ON ISOLATED LOG CANONICAL CENTERS

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Abstract. In this paper, we show that the depth of an isolated log canonical center is determined by the cohomology of the -1 discrepancy divisors over it. A similar result also holds for normal isolated Du Bois singularities.

1. Introduction

Singularities play a significant role in the minimal model program (mmp). Among the different types of singularities, Kawamata log terminal (klt) and log canonical (lc) are of particular importance. Many fundamental theorems are first proved in the klt case, then extended to the lc case. And it is expected that lc should be the largest class of singularities for which one can run mmp.

One of the major differences between klt and lc is that klt singularities are rational singularities, lc singularities are Du Bois [10] but in general not rational. So it is interesting and important to know how far lc is from being rational. Since rational implies Cohen-Macaulay, we can also ask if the variety $X$ is Cohen-Macaulay at some given point $p$. Or more precisely, we can calculate $\text{depth}_p(O_X)$.

There are some known results regarding this direction. For example, Fujino shows that given a lc pair $(X, \Delta)$ of dimension at least three, then $\text{depth}_p(O_X) \geq \min\{3, \text{codim}_p X\}$ if $\bar{p}$ is not a lc center (Theorem 4.21 in [4]), which is first proved by Alexeev assuming that $p$ is a closed point and $X$ is projective (Lemma 3.2 in [1]). Kollár and Kovács generalized this result in [12] and [16], respectively, but still under the assumption that $\bar{p}$ is not a lc center. (See also [2] for result about closed points.)

In this paper, we investigate a case when $\bar{p}$ is a lc center. Assume that $p$ is an isolated lc center, after localization we assume $p$ is a closed point. It turns out that there is a delicate relation between $\text{depth}_p(O_X)$ and the cohomology group of the exceptional divisors over $p$. More precisely, given a log canonical pair $(X, \Delta)$ and an isolated lc center $p \in X$ which is a closed point, we take a log resolution $f: Y \to X$ such that

$$K_Y = f^*(K_X + \Delta) + A - B - E.$$ 

Here $A, B$ are effective and $[B] = 0$, $E$ is the reduced divisor such that $f(E) = p$. Then we have the following,

**Theorem 1.1.** (= Corollary [3.2]) For any integer $3 \leq t \leq n$, we have $\text{depth}_p(O_X) \geq t$ if and only if $H^{i-1}(E, O_E) = 0, \forall 1 < i < t$. (Note that by assumption $X$ is normal, so we know $\text{depth}_p(O_X)$ is at least two.)

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This result generalizes Proposition 4.7 in [5], which gives a necessary and sufficient condition for an index one isolated log canonical singularity to be Cohen-Macaulay.

We prove this theorem by showing that the local cohomology $H^i_p(O_X)$ is the Matlis dual of $H^{n-i}(E, K_E)$. The same method applies to isolated Du Bois singularities, (see section 3.2). In the Du Bois case, $E$ denotes the reduced exceptional divisors.

The most crucial ingredient of the proof is Kovács vanishing theorem, which says that $R^i f_* O_Y(-E) = 0$, $\forall i > 0$. With this theorem, we see that $f_* O_Y(-E)$ is quasi isomorphic to $R f_* O_Y(-E)$. By this quasi isomorphism and Grothendieck duality, we are able to see the relation between the local cohomology of $X$ and cohomology of $O_E$. Because of the significant role of Kovács’s theorem in this paper, we give a quick proof of it in the last section. This proof, based on Fujino’s idea, only uses Grothendieck duality and Kawamata-Viehweg vanishing theorem instead of the notion of Du Bois pair in Kovács’s original paper.

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2. PRELIMINARIES

Given a pair $(X, \Delta)$, where $X$ is a normal variety and $\Delta$ is a $\mathbb{Q}$–linear combination of Weil divisors so that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Take a log resolution $f : Y \to X$, such that the exceptional locus and the strict transform $f^{-1}_* \Delta$ are simple normal crossing divisors. We say the pair $(X, \Delta)$ is log canonical if

$$K_Y = f^*(K_X + \Delta) + A - B - E,$$

where $A, B$ are effective, $\lfloor B \rfloor = 0$ and $E$ is reduced. We say $(X, \Delta)$ is log terminal if $E$ is empty.

In this paper we consider log canonical pair, $(X, \Delta)$. A sub variety $W \subset X$ is called log canonical center, if there is a log resolution as above, and some component $E' \subset E$ such that $f(E') = W$.

We recall Kovács vanishing theorem.

**Theorem 2.1.** (Theorem 1.2 in [17]) Let $(X, \Delta)$ be a log canonical pair and let $f : Y \to X$ be a proper birational morphism from a smooth variety $Y$ such that $Ex(f) \cup Supp f^{-1}_* \Delta$ is a simple normal crossing divisor on $Y$. If we write

$$K_Y = f^*(K_X + \Delta) + \sum a_i E_i$$

and put $E = \sum a_i = -1 E_i$, then

$$R^i f_* O_Y(-E) = 0$$

for every $i > 0$.

This theorem is first proved by notion of Du Bois pair under the assumption that $X$ is $\mathbb{Q}$-factorial. The proof is then simplified in [6] without assuming $\mathbb{Q}$-factorial.

Now we recall the duality theorems which will be used in this paper. First we recall Grothendieck duality theorem (III.11.1, VII.3.4 in [7]). Let $f : Y \to X$
be a proper morphism between finite dimensional noetherian schemes. Suppose that both $X$ and $Y$ admit dualizing complexes, for example when they are quasi-projective varieties. Then for any $\mathcal{F}^* \in D_{coh}^-(Y)$, we have

$$Rf_* R\text{Hom}_Y(\mathcal{F}^*, \omega_Y^*) \cong R\text{Hom}_X(Rf_* \mathcal{F}^*, \omega_X^*)$$

Here $\omega_X^*$ is dualizing complex. Let $n$ be the dimension of $X$ and assume that $X$ is normal, then $h^{-n}(\omega_X^n) := \omega_X = \mathcal{O}_X(K_X)$, the extension of regular $n$-forms on smooth locus. In this paper we only consider normal varieties, so we will use $\omega_X$ and $K_X$ interchangeably. If $X$ is Cohen-Macaulay, then $h^i(\omega_X^n) = 0$, if $i \neq -n$, and $h^{-n}(\omega_X^n) = \omega_X$. Or equivalently, $\omega_X^n = \omega_X[n]$.

Now we recall local duality (V.6.2 in [7]). Suppose that $(R, p)$ is a local ring. An injective hull $I$ of the residue field $k = R/p$ is a an injective $R$ module $I$ such that for any non-zero submodule $N \subset I$ we have $N \cap k = 0$. (See [3] Proposition 3.2.2. for more discussion.) Matlis duality says that the functor $\text{Hom}(\cdot, I)$ is a faithful exact functor on the category of Noetherian $R$ modules.

**Theorem 2.2.** (Local duality) Let $(R, p)$ be a local ring and $\mathcal{F}^* \in D_{coh}^+(R)$. Then

$$R\Gamma_p(\mathcal{F}^*) \to R\text{Hom}(R\text{Hom}(\mathcal{F}^*, \omega_R^n), I)$$

is an isomorphism.

In particular, if we take $i$-th cohomology on both hand sides, we have

$$H^i_p(\mathcal{F}^*) \cong \text{Hom}(H^{-i}(R\text{Hom}(\mathcal{F}^*, \omega_R^n)), I)$$

The $-i$ comes from switching the cohomology functor $H^i(\cdot)$ and $Hom(\cdot, I)$.

3. **Main Results**

3.1. **Depth of LC center.** Given a log canonical pair $(X, \Delta)$, and an isolated lc center $p \in X$ which is a closed point. Without loss of generality, we assume $X$ is an affine space and $p$ is the only closed point. By definition, we have a log resolution $f : Y \to X$ such that

$$K_Y = f^*(K_X + \Delta) + A - B - E.$$

Here $A, B$ are effective and $|B| = 0$, $E$ is the reduced exceptional divisor such that $f(E) = p$.

**Theorem 3.1.** For $1 < i < n$, $H^i_p(X, \mathcal{O}_X)$ is dual to $H^{n-i}(E, K_E)$ by Matlis duality. For $i = n$, $H^n_p(X, \mathcal{O}_X)$ is dual to $f_* \mathcal{O}_Y(K_Y + E)$.

**Proof.** Push forward the following exact sequence on $Y$,

$$0 \to K_Y \to K_Y(E) \to K_E \to 0.$$

By Grauert-Riemenschneider vanishing, we have $R^{n-i} f_* \mathcal{O}_Y(K_Y + E) \cong H^{n-i}(E, K_E)$ for $i < n$. So to prove the statement, it suffices to prove the duality between $H^i_p(X, \mathcal{O}_X)$ and $R^{n-i} f_* \mathcal{O}_Y(K_Y + E) \cong H^{n-i}(E, K_E)$. To this end, we consider the quasi isomorphism $f_* \mathcal{O}_Y(-E) \cong \text{quasi} Rf_* \mathcal{O}_Y(-E)$ implied by Kovács vanishing theorem. Apply Grothendieck duality, we have

$$\text{RHom}(f_* \mathcal{O}_Y(-E), \omega_X^*) \cong \text{quasi} R\text{Hom}(Rf_* \mathcal{O}_Y(-E), \omega_X^*) \cong \text{quasi} Rf_* \omega_Y^*(E)$$

Take $-i$th cohomology, we have

$$\text{Ext}^{n-i}(f_* \mathcal{O}_Y(-E), \omega_X^*) \cong R^{n-i} f_* \mathcal{O}_Y(K_Y + E)$$

By Matlis duality, the left hand side is isomorphic to $\text{Hom}(H^i_p(f_*\mathcal{O}_Y(-E)), I)$, where $I$ is injective hull of $k$.

To prove the statement, we claim that $H^i_p(f_*\mathcal{O}_Y(-E)) \cong H^i_p(\mathcal{O}_X)$ for $i > 1$. This follows from the following exact sequence

$$0 \to f_*\mathcal{O}_Y(-E) \to \mathcal{O}_X \to \mathcal{O}_p \to 0,$$

and the fact that $H^i_p(\mathcal{O}_p) = 0$ iff $i > 0$.

\[\square\]

**Corollary 3.2.** For any integer $3 \leq t \leq n$, we have $\text{depth}_p\mathcal{O}_X \geq t$ if and only if $H^{n-i}(E, K_E) = 0$, $\forall 0 < i < n$.

*Proof.* In the proof of Theorem 3.1 we showed $H^i_p(f_*\mathcal{O}_Y(-E)) \cong H^i_p(\mathcal{O}_X)$ for $i > 1$. So for any integer $3 \leq t \leq n$, we have

$$\text{depth}_p\mathcal{O}_X \geq t \iff H^i_p(X, \mathcal{O}_X) = 0, \forall i < t$$

$$\iff H^i_p(X, f_*\mathcal{O}_Y(-E)) = 0, \forall i < t$$

$$\iff H^{n-i}(E, K_E) = 0, \forall 1 < i < t \text{ (Matlis duality and Equation (3.1))}$$

$$\iff H^{n-1}(E, K_E) = 0, \forall 1 < i < t \text{ (Serre Duality)}$$

\[\square\]

**Remark 3.3.** The cohomology group $H^i(E, \mathcal{O}_E)$ is independent of resolution, because $H^i(E, \mathcal{O}_E) \cong R^i f_*\mathcal{O}_Y$ by Kovács vanishing theorem. And that $R^i f_*\mathcal{O}_Y$ is well known to be independent of resolution.

**Corollary 3.4.** (Proposition 4.7 [5]) Given a closed isolated lc center $p$ of a pair $(X, \Delta)$, then $X$ is Cohen-Macaulay at $p$ if and only if $H^i(E, \mathcal{O}_E) = 0$, $\forall 0 < i < n-1$.

### 3.2. Normal isolated Du Bois singularity

The notion of Du Bois singularities is a generalization of the notion of rational singularities. For a proper scheme of finite type $X$ there exists a complex $\Omega^*_{\mathcal{X}}$, which is an analogue of De Rham complex. Roughly speaking, $X$ is said to have Du Bois singularities if the natural map $\mathcal{O}_X \to \Omega^0_{\mathcal{X}}$ is a quasi isomorphism. We refer the reader to [15] and the reference there for more discussions.

In this subsection we consider the case where $(X, p)$ is a normal isolated Du Bois singularity of dimension $n$, and $f : Y \to X$ is a log resolution such that $f$ is an isomorphism outside of $p$. We claim that the idea in the previous subsection can be applied to this case. The crucial fact we need is the following,

**Theorem 3.5.** (Theorem 6.1 in [15]) Take a log resolution $f : Y \to X$ as above, and let $E$ be the reduced preimage of $p$. Then $(X, p)$ is a normal Du Bois singularity if and only if the natural map

$$R^i f_*\mathcal{O}_Y \to R^i f_*\mathcal{O}_E$$

is an isomorphism for all $i > 0$.

This theorem implies that $R^i f_*\mathcal{O}_Y(-E) = 0$, $\forall i > 0$. That is

$$f_*\mathcal{O}_Y(-E) \cong_{\text{quasi}} R f_*\mathcal{O}_Y(-E)$$

Then exactly the same proof as in previous section yields
Theorem 3.6. Given \((X, p)\) is a normal isolated Du Bois singularity of dimension \(n\). For \(1 < i < n\), \(H_p^n(X, \mathcal{O}_X)\) is dual to \(H^{n-i}(E, K_E)\) by Matlis duality. For \(i = n\), \(H_p^n(X, \mathcal{O}_X)\) is dual to \(\mathcal{I}_Y(K_Y + E)\). In particular, \(f_*\mathcal{I}_Y(K_Y + E) \cong K_X\).

Then the corollaries in the previous section also hold.

Remark 3.7. The last statement has been proved in [8] (the Claim in Theorem 2.3).

4. Kovács vanishing theorem

In this section we follow Fujino’s idea to give a simple proof of Kovács vanishing theorem. First we prove a similar result for dlt pair which was proved by the notion of rational pair in [13]. One of the equivalent definitions of dlt singularities is that there is a log resolution (Szabó resolution [19]) \(f : Y \to X\) such that the discrepancy \(a(E; X, \Delta) > -1\) for any exceptional divisor \(E\) on \(Y\) (Theorem 2.44 in [11]).

Theorem 4.1. Let \((X, \Delta_X)\) be a dlt pair and let \(f : Y \to X\) be a Szabó resolution. Then we can write \(K_Y + \Delta_Y = f^*(K_X + \Delta_X) + A - B\), where \(A, B\) are effective exceptional divisors, \(|B| = 0\) and \(\Delta_Y\) is the strict transform of \(\Delta_X\). Then for any reduced subset \(\Delta' \subseteq \Delta_Y\), we have \(R^i f_*\mathcal{O}_Y(-\Delta') = 0\) for every \(i > 0\).

Proof. Write \(K_Y - f^*(K_X + \Delta_X) + \Delta_Y = A - B\),

Then

\[ [A] = K_Y - f^*(K_X + \Delta_X) + \Delta_Y + B + [A] - A, \]

which is \(f\)-exceptional and effective. Consider the following diagram of complexes,

\[
\begin{array}{ccc}
\mathcal{O}_Y(-\Delta') & \xrightarrow{\alpha} & Rf_*\mathcal{O}_Y(-\Delta') \\
& \downarrow{\beta} & \\
& Rf_*\mathcal{O}_Y([A] - \Delta') & \\
\end{array}
\]

Note that

\[ [A] - \Delta' = K_Y - f^*(K_X + \Delta_X) + \text{strict transform} + \delta, \]

where \(\delta\) is some effective simple normal crossing divisors such that \(|\delta| = 0\). So by Reid-Fukuda type vanishing \(R^i f_*\mathcal{O}_Y([A] - \Delta') = 0\) for \(i > 0\). On the other hand, Since \([A]\) is exceptional and \(\Delta'\) is strict transform, so \(f_*\mathcal{O}_Y([A] - \Delta') = f_*\mathcal{O}_Y(-\Delta')\). (Lemma 12 in [12]). So \(\beta\) is a quasi isomorphism.

Dualize this diagram we have

\[
\begin{array}{ccc}
\text{RHom}(f_*\mathcal{O}_Y(-\Delta'), \omega_X^*) & \xrightarrow{\alpha^*} & \text{RHom}(Rf_*\mathcal{O}_Y(-\Delta'), \omega_X^*) \\
& \downarrow{\beta^*} & \\
& \text{RHom}(Rf_*\mathcal{O}_Y([A] - \Delta'), \omega_X^*) & \\
\end{array}
\]

Apply Grothendieck duality we get the following composition,
By Reid-Fukuda type vanishing, the complex $Rf_*\omega_Y^*(\Delta')$ has vanishing higher cohomology. Note that $\beta^*$ is a quasi isomorphism, so it is in fact a composition of sheaf morphisms as following,

$$f_*\omega_Y(\Delta' - [A]) \xrightarrow{\gamma^*} Rf_*\omega_Y^*(\Delta') \xrightarrow{\alpha^*} R\text{Hom}(f_*\mathcal{O}_Y(-\Delta'), \omega_X^*)$$

With theorem 4.1, we can prove Kovács vanishing theorem following Fujino’s idea [9]. Consider the following maps,

$$Y \xrightarrow{h} Z \xrightarrow{g} X$$

where $g : (Z, \Delta_Z) \to (X, \Delta)$ is a dlt modification such that $K_Z + \Delta_Z = g^*(K_X + \Delta)$. And $h : Y \to Z$ is a Szabó resolution such that $K_Y = h^*(K_Z + \Delta_Z) + A - B - \Delta_Y$, where $\Delta_Y = h_*^{-1}\Delta_Z$.

We claim that $R^i f_*\mathcal{O}_Y(-[\Delta_Y]) = 0, \forall i > 0$. By theorem [4,1], $R^h_*\mathcal{O}_Y(-[\Delta_Y]) = 0, \forall i > 0$. Also note that $h_*\mathcal{O}_Y([A] - [\Delta_Y]) = h_*\mathcal{O}_Y(-[\Delta_Y]) = \mathcal{O}_Z(-[\Delta_Z])$. (lemma 12 in [12]). So by Leray spectral sequence, $R^i f_*\mathcal{O}_Y(-[\Delta_Y]) = R^i g_*\mathcal{O}_Z(-[\Delta_Z])$. The latter is zero for $i > 0$ by Theorem 1-2-5 and Remark 1-2-6 in [9].

Note that $f : Y \to X$ is not a log resolution. To fix the problem, we can blow up centers with simple normal crossing with $\text{Supp}(\Delta_Y + A + B)$. Say the blow up is $\pi : W \to Y$. There are two cases can happen: If we blow up klt locus, it is a Szabó resolution and then the divisor with $-1$ discrepancy is $\Delta_W = \pi_W^{-1}(\Delta_Y)$. Then $R^i \pi_*\mathcal{O}_W(-\Delta_W) = 0$ by theorem [4,1]. If we blow up centers inside non-klt locus, then the divisor with $-1$ discrepancy may be $\Delta_W = \pi_W^{-1}(\Delta_Y) + F$, where $F$ is the exceptional divisor produced by blow up. Then $R^i \pi_*\mathcal{O}_W(-\Delta_W) = 0$ by direct calculation. In any case we showed that the higher direct image is not changed by these two kinds of blowing up. So we can conclude Kovács vanishing theorem.

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