CHARACTERIZATION OF PROBABILITY MEASURES BASED ON
Q-INDEPENDENT GENERALIZED RANDOM FIELDS

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Abstract: Prakasa Rao (Studia Sci. Math. Hungar., 11 (1976) 277-282) studied a charac-
terization of probability distributions for linear functions of independent generalized random
fields. These results are extended to Q-independent generalized random fields. It is known
that independence of random variables implies Q-independence of them but the converse is
not true.

Key words: Generalized random field; Characteristic functional; Q-independence; Gaussian characteristic functional.

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1 Introduction

Rao (1971) obtained some characterizations of probability distributions on the real line
through linear functions of independent real-valued random variables. Some of these results
were extended to linear functions of independent generalized random fields in Prakasa Rao
(1976). Kagan and Szekely (2016) introduced the concept of Q-independence for real-valued
random variables. A characterization of probability distributions for Q-independent random
elements was presented in a collection of articles in Prakasa Rao (2016, 2017, 2018a,b,c)
and for Q-independent random variables taking values in a locally compact Abelian group
by Feldman (2017). We now extend the results in Prakasa Rao (1976) to Q-independent
generalized random fields. It is known that independence of random variables implies their
Q-independence but the converse is not true.

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2 Preliminaries

Let $\mathcal{X}$ be the space of all real-valued functions $\phi(x) = \phi(x_1, \ldots, x_n)$ of $n$ variables which are infinitely differentiable and have bounded supports. A sequence of functions $\{\phi_m, m \geq 1\}$ in $\mathcal{X}$ is said to converge to zero if there exists a constant $a$ such that $\phi_m$ vanishes for $||x|| \geq a$, and if, for every integer $q \geq 1$, the sequence $\{\phi_m^{(q)}, m \geq 1\}$ converges uniformly to zero where $||x|| = (x_1^2 + \ldots + x_n^2)^{1/2}$ and $\phi^{(q)}$ denotes the $q$-th derivative of $\phi$. Any continuous linear functional on $\mathcal{X}$ is called a generalized function.

A functional $\Phi$ defined on $\mathcal{X}$ is said to be a random functional if for every $\phi \in \mathcal{X}$ there is associated a real-valued random variable $\Phi(\phi)$. In other words, for every set of $m$ elements $\phi_i, 1 \leq i \leq m$ in $\mathcal{X}$, one can specify the probability that

$$a_i \leq \Phi(\phi_i) \leq b_i, 1 \leq i \leq m$$

for $-\infty < a_i < b_i < \infty, 1 \leq i \leq m$ and these probability distributions are consistent. The random functional $\Phi$ is said to be linear if for any two elements $\phi, \psi \in \mathcal{X}$, and for any two real numbers $\alpha, \beta$,

$$\Phi(\alpha \phi + \beta \psi) = \alpha \Phi(\phi) + \beta \Phi(\psi)$$

almost surely. A random functional $\Phi$ is said to be continuous if the convergence of the functions $\phi_{kj}$ to $\phi_j, 1 \leq j \leq m$ as $k \to \infty$ in $\mathcal{X}$ implies that for every bounded continuous function $f(x_1, \ldots, x_m)$,

$$\lim_{k \to \infty} \int_{R^m} f(x_1, \ldots, x_m) P_k(dx) = \int_{R^m} f(x_1, \ldots, x_m) P(dx)$$

where $P$ is the probability measure corresponding to the random vector $(\Phi(\phi_1), \ldots, \Phi(\phi_m))$ and $P_k$ is the probability measure corresponding to the random vector $(\Phi(\phi_{k1}), \ldots, \Phi(\phi_{km}))$.

Any continuous linear random functional on $\mathcal{X}$ is called a generalized random function. If the space $\mathcal{X}$ consists of functions of one variable, then the corresponding random functional is called a generalized random process. If the space $\mathcal{X}$ consists of functions of several variables, then the corresponding random functional is called a generalized random field.

Let $\Phi$ and $\Psi$ be two generalized random fields on $\mathcal{X}$. The generalized random fields $\Phi$ and $\Psi$ are said to be independent if the set of random variables $\{\Phi(\phi), \phi \in \mathcal{X}\}$ is independent of the set of random variables $\{\Psi(\phi), \phi \in \mathcal{X}\}$. This notion can be extended to any finite number of generalized random fields in an obvious manner. 
Let \( \Phi \) be a generalized random field. The functional

\[
L(\phi) = E[e^{i\Phi(\phi)}], \phi \in \mathcal{X}
\]

is called the characteristic functional of the generalized random field \( \Phi \). It can be shown that \( L(0) = 1, L(-\phi) = L(\bar{\phi}), L(\phi) \) is continuous in \( \phi \) and positive definite. Conversely, if \( L(.) \) is a positive-definite continuous functional on \( \mathcal{X} \) such that \( L(0) = 1 \), it can be shown that there exists a generalized random field \( \Phi \) on \( \mathcal{X} \) whose characteristic functional is \( L(.) \). Furthermore the correspondence between the characteristic functionals \( L(.) \) and the generalized random fields \( \Phi \) on \( \mathcal{X} \) is one to one.

Let \( \Phi_1, \ldots, \Phi_k \) be generalized random fields on the space \( \mathcal{X} \). The joint characteristic functional of the \( k \)-dimensional generalized random field \( (\Phi_1, \ldots, \Phi_k) \) is defined by

\[
L_{\Phi_1, \ldots, \Phi_k}(\phi_1, \ldots, \phi_k) = E[\exp\{i\Phi_1(\phi_1) + \ldots + i\Phi_k(\phi_k)\}], \phi_j, 1 \leq j \leq k \in \mathcal{X}.
\]

If the generalized random fields are independent, then it can be shown that

\[
L_{\Phi_1, \ldots, \Phi_k}(\phi_1, \ldots, \phi_k) = L_1(\phi_1) \ldots L_k(\phi_k)
\]

where \( L_j(.) \) is the characteristic functional of \( \Phi_j \) for \( 1 \leq j \leq k \).

A generalized random field \( \Phi \) on \( \mathcal{X} \) is said to be Gaussian if its characteristic functional is of the form

\[
L(\phi) = \exp(i \, m(\phi) - \frac{1}{2} B(\phi, \phi)), \phi \in \mathcal{X}
\]

where \( m(.) \) is a generalized function and \( B(\phi, \psi) = E[\Phi(\phi)\Phi(\psi)], \phi, \psi \in \mathcal{X} \). It is said to be degenerate if its characteristic functional is of the form

\[
L(\phi) = \exp(i \, m(\phi)), \phi \in \mathcal{X}
\]

where \( m(.) \) is a generalized function.

We refer the reader to Gelfand and Vilenkin (1964) for more details on generalized random fields and generalized random processes.

Denote by \( \Delta_h \) the finite difference operator

\[
\Delta_h f(\phi) = f(\phi + h) - f(h).
\]
a function \( f(\phi) \) defined on \( X \) is called a polynomial if

\[
\Delta_h^{n+1} f(\phi) = 0
\]

for some integer \( n \geq 1 \) and for all \( \phi, h \in X \). The minimal integer \( n \) for which this equality holds is called the degree of the polynomial \( f(.) \) defined on \( X \).

Let \( \Phi_i, 1 \leq i \leq k \) be generalized random fields on \( X \). Let \( \beta_i, 1 \leq i \leq k \) be nonzero real numbers. We define the process \( \beta_1 \Phi_1 + \ldots + \beta_k \Phi_k \) to be the process for which to every \( \phi \in X \) corresponds the random variable \( \Phi_1(\beta_1 \phi) + \ldots + \Phi_k(\beta_k \phi) \).

Let \( \Phi_1, \ldots, \Phi_k \) be generalized random fields. We say that they are \( Q \)-independent if their joint characteristic functional can be represented in the form

\[
L_{\Phi_1,\ldots,\Phi_k}(\phi_1,\ldots,\phi_k) = \Pi_{i=1}^k L_{\Phi_i}(\phi_i) \exp(q(\phi_1,\ldots,\phi_k)), \phi_1, 1 \leq i \leq k \in X
\]

where \( q(\phi_1,\ldots,\phi_k) \) is a continuous polynomial on the space \( X^k \) and \( q(0,\ldots,0) = 0 \).

Suppose that \( \Phi_i, i = 1, 2, 3 \) are independent Gaussian random fields. Then it is obvious that \( \eta_1 = \Phi_1 + \Phi_2 \) and \( \eta_2 = \Phi_1 + \Phi_3 \) are not independent random fields. However they are \( Q \)-independent. This can be seen by computing the joint characteristic functional of the bivariate generalized random field \( (\eta_1, \eta_2) \) and the characteristic functionals of the generalized random fields \( \eta_1 \) and \( \eta_2 \).

The following result is a consequence of the Marcinkeiwicz theorem (cf. Marcinkiewicz (1938)) for real-valued random variables.

**Theorem 2.1:** Let \( f(y) \) be the characteristic functional of a generalized random field \( \Phi \) on \( X \). If

\[
f(y) = \exp[P(y)], y \in X,
\]

where \( P(y) \) is a continuous polynomial in \( y \in X \), then \( P(y) \) is a polynomial of degree less than or equal to 2 and \( f(y) \) is the characteristic functional of a Gaussian random field which could be degenerate.

**Proof:** By the definition of the characteristic functional of the generalized random field \( \Phi \) on \( X \), it follows that

\[
E[\exp(i\Phi(y))] = \exp[P(y)], y \in X.
\]
Hence
\[ E[\exp(it\Phi(y))] = E[\exp(it\Phi(ty))] = \exp[P(ty)] \]
for any real \( t \in \mathbb{R} \). Since the function on the left side of the above equation is the characteristic function of the real valued random variable \( \Phi(y) \) it follows that the function on the right side of the equation has to be a polynomial of degree less than or equal to 2 in \( ty \) by the classical Marcinkiewicz theorem. Choosing \( t = 1 \), it follows that \( P(y) \) is a polynomial of degree less than or equal to 2 in \( y \) which in turn implies that the generalized random field is either degenerate or is Gaussian.

We now prove a theorem dealing with functional equations on the space \( \mathcal{X} \) which is of independent interest. Proof of the theorem is similar to that when the space \( \mathcal{X} \) is the set of real numbers (cf. Kagan et al. (1973)). Our presentation is similar to that in Feldman (2017) when the space \( \mathcal{X} \) is a locally compact Abelian group. We present the detailed proof for completeness.

**Theorem 2.2 :** Let \( \mathcal{X} \) be the space of infinitely differentiable functions. Consider the functional equation
\[ \sum_{j=1}^{n} \psi_j(u + b_j v) = P(u) + Q(v) + R(u, v), u, v \in \mathcal{X} \]
where \( b_1, \ldots, b_n \) are nonzero real numbers with \( b_i \neq b_j, 1 \leq j \leq n \) and \( \psi_j(u), 1 \leq j \leq n, P(u), Q(v) \) are functions on \( \mathcal{X} \) and \( R(u, v) \) is a polynomial on \( \mathcal{X} \times \mathcal{X} \). Then \( P(u) \) is a polynomial on \( \mathcal{X} \).

**Proof:** We use the finite difference method for proving the theorem. Let \( h_1 \) be an arbitrary element of \( \mathcal{X} \). Define \( k_1 = -b_n^{-1}h_1 \). Then \( h_1 + b_n k_1 = 0 \). Substitute \( u + h_1 \) for \( u \) and \( v + k_1 \) for \( v \) in the equation (2.2). Subtracting the equation (2.2) from the resulting equation, it follows that
\[ \sum_{j=1}^{n-1} \Delta_{\ell_1j} \psi_j(u + b_j v) = \Delta_{h_1}P(u) + \Delta_{k_1}Q(v) + \Delta_{(h_1,k_1)}R(u, v), u, v \in \mathcal{X} \]
where \( \ell_{1j} = h_1 + b_j k_1 = (b_j - b_n)k_1, j = 1, \ldots, n - 1 \). Let \( h_2 \) be an arbitrary element of \( \mathcal{X} \). Let \( k_2 = -b_{n-1}^{-1}h_2 \). Then \( h_2 + b_{n-1}k_2 = 0 \). Substitute \( u + h_2 \) for \( u \) and \( v + k_2 \) for \( v \) in the equation (2.3). Subtracting equation (2.3) from the resulting equation, it follows that
\[ \sum_{j=1}^{n-2} \Delta_{\ell_2j} \Delta_{\ell_1j} \psi_j(u + b_j v) = \Delta_{h_2} \Delta_{h_1}P(u) + \Delta_{k_2} \Delta_{k_1}Q(v) \]
\[ + \Delta_{(h_2,k_2)} \Delta_{(h_1,k_1)} R(u,v), u,v \in \mathcal{X}, \]

where \( \ell_{2j} = h_2 + b_j k_2 = (b_j - b_{n-1}) k_2, j = 1, \ldots, n - 2. \) Following similar arguments, we get the equation

(2.5)

\[ \Delta_{\ell_{n-1,1}} \Delta_{\ell_{n-2,1}} \ldots \Delta_{\ell_{1,1}} \psi_1(u + b_1 v) = \Delta_{h_{n-1}} \Delta_{h_{n-2}} \ldots \Delta_{h_1} P(u) \]

\[ + \Delta_{k_{n-1}} \Delta_{k_{n-2}} \ldots \Delta_{k_1} Q(v) \]

\[ + \Delta_{(h_{n-1},k_{n-1})} \Delta_{(h_{n-2},k_{n-2})} \ldots \Delta_{(h_1,k_1)} R(u,v), \]

for \( u,v \in \mathcal{X}, \) where \( h_m \) are arbitrary elements in \( \mathcal{X}, k_m = -b^{-1}_{m+1} h_m, m = 1,2,\ldots,n - 1, \ell_{mj} = h_m + b_j k_m = (b_j - b_{m+1} k_m, j = 1,2,\ldots,n - m. \) Let \( h_n \) be an arbitrary element of \( \mathcal{X}. \) Let \( k_n = -b^{-1}_1 h_n. \) Then \( h_n + b_1 k_n = 0. \) Substitute \( u + h_n \) for \( u \) and \( v + k_n \) for \( v \) in the equation (2.5). Subtracting the equation (2.5) from the resulting equation, we get that

(2.6)

\[ \Delta_{h_{n+1}} \Delta_{h_{n-1}} \ldots \Delta_{h_1} P(u) + \Delta_{k_{n}} \Delta_{k_{n-1}} \ldots \Delta_{k_1} Q(v) \]

\[ + \Delta_{(h_n,k_n)} \Delta_{(h_{n-1},k_{n-1})} \ldots \Delta_{(h_1,k_1)} R(u,v) = 0, u,v \in \mathcal{X}. \]

Let \( h_{n+1} \) be an arbitrary element of \( \mathcal{X}. \) Substitute \( h_{n+1} \) for \( u \) in the equation (2.6). Subtracting the equation (2.6) from the resulting equation, we obtain that

(2.7)

\[ \Delta_{h_{n+1}} \Delta_{h_{n-1}} \ldots \Delta_{h_1} P(u) \]

\[ + \Delta_{(h_n,k_n)} \Delta_{(h_{n-1},k_{n-1})} \ldots \Delta_{(h_1,k_1)} R(u,v) = 0, u,v \in \mathcal{X}. \]

Observe that, if \( h \) and \( k \) are arbitrary elements of the space \( \mathcal{X}, \) it follows that

(2.8)

\[ \Delta_{(h,k)}^{\ell+1} R(u,v) = 0, u,v \in \mathcal{X} \]

for some integer \( \ell \geq 0 \) since \( R(u,v) \) is a polynomial in \( (u,v) \) by hypothesis. Since \( h_m, m = 1,\ldots,n + 1 \) are arbitrary elements of the space \( \mathcal{X}, \) let us choose \( h_1 = \ldots = h_{n+1} = h \in \mathcal{X} \) in the equation (2.7) and apply the operator \( \Delta_{(h,k)}^{\ell+1} \) to both sides of the resulting equation. Applying the equation (2.8) now leads to the equation

(2.9)

\[ \Delta_{h}^{\ell+n+2} P(u) = 0, u \in \mathcal{X}. \]
Hence the function $P(u)$ is a polynomial of degree at most $\ell + n + 1$.

**Remarks:** Let $\ell$ be the degree of the polynomial $R(u, v)$ in Theorem 2.2. Following the methods in Kagan et al. (1973), it can be shown that the degree of the polynomial $P(u)$ in Theorem 2.1 does not exceed $\max(n, \ell)$ where $n$ is the number of functions in the left side of the functional equation (2.2).

Two generalized random fields $\Phi$ and $\Psi$ are said to be “determined up to a Gaussian generalized random field” if there exist a generalized random field $\Lambda$ such that $\Phi = \Psi + \Lambda$ almost surely. They are said to be determined up to “translation” if there exists a generalized function $m$ such that $\Phi = \Psi + m$ almost surely.

## 3 Main Results

We now prove a theorem characterizing generalized random fields up to Gaussian factors.

**Theorem 3.1:** Let $\Phi_i, 0 \leq i \leq 3$ be four $Q$-independent generalized random fields on $\mathcal{X}$ and let

\[
\begin{align*}
\Psi_1 &= \Phi_0 + \Phi_1 + \Phi_2 + \Phi_3 \\
\Psi_2 &= \beta_0 \Phi_0 + \beta_1 \Phi_1 + \beta_2 \Phi_2 + \beta_3 \Phi_3
\end{align*}
\]

where $\beta_i, 0 \leq i \leq 3$ are non-zero real numbers such that $\beta_i \neq \beta_j, 0 \leq i \neq j \leq 3$. Further suppose that the joint characteristic functional $H(\phi, \psi)$ of $(\Psi_1, \Psi_2)$ does not vanish. If $L_i(\phi)$ and $M_i(\phi)$ are two alternate possible characteristic functionals of the generalized random field $\Phi_i, 0 \leq i \leq 3$, then

\[
L_j(\phi) = M_j(\phi) \exp(i m_j(\phi) - \frac{1}{2} B_j(\phi, \phi)), 0 \leq j \leq 3
\]

for some generalized functions $m_j(\phi), 0 \leq j \leq 3$ and for some continuous bilinear Hermitian functionals $B_j(\phi, \psi), 0 \leq j \leq 3$.

**Proof:** Let $\Gamma_i, 0 \leq i \leq 3$ be $Q$-independent generalized random fields on $\mathcal{X}$ such that the two-dimensional generalized random field $(\Sigma_1, \Sigma_2)$ where

\[
\begin{align*}
\Sigma_1 &= \Gamma_0 + \Gamma_1 + \Gamma_2 + \Gamma_3
\end{align*}
\]
\[ \Sigma_2 = \beta_0 \Gamma_0 + \beta_1 \Gamma_1 + \beta_2 \Gamma_2 + \beta_3 \Gamma_3 \]

has the same joint characteristic functional \( H(\phi, \psi) \) as that of \( (\Psi_1, \Psi_2) \). Let \( L_i(\cdot) \) and \( M_i(\cdot), 0 \leq i \leq 3 \) be the characteristic functionals of \( \Phi_i \) and \( \Gamma_i, 0 \leq i \leq 3 \) respectively. From the \( \mathcal{Q} \)-independence of the generalized random fields \( \Phi_i, 0 \leq i \leq 3 \), it follows that

\[ H(\phi, \psi) = \Pi_{i=0}^{3} M_i(\phi + \beta_i \psi) \exp(P_1(\phi, \psi)), \phi, \psi \in \mathcal{X} \]

for some polynomial \( P_1(\phi, \psi) \). From the \( \mathcal{Q} \)-independence of the generalized random fields \( \Gamma_i, 0 \leq i \leq 3 \), it follows that

\[ H(\phi, \psi) = \Pi_{i=0}^{3} L_i(\phi + \beta_i \psi) \exp(P_2(\phi, \psi)), \phi, \psi \in \mathcal{X} \]

for some polynomial \( P_2(\phi, \psi) \). Hence

\[ (3.4) \quad H(\phi, \psi) = \Pi_{i=0}^{3} M_i(\phi + \beta_i \psi) \exp(P_1(\phi, \psi)) = \Pi_{i=0}^{3} L_i(\phi + \beta_i \psi) \exp(P_2(\phi, \psi)), \phi, \psi \in \mathcal{X}. \]

Since \( H(\phi, \psi) \neq 0 \) for all \( \phi, \psi \in \mathcal{X} \) by hypothesis, the equation given above implies that \( L_i(\phi + \beta_i \psi) \neq 0, 0 \leq i \leq 3 \) and \( M_i(\phi + \beta_i \psi) \neq 0, 0 \leq i \leq 3 \) for all \( \phi, \psi \in \mathcal{X} \). Let

\[ J_i(\phi) = \log \frac{L_i(\phi)}{M_i(\phi)}, 0 \leq i \leq 3 \]

where the logarithm is taken to be the continuous branch with \( J_i(0) = 0 \). The equation (3.4) implies that

\[ (3.5) \quad \sum_{i=0}^{3} J_i(\phi + \beta_i \psi) = P_1(\psi, \phi) - P_2(\psi, \phi), \phi, \psi \in \mathcal{X} \]

where \( P_1(\cdot, \cdot) \) and \( P_2(\cdot, \cdot) \) are polynomials. Since \( \beta_i \neq \beta_j, 0 \leq i \neq j \leq 3 \) and \( \beta_j \neq 0 \), applying arguments similar to those in the proof of Lemma 1.5.1 in Kagan et al. (1973), it follows that the functions \( J_i(\phi), i = 0, \ldots, 3 \) are polynomials in \( \phi \) on \( \mathcal{X} \). Hence there exists polynomials \( f_j(\phi) \) such that

\[ (3.6) \quad L_j(\phi) = M_j(\phi) \exp[f_j(\phi)], \phi \in \mathcal{X}, 0 \leq j \leq 3. \]

Note that the functional \( L_j(\cdot) \) on the left side of the equation (3.6) is a characteristic functional and it is non-vanishing by the equation (3.4). Hence the function on the right side of the equation is also a non-vanishing characteristic functional which in turn implies that
the functional \( \exp[f_\phi], \phi \in \mathcal{X} \) is a characteristic functional by the one-to-one correspondence between the probability measures and the characteristic functionals on the space \( \mathcal{X} \). An application of the Marcinkiewicz lemma (cf. Theorem 2.1) implies that the degree of the polynomial \( f_\phi(.) \) cannot exceed two. It can be shown that

\[
\tag{3. 7}
L_j(\phi) = M_j(\phi) \exp(i m_j(\phi) - \frac{1}{2} B_j(\phi, \phi)), 0 \leq j \leq 3
\]

for some generalized functions \( m_j(\phi), 0 \leq j \leq 3 \) and for some continuous bilinear Hermitian functional \( B_j(\phi, \psi), 0 \leq j \leq 3 \) by arguments similar to those in Prakasa Rao (1976), p.281.

The following theorem can be proved by arguments similar to those given above. We omit the details.

**Theorem 3.2:** Let \( \Phi_i, 0 \leq i \leq 2 \) be four \( Q \)-independent generalized random fields on \( \mathcal{X} \) and let

\[
\tag{3. 8}
\Psi_1 = \Phi_0 + \Phi_1 + \Phi_2
\]
\[
\Psi_2 = \beta_0 \Phi_0 + \beta_1 \Phi_1 + \beta_2 \Phi_2
\]

where \( \beta_i, 0 \leq i \leq 2 \) are non-zero real numbers such that \( \beta_i \neq \beta_j, 0 \leq i \neq j \leq 2 \). Further suppose that the joint characteristic functional \( H(\phi, \psi) \) of \( (\Psi_1, \Psi_2) \) does not vanish. If \( L_i(\phi) \) and \( M_i(\phi) \) are two alternate possible characteristic functionals of the generalized random field \( \Phi_i, 0 \leq i \leq 2 \), then

\[
\tag{3. 9}
L_j(\phi) = M_j(\phi) \exp(i m_j(\phi)), 0 \leq j \leq 2
\]

for some generalized functions \( m_j(\phi), 0 \leq j \leq 2 \).

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