On a type of semi-sub-Riemannian connection on a sub-Riemannian manifold

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Abstract The authors first in this paper define a semi-symmetric metric non-holonomic connection (called in briefly a semi-sub-Riemannian connection) on sub-Riemannian manifolds, and study the relations between sub-Riemannian connections and semi-sub-Riemannian connections. An invariant under a connection transformation $\nabla \rightarrow D$ is obtained. The authors then further deduce a sufficient and necessary condition that a sub-Riemannian manifold associated with a semi-sub-Riemannian connection is flat, and derive that a sub-Riemannian manifold with vanishing curvature with respect to semi-sub-Riemannnian connection $D$ is a group manifold if and only if it is of constant curvature.

Keywords Sub-Riemannian manifolds; Semi-sub-Riemannian connections; Schouten curvature tensors

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1 Introduction

The study of transformation in Riemannian geometry has experienced a long time. In 1924, A. Fridmann and J. A. Schouten [10] first introduced the concept of a semi-symmetric linear connection in a differential manifold, namely, a linear connection $\tilde{\nabla}$ is said to be a semi-symmetric connection if its torsion tensor $\tilde{T}$ is of the form

$$\tilde{T}(X, Y) = \pi(Y)X - \pi(X)Y, \forall X, Y \in \Gamma(TM),$$

where $\pi$ is of 1-form associated with vector $P$ on M, and $P$ is defined by $g(X, P) = \pi(X)$. In 1970, K. Yano [17] considered a semi-symmetric metric connection (that means a linear connection is both metric and semi-symmetric) on a Riemannian manifold and studied some of its properties. He pointed out that a Riemannian manifold is conformal flat if and only if it admits a semi-symmetric metric connection whose curvature

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tensor vanishes identically. He also proved that a Riemannian manifold is of constant curvature if and only if it admits a semi-symmetric metric connection for which the manifold is a group manifold, where a group manifold is a differential manifold admitting a linear connection $\hat{\nabla}$ such that its curvature tensor $\hat{R}$ vanishes and the covariant derivative of torsion tensor $\hat{T}$ with respect to $\hat{\nabla}$ is vanishing. Liang in his paper [14] discussed some properties of semi-symmetric metric connections and proved that the projective curvature tensor with respect to semi-symmetric metric connections coincides with the projective curvature tensor with respect to Levi-civita connection if and only if the characteristic vector is proportional to a Riemannian metric. The authors [23] introduced the concept of the projective semi-symmetric metric connection, found an invariant under the transformation of projective semi-symmetric connections and indicated that this invariant could degenerate into the Weyl projective curvature tensor under certain conditions, so the Weyl projective curvature tensor is an invariant as for the transformation of the special projective semi-symmetric connection. For the study of semi-symmetric metric connections, the authors have other interesting results [11, 19, 20, 21, 22, 24]. Recently, the authors in paper [18] even studied the theory of transformations on Carnot Caratheodory spaces, and obtained the conformal invariants and projective invariants on Carnot-Caratheodory spaces with the view of Felix Klein.

In 1990, N. S. Agache and M. R. Chafle [1] discussed a semi-symmetric non-metric connection on a Riemannian manifold. A semi-symmetric connection $\hat{\nabla}$ is said to be a semi-symmetric non-metric connection if it satisfies the conditions:

\[ \hat{\nabla}_X Y = \nabla_X Y + \pi(Y)X + g(X, Y)P, \quad \forall X, Y, Z \in \Gamma(TM), \]

\[ \hat{\nabla}_Z g(X, Y) = -2\pi(X)g(Y, Z) - 2\pi(Y)g(X, Z), \quad \forall X, Y, Z \in \Gamma(TM), \]

where $\nabla$ is Levi-civita connection. This semi-symmetric non-metric connection was further developed by U. C. De and S. C. Biswas [5], U. C. De and D. Kamily [6], N. S. Agashe and M. R. Chafle [1] defined the curvature tensor with respect to semi-symmetric non-metric connections, and proved the Weyl projective curvature tensor with respect to semi-symmetric non-metric connections is equal to the Weyl projective curvature tensor with respect to Levi-Civita connection. They further got a necessary and sufficient condition that a Riemannian manifold with vanishing Ricci tensor with respect to semi-symmetric non-metric connections being projectively flat if and only if the curvature tensor with respect to semi-symmetric non-metric connections is vanished. U. C. De and S. C. Biswas [5] discussed the semi-symmetric non-metric connection on Riemannian manifold by using the similar arguments, and obtained some properties of curvature tensors with respect to semi-symmetric non-metric connections, and proposed that two semi-symmetric non-metric connections would be equal under certain conditions.

The study of geometric analysis in sub-Riemannian manifolds has been an active field over the past several decades. The past decade has witnessed a dramatic and widespread expansion of interest and activity in sub-Riemannian geometry. In particular, round about 1993, since the formidable papers were published in succession, these works stimulate such research fields to present a scene of prosperity, and
demonstrate the abnormal importance of this topic. Sub-Riemannian manifolds, on
the one hand, are the natural development of Riemannian manifolds, and are the ba-
sic metric spaces on which one can consider the problems of geometric analysis; On
the other hand, sub-Riemannian manifolds have been found useful in the study of
theories and applications of Control theory, PDEs, Calculus of Variations, Mechanic,
Gauge fields, etc. The study of geometric analysis in sub-Riemannian manifolds
is carrying on the following two folds. The first fold is describing the geometric
properties of sub-Riemannian manifolds\cite{2, 7, 9, 12}; The second fold is devoted to
the analysis problem of Sub-Riemannian manifolds\cite{3, 13, 15}. In the past decades,
we have focused our attention on the sub-Riemannian geodesics, and got some inter-
esting and remarkable results. Although a sub-Riemannian manifold is an natural
generalization of a Riemannian manifold, there are some essential differences. One
of the essential differences is that there exists a kind of strange geodesics which
are minimal geodesics and topological stability, but does not satisfy the geodesics
equation. We call them singular geodesics. The existence of singular geodesics
shows the importance of sub-Riemannian geometry. The second difference is that
the endpoint mapping can be defined by the normal sub-Riemannian geodesic but
it is not diffeomorphic any more. On the other hand, the horizontal connection $\nabla^{H,\Sigma}$, used for instance for studying the minimal surface and isoperimetric problem in
sub-Riemannian manifolds, defined on hypersurface $\Sigma$ is in general not torsion free,
and therefore it is not Levi-Civita any more, so the horizontal second fundamental
form $II^{H,\Sigma}$ is not symmetric, which is also different from Riemannian case\cite{8}. In
this paper we will take the liberty of considering the geometries of sub-Riemannian
manifolds via a point of view of transform groups, our final purpose is to establish
the relevant geometries in the sense of transformative theories.

As it is well known, there exists a unique symmetric metric nonholonomic con-
nection (i.e. sub-Riemannian connection or horizontal connection in this paper) in
sub-Riemannian manifolds just as Levi-Civita connection in Riemannian manifolds.
According to the geometric characteristics of Levi-Civita connection, this symmet-
ric metric nonholonomic connection in sub-Riemannian manifolds can preserve the
inner product of any two horizontal vector fields when they transport along a hori-
zontal curve. However there may be existing a bad nonholonomic connection in a sub-
Riemannian manifold which can not preserve the torsion property, so it is urgent and
important to study a kind of nonholonomic metric connection that is not symmetric.
The problem of geometries and analysis of a semi-symmetric metric nonholonomic
connection emerges as the times require. The semi-symmetric nonholonomic metric
connection in this paper is just a special non-symmetric nonholonomic connection.
Taking into account that sub-Riemannian manifolds are a natural generalization of
Riemannian manifolds, we would ask whether we can consider the invariants from
symmetric metric nonholonomic connections to semi-symmetric metric nonholo-
nomic connection. Once we found the invariants under connection transformations,
we could study the property of an object connection through an original connection.
In order to study the geometric properties in sub-Riemannian manifold, the second
author first discussed the transformations in Carnot-Caratheodory spaces, and got
the conformal invariants and projective invariants, which can be regarded as an nat-
ural generalization of those conclusions in Riemannian manifolds. We in this paper
wish to use the unique nonholonomic connection to solve the posed problems above. To the author’s knowledge, the study of the semi-symmetric metric connection in sub-Riemannian manifolds is still a gap.

In this paper, we first define a semi-symmetric metric non-holonomic connection in sub-Riemannian manifolds, and derive the relations between a symmetric metric non-holonomic connection and a semi-symmetric metric non-holonomic connection, and get an invariant under the connection transformation $V \rightarrow D$. We further define the Weyl conformal curvature tensor $\tilde{C}^h_{ijk}$ and the Weyl projective curvature tensor $\bar{W}^h_{ijk}$ of semi-symmetric metric nonholonomic connections, and find that $\tilde{C}^h_{ijk}$ is no longer an invariant under the connection transformation from $V$ to $D$, which is obviously different from the Riemannian case. On the other hand, we also deduce a sufficient and necessary condition that a sub-Riemannian manifold admitting semi-symmetric metric connection is flat. At last, we consider a group manifold and find the Carnot group is an example of group manifolds, at the same time, we prove that, a sub-Riemannian manifold associated with a semi-symmetric metric connection is a group manifold if and only if the sub-Riemannian manifold is of constant curvature.

The organization of this paper is as follows. In section 2, we will recall and give the necessary information about Schouten curvature tensor and symmetric metric connection in sub-Riemannian manifold. Section 3 is devoted to the new definition and main Theorems.

2 Preliminaries

Let $M^n$ be an $n$-dimensional smooth manifold. For each point $p \in M^n$, there assigns a $\ell(2 < \ell < n)$-dimensional subspace $V^\ell(p)$ of the tangent space $T_pM$, then $V^\ell = \bigcup_{p \in M} V^\ell(p)$ forms a tangent sub-bundle of tangent bundles $TM = \bigcup_{p \in M} T_pM$, $V^\ell$ is called a $\ell$-dimensional distribution over $M^n$. For any point $p$, if there exists a neighbourhood $U$ and $\ell$ linearly independent vector fields $X_1, \cdots, X_\ell$ in $U$ such that for each point $q \in U, X_\ell(q), \cdots, X_1(q)$ is a basis of subspace $V^\ell(q)$, then we call $V^\ell$ the $\ell$-dimensional smooth distribution (called also a horizontal bundle), and $X_1, \cdots, X_\ell$ are called a local basis of $V^\ell$ in $U$. We also say that $X_1, \cdots, X_\ell$ generate $V^\ell$ in $U$. We denote by $V^\ell|_U = \text{Span}\{X_1, \cdots, X_\ell\}$.

Definition 2.1. We call $(M, V_0, g)$ a sub-Riemannian manifold with the sub-Riemannian structure $(V_0, g)$, if $V_0$ is a $\ell$-dimensional smooth distribution over $M^n$, and $g$ is a fibre inner product in $V_0$. Here $g$ is called a sub-Riemannian metric and $V_0$ is called a horizontal bundle. In general, $g$ can be regarded as some Riemannian metric $\langle \cdot, \cdot \rangle$, defined on tangent bundle $TM$, restricted to $V_0$.

Throughout the paper, we denote by $\Gamma(V_0)$ the $C^\infty(M)$ -module of smooth sections on $V_0$. Also, if not stated otherwise, we use the following ranges for indices: $i, j, k, h, \cdots \in \{1, \cdots, \ell\}, \alpha, \beta, \cdots \in \{\ell + 1, \cdots, n\}$. The repeated indices with one upper index and one lower index indicates summation over their range.
Definition 2.2. A nonholonomic connection on sub-bundle $V_0 \subset TM$ is a binary mapping $\nabla : \Gamma(V_0) \times \Gamma(V_0) \rightarrow \Gamma(V_0)$ satisfying the following:

\[
\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z,
\]

\[
\nabla_X(fY) = X(f)Y + f\nabla_X Y,
\]

\[
\nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z,
\]

where $X, Y, Z \in \Gamma(V_0)$, $f, g \in \mathcal{C}^\infty(M)$.

In order to study the geometry of $\{M, V_0, g\}$, we suppose that there exists a Riemannian metric $<\cdot, \cdot>$ and $V_1$ is taken as the complementary orthogonal distribution to $V_0$ in $TM$, then, there holds $V_0 \oplus V_1 = TM$. Here we call $V_1$ the vertical distribution. Denote by $X_0$ the projection of the vector field $X$ from $TM$ onto $V_0$, and by $X_1$ the projection of the vector field $X$ from $TM$ onto $V_1$.

Definition 2.3. The torsion tensor of nonholonomic connection $\nabla$ is defined by

\[
T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y], \forall X, Y \in \Gamma(V_0).
\]

From Definition 2.3, we know the torsion tensor of horizontal vector fields is still horizontal vector field, so we call it the horizontal torsion tensor.

Assume that $\{e_i\}, i = 1, \cdots, \ell$ is a basis of $V_0$, then the formulas $\nabla_{e_j}e_j = \{k_{ij}\}e_k$, $i, j, k = 1, \cdots, \ell$ define $\ell^3$ functions as $\{k_{ij}\}$, we call $\{k_{ij}\}$ the connection coefficients of the non-holonomic connection $\nabla$.

It is well known that the Lie bracket $[\cdot, \cdot]$ on $M$ is a Lie algebra structure of smooth tangent vector fields $\Gamma(TM)$, then it is easy to see that the following formula

\[
[e_i, e_j]_0 = \Omega^k_{ij}e_k,
\]

determine $\ell^3$ functions $\Omega^k_{ij}$.

About the existence of this class of connections defined on the horizontal bundle $V_0$, we have the same result as Riemannian case.

Theorem 2.1. \[4, 16\] Given a sub-Riemannian manifold $(M, V_0, g)$, then there exists a unique nonholonomic connection satisfying

\[
Zg(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y),
\]

\[
T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]_0 = 0.
\]

Remark 2.1. Similar to Riemannian manifolds, we also say that the non-holonomic connections with property (2.5) and (2.6) are metric and torsion-free, respectively. An nonholonomic connection satisfying (2.5) and (2.6) is called a sub-Riemannian connection or a horizontal connection. For a simplified proof of Theorem 2.1, one can see [18] for details. On the other hand, K. Yano [17] posed a proof with a method of projecting the Riemannian connection onto the distribution to derive Theorem 2.1 in the case of Riemannian manifolds.
Next we discuss the horizontal connection of Carnot group, which is a very important example of sub-Riemannian manifolds. If $G$ is Lie group with graded Lie algebra satisfying

\[ h = V_0 \oplus V_1 \oplus \cdots \oplus V_{r-1}, \]
\[ [V_0, V_j] = V_{j+1}, j = 1, 2, \ldots, r - 1, \] \hfill (2.7)

then we call $G$ a Carnot group. Let $\circ$ be the group law on $G$, then the left translation operator is $L_p : q \rightarrow p \circ q$, denote by $(L_p)_*$ the differential of $L_p$. Now we can define the horizontal subspace as

\[ HG_p = (L_p)_*(V_0), \]

for any point $p \in G$, and the horizontal bundle as

\[ HG = \bigcup_{p \in G} HG_p. \]

Then we further consider the vertical distribution on $G$ defined by

\[ VG_p = (L_p)_*(V_1 \oplus \cdots \oplus V_{r-1}), \]
\[ VG = \bigcup_{p \in G} VG_p. \]

Now, we fix a basis $X_1, \cdots, X_{\ell}$ formed by the left invariant vector fields, then, by (2.7), we deduce that

\[ [\Gamma(VG), X_k] \in \Gamma(VG), \] \hfill (2.8)

and fix the inner product $< \cdot, \cdot >$ in $TG$ such that the system of left-invariant vector fields $\{X_1, \cdots, X_\ell, Y_1, \cdots, Y_{n-\ell}\}$ is an orthonormal basis of $TG$, so there is a natural nonholonomic connection $\nabla$ on $HG$ satisfying

\[ \nabla_X Y = X(Y^i)X_i, \] \hfill (2.9)

where $Y = Y^i X_i$.

For sub-Riemannian manifolds, J. A. Shouten first considered the curvature problem of non-holonomic connections (see [4]), he defined a curvature tensor as follows:

**Definition 2.4.** A Shouten curvature tensor is a mapping $K : \Gamma(V_0) \times \Gamma(V_0) \rightarrow \Gamma(V_0)$ defined by

\[ K(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]_0} Z - [[X, Y], Z]_0, \] \hfill (2.10)

where $X, Y, Z \in \Gamma(V_0)$.

If $M$ is a Carnot group $G$, the Schouten curvature tensor, because of (2.8), is of the form

\[ K(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]_0} Z. \] \hfill (2.11)
Remark 2.2. It is easy to check that Definition 2.4 is well defined. In fact, we know that the following formulas are tenable.

\[
\begin{align*}
K(fX, Y)Z &= fK(X, Y)Z, \\
K(X, fY)Z &= fK(X, Y)Z, \\
K(X, Y)(fZ) &= fK(X, Y)Z,
\end{align*}
\]

For Shouten tensor, by using Jacobi identity of Poisson bracket and Definition 2.4, we have

\[
\begin{align*}
K(X, Y)Z &= -K(Y, X)Z, \\
K(X, Y)Z + K(Y, Z)X + K(Z, X)Y &= 0.
\end{align*}
\]

It is well known that there hold the following formulas for the curvature tensor \( R \) over Riemannian manifolds

\[
\begin{align*}
R(X, Y, Z, W) &= -R(Y, X, Z, W), \\
R(X, Y, Z, W) &= -R(X, Y, W, Z), \\
R(X, Y, Z, W) &= R(Z, W, X, Y).
\end{align*}
\]

We also define \((0,4)\)-tensor by \( K(X, Y, Z, W) = g(K(X, Y)Z, W) \), which satisfies the following

\[
\begin{align*}
K(X, Y, Z, W) &= -K(Y, X, Z, W), \\
K(X, Y, Z, W) + K(Y, Z, X, W) + K(Z, X, Y, W) &= 0.
\end{align*}
\]

However, since the horizontal distribution \( V_0 \) is not involutive, so the curvature tensor \( K \) does not satisfy \( K(X, Y, Z, W) = -K(X, Y, W, Z) \), we only obtain

\[
K(X, Y, Z, W) = -K(X, Y, W, Z) - g([[[X, Y]_1, W]_0, Z]) - g([[[X, Y]_1, W]_0, Z]) + [X, Y]_1 g(Z, W).
\]

When \( V_0 \) is involutive, i.e., \([X, Y]_1 = 0\), in this setting, we have the analogue similar to Riemannian curvature tensors.

Remark 2.3. Since the curvature tensor \( K \) does not satisfy properties (2.15), (2.16), so we can not give out the second Bianchi identity of Shouten curvature tensors similar to Riemannian curvature tensors.

Let \( \{e_i\} \) be a basis of \( V_0 \), we denote by

\[
\begin{align*}
K(e_i, e_j)e_k &= K^h_{ij}e_h, \\
\nabla e_i e_j &= \{^k_{ij}\}e_k, \\
[e_i, e_j]_0 &= \Omega^k_{ij}e_k, \\
[e_i, e_j]_1 &= M^{a}_{ij}e_a, \\
[[e_i, e_j]_1, e_k]_0 &= M^{h}_{ij}A_{ak}e_h.
\end{align*}
\]
Then we know that
\[ K_{ijk}^h = \epsilon_i(t^h_{|jk}) - \epsilon_j(t^h_{|ik}) + \{ e^c \}_{|jk}^{t^h} - \{ e^c \}_{|ik}^{t^h} - \Omega^e_{ij} t^h_{|ke} - M^a_{ij} \Lambda^h_{ak} \] (2.19)
Since \( V \) is torsion free, then we get
\[ \nabla_e e_j - \nabla_j e_i - [e_i, e_j]_0 = 0, \]
so we arrive at
\[ \{ k^e_{ij} \} - \{ k^e_{ji} \} = \Omega^e_{ij}, \] (2.20)
we further have
\[ [e_i, e_j] - \Omega^e_{ij} e_k = M^a_{ij} e_a. \] (2.21)
Especially, if the horizontal distribution \( V_0 \) is involutive, then we obtain
\[ K_{ijk}^h = \epsilon_i(t^h_{|jk}) - \epsilon_j(t^h_{|ik}) + \{ e^c \}_{|jk}^{t^h} - \{ e^c \}_{|ik}^{t^h} - \Omega^e_{ij} t^h_{|ke}. \] (2.22)
In this basis, (2.12), (2.13) can be rewritten, respectively, as
\[ K_{ijk}^h = -K_{jik}^h, \] (2.23)
\[ K_{ijk}^h + K_{jki}^h + K_{kij}^h = 0, \] (2.24)
We call (2.13), (2.18) and (2.24) the first Bianchi identity of sub-Riemannian connection \( \nabla \).
In (2.24), by taking \( j = h = e \) and using (2.23), we get
\[ K_{kic} = K_{kic} - K_{ick}, \] (2.25)
It is clear that \( K_{kic} \) is an anti-symmetric \((0,2)\) tensor, which is different from Riemannian case. So
\[ 0 = K_{kic} g^k_i + K_{kic} g^k_i = K_{kic} g^k_i + K_{kic} g^k_i = 2K_{kic} g^k_i. \]
Now multiplying \( g^{ki} \) at both side of (2.25), then \( g^{ki} K_{kic} - K_{ick} g^{ki} = 0 \). Similar to the case of Riemannian manifolds, we call \( K = g^{ki} K_{kic} \) the scalar curvature of Shouten curvature tensors.

3 Main Theorems and Proofs

Theorem 2.1 shows that there exists unique metric and torsion free nonholonomic connection in sub-Riemannian manifolds, while there also exist other some nonholonomic connections which is not compatible with sub-Riemannian metric any more, nor is torsion free. For the first time, we introduce a very important nonholonomic connection-semi-sub-Riemannian connection. Roughly speaking, a semi-sub-Riemannian connection is a nonholonomic connection with non-vanishing torsion tensor which is compatible with sub-Riemannian metric. More precisely, let \( D \) be another non-holonomic connection on \( M \) and the coefficients be \( \Gamma^k_{ij} \). \( D \) is said to be a metric connection if it satisfies
\[ (D_Z g)(Y, Z) = Zg(X, Y) - g(D_Z X, Y) - g(X, D_Z Y) = 0, \forall X, Y, Z \in V_0, \] (3.1)
Now we give a new definition below
Definition 3.1. A nonholonomic connection is called a semi-sub-Riemannian connection, if it is metric and its torsion tensor satisfies

\[ T(X, Y) = D_X Y - D_Y X - [X, Y]_0 = \pi(Y)X - \pi(X)Y, \forall X, Y, Z \in V_0, \]  

where \( \pi \) is a smooth 1-form.

For the semi-sub-Riemannian connection \( D \), recurrent \( X, Y, Z \in V_0 \) in (3.1), and by a direct computation, we get

\[ D_X Y = \nabla_X Y + \pi(Y)X - g(X, Y)P, \]  

where \( P \) is a vector field defined by \( g(P, X) = \pi(X) \).

Remark 3.1. (3.3) is also called semi-symmetric connection transformation of \( \nabla \). It is easy to check the semi-symmetric connection transformation of metric torsion-free nonholonomic connection is still a metric connection by (3.3). This transformation will change horizontal curves into horizontal curves, however it is not true for the horizontal curves paralleling with itself (i.e. normal geodesics), we will discuss the connection transformations that conserve the normal geodesics in forthcoming papers.

In local frame \( \{ e_i \} \), denote by \( \pi(e_i) = \pi_i, \pi^i = g^{ij} \pi_j \), then we know

\[ \Gamma^k_{ij} = \{^k_{ij}\} + \delta^k_i \pi_j - g^{ij} \pi^k, \]  

we define the Schouten curvature tensor of semi-sub-Riemannian connection \( D \) is

\[ R^h_{ijk} = e_i (\Gamma^h_{jk}) - e_j (\Gamma^h_{ik}) + \Gamma^e_{jk} \Gamma^h_{ie} - \Gamma^e_{ik} \Gamma^h_{je} - \tilde{\Omega}^h_{ij} \Gamma^h_{ke} - \tilde{M}^a_{ij} \tilde{\Lambda}^h_{ak}, \]  

where

\[
\begin{align*}
[e_i, e_j]_0 &= \tilde{\Omega}^k_{ij} e_k, \\
[e_i, e_j]_1 &= \tilde{M}^a_{ij} e_a, \\
[[e_i, e_j], e_k]_0 &= \tilde{M}^a_{ij} \tilde{\Lambda}^h_{ak} e_a,
\end{align*}
\]

then by using (2.20), (2.21) and (3.4), we have

\[ \begin{align*}
\tilde{\Omega}^k_{ij} &= \Omega^k_{ij} \\
\tilde{M}^a_{ij} &= M^a_{ij} \\
\tilde{\Lambda}^h_{ak} &= \Lambda^h_{ak}
\end{align*} \]  

Substituting (3.4) and (3.6) into (3.5) and by straightway computation, we can get the relation between the Schouten curvature tensor of \( D \) and \( V \) as follows

\[ R^h_{ijk} = K^h_{ijk} + \delta^h_i \pi_j k - \delta^h_j \pi_i k + \pi^h_j g_{ik} - \pi^h_i g_{jk}, \]  

where

\[ \pi_{ik} = \nabla_i \pi_k - \pi_i \pi_k + \frac{1}{2} g_{ik} \pi_h \pi^h, \]  

\[ (3.8) \]
\[\pi_i^j = \pi_{jk}g^{jh} = \nabla_j\pi^i - \pi_i\frac{1}{2}\delta_j^i\pi_h^h, \quad (3.9)\]
\[\nabla_i\pi_j = e_i(\pi_j) - \left(\frac{k}{ij}\right)\pi_k. \quad (3.10)\]

Here we call \(\pi_{ij}\) the characteristic tensor of \(D\), and \(\alpha = \pi_{ij}g^{ij} = \pi_i^i\). Contracting \(j\) and \(h\) in (3.7), we have
\[R_{ijk}^e = K_{ijk}^e + (\ell - 2)\pi_{ik} + \alpha g_{ik}. \quad (3.11)\]

Multiplying (3.11) by \(g^{jk}\) we get
\[R = K + 2(\ell - 1)\alpha, \quad (3.12)\]
so there is
\[\alpha = \frac{R - K}{2(\ell - 1)}. \quad (3.13)\]

Substituting (3.13) into (3.11) we have
\[\pi_{ik} = \frac{1}{\ell - 2}(R_{ik}^e - K_{ik}^e) - \frac{R - K}{2(\ell - 1)}g_{ik}, \quad (3.14)\]
\[\pi_i^h = \frac{1}{\ell - 2}(R_{ik}^e - K_{ik}^e)g^{jh} - \frac{R - K}{2(\ell - 1)}\delta_i^h, \quad (3.15)\]
then substituting (3.14), (3.15) into (3.7), we get
\[R_{ijk}^h - \frac{1}{\ell - 2}\{\delta_j^h(R_{ik}^e - \frac{R}{2(\ell - 1)}g_{ik}) - \delta_i^h(R_{jk}^e - \frac{R}{2(\ell - 1)}g_{jk})\}
- \ g_{ik}(R_{je}^e g^{jh} - \frac{R}{2(\ell - 1)}\delta_j^h) + g_{jk}(R_{ie}^e g^{jh} - \frac{R}{2(\ell - 1)}\delta_i^h))
= K_{ijk}^h - \frac{1}{\ell - 2}\{\delta_j^h(K_{ik}^e - \frac{R}{2(\ell - 1)}g_{ik}) - \delta_i^h(K_{jk}^e - \frac{K}{2(\ell - 1)}g_{jk})\}
+ \ g_{ik}(K_{je}^e g^{jh} - \frac{K}{2(\ell - 1)}\delta_j^h) - g_{jk}(K_{ie}^e g^{jh} - \frac{K}{2(\ell - 1)}\delta_i^h)). \quad (3.16)\]

Let
\[\delta_{ijk}^h = R_{ijk}^h - \frac{1}{\ell - 2}\{\delta_j^h(R_{ik}^e - \frac{R}{2(\ell - 1)}g_{ik}) - \delta_i^h(R_{jk}^e - \frac{R}{2(\ell - 1)}g_{jk})\}
- \ g_{ik}(R_{je}^e g^{jh} - \frac{R}{2(\ell - 1)}\delta_j^h) + g_{jk}(R_{ie}^e g^{jh} - \frac{R}{2(\ell - 1)}\delta_i^h))
= R_{ijk}^h - \frac{1}{\ell - 2}\{\delta_j^h(R_{ik}^e - \delta_i^hR_{jk}^e + g_{ik}g^{jh}R_{je}^e - g_{jk}g^{jh}R_{ie}^e\}
+ \frac{R}{(\ell - 1)(\ell - 2)}(g_{ik}\delta_j^h - g_{jk}\delta_i^h),\]
\[ S^h_{ijk} = K^h_{ijk} - \frac{1}{\ell - 2} \{ \delta^h_i (K^e_{iek} - \frac{K}{2(\ell - 1)}g_{ik}) - \delta^h_i (K^e_{jek} - \frac{K}{2(\ell - 1)}g_{jk}) \\
+ g_{ik}(K^e_{jef}g^{fh} - \frac{K}{2(\ell - 1)}\delta^h_i) - g_{jk}(K^e_{ief}g^{fh} - \frac{K}{2(\ell - 1)}\delta^h_i)) \}
+ \frac{1}{\ell - 2} \{ \delta^h_i K^e_{iek} - \delta^h_i K^e_{jek} + g_{ik}g^{fh}K^e_{jef} - g_{jk}g^{fh}K^e_{ief} \}
+ \frac{1}{\ell - 1}(g_{ik}\delta^h_j - g_{jk}\delta^h_i). \] (3.17)

Therefore we have the following

**Theorem 3.1.** \( S^h_{ijk} = \bar{S}^h_{ijk}, \) namely, \( S^h_{ijk} \) is an invariant under the nonholonomic connection transformation \( \nabla \rightarrow D. \)

It is well known that one of differences between sub-Riemannian geometry and Riemannian case is that there exists a kind of singular geodesics, which does not satisfy the geodesic equation, in sub-Riemannian geometry, so when we consider the projective transformation of \( \nabla, \) we should modify that, if semi-sub-Riemannnian connection \( D \) and sub-Riemannian connection \( \nabla \) has the same normal geodesics, we call it the projective transformation of \( \nabla. \) Therefore the Weyl projective transformation of \( \nabla \) conserves the normal geodesics invariant.

Recall the conformal curvature tensor and projective curvature tensor (see [18]) of sub-Riemannian connection \( \nabla \) are respectively,

\[ C^h_{ijk} = K^h_{ijk} - \frac{1}{\ell - 2} \{ \delta^h_i (K^e_{iek} - \frac{1}{\ell}K^e_{ike} - \frac{K}{2(\ell - 1)}g_{ik}) \\
- \delta^h_i (K^e_{jek} - \frac{1}{\ell}K^e_{jke} - \frac{K}{2(\ell - 1)}g_{jk}) \\
+ g_{ik}(K^e_{jef}g^{fh} - \frac{1}{\ell}K^e_{jef}g^{fh} - \frac{K}{2(\ell - 1)}\delta^h_i) \\
- g_{jk}(K^e_{ief}g^{fh} - \frac{1}{\ell}K^e_{ief}g^{fh} - \frac{K}{2(\ell - 1)}\delta^h_i) \}
+ \frac{1}{\ell} \delta^h_i K^e_{i j e}, \]

\[ W^h_{ijk} = K^h_{ijk} - \frac{1}{\ell - 1}(\delta^h_i K^e_{iek} - \delta^h_i K^e_{jek}). \]

For the semi-sub-Riemannnian connection \( D, \) we define the Weyl conformal
curvature tensor and the projective curvature tensor, respectively, by

\[ \bar{C}_{ijk} = \bar{R}_{ijk} - \frac{1}{\ell - 2} \left[ \delta^h_i (R^e_{jek} - \frac{1}{\ell} R^e_{jke} - \frac{R}{2(\ell - 1)} g_{jk}) \right. \]

\[ \left. - \delta^h_i (R^e_{jck} - \frac{1}{\ell} R^e_{jke} - \frac{R}{2(\ell - 1)} g_{jk}) \right] \]

\[ + g_{ik} (R^e_{jcf} g^{fh} - \frac{1}{\ell} R^e_{jcf} g^{fh} - \frac{R}{2(\ell - 1)} \delta^h_i) \]

\[ - g_{jk} (R^e_{icf} g^{fh} - \frac{1}{\ell} R^e_{icf} g^{fh} - \frac{R}{2(\ell - 1)} \delta^h_i) \]

\[ + \frac{1}{\ell} \delta^h_k R^e_{jcf}, \] (3.18)

\[ \bar{W}_{ijk} = \bar{R}_{ijk} - \frac{1}{\ell - 1} (\delta^h_i R^e_{jck} - \delta^h_i R^e_{jck}). \] (3.19)

**Remark 3.2.** By using (3.7) and (3.11), we get

\[ \bar{C}_{ijk} = C_{ijk} - \frac{1}{\ell} (\delta^h_j \pi_{ik} - \delta^h_i \pi_{jk} + g_{ik} \pi^h_j - g_{jk} \pi^h_i) \]

\[ - \frac{2\alpha}{\ell(\ell - 2)} (\delta^h_j g_{ik} - \delta^h_i g_{jk}) - \frac{\ell - 2}{\ell} \delta^h_k \pi_{ij} - \frac{\alpha}{\ell} \delta^h_k g_{ij}, \]

\[ \bar{W}_{ijk} = W_{ijk} + \frac{1}{\ell - 1} (\delta^h_j \pi_{ik} - \delta^h_i \pi_{jk}) + (g_{ik} \pi^h_j - g_{jk} \pi^h_i) \]

\[ - \frac{\alpha}{\ell - 1} (\delta^h_j g_{ik} - \delta^h_i g_{jk}). \]

Therefore unlike the Riemannian case, here the Weyl conformal curvature tensor \( C_{ijk} \) is no longer an invariant under the connection transformation from sub-Riemannian connection \( \nabla \) to semi-sub-Riemannian connection \( D \).

Now we assume that \( \bar{C}_{ijk} = C_{ijk} \), then

\[ (\delta^h_j \pi_{ik} - \delta^h_i \pi_{jk} + g_{ik} \pi^h_j - g_{jk} \pi^h_i) + \frac{2\alpha}{\ell - 2} (\delta^h_j g_{ik} - \delta^h_i g_{jk}) + (\ell - 2) \delta^h_k \pi_{ij} + \alpha \delta^h_k g_{ij} = 0. \]

Contracting the above equation by \( k = h \), we obtain

\[ (\ell - 2) \pi_{ij} + \alpha g_{ij} = 0, \]

multiplying \( g^{ij} \) on both side of above equation, further we get \( \pi = 0 \). The inverse is also true, so we have the following result.

**Theorem 3.2.** The semi-sub-Riemannian connection \( D \) and the sub-Riemannian connection \( \nabla \) have the same conformal curvature tensor if and only if \( \alpha \) is vanishing.

Then we assume that \( \bar{W}_{ijk} = W_{ijk} \), hence we have

\[ \frac{1}{\ell - 1} (\delta^h_j \pi_{ik} - \delta^h_i \pi_{jk}) + (g_{ik} \pi^h_j - g_{jk} \pi^h_i) - \frac{\alpha}{\ell - 1} (\delta^h_j g_{ik} - \delta^h_i g_{jk}) = 0. \] (3.20)
By multiplying $g_{jk}$ in (3.20), we get

$$\pi_i^h = \frac{\alpha}{\ell} \delta_i^h, \text{ or } \pi_{jh} = \frac{\alpha}{\ell} g_{jh}.$$  

This implies the following

**Theorem 3.3.** The semi-sub-Riemannian connection $D$ and the horizontal connection $\nabla$ have the same projective curvature tensor if and only if the characteristic tensor is proportional to a metric tensor.

**Proof.** We just prove the sufficiency of Theorem 3.3. Let $\pi_i^l = \lambda \delta_i^l$, then $\pi = \pi_i^l = \lambda \ell$, and $\pi_{ij} = \lambda g_{ij}$. Substituting these equations above into the second formula in Remark 3.2, we get $W_{ijk}^h = W_{ijk}^h$. This ends the proof of Theorem 3.3. □

Theorem 3.3 implies the connection transformations from sub-Riemannian connection $\nabla$ to semi-sub-Riemannian connection $D$ that change normal geodesics into normal geodesics also conserve the projective curvature tensor invariant under certain conditions.

**Remark 3.3.** By comparing the tensor $S_{ijk}^h$ with the conformal curvature tensor $\bar{C}_{ijk}^h$ defined by (3.18), we find that

$$\bar{C}_{ijk}^h = \bar{S}_{ijk}^h + \frac{1}{\ell(\ell - 2)} \left( \delta_i^h R_{jke}^e - \delta_i^h g_{jke} + g_{ik} R_{jfe}^e g_{fh} - g_{jk} R_{ife}^e g_{fh} \right) + \frac{1}{\ell} \delta_i^h R_{jke}^e \tag{3.21}$$

Given that $K_{ije}^e = 0$, then $R_{ije}^e = 0$ (for any $i, j$), so $\bar{C}_{ijk}^h = \bar{S}_{ijk}^h$. Hence Theorem 3.1 implies that a geometric characteristic of tensor $S_{ijk}^h$ is conformal invariant tensor under certain conditions.

Now we assume $R_{ijk}^h = K_{ijk}^h$, then

$$\delta_i^h \pi_{jk} - \delta_i^h \pi_{jk} + \pi_{ij}^h g_{jk} - \pi_i^h g_{jk} = 0. \tag{3.22}$$

Contracting the equation (3.22) with $i$ and $h$, we get

$$(2 - \ell) \pi_{jk} - \alpha g_{jk} = 0. \tag{3.23}$$

Multiplying the equation (3.23) by $g_{jk}$ we get

$$2(\ell - 1) \alpha = 0,$$

and $\ell > 2$, therefore $\alpha = 0$; the converse is also true, thus we have

**Theorem 3.4.** The semi-sub-Riemannian connection $D$ and the sub-Riemannian connection $\nabla$ have the same Schouten curvature tensor if and only if $\alpha$ is vanishing.
A geometric characteristic of Theorem 3.4 is the connection transformations from sub-Riemannian connection $\nabla$ to semi-sub-Riemannian connection $D$ conserve the Schouten curvature tensor invariant under certain conditions.

Now we consider the case of $R^h_{ijk} = 0$, that is, there hold

$$K^h_{ijk} = \delta^h_j \pi_{jk} - \delta^h_i \pi_{ik} + \pi^h_i g_{jk} - \pi^h_j g_{ik},$$  \hspace{1cm} (3.24)

let $j = h = e$, we obtain

$$K^e_{iek} = (2 - \ell) \pi_{ik} - \alpha g_{ik},$$  \hspace{1cm} (3.25)

Multiplying the equation (3.25) by $g^{ik}$ we get

$$K = K^e_{iek} g^{ik} = 2(1 - \ell) \alpha,$$

So we have

$$\alpha = \frac{K}{2(1 - \ell)}.$$  \hspace{1cm} (3.26)

Substituting (3.26) into (3.25), we get

$$\pi_{ik} = \frac{1}{2 - \ell} \left( K^e_{iek} - \frac{K}{2(1 - \ell)} g^{ik} \right),$$  \hspace{1cm} (3.27)

Similarly, we substitute (3.27) into (3.24), we have

$$K^h_{ijk} = -\frac{1}{\ell - 2} (\delta^h_j K^e_{jek} - \delta^h_i K^e_{iek} + g_{jk} K^e_{iek} g^{fh} - g_{ik} K^e_{jfe} g^{fh})$$  \hspace{1cm} 

$$+ \frac{K}{(\ell - 2)(\ell - 1)} (g_{jk} \delta^h_i - g_{ik} \delta^h_j)$$  \hspace{1cm} (3.28)

By using (3.17), equation (3.28) is equivalent to $S^h_{ijk} = 0$. This implies the following

**Theorem 3.5.** The sub-Riemannian manifold $(M, V_0, g)$ associated with a semi-sub-Riemannian connection $D$ is flat (i.e. $R^h_{ijk} = 0$) if and only if the tensor $S^h_{ijk}$, defined by (3.17), of sub-Riemannian connection $\nabla$ is vanishing and $\pi_{ik} = \frac{1}{2 - \ell} \left( K^e_{iek} - \frac{K}{2(1 - \ell)} g^{ik} \right)$.

**Proof.** Here just to prove the sufficiency. If $\pi_{ik} = \frac{1}{2 - \ell} \left( K^e_{iek} - \frac{K}{2(1 - \ell)} g^{ik} \right)$, then $\alpha = \frac{K}{2(1 - \ell)}$, so $K^e_{iek} = (2 - \ell) \pi_{ik} - \alpha g_{ik}$, and

$$R^e_{iek} = K^e_{iek} + (\ell - 2) \pi_{ik} + \alpha g_{ik} = 0$$

By the first Bianchi identity, we know

$$R^e_{jke} = R^e_{kej} - R^e_{iek} = 0,$$

and

$$C^h_{ijk} = S^h_{ijk} + \frac{1}{\ell(\ell - 2)} (\delta^h_j R^e_{jke} - \delta^h_i R^e_{jke} + g_{jk} R^e_{jfe} g^{fh} - g_{ik} R^e_{jfe} g^{fh}) + \frac{1}{\ell} \delta^h_i R^e_{jke}$$  \hspace{1cm} 

$$= S^h_{ijk} = 0$$

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Therefore, we have

\[
R^h_{ijk} = \tilde{C}^h_{ijk} + \frac{1}{\ell - 2} \left( \delta^h_i (R^e_{jke} - \frac{1}{\ell} R^e_{jke}) - \frac{R}{2(\ell - 1)} g^h_{jk} \right)
\]

\[
+ \delta^h_i (R^e_{jke} - \frac{1}{\ell} R^e_{jke}) - \frac{R}{2(\ell - 1)} g^h_{jk}
\]

\[
- g_{ik} (R^e_{jef} g^{fh} - \frac{1}{\ell} R^e_{jfe} g^{fh} - \frac{R}{2(\ell - 1)} \delta^h_i)
\]

\[
+ g_{jk} (R^e_{jef} g^{fh} - \frac{1}{\ell} R^e_{jfe} g^{fh} - \frac{R}{2(\ell - 1)} \delta^h_i))
\]

\[- \frac{1}{\ell} \delta^h_k R^e_{ije} = 0.
\]

This completes the proof of Theorem 3.5.

We now assume that, for any \(X, Y, Z \in V_0\), there are

\[
R(X, Y)Z = 0,
\]

\[
(\nabla_X T)(Y, Z) = 0.
\]

A manifold satisfying these two conditions is called a group manifold with respect to \(\nabla\).

**Example 3.1.** Carnot group \(G\) is a group manifold with respect to \(\nabla\) defined by (2.9).

In fact, let \(X = X^i X_i, Y = Y^j X_j, Z = Z^k X_k\) and by (2.9) and (2.11), then the horizontal curvature tensor can be given exactly as

\[
K(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z
\]

\[
= X^i X_i(Y^j) X_j(Z^k) X_k + Y^j X^i X_i(Z^k) X_k - Y^j X^i (X^j) X_i (Z^k) X_k
\]

\[
- Y^j X^i X_j (Z^k) X_k - X^i (X^j) X_j (Z^k) X_k + Y^j X_i (X^j) X_i (Z^k) X_k
\]

\[
= 0.
\]

On the other hand, the horizontal torsion tensor of horizontal vector fields of \(Z, Y\) is

\[
T(Y, Z) = \nabla_Y Z - \nabla_Z Y - [Y, Z]_0
\]

\[
= Y^j X_j (Z^k) X_k - Z^k X_k (Y^j) X_j - Y^j X_j (Z^k) X_k + Z^k X_k (Y^j) X_j = 0
\]

\[
T(\nabla_X Y, Z) = \nabla_{\nabla_X Y} Z - \nabla_Z \nabla_X Y - [\nabla_X Y, Z]_0
\]

\[
= X(Y^j) \nabla_X Z - Z(X(Y^j)) X_j - X(Y^j) \nabla_Z X_j - X(Y^j) X_j (Z^k) X_k
\]

\[
+ Z(X(Y^j)) X_j - X(Y^j) [X_j, Z]_0
\]

\[
= 0,
\]

so one has

\[
(\nabla_X T)(Y, Z) = \nabla_X T(Y, Z) - T(\nabla_X Y, Z) - T(Y, \nabla_X Z) = 0.
\]
If \( D \) is a semi-sub-Riemannian connection, then we have
\[
(D_X T)(Y, Z) = 0 \iff (D_X \pi)(Z)Y - (D_X \pi)(Y)Z = 0,
\]
(3.29)

In a local frame \( \{e_i\} \), by taking \( X = e_i, Y = e_j, Z = e_k \) then we get
\[
0 = \{D_{e_i} \pi_k - \pi(D_{e_i} e_k)\}e_j - \{D_{e_i} \pi_j - \pi(D_{e_i} e_j)\}e_k
\]
\[
= \{e_i(\pi_k) - \Gamma^e_{jk} \pi_e\}e_j - \{e_i(\pi_j) - \Gamma^e_{ij} \pi_e\}e_k,
\]

Thus we know
\[
e_i(\pi_j) - \Gamma^e_{ij} \pi_e = 0,
\]
substituting (3.4) into (3.30) and using (3.8) we deduce
\[
\pi_{ij} = -\frac{1}{2} g_{ij} \pi_e \pi^e,
\]
(3.31)

By virtue of Theorem 3.3, we have \( W^h_{ijk} = \tilde{W}^h_{ijk} \). Since \( \tilde{R}^h_{ijk} = 0 \), we get \( \tilde{R}^e_{iek} = 0 \), then \( \tilde{W}^h_{ijk} = 0 \), so \( W^h_{ijk} = 0 \). This implies the following

**Proposition 3.6.** If sub-Riemannian manifold \( (M, V_0, g) \) is a group manifold with respect to the semi-sub-Riemannian connection \( D \), then \( M \) is projective flat.

Then substituting (3.31) into (3.24) we get
\[
K^h_{ijk} = \pi_e \pi^e (\delta^h_{jk} g_{ik} - \delta^h_{ij} g_{jk}).
\]

It is not hard to see by a direct checking up on a few things that the converse is also true, hence we obtain

**Theorem 3.7.** A sub-Riemannian manifold \( (M, V_0, g) \) with vanishing curvature with respect to semi-sub-Riemannian connection \( D \) is a group manifold if and only if \( M \) is of constant curvature.

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### References

[1] Agache, N.S., Chafle, M.R.: A semi-symmetric non-metric connection on a Riemannian manifold. Indian J. Pure Appl. Math. **23**(6), 399-409(1992)
[2] Bellaiche, A.: The tangent space in sub-Riemannian geometry. Progr. Math. **144**, 1-84(1996)

[3] Capogna, L., Lin, F.H.: Legendrian energy minimizers, Part I: Heisenberg group target. Cal. Var. **12**(2), 145-171(2001)

[4] Cantrijn, F., Langerock, B.: Generalized connections over a vector bundle map. Differential Geom. Appl. **18**(3), 295-317(2003)

[5] De, U.C., Biswas, S.C.: On a type of semi-symmetric non-metric connection on a Riemannian manifold. Istanbul Univ. Mat.Derg. **55/56**, 237-243(1996/1997)

[6] De, U.C., Kamila, D.: On a type of semi-symmetric non-metric connection on a Riemannian manifold. J. Indian Inst.Sci. **75**, 707-710(1995)

[7] Danielli, D., Garofalo, N., Nhieu, D.M.: Minimal surfaces, surfaces of constant mean curvature and isoperimetry in Carnot groups, 2001 Preprint

[8] Danielli, D., Garofalo, N., Nhieu, D.M.: Sub-Riemannian calculus on hypersurfaces in Carnot groups. Adv. Math. **215**, 292-378(2007)

[9] Franchi, B., Serapioni, R., Cassano, F.S.: Rectifiability and perimeter in Heisenberg group. Math. Ann. **321**(3), 479-531(2001)

[10] Friedmann, A., Schouten, J. A. Über die Geometrie der Halbsymmerischen Übertragung, Math. Z. **21**(1924)211-233

[11] Fu, F.Y., Yang, X.P., Zhao, P.B.: Geometrical and physical characteristics of a class conformal mapping. J. Geom. Phys. **62**(6), 1467-1479(2012)

[12] Garofalo, N., Nhieu, D.M.: Isoperimetric and Sobolev inequalities for Carnot-Caratheodory spaces and the existence of minimal surfaces. Comm. Pure Appl. Math. **49**, 1081-1144(1996)

[13] Jost, J., Xu, C.J.: Sub-elliptic harmonic maps, Amer. Math. Soc. **350**(11), 4611-4649(1998)

[14] Liang, Y.X.: Some properties of the semi-symmetric metric connection. J. of Xiamen University (Natural Science). **30**(1), 22-24(1991)

[15] Liu, W.S., Sussmann, H.J.: Shortest paths for Sub-Riemannian Metrics on Rank-two Distribution. Mem. Amer. Math. Soc. **118**(1995)

[16] Tan, K.H., Yang, X.P.: On some sub-Riemannian objects of hypersurfaces in sub-Riemannian manifolds, Bull. Austral. Math. Soc. **10**, 177-198(2004)

[17] Yano, K.: On semi-symmetric metric connection, Rev. Roum. Math. Pureset Appl. **15**, 1579-1586(1970)

[18] Zhao, P.B., Jiao, L.: Conformal transformations on Carnot Caratheodory spaces, Nihonkal Mathematical Journal. **17**(2), 167-185(2006)
[19] Zhao, P.B.: The invariant of projective transformation of semi-symmetric metric- recurrent connections and curvature tensor expression, Journal of Engineering Mathematics. 17(1), 105-108(2000)

[20] Zhao, P.B., Song, H. Z.: An invariant of the projective semi-symmetric connection. Chinese Quarterly J. of Math. 17(4), 48-52(2001)

[21] Zhao, P.B.: Some properties of projective semi-symmetric connections, International Mathematical Forum. 3(7), 341-347(2008)

[22] Zhao, P.B.: On F-semi-symmetric connection and F-curvature tensor, J. of University of Science and Technology of Suzhou. 21(5), 1-7(2004)

[23] Zhao, P.B., Song, H.Z., Yang, X.P.: Some invariant properties of the semi-symmetric metric recurrent connection and curvature tensor expressions. Chinese Quarterly J. of Math. 19(4), 355-361(2004)

[24] Zhao, P.B., Shangguan, L.X.: On semi-symmetric connection, J. of Henan Normal University(Natural Science). 19(4), 13-16(1994)