INTRODUCTION TO GAUGE THEORIES

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Abstract

These lectures present an elementary introduction to quantum

gauge fields. The first aim is to show how, in the tree approxima-

tion, gauge invariance follows from covariance and unitarity. This leads to

the standard construction of the Lagrangian by means of covariant
derivatives in a form that unifies the massive and the massless case.

Having so identified the classical theory, the Faddeev-Popov quanti-

zation method is introduced and the BRS invariance of the resulting

action is discussed.

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1 Introduction

The recent developments in high energy physics have put great emphasis on gauge theories; indeed the general theory of fundamental interactions is completely formulated in this framework. The importance of the role of gauge invariance has obscured the reasons that have historically justified the introduction and the development of gauge theories as consistent field theories. Indeed they are very often justified on the basis of a "symmetry principle" that has to be accepted as a fundamental principle in nature.

As a matter of fact great collective efforts have been needed to identify gauge theories as the natural, and in a sense unique, quantum theories of vector fields. It is the primary role of vector currents hiding associate vector fields and the ensuing discovery of the corresponding bosons that has put gauge theories in their actual preeminent position in High Energy Physics. The progresses in the understanding of the fundamental nature of strong interactions is also based on gauge theories in spite of the lack of associate observable charges and vector particles. The hypothesis of confinement has in fact extended the range of gauge theories opening the possibility that the vector fields give an adequate description only of the short distance properties of strong interactions [1]. They should remaining unobservable together with the associated charges since at large times and distances their presence is hidden by a catastrophic growth of the strength of the coupling,. In spite of the lack of vector field asymptotic states, gauge invariance appears to play an essential role in the control of the strength of interactions at short distances. The scale properties of scattering amplitudes at large momenta giving strong indications of a weakening of the coupling strength single out a gauge theory (QCD) among the possible field theoretical models.

On the basis of these considerations I have tried, in the first part of these lectures, to give an introduction to vector field theories based on the fulfillment of physical consistency criteria among which very important is the unitarity of scattering amplitudes. The main lines of this analysis are the same followed by Yang and Mills in their fundamental paper [2]. I am just presenting a translation of the same analysis into the modern functional language.

The second main part of the lectures are devoted to the construc-
tion of gauge theories as fundamental local field theories. This will be done discussing the structure of the functional measure involved into the Feynman functional integral form of the vacuum-to-vacuum transition amplitude.

As a matter of fact there are two complementary measures that are commonly taken into account in the construction of gauge theories. The first one is the Wilson measure that is based on the lattice regularization. This is a regularization method of field theories in which the space-time points are identified with those of a periodic four-dimensional lattice. The number of these points being finite, the functional integral is reduced to an ordinary integral. The Wilson measure does not involve vector fields; it is suitable for the description of the theory in the strong coupling regime, that is, in the case of QCD, at large distances. We shall not discuss Wilson measure but those belonging to the second class, the vector field theories. These are the natural generalisation of quantum electrodynamics and the basis of all the perturbative calculations; in QCD they are suitable for a description of the theory at short distances and responsible for the introduction of important concepts, as e.g. that of ”gluon”.

The lectures are organized as follows: in the first lecture the main tools and results of field theory in its functional formulation are reminded. In the second lecture vector field theories are introduced presenting the arguments leading to gauge invariance. In the third lecture the construction of vector effective field theories is described. In the fourth lecture we discuss the Higgs mechanism giving origin to massive vector particles. The fifth lecture is devoted to the construction of gauge theories in Feynman functional framework. In the last lecture we describe the symmetry properties of this theory.

2 The S-matrix

In quantum field theory the scattering amplitudes are computed by means of the reduction formula. This can be simply written using the Green functional generator of the theory that is defined according:

\[ Z[j] \equiv e^{\frac{j}{\hbar}Z_{\Omega}[j]} = \Omega, T\left( e^{\frac{j}{\hbar} \int d^4x \sum_{a} \phi_{a}(x)j_{a}(x)} \right) \Omega >. \]  (1)
where $\phi_\alpha$ and $j_\alpha$ label a set of quantized fields and corresponding sources with, in general, different Lorentz covariances. $Z_c$ is the connected functional. Whenever one is interested in computing matrix elements between scattering states of composite operators, that is of operators that are given by non-linear functions of fields, this is done by introducing a "source" $\zeta_i$ for each operator $O_i$ and extending the definition of $Z$, and hence of $Z_c$, according:

$$Z[j, \zeta] = \langle \Omega, T \left( e^{i \int d^4 x \left[ \sum_\alpha \phi_\alpha(x) j_\alpha(x) + \sum_i \zeta_i(x) O_i(x) \right] } \right) \Omega \rangle. \quad (2)$$

The "connected" $n$-point functions are given by the $n$-th functional derivatives of $Z_c$; in particular the two-point function is:

$$\frac{\delta^2}{\delta j_\alpha(x) \delta j_\beta(0)} Z_c|_{j=0} \equiv \Delta^{\alpha\beta}(x). \quad (3)$$

In this summary of the main tools and results of the quantum field theory of scattering we shall limit ourselves for simplicity to the case of massive fields; indeed with massless fields one has long range forces that change drastically the nature of the scattering states. In the massive case the asymptotic particles correspond to poles of the Fourier transformed Green functions. In particular we can separate from $\Delta$ the asymptotic propagator $\Delta_{as}$:

$$\Delta^{\alpha\beta}(x) = \sum_\lambda \int \frac{dp}{(2\pi)^4} \frac{e^{i p x}}{m^2_\lambda - p^2 - i0^+} \Gamma^\alpha_\lambda(p) + R(x) \equiv \Delta_{as}^{\alpha\beta}(x) + R(x), \quad (4)$$

where the Fourier transform of $R$ has no pole in $p^2$. It is clear that the asymptotic propagator is by no means unique since $\Gamma^\alpha_\lambda(p)$ is defined up to a polynomial in $p^2$ vanishing at $m^2_\lambda$; however this lack of uniqueness does not affect the $S$ matrix that is obtained through the LSZ reduction formulae.

Given $\Gamma^\alpha_\lambda(p)$ one introduces the asymptotic free fields $\phi_{in}$ with the commutation relations:

$$\left[ \phi^{(+)\alpha}_{in}(x), \phi^{(-)\beta}_{in}(0) \right] = \sum_\lambda \int \frac{dp}{(2\pi)^4} e^{i p x} \theta(p^0) \delta(p^2 - m^2_\lambda) \Gamma^\alpha_\lambda(p), \quad (5)$$

and the asymptotic wave operator:

$$K_{\alpha\gamma}(\partial) \Delta_{as}^{\gamma\beta}(x) = \delta_\alpha^\beta \delta(x), \quad (6)$$
the $S$ matrix is given in the asymptotic Fock space by:

$$S = e^{\frac{i}{\hbar} \int d^4x \phi^a(x) K_{\alpha\beta}(\partial) \frac{\delta}{\delta j^\alpha(x)} :Z\big|_{j=0} :e^\Sigma :Z\big|_{j=0} . }$$

(7)

Thus the whole dynamical information is contained into the Green functional $Z$. This is computed by means of the Feynman formula:

$$Z[j, \zeta] = \int d\mu e^{\frac{i}{\hbar} \int d^4x \left[ \phi(x) j(x) + \sum_i \zeta_i(x) O_i(x) \right]} ,$$

(8)

in terms of the functional measure of the theory $d\mu$ that is related to the "bare" action $S$ by the heuristic relation;

$$d\mu = N \prod_x d\phi(x) e^{\frac{i\delta(S)}{\hbar}} .$$

(9)

$N$ is a normalization factor implementing the normalization condition: $Z[0,0] = 1$.

The heuristic formula (8) acquires a well definite meaning after a regularization-renormalization procedure which is systematically known only at the perturbative level. The standard procedure in perturbative QCD is based on dimensional regularization. Due to obvious time limitations, we shall disregard this, however important, step of the construction.

It is reminded above that the functionals $Z$ and its connected part $Z_c$ are directly related to the scattering amplitudes; the quantum analog of the action is given by the Legendre transform of $Z_c$, the proper functional $\Gamma$ [5]. Perturbatively this is the functional generator of the 1-particle-irreducible amplitudes, that is of the amplitudes corresponding to Feynman diagrams that cannot be divided into two disconnected parts by cutting a single line, it is often called the effective action, although this name is also shared by completely different objects.

To introduce the proper functional one defines the field functional:

$$\phi[j, \zeta, x] \equiv \frac{\delta}{\delta j(x)} Z_c[j, \zeta] - \frac{\delta}{\delta j(0)} Z_c[0, 0] ,$$

(10)

then, assuming that the inverse functional $j[\phi, \zeta, x]$ be uniquely defined, one has:

$$\Gamma[\phi, \zeta] \equiv Z_c[j[\phi, \zeta], \zeta] - \int dx \left( \phi(x) + \frac{\delta}{\delta j(0)} Z_c[0, 0] \right) j[\phi, \zeta, x] .$$

(11)
It is easy to verify that:

$$\frac{\delta}{\delta \phi(x)} \Gamma [\phi [j, \zeta], \zeta] = -j [\phi, \zeta, x], \quad (12)$$

and

$$\frac{\delta}{\delta \zeta(x)} \Gamma [\phi, \zeta] |_{\phi=\phi[j,\zeta]} = \frac{\delta}{\delta \zeta(x)} Z_c [j, \zeta]. \quad (13)$$

Therefore:

$$\frac{\delta^2}{\delta \phi \delta \phi'} \Gamma [\phi, \zeta] |_{\phi=\phi[j,\zeta]} = -\left[ \frac{\delta^2}{\delta j \delta j'} Z_c [j, \zeta] \right]^{-1}. \quad (14)$$

That is: the second field-derivative of $\Gamma$ gives the "full" wave operator (not to be mistaken with the asymptotic one defined in (11)). Notice that the $-\,\text{sign in (12)}$ refers to bosonic fields while in the fermionic case one has the opposite sign.

Perturbation theory consists in the construction of $\Gamma$, $Z_c$ and $S$ as (formal) power series in $\bar{\hbar}$. The fundamental tool of this construction is the saddle point method. Considering the leading steepest descent contribution to the integral (8) we get:

$$Z_c = [S + j \phi + \zeta O] |_{\delta S \over \delta \phi|_{\bar{\hbar}=0} = 0, \bar{\hbar}=0} + O(\bar{\hbar}), \quad (15)$$

and hence, if the equation $\delta S |_{\bar{\hbar}=0} = 0$ has the unique solution: $j = 0$, we have:

$$\Gamma = [S + \zeta O]|_{\bar{\hbar}=0} + O(\bar{\hbar}). \quad (16)$$

Therefore in the "classical limit" $\Gamma$ coincides with the action extended to take into account the source terms of the composite operators.

This concludes our general introduction to the functional approach to field theory. In the following sections we shall apply the above results to vector field theories.

### 3 Vector fields and gauge invariance

In the study of vector field theories one has to distinguish the case of massive from that of massless fields and theories in which the vector fields are self-coupled (non-abelian) from those without self couplings.
(abelian). In particular, abelian theories behave at large times as free ones, therefore describing scattering processes of spin 1 particles. The massive case differs from the massless one for the asymptotic state counting since massless vector particles have two helicity states while the massive one have three of them.

The non-abelian situation is much more involved; indeed the massive case originates from the so-called Higgs mechanism, corresponding to an highly non-trivial structure of the vacuum state. The vector particles are analogous to quanta of the plasma oscillations in a medium containing free charged particles; hence the vacuum is analogous to a condensate of "charged particles". In much the same way as in the abelian case, in the Higgs theory the coupling weakens at large distances and hence one has a scattering theory involving asymptotic spin 1 states.

In a non-abelian massless vector field theory one faces a completely different situation. Indeed, if the number of matter fields is small, the coupling tends to weaken at short distances, while it is supposed to intensify at large distances. This phenomenon, that would lead to the confinement of the charged states, and hence no scattering theory for them, is interpreted from the fundamental point of view as a condensation of "magnetic monopoles" rather than charges. This would be a mechanism dual to superconductivity [9], confining the field strength into thin flux tubes and hence giving a constant attractive force between two opposite charged particles. Contrary to the massive case, in this situation vector fields would play a role only at short distances where they are associate with a very peculiar force law. For our purposes it is convenient to limit our analysis, at least at the beginning, to the massive case in which it is possible to exploit the whole apparatus described in the previous section.

Let us therefore consider a system of vector fields $A^\alpha_\mu$ that are assumed to have non-trivial asymptotic limit leading to scattering. Let the corresponding asymptotic fields be:

$$A^{(in)\alpha}_\mu(x) = \frac{1}{(2\pi)^{3/2}} \sum_{\lambda=1}^{3} \int \frac{d^3 p}{\sqrt{2E_p}} \left( \epsilon_{\lambda,\vec{p},\mu} A^{(in)\alpha}_{\lambda\vec{p}} e^{-\frac{i\vec{p} \cdot \vec{x}}{\bar{\hbar}}} + \text{hermitian conjugate} \right),$$

(17)

with the polarization vectors $\epsilon$ satisfying:

$$\epsilon_{\lambda,\vec{p},\mu} \epsilon_{\lambda',\vec{p}'}^{\mu'} = -\delta_{\lambda\lambda'} ,$$

(18)
\[ \epsilon_{\lambda, \vec{p}, \mu} p^\mu = 0. \]  

(19)

Notice that:

\[ \partial_\mu A^{(in)\alpha\mu}(x) = -\frac{i}{(2\pi)^{3/2}} \sum_{\lambda=1}^{3} \int \frac{d^3 p}{\sqrt{2E_p}} \left( \epsilon_\lambda^\mu \epsilon_{\lambda, \vec{p}}^\alpha A_{\lambda, \vec{p}}^{(in)} e^{-i p x} - \text{h.c.} \right) = 0. \]

(20)

Here and in the following to simplify the formulae, we assume mass degeneracy of the vector field components.

From (17) the asymptotic propagator is easily computed:

\[ \Delta_{\alpha\beta}^{(as) \mu \nu}(x) = \int \frac{dp}{(2\pi)^4} e^{i p x} \left[ \frac{p_{\mu} p_{\nu}}{m^2} - g_{\mu\nu} \right] \frac{m^2 - p^2 - i0_+}{m^2} \delta^{\alpha\beta}. \]

(21)

The two-point function is:

\[ \Delta_{\mu\nu}(x) = \Delta_{\alpha\beta}^{(as) \mu \nu}(x) + R_{\mu\nu}(x), \]

(22)

where \( R \) accounts for the contributions of many particle states in the Lehmann decomposition of \( \Delta \).

The main difficulty with this theory comes from the high momentum behavior of the Fourier transform of the asymptotic propagator that tends to an homogeneous function of degree zero:

\[ \tilde{\Delta}_{\alpha\beta}^{(as) \mu \nu}(p) \longrightarrow -\frac{p_{\mu} p_{\nu}}{p^2 m^2} \delta^{\alpha\beta}, \]

(23)

instead of \(-2\) that is expected for dimensional reason. The same behavior characterizes the two-point function \( \Delta \) at least not too far from the perturbative regime. It is fairly obvious that the propagator of degree zero is going to produce cross sections increasing proportionally to the second power of the center-of-mass energy \( \tilde{S} \) in much the same way as this happens in the Fermi theory of weak interactions. This induces a violation of \( S \)-matrix unitarity; therefore, in order the theory to be consistent, one has to assume a decoupling mechanism for the "longitudinal" components of the vector field. We are going to show that this leads directly to gauge invariance.

A convenient, however elaborate, way to analyze this decoupling mechanism is based on the introduction of artificial longitudinal asymptotic states by adding in (17) a further set of polarization states corresponding to \( \lambda = 0 \) with:

\[ \epsilon_{0, \vec{p}, \mu} \sim p^\mu. \]

(24)
A direct way to do this is to add to the proper functional \( \Gamma \) the term:

\[
- \int d^4 x \frac{\xi}{2} \sum_\alpha [\partial_\mu A^{\alpha \mu}(x)]^2 .
\]  

(25)

The Fourier transformed two-point vertex before the introduction of (25) is:

\[
\tilde{\Gamma}^{(2)\alpha \beta}_{\mu \nu}(p) = \delta^{\alpha \beta} \left[ \frac{p_\mu p_\nu}{p^2} - g_{\mu \nu} \right] \left( p^2 - m^2 \right) \left( 1 + A(p^2) \right) + \frac{p_\mu p_\nu m^2}{p^2} \left( 1 + B(p^2) \right) ,
\]  

(26)

with \( A(0) = B(0) \) since (26) must be regular at \( p^2 = 0 \) and, owing to (21), \( A(m^2) = B(m^2) = 0 \). From (26) we can compute the Fourier transformed two-point function after introduction of (25) getting:

\[
\tilde{\Delta}^{(2)\alpha \beta}_{\mu \nu}(p) = \delta^{\alpha \beta} \left[ \frac{p_\mu p_\nu}{m^2 - p^2} (1 + A(p^2)) + \frac{p_\mu p_\nu}{m^2 - p^2} (1 + B(p^2)) \right] .
\]  

(27)

(27) has a pole at \( p^2 = m^2 \) in much the same way as the Fourier transform of (21), however its degree at high momentum is \(-2\) and there is a second pole at \( p^2 = m^2 \xi \left( 1 + B(p^2) \right) \). Considering the asymptotic propagator corresponding to this second pole, we see that this can be identified with that of the derivative of a scalar field, however with the wrong sign. This shows that the introduction into the vertex functional of the term (25) is just a mathematical trick void of any physical meaning whose only role is to ”regularize” the short distance properties of the two-point function. Indeed the wrong sign indicates that the modified proper functional does not correspond to any quantum field theory whose asymptotic state space be a Hilbert space; the sign is however consistent with a theory in an indefinite metric space.

Our purpose is to prove that the fake theory with the term (25) leads to an \( S \) matrix independent of \( \xi \), which is equivalent to the wanted decoupling of the ”longitudinal” components of the vector fields. Since, this theory is expected to give cross sections less divergent at high energies, and hence compatible with unitarity, and since, in the limit \( \xi \to 0 \), one gets back the original scattering amplitudes, the \( \xi \)-independence of the \( S \) matrix of the fake theory excludes the feared violations of unitarity in the original one.

The decoupling of the ”longitudinal” components of the vector
fields is equivalent to the equation:

\[ \partial_{\mu} A^{\alpha \mu}(x) \mid_{\phi=\frac{\delta Z}{\delta \phi^j}, j=K\phi_{in}} := \partial_{\mu} A^{\alpha \mu}(x)_{in} . \]  

(28)

Now, using the field equation (12) on the mass-shell, we have:

\[ \left[ \xi \partial_{\mu} \partial_{\nu} A^{\alpha \nu}(x) + \frac{\delta \Gamma}{\delta A^{\alpha \mu}(x)} \right] \mid_{\phi=\frac{\delta Z}{\delta \phi^j}, j=K\phi_{in}} := 0 , \]  

(29)

whence we see that (28) is equivalent to:

\[ \partial_{\mu} \frac{\delta \Gamma}{\delta A^{\alpha \mu}(x)} \mid_{\phi=\frac{\delta Z}{\delta \phi^j}, j=K\phi_{in}} := 0 , \]  

(30)

in the subspace of the asymptotic space in which \( \partial_{\mu} A^{\alpha \mu}(x)_{in} = 0 \).

In the forthcoming formulae we shall forget the Wick ordering and understand the mass shell prescription appearing above whenever we shall write an identity using the identity symbol \( = \). We shall instead use the equivalence symbol \( \equiv \) in the case of relations holding true even outside the mass shell.

(30) shows that \( \frac{\delta \Gamma}{\delta A^{\alpha \mu}(x)} \) is a conserved current. After Nöether theorem, given a system of conserved currents \( I^{\mu \alpha} \), one can always find a set of infinitesimal field transformations:

\[ \phi^i(x) \Rightarrow \phi^i(x) + \epsilon_{\alpha} P_{\alpha}^i(x) \]  

(31)

leaving \( \Gamma \) invariant and hence such that:

\[ P_{\alpha}^i(x) \frac{\delta \Gamma}{\delta \phi^i} \equiv \partial_{\mu} I^{\mu \alpha} \equiv \partial_{\mu} \frac{\delta \Gamma}{\delta A^{\alpha \mu}} . \]  

(32)

In (31) the symbols \( P_{\alpha}^i \) stay for generic field functionals and once again we have used \( \phi \) as a collective symbol for all the fields.

Adding to \( \Gamma \) the \( \xi \) term and applying to the result the transformations (31) we get on the mass shell:

\[ - \xi \partial_{\mu} A^{\beta \mu} \partial_{\nu} P_{\alpha}^{\beta \nu} + P_{\alpha}^i \frac{\delta \Gamma}{\delta \phi^i} = 0 , \]  

(33)

and hence, on account of (29) and (32) we find:

\[ \xi \left[ \partial^2 \delta_{\alpha}^{\beta} + \partial_{\alpha}^\nu P_{\alpha}^{\beta \nu} \right] \partial_{\mu} A_{\beta \mu} = 0 , \]  

(34)
in the asymptotic subspace in which $\partial_\mu A^{\alpha \mu}(x)_{in} = 0$.

Therefore, provided that the kernel of the differential operator \[ \partial^2 \delta^\beta_\alpha + \partial^\nu P^\beta_\alpha \] be trivial, we can conclude that (32) implies (28) and hence the $\xi$-independence of the $S$ matrix. Notice that we can write (32) in the form:

\[
\left( \partial_\mu \frac{\delta}{\delta A^{\alpha \mu}_i} - P^i_\beta \frac{\delta}{\delta \phi^i_\beta} \right) \Gamma \equiv X_\alpha(x) \Gamma \equiv 0 ,
\]

(35)

It is clear that (35) corresponds to an invariance property of $\Gamma$. Considering in particular the case in which the fields $\phi$ correspond to the vectors and to a system of scalars $\varphi^a$, that is:

\[
\phi^i \equiv \left( A^{\beta \mu}, \varphi^a \right) ,
\]

(36)

and:

\[
P^i_\alpha \equiv \left( P^{\beta \mu}_\alpha, P^a_\alpha \right) ,
\]

(37)

(35) prescribes the invariance of $\Gamma$ under the system of infinitesimal transformations:

\[
A^{\alpha \mu}(x) \rightarrow A^{\alpha \mu}(x) + \partial^\mu \Lambda^\alpha(x) + \Lambda^\beta(x) P^{\alpha \mu}_\beta(x) ,
\]

(38)

and

\[
\varphi^a(x) \rightarrow \varphi^a(x) + \Lambda^\beta(x) P^a_\beta(x) .
\]

(39)

We shall call these transformations ”gauge transformations” and the invariance condition (32) ”gauge invariance condition”.

To push farther our analysis we assume that the above described invariance be minimal, in the sense that no further condition is needed for the theory to be consistent. The meaning of this hypothesis is clarified in mathematical terms requiring that the system of differential operators $X_\alpha(x)$ be in involution, that is:

\[
[X_\alpha(x), X_\beta(y)] = \int dz F^\gamma_{\alpha \beta}(x, y, z) X_\gamma(z) .
\]

(40)

Indeed, according to Frobenius theorem, the commutation relations (40) are necessary and sufficient conditions for the system (33) be integrable.

Our aim is now to use (40) to get further information on the nature of the transformations (38) and (39). In the framework of an
effective field theory one studies the low energy properties of a quantum system; if all the particles are massive, in the low energy regime, the vertices of the theory are analytical functions of the momenta; this implies that they can be approximated by polynomials and hence $\Gamma$ is approximately a local functional. The first non-trivial approximation contains only the terms whose coefficients have non-negative mass dimension; further terms with coefficient of increasingly high negative mass dimension are needed if one studies processes of increasing energy. The same considerations hold true for $P^\alpha_i$ whose first approximation, on account of Lorentz invariance, is:

$$P^\beta\mu(x) = C^\beta_\alpha A^\mu_\alpha(x) ,$$  

$$P^\alpha_\alpha(x) = v^a_\alpha + t^{ab}_\alpha \varphi^b(x) .$$  

Let us consider, for the moment the case in which the gauge transformations act homogeneously on the scalar fields, that is:

$$v^a_\alpha = 0 ,$$  

and hence:

$$X_\alpha(x) = \partial_\mu \frac{\delta}{\delta A^\alpha_\mu(x)} - C_\alpha^\beta A^\gamma_\mu(x) \frac{\delta}{\delta A^\gamma_\mu(y)} - t^{ab}_\alpha \varphi^b \frac{\delta}{\delta \varphi^a} .$$  

In this situation (40) is written:

$$[X_\alpha(x), X_\beta(y)] = \left[ \partial_\mu \frac{\delta}{\delta A^\alpha_\mu(x)} C_\beta^\gamma A^\delta_\mu(y) \frac{\delta}{\delta A^\delta_\nu(y)} \right] + \left[ C_\alpha^\beta A^\delta_\mu(x) \frac{\delta}{\delta A^\gamma_\mu(y)} C_\beta^\gamma A^\delta_\nu(y) \frac{\delta}{\delta A^\delta_\nu(y)} \right] +$$

$$+ \left[ t^{ab}_\alpha \varphi^b \frac{\delta}{\delta \varphi^a} , t^{cd}_\beta \varphi^d \frac{\delta}{\delta \varphi^c} \right] .$$  

The first two terms in the right-hand side give:

$$1 - 2 = -\partial_\mu \delta(x-y) \left( C^\gamma_\beta A^\gamma_\mu + C^\gamma_\alpha A^\gamma_\mu \right) \frac{\delta}{\delta A^\nu_\mu(y)} + \delta(x-y) \left( C^\gamma_\alpha \partial_\mu \frac{\delta}{\delta A^\gamma_\mu(x)} \right) .$$  

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On account of (40) the first term in the right-hand side has to vanish since in no way it can appear in the right-hand side of this equation. Thus we have:

$$C_{\beta\gamma}^\alpha + C_{\alpha\beta}^\gamma = 0$$  \hspace{1cm} (47)

taking into account the surviving term in (46) we see that:

$$F_{\alpha\beta}^\gamma (x, y; z) = \delta(x - z)\delta(y - z)C_{\alpha\beta}^\gamma$$  \hspace{1cm} (48)

and hence (40) is written:

$$[X_\alpha (x), X_\beta (y)] = \delta(x - y)C_{\alpha\beta}^\gamma X_\gamma (x) .$$  \hspace{1cm} (49)

The third term in (45) is:

$$(3) = \delta(x - y) \left( C_{\alpha\delta}^{\gamma} C_{\beta\gamma}^\eta - C_{\beta\delta}^{\eta} C_{\alpha\gamma}^\eta \right) A_{\mu}^{\delta} \frac{\delta}{\delta A_{\mu}^{\eta}} ,$$  \hspace{1cm} (50)

and consistency with (49) requires:

$$C_{\alpha\delta}^{\gamma} C_{\beta\gamma}^\eta + C_{\beta\delta}^{\eta} C_{\alpha\gamma}^\eta + C_{\beta\alpha}^{\gamma} C_{\gamma\delta}^\eta .$$  \hspace{1cm} (51)

This is a Jacobi identity; it proves that $C_{\beta\gamma}^\alpha$ are the structure constants of a Lie group the we shall call the ”gauge group”. Coming to the fourth term we get:

$$(4) = t^{ab}_{\alpha} t^{ca}_{\beta} \varphi^b \frac{\delta}{\delta \varphi^c} - (\alpha \leftrightarrow \beta) = -C_{\alpha\beta}^{\gamma} t^{ab}_{\alpha} t^{\gamma c}_{\beta} \frac{\delta}{\delta \varphi^c}$$  \hspace{1cm} (52)

and from (43):

$$[t_{\alpha}, t_{\beta}] = C_{\alpha\beta}^{\gamma} t^{\gamma}_{\gamma} .$$  \hspace{1cm} (53)

This identifies the matrices $t$ with the infinitesimal generators of a representation of the gauge group.

We have thus characterized completely the structure of the gauge generators identifying the coefficients $C$ with the structure constants of the gauge group and the matrices $t$ with a representation of the corresponding Lie algebra.

If we now relax condition (43) replacing:

$$X_\alpha \rightarrow X_\alpha - \epsilon^{a}_{\alpha} \frac{\delta}{\delta \varphi^a} .$$  \hspace{1cm} (54)
we insert in the right-hand side of (45) the further term:

\[
[X_\alpha(x), X_\beta(y)] = ... + \left[ v_\alpha^a \frac{\delta}{\delta \phi^a}, t_\beta^{cd} \phi^d \frac{\delta}{\delta \phi^c} \right] - (\alpha \leftrightarrow \beta),
\]

then, applying (49), we get:

\[
t_\beta v_\alpha - t_\alpha v_\beta = C^\gamma_{\beta \alpha} v_\gamma.
\]

In the standard situation in which the gauge group is semisimple and the representation \( t \) is unitary, one can show by purely algebraic means that (56) implies:

\[
v_\alpha^a = t_{ab} v_b ,
\]

that is:

\[
P_\alpha^a(x) = t_{ab}^a (\phi^b + v^b) .
\]

Making use of scale and dimensional arguments the same result can be extended to any compact gauge group. When the matter field representation of the gauge group is unitary it is convenient to replace the antihermitian matrices \( t \) by hermitian ones:

\[
t_\alpha = i \tau_\alpha ,
\]

writing (53) according:

\[
[\tau_\alpha, \tau_\beta] = -i C^\gamma_{\alpha \beta} \tau_\gamma \equiv i f_{\alpha \beta}^\gamma \tau_\gamma .
\]

We shall conventionally adopt the normalization condition:

\[
Tr (\tau_\alpha \tau_b) = \delta_{ab} ,
\]

With these symbols the infinitesimal gauge transformations in the "inhomogeneous" case are written:

\[
\begin{align*}
\delta A_\mu^\alpha &= \partial_\mu \Lambda^\alpha - \Lambda^\beta A_\mu^\gamma f_{\beta \gamma}^\alpha \\
\delta \phi^a &= i \Lambda^\alpha \tau_{ab}^\alpha (\phi^b + v^b)
\end{align*}
\]

The homogeneous form is obtained simply setting \( v = 0 \)
We now show how it is possible to construct the vertex functional of a gauge invariant effective field theory assuming the gauge transformations (62). The fundamental tool for this construction is given by the covariant derivative operator; this is a space-time partial derivative commuting with the gauge transformations. Let us consider the gauge variation of the usual space-time partial derivative of the scalar field; this is given by:

$$\delta \partial_\mu \phi = i \Lambda^\alpha \tau_\alpha \partial_\mu \phi + i \partial_\mu \Lambda^\alpha \tau_\alpha \phi ,$$  \hspace{1cm} (63)

from which it is fairly clear that this derivative is not covariant. However consider:

$$\delta \left( \partial_\mu \varphi - i A^\alpha_\mu \tau_\alpha (\varphi + v) \right) = i \Lambda^\alpha \tau_\alpha \varphi + i \partial_\mu \Lambda^\alpha \tau_\alpha (\varphi + v) + i \Lambda^\beta A^\gamma_\mu f^\alpha_{\beta \gamma} \tau_\alpha (\varphi + v) + A^\alpha_\mu \tau_\alpha \Lambda^\beta \tau_\beta (\varphi + v)$$

$$= i \Lambda^\alpha \tau_\alpha \left( \partial_\mu \varphi - i A^\beta_\mu \tau_\beta (\varphi + v) \right) .$$ \hspace{1cm} (64)

This shows that the derivative:

$$D_\mu (\varphi + v) \equiv \partial_\mu \varphi - i A^\alpha_\mu \tau_\alpha (\varphi + v) ,$$ \hspace{1cm} (65)

is covariant, since $D_\mu (\varphi + v)$ transforms as $\varphi + v$. The same property holds true for a multiple $D$ derivative.

Since the action of a gauge transformation on the scalar fields is unitary, and hence there is a scalar product $(.,.)$ such that

$$(\varphi + v, \delta (\varphi + v)) + (\delta (\varphi + v), \varphi + v) = 0 ,$$ \hspace{1cm} (66)

the scalar products:

$$(D_{\mu_1} \cdot D_{\mu_n} (\varphi + v), D_{\nu_1} \cdot D_{\nu_m} (\varphi + v)) ,$$ \hspace{1cm} (67)

are gauge invariant.

Therefore the functional:

$$\Gamma_s \equiv \int d^3x \left[ (D_\mu (\varphi + v), D^\mu (\varphi + v)) - \frac{\lambda}{4!} [(\varphi + v, \varphi + v) - (v, v)]^2 \right] ,$$ \hspace{1cm} (68)

is a natural term of the first approximation gauge invariant vertex functional; notice that the coefficient $\lambda$ is dimensionless and:

$$\frac{\delta \Gamma_s}{\delta \varphi} |_{\varphi = 0} = 0 ,$$ \hspace{1cm} (69)
as it is required by (10) and (12).

If the $\tau$ representation of the gauge group is reducible, every irreducible component of it carries an invariant scalar product and hence the second term of $\Gamma_s$ is replaced by a polynomial of second degree in these scalar products, bounded from above and satisfying (3).

Furthermore, consider the commutator:

$$[D_\mu, D_\nu] (\varphi + v) = \left[ \partial_\mu - i A_\mu^\alpha \tau_\alpha, \partial_\nu - i A_\nu^\beta \tau_\beta \right] (\varphi + v) = -i \tau_\alpha \left[ \partial_\mu A_\nu^\alpha - \partial_\nu A_\mu^\alpha + A_\mu^\beta A_\nu^\gamma f_\alpha^{\beta\gamma} \right] (\varphi + v) \equiv i \tau_\alpha G_\mu^\alpha (\varphi + v) \equiv G_\mu^\nu (\varphi + v).$$  

(70)

$G_\mu^\nu$ is the "field strength" of our gauge theory.

It is clear that:

$$\delta G_\mu^\nu = i \left[ A_\mu^\beta \tau_\beta, G_\mu^\nu \right],$$  

(71)

since, according to (53), an analogous equation holds true for any product of covariant derivatives; thus the terms of the form:

$$Tr \left( [D_{\eta_1}, \ldots, D_{\eta_k}, G_{\mu_1\nu_1}] \ldots [D_{\eta_j}, \ldots, D_{\eta_l}, G_{\mu_j\nu_j}] \ldots \right),$$  

(72)

are gauge invariant. In particular the functional:

$$\Gamma_g \equiv -k \int d^4x Tr (G_\mu^\nu G_\mu^\nu),$$  

(73)

is the natural candidate for a first approximation to the gauge field vertex functional of the effective theory. If the compact gauge group is the direct product of a set of invariant factors (either abelian or simple), the field strength decomposes into the sum of the contributions of the single factors:

$$G_\mu^\nu = \sum_I G_\mu^\nu^{(I)},$$  

(74)

and (73) is replaced by the sum of the contributions of the factors, each multiplied by an independent coefficient $k_I$.

The sum of (73) and (58) gives the complete effective field approximation vertex functional. To have a better understanding of the physical content of our theory we now discuss in some detail an example in which the gauge group is $SU(2)$ and the scalar representation
is the fundamental one:

\[
\varphi = \begin{pmatrix} \varphi_+ \\ \varphi_- \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi_2 + i\varphi_1 \\ \varphi_4 + i\varphi_3 \end{pmatrix}.
\]  

(75)

The field strength is:

\[
G_{\mu\nu} = \partial_\mu A_\nu^j - \partial_\nu A_\mu^j + \sqrt{2}\epsilon_{ijk} A_\mu^i A_\nu^k.
\]  

(76)

Choosing the constant \( k \) in (73) according:

\[
k = \frac{1}{4g^2},
\]  

(77)

and substituting the isovector potential:

\[
\vec{A}_\mu \rightarrow g \vec{A}_\mu,
\]  

(78)

(we use arrows to single out isovectors) we have, in the inhomogeneous case, the effective vertex functional:

\[
\Gamma = \int d^4x \left[ \left( \partial_\mu \varphi^\dagger - i (\varphi + v)^\dagger g\vec{A}_\mu \cdot \vec{\sigma} \right) \left( \partial^\mu \varphi + ig \vec{A}_\mu \cdot \vec{\sigma} (\varphi + v) \right) - \frac{\lambda}{4!} \left[ (\varphi^\dagger, \varphi + v) - (v^\dagger, v) \right]^2 - \frac{1}{4} \left( \partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu \right) \cdot \left( \partial^\mu \vec{A}^\nu - \partial^\nu \vec{A}^\mu \right) - g\sqrt{2} \left( \partial_\mu \vec{A}_\nu \cdot \vec{A}_\mu \wedge \vec{A}_\nu \right) - \frac{g^2}{2} \left( \vec{A}_\mu^2 \vec{A}_\nu^2 - (\vec{A}_\mu \cdot \vec{A}_\nu)^2 \right) \right].
\]  

(79)

It is interesting to compare this functional with the effective vertex functional in the case \( v = 0 \). Indeed in this situation one has:

\[
\Gamma = \int d^4x \left[ \left( \partial_\mu \varphi^\dagger - i\varphi^\dagger g\vec{A}_\mu \cdot \vec{\sigma} \right) \left( \partial^\mu \varphi + ig \vec{A}_\mu \cdot \vec{\sigma} \right) - \frac{m^2}{2} (\varphi^\dagger \varphi)^2 - \frac{\lambda}{4!} (\varphi^\dagger \varphi)^4 - \frac{1}{4} \left( \partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu \right) \cdot \left( \partial^\mu \vec{A}^\nu - \partial^\nu \vec{A}^\mu \right) - g\sqrt{2} \left( \partial_\mu \vec{A}_\nu \cdot \vec{A}_\mu \wedge \vec{A}_\nu \right) - \frac{g^2}{2} \left( \vec{A}_\mu^2 \vec{A}_\nu^2 - (\vec{A}_\mu \cdot \vec{A}_\nu)^2 \right) \right].
\]  

(80)

Notice that the second term in the right-hand side of (73) has been replaced by two independent term in (80) containing the new parameter \( m^2 \); as a matter of fact this new parameter compensates the loss
of $v$. Owing to gauge invariance that makes equivalent all the choices of $v$ with the same norm, $v$ is equivalent to a single parameter.

To study the particle content of these theories we select the two-point vertex generators. In the inhomogeneous case we choose a particular $v$ according:

$$v = \begin{pmatrix} 0 \\ F/\sqrt{2} \end{pmatrix}, \quad (81)$$

with $F$ real and positive. Introducing the symbol $\vec{\varphi}$ for $(\varphi_1, \varphi_2, \varphi_3)$ we get:

$$\Gamma_2 = \int d^4x \left[ \frac{1}{2} \left( \vec{A}_\mu (g^{\mu\nu} \partial^2 - \partial\mu \partial\nu) \cdot \vec{A}_\nu \right) + \frac{F}{2} g \left( \vec{A}_\mu \partial\mu \cdot \vec{\varphi} \right) + \frac{(\partial \varphi)^2}{2} + \frac{F^2 g^2 A^2}{8} + \frac{(\partial \varphi_4)^2}{2} - \lambda F^2 \varphi_4^2 \right]. \quad (82)$$

that can be simplified by the substitution:

$$\vec{A} + \frac{2}{gF} \partial \vec{\varphi} \to \vec{A}, \quad (83)$$

leading to:

$$\Gamma_2 = \int d^4x \left[ -\frac{1}{2} \left( \vec{A}_\mu (g^{\mu\nu} \partial^2 - \partial\mu \partial\nu) \cdot \vec{A}_\nu \right) + \frac{F^2 g^2 A^2}{8} + \frac{(\partial \varphi_4)^2}{2} - \lambda F^2 \varphi_4^2 \right]. \quad (84)$$

In the homogeneous case we have directly:

$$\Gamma_2 = \int d^4x \left[ -\frac{1}{2} \left( \vec{A}_\mu (g^{\mu\nu} \partial^2 - \partial\mu \partial\nu) \cdot \vec{A}_\nu \right) + (\partial \varphi)^2 + (\partial \varphi_4)^2 - m^2 (\varphi^2 + \varphi_4^2) \right]. \quad (85)$$

To both functionals we add the auxiliary term introducing for simplicity the Landau gauge choice: $\xi \to \infty$ that corresponds to the constraint:

$$\partial^\mu \vec{A}_\mu = 0. \quad (86)$$

It is clear that (84) describes a free isovector-vector field with mass $m = \sqrt{2} \lambda F$ and a neutral scalar field with mass $M = \frac{gF}{2}$. Indeed the
vector field is quantized by a decomposition strictly analogous to that in (17), with the choice (19) of the polarization vectors.

In the homogeneous case we introduce the decomposition:

$$\vec{A}(x) = \frac{1}{(2\pi)^{3/2}} \sum_{\lambda} \int \frac{d^3p}{\sqrt{2p}} \left( \epsilon_{\lambda,\vec{p}}^\mu \vec{A}_{\lambda} e^{-ipx} + \text{hermitian conjugate} \right),$$

where $\epsilon_{\lambda,\vec{p}}^\mu$ give a four-vector basis orthogonal to $p_\mu$, that is:

$$\begin{align*}
\epsilon_{3,\vec{p}}^\mu &= \frac{p_\mu}{p_0} \\
\epsilon_{i,\vec{p}}^\mu (i=1,2) &= \begin{cases} (\epsilon_i, \epsilon_j) = -\delta_{ij} \\ (\epsilon_i, p) = (\epsilon_i, \vec{p}) = 0 \end{cases} \tag{88}
\end{align*}$$

where $\vec{p}$ is the image of $p$ after a parity reflection. The two polarization vectors $\epsilon_i$ for $i=1,2$ correspond to isovector states with helicity $\lambda = \pm 1$. On the contrary $\epsilon_3$ does not correspond to any dynamical degree of freedom, indeed, owing to gauge invariance, the corresponding term of the vector field does not contribute to the action. Thus the third polarisation corresponds to the freedom of redefining the vector field by a gauge transformation. Notice that in the massless case gauge invariance is needed to guarantee the decoupling of the states with this polarization. Furthermore the functional (85) describes four real scalar fields with mass $m$.

Therefore, comparing the helicity states of the two theories, we find two isovector states with non-vanishing helicities in both cases. They are however massless in the homogeneous case and massive in the inhomogeneous one. There are furthermore four massive, helicity zero, states in both theories.

Notice that, from the point of view of symmetry, both theories have gauge invariant quantum actions, however in the inhomogeneous case the vacuum state, that corresponds to the vanishing field configuration, is not gauge invariant. In this situation one speaks of ”spontaneous symmetry breakdown”. In conclusion, the spontaneous breakdown of gauge invariance has made the vector field massive thus transferring an isotriplet of helicity zero states from the scalar to the vector fields. This is the ”Higgs mechanism".
It is quite clear from the above construction that, if the gauge symmetry remains unbroken, the vector fields are massless.

To conclude our analysis of an effective vector field, let us notice that the use we have made of an effective theory in the massless case is questionable; indeed as we have said above, the use of an effective approximation to describe the vertex functional is justified by analyticity arguments at vanishing momenta that do not hold in the massless case. Indeed in this case one has to take into account the singularities corresponding to the intermediate states of massless particles. In the confined theory it is believed that these "infrared" singularities wash out the vector asymptotic states.

4 The functional integral construction

The functional integral construction of field theories is based on the identification of suitable measures in the functional space of fields analogous to the heuristic form $(\mathcal{H})$. At first sight the choice of this measure should be completely determined on the basis of equation $(\mathcal{E})$ that implies that the classical action is identified with the classical approximation to the vertex functional; therefore the quantization criterion should be the gauge invariance of $S$. However this possibility is frustrated by the fact that if the measure is gauge invariant the corresponding Feynman integral does not exist. In perturbation theory this is due to the lack of uniqueness of the propagator. Considering as reference example a $SU(2)$ pure Yang-Mills theory, whose action corresponds to $(\mathcal{S}4)$ deprived of the scalars, we have the Lagrangian density:

$$\mathcal{L} = -\frac{1}{4} \overrightarrow{G}_{\mu\nu} \overrightarrow{G}^{\mu\nu},$$ (89)

and the propagator should be defined by the equation:

$$(g^{\mu\rho} \partial^2 - \partial^\mu \partial^\rho) \Delta_{\rho\nu}(x) = -\delta^\mu_\nu \delta(x),$$ (90)

it is however not unique, since for any function $F(x)$: $(g^{\mu\rho} \partial^2 - \partial^\mu \partial^\rho) \partial_\rho F(x) = 0$.

It is perhaps worth reminding here that the functional integral construction of $Z$ is based on the so-called Wick rotation. The theory is transformed to imaginary time; correspondingly the measure
$e^{iS}$ turns into $e^{-\bar{S}_e}$, where $S_e$ is bounded from below and it is Euclidean rather than Lorentz invariant. In this Euclidean formulation the Feynman integral can be defined if $S_e$ increases rapidly enough in all the directions of the field functional space. The identification of the measure should lead to a well defined functional generator of the Euclidean Green functions that turn out to be the analytic continuation of the physical (Minkowskian) ones. Therefore, in reality, when one speaks of the Feynman functional integral, one has in mind the generator of the Euclidean theory. From the perturbative point of view the whole procedure of Wick rotation combined with analytic continuation is in a sense trivial since the Feynman amplitudes are explicit analytic functions of the momenta.

Coming back to our theory, that we convert into its Euclidean version, gauge invariance makes the measure constant along the paths in the field functional space spanned by the transformations (62) that are called ”orbit” of a given configuration. The lack of convergence in the orbit directions makes the functional integral ill-defined. One could suggest here to provide the wanted convergence introducing into $S$ the $\xi$ term (25) corresponding in perturbation theory to a particular choice of the propagator, that is:

$$\Delta^{\mu\nu} = \frac{1}{(2\pi)^4} \int dk \frac{e^{ikx}}{k^2 + i0^+} (g^{\mu\nu} + (\xi^{-1} - 1) \frac{k^\mu k^\nu}{k^2}) . \quad (91)$$

As a matter of fact, this is only correct in the abelian case (QED). In the general case the introduction of a $\xi$ term is allowed into $\Gamma$ since, as we have shown above, the $S$ matrix remains $\xi$-independent. However this argument cannot be automatically extended to the case in which one inserts the $\xi$ term into the action $S$. Indeed the proof of the $\xi$-independence depends crucially on the Legendre transformation connecting $Z_c$ to $\Gamma$. This Legendre transformation can be interpreted as a development in Feynman tree diagrams, that is in diagrams without loops. On the contrary $Z_c$ is related to $S$ by a complete development into Feynman diagrams among which those with loops violate the $\xi$-independence.

The correct version of the $\xi$ term has been discovered by Faddeev and Popov [10]. We are now going to describe their result; however, instead of the $\xi$ term, we shall discuss the introduction of the term

$$\int d^4x \bar{\lambda} \partial^\mu \tilde{A}_\mu , \quad (92)$$
that corresponds to it in the limit: $\xi \to \infty$. In (92) there appears the auxiliary isovector-scalar field $\lambda$ that is called the Lautrup-Nakanishi field. Considering the variation of this term under an infinitesimal gauge transformation we get:

$$\int d^4x \lambda \partial^2 \Lambda,$$  

(93)
in the abelian case, where we have forgotten the arrows, and:

$$\int d^4x \lambda \left[ \partial^2 \bar{\Lambda} - g \sqrt{2} \partial_\mu (\bar{A}^\mu \wedge \bar{\Lambda}) \right] \equiv \int d^4x \lambda \partial D \bar{\Lambda},$$  

(94)
in the non-abelian one.

Let us for a moment interpret the gauge parameter $\Lambda$ as a field component. We see from (93) that in the abelian case it is a free field conjugate to $\lambda$ and hence it is not involved into the dynamics. On the contrary, in the non-abelian case (94), $\Lambda$ is coupled to the gauge field; therefore the introduction of the new term (92) modifies the dynamics in a substantial way thus violating the decoupling theorem (the $\xi \to \infty$ limit of the $\xi$ independence).

Naively we can say that Faddeev-Popov have cured this sickness compensating the two fake degrees of freedom ($\lambda$ and $\bar{\Lambda}$) by the introduction of their compensating antiquantized images.

In the case of ordinary integrals one can understand this compensation in the following way: consider a (finite) system of Grassmann (anticommuting) variables $\zeta_i$ and $\bar{\zeta}_j$:

$$\begin{cases}
\zeta_i \bar{\zeta}_j + \bar{\zeta}_j \zeta_i = 0 \\
\zeta_i \zeta_j + \zeta_j \zeta_i = 0 \\
\bar{\zeta}_i \bar{\zeta}_j + \bar{\zeta}_j \bar{\zeta}_i = 0
\end{cases}$$

(95)

and notice that a generic function of then is linear in each variable, that is, concerning e.g. $\zeta_k$, a generic function can be written in the form:

$$F(\zeta, \bar{\zeta}) = A_k(\zeta, \bar{\zeta}) + \zeta_k B_k(\zeta, \bar{\zeta}).$$

(96)
We define the partial derivative:

\[ \partial_{\zeta_k} F(\zeta, \bar{\zeta}) \equiv B_k(\zeta, \bar{\zeta}). \quad (97) \]

\( \partial_{\bar{\zeta}_i} \) is defined analogously. It is clear that:

\[ \partial_{\zeta_i} \zeta_j + \zeta_j \partial_{\zeta_i} = \delta_{ij}, \]
\[ \partial_{\zeta_i} \bar{\zeta}_j + \bar{\zeta}_j \partial_{\zeta_i} = 0, \quad (98) \]

The set of these partial derivatives generates a new Grassmann algebra analogous and conjugate to (95). Owing to the fact that \( \partial_{\zeta_k} F \) is independent of \( \zeta_k \) it is possible, and perhaps natural, to define the Berezin integral:

\[ \int d\zeta_k F(\zeta, \bar{\zeta}) \equiv \sqrt{\pi - 1} \partial_{\zeta_k} F(\zeta, \bar{\zeta}) = \sqrt{\pi - 1} B_k(\zeta, \bar{\zeta}). \quad (99) \]

Now, given a generic matrix \( M \) we can compute the Berezin integral:

\[ \int d\zeta d\bar{\zeta} e^{-\bar{\zeta}_i M_{ij} \zeta_j} \equiv \int d\zeta_1 \ldots d\zeta_n d\bar{\zeta}_1 \ldots d\bar{\zeta}_n e^{\bar{\zeta}_1 M_{ij} \zeta_j} \]
\[ = \int d\zeta_1 \ldots d\zeta_n d\bar{\zeta}_1 \ldots d\bar{\zeta}_n \frac{(\bar{\zeta} M \zeta)^n}{n!} = \frac{\det M}{\pi^n}. \quad (100) \]

If \( M \) is positive (it is diagonalizable with positive eigenvalues) we have also:

\[ \int \prod_i dz_i d\bar{z}_i e^{-\bar{z}_i M_{ij} z_j} = \frac{\pi^n}{\det M}, \quad (101) \]

therefore, combining (100) and (101) we have:

\[ \int \prod_{i=1}^n dz_i d\bar{z}_j e^{-\bar{z}_i M_{im} z_m} \cdot \int \prod_{j=1}^n d\zeta_j d\bar{\zeta}_j e^{-\bar{\zeta}_k M_{kn} \zeta_n} = 1. \quad (102) \]

This equation is naturally extended to the functional integrals: considering in particular the bosonic field (ordinary functional variables) \( \bar{\lambda} \) and \( \Lambda \) and the corresponding fermionic images (Grassmannian functional variables) \( \bar{\omega} \) and \( \omega \) one has:

\[ \int \prod_x d\bar{\lambda}(x) d\bar{\Lambda}(x) d\bar{\omega}(x) d\omega(x) e^{\int d^4 [\lambda \partial_\mu D^\mu \Lambda + i \bar{\omega} \partial_\mu D^\mu \omega]} = 1. \quad (103) \]

We have assumed that the operator \( \partial D \) be negative definite. As a matter of fact Gribov [7] has shown that this is not true for all choices of
vector fields; it is certainly true for small fields and hence our formula is correct at least in perturbation theory.

The Faddeev-Popov trick consists in the choice of the action:

$$S_e = \int d^4x \left[ \frac{1}{4} \vec{G}_{\mu\nu} \cdot \vec{G}^{\mu\nu} - i \vec{\lambda} \cdot \partial_\mu \vec{A}^\mu - \vec{\varpi} \cdot \partial_\mu D^\mu \vec{\varpi} \right], \quad (104)$$

and hence in that of the Green functional:

$$Z = \int d\mu C e^{i \int dx \left[ i \vec{\lambda} \cdot \partial_\mu \vec{A}^\mu - i \vec{\varpi} \cdot \partial_\mu D^\mu \vec{\varpi} \right]} e^{i \int dx \left[ \vec{J} \cdot \vec{A} + \vec{J} \cdot \vec{\lambda} + \vec{\eta} \cdot \vec{\varpi} + \vec{\bar{\eta}} \cdot \vec{\bar{\varpi}} \right]}, \quad (105)$$

where we have defined the functional measure:

$$d\mu C \equiv N \prod_x d\vec{\lambda}(x)d\vec{A}(x)d\vec{\varpi}(x)d\vec{\bar{\varpi}}(x)e^{\int dx \left[ -\frac{1}{4} \vec{G}_{\mu\nu} \cdot \vec{G}^{\mu\nu} \right]} . \quad (106)$$

According to (103) the introduction of two antiquantized isovector fields $\vec{\varpi}$ and $\vec{\bar{\varpi}}$ exactly compensates in the functional integral the contribution of the gauge degrees of freedom of the vector fields and of the Nakanishi-Lautrup field. In a more conventional language one says that the contributions to the transition probabilities of the states of pairs $\vec{\varpi}$-$\vec{\bar{\varpi}}$, that have negative norm, exactly compensate those of pairs of fake particles corresponding to $\vec{\lambda}$ and $\vec{\Lambda}$.

From a mathematical point of view this consists in the insertion into the functional measure of an invariant $\delta$ function implementing the gauge condition $\partial \vec{A} = 0$ [11].

5 Symmetry properties of the Green functional

In order the Faddeev-Popov trick to work correctly it is crucial that the third term in the action (104) correspond exactly to the gauge variation of the second one. To control this correspondence is a difficult task since the action has to be modified by the regularization and renormalization procedure that we do not discuss in these lectures. It is therefore essential to have a renormalization criterion that guarantees the persistence of the above correspondence. This criterion is identified with the invariance condition of the action under a
non-linear system of transformations (BRST) that, for the above action, are written:

\[
\begin{align*}
\vec{A}_\mu &\to \vec{A}_\mu - \varepsilon D_\mu \vec{\omega} \equiv \vec{A}_\mu - \varepsilon s \vec{A}_\mu \\
\vec{\omega} &\to \vec{\omega} - i\varepsilon \vec{\lambda} \equiv \vec{\omega} - \varepsilon s \vec{\omega} \\
\vec{\omega} &\to \vec{\omega} - g \frac{\varepsilon}{\sqrt{2}} \vec{\omega} \wedge \vec{\omega} \equiv \vec{\omega} - \varepsilon s \vec{\omega} ,
\end{align*}
\]

(107)

where \( \varepsilon \) is a Grassmannian parameter. It is easy to verify that the \( s \) operator defined in (107) together with the further condition: \( s\vec{\lambda} = 0 \) is nilpotent:

\[
s^2 \equiv 0 .
\]

(108)

Indeed, for example:

\[
s\vec{\omega} \propto s(\vec{\omega} \wedge \vec{\omega}) \propto (\vec{\omega} \wedge \vec{\omega}) \wedge \vec{\omega} = 0 ,
\]

(109)

The invariance of the action is verified considering that the \( s \) operator acts on the vector field in much the same way as an infinitesimal gauge transformation and hence:

\[
s\vec{G}_{\mu\nu} \cdot \vec{G}^{\mu\nu} = 0 .
\]

(110)

Furthermore we have:

\[
s\vec{\lambda} \cdot \partial_\mu \vec{A}_\mu = \varepsilon \vec{\lambda} \cdot \partial_\mu D^\mu \vec{\omega} ,
\]

(111)

and on account of the nilpotency of \( s \):

\[
s\vec{\omega} \cdot \partial_\mu D^\mu \vec{\omega} = -i\varepsilon \vec{\lambda} \cdot \partial_\mu D^\mu \vec{\omega} .
\]

(112)

One has to keep in mind that nilpotency is an essential condition to extend BRST symmetry at the renormalized level.

In terms of the \( s \) operator the Faddeev-Popov insertion (the second and third terms in (104)) is written \( s \int d^4x \left( \vec{\omega} \cdot \partial \vec{A} \right) \). The invariance of this expression is an obvious consequence of the nipotency of \( s \). The Faddeev-Popov insertion can be generalized reintroducing the \( \xi \) parameter according:

\[
s \int d^4x \vec{\omega} \cdot \left( \partial \vec{A} + \frac{i}{2\xi} \vec{\lambda} \right) .
\]

(113)
Indeed the introduction of the new term corresponds to the substitution:

\[ S_e \rightarrow S_e + \frac{1}{2\xi} \int d^4x \, \lambda^2 . \]  

(114)

Separating a quadratic term in \( \vec{\lambda} + i \xi \partial \vec{A} \), this action corresponds to the Lagrangian density:

\[ \mathcal{L} = \frac{1}{4} G_{\mu\nu} G^{\mu\nu} + \frac{\zeta}{2} \partial \vec{\lambda} \cdot \partial \vec{\lambda} - \partial_\mu \vec{\omega} D^\mu \vec{\omega} . \]  

(115)

Let us assume that the functional measure \( \mathcal{M} \) be \( s \)-invariant; this, of course, depends on the regularization-renormalization procedure. Given any integrable functional \( \Xi \) one has the Slavnov-Taylor identity [13]:

\[ \int d\mu e^{-s \int d^4x \, \vec{\omega} \cdot (\partial \vec{A} + \frac{i}{2} \vec{\lambda})} s\Xi = 0 , \]  

(116)

indeed:

\[ \int d\mu e^{-s \int d^4x \, \vec{\omega} \cdot (\partial \vec{A} + \frac{i}{2} \vec{\lambda})} \Xi = \int d\mu s \left[ e^{-s \int d^4x \, \vec{\omega} \cdot (\partial \vec{A} + \frac{i}{2} \vec{\lambda})} \Xi \right] = 0 , \]  

(117)

the last identity following from the invariance of the measure.

The meaning of this identity is the vanishing of all the correlators between \( s \)-invariant operators and elements of the image of \( s \), that appear above as \( s\Xi \). Indeed, if \( s\Sigma = 0 \), one has:

\[ \int d\mu e^{-s \int d^4x \, \vec{\omega} \cdot (\partial \vec{A} + \frac{i}{2} \vec{\lambda})} s\Sigma = \int d\mu e^{-s \int d^4x \, \vec{\omega} \cdot (\partial \vec{A} + \frac{i}{2} \vec{\lambda})} \Xi = 0 . \]  

(118)

Now it is not difficult to understand how BRST symmetry is related to \( \xi \)-independence. The observables of gauge theories correspond to gauge and hence \( s \) invariant operators; that is: if the operator \( \Omega \) is observable and hence physically meaningful, then: \( s\Omega = 0 \). As a matter of fact \( s \)-invariance, modulo the image of \( s \), gives a generalized definition of observables in gauge theories.

Using (116) we can prove the \( \xi \)-independence of the vacuum expectation value of \( \Omega \). Indeed computing the \( \xi \)-derivative:

\[ \partial_\xi \int d\mu e^{-s \int d^4x \, \vec{\omega} \cdot (\partial \vec{A} + \frac{i}{2} \vec{\lambda})} \Omega \]

\[ = \frac{i}{2\xi^2} \int d\mu e^{-s \int d^4x \, \vec{\omega} \cdot (\partial \vec{A} + \frac{i}{2} \vec{\lambda})} \Omega s \int d^4x \vec{\omega} \cdot \vec{\lambda} \]

\[ = \frac{i}{2\xi^2} \int d\mu e^{-s \int d^4x \, \vec{\omega} \cdot (\partial \vec{A} + \frac{i}{2} \vec{\lambda})} s \left( \Omega \int d^4x \vec{\omega} \cdot \vec{\lambda} \right) = 0 . \]  

(119)
The last identity following from the Slavnov-Taylor identity.

This result generalizes that of the $\xi$-independence of the $S$ matrix since, modulo infrared effects, the sources of the asymptotic field of the vector and matter particles, that we have written in the form: $K\phi_{in}$, are observable operators. Using the Slavnov-Taylor identity it is also possible to give a direct proof of the $S$-matrix unitarity [11].

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