Derivation of a transfer function model for a high pressure pipeline

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Abstract

In this report a lumped transfer function model for High Pressure Natural Gas Pipelines is derived. Starting with a partial nonlinear differential equation (PDE) model a high order continuous state space (SS) linear model is obtained using a finite difference method. Next, from the SS representation an infinite order transfer function (TF) model is calculated. In the end, this TF is approximated by a compact non-rational function.
1 Introduction

In this report we investigate the problem of the representation of a high pressure gas pipeline by a compact non rational transfer function model. This model is used to simulate mass flow and pressure in a small high pressure pipeline, and although this is a simple model with few parameters, it seems to have an accuracy comparable to the SIMONE\textsuperscript{\textregistered} simulator. Since this kind of models are suitable to control design and are well understood by control practitioners, it is our intention to apply them to gas leakage detection and gas network control.

2 STATE-SPACE DISCRETE-IN-SPACE MODEL

The gas dynamics within the pipes is represented by a set of partial differential equations (PDE). If we neglect the viscous and the turbulent effects of the flow and assume small temperature changes within the gas and small heat exchanges with the surroundings of the pipeline, it can be described by the one-dimensional hyperbolic model

\[
\begin{align*}
\frac{\partial q(\ell, t)}{\partial t} &= -A \frac{\partial p(\ell, t)}{\partial \ell} - \frac{f_c c^2 q^2(\ell, t)}{2DA}, \\
\frac{\partial p(\ell, t)}{\partial t} &= -\frac{c^2}{A} \frac{\partial q(\ell, t)}{\partial \ell},
\end{align*}
\]

where \( \ell \) is space, \( t \) is time, \( p \) is edge pressure-drop, \( q \) is mass flow, \( A \) is the cross-sectional area, \( D \) is the pipe diameter, \( c \) is the isothermal speed of sound, and \( f_c \) is the friction factor.

In this research we linearised model (1) around the operational levels \((p_m(\ell), q_m)\), where we assume a constant flow rate, and from the first equation of (1)

\[
p_m(\ell) = \sqrt{p_m^2(\ell_0) - \frac{f_c c^2}{2DA} q_m^2(\ell - \ell_0)}.
\]

Hence we set \( p(\ell, t) = p_m(\ell) + \Delta p(\ell, t) \) and \( q(\ell, t) = q_m + \Delta q(\ell, t) \), where \( \Delta p(\ell, t) \) and \( \Delta q(\ell, t) \) are deviations from the pressure/flow operational levels, respectively. Then

\[
\begin{align*}
\frac{q^2(\ell, t)}{p(\ell, t)} &= \frac{(q_m + \Delta q(\ell, t))^2}{p_m(\ell) + \Delta p(\ell, t)} = \\
&= \frac{q_m^2}{p_m(\ell)} + 2 \frac{q_m}{p_m(\ell)} \Delta q(\ell, t) - \frac{q_m^2}{p_m^2(\ell)} \Delta p(\ell, t).
\end{align*}
\]

The third term may be neglected since the distribution networks operate at very high pressure, ca. 80 bar. Then we substitute the remaining in the first equation

\[
\frac{\partial q(\ell, t)}{\partial t} = -A \frac{\partial p(\ell, t)}{\partial \ell} - \frac{f_c c^2 q_m}{2DA} (q_m + 2\Delta q(\ell, t)).
\]

Assuming small oscillations, \( \Delta q(\ell, t) \approx 2\Delta q(\ell, t) \), we may have \((q_m + 2\Delta q(\ell, t)) \approx q(\ell, t)\) and obtain the following linearized model:

\[
\begin{align*}
\frac{\partial q(\ell, t)}{\partial t} &= -A \frac{\partial p(\ell, t)}{\partial \ell} - 2\alpha q(\ell, t) \\
\frac{\partial p(\ell, t)}{\partial t} &= -\frac{c^2}{A} \frac{\partial q(\ell, t)}{\partial \ell},
\end{align*}
\]

(2)
\[
\alpha = \frac{f_c c^2}{4DA p_m}.
\] (3)

Next, decompose the pipeline into sections \( L_i = [\ell_{i-1}, \ell_i], \ i = 1, 2, \ldots, N, \) where \( \ell_0 = 0, \ell_N = L \) and \( L \) is the length of the pipeline. We assume the massflow to be the same in each section and accordingly define the following notation:

\[
q_0(t) = q(0, t) \\
q_i(t) = q(\ell, t), \ \ell_{i-1} < \ell < \ell_i, \ i = 1, 2, \ldots, N \\
q_{N+1}(t) = q(L, t) \\
p_i(t) = p(\ell_i, t), \ i = 0, 1, \ldots, N.
\] (4)

Making

\[
\left. \frac{\partial \cdot (\ell, t)}{\partial \ell} \right|_{\ell = \ell_i} \approx \frac{\cdot (\ell_i, t) - \cdot (\ell_{i-1}, t)}{\ell_i - \ell_{i-1}}, \ i = 1, 2, \ldots, N,
\] (5)

we can now approximate the linearized PDE (2) by the following discrete-in-space model

\[
\dot{q}_i(t) = A \frac{\Delta \ell}{c^2} [p_{i-1}(t) - p_i(t)] - 2\alpha q_i(t), \ i = 1, \ldots, N
\] (6)

\[
\dot{p}_{j-1}(t) = c^2 \frac{A \Delta \ell}{c^2} [q_{j-1}(t) - q_j(t)], \ j = 1, \ldots, N + 1.
\]

where

\[
\Delta \ell = \ell_{i+1} - \ell_i = \frac{L}{N}, \ i = 0, \ldots, N - 1.
\] (7)

The pipe can then be described by the following state-space model:

\[
\dot{x}_1(t) = -\frac{c^2}{A \Delta \ell} x_{N+2}(t) + \frac{c^2}{A \Delta \ell} u_1(t) \\
\dot{x}_i(t) = \frac{c^2}{A \Delta \ell} x_{N+i}(t) - \frac{c^2}{A \Delta \ell} x_{N+i+1}(t) \\
\dot{x}_{N+1}(t) = \frac{c^2}{A \Delta \ell} x_{2N+1}(t) - \frac{c^2}{A \Delta \ell} u_2(t) \\
\dot{x}_{N+1+j}(t) = A \frac{\Delta \ell}{c^2} x_j(t) - \frac{A}{\Delta \ell} x_{j+1}(t) - 2\alpha x_{N+1+j}(t) \\
y_1(t) = x_1(t) \\
y_2(t) = x_{N+1}(t),
\] (8)

where \( i = 1, \ldots, N, \ j = 1, \ldots N \) and also

\[
u(t) = [q_0(t) \quad q_{N+1}(t)]^T = [u_1(t) \quad u_2(t)]^T \\
x(t) = [p_0(t) \quad \cdots \quad p_N(t) \quad q_1(t) \quad \cdots \quad q_N(t)]^T \\
y(t) = [p_0(t) \quad p_N(t)]^T = [y_1(t) \quad y_2(t)]^T.
\] (9)
In matrix notation:

\[ \dot{x}(t) = Ax(t) + Bu(t) \]
\[ y(t) = Cx(t). \]  

(10)

Partition \( A \) as:

\[ A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \]

(11)

where

\[ A_{11} = 0_{(N+1) \times (N+1)} \]

(12)

\[ A_{12} = \begin{bmatrix} -\frac{c^2}{\Delta \ell} & 0 & \cdots & 0 & 0 \\ -\frac{c^2}{\Delta \ell} & -\frac{c^2}{\Delta \ell} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\frac{c^2}{\Delta \ell} & -\frac{c^2}{\Delta \ell} \\ 0 & 0 & \cdots & 0 & -\frac{c^2}{\Delta \ell} \end{bmatrix} \in \mathbb{R}^{(N+1) \times N} \]

(13)

\[ A_{21} = \begin{bmatrix} \frac{A}{\Delta \ell} & \frac{A}{\Delta \ell} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{A}{\Delta \ell} & \frac{A}{\Delta \ell} \end{bmatrix} \in \mathbb{R}^{N \times (N+1)} \]

(14)

\[ A_{22} = -2\alpha I_N \]

(15)

\[ B = \frac{c^2}{\Delta \ell} \begin{bmatrix} e_1 \\ -e_{N+1} \end{bmatrix} \]

(16)

\[ C = \begin{bmatrix} e_1 \\ e_{N+1} \end{bmatrix}^T \]

(17)

where \( e_i \) is the \( i^{th} \) vector of the canonical orthonormal basis, i.e., a vector with the \( i^{th} \) component equal to one and the others equal to zero.

### 3 Spectral analysis of \( A \)

In order to learn more about the system (3), we analyse the spectrum of matrix \( A \). Therefore the following theorem:

**Theorem 1** The eigenvalues of \( A \) defined in (12)-(15) are

\[ \lambda_0 = 0 \]

(18)

\[ \lambda_{\pm k} = -\frac{f_c c^2 Q_m}{4DA P_m} \pm j \sqrt{\left( 2 \frac{c}{\Delta \ell} \sin \left( \frac{k\pi}{2(N+1)} \right) \right)^2 - \left( \frac{f_c c^2 Q_m}{4DA P_m} \right)^2}, \ k = 1, \ldots, N \]

(19)
Proof: In the Appendix it was proven that matrix $\bar{A}$ is equivalent to $A$ up to a similarity transformation. Consequently they have the same eigenvalues and using (124), we have:

$$\det(sI_{2N+1} - A) = \det(sI_{2N+1} - \bar{A}) = s\det \begin{bmatrix} sI_N - \bar{A}_{11} & -\bar{A}_{12} \\ -\bar{A}_{21} & sI_N - \bar{A}_{22} \end{bmatrix} = 0$$

And this is equivalent to $s = 0$ and $\det \begin{bmatrix} sI_N - \bar{A}_{11} & -\bar{A}_{12} \\ -\bar{A}_{21} & sI_N - \bar{A}_{22} \end{bmatrix} = 0$. Therefore, $A$ has a zero eigenvalue, that is, $\lambda_0 = 0$.

From Fact 2.13.10 in [1, pp. 62–63], we have that for arbitrary matrices $A$, $B$, $C$ and $D \in \mathbb{R}^{N \times N}$ such that $AB = BA$ then

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(DA - CB).$$

Thus, if we take

$$A = sI_N - \bar{A}_{11}$$
$$B = -\bar{A}_{12}$$
$$C = -\bar{A}_{21}$$
$$D = sI_N - \bar{A}_{22}$$

we see that $(sI_N - \bar{A}_{11})(-\bar{A}_{12}) = (-\bar{A}_{12})(sI_N - \bar{A}_{11}) \Rightarrow AB = BA$ because $sI_N - \bar{A}_{11}$ is a diagonal matrix. Consequently

$$\det \begin{bmatrix} sI_N - \bar{A}_{11} & -\bar{A}_{12} \\ -\bar{A}_{21} & sI_N - \bar{A}_{22} \end{bmatrix} = \det((sI_N - \bar{A}_{22})(sI_N - \bar{A}_{11}) - \bar{A}_{21}\bar{A}_{12}).$$

Given that $\bar{A}_{11} = 0_{N \times N}$ and $\bar{A}_{22} = -2\alpha I_N$, then

$$\det \begin{bmatrix} sI_N - \bar{A}_{11} & -\bar{A}_{12} \\ -\bar{A}_{21} & sI_N - \bar{A}_{22} \end{bmatrix} = \det((s^2 + 2\alpha s)I_N - \bar{A}_{21}\bar{A}_{12})$$
$$= \det((s^2 + 2\alpha s + \alpha^2)I_N - \bar{A}_{21}\bar{A}_{12} - \alpha^2I_N) = \det((s + \alpha)^2I_N - \bar{A}_{21}\bar{A}_{12} - \alpha^2I_N).$$

If we define the following the change of variable:

$$S = (s + \alpha)^2$$

then we can write

$$\det \begin{bmatrix} sI_N - \bar{A}_{11} & -\bar{A}_{12} \\ -\bar{A}_{21} & sI_N - \bar{A}_{22} \end{bmatrix} = \det(SI_N - (\bar{A}_{21}\bar{A}_{12} + \alpha^2I_N))$$

(21)

From this equation, the eigenvalues of $\bar{A}_{21}\bar{A}_{12} + \alpha^2I_N$ that we denote by $\Lambda (\bar{A}_{21}\bar{A}_{12} + \alpha^2I_N)$, are the values of $S = (s + \alpha)^2$ that also set $\det(sI_N - A)$ to zero. From (21), $\Lambda (\bar{A}_{21}\bar{A}_{12} + \alpha^2I_N) = (\Lambda(A) + \alpha)^2$, where $\Lambda(A)$ denotes the non zero eigenvalues of $A$.

From the eigenvalues properties,

$$\Lambda (\bar{A}_{21}\bar{A}_{12} + \alpha^2I_N) = \Lambda (\bar{A}_{21}\bar{A}_{12}) + \alpha^2.$$  

(22)
The product $\bar{A}_{21}\bar{A}_{12}$ is

$$\bar{A}_{21}\bar{A}_{12} = \left( \frac{c}{\Delta \ell} \right)^2 \begin{bmatrix}
-2 & 1 & 0 & \cdots & 0 & 0 \\
1 & -2 & 1 & \cdots & 0 & 0 \\
0 & 1 & -2 & \ddots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ddots & -2 & 1 \\
0 & 0 & 0 & \cdots & 1 & -2
\end{bmatrix}.$$  

Using Fact 5.10.25 in [1, pp. 200]

$$\Lambda(\bar{A}_{21}\bar{A}_{12}) = -2 \left( \frac{c}{\Delta \ell} \right)^2 \left( 1 - \cos \left( \frac{k\pi}{N+1} \right) \right), \quad k = 1, \ldots, N \quad (23)$$

Then

$$\Lambda(\bar{A}_{21}\bar{A}_{12} + \alpha^2 I_N) = -2 \left( \frac{c}{\Delta \ell} \right)^2 \left( 1 - \cos \left( \frac{k\pi}{N+1} \right) \right) + \alpha^2, \quad k = 1, \ldots, N \quad (24)$$

are the values of $S = (s + \alpha)^2$ that set the characteristic equation of $A$ to zero.

Consequently

$$(\Lambda(A) + \alpha)^2 = \Lambda(\bar{A}_{21}\bar{A}_{12} + \alpha^2 I_N)$$

and from (20)–(24) the eigenvalues of $A$ are $\Lambda(A) = -\alpha \pm \sqrt{-2 \left( \frac{c}{\Delta \ell} \right)^2 \left( 1 - \cos \left( \frac{k\pi}{N+1} \right) \right) + \alpha^2}$.

That is

$$\Lambda(A) = -\alpha \pm j \sqrt{4 \left( \frac{c}{\Delta \ell} \right)^2 \sin^2 \left( \frac{k\pi}{2(N+1)} \right) - \alpha^2}.$$  

Recalling the definition of $\alpha$ in equation (3), we have:

$$\Lambda(A) = -\frac{f_c c^2 Q_m}{4DA P_m} \pm j \sqrt{\left( \frac{2 c}{\Delta \ell} \sin \left( \frac{k\pi}{2(N+1)} \right) \right)^2 - \left( \frac{f_c c^2 Q_m}{4DA P_m} \right)^2}, \quad k = 1, \ldots, N$$  

and this completes the proof.

The asymptotic case of the nonzero eigenvalues is reported in the following corollary

**Corollary 1** If $N \to \infty$ then the eigenvalues of $A$ are

$$\lambda_0 = 0 \quad (25)$$

$$\lambda_{\pm k} = -\frac{f_c c^2 Q_m}{4DA P_m} \pm j \sqrt{\left( \frac{k\pi}{T_d} \right)^2 - \left( \frac{f_c c^2 Q_m}{4DA P_m} \right)^2}, \quad k = 1, 2, \ldots \quad (26)$$
where $L$ is the pipe length and $T_d$ the time that a mass pressure takes to cross the pipeline, between its boundaries, at a constant speed $c$.

**Proof:** $\lim_{N \to \infty} \lambda_0 = 0$ is trivial

Since $\Delta \ell$ is given by

$$\Delta \ell = \frac{L}{N}$$

then

$$\frac{c}{\Delta \ell} \sin \left( \frac{k\pi}{2(N+1)} \right) = \frac{cN}{L} \sin \left( \frac{k\pi}{2(N+1)} \right).$$

Taking the limit when $N \to \infty$,

$$\lim_{N \to \infty} \frac{cN}{L} \sin \left( \frac{k\pi}{2(N+1)} \right) = \frac{c}{2L}k\pi$$

and, consequently,

$$\lim_{N \to \infty} \lambda_{\pm k} = -\frac{f c^2 Q_m}{4DAP_m} \pm j \sqrt{\left( \frac{c}{2L} k\pi \right)^2 - \left( \frac{f c^2 Q_m}{4DAP_m} \right)^2}, \ k = 1, 2, \ldots$$

and this completes the proof.

The zero eigenvalue means that there is an integrator in the pipeline model.

It has associated an eigenvector $v_0$, which defines a direction in the state-space where the pipeline behaves like a pure integrator. The following lemma gives the value of this eigenvector:

**Lemma 1** Consider $A$ as in (12)-(15). Then

$$v_0 = e_1 + e_2 + \cdots + e_{N+1}$$

where $e_i$ is the $i$th vector of the canonical orthonormal base in $\mathbb{R}^{2N+1}$. is the eigenvector associated to the zero eigenvalue, i.e., $Av_0 = 0$.

**Proof:** Let us denote $v_0$ as

$$v_0 = \begin{bmatrix} v \\ 0_N \end{bmatrix}$$

where $0_N$ is the zero vector in $\mathbb{R}^N$ and

$$v = e_1 + e_2 + \cdots + e_{N+1} \in \mathbb{R}^{N+1}$$

where $e_i$ is the $i$th vector of the canonical orthonormal basis in $\mathbb{R}^{N+1}$. Using (11) and (28)

$$Av_0 = \begin{bmatrix} A_{11}v \\ A_{21}v \end{bmatrix}$$

(30)
Given that $A_{11} = 0_{(N+1) \times (N+1)}$ then $A_{11}v = 0$. From (14) and (29)

$$A_{21}v = \begin{bmatrix}
\frac{A}{\Delta t} & \frac{A}{\Delta t} \\
\frac{A}{\Delta t} & \frac{A}{\Delta t} \\
\vdots & \\
\frac{A}{\Delta t} & \frac{A}{\Delta t}
\end{bmatrix} = 0_N$$

and this completes the proof.

\[\square\]

**Remark 1** In Lemma 1 to prove the existence of the eigenvalue associated to the zero eigenvalue we only used the submatrices $A_{11}$ and $A_{12}$. Therefore, we can say that this eigenvalue is generated by these submatrices. The nonlinearity of the model is only expressed by matrix $A_{22}$. For this reason, if we decompose the full nonlinear model into a linear subsystem in cascade with a nonlinear one, the zero eigenvalue would appear in the linear subsystem indicating the presence of an integrator in the full model. Also, $A_{11}$ and $A_{12}$ depend neither on $p_m$ nor on $q_m$.

### 4 Transfer functions characterisation

We determine the transfer function. Recall that the massflow at the boundaries were chosen to be our inputs and the pressure at the boundaries our outputs.

To start, we apply the Laplace transform to (10) and obtain:

$$Y(s) = C(sI - A)^{-1}BU(s)$$

where $Y(s) = \begin{bmatrix} Y_1(s) & Y_2(s) \end{bmatrix}^T$ and $U(s) = \begin{bmatrix} U_1(s) & U_2(s) \end{bmatrix}^T$, and $F(s)$ denotes the Laplace transform of $f(t)$. That is

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix}.$$ 

Also:

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} C_1(sI - A)^{-1}B_1 \\ C_2(sI - A)^{-1}B_2 \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix}$$

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} C_1(sI - A)^{-1}B_1 & C_1(sI - A)^{-1}B_2 \\ C_2(sI - A)^{-1}B_1 & C_2(sI - A)^{-1}B_2 \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix}$$

(31)
and also

\[
G_{11}(s) = \frac{Y_1(s)}{U_1(s)}_{U_2(s)=0} = \frac{P_0(s)}{Q_0(s)}_{Q_{N+1}(s)=0}
\]

\[
G_{22}(s) = \frac{Y_2(s)}{U_2(s)}_{U_1(s)=0} = \frac{P_N(s)}{Q_{N+1}(s)}_{Q_0(s)=0}
\]

\[
G_{12}(s) = \frac{Y_1(s)}{U_2(s)}_{U_1(s)=0} = \frac{P_0(s)}{Q_{N+1}(s)}_{Q_0(s)=0}
\]

\[
G_{21}(s) = \frac{Y_2(s)}{U_1(s)}_{U_2(s)=0} = \frac{P_N(s)}{Q_0(s)}_{Q_{N+1}(s)=0}
\]  \hspace{1cm} (32)

4.1 Transfer function \(G_{11}\)

When we select this transfer function, it means that we are interested in the transfer function between the pressure and massflow at the intake node, i.e. \(\frac{Y_1(s)}{U_1(s)}\) when \(U_2(s) = 0\) that is:

\[
G_{11}(s) = \frac{P_0(s)}{Q_0(s)}_{Q_1(s)=0} = \frac{Y_1(s)}{U_1(s)}_{U_2(s)=0}
\]

and hence:

\[
G_{11}(s) = C_1 (sI - A)^{-1} B_1,
\]  \hspace{1cm} (33)

where \(B_1\) and \(C_1\) are the first column and first row of \(B\) and \(C\) in (16)–(17), respectively. \(G_{11}(s)\) is a rational function whose poles are the eigenvalues of \(A\).

The following theorem states the zeros of this transfer function.

**Theorem 2** The zeros of \(G_{11}(s)\) are

\[
z_{\pm k} = -\frac{f_c c^2 Q_m}{4DAP_m} \pm j \sqrt{\left(\frac{c}{\Delta t} \sin \left(\frac{(2k - 1)\pi}{2(2N + 1)}\right)\right)^2 - \left(\frac{f_c c^2 Q_m}{4DAP_m}\right)^2}, \quad k = 1, \ldots, N
\]  \hspace{1cm} (34)

**Proof:** To do this we recall the result from [2, pp. 284], we have that the zeros of the transfer function \(G_{11}(s)\) are the zeros of the following polynomial

\[
\frac{sI_{(2N+1)} - A}{C_1} - B_1 = 0.
\]  \hspace{1cm} (35)
Recalling that $C_1 = e_1 \in \mathbb{R}^{(2N+1)}$ and $B_1 = \frac{c^2}{A\Delta \ell} e_1 \mathbb{R}^{(2N+1)}$ (see equations (16)–(17)), we have:

$$\left| \frac{sI_{(2N+1)}}{e_1} - A \right| = 0 \iff (36)$$

$$\iff \left| \begin{array}{ccc} sI_{N+1} - A_{11} & -A_{12} \\ -A_{21} & -A_{22} \end{array} \right| = 0 \iff (37)$$

using the definition of $A_{11}$ and $A_{22}$ in (12) and (15)

$$\iff \left| \begin{array}{ccc} sI_N & -A_{12} \\ -A_{21} & (s + 2\alpha)I_N \end{array} \right| = 0. \iff (38)$$

We develop this determinant first along the last column and next along the last row, and obtain:

$$\left| \begin{array}{cc} sI_N & -\bar{A}_{12} \\ -\bar{A}_{21} & (s + 2\alpha)I_N \end{array} \right| = 0 \iff (39)$$

where $\bar{A}_{12}$ denotes matrix $A_{12}$ without the 1st row and $\bar{A}_{21}$ denotes matrix $A_{21}$ without the 1st column.

Next, as $sI_N (\bar{A}_{12}) = (\bar{A}_{12}) sI_N$, we apply again Fact 2.13.10 in [1, pp. 62–63], which states that for the arbitrary matrices $A$, $B$, $C$ and $D \in \mathbb{R}^{N \times N}$ such that $AB = BA$ then

$$\det \left[ \begin{array}{cc} A & B \\ C & D \end{array} \right] = \det(DA - CB)$$

and

$$\left| \begin{array}{cc} sI_N & -\bar{A}_{12} \\ -\bar{A}_{21} & (s + 2\alpha)I_N \end{array} \right| = \left| s(s + 2\alpha)I_N - \bar{A}_{21}\bar{A}_{12} \right| \iff (40)$$

$$= \left| (s^2 + 2\alpha s + \alpha^2)I_N - (\bar{A}_{21}\bar{A}_{12} + \alpha^2) \right| \iff (41)$$

$$= \left| (s + \alpha)^2 I_N - (\bar{A}_{21}\bar{A}_{12} + \alpha^2) \right| \iff (42)$$

We do the same change of variable as before

$$S = (s + \alpha)^2 \iff (43)$$

and then can write:

$$\left| \begin{array}{cc} sI_N & -\bar{A}_{12} \\ -\bar{A}_{21} & (s + 2\alpha)I_N \end{array} \right| = \left| SI_N - (\bar{A}_{21}\bar{A}_{12} + \alpha^2) \right| \iff (44)$$

Next, we calculate the spectrum of matrix $(\bar{A}_{21}\bar{A}_{12} + \alpha^2)$, that is:

$$\Lambda(\bar{A}_{21}\bar{A}_{12} + \alpha^2) = \Lambda(\bar{A}_{21}\bar{A}_{12}) + \alpha^2$$
Now, we calculate the product:

\[
\tilde{A}_{21} \tilde{A}_{12} = \begin{bmatrix}
-\frac{A}{\Delta \ell} & 0 & \cdots & 0 & 0 \\
\frac{A}{\Delta \ell} & \frac{A}{\Delta \ell} & \cdots & 0 & 0 \\
0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \frac{A}{\Delta \ell} & -\frac{A}{\Delta \ell} & 0
\end{bmatrix} \begin{bmatrix}
\frac{c^2}{\Delta \ell} & -\frac{c^2}{\Delta \ell} & 0 & 0 & \cdots & 0 & 0 \\
0 & \frac{c^2}{\Delta \ell} & -\frac{c^2}{\Delta \ell} & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & \ddots & \ddots & \ddots \\
\end{bmatrix}
\]
then from (43):

\[ z_{\pm k} = -\alpha \pm j \sqrt{2 \left( \frac{c}{\Delta \ell} \right)^2 \left( 1 - \cos \left( \frac{(2k - 1)\pi}{2N + 1} \right) \right)} - \alpha^2 \]

\[ = -\frac{f_c c^2 Q_m}{4DAP_m} \pm j \sqrt{4 \left( \frac{c}{\Delta \ell} \right)^2 \sin^2 \left( \frac{(2k - 1)\pi}{2(2N + 1)} \right) - \left( \frac{f_c c^2 Q_m}{4DAP_m} \right)^2} \]

\[ = -\frac{f_c c^2 Q_m}{4DAP_m} \pm j \sqrt{\left( \frac{2}{\Delta \ell} \sin \left( \frac{(2k - 1)\pi}{2(2N + 1)} \right) \right)^2 - \left( \frac{f_c c^2 Q_m}{4DAP_m} \right)^2}, \quad k = 1, \ldots, N \]

and this completes the proof.

The following corollary resolves the asymptotic case.

**Corollary 2** If \( N \to \infty \) then the zero of \( G_{11}(s) \) are

\[ z_{\pm k} = -\frac{f_c c^2 Q_m}{4DAP_m} \pm j \sqrt{\left( \frac{(2k - 1)\pi}{2T_d} \right)^2 - \left( \frac{f_c c^2 Q_m}{4DAP_m} \right)^2}, \quad k = 1, 2, \ldots \] (49)

where \( L \) is the pipe length and \( T_d \) the time that a particle of gas takes to cross the pipeline between its boundaries, at a constant speed \( c \).

**Proof:** Since \( \Delta \ell \) is given by

\[ \Delta \ell = \frac{L}{N} \]

then

\[ \frac{c}{\Delta \ell} \sin \left( \frac{(2k - 1)\pi}{2(2N + 1)} \right) = \frac{c N}{L} \sin \left( \frac{(2k - 1)\pi}{2(2N + 1)} \right). \]

Taking the limit when \( N \to \infty \),

\[ \lim_{N \to \infty} \frac{c N}{L} \sin \left( \frac{(2k - 1)\pi}{2(2N + 1)} \right) = \lim_{N \to \infty} \frac{c N (2k - 1) \pi}{2(2N + 1) L} \lim_{N \to \infty} \frac{\sin \left( \frac{(2k - 1)\pi}{2(2N + 1)} \right)}{\frac{(2k - 1) \pi}{2(2N + 1)}} = \frac{c (2k - 1) \pi}{4L} \]

and, consequently,

\[ \lim_{N \to \infty} z_{\pm k} = -\frac{f_c c^2 Q_m}{4DAP_m} \pm j \sqrt{\left( \frac{c (2k - 1) \pi}{2L} \right)^2 - \left( \frac{f_c c^2 Q_m}{4DAP_m} \right)^2}, \quad k = 1, 2, \ldots \]

Given that \( T_d = \frac{L}{c} \) then

\[ \lim_{N \to \infty} z_{\pm k} = -\frac{f_c c^2 Q_m}{4DAP_m} \pm j \sqrt{\left( \frac{(2k - 1)\pi}{2T_d} \right)^2 - \left( \frac{f_c c^2 Q_m}{4DAP_m} \right)^2}, \quad k = 1, 2, \ldots \]

and this completes the proof.
Corollary 3 Transfer function $G_{11}$ has the following form:

$$G_{11} = \frac{K_G \prod_{k=1}^{\infty} \left( \frac{s^2}{z_k z_{-k}} + s \left( \frac{1}{z_k} + \frac{1}{z_{-k}} \right) + 1 \right)}{s \prod_{k=1}^{\infty} \left( \frac{s^2}{\lambda_k \lambda_{-k}} + s \left( \frac{1}{\lambda_k} + \frac{1}{\lambda_{-k}} \right) + 1 \right)}$$ (50)

where $z_k = \alpha + j\beta_k$ and $z_{-k} = \alpha - j\beta_k$, as well as $\lambda_k = \alpha + jb_k$ and $\lambda_{-k} = \alpha - jb_k$, as defined in Corollary 2 and $\lambda_i$ are defined in Corollary 1.

Proof: From Corollary 2 and Corollary 1 we can write:

$$G_{11} = \frac{K_G \prod_{k=1}^{\infty} \left( \frac{s}{-z_k} + 1 \right) e^{\frac{\beta_k}{2}} \left( \frac{s}{-z_{-k}} + 1 \right) e^{-\frac{\beta_k}{2}}}{s \prod_{k=1}^{\infty} \left( \frac{s}{-\lambda_k} + 1 \right) e^{\frac{b_k}{2}} \left( \frac{s}{-\lambda_{-k}} + 1 \right) e^{-\frac{b_k}{2}}}$$ (51)

and expression (50) follows immediately, after calculating the products:

$$\left( \frac{s}{-z_k} + 1 \right) \left( \frac{s}{-z_{-k}} + 1 \right) \text{ and } \left( \frac{s}{-\lambda_k} + 1 \right) \left( \frac{s}{-\lambda_{-k}} + 1 \right).$$

To complete the transfer function characterisation we need to compute the gain $K_G$.

Theorem 3 Consider $B_1$, the first column of $B$ defined in (16), written in the base

$$\{v_{-N}, \ldots, v_{-1}, v_0, v_1, \ldots, v_N\}$$

where $v_i, i = -N, \ldots, -1, 0, 1, \ldots, N$, are the eigenvectors of $A$. If $\vartheta_0$ is the component of $B_1$ along $v_0$ then

$$K_G = \vartheta_0.$$

Proof: Denote the zero eigenvalue of $A$ as $\lambda_0$ and $\lambda_i, i = -N, \ldots, -1, 1, \ldots, N$ the complex eigenvalues, i.e. $\lambda_{-i} = \lambda_i^*$ where $^*$ means the conjugate eigenvalue. Since all eigenvalues have multiplicity one there are $2N+1$ independent eigenvectors $v_i, i = -N, \ldots, N$, respectively associated to each eigenvalue $\lambda_i$. Thus $v_i$ and $(s - \lambda_i), i = -N, \ldots, -1, 0, 1, \ldots, N$ are, respectively, the eigenvectors and the eigenvalues of $sI - A$. Consequently,

$$(sI - A)^{-1} v_i = \frac{1}{s - \lambda_i} v_i$$ (52)

i.e., $\frac{1}{s - \lambda_i}, i = -N, \ldots, N$ are eigenvalues of $(sI - A)^{-1}$ associated to the eigenvectors $v_i$. Then we
can write:

\[
\Lambda = \begin{bmatrix}
\frac{1}{s - \lambda_{-N}} & 0 & \cdots & 0 & \cdots & 0 \\
0 & \frac{1}{s - \lambda_{-N+1}} & \cdots & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \cdots & \frac{1}{s - \lambda_{0}} & \cdots & 0 \\
\vdots & \vdots & \cdots & 0 & \ddots & 1 \\
\vdots & \vdots & \cdots & \cdots & \cdots & \frac{1}{s - \lambda_{N}}
\end{bmatrix}
\]  

(53)

Also define the similarity matrix \( T \), considering the \( 2N + 1 \) independent eigenvectors \( v_i \):

\[
T = \begin{bmatrix} v_{-N} & \cdots & v_{-1} & v_0 & v_1 & \cdots & v_N \end{bmatrix}
\]  

(54)

And we can write:

\[
(SI - A)^{-1} = T\Lambda T^{-1}.
\]  

(55)

If we decompose \( B_1 \) into directions \( v_i \), i. e.,

\[
B_1 = \partial_{-N}v_{-N} + \cdots + \partial_{-1}v_{-1} + \partial_0v_0 + \partial_1v_1 + \cdots + \partial_Nv_N = T\begin{bmatrix} \partial_{-N} \\ \vdots \\ \partial_N \end{bmatrix}
\]  

(56)

then we can express the transfer function \( G_{11}(s) \) as

\[
G_{11}(s) = C_1(sI - A)^{-1}B_1 = C_1T\Lambda T^{-1} \begin{bmatrix} \partial_{-N} \\ \vdots \\ \partial_N \end{bmatrix} = C_1T\Lambda \begin{bmatrix} \partial_{-N} \\ \vdots \\ \partial_N \end{bmatrix} = C_1 \left( \frac{\partial_{-N}}{s - \lambda_{-N}}v_{-N} + \cdots + \frac{\partial_{-1}}{s - \lambda_{-1}}v_{-1} + \frac{\partial_0}{s}v_0 + \frac{\partial_1}{s - \lambda_1}v_1 + \cdots + \frac{\partial_N}{s - \lambda_N}v_N \right).
\]  

(57)

After multiplying (57) by \( s \) one obtains:

\[
K_G = C_1 \lim_{s \to 0} \left( \frac{\partial_{-N}s}{s - \lambda_{-N}}v_{-N} + \cdots + \frac{\partial_{-1}s}{s - \lambda_{-1}}v_{-1} + \frac{\partial_0s}{s}v_0 + \frac{\partial_1s}{s - \lambda_1}v_1 + \cdots + \frac{\partial_Ns}{s - \lambda_N}v_N \right) = \partial_0C_1v_0 = \partial_0,
\]

because \( C_1 = e_1^T \) and \( v_0 = e_1 + e_2 + \cdots + e_{N+1} \). If we knew all the eigenvectors we could straightforwardly determine \( \partial_0 \). But, only \( v_0 \) is known and it is not so immediate to compute \( \partial_0 \). The next lemma is of good help to solve this problem.

**Lemma 2**: If \( A \in \mathbb{R}^{n \times n} \) is a singular matrix, \( v_0 \) its eigenvector associated to the zero eigenvalue and \( A^Tv_0 = 0 \), then \( v_0 \) is orthogonal to the remaining eigenvectors \( v_i \), \( i \neq 0 \), of \( A \).

**Proof**: Let \( v_i \) with \( i \neq 0 \) the eigenvector of \( A \) associated to the eigenvalue \( \lambda_i \neq 0 \). By the eigenvector definition

\[
Av_i = \lambda_i v_i \Rightarrow v_i = \frac{1}{\lambda_i} Av_i.
\]  

(58)
Now, using this equation, we compute the internal product between $v_i$ and $v_0$,
\[
\langle v_i, v_0 \rangle = v_i^T v_0 = \left( \frac{1}{\lambda_i} A v_i \right)^T v_0 = \frac{1}{\lambda_i} v_i^T A^T v_0 = 0
\]  
from which conclude that $v_0$ and $v_i$ are orthogonal.

\[\square\]

**Corollary 4** Consider $A$ as defined in \([12] - [15]\). Its eigenvectors $v_i$, $i = -N, \ldots, 1, 1, \ldots, N$, are orthogonal to $v_0$.

**Proof:** Recalling the definitions of $v_0$ and $A_{11}$ in equations \([28]\) and \([12]\), respectively, then:
\[
A^T v_0 = \begin{bmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T & A_{22}^T \end{bmatrix} v_0 = \begin{bmatrix} 0_{(N+1) \times (N+1)} \\ A_{12}^T v \end{bmatrix}
\]

Computing
\[
A_{12}^T v = \begin{bmatrix} -\frac{c^2}{\Delta \ell} & 0 & 0 & \cdots & 0 & 0 \\ \frac{c^2}{\Delta \ell} - \frac{c^2}{\Delta \ell} & \ddots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{c^2}{\Delta \ell} - \frac{c^2}{\Delta \ell} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ \vdots \\ \vdots \\ 1 \\ 1 \end{bmatrix}
\]

\[
= \begin{bmatrix} -\frac{\Delta \ell}{c^2} & \frac{\Delta \ell}{c^2} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{\Delta \ell}{c^2} & \frac{\Delta \ell}{c^2} & \cdots & \vdots & \vdots \\ \vdots & 0 & -\frac{\Delta \ell}{c^2} & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{\Delta \ell}{c^2} & \frac{\Delta \ell}{c^2} \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ \vdots \\ \vdots \\ 1 \\ 1 \end{bmatrix}
\]

\[
= \begin{bmatrix} \frac{\Delta \ell}{c^2} - \frac{\Delta \ell}{c^2} \\ \frac{\Delta \ell}{c^2} - \frac{\Delta \ell}{c^2} \\ \vdots \\ \frac{\Delta \ell}{c^2} - \frac{\Delta \ell}{c^2} \end{bmatrix} = 0,
\]
Then by Lemma 2 we have the expected result.

**Corollary 5** \( K_G = \frac{c^2}{AL} \).

**Proof:** Decompose \( B_1 \) as

\[
B_1 = P_{v_0} B_1 + P_{v_0}^\perp B_1
\]

where \( P_w \) is the orthogonal projection into \( w \) operator and \( w^\perp \) denotes the orthogonal complement of \( w \).

From the orthogonality condition between \( v_0 \) and \( v_i, i = -N, \ldots, -1, 1, \ldots, N \),

\[
P_{v_0} B_1 = \vartheta_0 v_0.
\]

Given that

\[
P_{v_0} B_1 = (v_0^T v_0)^{-1} v_0^T B_1 v_0,
\]

then

\[
\vartheta_0 = (v_0^T v_0)^{-1} v_0^T B_1 = \frac{c^2}{\mathcal{A}(N + 1) \Delta \ell}.
\]

Replacing \( \Delta \ell \) by \( \frac{L}{N} \) we find

\[
\vartheta_0 = \frac{c^2 N}{\mathcal{A}(N + 1) L}.
\]

When \( N \to \infty \Rightarrow \frac{N}{N + 1} \to 1 \),

\[
\vartheta_0 = \frac{c^2}{AL}.
\]

and \( K_G = \vartheta_0 = \frac{c^2}{AL} \).

**Corollary 6** Transfer function \( G_{11} \) has the following form, according to Corollary 3:

\[
G_{11} = \frac{c^2}{AL} \prod_{k=1}^{\infty} \left( \frac{s^2}{z_k z_{-k}} + s \left( \frac{1}{z_k} + \frac{1}{z_{-k}} \right) + 1 \right)
\]

where \( z_k \) are defined in Corollary 2 and \( \lambda_k \) are defined in Corollary 7

**4.2 Transfer function** \( G_{22} \)

Next, we determine the transfer function

\[
G_{22} = \frac{Y_2(s)}{U_2(s)} = \frac{P_N(s)}{Q_N(s)}
\]
when \( U_1(s) = 0 \). That is the ratio between the pressure and the massflow at the offtake node. From (31):

\[
G_{22}(s) = C_2 (s I - A)^{-1} B_2,
\]

where \( C_2 \) is the second column of \( C \) and \( B_2 \) is the second row of \( B \).

Similarly to what happens with \( G_{11}(s) \), \( G_{22}(s) \) is a rational function whose poles are the eigenvalues of \( A \). Following the same methodology as for \( G_{11}(s) \), we would like to calculate the zeros of \( G_{22}(s) \) in order to investigate pole-zero cancelations.

**Theorem 4** The zeros of \( G_{22}(s) \) are

\[
z_k = -\frac{f c^2 Q_m}{4 D_A P_m} \pm j \sqrt{\left(\frac{2 c}{\Delta \ell} \sin \left(\frac{(2k - 1) \pi}{2N + 1}\right)\right)^2 - \left(\frac{f c^2 Q_m}{4 D_A P_m}\right)^2}, \quad k = 1, \ldots, N
\]

**Proof:** The proof is very similar to the one of Theorem 2. Again, we recall the result from [2, pp. 284] that states that the zeros of the transfer function (67) are the zeros of the following polynomial:

\[
\left|\begin{array}{c|c}
 sI(2N + 1) - A & -B_2 \\
 C_2 & 0 \\
\end{array}\right| = 0
\]

(70)

Recall the definitions of \( C_2 = e_{N+1} \in \mathbb{R}^{(2N+1)} \) and \( B_2 = -\frac{c^2}{A \Delta \ell} e_{N+1} \mathbb{R}^{(2N+1)} \) and the proof follows exactly as for Theorem 2.

We have:

\[
\left|\begin{array}{c|c}
 sI(2N + 1) - A & e_{N+1} \\
 e_{N+1}^T & 0 \\
\end{array}\right| = 0 \quad \Leftrightarrow \\
\left|\begin{array}{c|c|c|c|}
 sI_{N+1} - A_{11} & -A_{12} & 0 & \vdots \\
 -A_{21} & sI - A_{22} & 0 & \vdots \\
 0 & \vdots & 1 & \vdots \\
 0 & \vdots & 0 & 1 \\
\end{array}\right| = 0 \\
\Leftrightarrow \\
\left|\begin{array}{c|c|c|c|}
 sI_{N+1} & -A_{12} & 0 & \vdots \\
 -A_{21} & (s + 2\alpha) I_N & 0 & \vdots \\
 0 & \vdots & 1 & \vdots \\
 0 & \vdots & 0 & 1 \\
\end{array}\right| = 0.
\]

(72)
We develop this determinant first along the last column and next along the last row, and obtain:

\[
\begin{vmatrix}
  sI_N & -\bar{A}_{12} \\
  -\bar{A}_{21} & (s + 2\alpha)I_N
\end{vmatrix} = 0
\]

(75)

where \( \bar{A}_{12} \) denotes matrix \( A_{12} \) without the 1st column and \( \bar{A}_{21} \) denotes matrix \( A_{21} \) without the 1st row. Next, as \( sI_N (\bar{A}_{12}) = (\bar{A}_{12}) sI_N \), we apply again Fact 2.13.10 in [1, pp. 62–63], that states that for the arbitrary matrices \( A, B, C \) and \( D \in \mathbb{R}^{N \times N} \) such that \( AB = BA \) then

\[
\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(\text{DA} - \text{CB})
\]

and

\[
\begin{vmatrix}
  sI_N & -\bar{A}_{12} \\
  -\bar{A}_{21} & (s + 2\alpha)I_N
\end{vmatrix} = |s(s + 2\alpha)I_N - \bar{A}_{21}\bar{A}_{12}|
\]

(76)

\[
= |(s^2 + 2\alpha s + \alpha^2)I_N - (\bar{A}_{21}\bar{A}_{12} + \alpha^2)|
\]

(77)

\[
= |(s + \alpha)^2 I_N - (\bar{A}_{21}\bar{A}_{12} + \alpha^2)|.
\]

(78)

We do the usual change of variable

\[
S = (s + \alpha)^2
\]

(79)

and then can write:

\[
= |SI_N - (\bar{A}_{21}\bar{A}_{12} + \alpha^2)|
\]

(80)

Next, we calculate the spectrum of matrix \((\bar{A}_{21}\bar{A}_{12} + \alpha^2)\), that is:

\[
\Lambda (\bar{A}_{21}\bar{A}_{12} + \alpha^2) = \Lambda (\bar{A}_{21}\bar{A}_{12}) + \alpha^2
\]

Now, we calculate the product:

\[
\bar{A}_{21}\bar{A}_{12} = \begin{bmatrix}
-\frac{A}{\Delta \ell} & 0 & \ldots & 0 & 0 \\
\frac{A}{\Delta \ell} & -\frac{A}{\Delta \ell} & \ldots & \vdots & \vdots \\
0 & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \frac{A}{\Delta \ell} & -\frac{A}{\Delta \ell} & \vdots \\
0 & 0 & 0 & 0 & \frac{A}{\Delta \ell}
\end{bmatrix}
\begin{bmatrix}
\frac{c^2}{\Delta \ell} & -\frac{c^2}{\Delta \ell} & 0 & 0 & \ldots & 0 & 0 \\
0 & \frac{c^2}{\Delta \ell} & -\frac{c^2}{\Delta \ell} & 0 & \ldots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \frac{c^2}{\Delta \ell} & 0 & \frac{c^2}{\Delta \ell} & -\frac{c^2}{\Delta \ell}
\end{bmatrix}
\]

\[
= \left(\frac{c}{\Delta \ell}\right)^2 
\begin{bmatrix}
-2 & 1 & 0 & \ldots & 0 & 0 \\
1 & -2 & 1 & \ldots & 0 & 0 \\
0 & 1 & -2 & \ddots & \ddots & \ddots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \ddots & -2 & 0 \\
0 & 0 & 0 & \ldots & 1 & -1
\end{bmatrix}
\in \mathbb{R}^N
\]


Then

\[
|sI_N - \tilde{A}_{21}\tilde{A}_{12}| = (\frac{c}{\Delta_\ell})^2 \Lambda(M_N)
\]

where

\[
M_N = \begin{bmatrix}
-2 & 1 & 0 & \cdots & 0 & 0 \\
1 & -2 & 1 & \cdots & 0 & 0 \\
0 & 1 & -2 & \ddots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ddots & -2 & 1 \\
0 & 0 & 0 & \cdots & 1 & -1
\end{bmatrix} \in \mathbb{R}^{N \times N}.
\] (81)

From [4, pp. 72]

\[
\Lambda(M_N) = -2 + 2 \cos \left(\frac{(2k-1)\pi}{2N+1}\right), \quad k = 1, 2, 3, \ldots, N.
\] (82)

and from here the proof follows exactly as for the calculus of the zeros of \(G_{11}(s)\), and we can see that are the same.

Likewise follows for the asymptotic case. As we can see from the definition of the transfer function (33) and (68) as well as from the definition of the \(B_i, C_i, i = 1, 2\) we have

\[
G_{11}(s) = -G_{22}(s)
\] (83)

Therefore, its zeros will be necessarily coincident.

\[\square\]

### 4.3 Transfer function \(G_{12}\)

According to (31), consider now the transfer function

\[
G_{12}(s) = \frac{Y_1(s)}{U_2(s)} = \frac{P_0(s)}{Q_{N+1}(s)}
\] (84)

with \(U_1(s) = Q_0(s) = 0\), or equivalently:

\[
G_{12}(s) = C_1(sI - A)^{-1}B_2,
\] (85)

where \(C_1\) is the first column of \(C\) and \(B_2\) is the second row of \(B\).

**Theorem 5** *The transfer function \(G_{12}(s)\) has no zeros.*

**Proof:** According to [2, pp. 284], we have that the zeros of the transfer function (84) are the zeros of the following polynomial:

\[
\begin{vmatrix}
    sI_{(2N+1)} - A & -B_2 \\
    C_1 & 0
\end{vmatrix}
\]
and recalling that $C_1 = e_1 \in \mathbb{R}^{(2N+1)}$ and $B_2 = -e_{N+1} \mathbb{R}^{(2N+1)}$, we have:

$$\left| sI_{(2N+1)} - A \begin{pmatrix} e_{N+1} \\ e_1 \end{pmatrix} \right| = 0 \iff$$

$$\left| \begin{array}{cc} sI_{N+1} - A_{11} & -A_{12} \\ -A_{21} & sI_N - A_{22} \end{array} \right| = 0 \iff$$

(86)

And from the definition of $A_{11}$ and $A_{22}$ in (12) and (15)

(87)

We develop this determinant first along the last column and next along the last row, and obtain:

$$\left| \begin{array}{ccccc} 0 & 0 & 0 & \cdots & 0 \\ s & 0 & 0 & \cdots & 0 \\ 0 & s & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & s \\ \varrho & 0 & 0 & \cdots & 0 \\ -\varrho & \varrho & 0 & \cdots & 0 \\ 0 & -\varrho & \varrho & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & -\varrho \\ -\varrho & \varrho & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{array} \right| = 0 \iff$$

(88)

$$\gamma = -\frac{c^2}{A\Delta \ell} \quad \text{and} \quad \varrho = -\frac{A}{\Delta \ell}.$$
We don’t worry about the signs of the cofactor, since our aim is to determine the zeros of the determinant. Now, we develop this determinant first along the first line, and we obtain:

\[
\begin{vmatrix}
  s & 0 & 0 & \cdots & 0 & 0 & 0 \\
  0 & s & 0 & \cdots & \vdots & \vdots & \vdots \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
  \vdots & \vdots & \vdots & s & 0 & 0 & 0 \\
  0 & 0 & 0 & \cdots & 0 & s & 0 \\
  \gamma & 0 & \cdots & 0 & 0 & 0 & 0 \\
\end{vmatrix}
= \begin{vmatrix}
  \gamma & 0 & \cdots & 0 & 0 & 0 & 0 \\
  0 & \gamma & \cdots & \vdots & \vdots & \vdots & \vdots \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
  \vdots & \vdots & \vdots & -\gamma & \gamma & 0 & 0 \\
  0 & 0 & 0 & \cdots & 0 & -\gamma & \gamma \\
  (s + 2\alpha) & 0 & \cdots & \vdots & \vdots & \vdots & \vdots \\
\end{vmatrix}
\]

Next, we develop along column–N:

\[
\begin{vmatrix}
  s & 0 & 0 & \cdots & 0 & 0 \\
  0 & s & 0 & \cdots & \vdots & \vdots \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
  \vdots & \vdots & \vdots & s & 0 & 0 \\
  0 & 0 & 0 & \cdots & 0 & s \\
  \gamma & 0 & \cdots & 0 & 0 & 0 \\
\end{vmatrix}
= \begin{vmatrix}
  \gamma & 0 & \cdots & 0 & 0 & 0 \\
  0 & \gamma & \cdots & \vdots & \vdots & \vdots \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
  \vdots & \vdots & \vdots & -\gamma & \gamma & 0 \\
  0 & 0 & 0 & \cdots & 0 & -\gamma & \gamma \\
  (s + 2\alpha) & 0 & \cdots & \vdots & \vdots & \vdots \\
\end{vmatrix}
\]

Here the matrix is of dimension \(2(N - 1) \times 2(N - 1)\).

Swap the first row of blocks with the second one:
Swap the first column of blocks with the second one:

\[
\begin{vmatrix}
\begin{array}{cccccc|cccccc}
0 & 0 & \cdots & 0 & 0 & 0 & \phi & 0 & \cdots & 0 & 0 \\
(s+2\alpha) & 0 & \cdots & 0 & 0 & 0 & -\phi & \phi & \cdots & 0 & 0 \\
0 & (s+2\alpha) & \cdots & 0 & 0 & 0 & 0 & -\phi & \phi & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & (s+2\alpha) & 0 & 0 & 0 & \cdots & -\phi & \phi \\
\end{array}
\end{vmatrix}
= 0
\]

Again develop the determinant along the first row:

\[
\begin{vmatrix}
\begin{array}{cccccc|cccccc}
(s+2\alpha) & 0 & \cdots & 0 & 0 & 0 & \phi & 0 & \cdots & 0 & 0 \\
0 & (s+2\alpha) & \cdots & 0 & 0 & 0 & -\phi & \phi & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & (s+2\alpha) & 0 & 0 & 0 & \cdots & -\phi & \phi \\
\end{array}
\end{vmatrix}
= 0
\]

Again along column–\((N-1)\)

\[
\begin{vmatrix}
\begin{array}{cccccc|cccccc}
(s+2\alpha) & 0 & \cdots & 0 & 0 & 0 & \phi & 0 & \cdots & 0 & 0 \\
0 & (s+2\alpha) & \cdots & 0 & 0 & 0 & -\phi & \phi & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & (s+2\alpha) & 0 & 0 & 0 & \cdots & -\phi & \phi \\
\end{array}
\end{vmatrix}
= 0
\]

Therefore, we find a pattern. To write the pattern, we define:

\[
\begin{align*}
\tilde{I}_N &= \text{identity matrix whose first row and column–}\(N\) are all zeros} \\
\tilde{A}_{12} &= \text{matrix } A_{12} \text{ without the last row} \\
\tilde{A}_{12,N} &= \text{matrix of order } N \text{ and with the same pattern as } \tilde{A}_{12} \\
\tilde{A}_{21} &= \text{matrix } A_{21} \text{ without the first column} \\
\tilde{A}_{21,N} &= \text{matrix of order } N \text{ and with the same pattern as } \tilde{A}_{21}
\end{align*}
\]
with this notation, we can write the determinant (88) as:

\[
\begin{vmatrix}
  s\tilde{I}_{N-i} & A_{12N-i} \\
  \tilde{A}_{21N-i} & (s + 2\alpha)I_{N-i}
\end{vmatrix}
\]

with \( i = 0 \), it becomes:

\[
\begin{vmatrix}
  s\tilde{I}_{N-i} & A_{12N-i} \\
  \tilde{A}_{21N-i} & (s + 2\alpha)I_{N-i}
\end{vmatrix}
\]

Define an iteration as:

1. Develop the determinant in cofactors along the first row
2. Develop the determinant in cofactors along column \((N - i)\)
3. Switch the first row of blocks with the second one
4. Switch the first column of blocks with the second one

Then, we obtain:

\[
\begin{vmatrix}
  \gamma_{i+1} & (s + 2\alpha)I_{N-(i+1)} \\
  \tilde{A}_{12N-(i+1)} & sI_{N-(i+1)}
\end{vmatrix}
\]

Iterate again and obtain:

\[
\begin{vmatrix}
  \gamma_{i+2} & s\tilde{I}_{N-(i+2)} \\
  \tilde{A}_{12N-(i+2)} & (s + 2\alpha)I_{N-(i+2)}
\end{vmatrix}
\]

Also, considering \( N = 1 \) we write (88) as:

\[
\begin{vmatrix}
  s & 0 & -\gamma & 0 \\
  0 & s & \gamma & 1 \\
  \theta & -\theta & (s - \alpha) & 0 \\
  1 & 0 & 0 & 0
\end{vmatrix}
\]

Similarly to what we have done for the general case, we develop the determinant first along the last column

\[
\begin{vmatrix}
  s & 0 & -\gamma \\
  \theta & \theta & (s - \alpha) \\
  1 & 0 & 0
\end{vmatrix}
\]

and next along the last row:

\[
\begin{vmatrix}
  0 & -\gamma \\
  \theta & (s - \alpha)
\end{vmatrix} = -\theta \gamma \neq 0,
\]

and the proof that \( G_{12}(s) \) has no zeros is complete.

\( G_{12}(s) \) is a rational function whose poles are all the eigenvalues of \( A \), since this transfer functions has no zeros. Therefore, we can write:
Corollary 7 \( G_{12}(s) \) is given by

\[
G_{12}(s) = \frac{K_G}{s \left( \frac{s^2}{\lambda_k \lambda_{-k}} + s \left( \frac{1}{\lambda_k} + \frac{1}{\lambda_{-k}} \right) + 1 \right)}
\]

where \( \lambda_{\pm k} \) is given by (26).

5 Approximated Transfer functions

In this section we propose some approximations for the models of the transfer functions.

From Corollaries 1–2 and Theorem 3, \( G_{ij}(s), i, j = 1, 2 \), are meromorphic functions given by

\[
G_{11}(s) = \frac{K_G}{s} \prod_{k=1}^{\infty} K_k \frac{(s + \alpha)^2 + (2k - 1)^2 \omega_0^2 - \alpha^2}{(s + \alpha)^2 + 4k^2 \omega_0^2 - \alpha^2}
\]

\[
G_{21}(s) = \frac{K_G}{s} \prod_{k=1}^{\infty} \frac{4k^2 \omega_0^2}{(s + \alpha)^2 + 4k^2 \omega_0^2 - \alpha^2}
\]

\[
G_{22}(s) = -G_{11}(s)
\]

\[
G_{12}(s) = -G_{21}(s)
\]

where

\[
\omega_0 = \frac{\pi}{2T_d}
\]

and

\[
K_k = \left( \frac{2k}{2k - 1} \right)^2
\]

Natural gas is highly pressurized in transportation networks in order to expedite its flow. To ensure this, it must compressed periodically along the pipe. This is accomplished by compressor stations, which are usually placed at 60 Km to 250 Km intervals along the pipeline. As a result, the frequency \( \omega_0 \) always remains much greater than \( \alpha \). Taking this into account as well as the requirement that its factors have a DC gain set to 1, we define \( \tilde{K}_k \), and thence can approximate \( G_{ij}(s), i, j = 1, 2 \) by

\[
\tilde{G}_{11}(s) = \frac{K_G}{s} \prod_{k=1}^{\infty} \tilde{K}_k \frac{(s + \alpha)^2 + (2k - 1)^2 \omega_0^2 - \alpha^2}{(s + \alpha)^2 + 4k^2 \omega_0^2 - \alpha^2}
\]

\[
\tilde{G}_{21}(s) = \frac{K_G}{s} \prod_{k=1}^{\infty} \frac{\alpha^2 + 4k^2 \omega_0^2}{(s + \alpha)^2 + 4k^2 \omega_0^2}
\]

\[
\tilde{G}_{12}(s) = -\tilde{G}_{21}(s)
\]

\[
\tilde{G}_{22}(s) = -\tilde{G}_{11}(s)
\]

with

\[
\tilde{K}_k = \frac{\alpha^2 + 4k^2 \omega_0^2}{\alpha^2 + (2k - 1)^2 \omega_0^2}.
\]
If we define $S = s + \alpha$ we can write
\[\tilde{G}_{11}(s) = \tilde{G}_{11}(S) = \frac{K_G}{S - \alpha} \prod_{k=1}^{\infty} \frac{\tilde{K}_k}{S^2 + (2k - 1)^2\omega_0^2}\]
\[\tilde{G}_{21}(s) = \tilde{G}_{21}(S) = \frac{K_G}{S - \alpha} \prod_{k=1}^{\infty} \frac{\alpha^2 + 4k^2\omega_0^2}{S^2 + 4k^2\omega_0^2}.\] (96)

Theorem 10 considers auxiliary functions that lead to significant simplification in the representation of $\tilde{G}_{11}(S)$ and $\tilde{G}_{21}(S)$. However, before stating Theorem 10 we need to prove some intermediate results:

**Theorem 6** The function
\[f(s) = \frac{e^{-sT_d}}{1 - e^{-2sT_d}}\] (97)
be expanded as
\[f(s) = \sum_{k=-\infty}^{\infty} \frac{a_k}{s - \lambda_k}\] (98)
where $a_k$ is the residual of $f(s)$ at $s = \lambda_k$, i.e.
\[a_k = \frac{(-1)^k}{2T_d}.\]

Since condition (129) holds (see appendix B) then the expansion (98) exists and the residuals $a_k$ are given by

**Proof:**
\[a_k = \frac{1}{2j\pi} \int_{C_k} f(s) ds = \lim_{s \to \lambda_k} (s - \lambda_k)f(s) = \lim_{s \to \lambda_k} \frac{(s - \lambda_k)e^{-T_ds}}{1 - e^{-2T_ds}} = \frac{e^{k\pi T_d}}{2T_d} = e^{j\pi} = (-1)^k.\]

**Theorem 7** The function
\[g(s) = \prod_{k=-\infty, k \neq 0}^{\infty} \frac{(-j\frac{k\pi}{T_d})}{(s - jk\frac{\pi}{T_d})}\] (99)
can be written as:
\[2T_d \frac{e^{-T_ds}}{1 - e^{-2T_ds}} = 2T_d f(s).\]
**Proof:** Since \( g(s) \) is proper it can be expanded as a Laurent series

\[
g(s) = \sum_{k=-\infty}^{\infty} \frac{b_k}{s - \lambda_k}
\]

where \( b_k \) is the residual of \( g(s) \) at \( s = \lambda_k = \frac{jk \pi}{T_d} \). In order to compute this residual we rewrite \( g(s) \) as

\[
g(s) = \lim_{M \to \infty} \frac{\prod_{k=-M}^{M} (-jk \frac{\pi}{T_d})}{\prod_{k=-M}^{M} (s - jk \frac{\pi}{T_d})}
\]

The residual \( b_k \) is then given by

\[
b_k = \frac{1}{2j\pi} \oint_{C_k} g(s) ds = \lim_{s \to \lambda_k} (s - \lambda_k)g(s) = \lim_{M \to \infty} \frac{\prod_{m=-M}^{M} \left( jm \frac{\pi}{T_d} \right)}{\prod_{m=-M}^{M-1} \left( j(k - m) \frac{\pi}{T_d} \right) \prod_{m=k+1}^{M} \left( j(k - m) \frac{\pi}{T_d} \right)}
\]

Given that

\[
\left( -jm \frac{\pi}{T_d} \right) \left( jm \frac{\pi}{T_d} \right) = -j^2 \left( \frac{m\pi}{T_d} \right)^2 = \left( \frac{m\pi}{T_d} \right)^2
\]

we can rewrite \( b_k \) as

\[
b_k = \lim_{M \to \infty} \frac{\prod_{m=1}^{M} \left( \frac{m\pi}{T_d} \right)^2}{\prod_{m=-M}^{M-1} \left( j(k - m) \frac{\pi}{T_d} \right) \prod_{m=k+1}^{M} \left( j(k - m) \frac{\pi}{T_d} \right)}
\]
If now we replace define \( \ell = k - m \) we get

\[
\begin{align*}
    b_k &= \lim_{M \to \infty} \frac{\prod_{m=1}^{k} \left( \frac{\pi}{T_d} \right)^2_{m}}{\prod_{\ell=1}^{k+M} \left( j\ell \frac{\pi}{T_d} \right)_{\ell=-\left(M-k\right)}} = \lim_{M \to \infty} \frac{\prod_{m=1}^{M} \left( \frac{m \pi}{T_d} \right)^2}{\prod_{\ell=1}^{k+M} \left( \frac{\ell \pi}{T_d} \right)_{\ell=-\left(M-k\right)}} \\
    &= \lim_{M \to \infty} \frac{\prod_{m=1}^{M} \left( m \pi \right)^2_{T_d}}{(-1)^M \prod_{\ell=1}^{k+M} \left( \ell \frac{\pi}{T_d} \right)_{\ell=-1}^{M-k} \left( -\ell \frac{\pi}{T_d} \right)} \\
    &= \lim_{M \to \infty} \frac{\prod_{\ell=1}^{M} \left( \ell \frac{\pi}{T_d} \right)_{\ell=-1}^{M-k} \left( \ell \frac{\pi}{T_d} \right)}{(-1)^{M-k} \prod_{\ell=1}^{k+M} \left( \ell \frac{\pi}{T_d} \right)_{\ell=-1}^{M-k} \left( \ell \frac{\pi}{T_d} \right)} \\
    &= \lim_{M \to \infty} \left( -1 \right)^k \frac{\left(M - (k-1)\right) \left[M - (k-2)\right] \ldots \left(M - 1\right) M}{\left(M + 1\right) \left(M + 2\right) \ldots \left(M + (k-1)\right) \left(M + k\right)} \\
    &= \lim_{M \to \infty} \left( -1 \right)^k \frac{M - (k-1) \frac{M - (k-2)}{M + 1} \ldots \frac{M - 1}{M + 1}}{M + (k-1) \ldots \frac{M - 2}{M + 1} \ldots \frac{M - 1}{M + 1}} \\
    &= \left( -1 \right)^k \lim_{M \to \infty} \frac{M - (k-1)}{M + k} \lim_{M \to \infty} \frac{M - (k-2)}{M + (k-1)} \ldots \lim_{M \to \infty} \frac{M - 1}{M + 2} \lim_{M \to \infty} \frac{M}{M + 1} = \\
    &= \left( -1 \right)^k \prod_{\ell=1}^{k} \lim_{M \to \infty} \frac{M - (\ell - 1)}{M + \ell}
\end{align*}
\]

Since we can always make \( M \) infinitely greater than \( k \) then

\[
\lim_{M \to \infty} \frac{M - (\ell - 1)}{M + \ell} = 1, \quad \forall \ell = 1, \ldots, k
\]

and, consequently,

\[
b_k = \left( -1 \right)^k = 2T_d a_k
\]

and we conclude that

\[
g(s) = \frac{\prod_{k=-\infty, k\neq 0}^{\infty} \left(-j k \frac{\pi}{T_d}\right)}{\prod_{k=-\infty}^{\infty} \left( s - j k \frac{\pi}{T_d}\right)} = 2T_d \frac{e^{-T_d s}}{1 - e^{-2T_d s}} \quad \text{:=} f(s)
\]
and the proof that the expansion of function \( f(s) \) in Laurent series is possible is done in Appendix B.

\[ \square \]

**Theorem 8** The function

\[
v(s) = \frac{e^{-sT_d}}{1 + e^{-2sT_d}}
\]

(100)

can expanded as

\[
v(s) = \sum_{k=-\infty}^{\infty} \frac{c_k}{s - \lambda_k}
\]

where \( c_k \) is the residual of \( v(s) \) at \( s = \lambda_k \):

\[
c_k = \frac{j(-1)^{(k+1)}}{2T_d}
\]

**Proof:** This function has poles at

\[
\lambda_k = j(2k - 1)\frac{\pi}{2T_d}, \quad k = -\infty, \ldots, -1, 0, 1, \infty
\]

We can prove that condition (129) holds for \( v(s) \) exactly the same way we did for \( f(s) \). So, we can expand \( v(s) \) as

\[
v(s) = \sum_{k=-\infty}^{\infty} \frac{c_k}{s - \lambda_k}
\]

where \( c_k \) is the residual of \( v(s) \) at \( s = \lambda_k \), i.e.

\[
c_k = \lim_{s \to \lambda_k} \frac{s - \lambda_k}{2T_d} = \lim_{s \to \lambda_k} \frac{(s - \lambda_k)e^{-Ts_d}}{1 + e^{-2Ts_d}} = \frac{j(-1)^{(k+1)}}{2T_d}
\]

\[ \square \]

**Theorem 9** The function

\[
w(s) = \prod_{k=-\infty, k \neq 0}^{\infty} \left( -j(2k - 1)\frac{\pi}{2T_d} \right)
\]

\[
\prod_{k=-\infty}^{\infty} \left( s - j(2k - 1)\frac{\pi}{2T_d} \right)
\]

(101)

can be written as:

\[
w(s) = 2\frac{e^{-Ts_d}}{1 + e^{-2Ts_d}} = 2v(s).
\]
Proof:

Since \( w(s) \) is proper it can be expanded as

\[
w(s) = \sum_{k=-\infty}^{\infty} \frac{d_k}{s-\lambda_k}
\]

where \( d_k \) is the residual of \( w(s) \) at \( s = \lambda_k = j(2k-1)\frac{\pi}{2T_d} \). In order to compute this residual we rewrite \( w(s) \) as

\[
w(s) = \lim_{M \to \infty} \frac{\prod_{k=-M+1}^{M} \left(-j(2k-1)\frac{\pi}{2T_d}\right)}{\prod_{k=-M+1}^{M} \left(s-j(2k-1)\frac{\pi}{2T_d}\right)}.
\]

The residual \( d_k \) is then given by

\[
d_k = \lim_{s \to \lambda_k} (s-\lambda_k)w(s) = \lim_{M \to \infty} \frac{\prod_{m=-M+1}^{M} \left(j(2m-1)\frac{\pi}{2T_d}\right)}{\prod_{m=-M+1}^{M} \left(j(k-m)\frac{\pi}{T_d}\right) \prod_{m=k+1}^{M} \left(j(k-m)\frac{\pi}{T_d}\right)}.
\]

Given that the numerator can be expressed as the product of two complex factors we can write \( d_k \) as

\[
d_k = \lim_{M \to \infty} \frac{\prod_{m=1}^{M} \left(2m-1\right)^2}{\prod_{m=-M+1}^{M} \left(j(k-m)\frac{\pi}{T_d}\right) \prod_{m=k+1}^{M} \left(j(k-m)\frac{\pi}{T_d}\right)}.
\]

If now we replace define \( \ell = k - m \) we get

\[
d_k = \lim_{M \to \infty} \frac{\prod_{m=1}^{M} \left(2m-1\right)^2}{\prod_{\ell=1}^{k-M-1} \left(j\ell\frac{\pi}{T_d}\right) \prod_{\ell=-(M-k)}^{-(M-k)} \left(j\ell\frac{\pi}{T_d}\right)} = \lim_{M \to \infty} \frac{\prod_{m=1}^{M} \left(2m-1\right)^2}{j^{2M-1} \prod_{\ell=1}^{k-M-1} \left(\ell\frac{\pi}{T_d}\right) \prod_{\ell=-(M-k)}^{-(M-k)} \left(\ell\frac{\pi}{T_d}\right)}
\]

\[
= \lim_{M \to \infty} \frac{(-j)^{2M} j^{-1} \prod_{\ell=1}^{k-M-1} \left(\ell\frac{\pi}{T_d}\right) \prod_{\ell=1}^{M-k} \left(-\ell\frac{\pi}{T_d}\right)}{(-1)^M \prod_{\ell=1}^{k-M-1} \left(\ell\frac{\pi}{T_d}\right) \prod_{\ell=1}^{M-k} \left(\ell\frac{\pi}{T_d}\right)} = \frac{-j}{(-1)^{2M-k} \frac{T_d}{T_d}} = \frac{j(-1)^{k+1}}{T_d}
\]
and we conclude that

\[ w(s) = \prod_{k=-\infty}^{\infty} \frac{(-j(2k + 1)\frac{\pi}{2T_d})}{(s - j(2k + 1)\frac{\pi}{T_d})} = 2\frac{e^{-T_d s}}{1 + e^{-2T_d s}} = 2v(s) \]

\[ \square \]

**Theorem 10** Consider the following functions

\[
G_e(S) = \prod_{k=-\infty, k\neq 0}^{\infty} \frac{(-j k \frac{\pi}{T_d})}{(S - j k \frac{\pi}{T_d})}
\]

\[
G_o(S) = \prod_{k=-\infty}^{\infty} \frac{(-j(2k - 1)\frac{\pi}{2T_d})}{(S - j(2k - 1)\frac{\pi}{2T_d})}
\]

\[
F_e(s) = \frac{e^{-T_d s}}{1 - e^{-2T_d s}}
\]

\[
F_o(s) = \frac{e^{-S T_d}}{1 + e^{-2S T_d}}
\]

then

\[
G_e(S) = 2T_d F_e(S)
\]

\[
G_o(S) = 2F_o(S)
\]

**Proof:** To prove the results in Theorem 7 we consider \( g(s) = G_e(s) \) and \( f(s) = F_e(S) \) and also in Theorem 9 we consider \( w(s) = G_o(s) \) and \( v(s) = F_o(S) \).

\[ \square \]

\( G_e(S) \) and \( G_o(S) \) may also be written as

\[
G_e(S) = \prod_{k=1}^{\infty} \frac{4k^2 \omega_0^2}{(S^2 + 4k^2 \omega_0^2)}
\]

\[
G_o(S) = \prod_{k=1}^{\infty} \frac{(2k - 1)^2 \omega_0^2}{(S^2 + (2k - 1)^2 \omega_0^2)}
\]
where \( \omega_0 \) is defined in equation (92). As a result,

\[
\prod_{k=1}^{\infty} \frac{S^2 + (2k-1)^2 \omega_0^2}{S^2 + 4k^2 \omega_0^2} = \frac{S G_e(S)}{G_o(S)} \prod_{k=1}^{\infty} \frac{1}{K_k}
\]

\[
\prod_{k=1}^{\infty} \frac{1}{S^2 + 4k^2 \omega_0^2} = \frac{S G_e(S)}{G_o(S)} \prod_{K=1}^{\infty} \frac{1}{4k^2 \omega_0^2}
\]

with \( K_k \) being defined in equation (93). Now, using these equations and theorem 10 we can write

\[
\prod_{k=1}^{\infty} \hat{K}_k \frac{S^2 + (2k-1)^2 \omega_0^2}{S^2 + 4k^2 \omega_0^2} = \hat{K}_{11} \frac{S G_e(S)}{G_o(S)} = \bar{K}_{11} T_d \frac{S F_e(S)}{F_o(S)}
\]

\[
\prod_{k=1}^{\infty} \frac{\alpha^2 + 4k^2 \omega_0^2}{4k^2 \omega_0^2} = \bar{K}_{12} F_e(S) = 2 \bar{K}_{12} T_d G_e(S)
\]

with

\[
\hat{K}_{11} = \prod_{k=1}^{\infty} \frac{\hat{K}_k}{K_k}
\]

\[
\bar{K}_{12} = \prod_{k=1}^{\infty} \frac{\alpha^2 + 4k^2 \omega_0^2}{4k^2 \omega_0^2}
\]

we thus can rewrite \( \hat{G}_{11}(S) \) and \( \hat{G}_{21}(S) \) as

\[
\hat{G}_{11}(S) = \frac{K_G T_d \bar{K}_{11} \left( s + \alpha \right) \left( 1 + e^{-2sT_d} \right)}{(S - \alpha) \left( 1 - e^{-2sT_d} \right)}
\]

\[
\hat{G}_{21}(S) = \frac{2K_G T_d \bar{K}_{12} s e^{-sT_d}}{(S - \alpha) \left( 1 - e^{-2sT_d} \right)}.
\]

We can compute \( \hat{G}_{11}(s) \) and \( \hat{G}_{21}(s) \) from these equations by replacing \( S \) with \( s + \alpha \):

\[
\hat{G}_{11}(s) = \frac{K_G T_d \bar{K}_{11} \left( s + \alpha \right) \left( 1 + e^{-2\alpha T_d e^{-2sT_d}} \right)}{s \left( 1 - e^{-2\alpha T_d e^{-2sT_d}} \right)}
\]

\[
\hat{G}_{21}(s) = \frac{2K_G T_d \bar{K}_{12} \left( s + \alpha \right) e^{-\alpha T_d e^{-sT_d}}}{s \left( 1 - e^{-2\alpha T_d e^{-2sT_d}} \right)}.
\]

Given that \( \hat{G}_{11}(s) \) and \( \hat{G}_{21}(s) \) were defined in such a way that

\[
\lim_{s \to 0} s \hat{G}_{11}(s) = \lim_{s \to 0} s \hat{G}_{21}(s) = \lim_{s \to 0} s G_{11}(s) = \lim_{s \to 0} s G_{21}(s) = K_G,
\]

we can rewrite (113) as

\[
\hat{G}_{11}(s) = K_{11} \frac{(s + \alpha) \left( 1 + e^{-2\alpha T_d e^{-2sT_d}} \right)}{s \left( 1 - e^{-2\alpha T_d e^{-2sT_d}} \right)}
\]

\[
\hat{G}_{21}(s) = K_{21} \frac{(s + \alpha) e^{-sT_d}}{s \left( 1 - e^{-2\alpha T_d e^{-2sT_d}} \right)}.
\]
\begin{align*}
K_{11} &= \frac{K_G (1 - e^{-2\alpha T_d})}{\alpha (1 + e^{-2\alpha T_d})} \\
K_{21} &= \frac{K_G (1 - e^{-2\alpha T_d})}{\alpha}.
\end{align*}

(116)

6 Case study

In this section we study a small pipeline using the lumped linear model derived above. The pipe has a length \( L = 35 \text{Km} \) and a diameter \( D = 793 \text{mm} \). The friction factor is \( f_c = 0.0079 \) and the isothermal speed of sound is \( c = 300 \text{m/sec} \). We considered \( q_m = 90 \text{Kg/sec} \) and \( p_m = 80 \text{bar} = 8 \times 10^6 \) Pascal as the mass-flow and pressure nominal values. The \( \alpha \) and \( K_G \) parameters were calculated from equations (3) and Corollary 5, respectively, and the values of 0.0051 and 5.064 were obtained. We computed the frequency responses (FR) of \( G_{11}(s) \) and \( G_{21}(s) \) from equations (91) using truncated approximations of order \( n \), \( G_{11}^{(n)}(s) \) and \( G_{11}^{(n)}(s) \), respectively, in a frequency bandwidth, \( BW = [10^{-4}\text{rad/sec}, 1 \text{rad/sec}] \).

Figure 1: Gas pipeline topology.

Figures 2 and 3 display the respective Bode diagrams and compare them with their approximations \( \hat{G}_{11}(s) \) and \( \hat{G}_{21}(s) \). \( G_{11}(s) \) converges very fast. The FR of a truncated approximation with two hundred poles and zeros had already converged to its limit in the whole frequency interval \( BW \). Figure 2 shows that there are no significant differences between \( G_{11}(s) \) and \( \hat{G}_{11}(s) \). Consequently, \( G_{11}(s) \) can be substituted by its approximation without loss of accuracy and with a significant reduction of the computational costs.

The convergence of \( G_{21}(s) \) is much slower. A two thousand order approximation didn’t converge in the whole bandwidth \( BW \). We can subsequently conclude that a truncated approximation of equation (91) needs too many factors leading, therefore, to high order transfer functions with high computational costs.

However, from Figures 3 and 4 one can conclude that \( G_{21}(s) \) can be also substituted by its approximation. In Figure 4 the Bode diagrams of \( \hat{G}_{21}(s) \) and \( G_{21}^{(2000)}(s) \) are compared with a better resolution, i.e. the phase is restricted to the range \((-180 \text{ degrees}, 180 \text{ degrees})\). The phase discontinuities are due to the phase-crossing...
Figure 2: Bode plots of $G_{11}(s)$ and the approximation $\hat{G}_{11}(s)$.

Figure 3: Bode plots of both $G_{12}^{(n)}(s)$, $n = 200, 100, 2000$ and the approximation $\hat{G}_{12}(s)$. 
of the odd multiples of 180 degrees which are converted from -180 degrees to 180 degrees. Also notice that the Bode diagram of the finite approximation $G_{12}^{(2000)}(s)$ converges to $G_{12}(s)$ almost in the whole frequency BW.

This pipeline was simulated taking two normal days operation data as the input and output mass-flows. The simulation was performed with the previous referred SIMONE® simulator. Figure 5 shows the input and output mass-flows on both days.

The intake and offtake massflows are denoted by $q_1(t)$ and $q_2(t)$, respectively.

Figures 6 and 7 compare the input and output pressures simulated with the lumped transfer function model

$$
P_1(s) = \tilde{G}_{11}(s)Q_1(s) - \tilde{G}_{21}(s)Q_2(s)$$
$$P_2(s) = \tilde{G}_{21}(s)Q_1(s) - \tilde{G}_{22}(s)Q_2(s)
$$

with the ones simulated with SIMONE® for the two days. The intake and offtake pressures are denoted by $p_1(t)$ and $p_2(t)$, respectively. We can see that the results are better for the first day. This is expectable, since the mass-flows and pressures of the second day data present stronger deviations from the nominal values used for the calculation of the $\alpha$ parameter. As a matter of fact, on the second day the quotient $(q_1(t) + q_2(t)) / (p_1(t) + p_2(t))$ has a root mean square deviation from the nominal value of about 50%.
Figure 6: Input and Output pressures, $P_1$, and $P_2$, respectively, simulated by the lumped model (117) (solid red line) and by SIMONE using the first day data.

Figure 7: Input and Output pressures, $P_1$, and $P_2$, respectively, simulated by the lumped model (117) (solid red line) and by SIMONE using the second day data.
(this deviation is of 30% in the first day). Yet, in both cases, the model has well captured the dynamics of the system and, therefore, it seems to be a valuable tool for gas leakage detection and gas networks controller design.

A Change of variables in an integral model for a short gas pipeline

In this section, a change in the state-space variables of the integral model is performed. The purpose is to obtain a simpler system matrix in order to simplify the determination of the eigenvalues of the system.

Consider model (10) with the respective matrices defined according to (12)–(17).

Consider for this system the following change of variables:

\[
\begin{align*}
  z_2 &= x_2 + x_2 + \cdots + x_N + x_{N+1} \\
  z_2 &= x_2 - x_2 \\
  z_3 &= x_2 - x_3 \\
  \vdots \\
  z_{N+1} &= x_N - x_{N+1} \\
  z_{N+2} &= x_{N+2} \\
  z_{N+3} &= x_{N+3} \\
  \vdots \\
  z_{2N+1} &= x_{2N+1} 
\end{align*}
\]  

(118)

and we obtain the following realisation

\[
\begin{align*}
  \dot{z}_2(t) &= \frac{c^2}{A\Delta \ell} u_2(t) - \frac{c^2}{A\Delta \ell} u_2(t) \\
  \dot{z}_2(t) &= -2 \frac{c^2}{A\Delta \ell} z_{N+2}(t) + \frac{c^2}{A\Delta \ell} z_{N+3}(t) \\
  \dot{z}_3(t) &= \frac{c^2}{A\Delta \ell} z_{N+2}(t) - 2 \frac{c^2}{A\Delta \ell} z_{N+3}(t) + \frac{c^2}{A\Delta \ell} z_{N+4}(t) \\
  \vdots \\
  \dot{z}_i(t) &= \frac{c^2}{A\Delta \ell} z_{N+i-1}(t) - 2 \frac{c^2}{A\Delta \ell} z_{N+i}(t) + \frac{c^2}{A\Delta \ell} z_{N+i+1}(t) \\
  \dot{z}_{N+1}(t) &= \frac{c^2}{A\Delta \ell} z_{2N}(t) - 2 \frac{c^2}{A\Delta \ell} z_{2N+1}(t) + \frac{c^2}{A\Delta \ell} u_2(t) \\
  \dot{z}_{N+2}(t) &= \frac{A}{\Delta \ell} z_2(t) - \frac{f_c c^2 Q_{m1}}{2DA P_{m1}} z_{N+2}(t) \\
  \dot{z}_{N+3}(t) &= \frac{A}{\Delta \ell} z_3(t) - \frac{f_c c^2 Q_{m1}}{2DA P_{m1}} z_{N+3}(t) \\
  \vdots 
\end{align*}
\]  

(119)
\[
\dot{z}_{N+i}(t) = \frac{A}{\Delta \ell} z_i(t) - \frac{f_c c^2 Q_{m1}}{2DA P_{m1}} z_{N+i}(t)
\]

\[
\vdots
\]

\[
\dot{z}_{2N+1}(t) = \frac{A}{\Delta \ell} z_{N+1}(t) - \frac{f_c c^2 Q_{m1}}{2DA P_{m1}} z_{2N+1}(t)
\]

\[
y_2(t) = \frac{1}{N + 1} z_2(t) + \frac{N}{N + 1} z_2(t) + \frac{N - 1}{N + 1} z_3(t) + \cdots + \frac{1}{N + 1} z_{N+1}(t)
\]

\[
y_2(t) = \frac{1}{N + 1} z_2(t) - \frac{1}{N + 1} z_2(t) - \frac{2}{N + 1} z_3(t) - \cdots - \frac{N}{N + 1} z_{N+1}(t)
\]

and in matricial form, we have:

\[
\dot{z}(t) = \tilde{A} z(t) + \tilde{B} u(t)
\]

\[
y(t) = \tilde{C} z(t)
\]

with

\[
\tilde{A} = \begin{bmatrix}
0 & 0_{1 \times N} & 0_{1 \times N} \\
0_{N \times 1} & \tilde{A}_{11} & \tilde{A}_{12} \\
0_{N \times 1} & \tilde{A}_{21} & \tilde{A}_{22}
\end{bmatrix}
\]

\[
\tilde{A}_{11} = 0_{N \times N}
\]

\[
\tilde{A}_{12} = \begin{bmatrix}
-\frac{2}{A \Delta \ell} c^2 & -\frac{c^2}{A \Delta \ell} & 0 & \cdots & 0 & 0 & 0 \\
-\frac{c^2}{A \Delta \ell} & -\frac{2}{A \Delta \ell} c^2 & -\frac{c^2}{A \Delta \ell} & \cdots & \vdots & \vdots & \vdots \\
0 & -\frac{c^2}{A \Delta \ell} & -\frac{2}{A \Delta \ell} c^2 & \cdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \cdots & -\frac{c^2}{A \Delta \ell} & -\frac{2}{A \Delta \ell} c^2 & \cdots \\
0 & 0 & 0 & \cdots & 0 & -\frac{c^2}{A \Delta \ell} & -\frac{2}{A \Delta \ell}
\end{bmatrix}
\in \mathbb{R}^{N \times N}
\]
\[ \bar{A}_{21} = \frac{A}{\Delta \ell} I_N, \quad I_N - \text{identity matrix } N \times N \]
\[ \bar{A}_{22} = -\frac{f c^2 Q_m}{2 D \Delta P_m} I_N \]
\[ \bar{B} = \frac{c^2}{\Delta \ell} \begin{bmatrix} e_2 & -e_2 + e_{N+1} \end{bmatrix} \in \mathbb{R}^{(2N+1) \times 2} \]
\[ \bar{C} = \begin{bmatrix} 1 & N & N-1 & \cdots & N+1 \\ N+1 & N & \cdots & N+1 \\ N+1 & N+1 & \cdots & N+1 \\ & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \in \mathbb{R}^{2 \times (2N+1)} \]

Matrices \( A \) and \( \bar{A} \) have the same spectrum, however its calculation seems to be easier if we use matrix \( \bar{A} \).

### B Rational expansion of meromorphic functions

Let \( f(s) \) be a function meromorphic in the finite complex plane with poles at \( \lambda_1, \lambda_2, \ldots \), and let \( (\Gamma_1, \Gamma_2, \ldots) \) be a sequence of simple closed curves such that:

- The origin lies inside each curve \( \Gamma_k \).
- No curve passes through a pole of \( f \).
- \( \Gamma_k \) lies inside \( \Gamma_{k+1} \) for all \( k \)
- \( \lim_{k \to \infty} d(\Gamma_k) = \infty \), where \( d(\Gamma_k) \) gives the distance from the curve to the origin

Suppose also that there exists an integer \( p \) such that

\[
\lim_{k \to \infty} \oint_{\Gamma_k} \left| \frac{f(s)}{s^p} \right| |ds| < \infty.
\]

Denoting the principal part of the Laurent series of \( f \) about the point \( \lambda_k \) as \( PP[f(s); s = \lambda_k] \), we have, if \( p < 0 \),

\[
f(z) = \sum_{k=0}^{\infty} PP[f(z); z = \lambda_k].
\]

**Theorem 11** Consider function \( f(s) = \frac{e^{-T_s}}{1 - e^{-2T_s}} \). There exists an integer \( p \) such that

\[
\lim_{k \to \infty} \oint_{\Gamma_k} \left| \frac{f(s)}{s^p} \right| |ds| < \infty.
\]

**Proof:** This function has poles at

\[
\lambda_k = j2k \frac{\pi}{2T_d} = jk \frac{\pi}{T_d}, \quad k = -\infty, \ldots, -1, 0, 1, \infty
\]

The contours \( \Gamma_k \) will be squares vertices at \( \pm(2k-1) \frac{\pi}{2T_d} \pm j(2k-1) \frac{\pi}{2T_d} \), \( k > 1 \), traversed counterclockwise, which are easily seen to satisfy the necessary conditions. To see what are the terms of the Laurent
series expansion of $f(s)$ we need do see for which $p$ the condition

$$\lim_{k \to \infty} \oint_{\Gamma_k} \left| f(s) \right| ds < \infty$$

(129)

holds. We can partition this integral as

$$\oint_{\Gamma_k} \left| f(s) \right| ds = \oint_{\Gamma_{k1}} \left| f(s) \right| ds + \oint_{\Gamma_{k2}} \left| f(s) \right| ds + \oint_{\Gamma_{k3}} \left| f(s) \right| ds + \oint_{\Gamma_{k4}} \left| f(s) \right| ds$$

where

$$\begin{align*}
\Gamma_{k1} &= \left\{ s : s = (2k-1) \frac{\pi}{2T_d} + j\omega, \omega \uparrow, \omega \in \left[-(2k-1) \frac{\pi}{2T_d}, (2k-1) \frac{\pi}{2T_d}\right] \right\} \\
\Gamma_{k2} &= \left\{ s : s = \sigma + j2(k-1) \frac{\pi}{2T_d}, \sigma \downarrow, \sigma \in \left[-(2k-1) \frac{\pi}{2T_d}, (2k-1) \frac{\pi}{2T_d}\right] \right\} \\
\Gamma_{k3} &= \left\{ s : s = -(2k-1) \frac{\pi}{2T_d} + j\omega, \omega \downarrow, \omega \in \left[-(2k-1) \frac{\pi}{2T_d}, (2k-1) \frac{\pi}{2T_d}\right] \right\} \\
\Gamma_{k4} &= \left\{ s : s = \sigma - j(2k-1) \frac{\pi}{2T_d}, \sigma \uparrow, \omega \in \left[-(2k-1) \frac{\pi}{2T_d}, (2k-1) \frac{\pi}{2T_d}\right] \right\} .
\end{align*}$$

Notice that

- For $s \in \Gamma_{k1}$, $|ds| = d\omega$.
- For $s \in \Gamma_{k2}$, $|ds| = -d\sigma$.
- For $s \in \Gamma_{k3}$, $|ds| = -d\omega$.
- For $s \in \Gamma_{k4}$, $|ds| = d\sigma$. 
Now we can write

\[
\lim_{k \to \infty} \int_{T_k} |f(s)| ds = \int_{-\pi}^{\pi} f\left((2k-1) \frac{\pi}{2T_d} + j\omega\right) d\omega - \int_{-\pi}^{\pi} f\left(\sigma + j(2k-1) \frac{\pi}{2T_d}\right) d\sigma
\]

Next we analyze the four terms of this integral

**First term**

\[
\int_{-\pi}^{\pi} f\left((2k-1) \frac{\pi}{2T_d} + j\omega\right) d\omega = \int_{-\pi}^{\pi} e^{-(2k-1) \frac{\pi}{2T_d}} e^{-j\omega T_d} \left((2k-1) \frac{\pi}{2T_d} + j\omega\right)^p \left(1 - e^{-(2k-1)\pi e^{-j\omega T_d}}\right) d\omega.
\]

Given that

\[
\lim_{k \to \infty} e^{-(2k-1) \frac{\pi}{2T_d}} = 0
\]

then

\[
\lim_{k \to \infty} \int_{-\pi}^{\pi} f\left((2k-1) \frac{\pi}{2T_d} + j\omega\right) d\omega = \lim_{k \to \infty} \int_{-\pi}^{\pi} e^{-(2k-1) \frac{\pi}{2T_d}} \left((2k-1) \frac{\pi}{2T_d} + j\omega\right)^p \left(1 - e^{-(2k-1)\pi e^{-j\omega T_d}}\right) d\omega = 0.
\]

**Second term**

\[
\int_{-\pi}^{\pi} f\left(\sigma + j(2k-1) \frac{\pi}{2T_d}\right) d\sigma = \int_{-\pi}^{\pi} e^{-\sigma T_d} e^{-j(2k-1) \frac{\pi}{2T_d}} \left(\sigma + j(2k-1) \frac{\pi}{2T_d}\right)^p \left(1 - e^{-2\sigma T_d e^{-j(2k-1)\pi}}\right) d\sigma
\]

\[
= \int_{-\pi}^{\pi} \left(\sigma + j(2k-1) \frac{\pi}{2T_d}\right)^p \left(1 - e^{-2\sigma T_d (-1)^k}\right) d\sigma + \int_{0}^{\pi} e^{-\sigma T_d} \left(\sigma + j(2k-1) \frac{\pi}{2T_d}\right)^p \left(1 - e^{-2\sigma T_d (-1)^k}\right) d\sigma
\]

Notice that

\[
\int_{-\pi}^{\pi} \left(\sigma + j(2k-1) \frac{\pi}{2T_d}\right)^p \left(1 - e^{-2\sigma T_d (-1)^k}\right) d\sigma \leq \int_{-\pi}^{\pi} e^{-\sigma T_d} \left(\sigma + j(2k-1) \frac{\pi}{2T_d}\right)^p \left(1 - e^{-2\sigma T_d (-1)^k}\right) d\sigma
\]

On the other hand, for \( \sigma < 0 \)

\[
|1 - e^{-\sigma T_d}| = e^{[\sigma |T_d| - 1} < e^{[\sigma T_d]} = e^{-\sigma T_d}.
\]
As a result

\[ \int_{-(2k-1) \pi T_d}^{0} \left| \frac{e^{-\sigma T_d}}{\left(\sigma + j(2k-1) \frac{\pi}{2T_d}\right)^p (1 - e^{-2\sigma T_d})} \right| d\sigma < \int_{-(2k-1) \pi T_d}^{0} \left| \frac{e^{-\sigma T_d}}{\left(\sigma + j(2k-1) \frac{\pi}{2T_d}\right)^p e^{-2\sigma T_d}} \right| d\sigma = \]

\[ = \int_{-(2k-1) \pi T_d}^{0} \left| \frac{e^{\sigma T_d}}{\left(\sigma + j(2k-1) \frac{\pi}{2T_d}\right)^p} \right| d\sigma \]

Making \( k \to \infty \),

\[ \lim_{k \to \infty} \int_{-(2k-1) \pi T_d}^{0} \left| \frac{e^{\sigma T_d}}{\left(\sigma + j(2k-1) \frac{\pi}{2T_d}\right)^p} \right| d\sigma = M_1. \]

Notice also that

\[ \int_{0}^{(2k-1) \pi T_d} \left| \frac{e^{-\sigma T_d}}{\left(\sigma + j(2k-1) \frac{\pi}{2T_d}\right)^p (1 - e^{-2\sigma T_d})} \right| d\sigma \leq \int_{0}^{(2k-1) \pi T_d} \left| \frac{e^{-\sigma T_d}}{\left(\sigma + j(2k-1) \frac{\pi}{2T_d}\right)^p (1 - e^{-2\sigma T_d})} \right| d\sigma \]

For \( \sigma > 0 \)

\[ 1 - e^{-\sigma T_d} < 1. \]

Consequently

\[ \int_{0}^{(2k-1) \pi T_d} \left| \frac{e^{-\sigma T_d}}{\left(\sigma + j(2k-1) \frac{\pi}{2T_d}\right)^p (1 - e^{-2\sigma T_d})} \right| d\sigma < \int_{0}^{(2k-1) \pi T_d} \left| \frac{e^{-\sigma T_d}}{\left(\sigma + j(2k-1) \frac{\pi}{2T_d}\right)^p} \right| d\sigma \]

Making \( k \to \infty \) again

\[ \lim_{k \to \infty} \int_{0}^{(2k-1) \pi T_d} \left| \frac{e^{-\sigma T_d}}{\left(\sigma + j(2k-1) \frac{\pi}{2T_d}\right)^p} \right| d\sigma = M_1 \]

and we conclude that

\[ \lim_{k \to \infty} \int_{-(2k-1) \pi T_d}^{0} \left| f \left( \sigma + j(2k-1) \frac{\pi}{2T_d} \right) \right| d\sigma < 2M_1 < \infty. \]

**Third term** This term is similar to the first one and using the same arguments we can prove that

\[ \lim_{k \to \infty} \int_{-(2k-1) \pi T_d}^{0} \left| f \left( -\left(2k-1 \right) \frac{\pi}{2T_d} + j\omega \right) \right| d\omega = 0. \]

In similar way that we did for the second term we can prove that

\[ \int_{-(2k-1) \pi T_d}^{0} \left| f \left( \sigma - j(2k-1) \frac{\pi}{2T_d} \right) \right| d\sigma < 2M_1 \]
We can now conclude that condition (129) holds for any $p < 0$.

Since there exists an integer $p$ such that

$$\lim_{k \to \infty} \oint_{\Gamma_k} \left| \frac{f(s)}{s^p} \right| ds < \infty.$$ 

Then, denoting the principal part of the Laurent series of $f$ about the point $\lambda_k$ as $PP[f(s); s = \lambda_k]$, we have, if $p < 0$.

$$f(z) = \sum_{k=0}^{\infty} PP(f(z); z = \lambda_k).$$

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