A counterexample to Durfee’s conjecture

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Abstract. An old conjecture of Durfee [Durfee1978] bounds the ratio of two basic invariants of complex isolated complete intersection surface singularities: the Milnor number and the singularity (or geometric) genus. We give a counterexample for the case of non-hypersurface complete intersections, and we formulate a weaker conjecture valid in arbitrary dimension and codimension. This weaker bound is asymptotically sharp. In this note we support the validity of the new proposed inequality by its verification in certain (homogeneous) cases. In our subsequent paper we will prove it for several other cases and we will provide a more comprehensive discussion.

1. Introduction

Let \((X,0) \subset (\mathbb{C}^N,0)\) be the analytic germ of a complex isolated complete intersection singularity (ICIS). Among various singularity invariants the two most basic ones are:

- the Milnor number \(\mu\), which measures the change of the local topological/homological Euler characteristic in smoothing deformation;
- the singularity genus (called also geometric genus) \(p_g\), which measures the change of the local analytic Euler characteristic (Todd index) when we replace \((X,0)\) by its resolution.

Their definitions and some of their properties are given in §2.

The relation of these two invariants was investigated intensively giving rise to several open problems as well. In particular, in [Durfee1978] two conjectures were formulated:

- **strong inequality**: if \((X,0)\) is an isolated complete intersection surface singularity, then \(\mu \geq 6p_g\);
- **weak inequality**: if \((X,0)\) is a normal surface singularity (not necessarily ICIS) which admits a smoothing with Milnor number \((\text{second Betti number of the fiber}) \mu, \text{then } \mu + \mu_0 \geq 4p_g\) (where \(\mu_0\) is the rank of the kernel of the intersection form).

Quite soon a counterexample to the weak conjecture was given in [Wahl1981, page 240] providing a normal surface singularity (not ICIS) with \(\mu = 3, \mu_0 = 0\) and \(p_g = 1\).

On the other hand, the ‘strong inequality’ valid for an ICIS was believed to be true (though hard to prove) and was verified in many particular hypersurfaces. For example, [Tomari1993] proved \(8p_g < \mu\) for \((X,0)\) of multiplicity 2, [Ashikaga1992] proved \(6p_g \leq \mu - 2\) for \((X,0)\) of multiplicity 3, [Xu-Yau1993] proved \(6p_g \leq \mu - \text{mult}(X,0) + 1\) for quasi-homogeneous singularities, [Némethi98, Némethi99] proved \(6p_g \leq \mu\) for suspension type singularities \(\{g(x,y) + z^k = 0\} \subset (\mathbb{C}^3,0)\), [Melle-Hernández2000] proved \(6p_g \leq \mu\) for absolutely isolated singularities.

Moreover, for arbitrary dimension, [Yau-Zhang2006] proved the inequality \(\mu \geq (n+1)!p_g\) for isolated weighted-homogeneous hypersurface singularities in \((\mathbb{C}^{n+1},0)\). The natural expectation was that the same bound \((n+1)!\) holds for any ICIS of any dimension \(n\) and any codimension.

In §3 we give a counterexample to the strong conjecture in any codimension \(r \geq 2\). In fact, for \(r \geq 2\), the bound \((n+1)!\) is wrong even asymptotically. Therefore, in §4 we propose a weaker bound. It is based on the Stirling number of the second kind: \(\left\{\frac{n+r}{r}\right\} := \frac{1}{r!} \sum_{j=0}^{r} (-1)^j \binom{r}{j} (r-j)^{n+r},\) cf. [Abramowitz-Stegun, §24.1.4].

**Conjecture.** Let \((X,0) \subset (\mathbb{C}^N,0)\) be an ICIS of dimension \(n\) and codimension \(r = N - n\). Then

- for \(n = 2\) and \(r = 1\) one has \(\mu \geq 6p_g\),
- for \(n = 2\) and arbitrary \(r\) one has \(\mu > 4p_g\),

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for $n \geq 3$ and fixed $r$ one has $\mu \geq C_{n,r} \cdot p_g$, where the coefficient $C_{n,r}$ is defined by

$$C_{n,r} := \frac{(n+r-1)(n+r)!}{\left\{ \begin{smallmatrix} n+r \cr r \end{smallmatrix} \right\} r!}.$$

Note that for any curve singularity (i.e. $n = 1$), $p_g$ is the delta invariant $\delta$ and $\mu \leq 2\delta$, (with equality exactly for irreducible ones); see example 2.1. For $n = 2$ the inequality $\mu(X,0) \geq C_{n,r} \cdot p_g(X,0)$, in general, is not satisfied (see Proposition 4.2 and the comments following it).

The bound of the conjecture is asymptotically sharp, i.e. for any fixed $n$ and $r$ there exists a sequence of isolated complete intersections for which the ratio $\frac{1}{p_g}$ tends to $C_{n,r}$. In our subsequent longer article [Kerner-Némethi] we verify it for several cases; here we exemplify only some ‘homogeneous’ situations. Moreover, we list some elementary properties of the sequence $\{C_{n,r}\}_{n,r}$, cf. §4. For example, we show:

$$n + 1 = C_{n,1} \geq C_{n,2} = \frac{(n+2)!(n+1)}{2n+2 - 2} \geq \prod_{r} \lim_{r \to \infty} C_{n,r} = 2^n.$$  

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2. BACKGROUND

2.1. The Milnor number. Let $(X,0) \subset (\mathbb{C}^N,0)$ be an isolated complete intersection of dimension $n \geq 1$ defined as the germ of the zero set of an analytic germ $f : (\mathbb{C}^N,0) \to (\mathbb{C}^r,0)$. Let $B \subset \mathbb{C}^N$ be a small ball centered at the origin. By the classical theory (see e.g. [Looijenga-book]) the ‘Milnor fiber’ $f^{-1}(\epsilon) \cap B$ ($0 < \epsilon \ll \text{radius of } B$) is smooth and has the homotopy type of a bouquet of $n$–spheres. Their number $\mu$ is called the Milnor number of $(X,0)$.

For general formulæ of $\mu$ see e.g. [Milnor-book] and [Looijenga-book, §9.A].

2.2. The singularity (geometric) genus. Let $(X,0) \subset (\mathbb{C}^N,0)$ be an ICIS as above with $n \geq 1$. Let $(\tilde{X}, E) \to (X,0)$ be one of its resolutions, i.e. a birational morphism with $\tilde{X}$ smooth, $\tilde{X} \setminus E \cong X \setminus \{0\}$ and $\tilde{X} \setminus E = \tilde{X}$. The singularity genus reflects the cohomological non-triviality of the structure sheaf on the germ $(\tilde{X}, E)$, [Artin1966]:

$$p_g(X,0) := h^{n-1}\mathcal{O}_{\tilde{X}}.$$  

This number does not depend on the choice of the resolution.

Example 2.1. • If $(X,0)$ is a (reduced) curve singularity then the analogue of singularity genus is classical delta invariant: $\delta = dim_c \mathcal{O}_{\tilde{X}} / \mathcal{O}_{(X,0)}$. It is related to the Milnor number via $2\delta = \mu + \gamma - 1$, where $\gamma$ is the number of local branches; cf. [Milnor-book], [Buchweitz-Greuel1980], [Looijenga1986].

• Assume that $(X,0)$ is a normal surface singularity which admits a smoothing. If $F$ is the Milnor fiber of the smoothing, let $(\mu_+ + \mu_0, \mu_-)$ be the Sylvester invariants of the symmetric intersection form in the middle integral homology $H_2(F, \mathbb{Z})$. Then $2p_g = \mu_0 + \mu_+ + \mu_-$, [Durfee1978, Proposition 3.1].

• Let $(X,0)$ be a Gorenstein normal surface singularity which admits a smoothing. Then, in fact, the Milnor number of the smoothing is independent of the smoothing. Indeed, if $(\tilde{X}, E) \to (X,0)$ is any resolution with relative canonical class $K_{\tilde{X}}$, then $\mu + 1 = \chi_{top}(E) + K_{\tilde{X}}^2 + 12p_g$. This formula was proved in [Laufer1977] for hypersurface surface singularities and in [Looijenga1986] for normal Gorenstein smoothable singularities.

• Let $(X,0) \subset (\mathbb{C}^{n+1},0)$ be an isolated hypersurface singularity. Let $Sp(X,0) \subset (-1,n)$ be its singularity spectrum. Then $p_g$ is the cardinality of $Sp \cap (-1,0]$, cf. [Saito1981, Steenbrink1981].

For the general introduction to singularities we refer to [AGLV-book], [Dimca-book], [Looijenga-book], [Seade-book].

3. THE COUNTEREXAMPLE

Let $Y = \{f_1 = \cdots = f_r = 0\} \subset \mathbb{P}^{N-1}$ be a smooth projective complete intersection defined by homogeneous polynomials of degrees $deg(f_i) = p_i$. Define the corresponding ICIS as the cone over $Y$: $(X,0) = Cone(Y) \subset (\mathbb{C}^N,0)$.

Proposition 3.1. (1) $\mu = \left( \prod_{i=1}^r p_i \right) \sum_{j=0}^n (-1)^j \left\{ \sum_{(k_1, \ldots, k_r), \sum k_i \geq 0} \prod_{i=1}^r (p_i - 1)^{k_i} \right\} - (-1)^n$,

(2) $p_g = \sum_{(k_1, \ldots, k_r), \sum k_i = n-j} \prod_{i=1}^r \left( \frac{p_i}{k_i+1} \right)$. 

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For a systematic study of the Milnor number of weighted homogeneous complete intersections the reader is invited to consult [Greuel1975, Greuel-Hamm1978]. Additionally, several other formulae can be found in the literature, see e.g. [Hamm1986, Hamm2011]. For formulae regarding the geometric genus we refer to [Khovanskii1978, Morales1985, Hamm2011]. These formulae are usually rather different than ours considered above. Nevertheless, the above expressions can be derived from them: here for the Milnor number we will use [Greuel-Hamm1978], and for the geometric genus [Morales1985].

Proof. (1) We will determine the Euler characteristic \( \chi = (-1)^n \mu + 1 \) of the Milnor fiber. For a power series \( Z := \sum_{i \geq 0} a_i x^i \) write \( [Z]_n \) for the coefficient \( a_n \) of \( x^n \). Also, set \( P := \prod_{i=1}^n p_i \). Then, by formula 3.7(c) of [Greuel-Hamm1978]

\[
\chi = P \cdot \left[ \frac{(1 + x)^N}{\prod_i (1 + p_i x)} \right]_n.
\]

Rewrite \( 1 + p_i x \) as \((1 + x)\left(1 - \frac{(1 - p_i) x}{1 + x}\right)\), hence

\[
\left[ \frac{(1 + x)^N}{\prod_i (1 + p_i x)} \right]_n = \left[ (1 + x)^n \cdot \prod_{i, k_i \geq 0} \left( \frac{(1 - p_i) x}{1 + x} \right)^{k_i} \right]_n = \sum_{k_1, \ldots, k_n \geq 0} x^{\sum k_i (1 + x)^{n - \sum k_i} (1 - p_i)^{k_i}} \prod_{i} (1 - p_i)^{k_i}.
\]

(2) By Theorem 2.4 of [Morales1985] (and computation of the lattice point under the 'homogeneous Newton diagram') we get

\[
p_g = \left( \frac{\sum k p_k}{N} \right) - \sum_{1 \leq i \leq r} \left( \frac{\sum k p_k}{N} - p_i \right) + \sum_{1 \leq i < j \leq r} \left( \frac{\sum k p_k}{N} - p_i - p_j \right) - \ldots
\]

Using the Taylor expansion \( \frac{1}{(1-z)^n} = \sum_{j=0}^\infty \frac{\binom{j}{n-1}}{N^j} N^{-1} \), the right hand side of (3) is \( \left[ \prod_i \left(1 - z^{p_i}\right)^{N^i} \right]_{\sum k p_k - N} \). Thus

\[
p_g = rez_{z=0} F(z), \quad \text{where} \quad F(z) := \frac{\prod_i \left(1 - z^{p_i}\right)^{N^i}}{\sum_{j=0}^\infty \frac{\binom{j}{n-1}}{N^j} N^{-1}}.
\]

Note that \( rez_{z=0} F(z) = 0 \) since \( F(1/z)/z^2 \) is regular at zero. Since \( F(z) \) has poles at \( z = 0 \) and \( z = 1 \) only, and \( \sum_{pt} rez_{z=pt} F(z) = 0 \), we have \( p_g = -rez_{z=1} F(z) \). By the change of variables \( z \mapsto 1/z \) we get

\[
p_g = rez_{z=1} \prod_{i=1}^r \frac{\left(1 - z^{p_i}\right)}{(z - 1)^{N+1}}.
\]

Since \( z^n - 1 = \sum_{j=0}^{n-1} \binom{n}{j} (z - 1)^j \), we obtain

\[
p_g = rez_{z=1} \frac{1}{(z - 1)^n+1} \prod_{i=1}^r \prod_{k_i \geq 0} \left( p_i k_i + 1 \right) \left( z - 1 \right)^{k_i} = \sum_{k_1, \ldots, k_r \geq 0} \sum_{k_1 + \cdots + k_r = n} \binom{p_1}{k_1+1} \cdots \binom{p_r}{k_r+1}.
\]

This ends the proof. \( \blacksquare \)

Example 3.2. Consider the particular case \( p_1 = \cdots = p_r = p \). Then the formulae of Proposition 3.1 read as

\[
\mu = (-1)^n \left( p^r \sum_{j=0}^{n-1} (1 - p)^j \left( \binom{j+r-1}{j} - 1 \right) \right), \quad \text{(see also [Greuel-Hamm1978, 3.10(b)])},
\]

(4)

\[
p_g = \sum_{k_1, \ldots, k_r \geq 0} \prod_{k_i=n}^r \binom{p_i}{k_i+1}.
\]

In some low dimensional cases we have:

- For \( n = 1 \): \( p_g = \frac{r(p-1)p^r}{12r} \)
- For \( n = 2 \): \( p_g = \frac{r(p-1)p^r}{24r^2} \left( r(p-1) + \frac{p-5}{3} \right) \)
- For \( n = 3 \): \( p_g = \frac{r(p-1)p^r}{72r^3} (pr - 2 - r)(pr - 3 + p - r) \).

Analyzing (4) one sees that in the case of \( \mu \), the leading term in \( p \) (for \( p \) large) comes from the last summand \( (j = n) \) and \( \mu = p^N(N-1) + O(p^{N-1}) \).
The leading term for $p_g$ is more complicated. For $p \gg 0$ one has:

$$p_g = p^N \cdot \sum_{(k_1, \ldots, k_r), \ k_i \geq 1} \prod_{i=1}^r \frac{1}{(k_i + 1)!} + O(p^{N-1}).$$

Thus, asymptotically, $\mu_{p_g} = \frac{(N-1)}{4p_g} = \frac{r + 1}{r + 4} + O\left(\frac{1}{p}\right)$, which in low dimensions gives:

$$n = N - r = 2 : \quad \mu_{p_g} = \frac{r + 1}{r + 4} + O\left(\frac{1}{p}\right); \quad n = N - r = 3 : \quad \mu_{p_g} = \frac{r + 2}{r + 3} + O\left(\frac{1}{p}\right).$$

Since $4(r + 1)/(r + 4) < 6$ for all $r \geq 2$, in the case $n = 2$ the strong Durfee bound is violated even asymptotically — that is, for any $p$ sufficiently large — whenever $r \geq 2$. Similarly, for $n = 3, 8(r + 2)/r < 4!$ whenever $r \geq 2$.

4. The New Conjecture with the New Weaker Bound

In the previous section we have seen that for the singularity which is the cone over a smooth projective complete intersection with $p_1 = \cdots = p_r$, the ratio $\mu/p_g$ tends to the numerical factor

$$C_{n,r} := \frac{(N-1)}{\sum_{(k_1, \ldots, k_r), \ k_i \geq 1} \prod_{i=1}^r \frac{1}{(k_i + 1)!}}.$$

Next we list some properties of these numbers.

**Lemma 4.1.** (I) $C_{n,r} = \frac{(n+r)!}{n + r \choose r} r!$

(II) $(n + 1)! = C_{n,1} \geq C_{n,2} \geq \cdots \geq \lim_{r \to \infty} C_{n,r} = 2^n$.

**Proof.** (I) Let $S_{n,r} := \sum_{(k_1, \ldots, k_r), \ k_i \geq 1} \prod_{i=1}^r \frac{1}{(k_i + 1)!}$. First we prove

$$r^{n+r} \frac{1}{(n+r)!} = S_{n,r} = r S_{n+1,r-1} + \binom{r}{2} S_{n+2,r-2} + \cdots.$$

Consider the expansion $r^{n+r} = (1 + \cdots + 1)^{n+r} = \sum_{k_1, \ldots, k_r \geq 0} \prod_{i=1}^r \left(\frac{1}{k_i + 1}\right)^{n+r}$. This gives

$$r^{n+r} \frac{1}{(n+r)!} = \sum_{k_1, \ldots, k_r \geq 0} \frac{1}{\prod_{i=1}^r (k_i + 1)!}.$$

For each $0 \leq j \leq r$ and subset $I(j) \in \{1, \ldots, r\}$ of cardinality $j$ set $K_{I(j)} := \{k = (k_1, \ldots, k_r) \text{ with } k_i = 0 \text{ for } j \in I(j)\}$. Then the sum $\sum_k$ from the right hand side of (8) can be replaced by $\sum_{j=0}^r \sum_{I(j)} \sum_{k \in K_{I(j)}}$. Since for each fixed $I(j)$

$$\sum_{k \in K_{I(j)}} \frac{1}{\prod_{i=1}^r k_i!} = S_{n+j,r-j}$$

(by a change $k_j \mapsto k_j - 1$ for those indices when $k_j \neq 0$), and for each $j$ there are $\binom{r}{j}$ subsets $I(j)$ of cardinality $j$, (7) follows.

Now, we prove that $S_{n,r} = \sum_{i=0}^r (r-i)^{n+r}(-1)^i \binom{r}{i}$. Note that the recursion (7) defines $S_{n,r}$ uniquely. Hence it is enough to verify that the proposed solution satisfies this recursive relation; that is, $r^{n+r} = \sum_{j=0}^r \binom{r}{i} \sum_{j=0}^r (j-i)^{n+r}(-1)^i \frac{1}{r!}$. Set $\tilde{j} = j - i$. Then the sum can be rewritten as $\sum_{j=0}^r \tilde{j}^{n+r} \binom{r}{i} \sum_{j=0}^r (j-i)^{n+r}(-1)^i \frac{1}{r!}$. Note that $\sum_{j=0}^r (j-i)^{n+r}(-1)^i \frac{1}{r!} = (1 - 1)^{n+r} = 0$ unless $r = \tilde{j}$. Hence the above formula for $S_{n,r}$ follows.

This formula can be compared with that one satisfied by the Stirling numbers (see introduction), hence we get $S_{n,r} = \sum_{i=0}^r \frac{r^i}{r!} \binom{r}{i}$, which proves the statement for $C_{n,r}$. 


(II) The limit of \( C_{n,r} \) is computed using the asymptotics of Stirling numbers of the second kind, \([\text{Abramowitz-Stegun, } \S 24.1.4] \): \( \frac{n + r}{r} \sim 2n \). This gives: \( C_{n,r} \sim 2^n \frac{(n+r-1)!}{(r-1)!r!} \rightarrow 2^n \).

The following proof showing that \( C_{n,r} \) is non-increasing sequence in \( r \) was communicated to us by D. Moews.

Write the inequality \( C_{n,r} \geq C_{n,r+1} \) in terms of Stirling numbers of the second kind:

\[
\left\{ \begin{array}{l}
\frac{n + r + 1}{r + 1} \\
\end{array} \right\} r(r + 1) \geq \left\{ \begin{array}{l}
\frac{n + r}{r} \\
\end{array} \right\} (n + r)(n + r + 1).
\]

Set \( N := n + r \) as usual, and write the inequality in terms of generating functions:

\[
\frac{r(r + 1)}{x} \sum_{N \geq r} \left\{ \begin{array}{l}
\frac{N + 1}{r + 1} \\
\end{array} \right\} \frac{x^{N+1}}{(N + 1)!} \geq x \partial_x \sum_{N \geq r} \left\{ \begin{array}{l}
\frac{N}{r} \\
\end{array} \right\} \frac{x^N}{N!}
\]

with the convention: \( \sum a_n x^n \geq \sum b_n x^n \) if and only if \( a_n \geq b_n \) for any \( n \geq 0 \).

Recall that Stirling numbers are the coefficients of Taylor series: \( \frac{(e^x - 1)^r}{r!} = \sum_{N \geq r} \left\{ \begin{array}{l}
\frac{N}{r} \\
\end{array} \right\} \frac{x^N}{N!} \). Therefore the inequality to be proved can be rewritten as: \( \frac{(e^x - 1)^{r+1}}{(r+1)!} \geq \frac{x^r (e^x - 1)^{r-1}}{(r-1)!} \). We claim that this inequality will follow from \( (e^x - 1)^2 \geq x^2 e^x \). Indeed, if \( \sum a_n x^n \geq \sum b_n x^n \) and the Taylor expansion of \( g(x) \) has only positive coefficients, then \( g(x) \sum a_n x^n \geq g(x) \sum b_n x^n \). Hence it is enough to prove \( (e^x - 1)^2 \geq x^2 e^x \). For the same reason, this last statement will follow from \( (e^x - 1) \geq x e^{x/2} \), which is immediate. 

The coefficients \( C_{n,r} \) satisfies the following identities and inequalities (some more will be listed in [Kerner-Némethi], where the general inequality \( \mu \geq C_{n,r} \) will also be discussed).

**Proposition 4.2.** Let \( (X, 0) = \cap_{i=1}^n \{ f_i(x_1, \ldots, x_N) = 0 \} \subset (\mathbb{C}^N, 0) \) be an isolated complete intersection singularity, where each \( f_i \) is homogeneous of degrees \( p_i \). Set \( P := \prod_{i=1}^n p_i \). Then:

For \( n = 1 \): \( \mu + P - 1 = C_{1,r} \cdot p_g = 2p_g \). (Here \( p_g := \delta_j \).)

For \( n = 2 \): \( \mu + P \cdot \left( \frac{r-1}{3r+1} \sum_{i=1}^r (p_i - 1) - \sum_{i<j} \frac{(p_i - p_j)^2}{3r+1} \right) + 1 = C_{2,r} \cdot p_g = 4 \frac{r+1}{r+2} \cdot p_g \).

For \( r = 1 \): \( \mu - C_{n,1} \cdot p_g = (p-1)N - \frac{p}{(p-N)!} \geq 0 \). In particular, \( \mu \geq (n+1)! \cdot p_g \).

For \( n = 3 \): \( \mu > C_{3,r} \cdot p_g \).

We wish to emphasize the equation \( \mu + P \cdot E + 1 = C_{2,r} \cdot p_g \) in the case \( n = 2 \). Note that the expression \( E \) is \(-1\) (hence negative) if \( r = 1 \), but for \( r \geq 2 \) and some choices of \( p_i \)'s (e.g. whenever they are all equal) \( E \) might be positive.

**Proof.** By Proposition 3.1 we have:

\[
\mu = P \cdot \sum_{j=0}^{n} (-1)^j \left( \sum_{\{k_i \geq 0\} \sum_{k_i=k-j} \prod_{i=1}^{r} (p_i - 1)^{k_i} \right) - (-1)^n, \quad p_g = P \cdot \sum_{\{k_i \geq 0\} \sum_{k_i=k-j} \prod_{i=1}^{r} (p_i - 1)^{k_i} \frac{1}{k_i+1}.
\]

We can assume \( p_i > 1 \), otherwise some of defining hypersurfaces are hyperplanes, so this reduces to a singularity of the same dimension and smaller codimension.

**Case n=1.** Use

\[
\mu = P \cdot \left( \sum_{i=1}^{r} (p_i - 1) - 1 \right) + 1, \quad p_g = P \cdot \sum_{i=1}^{r} (p_i - 1)^{r+1}, \quad C_{1,r} = 2.
\]

**Case n=2.** We have

\[
\mu = P \cdot \left( \sum_{i=1}^{r} (p_i - 1)(p_i - 2) + \sum_{i=1}^{r} (p_i - 1)(p_j - 1) + 1 \right) - 1,
\]

\[
p_g = P \cdot \left( \sum_{i=1}^{r} (p_i - 1) \frac{1}{2} + \sum_{i<j} \frac{(p_i - 1)(p_j - 1)}{2} \right), \quad C_{2,r} = 4 \frac{r+1}{r+2}.
\]

Hence

\[
\mu - C_{2,r} p_g = P \cdot \left( \sum_{i=1}^{r} (p_i - 1)(p_i - 2)(p_i - 1) \frac{r+1}{3r+1} - \sum_{i=1}^{r} (p_i - 1)(p_j - 1) + 1 \right) - 1 =
\]

\[
P \cdot \left( \sum_{i<j} \frac{(p_i - 1)^2 - 2(p_i - 1)(p_j - 1) + (p_i - 1)^2}{3r+1} - \sum_{i=1}^{r} (p_i - 1) \frac{r+1}{3r+1} + 1 \right) - 1,
\]
giving the needed equality. Note that from here one also obtains for \( r = 1 \) and \( n = 2 \):
\[
6p_g = \mu - P + 1 \leq \mu.
\]

**Case** \( r = 1 \) is obvious.

**Case** \( n = 3 \). It is enough to prove:
\[
\sum_{j=0}^{3} (-1)^j \left( \sum_{\{k_i \geq 0\}, j=1}^{r} (p_i - 1)^k_i \right) > \frac{8(r + 2)}{r} \sum_{\{k_i \geq 0\}, \sum_i k_i = 3} \prod_{i=1}^{r} \left( p_i - 1 \right) \frac{1}{k_i + 1}
\]

Present the difference as follows
\[
\sum_{\{k_i \geq 0\}, j=1}^{r} \prod_{i=1}^{r} (p_i - 1)^k_i \left( 1 - \frac{8(r + 2)}{r(k_i + 1)!} \right) + \frac{8(r + 2)}{r} \sum_{\{k_i \geq 0\}, \sum_i k_i = 3} \prod_{i=1}^{r} \left( p_i - 1 \right) \frac{1}{k_i + 1} - \left( p_i - 1 \right) \frac{1}{k_i + 1} + \sum_{j=1}^{3} (-1)^j \left( \sum_{\{k_i \geq 0\}, j=1}^{r} (p_i - 1)^k_i \right)
\]

First we prove that \( \text{Part I} \geq 0 \).
\[
\text{Part I} = \frac{2r - 2}{3r} \sum_i (p_i - 1)^3 + \frac{r - 4}{3r} \sum_{i \neq j} (p_i - 1)^2(p_j - 1) - \frac{2}{r} \sum_{i < j < k} (p_i - 1)(p_j - 1)(p_k - 1).
\]

Now use the elementary inequalities (the algebraic mean compared to the geometric mean)
\[
\sum_{i \neq j} a_i a_j \geq \frac{1}{r} \sum_{i \neq j} a_i a_j, \quad \sum_{i \neq j, 2k < n} a_i^k a_j^{n-k} = \sum_{i \neq j, m \neq m} \frac{(n-k-1)a_i^{a-n-k}a_j^{a-n-k}}{(n-k)(r-2)} \geq \sum_{i \neq j, m \neq m} \frac{a_i^{a-n-k, m}}{r-2}
\]
to get:
\[
\text{Part I} \geq \frac{2 + r - 4}{3r} \sum_i (p_i - 1)^2(p_j - 1) - \frac{2}{r} \sum_{i < j < k} (p_i - 1)(p_j - 1)(p_k - 1) \geq 0.
\]

Now simplify \( \text{Part II} \):
\[
\text{Part II} = \frac{8(r + 2)}{r} \left( \sum_i (p_i - 1)(3p_i - 5) \frac{1}{4!} + \sum_{i \neq j} (p_i - 1)(p_j - 1) \frac{1}{2} \right).
\]

Thus, by direct check one gets: \( \text{Part II} - \text{Part III} > 0 \). Hence the statement.

Finally, we add a slightly weaker inequality, but which is valid for any \( n \) (including \( n = 2 \)) and any \( r \). Since \( \binom{n-1}{r} \) is the number of terms in the sum from the denominator of \( C_{n,r} \), we obtain that \( C_{n,r} \geq \min_{\sum k_i r(k_i + 1)!} \left\{ \prod_{i=1}^{r} k_i + 1 \right\} \).

**Proposition 4.3.** Let \((X,0)\) as in Lemma 4.2 and \( n \geq 2 \). Then
\[
\mu \geq \min_{\sum k_i r(k_i + 1)!} \left\{ \prod_{i=1}^{r} k_i + 1 \right\} \cdot p_g \geq 2^n \cdot p_g.
\]

Moreover, if \( n > r \), let \( n = n_1 + r_1 \) for \( n_1, r_1 \geq 0, r_1 < r \). Then \( \mu \geq \left( (n_1 + 1)! \right) ^{r_1} \left( \frac{n_1}{r} \right) ^{r_1} \left( n_1 + 2 \right)! \)

In particular, for \( r = 1 \): \( \mu \geq (n + 1)!p_g \). For \( r = 2 \) and \( n \)-odd: \( \mu \geq (|\frac{n}{2} | + 1)!((|\frac{n}{2} | + 1)! + 2)1p_g \). For \( r = 2 \) and \( n \)-even: \( \mu \geq (|\frac{n}{2} + 1!! | )^2 p_g \).
Proof. Step 1. We claim that (by Proposition 3.1) it is enough to prove the inequality

\[ L.H.S. := P \cdot \left( \sum_{j=0}^{n} (-1)^j \sum_{k \geq 0, k \leq n-j} \prod_{i=1}^{r} (p_i - 1)^{k_i} \right) - (-1)^n \sum_{k_1, \ldots, k_r \geq 0} \prod_{i=1}^{r} \left( \frac{p_i}{k_i + 1} \right) (k_i + 1) =: R.H.S. \]

Clearly this imply the first inequality of the proposition, while the second follows from \((k+1)! \geq 2^k\).

Step 2. The left hand side can be written as

\[ L.H.S. = (p_r - 1)(\prod_{i=1}^{r-1} p_i)C_n(p_1, \ldots, p_r) + (p_{r-1} - 1)(\prod_{i=1}^{r-2} p_i)C_n(p_1, \ldots, p_{r-1}) + \ldots + (p_1 - 1)(p_1 - 1)^n, \]

where \(C_n(p_1, \ldots, p_r) := \sum_{\{k_i \geq 0\}} \sum_{k_1 + \cdots + k_r = n} (\prod_{i=1}^{r} (p_i - 1)^{k_i})\). This is expansion in terms with decreasing \(r\), hence it is natural to expand the right hand side similarly.

Let \(D_n(p_1, \ldots, p_r) := \sum_{k_1, \ldots, k_r \geq 0} \prod_{i=1}^{r} (p_i - 1)^{k_i} (k_i)!\). So \(D_1(p_1) = (p_1 - 1)n\). Write the \(R.H.S.\) as follows:

\[ \sum_{k_1, \ldots, k_r \geq 0} \prod_{i=1}^{r} (p_i - 1)^{k_i}(k_i)! = (p_r - 1)D_r(p_1, \ldots, p_r) + (p_{r-1} - 1)D_{r-1}(p_1, \ldots, p_{r-1}) + \ldots + p_1D_1(p_1). \]

Thus it is enough to prove the inequality for each pair of terms in these expansions:

\[ \forall r \geq 1, n \geq 2 : \quad (p_r - 1)\prod_{i=1}^{r-1} p_i C_n(p_1, \ldots, p_r) \geq (p_{r-1} - 1)\prod_{i=1}^{r-2} p_i C_n(p_1, \ldots, p_{r-1}) - (p_{r-1} - 1)D_{r-1}(p_1, \ldots, p_{r-1}). \]

For example, for \(r = 1\) we have: \((p_1 - 1)^{n+1} > p_1(p_1 - 1)^n\). Present the right hand side of (25) in the form

\[ (p_r - 1)(\prod_{i=1}^{r-1} p_i)D_n(p_1, \ldots, p_r) + (p_{r-1} - 1)(\prod_{i=1}^{r-2} p_i)(D_n(p_1, \ldots, p_{r-1}) - D_n(p_1, \ldots, p_{r-1})). \]

Note that

\[ D_n(p_1, \ldots, p_r) - D_n(p_1, \ldots, p_{r-1}) = \sum_{k_1, \ldots, k_r \geq 0} \prod_{i=1}^{r} \left( \frac{p_i - 1}{k_i} \right)(k_i)! = (p_r - 1)D_{n-1}(p_1, \ldots, p_{r-1}, (p_r - 1)). \]

Hence we have to prove the inequality: \(C_n(p_1, \ldots, p_r) \geq D_n(p_1, \ldots, p_r) + D_{n-1}(p_1, \ldots, p_{r-1}, (p_r - 1))\). In fact, we will prove by induction on \(r\) the stronger inequality:

\[ C_n(p_1, \ldots, p_r) \geq D_n(p_1, \ldots, p_r) + D_{n-1}(p_1, \ldots, p_r). \]

Step 3. Both parts are summations over \((k_1, \ldots, k_r)\), expand them in \(k_r\). Then we have to prove:

\[ \sum_{k_r = 0}^{n} (p_r - 1)^{k_r} C_{n-k_r}(p_1, \ldots, p_{r-1}) - \frac{(p_r - 1)^{k_r}}{k_r} D_{n-k_r}(p_1, \ldots, p_{r-1}) \geq \sum_{k_r = 1}^{n} \frac{(p_r - 1)^{k_r}}{k_r} D_{n-k_r}(p_1, \ldots, p_{r-1}). \]

For \(k_r \geq 2\) one has: \((p_r - 1)^{k_r} \geq (p_r - 1)^{k_r - 1}k_r! + (p_r - 1)^{k_r - 1}k_r - 1)\), by direct check. And \(C_j(p_1, \ldots, p_{r-1}) \geq D_j(p_1, \ldots, p_{r-1})\), for all values of \(j, r\). Therefore we only need to check the terms for \(k_r = 0, 1\), i.e. it is enough to prove:

\[ (p_r - 1) \left( C_{n-1}(p_1, \ldots, p_{r-1}) - D_{n-1}(p_1, \ldots, p_{r-1}) \right) + C_n(p_1, \ldots, p_r) - D_n(p_1, \ldots, p_{r-1}) \geq D_{n-1}(p_1, \ldots, p_{r-1}). \]

Again, \(C_{n-1}(p_1, \ldots, p_{r-1}) \geq D_{n-1}(p_1, \ldots, p_{r-1})\), so the initial inequality reduces to \(C_n(p_1, \ldots, p_{r-1}) \geq D_n(p_1, \ldots, p_{r-1}) + D_{n-1}(p_1, \ldots, p_{r-1})\). This completes the induction step from \((r - 1)\) to \(r\).

Finally, for \(r = 1\) the initial inequality is: \((p_1 - 1)^n \geq (p_1 - 1)^n \cdot n! + (p_1 - 1)^n \cdot (n - 1)\). By direct check it is true for \(n \geq 2\).
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