The Pairing interaction in nuclei: comparison between exact and approximate treatments

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Abstract

As a model for a deformed nucleus the many level pairing model (picket fence model with 100 levels) is considered in four approximations and compared to the exact solution given by Richardson long time ago. It is found that, as usual, the number projected BCS method improves over standard BCS but that it is much less accurate than the more sophisticated many-body-approaches which are Coupled Cluster Theory (CCT) in its SUB2 version or Self-Consistent Random Phase Approximation (SCRPA).
I. INTRODUCTION

The importance of two nucleon pair correlations in the ground state and low lying excited states of nuclei has been known for a long time [1]. The application to nuclear systems [2] of the concepts used in the description of superconductivity in solids was made immediately after the BCS theory has appeared [3]. During the sixties it was realized that the pairing interaction was relevant in the description of two particle transfer reactions in normal and superconducting nuclei [4,5]. This interaction, which is usually thought to represent the short range part of the bare nucleon interaction, was treated by many authors in a phenomenological and schematic way. Nevertheless, it has been found recently [6] that the pairing interaction is an important ingredient of the shell model interaction derived from realistic forces and used in large scale shell model calculations (the other two important ingredients being the quadrupole-quadrupole and the monopole-monopole interactions). It is also known [7] that to preserve the short range character of the force, it is necessary to use a large number of shells. Unfortunately the most simple theory for pairing in finite nuclei, namely the mean field BCS approach, is rather limited in its application, since particle number fluctuations are very strong. Therefore more sophisticated approaches such as particles number projection or the explicit introduction of quantal fluctuations like BCS-QRPA approach [8] and other more elaborated theories have to be considered.

In a series of papers between 1963 and 1968 Richardson [9] obtained the exact solution of the pairing hamiltonian providing an analytic form for the eigenvalues and eigenvectors. These papers have found a revival in the framework of ultrasmall metallic grains [10] where it was necessary to go beyond the existing approximations to explain the disappearance of superconductivity as the size of the grain [11,12] decreases. Subsequently, the exact solutions have been generalized [13] and applied to other systems like Bose condensates [14], interacting boson models [15] and nuclear superconductivity [16].

The purpose of the present paper is to test on the exact solution for a large scale case the precision of some well known approximations like number projected BCS (PBCS) [17],
Coupled Cluster Theory (CCT) \[18\], and Self-Consistent RPA (SCRPA) \[19\]. In reality the possibility of applying these approximations depends on details of the nuclear residual interaction. In general these approximations can deal in an appropriate way with the long range part of the interaction, that can be thought of, in a simplified way, as a particle-hole interaction (as for example the quadrupole-quadrupole one), but they may have problems in dealing with the short range part that can be represented by the pairing interaction. The possibility or convenience of using each of these methods depends on the strength of the pairing interaction as well as the set of single particle levels that is considered. For example for very strong pairing (which is equivalent to a single shell) it is known that all the particles participate in the ground state wave function, and therefore one will need a quite large number of particle-holes over the Hartree-Fock (HF) groundstate to describe properly the paired state, on the other hand for a weak pairing interaction the ground state wave function will be almost the HF one.

The paper is organized as follows. In Sect. II we present the picket fence model and sketch the main steps for its exact solution as given by Richardson. In Sect. III we outline the various approximate methods to treat the model and in Sect. IV we give the results together with a discussion. We end with the conclusions in Sect V.

II. THE MODEL

The picket fence model mimics superfluid correlations in a deformed nucleus where the level density can be considered as more or less constant and the levels are two-fold degenerate (for one sort of nucleons). As mentioned in the introduction, the model has been solved exactly in the early sixties by Richardson \[9\] for practically any number of levels. The latter feature makes the model very interesting because one can treat situations, very frequent in practice, which can not be mastered by ordinary diagonalization techniques. For example we here will treat the case of hundred particles distributed in hundred levels corresponding
to a dimension of the hamiltonian matrix of $10^{29}$, well beyond any diagonalization technique. The model has not been used very much in nuclear physics, probably because of its schematic character. However, recently, its properties have been exploited in rather great detail in the context of ultra small superconducting metallic grains [10]. We here will employ the model in order to assess the quality of commonly used approximation schemes for nuclear pair correlations. We will consider fermion creation $a^\dagger_{\alpha m}$ and annihilation $a_{\alpha m}$ operators defined in a discrete basis labelled by the quantum numbers $\{\alpha m\}$. This basis can be referred to the single particle states of an external potential, the single particle energies depend on the quantum numbers $\alpha, m$.

The three operators

$$n_{\alpha m} = a^\dagger_{\alpha m} a_{\alpha m} \quad , \quad A^\dagger_{\alpha m} = a^\dagger_{\alpha m} a^\dagger_{\bar{\alpha} m} = (A_{\alpha m})^\dagger$$

close the commutator algebra

$$[n_{\alpha m}, A^\dagger_{\beta n}] = 2\delta_{\alpha\beta}\delta_{mn} A^\dagger_{\alpha m} \quad , \quad [A_{\alpha m}, A^\dagger_{\beta n}] = \delta_{\alpha\beta}\delta_{mn} (1 - n_{\alpha m})$$

In Eq. (1) the pair operator $A^\dagger_{\alpha m}$ creates a pair of particles in the time reversal states $\{\alpha m, \alpha \bar{m}\}$ where $a^\dagger_{\alpha m}$ creates a particle in the time reversed state of $a^\dagger_{\alpha m}$. We will work with nucleons interacting via a pure pairing force and for simplicity we will represent by a single letter $k$ the quantum numbers $\alpha m$ (and when there is no possibility of confusion it will represent the pair $\{\alpha m, \alpha \bar{m}\}$). Therefore the Hamiltonian that we will consider is

$$H = \sum_k \varepsilon_k n_k + G \sum_{kk'} A^\dagger_{k} A_{k'}$$

where the $\varepsilon_k$ are the single particle energies.

The exact solution of this model has been obtained long ago by Richardson [9]. We will here briefly outline the method, giving the equations to be used later on in the numerical applications.

Richardson [3] has shown that the exact eigenstates of the Hamiltonian (3) with $M$ pairs can be written as
\[ |\Psi\rangle = \prod_{i=1}^{M} B_{i}^{\dagger} |\varphi_{\nu}\rangle \]  

(4)

where there are \( \nu \) unpaired nucleons. The state \( |\varphi_{\nu}\rangle \) describing the unpaired sector of \( |\Psi\rangle \) is defined by the action of the operators \( A \) and \( n \) as

\[ A_{k} |\varphi_{\nu}\rangle = 0 \quad , \quad n_{k} |\varphi_{\nu}\rangle = \nu_{k} |\varphi_{\nu}\rangle \]  

(5)

where \( \nu_{k} = 1 \) if there is one particle blocking the state \( k \) and \( \nu_{k} = 0 \) elsewhere.

The operator \( B_{i}^{\dagger} \) in (4) creates a collective pair

\[ B_{i}^{\dagger} = \sum_{k=1}^{\Omega} \frac{1}{2\varepsilon_{k} - E_{i}} A_{k}^{\dagger} \]  

(6)

where \( \Omega \) is the number of single particle levels in the valence space. The form of the amplitudes in (4) were suggested by the one pair diagonalization of the pairing Hamiltonian (3). The pair energies \( E_{i} \) are unknown parameters to be determined by the eigenvalue condition.

\[ H |\Psi\rangle = E |\Psi\rangle \]  

(7)

After a long but straightforward derivation one arrives at the set of \( M \) nonlinear equations for the \( M \) pair energies

\[ 1 - 2G \sum_{j(\neq i)=1}^{M} \frac{1}{E_{j} - E_{i}} + G \sum_{k=1}^{\Omega} \frac{(1 + 2\nu_{k})}{2\varepsilon_{k} - E_{i}} = 0 \]  

(8)

while the energy eigenvalue is

\[ E = \sum_{i=1}^{M} E_{i} + \sum_{k=1}^{\Omega} \varepsilon_{k} \nu_{k} \]  

(9)

The pair energies \( E_{i} \) are the roots of the set of \( M \) coupled equations (8). There are as many independent solutions as states in the Hilbert space of \( M \) pairs. The different solutions, each one corresponding to an eigenstate of the pairing hamiltonian, can be classified in the limit of \( G \to 0 \) as the different possible configurations of \( M \) pairs in \( \Omega \) levels, and then let them evolve adiabatically by solving the equations (8) for increasing values of \( G \).
The occupation probabilities are obtained by means of the Gellman-Feynman theorem, minimizing the energy with respect to the single particle energies

\[ n_k = \frac{\partial E}{\partial \varepsilon_k} = \nu_k + \sum_{i=1}^{\Omega} \frac{\partial E_i}{\partial \varepsilon_k} \]  

(10)

Differentiating (8) with respect to \( \varepsilon_k \), the occupation numbers can be expressed as

\[ n_k = \nu_k + 2 \sum_{i=1}^{\Omega} \frac{(1 + 2\nu_k)}{(2\varepsilon_k - E_i)^2} D_i \]  

(11)

where the \( D_i \) should satisfy the system of equations

\[
\begin{bmatrix}
\sum_{k=1}^{\Omega} \frac{(1 + 2\nu_k)}{(2\varepsilon_k - E_i)^2} + 4 \sum_{j(\neq i)=1}^{M} \frac{1}{(E_j - E_i)^2} \\
\sum_{j(\neq i)=1}^{M} \frac{1}{(E_j - E_i)^2} D_i - 4 \sum_{j(\neq i)=1}^{M} \frac{1}{(E_j - E_i)^2} D_j
\end{bmatrix} = 1
\]  

(12)

The above equations are used to establish the exact solution with, in the case considered here, a hundred levels with a hundred of particles.

III. APPROXIMATE SOLUTIONS

We will study some approximations that are written in terms of particular particle-hole excitations on a reference HF state. For simplicity we will consider the case when the shells are half filled, i.e. the number of pairs of particles \( M \) will satisfy \( \Omega = 2M \). In the weak interaction limit the separation between the energy levels is much greater than the gap. The physics of this regime can be given in terms of the fluctuations around the HF state

\[ |HF\rangle = \prod_{h=1}^{M} A_h^+ |0\rangle \]  

(13)

where \( h \) (\( p \)) refers to single particle states that are occupied (unoccupied) in the limit \( G = 0 \).

A. Variational treatments

We will first consider different variational treatments. The simplest one is the standard BCS treatment. The next approximation that we will consider is the number projected
(before variation) PBCS wave function where the ground state is assumed to be a condensate of pairs of fermions. It is written as

$$|PBCS⟩ = \frac{1}{\sqrt{Z_{M,\omega}}} [\Gamma^+]^M |0⟩ \quad (14)$$

where

$$\Gamma^+ = \sum_{k=1}^{\Omega} \lambda_k A_k^+ = \sum_{h=1}^{M} \lambda_h A_h^+ + \sum_{p=M+1}^{\Omega} \lambda_p A_p^+ = \Gamma_h^+ + \Gamma_p^+ \quad (15)$$

$$Z_{M,\omega} = <0| [\Gamma]^M [\Gamma^+]^M |0⟩ \quad (16)$$

and $|0⟩$ is the vacuum for the creation operator of the nucleons. In general one minimizes the energy by changing the variational parameters $\lambda_k$. In PBCS $\lambda_k$ can be written in terms of the $v_k$ and $u_k$ parameters as $\lambda_k = \frac{v_k}{u_k}$ with $v_k^2 + u_k^2 = 1$.

In Ref. [12] the ground state energy was evaluated in terms of these $\lambda_k$ coefficients using the auxiliary quantities

$$Z_{N,\Omega} = <0| [\Gamma]^N [\Gamma^+]^N |0⟩ \quad (17)$$

$$S_i^N = <0| [\Gamma]^N A_i^+ [\Gamma^+]^{N-1} |0⟩ \quad (18)$$

$$Z_{ij}^N = <0| [\Gamma]^{N-1} A_i A_j^+ [\Gamma^+]^{N-1} |0⟩ \quad (19)$$

$$T_{ij}^N = <0| [\Gamma]^{N-2} A_i A_j [\Gamma^+]^N |0⟩ \quad (20)$$

and

$$S_i^N = \frac{S_i^N}{Z^N}; \quad T_{ij}^N = \frac{T_{ij}^N}{Z^N} \quad (21)$$

The ground state energy is then written as

$$E_{gs} = 2M \sum_i (2\epsilon_i - \mu) \lambda_i \hat{S}_i^M + G \sum_{ij} \lambda_j \hat{S}_i^M - GM(M-1) \sum_{ij} \lambda_i^2 \hat{T}_{ij}^M \quad (22)$$
The auxiliary coefficients are determined by recurrence relations using the fact that
\[ Z_0 = 1 ; \quad Z_1 = \sum_i \lambda_i^2 \quad \text{and} \quad \hat{S}_n^N = \frac{\hat{S}}{Z_1}. \]

The pair creation operator has two parts: one \((\Gamma_p^+)\) creates two particles above the Fermi sea while the other part \((\Gamma_h^+)\) creates two particles below the Fermi sea. In Ref. [11] it is shown that if one defines the normalized states

\[ |K\rangle = \frac{1}{Z_{K,\Omega/2}} \left( \Gamma_p^+ \Gamma_h^+ \right)^K |HF\rangle \]

it is possible to write down the PBCS state as

\[ |PBCS\rangle = \sum_K \Psi_{PBCS}^K |K\rangle \]

where

\[ \Psi_{PBCS}^K = \frac{\left( (\Omega/2)! \right)^2 Z_{K,\Omega/2}}{\sqrt{Z_{\Omega/2} Z_{\Omega/2}} (K!)^2} \]

and therefore the wave function can be written as

\[ |PBCS\rangle = \mathcal{A}_\Omega \sum_K \frac{\left( \Gamma_p^+ \Gamma_h^+ \right)^K}{(K!)^2} |HF\rangle \]

For details on this derivation see [11].

The variational parameters in this wave function are the amplitudes \(\lambda_k\). It must be taken into account that \(\mathcal{A}_\Omega\) as well as the operators \(\Gamma_p^+\) and \(\Gamma_h\) are well defined functions of these parameters.

We also used another variational wave function with a structure similar to the \(exp(S_2)\) type(see below), i.e.

\[ |Exp\rangle = \mathcal{B}_\Omega \sum_K \frac{\left( \Gamma_p^+ \Gamma_h^+ \right)^K}{K!} |HF\rangle = \mathcal{B}_\Omega \exp \left( \Gamma_p^+ \Gamma_h^+ \right) |HF\rangle \]

In this case the dependence on the parameters \(\lambda_k\) appears through the structure of \(\Gamma_p^+\) and \(\Gamma_h\) and also in an indirect way in the normalization constant \(\mathcal{B}_\Omega\).
B. The Coupled Cluster Theory

The CCT has been proven in the past to be a highly performant method for the calculation of correlation functions \[18\]. It has, however, never been tested for pairing model hamiltonians which is an interesting study case because of its exact solvability, even for very large number of particles.

The Hamiltonian of the picket fence model can be written in the particle-hole basis as

\[
H = \sum_p \varepsilon'_p n_p + \sum_h \varepsilon'_h n_h - G \left\{ \sum_{p \neq p'} A_p^\dagger A_{p'} + \sum_{h \neq h'} A_h^\dagger A_{h'} + \sum_{ph} [A_p^\dagger A_h + A_h^\dagger A_p] \right\}
\]  

(28)

where

\[
\varepsilon'_p = \varepsilon_p - G/2 \quad \text{and} \quad \varepsilon'_h = \varepsilon_h - G/2
\]

The unnormalized CCT wave function in the SUB2 approximation \[18\] is

\[
|\Psi\rangle = e^{S^2} |HF\rangle , \quad S^2 = \sum_{ph} x_{ph} A_p^\dagger A_h
\]  

(29)

where the HF Slater determinant is given in (13). We have stopped at the one p-pair one h-pair, i.e. at the SUB2 level for reasons given below. The aim of the CCT is to determine the parameters \(x_{ph}\) and the ground state energy. Acting with the Hamiltonian on the wave function we have

\[
H |\Psi\rangle = E |\Psi\rangle = E e^{S^2} |HF\rangle
\]  

(30)

The key point of the CCT is to multiply (30) with \(e^{-S^2}\) from the left. Then

\[
e^{-S^2} H |\Psi\rangle = E |HF\rangle
\]  

(31)

Projecting on the HF bra

\[
E = \langle HF | e^{-S^2} H e^{S^2} |HF\rangle ,
\]  

(32)

taking into account that
\[ S_2^+ |HF] = \langle HF| S_2 = 0 \quad (33) \]

Equation (32) is reduced to
\[ E = \langle HF| He^{S_z^2} |HF \rangle \quad (34) \]

Having in mind the form of the Hamiltonian (28), the groundstate energy is
\[ E = E_{HF} - G \sum_{ph} x_{ph} \quad (35) \]

The amplitudes \( x_{ph} \) are determined from the set of equations
\[ \langle HF| A_h^\dagger A_p e^{-S_z^2} He^{S_z^2} |HF \rangle = 0 \quad (36) \]

which follows immediately after (31)

Eq. (36) can be expanded as
\[ \langle HF| A_h^\dagger A_p (1 - S_z) H (1 + S_z + S_z^2/2) |HF \rangle = 0 \quad (37) \]

The different terms are
\[ \langle HF| A_h^\dagger A_p H |HF \rangle = -G \]
\[ \langle HF| A_h^\dagger A_p (-S_z) H |HF \rangle = -x_{ph} E_{HF} \]
\[ \langle HF| A_{h_1}^\dagger A_{h_2}^\dagger A_{h_3} A_{h_4} |HF \rangle = (1 - \delta_{h_1 h_2}) (\delta_{h_1 h_4} \delta_{h_2 h_3} + \delta_{h_1 h_3} \delta_{h_2 h_4}) \]
\[ \langle HF| A_h^\dagger A_p HS_2 |HF \rangle = 2 \left( \varepsilon'_p - \varepsilon'_h \right) x_{ph} + 2x_{ph} \sum_{h'} \varepsilon'_{h'} - G \left[ \sum_{p' \neq p} x_{p'h} + \sum_{h' \neq h} x_{ph'} \right] \]
\[ \langle HF| A_h^\dagger A_p (-S_z) HS_2 |HF \rangle = G x_{ph} \sum_{p' \neq h'} x_{p'h'} \]
\[ \frac{1}{2} \langle HF| A_h^\dagger A_p HS_2^2 |HF \rangle = -G \left( x_{ph} \sum_{p' \neq h} x_{p'h'} + \sum_{p' \neq h, h' \neq h} x_{ph} x_{p'h'} \right) \]

And therefore it is possible to write the equation for \( x_{ph} \) as
\[ 2 \left( \varepsilon'_p - \varepsilon'_h \right) x_{ph} + 2Gx_{ph} \sum_{h'} x_{p'h'} + 2Gx_{ph} \sum_{p'} x_{p'h} + 2Gx_{ph}^2 - G \]
\[ - G \sum_{p'} x_{p'h} - G \sum_{h'} x_{p'h'} + Gx_{ph} - G \sum_{p'h'} x_{p'h} x_{p'h'} = 0 \quad (38) \]

This equation can be solved numerically.
C. Self Consistent RPA

The SCRPA for the Picket-Fence model has been developed in great detail in Ref. [21,22]. Here we give a brief summary. The basic ingredients of the SCRPA approach in the particle-particle channel are the two particle addition operator

$$A_\mu^\dagger = \sum_p X_\mu^p \overline{Q}_p - \sum_h Y_\mu^h \overline{Q}_h ,$$

(39)

and the removal operator

$$R_\lambda^\dagger = -\sum_p Y_\lambda^p \overline{Q}_p + \sum_h X_\lambda^h \overline{Q}_h ,$$

(40)

where $\overline{Q}_p = A_p/\sqrt{1 - \langle n_p \rangle}$ and $\overline{Q}_h = -A_h^\dagger/\sqrt{\langle n_h \rangle - 1}$. Where the expectation values are referred to the SCRPA vacuum defined as

$$A_\mu |SCRPA\rangle = R_\lambda |SCRPA\rangle = 0$$

(41)

and the collective RPA excitations are

$$|N + 2\rangle_\mu = A_\mu^\dagger |SCRPA\rangle , \quad |N - 2\rangle_\lambda = R_\lambda^\dagger |SCRPA\rangle$$

(42)

The equation of motion method applied to these operators leads directly to the SCRPA equations

$$\begin{pmatrix} A & B \\ -B & C \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = E \begin{pmatrix} X \\ Y \end{pmatrix} ,$$

(43)

where

$$A_{\mu\nu'} = \langle 0| [\overline{Q}_\mu, [H, \overline{Q}_{\nu'}]] |0\rangle$$

$$= \delta_{\mu\nu'} \left\{ 2\varepsilon_p + G + 2G \frac{G}{1 - \langle n_p \rangle} \left\langle \left( \sum_{p_1} A_{p_1}^\dagger + \sum_{h_1} A_{h_1}^\dagger \right) A_p \right\rangle \right\} - G \frac{\left\langle (1 - n_p)(1 - n_{\nu'}) \right\rangle}{\sqrt{(1 - \langle n_p \rangle)(1 - \langle n_{\nu'} \rangle)}} ,$$

(44)
\[ B_{ph} = \langle 0 | [Q_p, [H, \overline{Q}_h^\dagger]] | 0 \rangle = G \frac{(1 - n_p)(n_h - 1)}{\sqrt{(1 - \langle n_p \rangle)(\langle n_h \rangle - 1})}, \tag{44} \]

\[ C_{hh'} = \langle 0 | [\overline{Q}_h, [H, \overline{Q}_{h'}^\dagger]] | 0 \rangle \]

\[ = \delta_{hh'} \left\{ -2\varepsilon_h + G - 2 G \left( \frac{1}{\langle n_h \rangle - 1} \right) \left( A_h (\sum_{p_i} A_{p_i}^\dagger + \sum_{h_i} A_{h_i}^\dagger) \right) \right\} + G \frac{\langle (n_h - 1)(n_{h'} - 1) \rangle}{\sqrt{(\langle n_h \rangle - 1)(\langle n_{h'} \rangle - 1)}}. \]

Since the amplitudes \( X \) and \( Y \) form a complete orthonormal set of eigenvectors in (43) one can invert the Bogoliubov transformation of fermion pair operators (39,40) and all expectation values in (44) can be expressed in terms of the RPA amplitudes and of the number operators expectation values \( \langle n_p \rangle, \langle n_h \rangle, \langle n_{p'p'} \rangle, \langle n_{p'nh} \rangle, \) and \( \langle n_{nh'h'} \rangle \). For the particular case of the Picket Fence models these expectation values can be calculated exactly within the SCRPA approximation as shown in [21]. In this way the SCRPA constitutes a closed set of equations without any further approximation than the definition of the collective operators (39,40) and the corresponding vacuum condition (41).

Knowing these expectation values we can evaluate the SCRPA ground state energy:

\[ \langle H \rangle = \sum_p \varepsilon_p' \langle n_p \rangle + \sum_h \varepsilon_h' \langle n_h \rangle - G \left\{ \sum_{p \neq p'} \langle A_p^\dagger A_{p'} \rangle + \sum_{h \neq h'} \langle A_h^\dagger A_{h'} \rangle + \sum_{ph} \langle A_p^\dagger A_h + A_h^\dagger A_p \rangle \right\} \tag{45} \]

Assuming that the single particle energies \( \varepsilon_i \) are all equally spaced, separated by an energy gap \( \varepsilon \), we have \( \varepsilon_i' = \varepsilon_i - \varepsilon / 2 + G / 2 \). In this case the SCRPA correlation energy is

\[ E_{corr}^{SCRPA} = \langle H \rangle + \varepsilon M^2. \tag{46} \]

**D. Results and discussion**

We will study the approximate descriptions of the pairing interaction in the deformed nuclear region characterized by a constant density of levels near the Fermi surface. This situation, therefore, can be represented by a set of equally spaced levels with the appropriate density. We have used 100 levels with a constant level spacing of 300 keV and with 100 nucleons (half filling). This represents typical values of the level density and neutron numbers.
in the rare earth region \((A \simeq 170)\). As in this region the gap has a value of the order of \(\Delta \simeq 0.8\) MeV the physical value of the pairing interaction \(G \simeq 0.1\) MeV. For this level density and number of particles the critical pairing strength of the model in the BCS approximation turns out to be \(G_c \simeq 0.055\) MeV.

The aim here is to compare the quality of different approximations to treat the pairing problem which are outlined in the text. We display in Fig. 1 the ground state energy obtained using the various methods discussed in the previous section (only the correlation energy is displayed to isolate the effects due to the interaction). All the correlations energies are given in terms of the exact energy. Standard BCS approximation provides a rather poor description. The numerical results do not appear in Fig. 1 because they are out of scale. A strong improvement over BCS is obtained with the number projection before variation, i.e. the PBCS procedure. Still quite a bit better works the Exp method for moderate values of \(G\), described at the end of section III.A, with the factorisable ansatz in the exponential. Both curves show a typical structure: for small \(G\) there is a linear regime which can be qualified as the perturbative regime. It is followed by a part with negative curvature, characterized by precritical fluctuations, before the superfluid regime develops after the minimum. A detailed study of the two former regimes has been performed in ref. [20]. The figure also shows a clear indication that the PBCS approximation approaches the exact groundstate energy in the large \(G\) limit while this is not the case for the Exp method. Both approximations underbind, as it should be for a strictly variational theory in the sense of Raleigh-Ritz. On the contrary CCT \((expS_2)\) and SCRPA overbind because neither CCT nor SCRPA in general correspond to a Raleigh-Ritz theory. However, in absolute values both of the latter theories work extremely well. It should be pointed out that since SCRPA is a theory for two body correlation functions, we only can go in CCT up to the \(SUB2\) approximation, for consistency. Going to higher approximations, we should also include higher than two-body correlations and SCRPA and CCT would not be on the same level of approximation.

We only have worked in the normal particle basis for CCT and SCRPA and therefore the iterative solution of the eqs (38) and (44) did not converge any longer beyond \(G/G_c \sim 1.3\).
We know from experience in other models [19] that around the mean field phase transition point one has to change to the "deformed" basis which means to the quasiparticle basis in our case. For PBCS and Exp the error in the correlation energy in the superfluid phase decreases for $G \geq G_c$ and therefore the correlation energy has its maximal error in the transition region as it is to be expected. For the picket fence model we have not yet worked out the SCRPA in the superfluid phase and we are not aware of any attempt to apply CCT in this regime. As mentioned before, both curves in Fig. 1 stop at the point where we do not find a numerical solutions of the corresponding equations any more. We, however, conjecture that the end points of both curves represent the maximal error and continuing the calculation in the superfluid phase the error would start decreasing again. We see that the errors in $expS_2$ and SCRPA are, in the worst case, only of 5% and 2% respectively. These errors are much smaller than PBCS and Exp which are of the order of 15% − 20%.

The very small errors of $expS_2$ and SCRPA is a very satisfying result which confirms earlier positive results with these theories for correlation functions in other cases. The factor two improvement of SCRPA over $expS_2$ for the correlation energy in the transitional region has already been found in another model study [19] but this may be accidental. Grossly speaking both methods are of similar characteristics and accuracy for the correlation energy in the normal phase. The main advantage we see in SCRPA is that excitation energies and correlation functions are obtained simultaneously from the same theory. The SCRPA excitation energies also turn out to be very accurate in the present model (see ref. [21]). In CCT the excitation energies have to be constructed separately putting new ingredients into the theory.

In conclusion in this work we have compared four methods for the calculation of energies in the pairing case with parameters typical for deformed nuclei. This study was performed in the picket fence model with a model space of a hundred levels. The exact solution could be obtained owing to the method proposed by Richardson long time ago, whereas a brute force diagonalization is far beyond the limits of present computers. We found that the $expS_2$ and the SCRPA methods are quite superior to the other variational methods in the normal
phase. The results obtained in this work might stimulate further efforts to extend both approximations to the superfluid regime and more realistic forces.

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Figure Captions

Figure 1: Ratio between the approximate and the exact correlation energies for equally spaced levels as a function of the pairing strength for the four approximations discussed in Section III.
