ON LIE ALGEBRAS RESPONSIBLE FOR ZERO-CURVATURE REPRESENTATIONS AND BÄCKLUND TRANSFORMATIONS OF (1 + 1)-DIMENSIONAL SCALAR EVOLUTION PDES

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Abstract. Zero-curvature representations (ZCRs) are one of the main tools in the theory of integrable PDEs. In particular, Lax pairs for (1 + 1)-dimensional PDEs can be interpreted as ZCRs.

In [arXiv:1303.3575], for any (1 + 1)-dimensional scalar evolution equation $\mathcal{E}$, we defined a family of Lie algebras $\mathbb{F}(\mathcal{E})$ which are responsible for all ZCRs of $\mathcal{E}$ in the following sense. Representations of the algebras $\mathbb{F}(\mathcal{E})$ classify all ZCRs of the equation $\mathcal{E}$ up to local gauge transformations. Also, using these algebras, one obtains necessary conditions for existence of a Bäcklund transformation between two given equations. The algebras $\mathbb{F}(\mathcal{E})$ are defined in [arXiv:1303.3575] in terms of generators and relations.

In this approach, ZCRs may depend on partial derivatives of arbitrary order, which may be higher than the order of the equation $\mathcal{E}$. The algebras $\mathbb{F}(\mathcal{E})$ generalize Wahlquist-Estabrook prolongation algebras, which are responsible for a much smaller class of ZCRs.

In this preprint we prove a number of results on $\mathbb{F}(\mathcal{E})$ which were announced in [arXiv:1303.3575]. We present applications of $\mathbb{F}(\mathcal{E})$ to the theory of Bäcklund transformations in more detail and describe the explicit structure (up to non-essential nilpotent ideals) of the algebras $\mathbb{F}(\mathcal{E})$ for a number of equations of orders 3 and 5.

1. Introduction and the main results

1.1. Zero-curvature representations and the algebras $\mathbb{F}^p(\mathcal{E}, a)$. Zero-curvature representations and Bäcklund transformations belong to the main tools in the theory of integrable PDEs (see, e.g., [40, 8, 30]). This preprint is part of a research program on investigating the structure of zero-curvature representations (ZCRs) and Bäcklund transformations (BTs) for partial differential equations (PDEs) of various types.

In this preprint we present a number of results on ZCRs and BTs for (1 + 1)-dimensional scalar evolution equations

$$(1) \quad u_t = F(x, t, u_0, u_1, \ldots, u_d), \quad u = u(x, t),$$

where we use the notation

$$(2) \quad u_t = \frac{\partial u}{\partial t}, \quad u_0 = u, \quad u_k = \frac{\partial^k u}{\partial x^k}, \quad k \in \mathbb{Z}_{\geq 0}.$$

The number $d \geq 1$ in (1) is such that the function $F$ may depend only on $x$, $t$, $u_k$ for $k \leq d$.  

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In the future we plan to update this preprint at arxiv.org, in order to present more results and more details in proofs.

Let \( g \) be a finite-dimensional Lie algebra. For an equation of the form (1), a zero-curvature representation (ZCR) with values in \( g \) is given by \( g \)-valued functions
\[
A = A(x,t,u_0,u_1,\ldots,u_p), \quad B = B(x,t,u_0,u_1,\ldots,u_{p+d-1})
\]
satisfying
\[
D_x(B) - D_t(A) + [A,B] = 0. \tag{4}
\]

The total derivative operators \( D_x, D_t \) in (4) are
\[
D_x = \frac{\partial}{\partial x} + \sum_{k \geq 0} u_{k+1} \frac{\partial}{\partial u_k}, \quad D_t = \frac{\partial}{\partial t} + \sum_{k \geq 0} D_k^x F(x,t,u_0,u_1,\ldots,u_d) \frac{\partial}{\partial u_k}.
\]

The number \( p \) in (3) is such that the function \( A \) may depend only on the variables \( x, t, u_k \) for \( k \leq p \). Then equation (1) implies that the function \( B \) may depend only on \( x, t, u_{k'} \) for \( k' \leq p + d - 1 \).

Such ZCRs are said to be of order \( \leq p \). In other words, a ZCR given by \( A, B \) is of order \( \leq p \) iff \( \frac{\partial A}{\partial u_l} = 0 \) for all \( l > p \).

**Remark 1.** The right-hand side \( F = F(x,t,u_0,u_1,\ldots,u_d) \) of (1) appears in condition (4), because \( F \) appears in the formula for the operator \( D_t \) in (3). Note that (1) can be written as \([D_x + A, D_t + B] = 0\), because \([D_x, D_t] = 0\). See also Remark 3 below for another interpretation of equation (4).

**Remark 2.** When we consider a function \( Q = Q(x,t,u_0,u_1,\ldots,u_l) \) for some \( l \in \mathbb{Z}_{\geq 0} \), we always assume that this function is analytic on an open subset of the manifold with the coordinates \( x, t, u_0, u_1, \ldots, u_l \). For example, \( Q \) may be a meromorphic function, because a meromorphic function is analytic on some open subset of the manifold. In particular, this applies to the functions (3).

Without loss of generality, one can assume that \( g \) is a Lie subalgebra of \( gl_N \) for some \( N \in \mathbb{Z}_{>0} \), where \( gl_N \) is the algebra of \( N \times N \) matrices with entries from \( \mathbb{R} \) or \( \mathbb{C} \). So our considerations are applicable to both cases \( gl_N = gl_N(\mathbb{R}) \) and \( gl_N = gl_N(\mathbb{C}) \). And we denote by \( GL_N \) the group of invertible \( N \times N \) matrices.

Let \( K \) be either \( \mathbb{C} \) or \( \mathbb{R} \). Then \( gl_N = gl_N(K) \) and \( GL_N = GL_N(K) \). In this preprint, all algebras are supposed to be over the field \( K \).

**Remark 3.** So we suppose that functions \( A, B \) in (1) take values in \( g \subset gl_N \). Then condition (4) implies that the auxiliary linear system
\[
\partial_x(W) = -AW, \quad \partial_t(W) = -BW
\]
is compatible modulo (1). Here \( W = W(x,t) \) is an invertible \( N \times N \) matrix-function.

We need to consider also gauge transformations, which act on ZCRs and can be described as follows.

Let \( G \subset GL_N \) be the connected matrix Lie group corresponding to the Lie algebra \( g \subset gl_N \). (That is, \( G \) is the connected immersed Lie subgroup of \( GL_N \) corresponding to the Lie subalgebra \( g \subset gl_N \).) A gauge transformation is given by an invertible matrix-function \( G = G(x,t,u_0,u_1,\ldots,u_l) \) with values in \( G \).

For any ZCR (3), (4) and any gauge transformation \( G = G(x,t,u_0,u_1,\ldots,u_l) \), the functions
\[
\tilde{A} = GAG^{-1} - D_x(G) \cdot G^{-1}, \quad \tilde{B} = GBG^{-1} - D_t(G) \cdot G^{-1}
\]
satisfy \( D_x(\tilde{B}) - D_t(\tilde{A}) + [\tilde{A},\tilde{B}] = 0 \) and, therefore, form a ZCR. Moreover, since \( A, B \) take values in \( g \) and \( G \) takes values in \( G \), the functions \( \tilde{A}, \tilde{B} \) take values in \( g \).

The ZCR (7) is said to be gauge equivalent to the ZCR (3), (4). For a given equation (1), formulas (7) determine an action of the group of gauge transformations on the set of ZCRs of this equation.

Recall that the infinite prolongation \( E \) of equation (1) is an infinite-dimensional manifold with the coordinates \( x, t, u_k \) for \( k \in \mathbb{Z}_{\geq 0} \). The precise definition of the manifold \( E \) is given in Section 2.2 and is further clarified in Section 3.
Recall that $\mathbb{K}$ is either $\mathbb{C}$ or $\mathbb{R}$. We suppose that the variables $x, t, u_k$ take values in $\mathbb{K}$. A point $a \in \mathcal{E}$ is determined by the values of the coordinates $x, t, u_k$ at $a$. Let

$$a = (x = x_a, t = t_a, u_k = a_k) \in \mathcal{E}, \quad x_a, t_a, a_k \in \mathbb{K}, \quad k \in \mathbb{Z}_{\geq 0},$$

be a point of $\mathcal{E}$. In other words, the constants $x_a, t_a, a_k$ are the coordinates of the point $a \in \mathcal{E}$ in the coordinate system $x, t, u_k$.

For each $p \in \mathbb{Z}_{\geq 0}$ and each $a \in \mathcal{E}$, the paper [14] defines a Lie algebra $\mathbb{F}^p(\mathcal{E}, a)$ so that the following property holds. For any finite-dimensional Lie algebra $\mathfrak{g}$, on a neighborhood of $a \in \mathcal{E}$, any $\mathfrak{g}$-valued ZCR (3), (4) of order $\leq p$ is locally gauge equivalent to the ZCR arising from a homomorphism $\mathbb{F}^p(\mathcal{E}, a) \rightarrow \mathfrak{g}$. (We suppose that the $\mathfrak{g}$-valued functions (3) are defined on a neighborhood of $a \in \mathcal{E}$.)

The algebra $\mathbb{F}^p(\mathcal{E}, a)$ is defined in [14] in terms of generators and relations, using a normal form for ZCRs with respect to the action of the group of local gauge transformations. The definition of $\mathbb{F}^p(\mathcal{E}, a)$ from [14] is recalled in Section 3 of the present preprint. (To clarify the main idea, in Example 1 below we consider the case $p = 1$.)

According to Section 3, the algebras $\mathbb{F}^p(\mathcal{E}, a)$ for $p \in \mathbb{Z}_{\geq 0}$ are arranged in a sequence of surjective homomorphisms

$$\cdots \rightarrow \mathbb{F}^p(\mathcal{E}, a) \rightarrow \mathbb{F}^{p-1}(\mathcal{E}, a) \rightarrow \cdots \rightarrow \mathbb{F}^1(\mathcal{E}, a) \rightarrow \mathbb{F}^0(\mathcal{E}, a).$$

The family of Lie algebras $\mathbb{F}(\mathcal{E})$ mentioned in the abstract of this preprint consists of the algebras $\mathbb{F}^p(\mathcal{E}, a)$ for all $p \in \mathbb{Z}_{\geq 0}, a \in \mathcal{E}$.

**Remark 5.** Consider the case when $p = 0$ and the functions $F, A, B$ do not depend on $x, t$. Then formulas (3), (4) become

$$\begin{align*}
A &= A(u_0), \\
B &= B(u_0, u_1, \ldots, u_{d-1}), \\
D_x(B) - D_t(A) + [A, B] &= 0.
\end{align*}$$

ZCRs of the form (9) can be studied by the Wahlquist-Estabrook prolongation method (WE method for short).

Namely, for a given equation of the form $u_t = F(u_0, u_1, \ldots, u_d)$, the WE method constructs a Lie algebra so that $\mathfrak{g}$-valued ZCRs of the form (9) correspond to homomorphisms from this algebra to $\mathfrak{g}$ (see, e.g., [5] [16] [18] [39]). It is called the Wahlquist-Estabrook prolongation algebra. Note that in (9) the function $A = A(u_0)$ depends only on $u_0$.

The WE method does not use gauge transformations in a systematic way. In the classification of ZCRs (9) this is acceptable, because the class of ZCRs (9) is relatively small.

The class of ZCRs (3), (4) is much larger than that of (9). As is shown in [14], gauge transformations play a very important role in the classification of ZCRs (3), (4). Because of this, the classical WE method does not produce satisfactory results for (3), (4), especially in the case $p > 0$.

It is proved in [14] that, if the function $F$ in (4) does not depend on $x, t$, then the algebra $\mathbb{F}^0(\mathcal{E}, a)$ is isomorphic to a certain subalgebra of the Wahlquist-Estabrook prolongation algebra for (11). We recall this result in Section 3 and use it for computation of $\mathbb{F}^0(\mathcal{E}, a)$ for some equations.

**Example 1.** To clarify the definition of $\mathbb{F}^p(\mathcal{E}, a)$, let us consider the case $p = 1$. To this end, we fix an equation (11) and study ZCRs of order $\leq 1$ of this equation.

According to Theorem 3 in Section 3 any ZCR of order $\leq 1$

$$\begin{align*}
A &= A(x, t, u_0, u_1), \\
B &= B(x, t, u_0, u_1, \ldots, u_d), \\
D_x(B) - D_t(A) + [A, B] &= 0
\end{align*}$$

on a neighborhood of $a \in \mathcal{E}$ is gauge equivalent to a ZCR of the form

$$\begin{align*}
\tilde{A} &= \tilde{A}(x, t, u_0, u_1), \\
\tilde{B} &= \tilde{B}(x, t, u_0, u_1, \ldots, u_d),
\end{align*}$$

on a neighborhood of $\tilde{a} \in \tilde{\mathcal{E}}$ is gauge equivalent to a ZCR of the form

$$\begin{align*}
\tilde{A} &= \tilde{A}(x, t, u_0, u_1), \\
\tilde{B} &= \tilde{B}(x, t, u_0, u_1, \ldots, u_d),
\end{align*}$$
Moreover, according to Theorem 4, for any given ZCR of the form (10), on a neighborhood of \( a \in \mathcal{E} \) there is a unique gauge transformation \( G = G(x, t, u_0, \ldots, u_l) \) such that the functions \( \tilde{A} = GAG^{-1} - D_x(G) \cdot G^{-1}, \tilde{B} = GBC^{-1} - D_t(G) \cdot G^{-1} \) satisfy (11), (12), (13) and \( G(x, a^0, a_1, \ldots, a_l) = \text{Id} \), where \( \text{Id} \in \text{GL}_N \) is the identity matrix.

(In the case of ZCRs of order \( \leq 1 \), this gauge transformation \( G \) depends on \( x, t, u_0 \), so \( G = G(x, t, u_0) \). In a similar result about ZCRs of order \( \leq p \), which is described in Theorem 4, the corresponding gauge transformation depends on \( x, t, u_0, \ldots, u_{p-1} \).)

Therefore, we can say that properties (13) determine a normal form for ZCRs (10) with respect to the action of the group of gauge transformations on a neighborhood of \( a \in \mathcal{E} \).

A similar normal form for ZCRs (3), (4) with arbitrary \( m \geq 1 \) (including a normal form for ZCRs with respect to the action of gauge transformations and the main properties of ZCRs) is described in Theorem 4 and Remark 22.

Since the functions \( \tilde{A}, \tilde{B} \) from (11), (13) are analytic on a neighborhood of \( a \in \mathcal{E} \), these functions are represented as absolutely convergent power series

\[
\tilde{A} = \sum_{l_1, l_2, i_0, i_1 \geq 0} (x - x_a)^{i_1} (t - t_a)^{l_2} (u_0 - a_0)^{i_0} (u_1 - a_1)^{i_1} \cdot \tilde{A}_{i_0, i_1}^{l_1, l_2},
\]

\[
\tilde{B} = \sum_{l_1, l_2, j_0, \ldots, j_d \geq 0} (x - x_a)^{i_1} (t - t_a)^{l_2} (u_0 - a_0)^{j_0} \ldots (u_d - a_d)^{j_d} \cdot \tilde{B}_{j_0 \ldots j_d}^{l_1, l_2}.
\]

Here \( \tilde{A}_{i_0, i_1}^{l_1, l_2} \) and \( \tilde{B}_{j_0 \ldots j_d}^{l_1, l_2} \) are elements of a Lie algebra, which we do not specify yet.

Using formulas (14), (15), we see that properties (13) are equivalent to

\[
\tilde{A}_{i_0, i_1}^{l_1, l_2} = \tilde{A}_{i_0, 0}^{l_1, 0} = \tilde{B}_{0, 0}^{0, 0} = 0 \quad \forall l_1, l_2, i_0 \in \mathbb{Z}_{\geq 0}.
\]

To define \( \mathbb{F}(a) \), we regard \( \tilde{A}_{i_0, i_1}^{l_1, l_2}, \tilde{B}_{j_0 \ldots j_d}^{l_1, l_2} \) from (14), (15) as abstract symbols. By definition, the algebra \( \mathbb{F}(a) \) is generated by the symbols \( \tilde{A}_{i_0, i_1}^{l_1, l_2}, \tilde{B}_{j_0 \ldots j_d}^{l_1, l_2} \) for \( l_1, l_2, i_0, i_1, j_0, \ldots, j_d \in \mathbb{Z}_{\geq 0} \). Relations for these generators are provided by equations (12), (10). A more detailed description of this construction is given in Section 3.

Applications of \( \mathbb{F}(a) \) to the theory of Bäcklund transformations are presented in Section 4.2 and in Sections 7, 8. In Section 5 we describe the structure of \( \mathbb{F}(a) \) for some equations of orders 3 and 5, including the Krichever-Novikov equation and a 5th-order equation from [9]. The algebra \( \mathbb{F}(a) \) and the Wahlquist-Estabrook prolongation algebra for the 5th-order equation from [9] are studied in Section 4.

For completeness, in Theorem 8 we recall a result from [14] which describes the structure of \( \mathbb{F}(a) \) for the KdV equation.

Remark 6. It is possible to introduce an analog of \( \mathbb{F}(a) \) for multicomponent evolution PDEs

\[
\frac{\partial u^i}{\partial t} = F^i(x, t, u^1, \ldots, u^m, u_1^1, \ldots, u_1^m, \ldots, u_d^1, \ldots, u_d^m),
\]

\[
u^i = u^i(x, t), \quad u_k^i = \frac{\partial^k u^i}{\partial x^k}, \quad i = 1, \ldots, m.
\]

In this preprint we study only the scalar case \( m = 1 \). For \( m > 1 \) one gets interesting results as well, but the case \( m > 1 \) requires much more computations, which will be presented elsewhere. Some results for \( m > 1 \) (including a normal form for ZCRs with respect to the action of gauge transformations and the main properties of \( \mathbb{F}(a) \) in the multicomponent case) are sketched in the preprints [13] [15].

Remark 7. Some other approaches to the study of the action of local gauge transformations on ZCRs can be found in [21] [22] [23] [32] [33] [35] and references therein. For a given ZCR with values in a matrix Lie algebra \( \mathfrak{g} \), the papers [21] [22] [32] define certain \( \mathfrak{g} \)-valued functions, which transform by conjugation
when the ZCR transforms by gauge. Applications of these functions to construction and classification of some types of ZCRs are described in [21, 22, 23, 32, 33, 35].

To our knowledge, the theory of [21, 22, 23, 32, 33, 35] does not produce any infinite-dimensional Lie algebras responsible for ZCRs. So this theory does not contain the algebras \( \mathbb{F}^p(\mathcal{E}, a) \).

1.2. Bäcklund transformations.

Remark 8. In the study of Bäcklund transformations we use the geometric approach to PDEs by means of infinite jet spaces [3, 19, 27], which can be outlined as follows.

Let \( \mathcal{M} \) be a manifold. Let \( n \) be a nonnegative integer such that \( n \leq \text{dim} \mathcal{M} \). Recall that an \( n \)-dimensional distribution \( \mathcal{D} \) on \( \mathcal{M} \) is an \( n \)-dimensional subbundle of the tangent bundle \( T\mathcal{M} \). In other words, to define an \( n \)-dimensional distribution \( \mathcal{D} \) on \( \mathcal{M} \), we choose an \( n \)-dimensional subspace \( \mathcal{D}_a \subset T_a \mathcal{M} \) for each point \( a \in \mathcal{M} \) such that \( \mathcal{D}_a \) depends smoothly on \( a \). Here \( T_a \mathcal{M} \) is the tangent space of the manifold \( \mathcal{M} \) at \( a \in \mathcal{M} \). We need the case when \( \mathcal{M} \) is infinite-dimensional. The precise definitions of infinite-dimensional manifolds and \( n \)-dimensional distributions on them are given in Section 2.1.

A submanifold \( \mathcal{S} \subset \mathcal{M} \) is an integral submanifold of the distribution \( \mathcal{D} \) if \( T_a \mathcal{S} \subset \mathcal{D}_a \) for each \( a \in \mathcal{S} \), where \( T_a \mathcal{S} \) is the tangent space of \( \mathcal{S} \) at \( a \in \mathcal{S} \).

Consider a PDE for functions \( u^i = u^i(x_1, \ldots, x_n) \), \( i = 1, \ldots, m \),

\[
F_\alpha \left( x_1, \ldots, x_n, u^1, \ldots, u^m, \frac{\partial^k u^j}{\partial x_{i_1} \cdots \partial x_{i_k}}, \ldots \right) = 0, \quad \alpha = 1, \ldots, q.
\]

Geometrically, an \( m \)-component vector-function \( (u^1(x_1, \ldots, x_n), \ldots, u^m(x_1, \ldots, x_n)) \) corresponds to a section of a fiber bundle \( \pi: E \to B \) with \( m \)-dimensional fibers. Here \( B \) is an \( n \)-dimensional manifold with coordinates \( x_1, \ldots, x_n \). Then \( u^1, \ldots, u^m \) can be regarded as coordinates in the fibers of the bundle \( \pi \).

Let \( J^\infty \) be the manifold of infinite jets of local sections of the bundle \( \pi \). A geometric coordinate-independent definition of \( J^\infty \) can be found in [3]. We recall that \( x_i, u^j \), and all partial derivatives of \( u^j \) play the role of coordinates for the manifold \( J^\infty \).

Let \( \mathcal{E} \subset J^\infty \) be the subset of infinite jets satisfying the PDE (17) and all its differential consequences. (A detailed definition of \( \mathcal{E} \) is given in Section 2.2.)

On the manifold \( J^\infty \), one has the \( n \)-dimensional distribution called the Cartan distribution [3]. Integral submanifolds of this distribution provide a geometric interpretation for solutions of the PDE. Namely, solutions of the PDE correspond to \( n \)-dimensional integral submanifolds \( \mathcal{S} \subset J^\infty \) satisfying \( \mathcal{S} \subset \mathcal{E} \).

In coordinates, the Cartan distribution is spanned by the total derivative operators \( D_{x_i}, i = 1, \ldots, n \), which are regarded as vector fields on \( J^\infty \). The explicit formula for \( D_{x_i} \) is (17) in Section 2.2, where one uses the notation (15). In coordinates, the subset \( \mathcal{E} \subset J^\infty \) consists of the points \( a \in J^\infty \) that obey the equations \( F_\alpha = 0 \) and \( D_{x_{i_1}} \cdots D_{x_{i_s}} (F_\alpha) = 0 \) for all \( \alpha, s, i_1, \ldots, i_s \).

If the PDE satisfies some non-degeneracy conditions, then the set \( \mathcal{E} \) is a nonsingular submanifold of \( J^\infty \) and the Cartan distribution is tangent to \( \mathcal{E} \), which gives an \( n \)-dimensional distribution on \( \mathcal{E} \). Then \( \mathcal{E} \) is called nonsingular.

These non-degeneracy conditions are satisfied on an open dense subset of \( J^\infty \) for practically all PDEs in applications. (If there are some singular points in \( \mathcal{E} \), one can exclude these points from consideration and study only the nonsingular part of \( \mathcal{E} \), which is usually open and dense in \( \mathcal{E} \).) In particular, as is shown in Example 5, for any \((1 + 1)\)-dimensional evolution PDE the set \( \mathcal{E} \) is nonsingular.

In what follows, we always assume that \( \mathcal{E} \) is nonsingular in the above-mentioned sense. We often identify a PDE with the corresponding manifold \( \mathcal{E} \). So we can speak about a PDE \( \mathcal{E} \). Thus, in this geometric approach, a PDE is regarded as a manifold \( \mathcal{E} \) with an \( n \)-dimensional distribution (the Cartan distribution) such that solutions of the PDE correspond to \( n \)-dimensional integral submanifolds, where \( n \) is the number of independent variables in the PDE. A more detailed description of this approach is given in Section 2.

To clarify the main idea, in Examples 2, 3 below we describe the construction of \( \mathcal{E} \) for the KdV and sine-Gordon equations. These examples are well known, but it is instructive to discuss them. The general construction of \( \mathcal{E} \) for arbitrary PDEs is presented in Section 2.2.
Suppose that two PDEs $\mathcal{E}^1$ and $\mathcal{E}^2$ are isomorphic (i.e., $\mathcal{E}^1$ can be obtained from $\mathcal{E}^2$ by an invertible change of variables, and vice versa). Then the corresponding manifolds $\mathcal{E}^1$ and $\mathcal{E}^2$ are connected by a diffeomorphism that preserves the Cartan distribution. Therefore, the manifold of infinite jets and the Cartan distribution associated with a PDE are the right objects to study if one is interested in properties that are invariant with respect to changes of variables. (As has been said above, we identify a PDE with the corresponding manifold of infinite jets. So here, for $i = 1, 2$, a PDE $\mathcal{E}^i$ and the corresponding manifold are denoted by the same symbol $\mathcal{E}^i$.)

**Remark 9.** In the present preprint all manifolds and maps of manifolds are supposed to be analytic. In fact some analogous results can be proved for smooth manifolds as well, but the smooth case requires some extra technical considerations, which will be described elsewhere.

Several more conventions and assumptions that are used in the preprint are described in Section 1.3.

**Example 2.** In Remark 8 we have discussed the construction of a manifold $\mathcal{E}$ and the Cartan distribution on $\mathcal{E}$ for a given PDE. Here we describe this construction for the KdV equation $u_t - u_{xxx} - 6uu_x = 0$, where partial derivatives of $u = u(x, t)$ are denoted by subscripts.

Consider the space $\mathbb{R}^2$ with coordinates $(x, t)$, the space $\mathbb{R}^3$ with coordinates $(x, t, u)$, and the bundle $\pi: \mathbb{R}^3 \to \mathbb{R}^2$ such that $\pi((x, t, u)) = (x, t)$. A function $u(x, t)$ defined on an open subset of $\mathbb{R}^2$ can be regarded as a local section of the bundle $\pi$.

Let $J^\infty$ be the manifold of infinite jets of local sections of the bundle $\pi$. Then $J^\infty$ can be viewed as the infinite-dimensional manifold with coordinates

$$x, \ t, \ u, \ u_x, \ u_t, \ u_{xx}, \ u_{xt}, \ u_{tt}, \ ...$$

All partial derivatives of $u$ are included in (18). Here (18) are regarded as $\mathbb{R}$-valued variables, which play the role of coordinates for the manifold $J^\infty$. A detailed definition of such infinite-dimensional manifolds is given in Section 2.

The total derivative operators

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xt} \frac{\partial}{\partial u_t} + u_{xxx} \frac{\partial}{\partial u_{xx}} + u_{xtt} \frac{\partial}{\partial u_{xt}} + u_{ttt} \frac{\partial}{\partial u_{tt}} + ...,$$

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{xt} \frac{\partial}{\partial u_x} + u_{tt} \frac{\partial}{\partial u_t} + u_{xtt} \frac{\partial}{\partial u_{xt}} + u_{ttt} \frac{\partial}{\partial u_{tt}} + ...$$

can be viewed as vector fields on $J^\infty$.

Consider the differential consequences of the KdV equation

$$u_t - u_{xxx} - 6uu_x = 0, \quad D_x(u_t - u_{xxx} - 6uu_x) = u_{xt} - u_{xxx} - 6u_xu_x - 6uu_{xx} = 0,$$

$$D_t(u_t - u_{xxx} - 6uu_x) = u_{tt} - u_{xxx} - 6u_tu_x - 6uu_{xt} = 0,$$

$$D_x^{k_1} D_t^{k_2} (u_t - u_{xxx} - 6uu_x) = 0, \quad k_1, k_2 \in \mathbb{Z}_{\geq 0}.$$  

Here (21), (22), (23) are regarded as equations on the manifold $J^\infty$ with coordinates (18). Then $\mathcal{E} \subset J^\infty$ is the submanifold of the points $a \in J^\infty$ that satisfy equations (21), (22), (23).

The vector fields (19), (20) are tangent to the submanifold $\mathcal{E} \subset J^\infty$. Hence the vector fields $D_x, D_t$ can be restricted to $\mathcal{E}$, which gives the 2-dimensional Cartan distribution on $\mathcal{E}$.

Using equations (21), (22), (23), one can uniquely express each of the coordinates (18) in terms of the following coordinates

$$x, \ t, \ u, \ u_x, \ u_{xx}, \ u_{xxx}, \ ..., \ u_{kx}, \ ..., \ k \in \mathbb{Z}_{\geq 0}.$$  

Therefore, (24) can be viewed as coordinates on the manifold $\mathcal{E}$. So $\mathcal{E}$ is isomorphic to the space $\mathbb{R}^\infty$ with coordinates (24). However, geometry of the Cartan distribution on $\mathcal{E}$ is highly nontrivial. Solutions of the KdV equation correspond to 2-dimensional integral submanifolds of the Cartan distribution on $\mathcal{E}$. (Note that the Frobenius theorem on integral submanifolds of involutive distributions is not applicable here, because $\mathcal{E}$ is infinite-dimensional.)

The KdV equation is a $(1 + 1)$-dimensional evolution PDE. A detailed description of $\mathcal{E}$ for $(1 + 1)$-dimensional evolution PDEs is given in Example 5.
Example 3. Let us describe the construction of $E$ for the sine-Gordon equation $u_{xt} - \sin u = 0$. Consider the differential consequences of this equation

$$(25) \quad u_{xt} - \sin u = 0, \quad D_x(u_{xt} - \sin u) = u_{xxt} - u_x \cos u = 0,$$

$$(26) \quad D_t(u_{xt} - \sin u) = u_{xtt} - u_t \cos u = 0, \quad D_x^{k_1} D_t^{k_2}(u_{xt} - \sin u) = 0, \quad k_1, k_2 \in \mathbb{Z}_{\geq 0}.$$  

In Example 2 we have introduced the manifold $J^\infty$ with coordinates (18). We regard (25), (26) as equations on the manifold $J^\infty$. Then $E \subset J^\infty$ is the submanifold of the points $a \in J^\infty$ that satisfy equations (25), (26).

The vector fields (19), (20) are tangent to the submanifold $E \subset J^\infty$. Hence the vector fields $D_x$, $D_t$ can be restricted to $E$, which gives the 2-dimensional Cartan distribution on $E$.

Using equations (25), (26), one can uniquely express each of the coordinates (18) in terms of the following coordinates

$$(27) \quad x, \ t, \ u, \ u_x, \ u_t, \ u_{xx}, \ u_{tt}, \ u_{xxx}, \ u_{xxt}, \ldots, \ u_{kk}, \ u_{kt}, \ldots \quad k \in \mathbb{Z}_{\geq 0}.$$  

Therefore, (27) can be viewed as coordinates on the manifold $E$.

Bäcklund transformations (BTs) are a well-known tool to construct new solutions for PDEs from known solutions (see, e.g., [29, 30] and references therein). Applying BTs to trivial solutions, one can often obtain interesting solutions. Also, using BTs, one can sometimes transform complicated PDEs to simpler ones.

In this subsection we outline the main idea of the notion of BTs. A more detailed description of BTs is given in Section 2.3.

According to Remark 8, a PDE can be regarded as a manifold $E$ with an $n$-dimensional distribution (the Cartan distribution) such that solutions of the PDE correspond to $n$-dimensional integral submanifolds, where $n$ is the number of independent variables in the PDE.

Let $E^1$ and $E^2$ be PDEs. The PDEs $E^1$ and $E^2$ are connected by a Bäcklund transformation if there is another PDE $E^3$ with maps

$$(28) \quad \tau_1: E^3 \rightarrow E^1, \quad \tau_2: E^3 \rightarrow E^2$$

such that for each $i = 1, 2$ one has the following properties:

- For any solution $s$ of the PDE $E^3$, applying the map $\tau_i: E^3 \rightarrow E^i$ to $s$, we get a solution $\tau_i(s)$ of the PDE $E^i$.

- For any solution $s_i$ of the PDE $E^i$, the preimage $\tau_i^{-1}(s_i)$ is a family of $E^3$ solutions depending on a finite number of parameters.

In other words, a Bäcklund transformation (BT) between the PDEs $E^1$ and $E^2$ is given by a PDE $E^3$ and maps $\tau_i: E^3 \rightarrow E^i$, $i = 1, 2$, satisfying the above properties.

Following A. M. Vinogradov and I. S. Krasilshchik [38, 18], in Section 2.3 we formulate the above properties more precisely, using the geometry of the manifolds $E^1$, $E^2$, $E^3$ and the corresponding Cartan distributions. The main idea is that for each $i = 1, 2$ the map $\tau_i: E^3 \rightarrow E^i$ must be a surjective submersion with finite-dimensional fibers and must preserve the Cartan distribution in a certain sense. This implies the above properties for solutions, which are regarded as integral submanifolds. See Section 2.3 for more details. To our knowledge, this definition of BTs covers all known examples of BTs for $(1+1)$-dimensional PDEs.

Remark 10. Using a BT (28), one can obtain solutions of $E^2$ from solutions of $E^1$ (and vice versa) as follows:

Step 1. Take a solution $s_1$ of the PDE $E^1$ and compute its preimage $\tau_1^{-1}(s_1)$ under the map $\tau_1: E^3 \rightarrow E^1$. Then $\tau_1^{-1}(s_1)$ is a family of solutions of the PDE $E^3$.

Step 2. Apply the map $\tau_2: E^3 \rightarrow E^2$ to the family $\tau_1^{-1}(s_1)$. Then $\tau_2(\tau_1^{-1}(s_1))$ is a family of solutions of the PDE $E^2$. 
So, from a given solution \( s_1 \) of the PDE \( \mathcal{E}^1 \), one obtains the family \( \tau_2(\tau_1^{-1}(s_1)) \) of solutions of the PDE \( \mathcal{E}^2 \). Similarly, from a given solution \( s_2 \) of the PDE \( \mathcal{E}^2 \), one obtains the family \( \tau_1(\tau_2^{-1}(s_2)) \) of solutions of the PDE \( \mathcal{E}^1 \).

If \( \mathcal{E}^1 = \mathcal{E}^2 \) and \( \tau_1 \neq \tau_2 \), then in this way one obtains new solutions for \( \mathcal{E}^1 \) from known solutions.

We write a BT (28) as the following diagram

(29)

Example 4. A well-known BT for the KdV equation can be written as follows

(30)

\[
\begin{align*}
u_t &= v_{xxx} - 6v^2v_x + 6\lambda v_x \\
u &= v_x - v^2 + \lambda \\
u_t &= u_{xxx} + 6uu_x \\
u &= \lambda
\end{align*}
\]

where \( \lambda \in \mathbb{K} \) is a constant. Comparing (30) with (29), we see that in the BT (30) one has the following.

- \( \mathcal{E}^1 = \mathcal{E}^2 \) is the KdV equation \( u_t = u_{xxx} + 6uu_x \).
- \( \mathcal{E}^3 \) is the equation \( v_t = v_{xxx} - 6v^2v_x + 6\lambda v_x \).
- Applying the map \( \tau_1: \mathcal{E}^3 \to \mathcal{E}^1 \) to a solution \( v = v(x, t) \) of \( \mathcal{E}^3 \), we get the solution \( u = v_x - v^2 + \lambda \) of \( \mathcal{E}^1 \). This is the well-known Miura transformation.
- Applying the map \( \tau_2: \mathcal{E}^3 \to \mathcal{E}^2 \) to a solution \( v = v(x, t) \) of \( \mathcal{E}^3 \), we get the solution \( u = -v_x - v^2 + \lambda \) of \( \mathcal{E}^1 \).

Remark 11. If \( \mathcal{E}^1, \mathcal{E}^2, \mathcal{E}^3 \) in a BT (29) are evolution equations, then this BT is said to be of Miura type.

Note that, in general, \( \mathcal{E}^3 \) in a BT (29) is not necessarily an evolution equation, even if \( \mathcal{E}^1, \mathcal{E}^2 \) are evolution equations. For example, in V. E. Adler’s BT for the Krizchev-Novikov equation [11], \( \mathcal{E}^1 \) and \( \mathcal{E}^2 \) are evolution equations (isomorphic to the Krizchev-Novikov equation), but the equation \( \mathcal{E}^3 \) is not evolution.

We are going to show that the algebras \( \mathbb{F}^p(\mathcal{E}, a) \) help to obtain necessary conditions for existence of a Bäcklund transformation between two given evolution equations.

For each \( p \in \mathbb{Z}_{>0} \), consider the surjective homomorphism \( \varphi_p: \mathbb{F}^p(\mathcal{E}, a) \to \mathbb{F}^{p-1}(\mathcal{E}, a) \) from (8).

Let \( \mathbb{F}(\mathcal{E}, a) \) be the inverse limit of the sequence (8). An element of \( \mathbb{F}(\mathcal{E}, a) \) is given by a sequence \( (c_0, c_1, c_2, \ldots) \), where \( c_p \in \mathbb{F}^p(\mathcal{E}, a) \) and \( \varphi_p(c_p) = c_{p-1} \) for all \( p \).

Since (8) consists of homomorphisms of Lie algebras and \( \mathbb{F}(\mathcal{E}, a) \) is the inverse limit of (8), the space \( \mathbb{F}(\mathcal{E}, a) \) is a Lie algebra as well. If \( (c_0, c_1, c_2, \ldots) \) and \( (c'_0, c'_1, c'_2, \ldots) \) are elements of \( \mathbb{F}(\mathcal{E}, a) \), where \( c_p, c'_p \in \mathbb{F}^p(\mathcal{E}, a) \), then the corresponding Lie bracket is

\[
[(c_0, c_1, c_2, \ldots), (c'_0, c'_1, c'_2, \ldots)] = ([c_0, c'_0], [c_1, c'_1], [c_2, c'_2], \ldots) \in \mathbb{F}(\mathcal{E}, a).
\]

For each \( k \in \mathbb{Z}_{>0} \), we have the homomorphism

(31)

\[
\rho_k: \mathbb{F}(\mathcal{E}, a) \to \mathbb{F}^k(\mathcal{E}, a), \quad \rho_k((c_0, c_1, c_2, \ldots)) = c_k.
\]

Since the homomorphisms (8) are surjective, \( \rho_k \) is surjective as well.

We define a topology on the algebra \( \mathbb{F}(\mathcal{E}, a) \) as follows. For every \( k \in \mathbb{Z}_{>0} \) and every \( v \in \mathbb{F}^k(\mathcal{E}, a) \), the subset \( \rho_k^{-1}(v) \subset \mathbb{F}(\mathcal{E}, a) \) is, by definition, open in \( \mathbb{F}(\mathcal{E}, a) \). Such subsets form a base of the topology on \( \mathbb{F}(\mathcal{E}, a) \).

The meaning of the topology on \( \mathbb{F}(\mathcal{E}, a) \) is clarified by the following lemma.
Lemma 1. Let $\mathcal{L}$ be a Lie algebra. Consider a homomorphism $\psi: \mathbb{F}(\mathcal{E}, a) \to \mathcal{L}$. The subset $\ker \psi \subset \mathbb{F}(\mathcal{E}, a)$ is open in $\mathbb{F}(\mathcal{E}, a)$ iff the homomorphism $\psi: \mathbb{F}(\mathcal{E}, a) \to \mathcal{L}$ is of the form

\[(32)\quad \mathbb{F}(\mathcal{E}, a) \xrightarrow{\rho^k} \mathbb{F}^k(\mathcal{E}, a) \to \mathcal{L}\]

for some $k \in \mathbb{Z}_{\geq 0}$ and some homomorphism $\mathbb{F}^k(\mathcal{E}, a) \to \mathcal{L}$.

Proof. Suppose that $\ker \psi$ is open in $\mathbb{F}(\mathcal{E}, a)$. Since the subsets

\[(33)\quad \rho_k^{-1}(v) \subset \mathbb{F}(\mathcal{E}, a), \quad k \in \mathbb{Z}_{\geq 0}, \quad v \in \mathbb{F}^k(\mathcal{E}, a),\]

form a base of the topology on $\mathbb{F}(\mathcal{E}, a)$, for any element $w \in \ker \psi$ there are $k \in \mathbb{Z}_{\geq 0}$ and $v \in \mathbb{F}^k(\mathcal{E}, a)$ such that

\[(34)\quad w \in \rho_k^{-1}(v) \subset \ker \psi.\]

Let $w = 0$ be the zero element in $\ker \psi$. Then from $(34)$ we see that $v = \rho_k(w) = \rho_k(0)$ is the zero element in $\mathbb{F}^k(\mathcal{E}, a)$, and

\[(35)\quad \ker \rho_k \subset \ker \psi.\]

Relation $(35)$ implies that the homomorphism $\psi: \mathbb{F}(\mathcal{E}, a) \to \mathcal{L}$ is of the form $(32)$ for some homomorphism $\mathbb{F}^k(\mathcal{E}, a) \to \mathcal{L}$.

Conversely, if $\psi: \mathbb{F}(\mathcal{E}, a) \to \mathcal{L}$ is of the form $(32)$, then $\ker \psi = \rho_k^{-1}(Z)$, where $Z \subset \mathbb{F}^k(\mathcal{E}, a)$ is the kernel of the homomorphism $\mathbb{F}^k(\mathcal{E}, a) \to \mathcal{L}$ from $(32)$. According to the definition of the topology on $\mathbb{F}(\mathcal{E}, a)$, the relation $\ker \psi = \rho_k^{-1}(Z)$ implies that $\ker \psi$ is open in $\mathbb{F}(\mathcal{E}, a)$. \qed

According to Lemma $1$ the introduced topology on $\mathbb{F}(\mathcal{E}, a)$ allows us to remember which homomorphisms $\psi: \mathbb{F}(\mathcal{E}, a) \to \mathcal{L}$ are of the form $(32)$.

Definition 1. A Lie subalgebra $H \subset \mathbb{F}(\mathcal{E}, a)$ is called tame if there are $k \in \mathbb{Z}_{\geq 0}$ and a subalgebra $\mathfrak{h} \subset \mathbb{F}^k(\mathcal{E}, a)$ such that $H = \rho_k^{-1}(\mathfrak{h})$. Since $\rho_k: \mathbb{F}(\mathcal{E}, a) \to \mathbb{F}^k(\mathcal{E}, a)$ is surjective, the codimension of $H$ in $\mathbb{F}(\mathcal{E}, a)$ is equal to the codimension of $\mathfrak{h}$ in $\mathbb{F}^k(\mathcal{E}, a)$.

Remark 12. It is easy to prove that a subalgebra $H \subset \mathbb{F}(\mathcal{E}, a)$ is tame iff $H$ is open and closed in $\mathbb{F}(\mathcal{E}, a)$ with respect to the topology on $\mathbb{F}(\mathcal{E}, a)$.

The following theorem is proved in Section $7$ using some results of $12$.

Theorem 1 (Section $7$). Let $\mathcal{E}^1$ and $\mathcal{E}^2$ be $(1 + 1)$-dimensional scalar evolution equations. For each $i = 1, 2$, the symbol $\mathcal{E}^i$ denotes also the infinite prolongation of the corresponding equation.

Suppose that $\mathcal{E}^1$ and $\mathcal{E}^2$ are connected by a Bäcklund transformation. Then for each $i = 1, 2$ there are a point $a_i \in \mathcal{E}^i$ and a tame subalgebra $H_i \subset \mathbb{F}(\mathcal{E}^i, a_i)$ such that

- $H_i$ is of finite codimension in $\mathbb{F}(\mathcal{E}^i, a_i)$,
- $H_i$ is isomorphic to $H_2$, and this isomorphism is a homeomorphism with respect to the topology induced by the embedding $H_i \subset \mathbb{F}(\mathcal{E}^i, a_i)$.

Theorem $1$ provides a powerful necessary condition for two given evolution equations to be connected by a Bäcklund transformation (BT). For example, Theorem $2$ below is obtained in Section $8$ by means of Theorem $1$.

For any constants $e_1, e_2, e_3 \in \mathbb{C}$, consider the Krichever-Novikov equation $[20, 37]$

\[(36)\quad \text{KN}(e_1, e_2, e_3) = \left\{ u_t = u_{xxx} - \frac{3}{2} \frac{u_{xx}^2}{u_x} + \frac{(u - e_1)(u - e_2)(u - e_3)}{u_x}, \quad u = u(x, t) \right\}\]

and the algebraic curve

\[(37)\quad \text{C}(e_1, e_2, e_3) = \left\{ (z, y) \in \mathbb{C}^2 \mid y^2 = (z - e_1)(z - e_2)(z - e_3) \right\}.\]

If $e_1 \neq e_2 \neq e_3 \neq e_1$ then the curve $(37)$ is elliptic.
Theorem 2 (Section 3). Let \( e_1, e_2, e_3, e_1', e_2', e_3' \in \mathbb{C} \) such that \( e_1 \neq e_2 \neq e_3 \neq e_1', e_2' \neq e_2' \neq e_3' \neq e_1' \).

If the curve \( C(e_1, e_2, e_3) \) is not birationally equivalent to the curve \( C(e_1', e_2', e_3') \), then the equation \( \text{KN}(e_1, e_2, e_3) \) is not connected with the equation \( \text{KN}(e_1', e_2', e_3') \) by any Bäcklund transformation (BT).

Also, if \( e_1 \neq e_2 \neq e_3 \neq e_1' \), then \( \text{KN}(e_1, e_2, e_3) \) is not connected with the KdV equation by any BT.

BTs of Miura type (differential substitutions) for \( (36) \) were studied in [24, 37]. According to [24, 37], the equation \( \text{KN}(e_1, e_2, e_3) \) is connected with the KdV equation by a BT of Miura type iff \( e_i = e_j \) for some \( i \neq j \).

Theorems [1, 2] consider the most general class of BTs, which is much larger than the class of BTs of Miura type studied in [24, 37].

If \( e_1 \neq e_2 \neq e_3 \neq e_1' \) and \( e_1' \neq e_2' \neq e_3' \neq e_1' \), the curves \( C(e_1, e_2, e_3) \) and \( C(e_1', e_2', e_3') \) are elliptic. To clarify the first statement of Theorem 2, we need to recall the well-known classification of elliptic curves \( (37) \) up to birational equivalence.

Let \( e_1, e_2, e_3, e_1', e_2', e_3' \in \mathbb{C} \) such that \( e_1 \neq e_2 \neq e_3 \neq e_1' \) and \( e_1' \neq e_2' \neq e_3' \neq e_1' \).

The set \( \{e_1, e_2, e_3\} \) is affine-equivalent to the set \( \{e_1', e_2', e_3'\} \) if there are \( b_1, b_2 \in \mathbb{C} \), \( b_1 \neq 0 \), such that \( b_1 e_i + b_2 \in \{e_1', e_2', e_3'\} \) for all \( i = 1, 2, 3 \). In other words, the affine map \( g : \mathbb{C} \to \mathbb{C} \) given by \( g(z) = b_1 z + b_2 \) satisfies \( \{g(e_1), g(e_2), g(e_3)\} = \{e_1', e_2', e_3'\} \). Here \( \{g(e_1), g(e_2), g(e_3)\} \) and \( \{e_1', e_2', e_3'\} \) are unordered sets.

Consider the elliptic curves \( C(e_1, e_2, e_3) \) and \( C(e_1', e_2', e_3') \) given by \( (37) \). If \( e_1, e_2, e_3 \) is affine-equivalent to \( e_1', e_2', e_3' \) then the curve \( C(e_1, e_2, e_3) \) is isomorphic to the curve \( C(e_1', e_2', e_3') \). Indeed, if \( e_1, e_2, e_3 \) is affine-equivalent to \( e_1', e_2', e_3' \), then the equation \( y^2 = (z - e_1)(z - e_2)(z - e_3) \) can be transformed to the equation \( y^2 = (z - e_1')(z - e_2')(z - e_3') \) by a change of variables of the form

\[
\begin{align*}
    z &\mapsto c_1 z + c_2, \\
    y &\mapsto c_3 y, \\
    c_1, c_2, c_3 &\in \mathbb{C}, \\
    c_1 c_3 &\neq 0.
\end{align*}
\]

Clearly, the set \( \{e_1, e_2, e_3\} \) is affine-equivalent to \( \{0, e_2 - e_1, e_3 - e_1\} \), and the set \( \{0, e_2 - e_1, e_3 - e_1\} \) is affine-equivalent to \( \{0, 1, \frac{e_3 - e_1}{e_2 - e_1}\} \). Hence the curve \( C(e_1, e_2, e_3) \) is isomorphic to the curve \( C\left(0, 1, \frac{e_3 - e_1}{e_2 - e_1}\right) \). Therefore, in order to classify elliptic curves \( (37) \) up to birational equivalence, it is sufficient to consider the curves \( C(0, 1, v) \), where \( v \in \mathbb{C}, v \notin \{0, 1\} \).

The next proposition is well known (see, e.g., [10]).

Proposition 1 ([10]). Recall that, for any \( e_1, e_2, e_3 \in \mathbb{C} \), the algebraic curve \( C(e_1, e_2, e_3) \) is given by \( (37) \).

Let \( v_1, v_2 \in \mathbb{C} \) such that \( v_i \notin \{0, 1\} \) for \( i = 1, 2 \).

The curves \( C(0, 1, v_1) \) and \( C(0, 1, v_2) \) are birationally equivalent iff one has

\[
\frac{(v_1)^2 - v_1 + 1}{(v_1)2(v_1 - 1)^2} = \frac{(v_2)^2 - v_2 + 1}{(v_2)2(v_2 - 1)^2}.
\]

The numbers \( v_1, v_2 \) satisfy \( (38) \) iff the set \( \{0, 1, v_1\} \) is affine-equivalent to the set \( \{0, 1, v_2\} \).

Using the above results on elliptic curves, we can reformulate the first statement of Theorem 2 as follows.

Theorem 3. Let \( e_1, e_2, e_3, e_1', e_2', e_3' \in \mathbb{C} \) such that \( e_1 \neq e_2 \neq e_3 \neq e_1' \) and \( e_1' \neq e_2' \neq e_3' \neq e_1' \). Consider the Krichever-Novikov equations \( \text{KN}(e_1, e_2, e_3), \text{KN}(e_1', e_2', e_3') \) given by \( (36) \).

If the numbers

\[
\begin{align*}
    v_1 &= \frac{e_3 - e_1}{e_2 - e_1}, \\
    v_2 &= \frac{e_3' - e_1'}{e_2' - e_1'}
\end{align*}
\]

satisfy

\[
\frac{(v_1)^2 - v_1 + 1}{(v_1)2(v_1 - 1)^2} \neq \frac{(v_2)^2 - v_2 + 1}{(v_2)2(v_2 - 1)^2},
\]

then the equation \( \text{KN}(e_1, e_2, e_3) \) is not connected with the equation \( \text{KN}(e_1', e_2', e_3') \) by any BT.
Proof. As has been shown above, the curve $C(e_1, e_2, e_3)$ is isomorphic to the curve $C\left(0, 1, \frac{e_3 - e_1}{e_2 - e_1}\right)$. Similarly, the curve $C(e'_1, e'_2, e'_3)$ is isomorphic to the curve $C\left(0, 1, \frac{e_3 - e'_1}{e_2 - e'_1}\right)$.

By Proposition 1, if the numbers (39) satisfy (40), then $C\left(0, 1, \frac{e_3 - e_1}{e_2 - e_1}\right)$ is not birationally equivalent to $C\left(0, 1, \frac{e_3 - e'_1}{e_2 - e'_1}\right)$ and, therefore, $C(e_1, e_2, e_3)$ is not birationally equivalent to $C(e'_1, e'_2, e'_3)$.

By the first statement of Theorem 2 if $C(e_1, e_2, e_3)$ is not birationally equivalent to $C(e'_1, e'_2, e'_3)$ then $KN(e_1, e_2, e_3)$ is not connected with $KN(e'_1, e'_2, e'_3)$ by any BT. \qed

1.3. Abbreviations, conventions, and notation. The following abbreviations, conventions, and notation are used in this preprint.

ZCR = zero-curvature representation, WE = Wahlquist-Estabrook, BT = Bäcklund transformation.

The symbols $\mathbb{Z}_{>0}$ and $\mathbb{Z}_{\geq 0}$ denote the sets of positive and nonnegative integers respectively.

$K$ is either $\mathbb{C}$ or $\mathbb{R}$. All vector spaces and algebras are supposed to be over the field $K$. We denote by $\mathfrak{gl}_N$ the algebra of $N \times N$ matrices with entries from $K$ and by $GL_N$ the group of invertible $N \times N$ matrices.

By the standard Lie group – Lie algebra correspondence, for every Lie subalgebra $\mathfrak{g} \subset \mathfrak{gl}_N$ there is a unique connected immersed Lie subgroup $\mathcal{G} \subset GL_N$ whose Lie algebra is $\mathfrak{g}$. We call $\mathcal{G}$ the connected matrix Lie group corresponding to the matrix Lie algebra $\mathfrak{g} \subset \mathfrak{gl}_N$.

We use the notation (2) for partial derivatives of a $K$-valued function $u(x, t)$. Our convention about functions of the variables $x, t, u_k$ is described in Remark 2. We use also the assumptions described in Remark 3.

2. A geometric approach to PDEs and Bäcklund transformations

In this section we recall a geometric approach to PDEs and Bäcklund transformations by means of infinite jet spaces, in the analytic case. In the smooth case, a similar approach is presented in [3].

2.1. Infinite-dimensional manifolds. Recall that $K$ is either $\mathbb{C}$ or $\mathbb{R}$. By definition, the space $K^\infty$ with coordinates $z_i, i \in \mathbb{Z}_{>0}$, is the space of infinite sequences

$$
(z_1, z_2, z_3, \ldots, z_k, z_{k+1}, \ldots), \quad z_i \in K.
$$

For each $l \in \mathbb{Z}_{>0}$, one has the map

$$
\rho_l: K^\infty \to K^l, \quad \rho_l(z_1, z_2, z_3, \ldots, z_k, z_{k+1}, \ldots) = (z_1, z_2, \ldots, z_l).
$$

The topology on $K^\infty$ is defined as follows.

We have the standard topology on $K^l$. For any $l \in \mathbb{Z}_{>0}$ and any open subset $V \subset K^l$, the subset $\rho_l^{-1}(V) \subset K^\infty$ is, by definition, open in $K^\infty$. Such subsets form a base of the topology on $K^\infty$. In other words, we consider the smallest topology on $K^\infty$ such that the maps $\rho_l, l \in \mathbb{Z}_{>0}$, are continuous.

By definition, a continuous $K$-valued function $g$ on an open subset $U \subset K^\infty$ is analytic if for each point $a \in U$ there is a neighborhood $U_a \subset U$, $a \in U_a$, such that $g|_{U_a}$ depends analytically on a finite number of the coordinates $z_i, i \in \mathbb{Z}_{>0}$.

Let $U, \tilde{U} \subset K^\infty$ be open subsets. A continuous map $\tau: U \to \tilde{U}$ is said to be analytic if, for any open subset $V \subset \tilde{U}$ and any analytic function $f: V \to K$, the function $\tau^*(f): \tau^{-1}(V) \to K$ is analytic. (Essentially, this means that in coordinates the map $\tau$ is given by analytic functions.) Here $\tau^*(f)$ is defined by the standard formula

$$
\tau^*(f)(a) = f(\tau(a)), \quad a \in \tau^{-1}(V).
$$

To describe a geometric approach to PDEs and Bäcklund transformations, we need to consider analytic infinite-dimensional manifolds modelled on $K^\infty$. This is the analytic analog of the class of smooth infinite-dimensional manifolds described in [2].
So, in the present preprint, an *infinite-dimensional manifold* is a Hausdorff topological space $M$ such that

- for each point $a \in M$ there is a neighborhood homeomorphic to an open subset of $\mathbb{K}^\infty$, which is called a *coordinate chart*,
- the transition maps between overlapping coordinate charts are analytic.

As usual, using coordinate charts, one introduces local coordinates on a neighborhood of each point $a \in M$. By definition, a continuous $\mathbb{K}$-valued function $f$ on an open subset of $M$ is *analytic* if $f$ is analytic in local coordinates. An analytic function on a connected coordinate chart may depend only on a finite number of the coordinates.

One can also define germs of analytic functions in the standard way. For $a \in M$, we denote by $F_M(a)$ the algebra of germs of analytic functions at $a$.

A *tangent vector* at a point $a \in M$ is a $\mathbb{K}$-linear map $v: F_M(a) \to \mathbb{K}$ satisfying

$$v(g_1g_2) = v(g_1) \cdot g_2(a) + g_1(a) \cdot v(g_2)$$

for all $g_1, g_2 \in F_M(a)$. The *tangent space* $T_aM$ is the vector space of all tangent vectors at $a$.

Using analytic functions, one can introduce the notion of vector fields on open subsets of $M$ in the standard way. (In the language of sheaves, the sheaf of vector fields on $M$ is the sheaf of derivations of the sheaf of analytic functions on $M$.)

In particular, a vector field $X$ on $M$ determines a derivation $X: F_M(a) \to F_M(a)$ of the algebra $F_M(a)$ for each $a \in M$. This derivation determines the tangent vector $X|_a \in T_aM$ which is the following map

$$X|_a : F_M(a) \to \mathbb{K}, \quad X|_a(f) = X(f)(a) \quad \forall f \in F_M(a).$$

On an open coordinate chart $U \subset M$ with coordinates $z_i$, $i \in \mathbb{Z}_{>0}$, a vector field $X$ can be written as the sum $X = \sum_{i=1}^\infty f_i \frac{\partial}{\partial z_i}$, where $f_i = X(z_i)$ are analytic functions. Then $X|_a = \sum_{i=1}^\infty f_i(a) \frac{\partial}{\partial z_i}$.

Also, the notion of submanifolds of $M$ can be defined in the standard way.

Let $n \in \mathbb{Z}_{>0}$. To define an $n$-*dimensional distribution* $\mathcal{D}$ on $M$, we need to choose an $n$-dimensional subspace $\mathcal{D}_a \subset T_aM$ for each point $a \in M$ such that $\mathcal{D}_a$ depends analytically on $a$ in the following sense. For any $a \in M$, there are an open subset $U_a \subset M$ and vector fields $X_1, \ldots, X_n$ on $U_a$ such that $a \in U_a$ and for each point $b \in U_a$ the tangent vectors $X_1|_b, \ldots, X_n|_b \in T_bM$ determined by $X_1, \ldots, X_n$ span the space $\mathcal{D}_b$. (That is, the vectors $X_1|_b, \ldots, X_n|_b$ form a basis for the space $\mathcal{D}_b \subset T_bM$.) Then $\mathcal{D}$ is the collection of the subspaces $\mathcal{D}_a \subset T_aM$ for all $a \in M$.

**Remark 13.** If $M$ is finite-dimensional, then the above properties mean that $\mathcal{D}$ is an $n$-dimensional subbundle of the tangent bundle of $M$.

Let $S \subset M$ be a submanifold. Then for each $a \in S$ we have $T_aS \subset T_aM$, where $T_aS$ is the tangent space of $S$ at $a \in S$. A vector $v \in T_aM$ is *tangent to $S$* if $v \in T_aS \subset T_aM$. This means the following.

For $a \in S$, we denote by $\mathcal{I}_S(a) \subset F_M(a)$ the subspace of germs of analytic functions that vanish on $S$. So a germ $g \in F_M(a)$ belongs to $\mathcal{I}_S(a)$ iff the restriction of $g$ to $S$ is zero. A vector $v \in T_aM$ is tangent to $S$ iff $v(g) = 0$ for all $g \in \mathcal{I}_S(a)$. Here we use the fact that $v$ is a map $v: F_M(a) \to \mathbb{K}$ satisfying (13), according to the definition of $T_aM$.

Consider again a distribution $\mathcal{D}$ determined by subspaces $\mathcal{D}_a \subset T_aM$, $a \in M$. A submanifold $S \subset M$ is an *integral submanifold* of the distribution $\mathcal{D}$ if $T_aS \subset \mathcal{D}_a$ for each $a \in S$.

A vector field $X$ belongs to $\mathcal{D}$ if $X|_a \in \mathcal{D}_a$ for all $a \in M$. The distribution $\mathcal{D}$ is said to be *involutive* if, for any vector fields $X, Y$ belonging to $\mathcal{D}$, the commutator $[X, Y]$ belongs to $\mathcal{D}$ as well. Note that, if $M$ is infinite-dimensional, the Frobenius theorem on integral submanifolds of involutive distributions is not applicable.

Let $M^1, M^2$ be (possibly infinite-dimensional) manifolds. A continuous map $\tau: M^1 \to M^2$ is said to be analytic if, for any open subset $V \subset M^2$ and any analytic function $f: V \to \mathbb{K}$, the function $\tau^*(f): \tau^{-1}(V) \to \mathbb{K}$ is analytic. Here $\tau^*(f)$ is defined by (42).
Let $\tau \colon M^1 \to M^2$ be an analytic map. Let $a \in M^1$. According to our notation, $\mathcal{F}_{M^1}(a)$ is the algebra of germs of analytic functions at $a \in M^1$, and $\mathcal{F}_{M^2}(\tau(a))$ is the algebra of germs of analytic functions at $\tau(a) \in M^2$. One has the pull-back homomorphism $\tau^* \colon \mathcal{F}_{M^2}(\tau(a)) \to \mathcal{F}_{M^1}(a)$.

The differential of $\tau$ at $a$ is the $\mathbb{K}$-linear map $\tau_*|_a : T_a M^1 \to T_{\tau(a)} M^2$ defined as follows. A tangent vector $v \in T_a M^1$ is a $\mathbb{K}$-linear map $v : \mathcal{F}_{M^1}(a) \to \mathbb{K}$ satisfying (43) for all $g_1, g_2 \in \mathcal{F}_{M^1}(a)$. We define the $\mathbb{K}$-linear map

$$
\tau_*|_a(v) : \mathcal{F}_{M^2}(\tau(a)) \to \mathbb{K}, \quad \tau_*|_a(v)(g) = v(\tau^*(g)), \quad g \in \mathcal{F}_{M^2}(\tau(a)).
$$

Then $\tau_*|_a(v)(h_1 h_2) = \tau_*|_a(v)(h_1) \cdot h_2(\tau(a)) + h_1(\tau(a)) \cdot \tau_*|_a(v)(h_2)$ for all $h_1, h_2 \in \mathcal{F}_{M^2}(\tau(a))$, which means that $\tau_*|_a(v) \in T_{\tau(a)} M^2$.

As has been said in Remark 9 in this preprint all manifolds and maps of manifolds are supposed to be analytic.

**Definition 2.** Let $M^1$, $M^2$ be (possibly infinite-dimensional) manifolds. Let $q \in \mathbb{Z}_{\geq 0}$. A map $\varphi : M^2 \to M^1$ is called a bundle with $q$-dimensional fibers if

- the map $\varphi$ is surjective,
- for any point $a \in M^2$ there are a neighborhood $U \subset M^2$ and a manifold $W$ of dimension $q$ such that $\varphi(U)$ is open in $M^1$ and one has the commutative diagram

$$
\begin{array}{ccc}
U & \xrightarrow{\xi} & \varphi(U) \times W \\
\varphi \downarrow & & \downarrow \\
\varphi(U) & & \\
\end{array}
$$

where $\xi$ is an analytic diffeomorphism.

For $b \in M^1$ the subset $\varphi^{-1}(b) \subset M^2$ is called the fiber of $\varphi$ over $b$.

**Remark 14.** The introduced notion is different from the standard concept of locally trivial bundle, because in our case fibers over different points are not necessarily isomorphic to each other.

If $M^1$, $M^2$ are finite-dimensional manifolds then a bundle $M^2 \to M^1$ is the same as a surjective submersion.

### 2.2. Jet spaces and PDEs.

Fix $m, n \in \mathbb{Z}_{\geq 0}$. Let $J^\infty$ be the space of infinite jets of $m$-component vector functions $(u^1(x_1, \ldots, x_n), \ldots, u^m(x_1, \ldots, x_n))$. Equivalently, one can say that $J^\infty$ is the space of infinite jets of local sections of the bundle

$$
\pi : \mathbb{K}^{n+m} \to \mathbb{K}^n, \quad (x_1, \ldots, x_n, u^1, \ldots, u^m) \mapsto (x_1, \ldots, x_n).
$$

For an element $\mu \in \mathbb{Z}_{\geq 0}^n$, we denote by $\mu_i \in \mathbb{Z}_{\geq 0}$, $i = 1, \ldots, n$, the $i$-th component of $\mu$. That is, $\mu = (\mu_1, \ldots, \mu_n)$. Also, we set $|\mu| = \mu_1 + \cdots + \mu_n$.

We use the following notation for partial derivatives of functions $u^j = u^j(x_1, \ldots, x_n)$, $j = 1, \ldots, m$,

$$
u^j = \left. \frac{\partial^{\mu_1+\cdots+\mu_n}}{\partial x_1^{\mu_1} \cdots \partial x_n^{\mu_n}} \right|_{0} u^j, \quad \mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}_{\geq 0}^n, \quad u^j_{0, \ldots, 0} = u^j, \quad j = 1, \ldots, m.
$$

Then $J^\infty$ can be identified with the space $\mathbb{K}^\infty$ with the coordinates

$$
x_1, \ldots, x_n, \quad u^j, \quad \mu \in \mathbb{Z}_{\geq 0}^n, \quad j = 1, \ldots, m.
$$

This allows us to say that $J^\infty$ is an infinite-dimensional manifold with coordinates (46). In this approach, $u^j_\mu$ is regarded as a $\mathbb{K}$-valued variable, which belongs to the set of coordinates (46) of the manifold $J^\infty$.

The topology on $J^\infty$ can be described as follows. For each $k \in \mathbb{Z}_{\geq 0}$, consider the space $J^k$ with the coordinates

$$
x_1, \ldots, x_n, \quad u^j_\mu, \quad \tilde{\mu} \in \mathbb{Z}_{\geq 0}^n, \quad |\tilde{\mu}| \leq k, \quad j = 1, \ldots, m.
$$

One has the natural projection $p_k : J^\infty \to J^k$ that “forgets” the coordinates $u^j_\mu$ with $|\mu| > k$. Since $J^k$ is finite-dimensional, we have the standard topology on $J^k$. For any $k \in \mathbb{Z}_{\geq 0}$ and any open subset $V \subset J^k$,
the subset \( p_k^{-1}(V) \subset J^\infty \) is open in \( K^\infty \). Such subsets form a base of the topology on \( J^\infty \). An analytic function on a connected open subset of \( J^\infty \) depends on a finite number of the coordinates \([16]\).

For every \( \mu \in \mathbb{Z}_{\geq 0}^n \) and \( i \in \{1, \ldots, n\} \), we denote by \( \mu + 1_i \) the element of \( \mathbb{Z}_n^n \) whose \( i \)-th component is equal to \( \mu_i + 1 \) and \( l \)-th component is equal to \( \mu_l \) for any \( l \neq i \). That is,

\[
\mu + 1_i = (\mu_1, \ldots, \mu_{i-1}, \mu_i + 1, \mu_{i+1}, \ldots, \mu_n)
\]

For example, \( \mu + 1_n = (\mu_1, \ldots, \mu_{n-1}, 1, \mu_n + 1) \).

The total derivative operators

\[
D_{x_i} = \frac{\partial}{\partial x_i} + \sum_{j=1, \ldots, m, \mu \in \mathbb{Z}_{\geq 0}^n} u^j_{\mu+1_i} \frac{\partial}{\partial u^j_\mu}, \quad i = 1, \ldots, n,
\]

can be regarded as vector fields on the manifold \( J^\infty \).

It is easily seen that, for each point \( a \in J^\infty \), the corresponding tangent vectors

\[
D_{x_i}\big|_a \in T_a J^\infty, \quad i = 1, \ldots, n,
\]

are linearly independent. Here \( T_a J^\infty \) is the tangent space of the manifold \( J^\infty \) at \( a \in J^\infty \).

Let \( C_a \subset T_a J^\infty \) be the \( n \)-dimensional subspace spanned by the vectors \([15]\). It is called the Cartan subspace at \( a \in J^\infty \). The Cartan distribution \( \mathcal{C} \) on \( J^\infty \) is the \( n \)-dimensional distribution which consists of the Cartan subspaces \( C_a, a \in J^\infty \). So the Cartan distribution \( \mathcal{C} \) is spanned by the vector fields \([17]\). Geometric coordinate-independent definitions of \( J^\infty \) and the Cartan distribution can be found in \([3]\).

Consider a PDE for functions \( u^i = u^i(x_1, \ldots, x_n), i = 1, \ldots, m \),

\[
F_\alpha(x_i, u^j_\mu) = 0, \quad \alpha = 1, \ldots, q,
\]

where \( u^j_\mu \) is given by \([13]\), and \( q \in \mathbb{Z}_{>0} \). Here \( F_\alpha(x_i, u^j_\mu) \) depends on a finite number of the variables \([16]\) and can be viewed as a function on an open subset of \( J^\infty \).

**Remark 15.** We assume that \( F_\alpha(x_i, u^j_\mu) \) is an analytic function on an open subset of \( J^\infty \). For example, \( F_\alpha(x_i, u^j_\mu) \) may be a meromorphic function, because a meromorphic function is analytic on some open subset of \( J^\infty \).

Since \( F_\alpha = F_\alpha(x_i, u^j_\mu) \) is a function on an open subset of \( J^\infty \), we can consider the functions \( D_{x_1}^{k_1} \ldots D_{x_n}^{k_n}(F_\alpha) \) for \( k_1, \ldots, k_n \in \mathbb{Z}_{\geq 0} \). The equations \( D_{x_1}^{k_1} \ldots D_{x_n}^{k_n}(F_\alpha) = 0 \) are differential consequences of \([19]\). To understand the meaning of these equations, it is instructive to look at the differential consequences \([21], [22], [23]\) of the KdV equation.

Let \( \mathcal{E} \subset J^\infty \) be the subset of the points \( a \in J^\infty \) that obey the equations

\[
F_\alpha(a) = 0, \quad D_{x_1}^{k_1} \ldots D_{x_n}^{k_n}(F_\alpha)(a) = 0, \quad k_1, \ldots, k_n \in \mathbb{Z}_{\geq 0}, \quad \alpha = 1, \ldots, q.
\]

If the PDE \([19]\) satisfies some non-degeneracy conditions, then the set \( \mathcal{E} \) is a nonsingular submanifold of \( J^\infty \) and the Cartan distribution \( \mathcal{C} \) is tangent to \( \mathcal{E} \), which gives an \( n \)-dimensional distribution on \( \mathcal{E} \). Then \( \mathcal{E} \) is called nonsingular. The restriction of the distribution \( \mathcal{C} \) to \( \mathcal{E} \) is denoted by the same symbol \( \mathcal{C} \) and is called the Cartan distribution on \( \mathcal{E} \).

These non-degeneracy conditions are satisfied on an open dense subset of \( J^\infty \) for practically all PDEs in applications. (If there are some singular points in \( \mathcal{E} \), one can exclude these points from consideration and study only the nonsingular part of \( \mathcal{E} \), which is usually open and dense in \( \mathcal{E} \).) In particular, as is shown in Example \([5]\) for any \((1 + 1)\)-dimensional evolution PDE the set \( \mathcal{E} \) is nonsingular.

**Remark 16.** If \( F_\alpha(x_i, u^j_\mu) \) depends polynomially on \( x_i, u^j_\mu \), then equations \([50]\) are algebraic, and \( \mathcal{E} \) is an algebraic variety in \( J^\infty \). The case of the KdV equation discussed in Example \([2]\) is of this kind.

In general, we assume that \( \mathcal{E} \) is an analytic submanifold of an open subset of \( J^\infty \).

In what follows, we always assume that \( \mathcal{E} \) is nonsingular in the above-mentioned sense. According to Remark \([17]\), solutions of the PDE correspond to \( n \)-dimensional integral submanifolds of the Cartan distribution.
We often identify a PDE with the corresponding manifold $E$. So we can speak about a PDE $E$. Thus, in this geometric approach, a PDE is regarded as a manifold $E$ with an $n$-dimensional distribution (the Cartan distribution $C$) such that solutions of the PDE correspond to $n$-dimensional integral submanifolds, where $n$ is the number of independent variables in the PDE.

Thus we can say that the pair $(E, C)$ is a PDE. To simplify notation, we sometimes say that $E$ is a PDE, not mentioning the distribution $C$ explicitly.

**Remark 17.** Let
\begin{equation}
\hat{U}(x_1, \ldots, x_n) = (u^1(x_1, \ldots, x_n), \ldots, u^n(x_1, \ldots, x_n))
\end{equation}
be an $m$-component vector function defined on an open subset $V \subset \mathbb{R}^n$. Consider the corresponding map
\begin{equation}
\hat{U}^\infty: V \to J^\infty, \quad (x_1, \ldots, x_n) \mapsto \left( x_1, \ldots, x_n, \, w^1_\mu = \frac{\partial |\mu| u^1}{\partial x^{\mu_1}_1 \ldots \partial x^{\mu_n}_n} \right) \in J^\infty.
\end{equation}
In other words, $\hat{U}^\infty(x_1, \ldots, x_n)$ is the infinite jet of the vector function (51) at the point $(x_1, \ldots, x_n) \in V$. It easy to check that the vector fields (17) are tangent to the $n$-dimensional submanifold $\hat{U}^\infty(V) \subset J^\infty$, where $\hat{U}^\infty(V)$ is the image of the map (52). Indeed, for each $i = 1, \ldots, n$, the differential of $\hat{U}^\infty$ maps the vector field $\frac{\partial}{\partial x_i}$ to the vector field $D_{x_i}$ restricted to $\hat{U}^\infty(V)$.

Hence $\hat{U}^\infty(V)$ is an integral submanifold of the Cartan distribution on $J^\infty$. It is known that any $n$-dimensional integral submanifold of the Cartan distribution is locally of this type.

We have $\hat{U}^\infty(V) \subset E$ iff (51) is a solution of the PDE (49). Therefore, solutions of the PDE (49) correspond to $n$-dimensional integral submanifolds $S \subset E$ of the Cartan distribution on $E$.

**Example 5.** Consider the case $n = 2$. Then we have two independent variables, which are denoted by $x, t$. We use the following notation for partial derivatives of functions $u^j(x, t)$, $j = 1, \ldots, m$,
\begin{equation}
\frac{\partial u^j}{\partial \mu_1, \mu_2} = \frac{\partial^{\mu_1 + \mu_2} u^j}{\partial x^{\mu_1} \partial t^{\mu_2}}, \quad \mu_1, \mu_2 \in \mathbb{Z}_{\geq 0}, \quad u^j_{0,0} = u^j, \quad j = 1, \ldots, m.
\end{equation}
In this notation, the total derivative operators are written as
\begin{equation}
D_x = \frac{\partial}{\partial x} + \sum_{j=1, \ldots, m} \frac{\partial u^j_{\mu_1+1, \mu_2}}{\partial \mu_1, \mu_2} \frac{\partial}{\partial u^j_{\mu_1, \mu_2}}, \quad D_t = \frac{\partial}{\partial t} + \sum_{j=1, \ldots, m} \frac{\partial u^j_{\mu_1+1, \mu_2+1}}{\partial \mu_1, \mu_2} \frac{\partial}{\partial u^j_{\mu_1, \mu_2}}.
\end{equation}
In this case, $J^\infty$ is the infinite-dimensional manifold with the coordinates $x, t, u^j_{\mu_1, \mu_2}$. The Cartan distribution on $J^\infty$ is 2-dimensional and is spanned by the vector fields $D_x, D_t$ given by (54).

Consider an $m$-component evolution PDE
\begin{equation}
u^{i}_{0,1} = F^i(x, t, u^j_{0,0}, u^j_{1,0}, \ldots, u^j_{d,0}), \quad i, j = 1, \ldots, m,
\end{equation}
where, according to (53), one has $u^{0}_{0,1} = \frac{\partial u^i}{\partial t}$ and $u^{j}_{k,0} = \frac{\partial^k u^j}{\partial x^k}$ for $k \in \mathbb{Z}_{\geq 0}$. The number $d \in \mathbb{Z}_{\geq 0}$ in (55) is such that the functions $F^i$ may depend only on $x, t, u^j_{k,0}$ for $k \leq d$.

The PDE (55) is said to be $(1 + 1)$-dimensional, because in (55) we have one “space variable” $x$ and one “time variable” $t$.

The infinite prolongation $E \subset J^\infty$ of (55) is determined by the equations
\begin{equation}
D_{x}^{n_1} D_{t}^{n_2} (\hat{u}^{i}_{0,1} - F^i(x, t, u^j_{0,0}, u^j_{1,0}, \ldots, u^j_{d,0})) = 0, \quad i = 1, \ldots, m, \quad n_1, n_2 \in \mathbb{Z}_{\geq 0}.
\end{equation}
Using equations (56), for all $k_1 \geq 0, k_2 \geq 1$, and $j = 1, \ldots, m$ we can uniquely express $u^j_{k_1, k_2}$ in terms of
\begin{equation}
x, \quad t, \quad u^i_{k_0}, \quad i = 1, \ldots, m, \quad k \in \mathbb{Z}_{\geq 0}.
\end{equation}
Therefore, (57) can be regarded as coordinates on the submanifold $E \subset J^\infty$. 
It is easily seen that the vector fields $D_x, D_t$ are tangent to the submanifold $\mathcal{E} \subset J^\infty$. According to (56), the restriction of the function $D_t(u^i_{k,0}) = u^i_{k,1}$ to $\mathcal{E}$ is equal to $D^k_x(F^i(x,t,u^i_{0,0},\ldots,u^i_{d,0}))$. This implies that the restrictions $D_x|_{\mathcal{E}}, D_t|_{\mathcal{E}}$ of $D_x, D_t$ to $\mathcal{E}$ are written as

$$
(58) \quad D_x|_{\mathcal{E}} = \frac{\partial}{\partial x} + \sum_{i=1,\ldots,m, k \geq 0} u^i_{k+1,0} \frac{\partial}{\partial u^i_{k,0}}, \quad D_t|_{\mathcal{E}} = \frac{\partial}{\partial t} + \sum_{i=1,\ldots,m, k \geq 0} D^k_x(F^i(x,t,u^i_{0,0},\ldots,u^i_{d,0})) \frac{\partial}{\partial u^i_{k,0}}.
$$

Here we use the fact that (57) are regarded as coordinates on $\mathcal{E}$. So we see that for any (1+1)-dimensional evolution PDE (55) the set $\mathcal{E}$ is nonsingular.

Now consider the scalar case $m = 1$. We set $u = u^i, u_k = u^i_{k,0}, F = F^1$. Then (57) becomes

$$
(59) \quad x, \quad t, \quad u_k, \quad k \in \mathbb{Z}_{\geq 0},
$$

and the PDE (55) can be written as (11). Furthermore, formulas (58) become (5). (To simplify notation, in formulas (5) we have written $D_x, D_t$ instead of $D_x|_{\mathcal{E}}, D_t|_{\mathcal{E}}$.)

Therefore, we see that the infinite prolongation $\mathcal{E}$ of the scalar evolution equation (11) is an infinite-dimensional manifold with the coordinates (59), and the Cartan distribution on $\mathcal{E}$ is spanned by the vector fields (5).

2.3. Coverings and Bäcklund transformations of PDEs. Following A. M. Vinogradov and I. S. Krasilshchik [38, 19], we are going to give a geometric definition of Bäcklund transformations, using the notion of coverings of PDEs.

Before defining coverings of PDEs, we need to recall the classical notion of coverings in topology, which we call topological coverings, in the case of finite-dimensional manifolds.

Let $M^1, M^2$ be finite-dimensional manifolds. Suppose that $M^1$ is connected. Then a map $\varphi: M^2 \rightarrow M^1$ is a topological covering if $\varphi$ is a locally trivial bundle with discrete (0-dimensional) fibers. In general, when $M^1$ is not necessarily connected, a map $\varphi: M^2 \rightarrow M^1$ is a topological covering if $\varphi$ is a locally trivial bundle with trivial fibers over each connected component of $M^1$.

Coverings of PDEs are defined in Definition 3. They are sometimes called differential coverings, in order to distinguish them from topological ones. Relations between differential coverings and topological coverings are discussed in Remark 19.

Definition 3. Let $(\mathcal{E}^1, \mathcal{C}^1)$ and $(\mathcal{E}^2, \mathcal{C}^2)$ be PDEs, where $\mathcal{E}^i$ is a (possibly infinite-dimensional) manifold and $\mathcal{C}^i$ is the Cartan distribution on $\mathcal{E}^i$ for each $i = 1, 2$. So for each $a_i \in \mathcal{E}^i$ we have the Cartan subspace $\mathcal{C}^i_{a_i} \subset T_{a_i}\mathcal{E}^i$, and the distribution $\mathcal{C}^i$ is determined by these subspaces.

A map $\tau: \mathcal{E}^2 \rightarrow \mathcal{E}^1$ is a (differential) covering if $\tau$ is a bundle with $q$-dimensional fibers for some $q \in \mathbb{Z}_{\geq 0}$ such that for any $a \in \mathcal{E}^2$ the restriction of $\tau|_{a_i}$ to the subspace $\mathcal{C}^2_{a} \subset T_{a}\mathcal{E}^2$ is an isomorphism onto the subspace $\mathcal{C}^1_{\tau(a)} \subset T_{\tau(a)}\mathcal{E}^1$. (So one has $\tau|_{a_i}(\mathcal{C}^2_{a}) = \mathcal{C}^1_{\tau(a)}$ and $\mathcal{C}^2_{a} \cap \ker \tau|_{a_i} = 0$.) In particular, $\dim\mathcal{C}^2_{a} = \dim\mathcal{C}^1_{\tau(a)}$, so the dimension of the distribution $\mathcal{C}^2$ is equal to the dimension of the distribution $\mathcal{C}^1$.

Note that even local classification of differential coverings is highly nontrivial due to different possible configurations of the distributions.

Remark 18. Definition 3 implies the following. If $(\mathcal{E}^1, \mathcal{C}^1), (\mathcal{E}^2, \mathcal{C}^2)$ are PDEs and $\tau: \mathcal{E}^2 \rightarrow \mathcal{E}^1$ is a (differential) covering, then $\tau$ maps integral submanifolds of the distribution $\mathcal{C}^2$ to integral submanifolds of the distribution $\mathcal{C}^1$. Therefore, $\tau$ maps solutions of the PDE $(\mathcal{E}^2, \mathcal{C}^2)$ to solutions of the PDE $(\mathcal{E}^1, \mathcal{C}^1)$.

Let $n$ be the dimension of the distribution $\mathcal{C}^1$, which is equal to the dimension of the distribution $\mathcal{C}^2$. That is, $n = \dim\mathcal{C}^i_a$ for each $i = 1, 2$ and all $a \in \mathcal{E}^i$. Solutions of the PDE $(\mathcal{E}^i, \mathcal{C}^i)$ correspond to $n$-dimensional integral submanifolds of the Cartan distribution $\mathcal{C}^i$.

Since the Cartan distribution is involutive, for each $n$-dimensional integral submanifold $\mathcal{S} \subset \mathcal{E}^1$ of the distribution $\mathcal{C}^1$, the preimage $\tau^{-1}(\mathcal{S})$ is foliated by $n$-dimensional integral submanifolds of the distribution $\mathcal{C}^2$.

Therefore, locally, for each solution $s$ of the PDE $(\mathcal{E}^1, \mathcal{C}^1)$, the preimage $\tau^{-1}(s)$ is a family of solutions of the PDE $(\mathcal{E}^2, \mathcal{C}^2)$ depending on $q$ parameters, where $q$ is the dimension of fibers of $\tau$. 
Remark 19. Let us show that usual topological coverings of finite-dimensional manifolds are a special case of differential coverings.

Let $E^1, E^2$ be finite-dimensional manifolds and $\psi: E^2 \to E^1$ be an analytic map which is a topological covering. (As has been said in Remark 9 in this preprint all manifolds and maps of manifolds are supposed to be analytic.) Then $\psi$ becomes a differential covering if, for each $i = 1, 2, 3$, we consider the distribution $C_i$ equal to the whole tangent bundle of $E^i$ so that $C_a = T_a E^i$ for all $a \in E^i$.

It is shown in [38, 13] that the classical notion of Bäcklund transformations can be formulated geometrically as follows.

Definition 4. Let $(E^1, C^1)$ and $(E^2, C^2)$ be PDEs. A Bäcklund transformation (BT) between $(E^1, C^1)$ and $(E^2, C^2)$ is given by another PDE $(E^3, C^3)$ and a pair of coverings

\[(60)\]

\[\tau_1: E^3 \to E^1, \quad \tau_2: E^3 \to E^2.\]

In other words, $(E^1, C^1)$ and $(E^2, C^2)$ are connected by a BT if there are a PDE $(E^3, C^3)$ and differential coverings (60). Here $C^i$ is the Cartan distribution on the manifold $E^i$ for each $i = 1, 2, 3$.

To simplify notation, we sometimes say that $E^i$ is a PDE, not mentioning the distribution $C^i$ explicitly.

According to Remark 10, a BT (60) helps to obtain solutions of the PDE $E^2$ from solutions of the PDE $E^1$, and vice versa.

Remark 20. According to Definitions 3, 4 we consider BTs which consist of coverings with finite-dimensional fibers. To our knowledge, all known examples of BTs for $(1 + 1)$-dimensional PDEs can be formulated in this way.

For PDEs of other types (multidimensional PDEs), sometimes one needs to consider BTs with infinite-dimensional fibers, which are not studied in this preprint.

3. THE ALGEBRAS $\mathbb{F}^p(E, a)$

Recall that $x, t, u_k$ take values in $K$, where $K$ is either $C$ or $R$. Let $K^\infty$ be the infinite-dimensional space with the coordinates $x, t, u_k$ for $k \in \mathbb{Z}_{\geq 0}$. The topology on $K^\infty$ is defined as follows.

For each $l \in \mathbb{Z}_{\geq 0}$, consider the space $K^{l+3}$ with the coordinates $x, t, u_k$ for $k \leq l$. One has the natural projection $\pi_l: K^\infty \to K^{l+3}$ that “forgets” the coordinates $u_{k'}$ for $k' > l$.

Since $K^{l+3}$ is a finite-dimensional vector space, we have the standard topology on $K^{l+3}$. For any $l \in \mathbb{Z}_{\geq 0}$ and any open subset $V \subset K^{l+3}$, the subset $\pi_l^{-1}(V) \subset K^\infty$ is, by definition, open in $K^\infty$. Such subsets form a base of the topology on $K^\infty$. In other words, we consider the smallest topology on $K^\infty$ such that the maps $\pi_l, l \in \mathbb{Z}_{\geq 0}$, are continuous.

The infinite prolongation $\mathcal{E}$ of an evolution equation (1) has been defined in Example 5. Equivalently, the manifold $\mathcal{E}$ can described as follows. The function $F(x, t, u_0, \ldots, u_d)$ from (1) is defined on some open subset $U \subset K^{d+3}$. The infinite prolongation $\mathcal{E}$ of equation (1) can be identified with the set $\pi_d^{-1}(U) \subset K^\infty$.

So $\mathcal{E} = \pi_d^{-1}(U)$ is an open subset of the space $K^\infty$ with the coordinates $x, t, u_k$ for $k \in \mathbb{Z}_{\geq 0}$. The topology on $\mathcal{E}$ is induced by the embedding $\mathcal{E} \subset K^\infty$.

Example 6. Using the notation (2), for any constants $e_1, e_2, e_3 \in K$ we can rewrite the Krichever-Novikov equation (36) as follows

\[(61)\]

\[u_t = F(x, t, u_0, u_1, u_2, u_3),\]

\[(62)\]

\[F(x, t, u_0, u_1, u_2, u_3) = u_3 - \frac{3}{2} \frac{(u_2)^2}{u_1} + \frac{(u_0 - e_1)(u_0 - e_2)(u_0 - e_3)}{u_1}.\]

Since the right-hand side of (61) depends on $u_k$ for $k \leq 3$, we have here $d = 3$.

Let $K^6$ be the space with the coordinates $x, t, u_0, u_1, u_2, u_3$. According to (62), the function $F$ is defined on the open subset $U \subset K^6$ determined by the condition $u_1 \neq 0$. 
Recall that $\mathbb{K}^\infty$ is the space with the coordinates $x, t, u_k$ for $k \in \mathbb{Z}_{\geq 0}$. We have the map $\pi_3: \mathbb{K}^\infty \to \mathbb{K}^6$ that "forgets" the coordinates $u_{k'}$ for $k' > 3$. The infinite prolongation $\mathcal{E}$ of equation (3) is the following open subset of $\mathbb{K}^\infty$

$$\mathcal{E} = \pi_3^{-1}(\mathbb{U}) = \{(x, t, u_0, u_1, u_2, \ldots) \in \mathbb{K}^\infty \mid u_1 \neq 0\}. $$

Consider again an arbitrary scalar evolution equation (1). As has been said above, the infinite prolongation $\mathcal{E}$ of (1) is an open subset of the space $\mathbb{K}^\infty$ with the coordinates $x, t, u_k$ for $k \in \mathbb{Z}_{\geq 0}$.

A point $a \in \mathcal{E}$ is determined by the values of the coordinates $x, t, u_k$ at $a$. Let $a = (x = x_a, t = t_a, u_k = a_k) \in \mathcal{E}$, $x_a, t_a, a_k \in \mathbb{K}$, $k \in \mathbb{Z}_{\geq 0}$, be a point of $\mathcal{E}$. In other words, the constants $x_a, t_a, a_k$ are the coordinates of the point $a \in \mathcal{E}$ in the coordinate system $x, t, u_k$.

Let $N \in \mathbb{Z}_{>0}$. Recall that we denote by $\mathfrak{gl}_N$ the algebra of $N \times N$ matrices with entries from $\mathbb{K}$ and by $\text{GL}_N$ the group of invertible $N \times N$ matrices. Let $\text{Id} \in \text{GL}_N$ be the identity matrix.

Let $\mathfrak{g} \subset \mathfrak{gl}_N$ be a matrix Lie algebra. (So $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}_N$.) There is a unique connected immersed Lie subgroup $\mathcal{G} \subset \text{GL}_N$ whose Lie algebra is $\mathfrak{g}$. We call $\mathcal{G}$ the connected matrix Lie group corresponding to the matrix Lie algebra $\mathfrak{g} \subset \mathfrak{gl}_N$.

For any $l \in \mathbb{Z}_{\geq 0}$, a matrix-function $G = G(x, t, u_0, u_1, \ldots, u_l)$ with values in $\mathcal{G}$ is called a gauge transformation. Equivalently, one can say that a gauge transformation is given by a $\mathcal{G}$-valued function $G = G(x, t, u_0, u_1, \ldots, u_l)$.

In this section, when we speak about ZCRs, we always mean that we speak about ZCRs of equation (1). For each $i = 1, 2$, let

$$A_i = A_i(x, t, u_0, u_1, \ldots), \quad B_i = B_i(x, t, u_0, u_1, \ldots), \quad D_x(B_i) - D_t(A_i) + [A_i, B_i] = 0$$

be a $\mathfrak{g}$-valued ZCR. The ZCR $A_1, B_1$ is said to be gauge equivalent to the ZCR $A_2, B_2$ if there is a gauge transformation $G = G(x, t, u_0, \ldots, u_l)$ such that

$$A_1 = GA_2G^{-1} - D_x(G) \cdot G^{-1}, \quad B_1 = GB_2G^{-1} - D_t(G) \cdot G^{-1}. $$

**Remark 21.** For any $l \in \mathbb{Z}_{\geq 0}$, when we consider a function $Q = Q(x, t, u_0, u_1, \ldots, u_l)$ defined on a neighborhood of $a \in \mathcal{E}$, we always assume that the function is analytic on this neighborhood. For example, $Q$ may be a meromorphic function defined on an open subset of $\mathcal{E}$ such that $Q$ is analytic on a neighborhood of $a \in \mathcal{E}$.

Let $s \in \mathbb{Z}_{\geq 0}$. For a function $M = M(x, t, u_0, u_1, u_2, \ldots)$, the notation $M \big|_{u_k = a_k, \ k \geq s}$ means that we substitute $u_k = a_k$ for all $k \geq s$ in the function $M$.

Also, sometimes we need to substitute $x = x_a$ or $t = t_a$ in such functions. For example, if $M = M(x, t, u_0, u_1, u_2, u_3)$, then

$$M \big|_{x = x_a, \ u_k = a_k, \ k \geq 2} = M(x_a, t, u_0, u_1, a_2, a_3).$$

The following result is obtained in [14].

**Theorem 4 ([14]).** Let $N \in \mathbb{Z}_{\geq 0}$ and $p \in \mathbb{Z}_{\geq 0}$. Let $\mathfrak{g} \subset \mathfrak{gl}_N$ be a matrix Lie algebra and $\mathcal{G} \subset \text{GL}_N$ be the connected matrix Lie group corresponding to $\mathfrak{g} \subset \mathfrak{gl}_N$.

Let

$$A = A(x, t, u_0, \ldots, u_p), \quad B = B(x, t, u_0, \ldots, u_{p+d-1}), \quad D_x(B) - D_t(A) + [A, B] = 0$$

be a ZCR of order $\leq p$ such that the functions $A, B$ are analytic on a neighborhood of $a \in \mathcal{E}$ and take values in $\mathfrak{g}$.

Then on a neighborhood of $a \in \mathcal{E}$ there is a unique gauge transformation $G = G(x, t, u_0, \ldots, u_l)$ such that $G(a) = \text{Id}$ and the functions

$$\tilde{A} = GAG^{-1} - D_x(G) \cdot G^{-1}, \quad \tilde{B} = GBG^{-1} - D_t(G) \cdot G^{-1}$$

are a ZCR of order $\leq p$.
satisfy

\[
\frac{\partial \tilde{A}}{\partial u_s} \bigg|_{u_k=a_k, \; k\geq s} = 0 \quad \forall \; s \geq 1,
\]

(66)

\[
\tilde{A} \bigg|_{u_k=a_k, \; k\geq 0} = 0,
\]

(67)

\[
\tilde{B} \bigg|_{x=x_a, \; u_k=a_k, \; k\geq 0} = 0.
\]

(68)

Furthermore, one has the following.

- The function \( G \) depends only on \( x, t, u_0, \ldots, u_{p-1} \). (In particular, if \( p = 0 \) then \( G \) depends only on \( x, t \).)
- The function \( G \) is analytic on a neighborhood of \( a \in \mathcal{E} \).
- The functions (65) take values in \( \mathfrak{g} \) and satisfy

\[
\tilde{A} = \tilde{A}(x, t, u_0, u_1, \ldots, u_p), \quad \tilde{B} = \tilde{B}(x, t, u_0, u_1, \ldots, u_{p+d-1}),
\]

(69)

\[
D_x(\tilde{B}) - D_t(\tilde{A}) + [\tilde{A}, \tilde{B}] = 0.
\]

(So the functions (65) form a \( \mathfrak{g} \)-valued ZCR of order \( \leq p \).)

Note that, according to our definition of gauge transformations, the function \( G \) takes values in \( \mathcal{G} \). The property \( G(a) = \text{Id} \) means that \( G(x_a, t_a, a_0, \ldots, a_{p-1}) = \text{Id} \).

Fix a point \( a \in \mathcal{E} \) given by (63), which is determined by constants \( x_a, t_a, a_k \).

A ZCR

\[
A = A(x, t, u_0, u_1, \ldots), \quad B = B(x, t, u_0, u_1, \ldots), \quad D_x(B) - D_t(A) + [A, B] = 0
\]

is said to be \( a \)-normal if \( A, B \) satisfy the following equations

\[
\frac{\partial A}{\partial u_s} \bigg|_{u_k=a_k, \; k\geq s} = 0 \quad \forall \; s \geq 1,
\]

(72)

\[
A \bigg|_{u_k=a_k, \; k\geq 0} = 0,
\]

(73)

\[
B \bigg|_{x=x_a, \; u_k=a_k, \; k\geq 0} = 0.
\]

(74)

**Remark 22.** For example, the ZCR \( \tilde{A}, \tilde{B} \) described in Theorem 4 is \( a \)-normal, because \( \tilde{A}, \tilde{B} \) obey (66), (67), (68). Theorem 4 implies that any ZCR on a neighborhood of \( a \in \mathcal{E} \) is gauge equivalent to an \( a \)-normal ZCR. Therefore, following [14], we can say that properties (72), (73), (74) determine a normal form for ZCRs with respect to the action of the group of gauge transformations on a neighborhood of \( a \in \mathcal{E} \).

**Remark 23.** The functions \( A, B, G \) considered in Theorem 4 are analytic on a neighborhood of \( a \in \mathcal{E} \). Therefore, the \( \mathfrak{g} \)-valued functions \( \tilde{A}, \tilde{B} \) given by (65) are analytic as well.

Since \( \tilde{A}, \tilde{B} \) are analytic and are of the form (69), these functions are represented as absolutely convergent power series

\[
\tilde{A} = \sum_{l_1, l_2, i_0, \ldots, i_p \geq 0} (x - x_a)^{l_1} (t - t_a)^{l_2} (u_0 - a_0)^{i_0} \ldots (u_p - a_p)^{i_p} \cdot \tilde{A}_{i_0 \ldots i_p}^{l_1 l_2},
\]

(75)

\[
\tilde{B} = \sum_{l_1, l_2, j_0, \ldots, j_{p+d-1} \geq 0} (x - x_a)^{l_1} (t - t_a)^{l_2} (u_0 - a_0)^{j_0} \ldots (u_{p+d-1} - a_{p+d-1})^{j_{p+d-1}} \cdot \tilde{B}_{j_0 \ldots j_{p+d-1}}^{l_1 l_2},
\]

\[
\tilde{A}_{i_0 \ldots i_p}^{l_1 l_2}, \; \tilde{B}_{j_0 \ldots j_{p+d-1}}^{l_1 l_2} \in \mathfrak{g}.
\]

For each \( k \in \mathbb{Z}_{>0} \), we set

\[
\forall_k = \left\{ (i_0, \ldots, i_k) \in \mathbb{Z}_{\geq 0}^{k+1} \mid \exists \; r \in \{1, \ldots, k\} \text{ such that } i_r = 1, \; i_q = 0 \; \forall \; q > r \right\}.
\]

(77)
In other words, for \( k \in \mathbb{Z}_{>0} \) and \( i_0, \ldots, i_k \in \mathbb{Z}_{\geq 0} \), one has \((i_0, \ldots, i_k) \in \mathcal{V}_k\) iff there is \( r \in \{1, \ldots, k\}\) such that \((i_0, \ldots, i_{r-1}, i_r, i_{r+1}, \ldots, i_k) = (i_0, \ldots, i_{r-1}, 1, 0, \ldots, 0)\). Set also \( \mathcal{V}_0 = \emptyset \). So the set \( \mathcal{V}_0 \) is empty.

Using formulas (75), (76), we see that properties (66), (67), (68) are equivalent to
\[
(84) \quad \hat{A}^{l_1,l_2}_{0,0} = \hat{B}^{l_1,l_2}_{0,0} = 0, \quad \hat{A}^{l_1,l_2}_{i_0,\ldots,i_p} = 0, \quad (i_0, \ldots, i_p) \in \mathcal{V}_p, \quad l_1, l_2 \in \mathbb{Z}_{\geq 0}.
\]

Remark 24. The main idea of the definition of the Lie algebra \( \mathbb{F}^p(\mathcal{E}, a) \) can be informally outlined as follows. According to Theorem 4 and Remark 23, any ZCR (64) of order \( \leq p \) is gauge equivalent to a ZCR given by functions \( \hat{A}, \hat{B} \) that are of the form (75), (76) and satisfy (70), (78).

To define \( \mathbb{F}^p(\mathcal{E}, a) \), we regard \( \hat{A}^{l_1,l_2}_{i_0,\ldots,i_p}, \hat{B}^{l_1,l_2}_{j_0,\ldots,j_p+d-1} \) from (75), (76) as abstract symbols. By definition, the algebra \( \mathbb{F}^p(\mathcal{E}, a) \) is generated by the symbols \( \hat{A}^{l_1,l_2}_{i_0,\ldots,i_p}, \hat{B}^{l_1,l_2}_{j_0,\ldots,j_p+d-1} \) for \( l_1, l_2, i_0, \ldots, i_p, j_0, \ldots, j_{p+d-1} \in \mathbb{Z}_{\geq 0} \). Relations for these generators are provided by equations (70), (78). The details of this construction are presented below.

Consider formal power series
\[
(79) \quad \mathcal{A} = \sum_{l_1,l_2,i_0,\ldots,i_p \geq 0} (x - x_a)^{l_1}(t - t_a)^{l_2}(u_0 - a_0)^{i_0} \cdots (u_p - a_p)^{i_p} \cdot \hat{A}^{l_1,l_2}_{i_0,\ldots,i_p},
\]
\[
(80) \quad \mathcal{B} = \sum_{l_1,l_2,j_0,\ldots,j_{p+d-1} \geq 0} (x - x_a)^{l_1}(t - t_a)^{l_2}(u_0 - a_0)^{j_0} \cdots (u_{p+d-1} - a_{p+d-1})^{j_{p+d-1}} \cdot \hat{B}^{l_1,l_2}_{j_0,\ldots,j_{p+d-1}},
\]
where
\[
(81) \quad \hat{A}^{l_1,l_2}_{i_0,\ldots,i_p}, \quad \hat{B}^{l_1,l_2}_{j_0,\ldots,j_{p+d-1}}, \quad l_1, l_2, i_0, \ldots, i_p, j_0, \ldots, j_{p+d-1} \in \mathbb{Z}_{\geq 0},
\]
are generators of a Lie algebra, which is described below.

We impose the equation
\[
(82) \quad D_x(\mathcal{B}) - D_t(\mathcal{A}) + [\mathcal{A}, \mathcal{B}] = 0,
\]
which is equivalent to some Lie algebraic relations for the generators (81). Also, we impose the following condition
\[
(83) \quad \hat{A}^{l_1,l_2}_{0,0} = \hat{B}^{l_1,l_2}_{0,0} = 0, \quad \hat{A}^{l_1,l_2}_{i_0,\ldots,i_p} = 0, \quad (i_0, \ldots, i_p) \in \mathcal{V}_p, \quad l_1, l_2 \in \mathbb{Z}_{\geq 0}.
\]

The Lie algebra \( \mathbb{F}^p(\mathcal{E}, a) \) is defined in terms of generators and relations as follows. The algebra \( \mathbb{F}^p(\mathcal{E}, a) \) is given by the generators (81), relations (83), and the relations arising from (82).

Note that condition (83) is equivalent to the following equations
\[
(84) \quad \frac{\partial \mathcal{A}}{\partial u_s} \bigg|_{u_k = a_k, k \geq s} = 0 \quad \forall s \geq 1,
\]
\[
(85) \quad \mathcal{A} \bigg|_{u_k = a_k, k \geq 0} = 0,
\]
\[
(86) \quad \mathcal{B} \bigg|_{x = x_a, u_k = a_k, k \geq 0} = 0.
\]

Remark 25. According to (14), the algebra \( \mathbb{F}^p(\mathcal{E}, a) \) is responsible for ZCRs of order \( \leq p \) in the following sense. For any finite-dimensional matrix Lie algebra \( \mathfrak{g} \), it is shown in (14) that \( \mathfrak{g} \)-valued ZCRs of order \( \leq p \) on a neighborhood of \( a \in \mathcal{E} \) are classified (up to local gauge equivalence) by homomorphisms \( \mu: \mathbb{F}^p(\mathcal{E}, a) \to \mathfrak{g} \).

Suppose that \( p \geq 1 \). As has been said above, the algebra \( \mathbb{F}^p(\mathcal{E}, a) \) is given by the generators \( \hat{A}^{l_1,l_2}_{i_0,\ldots,i_p}, \hat{B}^{l_1,l_2}_{j_0,\ldots,j_{p+d-1}} \) and the relations arising from (82), (83). Similarly, the algebra \( \mathbb{F}^{p-1}(\mathcal{E}, a) \) is given by the generators \( \hat{A}^{l_1,l_2}_{i_0,\ldots,i_{p-1}}, \hat{B}^{l_1,l_2}_{j_0,\ldots,j_{p+d-2}} \) and the relations arising from
\[
(87) \quad D_x(\mathcal{B}) - D_t(\mathcal{A}) + [\mathcal{A}, \mathcal{B}] = 0,
\]
\[
\hat{A}^{l_1,l_2}_{0,0} = \hat{B}^{l_1,l_2}_{0,0} = 0, \quad \hat{A}^{l_1,l_2}_{i_0,\ldots,i_{p-1}} = 0, \quad (i_0, \ldots, i_{p-1}) \in \mathcal{V}_{p-1}, \quad l_1, l_2 \in \mathbb{Z}_{\geq 0}.
\]
Theorem 5

Let \( \mathfrak{A} \subset \mathfrak{M}_q \) be the subalgebra generated by the elements

\[
(\text{ad} \mathfrak{A}_0)^k(\mathfrak{A}_i), \quad k \in \mathbb{Z}_{\geq 0}, \quad i \in \mathbb{Z}_{>0}.
\]

Then the map \((\text{ad} \mathfrak{A}_0)^k(\mathfrak{A}_i) \mapsto k! \cdot A_i^{k,0}, k \in \mathbb{Z}_{\geq 0}\) determines an isomorphism between \( \mathfrak{A} \) and \( \mathbb{F}^0(\mathcal{E}, a) \).

(Note that for \( k = 0 \) we have \((\text{ad} \mathfrak{A}_0)^0(\mathfrak{A}_i) = \mathfrak{A}_i \), hence \( \mathfrak{A}_i \in \mathfrak{A} \) for all \( i \in \mathbb{Z}_{>0} \).)
Definition 5. Let $\mathfrak{L}$ be a Lie algebra. A formal ZCR of Wahlquist-Estabrook type with coefficients in $\mathfrak{L}$ is given by formal power series

$$M = \sum_{i \geq 0} (u_0 - a_0)^i \cdot M_i, \quad N = \sum_{j_0, \ldots, j_{d-1} \geq 0} (u_0 - a_0)^{j_0} \ldots (u_{d-1} - a_{d-1})^{j_{d-1}} \cdot N_{j_0 \ldots j_{d-1}},$$

satisfying

$$D_x(N) - D_t(M) + [M, N] = 0.$$  \hfill (96)

Example 7. Formulas (91) and equation (93) determine a formal ZCR of Wahlquist-Estabrook type with coefficients in $\mathfrak{W}_a$.

The next lemma follows from the definition of the WE algebra $\mathfrak{W}_a$.

Lemma 2. Any formal ZCR of Wahlquist-Estabrook type (95), (96) with coefficients in $\mathfrak{L}$ determines a homomorphism $\mathfrak{W}_a \to \mathfrak{L}$ given by $\mathcal{A}_i \mapsto M_i$, $\mathcal{B}_{j_0 \ldots j_{d-1}} \mapsto N_{j_0 \ldots j_{d-1}}$.

The following scalar evolution equation was studied by A. P. Fordy [9] in connection with the Hénon-Heiles system

$$u_t = u_5 + (4\alpha - 6\beta)u_1u_2 + (8\alpha - 2\beta)u_0u_3 - 20\alpha\beta u_0^2 u_1.$$  \hfill (97)

Here $u = u(x,t)$ is a $\mathbb{K}$-valued function, and $\alpha, \beta$ are arbitrary constants. (In [9] these constants are denoted by $a, b$, but we use the symbol $a$ for a different purpose.)

We are going to present some results on the structure of the WE algebra and the algebra $\mathbb{R}^0(\mathcal{E}, a)$ for equation (97).

Remark 26. If $\alpha = \beta = 0$ then (97) is the linear equation $u_t = u_5$. Since we intend to study nonlinear PDEs, in what follows we suppose that at least one of the constants $\alpha, \beta$ is nonzero.

The following facts were noticed in [9].

- If $\alpha + \beta = 0$ then (97) is equivalent to the Sawada-Kotera equation. (That is, if $\alpha + \beta = 0$ then (97) can be transformed to the Sawada-Kotera equation by scaling of the variables. As has been said above in Remark 26, we assume that at least one of the constants $\alpha, \beta$ is nonzero.)
- If $6\alpha + \beta = 0$ then (97) is equivalent to the 5th-order flow in the KdV hierarchy.
- If $16\alpha + \beta = 0$ then (97) is equivalent to the Kaup-Kupershmidt equation.

So in the cases $\alpha + \beta = 0, 6\alpha + \beta = 0, 16\alpha + \beta = 0$ equation (97) is equivalent to a well-known integrable equation. In this preprint we consider the case

$$\alpha + \beta \neq 0, \quad 6\alpha + \beta \neq 0, \quad 16\alpha + \beta \neq 0.$$  \hfill (98)

Let $\mathfrak{L}$ be a Lie algebra. According to Definition 5, a formal ZCR of Wahlquist-Estabrook type with coefficients in $\mathfrak{L}$ for equation (97) is given by formal power series

$$M = \sum_{i \geq 0} (u_0 - a_0)^i \cdot M_i, \quad N = \sum_{j_0, j_1, j_2, j_3, j_4 \geq 0} (u_0 - a_0)^{j_0}(u_1 - a_1)^{j_1}(u_2 - a_2)^{j_2}(u_3 - a_3)^{j_3}(u_4 - a_4)^{j_4} \cdot N_{j_0, j_1, j_2, j_3, j_4},$$

satisfying (96), where

$$D_t = \frac{\partial}{\partial t} + \sum_{k \geq 0} D^k_x(u_5 + (4\alpha - 6\beta)u_1u_2 + (8\alpha - 2\beta)u_0u_3 - 20\alpha\beta u_0^2 u_1) \frac{\partial}{\partial u_k}.$$  \hfill (101)
Lemma 3. Suppose that \( \alpha, \beta \) obey (98). Then power series (99), (100) satisfy (96) with (101) iff \( M, N \) are of the form
\[
M = A_1 u_0 + A_0, \tag{102}
\]
\[
N = A_1 u_4 - [M, A_1] u_3 + (8\alpha - 2\beta) A_1 u_0 u_2 + [M, [M, A_1]] u_2 - 2(\alpha + \beta) A_1 u_1^2 - \frac{1}{2} [A_1, [A_0, A_1]] u_1^2 +
+ ((2\beta - 8\alpha) [A_0, A_1] - [M, [A_1, [A_0, A_1]]]) u_0 u_1 - [M, [A_0, [A_0, A_1]]] u_1 - \left( \frac{20}{3} \alpha \beta A_1 + \frac{2\beta}{3} [A_1, [A_0, A_1]] \right) u_0^2 +
+ \left( \frac{1}{2} [A_1, [A_0, [A_0, A_1]]] - \beta [A_0, [A_0, A_1]] \right) u_0^2 + [A_0, [A_0, [A_0, A_1]]] u_0 + Y,
\]
where \( A_0, A_1, Y \in \mathfrak{L} \) obey
\[
\begin{align*}
[A_1, [A_1, [A_0, A_1]]] &= 0, \\
4\alpha [A_0, A_1] + [A_0, [A_1, [A_0, A_1]]] &= 0, \\
[A_1, [A_1, [A_0, [A_0, A_1]]]] &= 0, \\
[A_1, [A_0, [A_0, [A_0, A_1]]]] - \beta [A_0, [A_0, A_1]] + \frac{1}{2} [A_0, [A_1, [A_0, [A_0, A_1]]]] &= 0, \\
[A_1, Y] + [A_0, [A_0, [A_0, [A_0, A_1]]]] &= 0, \\
[A_0, Y] &= 0.
\end{align*}
\]

Proof. Substituting (99), (100), (101) in equation (96), we get
\[
\sum_{k=0}^{4} u_{k+1} \frac{\partial N}{\partial u_k} = (u_5 + (4\alpha - 6\beta) u_1 u_2 + (8\alpha - 2\beta) u_0 u_3 - 20\alpha \beta u_0^2 u_1) \frac{\partial M}{\partial u_0} + [M, N] = 0,
\]
\[
M = M(u_0), \quad N = N(u_0, u_1, u_2, u_3, u_4).
\]

Analyzing equation (109), it is easy to obtain the equation \( \frac{\partial^3 M}{\partial u_0^3} = 0 \), which implies that \( M \) is of the form
\[
M = A_2 u_0^2 + A_1 u_0 + A_0, \quad A_0, A_1, A_2 \in \mathfrak{L}.
\]

Further analysis of (109) gives
\[
-2(6\alpha + \beta) A_2 = 0.
\]

Combining (111) with (98), we get \( A_2 = 0 \). Then (110) becomes (102). Using formula (102), one can deduce (103)–(108) from (109) by a straightforward computation. \( \square \)

Theorem 6. Suppose that \( \alpha, \beta \) obey (98). Let \( \mathfrak{R} \) be the Lie algebra given by generators \( A_0, A_1, Y \) and relations (102), (105), (106), (107), (108). The WE algebra \( \mathfrak{W}_a \) for equation (97) is isomorphic to \( \mathfrak{R} \). Identifying \( \mathfrak{W}_a \) with \( \mathfrak{R} \), we can assume \( A_0, A_1, Y \in \mathfrak{W}_a \).

To describe the structure of the Lie algebra \( \mathfrak{M}_a \cong \mathfrak{R} \), we need to consider separately two cases: the case \( \alpha \neq 0 \) and the case \( \alpha = 0 \).

- **Suppose that \( \alpha \neq 0 \).** Then \( \mathfrak{W}_a \) is isomorphic to the direct sum of the 3-dimensional simple Lie algebra \( \mathfrak{sl}_2(\mathbb{K}) \) and the 3-dimensional abelian Lie algebra \( \mathbb{K}^3 \). That is, \( \mathfrak{W}_a \cong \mathfrak{sl}_2(\mathbb{K}) \oplus \mathbb{K}^3 \).

  The subalgebra \( \mathfrak{sl}_2(\mathbb{K}) \subset \mathfrak{W}_a \) is spanned by the elements
\[
E_1 = [A_0, A_1], \quad E_2 = [A_0, [A_0, A_1]], \quad E_3 = [A_1, [A_0, A_1]].
\]

  The subalgebra \( \mathbb{K}^3 \subset \mathfrak{W}_a \) is spanned by the elements \( Y, Z_0, Z_1 \), where \( Z_0, Z_1 \) are given by
\[
Z_0 = -4\alpha A_0 + [A_0, [A_0, A_1]],
\]
\[
Z_1 = 4\alpha A_1 + [A_1, [A_0, A_1]].
\]

- **Suppose that \( \alpha = 0 \).** Then the Lie algebra \( \mathfrak{W}_a \) is nilpotent, and \( \dim \mathfrak{W}_a \leq 6 \).
Using (121) and (115), we get

Using the Jacobi identity and the obtained relations, one gets

Proof.

From (120) we get

Let be the subalgebra generated by . Then (119) yields

From (120) we get

Using (116) and the Jacobi identity, we can rewrite (106) as follows

Using (121) and (115), we get

Using formulas (102), (103) determine a formal ZCR of Wahlquist-Estabrook type with coefficients in . By Lemma 2, this gives a homomorphism  \( \mathfrak{M}_a \to \mathfrak{g} \).

Lemma 3 implies that, for any Lie algebra \( \mathfrak{L} \), any formal ZCR of Wahlquist-Estabrook type with coefficients in \( \mathfrak{L} \) gives a homomorphism \( \mathfrak{g} \to \mathfrak{L} \). Applying this to the formal ZCR of Wahlquist-Estabrook type with coefficients in \( \mathfrak{M}_a \) described in Example 1, we get a homomorphism \( \mathfrak{g} \to \mathfrak{M}_a \).

It is easily seen that the constructed homomorphisms \( \mathfrak{M}_a \to \mathfrak{g} \) and \( \mathfrak{g} \to \mathfrak{M}_a \) are inverse to each other. Hence we can identify \( \mathfrak{M}_a \) with \( \mathfrak{g} \) and assume \( A_0, A_1, Y \in \mathfrak{M}_a \).

To prove the other statements of the theorem, we need to deduce some consequences from relations (104), (105), (106), (107), (108).

Using (104) and the Jacobi identity, we obtain

Using the Jacobi identity and the obtained relations, one gets

Relations (104) imply

where \( Z_1 \) is given by (114).

Let \( \mathfrak{F} \subset \mathfrak{M}_a \) be the subalgebra generated by \( A_0, A_1 \). Then (119) yields

From (120) we get

Using (116) and the Jacobi identity, we can rewrite (106) as follows

Using (121) and (115), we get

Using (122) \( 0 = [A_1, [A_0, [A_0, [A_0, A_1]]]] \) \( - \beta[A_0, [A_0, [A_0, A_1]]] + \frac{1}{2}[A_0, [A_1, [A_0, [A_0, A_1]]]] = \)

(122) \( 0 = [A_1, [A_0, [A_0, [A_0, A_1]]]] - \beta[A_0, [A_0, [A_0, A_1]]] + \frac{1}{2}[A_0, [A_1, [A_0, [A_0, A_1]]]] = \)

Using (121) and (115), we get

Using (122) \( 0 = [A_1, [A_0, [A_0, [A_0, A_1]]]] - \beta[A_0, [A_0, [A_0, A_1]]] + \frac{1}{2}[A_0, [A_1, [A_0, [A_0, A_1]]]] = \)

Using (123) \( A_1, \frac{5}{2}[[A_1, A_0], [A_0, [A_0, A_1]]] - (6\alpha + \beta)[A_0, [A_0, A_1]] = \)
\[
\frac{5}{2}[[A_1, [A_1, A_0]], [A_0, [A_0, A_1]]] + \frac{5}{2}[[A_1, A_0], [A_1, [A_0, [A_0, A_1]]]] - (6\alpha + \beta)[A_1, [A_0, [A_0, A_1]]] = \\
10\alpha[A_1, [A_0, [A_0, A_1]]] + (24\alpha^2 + 4\alpha\beta)[A_0, A_1] = (4\alpha\beta - 16\alpha^2)[A_0, A_1].
\]

Set
\[
\tilde{Z} = \frac{5}{2}[[A_1, A_0], [A_0, [A_0, A_1]]] - (6\alpha + \beta)[A_0, [A_0, A_1]] + (4\alpha\beta - 16\alpha^2)A_0.
\]

Relations (122), (123) imply
\[
[A_0, \tilde{Z}] = 0, \quad [A_1, \tilde{Z}] = 0.
\]

Since the algebra \(\mathfrak{h}\) is generated by \(A_0, A_1\), relations (125) yield
\[
[C, \tilde{Z}] = \left[ C, \frac{5}{2}[[A_1, A_0], [A_0, [A_0, A_1]]] - (6\alpha + \beta)[A_0, [A_0, A_1]] + (4\alpha\beta - 16\alpha^2)A_0 \right] = 0 \quad \forall C \in \mathfrak{h}.
\]

From (126) we get
\[
[C, [[A_1, A_0], [A_0, [A_0, A_1]]]] = \left[ C, \frac{2}{5}(6\alpha + \beta)[A_0, [A_0, A_1]] + \frac{2}{5}(16\alpha^2 - 4\alpha\beta)A_0 \right] \quad \forall C \in \mathfrak{h}.
\]

From (108), (107) one has
\[
[Y, A_0] = 0, \quad [Y, A_1] = [A_0, [A_0, [A_0, [A_0, A_0]]]].
\]

Using (128) and the Jacobi identity, we obtain
\[
[Y, Z_1] = [Y, 4\alpha A_1 + [A_1, [A_0, A_1]]] = \\
4\alpha[A_0, [A_0, [A_0, [A_0, [A_0, A_1]]]]] + [[A_0, [A_0, [A_0, [A_0, A_1]]]], [A_0, A_1]] + \\
+ [A_1, [A_0, [A_0, [A_0, [A_0, A_1]]]]] = \\
4\alpha[A_0, [A_0, [A_0, [A_0, A_1]]]] - [[A_0, A_1], [A_0, [A_0, [A_0, A_1]]]] + \\
+ [[A_1, A_0], [A_0, [A_0, [A_0, A_1]]]] + [A_0, [A_1, [A_0, [A_0, [A_0, A_1]]]]] = \\
4\alpha[A_0, [A_0, [A_0, [A_0, A_1]]]] - 2[[A_0, A_1], [A_0, [A_0, [A_0, A_1]]]] + \\
+ [A_0, [A_1, [A_0, [A_0, [A_0, A_1]]]]] = \\
4\alpha[A_0, [A_0, [A_0, [A_0, A_1]]]] - 2[A_0, [A_0, [A_0, [A_0, A_1]]]] + \\
+ 2[A_0, [A_0, [A_0, A_1]]] + [A_0, [A_1, [A_0, [A_0, [A_0, A_1]]]]] = \\
4\alpha[A_0, [A_0, [A_0, [A_0, A_1]]]] + 2[A_0, [A_0, [A_0, A_1]]] + \\
- 2[A_0, [A_0, [A_0, [A_0, A_1]]]] + 2[A_0, [A_0, [A_0, [A_0, A_1]]]] + \\
+ [A_0, [A_1, [A_0, [A_0, [A_0, A_1]]]]].
\]

Substituting (118) in the last term of (129), one gets
\[
[Y, Z_1] = 4\alpha[A_0, [A_0, [A_0, [A_0, A_1]]]] + 2[A_0, [A_0, [A_0, A_1]]] + \\
- 2[A_0, [A_0, [A_0, [A_0, A_1]]]] + 2[A_0, [A_0, [A_0, A_1]]] + \\
+ [A_0, [A_0, [A_0, A_1]]] + 3[A_0, [A_0, [A_0, A_1]]] + \\
- 4\alpha[A_0, [A_0, [A_0, A_1]]] = \\
= 5[A_0, [A_0, [A_0, A_1]]] - 5[A_0, [A_0, [A_0, A_1]]] = \\
= 5[A_0, [A_0, [A_0, A_1]]] - 5[A_0, [A_0, [A_0, A_1]]] = \\
= 5[[A_0, [A_0, A_1]], [A_0, [A_0, [A_0, A_1]]]] - 5[A_0, [A_0, [A_0, A_1]]] = 0.
\]

Using (120), (128), we obtain
\[
[A_1, [Y, Z_1]] = [[A_1, Y], Z_1] + [Y, [A_1, Z_1]] = 0.
\]

Since \([A_1, [Y, Z_1]] = 0\), applying \ad \ A_1 to (130), we get
\[
[A_1, [[A_0, [A_0, A_1]], [A_0, [A_0, [A_0, A_1]]]]] - [A_1, [A_0, [A_0, [A_0, A_1]]]] = 0.
\]
Let us simplify the left-hand side of (132). Using (127), (115), (117), (118), and the Jacobi identity, we obtain

\[ (133) \quad [A_1, [[A_0, [A_0, A_1]], [A_0, [A_0, [A_0, A_1]]]]] = -4\alpha [[A_0, A_1], [A_0, [A_0, [A_0, A_1]]]] + ([A_0, [A_0, A_1]], [A_1, [A_0, [A_0, [A_0, A_1]]]]) = -4\alpha [[A_0, A_1], [A_0, [A_0, [A_0, A_1]]]] + \frac{4}{5}(\alpha + \beta)[A_0, [A_0, A_1]], [A_0, [A_0, [A_0, A_1]]]], \]

Substituting (133), (134) in (132), one gets the relation

\[ (135) \quad -4\alpha [[A_0, A_1], [A_0, [A_0, [A_0, A_1]]]] + \left(\frac{16}{5}\alpha + \frac{6}{5}\beta\right)[A_0, [A_0, A_1]], [A_0, [A_0, A_1]]] + \frac{4}{25}(6\alpha + \beta)(8\alpha + 3\beta)[A_0, [A_0, [A_0, A_1]]] = 0. \]

Applying \text{ad} \, A_1 to (135), we obtain

\[ (136) \quad -4\alpha [A_1, [[A_0, A_1], [A_0, [A_0, [A_0, A_1]]]]] + \left(\frac{16}{5}\alpha + \frac{6}{5}\beta\right)[A_1, [[A_0, A_1], [A_0, [A_0, A_1]]]] + \frac{4}{25}(6\alpha + \beta)(8\alpha + 3\beta)[A_1, [A_0, [A_0, [A_0, A_1]]]] = 0. \]

Let us simplify the left-hand side of (136). Using (121), (127), (115), (116), and the Jacobi identity, we get

\[ (137) \quad [A_1, [[A_0, A_1], [A_0, [A_0, [A_0, A_1]]]]] = -4\alpha [A_1, [A_0, [A_0, [A_0, A_1]]]] + \left(\frac{16}{5}\alpha + \frac{6}{5}\beta\right)[A_1, [A_0, [A_0, [A_0, A_1]]]] + \frac{4}{25}(6\alpha + \beta)(8\alpha + 3\beta)[A_1, [A_0, [A_0, [A_0, A_1]]]] = -16\alpha(\alpha + \beta)[A_0, [A_0, [A_0, A_1]]] + \frac{4}{5}(\alpha + \beta)[A_0, [A_0, [A_0, A_1]]] = -16\alpha(\alpha + \beta)[A_0, [A_0, [A_0, A_1]]] - \frac{8}{25}(\alpha + \beta)(6\alpha + \beta)[A_0, [A_0, [A_0, A_1]]]. \]

\[ (138) \quad [A_1, [[A_0, A_0, A_1], [A_0, [A_0, A_1]]]] = -4\alpha [[A_0, A_1], [A_0, [A_0, A_1]]] + \left(\frac{16}{5}\alpha + \frac{6}{5}\beta\right)[A_1, [A_0, [A_0, A_1]]] + \frac{4}{25}(6\alpha + \beta)(8\alpha + 3\beta)[A_1, [A_0, [A_0, A_1]]] = -4\alpha [[A_0, A_1], [A_0, [A_0, A_1]]] + \frac{2}{5}(16\alpha^2 - 4\alpha\beta)[A_0, [A_0, [A_0, A_1]]] = \frac{8}{5}\alpha(6\alpha + \beta)[A_0, [A_0, [A_0, A_1]]] - \frac{2}{5}(16\alpha^2 - 4\alpha\beta)[A_0, [A_0, [A_0, A_1]]]. \]

Substituting (137), (138), (117) in (136), one obtains

\[ (139) \quad \frac{48}{125}(\alpha + \beta)(6\alpha + \beta)(16\alpha + \beta)[A_0, [A_0, [A_0, A_1]]] = 0. \]

From (98) and (139) it follows that

\[ (140) \quad [A_0, [A_0, [A_0, A_1]]] = 0. \]
From (128), (140) one gets
\[(141)\] \[Y, A_0] = 0, \quad [Y, A_1] = 0.\]
Since the algebra \(\mathfrak{w}_a \cong \mathfrak{k}\) is generated by \(A_0, A_1, Y\), relations (141) yield
\[(142)\] \[[Y, C] = 0 \quad \forall C \in \mathfrak{w}_a.\]

Relations (104), (140) imply
\[(143)\] \[[A_1, Z_0] = 0, \quad [A_0, Z_0] = 0,\]
where \(Z_0\) is given by (113). From (142) we obtain
\[(144)\] \[[Y, Z_0] = 0, \quad [Y, Z_1] = 0,\]
where \(Z_1\) is given by (114). Since the algebra \(\mathfrak{w}_a \cong \mathfrak{k}\) is generated by \(A_0, A_1, Y\), relations (119), (142), (143), (144) yield
\[(145)\] \[[Y, C] = 0, \quad [Z_0, C] = 0, \quad [Z_1, C] = 0 \quad \forall C \in \mathfrak{w}_a.\]
Therefore,
\[(146)\] \(Y, Z_0, Z_1\) belong to the center of the Lie algebra \(\mathfrak{w}_a\).

Lemma 4. Let \(\mathfrak{g} \subset \mathfrak{w}_a\) be the vector subspace spanned by the elements \(E_1, E_2, E_3\) given by (112). Then \(\mathfrak{g}\) is a Lie subalgebra of \(\mathfrak{w}_a\).

Proof. Using relations (104), (115), (130) and the Jacobi identity, one gets
\[(147)\] \[E_2, E_1] = [[A_0, [A_0, A_1]], [A_0, A_1]] = [A_0, [[A_0, [A_0, A_1]], A_1]] = -[A_0, [A_1, [A_0, A_1]]] = 4\alpha[A_0, [A_0, A_1]] = 4\alpha E_2,\]
\[(148)\] \[E_3, E_1] = [[A_1, [A_0, A_1]], [A_0, A_1]] = [A_1, [A_0, [A_0, A_1]]] = -4\alpha[A_1, [A_0, A_1]] = -4\alpha E_3,\]
\[(149)\] \[E_3, E_2] = [[A_1, [A_0, A_1]], [A_0, [A_0, A_1]]] = [A_0, [[A_1, [A_0, A_1]], [A_0, A_1]]] = -4\alpha[A_0, [A_1, [A_0, A_1]]] = 16\alpha^2 E_1.\]

Now we continue the proof of Theorem 6. Relations (104), (115), (130), (142) imply that the algebra \(\mathfrak{w}_a\) is equal to the linear span of the elements
\[(150)\] \(A_0, A_1, Y, [A_0, A_1], [A_0, [A_0, A_1]], [A_1, [A_0, A_1]].\)

Therefore, for any \(\alpha \in \mathbb{K}\), we have \(\dim \mathfrak{w}_a \leq 6\). Now we are going to consider separately the case \(\alpha \neq 0\) and the case \(\alpha = 0\).

The case \(\alpha \neq 0\).

Consider the space \(\mathbb{K}\) with coordinate \(w\). Let \(\mathfrak{L}\) be the 3-dimensional Lie algebra spanned by the following vector fields
\[\frac{\partial}{\partial w}, \quad w\frac{\partial}{\partial w}, \quad w^2\frac{\partial}{\partial w}\]
on \(\mathbb{K}\). It is well known that \(\mathfrak{L}\) is isomorphic to \(\mathfrak{sl}_2(\mathbb{K})\).

Consider the following elements of \(\mathfrak{L}\)
\[(151)\] \[A_0 = -2\alpha w^2\frac{\partial}{\partial w}, \quad A_1 = -\frac{\partial}{\partial w}, \quad Y = 0.\]
Recall that the Lie algebra \(\mathfrak{w}_a \cong \mathfrak{k}\) is given by the generators \(A_0, A_1, Y \in \mathfrak{w}_a\) and relations (104), (105), (106), (107), (108). The vector fields \(\tilde{A}_0, \tilde{A}_1, \tilde{Y} \in \mathfrak{L}\) satisfy (104), (105), (106), (107), (108). Therefore, we can consider the homomorphism
\[(152)\] \(\varphi: \mathfrak{w}_a \rightarrow \mathfrak{L},\)

\[(153) \quad \varphi(A_0) = \tilde{A}_0 = -2\alpha w^2 \frac{\partial}{\partial w}, \quad \varphi(A_1) = \tilde{A}_1 = -\frac{\partial}{\partial w}, \quad \varphi(Y) = \tilde{Y} = 0.\]

Let \(Q_0, Q_1, Q_2\) be a basis of the abelian Lie algebra \(\mathbb{K}^3\). So \([Q_i, Q_j] = 0\) for all \(i, j = 0, 1, 2\). Set
\[(154) \quad \tilde{A}_0 = Q_0, \quad \tilde{A}_1 = Q_1, \quad \tilde{Y} = Q_2.\]
The elements \(\tilde{A}_0, \tilde{A}_1, \tilde{Y} \in \mathbb{K}^3\) satisfy (103), (105), (106), (107), (108), because \([\tilde{A}_0, \tilde{A}_1] = 0, [\tilde{A}_0, \tilde{Y}] = 0, [\tilde{A}_1, \tilde{Y}] = 0\). Therefore, we have the homomorphism
\[(155) \quad \psi: \mathfrak{W}_a \rightarrow \mathbb{K}^3, \]
\[(156) \quad \psi(A_0) = \tilde{A}_0 = Q_0, \quad \psi(A_1) = \tilde{A}_1 = Q_1, \quad \psi(Y) = \tilde{Y} = Q_2.\]

Consider also the homomorphism
\[(157) \quad \rho: \mathfrak{W}_a \rightarrow \mathfrak{L} \oplus \mathbb{K}^3, \quad \rho(C) = \varphi(C) + \psi(C), \quad C \in \mathfrak{W}_a.\]

Using (153), (156), (157), we get
\[(158) \quad \rho([A_0, A_1]) = -4\alpha w \frac{\partial}{\partial w}, \quad \rho([A_0, [A_0, A_1]]) = -8\alpha^2 w^2 \frac{\partial}{\partial w}, \quad \rho([A_1, [A_0, A_1]]) = 4\alpha \frac{\partial}{\partial w},\]
\[(159) \quad \rho(Y) = Q_2, \quad \rho(Z_0) = 4\alpha Q_0, \quad \rho(Z_1) = 4\alpha Q_1,\]
where \(Z_0, Z_1\) are given by (113), (114).

As we assume \(\alpha \neq 0\), formulas (158), (159) imply that the homomorphism (157) is surjective. Since \(\dim (\mathfrak{L} \oplus \mathbb{K}^3) = 6\) and \(\dim \mathfrak{W}_a \leq 6\), we see that the homomorphism (157) is an isomorphism. Then, as \(\mathfrak{L} \cong \mathfrak{sl}(\mathbb{K})\), we obtain
\[(160) \quad \mathfrak{W}_a \cong \mathfrak{L} \oplus \mathbb{K}^3 \cong \mathfrak{sl}(\mathbb{K}) \oplus \mathbb{K}^3.\]

Property (146), Lemma 4 and formulas (158), (159) imply that the subalgebra
\[\mathfrak{sl}(\mathbb{K}) \subset \mathfrak{W}_a \cong \mathfrak{sl}(\mathbb{K}) \oplus \mathbb{K}^3\]
is spanned by the elements (112), and the subalgebra
\[\mathbb{K}^3 \subset \mathfrak{W}_a \cong \mathfrak{sl}(\mathbb{K}) \oplus \mathbb{K}^3\]
is spanned by the elements \(Y, Z_0, Z_1\).

**The case \(\alpha = 0\).**

For any vector subspace \(V \subset \mathfrak{W}_a\), we can consider the vector subspace \([\mathfrak{W}_a, V] \subset \mathfrak{W}_a\) spanned by the elements of the form \([A, B]\), where \(A \in \mathfrak{W}_a\) and \(B \in V\).

As has been shown above, the algebra \(\mathfrak{W}_a\) is equal to the linear span of the elements (150). Combining this fact with relations (104), (125), (140), (142), (147), (148), (149) and the assumption \(\alpha = 0\), we get the following.

- The subalgebra \(\mathfrak{W}_a^1 = [\mathfrak{W}_a, \mathfrak{W}_a] \subset \mathfrak{W}_a\) is equal to the linear span of the elements

\([A_0, A_1], [A_0, [A_0, A_1]], [A_1, [A_0, A_1]]\).

- The subalgebra \(\mathfrak{W}_a^2 = [\mathfrak{W}_a, \mathfrak{W}_a^1] \subset \mathfrak{W}_a\) is equal to the linear span of the elements

\([A_0, [A_0, A_1]], [A_1, [A_0, A_1]]\).

- One has \([\mathfrak{W}_a, \mathfrak{W}_a^2] = 0\), hence \(\mathfrak{W}_a\) is nilpotent.

\(\square\)

**Theorem 7.** Let \(\mathcal{E}\) be the infinite prolongation of equation (97). Let \(a \in \mathcal{E}\). Then one has the following.

- If \(\alpha, \beta\) satisfy (98) and \(\alpha \neq 0\), then the algebra \(\mathbb{F}^0(\mathcal{E}, a)\) is isomorphic to the direct sum of the 3-dimensional simple Lie algebra \(\mathfrak{sl}(\mathbb{K})\) and an abelian Lie algebra of dimension \(\leq 3\).

- If \(\alpha = 0\) and \(\beta \neq 0\), the Lie algebra \(\mathbb{F}^0(\mathcal{E}, a)\) is nilpotent and is of dimension \(\leq 6\).

**Proof.** Let \(\mathfrak{W}_a\) be the WE algebra of equation (97). According to Theorem 5, the algebra \(\mathbb{F}^0(\mathcal{E}, a)\) is isomorphic to the subalgebra \(\mathfrak{K} \subset \mathfrak{W}_a\) defined in Theorem 5. Applying Theorem 5 to the description of \(\mathfrak{W}_a\) presented in Theorem 6, we get the statements of Theorem 7. \(\square\)
5. The structure of $\mathbb{F}^p(\mathcal{E}, a)$ for some equations of orders 3 and 5

Recall that $\mathbb{K}$ is either $\mathbb{C}$ or $\mathbb{R}$. Consider the infinite-dimensional Lie algebra $\mathfrak{sl}_2(\mathbb{K}[\lambda]) \cong \mathfrak{sl}_2(\mathbb{K}) \otimes_{\mathbb{K}} \mathbb{K}[\lambda]$, where $\mathbb{K}[\lambda]$ is the algebra of polynomials in $\lambda$. Recall that we use the notation [2].

The following result is proved in [14].

**Theorem 8** ([14]). Let $\mathcal{E}$ be the infinite prolongation of the KdV equation $u_t = u_3 + u_0 u_1$. Let $a \in \mathcal{E}$. For each $p \in \mathbb{Z}_{>0}$, consider the surjective homomorphism $\varphi_p : \mathbb{F}^p(\mathcal{E}, a) \to \mathbb{F}^{p-1}(\mathcal{E}, a)$ from (88).

For each $k \in \mathbb{Z}_{>0}$, let $\psi_k : \mathbb{F}^k(\mathcal{E}, a) \to \mathbb{F}^0(\mathcal{E}, a)$ be the composition of the homomorphisms

$$\mathbb{F}^k(\mathcal{E}, a) \to \mathbb{F}^{k-1}(\mathcal{E}, a) \to \cdots \to \mathbb{F}^1(\mathcal{E}, a) \to \mathbb{F}^0(\mathcal{E}, a)$$

from (88). Then one has the following.

- The algebra $\mathbb{F}^0(\mathcal{E}, a)$ is isomorphic to the direct sum of $\mathfrak{sl}_2(\mathbb{K}[\lambda])$ and a 3-dimensional abelian Lie algebra.
- For each $p \in \mathbb{Z}_{>0}$, the kernel of $\varphi_p$ is contained in the center of the Lie algebra $\mathbb{F}^p(\mathcal{E}, a)$, that is,

$$[v_1, v_2] = 0 \quad \forall v_1 \in \ker \varphi_p, \quad \forall v_2 \in \mathbb{F}^p(\mathcal{E}, a).$$

- The kernel of $\psi_k$ is nilpotent.

**Remark 27.** In the proof of this theorem in [14], we use the fact that the explicit structure of the WE algebra for the KdV equation is known [6] [7] and contains $\mathfrak{sl}_2(\mathbb{K}[\lambda])$.

Let $\mathcal{L}, \mathcal{L}_1, \mathcal{L}_2$ be Lie algebras. One says that $\mathcal{L}_1$ is obtained from $\mathcal{L}$ by central extension if there is an ideal $\mathcal{I} \subset \mathcal{L}_1$ such that $\mathcal{I}$ is contained in the center of $\mathcal{L}_1$ and $\mathcal{L}_1/\mathcal{I} \cong \mathcal{L}$. Note that $\mathcal{I}$ may be of arbitrary dimension.

We say that $\mathcal{L}_2$ is obtained from $\mathcal{L}$ by applying several times the operation of central extension if there is a finite collection of Lie algebras $\mathfrak{g}_0, \mathfrak{g}_1, \ldots, \mathfrak{g}_k$ such that $\mathfrak{g}_0 \cong \mathcal{L}$, $\mathfrak{g}_k \cong \mathcal{L}_2$ and $\mathfrak{g}_i$ is obtained from $\mathfrak{g}_{i-1}$ by central extension for each $i = 1, \ldots, k$.

**Remark 28.** For the KdV equation, Theorem 8 implies that $\mathbb{F}^0(\mathcal{E}, a)$ is contained in $\mathfrak{sl}_2(\mathbb{K}[\lambda])$ by central extension, and $\mathbb{F}^p(\mathcal{E}, a)$ is obtained from $\mathbb{F}^{p-1}(\mathcal{E}, a)$ by central extension for each $p \in \mathbb{Z}_{>0}$. Therefore, for each $k \in \mathbb{Z}_{>0}$, the algebra $\mathbb{F}^k(\mathcal{E}, a)$ for the KdV equation is obtained from $\mathfrak{sl}_2(\mathbb{K}[\lambda])$ by applying several times the operation of central extension.

**Lemma 5.** Let $\mathcal{E}$ be the infinite prolongation of equation (97). Let $a \in \mathcal{E}$. Then one has the following.

1. The kernel of the surjective homomorphism $\varphi_1 : \mathbb{F}^1(\mathcal{E}, a) \to \mathbb{F}^0(\mathcal{E}, a)$ from (88) is contained in the center of the Lie algebra $\mathbb{F}^1(\mathcal{E}, a)$, that is,

$$[v_1, v_2] = 0 \quad \forall v_1 \in \ker \varphi_1, \quad \forall v_2 \in \mathbb{F}^1(\mathcal{E}, a).$$

(In particular, this implies that the Lie algebra $\mathbb{F}^1(\mathcal{E}, a)$ is obtained from $\mathbb{F}^0(\mathcal{E}, a)$ by central extension.)

2. If $\alpha, \beta$ satisfy (98) and $\alpha \neq 0$, then the algebra $\mathbb{F}^0(\mathcal{E}, a)$ is isomorphic to the direct sum of the 3-dimensional simple Lie algebra $\mathfrak{sl}_2(\mathbb{K})$ and an abelian Lie algebra of dimension $\leq 3$.

3. If $\alpha = 0$ and $\beta \neq 0$, the Lie algebra $\mathbb{F}^0(\mathcal{E}, a)$ is nilpotent and is of dimension $\leq 6$.

**Proof.** Equation (97) belongs to the following class of equations

$$u_t = u_5 + f(x, t, u_0, u_1, u_2, u_3).$$

For equations of the form (162), it is shown in [14] that the kernel of the homomorphism

$$\varphi_p : \mathbb{F}^p(\mathcal{E}, a) \to \mathbb{F}^{p-1}(\mathcal{E}, a)$$

from (88) is contained in the center of the Lie algebra $\mathbb{F}^0(\mathcal{E}, a)$ for all $p \geq 2$, that is,

$$[v_1, v_2] = 0 \quad \forall v_1 \in \ker \varphi_p, \quad \forall v_2 \in \mathbb{F}^p(\mathcal{E}, a).$$

For equation (97), the arguments used in the proof of (163) in [14] work also in the case $p = 1$, so we get (161). The statements about $\mathbb{F}^0(\mathcal{E}, a)$ have been proved in Theorem [7].
Using Lemma [5] in [14] we prove the following result.

**Theorem 9** ([14]). Let \( E \) be the infinite prolongation of equation (97). Let \( a \in E \). Then one has the following.

- For any \( p \in \mathbb{Z}_{\geq 0} \), the kernel of the surjective homomorphism \( \mathbb{F}^p(E, a) \to \mathbb{F}^0(E, a) \) from (8) is nilpotent. The algebra \( \mathbb{F}^p(E, a) \) is obtained from the algebra \( \mathbb{F}^0(E, a) \) by applying several times the operation of central extension.
- If (98) holds and \( \alpha \neq 0 \), then \( \mathbb{F}^0(E, a) \) is isomorphic to the direct sum of \( \mathfrak{s}_2(\mathbb{K}) \) and an abelian Lie algebra of dimension \( \leq 3 \), and for each \( p \in \mathbb{Z}_{\geq 0} \) there is a surjective homomorphism \( \mathbb{F}^p(E, a) \to \mathfrak{s}_2(\mathbb{K}) \) with nilpotent kernel.
- If \( \alpha = 0 \) and \( \beta \neq 0 \), the Lie algebra \( \mathbb{F}^p(E, a) \) is nilpotent for all \( p \in \mathbb{Z}_{\geq 0} \).

In the rest of this section we assume \( \mathbb{K} = \mathbb{C} \). To study \( \mathbb{F}^p(E, a) \) for the Krichever-Novikov equation (36), we need some auxiliary constructions.

Let \( \mathbb{C}[v_1, v_2, v_3] \) be the algebra of polynomials in the variables \( v_1, v_2, v_3 \). Let \( e_1, e_2, e_3 \in \mathbb{C} \) such that \( e_1 \neq e_2 \neq e_3 \neq e_1 \). Consider the ideal \( \mathcal{I}_{e_1,e_2,e_3} \subset \mathbb{C}[v_1, v_2, v_3] \) generated by the polynomials
\[
(164) \quad v_i^2 - v_j^2 + e_i - e_j, \quad i, j = 1, 2, 3.
\]

Set
\[
(165) \quad E_{e_1,e_2,e_3} = \mathbb{C}[v_1, v_2, v_3]/\mathcal{I}_{e_1,e_2,e_3}.
\]

In other words, \( E_{e_1,e_2,e_3} \) is the commutative associative algebra of polynomial functions on the algebraic curve in \( \mathbb{C}^3 \) defined by the polynomials (164). (This curve is given by the equations \( v_i^2 - v_j^2 + e_i - e_j = 0 \), \( i, j = 1, 2, 3 \), in the space \( \mathbb{C}^3 \) with coordinates \( v_1, v_2, v_3 \).

Since we assume \( e_1 \neq e_2 \neq e_3 \neq e_1 \), this curve is nonsingular, irreducible and is of genus 1, so this is an elliptic curve. It is known that the Landau-Lifshitz equation and the Krichever-Novikov equation possess \( \mathfrak{so}_3(\mathbb{C}) \)-valued ZCRs parametrized by points of this curve [36] [38] [22] [31]. (For the Krichever-Novikov equation, the paper [26] presents a ZCR with values in the Lie algebra \( \mathfrak{s}_2(\mathbb{C}) \cong \mathfrak{so}_3(\mathbb{C}) \).

We have the natural surjective homomorphism \( \rho : \mathbb{C}[v_1, v_2, v_3] \to \mathbb{C}[v_1, v_2, v_3]/\mathcal{I}_{e_1,e_2,e_3} = E_{e_1,e_2,e_3} \). Set
\[
(166) \quad \hat{v}_i = \rho(v_i) \in E_{e_1,e_2,e_3} \text{ for } i = 1, 2, 3.
\]

Consider also a basis \( \alpha_1, \alpha_2, \alpha_3 \) of the Lie algebra \( \mathfrak{so}_3(\mathbb{C}) \) such that
\[
[\alpha_1, \alpha_2] = \alpha_3, \quad [\alpha_2, \alpha_3] = \alpha_1, \quad [\alpha_3, \alpha_1] = \alpha_2.
\]

We endow the space \( \mathfrak{so}_3(\mathbb{C}) \otimes \mathbb{C} E_{e_1,e_2,e_3} \) with the following Lie algebra structure
\[
[\alpha \otimes h_1, \beta \otimes h_2] = [\alpha, \beta] \otimes h_1 h_2, \quad \alpha, \beta \in \mathfrak{so}_3(\mathbb{C}), \quad h_1, h_2 \in E_{e_1,e_2,e_3}.
\]

Denote by \( \mathfrak{R}_{e_1,e_2,e_3} \) the Lie subalgebra of \( \mathfrak{so}_3(\mathbb{C}) \otimes \mathbb{C} E_{e_1,e_2,e_3} \) generated by the elements
\[
\alpha_i \otimes \hat{v}_i \in \mathfrak{so}_3(\mathbb{C}) \otimes \mathbb{C} E_{e_1,e_2,e_3}, \quad i = 1, 2, 3.
\]

Since \( \mathfrak{R}_{e_1,e_2,e_3} \subset \mathfrak{so}_3(\mathbb{C}) \otimes \mathbb{C} E_{e_1,e_2,e_3} \), we can regard elements of \( \mathfrak{R}_{e_1,e_2,e_3} \) as \( \mathfrak{so}_3(\mathbb{C}) \)-valued functions on the elliptic curve in \( \mathbb{C}^3 \) determined by the polynomials (164).

Set \( z = \hat{v}_1^2 + e_1 \). Since \( \hat{v}_1^2 + e_1 = \hat{v}_2^2 + e_2 = \hat{v}_3^2 + e_3 \) in \( E_{e_1,e_2,e_3} \), we have
\[
(167) \quad z = \hat{v}_1^2 + e_1 = \hat{v}_2^2 + e_2 = \hat{v}_3^2 + e_3.
\]

It is easily seen (and is shown in [31]) that the following elements form a basis for \( \mathfrak{R}_{e_1,e_2,e_3} \)
\[
(168) \quad \alpha_i \otimes \hat{v}_j z^l, \quad \alpha_i \otimes \hat{v}_j \hat{v}_k z^l, \quad i, j, k \in \{1, 2, 3\}, \quad j < k, \quad j \neq i \neq k, \quad l \in \mathbb{Z}_{\geq 0}.
\]

As the basis (168) is infinite, the Lie algebra \( \mathfrak{R}_{e_1,e_2,e_3} \) is infinite-dimensional.

It is shown in [31] that the Wahlquist-Estabrook prolongation algebra of the Landau-Lifshitz equation is isomorphic to the direct sum of \( \mathfrak{R}_{e_1,e_2,e_3} \) and a 2-dimensional abelian Lie algebra. According to Theorem [10] below, the algebra \( \mathfrak{R}_{e_1,e_2,e_3} \) shows up also in the structure of \( \mathbb{F}^p(E, a) \) for the Krichever-Novikov equation.

**Theorem 10.** For any \( e_1, e_2, e_3 \in \mathbb{C} \), consider the Krichever-Novikov equation KN\((e_1, e_2, e_3)\) given by (36). Let \( E \) be the infinite prolongation of this equation. Let \( a \in E \). Then one has the following.
The algebra $\mathbb{F}^0(\mathcal{E}, a)$ is zero.

For each $p \geq 2$, the kernel of the surjective homomorphism $\varphi_p: \mathbb{F}^p(\mathcal{E}, a) \to \mathbb{F}^{p-1}(\mathcal{E}, a)$ from (88) is contained in the center of the Lie algebra $\mathbb{F}^p(\mathcal{E}, a)$, that is,

$$[v_1, v_2] = 0 \quad \forall v_1 \in \ker \varphi_p, \quad \forall v_2 \in \mathbb{F}^p(\mathcal{E}, a).$$

(In particular, this implies that the algebra $\mathbb{F}^p(\mathcal{E}, a)$ is obtained from $\mathbb{F}^{p-1}(\mathcal{E}, a)$ by central extension.)

The kernel of the surjective homomorphism $\mathbb{F}^p(\mathcal{E}, a) \to \mathbb{F}^1(\mathcal{E}, a)$ from (88) is nilpotent.

If $e_1 \neq e_2 \neq e_3 \neq e_1$, then $\mathbb{F}^1(\mathcal{E}, a) \cong \mathfrak{K}_{1, e_2, e_3}$ and for each $p \geq 2$ the algebra $\mathbb{F}^p(\mathcal{E}, a)$ is obtained from $\mathfrak{K}_{e_1, e_2, e_3}$ by applying several times the operation of central extension.

Proof. In this version of the preprint we present only a sketch of the proof. A more detailed proof will be added later in an updated version.

Using the notation (2), we can rewrite the Krichever-Novikov equation (36) in the form (61), (62). According to the definition of $D_t$, for this equation we have

$$D_t = \frac{\partial}{\partial t} + \sum_{k \geq 0} D_k x \left( u_3 - \frac{3}{2} \left( u_2 \right)^2 \frac{\partial}{\partial u_1} + \frac{u_0 - e_1}{u_1} \left( (u_0 - e_2)(u_0 - e_3) \right) \frac{\partial}{\partial u_k} \right).$$

According to the definition of the algebras $\mathbb{F}^p(\mathcal{E}, a)$ in the case $p = 0$, $d = 3$, for the Krichever-Novikov equation, the algebra $\mathbb{F}^0(\mathcal{E}, a)$ is generated by the elements

$$A_{i_0}^{l_1, l_2} = 0, \quad B_{j_0 j_1 j_2}^{l_1, l_2} = 0, \quad l_1, l_2, i_0, j_0, j_1, j_2 \in \mathbb{Z}_{\geq 0}.$$ 

Relations (83) in the case $p = 0$, $d = 3$ say that

$$A_{i_0}^{l_1, l_2} = 0, \quad B_{j_0 j_1 j_2}^{l_1, l_2} = 0, \quad \forall l_1, l_2.$$

Since equation (36) is invariant with respect to the change of variables $x \mapsto x - x_a$, $t \mapsto t - t_a$, we can assume $x_a = t_a = 0$ in (63). In view of (170), (171), and $x_a = t_a = 0$, in the case $p = 0$, $d = 3$ the power series (79), (80), (82) are written as

$$A_{i_0}^{l_1, l_2} = 0, \quad B_{j_0 j_1 j_2}^{l_1, l_2} = 0, \quad \forall l_1, l_2.$$

A straightforward study of (171), (171) shows that equations (171), (171) imply $A_{i_0}^{l_1, l_2} = 0$ for all $l_1, l_2, i_0, j_0, j_1, j_2$. Hence $\mathbb{F}^0(\mathcal{E}, a) = 0$.

For the Krichever-Novikov equation $KN(e_1, e_2, e_3)$ given by (36), the algebra $\mathbb{F}^1(\mathcal{E}, a)$ is responsible for ZCRs of the form

$$A = A(x, t, u_0, u_1), \quad B = B(x, t, u_0, u_1, u_2, u_3), \quad D_x(B) - D_t(A) + [A, B] = 0.$$

For this equation, the paper [17] constructed a somewhat similar Lie algebra which is responsible for ZCRs of the form

$$A = A(u_0, u_1), \quad B = B(u_0, u_1, u_2, u_3), \quad D_x(B) - D_t(A) + [A, B] = 0.$$

In the case $e_1 \neq e_2 \neq e_3 \neq e_1$ it is shown in [17] that this Lie algebra is isomorphic to the direct sum of $\mathfrak{K}_{e_1, e_2, e_3}$ and a 2-dimensional abelian Lie algebra. Similarly to this result, one can prove that in the case $e_1 \neq e_2 \neq e_3 \neq e_1$ we have $\mathbb{F}^1(\mathcal{E}, a) \cong \mathfrak{K}_{e_1, e_2, e_3}$. For each $p \in \mathbb{Z}_{>0}$, consider the surjective homomorphism $\varphi_p: \mathbb{F}^p(\mathcal{E}, a) \to \mathbb{F}^{p-1}(\mathcal{E}, a)$ from (88). For equations of the form $u_t = u_3 + f(x, t, u_0, u_1)$, property (169) is proved in [14] for all $p \geq 1$. For the
Krichever-Novikov equation, the arguments from [14] (with some small modifications) allow one to prove property (169) for all \( p \geq 2 \). In particular, this means that, for each \( p \geq 2 \), the algebra \( \mathbb{F}^p(\mathcal{E}, a) \) is obtained from the algebra \( \mathbb{F}^{p-1}(\mathcal{E}, a) \) by central extension.

So we have property (169) for all \( p \geq 2 \). It is easily seen that this implies that the kernel of the homomorphism \( \mathbb{F}^p(\mathcal{E}, a) \to \mathbb{F}^1(\mathcal{E}, a) \) from (88) is nilpotent, because this homomorphism is equal to the composition of the homomorphisms

\[
\mathbb{F}^p(\mathcal{E}, a) \xrightarrow{\phi_p} \mathbb{F}^{p-1}(\mathcal{E}, a) \xrightarrow{\phi_{p-1}} \cdots \xrightarrow{\phi_2} \mathbb{F}^1(\mathcal{E}, a) \xrightarrow{\phi_1} \mathbb{F}^0(\mathcal{E}, a)
\]

from (88).

As has been shown above, in the case \( e_1 \neq e_2 \neq e_3 \neq e_1 \) we have \( \mathbb{F}^1(\mathcal{E}, a) \cong \mathfrak{R}_{e_1,e_2,e_3} \), and for each \( p \geq 2 \) the algebra \( \mathbb{F}^p(\mathcal{E}, a) \) is obtained from the algebra \( \mathbb{F}^{p-1}(\mathcal{E}, a) \) by central extension. This implies that, in the case \( e_1 \neq e_2 \neq e_3 \neq e_1 \), for each \( p \geq 2 \) the algebra \( \mathbb{F}^p(\mathcal{E}, a) \) is obtained from \( \mathfrak{R}_{e_1,e_2,e_3} \) by applying several times the operation of central extension. \( \square \)

6. Some algebraic constructions

In this section we present some auxiliary algebraic constructions and results, which will be needed in our study of Bäcklund transformations in the next sections.

6.1. Lie algebras with topology and quasi-solvable elements. As has been said in Section 1.3, all vector spaces and algebras are supposed to be over the field \( \mathbb{K} \). Since \( \mathbb{K} \) is either \( \mathbb{C} \) or \( \mathbb{R} \), we have the standard topology on \( \mathbb{K} \).

This allows us to speak about Lie algebras with topology. A Lie algebra \( \mathcal{L} \) with topology is a topological vector space \( \mathcal{L} \) over \( \mathbb{K} \) with a Lie bracket such that the Lie bracket is continuous with respect to the topology on \( \mathcal{L} \).

Example 8. Let \( \mathcal{E} \) be the infinite prolongation of an evolution equation. Let \( a \in \mathcal{E} \). In Section 1.2 we have defined the Lie algebra \( \mathbb{F}(\mathcal{E}, a) \) and the topology on \( \mathbb{F}(\mathcal{E}, a) \). It is easy to check that \( \mathbb{F}(\mathcal{E}, a) \) is a Lie algebra with topology in the above sense.

Let \( \mathcal{L} \) be a Lie algebra with topology. For any ideal \( \mathcal{I} \subset \mathcal{L} \), we denote by \( \pi_3 \) the natural surjective homomorphism \( \pi_3: \mathcal{L} \to \mathcal{L}/\mathcal{I} \).

An ideal \( \mathcal{I} \subset \mathcal{L} \) is called an open ideal if \( \mathcal{I} \) is open in \( \mathcal{L} \) with respect to the topology on \( \mathcal{L} \).

An element \( w \in \mathcal{L} \) is said to be quasi-solvable if, for any open ideal \( \mathcal{I} \subset \mathcal{L} \), the ideal generated by \( \pi_3(w) \) in \( \mathcal{L}/\mathcal{I} \) is solvable. (Note that the ideal generated by \( w \) in \( \mathcal{L} \) is not necessarily solvable, and we do not consider any topology on \( \mathcal{L}/\mathcal{I} \).)

Let \( \mathcal{Q}(\mathcal{L}) \subset \mathcal{L} \) be the subset of all quasi-solvable elements of \( \mathcal{L} \). It is easily seen that \( \mathcal{Q}(\mathcal{L}) \) is an ideal of the Lie algebra \( \mathcal{L} \). We set \( \mathcal{G}(\mathcal{L}) = \mathcal{L}/\mathcal{Q}(\mathcal{L}) \). We do not consider any topology on \( \mathcal{G}(\mathcal{L}) \).

Lemma 6. Consider the infinite-dimensional Lie algebra \( \mathfrak{sl}_2(\mathbb{K}[\lambda]) \cong \mathfrak{sl}_2(\mathbb{K}) \otimes_{\mathbb{K}} \mathbb{K}[\lambda] \). Let \( \mathfrak{H} \subset \mathfrak{sl}_2(\mathbb{K}[\lambda]) \) be a subalgebra of finite codimension. Then any solvable ideal of the Lie algebra \( \mathfrak{H} \) is zero.

Proof. For each \( c \in \mathbb{K} \), we have the surjective homomorphism \( \eta_c: \mathfrak{sl}_2(\mathbb{K}[\lambda]) \cong \mathfrak{sl}_2(\mathbb{K}) \otimes_{\mathbb{K}} \mathbb{K}[\lambda] \to \mathfrak{sl}_2(\mathbb{K}), \quad \eta_c(y \otimes f(\lambda)) = f(c)y, \quad y \in \mathfrak{sl}_2(\mathbb{K}), \quad f(\lambda) \in \mathbb{K}[\lambda]. \)

Let \( b_1, b_2, b_3 \) be a basis of \( \mathfrak{sl}_2(\mathbb{K}) \). Since \( \mathfrak{H} \) is of finite codimension in \( \mathfrak{sl}_2(\mathbb{K}[\lambda]) \), there are nonzero polynomials \( f_i(\lambda) \in \mathbb{K}[\lambda], i = 1, 2, 3, \) such that

\[
(177) \quad b_i \otimes f_i(\lambda) \in \mathfrak{H}, \quad i = 1, 2, 3.
\]

Suppose that there is a nonzero solvable ideal \( I \subset \mathfrak{H} \). Consider a nonzero element \( \gamma \in I \). We have

\[
\gamma = b_1 \otimes g_1(\lambda) + b_2 \otimes g_2(\lambda) + b_3 \otimes g_3(\lambda)
\]

for some \( g_1(\lambda), g_2(\lambda), g_3(\lambda) \in \mathbb{K}[\lambda] \). As \( \gamma \neq 0 \), there is \( k \in \{1, 2, 3\} \) such that the polynomial \( g_k(\lambda) \) is nonzero.

Let \( c \in \mathbb{K} \) such that \( f_i(c) \neq 0, i = 1, 2, 3, \) and \( g_k(c) \neq 0 \). Then

\[
(178) \quad \eta_c(\mathfrak{H}) = \mathfrak{sl}_2(\mathbb{K}), \quad \eta_c(\gamma) \neq 0.
\]
Since $\gamma$ belongs to a solvable ideal of $\mathfrak{H}$, relations (178) imply that the element $\eta_{\epsilon}(\gamma)$ generates a nonzero solvable ideal in $\mathfrak{s}\mathfrak{l}_2(\mathbb{K})$. This contradicts the fact that $\mathfrak{s}\mathfrak{l}_2(\mathbb{K})$ is a simple Lie algebra. \hfill \Box

**Remark 29.** Taking $\mathfrak{H} = \mathfrak{s}\mathfrak{l}_2(\mathbb{K}[\lambda])$ in Lemma 6 we see that there are no nonzero solvable ideals in $\mathfrak{s}\mathfrak{l}_2(\mathbb{K}[\lambda])$.

**Lemma 7.** In this lemma we assume $\mathbb{K} = \mathbb{C}$. Let $e_1, e_2, e_3 \in \mathbb{C}$ such that $e_1 \neq e_2 \neq e_3 \neq e_1$. Consider the surjective homomorphism $\mathfrak{H} \to \mathfrak{H}$ and $\mathfrak{F}$ in Section 6.2. Associative algebras related to Lie algebras.

Let $\mathfrak{H} \subset \mathfrak{R}_{e_1, e_2, e_3}$ be a subalgebra of finite codimension. Then any solvable ideal of the Lie algebra $\mathfrak{H}$ is zero.

**Proof.** In Section 5 we have described the explicit structure of $\mathfrak{R}_{e_1, e_2, e_3}$. Using this description, one can prove Lemma 7 similarly to Lemma 6. \hfill \Box

**Remark 30.** Taking $\mathfrak{H} = \mathfrak{R}_{e_1, e_2, e_3}$ in Lemma 7 we see that there are no nonzero solvable ideals in $\mathfrak{R}_{e_1, e_2, e_3}$.

**Example 9.** Let $\mathcal{E}$ be the infinite prolongation of the KdV equation. According to Theorem 8 one has

$$\mathbb{F}^0(\mathcal{E}, a) \simeq \mathfrak{s}\mathfrak{l}_2(\mathbb{K}[\lambda]) \oplus \mathbb{K}^3,$$

where $\mathbb{K}^3$ is a 3-dimensional abelian Lie algebra.

We have the surjective homomorphism $\rho_0: \mathbb{F}(\mathcal{E}, a) \to \mathbb{F}^0(\mathcal{E}, a)$ defined by (31) in the case $k = 0$. Consider the surjective homomorphism $\psi: \mathbb{F}(\mathcal{E}, a) \to \mathfrak{s}\mathfrak{l}_2(\mathbb{K}[\lambda])$ equal to the composition of

$$\mathbb{F}(\mathcal{E}, a) \xrightarrow{\rho_0} \mathbb{F}^0(\mathcal{E}, a) \simeq \mathfrak{s}\mathfrak{l}_2(\mathbb{K}[\lambda]) \oplus \mathbb{K}^3 \to \mathfrak{s}\mathfrak{l}_2(\mathbb{K}[\lambda]).$$

The definition of the topology on $\mathbb{F}(\mathcal{E}, a)$, Lemma 1, Theorem 8, and Remark 29 imply that an element $w \in \mathbb{F}(\mathcal{E}, a)$ is quasi-solvable iff $w \in \ker \psi$. This yields $\mathcal{G}(\mathbb{F}(\mathcal{E}, a)) \simeq \mathfrak{s}\mathfrak{l}_2(\mathbb{K}[\lambda])$.

**Example 10.** Let $e_1, e_2, e_3 \in \mathbb{C}$ such that $e_1 \neq e_2 \neq e_3 \neq e_1$. Let $\mathcal{E}$ be the infinite prolongation of the Krichchev-Novikov equation $\text{KN}(e_1, e_2, e_3)$ given by (35). According to Theorem 10 one has $\mathbb{F}^0(\mathcal{E}, a) = 0$ and $\mathbb{F}^1(\mathcal{E}, a) \simeq \mathfrak{R}_{e_1, e_2, e_3}$.

We have the surjective homomorphism $\rho_1: \mathbb{F}(\mathcal{E}, a) \to \mathbb{F}^1(\mathcal{E}, a)$ defined by (31) in the case $k = 1$. Consider the surjective homomorphism $\mu: \mathbb{F}(\mathcal{E}, a) \to \mathfrak{R}_{e_1, e_2, e_3}$ equal to the composition of

$$\mathbb{F}(\mathcal{E}, a) \xrightarrow{\rho_1} \mathbb{F}^1(\mathcal{E}, a) \simeq \mathfrak{R}_{e_1, e_2, e_3}.$$

The definition of the topology on $\mathbb{F}(\mathcal{E}, a)$, Lemma 1, Theorem 10, and Remark 30 imply that an element $w \in \mathbb{F}(\mathcal{E}, a)$ is quasi-solvable iff $w \in \ker \mu$. This yields $\mathcal{G}(\mathbb{F}(\mathcal{E}, a)) \simeq \mathfrak{R}_{e_1, e_2, e_3}$.

### 6.2. Associative algebras related to Lie algebras.

Let $\mathcal{L}$ be a Lie algebra. (In this subsection we do not consider any topology on $\mathcal{L}$.) Consider a linear map $g: \mathcal{L} \to \mathcal{L}$ satisfying

$$g([p_1, p_2]) = [g(p_1), p_2] = [p_1, g(p_2)] \quad \forall p_1, p_2 \in \mathcal{L}. \tag{179}$$

Property (179) is equivalent to

$$g \circ \text{ad}(p_1) = \text{ad}(p_1) \circ g \quad \forall p_1 \in \mathcal{L}, \tag{180}$$

where the map $\text{ad}(p_1): \mathcal{L} \to \mathcal{L}$ is given by the standard formula $\text{ad}(p_1)(p_2) = [p_1, p_2]$ for all $p_2 \in \mathcal{L}$. Relation (180) means that the map $g: \mathcal{L} \to \mathcal{L}$ is an intertwining operator for the adjoint representation of $\mathcal{L}$.

Such operators are often used in the study of integrable PDEs with Lax pairs (e.g., for construction of Poisson structures [28] and symmetry recursion operators [4]).

Instead of operators $g: \mathcal{L} \to \mathcal{L}$, we need to consider linear maps $h: \mathfrak{H} \to \mathcal{L}$, where $\mathfrak{H} \subset \mathcal{L}$ is a Lie subalgebra. We fix the Lie algebra $\mathcal{L}$ and study linear maps defined on Lie subalgebras of $\mathcal{L}$ of finite codimension, as follows.

An *admissible pair* is a pair $(\mathfrak{H}, \mathfrak{Y})$, where $\mathfrak{H} \subset \mathcal{L}$ is a Lie subalgebra of finite codimension and $h: \mathfrak{H} \to \mathcal{L}$ is a linear map satisfying $h([p_1, p_2]) = [h(p_1), p_2] = [p_1, h(p_2)]$ for any $p_1, p_2 \in \mathfrak{H}$. 
Let \((\hat{h}, \hat{\mathfrak{F}})\) be another admissible pair. So \(\hat{\mathfrak{H}} \subset \mathfrak{L}\) is a subalgebra of finite codimension and \(\hat{h} : \hat{\mathfrak{F}} \to \mathfrak{L}\) is a linear map satisfying \(\hat{h}([p_1, p_2]) = [\hat{h}(p_1), p_2] = [p_1, \hat{h}(p_2)]\) for any \(p_1, p_2 \in \hat{\mathfrak{F}}\).

Admissible pairs \((h, \mathfrak{F})\) and \((\hat{h}, \hat{\mathfrak{F}})\) are called equivalent if there is a subalgebra \(\mathfrak{U} \subset \mathfrak{F} \cap \hat{\mathfrak{F}}\) of finite codimension such that \(h(w) = \hat{h}(w)\) for all \(w \in \mathfrak{U}\). It is easy to check that this is indeed an equivalence relation. For each admissible pair \((h, \mathfrak{F})\), we denote by \([(h, \mathfrak{F})]\) the corresponding equivalence class. So \((h, \mathfrak{F})\) and \((\hat{h}, \hat{\mathfrak{F}})\) are equivalent iff \([(h, \mathfrak{F})] = [(\hat{h}, \hat{\mathfrak{F}})]\).

Let \(\mathbb{P}(\mathfrak{L})\) be the set of such equivalence classes. So for each admissible pair \((h, \mathfrak{F})\) we have \([(h, \mathfrak{F})] \in \mathbb{P}(\mathfrak{L})\).

Note that, for any subalgebra \(\mathfrak{F} \subset \mathfrak{L}\) of finite codimension, the pair \((0, \mathfrak{F})\) is admissible, where 0: \(\mathfrak{F} \to \mathfrak{L}\) is the zero map. For any other subalgebra \(\mathfrak{F} \subset \mathfrak{L}\) of finite codimension, we have \([(0, \mathfrak{F})] = [(0, \mathfrak{F})]\), because \((0, \mathfrak{F})\) and \((0, \mathfrak{F})\) are equivalent. (To show that \((0, \mathfrak{F})\) and \((0, \mathfrak{F})\) are equivalent, one can take \(\mathfrak{U} = \mathfrak{F} \cap \mathfrak{F}\).)

As has been said in Section 1.3, all algebras are supposed to be over the field \(\mathbb{K}\). The set \(\mathbb{P}(\mathfrak{L})\) has a natural structure of associative algebra over \(\mathbb{K}\), which is defined as follows.

- For an admissible pair \((h, \mathfrak{F})\) and an element \(c \in \mathbb{K}\), we set \(c \cdot [(h, \mathfrak{F})] = [(ch, \mathfrak{F})]\).
- For admissible pairs \((h_1, \mathfrak{F}_1)\) and \((h_2, \mathfrak{F}_2)\), the sum and the product of the corresponding elements \([(h_1, \mathfrak{F}_1)], [(h_2, \mathfrak{F}_2)]\) of \(\mathbb{P}(\mathfrak{L})\) are defined as follows

\[
[(h_1, \mathfrak{F}_1)] + [(h_2, \mathfrak{F}_2)] = [(h_1 + h_2, \mathfrak{F}_1 \cap \mathfrak{F}_2)], \quad [(h_1, \mathfrak{F}_1)] \cdot [(h_2, \mathfrak{F}_2)] = [(h_1 \circ h_2, \mathfrak{F})],
\]

\(\mathfrak{F} = \{ w \in \mathfrak{F}_1 \cap \mathfrak{F}_2 \mid h_2(w) \in \mathfrak{F}_1 \}\). Here the map \(h_1 \circ h_2 : \hat{\mathfrak{F}} \to \mathfrak{L}\) is given by the formula \((h_1 \circ h_2)(w) = h_1(h_2(w))\) for \(w \in \hat{\mathfrak{F}}\). Since \(h_2(w) \in \mathfrak{F}_1\) for all \(w \in \hat{\mathfrak{F}}\), the element \(h_1(h_2(w)) \in \mathfrak{L}\) is well defined. It is easy to check that the pair \((h_1 \circ h_2, \hat{\mathfrak{F}})\) is admissible.

Note that \([(0, \mathfrak{F})] \in \mathbb{P} (\mathfrak{L})\) is the zero element in the algebra \(\mathbb{P}(\mathfrak{L})\). As has been shown above, the equivalence class \([(0, \mathfrak{F})]\) is the same for any subalgebra \(\mathfrak{F} \subset \mathfrak{L}\) of finite codimension.

So for any Lie algebra \(\mathfrak{L}\) we have defined the associative algebra \(\mathbb{P}(\mathfrak{L})\). Clearly, if \(\mathfrak{L}\) is finite-dimensional then \(\mathbb{P}(\mathfrak{L}) = 0\). So \(\mathbb{P}(\mathfrak{L})\) can be nontrivial only for infinite-dimensional Lie algebras \(\mathfrak{L}\).

In the rest of this subsection we assume \(\mathbb{K} = \mathbb{C}\). Let \(e_1, e_2, e_3 \in \mathbb{C}\) such that \(e_1 \neq e_2 \neq e_3 \neq e_1\).

Recall that \(E_{e_1, e_2, e_3}\) defined by (163) is the commutative associative algebra of polynomial functions on the algebraic curve in \(\mathbb{C}^2\) defined by the polynomials (164). Since we assume \(e_1 \neq e_2 \neq e_3 \neq e_1\), the algebra \(E_{e_1, e_2, e_3}\) is an integral domain. (That is, the product of any two nonzero elements of \(E_{e_1, e_2, e_3}\) is nonzero.)

Let \(F_{e_1, e_2, e_3}\) be the fraction field of the ring \(E_{e_1, e_2, e_3}\). So elements of \(F_{e_1, e_2, e_3}\) are fractions of the form \(b/c\), where \(b, c \in E_{e_1, e_2, e_3}\) and \(c \neq 0\).

Recall that the function field of an algebraic curve is the field of rational functions on this curve. The element \(z \in E_{e_1, e_2, e_3}\) is given by (167). Let \(Q_{e_1, e_2, e_3} \subset F_{e_1, e_2, e_3}\) be the subfield generated by the elements \(z\) and \(y = \sqrt{v_1v_2v_3}\). Then \(y^3 = (z - e_1)(z - e_2)(z - e_3)\), and it is easily seen that the field \(Q_{e_1, e_2, e_3}\) is isomorphic to the function field of the elliptic curve (37). So elements of \(Q_{e_1, e_2, e_3}\) can be identified with rational functions on the curve (37).

The infinite-dimensional Lie algebra \(\mathfrak{R}_{e_1, e_2, e_3}\) has been described in Section 5.

**Theorem 11.** For any Lie subalgebra \(L \subset \mathfrak{R}_{e_1, e_2, e_3}\) of finite codimension, the associative algebra \(\mathbb{P}(L)\) is commutative and is isomorphic to the field \(Q_{e_1, e_2, e_3}\).

**Proof.** The space \(\mathfrak{so}_3(\mathbb{C}) \otimes F_{e_1, e_2, e_3}\) has the \(F_{e_1, e_2, e_3}\)-module structure given by

\[
f_1 \cdot (w \otimes f_2) = w \otimes f_1f_2,\quad w \in \mathfrak{so}_3(\mathbb{C}), \quad f_1, f_2 \in F_{e_1, e_2, e_3}.
\]

Since \(F_{e_1, e_2, e_3} \subset F_{e_1, e_2, e_3}\), one has the natural inclusions of Lie algebras

\[
\mathfrak{R}_{e_1, e_2, e_3} \subset \mathfrak{so}_3(\mathbb{C}) \otimes E_{e_1, e_2, e_3} \subset \mathfrak{so}_3(\mathbb{C}) \otimes F_{e_1, e_2, e_3}.
\]
For each \( f \in F_{e_1,e_2,e_3} \) consider the map

\[
G_f: \mathfrak{r}_{e_1,e_2,e_3} \to \mathfrak{so}_3(\mathbb{C}) \otimes F_{e_1,e_2,e_3}; \quad G_f(p) = f \cdot p, \quad p \in \mathfrak{r}_{e_1,e_2,e_3}.
\]

Obviously,

\[
(181) \quad G_f([p_1,p_2]) = [G_f(p_1),p_2] = [p_1,G_f(p_2)] \quad \forall p_1, p_2.
\]

Recall that

\[
(182) \quad z = \hat{v}_1^2 + e_1 = \hat{v}_2^2 + e_2 = \hat{v}_3^2 + e_3, \quad y = \hat{v}_1\hat{v}_2\hat{v}_3.
\]

Recall that the elements (168) form a basis for \( \mathfrak{r}_{e_1,e_2,e_3} \). Let \( d_1(y,z) \) be a polynomial in \( y, z \) and \( d_2(z) \neq 0 \) be a polynomial in \( z \). Using the basis (168), one gets that

\[
G_{d_1(y,z)}(\mathfrak{r}_{e_1,e_2,e_3}) \subset \mathfrak{r}_{e_1,e_2,e_3}, \quad G_{d_2(z)}(\mathfrak{r}_{e_1,e_2,e_3}) \subset \mathfrak{r}_{e_1,e_2,e_3},
\]

and the space \( G_{d_2(z)}(\mathfrak{r}_{e_1,e_2,e_3}) \) is of finite codimension in \( \mathfrak{r}_{e_1,e_2,e_3} \). Using this property and the assumption codim \( L < \infty \), we obtain that

\[
(183) \quad \text{the subspace} \quad \tilde{L} = \{ w \in L \mid G_{d_1(y,z)}(w) \in G_{d_2(z)}(L) \} \quad \text{is of finite codimension in} \ L.
\]

Since \( y^2 = (z - e_1)(z - e_2)(z - e_3) \), any element \( f \in Q_{e_1,e_2,e_3} \) can be presented as a fraction of such polynomials \( f = \frac{d_1(y,z)}{d_2(z)} \). Then from property (183) it follows that the subspace

\[
L_f = \{ w \in L \mid G_f(w) \in L \}
\]

is of finite codimension in \( L \). Relation (183) implies that \( L_f \) is a Lie subalgebra of \( L \). Therefore, the pair \((G_f, L_f)\) determines an element of \( \mathbb{I}T(L) \), and we obtain the embedding

\[
\Psi: Q_{e_1,e_2,e_3} \hookrightarrow \mathbb{I}T(L), \quad \Psi(f) = [(G_f, L_f)].
\]

It remains to show that the map \( \Psi \) is surjective.

Let \((h,H)\) \( \in \mathbb{I}T(L) \), where \( H \subset L \) is a subalgebra of finite codimension and

\[
(184) \quad h: H \to L, \quad h([p_1,p_2]) = [h(p_1),p_2] = [p_1,h(p_2)] \quad \forall p_1, p_2 \in H.
\]

Let \( \mathfrak{r}^i \subset \mathfrak{r}_{e_1,e_2,e_3} \) be the subspace spanned by the elements (168) for fixed \( i = 1, 2, 3 \). Then \( \mathfrak{r}_{e_1,e_2,e_3} = \mathfrak{r}^1 \oplus \mathfrak{r}^2 \oplus \mathfrak{r}^3 \) as vector spaces, and

\[
(185) \quad \forall w \in \mathfrak{r}^i \quad \text{there is a unique} \ f \in Q_{e_1,e_2,e_3} \ \text{such that} \ w = \alpha_i \otimes \hat{v}_i f.
\]

Set \( H^i = \mathfrak{r}^i \cap H \). Due to properties (166), (185), the space \( \tilde{H} = H^1 + H^2 + H^3 \) is a Lie subalgebra of \( H \). Since \( H \) is of finite codimension in \( \mathfrak{r}_{e_1,e_2,e_3} \), the subalgebra \( \tilde{H} \) is of finite codimension in \( H \).

Let \( w_i \in H^i, \ w_i \neq 0, \ i = 1, 2, 3 \). Then \( h(w_i), w_i) = h([w_i,w_i]) = 0 \). From (166), (185) it follows that \( h(w_i) = f_i \cdot w_i \) for some \( f_i \in Q_{e_1,e_2,e_3} \). Then

\[
(186) \quad h([w_1,w_2]) = h([w_1],w_2) = [w_1,h(w_2)] = f_1 \cdot [w_1,w_2] = f_2 \cdot [w_1,w_2].
\]

Since, by properties (166), (185), one has \( [w_1,w_2] \neq 0 \), relation (186) implies \( f_1 = f_2 \). Similarly, one shows that \( f_1 = f_2 = f_3 \).

Therefore, for any other nonzero elements \( w'_i \in H^i \), we also get \( h(w'_i) = f'_i \cdot w'_i \) for some \( f'_i \in Q_{e_1,e_2,e_3} \). Similarly to (186), one obtains \( h([w_1,w'_2]) = f' \cdot [w_1,w'_2] = f' \cdot [w_1,w'_2] \), which implies \( f' = f_1 \).

Thus there is a unique \( f' \in Q_{e_1,e_2,e_3} \) such that \( h_{\mid \tilde{H}} = G_{f'}\mid \tilde{H} \). Therefore, \( [(h,H)] = [(G_{f'}, \tilde{H})] \) in \( \mathbb{I}T(L) \), that is, \( [(h,H)] = \Psi(f') \). \( \Box \)

Similarly to Theorem 11 one proves the following result.

**Theorem 12.** For any Lie subalgebra \( L \subset \mathfrak{sl}_2(\mathbb{C}[\lambda]) \) of finite codimension, the associative algebra \( \mathbb{I}T(L) \) is commutative and is isomorphic to the field of rational functions in \( \lambda \).
7. Necessary Conditions for Existence of Bäcklund Transformations

Recall that, for every topological space $X$ and every point $a \in X$, one has the fundamental group $\pi_1(X, a)$, which provides important information about the space $X$. The preprint \cite{12} introduces an analog of fundamental groups for PDEs. However, the “fundamental group of a PDE” is not a group, but a certain system of Lie algebras, which are called fundamental Lie algebras.

According to Remark 8 and Section 2.2, a PDE can be viewed as a manifold $\mathcal{E}$ with an $n$-dimensional distribution (the Cartan distribution) such that solutions of the PDE correspond to $n$-dimensional integral submanifolds, where $n$ is the number of independent variables in the PDE. To simplify notation, we do not mention the Cartan distribution explicitly.

For every PDE $\mathcal{E}$ and every point $a \in \mathcal{E}$, the preprint \cite{12} defines a Lie algebra $\mathbb{F}(\mathcal{E}, a)$, which is called the fundamental Lie algebra of the PDE $\mathcal{E}$ at the point $a \in \mathcal{E}$. In general, $\mathbb{F}(\mathcal{E}, a)$ can be infinite-dimensional. The definition of $\mathbb{F}(\mathcal{E}, a)$ in \cite{12} is coordinate-free and uses geometry of the manifold $\mathcal{E}$ and the Cartan distribution. According to \cite{12}, the Lie algebra $\mathbb{F}(\mathcal{E}, a)$ has a natural topology.

The definition of $\mathbb{F}(\mathcal{E}, a)$ in \cite{12} is applicable to PDEs with any number of variables. According to \cite{12}, if $\mathcal{E}$ is a $(1+1)$-dimensional evolution PDE, then the fundamental Lie algebra $\mathbb{F}(\mathcal{E}, a)$ introduced in \cite{12} is isomorphic to the Lie algebra $\mathbb{F}(\mathcal{E}, a)$ defined in Section 1.2 as the inverse limit of the sequence \cite{88}, which is equal to the sequence \cite{88}.

We need to recall a well-known property of topological coverings. Let $\tau: M' \to M$ be a topological covering, where $M$ and $M'$ are finite-dimensional manifolds. Let $a' \in M'$. Consider the point $\tau(a') \in M$. Then the fundamental group $\pi_1(M', a')$ is isomorphic to a subgroup of the fundamental group $\pi_1(M, a)$.

One has an analogous property for differential coverings of PDEs. The following proposition is proved in \cite{12}.

**Proposition 2** \cite{12}. Let $\tau: \mathcal{E}' \to \mathcal{E}$ be a differential covering, where $\mathcal{E}$ and $\mathcal{E}'$ are PDEs. We suppose that the fibers of $\tau$ are finite-dimensional.

Let $a' \in \mathcal{E}'$. Consider the point $\tau(a') \in \mathcal{E}$. Consider the fundamental Lie algebra $\mathbb{F}(\mathcal{E}', a')$ of $\mathcal{E}'$ at $a' \in \mathcal{E}'$ and the fundamental Lie algebra $\mathbb{F}(\mathcal{E}, \tau(a'))$ of $\mathcal{E}$ at $\tau(a') \in \mathcal{E}$. According to the definition of the fundamental Lie algebras, we have a topology on $\mathbb{F}(\mathcal{E}', a')$ and a topology on $\mathbb{F}(\mathcal{E}, \tau(a'))$.

Then one has an embedding

$$\varphi: \mathbb{F}(\mathcal{E}', a') \to \mathbb{F}(\mathcal{E}, \tau(a'))$$

such that

- the subalgebra $\varphi(\mathbb{F}(\mathcal{E}', a')) \subset \mathbb{F}(\mathcal{E}, \tau(a'))$ is of finite codimension in $\mathbb{F}(\mathcal{E}, \tau(a'))$,
- the subalgebra $\varphi(\mathbb{F}(\mathcal{E}', a'))$ is open and closed in $\mathbb{F}(\mathcal{E}, \tau(a'))$ with respect to the topology on $\mathbb{F}(\mathcal{E}, \tau(a'))$,
- the isomorphism $\varphi: \mathbb{F}(\mathcal{E}', a') \cong \varphi(\mathbb{F}(\mathcal{E}', a'))$ is a homeomorphism with respect to the topologies on $\mathbb{F}(\mathcal{E}', a')$ and $\varphi(\mathbb{F}(\mathcal{E}', a'))$.

For a $(1 + 1)$-dimensional scalar evolution equation $\mathcal{E}$ and a point $a \in \mathcal{E}$, the notion of tame Lie subalgebra $H \subset \mathbb{F}(\mathcal{E}, a)$ has been defined in Definition 11 and discussed in Remark 12. Now we can prove Theorem 11 which is repeated below.

**Theorem 13.** Let $\mathcal{E}^1$ and $\mathcal{E}^2$ be $(1 + 1)$-dimensional scalar evolution equations. For each $i = 1, 2$, the symbol $\mathcal{E}^i$ denotes also the infinite prolongation of the corresponding equation. (So on the manifold $\mathcal{E}^i$ we have the Cartan distribution spanned by the total derivative operators.) Suppose that $\mathcal{E}^1$ and $\mathcal{E}^2$ are connected by a Bäcklund transformation. Then for each $i = 1, 2$ there are a point $a_i \in \mathcal{E}^i$ and a tame subalgebra $H_i \subset \mathbb{F}(\mathcal{E}^i, a_i)$ such that

- $H_i$ is of finite codimension in $\mathbb{F}(\mathcal{E}^i, a_i)$,
- $H_i$ is isomorphic to $H_2$, and this isomorphism is a homeomorphism with respect to the topology induced by the embedding $H_i \subset \mathbb{F}(\mathcal{E}^i, a_i)$.

**Proof.** According to Definition 11 if $\mathcal{E}^1$ and $\mathcal{E}^2$ are connected by a Bäcklund transformation, then there are a PDE $\mathcal{E}^3$ and coverings \cite{60}.
Let \( a \in \mathcal{E}^3 \). We set \( a_i = \tau_i(a) \) for each \( i = 1, 2 \). Applying Proposition 2 to the covering \( \tau_i: \mathcal{E}^3 \rightarrow \mathcal{E}^i \) and using Remark 12, we get an embedding
\[
\varphi_i: F(\mathcal{E}^3, a) \hookrightarrow F(\mathcal{E}^i, a_i)
\]
such that \( \varphi_i(F(\mathcal{E}^3, a)) \) is a tame Lie subalgebra of \( F(\mathcal{E}^i, a_i) \) of finite codimension and the isomorphism \( \varphi: F(\mathcal{E}^3, a) \cong \varphi_i(F(\mathcal{E}^3, a)) \) is a homeomorphism with respect to the topologies on \( F(\mathcal{E}^3, a) \) and \( \varphi_i(F(\mathcal{E}^3, a)) \).

Then the subalgebras \( H_i = \varphi_i(F(\mathcal{E}^3, a)) \subset F(\mathcal{E}^i, a_i), i = 1, 2 \), satisfy all the required properties. In particular, the isomorphism
\[
\varphi_2 \circ \varphi_1^{-1}: H_1 = \varphi_1(F(\mathcal{E}^3, a)) \cong H_2 = \varphi_2(F(\mathcal{E}^3, a))
\]
is a homeomorphism. \( \square \)

**Remark 31.** In Theorem 13 we say that \( \mathcal{E}^i \) is the infinite prolongation of a \((1+1)\)-dimensional scalar evolution equation, for each \( i = 1, 2 \). Actually, the result and proof of Theorem 13 remain valid if \( \mathcal{E}^i \) is an open subset of the infinite prolongation of a \((1+1)\)-dimensional scalar evolution equation.

Theorem 13 provides a powerful necessary condition for two given evolution equations to be connected by a Bäcklund transformation. Using Theorem 13 in Section 8 we prove Theorem 14 which describes some non-existence results for Bäcklund transformations.

8. Some non-existence results for Bäcklund transformations

In this section we assume \( \mathbb{K} = \mathbb{C} \). Recall that, for any \( e_1, e_2, e_3 \in \mathbb{C} \), the Krichever-Novikov equation \( \text{KN}(e_1, e_2, e_3) \) is given by (36), and the algebraic curve \( \text{C}(e_1, e_2, e_3) \) is given by (37). Now we can prove Theorem 2 which is repeated below.

**Theorem 14.** Let \( e_1, e_2, e_3, e'_1, e'_2, e'_3 \in \mathbb{C} \) such that
\[
e_1 \neq e_2 \neq e_3 \neq e_1, \quad e'_1 \neq e'_2 \neq e'_3 \neq e'_1.
\]

If the curve \( \mathcal{C}(e_1, e_2, e_3) \) is not birationally equivalent to the curve \( \mathcal{C}(e'_1, e'_2, e'_3) \), then the equation \( \text{KN}(e_1, e_2, e_3) \) is not connected with the equation \( \text{KN}(e'_1, e'_2, e'_3) \) by any Bäcklund transformation (BT).

Also, if \( e_1 \neq e_2 \neq e_3 \neq e_1 \), then \( \text{KN}(e_1, e_2, e_3) \) is not connected with the KdV equation by any BT.

**Proof.** Let \( \mathcal{E} \) be the infinite prolongation of a \((1+1)\)-dimensional scalar evolution equation. For each \( a \in \mathcal{E} \), the notion of a tame Lie subalgebra \( H \subset F(\mathcal{E}, a) \) has been defined in Section 1.2. Using the topology on \( F(\mathcal{E}, a) \) described in Section 1.2 on any tame Lie subalgebra \( H \subset F(\mathcal{E}, a) \) we have the topology induced by the embedding \( H \subset F(\mathcal{E}, a) \).

In Section 6.1 for any Lie algebra \( \mathcal{L} \) with topology we have defined the Lie algebra \( \mathcal{S}(\mathcal{L}) = \mathcal{L}/Q(\mathcal{L}) \), where \( Q(\mathcal{L}) \) is the ideal of quasi-solvable elements in \( \mathcal{L} \). In particular, we can consider \( \mathcal{S}(H) \) for a tame Lie subalgebra \( H \subset F(\mathcal{E}, a) \).

We suppose that \( e_1, e_2, e_3, e'_1, e'_2, e'_3 \in \mathbb{C} \) obey (187). Let \( \mathcal{E}_{KdV} \) be the infinite prolongation of the KdV equation. Let \( \mathcal{E}_{e_1,e_2,e_3}, \mathcal{E}_{e'_1,e'_2,e'_3} \) be the infinite prolongations of the equations \( \text{KN}(e_1, e_2, e_3), \text{KN}(e'_1, e'_2, e'_3) \), respectively.

**Lemma 8.** Let \( a \in \mathcal{E}_{KdV} \). For any tame Lie subalgebra \( H \subset F(\mathcal{E}_{KdV}, a) \) of finite codimension, the Lie algebra \( \mathcal{S}(H) \) is isomorphic to a Lie subalgebra of \( \mathfrak{sl}_2(\mathbb{C}[\lambda]) \) of finite codimension.

**Proof.** Set \( \mathcal{E} = \mathcal{E}_{KdV} \). In Example 9 we have defined the surjective homomorphism
\[
\psi: F(\mathcal{E}, a) \rightarrow \mathfrak{sl}_2(\mathbb{K}[\lambda]).
\]
In this section we assume \( \mathbb{K} = \mathbb{C} \), so \( \mathfrak{sl}_2(\mathbb{K}[\lambda]) = \mathfrak{sl}_2(\mathbb{C}[\lambda]) \). Since \( H \) is of finite codimension in \( F(\mathcal{E}, a) \), the Lie subalgebra \( \psi(H) \subset \mathfrak{sl}_2(\mathbb{C}[\lambda]) \) is of finite codimension in \( \mathfrak{sl}_2(\mathbb{C}[\lambda]) \).

In Example 9 we have shown that an element \( w \in F(\mathcal{E}, a) \) is quasi-solvable iff \( w \in \ker \psi \). Similarly, the definition of \( \psi \), the definition of the topology on \( F(\mathcal{E}, a) \) and \( H \), Lemma 11 Theorem 8 and Lemma 6 imply that an element \( \tilde{w} \in H \) is quasi-solvable iff \( \tilde{w} \in \ker \psi \cap H \). Therefore, \( \mathcal{S}(H) \) is isomorphic to \( \psi(H) \). \( \square \)
Lemma 9. Let $a \in \mathcal{E}_{e_1,e_2,e_3}$. For any tame Lie subalgebra $H \subset \mathbb{F}(\mathcal{E}_{e_1,e_2,e_3}, a)$ of finite codimension, the Lie algebra $\mathcal{G}(H)$ is isomorphic to a Lie subalgebra of $\mathfrak{R}_{e_1,e_2,e_3}$ of finite codimension.

Proof. Set $\mathcal{E} = \mathcal{E}_{e_1,e_2,e_3}$. In Example 10 we have defined the surjective homomorphism 
$$
\mu : \mathbb{F}(\mathcal{E}, a) \to \mathfrak{R}_{e_1,e_2,e_3}.
$$
Since $H$ is of finite codimension in $\mathbb{F}(\mathcal{E}, a)$, the Lie subalgebra $\mu(H) \subset \mathfrak{R}_{e_1,e_2,e_3}$ is of finite codimension in $\mathfrak{R}_{e_1,e_2,e_3}$.

In Example 10 we have shown that an element $w \in \mathbb{F}(\mathcal{E}, a)$ is quasi-solvable iff $w \in \ker \mu$. Similarly, the definition of $\mu$, the definition of the topology on $\mathbb{F}(\mathcal{E}, a)$ and $H$, Lemma 9, Theorem 10, and Lemma 7 imply that an element $\tilde{w} \in H$ is quasi-solvable iff $\tilde{w} \in \ker \mu \cap H$. Therefore, $\mathcal{G}(H)$ is isomorphic to $\mu(H)$.

Suppose that $\text{KN}(e_1, e_2, e_3)$ and $\text{KN}(e'_1, e'_2, e'_3)$ are connected by a BT. Then, by Theorem 13 there are points $a_1 \in \mathcal{E}_{e_1,e_2,e_3}$, $a_2 \in \mathcal{E}_{e'_1,e'_2,e'_3}$ and tame subalgebras $H_1 \subset \mathbb{F}(\mathcal{E}_{e_1,e_2,e_3}, a_1)$, $H_2 \subset \mathbb{F}(\mathcal{E}_{e'_1,e'_2,e'_3}, a_2)$ of finite codimension such that $H_1$ is isomorphic to $H_2$, and this isomorphism is a homeomorphism. Then $\mathcal{G}(H_1) \cong \mathcal{G}(H_2)$, which yields

$$
(188) \quad \mathbb{I}(\mathcal{G}(H_1)) \cong \mathbb{I}(\mathcal{G}(H_2)).
$$

By Lemma 9, $\mathcal{G}(H_1)$ is isomorphic to a Lie subalgebra of $\mathfrak{R}_{e_1,e_2,e_3}$ of finite codimension, and $\mathcal{G}(H_2)$ is isomorphic to a Lie subalgebra of $\mathfrak{R}_{e'_1,e'_2,e'_3}$ of finite codimension.

Therefore, by Theorem 11, $\mathbb{I}(\mathcal{G}(H_1))$ is isomorphic to $Q_{e_1,e_2,e_3}$, and $\mathbb{I}(\mathcal{G}(H_2))$ is isomorphic to $Q_{e'_1,e'_2,e'_3}$. Combining this with (188), we get

$$
(189) \quad Q_{e_1,e_2,e_3} \cong Q_{e'_1,e'_2,e'_3}.
$$

Since $Q_{e_1,e_2,e_3}$ is isomorphic to the field of rational functions on the curve $C(e_1, e_2, e_3)$, and $Q_{e'_1,e'_2,e'_3}$ is isomorphic to the field of rational functions on the curve $C(e'_1, e'_2, e'_3)$, the isomorphism (189) implies that $C(e_1, e_2, e_3)$ is birationally equivalent to $C(e'_1, e'_2, e'_3)$.

Therefore, if $C(e_1, e_2, e_3)$ is not birationally equivalent to $C(e'_1, e'_2, e'_3)$, then $\text{KN}(e_1, e_2, e_3)$ is not connected with $\text{KN}(e'_1, e'_2, e'_3)$ by any BT. So we have proved the first statement of Theorem 14.

To prove the second statement of this theorem, we suppose that, for some $e_1, e_2, e_3 \in \mathbb{C}$ satisfying $e_1 \neq e_2 \neq e_3 \neq e_1$, the equation $\text{KN}(e_1, e_2, e_3)$ is connected with the KdV equation by a BT.

Then, by Theorem 13 there are points $\tilde{a}_1 \in \mathcal{E}_{e_1,e_2,e_3}$, $\tilde{a}_2 \in \mathcal{E}_{KdV}$ and tame subalgebras $\tilde{H}_1 \subset \mathbb{F}(\mathcal{E}_{e_1,e_2,e_3}, \tilde{a}_1)$, $\tilde{H}_2 \subset \mathbb{F}(\mathcal{E}_{KdV}, \tilde{a}_2)$ of finite codimension such that $\tilde{H}_1$ is isomorphic to $\tilde{H}_2$, and this isomorphism is a homeomorphism. Then $\mathcal{G}(\tilde{H}_1) \cong \mathcal{G}(\tilde{H}_2)$, which yields

$$
(190) \quad \mathbb{I}(\mathcal{G}(\tilde{H}_1)) \cong \mathbb{I}(\mathcal{G}(\tilde{H}_2)).
$$

By Lemma 9, $\mathcal{G}(\tilde{H}_1)$ is isomorphic to a Lie subalgebra of $\mathfrak{R}_{e_1,e_2,e_3}$ of finite codimension. According to Theorem 11, this implies that $\mathbb{I}(\mathcal{G}(\tilde{H}_1))$ is isomorphic to the field $Q_{e_1,e_2,e_3}$, which is isomorphic to the field of rational functions on $C(e_1, e_2, e_3)$.

Let $\mathcal{C}(\lambda)$ be the field of rational functions in $\lambda$. By Lemma 8, $\mathcal{G}(\tilde{H}_2)$ is isomorphic to a Lie subalgebra of $\mathfrak{sl}_2(\mathbb{C}[\lambda])$ of finite codimension. By Theorem 12, this implies that $\mathbb{I}(\mathcal{G}(\tilde{H}_2))$ is isomorphic to $\mathbb{C}(\lambda)$. The isomorphisms $\mathbb{I}(\mathcal{G}(\tilde{H}_1)) \cong Q_{e_1,e_2,e_3}$, $\mathbb{I}(\mathcal{G}(\tilde{H}_2)) \cong \mathbb{C}(\lambda)$, and (190) yield that $Q_{e_1,e_2,e_3}$ is isomorphic to $\mathbb{C}(\lambda)$, but this contradicts to the fact that the elliptic curve $C(e_1, e_2, e_3)$ is not birationally equivalent to the rational curve $\mathbb{C}$ with coordinate $\lambda$. The obtained contradiction shows that $\text{KN}(e_1, e_2, e_3)$ is not connected with the KdV equation by any BT. □

Remark 32. As we have shown in Theorem 3, the first statement of Theorem 14 (which is the same as the first statement of Theorem 2) implies the following. If the numbers (39) satisfy (40), then the equation $\text{KN}(e_1, e_2, e_3)$ is not connected with the equation $\text{KN}(e'_1, e'_2, e'_3)$ by any BT.
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