Online Tracking of a Predictable Drifting Parameter of a Time Series

Eduard Belitser
Department of Mathematics, VU University Amsterdam

Paulo Serra
Institute for Mathematical Stochastics, University of Göttingen

May 7, 2014

Abstract

We propose an online algorithm for tracking a multidimensional time-varying parameter of a time series, which is also allowed to be a predictable process with respect to the underlying time series. The algorithm is driven by a gain function. Under assumptions on the gain, we derive uniform non-asymptotic error bounds on the tracking algorithm in terms of chosen step size for the algorithm and the variation of the parameter of interest. We also outline how appropriate gain functions can be constructed. We give several examples of different variational setups for the parameter process where our result can be applied. The proposed approach covers many frameworks and models (including the classical Robbins-Monro and Kiefer-Wolfowitz procedures) where stochastic approximation algorithms comprise the main inference tool for the data analysis. We treat in some detail a couple of specific models.

Keywords: on-line tracking; predictable drifting parameter; recursive algorithm; stochastic approximation procedure; time series; time-varying parameter.

Contents

1 Introduction 2
2 Preliminaries 5
3 Main results 9
4 Construction of gain functions 16
  4.1 Signal + noise setting 16
  4.2 Robbins-Monro setting: tracking roots 17
  4.3 Kiefer-Wolfowitz setting: tracking maxima 18
  4.4 Tracking conditional quantiles 21
  4.5 Gain function based on score 21
1 Introduction

When one analyzes data that arrive sequentially over time, it is important to detect changes in the underlying model which can then be adjusted accordingly. Such problems arise in many engineering (signal processing, speech recognition, communication systems), econometric and biomedical applications and can be found in an extensive literature widely scattered in these fields. Inference on time-varying parameters in stochastic systems is therefore of fundamental interest in sequential analysis.

Consider an \( X \)-valued time series \( \{X_k, k \in \mathbb{N}_0\} \), \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \), \( X \subseteq \mathbb{R}^l \), \( l \in \mathbb{N} \), such that at time moment \( k = 0 \) the first observation \( X_0 \sim \mathbb{P}_0 \) and subsequently at each time moment \( k \in \mathbb{N} \) a new observation \( X_k \) arrives according to the model \( X_k | X_{k-1} \sim \mathbb{P}_k (\cdot | X_{k-1}) \), where \( X_{k-1} = (X_0, X_1, \ldots, X_{k-1}) \). Suppose we are interested in certain characteristics of the conditional distribution of \( X_k \) given the past \( X_{k-1} = x_{k-1} \):

\[
A_k(\mathbb{P}_k (\cdot | x_{k-1})) = \theta_k(x_{k-1}).
\]

Here \( A_k \) is an operator mapping conditional distributions \( \mathbb{P}_k (\cdot | x_{k-1}) \) into measurable \( \Theta \)-valued functions \( \theta_k(x_{k-1}), x_{k-1} \in X^k \), \( \Theta \) is a compact subset of \( \mathbb{R}^d \). The goal is to estimate (or to track) \( \theta_k(X_{k-1}) \) at time instant \( k \in \mathbb{N}_0 \), based on the data \( X_k \) (and prior information) available by that time moment.

The traditional parametric formulation is the most simple particular case of the above setting: the observations are independent and the parameter \( \theta \in \Theta \) is a constant vector. The simplest nonparametric formulation deals again with independent observations and time-varying parameter \( \theta_k \in \Theta, k \in \mathbb{N}_0 \) (cf. \([4, 5, 7]\)). Modeling observations by a Markov chain with a time varying parameter of the transition law would add a next level of complexity (cf. for the autoregressive model in \([4, 18]\)).

The proposed time series formulation admits an arbitrary dependence structure between the observations. Another important and peculiar feature of our approach is that the multidimensional parameter \( \theta_k \in \Theta \subseteq \mathbb{R}^d \), \( k \in \mathbb{N} \), besides being time-varying, is also allowed to depend on the past of the time series. It is thus a predictable process with respect to the natural filtration: \( \theta_k = \theta_k(X_{k-1}) \). An example of such characteristics is the conditional expectation \( \theta_k(X_{k-1}) = E[X_k | X_{k-1}] \). This time series formulation, with time-varying parameter of interest which is also allowed to depend on the past, represents
the most general sequential framework, independent observations and Markov chains are typical examples of models that fit into this framework.

Since the data arrives in a successive manner, conventional methods based on samples of fixed size are not easy to use. A more appropriate approach is based on sequential methods, stochastic recursive algorithms, which allow fast updating of parameter or state estimates at each instant as new data arrive and therefore can be used to produce an “online” inference, that is, during the operation of the system. Stochastic recursive algorithms, also known as stochastic approximation, take many forms and have numerous applications in the biomedical, socio-economic and engineering sciences, which highlights the interdisciplinary nature of the subject.

There is a vast literature on stochastic approximation beginning with the seminal papers of Robbins and Monro [20] and of Kiefer and Wolfowitz [13]. There is a big variety of techniques in the area of stochastic approximation which have been developed and inspired by the applications from other fields. We mention here the books [24, 23, 22, 14, 17, 8, 16].

A classical topic in adaptive control concerns the problem of tracking drifting parameters of a linear regression model, or somewhat equivalently, tracking the best linear fit when the parameters change slowly. This problem also occurs in communication theory for adaptive equalizers and noise cancellation, etc., where the signal, noise, and channel properties change with time. Successful stochastic approximation schemes for tracking in the time-varying case were given by [9, 11, 15, 16] (see further references therein).

Coming back to our time series model, the problem of tracking a time-varying parameter that is a functional of conditional distribution of the current observation given the past, is clearly unfeasible, especially in such general formulation, without some conditions on the model. In general, some knowledge about the structure of underlying time series and some control over the variability of the parameter of interest over time are needed. Interestingly, in this seemingly very general time series framework, we actually do not require the (full) knowledge of the observational model. Instead, all we do need is to be able to compute a so-called gain vector at each time moment \( k \in \mathbb{N} \), which is a certain (vector) function of the previous estimate of the parameter \( \theta_k(X_{k-1}) \), new observation \( X_k \) and prehistory \( X_{k-1} \).

The essential property of such gain vector is that it, roughly speaking, “pushes” in the right direction of the current value of true parameter to track. Although the assumption about the existence of that gain vector seems to be rather strong, we demonstrate on a number of interesting examples when such an assumption indeed holds. Basically, in case of Markov chain observations, if the form of transition density is known as function of the underlying parameter and it satisfies certain regularity assumptions, then the gain vector can always be constructed, for example, as a score function corresponding to the conditional maximum likelihood method. Under appropriate regularity conditions (the existence of the conditional Fisher information and \( L_2 \)-differentiability of the conditional log likelihood), such a score function has always the property of gain vector at least locally.

A gain function, together with a step sequence and new observations from the model, can be used to adjust the current approximation of the drifting parameter, resulting in a tracking algorithm. To ease the verification of our assumptions on the gain function, we formulate them in two equivalent forms. Under some assumptions on the gain vectors, we establish a uniform non-asymptotic bound the \( L_1 \) error of the resulting tracking algorithm,
in terms of the variation of the drifting parameter. Under the extra assumption that the
gain function is bounded, we can strengthen this result to a uniform bound on the $L_p$ error
(and then an almost sure bound). These error bounds constitute our main result and they
also guide us in the choice of the step size for the algorithm. Some extensions are also
presented where we allow for approximation terms and approximate gains.

Based on our main result, we specify the appropriate choice for the step sequence in
three different variational setups for the drifting parameter. We treat first the simple case
of a constant parameter. Although we are mainly concerned with tracking time-varying
parameters, our algorithm is still of interest in the constant parameter case since it should
result in an algorithm which is both recursive and robust. We also consider a setup where
the parameter is stabilizing. This covers both the case where the parameter is converging
and where we sample the signal with increasing frequency. The third variational setup
covers the important case of tracking smooth signals. This setup is somewhat different in
that we make observations with a certain frequency from an underlying continuous time
process which is indexed by a parameter changing like a Lipschitz function. Our result
can then either be interpreted as a uniform, non-asymptotic result for each fixed sampling
frequency or as an asymptotic statement in the observation frequency.

Examples are also given for different possible gain functions. These fall into two cat-
egories: general, score based gain functions for tracking multidimensional parameters in
regular models and specialized gains for tracking more specific quantities. The latter in-
clude gains to track level sets or maxima of drifting functions (extending the classical
Robbins-Monro and Kiefer-Wolfowitz algorithms) and gains to track drifting conditional
quantiles. We also propose modifications for a given gain function (rescaling, truncation,
projection) which can be used to design gains tailored specifically to verify our assumptions.

We illustrate our method by treating some concrete applications of the proposed al-
gorithm, in particular, we elaborate on the problem of tracking drifting parameters in
autoregressive models. Results on tracking algorithms for these models already exist in
the literature (cf. [4, 18]) and we can derive similar results by choosing an appropriate
gain function. Using our approach, obtaining error bounds on the resulting tracking algo-
rythm reduces to verifying our assumptions for the chosen gain function which considerably
simplifies the derivation of results.

This paper is structured as follows. In Section 2 we summarize the notation that will
be used thought the paper, as well as our model and two equivalent formulations for our
assumptions. Section 3 contains our main result and respective proof as well as some
straightforward extensions of the main result. The construction and modification of gain
functions for different models and different parameters of interest is explained in Section 4
Section 5 contains three examples of variational setups for the time-varying parameter for
which we specify the tracking error implied by our main result. We collect in Section 6
some examples of applications. Section 7 contains the proofs for our lemmas.
2 Preliminaries

First we introduce some notation that we are going to use throughout the paper. All vectors are always column vectors. We use bold letters to represent matrices and families of vectors, uppercase letters for families of random vectors and matrices; denote \( x_n = (x_0, x_1, \ldots, x_n) \) for a set of vectors \( x_0, x_1, \ldots, x_n \). For \( k_0, k \in \mathbb{Z}, N_k = \{ l \in \mathbb{Z} : l \geq k \} \) (of course, \( N = N_1 \)), \( N_{k_0, k} = \{ l \in \mathbb{Z} : k_0 \leq l \leq k \} \). We use interchangeably \( \{ a_i, i \in I \} = \{ a_i \}_{i \in I} = (a_i)_{i \in I} \) for \( I \subseteq \mathbb{N}_0 \). For vectors \( x, y \in \mathbb{R}^d \), denote by \( \|x\| = \|x\|_2 \) and \( \langle x, y \rangle = x^T y \) the usual Euclidean norm and the inner product in \( \mathbb{R}^d \), respectively, and by \( \|x\|_p \) the \( l_p \) norm (with \( p \geq 1 \)) on vectors in \( \mathbb{R}^d \). For an event \( A \), we denote by \( \mathbb{I}\{A\} \) the indicator of the event \( A \). For a symmetric \( (d \times d) \)-matrix \( M \), let \( \Lambda_1(M) \) and \( \Lambda(d)(M) \) be the smallest and the largest eigenvalues of \( M \) respectively. Denote by \( O \) denote the zero matrix and by \( I \) the identity matrix, whose dimensions will be determined by the context. Besides, we adopt the convention that \( \sum_{i \in \emptyset} A_i = O \) and \( \prod_{i \in \emptyset} B_i = I \) for matrices \( A_i \) and \( B_i \). When applied to matrices, \( \| \cdot \|_p \) represents the matrix norm induced by the \( l_p \) vector norm: \( \| M \|_p = \max_{\|x\|=1} \| Mx \|_p \).

Assume that by time \( n \in \mathbb{N}_0 \), time series data \( X_n = (X_0, X_1, \ldots, X_n) \) (which may not be fully observable) occur according to the following model:

\[
X_0 \sim \mathbb{P}_0, \quad X_k \mid X_{k-1} \sim \mathbb{P}_k(\cdot | X_{k-1}), \quad k \in \mathbb{N}. \tag{1}
\]

Here vector \( X_k \) takes value in some set \( \mathcal{X}_k \subseteq \mathbb{R}^{l_k} \) with \( l_k \in \mathbb{N} \), i.e., \( \mathbb{P}(X_k \in \mathcal{X}_k) = 1 \), \( k \in \mathbb{N}_0 \). By \( \mathbb{P}_k(\cdot | X_{k-1}) \) we denote the conditional distribution of \( X_k \) given \( X_{k-1} \) and \( \mathcal{X}_k^{k+1} = \mathcal{X}_k \times \mathcal{X}_{k+1} \) for \( k \in \mathbb{N}_0 \), with \( \mathcal{X}_0 = \emptyset \). Thus, \( X_k \) takes values in \( \mathcal{X}_k \). Clearly, the distribution of \( X_n, n \in \mathbb{N}_0 \), is given by

\[
\mathbb{P}^{(n)} = \mathbb{P}^{(n)}(x_n) = \prod_{k=0}^n \mathbb{P}_k(x_k | x_{k-1}), \quad x_k \in \mathcal{X}_k, \quad k = 0, 1, \ldots, n,
\]

where \( \mathbb{P}_0(x_0 | x_{-1}) \) should be understood as \( \mathbb{P}_0(x_0) \). For the sake of consistent notation, \( x_{-1} \) (also \( x_{-2} \) etc.) means void variables from now on.

Introduce an increasing sequence of \( \sigma \)-algebras \( \{ \mathcal{F}_k \}_{k \in \mathbb{N}_1} \) (i.e., \( \mathcal{F}_{k_1} \subseteq \mathcal{F}_{k_2} \) if \( k_1 \leq k_2 \)) such that \( \mathcal{F}_k \subseteq \sigma(\mathcal{X}_k) \), \( k \in \mathbb{N}_0 \), where \( \sigma(\mathcal{X}_k) \) denote the \( \sigma \)-algebra generated by \( \mathcal{X}_k \) and \( \mathcal{F}_{-1} \) is a \( \sigma \)-algebra such that \( \mathcal{F}_{-1} \subseteq \mathcal{F}_0 \). Unless otherwise specified, we assume that \( \mathcal{F}_k = \sigma(\mathcal{X}_k) \). As is discussed in Remark 4 below, the case \( \mathcal{F}_k \subset \sigma(\mathcal{X}_k) \) is not conceptually new because it can be reduced to the situation of the time series \( \{ Z_k, k \in \mathbb{N}_0 \} \) with \( Z_k = h_k(X_k) \) for some \( \sigma(\mathcal{X}_k), \mathcal{F}_k \)-measurable functions \( h_k \)'s such that \( \mathcal{F}_k = \sigma(Z_k), k \in \mathbb{N}_0 \).

Now we describe the statistical model which is a family probability measures for the observed data. For each \( k \in \mathbb{N}_0 \), let \( \mathcal{P}_k \) be a given family of conditional probability measures of \( X_k \) given \( X_{k-1} = x_{k-1} \in \mathcal{X}^{k-1} \). Then the underlying statistical model is determined by imposing \( \mathbb{P}_k(\cdot | x_{k-1}) \in \mathcal{P}_k, k \in \mathbb{N}_0 \). Thus, at time \( n \in \mathbb{N}_0 \), the underlying (growing) statistical model is \( \mathbb{P}^{(n)} \in \mathcal{P}^{(n)} = \left\{ \prod_{k=0}^n \mathbb{P}_k(x_k | x_{k-1}) : \mathbb{P}_k(\cdot | x_{k-1}) \in \mathcal{P}_k \right\} \).

For some compact subset \( \Theta \) of \( \mathbb{R}^d \), denote by \( \mathcal{B}_\Theta \) the Borel \( \sigma \)-algebra on \( \Theta \) and by \( \mathcal{M}_k \) the set of \( \Theta \)-valued \( (\mathcal{F}_k, \mathcal{B}_\Theta) \)-measurable functions on \( \mathcal{X}_k, k \in \mathbb{N}_0 \). Consider a sequence of
operators \( A_k : \mathcal{P}_k \mapsto \mathcal{M}_{k-1}, k \in \mathbb{N}_0 \), so that for a \( \mathbb{P}_k(\cdot|x_{k-1}) \in \mathcal{P}_k \)

\[ A_k(\mathbb{P}_k(\cdot|x_{k-1})) = \theta_k(x_{k-1}), \quad x_{k-1} \in \mathcal{X}^{k-1}, \]

with \( \theta_k(x_{k-1}) \in \mathcal{M}_{k-1} \). We will often abbreviate \( \theta_k = \theta_k(x_{k-1}) \), remembering that this is a measurable function of \( x_{k-1} \). For \( k = 0 \), since \( x_{-1} \) is void, \( \theta_0(x_{-1}) = \theta_0 \in \Theta \) means a constant.

Our goal is to design an online algorithm for tracking the drifting parameter of interest \( \theta_k(X_{k-1}) \) at time moment \( k \in \mathbb{N}_0 \) on the basis of the data \( X_k \) observed by that time moment. The time-varying parameter \( \theta_k = \theta_k(X_{k-1}), k \in \mathbb{N}_0 \), is thus allowed to depend on the past of the time series, i.e., it is a predictable process with respect to the filtration \( \{\mathcal{F}_k\}_{k \in \mathbb{N}_1} \). Recall further that \( \theta_k \) is assumed to take values in some compact subset \( \Theta \) of \( \mathbb{R}^d \), to be precise, \( \mathbb{P}(\theta_k(X_{k-1}) \in \Theta) = 1 \) for all \( k \in \mathbb{N}_0 \). Denote

\[ \sup_{\theta \in \Theta} \|\theta\|^2 = C_\Theta. \tag{2} \]

At time \( k \), given \( X_k \), the model \( \mathcal{P}_{k+1} \) contains all the relevant information about the next observation. Actually, we do not consider the model to be (completely) known. Instead, we assume that our prior knowledge about the model is formalized as follows: for each \( k \in \mathbb{N}_0 \), we have certain \( (\mathcal{B}_0 \times \mathcal{F}_k, \mathcal{B}_k) \)-measurable functions \( G_k \) at our disposal (which we call gain functions or gain vectors or just gains), \( G_k : \mathbb{R}^d \times \mathcal{X}^k \mapsto \mathbb{R}^d \). We use these gain functions to construct a recursive algorithm for tracking the sequence \( \theta_k = \theta_k(X_{k-1}) \in \Theta \subset \mathbb{R}^d \) from the observations \( \{ \theta \} \):

\[ \hat{\theta}_{k+1} = \hat{\theta}_k + \gamma_k G_k(\hat{\theta}_k, X_k), \quad k \in \mathbb{N}_0, \tag{3} \]

for some positive sequence \( \gamma_k \leq \Gamma \) and some (arbitrary) initial value \( \hat{\theta}_0 \in \Theta \subset \mathbb{R}^d \), measurable with respect to \( \mathcal{F}_1 \). Since \( \theta_k = \hat{\theta}_k(X_{k-1}) \) is \( \mathcal{F}_{k-1} \)-measurable, then \( \{ \theta_k \}_{k \in \mathbb{N}_0} \) is predictable with respect to the filtration \( \{\mathcal{F}_k\}_{k \in \mathbb{N}_1} \). Notice that \( \hat{\theta}_0 \) can be a random vector if \( \mathcal{F}_1 \) is not the trivial \( \sigma \)-algebra.

Of course, it is not to be expected that the tracking algorithm \( \{3\} \) performs well for arbitrary gains. Intuition suggests that the gain \( G_k \) should “push” \( \theta_k \) in the direction of \( \theta_k \). The following conditions formalize this requirement.

(A1) For all \( k \in \mathbb{N}_0 \), the quantity, which we call (conditional) average gain,

\[ g_k(\hat{\theta}_k, \theta_k) = g_k(\hat{\theta}_k, \theta_k|X_{k-1}) = \mathbb{E}[G_k(\hat{\theta}_k, X_k)|\mathcal{F}_{k-1}] \tag{4} \]

is well defined (recall that \( \theta_k = \theta_k(X_{k-1}) = A_k(\mathbb{P}_k) \) and \( \hat{\theta}_k \) is defined by \( \{3\} \)) and there exist a \( \mathcal{F}_{k-1} \)-measurable symmetric positive definite matrix \( M_k = M_k(X_{k-1}) \) (its entries are \( \mathcal{F}_{k-1} \)-measurable functions) such that, almost surely

\[ g_k(\hat{\theta}_k, \theta_k|X_{k-1}) = -M_k(\hat{\theta}_k - \theta_k), \tag{5} \]

\[ \lambda_1 \leq \mathbb{E}[\Lambda_{(t)}(M_k)|\mathcal{F}_{k-2}], \quad \lambda_1 \leq \Lambda_{(d)}(M_k) \leq \lambda_2, \tag{6} \]

for some fixed constants \( 0 < \lambda_1 \leq \lambda_2 < \infty \).
There exists a constant $C_g > 0$ such that
\begin{equation}
\mathbb{E}\|G_k(\hat{\theta}_k, X_k) - g_k(\hat{\theta}_k, \theta_k|X_{k-1})\|^2 \leq C_g, \quad k \in \mathbb{N}_0. \tag{7}
\end{equation}

As one can see from (4), a $\sigma$-algebra $\mathcal{F}_{-2}$, such that $\mathcal{F}_{-2} \subseteq \mathcal{F}_{-1}$, is also needed. Without loss of generality, assume that $\mathcal{F}_{-2} = \{\emptyset, \mathcal{X}_0\}$, the trivial $\sigma$-algebra on $\mathcal{X}_0$. We will often use shorthand notation: $G_k = G_k(\hat{\theta}_k, X_k)$ and $g_k = g_k(\hat{\theta}_k, \theta_k|X_{k-1})$.

**Remark 1.** Condition (A1) means, in a way, that the gain $G_k(\vartheta, X_k)$ shifts any $\vartheta = \hat{\vartheta}(X_{k-1})$, on average, towards the “true” value $\theta_k = \theta_k(X_{k-1})$. This elucidates the idea of algorithm (3). Suppose at time instant $k \in \mathbb{N}$ an observer had a reasonable estimator $\hat{\theta}_{k-1} = \hat{\theta}_{k-1}(X_{k-1})$ of the “old” value of the parameter of interest $\theta_{k-1} = \theta_{k-1}(X_{k-2})$, and a new data vector $X_k$ arrives. Then the available data is $X_k = (X_k, X_{k-1})$ and the observer can construct an estimator $\hat{\theta}_k = \hat{\theta}_k(X_k)$ of the new value of the parameter $\theta_k = \theta_k(X_{k-1})$ by calculating the gain $G_k(\hat{\theta}_k, X_k)$ and using a rescaled (by a step size $\gamma_k > 0$) version of it to update the “old” estimator $\hat{\theta}_{k-1}$ towards $\theta_k = \theta_k(X_{k-1})$.

The upper bound in relation (6) means that the gain is of a bounded magnitude, and the lower bound has the meaning of the so-called persistence of excitation as it is termed in control theory literature.

**Remark 2.** Assumption (A2) is trivially satisfied if the gain vectors $G_k(\vartheta, X_k)$ are almost surely uniformly bounded. This is not so difficult to arrange, for example, by dividing the gain vector by a multiple of its length or by truncating. In doing so, we make the resulting gain vector bounded, whereas retaining its direction. We discuss these approaches in more detail in Section 4.

Assumption (A2) is also satisfied if
\begin{equation}
\mathbb{E}\|G_k(\hat{\theta}_k, X_k)\|^2 \leq C_g, \quad k \in \mathbb{N}_0. \tag{8}
\end{equation}

Indeed, for a random vector $X$ with a finite second moment of its norm and any $\sigma$-algebra $\mathcal{F}$, $\mathbb{E}\|X - \mathbb{E}(X|\mathcal{F})\|^2 = \mathbb{E}\|X\|^2 - \mathbb{E}\|\mathbb{E}(X|\mathcal{F})\|^2 \leq \mathbb{E}\|X\|^2$. Combining this with (3) yields $\mathbb{E}\|G_k - g_k\|^2 \leq \mathbb{E}\|G_k\|^2 \leq C_g, \quad k \in \mathbb{N}_0$.

On the other hand, from (2), (A1) and (A2) it follows that
\begin{align*}
\mathbb{E}\|G_k\|^2 &= \mathbb{E}\|G_k - g_k + g_k\|^2 \\
&\leq 2C_g + 2\mathbb{E}\|g_k\|^2 \\
&\leq 2C_g + 2\lambda_2^2 \mathbb{E}\|\hat{\theta}_k - \theta_k\|^2 \\
&\leq 2C_g + 4\lambda_2^2 C_\vartheta + 4\lambda_2^2 \mathbb{E}\|\hat{\theta}_k\|^2, \quad k \in \mathbb{N}_0. \tag{9}
\end{align*}

This relation and Lemma 2 below (thus the conditions of Lemma 2 must hold) will in turn imply the uniform bound (8).

**Remark 3.** Generally, there is no universal way to find gain vectors which satisfy conditions (A1) and (A2). In many practical situations, the model $\{\mathcal{P}_k, k \in \mathbb{N}_0\}$ is typically specified and it is an art to find gain vectors which satisfy (A1) and (A2); we discuss this issue in more detail in Section 4. The assumptions above look somewhat unnatural and cumbersome because they are assumed to hold for all $k \in \mathbb{N}$, whereas functions involved in the conditions depend in general on $X_{k-1}$ whose dimension increases unlimitedly as $k$.
increases. However, the assumptions become reasonable in the important case of Markov chain observations \( \{X_k, k \in \mathbb{N}_0\} \) of order, say, \( p \). In this case, for any \( k \in \mathbb{N} \) we can use vector of bounded dimension \( (X_{k-p}, \ldots, X_{k-1}) \) instead of \( X_{k-1} \) (of growing dimension) in all the quantities from conditions (A1) and (A2).

Independent observations is a next simplification, also important in many practical applications. In this case there is no past involved in the function \( \theta_k, k \in \mathbb{N}_0 \), it will only be a function of time.

**Remark 4.** Suppose that conditions (A1) and (A2) hold for the filtration \( \mathcal{F}_{k-1} \cup \{\sigma(X_k)\}_{k \in \mathbb{N}_0} \) and for some measurable gain functions \( G_k(\theta_k, X_k), k \in \mathbb{N}_0 \), but the parameter sequence \( \{\theta_k\}_{k \in \mathbb{N}_0} \) is predictable with respect to a coarser filtration \( \{\mathcal{F}_k\}_{k \in \mathbb{N}_1} \) i.e., \( \sigma(Z_k) = \mathcal{F}_k \subset \sigma(X_k), k \in \mathbb{N}_0 \), where \( Z_k = h_k(X_k) \) for some measurable \( h_k \)'s. For example, the vector \( X_k \) consist of two subvectors \( Z_k \) and \( Y_k \) (i.e., \( X_k = (Z_k, Y_k) \)) and \( \mathcal{F}_k = \sigma(Z_k) \) with \( Z_k = (Z_0, \ldots, Z_k) \) (think of \( Y_k \) as unobservable part of \( X_k \) and \( Z_k \) as observable). Then, by the tower property of the conditional expectation, conditions (A1) and (A2) hold for the filtration \( \{\mathcal{F}_k\}_{k \in \mathbb{N}_1} \) as well if we take the new gain function \( G_k(\theta_k, Z_k) = E_1[G_k(\theta_k, X_k) | \mathcal{F}_k] \), \( k \in \mathbb{N}_0 \). In fact, this means that the case of a coarser filtration \( \{\mathcal{F}_k\}_{k \in \mathbb{N}_1} \), with respect to which the parameter sequence \( \{\theta_k\}_{k \in \mathbb{N}_0} \) is predictable, can be reduced to the above described setup in terms of “new time series” \( Z_k \), \( k \in \mathbb{N}_0 \), and the filtration \( \{\mathcal{F}_k\}_{k \in \mathbb{N}_1} \) generated by this new time series (or other way around). Thus, if the parameter sequence \( \{\theta_k\}_{k \in \mathbb{N}_0} \) is known to be predictable with respect to a coarser filtration \( \{\mathcal{F}_k\}_{k \in \mathbb{N}_1} \), we could have considered this coarser filtration and the corresponding time series \( \{Z_k\}_{k \in \mathbb{N}_0} \) from the very beginning, and impose conditions (A1) and (A2) in terms of \( \{Z_k\}_{k \in \mathbb{N}_0} \) and \( \{\mathcal{F}_k\}_{k \in \mathbb{N}_1} \). In fact, the coarser the filtration \( \{\mathcal{F}_k\}_{k \in \mathbb{N}_1} \), the weaker the conditions.

On the other hand, if the parameter sequence \( \{\theta_k\}_{k \in \mathbb{N}_0} \) is known to be predictable with respect to a finer filtration \( \{\mathcal{F}_k\}_{k \in \mathbb{N}_1} \) (i.e., \( \sigma(Z_k) = \mathcal{F}_k \supset \sigma(X_k), k \in \mathbb{N}_0 \)), then the key relation (5) will most likely not hold since the expression on the left hand side is \( \sigma(X_k) \)-measurable and the expression on the right hand side is \( \mathcal{F}_k \)-measurable. However, in such situation (5) may still hold with some (small) error \( \eta_k \), we address this issue in Remark 12 below. Intuitively, the information of what we observe should match (or, at least, not be less than) the information of what we want to track.

**Remark 5.** Consider one particular case of our general setting. At time \( n \in \mathbb{N}_0 \), we observe \( X_n = (X_0, X_1, \ldots, X_n) \) such that

\[
X_0 \sim \mathbb{P}_{\Theta_0}, \quad X_k | X_{k-1} \sim \mathbb{P}_{\theta_k} (\cdot | X_{k-1}), \quad \theta_k \in \Theta \subset \mathbb{R}^d, \quad k \in \mathbb{N}.
\]  
(10)

The model in this case is \( \mathcal{P}_k = \mathcal{P}_k(\Theta) = \{\mathbb{P}_{\theta}(\cdot | x_{k-1}) : \theta \in \Theta, x_{k-1} \in \mathcal{X}^{k-1}\} \) and the operator \( A_k(\mathbb{P}_{\theta_k}(\cdot | x_{k-1})) = \theta_k \). This is a convenient formulation when the time series model is parametrized by a time-varying parameter which we would like to recover by using an online tracking algorithm. Also in this case we can actually allow the parameter \( \theta_k \) to depend on the past of the time series, i.e., \( \theta_k = \theta_k(X_{k-1}) \) so that the sequence \( \{\theta_k, k \in \mathbb{N}_0\} \) is predictable with respect to the filtration \( \{\mathcal{F}_k\}_{k \in \mathbb{N}_1} \).

Condition (A1) can be reformulated as condition (A1) below, which gives some intuition about the role of the average gain \( g_k \) defined in (A1) and which may, in certain situations, be easier to verify.
For \( \lambda R \) such that \( \Lambda_{1}(X_{k-1}), \Lambda_{d}(X_{k-1}) \) and constants \( 0 < \lambda_1 \leq \lambda_2 < \infty, L > 0 \) such that \( \|g_k(\hat{\theta}_k, \theta_k|X_{k-1})\| \leq L\|\hat{\theta}_k - \theta_k\| \) and

\[
\Lambda_{1}(X_{k-1})\|\hat{\theta}_k - \theta_k\|^2 \leq -(\hat{\theta}_k - \theta_k)^T g_k(\hat{\theta}_k, \theta_k|X_{k-1}) \leq \Lambda_{d}(X_{k-1})\|\hat{\theta}_k - \theta_k\|^2,
\]

with \( \lambda_1 \leq E[\Lambda_{1}(M_k)|\mathcal{F}_{k-2}] \) and \( \lambda_1 \leq \Lambda_{d}(M_k) \leq \lambda_2 \).

In view of the lemma below, if (A1) holds, then (A1) will also hold (and vice versa); the values of the constants \( \lambda_1 \) and \( \lambda_2 \) appearing in the assumptions are different, though. The proof of this lemma is deferred to Section 7.

**Lemma 1.** Let \( x, y \in \mathbb{R}^d \). If there exists a symmetric positive definite matrix \( M \) such that \( y = Mx \) and \( 0 < \lambda_1 \leq \lambda_{1}(M) \leq \lambda_{d}(M) \leq \lambda_2 < \infty \) for some \( \lambda_1, \lambda_2 \in \mathbb{R} \), then \( 0 < \lambda_1' \|x\|^2 \leq \langle x, y \rangle \leq \lambda_2' \|x\|^2 < \infty \) and \( \|y\| \leq C\|x\| \) for some \( \lambda_1', \lambda_2', C \in \mathbb{R} \) (depending only on \( \lambda_1, \lambda_2 \)) such that \( 0 < \lambda_1' \leq \lambda_2' < \infty \) and \( C > 0 \).

Conversely, if \( 0 < \lambda_1' \|x\|^2 \leq \langle x, y \rangle \leq \lambda_2' \|x\|^2 < \infty \) and \( \|y\| \leq C\|x\| \) for some \( \lambda_1', \lambda_2', C \in \mathbb{R} \) such that \( 0 < \lambda_1' \leq \lambda_2' < \infty \) and \( C > 0 \), then there exists a symmetric positive definite matrix \( M \) such that \( y = Mx \) and \( 0 < \lambda_1 \leq \lambda_{1}(M) \leq \lambda_{d}(M) \leq \lambda_2 < \infty \) for some constants \( \lambda_1, \lambda_2 \in \mathbb{R} \) depending only on \( \lambda_1', \lambda_2' \) and \( C \).

## 3 Main results

We start with a lemma which we will need in the proof of the main result. Heuristically, since the gain vector \( G_k(\hat{\theta}_k, X_k) \) moves, on average, \( \hat{\theta}_k \) towards \( \theta_k \) and the sequence \( \theta_k \in \Theta \) is bounded (since \( \Theta \) is compact), the resulting estimating sequence \( \hat{\theta}_k \) should also be well-behaved. The following lemma states that the second moment of \( \hat{\theta}_k \) is uniformly bounded in \( k \in \mathbb{N}_0 \) for sufficiently small \( \gamma_k \).

**Lemma 2.** Let assumptions (A1) and (A2) hold. Then for sufficiently small \( \gamma_k \) there exists a constant \( C_{\Theta} \) such that

\[
E\|\hat{\theta}_k\|^2 \leq C_{\Theta}, \quad k \in \mathbb{N}_0.
\]

The proof of this lemma is given in Section 4. In fact, it is enough to assume that \( \gamma_k \) is sufficiently small for all \( k \geq N \) for some fixed \( N \in \mathbb{N} \). This is the case if \( \gamma_k \to 0 \) as \( k \to \infty \), which is typically assumed. This lemma will be used in the proof of the main theorem below. From now on we assume that the sequence \( \gamma_k \) is such that Lemma 2 holds.

The following theorem is our main result, it provides a non-asymptotic upper bound on the quality of the tracking algorithm (3) in terms of of the algorithm step sequence \( \{\gamma_i, i \in \mathbb{N}_0\} \) and oscillation of the process to track \( \{\theta_i, i \in \mathbb{N}_0\} \) between arbitrary time moments \( k_0, k \in \mathbb{N}_0, k_0 \leq k \).

**Theorem 1.** Let assumptions (A1) and (A2) hold, the tracking sequence \( \hat{\theta}_k \) be defined by (3) and \( \delta_k = \delta_k(X_{k-1}) = \hat{\theta}_k - \theta_k, k \in \mathbb{N}_0 \). Then for any \( k_0, k \in \mathbb{N}_0 \) and sequence
\{\gamma_k, k \in \mathbb{N}_0\}$ (satisfying the conditions of Lemma 2) such that $k_0 \leq k$ and $\gamma_i \lambda_2 \leq 1$ for all $i \in \mathbb{N}_{k_0,k}$, the following relation holds:

$$E|\delta_{k+1}| \leq C_1 \exp \left\{ -\frac{\lambda}{2} \sum_{i=k_0}^k \gamma_i \right\} + C_2 \left[ \sum_{i=k_0}^k \gamma_i^2 \right]^{1/2} + C_3 \max_{k_0 \leq i \leq k} E|\theta_{i+1} - \theta_{k_0}|,$$

(11)

where $C_1 = \sqrt{2(C_\Theta + C_\Theta^2)/2}$, $C_2 = C^g_2(1 + \lambda_2/\lambda_1)$, $C_3 = (1 + \lambda_2/\lambda_1)$, constants $\lambda_1, \lambda_2, C_g$ are from assumptions (A1) and (A2), $C_\Theta$ is defined by (2) and $\Theta_\Theta$ is from Lemma 2.

Remark 6. By using (11), one can derive a bound for $E|\delta_{k+1}|_p$, with $p \geq 1$. Indeed, as $\|x\|_s \leq \|x\|_r \leq d^{1/r-1/s}\|x\|_s$ for any $x \in \mathbb{R}^d$ and $s \geq r \geq 1$, we obtain that $\|\delta_{k+1}\|_p \leq \|\delta_{k+1}\|_2 = \|\delta_{k+1}\|$ for $p \geq 2$ and $\|\delta_{k+1}\|_p \leq d^{1/p-1/2}\|\delta_{k+1}\|_2 \leq d^{1/2}\|\delta_{k+1}\|$ for $1 \leq p < 2$.

Proof. For the sake of brevity, denote $\Delta_k = \Delta_k(X_k) = \theta_k - \theta_{k+1}$, $G_k = G(\hat{\theta}_k, X_k)$, $g_k = g(\hat{\theta}_k, \theta_k|X_{k-1})$ and $D_k = G_k - g_k$, $k \in \mathbb{N}_0$. We have

$$E[D_k|F_{k-1}] = E[(G_k - g_k)|F_{k-1}] = g_k - g_k = 0, \quad k \in \mathbb{N}_0.$$

It follows that $\{D_k, k \in \mathbb{N}_0\}$ is a (vector) martingale difference sequence with respect to the filtration $\{F_k\}_{k \in \mathbb{N}_1}$.

Rewrite the algorithm equation (3) as

$$\delta_{k+1} = \delta_k + \Delta \theta_k + \gamma_k D_k + \gamma_k g_k, \quad k \in \mathbb{N}_0.$$

In view of (A1), the decomposition $g_k = -M_k \delta_k$ holds almost surely, with an $F_{k-1}$-measurable symmetric positive definite matrix $M_k$, so that

$$\delta_{k+1} = \Delta \theta_k + \gamma_k D_k + (I - \gamma_k M_k) \delta_k, \quad k \in \mathbb{N}_0.$$

By iterating the above relation, we obtain that for any $k_0 = 0, \ldots, k$

$$\delta_{k+1} = (I - \gamma_k M_k)(I - \gamma_{k-1} M_{k-1}) \delta_{k-1} + \Delta \theta_k + \gamma_k D_k$$

$$+ (I - \gamma_k M_k)(\Delta \theta_{k-1} + \gamma_{k-1} D_{k-1})$$

$$= \left[ \prod_{i=k_0}^k (I - \gamma_i M_i) \right] \delta_{k_0} + \sum_{i=k_0}^k \left[ \prod_{j=i+1}^k (I - \gamma_j M_j) \right] (\Delta \theta_i + \gamma_i D_i).$$

(12)

Denote $A_i = \sum_{j=k_0}^i \gamma_j D_j$, $B_i = \sum_{j=k_0}^i \Delta \theta_j$ and $H_i = A_i + B_i$. Applying the Abel transformation (Lemma 3) to the second term of the right hand side of (12) yields

$$\sum_{i=k_0}^k \left[ \prod_{j=i+1}^k (I - \gamma_j M_j) \right] (\Delta \theta_i + \gamma_i D_i) = H_k - \sum_{i=k_0}^{k-1} \gamma_{i+1} M_{i+1} \left[ \prod_{j=i+2}^k (I - \gamma_j M_j) \right] H_i.$$ 

(13)

In particular, note that if we take $d = 1$, $M_i = \lambda_1$ and $\Delta \theta_j = 0$ for $j = k_0, \ldots, k$, $D_{k_0} = 1$ and $D_j = 0$ for $j = k_0 + 1, \ldots, k$, we derive that (since $0 \leq \gamma_j \lambda_1 \leq 1$ for $j = k_0, \ldots, k$)

$$\sum_{i=k_0}^{k-1} \lambda_1 \gamma_{i+1} \prod_{j=i+2}^k (1 - \gamma_j \lambda_1) = 1 - \prod_{j=k_{0}+1}^k (1 - \gamma_j \lambda_1) \leq 1,$$

(14)
which we will use later.

Using (13), we can rewrite our expansion of $\delta_{k+1}$ in (12) as follows:

$$
\delta_{k+1} = \left[ \prod_{i=k_0}^{k} (I - \gamma_i M_i) \right] \delta_{k_0} + H_k - \sum_{i=k_0}^{k-1} \gamma_{i+1} M_{i+1} \left[ \prod_{j=i+2}^{k} (I - \gamma_j M_j) \right] H_i.
$$

The previous display, the Minkowski inequality and the sub-multiplicative property of the operator norm ($\|AB\| \leq \|A\|\|B\|$) imply that

$$
\|\delta_{k+1}\| \leq \|\delta_{k_0}\| \prod_{i=k_0}^{k} \|I - \gamma_i M_i\| + \|H_k\|
+ \sum_{i=k_0}^{k-1} \gamma_{i+1} \|M_{i+1}\| \|H_i\| \prod_{j=i+2}^{k} \|I - \gamma_j M_j\|.
$$

(15)

In view of (A1) and the condition $\gamma_i \lambda_2 \leq 1$ for $i = k_0, \ldots, k$, $\gamma_i \Lambda_d(M_i) \leq \gamma_i \lambda_2 < 1$, $i = k_0, \ldots, k$, almost surely. Hence $0 \leq \gamma_i \Lambda_d(M_i) \leq \gamma_i \Lambda_d(M_i) \leq 1$, $i = k_0, \ldots, k$, almost surely. This, Lemma 3 and the fact (see (3) from (A1)) that $0 < \lambda_1 \leq \mathbb{E}[\Lambda_1(M_i) F_{k-2}]$, $i = k_0, \ldots, k$, almost surely, imply that

$$
\mathbb{E} \left[ \prod_{i=k_0}^{k} (I - \gamma_i M_i) \right] \leq \mathbb{E} \left[ \prod_{i=k_0}^{k} (1 - \gamma_i \Lambda_1(M_i)) \right] \leq \mathbb{E} \left[ \prod_{i=k_0}^{k} (1 - \gamma_i \Lambda_1(M_i)) \right] F_{k-2}.
$$

(16)

By (2) and Lemma 2 we have $\mathbb{E}\|\delta_{k_0}\|^2 \leq 2(\mathbb{E}\|\theta_{k_0}\|^2 + \mathbb{E}\|\hat{\theta}_{k_0}\|^2) \leq 2(C_\Theta + \tilde{C}_\Theta) = C_1^2$. Using this fact, the Cauchy-Schwartz inequality, (16) and the elementary inequality $1 + x \leq e^x$, $x \in \mathbb{R}$, leads to

$$
\mathbb{E} \left[ \|\delta_{k_0}\| \prod_{i=k_0}^{k} \|I - \gamma_i M_i\| \right] \leq \left[ \mathbb{E}\|\delta_{k_0}\|^2 \mathbb{E} \prod_{i=k_0}^{k} \|I - \gamma_i M_i\|^2 \right]^{1/2}
\leq C_1 \prod_{i=k_0}^{k} (1 - \gamma_i \lambda_1)^{1/2} \leq C_1 \exp \left\{ - \frac{\lambda_1}{2} \sum_{i=k_0}^{k} \gamma_i \right\}.
$$

(17)

Let $D_{kl}$ denote the $l$-th coordinate of the vector $D_k$. Clearly, for each $l = 1, \ldots, d$, \{D_{kl}, k \in N_0\} is a martingale difference with respect to the filtration \{F_k\}_{k \in \mathbb{N}_0}. Using (7) from (A2) and the fact that martingale difference terms are uncorrelated, we derive that for all $i = k_0, \ldots, k$

$$
\mathbb{E}\|A_i\|^2 = \mathbb{E} \sum_{l=1}^{d} \left( \sum_{j=k_0}^{i} \gamma_j D_{jl} \right)^2 = \sum_{l=1}^{d} \sum_{j=k_0}^{i} \gamma_j^2 \mathbb{E} D_{jl}^2 = \sum_{j=k_0}^{i} \gamma_j^2 \mathbb{E}\|D_j\|^2 \leq C_g \sum_{j=k_0}^{k} \gamma_j^2.
$$
Denote for brevity $\Gamma_{k_0,k}^2 = \sum_{i=k_0}^{k} \gamma_i^2$, so that $E\|A_i\|^2 \leq C_g \gamma_i^2$, $i = k_0, \ldots, k$.

The obtained relation for $E\|A_i\|^2$, together with the Minkowski and Hölder inequalities, imply that for all $i = k_0, \ldots, k$

$$E\|H_i\| \leq E\|A_i\| + E\|B_i\| \leq \left( E\|A_i\|^2 \right)^{1/2} + E\|B_i\| \leq C_{g}^{1/2} \Gamma_{k_0,k} + E\|B_i\|. \quad (18)$$

Notice that $\|M_{i+1}\|\|H_i\|$ from (15) is $\mathcal{F}_j$-measurable for all $j \geq i$. Therefore, by (16) from (A1), (18) and Lemma 3

$$E \left[ \|M_{i+1}\|\|H_i\| \prod_{j=i+2}^{k} \|I - \gamma_j M_j\| \right] = E E \left[ \|M_{i+1}\|\|H_i\| \prod_{j=i+2}^{k} \|I - \gamma_j M_j\| \bigg| \mathcal{F}_{k-2} \right]$$

$$= E \left[ E \left[ 1 - \gamma_k A(1) (M_k) \big| \mathcal{F}_{k-2} \right] \|M_{i+1}\|\|H_i\| \prod_{j=i+2}^{k-1} \left( 1 - \gamma_j A(1)(M_j) \right) \right] \leq \ldots$$

$$\leq E \left( \|M_{i+1}\|\|H_i\| \right) \prod_{j=i+2}^{k} \left( 1 - \gamma_j \lambda_1 \right) \leq \lambda_2 E\|H_i\| \prod_{j=i+2}^{k} \left( 1 - \gamma_j \lambda_1 \right)$$

$$\leq \lambda_2 \left[ C_{g}^{1/2} \Gamma_{k_0,k} + E\|B_i\| \right] \prod_{j=i+2}^{k} \left( 1 - \gamma_j \lambda_1 \right). \quad (19)$$

Now we take the expectation of relation (15) and use relations (17), (18), (19) and (14) to derive that $E\|\delta_{k+1}\|$ is bounded from above by

$$E \left[ \|\delta_{k_0}\| \prod_{i=k_0}^{k} \|I - \gamma_i M_i\| \right] + E\|H_k\| + \sum_{i=k_0}^{k-1} \gamma_{i+1} E \left[ \|M_{i+1}\|\|H_i\| \prod_{j=i+2}^{k} \|I - \gamma_j M_j\| \right]$$

$$\leq C_1 \exp \left\{ -\frac{\lambda_1}{2} \sum_{i=k_0}^{k} \gamma_i \right\} + E\|H_k\| + \lambda_2 \sum_{i=k_0}^{k-1} \gamma_{i+1} E\|H_i\| \prod_{j=i+2}^{k} \left( 1 - \gamma_j \lambda_1 \right)$$

$$\leq C_1 \exp \left\{ -\frac{\lambda_1}{2} \sum_{i=k_0}^{k} \gamma_i \right\} + E\|H_i\| \left[ 1 + \sum_{i=k_0}^{k-1} \lambda_2 \gamma_{i+1} \prod_{j=i+2}^{k} \left( 1 - \gamma_j \lambda_1 \right) \right]$$

$$\leq C_1 \exp \left\{ -\frac{\lambda_1}{2} \sum_{i=k_0}^{k} \gamma_i \right\} + \left[ C_{g}^{1/2} \Gamma_{k_0,k} + \max_{k_0 \leq i \leq k} E\|\theta_{i+1} - \theta_{k_0}\| \right] \left[ 1 + \frac{\lambda_2}{\lambda_1} \right],$$

where we also used in the last bound that $B_i = \sum_{j=k_0}^{i} \Delta \theta_j = \theta_{i+1} - \theta_{k_0}$, $i = k_0, \ldots, k$, is a telescopic sum. This completes the proof of the theorem. \(\square\)

**Remark 7.** At this stage, it may not be clear how the non-asymptotic bound from Theorem \(\text{(1)}\) can be utilized. The obtained result is not useful unless we assume some sort of
damping of the oscillations of the parameter process \( \{\theta_k, k \in \mathbb{N}_0\} \). Looking ahead, in Section 5 we impose certain settings for damping of the parameter process oscillations (either “stabilizing” in time or increasing the observation frequency) and derive results in various asymptotic regimes by using our main Theorem 1.

**Remark 8.** If we assume a slightly stronger version of (6) in (A1),

\[ 0 < \lambda_1 \leq \Lambda_{(1)}(M_i) \leq \Lambda_{(d)}(M_i) \leq \lambda_2, \quad k \in \mathbb{N}_0, \]

almost surely, then a slightly better version of bound (17) holds:

\[
E \left[ \|\delta_{k_0} \| \prod_{i=k_0}^k \|I - \gamma_i M_i\| \right] \leq \bar{C}_1 \prod_{i=k_0}^k (1 - \gamma_i \lambda_1) \leq \bar{C}_1 \exp \left\{ - \lambda_1 \sum_{i=k_0}^k \gamma_i \right\},
\]

since \( E[\|\delta_{k_0}\| \leq E[\|\hat{\theta}_{k_0}\| + E[\|\theta_{k_0}\| \leq \bar{C}_1^{1/2} + \bar{C}_1^{1/2} = \bar{C}_1, \text{ by (2)} \text{ and Lemma 2.} \]

We can derive a bound alternative to (17), which leads to slightly better constants in the first term of the right hand side of (17). Indeed, \( \delta_k \) is \( \mathcal{F}_{k-1} \)-measurable, and, instead of (17), we derive

\[
E \left[ \|\delta_{k_0} \| \prod_{i=k_0}^k \|I - \gamma_i M_i\| \right] = E \left[ \|\delta_{k_0} \| \prod_{i=k_0}^{k-1} \left( 1 - \gamma_i \Lambda_{(1)}(M_i) \right) \right] \left( 1 - \gamma_k \lambda_1 \right) \leq \ldots
\]

\[
\leq E \left[ \|\delta_{k_0} \| \left( 1 - \gamma_k \Lambda_{(1)}(M_{k_0}) \right) \right] \prod_{i=k_0+1}^k (1 - \gamma_i \lambda_1) \leq E \left[ \|\delta_{k_0}\| \right] \prod_{i=k_0+1}^k (1 - \gamma_i \lambda_1)
\]

\[
\leq \left( \bar{C}_1^{1/2} + \bar{C}_1^{1/2} \right) \prod_{i=k_0+1}^k \left( 1 - \gamma_i \lambda_1 \right).
\]

**Remark 9.** If we assume that \( \gamma_i \lambda_2 \leq 1 \) for all \( i \in \mathbb{N}_0 \) and \( \sum_{i=1}^{\infty} \gamma_i^2 < \infty \), then we can prove Lemma 2 in another way: first take the expectation of the second power of the relation (15) with \( k_0 = 0 \) to establish that \( E[|\delta_k|^2] < C \) is uniformly bounded in \( k \in \mathbb{N}_0 \), and then \( E[|\hat{\theta}_k|^2] \leq 2E[|\delta_k|^2] + 2E[|\theta_k|^2] \leq 2(C + C_0), \text{ by (2)} \).

**Remark 10.** One can try to establish a version of Theorem 1 where, instead of (6), one assumes

\[ \lambda_1 \leq \Lambda_{(1)}(E[M_k|\mathcal{F}_{k-2}]) \leq \Lambda_{(d)}(E[M_k|\mathcal{F}_{k-2}]) \leq \lambda_2 \quad \text{almost surely.} \quad (20) \]

The point is that there may be situations with certain gain functions when (6) does not hold but (20) does; see Remark 21 below. The idea of the proof would be to first introduce \( \tilde{M}_i = E[M_i|\mathcal{F}_{i-2}], \ i \in \mathbb{N}_0 \), and then, beginning with the relation (12), work with the representation \( M_i = \tilde{M}_i + M_i - \tilde{M}_i \) instead of just \( M_i \), using the relation (20) for \( M_i \) and the fact that \( \{M_k - \tilde{M}_k, k \in \mathbb{N}_0\} \), is a (matrix) martingale difference sequence with respect to the filtration \( \{\mathcal{F}_{k-1}\}_{k \in \mathbb{N}_1} \). We will not pursue this here.
Imposing somewhat stronger versions of conditions (A1) and (A2) enables us to derive a similar non-asymptotic bound for the expectation of $\|\delta_{k+1}\|_p^p$ for all $p \geq 1$. Of course, the bigger $p$, the bigger the constants involved in the bound. The next theorem is a strengthened version of the previous result.

**Theorem 2.** Suppose that the conditions of Theorem 1 are fulfilled. If, in addition (to assumption (A1)), $\Lambda_{(1)}(M_i) \geq \lambda_1$ and $\|G_i(\hat{\theta}_i, X_i)\| \leq \bar{G}$ (instead of (A2)) almost surely for all $i = k_0, \ldots, k$, then for any $p \geq 1$

$$
\mathbb{E}\|\delta_{k+1}\|_p^p \leq C'_1 \mathbb{E}\|\delta_{k_0}\|_p^p \exp \left\{ -p\lambda_1 \sum_{i=k_0}^k \gamma_i \right\} \\
+ C'_2 \left[ \sum_{i=k_0}^k \gamma_i^2 \right]^{p/2} + C'_3 \max_{k_0 \leq i \leq k} \mathbb{E}\|\theta_{i+1} - \theta_{k_0}\|_p^p, 
$$

where $C'_1 = 3^{p-1}K_p$, $C'_2 = 3^{p-1}2^p dB_p \bar{G}^p (1 + K_p^2 \lambda_2 / \lambda_1)^p$, $C'_3 = 3^{p-1} (1 + K_p^2 \lambda_2 / \lambda_1)^p$ and $K_p = K_p(d)$ is the constant from Lemma 3.

**Proof.** Now we have stronger versions of assumptions (A1) and (A2):

$$
0 < \lambda_1 \leq \Lambda_{(1)}(M_i) \leq \Lambda_{(d)}(M_i) \leq \lambda_2, \quad \|G_i(\hat{\theta}_i, X_i)\| \leq \bar{G}, \quad i = k_0, \ldots, k, 
$$

hold almost surely. Along the same lines as for (15), by using Lemma 3, (22), (14) and the elementary inequality $1 - x \leq e^{-x}$, we obtain that

$$
\|\delta_{k+1}\|_p \leq K_p \|\delta_{k_0}\|_p \prod_{i=k_0}^k (1 - \gamma_i \lambda_1) + \max_{k_0 \leq i \leq k} \|C_i\|_p \left[ 1 + K_p^2 \sum_{i=k_0}^{k-1} \gamma_i \lambda_2 \prod_{j=i+2}^k (1 - \gamma_i \lambda_1) \right] \\
\leq K_p \|\delta_{k_0}\|_p \exp \left\{ -\lambda_1 \sum_{i=k_0}^k \gamma_i \right\} + \left[ 1 + \frac{K_p^2 \lambda_2}{\lambda_1} \right] \left( \max_{k_0 \leq i \leq k} \|A_i\|_p + \max_{k_0 \leq i \leq k} \|B_i\|_p \right) 
$$

almost surely, where constant $K_p = K_p(d)$ is from Lemma 3. Take now the $p$-th power of both sides of the inequality and apply the Hölder inequality $|\sum_{i=1}^m a_i|^p \leq m^{p-1} \sum_{i=1}^m |a_i|^p$ for $m = 3$ to get

$$
\|\delta_{k+1}\|_p^p \leq 3^{p-1}K_p^p \|\delta_{k_0}\|_p^p \exp \left\{ -p\lambda_1 \sum_{i=k_0}^k \gamma_i \right\} \\
+ 3^{p-1} \left( 1 + \frac{K_p^2 \lambda_2}{\lambda_1} \right)^p \left( \max_{k_0 \leq i \leq k} \|A_i\|_p^p + \max_{k_0 \leq i \leq k} \|B_i\|_p^p \right).
$$

Recall that the sequence $\{ \sum_{j=k_0}^i \gamma_j D_j, i \geq k_0 \}$ is a martingale with respect to the filtration $\{\mathcal{F}_i, i \geq k_0 \}$ and that the coordinates of $D_j$ verify $|D_j| \leq 2\|G_j\| \leq 2\bar{G}$ almost
surely, \( l = 1, \ldots, d, \ j = k_0, \ldots, k \). Applying the maximal Burkholder inequality for \( p > 1 \) and the Davis inequality for \( p = 1 \) (cf. [21]) yields

\[
\mathbb{E} \max_{k_0 \leq r \leq k} \left\| A_r \right\|_p^p = \mathbb{E} \max_{k_0 \leq r \leq k} \left[ \sum_{l=1}^{d} \left( \sum_{j=k_0}^{r} \gamma_j D_{jl} \right) \right]_p^p \leq \sum_{j=k_0}^{i} \mathbb{E} \max_{k_0 \leq r \leq k} \left[ \sum_{j=k_0}^{r} \gamma_j D_{jl} \right]_p^p
\]

\[
\leq B_p \sum_{i=1}^{d} \mathbb{E} \left[ \sum_{j=k_0}^{k} \gamma_j^2 D_{jl}^2 \right]^{p/2} \leq dB_p 2^p G_p \left[ \sum_{j=k_0}^{k} \gamma_j^2 \right]^{p/2},
\]

for some constant \( B_p \). One can take \( B_p = ((18p^5/2)/(p - 1)^{3/2})^p \) for \( p > 1 \), cf. [21]. The second inequality of the theorem now follows by taking expectations on both sides of the bound on \( \left\| \delta_{k+1} \right\|_p^p \) above and by using the last inequality.

**Remark 11.** One can derive a similar result for the \( \mathbb{E} \left\| \delta_{k+1} \right\|_p^p \), by simply taking the \( p \)-th power of the inequality (13) and then proceeding in the same way as in the proof of Theorem 2 with minor modifications in the argument for the martingale \( A_i \).

Once a bound on \( \mathbb{E} \left\| \delta_{k+1} \right\|_p^p \) is established, one can use it for proving Theorem 2 in another way. Namely, since \( \left\| x \right\|_s \leq \left\| x \right\|_r \leq d^{1/r - 1/s} \left\| x \right\|_s \) for any \( x \in \mathbb{R}^d \) and \( s \geq r \geq 1 \), \( \left\| \delta_{k+1} \right\|_p \leq R_{2k}^p \left\| \delta_{k+1} \right\|_s \), with \( R_{2k}^p = 1 \) if \( p \geq 2 \) and \( R_{2k}^p = d^{(2-p)/(2p)} \leq d^{1/2} \) if \( 1 \leq p < 2 \). Thus, a bound for \( \mathbb{E} \left\| \delta_{k+1} \right\|_p^p \) will immediately follow from the obtained bound for \( \mathbb{E} \left\| \delta_{k+1} \right\|_s^p \). The bound will be of the same form as in Theorem 2 but with different constants \( C_1, C_2, C_3 \).

**Remark 12.** Consider the following situation, which we will call Case I. Suppose we are not interested in tracking the, say, natural parameter \( \theta_k \) of the model, but rather some other time-varying parameter \( \theta_k^* \), which is also assumed to be predictable with respect to the filtration \( \{ \mathcal{F}_k \}_{k \in \mathbb{N}_0} \). Denote \( \Delta \theta_k^* = \theta_k^* - \theta_{k+1}^* \), \( k \in \mathbb{N}_0 \). The difference \( \varepsilon_k = \theta_k - \theta_k^* \), \( k \in \mathbb{N}_0 \), can be seen as an approximation error. Similar to (12), the following expansion can be derived for the quantity \( \delta_k^* = \theta_k - \theta_k^* \):

\[
\delta_{k+1} = \delta_k^* + \Delta \theta_k^* + \gamma_k D_k - \gamma_k M_k (\theta_k - \theta_k^*) = \Delta \theta_k^* + \gamma_k M_k \varepsilon_k + \gamma_k D_k + (I - \gamma_k M_k) \delta_k^*
\]

\[
= \left[ \prod_{i=k_0}^{k} (I - \gamma_i M_i) \right] \delta_{k_0}^* + \sum_{i=k_0}^{k} \left[ \prod_{j=i+1}^{k} (I - \gamma_j M_j) \right] (\Delta \theta_i^* + \gamma_i M_i \varepsilon_i + \gamma_i D_i).
\]

Now consider Case II: we want to track the natural parameter \( \theta_k \) but the average gain makes an error \( \eta_k \), i.e., \( g_k = -M_k (\theta_k - \theta_k^*) + \eta_k \), \( k \in \mathbb{N}_0 \). The error term \( \eta_k \) may be random but must be measurable with respect to \( \mathcal{F}_{k-1} \). Again, similar to (12), we can derive

\[
\delta_{k+1} = \left[ \prod_{i=k_0}^{k} (I - \gamma_i M_i) \right] \delta_{k_0} + \sum_{i=k_0}^{k} \left[ \prod_{j=i+1}^{k} (I - \gamma_j M_j) \right] (\Delta \theta_i + \gamma_i \eta_i + \gamma_i D_i).
\]

Now notice that Case I can actually be reduced to Case II by putting in the last relation \( \delta_i = \theta_i - \theta_i^* \) and \( \eta_i = M_i \varepsilon_i \) (where \( \varepsilon_i = \theta_i - \theta_i^* \)), \( i \in \mathbb{N}_0 \). Therefore, consider only Case II from now on.
Under the conditions of Theorem 1, in the same way as for (11), we can derive the following bound:

\[
E\|\delta_{k+1}\| \leq C_1 \exp \left\{ -\lambda_1 \sum_{i=k_0+1}^{k} \gamma_j \right\} + C_2 \left[ \sum_{i=k_0}^{k-1} \gamma_i^2 \right]^{1/2} + C_3 \max_{k_0 \leq i \leq k} \|\theta_{i+1} - \theta_{k_0}\| + C_4 \sum_{i=k_0}^{k} \gamma_i \|\eta_i\|.
\] (23)

Similarly, under the conditions of Theorem 2,

\[
E\|\delta_{k+1}\|^p \leq C'_1 \exp \left\{ -p\lambda_1 \sum_{i=k_0}^{k} \gamma_j \right\} + C'_2 \left[ \sum_{i=k_0}^{k-1} \gamma_i^2 \right]^{p/2} + C'_3 \max_{k_0 \leq i \leq k} \|\theta_{i+1} - \theta_{k_0}\|^p + \sum_{i=k_0}^{k} \gamma_i \|\eta_i\|^p.
\] (24)

Clearly, (23) and (24) generalize the bounds of Theorems 1 and 2, where we had \(\eta_i = 0\), \(i \in N_0\).

In Case I, we have \(\delta_i = \hat{\theta}_i - \theta^*_i\) and \(\eta_i = M_i \varepsilon_i\) with \(\varepsilon_k = \theta_k - \theta^*_k\), \(i \in N_0\), in relations (23) and (24). Noting that \(\|\eta_i\|_p = \|M_i \varepsilon_i\|_p < \lambda_2 K \|\varepsilon_i\|_p\) for all \(p \geq 1\) and \(i \in N_0\), we can rewrite bounds (23) and (24) in terms of \(\|\varepsilon_i\|_p\) instead of \(\|\eta_i\|_p\) with appropriate adjustments of corresponding constants.

### 4 Construction of gain functions

Any gain function for which conditions (A1) and (A2) hold may be used with our algorithm, and whether a particular gain function is suitable or not depends on the model under study and the quantity that we wish to track. For certain types of models and quantities to track, there are natural choices for the gain function. Many different settings are investigated in the literature. In this section we consider the construction of appropriate gain functions to be used in the algorithm (3) in several traditional settings. In particular, we relate our general approach to well known classical procedures such as Robbins-Monro and Kiefer-Wolfowitz algorithms and outline possible extensions.

#### 4.1 Signal + noise setting

The traditional ‘signal+noise’ situation can be represented by the following observation model:

\[ X_k = \theta_k + \xi_k, \quad k \in N_0, \]

where \(l = d\), \(\{\theta_k\}_{k \in N_0}\) is a predictable process \(\theta_k = \theta_k(X_{k-1})\) we are interested in tracking, \(\{\xi_k\}_{k \in N_0}\) is a martingale difference noise, with respect to the filtration \(\{\mathcal{F}_k\}_{k \in N-1}\).
We use the algorithm (3) for tracking $\theta_k$, and in this case we can simply take the following

$$G_k(\hat{\theta}_k, X_k) = - (\hat{\theta}_k - X_k), \quad k \in \mathbb{N}_0,$$

since

$$g_k(\hat{\theta}_k, \theta_k|X_{k-1}) = \mathbb{E}[G_k(\hat{\theta}_k, X_k)|X_{k-1}] = - (\hat{\theta}_k - \theta_k), \quad k \in \mathbb{N}_0,$$

i.e., $M_k(X_{k-1}) = I$. Clearly, condition (A1) holds and condition (A2) follows as well if we assume $\mathbb{E}[\xi_k]^2 \leq c, k \in \mathbb{N}_0$. Indeed, according to (3), it is enough to show the boundedness of the second moment of $G_k$:

$$\mathbb{E}[|G_k(\hat{\theta}_k, X_k)|^2] \leq 3 \left[ \mathbb{E}[\hat{\theta}_k]^2 + \mathbb{E}[\theta_k]^2 + \mathbb{E}[\xi_k]^2 \right] \leq C, \quad k \in \mathbb{N}_0,$$

by virtue of the Hölder inequality, Lemma 2 and (2). The classical nonparametric regression model fits into this framework so that our results can be applied. For example, the simplest nonparametric regression model with an equidistant design on $[0,1]$ is as follows: $X_k = \theta(k/n) + \xi_k, k = 1, \ldots, n$, with independent noises $\xi_k$’s, $\mathbb{E}\xi_k = 0, \mathbb{E}\xi_k^2 = \sigma^2$; we will return to this issue in subsection 5.3.

**Remark 13.** Possibly, $\mathbb{E}[X_k|X_{k-1}] = \varphi(\theta_k)$ for some smooth function $\varphi$. In this case, one should consider $G_k(\hat{\theta}_k, X_k) = - (\varphi(\hat{\theta}_k) - X_k), \in \mathbb{N}_0$, so that $g_k(\hat{\theta}_k, \theta_k|X_{k-1}) = - (\varphi(\hat{\theta}_k) - \varphi(\theta_k)), k \in \mathbb{N}_0$, which should be comparable to $- (\hat{\theta}_k - \theta_k)$. Autoregressive models, for example, fall into this category (cf. Section 6.4).

### 4.2 Robbins-Monro setting: tracking roots

Let us turn to more dynamical situations where the observations themselves depend on our tracking sequence. In their seminal paper, [20] studied the problem of finding the unique $\alpha$-root $\theta$ of a monotone function $f$, i.e., the equation $f(x) = \alpha$ has a unique solution at $x = \theta$. The function $f$ can be observed at any point $x$ but with noise $\xi$: $X(x) = F(x, \xi)$ so that $\mathbb{E}F(x, \xi) = f(x)$. A stochastic approximation algorithm of design points converging to $\theta$ is known as classical Robbins-Monro procedure. We now illustrate how this also fits into our general tracking algorithm scheme.

In fact, the following model essentially extends the original setup of [20]. Suppose there is a time series $\{Y_k, k \in \mathbb{N}_0\}$ (with $Y_k$ taking values in $\mathcal{Y}^k$) running at the background, which is not (fully) observable. Instead, some other $d$-dimensional (related) time series $\{X_k, k \in \mathbb{N}_0\}$ is observed, which we introduce below. As usual, let $\mathcal{F}_k = \sigma(X_k), k \in \mathbb{N}_0$. Further, for a sequence of functions $f_k : \mathbb{R}^d \times \mathcal{Y}^k \mapsto \mathbb{R}^d$, let a $d$-dimensional measurable function $\theta_k = \theta_k(X_{k-1})$ be the unique solution of the equation $\alpha_k(X_{k-1}) = \tilde{f}_k(\theta_k, X_{k-1})$, where $\tilde{f}_k(\theta_k, X_{k-1}) = \mathbb{E}[f_k(\theta_k, Y_k)|\mathcal{F}_{k-1}]$, for some measurable function $\alpha_k(X_{k-1}), k \in \mathbb{N}_0$. (Here $Y_k$ may contain $X_{k-1}$.)

The goal is to track the sequence $\{\theta_k\}_{k \in \mathbb{N}_0}$. At a time moment $k \in \mathbb{N}_0$, we observe the noise corrupted value of $f_k(\hat{\theta}_k, Y_k)$ at some design point $\hat{\theta}_k$ (which can be picked on the basis of the previous observations $X_{k-1}$, i.e., $\hat{\theta}_k = \hat{\theta}_k(X_{k-1})$):

$$X_k = f_k(\hat{\theta}_k, Y_k) = \tilde{f}_k(\theta_k, X_{k-1}) + \xi_k, \quad k \in \mathbb{N}_0,$$
where \( \{\xi_k\}_{k \in \mathbb{N}_0} \) is a martingale difference noise sequence with respect to the filtration \( \{\mathcal{F}_k\}_{k \in \mathbb{N}_0} \) (indeed, let simply \( \xi_k = f_k(\hat{\theta}_k, Y_k) - f_k(\theta_k, X_{k-1}) \)). Of course, we could assume a more general model \( X_k = f_k(\theta_k, Y_k) + \xi_k, k \in \mathbb{N}_0 \), but this would not have made any principal difference, since variable \( \xi_k \) can be incorporated into the vector \( Y_k \).

Let the design points \( \{\theta_k, k \in \mathbb{N}_0\} \) in (20) be determined by the algorithm (3) and we want this algorithm to track \( \theta_k \). Theorem 14 is applicable if the gain \( G_k \) in (3) satisfies (A1) and (A2). As in the Robbins-Monro algorithm, the gain is taken to be

\[
G_k(\hat{\theta}_k, X_k) = -(X_k - \alpha_k(X_{k-1})), \quad k \in \mathbb{N}_0.
\]

Then \( g_k(\hat{\theta}_k, \theta_k, \mathcal{F}_{k-1}) = -(\bar{f}_k(\hat{\theta}_k, X_{k-1}) - \alpha_k(X_{k-1})) \), \( k \in \mathbb{N}_0 \), and (A2) is fulfilled if, for example, \( \mathbb{E}\|f_k(\hat{\theta}_k, Y_k)\|^2 \leq c \) and \( \mathbb{E}\|\xi_k\|^2 \leq C, k \in \mathbb{N}_0 \). Condition (A1), or equivalently (A1), is fulfilled if, for some \( 0 < \lambda_1 \leq \lambda_2 \),

\[
\lambda_1\|\hat{\theta}_k - \theta_k\|^2 \leq (\hat{\theta}_k - \theta_k)^T(\bar{f}_k(\hat{\theta}_k, X_{k-1}) - \bar{f}_k(\theta_k, X_{k-1})) \leq \lambda_2\|\hat{\theta}_k - \theta_k\|^2, \quad k \in \mathbb{N}_0,
\]

almost surely. In the last display, one should recognize the usual regularity requirements for the function \( f \) in the classical Robbins-Monro setting: \( d = 1, \alpha_k(X_{k-1}) = \alpha_k \) and \( f_k(\theta, Y_k) = f(\theta) \) (so that \( \theta_k = \theta \) is the non-random solution of the equation \( \alpha = f(\theta) \)); \( \lambda_1 \leq (f(\theta) - \alpha)/(\alpha - \theta) \leq \lambda_2 \). In multidimensional case, this can be seen as a generalized identifiability requirement for the sequence \( \{\theta_k, k \in \mathbb{N}_0\} \). For example, if \( \bar{f}_k(\theta, x_{k-1}) \) is a differentiable mapping in \( \theta \in \Theta \) for each \( x_{k-1} \in \mathbb{X}^{k-1} \), then a sufficient condition for (A1) is positive definiteness of the Jacobian matrix of \( \bar{f}_k(\theta, x_{k-1}) \) (with respect to \( \theta \)), uniformly in \( x_{k-1} \in \mathbb{X}^{k-1} \) and over the support of \( \theta_k \). One can possibly relax this to a vicinity of the root \( \theta_k \) under other appropriate conditions which guarantee that \( \hat{\theta}_k \) eventually gets into a neighborhood of \( \theta_k \).

**Remark 14.** A particular example is \( \alpha_k = \bar{f}_k(\theta_k) = \mathbb{E}[f_k(\theta_k, Z_k)|\mathcal{F}_{k-1}] \), where \( Z_k \) is a subvector of \( Y_k \), independent of \( \mathcal{F}_{k-1} \).

### 4.3 Kiefer-Wolfowitz setting: tracking maxima

Another classical example is the algorithm of [13] for successive estimating the maximum of a function \( f \) which can be observed at any point, but gets corrupted with a martingale difference noise (similarly, one can formulate the problem of tracking minima of a sequence of functions). The algorithm is based on a gradient-like method, the gradient of \( f \) being approximated by using finite differences. There are many modifications of the procedure, including multivariate extensions, and they are all based on estimates of the gradient of \( f \).

The following scheme essentially contains many such procedures considered in the literature and even extends them to a time-varying predictable maxima process \( \{\theta_k, k \in \mathbb{N}_0\} \).

As in the previous subsection, suppose there is a time series \( \{Y_k, k \in \mathbb{N}_0\} \), with \( Y_k \) taking values in \( \mathbb{Y}^k \), running in the background, which is not (fully) observable. Instead, some other related time series \( \{X_k, k \in \mathbb{N}_0\} \) is observed, which we introduce below. Let \( \mathcal{F}_k = \sigma(X_k), k \in \mathbb{N}_0 \). Suppose we are given a sequence of measurable functions \( F_k : \)
\( \Theta \times \mathcal{Y}^k \mapsto \mathbb{R}, \Theta \subset \mathbb{R}^d \), \( k \in \mathbb{N}_0 \), such that the function \( \tilde{F}_k(\vartheta, X_{k-1}) = \mathbb{E}[F_k(\vartheta, Y_k)|F_{k-1}] \) has a unique maximum \( \hat{\vartheta}_k = \hat{\vartheta}_k(X_{k-1}) \) on \( \Theta \), i.e.,
\[
\max_{\vartheta \in \mathbb{R}^d} \tilde{F}_k(\vartheta, X_{k-1}) = \max_{\vartheta \in \Theta} \tilde{F}_k(\vartheta, X_{k-1}) = \tilde{F}_k(\hat{\vartheta}_k, X_{k-1}), \quad k \in \mathbb{N}_0.
\]

Again, \( Y_k \) may contain \( X_{k-1} \).

We want to track the sequence \( \{\theta_k\}_{k \in \mathbb{N}_0} \). For that we use the sequence of design points \( \{\hat{\vartheta}_k, k \in \mathbb{N}_0\} \) defined by the tracking algorithm \( 49 \), with \( F_k \)-measurable gain functions \( G_k \) to be specified later. At a time moment \( k \in \mathbb{N}_0 \), we observe the so called “noisy approximate gradients” at those design points:
\[
X_k = f_k(\hat{\vartheta}_k, Y_k) + \xi_k, \quad k \in \mathbb{N}_0,
\]
where \( \mathbb{E}[|f_k(\hat{\vartheta}_k, Y_k)|]^2 \leq c \) and \( \mathbb{E}[|\xi_k|^2] \leq C \) for all \( k \in \mathbb{N}_0 \), and \( \{\xi_k\}_{k \in \mathbb{N}_0} \) is a martingale difference noise sequence with respect to the filtration \( \{F_k\}_{k \in \mathbb{N}_1} \). The \( d \)-dimensional approximate gradient \( f_k(\hat{\vartheta}_k, Y_k) \) is not necessarily the exact pathwise \( \vartheta \)-derivative of \( F_k(\vartheta, Y_k) \) at \( \hat{\vartheta}_k \) (i.e., \( f_k(\vartheta, Y_k) = \nabla_{\vartheta} F_k(\vartheta, Y_k) \)) but such that
\[
\mathbb{E}[f_k(\hat{\vartheta}_k, Y_k)|F_{k-1}] = \tilde{f}_k(\hat{\vartheta}_k, X_{k-1}) = -M_k(\hat{\vartheta}_k - \vartheta_k) + \eta_k, \quad k \in \mathbb{N}_0,
\]
amost surely, where a symmetric positive definite matrix \( M_k = M_k(X_{k-1}) \) satisfies conditions \( 3 \) and \( \eta_k = \eta_k(X_{k-1}) \) is some predictable approximation error. Of course, such a representation \( 28 \) is always possible: simply take \( \eta_k = \tilde{f}_k(\hat{\vartheta}_k, X_{k-1}) + M_k(\hat{\vartheta}_k - \vartheta_k) \) for some symmetric positive definite matrix \( M_k \) satisfying \( 3 \); useful ones are those for which the \( \eta_k \)’s are under control – basically, the \( \eta_k \)’s should be small.

As a choice for the gain, take now \( G_k(\hat{\vartheta}_k, X_k) = X_k \), so that \( g_k(\hat{\vartheta}_k, \vartheta_k|X_{k-1}) = \mathbb{E}[f_k(\hat{\vartheta}_k, Y_k)|F_{k-1}] = \tilde{f}_k(\hat{\vartheta}_k, X_{k-1}) = -M_k(\hat{\vartheta}_k - \vartheta_k) + \eta_k, \) \( k \in \mathbb{N}_0 \). Clearly, (A2) holds in view of moment conditions on the quantities in \( 27 \), however (A1) is not satisfied in general since there is an approximation (possibly nonzero) term \( \eta_k \) involved. Yet, we are in the position of Remark \( 12 \) and thus the bound \( 29 \) for the tracking error holds in this case. This bound is however useful only if the approximation errors \( \eta_k \)’s get sufficiently small as \( k \) gets bigger. The most desirable situation is when \( \eta_k = 0, \) \( k \in \mathbb{N}_0 \).

For each particular model of form \( 27 \), one needs to determine conditions that should be imposed on the approximate gradients \( f_k \)’s in order to be able to claim a reasonable quality of the tracking algorithm by using our general result. Conditions on approximate gradients \( f_k \)’s from \( 27 \) which provide control on the magnitude of the approximation errors \( \eta_k \)’s are comparable to the ones proposed in many papers. Examples can be found in \( 1, 2 \); see further references therein. Commonly, a finite difference form of the gradient estimate is used as noisy approximate gradient. Below we outline two settings.

First consider the following situation which is very close to the classical Kiefer-Wolfowitz setting: \( F_k(\vartheta, Y_k) = F_k(\vartheta, Z_k) \) for some subvector \( Z_k \) of \( Y_k \), independent of \( X_{k-1} \) defined below and we wish to maximize the function \( \mathbb{E}F_k(\vartheta, Z_k) = \tilde{F}_k(\vartheta) \). For simplicity, let \( F_k(\vartheta, Z_k) = F(\vartheta, Z_k) \) and all \( Z_k \)’s are identically distributed (although the generalization to the time-varying case is straightforward) so that \( \mathbb{E}F(\vartheta, Z_k) = \bar{F}(\vartheta) \) is to be maximized:
\[
\max_{\vartheta \in \Theta} \bar{F}(\vartheta) = F(\vartheta). \quad \text{Let} \{e_k\}_{k \in \mathbb{N}_0} \text{ be a positive sequence,} \{e_i, i = 1, \ldots, d\} \text{ be the}
standard orthonormal basis vectors in \( \mathbb{R}^d \), \( Z_{k,i}^+ \) and \( Z_{k,i}^- \) have the same distribution as \( Z_k \), \( i = 1, \ldots, d \). Denote \( Z_k = (Z_{k,1}^+, \ldots, Z_{k,d}^+)^T \), \( F(\hat{\theta}_k + c_k e, Z_k) = (F(\hat{\theta}_k + c_k e_1, Z_{k,1}^+), \ldots, F(\hat{\theta}_k + c_k e_d, Z_{k,d}^+))^T \), likewise for \( F(\hat{\theta}_k - c_k e, Z_k^-) \) and \( F(\hat{\theta}_k \pm c_k e) \). The observations are the noisy finite difference estimates of the gradient:

\[
X_k^\pm = F(\hat{\theta}_k \pm c_k e, Z_k^\pm) + \xi_k^\pm, \quad k \in \mathbb{N}_0.
\]

Here \( \{\xi_k^\pm\}_{k \in \mathbb{N}_0} \) is a martingale difference noise sequence with respect to the filtration \( \{\mathcal{F}_k\}_{k \in \mathbb{N}_0} \), \( \hat{\theta}_k \) denotes the \( k \)th estimate of the maximum point \( \theta \) according to the algorithm \( \theta \) with the gain \( G_k(\hat{\theta}_k, X_k) = \frac{X_k^+ - X_k^-}{2c_k} \). Then, under some regularity conditions,

\[
g_k(\hat{\theta}_k, \theta_k | X_{k-1}) = \frac{F(\hat{\theta}_k + c_k e) - F(\hat{\theta}_k - c_k e)}{2c_k} = \nabla F(\hat{\theta}_k) + \eta_k = -M_k(\hat{\theta}_k - \theta) + \eta_k,
\]

where the magnitude of \( \eta_k \) is controlled by \( c_k \). Usually \( c_k \to 0 \) as \( k \to \infty \) in an appropriate way. To ensure that \( \nabla F(\hat{\theta}_k) = -M_k(\hat{\theta}_k - \theta) \) (possibly with a small approximation error) for some positive definite matrix \( M_k \) satisfying \( \theta \), concavity of \( \bar{F} \) is typically required, either global or over a compact set which is known to include the maximum location \( \theta \).

For example, if function \( \bar{F} \) is sufficiently smooth and strongly concave, then by Taylor’s expansion \( -M_k = H(F(\theta_k^*)) \), the Hessian matrix of \( \bar{F} \) at some point \( \theta_k^* \) between \( \hat{\theta}_k \) and \( \theta \), the relations \( \theta \) are fulfilled and the approximation error \( \eta_k \) is small if \( c_k \) is small.

Another approach (due to \( \theta \)) is based on random direction instead of the unit basis vectors. We use the same notations as in the previous setting with one simplification: assume now that there are no vectors \( Z_k \)’s involved in the model so that \( \bar{F}(\hat{\theta}) = F(\hat{\theta}) \). Let \( \{D_k, k \in \mathbb{N}\} \) denote a sequence of independent (\( D_k \) is also assumed to be independent of \( X_{k-1} \)) random unit vectors in \( \mathbb{R}^d \). At time moment \( k \in \mathbb{N}_0 \) we observe

\[
X_k^\pm = F(\hat{\theta}_k \pm c_k D_k) + \xi_k^\pm, \quad k \in \mathbb{N}_0,
\]

where the tracking sequence \( \theta_k \) is defined by the algorithm \( \theta \) with the gain function \( G_k(\hat{\theta}_k, X_k^+, X_k^-, D_k) = D_k \frac{X_k^+ - X_k^-}{2c_k} \).

**Remark 15.** Notice that one step in the previous (classical Kiefer-Wolfowitz) observation scheme requires in essence \( 2d \) observations in design points \( \theta_k \pm c_k e_i, \ i = 1, \ldots, d \), whereas only two measurements must be made in the case of the above random direction observation scheme. This property was the main motivation for the random direction method introduced by \( \theta \).

Then, under some regularity conditions,

\[
g_k(\hat{\theta}_k, \theta_k | X_{k-1}) = \mathbb{E} \left[ D_k \frac{F(\hat{\theta}_k + c_k D_k) - F(\hat{\theta}_k - c_k D_k)}{2c_k} \mid \mathcal{F}_{k-1} \right]
\]

\[
= \mathbb{E} \left[ D_k D_k^T \right] \nabla F(\hat{\theta}_k) + \eta_k = -M_k(\hat{\theta}_k - \theta_k) + \eta_k,
\]

where \( M_k = -\mathbb{E} \left[ D_k D_k^T \right] H(F)(\theta_k^*) \) and again the magnitude of \( \eta_k \) is controlled by \( c_k \). The relations \( \theta \) hold if, for example, we assume that the random directions were chosen in such a way that \( \mathbb{E} \left[ D_k D_k^T \right] \) are positive definite matrices and the Hessian \( H(F)(\theta_k^*) \) is negative definite.
Remark 16. A particular choice of function $F_k$ is $F_k(\theta, V_k) = l(\theta, V_k)$, $k \in \mathbb{N}_0$, where $V_k$'s is a sequence of observations with values on a measurable space $\mathcal{Y}_k$ and $l : \Theta \times \mathcal{Y}_k \rightarrow \mathbb{R}_+$ is a loss function. Then $EF_k(\theta, V_k)$ is the prediction risk of the predictor given by $\theta$. Classical examples are least squares and logistic regression (cf. [2]): $F_k(\theta, V_k) = \frac{1}{2}(x_k^T \theta - y_k)^2$ or $F_k(\theta, V_k) = \log[1 + \exp(-y_k x_k^T \theta)]$, where $V_k = (x_k, y_k)$, $x_k \in \Theta$ and $y_k \in \mathbb{R}$, or $y_k \in \{-1, 1\}$ for logistic regression.

4.4 Tracking conditional quantiles

Consider one more example. Suppose $\mathcal{X} \subset \mathbb{R}$ and we would like to track the conditional quantile of our observed time series $\{X_k, k \in \mathbb{N}_0\}$, i.e., $\theta_k = \theta_k(X_{k-1})$ such that $\theta_k = \inf \{x \in \mathcal{X} : F_k(x|X_{k-1}) \geq \alpha_k\}$, where the levels $\alpha_k \in (0, 1)$ are of our choice and $F_k(x|X_{k-1})$ is the conditional distribution function of $X_k$ given the past $X_{k-1}$. Assume that this conditional distribution posses a density $f_k(x|X_{k-1})$. In this case it makes sense to use $G_k(\hat{\theta}_k, X_k) = \alpha_k - \mathbb{I}\{X_k - \hat{\theta}_k \leq 0\}$ in the algorithm [3] for tracking $\theta_k$, since

$$g_k(\hat{\theta}_k, \theta_k|X_{k-1}) = -(F_k(\hat{\theta}_k|X_{k-1}) - \alpha_k) \approx -f_k(\theta_k|X_{k-1})(\hat{\theta}_k - \theta_k),$$

for some $\theta_k^*$ between $\hat{\theta}_k$ and $\theta_k$. Under some mild conditions Theorem 1 is applicable. Note also that the algorithm based on this gain function only requires knowledge of the values of the indicators $\mathbb{I}\{X_k - \hat{\theta}_k \leq 0\}$ which means that we may still track the required quantiles without explicitly observing $X_k$. This problem is treated in detail for the case of independent observations in [3].

4.5 Gain function based on score

For certain models it may not be obvious how gain functions can be constructed, especially when tracking multi-dimensional parameters. It is therefore important to have a general procedure that can be used to construct candidate gain functions that can either be used directly or, if needed, modified to verify (A1) and (A2).

In this subsection we assume that we are in the framework of Remark 5 i.e., we are dealing with a parameterized model $\mathcal{P}_k = \mathcal{P}_k(\Theta) = \{\mathbb{P}_\theta(\cdot|x_{k-1}) : \theta \in \Theta, x_{k-1} \in \mathcal{X}^{k-1}\}$. Assume further that for each $k \in \mathbb{N}$, each distribution from the family of conditional distributions $\mathcal{P}_k$ has a density with respect to some $\sigma$-finite dominating measure and denote this conditional density by $p_\theta(x|x_{k-1})$, $\theta = (\vartheta_1, \ldots, \vartheta_d) \in \Theta \subset \mathbb{R}^d$. Assume also that there is a common support $\mathcal{X}$ for these densities, and that for any $x \in \mathcal{X}$ and $\vartheta \in \Theta \subset \mathbb{R}^d$, the partial derivatives $\partial p_\theta(x|X_{k-1})/\partial \vartheta_i$, $i = 1, \ldots, d$, exist and are finite, almost surely. As before, the “true” value of the time-varying parameter at time moment $k \in \mathbb{N}_0$ is denoted by $\theta_k = \theta_k(X_{k-1})$. Under these assumptions, the conditional gradient vector $\nabla_\vartheta \log p_\theta(x|X_{k-1})$ and the random matrices $I_k(\vartheta|X_{k-1})$, $k \in \mathbb{N}_0$, with entries

$$I_{k,i,j}(\vartheta|X_{k-1}) = \mathbb{E}_\vartheta\left[\frac{\partial p_\theta(x|X_{k-1})}{\partial \vartheta_i} \cdot \frac{\partial p_\theta(x|X_{k-1})}{\partial \vartheta_j}\right], \quad i, j = 1, \ldots, d,$$

can be defined, almost surely. A possible gain function for the algorithm [3] is simply the
conditional score of the model, i.e., the gradient vector
\[ G_k(\vartheta, X_k) = \nabla_{\vartheta} \log p_{\vartheta}(X_k | X_{k-1}). \]  
(29)
If \( I_k(\vartheta|X_{k-1}) \) is almost surely non-singular in point \( \hat{\vartheta}_k \), then one might also consider
\[ G_k(\vartheta, X_k) = I^{-1}_k(\vartheta|X_{k-1}) \nabla_{\vartheta} \log p_{\vartheta}(X_k | X_{k-1}). \]  
(30)

We now outline some heuristic arguments why these choices are reasonable. Take \( \vartheta, \theta \in \mathbb{R}^d \). It is not uncommon for the Kullback-Leibler divergence \( K(P_{\vartheta}(\cdot|X_{k-1}), P_{\vartheta}(\cdot|X_{k-1})) \) to be a quadratic form in the distance between the parameters \( \vartheta \) and \( \theta \), i.e., equal to a multiple of \((\vartheta - \theta)^T M (\vartheta - \theta)\) for some (eventually random) positive semi-definite matrix \( M \). Actually, this is also in general true under some regularity conditions, at least locally, in a vicinity of the “true” \( \theta \). For example, suppose that we can interchange integration and differentiation and that \( M \) does not depend on \( \vartheta \), then
\[
g_k(\vartheta, \theta | X_{k-1}) = \int \nabla_{\vartheta} \log p_{\vartheta}(x | X_{k-1}) dP_{\vartheta}(x | X_{k-1}) = \nabla_{\vartheta} \int \log p_{\vartheta}(x | X_{k-1}) dP_{\vartheta}(x | X_{k-1}) \\
= \nabla_{\vartheta} \left( \int \log \frac{p_{\vartheta}(x | X_{k-1})}{p_{\theta}(x | X_{k-1})} dP_{\vartheta}(x | X_{k-1}) + \int \log p_{\theta}(x | X_{k-1}) dP_{\theta}(x | X_{k-1}) \right) \\
= \nabla_{\vartheta} \int \log \frac{p_{\vartheta}(x | X_{k-1})}{p_{\theta}(x | X_{k-1})} dP_{\vartheta}(x | X_{k-1}) = - \nabla_{\vartheta} K(P_{\vartheta}(\cdot | X_{k-1}), P_{\theta}(\cdot | X_{k-1})) \\
= - \nabla_{\vartheta} (\vartheta - \theta)^T M (\vartheta - \theta). \]  
(31)

The score in principle depends on the past of the time series \( X_{k-1} \) and the previous argument might only be valid for a certain subset of values \( X_{k-1} \) in \( \mathcal{X}^{k-1} \). This dependence could prevent (A1) from holding. In such cases, using a gain of the form (30) might be a good alternative since the matrix \( I^{-1}_k(\vartheta|X_{k-1}) \) acts as an appropriate scaling factor.

The dependence of the gain function on the past of the time series is in fact one of the main issues one has to deal with when checking (A1) and (A2). On one hand, to ensure that the gain function has, on average, the right direction, as required by (4), the gain will often need to depend on previous observations. This might, however, affect either the range or the variance of the gain. Gain functions, such as (29) and (30), can be modified, or rescaled, to ensure that the respective conditional expectation \( g_k(\vartheta, \theta | X_{k-1}) \) verifies the assumptions of Theorem \( \text{[\ref{Theorem}]} \). One can for example truncate certain entries or factors in both \( G_k(\vartheta, X_k) \) and \( I_k(\vartheta|X_{k-1}) \) to ensure that the resulting \( g_k(\vartheta, \theta | X_{k-1}) \) meets the required assumptions. Another possibility is to rescale, or directly truncate, the length of a given gain vector and consider, for example, one of the following gains
\[
\hat{G}_k(\vartheta, X_k) = \frac{G_k(\vartheta, X_k)}{1 + \| G_k(\vartheta, X_k) \|}, \\
\check{G}_k(\vartheta, X_k) = G_k(\vartheta, X_k) \left[ 1 + \frac{\kappa - \| G_k(\vartheta, X_k) \|}{\| G_k(\vartheta, X_k) \|} I\{ \| G_k(\vartheta, X_k) \| \geq \kappa \} \right], \\
\bar{G}_k(\vartheta, X_k) = G_k(\vartheta, X_k) \min \left\{ \frac{s_k(X_{k-1})}{s_k(X_{k-1})}, \kappa \right\},
\]
for $G_k$ an arbitrary gain function, $\kappa > 0$ and some functions $s_k: \mathcal{X}^{k-1} \to \mathbb{R}_+$. Note that $\tilde{G}_k, \hat{G}_k$ and $\check{G}_k$ all preserve the direction of $G_k$ and have norm bounded by respectively $1$, $\kappa$ and the norm of $G_k$, almost surely.

The gain $\hat{G}_k$ is a rescaling of $G_k$ for situations when the corresponding conditional gain $g_k$ is of the form $g_k(\vartheta, \theta | X_{k-1}) = -s(X_{k-1}) M_k(\vartheta - \theta)$, where $M_k$ has eigenvalues as prescribed by (A1). Consequently the conditional rescaled gain is

$$\check{g}_k = -\min\{s(X_{k-1}), \kappa\} M_k(\vartheta - \theta),$$

so that the largest eigenvalue of the matrix $\min\{s(X_{k-1}), \kappa\} M_k$ is almost surely upper bounded. As to the lower bound, in certain situations it will be possible to show that $E[\min\{s(X_{k-1}), \kappa\} \Lambda(1)(M_k) | X_{k-2}] \geq c\lambda_1$ almost surely, for some $0 < c \leq 1$ and sufficiently large $\kappa$, by using the fact that $E[\Lambda(1)(M_k) | X_{k-2}] \geq \lambda_1$ almost surely. This would establish condition (A1) for the rescaled gain $\check{G}_k$. Since $\min(x, \kappa)/x \leq 1$ for all $x \in \mathbb{R}_+$, then

$$E[\|G_k - \check{g}_k\|^2 | X_{k-1}] = E\left[\left(\frac{\min\{s(X_{k-1}), \kappa\}}{s(X_{k-1})}\right)^2 E[\|g_k\|^2 | X_{k-1}]\right] \leq E[\|G_k - g_k\|^2].$$

Thus, if $G_k$ verifies (A2), then so does $\check{G}_k$.

Another possible modification one might consider is to truncate the iterates of the our algorithm [3]. This might be motivated by practical considerations in the case where the parameter being tracked has some physical meaning and is bounded for that reason. The algorithm should be restricted as well. We can then consider an algorithm of the form

$$\hat{\theta}_{k+1} = \Pi_{\Theta}(\hat{\theta}_k + \gamma_k G_k(\hat{\theta}_k, X_k)), \quad k \in \mathbb{N}_0,$$

where $\Pi_{\Theta}(\cdot)$ acts as a projection on a convex compact set $\tilde{\Theta} \supset \Theta$: $\Pi_{\Theta}(\cdot)$ is an identity on $\Theta$ and maps any point from $\tilde{\Theta}$ to the closest point in $\Theta$.

We provide concrete examples of gain functions later in Section 6. In Section 5 we present some examples of different types of parameter variation such that our algorithm is capable of adequately tracking the time-varying parameter.

5 Variational setups for the drifting parameter

It is clear – and in fact explicit in (11) and (21) – that the changes in the parameter have a non-negligible contribution to the accuracy of our tracking algorithm. This is reasonable since, if the parameter changes arbitrarily in-between observations, we should not expect it to the “trackable”. We should then specify how the parameter is allowed to vary and, based on that assumption, pick an appropriate sequence $\gamma_k$ which minimizes the general bounds in (11) or (21). In this section, we specify different settings for the variation of the parameter to be tracked. These settings refer only to how the parameter is assumed to change and are unrelated to the actual model in question; examples of specific models can be found in Section 6.

To avoid overloaded notations, we use letters $C$ and $c$ for constants whose values are not important to us and which can be different in different expressions.
5.1 Static parameter

We assume in this section that \( \theta_i(X_{i-1}) = \theta_0, \ i \in \mathbb{N}_0 \), almost surely, for some unknown \( \theta_0 \in \Theta \) so that \( \Delta \theta_i = 0 \) (zero vector) for all \( i \in \mathbb{N}_0 \), almost surely, and we are actually in a parametric setup. In this case the second terms in both (11) and (21) obviously vanish.

Take then \( \gamma_i = C_\gamma i^{-1} \log i \) and for \( q \in (0, 1), \ n_0 = \lfloor qn \rfloor \), where \( \lfloor a \rfloor \) is the whole part of \( a \in \mathbb{R} \). Let \( n \geq 2/q = N_q \) such that \( n_0 \geq 2 \). For any \( c > 0 \) there is a large enough \( C_\gamma \) such that, for all \( n \geq N_q \),

\[
\sum_{i=n_0}^{n} \gamma_i \geq C_\gamma \log n_0 \sum_{i=n_0}^{n} \frac{1}{i} \geq C_\gamma \log n_0 \sum_{i=n_0}^{n} \log \left( 1 + \frac{1}{i} \right) = C_\gamma \log n_0 \log \left( \frac{n+1}{n_0} \right) \geq c \log n.
\]

Moreover, it is easy to see that, under the conditions of Theorem 2, \( \mathbb{E}\|\delta_{n_0}\|_p^p \leq C_0 n_0^p \). Thus, in both (11) and (21) the first term can be upper bounded by \( Cn^{-p} \) for any \( c > 0 \) by taking sufficiently large \( C_\gamma \). Next note that \( \left( \sum_{i=n_0}^{n} \gamma_i \right)^{p/2} \leq C(n^{-1/2} \log n)^p \). We conclude that, for a sufficiently large \( C_\gamma \), we can rewrite (11) and (21) as respectively,

\[
\max_{n \geq N_q} \mathbb{E}\sqrt{n \log n} \|\delta_n\| \leq C \quad \text{and} \quad \max_{n \geq N_q} \mathbb{E}\left[ \frac{\sqrt{n \log n}}{p} \|\delta_n\|_p^p \right] \leq C, \quad p \geq 1. \tag{34}
\]

If we let \( n \to \infty \), this is almost (up to a log factor) parametric convergence rate, the log-factor in the rate cannot be avoided and is in some sense a price for the recursiveness of the algorithm.

If we are in the situation of Theorem 2 then by taking \( p > \epsilon^{-1} \) (where \( \epsilon > 0 \) is some small fixed number) and by using Markov’s inequality and the second bound in the previous display, we derive that

\[
\sum_{n=1}^{\infty} P(n^{1/2-\epsilon} \|\hat{\theta}_n - \theta_0\|_1 > c) \leq \sum_{n=1}^{\infty} P\left( d^{\frac{p-1}{2}} n^{1/2-\epsilon} \|\hat{\theta}_n - \theta_0\|_p > c \right)
\]

\[
\leq \sum_{n=1}^{\infty} \frac{d^{p-1} n^{p/2-p} \mathbb{E}\|\delta_n\|_p^p}{c^p} \leq C \sum_{n=1}^{\infty} \frac{(\log n)^p}{n^{p\epsilon}} < \infty. \tag{35}
\]

In view of the Borel-Cantelli Lemma, it follows that \( \|\hat{\theta}_n - \theta_0\|_1 \to 0 \) as \( n \to 0 \) with probability 1 at a rate \( n^{1/2-\epsilon} \).

Remark 17. The particular setup presented in this section, where the parameter is fixed, might seem out of place since we are mainly concerned with tracking time-changing parameters. We would like to point out that recursive algorithms in parametric situation can also be useful; for example, the classical Robbins-Monro and Kiefer-Wolfowitz algorithms deal with the parametric case. Recursive procedures often produce estimates in a fast, straightforward fashion. This is an advantage especially over “offline” estimators obtained, say, as solutions to a certain system, which require iterative likelihood or least squares optimization or are obtained via other indirect methods, a situation which is common when dealing with Markov models (cf. Section 6.4).
5.2 Stabilizing parameter

Suppose now that the parameter we want to track is stabilizing. This situation might arise if the expectation of the sequence of values that the parameter takes is converging to some limiting value. It could also be the case that the data is being sampled with increasing frequency from an underlying, continuous time process which depends on a parameter varying continuously; in this case, the parameter varies less because it has less time to change. Regardless, we assume that $\Delta \theta_i = \theta_i(X_{i-1}) - \theta_{i+1}(X_i)$ verifies

$$E\|\Delta \theta_i\|_p^p \leq \rho_i^p, \quad i \in \mathbb{N}_0,$$

for $p \geq 1$ and some positive sequence $\rho_i$. Assume that $\rho_i = c_i \beta^{-\beta}$ for some $c_i > 0$ and $\beta \geq 0$.

Consider first the case $\beta \geq 3/2$. In this case, the variation of the parameter vanishes so quickly that we are essentially in the setup of the previous section, i.e., as if the parameter is constant. Indeed, take $\gamma_i$ and $n_0$ as in the previous section. The first and second terms in both (11) and (21) can be bounded in the same way as in the previous section. Using the relations between norms from Remark 6 we upper bound the third term in (11) by a multiple of

$$E \sum_{i=n_0}^n \|\Delta \theta_i\| \leq c(n-n_0)\rho_{n_0} \leq C(n-n_0)n^{-\beta} \leq Cn^{-1/2}. \quad (36)$$

Using the Hölder inequality, we upper bound the third term in (21) by a multiple of

$$E \left( \sum_{i=n_0}^n \|\Delta \theta_i\|_p \right)^p \leq (n-n_0)^{p-1} \sum_{i=n_0}^n E\|\Delta \theta_i\|_p^p \leq C(n-n_0)^p \rho_{n_0}^p \leq c[n(n-n_0)n_0^{-\beta}]^p \leq Cn^{-(\beta-1)p} \leq Cn^{-p/2}. \quad (37)$$

Clearly, in both (11) and (21) the third term is of a smaller order than the second term. Thus, the relations (33) remain valid for the case $\beta \geq 3/2$.

Consider now the case $0 < \beta < 3/2$. Let $\gamma_i = C_\gamma (\log i)^{1/3} i^{-2\beta/3}$, $n_0 = n - n^{2\beta/3}(\log n)^{2/3}$. By using the elementary inequality $(1+x)^\alpha \leq 1 + \alpha x$ for $0 < \alpha < 1$ and $x \geq -1$, we obtain that for any $c > 0$ there is a sufficiently large constant $C_\gamma > 0$ such that

$$\sum_{i=n_0}^n \gamma_i \geq C_\gamma (\log n_0)^{1/3} \sum_{i=n_0}^n \frac{1}{i^{2\beta/3}} \geq C_\gamma (\log n_0)^{1/3} \int_{n_0}^{n} dx \frac{1}{x^{2\beta/3}}$$

$$= C_\gamma (\log n_0)^{1/3} \left[ n^{1-2\beta/3} - n^{1-2\beta/3} (1 - n^{2\beta/3-1}(\log n)^{2/3})^{1-2\beta/3} \right]$$

$$\geq C_\gamma (\log n_0)^{1/3} \left[ n^{1-2\beta/3} - n^{1-2\beta/3} (1 - n^{2\beta/3-1}(\log n)^{2/3}(1-2\beta/3)) \right]$$

$$= C_\gamma (\log n_0)^{1/3} (\log n)^{2\beta/3} \geq c \log n$$

for sufficiently large $n$, i.e., $n \geq N_1 = N_1(\beta)$. This yields the same upper bound for the first term in (11) and (21) as for the static parameter, namely, $Cn^{-c}$ for any $c > 0$ by
taking sufficiently large $C_\gamma$. Let us bound now the second term in (11) and (21):

$$\left(\sum_{i=n_0}^{n} \gamma_i^2\right)^{1/2} \leq C((\log n)^{2/3} n_0^{-4/3} (n - n_0))^{1/2} \leq c(\log n)^{2/3} n^{-\beta/3}$$

for $n \geq N_2 = N_2(\beta)$. For sufficiently large $n$ (i.e., $n \geq N_3 = N_3(\beta)$) the third terms in (11) and (21) are bounded similarly to (36) and (37) by, respectively,

$$\mathbb{E}\left(\sum_{i=n_0}^{n} \|\Delta \theta_i\|\right)^p \leq c((n - n_0)n_0^{-\beta})^p \leq C((\log n)^{2/3}n^{-\beta/3})^p.$$

Finally we obtain that for $0 < \beta < 3/2$ and sufficiently large constant $C_\gamma$ in the algorithm step $\gamma_i = C_\gamma(\log i)^{1/3} i^{-2\beta/3}$, (11) and (21) can be rewritten as respectively

$$\max_{n \geq N_\beta} \mathbb{E}\left(\frac{n^{\beta/3}}{(\log n)^{2/3}} \|\delta_n\| \right)_2 \leq C \quad \text{and} \quad \max_{n \geq N_\beta} \mathbb{E}\left(\frac{n^{\beta/3}}{(\log n)^{2/3}} \|\delta_n\| \right)_p \leq c,$$

where $N_\beta = \max(N_1, N_2, N_3)$ is the burn-in period of the algorithm.

**Remark 18.** If we choose $\gamma_i = C_\gamma(\log i)^{\alpha_1 i^{-\alpha}}$ and $n_0 = n - n^\alpha(\log n)^{\alpha_2}$, $0 < \alpha < 1$, $\alpha_1, \alpha_2 \geq 0$, $\alpha_1 + \alpha_2 \geq 1$ in case $0 < \beta < 3/2$, then we get the following bound of the convergence rate: for sufficiently large $n$ and sufficiently large constant $C_\gamma$

$$\mathbb{E}\|\delta_n\|_p \leq C \left(n^{-\min\{\beta - \alpha, \alpha_2\}} (\log n)^{\max\{\alpha_2, \alpha_1 + \alpha_2/2\}} \right)^p.$$ 

Thus, the choices $\alpha = 2\beta/3$, $\alpha_1 = 1/3$, $\alpha_2 = 2/3$ are optimal in the sense of the minimum of the right-hand side of the above inequality.

**Remark 19.** Much in the same way as for (34), we can establish that for any $\epsilon > 0$, $\lim_{n \to \infty} n^{\beta/3 - \epsilon} \|\delta_n\|_1 = 0$ with probability 1.

Finally, consider the case $\beta = 0$, i.e., we assume the following weak requirement: $\mathbb{E}\|\Delta \theta_i\|_p \leq c$, $i \in N_0$, for some uniform constant $c$. Take $n - n_0 = N$, $\gamma_i = \gamma$ for some $N \in \mathbb{N}$, $\gamma > 0$. Then Theorem 1 implies that

$$\max_{n \geq N} \mathbb{E}\|\delta_n\|_p^2 \leq C_1 e^{-pN\gamma} + C_2 N^{p/2} + C_3 N^p c = D.$$

We thus have that the algorithm will track down the parameter in the proximity of size $D$, which we can try to minimize by choosing appropriate constants $N$ and $\gamma$. 

26
5.3 Lipschitz signal with asymptotics in the sampling frequency

We consider now a different setup where we assume that the parameter is changing, on average, like a Lipschitz function. In this setup we let the time series \( X_t, t \in [0, 1] \), which we observe with frequency \( n \). This means that we deal with a triangular sequence of models, i.e., for each \( n \in \mathbb{N} \) we have a different model, namely,

\[
X_0^n \sim P_{\theta_0^n}, \quad X_k^n | X_{k-1}^n \sim P_{\theta_k^n}(\cdot | X_{k-1}^n), \quad k \leq n \in \mathbb{N},
\]

where the parameter \( \theta_k^n = \theta_k^n(X_{k-1}^n) \) verifies, for some \( p \geq 1, \kappa_{d,p} < \infty, 0 < \beta \leq 1, \)

\[
\mathbb{E}\|\theta_k^n(X_{k-1}^n) - \theta_{k_0^n}(X_{k_0^n-1})\|_p^p \leq \kappa_{d,p}^p \left( \frac{k-k_0}{n} \right)^{\beta p}.
\]

Assume for example that \( \theta_k^n(X_{k-1}^n) = \vartheta(k/n) \), where \( \vartheta(\cdot) \in \mathcal{L}(L, \beta) = \{ g(\cdot) : \| g(t_1) - g(t_2) \|_1 \leq L |t_1 - t_2|^{\beta}, t_1, t_2 \in [0, 1] \} \) for some \( 0 < \beta \leq 1 \) and \( L > 0 \), a space of vector valued Lipschitz functions.

Let \( \gamma_k = C_\gamma (\log n)^{(2\beta - 1)/(2\beta + 1)} n^{-2\beta/(2\beta + 1)} \) (constant in \( k \)) for \( k = 1, \ldots, n, \) and

\[
k_0 = k_0(n) = k - (\log n)^{2/(2\beta + 1)} n^{2\beta/(2\beta + 1)},
\]

for \( k \geq K_n = (\log n)^{2/(2\beta + 1)} n^{2\beta/(2\beta + 1)} \). Note that for \( K_n/n \to 0 \) as \( n \to \infty \) for any \( 0 < \beta \leq 1 \). We have

\[
\sum_{i=k_0}^{k} \gamma_i = C_\gamma (\log n)^{(2\beta - 1)/(2\beta + 1)} n^{2\beta/(2\beta + 1)} (k - k_0) \geq C_\gamma \log n,
\]

so that once again the first term in (11) and (21) can be upper bounded by \( Cn^{-c} \) for any \( c > 0 \) by taking sufficiently large \( C_\gamma \). As to the second term, we evaluate

\[
\left( \sum_{i=k_0}^{k} \gamma_i^2 \right)^{1/2} \leq C(n \log n)^{2\beta - 1/(2\beta + 1)} n^{-2\beta/(2\beta + 1)} \frac{2\beta}{2\beta + 1} (k - k_0)^{1/2} = C(n \log n)^{2\beta/(2\beta + 1)} n^{-\beta/(2\beta + 1)}.
\]

From our assumption on the variation of the parameter, we have

\[
\max_{k_0 \leq i \leq k} \mathbb{E}\|\theta_{i+1} - \theta_{i}^{n}\|_p^p \leq C \left( \frac{k-k_0}{n} \right)^{\beta p} \leq C \left( \log n \right)^{2\beta/(2\beta + 1)} n^{-\beta/(2\beta + 1)}.
\]

Combining these three bounds, we get that (11) (we also need the relations between norms from Remark 6) and (21) imply

\[
\sup_{\vartheta \in \mathcal{L}(L, \beta)^{1 \geq K_n}} \max_{1 \geq K_n} \mathbb{E}\|\delta_\vartheta\| \leq C(n \log n)^{2\beta/(2\beta + 1)} n^{-\beta/(2\beta + 1)},
\]

\[
\sup_{\vartheta \in \mathcal{L}(L, \beta)^{1 \geq K_n}} \max_{p \geq 1} \mathbb{E}\|\delta_\vartheta\|_p^p \leq C \left( \log n \right)^{2\beta/(2\beta + 1)} n^{-\beta/(2\beta + 1)}.
\]
Remark 20. If we consider step sizes of the form \( \gamma_k = C_\gamma (\log n)^{\alpha_1} n^{-\alpha_2} \), the above proposed choices of \( \alpha_1 \) and \( \alpha_2 \) are optimal in the sense of tracking error minimum.

Remark 21. Note that the obtained convergence rate (the asymptotic regime: the observation frequency \( n \to \infty \)) coincides, up to a log factor, with the minimax rate of convergence in the problem of estimating nonparametric regression function over Lipschitz functional class \( \mathcal{L}(L, \beta) \).

6 Some applications of the main result

In this section we present some examples of particular models to which our algorithm may be applied. We start with two toy examples and present thereafter some more involved examples. The toy examples illustrate the type of results that can be obtained from our main result and its extensions, how a gain function can be picked and modified, and how conditions (A1) and (A2) are checked.

6.1 Tracking the intensity function of a Poisson process

Suppose we are monitoring \( n \in \mathbb{N} \) independent Poisson processes on \([0, 1]\) with unknown intensity function \( \lambda(\cdot) \). This is equivalent to observing \( N(t) = N(t, n) \), a Poisson process with intensity \( n \lambda(t) \), \( 0 \leq t \leq 1 \). We would like to track the intensity function \( \lambda(\cdot) \) which is assumed to be upper bounded by \( L \).

Assume that we observe the process with frequency \( n \), in that our observations are \( X^n_k = N(k/n) \), so that for each \( n \in \mathbb{N} \) we have a Markov model

\[
X^n_0 = 0, \quad X^n_{k+1} | X^n_k \sim P_{\theta^n_k}(\cdot | X^n_k) = P_{\theta^n_k}(\cdot - X^n_k), \quad k = 1, \ldots, n,
\]

where \( P_{\theta}(\cdot) \) represents a Poisson law with parameter \( \theta \in \mathbb{R}^+ \). From now on, we will skip the dependence on \( n \) for notational simplicity: write \( X_k \) instead of \( X^n_k \), \( \theta_k \) instead of \( \theta^n_k \) etc. Introduce the conditional, shifted Poisson mass function given by

\[
p_{\theta}(x|y) = \frac{e^{-\theta} \theta^{x-y}}{(x-y)!}, \quad x, y \in \mathbb{N}, \ x \geq y.
\]

The moving parameter is given by \( \theta_k = \theta^n_k = \int_{(k-1)/n}^{k/n} n \lambda(t) dt \), \( k = 1, \ldots, n \), which is the average of function \( \lambda(t) \) over the interval \( [(k-1)/n, k/n] \). Assume that \( \lambda(t) \) is continuous, then \( \theta^n_k \approx \lambda(k/n) \) for \( n \) large enough.

Consider now the gain function \( G_k \) of the type (30) for the algorithm (3) so that

\[
G_k(\hat{\theta}_k, X_k) = X_k - X_{k-1} - \hat{\theta}_k, \quad g_k(\hat{\theta}_k, \theta_k | X_{k-1}) = \mathbb{E}[X_k - X_{k-1} - \hat{\theta}_k | X_{k-1}] = -(\hat{\theta}_k - \theta_k), \quad (39)
\]

It follows that

\[
\mathbb{E}[G_k(\hat{\theta}_k, X_k) - g_k(\hat{\theta}_k, \theta_k | X_{k-1})]^2 \leq 2\mathbb{E}|X_k - X_{k-1}|^2 + 2|\theta_k|^2 \leq C,
\]
since \( \sup_{t \in [0,1]} \lambda(t) \leq L \). We thus conclude that the gain function (39) satisfies both (A1) and (A2).

This gain function can now be used for the three setups outlined in Section 5 and we can attain the rates indicated there. For a constant intensity function \( \lambda(\cdot) = \theta, \ 0 < \theta \leq L \), the algorithm will simply estimate the parameter of the underlying homogeneous Poisson process \( \theta \), because we matched the sampling frequency \( 1/n \) with the sample size \( n \). If we had sampled the process with frequency, say, \( 2/n \), then \( \theta_k = 2\theta \) and the algorithm would track \( 2\theta \) and not \( \theta \). The tracking sequence would then have to be rescaled by a factor \( 1/2 \) to obtain a tracking sequence for \( \theta \) itself.

In the setup where we assume that the parameter is stabilizing, take \( n = 1 \) so that \( \theta_k = \int_{k-1}^{k} \lambda(t) \, dt \) is the mean number of events per time unit \( [(k-1)/n, k/n] \). Note that

\[
|\Delta \theta_k| = \left| \int_{k-1}^{k} \lambda(t) \, dt - \int_{k}^{k+1} \lambda(t) \, dt \right| = |\theta_k - \theta_{k+1}|,
\]

and the average number of events per time unit will stabilize in time if, for example, \( \lambda(t) \to \lambda \) as \( t \to \infty \). The algorithm will then track the mean number of events per time unit.

We can also assume that the intensity function \( \lambda(\cdot) \) belongs to \( \mathcal{L}(L, \beta) = \{ g(\cdot) : |g(t_1) - g(t_2)| \leq L |t_1 - t_2|^\beta, t_1, t_2 \geq 0 \} \) for some \( 0 < \beta \leq 1 \) and \( L > 0 \). Let \( \vartheta_k = \vartheta_k^n = \lambda(k/n) \), \( k, n \in \mathbb{N} \). It follows that

\[
|\Delta \vartheta_k| = |\lambda(k/n) - \lambda((k + 1)/n)| \leq L n^{-\beta},
\]

\[
|\vartheta_k - \vartheta_k^n| = \left| \int_{(k-1)/n}^{k/n} n \lambda(t) \, dt - \lambda(k/n) \right| \leq n \int_{(k-1)/n}^{k/n} |\lambda(t) - \lambda(k/n)| \, dt \leq L n^{-\beta}.
\]

The tracking sequence based on the gain (39) will then track the sequence \( \vartheta_k = \lambda(k/n) \), \( k, n \in \mathbb{N} \) (as well as \( \theta_k \)) with the asymptotics seen in Section 5 (cf. Remark 12).

### 6.2 Tracking the mean function of a conditionally Gaussian process

Assume that we observe, with fixed frequency \( n \in \mathbb{N} \), a process \( X(t), \ t \geq 0 \), taking values on \( \mathcal{X} \subset \mathbb{R}^d, \ d \in \mathbb{N} \). The observations available up to time moment \( k/n \) is a random vector \( X_k = X^n_k = (X_0, X_1, \ldots, X_k) \), with \( X_k = X^n_k = X(k/n) \). We again skip the dependence on \( n \), although all the quantities below do depend on \( n \). The increments \( X_k - X_{k-1} \) are assumed to be conditionally Gaussian in the sense that given the past of the process, each increment has a multivariate normal distribution:

\[
X_0 \sim \mathcal{N}(\theta_0, \Sigma_0), \quad X_{k+1}|X_k \sim \mathcal{N}(\theta_k(X_{k-1}), \Sigma_k(X_{k-1})), \quad k = 1, \ldots, n.
\]

The dependence on the past in the model comes from the fact that both the mean and the covariance processes of the above conditional distributions are predictable, i.e., \( \theta_k = \theta_k(X_{k-1}) \) and \( \Sigma_k = \Sigma_k(X_{k-1}) \), \( k \in \mathbb{N}_0 \), with respect to the filtration \( \{ \mathcal{F}_k \}_{k \in \mathbb{N}_0} \).

If the covariance structure of the process is known, we can use the gain (29) which verifies

\[
G_k(\vartheta, X_k) = \Sigma_k^{-1}(X_k - \vartheta) \quad \text{and} \quad g_k(\vartheta, \theta_k|X_{k-1}) = -\Sigma_k^{-1}(\vartheta - \theta_k).
\]
For this gain, we assume that almost surely
\[ 0 < \lambda_1 \leq \Lambda(1)(\Sigma_k) \leq \Lambda(d)(\Sigma_k) \leq \lambda_2 < \infty, \quad k \in \mathbb{N}_0, \]
for some positive \( \lambda_1 < \lambda_2 \). We then obtain that
\[ \mathbb{E} \| G_k(\hat{\theta}_k, X_k) - g_k(\hat{\theta}_k, \theta_k|X_{k-1}) \|^2 = \mathbb{E} \| \Sigma_k^{-1}(X_k - \theta_k) \|^2 \leq \left( \lambda_2/\lambda_1 \right)^2 = C, \]
and assumptions (A1) and (A2) are thus met for the gain from (40).

Now suppose that the covariance matrix of the process is unknown or difficult to invert. Then we can use the gain (30), so that
\[ G_k(\vartheta, X_k) = X_k - \vartheta, \quad g_k(\vartheta, \theta_k|X_{k-1}) = -(\vartheta - \theta_k). \] (41)
Clearly, assumptions (A1) and (A2) are again met for the gain from (41) if \( \Lambda(d)(\Sigma_k) \leq C \) for some \( C > 0, k \in \mathbb{N}_0 \), almost surely.

The results of Section 5 can be applied to the algorithm based on the gain functions presented above for all three considered asymptotic regimes: constant parameter process, stabilizing (on average) process and Lipschitz on average.

**Remark 22.** Although designed for different frameworks, it is interesting to compare the above resulting tracking algorithm with the famous *Kalman filter*. For simplicity, consider the one dimensional situation. Suppose we observe
\[ X_k = \theta_k + \xi_k, \quad \xi_k \sim N(0, \sigma^2_\xi), \quad k \in \mathbb{N}, \] (42)
where the parameter of interest \( \theta_k \), evolves according to
\[ \theta_k = \theta_{k-1} + \delta_k \varepsilon_k \quad \varepsilon_k \sim N(0, 1), \quad k \in \mathbb{N}, \]
with \( \theta_0 \sim N(m_0, \sigma^2_0) \). At each step, the initial state and the noises \( \theta_0, \xi_1, \ldots, \xi_k, \varepsilon_1, \ldots, \varepsilon_k \) are assumed to be mutually independent. One can show (by combining both prediction and update steps) that the Kalman filter in this case reduces to
\[ \dot{\theta}_k = \dot{\theta}_{k-1} + \gamma_k (X_k - \dot{\theta}_{k-1}), \quad \dot{\theta}_0 = m_0, \quad k \in \mathbb{N}, \] (43)
\[ \gamma_k = \frac{\gamma_{k-1} + \delta^2_k/\sigma^2_\xi}{\gamma_{k-1} + \delta^2_k/\sigma^2_\xi + 1}, \quad \gamma_0 = \frac{\sigma^2_0}{\sigma^2_\xi}, \quad k \in \mathbb{N}. \] (44)
We also derive the exact expression for the mean squared error of the algorithm:
\[ \mathbb{E}(\dot{\theta}_k - \theta_k)^2 = \sigma^2_\xi \gamma_k, \quad k \in \mathbb{N}. \]

Coming back to our framework, suppose we have observations (42) with predictable process \( \{\theta_k\}_{k \in \mathbb{N}_0} \) such that \( \mathbb{E}(\dot{\theta}_k - \theta_{k-1})^2 \leq \delta^2_k, k \in \mathbb{N} \); cf. Section 5.2. Then the Kalman filter (43) coincides with our tracking algorithm with the gain (40) and a particular choice of the step sequence \( \dot{\gamma}_k \) given by (44). One should keep in mind that the two frameworks are different, but it would still be interesting to compare the convergence rates for some.
particular settings for stabilizing the parameter $\theta_k$. For example, one can consider $\delta_k = c k^{-\beta}$, $0 < \beta < 3/2$, as in Section 5.2. The above Kalman filter setting has more structure and we expect therefore that the rate in this case (which is of order $\sqrt{n}$, with $\gamma_n$ defined by (44)) should be faster than the rate $\left( \log n \right)^{2/3} / n^{\beta/3}$ obtained in Section 5.2 for our general framework. We were however unable to solve the recursive rational difference equation (44) for $\delta_k = c k^{-\beta}$. Note that the trivial case $\delta_k = 0$ leads to the situation of a constant parameter $\theta_k = \theta$ and the sample mean $\bar{\theta}_k = \bar{X}_k$ as an estimator for that parameter.

6.3 Tracking an ARCH(1) parameter

Consider the following ARCH(1) model with drifting parameter $X_k = (1 + \theta_k X_{k-1}^2)^{1/2} \epsilon_k$, $k \in \mathbb{N}$, \hspace{1cm} (45)

where $|X_0| \leq 1$ almost surely, $\{\theta_k\}_{k \in \mathbb{N}}$ is predictable and $\{\epsilon_k\}_{k \in \mathbb{N}}$ is a martingale difference noise with respect to the filtration $\{\mathcal{F}_k\}_{k \in \mathbb{N}}$, $\mathbb{E}[\epsilon_k^2 | \mathcal{F}_{k-1}] = \sigma_k^2$ for some known $\sigma_k^2$, $k \in \mathbb{N}$. Without loss of generality assume $\sigma_k^2 = 1$. Assume further that $0 \leq \theta_k \leq C^{1/2}_0$ and $\mathbb{E}[\epsilon_k^2 | \mathcal{F}_{k-1}] \leq \rho$, $k \in \mathbb{N}$, for some $\rho > 0$.

Consider the gain function

$$G_k(\vartheta, X_k) = \min \left( \frac{X_{k-1}^2}{X_{k-1}^2}, T \right) (X_k^2 - 1 - \vartheta X_{k-1}^2),$$ \hspace{1cm} (46)

for some truncating constant $T > 0$. Since $\mathbb{E}X_k = 0$ and $\mathbb{E}[X_k^2 | X_{k-1}] = 1 + \theta_k X_{k-1}^2$,

$$g_k(\vartheta, \theta_k | X_{k-1}) = \mathbb{E}\left[ \min \left( \frac{X_{k-1}^2}{X_{k-1}^2}, T \right) (X_k^2 - 1 - \vartheta X_{k-1}^2) | \mathcal{F}_{k-1} \right] = -\min(X_{k-1}^2, T)(\vartheta - \theta_k).$$

We have that $\min(X_{k-1}^2, T) \leq T$ almost surely. Besides,

$$\mathbb{E}\left[ \min\{X_{k-1}^2, T\} | X_{k-2} \right] \geq \mathbb{E}\left[ \min\{1 + \theta_{k-1} X_{k-2}^2\} | X_{k-2} \right]$$

Using the Hölder inequality and the facts that $\min(a, b) = (a + b)/2 - |a - b|/2$ and $|a + b|^{1/2} \leq |a|^{1/2} + |b|^{1/2}$, it is straightforward to check that

$$2\mathbb{E}\left[ \min(\epsilon_k^2, T) | X_{k-2} \right] \geq \mathbb{E}\left[ T + \epsilon_k^2 - \epsilon_k^2 - T | X_{k-2} \right]$$

as long as $T^2 \geq \mathbb{E}[\epsilon_k^4 | X_{k-2}] / 4$, $k \in \mathbb{N}$. For example, we can take $T \geq \sqrt{p}/2$. We conclude that (A1) holds for the gain (46).
To ensure (A2), we evaluate
\[ E(G_k - g_k)^2 = E \left( \left( \frac{\min(X_{k-1}^2, T)}{X_{k-1}^2} \right)^2 (X_k^2 - 1 - \theta_k X_{k-1}^2)^2 \right) \]
\[ \leq 3 E \left( \left( \frac{\min(X_{k-1}^2, T)}{X_{k-1}^2} \right)^2 ((2 + 2\theta_k^2 X_k^2)\epsilon_k^2 + 1 + \theta_k^2 X_{k-1}^2) \right) \]
\[ \leq 9 + 3 E \left( \left( \frac{\min(X_{k-1}^2, T)}{X_{k-1}^2} \right)^2 (2C\Theta\rho + C\Theta) X_{k-1}^4 \right) \]
\[ \leq 9 + 3 C\Theta T^2(2\rho + 1). \]

6.4 Tracking an AR(d) parameter

In this section we use the notation \( X_{k,d} = (X_k, X_{k-1}, \ldots, X_{k-(d-1)}) \) for the vector of the \( d \) consecutive observations ending with \( X_k \).

Consider an autoregressive model with \( d \) time varying parameters:
\[ X_k = \sum_{i=1}^{d} \theta_k,i X_{k-i} + \xi_k = \theta_k^T X_{k-1,d} + \xi_k, \quad k \in \mathbb{N}, \]  
\[ = (47) \]
where \( \theta_k = (\theta_{k,1}, \ldots, \theta_{k,d}) \) is \( F_{k-d-1} \)-measurable, \( \{\xi_k\}_{k \in \mathbb{N}} \) is a martingale difference noise with respect to the filtration \( \{F_k\}_{k \in \mathbb{N}_0} \) such that \( E|\xi_k|^2 \leq C, k \in \mathbb{N} \), starting random vector \( X_{0,d} \) is given and such that \( E\|X_{0,d}\|^2 \leq c \), for some \( C, c > 0 \).

For \( \vartheta = (\vartheta_1, \ldots, \vartheta_d) \), associate with the AR(d) model its polynomial
\[ t(z, \vartheta) = 1 - \sum_{i=1}^{d} \vartheta_i z^i, \quad z \in \mathbb{C}. \]  
\[ = (48) \]
It is well known that an AR(d) model with autoregressive parameters \( \vartheta \) is stationary if, and only if, the (complex) zeros of the polynomial \( t(z, \vartheta) \) are outside the unit circle. This motivates the definition of the parameter sets \( \Theta(\rho) \) for some \( 0 < \rho < 1 \):
\[ \Theta(\rho) = \{ \vartheta \in \mathbb{R}^d : \text{for all } |z| < \rho^{-1}, t(z, \vartheta) \neq 0 \}; \]  
\[ = (49) \]
\[ \text{cf.} \] [18] who also showed that the following embeddings hold:
\[ B_\infty((\rho^{-2} + \cdots + \rho^{-2d})^{-1/2}) \subseteq \Theta(\rho) \subseteq B_\infty((1 + \rho)^d - 1), \]
where \( B_\infty(r) = \{ \vartheta \in \mathbb{R}^d : \max_{1 \leq i \leq d} |\vartheta_i| \leq r \} \) is a uniform ball around zero in \( \mathbb{R}^d \) with radius \( r > 0 \). This gives some feeling about the size of the parameter set \( \Theta(\rho) \) and implies in particular that the set \( \Theta(\rho) \) is non-empty and bounded for all \( \rho \in (0, 1) \).

The AR(d) model (47) can also be described by the following inhomogeneous difference equation
\[ X_{k,d} = C(\theta_k) X_{k-1,d} + I e_1 \xi_k, \]  
\[ = (50) \]
where $\mathbf{e}_1 = (1, 0, \ldots, 0) \in \mathbb{R}^d$ and, for any $\vartheta \in \mathbb{R}^d$, $C(\vartheta)$ is the square matrix of order $d$

$$C(\vartheta) = \begin{bmatrix} \vartheta_1 & \vartheta_2 & \cdots & \vartheta_{d-1} & \vartheta_d \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$ \hspace{1cm} (51)

This matrix is usually called the \textit{companion matrix} to the autoregressive polynomial $t(z, \vartheta)$; it is also sometimes called the \textit{state transition matrix}. One can show that the eigenvalues of $C(\vartheta)$ are exactly the reciprocals of the zeros of $t(z, \vartheta)$. This means that the absolute values of the eigenvalues of $C(\vartheta)$ for $\vartheta \in \Theta(\rho)$ are all at most $\rho < 1$. This in turn implies that for any sequence of vectors $\theta_d, \theta_{d+1}, \ldots \in \Theta(\rho)$, the pair of sequences $(C(\theta_d), C(\theta_{d+1}), \ldots, (I, I, \ldots))$ forms a so called \textit{exponentially stable} pair (cf. [1]). Among other things, this gives us that so long as the $p$-th moments of both the initial $X_{0,d}$ and the noise terms $\xi_k$ are bounded, then the $p$-th moments of all $X_k$, $k \in \mathbb{N}$, will be bounded as well (cf. Proposition 10 of [18]).

In [18] the model (17) is considered with nonrandom but time varying $\theta_k = \theta(k/n) = (\theta_1(k/n), \ldots, \theta_d(k/n))$ for some smooth function $\theta(t) \in \mathbb{R}^d$, $t \in [0, 1]$. One has a triangular array of models and the studied asymptotics is in sampling frequency $n \to \infty$; cf. Section 5.3. The considered gain function is an appropriately rescaled version of the gain from Remark 13, namely,

$$G_k(\vartheta, X_k) = \left( X_{k} - \vartheta^T X_{k-1,d} \right) \frac{X_{k-1,d}}{1 + \mu \|X_{k-1,d}\|^2}.$$ \hspace{1cm} (52)

for an appropriately chosen $\mu > 0$ depending on the observation frequency $n$.

\textbf{Remark 23.} One should mind the difference in indexing in our algorithm (3) and algorithm (3) from [18]. This is not an issue since we can make the correspondence between the algorithms exact by treating $\hat{\theta}_{k+1}$ as an estimate of $\theta_k$ rather than of $\theta_{k+1}$, the error can be absorbed into the third term of the right hand side of (11).

Although assumption (A2) is trivially satisfied, our general Theorem 1 cannot be applied for $d \geq 2$ because assumption (A1) does not hold. Indeed,

$$g_k(\hat{\theta}_k, \theta_k | X_{k-1}) = \frac{-X_{k-1,d}X_{k-1,d}^T}{1 + \mu \|X_{k-1,d}\|^2} (\hat{\theta}_k - \theta_k) = -M_k (\hat{\theta}_k - \theta_k),$$

and the matrix $M_k$ is of the form $\alpha xy^T$ for some $\alpha > 0$ and column vectors $x, y \in \mathbb{R}^d$. But the matrix $xy^T$ has $d - 1$ zero eigenvalues and one eigenvalue $y^T x$, so that always $\Lambda_{(1)}(M_k) = 0$ and thus (2) does not hold.

33
Remark 24. On the other hand, the authors of [18] do manage to establish a convergence results for the gain function (52). It is instructive to understand where the difference in the two approaches is. Careful inspection of the proofs in [18] reveals that the analogue of the lower bound (6), the persistence of excitation condition, is established in Lemma 17 (p. 2627 of [18]). The basic difference is that the quantity to bound from below in our case is the conditional expectation of the smallest eigenvalue of the matrix $M_k$, whereas in [18] it is the smallest eigenvalue of the conditional expectation of the matrix $M_k$. For the gain function (52), the lower bound for the former is zero (as is demonstrated above) and it is positive for the latter (cf. Lemma 17 of [18]). A way to fix this would be to establish a version of the general theorem, where (20) is assumed instead of (6), see also Remark 10. We do not consider this here.

Consider the case $d = 1$ and gain (52) for which Theorem 1 can be applied. Assume that $EX_0^2$ is bounded, $E(\xi_k|F_{k-1}) = 0$ and $E(\xi_k^2|F_{k-1}) = \sigma^2 > 0$, $k \in \mathbb{N}$. We have

$$G_k(\hat{\theta}_k, X_k) = \frac{X_k - \hat{\theta}_k X_{k-1}}{1 + \mu X_{k-1}^2}, \quad g_k(\hat{\theta}_k, \theta_k|X_{k-1}) = -M_k(\hat{\theta}_k - \theta_k),$$

where $M_k = \frac{X_k^2}{1 + \mu X_{k-1}^2} \leq \frac{1}{\mu}$. Besides, if $|X_{k-1}|^2 \geq c$, then $M_k \geq \frac{1}{c^2 + \mu c}$, and if $|X_{k-1}|^2 < c$, then

$$E(M_k|F_{k-2}) \geq \frac{E(|X_{k-1}|^2|F_{k-2})}{1 + \mu c} = \frac{X_{k-2}^2 \hat{\theta}_{k-2}^2 + \sigma^2}{1 + \mu c} \geq \frac{\sigma^2}{1 + \mu c}.$$  

Condition (A1) is fulfilled. As to condition (A2),

$$E[G_k^2 - g_k^2] = E \left[ \frac{X_{k-1}^2 \xi_k^2}{(1 + \mu X_{k-1}^2)^2} \right] \leq \sigma^2 \max_{u > 0} \left\{ \frac{u}{(1 + \mu u)^2} \right\} = \frac{\sigma^2}{4\mu}.$$  

Both (A1) and (A2) are thus satisfied. Interestingly, there is no issue of stability in this case: we do not have to assume that $|\theta_k| \leq \rho < 1$, $k \in \mathbb{N}$, almost surely. Just almost sure boundedness $|\theta_k|^2 \leq C\theta$, $k \in \mathbb{N}$, for a constant $C\theta$, is sufficient.

Consider now another gain function for the case $d = 1$. This time we assume that $E[\xi_k|X_{k-1}] = E[\xi_k^3|X_{k-1}] = 0$, $E[\xi_k^2|X_{k-1}] = \sigma^2 > 0$, and $E[\xi_k^4|X_{k-1}] = c\sigma^4$, $k \in \mathbb{N}$, for some constant $0 < c < 5$. The proposed gain and the corresponding average gain are as follows:

$$G_k(\theta, X_k) = \frac{\min\{X_{k-1}^2, T\}}{X_{k-1}}(X_k X_{k-1} - \theta X_{k-1}^2),$$

$$g_k(\hat{\theta}_k, \theta_k|X_{k-1}) = -\min\{X_{k-1}^2, T\}(\hat{\theta}_k - \theta_k) = -M_k(\hat{\theta}_k - \theta_k), \quad (53)$$

with some $T \geq (9 - c)\sigma^2/4$. Note that this is a rescaled gain function of type $G$ from Section 3. Clearly, $M_k \leq T$ and, according to Lemma 5

$$E[M_k|X_{k-2}] = E[\min\{X_{k-1}^2, T\}|X_{k-2}] \geq \frac{(5 - c)\sigma^2}{4}, \quad (54)$$

34
so that (A1) holds. Assumption (A2) also holds since

$$
E|G_k(\hat{\theta}_k, X_k) - g_k(\hat{\theta}_k, \theta_k|X_{k-1})|^2 = E\left[\min\{X_{k-1}^2, T\}^2 \xi_k^2 \over X_{k-1}^2\right] \leq \max\{T^2, 1\} \sigma^2.
$$

Finally consider a version of general AR(d) model. We will only outline the main steps, leaving out the details. Assume that the noise terms $\xi_k$ in (47) form a Gaussian white noise sequence with mean zero and variance $\sigma^2 > 0$ and that the parameter process $\{\theta_t\}_{t \in \mathbb{N}}$ is constant within the batch of $d$ consecutive observations. For a $(2d - 1)$-dimensional vector $m = (m_{-(d-1)}, \ldots, m_{-1}, m_0, m_1, \ldots, m_{d-1})$, introduce the Toeplitz matrix $T(m) = (m_{ij})$ associated with that vector whose entries are $m_{ij} = m_{i-j}$, $i, j = 1, \ldots, d$, so that this matrix has constant (from left to right) diagonals. Thus, $m$ is the column vector formed by starting at the top right element of $T(m)$, going backwards along the top row of $T(m)$ and then down the left column of $T(m)$. Denote $\vartheta = (\vartheta_1, \ldots, \vartheta_d)$ and introduce $A(\vartheta) = T(a(\vartheta))$ and $B(\vartheta) = T(b(\vartheta))$, the Toeplitz matrices created from the vectors $a(\vartheta) = (-\vartheta_{d-1}, \ldots, -\vartheta_1, 1, 0, \ldots, 0)$ and $b(\vartheta) = (0, \ldots, 0, \vartheta_d, \vartheta_{d-1}, \ldots, \vartheta_1)$ respectively. Under the imposed assumptions, we can rewrite the model (47) as follows:

$$
A(\theta_k)X_{k,d} = B(\theta_k)X_{k-d,d} + \xi_{k,d}.
$$

The matrix $A(\theta_k)$ is upper triangular with a diagonal consisting of ones, whence invertible. From this point on, we regard vector $X_{dk,d}$, $k \in \mathbb{N}$, as an observation at time moment $k$ so that we can specify our observation model in terms of conditional distribution of $X_{k,d}$ given $X_{k-d}$:

$$
X_{k,d}|X_{k-d} \sim N(A^{-1}(\theta_k)B(\theta_k)X_{k-d,d}, \sigma^2 A^{-1}(\theta_k)(A^{-1}(\theta_k))^T),
$$

where $\{\theta_k\}_{k \in \mathbb{N}}$ is a predictable process with respect to the filtration $\{F_{kd}\}_{k \in \mathbb{N}}$. Notice that the observation process is of a Markov structure.

**Remark 25.** Even if the normality of the noise is assumed in the model (47), the models (47) and (56) still differ since in general the parameter process $\{\theta_k\}_{k \in \mathbb{N}}$ varies also within the batches of $d$ observations in the model (47). However, this is not an issue. Indeed, even though the parameter is allowed to vary within each batch of $d$ observation, we still can use the gain function (which we derive below) as if the parameter process is constant within the batches and establish an upper bound of type (1) for the quality of such a procedure. The error that is made by pretending that the parameter is constant within the batches can be absorbed into the third term of the right hand side of (1).

In this case we propose a gain of the type (29):

$$
G_{dk}(\theta, X_{dk}) = \nabla_\theta \log p_\theta(X_{k,d}|X_{d(k-1)}),
$$

where $p_\theta(\cdot|X_{d(k-1)}) = p_\theta(\cdot|X_{d(k-1),d})$ is the conditional density of (56). Thus, the tracking sequence is updated with batches of $d$ observations from the autoregressive process. Below, to ease the notation, we will often write $X$ and $Y$ instead of $X_{dk,d}$ and $X_{d(k-1),d}$, respectively. As explained in Section 4, the corresponding average gain $g_{dk}$ can be found as minus
the gradient of the Kullback-Leibler divergence between two conditional distributions with two different parameters. This observation is particularly useful if we are able to write this Kullback-Leibler divergence as an appropriate quadratic form. The Kullback-Leibler divergence between two $d$-dimensional multivariate normal distributions $\mathbb{P}_0 = N(\mu_0, \Sigma_0)$ and $\mathbb{P}_1 = N(\mu_1, \Sigma_1)$ is given by

$$K(\mathbb{P}_0, \mathbb{P}_1) = \frac{1}{2} \left( \log \frac{\det(\Sigma_1)}{\det(\Sigma_0)} + \text{tr}(\Sigma_1^{-1}\Sigma_0) - d + (\mu_1 - \mu_0)^T \Sigma_1^{-1}(\mu_1 - \mu_0) \right). \quad (58)$$

Let $\theta, \vartheta \in \mathbb{R}^d$, i.e., $\theta = (\theta_1, \ldots, \theta_d)$ (not to be confused with the vectors $\theta_k$, $k \in \mathbb{N}_0$) and $\vartheta = (\vartheta_1, \ldots, \vartheta_d)$. According to (56), $\mu(\theta, \vartheta) = \mathbf{A}^{-1}(\theta)\mathbf{B}(\theta)\vartheta$ and $\Sigma(\theta) = \sigma^2 \mathbf{A}^{-1}(\theta)\mathbf{A}^T(\theta)$. Now we compute $K(N(\mu(\theta, \vartheta), \Sigma(\theta)), N(\mu(\vartheta, \vartheta), \Sigma(\vartheta)))$. Let $\mathbf{S} = \mathbf{T}(s)$ be the Toeplitz matrix associated with the vector $s = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^{2d-1}$ where 1 is in the $(d-1)$-th position. Matrix $\mathbf{S}$ has ones above the main diagonal and zeros elsewhere and is sometimes called upper shift matrix. For $i = 2, \ldots, d-1$, the powers $\mathbf{S}^i$ are the Toeplitz matrices associated with the vectors $(0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^{2d-1}$ where 1 occupies the $(d-i)$-th position, $\mathbf{S}^d = 0$, the zero matrix of order $d$, and $\mathbf{S}^0$ should be read as $\mathbf{I}$, the identity matrix of order $d$. It follows that $\mathbf{A}(\vartheta) = \mathbf{I} - \mathbf{S}\vartheta_1 - \cdots - \mathbf{S}^d\vartheta_d$, so that

$$\mathbf{A}(\vartheta) - \mathbf{A}(\theta) = \mathbf{S}(\theta_1 - \vartheta_1) + \mathbf{S}^2(\theta_2 - \vartheta_2) + \cdots + \mathbf{S}^d(\theta_d - \vartheta_d),$$

and

$$\mathbf{A}(\vartheta)\mathbf{A}^{-1}(\theta) = \mathbf{I} + \mathbf{S}\mathbf{A}^{-1}(\theta)(\theta_1 - \vartheta_1) + \mathbf{S}^2\mathbf{A}^{-1}(\theta)(\theta_2 - \vartheta_2) + \cdots + \mathbf{S}^d\mathbf{A}^{-1}(\theta)(\theta_d - \vartheta_d).$$

For all $\vartheta \in \mathbb{R}^d$, the matrices $\mathbf{A}(\vartheta)$ have all eigenvalues equal to one (so do their inverses), hence $\det(\Sigma(\vartheta)) = \sigma^{2d}$ and we conclude that the logarithm in (58) is zero. Also, using basic properties of the trace and the representation for $\mathbf{A}(\theta)\mathbf{A}^{-1}(\vartheta)$ derived above,

$$\text{tr} \left[ \Sigma^{-1}(\vartheta) \Sigma(\theta) \right] - d = \text{tr} \left[ (\mathbf{A}^{-1}(\vartheta)\mathbf{A}^{-T}(\vartheta))^{-1}(\mathbf{A}^{-1}(\theta)\mathbf{A}^{-T}(\theta)) \right] - d = \text{tr} \left[ \mathbf{A}^{-1}(\vartheta)\mathbf{A}^{-T}(\vartheta) \right] - d = \text{tr} \left[ \mathbf{A}^{-1}(\theta)\mathbf{A}^{-T}(\theta) \right] - d = 2 \sum_{i=1}^d \text{tr} \left[ \mathbf{S}^i\mathbf{A}^{-1}(\theta) \right] (\theta_i - \vartheta_i) + \sum_{i=1}^d \sum_{j=1}^d \text{tr} \left[ \mathbf{A}^{-T}(\theta)(\mathbf{S}^i)^T\mathbf{S}^j\mathbf{A}^{-1}(\theta) \right] (\theta_i - \vartheta_i)(\theta_j - \vartheta_j).$$

Since the inverse of an upper-triangular matrix is upper-triangular, $\text{tr} \left[ \mathbf{S}^i\mathbf{A}^{-1}(\theta) \right] = 0$, for all $i = 1, \ldots, d$ and $\vartheta \in \mathbb{R}^d$. For any $(n \times m)$-matrix $\mathbf{M}$, denote by $\text{vect}(\mathbf{M})$ the column vector containing the $nm$ entries of $\mathbf{M}$ in any (fixed) order. Let $v_i(\theta) = \text{vect} \left( \mathbf{S}^i\mathbf{A}^{-1}(\theta) \right)$, $i = 1, \ldots, d$, and note that $v_d(\theta)$ is always a zero vector. Note that $\text{tr} \left[ \mathbf{A}^{-T}(\theta)(\mathbf{S}^i)^T\mathbf{S}^j\mathbf{A}^{-1}(\theta) \right] = v_i^T(\theta)v_j(\theta)$, $i, j = 1, \ldots, d$. We conclude that the previous display can be written as

$$\text{tr} \left[ \Sigma^{-1}(\vartheta) \Sigma(\theta) \right] - d = (\vartheta - \theta)^T \left[ v_1(\theta)v_2(\theta) \ldots v_d(\theta) \right]^T \left[ v_1(\theta)v_2(\theta) \ldots v_d(\theta) \right] (\vartheta - \theta),$$

where the matrices on the right are computed by columns.
Consider now the quadratic form in the Kullback-Leibler divergence (58). For any \( \theta, \vartheta, Y \in \mathbb{R}^d \),
\[
(\mu(\vartheta, Y) - \mu(\theta, Y))^T \Sigma^{-1}(\vartheta)(\mu(\vartheta, Y) - \mu(\theta, Y)) = \sigma^{-2} Y^T (B(\vartheta) - A(\vartheta)A^{-1}(\vartheta)(B(\theta) \Sigma)B(\vartheta))Y.
\]
Since \( B(\vartheta) = (S^{d-1})^T \vartheta_1 + \ldots + S^T \vartheta_{d-1} + I \vartheta_d \), we have
\[
B(\vartheta) - A(\vartheta)A^{-1}(\vartheta)B(\theta) = C_1(\theta)(\vartheta_1 - \theta_1) + C_2(\theta)(\vartheta_2 - \theta_2) + \ldots + C_d(\theta)(\vartheta_d - \theta_d),
\]
where \( C_i(\theta) = (S^{d-1})^T + S^i A^{-1}(\theta)B(\theta), i = 1, \ldots, d \); notice also that \( C_d(\theta) = I \). Then
\[
(B(\vartheta) - A(\vartheta)A^{-1}(\vartheta)B(\theta))Y = [C_1(\theta)Y \ldots C_d(\theta)Y](\vartheta - \theta).
\]
Summarizing, we obtained that
\[
K(N(\mu(\theta, Y), \Sigma(\theta)), N(\mu(\vartheta, Y), \Sigma(\vartheta))) = \frac{1}{2} (\vartheta - \theta)^T M(\vartheta - \theta),
\]
with
\[
M = M(\theta, Y) = [v_1(\theta)v_2(\theta) \ldots v_d(\theta)]^T [v_1(\theta)v_2(\theta) \ldots v_d(\theta)] + \sigma^{-2} [C_1(\theta)Y \ldots C_d(\theta)Y]^T [C_1(\theta)Y \ldots C_d(\theta)Y].
\]
(59)
According to (51), we can derive the expression for the average gain:
\[
g_{d,k}(\vartheta, \theta|X_{d(k-1)}) = -M(\theta, X_{d(k-1)}, \vartheta)(\vartheta - \theta).
\]
where \( M \) is given by (59). Note that the matrix \( M \) does not depend on \( \vartheta \) and is clearly positive semi-definite. We evaluate now its eigenvalues. In the representation (59), the first matrix in the sum is positive semi-definite but has at least one zero eigenvalue. It is also clear that the entries of this matrix are polynomials of the coordinates of \( \theta \), so that, if \( \theta \in \Theta \) for a bounded set \( \Theta \), then the largest eigenvalue of this matrix is upper bounded, uniformly over \( \Theta \), by some constant, say, \( K_1 \). As to the second (also positive semi-definite) matrix in the sum of matrices from (59), note that
\[
\text{tr} \left( \left[ C_1(\theta)Y \ldots C_d(\theta)Y \right]^T \left[ C_1(\theta)Y \ldots C_d(\theta)Y \right] \right) = Y^T C_1^T(\theta)C_1(\theta)Y + \ldots + Y^T C_d^T(\theta)C_d(\theta)Y.
\]
The entries of the matrices \( C_i^T(\theta)C_i(\theta), i = 1, \ldots, d, \) are polynomials in \( \theta_1, \ldots, \theta_d \) which are bounded uniformly over a bounded set \( \Theta \). Recall also that the trace of a matrix is equal to the sum of its eigenvalues. We conclude that \( \Lambda_{d,k}(M(\theta, Y)) \leq K_1 + K_2|Y|^2 \) uniformly in \( \theta \in \Theta \) for any bounded \( \Theta \subset \mathbb{R}^d \).
To derive a lower bound on the smallest eigenvalue of the matrix \( M(\theta, Y) \), note that this matrix can be rewritten in the form

\[
\begin{bmatrix}
  v_{1,1}(\theta) & \cdots & v_{1,d-1}(\theta) & 0 \\
  \vdots & \ddots & \vdots & \vdots \\
  v_{d-1,1}(\theta) & \cdots & v_{d-1,d-1}(\theta) & 0 \\
  0 & \cdots & 0 & Y^T Y
\end{bmatrix} + \begin{bmatrix}
  c_{1,1}(\theta) & \cdots & c_{1,d-1}(\theta) & c_{1,d}(\theta) \\
  \vdots & \ddots & \vdots & \vdots \\
  c_{d-1,1}(\theta) & \cdots & c_{d-1,d-1}(\theta) & c_{d-1,d}(\theta) \\
  c_{d,1}(\theta) & \cdots & c_{d,d-1}(\theta) & 0
\end{bmatrix}
\]

for \( v_{i,j}(\theta) = v_i^T(\theta)v_j(\theta) \) and \( c_{i,j}(\theta) = \sigma^{-2}Y^T C_i^T(\theta)c_j(\theta)Y \), where we swapped the \((d, d)\)-th entries of the matrices in the sum from (59). We used also that \( v_d(\theta) = 0 \) and \( C_d(\theta) = I \).

Note that the top left matrices in the block matrices above are Gram matrices and therefore positive semidefinite. The matrix \( [v_{i,j}(\theta)]_{i,j=1,\ldots,d-1} \) is the Gram matrix associated with the vectors \( v_1(\theta), \ldots, v_{d-1}(\theta) \). Since \( A^{-1}(\theta) \) is a triangular matrix with 1’s in its main diagonal, it follows that these vectors are linearly independent. Hence the associated Gramian is actually positive definite for each \( \theta \). The determinant of this Gramian is a polynomial in the entries of the matrix which in turn are polynomials in \( \theta_1, \ldots, \theta_d \). If \( \theta \) lies in a compact set \( \Theta \), the infimum of the determinant of this matrix over \( \theta \in \Theta \) must be lower bounded by some positive constant, say \( K_3 \). Using the same reasoning, we conclude that its determinant is upper bounded by some constant \( K_4 \). A lower bound on the smallest eigenvalue can then be obtained by noting that for any positive definite matrix \( M \) of order \( d \),

\[
\lambda_{(1)}(M) \geq \frac{\det(M)}{\chi_{d-1}^2(M)} \geq \frac{K_3}{K_4^{d-1}} = K_5 > 0.
\]

We conclude that the smallest eigenvalue of the block matrix on the left is at least \( \min(K_5, ||Y||^2) \). The block matrix on the right is clearly positive semidefinite. We conclude that the smallest eigenvalue of the matrix \( \Lambda_{(1)}(M(\theta, Y)) \geq \min\{K_5, ||Y||^2\} \) by using Weyl’s Monotonicity Theorem, see for example [3]. This means that \( \Lambda_{(1)}(M(\theta, Y)) \geq ||Y||^2 \) for all \( ||Y|| \leq K_5 \). Swapping back the \((d, d)\)-th entries of the matrices in the sum from (59), we see that there must exist an \( Y \) such that \( ||Y|| \leq K_5 \) and for which the smallest eigenvalue of the second matrix in the sum from (59) is bounded from below by \( c_1||Y||^2 \) for some \( c_1 > 0 \). By using renormalization arguments, we conclude that \( \Lambda_{(1)}(M(\theta, Y)) \geq c_2||Y||^2 \) for some \( c_2 > 0 \).

Condition (A2) is not difficult to check. The gain (57) can be written in the following form

\[
G_{dk}(\theta, X_{dk}) = -\sigma^{-2}(A(\theta)X_{dk,d} - B(\theta)X_{d(k-1),d})^T \frac{\partial (A(\theta)X_{dk,d} - B(\theta)X_{d(k-1),d})}{\partial \theta},
\]

where \( \partial/\partial \theta \) represents the Jacobian operator. To verify (A2), it suffices to check that the expectation of the norm of \( G_{dk} \) is bounded. We omit the details but it is clear from the expression derived above that the norm of the gain function squared is a polynomial of degree four in the coordinates of \( X_{dk,2d} \). We have already mentioned that if the initial values for the autoregressive process and the noise terms have uniformly bounded second moments, then this transfers to the each observation \( X_k \), provided that the sequence of parameters of the model, \( \theta_k \), lives in the parameter set \( \Theta(\rho) \) for some \( \rho < 1 \).
For the matrix $M$ defined by (59), we established above that almost surely
\[ c\|Y\|^2 \leq \Lambda_{(1)}(M(\theta, Y)) \leq \Lambda_{(d)}(M(\theta, Y)) \leq K_1 + K_2\|Y\|^2 \]
uniformly in $\theta \in \Theta$ for any bounded $\Theta \subset \mathbb{R}^d$. We can get rid of the dependence on $\|Y\|^2$ (recall that $Y = X_{d(k-1), d}$) by using the rescaled gain $\tilde{G}_{dk}$ defined in Section 4
\[ \tilde{G}_{dk}(\vartheta, X_{dk}) = \frac{\min\{\kappa, \|X_{d(k-1), d}\|^2\}}{\|X_{d(k-1), d}\|^2} G_{dk}(\vartheta, X_{dk}), \quad \kappa > 0. \]
Then $\tilde{g}_{dk}(\vartheta, \theta|X_{d(k-1)}) = -\tilde{M}(\theta, X_{d(k-1), d})(\vartheta - \theta)$, where
\[ \tilde{M}(\theta, X_{d(k-1), d}) = \frac{\min\{\kappa, \|X_{d(k-1), d}\|^2\}}{\|X_{d(k-1), d}\|^2} M(\theta, X_{d(k-1), d}) \]
and $M$ is defined by (59). The argument in (62) shows that (A2) still holds for this rescaled gain. Next, $\Lambda_{(d)}(\tilde{M}(\theta, X_{d(k-1), d})) \leq C$ almost surely by construction. Thus to establish (A1), we need to verify that $\mathbb{E}[\Lambda_{(1)}(\tilde{M}(\theta, X_{d(k-1), d})|X_{d(k-2)})] \geq c$. One can then proceed as in (54) (and Lemma 5) to show that for an appropriately large $\kappa$, $\mathbb{E}\left[ \min\{\kappa, \|X_{d(k-1), d}\|^2\}|X_{d(k-2)}\right] > c$, we omit this derivation.

**Remark 26.** One could drop the requirement for the errors to be Gaussian and still use the same gain $G_{dk}$. We expect the same results to hold, under appropriate moment assumptions. Instead of using the Kullback-Leibler representation (59), one has to work with quantities $G_{dk}$ and $g_{dk}$ directly and assure the validity of (A1) and (A2) based on moment assumptions on the error terms (and possibly the initial conditions) as we did in the one dimensional case.

### 7 Proofs of the lemmas

**Proof of Lemma 7** First suppose that $y = Mx$ for some symmetric positive definite matrix $M$ such that $0 < \lambda_1 \leq \lambda_{(1)}(M) \leq \lambda_{(d)}(M) \leq \lambda_2 < \infty$. Then $\langle x, y \rangle = x^T Mx$ and therefore
\[ 0 < \lambda_1 \|x\|^2 \leq \lambda_{(1)}(M)\|x\|^2 \leq \langle x, y \rangle \leq \lambda_{(d)}(M)\|x\|^2 \leq \lambda_2 \|x\|^2 \]
and
\[ \|y\|^2 = \langle y, y \rangle = x^T M^T Mx = x^T M^2 x \leq \lambda_2^2 \|x\|^2. \]
Now we prove the converse assertion. Suppose $x, y \in \mathbb{R}^d$ and $0 < \lambda_1 \|x\|^2 \leq \langle x, y \rangle \leq \lambda_2 \|x\|^2 < \infty$ for some $\lambda_1', \lambda_2' \in \mathbb{R}$ such that $0 < \lambda_1' \leq \lambda_2' < \infty$ and that $\|y\| \leq C\|x\|$.
Let $V = \{v = ax + by : a, b \in \mathbb{R}\}$ be the linear space spanned by $x$ and $y$. First consider the case $\dim(V) = 1$, i.e., $y = \alpha x$ for some $\alpha \in \mathbb{R}$. Then $\langle y, x \rangle = \alpha \|x\|^2$ so that $0 < \lambda_1' \leq \alpha \leq \lambda_2' < \infty$. Thus $y = \alpha x = Mx$ with symmetric and positive $M = \alpha I$ so that $0 < \lambda_1' \leq \alpha = \lambda_{(1)}(M) = \lambda_{(d)}(M) \leq \lambda_2' < \infty$. 

39
Now consider the case $\dim(V) = 2$. Let $e_1 = x/\|x\|$ and $\{e_1, e_2\}$ be an orthonormal basis of $V$. Then

$$x = \|x\|e_1$$
$$y = \alpha e_1 + \beta e_2.$$

The conditions $\lambda_1\|x\|^2 \leq \langle x, y \rangle = \alpha \|x\| \leq \lambda_2\|x\|^2$ and $\|y\| = \sqrt{\alpha^2 + \beta^2} \leq C\|x\|$ imply that

$$\lambda_1\|x\| \leq \alpha \leq \min\{\lambda_2, C\}\|x\|, \quad \|\beta\| \leq C\|x\|.$$

Let $e_2$ be chosen in such a way that $\beta > 0$ (which is always possible.) Now, we change the basis of $V$ as follows:

$$e_1' = \cos(\theta)e_1 - \sin(\theta)e_2,$$
$$e_2' = \sin(\theta)e_1 + \cos(\theta)e_2.$$

We thus rotate the basis $\{e_1, e_2\}$ by the angle $\theta$. In these new basis we have

$$x = \|x\|\cos(\theta)e_1' + \|x\|\sin(\theta)e_2' = \alpha e_1' + \beta e_2',$$
$$y = (\alpha \cos(\theta) - \beta \sin(\theta))e_1' + (\alpha \sin(\theta) + \beta \cos(\theta))e_2' = \alpha_y e_1' + \beta_y e_2'.$$

Recall that $\alpha, \beta > 0$. Take $\theta \in (0, \pi/2)$ such that $\alpha \cos(\theta) - \beta \sin(\theta) = \frac{1}{2}\alpha \cos(\theta)$, i.e.,

$$\tan(\theta) = \frac{\alpha_y}{\beta_y}.$$ Then we have that

$$\frac{\lambda_1}{2} \leq \frac{\alpha}{\|x\|} = \frac{\alpha_y}{\alpha_x} \leq \frac{\min\{\lambda_2, C\}}{2}, \quad \lambda_1 \leq \frac{\alpha}{\|x\|} \frac{\beta_y}{\beta_x} \leq \frac{\alpha}{\|x\|} + \frac{2\beta^2}{\alpha \|x\|} \leq \min\{\lambda_2, C\} + \frac{2C^2}{\lambda_1}.$$

Take then $\lambda_1 = \lambda_1'/2$ and $\lambda_2 = \min\{\lambda_2, C\} + 2C^2/\lambda_1'$. Let $\{e_3', \ldots, e_d'\}$ be the orthonormal basis of $V^\perp$, so that $b = \{e_1', e_2', e_3', \ldots, e_d'\}$ is an orthonormal basis of $\mathbb{R}^d$. Take

$$\tilde{M} = \begin{bmatrix} D & O \\ O & I_{d-2} \end{bmatrix}$$
with $D = \begin{bmatrix} \alpha_y/\alpha_x & 0 \\ 0 & \beta_y/\beta_x \end{bmatrix}$.

where the $O$'s indicate null matrices of the appropriate dimensions. We then have $y = \tilde{M}x$ in the basis $b$ and $\lambda_1 \leq \lambda_{(1)}(\tilde{M}) \leq \lambda_{(d)}(\tilde{M}) \leq \lambda_2$. We can finally obtain $M$ by using the orthogonal matrix $T$ to change the basis $b$ to the canonical basis of $\mathbb{R}^d$ as $M = T^{-1}\tilde{M}T$. Clearly, $M$ has the same eigenvalues as $\tilde{M}$ and is symmetric. \hfill $\square$

**Proof of Lemma** For the sake of brevity, we use the notations $\theta_k = \theta_k(X_{k-1})$, $G_k = G(\tilde{\theta}_k, X_k|X_{k-1})$ and $g_k = g(\tilde{\theta}_k, \theta_k|X_{k-1})$, $k \in \mathbb{N}_0$.

Recall that $\Theta$ is compact and $\sup_{\theta \in \Theta} \|\theta\|_2^2 \leq C_\Theta$. By iterating (2), it is easy to see that $\mathbb{E}\|\tilde{\theta}_k\|^2 < \infty$ for each $k \in \mathbb{N}_0$. First assume $\mathbb{E}\|\tilde{\theta}_k\|^2 \leq K\Theta$, for some $K > 0$ to be chosen later. By (3), we obtain $\mathbb{E}\|G_k\|^2 \leq 2C_\Theta + 4\lambda_2^2 C_\Theta + 4\lambda_2^2 K\Theta = \bar{C}_g$, which implies, in view of (2) and the fact that $\gamma_k \leq \Gamma$,

$$\mathbb{E}\|\tilde{\theta}_{k+1}\|^2 \leq 2\mathbb{E}\|\tilde{\theta}_k\|^2 + 2\gamma_k^2 \mathbb{E}\|G_k\|^2 \leq 2KC_\Theta + 2\Gamma^2 \bar{C}_g = C_2.$$
Next, consider the case $E\|\hat{\theta}_k\|^2 > KC_\Theta$, which, together with (2), implies that $E\|\hat{\theta}_k\|^2 > K\|\theta_k\|^2$. Recall that, in view of (A1), $M_k$ is a symmetric positive definite matrix such that $0 < \lambda_1 \leq \Lambda_{(d)}(M_k) \leq \lambda_2 < \infty$ almost surely. Therefore we obtain that, almost surely,

$$\hat{\theta}_k^T M_k \hat{\theta}_k = \Lambda_{(d)}(M_k)\|\hat{\theta}_k\|^2 \geq \lambda_1\|\hat{\theta}_k\|^2$$

and, by the Cauchy-Schwarz inequality,

$$\hat{\theta}_k^T M_k \hat{\theta}_k \leq \|\hat{\theta}_k^T M_k \hat{\theta}_k\| \leq (\hat{\theta}_k^T M_k \hat{\theta}_k)^{1/2} (\hat{\theta}_k^T M_k \hat{\theta}_k)^{1/2} \leq \lambda_2\|\hat{\theta}_k\||\theta_k||.$$

By using the last two relation, (2), (5), (7) and (3), we evaluate $E\|\hat{\theta}_{k+1}\|^2$:

$$E\|\hat{\theta}_{k+1}\|^2 \leq E\|\hat{\theta}_k\|^2 + 2\gamma_k E\|\hat{\theta}_k^T E(G_k|X_{k-1})\|^2 + \gamma_k C_g\|\theta_k\|^2$$

$$\leq E\|\hat{\theta}_k\|^2 - 2\gamma_k E(\hat{\theta}_k^T M_k (\hat{\theta}_k - \theta_k)) + \gamma_k C_g$$

$$\leq E\|\hat{\theta}_k\|^2 - 2\gamma_k \max\|\lambda_1 E\|\theta_k\|^2 - E(\hat{\theta}_k^T M_k \theta_k)\| \leq E\|\hat{\theta}_k\|^2 - 2\gamma_k \max\|\lambda_1 E\|\theta_k\|^2 - \lambda_2 E(|\hat{\theta}_k||\theta_k|)) + \gamma_k C_g E\|\hat{\theta}_k\|^2,$$

with $C_3 = C_g/(KC_\Theta)$.

Since $E\|\hat{\theta}_k\|^2 > K E\|\theta_k\|^2$, it follows that $E(\|\hat{\theta}_k||\theta_k||) < (E\|\hat{\theta}_k\|^2 E\|\theta_k\|^2)^{1/2} \leq \frac{E\|\hat{\theta}_k\|^2}{\sqrt{K}}$. Using this, we proceed by bounding the previous display as follows:

$$\leq E\|\hat{\theta}_k\|^2 - 2\gamma_k E\|\hat{\theta}_k\|^2 (\lambda_1 - \frac{\lambda_2}{\sqrt{K}}) + \gamma_k C_3 E\|\hat{\theta}_k\|^2$$

$$= E\|\hat{\theta}_k\|^2 - 2\gamma_k E\|\hat{\theta}_k\|^2 (\lambda_1 - \frac{\lambda_2}{\sqrt{K}} - \frac{C_3 \gamma_k}{2}) \leq E\|\hat{\theta}_k\|^2,$$

for sufficiently large $K$ and sufficiently small $\gamma_k$. Thus, for sufficiently large $K$ and sufficiently small $\gamma_k$, $E\|\hat{\theta}_{k+1}\|^2 \leq C_2$ with $C_2$ as defined above.

**Lemma 3.** Let $M$ be a symmetric positive definite matrix of order $d$, $p \geq 1$ and a constant $\gamma > 0$ be such that $\gamma \Lambda_{(d)}(M) < 1$. Then $\|I - \gamma M\| = 1 - \gamma \Lambda_{(1)}(M)$ and

$$0 < 1 - \gamma \Lambda_{(d)}(M) = \Lambda_{(1)}(I - \gamma M) \leq \Lambda_{(d)}(I - \gamma M) = 1 - \gamma \Lambda_{(1)}(M) < 1.$$

Besides, $\|M\|_p = K_p(d)\|M\|_p = K_p(d)\Lambda_{(d)}(M)$ for some constant $K_p(d) > 0$.

**Proof.** Let $\lambda_i$ be the eigenvalues of $M$, so that the matrix $I - \gamma M$ has eigenvalues $1 - \gamma \lambda_i$, $i = 1, \ldots, d$. Since $\gamma \Lambda_{(d)}(M) < 1$, then, for all $i = 1, \ldots, d$, $0 < \gamma \Lambda_{(1)}(M) \leq \gamma \lambda_i \leq \gamma \Lambda_{(d)}(M) < 1$, implying $1 - \gamma \Lambda_{(1)}(M) > 1 - \gamma \lambda_i > 1 - \gamma \Lambda_{(d)}(M) > 0$, so that $\|I - \gamma M\| = \max\|1 - \gamma \lambda_i\| = 1 - \gamma \Lambda_{(1)}(M) < 1$. The first two assertions follow.

It remains to prove the last assertion. For $x \in \mathbb{R}^d$, let $R^0_p(d) = \max_{x \neq 0} ||x||_p/||x||_2$ and $R^2_p(d) = \max_{x \neq 0} ||x||_2/||x||_p$. According to Theorem 5.6.18 from [12],

$$\max_{M \neq 0} \frac{\|M\|_p}{\|M\|_2} = R^0_p(d) R^2_p(d) = K_p(d).$$

Recall that $\|M\|_2 = \|M\|_2 = \lambda_{(d)}(M)$ and $\|x\|_s = \|x\|_r \leq d^{1/r-1/s} \|x\|_s$ for any $x \in \mathbb{R}^d$ and $s \geq r \geq 1$. From this relation it is easy to get the following bounds: $R^0_p(d) \leq 1$ if $p \geq 2$, $R^0_p(d) \leq d^{(2-p)/(2p)}$ if $1 \leq p < 2$; $R^2_p(d) \leq d^{(p-2)/(2p)}$ if $p \geq 2$, $R^2_p(d) \leq 1$ if $1 \leq p < 2$. These bounds imply that $K_p(d) \leq d^{(p-2)/(2p)}$ if $p \geq 2$ and $K_p(d) \leq d^{(2-p)/(2p)} \leq d^{1/2}$ if $1 \leq p < 2$. This completes the proof of the lemma. \qed
Lemma 4 (Abel transformation). Suppose $d_1, d, k_0, k \in \mathbb{N}$ and $k_0 \leq k$. Let $B_i$ be $(d_1 \times d)$-matrices, $a_i \in \mathbb{R}^d$ and $A_i = \sum_{j=k_0}^i a_j$, $i = k_0, \ldots, k$. Then

$$\sum_{i=k_0}^k B_i a_i = \sum_{i=k_0}^{k-1} (B_i - B_{i+1}) A_i + B_k A_k.$$ 

Proof. We prove this by induction in $k$. For $k = k_0$ we simply have $B_{k_0} a_{k_0} = B_{k_0} A_{k_0} = B_{k_0} a_{k_0}$ and the assertion holds true. Assume that the equality holds for $k = n$ and let us prove the result for $k = n + 1$. We have

$$\sum_{i=k_0}^{n+1} B_i a_i = \sum_{i=k_0}^n B_i a_i + B_{n+1} a_{n+1} = \sum_{i=k_0}^{n-1} (B_i - B_{i+1}) A_i + B_n A_n + B_{n+1} a_{n+1}$$

$$= \sum_{i=k_0}^n (B_i - B_{i+1}) A_i - (B_n - B_{n+1}) A_n + B_n A_n + B_{n+1} a_{n+1}$$

$$= \sum_{i=k_0}^n (B_i - B_{i+1}) A_i + B_{n+1} A_{n+1}. \quad \square$$

Lemma 5. Consider an AR(1)-model with a measurable $\theta_k = \theta_k(X_{k-1})$:

$$X_k = X_{k-1} \theta_k + \xi_k, \quad k \in \mathbb{N},$$

where $E[\xi_k | X_{k-1}] = E[\xi_k^2 | X_{k-1}] = 0$, $E[\xi_k^2 | X_{k-1}] = \sigma^2 > 0$, and $E[\xi_k^4 | X_{k-1}] = c \sigma^4$, $k \in \mathbb{N}$, for some constant $0 < c < 5$. Then, for any $T$ such that $T \geq (9 - c)\sigma^2/4$,

$$E[\min(X_k^2, T) | X_{k-1}] \geq \frac{(5 - c)\sigma^2}{4}, \quad k \in \mathbb{N}.$$ 

Proof. We compute

$$E[X_k^2 | X_{k-1}] = X_{k-1}^2 \theta_k^2 + 2X_{k-1} \theta_k E[\xi_k | X_{k-1}] + E[\xi_k^2 | X_{k-1}] = X_{k-1}^2 \theta_k^2 + \sigma^2,$$

$$E[X_k^4 | X_{k-1}] = X_{k-1}^4 \theta_k^4 - 4X_{k-1}^3 \theta_k^2 E[\xi_k | X_{k-1}] + 6X_{k-1}^2 \theta_k^2 E[\xi_k^2 | X_{k-1}]$$

$$- 4X_{k-1} \theta_k E[\xi_k^3 | X_{k-1}] + E[\xi_k^4 | X_{k-1}] = X_{k-1}^4 \theta_k^4 + 6X_{k-1}^2 \theta_k^2 \sigma^2 + c \sigma^4.$$

For $a, b \in \mathbb{R}$ we have $\min(a, b) = (a + b)/2 - |a - b|/2$. Using this relation, conditional version of Jensen’s inequality and the last display, we derive:

$$E[\min(X_k^2, \rho \sigma^2) | X_{k-1}] = \frac{1}{2} E[X_k^2 + \rho \sigma^2 - |X_k^2 - \rho \sigma^2| | X_{k-1}]$$

$$\geq \frac{1}{2} \left[ X_{k-1}^2 \theta_k^2 + (\rho + 1)\sigma^2 - \left( E[(X_k^2 - \rho \sigma^2)^2 | X_{k-1}] \right)^{1/2} \right],$$
for \( \rho > 0 \). We now have, by plugging in the expressions derived above and simplifying,

\[
E \left[ (X_k^2 - \rho \sigma^2)^2 | X_{k-1} \right] = E \left[ X_k^4 | X_{k-1} \right] - 2\rho \sigma^2 E \left[ X_k^2 | X_{k-1} \right] + \rho^2 \sigma^4
= X_{k-1}^4 \theta_k^4 + 2(3 - \rho)X_{k-1}^2 \theta_k^2 \sigma^2 + (c - 2\rho + \rho^2)\sigma^4 = \left( X_{k-1}^2 \theta_k^2 + \frac{c + 3}{4} \sigma^2 \right)^2,
\]

if we pick \( \rho = (9 - c)/4 > 1 \). Combining the previous two displays, we conclude that

\[
E \left[ \min \left( X_k^2, (9 - c)\sigma^2/4 \right) | X_{k-1} \right] \geq \frac{(5 - c)\sigma^2}{4}, \quad k \in \mathbb{N},
\]

and the statement of the lemma follows.

\[\square\]

References

[1] Panos J. A. and Michel, A. N. (2007). A Linear Systems Primer. Basel: Birkhäuser.

[2] Bach F. and Moulines, E. (2011). Non-asymptotic analysis of stochastic approximation algorithms for machine learning. NIPS, Spain.

[3] Bathia, R. (1997). Matrix Analysis New York: Springer.

[4] Belitser, E. (2000). Recursive estimation of a drifting autoregressive parameter. Ann. Statist. 28(3): 860–870.

[5] Belitser, E. N. and Korostelev, A. P. (1992). Pseudovalues and minimax filtering algorithms for the nonparametric median. Adv. in Sov. Math. 12, 115–124.

[6] Belitser, E. and van de Geer, S. (2000). On robust recursive nonparametric curve estimation. High dimensional probability II. Birkhäuser, Progr. Probab. 47, 391–404.

[7] Belitser, E. and Serra, P. (2013). On properties of the algorithm for pursuing a drifting quantile. Automation and Rem. Control, 74(4): 613–627.

[8] Benveniste, A, Metivier, M. and Priouret, P. (1990). Adaptive Algorithms and Stochastic Approximation. Berlin: Springer.

[9] Brossier, J.-M. (1992). Égalization Adaptive et Estimation de Phase: Application aux Communications Sous-Marines. PhD thesis, Institut National Polytechnique de Grenoble.

[10] Chow, Y. S. and Teicher, H. (1988). Probability Theory. Independence, Interchangeability, Martingales. Springer texts in Statistics. New York: Springer Verlag, second edition.

[11] Delyon, B. and Juditsky, A. (1995). Asymptotical study of parameter tracking algorithms. SIAM J. Control and Optimization, 33(1), 323–345.
[12] Horn, R. A. and Charles, R. J. (1991). *Topics in Matrix Analysis*. Cambridge: Cambridge University Press.

[13] Kiefer, J. and Wolfowitz, J. (1952). Stochastic estimation of the maximum of a regression function. *Ann. Math. Statist.*, 23(3), 462–466.

[14] Kushner, H. J. and Clark, D. S. (1978). *Stochastic Approximation for Constrained and Unconstrained Systems*. New York: Springer Verlag.

[15] Kushner, H. J. and Yang, J. (1995). Analysis of adaptive step-size sa algorithms for parameter tracking. *IEEE Trans. Autom. Control*, 40, 1403–1410.

[16] Kushner, H. J and Yin, G. (2003). *Stochastic Approximation and Recursive Algorithms and Applications*. Berlin: Springer-Verlag.

[17] Ljung, L. and Söderström, T. (1983). *Theory and Practice of Recursive Identification*. Cambridge: MIT Press, MA.

[18] Moulines, E, Priouret, P. and Roueff, F. (2005). On recursive estimation for time varying autoregressive processes. *Ann. Statist.*, 33(6), 2610–2654.

[19] Nevelson, M. B. and Khasminskii, R. Z. (1976). *Stochastic Approximation and Recursive Estimation*, volume 47 of *Translation of Mathematical Monographs*. American Mathematical Society.

[20] Robbins, H. and Monro, S. (1951). A stochastic approximation method. *Ann. Math. Statist.*, 22(3), 400–407.

[21] Shiryaev, A. N. (1996). *Probability*. New York: Springer, second edition.

[22] Spall, J. C. (1992). Multivariate Stochastic Approximation Using a Simultaneous Perturbation Gradient Approximation. *IEEE Trans. Autom. Control* 37(3), 332–341.

[23] Tsypkin, Y. Z. (1971). *Adaptation and Learning in Automatic Systems*. New York: Academic Press.

[24] Wasan, M. T. (1969). *Stochastic Approximation*. Cambridge: Cambridge University Press.