Probabilistic Description of Traffic Breakdowns

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We analyze the characteristic features of traffic breakdowns. To describe this phenomenon we apply to the probabilistic model regarding the jam emergence as the formation of a large car cluster on highway. In these terms the breakdown occurs through the formation of a certain critical nucleus in the metastable vehicle ow, which enables us to con ne us to the cluster model. We assume that, rst, the growth of the car cluster is governed by attachment of cars to the cluster whose rate is mainly determined by the mean headway distance between the cars in the vehicle ow and, may be, also by the headway distance in the cluster. Second, the cluster dissolution is determined by the car escape from the cluster whose rate depends on the cluster size directly. The latter is justified using the available experimental data for the correlation properties of the synchronized mode. We write the appropriate master equation converted then into the Fokker-Planck equation for the cluster distribution function and analyze the formation of the critical car cluster due to the climb over a certain potential barrier. The further cluster growth irreversibly gives rise to the jam formation. Numerical estimates of the obtained characteristics and the experimental data of the traffic breakdowns are compared. In particular, we draw a conclusion that the characteristic time scale of the breakdown phenomenon should be about one minute and explain the case why the traffic volume interval inside which traffic breakdown is observed is sufficiently wide.

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I. INTRODUCTION: TRAFFIC BREAKDOWN AS THE NUCLEATION PHENOMENON

The spontaneous formation of traffic jams on highways has attracted attention for the last years because of two reasons. The former is the importance of this problem for traffic engineering especially concerning the feasibility of attaining the limited capacity of traffic networks and quantifying it. The latter is that the jam formation nicely exemplifies the existence of various phase states and transitions between them in statistical systems comprising elements with motivated behavior, which is a novel branch of physics. According to the latter notion proposed by Kerner (see, e.g., Ref. [1]) based on the experimental data [2,3,13] the jam formation is of a complex nature. In particular, it proceeds mainly through the sequence of two phase transitions: free ow ow (F) ! synchronized mode (S) ! stop-and-go pattern (J) . Both of these transitions are of the first order, i.e., they exhibit breakdown, hysteresis, and nucleation effects [1]. The F ! J transition can occur directly if only the synchronized mode is suppressed by a road heterogeneity [3]. The recent analysis of single-vehicle data by Neubert et al. [14], in particular, confirmed these features and also discovered fundamental microscopic properties distinguishing the synchronized mode from the other traffic states.

Theoretical description of the jam formation is far from being developed well because the synchronized mode possesses extremely complex structure [1]. For example, it comprises a certain continuum of quasi-stable states, so, matches a whole two-dimensional region on the phase plane \[ vehicle density ( traffic volume ) \] in contrast to the free ow state. However, tackling the question of how to regulate traffic ow on highways, for example, by controlling the speed limitation in order to prevent the jam formation may rough out the problem. Indeed, for this purpose it is sufficient to analyze the conditions giving rise to jams rather than the jam evolution itself. Such a standpoint is justified, partly, by the aforementioned phase transitions being of the first order. The free ow state, presumably, should have the feasibility to exist at the given car density or inside its certain neighborhood. This might be the necessary requirement for jam formation at a prescribed vehicle density, or for appearance of the second jam phase and the synchronized mode at a prescribed traffic volume. Second, the jam formation proceeds via the nucleation mechanism but not in a regular manner. Therefore, the key point in the emergence of a jam is the random occurrence of its critical nucleus inside the...
At the first glance the problem seems hopeless until the model of the synchronized mode is developed. Nevertheless, there are some circumstances enabling us to make a step towards this problem right now (see also Ref. [13]). The matter is that the F! S transition is of another nature than the S! J transition. The former is due to a sharp decrease in the overtaking frequency, giving rise to the synchronized mode, whereas the latter is caused by the pinch effect (see, e.g., Ref. [1, 2, 3]). Thereby the main case of the F! S transition is not trac tions in the vehicle density and velocity but in other characteristic parameters of the tra c ow (cf. also Ref. [4]). By contrast, just these trac tions give rise to the jam in the synchronized tra c ow. As a result, the threshold of the F! S transition turns out to be remarkably lower than that of the jam ation and attained at lower values of the vehicle density. So, the generation rate of critical nuclear for the former transition has to be large in comparison with the latter one. Thus, on time scales characterizing the occurrence of the jam critical nucleus, the tra c ow is in an ergodic state in the synchronized tra c ow. As a result, the threshold of the F! S transition turns out to be remarkably lower than that of the jam ation and attained at lower values of the vehicle density. So, the generation rate of critical nuclear for the former transition has to be large in comparison with the latter one. Thus, on time scales characterizing the occurrence of the jam critical nucleus, the tra c state with respect to the transitions between the free ow and synchronized mode is quasistationary. Therefore the formation of a jam critical nucleus is the leading nonequilibrium process limiting the tra c breakdown. The latter feature allows us to consider the problem solely to the jam nucleus generation and to regard the synchronized mode and the free ow phase (if they coexist in the case under consideration) as one tra c state. Moreover, since a jam actually inside the synchronized mode where the vehicle motion at di erent lanes is strongly correlated we may apply to a single-lane road approximation that treats all the cars moving at di erent lanes as a multilane highway and be ng neighboring across the highway as a single e ective macrovehicle consisting of m any cars. The macrovehicle concept is partly justi ed by the empirically observed fact that trac tions in the downstream ow leaving a tra c ow has to include m any vehicles. This feature is also pointed to by the observed breakdown near a jam p-on occurring each time a large vehicle cluster entered the tra c ow. 

The jam ation manifests itself in the tra c breakdown, i.e., in a sharp drop of the tra c ow to a substantially lower value. Detecting these events one can get the rate of the critical nucleus generation depending on the road conditions and the tra c ow state. In this way the main focus is shifted to experimental and theoretical analysis of the probabilistic features of jam ation regarding the characteristic means values of the tra c ow as phenom enological param eters [5, 9, 18]. Such a probabilistic description of the tra c breakdown is the main purpose of the present paper.

Processes similar to the tra c breakdown are widely met in physical systems. For example, water condensation in supersaturated vapor proceeds via formation of small atom clusters of critical size. Keeping in mind this analogy between the tra c breakdown and the phase transitions in physical systems, Ahnke et al. [5, 6] proposed a kinetic approach based on the stochastic master equation describing the jam ation in terms of the attachment of individuals to their cluster. However, the particular form of the developed master equation does not allow for the jam ation being the rst order phase transition and, thus, the tra c breakdown. In the present paper we generalize this kinetic approach to describe the latter phenomena.

It should be pointed out the real structure of congested tra c near a highway bottleneck is su ciently complex, it contains the region of synchronized mode located in the close proximity of the bottleneck, the preceding upstream region of moving narrow jams transformed into wider jams [6]. However it is quite reasonable to consider this structure as being induced by the tra c breakdown processes arising inside the "head" of this complex jam, in the region of synchronized mode adjacent the bottleneck. Therefore the main characteristics of the breakdown phenomenon may be related to intrinsic processes taking place inside the synchronized mode on not large spatial scales. The latter justi es our attempt to describe tra c breakdown ignoring the complex spatial structure of the metastable tra c state inside which critical jam nuclei originate.

Processes similar to the tra c breakdown are widely met in physical systems. For example, water condensation in supersaturated vapor proceeds via formation of small atom clusters of critical size. Keeping in mind this analogy between the tra c breakdown and the phase transitions in physical systems, Ahnke et al. [5, 6] proposed a kinetic approach based on the stochastic master equation describing the jam ation in terms of the attachment of individual cars to their cluster. However, the particular form of the developed master equation does not allow for the jam ation being the rst order phase transition and, thus, the tra c breakdown. In the present paper we generalize this kinetic approach to describe the latter phenomena.

However, before passing directly to the model statement we recall the experimental data enabling us to estimate the characteristic size \( n_0 \) of vehicle clusters that are small enough so the behavior of drivers inside them seem to be special. From our point of view the multilane correlations in the vehicle motion are due to the drivers taking into account the behavior of all the cars, including also the cars at the neighboring lanes, that are inside the region accessible to observation. Therefore the synchronized mode has to exhibit strong correlations in

![Image](544x690 to 504x715)
The free phase is also specified by the correspondence speed function $P(n; t)$ for the cluster to be of size $n_0$ at time $t$. Then following M. Ahn et al. [3] we write the following master equation governing the cluster growth

$$
\theta_{0} P(n; t) = w_{+}(n) P(n+1; t) + w(n) P(n+1; t),
$$

where the cluster size $n$ meets the inequality $1 
\le n \le N$ and $w_{+}(n)$ and $w(n)$ are the transition rates illustrated in Fig. 2 and depending generally on the cluster size $n$. The formation and dissolution of the maximum possible cluster containing all the cars is described by the equation

$$
\theta_{c} P(n; t) = w_{+}(N) P(n+1; t) - w(n) P(n; t),
$$

whereas the emergence of the jam seed, the cluster consisting of one car called below precluster, obeys the equation

$$
\theta_{p} P(n; t) = w_{+}(0) P(1; t) - w(n) P(0; t).
$$

Here the function $P(0; t)$ is the probability of no cluster on the road. At the initial time $t = 0$ no cluster is assumed to be on the road:

$$
P(n; 0) = \delta_{n0}.
$$

where $\delta_{n0}$ is Kronecker's symbol. The system of equations (3) subject to the initial condition (4) makes up the probabilistic description of the cluster formation.

Special attention should be paid to the question as to what the precluster is. The model proposes the following. When there is no cluster on the road, i.e. all the cars move freely the velocity of one of them can randomly drop down to the velocity $v_{\text{clust}}$ in the cluster. Such a car is regarded as the precluster, a size one cluster. When a precluster has arisen its further evolution follows the scheme shown in Fig. 2. The precluster concept may be justified by recalling the problem we deal with initially, i.e. the breakdown processes in multilane traffic. The cars under consideration actually match all vehicle clusters of the synchronized model, i.e. a car moving in a lane has no communication with the cars ahead and behind it. Therefore the precluster is actually as a certain small cluster of the synchronized model. Knowing in mind the relatively low threshold of the $F$! $S$ transition we will assume the precluster generation as well as the precluster dissipation to be intensive processes so that the free flow phase, $n = 0$, and the precluster state, $n = 1$, come into quasi-equilibrium on time scales needed for the critical cluster nucleus to arise. In particular, in no case the precluster emergence lim its the cluster evolution, so, the particular details of the precluster formation have no substantial effect on the traffic breakdown.

At the next step we should specify the transition rates $w_{+}(n)$ and $w(n)$. Let us apply to the optimality

\begin{align*}
\begin{array}{c}
n + 1 \\
\mid \mid \mid \mid \\
\mid \mid \mid \mid \\
n \\
\mid \mid \mid \mid \\
n - 1 \\
\end{array}
\end{align*}

\begin{align*}
\begin{array}{c}
p + 1 \\
\mid \mid \mid \mid \\
\mid \mid \mid \mid \\
p \\
\mid \mid \mid \mid \\
p - 1 \\
\end{array}
\end{align*}

FIG. 2: Schematic illustration of the cluster transformations.
model assuming the velocity \( v \) of the \( \text{free} \) moving cars as well as the clustered cars to be determined directly by the corresponding headway distance \( h \) according to the formula

\[
v = \#(t) = v_{\text{max}} \frac{h^p}{h^p + D_{\text{opt}}};
\]

where the value \( D_{\text{opt}} \) is the headway distance at which drivers feel them selves \( \text{free} \) and their velocity attains the maximum \( v_{\text{max}} \). The parameter \( p > 1 \) allows for possible forms of the function \( \#(n) \), the greater the value of \( p \), the sharper the dependence \( \#(n) \). Case \( p = 2 \) is often used [13, 14]. A car attaches itself to the cluster as fast as the distance to the last car in the cluster decreases down to the cluster headway \( h_{\text{clust}} \), enabling us to write the following ansatz for the attachent rate to the cluster of size \( n \)

\[
w_{+}(n) = w_{+,0}^\text{ov}(n) = \frac{\#h_{\text{free}}(n)}{\#h_{\text{free}}(n)} \frac{h_{\text{free}}(n)}{h_{\text{clust}}};
\]

Applying to a simple geometrical consideration and assuming \( N = 1 \) as well as \( n \geq 1 \) we get the relationship (illustrated also in Fig. 3)

\[
h_{\text{free}}(n) = h_{\text{clust}} + (h_{\text{car}} + h_{\text{clust}}) \left( \frac{1}{n} \right);
\]

which together with (6) gives the attachment rate as a function of the cluster size \( n \). Here we have introduced the following terms on parameters: \( N = \text{L} \) being the maximum value of the cluster density on the road, its maximum possible value \( 1 = (h_{\text{car}} + h_{\text{clust}}) \) for the given road, and the relative volume \( n = N \) of the cluster with respect to the initial volume of the \( \text{free} \) state.

In order to compare the cluster growth due to the car attachment with the precluster generation we specify its rate in terms of

\[
w_{+}(0) = w_{+,0}^\text{ov}(0);
\]

where \( w_{+}(0) \) is a phenomenological factor and \( w_{+,0}^\text{ov}(0) \) is the rate of the precluster generation for which we assume the factor to be about unit \( 1 \), or at least not to be small enough to limit the cluster formation, so its particular numerical value is of no importance.

The rate of the cars escaping from the cluster at its downstream front is written as (see also Fig. 4)

\[
w_{-}(n) = \frac{1}{\#(n)} = \frac{1}{1} + \frac{(n)}{0};
\]

where the value \( (n) \) can be interpreted as the characteristic time needed for the first car in the cluster to leave it and to go out from its downstream boundary at a distance about the headway distance \( h_{\text{free}}(n) \) in the current \( \text{free} \) ow state. The function \( (n) \) allows for the dependence of the escape rate \( w_{-}(n) \) on the cluster size \( n \). We note that expression \( (n) \) is the main original part of the model under consideration.

When the cluster size is sufficiently large, \( n \gg 1 \), it is reasonable to regard the characteristic time \( (n) \) as a constant (i.e. \( (n) = 0 \) for \( n = 1 \)) as was done in papers [13, 15] for all the values of \( n \).

For small clusters the \( (n) \)-dependence, however, requires a special attention. The matter is that the car attachment rate \( w_{+}(n) \) is considered to be directly determined by the local characteristics of the \( \text{free} \) ow phase and the car cluster. Thus the dependence of the attachment rate \( w_{+}(n) \) on the cluster size \( n \) arises via the headway distance \( h_{\text{free}}(n) \) in the \( \text{free} \) ow being a function of \( n \), i.e. \( w_{+}(n) = w_{+} h_{\text{free}}(n); h_{\text{clust}} \). Therefore the attachment rate is actually an explicit function \( w_{+}(n) \) of the mean car density \( \text{and the cluster relative volume only} \) and, so, exhibits minor variations on scales \( n \).

As will be seen below, exactly this feature is essential rather than the particular form of \( w_{+}(n) \) given here for the niteness sake only, because to describe tra c breakdown at least one of the kinetic coefficients \( w_{+}(n) \) and \( w_{-}(n) \) has to be a direct function of the cluster size \( n \) for its relatively small values corresponding to the formation of the cluster critical nucleus. We associate this dependence with the escaping rate \( w_{-}(n) \) that, in contrast to the attachment rate \( w_{+}(n) \), exhibits substantial variations in the region \( n < n_0 \) ~ 20.

The parameter \( n_0 \) actually divides the car clusters into the large cluster group, \( n > n_0 \), for which the escaping rate is constant, \( (n) = 0 \), and the group of small clus-
ters, \( n < n_0 \), whose dissolution is acted substantially by the size \( n \). This assumption is based on the fact that there should be a variety of possible manoeuvres for a driver to escape from a sufficiently small cluster on a multilane highway when the lanes are not too crowded.

Expression (13) takes into account this effect via the function \( (n) \) running from 1 to 0 as the cluster size \( n \) increases, so, \( (1) = 1 \) and \( (n) = 0 \) as \( n = 1 \). In particular, for a small neighborhood of the precursor size, \( n = 1 \), the value \( (1) \) gives us actually the lineage of the precursors and is assumed to be less than the escaping time from a large cluster, i.e., \( 0 < 1 \). Naturally, for the case of no cluster on the road we have to set \( w(0) = 0 \). The main results will be obtained below actually applying to the general properties of the dependence \( w(n) \), however for simplicity sake, we will adopt the following ansatz for \( n = 1 \)

\[
(1) = [x]_{q} = \frac{w}{x_n} = \frac{1}{(1 + x)^q}; \tag{10}
\]

where the exponent \( q > 1 \) is regarded as a given constant.

We point out once more that the dependence of the characteristic time \( (n) \) on the cluster size is crucial because it is responsible for the existence of the metastable \"free\" ow phase.

The system of equations (13) subject to the initial condition (15) with the relationships (1), (2), and (3) form a proposed probabilistic model for the car aggregation. Within this model we will analyze the characteristic features of the large cluster emergence and the form of the fundamental diagram, i.e., the \"flow volume (car density)\" relation in the vicinity of traffic breakdown. In particular, in the adopted term \( c_{\text{free}} \) for the given traffic state, i.e., when a cluster of size \( n \) arises on the road is written as (15)

\[
j(n) = (1) \# [c_{\text{free}}] + \# [c_{\text{clust}}]; \tag{11}
\]

Averaging expression (11) with respect to the distribution \( P(n; t) \) we will get the fundamental diagram \( j = j(t) \).

B. Equilibrium distribution

To clarify the characteristic features of the cluster formation let us analyze, rst, the stationary cluster sizes distribution \( \mathcal{P}_{eq}(n) \). The system of equations (13) subject to the initial condition (15) admits the stationary solution \( \mathcal{P}_{eq}(n) \) meeting the zero \"probability\" ux in the cluster size space:

\[
\mathcal{P}_{eq}(n) = 1 \quad \text{w} (n) \mathcal{P}_{eq}(n) = 0;
\]

hence we see that

\[
\frac{\mathcal{P}_{eq}(n)}{\mathcal{P}_{eq}(1)} = \frac{w(n)}{w(1)};
\]

enabling us to write the expression

\[
\mathcal{P}_{eq}(n) \exp \left( \frac{1}{2} \mathcal{P}_{eq}(n) \right) ;
\]

where the function \( (n) \) (called below the car growth potential) is specified for \( n = 2 \) by the formula

\[
(1)^{\mathcal{P}_{eq}(n)} = \frac{h_{0}^{n}}{1 + 1} w(n) \mathcal{P}_{eq}(n) \tag{12}
\]

Both of the terms in (12) vary weakly as the argument \( n \) changes by one, enabling us to convert sum (12) into an integral with respect to the cluster size \( n \) treated as a continuous variable:

\[
(1) = \int_{0}^{\infty} \mathcal{P}_{eq}(n) \exp \left( \frac{1}{2} \mathcal{P}_{eq}(n) \right) \tag{13}
\]

where

\[
\begin{align*}
Z_{0}(n) &= \int_{0}^{\infty} \mathcal{P}_{eq}(n) \exp \left( \frac{1}{2} \mathcal{P}_{eq}(n) \right) \tag{14} \\
0(n) &= \int_{0}^{\infty} \mathcal{P}_{eq}(n) \exp \left( \frac{1}{2} \mathcal{P}_{eq}(n) \right) \tag{15}
\end{align*}
\]

The former term in (12) or (13), i.e. the component \( 1(n) \) called below the growth potential, mainly characterizes whether a stable car cluster can arise on the road under the given conditions and specifies its size because it exhibits substantial variations on large scales exceeding substantially the size \( n_0 \). By contrast the latter one, the component \( 0(n) \), describes the formation of the critical cluster nucleus and, so, the breakdown phenom enon. Indeed, as follows from (10) and (13) the potential \( 0(n) \) is constant for \( n = n_0 \) and, thus, cannot a ect the growth of a large cluster already formed on the road. Besides, within the continuum approximation we have ignored the details of the cluster distribution in the region including both the points \( n = 0 \) and \( n = 1 \) and expand the cluster space \( n = 1 \) to the whole axis \( n = 0 \).

Let us, rst, analyze the condition of the cluster emergence. Applying to Fig. 1 we can see that a large cluster can arise on the road, in principle, if there exists a value of the headway distance \( h_c \) meeting the equality

\[
1 \mathcal{P}_{eq}(h_c) = 1; \tag{16}
\]

which will be assumed to hold beforehand. In particular, within approximation (2) together with (13) for \( D_{\text{opt}} \) \( h_{clust} \) and \( p = 2 \) this assumption holds if \( 1 \mathcal{P}_{eq} > 2D_{\text{opt}} \) and the critical headway reads

\[
h_{c}(2) = \frac{1}{2} \mathcal{P}_{eq} \mathcal{P}_{eq}(h_c) = \frac{1}{2} \mathcal{P}_{eq}(h_{c}) = \frac{1}{2} \mathcal{P}_{eq}(h_{c}) \tag{17}
\]

whereas for \( p = 1 \) and \( 1 \mathcal{P}_{eq} > D_{\text{opt}} \) we have

\[
h_{c}(1) = 1 \mathcal{P}_{eq} \mathcal{P}_{eq}(h_c) = 1 \mathcal{P}_{eq}(h_{c}) = 1 \mathcal{P}_{eq}(h_{c}) \tag{18}
\]
The \textit{free} ow phase will be stable if the initial headway distance $h_{\text{free}}(0) > h_c$ and unstable otherwise.

Let us justify these statements. The growth potential \((n)\) is actually the sum of $\ln \left( w(0) \ln w(n) \right)$ over $n$ (see formula (14)). So, in the region where the integrand of (14) is less than 1 and the potential \((n)\) is an increasing function of $n$, the cluster dissolution is more intensive than the car attachment. Under these conditions the cluster size on the average decreases in time. The same concerns the time dependence of the headway distance $h_{\text{free}}(n)$ in the \textit{free} ow phase because the value of $h_{\text{free}}(n)$ decreases as the cluster becomes smaller (Fig. 3), which is also illustrated by arrows in Fig. 8. Since \(W^{\text{ov}}[h] < 1\) for $h > h_c$ any randomly arising cluster tends to disappear and, consequently, the \textit{free} ow phase is stable when $h_{\text{free}}(0) > h_c$. In this case the potential \((n)\) possesses one minimum located at the boundary point $n = 0$ (or $n = 1$ what is the same in the continuum description).

Otherwise, $h_{\text{free}}(0) < h_c$, there is a region $h_{\text{free}}(0) < h < h_c$ where $\ln w^{\text{ov}}[h] > 1$ and the car attachment rate exceeds that of the cluster dissolution and a cluster occurring in the corresponding \textit{free} ow state tends to grow, inducing the further increase in the headway distance $h_{\text{free}}(n)$. In this case the \textit{free} ow phase is unstable and the cluster will continue to grow until the value of $h_{\text{free}}(n)$ reaches the critical point $h_c$, where the car attachment and the cluster dissolution balance each other. Hence it follows, in particular, that the developed cluster is of the size $n_{\text{clust}}$ obeying the equation

$$h_{\text{free}}(n_{\text{clust}}) = h_c$$  \hspace{1cm} (17)

and the \(n\) has a minimum at the internal point $n = n_{\text{clust}}$. In the present paper we shall ignore the existence of another region where the equality \(W^{\text{ov}}[h] < 1\) also holds for very dense traffic, which has been considered in papers [19, 20].

Relationship (7) enables us to rewrite the instability conditions in terms of the mean car density. The critical value $c$ of the car density is the solution of the equation $h_{\text{free}}(0) = h_c$, whence we immediately get

$$c = \frac{\kappa_{\text{car}} + h_{\text{clust}}}{\kappa_{\text{car}} + h_c}.$$  \hspace{1cm} (18)

Then the stable state of the \textit{free} ow phase corresponds to the inequality $c < c_l$ and it loses the stability when $c > c_l$. In the latter case a large cluster arises on the road whose size $n_{\text{clust}}(\cdot) = \text{clust}(\cdot)N$, and relative volume

$$\text{clust}(\cdot) = \frac{h_c + \kappa_{\text{car}}}{h_c + h_{\text{clust}}} c_l.$$  \hspace{1cm} (19)

In the given analysis we have ignored the dependence of the cluster dissolution rate $w(0)$ on the size $n$ and, thereby, the considered picture describes actually the \textit{free} ow (cluster transition of the second order. It does not allow for the metastable state of the \textit{free} ow phase and corresponds to the continuous transition from the traffic state of no cluster on the road to the formation of a certain cluster whose relative volume changes continuously from zero as the car density penetrates deeper into the instability region (see formula (19)). Consequently, this approximation cannot explain the traffic breakdown and on the phase diagram matches solely the stable branches \textit{S} and \textit{C} of the \textit{free} ow and the traffic with a developed cluster, respectively (Fig. 6).

Nevertheless, exactly the given approximation describes the stable branches of the fundamental diagram and, moreover, the metastable branch is a continuation of the branch \textit{F} into the instability region. Keeping the latter in mind we present also the expression specifying these branches:

$$\left\{ \begin{array}{ll}
\dot{c}_c (\cdot) = \# [h_c + \theta_c + \lambda_{\text{car}}(c_l) = \# [h_c + \theta_c] + \lambda_{\text{car}}(c_l) = ] & \text{if} < c_l; \\
\dot{c}_1 (\cdot) G (c_l) = c_l & \text{if} > c_l;
\end{array} \right.$$  \hspace{1cm} (20)

where the constants

$$\dot{c}_1 = \text{cl} \# [h_c];$$

$$G = \left( \frac{\theta_c + \lambda_{\text{car}}}{h_c + h_{\text{clust}}} \right) \lim_{\# [h_c] \to \# [h_{\text{clust}}]}.$$  \hspace{1cm} (21)
It should be noted that in obtaining this expression we have substituted the maximum probability value \( n_{\text{max}} \) of the cluster size into expression (1), instead of averaging it over the distribution \( P(\sigma) \). The latter is justifiably because the effect of the cluster size fluctuations is ignorable due to \( N \).

Now we analyze possible metastable states of the "free" ow phase. In order to do this we should take into account both the component of the growth potential \( \omega \). Since the function \( \omega(n) \) exhibits remarkable variations in the region \( n < n_{0} \) only and, thus, the size \( n_{c} \) of the critical nucleus also belongs to this region we may consider clusters whose size \( n \) is much less than the whole cluster size \( n_{\text{clust}} \) attained after the instability development. In addition for the sake of simplicity we will regard the value \( (\lambda_{1})_{0} = 1 \) as a small parameter, which enables us to examine solely a small neighborhood of the instability boundary, \( 0 < n_{c} \).

In this case the value of \( \omega(\omega(\text{n}_\text{free}(n))) \) is practically constant and can be approximated by the expression

\[
\ln \omega(\omega(\text{n}_\text{free}(n))) = \ln(\omega(\text{n}_\text{free}(n))) - \ln(n_{c}) = g \frac{c_{1}}{c_{1}}
\]

where the coe cient

\[
g = \left( \frac{\sigma_{\text{air}} + h_{c}}{h_{c}} \right) \frac{d \ln \omega(\omega(\text{n}_\text{free}(n)))}{d \ln h} \frac{d \ln(n_{c})}{d \ln h} = \left( \frac{\sigma_{\text{air}} + h_{c}}{h_{c}} \right) n_{0}^{q} \frac{c_{1}}{c_{1}}
\]

is about unity, \( g = 1 \), in the general case. In particular, for the stepwise dependence \( \sigma(n) \) (if we set \( p = 1 \) in expression (5)) and \( \sigma_{\text{opt}} = \sigma_{\text{air}} + h_{c} \) we have the rigorous equality \( g = 1 \). Expression (23) together with formula (10) allow us to represent the dependence of the growth potential \( \omega(n) \) on the cluster size \( n \) as

\[
\frac{d}{dn}(n) = \left( \frac{1}{0} \right) (n) g \frac{c_{1}}{c_{1}}
\]

The first term on the right-hand side of (21) is due to the increase in the cluster dissolution rate for \( n < n_{0} \), whereas the latter one is proportional to the cluster growth rate in the region of large values of \( n \). The resulting value of the derivative \( d(n) =dn \) characterizes the direction of the cluster evolution. If it is positive, \( d(n) =dn > 0 \), i.e. the potential \( \omega(n) \) is an increasing function of \( n \) the cluster dissolution is the dominant process and the size of the cluster tends to dissipate. Otherwise, i.e. when \( d(n) =dn < 0 \) the cluster size grows.

The function attains its maximum at \( n = n_{0} \), so, according to (21) the derivative \( d(n) =dn \) is negative for all the possible values of the cluster size under consideration \( 0 < n_{\text{clust}} \) when

\[
> c_{2} = \left( \frac{1}{0} \right) g + 1
\]

In this case the "free" ow phase becomes absolutely unstable. Under the opposite condition, \( c_{1} < < c_{2} \) there is a certain value \( n_{c} \) at which the derivative \( d(n) =dn \) changes the sign (Fig. 7). Setting the left-hand side of (23) equal to zero we get the relationship

\[
(n_{c}) = \left( \frac{1}{0} \right) n_{0}^{q} \frac{c_{1}}{c_{1}}
\]

which together with ansatz (13) gives the estimate

\[
n_{c} = n_{0}^{q} \frac{c_{2}}{c_{1}} \frac{1}{q} \frac{3}{5}
\]
random fluctuations, then it will grow and a large cluster of size \( n_{\text{clust}} \) arises on the road because \( d(n) = \frac{dn}{n} < 0 \) in the region \( n > n_c \).

In other words, we have shown that the dependence of the dissolution rate \( w(n) \) on the cluster size \( n < n_0 \) makes the \( \text{free"} \) ow phase metastable when the car density belongs to the interval \( 2 (c_1; c_2) \) (branch \( \text{in"} \)) in Fig. 3. The formation of a large cluster, \( n > n_0 \), proceeds via generation of the critical nucleus whose size \( n_c \) is estimated by expression (2). In order to nd the generation rate of the critical nucleus and, thus, the breakdown frequency we should consider the transient processes in the cluster growth, which is the subject of the next section.

C. Continuum approximation. The breakdown probability

In order to apply well-developed techniques of the escaping theory (see, e.g., [3]) to the analysis of the traffic breakdown probability we approximate the discrete master equation (1) by the corresponding Fokker-Planck equation. It is feasible because in the case under consideration the kinetic coefficients \( w(n), w(n), \) rst, vary smoothly on scales about unity and, second, are approximately equal to each other, \( w(n) = \bar{w}(n) \). The latter conditions enable us to treat the argument \( n \) as a continuous variable and to expand the functions \( w(n), w(n), P(n; t) \) into the Taylor's series. In this way and, in addition, taking into account expression (3) we reduce equation (1) to the following Fokker-Planck equation

\[
\begin{align*}
1 \partial_t P(n; t) &= \bar{w}(n) P(n; t) + P(n; t) \Delta \bar{w}(n),
\end{align*}
\]

where the potential \( \bar{w}(n) \) is given by formula (13) in the general form. However, in the case under consideration the ratios \( n/N, (c_1) = c_1 \) and \( (c_2) = 0 \) are regarded to be sufficiently small. It is possible to expand the potential \( \bar{w}(n) \) in the tree parameters and to retain the leading term only. In this way we get

\[
\begin{align*}
\bar{w}(n) &= \left( \frac{1}{n_0} \right) n_0 - \left( \frac{1}{n_0} \right) n_0, \\
\text{for} \quad n \ll n_0, \\
\text{and we have set} \quad \kappa_0 = 0.
\end{align*}
\]

Equation (3) transforms into the boundary condition at infinitely distant points that is imposed on the probability \( u_x \)

\[
J(n) = \partial_x P(n; t) + P(n; t) \partial_n (n) = 0.
\]

and requires it to be equal to zero, \( J(1) = 0 \). Equation (3) describing the precluster generation is reduced, in turn, to the zero boundary condition imposed on the probability \( u_x J(n) \) formally at \( n = 0 \), i.e. \( J(0) = 0 \). The latter is justified by the assumed quasi-equilibrium between the \( \text{free"} \) ow phase and the preclusters. And, finally, the initial condition (4) can be rewritten as

\[
Z_1 \partial_P P(n; t) = 1:
\]

where the function \( P(n; t) = \frac{Z n_{\text{clust}}}{n_{\text{clust}}} \) dx \( \frac{d[k]}{dx} \ ;
\]

In particular, for ansatz (10) with the exponent \( q = 2 \) expression (27) becomes

\[
c' \left( \frac{1}{n_0} \right) n_0 \frac{\bar{w}(n)}{2(1 + \epsilon)} ;
\]

mo\(\text{\text{ncever, in the limit } x_c \to 1 \text{ we have}
\]

\[
! [\epsilon] = \frac{1}{2} \frac{d[k]}{dx} ;
\]

as follows from expression (28), and the general formula for the potential \( \epsilon \) can be written as

\[
c' \left( \frac{1}{n_0} \right) n_0 x_c^2 ;
\]

In the same limit expression (29) gives us

\[
x_c' \left( \frac{c_2}{c_1} \right) \frac{\bar{w}(n)}{2r};
\]

The main much deeper minimum of the potential \( \bar{w}(n) \) is located at \( n = n_{\text{clust}} = n_0 \).

We have demonstrated that a precluster must climb over the potential barrier \( \epsilon \) at the point \( n_c \) to convert into a large stable cluster. It is implemented through random fluctuations carrying the cluster size up to the critical value \( n_c \). In this term s the tra c breakdown is the classical escaping from a potential well described by the Fokker-Planck equation (22). The latter analogy enables us to write down the estimate for the frequency \( \epsilon_0 \) of the tra c breakdown processes depending on the
A given vehicle density in the "free" ow state. Namely, as shown in Appendix

\[
\text{bd} = \frac{1}{2} n_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{3}{4} (1 - k_c) \frac{1}{2} J \frac{1}{2} n_0 ! k_c \right) ;
\]

which is well justified for the car density belonging to the interval \( c_1 < c < c_2 \) except for a certain sufciently small neighborhoods on the critical points \( c_1, c_2 \). Ansatz [10] with the exponent \( q = 2 \) together with formula [23] enables us to rewrite expression [22] as

\[
\text{bd} = \frac{1}{2} n_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \end{pmatrix} (1 - k_c) \frac{1}{2} J \frac{1}{2} n_0 ! k_c \right) ;
\]

Here we have introduced the quantity

\[
\tau = \frac{c_1}{c_2 - c_1} \tag{34}
\]

treated as a dimensionless overcriticality measure showing how deep the system penetrates into the metastability region, \( n_0 = 0 \) corresponds to the value \( c_1 \) of the vehicle density where a jam can emerge in principle and \( = 1 \) matches the vehicle density \( c_2 \) after exceeding which no tra c states except for jams exist at all (Fig. 1).

D. Frequency of tra c breakdown during a xed time interval

Experimentally tra c breakdown is typically analyzed detecting a signi cant drop in the vehicle speed during a certain xed time interval \( T_{\text{obs}} \) about several minutes and then drawing the relative frequency of these events vs the tra c volume \( n_0 \) [1, 0, 1, 1, 2]. In order to compare this representation with the obtained results let us consider them in more details.

As follows from expression [22] the density interval \( (c_2, c_1) \) inside which the tra c jam emerges by the nucleation mechanism is of the thickness

\[
(c_2 - c_1) = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ;
\]

According to the experimental data [3, 13, 11, 12, 3] the thickness of the tra c volume interval inside which the tra c breakdown occurs onetr of the probabilistic behavior is about its low boundary in magnitude. So we have to regard the ratio \( (1, 0) \) as also a value about unity:

\[
(1, 0) = 1 ;
\]

Thereby, setting \( n_0 = 20 \) we get the conclusion that in the general case where \( n_2, n_1 \) the potential barrier \( c_1, c_5 \) corresponding to the exponential factor \( \exp (c g) \) is \( 0.7 \times 10^2 \). Then setting \( r = 2 \times 10^2 \) sec and estimating the preceding cofactor as \( L \approx (2 + 20 n_0) \) we nd the characteristic rate of the tra c breakdown being about \( 1/50 \) min/s in the general case. So the real tra c breakdown events seem to be observed in cases where the vehicle density comes to the upper boundary \( c_2 \). The latter allows us to con ne our analysis formally to the limit case

\[
(1, 0) (1, 0) ;
\]

Then estimating the probability \( F_{\text{bd}} \) of detecting a tra c breakdown during the observation time interval \( T_{\text{obs}} \) as \( F_{\text{bd}} = T_{\text{obs}} / T_{\text{bd}} \) we obtain from [22] the expression

\[
F_{\text{bd}}(t) = \frac{T_{\text{bd}}}{T_{\text{obs}}} 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{3}{4} (1 - k_c) \frac{1}{2} J \frac{1}{2} n_0 ! k_c \right) ;
\]

where

\[
c = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ;
\]

and we have introduced the time scale

\[
T_{\text{bd}} = \frac{2 \frac{1}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} n_0 (1 - k_c) \frac{1}{2} J \frac{1}{2} n_0 ! k_c \right) ;
\]

giving us the characteristic time of the breakdown emergence. In deriving [37] we have also taken into account from [44] [10, 30, 31] and remained directly number 2 as cofactor because the maximum of the function \( z^{1/2} \text{exp}(z) \) is about 0.43. Naturally, we have to consider ourselves to such values of the vehicle density for which \( F_{\text{bd}} \) is 1 because tra c ow with higher values of the vehicle density cannot exist on these time scales. For the following values \( r = 2 \), the ratio \( (1, 0) = 1, i, 1, 2 \) sec, and \( n_0 = 20 \) expression [37] gives us the estimate \( T_{\text{bd}} \) in the characteristic breakdown time. It should be pointed out that the latter estimate does not contradict the evaluation of the breakdown rate given at the beginning of the present section because it holds only in the region \( c_2 \).

Figure 8 illustrates the obtained results depicting the breakdown probability for different values of the observation time \( T_{\text{obs}} \) measured in units of \( T_{\text{bd}} \). The depth of penetrating into the metastability region. It should be pointed out that in drawing Fig. 8 we have applied to some result of 37] rather than [33] in order to have a possibility to go out of the frame works of the form all in [33]. The latter is considered here to clarify the obtained results only. To make the form of the \( t_{\text{bd}}(t) \)-dependence more evident we apply again to the form all in case [39] assuming the ratio \( m = T_{\text{obs}} / T_{\text{bd}} \) to be a large parameter. Than analyzing a small neighborhood of the point
The trac breakdown probability vs the depth $m$ is specified by the equality
\[
\frac{2 \ln (2m)}{(1 - m)n_0} \frac{1}{m}
\]
(40)
where the vehicle density scale
\[
\rho_s = (c_2 - c_1) \frac{r_0}{2 \ln (2m)}(1 - m)n_0
\]
(42)
Therefore, in a rough approximation the $F_{bd}(\rho)$ dependence is a simple exponential function whose scale $\rho_s$ is approximately a constant value (because the function $m$ shows week variations for $m < 1$). Changing the observation duration $T_{obs}$ practically shift the cut-off point $m$ only.

III. COMPARISON WITH EXPERIMENTAL RESULTS

Experimental investigations of the trac breakdown regarded as a probabilistic phenomenon have been carried out by several authors (see, e.g., Ref. [3], [12], [13]). Elefteriadou et al. [3] actually pointed out the fact that trac breakdown at ramp merges with breakdown probability occurs randomly without precise relation to a certain xed value of trac volume. A more detailed analysis of the breakdown probability has been fulfilled in papers [10, 11, 12, 13]. These observations show that trac breakdown can occur inside a wide interval of trac volume from about 1500 veh/h/l (vehicles per hour per lane) up to 3000 veh/h/l. The real dynamics of trac breakdown near bottlenecks and the developed structure of the congestion in the vicinity of highway bottlenecks is locally metastable under the discharged downstream trac ow of volume $\rho_s$ varying in the same interval. The latter enables us to estimate the delay time $\tilde{\epsilon}$ playing a significant role in the presented model. In fact, ignoring the velocity $\#(\rho_{car})$ of cars in the cluster as well as the headway distance $h_{car}$, we get that the lower boundary of the metastability region meets the trac volume $\rho_s$.

\[
j = c_1 \left( \frac{k}{n_0} \right) \ln \left( \frac{n_0}{\rho_s} \right) = \frac{1}{1800} \text{veh/h/l}
\]
when we immediately estimate $\tilde{\epsilon}$ as 2 sec, which is in agreement with the value adopted previously in papers [14, 15]. In this section, as it has been done in the previous one, we use the estimates of the quantities $n_0$, 20 according to the experimental data depicted in Fig. 1, set $(1, 0) = \rho, \tilde{\epsilon}$ from the general consideration.

In order to compare the obtained results and the available experimental data we have applied to the latest materials presented in detail by Lorenz & Elefteriadou [2]. The breakdown phenomenon was investigated in trac ow near two bottlenecks of Highway 401, one of the primary Toronto trac arteries. The detectors were located right after the on-ramp within several hundred meters downstream. So the dynamics of trac breakdown observed at these places seems to be main due to local internal properties of trac ow discussed in the present paper. The complex spatial structure of the induced congested phase including moving wide and narrow jams reported by Kerner [13] should emerge above the detectors upstream. We consider in detail the data obtained for one these bottleneck (site A " in paper [2]). The paired detectors were located in each of the three lanes and were instrumented to provide vehicle count and speed estimates continuously at 20-second intervals. A breakdown event was xed via the velocity drop below 90 km/h, the middle point of a certain gap in the velocity exhibits visually separating the congested and free trac ow states. Besides, only those disturbances that caused the averaged speed over all the lanes to drop below 90 km/h for a period of 1 min or m ore were considered a true breakdown. The latter enabled the authors to filter out large amplitude fluctuations in the mean vehicle velocity not leading to the trac breakdown. In Fig. 1, exhibits the obtained probability (relative frequency) of the trac breakdown events during 5-minute and 15-minute intervals vs the trac volume partitioned within 100 veh/h/l steps. We note that Fig. 1 does not show the available 1-minute interval data because the corresponding breakdown probability is not signi cant for all the observed values of trac volume except for the upper boundary.
We have considered the traffic breakdown phenomenon regarded as a random process developing via the nucleation mechanism. The origin of critical jam nuclei proceeds in a metastable phase of traffic ow and seems to be located inside a not too large region on a highway, for example, in the close proximity of a highway bottleneck. The induced complex structure of the congested traffic phase is located upstream the bottleneck. Keeping these properties in mind we have applied to the probabilistic model regarding the jam emergence as the development of a large car cluster on highway. In these terms the traffic breakdown proceeds through the formation of a certain car of critical size in the metastable vehicle ow, which enabled us to confine ourselves to the single cluster model.

We assumed that, rst, the growth of the car cluster is governed by attachment of cars to the cluster whose rate is mainly determined by the mean headway distance between the cars in the vehicle ow and, may be, also by the headway distance in the cluster. Second, the cluster dissolution is determined by the car escape from the cluster whose rate depends on the cluster size directly. To justify the latter assumption we apply to the modern notion of the traffic ow structure (see Ref. [12]). Namely, the jam emergence goes mainly through the sequence of two phase transitions: free ow ! synchronized ow! stop-and-go pattern [3]. Both of these transitions are of the rst order, i.e., they exhibit breakdown, hysteresis, and nucleation effects [3]. Therefore considering the nal stage of the jam emergence we have to regard the synchronized ow as the metastable phase exactly inside which a critical jam nucleus appears due to random fluctuations. The synchronized ow is characterized by strong multi-lane correlations in the ow motion and, as a result, all the vehicles in a certain active cluster spanning over all the highway lanes move as a whole. So the proposed probabilistic description deals with actually macrovehicles comprising any individual cars. The available single-vehicle experimental data present the correlation characteristics of the synchronized ow which have enabled us to estimate the characteristic dimension $n_0 = 20(30)$ of the car cluster entering the dependence of the car detachment rate on the cluster size. Namely for all car clusters, $n < n_0$, the characteristic detachment time $t_0$ should be substantially less than this time for large clusters, $n = n_0$.

We have written the appropriate master equation for the cluster distribution function and analyze the formation of the critical car cluster due to the clump over a certain potential barrier. The inequality $n_0 = 1$ has opened us the way to convert from the discrete master equation to the appropriate Fokker-Plank equation and nd all the required characteristics of the traffic breakdown.

The obtained results were compared with the available experimental data and, in detail, with the probability of traffic breakdown in the vicinity of bottlenecks vs the traffic ow volume presented by Lorenz & Elefteriadou [12]. It turned out that the theoretical curves can be ftted closely to the given experimental data using values of the main parameters chosen based on the general properties of the traffic ow not related directly to the breakdown dynamics. In particular, rst, we have dem onstrated that the characteristic internal time scale $t_{bd}$ of the breakdown development is about $t_{bd} = n_0$ (we recall that $t_{bd}$ is the characteristic time during which a car can individually leave a cluster). Hence we get the estimate of the breakdown time scale about one minute. The latter justifies the widely used probabilistic technique of the breakdown investigation based on xing this event.
during a time interval of several minutes. Second, the proposed model explains why the trace breakdown as a probabilistic phenomenon is observed inside a sufficiently wide interval of the trace volume $n_{\text{vol}}$, namely, the thickness 4 of this layer can attain its low boundary $n_{\text{bd}}$ in magnitude. The matter is that $4 = n_{\text{bd}} = n_{\text{vol}}$.

Concluding the above we state that trace breakdown is a mesoscopic process, as it must be for the synchronized modes, whose characteristic spatial and temporal scales correspond to carrier clusters made of a large number of vehicles.

\section*{Appendix A: Escaping Rate from a Boundary Well}

In section II C we have obtained the Fokker-Planck equation \( \frac{\partial}{\partial t} \langle F(t) \rangle = \frac{1}{2} \frac{\partial^2}{\partial n^2} \langle F(t) \rangle \) governing the evolution of carrier clusters treated as random wandering in the space of their size $n$. It has turned out that near the threshold the precluster domain is separated from the large cluster region by a potential barrier, so that the formation of the clustered phase should proceed through the nucleation mechanism \( F(t) \). In other words, for a large cluster to emerge on the road its critical nucleus $n_c$ has to arise via random fluctuations of the cluster size in the precluster region. Thereby in order to describe the cluster formation we need the expression specifying the rate of the critical nucleus generation, being the subject of the present appendix.

Mathematically the description of the critical nucleus generation is equivalent to the problem of a particle escaping from the corresponding potential barrier \( F(t) \). Thereby the rate of the critical nucleus generation, i.e. the frequency of the trace breakdown, is represented in terms of the probability density $F(t)$ for this particle to escape from the potential well at a given time $t$ provided initially, $t = 0$, it has been placed near the local minimum (here $n = 0$). Namely

\[ \Omega_c = \frac{1}{2} \frac{\partial F(t)}{\partial n} = \frac{1}{2} \frac{\partial^2}{\partial n^2} F(t), \]  

where the value $0$ of the argument means that we consider time scales exceeding substantially the duration of all the transient processes during which the distribution of the particle inside the potential well attains locally quasi-equilibrium.

Since the potential relief under consideration is rather special we prefer to recall briefly the way of deriving the probability $F(t)$ referring a reader to the specific literature (see, e.g., book [1]) for details.

The concept of potential well implies that the barrier is sufficiently high, i.e. $n = 1$, therefore the particle can climb over it due to random fluctuations lifting the particle to points at the potential barrier where $n = 1$. If such an event does not lead to the escape of the particle it will drift back to the neighborhood of the local minimum $n = 0$ whose thickness is specified by the inequality $(n) < 1$. Therefore the subsequent attempts of escaping may be considered as being mutually independent. A filter the particle has climbed over the barrier the force $F(t) = n$ carries it away to distant points, making the return impossible. So from this point of view we may refer to the particle being inside the potential well or having escaped from it as two possible states without specifying the particular position. Therefore the probability $P(t)$ that the particle remains inside the well at time $t$ if it has being placed in it at the time $t_0$ obeys the equation:

\[ P(t) = P(t_0) \exp \left( - \frac{t - t_0}{\tau_{\text{life}}} \right) \]  

for $0 < t < t_0$.

We hence get the general expression for the function $P(t)$

\[ P(t) = \exp \left( - \frac{t}{\tau_{\text{life}}} \right) ; \]

where $\tau_{\text{life}}$ is a certain constant specified by the particular properties of a potential well. The latter formula gives us immediately the general form of the escape probability

\[ F(t) = \frac{dP(t)}{dt} = \frac{1}{\tau_{\text{life}}} \exp \left( - \frac{t}{\tau_{\text{life}}} \right) ; \]  

(A2)

In order to find the lifetime $\tau_{\text{life}}$ we will deal with the Laplace transform $F_L(s)$ of the escape probability $F(t)$

\[ F_L(s) \equiv \int_0^\infty e^{-st} F(t) dt = \frac{1}{1 + s \tau_{\text{life}}} ; \]  

(A3)

whence it follows that in the expansion of $F_L(s)$ with respect to $s$ around the point $s = 0$

\[ F_L(s) = 1 - s \tau_{\text{life}} + \cdots ; \]  

(A4)

the first order term directly contains the desired lifetime as the coefficient.
Following the standard approach [1] we reduce the escaping problem to the \( r st \) passage time probability. In other words, we assume the particle never to come back to the potential well if it after climbing the barrier reaches points where \( c \ (n) > 1 \) (Eq. [1]). The particle may be withdrawn from the consideration or, what is the same, it will be trapped when reaches for the \( r st \) time any fixed point \( n \) in this region. The time \( t \) takes for the particle to reach the point \( n \) after overcoming the barrier at the critical point \( n_c \) is ignorable in comparison with the characteristic width of critical fluctuations. Thereby the function \( F(t) \) speciﬁes actually the probability of passing (reaching) the point \( n \) for the \( r st \) time at the time \( n \) on entt. This construction enables us to introduce a more detailed relative function \( F(n; t) \) giving the probability for the particle initially placed at the point \( 0 < n < n_c \) to reach \( r st \) the right boundary \( n \) of the region under consideration at the time \( n \) on entt. The left boundary \( n = 0 \) is permeable for the particle. Then using the standard technique based on the backward Fokker-Planck equation conjugated with Eq. (25) we give the following equation for the function \( F_L(n; s) \)

\[
sF_L = t e^2 F_L \left[ 0, (n) \right] \left[ 0, F_L \right] \tag{A 5}
\]

subject to the boundary conditions

\[
F_L(0; s) = F_L(\infty; s) = 1 \tag{A 6}
\]

which directly follows that the \( r st \) order term \( ' (n) \) in the expansion of the Laplace transform \( F_L(n; s) \) with respect to \( s \)

\[
F(n; s) = 1 \quad \text{s' (n)};
\]

obeys in turn the equation

\[
\theta^2_1 (n) \left[ 0, (n) \right] \left[ 0, \theta_1 (n) \right] = 1 \tag{A 7}
\]

subject to the boundary conditions

\[
\theta_1 ^\prime (0) = 0 \quad \text{and} \quad ' (n) = 0 \tag{A 8}
\]

The solution of the system \([A 7] \) and \([A 8] \) has the form

\[
' (n) = \sum_{n}^{Z} d n e^0 (n) \quad 0 \quad d n e^0 (n) \tag{A 9}
\]

and the value \( ' (0) \) gives us the desired lifetime:

\[
\lim_{n \to 0} = ' (0) \tag{A 10}
\]

Inside the potential well the function \( ' (n) \) takes practically a constant value mainly contributed by the points \( n^0 \) belonging to the well bottom, i.e., to the region \( (n) < 1 \) and by the points \( n^0 \) located near the top of the potential barrier where \( (n_c) < 1 \). This feature leads us immediately to the approximation

\[
\lim_{n \to 0} \frac{P}{n^0} \left[ \theta_n (n) \right] \left[ \theta_n (0) \right] e^{n^0} ;\tag{A 11}
\]

which is the main result of the present appendix.

In particular, for the potential \( (n) \) speciﬁed by expression \([2] \) or \([2] \) formula \([A 11] \) gives

\[
\lim_{n \to 0} \frac{P}{n^0} \left[ \theta_n (0) \right] \left[ \theta_n (0) \right] e^{n^0} ;\tag{A 12}
\]

Formulas \([A 3] \), \([A 4] \), \([A 5] \), and \([2] \) give us expression \([A 12] \).

[1] B.S. Kemer, "The physics of trafﬁc." Physics World, August, 25(30) (1999).
[2] B.S. Kemer, "Congested trafﬁc: observation and theory." Transport. Res. Record 1678, 160(167) (1999).
[3] B.S. Kemer, "Experimental features of the emergence of moving jams in free trafﬁc." J. Phys., A33, L221-222 (2000).
[4] B.S. Kemer and H. Rehborn, "Experimental features and characteristics of trafﬁc jams." Phys. Rev. E 53, R1297/R1300 (1996).
[5] B.S. Kemer and H. Rehborn, "Experimental properties of complexity in trafﬁc." Phys. Rev. E 53, R4275/R4278 (1996).
[6] B.S. Kemer and H. Rehborn, "Experimental properties of phase transitions in trafﬁc." Phys. Rev. Lett. 79, 4030/4033 (1997).
[7] B.S. Kemer, "Experimental features of self-organization in trafﬁc." Phys. Rev. Lett. 81, 3797(3800) (1999).
[8] L. Neubert, L. Santen, L., A. Schadschneider, and M. Schreckenberg, "Single-vehicle data of highway trafﬁc: A statistical analysis." Phys. Rev. E 60, 6480(6490) (1999).
[9] L. E. Lefterdou, R. F. Rehm, and W. R. M. She, "Probabilistic approach to the breakdown phenomenon in free trafﬁc." Transport. Res. Record 1484, 80(89) (1995).
[10] R. Kuhn, and N. Ansset, "New methods for determining critical sections on freeways." Transport Research Board, 7th Annual Meeting, January 12-16, 1997, Washington D.C., Paper # 970672 (1997).
[11] B. Persaud, S. Yagar, and R. Brownlee, "Exploration of the breakdown phenomenon in free traffic." Transport Research Record 1364, 64(69) (1998).
[12] M. Lorenz and L. Lefterdou, "A probabilistic approach to determining freeway capacity and breakdown." Transport Research Circular E-C 018, 84(95) (2000).
[13] B.S. Kerner, "Theory of breakdown phenomenon at highway bottlenecks." Transport. Res. Rec 1710, 136(144) (2000).

[14] I.A. Lubachevsky and R. Mahnke, "Order-parameter model for unstable multilane traffic," Phys. Rev. E 62, 6082-6093 (2000); e-print: cond-mat/9910268.

[15] R. Mahnke and N. Pierre, "Stochastic master-equation approach to aggregation in freewary traffic," Phys. Rev. E 56, 2666-2671 (1997).

[16] R. Mahnke and J. Kaupuzs, "Stochastic theory of freewary traffic," Phys. Rev. E 59, 117-125 (1999).

[17] D. Helbing, A. Hennecke, V. Shvetsov, and M. Treiber, "Macroscopic traffic simulation based on a gas-kinetic, non-local traffic model," Transportation Research, B 35, 183(211) (2001).

[18] P. Schick and R. Kuhne, "Untersuchungen zum Verkehrslauf an Streckenbein uzungslagen" (Investigation on the benefits of corridor-control system in terms of traffic). In: Arbeiten aus dem Institut fur Straen-und Verkehrsweisen 2000/2001 (Research at the Institute of Road and Transportation Studies), Serie Veroffentlichungen aus dem Institut fur Straen- und Verkehrswesen (Publications of the Institute of Road and Transportation Studies), (Universitat Stuttgart, Juli 2001) # 28, pp.1(22).

[19] C.W. Gardiner, Handbook of Stochastic Methods for Physics, Chemistry, and the Natural Sciences (Springer, Berlin, 1994).