SOME RESULTS ON COSYMPLECTIC MANIFOLDS

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Abstract. We obtain a generalization of the Kodaira-Morrow stability theorem for cosymplectic structures. We investigate cosymplectic geometry on Lie groups and on their compact quotients by uniform discrete subgroups. In this way we show that a compact solvmanifold admits a cosymplectic structure if and only if it is a finite quotient of a torus.

1. Introduction

An odd-dimensional counterpart of a Kähler manifold is given by a cosymplectic manifold, which is locally a product of a Kähler manifold with a circle or a line. Indeed, a cosymplectic structure on a \((2n + 1)\)-dimensional manifold \(M\) is a normal almost contact metric structure \((J, \xi, \alpha, g)\) on \(M\) such that the 1-form \(\alpha\) and the fundamental 2-form \(\omega\) are closed (see \([4, 5, 6]\)).

In the context of CR geometry, cosymplectic structures can be also viewed as a special class of Levi-flat CR-structures since the foliation \(\ker \alpha\), endowed with the complex structure \(J\), defines a complex subbundle \(\mathcal{F}_C\) of the complexified tangent bundle \(TM \otimes \mathbb{C}\) satisfying \(\mathcal{F}_C \cap \overline{\mathcal{F}_C} = 0\) and such that the spaces of sections \(\Gamma(\mathcal{F}_C)\) and \(\Gamma(\mathcal{F}_C \oplus \overline{\mathcal{F}_C})\) are both closed under the Lie brackets (for more details on CR structures see for instance \([15]\)).

Trivial examples of cosymplectic manifolds are given by a product of a \(2n\)-dimensional Kähler manifold with a 1-dimensional manifold and by products of the \((2m + 1)\)-dimensional real torus \(T^{2m+1}\) with the \(r\)-dimensional complex projective space \(\mathbb{CP}^r\), where \(m, r \geq 0\) and \(m + r = n\). Not many examples of non-trivial cosymplectic manifolds are known. In \([6]\) an example of 3-dimensional compact cosymplectic manifold not topologically equivalent to a global product of a compact Kähler manifold with the circle was given. Moreover, in \([23]\) Marrero and Padron found some examples of \((2n + 1)\)-dimensional compact cosymplectic manifolds which are not topologically equivalent to the standard example \(T^{2m+1} \times \mathbb{CP}^r\). These manifolds are constructed as suspensions with fibre a compact Kähler manifold of representations defined by Hermitian isometries and they are compact solvmanifolds, i.e. compact quotients of a simply-connected solvable Lie group.
by a uniform discrete subgroup. We will show that in fact such solvmanifolds are finite quotients of tori.

The presence of so few cosymplectic compact examples is in part due to many topological restrictions. Indeed, in [5, 6, 16, 7, 22] some topological properties of compact normal contact metric manifolds and compact cosymplectic manifolds are shown. For instance, if \((M, J, \xi, \alpha, g)\) is a compact \((2n + 1)\)-dimensional cosymplectic manifold, then the Betti numbers \(b_i(M)\) satisfy the conditions:

1. \(b_i(M)\) are non-zero for all \(0 \leq i \leq 2n + 1\);
2. \(b_0(M) \leq b_1(M) \leq \cdots \leq b_n(M) = b_{n+1}(M)\);
3. \(b_{n+1}(M) \geq b_{n+2}(M) \geq \cdots \geq b_{2n+1}(M)\).

Moreover, like for the Kähler case, a compact cosymplectic manifold \(M\) is formal in the sense of Sullivan [29].

In order to find new cosymplectic structures we will study small deformations of cosymplectic structures and determine which types of Lie groups are cosymplectic.

More precisely, in the first part of the paper we will examine how a cosymplectic structure \((J, \xi, \alpha, g)\) on a compact \((2n + 1)\)-dimensional manifold changes under small deformations of the complex structure \(J\) on \(\ker \alpha\). To this aim we will interpret a cosymplectic structure in terms of smooth foliations on the manifold. Indeed, if \((M, J, \xi, \alpha, g)\) is a cosymplectic manifold, then the pair \((\ker \alpha, \omega)\) defines a Kähler-Riemann foliation on \(M\). Therefore it is possible to apply the results obtained in [1, 13] about Kähler-Riemann foliations, that we will review in Section 2 and 3. By using similar methods to the ones introduced by Kodaira and Morrow for the stability theorem of Kähler manifolds (see Theorem 4.6 of [21]), in Section 5 we will obtain the following

**Theorem 1.1.** Let \((M, J, \xi, \alpha, g)\) be a compact cosymplectic manifold and let \(J_t\) be a family of endomorphisms of \(TM\) depending differentiably on \(t\) and satisfying

\[
J_t^2 = -\text{Id} + \alpha \otimes \xi, \quad J_0 = J, \quad N_{J_t} = 0,
\]

where \(N_{J_t}\) denotes the Nijenhuis tensor of \(J_t\). Then there exists, for \(|t|\) small, a Riemannian metric \(g_t\) on \(M\) such that \(g_0 = g\) and \((J_t, \xi, \alpha, g_t)\) defines a cosymplectic structure on \(M\).

In the last two sections we will study left-invariant cosymplectic structures on Lie groups and on their compact quotients by uniform discrete subgroups. Summing up all the results we will get the following

**Theorem 1.2.** There exists a one-to-one correspondence between cosymplectic and Kähler Lie algebras.

A cosymplectic unimodular Lie group is necessarily flat and solvable. Moreover, if a solvmanifold admits a cosymplectic structure, then it is a finite quotient of a torus.
Although the first part of this theorem comes from a result of Dacko (see [8]), for sake of completeness we prove it in section 6 (see the proof of Theorem 6.1). More precisely, from [8], we have that if a $2n$-dimensional Kähler Lie algebra has a particular type of derivation, then one can construct a cosymplectic Lie algebra of dimension $2n + 1$. Since by [14] any solvable Kähler Lie algebra can be obtained by using modifications and normal $J$-algebras, it turns out that the previous construction can be applied to solvable Kähler Lie algebras.

The fact that a cosymplectic unimodular Lie group is flat and solvable follows from the results in [22] and [18] about symplectic and Kähler Lie groups. Therefore it is natural to investigate which types of solvmanifolds admit a cosymplectic structure. Like in the Kähler case (see [19]) the cosymplectic condition imposes strong restrictions.

2. Riemannian foliations

In this section we will recall some properties about Riemannian foliations that we will use in the next sections (for more details see for instance [1, 25, 26, 28]).

**Definition 2.1.** A smooth foliation $\mathcal{F}$ on a compact Riemannian manifold $(M, g)$ is called a Riemannian foliation if the Riemannian metric $g$ is bundle-like, i.e. if the normal plane field to $\mathcal{F}$ is totally geodesic.

In other words a Riemannian foliation is a foliation which is locally defined by Riemannian submersions. Such foliations have been introduced by Reinhart [28] and studied subsequently by Molino [25, 26].

Consider now a $\mathbb{Z}$-graded Riemannian vector bundle $E$ on $M$ and denote by $\Gamma(E)$ the space of the its smooth sections. Let $d: \Gamma(E) \to \Gamma(E)$ be a first-order leafwise elliptic differential operator and denote by $\delta$ the formal adjoint of $d$ with respect to the Riemannian metric $g$. Then the Laplacian

$$\Delta := (d + \delta)^2$$

is a self-adjoint differential operator on $\Gamma(E)$. We recall that a leafwise differential operator on $\Gamma(E)$ is a differential operator whose local expressions only contain derivatives along leaf directions of $\mathcal{F}$ and it can be restricted to the leaves. If in addition the restriction to the leaves is elliptic, then the leafwise differential operator is called leafwise elliptic. Moreover, a transversely elliptic differential operator is a differential operator whose leading symbol is an isomorphism at non-trivial covectors normal to the leaves, where by normal to the leaves we mean that the covector vanishes on vectors tangent to the leaves.

In the case of transversely elliptic differential operators Alvarez Lópex and Kordyukov in [1] proved that if $A: \Gamma(E) \to \Gamma(E)$ is a transversely elliptic first order differential operator and there exist morphisms $G, H, K$ and $L$ such that

$$Ad \pm dA = Gd + dH, \quad A\delta \pm \delta A = K\delta + \delta L,$$
then the orthogonal projection $\pi: \Gamma(E) \to \ker \Delta$ is a continuous operator on $\Gamma(E)$ and

$$\Gamma(E) = \ker \Delta \oplus \overline{\im \delta},$$

where the bar denotes the closure in the Fréchet space $\Gamma(E)$.

In this way one gets a natural splitting of $\Gamma(E)$ in terms of the Laplacian $\Delta$ and by [1] it is possible to apply the previous result to the case of Riemannian foliations, constructing a leafwise differential complex of $\mathcal{F}$ as follows.

Let $\Omega$ be the de Rham cohomology algebra of $M$. Denote by $T\mathcal{F}^*$ the dual bundle of the tangent bundle $T\mathcal{F}$ with respect to the duality induced by the metric $g$ and by $T\mathcal{F}^\perp$ the bundle orthogonal to $T\mathcal{F}$. One has the following bigrading of the algebra $\Omega$

$$\Omega^{u,v} = \Gamma\left(\bigwedge^u T\mathcal{F}^{\perp,*} \otimes \bigwedge^v T\mathcal{F}^*\right), \quad u, v \in \mathbb{Z}.$$ 

Since

$$d(\Omega^{u,v}) \subseteq \Omega^{u+1,v} \oplus \Omega^{u,v+1} \oplus \Omega^{u+2,v-1},$$

then the de Rham differential $d$ and its codifferential $\delta$ decompose respectively as

$$d = d_{0,1} + d_{1,0} + d_{2,-1}, \quad \delta = \delta_{0,-1} + \delta_{-1,0} + \delta_{-2,1},$$

where the double subindices denote the corresponding bidegrees of the bihomogeneous components. Each $\delta_{i,j}$ is the formal adjoint of $d_{-i,-j}$ and if we denote by $D_0 = d_{0,1} + \delta_{0,-1}$, then $D_0$ and the Laplacian $\Delta_0 = D_0^2$ are leafwise elliptic and symmetric differential operators.

By applying [1] to $(\Omega, d_{0,1})$ Alvarez Lópex and Kordyukov showed the following result

**Theorem 2.2** ([1], Theorem B). Assume that $\mathcal{F}$ is a Riemannian foliation on a compact manifold $M$ with a bundle-like metric; then, with the notation stated above, the following leafwise Hodge decomposition

$$(3) \quad \Omega = \ker \Delta_0 \oplus \overline{\im \Delta_0} = (\ker d_{0,1} \cap \ker \delta_{0,-1}) \oplus \overline{\im d_{0,1}} \oplus \overline{\im \delta_{0,-1}}$$

holds.

Note that the operator

$$d_{0,1}: \Omega^{0,v} \to \Omega^{0,v+1}$$

can be canonically identified with the de Rham derivative $d_{\mathcal{F}}$ on the leaves, or equivalently it can be defined by

$$(d_{\mathcal{F}} \alpha)|_l = d(\alpha|_l),$$

for any $\alpha \in \Omega^{0,v}$ and any leaf $l$ of $\mathcal{F}$. Moreover, $(\Omega, d_{0,1})$ can be considered, up to a sign, as the leafwise de Rham complex of $\mathcal{F}$ with coefficients in the vector bundle $\Lambda(TM/T\mathcal{F})^*$.

Furthermore we denote by

$$\delta_{\mathcal{F}} = -*_{\mathcal{F}} d_{\mathcal{F}}^*_{\mathcal{F}}, \quad \Delta_{\mathcal{F}} = (d_{\mathcal{F}} + \delta_{\mathcal{F}})^2,$$
where $\ast_F$ denotes the Hodge star operator restricted to $AT^*F$. In particular by \[1\] Corollary C\] one has that, if $\mathcal{F}$ is a Riemannian foliation on a compact manifold with a bundle-like metric and let $V = \mathbb{R}$ or $\mathbb{C}$, then with respect to such a metric we have the leafwise Hodge decomposition

$$\Omega^{0,v}(V) = \ker \Delta_F \oplus \text{im} \Delta_F = (\ker d_F \cap \ker \delta_F) \oplus \text{im} d_F \oplus \text{im} \delta_F,$$

where $(\Omega^{0,v}(V), d_F)$ is the leafwise de Rham complex with coefficients in $V$.

### 3. Complex and Kähler-Riemannian Foliations

In this section we will deal with special types of complex Riemannian foliations, called Kähler-Riemann. First of all we recall some basic facts of complex geometry:

A complex manifold can be defined as a pair $(M, J)$, where $M$ is a $2n$-dimensional smooth manifold and $J \in \text{End}(TM)$ satisfies

$$J^2 = -\text{Id}, \quad [JX, JY] = J[JX, Y] + J[X, JY] + [X, Y], \quad \text{for } X, Y \in \Gamma(TM).$$

The complex structure $J$ induces the natural splitting $TM \otimes \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$, where

$$T^{1,0}_x M = \{v \in T_x M \otimes \mathbb{C} : Jv = iv\};$$
$$T^{0,1}_x M = \{v \in T_x M \otimes \mathbb{C} : Jv = -iv\}.$$

Moreover, if $\Omega^v$ denotes the vector bundle of complex differential forms on $M$, then

$$\Omega^1 = \Omega^{1,0} \oplus \Omega^{0,1},$$
where $\Omega^{1,0}$ and $\Omega^{0,1}$ are the dual vector bundles of $T^{0,1}M$ and $T^{0,1}M$, respectively. Consequently $\Omega^v$ splits in

$$\Omega^v = \bigoplus_{r+s=v} \Omega^{r,s}$$
where

$$\Omega^{r,s} = \Omega^{r,0} \otimes \Omega^{0,s}$$
and

$$\Omega^{r,0} = \bigwedge^r \Omega^{1,0}, \quad \Omega^{0,s} = \bigwedge^r \Omega^{0,1}.$$

Now we consider the case of smooth foliations. Let $\mathcal{F}$ be a smooth $2n$-dimensional foliation on a compact Riemannian manifold $(M, g)$. An almost complex structure on $\mathcal{F}$ is a section of the bundle $\text{End}(T\mathcal{F})$ of the endomorphisms of $T\mathcal{F}$ satisfying $J^2 = -\text{Id}$. If further the Nijenhuis tensor $N_J(X, Y) := [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y]$, for $X, Y \in \Gamma(T\mathcal{F})$ vanishes, then $J$ is said to be a complex structure on $\mathcal{F}$ and the pair $(\mathcal{F}, J)$ is called a complex foliation.

A Riemannian metric $g$ is compatible with a complex foliation $(\mathcal{F}, J)$ if

$$g(JX, JY) = g(X, Y), \quad \text{for any } X, Y \in \Gamma(T\mathcal{F}).$$
Let us consider now a complex foliation \((\mathcal{F}, J)\) on Riemannian manifold \((M, g)\) and we assume that the metric \(g\) is compatible with \(J\). The tangent bundle to \(\mathcal{F}\) decomposes as

\[ T\mathcal{F} = T\mathcal{F}^{1,0} \oplus T\mathcal{F}^{0,1} \]

and consequently, if we denote by \(\Omega^{0,v} = \Omega^{0,v}(\mathbb{C})\) the bundle of complex differential forms on \(\mathcal{F}\), we have the splitting

\[ \Omega^{0,v} = \bigoplus_{r+s=v} \Omega^{0,r,s}. \]

Therefore, for the complex de Rham algebra \(\Omega\) of \(M\), we get the decomposition

\[ \Omega^p = \bigoplus_{u+v=p} \Omega^{u,v} = \bigoplus_{u+r+s=p} \Omega^{u,r,s}, \]

where the spaces \(\Omega^{u,v}\) are defined by (2).

We denote by

\[ d_{i,j,k} : \Omega^{u,r,s} \to \Omega^{u+i,r+j,s+k} \]

\[ \delta_{i,j,k} : \Omega^{u,r,s} \to \Omega^{u+i,r+j,s+k} \]

the components of the operators \(d\) and \(\delta\) with respect to the above decomposition.

A complex structure \(J\) on a smooth foliation \(\mathcal{F}\) induces a natural complex structure on the leafs of \(\mathcal{F}\). Furthermore the exterior derivative \(d_{\mathcal{F}}\) along the leaves splits as

\[ d_{\mathcal{F}} = \partial_{\mathcal{F}} + \bar{\partial}_{\mathcal{F}}, \]

where \(\bar{\partial}_{\mathcal{F}}\) and \(\partial_{\mathcal{F}}\) are defined by the relations

\[ (\partial_{\mathcal{F}} \phi)|_l = \partial(\phi|_l), \quad (\bar{\partial}_{\mathcal{F}} \phi)|_l = \bar{\partial}(\phi|_l), \]

for any \(\phi \in \Omega^{0,v}\) and any leaf \(l\) of \(\mathcal{F}\). Note that \(\partial_{\mathcal{F}}\) and \(\bar{\partial}_{\mathcal{F}}\) can be respectively identified with the operators

\[ d_{0,1,0} : \Omega^{0,u,v} \to \Omega^{0,u+1,v}, \]

\[ d_{0,0,1} : \Omega^{0,u,v} \to \Omega^{0,u+1,v}. \]

Let \(\vartheta_{\mathcal{F}}\) and \(\varphi_{\mathcal{F}}\) be the formal adjoints of \(\bar{\partial}_{\mathcal{F}}\) and \(\partial_{\mathcal{F}}\), respectively. Then the maps

\[ \Box_{\mathcal{F}} = \bar{\partial}_{\mathcal{F}} \varphi_{\mathcal{F}} + \partial_{\mathcal{F}} \vartheta_{\mathcal{F}} : \Omega^{0,r,s} \to \Omega^{0,r,s}, \]

\[ \Box_{\mathcal{F}} = \partial_{\mathcal{F}} \vartheta_{\mathcal{F}} + \bar{\partial}_{\mathcal{F}} \varphi_{\mathcal{F}} : \Omega^{0,r,s} \to \Omega^{0,r,s}, \]

are elliptic leafwise differential operators.

We recall now the definition and some properties of Kähler-Riemann foliations (for more details on this topic see for instance [13]).
Definition 3.1. Let \((M, g)\) be a compact Riemannian manifold. A Kähler foliation on \(M\) is a complex foliation \((\mathcal{F}, J)\) endowed with a real 2-form \(\omega \in \Omega^{0,1}\) such that
\[
d_{\mathcal{F}} \omega = 0, \quad \omega(X, Y) = g(JX, Y).
\]
for any \(X, Y \in \Gamma(TM)\).
If the metric \(g\) is bundle-like with respect to \(\mathcal{F}\), then \((\mathcal{F}, J, \omega)\) is called a Kähler-Riemann foliation.

For a Kähler-Riemann foliation \(\mathcal{F}\) one has the following Kähler identities
\[
\Delta_{\mathcal{F}} = 2\Box_{\mathcal{F}} = 2\Box_{\mathcal{F}}.
\]
Moreover, for the cohomology groups \(\mathcal{H}^v_{d_{\mathcal{F}}} = \ker \Delta_{\mathcal{F}} \cap \Omega^0, v\) we get the splitting
\[
\mathcal{H}^v_{d_{\mathcal{F}}} = \bigoplus_{r+s=v} \mathcal{H}^{r,s}_{\Box_{\mathcal{F}}},
\]
where
\[
\mathcal{H}^{r,s}_{\Box_{\mathcal{F}}} := \ker \Box_{\mathcal{F}} \cap \Omega^{0,r,s}.
\]
In particular if \(\mathcal{H}^v_{d_{\mathcal{F}}}\) is finite dimensional, then \(\mathcal{H}^{r,s}_{\Box_{\mathcal{F}}}\) have finite dimension for any \(r, s\) such that \(r + s = v\).

4. Cosymplectic structures

Cosymplectic structures have been introduced as an odd-dimensional analogous of Kähler manifolds and they can be interpreted in terms of Kähler-Riemann foliations.

Let start to recall the definition of a normal almost contact metric structure.

Definition 4.1. Let \(M\) be a \((2n + 1)\)-dimensional manifold. An almost contact metric structure on \(M\) consists of a quadruple \((J, \xi, \alpha, g)\), where \(J\) is an endomorphism of \(TM\), \(\xi\) is a tangent vector field, \(\alpha\) is a 1-form and \(g\) is a Riemannian metric on \(M\) satisfying the conditions
\[
J^2 = -\text{Id} + \alpha \otimes \xi, \quad \alpha(\xi) = 1,
\]
\[
g(JX, JY) = g(X, Y) - \alpha(X)\alpha(Y), \quad X, Y \in \Gamma(TM).
\]
An almost contact metric structure \((J, \xi, \alpha, g)\) is said to be normal if the endomorphism \(J\) satisfies
\[
N_J(X, Y) = 2d\alpha(X, Y)\xi, \quad \text{for any } X, Y \in \Gamma(TM),
\]
where, in this case, \(N_J\) is the Nijenhuis tensor defined by
\[
N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] + J^2[X, Y].
\]
An almost contact metric structure is called almost cosymplectic if $d\alpha = 0, d\eta = 0$. Almost cosymplectic structures with Kähler leaves were studied in [9].

Let $(M,J,\xi,\alpha,g)$ be an almost contact metric manifold such that $d\alpha = 0$, then $F = \ker \alpha$ is a smooth foliation on $M$ and the endomorphism $J$ induces an almost complex structure on $F$ which is integrable if and only if the almost contact metric structure $(J,\xi,\alpha,g)$ is normal. Therefore if $(M,J,\xi,\alpha,g)$ is a normal almost contact metric manifold such that $d\alpha = 0$, then $\ker \alpha$ is a complex foliation on $M$ and the integral curves of $\xi$ are geodesics, and, thus, $g$ is a bundle-like metric on $M$.

Moreover, for the operators $\Box_F, \Box_F$ defined by (6) and for the map $\Delta_\perp = \delta_{-1,0}d_{1,0} + d_{1,0}\delta_{-1,0}$ we can prove the following

**Proposition 4.2.** Let $(M,J,\xi,\alpha,g)$ be a compact normal almost contact metric manifold such that $d\alpha = 0$ and let $F = \ker \alpha$. Then the leaf-wise differential operators

\[
\delta_{-1,0}d_{1,0} + \Box_F : \Omega^{0,r,s} \rightarrow \Omega^{0,r,s},
\]

\[
\delta_{-1,0}d_{1,0} + \Box_F : \Omega^{0,r,s} \rightarrow \Omega^{0,r,s}
\]

are strongly elliptic and self-adjoint.

**Proof.** First of all we note that $\delta_{-1,0}$ does not act on $\Omega^{0,v}$. Hence the operator $\delta_{-1,0}d_{1,0}$ acts on $\Omega^{0,v}$ as $\Delta_\perp$ and consequently $\delta_{-1,0}d_{1,0} + \Box_F$ and $\delta_{-1,0}d_{1,0} + \Box_F$ are both self-adjoint on $\Omega^{0,v}$.

Furthermore, we can find a system $\{x,z^1,\ldots,z^n\}$ of local coordinates around each point $p$ of $M$ such that

\[
\xi = \frac{\partial}{\partial x}
\]

and $\{z^1,\ldots,z^n\}$ are holomorphic coordinates for the leaf passing trough $p$.

Then, we get the following local expressions

\[
\delta_{-1,0}d_{1,0} + \Box_F = -\frac{\partial^2}{\partial x^2} - \sum_{\mu,\beta=1}^{n} g^{\mu\beta} \frac{\partial^2}{\partial z^\beta \partial z^\mu} + \text{lower order terms},
\]

\[
\delta_{-1,0}d_{1,0} + \Box_F = -\frac{\partial^2}{\partial x^2} - \sum_{\mu,\beta=1}^{n} g^{\mu\beta} \frac{\partial^2}{\partial z^\beta \partial z^\mu} + \text{lower order terms},
\]

where $g^{\mu\beta}$ denotes the inverse of the matrix representing the restriction of the metric $g$ to the leaf passing trough $p$. From the previous local formulae it follows that the operators $\delta_{-1,0}d_{1,0} + \Box_F$ and $\delta_{-1,0}d_{1,0} + \Box_F$ are strongly elliptic.

In general the groups $\Pi_{d_F}^{\nu}$ introduced in the previous section are infinite dimensional. We will show that they are finite dimensional for this special type of normal almost contact structures. Indeed, we have the following
Proposition 4.3. Let \((M, J, \xi, \alpha, g)\) be a \((2n+1)\)-dimensional compact normal almost contact metric manifold satisfying \(d\alpha = 0\) and let \(\mathcal{F} = \ker \alpha\). Then the cohomology groups \(\mathcal{H}^v_{\mathcal{F}}\) have finite dimension for any \(v = 0, \ldots, 2n\).

Proof. Let \(\Theta : \Omega^{0,v} \to \Omega^{1,v}\) be the injective map
\[
\phi \mapsto \alpha \wedge \phi.
\]

In order to prove the proposition, it is sufficient to show that \(\Theta\) induces an injective map between \(\mathcal{H}^v_{\mathcal{F}}\) and the de Rham cohomology group \(H^{v+1}(M)\). Indeed, it is sufficient to prove that \(\Theta\) takes \(\Delta_{\mathcal{F}}\)-harmonic forms to \(\Delta\)-harmonic forms. Let \(\phi \in \mathcal{H}^{0,v}_{\mathcal{F}}\). Then \(\phi \in \Omega^{0,v}\) and satisfies the conditions
\[
\begin{aligned}
d_{\mathcal{F}} \phi &= 0, \\
\delta_{\mathcal{F}} \phi &= 0,
\end{aligned}
\]

which are equivalent to
\[
\begin{aligned}
d\phi \wedge \alpha &= 0, \\
d^*_{\mathcal{F}} \phi &= 0,
\end{aligned}
\]

where \(*_{\mathcal{F}}\) is the star Hodge operator on \(\oplus_v \Omega^{0,v}\). Since \(\alpha\) is closed and \(\phi\) belongs to \(\Omega^{0,v}\), we have
\[
d\phi \wedge \alpha = d(\phi \wedge \alpha), \quad *_{\mathcal{F}} \phi = *(\phi \wedge \alpha),
\]
being \(*\) the Hodge star operator associated to \(g\). Hence,
\[
\Delta_{\mathcal{F}} \phi = 0 \iff \Delta(\phi \wedge \alpha) = 0
\]

which ends the proof. \(\square\)

Let \((M, J, \xi, \alpha, g)\) be an almost contact metric manifold. Then we can define the fundamental form on \(M\) as the 2-form given by
\[
\omega(\cdot, \cdot) := g(J\cdot, \cdot).
\]

The 2-form \(\omega\) is \(J\)-invariant and satisfies the condition \(\alpha \wedge \omega^n \neq 0\) everywhere.

Definition 4.4. A normal almost contact metric structure \((J, \xi, \alpha, g)\) is said to be a cosymplectic structure if the pair \((\alpha, \omega)\) satisfies
\[
da\alpha = 0, \quad d\omega = 0.
\]

In terms of smooth foliations note that if \((M, J, \xi, \alpha, g)\) is a cosymplectic manifold, then the pair \((\ker \alpha, \omega)\) defines a Kähler-Riemann foliation on \(M\) (see also [20]). Therefore, the results of the previous sections about Kähler-Riemann foliations can be applied.
5. **Infinitesimal deformations of cosymplectic structures**

The aim of this section is to prove Theorem [1.1], which is the analogous of Kodaira-Morrow theorem [21] for Kähler manifolds.

In order to prove the theorem, we review some properties about elliptic operators (for more details see [21] and the references therein).

Let $M$ be a compact and oriented manifold, $B$ be a complex vector bundle on $M \times (-\lambda, \lambda)$ and $B_t := B_{\mid M \times \{t\}}$, with $t \in (-\lambda, \lambda)$. Then $\{B_t\}$ is a family of complex vector bundles on $M$.

Let $\psi_t \in \Gamma(B_t)$ be a family of smooth sections of $B_t$. Then we say that $\psi_t$ depends differentiably on $t$ if there exists a section $\psi$ of $B$ such that $\psi_t = \psi_{\mid M \times \{t\}}$.

Assume that every $B_t$ is equipped with a Hermitian metric $h_t$ on the fibres such that the family $\{h_t\}$ depends differentiably on $t$. We recall that a family of linear operators $A_t: \Gamma(B_t) \to \Gamma(B_t)$ depends differentiably on $t$ if the following property holds: if $\psi_t \in \Gamma(B_t)$ depends differentiably on $t$, then also $A_t \psi_t$ depends differentiably on $t$.

Consider now a family of strongly elliptic self-adjoint operators $E_t: \Gamma(B_t) \to \Gamma(B_t)$ depending differentiably on $t$.

We recall the following

**Theorem 5.1.** [21] Theorem 4.3 Let $\mathbb{F}_t$ be the kernel of $E_t$. Then, the map $t \mapsto \dim \mathbb{F}_t$ is upper semicontinuous, i.e. given a $t_0$ there exists a sufficiently small $\epsilon$ such that $\dim \mathbb{F}_t \leq \dim \mathbb{F}_{t_0}$ for $|t - t_0| < \epsilon$.

If we denote by

$$F_t: \Gamma(B_t) \to \mathbb{F}_t$$

the orthogonal projection with respect to $h_t$ and by

$$G_t: \Gamma(B_t) \to \Gamma(B_t)$$

the Green operator associated to $E_t$, then we have the following

**Theorem 5.2.** [21] Theorem 4.5 If the dimension of $\mathbb{F}_t$ is independent on $t$, then for $|t|$ sufficient small, $F_t$ and $G_t$ depend differentiably on $t$.

Now we apply the previous results in the context of cosymplectic structures in order to prove Theorem [1.1].

Let $(J, \xi, \alpha, g)$ be a cosymplectic structure on a compact manifold $M$ and let $\{J_t\}_{t \in (-\lambda, \lambda)}$ be a differentiable family of endomorphisms of $TM$ satisfying

$$J_t^2 = -\text{Id} + \alpha \otimes \xi, \quad N_{J_t} = 0, \quad J_0 = J.$$

The Riemannian metric

$$\bar{g}(\cdot, \cdot) := \frac{1}{2} \left( g(\cdot, \cdot) + g(J\cdot, J\cdot) \right),$$
is compatible with $J_t$ and such that

$$\tilde{g}_t(J_t \cdot, J_t \cdot) = \tilde{g}_t(\cdot, \cdot) - \alpha(\cdot) \alpha(\cdot).$$

Therefore, $(J_t, \xi, \alpha, \tilde{g}_t)$ defines a normal almost contact metric structure on $M$ for any $t$.

Moreover, by using the fact that $\mathcal{F}_t = (\ker \alpha, J_t)$ defines a complex foliation on $M$ for any $t$, and by using the decomposition (5) for the complex de Rham algebra, we get the following splitting

$$\Omega^{u,v} = \bigoplus_{r+s=v} \Omega^{u,r,s}_t,$$

where by $\Omega^{u,r,s}_t$ we denote the vector space of complex forms in $\Omega^{u,r+s}_t$ which are of type $(u, r, s)$ with respect to $J_t$.

Therefore for the differential $d_{\mathcal{F}_t}$ we have the natural decomposition

$$d_{\mathcal{F}_t} = \partial_{\mathcal{F}_t} + \overline{\partial}_{\mathcal{F}_t}.$$ 

In the sequel we will denote $\partial_{\mathcal{F}_t}$ and $\overline{\partial}_{\mathcal{F}_t}$ respectively by $\partial_t$ and $\overline{\partial}_t$.

Consider the family of linear operators

$$\widetilde{E}_t: \Omega_0^{0,r,s} \to \Omega_0^{0,r,s}$$

defined by

$$\widetilde{E}_t = \partial_t \overline{\partial}_t \partial_t \overline{\partial}_t + \partial_t \overline{\partial}_t \partial_t \overline{\partial}_t + \partial_t \overline{\partial}_t \partial_t \overline{\partial}_t + \partial_t \overline{\partial}_t \partial_t \overline{\partial}_t + \partial_t \overline{\partial}_t \partial_t \overline{\partial}_t,$$

where $\overline{\partial}_t$ and $\partial_t$ are the $\tilde{g}_t$-adjoints respectively of $\partial_t$ and $\overline{\partial}_t$. Note that $\widetilde{E}_t$ is self-adjoint with respect to the Hermitian metric $\tilde{g}_t$.

Adapting the proof of Proposition 4.3 of [21] for our case we can prove

**Proposition 5.3.** $\widetilde{E}_0 = \square_0 \square_0 + \partial_0 \overline{\partial}_0 + \overline{\partial}_0 \partial_0$ and $\widetilde{E}_t \phi = 0$ if and only if

$$\partial_t \overline{\partial}_t \partial_t \overline{\partial}_t \phi = \overline{\partial}_t \partial_t \phi = \partial_t \overline{\partial}_t \phi = 0.$$

Now we introduce the new operators $E_t: \Omega_0^{0,u,v} \to \Omega_0^{0,u,v}$ defined by

$$E_t = \widetilde{E}_t + \delta_{-1,0} d_{1,0} \delta_{-1,0} d_{1,0} + \delta_{-1,0} d_{1,0}. $$

We have the following

**Proposition 5.4.** The operator $E_t$ is strongly elliptic and self-adjoint. A form $\phi \in \Omega_0^{0,r,s}$ belongs to the space $\mathcal{F}_t^{r,s} = \ker E_t \cap \Omega_0^{0,r,s}$ if and only if

$$\partial_t \overline{\partial}_t \phi = \partial_t \phi = \overline{\partial}_t \phi = d_{1,0} \phi = 0.$$

Moreover, $\phi \in \mathcal{F}_0^{r,s}$ if and only if

$$\square_0 \phi = d_{1,0} \phi = 0.$$

**Proof.** Around any $p \in M$ we can find a system $\{x, z^1, \ldots, z^n\}$ of local coordinates, such that $\xi = \partial / \partial x$ and $\{z^1, \ldots, z^n\}$ are holomorphic coordinates
for the leaf passing through $p$. By a direct computation one can show that $E_t$ can be locally expressed in terms of previous coordinates as

$$E_t = \frac{\partial^4}{\partial x^4} + \sum_{i,j,k,l} \tilde{g}_t^{i} \tilde{g}_t^{j} \frac{\partial^4}{\partial z^i \partial \bar{z}^j \partial z^k \partial \bar{z}^l} + \text{lower order terms}.$$

Hence $E_t$ is strongly elliptic. Moreover

$$\tilde{g}_t(E_t \phi, \phi) = \tilde{g}_t(\partial_t \bar{\partial}_t \phi, \bar{\partial}_t \phi) + \tilde{g}_t(\partial_t \bar{\partial}_t \phi, \partial_t \phi) + \tilde{g}_t(\partial_t \partial_t \phi, \partial_t \phi) + \tilde{g}_t(\bar{\partial}_t \partial_t \phi, \partial_t \phi) + \tilde{g}_t(\partial_t \bar{\partial}_t \phi, \bar{\partial}_t \phi) + \tilde{g}_t(\partial_t \partial_t \phi, \bar{\partial}_t \phi) + \tilde{g}_t(\bar{\partial}_t \partial_t \phi, \partial_t \phi) + \tilde{g}_t(\partial_t \bar{\partial}_t \phi, \bar{\partial}_t \phi) + \tilde{g}_t(\partial_t \partial_t \phi, \partial_t \phi) + \tilde{g}_t(\bar{\partial}_t \partial_t \phi, \bar{\partial}_t \phi) + \tilde{g}_t(\bar{\partial}_t \partial_t \phi, \partial_t \phi) + \tilde{g}_t(\bar{\partial}_t \partial_t \phi, \partial_t \phi) + \tilde{g}_t(\partial_t \bar{\partial}_t \phi, \partial_t \phi) + \tilde{g}_t(\partial_t \partial_t \phi, \bar{\partial}_t \phi) + \tilde{g}_t(\partial_t \bar{\partial}_t \phi, \bar{\partial}_t \phi) + \tilde{g}_t(\partial_t \partial_t \phi, \bar{\partial}_t \phi) + \tilde{g}_t(\bar{\partial}_t \partial_t \phi, \partial_t \phi) + \tilde{g}_t(\bar{\partial}_t \partial_t \phi, \bar{\partial}_t \phi) + \tilde{g}_t(\bar{\partial}_t \partial_t \phi, \partial_t \phi) ,$$

which implies $E_t \phi = 0$ if and only if $\partial_t \bar{\partial}_t \phi = \partial_t \phi = \bar{\partial}_t \phi = d_{1,0} \phi = 0$. Finally, the Kähler identities (7) imply the last part of the proposition. 

For the kernel $\mathbb{F}_t^{r,s}$ of $E_t$ we can show

**Proposition 5.5.** Let $Z_t^{r,s} := \ker d \cap \Omega_t^{0,r,s}$. Then

$$Z_t^{r,s} = (\partial_t \bar{\partial}_t \Omega_t^{0,r-1,s-1} \cap \ker d_{1,0}) \oplus \mathbb{F}_t^{r,s}.$$

**Proof.** Obviously we have

$$(\ker d_{1,0} \cap \partial_t \bar{\partial}_t \Omega_t^{0,r-1,s-1}) \oplus \mathbb{F}_t^{r,s} \subseteq Z_t^{r,s}$$

and

$$\partial_t \bar{\partial}_t \Omega_t^{0,r-1,s-1} \cap \mathbb{F}_t^{r,s} = \{0\}.$$

On the other hand, for any $\psi \in Z_t^{r,s}$, one has

$$\psi = E_t G_t \psi + F_t \psi = \bar{\partial}_t \nu + \partial_t \beta + \bar{\partial}_t \gamma + d_{-1,0} \mu + F_t \psi ,$$

where $F_t$ is the projection on the kernel of $E_t$ and $G_t$ is the Green operator associated to $E_t$.

Since $d \psi = 0$, we have

$$d(\partial_t \beta + \bar{\partial}_t \gamma + d_{-1,0} \mu) = 0.$$

Furthermore, if $\sigma = \partial_t \beta + \bar{\partial}_t \gamma + d_{-1,0} \mu$, then

$$\tilde{g}_t(\sigma, \sigma) = \tilde{g}_t(\partial_t \beta, \sigma) + \tilde{g}_t(\bar{\partial}_t \gamma, \sigma) + \tilde{g}_t(\delta_{-1,0} \gamma, \sigma) + \tilde{g}_t(\partial_t \beta, \bar{\partial}_t \gamma) + \tilde{g}_t(\partial_t \beta, d_{1,0} \sigma) + \tilde{g}_t(\bar{\partial}_t \gamma, d_{1,0} \sigma) = 0. $$

Hence $\sigma = 0$ and the proposition follows. 

As a consequence we can prove
Proposition 5.6. \( \dim \mathbb{F}^{1,1}_t = \dim \mathbb{F}^{1,1}_0 \), for \( |t| \) small. Moreover, for \( |t| \) sufficiently small the orthogonal projection \( F_t : \Omega^{0,2} \to \ker E_t \cap \Omega^{0,2} \) depends differentiably on \( t \).

Proof. By Proposition [5.5] we have
\[
Z_t^{1,1} = (\partial_t \overline{\partial}_t \Omega^0 \cap \ker d_{1,0}) \oplus \mathbb{F}^{1,1}_t.
\]
Therefore we get
\[
\mathbb{F}^{1,1}_t = \frac{Z_t^{1,1}}{(\partial_t \overline{\partial}_t \Omega^0 \cap \ker d_{1,0})}.
\]
Moreover
\[
\partial_t \overline{\partial}_t \Omega^0 \cap \ker d_{1,0} \subseteq \frac{d_F \Omega^{0,1} \cap \ker d_{1,0}}{d_F \Omega^{0,1} \cap \ker d_{1,0}}
\]
and
\[
\mathbb{F}^{1,1}_t \subseteq \frac{Z_t^{1,1} + (d_F \Omega^{0,1} \cap \ker d_{1,0})}{d_F \Omega^{0,1} \cap \ker d_{1,0}},
\]
where \( \Omega^{0,1} = \Omega_t^{1,0} \oplus \Omega_t^{0,1} \) and the bar denotes the Fréchet closure in the space
\[
\Omega^{0,2} = \Omega_t^{2,0} \oplus \Omega_t^{0,1} \oplus \Omega_t^{0,2}.
\]
Since \( F \) is a Riemannian foliation, by Theorem [2.2] we have the isomorphism
\[
\ker d_F \cap \Omega^{0,2} \cap \ker d_{1,0} \cong \ker \Delta_F \cap \Omega^{0,2} \cap \ker d_{1,0}.
\]
Now we observe that since \( d : \Omega^{0,v} \to \Omega^{1,v} \oplus \Omega^{0,v+1} \) splits as \( d = d_{1,0} + d_F \) and the operator \( \delta_{-1,0} \) does not act on \( \Omega^{0,v} \), then the Laplace operator \( \Delta = d\delta + \delta d \) reduces on \( \Omega^{0,v} \) to
\[
\Delta = (\delta_{-1,0} + \delta_0)(d_{1,0} + d_F) + (d_{1,0} + d_F)(\delta_{0,-1} + \delta_0).
\]
Moreover, a form \( \phi \in \Omega^{0,v} \) belongs to \( \ker \Delta \) if and only if it belongs to the space \( \ker d_{1,0} \cap \ker \Delta_F \). Hence the cohomology groups
\[
H^{0,v}(M) := \ker \Delta \cap \Omega^{0,v}
\]
can be identified with the spaces
\[
H^{0,v}(M) = \ker (\Delta_F + d_{1,0}) \cap \Omega^{0,v} = \mathcal{T}_{d_F} \cap \ker d_{1,0}.
\]
Now we observe that the sequence
\[
\frac{Z_t^{1,1} + (d_F \Omega^{0,1} \cap \ker d_{1,0})}{d_F \Omega^{0,1} \cap \ker d_{1,0}} \cong \frac{\ker d \cap \Omega^{0,2}}{d_F \Omega^{0,1} \cap \ker d_{1,0}} \cong \ker \Delta \cap \Omega^{0,2}
\]
leads to
\[
\ker \square_t \cap \ker d_{1,0} \cap \Omega_t^{0,2} \oplus \ker \square_t \cap \ker d_{1,0} \cap \Omega_t^{0,2}
\]
is exact, where the map $\pi_t$ is defined by

$$\pi_t(\psi) = \pi_t(\psi_t^{0,2,0} + \psi_t^{0,1,1} + \psi_t^{0,0,2}) = \psi_t^{0,2,0} + \psi_t^{0,0,2},$$

and $\psi_t^{u,r,s}$ denotes the component of $\psi$ in $\Omega_t^{u,r,s}$.

Since $(F, J_0)$ is a Kähler-Riemann foliation, one has the Kähler identities

$$(7)$$

and in particular

$$H_2^2 = \ker d_1 \otimes H_1^0 \oplus \ker d_0 \otimes \ker d_1 \otimes \ker d_0.$$

Moreover

$$H_0^0(M) = \ker d_1 \otimes \ker d_0 \otimes \ker d_1 \otimes \ker d_0.$$

By Proposition 4.2, the operator $\delta_{-1,0} d_1 + \Box_t$ is strongly elliptic for every $t$; hence we can apply Theorem 5.1 obtaining

$$\dim \left( \ker(\Box_t + d_1) \cap \Omega_t^{0,0,2} \right) < \infty, \text{ for } |t| \text{ small}.$$

Let

$$b_0,\nu(M) := \dim H^{0,\nu}(M),$$

$$h_{r,s}^t := \dim \left( \ker(\Box_t + d_1) \cap \Omega_t^{0,r,s} \right) = \dim \left( \ker(\Box_t + d_1) \cap \Omega_t^{0,s,r} \right).$$

Then, by the exactness of (14), we have

$$\dim F_t^{1,1} \geq b_{0,2}(M) - 2h_{0,2}^t$$

and furthermore, since $(J, \xi, \alpha, g)$ is cosymplectic,

$$b_{0,2}(M) = h_{0,0}^0 + h_{0,1,1}^0 + h_{0,2}^0 = h_{1,1}^0 + 2h_{0,2}^0.$$

By equations (12) and (16) we obtain

$$\dim F_0^{1,1} = h_{0,1}^0 = b_{0,2}(M) - 2h_{0,2}^0.$$

As a consequence of Theorem 5.1 we have

$$\dim F_t^{1,1} \leq \dim F_0^{1,1}, \text{ for } |t| \text{ small}.$$

Since

$$\delta_{-1,0} d_1 + \Box_t : \Omega_t^{0,r,s} \to \Omega_t^{0,r,s}$$

is strongly elliptic, then the map $t \mapsto h_{0,2}^t$ is upper semicontinuos and

$$h_{0,2}^t \leq h_{0,2}^0, \text{ for } |t| \text{ small}.$$

Therefore

$$\dim F_0^{1,1} \geq b_{0,2}(M) - 2h_{0,2}^0 \geq b_{0,2}(M) - 2h_{0,2}^0 = \dim F_0^{1,1}.$$

which implies the proposition. \qed

By using Proposition 5.10 we are now able to prove Theorem 1.1.
Proof of Theorem 1.1. Let
\[ \tilde{\omega}_t(\cdot, \cdot) := \tilde{g}_t(J_t \cdot, \cdot) \]
be the fundamental 2-form associated to \((J_t, \tilde{g}_t)\). This form cannot be closed, but if we consider the new real 2-form
\[ \omega_t := \frac{1}{2} \left( F_t(\tilde{\omega}_t) + F_t(\tilde{\omega}_t) \right), \]
on \(M\), we have that \(\omega_t\) is closed and satisfies \(\omega_0 = \omega\). Since the orthogonal projection \(F_t\) depends differentiably on \(t\), then \(\alpha \wedge \omega^n_t \neq 0\) for \(|t|\) sufficiently small.

Moreover, if we introduce the new Riemannian metric
\[ g_t(X,Y) := \omega_t(X,J_t Y) + \alpha(X) \alpha(Y), \quad X,Y \in \Gamma(TM), \]
then for \(|t|\) small \((J_t, \xi, \alpha, g_t)\) defines a cosymplectic structure on \(M\) such that \(g_0 = g\) and the theorem follows. \(\square\)

6. Cosymplectic structures on Lie groups and new examples

In this section we set a one-to-one correspondence between cosymplectic and Kähler Lie algebras. More precisely, we show that if a Kähler Lie algebra is endowed with a particular type of derivation, then one can construct a cosymplectic Lie algebra.

We recall that a cosymplectic structure on a real Lie algebra \(g\) of dimension \(2n + 1\) is a quadruple \((J, \xi, \alpha, g)\), where \(\xi\) is a non-zero vector of \(g\), \(\alpha\) is the dual 1-form of \(\xi\), \(J\) is an endomorphism of \(g\), \(g\) is an inner product on \(g\) satisfying the conditions
\[ d\alpha = 0, \quad J^2 = -\text{Id} + \alpha \otimes \xi, \quad N_J = 0, \]
\[ g(J\cdot, J\cdot) = g(\cdot, \cdot) - \alpha \otimes \alpha(\cdot, \cdot), \quad d\omega = 0, \]
being \(\omega\) the 2-form defined by \(\omega(\cdot, \cdot) := g(J\cdot, \cdot)\).

If \(G\) is the simply connected Lie group with Lie algebra \(g\), then a left-invariant cosymplectic structure on \(G\) is equivalent to a cosymplectic structures on its Lie algebra \(g\).

For cosymplectic Lie algebras we can prove the following

**Theorem 6.1.** Cosymplectic Lie algebras of dimension \(2n + 1\) are in one-to-one correspondence with \(2n\)-dimensional Kähler Lie algebras equipped with a skew-adjoint derivation \(D\) commuting with its complex structure.

**Proof.** Let \((J, \xi, \alpha, g)\) be a cosymplectic structure on a \((2n + 1)\)-dimensional Lie algebra \(g\) and let \(h = \ker \alpha\). Since \(\alpha\) is closed, \(h\) is a Lie subalgebra of \(g\) carrying a Kähler structure induced by the pair \((g, J)\). Furthermore, since \(\alpha(\xi) = 1\), the vector \(\xi\) does not belong to the commutator \(g^1 = [g, g]\).

Therefore we have
\[ [\xi, h] \subseteq h, \quad [h, h] \subseteq h \]
and, consequently, \(g\) is the semidirect sum
\[ g = \mathbb{R} \xi \oplus_{\text{ad} \xi} h. \]
Since the fundamental 2-form $\omega$ associated to the cosymplectic structure is closed and $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, then one has

$$d\omega(\xi, X_1, X_2) = -\omega([\xi, X_1], X_2) + \omega([\xi, X_2], X_1) = 0,$$

for any $X_1, X_2 \in \mathfrak{h}$. Moreover, by definition of $\omega$ we have

$$g(\text{Jad}_\xi(X_1), X_2) = g(\text{Jad}_\xi(X_2), X_1),$$

for any $X_1, X_2 \in \mathfrak{h}$ and thus $\text{Jad}_\xi = \text{ad}_\xi J$ is a self-adjoint endomorphism of $(\mathfrak{h}, g)$. Therefore, $\text{ad}_\xi$ is skew-adjoint and commutes with $J$ on $\mathfrak{h}$.

On the other hand, if $\mathfrak{h}$ be a $2n$-dimensional Lie algebra endowed with a Kähler structure $(g, J)$ and a skew-adjoint derivation $D$ commuting with $J$, then we can construct a cosymplectic Lie algebra as follows. We consider the vector space $\mathfrak{g} = \mathbb{R}\xi \oplus \mathfrak{h}$ and we define a Lie bracket $[\cdot, \cdot]$ on $\mathfrak{g}$ by the relations

$$[X, Y] = [X, Y]_\mathfrak{h}, \quad [\xi, X] = D(X),$$

for any $X, Y \in \mathfrak{h}$, being $[\cdot, \cdot]_\mathfrak{h}$ the Lie bracket on $\mathfrak{h}$. We extend the endomorphism $J$ on $\mathfrak{g}$ by putting $J\xi = 0$ and we consider the inner product $g$ on the Lie algebra $\mathfrak{g}$ defined as the unique metric extension of the inner product on $\mathfrak{h}$ for which $\xi$ is unitary and orthogonal to $\mathfrak{h}$. Finally, as 1-form $\alpha$ on $\mathfrak{g}$ we take the dual to $\xi$. Then $(\mathfrak{g}, J, \xi, \alpha, g)$ is a cosymplectic Lie algebra. □

**Remark 6.2.** We remark that this last Theorem is proved in a more general contest in [8].

**Example 6.3.** An example of cosymplectic Lie algebra was given in [23]. In this case the Lie algebra $\mathfrak{g}$ is solvable, with structure equations:

$$[X_i, Z] = \frac{3}{2} \pi Y_i, \quad [Y_i, Z] = -\frac{3}{2} \pi X_i, \quad i = 1, \ldots, n$$

and the other Lie brackets are zero. With respect to the previous theorem we have that the Kähler Lie algebra $\mathfrak{h}$ is the abelian Lie algebra $\text{span}\{X_1, Y_1, \ldots, X_n, Y_n\}$ and that $\xi = Z$. The corresponding Lie group is unimodular and admits a compact quotient by a uniform discrete subgroup. Moreover, the induced left-invariant Riemannian metric on the compact quotient is flat.

We recall by [24] that if a Lie group admits a compact quotient by a uniform discrete subgroup, then the Lie group is unimodular, i.e. $\text{tr} \text{ad}_X = 0$ for any $X$ in the Lie algebra of $G$.

By [22], if a unimodular Lie group $G$ admits a left-invariant symplectic structure, then it has to be solvable. Moreover, by [18] a Kähler unimodular Lie group is flat. Hence, we have the following

**Proposition 6.4.** A cosymplectic unimodular Lie group is necessarily flat and solvable.
Therefore, it is interesting to see if there exist non-flat cosymplectic Lie algebras. By the previous Proposition the corresponding solvable Lie group will be not unimodular and therefore will do not admit any compact quotient. We will construct new cosymplectic solvable Lie algebras by applying Theorem 6.1 to solvable Kähler Lie algebras which admit a derivation with the previous property.

Many results are known about Kähler Lie groups (see for instance [14]). Basic examples are flat Kähler Lie groups and solvable Kähler Lie groups acting simply transitively on a bounded domain in $\mathbb{C}^n$ by biholomorphisms (see [17]). A classification of solvable Kähler Lie groups, up to (a holomorphic isometry) is obtained in [27].

We recall that by [10, 27] a normal $J$-algebra $a$ is in general a real solvable Lie algebra endowed with an integrable almost complex structure $J$ and a linear form $\mu$ on $a$ (called admissible) such that

\[ \mu([JX,JY]) = \mu([X,Y]), \quad \mu([JZ,Z]) > 0, \]

for all $X, Y \in a$ and $Z \neq 0 \in a$. By the previous conditions the eigenvalues of the adjoint representation of $a$ are real, i.e. $a$ is completely solvable. Moreover, the bilinear form $\langle X, Y \rangle = \mu([JX,Y])$ determines a Kähler metric on $a$.

Given a solvable Kähler Lie algebra $(\mathfrak{h}, J, g)$, a “weak modification map” is a linear map $D : \mathfrak{h} \to \text{Der} \mathfrak{h}$ such that

1. the derivation $D(X)$ is skew-adjoint with respect to $g$;
2. $[D(X), J] = 0$;
3. $[D(X), D(Y)] = D([X,Y]) = 0$;
4. $D(D(X)Y - D(Y)X) = 0$,

for any $X, Y \in \mathfrak{h}$. One may define a new Lie bracket on $(\mathfrak{h}, J, g)$ by

\[ (X,Y) = [X,Y] + D(X)Y - D(Y)X, \]

which gives to $\mathfrak{h}$ a structure of Lie algebra and $(\mathfrak{h}_D, (\cdot, \cdot), J, g)$ is called a modification of $\mathfrak{h}$ (see [13]).

In [14] Dorfmeister proved that any solvable Kähler Lie algebra $(\mathfrak{g}, J, g)$ is a modification of the semidirect products $\mathfrak{a} \oplus \mathfrak{b}$, where $\mathfrak{a}$ is an abelian ideal and $\mathfrak{b}$ is a normal $J$-algebra. Therefore, by using the previous result we can construct new cosymplectic Lie algebras.

**Example 6.5.** By [14] there exist solvable Kähler Lie algebras $(\mathfrak{s}, J, g)$ admitting a skew-adjoint derivation commuting with their complex structure $J$. Examples of such Lie algebras can be written in the form:

\[ \mathfrak{s} = \mathfrak{v} + \mathfrak{s}', \]

where $\mathfrak{v} = \mathbb{R}X_0 + \mathbb{R}JX_0 + u$ and $\mathfrak{s}'$ are both $J$-invariant subalgebras of $\mathfrak{s}$, such that $[JX_0, X_0] = X_0$, $[X_0, \mathfrak{s}'] = 0$ and the eingenvalues of $\text{ad}_{JX_0}$ restricted to $u$ have real part equal to $\frac{1}{2}$. Moreover $X_0, JX_0, u$ and $\mathfrak{s}'$ are pairwise
orthogonal. Therefore, by [14, Sections 5.1 and 5.2] \( \mathfrak{s} \) admits a self-adjoint derivation \( D \) commuting with \( J \), defined by

\[
D(aX_0 + bJX_0 + U + X') = [JX_0, U] - \frac{1}{2} U + [JX_0, X'],
\]

for any \( a, b \in \mathbb{R} \) and \( U \in \mathfrak{u}, X' \in \mathfrak{s}' \).

Therefore by Theorem [6.1] the Lie algebra \( \mathbb{R}_\xi \oplus_D \mathfrak{s} \) has a cosymplectic structure.

7. Cosymplectic structures on solvmanifolds

In [23] some examples of cosymplectic solvmanifolds are constructed as suspensions with fibre a \( 2n \)-dimensional compact Kähler manifold \( N \) (a torus) of representations defined by Hermitian isometries. Indeed, they consider a Hermitian isometry \( f \) of the torus \( N \) and define on the product \( N \times \mathbb{R} \) the free and properly discontinuous action of \( \mathbb{Z} \) given by

\[
(n, (x, t)) \mapsto (f^n(x), t - n),
\]

for any \( n \in \mathbb{Z} \) and \( (x, t) \in N \times \mathbb{R} \). The orbit space \( N \times \mathbb{R}/\mathbb{Z} \) is a compact \((2n + 1)\)-dimensional manifold endowed with the cosymplectic structure induced by the natural one on \( N \times \mathbb{R} \). We will show that the solvmanifolds constructed in this way are finite quotient of a torus and the metrics associated to the cosymplectic structures are flat.

In this section we will prove that any solvmanifold admitting a cosymplectic structure, even not left-invariant, is a finite quotient of a torus. By considering the universal covering \( \tilde{G} \), it is known (see [2]) that a solvmanifold has as finite (normal) covering a solvmanifold of the form \( \tilde{G}/\Gamma \), with discrete isotropy subgroup \( \Gamma \). Therefore, we will consider as solvmanifold a compact quotient of a simply-connected solvable Lie group \( G \) by a uniform discrete subgroup \( \Gamma \).

In [19] Hasegawa proved that a solvmanifold admits a Kähler structure if and only if it is a finite quotient of complex torus which has a structure of a complex torus bundle over a complex torus. By using this result we are able to prove the following

**Theorem 7.1.** A solvmanifold has a cosymplectic structure if and only if it is a finite quotient of torus which has a structure of a torus bundle over a complex torus.

**Proof.** Let \( M = G/\Gamma \) be a \((2n + 1)\)-dimensional solvmanifold endowed with a cosymplectic structure. Then the product \( M \times S^1 \) is a solvmanifold of the form \( G \times \mathbb{R}/(\Gamma \times \mathbb{Z}) \) and has a Kähler structure. By [19] \( M \) is thus a finite quotient of a torus and \( \Gamma \times \mathbb{Z} \) is an extension of a free abelian group of rank \( 2l \) by the free abelian group of rank \( 2k \), where \( 2k \) is the first Betti number of \( M \times S^1 \), with \( k + l = n + 1 \).

Therefore we have

\[
0 \rightarrow \mathbb{Z}^{2l - 1} \rightarrow \Gamma \rightarrow \mathbb{Z}^{2k} \rightarrow 0,
\]
where the maximal normal free abelian subgroup of rank $2n + 1$ with finite index in $\Gamma$ is of the form $\mathbb{Z}^{2l-1} \times s_1 \mathbb{Z} \times s_2 \mathbb{Z} \times \cdots \times s_{2k} \mathbb{Z}$ and the holonomy group of the Bieberbach group $\Gamma$ is $\mathbb{Z}_{s_1} \times \mathbb{Z}_{s_2} \times \cdots \times \mathbb{Z}_{s_{2k}}$, where some of the factor $\mathbb{Z}_{s_i}$ may be trivial. This follows from the fact that by [19] $M \times S^1 = \mathbb{C}^{n+1}/(\Gamma \times \mathbb{Z})$, where $\Gamma \times \mathbb{Z}$ is a Bieberbach group with holonomy $\mathbb{Z}_{s_1} \times \mathbb{Z}_{s_2} \times \cdots \times \mathbb{Z}_{s_{2k}}$. Since the action of the holonomy group on $\mathbb{C}^{n+1}/(\Gamma \times \mathbb{Z})$ is holomorphic, $M \times S^1$ is a holomorphic fiber bundle over the complex torus $\mathbb{C}^l/\mathbb{Z}^{2l}$.

In this way $M$ is a fiber bundle over the complex torus $\mathbb{C}^k/\mathbb{Z}^{2k}$ with fiber the torus $\mathbb{R}^{2l-1}/\mathbb{Z}^{2l-1}$. □

As a direct consequence of previous theorem we get the following

**Corollary 7.2.** A solvmanifold $M = G/\Gamma$ of completely solvable type has a cosymplectic structure if and only if it is a torus.

**Acknowledgements** We would like to thank Ernesto Buzano for useful comments. We also would like to thank the referee for valuable remarks and suggestions for a better presentation of the results.

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