LAPLACIANS ON A FAMILY OF QUADRATIC JULIA SETS II

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Abstract. This paper continues the work started in [4] to construct $P$-invariant Laplacians on the Julia sets of $P(z) = z^2 + c$ for $c$ in the interior of the Mandelbrot set, and to study the spectra of these Laplacians numerically. We are able to deal with a larger class of Julia sets and give a systematic method that reduces the construction of a $P$-invariant energy to the solution of nonlinear finite dimensional eigenvalue problem. We give the complete details for three examples, a dendrite, the airplane, and the Basilica-in-Rabbit. We also study the spectra of Laplacians on covering spaces and infinite blowups of the Julia sets. In particular, for a generic infinite blowups there is pure point spectrum, while for periodic covering spaces the spectrum is a mixture of discrete and continuous parts.

1. Introduction. This is a continuation of [4], which we will refer to as Part I, which dealt with the construction of Laplacians and the study of their eigenvalues and eigenfunctions for a family of quadratic Julia sets that includes the Basilica and Rabbit. Here we will enlarge the family of Julia sets, in the process clarifying some aspects of the construction in Part I. In particular we will look in detail at three new examples, a dendrite, the Airplane, and the Basilica-in-Rabbit. We also study the spectra of Laplacians on covering spaces and blow-ups [9] of the Julia sets. In particular, for generic infinite blow-ups we show that there is pure point spectrum, extending results of Teplyaev [11] for infinite blow-ups of the Sierpinski gasket. For infinite periodic covering spaces, on the other hand, the spectrum is a mixture of point and continuous parts. As far as we are aware, the only other work

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on Laplacians on Julia sets is the paper [8] of Rogers and Teplyaev that deals only with the Basilica.

The method we use to construct Laplacians on Julia sets may be called the method of identifications. Each Julia set is parametrized by a circle with points that are identified. Since we already have a Laplacian on the circle (the second derivative), we might hope to simply transfer the Laplacian to the Julia set by restricting to functions that respect the identifications. In other words, if we denote the identifications by $t \sim t'$, then we require $u(t) = u(t')$ for all identified points. Of course this naive approach fails, because no nontrivial functions that respect the identifications belong to the domain of the second derivative. Instead we follow an idea used very successfully by Kigami ([5], [10]) to define Laplacians on other types of fractals. We split the Laplacian into two pieces, a measure $\mu$ and an energy $E(u, v)$. In the case of the circle the measure is just standard Lebesgue measure, and the energy is

$$E(u, v) = \int_0^1 u'(t)v'(t)dt \quad (1)$$

The Laplacian is then defined via the weak formulation

$$E(u, v) = -\int_0^1 v \Delta u \, d\mu. \quad (2)$$

Here the natural domain of the energy is the Sobolev space $H^1$, and we require (2) to hold for all $v \in H^1$ (we parametrize the circle by the interval $[0, 1]$ with endpoints identified). If we require $\Delta u$ to be continuous then (2) characterizes the second derivative on the domain $C^2$, but if we only require $\Delta u$ to be $L^2$ then (2) characterizes the second derivative on the Sobolev space $H^2$. We note that although the integral in (1) and (2) both involve Lebesgue measure, we need to think of them as independent objects. In particular, for the energy we construct on Julia sets, there will be no analog to the right side of (1).

Each quadratic Julia set $J_c$ is defined in terms of the dynamics of the polynomial $P_c(z) = z^2 + c$ in the complex plane, where $c$ is an arbitrary complex parameter. For simplicity of notation we drop the subscripts $c$ when we are discussing only a single choice of $c$. The parameterization of $J$ by the circle $C$, which we denote $C \xrightarrow{\psi} J$, is called the external ray parameterization, and it has the notable property that it intertwines the action of $P$ on $J$ with the doubling map $t \to 2t \pmod{1}$ on $C$. The map $\psi$ creates identifications of points in $C$ via $t \sim t'$ if and only if $\psi(t) = \psi(t')$. It is a straightforward matter to transfer the measure $\mu$ from $C$ to $J$, and we continue to denote it by $\mu$. Since $\mu$ on $C$ is invariant under doubling, it follows that $\mu$ on $J$ is invariant under the action of $P$. Again, although we are naively tempted to transfer the energy to $J$ by using (1) restricted to functions respecting the identifications, this also fails because there are no nontrivial functions in $H^1$ that respect identifications. Instead, our approach, also inspired by Kigami, is to consider a sequence of discrete approximations to the energy on $C$, transfer them to $J$, and then take a renormalized limit. Specifically, we consider a sequence of finite subsets $X^{(m)}$ of $C$, each giving rise to a graph energy

$$E_m(u) = \sum \frac{|u(t_{j+1}) - u(t_j)|^2}{|t_{j+1} - t_j|} \quad (3)$$
(for simplicity we deal with the quadratic form with \(u = v\) rather than the general bilinear form) such that

\[
\lim_{m \to \infty} E_m(u) = \mathcal{E}(u) \quad \text{on } C
\]  

(4)

But once we impose the condition that \(u\) respect the identifications the simple limit in (4) will not make sense. In Part I we considered multiplying \(E_m\) by a renormalization factor \(r^{-m}\) for a suitable choice of \(r\) before taking the limit. We found, however, that \(r^{-m}E_m(u)\) exhibits a periodic behavior (the period depends on the Julia set), and we overcame this difficulty by the ad hoc decision to take a suitable average over the period. This approach does not seem to extend beyond the specific family of Julia sets considered in Part I.

Here we break up the sum (3) into a finite collection of sums based on the structure of the sets \(X^{(m)}\), and choose \(E_m(u)\) to be a linear combination of these subsums. We find a relation between \(E_m(u)\) and \(E_{m+1}(\tilde{u})\), where \(\tilde{u}\) denotes the extension of \(u\) from \(X^{(m)}\) to \(X^{(m+1)}\) that minimizes \(E_{m+1}(\tilde{u})\). This leads to a finite dimensional eigenvalue problem where the renormalization factor is the eigenvalue and the coefficients of the linear combination correspond to the eigenvector. Since we want everything to be positive, this suggests that the Perron-Frobenius eigenvector and eigenvalue should lead to the desired renormalized energy. This turns out to be an oversimplification, however, because the matrix in the eigenvalue problem varies with the eigenvector, so we are dealing with a nonlinear problem. In the three examples we study in detail, it turns out that we can replace the nonlinear problem with an equivalent linear problem where the Perron-Frobenius theorem applies, and in fact we explicitly solve for the positive eigenvalue and eigenvector. We are at present unable to determine for which Julia sets this linearization idea works. We leave this as a mystery for the future.

Once we have solved the renormalization problem for the energy, defining \(E_m(u)\), we pass to the limit

\[
\mathcal{E}(u) = \lim_{m \to \infty} E_m(u)
\]

(5)

to define an energy \(\mathcal{E}\) that satisfies the invariance

\[
\mathcal{E}(u \circ P) = \left(\frac{4}{r}\right) \mathcal{E}(u)
\]

(6)

under the action of the polynomial on \(J\). We then define a Laplacian by (2), and it satisfies the invariance condition

\[
\Delta(u \circ P) = \frac{4}{r} (\Delta u) \circ P.
\]

(7)

In this way all three constructs: measure, energy and Laplacian, are related to the dynamics of the polynomial on the Julia set. We should point out that the measure we are using is what is known as the equilibrium measure. Although it is \(P\)-invariant, one could argue that it does not encode all the information concerning the dynamics, since the action of \(P\) as a conformal mapping of \(J\) distorts distances by a different factor \((\left|P'(z)\right|^{1/2})\) at different points of \(J\). There is another measure, called the conformal measure which takes into account this distortion factor (raised to a power that may be identified with the dimension of \(J\)). In Part I we discussed algorithms to approximate the conformal measure, and then constructed two Laplacians, an equilibrium Laplacian and a conformal Laplacian, using (2) with the two different measures. What we discovered was that the spectrum of the equilibrium Laplacian displays a rich and fascinating structure, whereas the spectrum of the conformal
Laplacian did not reveal any obvious structure. Therefore, in this paper we made the decision to consider only the equilibrium Laplacian. As a consequence, our Laplacian depends only on the topological structure of the Julia set, so that all Julia sets for \( c \) belonging to a fixed component of the Mandelbrot set have equivalent Laplacians. In particular, for \( c \) in the main cardioid, the Laplacian is just the usual second derivative on the circle for an appropriate parameterization. It should be pointed out that Kigami’s construction of Laplacians on other fractals also shares this property of depending only on the topology and not the geometry of the fractal.

We now outline the contents of this paper. In section 2 we describe the parameter identifications on \( C \) that describe the Julia set. We consider only Julia sets corresponding to rational angles \( \theta \) in the external ray description of the boundary of the Mandelbrot set. For these Julia sets there is a very concrete algorithm to find the identifications. We do not assume the reader is an expert in the theory of quadratic Julia sets, so we give a complete description of the algorithm. We then describe the construction of a nested sequence \( X^{(m)} \) of parameter points. We are particularly interested in the subdivision rules that describe how an interval in \( X^{(m)} \) (meaning \([t_j, t_{j+1}]\) for consecutive points in \( X^{(m)} \)) subdivides into intervals in \( X^{(m+1)} \). In some cases we observe a dichotomy: either the identifications of points after subdivision are all internal to the interval, or there is a pairing of the interval with another interval and all identifications are made among the subdivision points of the paired intervals. We note that pairing did not occur in any of the examples in Part I. Roughly speaking, paired intervals give parameterizations of regions of the Julia set that may be approached externally from two different directions. In section 3 we study the energy renormalization problem discussed above and reduce it to a finite dimensional nonlinear eigenvalue problem. In section 4 we work out the details for our three new examples. In section 5 we describe algorithms to approximate the spectrum of the Laplacian. We use two different algorithms, based on the finite difference method (FDM) and finite element method (FEM) for ODE’s (Part I used only FDM). These give us approximations from above and below for the eigenvalues, and so provide good a posteriori estimates for the errors. We note that the Julia sets in our three examples are more complicated than those studied in Part I, so it is not feasible to carry out the computation to the same level of depth \( (m) \). We present a sample of the numerical results (more data may be found at the website [2]). In section 6 we discuss the structure of the spectral data obtained for the three examples. One striking observation is that for the dendrite, the zeroes of the eigenfunctions follow the same pattern as is found in the classical Sturm-Liouville theory.

The second part of this paper deals with covering spaces and blow-ups of Julia sets. In section 7 we give the definition of covering spaces and covering maps. Note that the standard theory is usually based on the hypothesis that the base space is locally simply connected, but this is usually not the case for Julia sets. Nevertheless, the definition follows the expected pattern. We are interested in covering spaces and maps that preserve the local “geometry” of \( \mathcal{J} \), but here we are referring to the geometry of the parameter circle \( C \) rather than the geometry of \( \mathcal{J} \) embedded in the complex plane. Nevertheless, if we have a covering map of \( \mathcal{J} \) in the topological sense we can simply lift the geometry from \( \mathcal{J} \) to the covering space via the covering map. For a covering space \( \tilde{\mathcal{J}} \) we show how to describe it in terms of a covering space \( \tilde{C} \) of the parameter circle (note that \( \tilde{C} \) does not have to be connected as the identifications made to construct \( \tilde{\mathcal{J}} \) can make \( \tilde{\mathcal{J}} \) connected). Some interesting
questions we are unable to answer are how to decide whether or not two covering maps are equivalent, and whether or not there exist isometric covering spaces that have inequivalent covering maps. In section 8 we restrict attention to double covers of the Basilica, and show how covering maps may be characterized by combinatorial data, namely the type of covering of each of the circles contained in \( J \). This analysis may be extended to other Julia sets, but does not extend to higher order covering maps. In section 9 we study infinite blowups of Julia sets. In many cases these blowups are special cases of covering spaces, but for example the dendrite has no nontrivial covering spaces, but the blowups exist. There are uncountably many distinct blowups. We define what is meant by a generic blow up. Unlike the case of the Sierpinski gasket treated in [11] our spaces have no boundary, so the nongeneric blowups cannot be characterized by having boundary points. It is easy to lift the Laplacian to either covering spaces or blowups. The main result of this section is that the Laplacian on a generic infinite blowup has pure point spectrum, meaning that there is an orthonormal basis of \( L^2 \) eigenfunctions, and each eigenspace has infinite multiplicity. In section 10 we study the Laplacian on periodic covering spaces. In the case of the circle, the only periodic covering space is the line, and the spectrum of the Laplacian is continuous, with the eigenfunctions given by complex exponentials. In the general case the spectrum is a mix of continuous and discrete parts. In particular there exist eigenfunctions of compact support. In Section 11, we give an algorithm to compute the spectrum and report some numerical results for the Basilica. We refer the reader to [5] and [10] for earlier work on defining Laplacians on fractals, and to [7] for the description of Julia sets as circles with identifications.

2. Parameter identifications. The key to understanding the structure of our Julia sets \( J \) is the parameter map from the circle ( we take it to have length 1 ) to \( J \) that intertwines the action of the polynomial \( P(z) = z^2 + c \) on \( J \) and the doubling map \( t \to 2t \) on the circle. We will abuse notation by avoiding a symbol for the parameter map and allowing \( t \) to represent both the parameter value and the corresponding point on \( J \). Of course the parameter map is not one to one, so our first task is to understand the identifications that we will write \( t_1 \sim t_2 \sim \cdots \sim t_n \) when \( t_1 \) and \( t_2 \) correspond to the same point in \( J \). Functions \( u \) on \( J \) will be represented by functions \( u(t) \) on the circle that respect the identifications: \( u(t_1) = u(t_2) \) if \( t_1 \sim t_2 \).

The goal of this section is to describe a recursive algorithm for obtaining a nested sequence \( X^{(0)} \subseteq X^{(1)} \subseteq \cdots \) of finite sets of points in the circle with identifications that are consistent from level to level. Each \( X^{(m)} \) will contain \( 2^m \) distinct parameter values that will split into \( 2^m N_0 \) equivalence classes under identification (the specific values of \( N_0 \) and \( N \) will depend on the specific Julia set). So we will regard \( X^{(m)} \) as a discrete subset of \( J \) with \( 2^m N_0 \) elements, and we will take them to be the vertices of a graph whose edges are just the neighboring edges along the circles. In other words, if a vertex in the graph is the equivalence class containing parameter value \( t_1 \sim t_2 \sim \cdots \sim t_n \), then there will be \( 2n \) edges joining each \( t_j \) to the nearest neighbors in \( X^{(m)} \). Because of identifications, this graph may have self-edges (if one of the neighbors is in the original equivalence class) or multiple edges (if two different neighbors are equivalent). Also, we will assign conductances to each of the edges on the circle, equal to the length of the edge. Note that this implies that the sum of all conductances over all edges is equal to 1, the length of the circle.
The structure of \( X^{(m)} \) as a subset of the circle will be determined by very simple rules from \( X^{(0)} \). The identifications will be more complicated and will be described later. In all our examples, the points in \( X^{(0)} \) will be the rational numbers of the form \( \frac{j}{2^m(2^k - 1)} \) with \( j \in J \) for some choices of \( k \) and a subset \( J \) of \( [1, 2^k - 1] \) that is fixed for the Julia set. Note that this means that the points in \( X^{(0)} \) are all periodic of period \( k \) (or a divisor of \( k \)) under the doubling map. We will write \( t_1 \equiv t_2 \) for congruence mod 1 on the circle (in other words, the circle is \( \mathbb{R}/\mathbb{Z} \)) and \( j_1 \equiv j_2 \) for congruence mod \( 2^k - 1 \). We will also require that

\[
2J \equiv J
\]

(meaning that for each \( j_1 \in J \) there exists a unique \( j_2 \in J \) such that \( 2j_1 \equiv j_2 \)). This is equivalent to the statement that the doubling map preserves \( X^{(0)} \). As a subset of \( J \), \( X^{(0)} \) is preserved by the action of \( P \). For example, if \( k = 3 \), then \( J = \{1, 2, 4\} \) and \( J = \{1, 2, 3, 4, 5, 6\} \) satisfy (8). Both of these choices will appear in some of the explicit examples we study below. We will sometimes write \( J = \{j_1, j_2, \ldots, j_N\} \) in increasing order, and set \( j_0 = j_N \).

We may now define

\[
X^{(m)} = \left\{ \frac{j}{2^m(2^k - 1)} : j \in J \right\} = \left\{ \frac{j_n + l(2^k - 1)}{2^m(2^k - 1)} : j_n \in J, 0 \leq l \leq 2^m - 1 \right\}
\]

Note that \( X^{(m)} \) is the inverse image of \( X^{(m-1)} \) under the doubling map, and because of (9) the sets \( X^{(m)} \) are nested. We note that the graph structure of \( X^{(0)} \) consists of edges that we identify with the intervals \( I_n = \left[ \frac{j_n}{2^m(2^k - 1)}, \frac{j_{n+1}}{2^m(2^k - 1)} \right] \) for \( 0 \leq n \leq N - 1 \). Similarly, we identify the edges in \( X^{(m)} \) with the intervals \( I_{n,l}^{(m)} = \left[ \frac{j_n + l(2^k - 1)}{2^m(2^k - 1)}, \frac{j_{n+1} + l(2^k - 1)}{2^m(2^k - 1)} \right] \) (note that if \( n = N - 1 \), the right endpoint is \( \frac{j_{n+1} + l(2^k - 1)}{2^m(2^k - 1)} \)). We denote the intervals with the same \( n \) value as type \( n \) intervals, and write \( I_n^{(m)} \) for the set of all type \( n \) intervals at level \( m \).

It is important that we understand the subdivision rules that tell us how a type \( n \) interval at level \( m \) splits into a union of intervals of different types at level \( m+1 \). We observe that these rules depend only on the type \( n \), not on the particular interval in \( I_n^{(m)} \) or the level \( m \). Note that the endpoints of \( I_{n,l}^{(m)} \) are points in \( X^{(m+1)} \) by the nesting property. Indeed,

\[
\frac{j_n + l(2^k - 1)}{2^m(2^k - 1)} = \frac{2j_n + 2l(2^k - 1)}{2^{m+1}(2^k - 1)}
\]

and \( 2j_n \equiv j_{n'} \) for some unique \( n' \). So at level \( m + 1 \), \( I_{n,l}^{(m)} \) splits into a union of intervals of consecutive types \( n', n' + 1, \ldots, n'' \) where \( 2j_{n+1} \equiv j_{n'' + 1} \). It may happen that \( n'' = n' \), in which case the interval does not really split but gets passed along whole, but its type changes. (In many of the examples, this is the most common behavior, with only a few \( n \)-types splitting.) We note that the subdivision rules do not depend on the identifications. In our example with \( k = 3 \) and \( J = \{1, 2, 4\} \), the 1-type becomes a 2-type, the 2-type becomes a 3-type, and the 3-type splits into a 1-type, a 2-type, a 3-type and a 1-type in consecutive order.

Now we describe the identifications. Each Julia set we consider (actually an equivalence class up to topological homeomorphism) is described by a parameter
\(\theta\) in the circle identified with the external ray parameter where the ray hits the Mandelbrot set at the root of the component that contains \(c\) [3]. Typically there will be more than one such parameter, but all yield the same identifications. We will only deal with rational values of \(\theta\). Once \(\theta\) is fixed, we subdivide the circle into two half-circles \(A\) and \(B\), where \(B = \left[ \frac{0}{2}, \frac{0}{2} + \frac{1}{2} \right] \) (or \(\left[ \frac{\theta}{2}, \frac{\theta}{2} + \frac{1}{2} \right] \) and \(A = \left[ \frac{\theta}{2} + \frac{1}{2}, \frac{\theta}{2} \right] \) (or \(\left[ \frac{\theta}{2} + \frac{1}{2}, \frac{\theta}{2} \right] \)). The decision where to place \(\frac{\theta}{2}\) or \(\frac{\theta}{2} + \frac{1}{2}\) is rather subtle, and we will not discuss it here. In some cases it affects the identifications. In all the examples we discuss the correct choice is known.

For any value of \(t\) in the circle, we define the kneading sequence \(\sigma(t)\) to be a sequence of symbols \(\sigma_j(t) = A\) or \(B\), with \(\sigma_j(t) = A\) if and only if \(2^jt\mod 1 \in A\) for \(j = 0, 1, 2, \ldots\) Then we identify \(t_1 \sim t_2\) if and only if \(\sigma(t_1) = \sigma(t_2)\). This has the following simple consequence: if \(t_1 \sim t_2\) then \(\frac{1}{2}t_1 \sim \frac{1}{2}t_2\) or \(\frac{1}{2}t_2 + \frac{1}{2}\) according to which of the two belongs to the same half-circle as \(\frac{1}{2}t_1\). Note that one or the other but not both get identified, since \(\frac{1}{2}t_2\) and \(\frac{1}{2}t_2 + \frac{1}{2}\) differ by exactly \(\frac{1}{2}\). The explanation for this identification rule is that since \(t_1 \sim t_2, \frac{1}{2}t_1\) and \(\frac{1}{2}t_2\) (or \(\frac{1}{2}t_2 + \frac{1}{2}\)) have the same kneading sequence after \(\sigma_0\), so we only need to check if \(\sigma_0\left(\frac{1}{2}t_1\right) = \sigma_0\left(\frac{1}{2}t_2\right)\) or \(\sigma_0\left(\frac{1}{2}t_2 + \frac{1}{2}\right)\).

So we may state the following recursive definition for identifications in \(X^{(m)}\). Suppose we know all identifications in \(X^{(m)}\) and wish to find all identifications in \(X^{(m+1)}\). For each equivalence class \(t_1 \sim t_2 \sim \cdots \sim t_n\) in \(X^{(m)}\), the points \(\left\{ \frac{1}{2}t_j, \frac{1}{2}t_j + \frac{1}{2} \right\}\) split into two equivalence classes in \(X^{(m+1)}\), one containing \(\frac{1}{2}t_j\) or \(\frac{1}{2}t_j + \frac{1}{2}\) according to which one is in \(A\). We have consistency across levels: some of the points in \(X^{(m+1)}\) will also belong to \(X^{(m)}\) (in fact exactly half of them), but they will belong to the same equivalence class on both levels. (This is true because the identifications are described by an algorithm that is independent of level.)

To get started we need to construct \(X^{(0)}\) and sort it into equivalence classes. We can do this by computing the kneading sequences for all rationals of the form \(\frac{j}{2^k}\). Since these points are periodic under doubling, they have periodic kneading sequences (the period is a divisor of \(k\)), and moreover they are the only points with periodic kneading sequences of such period. This yields an algorithm for finding all identifications among such points. More explicitly, we assemble a disjoint list of all orbits \(\{j, 2j, 2^2j, \ldots, 2^{k-1}j\}\) mod \(2^k - 1\) (some of these orbits will have length dividing \(k\)), and compute the associated kneading sequences, \(\{\sigma_0\left(\frac{j}{2^k-1}\right), \sigma_1\left(\frac{j}{2^k-1}\right), \ldots, \sigma_{k-1}\left(\frac{j}{2^k-1}\right)\}\). We then cyclically permute the kneading sequences, and if we see any repetitions we identify the corresponding points in the orbit. For example, if one kneading sequence is \(BBB\) we identify all 3 points in the corresponding orbit. On the other hand, if one kneading sequence is \(BAB\) and another is \(ABB\), we identify 3 pairs of points across the two orbits.

The default choice for \(J\) is the set of all \(j\) for which \(\frac{j}{2^k-1}\) gets identified with at least one other point. In some cases (this will happen in the airplane example below) we may choose for \(J\) a subset of the above, provided we always take all points in an equivalence class. The motivation for such an incomplete choice is that we get just as good an approximation to \(J\) with fewer points, thus simplifying the computations. It is not hard to see that we will never choose \(0 \in J\). Indeed, the kneading sequence of 0 is \(AA \cdots A\). Any other orbit together with 0 is too large to fit in any half-circle, in particular the \(A\) half-circle, so the kneading sequence of the orbit must contain at least one \(B\).
How do we choose $k$? If we choose $k$ too small, we may not find identifications at all. So we may start with $k = 2$ and run the above algorithm, increasing $k$ by 1, until we find identifications. Do we stop at the smallest value of $k$ which produces identifications? Not necessarily. What we need to check is that the approximations to $\mathcal{J}$ that we obtain capture the important structure of $\mathcal{J}$. For this we need a picture of the Julia set and a bit of common sense, so we cannot describe this step as an algorithm. For example, if the Julia set contains triple points but the smallest $k$ only produces double identifications, we want to increase $k$ at least until we obtain triple identifications. In the basilica-in-rabbit example below, we want both double and triple identifications, but for $k = 3$ we just get triple identifications (in this example $\theta = \frac{10}{63}$, which suggests we take $k = 6$).

We conclude this section with a careful discussion of the interaction between subdivision rules and identifications. We will give the explicit details for three specific examples in section 4. Our discussion here sets the stage for the story of energy renormalization in section 3. We note that everything we do here is completely algorithmic.

Suppose we fix an interval $I$ for type $n$ at level $m$. Suppose $I$ subdivides into a union of intervals at level $m + 1$, $I = [t_0, t_1] \cup [t_1, t_2] \cup \ldots \cup [t_{j-1}, t_j]$, so $t_0$ and $t_j$ are the endpoints of $I$, hence in $X^{(m)}$, while $t_1, \ldots, t_{j-1}$ are points in $X^{(m+1)}$. We would like to know all the identifications of these new points. We will say that the identifications are internal if none of these points is identified with any point outside $I$. These mean that all these points in $\{t_1, \ldots, t_{j-1}\}$ are identified with other points in this set, but more than that, there are no identifications going outside this set. Note that we are not saying anything about the identification of the boundary points $t_0$ and $t_j$. We say that $I$ is a loop if $t_0$ and $t_j$ are identified (note that they may also be identified with other points). It is easy to see that if $I$ is a loop then all identifications are internal, since identification arcs may not cross, but the converse is not true.

Another possible behavior is what we call pairing. Suppose $I' = [t'_0, t'_1] \cup [t'_1, t'_2] \cup \ldots \cup [t'_{j-1}, t'_j]$ is another interval of type $n$ and level $m$. We say that $I$ and $I'$ are paired if $t'_0 \sim t_0$, $t'_j \sim t_j$, and this implies that the points $\{t_1, \ldots, t_{j-1}, t'_1, \ldots, t'_{j-1}\}$ are only identified with points within this set. Specific illustrations of pairing may be found in section 4.

In some cases these are the only two possibilities. We then say that the subdivision dichotomy holds. The following is the key observation. Suppose $I$ does not contain either $\frac{\theta}{2}$ or $\frac{\theta}{2} + \frac{1}{2}$. Then all the points $\{t_1, \ldots, t_{j-1}\}$ have the same initial value in the kneading sequence, $\sigma_0(t_1) = \ldots = \sigma_0(t_{j-1})$. So they get identified according to the identifications of $\{2t_1, \ldots, 2t_{j-1}\}$ and these are the subdivision points of the interval $2I$ on level $m - 1$ (also having $n$-type). If $2I$ has all identifications internal, it follows that the same is true for $I$. On the other hand, if $2I$ is paired with another interval, that we may call $2I'$, then either $I$ is paired with $I'$ or $I' + \frac{1}{2}$, depending on which lies in the same half-circle ($A$ or $B$). (Note that $\frac{\theta}{2}$ or $\frac{\theta}{2} + \frac{1}{2}$ cannot lie in $I'$, as the identification of $2I$ and $2I'$ would imply the same for $I$, in contradiction to our assumption.)

We may use this as the basis of an induction argument. We will need to verify that the dichotomy holds for all the intervals in $X^{(1)}$, and for all intervals of any order that contain either $\frac{\theta}{2}$ or $\frac{\theta}{2} + \frac{1}{2}$. (We will see in a moment that this only requires a finite number of verifications when $\theta$ is rational.) It then follows that the dichotomy holds in general.
We can be more specific about the identifications under subdivision. Suppose \( I \) is an \( n \)-type interval in \( X^{(m)} \) that subdivides with all identifications internal. Then under repeated doubling we will eventually arrive at the interval \( \left[ \frac{j_{n'}}{2^{2n-1}}, \frac{j_{n'+1}}{2^{2n-1}} \right] \) of \( X^{(0)} \) that subdivides in \( X^{(1)} \) into \( \left[ \frac{j_{n'}}{2^{2n-1}}, \frac{j_{n'+1}}{2^{2n-1}} \right] \cup \cdots \cup \left[ \frac{j_{n''}}{2^{2n-1}}, \frac{j_{n''+1}}{2^{2n-1}} \right] \), where \( j_{n'} = 2j_n \) and \( j_{n''} = 2j_{n+1} \). Then the internal identification among the points \( \left\{ \frac{j_{n'+1}}{2^{2n-1}}, \cdots, \frac{j_{n''}}{2^{2n-1}} \right\} \) determine the identical identifications in the \( X^{(m+1)} \) points in \( I \).

On the other hand, suppose \( I \) and \( \tilde{I} \) are paired \( n \)-type and \( \tilde{n} \)-type intervals in \( X^{(m)} \). There are two possibilities. The first is that under repeated doubling, we never encounter an interval containing \( \frac{\theta}{2} \) or \( \frac{\theta}{2} + \frac{1}{2} \). In that case we end up with paired intervals in \( X^{(0)} \), \( \left[ \frac{j_{n}}{2^{2n-1}}, \frac{j_{n+1}}{2^{2n-1}} \right] \) and \( \left[ \frac{j_{\tilde{n}}}{2^{2\tilde{n}-1}}, \frac{j_{\tilde{n}+1}}{2^{2\tilde{n}-1}} \right] \), and then the identifications among their subdivisions in \( X^{(1)} \) leads to the same identifications among \( I \) and \( \tilde{I} \) in \( X^{(m+1)} \). Finally, suppose at some point in the doubling process we end up with intervals \( I' \) and \( I' + \frac{1}{2} \) that contain \( \frac{\theta}{2} \) and \( \frac{\theta}{2} + \frac{1}{2} \). Suppose that \( \frac{\theta}{2} \) lies in the interior of \( I' \). Then by hypothesis either \( I' \) subdivides internally or it is paired. If it subdivides internally then this yields an internal subdivision of \( I \). If \( I' \) is paired, it must be paired with \( I' + \frac{1}{2} \), and this yields a paired subdivision of \( I \) and \( \tilde{I} \), (and \( n = \tilde{n} \)). If \( \frac{\theta}{2} \) is an endpoint of \( I' \), then either \( I' \) lies entirely in a single half \( A \) of \( B \), in which case the behavior is the same as when \( \frac{\theta}{2} \) is not in \( I' \), or \( \frac{\theta}{2} \) is in one half and the rest of \( I' \) is in the other half, in which case the behavior is the same as when \( \frac{\theta}{2} \) is in the interior of \( I' \).

We also note that there is a periodicity in the location of \( \frac{\theta}{2} \). Note that the rational number \( \theta \) may be written \( \theta = \frac{a}{2m+b} \) for some integers \( a, b \) and \( p \). The condition that \( \frac{\theta}{2} \) belong to \( I^{(m)}_{n,l} \) is equivalent to

\[
j_n \leq p2^{m-a-1} \left( \frac{2^k - 1}{2^b - 1} \right) - l(2^k - 1) \leq j_{n+1}.
\]

But

\[
p2^{m-a-1} \left( \frac{2^k - 1}{2^b - 1} \right) = 2^b p2^{m-a-1} \left( \frac{2^k - 1}{2^b - 1} \right) - p2^{m-a-1}(2^k - 1)
\]

so

\[
j_n \leq p2^{m+b-a-1} \left( \frac{2^k - 1}{2^b - 1} \right) - l'(2^k - 1) \leq j_{n+1} \quad \text{for} \quad l' = l + p2^{m-a-1}
\]

and this is an integer if \( m \geq a + 1 \). Thus \( \frac{\theta}{2} \) is in \( I^{(m+b)}_{n,l} \). Moreover, the relative location of \( \frac{\theta}{2} \) in \( I^{(m)}_{n,l} \) and \( I^{(m+b)}_{n,l} \) is the same (meaning \( \frac{a-c}{d-c} \) for an interval \([c, d]\)).

For \( I^{(m)}_{n,l} \) this is

\[
\frac{\theta}{2} - \frac{j_n + (2^k - 1)}{2m(2^k - 1)} = p2^{m-a-1} \left( \frac{2^k - 1}{2^b - 1} \right) - (j_n + l(2^k - 1))
\]

\[
\frac{j_{n+1} - j_n}{2m(2^k - 1)}
\]

while for \( I^{(m+b)}_{n,l} \) this is

\[
p2^{m+b-a-1} \left( \frac{2^k - 1}{2^b - 1} \right) - (j_n + l'(2^k - 1))
\]

\[
\frac{j_{n+1} - j_n}{2m(2^k - 1)}
\]
and the two are equal by the same computation as above. Thus the identifications in the intervals $I_{n,l}^{m+b}$ and $I_{n,l}^{m+b} + \frac{1}{2}$ when it subdivides are the same as for $I_{n,l}^{(m)}$ and $I_{n,l}^{(m)} + \frac{1}{2}$.

3. **Energy renormalization.** We will construct a $P$-invariant energy on $J$ by taking an appropriately renormalized limit of graph energies on $X^{(m)}$. The simplest expression for such a graph energy would be

$$E^{(m)}(u) = \sum_{x \sim y} \frac{|u(x) - u(y)|^2}{|x - y|}. \quad (10)$$

However, this choice does not yield a useful relationship between $E^{(m)}(u)$ and $E^{(m+1)}(\tilde{u})$, where $\tilde{u}$ is the harmonic ($E^{(m+1)}$-energy minimizing) extension of $u$ from $X^{(m)}$ to $X^{(m+1)}$. It turns out that we do better if we sort the contributions to the energy according to the different types of intervals. Define

$$E^{(m)}_n(u) = \sum_{i=0}^{2^m-1} |u\left(\frac{j_{n+1} + (2^k - 1)}{2^{n+1}}\right) - u\left(\frac{(2^k - 1)}{2^{n+1}}\right)|^2. \quad (11)$$

Note that this is just the contribution to $(10)$ from intervals of type $n$. In the general case we may have to further sort the terms in $(11)$, but to simplify the discussion we will postpone discussing that possibility until later. In turns out that $(11)$ is sufficient in the three examples we discuss in section 4.

Note that

$$E^{(m)}_n(u \circ P) = 4E^{(m-1)}_n(u). \quad (12)$$

Indeed each interval that contributes to $E^{(m-1)}_n(u)$ corresponds to two intervals that contribute to $E^{(m)}_n(u \circ P)$, but since they are half the size, their contribution is doubled. We are using the fact that if $I$ is an $n$-type interval in $X^{(m-1)}$, then $\frac{1}{2}I$ and $\frac{1}{2}I + \frac{1}{2}$ are also $n$-type intervals in $X^{(m)}$. The significance of $(12)$ is that limits of appropriate linear combinations of $E^{(m)}_n$ may produce $P$-invariant energies; we don’t have to limit ourselves to multiples of $(10)$.

Consider a general linear combination

$$E^{(m)}_b(u) = \sum b_n E^{(m)}_n(u) \quad (13)$$

where $b = \{b_n\}$ is a set of positive weights. From the point of view of the graph $X^{(m)}$, we are changing the conductances of the edges, multiplying each type $n$ edge conductance by $b_n$, or equivalently dividing the length of the interval by $b_n$. We are only interested in the weights up to a constant multiple, so we may normalize by any desirable convention (such as average value = 1). We will also have to renormalize the energies, taking

$$\mathcal{E}^{(m)}(u) = r^{-m} E^{(m)}_b(u) \quad (14)$$

for an appropriate constant $r$. Of course the values of $b$ and $r$ will have to be determined carefully by solving a certain eigenvalue problem for a finite matrix. Suppose $u$ is defined on points in $X^{(m-1)}$. We define the harmonic extension $\tilde{u}$ to $X^{(m)}$ to be the extension that minimizes the energy $E^{(m)}_b$ (or equivalently $\mathcal{E}^{(m)}$). Note, however, that this is not necessarily the same thing as minimizing the simple
expression (10), although for some examples, this turns out to be the case. What we want to achieve is the identity
\[ E_b^{(m)}(\tilde{u}) = rE_b^{(m-1)}(u). \]  
(15)
If this is true, then the definition (14) makes
\[ E^{(m)}(\tilde{u}) = E^{(m-1)}(u) \]
and since \( \tilde{u} \) is the energy minimizer, we see that \( E^{(m)}(u) \) is always monotone increasing with \( m \), so
\[ E(u) = \lim_{m \to \infty} E^{(m)}(u) \]
is always well defined as an extended real number, and we may define \( \text{dom } E = \{ u : E(u) < \infty \} \). It is then a routine matter to show all functions in \( \text{dom } E \) are continuous, only the constants have \( E(u) = 0 \), and \( \text{dom } E \) modulo constants is a Hilbert space with norm \( E(u)^2 \). The inner product \( E(u,v) \) may be defined by polarization, or by the limit of the bilinear form \( E^{(m)}(u,v) \) defined in the usual manner. Also, from (12) and (14) we obtain
\[ E(u \circ P) = \frac{4}{P}E(u), \]
the \( P \)-invariance condition on the energy.

We now look at the condition (15) more explicitly. We begin with two simple observations. The first is that the harmonic extension problem is entirely local. Consider an interval \( I \) in \( X^{(m-1)} \) that subdivides internally in \( X^{(m)} \). Then \( u \) already has values at the endpoints of \( I \), and we must assign values for \( \tilde{u} \) at the \( X^{(m)} \) points interior to \( I \) in order to minimize the contribution to the energy from the intervals in \( I \). Similarly, if \( I \) is paired with \( I' \), then \( u \) assumes the same values at the endpoints of \( I \) and \( I' \) and the extension \( \tilde{u} \) needs to be defined at the interior points of \( I \) and \( I' \) to minimize the contribution to the energy from these two intervals. The second observation is that if \( I \) is a loop, then it contributes zero to the energy on level \( m \), and the obvious harmonic extension makes \( \tilde{u} \) constant on the loop, so the contribution to the energy on level \( m \) is still zero. In other words, we can ignore loops.

We make the following simplifying assumption: every interval of type \( n \), that is not a loop, subdivides in the same way. So if \( I \) subdivides internally, we have
\[ I = \left[ \frac{j_{n} + l(2^k - 1)}{2^{m-1}(2^k - 1)}, \frac{j_{n+1} + l(2^k - 1)}{2^{m-1}(2^k - 1)} \right] \]
\[ = \left[ \frac{j_{n'} + l'(2^k - 1)}{2^{m}(2^k - 1)}, \frac{j_{n'+1} + l'(2^k - 1)}{2^{m}(2^k - 1)} \right] \cup \cdots \cup \left[ \frac{j_{n''} - l'(2^k - 1)}{2^{m}(2^k - 1)}, \frac{j_{n''+1} + l'(2^k - 1)}{2^{m}(2^k - 1)} \right] \]
with identifications among the junction points that are the same as \( X^{(1)} \). The identifications force \( \tilde{u} \) to assume the same value at identified points. The contribution to \( E_b^{(m)}(\tilde{u}) \) from these intervals is
\[ 2^{m}(2^k - 1) \left[ \frac{b_{n'}}{j_{n'+1} - j_{n'}} \left( \tilde{u} \left( \frac{j_{n'+1} + l'(2^k - 1)}{2^{m}(2^k - 1)} \right) - \tilde{u} \left( \frac{j_{n'} + l'(2^k - 1)}{2^{m}(2^k - 1)} \right) \right)^2 + \cdots \right. \]
\[ + \left. \frac{b_{n''} - l'}{j_{n''} - j_{n''-1}} \left( \tilde{u} \left( \frac{j_{n''} + l'(2^k - 1)}{2^{m}(2^k - 1)} \right) - \tilde{u} \left( \frac{j_{n''-1} + l'(2^k - 1)}{2^{m}(2^k - 1)} \right) \right)^2 \right]. \]  
(19)
Many of the summands will be zero, either because the interval is a loop or because it is located between identified points. (Again it is obvious that the harmonic extension is constant between identified points.) Thus there exists a set \( J_n \subseteq J \) of values \( j_p \) for \( n' \leq p \leq n'' - 1 \) that corresponds to values where \( \tilde{u} \) may be chosen without constraints, and so that

\[
2^m (2^k - 1) \sum_{p \in J_n} \left( \frac{b_p}{j_p - j_p} \right) \left( \frac{2m}{2m - 1} \right) \left( \frac{2m}{2m - 1} \right)^2
\]

(20)

(where \( \tilde{p} \) is the term in \( J_n \) after \( p \)) is the same as (19). (Note that both \( n' \) and \( n'' \) are in \( J_n \), so the values of \( u \) at the endpoints of \( I \) are involved in (20).) Now minimizing (20) is just a linear problem, with solution

\[
\tilde{u} \left( \frac{j_p + l'(2^k - 1)}{2^m (2^k - 1)} \right) = u \left( \frac{j_n + l(2^k - 1)}{2^m - 1(2^k - 1)} \right) + \frac{c_p}{C_n} \left( u \left( \frac{j_{n+1} + l(2^k - 1)}{2^m - 1(2^k - 1)} \right) - u \left( \frac{j_n + l(2^k - 1)}{2^m - 1(2^k - 1)} \right) \right)
\]

(21)

for

\[
c_p = \sum_{q < p} \frac{j_q - j_q}{b_q} \quad \text{and} \quad C_n = \sum_{q \in J_n} \frac{j_q - j_q}{b_q}
\]

(22)

This means that (20) consists of the sum of terms

\[
\frac{2^m (2^k - 1)}{C_n^2} \left( \frac{j_p - j_p}{b_p} \right) \left( \frac{2m}{2m - 1(2^k - 1)} \right) \left( \frac{2m}{2m - 1(2^k - 1)} \right)^2
\]

(23)

for \( p \in J_n \), and each term in (23) contributes to \( E_p^{(m)}(\tilde{u}) \). If we sum over all type \( u \) intervals in \( X^{(m-1)} \), we see that the energy \( E_n^{(m-1)}(u) \) gets sent to energies \( E_p^{(m)}(\tilde{u}) \) for \( p \in J_n \) multiplied by the factor \( M_{pn} \) given by

\[
M_{pn} = \frac{2}{C_n^2} \frac{(j_{n+1} - j_n)(j_p - j_p)}{b_p}.
\]

(24)

(Note that the same \( p \) value may occur more than once in (20), in which case \( M_{pn} \) is (24)multiplied by the number of occurrences.)

The story for paired intervals is virtually the same, except that each \( p \)-type interval appears in \( I \) and \( I' \). Summing over \( n \), we obtain

\[
E_p^{(m)}(\tilde{u}) = \sum_n M_{pn} E_n^{(m-1)}(u)
\]

(25)

In view of (13) this means

\[
E_p^{(m)}(\tilde{u}) = \sum_p \sum_n b_p M_{pn} E_n^{(m-1)}(u)
\]

(26)

so (15) is the eigenvalue equation.

\[
r b_n = \sum_n b_p M_{pn}.
\]

(27)

In other words, the row vector \( b \) is a left eigenvector of the matrix \( M \) with eigenvalue \( r \). Since we want the entries of \( b \) to be positive, we take the Perron-Frobenius eigenvector for the nonnegative matrix \( M \) (presumably \( M \) is always irreducible). However, the entries of the matrix \( M \) also depend on \( b \), so (27) is not simply a linear eigenvalue equation. In the examples that we work out in detail in the next
section, we can transform (27) into a linear system that may be solved explicitly.
In the examples discussed in Part I, some power of the matrix is a multiple of the
identity, which leads to the observation that there exists a multidimensional space of
ergies invariant under a power of \( P \). This allowed the formation of a \( P \)-invariant
ergency by averaging consecutive energies of the form (10). It turns out that the
method used in Part I is equivalent to the method discussed here. It seems that
these examples are rather special, and this type of behavior should not be expected
more generally.

If we drop the simplifying assumption, there may be more than one way for an
interval of type \( n \) to subdivide, but the number of choices will be finite. To handle
this situation, we have to further partition the energy \( E_n^m(u) \) into terms arising
from intervals with different subdivision identifications. There are essentially no
new ideas here, but the challenge of devising a general notation is rather annoying,
so we omit the details.

4. Three examples. Before discussing the three new examples, we mention briefly
how the examples discussed in Part I fit into the general scheme described here.
Those examples correspond to \( \theta = \frac{1}{2k-1} \), with \( J = \{1, 2, 2^2, \ldots, 2^{k-1}\} \) and all
points \( \left\{ \frac{j\theta}{2k-1} \right\} \) in \( X(0) \) identified. The subdivision rule is that each \( n \)-type interval
in \( X(m) \) for \( n \leq k-1 \) is undivided in \( X(m) \) and becomes an \((n+1)\)-type interval.
The \( k \)-type interval \( I = \left\{ \frac{2^{k-1}+l(2^{k-1})}{2^{m-1}(2^{k-1})}, \frac{1+(l+1)(2^{k-1})}{2^{m-1}(2^{k-1})} \right\} \) subdivides at the \( k \) identified
points \( \left\{ \frac{2^l+(l+1)(2^{k-1})}{2^{m}(2^{k-1})} \right\} \), so that leaves just two intervals of type 1 at the ends of
\( I \). Thus, there are only two terms in (20) and they both contribute to \( E_1^m(\tilde{u}) \). In
this case, since there is only one \( n \)-type interval that subdivides, the matrix \( M \) is
independent of \( b \) and has the form

\[
\begin{pmatrix}
0 & \cdots & 0 & 2^{k-1} \\
1 & \ddots & 0 \\
\vdots & & \ddots & \\
0 & \cdots & 1 & 0
\end{pmatrix}
\]  

(28)

Note that \( M^k = 2^{k-1}I \), so \( r = 2^{(\frac{k-1}{k})} \) and \( b = (1, 2^{\left(\frac{1}{k}\right)}, 2^{\left(\frac{2}{k}\right)}, \ldots, 2^{\left(\frac{k-1}{k}\right)}) \).

4.1. Example 1: \( \theta = \frac{1}{6}, c = i \), the Dendrite (actually this is just one of many Julia
sets that are topological dendrites). The dendrite resembles a Rabbit that has been
squashed so that all circles become line segments.

We take \( k = 3 \), and note that the cycle \( \frac{1}{7}, \frac{2}{7}, \frac{4}{7} \) contain points only in \( B = \left( \frac{1}{12}, \frac{7}{12} \right) \),
so these points are identified and we take them for \( X(0) \). (The cycle \( \frac{3}{7}, \frac{6}{7}, \frac{5}{7} \) has
kneading sequence \( B, A, A \) and so has no identifications.) It is clear from Figure
1(a) that the Dendrite has triple points, but no points of higher multiplicity. So
already at \( k = 3 \) we have captured this feature of the Julia set. However, a generic
point has multiplicity two, and this choice of \( X(0) \) means that \( X^m \) will not have
any double identifications. Is this a problem? In our judgment it is not, because
in fact what is happening is that the triple identifications in \( X^m \) imply enough
double identifications just by ignoring one of the three identified parameter values.
By increasing \( k \) we could produce double identified points, but it seems unlikely that
adding them to $X^{(0)}$ would reveal anything new in the numerical approximation of eigenvalues and eigenfunctions.

With this choice of $X^{(0)}$, the points in $X^{(m)}$ will be the same as for the Rabbit ($\theta = \frac{1}{2}$), $J = \{1, 2, 4\}$ but the identifications will be different. Figure 1(b) shows the identifications on $X^{(4)}$. On $X^{(1)}$, $X^{(2)}$, and $X^{(3)}$, all identifications are internal, but the intervals $[\frac{1}{14}, \frac{1}{7}]$ and $[\frac{4}{7}, \frac{9}{14}]$ in $X^{(3)}$ contain the division points $\frac{1}{12}$ and $\frac{7}{12}$ so they will pair and subdivide in $X^{(4)}$ as shown in Figure 1(c). Note that these are intervals of type 3 in $X^{(3)}$. Also $X^{(3)}$ contains six other intervals of type 3, but they are all loops. After subdivision, the new type 3 intervals created in $X^{(4)}$ are loops. As the discussion at the end of section 3 shows ($\theta = \frac{1}{2}$ so $b = 2$), the same pattern repeats in the intervals in $X^{(m-1)}$ for $m$ even that contain $\frac{1}{12}$ and $\frac{7}{12}$ (the passage from $X^{(m)}$ to $X^{(m+1)}$ does not subdivide those intervals). So, we can summarize the subdivision rules for intervals as follows:

(a) type 1 intervals do not subdivide, and become type 2;
(b) type 2 intervals do not subdivide, and become type 3;
(c) type 3 intervals that are not loops are paired and subdivide (as shown in figure 1(c)) into two intervals of type 1, one interval of type 2, and one loop of type 3.

To compute the harmonic extension in the (c) subdivision we note that it is equivalent to dividing an interval of length \( \frac{b_k}{b_1} + \frac{b_2}{b_2} \) into two intervals of length \( \frac{1}{b_1} \) on the sides and one interval of length \( \frac{1}{b_2} \) in the center, and the function is interpolated linearly. If, for simplicity, the original values were \((0, 1)\) then after subdivision the values are \(\left(0, \frac{b_k}{2(b_1 + b_2)}, \frac{2b_1 + b_2}{2(b_1 + b_2)}, 1\right)\). The matrix \(M\) from (24) thus has the form

\[
\begin{pmatrix}
0 & 0 & \frac{4b_1^3}{(b_1 + b_2)^2} \\
1 & 0 & \frac{4b_2^3}{(b_1 + b_2)^2} \\
0 & 1 & 0
\end{pmatrix}
\]

The eigenvalue equation (27) reduces to the equations

\[b_2 = rb_1, \quad b_3 = rb_2, \quad \text{and} \quad \frac{4b_1b_2}{b_1 + b_2} = rb_3.\]

The solution has \(b = (1, r, r^2)\) where \(r\) satisfies \(r^4 + r^3 - 4r = 0\), and since we want \(r > 0\) we can cancel \(r\) to obtain the cubic \(r^3 + r^2 - 4 = 0\), which has a unique positive solution \(r = 1.3146\ldots\)

4.2. Example 2: \(\theta = \frac{3}{7}, c = -\frac{7}{4}\), the Airplane. This resembles the Basilica, except that the circles do not touch, but are separated by a countable collection of very small circles. (See Figure 2(a))

In this example \(A = \left[\frac{5}{14}, \frac{3}{14}\right]\) and \(B = \left[\frac{3}{14}, \frac{5}{7}\right]\). Again we take \(k = 3\) because there are no identifications with \(k = 2\). The cycles \(\frac{1}{7}, \frac{2}{7}, \frac{4}{7}\) and \(\frac{3}{7}, \frac{6}{7}, \frac{5}{7}\) have kneading sequence \(A, B, B\) and \(B, A, B\), so \(X^{(0)}\) identifies \(\frac{1}{7} \sim \frac{6}{7}, \frac{2}{7} \sim \frac{5}{7}, \frac{3}{7} \sim \frac{4}{7}\). This gives us double identifications, and since the Figure 2(a) shows no triple identifications, it seems to be a reasonable starting point. Note that we have \(J = \{1, 2, 3, 4, 5, 6\}\). Figure 2(b) shows \(X^{(1)}\) with identifications.

Since the division points \(\frac{3}{14}\) and \(\frac{5}{7}\) both belong to \(X^{(1)}\), we will never encounter an interval in \(X^{(m)}\) for \(m > 1\) containing them in its interior, so the subdivision rule for \(X^{(0)} \rightarrow X^{(1)}\) is simply replicated at all levels. We may describe the rules as follows:

(a) intervals of type 1 (\(\left[\frac{1}{7}, \frac{2}{7}\right]\)) and type 5 (\(\left[\frac{3}{7}, \frac{6}{7}\right]\)) are paired and subdivide into a paired type 2 (\(\left[\frac{2}{14}, \frac{3}{14}\right]\)) and type 4 (\(\left[\frac{11}{14}, \frac{12}{14}\right]\)) interval, and two paired type 3 intervals (\(\left[\frac{3}{14}, \frac{4}{14}\right]\) and \(\left[\frac{10}{14}, \frac{11}{14}\right]\));
(b) intervals of type 2 (\(\left[\frac{3}{7}, \frac{4}{7}\right]\)) and type 4 (\(\left[\frac{4}{7}, \frac{5}{7}\right]\)) are paired and subdivide into a paired type 1 (\(\left[\frac{5}{14}, \frac{6}{14}\right]\)) and type 5 (\(\left[\frac{6}{14}, \frac{7}{14}\right]\)) interval and a paired type 2 (\(\left[\frac{9}{14}, \frac{10}{14}\right]\)) and type 4 (\(\left[\frac{3}{7}, \frac{5}{7}\right]\)) interval;
(c) intervals of type 3 (\(\left[\frac{5}{7}, \frac{6}{7}\right]\)) do not subdivide and become intervals of type 6;
(d) intervals of type 6 (\(\left[\frac{9}{14}, \frac{1}{14}\right]\)) subdivide into a loop of type 6 (\(\left[\frac{13}{14}, \frac{1}{14}\right]\)) and a paired type 1 (\(\left[\frac{1}{14}, \frac{2}{14}\right]\)) and type 5 (\(\left[\frac{12}{14}, \frac{13}{14}\right]\)) interval.

We may simplify matters by lumping together intervals of type 1 and 5, and those of type 2 and 4, since they are always paired. So we only have to consider type 1, 2, 3, 6 and \(M\) becomes a \(4 \times 4\) matrix. (In other words, \(b_1 = b_5\) and \(b_2 = b_4\) by the pairing symmetry.) Note that in the subdivision of type (d), an interval is split into two intervals of length \(\frac{1}{4}\) of the length of the original interval (ignoring the
loop), and both are of the same type so the energy is multiplied by 2 (independent of the coefficients $b$). On the other hand, in the subdivisions of type (a) and (b), the two intervals have lengths $\frac{1}{2}$ of the length of the original interval, but they have different types so in computing the harmonic extension, the lengths are divided by the $b$ coefficients. Thus the matrix $M$ has the form

$$
\begin{pmatrix}
0 & \frac{2b_2^2}{(b_1+b_2)^2} & 0 & 2 \\
\frac{2b_2^2}{(b_2+b_3)^2} & \frac{2b_3^2}{(b_1+b_3)^2} & 0 & 0 \\
\frac{2b_2^2}{(b_2+b_3)^2} & \frac{2b_3^2}{(b_1+b_2)^2} & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
$$

The eigenvalue equation (27) reduces to the equations

$$
2b_2b_3 = rb_1, \quad 2b_1b_2 = rb_2, \quad b_6 = rb_3 \quad \text{and} \quad 2b_1 = rb_6.
$$

Using the last two equations to eliminate the variables $b_1$ and $b_6$, the first two equations reduce (assuming $r \neq 0$) to

$$
\begin{pmatrix}
2 - \frac{r^2}{2} & \frac{r^2}{2} \\
1 & \frac{r^2}{2} - r
\end{pmatrix}
\begin{pmatrix}
b_2 \\
b_3
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \end{pmatrix}
$$
Thus we may take \((b_1, b_2, b_3, b_6) = (\frac{r^2}{2}, r - \frac{r^2}{2}, 1, r)\) where \(r\) makes the determinant of the \(2 \times 2\) matrix vanish (we also need \(0 < r < 2\) to make the solution positive). This leads to the 4th degree equation
\[
\frac{r^4}{4} - \frac{r^3}{2} - \frac{r^2}{2} - r + 2 = 0,
\]
which has a unique solution \(r = 1.09827\ldots\) in \(0 < r < 2\).

4.3. Example 3: \(\theta = \frac{10}{63}\), \(c = -0.11 + 0.86i\), the Basilica-in-Rabbit. This example is a tuning of the Basilica \((\theta = \frac{1}{7})\) in the Rabbit \((\theta = \frac{1}{5})\), that is obtained by replacing the circles in the Rabbit by little Basilicas [3]. (See Figure 3(a))

This example has triple points from the Rabbit and double points from the Basilica, so it is important to choose \(X^{(0)}\) to have both double and triple identifications. We take \(k = 6\). The triple identification already occurs at \(k = 3\), but because 7 divides 63, it also shows up when \(k = 6\). In this example \(A = (\frac{10}{126}, \frac{73}{126})\) and \(B = (\frac{10}{126}, \frac{73}{126})\). The cycle \(\frac{9}{63}, \frac{18}{63}, \frac{36}{63}\) has kneading sequence \(B, B, B\) so we identify \(\frac{9}{63}, \frac{18}{63}, \frac{36}{63}\) (this is of course \(\frac{1}{7}, \frac{2}{7}, \frac{4}{7}\) as in the Rabbit). The cycle \(\frac{5}{63}, \frac{10}{63}, \frac{20}{63}, \frac{40}{63}, \frac{17}{63}, \frac{34}{63}\) has kneading sequence \(A, B, B, A, B, B\), so we obtain the three double identifications \(\frac{5}{63}, \frac{10}{63}, \frac{17}{63}, \frac{34}{63}\). This yields the complete descriptions of \(X^{(0)}\) with \(J = \{5, 9, 10, 17, 18, 20, 34, 36, 40\}\). Figure 3(b) shows \(X^{(1)}\) with identifications.

Just like in example 2, the division points \(\frac{10}{126}\) and \(\frac{73}{126}\) belong to \(X^{(1)}\), so the subdivision rule for \(X^{(0)} \rightarrow X^{(1)}\) is replicated on all levels. In this case there are 9 types of intervals, but among these there are 3 pairs, \((1, 8), (2, 4),\) and \((5, 7)\), so we only have to examine 6 cases, only two of which involve subdivision. The results are summarized in Table 1. The 3 pairings are indicated on the left.

(Note that two intervals of type 3 are paired with each other after the subdivision of intervals of type 1 and type 8). The third column gives the intervals in units of \(\frac{1}{126}\) in \(X^{(0)}\), and the fourth column gives the subdivision in \(X^{(1)}\) in the same units. Note that the interval \([\frac{81}{126}, \frac{9}{126}]\) is actually a union of several different types of intervals, but because the endpoints are identified none of these subintervals contributes to
Table 1. Basilica-in-Rabbit Subdivision Rules

| n | $j_n$ | $2j_n, 2j_{n+1}$ | subdivision types |
|---|---|---|---|
| 1 | 5 | 10, 18 | $10, 17 \cup 17, 18$ | 3 \cup 4 |
| 2 | 9 | 18, 20 | 18, 20 | 5 |
| 3 | 10 | 20, 34 | 20, 34 | 6 |
| 4 | 17 | 34, 36 | 34, 36 | 7 |
| 5 | 18 | 36, 40 | 36, 40 | 8 |
| 6 | 20 | 40, 68 | 40, 68 | 9 |
| 7 | 34 | 68, 72 | 68, 72 | 1 |
| 8 | 36 | 72, 80 | $72, 73 \cup 73, 80$ | 2 \cup 3 |
| 9 | 40 | 80, 10 | $80, 81 \cup 81, 9 \cup 9, 10$ | 2 \cup 4 |

The energy. Thus intervals of type 9 subdivide into just a pair of intervals of types 2 and 4. Since the length of the original interval is multiplied by $\frac{56}{7}$ in the subdivided intervals, the energy is multiplied by 28, regardless of the choice of $b$. On the other hand, when a $(1, 8)$ pair subdivides into a $(3, 3)$ pair and a $(2, 4)$ pair, the length is reduced by factors of $\frac{7}{8}$ and $\frac{1}{8}$. The $6 \times 6$ matrix $M$ becomes

$$
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
8b_2^2 & 0 & 0 & 0 & 28 \\
(7b_3^2 + b_3)^2 & 0 & 0 & 0 & 0 \\
56b_2^2 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
$$

The eigenvalue equation (27) reduces to the equations

$$
\frac{8b_2b_3}{7b_2 + b_3} = rb_1, \quad b_5 = rb_2, \quad b_6 = rb_3, \quad b_1 = rb_5, \quad b_9 = rb_6 \quad \text{and} \quad 28b_2 = rb_9.
$$

We may choose $b_3 = 1$, and all but the first equation yield $b = \left( \frac{r^6}{28}, \frac{r^3}{28}, 1, \frac{r^4}{28}, r, r^2 \right)$. The first equation says (after canceling $r^3$).

$$
\frac{r^6}{4} + r^3 - 8 = 0
$$

and the solution is $r = 4^{\frac{1}{3}}$ with $b = \left( \frac{4^{\frac{2}{3}}}{7}, \frac{1}{7}, 1, \frac{4^{\frac{1}{3}}}{7}, 4^{\frac{1}{3}}, 4^{\frac{2}{3}} \right)$.

5. Eigenvalues and eigenfunctions. In this section we describe how to approximate the spectrum of a $P$-invariant Laplacian on $\mathcal{J}$. This Laplacian is constructed from the $P$-invariant energy defined in section 3 and the $P$-invariant probability measure $\mu$ (often called the equilibrium measure) by

$$
\mathcal{E}(u, v) = -\int_{\mathcal{J}} (\Delta u)vd\mu \quad \text{for all} \quad v \in \text{dom } \mathcal{E}
$$

The invariance condition

$$
\int_{\mathcal{J}} f \circ Pd\mu = \int_{\mathcal{J}} fd\mu
$$
of the equilibrium measure together with (18) yields the invariance condition
\[ \Delta (u \circ P) = \frac{4}{r} (\Delta u) \circ P \] (32)
for this Laplacian. We are interested in solutions of the eigenvalue equation
\[ -\Delta u = \lambda u, \] (33)
which, in view of (30), becomes
\[ E(u, v) = \lambda \int Juvd\mu \] for all \( v \in \text{dom} E \) (34)
We describe two methods to approximate solution of (34), the finite difference method and the finite element method. Both methods replace the left-side of (34) by \( E_m(u, v) \) and let \( v \) vary over piecewise harmonic functions on \( X^{(m)} \). They differ in the treatment of the right-hand side. We let \( \mu_m \) denote the discrete measure on \( X^{(m)} \) that approximates \( \mu \) by assigning to each \( t \in X^{(m)} \) the average of the lengths of the two \( X^{(m)} \) intervals that have endpoint \( t \), and then assigning to an equivalence class of identified points the sum of the weights assigned to each point in the equivalence class. Note that this is just the obvious discrete approximation to the uniform measure on the parameter circle. For the finite difference method we approximate
\[ \int_J uv d\mu \approx \sum_{t_j \in X^{(m)}} u(t_j) v(t_j) \mu_m(t_j) \] (35)
(we can interpret the sum on the right either as a sum over individual points or as a sum over equivalence classes). For the finite element method we approximate
\[ \int_J uv d\mu \approx \int_0^1 \tilde{u}(t) \tilde{v}(t) dt \] (36)
where \( \tilde{u} \) and \( \tilde{v} \) are piecewise linear functions on the parameter circle that agree with \( u \) and \( v \) at the points in \( X^{(m)} \). Note that \( \tilde{u} \) and \( \tilde{v} \) do not respect identifications at levels higher than \( m \), so they cannot be interpreted as functions on \( J \). Nevertheless, we feel justified in using the terms “finite element method” because this is exactly what that method would do using piecewise linear splines on the circle. Anyway, we don’t know how to compute integrals of products of piecewise harmonic functions on \( J \), so (36) is the best we can do.
We can make (36) completely explicit by noting that
\[ \frac{1}{b-a} \int_a^b f(t)g(t) dt = \frac{1}{3} [f(a)g(a) + f(b)g(b)] + \frac{1}{6} [f(a)g(b) + f(b)g(a)] \]
if \( f \) and \( g \) are linear on \([a, b]\). This means (36) says
\[ \int_J uv d\mu \approx \frac{1}{3} \sum_j (t_{j+1} - t_j - 1) u(t_j) v(t_j) + \frac{1}{6} \sum_j (t_{j+1} - t_j)(u(t_{j+1}) v(t_j) + u(t_j) v(t_{j+1})) \] (37)
Here the sum is over all points in \( X^{(m)} \). It is completely straightforward to express (37) as a sum over equivalence classes, but the notation becomes quite awkward in the general case. We will work out the details for each of our three examples below.
Let \( \{x_j\} \) denote the equivalence classes of points in \( X^{(m)} \), so \( u(x_j) \) and \( v(x_j) \) are well defined. For any fixed \( l \) choose \( v_l \) to be the piecewise harmonic function
satisfying
\[ v_l(x_j) = \delta_{jl} \]
on \( X^{(m)} \). Then our approximation to (34) for \( v = v_l \) becomes
\[ \mathcal{E}^{(m)}(u, v_l) = \lambda \mu_m(x_l) u(x_l) = \lambda \left( \frac{1}{2} \sum_{t_j \in x_l} (t_{j+1} - t_j) \right) u(x_l) \]  
(38)
for the finite difference method, and
\[ \mathcal{E}^{(m)}(u, v_l) = \lambda \left[ \left( \frac{1}{3} \sum_{t_j \in x_l} (t_{j+1} - t_{j-1}) \right) u(x_l) \right. \\
+ \left. \frac{1}{6} \sum_{t_j \in x_l} \left( (t_{j+1} - t_j) u(t_{j+1}) + (t_j - t_{j-1}) u(t_{j-1}) \right) \right] \]  
(39)
for the finite element method. The left side of (38) and (39) may be expressed as
\[ m \sum_{t_j \in x_l} (b_{n(j)} \left( \frac{u(t_j) - u(t_{j+1})}{t_{j+1} - t_j} \right) + b_{n(j-1)} \left( \frac{u(t_j) - u(t_{j-1})}{t_j - t_{j-1}} \right)) \]  
(40)
where \( n(j) \) denotes the \( n \)-type of \( t_j \). So in both cases, we obtain a matrix generalized eigenvalue problem for the vector \( \{u(x_j)\} \) of the form
\[ Eu = \lambda Gu \]  
(41)
where \( E \) and \( G \) are sparse symmetric matrices. In the case of the finite difference method, the matrix \( G \) is diagonal. In all the examples studied in Part I, \( G \) was a multiple of the identity.

The advantage of using two different methods to do the approximation is that the finite element method always gives an overapproximation, while the finite difference method tends to give an underapproximation. The difference between the two eigenvalue approximations thus gives us an indication of the accuracy of the approximation, and we may take the average of the two for a more accurate approximation.

For an explicit description of the matrix equation (41) for the three examples discussed in section 4 see the website [2].

We present some of the numerical results we obtained in each of the three examples. We observe that the eigenvalues converge quite well for the dendrite example, but rather poorly for the other two examples. This may be symptomatic of slow convergence due to the complicated nature of the Julia sets. It is also possible that there are coding errors in the programs, but we have been unable to find any. On the other hand, the eigenfunctions converge well in all three examples.

We present Tables of the values of the first 40 eigenvalues at the two highest levels that we are able to compute, giving the results for both FDM and FEM, and the average of the two. We present graphs of the first 16 eigenfunctions on the parameter circle at the highest level. There is no discernible difference between the graphs using FDM and FEM. We also present graphs of certain eigenfunctions that are discussed in the next section.

We present graphs of the eigenvalue counting function
\[ N(t) = \# \{ \lambda_j : \lambda_j \leq t \} . \]  
(42)
As discussed in [8] and Part I, the results of [6] imply that the Weyl ratio
\[ W(t) = \frac{N(t)}{t^\alpha} \quad \text{for} \quad \alpha = \frac{\log(4/r)}{\log 2} \] (43)
is bounded above and away from zero and is asymptotically multiplicatively periodic of period \(4/r\). We present graphs of \(W(t)\) as a function of \(\log(t)\) that show the beginnings of this periodic behavior.

There is a strong resemblance among the graphs of eigenfunctions of the Basilica and Airplane on the one hand, and of the Rabbit, Dendrite and Basilica-in-Rabbit on the other hand, especially for low eigenvalues. To illustrate this we present side-by-side graphs of the first eigenfunctions in each of these families.

![Figure 4. Eigenfunctions #117, 234 and 235 of Dendrite on Level 11](image-url)
Table 2. Dendrite Eigenvalues on Levels 10 and 11

| #  | FDM Level 10 | FEM Level 10 | Avg Level 10 | FDM Level 11 | FEM Level 11 | Avg Level 11 |
|----|-------------|-------------|-------------|-------------|-------------|-------------|
| 1  | 56.0714     | 56.0722     | 56.0718     | 56.0713     | 56.0715     | 56.0714     |
| 2  | 170.6108    | 170.6146    | 170.6137    | 170.6113    | 170.6149    | 170.6124    |
| 3  | 212.6105    | 212.6658    | 212.6607    | 212.6645    | 212.6626    |
| 4  | 519.1221    | 519.1564    | 519.1263    | 519.1491    | 519.1377    |
| 5  | 561.6728    | 561.7118    | 561.6752    | 561.7021    | 561.6887    |
| 6  | 647.0759    | 647.1723    | 647.1241    | 647.0710    | 647.089     |
| 7  | 1214.6468   | 1214.8338   | 1214.6792   | 1214.7992   | 1214.7392   |
| 8  | 1579.5737   | 1579.6654   | 1579.5591   | 1579.7675   | 1579.6633   |
| 9  | 1682.2958   | 1682.9751   | 1682.6355   | 1682.5465   | 1682.4311   |
| 10 | 1709.0051   | 1709.3572   | 1709.0302   | 1709.2675   | 1709.1498   |
| 11 | 1785.3534   | 1785.9464   | 1785.6355   | 1785.884    | 1785.7598   |
| 12 | 1968.7232   | 1969.7534   | 1969.2383   | 1969.1488   |
| 13 | 2023.1363   | 2024.2335   | 2023.6849   | 2023.327    | 2023.8404   |
| 14 | 3695.6182   | 3697.3323   | 3695.8672   | 3697.0057   | 3696.4365   |
| 15 | 3718.5839   | 3722.0177   | 3718.7775   | 3719.9444   | 3719.3609   |
| 16 | 4804.8679   | 4808.9988   | 4806.249    | 4808.0006   | 4807.1248   |
| 17 | 5117.503    | 5124.682    | 5118.8067   | 5120.8736   | 5119.8401   |
| 18 | 5123.2996   | 5126.8996   | 5124.5817   | 5126.5955   | 5125.6206   |
| 19 | 5198.7497   | 5202.4506   | 5200.0764   | 5202.1899   | 5201.4746   |
| 20 | 5265.6111   | 5289.3641   | 5286.7441   | 5289.0228   | 5287.8835   |
| 21 | 5432.0781   | 5435.9421   | 5432.9386   | 5434.4042   | 5433.1894   |
| 22 | 5836.7946   | 5841.2615   | 5837.7525   | 5840.6836   | 5839.218    |
| 23 | 5989.4529   | 5994.1247   | 5990.3338   | 5993.4685   | 5991.9011   |
| 24 | 6122.8497   | 6127.7276   | 6123.7361   | 6127.033    | 6125.3847   |
| 25 | 6105.0169   | 6159.9436   | 6155.8993   | 6159.2378   | 6157.5686   |
| 26 | 9843.359    | 9855.6523   | 9848.9058   | 9853.5658   | 9849.7358   |
| 27 | 11243.1926  | 11258.7171  | 11244.8447  | 11255.276   | 11250.6094  |
| 28 | 11277.3357  | 11293.1307  | 11279.2655  | 11289.6452  | 11284.4254  |
| 29 | 11313.0356  | 11328.7636  | 11314.7236  | 11325.1718  | 11319.9477  |
| 30 | 12754.773   | 12793.5591  | 12774.166   | 12775.2599  | 12768.9364  | 12762.0981  |
| 31 | 14602.0822  | 14672.4107  | 14647.242   | 14620.0149  | 14639.6768  | 14629.8458  |
| 32 | 14686.1209  | 14736.7914  | 14711.4562  | 14683.9652  | 14703.77    | 14693.8676  |
| 33 | 14787.4904  | 14857.6345  | 14828.1619  | 14805.1917  | 14805.2197  | 14795.2057  |
| 34 | 15292.7718  | 15362.6109  | 15319.6914  | 15289.7437  | 15310.7155  | 15300.2296  |
| 35 | 15574.8882  | 15602.6274  | 15571.2856  | 15593.1295  | 15582.2075  |
| 36 | 15585.9434  | 15631.5469  | 15613.7406  | 15582.3635  | 15604.2052  | 15593.2844  |
| 37 | 15592.5055  | 15648.1445  | 15620.325   | 15588.9232  | 15610.7703  | 15599.8468  |
| 38 | 15790.0337  | 15846.8353  | 15818.4249  | 15786.0509  | 15808.4858  | 15797.2683  |
| 39 | 15822.5461  | 15879.5533  | 15851.0497  | 15818.4991  | 15841.0206  | 15829.7599  |

(a) Eigenvalue counting function for Dendrite on Level 11

(b) Weyl Ratio for Dendrite on Level 11
Figure 5. Eigenfunctions #1 – 16 of Dendrite on Level 11
Table 3. Airplane Eigenvalues on Levels 9 and 10

| #  | FDM Level 9 | FEM Level 9 | Avg Level 9 | FDM Level 10 | FEM Level 10 | Avg Level 10 |
|----|-------------|-------------|-------------|-------------|-------------|-------------|
| 1  | 41.0278     | 41.0279     | 41.0279     | 46.0982     | 46.0983     | 46.0983     |
| 2  | 134.7372    | 134.7384    | 134.7378    | 149.4271    | 149.4275    | 149.4273    |
| 3  | 335.2911    | 335.2981    | 335.2946    | 374.4527    | 374.4551    | 374.4539    |
| 4  | 444.3255    | 444.3379    | 444.3317    | 490.7254    | 490.7296    | 490.7275    |
| 5  | 1002.1231   | 1002.1870   | 1002.1550   | 1136.3817   | 1136.4039   | 1136.3928   |
| 6  | 1094.0517   | 1094.1277   | 1094.0897   | 1221.1608   | 1221.1866   | 1221.1737   |
| 7  | 1324.2008   | 1324.3106   | 1324.2557   | 1584.1267   | 1584.1692   | 1584.1479   |
| 8  | 1426.9224   | 1427.0503   | 1426.9863   | 1613.4262   | 1613.4709   | 1613.4485   |
| 9  | 1443.5502   | 1443.6811   | 1443.6157   | 1618.2744   | 1618.3194   | 1618.2969   |
| 10 | 1246.9224   | 1247.0503   | 1246.9863   | 1613.4262   | 1613.4709   | 1613.4485   |
| 11 | 1346.2008   | 1346.3106   | 1346.2557   | 1584.1267   | 1584.1692   | 1584.1479   |
| 12 | 1426.9224   | 1427.0503   | 1426.9863   | 1613.4262   | 1613.4709   | 1613.4485   |
| 13 | 1443.5502   | 1443.6811   | 1443.6157   | 1618.2744   | 1618.3194   | 1618.2969   |
| 14 | 1246.9224   | 1247.0503   | 1246.9863   | 1613.4262   | 1613.4709   | 1613.4485   |
| 15 | 1346.2008   | 1346.3106   | 1346.2557   | 1584.1267   | 1584.1692   | 1584.1479   |
| 16 | 1426.9224   | 1427.0503   | 1426.9863   | 1613.4262   | 1613.4709   | 1613.4485   |
| 17 | 1443.5502   | 1443.6811   | 1443.6157   | 1618.2744   | 1618.3194   | 1618.2969   |
| 18 | 1246.9224   | 1247.0503   | 1246.9863   | 1613.4262   | 1613.4709   | 1613.4485   |
| 19 | 1346.2008   | 1346.3106   | 1346.2557   | 1584.1267   | 1584.1692   | 1584.1479   |
| 20 | 1426.9224   | 1427.0503   | 1426.9863   | 1613.4262   | 1613.4709   | 1613.4485   |
| 21 | 1443.5502   | 1443.6811   | 1443.6157   | 1618.2744   | 1618.3194   | 1618.2969   |
| 22 | 1246.9224   | 1247.0503   | 1246.9863   | 1613.4262   | 1613.4709   | 1613.4485   |
| 23 | 1346.2008   | 1346.3106   | 1346.2557   | 1584.1267   | 1584.1692   | 1584.1479   |
| 24 | 1426.9224   | 1427.0503   | 1426.9863   | 1613.4262   | 1613.4709   | 1613.4485   |
| 25 | 1443.5502   | 1443.6811   | 1443.6157   | 1618.2744   | 1618.3194   | 1618.2969   |
| 26 | 1246.9224   | 1247.0503   | 1246.9863   | 1613.4262   | 1613.4709   | 1613.4485   |
| 27 | 1346.2008   | 1346.3106   | 1346.2557   | 1584.1267   | 1584.1692   | 1584.1479   |
| 28 | 1426.9224   | 1427.0503   | 1426.9863   | 1613.4262   | 1613.4709   | 1613.4485   |
| 29 | 1443.5502   | 1443.6811   | 1443.6157   | 1618.2744   | 1618.3194   | 1618.2969   |
| 30 | 1246.9224   | 1247.0503   | 1246.9863   | 1613.4262   | 1613.4709   | 1613.4485   |
| 31 | 1346.2008   | 1346.3106   | 1346.2557   | 1584.1267   | 1584.1692   | 1584.1479   |
| 32 | 1426.9224   | 1427.0503   | 1426.9863   | 1613.4262   | 1613.4709   | 1613.4485   |

(a) Eigenvalue counting function for Airplane on Level 10

(b) Weyl Ratio for Airplane on Level 10
Figure 6. Eigenfunctions #1 – 16 of Airplane on Level 10
Table 4. Basilica-in-Rabbit Eigenvalues on Levels 9 and 10

|   | FDM Level 9 | FDM Level 10 | FEM Level 9 | FEM Level 10 | Avg Level 9 | Avg Level 10 |
|---|-------------|-------------|-------------|-------------|-------------|-------------|
| 1 | 20.8140     | 20.8142     | 23.0214     | 23.0216     | 23.0171     | 23.0215     |
| 2 | 51.2857     | 51.2872     | 52.4479     | 52.4490     | 52.4484     |             |
| 3 | 62.5232     | 62.5244     | 66.0223     | 66.0235     | 66.0229     |             |
| 4 | 120.5270    | 120.5325    | 129.2318    | 129.2357    | 129.2338    |             |
| 5 | 129.2171    | 129.2241    | 139.9977    | 140.0044    | 140.0020    |             |
| 6 | 143.4825    | 143.4923    | 157.5485    | 157.5550    | 157.5517    |             |
| 7 | 270.4062    | 270.4750    | 294.1762    | 294.2018    | 294.1890    |             |
| 8 | 281.0405    | 281.1124    | 303.7090    | 303.7230    |             |             |
| 9 | 307.1024    | 307.1773    | 325.0018    | 325.0194    |             |             |
| 10| 307.8939    | 307.9688    | 325.6067    | 325.6243    |             |             |
| 11| 329.9127    | 330.0174    | 329.6164    | 329.6336    |             |             |
| 12| 352.3946    | 352.4692    | 352.4319    | 352.4592    |             |             |
| 13| 353.5241    | 353.5993    | 360.4646    | 360.4892    |             |             |
| 14| 682.4605    | 682.7039    | 681.3809    | 681.4675    |             |             |
| 15| 694.0068    | 694.2530    | 693.5810    | 693.4929    |             |             |
| 16| 710.0274    | 710.2849    | 708.1777    | 708.2684    |             |             |
| 17| 728.0126    | 728.2919    | 729.3874    | 729.4790    |             |             |
| 18| 769.6702    | 769.8972    | 773.8497    | 773.9440    |             |             |
| 19| 770.4506    | 770.7686    | 774.7785    | 774.8728    |             |             |
| 20| 771.3615    | 771.6806    | 775.8441    | 775.9384    |             |             |
| 21| 833.8418    | 834.2317    | 834.0368    | 834.1598    |             |             |
| 22| 833.8418    | 834.2317    | 834.0368    | 834.1598    |             |             |
| 23| 849.1411    | 849.5708    | 849.3560    | 849.3363    |             |             |
| 24| 869.1031    | 869.5503    | 869.3267    | 869.0728    |             |             |
| 25| 871.4533    | 871.8988    | 889.5820    | 889.6764    |             |             |
| 26| 873.1355    | 873.5894    | 890.8249    | 890.9196    |             |             |
| 27| 919.8287    | 920.3795    | 920.1041    | 921.2637    |             |             |
| 28| 919.4002    | 919.4107    | 917.1690    | 917.3583    |             |             |
| 29| 1727.4208   | 1729.4375   | 1728.4921   | 1729.6272   |             |             |
| 30| 1734.0072   | 1745.0819   | 1744.0445   | 1745.1547   |             |             |
| 31| 1757.6784   | 1759.7953   | 1758.7369   | 1759.9092   |             |             |
| 32| 1764.9966   | 1767.1319   | 1766.0642   | 1767.1073   |             |             |
| 33| 1798.9455   | 1801.1878   | 1800.0666   | 1801.5697   |             |             |
| 34| 1812.7373   | 1815.0441   | 1813.8908   | 1814.2121   |             |             |
| 35| 1843.6160   | 1846.0678   | 1844.8149   | 1845.8186   |             |             |
| 36| 1936.8557   | 1939.7587   | 1938.3072   | 1939.2803   |             |             |
| 37| 1945.1900   | 1948.1278   | 1946.6589   | 1947.4473   |             |             |
| 38| 1947.4473   | 1950.3943   | 1948.9208   | 1949.3941   |             |             |
| 39| 1947.4778   | 1950.4249   | 1948.9513   | 1949.4139   |             |             |
| 40| 1948.6587   | 1951.6097   | 1950.1342   | 1952.3973   |             |             |

(a) Eigenvalue counting function for Basilica-in-Rabbit on Level 8

(b) Weyl Ratio for Basilica-in-Rabbit on Level 8
Figure 7. Eigenfunctions #1 – 16 Basilica-in-Rabbit
Figure 8. Eigenfunctions of Rabbit, Dendrite and Basilica-in-Rabbit.
Figure 9. Eigenfunctions of Basilica and Airplane.
6. Structure of the spectrum. Based on the data described in section 5, we make some observations, mostly speculations, about the eigenvalues and eigenfunctions in each of the three examples.

Example 1 (Dendrite). The most striking observation is that for any \( n \), the derived eigenfunction \( u_n(2x) \) is exactly \( u_{2n}(x) \). (This requires that we label the constant as \( u_0 \), not \( u_1 \).) This is in contrast to the examples studied in Part I and the other two examples below, where \( u_n(2x) = u_m(x) \) for \( m \) close to \( 2n \), but not always equal to \( 2n \). Since we see the same behavior for Fourier cosine series on an interval, we speculate that it has to do with the fact that the dendrite is a contractible topological space. Indeed \( u_n(2x) = u_{2n}(x) \) is an immediate consequence of the following conjecture.

Conjecture 1. (i) \( u_n \) has exactly \( n \) zeros on the dendrite located at points corresponding to two parameter values (so \( u_n \) has \( 2n \) zeros on the parameter circle), and the zeros of \( u_n \) are located in between the zeros of \( u_{n-1} \).
(ii) The same will be true for any Julia set that is contractible.

Note that (i) is true for the eigenfunctions of regular Sturm-Liouville differential equations on an interval, so it is possible that some arguments from Sturm-Liouville theory might be adopted to our example. On the other hand it should also be noted that many standard tools in Sturm-Liouville theory, such as the Wronskian, do not appear to have any analogs here.

Another easy consequence of the conjecture is that even and odd functions under the parameter space isometry \( t \to t + \frac{1}{2} \) will alternate. In other words,

\[
\begin{align*}
\{ & u_{2k} \left( t + \frac{1}{2} \right) = u_{2k} (t) \\
& u_{2k+1} \left( t + \frac{1}{2} \right) = -u_{2k+1} (t)
\end{align*}
\]

(44)

Again this was not the case for the examples studied in Part I. However, the dendrite appears to have only the single symmetry \( t \to t + \frac{1}{2} \) (\( z \to -z \) in the plane) rather than the pair of symmetries shared by the examples in Part I and the other two examples below.

The data suggests that the dendrite has large spectral gaps. (See [10] for a discussion of general discussion of spectral gaps and their significance.)

Conjecture 2. \( \frac{\lambda_{n+1}}{\lambda_n} \geq 1.5 \) for \( n = \frac{5.4^k - 2}{3} \) or \( \frac{5.4^k - 2}{6} \). Evidence for the conjecture is given in Table 4(a), using eigenvalues computed on level 11. Note that computational error is becoming apparent by \( n = 853 \).

We also have experimental evidence for eigenspaces of multiplicity greater than one. However, among the first 1000 eigenvalues, we only found one sequence of eigenvalues with multiplicity: \( \lambda_{234} = \lambda_{235}, \lambda_{468} = \lambda_{469} = \lambda_{470} \) and \( \lambda_{936} = \lambda_{937} = \lambda_{938} = \lambda_{939} = \lambda_{940} = \lambda_{941} \). Note that Conjecture 6.1 implies that once we know \( \lambda_{234} = \lambda_{235} \), all the other equalities must hold except \( \lambda_{941} \). The pattern of multiplicities is the same as observed in Part I for the Basilica, but there it was explained on the basis of symmetry and small support of the eigenfunctions. Here the numerical evidence is that the support of the eigenfunctions is the whole dendrite. In Figure 4 we show the eigenfunctions \( u_{117}, u_{234} \) and \( u_{235} \) (note that \( u_{234}(t) = u_{117}(2t) \)). From the graphs it appears that these eigenfunctions have regions where they are identically zero, but the numerical data indicates that they are only close to zero (on the order of \( 10^{-7} \)). At present we have no explanation for why the multiplicities occur. It is remotely possible that the values \( \lambda_{234} \) and \( \lambda_{235} \) are not actually equal
but only very close, since we see a difference of the order of $10^{-6}$ for values around $3 \cdot 10^5$. We attribute this to round-off error, because the results are consistent on levels 10 and 11 for both FDM and FEM algorithms. The equality of $\lambda_{941}$ (only valid on level 11) would not be expected in the case of close but unequal eigenvalues.

A closely related issue is the existence of eigenvalue clusters: for every $\epsilon > 0$ and $N$ there exists an interval of length at most $\epsilon$ containing $N$ distinct eigenvalues. Again, if Conjecture 6.1 holds, it would almost suffice to show that there exist pairs of distinct eigenvalues differing by at most $\epsilon$ (the distinctness of the eigenvalues in the clusters would not follow). It is rather difficult to obtain convincing experimental evidence for clusters, but we note for example that $\lambda_{157} = 147043.63645$ and $\lambda_{158} = 147043.64265$ with a difference of .0062. Of course if $\lambda_{234} \neq \lambda_{235}$ then those eigenvalues would generate a much tighter cluster. However, looking at the experimental evidence for the Vicsek sets [1], the only fractal for which eigenvalue clusters have been proven to exist, we are not led to expect such extremely close eigenvalues so low in the spectrum.

Both the Airplane and the Basilica-in-Rabbit have two reflectional symmetries, just like the examples in Part I, that we call horizontal and vertical. This is apparent for the Airplane in Figure 2(a) where these are just reflections in the y-axis and x-axis (the same is true in the parameter circle in Figure 2(b)). For the Basilica-in-Rabbit it is more subtle (as it is for the Rabbit in Part I). These result in eigenfunctions that vanish on certain open subsets of $\mathcal{J}$, and also eigenspaces of high multiplicity. Following the terminology in Part I, we separate eigenfunctions into primitive ($u(t)$ is not of the form $v(2t)$ for another eigenfunction $v$) and derived ($u(t) = v(2t)$), and similarly eigenvalues are primitive if all eigenfunctions in the associated eigenspace are primitive, otherwise derived (note that the eigenspace may contain both primitive and derived eigenfunctions). The primitive eigenfunctions may be separated into horizontal (skew-symmetric with respect to the horizontal reflection and symmetric with respect to the vertical reflection) and vertical (skew-symmetric with respect to the vertical reflection and symmetric with respect to the horizontal reflection). For example, in the Airplane eigenvalues 1, 3, 5 are primitive horizontal, eigenvalue 7 is primitive vertical, eigenvalue 2 is derived from 1 and 4 is derived from 2, while the derived eigenspace from 7 has multiplicity two (14 and 15). These facts are readily apparent from the graphs.

The argument in Theorem 7.5 of Part I shows that eigenfunctions associated to primitive eigenvalues must have support on proper subsets of $\mathcal{J}$ determined by the symmetry type. (In the graphs this shows up as the function vanishing on certain intervals in the parameter circle.) In the horizontal case the support is rather large, and has no multiplicity consequence. In the vertical case the support is small enough that the support of the derived eigenfunctions becomes disconnected, and this allows higher multiplicity in the derived eigenspaces. Then we obtain a pattern of lower bounds for multiplicities in the sequence of derived vertical eigenspaces. Our data confirms these lower bounds are the actual multiplicities, but nothing rules out higher multiplicities occurring higher up in the spectrum. We describe these patterns in detail for each example.

For the Airplane, a horizontal primitive eigenfunction must vanish on intervals $[\frac{13}{56}, \frac{15}{56}]$ and $[\frac{41}{56}, \frac{43}{56}]$. However, a vertical primitive eigenfunction must vanish on the larger intervals $[\frac{2}{28}, \frac{2}{28}]$ and $[\frac{14}{28}, \frac{14}{28}]$, so the support is $[\frac{13}{56}, \frac{15}{56}] \cup [\frac{14}{28}, \frac{14}{28}]$ (this parametrizes the central blob in Figure 2(a)). The derived eigenfunction is then supported in $[\frac{2}{28}, \frac{2}{28}] \cup [\frac{10}{28}, \frac{14}{28}] \cup [\frac{17}{28}, \frac{18}{28}] \cup [\frac{24}{28}, \frac{28}{28}]$. But this support splits into the
two connected components \([\frac{3}{28}, \frac{4}{28}] \cup [\frac{24}{28}, \frac{25}{28}]\) and \([\frac{10}{28}, \frac{11}{28}] \cup [\frac{17}{28}, \frac{18}{28}]\) (parametrizing the two next largest blobs in Figure 2(a)). Thus the first derived eigenspace has multiplicity 2. Similarly the second derived eigenspace has multiplicity 4, since the support splits into components \([\frac{5}{28}, \frac{6}{28}]\), \([\frac{10}{28}, \frac{11}{28}]\), \([\frac{15}{28}, \frac{16}{28}]\), \([\frac{17}{28}, \frac{18}{28}]\), \([\frac{25}{28}, \frac{26}{28}]\), \([\frac{29}{28}, \frac{30}{28}]\), \([\frac{32}{28}, \frac{33}{28}]\), \([\frac{35}{28}, \frac{36}{28}]\), and \([\frac{42}{28}, \frac{43}{28}]\). The pattern changes with the third derived eigenspace as one of the components is made up of four intervals \([\frac{24}{172}, \frac{25}{172}] \cup [\frac{31}{172}, \frac{32}{172}] \cup [\frac{80}{172}, \frac{81}{172}] \cup [\frac{87}{172}, \frac{88}{172}]\) (note that this is a subset of the support of the vertical primitive eigenfunction, corresponding in Figure 2(a) to the central blob with its top and bottom decorations removed). Thus the multiplicity is 7. The pattern then repeats, and in fact the multiplicities are exactly the same as for the Rabbit (see Theorem 7.3 with \(k = 3\) in Part I).

For the Basilica-in-Rabbit, the horizontal primitive eigenfunctions vanish on the intervals \([\frac{81}{16-63}, \frac{143}{16-63}]\) and \([\frac{565}{16-63}, \frac{647}{16-63}]\). The vertical primitive eigenfunctions have support \([\frac{88}{16-63}, \frac{592}{16-63}] \cup [\frac{1063}{16-63}, \frac{610}{16-63}]\). This is the central boxed region in Figure 10. The pattern of multiplicities is slightly different, however. The first and second derived eigenspaces have multiplicities 2 and 4, and the supports are indicated by boxes in Figure 10. However, in the third derived eigenspace the support breaks up into the 8 components shown, yielding a multiplicity of 8. The parameter values of the support are (in units of \(\frac{1}{16-63}\)) \([\frac{1}{10}, \frac{17}{10}]\cup[\frac{966}{16-63}, \frac{972}{16-63}], [\frac{74}{16-63}, \frac{80}{16-63}]\cup[\frac{641}{16-63}, \frac{647}{16-63}], [\frac{137}{16-63}, \frac{143}{16-63}]\cup[\frac{578}{16-63}, \frac{584}{16-63}], [\frac{200}{16-63}, \frac{206}{16-63}]\cup[\frac{263}{16-63}, \frac{269}{16-63}], [\frac{326}{16-63}, \frac{332}{16-63}]\cup[\frac{389}{16-63}, \frac{395}{16-63}], [\frac{452}{16-63}, \frac{458}{16-63}]\cup[\frac{515}{16-63}, \frac{521}{16-63}], [\frac{704}{16-63}, \frac{710}{16-63}]\cup[\frac{767}{16-63}, \frac{773}{16-63}], [\frac{830}{16-63}, \frac{836}{16-63}]\cup[\frac{903}{16-63}, \frac{909}{16-63}]\).

When we iterate again it is only at the sixth derived eigenspace that we get four intervals contained in the original support, \([\frac{704}{128-63}, \frac{710}{128-63}] \cup [\frac{1082}{128-63}, \frac{1088}{128-63}] \cup [\frac{473}{128-63}, \frac{474}{128-63}] \cup [\frac{5114}{128-63}, \frac{5129}{128-63}]\) that combine into a single connected component in \(\mathcal{J}\). Thus the multiplicities for the first six derived eigenspaces are 2, 4, 8, 16, 32, 63 and the pattern repeats thereafter. In our data we are only able to detect multiplicities 8 on level 7 and 16 on level 8.

There is strong evidence for the existence of spectral clustering in both examples. The evidence for spectral gaps is less conclusive, and is presented in Tables 4(b) and 4(c) below.

7. Covering spaces and covering maps. In this section we discuss the general theory of covering spaces and covering maps. Let \(\mathcal{J}\) denote a Julia set and \(\tilde{\mathcal{J}}\) denote a covering space with \(\pi : \tilde{\mathcal{J}} \to \mathcal{J}\) a covering map. We want to restrict our attention to maps that locally preserve the geometry of \(\mathcal{J}\) that it inherits from the circle parameterization. We will refer to this as the intrinsic geometry. Note that this is not the same as the geometry \(\mathcal{J}\) inherits from its embedding in the complex plane. Let us write \(\psi : C \to \mathcal{J}\) for the parametric map from the circle to the Julia set. Suppose \(U_1\) and \(U_2\) are the open subsets of \(\mathcal{J}\), and \(g : U_1 \to U_2\) is a homeomorphism. We say \(g\) is an intrinsic isometry (or just isometry for short) if

\[
\psi^{-1} \circ g \circ \psi : \psi^{-1}(U_1) \to \psi^{-1}(U_2)
\]

is an isometry with respect to the usual geometry of the circle. Since \(\psi\) is not one-to-one, \(\psi^{-1}\) is multi-valued, so the correct interpretation of this condition is that there exists an isometry \(\tilde{g} : \psi^{-1}(U_1) \to \psi^{-1}(U_2)\) such that the diagram pictured below commutes:

\[
\begin{array}{ccc}
U_1 & \xrightarrow{\psi} & U_2 \\
\downarrow \psi & & \uparrow \psi \\
\psi^{-1}(U_1) & \xrightarrow{\tilde{g}} & \psi^{-1}(U_2)
\end{array}
\]
Figure 10. The supports of the vertical (0) primitive eigenfunctions and the first (1), second (2) and third (3) derived eigenfunctions on the Basilica-in-Rabbit.

(a) Dendrite Spectral Gaps

| n   | n+1 | λ_n | λ_{n+1} | \frac{\lambda_{n+1}}{\lambda_n} |
|-----|-----|-----|---------|----------------------------------|
| 1   | 2   | 66  | 170     | 2.44                             |
| 3   | 4   | 212 | 519     | 2.45                             |
| 6   | 7   | 647 | 1214    | 1.88                             |
| 13  | 14  | 2025| 3696    | 1.83                             |
| 26  | 27  | 6157| 9849    | 1.60                             |
| 53  | 54  | 18938| 29588  | 1.58                             |
| 106 | 107 | 57370| 91383  | 1.58                             |
| 213 | 214 | 176644| 280314 | 1.59                             |
| 426 | 427 | 548157| 862974 | 1.57                             |
| 852 | 853 | 1802949| 3150718| 1.75                             |

(b) Airplane Spectral Gaps

| n   | n+1 | λ_n | λ_{n+1} | \frac{\lambda_{n+1}}{\lambda_n} |
|-----|-----|-----|---------|----------------------------------|
| 1   | 2   | 46  | 149     | 3.21                             |
| 2   | 3   | 149 | 374     | 2.51                             |
| 4   | 5   | 490 | 1136    | 2.32                             |
| 9   | 10  | 1618| 3649    | 2.26                             |
| 18  | 19  | 5257| 11587   | 2.20                             |
| 36  | 37  | 17551| 35911   | 2.05                             |
| 73  | 74  | 99561| 207849  | 1.84                             |
| 146 | 147 | 192810| 341602  | 1.77                             |
| 292 | 293 | 684821| 131829  | 1.20                             |

(c) Basilica-in-Rabbit Spectral Gaps

| n   | n+1 | λ_n | λ_{n+1} | \frac{\lambda_{n+1}}{\lambda_n} |
|-----|-----|-----|---------|----------------------------------|
| 1   | 2   | 230 | 962     | 2.35                             |
| 3   | 4   | 669 | 1311    | 1.96                             |
| 6   | 7   | 1649| 3184    | 1.93                             |
| 13  | 14  | 4074| 8060    | 1.98                             |
| 27  | 28  | 10880| 20789   | 1.86                             |
| 53  | 54  | 31344| 59622   | 1.90                             |
| 106 | 107 | 12172| 22634   | 1.85                             |
| 213 | 214 | 33974| 64608   | 1.90                             |
| 426 | 427 | 86952| 153904  | 1.78                             |
| 852 | 853 | 183110| 333740  | 1.76                             |

Table 5. Spectral Gaps

Let \( \hat{J} \) be a connected, locally compact, Hausdorff topological space. Note that the assumption that \( \hat{J} \) be connected could be dropped at the expense of allowing a lot of uninteresting examples. Let \( \pi : \hat{J} \rightarrow J \) be a continuous map. The requirement that \( \pi \) be a covering map in the topological sense is that each point \( x \in J \) has a
connected neighborhood $U$ such that $\pi^{-1}(U)$ has components $V_j$,
\[
\pi^{-1}(U) = \bigcup_j V_j
\]
where the union is either finite or countable, and such that $\pi : V_j \to U$ is a homeomorphism onto. Since $\mathcal{J}$ is compact we can cover $\mathcal{J}$ with a finite number of such neighborhoods. We would like to use $\pi^{-1}$ to lift the geometry from $\mathcal{J}$ to $\tilde{\mathcal{J}}$.

Write $[\pi_j]$ for the restriction of $\pi$ to $V_j$. We can define a map 
\[
\tilde{\psi}_j : \psi^{-1}(U) \to V_j
\]
by $\tilde{\psi}_j = [\pi_j]^{-1} \circ \psi$; in other words, by the commutative diagram

\[
\begin{array}{ccc}
V_j & \xrightarrow{\tilde{\psi}_j} & [\pi_j] \\
\downarrow & & \downarrow \\
\psi^{-1}(U) & \xrightarrow{\psi} & U
\end{array}
\]

Then $\tilde{\psi}_j$ transfers the geometry of $\psi^{-1}(U)$ in the circle to $V_j$. We need to check compatibility when the sets $V_j$ overlap, but this is routine because all the maps $[\pi_j]$ are defined by restricting the single map $\pi$. Note that $\tilde{\psi}_j$ respects identifications: if $t_1 \sim t_2$ in $\psi^{-1}(U)$, meaning $\psi(t_1) = \psi(t_2)$ in $U$, then $\tilde{\psi}(t_1) = \tilde{\psi}(t_2)$ in $V_j$.

We define $\tilde{\mathcal{J}}$ with the lifted geometry to be a geometric covering space, and $\pi : \tilde{\mathcal{J}} \to \mathcal{J}$ to be a geometric covering map. In order for this definition to be more than a tautology, we need to define isometry and equivalence for geometric covering spaces and geometric covering maps. In order for a homeomorphism $\tilde{h} : \tilde{\mathcal{J}}_1 \to \tilde{\mathcal{J}}$ to be an isometry we require that $\tilde{h}$ preserve the lifted geometries. In other words, if we restrict $\tilde{h}$ to some $V_j$, (a component of $\pi^{-1}(U)$), then there exists an isometry $h_j : \psi^{-1}(U) \to \psi^{-1}(U')$ for some $U'$ and some component $V'_j$ of $\pi^{-1}(U')$ such that diagram below commutes:

\[
\begin{array}{ccc}
V_j & \xrightarrow{h_j} & V'_j \\
\tilde{\psi}_j & \downarrow & \tilde{\psi}'_j \\
\psi^{-1}(U) & \xrightarrow{\psi} & \psi^{-1}(U')
\end{array}
\]

We define two geometric covering maps $\pi_1 : \tilde{\mathcal{J}}_1 \to \mathcal{J}$ and $\pi_2 : \tilde{\mathcal{J}}_2 \to \mathcal{J}$ to be equivalent if there exist isometries $\tilde{h}_1 : \tilde{\mathcal{J}}_1 \to \tilde{\mathcal{J}}_2$ and $g : \mathcal{J} \to \mathcal{J}$ such that following diagram commutes:

\[
\begin{array}{ccc}
\tilde{\mathcal{J}}_1 & \xrightarrow{\tilde{h}_1} & \tilde{\mathcal{J}}_2 \\
\downarrow & & \downarrow \\
\mathcal{J} & \xrightarrow{g} & \mathcal{J}
\end{array}
\]

Note that the isometry $g$ in the definition of equivalence must be a global isometry of $\mathcal{J}$, while the isometries $h_j$, in the definition of isometric covering spaces need only be locally defined. It seems likely that there are isometric geometric covering spaces whose geometric covering maps are not equivalent, although we have not found an explicit example. We also note that some definitions of equivalence would require $g$ to be the identity, but we find that this choice of definition would result in vastly too many inequivalent geometric covering maps.
Next we give a method to describe geometric covering maps. The method, called the parameter space construction, is to create a parameter map \( \tilde{\psi} : \tilde{C} \to \tilde{J} \) for a suitable parameter space \( \tilde{C} \). In particular we want a commutative diagram

\[
\begin{array}{ccc}
\tilde{C} & \xrightarrow{\tilde{\psi}} & \tilde{J} \\
\pi' \downarrow & & \downarrow \pi \\
C & \xrightarrow{\psi} & J
\end{array}
\]

where \( \pi' : \tilde{C} \to C \) is a suitable geometric covering map of \( C \). Of course here we must remember that \( C \) is a circle with identifications, and \( \pi' \) must map identified points to identified points. Without taking into account the identifications, \( \tilde{C} \) does not have to be connected. Thus \( \tilde{C} \) is a union of circles of integer circumference and lines, and \( \pi' \) is the mapping \( t \mapsto t(mod1) \) where \( t \) is the natural parameter on any of the circles or lines. Of course if \( J \) is compact then \( \tilde{C} \) will just be a finite union of circles.

The idea of the construction of \( \tilde{C} \) and the maps \( \tilde{\psi} \) and \( \pi' \) is to cut up the circle into small intervals that are mapped into the neighborhood \( U \) by \( \psi \), take \( N \) copies of each interval (where \( N \) is the degree of the covering map), and reassemble them into \( \tilde{C} \) using identifications in \( \tilde{J} \). Without loss of generality we may take the \( X^{(m)} \) intervals for some fixed \( m \) (this will of course depend on the covering map). Let us denote these intervals \( \{I_j\} \). We assume the subdivision dichotomy holds for \( J \). Then some of the intervals will be paired. We assume that every pair \( (I_j, I_{j'}) \) has the property that both intervals are mapped by \( \psi \) to a single \( U \) neighborhood (this can be achieved by taking \( m \) large enough).

For each \( I_j \) we construct \( N \) identical copies \( \{I_j^{(l)}\} \). If \( I_j \) corresponds to parameter values \( a_j \leq t \leq b_j \) then \( I_j^{(l)} \) will be parameterized by \( a_j + k_l \leq t \leq b_j + k_l \), where \( k_l \) is some integer. If \( I_j \) is not paired, so it subdivides internally, then all identifications in \( I_j \) are carried over to each \( I_j^{(l)} \). If \( (I_j, I_{j'}) \) is a pair, then we will have to decide how to match \( l \) and \( l' \) so that \( (I_j^{(l)}, I_{j'}^{(l')}) \) are paired, and then all identifications among the original pair \( (I_j, I_{j'}) \) are carried over to the pairs \( (I_j^{(l)}, I_{j'}^{(l')}) \). We will also have to decide, for consecutive intervals \( I_j, I_{j+1} \), how to pair up \( l \) and \( l' \) so that \( I_j^{(l)} \) and \( I_{j+1}^{(l')} \) have an endpoint in common in \( \tilde{C} \). These decisions will complete the description of \( \tilde{C} \) as

\[
\tilde{C} = \bigcup_l \bigcup_j I_j^{(l)}
\]

All these decisions are made by looking at what happens in \( \tilde{J} \). Each \( I_j \) has the property that \( \pi^{-1}\psi(I_j) \) splits into \( N \) connected components, each isometric to \( \psi(I_j) \) in \( J \). We associate each \( I_j^{(l)} \) to one of these components in \( \tilde{J} \). At this stage we do not impose any structure to the index \( l \) and the components in \( \tilde{J} \). Then we define \( \tilde{\psi} \) on \( I_j^{(l)} \) to map to the associated component of \( \pi^{-1}\psi(I_j) \) in \( \tilde{J} \) and \( \pi' : I_j^{(l)} \to I_j \) to be \( t \mapsto t(mod1) \) so that the diagram below commutes:

\[
\begin{array}{ccc}
I_j^{(l)} & \xrightarrow{\tilde{\psi}} & \tilde{J} \\
\pi' \downarrow & & \downarrow \pi \\
I_j & \xrightarrow{\psi} & \psi(I_j)
\end{array}
\]
If \( I_j \) and \( I_{j+1} \) are consecutive intervals that intersect at a point, then we identify the corresponding endpoints of \( I_j^{(t)} \) and \( I_{j+1}^{(t')} \) exactly when \( \tilde{\psi}(I_j^{(t)}) \) and \( \tilde{\psi}(I_{j+1}^{(t')}) \) intersect at the corresponding endpoints in \( \tilde{\mathcal{J}} \), and we adjust the parameters by an integer so the parameter values at the corresponding endpoints of \( I_j^{(t)} \) and \( I_{j+1}^{(t')} \) are equal (not just equal mod 1). Because \( \pi : \tilde{\mathcal{J}} \to \mathcal{J} \) is a covering map there will be a unique choice of \( t' \) for each \( I_j^{(t)} \). In this way we construct \( \tilde{\mathcal{C}} \) as a union of circles of integer circumference and lines. Moreover we obtain the desired global commutative diagram by piecing together the local commutative diagrams.

We then have to make the identifications on \( \tilde{\mathcal{C}} \). Suppose \( \{t_1, t_2, ..., t_n\} \) is a set of identified points in \( X^{(m)} \). Each \( t_i \) is the common endpoint of consecutive intervals \( I_{j_i}, I_{j_i+1} \), and so we obtain \( N \) values \( \{t^{(i)}_1\} \) as common endpoints of consecutive intervals \( \{l_1^{(i)}, l_2^{(i)}, ..., l_n^{(i)}\} \) exactly when the image points under \( \tilde{\psi} \) are equal. In this way we obtain \( N \) equivalence classes of points in \( \tilde{\mathcal{C}} \) that cover (under \( \pi' \)) the original equivalence class in \( C \). Note that these identifications need not be confined to the individual circles and lines that make up \( \tilde{\mathcal{C}} \), and indeed since \( \tilde{\mathcal{C}} \) must be connected there will have to be identifications connecting the different circles and lines (if there are more than one of these). Having thus obtained all the identifications in \( (\pi')^{-1}(X^{(m)}) \), we fill in all the other identifications in the obvious way. If \( I_j \) is not paired in \( X^{(m)} \), then we make all identifications internally in each \( I_j^{(t)} \) when we subdivide. If \( (I_j, I_{j+1}) \) are paired in \( X^{(m)} \), then we need to check that if one endpoints of \( I_j^{(t)} \) and \( I_{j+1}^{(t')} \) are mapped to the same point in \( \tilde{\mathcal{J}} \) by \( \tilde{\psi} \), then the same is true for the other endpoint. This follows by the assumption that \( I_j \) and \( I_{j'} \) get mapped under \( \psi \) to the same \( U \) neighborhood. Then we pair \( I_j^{(t)} \) and \( I_{j'}^{(t')} \) in \( \tilde{\mathcal{C}} \) and make identifications on them identical to the \( I_j, I_{j'} \) identifications. Again there is no necessity that the paired intervals in \( \tilde{\mathcal{C}} \) lie in the same circle or line.

It is a routine matter to check that the commutative diagram respects identifications, and that \( \pi' : \tilde{\mathcal{C}} \to \mathcal{C} \) is a geometric covering map of degree \( N \). Now we may attempt to reverse the construction. Choose a value for \( m \), a degree \( N \), and build any geometric covering map \( \pi' : \tilde{\mathcal{C}} \to C \) by covering the \( X^{(m)} \) points, respecting the pairings, and make identifications on the points \( (\pi')^{-1}(X^{(m)}) \) to cover the identifications on \( X^{(m)} \) and the interval pairings. Then complete \( \pi' \) to all of \( \tilde{\mathcal{C}} \) as indicated above, and define \( \tilde{\mathcal{J}} \) to be \( \tilde{\mathcal{C}} \) modulo the identifications (we need to assume that \( \tilde{\mathcal{C}} \) is connected). The commutative diagram then may be used to define \( \pi : \tilde{\mathcal{J}} \to \mathcal{J} \).

Do we in fact obtain a covering map? Not necessarily. For example, the Dendrite is contractible and has no nontrivial covering spaces, but we can easily construct covering maps of \( X^{(1)} \) and then continue to \( X^{(m)} \) for any finite \( m \). It is only when we pass to the limit as \( m \to \infty \) that the covering map property breaks down, essentially because the pairings force identifications that make it impossible for \( \pi^{-1}(U) \) to break into disconnected pieces, or force \( \tilde{\mathcal{J}} \) to be disconnected. A related issue is to decide when two such constructions lead to equivalent covering spaces. We will not attempt to answer these questions in general, but will discuss them for specific Julia sets in the next section.

8. **Double covers.** In this section we show that double covers can be characterized by certain combinatorial data. For simplicity we restrict to the case when \( \mathcal{J} \) is the
Basilica, but a similar analysis pertains to all the examples in part I. Since \( \pi \) is two-to-one, the same is true for \( \pi' \), so \( \tilde{C} \) is either a circle of length 2 or two disjoint circles of length 1. For each topological circle \( C' \) in \( \tilde{J} \), the preimage under \( \pi \) is similarly either a single circle of twice the length or two circles of the same length. We define the length function \( l(C') \) to be 2 in the first case and 1 in the second case. In other words, \( l(C') = 2 \) means \( \pi \) on \( \pi^{-1}(C') \) is a double cover of \( C' \), while \( l(C') = 1 \) means \( \pi \) on \( \pi^{-1}(C') \) consists of two single covers. In order for \( \pi \) to be a local isometry we must have \( l(C') = 1 \) for all sufficiently small circles. So there is a level \( m \) such that all circles with \( l(C') = 2 \) appear in the \( X^{(m)} \) decomposition. We fix this level \( m \) for the remainder of this discussion.

Although the circles \( C' \) in \( \tilde{J} \) are parameterized by Cantor sets in \( C \), in the \( X^{(m)} \) decomposition we may associate with \( C' \) a union of \( 2^l \) intervals that contain the Cantor set. In Figure 11 we illustrate the cases \( m = 1, 2, 3 \). Note that the number of intervals in \( X^{(m)} \) parameterizing a fixed circle \( C' \) will vary with \( m \). The total number of circles is \( 2^m + 1 \).

Given a length function on level \( m \), we will see that there is a unique double cover with that length function. The only restriction on the length function is that it not be identically one, in order that \( \tilde{J} \) be connected. We define the parity of a length function to be the parity of the number of circles with \( l(C') = 2 \). We will see that odd parity corresponds to the case that \( \tilde{C} \) is a single circle of length 2, and even parity corresponds to the case that \( \tilde{C} \) is a pair of circles of length 1. Of course different length functions may correspond to covering maps that are equivalent.

Suppose that we have chosen the nature of \( \tilde{C} \). Using the interval parametrization from the previous section, if \( \tilde{C} \) is a circle of length 2 it is parameterized by \([0, 2)\), and if it is a two circles of length 1 they are parameterized by \([0, 1)\) and \([1, 2)\) respectively. Abusing notation we denote the lifts of \( x \in X^{(m)} \) by \( x \) and \( x + 1 \) in \( \tilde{C} \). Now let \( x_j \) and \( x_k \) be two points that are identified in \( X^{(m)} \), so \( \psi(x_j) = \psi(x_k) \). The covering map is determined by the pairwise identifications made between the lifts \( \{x_j, x_j + 1\} \) and \( \{x_k, x_k + 1\} \) of such points. Specifically we must have one of

1. \( x_j \sim x_k \) and \( x_j + 1 \sim x_k + 1 \)
2. \( x_j \sim x_k + 1 \) and \( x_j + 1 \sim x_k \)

If \( \tilde{C} \) consists of two circles then we cannot make the choice 1 for all pairs or \( \tilde{J} \) will not be connected, but all other choices are permissible. In Figure 12 we illustrate the seven covers for \( m = 1 \) that we obtain, together with their length functions. Note that all allowable length functions appear with the correct parities. We also note the equivalences of covers 1 and 2 and covers 5 and 6.

We now outline an algorithm that constructs a cover with prescribed length functions. The intervals on \( X^{(m)} \) are alternately of lengths \( \frac{1}{3^m} \) and \( \frac{2}{3^m} \). There are \( 2^{m-1} \) intervals of length \( \frac{2}{3^m} \) with identified endpoints \( (k = j + 1) \), and each parameterizes a circle \( C'' \) in \( \tilde{J} \), (for \( m = 3 \) these are the circles \( C''_6, C''_7, C''_8, C''_9 \)). For these endpoints we make the choice (1) if we want \( l(C'') = 1 \) and (2) if we want \( l(C'') = 2 \). Next we consider intervals of length \( \frac{1}{3^m} \) that are adjacent to those already considered such that two of them parameterize a circle \( C'' \) in \( \tilde{J} \) (for \( m = 3 \) these are the circles \( C'_4, C'_5 \)). Of the four endpoints of the two intervals, we have already decided the identifications in \( \tilde{C} \) for two of them. For the remaining two, we will choose either (1) or (2) in order to achieve the desired value for \( l(C'') \), as the two choices yield different values for \( l(C'') \). We then move on to pairs of intervals of length \( \frac{2}{3^m} \) adjacent to the previous intervals that correspond to a circle \( C'' \).
in $\mathcal{J}$ (for $m = 3$ these are the intervals $C_1', C_3'$). Again we have already decided identifications for one pair of endpoints, and so we choose identifications of type (1) or (2) for the remaining pair to achieve the desired values for $l(C')$.

The process continues inductively to consider groupings of $2^k$ intervals that together parameterize a circle $C''$ in $\mathcal{J}$ and are adjacent to intervals already considered. Of the endpoints, all but two have already had identifications in $\tilde{C}$ decided in earlier stages of the algorithm, so we decide between (1) and (2) for this pair based on the desired value of $l(C'')$. Note that when we arrive at the last set of intervals corresponding to the central circle $C_2'$, all the identifications of endpoints have already

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure11.png}
\caption{$C$ on the left and $\mathcal{J}$ on the right.}
\end{figure}
been made. But now we make the choice of the nature of $\tilde{C}$ to arrive at the correct value of $l(C'_2)$, and this decision is forced by parity considerations.

We note that the success of the above algorithm is related to the tree-like structure of the connectivity of the circles $C'$ in any finite approximation to $J$.

9. **Blowups.** Fix a Julia set $J$ parameterized by the circle $\mathbb{R}/\mathbb{Z}$ with identifications. It is convenient here to realize the circle as the interval $[-\frac{1}{2}, \frac{1}{2}]$ with the endpoints
identified. We may blowup \( J \) by any positive integer factor, but for simplicity we will just consider blowups by factors \( 2^m \). Then the blowup \( B_{2^m}(J) \) is parametrized by the interval \([−2^{m−1}, 2^{m−1}]\) with endpoints identified, and other points \( x, y \) are identified in \( B_{2^m}(J) \) if and only if \( 2^{-m}x, 2^{-m}y \) are identified in \( J \). Note that if \( x \) and \( y \) are in \((−\frac{1}{2}, \frac{1}{2})\) and are identified in \( B_{2^m}(J) \) then they are identified in \( J \) (the doubling map in \( J \) preserves identification). The converse is not true, as there are points identified in \( J \) that are not identified in \( B_{2^m}(J) \). In some cases \( B_{2^m}(J) \) is a covering space of \( J \) with covering map \( t \to t \mod 1 \) (this is true for the Basilica and all examples discussed in part I), but in any event it is a fractafold based on \( J \) (see [10] sec 5.4).

The Laplacian \( \Delta \) on \( J \) lifts to a Laplacian (also denoted \( \Delta \)) on \( B_{2^m}(J) \), and the spectrum transfers in the obvious way: there is a one-to-one correspondence between eigenfunctions \( u \) on \( J \) with eigenvalue \( \lambda \) and eigenfunctions \( u'(t) = u(2^{-m}t) \) on \( B_{2^m}(J) \) with eigenvalue \( (\frac{\lambda}{2})^m \). Of course if \( u \) is an \( m \)-fold derived eigenfunction on \( J \) (\( u(t) = v(2^m t) \) for some eigenfunction \( v \)), then \( u'(t) \) is simply \( v(t) \) extended to be periodic of period 1. Thus the spectrum of \( J \) is contained in the spectrum of \( B_{2^m}(J) \), and more generally we have the infinite tower of containments

\[
\Lambda(J) \subseteq \Lambda(B_2(J)) \subseteq \Lambda(B_2^2(J)) \subseteq \ldots \tag{45}
\]

We are mainly interested in infinite blowups which may be thought of as limits as \( m \to \infty \) of \( B_{2^m}(J) \). However, before describing those we need to allow integer rotations in the finite blowup. These rotations produce isometric spaces, but allow us to take limits in uncountably many different ways. If \( l \) denotes any integer, define the translate \( B_{2^m}^{(l)}(J) \) parameterized by \([-2^{m−1}, 2^{m−1}]\) by identifying \( x \) and \( y \) whenever \( x−l \) and \( y−l \mod(2^m) \) are identified in \( B_{2^m}(J) \). Since the definition depends only on \( l \mod(2^m) \), we may restrict \( l \) to lie in \([0, 2^m−1]\).

Let \( \epsilon = \{\epsilon_0, \epsilon_1, \ldots\} \) where each \( \epsilon_j \) is 0 or 1. Define \( l_m = \sum_{j=0}^{m−1} \epsilon_j 2^j \). Note that \( l_m' = l_m \mod(2^m) \) for any \( m' > m \), so \( B_{2^m}^{(l_m)}(J) \) and \( B_{2^m}^{(l_m')} \) are identical. We want to define \( B_\infty^\epsilon(J) \) as the inductive limit of \( B_{2^m}^{(l_m)}(J) \) as \( m \to \infty \). In other words, \( B_\infty^\epsilon(J) \) is parameterized by \( \mathbb{R} \), and \( x \) is identified with \( y \) if and only if \( 2^{-m}(x−l_m) \) and \( 2^{-m}(y−l_m) \) are identified in \( J \) for all sufficiently large \( m \). Figure 13 shows part of \( B_\infty^\epsilon(J) \) when each \( \epsilon_j = 0 \) for the Basilica, and Figure 14 shows the same for \( \{\epsilon_j\} = \{0, 1, 0, \ldots\} \).

Now we restrict attention to the examples \( \theta = \frac{1}{2^{k+1}} \) considered in Part I. These examples have vertical symmetries \( \rho_v \) and a central vertical segment (CVS) parameterized by \([\frac{1}{2^{2k+1}}, \frac{2}{2^{2k+1}}] \cup [\frac{2}{2^{2k+1}}, \frac{2^{k+1}}{2^{2k+1}}]\).

**Lemma 9.1.** Any function on \( J \) supported on CVS and skew-symmetric with respect to \( \rho_v \) is an infinite linear combination of eigenfunctions with the same support and symmetry properties.

**Proof.** Since \( \rho_v \) commutes with the Laplacian it is clear that a skew-symmetric function is an infinite linear combination of skew-symmetric eigenfunctions. The key observation is that if \( u \) is a skew-symmetric eigenfunction, then \( R(u) \) defined to be the restriction of \( u \) to CVS and extended to be zero outside is also a skew-symmetric eigenfunction. (See Theorem 7.5 of part I.) If we apply the \( R \) operator to our representation in terms of skew-symmetric eigenfunctions, we obtain a representation in terms of skew-symmetric eigenfunctions with the desired support. \( \square \)
We can use this observation to show that a generic blowup has pure point spectrum. Here we only need to define generic by the condition that \( \epsilon_j \neq 0 \) for infinitely many \( j \). We use an idea due to Teplyaev [11]. Let \( \Sigma^* \) denote the part of the spectrum of \( J \) corresponding to eigenfunctions described in Lemma 9.1. (Note that for \( \lambda \in \Sigma^* \) we do not require that all eigenfunctions in the \( \lambda \)-eigenspace have the required symmetry and support properties, just at least some.) Recall that \( \Sigma^* \) is preserved by multiplication by \( 2^{1+\frac{1}{k}} \),

\[
2^{1+\frac{1}{k}} \Sigma^* \subseteq \Sigma^*.
\] (46)

Define \( \Sigma^*_\infty \) by closing \( \Sigma^* \) by division by \( 2^{1+\frac{1}{k}} \),

\[
\Sigma^*_\infty = \bigcup_{n=0}^{\infty} 2^{-n(1+\frac{1}{k})} \Sigma^*.
\] (47)

It is clear that each \( \lambda \in \Sigma^*_\infty \) is an eigenvalue in the spectrum of \( B^{(\epsilon)}(J) \) with infinite multiplicity and compactly supported eigenfunctions, regardless of which blowup we take. Indeed we just need to find infinitely many copies of the \( n \)-fold blowup of CVS in \( B^{(\epsilon)}(J) \) and place a copy there of the eigenfunctions supported on CVS and blown up \( n \) times (multiplying the eigenvalue by \( 2^{-n(1+\frac{1}{k})} \)). So \( \Sigma^*_\infty \) is always a portion of the spectrum of \( B^{(\epsilon)}(J) \) corresponding to compactly supported eigenfunctions. Of course it is not the whole spectrum, since the closure \( \text{cl}(\Sigma^*_\infty) \) is in the spectrum, and the closure has the structure of a Cantor set. For a generic blowup, this is the whole story.
Figure 14. \( \{ \epsilon_j \} = \{ 0, 1, 0, 0, 1, 0, 0, 1, 0, \ldots \} \)

**Theorem 9.2.** For a generic blowup for \( \theta = \frac{1}{2^{1/2}} \), there is a countable collection of compactly supported eigenfunctions with eigenvalues in \( \Sigma^* \infty \) (each eigenvalue with infinite multiplicity) that form an orthonormal basis of \( L^2(B^\infty(J)) \). In particular, the spectrum is \( \text{cl}(\Sigma^\infty) \).

**Proof.** The idea is that we can find an increasing sequence of domains

\[ D_1 \subseteq D_2 \subseteq D_3 \subseteq \ldots \]

whose union is all of \( B^\infty(J) \), and each \( D_j \) is half of a blowup \( B_j \) of CVS. This is not possible for nongeneric blowups, but if \( \epsilon_{ij} = 1 \) we may choose \( D_j \) at level \( i_j \). Suppose \( u \) is in \( L^2 \) and orthogonal to all the eigenfunctions associated to \( \Sigma^\infty \) above. Then by Lemma 9.1 \( u \) restricted to \( B_j \) must be symmetric with respect to the blowup of \( \rho_v \). For each \( j \) and every \( k > j \) the vertical reflection on \( B_k \) copies \( D_j \) to a set \( D_{j,k} \) that is disjoint from \( D_j \). The collection \( D_{j,k}, k > j \) constructed in this manner consists of infinitely many disjoint copies of \( D_j \), and \( u \) must be the
same on all these copies by the symmetry. For \( u \) to be in \( L^2 \) we must have \( u|_{D_j} \equiv 0 \), and since \( \bigcup D_j \) exhausts \( B_{\infty}(\mathcal{J}) \) it follows that \( u \equiv 0 \).

The same results are valid for the airplane and basilica-in-rabbit examples, but not for the dendrite.

10. Periodic covering maps. In this section we consider periodic covering maps \( \pi : \tilde{\mathcal{J}} \to \mathcal{J} \), defined by the requirement that the parameter map \( t \mapsto t + 1 \) induces an isometry on \( \tilde{\mathcal{J}} \). Of course there are infinitely many periodic covering spaces. Our goal is to understand the spectrum of the Laplacian on a periodic covering space, and how it depends on the structure of the covering space. For simplicity we discuss the case when \( \mathcal{J} \) is the Basilica, but a similar discussion is valid more generally. The parameter space is then either \( \mathbb{R} \) or countably many copies of a circle of length one. We will call these case 1 and case 2.

Choose a depth level \( m \). Then \( X^{(m)} \) consists of \( 2^m \) pairs \( (x_j \sim y_j) \) of identified points in \([0, 1]\). In the parameter space \( \tilde{\mathcal{J}} \) we will identify \( x_j \sim y_j + k_j \) where \( k_j \) is an integer. Then by periodicity we identify \( l + x_j \sim l + y_j + k_j \) for any integer \( l \), and this completely determines the identifications on \( \mathbb{Z} + X^{(m)} \). Then by induction we may obtain identifications on \( \mathbb{Z} + X^{(m')} \) for any \( m' \geq m \) by the local subdivision rule: if \( t_1 \sim t_2 \) are consecutive identified points in \( X^{(m')} \setminus X^{(m' - 1)} \) we identify \( l + t_1 \sim l + t_2 \), except when \( t_1 = \frac{1}{3 \cdot 2^m} \) and \( t_2 = 1 - \frac{1}{3 \cdot 2^m} \) in which case we identify \( l + t_1 \sim l + t_2 - 1 \) in case 1 and \( l + t_1 \sim l + t_2 \) in case 2. Note that in case 2 we are doing all identifications at levels \( m' > m \) within each circle of length 1.

Note that we are free to choose the integers \( \{k_j\} \) in any way, except that in case 2 we may not choose all \( k_j = 0 \), for that would lead to a disconnected \( \tilde{\mathcal{J}} \). Of course different choices of \( \{k_j\} \) may lead to equivalent covering maps. In Figure 12 we show the 7 simplest covering spaces for \( m = 1 \), subdivided to level \( m' = 2 \) : \( \left( \frac{1}{3} \sim \frac{1}{3} + k_1, \frac{1}{3} \sim \frac{2}{3} + k_2 \right) \).

It is clear that Covers 1 and 2 are equivalent and Covers 5 and 6 are equivalent (both equivalences correspond to \( t \mapsto t + \frac{1}{2} \)). We also note that \( t \mapsto t + \frac{1}{2} \) is an isometry for Covers 3 and 4. If we factor out by the map \( t \mapsto t + 2 \) in the parameter space we obtain a double cover \( \tilde{\mathcal{J}}_2 \). In Figure 15 we show the double covers (at level \( m = 1 \)) for each of the examples in Figure 12.

Note that 1 is the double blowup of \( \mathcal{J} \). Also, 3 and 4 have a \( D_4 \) symmetry group. Although there are infinitely many periodic coverings for \( m = 1 \), when we reduce to the double cover we must obtain one of these seven cases.

We follow a well-known method for analyzing the spectrum of any operator with a group of symmetries isomorphic to \( \mathbb{Z} \). This requires that we understand the eigenfunctions of \( \Delta \) on \( \tilde{\mathcal{J}} \) that satisfy the identity

\[
    u(t + 1) = e^{2\pi i \theta} u(t)
\]

for any fixed choice of \( \theta \in [0, 1) \). We call this the \( \theta \)-spectrum. (Note that the use of the variable \( \theta \) here is entirely unrelated to the use of \( \theta \) as the parameter that defines the Julia set.) We reduce the problem to finding eigenfunctions of \( \Delta \) on \( \mathcal{J} \) allowing a discontinuity at \( t = 0 \sim 1 \). It is clear that continuous functions satisfying (48) on \( \tilde{\mathcal{J}} \) are determined by their restriction to \([0, 1]\) with the condition

\[
    u(1) = e^{2\pi i \theta} u(0)
\]
1. Case 1: $k_1 = -1, k_2 = -1$

2. Case 1: $k_1 = 0, k_2 = 0$

3. Case 1: $k_1 = 0, k_2 = -1$

4. Case 1: $k_1 = -1, k_2 = 0$

5. Case 2: $k_1 = 0, k_2 = 1$

6. Case 2: $k_1 = 1, k_2 = 0$

7. Case 2: $k_1 = 1, k_2 = 1$

Figure 15.
holding. The identification \( u(l + x_j) = u(l + y_j + k_j) \) on \( \mathbb{Z} + X^{(m)} \) means
\[
  u(x_j) = e^{2\pi i k_j \theta} u(y_j)
\]
for \( x_j \sim y_j \) in \( X^{(m)} \). For further refinements we have
\[
  u(t_1) = u(t_2)
\]
for all \( t_1 \sim t_2 \) in \( X^{(m')} \setminus X^{(m)} \) in case 2, while in case 1 (51) holds except that
\[
  u \left( \frac{1}{3 \cdot 2^m} \right) = e^{-2\pi i \theta} u \left( 1 - \frac{1}{3 \cdot 2^m} \right).
\]
(52)

Note that (52) yields (49) in the limit as \( m' \to \infty \). Since the points 0 and 1 are not in \( X^{(m')} \) for any \( m' \), we do not have to deal with (49) at all (in case 2 the values \( u(t) \) for \( t \) an integer are ambiguously defined).

To approximate eigenfunctions on \( \tilde{J} \) satisfying (48) we assign values \( u(t) \) for all \( t \in X^{(m')} \) consistent with (50), (51), and (52) (in case 1) for some \( m' > m \). The approximate eigenvalue equation on \( \tilde{J} \)
\[
  -\tilde{\Delta}_{m'} u(x) = \lambda u(x)
\]
for \( x \in \mathbb{Z} + X^{(m')} \) says explicitly
\[
  2^{3m'} \left( \sum_{z} \frac{u(x) - u(z)}{|x - z|} + \sum_{z'} \frac{u(x') - u(z')}{|x' - z'|} \right) = \lambda u(x)
\]
where \( x \sim x' \) in \( \mathbb{Z} + X^{(m')} \). For identified pairs \( x_j \sim y_j \) in \( X^{(m)} \) this says
\[
  2^{2m'} \left( \sum_{z} \frac{u(x_j) - u(z)}{|x_j - z|} + \sum_{z'} \frac{e^{-2\pi i k_j \theta} (u(y_j) - u(z'))}{|y_j - z'|} \right) = \lambda u(x_j)
\]
where \( z \) and \( z' \) vary over the two neighboring vertices to \( x_j \) and \( y_j \) in \( X^{(m')} \). Note that the equation at \( y_j \) is identical (just multiplied by \( e^{2\pi i k_j \theta} \)). For identified points \( t_1 \sim t_2 \) in \( X^{(m')} \setminus X^{(m)} \) for which (51) holds the equation is
\[
  2^{2m'} \left( \sum_{z} \frac{u(t_1) - u(z)}{|t_1 - z|} + \sum_{z'} \frac{u(t_2) - u(z')}{|t_2 - z'|} \right) = \lambda u(t_1)
\]
where \( z \) and \( z' \) are neighbors of \( t_1 \) and \( t_2 \) in \( X^{(m')} \). Finally, in case 1 for the points \( \frac{1}{3 \cdot 2^m} \) and \( 1 - \frac{1}{3 \cdot 2^m} \) the equation is
\[
  2^{3m'} \left( 2 u \left( \frac{1}{3 \cdot 2^m} \right) - u \left( \frac{1}{3 \cdot 2^m - 1} \right) - e^{-2\pi i \theta} u \left( 1 - \frac{1}{3 \cdot 2^m - 1} \right) \right) = \lambda u \left( \frac{1}{3 \cdot 2^m} \right)
\]
(57)

We note that the equations (55), (56), (57) are equivalent to the eigenvector equations for a Hermitian matrix, so the eigenvalues are all real (in fact nonnegative). If \( u \) is an eigenfunction associated to \( \theta \), then \( \hat{u} \) is an eigenfunction associated to \( 1 - \theta \) with the same eigenvalue. When \( \theta = \frac{1}{2} \) then \( e^{2\pi i \theta} = -1 \) and the eigenfunctions may be taken to be real valued. When \( \theta = 0 \) the functions satisfying (48) are just functions on \( J \) lifted to be periodic, so the eigenfunctions and eigenvalues agree with those on \( J \). When \( \theta = \frac{1}{4} \) the functions satisfying (48) may be regarded as functions on the double cover \( \tilde{J}_2 \) satisfying the skew symmetry condition
\[
  u(x + 1) = -u(x).
\]
(58)
Of course the functions corresponding to $\theta = 0$ may also be regarded as functions on $\tilde{\mathcal{J}}_2$ satisfying the symmetry condition

$$u(x + 1) = u(x),$$

and so the eigenspaces corresponding to $\theta = 0$ and $\theta = \frac{1}{2}$ together generate the eigenspaces on $\tilde{\mathcal{J}}_2$. This is especially interesting for cover 1 (hence also the equivalent cover, 2), since in that case $\tilde{\mathcal{J}}_2$ is the double blowup so we know its spectrum exactly: the eigenvalues of $\tilde{\mathcal{J}}_2$ are exactly the eigenvalues of $\mathcal{J}$ multiplied by $2^{-\frac{3}{2}}$, and if $u(x)$ is an eigenfunction on $\mathcal{J}$ with eigenvalue $\lambda$, then $u\left(\frac{1}{2}x\right)$ is an eigenfunction on $\tilde{\mathcal{J}}_2$ with eigenvalue $2^{-\frac{3}{2}}\lambda$. Indeed if $u$ is a derived eigenfunction on $\mathcal{J}$, meaning $u(x) = u_1(2x)$ for some eigenfunction $u_1$, then $u\left(\frac{1}{2}x\right)$ corresponds to $\theta = 0$. Then the primitive eigenfunctions on $\mathcal{J}$ give rise to the $\theta = \frac{1}{2}$ case.

Another interesting observation about $\theta = \frac{1}{2}$ concerns covers 3 and 4 that have $\tilde{\mathcal{J}}_2$ with a $D_4$ symmetry group: the eigenspaces of $\tilde{\mathcal{J}}_2$ must split according to the irreducible representations of $D_4$ (four one-dimensional representations and one two-dimensional representation). It is easy to see that the eigenfunctions transforming according to the one-dimensional representations must satisfy (59), hence only the two-dimensional representations contribute to the $\theta = \frac{1}{2}$ spectrum. We conclude that the multiplicities of all $\theta = \frac{1}{2}$ eigenspaces must be even numbers.

Because $\mathcal{J}$ has localized eigenfunctions, there will also be localized eigenfunctions on $\tilde{\mathcal{J}}$. These will give rise to a discrete spectrum with countable multiplicity. Specifically, let $\text{supp}(u)$ denote the support of an eigenfunction $u$ on $\mathcal{J}$. If $\pi^{-1}(\text{supp}(u))$ splits into countably many connected components, then we may take $\tilde{u} = u \circ \pi$ on one component and extend it to be zero everywhere else, and then $\tilde{u}$ and all its integer translates will be compactly supported eigenfunctions with the same eigenvalue as $u$. These eigenvalues will show up in the $\theta$-spectrum for all values of $\theta$, as we can use (48) to extend $u \circ \pi$ on $[0, 1]$ to the whole parameter space. For example, if $u$ is a primitive vertical eigenfunction on $\mathcal{J}$ then its support is contained in $\left[\frac{1}{6}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{5}{6}\right]$. The splitting of $\pi^{-1}$ of this set occurs in covers 1, 2, and 7, but not in the others. For the first derived eigenvalues, we have a two-dimensional eigenspace, and we can find eigenfunctions with supports in $\left[\frac{1}{12}, \frac{\pi}{6}\right] \cup \left[\frac{\pi}{6}, \frac{11}{12}\right]$ and $\left[\frac{1}{3}, \frac{7}{12}\right] \cup \left[\frac{7}{12}, \frac{7}{3}\right]$. In covers 1, 2, 5, 6, exactly one of these sets satisfies the $\pi^{-1}$ splitting while in covers 3 and 4 both do (for cover 7 neither of them do).

There is a standard procedure to reduce the study of the spectrum of $\hat{\Delta}$ on $\tilde{\mathcal{J}}$ to the amalgam of all the $\theta$-spectra. There is a dense subspace of $L^2(\tilde{\mathcal{J}})$ consisting of compactly supported functions in the domain of $\hat{\Delta}$. For such functions define

$$u_\theta(t) = \sum_{k \in \mathbb{Z}} u(t + k)e^{-2\pi ik\theta}.$$

Because $u$ is compactly supported the sum is finite, so $u_\theta$ belongs to the domain of $\hat{\Delta}$ (it is of course no longer compactly supported) and satisfies (48). Let $\left\{f_j^{(\theta)}(t)\right\}$ be an orthonormal basis of eigenfunctions of the $\theta$-spectrum with eigenvalues $\lambda_j^{(\theta)}$. Then

$$u_\theta(t) = \sum_j \left(\int_0^1 u_\theta(s)f_j^{(\theta)}(s)ds\right)f_j^{(\theta)}(t).$$
On the other hand we can invert (60) to obtain

\[ u(t) = \int_0^1 u_\theta(t)d\theta. \]  

(62)

Substituting (61) into (62) we obtain

\[ u(t) = \int_0^1 \left[ \sum_j \left( \int_0^1 u_\theta(s)f_j^{(\theta)}(s)ds \right) f_j^{(\theta)}(t) \right] d\theta, \]  

(63)

and then substituting (60) for \( u_\theta(s) \) in (63) yields

\[ u(t) = \sum_j \int_0^1 \left( \int_{-\infty}^{\infty} u(s)f_j^{(\theta)}(s)ds \right) f_j^{(\theta)}(t)d\theta, \]  

(64)

which we may interpret as the spectral resolution of \( \tilde{\Delta} \) on \( \tilde{J} \). By choosing the indexing carefully we can sort the \( j \) sum into disjoint groupings \( j \in I_d \) (the discrete indices) for which \( \lambda_j^{(\theta)} = \lambda_j \) is a constant independent of \( \theta \), and \( j \in I_c \) (the continuous indices) for which \( \lambda_j^{(\theta)} \) varies over an interval \( A_j \). Then for \( j \in I_d \),

\[ \int_0^1 \left( \int_{-\infty}^{\infty} u(s)f_j^{(\theta)}(s)ds \right) f_j^{(\theta)}(t)d\theta, \]

(65)

projects \( u \) onto the infinite dimensional space of \( L^2 \) eigenfunctions with eigenvalue \( \lambda_j \), so \( \Lambda_d = \{ \lambda_j : j \in I_d \} \) is the discrete part of the spectrum. The continuous part of the spectrum is \( \Lambda_c = \bigcup_{j \in I_c} A_j \). Note that the intervals may overlap, but there are at most a finite number of overlaps in any bounded region. For any reasonable subset \( B \subseteq \Lambda_c \), the spectral projection \( P_B \) onto \( B \) associated with the continuous spectrum is given by

\[ P_B u(t) = \sum_{j \in \Lambda_c} \int_{\lambda_j^{-1}(B)} \left( \int_{-\infty}^{\infty} u(s)f_j^{(\theta)}(s)ds \right) f_j^{(\theta)}(t)d\theta. \]  

(66)

11. Spectra of periodic covering spaces. In this section we present some of our results computing the \( \theta \)-spectrum of the seven covering spaces discussed in section 10. Because of equivalences we only deal with covers 1,3,4,5,7. For each cover the \( \theta \)-spectrum consists of eigenvalues \( \{ \lambda_j(\theta) \} \) in increasing order. Because the variability of each \( \lambda_j(\theta) \) may be quite different, it is difficult to visualize the spectrum as a whole (strictly speaking just some lower segment of the spectrum), so instead we provide several cross-sections of several \( j \) values together. These are shown in Figures 16-18. The horizontal lines correspond to the discrete spectrum, and the other lines correspond to the continuous spectrum. Note that all lines are symmetric about \( \theta = \frac{1}{2} \). Some lines are not horizontal but are very close to horizontal, because \( \lambda_j(\theta) \) varies only a small amount with \( \theta \). The data presented shows that there are gaps in the spectrum, and that the discrete eigenvalues may occur in the interior of the continuous spectrum, at an endpoint of an interval of the continuous spectrum, or in a gap. Except in the case of higher multiplicities when \( \lambda_j(\theta) \) is constant, all multiplicities greater than one (seen as crossings of \( \lambda_j(\theta) \)) occur at \( 0, \frac{1}{2}, 1 \). We do not know if this phenomenon persists higher up in the spectrum or for all covers. The graphs of \( \lambda_j(\theta) \) appear to be relatively smooth (except at \( \theta = \frac{1}{2} \) when higher multiplicities occur). If this is valid it suggests that the continuous spectrum is absolutely continuous.
It is easy to explain the cases when $\lambda_j(\theta)$ is constant. In other words, the eigenvalue is independent of $\theta$. These correspond to eigenfunctions on $\mathcal{J}$ that have small enough support, and the identifications in the cover keep the support isolated. Consider a vertical primitive eigenfunction which has support $[\frac{1}{12}, \frac{1}{6}] \cup [\frac{5}{12}, \frac{5}{6}]$ (the smallest corresponding eigenvalue is $\lambda_3$). In covers 1,2 and 7 the covering supports remain isolated, but not in other covers. Therefore we see eigenvalues independent of $\theta$ in those three coverings. Next consider the first derived eigenspace, which has multiplicity 2 (the smallest corresponding eigenvalues are $\lambda_7$ and $\lambda_9$). We may choose eigenfunctions with supports in $\left[\frac{1}{12}, \frac{1}{6}\right] \cup \left[\frac{5}{12}, \frac{5}{6}\right]$ and $\left[\frac{1}{3}, \frac{5}{12}\right] \cup \left[\frac{7}{12}, \frac{5}{3}\right]$. For the first support, the identifications keep it isolated in covers 2,3 and 6, while for the second it is covers 1,3 and 5. So in cover 3 we see multiplicity two, while in covers 1,2,5 and 6 we see multiplicity one (there is no consistent pattern as to whether it is the larger or the smaller eigenvalue that is independent of $\theta$).

Next we consider a second derived vertical eigenspace, which has multiplicity 3 (the smallest corresponding eigenvalues are $\lambda_{14}, \lambda_{15}, \lambda_{16}$). The three corresponding supports and the covers that keep them isolated are the following:

- $\left[\frac{1}{24}, \frac{1}{12}\right] \cup \left[\frac{11}{12}, \frac{23}{24}\right]$: covers:1,2,3,4,6,7
- $\left[\frac{5}{12}, \frac{11}{12}\right] \cup \left[\frac{13}{24}, \frac{7}{12}\right]$: covers:1,2,3,4,5,7
- $\left[\frac{1}{6}, \frac{5}{24}\right] \cup \left[\frac{7}{24}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{17}{24}\right] \cup \left[\frac{19}{24}, \frac{5}{6}\right]$: covers:1,2,5,6,7

Therefore we see eigenvalues independent of $\theta$ with multiplicity two for covers 3,4,5 and 6 and multiplicity three for covers 1,2 and 7. We can similarly analyze all higher derived vertical eigenspaces. We just report the multiplicities for a third derived vertical eigenspace, which has multiplicity 6 (the smallest corresponding eigenvalues are $\lambda_{27}, \lambda_{28}, \lambda_{29}, \lambda_{30}, \lambda_{31} \text{ and } \lambda_{32}$):

- multiplicity = 4 for covers 4 and 7
- multiplicity = 5 for covers 1,2,5 and 6
- multiplicity = 6 for cover 3.

It is also interesting to look at the graphs of the corresponding eigenfunctions for $\theta = \frac{1}{2}$. For the sake of comparison we show the graphs of the first sixteen eigenfunctions for $\theta = 0$ (these are the same for all covers, and just coincide with the eigenfunctions on $\mathcal{J}$) in Figure 19. Then in Figures 20-24 we show the $\theta = \frac{1}{2}$ eigenfunctions for the five different covers. Note that (10.1) says $u(t + 1) = -u(t)$ in this case, which implies that those functions are all real-valued, and in fact they may be identified with eigenfunctions on the corresponding double cover with the above skew-symmetry. As mentioned in section 10, for covers 3 and 4 all eigenspaces have even multiplicity for $\theta = \frac{1}{2}$. We note that for cover 7 many eigenspaces have multiplicity 3. This may be explained as follows:

The $\theta = \frac{1}{4}$ eigenfunctions for cover 7 may be sorted according to symmetry types with respect to the horizontal ($x \mapsto \frac{1}{2} - x(mod\ 1)$) and vertical ($x \mapsto 1 - x$) reflections. We write $H^\pm$ and $V^\pm$ to indicate the symmetry type. Suppose $u(x)$ has type $H^+ V^\pm$. We claim that $u$ vanishes on $[\frac{1}{6}, \frac{1}{3}]$ and $[\frac{5}{6}, \frac{5}{3}]$. Indeed, $\frac{5}{24} \sim \frac{5}{25}$ so $u\left(\frac{5}{24}\right) = u\left(\frac{5}{25}\right)$. But the $H^-$ symmetry implies $u\left(\frac{5}{24}\right) = -u\left(\frac{5}{25}\right)$, so $u$ vanishes at $\frac{5}{24}$ and $\frac{5}{25}$. Similarly the $V^+$ symmetry implies $u\left(\frac{1}{6}\right) = u\left(\frac{1}{5}\right)$ and $u\left(\frac{1}{3}\right) = u\left(\frac{2}{3}\right)$, while the cover identifications require $u\left(\frac{8}{24}\right) = -u\left(\frac{8}{25}\right)$ and $u\left(\frac{4}{3}\right) = -u\left(\frac{4}{3}\right)$, so $u$
vanishes at \( \frac{1}{6} \) and \( \frac{1}{3} \). Once we have \( u \) vanishing at the points \( \frac{1}{6}, \frac{5}{24}, \frac{7}{24}, \frac{1}{3} \) the eigenvalue equations forces \( u \) to vanish on the whole interval \( \left[ \frac{1}{6}, \frac{1}{3} \right] \), and a similar argument works for \( \left[ \frac{2}{3}, \frac{5}{3} \right] \).

Once we have \( u \) vanishing on these intervals, we may multiply \( u \) by -1 on the interval \( \left[ \frac{1}{3}, \frac{2}{3} \right] \) to obtain another eigenfunction \( \tilde{u} \) of symmetry type \( H^+V^- \) with the same eigenvalue. We can also rotate \( u \) by \( \frac{1}{4} \left( u \left( x - \frac{1}{4} \right) \right) \) to obtain a third eigenfunction of symmetry type \( H^+V^- \). This shows that every \( H^+V^- \) eigenfunction belongs to an eigenspace of multiplicity at least 3. It is also clear that a linear combination of \( u \) and \( \tilde{u} \) will be supported on \( \left[ \frac{1}{3}, \frac{2}{3} \right] \), and this eigenfunction also exists in cover 5. Thus every multiplicity 3 eigenspace in cover 7 corresponds to a multiplicity 1 eigenspace in cover 5.

Another interesting set of coincidences between \( \theta = \frac{1}{2} \) eigenfunctions of different covers may be observed in covers 1 and 4. If \( u \) is a vertical primitive eigenfunction on \( \mathcal{J} \), then \( u (\frac{1}{2} x) \) is a \( \theta = \frac{1}{2} \) eigenfunction on cover 1, and it has support \( \left[ \frac{1}{3}, \frac{2}{3} \right] \cup \left[ \frac{4}{3}, \frac{5}{3} \right] \). This may then be interpreted as a \( \theta = \frac{1}{2} \) eigenfunction on cover 4. Note that the eigenspace for cover 4 has multiplicity 2 (another distinct eigenfunction in this eigenspace may be identified with an eigenfunction from cover 2).

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Figure 16. Covers 1 and 3 Eigenvalues. Column 1 (bottom to top): Cover 1. Plot 1: 1-8. Plot 2: 9-16. Plot 3: 17-24. Plot 4: 25-30. Column 2 (bottom to top): Cover 3. Plot 1: 1-8. Plot 2: 9-16. Plot 3: 17-24. Plot 4: 25-30.
Figure 17. Covers 4 and 5 Eigenvalues. Column 1 (bottom to top): Cover 4. Plot 1: 1-8. Plot 2: 9-16. Plot 3: 17-24. Plot 4: 25-30. Column 2 (bottom to top): Cover 5. Plot 1: 1-8. Plot 2: 9-16. Plot 3: 17-24. Plot 4: 25-30.
Figure 18. Cover 7 Eigenvalues. (bottom to top) Plot 1: 1-8. Plot 2: 9-16. Plot 3: 17-24. Plot 4: 25-30.
Figure 19. First 16 Eigenfunctions ordered by row: $\theta = 0$
Figure 20. First 12 Eigenfunctions for Cover 1 ordered by row: $\theta = \frac{1}{2}$
Figure 21. First 12 Eigenfunctions for Cover 3 ordered by row: $\theta = \frac{1}{2}$. 
Figure 22. First 12 Eigenfunctions for Cover 4 ordered by row:
\[ \theta = \frac{1}{2} \]
Figure 23. First 12 Eigenfunctions for Cover 5 ordered by row: $\theta = \frac{1}{2}$. 
Figure 24. First 12 Eigenfunctions for Cover 7 ordered by row: $\theta = \frac{1}{2}$.