REGULARITY ESTIMATES FOR ELLIPTIC NONLOCAL OPERATORS

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ABSTRACT. We study weak solutions to nonlocal equations governed by integrodifferential operators. Solutions are defined with the help of symmetric nonlocal bilinear forms. Throughout this work, our main emphasis is on operators with general, possibly singular, measurable kernels. We obtain regularity results which are robust with respect to the differentiability order of the equation. Furthermore, we provide a general tool for the derivation of Hölder a-priori estimates from the weak Harnack inequality. This tool is applicable for several local and nonlocal, linear and nonlinear problems on metric spaces. Another aim of this work is to provide comparability results for nonlocal quadratic forms.

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1. INTRODUCTION

The aim of this work is to develop a local regularity theory for general nonlocal operators. The main focus is on operators that are defined through families of measures, which might be singular. The main question that we ask is the following. Given a function \( u : \mathbb{R}^d \to \mathbb{R} \) satisfying

\[
\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^d \setminus B_\varepsilon(x)} (u(y) - u(x)) \mu(x, dy) = f(x) \quad (x \in D),
\]

(1.1)

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which properties of \( u \) can be deduced in the interior of \( D \)? Here \( D \subset \mathbb{R}^d \) is a bounded open set and the family \( (\mu(x, \cdot))_{x \in D} \) of measures satisfies some assumptions to be discussed later in detail. The measures \( \mu(x, \cdot) \) are assumed to have a singularity for sets \( A \subset \mathbb{R}^d \) with \( x \in \overline{A} \). As a result, the operators of the form (1.1) are not bounded integral operators but integrodifferential operators. For this reason we are able to prove regularity results which resemble results for differential operators. One aim of this work is to address an important conjecture in this field:

**Conjecture:** Assume \( \mu(x, dy) \) is uniformly (w.r.t. the variable \( x \)) comparable on small scales (w.r.t. the variable \( y \)) to \( \nu^\alpha(dy - \{x\}) \) for some \( \alpha \)-stable measure \( \nu^\alpha \) and

\[
\inf_{\xi \in \mathbb{S}^{d-1}} \min_{B_1} |\langle h, \xi \rangle|^2 \nu^\alpha(dh) > 0
\]

for some \( \alpha \in (0, 2) \). Then solutions to (1.1) satisfy uniform H"older regularity estimates in the interior of \( D \).

This conjecture has received significant attention over the last years and we give a small overview of results below. Note that, assuming comparability of measures rather than of corresponding densities allows for a much wider class of cases that can be treated. In this work we provide a structural approach to this problem. We give an affirmative answer if \( \mu(x, \cdot) \) is absolutely continuous on \( \mathbb{R}^d \) or on sufficiently many subspaces. Note that it is well known how to treat functions \( f \) in (1.1). Thus we will concentrate on the case \( f = 0 \).

In order to approach the question raised above, we need to establish the following results:

- weak Harnack inequality,
- implications of the weak Harnack inequality,
- comparability results for nonlocal quadratic forms.

The last topic needs to be included because our concept of solutions involves quadratic forms related to \( \mu(x, dy) \). We present the main results in Subsection 1.3, Subsection 1.4, and in Subsection 1.5. The following two subsections are devoted to the set-up and our main assumptions.

### 1.1 Function spaces

Before we can formulate the first result we need to set up quadratic forms and function spaces. Let \( \mu = (\mu(x, \cdot))_{x \in \mathbb{R}^d} \) be a family of measures on \( \mathbb{R}^d \) which is symmetric in the sense that for every set \( A \times B \subset \mathbb{R}^d \times \mathbb{R}^d \setminus \text{diag} \)

\[
\int_A \int_B \mu(x, dy) dx = \int_B \int_A \mu(x, dy) dx.
\]  
(1.2)

We furthermore require

\[
\sup_{x \in \mathbb{R}^d} \min_{\mathbb{R}^d} (|x - y|^2, 1) \mu(x, dy) < +\infty.
\]  
(1.3)

**Example 1.** An important example satisfying the above conditions is given by

\[
\mu_\alpha(x, dy) = (2-\alpha)|x - y|^{-d-\alpha} dy \quad (0 < \alpha < 2).
\]  
(1.4)

The choice of the factor \((2-\alpha)\) will be discussed below in detail, see Subsection 1.2 and Section 2.
For a given family $\mu$ and a real number $\alpha \in (0, 2)$ we consider the following quadratic forms on $L^2(D) \times L^2(D)$, where $D \subset \mathbb{R}^d$ is some open set:

$$\mathcal{E}_\mu^D(u, u) = \int_D \int_D (u(y) - u(x))^2 \mu(x, dy) dx.$$  \hspace{1cm} (1.5)

We denote by $H^{\alpha/2}(\mathbb{R}^d)$ the usual Sobolev space of fractional order $\alpha/2 \in (0, 1)$ with the norm

$$\|u\|_{H^{\alpha/2}(\mathbb{R}^d)} = \left( \|u\|_{L^2(\mathbb{R}^d)}^2 + \mathcal{E}_{\mu}^\alpha(u, u) \right)^{1/2}. \hspace{1cm} (1.6)$$

If $D \subset \mathbb{R}^d$ is open and bounded, then by $H^{\alpha/2}_D(\mathbb{R}^d)$ we denote the Banach space of functions from $H^{\alpha/2}(\mathbb{R}^d)$ which are zero almost everywhere on $D^c$. $H^{\alpha/2}(D)$ shall be the space of functions $u \in L^2(D)$ for which

$$\|u\|_{H^{\alpha/2}(D)} = \|u\|_{L^2(D)} + \int_D \int_D (u(y) - u(x))^2 \mu_\alpha(x, dy) dx$$

is finite. Note that, for domains $D$ with a Lipschitz boundary, $H^{\alpha/2}_D(\mathbb{R}^d)$ can be identified with the closure of $C_0^\infty(D)$ with respect to the norm of $H^{\alpha/2}(D)$. In general, these two objects might be different, though. By $V^{\alpha/2}_D = V^{\alpha/2}_D(\mathbb{R}^d)$ we denote the space of all measurable functions $u : \mathbb{R}^d \to \mathbb{R}$ for which the quantity

$$\int_D \int_{\mathbb{R}^d} \frac{(u(y) - u(x))^2}{|x-y|^{d+\alpha}} dx dy \hspace{1cm} (1.7)$$

is finite, which implies finiteness of the quantity $\int_{\mathbb{R}^d} \frac{u(x)^2}{(1+|x|)^{d+\alpha}} dx$. The function space $V^{\alpha/2}_D$ is a Hilbert space with the scalar product

$$(u, v)_{V^{\alpha/2}_D} = \int \frac{u(x)v(x)}{(1+|x|)^{d+\alpha}} dx + \int_D \int_{\mathbb{R}^d} \frac{(u(y) - u(x))(v(y) - v(x))}{|x-y|^{d+\alpha}} dx dy. \hspace{1cm} (1.8)$$

The proof is similar to the one of [25, Lemma 2.3] and the one of [31, Proposition 3.1]. If the scalar product (1.8) is defined with the expression replaced $\int \frac{u(x)v(x)}{(1+|x|)^{d+\alpha}} dx$ by $\int_D u(x)v(x) dx$, then the Hilbert space is identical. The following continuous embeddings trivially hold true:

$$H^{\alpha/2}_D(\mathbb{R}^d) \hookrightarrow H^{\alpha/2}(\mathbb{R}^d) \hookrightarrow V^{\alpha/2}_D(\mathbb{R}^d).$$

We make use of function spaces generated by general $\mu$ in the same way as above. Let $H^\mu(\mathbb{R}^d)$ be the vector space of functions $u \in L^2(\mathbb{R}^d)$ such that $\mathcal{E}^\mu(u, u) = \mathcal{E}_{\mu}^\alpha(u, u)$ is finite. If $D \subset \mathbb{R}^d$ is open and bounded, then by $H^\mu_D = H^\mu_D(\mathbb{R}^d)$ we denote the space of functions from $H^\mu(\mathbb{R}^d)$ which are zero almost everywhere on $D^c$. By $V^\mu_D = V^\mu_D(\mathbb{R}^d)$ we denote the space of all measurable functions $u : \mathbb{R}^d \to \mathbb{R}$ for which the quantity

$$\int_D \int_{\mathbb{R}^d} (u(y) - u(x))^2 \mu(x, dy) dx \hspace{1cm} (1.9)$$

is finite. Now we are in a position to present and discuss our main results.
1.2. Main Assumptions. Let us formulate our main assumptions on \((\mu(x, \cdot))_{x \in D}\). Given \(\alpha \in (0, 2)\) and \(A \geq 1\), the following condition is an analog of (A') for nonlocal energy forms:

For every ball \(B_\rho(x_0)\) with \(\rho \in (0, 1)\), \(x_0 \in B_1\) and every \(v \in H^{\alpha/2}(B_\rho(x_0))\):

\[
A^{-1} E^\mu_{B_\rho(x_0)}(v, v) \leq E^{\mu_\alpha}_{B_\rho(x_0)}(v, v) \leq A E^{\mu}_{B_\rho(x_0)}(v, v). \tag{A}
\]

Condition (A) says that, locally in the unit ball, the energies \(E^\mu\) and \(E^{\mu_\alpha}\) are comparable on every scale. Note that this does not imply pointwise comparability of the densities of \(\mu\) and \(\mu_\alpha\). We also need to assume the existence of cut-off functions. Let \(\alpha \in (0, 2)\) and \(B \geq 1\).

For \(0 < \rho \leq R \leq 1\) and \(x_0 \in B_1\) there is a nonnegative measurable function \(\tau : \mathbb{R}^d \rightarrow \mathbb{R}\) with \(\text{supp}(\tau) \subset \overline{B_{R+\rho}(x_0)}\), \(\tau(x) \equiv 1\) on \(B_R(x_0)\), \(\|\tau\|_\infty \leq 1\), and

\[
\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} (\tau(y) - \tau(x))^2 \mu(x, dy) \leq B \rho^{-\alpha}. \tag{B}
\]

In most of the cases (B) does not impose an additional restriction because the standard cut-off function \(\tau(x) = \max(0, 1 + \min(0, \frac{R-(x-x_0)}{\rho}))\) is an appropriate choice. It is an interesting question whether, under assumptions (1.2), (1.3) and (A), this standard choice would be possible in (B). Note that, condition (B) becomes \(|\nabla \tau|^2 \leq B \rho^{-2}\) when \(\alpha \rightarrow 2-\) and \(\mu(x, dy)\) is as in Example 1.

For every \(\alpha \in (0, 2)\), the family of measures \(\mu_\alpha\) given in Example 1 satisfies the above conditions for some constants \(A, B \geq 1\). The normalizing constant \((2 - \alpha)\) in the definition of \(\mu_\alpha\) has the effect that the constants \(A, B \geq 1\) can be chosen independently of \(\alpha\) for \(\alpha \rightarrow 2-\). Since in this work we do not care about the behavior of constants for \(\alpha \rightarrow 0+\), in our examples we will use factors of the form \(2 - \alpha\). Let us look at more examples.

**Example 2.** Assume \(0 < \beta \leq \alpha < 2\). Let \(f, g : \mathbb{R}^d \rightarrow [1, 2]\) be measurable and symmetric functions. Set

\[
\mu(x, dy) = f(x, y) \mu_\alpha(x, dy) + g(x, y) \mu_\beta(x, dy).
\]

Then \(\mu\) satisfies (1.2), (1.3), (A), and (B) with exponent \(\alpha\). This simply follows from

\[
\frac{1}{|x-y|^{d+\alpha}} \leq \frac{1}{|x-y|^{d+\beta}} + \frac{1}{|x-y|^{d+\alpha}} \leq \frac{2}{|x-y|^{d+\alpha}} \quad (x, y \in B_1(x_0), x_0 \in \mathbb{R}^d).
\]

For the verification of (B) we may choose the standard Lipschitz-continuous cutoff function.

Here is an example with some kernels which are not rotationally symmetric.

**Example 3.** Assume \(\alpha_0 \in (0, 2)\). \(0 < \lambda \leq \Lambda\), \(v \in S^{d-1}\) and \(\theta \in [0, 1]\). Set \(M = \{h \in \mathbb{R}^d \mid \|\frac{h}{|h|}, v\| \geq \theta\}\). Let \(k : \mathbb{R}^d \times \mathbb{R}^d \rightarrow [0, \infty]\) be any measurable function satisfying

\[
\lambda 1_M(x-y) \frac{(2-\alpha)}{|x-y|^{d+\alpha}} \leq k(x, y) \leq \Lambda \frac{(2-\alpha)}{|x-y|^{d+\alpha}} \tag{1.10}
\]

for some \(\alpha \in [\alpha_0, 2]\) and for almost every \(x, y \in \mathbb{R}^d\). Set \(\mu(x, dy) = k(x, y)dy\). Then, as we will prove, there are \(A \geq 1, B \geq 1\), independent of \(\alpha\), such that (A) and (B) hold.

The following example of a family of measures falls into our framework. Note that the measures do not possess a density with respect to the \(d\)-dimensional Lebesgue measure.
Example 4. Assume $\alpha_0 \in (0, 2)$, $\alpha_0 \leq \alpha < 2$. Set
\[
\mu(x, dy) = (2 - \alpha) \sum_{i=1}^{d} |x_i - y_i|^{-1 - \alpha} \delta_{(x_i)}(dy_i).
\] (1.11)

Again, as we will prove, there are $A \geq 1, B \geq 1$, independent of $\alpha$, such that (A) and (B) hold. Note that $\mu(x, A) = 0$ for every set $A$ which has an empty intersection with any of the $d$ lines $\{x + te_i | t \in \mathbb{R}\}$.

Let us now formulate our results.

1.3. The Weak Harnack Inequality. Given functions $u, v : \mathbb{R}^d \to \mathbb{R}$ we define the quantity
\[
\mathcal{E}^\mu(u, v) = \iint_{\mathbb{R}^d \times \mathbb{R}^d} (u(y) - u(x))(v(y) - v(x)) \mu(x, dy) dx,
\] (1.12)
if it is finite. We write $\mathcal{E}$ instead of $\mathcal{E}^\mu$ when it is clear resp. irrelevant which measure $\mu$ is used. One aim of this work is to study properties of functions $u$ satisfying $\mathcal{E}(u, \phi) \geq 0$ for every nonnegative test function $\phi$. Note that $\mathcal{E}^\mu(u, \phi)$ is finite for $u \in V^\mu_{B_1}$, $\phi \in H^\mu_{B_1}(\mathbb{R}^d)$ for any open set $D \subset \mathbb{R}^d$. This follows from the definition of these function spaces, the Cauchy-Schwarz inequality and the following decomposition:
\[
\mathcal{E}^\mu(u, \phi) = \iint_{D \times D} (u(y) - u(x))(\phi(y) - \phi(x)) \mu(x, dy) dx + 2 \iint_{D \times D} (u(y) - u(x))(\phi(y) - \phi(x)) \mu(x, dy) dx.
\]

Here is our first main result.

Theorem 1.1 (Weak Harnack Inequality). Assume $0 < \alpha_0 < 2$ and $A \geq 1, B \geq 1$. Let $\mu$ satisfy (A), (B) for some $\alpha \in [\alpha_0, 2)$. Assume $f \in L^{q/\alpha}(B_1)$ for some $q > d$. Let $u \in V^\mu_{B_1}(\mathbb{R}^d)$, $u \geq 0$ in $B_1$, satisfy $\mathcal{E}^\mu(u, \phi) \geq (f, \phi)$ for every nonnegative $\phi \in H^\mu_{B_1}(\mathbb{R}^d)$. Then
\[
\inf_{B_1} u \geq c \left( \int_{B_1} u(x)^{p_0} dx \right)^{1/p_0} - \sup_{x \in B_1} \int_{\mathbb{R}^d \setminus B_1} u^{-}(z) \mu(x, dz) - \|f\|_{L^{q/\alpha}(B_1)},
\] (1.13)
with constants $p_0, c \in (0, 1)$ depending only on $d, \alpha_0, A, B$. In particular, $p_0$ and $c$ do not depend on $\alpha$.

Note that, below we explain a local counterpart to this result, which relates to the limit $\alpha \to 2-$, cf. Theorem 1.6.

Remark. It is remarkable that (A) and (B) do not imply a strong formulation of the Harnack inequality. Both, Example 4 and Example 3 provide cases in which the classical strong formulation fails. See the discussion in [22, Appendix A.1] and the concrete examples in [6, p. 148] and [3, Sec. 3]. The nonlocal term, i.e. the integral of $u^{-}$ in (1.13) is unavoidable since we do not assume nonnegativity of $u$ in all of $\mathbb{R}^d$. 
1.4. **Regularity estimates.** A separate aim of our work is to provide consequences of the (weak) Harnack inequality. Before we explain this in a more abstract fashion let us formulate a regularity result, which will be derived from Theorem 1.1 and which is one of the main results of this work. We need an additional mild assumption on the decay of the kernels considered.

Given $\alpha \in (0, 2)$ we assume that for some constants $\chi > 1$, $C \geq 1$

$$
\mu(x, \mathbb{R}^d \setminus B_r, \chi)(x) \leq Cr^{-\alpha \chi - j} \quad (x \in B_1, 0 < r \leq 1, j \in \mathbb{N}_0).
$$

(D)

Condition (D) rules out kernels with very heavy tails for large values of $|x - y|$. For example, $\mu$ given by $\mu(x, dy) = k(x, y)dy$ with $k(x, y) = |x - y|^{-d-1} + |x - y|^{-d} \ln(2 + |x - y|)^{-2}$ does not satisfy (D).

Here is our main regularity result.

**Theorem 1.2.** Let $\alpha_0 \in (0, 2), \gamma > 0$ and $A \geq 1, B \geq 1$. Let $\mu$ satisfy (A), (B) and (D) for some $\alpha \in [\alpha_0, 2]$. Assume $u \in V^\mu(B_1)$ satisfies $\mathcal{E}(u, \phi) = 0$ for some $x_0 \in \mathbb{R}^n$ and every $\phi \in H^\mu_{B_1}(\mathbb{R}^d)$. Then the following Hölder estimate holds for almost every $x, y \in B_1$:

$$
|u(x) - u(y)| \leq c\|u\|_{\infty}|x - y|^{\beta},
$$

(1.14)

where $c \geq 1$ and $\beta \in (0, 1)$ are constants which depend only on $d, \alpha_0, A, B, C, \gamma$. In particular, $c$ and $\beta$ do not depend on $\alpha$.

This result contrasts the corresponding result for differential operators, see Theorem 1.7 below.

The main tool for the proof of Theorem 1.2 is the weak Harnack inequality, Theorem 1.1. The Harnack inequality itself is an interesting object of study for nonlocal operators. In Section 2 we have explained different formulations of the Harnack inequality for nonlocal operators satisfying a maximum principle. A separate aim of this article is to prove a general tool that allows to deduce regularity estimates from the Harnack inequality for nonlocal operators. This step was subject to discussion of many recent articles in the field. We choose the set-up of a metric measure space so that this tool can be of future use in different contexts.

In the first decades after publication the Harnack inequality itself did not attract as much of attention as the resulting convergence theorems. This changed when J. Moser in 1961 showed that the inequality itself leads to a-priori estimates in Hölder spaces. His result can be formulated in a metric measure space $(X, d, m)$ as follows. For $r > 0, x \in X$, set $B_r(x) = \{y \in X \mid d(y, x) < r\}$.

For every $x \in X$ and $r > 0$ let $S_{x,r}$ denote a family of measurable functions on $X$ satisfying the following conditions:

$$
\begin{align*}
  r > 0, u \in S_{x,r}, a \in \mathbb{R} & \quad \Rightarrow \quad au \in S_{x,r}, (u + 1) \in S_{x,r}, \\
  B_r(x) \subset B_a(y) & \quad \Rightarrow \quad S_{y,\delta} \subset S_{x,r}.
\end{align*}
$$

An example for $S_{x,r}$ is given by the set of all functions $u : \mathbb{R}^d \to \mathbb{R}$ satisfying some (possibly nonlinear) appropriate partial differential or integro-differential equation in a ball $B_r(x)$.

**Theorem 1.3** (compare [27]). Assume $X$ is separable. Let $x_0 \in X$ and $S_{x,r}$ be as above. Assume that there is $c \geq 1$ such that for $r > 0$,

$$
(u \in S_{x_0,r}) \land (u \geq 0 \text{ in } B_r(x_0)) \quad \Rightarrow \quad \sup_{x \in B_{1/2}(x_0)} u \leq c \inf_{x \in B_{1/2}(x_0)} u.
$$

(1.15)
Then there exist $\beta \in (0, 1)$ such that for $r > 0$, $u \in \mathcal{S}_{x_0, r}$ and almost every $x \in B_r(x_0)$
\[
|u(x) - u(x_0)| \leq 3\|u - u(x_0)\|_{\infty} \left(\frac{d(x, x_0)}{r}\right)^\beta.
\]

Recall that 'sup' denotes the essential supremum and 'inf' the essential infimum. With the help of this theorem, regularity estimates can be established for various linear and nonlinear differential equations, see [15]. One aim of this article is to show that (1.15) can be relaxed significantly by allowing some global terms of $u$ to show up in the Harnack inequality. Already in Section 2 we have seen that they naturally appear.

For $x \in X, r > 0$ let $\nu_{x, r}$ be a measure on $\mathcal{B}(X \setminus \{x\})$, which is finite on all sets $M$ with $\text{dist}(\{x\}, M) > 0$. We assume that for some $c \geq 1, \chi > 1$, and for every $j \in \mathbb{N}_0$, $x \in X$ and $0 < r \leq 1$
\[
\nu_{x, r}(X \setminus B_r(x)) \leq c\chi^{-j}.
\]  

We further assume that, given $K > 1$ there is $c \geq 1$ such that for $0 < r \leq R \leq Kr, x \in X, M \subset X \setminus B_r(x)$
\[
\nu_{x, R}(M) \leq c\nu_{x, r}(M).
\]  

Conditions (1.16) and (1.17) will trivially hold true in the applications that are of importance to us.

**Example 5.** Let $\alpha \in (0, 2)$. For $x \in \mathbb{R}^d, r > 0$ and $A \in \mathcal{B}(\mathbb{R}^d \setminus \{x\})$ set
\[
\nu_{x, r}(A) = r^\alpha \mu_\alpha(x, A) = r^\alpha \alpha(2 - \alpha) \int_A |x - y|^{-d - \alpha} dy.
\]  

Then $\nu_{x, r}$ satisfies conditions (1.16), (1.17).

In Section 5 we discuss this condition in detail. A standard example for us is Example 5. The following result extends Theorem 1.3 to situations with nonlocal terms. It is an important tool in the theory of nonlocal operators.

**Theorem 1.4.** Let $x_0 \in X, r_0 > 0$ and $\lambda > 1, \sigma > 1, \theta > 1$. Let $\mathcal{S}_{x, r}$ and $\nu_{x, r}$ be as above. Assume that conditions (1.16), (1.17) are satisfied. Assume that there is $c \geq 1$ such that for $0 < r \leq r_0$,
\[
\left\{ \begin{array}{l}
(u \in \mathcal{S}_{x_0, r}) \land (u \geq 0 \text{ in } B_r(x_0)), \\
\Rightarrow \left\{ \begin{array}{l}
\frac{\int_{B_r(x_0)} u(x)^p m(dx)}{\inf_{x \in B_r(x_0)} u} \leq c \inf_{x \in B_r(x_0)} u + c \sup_{x \in B_r(x_0)} \int u^{-}(z) \nu_{x, r}(dz).
\end{array} \right.
\end{array} \right.
\]  

Then there exist $\beta \in (0, 1)$ such that for $0 < r \leq r_0, u \in \mathcal{S}_{x_0, r}$
\[
\text{osc}_{B_r(x_0)} u \leq 2\theta^\beta \|u\|_{\infty} \left(\frac{2}{r}\right)^\beta
\]  

(0 < r \leq r),

where $\text{osc}_M u := \sup_M u - \inf_M u$ for $M \subset X$.

Note that, in Lemma 5.1 we provide several conditions that are equivalent to (1.16).
1.5. Comparability of nonlocal quadratic forms. With regard to Theorem 1.2 one major problem is to provide conditions on $\mu$ which imply (A). Let us formulate our results in this direction.

Since $\mu = (\mu(x, \cdot))_{x \in \mathbb{R}^d}$ is a family of measures we need to impose a condition that fixes a uniform behavior of $\mu$ with respect to $x$. In our setup this condition implies that the integrodifferential operator from (1.1) is comparable to a translation invariant operator - most often the generator of an $\alpha$-stable process. We assume that there are measures $\nu_*$ and $\nu^*$ such that

$$
\int f(x, x + z) \nu_*(dz) \leq \int f(x, y) \mu(x, dy) \leq \int f(x, x + z) \nu^*(dz)
$$

for every measurable function $f : \mathbb{R}^d \to [0, \infty]$ and every $x \in \mathbb{R}^d$. For a measure $\nu$ on $\mathbb{R}^d$ such that $\nu(\{0\}) = 0$ and a set $B \subset \mathbb{R}^d$ we define, abusing the previous notation slightly,

$$
\mathcal{E}_B^\nu(u, v) = \int_B \int_{\mathbb{R}^d} (u(x) - u(x + z))(v(x) - v(x + z)) \mathbf{1}_B(x + z) \nu(dz) dx.
$$

(1.21)

Note that (T) implies for every $u \in L^2(B)$

$$
\mathcal{E}_B^\nu(u, u) \leq \mathcal{E}_B^\nu(u, u) \leq \mathcal{E}_B^\nu(u, u).
$$

Let $\overline{\nu}(A) = \nu(-A)$. It is easy to check that $\mathcal{E}^\nu = \mathcal{E}^{\overline{\nu}}$. Hence we may and do assume that the measures $\nu_*$, $\nu^*$ are symmetric, i.e., $\nu_*(A) = \nu_*(-A)$ and $\nu^*(A) = \nu^*(-A)$.

We say that a measure $\nu$ on $\mathcal{B}(\mathbb{R}^d)$ satisfies the upper-bound assumption (U) if for some $C_U > 0$

$$
\int_{\mathbb{R}^d} (r \wedge |z|)^2 \nu(dz) \leq C_U r^{2-\alpha} \quad (0 < r \leq 1).
$$

We say that a measure $\nu$ on $\mathcal{B}(\mathbb{R}^d)$ satisfies the scaling assumption (S) if for some $a > 1$

$$
\int_{\mathbb{R}^d} f(y) \nu(dy) = a^{-\alpha} \int_{\mathbb{R}^d} f(ay) \nu(dy),
$$

for every measurable function $f : \mathbb{R}^d \to [0, \infty]$ with supp $f \subset B_1$. For a linear subspace $E \subset \mathbb{R}^d$, let $H_E$ denote the dim$(E)$-dimensional Hausdorff measure supported on $E$.

We say that a measure $\nu$ on $\mathcal{B}(\mathbb{R}^d)$ satisfies the nondegeneracy assumption (ND) if for some $n \in \{1, \ldots, d\}$

$$
\nu = \sum_{k=1}^n f_k H_{E_k} \quad \text{for some linear subspaces } E_k \subset \mathbb{R}^d \text{ and densities } f_k
$$

(ND)

with $\text{lin}(\cup_k E_k) = \mathbb{R}^d$ and $\int_{B_1} f_k dH_{E_k} > 0$ for $k = 1, \ldots, n$.

Here is our result on local comparability of nonlocal energy forms:

**Theorem 1.5.** Let $\mu = (\mu(x, \cdot))_{x \in \mathbb{R}^d}$ be a family of measures on $\mathcal{B}(\mathbb{R}^d)$ satisfying (1.2). Assume that there exist measures $\nu_*$ and $\nu^*$ for which (T) and (U) hold with $\alpha_0 \in (0, 2)$ and $C_U > 0$.

Assume that $\nu_*$ satisfies (ND) and each measure $f_k H_{E_k}$ satisfies (S) for some fixed $a > 1$. Then there are $A \geq 1$, $B \geq 1$ such that (A) and (B) hold. One can choose $B = 4C_U$ but the constant $A$ depends also on $a$, the measure $\nu_*$ and on $\alpha_0$.

The result is robust in the following sense: If $\mu^\alpha = (\mu^\alpha(x, \cdot))_{x \in \mathbb{R}^d}$ satisfies (1.2) and (T) with measures $(\nu_*)^\alpha$ and $(\nu^*)^\alpha$, $\alpha_0 \leq \alpha < 2$, that are defined with the help of $\nu_*$ and $\nu^*$ as in Definition 6.5, then (A) holds with a constant $A$ independent of $\alpha \in [\alpha_0, 2]$. 
1.6. Related results.

It is instructive to compare or results with two key results for differential operators in divergence form. Let \( (A(x))_{x \in \mathbb{R}^d} \) be a family of \( d \times d \)-matrices. Given a subset \( D \subset \mathbb{R}^d \) we introduce a bilinear form \( A_D \) by \( A_D(u, v) = \int_D (\nabla u(x), A(x) \nabla u(x)) \, dx \) for \( u \) and \( v \) from the Sobolev space \( H^1(D) \). Instead of \( A_{\mathbb{R}^d} \) we write \( A \). The following theorem is at the heart of the theory named after E. DeGiorgi, J. Moser and J. Nash, see [15, Ch. 8.8-8.9]:

**Theorem 1.6** (Weak Harnack Inequality). Let \( \Lambda > 1 \). Assume that for all balls \( B \subset \mathbb{B}_1 \) and all functions \( v \in H^1(B) \)

\[
\Lambda^{-1} A_B(u, u) \leq \int_B |\nabla u|^2 \leq \Lambda A_B(u, u). \tag{A'}
\]

Assume \( f \in L^{q/2}(B_1) \) for some \( q > d \). Let \( u \in H^1(B_1) \) satisfy \( u \geq 0 \) in \( B_1 \) and \( A_{B_1}(u, \phi) \geq (f, \phi) \) for every nonnegative \( \phi \in H^1_0(B_1) \). Then

\[
\inf_{B_1} u \geq ( \int_{B_2} u(x)^{p_0} \, dx )^{1/p_0} - \|f\|_{L^{q/2}(B_{2R})},
\]

with constants \( p_0, c \in (0, 1) \) depending only on \( d \) and \( \Lambda \).

**Remark.** This by now classical result can be seen as the limit case of Theorem 1.1 for \( \alpha \to 2^- \). Condition \( (A') \) implies that the differential operator \( \text{div}(A(\cdot) \nabla u) \) is uniformly elliptic and obviously describes a limit situation of \( (A) \). One might object that the nonlocal term in (1.13) is unnatural but in fact, it is not. In Section 2 we explain this phenomenon in detail for the fractional Laplace operator.

If \( u \) is not only a supersolution but a solution in Theorem 1.6, then one obtains a classical Harnack inequality: \( \sup_{B_{3 \frac{1}{4}}} u \leq c \inf_{B_{\frac{1}{4}}} u \). Either one, the Harnack inequality and the weak Harnack inequality, imply Hölder a-priori regularity estimates:

**Theorem 1.7.** Assume condition \( (A') \) holds true. There exist \( c \geq 1, \beta \in (0, 1) \) such that for every \( u \in H^1(B_1) \) satisfying \( A(u, \phi) = 0 \) for every \( \phi \in H^1_0(B_1) \) the following Hölder estimate holds for almost every \( x, y \in B_{\frac{1}{4}} \):

\[
|u(x) - u(y)| \leq c\|u\|_\infty |x - y|^{\beta}. \tag{1.22}
\]

The constants \( \beta, c \) depend only on \( d \) and \( \Lambda \).

After having recalled corresponding results for local differential operators, let us review some related results for nonlocal problems. Note that we restrict ourselves to nonlocal equations related to bilinear forms resp. distributional solutions.

**Theorem 1.2** has already been proved under additional assumptions. If \( \mu(x, \cdot) \) has a density \( k(x, \cdot) \) which satisfies some isotropic lower bound, e.g. for some \( c_0 > 0, \alpha \in (0, 2) \)

\[
\mu(x, dy) = k(x, y) \, dy, \quad k(x, y) \geq c_0 |x - y|^{-d-\alpha} \quad (|x - y| \leq 1),
\]

then **Theorem 1.2** is proved in resp. follows from the works [24, 4, 9, 8]. In these works the constant \( c \) in (1.14) depends on \( \alpha \in (0, 2) \) with \( c(\alpha) \to +\infty \) for \( \alpha \to 2^- \). The current work follows the strategy laid out in [20] which, on the one hand, allows the constants to be independent of \( \alpha \) for \( \alpha \to 2^- \) and, on the other hand, allows to treat general measures. See [14] and [23] for corresponding results in the parabolic case.
The articles [10], [11] study Hölder regularity estimates and Harnack inequalities for nonlinear equations. Moreover, the results therein provide boundedness of weak solutions. In [10], [11] the measures \( \mu(x, dy) \) are assumed to be absolutely continuous with respect to the Lebesgue measure. Another difference to the present article is that our local regularity estimates require only local conditions on the data and on the operator. Note that our study of implications of (weak) Harnack inequalities in Section 5 allows for nonlinear problems in metric measure spaces and could be used to deduce the regularity results of [11] from results in [10].

To our best knowledge there has been no contribution addressing the question of comparability of quadratic nonlocal forms, cf. Section 6. This question becomes important when studying very irregular kernels as in [33, Section 4].

The conjecture mentioned in the beginning of the introduction has recently been established in the translation invariant case, i.e., when \( \mu(x, dy) = \nu^\alpha(dy - \{x\}) \) for some \( \alpha \)-stable measure \( \nu^\alpha \), cf. [30]. The methods of [30] seem not to be applicable in the general case, though.

Related questions on nonlocal Dirichlet forms on metric measure spaces are currently investigated by several groups. We refer to the exposition in [16] for a discussion of results regarding the fundamental solution.

1.7. Notation. Throughout this article, ”inf” denotes the essential infimum, ”sup” the essential supremum. By \( S^{d-1} = \{x \in \mathbb{R}^d | |x| = 1\} \) we denote the unit sphere. We define the Fourier transform as an isometry of \( L^2(\mathbb{R}^d) \) determined by

\[
\hat{u}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} u(x) e^{-i\xi \cdot x} \, dx, \quad u \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d).
\]

1.8. Structure of the article. The paper is organized as follows. In Section 2 we study the Harnack inequality for the Laplace and the fractional Laplace operator. We explain how one can formulate a Harnack inequality without assuming the functions under consideration to be nonnegative. In Section 3 we provide several auxiliary results and explain how the inequality \( \mathcal{E}(u, \phi) \geq (f, \phi) \) is affected by rescaling the family of measures \( \mu \). In Section 4 we prove Theorem 1.1 under assumptions (A) and (B) adapting the approach by Moser to nonlocal bilinear forms. Subsection 5.1 provides the proof of Theorem 1.2. We first prove a general tool which allows to deduce regularity results from weak Harnack inequalities, see Corollary 5.2. Then Theorem 1.2 follows immediately. In Section 6 we study the question which conditions on \( \mu \) are sufficient for conditions (A) and (B) to hold true. In addition, we provide two examples of quite irregular kernels satisfying (A) and (B).

2. HARNACK INEQUALITIES FOR THE LAPLACE AND THE FRACTIONAL LAPLACE OPERATOR

We establish a formulation of the Harnack inequality which does not require the functions to be nonnegative. This reformulation is especially interesting for nonlocal problems but our formulation seems to be new even for harmonic functions in the classical sense, see Theorem 2.5. For \( \alpha \in (0, 2) \) and \( u \in C^2_\alpha(\mathbb{R}^d) \) the fractional power of the Laplacian can be defined as follows:

\[
\Delta^{\alpha/2} u(x) = C_{\alpha, d} \lim_{\epsilon \to 0^+} \int_{|y-x| > \epsilon} \frac{u(y) - u(x)}{|y-x|^{d+\alpha}} \, dy = C_{\alpha, d} \int_{\mathbb{R}^d} \frac{u(x+h) - 2u(x) + u(x-h)}{|h|^{d+\alpha}} \, dh.
\]
where $C_{\alpha,d} = \frac{\Gamma((d+\alpha)/2)}{2^{-\alpha\pi^{d/2}}\Gamma(\alpha/2)}$. For later purposes we note that with some constant $c > 0$ for every $\alpha \in (0,2)$

$$c \alpha(2-\alpha) \leq C_{\alpha,d} \leq \frac{\alpha(2-\alpha)}{c}. \quad (2.2)$$

The use of the symbol $\Delta^{\alpha/2}$ and the term “fractional Laplacian” are justified because of $(-\Delta)^{\alpha/2}u(\xi) = |\xi|^\alpha \hat{u}(\xi)$ for $\xi \in \mathbb{R}^d$ and $u \in C_c^\infty(\mathbb{R}^d)$. Note that we write $\Delta^{\alpha/2}u$ instead of $-(-\Delta)^{\alpha/2}u$ which would be more appropriate. The potential theory of these operators was initiated in [29]. The following Harnack inequality can be easily established using the corresponding Poisson kernels.

**Theorem 2.1.** There is a constant $c \geq 1$ such that for $\alpha \in (0,2)$ and $u \in C(\mathbb{R}^d)$ with

$$\Delta^{\alpha/2}u(x) = 0 \quad (x \in B_1), \quad (2.3)$$

$$u(x) \geq 0 \quad (x \in \mathbb{R}^d), \quad (2.4)$$

the following inequality holds:

$$u(x) \leq cu(y) \quad (x, y \in B_{\frac{1}{2}}). \quad (2.5)$$

Note that $\Delta^{\alpha/2}u(x) = 0$ at a point $x \in \mathbb{R}^d$ requires that the integral in (2.1) converges. Thus some additional regularity of $u \in C(\mathbb{R}^d)$ is assumed implicitly. Since $\Delta^{\alpha/2}$ allows for shifting and scaling, the result holds true for $B_1, B_{\frac{1}{2}}$ replaced by $B_R(x_0), B_{\frac{1}{2}R}(x_0)$ with the same constant $c$ for arbitrary $x_0 \in \mathbb{R}^d$ and $R > 0$.

**Theorem 2.1** formulates the Harnack inequality in the standard way for nonlocal operators. The function $u$ is assumed to be nonnegative in all of $\mathbb{R}^d$. In the following we discuss the necessity of this assumption and possible alternatives. The following result proves that this assumption cannot be dropped completely.

**Theorem 2.2.** Assume $\alpha \in (0,2)$. Then there exists a bounded function $u \in C(\mathbb{R}^d)$, which is infinitely many times differentiable in $B_1$ and satisfies

$$\Delta^{\alpha/2}u(x) = 0 \quad (x \in B_1),$$

$$u(x) > 0 \quad (x \in B_1 \setminus \{0\}),$$

$$u(0) = 0.$$

Therefore, the classical local formulation of the Harnack inequality as well as the local maximum principle fail for the operator $\Delta^{\alpha/2}$.

A complicated and lengthy proof can be found in [18]. An elegant way to construct a function would be to mollify $v(x) = (1 - |x|^2)^{-1+\frac{\alpha}{d}}$ for $x \in B_1$. Here we provide a short proof\(^1\) which includes a helpful observation on radial functions.

For an open set $D \subset \mathbb{R}^d$, $x \in D$, $0 < \alpha \leq 2$ and $v : \mathbb{R}^d \to \mathbb{R}$ ($0 < \alpha < 2$) resp. $v : \overline{D} \to \mathbb{R}$ ($\alpha = 2$) we write

$$H_\alpha(v|D)(x) = \int_{y \notin D} P_\alpha(x,y)v(y) \, dy = \begin{cases} \int_{\mathbb{R}^d \setminus D} P_\alpha(x,y)v(y) \, dy & (0 < \alpha < 2) \\ \int_{\partial D} P_\alpha(x,y)v(y) \, dy & (\alpha = 2) \end{cases}. \quad (2.5)$$

\(^1\)We owe the idea to this proof to Wolfhard Hansen.
Note that for $R > 0$ and $f : \mathbb{R}^d \setminus B_R(0) \to \mathbb{R}$

$$H_\alpha(f|B_R(0))(x) = \begin{cases} f(x) & (|x| \geq R), \\ c_\alpha (R^2 - |x|^2)^{\alpha/2} \int_{|y| > R} \frac{f(y) \, dy}{(|y|^2 - R^2)^{\alpha/2}|x-y|^d} & (|x| < R), \end{cases}$$

where $c_\alpha = \pi^{-d/2 - 1} \Gamma(\frac{d}{2}) \sin \frac{\pi \alpha}{2}$. For a function $\phi : [0, \infty) \to [0, \infty)$ we set

$$h_\alpha^\phi := H_\alpha(\phi \circ \cdot \mid |B_R(0)|).$$

**Proposition 2.3.** For all $0 < |x| < R$

$$h_\alpha^\phi(x) = \frac{\sin \frac{\pi x}{2\alpha}}{\pi} \int_0^\infty \phi(\sqrt{R^2 + s(R^2 - |x|^2)})\frac{ds}{(s+1)s^{\alpha/2}}. \tag{2.6}$$

**Proof.** Let us fix $R > 0$ and $x \in B_R(0)$. Using polar coordinates we obtain

$$h_\alpha^\phi(x) = c_\alpha (R^2 - |x|^2)^{\alpha/2} \int_R^\infty \int_{\rho S^{d-1}} |x-y|^{-d} \, \sigma(dy) \, \frac{\phi(\rho)}{(\rho^2 - R^2)^{\alpha/2}}. \tag{2.6}$$

By the classical Poisson formula

$$\int_{S^{d-1}} \frac{1 - |w|^2}{|w-y|^d} \, \sigma(dy) = |S^{d-1}| \quad (|w| < 1),$$

hence

$$\int_{\rho S^{d-1}} |x-y|^{-d} \, \sigma(dy) = \rho^{-1} \int_{S^{d-1}} |\frac{x}{\rho} - y|^{-d} \, \sigma(dy) = \rho^{-1} |S^{d-1}| (1 - \frac{|x|^2}{\rho^2})^{-1}$$

$$= \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})} \frac{\rho}{\rho^2 - |x|^2}. \quad \text{(2.6)}$$

Plugging this into (2.6) yields

$$h_\alpha^\phi(x) = \frac{c_\alpha \pi^{d/2}}{\Gamma(\frac{d}{2})} (R^2 - |x|^2)^{\alpha/2} \int_R^\infty \frac{2\rho \phi(\rho) \, d\rho}{(\rho^2 - |x|^2)(\rho^2 - R^2)^{\alpha/2}}. \tag{2.6}$$

The simple substitution $s = (\rho^2 - R^2)/(R^2 - |x|^2)$ leads to

$$\int_R^\infty \frac{2\rho \phi(\rho) \, d\rho}{(\rho^2 - |x|^2)(\rho^2 - R^2)^{\alpha/2}} = \frac{1}{(R^2 - |x|^2)^{\alpha/2}} \int_0^\infty \phi(\sqrt{R^2 + s(R^2 - |x|^2)}) \frac{ds}{(s+1)s^{\alpha/2}}.$$ 

Thus the assertion follows. \qed

**Theorem 2.2** now follows directly from the following corollary.

**Corollary 2.4.** Let $R > 0$ and suppose that $\phi$ is decreasing on $[R, \infty)$ such that $\phi(s) < \phi(r)$ for some $R < r < s$. Then

$$h_\alpha^\phi(x) < h_\alpha^\phi(y), \quad \text{whenever } 0 \leq x < y < R.$$ 

In particular, $u := h_\alpha^\phi - h_\alpha^\phi(0)$ is a bounded function on $\mathbb{R}^d$ which is $\alpha$-harmonic on $B_R(0)$ and satisfies $0 = u(0) < u(y)$ for every $y \in B_R(0)$.

In Theorem 2.1 the function $u$ is assumed to be nonnegative in all of $\mathbb{R}^d$. It is not plausible that the assertion should be false for functions $u$ with small negative values at points far from the origin. A similar question can be asked for classical harmonic functions. If $u$ is positive and large on a large part of $\partial B_1$, it should not matter for the Harnack inequality on $B_{\frac{1}{2}}$ if $u$ is negative with small absolute values on a small part of $\partial B_1$. Another motivation
for a different formulation of the Harnack inequality is that Theorem 2.1 does not allow to use Moser’s approach to regularity estimates, Theorem like Theorem 1.3 in a straightforward manner.

Let us give a new formulation of the Harnack\textsuperscript{2} inequality that does not need any sign assumption on $u$. It is surprising that that this formulation seems not to have been established since Harnack’s textbook in 1887. We treat the classical local case $\alpha = 2$ together with the nonlocal case $\alpha \in (0, 2)$.

**Theorem 2.5.** (Harnack inequality for $\Delta^{\alpha/2}$, $0 < \alpha \leq 2$)

1. There is a constant $c \geq 1$ such that for $0 < \alpha \leq 2$ and $u \in C(\mathbb{R}^d)$ satisfying
   \[ \Delta^{\alpha/2}u(x) = 0 \quad (x \in B_1), \tag{2.7} \]
   the following estimate holds for every $x, y \in B_{1/2}$:
   \[ c\left( u(y) - H_\alpha(u^+|B_1)(y) \right) \leq u(x) \leq c\left( u(y) + H_\alpha(u^-|B_1)(y) \right). \tag{2.8} \]

2. There is a constant $c \geq 1$ such that for $0 < \alpha \leq 2$ and every function $u \in C(\mathbb{R}^d)$, which satisfies (2.7) and is nonnegative in $B_1$, the following inequality holds for every $x, y \in B_{1/2}$:
   \[ u(x) \leq c(u(y) + \alpha(2 - \alpha) \int_{\mathbb{R}^d \setminus B_1} \frac{u^-(z)}{|z|^{d + \alpha}} \, dz). \tag{2.9} \]

**Proof of Theorem 2.5.** The decomposition $u = u^+ - u^-$ and an application of Theorem 2.1 gives

\[
\begin{align*}
  u(x) &= H_\alpha(u|B_1)(x) \leq H(u^+|B_1)(x) \leq cH_\alpha(u^+|B_1)(y) \\
  &= cH_\alpha(u|B_1)(y) + cH_\alpha(u^-|B_1)(y) = cu(y) + cH_\alpha(u^-|B_1)(y),
\end{align*}
\]

which proves the second inequality in (2.8). The first one is proved analogously.

Inequality (2.9) is proved as follows. Assume $u$ is nonnegative in $B_1$. Using the same strategy as above we obtain for some $c_1, c_2 > 0$ and $c = \max(c_1, c_2)$

\[
\begin{align*}
  u(x) &\leq c_1 H_\alpha(u|B_{1/4})(y) + c_1 H_\alpha(u^-|B_{1/4})(y) \\
  &\leq c_1 u(y) + c_2 \alpha(2 - \alpha) \int_{\mathbb{R}^d \setminus B_1} \frac{u^-(z)}{|z|^{d - (4/3)^2 \alpha|^2}/z - y|d} \, dz \\
  &\leq cu(y) + c\alpha(2 - \alpha) \int_{\mathbb{R}^d \setminus B_1} \frac{u^-(z)}{|z|^{d + \alpha}} \, dz.
\end{align*}
\]

The proof of the theorem is complete. Note that different versions of this result have been announced in [21]. \qed

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\textsuperscript{2}The second author would like to use the opportunity to correct an error in [19] concerning the name Harnack. The correct name of the mathematician Harnack is Carl Gustav Axel Harnack. His renowned twin brother Carl Gustav Adolf carried the last name “von Harnack” after being granted the honor.
Let us make some observations:

1. There is no assumption on the sign of \( u \) needed for (2.8). Inequality (2.8) does hold in the classical case \( \alpha = 2 \), too.
2. If \( u \) is nonnegative in all of \( \mathbb{R}^d (\alpha \in (0,2)) \) or nonnegative in \( B_1 (\alpha = 2) \), then the second inequality in (2.8) reduces to the well-known formulation of the Harnack inequality.
3. If \( u \) is nonnegative in \( B_1 \), then (2.9) reduces for \( \alpha \to 2 \) to the original Harnack inequality.
4. For the above results, one might want to impose regularity conditions on \( u \) such that \( \Delta^{\alpha/2} u(x) \) exists at every point \( x \in B_1 \), e.g. \( u|_{B_1} \in C^2(B_1) \) and \( u(x)/(1+|x|^{d+\alpha}) \in L^1(\mathbb{R}^d) \). However, the assumption that the integral in (2.1) converges, is sufficient.

The proof of Theorem 2.5 does not use the special structure of \( \Delta^{\alpha/2} \). The proof only uses the decomposition \( u = u^+ - u^- \) and the Harnack inequality for the Poisson kernel. Roughly speaking, it holds for every linear operator that satisfies a maximum principle. One more abstract way of formulating this result in a general framework is as follows:

**Lemma 2.6.** Let \( (X, W) \) be a balayage space (see [5]) such that \( 1 \in W \). Let \( V, W \) be open sets in \( X \) with \( V \subset W \). Let \( c > 0 \). Suppose that, for all \( x, y \in V \) and \( h \in \mathcal{H}^+(V) \),

\[
u(x) \leq cu(y).
\]

Then \( \varepsilon^V_x \leq c \varepsilon^V_y \) and, for every \( u \in \mathcal{H}_b(W) \),

\[
u(x) \leq cu(y) + c \int u^- \, d\varepsilon^V_y.
\]

Here, \( \mathcal{H}_b(A) \) denotes the set of bounded functions which are harmonic in the Borel set \( A \). Functions in \( \mathcal{H}^+(A) \), in addition, are nonnegative.

**Proof.** Since, for every positive continuous function \( f \) with compact support the mapping \( f \mapsto \varepsilon^V_x(f) \) belongs to \( \mathcal{H}^+(V) \), the first statement follows. Let \( u \in \mathcal{H}_b(W) \). Then \( u(x) = \varepsilon^V_x(u) \), \( u(y) = \varepsilon^V_y(u) \) and hence

\[
u(x) \leq \varepsilon^V_x(u^+) \leq c \varepsilon^V_y(u^+) = c \varepsilon^V_y(u + u^-) = cu(y) + c \int u^- \, d\varepsilon^V_y.
\]

\( \square \)

### 3. Functional inequalities and scaling property

In this section we collect several auxiliary results. In particular, we will need some properties of the Sobolev spaces \( H^{\alpha/2}(D) \). The following fact about extensions has an elementary proof, see [12]. However, one has to go through it and see that the constants do not depend on \( \alpha \), provided one has the factor \( (2 - \alpha) \) in front of the Gagliardo norm, cf. (1.4) and (1.6).

**Fact 3.1 (Extension).** Let \( D \subset \mathbb{R}^d \) be a bounded Lipschitz domain, and let \( 0 < \alpha < 2 \). Then there exists a constant \( c = c(d, D) \), which is independent of \( \alpha \), and an extension operator \( E : H^{\alpha/2}(D) \to H^{\alpha/2}(\mathbb{R}^d) \) with norm \( \|E\| \leq c \).

Furthermore, we will need the following Poincaré inequality, cf. [28].

**Fact 3.2 (Poincaré I).** Let \( D \subset \mathbb{R}^d \) be a bounded Lipschitz domain, and let \( 0 < \alpha_0 \leq \alpha < 2 \). Then there exists a constant \( c = c(d, \alpha_0, D) \), which is independent of \( \alpha \), such that

\[
\| u - \frac{1}{|D|} \int_D u \, dx \|_{L^2(D)}^2 \leq c \mathcal{E}^D_{\alpha_0}(u, u) \quad (u \in H^{\alpha/2}(D)).
\]

(3.1)
The following results, Fact 3.3 and Fact 3.4, are standard for fixed $\alpha$. For $\alpha \to 2$ they follow from results in [7],[26], [28]. They are established in the case when $B_r(x)$ denotes the cube of all $y \in \mathbb{R}^d$ such that $|y_i - x_i| < r$ for any $i \in \{1, \ldots, d\}$. They hold true for balls likewise.

**Fact 3.3** (Poincaré II). Assume $\alpha_0, \varepsilon > 0$ and $0 < \alpha_0 \leq \alpha < 2$. There exists a constant $c$, which is independent of $\alpha$, such that for $B_R = B_R(x_0)$

$$u \in H^{\alpha/2}(B_R), \quad |B_R \cap \{u = 0\}| \geq \varepsilon |B_R|$$

imply

$$\int_{B_R} (u(x))^2 \, dx \leq c R^\alpha \int_{B_R B_R} \frac{(u(y) - u(x))^2}{|x - y|^{d+\alpha}} \, dy \, dx. \tag{3.2}$$

**Fact 3.4** (Sobolev embedding). Assume $d \in \mathbb{N}, d \geq 2, R_0 > 0$, and $0 < \alpha_0 \leq \alpha < 2$, $q \in [1, \frac{2d}{d-\alpha}]$. Then there exists a constant $c$, which is independent of $\alpha$, such that for $R \in (0, R_0)$ and $u \in H^{\alpha/2}(B_R)$

$$\left( \int_{B_R} |u(x)|^{\frac{2d}{d-\alpha}} \, dx \right)^{\frac{d-\alpha}{d}} \leq c \int_{B_R B_R} \frac{(u(y) - u(x))^2}{|x - y|^{d+\alpha}} \, dy \, dx + c R^{-\alpha + \frac{2(d-\alpha)}{q}} \left( \int_{B_R} |u(x)|^q \, dx \right)^{\frac{2}{q}}.$$

When studying nonlocal bilinear forms on bounded sets, it is natural to work with function spaces which impose some regularity of the functions across the boundary. These spaces seem not be part of the standard literature which is why we provide a small introduction.

We often make use of scaling and translations. Our main assumptions, conditions (A) and (B) assure a certain behavior of the family of measures $\mu$ with respect to the unit ball $B_1 \subset \mathbb{R}^d$.

Let us formulate these conditions with respect to general balls $B_r(x_0) \subset \mathbb{R}^d$.

Given $\xi \in \mathbb{R}^d, r > 0, A \geq 1$, we say that $\mu$ satisfies $(A;\xi,r)$ if:

For every ball $B_{\rho}(x_0)$ with $\rho \in (0, r), x_0 \in B_r(\xi)$ and every $v \in H^{\alpha/2}(B_{\rho}(x_0))$

$$A^{-1} \mathcal{E}_{B_{\rho}(x_0)}^\mu(v,v) \leq \mathcal{E}_{B_{\rho}(x_0)}^{\mu_{\rho_0}}(v,v) \leq A \mathcal{E}_{B_{\rho}(x_0)}^{\mu}(v,v). \tag{A;\xi,r}$$

Given $\xi \in \mathbb{R}^d, r > 0, B \geq 1$, we say that $\mu$ satisfies $(B;\xi,r)$ if:

For $0 < \rho \leq R \leq r$ and $x_0 \in B_r(\xi)$ there is a nonnegative measurable function $\tau : \mathbb{R}^d \to \mathbb{R}$ with supp($\tau$) $\subset B_{R+\rho}(x_0), \tau(x) \equiv 1$ on $B_R(x_0), ||\tau||_\infty \leq 1$, and

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} (\tau(y) - \tau(x))^2 \mu(x,dy) \leq B \rho^{-\alpha}. \tag{B;\xi,r}$$

Let us explain how the operator under consideration behaves with respect to rescaled functions.

**Lemma 3.5** (Scaling property). Assume $\xi \in \mathbb{R}^d$ and $r \in (0,1)$. Let $u \in V^\mu_{B_r(\xi)}(\mathbb{R}^d)$ satisfy $\mathcal{E}^\mu(u,\phi) \geq (f,\phi)$ for every nonnegative $\phi \in H^{\mu}_{B_r(\xi)}(\mathbb{R}^d)$. Define a diffeomorphism $J$ by $J(x) = rx + \xi$. Define rescaled versions $\tilde{f}$, $\tilde{u}$ of $u$ and $f$ by $\tilde{u}(x) = u(J(x))$ and $\tilde{f}$ by $\tilde{f}(x) = r^\alpha f(J(x))$.

(1) Then $\tilde{u}$ satisfies for all nonnegative $\phi \in H^{\tilde{\mu}}_{B_1}(\mathbb{R}^d)$

$$\mathcal{E}^{\tilde{\mu}}(\tilde{u},\phi) = \int_{\mathbb{R}^d} \left( \tilde{u}(y) - \tilde{u}(x) \right) \left( \phi(y) - \phi(x) \right) \tilde{\mu}(x,dy) \, dx \geq (\tilde{f},\phi),$$
where
\[ \tilde{\mu}(x, dy) = r^\alpha \mu_{j-1}(J(x), dy) \] and \( \mu_{j-1}(z, A) = \mu(z, J(A)) \).

(2) Assume \( \mu \) satisfies conditions (A; \( \xi, x \)), (B; \( \xi, r \)) for some \( \alpha \in (0, 2) \) and \( A \geq 1, B \geq 1 \), \( \xi \in \mathbb{R}^d, r > 0 \). Then the family of measures \( \tilde{\mu} = \tilde{\mu}(\cdot, dy) \) satisfies assumptions (A) and (B) with the same constants.

**Remark.** The condition (D) is affected by scaling in a non-critical way. We deal with this phenomenon further below in Section 4 and Subsection 5.1.

**Proof.** For the proof of the first statement, let \( \phi \in \mathcal{H}_{B_1}(\mathbb{R}^d) \) be a nonnegative test function. Define \( \phi_r \in \mathcal{H}_{B_r}(\xi)(\mathbb{R}^d) \) by \( \phi_r = \phi \circ J^{-1} \). Then

\[
\int \left( \tilde{\mu}(y) - \tilde{\mu}(x) \right) (\phi(y) - \phi(x)) \tilde{\mu}(x, dy) dx
\]

\[
= r^\alpha \int \left( u(J(y)) - u(J(x)) \right) (\phi_r(J(y)) - \phi_r(J(x)) \mu_{j-1}(J(x), dy) dx
\]

\[
= r^{\alpha-d} \int \left( u(J(y)) - u(x) \right) (\phi_r(J(y)) - \phi_r(x)) \mu_{j-1}(x, dy) dx
\]

\[
\geq r^{\alpha-d} \int f(x) \phi_r(x) dx = \int r^\alpha f(J(x)) \phi(x) dx = \int \tilde{f}(x) \phi(x) dx,
\]

which is what we wanted to prove. Let us now prove that \( \tilde{\mu} \) inherits properties (A), (B) from \( \mu \) with the same constants \( A \) and \( B \). Let us only consider the case \( \xi = 0 \). In order to verify condition (A) we need to consider an arbitrary ball \( B_\rho(x_0) \) with \( \rho \in (0, 1) \) and \( x_0 \in B_1 \). Let us simplify the situation further by assuming \( x_0 = 0 \). The general case can be proved analogously. Thus, we assume \( r \in (0, 1) \) and \( u \in H^{\alpha/2}(B_\rho) \). The estimate \( \mathcal{E}_{B_\rho}(u, u) \leq A \mathcal{E}^{\mu_o}_{B_\rho}(u, u) \) can be derived as follows. Define a function \( \tilde{u} \in \mathcal{H}^{\alpha/2}(B_\rho) \) by \( \tilde{u} = u \circ J^{-1} \). Then

\[
\mathcal{E}_{B_\rho}(u, u) = \int_{B_\rho} \int_{B_\rho} (u(y) - u(x))^2 \tilde{\mu}(x, dy) dx = r^\alpha \int_{B_\rho} \int_{B_\rho} (\tilde{u}(J(y)) - \tilde{u}(J(x)))^2 \mu_{j-1}(J(x), dy) dx
\]

\[
= r^{\alpha-d} \int_{B_\rho} \int_{B_\rho} (\tilde{u}(J(y)) - \tilde{u}(x))^2 \mu_{j-1}(x, dy) dx
\]

\[
\leq A r^{\alpha-d} \int_{B_\rho} \int_{B_\rho} \frac{(\tilde{u}(y) - \tilde{u}(x))^2}{|x - y|^{d+\alpha}} dy dx
\]

\[
= A r^{-2d} \int_{B_\rho} \int_{B_\rho} \frac{(u(J^{-1}(y)) - u(J^{-1}(x)))^2}{|J^{-1}(x) - J^{-1}(y)|^{d+\alpha}} dy dx = A \int_{B_\rho} \int_{B_\rho} \frac{(u(y) - u(x))^2}{|x - y|^{d+\alpha}} dy dx,
\]

which proves our claim. The estimate \( \mathcal{E}^{\mu_o}_{B_\rho}(u, u) \leq A \mathcal{E}_{B_\rho}(u, u) \) follows in the same way.

In order to check condition (B) for \( \tilde{\mu} \) we proceed as follows. Again, we assume \( x_0 = 0, r \in (0, 1) \). The general case can be proved analogously. Assume \( \tilde{r} : \mathbb{R}^d \to \mathbb{R} \) satisfy
supp(\hat{\tau}) \subset \overline{B_{rR+\rho}}, \hat{\tau} \equiv 1 \text{ on } B_r R \text{ and }
\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} (\hat{\tau}(y) - \hat{\tau}(x))^2 \mu(x, dy) \leq B(r\rho)^{-\alpha}
\iff \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} (\hat{\tau}(y) - \hat{\tau}(J(x)))^2 \mu(J(x), dy) \leq B(r\rho)^{-\alpha}.

Such a function \hat{\tau} exists because, by assumption, \mu satisfies (B;\xi;x). Next, define \tau = \hat{\tau} \circ J. Then \tau satisfies supp(\tau) \subset \overline{B_{R+\rho}}, \tau \equiv 1 \text{ on } B_R \text{ and, by a change of variables, }
\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} (\tau(y) - \tau(x))^2 \tilde{\mu}(x, dy) = r^\alpha \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} (\hat{\tau}(J(y)) - \hat{\tau}(J(x)))^2 \mu_{J^{-1}}(J(x), dy)
\quad = r^\alpha \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} (\hat{\tau}(y) - \hat{\tau}(J(x)))^2 \mu(J(x), dy) \leq B r^{-\alpha},

which shows that \tilde{\mu} satisfies (B) with the constant B. The proof of the lemma is complete. \(\square\)

4. THE WEAK HARNACK INEQUALITY FOR NONLOCAL EQUATIONS

The main aim of this section is to provide a proof of the weak Harnack inequality Theorem 1.1. The key result of this section is the corresponding result for supersolutions that are nonnegative in all of \(\mathbb{R}^d\).

**Theorem 4.1.** Assume \(f \in L^{q/\alpha}(B_1)\) for some \(q > d, \alpha \in [\alpha_0, 2)\). There are positive reals \(p_0, c\) such that for every \(u \in V^\mu_{B_1}(\mathbb{R}^d)\) with \(u \geq 0\) in \(\mathbb{R}^d\) satisfying
\[\mathcal{E}(u, \phi) \geq (f, \phi)\]
the following holds:
\[\inf_{B_1 \cup} u \geq c \left( \int_{B_1} u(x)^{p_0} \, dx \right)^{1/p_0} - \|f\|_{L^{q/\alpha}(B_{1r})}.\]

The constants \(p_0, c\) depend only on \(d, \alpha_0, A, B\). They are independent of \(\alpha \in [\alpha_0, 2)\).

**Remark.** All results in this section are robust with respect to \(\alpha \in [\alpha_0, 2)\), i.e. constants do not depend on \(\alpha\).

The main application of this result is the proof of Theorem 1.1.

**Proof.** Set \(u = u^+ - u^-\). The assumptions imply for any nonnegative \(\phi \in H^\mu_{B_1}(\mathbb{R}^d)\)
\[\mathcal{E}(u^+, \phi) \geq \mathcal{E}(u^-, \phi) + (f, \phi) = \int_{B_1} \phi(x) \bigl( f(x) - 2 \int_{\mathbb{R}^d \setminus B_1} u^-(y) \mu(x, dy) \bigr) \, dx,\]
i.e. \(u^+\) satisfies all assumptions of Theorem 4.1 with \(q = +\infty\) and \(\tilde{f} : B_1 \to \mathbb{R}\) defined by
\[\tilde{f}(x) = f(x) - 2 \int_{\mathbb{R}^d \setminus B_1} u^-(y) \mu(x, dy).\]
The assertion of the theorem is true if \( \sup_{x \in B(\xi)} \int_{E} u^{-}(y) \mu(x, dy) \) is infinite. Thus we can assume this quantity to be finite. Theorem 4.1 now implies

\[
\inf_{B_{\xi}^{4}(x_0)} u \geq c( \int_{B_{\xi}^{4}(x_0)} u(x)^{p_0} \, dx)^{1/p_0} - c_2 \sup_{x \in B_{\xi}^{0}(\xi)} \left( \int_{R^{d}\setminus B_{\xi}} u^{-}(y) \mu(x, dy) \right) - \|f\|_{L^{q/\alpha}(B_{\xi}^{4})},
\]

for some positive constants \( c_1, c_2 \). The proof is complete. \( \square \)

By scaling and translation, we obtain the following corollary.

**Corollary 4.2.** Let \( x_0 \in \mathbb{R}^{d}, R \in (0, 1) \). Assume \( \mu \) is a family of measures satisfying \( (A; \xi, r) \) and \( (B; \xi, r) \). Assume \( u \in V_{BR}^{\mu}(\mathbb{R}^{d}) \) satisfies \( u \geq 0 \) in \( B_{R}(x_0) \) and \( \mathcal{E}(u, \phi) \geq 0 \) for every nonnegative \( \phi \in H^{\mu}_{BR}(\mathbb{R}^{d}) \). Then

\[
\inf_{B_{\xi}^{4}(x_0)} u \geq c( \int_{B_{\xi}^{4}(x_0)} u(x)^{p_0} \, dx)^{1/p_0} - R^{\alpha} \sup_{x \in B_{BR}(x_0)} \int_{R^{d}\setminus BR_{R}(x_0)} u^{-}(y) \mu(x, dy),
\]

with positive constants \( p_0, c \) which depend only on \( d, \alpha, A, B \). In particular, they are independent of \( \alpha \in [\alpha_0, 2) \).

Let us proceed to the proof of Theorem 4.1.

**Remark.** Without further mentioning we assume that \( \mu \) is a family of measures that satisfies \( (A) \) and \( (B) \) for some \( A \geq 1, B \geq 1 \) and \( \alpha \leq \alpha < 2 \). The constants in the assertions below depend, among other things, on \( A, B, \) and \( \alpha_0 \). They do not depend on \( \alpha \), though.

Let us first establish several auxiliary results. Our approach is closely related to the approach in [20] from where we borrow the following technical lemma, cf. [20, Lemma 2.5].

**Lemma 4.3.** Let \( a, b > 0, p > 1 \) and \( \tau_1, \tau_2 \geq 0 \). Then

\[
(b - a)(\tau_1^{p+1}a^{-p} - \tau_2^{p+1}b^{-p}) \geq \frac{\tau_1\tau_2}{p+1}\left((\frac{b}{a})^{\frac{p+1}{p}} - (\frac{a}{b})^{\frac{p+1}{p}}\right)^2 - \max\{4, \frac{6p-5}{2}\}(\tau_2 - \tau_1)^2((\frac{b}{a})^{-p+1} + (\frac{a}{b})^{-p+1}).
\]  

(4.1)

The next result is an extension of corresponding results in [20] and [2].

**Lemma 4.4.** Assume \( 0 < \rho < r < 1 \) and \( z_0 \in B_{1} \). Set \( B_{r} = B_{r}(z_0) \). Assume \( f \in L^{q/\alpha}(B_{2r}) \) for some \( q > d \). Assume \( u \in V_{B_{2r}}^{\mu}(\mathbb{R}^{d}) \) is nonnegative in \( \mathbb{R}^{d} \) and satisfies

\[
\mathcal{E}(u, \phi) \geq (f, \phi) \quad \text{for any nonnegative } \phi \in H^{\mu}_{B_{2r}}(\mathbb{R}^{d})
\]

\[
u(x) \geq \varepsilon \quad \text{for almost all } x \in B_{2r} \text{ and some } \varepsilon > 0.
\]

Then

\[
\int_{B_{r}, B_{r}} \left( \sum_{k=1}^{\infty} \frac{(\log u(y) - \log u(x))^{2k}}{(2k)!} \right) \mu(x, dy)dx \leq cp^{-\alpha}|B_{r+\rho}| + \varepsilon^{-1}\|f\|_{L^{q/\alpha}(B_{r+\rho})}\|1\|_{L^{q/(q-\alpha)}(B_{r+\rho})},
\]

(4.2)

(4.3)

where \( c > 0 \) is independent of \( u, x_0, r, \rho, f, \varepsilon, \alpha \).
Note that for $\varepsilon \geq c_1(r + \rho)^{\delta}\|f\|_{L^q(\mathbb{B}_{r+p})}$ with $\delta = \alpha \left( \frac{2-q}{q} \right)$ one obtains
\[
\iint_{B_r, B_r} \left( \sum_{k=1}^{\infty} \frac{\log u(y) - \log u(x)}{(2k)!} \right)^{2k} \mu(x, dy)dx \leq c_2 \rho^{-\alpha} |B_{r+p}|. \tag{4.4}
\]
From the above lemma it will be deduced that $\log u \in \text{BMO} (B_1)$ where BMO ($B_1$) contains all functions of bounded mean oscillations \cite{17}.

\textbf{Proof.} The proof uses several ideas developed in \cite{2}. Let $\tau : \mathbb{R}^d \to \mathbb{R}$ be a function according to (B), i.e. more precisely we assume
\[
\begin{cases}
\text{supp}(\tau) \subset \overline{B_{r+\rho}} \subset B_{2r}, \|\tau\|_\infty \leq 1, \tau \equiv 1 \text{ on } B_r, \\
\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} (\tau(y) - \tau(x))^2 \mu(x, dy) \leq B\rho^{-\alpha}.
\end{cases}
\]
Then
\[
\iint_{\mathbb{R}^d \times \mathbb{R}^d} (\tau(y) - \tau(x))^2 \mu(x, dy)dx
\]
\[
= \iint_{B_{r+\rho} \times B_{r+\rho}} (\tau(y) - \tau(x))^2 \mu(x, dy)dx + 2 \iint_{B_{r+\rho} \setminus B_{r+\rho}} (\tau(y) - \tau(x))^2 \mu(x, dy)dx
\]
\[
\leq 2 \iint_{B_{r+\rho} \times \mathbb{R}^d} (\tau(y) - \tau(x))^2 \mu(x, dy)dx
\]
\[
\leq 2|B_{r+\rho}| \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} (\tau(y) - \tau(x))^2 \mu(x, dy)
\]
\[
\leq 2c\rho^{-\alpha} |B_{r+\rho}|. \tag{4.5}
\]
We choose $\phi(x) = -\tau^2(x)u^{-1}(x)$ as a test function. Denote $B_{r+\rho}$ by $B$. We obtain
\[
(f, \phi) \geq \iint_{\mathbb{R}^d \times \mathbb{R}^d} (u(y) - u(x)) (\tau^2(x)u^{-1}(x) - \tau^2(y)u^{-1}(y)) \mu(x, dy) dx
\]
\[
= \iint_{BB} \tau(x)\tau(y) \left( \frac{\tau(x)u(y)}{\tau(y)u(x)} + \frac{\tau(y)u(x)}{\tau(x)u(y)} - \frac{\tau(y)}{\tau(x)} - \frac{\tau(x)}{\tau(y)} \right) \mu(x, dy) dx
\]
\[
+ 2 \iint_{BB^c} (u(y) - u(x)) (\tau^2(x)u^{-1}(x) - \tau^2(y)u^{-1}(y)) \mu(x, dy) dx
\]
\[
+ \iint_{B^cB^c} (u(y) - u(x)) (\tau^2(x)u^{-1}(x) - \tau^2(y)u^{-1}(y)) \mu(x, dy) dx. \tag{4.6}
\]
Setting $A(x, y) = \frac{u(y)}{u(x)}$ and $B(x, y) = \frac{\tau(y)}{\tau(x)}$ we obtain
\[
\iint_{BB} \tau(x)\tau(y) \left( \frac{A(x, y)}{B(x, y)} + \frac{B(x, y)}{A(x, y)} - B(x, y) - \frac{1}{B(x, y)} \right) \mu(x, dy) dx
\]
The third term on the right-hand side can be estimated as follows:

\[
\begin{align*}
2 \int_{B_{r+\rho}^c} & (u(y) - u(x)) \left( \tau^2(x) u^{-1}(x) - \tau^2(y) u^{-1}(y) \right) \mu(x, dy) dy \\
&= 2 \int_{B_{r+\rho}^c} (u(y) - u(x)) \left( - \tau^2(y) u^{-1}(y) \right) \mu(x, dy) dy \\
&= 2 \int_{B_{r+\rho}^c} \int_{B_{r+\rho}^c} \frac{\tau^2(y) u(x) - \tau^2(x) u(y)}{u(y)} \mu(x, dy) dx - 2 \int_{B_{r+\rho}^c} \int_{B_{r+\rho}^c} \tau^2(y) \mu(x, dy) dx \\
&\geq -2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (\tau(y) - \tau(x))^2 \mu(x, dy) dx ,
\end{align*}
\]

where we applied (4.5) and the fact that for positive real \(a, b\)

\[
\frac{(a - b)^2}{ab} = (a - b) (b^{-1} - a^{-1}) = (\log a - \log b)^2 + 2 \sum_{k=2}^{\infty} \frac{(\log a - \log b)^{2k}}{(2k)!} .
\]
where we used nonnegativity of $u$ in $\mathbb{R}^d$. Therefore,

$$\int \int_{B_\rho \times B_\rho} \left( \sum_{k=1}^{\infty} \frac{(\log u(y) - \log u(x))^{2k}}{(2k)!} \right) \mu(x, dy) dx$$

$$\leq 2 \int \int_{\mathbb{R}^d \times \mathbb{R}^d} (\tau(x) - \tau(y))^2 \mu(x, dy) dx + \|f\|_{L^{q/\alpha}(B_{\rho+\rho})} \|u^{-1}\|_{L^{q/(q-\alpha)}(B_{\rho+\rho})}.$$

(4.9)

The proof is complete after the trivial observation $|u^{-1}| \leq \varepsilon^{-1}$. \qed

**Lemma 4.5.** Assume $0 < R < 1$ and $f \in L^{q/\alpha}(B_{\rho/4})$ for some $q > d$. Assume $u \in V_{B_{2\rho}}^\mu(\mathbb{R}^d)$ is nonnegative in $\mathbb{R}^d$ and satisfies

$$\mathcal{E}(u, \phi) \geq (f, \phi) \text{ for any nonnegative } \phi \in H_{B_{2\rho}}^\mu(\mathbb{R}^d),$$

$$u(x) \geq \varepsilon \quad \text{for almost all } x \in B_{2\rho} \text{ and some } \varepsilon > \frac{1}{4} R^d \|f\|_{L^{q/\alpha}(B_{4\rho})},$$

where $\delta = \alpha^{\frac{d-d}{q}}$. Then there exist $\overline{\rho} \in (0, 1)$ and $\overline{c} > 0$ such that,

$$\left( \int_{B_{\overline{\rho}}} u(x)^{\overline{\rho}} \, dx \right)^{1/\overline{\rho}} \leq c \left( \int_{B_{\overline{\rho}}} u(x)^{-\overline{\rho}} \, dx \right)^{-1/\overline{\rho}},$$

(4.10)

where $c$ and $\overline{\rho}$ are independent of $x_0, R, u, \varepsilon$, and $\alpha$.

**Proof.** The main idea is to prove $\log u \in \text{BMO}(B_R)$. Choose $z_0 \in B_R$ and $r > 0$ such that $B_{2r}(z_0) \subset B_{\rho/4}$. Set $\rho = r$. Lemma 4.4 and Assumption (A) imply

$$\int \int_{B_r(z_0) \times B_r(z_0)} \frac{(\log u(y) - \log u(x))^2}{|x - y|^{d+\alpha}} \, dy \, dx$$

$$\leq \int \int_{B_r(z_0) \times B_r(z_0)} (\log u(y) - \log u(x))^2 \mu(x, dy) dx \leq c_1 r^{d-\alpha}.$$

Application of the Poincaré inequality, Fact 3.2, and the scaling property (3.3) leads to

$$\int_{B_r(z_0)} |\log u(x) - [\log u]_{B_r(z_0)}|^2 \, dx \leq c_2 r^d,$$

(4.11)

where $[\log u]_{B_r(z_0)} = [B_r(z_0)]^{-1} \int_{B_r(z_0)} \log u = \int_{B_r(z_0)} \log u$. From here

$$\int_{B_r(z_0)} |\log u(x) - [\log u]_{B_r(z_0)}| \, dx \leq \left( \int_{B_r(z_0)} |\log u(x) - [\log u]_{B_r(z_0)}|^2 \, dx \right)^{1/2} |B_r(z_0)|^{1/2} \leq c_3 r^d.$$

An application of the John-Nirenberg embedding, see [15, Chapter 7.8], then gives

$$\int_{B_R} e^{\overline{\rho}|\log u(y) - [\log u]_{B_{\overline{\rho}}}|} \, dy \leq c_4 R^d,$$
where $\overline{p}$ and $c_4$ depend only on $d$ and $c_3$. One obtains
\[
\left( \int_{B_R} u(y) \overline{p} \, dy \right) \left( \int_{B_R} u(y)^{-p} \, dy \right) = \int_{B_R} e^{\overline{p} \log u(y) - [\log u]_{B_R}} \, dy \times \int_{B_R} e^{-\overline{p} \log u(y) - [\log u]_{B_R}} \, dy \leq c_4^2 R^{2d}.
\]
The above inequality proves assertion (4.10). Lemma 4.5 is proved.

The next result allows us to apply Moser’s iteration for negative exponents. It is a purely local result although the Dirichlet form is nonlocal.

**Lemma 4.6.** Assume $x_0 \in B_1$ and $0 < 4\rho < R < 1 - \rho$. Set $B_R = B(x_0, R)$. Assume $f \in L^{q/\alpha}(B_{2R})$ for some $q > d$. Assume $u \in V^\mu_{B_{2R}}(\mathbb{R}^d)$ satisfies
\[
\mathcal{E}(u, \phi) \geq (f, \phi) \text{ for any nonnegative } \phi \in H^\mu_{B_R}(\mathbb{R}^d),
\]
\[
u(x) \geq \varepsilon \quad \text{for almost all } x \in B_R \text{ and some } \varepsilon > R^d \|f\|_{L^{q/\alpha}(B_{2R})},
\]
where $\delta = \alpha \left( \frac{q-d}{q} \right)$. Then for $p > 1$
\[
\|u^{-1}\|_{L^{(p-1)\frac{d}{\alpha}}(B_R)}^{p-1} \leq c \left( \max\{\frac{p-1}{2}, \frac{6(p-1)^2}{16}\} \right) \rho^{-\alpha} \|u^{-1}\|_{L^{p-1}(B_{R+\rho})}^{p-1},
\]
where $c > 0$ is independent of $u, x_0, R, \rho, p, \varepsilon,$ and $\alpha$.

Note that the result does not require $u$ to be nonnegative in all of $\mathbb{R}^d$.

**Proof.** Let $\tau : \mathbb{R}^d \to \mathbb{R}$ be a function according to assumption (B), i.e.
\[
\begin{align*}
supp(\tau) & \subset B_{R+\rho} \subset B_{2R}, \quad \|\tau\|_\infty \leq 1, \forall x \in B_R : \tau(x) = 1, \\
\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} (\tau(y) - \tau(x)) \mu(x, dy) & \leq B \rho^{-\alpha}.
\end{align*}
\]
The assumptions of the lemma imply
\[
\mathcal{E}(u, -\tau^{p+1}u^{-p}) \leq (f, -\tau^{p+1}u^{-p}),
\]
leading via Lemma 4.3 and the choice $a = u(x), b = u(y), \tau_1 = \tau(x), \tau_2 = \tau(y)$ to
\[
\begin{align*}
\iint_{\mathbb{R}^d} & \tau(x)\tau(y) \left[ \left( \frac{u(y)}{\tau(y)} \right)^{\frac{p+1}{2}} - \left( \frac{u(x)}{\tau(x)} \right)^{\frac{p+1}{2}} \right]^2 \mu(x, dy) dx \\
& \leq c_1(p) \iint_{\mathbb{R}^d} \tau(y - \tau(x))^2 \left[ \left( \frac{u(y)}{\tau(y)} \right)^{-p+1} + \left( \frac{u(x)}{\tau(x)} \right)^{-p+1} \right] \mu(x, dy) dx + (f, -\tau^{p+1}u^{-p}),
\end{align*}
\]
(4.13)
where $c_1(p) = \max\{\frac{p-1}{2}, \frac{6(p-1)^2}{16}\}$. The left-hand side can trivially be estimated from below like this:

$$
\iint_{\mathbb{R}^d} \tau(x) \tau(y) \left[ \left( \frac{u(y)}{\tau(y)} \right)^{\frac{p+1}{2}} - \left( \frac{u(x)}{\tau(x)} \right)^{\frac{p+1}{2}} \right]^2 \mu(x, dy) dx 
\geq \iint_{B_R B_R} \left( \frac{u(y)}{\tau(y)} \right)^{\frac{p+1}{2}} - \left( \frac{u(x)}{\tau(x)} \right)^{\frac{p+1}{2}} \mu(x, dy) dx.
$$

Using symmetry, the first term on the right-hand side in Equation 4.13 is estimated from above as follows:

$$
2c_1(p) \iint_{\mathbb{R}^d} (\tau(y) - \tau(x))^2 \tau(x)^{p-1} u(x)^{-p+1} \mu(x, dy) dx 
\leq 2c_1(p) \int_{B_{R+\rho}} u(x)^{-p+1} \left( \int_{B_R} \tau(y) - \tau(x))^2 \mu(x, dy) dx \right) \leq c_2(p) \rho^{-\alpha} \int_{B_{R+\rho}} u(x)^{-p+1}.
$$

It remains to estimate $|(f, -\tau^{p+1} u^{-p})|$ from above. For any $\alpha > 0$ we have

$$
|\omega^{p+1} u^{-p} - \tau^{p+1} u^{-p}| \leq \varepsilon^2 \left( 2^{p+1} \tau^{p-1} u^{-p+1} \right) \leq \varepsilon^2 \left( 2 \tau^2 \|\tau^{p-1} u^{-p+1}\|_{q/(q-\alpha)} \right)
= \varepsilon^2 \left( 2 \|\tau\|_{q/(q-\alpha)} \right)^{\frac{p+1}{2}} \|\tau\|_{\frac{p+1}{2}} \|\tau\|_{q/(q-\alpha)}
\leq \varepsilon^2 \left( 2 \|\tau\|_{q/(q-\alpha)} \right)^{\frac{p+1}{2}} \|\tau\|_{\frac{p+1}{2}} \|\tau\|_{q/(q-\alpha)}
\leq (2R)^{-\alpha} \frac{\omega^d}{\varepsilon^2} \|\tau\|^{p-1} \|\tau\|_{d/(d-\alpha)} + R^{-\alpha} \frac{\omega^d}{\varepsilon^2} \|\tau\|^{p-1} \|\tau\|_{d/(d-\alpha)}.
$$

We choose $\alpha = \rho R^{-\alpha} \frac{\omega^d}{\varepsilon^2}$ for some $\omega > 0$ and obtain

$$
|\omega^{p+1} u^{-p} - \tau^{p+1} u^{-p}| \leq \varepsilon \left( \|\tau\|^{p-1} \|\tau\|_{d/(d-\alpha)} + \omega^{-d/(q-d)} R^{-\alpha} \|\tau\|^{p-1} \right).
$$

Combining these estimates we obtain from (4.13) for any $p > 1$ and any $\omega > 0$

$$
\iint_{B_{R+\rho} B_{R+\rho}} \left[ \frac{u(y)}{\tau(y)} \right)^{\frac{p+1}{2}} - \left( \frac{u(x)}{\tau(x)} \right)^{\frac{p+1}{2}} \mu(x, dy) dx 
\leq c_3 \left( \|\omega\|^{\frac{d}{\varepsilon^2}} + \|\omega\|^{\frac{d}{\varepsilon^2}} \right) \rho^{-\alpha} \int_{B_{R+\rho}} u(x)^{-p+1} dx + \omega \|\mu|^{-p+1} \|_{L^{\frac{d}{\varepsilon^2}}(B_{R+\rho}).
$$

Next, we use Assumption (A) and apply the Sobolev inequality, Fact 3.4, to the left-hand side. Choosing $\omega$ small enough and subtracting the term $\omega \|\mu|^{-p+1} \|_{L^{\frac{d}{\varepsilon^2}}(B_{R+\rho})}$ from both sides, we prove the assertion of the lemma. \qed

Lemma 4.6 provides us with an estimate which can be iterated. As a result of this iteration we obtain the following corollary.

**Corollary 4.7.** Assume $x_0 \in B_1$, $0 < R < 1/2$, and $0 < \eta < 1 < \Theta$. Set $B_R = B_R(x_0)$. Assume $f \in L^{q/(\alpha)}(B_{R})$ for some $q > d$. Assume $u \in V^\mu_{B_{2R}}(\mathbb{R}^d)$ satisfies

$$
\mathcal{E}(u, \phi) \geq (f, \phi) \text{ for any nonnegative } \phi \in H^\mu_{B_{2R}}(\mathbb{R}^d)
$$

$$
u(x) \geq \varepsilon \text{ for almost all } x \in B_{\Theta R} \text{ and some } \varepsilon > (\Theta R)^\delta \|f\|_{L^{q/(\alpha)}(B_{R+1+\rho})}.\]
where $\delta = \alpha\left(\frac{d-d_q}{q}\right)$. Then for any $p_0 > 0$

$$\inf_{x \in B_R(x_0)} u(x) \geq c \left( \frac{1}{\int_{B_{R_0}(x_0)} u(x)^{-p_0} \, dx} \right)^{\frac{1}{p_0}},$$

(4.14)

where $c > 0$ is independent of $u, x_0, R, \varepsilon$, and $\alpha$.

Proof. The idea of the proof is to apply Lemma 4.6 to radii $R_k, \rho_k$ with $R_k \searrow \eta R$ and $\rho_k \searrow 0$ for $k \to \infty$. For each $k$ one chooses an exponent $p_k > 1$ with $p_k \to \infty$ for $k \to \infty$. Because of Assumption (A) we can apply the Sobolev inequality, Fact 3.4, to the left-hand side in (4.12). Next, one iterates the resulting inequality as in [27], see also Chapter 8.6 in [15]. The only difference to the proof in [27] is that the factor $d^d$ now becomes $d^d \alpha$. The assertion then follows from the fact

$$\left( \int_{B_{R_k}(x_0)} u^{-p_0} \, dx \right)^{\frac{1}{p_k}} \to \inf_{B_{\rho R}(x_0)} u \quad \text{for} \quad k \to \infty. \quad \Box$$

Let us finally prove Theorem 4.1.

Proof of Theorem 4.1. Define $\overline{\nu} = u + \|f\|_{L^{\infty}(B_{1R})}$ and note that $\nu'(u, \phi) = \nu'(\overline{\nu}, \phi)$ for any $\phi$. We apply Lemma 4.5 for $R = 3/4$ and obtain that there exist $\overline{\nu} \in (0, 1)$ and $c > 0$ such that

$$\left( \int_{B_{\frac{3}{4}}} \overline{\nu}(x)^p \, dx \right)^{1/p} \leq c \left( \int_{B_{\frac{3}{4}}} u(x)^{-p} \, dx \right)^{-1/p}.$$

Next, we apply Corollary 4.7 with $R = 3/4$, $\eta = 2/3$ and $\Theta = 4/3$. Together with the estimate from above we obtain

$$\inf_{B_{\frac{1}{2}}} u \geq c \left( \frac{1}{\int_{B_{\frac{3}{4}}} \overline{\nu}(x)^p \, dx} \right)^{\frac{1}{p}},$$

(4.15)

which, after recalling the definition of $\overline{\nu}$, proves Theorem 4.1. \hfill \Box

5. The weak Harnack inequality implies Hölder estimates

The aim of this section is to provide the proof of Theorem 1.4. As is explained in Subsection 1.4 it is well known that the Harnack inequality or the weak Harnack inequality imply regularity estimates in Hölder spaces. Here we are going to establish such a result for quite general nonlocal operators in the framework of metric measure spaces.

We begin with a short study of condition (1.16). The standard example that we have in mind is given in Example 5. Let $(X, d, m)$ be a metric measure space. For $R > r > 0, x \in X$, set

$$B_r(x) = \{ y \in X | d(y, x) < r \}, \quad A_{r,R}(x) = B_R(x) \setminus B_r(x).$$

(5.1)

Lemma 5.1. For $x \in X, r > 0$ let $\nu_{x,r}$ be a measure on $B(X \setminus \{ x \})$, which is finite on all sets $M$ with $\text{dist}((x), M) > 0$. Then the following conditions are equivalent:

(1) For some $\chi > 1$, $c \geq 1$ and all $x \in X, 0 < r \leq 1, j \in \mathbb{N}_0$

$$\nu_{x,r}(X \setminus B_r(x)) \leq c \chi^{-j}.$$
(2) Given \( \theta > 1 \), there are \( \chi > 1 \), \( c \geq 1 \) such that for all \( x \in X, 0 < r \leq 1, j \in \mathbb{N}_0 \)
\[ \nu_{x,r}(X \setminus B_{r \theta^j}(x)) \leq c \chi^{-j}. \]

(3) Given \( \theta > 1 \), there are \( \chi > 1 \), \( c \geq 1 \) such that for all \( x \in X, 0 < r \leq 1, j \in \mathbb{N}_0 \)
\[ \nu_{x,r}(A_{r \theta^j}, r \theta^{j+1})(x)) \leq c \chi^{-j}. \]

(4) Given \( \sigma > 1, \theta > 1 \) there are \( \chi > 1 \), \( c \geq 1 \) such that for all \( x \in X, 0 < r \leq 1, j \in \mathbb{N}_0 \)
and \( y \in B_{\sigma}(x) \)
\[ \nu_{y,r'}(A_{r \theta^j}, r \theta^{j+1})(x)) \leq c \chi^{-j}, \] where \( r' = r(1 - \frac{1}{\sigma}) \).

(5.2) If, in addition to any of the above conditions, (1.17) holds, then (5.2) can be replaced by
\[ \nu_{y,r}(A_{r \theta^j}, r \theta^{j+1})(x)) \leq c \chi^{-j}. \] (5.3)

Proof. In \( \theta > 2 \), the implication (1) \( \Rightarrow \) (2) trivially holds true. For \( \theta < 2 \) it can be obtained by adjusting \( \chi \) appropriately. The proof of (2) \( \Rightarrow \) (1) is analogous. The implication (2) \( \Rightarrow \) (3) trivially holds true. The implication (3) \( \Rightarrow \) (2) follows from
\[ \nu_{x,r}(X \setminus B_{r \theta^j}(x)) = \sum_{k=j}^{\infty} \nu_{x,r}(A_{r \theta^k}, r \theta^{k+1})(x)) \leq c \sum_{k=j}^{\infty} \chi^{-k} = c \frac{\chi^{j+1}}{\chi - 1} \chi^{-j}. \]
The implication (4) \( \Rightarrow \) (3) trivially holds true. Instead of (3) \( \Rightarrow \) (4) we explain the proof of (2) \( \Rightarrow \) (4). Fix \( \sigma > 1, \theta > 1, x \in X, r > 0, j \in \mathbb{N}_0 \) and \( y \in B_{\sigma}(x) \). Set \( r' = r(1 - \frac{1}{\sigma}) \). Then \( X \setminus B_{r \theta^j}(x) \subset X \setminus B_{r \theta^j}(y) \). Thus
\[ \nu_{y,r'}(X \setminus B_{r \theta^j}(x)) \leq \nu_{y,r'}(X \setminus B_{r \theta^j}(y)) \leq c \chi^{-j}. \]

Remark. Note that the conditions above imply that, given \( j \in \mathbb{N}_0 \) and \( x \in X \), the quantity \( \lim_{r \to 0^+} \nu_{x,r}(X \setminus B_{r \theta^j}(x)) \) is finite.

Remark. Let \( x \in X, A \in \mathcal{B}(X \setminus \{x\}) \) with \( \text{dist}({x},A) > 0 \). In the applications that are of interest to us, the function \( r \mapsto \nu_{x,r}(A) \) is strictly increasing with \( \nu_{x,0}(A) = 0 \).

Proof of Theorem 1.4. The proof follows closely the strategy of [27], see also [32]. In the sequel of the proof, let us write \( B_t \) instead of \( B_t(x_0) \) for \( t > 0 \). Fix \( r \in (0, r_0) \) and \( u \in \mathcal{S}_{x_0, r} \). Let \( c_1 \geq 1 \) be the constant in (1.19). Set \( \kappa = (2c_1 2^{1/p})^{-1} \) and
\[ \beta = \ln\left(\frac{2}{2^{1/p}}\right) / \ln(\theta) = (1 - \frac{\kappa}{2}) = \theta^{-\beta}. \]
Set \( M_0 = \|u\|_\infty, m_0 = \inf_x u(x) \) and \( M_{-n} = M_0, m_{-n} = m_0 \) for \( n \in \mathbb{N} \). We will construct an increasing sequence \( (m_n) \) and a decreasing sequence \( (M_n) \) such that for \( n \in \mathbb{Z} \)
\[ m_n \leq u(z) \leq M_n \quad \text{for almost all} \; z \in B_{r \theta^{-n}}, \]
\[ M_n - m_n \leq K\theta^{-n\beta} \quad \text{for almost all} \; z \in B_{r \theta^{-n}}, \] (5.4)
where \( K = M_0 - m_0 \in [0, 2]\|u\|_\infty \). Assume there is \( k \in \mathbb{N} \) and there are \( M_0, m_k \) such that (5.4) holds for \( n \leq k - 1 \). We need to choose \( m_n, M_n \) such that (5.4) still holds for \( n = k \). Then the assertion of the lemma follows by complete induction. For \( z \in X \) set
\[ v(z) = \left( u(z) - \frac{M_{k-1} + m_{k-1}}{2} \right) \frac{2\theta^{(k-1)\beta}}{K}. \]
The definition of $v$ implies $v \in S_{x_0,r}$ and $|v(z)| \leq 1$ for almost any $z \in B_{r\theta^{-k}(1)}$. Our next aim is to show that (1.19) implies that either $v \leq 1 - \kappa$ or $v \geq -1 + \kappa$ on $B_{r\theta^{-k}}$. Since our version of the Harnack inequality contains nonlocal terms we need to investigate the behavior of $v$ outside of $B_{r\theta^{-k}(1)}$. Given $z \in X$ with $d(z,x_0) \geq r\theta^{-k(1)}$ there is $j \in \mathbb{N}$ such that

$$r\theta^{-k+j} \leq d(z,x_0) < r\theta^{-k+j+1}.$$  

For such $z$ and $j$ we conclude

$$\frac{K}{2\theta^{(k-1)\beta}} v(z) = \left( u(z) - \frac{M_{k-1} + m_{k-1}}{2} \right) \leq \left( M_{k-j-1} - m_{k-j-1} + M_{k-1} - m_{k-1} - \frac{M_{k-1} + m_{k-1}}{2} \right)$$

$$\leq \left( M_{k-j-1} - m_{k-j-1} - \frac{M_{k-1} - m_{k-1}}{2} \right) \leq \left( K\theta^{-(k-j-1)\beta} - \frac{K}{2}\theta^{-(k-1)\beta} \right),$$

i.e. $v(z) \leq 2\theta^\beta - 1 \leq 2\left( \frac{d(z,x_0)}{r\theta^{(k-1)}} \right)^\beta - 1,$ (5.5)

and

$$\frac{K}{2\theta^{(k-1)\beta}} v(z) = \left( u(z) - \frac{M_{k-1} + m_{k-1}}{2} \right) \geq \left( m_{k-j-1} - M_{k-j-1} + M_{k-1} - m_{k-1} - \frac{M_{k-1} + m_{k-1}}{2} \right)$$

$$\geq \left( - (M_{k-j-1} - m_{k-j-1}) + \frac{M_{k-1} - m_{k-1}}{2} \right) \geq \left( - K\theta^{-(k-j-1)\beta} + \frac{K}{2}\theta^{-(k-1)\beta} \right),$$

i.e. $v(z) \geq 1 - 2\theta^\beta \geq 1 - 2\left( \frac{d(z,x_0)}{r\theta^{(k-1)}} \right)^\beta.$

Now there are two cases:

Case 1: $m(\{x \in B_{r\theta^{-(k+1)/\lambda}} | v(x) \leq 0 \}) \geq \frac{1}{2}m(B_{r\theta^{-k+1}/\lambda})$

Case 2: $m(\{x \in B_{r\theta^{-(k+1)/\lambda}} | v(x) > 0 \}) \geq \frac{1}{2}m(B_{r\theta^{-k+1}/\lambda})$

We work out details for Case 1 and comment afterwards on Case 2. In Case 1 our aim is to show $v(z) \leq 1 - \kappa$ for almost every $z \in B_{r\theta^{-k}}$ and some $\kappa \in (0,1)$. Because then for almost any $z \in B_{r\theta^{-k}}$

$$u(z) \leq \frac{(1-\kappa)K}{2}\theta^{-(k-1)\beta} + \frac{M_{k-1} + m_{k-1}}{2}$$

$$= \frac{(1-\kappa)K}{2}\theta^{-(k-1)\beta} + \frac{M_{k-1} - m_{k-1}}{2} + m_{k-1}$$

$$= m_{k-1} + \frac{(1-\kappa)K}{2}\theta^{-(k-1)\beta} + \frac{1}{2}K\theta^{-(k-1)\beta}$$

$$\leq m_{k-1} + K\theta^{-k\beta}.$$ (5.6)

We then set $m_k = m_{k-1}$ and $M_k = m_k + K\theta^{-k\beta}$ and obtain, using (5.6), $m_k \leq u(z) \leq M_k$ for almost every $z \in B_{r\theta^{-k}}$, what needs to be proved.

Consider $w = 1 - v$ and note $w \in S_{x_0, r\theta^{-(k-1)}}$ and $w \geq 0$ in $B_{r\theta^{-(k-1)}}$. We apply (1.19) and obtain

$$\left( \int_{B_{r\theta^{-k+1}/\lambda}(x_0)} w^p dm \right)^{1/p} \leq c_1 \inf_{B_{r\theta^{-k}}} w + c_1 \sup_{x \in B_{r\theta^{-k+1}/\lambda}} \int_X w^{-}(z) v_{x, r\theta^{-(k-1)}} (dz),$$ (5.7)
In Case 1 the left-hand side of (5.7) is bounded from below by \((\frac{1}{2})^{1/p}\). This, the estimate (5.5) on \(v\) from above leads to
\[
\inf_{B_{r\theta-k}} w \geq (c_1 2^{1/p})^{-1} \sup_{x \in B_{r\theta-k+1/\rho}} \int_X w^-(z) \nu_{x,r\theta^{-((k-1)})} (dz) \\
\geq (c_1 2^{1/p})^{-1} \sum_{j=1}^{\infty} \sup_{x \in B_{r\theta-k+1/\rho}} \int A_{r\theta-k+j, r\theta-k+j+1}(x_0) (1-v(z))^{-\nu_{x,r\theta^{-((k-1)})}} (dz) \\
\geq (c_1 2^{1/p})^{-1} - \sum_{j=1}^{\infty} (2^{j\beta} - 2) \eta_{x_0,r\theta,j,k},
\]
where \(\eta_{x_0,r\theta,j,k} = \sup_{x \in B_{r\theta-k+1/\rho}} \nu_{x,r\theta^{-((k-1)})} (A_{r\theta-k+j, r\theta-k+j+1}(x_0))\). Now, (5.3) implies that \(\eta_{x_0,r\theta,j,k} \leq c_1^{-j-1}\). Thus we obtain
\[
\inf_{B_{r\theta-k}} w \geq (c_1 2^{1/p})^{-1} - 2c_1 \sum_{j=1}^{\infty} (2^{j\beta} - 1) \chi^{-j-1}.
\]
(5.8)

Note that \(\sum_{j=1}^{\infty} 2^{j\beta} \chi^{-j-1} < \infty\) for \(\beta > 0\) small enough, i.e. there is \(l \in \mathbb{N}\) with
\[
\sum_{j=l+1}^{\infty} (2^{j\beta} - 1) \chi^{-j-1} \leq \sum_{j=l+1}^{\infty} 2^{j\beta} \chi^{-j-1} \leq (16c_1)^{-1}.
\]
Given \(l\) we choose \(\beta > 0\) smaller (if needed) in order to assure
\[
\sum_{j=1}^{l} (2^{j\beta} - 1) \chi^{-j-1} \leq (16c_1)^{-1}.
\]
The number \(\beta\) depends only on \(c_1, c, \chi\) from (5.3) and on \(\theta\). Thus we have shown that \(w \geq \kappa\) on \(B_{r\theta-k}\) or, equivalently, \(v \leq 1 - \kappa\) on \(B_{r\theta-k}\).

In Case 2 our aim is to show \(v(x) \geq -1 + \kappa\). This time, set \(w = 1 + v\). Following the strategy above one sets \(M_k = M_{k-1}\) and \(m_k = M_k - K \theta^{-k\beta}\) leading to the desired result.

Let us show how (5.4) proves the assertion of the lemma. Given \(\rho \leq r\), there exists \(j \in \mathbb{N}_0\) such that
\[
r \theta^{-j-1} \leq \rho \leq r \theta^{-j}.
\]
From (5.4) we conclude
\[
\osc_{B_{\rho}} u \leq \osc_{B_{\theta^{-j}}} u \leq M_j - m_j \leq 2\theta^\beta \|u\|_\infty \left(\frac{\rho}{r}\right)^\beta.
\]
\(\square\)

**Corollary 5.2.** Let \(\Omega = B_{r_0}(x_0) \subset X\) and let \(\sigma, \theta, \lambda > 1\). Let \(S_{x,r}\) and \(\nu_{x,r}\) be as above. Assume that conditions (1.16), (1.17) are satisfied. Assume that there is \(c \geq 1\) such that for \(0 < r \leq r_0\),
\[
\left\{
\begin{array}{l}
(B_r(x) \subset \Omega) \land (u \in S_{x,r}) \land (u \geq 0 \text{ in } B_r(x)), \\
\Rightarrow \left( \int_{B_{\frac{r}{\theta}}(x)} u(\xi)^p m(d\xi) \right)^{1/p} \leq c \inf_{B_{\frac{r}{\theta}}(x)} u + c \sup_{\xi \in B_{\frac{r}{\theta}}(x)} \int_{B_{\frac{r}{\theta}}(x)} u^-(z) \nu_{\xi,r}(dz) \right.
\end{array}
\right\}
\]
(5.9)
Then there exist $\beta \in (0,1)$ such that for every $u \in \mathcal{S}_{x_0,r_0}$ and almost every $x, y \in \Omega$

$$|u(x) - u(y)| \leq 16\theta^3\|u\|_\infty \left(\frac{d(x,y)}{d(x,\Omega^c) \lor d(y,\Omega^c)}\right)^\beta.$$  \hfill (5.10)

**Proof.** By symmetry, we may assume that $r := d(y,\Omega^c) \geq d(x,\Omega^c)$. Furthermore, it is enough to prove (5.10) for pairs $x, y$ such that $d(x,y) < r/8$, as in the opposite case the assertion is obvious.

We fix a number $\rho \in (0, r_0/4)$ and consider all pairs of $x, y \in \Omega$ such that

$$\frac{\rho}{2} \leq d(x,y) \leq \rho.$$  \hfill (5.11)

We cover the ball $B_{r_0-4\rho}(x_0)$ by a countable family of balls $\tilde{B}_i$ with radii $\rho$. Without loss of generality, we may assume that $\tilde{B}_i \cap B_{r_0-4\rho}(x_0) \neq \emptyset$. Let $B_i$ resp. $B_i^*$ denote the balls with the same center as the ball $\tilde{B}_i$ and the radius $2\rho$ resp. the maximal radius that allows for $B_i^* \subset \Omega$.

Let $x, y \in \Omega$ satisfy (5.11). From $r > 8d(x,y) \geq 4\rho$ it follows that $y \in B_{r_0-4\rho}(x_0)$, therefore $y \in B_i$ for some index $i$. We observe that both $x$ and $y$ belong to $B_i$. We apply Theorem 1.4 to $x_0$ and $r_0$ being the center and radius of $B_i^*$, respectively, and obtain

$$\begin{align*}
\text{osc}_{B_i} u &\leq 2\theta^3\|u\|_\infty \left(\frac{\text{radius}(B_i)}{\text{radius}(B_i^*)}\right)^\beta \\
&\leq 2\theta^3\|u\|_\infty \left(\frac{\rho}{r-\rho}\right)^\beta.
\end{align*}$$

Hence (5.10) holds, provided $x$ and $y$ are such that $|u(x) - u(y)| \leq \text{osc}_{B_i} u$.

By considering $\rho = r_02^{-j}$ for $j = 3, 4, \ldots$, we prove (5.10) for almost all $x$ and $y$ such that $d(x,y) \leq r_0/8$, hence the proof is finished. $\square$

### 5.1. Proof of Theorem 1.2

We are now going to use the above results and prove one of our main results.

**Proof of Theorem 1.2.** The proof of Theorem 1.2 follows from Corollary 4.2 and Corollary 5.2. The proof is complete once we can apply Corollary 5.2 for $x_0 = 0$ and $r_0 = \frac{1}{2}$. Assume $0 < r \leq r_0$ and $B_r(x) \subset B_{\frac{1}{2}}$. Let $\mathcal{S}_{x,r}$ be the set of all functions $u \in V^\mu_{B_r(x)}(\mathbb{R}^d)$ satisfying $\mathcal{E}(u,\phi) = 0$ for every $\phi \in H^\mu_{B_r(x)}(\mathbb{R}^d)$. Assume $u \in \mathcal{S}_{x,r}$ and $u \geq 0$ in $B_r(x)$. Then Corollary 4.2 implies

$$\inf_{B_{\frac{1}{2}}(x)} u \geq c \left( \int_{B_{\frac{1}{2}}(x)} u(x)^{p_0} \, dx \right)^{1/p_0} - r^\alpha \sup_{y \in B_{1/2}(x)} \int_{\mathbb{R}^d \setminus B_r(x)} u^-(z) \mu(y,dz),$$

with positive constants $p_0, c$ which depend only on $d, \alpha_0, A, B$. Set $\theta = 4, \lambda = 2, \sigma = \frac{16}{15}$. Let $\nu_{x,r}$ be the measure on $\mathbb{R}^d \setminus B_r(x)$ defined by

$$\nu_{x,r}(A) = r^\alpha \mu(x, A).$$

The condition (1.17) obviously holds true. The condition (1.16) follows from (D). Thus we can apply Corollary 5.2 for $x_0 = 0$ and $r_0 = \frac{1}{2}$ and obtain the assertion of Theorem 1.2. The proof is complete. $\square$
6. Local comparability results for nonlocal quadratic forms

The aim of this section is to prove Theorem 1.5. The assertion of this result is that (A) and (B) hold true under certain assumptions on $\mu(\cdot, dy)$, see Subsection 1.5. It is easy to prove that (T) and (U) imply (B) with a constant $B \geq 1$ independent of $\alpha \in (\alpha_0, 2)$: Let $\tau \in C^\infty(\mathbb{R}^d)$ be a function satisfying $\text{supp}(\tau) = \overline{B_{R+r}}, \tau \equiv 1$ on $B_R$, $0 \leq \tau \leq 1$ on $\mathbb{R}^d$ and $|\tau(x) - \tau(y)| \leq 2r^{-1}|x - y|$ for all $x, y \in \mathbb{R}^d$. In particular, we have then $|\tau(x) - \tau(y)| \leq (2r^{-1}|x - y|) \wedge 1$. For every $x \in \mathbb{R}^d$ we obtain

$$
\int_{\mathbb{R}^d} (\tau(x) - \tau(y))^2 \mu(x, dy) \leq \int_{\mathbb{R}^d} \left((4r^{-2}|z|^2) \wedge 1\right) \nu^*(dz) = 4r^{-2} \int_{\mathbb{R}^d} (|z|^2 \wedge \frac{r^2}{4}) \nu^*(dz) \leq 2^\alpha C_U r^{-\alpha} \leq 4C_U r^{-\alpha}.
$$

Thus we only need to concentrate on proving (A). The upper bound can be established quite easily, so we do this first.

6.1. Upper bound in (A). Let us formulate and prove the following comparability result.

**Proposition 6.1.** Assume that $\nu$ satisfies (U) with the constant $C_U$ and let $0 < \alpha_0 \leq \alpha < 2$. If $D \subset \mathbb{R}^d$ is a bounded Lipschitz domain, then there exists a constant $c = c(\alpha_0, d, C_U, D)$ such that

$$
\mathcal{E}_D(u, u) \leq c \mathcal{E}_D^{\alpha_0}(u, u), \quad u \in H^{\alpha/2}(D).
$$

The constant $c$ may be chosen such that (6.1) holds for all balls $D = B_r$ of radius $r < 1$, and for all $\alpha \in [\alpha_0, 2)$.

**Proof.** By $E$ we denote the extension operator from $H^{\alpha/2}(D)$ to $H^{\alpha/2}(\mathbb{R}^d)$, see Fact 3.1. By subtracting a constant, we may and do assume that $\int_D u dx = 0$. We have by Plancherel formula and Fubini theorem

$$
\mathcal{E}_D^{\nu}(u, u) = \int_D \int_{D-y} (u(y+z) - u(y))^2 \nu(dz) dy
$$

$$
\leq \int_D \int_{B_{\text{diam}(D)}(0)} (Eu(y+z) - Eu(y))^2 \nu(dz) dy
$$

$$
\leq \int_{B_{\text{diam}(D)}(0)} \int_{\mathbb{R}^d} (Eu(y+z) - Eu(y))^2 dy \nu(dz)
$$

$$
= \int_{\mathbb{R}^d} \left( \int_{B_{\text{diam}(D)}(0)} |e^{i\xi z} - 1|^2 \nu(dz) \right) |\widehat{Eu}(\xi)|^2 d\xi
$$

$$
= \int_{\mathbb{R}^d} \left( \int_{B_{\text{diam}(D)}(0)} 4 \sin^2 \left( \frac{\xi \cdot z}{2} \right) \nu(dz) \right) |\widehat{Eu}(\xi)|^2 d\xi.
$$

For $|\xi| > 2$ we obtain, using (U)

$$
\int 4 \sin^2 \left( \frac{\xi \cdot z}{2} \right) \nu(dz) \leq |\xi|^2 \int (|z|^2 \wedge 4|\xi|^2) \nu(dz) \leq 4C_U |\xi|^\alpha,
$$

and for $|\xi| \leq 2$

$$
\int 4 \sin^2 \left( \frac{\xi \cdot z}{2} \right) \nu(dz) \leq 4 \int \left( \left| \frac{\xi \cdot z}{2} \right| \wedge 1 \right) \nu(dz) \leq 4C_U.
$$
Thus \[
\mathcal{E}_D^\nu(u, u) \leq c \int_{\mathbb{R}^d} (|\xi|^\alpha + 1) |\hat{E}u(\xi)|^2 d\xi
\leq c \|E\|_{L^2(\mathbb{R}^d)}^2 \leq c \|u\|_{H^\alpha/2(\mathbb{R}^d)}^2 = c(\mathcal{E}_D^{\nu_0}(u, u) + \|u\|_{L^2(\mathbb{R}^d)}^2)
\] (6.5)
with \(c = c(d, C_U, D)\). Since \(\int_D u\, dx = 0\), we have by Fact 3.2
\[
\mathcal{E}_D^{\nu_0}(u, u) \geq c(\alpha_0, d, D) \int_D u^2(x)\, dx
\]
and this together with (6.5) proves (6.1).

By scaling, the last assertion of the Theorem is satisfied with a constant \(c = c(\alpha_0, d, C_U, B_1)\).

Proof of Theorem 1.5 – upper bound in (A). The second inequality in (A) follows from Proposition 6.1. We note that the constant in this inequality is robust under the mere assumption that \(\alpha\) is bounded away from zero.

6.2. Lower bound in (A). The main difficulty in establishing the lower bound in (A) is that the measures might be singular. We will introduce a new convolution-type operation that, on the one hand, smoothes the support of the measures and, on the other hand, interacts nicely with our quadratic forms. The main result of this subsection is Proposition 6.10.

For \(\lambda < 1 \leq \eta\) and \(\alpha \in (0, 2)\) let
\[
g^0_\lambda(y, z) = \frac{1}{2 - \alpha} |y + z|^{\alpha} \mathbb{I}_{A^\eta_0}(y) \mathbb{I}_{A^\eta_0}(z), \quad y, z \in \mathbb{R}^d,
\] (6.6)
where
\[A_\lambda = B(0, \eta r) \setminus B(0, \lambda r).\]

Definition 6.2. For measures \(\nu_1, \nu_2\) on \(\mathcal{B}(\mathbb{R}^d)\) satisfying (U) with some \(\alpha \in (0, 2)\), define a new measure \(\nu_1 \bowtie \nu_2\) on \(\mathcal{B}(\mathbb{R}^d)\) by
\[
\nu_1 \bowtie \nu_2(E) = \int E \cap B_2(\eta(y + z)) g^0_\lambda(y, z) \nu_1(dy) \nu_2(dz),
\]
i.e.,
\[
\int f(x)\nu_1 \bowtie \nu_2(dx) = \int (f \circ \mathbb{I}_{B_2}(\eta(y + z))) g^0_\lambda(y, z) \nu_1(dy) \nu_2(dz),
\]
for every measurable function \(f : \mathbb{R}^d \to [0, \infty]\).

This definition is tailored for our applications and needs some explanations. We consider \(\nu_1 \bowtie \nu_2\) only for measures \(\nu_j\) which satisfy (U) with some \(\alpha \in (0, 2)\) for \(j \in \{1, 2\}\). This \(\alpha\) equals the exponent \(\alpha\) in the definition of \(g^0_\lambda\). The above definition does not require \(\nu_j\) to satisfy (S) but most often, this will be the case. Note that Definition 6.2 is valid for any choice \(\lambda < 1 \leq \eta\). However, it will be important to choose \(\lambda\) small enough and \(\eta\) large enough. The precise bounds depend on the number \(a\) from (S), see Proposition 6.10. Before we explain and prove the rather technical details, let us treat an example.

Let us study Example 4 in \(\mathbb{R}^2\). Assume \(\alpha \in (0, 2)\) and
\[
\nu_1(dh) = (2 - \alpha)|h_1|^{-1-\alpha}dh_1\delta_{\{0\}}(dh_2),
\nu_2(dh) = (2 - \alpha)|h_2|^{-1-\alpha}dh_2\delta_{\{0\}}(dh_1).
\]
Both measures are one-dimensional \(\alpha\)-stable measures which are orthogonal to each other. The factor \((2 - \alpha)\) ensures that for \(\alpha \to 2-\) the measures do not explode. Let us show that \(\nu_1 \bowtie \nu_2\)
is already absolutely continuous with respect to the two-dimensional Lebesgue measure. For $E \subset B_2$; by the Definition 6.2 and the Fubini theorem

$$
\nu_1 \wedge \nu_2 (E) = (2 - \alpha) \iiint |y + z|^\alpha \mathbb{1}_E(\eta(y + z)) \mathbb{1}_{A_{|y+z|}}(y) \mathbb{1}_{A_{|y+z|}}(z) |y_1|^{-\alpha}|z_2|^{-\alpha} \ldots \delta_{[0]}(dy_2) \delta_{[0]}(dz_2) dy_1 dz_2
$$

$$
= (2 - \alpha) \int |(y_1, z_2)|^\alpha \mathbb{1}_E(\eta(y_1, z_2)) \mathbb{1}_{A_{|y_1-z_2|}}(y_1, 0) \mathbb{1}_{A_{|y_1-z_2|}}(0, z_2) |y_1|^{-\alpha}|z_2|^{-\alpha} dy_1 dz_2
$$

$$
= (2 - \alpha) \int \mathbb{1}_E(\eta x) \mathbb{1}_{A_{|x|}}(x_1, 0) \mathbb{1}_{A_{|x|}}(0, x_2) |x|^\alpha |x_1|^{-\alpha} |x_2|^{-\alpha} dx_1 dx_2.
$$

The above computation shows that the measure $\nu_1 \wedge \nu_2$ is absolutely continuous with respect to the two-dimensional Lebesgue measure, because $\nu_1 \wedge \nu_2 (\mathbb{R}^d \setminus B_2) = 0$. Let us look at the density more closely.

So far, we have not specified $\lambda$ and $\eta$ in the definition of $g_\eta^\alpha$. If $\lambda < 1$ is too large (in this particular case, if $\lambda > 1/\sqrt{2}$), then $\mathbb{1}_{A_{|x|}}(x_1, 0) \mathbb{1}_{A_{|x|}}(0, x_2) = 0$ for all $x \in \mathbb{R}^2$. If $\lambda$ is sufficiently small, then the support of the function $\mathbb{1}_{A_{|x|}}(x_1, 0) \mathbb{1}_{A_{|x|}}(0, x_2)$ is a double-cone centered around the diagonals $\{x \in \mathbb{R}^2||x_1| = |x_2|\}$. Let us denote this support by $M$. Note that on $M$ the function $|x|^\alpha |x_1|^{-\alpha} |x_2|^{-\alpha}$ is comparable to $|x|^{-2\alpha}$. Thus indeed the quantity $\nu_1 \wedge \nu_2$ is comparable to an $\alpha$-stable measure in $\mathbb{R}^2$. If we continue the procedure and define

$$
\tilde{\nu} = (\nu_1 \wedge \nu_2) \wedge (\nu_1 \wedge \nu_2),
$$

then we can make use of the fact that $\nu_1 \wedge \nu_2$ is already absolutely continuous with respect to the two-dimensional Lebesgue measure. Note that, if $\mu_j = h_j \, dx$, then $\mu_1 \wedge \mu_2$ has a density $h_1 \wedge h_2$ with respect to the Lebesgue measure given by

$$
h_1 \wedge h_2(\eta y) = \frac{\eta^{-\alpha} |y|^\alpha}{2 - \alpha} \int \mathbb{1}_{A_{|y|}}(y - z) \mathbb{1}_{A_{|y|}}(z) h_1(y - z) h_2(z) dz, \quad \eta y \in B_2. \quad (6.7)
$$

In this way we conclude that $\tilde{\nu}$ has full support and is comparable to a rotationally symmetric $\alpha$-stable measure in $\mathbb{R}^2$. With this observation we end our study of Definition 6.2 in light of Example 4.

Before we proceed to the proofs, let us informally explain the idea behind Definition 6.2 and our strategy. In the inner integral defining

$$
\mathcal{E}_B^\nu (u, u) = \int_B \int_{\mathbb{R}^d} \left( u(x) - u(x + h) \right)^2 \mathbb{1}_B(x + h) \nu(dh) \, dx
$$

we take into account squared increments $(u(x) - u(x + h))^2$ in these directions $h$, which are charged by the measure $\nu$ and such that $x + h$ is still in $B$. By changing the variables, we see that we also have squared increments $(u(x + h) - u(x + h + z))^2$, again in directions $z$, which are charged by the measure $\nu$ and such that $x + h + z$ is still in $B$. This allows us to estimate the integral $\mathcal{E}_B^\nu (u, u)$ from below by a similar integral with $\nu$ replaced by some kind of a convolution of $\nu$ with itself. Measure $\nu \wedge \nu$ turns out to be the right convolution for this purpose, see Lemma 6.8.

In the definition of $\nu \wedge \nu$, function $g_\eta^\alpha$ vanishes if $|y|$ or $|z|$ is bigger than $\eta |y + z|$ or smaller than $\lambda |y + z|$. This means, in our interpretation, that we consider only those pairs of jumps which are comparable with the size of the whole two-step jump (and in particular, the jumps must be comparable with each other).
To conclude these informal remarks on the definition of \( \nu_1 \odot \nu_2 \) let us note that if \( \nu_1 \) and \( \nu_2 \) have 'good properties', then so has \( \nu_1 \odot \nu_2 \) (see Lemma 6.3 and Lemma 6.7) and that \( \mathcal{E}_B^{\nu_1 \odot \nu_2} (u, u) \) can be estimated from above by \( \mathcal{E}_B^{\nu_1} (u, u) \) (see Lemma 6.8). This allows us to reduce the problem of estimating \( \mathcal{E}_B^{\nu_1} (u, u) \) from below to estimating \( \mathcal{E}_B^{\nu_1 \odot \nu_2} (u, u) \) from below, and this turns out to be easier, since the \( \odot \)-convolution makes the measure more 'smooth', see Proposition 6.10.

**Lemma 6.3.** If two measures \( \nu_j \) for \( j \in \{1, 2\} \) satisfy the scaling assumption (S) for some \( a > 1 \), then so does the measure \( \nu_1 \odot \nu_2 \) for the same constant \( a \).

**Proof.** If \( \text{supp } f \subset B_1 \), then
\[
\int f(ax) \nu_1 \odot \nu_2 (dx) = \iint f(\eta a(y + z)) 1_{B_1}(\eta(y + z)) g^\eta(y, z) \nu_1(dy) \nu_2(dz) = a^{-a} \iint f(\eta(ay + az)) g^\eta(ay, az) \nu_1(dy) \nu_2(dz),
\]
because \( g^\eta(y, z) = a^{-a} g^\eta(ay, az) \). We observe that the function \( (y, z) \mapsto f(\eta(y + z)) g^\eta(y, z) \) vanishes outside \( B_1 \times B_1 \). Hence we may apply (S) twice to obtain
\[
\int f(ax) \nu_1 \odot \nu_2 (dx) = a^a \iint f(\eta(y + z)) g^\eta(y, z) \nu_1(dy) \nu_2(dz) = a^a \int f(x) \nu_1 \odot \nu_2(dx). \quad \square
\]

Next, we establish conditions which are equivalent to (U). We say that a measure \( \nu \) on \( B(\mathbb{R}^d) \) satisfies the upper-bound assumption (U0) if for some \( C_0 > 0 \)
\[
\int_{\mathbb{R}^d} (|z|^2 \wedge 1) \nu(dz) \leq C_0. \tag{U0}
\]

We say that a measure \( \nu \) on \( B(\mathbb{R}^d) \) satisfies the upper-bound assumption (U1) if there exists \( C_1 > 0 \) such that for every \( r \in (0, 1) \)
\[
\int_{B_r(0)} |z|^2 \nu(dz) \leq C_1 r^{2-\alpha}. \tag{U1}
\]

**Lemma 6.4.**
\[
(U) \iff (U0) \wedge (U1).
\]

If the constants \( C_0, C_1 \) are independent of \( \alpha \in [\alpha_0, 2) \), then so is \( C_U \), and vice versa.

**Proof.** The implications \( (U) \Rightarrow (U1) \) and \( (U) \Rightarrow (U0) \) are obvious, we may take \( C_0 = C_1 := C_U \). Let us now assume that \( (U1) \) and \( (U0) \) hold true. Fix \( 0 < r \leq 1 \). We consider \( n = 0, 1, 2, \ldots \) such that \( 2^{n+1} r \leq 1 \) (the set of such \( n \)'s is empty if \( r > \frac{1}{2} \)). We have by (U1)
\[
\int_{2^n r \leq |z| < 2^{n+1} r} \nu(dz) \leq 2^{-2n} r^{-2} \int_{2^n r \leq |z| < 2^{n+1} r} |z|^2 \nu(dz)
\leq 2^{-2n} r^{-2} C_1 2^{(n+1)(2-\alpha)} r^{2-\alpha} = 2^{-na} 2^{-\alpha} C_1 r^{-\alpha}.
\]

After summing over all such \( n \) we obtain
\[
\int_{r \leq |z| < 1/2} \nu(dz) \leq \frac{2^{2-\alpha} C_1}{1-2^{-\alpha}} r^{-\alpha}.
\]

Finally
\[
\int_{1/2 \leq |z|} \nu(dz) \leq 4 \int_{\mathbb{R}^d} (|z|^2 \wedge 1) \nu(dz) \leq 4C_0 \leq 4C_0 r^{-\alpha}.
\]

Combining the two inequalities above and (U1) we get (U) with \( C_U = (\frac{2^{2-\alpha}}{1-2^{-\alpha}} + 1)C_1 + 4C_0 \). \quad \square
The following definition interpolates between measures \( \nu \) which are related to different values of \( \alpha \in (0, 2) \). Such a construction is important for us because we want to prove comparability results which are robust in the sense that constants stay bounded when \( \alpha \to 2^- \).

**Definition 6.5.** Assume \( \nu^{\alpha_0} \) is a measure on \( B(\mathbb{R}^d) \) satisfying (U) or (S) for some \( \alpha_0 \in (0, 2) \). For \( \alpha_0 \leq \alpha < 2 \) we define a new measure \( \nu^{\alpha,\alpha_0} \) by

\[
\nu^{\alpha,\alpha_0} = \frac{2 - \alpha}{2 - \alpha_0} \nu^{0_0 - \alpha_0} \nu^{\alpha_0}(dx) \quad \text{if } \alpha > \alpha_0 \quad \text{and by} \quad \nu^{\alpha_0,\alpha_0} = \nu^{\alpha_0}.
\]

(6.8)

To shorten notation we write \( \nu^\alpha \) instead of \( \nu^{\alpha,\alpha_0} \) whenever there is no ambiguity.

The above definition is consistent in the following ways. On the one hand, the first part of (6.8) holds true for \( \alpha = \alpha_0 \). On the other hand, for \( 0 < \alpha_0 < \alpha < \beta < 2 \), the following is true:

\[ \nu^{\beta,\alpha_0} = (\nu^{\alpha,\alpha_0})^{\beta,\alpha}. \]

This requires that \( \nu^{\alpha,\alpha_0} \) itself satisfies (U) or (S) which is established in the following lemma.

**Lemma 6.6.** Assume \( \nu^{\alpha_0} \) satisfies (U) with some \( \alpha_0 \in (0, 2) \), \( C_U > 0 \) or condition (S) with some \( \alpha_0 \in (0, 2) \), \( a > 1 \). Assume \( \alpha_0 \leq \alpha < 2 \) and \( \nu^\alpha \) as in **Definition 6.5**.

(a) If \( \nu^{\alpha_0} \) satisfies (U), then for every \( 0 < b < 1 \), \( 0 < r \leq 1 \)

\[
\int_{B_r \setminus |z| < r} |z|^2 \nu^\alpha (dz) \leq \frac{2 - \alpha}{2 - \alpha_0} C_U b^{\alpha_0 - \alpha} r^{2 - \alpha},
\]

(6.9)

\[
\int_{B_r} \nu^\alpha (dz) \leq \frac{2 - \alpha}{2 - \alpha_0} C_U r^{-\alpha}.
\]

(6.10)

(b) If \( \nu^{\alpha_0} \) satisfies (U), then \( \nu^\alpha \) satisfies (U) with exponent \( \alpha \) and constant \( 13 C_U (2 - \alpha_0)^{-1} \).

In particular, the constant does not depend on \( \alpha \).

(c) If \( \nu^{\alpha_0} \) satisfies (S), then \( \nu^\alpha \) satisfies (S) with exponent \( \alpha \).

**Proof.** Let \( 0 < r \leq 1 \) and \( 0 < b < 1 \). To prove (a), we derive,

\[
\int_{B_r \setminus |z| < r} |z|^2 \nu^\alpha (dz) = \frac{2 - \alpha}{2 - \alpha_0} \int_{B_r \setminus |z| < r} |z|^{2 + \alpha_0 - \alpha} \nu^{\alpha_0}(dz) \leq \frac{2 - \alpha}{2 - \alpha_0} \left( br \right)^{\alpha_0 - \alpha} \int_{B_r} |z|^2 \nu^{\alpha_0}(dz)
\]

\[
\leq \frac{2 - \alpha}{2 - \alpha_0} b^{\alpha_0 - \alpha} C_U r^{2 - \alpha},
\]

which proves (6.9). Furthermore,

\[
\int_{B_r} \nu^\alpha (dz) = \frac{2 - \alpha}{2 - \alpha_0} \int_{B_r} |z|^{\alpha_0 - \alpha} \nu^{\alpha_0}(dz) \leq \frac{2 - \alpha}{2 - \alpha_0} r^{\alpha_0 - \alpha} C_U r^{-\alpha_0}
\]

and (6.10) follows. To prove part (b), we use (6.9) and conclude

\[
\int_{B_r} |z|^2 \nu^\alpha (dz) = \sum_{n=0}^{\infty} \int_{\frac{r}{2^{n+1}}}^{\frac{r}{2^n}} |z|^2 \nu^\alpha (dz) \leq \frac{2 - \alpha}{2 - \alpha_0} C_U 2^{\alpha - \alpha_0} r^{2 - \alpha} \sum_{n=0}^{\infty} 2^{\alpha (2 - 2)}
\]

\[
= C_U 2^{\alpha - \alpha_0} r^{2 - \alpha} \leq \frac{32 C_U}{3(2 - \alpha_0)^{2 - \alpha}},
\]

since the function \( x \mapsto \frac{x}{1 - 2^{-x}} \) is increasing. Furthermore, by (6.10),

\[
\int_{B_r} r^2 \nu^\alpha (dz) \leq \frac{2 C_U}{2 - \alpha_0} r^{2 - \alpha},
\]

therefore (b) follows. Finally, part (c) is obvious. \( \square \)
Lemma 6.7. Assume $\nu_j^{\alpha_0}$ for $j \in \{1, 2\}$ satisfies (U) with some $\alpha_0 \in (0, 2)$, $C_U > 0$. Assume $\alpha_0 \leq \alpha < 2$ and $\nu_j^{\alpha}$ as in Definition 6.5. Then the measure $\nu_1^{\alpha} \lor \nu_2^{\alpha}$ satisfies (U) with the same exponent $\alpha$ and a constant depending only on $\alpha_0$, $C_U$, $\lambda$, and $\eta$.

**Proof.** By Lemma 6.4, it suffices to show that $\nu_1^{\alpha} \lor \nu_2^{\alpha}$ satisfies (U0) and (U1). For $0 < r \leq 1$ we derive

$$
\int_{B_r} |x|^2 \nu_1^{\alpha} \lor \nu_2^{\alpha}(dx) \leq \frac{1}{2 - \alpha} \int_{\lambda |y+z| \leq |\eta|, |z| \leq \eta |y+z|} |\eta(y+z)|^2 \mathbb{1}_{B_r}(\eta(y+z)) |y+z|^\alpha \nu_1^{\alpha}(dy) \nu_2^{\alpha}(dz)
$$

$$
\leq \frac{1}{2 - \alpha} \int_{\lambda |y+z| \leq |\eta|, |z| \leq \eta |y+z| < r} \nu_1^{\alpha}(dy) \nu_2^{\alpha}(dz)
$$

$$
\leq \frac{1}{2 - \alpha} \eta^2 \lambda^{2+\alpha} \int_{B_r} |z|^\alpha \int_{\lambda \leq |\eta|, |z| < \eta |y+z|} |y|^2 \nu_1^{\alpha}(dy) \nu_2^{\alpha}(dz) \leq \frac{\eta^4(C_U)^2}{\lambda^4(2-\alpha_0)^2},
$$

where in the last passage we used parts (b) and (a) of Lemma 6.6. Furthermore, by (6.10),

$$
\int_{\mathbb{R}^d \setminus B_1} \nu_1^{\alpha} \lor \nu_2^{\alpha}(dx) \leq \frac{1}{2 - \alpha} \int_{\lambda |y+z| \leq |\eta|, |z| \leq \eta |y+z|} \mathbb{1}_{B_2 \setminus B_1}(\eta(y+z)) |y+z|^\alpha \nu_1^{\alpha}(y) \nu_2^{\alpha}(dz) \leq \frac{2^\alpha}{2 - \alpha} \int_{\lambda \leq |y|, |z|} \nu_1^{\alpha}(y) \nu_2^{\alpha}(dz) \leq \frac{8(C_U)^2 \eta^4}{\lambda^4(2-\alpha_0)^2}.
$$

The following lemma shows that the quadratic form w.r.t. $\nu_1 \lor \nu_2$ is dominated by the sum of the quadratic forms w.r.t. $\nu_1$ and $\nu_2$. Some enlargement of the domain is needed which is taken care of in Lemma 6.9 by a covering argument.

**Lemma 6.8.** Assume $\nu_j^{\alpha_0}$ for $j \in \{1, 2\}$ satisfies (U) and (S) with some $\alpha_0 \in (0, 2)$, $\alpha > 1$, and $C_U > 0$. Assume $\alpha_0 \leq \alpha < 2$ and $\nu_j^{\alpha}$ as in Definition 6.5. Let $\eta = \alpha^k > 1$ for some $k \in \mathbb{Z}$. For $B = B_r(x_0)$ let us denote $B^* = B_{3\eta r}(x_0)$. Then with $c = 4C_U \eta^6 \alpha^{-4}$ it holds,

$$
\mathcal{E}^{\nu_1 \lor \nu_2}_{B}(u, u) \leq c(\mathcal{E}^{\nu_1}_{B}(u, u) + \mathcal{E}^{\nu_2}_{B}(u, u))
$$

for any measurable function $u$ on $B_1$ and any $B$ such that $B^* \subset B_1$.

**Proof.** Let $B = B_r(x_0)$ be such that $B^* \subset B_1$. In particular, this means that $r \leq 1/(3\eta)$. By definition, we obtain

$$
\mathcal{E}^{\nu_1 \lor \nu_2}_{B}(u, u) = \int (u(x) - u(x+z))^2 \mathbb{1}_{B}(x) \mathbb{1}_{B}(x+z) \nu_1 \lor \nu_2(dz) dx
$$

$$
\leq \int \int (u(x) - u(x+\eta(y+z)))^2 \mathbb{1}_{B}(x) \mathbb{1}_{B}(x+\eta(y+z)) g_\lambda(y,z) \nu_1(dy) \nu_2(dz) dx
$$

$$
\leq 2 \int \int [(u(x) - u(x+\eta y))^2 + (u(x+\eta y) - u(x+\eta(y+z)))^2]
$$

$$
\times \mathbb{1}_{B}(x) \mathbb{1}_{B}(x+\eta(y+z)) g_\lambda(y,z) \nu_1(dy) \nu_2(dz) dx
$$

$$
= 2[I_1 + I_2].
$$

We may assume that $|\lambda y + z| \leq |z| < \eta |y + z| \leq 2r$ and $|\lambda y + z| \leq |y| < \eta |y + z| \leq 2r$, as otherwise the expression $\mathbb{1}_{B}(x) \mathbb{1}_{B}(x+\eta(y+z)) g_\lambda(y,z)$ would be zero. Since $2r \leq 1$, it follows
that \( \frac{|y|}{\eta} < |z| \leq \frac{\eta|y|}{\lambda} \wedge 1 \). Therefore, by changing the order of integration,

\[
 I_1 \leq \int_B \int_{B_{2r}} \int_{\frac{|x|}{\eta} \vee |y+z| \leq |z| \leq \frac{\eta|y|}{\lambda} \wedge 1} (u(x) - u(x + \eta y))^2 |y + z|^\alpha \nu_2(dz) \nu_1(dy) \, dx.
\]

We estimate the inner integral above,

\[
 J_1 \leq \int \frac{|y + z|^\alpha \nu_2(dz)}{\int_{\frac{|x|}{\eta} \vee |y+z| \leq |z| \leq \frac{\eta|y|}{\lambda} \wedge 1}} \leq \int \frac{|z|^\alpha |z|^{2-\alpha} \nu_2(dz)}{\lambda^\alpha} \leq \frac{\eta^4 C_U}{\lambda^4}.
\]

Coming back to \( I_1 \) we obtain,

\[
 I_1 \leq \frac{\eta^4 C_U}{\lambda^4} \int \frac{(u(x) - u(x + \eta y))^2 \nu_1(dy)}{\int_{B_{2r}}} \leq \frac{\eta^4 C_U}{\lambda^4} ||u||^2_{B_{2r}}(u, u),
\]

where we used (S) and the fact that \( B_{2r} \subset B_1 \).

Finally, in order to estimate \( I_2 \), we first change variables \( x = w - \eta y \),

\[
 I_2 \leq \int_{B^*} \int_{B_{2r}} \int (u(w) - u(w + \eta z))^2 \eta^2 \nu_1(dy) \nu_2(dz) \, dw \leq \int_{B^*} \int_{B_{2r}} \int (u(w) - u(w + \eta z))^2 \nu_1(dy) \nu_2(dz) \, dw \leq \int_{B^*} \int_{B_{2r}} \int (u(w) - u(w + \eta z))^2 \nu_1(dy) \nu_2(dz) \, dw.
\]

By symmetry, the following integral may be estimated exactly like \( J \) before,

\[
 \int \frac{|y + z|^\alpha \nu_1(dy)}{\int_{\frac{|x|}{\eta} \vee |y+z| \leq |z| \leq \frac{\eta|y|}{\lambda} \wedge 1}} \leq \frac{\eta^4 C_U}{\lambda^4}.
\]

This leads to an estimate

\[
 I_2 \leq \frac{\eta^4 C_U}{\lambda^4} \int \frac{(u(w) - u(w + \eta z))^2 \nu_2(dz)}{\int_{B_{2r}}} \leq \frac{\eta^4 C_U}{\lambda^4} ||u||^2_{B_{2r}}(u, u),
\]

where we used (S) and the fact that \( B_{2r} \subset B_1 \). The result follows from (6.13) and the obtained estimates of \( I_1 \) and \( I_2 \).

\[\square\]

**Lemma 6.9.** Let \( 0 < \alpha < \alpha < 2, r_0 > 0, \kappa \in (0, 1), \) and \( \nu \) be a measure on \( B(\mathbb{R}^d) \). For \( B = B_r(x), x \in \mathbb{R}^d, r > 0, \) we set \( B^* = B_{\frac{r}{\kappa}}(x) \). Suppose that for some \( c_{\nu} > 0 \)

\[
 \mathcal{E}^{\nu}_{B^*}(u, u) \geq c_{\nu} \mathcal{E}^{\nu}_{B}(u, u),
\]

for every \( 0 < r \leq r_0 \), every \( u \in L^2(B_{r_0}) \), and for every ball \( B \subset B_{r_0} \) of radius \( \kappa r \). Then there exists a constant \( c = c(d, \alpha_0, \kappa) \), such that for every ball \( B \subset B_{r_0} \) of radius \( r \leq r_0 \) and every \( u \in L^2(B_{r_0}) \)

\[
 \mathcal{E}^{\nu}_{B}(u, u) \geq c_{\nu} \mathcal{E}^{\nu}_{B}(u, u).
\]

**Proof.** Fix some \( 0 < r \leq r_0 \) and a ball \( D \) of radius \( r \). We take \( B \) to be a family of balls with the following properties.
For some \( c = c(d) \) and any \( x, y \in D \), if \( |x - y| < c \text{dist}(x, D^c) \), then there exists \( B \in \mathcal{B} \) such that \( x, y \in B \).

(ii) For every \( B \in \mathcal{B}, B^* \subset D \).

(iii) Family \( \{B_j^*\}_{B \in \mathcal{B}} \) has the finite overlapping property, that is, each point of \( D \) belongs to at most \( M = M(d) \) balls \( B^* \), where \( B \in \mathcal{B} \).

Such a family \( \mathcal{B} \) may be constructed by considering Whitney decomposition of \( D \) into cubes and then covering each Whitney cube by an appropriate family of balls.

We have

\[
E_D^\nu(u, u) \geq \frac{1}{M^2} \sum_{B \in \mathcal{B}} \int_{B^*} (u(x) - u(x + y))^2 \nu(dy) \, dx \\
\geq \frac{c^\nu}{M^2} (2 - \alpha) \sum_{B \in \mathcal{B}} \int_B (u(x) - u(y))^2 |x - y|^{-d - \alpha} \, dy \, dx \\
\geq \frac{c^\nu}{M^2} (2 - \alpha) \int_D \int_{|x - y| < \text{dist}(x, D^c)} (u(x) - u(y))^2 |x - y|^{-d - \alpha} \, dy \, dx. \tag{6.14}
\]

By [13, Proposition 5 and proof of Theorem 1], we may estimate

\[
\int_D \int_{|x - y| < \text{dist}(x, D^c)} (u(x) - u(y))^2 |x - y|^{-d - \alpha} \, dy \, dx \\
\geq c(\alpha, d) \int_D \int_D (u(x) - u(y))^2 |x - y|^{-d - \alpha} \, dy \, dx. \tag{6.15}
\]

with some constant \( c(\alpha, d) \). We note that in [13, proof of Theorem 1] the constant depends on the domain in question, but in our case, by scaling, we can take the same constant independent of the choice of the ball \( D \). One may also check that \( c(\alpha, d) \) stays bounded when \( \alpha \in [\alpha_0, 2) \).

By (6.14) and (6.15) the lemma follows. \( \square \)

For a linear subspace \( E \subset \mathbb{R}^d \), we denote by \( H_E \) the \((\dim E)\)-dimensional Hausdorff measure on \( \mathbb{R}^d \) with the support restricted to \( E \). In particular, \( H_{\{0\}} = \delta_{\{0\}} \), the Dirac delta measure at 0.

**Proposition 6.10.** Let \( E_1, E_2 \subset \mathbb{R}^d \) be two linear subspaces with \( E_1, E_2 \neq \{0\} \). Assume that \( \nu_j, j \in \{1, 2\} \), are measures on \( \mathcal{B}(\mathbb{R}^d) \) of the form \( \nu_j = f_j H_{E_j} \) satisfying \( \nu_j(B_1) > 0 \), \( (U) \), and \( (S) \) with \( \alpha_0 \in (0, 2) \), \( C_U > 0 \) and \( a > 1 \). Then the following is true:

1. \( \nu_1 \wedge \nu_2 \) is absolutely continuous with respect to \( H_{E_1 + E_2} \) and satisfies \((U)\) and \((S)\).
2. If \( \eta \geq \frac{\delta^2}{a^{-1}} \) and \( \lambda \leq \frac{1}{\alpha + 1} \), then \( \nu_1 \wedge \nu_2(B_1) > 0 \).
3. If \( \nu_1^{\alpha_0} = \nu_j \) and \( \nu_2^{\alpha} \) is defined as in Definition 6.5 for \( \alpha_0 \leq \alpha < 2 \), then

\[
\nu_1^{\alpha_0} \wedge \nu_2^{\alpha} \geq \eta^{-2}(\nu_1^{\alpha_0} \wedge \nu_2^{\alpha})^{\alpha}. \tag{6.16}
\]

**Proof.** Properties \((U)\) and \((S)\) follow from Lemma 6.7 and Lemma 6.3, respectively. Let \( E = E_1 \cap E_2 \) and let \( F_j \) be linear subspaces such that \( E_j = E \oplus F_j \), where \( j = 1, 2 \). For \( y \in E_1 \) let us write \( y = Y + \tilde{y} \), where \( Y \in E \) and \( \tilde{y} \in F_1 \); similarly, for \( z \in E_2 \) we write \( z = Z + \tilde{z} \), where
Z \in E$ and $\hat{z} \in F_2$. Then for $A \subset B_2$

$\nu_1 \llcorner \nu_2(A) = \iiint A(\eta(Y + \hat{y} + Z + \hat{z})) \eta^\|_\lambda(Y + \hat{y}, Z + \hat{z}) \times f_1(Y + \hat{y}) f_2(Z + \hat{z}) H_E(dY) H_E(dZ) H_{F_1}(d\hat{y}) H_{F_2}(d\hat{z})$

$= \iiint A(\eta(W + \hat{y} + \hat{z})) \left( \int \eta^\|_\lambda(Y + \hat{y}, W - Y + \hat{z}) f_1(Y + \hat{y}) f_2(W - Y + \hat{z}) H_E(dY) \right) H_E(dW) H_{F_1}(d\hat{y}) H_{F_2}(d\hat{z})$

and since $\nu_1 \llcorner \nu_2(\mathbb{R}^d \setminus B_2) = 0$, the desired absolute continuity follows.

To show non-degeneracy, let $G_n := B_{a - n} \setminus B_{a - n - 1}$. By scaling property (S) it follows that $\nu_j(G_{n+1}) = \alpha^n \nu_j(G_n)$, therefore $\nu_j(G_n) > 0$ for each $n = 0, 1, \ldots$. Hence

$$\nu_1 \llcorner \nu_2(B_1) \geq \frac{1}{2 - \alpha_0} \int_{G_0} \int_{G_n} 1_{B_1}(\eta(y + z))(\eta^\|_\lambda(y)) 1_{A_{y + \hat{z}}}(|y + z|^{\alpha} \nu_1(dy) \nu_2(dz)).$$

For $(y, z) \in G_{n+2} \times G_n$ it holds that $\frac{a - 1}{2\alpha_0^2}(|y| + |z|) \leq |y + z| \leq (a^3 + 1)(|y| + |z|)$ and also $\eta(y + z) \in B_1$, provided $n$ is large enough. Therefore $\nu_1 \llcorner \nu_2(B_1) > 0$, if $\eta \geq a^{\alpha_0}$ and $\lambda \leq \frac{1}{a + \alpha}$.

To prove the last part of the lemma, we calculate first the most inner integral in (6.17) corresponding to $\nu_1 \llcorner \nu_2^{\alpha_0}$. It equals

$$L := \int \eta^\|_\lambda(Y + \hat{y}, W - Y + \hat{z}) f_1^{\alpha_0}(Y + \hat{y}) f_2^{\alpha_0}(W - Y + \hat{z}) H_E(dY)$$

$$= \frac{2 - \alpha}{2 - \alpha_0} \int |W + \hat{y} + \hat{z}|^{\alpha_0} |Y + \hat{y}|^{\alpha_0 - \alpha} |W - Y + \hat{z}|^{\alpha_0 - \alpha} 1(\ldots) \times f_1^{\alpha_0}(Y + \hat{y}) f_2^{\alpha_0}(W - Y + \hat{z}) H_E(dY),$$

where we used an abbreviation

$$1(\ldots) := 1_{A_{y + \hat{y} + \hat{z}}}(Y + \hat{y}) 1_{A_{y + \hat{y} + \hat{z}}}(W - Y + \hat{z}).$$

On the other hand, the most inner integral in (6.17) corresponding to $(\nu_1 \llcorner \nu_2^{\alpha_0})^{\alpha}$ is

$$R := \frac{2 - \alpha}{2 - \alpha_0} (\eta|W + \hat{y} + \hat{z}|)^{\alpha - \alpha_0}$$

$$\times \int \eta^\|_\lambda(Y + \hat{y}, W - Y + \hat{z}) f_1^{\alpha_0}(Y + \hat{y}) f_2^{\alpha_0}(W - Y + \hat{z}) H_E(dY)$$

$$= \frac{(2 - \alpha)(\alpha_0 - \alpha)}{(2 - \alpha_0)^2} \int |W + \hat{y} + \hat{z}|^{2\alpha_0 - \alpha} 1(\ldots) f_1^{\alpha}(Y + \hat{y}) f_2^{\alpha}(W - Y + \hat{z}) H_E(dY).$$

Inequality (6.16) follows now from the following estimate,

$$|Y + \hat{y}|^{\alpha_0 - \alpha} |W - Y + \hat{z}|^{\alpha_0 - \alpha} 1(\ldots) \geq (\eta|W + \hat{y} + \hat{z}|)^{2(\alpha_0 - \alpha)} 1(\ldots)$$

and the fact that both sides of (6.16) are zero on $\mathbb{R}^d \setminus B_2$. \hfill \Box

\textbf{Proof of Theorem 1.5 – lower bound in (A).} We recall from Subsection 1.5 that we may and do assume that $f_k$ are symmetric, i.e., $f_k(x) = f_k(-x)$ for all $x$. By Proposition 6.10 it follows that the measure

$$\nu := (f_1 H_{E_1}) \llcorner (f_2 H_{E_2}) \llcorner \ldots \llcorner (f_n H_{E_n})$$

satisfies (U) and (S) and has a density $h$ with respect to the Lebesgue measure on $B(\mathbb{R}^d)$ with $\int_{B_1} h(x) dx > 0$, if $\eta$ is large enough and $\lambda$ small enough. We will show that the measure $\nu \llcorner \nu$...
possesses a density $h^\triangledown$ with $h^\triangledown(x) \geq c|x|^{-d-\alpha_0}$ for all $x \in B_1 \setminus \{0\}$ and some positive constant $c$ to be specified. This, together with the preliminary results, will establish the assertion.

Condition (S) for $\nu$ implies that $h(ax) = a^{-d-\alpha_0} h(x)$ if $x \in B_{1/a}$. Therefore $\int_{G_0} h(x) \, dx > 0$, where $G_0 = B_1 \setminus B_{1/a}$. Define $h_{G_0}(x) = h(x) \mathbb{1}_{G_0}(x) \wedge 1$. The function

$$x \mapsto h_{G_0} * h_{G_0}(x) = \int h_{G_0}(y-x) h_{G_0}(y) \, dy$$

is continuous and strictly positive at 0. Thus there exists $\delta \in (0, (2a)^{-1})$ and $\varepsilon > 0$ such that

$$h_{G_0} * h_{G_0}(x) \geq \varepsilon \quad \text{for } x \in B_\delta.$$

We consider the measure $\nu^\triangledown \nu$, it has a density $h^\triangledown$ with respect to the Lebesgue measure on $\mathcal{B}(B_2)$ given by formula, cf. (6.7),

$$h^\triangledown(x) = \eta^{-2d} \int g_\lambda^\nu \left( \frac{x-w}{\eta} \right) h \left( \frac{x-w}{\eta} \right) \, dw \geq \eta^{2\alpha_0} \int_{G_0} g_\lambda^\nu \left( \frac{x-w}{\eta} \right) \mathbb{1}_{G_0}(x-w) h(w) h(x-w) \, dw \geq \frac{\eta^{2\alpha_0}}{2 - \alpha_0} \int_{G_0} |x|^\alpha \mathbb{1}_{A|x|}(w) \mathbb{1}_{A|x|}(x-w) h_0(x-w) \, dw.$$

Suppose $\eta \geq a^2/\delta$ and $\lambda \leq 1/(a\delta)$. Then for $x \in B_\delta \setminus B_{\delta/a^2}$ and $w \in G_0$ such that $x - w \in G_0$ it holds

$$\mathbb{1}_{A|x|}(w) \mathbb{1}_{A|x|}(x-w) = 1.$$

This leads to the following estimate

$$h^\triangledown(x) \geq \frac{\eta^{\alpha_0} \delta^{\alpha_0} a^{-2\alpha_0}}{2 - \alpha_0} h_{G_0} * h_{G_0}(x) \geq \frac{\varepsilon}{2 - \alpha_0}, \quad \text{for } x \in B_\delta \setminus B_{\delta/a^2}.$$

For $x \in B_1 \setminus \{0\}$ let $k \in \mathbb{Z}$ be such that $\frac{\delta}{a^2} < |x|^k < \delta < |x|^k+1$. Then, by scaling (S),

$$h^\triangledown(x) = a^{k(d+\alpha_0)} h^\triangledown(ax^k) \geq a^{k(d+\alpha_0)} \frac{\delta^{d+\alpha_0} \varepsilon}{a^{2d+2\alpha_0} (2 - \alpha_0)} |x|^{-d-\alpha_0}.$$

Now from Lemma 6.8 and Lemma 6.9 it follows that for any $B \subset B_1$

$$\varepsilon_{B,\alpha_0}^\nu(u, u) \leq c \varepsilon_{B,\alpha_0}^\nu(u, u), \quad (6.18)$$

with $c = c((f_j), (E_j))$.

Finally, to obtain a robust result, we observe that by (6.16)

$$\left( \nu_k \right)^\alpha \circ \cdots \circ \left( \nu_k \right)^\alpha \geq \eta^{-(2n-1)} \left( \nu_k \circ \cdots \circ \nu_k \right)^\alpha \geq \frac{\eta^{2(2n-1)}}{2 - \alpha} |x|^{\alpha_0 - \alpha \frac{\delta^{d+\alpha_0} \varepsilon}{a^{2d+2\alpha_0} (2 - \alpha_0)}} \mathbb{1}_{B_1}(x) \, dx.$$

This together with Lemma 6.8 and Lemma 6.9 gives us

$$\varepsilon_{B,\alpha_0}^\nu(u, u) \leq c \varepsilon_{B,\alpha_0}^\nu(u, u),$$

with the constant $c$ not depending on $\alpha \in [\alpha_0, 2]$. \qed

Let us show that the assumptions of Theorem 1.5 are not necessary for (A) and (B) to hold. This is true because the condition (A) relates to integrated quantities but does not require pointwise bounds on the density of $\mu(x, dy)$.
Example 6. Let $b \in (0, 1)$ and

$$
\Gamma = \{(x_1, x_2) \in \mathbb{R}^2 | |x_2| \geq |x_1|^b \text{ or } |x_1| \geq |x_2|^b\}.
$$

We consider the following function

$$
k(z) = (2-\alpha) \mathbb{1}_{\Gamma \cap B_1(z)} |z|^{-2-\beta}, \quad z \in \mathbb{R}^2,
$$

where $\beta = \alpha - 1 + 1/b$, see Figure 1. As we will show, for such a function $k$ conditions (A) and (B) are satisfied. We have, for $0 < r < 1$

$$
\int_{B_r} |z|^2 k(z) \, dz \leq 8(2-\alpha) \int_0^r \int_0^{x_1/b} (x^2 + y^2)^{-\beta/2} \, dy \, dx
$$

$$
\leq 8(2-\alpha) \int_0^r \int_0^{x_1/b} x^{-\beta} \, dx \, dy = 8r^{2-\alpha},
$$

hence $k$ satisfies (U1) with $C_1 = 8$. Since (U0) is clear, from Lemma 6.4 we conclude that $k$ satisfies (U).

Let

$$
P = \{x \in B_{1/4} | 0 < x_1 < x_2 < 2x_1\}
$$

and for $y = (x_1, x_2) \in P$, let

$$
E_y = [x_1, x_1 + x_1^{1/b}] \times [-x_1^{1/b}, 0].
$$

It is easy to check that if $y \in P$ and $z \in E_y$, then

$$
\frac{|y|}{3} \leq |z| \leq 4|y|, \quad \frac{|y|}{3} \leq |y - z| \leq 4|y| \quad \text{and} \quad z, y - z \in \Gamma \cap B_1.
$$
Let \( \eta = 4 \) and \( \lambda = \frac{1}{3} \). Then for \( y \in P \)
\[
\kappa \cap k(\eta y) = \left| y \right|^{\alpha} \left| y - z \right|^{2(\alpha - 2)} \left( \left| y - z \right|^{2-\beta} \right) d\nu \leq C(\alpha, 2, C_0, a, \eta, a)(\alpha - 2) \left| y - z \right|^{2-\beta} d\nu.
\]
where
\[
C(\alpha, 2, C_0, a, \eta, a)(\alpha - 2) \left| y - z \right|^{2-\beta} d\nu.
\]

In the following example we provide a condition that implies comparability of corresponding quadratic forms but which is not covered by Theorem 1.5.

**Example 7.** For a measure \( \nu \) on \( B(\mathbb{R}^d) \) with a density \( k \) with respect to the Lebesgue measure we formulate the following condition:

There exist \( a > 1 \) and \( C_2, C_3 > 0 \) such that every annulus \( B_{a-n+1} \setminus B_{a-n} \) \((n = 0, 1, \ldots)\) contains a ball \( B_n \) with radius \( C_2a^{-n} \), such that

\[
k(z) \geq C_3(2-\alpha)\left| z \right|^{-d-\alpha}, \quad z \in B_n,
\]

(6.21)

The following proposition provides a substitute for Theorem 1.5.

**Proposition 6.11.** Let \( a > 1, a_0 \in (0, 2), \alpha \in [a_0, 2), \) and \( C_U, C_2, C_3 > 0 \). Let \( \mu = (\mu(x, \cdot))_{x \in \mathbb{R}^d} \) be a family of measures on \( \mathbb{R}^d \) which satisfies (1.2). Furthermore, we assume that there exist measures \( \nu_* \) and \( \nu^* \) with property (T), such that (U) and (6.21) hold with exponent \( \alpha \) and the constants \( C_U, C_2, C_3 \). Then there is \( A = A(a, a_0, C_U, C_0, C_3) \geq 1 \) not depending on \( \alpha \) such that (A) hold.

**Proof.** We fix \( \lambda < 2/C_2 \wedge 1 \) and \( \eta \geq 2a^2/C_2 \wedge 1 \). Let for some \( n \in \{0, 1, \ldots, \} \)
\[
\frac{C_2}{2} a^{-n-1} \leq \left| y \right| \leq \frac{C_2}{2} a^{-n},
\]

and assume that \( \eta y \in B_2 \). By formula (6.7), we obtain
\[
k \cap k(\eta y) \geq \frac{\eta^{-d} \left| y \right|^{\alpha}}{2 - \alpha} \int_{A_{\left| y \right|}} 1_{A_{\left| y \right|}}(y - z) 1_{A_{\left| y \right|}}(z) k(y - z) k(z) dz.
\]

Let us denote by \( B_n^o \) the ball concentric with \( B_n \), but with radius \( C_2a^{-n}/2 \) (that is, \( B_n^o \) is twice smaller than \( B_n \)). We observe that if \( z \in B_n^o \), then \( y - z \in B_n \). Furthermore, by our choice of \( \lambda \) and \( \eta \) it follows that
\[
\lambda \left| y \right| < \left| y - z \right| < \eta \left| y \right|, \quad \lambda \left| y \right| < \left| z \right| < \eta \left| y \right|, \quad \text{if } z \in B_n^o,
\]

that is, \( y - z, z \in A_{\left| y \right|} \) for \( z \in B_n^o \). Hence
\[
k \cap k(\eta y) \geq \frac{\eta^{-d} \left| y \right|^{\alpha} C_2^2 (2 - \alpha)^2}{2^{d+2_\alpha} a^d} \int_{B_n^o} \left| y - z \right|^{-d-\alpha} \left| z \right|^{-d-\alpha} dz
\]
\[
\geq C(\alpha, 2, a, C_2, C_3, \eta, a)(\alpha - 2) \left| y \right|^{-d-\alpha},
\]

or, equivalently, for \( w \in B_2 \)
\[
k \cap k(w) \geq C(\alpha, 2, a, C_2, C_3, \eta, a)(\alpha - 2) \left| w \right|^{-d-\alpha}.
\]
By Lemma 6.8 and Lemma 6.9 we conclude that the lower estimate in (A) holds. The upper estimate is in turn a consequence of Proposition 6.1.

7. Global comparability results for nonlocal quadratic forms

In this section we provide a global comparability result, i.e. we study comparability in the whole \( \mathbb{R}^d \). This result is not needed for the other results in this article, however it contains an interesting and useful observation.

**Proposition 7.1.** Assume \((U)\) holds. Then there exists a constant \( c = c(\alpha, d, C_U) \) such that
\[
\mathcal{E}_\mu(u, u) \leq c(\mathcal{E}^{\mu_\alpha}(u, u) + \|u\|_{L^2(\mathbb{R}^d)}^2)
\]
for every \( u \in L^2(\mathbb{R}^d) \). (7.1)

Furthermore, if \((U)\) is satisfied for all \( r > 0 \), then for every \( u \in L^2(\mathbb{R}^d) \)
\[
\mathcal{E}_\mu(u, u) \leq c \mathcal{E}^{\mu_\alpha}(u, u).
\]
If the constant \( C_U \) in \((U)\) is independent of \( \alpha \in (\alpha_0, 2) \), where \( \alpha_0 > 0 \), then so are the constants in (7.1) and (7.2).

**Proof.** By \( E \) we denote the identity operator from \( H^{\alpha/2}(\mathbb{R}^d) \) to itself. One easily checks that the proof of Proposition 6.1 from (6.2) until (6.5) works also in the present case of \( D = \mathbb{R}^d \). Hence (7.1) follows.

To prove (7.2) we observe that if \((U)\) holds for all \( r > 0 \), then also (6.4) holds for all \( \xi \neq 0 \), we plug it into (6.3) and we are done. □

We consider the following condition.
\((K2,r_0)\) There exists \( c_0 > 0 \) such that for all \( h \in S^{d-1} \) and all \( 0 < r < r_0 \)
\[
\int_{\mathbb{R}^d} r^2 \sin^2 \left( \frac{h \cdot z}{r} \right) \nu_\alpha(dz) \geq c_0 r^{2-\alpha}.
\]
Clearly (6.21) implies \((K2,r_0)\) for \( r_0 = 1 \), and if \( C_3 \) is independent of \( \alpha \in (\alpha_0, 2) \), where \( \alpha_0 > 0 \), then so is \( c_0 \). Condition \((K2,r_0)\) is also satisfied if for all \( h \in S^{d-1} \) and all \( 0 < r < r_0 \)
\[
\int_{B_r(0)} |h \cdot z|^2 \nu_\alpha(dz) \geq c_2 r^{2-\alpha}.
\]
(7.4)

We note that (7.5) under condition (7.4) has been proved in [1] by Abels and Husseini. The following theorem extends their result by giving a characterization of kernels \( \nu_\alpha \) admitting comparability (7.5). We stress that \( r_0 = \infty \) is allowed, and in such a case we put \( \frac{1}{r_0} = 0 \).

**Theorem 7.2.** Let \( 0 < r_0 \leq \infty \). If \((K2,r_0)\) holds, then
\[
\mathcal{E}^{\mu_\alpha}(u, u) \leq \frac{1}{c_0} \mathcal{E}_\mu(u, u) + \frac{2\alpha}{r_0^2} \|u\|_{L^2}^2, \quad u \in C_1^1(\mathbb{R}^d).
\]
(7.5)

Conversely, if for some \( c < \infty \)
\[
\mathcal{E}^{\mu_\alpha}(u, u) \leq c \mathcal{E}^{\nu_\alpha}(u, u) + \frac{2\alpha}{r_0^2} \|u\|_{L^2}^2, \quad u \in \mathcal{S}(\mathbb{R}^d),
\]
(7.6)

then \((K2,r_0)\) holds.
Proof. Recalling that \((u(\cdot + z))^\wedge(\xi) = e^{i\xi \cdot z} \hat{u}(\xi)\) and using Plancherel formula we obtain
\[
\mathcal{E}^\mu(u, u) \geq \iint (u(x) - u(x + z))^2 \, dx \, \nu_*(dz) \\
= \int |e^{i\xi \cdot z} - 1|^2 |\hat{u}(\xi)|^2 \, d\xi \, \nu_*(dz) \\
= \int \left( \int 4 \sin^2 \left( \frac{\xi \cdot z}{2} \right) \nu_*(dz) \right) |\hat{u}(\xi)|^2 \, d\xi.
\]
(7.7)

If \((K2,r_0)\) holds, then for all \(|\xi| > 2/r_0\)
\[
\int 4 \sin^2 \left( \frac{\xi \cdot z}{2} \right) \nu_*(dz) \geq \frac{4c_0}{2^\alpha}|\xi|^\alpha \geq c_0|\xi|^\alpha.
\]
For \(|\xi| \leq 2/r_0\) we have \(|\xi|^\alpha \leq (2/r_0)^\alpha\). Inequality (7.5) follows from
\[
\frac{\mathcal{A}_{d-a}}{2^\alpha(2-a)} \mathcal{E}^\alpha_{Re}(u, u) = \int_{\mathbb{R}^d} |\xi|^\alpha |\hat{u}(\xi)|^2 \, d\xi.
\]
(7.8)

Now we prove the converse. Assume (7.6). By (7.7), the right hand side of (7.6) equals
\[
\int \left( c \int 4 \sin^2 \left( \frac{\xi \cdot z}{2} \right) \nu_*(dz) + \frac{2^\alpha}{r_0^\alpha} \right) |\hat{u}(\xi)|^2 \, d\xi,
\]
hence by (7.8) and (7.6) we obtain that
\[
c \int 4 \sin^2 \left( \frac{\xi \cdot z}{2} \right) \nu_*(dz) + \frac{2^\alpha}{r_0^\alpha} \geq |\xi|^\alpha, \quad \text{for a.e. } \xi \in \mathbb{R}^d.
\]
(7.9)

By continuity of the function
\[
\mathbb{R}^d \setminus \{0\} \ni \xi \mapsto \int 4 \sin^2 \left( \frac{\xi \cdot z}{2} \right) \nu_*(dz),
\]
(7.9) holds for all \(\xi \in \mathbb{R}^d\). For \(|\xi| \geq 2^{1+1/\alpha}r_0^{-1}\) we have by (7.9)
\[
c \int 4 \sin^2 \left( \frac{\xi \cdot z}{2} \right) \nu_*(dz) \geq \frac{|\xi|^\alpha}{2},
\]
and hence \((K2,2^{-1/\alpha}r_0)\) holds with \(c_0 = 2^{\alpha-3}c^{-1}\). Since
\[
\sin^2 \left( \frac{h \cdot z}{2r} \right) \geq \frac{1}{4} \sin^2 \left( \frac{h \cdot z}{r} \right),
\]
also \((K2,r_0)\) holds with some constant \(c_0\). \(\square\)

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