System Level Synthesis Beyond Finite Impulse Response Using Approximation by Simple Poles

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Abstract—Optimal linear feedback control design is valuable but challenging. The system level synthesis approach uses a reparameterization to expand the class of problems that can be solved using convex reformulations, among other benefits. However, to solve system level synthesis problems prior work relies on finite impulse response approximations that lead to deadbeat control, and that can experience infeasibility and increased suboptimality, especially in systems with large separation of time scales. This work develops a new technique by combining system level synthesis with a new approximation based on simple poles. The result is a new design method which does not result in deadbeat control, is convex and tractable, always feasible, can incorporate prior knowledge, and works well for systems with large separation of time scales. A general suboptimality result is provided which bounds the approximation error based on the geometry of the pole selection. The bound is then specialized to a particularly interesting pole selection to obtain a non-asymptotic convergence rate. An example demonstrates superior performance of the method.

I. INTRODUCTION

Optimal control design for linear time invariant systems has been thoroughly studied, but remains a challenging problem. Key difficulties arise from the infinite dimensionality of the Hardy space of stabilizing controllers, as well as the nonconvexity of the resulting optimization problem for the design.

One of the most celebrated approaches for optimal control design is the Youla parameterization [1], which parameterizes all stabilizing controllers and leads to a convex reformulation in terms of the Youla parameter. Recently, new optimal control design techniques have been introduced, known as system level synthesis (SLS) [2], [3] and input-output parameterization (IOP) [4], which also parameterize all stabilizing controllers and include closed-loop system responses as decision variables. However, they have some advantages over Youla parameterization, including a larger class of optimal control problems which admit convex representations, and the ability to preserve structure from the closed-loop transfer functions in the internal controller realizations. For this work, we focus on state feedback controllers, and so restrict our attention to SLS rather than IOP since the latter focuses on output feedback.

The SLS reparameterization results in a convex but infinite dimensional optimization problem. In order to solve it, prior work [5] has assumed that the closed-loop responses are finite impulse responses (FIR) in order to arrive at a tractable finite dimensional optimization problem. However, this results in deadbeat control (DBC), which often experiences poorly damped oscillations between discrete sampling times that can even persist in steady state, as well as lack of robustness to model uncertainty and parameter variations because of the high control gains required to reach the origin in finite time [6]. We therefore denote SLS with the FIR approximation by DBC for the remainder of the paper.

With DBC, the number of poles in the closed-loop transfer functions is equal to the length of the FIR, hence, potentially resulting in large numbers of poles that can lead to high computational complexity for the control design, lack of robustness in the resulting controller, and implementation challenges in practice [7, Chapter 19]. This is especially problematic when the optimal solution has a long settling time, such as in systems with large separation of time scales, where short sampling times are needed to capture the fast dynamics, which are also coupled with much slower dynamics. This leads to closed-loop impulse responses settling only after a large number of time steps. Additionally, if the closed-loop response is FIR in discrete time then all poles must be at the origin. This results in infeasibility in case of stable but uncontrollable poles in the plant. To resolve this, DBC introduces a slack variable enabling constraint violation in this case, which however leads to additional suboptimality [5]. Furthermore, in this case DBC results in a quasi-convex formulation, which requires an iterative approach such as golden section search to solve, rather than solving with a single convex optimization [5].

This work presents a new control design method to address these limitations. The key idea is to preserve the SLS reparameterization, but to choose a different finite dimensional approximation. In particular, we propose to approximate the optimal closed-loop transfer functions using a collection of simple poles, which we call the simple pole approximation (SPA). This approach is not FIR, so it does not suffer from the drawbacks of deadbeat control. Moreover, the number of poles is independent of the settling time of the optimal closed-loop responses, and therefore SPA even works well for systems with large separation of timescales. It results in a convex and tractable optimization formulation for the design, and requires only a small number of poles. As the poles of the plant can be automatically incorporated, feasibility is guaranteed for stabilizable systems. Therefore, no slack variables or constraint violations are required, and additional suboptimality resulting from these can be avoided. Furthermore, there is no need to use iterative methods such as golden section search since the optimization is convex and can be solved with a single
semidefinite program. Finally, if prior information is known about the optimal solution, such as the locations of some of the optimal poles (e.g., for model matching [8], model reference control [9], design based on the internal model principle [10], expensive control [11, Theorem 3.12(b)], etc.), then these can be incorporated directly into the design for improved performance.

The main contributions of this work are the development of this novel control design method, a simple pole approximation theorem, and a general suboptimality certificate which shows the convergence of the approximation to the optimal solution based on the geometry of the pole selection. This general certificate is then specialized for a particular choice of poles based on the geometry of an Archimedean spiral, which is a powerful heuristic for obtaining approximately uniform selections of poles over the unit disk [12, Section 5]. An example shows superior performance of SPA over DBC, and is fully reproducible with all code publicly available [13].

The remainder of this paper is organized as follows. Section II provides some notation and the setup of the optimal control problem. Section III reviews some general results of SLS. Section IV provides the development of the SPA method as well as the suboptimality certificates. Section V provides an illustrative example comparing DBC and SPA. Section VI gives the proofs of the theoretical results. Finally, Section VII offers concluding remarks.

II. PRELIMINARIES AND PROBLEM SETUP

A. Notation

We will use the following notation throughout the paper. Let \( \mathbb{D} \) be the open unit disk in the complex plane, \( \overline{\mathbb{D}} \) the closed unit disk, and \( \partial \mathbb{D} \) the unit circle. Let \( \overline{B}_r \) be the closed ball of radius \( r \) centered at the origin. Let \( S \subseteq \mathbb{C} \) and any \( \overline{S} = \{ z \in \mathbb{C} : |z| \leq 1 \} \). Let \( \mathbb{E} \) denote the complex conjugate of \( z \), and let \( \arg(z) \) and \( \Im(z) \) denote the real and imaginary components of \( z \), respectively. For any positive integer \( m \), let \( \mathbb{E} = \{ 1, 2, ..., m \} \). For any set \( S \), let \( |S| \) be the cardinality of set \( S \) (i.e. the number of elements it contains).

For a matrix \( M \), let \( \|M\|_2 \) denote the spectral norm of \( M \), and let \( \|M\|_F \) denote the Frobenius norm of \( M \). Let \( M_{i,j} \) denote the entry of \( M \) at row \( i \) and column \( j \).

Let \( \mathcal{H}_\infty \) be the Hardy space of real, rational, proper, and stable transfer functions in discrete time (see \cite{7, 14} for a more detailed discussion). Let \( \frac{1}{z} \mathcal{H}_\infty \subseteq \mathcal{H}_\infty \) consist of the strictly proper transfer functions in \( \mathcal{H}_\infty \). For \( S \in \mathcal{H}_\infty \), define the norms

\[
\|S\|_{\mathcal{H}_\infty} = \sup_{z \in \partial \mathbb{D}} |S(z)|, \quad \|S\|_{\mathcal{H}_\infty}^2 = \frac{1}{2\pi} \int_{\partial \mathbb{D}} |S(z)|^2 |dz|.
\]

For any \( S \in \mathcal{H}_\infty \) and positive integer \( k \), let \( \mathcal{S}(k) \) denote the time-domain impulse response of \( S \) at time \( k \). For \( S \in \mathcal{H}_\infty \), by Parseval’s theorem it can be shown that

\[
\|S\|_{\mathcal{H}_\infty}^2 = \sum_{k=1}^{\infty} \|\mathcal{S}(k)\|^2 = \lim_{T \to \infty} \|\mathcal{T}(S)\|^2_{\mathcal{T}}.
\]

where \( \mathcal{T}(S) := [S(1)^T \quad S(2)^T \ldots S(T)^T]^T \), which can be well approximated numerically using \( T \) finite. For any \( S \in \mathcal{H}_\infty \), let \( \mathcal{E}(S) \) denote the convolution operator of \( S \) so that for any input signal \( u(k) \) and nonnegative integer \( n \), \( \mathcal{E}(S)(u)[n] = \sum_{k=1}^{\infty} \mathcal{S}(k) u(n-k) \). Note that for \( S \in \mathcal{H}_\infty \), since \( \| \cdot \|_{\mathcal{H}_\infty} \) is the induced \( L_2 \to L_2 \) norm, it can be shown that

\[
\|S\|_{\mathcal{H}_\infty}^2 = \lim_{T \to \infty} \|\mathcal{E}(S)(u)[2] = \|\mathcal{E}(S)(u)[2] = \lim_{T \to \infty} \|\mathcal{E}(T)(S)\|^2_{\mathcal{T}}
\]

which can be well approximated numerically using \( T \) finite.

B. Problem Setup

Consider the following LTI system in discrete time

\[
x(k+1) = Ax(k) + Bu(k) + Kw(k) \\
y(k) = Cx(k) + Du(k)
\]

where \( x(k) \in \mathbb{R}^n, u(k) \in \mathbb{R}^r, w(k) \in \mathbb{R}^s, \) and \( y(k) \in \mathbb{R}^m \) are the state, controller input, disturbance input, and performance output vectors at time step \( k \), respectively. Let \( \sigma \) be the plant poles (i.e. the eigenvalues of \( A \)). For signals \( u(z) \) and \( y(z) \) in the \( z \)-domain, let \( T_{u \to y}(z) \) denote the transfer function from \( u(z) \) to \( y(z) \). Consider a linear (possibly dynamic) state feedback control law of the form \( u(z) = K(z)x(z) \) where \( K \in \mathcal{H}_\infty \), and let \( T_{u \to y}(z) \) be some desired closed-loop transfer function for model reference or model matching control (note that we can set \( T_{\text{desired}}(z) = 0 \) if desired). The goal is to choose a controller \( K(z) \) that is a solution to the mixed \( \mathcal{H}_2 \mathcal{H}_\infty \) \cite{7, 14} optimal control problem given by

\[
\arg \min_{K(z)} \|T_{u \to y}(z) - T_{\text{desired}}(z)\|_{\mathcal{H}_2} + \lambda \|T_{u \to y}(z) - T_{\text{desired}}(z)\|_{\mathcal{H}_\infty}
\]

s.t. \( T_{u \to x}(z), T_{w \to u}(z) \in \frac{1}{z} \mathcal{H}_\infty \),

where \( \lambda \in \mathbb{R} \) is constant. As \( T_{u \to y}(z) \) is nonconvex in \( K(z) \), (2) is known to be a challenging problem. We make the following feasibility assumptions:

(A1) A solution to (2) exists, i.e. \( (A, B) \) is stabilizable, and the optimal closed-loop transfer functions are rational (hence they have finitely many poles).

While one can construct pathological examples where this assumption does not hold (e.g., a controllable SISO system with \( y = x \) and \( T_{\text{desired}}(z) = e^z \), in the standard mixed \( \mathcal{H}_2 \mathcal{H}_\infty \) setting Assumption A1 is satisfied automatically \cite{15}.

By Assumption A1 there exists an optimal solution \( (T_{u \to x}^{*}, T_{w \to u}^{*}) \) to (2). As \( T_{u \to x}^{*}, T_{w \to u}^{*} \in \frac{1}{z} \mathcal{H}_\infty \), we can write their partial fraction decomposition as

\[
T_{u \to x}(z) = \sum_{q \in \Omega} \sum_{j=1}^{m_{q}^{*}} H_{(q,j)}^{*} \frac{1}{z - q} \\
T_{w \to u}(z) = \sum_{q \in \hat{\Omega}} \sum_{j=1}^{m_{q}^{*}} G_{(q,j)}^{*} \frac{1}{z - q} \]

where \( \Omega \) and \( \hat{\Omega} \) are finite sets of stable poles closed under complex conjugation, \( H_{(q,j)}^{*} \) and \( G_{(q,j)}^{*} \) are coefficient matrices, and \( m_{q}^{*} \) and \( m_{q}^{*} \) are the multiplicities of the pole \( q \) in
It will be shown (in the proof of Lemma 3) that the following relationship between the poles $Ω$ and $\hat{Ω}$ holds: $\hat{Ω} \subset Ω \cup σ$. Thus, each pole of $T_{w→x}$ must be a pole of at least one of $T_{w→u}$ and the plant.

III. REVIEW OF SYSTEM LEVEL SYNTHESIS (SLS)

A. System Level Parameterization

A recent approach was proposed to solve problem (2) for the special case, where $y = [(Qx)^T (Ru)^T]^T$ for constant matrices $Q$ and $R$, $T_{\text{desired}}(z) = 0$, and $B = I$. This approach is known as system level synthesis (SLS) [5], and the key idea is to reparameterize the control design in terms of the closed-loop transfer functions $Φ_x(z) = T_{w→x}(z)$ and $Φ_u(z) = T_{w→u}(z)$. This transforms (2) into an infinite dimensional convex optimization problem at the price of an additional affine constraint (further details are given in [5]). The result is

\[
\min_{Φ_x(z),Φ_u(z)} \left\{ \left| \left[ \begin{array}{cc} Q & 0 \\ 0 & R \end{array} \right] \left[ \begin{array}{c} Φ_x(z) \\ Φ_u(z) \end{array} \right] \right|_2 + λ \left| \left[ \begin{array}{cc} Q & 0 \\ 0 & R \end{array} \right] \left[ \begin{array}{c} Φ_x(z) \\ Φ_u(z) \end{array} \right] \right|_2 \right\}_{z ∈ Ω} \\
\text{s.t.} \quad (I - A)Φ_x(z) - BΦ_u(z) = I \\
Φ_x(z), Φ_u(z) ∈ \frac{1}{z} \mathbb{R}^\mathcal{H}_∞. \tag{4}
\]

B. Finite Impulse Response Approximation

To obtain a tractable optimization problem, the FIR approximation is made for the closed-loop transfer functions in [5], i.e., $Φ_x(z) = \sum_{i=1}^{T} G_i \frac{1}{z}$ and $Φ_u(z) = \sum_{i=1}^{T} H_i \frac{1}{z}$, where $G_i$ and $H_i$ are coefficient matrices. If the plant is uncontrollable, then it has stable poles which cannot be removed with feedback, so it is infeasible to achieve FIR closed-loop transfer functions. To maintain feasibility, DBC introduces a slack variable $V$ that allows the affine constraints to be violated. However, the objective becomes non-convex as a result, so DBC uses a quasi-convex upper bound of the objective [5]. The resulting control design is quasi-convex and finite dimensional, and can be solved using methods such as golden section search. The true (i.e., realized) closed-loop responses are then given by $T_{w→x}(z) = Φ_x(z)(I + \frac{V}{z})^{-1}$ and $T_{w→u}(z) = Φ_u(z)(I + \frac{V}{z})^{-1}$ [5].

C. System Level Synthesis Certificates

Let $J^*$ and $J(T)$ be the optimal costs of (4) and of DBC with an FIR of length $T$, respectively. Let $(Φ_x^*, Φ_u^*)$ be an optimal solution to (4). Then there exist constants $C_*, ρ_* > 0$ such that $|Φ_x^*(k)| ≤ C_*, ρ_*^k$ for all $k ≥ 0$. Then $C_*, ρ_* < 1$ is sufficient for DBC to be feasible and to satisfy the following suboptimality bound [5, Theorem 4.7] for some $c > 0$, which is shown as a relative error bound for ease of comparison

\[
\frac{J(T) - J^*}{J^*} ≤ C_* ρ_*^T \frac{1 + \frac{λc}{1 - ρ_*}}{1 - C_* ρ_*^T}. \tag{5}
\]

Note that when $ρ_*$ is small (i.e. the optimal closed-loop response is slow), such as for systems with large separation of time scales, $C_* ρ_*^T < 1$ may require large $T$, the convergence rate of $C_* ρ_*^T$ in (5) is slow, and the term $\frac{1}{1 - C_* ρ_*^T}$ in (5) (which arises from the slack variable $V$) will slow down convergence further. In addition, it is unclear how prior knowledge about optimal poles could be used to improve the convergence rate in (5). Finally, as DBC results in deadbeat control, it suffers from the limitations discussed in the introduction.

IV. MAIN RESULTS

A. Approximation by Simple Poles

We motivate our novel design method (which will be presented in Section IV-B) with a result regarding the approximation of transfer functions in $\frac{1}{z} \mathbb{R}^\mathcal{H}_∞$ by transfer functions with only simple poles (i.e. poles with multiplicity no greater than one). This result may also be of independent interest.

Let $S ∈ \frac{1}{z} \mathbb{R}^\mathcal{H}_∞$ and let $Ω$ be the poles of $S$. For each pole $q ∈ Ω$, let $m_q$ be its multiplicity in $S$. Let $P$ be a set of poles, hereafter referred to as approximating poles, which will be used to construct a transfer function for approximating $S$ (i.e. $S ≈ \sum_{p ∈ P} G_p \frac{1}{z-p}$). The key idea is that for each pole $q ∈ Ω$, an approximating transfer function is constructed to approximate $q$’s contribution to the partial fraction decomposition of $S$. The poles of this approximation are selected to be the $m_q$ closest poles in $P$ to $q$, which we denote by $P(q)$. Then, the overall approximating transfer function for $S$ is obtained by summing over the individual approximating transfer functions.

To evaluate the accuracy of the approximation, for each $q ∈ Ω$ let $d(q)$ be the distance from $q$ to the furthest of the $m_q$ simple poles being used to approximate it, i.e. $d(q) = \max_{p ∈ P(q)} |z-p|$. Let $D(P)$ be the maximum of these distances over all the poles in $Ω$, i.e. $D(P) = \max_{p ∈ Ω} d(q)$. Then $D(P)$ represents the largest distance between approximating poles in $P$ and the poles in $Ω$ they are being used to approximate, so it measures the worst case error in this pole approximation. Theorem 1 provides an approximation error bound of the simple pole approximation which is linear in $D(P)$. Thus, Theorem 1 shows that the simple pole approximating transfer function converges to $T$ at least linearly with $D(P)$, and, therefore, that this bound on the convergence rate depends purely on the geometry of the pole selection.

Before presenting Theorem 1, we make the following assumptions:

(A2) There exists some $r ∈ (0, 1)$ such that $P ⊂ B_r$.

(A3) $|P| ≥ m_{\text{max}} := \max_{q ∈ Ω} m_q$.

(A4) $p ∈ P$ implies that $P(q)$.

(A5) $D(P) < 1$.

Assumption A2 ensures that $P$ consists of stable poles, Assumption A3 that the size of $P$ is at least as large as $m_{\text{max}}$. Assumption A4 that $P$ can be used to construct a transfer function with real coefficients (as will be seen in the proof of Theorem 1), and Assumption A5 that $P$ is not excessively far from $Ω$. Note that Assumption A5 can be satisfied with only two pairs of complex conjugate poles if $S$ has only simple poles, so it tends not to be restrictive in practice.

Theorem 1. Let $S ∈ \frac{1}{z} \mathbb{R}^\mathcal{H}_∞$ and let $P$ be a set of poles satisfying Assumptions A2-A5. Then for $r ∈ \{2, ∞\}$ there exist constants $K_∗ = K_∗(S, r) > 0$ and constant matrices $\{G_p\}_{p ∈ P}$ such that $\sum_{p ∈ P} G_p \frac{1}{z-p} ∈ \frac{1}{z} \mathbb{R}^\mathcal{H}_∞$ and

\[
\left| \sum_{p ∈ P} G_p \frac{1}{z-p} - S \right|_{2r} ≤ K_∗ D(P) \tag{6}
\]
B. New Solution Method: Simple Pole Approximation (SPA)

To introduce our new design method, recall that (4) gives the SLS reparameterization for a special case of (2). Therefore, we begin by reformulating (2) using the SLS reparameterization, which results in the following convex but infinite dimensional optimization problem which is a strict generalization of the SLS formulation in (4):

$$
\begin{align*}
\min_{\Phi_x(z), \Phi_u(z)} & \left\| C \Phi_x(z) \bar{B} + D \Phi_u(z) \bar{B} - T_{\text{desired}}(z) \right\|_{\mathcal{H}_2} \\
\text{s.t.} & \quad (zI - A) \Phi_x(z) - B \Phi_u(z) = I \\
& \quad \Phi_x(z), \Phi_u(z) \in \mathcal{R}\mathcal{H}_\infty.
\end{align*}
$$

(7)

To obtain a tractable optimization problem, we approximate $\Phi_x$ and $\Phi_u$ using $\mathcal{P}$ and $\sigma$ by

$$
\begin{align*}
\Phi_u(z) &= \sum_{p \in \mathcal{P}} H_p \frac{1}{z - p} \\
\Phi_x(z) &= \sum_{p \in \mathcal{P}} G_p \frac{1}{z - p} + \sum_{q \in \sigma} \sum_{i=1}^{m_q+1} G(q,i) \frac{1}{(z - q)^i}
\end{align*}
$$

(8)

where $H_p$, $G_p$, and $G(q,i)$ are coefficient matrices. We refer to this as the simple pole approximation (SPA) since all of the poles other than the poles of the plant in $\Phi_x$ are simple. As we will see, the poles in $\sigma$ are included in $\Phi_x$ with multiplicities potentially greater than one in order to ensure feasibility in case the plant is stabilizable but not controllable. However, it is not necessary (though possible if desired) to include poles with multiplicity greater than one in $\Phi_u$, which is why there is an asymmetry in the approximations of $\Phi_u$ and $\Phi_x$ in (8). Note that the coefficients for the plant poles range from 1 to $m_q+1$ in $\Phi_x$, since if $\Phi_u$ has a pole at the same location as the plant, and because $\Phi_u$ has only simple poles, it is possible to increase the multiplicity of this pole by one.

Although it is possible to select any poles in $\mathcal{P}$ for the SPA method, we provide several recommendations that often lead to improved performance. First, we suggest to include the poles of the plant $\sigma$ in $\mathcal{P}$ to allow the design to fully or partially cancel out the controllable modes of the plant if it proves advantageous to do so. In addition, there are often some poles of the optimal solution which are known a priori, such as discussed in Section I. In such cases, including these optimal poles in $\mathcal{P}$ can lead to a dramatic improvement in performance. In the general case in which no further information about the remaining optimal poles is known, we provide a particular suggestion in Section IV-D for the selection of the remaining poles in $\mathcal{P}$.

For any $q \in \sigma$, let $\tilde{m}_q = 1$ if $q \in \mathcal{P}$ and $\tilde{m}_q = 0$ otherwise. Then the SPA of (8) applied to (7) results in the following optimal control design problem, consisting of the objective

$$
\begin{align*}
\min_{H_p, G_p, G(q,i)} & \quad \left\| \mathcal{J}(C \Phi_x \bar{B}) + \mathcal{J}(D \Phi_u \bar{B}) - \mathcal{J}(T_{\text{desired}}) \right\|_F \\
& + \lambda \left\| C(\Phi_x \bar{B}) + C(D \Phi_u \bar{B}) - C(T_{\text{desired}}) \right\|_{\mathcal{H}_2},
\end{align*}
$$

subject to the following SLS constraints (whose form given below is derived in the proof of Lemma 3):

$$
\begin{align*}
G(q,2) + (qI - A)G(q,1) - BH_q &= 0, \forall q \in \sigma \cap \mathcal{P} \\
(pI - A)G_p - BH_p &= 0, \forall p \in \mathcal{P} - \sigma \\
G(q,i+1) + (qI - A)G(q,i) &= 0, \forall q \in \sigma, i \in \{1 + \tilde{m}_q, \ldots, m_q\} \\
\sum_{p \in \mathcal{P} - \sigma} G_p + \sum_{q \in \sigma} \sum_{i=1}^{m_q+1} p^{k-i} \left( k - 1 \right) G(q,i) &= I
\end{align*}
$$

(10)

and the impulse responses

$$
\begin{align*}
\mathcal{J}(\Phi_u)(k) &= \sum_{p \in \mathcal{P}} p^{k-1} H_p \\
\mathcal{J}(\Phi_x)(k) &= \sum_{p \in \mathcal{P} - \sigma} p^{k-1} G_p + \sum_{q \in \sigma} \sum_{i=1}^{m_q+1} p^{k-i} \left( k - 1 \right) G(q,i)
\end{align*}
$$

(11)

It is straightforward to see that in (10)-(11) the SLS constraints and the impulse responses are affine and linear, respectively, in the coefficients $H_p$, $G_p$, and $G(q,i)$. As the impulse responses $\mathcal{J}$ and convolution operators $\mathcal{C}$ appearing in the objective (9) are linear in the impulse responses of $\Phi_x$ and $\Phi_u$, this implies that the terms inside the norms $|| \cdot ||_F$ and $|| \cdot ||_2$ are affine in the coefficients $H_p$, $G_p$, and $G(q,i)$. Therefore, since $|| \cdot ||_F$ and $|| \cdot ||_2$ are convex, the SPA control design (9)-(11) is convex.

As representations of $\mathcal{J}$ and $\mathcal{C}$ would require matrices of infinite size, in order to evaluate the norms $|| \cdot ||_F$ and $|| \cdot ||_2$ in the objective (9) in practice, we introduce a finite $T > 0$ and replace all instances of $\mathcal{J}$ and $\mathcal{C}$ in (9) by $\mathcal{J}_T$ and $\mathcal{C}_T$, respectively. Then these norms become the standard Frobenius and spectral matrix norms, so (9)-(11) can be formulated as a tractable semidefinite program (SDP). As the dimension of (9)-(11) is independent of the time horizon $T$, in practice one can take $T$ sufficiently large such that the Frobenius and spectral norms in the objective approximate arbitrarily well the true $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms, respectively.

Since the uncontrollable poles of the plant are included in $\Phi_x(z)$, feasibility is ensured whenever $(A,B)$ is stabilizable. As feasibility is guaranteed in this case and (9)-(11) is convex, unlike with DBC there is no need to introduce a slack variable or use iterative unimodal optimization methods for SPA. Instead, SPA can be solved with a single convex optimization (a SDP), and then the closed-loop responses are given by the linear expressions $T_{w \rightarrow u}(z) = \Phi_x(z) \bar{B}$ and $T_{w \rightarrow y}(z) = \Phi_u(z) \bar{B}$, which do not require inverting transfer functions as in Section III-B for DBC. Furthermore, note that the poles in $\mathcal{P}$ can be chosen to lie anywhere within the open unit disk, so this method does not result in FIR closed-loop transfer functions and, hence, avoids deadbeat control.

C. Suboptimality Bounds

By Assumption A1, there exists an optimal solution to (7), call it $(\Phi^*_x, \Phi^*_u)$, and its partial fraction decomposition is given by (3) where $T_{w \rightarrow u} = \Phi^*_u$ and $T_{w \rightarrow x} = \Phi^*_x$. Let $\Omega$ be the poles of $\Phi^*_x$, $m_q^*$ the multiplicity of pole $q$ in $\Phi^*_x$, and $\mathcal{P}(q) \subset \mathcal{P}$ the
m^*_q$ closest poles in $\mathcal{P}$ to $q$. Note that for any pole $q \in \Omega$, its approximating poles are $\mathcal{P}(q)$, so $d(\mathcal{P}(q), z)$ measures the minimum distance between the approximating poles of $q$ and the plant pole $z \in \sigma$. Taking minimums over all the poles in $\Omega$ and in the plant, we let $\delta = \min_{q \in \Omega} \min_{z \in \sigma, z \neq q} d(\mathcal{P}(q), z)$. Then $\delta$ represents the minimum of the distances between each pole $z$ of the plant and the poles in $\mathcal{P}$ being used to approximate all poles in $\mathcal{P}$ other than $z$.

Our main result shows that the relative error of the SPA method decays at least linearly with $D(\mathcal{P})$.

**Theorem 2** (Global Suboptimality Bound). Let $J^*$ denote the optimal cost of (7), and let $J(\mathcal{P})$ denote the optimal cost of (9)-(11) for any choice of $\mathcal{P}$. Suppose Assumption A1 is met, and $\mathcal{P}$ satisfies Assumptions A2-A5. Then there exists a constant $K = K(\Omega, G^*_q(q,j), H^*_q(q,j), r, \delta) > 0$ such that

$$
\frac{J(\mathcal{P}) - J^*}{J^*} \leq \frac{K}{D(\mathcal{P})}.
$$

While the DBC suboptimality bound in (5) only holds for $T$ sufficiently large such that $||\Phi^*_T(T)||_2 \leq C_\rho^2 < 1$, the SPA bound in (12) does not have this requirement (the condition that $D(\mathcal{P}) < 1$ can be satisfied with only two pairs of complex conjugate poles if the optimal solution has only simple poles, so it tends not to be restrictive in practice). Furthermore, the DBC bound includes a term $\frac{1}{C_\rho^2}$ resulting from the slack variable, whereas the SPA bound has no such term because it does not need a slack variable. Finally, the convergence for the DBC bound depends on the rate of decay of the optimal closed-loop impulse response, whereas the SPA bound convergence depends on the geometry of the pole selection - in particular, on how far the poles in $\mathcal{P}$ are from the optimal closed-loop poles. Therefore, SPA is preferable when the optimal impulse response takes long to decay, such as in stabilizable systems with large separation of time scales.

In addition, as discussed in Section IV-B, if prior knowledge is available about the location of some of the optimal poles $q$, then these optimal poles which are known a priori should be included in $\mathcal{P}$, which will result in a reduced value for $\bar{d}(q)$. Typically this will have the effect of decreasing both $D(\mathcal{P})$ and $\tilde{K}$ in (12), significantly reducing the relative error of SPA. In contrast, it is not clear how such prior knowledge could be included with DBC to reduce the relative error given by (5).

### D. Archimedes Spiral Pole Selection

Once the prior information about the optimal and plant poles have been incorporated into $\mathcal{P}$, if no information about the remaining optimal poles is known in advance, a natural choice for these remaining poles in $\mathcal{P}$ would be to distribute the poles evenly over the unit disk so as to minimize $D(\mathcal{P})$. However, finding an exactly even pole distribution over the unit disk is equivalent to finding the minimum energy configuration of a collection of identical point charges over a disk. This is a nonconvex optimization problem that has never been solved in the general case, requires high computational effort to solve even for smaller numbers of poles, and for which when solutions can be obtained they are not guaranteed to be globally optimal [16]. Therefore, rather than attempting to find exactly even pole distributions, we resort to finding approximate solutions instead. These approximations also have the advantage that, unlike for the exact solution methods, they can be used to derive suboptimality bounds with a convergence rate based on the geometry of the pole selection.

One class of heuristic techniques that has been used to generate approximately evenly spaced points over a disk, and that has worked well in practice in a variety of different fields, is to select points along a spiral [12], [17], [18], transforming the pole selection problem from two dimensions into one. After selecting a particular spiral, typically specified in polar coordinates by $r = r(\theta)$, appropriate points can then be chosen along the spiral to yield an approximately even distribution.

Therefore, we choose the poles $\mathcal{P}$ along an Archimedes spiral according to the selection proposed in [12, Section 5.1] and as shown in Fig. 1. This yields a powerful heuristic for minimizing $D(\mathcal{P})$ which tends to work well in practice (see for example Section V), and for which it is possible to derive suboptimality bounds with a particular convergence rate based on the spiral geometry (see Corollary 1). The spiral is then reflected over the imaginary axis to ensure that $\mathcal{P}$ is closed under complex conjugation. Letting $m$ be the total number of poles in $\mathcal{P}$, the poles chosen for this selection (as shown in Fig. 1) are given by

$$
\theta_k = 2\sqrt{\pi} k, \quad r_k = \sqrt{\frac{k}{m^2 + 1}}, \quad p_k = (r_k, \theta_k), \quad p_{-k} = (r_k, -\theta_k)
$$

for $k \in \{1, ..., \frac{m}{2}\}$. Corollary 1 shows that, for the selection of (13), the relative error of SPA converges to zero at a rate of the inverse square root of the number of poles. This implies that as $|\mathcal{P}|$ increases, the solution of SPA converges to the globally optimal solution of (7) at this same rate.

**Corollary 1** (Spiral Suboptimality Bound). Consider the setup of Theorem 2. For any positive even integer $m$, let $\mathcal{P}_m$ denote the selection of $m$ poles along an Archimedes spiral given by $p_k$ in (13) for $k \in \{-\frac{m}{2}, ..., -1, 1, ..., \frac{m}{2}\}$. Then there exists a constant $\tilde{K} = \tilde{K}(\Omega, G^*_q(q,j), H^*_q(q,j)) > 0$ such that

$$
\frac{J(\mathcal{P}_m) - J^*}{J^*} \leq \frac{\tilde{K}}{\sqrt{m + 2}}.
$$

Fig. 1. Archimedes spiral (green) with pole selection (red) to distribute the poles and their complex conjugates approximately evenly over the unit disk.
V. NUMERICAL EXAMPLE

To compare DBC and SPA, we consider the example of using a power converter to provide regulation services to the power grid, which arises naturally as a result of interfacing renewable generation such as photovoltaics or wind turbines to the grid, and is needed to ensure stable grid operation [19]. This example served as the motivation to develop the SPA method, because of the inadequate performance of DBC resulting from the large separation of time scales in power systems containing power converter interfaced devices [20]. The goal is to design the outer control loop of the power grid so that it provides frequency and voltage control to the power grid. Let \( w \) represent the frequency and voltage magnitude of the power grid at the point of connection, and let \( y \) represent the power output of the converter. Then this can be formulated as an optimal control problem for a system of the form (1) with matrices given by

\[
A = \begin{bmatrix}
0.988 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.9 \\
0 & 0 & 0 & 0.955 & 0 \\
0 & 0 & 0 & 0 & 0.01 \\
0 & 0 & 0 & 0.005 & 0 \\
0 & 0 & 0 & 0 & 0.1 \\
\end{bmatrix}, \quad B = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}, \quad D = 0.01I
\]

\[
\hat{B} = \begin{bmatrix}
-0.0001 & 0 & 0 & 0 & 0 \\
0.0001 & 0 & 0 & 0 & 0 \\
0 & 0.0066 & 0 & 0 & 0 \\
0 & 0.0001 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.01 & 0 \\
0 & 0 & 0 & 0 & 0.001 \\
0 & 0 & 0 & 0 & 0.1 \\
0 & 0 & 0 & 0 & 0.005 \\
\end{bmatrix}, \quad C = \begin{bmatrix}
0 & 0.829 & -0.428 & 1.02 & 0 \\
0 & 0.428 & 0.829 & 0 & -1.02 \\
\end{bmatrix}
\]

\[
T_{\text{desired}}(z) = \begin{bmatrix}
T^w_{\text{desired}}(z) \\
0
\end{bmatrix}
\]

With \( \lambda = 0 \), the above choice of matrices implies that the objective is to minimize

\[
\left\| \begin{bmatrix}
T^w_{\text{desired}}(z) - T^y_{\text{desired}}(z) \\
0.01 & T_{\text{desired}}(z)
\end{bmatrix} \right\|_{\mathcal{H}_2}^2
\]

For this example, we let \( t \in \mathbb{R} \) denote continuous time and \( k \in \mathbb{Z} \) denote discrete sampling time steps, where the sample time is \( h = 0.001 \) seconds. Such a high sampling rate is chosen to avoid aliasing for the fast dynamics of the power converter.

To solve (2), for DBC we use golden section search as suggested in [5, p. 380]. This requires solving SDPs iteratively to find the variables \( \Phi \) and \( \Phi_y \). Then, \( T_{\text{desired}}(z) = CT_{\text{desired}}(z)\hat{B} \) can be recovered from \( \Phi_z(z) \) as in Section III-B, which requires inverting \( (I + \frac{1}{2}z) \). For SPA, we let \( \Phi_z \) consist of the poles of the plant and \( T_{\text{desired}} \), and select the remaining poles from the Archimedes spiral as given by (13). Then, to solve (2) using SPA we only need to solve a single SDP, and then \( T_{\text{desired}}(z) = C\Phi_z(z)\hat{B} \) so no inversion of transfer functions is necessary. To solve the SDPs in each case, Matlab was used with YALMIP and the solver MOSEK. This control design implementation is available online [13].

The DBC and SPA control design approaches are run for varying numbers of poles. For DBC, the problem is infeasible for 30 or less poles because no solution exists in which \( \|\Phi_z(30)\|_2 < 1 \). For 31 poles, the DBC golden section search algorithm converges in 16 iterations, and for 300 poles it converges in 7 iterations, where each iteration requires the solution of a SDP. Recovering \( T_{\text{desired}}(z) \) from the DBC solution requires inverting a transfer function, as shown in Section III-B, but for large numbers of poles this leads to out of memory errors and numerical errors (e.g., unstable poles that should not exist). Therefore, the DBC impulse response and step response figures show only \( C\Phi_z(z)\hat{B} \) instead of the true system response \( T_{\text{desired}}(z) \), so the true DBC results are actually worse than the DBC results shown in these figures. The SPA method is feasible for any number of poles, and requires only one SDP for each number of poles. It is run for 7 and 15 poles, and \( T_{\text{desired}}(z) = C\Phi_z(z)\hat{B} \) for SPA so the true system responses \( T_{\text{desired}}(z) \) are easily recovered.

The impulse responses for the solutions of DBC, SPA, and the desired transfer function are shown in Fig. 2. For DBC, the impulse response is close to the desired impulse response only for the first 0.031 seconds or 0.300 seconds for 31 and 300 poles, respectively, after which the impulse response becomes zero (an undesirable but inevitable feature of DBC). However, the desired impulse response takes several seconds to decay, so overall the matching is very poor for DBC, with 300 poles only slightly better than with 31 poles. In contrast, for SPA the impulse response shows a large initial mismatch during the first few milliseconds, but after this the matching is much closer, with the 15 pole solution showing significantly better matching than the 7 pole case. Note that the large initial impulse responses of the SPA method could be reduced with additional convex constraints or by adding fast poles, neither of which are included here. From the impulse responses it is clear that SPA achieves much closer matching to the desired transfer function, and with orders of magnitude fewer poles.

The step responses for the solutions of DBC, SPA, and the desired transfer function are shown in Fig. 3. For DBC, the step responses deviate greatly from the desired step response, although the solution with 300 poles is closer than with 31 poles. With SPA the step responses are close to the desired step response, with the 15 pole solution showing closer matching during the initial few seconds than the 7 pole case. However, even with SPA there is a small steady state error. Note that this could be removed either by imposing convex constraints on the DC gain directly, or by changing the objective from the \( \mathcal{H}_2 \) norm to the difference in the step responses (which is
For any positive integer provide useful identities for developing approximation error approximating \( T_{\text{desired}} \) and the optimal poles. The next two lemmas the repeated pole they are approximating. In other words, repeated poles, and that the error in the approximation can poles can be used to approximate transfer functions with responses it is clear that SPA results in much closer matching also convex, although neither is pursued here. From the step results it is clear that SPA results in much closer matching with the desired transfer function behavior than DBC, and with far fewer poles.

VI. PROOFS

A. Proof of Theorem 1

The first step in proving Theorem 2 is to show that simple poles can be used to approximate transfer functions with repeated poles, and that the error in the approximation can be bounded by the distance of the approximating poles from the repeated pole they are approximating. In other words, we will show that the error depends on the geometry of the approximating and the optimal poles. The next two lemmas provide useful identities for developing approximation error bounds for a single repeated pole in the SISO case.

**Lemma 1.** For any positive integer \( m \), let \( p_1, \ldots, p_m, q \in \mathbb{D} \), and let \( I_m = \{1, 2, \ldots, m\} \). Then

\[
q^m - \prod_{i=1}^{m} p_i \leq \sum_{k=1}^{m} \sum_{S \subseteq I_m, |S| = k} |q|^{m-k} \prod_{i \in S} (p_i - q).
\]

**Proof of Lemma 1.** We compute

\[
\prod_{i=1}^{m} p_i = \prod_{i=1}^{m} ((p_i - q) + q) \quad \text{distributive property}
\]

\[
= \sum_{k=0}^{m} \sum_{S \subseteq I_m, |S| = k} m \binom{m}{k} (p_i - q)^k \prod_{i \in S} \]

which implies that

\[
q^m - \prod_{i=1}^{m} p_i \quad \text{above identity} \quad q^m - \sum_{k=0}^{m} \sum_{S \subseteq I_m, |S| = k} m \binom{m}{k} (p_i - q)^k \prod_{i \in S}
\]

by the triangle inequality. \( \square \)

**Lemma 2.** For any positive integer \( m \), let \( p_1, \ldots, p_m, q \in \mathbb{D} \), and let \( z \in \partial \mathbb{D} \). Then there exist constants \( c_1, \ldots, c_m \) such that

\[
\left| \sum_{i=1}^{m} c_i \frac{1}{z - p_i} - \frac{1}{(z - q)^m} \right| \leq \frac{1}{|\partial \mathbb{D}|} \sum_{i=1}^{m} |q|^{m-k} \prod_{j \in T} |p_j - q|
\]

\[
\leq \frac{d(q, \partial \mathbb{D})^m}{|\partial \mathbb{D}|} \prod_{i=1}^{m} d(p_i, \partial \mathbb{D})
\]

**Proof of Lemma 2.** First we show there exist constants \( \{c_i\}_{i=1}^{m} \) such that the following partial fraction decomposition holds:

\[
\sum_{i=1}^{m} c_i \frac{1}{z - p_i} = \prod_{i=1}^{m} (z - p_i).
\]

Multiplying both sides by the product \( \prod_{i=1}^{m} (z - p_i) \) and evaluating at \( z = p_i \) gives

\[
c_i = \frac{1}{\prod_{j \neq i} (p_i - p_j)}
\]

for all \( i \in I_m \), which satisfy (16). So, after choosing constants \( \{c_i\}_{i=1}^{m} \) by (17), to prove the claim it suffices to show that \( \prod_{i=1}^{m} (z - p_i) - \frac{1}{(z - q)^m} \) satisfies the inequality of Lemma 2. We compute

\[
\frac{1}{\prod_{i=1}^{m} (z - p_i)} - \frac{1}{(z - q)^m} \leq \prod_{i=1}^{m} \frac{(z - q)^m - \prod_{i=1}^{m} (z - p_i)}{(z - q)^m}
\]

Then

\[
(z - q)^m - \prod_{i=1}^{m} (z - p_i)
\]

\[
= \sum_{i=1}^{m} \binom{m}{k} (z - q)^{m-k} \prod_{j \neq i} (z - p_j)
\]

\[
\text{canceling terms}
\]

\[
\text{distributive property}
\]

\[
\text{difference of sums}
\]

where the last equality follows since the number of terms in \( \sum_{S \subseteq I_m, |S| = k} \) is the number of ways to select \( k \) poles out of \( m \).
poles, which is \( \binom{m}{n} \). Then, recalling that \( z \in \partial \mathbb{D} \) so \( |z| = 1 \), and applying Lemma 1 to poles \( -q \) and \( \{-p_i\}_{i=1}^k \) for each set \( S \) in the sum yields
\[
\left| (z - q)^m \prod_{i=1}^m (z - p_i) \right| \leq \sum_{k=1}^{m} \sum_{S \subseteq I_m, |S| = k} \sum_{|j|=i} (-q)^k - \prod_{j \in S} (-p_j) \left( \binom{m}{k} \prod_{i=1}^k (z - p_i) \right).
\]

Furthermore, since \( z \in \partial \mathbb{D} \)
\[
\left| (z - q)^m \prod_{i=1}^m (z - p_i) \right| \geq d(q, \partial \mathbb{D})^m \prod_{i=1}^m d(p_i, \partial \mathbb{D}).
\]

Thus, taking the absolute value of (18) and combining with (19)-(20) we obtain the desired bound. \( \square \)

Corollary 2 provides an approximation error bound for a single repeated pole in the SISO case.

**Corollary 2.** Let \( p_1, \ldots, p_m \in \partial \mathbb{B} \), for some \( r \in (0, 1) \), \( q \in \mathbb{D} \), \( z \in \partial \mathbb{D} \), and \( \mathcal{P} = \bigcup_{i=1}^{m} \). Let \( \hat{d}(q) = \max_{|p-j|} \) and assume that \( \hat{d}(q) < 1 \). Let \( \bullet \in \{q, \infty\} \). Then there exist constants \( c_1, \ldots, c_m \) such that
\[
\left\| \sum_{i=1}^{m} c_i \frac{1}{z - p_i} - \frac{1}{(z - q)^m} \right\|_{H_\bullet} \leq \left( \frac{|q| + 2}{d(q, \partial \mathbb{D})^m (1 - r)^m} \right) \hat{d}(q).
\]

**Proof of Corollary 2.** To prove the claim, we upper bound the right-hand side of (15). We compute
\[
\sum_{k=1}^{m} \sum_{S \subseteq I_m, |S| = k} \sum_{i=1}^{k} \binom{k}{i} (-q)^i \hat{d}(q)^i |S| = k \sum_{|j|=i} \prod_{j \in T} |p_j - q|
\]

by \( d(q) \)
\[
\sum_{k=1}^{m} \sum_{S \subseteq I_m, |S| = k} \sum_{|j|=i} \binom{k}{i} (-q)^i \hat{d}(q)^i |S| = k \sum_{|j|=i} \prod_{j \in T} |p_j - q|
\]

combining equal terms
\[
\sum_{k=1}^{m} \sum_{S \subseteq I_m, |S| = k} \binom{k}{i} \sum_{|j|=i} \left( (|q| + \hat{d}(q))^k - |q|^k \right)
\]

binomial theorem
\[
\sum_{k=1}^{m} \binom{m}{k} \left( (|q| + \hat{d}(q))^k - |q|^k \right)
\]

binomial theorem
\[
\left( (|q| + \hat{d}(q) + 1)^m - 1 \right) - \left( (|q| + 1)^m - 1 \right)
\]

canceling
\[
\frac{1}{d(q)} \left( (|q| + 1 + \hat{d}(q))^m - (|q| + 1)^m \right)
\]

binomial theorem
\[
\sum_{k=0}^{m} \binom{m}{k} (|q| + 1)^m - k \hat{d}(q)^k \frac{1}{d(q)} \left( (|q| + 1)^m - k \hat{d}(q)^k \right)
\]

canceling
\[
\sum_{k=1}^{m} \binom{m}{k} (|q| + 1)^m - k \hat{d}(q)^k \frac{1}{d(q)} \left( (|q| + 1)^m - k \hat{d}(q)^k \right)
\]

since \( \hat{d}(q) \leq \hat{d}(q) \) because \( \hat{d}(q) < 1 \). Note that
\[
d(q, \partial \mathbb{D})^m \prod_{i=1}^m d(p_i, \partial \mathbb{D}) \geq d(q, \partial \mathbb{D})^m (1 - r)^m
\]

in the sum yields
\[
\left\| \sum_{i=1}^{m} c_i \frac{1}{z - p_i} - \frac{1}{(z - q)^m} \right\|_{H_\bullet} \leq \left( \frac{|q| + 2}{d(q, \partial \mathbb{D})^m (1 - r)^m} \right) \hat{d}(q).
\]

Taking the supremum over \( z \in \partial \mathbb{D} \) yields
\[
\left\| \sum_{i=1}^{m} c_i \frac{1}{z - p_i} - \frac{1}{(z - q)^m} \right\|_{H_\infty} \leq \left( \frac{|q| + 2}{d(q, \partial \mathbb{D})^m (1 - r)^m} \right) \hat{d}(q).
\]

Finally, for any \( T \in \mathcal{P} \) SISO, \( ||T||_{H_2} \leq ||T||_{H_\infty} \), which, combined with the above inequality, yields the desired bound. \( \square \)

Theorem 1 extends the approximation error bound to an arbitrary number of (possibly repeated) poles and to the MIMO case.

**Proof of Theorem 1.** The proof begins by writing the partial fraction decomposition of \( T \), and for each pole \( q \) of \( T \) using the SISO approximation of Corollary 2 to approximate each SISO term \( T_{\{q\}}^1 \) with the \( j \) nearest poles in \( \mathcal{P} \). These are then combined with the MIMO coefficients in the partial fraction decomposition of \( T \) to yield the approximating transfer function \( \sum_{q \in \mathcal{P}} G_q z^{-q} \). Care must be taken to ensure symmetry between approximations of complex conjugate poles in \( T \) so that the approximating transfer function has real coefficients. Next, using the SISO approximation error bounds of Corollary 2, the main approximation error bounds of the corollary are derived. Finally, it is shown that, by construction, the approximating transfer function does indeed have real coefficients, and therefore belongs to \( \frac{1}{2} \mathcal{H}_\infty \).
We begin by writing the partial fraction decomposition of $T$ and constructing the approximating transfer function. Let $\Omega_R \subset \Omega$ denote the real poles of $T$, and let $\Omega_C \subset \Omega$ denote the remaining poles. Since $T \in \frac{1}{z} \mathcal{H}_\infty$, we can write its partial fraction decomposition with matrix-valued coefficients $C_{(q,j)}$ as

$$T(z) = \sum_{q \in \Omega_R} \sum_{j=1}^{m_q} C_{(q,j)} \frac{1}{z - q_j} + \sum_{q \in \Omega_C} \sum_{j=1}^{m_q} C_{(q,j)} \frac{1}{z - q_j}$$

(22)

For each $q \in \Omega$ and each $j \in I_{m_q}$, let $\mathcal{P}(q,j) \subset \mathcal{P}(q)$ denote the $j$ closest poles in $\mathcal{P}$ to $q$ (or at least one choice in case this is not unique) and choose constants $\left\{ c_{(p,j)} \right\}_{p \in \mathcal{P}(q,j)}$ as in Corollary 2 for approximating the pole $q$ with multiplicity $j$ by $\mathcal{P}(q,j)$. Then, choose $\mathcal{P}(q,j) = \mathcal{P}(q,j)$ (note that by symmetry, since $\mathcal{P}$ and $\Omega$ are both closed under complex conjugation, these are the $j$ closest poles in $\mathcal{P}$ to $\overline{q}$) and choose constants $\left\{ c_{(p,j)} \right\}_{p \in \mathcal{P}(q,j)}$ as in Corollary 2 for approximating the pole $\overline{q}$ with multiplicity $j$ by $\mathcal{P}(\overline{q},j)$. Note that in case $q$ is real, for each $j \in I_{m_q}$ this results in two sets of constants for approximating $q$: one for $\mathcal{P}(q,j)$ and one for $\mathcal{P}(\overline{q},j) = \mathcal{P}(q,j)$ (where we have abused notation for simplicity of presentation).

Then for each $p \notin \mathcal{P}(q,j)$, let $c_{p,q,j} = 0$, and for each $p \notin \mathcal{P}(\overline{q},j)$, let $c_{p,j} = 0$ for notational convenience. For each $q \in \Omega$ and each $j \in I_{m_q}$, let $d(q,j) = \max_{p \in \mathcal{P}(q,j)} |p - q|$. Then $d(q,j) \leq d(q) \leq D(\mathcal{P})$. By Corollary 2, this implies that for each $q \in \Omega$ and each $j \in I_{m_q}$

$$\left\| \sum_{p \in \mathcal{P}(q,j)} c_{p,q,j} \frac{1}{z - q_j} + \frac{1}{z - q_j} \right\|_{\mathcal{H}_\infty} \leq \left( \frac{|\overline{q} + 2|^j - (|q| + 1)^j}{d(q, \mathcal{P}(q,j) \hat{2}) (1 - r)^j} \right) \tilde{d}(q,j) \leq D(\mathcal{P})$$

(23)

and similarly for $\overline{q}$. Then for each $p \in \mathcal{P}$, define

$$G_p = \sum_{q \in \Omega_R} \sum_{j=1}^{m_q} C_{(q,j)} c_{p,q,j} + \frac{1}{2} \sum_{q \in \Omega_C} \sum_{j=1}^{m_q} C_{(q,j)} c_{p,j}$$

(24)

This completes the construction of the approximating transfer function $\sum_{p \in \mathcal{P}} G_p \frac{1}{z - p}$. Next we show that the approximating transfer function satisfies the desired approximation error bounds of (6). We compute

$$\left\| \sum_{p \in \mathcal{P}} G_p \frac{1}{z - p} - T \right\|_{\mathcal{H}_\infty} \leq \left( \sum_{q \in \Omega_R} \sum_{j=1}^{m_q} \left( \frac{1}{2} \sum_{p \in \mathcal{P}} c_{p,q,j} \frac{1}{z - q_j} + \frac{1}{z - q_j} \right) \right)$$

$$\leq \left( \sum_{q \in \Omega_R} \sum_{j=1}^{m_q} \left( \frac{1}{2} \sum_{p \in \mathcal{P}} c_{p,q,j} \frac{1}{z - q_j} + \frac{1}{z - q_j} \right) \right)$$

$$\leq \sum_{q \in \Omega_C} \sum_{j=1}^{m_q} \left( \frac{1}{2} \sum_{p \in \mathcal{P}} c_{p,q,j} \frac{1}{z - q_j} + \frac{1}{z - q_j} \right)$$

$$\leq \sum_{q \in \Omega_C} \sum_{j=1}^{m_q} \left( \frac{1}{2} \sum_{p \in \mathcal{P}} c_{p,q,j} \frac{1}{z - q_j} + \frac{1}{z - q_j} \right)$$

(22)\(24)

This gives the result for the $\mathcal{H}_\infty$ case. Let $s$ be the minimum of the dimensions of $T$, and note that

$$||T||_{\mathcal{H}_s} \leq \sqrt{s} ||T||_{\mathcal{H}_\infty} \leq \sqrt{s} K_\infty D(\mathcal{P}) = K_2 D(\mathcal{P})$$

where $K_2 = \sqrt{s} K_\infty$. This completes the result for the $\mathcal{H}_s$ case.

Finally, we show that $\sum_{p \in \mathcal{P}} G_p \frac{1}{z - p} \in \frac{1}{z} \mathcal{H}_\infty$. Clearly $\sum_{p \in \mathcal{P}} G_p \frac{1}{z - p}$ is strictly proper, rational, and stable, so it suffices to show it has real coefficients. Since $T$ has real coefficients it satisfies $T(z) = T(\overline{z})$ which implies, by matching coefficients in the partial fraction decomposition and since $\mathcal{P}(q,j) = \mathcal{P}(q,j)$ for all $q \in \Omega$ and $j \in I_m$

$$C_{(q,j)} = \overline{C_{(q,j)}}$$

(25)

for all $q \in \Omega_C$ and $j \in I_{m_q}$, and that

$$C_{(q,j)} = \overline{C_{(q,j)}}$$

(26)

for all $q \in \Omega_R$ and $j \in I_{m_q}$. Furthermore, using the definition of $\left\{ c_{p,j} \right\}_{p \in \mathcal{P}(q,j)}$ from (17) and since $\mathcal{P}(q,j) = \mathcal{P}(q,j)$ for all $q \in \Omega$ and $j \in I_m$, it is straightforward to verify that

$$c_{p,j} = \overline{c_{p,j}}$$

(27)

for all $q \in \Omega$ and $j \in I_{m_q}$. It is also straightforward to verify that for any complex-valued matrix $M$ and pole $p \in \mathbb{D}$, $M \frac{1}{z - p} + M^* \frac{1}{\overline{z} - \overline{p}}$ can be expressed as a rational transfer function matrix with real coefficients (we will refer to this later as fact (a)). Let $\Omega_+ \subset \Omega_C$ be the subset of poles with nonnegative imaginary component, and similarly for $\mathcal{P}_+ \subset \mathcal{P}$. Then

$$\sum_{p \in \mathcal{P}_+} G_p \frac{1}{z - p}$$
Under the conditions of Theorem 2, let
\[
\left( \frac{c^{(q,j)}_{P}}{2} + \frac{c^{(\tau,j)}_{P}}{2} \right) \frac{1}{z - p} + \frac{c^{(q,j)}_{P}}{2} \frac{1}{z - \bar{p}}
\]
regrouping terms
\[
\sum_{p \in \mathcal{P}, q \in \mathcal{Q}_{k}, j = 1}^{m_{q}} \left( C_{(q,j)} e^{(q,j)}_{P} \frac{1}{z - p} + C_{(\tau,j)} e^{(\tau,j)}_{P} \frac{1}{z - \bar{p}} \right)
\]
which, by fact (a) above and (26), is a sum of rational transfer functions with real coefficients, hence it is rational with real coefficients.

\[\text{Proof of Lemma 3.}\]
Let \( \mathcal{P} = \{p_{1}, ..., p_{m}\} \). We compute
\[
\left| \frac{(z - q)^{k}}{\prod_{i=1}^{m} (z - p_{i})} - (z - q)^{k-m} \right| = \left| \frac{(z - q)^{m} - \prod_{i=1}^{m} (z - p_{i})}{(z - q)^{m-k} \prod_{i=1}^{m} (z - p_{i})} \right|
\]
which, by fact (a) above and (26), is a sum of rational transfer functions with real coefficients, hence it is rational with real coefficients.

\[\text{Proof of Lemma 4.}\] Let \( \mathcal{P} = \{p_{1}, ..., p_{m}\} \). We compute
\[
\left| \frac{(z - q)^{k}}{\prod_{i=1}^{m} (z - p_{i})} - (z - q)^{k-m} \right| = \left| \frac{(z - q)^{m} - \prod_{i=1}^{m} (z - p_{i})}{(z - q)^{m-k} \prod_{i=1}^{m} (z - p_{i})} \right|
\]
Noting that the proofs of (19) and (21) are still valid for \( z \in \mathbb{D} \) (i.e. \( |z| \leq 1 \)), applying them here we have that
\[
\left| (z - q)^{m} - \prod_{i=1}^{m} (z - p_{i}) \right| \leq (|q| + 2)^{m} - (|q| + 1)^{m} \tilde{d}(q).
\]
Furthermore,
\[
\left| (z - q)^{m-k} \prod_{i=1}^{m} (z - p_{i}) \right| \geq d(z, q)^{m-k} d(z, \mathcal{P})^{m}.
\]
Combining these two inequalities implies that
\[
\frac{(z - q)^{k} - (z - q)^{k-m}}{\prod_{i=1}^{m} (z - p_{i})} \leq K \tilde{d}(q)
\]
where
\[
K = \frac{(|q| + 2)^{m} - (|q| + 1)^{m}}{d(z, q)^{m-k} d(z, \mathcal{P})^{m}}.
\]

\[\text{Proof of Lemma 5.}\] Let \( k \) be a nonnegative integer, \( m \) a positive integer and \( q, p_{1}, ..., p_{m} \in \mathbb{D} \). Choose constants \( c_{p_{j}} \) as in the proof of Lemma 2. Then
\[a.\] For \( k < m \)
\[
\sum_{i=1}^{m} (p_{i} - q)^{k} c_{p_{j}} \frac{1}{z - p_{i}} = \frac{(z - q)^{k}}{\prod_{i=1}^{m} (z - p_{i})}.
\]
\[b.\] For \( k \geq m \)
\[
\sum_{i=1}^{m} (p_{i} - q)^{k} c_{p_{j}} \frac{1}{z - p_{i}} = \frac{(z - q)^{k}}{\prod_{i=1}^{m} (z - p_{i})} - \sum_{i=0}^{k-m} b_{i}(z - q)^{i}
\]
where
\[
b_{i} = \sum_{j=1}^{m} (p_{j} - q)^{k-1-i} c_{p_{j}}.
\]

\[\text{Proof of Lemma 5.}\] First consider Case (a). Write the partial fraction decomposition
\[
\frac{(z - q)^{k}}{\prod_{i=1}^{m} (z - p_{i})} = \sum_{i=1}^{m} \kappa_{i} \frac{1}{z - p_{i}}.
\]
Multiplying both sides by \( \prod_{i=1}^{m} (z - p_{i}) \) and evaluating at \( z = p_{i} \) implies that
\[
\kappa_{i} = \frac{(p_{i} - q)^{k}}{\prod_{j=1}^{m} (p_{j} - p_{j})} = (p_{i} - q)^{k} c_{p_{i}}
\]
which completes the proof for Case (a).
Next consider Case (b). Write the partial fraction decomposition
\[
\frac{(z-q)^k}{\prod_{i=1}^{m} (z-p_i)} = \sum_{i=1}^{m} \frac{\kappa_i}{z-p_i} + \sum_{i=0}^{k-m} b_i (z-q)^i. \tag{30}
\]
Multiplying by \( \prod_{i=1}^{m} (z-p_i) \) and evaluating at \( z = p_i \) implies
\[
\kappa_i = \frac{(p_i-q)^k}{\prod_{j \neq i} (p_i-p_j)} = (p_i-q)^k c_{p_i}.
\]
Differentiating (30) \( i \) times with respect to \( z \) and evaluating at \( z = q \) implies that
\[
0 = -i! \sum_{j=1}^{m} (p_j-q)^{k-1-i} c_{p_j} + i! b_i
\]
for \( i \in \{0, \ldots, k-m\} \), so
\[
b_i = \sum_{j=1}^{m} (p_j-q)^{k-1-i} c_{p_j}.
\]

It is useful to introduce the notions of rising and falling factorials for the statement and proof of Lemma 6. Let \( m \) and \( n \) be integers. Define the rising factorial \( m^{(n)} = \prod_{k=0}^{n-1} (m+k) \) and the falling factorial \( m_{(n)} = \prod_{k=0}^{n-1} (m-k) \). Note that for \( m \) and \( n \) nonnegative, letting \( m! \) denote the standard factorial, we have \( m^{(n)} = \frac{(m+n-1)!}{(m-1)!} \) and \( m_{(n)} = \frac{m!}{(m-n)!} \). It is straightforward to verify the following facts for converting between rising and falling factorials, and for a binomial theorem for falling factorials:

**Fact 3.** \((-1)^n m^{(n)} = (-m)_n\)

**Fact 4.** \( \sum_{j=0}^{n} \binom{n}{j} m_j (m')_{n-j} = (m+m')_n \).

**Lemma 6.** Let \( k \) and \( m \) be positive integers, \( z \in \partial \mathbb{D} \), and \( q, \lambda, p_1, \ldots, p_m \in \mathbb{D} \). Choose constants \( c_{p_i} \) as in the proof of Lemma 2. Then

\[ a. \text{ There exist } K > 0 \text{ such that} \]
\[
\sum_{i=1}^{m} c_{p_i} \frac{(\lambda-p_i)^{-k}}{z-p_i} = \frac{(\lambda-z)^{-k}}{\prod_{i=1}^{m} (z-p_i)} - \frac{r(z)}{(\lambda-z)^k}
\]
\[
r(z) = \sum_{n=0}^{k-1} a_n (\lambda-z)^n
\]

and
\[
\lim_{z \to \lambda} \frac{d}{dz^l} \left( r(z) \prod_{i=1}^{m} (z-p_i) \right) = \begin{cases} 
-1, & l = 0 \\
0, & l \in \{1, \ldots, k-1\} \\
(-1)^{l+1} m^{(l)} + \epsilon \prod_{i=1}^{m} (\lambda-p_i), & l = k 
\end{cases}
\]
\[
|\epsilon| \leq K \hat{d}(q).
\]

\[ b. \text{ There exist } K'_0, \ldots, K'_{k-1} > 0 \text{ such that} \]
\[
\sum_{i=1}^{m} c_{p_i} \frac{(p_i-q)^m}{(\lambda-p_i)^k} \frac{1}{z-p_i} = \frac{(z-q)^m}{(\lambda-z)^k \prod_{i=1}^{m} (z-p_i)}
\]
\[
- \sum_{n=0}^{k-1} \frac{a_n}{(\lambda-z)^k-n}
\]
\[
|a_0 - 1| \leq K'_0 \hat{d}(q), \ |a_n| \leq K'_n \hat{d}(q), \ n \in \{1, \ldots, k-1\}.
\]

**Proof of Lemma 6.** For \( l \in \{0, m\} \), write the partial fraction decomposition
\[
\frac{(z-q)^l}{(\lambda-z)^k \prod_{i=1}^{m} (z-p_i)} = \sum_{i=1}^{m} \frac{1}{z-p_i} + \frac{r(z)}{(\lambda-z)^k}
\]
\[
r(z) = \sum_{n=0}^{k-1} a_n (\lambda-z)^n.
\]

Multiplying both sides by \( (\lambda-z)^k \prod_{i=1}^{m} (z-p_i) \) yields
\[
(z-q)^l = (\lambda-z)^k \sum_{i=1}^{m} \frac{\kappa_i}{z-p_i} + r(z) \prod_{i=1}^{m} (z-p_i).
\]

Evaluating (32) at \( z = p_i \) implies that
\[
\kappa_i = c_{p_i} (p_i-q)^l (\lambda-p_i)^{-k}.
\]

For \( n \) any nonnegative integer, define
\[
b_n = \left( \frac{d}{dz^n} r(z) \right) (\lambda) = (-1)^n n! a_n \tag{33}
\]
\[
d_n = \left( \frac{d}{dz^n} \prod_{i=1}^{m} (z-p_i) \right) (\lambda) = \sum_{v \in \mathbb{S}^m_{l}} \prod_{i=1}^{m} (\lambda-p_k) \tag{34}
\]
\[
e_n = \left( \frac{d}{dz^n} (z-q)^l \right) (\lambda) = t_n (\lambda-q)^{l-n} \tag{35}
\]

Note that
\[
\lim_{z \to \lambda} \frac{d}{dz^l} \left( r(z) \prod_{i=1}^{m} (z-p_i) \right) = \sum_{j=0}^{n} \binom{n}{j} d_j b_{n-j} \tag{36}
\]
for any nonnegative integer \( n \). Differentiating (32) \( n \) times with respect to \( z \), and evaluating at \( z = \lambda \) implies that
\[
e_n = \lim_{z \to \lambda} \frac{d}{dz^n} \left( r(z) \prod_{i=1}^{m} (z-p_i) \right) = \sum_{j=0}^{n} \binom{n}{j} d_j b_{n-j} \tag{37}
\]
for \( n \in \{0, \ldots, k-1\} \). Dividing by \( d_0 \) and solving for \( b_n \) implies
\[
b_n = \frac{e_n}{d_0} - \sum_{j=1}^{n} \binom{n}{j} d_j d_0^{-1} b_{n-j} \tag{38}
\]

Note that
\[
\frac{d}{dz} \frac{1}{\lambda-p_k} = \sum_{v \in \mathbb{S}^m_{l}} \prod_{i=1}^{m} \frac{1}{\lambda-p_k}.
\]
Define
\[\epsilon_n' = \frac{d_n}{d_0} - \frac{m_n}{(\lambda - q)^n} = \sum_{v \in \mathbb{N}^n \setminus \{v_i \neq v_j \text{ for } i \neq j\}} \left( \prod_{k \in v} \frac{1}{\lambda - p_k} - \frac{1}{(\lambda - q)^n} \right)\]
since the number of terms in the sum is \(m_n\). Thus, by Lemma 4 there exists \(k'_n > 0\) such that
\[\frac{d_n}{d_0} = m_n \frac{1}{(\lambda - q)^n} + \epsilon_n', \quad |\epsilon_n'| \leq k'_n \hat{d}(q). \quad (39)\]
Consider first Case (a): \(l = 0\). Then \(\epsilon_0 = 1\) and \(\epsilon_n = 0\) for \(n \in \{1, \ldots, k - 1\}\). By (37), this implies the desired result for \(n \in \{0, \ldots, k - 1\}\), so it suffices to prove the desired result for \(n = k\). By (38), \(b_0 = \frac{1}{d_0}\). We claim that there exists \(k_n > 0\) such that
\[-\sum_{j=1}^{n} \binom{n}{j} \frac{d_j}{d_0} b_{n-j} = \frac{(1)^m m(n)}{(\lambda - q)^{m+n}} + \epsilon_n, \quad |\epsilon_n| \leq k_n \hat{d}(q) \quad (40)\]
for all \(n \in \{1, \ldots, k\}\). Note that by (38), this implies that
\[b_n = (1)^m m(n) \frac{1}{(\lambda - q)^{m+n}} + \epsilon_n, \quad |\epsilon_n| \leq k_n \hat{d}(q) \quad (41)\]
for \(n \in \{1, \ldots, k - 1\}\). We prove (40) by strong induction. For the base case, first note that
\[b_0 = \frac{1}{d_0} = \frac{1}{(\lambda - q)^m} + \frac{1}{\prod_{i=1}^{m} (\lambda - p_i)} - \frac{1}{(\lambda - q)^m} \quad (42)\]
where such \(k_0 > 0\) exists by Lemma 4. Then for \(n = 1\) we have
\[-\frac{d_1}{d_0} b_0 = -\left( \frac{m}{\lambda - q} + \epsilon'_1 \right) \left( \frac{1}{(\lambda - q)^m} + \epsilon_0 \right) = -\frac{m}{(\lambda - q)^m} - \frac{m}{\lambda - q} \epsilon_0 - \frac{1}{(\lambda - q)^m} \epsilon'_1 - \epsilon_0 \epsilon'_1 = -\frac{mk_0}{(\lambda - q)^m} + \epsilon_1, \quad |\epsilon_1| \leq k_1 \hat{d}(q)\]
\[k_1 = \frac{m}{\lambda - q} + \frac{k'_0}{(\lambda - q)^m} + k_0 k'_1, \quad \epsilon_n = -\sum_{j=1}^{n} \binom{n}{j} \frac{d_j}{d_0} b_{n-j} = -\sum_{j=1}^{n} \binom{n}{j} \frac{m_j \frac{1}{(\lambda - q)^j}}{(\lambda - q)^{m+n-j}} + \epsilon'_n = \sum_{j=1}^{n} \binom{n}{j} \frac{m_j \epsilon_n - 1}{(\lambda - q)^j} + \epsilon_n \epsilon_n = \sum_{j=1}^{n} \binom{n}{j} \frac{m_j \epsilon_n - 1}{(\lambda - q)^j} + \epsilon_n \epsilon_n \]
so
\[|\epsilon_n| \leq k_n \hat{d}(q)\]
\[k_n = \sum_{j=1}^{n} \binom{n}{j} \left( \frac{m_j k_{n-j}}{(\lambda - q)^j} + \frac{m(n-j) k'_n}{(\lambda - q)^{m+n-j}} + k_{n-j} k'_n \right)\]

Thus, (40) holds. Note that \(b_k = 0\) since \(r(z)\) is a polynomial of order \(k - 1\). Therefore, by (36) and (40) we have that
\[\lim_{z \to q} \frac{d}{dz} \left( r(z) \prod_{i=1}^{m} (z - p_i) \right) = \sum_{j=0}^{k} \binom{k}{j} d_j b_{k-j} \]
\[= d_0 b_k + \sum_{j=1}^{k} \binom{k}{j} d_j b_{k-j} = \sum_{j=1}^{k} \binom{k}{j} d_j b_{k-j} \]
\[= (-d_0) (-\sum_{j=1}^{k} \binom{k}{j} \frac{d_j}{d_0} b_{k-j}) \]
\[= (-d_0) ((-1)^k m(k) + \epsilon_k), \quad |\epsilon_k| \leq k_k \hat{d}(q)\]
which yields the result for Case (a). Next consider Case (b): \(l = m\). Then by (38) and (33)
\[a_0 = b_0 = \frac{\epsilon_0}{d_0} = \frac{(\lambda - q)^m}{\prod_{i=1}^{m} (\lambda - p_i)} \]

For the induction step, assume that (40) holds for all \(j \in \{1, \ldots, n - 1\}\), which, together with (42), implies that (41) holds for all \(j \in \{0, \ldots, n - 1\}\). By (39) and (41) we have
\[-\sum_{j=1}^{n} \binom{n}{j} \frac{d_j}{d_0} b_{n-j} = -\sum_{j=1}^{n} \binom{n}{j} \left( m_j \frac{1}{(\lambda - q)^j} + \epsilon'_n \right) = -\sum_{j=1}^{n} \binom{n}{j} \frac{m_j \epsilon_n - 1}{(\lambda - q)^j} + \epsilon_n \]
\[k_n = \sum_{j=1}^{n} \binom{n}{j} \left( \frac{m_j k_{n-j}}{(\lambda - q)^j} + \frac{m(n-j) k'_n}{(\lambda - q)^{m+n-j}} + k_{n-j} k'_n \right)\]

so
\[\left| a_0 - 1 \right| = \left| \frac{(\lambda - q)^m}{\prod_{i=1}^{m} (\lambda - p_i)} - 1 \right| \leq k_0 \hat{d}(q) \]
where such \(k_0 > 0\) exists by Lemma 4. We claim that there exist \(k_n > 0\) such that
\[|b_n| \leq k_n \hat{d}(q) \quad (43)\]
for \(n \in \{1, \ldots, k - 1\}\). We prove (43) by strong induction. For the base case, note that by (38) and (39)
\[b_1 = \frac{\epsilon_1}{d_1} \frac{d_1}{d_0} b_0 = \frac{m(\lambda - q)^{m-1}}{(\lambda - q)^m} - \left( \frac{1}{\lambda - q} + \epsilon_1 \right) \frac{\epsilon_0}{d_0} \]
the result for Case (b). Therefore, (43) holds. Combining (43) with (31) and (33), we obtain:

\[ b_n = \frac{m_n(\lambda - q)^{m-n}}{d_0} - \frac{m_n(\lambda - q)^{m-n} + \epsilon'_m}{d_0} \]

which can be regrouped as:

\[ b_n = \frac{m_n(\lambda - q)^{m-n} - m_n(\lambda - q)^{m-n}}{d_0} - \frac{\epsilon'_m(\lambda - q)^m}{\prod_{i=1}^{m}(\lambda - p_i)} \]

For the inductive step, assume (43) holds for all \( j \in \{1, \ldots, n-1\} \). By (38), (39), (35), and the induction hypothesis, we have

\[ b_n = \frac{m_n(\lambda - q)^{m-n}}{d_0} - \frac{m_n(\lambda - q)^{m-n} + \epsilon'_m}{d_0} \]

which can be regrouped as:

\[ b_n = \frac{m_n(\lambda - q)^{m-n} - m_n(\lambda - q)^{m-n}}{d_0} - \frac{\epsilon'_m(\lambda - q)^m}{\prod_{i=1}^{m}(\lambda - p_i)} \]

\[ b_n = k_n d(q) \]

Thus, (43) holds. Combining (43) with (31) and (33) yields the result for Case (b).

For \( z \in \mathbb{C} \), let \( J(z) \) denote an elementary Jordan block with eigenvalue \( z \) whose dimension can be inferred from context.

**Corollary 3.** The Let \( k \) be a nonnegative integer, \( m \) a positive integer, \( z \in \partial \mathbb{D} \), and \( q, \lambda, p_1, \ldots, p_m \in \mathbb{D} \). Choose constants \( c_{p_i} \) as in the proof of Lemma 2. Then

a. There exists \( K > 0 \) such that

\[ \sum_{i=1}^{m} \frac{c_{p_i} (\lambda - p_i)^{-(k+1)}}{(\lambda - p_i)^{k+1}} = \left( m - 1 + k \right) (\lambda - q)^{-(m+k)} + \epsilon \]

\[ |\epsilon| \leq K d(q) \]

b. There exists \( K > 0 \) such that

\[ \left| \sum_{i=1}^{m} c_{p_i} (\lambda - p_i)^{m} J(\lambda - p_i)^{-1} \frac{1}{z - p_i} \right| \leq K d(q) \]
Let $\tilde{\lambda} = \frac{1}{\lambda - q}$ and $\tilde{\mu} = \frac{1}{\mu - q}$, and poles $p_j$, such that the following hold:

$$H_j^i = c_j^i \tilde{\lambda}^i, \quad G_j^i = -J(q - p_j^{-1})^{-1} \tilde{\lambda}^i BH_j^i.$$  \hspace{1cm} (44)

Then

$$\sum_{i=1}^{m} \sum_{j=1}^{m} G_j^i \frac{1}{z - p_j^i} + \tilde{\mu} \sum_{j=1}^{m+1} \sum_{i=1}^{m} J(0)^{k-1-l} \frac{1}{(z - q)^l}$$

$$= \sum_{i=1}^{m+1} \sum_{j=1}^{m} J(0)^{k-1-l} \frac{1}{(z - q)^l}$$

where such $K' > 0$ exists by Lemma 4. This proves Case (b).

\[ \blacksquare \]

**Lemma 7.** Let $\hat{m} \in \{0, 1\}$, let $m_q, m > 0$ be integers, and let $q \in \mathbb{D}$. Suppose that for each $i \in \{1, ..., m\}$ we have matrices $G_j^i$ and $H_j^i$, and for each $j \in \{1, ..., i\}$ we have matrices $H_j^i$ and $G_j^i$, and poles $p_j$, such that the following hold:

$$H_j^i = c_j^i \tilde{\lambda}^i, \quad G_j^i = -J(q - p_j^{-1})^{-1} \tilde{\lambda}^i BH_j^i.$$  \hspace{1cm} (44)

Then

$$\sum_{i=1}^{m} \sum_{j=1}^{m} G_j^i \frac{1}{z - p_j^i} + \hat{m} \sum_{j=2}^{m+1} \sum_{i=1}^{m} J(0)^{k-1-l} \frac{1}{(z - q)^l}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m+1} \sum_{i=1}^{m} J(0)^{k-1-l} \frac{1}{(z - q)^l}$$

where such $K' > 0$ exists by Lemma 4. This proves Case (b).

\[ \blacksquare \]
We compute

\[ \Phi(z) = \sum_{i=1}^{m} \sum_{j=1}^{l} G^i_j \frac{1}{z - p^i_j} + \hat{m} \sum_{i=2}^{m+1} J(0)^{l-2} \sum_{i=1}^{m} BH^i_j \frac{1}{(z - q)^l} - m \sum_{i=1}^{l} J(0)^{l-1} \sum_{i=1}^{m} G^i_j \frac{1}{(z - q)^l} \]

comparing terms in sum

\[ \sum_{i=1}^{m} \sum_{j=1}^{l} c^i_j J(0)^{k-1} BH^i_j \]

\[ \lambda_i \sum_{i=1}^{m} c^i_j \]

\[ \sum_{j=1}^{i} c^i_j (p^i_j - q)^k J(q - p^i_j)^{-1} BH^i_j \]

\[ \sum_{j=1}^{i} (p^i_j - q)^{k} G^i_j. \]

\[ \Phi(z) = \sum_{i=1}^{m} \sum_{j=1}^{l} G^i_j \frac{1}{z - p^i_j} + \hat{m} \sum_{i=2}^{m+1} J(0)^{l-2} \sum_{i=1}^{m} BH^i_j \frac{1}{(z - q)^l} - m \sum_{i=1}^{l} J(0)^{l-1} \sum_{i=1}^{m} G^i_j \frac{1}{(z - q)^l} \]

\[ \sum_{j=1}^{i} c^i_j (p^i_j - q)^k J(q - p^i_j)^{-1} BH^i_j \]

\[ \sum_{j=1}^{i} (p^i_j - q)^{k} G^i_j. \]

Also, we compute

\[ \sum_{i=2}^{m+1} J(0)^{l-2-k} BH^i_j \frac{1}{(z - q)^l} \]

\[ \sum_{j=1}^{i} c^i_j (p^i_j - q)^k \]

\[ \sum_{j=1}^{i} (p^i_j - q)^{k} G^i_j. \]
Furthermore, we have
\[
\hat{m} J(0)^{m_q-1} \sum_{i=1+\hat{m}}^m \frac{1}{(z-q)^{m_q+1}} \sum_{k=1+\hat{m}}^m \sum_{t'=0}^{m_q-i-1} \sum_{t''=0}^{m_q-1-i} J(0)^{t''} B H_i^* \frac{c_j^i (p_j^i - q)^{m_q-t'-t''}}{(z-q)^{m_q-k}}.
\]

This completes the proof of (45).

Next we derive the upper bound of the difference from (52). For \(i \in \{1 + \hat{m}, ..., m\}\), the \((m_q-1)\)th superdiagonal is given by
\[
\left( \sum_{j=1+\hat{m}}^m (-1)^i (q - p_j^i)^{-l+1} \frac{c_j^i (p_j^i - q)^{m_q}}{z - p_j^i} \right) \frac{1}{(z-q)^{m_q}} + \hat{m} c_j^i \frac{1}{(z-q)^{m_q}}.
\]

Substituting (51), (52), and (53) into (50) yields
\[
\Phi_x(z) = \left( \sum_{i=0}^{m_q-1} \frac{1}{(z-q)^{m_q+1}} \sum_{k=0}^{m_q-i-1} \frac{1}{(z-q)^{m_q-1} - k} \sum_{j \geq 1+\hat{m}} \frac{c_j^i (p_j^i - q)^{m_q-2-k}}{(z-q)^{m_q-k}} \right) B H_i^*.
\]

For \(i \in \{1 + \hat{m}, ..., m\}\) and \(l \in \{0, ..., m_q - i - 1\}\), the \(l\)th superdiagonal of the term multiplying \(B H_i^*\) in \(\Phi_x(z)\) is given by
\[
\left( \sum_{j=1+\hat{m}}^m (-1)^l (q - p_j^i)^{-l+1} \frac{c_j^i (p_j^i - q)^{m_q}}{z - p_j^i} \right) \frac{1}{(z-q)^{m_q}} + \sum_{k=0}^{m_q-i-1-l} \frac{1}{(z-q)^{m_q-1} - k} \sum_{j \geq 1+\hat{m}} \frac{c_j^i (p_j^i - q)^{m_q-2-k}}{(z-q)^{m_q-k}}.
\]
\[\sum_{k=0}^{m_q-i-1-l} \sum_{j=1+n}^{i} c_{j}(p_j^q - q)m_{q-2-k-l} \frac{1}{(z-q)^{m_q-k}}\]

simplifying
\[\sum_{j=1+n}^{i} \frac{1}{(z-q)^{l+1+m}}\]

\[-\sum_{k=0}^{m_q-i-1-l} \sum_{j=1+n}^{i} c_{j}(p_j^q - q)m_{q-2-k-l} \frac{1}{(z-q)^{m_q-k}}\]

cancelling
\[\sum_{j=1+n}^{i} \frac{1}{(z-q)^{l+1+m}}\]

where for Lemma 5(b) note that for \(\tilde{m} = 1\), \(c_j^q\) contains a factor of \(\frac{1}{p_j^q - q}\). Finally, if \(\tilde{m} = 1\), then for \(i = 1\) and any \(l \in \{0, ..., m_q - 1\}\), the \(l^{th}\) superdiagonal of the term multiplying \(B\) in \(\Phi_u(z)\) is given by
\[\tilde{m} = \frac{1}{(z-q)^{l+1+m}}\]

Thus, combining the cases above, by Lemma 4 for every \(i \in \{1, ..., m\}\) and \(l \in \{0, ..., m_q - 1\}\), each term in the \(l^{th}\) superdiagonal of the term multiplying \(B\) in (45) has difference from \(\frac{1}{(z-q)^{l+1+m}}\) bounded by \(K_{i,l}D(P)\) for some \(K_{i,l} > 0\).

Now we are ready to prove Lemma 3.

**Proof of Lemma 3.** The proof begins by selecting an optimal solution \(\Phi_u^e, \Phi_u^s\) to the infinite dimensional control design problem of (7), and constructing \(\Phi_u(z) = \sum_{p \in P} H_p \frac{1}{z-p}\) by Theorem 1 to approximate \(\Phi_u^e\). By Theorem 1, this immediately implies that the approximation error bounds for \(\Phi_u\) of (28) are satisfied. Next, \(\Phi_x(z)\) is defined as the unique transfer function that satisfies the SLS constraint in (7). The remainder of the proof will show that \(\Phi_x(z)\) is a feasible solution to (9)-(11), and that it satisfies the approximation error bounds of (29).

Towards that end, first it is shown that it suffices to work in coordinates in which \(\Lambda\) is in Jordan normal form. Next it is shown that, in these coordinates, the approximation error bounds and the SLS constraints decouple according to each elementary Jordan block in \(\Lambda\), so it suffices to prove the result for a single elementary Jordan block with eigenvalue \(\lambda\). Afterwards, it is shown that the SLS constraint uniquely determines the poles and multiplicities of \(\Phi_x\) from those of \(\Phi_u\) for any transfer functions \(\Phi_x, \Phi_u\) in \(\mathbb{C}_{\infty}\) that satisfy it. From the choice of \(\Phi_u\), this immediately implies that \(P\) is a feasible solution to (9)-(11).

Subsequently, for each pole \(q\) in \(\Phi_u^e\) that appears in \(\Phi_u^s\), by Theorem 1 there exist poles in \(\Phi_u\) for approximating the portion of \(\Phi_u^e\) corresponding to pole \(q\). By the relationship between \(\Phi_u\) and \(\Phi_x\) described above, we then consider the resulting poles that appear in \(\Phi_x\), and will show that the portion of \(\Phi_x\) corresponding to these poles closely approximates the portion of \(\Phi_u^e\) corresponding to the pole \(q\). To do so, we fix a pole \(q\) in \(\Phi_u^e\) and consider two cases: Case 1 where \(q \neq \lambda\), and Case 2 where \(q = \lambda\). For each of these cases we use the SLS constraints to determine the coefficients in the portions of \(\Phi_u^e\) and \(\Phi_x\) corresponding to pole \(q\) and the poles used to approximate it, respectively, and then bound the resulting approximation error. As \(q\) was arbitrary, this then yields the desired approximation error bounds for \(\Phi_x\) of (29).

First we obtain an optimal solution to the infinite dimensional control design problem, and use Theorem 1 to find \(\Phi_u\) which closely approximates \(\Phi_u^e\). Let \((\Phi_x^e, \Phi_x^s)\) be an optimal solution to (7), which exists by Assumption A1. By Theorem 1, there exist coefficient matrices \(\{H_p\}_{p \in P}\) such that, if we define \(\Phi_u(z) = \sum_{p \in P} H_p \frac{1}{z-p}\) then \(\Phi_u \in \mathbb{C}_{\infty}\), \(\|\Phi_u - \Phi_u^e\|_{\infty} \leq K_u D(P)\), and \(\|\Phi_u - \Phi_u^s\|_{\infty} \leq K_s D(P)\). Define \(\Phi_x(z) = (zI - A)^{-1}(\Phi_u(z) + I)\) and note that this implies \((\Phi_x, \Phi_u)\) satisfy the SLS constraint in (7) by construction.

Next we show that it suffices to work in Jordan normal form, and with a single elementary Jordan block. There exists matrices \(I\) in Jordan normal form and \(V\) invertible such that \(J = VAV^{-1}\). Fix \(z \in \partial D\) for the remainder of the proof. We will show that there exists \(K > 0\) such that
\[\|\Phi_x(z) - \Phi_x^e(z)\|_2 \leq K D(P).\] (54)

This will imply that
\[\|\Phi_x - \Phi_x^e\|_{\infty} = \sup_{z \in \partial D} \|V^{-1}(\Phi_x(z) - \Phi_x^e(z))\|_2 \leq \|V^{-1}\|_2 \sup_{z \in \partial D} \|\Phi_x(z) - \Phi_x^e(z)\|_2 \leq K_x D(P)\]
\[\max(\|\Phi_x - \Phi_x^e\|_{\infty}, \|\Phi_x - \Phi_x^s\|_{\infty}) \leq \sqrt{n} K_x D(P)\]
\[K_x = \|V^{-1}\|_2 K_u - K_s D(P)\]

So, to prove the lemma it suffices to show that (54) holds and that \((\Phi_x, \Phi_u)\) is a feasible solution to (9)-(11). Furthermore, letting \(J(\lambda)\) denote an elementary Jordan block with eigenvalue \(\lambda\) in \(J\), and \(M_{J(\lambda)}\) the restriction of the matrix \(M\) to the rows corresponding to the rows of \(J(\lambda)\) in \(J\), we have
\[
\|\Phi_x(z) - \Phi_x^e(z)\|_2 \leq \sum_{\lambda \in \sigma J(\lambda) \cap J} \|\Phi_x(z) - \Phi_x^e(z)\|_2.
\]

Thus, to prove (54) it suffices to show that for each elementary Jordan block \(J(\lambda)\) in \(J\) there exists a constant \(K_{J(\lambda)} > 0\) such that
\[
\|\Phi_x(z) - \Phi_x^e(z)\|_2 \leq K_{J(\lambda)} D(P).\] (55)

Preliminary the SLS constraint in (7) by \(V\) implies that \((zI - J)V\Phi_x - VB\Phi_u = V\), and note that this is satisfied by both \(V\Phi_x^e, \Phi_u\) and \(V\Phi_x^s, \Phi_u\). As \((zI - J)\) is block diagonal, this equation decouples into independent equations for each elementary Jordan block \(J(\lambda)\) in \(J\) given by \((zI - J(\lambda))V\Phi_x(z))\|_{J(\lambda)} \leq (VB)\|_{J(\lambda)} \Phi_u(z) = V\|_{J(\lambda)}\). Therefore, both our objective (55) and the SLS constraints become decoupled for each \(J(\lambda)\), so for the remainder of the proof we fix a particular \(\lambda \in \sigma J(\lambda)\) in \(J\). For notational convenience, for the remainder of the proof we abuse notation.
and let $\Phi_x, \Phi^*_x, B,$ and $V$ denote $(V\Phi_x)_{|J(\lambda)}$, $(V\Phi^*_x)_{|J(\lambda)}$, $(VB)_{|J(\lambda)}$, and $V_{|J(\lambda)}$, respectively. Then the objective (55) and the SLS constraint become

$$
\|\Phi_x(z) - \Phi^*_x(z)\|_2 \leq KD(p)
$$
(56)
\[
(zI - J(\lambda))\Phi_x(z) - B\Phi_u(z) = V.
\]
(57)

To complete the proof it suffices to show that there exists $K > 0$ such that (56) holds, and that $(\Phi_x, \Phi_u)$ is a feasible solution to (9)-(11).

Let $m_1$ denote the multiplicity of $\lambda$ in $J(\lambda)$. As $(zI - A)^{-1}$ is strictly proper real rational and $(B\Phi_x(z) + I)$ is proper real rational, their product $\Phi_x$ is strictly proper real rational. Therefore, $\Phi_x$ has a partial fraction decomposition which does not include any constant or polynomial terms, and in which all poles have finite multiplicity.

Now we derive the relationship between the poles and multiplicities of any pair of transfer functions which satisfy the SLS constraint. Let $(\Phi_x, \Phi_u)$ be any transfer functions which satisfy (57) and such that $\Phi_u \in \frac{1}{z} \mathbb{R} \mathcal{H}_\infty$ and $\Phi_x$ is strictly proper rational. Let $q$ be any pole of $\Phi_x$, $m_q = m_\lambda$ if $q = \lambda$ or $m_q = 0$ otherwise, and $m$ the multiplicity of $q$ in $\Phi_u$. Note that $m = 0$ if $q$ is not a pole of $\Phi_u$. Let $m$ be the multiplicity of $q$ in $\Phi_u$. Then the terms in the partial fraction decompositions of $\Phi_u$ and $\Phi_x$ corresponding to pole $q$ are given by $\sum_{i=1}^{m_q} H_i (z - q)^i$ and $\sum_{i=1}^{m} G_i (z - q)^i$, respectively. By uniqueness of the partial fraction decomposition, (57) therefore implies that

$$
G^{i+1} = J(\lambda - q)G^i + BH^i, \quad i \in \{1, \ldots, m\}
$$
(58)

$$
0 = J(\lambda - q)G^{m_\lambda}.
$$
(59)

First consider the case where $\lambda \neq q$. It is straightforward to verify the following fact:

Fact 2. If $J(\lambda)G = 0$ for $\lambda \neq 0$ then $G = 0$.

Then by Fact 2 and (60), $G^m = 0$. Proceeding downwards in $i$, repeated application of Fact 2 and (59) imply that $G^i = 0$ for $i \in \{m + 1, \ldots, m\}$. So, in this case the order of $q$ in $\Phi_x$ is $m = m_\lambda + m_q$. Next consider the case where $\lambda = q$. Then, by (59), $G^i = J(0)^i (m_\lambda + 1)^{m_\lambda + 1}$ for $i \in \{1, \ldots, m_\lambda - 1\}$. As $J(0)^{m_\lambda} = 0 = J(0)^{m_q}$, this implies that $G^i = 0$ for $i \geq m_\lambda + m_\lambda + 1$. So, in this case the order of $q$ in $\Phi_x$ is $m + m_q$.

Combining the above cases implies the following fact: that $\Phi_x$ only contains poles in $\sigma$ and $\Phi_u$, and that their multiplicities are given by $m + m_q$. Applying this fact to $(\Phi_x, \Phi_u)$ implies that $\Phi_x \in \frac{1}{z} \mathbb{R} \mathcal{H}_\infty$ and is a feasible solution to (9)-(11). Hence, to complete the proof it suffices to prove (56).

In what follows, we show that to prove (29) it suffices to fix a particular pole in $\Phi^*_x$, and to show that a certain portion of $\Phi_x$ closely approximates the portion of $\Phi^*_x$ corresponding to this pole. This is done by using the construction of Theorem 1 to approximate $\Phi^*_x$ by $\Phi_u$. Let $\mathcal{Q}$ denote the poles of $\Phi^*_x$. For each $q \in \mathcal{Q}$, its contribution to the partial fraction decompositions of $\Phi^*_x$ and $\Phi^*_u$ is given, respectively, by

$$
\sum_{i=1}^{m_q + m_q} G^*_i (z - q)^i, \quad \sum_{i=1}^{m} H^*_i (z - q)^i.
$$
(61)

by the above fact. Since $\Phi_u$ was constructed as in Theorem 1, the portion of $\Phi_u$ that was chosen to approximate the pole at $q$ in $\Phi^*_u$ is given by

$$
\sum_{i=1}^{m_q + m_q} H^*_i (z - q)^i, \quad H^*_j = c^*_j H^*_i
$$
(62)

for all $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, i\}$, where $\{c^*_j\}_{i=1}^{m_q + m}$ are the constants chosen in Corollary 2 for approximating the pole $q$ with multiplicity $i$ by the poles $\{p^*_j\}_{j=1}^{i}$. Let $\tilde{m} = 1$ if $q = \lambda$ and $\Phi_u$ contains a pole at $q$, and $\tilde{m} = 0$ otherwise. If $\tilde{m} = 1$, reorder the poles in $\{p^*_j\}_{j=1}^{i}$ for each $i \in \{1, \ldots, m\}$ such that $p^*_i = q$. Then the above fact implies that the portion of $\Phi_x$ corresponding to the above portion of $\Phi_u$ is given by

$$
\sum_{i=1}^{m_q + m_q} G_i (z - q)^i + \sum_{i=1+\tilde{m}}^{m} \sum_{j=1+\tilde{m}}^{m} G^*_j (z - p^*_j).
$$
(63)

Hence, from (63) and (61) we compute

$$
\|\Phi_x(z) - \Phi^*_x(z)\|_2
$$

$$
\leq \left\| \sum_{q \in \mathcal{Q}} \sum_{i=1}^{m_q + m_q} G_i (z - q)^i + \sum_{i=1+\tilde{m}}^{m} \sum_{j=1+\tilde{m}}^{m} G^*_j (z - p^*_j) \right\|_2
$$

$$
\leq \left\| \sum_{q \in \mathcal{Q}} \sum_{i=1+\tilde{m}}^{m_q + m_q} G_i (z - q)^i + \sum_{i=1+\tilde{m}}^{m} \sum_{j=1+\tilde{m}}^{m} G^*_j (z - p^*_j) \right\|_2
$$

$$
\leq \left\| \sum_{i=1+\tilde{m}}^{m_q + m_q} G^*_i (z - q)^i \right\|_2
$$

so, since $\mathcal{Q}$ is finite, to prove (56) it suffices to show that there exists $K_q > 0$ such that

$$
\left\| \sum_{i=1+\tilde{m}}^{m_q + m_q} G^*_i (z - q)^i \right\|_2 \leq K_q D(p)
$$
(64)

for each $q \in \mathcal{Q}$. Towards that end, fix $q \in \mathcal{Q}$ and for the remainder of the proof let $\Phi_x(z)$ and $\Phi^*_x(z)$ denote the contributions to $\Phi_x(z)$ and $\Phi^*_x(z)$ given by (63) and (61), respectively, as in (64). We consider two cases.

Case 1: $q \neq \lambda$. Substituting (63), (62), and (61) into (58)-(60) implies that

$$
-J(\lambda - q)G^m = BH^*_m
$$

$$
G^*_i = J(\lambda - q)^{-1} (G^*_i - BH^*_i), \quad i \in \{2, \ldots, m\}
$$

$$
-J(\lambda - p^*_i)G^*_j = BH^*_j, \quad i \in \{1, \ldots, m\}, \quad j \in \{1, \ldots, \}
$$

and all other coefficients in $\Phi_x$ and $\Phi^*_x$ are zero. The above implies that $G^*_l = - \sum_{i=1}^{m} J(\lambda - q)^{-i(1+l)-1} BH^*_l$ for all $l \in \{1, \ldots, m\}$. Define $G^*_l = - \sum_{i=1}^{m} J(\lambda - q)^{-i(1+l)-1} BH^*_l$ for all $l \in \{1, \ldots, m\}$.
\{1, \ldots, m\}$ and $i \in \{1, \ldots, m\}$, and note that $G^*_{i} = \sum_{i=1}^{m} G^*_{(i,i)}$.

Write $G^*_j = c_j^i (G^*_{(i,i)} + \Delta G^*_j)$. Then, by (62)

$$J(\lambda - q) G^*_{(i,i)} = -BH^*_{i} = -\frac{1}{c_j^i} BH^*_{j} = \frac{1}{c_j^i} J(\lambda - p_j^i) G^*_j$$

$$= J(\lambda - p_j^i) (G^*_{(i,i)} + \Delta G^*_j)$$

$$= (J(\lambda - q) - (p_j^i - q)) J(\lambda - p_j^i)^{-1} G^*_{(i,i)},$$

so $0 = -(p_j^i - q) G^*_{(i,i)} + J(\lambda - p_j^i) \Delta G^*_j$ and $\Delta G^*_j = (p_j^i - q) J(\lambda - p_j^i)^{-1} G^*_{(i,i)}$. In summary,

$$\Phi_x = \sum_{i=1}^{m} G^*_{(i,i)} \frac{1}{(z - q)^i} = \sum_{i=1}^{m} \sum_{j=1}^{i} G^*_{(i,i)} \frac{1}{(z - q)^j}$$

$$\setminus \setminus G^*_{(i,i)} = -J(\lambda - q)^{-(i+1)} BH^*_{i}, \quad l \in \{1, \ldots, m\}$$

$$\Phi_x = \sum_{i=1}^{m} \sum_{j=1}^{i} G^*_{(i,i)} \frac{1}{(z - q)^j}$$

$$\setminus \setminus G^*_j = c_j^i (I + (p_j^i - q) J(\lambda - p_j^i)^{-1}) G^*_{(i,i)}.$$

For any $i \in \{1, \ldots, m\}$ and $l \in \{2, \ldots, i\}$ write

$$J(\lambda - q) G^*_{(l-1,i)} = G^*_{(l-1,i)} + \Delta G.$$ Note that for $i \in \{1, \ldots, m\}$ and $l \in \{2, \ldots, i\}$, $J(\lambda - q) G^*_{(l-1,i)} = G^*_{(l-1,i)}$. Then

$$\setminus \setminus J(\lambda - q) G^*_{(l-1,i)} - J(\lambda - p_j^i)^{-1} G^*_j$$

$$\Delta G = (p_j^i - q) J(\lambda - p_j^i)^{-1} G^*_{(l-1,i)}$$

so $0 = -(p_j^i - q) G^*_{(l-1,i)} + J(\lambda - p_j^i) \Delta G$ and $\Delta G = (p_j^i - q) J(\lambda - p_j^i)^{-1} G^*_{(l-1,i)}$ which implies that

$$\setminus \setminus J(\lambda - p_j^i)^{-1} G^*_{(l,i)} = (I + (p_j^i - q) J(\lambda - p_j^i)^{-1}) G^*_{(l-1,i)}.$$ Applying this equation recursively implies that for $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, i\}$

$$G^*_j = c_j^i G^*_{(i,i)} + c_j^i (p_j^i - q) J(\lambda - p_j^i)^{-1} G^*_{(i,i)}$$

$$= c_j^i G^*_{(i,i)} + c_j^i (p_j^i - q) (I + (p_j^i - q) J(\lambda - p_j^i)^{-1}) G^*_{(i,i)}$$

$$= c_j^i G^*_{(i,i)} + c_j^i (p_j^i - q) G^*_{(i,i)}$$

$$\setminus \setminus c_j^i (p_j^i - q) J(\lambda - p_j^i)^{-1} G^*_{(i,i)} = \ldots$$

$$\setminus \setminus \sum_{j=1}^{i} c_j^i (p_j^i - q) J(\lambda - p_j^i)^{-1} G^*_{(i,i)}.$$ Therefore,

$$\Phi_x(z) = \sum_{i=1}^{m} \sum_{j=1}^{i} G^*_{(i,i)} \frac{1}{(z - q)^j}$$

$$\setminus \setminus \sum_{i=1}^{m} \sum_{j=1}^{i} c_j^i (p_j^i - q) J(\lambda - p_j^i)^{-1} G^*_{(i,i)} \frac{1}{(z - q)^j}$$

$$\setminus \setminus \sum_{i=1}^{m} \sum_{j=1}^{i} c_j^i (p_j^i - q) J(\lambda - p_j^i)^{-1} G^*_{(i,i)} \frac{1}{(z - q)^j}$$

$$\setminus \setminus \sum_{i=1}^{m} \sum_{j=1}^{i} c_j^i (p_j^i - q) J(\lambda - p_j^i)^{-1} G^*_{(i,i)} \frac{1}{(z - q)^j}.$$
\[
\left| \sum_{j=1}^{i} c_j^i J(\lambda - p_j^i)^{-1} - J(\lambda - q)^{-l} \right|_2 \leq \left| J(\lambda - q)^i G^*_{i,1} \right|_2
\]

Therefore, in order to prove (65) it suffices to show that
\[
\left| \sum_{j=1}^{i} c_j^i J(\lambda - p_j^i)^{-1} - J(\lambda - q)^{-l} \right|_2 \leq K'_i D(\mathcal{P}) \tag{66}
\]

for some constants \(K'_i > 0\). For \(l \in \{0, ..., m_q - 1\}\), the \(l\)th superdiagonal of \(\sum_{i=1}^{i} c_j^i J(\lambda - p_j^i)^{-1}\) is given by
\[
\sum_{j=1}^{i} c_j^i (-1)^j (\lambda - p_j^i)^{-l+1} = (-1)^j \sum_{j=1}^{i} c_j^i (\lambda - p_j^i)^{-l+1} = \left( \frac{1}{i} \right) \frac{(-1)^j}{(\lambda - q)^{l+1}} + \epsilon_{(i,l)}, \quad |\epsilon_{(i,l)}| \leq K_{(i,l)} D(\mathcal{P})
\]

where we evaluate the sum by Corollary 3(a). Consider the function \(f(x) = x^{-l}\) and note that \(f(J(\lambda - q)) = J(\lambda - q)^{-l}\). By [21, Theorem 11.1.1], for \(l \in \{0, ..., m_q - 1\}\), the \(l\)th superdiagonal of \(J(\lambda - q)^{-l}\) is \(f(J(\lambda - q)) = J(\lambda - q)^{-l}\) is given by
\[
\frac{1}{l} f^{(l)}(\lambda - q) = \left( \frac{1}{i} \right) \frac{(-1)^j}{(\lambda - q)^{l+1}} = \left( \frac{1}{i} \right) \frac{(-1)^j}{(\lambda - q)^{l+1}} + \epsilon_{(i,l)}, \quad |\epsilon_{(i,l)}| \leq K_{(i,l)} D(\mathcal{P})
\]

Thus, for each \(i \in \{1, ..., m\}\) and \(l \in \{0, ..., m_q - 1\}\), the difference between superdiagonal \(l\) of the matrix in (66) is \(\epsilon_{(i,l)}\), which satisfies \(|\epsilon_{(i,l)}| \leq K_{(i,l)} D(\mathcal{P})\). Therefore, by Fact 1 in the proof of Corollary 3, this implies that there exist \(K'_i > 0\) such that (66) holds.

Case 2: \(q = \lambda\). Let \(\hat{Q}\) denote the poles in \(\Phi_x\). Substituting (63), (62), and (61) into (58)-(60) and (57) implies that
\[
J(0) G_{m_q + m}^* = 0, \quad G_1^* = V - \sum_{q \neq \lambda} G_{(q,1)}^*
\]

\[
G_{i+1}^* = J(0) G_i^*, \quad i \in \{m + 1, ..., m_q + m - 1\}
\]

\[
G_{i+1}^* = J(0) G_i^* + B H_i^*, \quad i \in \{1, ..., m\}
\]

\[
J(0) G_{m_q + m + n} = 0, \quad G_1 = V - \sum_{q \neq \lambda} G_{(q,1)}
\]

\[
G_{i+1} = J(0) G_i, \quad i \in \{2, ..., m_q + m + n - 1\}
\]

\[
G_2 = J(0) G_1 + \tilde{m} \sum_{i=1}^{m} B H_i
\]

\[
- J(q - p_j^i) G_j = B H_j, \quad i \in \{1 + \tilde{m}, ..., m\},
\]

\[
\quad j \in \{1 + \tilde{m}, ..., i\}
\]

where \(G^*_{(q,1)}\) and \(G_{(q,1)}\) denote the coefficients of \(\frac{1}{z-q}\) in \(\Phi_x^*\) and \(\Phi_x\), respectively, for the pole \(q\). For \(l \in \{1, ..., m_q\}\), define
\[
\hat{G}_1 = G_1 + \sum_{i=1+n}^{m} \sum_{j=1+n}^{m} G_i^j, \quad \hat{G}_1 = J(0)^{l-1} G_1
\]

and \(\hat{G}_1^* = J(0)^{l-1} G_1^*\). By (65) from Case 1,
\[
\left| \hat{G}_1^* - G_1^* \right|_2 \leq \sum_{q \neq \lambda} \sum_{i=1}^{m_q} \left| G^*_{(q,1)} - \sum_{j=1}^{m_q} G_{(q,1)}^j \right|_2 \leq K_1 D(\mathcal{P})
\]

where \(K_i = ||J(0)^{l-1}||_2 K_1\). For \(l \in \{2, ..., m_q + \tilde{m}\}\) define
\[
\hat{G}_1 = G_1 - \tilde{m} \sum_{i=1+n}^{m} \sum_{j=1+n}^{m} G_i^j
\]

\[
\hat{G}_1 = G_1 - J(0)^{l-1} \sum_{i=1+n}^{m} \sum_{j=1+n}^{m} G_i^j
\]

\[
+j(0)^{l-2} \tilde{m} \sum_{i=1}^{m} B H_i
\]

\[
\hat{G}_1^* = G_1^* - \tilde{m} \sum_{i=1+n}^{m} \sum_{j=1+n}^{m} G_i^j
\]

\[
+ J(0)^{l-1} \sum_{i=1+n}^{m} \sum_{j=1+n}^{m} G_i^j
\]

\[
\hat{G}_1^* - \tilde{m} \sum_{i=1+n}^{m} \sum_{j=1+n}^{m} G_i^j
\]

\[
\left| \hat{G}_1^* - G_1^* \right|_2 \leq \sum_{q \neq \lambda} \sum_{i=1}^{m_q} \left| G^*_{(q,1)} - \sum_{j=1}^{m_q} G_{(q,1)}^j \right|_2 \leq K_1 D(\mathcal{P})
\]

Thus, for \(l \in \{1, ..., m_q\}\) we have
\[
||\hat{G}_1 - G_1||_2 \leq ||J(0)^{l-1}||_2 |\hat{G}_1 - G_1||_1 \leq K_1 D(\mathcal{P})
\]
+ \sum_{l=0}^{m} \sum_{k=0}^{m_q-1-l} J_0^l \sum_{i=1+n}^i c_i^j (p_j^i - q)^m_q - 2 - k - l \left( -q \right)^m_q - k \\
+ \sum_{l=1}^{m} J_0^l m_q - 1 \left( -q \right)^m_q + l \right) \cdot B H_i^*.

By (71) and (68) we have

\[ \Phi_x^* = \sum_{i=1}^m \sum_{l=0}^{m_q-1} J_0^l \cdot B H_i^* \cdot \frac{1}{(z - q)^{l+l+1}}. \]

Therefore, for \( i \in \{1, \ldots, m\} \) and \( l \in \{0, \ldots, m_q - 1\} \), the \( l \)th superdiagonal of the term multiplying \( B H_i^* \) in \( \Phi_x^* \) is given by \( \frac{1}{(z - q)^{l+l+1}} \).

Thus, by Lemma 7, for every \( j, j' \in \{1, \ldots, m_q\} \),

\[ \left| \left( \Phi_x(z) - \Phi_x^*(z) \right)_{(j,j')} \right| \leq \sum_{i=1}^m \sum_{l=0}^{m_q-1} K_{ij} D(P) \left| \left( B H_i^* \right)_{(j+l,j')} \right| \leq K(1+l) D(P), \]

which is the desired bound, and where \( K = K(\Omega, C, H, r, \delta) \) by the proofs of Theorem 1 and Lemma 3.

C. Proof of Corollary 1

This section will prove Corollary 1. To achieve an approximately uniform selection of poles over the unit disk, the basic idea is to choose a spiral whose windings are equally spaced apart, and then to select poles along this spiral such that the distance between any two successive poles is close to but less than the distance between the windings. Such a selection is shown in Fig. 1 and first appeared in [12, Section 5.1], although we are not aware of any prior uses for control design. Note that as the number of poles increases, the spiral also changes so that the distance between windings (and, hence, between successive poles) converges towards zero. The main challenge is to show that the chosen pole selection along this spiral does in fact possess these geometric properties.

Then, these will be used to show that as the number of poles increases, \( D(P) \) converges to zero at the same rate at which the distance between windings converges to zero.

Lemma 8 and Corollary 4 provide basic geometric facts about the Archimedes spiral as parameterized in (13). Lemma 8 appears without proof in [12, p. 18].

Lemma 8. The Archimedes spiral given in polar coordinates by \( r = c \theta \) for a fixed \( c > 0 \) has a constant distance between its windings of \( 2\pi c \).

Proof of Lemma 8. Two points on successive windings of the spiral are given by \( p_1 = (c \theta_1, \theta_1) \) and \( p_2 = (c \theta_1 + 2\pi, \theta_1 + 2\pi) \) for some \( \theta_1 \geq 0 \). Then, using the law of cosines to compute distances in polar coordinates yields

\[ d(p_1, p_2)^2 = (c \theta_1)^2 + (c \theta_1 + 2\pi)^2 - 2(c \theta_1)(c \theta_1 + 2\pi) \cos(\theta_1 + 2\pi - \theta_1) \]

so, taking the square root implies \( d(p_1, p_2) = 2\pi c \).

Corollary 4. For any \( z \in \mathbb{D} \), there exists a point \( p \) on the Archimedes spiral \( r = c \theta \) for a fixed \( c > 0 \) such that \( |p| \leq |z| \) and \( d(z, p) < 2\pi c \).

Proof of Corollary 4. Fix \( z = (r, \theta) \in \mathbb{D} \) with \( \theta \in [0, 2\pi) \). If \( |z| = 0 \) or \( z \) is in the spiral, then the distance to the spiral is zero. Therefore, assume that \( z \) is not in the spiral. Then there exists \( \theta_1 \geq 0 \) and an integer \( n \) such that \( r \in (c \theta_1, c(\theta_1 + 2\pi)) \) and \( \theta_1 = \theta + 2\pi n \). Let \( p_1 = (c \theta_1, \theta_1) \) and note that \( p_1 \) is in the spiral and \( |p_1| = c \theta_1 < r = |z| \). By the above, we have

\[ d(z, p_1)^2 = r^2 + (c \theta_1)^2 - 2r(c \theta_1) \cos(\theta + 2\pi n - \theta) = (r - c \theta_1)^2 < (c(\theta_1 + 2\pi) - c \theta_1)^2 = (2\pi c)^2. \]
By [12, Appendix] even integer $m$, let $c_m = \frac{1}{2\sqrt{\pi m}}$. Then every pole in the selection (13) lies along the Archimedes spiral $r = c_m \theta$, as shown in Fig. 1.

**Lemma 9.** For $x \in [1, \infty)$ and $m$ a positive even integer, let

$$g(x) = \frac{1}{\frac{m}{2} + 1} \left[2x + 1 - 2\sqrt{x(x + 1)} \cos(2\sqrt{\pi} (\sqrt{x + 1} - \sqrt{x}))\right].$$

Then $\lim_{x \to \infty} g(x) = (2\pi c_m)^2$ and $g$ is monotonically increasing over $x \in [1, \infty)$, so $g(x) < (2\pi c_m)^2$ for all $x \in [1, \infty)$.

**Proof of Lemma 9.** By [12, Appendix]

$$\lim_{x \to \infty} g(x) = \frac{\pi}{\frac{m}{2} + 1} = (2\pi c_m)^2.$$  

To prove that $g$ is monotonically increasing over $[1, \infty)$, it suffices to show that $g' = \frac{d}{dx} > 0$ over $[1, \infty)$. Taking a derivative of $g$ leads to

$$g'(x) = 2 - \frac{2x + 1}{\sqrt{x(x + 1)}} \cos(a_x) - a_x \sin(a_x), \quad a_x = 2\sqrt{\pi} (\sqrt{x + 1} - \sqrt{x}) = \frac{2\sqrt{\pi}}{\sqrt{x + 1} + \sqrt{x}}.$$

For $z \in (0, 1]$, define the function

$$h(z) = \left(\frac{m}{2} + 1\right) g' \left(\frac{1}{z}\right) = 2 - \frac{2 + z}{\sqrt{1 + z}} \cos(a_z) - a_z \sin(a_z), \quad a_z = 2\sqrt{\pi} \frac{\sqrt{z}}{\sqrt{1 + z} + 1}.$$

Taking a derivative leads to

$$h'(z) = \sqrt{\pi} \left(\frac{2 + z - \sqrt{1 + z}}{\sqrt{1 + z} + \sqrt{1 + z}}\right) \sin(a_z).$$

Taking another derivative leads to $h''$ as shown in Fig. 4. Note that for $z \in [0, 1]$, Fig. 4 shows that $h''(z) > 0$. This implies that $h'$ is monotonically increasing over $(0, 1]$. Using that $\lim_{z \to 0} \cos(a_z) = 1$, $\lim_{z \to 0} \sin(a_z) = 0$, and by L’Hospital’s rule we can evaluate the limit $\lim_{z \to 0} h'(z) = 0$. Thus, since $h'$ is monotonically increasing over $(0, 1]$ and satisfies $\lim_{z \to 0} h'(z) = 0$, this implies that $h'(z) > 0$ for all $z \in (0, 1]$. In turn, this implies that $h(z)$ is monotonically increasing over $z \in (0, 1]$. It is straightforward to see that $\lim_{z \to 0} h(z) = 0$ which, since $h(z)$ is monotonically increasing over $z \in (0, 1]$, implies that $h(z) > 0$ for $z \in (0, 1]$. Consequently, and since $x \in [1, \infty)$ implies that $\frac{1}{x} \in (0, 1]$, this implies that for every $x \in [1, \infty)$, $g'(x) = \frac{1}{x} \frac{1}{x - 1} h \left(\frac{1}{x}\right) > 0$. Thus, as $g'(x) > 0$ for all $x \in [1, \infty)$, $g$ is monotonically increasing over $[1, \infty)$.

$$\lim_{x \to \infty} g(x) = (2\pi c_m)^2,$$  

this implies that for $x \in [1, \infty)$ we have $g(x) < (2\pi c_m)^2$.

**Corollary 5.** For any even integer $m > 0$, selecting poles along the Archimedes spiral $r = c_m \theta$ according to (13) for $k \in \{1, ..., \frac{m}{2}\}$ implies that, for $k \in \{1, ..., \frac{m}{2} - 1\}$

$$\max\{d(p_k, p_{k+1}), d(p_{k-1}, p_{k+1}), d(p_k, p_{k-1})\} < 2\pi c_m \quad \lim_{m \to \infty} \lim_{m' \to m - 1} d(p_k, p_{k+1}) = 2\pi c_m.$$

Corollary 5 shows that in the limit, the distance between successive poles in the Archimedes spiral approaches $2\pi c_m$, which is the same the distance between successive windings of the spiral. So, in this sense the distance between successive poles is as close as possible to the distance between successive windings, which was the goal for the pole selection along the spiral in an attempt to approximate a uniform selection of poles over the unit disk.

**Proof of Corollary 5.** That $\lim_{m \to \infty} \lim_{k \to m - 1} d(p_k, p_{k+1}) = 2\pi c_m$ is proven in [12, Appendix]. Since $2\theta_1 - 2\pi = 4\sqrt{\pi} - 2\pi \in (0, \frac{\pi}{2})$, $\cos(2\theta_1 - 2\pi) > 0$ so

$$d(p_1, p_{-1})^2 = r_1^2 (2 - 2 \cos(2\theta_1)) = \frac{2 - 2 \cos(2\theta_1 - 2\pi)}{\frac{m}{2} + 1} < \frac{2}{\frac{m}{2} + 1} < \frac{\pi}{\frac{m}{2} + 1} = (2\pi c_m)^2.$$

So taking the square root implies that $d(p_1, p_{-1}) < 2\pi c_m$. For any $k \in \{1, ..., \frac{m}{2}\}$, $p_{-k} = p_k$, which implies by symmetry that $d(p_{-k}, p_{-(k+1)}) = d(p_k, p_{k+1})$ for all $k \in \{1, ..., \frac{m}{2} - 1\}$. Thus, to prove Corollary 5 it suffices to show that for all $k \in \{1, ..., \frac{m}{2} - 1\}$, $d(p_k, p_{k+1}) < 2\pi c_m$. Define $g(x)$ as in Lemma 9, and note that for all $k \in \{1, ..., \frac{m}{2} - 1\}$, $g(k) = d(p_k, p_{k+1})^2$. Since $k \in \{1, ..., \frac{m}{2} - 1\} \subset [1, \infty)$, by Lemma 9, $d(p_k, p_{k+1})^2 = g(k) < (2\pi c_m)^2$ so taking the square root implies that $d(p_k, p_{k+1}) < 2\pi c_m$.

**Corollary 6.** For any even integer $m > 0$, select poles along the Archimedes spiral $r = c_m \theta$ according to Corollary 5. Then for any $z \in \mathbb{D}$, there exists $k \in \{1, ..., \frac{m}{2}\}$ such that $d(z, p_k) < 4\pi c_m$ and either $|p_k| \leq |z|$ or $k = 1$.

**Proof of Corollary 6.** Fix $z \in \mathbb{D}$. By Corollary 4, there exists a point $p$ in the spiral $r = c_m \theta$ such that $|p| \leq |z|$ and $d(z, p) < 2\pi c_m$. Since $p$ is in the spiral, there exists $\hat{\theta} \geq 0$ such that $p = (c_m \theta, \hat{\theta})$. If $\hat{\theta} = \theta_k$ for some $k \in \{1, ..., \frac{m}{2}\}$, then $p = p_k$
We claim that
\[ \theta \]
Proof of Corollary 1. implies that
\[ \text{corollary follows since } \]
\[ \text{assume that } \]
\[ \text{there exists } \]
\[ \text{monotonically nondecreasing over } \theta \]
Case 1: suppose that
\[ m(\theta_k) = +1 \]
\[ \text{This will then imply, since } \]
\[ \text{for any } \]
\[ \text{this implies that the } \]
\[ \text{since } \]
\[ \text{we have } \]
\[ \text{where the final inequality follows from evaluation. Thus, } \]
\[ \text{so } \]
\[ \text{since } \theta > \theta_k \text{ and the other term is positive. Thus, } \]
\[ \text{Case 2: suppose } \theta < \theta_1. \text{ If } \theta_1 - \theta \geq \frac{\pi}{2} \text{ then } \]
\[ \text{since } \sqrt{2} \theta_1 \geq 2\pi \text{, taking the square root implies that } \]
\[ \text{Otherwise, if } \theta_1 - \theta \leq \frac{\pi}{2} \text{ then } \]
\[ \text{and } \]
\[ \text{the distance between } q \text{ and the closest pole in the spiral } \mathcal{P} \text{ is bounded by } 4\pi c_m. \text{ Traversing the spiral along neighboring poles (i.e. } p_k \text{ and } p_{k+1}, \text{ Corollary 5 implies that the distance from } q \text{ to each successive pole in the spiral cannot increase by more than } 2\pi c_m. \text{ This implies that the } m_q \text{ closest poles in the spiral to } q \text{ are all within a distance of } (m_q + 1)(2\pi c_m) \text{ from } q. \text{ As } (m_q + 1)(2\pi c_m) \text{ is bounded by a constant times } \sqrt{m+2}, \text{ this provides a bound on } D(\mathcal{P}). \text{ Then, this bound on } D(\mathcal{P}) \text{ resulting from the spiral geometry is combined with the suboptimality bound of Theorem 2 to yield the desired result. As we do not want the constant } K \text{ appearing in the suboptimality bound to depend on } m, \text{ additional work is required at the beginning of the proof to only choose poles from the spiral within a ball of fixed radius } \hat{r} \in (0, 1), \text{ and at the end of the proof to only choose poles from the spiral that are at least a fixed distance } \delta > 0 \text{ away from all poles of the plant which they are not approximating.} \]

We begin by finding the desired radius \( \hat{r} \in (0, 1) \) of the ball for intersecting with the spiral. Let the poles \( p_k \) and \( p_{k-1} \) be selected along the Archimedes spiral according to Corollary 5 for the even integer \( m \), and define \( \mathcal{P}_m = \bigcup_{k=1}^{m} \{p_k, p_{k-1}\}. \) Let \( m_{\text{max}} = \max_{q \in \Omega} m_q. \) Let \( r_q = \max_{p \in \mathcal{P}_{m_{\text{max}}}} |p| \) and let \( r_p = \max_{p \in \mathcal{P}_{m_{\text{max}}}} |p|. \) Let \( \hat{r} = \max\{r_q, r_p\} \) and note that this implies that \( \mathcal{Q}, \mathcal{P}_{m_{\text{max}}} \subset \overline{B}_{\hat{r}} \) and \( p_i \in \overline{B}_{\hat{r}} \) for any \( m \). Let \( \mathcal{P}_m = \mathcal{P}_m \cap \overline{B}_{\hat{r}}. \) By the definition of the pole selection in Corollary 5, and since \( \mathcal{P}_{m_{\text{max}}} \subset \overline{B}_1, \) for any \( m \geq m_{\text{max}} \) we have \( \mathcal{P}_m = \bigcup_{k=1}^{m_{\text{max}}} \{p_k, p_{k-1}\} \) for some \( k' \in \left[ \frac{m_{\text{max}}}{2}, \frac{m_{\text{max}}}{2} \right]. \) Thus, \( |\mathcal{P}_m| \geq m_{\text{max}}. \)

Next we consider a pole \( q \) of \( \Phi_z^o \), and bound the distance of its \( m_q \) closest approximating poles from the spiral. Let \( q \in \Omega. \) By Corollary 6, there exists \( k \in \{1, ..., \frac{m}{2}\} \) such that \( d(q, p_k) < 4\pi c_m \) and either \( |p_k| \leq |q| \) or \( k = 1. \) The latter implies that \( p_k \in \overline{B}_{\hat{r}}, \) so \( p_k \in \mathcal{P}_m. \) As \( 2k' \cdot |p_m| \geq m_{\text{max}} \geq m_q, \) there exists a subset of consecutive integers \( S_{(q,m)} \subset \{-k', -1, 1, ..., k\} \) (where we define \(-1\) and \(1\) to be consecutive for this purpose) such that \( k \in S_{(q,m)} \) and \( |S_{(q,m)}| = m_q. \) Let \( \mathcal{P}_m(q) \) denote the \( m_q \) closest poles in \( \mathcal{P}_m \) to \( q, \) and let \( l = \arg \max_{s \in S_{(q,m)}} |p_i - q|. \) Then by Corollary 5

This is the desired bound based on the spiral geometry.

Now we derive the desired \( \delta > 0 \) for ensuring the spiral poles are at least distance \( \delta \) from all poles of the plant which they are not approximating. Since \( \mathcal{P}_m(q) \to q \text{ as } m \to \infty, \)

\[ D(\mathcal{P}_m) = \max_{q \in \Omega} d(q) = (m_{\text{max}}+1)\sqrt{2\pi} \frac{1}{\sqrt{m+2}}. \] (73)
there exists \( \tilde{m} \) such that \( m \geq \tilde{m} \) implies that for all \( q \in \mathcal{Q} \) and \( \lambda \in \sigma \) with \( \lambda \neq q \),
\[
d(q, \lambda) \geq \frac{1}{2} d(q, \lambda).
\]
Define
\[
\delta = \min_{\lambda \in \sigma} \min_{q \neq \lambda} \left\{ \frac{1}{2} d(q, \lambda), \min_{m=2}^{\tilde{m}} d(p_m(q), \lambda) \right\} > 0.
\]
Then for any even integer \( m \), any \( q \in \mathcal{Q} \) and any \( \lambda \in \sigma \) with \( \lambda \neq q \),
\[
d(p_m(q), \lambda) \geq \delta > 0.
\]
Finally, we apply Theorem 2 and obtain our main result. As \( P_m \) is closed under complex conjugation, contained in \( B_r \)
with \( r \in (0,1) \), \( |P_m| \geq \max \) satisfies \( d(p_m(q), \lambda) \geq \delta > 0 \),
and \( D(p_m) = (m_{\max} + 1) \sqrt{2m + \frac{1}{\sqrt{m}}} < 1 \), by Theorem 2 and
(73) there exists \( K' > 0 \) such that
\[
\begin{align*}
\frac{J(p)}{J^*} & \leq K'D(p) < K'(m_{\max} + 1) \sqrt{2\pi} = \\
K & = K'(m_{\max} + 1) \sqrt{2\pi}.
\end{align*}
\]

VII. CONCLUSION

This work developed a new control design method – SPA – to address limitations of SLS with the FIR approximation (DBC)
for optimal control design. Unlike DBC, SPA does not result in deadbeat control, and feasibility is automatic so
it does not require the addition of slack variables or constraint violation which lead to additional suboptimality in (5).
Furthermore, SPA leads to a convex and tractable optimization problem for the control design consisting of a
single SDP program, as opposed to the iterative algorithm that DBC requires because it is only quasi-convex.

A general suboptimality bound was provided for SPA in (12) using the approximation result of (6) to relate the subopti-
mality to the geometry of the pole selection. While the DBC suboptimality bound only holds for a time horizon sufficiently
large that the optimal impulse response has already decayed and depends on the rate of this decay, the SPA bound does not
have this requirement. For the particular choice of a spiral, the SPA suboptimality bound shows a relative error convergence
rate of the inverse square root of the number of poles.

The example of power converter control design in Section V arises naturally from power grids with renewable generation,
and shows that SPA achieves much better matching with the optimal solution than DBC with orders of magnitude fewer
poles. This occurs because SPA works well for systems with large separation of time scales, and because it can incorporate
prior information including the locations of the desired poles.

Future work should address different geometric criteria for selecting poles resulting in favorable numerical properties,
extensions to state and input constraints, application of the SPA method to output feedback as in IOP, and extensions to
continuous-time control design.

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