Exact discreteness and mass gap of $N = 1$ symplectic Yang-Mills from M-theory.

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Abstract. In this note we summarize some of the results found recently in [1]. We show the pure discreteness of the non-perturbative quantum spectrum of a symplectic Yang-Mills theory defined on a Riemann surface of positive genus, living in a target space that, in particular, can be 4D. This theory corresponds to the membrane with central charges. The presence of the central charge induces a confinement in the phase at zero temperature. When the energy rises, the center of the group breaks and the theory enters in a quark-plasma phase after a topological transition corresponding to the $N = 4$ wrapped supermembrane.

1. Introduction

The non-perturbative quantization of String Theory is still an open problem which receives much of the attention of specialists. It can be reformulated in terms of the quantization of the M-theory in 11 dimensions which, in turns, reduces to finding the quantization of the basic ingredients: M2-brane or supermembrane and M5 brane. A further important open problem, not necessary connected with String Theory, is the non-perturbative quantization of Yang-Mills theories. Attempts along this direction include lattice QCD, twistors, gauge-gravity duality, spin chains, large N matrix models and canonical quantization.

The aim of this note is to draw to the attention of specialists, the fact that the membrane with central charges, which is the quantum equivalent of a symplectic noncommutative Yang-Mills theory [2],[3], has a purely discrete spectrum at the exact level of the theory. This is an extension of previous results found for the regularized supermembrane with central charges, ([4]-[9]).

The correspondence with an $N = 1$ Yang-Mills theory defined on a $(2 + 1)D$ Riemann surface of positive genus $g$ that can live in a target space of 4D, allows this theory to be of interest also outside the scope of String Theory. Admitting an interpretation in terms of SQCD, it consists of two different phases at zero temperature: a confined phase comprising glueballs in the bosonic sector and a quark-gluon plasma phase which has a microscopic origin in the M-theory. The quantum consistency of supermembranes with fixed central charges provides then an indirect proof of consistency of all these noncommutative gauge theories.

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2. The supermembrane with non-trivial central charge

Supermembranes are extended objects defined in terms of a base manifold, a Riemann surface $\Sigma$, which live in a Minkowski target space. The canonically reduced Hamiltonian in the light cone gauge \([10]\) has the expression

$$\int_{\Sigma} \sqrt{W} \left( \frac{1}{2} \left( \frac{P_M}{\sqrt{W}} \right)^2 + \frac{1}{4} (X^M, X^N)^2 + \text{Fermionic terms} \right)$$

where

$$\{X^M, X^N\}^2 = \frac{\epsilon_{ab}}{\sqrt{W(\sigma)}} \partial_a X^M \partial_b X^N. \quad (2)$$

restricted by the first class constraint,

$$\int_{\Sigma} \frac{P_M}{\sqrt{W}} X^M = 0. \quad (3)$$

which generates area preserving diffeomorphisms of $\Sigma$ for $\mathcal{C}$ any given closed path. Here and below $M,N = 1,\ldots,9$. The continuity of the spectrum of the above Hamiltonian at the $SU(N)$ regularized level was demonstrated in [11]. This property relies on two basic facts: supersymmetry and the presence of singular configurations with zero energy at a classical level.

Under compactifications, for example regard the target space as being $M_{10} \times S^1$, it is believed, [12], that the spectrum of the theory remains continuous. Therefore, the compactification procedure by itself, does not seem enough to change the spectral properties of the model.

We now impose some topological restrictions on the configuration space. These completely characterize the $D = 11$ supermembrane with non-trivial central charge generated by the wrapping on the compact sector of the target space. We will assume that the target space is $M_0 \times S^1 \times S^1$ and its base manifold $\Sigma$ of positive genus $g$, for simplicity, $g = 1$. We stress that these properties remain valid for any other target space of dimension less than 9 in particular $D = 4$. All maps from the base space $\Sigma$, must satisfy

$$\int_{C_i} dX^r = 2\pi S^i_r R^r, \quad r = 1,2; \quad \int_{C_i} dX^m = 0 \quad m = 3,\ldots,9 \quad (4)$$

for $i = 1,2$ and the topological condition,

$$Z = \int_{\Sigma} dX^r \wedge dX^s = e^{\epsilon_{rs}(2\pi^2 R_1 R_2)} n, \quad (5)$$

where $n = \det S^i_r$ is fixed, each entry $S^i_r$ is integer, and $R_1$ and $R_2$ denote the radii of the target component $S^1 \times S^1$. Note that [3] describe maps from $\Sigma$ to $S^1 \times S^1$ with $dX^m$ a non-trivial closed one-form. The only restriction upon these maps is the assumption that $n$ is fixed. The term on the left side of (5) describes the central charge of the supersymmetric algebra. Among all the maps from the torus $\Sigma$ to the target space satisfying (4), [5] there is a minimizer of the Hamiltonian. It corresponds to a minimal immersion from $\Sigma$ to the target space which implies that, for the case of flat target spaces, the worldvolume of the supermembrane is a calibrated submanifold.

Minimal immersions can also describe non-BPS minimal solutions, [8]. The theory results to be invariant under $SL(2,\mathbb{Z})$. The degrees of freedom are expressed in terms of $A_r$ and the discrete set of integers described by the harmonic one-forms. We can always fix these gauge transformations by

$$S^i_r = l^i \delta^i_r, \quad \delta^i_r (l^i)^2 = n. \quad \rightarrow \quad dX^r = 2\pi R^r l^i \delta^i_r dA_r. \quad (6)$$

After the gauge fixing there is a residual transformation \([1]\) $\mathbb{Z}(2)$ under which,

$$A_1 \rightarrow A_2 \quad A_2 \rightarrow -A_1. \quad (7)$$

This allows us to rewrite the Hamiltonian in terms of $X^m, m = 1,\ldots,7$ and $A_r, r = 1,2$. The resulting expression is:

$$H = \int_{\Sigma} \frac{1}{2} \sqrt{W} \left( P_m^2 + \Pi_t^2 + \frac{1}{2} \left( W(X^m, X^n)^2 + W(\theta_\Gamma X^m)^2 + \frac{1}{2} W(\theta_\Gamma)^2 \right) \right) + \int_{\Sigma} \left[ \frac{1}{8} \sqrt{W} n^2 - \Lambda(\theta_\Gamma \Pi_\Gamma + \{X^m, P_m\}) \right] + \int_{\Sigma} \sqrt{W} \left[ -\theta_\Gamma \Gamma_{\Gamma - \Gamma} \theta_\Gamma \sigma + \theta_\Gamma \Gamma_{\Gamma - \Gamma} \{X^m, \Psi\} + \Lambda \{\theta_\Gamma \Gamma_{\Gamma - \Gamma}, \Psi\} \right] \quad (8)$$
where (2, 4) \( \mathcal{D}_r \cdot = 2 \mathcal{D}_r \cdot + [A_r, \cdot] \), \( \mathcal{F}_{rs} = \mathcal{D}_r A_s - \mathcal{D}_s A_r + [A_r, A_s] \), and \( P_n \) and \( \Pi_r \) are the conjugate momenta to \( X^m \) and \( A_r \), respectively. \( \mathcal{D}_r \) and \( \mathcal{F}_{rs} \) are the covariant derivative and curvature of a symplectic noncommutative theory (2, 3), constructed from the symplectic structure \( \frac{\epsilon^{ab}}{\sqrt{\mathcal{W}}} \) introduced by the central charge. The last term represents its supersymmetric extension in terms of Majorana spinors. The relevant degrees of freedom in order to quantize the theory are the \( X^m \) and the gauge invariant part of \( A_r \), which are single valued over the base manifold. A SU\((N)\) regularization was obtained in [4] and the spectral properties of the spectrum were rigourously demonstrated at classical level, and at quantum level in several papers, [5], [6], [9].

3. On the spectrum of the exact theory

According to the results reported in [5], the bosonic regularized Hamiltonian of the \( D = 11 \) supermembrane with central charge, \( H^{B}_{sc, N} \), relates to its semi-classical approximation, \( H^{B}_{sc, N} \), by means of the following operator inequality:

\[
H^{B}_{sc, N} \geq C_N H^{B}_{sc, N}.
\]

Here \( N \) denotes the size of the truncation in the Fourier basis of \( \Sigma \) and \( C_N \) is a positive constant. A seemingly crucial step in the proof of (9) found in [3], relies heavily on the compactness of the unit ball of the configuration space which happens to be finite dimensional. The exact bosonic Hamiltonian however contains a configuration space which is infinite-dimensional, so that the unit ball is not compact. In [11] we show that the same operator relation holds true for the exact bosonic Hamiltonians. We overcome the difficulty of the analysis by carrying out a detailed analysis of each term involved in the expansion of the potential term of the exact bosonic Hamiltonian, \( H^{B} \).

Following the standard notation \( L^p \equiv L^p(\sigma) \) denotes the Banach space of all fields \( u \), such that \( \| u \|_p = \langle u^p \rangle^{1/p} < \infty \). Let

\[
\| u \|_{4,2} = (\| D_1 u \|^4 + \| D_2 u \|^4)^{1/4}.
\]

The fields \( X^m, A_r \) lie on the configuration space \( \mathcal{H}^{4,2} \) of functions \( u \in \mathcal{H}^1 \) such that \( \| u \|_{4,2} < \infty \). Note that the left hand side of (10) is a well defined norm in \( \mathcal{H}^{4,2} \), the later is a linear space, but we do not make any assumption about completeness. The potential, \( V \), of the bosonic sector of the supermembrane with central charges is well defined in \( \mathcal{H}^{4,2} \) as

\[
V = \langle \mathcal{D}_r X^m \mathcal{D}_r X^m + \frac{1}{4} \mathcal{F}_{rs} \mathcal{F}_{rs} \rangle.
\]

\( V \) is not well defined in \( \mathcal{H}^1 \) but in \( \mathcal{H}^{4,2} \).

By imposing the gauge fixing conditions, \( D_1 A_1 = 0 \) and \( D_1 A_2 = 0 \), the potential can be re-written as,

\[
V = \rho^2 + 2B + A^2
\]

where \( \rho^2 \) is the semiclassical potential term of \( H^{B}_{sc} \). Let \( \rho^2 \) be the potential term of \( H^{B}_{sc} \), so that

\[
\rho^2 = \langle D_1 X^m D_1 X^m + (D_1 A_1)^2 + (D_2 A_2)^2 \rangle.
\]

and

\[
B = \langle D_1 X^m \{ A_r, X^m \} + D_1 A_2 \{ A_1, A_2 \} \rangle
\]

\[
A = \langle \{ A_1, X^m \}^2 + \{ A_2, X^m \}^2 + \{ A_1, A_2 \}^2 + \{ X^m, X^m \}^2 \rangle.
\]

This allows us to show the following crucial identity, [11]: there exists a constant \( 0 < C \leq 1 \), such that

\[
V \geq C \rho^2, \quad \forall X^m, A_r \in \mathcal{H}^2.
\]

The latter is a consequence of the particular expression of the potential, properties as the compactness of the base manifold (not of the configuration space) and the Cauchy-Schwarz inequality.
In order to define rigorously the Laplacian in the non-compact infinite dimensional configuration space we have introduced, the Hamiltonian is expressed as

$$H^B = [V_{quartic} + V_{cubic} + (1 - C)V_{quadratic}] + [-\Delta + CV_{quadratic}]$$

where the first bracket acts multiplicatively on the Hilbert space of states, while the operator on the second bracket may be expressed in terms of creation and annihilation operators in the usual way. Alongside with (15), this expression ensures the operator identity

$$H^B \geq CH_{sc}^B,$$

analogous to (9).

4. Confinement of the theory

It was an original idea of G. ’t Hooft, [13] that permanent quark confinement occurs in a gauge theory if its vacuum condenses into a state which resembles a superconductor. His proposal was to consider the confinement of quarks as dual of the Meissner effect, where the role of magnetism and electricity are interchanged. In his approach he considered a nonabelian gauge theory that were seen as an abelian theory enriched with Dirac magnetic monopoles, see also [14]. The symplectic Yang-Mills naturally creates this effect. The mass contribution of the central charge, or, analogously, its correlated residual $Z(2)$ symmetry of the Hamiltonian, can be described in terms of the quadratic derivatives of the configuration fields $X^m$ and $A_r$. The derivatives of these fields correspond to mappings of the target space into $\Sigma$. These are induced by the minimal immersion which realises by $\hat{X}_r, r = 1, 2,$ and the harmonic fields over $\Sigma$,

$$D_r Y_A = \{\hat{X}_r, Y_A\} = \lambda_{rA} Y_B = \lambda_r A Y_A$$

where

$$\lambda_{rA}^B = \int d^2 \sigma \sqrt{\omega} \{\hat{X}_r, Y_A\} Y_B.$$

They correspond to a particular subset of the structure constants that mixes the harmonic and the exact forms, $s_{rA}^C$.

For the case of a torus, an explicit relation was found in [4]. The quadratic terms on the derivatives of the configuration variables, define a strictly positive function whose contribution to the overall Hamiltonian gives rise to a basin shaped potential. The latter diminishes the string-like spikes and provides a discrete spectrum, even for the supersymmetric model. The centre created by a discrete symmetry is a mechanism for providing mass to the monopoles [15]. The supermembrane theory when compactified in 4D can be interpreted as a theory modelling susy QCD in the spirit of [16]. It exhibits confinement in the phase at zero temperature since the theory becomes naturally the supermembrane with central charges, which has minimal energy.

The mass terms are determined by the elements of the centre $m(z)$ associated to the correlation length of the particles [17]. By rising the energy, the theory enter in the phase of asymptotic freedom described by the supermembrane without central charges. The phase transition is described by the breaking of the center of the group that becomes trivial. The phase transition of topological nature [18]-[20]. The particles behave as as if they were in a quark-gluon plasma. These quarks-gluons do not feel the topological effects, since the correlation length becomes infinite and the effective volume is zero. Along the commutative directions the quarks experiment no force.

5. Conclusions

In this note we have summarized the results of the recent manuscript [1]. The $D = 11$ supermembrane with central charges is quantum equivalent to the $N = 12 + 1$ Symplectic non-commutative Super Yang-Mills Theory defined in target spaces of dimension $D \leq 9$. The spectrum of the bosonic sector of the $D = 11$ theory has been demonstrated at exact level of the theory to be purely discrete, hence containing a mass gap. The theory exhibit confinement in the supermembrane with central charge phase. It enters in the asymptotic free phase through the spontaneous breaking of the center. This phase corresponds to the $N = 4$ wrapped supermembrane.
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References

[1] Boulton L, García del Moral M and Restuccia A (2006). Preprint hep-th/0609054
[2] Martín I, Ovalle J and Restuccia A 2001 Phys. Rev. D64 040061
[3] Martín I, Restuccia A 2002 Nucl. Phys. B622 240–256.
[4] García del Moral M and Restuccia A 2002 Phys. Rev. D66 045023
[5] Boulton L, García del Moral M, Martín I and Restuccia A 2002 Class. Quantum Grav. 19 2951-2959.
[6] Boulton L, García del Moral M and Restuccia A 2003 Nucl. Phys. B671 343–358.
[7] M.P. Garcia del Moral, A. Restuccia. Institute of Physics Conference Series 2005, Vol43,151
[8] Bellorin J and Restuccia A Nucl. Phys. B737 (2006) 190-208
[9] Boulton L and Restuccia A Nucl. Phys. B724 380-396,2005
[10] De Wit B, Hoppe j and Nicolai H 1988 Nucl. Phys. B305 545
[11] De Wit B, Lüsher and Nicolai H 1989 Nucl. Phys. B320 135–159.
[12] De Wit B, Peeters K and Plefka J 1998 Nucl. Phys. Proc Suppl 62 405–411.
[13] G. ’t Hooft Nucl. Phys. B138:1,1978; Nucl. Phys. B190:455,1981
[14] E. Witten Nucl. Phys. B149(1979) 285-320
[15] M. Pepe Nucl.Phys.Proc.Suppl.153:207-214,2006;
[16] A. Chodos, R.L. Jaffe, K.Johnson, Charles B. Thorn, V.F. Weisskopf Phys. Rev. D9:3471-3495,1974
[17] K. Holland, M. Pepe, U.J. Wiese Nucl. Phys. B694:35-58,2004
[18] J.Preskill, A. Vilenkin Phys. Rev. D47:2324-2342,1993
[19] A. S. Kronfeld, G. Schierholz, U.J. Wiese Nucl. Phys. B293:461,1987
[20] A.P. Balachandran, E. Batista, I.P. Costa e Silva, P. Teotonio-Sobrinho Int.J.Mod.Phys. A15:1629-1660,2000