ON THE DIRAC-MOTZKIN CONJECTURE FOR SUPERSOLVABLE LINE ARRANGEMENTS

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ABSTRACT. Given a full rank real arrangement $A$ of $n$ lines in the projective plane, the Dirac-Motzkin conjecture states that for $n$ sufficiently large, the number of simple intersection points of $A$ is greater than or equal to $n/2$. In this note we show that if $A$ is supersolvable, then the conjecture is true for any $n$.

1. INTRODUCTION

Let $A$ be a real line arrangement in $\mathbb{P}^2$. Suppose $\ell_1, \ldots, \ell_n \in \mathbb{R}[x, y, z]$ are the defining equations of the lines of $A$, and assume $\dim \mathbb{R} \text{Span}(\ell_1, \ldots, \ell_n) = 3$ (i.e., $A$ has full rank, equal to 3).

An intersection point $P$ of any two of the lines of $A$ is a singularity, denoted by $P \in \text{Sing}(A)$. The number of lines of $A$ that intersect at a singular point $P \in \text{Sing}(A)$ is called the multiplicity of $P$, denoted $m(P, A)$, or just $m_P$ if the line arrangement is understood. The multiplicity of $A$, denoted $m(A)$, is $m(A) = \max\{m_P \mid P \in \text{Sing}(A)\}$. Simple points are singular points of multiplicity 2, and the set of such points in $A$ will be denoted $\text{Sing}_2(A)$.

In [13], Sylvester proposed the following problem (we state here the projective dual of Sylvester’s original problem): if $A$ is a full rank real line arrangement in $\mathbb{P}^2$, then $\text{Sing}_2(A) \neq \emptyset$. In 1944, Gallai solved this problem (see [4]), which is now known in the literature as Sylvester-Gallai Theorem.

The Dirac-Motzkin Conjecture states that if $A$ is a full rank real arrangement of $n$ lines in $\mathbb{P}^2$, then, for $n$ sufficiently large, $|\text{Sing}_2(A)| \geq n/2$. The existence of an absolute constant $n_0$ for which $|\text{Sing}_2(A)| \geq n/2$ for all $n \geq n_0$ is justified in part by the results of Kelly and Moser ([9, Theorem 3.6]) and Csisma and Sawyer ([3, Theorem 2.15]) who proved $|\text{Sing}_2(A)| \geq 3n/7$, and $|\text{Sing}_2(A)| \geq 6n/13$ if $n > 7$, respectively, as well as by the examples where these two bounds are attained. For $n = 7$, the non-Fano arrangement ([9, Figure 3.1] or [3, Fig. 3]) has $|\text{Sing}_2(A)| = 3$, which clearly fails to be $\geq n/2$; a more complicated example of Crowe and McKee ([2] or [3, Fig. 4]) is an arrangement of $n = 13$ lines with $|\text{Sing}_2(A)| = 6$, so that again $6 \neq n/2$. Hence we must have $n_0 \geq 14$.

The Dirac-Motzkin Conjecture has been proven only recently by two of the leading mathematicians of the present time: Ben Green and Terence Tao (see [5]). One
must give them credit not only for proving this conjecture and for solving another famous problem (the Orchard Problem), but also for the extraordinary exposition of their ideas, with a beautiful historical summary of these two problems; our brief introduction is largely inspired by their paper.

Aside from the two examples above, there is no known line arrangement for which $|\text{Sing}_2(A)| < n/2$; this is the reason why Grünbaum conjectured that if $n \neq 7, 13$, then $|\text{Sing}_2(A)| \geq n/2$ (see [7]). Also, one aspect of the problem is to find configurations when this bound is attained (of course $n$ must be even), and Green and Tao give a nice class of such examples (called “Böröczky examples”), see [5, Proposition 2.1 (i)].

In the following notes we investigate all of the above questions for the special case of supersolvable real line arrangements. In general, a hyperplane arrangement is supersolvable if its intersection lattice has a maximal chain of modular elements ([12]). For the case of line arrangements $A \subset \mathbb{P}^2$, supersolvability is equivalent to the existence of a $P \in \text{Sing}(A)$ such that for any other $Q \in \text{Sing}(A)$, the line connecting $P$ and $Q$ belongs to $A$ (in other words, there is an intersection point $P$ that “sees” all the other intersection points through lines of $A$). Such a point $P$ will be called modular.

We are going to prove the following result and investigate some of the related consequences.

- Let $A \subset \mathbb{P}^2$ be a full rank supersolvable line arrangement. Then $|\text{Sing}_2(A)| \geq \frac{|A|}{2}$.

Supersolvable hyperplane arrangements are an important class of hyperplane arrangements, capturing a lot of topological (over $\mathbb{C}$ they are also known as fiber-type arrangements), combinatorial, and homological information (see [10] for lots of information). Because of their highly combinatorial content, these are the best understood arrangements, yet there are some questions and problems like the one we are interested in, which deserve the effort to be analyzed.

2. THE DIRAC-MOTZKIN CONJECTURE FOR SUPERSOLVABLE REAL LINE ARRANGEMENTS

In this section $A$ will be a full rank supersolvable real line arrangement in $\mathbb{P}^2$. Suppose $|A| = n$, and denote the multiplicity of $A$ with $m := m(A)$. We begin by reproving [14] Lemma 2.1.

**Lemma 2.1.** Let $A$ be a supersolvable line arrangement with a modular point $P$. Let $Q \in \text{Sing}(A)$ not modular. Then $m(P, A) > m(Q, A)$.

Therefore, any intersection point of maximum multiplicity $m$ is modular.

**Proof.** Let $s = m(Q, A)$ where the lines passing through $Q$ are $\{\ell_1, \ldots, \ell_s\} \subset A$. Since $Q$ is not modular, there exists a point $P' \in \text{Sing}(A)$ with $P' \notin \ell_i, i = 1, \ldots, s$. Because $P$ is modular, there is a line $\ell_{PP'} \in A$ through $P$ and $P'$. Since $P' \in \text{Sing}(A)$ there is another line $\ell \in A$ through $P'$ and not passing through $P$. $\ell$ intersects the $\ell_i$'s in $s$ points, say $\{P_1, \ldots, P_s\} \subset \text{Sing}(A)$. Since $P$ is modular,
there are lines \( \ell'_i \in \mathcal{A} \) through \( P \) and \( P_i \). Counting the number of lines through \( P \) we get \( m(P, \mathcal{A}) \geq s + 1 > s = m(Q, \mathcal{A}) \).

Let \( D \in \text{Sing}(\mathcal{A}) \) be such that \( m(D, \mathcal{A}) = m \). Since \( \mathcal{A} \) is supersolvable, it must have a modular point \( P \). If \( D \) is not modular, then from the first part we have \( m(P, \mathcal{A}) > m(D, \mathcal{A}) = m \), which contradicts the maximality of \( m \). \( \square \)

The next lemma is crucial in proving the main result. The condition that \( \mathbb{R} \) is our base field is necessary for the proof; we are wondering if the result is still true over the field of complex numbers.

**Lemma 2.2.** Let \( \mathcal{A} \) be a full rank supersolvable real line arrangement in \( \mathbb{P}^2 \). Let \( P \in \text{Sing}(\mathcal{A}) \) be a point of max multiplicity (and hence, by Lemma 2.1 a modular point). Then, any line of \( \mathcal{A} \) not passing through \( P \) has at least one simple singularity.

**Proof.** Define \( m = m(P, \mathcal{A}) \). After a linear change of variables, we may assume that \( P = [0, 0, 1] \) so that the lines passing through \( P \) are parallel and vertical. Suppose for contradiction that there exists \( \ell \in \mathcal{A} \) with \( P \notin \ell \), and \( \ell \cap \text{Sing}_2(\mathcal{A}) = \emptyset \). Since \( P \) is modular, we have \( \ell \cap \text{Sing}(\mathcal{A}) = \{P_1, \ldots, P_m\} \) where each \( P_i \) lies at the intersection of \( \ell \) and a line through \( P \), denoted \( \ell'_i \). We may assume that the \( P_i \)'s are ordered from left to right (i.e. from least homogenized first coordinate to largest).

For each \( i \), let \( \ell_i \in \mathcal{A} \) be another line through \( P \) different from \( \ell \) and \( \ell'_i \). Note that the \( \ell_i \) are distinct, since \( \ell \) is the unique line passing through the \( P_i \).

Let \( a_i \) be the slope of \( \ell_i \) for each \( i \). Note that each \( a_i \) is finite, since \( P = [0, 0, 1] \). We may assume that \( a_1 \geq 0 \).

Consider the case that all the \( a_i \) have the same sign. Let \( Q \) be the intersection point of \( \ell_1 \) and \( \ell_m \). If \( Q \) lies “above” \( \ell \), then \( Q \) necessarily lies to the “right” of \( \ell_m \) and hence, since \( a_m \geq 0 \), to the “right” of any of the \( \ell'_i \). Similarly, if \( Q \) lies “below” \( \ell \), then \( Q \) necessarily lies to the “left” of \( \ell_1 \) and hence, since \( a_1 \geq 0 \), to the “left” of any of the \( \ell'_i \). In either case, \( Q \) is not on a line through \( P \), contradicting that \( P \) is modular.

In the remaining case, let \( j \) be the least index in which \( a_j < 0 \). Let \( Q \) be the intersection point of \( \ell_{j-1} \) and \( \ell_j \). Then \( Q \) must lie “between” \( \ell'_{j-1} \) and \( \ell'_j \), and so cannot lie on a line through \( Q \). This again contradicts that \( P \) is modular. \( \square \)

**Corollary 2.3.** Let \( \mathcal{A} \) be a full rank supersolvable real line arrangement in \( \mathbb{P}^2 \). Then

\[
|\text{Sing}_2(\mathcal{A})| + m(\mathcal{A}) \geq |\mathcal{A}|.
\]

**Proof.** There are exactly \( |\mathcal{A}| - m(\mathcal{A}) \) lines of \( \mathcal{A} \) not passing through \( P \), and, by Lemma 2.2, each of these must have a simple singularity of \( \mathcal{A} \) on it. Because \( P \) is modular, such a simple point can occur only as the intersection of a line of \( \mathcal{A} \) through \( P \) and of a line of \( \mathcal{A} \) not through \( P \). Therefore \( |\text{Sing}_2(\mathcal{A})| \geq |\mathcal{A}| - m(\mathcal{A}) \). \( \square \)
Theorem 2.4. Let $\mathcal{A}$ be a full rank supersolvable real line arrangement in $\mathbb{P}^2$. Then

$$|\text{Sing}_2(\mathcal{A})| \geq \frac{|\mathcal{A}|}{2}.$$ 

Proof. Let $n := |\mathcal{A}|$, and let $m := m(\mathcal{A})$. Let $P \in \text{Sing}(\mathcal{A})$ with $m(P, \mathcal{A}) = m$. Then, by Lemma 2.1, $P$ is a modular point.

Case 1. Suppose $m \leq n/2$. Then $n - m \geq n/2$, and from Corollary 2.3 one has $|\text{Sing}_2(\mathcal{A})| \geq n/2$.

Case 2. Suppose $m > n/2$. In general, any $\mathcal{A} \subset \mathbb{P}^2$ with each line containing a simple singularity has $|\text{Sing}_2(\mathcal{A})| \geq |\mathcal{A}|/2$. Suppose that there is $\ell \in \mathcal{A}$ with no simple singularity on it. From Lemma 2.2 we necessarily have that $P \in \ell$.

Let $\ell \in \mathcal{A}$ with $P \notin \ell$. Let $\{Q\} = \ell \cap \ell'$. Then $m(Q, \mathcal{A}) \geq 3$.

Suppose $\ell' \cap \text{Sing}(\mathcal{A}) = \{Q_1, \ldots, Q_{m-1}, Q_m\}$, with $Q_m = Q$, and $m(Q_1, \mathcal{A}) = 2$ (from Lemma 2.2). Denote $n_i := m(Q_i, \mathcal{A})$, $i = 1, \ldots, m$. It is clear that

\[(1) \quad (n_1 - 1) + (n_2 - 1) + \cdots + (n_m - 1) = n - 1.\]

Case 2.2. Suppose $3n/4 \geq m > n/2$. If $n_i \geq 3$, for all $i = 2, \ldots, m$, then, from (1) above

$$n - 2 \geq 2(m - 1)$$

giving $n \geq 2m$; contradiction. Hence we can assume $n_2 = 2$ as well, so that $\ell'$ has at least two simple singularities on it. This leads to

$$|\text{Sing}_2(\mathcal{A})| \geq 2(n - m) \geq 2n - 3n/2 = n/2.$$ 

Case 2.3. Suppose $5n/6 \geq m > 3n/4$. From the previous case we can assume $n_1 = n_2 = 2$. Suppose $n_i \geq 3$, for all $i = 3, \ldots, m$. Then, from (1),

$$n - 3 \geq 2(m - 2)$$

giving $(n + 1)/2 \geq m$. This contradicts $m > 3n/4$ and the fact that $n \geq 3$ ($\mathcal{A}$ has rank 3). So we can assume $n_3 = 2$ as well, which means that $\ell'$ has at least three simple singularities on it. This leads to

$$|\text{Sing}_2(\mathcal{A})| \geq 3(n - m) \geq 3n - 5n/2 = n/2.$$ 

Case 2.4. Suppose $2u \cdot n \geq 2u - 2 \cdot n$, where $u$ is some integer $> 3$.

From the inductive hypothesis (since $m > 2u - 2 \cdot n$) we can assume $n_1 = \cdots = n_u = 2$. Suppose $n_i \geq 3$, for all $i = u, \ldots, m$. Then, from (1),

$$n - u \geq 2(m - u + 1)$$

giving $(n + u - 2)/2 \geq m$.

We have $n - 1 \geq m$, from the full rank hypothesis. Then $n - 1 > 2u - 2 \cdot n$, which gives $n > 2u - 2$. The inequality obtained before $(n + u - 2)/2 \geq m$, together with $m > 2u - 2 \cdot n$, leads to $u - 1 > n$. But we previously obtained
n > 2u − 2, a contradiction. Hence we can assume n_u = 2 as well, which means that ℓ’ has at least u simple singularities on it. This leads to

$$|\text{Sing}_2(\mathcal{A})| \geq u(n - m) \geq un - (2u - 1)n/2 = n/2.$$ 

□

Example 2.5. For m ≥ 3, the (even) Böröczky configuration of points is

$$X_{2m} := \{[\cos \frac{2\pi j}{m}, \sin \frac{2\pi j}{m}, 1] : 0 \leq j < m\} \cup \{-[\sin \frac{\pi j}{m}, \cos \frac{\pi j}{m}, 0] : 0 \leq j < m\}.$$ 

Let $\mathcal{A}_{2m} \subset \mathbb{P}^2$ be the real arrangement of the 2m lines dual to the points of $X_{2m}$.

The calculations done in the proof of [5, Proposition 2.1 (i)] show that $\mathcal{A}_{2m}$ is supersolvable. The dual lines to the points of $\Lambda_2$ all pass through the point $P := [0, 0, 1]$, and any two lines dual to two distinct points $[\cos \frac{2\pi j}{m}, \sin \frac{2\pi j}{m}, 1], [\cos \frac{2\pi j'}{m}, \sin \frac{2\pi j'}{m}, 1] \in \Lambda_1$ intersect in a point that belongs to the line with equation

$$-\sin \frac{\pi (j + j')}{m} \cdot x + \cos \frac{\pi (j + j')}{m} \cdot y = 0.$$ 

Since $0 \leq j, j' \leq m - 1$, then $0 < j + j' \leq 2m - 3$.

If $j + j' \leq m - 1$, then the above line is in $A_{2m}$, and also passes through $P$.

If $j + j' \geq m$, then consider $k := j + j' - m$, which satisfies $0 \leq k \leq m - 3 < m$.

Trig identities

$$\sin \frac{\pi (j + j')}{m} = -\sin \frac{\pi k}{m}, \cos \frac{\pi (j + j')}{m} = -\cos \frac{\pi k}{m}$$

give also that the line of equation listed above is in $A_{2m}$.

Everything put together show that $P$ is a modular point (of maximum multiplicity $m$), hence $A_{2m}$ is supersolvable and satisfies equality in the bound of our Theorem 2.4.

Theorem 2.6. Let $\mathcal{A}$ be a supersolvable, real line arrangement of max multiplicity $m \geq 3$ and cardinality $2m$. Then, $|\text{Sing}_2(\mathcal{A})| = m$ if and only if $\mathcal{A}$ and $A_{2m}$ are combinatorially equivalent (i.e., they have isomorphic Orlik-Solomon algebras).

Proof. We first analyze a bit the geometry of $A_{2m}$.

Let $L_j := V(\cos \frac{2\pi j}{m}x + \sin \frac{2\pi j}{m}y + z)$ for $j = 0, \ldots, m - 1$ and $L'_k := V(\sin \frac{\pi k}{m}x - \cos \frac{\pi k}{m}y)$ for $k = 0, \ldots, m - 1$ be the lines of $A_{2m}$; observe that
the modular point $P := [0, 0, 1]$ lies on all the $L_k'$s. We have the following circuits (i.e. minimal dependent sets):

$$\{L_j, L_{j'}, L'_{j+j'}\}, j < j' \text{ and } \{L_k, L'_{k'}, L'_{k''}\}, k < k' < k'',$$

where $j + j'$ denotes the reminder of the division of $j + j'$ by $m$. Note that if $j + j' \equiv j + j'' \mod m$, with $j', j'' \in \{0, \ldots, m - 1\}$, then $m | (j' - j'')$ and hence $j' = j''$ (since $|j' - j''| \leq m - 1$). So the intersection points on lines not passing through $P$ are either simple points or triple points.

**Claim 1.** There exists exactly one simple point on each $L_j$, namely the intersection of $L_j$ and $L_{2j'}$.

Suppose there exists $L_{j'}$ that passes through the same intersection point. Then we must have $j + j' = 2j'$, leading to $m | (j' - j)$. So $j' = j$.

As a consequence we obtain that if $m$ is even, then half of the lines through $P$ have exactly two simple points and half have no simple points, and if $m$ is odd, then each line through $P$ has exactly one simple point.

Let $A$ be a real supersolvable line arrangement with $m(A) = m$ consisting of $n = 2m$ lines. Suppose that $|Sing_2(A)| = m$. To show that $A$ and $A_{2m}$ are combinatorially equivalent we follow roughly the same ideas of Green and Tao, with the hope that for supersolvable arrangements the argument is more transparent. If for them the Cayley-Bacharach Theorem was the key ingredient in the proof, for us a simple plane geometry problem does the trick (see the proof of Claim 2 below).

Let $Q \in Sing(A)$ with $m(Q, A) = m$. Then, by Lemma 2.1 $Q$ is a modular point. Let $M_0', \ldots, M_{m-1}'$ be the lines of $A$ passing through $Q$. Also, denote by $M_0, \ldots, M_{m-1}$ the remaining $n - m = m$ lines of $A$.

By Lemma 2.2 each line $M_i$ has at least one simple point on it which is the intersection of this $M_i$ and one of the $M_i'$. Since we have $m$ simple points in total and $m$ lines not passing through $Q$, each $M_i$ has exactly one simple point on it. If an $M_i$ has a point with 4 or more lines of $A$ through it, then, from equation (11) in the proof of Theorem 2.4 we obtain $2(m - 1) < n - 2$, contradicting that $2m = n$. So all the other points on $M_i$ have multiplicity 3.

Let us consider some arbitrary line $M' \in A$ through $Q$. Suppose it has $u$ simple points and $v$ triple points. Then, the same equation (11) gives

$$m - 1 + u + 2v = n - m = m,$$

leading to $u + 2v = n - m = m$.

**Claim 2.** $u \leq 2$.

Suppose $u \geq 3$. Pick $M_1, M_2, M_3$ through 3 of the $u$ simple points on $M'$. If $M_1 \cap M_2 \cap M_3 = \{Q'\}$, then since $Q$ is modular and hence lies on another line passing through $Q'$, we get $m(Q', A) \geq 4$. Contradiction. So $M_1 \cap M_2 = \{Q_{1,2}\}, M_1 \cap M_3 = \{Q_{1,3}\}, M_2 \cap M_3 = \{Q_{2,3}\}$, with these three points distinct. Since $Q$ is modular, it must connect to these three points through some lines $M_i', M_i', M_i'$ respectively. Intersecting $M_i$ and $M_i'$, for $i = 1, 2, 3$, we obtain $Q_i$. The lines $M_1, M_2, M_3$ already have their simple points, so $m(Q_i, A) = \ldots$
where \(M\) simple point on of the following: we have the triangle \(\triangle(Q_{1,2}Q_{1,3}Q_{2,3})\) and a point \(Q\) not on any of its edges \(M_1, M_2, M_3\). Let \(Q_1 = QQ_{2,3} \cap M_1, Q_2 = QQ_{1,3} \cap M_2, Q_3 = QQ_{1,2} \cap M_3\). Then \(Q_1, Q_2, Q_3\) are not collinear.

Suppose one such extra line contains \(Q_1\) and \(Q_2\). Then it intersects \(M_3\) in a “new” point \(Q_3\). The extra line passing through \(Q_3\) should not contain either \(Q_1\), nor \(Q_2\) because it will make their multiplicity bump to 4. So this extra line intersecting \(Q_3\) intersects the lines \(M_1\) and \(M_2\) in two “new” points \(Q'_1\) and \(Q'_2\). In this case, each \(Q_i\) comes with its own extra line so we still obtain three “new” points \(Q'_1, Q'_2, Q'_3\) on \(M_1, M_2, M_3\), respectively. Now with these three non-collinear (by the same geometry argument as above) points replacing the three points \(Q_{1,j}\), we can continue the argument repeatedly until we exhaust the line through \(Q\), obtaining a contradiction.

Claim 2 tells us that if \(m\) is odd, then \(u = 1\), and hence each line through \(Q\) has exactly one simple point, and if \(m\) is even, then \(u = 0\) or \(u = 2\), and hence, keeping in mind that there are exactly \(m\) simple points, half of the lines through \(Q\) have exactly two simple points and half have no simple point.

To conclude the proof, we show that the Orlik-Solomon algebras are isomorphic via an isomorphism of NBC-bases (see \([10]\) for definitions and more information, but more specifically \([11]\)). The abbreviation NBC stands for non broken circuit (in \([11]\) Definition 2.1 it is called basic), and it means the following: Suppose one picks an ordering of the hyperplanes of an arrangement \(A = \{H_1, \ldots, H_n\}\). A broken circuit (under this chosen ordering) is a subset \(S \subseteq A\) such that there is \(H \in A\) with \(H\) smaller (with respect to the ordering) than all elements of \(S\) and such that \(S \cup \{H\}\) is a circuit. An NBC is an independent subset of \(A\) that does not contain a broken circuit.

\(A_{2m}\) is supersolvable, and suppose we order the lines of this arrangement as 
\[
L'_0 < L'_1 < \cdots < L'_{m-1} < L_0 < L_1 < \cdots < L_{m-1}.
\]

Then \([11]\) Theorem 3.22] (or the original \([1]\) Theorem 2.8]) says that the Orlik-Solomon algebra, \(OS(A_{2m})\) is quadratic with the quadratic NBC elements in its basis consisting of the following:
\[
\{(L'_i, L'_j) : 1 \leq i \leq m - 1\} \cup \{(L_j, L'_{j-2}) : 0 \leq j \leq m - 1\}.
\]

Similarly, \(A\) is supersolvable, and if we order its lines as
\[
M'_0 < M'_1 < \cdots < M'_{m-1} < M_0 < M_1 < \cdots < M_{m-1}
\]
then \(OS(A)\) is quadratic, with the quadratic NBC elements in its basis consisting of the following:
\[
\{(M'_0, M'_i) : 1 \leq i \leq m - 1\} \cup \{(M_j, M'_{\delta(j)}) : 0 \leq j \leq m - 1\},
\]
where \(M'_{\delta(j)}\) is the line passing through \(Q\) such that \(M_j \cap M'_{\delta(j)}\) is the unique simple point on \(M_j\).
Clearly the two Orlik-Solomon algebras are isomorphic.

3. Appendix: Connections to the degree of the reduced Jacobian scheme of a supersolvable line arrangement

In this section we assume $K$ to be any field of characteristic 0. Let $\mathcal{A} \subset \mathbb{P}^2_K$ be a full rank supersolvable arrangement of $n$ lines. Let $m := m(\mathcal{A})$ be the multiplicity of $\mathcal{A}$, and suppose $m \geq 3$. Let $P \in \text{Sing}(\mathcal{A})$ with $m(P, \mathcal{A}) = m$; therefore by Lemma 2.1, $P$ is modular.

In $R := \mathbb{K}[x, y, z]$, let $f$ be the defining polynomial of $\mathcal{A}$, let $g$ be the product of linear forms defining the $m$ lines of $\mathcal{A}$ passing through $P$, and let $h$ be the product of the linear forms defining the $n - m$ lines of $\mathcal{A}$ not passing through $P$. Without any loss of generality we can assume $f = g \cdot h$.

Denote $\mathcal{A}_h := V(h) \subset \mathbb{P}^2$; this is the line arrangement consisting of all the lines of $\mathcal{A}$ not passing through $P$. The following equivalent statement is immediate: $Q \in \text{Sing}(\mathcal{A}_h)$ if and only if $Q \in \text{Sing}(\mathcal{A})$ and $m(Q, \mathcal{A}) \geq 3$. Therefore, taking into account that $P$ is not a simple point of $\mathcal{A}$ ($m \geq 3$), we have the formula

$$|\text{Sing}_2(\mathcal{A})| = |\text{Sing}(\mathcal{A})| - |\text{Sing}(\mathcal{A}_h)| - 1.$$  

This is the main reason why the study of $|\text{Sing}(\mathcal{A})|$ goes together with the study of $|\text{Sing}_2(\mathcal{A})|$. In fact we have the following immediate result:

**Proposition 3.1.** Let $\mathcal{A}$ be a full rank supersolvable arrangement of $n$ lines, with max multiplicity $m \geq 3$. Then

$$|\text{Sing}_2(\mathcal{A})| \geq 2|\text{Sing}(\mathcal{A})| - m(n - m) - 2.$$  

Equality holds if and only if $\mathcal{A}_h$ is generic (i.e., $\text{Sing}_2(\mathcal{A}_h) = \text{Sing}(\mathcal{A}_h)$), and in this case $|\text{Sing}_2(\mathcal{A})| = (n - m)(2m - n + 1)$.

**Proof.** Let $P \in \text{Sing}(\mathcal{A})$ be such that $m(P, \mathcal{A}) = m$. Then $P$ is modular. Let $\ell \in \mathcal{A}$ be an arbitrary line passing through $P$. Suppose there are $s$ simple points on $\ell$, and $t$ multiple points on $\ell$, distinct from $P$, with multiplicities $n_1, \ldots, n_t \geq 3$. Equation (1) in the proof of Theorem 2.4 gives:

$$(m - 1) + s + (n_1 - 1) + \cdots + (n_t - 1) = n - 1.$$  

As $n_i \geq 3$ leads to $n - m - s \geq 2t$.

Summing over all the lines through $P$ implies

$$m(n - m) - |\text{Sing}_2(\mathcal{A})| \geq 2|\text{Sing}(\mathcal{A}_h)|.$$

Formula (2) above proves the assertion, and the “if and only if” statement. If $\mathcal{A}_h$ is generic, then $|\text{Sing}(\mathcal{A}_h)| = \binom{n-m}{2}$, and immediately one obtains the claimed formula. □
It is worth noting that for the arrangement(s) in Theorem 2.6, \(|\text{Sing}(A)| = \binom{m+1}{2} + 1\); calculate \(u + v\) for each line through \(P\), sum these and add 1 to account for \(P\). Then formula (2), together with \(|\text{Sing}_2(A)| = m\), gives
\[|\text{Sing}(A_h)| = \binom{m}{2},\]
meaning that \(A_h\) is generic (as \(|A_h| = n - m = m\)).

3.1. **An interesting example.** For any homogeneous polynomial \(\Delta \in R\), let \(J_\Delta = \langle \Delta_x, \Delta_y, \Delta_z \rangle \subset R\) be the Jacobian ideal of \(\Delta\). Since \(\Delta\) is a homogeneous polynomial, \(J_\Delta\) defines the scheme of the singular locus of the divisor \(V(\Delta) \subset \mathbb{P}^2\).

A singular point shows up in \(J_\Delta\) with a certain multiplicity (known in the literature as the Tjurina number), but we are interested only in the number of singular points, which is the degree of \(\sqrt{J_\Delta}\) (the defining ideal of the reduced Jacobian scheme).

The homological information of \(J_f\), where \(f\) is the defining polynomial of a supersolvable line arrangement, is very well understood. For example, the first syzygies module of \(J_f \subset R\) is a free \(R\)-module of rank 2, with basis elements having degrees \(m - 1\) and \(n - m\) (see [8], or [10]).

By [15, Theorem 2.2], there exists \((\alpha, \beta, \gamma)\) a syzygy on \(J_f\) with \(\{\alpha, \beta, \gamma\}\) forming a regular sequence. The degree of this syzygy is \(d\) (equal to \(m - 1\) or \(n - m\)).

Then, if \(m < n - 1\), by [15, Proposition 3.1],
\[|\text{Sing}(A)| \leq d^2 + d + 1.\]

When [15] appeared, there was a question as to if the bound can be attained. The next example shows that it can. This line arrangement came to surface as a counterexample to a conjecture that \(m(A) \geq |A|/2\) which we made about real supersolvable arrangements.

Consider the line arrangement \(A\) with defining polynomial
\[f = xyz(x - y)(x - z)(y - z)(x + y - z)(x - y + z)(x - y - z).\]

We have \(n = 9\) and \(m = 4\).

The calculations below were done with [16].

\[P = [1, 1, 0] \in \text{Sing}(A)\] with \(m(P, A) = 4\); the lines of equation \(z = 0, x - y = 0, x - y + z = 0, x - y - z = 0\) all go through \(P\). In fact, since \(z(x - y)(x - y + z)(x - y - z) \in \sqrt{J_f}\), by [14, Theorem 2.2], \(P\) is a modular point and \(A\) is supersolvable.

Calculations show that
\[
\begin{align*}
\frac{1}{10}x^2z - \frac{11}{30}xyz + \frac{1}{10}y^2z + \frac{1}{12}xz^2 + \frac{1}{12}yz^2 - \frac{1}{20}z^3, \\
\frac{1}{5}x^2y + \frac{1}{6}xy^2 - \frac{1}{10}y^3 + \frac{11}{15}xyz + \frac{1}{6}y^2z + \frac{1}{5}yz^2, \\
-\frac{3}{10}x^3 + \frac{1}{2}x^2y + \frac{3}{5}xy^2 + \frac{1}{2}x^2z - \frac{11}{5}xyz + \frac{3}{5}xz^2
\end{align*}
\]
is a syzygy on $J_f$ (i.e., on some linear combination of the partial derivatives of $f$). Its entries generate an ideal of height 3, hence they form a regular sequence. The degree of this syzygy is $3 = m - 1$.

By [15 Proposition 3.1], $|\text{Sing}(\mathcal{A})| \leq 3^2 + 3 + 1 = 13$, and calculations show that in fact we have equality.

REFERENCES

[1] A. Björner, G. Ziegler, Broken circuit complexes: factorizations and generalizations, J. Combin. Theory Ser. B 51 (1991), no. 1, 96126.

[2] D. W. Crowe, T. A. McKee, Sylvester's problem on collinear points, Math. Mag. 41 (1968), 30-34.

[3] J. Csima, E. T. Sawyer, There exist $6n/13$ ordinary points, Discrete Comput. Geom. 9 (1993), 187-202.

[4] T. Gallai, Solution to problem number 4065, American Math. Monthly 51 (1944), 169-171.

[5] B. Green, T. Tao, On sets defining few ordinary lines, Discrete Comput. Geom. 50 (2013), 409–468.

[6] D. Grayson, M. Stillman, Macaulay2, a software system for research in algebraic geometry, Available at [http://www.math.uiuc.edu/Macaulay2/]

[7] B. Grünbaum, The importance of being straight, Proc. 12th Internat. Sem. Canad. Math. Congress, Vancouver, 1969.

[8] M. Jambu, H. Terao, Free arrangements of hyperplanes and supersolvable lattices, Advances in Math. 52 (1984), 248-258.

[9] L. Kelly, W. Moser, On the number of ordinary lines determined by $n$ points, Canadian J. Math. 10 (1958), 210-219.

[10] P. Orlik, H. Terao, Arrangements of Hyperplanes, Grundlehren Math. Wiss., Bd. 300, Springer-Verlag, Berlin-Heidelberg-New York, 1992.

[11] K. Pearson, Cohomology of OS Algebras for Quadratic Arrangements, Lecturas Matematicas 22 (2001), 103–134.

[12] R. Stanley, Supersolvable lattices, Algebra Universalis 2 (1972), 214–217.

[13] J. Sylvester, Mathematical question 11851, Educational Times 46 (March 1893), 156.

[14] S. Tohaneanu, A computational criterion for supersolvability of line arrangements, Ars Combinatoria 117 (2014), 217–223.

[15] S. Tohaneanu, On freeness of divisors on $\mathbb{P}^2$, Communications in Algebra 41 (2013), 2916–2932.