COMPUTING MINIMAL FREE RESOLUTIONS OF RIGHT 
MODULES OVER NONCOMMUTATIVE ALGEBRAS

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Abstract. In this paper we propose a general method for computing a minimal free right resolution of a finitely presented graded right module over a finitely presented graded noncommutative algebra. In particular, if such module is the base field of the algebra then one obtains its graded homology. The approach is based on the possibility to obtain the resolution via the computation of syzygies for modules over commutative algebras. The method behaves algorithmically if one bounds the degree of the required elements in the resolution. Of course, this implies a complete computation when the resolution is a finite one. Finally, for a monomial right module over a monomial algebra we provide a bound for the degrees of the non-zero Betti numbers of any single homological degree in terms of the maximal degree of the monomial relations of the module and the algebra.

1. Introduction

A fundamental tool in the study of commutative and noncommutative algebraic structures consists in developing some homology (or cohomology) theory. In this context, the notion of free resolution, that is, of an exact sequence of free modules over an algebra is an important one. For instance, by means of the augmentation ideal of a graded algebra one defines the minimal resolution of the base field which is used to compute the graded homology of such algebra. Observe that in the noncommutative case there are different notions of free resolution depending on we consider one-sided or two-sided modules. In fact, for many applications the free right (or left) resolutions are the most effective ones and we will stick to this case in the present paper. It is important also to mention that owing to non-Noetherianity of a general (noncommutative finitely generated) algebra, even a finitely generated module may have that the corresponding right syzygy module is infinitely generated. There are of course important classes of algebras which are right Noetherian (finite dimensional, universal enveloping algebras, etc) or at least have finite homology. It was Anick [4] who proved that the algebras having a finite Gröbner basis for their ideal of relations hold finite Betti numbers for the minimal resolution of the base field. He proved this by constructing the so-called Anick’s resolution which extends the combinatorial minimal resolution that was introduced by Backelin [2] for the case of monomial algebras. One problem with the Anick’s construction is that it is generally a non-minimal resolution and hence one needs some extra algorithmic work to minimalize it. Beside this approach, many other

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ad hoc methods are used to compute the homology (cohomology in case of group algebras) of specific classes of algebras like modular group algebras, quiver algebras (path algebras), G-algebras (PBW algebras, algebras of solvable type), etc. For an overview about these methods, see for instance [5, 6, 11].

In this paper we follow a different path aiming to describe a general method for computing a minimal free right resolution of a finitely presented graded right module over a finitely presented graded (noncommutative) algebra by using syzygy computations for modules over commutative algebras. In case the resolution is infinite with respect to the Betti numbers or its length, the proposed method behaves algorithmically by bounding the degree of the syzygies to be computed. The idea to use commutative modules has its roots in a series of papers [7, 8, 9] where commutative analogues of noncommutative graded ideals has been introduced for the purpose of obtaining noncommutative Gröbner bases computations from commutative ones. In those papers such correspondence between ideals is called the letterplace correspondence. It is essentially based on the fact that a word \( w = x_{i_1} \cdots x_{i_d} \) can be represented as a set of commutative monomials \( \{ x_{i_1,s+1} \cdots x_{i_d,s+d} \} \) \( (s \geq 0) \) where the index \( s + k \) of the letterplace variable \( x_{i_k,s+k} \) corresponds to the place where the letter \( x_{i_k} \) may occur in a word containing \( w \). As a by-product one obtains, for instance, that the Hilbert function of a noncommutative graded algebra \( A \) can be obtained by computing the dimension of suitable components of a corresponding commutative algebra. A brief review of these ideas can be found in this paper in Section 4. The extension of them to submodules of graded free right \( A \)-modules is developed in Section 5 and 6. In Section 7 we analyze how such module letterplace correspondence behaves with respect to homomorphisms between free modules. This study culminates with Theorem 7.3 which provides the new method to obtain minimal free right resolutions. In view of this result we propose in Section 3 a process, preserving syzygies, which allows to reduce any grading of a free right module to the standard one where all degrees of a free basis are equal to 1. The previous Section 2 acquaints the reader to all noncommutative structures and basic notation that are used in the paper. In Section 8 one finds the detailed description of the computation of a finite minimal free right resolution of the base field of the universal enveloping algebra of a free nilpotent Lie algebra. In Section 9 we explain how similar complete computations can be performed in an algorithmic way by means of our method. In particular, for finitely presented monomial right modules over finitely presented monomial algebras we propose a bound for the degrees of the non-zero Betti numbers of any single homological degree. In Section 10 we explain some tricks that can be used to speedup computer calculations. By means of some of them, we propose also a comparison between the timing obtained by an experimental application of our technique to the example in Section 8 with the performance of an optimized library for G-algebras. Finally, in Section 11 we present some conclusions and ideas for further developments of the proposed approach.

### 2. Right syzygy modules

We start by introducing all noncommutative structures that we need in this paper. Let \( \mathbb{K} \) be any field and denote by \( F = \mathbb{K}\langle x_1, \ldots, x_n \rangle \) the free associative algebra which is freely generated by the finite set \( \{ x_1, \ldots, x_n \} \). In other words, the elements of \( F \) are noncommutative polynomials in the variables \( x_i \). We denote by \( W \subset F \) the subset of all monomials of \( F \), that is, the elements of \( W \) are words over
the alphabet \{x_1, \ldots, x_n\}. We endow \(F\) with the standard grading which defines \(\text{deg}(w)\) as the total degree of the monomial \(w \in W\). In other words, one has that \(\text{deg}(x_i) = 1\), for all variables \(x_i\). Denote by \(F = \bigoplus_{d \geq 0} F_d\) the decomposition of the algebra \(F\) in its homogeneous components. Let \(I \subset F\) be a graded two-sided ideal, that is, \(I = \sum_d I_d\) where \(I_d = I \cap F_d\) and consider the quotient graded algebra \(A = F/I\). Observe that if \(A'\) is a graded algebra which is finitely generated by the homogeneous set \(\{x'_1, \ldots, x'_n\}\) where \(\text{deg}(x'_i) = 1\) for all \(i\) then \(A'\) is clearly isomorphic to \(A\) by the graded algebra homomorphism \(\varphi : F \to A'\) such that \(x_i \mapsto x'_i\) and \(I = \text{Ker} \varphi\). In other words, \(I\) is the ideal of the relations that are satisfied by the generators \(x'_i\). By definition, the algebra \(A'\) or equivalently \(A\) is finitely presented when the ideal \(I\) is finitely generated. By abuse of notation, we denote also by \(x_i\) the cosets \(x_i + I\) which are the generators of the algebra \(A\).

Let \(\delta \geq 0\) be an integer and denote by \(A[-\delta] = \bigoplus_{d \geq 0} A[-\delta]_d\) the algebra \(A\) which is endowed with the grading induced by \(\delta\), that is, we put \(A[-\delta]_d = 0\), for any \(d < \delta\) and \(A[-\delta]_d = A_{d-\delta}\), for all \(d \geq \delta\). Fix an integer \(r > 0\) and some integers \(\delta_i \geq 0\) \((1 \leq i \leq r)\). We consider the direct sum \(\bigoplus_{1 \leq i \leq r} A[-\delta_i]\) which is a graded right \(A\)-module of finite rank \(r\). If \(\{e_1, \ldots, e_r\}\) is the canonical free basis of such module then \(\text{deg}(e_i) = \delta_i\), for all \(i\). A homogeneous element of \(\bigoplus_{1 \leq i \leq r} A[-\delta_i]\) of degree \(d\) is hence a right linear combination \(\sum_i e_i f_i (f_i \in A)\) where \(\delta_i + \text{deg}(f_i) = d\), for any \(i\).

**Definition 2.1.** Consider a right submodule \(M \subset \bigoplus_{1 \leq i \leq r} A[-\delta_i]\). We call \(M\) a graded submodule if \(M = \sum_d M_d\) where \(M_d = M \cap \bigoplus_{1 \leq i \leq r} A[-\delta_i]_d\). In this case, we define the quotient graded right module \(N = \bigoplus_{1 \leq i \leq r} A[-\delta_i]/M\) where the homogeneous component \(N_d\) is isomorphic to \(\bigoplus_{1 \leq i \leq r} A[-\delta_i]_d/M_d\), for all \(d \geq 0\). A finitely generated graded right module \(N' = \langle g_1, \ldots, g_r \rangle\) where \(g_i\) is a homogeneous element of degree \(\delta_i\) is clearly isomorphic to \(N\) by the graded right module homomorphism \(\varphi : \bigoplus_{1 \leq i \leq r} A[-\delta_i] \to N'\) such that \(e_i \mapsto g_i\) and \(M = \text{Ker} \varphi\). Since \(M\) is the submodule of the relations which are satisfied by the generators \(g_i\), then we have that the module \(N'\) or equivalently \(N\) is finitely presented when \(M\) is finitely generated.

Remark that even if we assume that \(A\) is finitely presented, this is generally not a right Noetherian algebra which implies that a right submodule \(M \subset A^e\) is usually infinitely generated. Of course, there are important cases when \(A\) is right (or left) Noetherian as the case when \(A\) is finite-dimensional or it is the universal enveloping algebra of a finite-dimensional Lie algebra, etc. (see [12]). For graded right modules one can define the notion of minimal (homogeneous) basis and a noncommutative version of the Nakayama’s lemma holds (see, for instance, [17]). It makes sense then to define the following invariants.

**Definition 2.2.** Let \(A = F/I\) be a finitely generated graded algebra which is endowed with the standard grading and consider a right submodule \(M \subset \bigoplus_{1 \leq i \leq r} A[-\delta_i]\). Let \(\{g_j\}\) be a (possibly infinite) minimal basis of \(M\). For all degrees \(d \geq 0\), since the homogeneous component \(M_d\) is finite dimensional we can define

\[
b_d(M) = \#\{g_j \mid \text{deg}(g_j) = d\}.
\]

The integers \(\{b_d(M)\}_{d \geq 0}\) are called the graded Betti numbers of \(M\). In addition, consider a finitely generated graded right module \(N = \bigoplus_{1 \leq i \leq r} A[-\delta_i]/M\). We define \(b_d(N)\) as the number of generators of degree \(d\) in a minimal basis of \(N\) and we call again \(\{b_d(N)\}_{d \geq 0}\) the graded Betti numbers of \(N\).
Assume now that \( \{g_1, \ldots, g_s\} \) is a finite homogeneous basis of a finitely generated graded right submodule \( M \subset \bigoplus_{1 \leq i \leq s} A[-\delta_i] \). If \( \Delta_j = \deg(g_j) \) then we consider the finitely generated graded free right module \( \bigoplus_{1 \leq j \leq s} A[-\Delta_j] \). By denoting \( \{\epsilon_1, \ldots, \epsilon_s\} \) its canonical basis we have therefore that \( \deg(\epsilon_j) = \Delta_j \). It is natural to define the graded right module homomorphism

\[
\varphi : \bigoplus_{1 \leq j \leq s} A[-\Delta_j] \rightarrow \bigoplus_{1 \leq i \leq r} A[-\delta_i], \quad \epsilon_j \mapsto g_j.
\]

Its image is clearly \( \text{Im} \varphi = M \) and its kernel is by definition

\[
K = \text{Ker} \varphi = \{ \sum_j \epsilon_j f_j \mid f_j \in A, \sum_j g_j f_j = 0 \}.
\]

We call the elements of the graded right submodule \( K \subset \bigoplus_{1 \leq j \leq s} A[-\Delta_j] \) the \textit{right syzygies of the basis} \( \{g_j\} \). We refer also to \( K \) as the \textit{right syzygy module} of \( M \) (with respect to \( \{g_j\} \)). Note that possible synonyms for the term “right syzygies” are \textit{right module relations} or \textit{right linear relations}.

The main concern of the present paper is to develop a method to compute a homogeneous basis of \( K \). We have to remark immediately that the right syzygy module \( K \) is not necessarily finitely generated again because \( A \) is generally not a right Noetherian algebra or at least a right coherent algebra \([16]\). Nevertheless, there are many cases when \( K \) is finitely generated that we will discuss in Section 9. For the moment, let us just observe that if \( M, K \) are not finitely generated then one can at least study homogeneous syzygies of homogeneous generators up to some fixed degree. A natural generalization of the notion of right syzygy module is the following one.

**Definition 2.3.** Let \( N = \bigoplus_{1 \leq i \leq r} A[-\delta_i]/M \) be a finitely generated graded right module. A graded free right resolution of \( N \) is by definition a (possibly infinite) exact sequence of graded right module homomorphisms

\[
0 \leftarrow N \leftarrow \bigoplus_{1 \leq i \leq r} A[-\delta_i] \leftarrow \bigoplus_{1 \leq j \leq s} A[-\delta_j'] \leftarrow \bigoplus_{1 \leq k \leq t} A[-\delta_k'] \leftarrow \ldots .
\]

Denote by \( M_i \) the kernel of the \((i + 1)\)-th map of the above sequence (including the first zero map). Note that \( M_0 = N \) and \( M_1 = M \). We call \( M_i \) the \( i \)-th right syzygy module of \( N \) (with respect to the resolution). We call this sequence a minimal resolution of \( N \) if all the finite homogeneous bases of \( M_i \) which are the images of the canonical bases of the free modules are minimal ones. In this case, we refer to the integer \( b_i(M_i) \) as the graded Betti number of \( N \) of homological degree \( i \) and (internal) degree \( d \).

It may happen that a resolution is finite because the second to last map is an injective one. In this case, one says that \( N \) admits a \textit{finite free right resolution}. For instance, the right modules over a \( G \)-algebra hold such resolutions \([11]\). Another interesting case is when there is a resolution with exactly \( m \) right syzygy modules (including \( M_0 = N \)) which are finitely presented, that is, \( M_m \) is finitely generated but infinitely related. In this case the right module \( N \) is called of type \((FP)_m\). Finally, if all resolutions have infinite length then \( N \) is by definition of type \((FP)_{\infty}\).

Observe that the modules over Noetherian commutative algebras either have finite resolutions or belong to type \((FP)_{\infty}\) (see, for instance, \([15]\)).
3. Module component homogenization

We introduce now a construction which provides that for the computation of the right syzygies of a graded right submodule \( M \subset \bigoplus_{1 \leq i \leq r} A[-\delta_i] \) one is always reduced to the case of the standard grading, that is, \( \delta_i = 0 \) for any \( i \) and hence \( M \subset A^r \). This is in fact an essential step in view of the method that we will propose in Section 7.

Let \( t \notin F = \mathbb{K}(x_1, \ldots, x_n) \) be a new variable and consider the free associative algebra \( \bar{F} = \mathbb{K}(x_1, \ldots, x_n, t) \) endowed with the standard grading. If \( A = F/I \) where \( I \subset F \) is a graded two-sided ideal then we denote by \( \bar{I} \) the extension of \( I \) in \( \bar{F} \), that is, \( \bar{I} \subset \bar{F} \) is the graded two-sided ideal generated by \( I \). Then, we define the quotient graded algebra \( \bar{A} = \bar{F}/\bar{I} \). Since \( I = I \cap F \), one has that \( A \) can be canonically embedded in \( \bar{A} \).

Fix now two integers \( \delta \geq \delta' \geq 0 \). We have an injective graded right \( A \)-module homomorphism \( \eta : A[-\delta] \rightarrow \bar{A}[-\delta'] \) which is defined as \( 1 \mapsto t^{\delta - \delta'} \). This map can be extended to graded right free modules in the following way. Consider the integers \( \delta_i \geq \delta'_i \geq 0 \) (\( 1 \leq i \leq r \)) and let \( \{e_i\}, \{\bar{e}_i\} \) be the canonical bases of the graded free right modules \( \bigoplus_{1 \leq i \leq r} A[-\delta_i], \bigoplus_{1 \leq i \leq r} \bar{A}[-\delta_i'] \) respectively. We define the injective graded right \( A \)-module homomorphism

\[
\eta : \bigoplus_i A[-\delta_i] \rightarrow \bigoplus_i \bar{A}[-\delta_i'], e_i \mapsto \bar{e}_i t^{\delta - \delta_i}.
\]

**Definition 3.1.** Let \( M \subset \bigoplus_i A[-\delta_i] \) be a graded right \( A \)-submodule. Denote by \( H(M) \subset \bigoplus_i \bar{A}[-\delta_i'] \) the graded right \( \bar{A} \)-submodule generated by \( \eta(M) \). We call \( H(M) \) a (partial) component homogenization of \( M \). In particular, if \( \delta'_i = 0 \) for any \( i \), that is, \( \bigoplus_i \bar{A}[-\delta_i'] = \bar{A}^r \) with the standard grading then we say that \( H(M) \) is the complete component homogenization of \( M \). On the other hand, if \( \delta'_i = \delta_i \) for all \( i \), that is, \( H(M) \) is the extension of an \( A \)-submodule of \( \bigoplus_i A[-\delta_i] \) to a \( \bar{A} \)-submodule of \( \bigoplus_i \bar{A}[-\delta_i] \) then we call \( H(M) \) the trivial component homogenization.

It is clear that \( \eta(\bigoplus_i A[-\delta_i]) = \bigoplus_i t^{\delta - \delta_i} A[-\delta_i] \) which is a graded right \( A \)-submodule of \( \bigoplus_i \bar{A}[-\delta_i'] \). This result can be generalized in the following way.

**Proposition 3.2.** Let \( M \subset \bigoplus_i A[-\delta_i] \) be a graded right submodule and denote \( M = H(M) \) and \( M' = M \cap \bigoplus_i t^{\delta - \delta_i} A[-\delta_i] \). Then, one has that \( \eta(M) = M' \). In particular, we have that \( \dim_k M_d = \dim_k M'_d \), for all \( d \geq 0 \).

**Proof.** Since \( \eta(\bigoplus_i A[-\delta_i]) = \bigoplus_i t^{\delta - \delta_i} A[-\delta_i] \), it is clear that \( \eta(M) \subset \bar{M}' \). Moreover, any element \( h' \in \bar{M} \) is such that \( h' = \sum g_j f'_j \) where \( g_j \in M \) and \( f'_j \in \bar{A} \). In other words, if \( h' = \sum g_j f_j \) (\( h' \in A \)) and \( g_j = \sum e_i g_{ij} \) (\( g_{ij} \in A \)) then \( h'_i = t^{\delta - \delta_i} \sum g_{ij} f'_j \). By assuming that \( h' \in \bar{M}' \), that is, \( h'_i = t^{\delta - \delta_i} A \) we obtain clearly that \( f'_j \in \bar{A} \) and therefore \( h' = \eta(h) \) where \( h = \sum g_j f'_j \in M \). \( \square \)

Since \( \eta \) is an injective map, note that the above result implies that the mapping \( M \rightarrow H(M) \) is also an embedding of the graded right \( A \)-submodules of \( \bigoplus_i A[-\delta_i] \) into the graded right \( \bar{A} \)-submodules of \( \bigoplus_i \bar{A}[-\delta_i] \). It is clear that the component homogenizations are exactly those graded right \( A \)-submodules \( \bar{M} \subset \bigoplus_i \bar{A}[-\delta_i] \) that are generated by \( \bar{M}' = \bar{M} \cap \bigoplus_i t^{\delta - \delta_i} A[-\delta_i] \). Moreover, since \( \eta \) is an injective graded right \( A \)-module homomorphism, we have that \( M = \eta^{-1}(M') \subset \bigoplus_i A[-\delta_i] \) is the unique graded right \( A \)-submodule such that \( H(M) = M \). We want now to study
how the correspondence $M \mapsto H(M)$ behaves with respect to minimal generating sets.

**Theorem 3.3.** Consider a graded right submodule $M \subset \bigoplus_i A[-\delta_i]$ and its component homogenization $\tilde{M} = H(M) \subset \bigoplus_i A[-\delta_i]$. Moreover, let $\{g_j\}$ be a set of homogeneous elements of $M$ and define $g'_j = \eta(g_j)$, for all $j$. Then, we have that $\{g_j\}$ is a (minimal) basis of $M$ if and only if $\{g'_j\}$ is a (minimal) basis of $\tilde{M}$. In particular, one has that $b_d(M) = b_d(\tilde{M})$, for all $d \geq 0$.

**Proof.** Assume that $\{g_j\}$ is a homogeneous basis of $M$. Since $\eta$ is a graded $A$-module homomorphism, one has that $\{g'_j\}$ is a homogeneous basis of the graded $A$-submodule $\tilde{M}' = \tilde{M} \cap \bigoplus_i A^k[-\delta_i]$. Because the graded $A$-submodule $\tilde{M}$ is generated by $\tilde{M}'$ we obtain immediately that $\{g'_j\}$ is a homogeneous basis of $\tilde{M}$.

Suppose now that $g'_j = \sum_{j>1} g'_j f'_j$ where $f'_j \in \tilde{A}$, that is, $\{g'_j\}$ is not a minimal basis. If $g_j = \sum_i \epsilon_i g_{ij} \ (g_{ij} \in \tilde{A})$ then one has that $\sum_{j>1} \epsilon'_i g_{ij} f'_j$. We conclude that $f'_j \in \tilde{A}$ and $g_1 = \sum_{j>1} g_j f'_j$. Finally, it is clear that by similar arguments one obtains also the necessary condition in the statement. \square

Note explicitly that the above result provides a method to obtain a minimal basis $\{g_j\}$ of the graded submodule $M$ starting from a minimal basis $\{g'_j\}$ of its component homogenization $\tilde{M} = H(M)$. In fact, since $\tilde{M}$ is generated by $\tilde{M}' = \eta(M)$, we can assume that $\{g'_j\} \subset \tilde{M}'$ and hence $g_j = \eta^{-1}(g'_j)$, for all $j$. In Section 2 we have already observed that we cannot always assume that these bases are finite sets but in what follows we will make this assumption.

Let $\{g_j\}$ be a finite homogeneous basis of a finitely generated graded right $A$-submodule $M \subset \bigoplus_i A[-\delta_i]$. Then, denote $\Delta_j = \deg(g_j)$ and consider the finitely generated graded free right $A$-module $\bigoplus_j A[-\Delta_j]$. If $\{\epsilon_j\}$ is its canonical basis then by definition we have that $\deg(\epsilon_j) = \Delta_j$, for any $j$. In a similar way, one defines the finitely generated graded free right $A$-module $\bigoplus_j A[-\Delta_j]$ with canonical basis $\{\epsilon_j\}$ such that $\deg(\epsilon_j) = \Delta_j$, for all $j$. One has therefore the graded right $A$-module homomorphism

$$\varphi : \bigoplus_j A[-\Delta_j] \rightarrow \bigoplus_i A[-\delta_i], \epsilon_j \mapsto g_j.$$  

By putting $g'_j = \eta(g_j)$, we consider also the graded right $A$-module homomorphism

$$H(\varphi) : \bigoplus_j \tilde{A}[-\Delta_j] \rightarrow \bigoplus_i \tilde{A}[-\delta_i], \epsilon_j \mapsto g'_j.$$  

Observe that by Theorem \ref{thm:homogenization}, we have that $\text{Im} H(\varphi) = H(\text{Im} \varphi) = H(M)$. We want now to understand how the correspondence $\varphi \mapsto H(\varphi)$ behaves with respect to the kernels. By denoting $\tilde{\varphi} = H(\varphi)$, we have by definition

$$K = \text{Ker} \varphi = \{ \sum_j \epsilon_j f_j \mid f_j \in \tilde{A}, \sum_j g_j f_j = 0 \},$$  

$$\tilde{K} = \text{Ker} \tilde{\varphi} = \{ \sum_j \tilde{\epsilon}_j f'_j \mid f'_j \in \tilde{A}, \sum_j \tilde{g}_j f'_j = 0 \}.$$  

In other words, the graded right $A$-submodule $K \subset \bigoplus_j \tilde{A}[-\Delta_j]$ is the right syzygy module of the basis $\{g_j\}$ and the graded right $\tilde{A}$-submodule $\tilde{K} \subset \bigoplus_j \tilde{A}[-\Delta_j]$ is the right syzygy module of $\{g'_j\}$.
Consider now the canonical embedding $\bigoplus_j A[-\Delta_j] \to \bigoplus_j \bar{A}[-\Delta_j]$ such that $\epsilon_j \mapsto \bar{\epsilon}_j$. By abuse of notation, we denote this injective graded right $A$-module homomorphism also by $\eta$ since it defines the trivial component homogenization $H(K) \subset \bigoplus_j \bar{A}[-\Delta_j]$ of the submodule $K \subset \bigoplus_j A[-\Delta_j]$. In fact, we will prove that $\bar{K} = H(K)$, that is, $\text{Ker } H(\varphi) = H(\text{Ker } \varphi)$. We start by considering the following key property.

**Proposition 3.4.** The following diagram is a commutative one

$$
\begin{array}{c}
\bigoplus_j A[-\Delta_j] \\
\eta \\
\downarrow \\
\bigoplus_j \bar{A}[-\Delta_j] \\
\eta
\end{array}
\xrightarrow{\varphi} 
\begin{array}{c}
\bigoplus_j \bar{A}[-\Delta'_j] \\
\eta
\end{array}
$$

**Proof.** Put $\bar{\varphi} = H(\varphi)$. For all $h = \sum_j \epsilon_j f_j \in \bigoplus_j A[-\Delta_j]$, we have to prove that $\eta(\bar{\varphi}(h)) = \bar{\varphi}(\eta(h))$. In fact, one has that $\eta(\varphi(h)) = \eta(\sum_j \eta(g_j) f_j) = \sum_j \eta(g_j) f_j = \sum_j g'_j f_j$. On the other hand, it holds that $\bar{\varphi}(\eta(h)) = \bar{\varphi}(\sum_j \epsilon_j f_j) = \sum_j g'_j f_j$. \(\square\)

**Theorem 3.5.** One has that $\bar{K} = H(K)$, that is, $\eta(K) = \bar{K}' = \bar{K} \cap \bigoplus_j A[-\Delta_j]$.

**Proof.** By Proposition 3.4 we have immediately that $\eta(K) \subset \bar{K}$ and therefore $H(K) \subset \bar{K}$. It is sufficient to prove that $\bar{K}$ has a generating set which is contained in $\eta(K)$. Consider $h' = \sum_j \epsilon_j f'_j \in \bar{K}$ any element of a minimal basis of $\bar{K}$ where $f'_j \in A_{d-\Delta_j}$, for some $d \geq 0$ and for all $j$. If $g_j = \sum_i \epsilon_i g_{ij}$ where $g_{ij} \in A_{\Delta_j - \delta}$, then $g'_j = \sum_i \epsilon_i t_d g_{ij}$ and hence in the algebra $\bar{A}$ we have that $t_d \sum_i g_{ij} f'_j = 0$ for all $i$. Recall now the graded algebras $A = F/I$ and $\bar{A} = \bar{F}/\bar{I}$ are defined by the graded two-sided ideal $I \subset F$ and $\bar{I} \subset \bar{F}$ where $\bar{I}$ is generated by $I = I \cap F$. Since $g_{ij} \in A$, we obtain therefore that $\sum_i g_{ij} f'_j = 0$ for any $i$. Moreover, because $h'$ belongs to a minimal basis one also has that $f'_j \in A$ for all $j$, that is, $h' = \eta(h)$ where $h = \sum_j \epsilon_j f'_j \in K$.

Observe explicitly that Theorem 3.3 together with Theorem 3.5 provides a method for obtaining the couple of graded Betti numbers sets $\{b_d(M)\}$ and $\{b_d(K)\}$ of a graded right submodule $M \subset \bigoplus_j A[-\delta_j]$ by its component homogenization $H(M) \subset \bigoplus_j \bar{A}[-\delta'_j]$. In the next sections we will show that this is essential to obtain these data via commutative analogues of such structures.

4. **Letterplace ideals**

In this section we show that graded two-sided ideals and hence noncommutative graded algebras have useful commutative counterparts. The results of this section has been introduced \[7\] \[9\] but we recall them here for the sake of completeness.

Consider the polynomial algebra $P = \mathbb{K}[x_{ij} \mid 1 \leq i \leq n, j \geq 1]$ in the infinite set of commutative variables $\{x_{ij}\}$. We assume that $P$ is endowed with the standard grading, that is, $\deg(x_{ij}) = 1$, for all $i, j$. Then, we define the monomial ideal

$$Q = \langle x_{ij} x_{kj} \mid 1 \leq i, k \leq n, j \geq 1 \rangle \subset P$$

and we consider the quotient graded algebra $R = P/Q$. A $\mathbb{K}$-linear basis of the algebra $R$ is given by (the cosets of) the monomials $x_{i_1 j_1} \cdots x_{i_d j_d}$ ($d \geq 0$) where $1 \leq i_1, \ldots, i_d \leq n$ and $1 \leq j_1 < \ldots < j_d$. If, as usual, $F = \mathbb{K}\langle x_1, \ldots, x_n \rangle$ then a graded $\mathbb{K}$-linear embedding $\iota : F \to R$ is defined as

$$x_{i_1} \cdots x_{i_d} \mapsto x_{i_1 1} \cdots x_{i_d d}$$
where \( x_{i_1} \cdots x_{i_d} \) is any monomial of \( W \). For any degree \( d \geq 0 \), the image \( \iota(F_d) \subset R_d \) can be described in the following way. Consider the finitely generated graded subalgebra \( P(d) = \mathbb{K}[x_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq d] \subset P \) and denote \( Q(d) = Q \cap P(d) \).

Then, we define the quotient graded algebra \( R(d) = P(d)/Q(d) \) which can be canonically embedded in \( R \). One has immediately that \( \iota(F_d) = R(d)_d \).

Consider now \( \sigma : R \to R \) the injective algebra endomorphism that is defined as \( x_{ij} \mapsto x_{ij+1} \), for all \( i, j \). For the mapping \( \iota \) one has the following key property. For all \( g \in F_d \) and \( f \in F \), we have that

\[
\iota(gf) = \iota(g)\sigma^d(\iota(f)).
\]

**Definition 4.1.** Let \( J \) be an ideal of \( R \). If \( \sigma(J) \subset J \) then we call \( J \) a \( \sigma \)-invariant ideal. In this case, if there is a subset \( G \subset J \) such that \( J \) is generated by \( \bigcup_{k \geq 0} \sigma^k(G) \) as an ideal of \( R \) then we say that \( J \) is \( \sigma \)-generated by \( G \).

**Definition 4.2.** Let \( I \subset F \) be a graded two-sided ideal. We define \( L(I) \subset R \) as the graded \( \sigma \)-invariant ideal which is \( \sigma \)-generated by \( \iota(I) \). We call \( L(I) \) the letterplace analogue of \( I \).

It is clear that \( L(F) = R \) and we have that \( \iota(F_d) = R(d)_d \), for all \( d \geq 0 \). This property can be extended to any graded two-sided ideal of \( F \) in the following way.

**Proposition 4.3.** Let \( I \subset F \) be a graded two-sided ideal and denote \( J = L(I) \). For any degree \( d \geq 0 \), one has that \( \iota(I_d) = J(d)_d \) where \( J(d) = J \cap R(d) \). In particular, we obtain that \( \dim_k I_d = \dim_k J(d)_d \).

**Proof.** It is clear that \( \iota(I_d) \subset J(d)_d \). Let now \( g_j \in I \) \((1 \leq j \leq s)\) be a homogeneous element of degree \( d_j \) and put \( g'_j = \iota(g_j) \in J \). Any element of the subspace \( J_d \subset J \) is a \( \mathbb{K} \)-linear combination of elements of type \( h' = \sum_j \sigma^{k_j}(g'_j)m'_j \) where \( m'_j \) is a monomial in \( R_{d-d_j} \). Moreover, all monomials of \( \sigma^{k_j}(g'_j) \) are of type

\[
x_{i_1k_1+j_1} \cdots x_{i_dk_d+j_d}
\]

with \( 1 \leq i_1, \ldots, i_d \leq n \). Assume now that \( h' \in J(d)_d = J_d \cap R(d) \). Since by definition \( R = P/Q \), it follows necessarily that \( m'_j = u'_j\sigma^{k_j+d_j}(v'_j) \) where \( u'_j \) is a monomial in \( R(k_j)_{k_j} \) and \( v'_j \) is a monomial in \( R(d-k_j-d_j)_{d-k_j-d_j} \).

By defining \( u_j = \iota^{-1}(u'_j), v_j = \iota^{-1}(v'_j) \), we conclude that

\[
h' = \sum_j \iota(u_j)\sigma^{k_j}(\iota(g_j))\sigma^{k_j+d_j}(\iota(v_j)) = \iota(\sum_j u_j g_j v_j) \in \iota(I_d).
\]

Since \( \iota \) is an injective map, the above result implies that the mapping \( I \mapsto L(I) \) is also an embedding that we call the (ideal) letterplace correspondence. It is easy to characterize the letterplace analogues among all graded \( \sigma \)-invariant ideals of \( R \).

**Proposition 4.4.** Let \( J \) be a graded \( \sigma \)-invariant ideal of \( R \) which is \( \sigma \)-generated by the graded subspace \( \bigoplus_d J(d)_d \) where \( J(d) = J \cap R(d) \). Then, there exists a (unique) graded two-sided ideal \( I \subset F \) such that \( J = L(I) \).

**Proof.** For all \( d \geq 0 \), denote \( I_d = \iota^{-1}(J(d)_d) \) and define \( I = \bigoplus_d I_d \). We have to show that \( I \subset F \) is a two-sided ideal. Then, consider \( f \in F_k, g \in I_d \) and \( h \in F_{d'} \).

One has clearly that

\[
\iota(fgh) = \iota(f)\sigma^k(\iota(g))\sigma^{k+d}(\iota(h)) \in J(k + d + d')_{k+d+d'}
\]
and we conclude that $fgh \in I_{k+d+d'}$. 

Owing to the above result, we call letterplace ideals all the graded $\sigma$-invariant ideals $J \subset R$ that are $\sigma$-generated by $\bigoplus \delta J(d)_d$. In other words, the letterplace correspondence establishes a bijection between all the graded two-sided ideals of $F$ and the class of their analogues which are the letterplace ideals of $R$.

5. Module letterplace embedding

As in Section 2, we consider a finitely generated graded algebra $A = F/I$ where $I \subset F$ is a graded two-sided ideal. Moreover, with the notation of Section 4 we have the (infinitely generated commutative) graded algebra $R = P/Q$ which is endowed with the algebra endomorphism $\sigma : x_{ij} \mapsto x_{ij+1}$. Then, we consider $J = L(I) \subset R$ the letterplace analogue of $I$ and we define the quotient graded algebra $S = R/J$. Since $J$ is the letterplace analogue of $I$, observe that $S$ is endowed with an algebra endomorphism induced by $\sigma$. By abuse of notation, we will denote this map also by $\sigma$. Moreover, we have a graded $\mathbb{K}$-linear embedding $A \rightarrow S$ induced by $\iota$ that we will denote again by this symbol.

Fix now an integer $\delta \geq 0$ and denote by $A[-\delta]$ the algebra $A$ which is endowed with the grading induced by $\delta$. In the same way, we define the graded algebra $S[-\delta]$. Consider now the graded $\mathbb{K}$-linear embedding $A[-\delta] \rightarrow S[-\delta]$ such that $f \mapsto \sigma^\delta(\iota(f))$. By abuse of notation, we denote this map also by $\iota$.

To the aim of describing the image $\iota(A[-\delta]_d) \subset S[-\delta]_d$ for any degree $d \geq 0$, we introduce the following objects. Consider the subalgebra

$$P(\delta, d) = \mathbb{K}[x_{ij} \mid 1 \leq i \leq n, \delta + 1 \leq j \leq d] \subset P$$

and define its ideal $Q(\delta, d) = Q \cap P(\delta, d)$. Then, we consider the quotient algebra $R(\delta, d) = P(\delta, d)/Q(\delta, d)$ which is canonically embedded in $R$. Finally, consider the ideal $J(\delta, d) = J \cap R(\delta, d)$ and define the quotient $S(\delta, d) = R(\delta, d)/J(\delta, d)$. If we use the grading induced by $\delta$ then we denote this algebra as $S(\delta, d)[-\delta]$. It is immediate to show that

$$\iota(A[-\delta]_d) = S(\delta, d)[-\delta]_d.$$

One can easily extend the above map $\iota$ to graded free modules in the following way. Consider the graded free right $A$-module $\bigoplus_{1 \leq i \leq r} A[-\delta_i]$ whose grading is defined by the integers $\delta_i \geq 0$. Denote $\{e_i\}$ its canonical basis and therefore one has that $\deg(e_i) = \delta_i$, for all $i$. In a similar way, we define $\bigoplus_{1 \leq i \leq r} S[-\delta_i]$ as the graded free $S$-module such that the elements of its canonical basis $\{e'_i\}$ have degrees $\deg(e'_i) = \delta_i$, for any $i$. Then, we consider the graded $\mathbb{K}$-linear embedding $\bigoplus_i A[-\delta_i] \rightarrow \bigoplus_i S[-\delta_i]$ such that, for any $f \in A$

$$e_i f \mapsto e'_i \sigma^{\delta_i}(\iota(f)).$$

By abuse of notation, we denote also this mapping as $\iota$ and we call it the module letterplace embedding. For all degrees $d \geq 0$, we have clearly that

$$\iota\left(\bigoplus_i A[-\delta_i]_d\right) = \bigoplus_i S(\delta_i, d)[-\delta_i]_d.$$

In particular, if $\delta_i = 0$ for all $i$ then one has that $\bigoplus_i A[-\delta_i] = A^r, \bigoplus_i S[-\delta_i] = S^r$ and we denote $S(d)^r = S(0, d)^r$, for any $d$. It holds hence that $\iota(A^r_d) \subset S(d)^r$. We will make use of this notation in Section 7. We conclude this section with the following key property.
Let \( g \in \bigoplus_1 A_{[-\delta_i]} \) be a homogeneous element of degree \( d \) and let \( f \in A \). Then, one has that \( \iota(gf) = \iota(g)\sigma^d(\iota(f)) \).

**Proof.** Denote \( g = \sum_i e_i g_i \) where \( g_i \in A_{[-\delta_i]_d} = A_{d-\delta_i} \). Then \( gf = \sum_i e_i g_i f \) and therefore

\[
\iota(gf) = \sum_i e'_i \sigma^d(\iota(g_i f)) = \sum_i e'_i \sigma^d(\iota(g_i) \sigma^d(\iota(f))) = \\
\sum_i e'_i \sigma^d(\iota(g_i)) \sigma^d(\iota(f)) = \iota(g) \sigma^d(\iota(f)).
\]

\( \square \)

6. Letterplace modules

Similarly to what we have done for ideals in Section 4, we introduce here the concept that a noncommutative module can be associated to a commutative one in a very meaningful way. One main difference with the case of graded two-sided ideals is that for graded right modules we do not need to require the \( \sigma \)-invariance of the commutative analogues. Note that this property is inherently associated to left structures. Note that this is a good motivation for us to prefer noncommutative right modules to left ones.

**Definition 6.1.** Let \( M \) be a graded right submodule of \( \bigoplus_{1 \leq i \leq r} A_{[-\delta_i]} \). We define \( L(M) \subset \bigoplus_{1 \leq i \leq r} S_{[-\delta_i]} \) as the graded submodule generated by \( \iota(M) \). We call \( L(M) \) the letterplace analogue of \( M \).

It is clear that \( L(\bigoplus_i A_{[-\delta_i]}) \) is exactly the free module \( \bigoplus_i S_{[-\delta_i]} \). Recall also that \( \iota(\bigoplus_i A_{[-\delta_i]_d} = \bigoplus_i S(\delta_i, d)[-\delta_i]_d \) for all degrees \( d \geq 0 \). This property can be extended to any graded right submodule of \( \bigoplus_i A_{[-\delta_i]} \) in the following way.

**Proposition 6.2.** Let \( M \subset \bigoplus_i A_{[-\delta_i]} \) be a graded right submodule and denote \( M' = L(M) \). For any \( d \geq 0 \), one has that \( \iota(M_d) = M'(d)_d \) where \( M'(d) = M' \cap \bigoplus_i S(\delta_i, d)[-\delta_i] \). In particular, we obtain that \( \dim_k M_d = \dim_k M'(d)_d \).

**Proof.** From \( \iota(\bigoplus_i A_{[-\delta_i]_d} = \bigoplus_i S(\delta_i, d)[-\delta_i]_d \) it follows immediately that \( \iota(M_d) \subset M'(d)_d \). Moreover, any element \( h' \in M'_d \) is such that \( h' = \sum_i g'_i f'_i \) where \( g'_i \) and \( f'_i \) are in \( M_d \). In other words, if \( h' = \sum_i \iota(d'_i) f'_i \) then \( d'_i = \sum_j e_j g_{ij} \in S(\delta_i, d) \). By observing that \( d'_i = \sum_j \sigma^{d_j} (\iota(g_{ij})) f'_j \), we conclude that \( f'_j \in S(\delta_j, d) \). Hence, \( h' = \sum_i h'_i f'_i \) and \( h = \sum_i g_j f'_i \in M_d \).

Owing to \( \iota \) is an injective map, by the above result we have that the mapping \( M \mapsto L(M) \) is also an embedding that we call the (module) letterplace correspondence. We want now to characterize the letterplace analogues among all graded submodules of \( \bigoplus_i S_{[-\delta_i]} \).

**Proposition 6.3.** Let \( M' \) be a graded submodule of \( \bigoplus_i S_{[-\delta_i]} \) which is generated by the graded subspace \( \bigoplus_d M'(d)_d \) where \( M'(d) = M' \cap \bigoplus_i S(\delta_i, d)[-\delta_i] \). Then, there exists a (unique) graded right submodule \( M \subset \bigoplus_i A_{[-\delta_i]} \) such that \( M' = L(M) \).

**Proof.** For all \( d \geq 0 \), denote \( M_d = \iota^{-1}(M'(d)_d) \) and define \( M = \bigoplus_d M_d \). We have to show that \( M \) is a right \( A \)-module. Then, consider \( g \in M_d \) and \( f \in A_d \).
One has clearly that $\iota(gf) = \iota(g)\sigma^d(\iota(f)) \in M'(d + d')_{d+d'}$ and we conclude that $gf \in M_{d+d'}$. 

Accordingly with this result, we call letterplace submodules all the graded submodules $M' \subset \bigoplus_i S[\delta_i]$ that are generated by $\bigoplus_i M'(d)_d$. Then, the letterplace correspondence defines a bijection between all the graded submodules of $\bigoplus_i A[-\delta_i]$ and the class of their analogues which are the letterplace submodules

In a similar way to the case of graded right $A$-modules, we have that minimal bases and graded Betti numbers are defined for graded $S$-modules. We analyze now how the correspondence $M \mapsto L(M)$ behaves with respect to such objects.

**Theorem 6.4.** Consider a graded right submodule $M \subset \bigoplus_i A[-\delta_i]$ and its letterplace analogue $M' = L(M) \subset \bigoplus_i S[\delta_i]$. Moreover, let $\{g_j\}$ be a set of homogeneous elements of $M$ and define $g'_j = \iota(g_j)$, for all $j$. Then, $\{g_j\}$ is a (minimal) basis of $M$ if and only if $\{g'_j\}$ is a (minimal) basis of $M'$. In particular, we have that $b_d(M) = b_d(M')$, for all $d \geq 0$. 

**Proof.** Assume that $\{g_j\}$ is a homogeneous basis of $M$. Since $M'$ is generated by $\iota(M) = \bigoplus_i M'(d)_d$, to prove that $\{g'_j\}$ is a basis of $M'$ it is sufficient to show that any homogeneous element $h' \in M'(d)_d$ is generated by $\{g'_j\}$. Denote $h = \iota^{-1}(h') \in M_d$ and hence $h = \sum_j g_j f_j$ where $f_j \in A_{d-\Delta_j}$ ($\Delta_j = \deg(g_j)$). Then, we have immediately that $h' = \sum_j g'_j \sigma^{\Delta_j}(\iota(f_j))$. 

Suppose now that $g'_1 = \sum_{j>1} g'_j f'_j$ with $f'_j \in S_{\Delta_1-\Delta_j}$. Because $g'_j = \iota(g_j) \in \bigoplus_i S(\delta_i, \Delta_j)[-\delta_i]_{\Delta_j}$, we have necessarily that $f'_j = \sigma^{\Delta_j}(f''_j)$ for some element $f''_j \in S(\Delta_1 - \Delta_j)_{\Delta_1-\Delta_j}$. By defining $f_j = \iota^{-1}(f''_j)$, we have therefore that $g_1 = \sum_{j>1} g_j f_j$. Finally, by similar arguments one proves that if $\{g'_j\}$ is a (minimal) basis of $M'$ then $\{g_j\}$ is a (minimal) basis of $M$. 

Observe that by Theorem 6.4 one has a method to obtain a minimal basis $\{g_j\}$ of the graded submodule $M$ starting from a minimal basis $\{g'_j\}$ of its letterplace analogue $M' = L(M)$. In fact, since $M'$ is generated by $\bigoplus_i M'(d)_d = \iota(M)$, one can assume that $\{g'_j\} \subset \bigoplus_i M'(d)_d$ and therefore $g_j = \iota^{-1}(g'_j)$, for all $j$. Note finally that such bases are not necessarily finite because both the finitely generated noncommutative algebra $A$ and the infinitely generated commutative algebra $S$ are generally not Noetherian ones. Of course, one can essentially restore finiteness just by considering the homogeneous generators up to some degree.

### 7. Letterplace Morphisms

Owing to the module component homogenization of Section 3, when considering submodules of a finitely generated graded free right $A$-module we can always assume that this is $A^r$ endowed with the standard grading. Such assumption will be in fact necessary to the main result of this section which is Theorem 7.3. Consider $M \subset A^r$ a finitely generated graded right submodule and let $\{g_j\} (1 \leq j \leq s)$ be a minimal basis of $M$. By putting $\Delta_j = \deg(g_j)$ for all $j$, we define the finitely generated graded free right $A$-module $\bigoplus_{1 \leq j \leq s} A[-\Delta_j]$. In other words, if $\{e_j\}$ is its canonical basis then by definition $\deg(e_j) = \Delta_j$. In a similar way, one defines $\bigoplus_{1 \leq j \leq s} S[-\Delta_j]$ as the finitely generated graded free $S$-module such that the elements of its canonical
basis \( \{ \epsilon'_j \} \) have degrees \( \deg(\epsilon'_j) = \Delta_j \). One defines therefore the graded right \( A \)-module homomorphism
\[
\varphi : \bigoplus_j A[-\Delta_j] \rightarrow A^r, \quad \epsilon_j \mapsto g_j
\]
such that \( \Im \varphi = M \). By putting \( g'_j = \iota(g_j) \), we consider also the graded \( S \)-module homomorphism
\[
L(\varphi) : \bigoplus_j S[-\Delta_j] \rightarrow S^r, \quad \epsilon'_j \mapsto g'_j.
\]
Note that Theorem 6.4 implies that \( \Im L(\varphi) = L(M) \). A critical point is now to understand how the correspondence \( \varphi \mapsto L(\varphi) \) behaves with respect to the kernels. By putting \( \varphi' = L(\varphi) \), we have that
\[
K = \ker \varphi = \{ \sum_j \epsilon_j f_j \mid f_j \in A, \sum_j g_j f_j = 0 \},
\]
\[
K' = \ker \varphi' = \{ \sum_j \epsilon'_j f_j \mid f_j \in S, \sum_j g'_j f_j = 0 \}.
\]
Recall that the elements of the graded right submodule \( K \subset \bigoplus_j A[-\Delta_j] \) are the right syzygies of the basis \( \{ g_j \} \). In a similar way, the elements of the graded submodule \( K' \subset \bigoplus_j S[-\Delta_j] \) are called the syzygies of the basis \( \{ g'_j \} \). We will prove that \( K' \) strictly contains the letterplace analogue \( L(K) \) in a meaningful way. Let us start with the following example.

**Example 7.1.** Consider the graded right ideal \( I = \langle x_1, \ldots, x_n \rangle \subset F \) and its letterplace analogue \( J = L(I) = \langle x_1, \ldots, x_{n1} \rangle \subset R \). We have then the maps \( \varphi : F[-1]^n \rightarrow F, \varphi(\epsilon_j) = x_j \) and \( \varphi' = L(\varphi) : R[-1]^n \rightarrow R, \varphi'(\epsilon'_j) = x_{j1} \). One has clearly that \( K = \ker \varphi = 0 \) and hence \( L(K) = 0 \). Moreover, from \( R = P/Q \) it follows that
\[
K' = \ker \varphi' = \langle \epsilon'_j x_{k1} \mid 1 \leq j, k \leq n \rangle.
\]
Clearly \( K' \) is not a letterplace submodule but we have that \( 0 = \iota(K_d) = K'(d)_d = K' \cap R(1, d)[-1]^n_d, \) for all \( d \geq 0 \).

From the above example we understand immediately that there is another set of natural syzygies in \( K' \). In fact, since \( g'_j = \iota(g_j) \in \iota(A^r_{\Delta_j}) = S(\Delta_j)^r_{\Delta_j} \) for any \( 1 \leq j \leq s \), we have immediately that
\[
C = \langle \epsilon'_j x_{k1} \mid 1 \leq j \leq s, 1 \leq k \leq n, 1 \leq l \leq \Delta_j \rangle \subset K'.
\]
Clearly, the given monomial basis of the graded submodule \( C \) is a minimal one. It is useful to consider the following property.

**Proposition 7.2.** The following diagram is a commutative one
\[
\begin{array}{ccc}
\bigoplus_j A[-\Delta_j] & \xrightarrow{\varphi} & A^r \\
\downarrow{\iota} & & \downarrow{\iota} \\
\bigoplus_j S[-\Delta_j] & \xrightarrow{L(\varphi)} & S^r
\end{array}
\]

**Proof.** If \( \varphi' = L(\varphi) \) then one has to prove that \( \iota(\varphi(h)) = \varphi'(\iota(h)) \), for all \( h = \sum_j \epsilon_j f_j \in \bigoplus_j A[-\Delta_j] \). In fact, by Proposition 5.1 we have that
\[
\iota(\varphi(h)) = \iota(\sum_j g_j f_j) = \sum_j \iota(g_j) \sigma^{\Delta_j}(\iota(f_j)) = \sum_j g'_j \sigma^{\Delta_j}(\iota(f_j)).
\]
On the other hand, one has immediately that
\[ \varphi'(\iota(h)) = \varphi'(\sum j \epsilon_j \sigma^{\Delta_j}(\iota(f_j))) = \sum j \epsilon_j \sigma^{\Delta_j}(\iota(f_j)). \]

**Theorem 7.3.** We have that \( K' = C + L(K) \). In particular, one obtains that \( b_d(K) = b_d(K') - b_d(C) \), for all \( d \geq 0 \).

**Proof.** From Proposition 7.2 it follows that \( L(K) \subseteq K' \) and we have already observed that \( C \subseteq K' \). Consider now \( h' = \sum_j \epsilon'_j f'_j \) any element of a minimal basis of \( K' \), where \( f'_j \in S_{d - \Delta_j} \), for some \( d \geq 0 \) and for all \( j \). Modulo \( C \subseteq K' \), we can clearly assume that there is \( d' \geq d \) such that \( f'_j \in S(\Delta_j, d') \) for each \( j \). Moreover, since \( h' \) is a minimal generator of \( K' \) and \( \sum_j \epsilon'_j f'_j = 0 \), we may choose \( d' = d \), that is, \( f'_j \in S(\Delta_j, d_{d - \Delta_j}) \) for all \( j \). We conclude that \( f'_j = \sigma^{\Delta_j}(\iota(f_j)) \) with \( f_j \in A_{d - \Delta_j} \) and therefore \( h' = \iota(h) \) where \( h = \sum_j \epsilon_j f_j \in K \).

Finally, from the above argument it follows also that \( b_d(K') = b_d(C) + b_d(L(K)) \) for any \( d \), where \( b_d(L(K)) = b_d(K) \) owing to Theorem 7.2. \( \square \)

Note that the graded Betti numbers \( \{b_d(C)\} \) are immediately obtained in the following way. For any \( d \geq 0 \), denote \( m_d = \#\{1 \leq j \leq s \mid \Delta_j = d\} \). We have clearly that \( b_0(C) = 0 \) and for each \( d \geq 0 \)
\[ b_{d+1}(C) = m_d \cdot n \cdot d. \]

By Theorem 6.4 and Theorem 7.3 one has therefore a letterplace method to compute the couple of graded Betti numbers sets \( \{b_d(M)\} \) and \( \{b_d(K)\} \) of a graded right submodule \( M \subset A' \). In case one has that \( M \subset \bigoplus_j A[-\delta_j] \) for some integers \( \delta_j \geq 0 \), recall that we can apply to \( M \) the module component homogenization described in Section 3. In particular, this homogenization is always needed for the kernel \( K = \text{Ker} \varphi \subset \bigoplus_j A[-\Delta_j] \) if one wants to compute a minimal free right resolution of \( N = \bigoplus_j A[-\delta_j]/M \). In the next section we study an example to show how this method works in practice.

8. An illustrative example

To the aim of illustrating the proposed methods by means of a concrete example, let us fix the field \( \mathbb{Q} \) of rational numbers and consider the free associative algebra in three variables \( F = \mathbb{Q}(x, y, z) \). Then, we define the graded two-sided ideal
\[ I = \langle [[a, b], c] \mid a, b, c \in \{x, y, z\} \rangle \subset F \]
where by definition \([a, b] = ab - ba\). The corresponding quotient graded algebra \( A = F/I \) is the universal enveloping algebra of the free nilpotent Lie algebra of class 2 which is freely generated by three variables. We want to compute the graded homology of \( A \), that is, a minimal free right resolution of its base field \( \mathbb{Q} \).

We start by considering the augmentation ideal
\[ M_1 = \langle x, y, z \rangle \subset A \]
By linear algebra, one can easily compute that a minimal basis of the ideal \( I \) is given by the following (noncommutative) polynomials

\[
\begin{align*}
-\{[z, y], z &= z^2 y - 2 z y z + y z^2, -\{[y, z], y &= z y^2 - 2 y z y + y^2 z, \\
[z, x], z &= z^2 x - 2 z x z + x z^2, [[x, z], y &= y z x - z x y + x y z - y x z, \\
\} = z y x - z x y - y x z + x y z, &-\{[y, x], y &= y^2 x - 2 y x y + x y^2, \\
[x, z], x &= z x^2 - 2 x z x + x^2 z, &-\{[x, y], x &= y x^2 - 2 x y x + x^2 y. \\
\}
\end{align*}
\]

This immediately implies that the right syzygy module \( M_2 \subset A[-1]^3 \) of the minimal basis \( \{x, y, z\} \) of \( M_1 \) has a minimal basis consisting of following homogeneous elements

\[
\begin{align*}
\begin{array}{c}
\epsilon_3 z y - 2 \epsilon_3 y z + \epsilon_2 z^2, \\
\epsilon_3 z x y - \epsilon_3 x z y + \epsilon_1 z y - \epsilon_2 x z, \\
\epsilon_3 y x - \epsilon_3 x y - \epsilon_2 x z + \epsilon_1 y z, \\
\epsilon_3 x^2 - 2 \epsilon_1 x z + \epsilon_1 x z, \\
\end{array}
\end{align*}
\]

We need now to compute the right syzygy module \( M_3 \subset A[-3]^8 \) of the above basis. Our approach is based on Theorem \([7,3]\) and hence we start by defining the polynomial algebra \( \mathcal{P} = \mathbb{Q}[x_j, y_j, z_j \mid j \geq 1] \) and the quotient algebra \( R = P/Q \) where \( Q = \langle a_j, b_j \mid a_j, b_j \in \{x_j, y_j, z_j\}, j \geq 1 \rangle. \) Then, the letterplace analogue \( \mathcal{J} = L(I) \) is defined as the graded \( \sigma \)-invariant ideal of \( R \) which is \( \sigma \)-generated by the (commutative) polynomials

\[
\begin{align*}
\begin{array}{c}
z_1 z_2 y_3 - 2 z_1 y_2 z_3 + y_2 z_1 z_3, z_1 y_2 z_3 - 2 y_2 z_1 y_3 + y_1 z_2 y_3, \\
z_1 z_2 x_3 - 2 z_1 x_2 z_3 + x_1 z_2 y_3, y_2 z_1 x_3 - z_1 x_2 y_3 + x_1 y_2 z_3 - y_1 x_2 z_3, \\
z_1 y_2 x_3 - z_1 x_2 y_3 + y_2 x_1 z_3, y_1 y_2 x_3 - y_1 x_2 y_3 + x_1 y_2 z_3, \\
z_1 x_2 x_3 - 2 x_1 x_2 z_3 + x_1 x_2 z_3, y_1 x_2 x_3 - 2 x_1 y_2 x_3 + x_1 x_2 y_3. \\
\end{array}
\end{align*}
\]

Observe now that we cannot immediately apply Theorem \([7,3]\) because the elements of the canonical basis of \( A[-1]^3 \) have degrees different from zero. Then, by the module component homogenization of Section 3, we transform the basis \([\mathcal{P}] \) into the set

\[
\begin{align*}
\begin{array}{c}
\tilde{\epsilon} z t y z - 2 \tilde{\epsilon} y t z y + \tilde{\epsilon} t z^2, \tilde{\epsilon} t y^2 - 2 \tilde{\epsilon} t y z + \tilde{\epsilon} t y z, \\
\tilde{\epsilon} t z x - \tilde{\epsilon} t x z + \tilde{\epsilon} t x z, \tilde{\epsilon} t x z - \tilde{\epsilon} t x z + \tilde{\epsilon} t x z, \\
\tilde{\epsilon} t y x - \tilde{\epsilon} t x y + \tilde{\epsilon} t x y, \tilde{\epsilon} t x y - \tilde{\epsilon} t x y + \tilde{\epsilon} t x y, \\
\tilde{\epsilon} t x^2 - 2 \tilde{\epsilon} t x z + \tilde{\epsilon} t x z, \tilde{\epsilon} t x^2 - 2 \tilde{\epsilon} t x z + \tilde{\epsilon} t x z. \\
\end{array}
\end{align*}
\]

which is a minimal basis of \( \mathcal{M}_2 = H(M_2) \subset \mathcal{A}^3 (\mathcal{A} = F/I \text{ where } F = \mathbb{Q}(x, y, z, t) \) and \( \mathcal{I} \) is the extension of \( I \) to \( F). \) Then, we consider the letterplace analogue \( \mathcal{M}_2 \subset \mathcal{S}^3 (\mathcal{S} = \mathcal{R}/\mathcal{J} \text{ where } \mathcal{R} \text{ corresponds to } F \text{ and } \mathcal{J} = L(I) \) which has the minimal basis

\[
\begin{align*}
\begin{array}{c}
\tilde{\epsilon}'_3 t_1 z_2 y_3 - 2 \tilde{\epsilon}'_3 t_1 y_2 z_3 + \tilde{\epsilon}'_2 t_1 z_2 z_3, \tilde{\epsilon}'_2 t_1 y_2 y_3 - 2 \tilde{\epsilon}'_2 t_1 z_2 y_3 + \tilde{\epsilon}'_2 t_1 y_2 z_3, \\
\tilde{\epsilon}'_3 t_1 z_2 x_3 - 2 \tilde{\epsilon}'_3 t_1 x_2 z_3 + \tilde{\epsilon}'_1 t_1 z_2 z_3, \tilde{\epsilon}'_2 t_1 z_2 x_3 - \tilde{\epsilon}'_2 t_1 x_2 y_3 + \tilde{\epsilon}'_1 t_1 z_2 y_3 - \tilde{\epsilon}'_2 t_1 x_2 z_3, \\
\tilde{\epsilon}'_3 t_1 y_2 x_3 - \tilde{\epsilon}'_2 t_1 y_2 z_3 - \tilde{\epsilon}'_2 t_1 x_2 z_3, \tilde{\epsilon}'_2 t_1 y_2 x_3 - \tilde{\epsilon}'_2 t_1 y_2 z_3 + \tilde{\epsilon}'_1 t_1 y_2 z_3, \tilde{\epsilon}'_2 t_1 y_2 x_3 - \tilde{\epsilon}'_2 t_1 y_2 z_3, \\
\tilde{\epsilon}'_3 t_1 x_2 x_3 - 2 \tilde{\epsilon}'_3 t_1 x_2 z_3 + \tilde{\epsilon}'_1 t_1 x_2 z_3, \tilde{\epsilon}'_3 t_1 x_2 x_3 - 2 \tilde{\epsilon}'_3 t_1 y_2 x_3 + \tilde{\epsilon}'_1 t_1 x_2 y_3. \\
\end{array}
\end{align*}
\]
In Section 9 we will show that one can algorithmically compute a minimal basis of the corresponding syzygy module $M'_3 \subset S^{-3}$, namely

$$
\begin{align*}
\hat{c}'_j x_k & , \hat{c}'_j y_k , \hat{c}'_j z_k , \hat{c}'_j t_k \\
(1 \leq j \leq 8, 1 \leq k \leq 3),
\end{align*}
$$

$$
\begin{align*}
\hat{c}_1 y_4 + \hat{c}'_3 z_4 + \hat{c}_5 x_4 + \hat{c}'_7 y_4 + \hat{c}_8 z_4 - 2 \hat{c}_4 z_4, \\
\hat{c}_5 x_4 + \hat{c}_7 z_4 + \hat{c}_8 x_4 + \hat{c}_8 y_4, \\
\hat{c}_1 z_4 x_5 - \hat{c}_3 z_4 y_5 - \hat{c}_1 x_4 z_5 - \hat{c}_4 z_4 z_5, \\
\hat{c}_2 z_4 x_5 - \hat{c}_3 y_4 y_5 + 2 \hat{c}_4 z_4 y_5 - \hat{c}_2 x_4 z_5 - \hat{c}_4 y_4 z_5, \\
\hat{c}_2 y_4 x_5 - 2 \hat{c}_2 x_4 y_5 + \hat{c}_3 y_4 y_5 - \hat{c}_1 y_4 y_5 + \hat{c}_4 z_4 y_5 - \hat{c}_4 y_4 z_5, \\
\hat{c}_3 y_4 y_5 - \hat{c}_5 z_4 x_5 - \hat{c}_3 x_4 y_5 + \hat{c}_5 x_4 z_5 + \hat{c}_8 z_4 z_5, \\
\hat{c}_1 y_4 x_5 - \hat{c}_6 z_4 x_5 + \hat{c}_5 x_4 y_5 + \hat{c}_4 z_4 y_5 - \hat{c}_4 y_4 z_5, \\
\hat{c}_4 x_4 z_5 + 2 \hat{c}_7 y_4 x_5 - \hat{c}_6 z_4 x_5 - \hat{c}_4 x_4 y_5 + \hat{c}_8 x_4 z_5, \\
\end{align*}
$$

By Theorem 7.3 one obtains that a minimal basis of the right syzygy module $\hat{M}_3 \subset A^{-3}$ of the minimal basis $[\hat{e}_j]$ is given by the following elements

$$
\begin{align*}
\hat{e}_1 y + \hat{e}_2 z, \hat{e}_1 x + \hat{e}_3 y + \hat{e}_5 z - 2 \hat{e}_4 z, \hat{e}_2 x - 2 \hat{e}_5 y + \hat{e}_4 y + \hat{e}_6 z, \\
\hat{e}_3 x + \hat{e}_7 z, \hat{e}_5 x + \hat{e}_4 x + \hat{e}_7 y + \hat{e}_8 z, \hat{e}_6 x + \hat{e}_8 y, \\
\hat{e}_1 x - \hat{e}_3 y - \hat{e}_1 x + 2 \hat{e}_3 y z - \hat{e}_4 z^2, \\
\hat{e}_2 x - \hat{e}_3 y^2 + 2 \hat{e}_4 y z - \hat{e}_2 x z - \hat{e}_4 y z, \\
\hat{e}_2 y x - 2 \hat{e}_2 x y + \hat{e}_5 y^2 - \hat{e}_4 y^2 + \hat{e}_6 y z - \hat{e}_6 y z, \\
\hat{e}_3 y x - \hat{e}_5 z x - \hat{e}_3 x y + 2 \hat{e}_5 x z + \hat{e}_8 z^2, \\
\hat{e}_4 y x - \hat{e}_6 z x + \hat{e}_5 x y - \hat{e}_4 x y + \hat{e}_8 y z - \hat{e}_6 y z, \\
\hat{e}_4 x^2 + 2 \hat{e}_7 y x - \hat{e}_8 x z - \hat{e}_7 x y + \hat{e}_8 x z.
\end{align*}
$$

From Theorem 8.3 it follows that $\hat{M}_3 = H(M_3)$ and therefore a minimal basis of $M_3$ is obtained simply by substituting in (8) the canonical basis $\{ \hat{e}_j \}$ of $A^{-3}$ with the canonical basis $\{ \hat{e}_j \}$ of $A^{-3}$.

For computing now a minimal basis of the right syzygy modules $M_4 \subset A^{-4}$ and $M_5 \subset A^{-5}$ of the minimal basis of $M_3$, by the same approach one has to consider the following minimal basis of a graded submodule $M'_3 \subset S^8$

$$
\begin{align*}
\hat{c}'_1 t_1 t_2 t_3 y_4 + \hat{c}'_2 t_1 t_2 t_3 z_4, & \hat{c}'_1 t_1 t_2 t_3 x_4 + \hat{c}'_2 t_1 t_2 t_3 y_4 + \hat{c}'_2 t_1 t_2 t_3 z_4 - 2 \hat{c}'_3 t_1 t_2 t_3 z_4, \\
\hat{c}'_2 t_1 t_2 t_3 x_4 - 2 \hat{c}'_3 t_1 t_2 t_3 y_4 + \hat{c}'_1 t_1 t_2 t_3 z_4 + \hat{c}'_2 t_1 t_2 t_3 x_4 + \hat{c}'_1 t_1 t_2 t_3 z_4, \\
\hat{c}'_1 t_1 t_2 t_3 x_4 + \hat{c}'_1 t_1 t_2 t_3 x_4 + \hat{c}'_1 t_1 t_2 t_3 y_4 + \hat{c}'_1 t_1 t_2 t_3 z_4, \\
\hat{c}'_1 t_1 t_2 t_3 x_4, \hat{c}'_1 t_1 t_2 t_3 y_4 - \hat{c}'_1 t_1 t_2 t_3 x_4 z_5 + 2 \hat{c}'_3 t_1 t_2 t_3 y_4 z_5 - \hat{c}'_1 t_1 t_2 t_3 y_4 z_5, \\
\hat{c}'_2 t_1 t_2 t_3 x_4 z_5 - \hat{c}'_3 t_1 t_2 t_3 y_4 z_5, \\
\hat{c}'_2 t_1 t_2 t_3 y_4 x_5 - 2 \hat{c}'_3 t_1 t_2 t_3 z_5, \hat{c}'_2 t_1 t_2 t_3 y_4 z_5 + \hat{c}'_1 t_1 t_2 t_3 z_5 + \hat{c}'_4 t_1 t_2 t_3 y_4 z_5 - \hat{c}'_4 t_1 t_2 t_3 x_4 z_5 + \hat{c}'_2 t_1 t_2 t_3 z_5 x_5 - \hat{c}'_3 t_1 t_2 t_3 y_4 z_5 - \hat{c}'_1 t_1 t_2 t_3 z_5 x_5 - \hat{c}'_4 t_1 t_2 t_3 y_4 z_5 - \hat{c}'_2 t_1 t_2 t_3 x_4 z_5 - \hat{c}'_3 t_1 t_2 t_3 y_4 z_5, \\
\hat{c}'_4 t_1 t_2 t_3 x_4 z_5 + 2 \hat{c}'_7 t_1 t_2 t_3 y_4 x_5 + \hat{c}'_2 t_1 t_2 t_3 z_5 x_5 - \hat{c}'_7 t_1 t_2 t_3 y_4 z_5 + \hat{c}'_6 t_1 t_2 t_3 x_4 z_5.
\end{align*}
$$
We can calculate that a minimal basis of the corresponding syzygy module $\tilde{M}_4' \subset \tilde{S}[4]^6 \oplus \tilde{S}[-5]^6$ is given by the following elements

$\begin{align*}
&\epsilon'_{j}x_k, \epsilon'_{j}y_k, \epsilon'_{j}z_k, \epsilon'_{j}t_k \ (1 \leq j \leq 6, 1 \leq k \leq 4), \\
&\epsilon'_{j}x_k, \epsilon'_{j}y_k, \epsilon'_{j}z_k, \epsilon'_{j}t_k \ (7 \leq j \leq 12, 1 \leq k \leq 5), \\
&\epsilon'_{9}x_6 + \epsilon'_{2}x_5y_6 - \epsilon'_{5}z_5y_6 + \epsilon'_{11}y_6 + \epsilon'_{6}y_5z_6, \epsilon'_{1}z_5x_6 - \epsilon'_{1}x_5z_6 - \epsilon'_{8}z_6, \\
&\epsilon'_{2}z_5x_6 - \epsilon'_{7}x_6 - \epsilon'_{4}z_5y_6 - \epsilon'_{2}x_5y_6 + 2\epsilon'_{4}y_5z_6 - \epsilon'_{5}z_5z_6 + \epsilon'_{10}z_6, \\
&\epsilon'_{3}z_5x_6 - \epsilon'_{9}x_6 - \epsilon'_{4}y_5y_6 + 2\epsilon'_{5}z_5y_6 - 2\epsilon'_{10}y_6 - \epsilon'_{3}x_5z_6 - \epsilon'_{5}y_5z_6 + \epsilon'_{6}z_5z_6 + \epsilon'_{1}z_6, \\
&\epsilon'_{1}y_5x_6 - 2\epsilon'_{1}x_5y_6 + \epsilon'_{2}y_5y_6 + \epsilon'_{3}z_5y_6 + \epsilon'_{8}y_6 - \epsilon'_{3}y_5z_6 - \epsilon'_{9}z_6, \\
&\epsilon'_{4}y_5x_6 - \epsilon'_{10}x_6 - \epsilon'_{4}x_5y_6 + \epsilon'_{5}x_5z_6 - \epsilon'_{12}z_6, \\
&\epsilon'_{5}y_5x_6 - 2\epsilon'_{6}z_5x_6 - \epsilon'_{11}x_6 - \epsilon'_{12}y_6 + \epsilon'_{6}x_5z_6, \\
&\epsilon'_{1}x_5z_6 - 2\epsilon'_{2}y_5x_6 - \epsilon'_{3}z_5x_6 - \epsilon'_{9}x_6 + \epsilon'_{2}x_5y_6 + \epsilon'_{10}y_6 - \epsilon'_{6}z_5z_6 - 2\epsilon'_{11}z_6.
\end{align*}$

(10)

We conclude that a minimal basis of $M_4$ is given by

$\begin{align*}
&\epsilon_9x + \epsilon_3xy - \epsilon_6zy + \epsilon_{11}y + \epsilon_6yz, \epsilon_1zx - \epsilon_7y - \epsilon_1xz - \epsilon_8z, \\
&\epsilon_2zx - \epsilon_7x - \epsilon_4zy - \epsilon_2xz + 2\epsilon_4yz - \epsilon_5z^2 + \epsilon_{10}z, \\
&\epsilon_3zx - \epsilon_4x^2 + 2\epsilon_5zy - 2\epsilon_{10}y - \epsilon_3zx - \epsilon_5yz + \epsilon_6z^2 + \epsilon_{11}z, \\
&\epsilon_1yx - 2\epsilon_1xy + \epsilon_2y^2 + \epsilon_3zy + \epsilon_8y - \epsilon_3yz - \epsilon_9z, \\
&\epsilon_4yx - \epsilon_{10}x - \epsilon_4xy + \epsilon_5xz - \epsilon_{12}z, \\
&\epsilon_5yx - 2\epsilon_6zx - \epsilon_{11}x - \epsilon_{12}y + \epsilon_6xz, \\
&\epsilon_1x^2 - 2\epsilon_2yx - \epsilon_3zx - \epsilon_8x + \epsilon_2xy + \epsilon_{10}y - \epsilon_6z^2 - 2\epsilon_{11}z.
\end{align*}$

(11)

By similar computations, we obtain a minimal basis of the right syzygy module $M_5 \subset A[-6]^8$ of the minimal basis (11) of $M_4$ which is

$\begin{align*}
&\epsilon_2yx - \epsilon_6zx - 2\epsilon_2xy + \epsilon_3y^2 + \epsilon_4zy + 2\epsilon_5xz + \epsilon_8yz + \epsilon_1z^2, \\
&\epsilon_2x^2 - 2\epsilon_3yx - \epsilon_6zx - \epsilon_4zx - \epsilon_3xy - \epsilon_6zy + \epsilon_8xz + 2\epsilon_6yz - \epsilon_7z^2, \\
&\epsilon_5x^2 + \epsilon_8yx + \epsilon_4yx + 2\epsilon_1zx - \epsilon_4xy + \epsilon_6y^2 - 2\epsilon_7zy - \epsilon_1xz + \epsilon_7yz.
\end{align*}$

(12)

Finally, a minimal basis of the right syzygy module $M_6 \subset A[-8]^3$ of the above minimal basis can be calculated as consisting of one single free element

$\epsilon_1x + \epsilon_2y - \epsilon_3z.$

(13)

All the computed minimal bases define therefore a finite minimal free right resolution of the base field $\mathbb{Q} \simeq A/M_1$ of the algebra $A$. Such resolution reads

$\begin{align*}
0 \leftarrow \mathbb{Q} \leftarrow A \leftarrow A[-1]^3 \leftarrow A[-3]^8 \leftarrow A[-4]^6 \oplus A[-5]^6 \leftarrow A[-6]^8 \\
\leftarrow A[-8]^3 \leftarrow A[-9] \leftarrow 0.
\end{align*}$

(14)
This exact sequence describes the homology of the graded algebra $A$ since one has that $b_i(M_i) = \dim_{\mathbb{Q}} \text{Tor}_{i,d}^A(\mathbb{Q}, \mathbb{Q})$ ($1 \leq i \leq 6, d \geq 0$). In particular, the corresponding graded Betti numbers table is

|   | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 3 | - | - | - | - | - |
| 1 | - | - | 8 | 6 | - | - | - |
| 2 | - | - | 6 | 8 | - | - | - |
| 3 | - | - | - | - | 3 | 1 | - |

Note that this is a standard way to represent such table. Recall that if a syzygy belongs to the kernel of $(i+1)$-th map in the resolution then $i$ is called the homological degree of the syzygy. Moreover, the (induced) degree of a syzygy is also called the internal degree and the slanted degree is by definition the difference between the internal and homological degree. In the above graded Betti numbers table the columns are then indexed by the homological degrees and the rows are indexed by the slanted degrees. By such table we conclude that the Castelnuovo-Mumford columns are then indexed by the homological degrees and the rows are indexed by the global (homological) dimension. Note that the finiteness of these numbers is due to the property of the algebra $A$ to have a PBW basis, that is, $A$ is a G-algebra [14]. Another reason for the right syzygy modules $M_i$ ($1 \leq i \leq 6$) to be finitely generated is that the ideal $I$ has a finite Gröbner basis [3] with respect to the graded lexicographic monomial ordering of $F$ and hence the corresponding Anick’s resolution [11] of $A = F/I$ consists of a finite number of chains for each homological degree. Note that such resolution is generally not of finite length or minimal but in fact in this case it coincides with the finite minimal resolution [14].

9. Finite computations

In this section we explain how the computation of noncommutative resolutions, as the one that we have just illustrated, can be obtained in an algorithmic way. Since our approach is to reduce such calculations to analogous ones for modules over polynomial algebras in commutative variables, the problem of being algorithmic essentially consists in working only with a finite number of such variables. We start by analyzing in general the amount of right syzygies that can be obtained with a finite number of letterplace variables. Then, we will show that there are some cases when a suitable large number of them provides the complete computation of a right syzygy module and iteratively of a finite number of such modules in a minimal right resolution.

Let $M' = L(M) \subset S'$ be the letterplace analogue of a finitely generated graded right submodule $M \subset A'$. Recall that $A = F/I$ and $S = R/J$ where $J = L(I)$. With the notation of Section 4, for all integers $d \geq 0$ we have that $P(d) = \mathbb{K}[x_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq d]$ and $R(d) = P(d)/Q(d)$ where $Q(d) = P(d) \cap Q$. Moreover, we put $J(d) = J \cap R(d)$ and $S(d) = R(d)/J(d)$. We finally define $M'(d) = M' \cap S(d)'$. Assume now that $M'$ is generated by $M'(d)$. If $\{g_j\} (1 \leq j \leq s)$ is a minimal basis of $M$ then this happens when the maximal degree in $\{g_j\}$ is bounded by $d$ and therefore the minimal basis $\{g'_j\}$ of $M' (g'_j = \iota(g_j))$ is in fact contained in $M'(d)$. Let $K$ be the right syzygy module of $\{g_j\}$ and let $K'$ be the corresponding syzygy module of $\{g'_j\}$. Put $\Delta_j = \deg(g_j) = \deg(g'_j) \leq d$. By Theorem [7,3] we know that $K' = C + L(K)$ where by definition $C = \langle c_{j,k}x_{kl} \mid 1 \leq j \leq s, 1 \leq k \leq n, 1 \leq l \leq \Delta_j \rangle$. 


By abuse of notation, we denote also by \{ε'_j\} the canonical basis of \(\bigoplus_j S(d)[−\Delta_j]\) and consider the graded right \(S(d)\)-module homomorphism

\[ ϕ'_j : \bigoplus_j S(d)[−\Delta_j] → S(d)^r, \ ε'_j → g'_j. \]

Define \(K'_id = \text{Ker} ϕ'_id\) the syzygy module of the minimal basis \(\{g'_j\}\) of \(M'(d)\). In other words, one has that \(K'_id = \{\sum_j ε'_j f'_j | f'_j ∈ S(d), \sum_j g'_j f'_j = 0\}\) and therefore

\[ K'_id = K' \cap \bigoplus_j S(d)[−\Delta_j]. \]

Note that \(C ⊂ K'_id\) since \(\Delta_j ≤ d\). By Theorem 7.3 one obtains immediately what follows.

**Proposition 9.1.** Let \(B ∪ \{h'_k\}\) be a minimal basis of \(K'_id\) where \(B\) is the given basis of the submodule \(C\). If \(h_k = r^{-1}(h'_k)\) then \(\{h_k\}\) is a minimal basis of \(K\) for all degrees \(≤ d\).

Since \(P(d)\) is a polynomial algebra in a finite number of commutative variables, observe that a minimal basis of the syzygy module \(K'_id\) can be computed algorithmically by any (commutative) computer algebra system (see, for instance, [3]). By iterating this method, it is then clear that one obtains the partial computation of a minimal free right resolution of \(N = A'/M\) up to degree \(d\).

We may ask now when \(K'\) is generated by \(K'_id\), that is, \(K\) is finitely generated in degrees \(≤ d\) and hence a complete minimal basis of \(K\) can be obtained by computing a minimal basis of \(K'_id\). This is a complicated issue because we have already observed that finitely presented noncommutative algebras are generally not right Noetherian (or even right coherent) and hence right syzygy modules are not always finitely generated.

Since the computation of syzygies is intimately related to the notion of Gröbner basis it is not surprising that to provide some cases when the right syzygy module is finitely generated we consider the monomial case. Let \(I ⊂ F\) be a finitely generated monomial two-sided ideal and consider the finitely presented monomial algebra \(A = F/I\). If \(M ⊂ \bigoplus_j A[−\delta_j]\) is a finitely generated monomial right submodule then we want to analyze the right syzygy module of a monomial basis of \(M\). For the sake of simplicity, we start with the simplest case, that is, \(M ⊂ A\) is a cyclic right ideal. Recall that \(W\) is the set of monomials of \(F\) and let \(w ∈ W \setminus I\). Assume that \(M\) is generated by the coset \(\bar{w} = w + I\). Put \(Δ = \text{deg}(\bar{w}) = \text{deg}(w)\) and consider the graded right \(A\)-module homomorphism

\[ ϕ : A[−Δ] → A, 1 ↦ \bar{w}. \]

If \(K = \text{Ker} \ ϕ\) then one has immediately that \(K = (I :_R w)/I\) where by definition

\[ (I :_R w) = \{f ∈ F | wf ∈ I\}. \]

The above set is a right ideal of \(F\) which is called the right colon ideal of \(I\) with respect to \(w\) (see, for instance, [13]). Because \(I\) is a monomial two-sided ideal and \(w ∈ W\), we have that \(I ⊂ (I :_R w)\) which is a monomial right ideal. Moreover, the assumption \(w ∉ I\) is equivalent to \((I :_R w) ≠ (1)\). A complete description of the right ideal \((I :_R w)\) is provided by the following result.
Proposition 9.2 ([11], Proposition 4.10). Let \{v_k\} \subset W be a monomial basis of I and put \(d_k = \deg(v_k)\). For all \(k\), we define the finitely generated monomial right ideal
\[
I_w(v_k) = \langle u_{kl} \mid wu_{kl} = t_{kl}v_k, t_{kl}, u_{kl} \in W, \deg(u_{kl}) < d_k \rangle.
\]
Then, one has that \((I :_R w) = \sum_k I_w(v_k) + I\).

Since we are assuming that \{v_k\} is a finite set and the right annihilator ideal \(K = \text{Ker} \varphi \subset A[-\Delta]\) of the coset \(\bar{w} = w + I \in A\) (\(\Delta = \deg(w)\)) is generated by the cosets modulo I of the elements of a monomial basis of \(\sum_k I_w(v_k)\), one obtains immediately what follows.

Lemma 9.3. The right ideal \(K \subset A[-\Delta]\) is finitely generated and monomial. Precisely, if \(d = \max\{d_k\}\) is the maximal degree of a finite monomial basis of I then the maximal (induced) degree of a minimal monomial basis of \(K\) is bounded by \(\Delta + d - 1\).

Let now \(M = \langle e_i \bar{w}_{ij} \rangle \subset \bigoplus_i A[-\delta_i]\) be a finitely generated monomial right submodule where \(\bar{w}_{ij} = w_{ij} + I\) and \(w_{ij} \in W \setminus I\). If \(\Delta_{ij} = \deg(e_i \bar{w}_{ij}) = \delta_i + \deg(w_{ij})\) and \(\{e_{ij}\}\) is the canonical basis of \(\bigoplus_{i,j} A[-\Delta_{ij}]\) then we consider the graded right \(A\)-module homomorphism
\[
\varphi : \bigoplus_{i,j} A[-\Delta_{ij}] \rightarrow \bigoplus_i A[-\delta_i], e_{ij} \mapsto e_i \bar{w}_{ij}.
\]
One has clearly that \(K = \text{Ker} \varphi = \bigoplus_{i,j} \epsilon_{ij}(I :_R w_{ij})/I\) which implies the following result.

Proposition 9.4. The right syzygy module \(K \subset \bigoplus_{i,j} A[-\Delta_{ij}]\) is finitely generated and monomial. Precisely, if \(d\) is the maximal degree of a finite monomial basis of I and \(\Delta = \max\{\Delta_{ij}\}\) then the maximal (induced) degree of a minimal monomial basis of \(K\) is bounded by \(\Delta + d - 1\).

By applying iteratively the above result we obtain what follows.

Theorem 9.5. Let \(A = F/I\) be a finitely presented monomial algebra and let \(M \subset \bigoplus_{1 \leq i \leq r} A[-\delta_i]\) be a finitely generated monomial submodule. There exists a minimal free right resolution of \(N = \bigoplus_{1 \leq i \leq r} A[-\delta_i]/M\) where all right syzygies modules \(M_i\) are finitely generated and monomial. Moreover, if \(b_k(I) = 0\) for \(k > d\) and \(b_k(M) = 0\) for \(k > \Delta\) then \(b_k(M_i) = 0\), for all \(k > \Delta + (i - 1)(d - 1)\). Note finally that such resolution is not necessarily of finite length.

A major application of the above result is when \(M = \langle x_1, \ldots, x_n \rangle \subset A\) is the augmentation ideal and hence \(b_k(M_i) = \dim_K \text{Tor}_k^A(K, K)\). In this case we have that \(\Delta = 1\) and an explicit combinatorial description of a minimal free right resolution of \(K \simeq A/M\) was introduced by Backelin [2] in terms of the monomial relations of \(A\). The Anick’s resolution is a (usually non-minimal) extension of that resolution to general algebras by means of a Gröbner basis of the ideal of relations. A by-product of these constructions is that the homology of monomial algebras bounds the homology of corresponding general algebras. Precisely, fix a monomial ordering of \(F\), that is, a multiplicatively compatible well-ordering of \(W\). If \(0 \neq f \in F\) then we denote by \(\text{lm}(f) \in W\) the greatest among the monomials of \(f\) with respect to such ordering. Let \(I \subset F\) be a two-sided ideal and define the monomial two-sided ideal \(\text{LM}(I) = \langle \text{lm}(f) \mid f \in I, f \neq 0 \rangle\). By definition, a Gröbner basis of \(I\) is a subset
\[ \{g_j\} \subset I \text{ such that } \{\text{Im}(g_j)\} \text{ is a (monomial) basis of } \text{LM}(I). \] Assume now that \( I \) is a graded ideal and consider the finitely generated graded algebras \( A = F/I \) and \( \hat{A} = F/\text{LM}(I) \).

**Theorem 9.6** ([1], Lemma 3.4). *For any homological degree \( i \) and internal degree \( k \), one has that \( \dim_{\mathbb{K}} \text{Tor}_{i,k}^A(\mathbb{K}, \mathbb{K}) \leq \dim_{\mathbb{K}} \text{Tor}_{i,k}^A(\mathbb{K}, \mathbb{K}) \).

The above result together with Theorem 9.5 provides that if \( I \) has a finite Gröbner basis, that is, \( \hat{A} \) is a finitely presented monomial algebra then \( A \) has a (possibly infinite) minimal free right resolution of the base field \( \mathbb{K} \) where all right syzygies modules \( M_i \) are finitely generated. These results imply also a bound \( d \) for the maximal degree in a finite number of minimal bases in the resolution. Then, by Proposition 9.1 we conclude that the proposed method is able to fully compute such bases by working with modules over the finitely generated (hence Noetherian) polynomial algebra \( P(d) \).

### 10. Tips and Tricks

Some additional tricks may be used to speedup the computation of a minimal right resolution by our approach. Especially for higher syzygies, one may have that the elements of the canonical basis of the free right module containing such syzygies have large degrees. Let us denote by \( d \) the minimum of these degrees. Owing to the module component homogenization we have clearly that the corresponding letterplace syzygies are all multiple of the monomial \( t_1 \cdots t_d \) and the letterplace variables \( x_{ij} \) occurring in these elements have the index \( j > d \). For instance, in the example of Section 8 one has that the letterplace encoding (after module component homogenization) of the minimal basis ([1]) of the right syzygy module \( M_4 \) is

\[
\begin{align*}
t_{1234}t_5^2(x_6 + x_5y_6 - x_6z_5y_6 + x_1t_5y_6 + x_6yz_5z_6), \\
t_{1234}(x_6z_5x_6 - x_5^2t_5y_6 - x_1t_5z_6 - x_4t_5z_6), \\
t_{1234}(x_4z_5x_6 - x_5^2t_5x_6 - x_4t_5y_6 - x_6z_5y_6 + 2x_2y_5z_6 - x_4z_5z_6 + x_1t_5z_6), \\
t_{1234}(x_4z_5x_6 - x_5^2t_5x_6 - x_4t_5y_6 + 2x_2y_5y_6 - 2x_4t_5y_6 - x_3t_5z_6 - x_5^2y_5z_6 + x_6z_5z_6 + x_1t_5z_6), \\
t_{1234}(x_4y_5x_6 + x_2y_5y_6 + x_3z_5y_6 + x_8t_5y_6 - x_3t_5z_6 - x_9t_5z_6), \\
t_{1234}(x_4y_5x_6 - x_4t_5x_6 - x_4t_5y_6 + x_5x_5z_6 - x_1t_5z_6), \\
t_{1234}(x_4y_5x_6 - 2x_2y_5x_6 - x_4t_5x_6 - x_4t_5y_6 - x_6z_5x_6 + x_1t_5x_6 + x_1t_5y_6 + x_6z_5z_6, \\
t_{1234}(x_1t_5x_6 - 2x_2y_5x_6 - x_4z_5x_6 - x_4t_5x_6 + x_2t_5y_6 + x_6z_5x_6 - x_6z_5z_6 - 2x_1t_5z_6).
\end{align*}
\]

For this example we have hence that \( d = 4 \). It is clear now that if we factor out the monomial \( t_1 \cdots t_d \) and shift back the letterplace variables by \( d \) then we obtain again a minimal basis whose syzygies imply the next right syzygies that one needs
for computing the resolution. For instance, the simplified minimal basis
\[
\begin{align*}
\tilde{c}_0 t_1 x_2 + \tilde{c}_4 x_1 y_2 - \tilde{c}_6^1 z_1 y_2 + \tilde{c}_1^1 t_1 y_2 + \tilde{c}_6^1 y_1 z_2, \\
\tilde{c}_4 z_1 x_2 - \tilde{c}_7 t_1 y_2 - \tilde{c}_1^1 x_1 z_2 - \tilde{c}_8^1 t_1 z_2, \\
\tilde{c}_2^1 z_1 x_2 - \tilde{c}_7 t_1 x_2 - \tilde{c}_4 z_1 y_2 - \tilde{c}_2^1 x_1 z_2 + 2\tilde{c}_4^1 y_1 z_2 - \tilde{c}_5^1 z_1 z_2 + \tilde{c}_1^1 t_1 z_2, \\
\tilde{c}_8^1 z_1 x_2 - \tilde{c}_8^1 t_1 x_2 - \tilde{c}_4^1 y_1 y_2 + 2\tilde{c}_4^1 z_1 y_2 - 2\tilde{c}_1^1 t_1 y_2 - \tilde{c}_5^1 z_1 z_2 - \tilde{c}_6^1 y_1 z_2 + \tilde{c}_6^1 z_1 z_2 \\
+ \tilde{c}_1^1 t_1 z_2, \\
\tilde{c}_1^1 y_1 x_2 - 2\tilde{c}_1^1 x_1 y_2 + \tilde{c}_4^1 y_1 y_2 + \tilde{c}_1^1 z_1 y_2 + \tilde{c}_4^1 t_1 y_2 - \tilde{c}_3^1 y_1 z_2 - \tilde{c}_6^1 t_1 z_2, \\
\tilde{c}_4^1 y_1 x_2 - \tilde{c}_1^1 t_1 x_2 - \tilde{c}_4^1 y_1 y_2 + \tilde{c}_5^1 x_1 z_2 - \tilde{c}_1^1 t_1 z_2, \\
\tilde{c}_5^1 y_1 x_2 - 2\tilde{c}_6^1 z_1 x_2 - \tilde{c}_1^1 t_1 x_2 - \tilde{c}_1^1 t_1 y_2 + \tilde{c}_6^1 t_1 z_2, \\
\tilde{c}_1^1 x_1 x_2 - 2\tilde{c}_2^1 y_1 x_2 - \tilde{c}_3^1 z_1 x_2 - \tilde{c}_8^1 t_1 x_2 + \tilde{c}_4^1 x_1 y_2 + \tilde{c}_1^1 t_1 y_2 - \tilde{c}_6^1 z_1 z_2 - 2\tilde{c}_1^1 t_1 z_2
\end{align*}
\]
has the following minimal syzygy basis
\[
\begin{align*}
\tilde{c}_j^1 x_k, \tilde{c}_j^1 y_k, \tilde{c}_j^1 z_k, \tilde{c}_j^1 t_k (1 \leq j \leq 8, 1 \leq k \leq 2), \\
\tilde{c}_2^1 y_3 x_4 - \tilde{c}_4^2 z_3 x_4 - 2\tilde{c}_2^1 x_3 y_4 + \tilde{c}_4^2 y_3 y_4 + \tilde{c}_4^2 z_3 y_4 + 2\tilde{c}_2^1 z_3 z_4 + \tilde{c}_4^2 y_3 z_4 + \tilde{c}_4^2 z_3 z_4, \\
\tilde{c}_4^2 x_3 x_4 - 2\tilde{c}_2^2 y_3 x_4 - \tilde{c}_4^2 z_3 x_4 - 2\tilde{c}_4^2 x_3 y_4 + \tilde{c}_2^2 y_3 y_4 + \tilde{c}_2^2 z_3 z_4 + \tilde{c}_4^2 x_3 x_4 \\
- \tilde{c}_4^2 z_3 x_4, \\
\tilde{c}_5^2 x_3 x_4 + \tilde{c}_8^2 y_3 x_4 + \tilde{c}_4^2 z_3 y_4 + 2\tilde{c}_1^2 z_3 x_4 - \tilde{c}_4^2 x_3 y_4 - \tilde{c}_6^2 y_3 y_4 - 2\tilde{c}_1^2 z_3 z_4 - \tilde{c}_4^2 x_3 z_4 + \tilde{c}_4^2 y_3 z_4.
\end{align*}
\]
It is clear that from these elements one obtains the minimal right syzygy basis (12). This trick generally speedups the computation because it reduces the number of needed letterplace variables which is usually fixed in advance in a computer algebra system. For instance, the minimal free right resolution of the example in Section 8 can be computed in 1 sec by applying this trick and just standard routines of Singular [3] for computing commutative syzygies. Note that this timing is in fact comparable with 0.1 sec that we have obtained with the optimized library PLURAL [11] for computations with G-algebras. We believe that this indicates that an effective implementation of our method would be feasible also for algebras which have no PBW basis, that is, that cannot be directly treated by computations over commutative variables. Note that the computational experiment has been performed on a laptop running Singular 4.0.3 with a four core Intel i3 at 2.20GHz and 16 GB RAM.

Since the computation of commutative syzygies which is needed in Proposition 9.1 is usually a by-product of a Gröbner basis calculation, another usable trick we may finally suggest is to use the monomial syzygies generating the submodule C as criteria to avoid useless S-polynomials. For more details about this idea to use syzygies as criteria we refer to [14].

11. Conclusions and future directions

In this paper we have shown that by extending the notion of letterplace correspondence to graded right modules over graded (noncommutative) algebras, one obtains a method for computing minimal free right resolutions of such modules. By bounding the degree of the right syzygies that one wants to compute in the
resolution, this method results in a feasible algorithm which requires only the computation of syzygies in (Noetherian) modules over finitely generated commutative algebras. Such calculations are provided by any (commutative) computer algebra system and hence our approach extends the availability of homological computations to general graded algebras which are not covered by ad hoc methods.

Of course, to improve the practical efficiency of the proposed algorithms it is necessary to develop an implementation of them in the kernel of a computer algebra system using, for instance, the tricks that we have explained in Section 10. As a further research direction, we can suggest to extend the letterplace correspondence to graded bimodules in order to obtain also Hochschild homology computations. Finally, we mention that free resolutions of nongraded modules are of course worthwhile of investigation and the letterplace correspondence, as proved in [9], can be suitably extended to the nongraded case.

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