Noncommutative QCD corrections to the gluonic decays of heavy quarkonia

Alberto Devoto,‡ Stefano Di Chiara, and Wayne W. Repko

1 Dipartimento di Fisica, Università di Cagliari and INFN, Sezione di Cagliari, Cagliari, Italy
2 Department of Physics and Astronomy, Michigan State University, East Lansing, MI 48824

(Dated: March 26, 2022)

We compute the Noncommutative QCD (NCQCD) contributions to the three gluon decay modes of heavy quarkonia. For triplet quarkonia (ortho-quarkonia), the NCQCD correction to the QCD three gluon decay mode, like the standard model contribution, is infrared finite. In the case of singlet quarkonia (para-quarkonia), whose QCD three gluon decay mode has infrared singularities which are removed using one-loop corrections to the two gluon mode, we find that NCQCD contribution is also infrared finite. The calculations are performed in the weak binding limit and do not require the introduction of additional effective couplings.

1. INTRODUCTION

Efforts to explore the physical implications of field theories formulated on noncommutative spaces have increased recently due to developments in string theories, which suggest that noncommutative field theories are well defined quantum theories. In noncommutative geometry, the coordinates $x^\mu$ obey the commutation relations

$$[x^\mu, x^\nu] = i\theta^{\mu\nu},$$

where $\theta^{\mu\nu} = -\theta^{\nu\mu}$. A noncommutative version of an ordinary field theory can be obtained by replacing all ordinary products with Moyal star products defined by

$$(f * g)(x) = \exp\left(\frac{i}{2}\theta^{\mu\nu}\partial_{x^\mu}\partial_{y^\nu}\right) f(x)g(y)\bigg|_{x=y}.$$  

Here, we use the generalization of the QCD Lagrangian

$$\mathcal{L} = \bar{\psi} \star D\psi - m\bar{\psi} \star \psi - \frac{1}{2g^2} \text{Tr} (F_{\mu\nu} \star F^{\mu\nu}),$$

where

$$D_\mu \psi = \partial_\mu \psi - iA_\mu \star \psi,$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu],$$

and

$$[A_\mu, A_\nu] = A_\mu \star A_\nu - A_\nu \star A_\mu.$$  

When supplemented with a gauge fixing term, including a ghost contribution, the Lagrangian Eq. (3) can be used to obtain a set of vertices and Feynman rules for perturbative calculations.

2. NCQCD CORRECTIONS TO THE LIFETIMES OF HEAVY ORTHO AND PARA QUARKONIA

In standard model QCD, the hadronic contributions to the widths of ground state quarkonia are attributed to gluonic decays. The pseudoscalar states, para-quarkonia, decay predominantly into two gluons while the vector states, ortho-quarkonia, being color singlet spin one states, must decay into three gluons. Unlike para-positronium, which cannot decay into three photons in ordinary QED due to charge conjugation symmetry, para-quarkonium can decay into three gluons. However, the three gluon mode is infrared singular and one-loop corrections to the two gluon mode must be included to obtain a finite contribution to the hadronic width.

By assuming that quarkonia are weakly bound it is possible to calculate the NCQCD correction to the three gluon lifetimes by computing the annihilation amplitudes for a noninteracting quark and antiquark at rest and supplying a factor of the square of the bound state wave function at the origin, $|\psi(0)|^2$, to account for the leading binding effect. There is no need to devise an effective interaction as in the case of the pion decay into three photons. The NCQCD amplitudes contributing to the three gluon corrections were calculated using the Feynman rules of Ref. [6, 7]. These rules contain contributions involving $\theta_{\mu\nu}$ of the form $k_1^i \theta_{\mu\nu} k_2^j$, where $k_1$ and $k_2$ are the momenta of two gluons. To ensure that the unitarity conditions $\theta_{\mu\nu}\theta^{\mu\nu} > 0$ and $\varepsilon_{\mu\nu\lambda\rho}\theta^{\mu\nu}\theta^{\lambda\rho} = 0$ are satisfied, we take $\theta_{0k} = -\theta_{k0}$ and write $\theta_{ij}$ as

$$\theta_{ij} = \frac{1}{\Lambda_{NC}^2} \varepsilon_{ijk} \theta_k,$$

where $\theta_k$ is a unit vector and $\Lambda_{NC}$ is the noncommutativity scale. We then have $k_1^i \theta_{\mu\nu} k_2^j = \theta (k_1 \times k_2)/\Lambda_{NC}^2$.

The amplitudes to be calculated are illustrated in
Fig. 1. Diagrams with the three gluon final state connected to the quark line by a single virtual gluon do not contribute. We computed the threshold helicity amplitudes with the aid of the symbolic manipulation program FORM and simplified the results using Mathematica. In what follows, we consider quarkonium in the “non-relativistic” approximation, taking the quark (and the anti-quark) 4-momentum $p^μ$ to be $p = (m, 0, 0, 0)$. The gluon 4-momenta are labelled $k_i$, with $i = 1, 2, 3$.

All the squared amplitudes can be expressed as functions of the parameters $s, t, u$ given by

$$s \equiv (k_1 + k_2)^2, \quad t \equiv (k_2 + k_3)^2, \quad u \equiv (k_3 + k_1)^2$$

which satisfy the identity

$$s + t + u = 4m^2.$$  

In terms of these variables the squared matrix element summed over the gluon helicities is, for the triplet state

$$|^3\mathcal{M}|^2 = \frac{128}{27} g^6 (5 + 4 \sin^2 \varphi)(s + t + u)$$

$$\times \frac{s^2 (t + u)^2 + t^2 (u + s)^2 + u^2 (s + t)^2}{(s + t)^2 (t + u)^2 (u + s)^2},$$

and for the singlet state

$$|^1\mathcal{M}|^2 = \frac{64}{9} g^6 (9 - 4 \sin^2 \varphi)(st + tu + us)^2$$

$$\times \frac{s^4 + t^4 + u^4 + (s + t + u)^4}{stu (s + t)^2 (t + u)^2 (u + s)^2},$$

where

$$\varphi = \frac{1}{2} k_1^\mu \theta_{\mu\nu} k_2^\nu.$$  

The integration over phase space for either state is straightforward and the expression for the width in terms of the scaled variables $x = s/4m^2$, $y = t/4m^2$ takes the form

$$\frac{d\Gamma_{Q \to 3g}}{d \cos \delta} = \frac{|\psi(0)|^2}{192(2\pi)^3} \int_0^1 dx \int_0^{1-x} dy |^3\mathcal{M}_{q\to 3g}|^2.$$  

The variable $\varphi$ can be expressed in terms of $x$, $y$ and the dimensionless scale parameter $z = m^2/\Lambda_{QCD}^2$ as

$$\varphi = \frac{1}{2} |k_1 \times k_2| \cos \delta = \sqrt{xy(1 - x - y)} z \cos \delta.$$  

2.1. Ortho-quarkonium

The ortho-quarkonium decay width can be separated into two terms, $\Gamma_{QCD}^{oQ \to 3g}$ and $\Gamma_{NCQCD}^{oQ \to 3g}$. The first, which is independent of $z$, gives, after completing the phase space integration, the standard QCD result

$$\Gamma_{QCD}^{oQ \to 3g} = \frac{40}{81} \alpha_3^3 (2\pi^2 - 9) \frac{|\psi(0)|^2}{m^2}.$$  

Using the symmetry of the integrand with respect to the 4-vectors $k_1$, $k_2$ and $k_3$ the ($z$ dependent) NCQCD contribution can be written

$$\frac{d\Gamma_{NCQCD}^{oQ \to 3g}}{d \cos \delta} = \frac{16}{27} \alpha_3^3 |\psi(0)|^2 \int_0^1 dx \int_0^{1-x} dy$$

$$\times \frac{x^2 \sin^2 \left(\frac{\sqrt{xy(1 - x - y)} z \cos \delta}{(x + y)^2(1 - y)^2}\right)}{2z \sqrt{xy(1 - x - y)}}.$$  

Integration over $d \cos \delta (-1 \leq \cos \delta \leq 1)$ gives

$$\Gamma_{NCQCD}^{oQ \to 3g} = \frac{16}{27} \alpha_3^3 |\psi(0)|^2 \int_0^1 dx \int_0^{1-x} dy$$

$$\times \frac{x^2}{2z \sqrt{xy(1 - x - y)}} \left(1 - \frac{\sin \left(\frac{2z \sqrt{xy(1 - x - y)} z \cos \delta}{(x + y)^2(1 - y)^2}\right)}{2z \sqrt{xy(1 - x - y)}}\right).$$  

Due to the presence of the square root in the argument of the sine function in Eq. (17) it is not possible to perform the integration analytically. However, rather than simply keeping the leading term in $z$, we investigated the behavior of the correction to all orders in $z$. The result, when combined with Eq. (15), has the form

$$\Gamma_{oQ \to 3g} = \frac{8}{81} \alpha_3^3 |\psi(0)|^2 \left(5\pi^2 - 9\right)$$

$$+ \frac{2}{9} \left(385 - 39\pi^2\right) z^2 f(z),$$  

where the behavior of $f(z)$ is illustrated in Fig. 2.

2.2. Para-quarkonium

The evaluation of the QCD contribution to Eq. (13) in the case of para-quarkonium is complicated by the
existence of infrared divergences. By requiring the variables \( x \) and \( y \) to satisfy \( x, y \geq \varepsilon \) and using the symmetry of the integrand, the QCD contribution can be written

\[
\Gamma_{pQ \rightarrow 3g}^{\text{QCD}} = \frac{4}{3} \alpha_s^3 \frac{\langle |\psi(0)|^2 \rangle}{m^2} \int_x^{1-2\varepsilon} \frac{dx}{x} \int_y^{1-\varepsilon-x} \frac{dy}{y} (1 + 3x^4) \left( x^2 - (x + y)(1 - y) \right)^2 \]

\[
\times \frac{xy(1 - x - y)(1 - x)^2(1 - y)^2(x + y)^2}{(1 - y)^2(1 - x)^2(1 - y)^2(x + y)^2},
\]

which can be integrated to give

\[
\Gamma_{pQ \rightarrow 3g}^{\text{QCD}} = \alpha_s^3 \frac{\langle |\psi(0)|^2 \rangle}{3m^2} \left[ 152 - 11\pi^2 \right.
\]

\[
+ 4 \log(\varepsilon) (11 + 6 \log(\varepsilon)) \right]. \tag{20}
\]

The infrared behavior of \( \Gamma_{pQ \rightarrow 3g}^{\text{QCD}} \) exhibited in Eq. (20) must be combined with the one-loop corrections to the two gluon decay to obtain a finite correction to the hadronic width \( \mathcal{G} \).

The evaluation of the non-commutative contribution involves an additional factor of \(-4\sin^2 \varphi\), which may be handled as in the ortho-quarkonium case to obtain

\[
\Gamma_{pQ \rightarrow 3g}^{\text{NCQCD}} = -\frac{8}{27} \alpha_s^3 \frac{\langle |\psi(0)|^2 \rangle}{m^2} \int_x^{1-2\varepsilon} \frac{dx}{x} \int_y^{1-\varepsilon-x} \frac{dy}{y} \times \left[ 1 - \sin \left( \frac{2\pi \sqrt{x y(1 - x - y)}}{2\pi \sqrt{x y(1 - x - y)}} \right) \right]. \tag{21}
\]

With the introduction of \( \sin^2 \varphi \), the integrand of Eq. (21) is no longer singular when \( x(1 - x - y) \rightarrow 0 \) and we may complete the integration with \( \varepsilon = 0 \). The NCQCD contribution then can be written

\[
\Gamma_{pQ \rightarrow 3g}^{\text{NCQCD}} = -\frac{4}{81} \alpha_s^3 \frac{\langle |\psi(0)|^2 \rangle}{m^2} \left( 37\pi^2 - \frac{5437}{15} \right) z^2 g(z), \tag{22}
\]

where \( g(z) \) is shown in Fig. 3.

3. DISCUSSION AND CONCLUSIONS

The inclusion of NCQCD corrections to the three gluon decay widths of ortho- and para-quarkonium does not change the magnitudes of their hadronic widths substantially for scales \( \Lambda_{\text{NC}} \) of order 1 TeV. That said, the NCQCD results are interesting in the sense of what they imply about the consistency of perturbative calculations in these models.

For ortho-quarkonium, the two gluon decay is forbidden by Yang’s theorem. Hence the three gluon decay cannot have any soft gluon singularities because there are no two gluon one-loop corrections available to cancel them. The absence of soft gluon singularities, which is, of course, a feature of the QCD three gluon decay width, persists when the small (positive) NCQCD correction is included. This is guaranteed by the form of the squared amplitude, Eq. (10).

The situation with para-quarkonium is somewhat more involved because, unlike para-positronium, which can only decay into two photons, both two gluon and three gluon decays are allowed in QCD. In this case, the three gluon decay has infrared singularities, which can be combined with the one-loop corrections to the two gluon decay to obtain a finite contribution to the hadronic width \( \mathcal{G} \). Here, too, the NCQCD correction is infrared finite, but only as a result of a cancellation provided by the NCQCD effective coupling. Interestingly, this correction is negative, and, while small for realistic values of \( z = m^2/\Lambda_{\text{NC}}^2 \), it cannot change the sign of the total width for any value of \( z \).

In summary, we are led to the conclusion that it is not necessary to invoke the smallness of \( z = m^2/\Lambda_{\text{NC}}^2 \) to obtain a sensible NCQCD correction to the hadronic decays of quarkonia. In principle, should we be presented with a fourth generation of very heavy quarks with lifetimes sufficiently long to produce quarkonia, corrections to their decay widths associated with non-commutative geometry could be calculated con-
sistently in the sense that they are finite for any value of \( z \).

Acknowledgments

One of us (WWR) wishes to thank INFN, Sezione di Cagliari and Dipartimento di Fisica, Università di Cagliari for support. This research was supported in part by the National Science Foundation under Grant PHY-02744789 and by M.I.U.R. (Ministero dell’Istruzione, dell’Università e della Ricerca) under Cofinanziamento PRIN 2001.

[1] P. Mathews, Phys. Rev. D 63, 075007 (2001), H. Grosse and Y. Liao, Phys. Rev. D 64, 115007 (2001), M. Chaichian, P. Presnajder, M. M. Sheikh-Jabbari and A. Tureanu, Eur. Phys. J. C 29, 413 (2003), C. E. Carlson, C. D. Carone and R. F. Lebed, Phys. Lett. B 518, 201 (2001), R. J. Szabo, Phys. Rept. 378, 207 (2003), I. Hinchliffe and N. Kersting, Phys. Lett. B 527, 115 (2002), R. Jackiw, Nucl. Phys. Proc. Suppl. 108, 30 (2002) [Phys. Part. Nucl. 33, S6 (2002 LNPMA,616,294-304.2003)], Y. Liao, JHEP 0111, 067 (2001), N. Mahajan, arXiv:hep-ph/0110148, Y. Liao, JHEP 0204, 042 (2002), J. L. Hewett, F. J. Petriello and T. G. Rizzo, arXiv:hep-ph/0201275, N. G. Deshpande and X.-G. He, Phys. Lett. B 533, 116 (2002), E. O. Itlan, New J. Phys. 4, 54 (2002), E. O. Itlan, Phys. Rev. D 66, 034011 (2002), C. E. Carlson, C. D. Carone and N. Zobin, Phys. Rev. D 66, 075001 (2002), N. Mahajan, Phys. Rev. D 68, 095001 (2003), X.-G. He, Eur. Phys. J. C 28, 557 (2003), W. Behr, N. G. Deshpande, G. Duplancic, P. Schupp, J. Trampetic and J. Wess, Eur. Phys. J. C 29, 441 (2003).

[2] For a recent review of NCQCD phenomenology, see: I. Hinchliffe, N. Kersting, and Y. L. Ma, arXiv:hep-ph/0205040.

[3] A. Connes, M. R. Douglas and A. Schwarz, JHEP 02, 003 (1998).

[4] M. R. Douglas and C. Hull, JHEP 02, 008 (1998).

[5] N. Seiberg and E. Witten, JHEP 09, 032 (1999).

[6] A. Armoni, Nuc. Phys. B 593, 229 (2001).

[7] L. Bonora and M. Salizzoni, Phys. Lett. B 504, 80 (2001).

[8] R. Barbieri, G. Curreri, E. d’Emilio and E. Remiddi, Nuc. Phys. B 154, 355 (1979).

[9] H. Grosse and Y. Liao, Phys. Lett. B 520, 63 (2001).

[10] M. Caravati, A. Devoto and W. W. Repko, Phys. Lett. B 556, 123 (2003).

[11] J. A. M. Vermaseren, arXiv:math-ph/0010025.

[12] The Mathematica Book, Wolfram Research, Inc., Champaign, IL.

[13] F. E. Close, An Introduction to Quarks and Partons, Academic Press, London (1979), p. 357.

[14] More precisely, one can unambiguously compute the ratio of the QCD radiatively corrected two gluon to two photon widths.\]