SUPERSYMMETRY AND SUPERCOHERENT STATES
OF A NONRELATIVISTIC FREE PARTICLE

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Coordinate atypical representation of the orthosymplectic superalgebra \(osp(2/2)\) in a Hilbert superspace of square integrable functions constructed in a special way is given. The quantum nonrelativistic free particle Hamiltonian is an element of this superalgebra which turns out to be a dynamical superalgebra for this system. The supercoherent states, defined by means of a supergroup displacement operator, are explicitly constructed. These are the coordinate representation of the known atypical abstract super group \(OSp(2/2)\) coherent states. We interpret obtained results from the classical mechanics viewpoint as a model of classical particle which is immovable in the even sector of the phase superspace and is in rectilinear movement (in the appropriate coordinate system) in its odd sector. © 1997 American Institute of Physics. [S0022-2488(97)00809-8]

I. INTRODUCTION

The supersymmetry in physics has been introduced in the quantum field theory for unifying of interactions of different kinds in a unique construction [1]. Supersymmetric formulation of quantum mechanics is due to the problem of spontaneous supersymmetry breaking [2]. Ideas of supersymmetry have been profitably applied to many nonrelativistic quantum mechanical problems since, and now there are no doubts that the supersymmetric quantum mechanics has its own right to exist (see for Ref. [2] a recent review). It is worth noticing that almost all papers are concerned with the stationary Schrödinger or Pauli equations. There are only a few papers dealing with nonstationary equations [4, 5].

The mathematical foundation of the conventional the quantum mechanics consists in the operator theory in a Hilbert space [6, 7]. The notion of Hilbert space is also indispensable in the construction of unitary representations of Lie groups. The space of square integrable on Lebesgue measure functions is one of the most important realization of the Hilbert space.

When we pass from the conventional quantum mechanics to the supersymmetric ones and from conventional Lie algebras and groups to superalgebras and supergroups we need the notion of Hilbert superspace. There are few works about Hilbert superspaces [8, 9]. It seems that mathematically rigorous and consistent theory of Hilbert superspaces and the theory of operators acting in these superspaces need to be developed.

In this paper for a particular case of nonrelativistic free particle we construct a Hilbert superspace which is a \(\mathbb{Z}_2\)-graded infinite-dimensional linear space equipped with a super-Hermitian form (superscalar product) and in some sense complete. Solutions of the free particle Schrödinger equation form a dense set in this superspace.

The notion of coherent states is widely used in the conventional quantum mechanics and mathematical physics [10-12]. Many definitions of coherent states exist [11]. The more suitable for generalization to the supersymmetric case is the one based the on group-theoretical approach developed by Perelomov for a wide class of conventional Lie groups [13]. This definition has a natural
generalization to Lie supergroups and superalgebras based on the notion of supergroup translation operator [13].

In a number of papers [14, 15] abstract representations of some simple Lie superalgebras are studied and with their help supercoherent states have recently been constructed [5, 13], [16]-[19] and underlying geometric structure has been envisaged [18, 19]. Nevertheless, application of these results to the quantum mechanics is not numerous [5, 13]. The supercoherent states of a charged spin-\(\frac{1}{2}\) particle in a constant magnetic field [13] and in a time-varying electromagnetic field [5] are explored and interpreted in a physical context. The role of Grassmann variables is clarified and insight is gained into the link between supercoherent states and the classical motion. The fermion monopole system which is known to have a dynamical \(OSp(1/2)\) supersymmetry is considered and their supercoherent states are obtained [13].

In this paper we show that not only the above-mentioned quantum systems may be interpreted in terms of supersymmetric notions but every nonrelativistic one-dimensional quantum system with quadratic in coordinate \(x\) Hamiltonian exhibits supersymmetric properties. We concentrate our attention on a simple but nontrivial case of the nonrelativistic free particle which in our interpretation has \(OSp(2/2)\) dynamical supersymmetry. More precisely, the space of solutions of the Schrödinger equation for the free particle is an atypical Lie \(osp(2/2)\)-module. We use the notion of supergroup both as a \(z\)-graded group and a superanalytic supermanifold [20]. The action of a supergroup operator translation is defined on a dense set in the Hilbert superspace. This operator maps the dense set from the Hilbert superspace onto the Grassmann envelope of the second kind of the Hilbert superspace. Being applied to a maximal symmetry vector (in our case the vacuum vector) this operator produces supercoherent states for the free particle which are the coordinate representation of the known atypical \(osp(2/2)\) coherent states. These states are parametrized by the points of the \(N = 1\) superunit disk \(\mathbb{D}^{(1|1)}\). The supermanyfold \(\mathbb{D}^{(1|1)}\) is a phase superspace of a classical system possessing remarkable property, namely, geometric quantization of this system gives a superholomorphic representation of the initial (i.e., free particle) quantum system. By these means we construct a classical mechanics system which corresponds to the nonrelativistic quantum free particle. Finally we interpret the obtained classical system as a classical particle which is immovable in the even sector of the phase superspace and is in rectilinear movement in its odd sector.

The paper is organized as follows. In Sec. II we summarize the well-known results about the representation in the Hilbert space of symmetry algebra of the free particle Schrödinger equation we need further. Section III includes two parts. In the first one we recall main notions about the superanalysis and in the second we construct a Hilbert superspace of square integrable functions. In Sec. IV we define the action of operators in constructed Hilbert superspace which are symmetry operators for the free particle Schrödinger equation and realize an atypical coordinate representation of the \(osp(2/2)\) superalgebra. In Sec. V the coherent states for the nonrelativistic free particle are constructed. In Sec. VI we discuss obtained results, compare them with the known ones, and interpret from the classical mechanics viewpoint.

II. SCHRODINGER ALGEBRA

In this section we summarize briefly the well-known constructions [21] for a representation in the Hilbert space of square integrable on Lebesgue measure functions of the Schrödinger algebra \(G_2\) which is a dynamical symmetry algebra for the nonrelativistic free particle.

Consider the free particle Schrödinger equation

\[
i\partial_t \chi(x,t) = h \chi(x,t), \quad h = -\partial_x^2, \quad \partial_x^2 = \partial_x \cdot \partial_x, \quad \partial_x = \partial / \partial_x.
\]  

Solutions of this equation pertaining to the space \(L_2(\mathbb{R})\) of square integrable functions on full real
axis with respect to Lebesgue measure are well known [21]:

\[ \chi_n(x, t) = \langle x \mid n, t \rangle = (-i)^n \sqrt{n!} \left[ \frac{\sqrt{2\pi}}{(1 + it)} \right]^{-1/2} \times \exp \left( -in \arctan t - \frac{x^2}{4 + 4it} \right) H e_n(z), \]

where \( H_n(z) \) are the Hermitean polynomials. Let \( L \) be lineal (linear hull) of the functions \{\( \chi_n(x, t) \)\}. Introduce the notations \( \psi_n(x, t) = \chi_{2n}(x, t) = \chi_{2n}(-x, t) \) and \( \phi_n(x, t) = \chi_{2n+1}(x, t) = -\chi_{2n+1}(-x, t) \), \( n = 0, 1, 2, \ldots \) and denote \( L^0 \) the lineal of even functions \{\( \psi_n(x, t) \)\} and \( L^1 \) the lineal of odd ones \{\( \phi_n(x, t) \)\}.

If we introduce the scalar product (Hermitian form) in \( L \) in a usual way,

\[ \langle \chi_1(x, t) \mid \chi_2(x, t) \rangle = \int_{-\infty}^{\infty} \overline{\chi_1(x, t)} \chi_2(x, t) dx, \]

where overline signifies the complex conjugation and the integral should be understood in the sense of Lebesgue, then the completion \( \overline{L} \) of the lineal \( L \) with respect to the measure induced by this scalar product gives the Hilbert space \( H \). We denote \( \langle \cdot \mid \cdot \rangle \) the restriction of the scalar product (3) on the lineals \( L^j \), \( j = 0, 1 \). The completion of the lineals \( L^j \) with respect to the norms generated by the appropriate scalar products produces the Hilbert subspaces \( H^j = \overline{L^j} \). It is clear that for the lineal \( L \) we have the orthogonal sum \( L = L^0 \oplus L^1 \) and for the space \( H \) we have the orthogonal decomposition \( H = H^0 \oplus H^1 \). By these means we obtain the well-known constructions [21] of the Hilbert space structure on the solutions of the Schrödinger equation (1).

Symmetry operators for the equation (1) are defined as usual as the operators which transform any solution of this equation to another solution of the same equation. These operators realize a Hilbert space structure on the solutions of the Schrödinger equation (1).

The operators (4) are defined on the dense set \( L \subset H \) and \( L \) is the Lie \( \mathcal{G}_2 \)-module. Moreover, every operator from (4) is skew symmetric with respect to the scalar product (3) and, consequently, with the help of the exponential mapping we may construct a group of unitary operators for (2).

Since we are not interested in the other representations of the algebra \( \mathcal{G}_2 \) we shall denote \( \mathcal{G}_2 = \text{span} \{K_{0, \pm 1}, K_{\pm 2}, K^0\} \) where span stands for the linear hull over the real number field \( \mathbb{R} \) if we want to have the real algebra \( \mathcal{G}_2(\mathbb{R}) \) and over the complex number field \( \mathbb{C} \) for its complex form \( \mathcal{G}_2(\mathbb{C}) \).

Let us pass in \( \mathcal{G}_2(\mathbb{C}) \) to another basis more suitable for our purpose. Consider \( a^\pm = \frac{1}{2}(iK_{-1} \pm K_1) \). Operators \( \{a^\pm, I\} \), where \( I \) is the identity operator, form the basis of the Heisenberg-Weil algebra \( w_1 \). Consider now the quadratic combinations of \( a^\pm \), i.e., \( k_\pm = 2(a^\pm)^2 \), \( k_0 = a^+a^- + a^-a^+ \). Since for every \( \chi \in \mathcal{L} \) equation (1) represents an operator identity \( \partial_x^2 = -i\partial_t \), operators \( k_0 \) and \( k_\pm \) acting in \( \mathcal{L} \) may be considered as of the first degree in \( \partial_x \) and \( \partial_t \). It is easy to see that these operators form another basis in subalgebra \( sl(2, \mathbb{C}) = \text{span} \{K^0, K_{\pm 2}\} \subset \mathcal{G}_2(\mathbb{C}) \) and \( \mathcal{G}_2(\mathbb{C}) = sl(2, \mathbb{C}) \# w_1(\mathbb{C}) \). Its real form \( sl(2, \mathbb{R}) \) is isomorphic to \( su(1,1) \). Moreover, it is apparent that \( \mathcal{L}^0 \) and \( \mathcal{L}^1 \) are irreducible Lie \( su(1,1) \)-modules of the weight \( 1/4 \) and \( 3/4 \) respectively. Note that the operators \( a^\pm \) map the lineals \( \mathcal{L}^0 \) and \( \mathcal{L}^1 \) one into another. We note as well the following conjugation properties of the operators from \( \mathcal{G}_2 \) with respect to the scalar product (3): \( (a^\pm)^\dagger = a^\mp \), \( k_0^\dagger = k_0 \), \( (k_\pm)^\dagger = k_\mp \), which hold on the lineal \( \mathcal{L} \).

To conclude this section we would like to notice that the procedure specified above may be applicable to a wide class of Hamiltonians of the form \( \hat{h} = -\partial_x^2 + A(t)x^2 + B(t)x + C(t) \). This assertion follows from the well-known fact that the nonstationary Schrödinger equation with this Hamiltonian has integrals of motion (symmetry operators) \( \tilde{a}^\pm \) that depend only on the derivative \( \partial_x \) and form a representation of the Heisenberg-Weil algebra \( w_1 \) [12]. Their quadratic combinations being apparently symmetry operators may be expressed only through the \( \partial_x \) and \( \partial_t \) on the
space of the solutions of the nonstationary Schrödinger equation. These operators form a representation of the algebra $su(1,1)$. The semidirect sum of representations of $su(1,1)$ and $w_1$ gives a representation of the Schrödinger algebra $G_2$. The Hilbert space structure on the solutions of the Schrödinger equation is introduced with the help of the well-known constructions of the discrete series representation of the Schrödinger algebra. The latter is built of two Lie irreducible $su(1,1)$ modules in the same way as above.

III. HILBERT SUPERSPACE OF SQUARE INTEGRABLE FUNCTIONS

A. Basic definitions

In what follows we shall use the Grassmann-valued analysis. It is worth mentioning that for some notions several distinct definitions exist in the literature. In these cases we give the definition we use.

The more suitable approach to the superanalysis for our purpose is the one described in Ref. 23 and based on the theory of functions in Banach spaces and the theory of Banach algebras. The basis notion in this approach is the notion of commutative Banach superalgebra introduced as follows.

Let $\Lambda$ be a $\mathbb{Z}_2$-graded linear space $\Lambda = \Lambda_0 \oplus \Lambda_1$. When an element $a \in \Lambda_0$, it is called even [parity $p(a) = 0$] and when $a \in \Lambda_1$ it is called odd [parity $p(a) = 1$]. The elements from $\Lambda_0$ and $\Lambda_1$ are called homogeneous. When the structure of associative algebra with unit $e \in \Lambda_0$ and even multiplication operation [i.e., $p(ab) = p(a) + p(b)$, mod 2 for homogeneous $a$ and $b$] is introduced in $\Lambda$ it is called superalgebra. Superalgebra $\Lambda$ is called commutative if supercommutator $[a,b] = ab - (-1)^{p(a)p(b)}ba = 0$ for homogeneous $a,b \in \Lambda$. Further, the commutative superalgebra $\Lambda = \Lambda_0 \oplus \Lambda_1$ is supposed to be a Banach space with the norm $\|f\| \leq \|f\| \cdot N\|g\|$, $f,g \in \Lambda$, $\|e\| = 1$. The components $\Lambda_0$ and $\Lambda_1$ are closed subspaces in $\Lambda$. When $\Lambda$ is defined over the real number field $\mathbb{R}$ we obtain the real superalgebra $\Lambda(\mathbb{R})$, and for the case of the complex number field $\mathbb{C}$ we obtain its complex form $\Lambda(\mathbb{C})$.

Given a real superalgebra $\Lambda(\mathbb{R})$ real superspace $\mathbb{R}_\Lambda^{m,n}$ of dimension $(m,n)$ over $\Lambda(\mathbb{R})$ is defined as follows:

$$\mathbb{R}_\Lambda^{m,n} = \Lambda_0 \otimes \ldots \otimes \Lambda_0 \otimes \Lambda_1 \otimes \ldots \otimes \Lambda_1$$

A complex superspace $\mathbb{C}_\Lambda^{m,n}$ over $\Lambda(\mathbb{C})$ is defined in the same way but with the help of the complex superalgebra $\Lambda(\mathbb{C})$. If for every point $X = (x, \theta) = (x_1, \ldots, x_m, \theta_1, \ldots, \theta_n) \in \mathbb{R}_\Lambda^{m,n}$ we introduce the norm $\|X\|^2 = |x|^2 + \|\theta\|^2 = \sum_{k=0}^m x_k^2 + \sum_{j=1}^n \theta_j^2$, then $\mathbb{R}_\Lambda^{m,n}$ becomes a Banach space. Every connected open set $\mathcal{O} \subset \mathbb{R}_\Lambda^{m,n}$ is called domain in $\mathbb{R}_\Lambda^{m,n}$.

Let us have two superspaces $\mathbb{R}_\Lambda^{m,n}$ and $\mathbb{R}_{\Lambda'}^{m',n'}$ with the norms $\| \cdot \|$ and $\| \cdot \|'$, $\Lambda \subseteq \Lambda'$, and a domain $\mathcal{O}$ in $\mathbb{R}_\Lambda^{m,n}$. Function $f(X) : \mathcal{O} \rightarrow \mathbb{R}_{\Lambda'}^{m',n'}$ is called continuous in the point $X \in \mathcal{O}$ if $\|f(X + H) - f(X)\|' \rightarrow 0$ when $\|H\| \rightarrow 0$. The same function is called superdifferentiable from the left in the point $X \in \mathcal{O}$ if elements $F_k(X) \in \mathbb{R}_{\Lambda'}^{m',n'}$, $k = 1, \ldots, m + n$, such that

$$f(X + H) = f(X) + \sum_{k=1}^{m+n} H_k F_k(X) + \tau(X,H),$$

where $\|\tau(X,H)\|' / \|H\| \rightarrow 0$ when $\|H\| \rightarrow 0$ exist. The functions $F_k(x)$ are called left partial superderivatives of $f$ with respect to $X_k$ in the point $X \in \mathcal{O}$:

$$F_k(x) = \frac{\partial f(X)}{\partial X_k}, \quad F_{m+j}(x) = \frac{\partial f(X)}{\partial X_{m+j}}, \quad k = 1, \ldots, m, \quad j = 1, \ldots n.$$
The expression $\sum_{k=1}^{m+n} H_k \frac{\partial f(X)}{\partial X}$ is called left superdifferential of the function $f(X)$ in the point $X$. One can find more details about superanalysis in Ref. [23].

**B. Hilbert superspace**

Consider the real superspace $\mathbb{R}^{1,1}_\Lambda$ defined over $\Lambda(\mathbb{R}) = \Lambda_0(\mathbb{R}) \otimes \Lambda_1(\mathbb{R})$ where $\Lambda_0(\mathbb{R}) = \mathbb{R}$ and $\Lambda_1(\mathbb{R})$ has two generators $\theta$ and $\overline{\theta}$ with the properties $\theta^2 = \overline{\theta}^2 = \theta \overline{\theta} + \overline{\theta} \theta = 0$, $x, \theta = \theta$. The complex superspace $\mathbb{C}^{1,1}_\Lambda$ is defined over $\Lambda(\mathbb{C}) = \Lambda_0(\mathbb{C}) \otimes \Lambda_1(\mathbb{C})$ where $\Lambda_0(\mathbb{C}) = \mathbb{C}$ and $\Lambda_1(\mathbb{C})$ has the same generators $\theta$ and $\overline{\theta}$.

Consider now functions from $\mathbb{R}^{1,1}_\Lambda$ to $\mathbb{C}^{1,1}_\Lambda$ of the following form: $\Psi^0 (t, x, \theta, \overline{\theta}) = \psi(x, t)$, $\psi(x, t) \in H^0$ and $\Psi^1 (t, x, \theta, \overline{\theta}) = \theta \varphi(x, t)$, $\varphi(x, t) \in H^1$. We shall designate the collection of the functions $\Psi^0 (t, x, \theta, \overline{\theta})$ and $\Psi^1 (t, x, \theta, \overline{\theta})$ as $H_{\overline{\theta}}$ and $H_{\theta}$, respectively. It follows from these constructions that $H_{\overline{\theta}}$ and $H_{\theta}$ are linear spaces (over the field $\mathbb{C}$), and $H_s = H_{\overline{\theta}} \oplus H_{\theta}$ is a $\mathbb{Z}_2$-graded linear space of functions. The elements from $H_{\overline{\theta}}$ and $H_{\theta}$ are called homogeneous with the parity $p(\Phi) = 0$ when $\Phi \in H_{\overline{\theta}}$ and $p(\Phi) = 1$ when $\Phi \in H_{\theta}$.

Define in the space $H_s$ scalar product (super Hermitian form) as follows:

$$
(\Phi_1 | \Phi_2) = \int_{\mathbb{R}} (t, x, \theta, \overline{\theta}) \Phi_2 (t, x, \theta, \overline{\theta}) \, e^{-\overline{\theta} \theta \, dx \, d\theta \, d\bar{\theta} \, d\bar{\theta}} \in \mathbb{C}.
$$

(5)

Since the integration in superspaces is developed in Ref. [23] for sufficiently smooth functions (it is a super generalization of various integral constructions based on Riemann integral and not on Lebesque integral) we should make more precise the sense of integral in (5). If functions $\Phi_1$ and $\Phi_2$ are defined by their homogeneous components $\Phi_l (x, \theta, \overline{\theta}) = \Phi_l^0 (x, \theta, \overline{\theta}) + \Phi_l^1 (x, \theta, \overline{\theta})$, $\Phi_l^0 (x, \theta, \overline{\theta}) = \chi_l^0 (x) \in H_{\overline{\theta}}$, and $\Phi_l^1 (x, \theta, \overline{\theta}) = \theta \chi_l^1 (x) \in H_{\theta}$, $l = 1, 2$, and functions $\chi_l^0 (x)$, $\chi_l^1 (x)$, $j = 0, 1$, are sufficiently smooth, then we may interpret the integral (5) in the sense defined in Ref. [23]. In our case this integral becomes equal to a product of two integrals. The first one is a conventional integral with respect to the variable $x$ and the second one is an integral with respect to the Grassmann variables $\theta$ and $\overline{\theta}$. The only integral with respect to the Grassmann variables different from zero is $\int \overline{\theta} \theta \, dx \, d\theta \, d\bar{\theta} \, d\bar{\theta} = 1$. Thus, for the integral (5) we obtain the expression

$$
(\Phi_1 | \Phi_2) = (\Phi_1^0 | \Phi_2^0)_0 + (\Phi_1^1 | \Phi_2^1)_1,
$$

(6)

$$
(\Phi_l^1 | \Phi_j^1)_j = i^l \langle \chi_l^0 | \chi_j^0 \rangle_{j}, \chi_l^0 \in H^0, l = 1, 2, j = 0, 1.
$$

We note that the spaces $H_{\overline{\theta}}$ and $H_{\theta}$ are mutually orthogonal with respect to the scalar product (5) and are complete in the sense we shall make more precise so that $\langle \cdot | \cdot \rangle_j$, $j = 0, 1$, are the restrictions of the scalar product (5) on the spaces $H_{\theta}$.

In the case when functions $\chi_l^0 \in L_2(\mathbb{R})$ are not sufficiently smooth for applying the definition of the integral given in Ref. [23], we directly apply the formula (5) for calculating the integral (5). We remind the reader that the scalar product $\langle \cdot | \cdot \rangle$ in $L_2(\mathbb{R})$ is defined with the help of the Lebesgue integral. We will notice that the formula (5) is in accord with the definition of the super-Hermitian form in the abstract Hilbert superspace given in Ref. [19]. The super-Hermitian form (5) is positive definite in the sense that the Hermitian forms $\langle \cdot | \cdot \rangle_j$, $j = 0, 1$, from which it is expressed are positive definite.

The super-Hermitian form generates a norm in $H_s$. For every $\Phi = \Phi^0 + \Phi^1 \in H_s$, $\Phi^0 = \chi^0 (x, t)$, and $\Phi^1 = \theta \chi^1 (x, t)$ we put by definition

$$
\| \Phi \|^2 = |(\Phi | \Phi)|^2 = || \chi^0 ||_{0}^2 + || \chi^1 ||_{1}^2,
$$

(7)

where $|| \cdot ||_j$ are the norms in $H^j$, $j = 0, 1$, generated by the appropriate scalar products. It is not difficult to see that the properties of the norm so defined correspond to the axioms of the conventional norm: (i) $\| \Phi \| \geq 0$, (ii) $\| \Phi \| = 0$ if and only if $\Phi = 0$, (iii) $c | \Phi = c | \cdot \| \Phi \|$,
\( \forall \theta \in \mathbb{C} \), (iv) \( \Phi_1 + \Phi_2 \| \leq \| \Phi_1 \| + \| \Phi_2 \| \). It follows that \( H_s \) is a normed space in the usual sense. Conditions (i), (iii), and (iv) mean that the norm is a convex functional in \( H_s \) (see, e.g., Ref. 24). Condition (ii) means that the set \( \{ \| \cdot \| \} \) formed from a single convex functional is sufficient for defining a (strong) topology in \( H_s \). The space \( H_s \) becomes a locally convex topological space. Just in this sense we shall understand the completeness of the space \( H_s \) which we shall call the Hilbert superspace. In fact, this signifies that the space \( H_s \) contains only linear functions of the variable \( \theta \) with the coefficients from \( H \). Since the functions \( \Psi_n^0(t, x, \theta, \overline{\theta}) = \psi_n(x, t) \) and \( \Psi_n^1(t, x, \theta, \overline{\theta}) = \theta \varphi_n(x, t) \) form bases in the spaces \( H_1^\perp \) and \( H_{1-1}^\perp \), respectively, we have obtained a separable Hilbert superspace. It is worth noticing that other definitions of the Hilbert superspace exist [9].

IV. ATYPICAL COORDINATE REPRESENTATION OF \( osp(2/2) \)

We may now define the action of operators in the space \( H_s \). Let us put \( K_0 = k_0 = h/2 \) and \( K_{\pm} = k_{\pm} \). These operators by definition act only on the variable \( x \) and do not affect the Grassmann variable \( \theta \). This signifies that they have the even parity. Since the operators \( k_0 \) and \( k_{\pm} \) are defined on the field \( \mathcal{L} = \mathcal{L}^0 \oplus \mathcal{L}^1 \), the operators \( K_0 \) and \( K_{\pm} \) are defined on the field \( \mathcal{L}_s = \mathcal{L}_s^\perp \oplus \mathcal{L}_s^\perp \) over the field \( \mathcal{C} \), where \( \mathcal{L}_s^\perp \) is the field over \( \mathcal{C} \) of the even functions \( \Psi^0_n(t, x, \theta, \overline{\theta}) \) and \( \mathcal{L}^1 \) is the field over \( \mathcal{C} \) of the odd functions \( \Psi^1_n(t, x, \theta, \overline{\theta}) \). It is clear that closure \( \mathcal{L}_s^\perp \) of the lineal \( \mathcal{L}_s^\perp \) with respect to the norm \( \| \cdot \| \) gives \( H_{1-1}^\perp \) and similar closure \( \mathcal{L}_s^\perp \) gives \( H_{1}^\perp \). Moreover, \( 2\mathcal{L}_s = \mathcal{L}_s^\perp \oplus \mathcal{L}_s^\perp = H_s \). Operators \( K_0 \) and \( K_{\pm} \) form a basis of subalgebra \( su(1,1) \) of the superalgebra under construction and lineals \( \mathcal{L}_s^\perp \) and \( \mathcal{L}_s^\perp \) are Lie irreducible \( su(1,1) \) modules.

We may define operators of the left multiplication by the variable \( \theta \) and the left differentiation \( \partial_\theta = \frac{\partial}{\partial \theta} \) (we define the left action of the operators on vectors) for the elements from \( \mathcal{L}_s \). It is clear that in our case \( \forall \Phi \in \mathcal{L}_s \), we have \( \partial_\theta \Phi \notin \mathcal{L}_s \) and \( \partial_\theta \Phi \notin \mathcal{L}_s \). Nevertheless, the same operators may be defined not only for the elements from \( \mathcal{L}_s \), but for every linear function of \( \theta \). Therefore operator \( B = \frac{1}{2} (\partial_\theta \partial_{\overline{\theta}} - \partial_{\overline{\theta}} \partial_\theta) \) is defined on the functions from \( \mathcal{L}_s \) and \( \mathcal{L}_s \) is its invariant space. Moreover, it is easy to see that \( B \Phi = -\frac{1}{4} (-1)^{p(\Phi)} \Phi \) for every homogeneous \( \Phi \in \mathcal{L}_s \).

Operators \( a^\pm \) map the lineals \( \mathcal{L}^0 \) and \( \mathcal{L}^1 \) one into another. With their help we construct the odd sector of the superalgebra under construction: \( V_\pm = \sqrt{2} a^\pm \theta \) and \( W_\pm = \sqrt{2} a^\pm \overline{\theta} \). Operators \( V_\pm \) map vectors from \( \mathcal{L}_s^\perp \) to vectors from \( \mathcal{L}_s^\perp \). Operators \( W_\pm \) realize the inverse mapping. In addition, \( V_\pm \Phi = 0 \forall \Phi \in \mathcal{L}_s^\perp \) and \( W_\pm \Phi = 0 \forall \Phi \in \mathcal{L}_s^\perp \).

It is an easy exercise to check that the set of operators \( \{K_0, K_{\pm}, B, V_\pm, W_\pm\} \) is closed with respect to the supercommutator \( [A, C] = AC - (-1)^{p(A)p(B)} CA \) and generalized Jacobi identity holds. The nonzero supercommutators are written as follows:

\[
\begin{align*}
[K_0, K_{\pm}] &= \pm K_{\pm}, \quad [K_-, K_+] = 2K_0, \quad [K_0, V_\pm] = \pm \frac{1}{2} V_\pm, \quad [K_0, W_\pm] = \pm \frac{1}{2} W_\pm, \\
[K_{\pm}, V_\pm] &= \mp V_\pm, \quad [K_{\pm}, W_\pm] = \mp W_\pm, \quad [B, V_\pm] = \frac{1}{2} V_\pm, \quad [B, W_\pm] = -\frac{1}{2} W_\pm, \\
[V_\pm, W_\pm] &= K_{\pm}, \quad [V_\pm, W_\pm] = K_0 + B.
\end{align*}
\]

Vector \( \Psi_0^0 \) has the properties

\[
\begin{align*}
K_0 \Psi_0^0 &= \frac{1}{4} \Psi_0^0, \quad B \Psi_0^0 = -\frac{1}{4} \Psi_0^0, \quad K_- \Psi_0^0 = V_- \Psi_0^0 = W_\pm \Psi_0^0 = 0.
\end{align*}
\]

It follows that the set of operators \( \{K_0, K_{\pm}, B, V_\pm, W_\pm\} \) realizes an atypical (coordinate) representation of the abstract orthosymplectic superalgebra \( osp(2/2) = osp(2/2)^\perp \oplus osp(2/2)^\perp \), where \( \mathfrak{sp}(2/2)^\perp = \text{span} \{K_0, K_{\pm}, B\} \) and \( \mathfrak{sp}(2/2)^\perp = \text{span} \{V_\pm, W_\pm\} \).

The nonrelativistic free particle Hamiltonian \( \hbar = -\partial_x^2 + (a^+ + a^-)^2 = \frac{1}{2} K_+ + \frac{1}{2} K_- + K_0 \) is an element of \( osp(2/2) \) superalgebra and, consequently, this algebra is a dynamical supersymmetry
algebra for this system. Moreover, its representation space $L_s$ is the space of solutions of the free particle Schrödinger equation

$$i\partial_t \Psi(t,x,\theta,\overline{\theta}) = h\Psi(t,x,\theta,\overline{\theta}).$$

Given the super-Hermitian forms (3) and (4) we define operator $A^+$ superadjoint to an $A$. An operator $A \in osp(2/2)$ is defined on the dense set $L_s$ in $H_s$. Then, for every homogeneous element $A \in osp(2/2)$, element $\Phi^*_1 \in H_s$ is uniquely defined by the equation

$$\left(\Phi^*_1 | \Phi_2\right) = (-1)^{p(\Phi_1)p(A)} \left(\Phi_1 | A\Phi_2\right), \quad \Phi_2 \in L_s,$$

for a homogeneous $\Phi_1 \in H_s$. Therefore we may put $\Phi^*_1 = A^+\Phi_1$. The domain of definition of operator $A^+$ is the collection of all $\Phi_1 \in H_s$ which verify the equation (8).

The Hilbert superspace introduced here is quite analogous to the conventional Hilbert space and we may use many conventional definitions (see, e.g., Refs. [6, 7, 24]). In particular, the definition of a closed operator remains unchanged. Then, since the operator $h$ is essentially self-adjoint in $H$, the operator $K_0$ is essentially self-adjoint in $H_s$. Operator $B$ is restricted and consequently closed in $H_s$. It is not difficult to see that $B^+ = B$. The operators $a$ and $a^+$ are defined in $L \subset H$ and are mutually conjugated. This involves the following conjugation properties $K_0 = K_0$, $K_0^+ = K_0$, $B^+ = B$, $V^+_L = iW_2$ and $W^+_L = iV_2$, valid in $L_s$. Moreover, $\forall A \in osp(2/2)$ the following relations hold in $L_s$: $(A^+)^+ = A$, $(AC)^+ = (-1)^{p(A)p(C)}A^+$, and $[A,C]^+ = -[A^+, C^+]$.

Given a 2 grading linear space $L_s = L_1 \oplus L_2$ [for example, the lineal $L_s$ or the superalgebra $osp(2/2)$] we should have the possibility to define the multiplication of the elements from $L_s$ on the elements from a complex commutative Banach superalgebra $\Lambda(\mathbb{C}) = \Lambda_0(\mathbb{C}) \oplus \Lambda_1(\mathbb{C})$. In particular, we need definition of the $\Lambda(\mathbb{C})$ envelope of the second kind $\tilde{L}_s$ of the space $L_s$. This definition is similar to the definition of the Grassmann envelope of the second kind of the space $L_s$ (Ref. [1]), where the role of a Grassmann algebra plays the algebra $\Lambda(\mathbb{C})$: $\tilde{L}_s = (\Lambda_0(\mathbb{C}) \otimes L_1) \oplus (\Lambda_1(\mathbb{C}) \otimes L_2) = \Lambda(\mathbb{C}) \otimes L_s$. The elements from $\Lambda(\mathbb{C})$ play the role of supernumbers, the elements from $\Lambda_0(\mathbb{C})$ play the role of c-numbers, and the elements from $\Lambda_1(\mathbb{C})$ play the role of a-numbers. This terminology corresponds to Ref. [8].

Let us make more precise the definition of the complex conjugation in $\tilde{L}_s$ and $\Lambda(\mathbb{C})$. Put by definition

$$\overline{\beta_1\beta_2} = \overline{\beta_1}\overline{\beta_2}, \quad \overline{\beta\Phi} = \overline{\beta}\overline{\Phi}, \quad \forall \Phi \in L_s, \quad \forall \beta, \beta_1, \beta_2 \in \Lambda(\mathbb{C}).$$

(9)

This definition differs from a one widely used in literature $\overline{\beta_1\beta_2} = \overline{\beta_2}\overline{\beta_1}, \quad \beta_1, \beta_2 \in \Lambda(\mathbb{C})$ (Refs. [14, 8]) which gives for the product $\beta\overline{\beta}$ a real value independently on the parity of $\beta$. We shall use the definition (9) since in the other case one faces some inconsistencies. In particular, the super Kähler two-form becomes neither real nor imaginary, and it is difficult to establish the correspondence between physical observables and self-superadjoint operators [14].

We shall use the expression (8) for calculating of the scalar product of the elements from $\tilde{H}_s$. Definition (8) realizes in this case the following mapping: $\tilde{H}_s \otimes \tilde{H}_s \rightarrow \Lambda_0(\mathbb{C})$. The rule of manipulation with the supernumbers in the scalar product

$$\left(\beta_1\Phi_1 | \beta_2\Phi_2\right) = (-1)^{p(\Phi_1)p(\beta_2)} \overline{\beta_1\beta_2} \left(\Phi_1 | \Phi_2\right),$$

where $\Phi_1$ and $\beta_2$ are homogeneous elements from $H_s$ and $\Lambda(\mathbb{C})$ and the rule of complex conjugation for the scalar product of homogeneous elements

$$\overline{\left(\Phi_1 | \Phi_2\right)} = (-1)^{p(\Phi_1)p(\Phi_2)} \left(\Phi_2 | \Phi_1\right).$$

follow from Eq. (8) as well.
V. FREE PARTICLE SUPERCOHERENT STATES

Supercoherent states are the direct generalization of coherent states for the conventional (non-super) Lie groups and algebras [10].

We follow the definition of supergroup given in Ref. [20]. An $(m,n)$-dimensional supergroup $G$ is both an abstract group and an $(m,n)$-dimensional superanalytic supermanifold $S_{\Lambda}^{m,n}$ with superanalytic mapping $G \otimes G \to G : (g_1, g_2) \to g_1 g_2^{-1}$. Superanalytic supermanifold $S_{\Lambda}^{m,n}$ is defined as a Hausdorff space with an atlas such that $S_{\Lambda}^{m,n}$ is locally homeomorphic to a flat superspace $R_{\Lambda}^{m,n}$ and the transition functions are superanalytic.

An operator of left translations on a supergroup has been used in Refs. [13, 18] for constructing supercoherent states for the algebra $osp(1,2)$ and in Ref. [19] for the algebra $osp(2,2)$. Using the same approach we pass in $osp(2,2)$ to super Hermitian base,

$$X_1 = K_0, \quad X_2 = B, \quad X_3 = K_+ + K_-, \quad X_4 = i(K_+ - K_-),$$

$$X_5 = V_+ - iW_-, \quad X_6 = V_- - iW_+, \quad X_7 = W_+ - iV_-, \quad X_8 = W_- - iW_+,$$

which has the property $X_j^+ = (-1)^{p(X_j)} X_j$. Further, the Grassmann envelop of $\overline{osp}(2,2)$ is considered. An arbitrary element $\overline{X}$ from $\overline{osp}(2,2)$ has the form

$$\overline{X} = \sum_{j=1}^{4} \xi_j^0 X_j + \sum_{j=1}^{4} \xi_j^1 X_{4+j}, \quad \xi_j^0 \in G^0(2), \quad \xi_j^1 \in G^1(2).$$

The set of left translations on the supergroup $OSp(2,2)$ is defined as follows [11]: $T(g) = \exp(i\overline{X})$, $g \in OSp(2,2)$.

The highest symmetry vector (fiducial state) in $H_s$ is $\Psi_0^0$. Its isotropy subalgebra consists of Cartan subalgebra of $osp(2,2,\mathbb{C})$ which is spanned by $\{B, K_0\}$, all lowering operators $\{K_-, V_-, W_-\}$, and one raising operator $W_+$. The latter is a consequence of the fact that we have obtained the atypical representation of the $osp(2,2)$ superalgebra [13]. Therefore, the $osp(2,2)$ coherent states in this case are the $osp(1,2)$ coherent states as well and are labeled by one complex parameter $z \in \mathbb{C}$, $|z| < 1$, and one Grassmann parameter $\alpha$. Since $z$ parametrizes the unit disc $D^{(1)}$, the corresponding supermanifold, realized in terms of coordinates $(z, \alpha)$, is called $N = 1$ superunit disc and denoted by $D^{(1)} = OSp(2,2)/U(1/1)$ where subgroup $U(1/1)$ has the generators $K_0, B$, and $W_\pm$. This reasoning leads to the following translation operator suitable in our case:

$$D'(z, \alpha) = \exp(z K_+ - \alpha V_+ - \frac{i}{2} W_-), \quad |z| < 1.$$  \hspace{1cm} (11)

where the use of the complex variables $(z, \alpha)$ instead of the real supernumbers $\xi_j^1$ has been made.

The action of operator (11) is defined on the elements from $\mathcal{L}_s \subset H_s$. It maps the lineal $\mathcal{L}_s$ onto $\mathcal{H}_s$ defined over $\Lambda(\mathbb{C}) = \Lambda_0(\mathbb{C}) \oplus \Lambda_1(\mathbb{C})$, where $\Lambda_0(\mathbb{C}) = \mathbb{C}$ and $\Lambda_1(\mathbb{C})$ is defined over $\mathbb{C}$ with the help of two Grassmann generators $\xi, \bar{\xi}$ and $\alpha, \overline{\alpha} \in \Lambda_1(\mathbb{C})$. Moreover, this operator preserves the value of the scalar product $(D'(z, \alpha) \Phi_1 | D'(z, \alpha) \Phi_2) = (\Phi_1 | \Phi_2) \forall \Phi_{1,2} \in \mathcal{L}_s$. This property characterizes a superisometric operator.

To rewrite operator (11) in the form of the ordered exponential factors we may use superextension [23] of the well-known Baker-Campbell-Hausdorff [26] relation. However, the existence of the relations $K_- \Psi_0^0 = 0$ and $W_- \Psi_0^0 = 0$ make it possible to use a simpler translation operator $D(z, \alpha) = \exp(z K_+ + \alpha V_+)$, instead. This operator does not preserve the scalar product. Therefore, we have to introduce a normalizing constant. Thus, for supercoherent states we obtain the following relation:

$$\Psi_{z\alpha}(t, x, \theta, \overline{\theta}) = N' \exp(z K_+ + \alpha V_+) \Psi_0^0(t, x, \theta, \overline{\theta}) = N \left( \psi_z(x, t) + \sqrt{2} \alpha \varphi_z(x, t) \right)$$  \hspace{1cm} (12)
where

\[ \psi_z(x,t) = \left( \frac{\sigma + i\alpha}{4\pi} \right)^{1/4} (\sigma + it)^{-1/2} \exp \left[ -\frac{x^2}{4(\sigma + it)} \right], \]

\[ \varphi_z(x,t) = a^+ \psi_z(x,t) = -\frac{ix}{\sigma + it} \psi_z(x,t), \]

\[ \sigma = \frac{1 - z}{1 + z}, \quad N = 1 + \frac{i\alpha z}{4(1 - z^2)}, \quad |z| < 1. \]

Function \( \psi_z(x,t) \) is the free particle coherent state obtained by applying the displacement operator for the algebra \( su(1,1) \) to the lowest vector \( \psi_0(x,t) \) of the representation with the weight \( k_0 = 1/4 \), and \( \varphi_z(x,t) \) is an analogous one (but non normalized to unity) corresponding to the weight \( k_1 = 3/4 \).

VI. DISCUSSION AND CONCLUDING REMARKS

The coherent states of the abstract orthosymplectic superalgebra \( osp(2/2) \) are studied in detail in Ref. [19]. If we expand the functions \( \psi_z(x,t) \) and \( \varphi_z(x,t) \) in terms of the basis functions \( \psi_n(x,t) \) and \( \varphi_n(x,t) \),

\[ \psi_z(x,t) = (1 - z\tau)^{1/4} \sum_{n=0}^{\infty} z^n \sqrt{\frac{\Gamma(n + \frac{1}{2})}{n!\Gamma(\frac{1}{2})}} \psi_n(x,t), \]

\[ \varphi_z(x,t) = \frac{1}{2} (1 - z\tau)^{1/4} \sum_{n=0}^{\infty} z^n \sqrt{\frac{\Gamma(n + \frac{1}{2})}{n!\Gamma(\frac{1}{2})}} \varphi_n(x,t), \]

we obtain the same formula that those given in Ref. [19] for the atypical abstract \( OSp(2/2) \) coherent states at \( \tau = 1/4 \) and \( b = -1/4 \). In that paper the geometric properties of the coherent states supermanifold are studied. It is established that their underlying geometries turn out to be those of supersymplectic \( OSp(2/2) \) homogeneous space possessing the super-Kähler structure; superunitary irreducible representation of \( OSp(2/2) \) supergroup in the super-Hilbert space of the superholomorphic in the superunit disc \( D^{(1|1)} \) (for the atypical in \( D^{(1|1)} \)) functions is explicitly constructed.

Given the supercoherent states (12) and the scalar product (5) we can calculate the classical observables in phase space \( D^{(1|1)} \). These are the covariant Berezin symbols of the \( osp(2/2) \) superalgebra generators: \( H^{cl} = \sum \alpha H |\alpha\rangle \langle \alpha|, H \in osp(2/2) \). Our calculation gives the following result:

\[ K^{cl}_0 = \frac{1 + |z|^2}{4(1 - |z|^2)} K_\alpha, \quad K^{cl}_+ = \frac{z}{2(1 - |z|^2)} K_\alpha, \quad K^{cl}_- = \frac{-\tau}{2(1 - |z|^2)} K_\alpha, \]

where

\[ K_\alpha = \left( 1 + \frac{i\alpha z}{1 - |z|^2} \right) \]

and

\[ V^{cl}_+ = \frac{i\alpha}{2(1 - |z|^2)}, \quad V^{cl}_- = \frac{i\alpha z}{2(1 - |z|^2)}, \]

\[ W^{cl}_+ = \frac{-\tau z}{2(1 - |z|^2)}, \quad W^{cl}_- = \frac{-\tau}{2(1 - |z|^2)}. \]
Note that the even quantities $K^c_0$ and $K^c_\pm$ completely coincide with those given in Ref. [13] at $\tau = 1/4$ and $b = -1/4$, but for the odd ones we have the different sign. This difference is due to the phase factor $-i^n$ in the basis functions (2).

Using the potential of the super-Kähler metric $f(z, \bar{z}, \alpha, \bar{\alpha}) = \log |(0, \bar{0})|^2$, we may calculate the supersimplectic form $\omega$ and then the Hamiltonian vector superfields $X_H$ associated to a classical observables $H^c$. The same supersymplectic form $\omega$ is used to define a Poisson superbracket in the space of smooth functions on $D^{(1|1)}$ and obtain by these means a Poisson superalgebra. All these quantities are the straightforward generalization of the usual (nonsuper) Hamiltonian mechanics (see, e.g., Ref. [24]), which in our case is the Hamiltonian mechanics of the free particle in $D^{(1|1)}$ phase superspace. The geometric quantization of this classical mechanics gives the quantum mechanics of the free particle we started from, but in the superholomorphic representation. The reader may find the detailed calculations in Ref. [13].

We will now discuss another interpretation of our results which is a generalization of the conventional (nonsuper) interpretation of the free particle squeezed states presented in Ref. [25]. Note that since $\psi_z(x)$ is an even function and $\varphi_z(x)$ is an odd one, we have $\langle \psi_z | x \rangle \varphi_z = 0$ and $\langle \varphi_z | x \rangle \psi_z = 0$ where $p = -i\partial/\partial x$. Using the expressions of $x$ and $p$ in terms of the operators $a^\pm$: $x = 2tp + 2i(a^+ - a^-)$ and $p = -(a^+ + a^-)$, we express the products $xp$ and $p\theta$ in terms of the superalgebra generators $V_\pm$:

$$p\theta = -\frac{1}{\sqrt{2}} (V_+ + V_-), \quad xp = 2tp + i\sqrt{2} (V_+ - V_-).$$

With the help of the expressions for $V_\pm$ (13) we find the expectation values of these quantities in the state $\Psi_{z\alpha}$:

$$\langle p\theta \rangle_{z\alpha} = p_0 \bar{\alpha}, \quad \langle xp \rangle_{z\alpha} = (2p_0 + x_0) \bar{\alpha},$$

where

$$x_0 = -\frac{1}{\sqrt{2}} \frac{1 - z}{(1 - z\bar{z})}, \quad p_0 = -\frac{i (1 + z)}{2\sqrt{2} (1 - z\bar{z})}.$$

If now we pass from the variables $z$ and $\bar{z}$ to $p_0$ and $x_0$ by putting $z = (ip_0 + \frac{1}{2}x_0) / (ip_0 - \frac{1}{2}x_0)$, we may conclude that the trajectory of a particle becomes a straight line in the odd sector of the superspace whereas in the even sector the particle is immovable because of the conditions $\langle \Psi_{z\alpha} | x \rangle \Psi_{z\alpha} = 0$ and $\langle \Psi_{z\alpha} | p \rangle \Psi_{z\alpha} = 0$.

In this paper a simpler example of the space of square integrable superfunctions is given. This space may be considered as a realization of a Hilbert super space. It is clear that in more complex cases we need to have a theory of measure for superspaces. In particular, to give a mathematically rigorous general concept of square integrable superfunctions the super generalization of the Lebesgue integral based on the Lebesgue measure is indispensable. We now have many interesting results obtained in supersymmetric quantum mechanics [3] but a mathematically rigorous and consistent base of this theory is far from completion.

As a final comment we note that our constructions of the Hilbert superspace $H_\omega$ are based on a natural grading of the conventional Hilbert space $H = H^0 \oplus H^1$. Therefore, these constructions are applicable not only to the free particle but to every system for which such a decomposition exists. In particular, minor modifications are necessary for obtaining a Hilbert superspace structure on the solutions of the Schrödinger equation with a Hamiltonian quadratic in $x$. Further, the representation of the $osp(2|2)$ superalgebra obtained in this paper is based on an infinite dimensional representation of the Schrödinger algebra $\mathfrak{g}_2$. It follows that every quantum system with the same symmetry algebra may be treated as a system possessing a dynamical $osp(2|2)$ supersymmetry. With the help of the operators $\bar{a}$ and $\bar{a}^+$ a representation of $osp(2|2)$ superalgebra suitable for this case may be constructed. Exponential mapping of the $OSp(2|2)$ generators gives superisometric supergroup operator translation which produces the supercoherent states of the system under consideration in the same way as it was made above.
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References

[1] Y. A. Golfand and E. P. Likhtman, JETP Lett. 13, 323 (1971); P. Ramond, Phys. Rev. D 3, 2415 (1971); A. Neveu and J. Schwarz, Nucl. Phys. B 31, 86 (1971).

[2] E. Witten, Nucl. Phys. B 188, 513 (1981); B 202, 253 (1982).

[3] F. Cooper, A. Khare and U. Sukhatme, Phys. Rep. 251, 267 (1995).

[4] V. G. Bagrov and B. F. Samsonov, Phys. Lett. A 210, 60 (1996).

[5] V. A. Kostelecky, V. I. Man’ko, M. M. Nieto and D. R. Truax, "Supersymmetry and a Time-Dependent Landau System," Preprint LA-UR-93-206 (Los Alamos, 1993) (Also available as e-print hep-th/9303068).

[6] J. V. Neumann von, *Mathematische grundlagen der quantenmechanik*. (Springer, Berlin, 1932).

[7] M. Reed and B. Simon, *Methods of Modern Mathematical Physics. 1. Functional Analysis*. (Academic, New York, 1972).

[8] B. S. DeWitt, *Supermanifolds*. 2nd ed. (Cambridge U. P., Cambridge, 1992).

[9] S. Naganachi and Y. Kobayashi, J. Math. Phys. 33, 4274 (1992).

[10] A. Perelomov, *Generalized Coherent States and Their Applications*. (Springer, Berlin, 1986).

[11] J. R. Klauder and B. -S. Skagerstam, *Coherent states: applications in physics and mathematical physics* (World Scientific, Singapore, 1985).

[12] I. A. Malkin and V. I. Man’ko, *Dinamical Symmetries and Coherent States of Quantum Systems*, (Nauka, Moskow, 1979).

[13] B. W. Fatyga, V. A. Kostelecky, M. M. Nieto and D. R. Truax, Phys. Rev. D 43, 1403 (1991).

[14] F. A. Berezin, *Introduction to Superanalysis* (Reidel, Dordrecht, 1987); Sov. J. Nucl. Phys. 29, 857 (1979); *The Method of Second Quantization*, (Academic, New York, 1966).

[15] V. Kac, Lect. Notes. in Math. 676, 597 (1978); M. Scheunert, ibid 716, 1 (1979); M. Scheunert, W. Nahm and V. Rittenberg, J. Math. Phys. 18, 155 (1977); K. Nishiyama, J. Alg. 141, 399 (1991).

[16] A. B. Balantekin, H. A. Schmitt, and B. R. Barret, J. Math. Phys. 29, 1634 (1988).

[17] A. B. Balantekin, H. A. Schmit, and P. Halse, J. Math. Phys. 30, 274 (1989).

[18] A. M. El Gradechi, J. Math. Phys. 34, 5951 (1993).

[19] A. M. El Gradechi and L. M. Nieto, "Supercoherent States, Super Kähler Geometry and Geometric Quantization," preprint CRM-1876, Montreal, 1994 (also available as e-print hep-th/9403103).

[20] A. Rogers, J. Math. Phys., 21, 1352 (1980); 22, 443 (1981); 22, 939 (1981).

[21] W. Miller, Jr., *Symmetry and Separation of Variables*, (Addison-Wesley, Reading, MA, 1977).
[22] D. A. Leites, Russ. Math. Surv. 35, 1 (1980).

[23] V. S. Vladimirov and I. V. Volovich, Dokl. Akad. Nauk SSSR. 269, 524 (1983); 273, 26 (1983); 276, 521 (1984); 285, 1042 (1985); Theor. Math. Phys. 59, 3 (1984); 60, 169 (1984).

[24] M. A. Naimark, Normed Rings. (GITTL, Moskow, 1956).

[25] V. A. Kostelecky, M. M. Nieto and D. R. Truax, J. Math. Phys. 27, 1419 (1986); V. A. Kostelecky and D. R. Truax, J. Math. Phys. 28, 2480 (1987); B. W. Fattyga, V. A. Kostelecky, and D. R. Truax, J. Math. Phys. 30, 291 (1989).

[26] J. E. Campbell, Proc. London Math. Soc. 34, 347 (1902); 35, 33 (1903); F. Hausdorff, Ber. Verh. Saechs. Akad. Wiss. Leipzig Math. Phys. K1. 2, 293, (1905); R. Gilmore, Lie Groups, Lie Algebras, and Some Their Applications (Wiley, New York, 1974).

[27] V. I. Arnold, Mathematical methods of classical mechanics. (Nauka, Moskow, 1989).

[28] M. M. Nieto and D. R. Truax, ”Displacement-operator squeezed states. II. Examples of time-dependent systems having isomorphic symmetry algebras,” preprint LA-UR-96-2756 (Los Alamos, 1996). (Also available as e-print quant-ph/9608009).