Abstract

The corrected capacity of a quantum channel is defined as the best one-shot capacity that can be obtained by measuring the environment and using the result to correct the output of the channel. It is shown that (i) all qubit channels have corrected capacity $\log 2$, (ii) a product of $N$ qubit channels has corrected capacity $N \log 2$, and (iii) all channels have corrected capacity at least $\log 2$. The question is posed of finding the channel with smallest corrected capacity in any dimension $d$.

1 Introduction and statement of results

Every quantum channel can be viewed as arising from the unitary interaction of a system with its environment. The resulting entanglement between system and environment is lost when the environment is ‘traced out’, thereby destroying the purity of the signal states and introducing noise into the system. Specifically, letting $S$ denote the system and $E$ its environment, the action of the channel $\Phi$
on $\mathcal{S}$ is obtained as
\[
\Phi(\rho) = \text{Tr}_\mathcal{E} \left[ U_{SE} (\rho \otimes \omega) U_{SE}^* \right]
\] (1)
where the unitary matrix $U_{SE}$ entangles the system and environment, and $\omega$ is a state in $\mathcal{E}$.

In a recent paper, Gregoratti and Werner [1] explored the extent to which the noise produced by the channel $\Phi$ could be removed by performing a measurement on the environment and using the result to correct the output state of the channel. To be specific, let $\{X_k\}$ denote a POVM acting on the environment. If the result 'k' is obtained from this measurement, then the output state is, up to normalization,
\[
\text{Tr}_\mathcal{E} \left[ (I \otimes X_k) U_{SE} (\rho \otimes \omega) U_{SE}^* \right] = A_k \rho A_k^*
\] (2)
where this expression defines the matrix $A_k$. If the measurement result is ignored, then the output state is just
\[
\sum_k A_k \rho A_k^* = \Phi(\rho).
\] (3)

In fact every Kraus representation of $\Phi$ arises in this way, as the effect of an unrecorded measurement on the environment. However, if the result of the measurement is recorded, then there is the possibility of making a correction to the output state, based on the measurement result. That is, one could apply a completely positive trace preserving map $R_k$ to the output, conditioned on receiving the result $k$ from the measurement. The resulting output state would then be
\[
\sum_k R_k(A_k \rho A_k^*).
\] (4)

Writing $\mathcal{A} = \{A_1, \ldots, A_N\}$ and $\mathcal{R} = \{R_1, \ldots, R_N\}$, this defines a new channel, which is a corrected version of $\Phi$, namely
\[
\Phi_{\mathcal{A},\mathcal{R}}(\cdot) = \sum_k R_k(A_k \cdot A_k^*).
\] (5)

This corrected channel $\Phi_{\mathcal{A},\mathcal{R}}$ may be less noisy than $\Phi$ if the maps $R_k$ are chosen well. For example, if $\Phi$ has a Kraus representation with operators $A_k = \sqrt{p_k} V_k$, where the $\{V_k\}$ are unitary and $\sum p_k = 1$, then by choosing $R_k(\cdot) = V_k^* (\cdot) V_k$ the channel can be corrected to the identity, that is $\Phi_{\mathcal{A},\mathcal{R}} = I$ in this
case. This is an extreme case of course, and in fact Gregoratti and Werner show that this can happen if and only if \( \Phi \) is such a 'random unitary' channel.

Nevertheless this example raises the question of determining the ‘best’ correction that can be achieved for a given channel \( \Phi \). We will use the 1-shot Shannon capacity of the corrected channel as a way to quantify ‘best’. That is, we consider the optimal combination of input states and output measurements for the channel \( \Phi_{A,R} \), in order to maximize the mutual information between input and output. This maximum mutual information is the Shannon capacity of the corrected channel \( C_{\text{Shan}}(\Phi_{A,R}) \). Furthermore, in order to find the overall best correction for \( \Phi \), we must maximize over choices of \( R \) to find the best correction for any Kraus representation of \( \Phi \), and then maximize this quantity over the choice of Kraus operators. Accordingly, we denote by \( \mathcal{K}(\Phi) \) the collection of all Kraus sets for \( \Phi \), that is all collections \( \{A_1, \ldots, A_N\} \) satisfying

\[
\sum_{k=1}^{N} A_k^* A_k = I
\]

and

\[
\Phi(\rho) = \sum_{k=1}^{N} A_k \rho A_k^*.
\]

Notice that different elements of \( \mathcal{K}(\Phi) \) may contain different numbers of matrices.

**Definition 1** The optimal corrected capacity for \( \Phi \) is

\[
C_{\text{corr}}(\Phi) = \sup_{A \in \mathcal{K}(\Phi)} \sup_{R} C_{\text{Shan}}(\Phi_{A,R}).
\]

It is nearly immediate, for example, that the optimal corrected capacity of a so-called classical-quantum (c-q) channel \([2]\) on \( \mathbb{C}^d \) is \( \log d \). By definition, a c-q channel \( \Phi \) can always be written in the form

\[
\Phi(\rho) = \sum_{k} \langle k | \rho | k \rangle \sigma_k
\]

for a set of density operators \( \{\sigma_k\} \) and orthonormal basis \( \{|k\rangle\} \). One possible choice for the operators \( R_k \) is then to set each to the constant map \( R_k(\sigma) = |k\rangle \langle k| \), in which case

\[
\Phi_{A,R}(\rho) = \sum_{k} |k\rangle \langle k| \rho |k\rangle \langle k|,
\]

which obviously has Shannon capacity \( \log d \).

We can now state our first result.
Theorem 2 For any qubit channel $\Phi$,

$$C_{\text{corr}}(\Phi) = \log 2$$ (11)

As (11) shows, every qubit channel can be corrected to a channel with full capacity by performing a measurement on the environment and correcting the output depending on the result. The fact that this is true for the completely noisy channel $\Phi(\rho) = 1/2 \mathbb{I}$ for example may seem surprising – however it reflects the fact that the information about the initial state is stored in the environment and in this case can be fully recovered by measurement.

Theorem 2 is proved by showing that for any qubit channel $\Phi$ it is possible to find two orthogonal input states which can be perfectly distinguished by making measurement-based corrections at the output. In fact this result holds in any dimension, and therefore provides the same lower bound on the optimal corrected capacity for any channel.

Theorem 3 For any channel $\Phi$,

$$C_{\text{corr}}(\Phi) \geq \log 2$$ (12)

Remarks

1) One could consider other measures of ‘best’ correction for a channel, for example the Holevo capacity. However, operationally this refers to making entangled measurements on outputs from multiple copies of the channel, and in this case it probably makes sense to also consider corrections which arise from entangled measurements on multiple copies of the environment, so this should be done in a more general setting.

2) For a qubit channel Theorem 2 says that it is always possible to achieve full transmission capacity by measuring the environment and applying corrections to the channel output. It follows that the same is true for a product of qubit channels, and furthermore this can be done by making independent measurements on the environment of each qubit.

3) In dimensions higher than two, the bound in (12) is certainly not tight. However it remains an open question to find a larger bound. For each dimension $d$ there is a worst-case channel (or channels) for which $C_{\text{corr}}(\Phi)$ takes its smallest value, so we could define

$$C_{\text{corr}}(d) = \inf \{C_{\text{corr}}(\Phi) : \Phi \text{ is CPT on } \mathbb{C}^d\}. \quad (13)$$

Then the question becomes: what are these channels, and what are these worst values?
2 Proof of Theorems

Theorem 2 is a special case of Theorem 3 and Theorem 3 can be deduced from the following result of Walgate et al [4]: any pair of orthogonal pure states in a bipartite system can be perfectly distinguished using LOCC. So if we use two orthogonal signal states $|\psi_1\rangle$ and $|\psi_2\rangle$ for the channel, then the entangled states $U|\psi_1\rangle$ and $U|\psi_2\rangle$ are orthogonal and hence can be perfectly distinguished by first measuring in $\mathcal{E}$, then using the result to select a measurement in $\mathcal{S}$. Hence the capacity of this corrected channel is at least log 2, and this proves the Theorem.

For completeness we include a direct proof of Theorem 3. The key idea is to find a Kraus representation $A_1,\ldots,A_N$ for $\Phi$ with the property that the first and second columns of every matrix $A_k$ are orthogonal, and to use the first two canonical basis vectors $|e_1\rangle$ and $|e_2\rangle$ as the signal states. Measuring the value ‘$k$’ on these states will produce either $A_k|e_1\rangle\langle e_1|A_k^*$ or $A_k|e_2\rangle\langle e_2|A_k^*$, and these are the projections onto the first and second column vectors of $A_k$ respectively. By assumption these are orthogonal, and therefore can be perfectly distinguished.

So the proof reduces to showing that every channel has a Kraus representation with this property. To show this, let $A_1,\ldots,A_N$ be any Kraus representation for $\Phi$, and define the $N \times N$ matrix $M(A)$ by

$$M(A)_{ij} = \text{Tr} A_i |e_1\rangle\langle e_2| A_j^* = \langle e_2| A_j^* A_i |e_1\rangle$$ (14)

So $M(A)_{ij}$ is the inner product of the first column of $A_i$ with the second column of $A_j$. Now let $V$ be any unitary $N \times N$ matrix, and define the matrices

$$B_i = \sum_{j=1}^{N} V_{ij} A_j.$$ (15)

Then $B_1,\ldots,B_N$ is also a Kraus representation for $\Phi$. Furthermore

$$M(B) = VM(A)V^*.$$ (16)

We now use the following interesting mathematical fact [3]: given the matrix $M(A)$, there is a unitary matrix $V$ so that all diagonal entries of $M(B)$ are equal. Since $\sum A_i^* A_i = I$ it follows that Tr $M(A) = 0$. Hence with this choice of $V$, all diagonal entries of $M(B)$ are zero. This means that for every matrix $B_i$, the first and second columns are orthogonal, and so $B_1,\ldots,B_N$ is the desired representation.

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