Box Suite Recommendation

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April 27, 2020

Abstract

An algorithm for recommending a suite of boxes for shipping a retailer’s online customer orders is presented. Keywords: box suite, fitting MILP, p-median problem, POPSTAR

1 Introduction

By selecting a cost-optimal suite of boxes for shipping its online customer orders, an online retailer such as Amazon, Walmart, or Target can save a significant amount of money through reduced shipping and material (i.e., cardboard, dunnage, and tape) costs. As a consequence of minimizing cost, the cost-optimal suite also reduces the box outer volume and quantity and weight of material shipped, thereby lowering the online retailer’s environmental carbon footprint. An algorithm is presented that recommends such a cost-optimal suite to an online retailer.

2 Notation

Z denotes the set of integers, N denotes the set of natural numbers, which is the same as the set of positive integers, and N₀ = {0} ∪ N denotes the set of nonnegative integers. If m, n ∈ N₀, {m : n} = {i ∈ N₀ : m ≤ i ≤ n} denotes the set of nonnegative integers greater than or equal to m and less than or equal to n. If N ∈ N, [N] = {1, 2, . . . , N} = {1 : N} denotes the set of natural numbers from 1 to N. R denotes the set of real numbers, R>0 denotes the set of positive real numbers, and R≥0 = {0} ∪ R>0 denotes the set of nonnegative real numbers. If N ∈ N, {aₙ}ₙ=1N ⊂ R, and w ∈ R, SEARCHSORTEDFIRST (aₙ)ₙ=1N, w) finds the index of the first value in {aₙ}ₙ=1N that is greater than or equal to w and assumes that the values in {aₙ}ₙ=1N are sorted in nondecreasing order. If w is greater than all values in {aₙ}ₙ=1N, then N + 1 is returned. The complexity of SEARCHSORTEDFIRST is O (log N) if binary search is used and is O (1) if the algorithm in [1] is used. If a₁, a₂, a₃ ∈ R, sort (a₁, a₂, a₃) returns a permutation (b₁, b₂, b₃) of (a₁, a₂, a₃) such that b₁ ≥ b₂ ≥ b₃. Having only 3 inputs, the complexity of sort is O (1). Given an array of integers W ⊂ N and an integer i ∈ N, push (W, i) appends i to the end of the array W. The complexity of push is O (1) if an appropriately implemented data structure is utilized to store the array of integers.

3 Algorithm

Problem Description and Inputs An online retailer needs a suite of p ∈ N boxes to ship its online customer orders. Moreover, k ∈ {0} ∪ [p − 1] = {0 : p − 1} boxes in the suite may be prescribed (or locked). For example, the online retailer may need some special boxes for shipping liquid-containing items such as liquid detergent. To save money, the online retailer wants such a suite that minimizes total shipping (shipping plus material) cost. One approach to designing this suite is to select I ∈ N historical customer shipments and J ∈ N candidate boxes, where J > p. Each historical customer shipment is assigned a unique index i ∈ [I], and each candidate box is assigned a unique index j ∈ [J]. The set of historical customer shipments should be a small, but statistically significant, randomly sampled subset of the online retailer’s past (e.g., within the previous year) online customer shipments. Each historical customer shipment i ∈ [I] consists of Nᵢ ∈ N₀ rectangular cartons with positive outer lengths, widths, and heights \((pᵢᵠ, qᵢᵠ, rᵢᵠ) \in \mathbb{R}^3_{≥0}\) for \(i \in [I]\) and \(n \in [Nᵢ]\), and \(Mᵢ \in N₀\) foldable items with positive outer lengths, widths, and heights \((sᵢᵠ, tᵢᵠ, uᵢᵠ) \in \mathbb{R}^3_{≥0}\) for \(i \in [I]\) and \(m \in [Mᵢ]\). For the purposes of

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Algorithm 1 Box Suite Recommendation Part I

Input: Box suite size \( p \). I shipments. Each shipment \( i \in [I] \) consists of \( N_i \) rectangular cartons with outer lengths, widths, and heights \( \{(p_{im}, q_{im}, r_{im})\}_{n=1}^{N_i} \) and \( M_i \) foldable items with outer lengths, widths, and heights \( \{(s_{im}, t_{im}, u_{im})\}_{m=1}^{M_i} \). J candidate boxes, where \( J > p \), sorted by nondecreasing inner volume with inner lengths, widths, and heights \( \{(x_j, y_j, z_j)\}_{j=1}^{J} \) and inner volumes \( \{V_j = x_jy_jz_j\}_{j=1}^{J} \). For \( j \in [J-1] \), \( V_j < V_{j+1} \). A subset \( T \subset [J] \) of the candidate boxes, where \( |T| = k \in \{0, 1, 2, \ldots, p-1\} \), must be in the box suite.

Output: A subset \( S^* \subset [J] \) of the candidate boxes such that \( S^* \) ships all the packable shipments with minimum cost, subject to the constraints \( |S^*| = p \) and \( T \subset S^* \). If such a subset does not exist, then \( \emptyset \) is returned.

1. for \( j = 1 \) to \( J \) do
   \( \triangleright \) Iterate over candidate boxes.
   2. \((\hat{x}_j, \hat{y}_j, \hat{z}_j) \leftarrow \text{Sort}(x_j, y_j, z_j)\) \( \triangleright \) Sort box inner dimensions in nonincreasing order.
   3. end for

4. for \( j = 1 \) to \( J \) do
   \( \triangleright \) Iterate over candidate boxes.
   5. \( \Theta_j \leftarrow \{j\} \) \( \triangleright \) \( \Theta_j \) stores the set of boxes into which box \( j \) nests.
   6. for \( k = j + 1 \) to \( J \) do
      \( \triangleright \) Iterate over equal or larger volume candidate boxes.
      7. if \( (\hat{x}_k \leq \hat{x}_j) \land (\hat{y}_k \leq \hat{y}_j) \land (\hat{z}_k \leq \hat{z}_j) \) then
         \( \triangleright \) If box \( j \) nests inside box \( k \).
         8. Push \( (\Theta_j, k) \)
      end if
   end for
   10. end for
11. end for

\( \triangleright \) Construct the \( I \times J \) fitting matrix \( B \) and determine the packable shipments.
12. \( I \leftarrow 0 \) \( \triangleright \) Initialize the number of packable shipments to 0.
13. \( W \leftarrow \emptyset \) \( \triangleright \) \( W \) stores the indices of shipments that are packable.
14. for \( i \) from 1 to \( I \) do
      \( \triangleright \) Iterate over shipments.
      16. \( V_i \leftarrow \sum_{n=1}^{N_i} p_{im} q_{im} r_{im} + \sum_{m=1}^{M_i} s_{im} t_{im} u_{im} \) \( \triangleright \) Liquid volume of shipment \( i \).
      17. \( j_0 \leftarrow \text{SearchSortedFirst}(\{V_j\}_{j=1}^{J}, V_i) \) \( \triangleright \) Find the smallest box whose inner volume \( \geq V_i \).
      18. if \( j_0 = J + 1 \) then continue \( \triangleright \) This shipment does fit into any box, so skip to the next shipment.
      19. end if
      20. if \( N_i = 1 \) then
         \( \triangleright \) Only foldable items in the shipment.
         21. \( B_{j_0,j} \leftarrow 1_{1 \times (J-j_0+1)} \)
         22. continue \( \triangleright \) Skip to the next shipment.
      23. end if
      24. for \( n = 1 \) to \( N_i \) do
         \( \triangleright \) Iterate over cartons in shipment \( i \).
         25. \( (\hat{p}_i, \hat{q}_i, \hat{r}_i) \leftarrow \text{Sort}(p_{im}, q_{im}, r_{im}) \) \( \triangleright \) Sort carton outer dimensions in nonincreasing order.
      26. end for
      27. \( \hat{p}_i \leftarrow \sum_{n=1}^{N_i} \hat{p}_i \) \( \triangleright \) outer \( \hat{p}_i \)\( \hat{q}_i \leftarrow \sum_{n=1}^{N_i} \hat{q}_i \) \( \triangleright \) outer \( \hat{q}_i \)\( \hat{r}_i \leftarrow \sum_{n=1}^{N_i} \hat{r}_i \) \( \triangleright \) outer \( \hat{r}_i \)
      28. \( \hat{p}_i \leftarrow \max_{1 \leq n \leq N_i} \hat{p}_i \) \( \triangleright \) inner \( \hat{p}_i \)
      29. \( \hat{q}_i \leftarrow \max_{1 \leq n \leq N_i} \hat{q}_i \) \( \triangleright \) inner \( \hat{q}_i \)
      30. \( \hat{r}_i \leftarrow \max_{1 \leq n \leq N_i} \hat{r}_i \) \( \triangleright \) inner \( \hat{r}_i \)
      31. for \( j = j_0 \) to \( J \) do
         \( \triangleright \) Iterate over candidate boxes whose inner volume \( \geq V_i \).
         32. if \( B_{j,j} = 1 \) then continue \( \triangleright \) Skip to the next candidate box.
         33. else if \( (\hat{p}_i \leq \hat{x}_j) \land (\hat{q}_i \leq \hat{y}_j) \land (\hat{r}_i \leq \hat{z}_j) \) then
            \( \triangleright \) Each carton in shipment \( i \) must fit in box \( j \).
            34. if \( N_i = 1 \) then \( B_{i,\Theta_j} \leftarrow 1_{1 \times |\Theta_j|} \) \( \triangleright \) Only 1 carton in the shipment.
            35. end if
            36. else if \( (\hat{p}_i \leq \hat{x}_j) \lor (\hat{q}_i \leq \hat{y}_j) \lor (\hat{r}_i \leq \hat{z}_j) \) then \( B_{i,\Theta_j} \leftarrow 1_{1 \times |\Theta_j|} \) \( \triangleright \) Try stacking.
            37. end if
            38. else \( \text{Solve the NP-complete fitting MILP, e.g. with CPLEX or Gurobi.} \)
            39. end if
         39. end if
      41. \( I \leftarrow I + 1 \) \( \triangleright \) \( I \) stores the number of packable shipments encountered so far.
      40. \( J_i \leftarrow \{j \in [J]: B_{i,j} = 1\} \) \( \triangleright \) Find the subset of candidate boxes into which shipment \( i \) fits.
      42. \( \text{Push}(W, i) \) \( \triangleright \) Add shipment \( i \) to the set of packable shipments.
      43. end if
end for
Algorithm 1 Box Suite Recommendation Part II

\[ C \leftarrow 0_{(I+k) \times J} \quad \text{▷ Preallocate memory for a } I + k \text{ by } J \text{ cost matrix.} \]

\[ \text{for } i = 1 \text{ to } \hat{I} \text{ do} \quad \text{▷ Iterate over packable shipments.} \]

\[ \text{for } j \in J \text{ do} \quad \text{▷ Iterate over the subset of candidate boxes into which packable shipment } i \text{ fits.} \]

\[ \text{Compute the cost } C_{ij} \text{ of shipping packable shipment } i \text{ (shipment } W_{ij} \text{) in candidate box } j. \]

\[ \text{end for} \]

\[ \text{end for} \]

\[ \Gamma \leftarrow \left\lfloor \sum_{i=1}^{I} \max_{j \in J} C_{ij} \right\rfloor + 1 \quad \text{▷ Set } \Gamma \text{ to } \infty \text{ or a sufficiently large positive real number.} \]

\[ \text{for } i = 1 \text{ to } \hat{I} \text{ do} \quad \text{▷ Iterate over packable shipments.} \]

\[ C_{ij} \leftarrow \Gamma_{1 \times (J-\{J_{i}\})} \quad \text{▷ Packable shipment } i \text{ ships with infinite cost in boxes into which it does not fit.} \]

\[ \text{end for} \]

\[ \text{for } i = \hat{I} + 1 \text{ to } \hat{I} + k \text{ do} \quad \text{▷ Iterate over fake shipments.} \]

\[ C_{ij} \leftarrow 0 \quad \text{▷ Fake shipment } i \text{ ships for free in box } T_{i-1}. \]

\[ \text{end for} \]

\[ S^* \leftarrow \arg \min \sum_{\{i=1\} \leq p} \min_{j \in S} C_{ij} \quad \text{▷ Solve the NP-hard } p \text{-median problem, e.g. with POPSTAR.} \]

\[ \Phi \leftarrow \sum_{\{i=1\} \leq p} \min_{j \in S^*} C_{ij} \quad \text{▷ Compute the cost of using } S^* \text{ to ship the packable shipments.} \]

\[ \text{if } \Phi \geq \Gamma \text{ then} \]

\[ \text{return } \emptyset \quad \text{▷ There is no feasible solution, so return the empty set.} \]

\[ \text{else} \]

\[ \text{return } S^* \quad \text{▷ Return a cost-optimal suite.} \]

\[ \text{end if} \]

packing, foldable items are assumed to be liquid so that they may be deformed to fit into arbitrarily-shaped empty spaces inside a box. The set of \( J \) candidate boxes should finely discretize the space of all possible boxes and must include the \( k \) locked boxes that must be in the suite. The indices of those \( k \) locked boxes are prescribed in the subset \( I \subset \{\hat{J}\} \), where \(|I| = k\). Each candidate box \( j \in \{\hat{J}\} \) is characterized by a positive inner length, width, and height \((x_j, y_j, z_j) \in \mathbb{R}_+^3\). Therefore, the inner volume of candidate box \( j \in \{\hat{J}\} \) is \( V_j = x_j y_j z_j \in \mathbb{R}_{>0} \). It is assumed that the candidate boxes are sorted by nondecreasing inner volume, which may be realized in \( O(J \log J) \), so that \( V_{j+1} \leq V_j \) for \( j \in \{\hat{J} - 1\} \). The optimal suite is obtained by selecting a subset \( S \subset \{\hat{J}\} \) of the \( J \) boxes, such that \(|S| = p\) and \( T \subset S \), that ships the \( I \) packable shipments, where \( I \leq I \) (since not all of the \( I \) shipments necessarily fit in the \( J \) candidate boxes), with minimum cost. Algorithm (1) gives a method for solving this optimization problem. The next few paragraphs describe the key parts of Algorithm (1).

**Fitting Matrix** The algorithm begins by sorting each box’s dimensions in nonincreasing order. Then, the algorithm determines whether each shipment fits into each candidate box, recording the result in the binary fitting matrix \( B \). It is simple to solve the fitting problem when there are 0 or 1 cartons in the shipment. There are brute force algorithms for solving the fitting problem for 2 and 3 cartons in the shipment, but these are omitted from the algorithm for conciseness. For 2 or more cartons in the shipment, the algorithm first attempts to stack the cartons along each of the box’s three orthogonal axes, to see if they fit. If simple stacking does not work, then the algorithm uses the fitting mixed-integer linear program (MILP) described in Section 4, a feasibility MILP, to solve the fitting problem. Since the fitting MILP must be solved via a third-party MILP solver, which may require a commercial license and which is computationally expensive, substantial run-time improvements can be realized by utilizing the brute force fitting algorithms for 2 and 3 cartons; the source code accompanying [2] provides implementations of the brute force algorithms for 2 and 3 cartons; however, the algorithms presented in that code must be modified to permit rotations.

Also note that in order for a shipment to fit into a candidate box, the box’s inner volume must be greater than or equal to the shipment’s liquid volume and each individual carton in the shipment must fit inside the box. For efficiency, the algorithm checks that these necessary conditions are satisfied first before attempting to solve the stacking problem or fitting MILP when there are 2 or more cartons in the shipment. Moreover, if brute force fitting algorithms for 2 and 3 cartons are available, these may be used to check that all pairs and all triples of cartons fit inside the box before attempting to solve the stacking problem or fitting MILP.

**Cost Matrix** Next, the algorithm removes shipments that could not be packed into any candidate box, leaving \( I \leq I \) packable shipments. For each packable shipment \( i \in \{I\} \), \( J \subset J \) denotes the set of candidate boxes into which packable shipment \( i \) fits. Then, the algorithm constructs the nonnegative cost matrix \( C \) which records the cost of shipping each packable shipment into each candidate box. If packable shipment \( i \in \{I\} \) fits in candidate box \( j \in \{J\} \) (i.e. if \( j \in J \)), the cost \( C_{ij} \in \mathbb{R}_{\geq 0} \) to ship packable shipment \( i \in \{I\} \) in
candidate box \( j \in [J] \) is computed. Note that in order to compute the cost for packable shipment \( i \in [I] \), the data for shipment \( W_i \in [I] \) is needed; that is, \( W_i \) is the shipment index of packable shipment \( i \). The data for shipment \( W_i \in [I] \) may include the outer dimensions and weights of each item, the shipping carrier (e.g. USPS, FedEx, or UPS), the shipping service (e.g. 1 day, 2 day, or 3 day), and the shipping zone, which is determined by the locations of the shipping store or warehouse and the customer. The cost is computed via a detailed formula, which is omitted here, that depends on the shipping cost charged by the carrier and service combination, the cost of the cardboard used to construct the box, the cost to transport the box blank from the box manufacturer to the retailer’s stores or warehouses, the cost of the dunnage used to fill the empty spaces between the packed shipment and the box’s interior, and the cost of the tape used to seal the top and bottom flaps of the box shut. More simply, if instead of minimizing cost, the online retailer wishes to minimize the material (cardboard, dunnage, and tape) used to ship packable shipment \( ij \), then, by construction of \( \Gamma \) and \( C \), there exists a solution to (3.3).

Let \( \Gamma = \mathbb{R}_{\geq 0} \cup \{\infty\} \) be \( p \) or a sufficiently large positive real number such as \( \sum_{i=1}^{\infty} \max_{j \in [J]} \Gamma_{ij} + 1 \in \mathbb{R}_{>0} \). \( \Gamma \) serves as a penalty constant to impose constraints on the solution suite \( S \). In order to ensure that the solution suite \( S \) ships all the packable shipments, \( C_{ij} \) is set to \( \Gamma / |J| \) if packable shipment \( i \) does not fit in candidate box \( j \), i.e. if \( j \in [J] \setminus J_i \). In order to force the boxes in \( T \) into the solution suite \( S \), \( k \) fake shipments are appended to the set of packable shipments and \( k \) rows are added to the bottom of \( C \), where each row, indexed by \( i \in [I + 1, I + 2, \ldots, I + k] \), stores the shipping costs for a fake shipment representing box \( T_{i-j} \). The cost of shipping the fake shipment indexed by \( i \in [I + 1, I + 2, \ldots, I + k] \) in box \( T_{i-j} \) is 0 and in all other boxes is \( \Gamma \). That is, \( C_{T_{i-j}} = 0 \) and \( C_{ij} = \Gamma \) for \( j \in [I] \setminus \{T_{i-j}\} \).

**Formulate the p-Median Problem** The optimization problem that must be solved is

\[
\underset{T \subseteq S \subseteq [J], |S| = p, J_i \cap S \neq \emptyset \forall i \in [I]}{\arg \min} \sum_{i=1}^{I} \min_{j \in S} C_{ij}. \tag{3.1}
\]

Instead, the following optimization problem is solved:

\[
\underset{S \subseteq [J], |S| = p, \cap S \neq \emptyset \forall i \in [I]}{\arg \min} \sum_{i=1}^{I} \min_{j \in S} C_{ij}. \tag{3.2}
\]

The optimization problem (3.2) is an instance of the p-median problem [3, 4, 5], or more precisely the p-facility location problem [6]. Given \( n \in \mathbb{N} \) customers, \( m \in \mathbb{N} \) candidate facilities, the nonnegative cost of serving each customer with each candidate facility, and a fixed \( p \in [m] \), the p-median problem is to find (or open) a subset of \( p \) facilities that serves all the customers with minimum cost, that is, that minimizes the sum of the costs of serving each customer with its minimum cost open facility. In (3.2), the shipments are the customers, the boxes are the candidate facilities, \( I + k \) is the number of fake shipments, and \( J = m \).

The equivalence of (3.1) and (3.2) is established as follows.

**Lemma 1.** (3.1) does not have a solution if and only if

\[
\min_{S \subseteq [J], |S| = p, \cap S \neq \emptyset \forall i \in [I]} \sum_{i=1}^{I} \min_{j \in S} C_{ij} \geq \Gamma. \tag{3.3}
\]

**Proof.** If

\[
\min_{S \subseteq [J], |S| = p, \cap S \neq \emptyset \forall i \in [I]} \sum_{i=1}^{I} \min_{j \in S} C_{ij} \geq \Gamma, \tag{3.4}
\]

then, by construction of \( \Gamma \) and \( C \), \( \forall S \subseteq [J] \) such that \( |S| = p, T \subseteq S \) or \( \exists i \in [I] \ni J_i \cap S = \emptyset \). To see this explicitly, suppose that \( \exists S^* \subseteq [J] \) such that \( |S^*| = p, T \subseteq S^* \), and \( \forall i \in [I] \ni J_i \cap S^* \neq \emptyset \). Then

\[
\sum_{i=1}^{I} \min_{j \in S^*} C_{ij} = \sum_{i=1}^{I} \min_{j \in S^*} C_{ij} + \sum_{i=1}^{I} \min_{j \in S^*} C_{ij} = \sum_{i=1}^{I} \min_{j \in S^*} C_{ij} + \sum_{i=1}^{I} \min_{j \in S^*} C_{ij} = \sum_{i=1}^{I} \max_{j \in S^*} C_{ij} \leq \sum_{i=1}^{I} \max_{j \in S^*} C_{ij} < \Gamma, \tag{3.5}
\]

which contradicts (3.4). The second equality in (3.5) holds because \( T \subseteq S^* \) implies that \( \min_{j \in S^*} C_{ij} = C_{T_{i-j}} = 0 \) \forall i \in [I + 1, I + 2, \ldots, I + k]. \) The third equality in (3.5) holds because \( S^* = (J_i \cap S^*) \cup (([I] \setminus J_i) \cap S^*) \), \( C_{ij} = \Gamma \forall J_i \cap S^* \neq \emptyset \forall i \in [I] \). In this case, there does not exist a solution to (3.1).

Conversely, if there does not exist a solution to (3.1), then \( \forall S \subseteq [J] \) such that \( |S| = p, T \subseteq S \) or \( \exists i \in [I] \ni J_i \cap S = \emptyset \). Therefore, by construction of \( C \), \( \forall S \subseteq [J] \) such that \( |S| = p, \exists i \in [I + k] \ni \min_{j \in S} C_{ij} = \Gamma \). Hence, \( \forall S \subseteq [J] \) such that \( |S| = p \),

\[
\sum_{i=1}^{I} \min_{j \in S} C_{ij} \geq \Gamma, \tag{3.6}
\]
so that
\[
\min_{S \subseteq [J], |S| = p} \sum_{i=1}^{i+k} \min_{j \in S} C_{ij} \geq \Gamma. \tag{3.7}
\]

The contrapositive of Lemma (1) gives the following corollary.

**Corollary.** (3.1) has a solution if and only if
\[
\min_{S \subseteq [J], |S| = p} \sum_{i=1}^{i+k} \min_{j \in S} C_{ij} < \Gamma. \tag{3.8}
\]

**Lemma 2.** If
\[
\min_{S \subseteq [J], |S| = p} \sum_{i=1}^{i+k} \min_{j \in S} C_{ij} < \Gamma, \tag{3.9}
\]

that is, if (3.1) has a solution, then
\[
\text{arg min}_{T \subseteq S^*} \sum_{i=1}^{I} \min_{j \in T} C_{ij} = \text{arg min}_{S \subseteq [J], |S| = p} \sum_{i=1}^{i+k} \min_{j \in S} C_{ij}, \tag{3.10}
\]

**Proof.** If the inequality (3.9) holds, then, by construction of \(C\), \(\forall S^* \in \arg \min_{S \subseteq [J], |S| = p} \sum_{i=1}^{i+k} \min_{j \in S} C_{ij}\) the following properties hold:
(a) \(T \subset S^*\) holds, then, by construction of \(C\), \(\forall S^* \in \arg \min_{S \subseteq [J], |S| = p} \sum_{i=1}^{i+k} \min_{j \in S} C_{ij}\) the following properties hold:
(b) \(J_i \cap S^* \neq \emptyset \forall i \in [I]\).

Consequently,
\[
\text{arg min}_{S \subseteq [J], |S| = p} \sum_{i=1}^{i+k} \min_{j \in S} C_{ij} = \text{arg min}_{T \subseteq S^*} \sum_{i=1}^{I} \min_{j \in T} C_{ij} = \text{arg min}_{T \subseteq S^*} \sum_{i=1}^{I} \min_{j \in S} C_{ij}.
\]

The first equality follows from properties (a) and (b). The third equality holds because \(T \subset S\) implies that \(\min_{j \in S} C_{ij} = C_{i_1} \eta_{i_1-1} = 0 \forall i \in \{I+1, I+2, \ldots, I+k\}\). The fourth equality holds because \(S = (J_i \cap S) \cup ([J_i] \setminus J_i) \cap S)\).

In summary, if (3.2) does not have a solution with cost less than \(\Gamma\), then (3.1) does not have a solution, and if (3.2) does have a solution with cost less than \(\Gamma\), then (3.1) has a solution and the solution set realized by (3.1) equals that realized by (3.2).

**Solving the \(p\)-Median Problem** The \(p\)-median problem is NP-hard [7]. There are many methods to solve the \(p\)-median problem including MILP, Lagrangian relaxation, heuristics (e.g., myopic algorithm, neighborhood search, exchange algorithm, Lin–Kernighan neighborhood exchange algorithm), metaheuristics (e.g., simulated annealing, genetic algorithm, tabu search, heuristic concentration, variable neighborhood search, ant colony optimization, and disperse construction), and hyper-heuristics [5, 8, 9]. POPSTAR [10] is a freely available \(p\)-median problem solver implemented in C++. POPSTAR solves the \(p\)-median problem via a hybrid metaheuristic that combines GRASP with path-relinking and the genetic algorithm [11]. Moreover, POPSTAR performs local searches via a fast implementation of the exchange algorithm [12]. Reference [8] comprehensively surveyed many methods, excluding hyper-heuristics, for solving the \(p\)-median problem and concluded that the overall algorithm implemented by POPSTAR is the best. dc2 [13] is a new, freely available \(p\)-median problem solver implemented in C. dc2 solves the \(p\)-median problem via a new metaheuristic called disperse construction and performs local searches via a fast implementation of the exchange algorithm. Unlike POPSTAR, dc2 is multithreaded and therefore is able to exploit the parallelism offered by multicore CPUs. POPSTAR and dc2 can solve instances of the \(p\)-median problem with several thousand customers, several thousand candidate facilities, and \(p \leq 20\) in a few minutes on a modern laptop. Several metaheuristics, including GRASP and the genetic algorithm, have been implemented on GPGPUs to solve the \(p\)-median problem [14, 15].
Validation  The box suite recommendation may be validated by packing the optimization shipment set and a much larger randomly sampled shipment set into the recommended suite $S^*$, where each shipment is assigned to the minimum cost box in the suite into which it fits. Several metrics, such as percentage of shipments packed into each box, percentage of total cost shipped by each box, percentage of all box outer volume shipped by each box, and percentage liquid void space, collected for both packings can be compared. If the metrics are similar, then the cost savings afforded by the recommended suite $S^*$ should be expected to hold on all shipments. An alternative validation method is to pack the optimization shipment set and a much larger randomly sampled shipment set into several suites $Q \cup S^*$, where each suite $S \in Q$ satisfies $T \subset S$, $|S| = p$, and $J_i \cap S \neq \emptyset \forall i \in [I]$, and ensure that the cost reductions (comparing the cost of each suite $S \in Q$ to the cost of $S^*$) predicted by the optimization shipment set agree with those predicted by the large shipment set. If the various metrics or cost reductions are dissimilar, then Algorithm (1) should be run again using a larger set of randomly sampled historical customer shipments.

Fine-Tuning  The recommended suite $S^*$ can be further refined (or fine-tuned) by running Algorithm (1) again on a new set of candidate boxes $J'$ obtained by taking small variations above and below the inner lengths, widths, and heights of the unlocked boxes $S^* \setminus T$ in the recommended suite $S^*$. Like the original set of candidate boxes $J$, the new set of candidate boxes $J'$ must also include the $k$ locked boxes.

Modifications to Handle Height-Oriented and Bottom-Resting Cartons  Some shipments may contain cartons that must be packed vertically, so that their height dimensions must be parallel to the box’s height dimension. Such cartons will be called height-oriented (HO). This constraint is quite common in online retail, e.g. liquid detergent often must be HO when packed into a shipping box to prevent spillage. When packed into a box, a HO carton may be rotated in only 2 (instead of 6) possible ways. In order to handle this additional constraint, Algorithm (1) must be modified in the following ways. Lines 1-11 in Algorithm (1) must be replaced with the pseudocode given in Algorithm (2), and lines 24-26 in Algorithm (1) must be replaced with the pseudocode given in Algorithm (3). HO cartons may have the additional constraint that they rest at the bottom of the box, to encourage stability and mitigate the possibility of tipping over. To handle this additional constraint, the Boolean condition in line 33 in Algorithm (1), which checks to see if the cartons can be stacked along any of the 3 box dimensions, must be replaced with the Boolean condition

$$\left( \tilde{p}_i \leq \tilde{x}_j \right) \lor \left( \tilde{q}_i \leq \tilde{y}_j \right) \lor \left( \tilde{H}_i \leq 1 \land \tilde{r}_i \leq \tilde{z}_j \right),$$  

(3.12)

where $H_i$ (see lines 1 and 5 in Algorithm (3)) denotes the number of HO cartons in shipment $i$, since stacking the cartons along the box’s height dimension is valid only if there are less than 2 HO cartons in the shipment. More generally, only a proper subset of HO cartons may need to rest at the bottom of the box or some non-HO cartons may need to rest at the bottom of the box. Such cartons will be called bottom-resting (BR). To handle this more general case, let $R_i$ denote the number of BR cartons in shipment $i$. Then the Boolean condition in line 33 in Algorithm (1) must be replaced with the Boolean condition

$$\left( \tilde{p}_i \leq \tilde{x}_j \right) \lor \left( \tilde{q}_i \leq \tilde{y}_j \right) \lor \left( R_i \leq 1 \land \tilde{r}_i \leq \tilde{z}_j \right),$$  

(3.13)

since stacking the cartons along the box’s height dimension is valid only if there are less than 2 BR cartons in the shipment. (3.12) is valid instead of (3.13) only if all HO cartons are BR and there are no non-HO cartons that are BR, in which case $H_i = R_i$. The paragraph “Special Packing Constraints” in the next section discusses how to enforce HO and BR packing constraints in the fitting MILP.
Algorithm 2 Box Suite Recommendation: HO Modification I

1: for $j = 1$ to $J$ do
   \(\triangleright\) Iterate over candidate boxes.
   $\triangleright$ Sort the length and width box inner dimensions in nonincreasing order for shipments with HO cartons.
   \((\bar{x}_j, \bar{y}_j) \leftarrow \text{SORT} (x_j, y_j)\)
   $\triangleright$ Sort box inner dimensions in nonincreasing order for shipments with no HO cartons.
   \((\bar{x}_j, \bar{y}_j, \bar{z}_j) \leftarrow \text{SORT} (x_j, y_j, z_j)\)
2: \begin{align*}
   \Psi_j &\leftarrow \{j\} \\
   \Sigma_j &\leftarrow \{j\}
\end{align*}
3: $\triangleright$ Determine into which candidate boxes each candidate box nests.
4: for $j = 1$ to $J$ do
   \(\triangleright\) Iterate over candidate boxes.
5: \begin{align*}
   \Psi_j &\leftarrow \{j\} \\
   \Sigma_j &\leftarrow \{j\}
\end{align*}
6: $\Psi_j$ stores the set of boxes into which box $j$ nests, only permitting the 2 HO rotations for nesting.
7: $\Sigma_j$ stores the set of boxes into which box $j$ nests, permitting any of the 6 possible rotations for nesting.
8: $\triangleright$ Iterate over equal or larger volume candidate boxes.
9: if $(\bar{x}_j \leq \bar{x}_k) \land (\bar{y}_j \leq \bar{y}_k) \land (\bar{z}_j \leq \bar{z}_k)$ then
   \(\triangleright\) If box $j$ nests inside box $k$ in a HO way.
10: \(\text{push} (\Psi_j, k)\)
11: end if
12: if $(\bar{x}_j \leq \bar{x}_k) \land (\bar{y}_j \leq \bar{y}_k) \land (\bar{z}_j \leq \bar{z}_k)$ then
   \(\triangleright\) If box $j$ nests inside box $k$.
13: \(\text{push} (\Sigma_j, k)\)
14: end if
15: end for
16: end for

Algorithm 3 Box Suite Recommendation: HO Modification II

1: $H_i \leftarrow 0$ \(\triangleleft\) $H_i$ records the number of HO cartons in shipment $i$.
2: for $n = 1$ to $N$ do
   \(\triangleright\) Iterate over cartons in shipment $i$.
3: if carton $n$ is HO then
   $\triangleright$ If carton $n$ must be HO when packed into a box.
4: \begin{align*}
   (\tilde{p}_{in}, \tilde{q}_{in}) &\leftarrow \text{SORT} (p_{in}, q_{in}) \\
   \tilde{r}_{in} &\leftarrow r_{in}
\end{align*}
5: $H_i \leftarrow H_i + 1$ \(\triangleright\) Increment $H_i$.
6: else
   $\triangleright$ If carton $n$ need not be HO when packed into a box.
7: \begin{align*}
   (\tilde{p}_{in}, \tilde{q}_{in}, \tilde{r}_{in}) &\leftarrow \text{SORT} (p_{in}, q_{in}, r_{in})
\end{align*}
8: end if
9: end for
10: if $H_i > 0$ then
   \(\triangleright\) If shipment $i$ contains a HO carton.
11: \begin{align*}
   \{\Theta_j\}_{j=j_0}^J &\leftarrow \{\Psi_j\}_{j=j_0}^J \\
   \{(\bar{x}_j, \bar{y}_j, \bar{z}_j)\}_{j=j_0}^J &\leftarrow \{(\bar{x}_j, \bar{y}_j, \bar{z}_j)\}_{j=j_0}^J
\end{align*}
12: else
   \(\triangleright\) If shipment $i$ does not contain a HO carton.
13: \begin{align*}
   \{\Theta_j\}_{j=j_0}^J &\leftarrow \{\Sigma_j\}_{j=j_0}^J \\
   \{(\bar{x}_j, \bar{y}_j, \bar{z}_j)\}_{j=j_0}^J &\leftarrow \{(\bar{x}_j, \bar{y}_j, \bar{z}_j)\}_{j=j_0}^J
\end{align*}
14: end if
4 Fitting MILP

Introduction  Can \( n \in \mathbb{N} \) rectangular cartons, with positive outer lengths, widths, and heights \( \{(p_i, q_i, r_i)\}_{i=1}^n \) where \( (p_i, q_i, r_i) \in \mathbb{R}_{>0}^3 \), fit in a box with positive inner length, width, and height \((x, y, z) \in \mathbb{R}_{>0}^3\), permitting orthogonal rotations of each carton? When packed into the box, it is assumed that a carton’s edges must be parallel to those of the box, so that there are 6 possible orthogonal rotations of each carton. A right-handed orthogonal 3D cartesian coordinate system is introduced in the box’s frame. The \( X \)-axis is parallel to the box length \( l \), the \( Y \)-axis is parallel to the box width \( w \), and the \( Z \)-axis is parallel to the box height \( h \). The axes intersect orthogonally at \((0, 0, 0)\), which coincides with the left-back-bottom (llb) corner of the box. Left to right is along the \( X \)-axis with 0 on the left and \( X \) on the right. Back to front is along the \( Y \)-axis with 0 at the back and \( Y \) at the front. Bottom to top is along the \( Z \)-axis with 0 at the bottom and \( Z \) at the top. Whether the \( n \) cartons fit into the box can be determined by checking the feasibility (or satisfiability) of a set of linear inequality constraints depending on a set of continuous and binary variables. These constraints and variables form a feasibility MILP called the fitting MILP. Since any feasibility MILP is NP-complete [16], the fitting MILP is NP-complete. The next few paragraphs present the constraints and variables that comprise the fitting MILP.

Orientation Constraints  For each carton \( i \in [n] \), there are 9 orientation binary variables that indicate in which of the 6 possible ways each carton is oriented in the box. \( l_{X_i}, l_{Y_i}, l_{Z_i}, \in \{0, 1\} \) indicate whether the length dimension \( p_i \) of carton \( i \) is parallel to the \( X \)-, \( Y \)-, or \( Z \)-axis, \( w_{X_i}, w_{Y_i}, w_{Z_i} \in \{0, 1\} \) indicate whether the width dimension \( q_i \) of carton \( i \) is parallel to the \( X \)-, \( Y \)-, or \( Z \)-axis, and \( h_{X_i}, h_{Y_i}, h_{Z_i} \in \{0, 1\} \) indicate whether the height dimension \( r_i \) of carton \( i \) is parallel to the \( X \)-, \( Y \)-, or \( Z \)-axis. The reader is referred to Figure 1 in [17] for a 3D illustration of the meaning of the orientation binary variables. Since each carton dimension is parallel to one and only one axis, there are six linear equality constraints on the orientation binary variables. The 6 constraints are linearly dependent with rank 5, so that only 5 are needed.

\[
\begin{align*}
l_{X_i} + l_{Y_i} + l_{Z_i} &= 1 \quad \forall i \in [n],
wx_i + wy_i + wz_i &= 1 \quad \forall i \in [n],
hx_i + hy_i + hz_i &= 1 \quad \forall i \in [n],
l_{X_i} + wx_i + hz_i &= 1 \quad \forall i \in [n],
l_{Y_i} + wy_i + hz_i &= 1 \quad \forall i \in [n],
l_{Z_i} + wz_i + hz_i &= 1 \quad \forall i \in [n].
\end{align*}
\]

(4.1)

Containment Constraints  For each carton \( i \in [n] \), \((x_i, y_i, z_i) \in \mathbb{R}_{>0}^3\) denotes the nonnegative coordinates of the llb corner of carton \( i \). Each carton \( i \in [n] \) must be contained in the box, which requires the following 6 linear inequality constraints.

\[
\begin{align*}
x_i &\geq 0 \quad \forall i \in [n],
y_i &\geq 0 \quad \forall i \in [n],
z_i &\geq 0 \quad \forall i \in [n],
x_i + p_i x_i + q_i w_i + r_i h_i &\leq x \quad \forall i \in [n],
y_i + p_i y_i + q_i w_i + r_i h_i &\leq y \quad \forall i \in [n],
z_i + p_i z_i + q_i w_i + r_i h_i &\leq z \quad \forall i \in [n].
\end{align*}
\]

(4.2)

Nonoverlapping Constraints  Every distinct pair of cartons \( i, k \in [n] \), with \( i < k \), cannot overlap. To enforce these constraints, 6 nonoverlapping binary variables indicate the relative position of pairs of cartons. \( a_{ik} = 1 \) implies that carton \( i \) is left of carton \( k \), \( b_{ik} = 1 \) implies that carton \( i \) is right of carton \( k \), \( c_{ik} = 1 \) implies that carton \( i \) is behind carton \( k \), \( d_{ik} = 1 \) implies that carton \( i \) is in front of carton \( k \), \( e_{ik} = 1 \) implies that carton \( i \) is below carton \( k \), and \( f_{ik} = 1 \) implies that carton \( i \) is on top of carton \( k \). The reader is referred to Figure 2 in [17] for a 3D illustration of the meaning of the nonoverlapping binary variables. In order for cartons \( i \) and \( k \) to be nonoverlapping, at least one of \( a_{ik}, b_{ik}, c_{ik}, d_{ik}, e_{ik}, \) and \( f_{ik} \) must be 1. The following 7 linear inequality constraints enforce nonoverlapping of every distinct pair of cartons \( i \) and \( k \).

\[
\begin{align*}
x_i + p_i x_k + q_i w_x + r_i h_x &\leq x_k + (1 - a_{ik}) x \quad \forall i, k \in [n], i < k,
x_k + p_i x_k + q_i w_x + r_i h_k &\leq x_i + (1 - b_{ik}) x \quad \forall i, k \in [n], i < k,
y_i + p_i y_k + q_i w_y + r_i h_y &\leq y_k + (1 - c_{ik}) y \quad \forall i, k \in [n], i < k,
y_k + p_i y_k + q_i w_y + r_i h_y &\leq y_i + (1 - d_{ik}) y \quad \forall i, k \in [n], i < k,
z_i + p_i z_k + q_i w_z + r_i h_z &\leq z_k + (1 - e_{ik}) z \quad \forall i, k \in [n], i < k,
z_k + p_i z_k + q_i w_z + r_i h_z &\leq z_i + (1 - f_{ik}) z \quad \forall i, k \in [n], i < k,
a_{ik} + b_{ik} + c_{ik} + d_{ik} + e_{ik} + f_{ik} &\geq 1 \quad \forall i, k \in [n], i < k.
\end{align*}
\]

(4.3)
Symmetry-Breaking Constraints: Identical Cartons Suppose that some of the $n$ cartons are identical, in the sense that they share the same sorted dimensions, and that each subset of identical cartons has been assigned consecutive indices in $[n]$. Let $V^* \subset [n]$ denote the subset of carton indices in the union of all subsets of identical cartons (i.e., cartons with identical sorted dimensions). Let $V \subset V^*$ denote the subset of carton indices such that $m \in V$ if and only if $\text{sort}(p_m, q_m, r_m) = \text{sort}(p_{m+1}, q_{m+1}, r_{m+1})$. For $m \in V$, the $X$–coordinates (or alternatively the $Y$– or $Z$–coordinates) of the lbb corners of identical cartons can be arranged in nondecreasing order.

$$x_m \leq x_{m+1} \quad \forall m \in V. \quad (4.4)$$

Symmetry-Breaking Constraints: Carton LBB Corner in First Orthant Let $\beta \in [n]$ be the index of a particular carton. For example, $\beta$ might be the index of the smallest volume carton. If $\beta \in V^*$, $\beta$ should be the smallest index of the subset of identical cartons in which $\beta$ lies, in order to be compatible with (4.4). Any feasible packing of the $n$ cartons into the box can be rearranged, through a finite sequence of reflections across the box’s 3 inner half-planes, to realize a feasible packing such that the lbb corner of the carton with index $\beta$ is located in the box’s first orthant $\{(u, v, w) \in \mathbb{R}_{\geq 0}^3 : 0 \leq u \leq \frac{x}{2}, 0 \leq v \leq \frac{y}{2}, 0 \leq w \leq \frac{z}{2}\}$.

$$x_{\beta} \leq \frac{x}{2}, \quad y_{\beta} \leq \frac{y}{2}, \quad z_{\beta} \leq \frac{z}{2}. \quad (4.5)$$

Comments on the Constraints The orientation (4.1), containment (4.2), and nonoverlapping (4.3) constraints are given in [18, 19, 17]. References [19, 17, 20, 21, 22] provide alternative formulations of these constraints, however the author found that a MILP solver is able to determine feasibility faster using the constraints (4.1), (4.2), and (4.3) compared to the other constraint formulations. The symmetry-breaking constraints (4.4) and (4.5) are new and have not appeared in the literature before and should benefit MILP formulations of related packing problems such as the knapsack container loading problem (KCLP) [23, 24, 25, 26, 27, 28, 29, 30], the three-dimensional bin packing problem (3D-BPP) [31], and the three-dimensional open-dimensional rectangular packing problem (3D-ODRPP) [22]. The symmetry-breaking constraints tend to help the MILP solver determine feasibility faster by reducing the number of feasible solutions and thereby reducing the size of the search tree. Altogether, the constraints (4.1)-(4.5) comprise the fitting MILP. The fitting MILP consists of $3n$ nonnegative continuous variables, $3n(n-1) + 9n$ binary variables, $5n$ linear equality constraints, and $5n(n-1) + 6n + |V| + 3$ linear inequality constraints.

Special Packing Constraints Some cartons must be packed in special ways, in which case the special packing constraints must be enforced without conflicting with the symmetry-breaking constraints (4.4)-(4.5). For example, some cartons cannot be stacked on top of other cartons, so that the $Z$–coordinates of their lbb corners must equal 0, in which case the constraint $z_{\beta} = 0$ must be added to the fitting MILP for each such carton $i$ and the third constraint $z_{\beta} \leq \frac{z}{2}$ in (4.5) must be removed since a feasible packing cannot be reflected across the vertical half-plane. In the final paragraph of the previous section, such cartons were called bottom-resting (BR).

As another example, some cartons must be packed vertically, so that their height dimensions must be parallel to the box’s $Z$–axis, in which case the constraint $h_{Z,i} = 1$ must be added to the fitting MILP for each such carton $i$. In the final paragraph of the previous section, such cartons were called height-oriented (HO). If there is at least one carton in the shipment that must be height-oriented, then the definition of identical cartons given earlier must be revised in order to construct the subset $V$ for the symmetry-breaking constraints (4.4). A pair of cartons $i, j \in [n]$, with $i \neq j$, is identical if and only if either of the following conditions is satisfied:

(i) they are both not height-oriented and $\text{sort}(p_i, q_i, r_i) = \text{sort}(p_j, q_j, r_j)$
(ii) they are both height-oriented, $\text{sort}(p_i, q_i) = \text{sort}(p_j, q_j)$, and $r_i = r_j$.

With this new definition of identical cartons, it is still assumed that each subset of identical cartons has been assigned consecutive indices in $[n]$ and $V^* \subset [n]$ denotes the subset of carton indices in the union of all subsets of identical cartons. In addition, $V \subset V^*$ denotes the subset of carton indices such that $m \in V$ if and only if cartons $m$ and $m + 1$ are identical in the new sense.

As a third example, stability may be required for cartons which do not rest on the box’s bottom, which requires additional constraints and elimination of the third constraint $z_{\beta} \leq \frac{z}{2}$ in (4.5) since a feasible stable packing cannot necessarily be reflected across the vertical half-plane to generate another feasible stable packing (reflecting a stable packing across the vertical half-plane may result in an unstable packing).

Solving the Fitting MILP A third-party MILP solver must be used to solve the fitting MILP. There are many MILP solvers available, but only Gurobi [32], CPLEX [33], MIPCL [34], and CBC [35] are mentioned here. Gurobi and CPLEX are regarded as the best available MILP solvers, though they are commercial and require an expensive license for non-academic use. As of this writing, there is a free edition of CPLEX that solves MILPs having less than 1000 variables and 1000 constraints, which means that it can be used to solve
the fitting MILP for shipments with less than 16 cartons (though the author’s tests showed that it worked for
shipments with less than 18 cartons). MIPCL and CBC are regarded as the best available free MILP solvers,
though their performance is quite inferior to any of the commercial MILP solvers.

**Benchmarking** 20425 historical customer shipments, consisting of less than 13 cartons, were fit into
4922 candidate boxes using CPLEX v12.10.0 and Gurobi v9.0.0 with and without the symmetry-breaking
constraints (4.4)-(4.5). Of the 20425 historical customer shipments, 26% (5330 shipments) consisted of 4 or
more cartons, for which the fitting MILP was solved by one of the MILP solvers since brute force fitting
algorithms were used for shipments consisting of 0, 1, 2, or 3 cartons. Slightly under 676000 fitting MILPs
had to be solved for each of the four MILP solver and constraint combinations. MILPs taking more than
5 seconds (s) to solve were terminated early. Table 4.1 shows the run times (measured in seconds) and the
number of MILPs terminated early due to the 5s time limit (TL) for each of the four combinations. Table 4.1
also shows the speedup and the percentage reduction in the number of MILPs terminated early due to the 5s
time limit afforded by using CPLEX v12.10.0 instead of Gurobi v9.0.0 and by using the symmetry-breaking
constraints (4.4)-(4.5). Table 4.1 shows that CPLEX v12.10.0 is faster than Gurobi v9.0.0, resulting in fewer
MILPs terminated early due to the 5s time limit. Table 4.1 also shows that the symmetry-breaking constraints
(4.4)-(4.5) yield faster run times, resulting in fewer MILPs terminated early due to the 5s time limit. The
symmetry-breaking constraint (4.5) alone does not improve run times much; however, (4.5) in concert with
(4.4) gives a bit of improvement over (4.4) alone. The benchmarking results in Table 4.1 were obtained using
Julia v0.6.4 [36, 37] and JuMP v0.18.5 [38] on an Intel Core i7-3930K CPU @ 3.20 GHz with 6 physical cores
(12 logical cores with hyper-threading).

|                  | CPLEX v12.10.0 | Gurobi v9.0.0 | CPLEX Speedup / %Δ # MILP TL |
|------------------|----------------|---------------|------------------------------|
| (4.1)-(4.3)      | 259251.983s / 31110 | 334261.906 / 40837 | 1.29 / -23.8%               |
| (4.1)-(4.3) & (4.4)| 231698.436s / 27606 | 283056.089s / 32600 | 1.22 / -15.3%               |
| (4.1)-(4.3) & (4.5)| 256932.555s / 30658 | 336187.430s / 40615 | 1.31 / -24.5%               |
| (4.1)-(4.3) & (4.4)-(4.5)| 220688.691s / 25832 | 272121.708s / 30902 | 1.23 / -16.4%               |

Table 4.1: Comparison of CPLEX v12.10.0 and Gurobi v9.0.0 with and without the symmetry-breaking
constraints (4.4)-(4.5). Both run times and number of MILPs terminated early due to the 5s time limit are compared.
CPLEX v12.10.0 is faster than Gurobi v9.0.0, resulting in fewer MILPs terminated early due to the 5s time limit.
The symmetry-breaking constraints (4.4)-(4.5) yield faster run times, resulting in fewer MILPs terminated early
due to the 5s time limit.

**An Additional Application** Aside from being used for box suite recommendation, an online retailer
can also use the fitting MILP to recommend a box from its box suite for shipping a customer’s order. Given a
suite of boxes and a customer’s order, the fitting MILP can be used to determine into which boxes the order
fits, from which the box having minimum total shipping (plus material) cost can be selected.

**5 Summary & Future Work**

This paper offers an algorithm that recommends an optimal suite of shipping boxes subject to being able
to 1) lock specific boxes in the suite and 2) pack certain items that must be height-oriented and/or bottom-
resting. The algorithm assumes that shipped items are either foldable (in which case they are modeled to
be liquid) or rigid (in which case they are modeled as 3D rectangular cartons). If not height-oriented, the
algorithm assumes that a rectangular carton must be oriented in 1 of 6 possible ways when packed into a
shipping box, so that its edges are parallel to those of the box. The fitting problem is formulated and solved
using MILP. New symmetry-reduction constraints are introduced that lower the run times of the fitting
MILP. By solving the fitting MILP, the algorithm determines the costs of shipping a set of shipments into
a set of candidate boxes. Then, given the cost matrix, the algorithm solves the p-median problem to select the
minimum cost subset of p boxes.

An avenue for further investigation is to model additional physically accurate constraints, such as cargo
stability, in the fitting problem [39, 40, 41]. In future work, instead of using the computationally expensive
MILP, it may be possible to solve the fitting problem more rapidly by using metaheuristics that solve the
KCLP [23, 24, 25, 26, 27, 28, 29, 30] or by using machine learning (e.g. by training a neural network on a set
of shipments and a fine set of candidate boxes, using the results of the fitting MILP or KCLP metaheuristics
as truth).
Acknowledgements

This research was funded by Target Corporation and by the Institute for Mathematics and its Applications at the University of Minnesota, Twin Cities. The author thanks his Target colleagues Kaveh Khodjasteh and Neil Witte for their leadership and organization, Chinmay Jethwa and Sunita Venkataraman for providing historical shipment data, Brian Ager for his leadership and organization and for providing copious data and input parameters, and Jake Streich for his expertise in packaging engineering. Brian Ager suggested the idea of optimizing the box suite based on a statistically significant, randomly sampled subset of the previous year’s customer shipments.

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