A note on generating functions

Suppose $A$ is an affine symplectic space. There is an affine-invariant view of generating functions of symplectic transformations of $A$. Namely, let $H$ be a function on $A$. At any point $x$ we take the vector $u_x$ defined by $d_xH = u_x\omega$ ($\omega$ is the symplectic form) and put it in $A$ so that $x$ lies in its middle. Then the map $\Phi_H$ sending the tails of $u_x$’s to their heads is a symplectic transformation:

![Diagram of symplectic transformation](image)

Notice that for infinitesimal $H$, this is the usual infinitesimal transformation generated by Hamiltonian $H$. The map $H \mapsto \Phi_H$ is a kind of Cayley transform: choosing an origin in $A$ (to turn it to a vector space) and restricting ourselves to quadratic forms, we get the usual Cayley transform $\mathfrak{sp} \to \text{Sp}$.

Symplectic transformations can be composed. The corresponding composition of generating functions is $H(x) = H_1(x_1) + H_2(x_2) + \text{symplectic area of } \triangle PQR$:

![Diagram of symplectic area](image)

Recall that the integral kernel of the Moyal product is $K(x_1, x_2, x) = \exp(\sqrt{-1} \times \text{symplectic area of } \triangle PQR/h)$. We may notice that $\exp(\sqrt{-1}H/h)$ is the classical part of the Moyal product of $\exp(\sqrt{-1}H_1/h)$ and $\exp(\sqrt{-1}H_2/h)$.

Let us have a look where these claims come from. A symplectic transformation of $A$ is (more-or-less) the same as a Lagrangian submanifold of $\tilde{A} \times A$ (the graph of the map). For each point $x \in A$ the symmetry with respect to $x$ is a symplectic map. Identity is also a symplectic map, so that we have many Lagrangian submanifolds of $\tilde{A} \times A$:

![Diagram of Lagrangian submanifolds](image)

In this way we have an isomorphism between $\tilde{A} \times A$ and $T^*A$. Explicitely (as one immediately sees from the picture), a pair $(P, Q) \in \tilde{A} \times A$ corresponds to $((P + Q)/2, (Q - P) \omega) \in T^*A$. Here the vector-and-its-midpoint picture appears.
Correspondence between generating functions and symplectic transformations is clear now: $dH$ is a Lagrangian submanifold of $T^*A$, and therefore of $\tilde{A} \times A$. Let us also have a look where the composition law comes from. $\tilde{A} \times A$ is a symplectic groupoid (the pair groupoid of $A$). The graph of its multiplication is a Lagrangian submanifold; using the identification of $T^*A$ and $\tilde{A} \times A$, it should be given by a closed 1-form on $A \times A \times A$; this 1-form is the differential of the function $(x_1, x_2, x) \mapsto \text{symplectic area of } \triangle PQR$. The composition of generating functions and its connection with Moyal product follows.

For the fun of it, let us make a similar construction, replacing $A$ by the sphere $S^2$ with the area 2-form. Again, symmetry with respect to a point is a symplectic map, therefore we locally have a similar identification between $S^2 \times S^2$ and $T^*S^2$; more precisely, there is an isomorphism between the subset of covectors in $T^*S^2$ of length less than 2 and $S^2 \times S^2$ with erased pairs of antipodal points. Explicitly, to a non-antipodal pair $(P, Q)$ we associate a point in $T S^2$ (and thus, via $\omega$, a point in $T^*S^2$) as on the picture:

$x$ is the midpoint of the shorter geodesic arc $PQ$ and $u \in T_x S^2$ appears by its orthogonal projection. This picture can be derived from the famous theorem of Archimedes, claiming that certain map between cylinder and sphere is area-preserving.

As a result, we have a similar picture of generating functions: for a function $H$ on $S^2$ and any point $x$ we take the vector $u_x$ defined by $d_x H = u_x \cdot \omega$, place it into the tangent plane so that $x$ is in its middle and project it into the sphere: $\Phi_H$ maps $P$ to $Q$. Composition rule looks as before (only triangles are spherical now).

Generally, this picture works with no changes for arbitrary symmetric symplectic space $M$. Using the symmetries we locally identify $\tilde{M} \times M$ with $T^*M$. Multiplication in this pair groupoid is again given by the symplectic area of a surface bounded by the geodesic triangle $PQR$ with $x_1, x_2, x$ being the midpoints of its sides. The identification between $\tilde{M} \times M$ and $T^*M$ is via a projection of $M$ into $T_x M$, as in the case $M = S^2$: Up to coverings, we embed $M$ into an affine space $A$. For any $x \in M$, the symmetry with respect to $x$ will be extended to an involution $\sigma_x$ of $A$; we project $M$ to $T_x M$ in the direction of $A^{\sigma_x}$ (the subspace of $A$ fixed by $\sigma_x$). Namely, since $M$ is a symmetric space, it is (a covering of) $G/G^\sigma$, where $G$ is a Lie group and $\sigma$ is an involutory automorphism of $G$. Let $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{p}$ be the decomposition of $\mathfrak{g}$ to $\pm 1$ eigenspaces of $d\sigma$ (to make $G/G^\sigma$ into a symmetric symplectic space, one has to specify a $G^\sigma$-invariant symplectic form on $\mathfrak{p}$). As a homogeneous symplectic space, $M$ can be embedded (up to coverings) into an affine space over $\mathfrak{g}^+$ via (non-equivariant) moment map. If $x \in M$ is fixed by $G^\sigma$, $T_x M$ is $\mathfrak{p}^*$ translated to $x$; we project $M$ to $T_x M$ in the direction of $\mathfrak{g}^{\sigma,*}$.

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