical regime $x_c > x_c^*(N_p, m)$ are stable to acoustic perturbation modes with azimuthal harmonic index $m$, whereas circulating fluid systems with supporting radii in the
physical regime \( x_c < x_c^*(N_p, m) \) are known to develop superradiant instabilities under sound perturbation modes with azimuthal harmonic index \( m \).

Below we shall explicitly demonstrate that, for certain values of the dimensionless physical parameters \( \{N_p, m\} \) which characterize the polytropic hydrodynamic vortex, the characteristic discrete set \( \{x_c(N_p, m; n)\}_{n=1}^{N_{\text{max}}} \) of dimensionless cylinder radii that can support the static (marginally-stable) acoustic resonant modes can be determined analytically.

### IV. GENERIC PROPERTIES OF THE POLYTROPIC HYDRODYNAMIC VORTEX

In this section we shall discuss three fundamental features of the polytropic hydrodynamic vortex: (1) the stability properties of the fundamental \( m = 1 \) sound modes, (2) the finite number of marginally-stable (static) acoustic modes, and (3) the existence of a lower bound on the dimensionless polytropic index \( N_p(m) \) of the marginally-stable acoustic resonances.

#### A. No marginally-stable sound modes for the fundamental \( m = 1 \) acoustic perturbations

We shall first prove that there are no marginally-stable (static) acoustic modes for the fundamental \( m = 1 \) perturbations of the polytropic hydrodynamic vortex. To this end, we note that the resonance condition (13) takes the remarkably simple form

\[
2F_1(0, N_p + 1; 2; x_c^{-2}) = 0
\]

for the \( m = 1 \) acoustic perturbation modes. Using the fact that \( 2F_1(0, b; c; z) = 1 \), one immediately deduces from (16) that, for the fundamental \( m = 1 \) acoustic perturbations, there are no static (marginally-stable) resonant modes. Furthermore, remembering that the static resonances mark the onset of superradiant instabilities in this physical system [5, 16, 17], our analysis in the present subsection indicates that the polytropic hydrodynamic vortex is stable to the fundamental \( m = 1 \) perturbation modes.

#### B. The number of static (marginally-stable) resonances is finite

As emphasized above, it has been proved [17] that for constant density fluids, which are characterized by the simple relation \( N_p = 0 \) [2, 17], there is an infinite countable set \( \{r_c(N_p = 0, m; n)\}_{n=1}^{\infty} \) of reflecting surface radii (with the property \( r_c(n \to \infty) \to 0 \) [17]) that can support the marginally-stable static acoustic modes.

On the other hand, as we shall now show, the polytropic hydrodynamic vortex in the physical regime \( N_p > 0 \) is characterized by a finite set of reflecting surface radii that can support the marginally-stable acoustic resonant modes. In particular, we point out that, for positive integer values of the composed physical parameter \( 20 \)

\[
N_c(N_p, m) = \frac{\sqrt{N_p^2 + m^2(1 + 2N_p)} - N_p - m}{2}, \quad (17)
\]

the characteristic resonance condition (13) is a polynomial equation of degree \( N_c \) in the dimensionless variable \( z \equiv x_c^{-2} \) which yields a finite number \( N_c \) of dimensionless cylinder radii that can support the marginally-stable (static) acoustic modes of the polytropic hydrodynamic vortex.

For non-integer values of the composed physical parameter \( N_c \) [see Eq. (17)], one can solve numerically the characteristic resonance condition (13) in order to determine the discrete set of static resonances which characterize the polytropic hydrodynamic vortex. Doing so, one finds (see Table II below) that, for \( N_c \notin \mathbb{N} \) with \( N_p \geq 1 \), the number \( N_r \) of resonances (or equivalently, the number \( N_r \) of supporting cylinder radii) is given by the simple relation \( N_r = \lfloor N_c \rfloor \).

In addition, taking cognizance of the fact that the supporting radii of the central reflecting cylinder are characterized by the dimensionless lower bound [see Eqs. (10) and (12)]

\[
x_c \geq 1, \quad (18)
\]

one may use the characteristic relation [31]

\[
2F_1(a, b; c; x_c = 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} \quad \text{for} \quad \Re(c - a - b) > 0
\]

\[
(19)
\]
of the hypergeometric function and the well known pole structure \[32\]

\[
\frac{1}{\Gamma(-n)} = 0 \quad \text{for} \quad n = 0, 1, 2, \ldots 
\]

(20)
of the Gamma functions to infer from the resonance equation \[13\] that, in the \(N_p < 1\) regime \[33\] and for positive integer values of the composed physical quantity \[34, 35\]

\[
\tilde{N}_c(N_p, m) \equiv \sqrt{\frac{N_p^2 + m^2(1 + 2N_p) + N_p - m}{2}} , \quad \text{for} \quad 0 < N_p < 1 ;
\]

(21)
a new resonant mode (characterized by the limiting dimensionless radius \(x_c = 1\)) is added to the discrete family of static (marginally-stable) resonant modes each time the integer \(\tilde{N}_c\) increases by one.

We therefore conclude that, as opposed to the \(N_p = 0\) case studied in \[7, 17\], the polytropic hydrodynamic vortex in the physical regime \(N_p > 0\) is characterized by a finite discrete set of dimensionless reflecting radii, \\{\(x_c(N_p, m; n)\)\}\(_{n=N_c}^{n=N_r}\), that can support the static (marginally-stable) acoustic resonances, where \[36–38\] [see Eqs. (17) and (21)]

\[
N_r(N_p, m) = \begin{cases} 
\left\lfloor \sqrt{\frac{N_p^2 + m^2(1 + 2N_p) + N_p - m}{2}} \right\rfloor & \text{for} \quad 0 < N_p < 1 ; \\
\left\lceil \sqrt{\frac{N_p^2 + m^2(1 + 2N_p) - N_p - m}{2}} \right\rceil & \text{for} \quad N_p \geq 1 .
\end{cases}
\]

(22)

In Table I we present the number of reflecting cylinder radii that can support the static resonant modes of the polytropic hydrodynamic vortex as deduced by directly solving numerically the resonance condition \[13\] for various values of the dimensionless physical parameters \(N_p\) and \(m\). It is worth pointing out that the numerical data presented in Table I agree with the compact analytical formula \(22\) for the (finite) number of acoustic resonant modes that characterize the polytropic hydrodynamic vortex in the physical regime \(N_p > 0\).

|  \(|N_p|  \) |  \(|m|  \) | \# of resonances |
|-------|-------|----------------|
|  \(\frac{1}{2}\)  | 1     | 0              |
|  \(\frac{1}{2}\)  | 2     | 0              |
|  \(\frac{1}{2}\)  | 3     | 0              |
|  \(\frac{1}{2}\)  | 4     | 1              |
|  \(\frac{1}{2}\)  | 5     | 1              |
| 1     | 1     | 0              |
| 1     | 2     | 1              |
| 1     | 3     | 1              |
| 1     | 4     | 1              |
| 1     | 5     | 2              |
| 3     | 1     | 0              |
| 3     | 2     | 1              |
| 3     | 3     | 2              |
| 3     | 4     | 2              |
| 3     | 5     | 3              |
| 5     | 1     | 0              |
| 5     | 2     | 1              |
| 5     | 3     | 2              |
| 5     | 4     | 3              |
| 5     | 5     | 4              |

TABLE I: Static (marginally-stable) acoustic resonant modes of the polytropic hydrodynamic vortex. We present, for various values of the dimensionless physical parameters \(N_p\) and \(m\), the number of reflecting cylinder radii that can support the static resonant modes as computed numerically from the resonance condition \[13\]. The numerically computed data agree with the compact analytical formula \(22\) which, in the \(N_p > 0\) regime, determines the (finite) number of static (marginally-stable) resonant modes that characterize the polytropic hydrodynamic vortex.

It is interesting to stress the fact that our results in the present subsection reveal the fact that the \(N_p \to 0\) limit of the polytropic hydrodynamic vortex is not continuous. In particular, while the \(N_p = 0\) case of constant density fluids
studied in [7, 17] is characterized, for every positive value of the azimuthal harmonic index $m \geq 1$, by an infinite countable set of marginally-stable acoustic resonances [17], in the $N_p \to 0^+$ limit of the polytropic hydrodynamic vortex and for a fixed value of the integer field parameter $m$ there are no static (marginally-stable) acoustic resonant modes [39].

C. Lower bound on the polytropic index $N_p(m)$ of the marginally-stable acoustic resonances

From the analytical formula (22) for the number $N_r(N_p, m)$ of marginally-stable acoustic resonant modes one deduces that, for a given value of the azimuthal harmonic index $m$, the static resonant modes of the polytropic hydrodynamic vortex can only exist in the regime [40, 41]

$$N_p(m) \geq N_p^{\text{min}}(m) = \frac{2(m+1)}{m(m+1)+2}.$$  \hspace{1cm} (23)

Equivalently, the inequality (23) provides, for a given value of the dimensionless polytropic index $N_p$ of the circulating fluid, the lower bound [42, 43]

$$m(N_p) \geq m^{\text{min}}(N_p) \equiv \left\{ \begin{array}{ll} \left\lfloor \frac{2-N_p+\sqrt{4+4N_p-7N_p^2}}{2N_p} \right\rfloor & \text{for } 0 < N_p < \frac{4}{7} ; \\
2 & \text{for } N_p \geq \frac{4}{7} \end{array} \right.$$  \hspace{1cm} (24)

on the values of the azimuthal harmonic indices that can trigger rotational instabilities in the polytropic hydrodynamic vortex. Interestingly, from the relation (24) one finds the simple asymptotic behavior

$$m^{\text{min}}(N_p \to 0) \to \infty ,$$  \hspace{1cm} (25)

which implies that only acoustic modes with large values $[m(N_p \ll 1) \geq \lceil 2/N_p \rceil \gg 1]$ of the azimuthal harmonic index $m$ can trigger rotational instabilities in the $N_p \ll 1$ regime.

V. UPPER BOUND ON THE SUPPORTING RADII OF THE MARGINALLY-STABLE (STATIC) RESONANT MODES

In the present section we shall explicitly prove that, for given dimensionless physical parameters $\{N_p, m\}$, one can derive an upper bound [which is stronger than the bound (14)] on the inner supporting radii $\{x_c(N_p, m; n)\}$ which characterize the marginally-stable resonant modes of the polytropic hydrodynamic vortex.

To this end, we shall first write the radial differential equation (7) in the form

$$x^2(x^2-1)\frac{d^2\Phi}{dx^2} + \left[2\alpha x(x^2-1) + 2N_p x^2\right] \frac{d\Phi}{dx} + \left[(\alpha^2 - \alpha - m^2 + \frac{1}{4})(x^2-1) + N_p(2\alpha - 1 + 2m^2)\right] \Phi = 0 ,$$  \hspace{1cm} (26)

where

$$\psi(x) = x^\alpha \cdot \Phi(x) \quad \text{with} \quad \alpha > 1/2 - m .$$  \hspace{1cm} (27)

Taking cognizance of the inner boundary condition (11) at the surface of the reflecting cylinder and the characteristic asymptotic behavior $\psi(x \to \infty) \to A \cdot x^{1/2-m}$ of the radial eigenfunction (9) [28], one deduces that the radial function $\Phi(x)$, which characterizes the marginally-stable (static) resonances of the polytropic hydrodynamic vortex, must have (at least) one extremum point, $x = x_{\text{peak}}$, with the properties [44]

$$\{ \Phi \neq 0 ; \quad \frac{d\Phi}{dx} = 0 ; \quad \frac{d^2\Phi}{dx^2} < 0 \} \quad \text{for} \quad x = x_{\text{peak}} ,$$  \hspace{1cm} (28)

where

$$x_{\text{peak}} \in (x_c, \infty) .$$  \hspace{1cm} (29)

Substituting the functional relations (28) of the radial eigenfunction $\Phi(x)$ into Eq. (26), one obtains the characteristic inequality

$$(\alpha^2 - \alpha - m^2 + \frac{1}{4})(x_{\text{peak}}^2 - 1) + N_p(2\alpha - 1 + 2m^2) > 0$$  \hspace{1cm} (30)
at the extremum point \(29\). Assuming
\[
\alpha(\alpha - 1) < m^2 - \frac{1}{4}, \tag{31}
\]
one finds from \(30\) the inequality
\[
x_{\text{peak}}^2 < 1 + N_p \cdot \frac{2m^2 - 1 + 2\alpha}{m^2 - \frac{1}{4} - \alpha(\alpha - 1)}. \tag{32}
\]
The strongest upper bound on the radial location of the extremum point \(x = x_{\text{peak}}\) can be obtained by minimizing, with respect to the dimensionless parameter \(\alpha\), the functional expression on the right-hand-side of \(32\). In particular, this expression is minimized for \(45\)
\[
\alpha(m) = \alpha^*(m) = \frac{1}{2} - m^2 + m\sqrt{m^2 - 1}, \tag{33}
\]
in which case one finds from Eqs. (29), (32), and (33) the upper bound
\[
x_c < \sqrt{1 + N_p \cdot \left(1 + \sqrt{1 - \frac{1}{m^2}}\right)} \tag{34}
\]
on the characteristic inner supporting radii of the polytropic hydrodynamic vortex.

It is worth noting that, taking cognizance of the characteristic relation \(3\) and defining the dimensionless radial coordinate
\[
\bar{x}_c \equiv \frac{r_c}{r_e}, \tag{35}
\]
one can express the analytically derived upper bound \(34\) in the form
\[
\bar{x}_c < \sqrt{1 + N_p \cdot \left(1 + \sqrt{1 - \frac{1}{m^2}}\right) \over 1 + 2N_p}. \tag{36}
\]

VI. ANALYTIC TREATMENT OF THE RESONANCE CONDITION FOR MARGINALLY-STABLE RESONANT MODES IN THE SMALL-RADIUS REGIME

In the present section we shall explicitly show that the resonance equation \(13\), which determines the unique family \(\{x_c(N_p, m; n)\}\) of dimensionless supporting radii that characterize the static (marginally-stable) resonant modes of the polytropic hydrodynamic vortex, is amenable to an analytical treatment in the small-radius regime
\[
x_c - 1 \ll 1. \tag{37}
\]
Taking cognizance of the characteristic limiting behavior [see Eq. 15.1.1 of \(27\)]
\[
_2F_1(a; b; c; z \to 0) \to 1 \tag{38}
\]
of the hypergeometric function and using Eqs. 6.1.15 and 15.3.6 of \(27\), one can express the resonance condition \(13\) in the form
\[
(1 - x_c^{-2})^{1-N_p} = \frac{2(1 - N_p)\Gamma(-N_p)\Gamma\left[\frac{N_p\sqrt{N_p^2 + m^2(1+2N_p)+m}}{2}\right] \Gamma\left[\frac{N_p\sqrt{N_p^2 + m^2(1+2N_p)+m}}{2}\right]}{m(m+1)\Gamma(N_p)\Gamma\left[- \frac{N_p\sqrt{N_p^2 + m^2(1+2N_p)-m}}{2}\right] \Gamma\left[- \frac{N_p\sqrt{N_p^2 + m^2(1+2N_p)-m}}{2}\right]} \quad \text{for} \quad x_c \to 1^+. \tag{39}
\]
A. The \( 0 < N_p < 1 \) regime

We shall first solve the analytically derived resonance equation (39) in the \( 0 < N_p < 1 \) regime, which corresponds to the strong inequality \((1 - x_c^{-2})^{1-N_p} \ll 1 \) [see Eqs. (37) and (39)]. Substituting into (39)

\[
N_p(m; n) = N^*_p \times (1 + \epsilon) \quad \text{with} \quad \epsilon \ll 1 ,
\]

where

\[
N^*_p(m; n) = \frac{2n(m+n)}{m(m+1)+2n} ; \quad n = 1, 2, 3, ... ,
\]

one obtains the linearized (small-\( \epsilon \)) resonance condition (40)

\[
(1-x_c^{-2})^{1-N^*_p} = \frac{2\alpha(-1)^n n!(1-N^*_p)^\Gamma(-N^*_p)\Gamma[N^*_p-\sqrt{N^*_p^2+m^2(1+2N^*_p)}-n] \Gamma(N^*_p-n)}{m(m+1)\Gamma(N^*_p)\Gamma[-\sqrt{N^*_p^2+m^2(1+2N^*_p)}-n] \Gamma(N^*_p-n)} \times \epsilon + O(\epsilon^2) ,
\]

where

\[
\alpha = \frac{1}{2} \left[ \frac{N^*_p(N^*_p+m^2)}{\sqrt{N^*_p^2+m^2(1+2N^*_p)}} - N^*_p \right] .
\]

From (42) one finds the resonant solution

\[
x_c(\epsilon, m; n) = 1 + \beta_1 \times \epsilon^{-N_p} \quad \text{for} \quad 0 < N_p < 1
\]

in the \( \epsilon \ll 1 \) regime [or equivalently, in the \( x_c - 1 \ll 1 \) regime, see (37)], where

\[
\beta_1 = \frac{1}{2} \left\{ \frac{N^*_p(N^*_p+m^2)}{\sqrt{N^*_p^2+m^2(1+2N^*_p)}} - N^*_p \right\} \times \frac{N^*_p}{m(m+1)\Gamma(N^*_p)\Gamma[-\sqrt{N^*_p^2+m^2(1+2N^*_p)}-n] \Gamma(N^*_p-n)} .
\]

B. The \( N_p > 1 \) regime

We shall next solve the resonance equation (39) in the \( N_p > 1 \) regime, which corresponds to the strong inequality \((1 - x_c^{-2})^{1-N_p} \gg 1 \) [see Eqs. (37) and (39)]. Substituting into (39)

\[
N_p(m; n) = N^\#_p \times (1 + \epsilon) \quad \text{with} \quad 0 < \epsilon \ll 1 ,
\]

where (47)

\[
N^\#_p(m; n) = \frac{2n(m+n)}{m(m-1)-2n} ; \quad n = 1, 2, 3, ... ,
\]

one obtains the linearized (small-\( \epsilon \)) resonance condition (48)

\[
(1-x_c^{-2})^{1-N^\#_p} = \frac{2(1-N^\#_p)^\Gamma(-N^\#_p) \Gamma(N^\#_p)\Gamma[-N^\#_p+\sqrt{N^\#_p^2+m^2(1+2N^\#_p)}-n] \Gamma(-N^\#_p-n)}{\bar{\alpha}m(m+1)(-1)^n n! \Gamma(N^\#_p)\Gamma[-N^\#_p+\sqrt{N^\#_p^2+m^2(1+2N^\#_p)}-n] \Gamma(-N^\#_p-n)} \times \epsilon^{-1} + O(1) ,
\]

where

\[
\bar{\alpha} = \frac{1}{2} \left[ \frac{N^\#_p(N^\#_p+m^2)}{\sqrt{N^\#_p^2+m^2(1+2N^\#_p)}} - N^\#_p \right] .
\]
that the largest (outermost) dimensionless supporting radius \( \bar{\rho} \) for the dynamic vortex, can be solved numerically. In particular, using numerical techniques, it was nicely demonstrated in [16] that the characteristic supporting radius \( \bar{\rho} \) for the dimensionless polytropic index \( \bar{N}_x \), increases monotonically with increasing values of the azimuthal harmonic index \( \bar{m} \) of the polytropic hydrodynamic vortex.

In the present section we shall provide an analytical treatment of the resonance condition: monotonic behavior of the outermost supporting radii \( x_c(\epsilon, m; n) \) of the polytropic hydrodynamic vortex, which is amenable to an analytical treatment in cases where \( \epsilon \ll 1 \) regime, where

\[
\beta_2 \equiv \frac{1}{2} \left\{ \frac{4(N_p^# - 1)\Gamma(-N_p^#)\Gamma[\sqrt{N_p^# + m^2(1 + 2N_p^#)} - n]}{m(m + 1)(-1)^n!\sqrt{N_p^#}} - N_p^#\Gamma(\bar{N}_p^#)\Gamma[-N_p^# + \sqrt{N_p^# + m^2(1 + 2N_p^#)} + n]\Gamma(-N_p^# - n) \right\}^{1 - N_p^#}.
\]

(51)

It is interesting to stress the fact that, taking cognizance of the analytically derived resonant formulas \( 44 \) and \( 50 \), one learns that, in general, the characteristic supporting radii \( \{x_c(\epsilon, m; n)\} \) of the polytropic hydrodynamic vortex have a non-trivial (non-linear) dependence on the dimensionless small parameter \( \epsilon \).

VII. ANALYTIC TREATMENT OF THE RESONANCE CONDITION: MONOTONIC BEHAVIOR OF THE OUTERMOST SUPPORTING RADII \( x_c^{\text{max}} \)

The resonance equation \( 13 \), which determines the static (marginally-stable) resonances of the polytropic hydrodynamic vortex, can be solved numerically. In particular, using numerical techniques, it was nicely demonstrated in [16] that the largest (outermost) dimensionless supporting radius \( x_c^{\text{max}}(N_p, m) \) [see Eq. \( 35 \)] of the central reflecting cylinder increases monotonically with increasing values of the azimuthal harmonic index \( m \) and decreasing values of the dimensionless polytropic index \( N_p \) of the circulating fluid. It is worth emphasizing again that the physical significance of the quantity \( x_c^{\text{max}}(N_p, m) \) stems from the fact that, for given dimensionless physical parameters \( \{N_p, m\} \), this outermost supporting radius marks the boundary between stable and superradiantly unstable configurations of the polytropic hydrodynamic vortex.

In the present section we shall provide an analytical explanation for the interesting monotonic behavior, first observed numerically in [16], which characterizes the outermost supporting radius \( x_c^{\text{max}}(N_p, m) \) of the marginally-stable (static) resonant modes of the polytropic hydrodynamic vortex. In particular, we shall use analytical techniques in order to demonstrate that the characteristic supporting radius \( x_c^{\text{max}} \) increases monotonically with increasing values of the azimuthal harmonic index \( m \) and decreasing values of the dimensionless polytropic index \( N_p \).

Interestingly, as we shall now demonstrate explicitly, the resonance condition \( 13 \), which determines the discrete family \( \{x_c(N_p, m; n)\} \) of inner supporting radii that characterize the marginally-stable resonant modes of the polytropic hydrodynamic vortex, is amenable to an analytical treatment in cases where

\[
\frac{1}{2} [N_p - \sqrt{N_p^2 + m^2(1 + 2N_p^#) + m}] = -n ; \quad n = 0, 1, 2, \ldots.
\]

(52)

In particular, in these cases, which correspond to the relation

\[
N_p(m; n) = \frac{2m(m + n)}{m(m - 1) - 2n} ;
\]

(53)

one finds that the resonance condition \( 13 \) becomes a polynomial equation of degree \( n \) in the dimensionless physical variable

\[
z_c = \frac{1}{x_c^2}.
\]

(54)

For example, in the first nontrivial \( 49 \) case, \( n = 1 \), the resonance condition \( 13 \) yields the simple linear equation

\[
\frac{m(1 - m) \cdot z_c + m(m - 1) - 2}{m(m - 1) - 2} = 0 ; \quad m \geq 3
\]

(55)

for the variable \( z_c \). Substituting \( 54 \) into \( 55 \), one finds the simple analytical expression \( 51 \)

\[
x_c = \sqrt{\frac{m(m - 1)}{(m - 2)(m + 1)}} ; \quad N_p = \frac{2}{m - 2} \quad \text{with} \quad m \geq 3 .
\]

(56)
Taking cognizance of the characteristic relation (3), one finds from (50) the compact expression

\[ \bar{x}_c = \sqrt{\frac{m(m-1)}{(m+1)(m+2)}} \ ; \ N_p = \frac{2}{m-2} \quad \text{with} \quad m \geq 3 \ . \] (57)

As another analytically solvable example, let us consider the \( n = 2 \) case [see Eq. (53)], in which case the resonance condition (13) yields the quadratic equation

\[ (m^5 + m^4 - 5m^3 - m^2 + 4m)z_c^2 + (-2m^5 + 14m^3 + 4m^2 - 16m)z_c + (m^5 - m^4 - 9m^3 + m^2 + 24m + 16) = 0 \]

for the dimensionless variable \( z_c \). Substituting (54) into (58), one obtains the two supporting radii (51, 52)

\[ x_c^\pm = \frac{m(m-1)(m+1)^2 + m^4 - m^2 + 4m}{(m^2 - m - 4)|m(m-1)(m+2) \pm \sqrt{m(m-1)(m^2 + 3m + 4)}|} \ ; \ N_p = \frac{4(m+2)}{m(m-1) - 4} \quad \text{with} \quad m \geq 3 \ . \] (58)

Taking cognizance of the dimensionless ratio (3), one finds from (59) the analytical expression

\[ \bar{x}_c^\pm = \frac{m(m-1)(m+1)^2 + m^4 - m^2 + 4m}{(m+3)(m+4)|m(m-1)(m+2) \pm \sqrt{m(m-1)(m^2 + 3m + 4)}|} \ ; \ N_p = \frac{4(m+2)}{m(m-1) - 4} \quad \text{with} \quad m \geq 3 \ . \] (59)

for the dimensionless supporting radii of the polytropic hydrodynamic vortex.

It is physically interesting to point out that the analytically derived expressions (57) and (60) reveal the fact that the dimensionless supporting radius \( \bar{x}_c^\text{max} \), which characterizes the marginally-stable static \( (\omega = 0) \) resonances of the polytropic hydrodynamic vortex, increases monotonically with increasing values of the azimuthal harmonic index \( m \) and decreasing values (53) of the dimensionless polytropic index \( N_p \) of the circulating fluid. It is important to stress the fact that this analytically demonstrated monotonic behavior of the characteristic supporting radius \( \bar{x}_c^\text{max} \) agrees with the numerical results recently presented in the interesting work of Oliveira, Cardoso, and Crispino (16).

**VIII. SUMMARY AND DISCUSSION**

Contrary to the asymptotically flat spinning Kerr black-hole spacetime, whose stability to massless bosonic perturbations relies on the characteristic absorptive properties of its horizon (6, 8), horizonless rotating spacetimes with ergoregions and inner reflecting boundary conditions may develop superradiant instabilities to co-rotating integer-spin (bosonic) fields (2, 12, 17). In particular, it has recently been demonstrated that the polytropic hydrodynamic vortex (16), an effective \( (2+1) \)-dimensional acoustic spacetime with an ergoregion of radius \( r_e \) and an inner reflecting boundary at \( r = r_c \) may develop exponentially growing superradiant instabilities (2, 16, 17).

In the present paper we have used analytical techniques in order to explore the physical and mathematical properties of the static acoustic resonances which characterize the polytropic hydrodynamic vortex. The physical significance of these marginally-stable sound modes stems from the fact that these static resonances mark the boundary between stable and superradiantly unstable configurations of the effective \( (2+1) \)-dimensional circulating fluid system (2, 16, 17). The main results derived in this paper and their physical implications are:

(1) It has been explicitly proved that, for a given value of the azimuthal harmonic index \( m \), the marginally-stable (static) resonant modes of the polytropic hydrodynamic vortex are restricted to the dimensionless physical regime [see Eq. (23)]

\[ N_p(m) \geq N_p^\text{min} \quad \text{with} \quad N_p^\text{min} = \frac{2(m+1)}{m(m+1) + 2} \ . \] (61)

This characteristic inequality implies, in particular, that in the \( 0 < N_p \ll 1 \) regime, only acoustic resonant modes with large azimuthal indices [see Eq. (24)]

\[ m \geq \left[ 2/N_p \right] \gg 1 \ , \] (62)

can trigger superradiant instabilities in the polytropic hydrodynamic vortex.
(2) We have proved that, for a given set \( \{N_p, m\} \) of the dimensionless physical parameters that characterize the polytropic hydrodynamic vortex, the compact inequality [see Eqs. (12) and (43)]

\[
\frac{r_c}{r_s} < \sqrt{(1 + N_p \cdot (1 + \frac{1}{m^2}))}
\]

provides an upper bound on the characteristic supporting radii of the marginally-stable (static) resonant modes of the system.

(3) It has been shown that the polytropic hydrodynamic vortex is characterized by a finite unique family of dimensionless cylinder radii, \( \{x_c(N_p, m; n)\}_{n=1}^{N_r} \), that can support the marginally-stable (static) acoustic resonant modes. In particular, in the \( N_p > 0 \) regime, one finds [30, 33, 38] [see Eq. (22)]

\[
N_r(N_p, m) = \begin{cases} 
\sqrt{N_p^2 + m^2(1 + 2N_p) + N_p - m} - \frac{m}{2} & \text{for } 0 < N_p < 1; \\
\sqrt{N_p^2 + m^2(1 + 2N_p) - N_p - m} - \frac{m}{2} & \text{for } N_p \geq 1.
\end{cases}
\]

(4) Interestingly, the fact that the polytropic hydrodynamic vortex in the physical regime \( N_p > 0 \) is characterized by a finite discrete set \( \{x_c(N_p, m; n)\}_{n=1}^{N_r} \) of dimensionless cylinder radii that can support the marginally-stable (static) resonant modes [see Eq. (43)] should be contrasted with the \( N_p = 0 \) case studied in [7, 17]. In particular, it has been explicitly proved in [17] that, for the \( N_p = 0 \) case, the effective \((2 + 1)\)-dimensional acoustic spacetime is characterized by an infinite countable set of cylinder radii, \( \{r_c(N_p = 0, m; n)\}_{n=1}^{\infty} \), that can support the marginally-stable resonant sound modes.

(5) Our analysis has revealed the intriguing fact that the \( N_p \to 0 \) limit of the polytropic hydrodynamic vortex is not continuous. In particular, while the \( N_p = 0 \) case of constant density fluids studied in [7, 17] is characterized by an infinite countable set of marginally-stable (static) resonant modes [17], in the \( N_p \to 0^+ \) limit of the polytropic hydrodynamic vortex and for a fixed value of the azimuthal harmonic index \( m \) there are no marginally-stable acoustic resonant modes [39].

(6) It has been explicitly shown that the resonance condition [13], which determines the static (marginally-stable) resonant modes of the polytropic hydrodynamic vortex, can be solved analytically in the dimensionless regime

\[
x_c - 1 \ll 1
\]

of small supporting radii. In particular, we have revealed the intriguing fact that, in the small-radii regime [65], the supporting radii \( \{x_c(\epsilon, m; n)\} \) of the polytropic hydrodynamic vortex are characterized by the non-trivial (non-linear) functional dependence [see Eqs. (44) and (50)]

\[
x_c(\epsilon) = 1 + \beta \cdot \epsilon^{1 - N_p},
\]

where the dimensionless deviation parameter \( \epsilon \) is defined in [40] and [46].

(7) We have explicitly shown that the characteristic resonance equation [13] of the polytropic hydrodynamic vortex can be solved analytically in the dimensionless regime

\[
N_p(m; n) = \frac{2n(m + n)}{m(m - 1) - 2n} ; \quad n = 0, 1, 2, \ldots.
\]

In particular, using analytical techniques, we have explicitly demonstrated that the dimensionless supporting radius \( x_c^{\text{max}} \) [see Eq. (35)], which characterizes the marginally-stable resonant modes of the polytropic hydrodynamic vortex, increases monotonically with increasing values of the azimuthal harmonic index \( m \) and decreasing values [53] of the dimensionless polytropic index \( N_p \) of the circulating fluid [see Eqs. (57) and (60)]. It is important to stress the fact that the characteristic monotonic behavior of the outermost supporting radius \( x_c^{\text{max}}(N_p, m) \), which has been explicitly demonstrated analytically in the present paper, agrees with the interesting numerical results presented recently in [16] for the polytropic hydrodynamic vortex.

ACKNOWLEDGMENTS

This research is supported by the Carmel Science Foundation. I thank Yael Oren, Arbel M. Ongo, Ayelet B. Lata, and Alona B. Tea for stimulating discussions.

[1] S. Chandrasekhar, *The Mathematical Theory of Black Holes*, (Oxford University Press, New York, 1983).
Note that the decomposition (6) describes cylindrical perturbation modes of the effective (2 + 1)-dimensional polytropic spacetime, where the azimuthal index $m$ is an integer.

[21] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover Publications, New York, 1970).

Using the characteristic asymptotic behavior $\psi_{1}(x, \rho) \sim x^{1/2} \phi_{m}(\rho)$, one finds that the radial eigenfunction $\phi_{m}(\rho)$ is well behaved (bounded) at spatial infinity: $\phi_{m}(\rho \rightarrow \infty) \sim \rho^{-1}$ [see Eq. (17)] for the static linearized perturbation modes of the polytropic hydrodynamic vortex.

Note that, in the physical regime $N_{p} > 0$ that we consider in this paper, the dimensionless physical parameter $N_{c}$ [see Eq. (17)] is non-negative by definition.
[30] Here \([x]\) is the ceiling function of the variable \(x\), which denotes the least integer that is greater than or equal to \(x\).

[31] Here we have used Eq. 15.1.20 of \([27]\).

[32] Here we have used Eq. 6.1.7 of \([27]\).

[33] Note that \(c − a − b \equiv 1 − \bar{N}_p\) in Eq. \([19]\) [see the resonance equation \([18]\)]. Thus, the requirement \(R(c − a − b) > 0\) in \([19]\) is satisfied for \(\bar{N}_p < 1\).

[34] Note that \(c − b \equiv 1 − \bar{N}_c\) in Eq. \([19]\) [see the resonance condition \([18]\)].

[35] Note that \(\bar{N}_c = \bar{N}_c + N_p\).

[36] Here \([x]\) is the floor function of the variable \(x\), which denotes the largest integer that is smaller than or equal to \(x\).

[37] Note that, in the \(0 < N_p < 1\) regime, we use the floor function \([\bar{N}_c]\) because, as the value of the dimensionless composed function \(\bar{N}_c\) gradually increases, a new resonant mode (which is characterized by the dimensionless relation \(x_c = 1\)) is added to the discrete set of static modes only when \(\bar{N}_c\) becomes a positive integer [see Eqs. \((19)\), \((20)\), and \((21)\)].

[38] Note that if \(\sqrt{N_p^2 + m^2(1 + 2N_p) + N_p − m}\) is an integer [that is, if \(\bar{N}_c\) is an integer [see Eq. \((21)\)]] with \(0 < N_p < 1\), then one of the \(N_c\) acoustic resonant modes of the polytropic hydrodynamic vortex [see the analytical formula \((22)\) for the number of marginally-stable resonances] is characterized by the dimensionless supporting radius \(x_c = 1\).

[39] Note, in particular, that for a fixed value of the azimuthal harmonic index \(m\), one finds \(N_i = \lfloor \sqrt{N_p^2 + m^2(1 + 2N_p) + N_p − m} \rfloor = 0\) [see Eq. \((23)\)] in the \(N_p \to 0^+\) limit of the polytropic hydrodynamic vortex.

[40] Note that one finds from Eq. \((22)\) the simple relations \(N_i = 0\) for \(N_p < N_p^{\text{min}}(m)\) and \(N_i = 1\) for \(N_p = N_p^{\text{min}}(m)\).

[41] Note that, in agreement with the required inequality \(N_p < 1\) [see Eq. \((22)\)], one finds from \((23)\) the relation \(N_p^{\text{min}}(m) < 1\) for all integer values of the azimuthal harmonic index \(m\).

[42] As emphasized above, static resonant modes of the polytropic hydrodynamic vortex that satisfy the equality \(2(m + 1)/[m(m + 1) + 2] = N_p\) correspond to the limiting value \(x_c = 1\) of the supporting cylinder radius.

[43] Note that, according to the analytical formula \((22)\), the regime \(N_p(m) < N_p^{\text{min}}(m)\) is characterized by \(N_i = 0\).

[44] Note, in particular, that the radial function \(\Phi(x)\) is characterized by the relations \(\Phi(x = x_c) = 0\) and \(\Phi(x \to \infty) \to 0\) [see \((24)\) and also Eqs. \((11)\) and \((27)\)].

[45] Note that the value of \(\alpha^2(m)\) as given in \((33)\) respects the assumed inequalities \((21)\) and \((31)\).

[46] Here we have used the relations \(1/\Gamma \left[ −N_p + m^2 \bar{c} + \sqrt{N_p^2 + m^2(1 + 2N_p) + N_p - m} \right] = 1/\Gamma \left[ −n + \bar{c} + \epsilon + O(\epsilon^2) \right] = (-1)^n n! \cdot \bar{c} + O(\epsilon^2)\) [see Eqs. \((20)\), \((21)\), \((41)\), and Eq. 6.1.34 of \([27]\)].

[47] The case \(N_p(m; n) = N_p^{\text{th}}\) [that is, the \(\epsilon = 0\) case] will be treated analytically in section VII.

[48] Here we have used the relations \(\Gamma \left[ N_p^{\text{th}} - \sqrt{N_p^2 + m^2(1 + 2N_p) + N_p - m} \right] = \Gamma \left[ −n + \bar{c} + \epsilon + O(\epsilon^2) \right] = [(-1)^n n! \cdot \bar{c} + O(1)]\) [see Eqs. \((20)\), \((21)\), \((41)\), and Eq. 6.1.34 of \([27]\)].

[49] Note that the case \(n = 0\) corresponds to the trivial relation \(N_p(m) \equiv 0\).

[50] As a consistency check we note that the value \(N_p = 2/(m - 2)\) respects the analytically derived lower bound \((25)\).

[51] Note that the requirement \(N_p > 0\) implies \(m \geq 3\) in this case.

[52] It is worth noting that the value \(N_p = 4(m + 2)/[[m(m - 1) - 4]\) respects the analytically derived lower bound \((25)\).

[53] Note that the cases \((62)\) that we have studied analytically in section VII correspond to a monotonically decreasing functional dependence of the physical parameter \(N_p(m)\) [see Eq. \((53)\) on the azimuthal harmonic index \(m\) of the resonant perturbation modes.