SHARP CONVERGENCE FOR DEGENERATE LANGEVIN DYNAMICS

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Abstract. In this paper, we study an ordinary differential equation with a degenerate global attractor at the origin, to which we add a white noise with a small parameter that regulates its intensity. Under general conditions, for any fixed intensity, as time tends to infinity, the solution of this stochastic dynamics converges exponentially fast in total variation distance to a unique equilibrium distribution. We suitably accelerate the random dynamics and show that the preceding convergence is sharp, that is, the total variation distance of the accelerated random dynamics and its equilibrium distribution tends to a decreasing profile, which corresponds to the total variation distance between the marginal of a stochastic differential equation that comes down from infinity and its corresponding equilibrium distribution. In particular, there is no cutoff phenomenon for this one-parameter family of random processes.

Contents

1. Introduction
   1.1. The degenerate Langevin dynamics
   1.2. Results
2. Proof of Theorem 1.1
   2.1. Heuristics
   2.2. The invariant probability measure
   2.3. Multi-scale analysis
   2.4. Coupling near the origin
   2.5. Bound via limit replacements
3. Proofs: details
   3.1. Proof of Corollary 1.1
   3.2. Proof of Corollary 1.2
   3.3. Proof of Lemma 2.2
   3.4. Proof of Proposition 2.1
   3.4.1. The continuous Markovian extension
   3.4.2. Convergence in total variation for fixed marginal
   3.5. Strict inequalities for the rescaled process
Appendix A. The Continuous Markovian extension
   A.1. Coming down from infinity
   A.1.1. Entrance condition for ODE’s
   A.1.2. Uniform $L^2$ bounds
   A.1.3. Integral expression
   A.2. Markov property of the extended family
   A.3. Continuity

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Appendix B. Uniform bounds

B.1. Uniform entrance in a compact
B.2. Uniform convergence in total variation distance

Appendix C. Complements

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1. Introduction

The study of random dynamical systems and their convergence to equilibrium is one of the most studied subject in probability theory and mathematical physics with a vast literature such as equilibrium selection, slow-fast systems, random Hamiltonian systems, small noise limit, small noise asymptotics for invariant densities, sharp estimates on transit and exit times, couplings and quantitative contraction rates for Langevin dynamics, convergence to equilibrium in Fokker–Planck equations, random attractors for stochastic dissipative systems, numerical computations of geometric ergodicity, multi-scale analysis, exponential loss of memory of the initial condition, ergodicity, regularity for Lyapunov exponents, metastability, large deviations, optimal transport, etc. See for instance, [4, 6, 20, 22, 23, 24, 27, 38, 41, 44, 48, 50, 54, 55, 61, 63, 66, 67, 69, 75, 76, 79, 85, 86, 88, 89].

The goal of this paper is to study the convergence to equilibrium in the so-called zero-noise limit for a family of stochastic small random perturbations of a given one-dimensional dynamical system. We consider an ordinary differential equation with a degenerate (non-hyperbolic) global attractor at the origin. Under appropriate conditions on the dynamics, as time increases, for any initial condition the solution of this differential equation tends to the origin polynomially fast. We then consider a perturbation of the deterministic dynamics by a Brownian motion of small intensity. This random dynamics possesses a unique invariant probability measure and for any initial condition, the solution converges in the total variation distance to such invariant probability measure as times increases. We prove that the convergence occurs gradually, that is, when the strength of the noise tends to zero, with a suitable scaling of time, the total variation distance between the marginal of the random dynamics and its equilibrium tends to a continuous decreasing profile. This fact implies no-cutoff phenomenon in the context of random processes, see for instance [12, 18, 31, 62].

1.1. The degenerate Langevin dynamics. In this subsection, we specify the degenerate Langevin dynamics that we consider in this paper. Here we say that a Langevin dynamics is degenerate when its vector field possesses a degenerate fixed (critical) point.

The Langevin dynamics was introduced by P. Langevin in 1908 in his seminal article [60]. It is perhaps one of the most popular models in molecular systems. For details on its history and phenomenological treatment, we refer to [31, 81] and the references therein.
Let $\varepsilon \in (0,1]$ be the parameter that controls the intensity of the noise and let $X^\varepsilon(x) := (X^\varepsilon_t(x), t \geq 0)$ be the unique strong solution of the one-dimensional Stochastic Differential Equation (for short SDE)

$$
\begin{aligned}
&\left\{
\begin{array}{ll}
dX^\varepsilon_t = -V'(X^\varepsilon_t)dt + \sqrt{\varepsilon}dB_t & \text{for } t \geq 0, \\
X^\varepsilon_0 = x,
\end{array}
\right.
\end{aligned}
$$

where $x \in \mathbb{R}$ is a deterministic initial condition, $B := (B_t, t \geq 0)$ is a one-dimensional standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $V : \mathbb{R} \to [0, \infty)$ is a given function that will be referred to as the potential. In order to avoid technicalities and since we want to be able to use Itô’s formula, we assume the following conditions for $V$.

**Hypothesis 1** (Regularity). We assume that the potential $V$ is a twice continuously differentiable, convex and even function with $V(0) = 0$.

Since $V$ is even and differentiable, we have $V'(0) = 0$. We recall that 0 is a degenerate fixed point when $V''(0) = 0$. More specifically, we assume the following local behavior at 0.

**Hypothesis 2** (Local behavior at the origin). There exist positive constants $C_0$ and $\alpha$ such that

$$
\limsup_{\lambda \to 0} \sup_{|z| \leq K} \left| \frac{V'(\lambda z)}{\lambda^{1+\alpha}} - C_0 |z|^{1+\alpha} \text{sgn}(z) \right| = 0 \quad \text{for any } K > 0,
$$

where $\text{sgn}(z) := z |z|^{-1} 1_{\{z \neq 0\}}$.

Finally, in order to control the growth of $V'$ around infinity and to ensure that (1.1) has a unique invariant probability measure, we assume the following growth condition.

**Hypothesis 3** (Growth at infinity). There exist $c_0, R_0 \in (0, \infty)$, and $\beta \in (-1, \infty)$ such that

$$
V'(z) \geq c_0 z^{1+\beta} \quad \text{for all } z \geq R_0.
$$

By Hypothesis 1 it follows that $V$ is convex, then Theorem 3.5 in [65, p.58] implies that the SDE (1.1) has a unique strong solution. Hence, $X^\varepsilon(x)$ is a well-defined stochastic process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Furthermore, Lemma 2.1 below yields that (1.1) possesses a unique invariant probability measure $\mu^\varepsilon$ with an explicit formula.

### 1.2. Results

In this section, before we state the main results of the paper, we recall the definition of the total variation distance and fix some conventions.

In the sequel, we adopt the convention that $\text{sgn}(0) \infty = 0$ and since $\varepsilon \in (0,1]$, for simplicity we write $\varepsilon \to 0$ instead of $\varepsilon \to 0^+$. We point out that for any $x \in \mathbb{R}$ and $t > 0$, the marginal $X^\varepsilon_t(x)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$. Then we measure the distance between the law of $X^\varepsilon_t(x)$ and its limiting distribution $\mu^\varepsilon$ by the total variation distance, defined by

$$
d_{\text{TV}}(\nu_1, \nu_2) := \sup_{\mathcal{F} \in \mathcal{F}} |\nu_1(F) - \nu_2(F)|
$$

for any $\nu_1$ and $\nu_2$ probability measures in the same measurable space $(\Omega, \mathcal{F})$. For convenience, we do not distinguish a random variable $X_1$ and its law $\mathbb{P}_{X_1}$ as an argument of $d_{\text{TV}}$. In other words, for random variables $X_1$ and $X_2$ and probability measure $\mu$ we write $d_{\text{TV}}(X_1, X_2)$ in place of $d_{\text{TV}}(\mathbb{P}_{X_1}, \mathbb{P}_{X_2})$, $d_{\text{TV}}(X_1, \mu)$ instead of $d_{\text{TV}}(\mathbb{P}_{X_1}, \mu)$. For further details on the total variation distance, we refer to [55, Ch. 2] or [83, Sec. 3.3].

The main result of this paper is the following.
Theorem 1.1 (Sharp convergence). Assume that Hypothesis 1, Hypothesis 2 and Hypothesis 3 hold true. For $\varepsilon \in (0,1]$ and $x \in \mathbb{R}$, let $X^\varepsilon(x)$ be the unique strong solution of (1.1) and denote by $\mu^\varepsilon$ its unique invariant probability measure. Define the scaling parameter

$$a_\varepsilon := \varepsilon^{-\frac{2}{\alpha+\alpha}}, \quad \text{where } \alpha > 0 \text{ is given in (1.1)}. \quad (1.2)$$

Then for any $t > 0$ it follows that

$$\lim_{\varepsilon \to 0} d_{TV}(X^\varepsilon_{a_\varepsilon t}(x), \mu^\varepsilon) = d_{TV}(Y_t(\text{sgn}(x)\infty), \nu) \in (0,1), \quad (1.3)$$

where $(Y_t(\text{sgn}(x)\infty), t \geq 0)$ is the solution of the SDE

$$\begin{cases} \d Y_t = -C_0|Y_t|^{1+\alpha}\text{sgn}(Y_t)dt + dW_t & \text{for } t > 0, \\ Y_0 = \text{sgn}(x)\infty, \end{cases} \quad (1.4)$$

$(W_t, t \geq 0)$ is a standard Brownian motion, $\nu$ is the unique invariant probability measure for (1.4), and the constant $C_0$ is defined in (1.1).

We point out that the solution of (1.4) comes down from infinity, that is, $Y_t \in \mathbb{R}$ for any $t > 0$. The continuous Markovian extension of the SDE (1.4) is done in detail using basic ODE/probabilistic techniques in Appendix A and here we only outline the main steps. First, based on a monotonic comparison, which follows from the synchronous coupling, and uniform second moment bounds for $x \in \mathbb{R}$, the SDE (1.4) can be extended to $\mathbb{R}$, see Section A.1. Then because $\pm \infty$ are entrance boundaries for the dynamics in $\mathbb{R}$, the extended family $(Y(x) := (Y_t(x), t \geq 0), x \in \mathbb{R})$ is Markovian, see Section A.2. The rigorous definition of (1.4) is given in Proposition 2.1 below. Moreover, Theorem 1.1 actually provides the essentially unique scale $(a_\varepsilon, \varepsilon \in (0,1))$ that captures the convergence to equilibrium. In fact, since the map $t \mapsto d_{TV}(Y_t(\text{sgn}(x)\infty), \nu)$ is non-increasing, any sequence $(t_\varepsilon, \varepsilon \in (0,1))$ for which

$$\lim_{\varepsilon \to 0} d_{TV}(X^\varepsilon_{t_\varepsilon}(x), \mu^\varepsilon) = d_{TV}(Y_t(\text{sgn}(x)\infty), \nu)$$

must satisfy

$$\lim_{\varepsilon \to 0} a_\varepsilon = t.$$

Roughly speaking, one generally expects that a one-parameter family of well-mixing stochastic processes will exhibit abrupt convergence of the marginals to the equilibrium distribution as a function of the parameter. This is known in the literature as the cutoff phenomenon introduced by [2] in the context of card shuffling. Actually, the notion of cutoff applies to a wide range of random models. In the discrete setting, the cutoff phenomenon has been proved for many different models such as Markovian shuffling cards, random walks on the hypercube, birth and death chains, sparse Markov chains, Glauber dynamics, SSEP dynamics, SEP in the circle, random walks in random regular graphs, Ornstein–Uhlenbeck processes, mean-zero field zero-range process, averaging processes, sampling chains, star transpositions, etc. For further details, see [2] [16] [19] [25] [33] [34] [35] [36] [45] [47] [57] [58] [59] [62] [64] [68] [70] [71] [72] [80] [82] [90] and the references therein. There are relatively few examples of Markov processes, taking values in continuous state-spaces for which the cutoff phenomenon has been studied, such processes include linear and nonlinear SDEs driven by small Lévy noise, Dyson–Ornstein–Uhlenbeck process, the biased adjacent walk on the simplex, Brownian motion on families of compact Riemannian manifolds, etc. For more details, we refer to [17] [8] [9] [11] [12] [13] [17] [26] [29] [56] [58].

An interesting example, which satisfies Hypothesis 1, Hypothesis 2 and Hypothesis 3 is the one-well potential $V_\alpha(z) = |z|^{2+\alpha}$, $z \in \mathbb{R}$ for some $\alpha > 0$. When $\alpha = 0$, we have that (1.1) corresponds
to the Ornstein–Uhlenbeck process which exhibits profile cutoff for \( x \neq 0 \) and it does not when \( x = 0 \), for further details see \[17\] and \[12\]. For \( \alpha > 0 \) we have \( V_\alpha'(0) = V''_\alpha(0) = 0 \) and hence Theorem 2.1 in \[12\] cannot be applied. In fact, in the degenerate case, for any initial condition the convergence to equilibrium is gradual. This is in stark contrast with the Ornstein–Uhlenbeck process and it is natural from the dynamical point of view, since the fixed point changes from hyperbolic to non-hyperbolic (degenerate). We point out that the techniques used in the proof of Theorem \[1.1\] are very different from the ones used in \[12\] to prove cutoff. This is mainly because the Hartman–Grobman theorem breaks down at degenerate points.

We now fix the notion of abrupt convergence to equilibrium (cutoff). For further details, see Definition 1.8 in \[9\] and also \[15\].

**Definition 1.1** (Cutoff phenomenon). We say that the one-parameter family of stochastic processes \( (X^\varepsilon(x), \varepsilon \in (0, 1]) \) exhibits cutoff when there exists a scale \( (t_\varepsilon(x), \varepsilon \in (0, 1]) \) such that \( t_\varepsilon \to \infty \) as \( \varepsilon \to 0 \) and

\[
\lim_{\varepsilon \to 0} \text{d}_{TV}(X^\varepsilon_{t_\varepsilon(x)}(x), \mu^\varepsilon) = \begin{cases} 
1 & \text{for } \delta \in (0, 1), \\
0 & \text{for } \delta \in (1, \infty).
\end{cases}
\]

(1.5)

A classical example of a Markov dynamics that does not exhibit cutoff phenomenon is the random walk on the circle \( \mathbb{Z}_n \), see Example 18.5 in \[32\] Ch.18, p.253 or \[28\] Thm. 2.2.1, p.55]. Numerical results yields that the cutoff phenomenon does not occur for the entropy in the sense of information theory, see \[87\]. It has been also proved that the “insect Markov chain” does not have cutoff, see \[32\]. Moreover, it has been showed the absence of cutoff for several classes of trees, including spherically symmetric trees, Galton–Watson trees of a fixed height, and sequences of random trees converging to the Brownian CRT, see \[30, 12\]. More recently, it is shown that the TASEP in the coexistence line does not have cutoff, see \[39\]. From Theorem \[1.1\] we deduce that the family \( (X^\varepsilon(x), \varepsilon \in (0, 1]) \) defined in \[1.1\] does not exhibit cutoff.

**Corollary 1.1** (No cutoff phenomenon). With the assumptions and notations of Theorem \[1.1\] for any \( x \in \mathbb{R} \), the family of processes \( (X^\varepsilon(x), \varepsilon \in (0, 1]) \) does not exhibit cutoff in the total variation distance as \( \varepsilon \to 0 \). In other words, there is no time scale \( (t_\varepsilon, \varepsilon \in (0, 1]) \) with \( t_\varepsilon \to \infty \) as \( \varepsilon \to 0 \) and

\[
\lim_{\varepsilon \to 0} \text{d}_{TV}(X^\varepsilon_{t_\varepsilon(x)}(x), \mu^\varepsilon) = 1_{(0,1)}(\delta) \quad \text{for any } \delta > 0, \delta \neq 1.
\]

The proof of Corollary \[1.1\] is given in Subsection \[3.1\].

Given the initial condition \( x \), one has a well-posed problem: for a prescribed tolerance \( \eta \in (0, 1) \) how much time \( t = t(\eta, x, \varepsilon) \) should we run the process \( X^\varepsilon(x) \) in order that \( d_{TV}(X^\varepsilon_t(x), \mu^\varepsilon) \leq \eta \). That is to say, the time required by the process \( X^\varepsilon(x) \) for the (total variation) distance to its invariant measure \( \mu^\varepsilon \) to be equal or smaller than \( \eta \). The latter is called \( \eta \)-mixing time. Due to its natural relevance, it has been extensively studied in many stochastic models, see for instance \[3, 5, 36, 43, 46, 47, 62, 74, 80, 84\] and the references therein. As a consequence of Theorem \[1.1\] we obtain the following asymptotic estimate for the mixing times associated to \( X^\varepsilon(x) \).

**Corollary 1.2** (Mixing time asymptotics). Suppose that all assumptions and notation made in Theorem \[1.1\] hold true. For any \( x \in \mathbb{R} \) and \( \eta \in (0, 1) \) the \( \eta \)-mixing time

\[
\tau_{mix}^\varepsilon(x)(\eta) := \inf\{t \geq 0 : d_{TV}(X^\varepsilon_t(x), \mu^\varepsilon) \leq \eta\}
\]

satisfies the limiting behavior

\[
\lim_{\varepsilon \to 0} \frac{\tau_{mix}^\varepsilon(x)(\eta)}{\delta_{\varepsilon}} = \inf\{t \geq 0 : d_{TV}(Y_t(sgn(x)\infty), \nu) \leq \eta\}.
\]

(1.6)
The proof of Corollary 1.2 is given in Section 3.2.

Structure of the paper. In Section 2 we explain the proof of Theorem 1.1 and in Section 3 we give the proof of the corollaries and complete details in the proof of Theorem 1.1. The Appendix is divided in three sections. Appendix A proves that the process defined in (1.4) admits a continuous Markovian extension. Appendix B is devoted to the proof of uniform bounds for the entrance on compact sets, a crucial estimate to control the coupling rate of the process with the equilibrium measure. Appendix C contains results of technical nature that we collect to make the presentation more self-contained.

2. Proof of Theorem 1.1

This section is divided in five parts. Firstly, we give an heuristic argument for (1.3). Secondly, we examine, for fixed \( \varepsilon \in (0, 1] \), convergence to the unique invariant measure of \( X^\varepsilon_t(x) \) as \( t \to \infty \). Thirdly, we perform a scale analysis to deduce \( (a_\varepsilon, \varepsilon) \in (0, 1] \). Fourthly, we introduce a localization argument which allows us to simplify the potential \( V \) under analysis. Finally, we state the key results used in the proof of (1.3).

2.1. Heuristics. Assume that \( V \) satisfies Hypothesis 1, Hypothesis 2 and Hypothesis 3. Let \( \psi(x) := (\psi_t(x), t \geq 0) \) be the solution of the Ordinary Differential Equation (for short ODE)

\[
\begin{align*}
\frac{d\psi_t}{dt} &= -V'(\psi_t)dt \quad \text{for} \quad t \geq 0, \\
\psi_0 &= x.
\end{align*}
\]

The intuitive reason to consider (2.1) is that, with high probability, at early stages of the random evolution (1.1), the process stays close to the deterministic evolution (2.1). By the contracting nature of the random evolution (1.1), only true if \( V''(x) > 0 \) for every \( x \neq 0 \), the noise gets dissipated and the process is driven to zero, falling back into the stream of the deterministic evolution. For this reason, (2.1) is actually a good approximation of (1.1) for a long period of time. By Hypothesis 2 for large times, what matters is the behavior of \( V \) at zero, \( V'(z) \sim C_0|z|^{1+\alpha}\text{sgn}(z) \), and the properly rescaled process should converge to \( Y(\text{sgn}(x)\infty) \) as \( \varepsilon \to 0 \), where \( Y(x) = (Y_t(x), t \geq 0) \) solves

\[
\begin{align*}
\frac{dY_t}{dt} &= -C_0|Y_t|^{1+\alpha}\text{sgn}(Y_t)dt + dW_t \quad \text{for} \quad t \geq 0, \\
Y_0 &= x.
\end{align*}
\]

2.2. The invariant probability measure. By the next lemma, (1.1) admits a unique invariant probability measure \( \mu^\varepsilon \).

Lemma 2.1 (Exponential ergodicity). Assume \( V \) satisfies Hypothesis 1 and Hypothesis 3. Let \( \varepsilon \in (0, 1] \) be fixed and for each \( x \in \mathbb{R} \) let \( X^\varepsilon_t(x) \) be the unique strong solution of (1.1). Then there exists a unique probability measure \( \mu^\varepsilon \) such that for any \( c > 0 \) there are positive constants \( C_1 = C_1(c, \varepsilon) \) and \( C_2 = C_2(c, \varepsilon) \) for which

\[
\text{d}_{\text{TV}}(X^\varepsilon_t(x), \mu^\varepsilon) \leq C_1 e^{-C_2 t} \left(e^{c|x|} + \int_\mathbb{R} e^{c|y|} \mu^\varepsilon(dy)\right) \quad \text{for all} \quad x \in \mathbb{R}, \ t \geq 0.
\]

Furthermore, \( \mu^\varepsilon \) is absolutely continuous with respect to the Lebesgue measure on \( \mathbb{R} \) and its density \( \rho^\varepsilon : \mathbb{R} \to (0, \infty) \) is given by

\[
\rho^\varepsilon(x) = \frac{e^{-\frac{2}{\varepsilon} V(x)}}{C_\varepsilon} \quad \text{with} \quad C_\varepsilon := \int_\mathbb{R} e^{-\frac{2}{\varepsilon} V(y)} dy.
\]

The proof of Lemma 2.1 is given in Appendix C.
2.3. Multi-scale analysis. Define, for \( t \geq 0 \),
\[
Y_{t}^{\varepsilon,x} := \frac{X_{t}^{\varepsilon}(x)}{b_{\varepsilon}}
\]
and let us determine the time and space scaling parameters \( a_{\varepsilon} > 0 \) and \( b_{\varepsilon} > 0 \). By Itô’s formula the stochastic process \((Y_{t}^{\varepsilon,x}, t \geq 0)\) has the same law as \((Y_{t}^{\varepsilon}(x), t \geq 0)\), where \(Y_{t}^{\varepsilon}(y) := (Y_{t}^{\varepsilon}(y), t \geq 0)\) is the solution of the SDE
\[
\begin{aligned}
\text{d}Y_{t}^{\varepsilon} & = -\frac{a_{\varepsilon}}{b_{\varepsilon}}V'(b_{\varepsilon}Y_{t}^{\varepsilon}) \text{d}t + \frac{\sqrt{a_{\varepsilon}}}{b_{\varepsilon}} \text{d}B_{t} & \text{for} & \ t \geq 0, \\
Y_{0}^{\varepsilon} & = y.
\end{aligned}
\]
Define \( a_{\varepsilon} \) and \( b_{\varepsilon} \) as the unique solution of the system
\[
\begin{aligned}
\sqrt{\varepsilon a_{\varepsilon}} b_{\varepsilon} & = 1, \\
a_{\varepsilon} b_{\varepsilon}^{\alpha} & = 1,
\end{aligned}
\]
where the constant \( \alpha \) is given in Hypothesis 2. The solution of (2.7) is given by
\[
a_{\varepsilon} = \varepsilon^{-\frac{\alpha}{2+\alpha}} \quad \text{and} \quad b_{\varepsilon} = \varepsilon^{-\frac{1}{2+\alpha}}.
\]
Condition (2.7) sets the scale analysis to a fixed magnitude of the noise \((\varepsilon a_{\varepsilon} = b_{\varepsilon}^{2})\) and a constant strength of the velocity field at the origin \(a_{\varepsilon} b_{\varepsilon}^{\alpha} = 1\). Note that any non-zero initial condition of the process \((X_{t}^{\varepsilon}, t \geq 0)\) gives rise to a diverging initial condition for \((Y_{t}^{\varepsilon}, t \geq 0)\) and hence the zero-noise limit of (2.6) requires a rigorous analysis at infinity.

2.4. Coupling near the origin. Let \( x \) be the initial condition of (1.1). In the sequel, we consider a convex potential \( \tilde{V} = \tilde{V}_{x} \) that satisfies
\[
\tilde{V}(z) = V(z) \quad \text{for any} \quad z \text{ with} \quad |z| \leq L,
\]
where \( L \) is any fixed number such that \( L^{2} \geq 1 + |x|^{2} \). Additionally, we assume the following growth condition.

**Hypothesis 3A** (Polynomial growth at infinity) : There exist positive constants \( c, C \) and \( R \) such that
\[
\begin{aligned}
\text{G1} & \quad \tilde{V}'(z) \geq cz^{1+\alpha} \quad \text{for} \quad z \geq R \\
\text{G2} & \quad |\tilde{V}'(z)| \leq Ce^{z^{2}} \quad \text{for} \quad |z| \geq R,
\end{aligned}
\]
where \( \alpha > 0 \) is given in Hypothesis 2.

Furthermore, note that \( \tilde{V} \) satisfies Hypothesis 1 and Hypothesis 2. The existence of \( \tilde{V} \) is guaranteed by Lemma C.1 in Appendix C.

For each \( \varepsilon \in (0,1] \) and \( x \in \mathbb{R} \) we consider the unique strong solution \( \tilde{X}^{\varepsilon}(x) := (\tilde{X}_{t}^{\varepsilon}(x))_{t \geq 0} \) of the SDE
\[
\begin{aligned}
\text{d}\tilde{X}^{\varepsilon}_{t} & = -\tilde{V}'(\tilde{X}^{\varepsilon}_{t}) \text{d}t + \sqrt{\varepsilon} \text{d}B_{t} & \text{for} & \ t \geq 0, \\
\tilde{X}^{\varepsilon}_{0} & = x.
\end{aligned}
\]
Since \( \tilde{V} \) is a convex function, Theorem 3.5 in [65, p.58] yields that the SDE (2.10) has a unique strong solution. Furthermore, Lemma 2.1 implies that (2.10) possesses a unique invariant probability measure \( \tilde{\mu}^{\varepsilon} \). Recall that \( \mu^{\varepsilon} \) is the unique invariant probability measure for (1.1) and for any
where \( \epsilon \in [0, 1] \) is defined in (1.2).

The proof of Lemma 2.2 is given in Subsection 3.3. By Lemma 2.2 it is enough to show Theorem 1.1 under Hypothesis 1, Hypothesis 2 and Hypothesis 3A.

2.5. Bound via limit replacements. From this point onwards, we assume that \( V \) satisfies Hypothesis 1, Hypothesis 2 and Hypothesis 3A.

Due to the scale invariance of the total variation distance (dTV(cX, cY) = dTV(X, Y) for any \( c \neq 0 \) and any pair of random variables \( X, Y \), (for a proof see Lemma A.1 in [14]), the distance \( d_{\alpha,t}^\epsilon(x) \) in (2.11) can be expressed in terms of \( Y_t^\epsilon \) and \( \nu^\epsilon \), a “scalar multiple” of \( \mu^\epsilon \). For convenience, we denote by \( X_t^\epsilon \) a random variable with the law \( \mu^\epsilon \) and by \( Y_\infty^\epsilon \) a random variable with law \( \nu^\epsilon \) which is the unique invariant probability measure for (1.4). With this notation, we have that \( \nu^\epsilon \) is the law of \( Y_\infty^\epsilon := b^\epsilon_1 X_\infty^\epsilon \) and therefore

\[
\begin{align*}
d_{\alpha,t}^\epsilon(x) &= d_{TV}(X_{\alpha,t}^\epsilon(x), X_\infty^\epsilon) = d_{TV}(b^\epsilon_1 X_{\alpha,t}^\epsilon(x), b^\epsilon_1 X_\infty^\epsilon) = d_{TV}(Y_t^{\epsilon,x}, Y_\infty^\epsilon).
\end{align*}
\]

where \( Y_t^{\epsilon,x} \) is given in (2.5). Now, by the triangle inequality we have

\[
\begin{align*}
d_{\alpha,t}^\epsilon(x) &\leq d_{TV}(Y_t^{\epsilon,x}, Y_t(\text{sgn}(x)\infty)) + d_{TV}(Y_t(\text{sgn}(x)\infty), Y_\infty^\epsilon) + d_{TV}(Y_\infty^\epsilon, Y_\infty^\epsilon),
\end{align*}
\]

where \( Y_t(\text{sgn}(x)\infty) := \lim_{t \to \infty} Y_t(\text{sgn}(x)r) \) and \( (Y_t(\text{sgn}(x)r), t \geq 0) \) is defined in (2.2). We stress that \( Y_t(\text{sgn}(x)\infty) \) is well-defined as we show in Proposition 2.1 below. Informally, the idea is that the drift dominates the noise and is strong enough to ensure that the process comes down from infinity. The triangle inequality also implies that

\[
\begin{align*}
d_{TV}(Y_t(\text{sgn}(x)\infty), Y_\infty^\epsilon) &\leq d_{TV}(Y_t(\text{sgn}(x)\infty), Y_t^{\epsilon,x}) + d_{\alpha,t}^\epsilon(x) + d_{TV}(Y_\infty^\epsilon, Y_\infty^\epsilon).
\end{align*}
\]

Combining (2.13) and (2.14) we obtain the following key estimate that we state as a lemma.

**Lemma 2.3 (Decoupling inequality).** Assume Hypothesis 1, Hypothesis 2 and Hypothesis 3A hold true. Then for any \( x \in \mathbb{R} \), \( \epsilon > 0 \) and \( t \geq 0 \) it follows that

\[
\begin{align*}
|d_{\alpha,t}^\epsilon(x) - d_{TV}(Y_t(\text{sgn}(x)\infty), Y_\infty^\epsilon)| &\leq d_{TV}(Y_t^{\epsilon,x}, Y_t(\text{sgn}(x)\infty)) + d_{TV}(Y_\infty^\epsilon, Y_\infty^\epsilon).
\end{align*}
\]

To complete the proof of (1.3) it suffices to show that the right-hand side of (2.15) tends to zero as \( \epsilon \to 0 \) and that for all \( t > 0 \) and \( x \in \mathbb{R} \)

\[
0 < d_{TV}(Y_t(\text{sgn}(x)\infty), Y_\infty^\epsilon) < 1.
\]

The proof of (2.16) is given in Subsection 3.5. The following proposition states that the right-hand side of (2.15) tends to zero as \( \epsilon \to 0 \).
Proposition 2.1. Assume Hypothesis 1, Hypothesis 2 and Hypothesis 3A hold true. The real valued process defined by (2.2) admits a continuous Markovian extension to $\mathbb{R} := \mathbb{R} \cup \{ \pm \infty \}$.

Furthermore, for all $x \in \mathbb{R}$ and $t > 0$ it follows that

\begin{equation}
\lim_{\varepsilon \to 0} d_{TV}(Y_t(\text{sgn}(x) \infty), Y^{\varepsilon,x}_t) = 0,
\end{equation}

where $(Y^{\varepsilon,x}_t, t \geq 0)$ is defined in (2.5). In addition, the following limit holds

\begin{equation}
\lim_{\varepsilon \to 0} d_{TV}(Y_{\infty}, Y^{\varepsilon}_{\infty}) = 0,
\end{equation}

where $Y_{\infty}$ and $Y^{\varepsilon}_{\infty}$ denote the unique invariant distributions of the dynamics given by (2.2) and (2.6), respectively.

The proof of Proposition 2.1 is given in Subsection 3.4.

Now, we are ready to prove Theorem 1.1, which is a consequence of what we have already stated up to here.

Proof of Theorem 1.1. Inequality (2.15) with the help of (2.16), (2.17) and (2.18) implies

\begin{equation}
\lim_{\varepsilon \to 0} d_{TV}(\mu_{\varepsilon}, \mu_{\infty}) = 0.
\end{equation}

This finishes the proof. \[\square\]

3. Proofs: details

In this section, we give the proof Corollary 1.1, Corollary 1.2 and complete the proofs of the statements in Section 2. To be more precise, the proof of Corollary 1.1 is given in Subsection 3.1, the proof of Lemma 2.2 is given in Subsection 3.3, the proof of Proposition 2.1 is given in Subsection 3.4 and the proof of inequality (2.16) in Subsection 3.5.

3.1. Proof of Corollary 1.1. By the Chapman–Kolmogorov equation, for any $x \in \mathbb{R}$ and $\varepsilon \in (0, 1]$ it follows that the map

\begin{equation}
t \mapsto d^\varepsilon_t(x) := d_{TV}(X^\varepsilon_t(x), \mu^\varepsilon)
\end{equation}

is non-increasing.

The following lemma provides a rather general technique to prove no cutoff. In rough terms, it states that if there exists a non-trivial behavior for a suitable scale, then there is no cutoff for any scale.

Lemma 3.1 (Scaling procedure). Let $x \in \mathbb{R}$ and assume that there is a sequence $(a_\varepsilon(x))_{\varepsilon \in (0, 1]}$ for which the following conditions hold true

i) $\lim_{\varepsilon \to 0} a_\varepsilon(x) = \infty$.

ii) For any $t > 0$

\begin{equation}
0 < \liminf_{\varepsilon \to 0} d^\varepsilon_{a_\varepsilon(x)t}(x) \leq \limsup_{\varepsilon \to 0} d^\varepsilon_{a_\varepsilon(x)t}(x) < 1.
\end{equation}

Then there is no cutoff for the family $(X^\varepsilon_t(x))_{\varepsilon \in (0, 1]}$ in total variation distance as $\varepsilon$ tends to zero.

Proof of Corollary 1.1. By Theorem 1.1 and Lemma 3.1, we obtain Corollary 1.1. Indeed, (1.2) and (1.3) in Theorem 1.1 imply i) and ii) of Lemma 3.1, respectively. \[\square\]
Proof of Lemma 3.1. Let \((\bar{a}_\varepsilon(x))_{\varepsilon \in (0,1]}\) be such that \(\bar{a}_\varepsilon(x) \to \infty\), as \(\varepsilon \to 0\). First, we assume that

\[
(3.2) \quad \limsup_{\varepsilon \to 0} \frac{\bar{a}_\varepsilon(x)}{a_\varepsilon(x)} < \infty,
\]

that is, there are constants \(C_1(x) > 0\) and \(\varepsilon_0(x) \in (0,1]\) such that \(\bar{a}_\varepsilon(x) \leq C_1(x)a_\varepsilon(x)\) for any \(\varepsilon \in (0,\varepsilon_0]\). By (3.1), for any \(\delta > 0\) and \(\varepsilon \in (0,\varepsilon_0]\) we have \(d_\text{TV}^{\varepsilon}_{C_1(x)}(a_\varepsilon(x)) \leq d_\text{TV}^{\varepsilon}_{\delta a_\varepsilon(x)}(x)\). In particular, for \(\delta > 1\) by ii) we have

\[
0 < \liminf_{\varepsilon \to 0} d_\text{TV}^{\varepsilon}_{\delta C_1(x)}(a_\varepsilon(x)) \leq \liminf_{\varepsilon \to 0} d_\text{TV}^{\varepsilon}_0(x).
\]

Hence, by (1.5) there is no cutoff at the scale \((\bar{a}_\varepsilon(x))_{\varepsilon \in (0,1]}\) for the family \((X^{\varepsilon}(x))_{\varepsilon \in (0,1]}\).

If (3.2) fails, then there exists a sequence \((\varepsilon_k, k \in \mathbb{N})\) such that \(\varepsilon_k \to 0\) as \(k \to \infty\) and

\[
\limsup_{k \to \infty} \frac{\bar{a}_{\varepsilon_k}(x)}{a_{\varepsilon_k}(x)} = \infty.
\]

In particular, there exists \(k_0 \in \mathbb{N}\) such that \(a_{\varepsilon_k}(x) \leq \bar{a}_{\varepsilon_k}(x)\) for all \(k \geq k_0\). By (3.1) and ii), for \(0 < \delta < 1\) we have

\[
\limsup_{k \to \infty} d_\text{TV}^{\varepsilon}_{\delta a_{\varepsilon_k}(x)}(x) \leq \limsup_{k \to \infty} d_\text{TV}^{\varepsilon}_{\delta \bar{a}_{\varepsilon_k}(x)}(x) \leq \limsup_{\varepsilon \to 0} d_\text{TV}^{\varepsilon}_0(x) < 1.
\]

Hence, by (1.5) there is no cutoff at the scale \((\bar{a}_\varepsilon(x))_{\varepsilon \in (0,1]}\) for the family \((X^{\varepsilon}(x))_{\varepsilon \in (0,1]}\). Since \((\bar{a}_\varepsilon(x))_{\varepsilon \in (0,1]}\) is any function that \(\bar{a}_\varepsilon(x) \to \infty\), as \(\varepsilon \to 0\), the proof is complete. \(\square\)

3.2. Proof of Corollary 1.2. For each \(x \in \mathbb{R}\) and \(t \geq 0\) we define

\[
(3.3) \quad G_x(t) := d_\text{TV}(Y_t(\text{sgn}(x)\infty), \nu).
\]

In addition, for each \(\eta \in (0,1)\) we set \(H_x(\eta) := \inf\{t \geq 0 : G_x(t) \leq \eta\}\). To conclude the proof we show that

\[
\limsup_{\varepsilon \to 0} \frac{\tau_{\text{mix}}^{\varepsilon,x}(\eta)}{a_\varepsilon} \leq H_x(\eta) \quad \text{and} \quad \liminf_{\varepsilon \to 0} \frac{\tau_{\text{mix}}^{\varepsilon,x}(\eta)}{a_\varepsilon} \geq H_x(\eta).
\]

We first remark that the function \(t \mapsto G_x(t)\) is continuous and strictly decreasing, see Lemma C.2. To prove the upper bound, let \(\gamma^* \in (0, \eta)\) be fixed and let \(t^* := t^*(\eta - \gamma^*, x) > 0\) be such that \(G_x(t^*) = \eta - \gamma^*\), and \(G_x(t) > \eta - \gamma^*\) for all \(t < t^*\). By Theorem 1.1 we have the existence of \(\varepsilon^* := \varepsilon^*(\eta, \gamma^*, x) > 0\) such that

\[
-\gamma^* < d_\text{TV} \left( X^{\varepsilon}_{t^*a_\varepsilon}(x), \mu^\varepsilon \right) - G_x(t^*) < \gamma^* \quad \text{for all} \ \varepsilon \in (0, \varepsilon^*).
\]

In particular, \(d_\text{TV} \left( X^{\varepsilon}_{t^*a_\varepsilon}(x), \mu^\varepsilon \right) \leq \eta\) for all \(\varepsilon \in (0, \varepsilon^*)\). Therefore, \(\tau_{\text{mix}}^{\varepsilon,x}(\eta) \leq t^* a_\varepsilon\) for all \(\varepsilon \in (0, \varepsilon^*)\), which implies

\[
\limsup_{\varepsilon \to 0} \frac{\tau_{\text{mix}}^{\varepsilon,x}(\eta)}{a_\varepsilon} \leq t^* = t^*(\eta - \gamma^*, x).
\]

Sending \(\gamma^* \to 0\) we obtain

\[
(3.4) \quad \limsup_{\varepsilon \to 0} \frac{\tau_{\text{mix}}^{\varepsilon,x}(\eta)}{a_\varepsilon} \leq t^*(\eta, x) = H_x(\eta).
\]

To prove the lower bound, let \(\gamma_* \in (0, 1 - \eta)\) be fixed and let \(t_* = t_*(\eta + \gamma_*, x) > 0\) be such that \(G_x(t_*) = \eta + \gamma_*\) and \(G_x(t) > \eta + \gamma_*\) for all \(t < t_*\). By Theorem 1.1 there is \(\varepsilon_* := \varepsilon_*(\eta, \gamma_*, x) > 0\) such that

\[
d_\text{TV} \left( X^{\varepsilon}_{t_*a_\varepsilon}(x), \mu^\varepsilon \right) > \eta \quad \text{for all} \ \varepsilon \in (0, \varepsilon_*).
\]
Therefore, \( \tau_{\text{mix}}^{\varepsilon,x}(\eta) \geq t_\ast a_\varepsilon \) for all \( \varepsilon \in (0, \varepsilon_\ast) \), which implies
\[
\liminf_{\varepsilon \to 0} \tau_{\text{mix}}^{\varepsilon,x}(\eta) a_\varepsilon \geq t_\ast = t_\ast(\eta + \gamma_\ast, x).
\]
Sending \( \gamma_\ast \to 0 \) we obtain
\[
(3.5) \quad \liminf_{\varepsilon \to 0} \frac{\tau_{\text{mix}}^{\varepsilon,x}(\eta)}{a_\varepsilon} \geq t_\ast(\eta, x) = H_x(\eta).
\]
By (3.4) and (3.5) we deduce (1.6). This finishes the proof. \( \Box \)

3.3. **Proof of Lemma 2.2.** By the triangle inequality we have
\[
(3.6) \quad d_{a_{it}}^\varepsilon(x) \leq d_{\text{TV}}(X_{a_{it}}^\varepsilon(x), \tilde{X}_{a_{it}}^\varepsilon(x)) + d_{a_{it}}^\varepsilon(x) + d_{\text{TV}}(\tilde{\mu}^\varepsilon, \mu^\varepsilon).
\]
Similarly,
\[
(3.7) \quad \tilde{d}_{a_{it}}^\varepsilon(x) \leq d_{\text{TV}}(\tilde{X}_{a_{it}}^\varepsilon(x), X_{a_{it}}^\varepsilon(x)) + d_{a_{it}}^\varepsilon(x) + d_{\text{TV}}(\mu^\varepsilon, \tilde{\mu}^\varepsilon).
\]
By (3.6) and (3.7) we obtain
\[
(3.8) \quad |d_{a_{it}}^\varepsilon(x) - \tilde{d}_{a_{it}}^\varepsilon(x)| \leq d_{\text{TV}}(X_{a_{it}}^\varepsilon(x), \tilde{X}_{a_{it}}^\varepsilon(x)) + d_{\text{TV}}(\mu^\varepsilon, \tilde{\mu}^\varepsilon) \quad \text{for all} \quad t \geq 0.
\]
By Lemma 3.2 and Lemma 3.3 below, we deduce that the right-hand side of (3.8) tends to zero as \( \varepsilon \to 0 \) and thereby conclude the proof of Lemma 2.2.

Recall the definition of \( \tilde{V} \) given in (2.9). In particular, note that \( L = L_\varepsilon \) is chosen such that \( L^2 \geq |x|^2 + 1 \).

**Lemma 3.2 (Convergence of rescaled process around origin).** For any \( x \in \mathbb{R} \) and \( t \geq 0 \) the following limit holds
\[
\lim_{\varepsilon \to 0} d_{\text{TV}}(X_{a_{it}}^\varepsilon(x), \tilde{X}_{a_{it}}^\varepsilon(x)) = 0,
\]
where \( (a_\varepsilon, \varepsilon \in [0, 1)) \) is defined in (1.2).

**Proof.** The proof follows the steps given in the proof of Proposition 4.1, item (ii), of [12]. Let \( \varepsilon \in (0, 1] \) be fixed. The variational formulation of the total variation distance yields
\[
d_{\text{TV}}(X_{a_{it}}^\varepsilon(x), \tilde{X}_{a_{it}}^\varepsilon(x)) \leq \mathbb{Q}(X_{a_{it}}^\varepsilon(x) \neq \tilde{X}_{a_{it}}^\varepsilon(x))
\]
for any coupling \( \mathbb{Q} \). Moreover, as \( |x| < L \) for the synchronous coupling \( \mathbb{P} \) (where processes are driven by the same noise), we have \( \tilde{X}_{s}^\varepsilon(x) = X_{s}^\varepsilon(x) \) for \( 0 \leq s < \tilde{\tau}^\varepsilon(x) \), where
\[
\tilde{\tau}^\varepsilon(x) := \inf\{s \geq 0 : |\tilde{X}^\varepsilon_s(x)| > L\}.
\]
Therefore,
\[
(3.9) \quad d_{\text{TV}}(X_{a_{it}}^\varepsilon(x), \tilde{X}_{a_{it}}^\varepsilon(x)) \leq \mathbb{P}(\tilde{\tau}^\varepsilon(x) \leq a_\varepsilon t) \quad \text{for any} \quad t \geq 0.
\]
Note that
\[
(3.10) \quad \mathbb{P}(\tilde{\tau}^\varepsilon(x) \geq a_\varepsilon t) = \mathbb{P}\left(\sup_{0 \leq s \leq a_\varepsilon t} |\tilde{X}^\varepsilon_s(x)| \leq L\right).
\]
Since \( \tilde{V} \) is convex, Itō’s formula yields \( \mathbb{P} \)-almost surely that
\[
(3.11) \quad |\tilde{X}^\varepsilon_t(x)|^2 \leq |x|^2 + \varepsilon t + \tilde{M}^\varepsilon_t(x) \quad \text{for all} \quad t \geq 0,
\]
where \( \tilde{M}^\varepsilon_t(x) := 2\sqrt{\varepsilon} \int_0^t \tilde{X}^\varepsilon_s(x) dB_s, \ t \geq 0 \). By a localization procedure, it follows that
\[
(3.12) \quad \mathbb{E}|\tilde{X}^\varepsilon_t(x)|^2 \leq |x|^2 + \varepsilon t \quad \text{for all} \quad t \geq 0
\]
and hence \((\tilde{M}_t^\epsilon(x), t \geq 0)\) is a true martingale. By (2.8) we have \(\epsilon a = \epsilon^{\frac{\alpha}{2}}\), which tends to zero as \(\epsilon \to 0\). Then for any \(t > 0\) fixed there exists \(\epsilon_0 = \epsilon_0(t, \alpha) > 0\) such that \(1 - \epsilon a t > 1/2\) for all \(\epsilon \in (0, \epsilon_0)\). By (3.11) for any \(\epsilon \in (0, \epsilon_0)\) we have

\[
\mathbb{P} \left( \sup_{0 \leq s \leq a t} \left| \tilde{X}_s^\epsilon(x) \right| \geq L \right) = \mathbb{P} \left( \sup_{0 \leq s \leq a t} \left| \tilde{M}_s^\epsilon(x) \right|^2 \geq L^2 \right) \\
\leq \mathbb{P} \left( \sup_{0 \leq s \leq a t} \left| \tilde{M}_s^\epsilon(x) \right| \geq (L^2 - |x|^2 - \epsilon a t) \right) \\
\leq \mathbb{P} \left( \sup_{0 \leq s \leq a t} \left| \tilde{M}_s^\epsilon(x) \right| \geq 1/2 \right) = \mathbb{P} \left( \sup_{0 \leq s \leq a t} \left| \tilde{M}_s^\epsilon(x) \right|^2 \geq 1/4 \right),
\]

where for the last inequality we used that \(|x|^2 + 1 < L^2\) and \(1 - \epsilon a t > 1/2\). Now, by Doob's submartingale inequality, Itô's isometry and (3.12) we have

\[
\mathbb{P} \left( \sup_{0 \leq s \leq a t} \left| \tilde{X}_s^\epsilon(x) \right| \geq L \right) \leq \mathbb{P} \left( \sup_{0 \leq s \leq a t} \left| \tilde{M}_s^\epsilon(x) \right|^2 \geq 1/4 \right) \leq 4 \mathbb{E}[|\tilde{M}_a^\epsilon(x)|^2] \\
= 16\epsilon \int_0^{a t} \mathbb{E}[\tilde{X}_s^\epsilon(x)]^2 \, ds \leq 16|x|^2 \epsilon a t + 8\epsilon^2 a^2 t^2
\]
for all \(\epsilon \in (0, \epsilon_0)\). By (3.9), (3.10) and (3.13) we deduce

\[
d_{TV}(X_{a t}^\epsilon(x), \tilde{X}_{a t}^\epsilon(x)) \leq 16|x|^2 \epsilon a t + 8\epsilon^2 a^2 t^2
\]
for any \(t \geq 0\), which implies the statement. \(\square\)

Recall the notation introduced above (2.12), that is, \(Y_\infty \overset{d}{=} \nu\), \(X_\infty \overset{d}{=} \mu^\epsilon\) and \(\tilde{X}_\infty \overset{d}{=} \tilde{\mu}^\epsilon\), where \(\overset{d}{=}\) denotes equality in the distribution sense. By Lemma 2.1 it follows that

\[
\nu(dz) = C^{-1} \exp(-2V_0(z)) \, dz,
\]
where \(C\) is a normalization constant, \(V_0(z) \overset{\text{def}}{=} (2 + \alpha)^{-1} C_0 |z|^{2 + \alpha}\) with \(\alpha\) and \(C_0\) defined in Hypothesis 2. Similarly, \(\mu^\epsilon\) and \(\tilde{\mu}^\epsilon\) are the densities of \(X_\infty^\epsilon\) and \(\tilde{X}_\infty^\epsilon\), and they are given by

\[
\mu^\epsilon(dz) = C^{-1}_\epsilon \exp \left(-\frac{2V(z)}{\epsilon} \right) \, dz \quad \text{and} \quad \tilde{\mu}^\epsilon(dz) = \tilde{C}^{-1}_\epsilon \exp \left(-\frac{\tilde{V}(z)}{\epsilon} \right) \, dz,
\]
respectively. By the change of variable theorem, the densities of \(\frac{X_\infty^\epsilon}{b_\epsilon}\) and \(\frac{\tilde{X}_\infty^\epsilon}{b_\epsilon}\) are given by

\[
b_\epsilon C^{-1}_\epsilon \exp \left(-\frac{2V(b_\epsilon z)}{\epsilon} \right) \, dz \quad \text{and} \quad b_\epsilon \tilde{C}^{-1}_\epsilon \exp \left(-\frac{\tilde{V}(b_\epsilon z)}{\epsilon} \right) \, dz,
\]
respectively, where \((b_\epsilon, \epsilon \in [0, 1])\) is defined in (2.8).

**Lemma 3.3** (Asymptotic coupling of the invariant measures). It follows that

\[
\lim_{\epsilon \to 0} d_{TV} \left( \frac{X_\infty^\epsilon}{b_\epsilon}, Y_\infty \right) = 0 \quad \text{and} \quad \lim_{\epsilon \to 0} d_{TV} \left( \frac{\tilde{X}_\infty^\epsilon}{b_\epsilon}, Y_\infty \right) = 0,
\]
where \((b_\epsilon, \epsilon \in [0, 1])\) is defined in (2.8). In particular,

\[
\lim_{\epsilon \to 0} d_{TV}(X_\infty^\epsilon, \tilde{X}_\infty^\epsilon) = 0.
\]
By (3.14), (3.15) and Scheffé’s lemma ([83, Lemma 3.3.2, p.95]), to conclude the proof of (3.16), it suffices to show

\[
\lim_{\varepsilon \to 0} \frac{b_\varepsilon}{C_\varepsilon} e^{-2V(b_\varepsilon z)} = \frac{1}{C} e^{-2V_0(z)} \quad \text{for any} \quad z \in \mathbb{R}
\]

and

\[
\lim_{\varepsilon \to 0} \frac{b_\varepsilon}{C_\varepsilon} e^{-2\tilde{V}(b_\varepsilon z)} = \frac{1}{C} e^{-2V_0(z)} \quad \text{for any} \quad z \in \mathbb{R}.
\]

The proofs of (3.17) and (3.18) are consequences of Lemma 3.4 and Lemma 3.5 below.

**Lemma 3.4 (Uniform convergence).** For any \( K > 0 \) it follows that

\[
\lim_{\varepsilon \to 0} \sup_{|z| \leq K} \left| \frac{V(b_\varepsilon z)}{\varepsilon} - V_0(z) \right| = 0
\]

and

\[
\lim_{\varepsilon \to 0} \sup_{|z| \leq K} \left| \frac{\tilde{V}(b_\varepsilon z)}{\varepsilon} - V_0(z) \right| = 0.
\]

where \( V_0(z) = (2 + \alpha)^{-1}C_0|z|^{2+\alpha} \) for any \( z \in \mathbb{R} \) with \( \alpha \) and \( C_0 \) defined in Hypothesis 2.

**Proof.** In the sequel, we show (3.19). Let \( K > 0 \) and \( \eta > 0 \) be fixed and define \( \tilde{\eta} := \eta K^{-1} > 0 \). By (2.8), \( b_\varepsilon \to 0 \), as \( \varepsilon \to 0 \). Now, by Hypothesis 2 there exists \( \varepsilon_0 = \varepsilon_0(K, \tilde{\eta}) > 0 \) such that for any \( |z| \leq K \) and \( \varepsilon < \varepsilon_0 \),

\[
C_0|z|^{1+\alpha}\text{sgn}(z) - \tilde{\eta} < \frac{V'(b_\varepsilon z)}{b_\varepsilon^{1+\alpha}} < C_0|z|^{1+\alpha}\text{sgn}(z) + \tilde{\eta}.
\]

If we integrate each term from 0 to \( x \) in the above inequality, use Hypothesis 1 and note that, by (2.8), \( b_\varepsilon^{2+\alpha} = \varepsilon \) we obtain that for any \( |z| \leq K \) and \( \varepsilon < \varepsilon_0 \)

\[
\sup_{|z| \leq K} \left| \frac{V(b_\varepsilon z)}{\varepsilon} - V_0(z) \right| \leq \tilde{\eta}K = \eta.
\]

Since \( \eta > 0 \) is arbitrary, the proof of (3.19) is complete.

By construction, we stress that \( \tilde{V} \) also satisfies Hypothesis 1 and Hypothesis 2. Hence the proof of (3.20) is analogous. \( \square \)

**Lemma 3.5 (Normalizing constants).** It follows that

\[
\lim_{\varepsilon \to 0} b_\varepsilon C_\varepsilon^{-1} = C
\]

and

\[
\lim_{\varepsilon \to 0} b_\varepsilon \tilde{C}_\varepsilon^{-1} = C.
\]

**Proof.** By the change of variables \( z \mapsto b_\varepsilon z \) we obtain that \( \frac{C}{b_\varepsilon} = \int_{\mathbb{R}} e^{-\frac{2}{\varepsilon}V(b_\varepsilon z)} \, dz \). Now, by Lemma 3.4 and Fatou’s lemma we have

\[
C = \int_{\mathbb{R}} e^{-2V_0(z)} \, dz \leq \liminf_{\varepsilon \to 0} \int_{\mathbb{R}} e^{-\frac{2}{\varepsilon}V(b_\varepsilon z)} \, dz = \liminf_{\varepsilon \to 0} \frac{C_\varepsilon}{b_\varepsilon}.
\]
Similarly, for \( \bar{V} \) we obtain

\[
(3.23) \quad C \leq \liminf_{\varepsilon \to 0} \frac{C_{\varepsilon}}{b_{\varepsilon}}.
\]

We next show that \( \limsup_{\varepsilon \to 0} \frac{C_{\varepsilon}}{b_{\varepsilon}} \leq C \). Note first that

\[
\limsup_{\varepsilon \to 0} \frac{C_{\varepsilon}}{b_{\varepsilon}} \leq \lim_{K \to \infty} \limsup_{\varepsilon \to 0} \int_{|z| \leq K} e^{-\frac{2}{4} V(b_{\varepsilon} z)} dz + \lim_{K \to \infty} \limsup_{\varepsilon \to 0} \int_{|z| > K} e^{-\frac{2}{4} V(b_{\varepsilon} z)} dz.
\]

By Lemma 3.4, the dominated convergence theorem, and the monotone convergence theorem we obtain that

\[
\lim_{K \to \infty} \limsup_{\varepsilon \to 0} \int_{|z| \leq K} e^{-\frac{2}{4} V(b_{\varepsilon} z)} dz = \lim_{K \to \infty} \int_{|z| \leq K} e^{-2V(0)z} dz = C.
\]

Similarly, for \( \bar{V} \) we obtain

\[
(3.24) \quad \lim_{K \to \infty} \limsup_{\varepsilon \to 0} \int_{|z| \leq K} e^{-\frac{2}{4} \bar{V}(b_{\varepsilon} z)} dz = C.
\]

It remains to show that

\[
(3.25) \quad \lim_{K \to \infty} \limsup_{\varepsilon \to 0} \int_{|z| > K} e^{-\frac{2}{4} V(b_{\varepsilon} z)} dz = 0.
\]

and

\[
(3.26) \quad \lim_{K \to \infty} \limsup_{\varepsilon \to 0} \int_{|z| > K} e^{-\frac{2}{4} \bar{V}(b_{\varepsilon} z)} dz = 0.
\]

In the sequel, we give the proof of (3.25), which we divide in two cases, depending on whether \( \beta \geq \alpha \) or \( \beta < \alpha \), where \( \alpha \) and \( \beta \) are defined in Hypothesis 2 and Hypothesis 3, respectively. Note first that by Hypothesis 2 for any \( \delta > 0 \) there is \( c_0(\delta) > 0 \) such that for any \( z \) with \( |z| \leq \delta \)

\[
(3.27) \quad |V'(z)| \geq c_0(\delta) |z|^{1+\alpha}.
\]

Assume that \( \beta \geq \alpha \). By Hypothesis 1 and Hypothesis 3 there is an \( R > 1 \) and \( c > 0 \) such that for any \( z \) with \( |z| \geq R_0 \),

\[
(3.28) \quad |V'(z)| \geq c |z|^{1+\beta} \geq c |z|^{1+\alpha}.
\]

By (3.27) and (3.28) there is a \( c_1(\delta) > 0 \) such that for any \( z \in \mathbb{R} \), \( |V'(z)| \geq c_1(\delta) |z|^{1+\alpha} \). Since \( V'(z) = V'(|z|) \text{sgn}(z) \) and \( V(0) = 0 \), if we compute the integral from 0 to \( z \) of both sides of (3.28) we obtain that there is a \( c(\delta) > 0 \) such that \( V(z) \geq c(\delta) |z|^{2+\alpha} \) for any \( z \in \mathbb{R} \). The preceding inequality implies that \( -V(b_{\varepsilon} z) \leq -c(\delta) b_{\varepsilon}^{2+\alpha} |z|^{2+\alpha} \) which together with \( b_{\varepsilon}^{2+\alpha} = \varepsilon \) yields that

\[
(3.29) \quad \lim_{K \to \infty} \limsup_{\varepsilon \to 0} \int_{|z| > K} e^{-\frac{2}{4} V(b_{\varepsilon} z)} dz = \lim_{K \to \infty} \limsup_{\varepsilon \to 0} \int_{|z| > K} e^{-2c(\delta)|z|^{2+\alpha}} dz = 0.
\]

This completes the case \( \beta \geq \alpha \).

Now, we assume that \( -1 < \beta < \alpha \). Since \( V \) satisfies Hypothesis 3 we may now take \( R_0 \geq [(2+\alpha)(2+\beta)^{-1}]^{-\alpha} \) and let \( \kappa_0(\delta) := \min \{ V'(z) |z|^{-1-\alpha} ; \delta \leq z \leq R_0 \} \), where \( \delta > 0 \) is given in (3.27). Note that \( \kappa_0(\delta) > 0 \) and that \( k_1(\delta) := \min \{ c_0(\delta), \kappa_0(\delta) \} > 0 \) is such that \( V'(z) \geq k_1(\delta) z^{1+\alpha} \) for
$z \in [0, R_0]$. Now, by Hypothesis \(3\) there is $c > 0$ such that $V'(z) \geq cz^{1+\beta}$ for any $z \geq R_0$ and therefore, for $\tilde{c} = \tilde{c}(\delta) := \min\{k_1(\delta), c\} > 0$

$$V'(z) \geq \begin{cases} \tilde{c} z^{1+\alpha} & \text{for } z \in [0, R_0], \\ \tilde{c} z^{1+\beta} & \text{for } z > R_0. \end{cases}$$

As $V(0) = 0$, integrating from 0 to $z$ in the both sides of the above inequality we obtain

$$V(z) \geq \begin{cases} \tilde{c} z^{2+\alpha}(2+\alpha)^{-1} & \text{for } z \in [0, R_0], \\ \tilde{c} \frac{R_0^{2+\alpha}}{2+\alpha} - \frac{R_0^{2+\beta}}{2+\beta} & \text{for } z > R_0. \end{cases}$$

Since $R_0 \geq [(2+\alpha)(2+\beta)^{-1}]^{\frac{1}{1-\alpha}}$, it follows that $\frac{R_0^{2+\alpha}}{2+\alpha} - \frac{R_0^{2+\beta}}{2+\beta}$ $\geq 0$, and because $V$ is an even function we deduce the existence of $\kappa = \kappa(\delta) > 0$ such that

$$V(z) \geq \begin{cases} \kappa |z|^{2+\alpha} & \text{for } |z| \leq R_0, \\ \kappa |z|^{2+\beta} & \text{for } |z| \geq R_0. \end{cases}$$

Since $b^{2+\alpha}_\varepsilon = \varepsilon$ and $b^{2+\beta/\varepsilon}_\varepsilon = b^{2-\beta-\alpha}_\varepsilon$ $\rightarrow \infty$ as $\varepsilon \to 0$, the dominated convergence theorem yields

$$\lim_{K \to \infty} \limsup_{\varepsilon \to 0} \int_{|z|>K} e^{-\frac{2}{\varepsilon} V(b_\varepsilon z)} \mathrm{d}z = \lim_{K \to \infty} \limsup_{\varepsilon \to 0} \int_{|z|>K} e^{-\frac{2}{\varepsilon} V(b_\varepsilon z)} (\mathbbm{1}_{|b_\varepsilon z| \leq R_0} + \mathbbm{1}_{|b_\varepsilon z| > R_0}) \mathrm{d}z
\leq \lim_{K \to \infty} \limsup_{\varepsilon \to 0} \left( \int_{|z|>K} e^{-2\kappa |z|^{2+\alpha}} \mathrm{d}z + \int_{|z|>K} e^{-2\kappa b^{2-\beta-\alpha}_\varepsilon |z|^{2+\beta}} \mathrm{d}z \right)
= \lim_{K \to \infty} \int_{|z|>K} e^{-2\kappa |z|^{2+\alpha}} \mathrm{d}z = 0.$$

This completes the case $-1 < \beta < \alpha$.

Combining (3.29) and (3.30), we obtain (3.28). This finishes the proof of (3.21).

In the sequel, we stress that (3.22) is just a consequence from above case $\beta \geq \alpha$. Indeed, by (3.23) and (3.24), it is enough to show (3.26). Since $\tilde{V}$ satisfies Hypothesis $3$ with $\beta = \alpha$, the proof is already covered in (3.29).

The proof of Lemma 3.5 is finished. 

\[3.4. \text{Proof of Proposition 2.1}\] The proof of Proposition 2.1 is divided in three parts, one for each claim of the proposition.

\[3.4.1. \text{The continuous Markovian extension.}\] The continuous Markovian extension of the SDE (2.2) is done in three steps. Their proofs are given in detail in Appendix A and here we only outline the main steps. First, based on a monotonic coupling and uniform moment bounds for $x \in \mathbb{R}$, equation (2.2) can be extended to $\mathbb{R}$, see Section A.1. Second, because $\pm \infty$ are exit boundaries for the dynamics in $\mathbb{R}$, the extended family $(Y(x), x \in \mathbb{R})$ is Markovian, see Section A.2. Finally, in Section A.3 we show that the extension is continuous in the sense that

$$\lim_{x \to \pm \infty} \mathrm{d}_{TV} (Y_t(x), Y_t(\pm \infty)) = 0.$$
3.4.2. Convergence in total variation for fixed marginal. In this section we show the limit \((2.17)\). For simplicity, we consider only the case when the initial condition \(x\) in \((1.1)\) is positive, the case when \(x\) is negative can be treated by an analogous argument, while the case \(x = 0\) is easier as no scaling of the initial condition is required and \((2.17)\) follows from the uniform convergence of the velocity fields, see \((3.34)\) below. To ease notation and clarify the limit procedures, we denote by \(F_0, F_\varepsilon\) the velocity fields of \((2.2)\) and \((2.6)\), respectively. That is, for any \(\varepsilon \in [0, 1]\) and \(z \in \mathbb{R}\) we define

\[
F_0(z) := -C_0|z|^{1+\alpha} \text{sgn}(z) \quad \text{and} \quad F_\varepsilon(z) := -\frac{a_\varepsilon}{b_\varepsilon} V'(b_\varepsilon z) = -\frac{V'(b_\varepsilon z)}{b_\varepsilon^{1+\alpha}}.
\]

To ease notation, denote by \(Y^\varepsilon_0(x)\) the solution of \((2.2)\). With this, \((Y^\varepsilon(x), \varepsilon \in [0, 1])\) solves

\[
\begin{cases}
\text{d}Y_t = F_\varepsilon(Y_t) \text{d}t + \text{d}B_t & \text{for} \quad t \geq 0, \\
Y_0 = x.
\end{cases}
\]

In what follows, we consider uniform bounds for \(Y^\varepsilon(x)\) with \(\varepsilon \in [0, 1]\) and we will take the limit of such processes as \(\varepsilon \to 0\). First, since \(b_\varepsilon \to 0\) as \(\varepsilon \to 0\), Hypothesis \([2]\) yields for all \(K > 0\)

\[
\lim_{\varepsilon \to 0} \sup_{|z| \leq K} |F_\varepsilon(z) - F_0(z)| = 0.
\]

Also, by Proposition \([A.1]\) it follows that, almost surely, for any \(\varepsilon \in [0, 1]\), the limit

\[
Y^\varepsilon_t(\infty) := \lim_{x \to \infty} Y^\varepsilon_t(x)
\]

exists and is finite for \(t > 0\). Next, by Lemma \([A.4]\) for all \(t > 0\) and \(\varepsilon \in [0, 1]\),

\[
\lim_{x \to \infty} \text{d}_{TV}(Y^\varepsilon_t(x), Y^\varepsilon_t(\infty)) = 0.
\]

Now, we fix \(\eta > 0\). By the uniform behavior at infinity, see Proposition \([B.1]\) it follows that for any \(a > 0\), there are \(b > 0\) and \(\delta \in (0, t)\) such that

\[
\sup_{\varepsilon \in [0, 1]} \mathbb{P}(Y^\varepsilon_\delta(\infty) \notin [a, b]) \leq \eta/8.
\]

By \((3.35)\), we may choose \(a > 0\) large enough so that

\[
\sup_{x \geq a} \text{d}_{TV}(Y^0_t(x), Y^0_\infty) \leq \eta/4.
\]

Now, given \(a, b\) and \(\delta \in (0, t)\) we claim that there is \(\varepsilon_0 = \varepsilon(\eta) > 0\) for which

\[
\sup_{0 \leq \varepsilon \leq \varepsilon_0} \text{d}_{TV}(Y^\varepsilon_{t-\delta}(x), Y^\varepsilon_{t-\delta}(x)) \leq \eta/4.
\]

The proof of \((3.38)\) is given in Appendix \([B.2]\). Now, let \(x_\varepsilon := xb_\varepsilon^{-1}\) and define \(\mu^\varepsilon_\delta\) to be the synchronous coupling (both SDEs are driven with the same noise) of \(Y^0_\delta(\infty)\) and \(Y^\varepsilon_\delta(x_\varepsilon)\). We write \(\mu^\varepsilon_\delta(A, B) := \mathbb{P}(Y^0_\delta(\infty) \in A, Y^\varepsilon_\delta(x_\varepsilon) \in B)\) for any \(A, B\) Borelian subsets of \(\mathbb{R}\). We may choose \(a > 0\) for which \((3.37)\) holds, then we choose \(b > a\) and \(\delta \in (0, t)\) such that \((3.36)\) and \((3.38)\) also hold true. With these choices, it follows that for any \(\varepsilon \in [0, 1]\)

\[
\mu^\varepsilon_\delta(\mathbb{R}^2 \setminus [a, b]^2) \leq \mathbb{P}(Y^0_\delta(\infty) \notin [a, b]) + \mathbb{P}(Y^\varepsilon_\delta(\infty) \notin [a, b]) \leq \eta/4.
\]
The disintegration inequality, see Proposition [C.1] and the triangle inequality for the total variation distance imply that for each \(x > 0\) and \(t > 0\) there is \(\varepsilon_0 > 0\) such that for \(\varepsilon \in [0, \varepsilon_0]\) we choose \(a > 0, b > a\) and \(\delta \in (0, t)\) for which \((3.36), (3.37),\) and \((3.38)\) hold true and therefore

\[
d_{TV}(Y_t^0(\infty), Y_t^\varepsilon(x_\varepsilon)) \leq \int_{\mathbb{R}^2} d_{TV}(Y_t^0(x), Y_t^\varepsilon(y)) \mu_\delta^\varepsilon(dx, dy) \leq \mu_\delta^\varepsilon(\mathbb{R}^2 \setminus [a, b]^2) + \int_{[a, b]^2} d_{TV}(Y_t^0(x), Y_t^\varepsilon(y)) \mu_\delta^\varepsilon(dx, dy)
\]

\[
\leq \eta/4 + \int_{[a, b]^2} d_{TV}(Y_t^0(\infty), Y_t^\varepsilon(y)) \mu_\delta^\varepsilon(dx, dy)
\]

\[
+ \int_{[a, b]^2} d_{TV}(Y_t^0(\infty), Y_t^\varepsilon(y)) \mu_\delta^\varepsilon(dx, dy)
\]

\[
\leq \eta/4 + \eta/4 + \eta/4 + \eta/4 = \eta,
\]

Recall \((2.5)\) and observe that \(d_{TV}(Y(\text{sgn}(x)\infty), Y_t^\varepsilon(x)) = d_{TV}(Y_t^0(\infty), Y_t^\varepsilon(x_\varepsilon))\). Since \(\eta > 0\) is arbitrary, the proof of \((2.17)\) is complete.

3.5. **Strict inequalities for the rescaled process.** In this section we show

\[
0 < d_{TV}(Y_t(\text{sgn}(x)\infty), Y_\infty) < 1 \quad \text{for any} \quad t > 0.
\]

First, we prove the upper bound and then we show the lower bound.

**The upper bound.** We first note that for any \(t > 0, x \in \mathbb{R}\), the marginal \(Y_t(x)\) has full support in \(\mathbb{R}\), see Proposition [C.2] in Appendix [C] for a proof. By Proposition [2.1] the family \((Y(x), x \in \mathbb{R})\) is Markovian, and hence, by semigroup property, \(Y_t(\infty)\) is equal in law to \(Y_{t/2}(Y_{t/2}(\infty))\) for any \(t > 0\). Since \(\mathbb{P}(Y_{t/2}(\infty) \in \mathbb{R}) = 1\) it follows by our previous discussion that \(Y_t(\infty)\) possesses a continuous density \(\rho_t : \mathbb{R} \to (0, \infty)\). Furthermore, the invariant distribution of \(Y\) represented by \(Y_\infty\) has explicit density function \(\rho : \mathbb{R} \to (0, \infty)\) which is given in \((3.14)\). To conclude that \(d_{TV}(Y_t(\infty), Y_\infty) < 1\) we note that

\[
d_{TV}(Y_t(\infty), Y_\infty) = 1 - \int_{\mathbb{R}} \min\{\rho_t(z), \rho(z)\} dz < 1.
\]

**The lower bound: injective evolution map.** To prove the lower bound, we first define the evolution map on the space of measures. Let \(\mathcal{P}\) be the space of probability measures on \(\mathbb{R}\) that are absolutely continuous with respect to the Lebesgue measure on \(\mathbb{R}\) and let \(C_b(\mathbb{R})\) be the set of bounded continuous functions on \(\mathbb{R}\). For each \(\mu \in \mathcal{P}\) and \(t \geq 0\) let \(\varphi(\mu, t)\) be the measure \(\varphi\) such that for every \(f \in C_b(\mathbb{R})\)

\[
\int f(x) d\varphi(x) := \int \mathbb{E}[f(Y_t(x))] d\mu(x).
\]

By Proposition [C.2] we have that \(\varphi(\mu, t) \in \mathcal{P}\) for all \(\mu \in \mathcal{P}\) and \(t > 0\). For fixed time \(t > 0\), the evolution map is injective in the sense that

\[
(3.40) \quad \text{if} \quad \mu, \mu' \in \mathcal{P}, \quad \mu \neq \mu' \quad \text{then for all} \quad t \geq 0 \quad \varphi(\mu, t) \neq \varphi(\mu', t).
\]
Moreover, by ergodicity of the dynamics, see Lemma 2.1, for all $t > 0$ the map $\mu \mapsto \varphi(\mu, t)$ admits a unique fixed point, that is, there is a unique $\nu \in \mathcal{P}$ such that
\begin{equation}
\varphi(\nu, t) = \nu \quad \text{for any } t \geq 0.
\end{equation}
Recall that we denote the law of $Y_\infty$ by $\nu$. By Proposition 2.1 and Proposition C.2 it follows that for all $\delta > 0$ the law of $Y_\delta(\text{sgn}(x)\infty)$ denoted by $\mu_\delta^\infty$ belongs to $\mathcal{P}$. By the Markov property of the extended process, see Proposition 2.1 we have for any $\delta \in (0, t)$
\begin{equation}
\mu_t^\infty = \varphi(\mu_\delta^\infty, t - \delta).
\end{equation}
Let $t > 0$ be fixed. We observe that there exists $\delta \in (0, t)$ such that $\mu_\delta^\infty \neq \nu$. By (3.40), (3.41) and (3.42) it follows that $\mu_t^\infty = \varphi(\mu_\delta^\infty, t - \delta) \neq \varphi(\nu, t - \delta) = \nu$ and hence
\begin{equation}
0 < d_{TV}(\mu_t^\infty, \nu) = d_{TV}(Y_t(\text{sgn}(x)\infty), Y_\infty).
\end{equation}

APPENDIX A. The Continuous Markovian extension

In this section we prove that the SDEs defined in (2.2) and (2.6), or equivalently in (3.33), with state space $\mathbb{R}$ may be extended to $\mathbb{R} := \mathbb{R} \cup \{\pm \infty\}$. Furthermore, we show that this extension is Markovian and that the family of transition kernels associated to it is continuous with respect to the initial condition, in the sense of (3.31).

For the extension, we consider $\mathbb{R}$ endowed with the Borel $\sigma$-algebra associated to the metric $d_\infty : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ defined by
\begin{equation}
d_\infty(x_1, x_2) := |\arctan(x_1) - \arctan(x_2)|,
\end{equation}
where $\arctan : \mathbb{R} \to \mathbb{R}$ is the continuous function defined by
\begin{equation}
\arctan(v) := \begin{cases}
-\pi/2 & \text{for } v = -\infty, \\
\int_{0}^{v} \frac{1}{1 + u^2} du & \text{for } v \in \mathbb{R}, \\
\pi/2 & \text{for } v = \infty.
\end{cases}
\end{equation}
Let $Y^\varepsilon(x) = (Y^\varepsilon_t(x), t \geq 0)$ be the unique strong solution of (3.33). For $x \in \mathbb{R}$ let $P^\varepsilon_x$ be the law induced by $Y^\varepsilon(x)$ on the space of real valued continuous functions $(C, \mathcal{C})$ and let $\overline{P}^\varepsilon_x$ be its law on the space of $\overline{C}$-valued continuous functions $(\overline{C}, \overline{\mathcal{C}})$. To complete the extension we define $\overline{P}^\varepsilon_x$ for $x \in \{-\infty, \infty\}$ as the law on $(\overline{C}, \overline{\mathcal{C}})$ induced by $Y^\varepsilon(\infty)$ and $Y^\varepsilon(-\infty)$ where for all $t \geq 0$
\begin{equation}
Y^\varepsilon(\infty) := \lim_{\delta \to \infty} Y^\varepsilon_\delta(x) \quad \text{and} \quad Y^\varepsilon(\infty) := \lim_{\delta \to \infty} Y^\varepsilon_\delta(x).
\end{equation}
The above extension is well-defined since, by the comparison lemma for SDE’s, see [49 Thm.1.1],
\begin{equation}
x \leq x' \Rightarrow \mathbb{P}(Y^\varepsilon_t(x) \leq Y^\varepsilon_t(x') \quad \forall t \geq 0) = 1.
\end{equation}
This section is divided into three subsections. In Subsection A.1 we prove trajectory properties of the above extension. In Subsection A.2 we prove that the extension is Markovian. Finally, in Subsection A.3 we prove that the extension is continuous with respect to the initial condition.

A.1. Coming down from infinity. We next explain when a solution to an SDE comes down from infinity. This is based on entrance conditions at the boundary for ODE’s. In fact, as we shall see, the stochastic processes we consider satisfy a uniform $L^2$ bound and so are finite for all positive times. This section is organized as follows: First we prove an entrance condition for ODE’s. Then we show the uniform bounds in $L^2$. Finally we define what is meant by the integral form of the solution when the initial condition is $\pm \infty$. We include an explanation of these standard techniques for completeness and to prepare for specific results we will need.
A.1.1. Entrance condition for ODE’s. Let
\[ \mathcal{L} := \left\{ G : \mathbb{R} \to \mathbb{R} \mid G \text{ is locally Lipschitz and} \right\} \]
\[ -\infty < \int_{R}^{\infty} \frac{1}{G(u)} \, du < 0 \text{ for some } R > 0 \]
be the space of velocity fields in which we are interested on.

Lemma A.1 (Entrance at infinity). Given any fixed \( G \in \mathcal{L} \) and any \( x \in \mathbb{R} \), let \( \psi(x) := (\psi_t(x), t \geq 0) \) be the unique solution of the differential equation
\[
\begin{cases}
\frac{d}{dt} \psi_t = G(\psi_t) & \text{for } t \geq 0, \\
\psi_0 = x.
\end{cases}
\]
Then for all \( t > 0 \), the limit \( \psi_t(\infty) := \lim_{x \to \infty} \psi_t(x) \) is well-defined and finite.

Proof. By the comparison lemma for ODE’s we have
\[ \forall t \geq 0, \ x_1, x_2 \in \mathbb{R}, \ x_1 \leq x_2 \Rightarrow \psi_t(x_1) \leq \psi_t(x_2). \]
Therefore, the limit \( \psi_t(\infty) := \lim_{x \to \infty} \psi_t(x) \) is well-defined but may be infinite. In the sequel, we show that \( \psi_t(\infty) < \infty \) for any \( t > 0 \). Since \( G \) is locally Lipschitz and satisfies (A.3), \( G(x) < 0 \) for all \( x \geq R \). Fix \( T > 0 \) and let \( L := \psi_T(R) \). By uniqueness of solutions, the map \( t \mapsto \psi_t(R) \) is decreasing and for all \( x \geq L \ G(x) < 0 \). Let \( F_{L,R} : [L, R] \to [0, T] \) be such that \( F_{L,R}(\psi_t(R)) = t \) for all \( t \in [0, T] \) and note that \( F'_{L,R}(u) = 1/G(u) \). Since \( F_{L,R}(R) = 0 \) and \( F_{L,R}(L) = T \), we obtain
\[ -T = \int_{L}^{R} F'_{L,R}(u) \, du = \int_{L}^{R} \frac{1}{G(u)} \, du. \]
Now let \( F_L : [L, \infty] \to [F_L(\infty), 0] \) be given by \( F_L(x) := \int_{L}^{x} \frac{1}{G(u)} \, du \) for \( x \in [L, \infty) \), and set \( F_L(\infty) := \lim_{x \to \infty} F_L(x) \). By (A.3) and (A.5), \( F_L(\infty) \in (-\infty, 0) \). By (A.4), for any \( x \in [R, \infty) \), \( t \leq T \), we have \( \psi_t(x) \geq L \) and \( \frac{d}{dt} F_L(\psi_t(x)) = 1 \). We thereby, conclude that
\[ F_L(\psi_t(x)) = t + F_L(x), \quad t \in [0, T]. \]
Again by (A.4) and the continuity of \( F_L \), we can take the limit \( x \to \infty \) in (A.6) to obtain
\[ F_L(\psi_t(\infty)) = t + F_L(\infty), \quad t \in [0, T]. \]
Therefore \( \psi_t(\infty) < \infty \) for any \( t > 0 \). \( \square \)

We may now define the extended ODE. By Lemma (A.1) \( \psi(\infty) := (\psi_t(\infty), t \geq 0) \) solves
\[
\begin{cases}
\frac{d}{dt} \psi_t = G(\psi_t) & \text{for } t > 0, \\
\psi_0 = \infty,
\end{cases}
\]
in the sense that \( \psi_0(\infty) = \infty \), and for any \( t_0 > 0 \), the following integral relation holds
\[ \psi_t(\infty) = \psi_{t_0}(\infty) + \int_{t_0}^{t} G(\psi_s(\infty)) \, ds \quad \text{for all } \quad t \geq t_0. \]
Equation (A.9) is a consequence of the Fundamental Theorem of Calculus. Indeed, by (A.7), for any \( s \geq t_0, \psi_s(\infty) := F^{-1}_L(s + F_L(\infty)) \in \mathbb{R} \) and taking derivatives on both sides of (A.7), we obtain
\[ \frac{d}{ds} \psi_s(\infty) = \frac{1}{F'_L(\psi_s(\infty))} = G(\psi_s(\infty)). \]
A.1.2. **Uniform $L^2$ bounds.** In what follows, we prove a second moment bound for any fixed \( t > 0 \) for \( (Y_t^\varepsilon(x), x \in \mathbb{R}) \).

**Lemma A.2 (L²-bound).** If \( \varepsilon \in [0, 1] \), then the process \( Y^\varepsilon(x) \) defined by (3.33) satisfies for any \( t > 0 \),

\[
(A.10) \quad \sup_{x \in \mathbb{R}} \mathbb{E}[(Y_t^\varepsilon(x))^2] < \infty.
\]

**Proof.** Fix \( \varepsilon \in [0, 1] \) and let \( G_\varepsilon : \mathbb{R} \to \mathbb{R} \) be given by \( G_\varepsilon(y) := yF_\varepsilon(y) \) for all \( y \in \mathbb{R} \) with \( F_\varepsilon \) defined in (3.32). Recall that \( V \) satisfies Hypothesis 1, Hypothesis 2 and Hypothesis 3A. Since \( V' \) is an odd function, it follows that \( G_\varepsilon(y) = G_\varepsilon(|y|) \leq 0 \) for all \( y \in \mathbb{R} \). By Hypothesis 2 and Hypothesis 3A (Condition (G1)) there is \( c_* > 0 \) for which

\[
(A.11) \quad G_\varepsilon(y) \leq -c_*|y|^{2+\alpha} \quad \text{for all} \quad y \in \mathbb{R}.
\]

By Itô’s formula, for \( t > 0 \), we have

\[
(A.12) \quad |Y_t^\varepsilon(x)|^2 = x^2 + 2 \int_0^t G_\varepsilon(Y_s^\varepsilon(x))ds + t + M_t^x,
\]

where \( M^x = (M_t^x, t \geq 0) \) is a local martingale given by

\[
(A.13) \quad M_t^x = 2 \int_0^t Y_s^\varepsilon(x)dB_s.
\]

Since \( G_\varepsilon(y) \leq 0 \) for all \( y \in \mathbb{R} \), a localization argument yields that, \( \mathbb{E}[(Y_t^\varepsilon(x))^2] \leq x^2 + t \) for any \( t \geq 0 \). As a consequence, we have that \( M^x \) is a true mean-zero martingale. Now, if we take expectation on both sides of equality (A.12), apply Fubini’s theorem and use (A.11) we obtain for all \( t > 0 \)

\[
(A.14) \quad \mathbb{E}[(Y_t^\varepsilon(x))^2] = x^2 + 2 \int_0^t \mathbb{E}[G_\varepsilon(Y_s^\varepsilon(x))]ds + t \leq x^2 - 2c_* \int_0^t \mathbb{E}[(Y_s^\varepsilon(x))^{2+\alpha}]ds + t.
\]

By Jensen’s inequality we obtain

\[
(A.15) \quad \mathbb{E}[(Y_t^\varepsilon(x))^{2+\alpha}] \geq (\mathbb{E}[(Y_t^\varepsilon(x))^2])^{1+\alpha/2} \quad \text{for all} \quad t \geq 0.
\]

By (A.14) and (A.15) if we denote \( \psi_t^\varepsilon(x) := \mathbb{E}[(Y_t^\varepsilon(x))^2] \) and let \( \widetilde{G}(y) := -2c_*|y|^{1+\alpha/2} + 1 \) for all \( y \in \mathbb{R} \) then we have that

\[
\frac{d}{dt}\psi_t^\varepsilon(x) \leq \widetilde{G}(\psi_t^\varepsilon(x)) \quad \text{for} \quad t \geq 0.
\]

Now we let \((\widetilde{\psi}_t(x), t \geq 0)\) be the solution of (A.8) for \( G = \widetilde{G} \) and with initial condition \( \widetilde{\psi}_0(x) = \psi_0^\varepsilon(x) = x^2 \). Observe that \( \widetilde{G} \in \mathcal{L} \), where \( \mathcal{L} \) is defined in (A.3). To conclude (A.10), we rely on monotonicity and Lemma A.1. Indeed, for any \( x \in \mathbb{R} \) and \( t > 0 \)

\[
(A.16) \quad \mathbb{E}[(Y_t^\varepsilon(x))^2] \leq \widetilde{\psi}_t(x) \leq \lim_{z \to \infty} \widetilde{\psi}_t(z) = \widetilde{\psi}_t(\infty) < \infty.
\]

□
A.1.3. Integral expression. Now, we examine the integral form of the limit process.

**Proposition A.1 (Integral form).** For any fixed \( \varepsilon \in [0, 1] \), let \( F_\varepsilon \) be as defined in (33.2). Then, the limit process \( Y_\varepsilon^x(sgn(x) \infty) := \lim_{r \to \infty} Y_\varepsilon^x(sgn(x) \cdot r) \) solves

\[
\begin{aligned}
   dY_t &= F_\varepsilon(Y_t)dt + dB_t \quad \text{for} \quad t > 0, \\
   Y_0 &= sgn(x) \infty,
\end{aligned}
\]

in the sense that almost surely \( \lim_{t \to 0} Y_t = sgn(x) \infty \) and for any \( 0 < t_0 < t \)

\( (A.17) \)

\[
Y_t = Y_{t_0} + \int_{t_0}^t F_\varepsilon(Y_s)ds + B_t - B_{t_0}.
\]

**Proof.** Assume without loss of generality that \( x > 0 \). By (A.2) \( Y_\varepsilon^x(r) \) increases with \( r \) and therefore the limit \( Y_\varepsilon^x(\infty) \) exists. By (A.10) it follows that \( \mathbb{P}(Y_\varepsilon^x(\infty) < \infty) = 1 \) for any \( t > 0 \). Given \( T > t_0 > 0 \), we claim that, for every \( \delta > 0 \)

\( (A.18) \)

\[
\lim_{r \to \infty} \mathbb{P} \left( \sup_{t \in [t_0, T]} |F_\varepsilon(Y_\varepsilon^x(\infty)) - F_\varepsilon(Y_\varepsilon^x(r))| > \delta \right) = 0.
\]

The proof of (A.18) is postponed to Lemma A.3 below. Now, note that, almost surely

\[
Y_\varepsilon^x(r) = Y_\varepsilon^x(0) + \int_{t_0}^t F_\varepsilon(Y_\varepsilon^x(r))ds + B_t - B_{t_0}.
\]

By (A.18) we may take the limit inside the above integral and therefore

\[
\mathbb{P} \left( Y_\varepsilon^x(\infty) = Y_{t_0}^x(\infty) + \int_{t_0}^t F_\varepsilon(Y_\varepsilon^x(\infty))ds + B_t - B_{t_0} \quad \forall \ t > t_0 \right) = 1.
\]

\( \square \)

**Lemma A.3.** For any \( T > t_0 > 0 \) and \( \delta > 0 \) the equality in (A.18) holds true.

**Proof.** We first note that (A.18) is a consequence of

\( (A.19) \)

\[
\lim_{A \to \infty} \mathbb{P} \left( \sup_{t \in [t_0, T]} |Y_\varepsilon^x(\infty)| > A \right) = 0
\]

and

\( (A.20) \)

\[
\forall \delta > 0 \quad \lim_{r \to \infty} \sup_{t \in [t_0, T]} \mathbb{P} \left( \sup_{t \in [t_0, T]} |Y_\varepsilon^x(\infty) - Y_\varepsilon^x(r)| > \delta \right) = 0.
\]

Indeed, as \( F_\varepsilon \) is locally Lipschitz, for any \( \delta > 0 \) and \( A > 0 \) there is \( \delta' = \delta'(\delta, A, \varepsilon) > 0 \) for which

\[
\mathbb{P} \left( \sup_{t \in [t_0, T]} |F_\varepsilon(Y_\varepsilon^x(\infty)) - F_\varepsilon(Y_\varepsilon^x(r))| > \delta \right)
\]

\[
\leq \mathbb{P} \left( \sup_{t \in [t_0, T]} |Y_\varepsilon^x(\infty) - Y_\varepsilon^x(r)| > \delta' \right) + \mathbb{P} \left( \sup_{t \in [t_0, T]} |Y_\varepsilon^x(\infty)| > A \right).
\]

By monotonicity and Lemma A.2 \( \lim_{r \to \infty} Y_\varepsilon^x(t) = Y_\varepsilon^x(\infty) \in \mathbb{R} \), for any \( t > 0 \). The pointwise limit, does not guarantee (A.19) and (A.20). In order to obtain the above uniform bounds we will show that the family \( (Y_\varepsilon^x(r), r \geq 0) \) is tight in the space of continuous paths \( C \). Tightness in \( C \) and pointwise convergence imply uniform convergence of the family and the bounds (A.19) and (A.20).
By Aldous’ tightness criterion, see [21, Thm. 16.10, p.178] or [53, Thm. 4.1.3, p.51] we only need to show that

(A.21) \[ \forall t \in [t_0, T] \lim_{A \to \infty} \sup_{r \in \mathbb{R}} \mathbb{P}(|Y_t^\varepsilon(r)| > A) = 0, \]

and that

(A.22) \[ \forall \eta > 0 \lim_{\delta \to 0} \sup_{r \in \mathbb{R}} \left( \sup_{|t-s| < \delta} |Y_t^\varepsilon(r) - Y_s^\varepsilon(r)| > \eta \right) = 0. \]

**Proof of (A.21).** By Lemma A.2 we have that

(A.23) \[ C_t^\varepsilon := \sup_{r \in \mathbb{R}} \mathbb{E}[|Y_t^\varepsilon(r)|^2] < \infty. \]

Therefore, by Chebyshev’s inequality, for any \( t \in [t_0, T] \)

\[ \sup_{r \in \mathbb{R}} \mathbb{P}(|Y_t^\varepsilon(r)| > A) \leq \sup_{r \in \mathbb{R}} \frac{\mathbb{E}[|Y_t^\varepsilon(r)|^2]}{A^2} \leq \frac{C_t^\varepsilon}{A^2} \to 0 \quad \text{as} \quad A \to \infty. \]

**Proof of (A.22).** We first write \( Y_t^\varepsilon(r) = \int_t^t F_\varepsilon(Y_u^\varepsilon(r)) \, du + B_t - B_s \). By the triangle inequality, and the continuity of Brownian motion to verify (A.22) it suffices to prove that for any \( \eta > 0 \)

\[ \lim_{\delta \to 0} \sup_{r \in \mathbb{R}} \left( \sup_{|t-s| \leq \delta} \int_t^t F_\varepsilon(Y_u^\varepsilon(r)) \, du > \eta \right) = 0. \]

Fix \( K > 0, \eta > 0 \), and let \( A_K^\varepsilon := \{\sup_{u \in [t_0, T]} |Y_u^\varepsilon(r)| > K\} \). Now note that there is \( \delta = \delta(K, \eta, \varepsilon) \) such that \( \delta \sup_{|y| \leq K} |F_\varepsilon(y)| \leq \eta \) and therefore for any \( K > 0, \eta > 0 \)

\[ \lim_{\delta \to 0} \sup_{r \in \mathbb{R}} \left( \sup_{|t-s| \leq \delta} \int_t^t F_\varepsilon(Y_u^\varepsilon(r)) \, du > \eta \right) \leq \sup_{r \in \mathbb{R}} \mathbb{P}(A_K^\varepsilon). \]

Since the left-hand side of the above inequality does not depend on \( K \) we obtain that

\[ \lim_{\delta \to 0} \sup_{r \in \mathbb{R}} \left( \sup_{|t-s| \leq \delta} \int_t^t F_\varepsilon(Y_u^\varepsilon(r)) \, du > \eta \right) \leq \lim_{K \to \infty} \sup_{r \in \mathbb{R}} \mathbb{P}(A_K^\varepsilon). \]

Hence, to obtain (A.22) it is enough to prove that \( \lim_{K \to \infty} \sup_{r \in \mathbb{R}} \mathbb{P}(A_K^\varepsilon) = 0 \). By (A.12) and (A.13), since \( G_\varepsilon(y) = yF_\varepsilon(y) \leq 0 \) for all \( y \in \mathbb{R} \) we have that

\[ \sup_{t \in [t_0, T]} |Y_t^\varepsilon(r)|^2 \leq |Y_{t_0}^\varepsilon(r)|^2 + \sup_{t \in [t_0, T]} \left| \int_{t_0}^t 2Y_s^\varepsilon(r) \, dB_s \right|. \]

The estimate on \( A_K^\varepsilon \) then becomes

\[ \mathbb{P}(A_K^\varepsilon) \leq \mathbb{P}(|Y_{t_0}^\varepsilon(r)|^2 + T > K^2/2) + \mathbb{P} \left( \sup_{t \in [t_0, T]} \left| \int_{t_0}^t 2Y_s^\varepsilon(r) \, dB_s \right| > K^2/2 \right). \]

First note that for \( K^2/2 > 2T \), by (A.23) we have

\[ \mathbb{P}( |Y_{t_0}^\varepsilon(r)|^2 + T > K^2/2 ) \leq \mathbb{P}( |Y_{t_0}^\varepsilon(r)|^2 > K^2/4 ) \leq 4 \frac{C_{t_0}^\varepsilon}{K^2}. \]
To conclude, note that by Doob’s submartingale inequality, Itô’s isometry and (A.23)
\[ \mathbb{P}\left( \sup_{t \in [t_0, T]} \int_{t_0}^t 2Y^\varepsilon_s(r) dB_s > K^2/2 \right) \leq \int_{t_0}^T 4\mathbb{E}[|Y^\varepsilon_s(r)|^2] ds \leq \int_{t_0}^T 16C_s^\varepsilon ds \leq \frac{16T \sup_{s \in [t_0, T]} C_s^\varepsilon}{K^4} \to 0 \quad \text{as} \quad K \to \infty, \]
where for the last passage we note that by (A.16), \( s \mapsto C_s^\varepsilon \) is bounded by a continuous function in \( (0, \infty) \).

\[ \square \]

A.2. Markov property of the extended family. To prove that the extended family obtained by (A.1) is Markovian, one needs to verify the conditions stated in Theorem 5.16 in [52, Ch 2, p.78]. These are the (i) compatible initial values, (ii) the measurability of the transition laws, and (iii) Markov property.

Note that for all \( x \in \mathbb{R}, \varepsilon > 0 \), and \( t > 0 \), \( \mathbb{P}(Y^\varepsilon_t(x) \in \{\pm \infty\}) = 0 \) so conditions (i)–(iii) are satisfied for all finite initial values. It only remains to see that (i)–(iii) is satisfied for initial values \( x \in \{-\infty, \infty\} \). We only consider \( x = \infty \), the case \( x = -\infty \) is analogous. For (i), note that
\[ \mathbb{P}(Y^\varepsilon_0(\infty) = \infty) = \lim_{R \to \infty} \mathbb{P}(Y^\varepsilon_0(\infty) > R) = \lim_{R \to \infty} \lim_{x \to \infty} \mathbb{P}(Y^\varepsilon_0(x) > R) = 1 \]
For (ii), Proposition A.1 implies that for any \( t_0 > 0 \) and \( T > 0 \), the process \( Y^\varepsilon(n) \) converges uniformly on the interval \([t_0, T]\) to \( Y^\varepsilon(\infty) \) as \( n \to \infty \). Therefore by the monotonicity in (A.2) and the continuity of probability, for all \( k \in \mathbb{N}, t_1 < \ldots < t_k \), and \( a_1, \ldots, a_k \in \mathbb{R} \) one has
\[ \lim_{n \to \infty} \mathbb{P}(Y^\varepsilon_{t_1}(n) > a_1, \ldots, Y^\varepsilon_{t_k}(n) > a_k) = \mathbb{P}(Y^\varepsilon_{t_1}(\infty) > a_1, \ldots, Y^\varepsilon_{t_k}(\infty) > a_k). \]
The measurability follows by extending the above using Dynkin’s \( \pi-\lambda \) theorem.

Condition (iii) is a consequence of (A.17) in Proposition A.1. Indeed, for any fixed \( s > 0 \), if we let \( (W_t := B_{t+s} - B_s, t \geq 0) \) we have that almost surely for any \( t > 0 \)
\[ Y^\varepsilon_{t+s}(\infty) = Y^\varepsilon_s(\infty) + \int_0^t F^\varepsilon_s(Y^\varepsilon_{u+s}(\infty)) \, du + W_t. \]
Now, if we let \( \mathbb{Y}_t := Y^\varepsilon_{t+s}(\infty) \), it follows that \( (\mathbb{Y}_t, t \geq 0) \) solves the SDE (3.33) with initial condition \( Y^\varepsilon_s(\infty) \). Furthermore, by Theorem 3.5 in [65, p.58], Equation (3.33) is well-posed in \( \mathbb{R} \) and, by Lemma A.2 \( Y^\varepsilon_s(\infty) \in \mathbb{R} \) almost surely for any \( s > 0 \). Therefore, with the help of Theorem 9.1 in [65, p.86] we conclude that
\[ \mathbb{P}(Y^\varepsilon_{t+s}(\infty) \in \mathbb{A} | Y^\varepsilon_s(\infty) = y) = \mathbb{P}(Y^\varepsilon_t(y) \in \mathbb{A}), \]
which yields (iii) and concludes that the extended family is Markovian.

A.3. Continuity. Let \( \varepsilon \in [0,1] \) be fixed. In this section we prove that the map \( x \mapsto Y^\varepsilon_t(x) \) is continuous with respect to the total variation distance for any \( t > 0 \). We first note that the map above is continuous in \( \mathbb{R} \), see Theorem 1.3 in [37] and Theorem 1.1 in [78] for a proof. Therefore, it only remains to verify the continuity at infinity. This is the content of the following lemma.

\[ \text{Lemma A.4 (Continuity in total variation). For any } \varepsilon \in [0,1] \text{ and any } t > 0 \]
\[ \lim_{x \to \infty} d_{TV}(Y^\varepsilon_t(x), Y^\varepsilon_t(\infty)) = 0 \quad \text{and} \quad \lim_{x \to -\infty} d_{TV}(Y^\varepsilon_t(x), Y^\varepsilon_t(-\infty)) = 0. \]
Proof. We only prove the case for which $x \to +\infty$. The case when $x \to -\infty$ follows from the symmetry of $V^\varepsilon$. Let $\mu^{\varepsilon,x}_s$ be the measure in $\mathbb{R}^2$ defined by

$$\mu^{\varepsilon,x}_s(dz_1, dz_2) = \mathbb{P}(Y^\varepsilon_t(x) \in dz_1, Y^\varepsilon_t(\infty) \in dz_2).$$

For $s \in (0, t)$ we define $f : \mathbb{R}^2 \to [0, 1]$ by $f(z_1, z_2) := d_{TV}(Y^\varepsilon_{t-s}(z_1), Y^\varepsilon_{t-s}(z_2))$. By the Markovian property of the extended family $(Y^\varepsilon_t(x), x \in \mathbb{R})$ and Proposition C.1 for any $K > 0$ we have that

$$d_{TV}(Y^\varepsilon_t(x), Y^\varepsilon_t(\infty)) \leq \int \mathbb{R}^2 f(z_1, z_2) \mu^{\varepsilon,x}_s(dz_1, dz_2) \leq \int \mathbb{R}^2 f(z_1, z_2) \mu^{\varepsilon,x}_s(dz_1, dz_2) + \mathbb{P}(|Y^\varepsilon_t(x)| > K) + \mathbb{P}(|Y^\varepsilon_t(\infty)| > K).$$

By Lemma A.2 and Chebyshev’s inequality it follows that

$$\limsup_{K \to \infty} \limsup_{x \to \infty} \mathbb{P}(|Y^\varepsilon_t(x)| > K) + \mathbb{P}(|Y^\varepsilon_t(\infty)| > K) = 0.$$

It suffices to show that for any $K > 0$

$$\limsup_{x \to \infty} \int_{|z_1|,|z_2| \leq K} f(z_1, z_2) \mu^{\varepsilon,x}_s(dz_1, dz_2) = 0. \quad (A.24)$$

We now define for any $\delta > 0$

$$\omega_{f,K}(\delta) := \max\{f(z_1, z_2) : |z_1|, |z_2| \leq K, |z_1 - z_2| \leq \delta\}.$$

Since $f$ is continuous and $f(z, z) = 0$, it follows that

$$\omega_{f,K}(\delta) < \infty \quad \text{and} \quad \lim_{\delta \to 0} \omega_{f,K}(\delta) = 0 \quad (A.25)$$

Given $\delta > 0$, consider the following split of the integral in (A.24),

$$\int_{|z_1|,|z_2| \leq K} f(z_1, z_2) \mu^{\varepsilon,x}_s(dz_1, dz_2) = \int_{|z_1|,|z_2| \leq K} f(z_1, z_2) \mathbb{1}_{|z_1 - z_2| \leq \delta} \mu^{\varepsilon,x}_s(dz_1, dz_2) + \int_{|z_1|,|z_2| \leq K} f(z_1, z_2) \mathbb{1}_{|z_1 - z_2| > \delta} \mu^{\varepsilon,x}_s(dz_1, dz_2) \leq \omega_{f,K}(\delta) + \eta(x, \delta),$$

where $\eta(x, \delta) := \mu^{\varepsilon,x}_s(|z_1 - z_2| > \delta) = \mathbb{P}(|Y^\varepsilon_s(x) - Y^\varepsilon_s(\infty)| > \delta)$. By (A.1), $\lim_{x \to \infty} \eta(x, \delta) = 0$ for any $\delta > 0$. To conclude the proof of Lemma A.4 we note that, by (A.25)

$$\limsup_{x \to \infty} \int_{|z_1|,|z_2| \leq K} f(z_1, z_2) \mu^{\varepsilon,x}_s(dz_1, dz_2) \leq \inf_{\delta > 0} \omega_{f,K}(\delta) = 0.$$

\[\square\]

**Appendix B. Uniform bounds**

In this section we prove the bounds (3.36) and (3.38).
B.1. Uniform entrance in a compact. The bound (3.36) is a consequence of the following proposition.

**Proposition B.1.** For any \( \eta > 0 \) and \( a > 0 \) there are \( b > 0 \) and \( \delta \in (0, \eta) \) such that

\[
\sup_{\varepsilon \in [0,1]} \mathbb{P}(Y_\delta^{\varepsilon}(\infty) \notin [a, b]) \leq \eta.
\]

Proposition B.1 is based on the two following statements, whose proofs are given afterwards.

(B.1) For any \( a > 0 \), \( \lim_{\delta \to 0} \sup_{\varepsilon \in [0,1]} \mathbb{P}(|Y_\delta^{\varepsilon}(\infty)| \leq a) = 0. \)

(B.2) For any \( \delta > 0 \), \( \lim_{b \to \infty} \sup_{\varepsilon \in [0,1]} \mathbb{P}(|Y_\delta^{\varepsilon}(\infty)| > b) = 0. \)

**Proof of Proposition B.1.** Given \( \eta > 0 \) and \( a > 0 \), by (B.1) there is \( \delta \in (0, \eta) \) such that \( \mathbb{P}(|Y_\delta^{\varepsilon}(\infty)| < a) < \eta/2 \) for any \( \varepsilon \in [0,1] \). Next, by (B.2), we choose \( b > 0 \) such that \( \mathbb{P}(|Y_\delta^{\varepsilon}(\infty)| > b) < \eta/2 \) for any \( \varepsilon \in [0,1] \). With this, we conclude that for every \( \varepsilon \in [0,1] \)

\[
\mathbb{P}(|Y_\delta^{\varepsilon}(\infty)| \notin [a, b]) = \mathbb{P}(|Y_\delta^{\varepsilon}(\infty)| > b) + \mathbb{P}(|Y_\delta^{\varepsilon}(\infty)| < a) < \eta.
\]

In what follows we prove (B.1) and (B.2).

**Proof of (B.1).** Fix any \( a > 0 \). For \( D > 0 \), let \( \Omega(\delta, D) := \{ \sup_{t \leq \delta} |B_t| \leq D \} \). Fix \( x := 2(a + D) \) and choose \( K := 2x \). Now let \( \sigma := \tau(a) \wedge \tau(K) \) where \( \tau(v) = \tau(a, x, \varepsilon) := \inf \{ t > 0 : Y_t^{\varepsilon}(x) = v \} \) for \( v \in \mathbb{R} \). Note that \( Y_\delta^{\varepsilon}(\infty) \geq Y_\delta^{\varepsilon}(x) \), and that almost surely

(B.3)

\[
Y_{\delta \wedge \sigma}^{\varepsilon}(x) = x - \int_0^{\delta \wedge \sigma} |F_\varepsilon(Y_s^{\varepsilon}(x))| \, ds + B_{\delta \wedge \sigma}.
\]

By Hypothesis (2) and (3.32) it follows that

(B.4)

\[
C(K) := \sup_{\varepsilon \in [0,1]} \sup_{|y| \leq K} \mathbb{E} |F_\varepsilon(y)| = \sup_{\varepsilon \in [0,1]} \sup_{|y| \leq K} \left| \frac{V'(b_\varepsilon y)}{b_\varepsilon^{1+\alpha}} \right| < \infty.
\]

Given \( D > 0 \), there is \( \eta > 0 \) such that \( \delta \in (0, \eta) \), implies \( \delta < \sigma \) on \( \Omega(\delta, D) \). Indeed, by (B.4) there is \( C = C(D) > 0 \) which allows (B.3) to be bounded by

\[
Y_{\delta \wedge \sigma}^{\varepsilon}(x) \geq x - C(\delta \wedge \sigma) - D \geq x - D - C \delta \quad \text{and} \quad Y_{\delta \wedge \sigma}^{\varepsilon}(x) \leq x + D < K.
\]

Since \( x - D > 2a \), for any \( \delta \in (0, C^{-1}a) \) it follows that \( Y_{\delta \wedge \sigma}^{\varepsilon}(x) \in (a, K) \). In conclusion, on the event \( \Omega(\delta, D) \), we have that \( Y_{\delta \wedge \sigma}^{\varepsilon}(\infty) \geq Y_{\delta \wedge \sigma}^{\varepsilon}(x) > a \). Since \( \mathbb{P}(\Omega(\delta, D)) \to 1 \) as \( \delta \to 0 \), the proof of (B.1) is complete.

**Proof of (B.2).** We start the proof with uniform \( L^2 \) bounds, that is, we prove that for any \( t > 0 \)

\[
\sup_{\varepsilon \in [0,1]} \mathbb{E}[|Y_t^{\varepsilon}(x)|^2] < \infty.
\]

For any \( x \in \mathbb{R} \) and \( t \geq 0 \), inequality (A.16) yields

\[
\mathbb{E}[|Y_t^{\varepsilon}(x)|^2] \leq \tilde{\psi}_t(x),
\]

where \( (\tilde{\psi}_t(x), t \geq 0) \) is the solution of

\[
\begin{aligned}
\frac{d}{dt} \tilde{\psi}_t(x) &= \tilde{G}(\psi_t(x)), \\
\tilde{\psi}_0(x) &= x
\end{aligned}
\]

\[
\tilde{G}(\psi_t(x)) = \frac{V'(b_\varepsilon \psi_t(x))}{b_\varepsilon^{1+\alpha}}
\]

\[
\psi_t(x) = \int_0^t \frac{V'(b_\varepsilon \psi_s(x))}{b_\varepsilon^{1+\alpha}} \, ds + \psi_0(x).
\]
with \( \tilde{G}(y) := -2c_s|y|^{1+\alpha/2} + 1 \) for all \( y \in \mathbb{R} \) for some \( c_s > 0 \). The monotone convergence theorem with the help of Lemma A.1 implies
\[
(\text{B.5}) \quad \mathbb{E} \left[ |Y_t^\varepsilon(\infty)|^2 \right] \leq \tilde{\psi}_t(\infty) < \infty.
\]
To conclude (B.2) note that (B.5) yields,
\[
\sup_{\varepsilon \in [0,1]} \mathbb{P}(|Y_0^\varepsilon(\infty)| > b) \leq \frac{\tilde{\psi}_t(\infty)}{b^2} \to 0 \quad \text{as} \quad b \to \infty.
\]

B.2. Uniform convergence in total variation distance. The bound (B.38) is a consequence of the following proposition.

**Proposition B.2.** For any \( t > 0 \), \( a > 0 \), \( b > a \) and \( \eta > 0 \) there is \( \varepsilon_0 > 0 \) for which
\[
(\text{B.6}) \quad \sup_{0 \leq \varepsilon < \varepsilon_0} \sup_{x \in [a,b]} d_{\text{TV}}(Y_t^0(x), Y_t^\varepsilon(x)) < \eta.
\]

**Proof.** By Theorem 5.1 in [51], we have
\[
\left| d_{\text{TV}}(Y_t^0(x), Y_t^\varepsilon(x)) \right|^2 \leq 2 \int_0^t \mathbb{E} \left[ \left| F_0(Y_s^0(x)) - F_\varepsilon(Y_s^\varepsilon(x)) \right|^2 \right] ds.
\]
To conclude (B.6) we show that
\[
(\text{B.7}) \quad \lim_{\varepsilon_0 \to 0} \sup_{0 \leq \varepsilon < \varepsilon_0} \sup_{x \in [a,b]} \sup_{s \in [0,t]} \mathbb{E} \left[ \left| F_0(Y_s^0(x)) - F_\varepsilon(Y_s^\varepsilon(x)) \right|^2 \right] = 0.
\]
First we define the event
\[
(\text{B.8}) \quad A_{M,\varepsilon} = A_{M,\varepsilon}(x,t) := \left\{ \sup_{s \in [0,t]} |Y_s^0(x)| \vee |Y_s^\varepsilon(x)| \leq M \right\}.
\]
Now, for any \( M > 0 \), we may write the expectation term in (B.7) as
\[
\mathbb{E} \left[ \left| F_0(Y_s^0(x)) - F_\varepsilon(Y_s^\varepsilon(x)) \right|^2 1_{A_{M,\varepsilon}} \right] + \mathbb{E} \left[ \left| F_0(Y_s^0(x)) - F_\varepsilon(Y_s^\varepsilon(x)) \right|^2 1_{A_{M,\varepsilon}}^c \right].
\]
Since \( M > 0 \) is arbitrary, to prove (B.7) it suffices to show that for any \( M > 0 \)
\[
(\text{B.9}) \quad \lim_{\varepsilon_0 \to 0} \sup_{0 \leq \varepsilon < \varepsilon_0} \sup_{x \in [a,b]} \sup_{s \in [0,t]} \mathbb{E} \left[ \left| F_0(Y_s^0(x)) - F_\varepsilon(Y_s^\varepsilon(x)) \right|^2 1_{A_{M,\varepsilon}}^c \right] = 0
\]
and that
\[
(\text{B.10}) \quad \lim_{M \to \infty} \sup_{\varepsilon \in [0,1]} \sup_{x \in [a,b]} \sup_{s \in [0,t]} \mathbb{E} \left[ \left| F_0(Y_s^0(x)) - F_\varepsilon(Y_s^\varepsilon(x)) \right|^2 1_{A_{M,\varepsilon}}^c \right] = 0.
\]
**Proof of (B.9).** Let \( \Delta_t^\varepsilon(x) := Y_t^\varepsilon(x) - Y_t^0(x) \). We recall that for each \( \varepsilon \in [0,1] \) the process \( Y_t^\varepsilon(x) = (Y_t^\varepsilon(x), t \geq 0) \) solves (3.33). Moreover, the processes \( (Y_t^\varepsilon(x), \varepsilon \in [0,1]) \) are coupled with the same noise, i.e., there is \( (B_t, t \geq 0) \) a Brownian motion under \( \mathbb{P} \) such that for all \( \varepsilon \in [0,1] \) we have almost surely that
\[
(\text{B.11}) \quad Y_t^\varepsilon(x) = x + \int_0^t F_\varepsilon(Y_s^\varepsilon(x)) ds + B_t \quad \text{for all} \quad t > 0.
\]
Now, we may use (B.11) to write
\[
\Delta_t^\varepsilon(x) = Y_t^\varepsilon(x) - Y_t^0(x) = \int_0^t \left[ F_\varepsilon(Y_u^\varepsilon(x)) - F_0(Y_u^0(x)) \right] du.
\]
By the mean value theorem we have
\[ \Delta^\varepsilon_s(x) = \int_0^s [F'_\varepsilon(Y^\varepsilon_u(x)) - F'_\varepsilon(Y^0_u(x)) - (F_0(Y^0_u(x)) - F'_\varepsilon(Y^0_u(x))] \, du \]
\[ = \int_0^s [F'_\varepsilon(\Theta^\varepsilon_u(x)) \Delta^\varepsilon_u(x) - (F_0(Y^0_u(x)) - F'_\varepsilon(Y^0_u(x))] \, du, \]
where \( \Theta^\varepsilon_u \in (Y^\varepsilon_u(x) \wedge Y^0_u(x), Y^\varepsilon_u(x) \vee Y^0_u(x)) \) for all \( u \geq 0 \). By the convexity of \( V \), it follows that \( F'_\varepsilon(\Theta^\varepsilon_u) \leq 0 \). By the chain rule and the fact that \( |x| \leq 1 + x^2 \) for all \( x \in \mathbb{R} \) we have that
\[ |\Delta^\varepsilon_s(x)|^2 = |\Delta^\varepsilon_0(x)|^2 + \int_0^s 2 \Delta^\varepsilon_u(x) d\Delta^\varepsilon_u(x) \]
\[ = 2 \int_0^s [F'_\varepsilon(\Theta^\varepsilon_u)|\Delta^\varepsilon_u(x)|^2 - \Delta^\varepsilon_u(x)(F_0(Y^0_u(x)) - F'_\varepsilon(Y^0_u(x))] \, du \]
\[ \leq 2 \int_0^s (1 + |\Delta^\varepsilon_u(x)|^2) |F_0(Y^0_u(x)) - F'_\varepsilon(Y^0_u(x))| \, du. \]

If we let \( \psi^\varepsilon_s(x) := \sup_{u \in [0,s]} |\Delta^\varepsilon_s(x)|^2 1_{A^c_{M,\varepsilon}} \) and \( K(M, \varepsilon) := \sup_{|z| \leq M} |F_0(z) - F'_\varepsilon(z)| \) we obtain that \( \psi^\varepsilon_s(x) \leq 2K(M, \varepsilon) \int_0^s (1 + \psi^\varepsilon_u(x)) \, du \) for any \( x \in \mathbb{R} \). Now, for any fixed \( M > 0 \), by Hypothesis \( 2 \) we have \( K(M, \varepsilon) \to 0 \) as \( \varepsilon \to 0 \). This implies that for any \( \eta > 0 \) and \( M > 0 \) there is \( \varepsilon_0 \) such that \( \sup_{\varepsilon \in [0,\varepsilon_0]} \sup_{x \in \mathbb{R}} \psi^\varepsilon_s(x) \leq \eta \). This completes the proof of (B.9).

**Proof of (B.10).** Since \((r_1 + r_2)^2 \leq 2(r_1^2 + r_2^2)\) for any \( r_1, r_2 \in \mathbb{R} \), the expectation in (B.10) can be bounded by \( 2\mathbb{E} \left[ |F_0(Y^0_s(x))|^4 1_{A^c_{M,\varepsilon}}(x,s) \right] + 2\mathbb{E} \left[ |F'_\varepsilon(Y^\varepsilon_s(x))|^4 1_{A^c_{M,\varepsilon}}(x,s) \right] \). It remains to show that
\[ \lim_{M \to \infty} \sup_{\varepsilon \in [0,1]} \sup_{x \in [a,b]} \sup_{s \in [0,t]} \mathbb{E} \left[ |F'_\varepsilon(Y^\varepsilon_s(x))|^4 1_{A^c_{M,\varepsilon}}(x,s) \right] = 0. \]

By Cauchy–Schwarz inequality, we have
\[ \mathbb{E} \left[ |F'_\varepsilon(Y^\varepsilon_s(x))|^4 1_{A^c_{M,\varepsilon}} \right] \leq \left( \mathbb{E} \left[ |F'_\varepsilon(Y^\varepsilon_s(x))|^8 \right] \right)^{1/2} \cdot \left( \mathbb{P} (A^c_{M,\varepsilon}) \right)^{1/2} \]
\[ \text{We now claim that} \]
\[ \sup_{\varepsilon \in [0,1]} \sup_{x \in [a,b]} \sup_{s \in [0,t]} \mathbb{E} \left[ |F'_\varepsilon(Y^\varepsilon_s(x))|^8 \right] < \infty, \]
and that, recall (B.8), for any fixed \( t > 0 \)
\[ \sup_{\varepsilon \in [0,1]} \sup_{x \in [a,b]} \mathbb{P}(A^c_{M,\varepsilon}(x,t)) \to 0 \quad \text{as} \quad M \to \infty. \]

From (B.12), (B.13) and (B.14) we conclude (B.10). It remains to prove (B.13) and (B.14).

**Proof of (B.13).** We first note that Hypothesis \( 2 \) and Hypothesis \( 3A \) (Condition \( G2 \)) imply that there is \( \tilde{c} > 0 \) such that for any \( z \in \mathbb{R} \)
\[ |V'(z)| \leq \tilde{c}|z|^{1+\alpha} \exp(z^2). \]
Therefore
\[ \mathbb{E} \left[ |F'_\varepsilon(Y^\varepsilon_s(x))|^8 \right] = \mathbb{E} \left[ |V''(b_\varepsilon Y^\varepsilon_s(x))b_\varepsilon^{1+\alpha}|^8 \right] \]
\[ \leq \tilde{c}^8 \mathbb{E} \left[ |Y^\varepsilon_s(x)|^{8(1+\alpha)} \left( 1 + \exp(8b_\varepsilon |Y^\varepsilon_s(x)|^2) \right) \right]. \]
Now, since \( b_\varepsilon \to 0 \) as \( \varepsilon \to 0 \) and \( |z|^{1+\alpha} \leq \exp(z^2) \) for all \( z \in \mathbb{R} \), there is \( \tilde{C} > 0 \) for which

\[
\mathbb{E} \left[ |F_\varepsilon(Y^\varepsilon_s(x))|^8 \right] \leq \tilde{C} \mathbb{E} \left[ \exp(|Y^\varepsilon_s(x)|^2) \right].
\]

To conclude, we now show that

\[
\text{(B.15)} \quad \sup_{\varepsilon \in [0,1]} \sup_{x \in [a,b]} \sup_{s \in [0,t]} \mathbb{E} \left[ \exp(|Y^\varepsilon_s(x)|^2) \right] = C(t, a, b) < \infty.
\]

Indeed, by Itô’s formula for \( H(z) = \exp(z^2), \ z \in \mathbb{R} \), we have that

\[
H(Y^\varepsilon_s(x)) = H(x) + \int_0^s H(Y^\varepsilon_u(x))(2Y^\varepsilon_u(x)F_\varepsilon(Y^\varepsilon_u(x)) + 2(Y^\varepsilon_u(x))^2 + 1) \, du + M_s,
\]

where \( (M_s, s \geq 0) \) is a local martingale. Recall that \( G_\varepsilon(z) = zF_\varepsilon(z) \) for all \( z \in \mathbb{R} \). By \text{(A.11)} we deduce that

\[
\sup_{\varepsilon \in [0,1]} \sup_{z \in \mathbb{R}} (2zF_\varepsilon(z) + 2z^2 + 1) = C < \infty.
\]

Therefore, if we let \( \tau_K = \tau(K, \varepsilon, x) := \inf \{ s > 0 : |Y^\varepsilon_s(x)| > K \} \) we obtain

\[
\text{(B.16)} \quad \mathbb{E} \left[ H(Y^\varepsilon_{s\wedge \tau_K}(x)) \right] \leq H(x) + C \int_0^s \mathbb{E} \left[ H(Y^\varepsilon_{u\wedge \tau_K}(x)) \right] \, du.
\]

Now, by Grönwall’s inequality we obtain for \( x \in [a,b] \) and \( s \in [0,t] \) that

\[
\mathbb{E} \left[ H(Y^\varepsilon_{s\wedge \tau_K}(x)) \right] \leq H(x) \exp(Cs) \leq (H(a) + H(b)) \exp(Ct).
\]

Since the constant \( C \) in \text{(B.16)} does not depend on \( \varepsilon \), Fatou’s lemma imply

\[
\text{(B.17)} \quad \sup_{\varepsilon \in [0,1]} \sup_{x \in [a,b]} \sup_{s \in [0,t]} \mathbb{E} \left[ \exp(|Y^\varepsilon_s(x)|^2) \right] = \sup_{\varepsilon \in [0,1]} \sup_{x \in [a,b]} \sup_{s \in [0,t]} \mathbb{E} \left[ \liminf_{K \to \infty} H(Y^\varepsilon_{s\wedge \tau_K}(x)) \right]
\]

\[
\leq \sup_{\varepsilon \in [0,1]} \sup_{x \in [a,b]} \limsup_{K \to \infty} \mathbb{E} \left[ H(Y^\varepsilon_{s\wedge \tau_K}(x)) \right]
\]

\[
\leq (H(a) + H(b)) \exp(Ct),
\]

which yields \text{(B.15)}. This completes the proof of \text{(B.13)}.

**Proof of (B.14).** Since \( zF_\varepsilon(z) = G_\varepsilon(z) \leq 0 \) for all \( z \in \mathbb{R} \), it follows from \text{(A.12)} that

\[
\sup_{s \in [0,t]} |Y^\varepsilon_s(x)|^2 \leq x^2 + t + \sup_{s \in [0,t]} |M^\varepsilon_{s,x}|,
\]

where \( M^\varepsilon_{s,x} = (M^\varepsilon_{s,x}, s \geq 0) \) is the local martingale given by \text{(A.13)}. Therefore, for any \( x \in [a,b] \) and any \( M \) such that \( M > a^2 + b^2 + t \)

\[
\mathbb{P} \left( \sup_{s \in [0,t]} |Y^\varepsilon_s(x)|^2 > M \right) \leq \mathbb{P} \left( x^2 + t + \sup_{s \in [0,t]} |M^\varepsilon_{s,x}| > M \right)
\]

\[
= \mathbb{P} \left( \sup_{s \in [0,t]} |M^\varepsilon_{s,x}|^2 > (M - x^2 - t)^2 \right)
\]

\[
\leq \frac{\mathbb{E} \left[ |M^\varepsilon_{t,x}|^2 \right]}{(M - x^2 - t)^2} = \frac{C_{\varepsilon,t,x}}{(M - x^2 - t)^2}.
\]
where the second inequality follows from Doob's $L^2$ submartingale inequality and by Itô's isometry, $C_{t,x} := 4 \int_0^t \mathbb{E} \left[ |Y_s^x(x)|^2 \right] ds$. Now, since $z^2 \leq \exp(z^2)$ for all $z \in \mathbb{R}$, by (B.17) it follows that

$$\sup_{\varepsilon \in [0,1]} \sup_{x \in [a,b]} \sup_{t \in [0,d]} C_{\varepsilon,t,x} = C'(t, a, b) < \infty.$$ 

Now, recall the definition in (B.8). Since $x^2 \leq a^2 + b^2$ for any $x \in [a, b]$, it follows that for any $M$ such that $M > a^2 + b^2 + t$

$$\sup_{\varepsilon \in [0,1]} \sup_{x \in [a,b]} \mathbb{P}(A_{\varepsilon,x}^c(x, t)) \leq \frac{C'(t, a, b)}{(M - a^2 - b^2 - t)^2}$$

and so, we obtain (B.14) and thereby conclude the proof of Proposition B.2.

**Appendix C. Complements**

In this section we include, for completeness of the exposition, a few results that have been used throughout the text with a brief explanation.

**Proof of Lemma 4.2** We apply Theorem 3.3.4 of [77, Ch. 3, p.110]. By Hypothesis 3 for all $|z| \geq R$ we have $-V'(z)|z |^{1+\kappa} \leq -c|z|^\rho - \kappa$ for any $\kappa \in (0, \rho)$, and therefore

$$\lim_{|z| \to \infty} \left( -V'(z)|z |^{-1-\kappa} \right) = -\infty < 0.$$ 

Hence, the field $-V'$ satisfies the drift condition eq. (3.3.4) in [77, Ch. 3, p.86]. By Theorem 3.3.4 of [55] for any $c > 0$ there are $C_1 = C_1(c, \kappa, \varepsilon) > 0$ and $C_2 = C_2(c, \kappa, \varepsilon) > 0$ such that

$$d_{TV}(X_t^{x}(x), X_t^{y}(y)) \leq C_1 e^{-C_2 t} e^{c|z|} e^{c|y|}$$

for any $x, y \in \mathbb{R}$, $t \geq 0$, which implies the existence of the invariant measure $\mu^c$. By Hypothesis 3 we have $\int_{\mathbb{R}} e^{c|z|} \mu^c(dz) < \infty$. Therefore, Theorem 3.3.4 in [55] yields [23]. Moreover, formula (2.4) follows from Proposition 4.2 in [77, p.110].

Let $C^2$ represent the set of twice continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$.

**Lemma C.1 (Existence of a regular potential).** Assume that $V$ satisfies Hypothesis 4 and Hypothesis 2 with $\alpha > 0$. For each $M > 0$, there exist an even $C^2$ convex function $V_M = V_{M,\alpha} : \mathbb{R} \to [0, \infty)$ and positive constants $c = c_{M,\alpha}$, $C = C_{M,\alpha}$ and $R = R_{M,\alpha}$ such that

\begin{equation}
V_M(z) = V(z) \quad \text{for } |z| \leq M \tag{C.1}
\end{equation}

and

\begin{equation}
V'_M(z) \geq cz^{1+\alpha} \quad \text{and} \quad |V_M'(z)| \leq Ce^{z^2} \quad \text{for all } z \geq R. \tag{C.2}
\end{equation}

In particular, the potential $V_M$ satisfies Hypothesis 4, Hypothesis 2 and Hypothesis 3A.

**Proof.** The proof follows by a standard mollifier procedure. We mimic the lines given in Proposition 4.10 of [12]. Let $g : \mathbb{R} \to [0, 1]$ be an increasing $C^\infty$-function such that $g(u) = 0$ for $u \leq 1/2$, $g(u) = 1$ for $u \geq 1$, and $g(u) \in (0, 1)$ for all $u \in (1/2, 1)$. Let $M > 0$ be fixed and define

$$G_M(u) = \left( 1 - g \left( \frac{u^2}{2M^2} \right) \right) V''(u) + g \left( \frac{u^2}{2M^2} \right) |u|^\alpha \quad \text{for all } u \in \mathbb{R}.$$ 

Observe that $G_M(u) = V''(u)$ for all $|u| \leq M$, and $G_M(u) = |u|^\alpha$ for all $|u| \geq \sqrt{2}M$. We note that $G_M$ is a non-negative continuous function and then we set $H_M(u) := \int_0^u G_M(y)dy$ for all $u \in \mathbb{R}$. Finally we define $V_M(z) := \int_0^{|z|} H_M(u)du$ for all $z$. Since $G_M$ is an even function, it follows
that \( H_M \) is odd and \( V_M \) is again even. Now, since \( V_M(0) = V(0) = 0 \), \( V'_M(0) = V'(0) = 0 \) and \( V''_M(z) = V''(z) \) for \( z \leq M \) it follows that \( V_M \) satisfies (C.1). Moreover, since there is \( C > 0 \) for which \(|u|^a \leq C \exp(u^2)\) for all \( u \in \mathbb{R} \) it follows that \( V_M \) satisfies (C.2).

**Proposition C.1** (Disintegration inequality). Suppose that \( \{X(x) = (X_t(x), t \geq 0), x \in S\} \) and \( Y = \{Y(y) = (Y_t(y), t \geq 0), y \in S\} \) are Markov families on the measurable space \((S, S)\) and defined on the same probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Then, for all \( r, s > 0, a, b \in S \), the following disintegration inequality for the total variation distance holds:

\[
\mathbb{D}_{TV}(X_{r+s}(a), Y_{r+s}(b)) \leq \int_{S^2} \mathbb{D}_{TV}(X_s(x), Y_s(y)) \mathbb{P}(X_r(a) \in dx, Y_r(b) \in dy).
\]

**Proof.** Write \( t = r + s \). Since the families \( \{X(x), x \in S\} \) and \( \{Y(y), y \in S\} \) are Markovian and are defined on the same probability space, for any \( a \in S \) and \( B \in S \) we have that

\[
\mathbb{P}(X_t(a) \in B) = \int_S \mathbb{P}(X_s(x) \in B) \mathbb{P}(X_r(a) \in dx) = \int_{S^2} \mathbb{P}(X_s(x) \in B) \mathbb{P}(X_r(a) \in dx, Y_r(a) \in dy).
\]

Similarly, we have that

\[
\mathbb{P}(Y_t(a) \in B) = \int_{S^2} \mathbb{P}(Y_s(y) \in B) \mathbb{P}(X_r(a) \in dx, Y_r(a) \in dy).
\]

Therefore, from the definition of total variation distance, together with (C.3) and (C.4) we obtain that

\[
\mathbb{D}_{TV}(X_t(a), Y_t(b)) = \sup_{B \in S} |\mathbb{P}(X_t(a) \in B) - \mathbb{P}(Y_t(b) \in B)|
\]

\[
= \sup_{B \in S} \left| \int_{S^2} \left( \mathbb{P}(X_s(x) \in B) - \mathbb{P}(Y_s(y) \in B) \right) \mathbb{P}(X_r(a) \in dx, Y_r(b) \in dy) \right|
\]

\[
\leq \int_{S^2} \mathbb{D}_{TV}(X_s(x), Y_s(y)) \mathbb{P}(X_r(a) \in dx, Y_r(b) \in dy).
\]

**Proposition C.2** (Support theorem for diffusions). For any \( x \in \mathbb{R} \) and \( \varepsilon \in [0, 1] \) let \( Y^\varepsilon(x) = (Y^\varepsilon_t(x), t \geq 0) \) be the solution of (3.33). For each fixed \( t > 0 \), the law of \( Y^\varepsilon_t(x) \) is absolutely continuous with respect to the Lebesgue measure and it has full support on \( \mathbb{R} \).

**Proof.** Fix \( \varepsilon \in [0, 1] \). Now, write for simplicity \( Y_t = Y^\varepsilon_t(x) \), \( F = F_\varepsilon \) and note that, almost surely, for every \( t \geq 0 \)

\[
Y_t = x + \int_0^t F(Y_s) \, ds + B_t.
\]

The proof is done in two steps. On the first step, following the ideas in [10], we prove that for any \( t > 0 \) the law of \( Y_t \) denoted by \( \mu_t \) is absolutely continuous with respect to the Lebesgue measure on \( \mathbb{R} \). Let \( \rho_t \) represent a density of \( \mu_t \), i.e. for any \( a, b \in \mathbb{R} \) with \( a < b \)

\[
\mathbb{P}(Y_t \in [a, b]) = \mu_t([a, b]) = \int_a^b \rho_t(z) \, dz.
\]
On the second step, we prove, with the help of the maximum principle in [73], that $\rho_{t+s}(z) > 0$ for all $z \in \mathbb{R}$. Since $t > 0$ and $s > 0$ are arbitrary, this completes the proof that $\mu_t(dz) = \rho_t(z)dz$ with $\rho_t(z) > 0$ for all $t > 0$, i.e. the law of $Y_t$ has full support.

We remark that a standard localization argument is not straightforward with the methods in [40]. Indeed, as the authors themselves say

“Our result might be deduced from [Aronson-1968] by a localization argument, however, we did not succeed in this direction.”

**Step 1.** We adapt to our case the proof of Theorem 2.1 in [40]. This means that $dY_t = b(Y_t)dt + \sigma(Y_t)dB_t$ with $b(z) = F(z)$ and $\sigma(z) = 1$ for all $z \in \mathbb{R}$. Since $b$ is not bounded by a linear function we cannot apply Theorem 2.1 in [40] directly. However, the field $F$ is convex and drives the trajectories towards the origin which allows us to obtain $L^2$ bounds and replicate the main steps in the proof. Moreover, the noise term is simpler and this allows us to ignore the auxiliary function $f_\delta$ defined in Lemma 1.2 in [40].

Now for $\delta \in (0,t)$, consider the random variable $Z_\delta := Y_{t-\delta} + B_t - B_{t-\delta}$. Note that for any $b \in \mathbb{R}$

(C.6) $|\mathbb{E}[\exp(ibZ_\delta)|F_{t-\delta}]| = |\exp(ibY_{t-\delta} - \delta b^2/2)| = \exp(-\delta b^2/2),$

where $(F_t, t \geq 0)$ is the natural filtration of the Brownian motion $B = (B_t, t \geq 0)$ and $i$ is the unit imaginary. By (C.5) it follows that

$$Y_t - Z_\delta = \int_{t-\delta}^{t} F(Y_s) \, ds.$$

By [13], there is $C = C_t$ such that $\sup_{s \in [0,t]} \mathbb{E}[|F(Y_s)|^2] \leq C^2$ and therefore by Jensen’s and Cauchy–Schwarz’s inequalities

(C.7) \begin{align*}
(\mathbb{E}[|Y_t - Z_\delta|]^2 &\leq \mathbb{E}[|Y_t - Z_\delta|^2] = \mathbb{E}
\left[\left(\int_0^\delta F(Y_{t-\delta+s}) \, ds\right)^2\right]
\leq \delta \int_0^\delta \mathbb{E}[|F(Y_{t-\delta+s})|^2] \leq C^2 \delta^2.\end{align*}

Let $\mu_t$ be the law of $Y_t$ and let $\hat{\mu}_t$ be the characteristic function of $\mu_t$ defined by $\hat{\mu}_t(b) := \mathbb{E}[\exp(ibY_t)]$. We note that for any $\delta \in (0,t)$ and $b \in \mathbb{R}$, by (C.6) and (C.7), we have

(C.8) \begin{align*}
|\hat{\mu}_t(b)| = |\mathbb{E}[\exp(ibY_t)]| &\leq |\mathbb{E}[\exp(ibZ_\delta)]| + |b| \mathbb{E}[|Y_t - Z_\delta|]
\leq \exp(-\delta b^2/2) + C |b| \delta.
\end{align*}

Let $R_t > 0$ be such that $(\log |b|)^2/b^2 < t$ when $|b| > R_t$. For each $b$ with $|b| \geq R_t$ we choose $\delta_b := (\log |b|)^2/b^2$ and so the bound in (C.8) implies that

$$|\hat{\mu}_t(b)| \leq \exp(-\delta_b b^2/2) + C |b| \delta_b = \exp(-(\log |b|)^2/2) + C(\log |b|)^2/b.$$

Since $|\hat{\mu}_t(b)| \leq 1$ for all $b \in \mathbb{R}$ it follows that $\int_{-\infty}^{\infty} |\hat{\mu}_t(b)|^2 \, db < \infty$ and so, by Lemma 1.1 in [40] it follows that $\mu_t$ has density in $\mathbb{R}$.

**Step 2.** Note that $\rho_{t+s}$ is the solution of

(C.9) \begin{align*}
Lu + F'u = 0
\end{align*}

at time $s$ with initial condition $\rho_t$ where $Lu := \partial_t u - \frac{1}{2}(\partial_x)^2 u + F \partial_x u$. By Theorem 3 in [73], $(\rho_{t+h}, h \geq 0)$ is a non-negative solution of (C.9) and therefore either $\rho_{t+s}(z) > 0$ for all $z \in \mathbb{R}$ or $\rho_{t+s}(z) = 0$ for all $z \in \mathbb{R}$, and since $\int_{\mathbb{R}} \rho_{t+s}(z) \, dz = 1$ it follows that $\rho_{t+s}(z) > 0$ for all $z \in \mathbb{R}$. This concludes the proof. □
We conclude this section with the following result.

**Lemma C.2.** For all $x \in \mathbb{R}$ the function $t \mapsto G_t(x)$ defined in (3.32) is continuous and strictly decreasing in $t$.

**Proof.** Let $x \in \mathbb{R}$ be fixed. By the triangle inequality for all $t > 0$ and $s > 0$ we have

$$|G_x(t) - G_x(s)| \leq d_{TV}(Y_t(\text{sgn}(x)\infty), Y_s(\text{sgn}(x)\infty)).$$

Then it is enough to show that the right-hand side of the preceding inequality tends to zero as $t \to s$. For short, we write $Y_u = Y_u(\text{sgn}(x)\infty)$, $u \geq 0$. By Proposition C.2 it follows that for every $t > 0$, the law of $Y_t$ is absolutely continuous with respect to the Lebesgue measure and has a full support density $\rho_t(y)$. Moreover, $(\rho_t(y))_{t \geq 0}$ solves the so-called Fokker–Planck equation

$$\partial_t \rho_t(y) = \frac{1}{2} \partial_y^2 \rho_t - \partial_y (F_0(y) \rho_t(y)),$$

where $F_0$ is as defined in (3.32), see for instance [83, Section 2.2]. Then $\lim_{t \to s} \rho_t(y) = \rho_s(y)$ for all $y \in \mathbb{R}$ and therefore by Scheffé’s lemma, see [83, Lemma 3.3.2, p. 95], we have $\lim_{t \to s} d_{TV}(Y_t, Y_s) = 0$. This completes the proof that $t \mapsto G_t(x)$ is continuous.

We now turn to the proof that $G_t(x)$ is strictly decreasing in $t$. Recall that

$$G_x(t) = d_{TV}(Y_t(\text{sgn}(x)\infty), \nu) \quad \text{for} \quad t \geq 0.$$

Let $x \in \mathbb{R}$ and $t > 0$ be fixed. By (3.39) we have $G_x(t) < 1$. For short let $\theta_{x,t} := G_x(t)$ and denote the law of $Y_t(\text{sgn}(x)\infty)$ by $\mu_{x,t}$. Let $(P_s)_{s \geq 0}$ be the semigroup associated to the Markov process $(Y_s(z), s \geq 0, z \in \mathbb{R})$ and note the invariance $P_s(\nu) = \nu$, $s \geq 0$. Since $\theta_{x,t}$ is the total variation distance between $Y_t(\text{sgn}(x)\infty)$ and $\nu$, there exists a coupling between $\mu_{x,t}$ and $\nu$ such that $\mu_{x,t} = (1 - \theta_{x,t})\nu + \theta_{x,t}\eta_{x,t}$, where $\eta_{x,t}$ is a probability measure on $\mathbb{R}$. By the semigroup property we have for any $s > 0$

$$d_{TV}(\mu_{x,t+s}, \nu) = d_{TV}(P_s(\mu_{x,t}), \nu) = d_{TV}((1 - \theta_{x,t})P_s(\nu) + \theta_{x,t}P_s(\eta_{x,t}), \nu) = d_{TV}((1 - \theta_{x,t})\nu + \theta_{x,t}P_s(\eta_{x,t}), \nu) = \theta_{x,t}d_{TV}(P_s(\eta_{x,t}), \nu).$$

Now, we claim that $d_{TV}(P_s(\eta_{x,t}), \nu) < 1$. Indeed, by disintegration we have

$$d_{TV}(P_s(\eta_{x,t}), \nu) \leq \int_{\mathbb{R}} d_{TV}(P_s(z), \nu)\eta_{x,t}(dz) = \int_{\mathbb{R}} G_z(s)\eta_{x,t}(dz).$$

Hence, $d_{TV}(P_s(\eta_{x,t}), \nu) = 1$ if and only if $G_z(s) = 1$ for $z$-almost surely with respect to the measure $\eta_{x,t}$. This yields a contradiction with (3.39) and hence the proof that the function $G_x$ is strictly decreasing is finished. \qed

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