Exact Einstein–scalar field solutions for formation of black holes in a cosmological setting

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Abstract

We consider self–interacting scalar fields coupled to gravity. Two classes of exact solutions to Einstein’s equations are obtained: the first class corresponds to the minimal coupling, the second one to the conformal coupling. One of the solutions is shown to describe a formation of a black hole in a cosmological setting. Some properties of this solution are described. There are two kinds of event horizons: a black hole horizon and cosmological horizons. The cosmological horizons are not smooth. There is a mild curvature singularity, which affects extended bodies but allows geodesics to be extended. It is also shown that there is a critical value for a parameter on which the solution depends. Above the critical point, the black hole singularity is hidden within a global black hole event horizon. Below the critical point, the singularity appears to be naked. The relevance to cosmic censorship is discussed.

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I Introduction

Two remarkable predictions of General Relativity, the expanding universe and black holes, have always been of main interest in the theory of gravitation. In fact, the expansion of the universe and the gravitational collapse of matter are very similar phenomena from the mathematical point of view [1]. Both of them yield the existence of regions in the space–time where matter has transplanckian energies. It is clear that classical General Relativity cannot be extrapolated there. What theory describes the early stage of the evolution of our universe and the final stage of the gravitational collapse? Will the notions of space and time survive in a future theory of Quantum Gravity at all? Or more specifically: what is left after a black hole has evaporated and what was before the Big Bang? These questions are among the most fundamental issues in contemporary theoretical physics.

Yet, there are many unresolved problems in the classical General Relativity. One of such problems is the dearth of exact solutions describing collapsing matter and formation of black holes. While many cosmological models are available in the market, very few dynamical solutions are known in (4D) black hole physics - the most popular are the Tolman solution [2] for the collapse of a dust and the Vaidya solution [3] for the collapse of radiation. Some recent numerical studies of different models of the gravitational collapse [4, 5] indicate that black hole formation may be accompanied with other interesting phenomena, such as echoing and critical behaviour. In view of this, it would be extremely important to have exact analytic solutions of Einstein’s equations which could be regarded as describing collapsing matter. This paper aims at obtaining such solutions.

We shall study two models of self–interacting scalar fields. In the first model the
scalar field is minimally coupled to gravity and the scalar potential has a Liouville form,
\[ V \propto \exp(-k\varphi). \] This model has been studied by numerous authors with a view to cosmological applications (see, for instance, Refs. [6, 7, 8]). The Liouville potential arises as an effective potential in some supergravity theories or in Kaluza–Klein theories after dimensional reduction to an effective 4–dimensional theory [6]. It also arises in higher–order gravity theories after a transformation to the Einstein frame [6].

In the second model the scalar field is conformally coupled to gravity. This model is less studied, though it was proved by Bekenstein [9] that there is a mapping between the space of Einstein–conformal scalar solutions and the space of Einstein–ordinary scalar solutions. Bekenstein found the mapping for massless scalar fields but the result can easily be extended to self–interacting fields [10]. It turns out that the counterpart of the ordinary–scalar Liouville potential is the following conformal–scalar potential:
\[ W \propto (1 - \zeta \phi)^{2+k/(2\zeta)}(1 + \zeta \phi)^{2-k/(2\zeta)}, \]
where \( \zeta = (4\pi G / 3)^{1/2}. \) The most interesting case appears to be \( k = \pm 2\zeta. \) In this case we find a solution of the Einstein–conformal scalar equations which is shown to describe the gravitational collapse of the scalar field in a cosmological setting.

This paper is organized as follows. In Sec. [II] we prove two theorems that allow one to generate new exact solutions of the Einstein–scalar field equations from static vacuum solutions of Einstein’s equations. As an application of the theorems we obtain spherically symmetric dynamical solutions in the two models mentioned above. The cosmological black hole solution is studied in Sec. [III]. Section [IV] contains some concluding remarks. Sign conventions for the metric and the Riemann tensor follow those of the book [11].
II Generating new solutions of the Einstein–scalar equations

A Minimal coupling

Let us consider the Einstein–scalar system described by the action

$$S_{m}[g_{\alpha\beta}, \phi] = \int d^4x \left(-g\right)^{1/2} \left[-(16\pi G)^{-1}R + \frac{1}{2}\phi,_{\alpha}\phi,_{\beta}g^{\alpha\beta} - V(\phi)\right], \quad (2.1)$$

with an exponential potential,

$$V(\phi) = V_o \exp(-k\phi). \quad (2.2)$$

Variations of the action with respect to $g^{\mu\nu}$ and $\phi$ yield the following equations:

$$R_{\mu\nu} = 8\pi G \left(\phi,_{\mu}\phi,_{\nu} - g_{\mu\nu}V_o \exp(-k\phi)\right), \quad (2.3)$$

$$\Box \phi = kV_o \exp(-k\phi). \quad (2.4)$$

Our first theorem is:

**Theorem 1** If $ds_o^2 = e^{2u}dt^2 - e^{-2u}h_{ij}dx^i dx^j$ - is a static vacuum solution of Einstein's equations, then

$$ds^2 = \exp(2\delta u + 8\alpha^2 aT)dT^2 - \exp(-2\delta u + 2aT)h_{ij}dx^i dx^j, \quad (2.5)$$

$$\phi = (4\pi G)^{-1/2} \left((4\alpha^2 + 1)^{-1/2}u + 2aT\right) \quad (2.6)$$

form a solution of equations (2.3) and (2.4), where the parameters $\alpha, \delta$ and $a$ are related to the parameters of the potential as follows:

$$k = 8\alpha(\pi G)^{1/2},$$

$$\delta = 2\alpha(4\alpha^2 + 1)^{-1/2}, \quad (2.7)$$

$$V_o = (8\pi G)^{-1}a^2(3 - 4\alpha^2).$$
Proof. For $a = 0$ (no potential term) the corresponding theorem was proved in [12, 13].

Consider thus the static solution of equations (2.3) and (2.4) with $V_o = 0$,

$$
\begin{align*}
    ds^2 &= \exp(2\delta u)dT^2 - \exp(-2\delta u)h_{ij}dx^i dx^j, \\
    \varphi &= (4\pi G)^{-1/2}(4\alpha^2 + 1)^{-1/2} u .
\end{align*}
$$

Under the conformal transformation,

$$
    g_{\mu\nu} = \exp(2\mu(t)) g_{\mu\nu} ,
$$

the Ricci tensor transforms as follows (see, for instance, [14])

$$
\begin{align*}
    R_{\infty} &= \bar{R}_{\infty} - 3\ddot{\mu} , \\
    R_{oi} &= \bar{R}_{oi} + 2\dot{\mu} \partial_i \log(g_{oo}) , \\
    R_{ij} &= \bar{R}_{ij} - (\ddot{\mu} + 2\dot{\mu}^2) g_{ij} g^{oo} .
\end{align*}
$$

Since (2.8) and (2.9) satisfy the Einstein equations,

$$
    \bar{R}_{\mu\nu} = 8\pi G\bar{\varphi}_\mu \bar{\varphi}_\nu ,
$$

it is easy to check that the metric tensor (2.10) and the scalar field

$$
    \varphi = \bar{\varphi} + \alpha(\pi G)^{-1/2}\mu(t)
$$

satisfy equation (2.3) if the function $\mu(t)$ obeys the following two equations:

$$
\begin{align*}
    \ddot{\mu} &= (1 - 4\alpha^2)\dot{\mu}^2 , \\
    (3 - 4\alpha^2)\dot{\mu}^2 &= 8\pi GV_o \exp(2(1 - 4\alpha^2)\mu) .
\end{align*}
$$

When $\alpha^2 \neq 3/4$, equation (2.15) implies equation (2.14) (the Bianchi identity [1]). If $\alpha^2 = 3/4$, equation (2.15) is trivial.

There are two distinct cases:
1. $\alpha^2 = 1/4$. The solution of equations (2.14) and (2.15) is $\mu(t) = at + \mu_o$. Taking into account equations (2.10) and (2.13), one immediately gets the solution (2.5)–(2.7).

2. $\alpha^2 \neq 1/4$. In this case the solution is $\mu(t) = (4\alpha^2 - 1)^{-1} \log |(4\alpha^2 - 1)at| + \mu_o$. Redefining the time coordinate, $(4\alpha^2 - 1)at = \pm \exp((4\alpha^2 - 1)aT)$, we arrive at equations (2.5)–(2.7).

To complete the proof we notice that equation (2.4) coincides, in our case, with the $(0,0)$–component of Einstein’s equations. Q.E.D.

As an application of the theorem we shall consider spherically symmetric solutions of equations (2.3) and (2.4). The corresponding vacuum metric is the Schwarzschild metric:

$$e^{2u} = 1 - 2m/r ,$$

(2.16)

$$h_{ij} dx^i dx^j = dr^2 + (1 - 2m/r)r^2(d\theta^2 + \sin^2 \theta d\varphi^2) .$$

(2.17)

Applying our theorem, we get, thus, the following solution of equations (2.3) and (2.4):

$$ds^2 = \exp(8\alpha^2 aT)(1 - 2m/r)^\delta dT^2$$

$$- \exp(2aT) \left( \frac{dr^2}{(1 - 2m/r)^\delta} + (1 - 2m/r)^{1-\delta}r^2(d\theta^2 + \sin^2 \theta d\varphi^2) \right) ,$$

(2.18)

$$\varphi = (4\pi G)^{-1/2} \left( 2\alpha aT + (4\alpha^2 + 1)^{-1/2}\frac{1}{2} \ln(1 - 2m/r) \right) .$$

(2.19)

The solution depends on three parameters, $m, a$ and $\alpha$ (see equation (2.7)). When $m = 0$, it represents an isotropic and homogeneous FRW solution first obtained in Ref. [15]. The static solution ($a = 0$) was obtained in Refs. [12, 13]. The case $\alpha^2 = 3/4$ (no potential term) was recently analyzed in [16].
Let us summarize some properties of the obtained solution. It follows from the proof (see equation (2.10)) that the metric is conformally static. As $r \to \infty$, it asymptotically approaches a spatially flat Robertson–Walker metric. If $\alpha = 0$, which corresponds to a massless scalar field in a universe with a cosmological constant $\Lambda = 3a^2$, the metric is asymptotically de Sitter. For any $\alpha$ (except in the case of purely de Sitter universe, $\alpha = m = 0$) there is a "big bang" singularity as $aT \to -\infty$. If $m \neq 0$, there are also time-like singularities at $r = 0$ and $r = 2m$. Thus, the solution may be interpreted as an inhomogeneous cosmological model.

B Conformal coupling

We now consider the theory described by the action

\[
S_c[g_{\alpha\beta}, \phi] = \int d^4x (\sqrt{-g})^{1/2} \left[ -(16\pi G)^{-1}R + \frac{1}{2} \phi_{,\alpha} \phi_{,\beta} g^{\alpha\beta} + \frac{1}{12} R \phi^2 - W(\phi) \right], \tag{2.20}
\]

where $W(\phi)$ is a potential. In the absence of the potential term or with a quartic potential, the action for the scalar field is known to be conformally invariant (see, for instance, Ref. [17]). The equations obtained by variations of the action (2.20) are more complicated than the ones for a minimally coupled scalar field. But owing to the theorems proved by Bekenstein [9], one can generate solutions for the conformal scalar field from known solutions for the minimal scalar field. The Bekenstein technique, originally formulated for massless fields, can easily be extended to include potential terms [10].

Let us summarize here the result: if $(g_{\alpha\beta}, \varphi)$ forms a solution of the Einstein–ordinary scalar field equations with a potential $V(\varphi)$ then the Einstein–conformal scalar field equations
with the potential
\[ W(\phi) = \varepsilon (1 - \zeta^2 \phi^2)^2 V\left( \frac{\zeta^{-1}}{2} \ln \left| \frac{1 + \zeta \phi}{1 - \zeta \phi} \right| \right) \]
are satisfied by the following two sets:

A. \( \varepsilon = 1 \),
\[
\phi = \zeta^{-1} \tanh(\zeta \varphi),
\]
\[
g_{\mu\nu} = \cosh^2(\zeta \varphi) \bar{g}_{\mu\nu};
\]

B. \( \varepsilon = -1 \),
\[
\phi = \zeta^{-1} \coth(\zeta \varphi),
\]
\[
g_{\mu\nu} = \sinh^2(\zeta \varphi) \bar{g}_{\mu\nu}.
\]

Here \( \zeta = (4\pi G/3)^{1/2} \).

Applying these mappings to our solution for the ordinary scalar field, equations (2.5) and (2.6), we obtain two solutions of the Einstein–conformal scalar field equations with the potential (compare equations (2.2), (2.7) and (2.21))
\[ W(\phi) = \varepsilon (8\pi G)^{-1} a^2 (3 - 4\alpha^2) |1 - \zeta \phi|^{2+2\sqrt{3}\alpha} |1 + \zeta \phi|^{2-2\sqrt{3}\alpha}. \] (2.24)

Thus, we have proved the theorem:

Theorem 2 If \( ds_o^2 = e^{2u} dt^2 - e^{-2u} h_{ij} dx^i dx^j \) - is a static vacuum solution of Einstein’s equations, then
\[ ds^2 = \frac{1}{4} (F + \varepsilon F^{-1})^2 \left\{ \exp(2\delta u + 8\alpha^2 aT) dT^2 - \exp(-2\delta u + 2aT) h_{ij} dx^i dx^j \right\}, \]
(2.25)
\[
\phi = \zeta^{-1} (F^2 - \varepsilon)(F^2 + \varepsilon)^{-1},
\]
(2.26)

where
\[ F = \exp \left[ (4\alpha^2 + 1)^{-1/2} u + 2\alpha aT \right] \] (2.27)
are a pair \((\varepsilon = \pm 1)\) of solutions of the Einstein–conformal scalar field equations with the potential \((2.24)\), the parameters \(\alpha\) and \(\delta\) being related by equation \((2.7)\).

It is evident that the metrics \((2.25)\) are also conformally static. When \(a = 0\), we get Bekenstein’s result \([9]\).

Let us now consider the case of spherical symmetry. The spherically symmetric solutions of the Einstein–conformal scalar field equations are given by expressions \((2.25)\)–\((2.27)\), \((2.16)\) and \((2.17)\). As in the case of the ordinary scalar field, each solution (for \(\varepsilon = \pm 1\)) depends on three parameters, \(m\), \(a\) and \(\alpha\). There is also a ”big bang” singularity as \(aT \to -\infty\). When \(m = 0\), \(\varepsilon = 1\), our solution reduces to the class of cosmological solutions found in Ref. \([10]\).

If \(m \neq 0\), each solution is singular at both \(r = 0\) and \(r = 2m\), except in the case \(\alpha = \frac{1}{2\sqrt{3}}\). The latter case corresponds (for \(a = 0\)) to Bekenstein’s black hole solution \([9, 18]\). In the next section we discuss this case. We will see that, for \(a \neq 0\), \(m \neq 0\), the solution describes the formation of a black hole in a cosmological setting.

### III Collapsing scalar field

#### A The solution

In this section we shall consider the solution found in the preceding section corresponding to \(\alpha = \frac{1}{2\sqrt{3}}\). It is a solution of the Einstein–conformal scalar field equations with the potential

\[
W(\phi) = (H/2\zeta)^2(1 + \zeta\phi)^3(1 - \zeta\phi) ,
\]  

\[(3.28)\]
where we denoted $H = 4a/3$. The solution written in the $(T, r)$–coordinates reads

$$ds^2 = \frac{1}{4} \left( \varepsilon e^{HT/2} (1 - 2m/r)^{1/2} + 1 \right)^2 \left\{ dT^2 - e^{HT} \left( \frac{dr^2}{1 - 2m/r} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right) \right\},$$

(3.29)

$$\phi = (3/4\pi G)^{1/2} \left( \varepsilon e^{HT/2} (1 - 2m/r)^{1/2} - 1 \right) \left( \varepsilon e^{HT/2} (1 - 2m/r)^{1/2} + 1 \right)^{-1},$$

(3.30)

To analyze the properties of the solution we first compute the curvature invariants. The Ricci scalar is found to be

$$R_\alpha^\alpha = -12H^2 \frac{3\varepsilon e^{HT/2} (1 - 2m/r)^{1/2} + 1}{(\varepsilon e^{HT/2} (1 - 2m/r)^{1/2} + 1)^3}. \quad (3.31)$$

If $\varepsilon = +1$, it is bounded for all $r$ and $T$. If $\varepsilon = -1$, the Ricci scalar blows up when $\sqrt{1 - 2m/r} = e^{-HT/2}$.

The Weyl tensor can be shown to belong to type D by Petrov [13], with the only invariant being

$$\lambda = \frac{m}{2rR^2}, \quad (3.32)$$

where

$$R = \frac{1}{2} e^{HT/2} r (\varepsilon e^{HT/2} (1 - 2m/r)^{1/2} + 1) \quad (3.33)$$

is the proper radius of the 2–sphere labeled by $(T, r)$.

It is seen that $\lambda$ blows up as $rR^2 \to 0$. The scalar $R_{\alpha\beta}R^{\alpha\beta}$ has the same kind of singularity. All the curvature invariants are finite at $r = 2m$, therefore, the apparent singularity of the metric at $r = 2m$ can be removed. We shall do this by introducing the new (isotropic) radial coordinate $\tilde{r}$, by the relation:

$$r = \left(1 + \frac{m}{2\tilde{r}}\right)^2 \tilde{r}, \quad 0 < \tilde{r} < \infty.$$

(3.34)
It is also convenient to introduce the new time variable,

$$\tau = -\varepsilon(2H)^{-1}e^{HT/2}.$$  \hspace{1cm} (3.35)

It is evident that the two solutions, with \(\varepsilon = +1\) and \(\varepsilon = -1\), correspond, in fact, to different coordinate patches of the manifold described below (we assume that \(m > 0, H > 0\) in what follows). We will see in Sec. III C that the causal structure of the space–time depends on the value of the dimensionless parameter \(mH\) (see also Table 1).

In the \((\tau, \bar{r})\)-coordinates the metric reads

$$ds^2 = \left(1 + \frac{m}{2\bar{r}} - 2H\tau(1 - \frac{m}{2\bar{r}})\right)^2 \times \left\{\frac{d\tau^2}{H^2\tau^2(1 + \frac{m}{2\bar{r}})^2} - H^2\tau^2(1 + \frac{m}{2\bar{r}})^2(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2)\right\}.$$  \hspace{1cm} (3.36)

The metric is now singular \((R = 0)\) at the surfaces

$$\bar{r}_s = \frac{m}{2}\frac{2H\tau + 1}{2H\tau - 1}.$$  \hspace{1cm} (3.37)

It is easy to see that the solution is invariant under the transformation:

$$\tau \leftrightarrow -\tau,$$

$$\bar{r} \leftrightarrow \frac{m^2}{4\bar{r}}.$$  \hspace{1cm} (3.38)

We shall describe the geometry of space–like surfaces of constant \(\tau\) for \(\tau > 0\) (see Fig. 1).

For \(0 < \tau < (2H)^{-1}\) the spatial sections are asymptotically flat, as \(\bar{r} \rightarrow 0\) and \(\bar{r} \rightarrow \infty\). The two asymptotic regions are connected by a "throat" whose 2–area reaches a minimal value at the surface

$$\bar{r}_{thr} = \frac{m}{2} \sqrt{\frac{1 + 2H\tau}{1 - 2H\tau}}.$$  \hspace{1cm} (3.39)

At time \(\tau = (2H)^{-1}\), the "right" asymptotic region \((\bar{r} \rightarrow \infty)\) shrinks to a singular point and the throat becomes infinitely long, with the area decreasing to zero when approaching the
Figure 1: Qualitative representation of the spatial geometry at different times \( \tau \): (a) \( 0 < \tau < (2H)^{-1} \), (b) \( \tau = (2H)^{-1} \), (c) \( \tau > (2H)^{-1} \). One dimension is suppressed. The "bound" photon orbit \( \bar{r} = m/2 \) is also shown.

"end" of the throat. As time develops further, the singularity cuts off more and more of the throat approaching the "point" \( \bar{r} = m/2 \) as \( \tau \to \infty \). As we shall show, an observer in the "left" universe \( 0 < \bar{r} < m/2 \) will see the appearance of a naked singularity if \( 0 < mH < 1/2 \). If \( mH > 1/2 \), the singularity is hidden within an event horizon which appears to be redshifted to observers outside the horizon. The horizon "freezes" at \( \bar{r} = m/2 \) as \( \tau \to \infty \), but its area increases with the cosmological expansion of the universe (see equation (3.49) bellow).

B Trajectories of test particles

Let us consider the motion of test particles in the metric (3.36). Since the metric is spherically symmetric, there is one constant of motion regarded as the angular momentum
(per unit rest mass) of a particle,

\[ L = R^2 \frac{d\phi}{ds}. \]  

(3.40)

Here \( s \) is an affine parameter along the particle trajectory, \( R \) is the proper radius \( (3.33) \). In the \((\tau, \bar{r})\)-coordinates,

\[ R^2 = H^2 \tau^2 \bar{r}^2 (1 + \frac{2m}{\bar{r}})^2 \left( 1 + \frac{2m}{\bar{r}} - 2H\tau (1 - \frac{2m}{\bar{r}}) \right)^2. \]  

(3.41)

Without loss of generality we may restrict attention to study of the equatorial motion, \( \theta = \pi/2 \). Since the metric is conformally static, there is another constant of motion for massless particles, the "energy" of a photon,

\[ E = \frac{R^2}{H^2 \tau^2 \bar{r}^2 (1 + \frac{2m}{\bar{r}})^4} d\tau. \]  

(3.42)

Therefore, for isotropic geodesics \((ds = 0)\) one gets:

\[ \frac{R^2 d\bar{r}}{\bar{r} ds} = \pm \sqrt{E^2 \bar{r}^2 (1 + \frac{2m}{\bar{r}})^4 - L^2}, \]  

(3.43)

where the upper sign (+) corresponds to photons moving from the "left" universe towards the "right" universe, the lower sign (−) to photons moving in the opposite direction. It is easy to show that photons with \( L^2 \leq 4m^2E^2 \) can pass through the throat and travel from one universe to another till the "point" where they hit the singularity (equation \( (3.37) \) for \( \tau > 0 \)). The orbit of a photon with \( L^2 \geq 4m^2E^2 \) will have a "turning" point, \( \bar{r}_o \), which obeys the equation:

\[ \bar{r}_o(1 + \frac{2m}{\bar{r}_o})^2 = |L/E|, \]  

(3.44)

the "light bending" effect.

The orbits can be explicitly found in two cases: \( L = 0 \) (radial light rays) and \( L^2 = \)
For radial light rays one easily gets:

\[ \bar{r}_* - r_o = \mp \frac{1}{H^2 \tau}, \]  

(3.45)

where \( r_o \) is a constant and

\[ \bar{r}_* = \bar{r} + m \ln(2\bar{r}/m) - \frac{m^2}{4\bar{r}}. \]  

(3.46)

If \( L = 2mE \), there is one unstable "bound" orbit,

\[ \bar{r} = m/2, \]  

\[ \varphi - \varphi_o = -(2mH\tau)^{-1}, \]  

(3.47)

with \( \tau \) being linear in the affine parameter,

\[ \tau = Es + \tau_o. \]  

(3.48)

Note that the orbit (3.47) is not closed since the proper radius, (3.41), is changing with time,

\[ R_{b}^2 = 4m^2H^2\tau^2. \]  

(3.49)

As \( \tau \to 0 \), the photon on the orbit (3.47), which lies inside the throat, reaches a singularity, \( R_{\alpha\beta}R^{\alpha\beta} \to \infty \). Photons with \( L = 2mE \) moving towards the throat will asymptotically approach the orbit (3.47) as \( \tau \to 0 \). The throat between the two universes is, thus, singular at time \( \tau = 0 \).

It follows from equations (3.42) and (3.43) that, for any \( L \) and \( E \), photons moving away from the throat (in both directions) will asymptotically approach the radial trajectories, (3.43), as \( \tau \to 0 \). All of them reach (one of) the cosmological horizons, \( R_{DS} = H^{-1} \), within a finite proper time, \( s \sim (\tau - \tau_o)/E \). We shall discuss the cosmological horizons in Sec. III D.

Timelike geodesics cannot be found analytically. It can be shown, however, that trajectories of massive test particles also approach the asymptotics (3.47), (3.46) as \( \tau \to 0 \).
Trapped surfaces and the black hole event horizon

The causal structure of the space–time reflects its physical properties. Apparent horizons are surfaces where light wave–fronts are momentarily ”frozen” \[1\]. They are given by the equation:

\[ g^{\alpha \beta} R_{,\alpha} R_{,\beta} = 0 , \]  

where \( R \) is the proper radius (3.41). Equation (3.50) can be solved exactly in our case. One can get, in the \((\tau, \bar{r})\)–coordinates,

\[ g^{\alpha \beta} R_{,\alpha} R_{,\beta} = \frac{\left( a_{2} H^{2} \tau^{2} + a_{1u} H \tau + a_{0} \right) \left( a_{2} H^{2} \tau^{2} + a_{1v} H \tau - a_{0} \right)}{2 H^{6} \bar{r}^{6} \left( 1 + \frac{m}{2 \bar{r}} \right)^{2} \left( 1 + \frac{m}{2 \bar{r}} - 2 H \tau \left( 1 - \frac{m}{2 \bar{r}} \right) \right)^{2}} , \]  

where

\[ a_{0} = H^{3} \bar{r} \left( m^{2} - 4 \bar{r}^{2} \right) , \]
\[ a_{2} = H^{4} \left( m - 2 \bar{r} \right) \left( m + 2 \bar{r} \right)^{3} , \]
\[ a_{1u} = 8 H^{3} m \bar{r}^{2} \left( 2 H m + 1 \right) + H^{3} \left( m - 2 \bar{r} \right)^{2} \left( \frac{1}{4} m^{2} H + 2 \bar{r} + 3 H m \bar{r} + H \bar{r}^{2} \right) , \]
\[ a_{1v} = 8 H^{3} m \bar{r}^{2} \left( 2 H m - 1 \right) + H^{3} \left( m - 2 \bar{r} \right)^{2} \left( \frac{1}{4} m^{2} H - 2 \bar{r} + 3 H m \bar{r} + H \bar{r}^{2} \right) . \]

The numerator on the right hand side of equation (3.51) has in general 4 roots:

\[ H \tau_{u,\pm} ( \bar{r} ) = \left( -a_{1u} \pm \sqrt{a_{1u}^{2} - 4 a_{0} a_{2}} \right) / \left( 2 a_{2} \right) , \]
\[ H \tau_{v,\pm} ( \bar{r} ) = \left( -a_{1v} \pm \sqrt{a_{1v}^{2} + 4 a_{0} a_{2}} \right) / \left( 2 a_{2} \right) . \]

We note here that the functions \( H \tau_{u,\pm} \) and \( H \tau_{v,\pm} \) are in fact functions of \( H \bar{r} \) and \( H m \). They also possess the symmetry (3.38), so we again consider the half plane, \( \tau > 0 \).

The meaning of the functions (3.53) and (3.54) is that \( \tau_{u,\pm} ( \bar{r} ) \) give the outer apparent horizons and \( \tau_{v,\pm} ( \bar{r} ) \) the inner apparent horizons \[1\]. We refer to radial light rays whose
trajectories obey the equation (compare equations (3.42) and (3.43))

\[
(1 + \frac{m}{2\bar{r}})^2 \frac{d\bar{r}}{d\tau} = \pm \frac{1}{H^{2\tau^2}},
\]

as to outgoing (−) and ingoing (+) rays, respectively.

The existence, in general, of two outer and two inner apparent horizons reflects the fact that our solution describes the formation of a black hole in a cosmological setting. Two (outer and inner) apparent horizons cover the black hole singularity given by equation (3.37) for \( \tau > 0 \). The other ones surround the "inner" \( 0 < \bar{r} < m/2 \) or "outer" \( \bar{r} > m/2 \) cosmological horizons which correspond to \( \tau = 0 \). We shall say more about the cosmological horizons in the next subsection. The situation described above is "normal", in the sense that one can clearly separate out the two phenomena, the cosmological expansion and the collapse of the scalar field with the formation of a black hole. If one changes the parameters \( m \) and/or \( H \) of the solution, the picture is also changed. We summarize here some properties of the apparent horizons (see Table 1).

The behaviour of the horizons depends on the value of the dimensionless parameter \( mH \). The functions \( \tau_{v \pm}(\bar{r}) \) are real for all \( \bar{r} > 0 \). If \( mH > 1/2 \), \( \tau_{v-}(\bar{r}) \) grows from zero (at \( \bar{r} = 0 \)), reaches a maximum at some point \( \bar{r}_m \), \( 0 < \bar{r}_m < m/2 \), and then again decreases to zero (at \( \bar{r} = m/2 \)). The function \( \tau_{v+}(\bar{r}) \) monotonically decreases from infinity (at \( \bar{r} = m/2 \)) to \((4H)^{-1}\) (as \( \bar{r} \to \infty \)).

If \( 0 < mH < 1/2 \), \( \tau_{v-}(\bar{r}) \) monotonically increases from zero (at \( \bar{r} = 0 \)) to infinity (at \( \bar{r} = m/2 \)), while \( \tau_{v+}(\bar{r}) \) changes from zero (at \( \bar{r} = m/2 \)) to \((4H)^{-1}\) (as \( \bar{r} \to \infty \)).

If \( mH = 1/2 \), the function \( \tau_{v-}(\bar{r}) \) is negative (\( \tau_{v+}(\bar{r}) \) is positive) for all \( \bar{r} > 0 \), but instead another inner apparent horizon appears which is situated at \( \bar{r} = m/2 \). The function
| parameter range | black hole singularity | black hole inner apparent horizon | black hole outer apparent horizon |
|-----------------|------------------------|-----------------------------------|----------------------------------|
| $0 < mH < 1/2$  | naked                  | does not globally exist            | does not globally exist          |
| $1/2 \leq mH < h_o$ | hidden within the global timelike at late times | inside the event horizon |
| $h_o \leq mH < 3/2$ | event horizon inside the event horizon |
| $mH \geq 3/2$    | event horizon inside the event horizon |

Table 1: Some properties of the cosmological black hole solution for different values of the parameter $mH$

$\tau_{v+}(\bar{r})$ monotonically grows from zero (at $\bar{r} = 0$), reaches the maximal value $(2H)^{-1}$ at $\bar{r} = m/2$ and then monotonically decreases to $(4H)^{-1}$ ($\bar{r} \rightarrow \infty$).

The value $1/2$ of the parameter $mH$ is thus critical for the behaviour of ingoing light rays. If $mH > 1/2$, all ingoing light rays emitted in the vicinity of the "inner" cosmological horizon first converge and then begin to diverge due to the cosmological expansion. The light rays that have passed the throat (at $\bar{r} = m/2$) will be again converging after some time and they will hit the singularity (where the area of the light–fronts vanishes) within a finite proper time (affine parameter).

There is another interesting property of the black hole inner apparent horizon ($\tau_{v+}$). It can be shown that it is spacelike everywhere (for $\bar{r} > m/2$) if $mH > 3/2$. If $1/2 < mH < 3/2$, there exist a "point", $\bar{r}_c$, such that the black hole inner apparent horizon is timelike for $m/2 \leq \bar{r} < \bar{r}_c$. This means that some ingoing light rays begin to converge before they enter
the black hole region (see the discussion below).

If $0 < mH \leq 1/2$, there are some (one if $mH = 1/2$) ingoing light–fronts whose area is non–increasing all the time. Such a behaviour is normal in asymptotically flat space–times, but it is anomalous in an expanding universe.

The fact that $mH = 1/2$ is a critical value where the solution qualitatively changes, becomes more evident if one studies the character of the singularity (see equation (3.37)). It is easy to show that, if $mH > 1/2$, the singularity is spacelike and hidden within the black hole event horizon. If $mH \leq 1/2$, the singularity becomes isotropic at some time $\tau_i$ ($\tau_i \to \infty$ if $mH = 1/2$), and it is timelike for $\tau > \tau_i$. Thus, some observers who have succeeded to escape to infinity will see the appearance of a naked singularity, if $0 < mH < 1/2$.

To locate the black hole event horizon, we shall consider outgoing radial light rays. They propagate along the trajectories (see equations (3.45) and (3.46)):

$$\bar{r} + m \ln\left(\frac{2\bar{r}}{m}\right) - \frac{m^2}{4\bar{r}} = \frac{1}{H^2\tau} + u,$$

(3.56)

where $u$ is a parameter. The left hand side of equation (3.56) is negative for $0 < \bar{r} < m/2$ and positive for $\bar{r} > m/2$. It is clear that all outgoing light rays with $u > 0$ hit the singularity within a finite time (see equation (3.37) for $\tau > 0$). On the other hand, if $mH \geq 1/2$, all outgoing light rays with $u < 0$ escape to future null infinity ($\tau \to \infty$). Therefore, the black hole event horizon is given by equation (3.56) with $u = 0$. If $mH \geq 1/2$, the black hole horizon is global and it separates the black hole region from the rest of the universe: any timelike observer or null ray in the black hole region inevitably falls to the singularity, while observers and null rays outside the black hole region can escape to infinity and will never see the singularity. If $0 < mH < 1/2$, there is no global black hole event horizon. For, as
we discussed, some observers at infinity can see the singularity in this case. Note here that future null infinity \((I^+)\) is spacelike since the metric (3.36) asymptotically \((\bar{r} \to 0)\) approaches the Robertson–Walker one (see Ref. [1]). The Penrose diagrams for different values of the parameter \(mH\) are shown in Fig. 2.

We shall now consider the outer apparent horizons, equation (3.53). The analysis of the functions \(\tau_{u\pm}(\bar{r})\) shows that they are real and positive for all \(\bar{r} > m/2\) if \(mH \geq h_o\), where \(h_o \simeq 1.1945\). The function \(\tau_{u-}(\bar{r})\) grows from zero (at \(\bar{r} = m/2\)), reaches its maximal value at some point and then again decreases to zero \((\bar{r} \to \infty)\), while \(\tau_{u+}(\bar{r})\) changes from infinity (at \(\bar{r} = m/2\)) to \((4H)^{-1}\) \((\bar{r} \to \infty)\). Thus, all outgoing light–fronts emitted in the vicinity of the "outer" cosmological horizon first decrease and then increase in area. The light rays with \(u > 0\) will then again converge and reach the singularity. Note that the black hole outer apparent horizon \((\tau_{u+})\) is spacelike and lies inside the event horizon, if \(mH \geq h_o\).

If \(0 < mH < h_o\), the functions \(\tau_{u\pm}(\bar{r})\) are real and positive for \(m/2 \leq \bar{r} \leq \bar{r}_1\) and for \(\bar{r} \geq \bar{r}_2\), where \(H\bar{r}_1\) and \(H\bar{r}_2\) depend on the value of the parameter \(mH\). At \(\bar{r} = \bar{r}_1\) and \(\bar{r} = \bar{r}_2\) the function \(\tau_{u-}(\bar{r})\) smoothly joins \(\tau_{u+}(\bar{r})\). Thus, in this case the singularity is not entirely covered by the outer apparent horizon and there are some (one if \(mH = h_o\)) outgoing light–fronts whose area is non–increasing all the time.

D The cosmological horizons

We saw in Sec. [1] that all ingoing (outgoing) light rays reach the inner (outer) cosmological horizon, \(R_{DS} = H^{-1}\), as \(\tau \to 0\). Since the metric (3.36) becomes asymptotically \((\bar{r} \to 0\) or \(\infty)\) de Sitter as \(\tau \to 0\), it would seem that one could analytically extend the metric beyond the cosmological horizons \((\tau = 0)\). As we shall show, this is not the case.
Figure 2: Penrose diagrams of the cosmological black hole solution for different values of the parameter $mH$: (a) $mH > \frac{3}{2}$, (b) $h_o < mH < \frac{3}{2}$, (c) $mH = h_o$, (d) $1/2 < mH < h_o$, (e) $mH = 1/2$, (f) $0 < mH < 1/2$. The dashed curves with long segments represent the inner apparent horizons ($\tau_v\pm$), the dashed curves with short segments the outer apparent horizons ($\tau_u\pm$). The bold curve is the black hole future singularity, the point $S$ is the initial ($\tau = 0, 0 < \bar{\bar{r}} < \infty$) singularity. The dotted lines represent the (singular) cosmological horizons. The solid curve going from the left edge ($\bar{v}$) to the right one represents the Cauchy hypersurface $\tau = (2H)^{-1}$. The left and right edges represent "left" ($\bar{\bar{r}} \to 0$) and "right" ($\bar{\bar{r}} \to \infty$) spatial infinities, respectively. The solid line is the black hole event horizon.
Let us write the metric in the "hyperbolic" coordinates \([19]\), \((\eta, \chi)\). They are given by the relations:

\[
\bar{r}_* = \frac{1}{2} \frac{\sin \chi}{\cos(\frac{\eta}{2}) \cos(\frac{\chi}{2})},
\]

\[
(H^2 \tau)^{-1} = \frac{1}{2} \frac{\sin \eta}{\cos(\frac{\eta}{2}) \cos(\frac{\chi}{2})},
\]

where \(\bar{r}_*\) is given by equation \((3.46)\). In the \((\eta, \chi)\)–coordinates the metric takes the form

\[
ds^2 = \frac{1}{H^2 \sin^2 \eta} (d\eta^2 - d\chi^2) - R^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \tag{3.58}
\]

where \(R\) is now a function of \(\eta\) and \(\chi\),

\[
R^2(\eta, \chi) = \frac{\sin^2 \chi}{H^2 \sin^2 \eta} \frac{\bar{r}_*(1 + \frac{m}{\bar{r}_*})^4}{\bar{r}_*^2}.
\]

The cosmological horizons correspond to \(\eta \pm \chi = \pi (2n + 1), n = 0, \pm 1, \cdots\). Unfortunately, the function \(R(\eta, \chi)\) is not analytic across the horizons. Indeed, from equations \((3.57),(3.59)\) one can get, for example \((\chi \neq \pi n)\):

\[
\frac{1}{R} \frac{\partial R}{\partial \chi} \rightarrow \mp \frac{m}{2} \cot \chi \ln(\cos(\frac{\eta + \chi}{2})) \text{, when } \eta \pm \chi \rightarrow \pi (2n + 1). \tag{3.60}
\]

Therefore, some components of the Riemann tensor (which involve the derivatives of \(R(\eta, \chi)\)) blow up, though all the curvature invariants remain finite at the cosmological horizons (see the discussion after equation \((3.33)\)). We shall prove here that such a situation occurs in any coordinate system, namely, some components of the Riemann tensor along geodesics blow up when the geodesics approach one of the cosmological horizons (this is a kind of the "parallelly propagated curvature singularity" \([20]\) or null singularity). In fact, we shall show that the metric is only \(C^0\) across the cosmological horizon. Owing to the spherical symmetry it is sufficient to consider the following form of the metric:

\[
ds^2 = g_{ab}(x^c)dx^a dx^b - R^2 (x^c)(d\theta^2 + \sin^2 \theta d\varphi^2), a, b, c = 0, 1. \tag{3.61}
\]
Consider any timelike or null geodesic, \(x^\alpha(s)\), and suppose that the 2–metric \(g_{ab}\) is analytic across the horizon. Thus, \(k^\alpha \equiv \frac{dx^\alpha}{ds}\) is finite there. For simplicity we assume the motion is equatorial (\(\theta = \pi/2\)). One can prove then that the following relation holds:

\[
k^\alpha k^\beta R_{\alpha\varphi\beta\varphi} = R \frac{d^2 R}{d\tau^2} .
\]

Equation (3.62) can be proved by considering, for example, the geodesic deviation equation [11].

On the other hand, it was mentioned in Sec. III that all particles approach the horizon along the trajectories (3.45), (3.46), with \(\tau\) being linear in the affine parameter, \(\tau \sim E(s - s_0)\). For such trajectories one gets (see equation (3.41)):

\[
R \sim H^{-1} \mp mH \tau \ln |mH^2 \tau| + O(\tau^2 \ln \tau) .
\]

From equations (3.62) and (3.63) we then obtain:

\[
k^\alpha k^\beta R_{\alpha\varphi\beta\varphi} \sim \mp \frac{mHE}{s - s_0} + O(\ln |s - s_0|) .
\]

Therefore, the components \(R_{\alpha\varphi\beta\varphi}\) (and \(R_{\alpha\beta\theta\theta}\)) blow up at the cosmological horizon. An extended body will rotate with an infinite angular velocity when it approaches the cosmological horizon.

IV Concluding remarks

One of the results presented in this work is the cosmological black hole solution which is discussed in Section III. It displays some interesting features. At earlier times (0 < \(\tau < (2H)^{-1}\)) there are two universes connected by a ”throat” which is traversable to light rays and observers. One of the universes then collapses and a future singularity is
formed (for $\tau < 0$ the solution describes a white hole in the cosmological setting). Depending on the value of the parameter, $mH$, there are two possibilities: either some observers in the second (expanding) universe will see the naked singularity ($0 < mH < 1/2$) or the singularity will be hidden within a global event horizon ($mH \geq 1/2$) which appears to be redshifted to external observers.

In the first case cosmic censorship [21] is manifestly violated. It is not a serious violation yet since the model we have considered is not generic. For instance, the stress–energy tensor obtained from the action (2.20) does not satisfy the (strong) energy condition [1]. Note, however, that another counterexample to the cosmic censorship conjecture was recently obtained in Ref. [22].

Another interesting feature of our solution is that the metric is not smooth across the cosmological horizons (if $m \neq 0$). The singularity is rather mild: radial timelike and null geodesics can pass through it without being affected, but an extended body will be torn to pieces by fast rotation. The lack of smoothness at some horizons was also observed in other exact solutions (see Refs. [22, 23]). In our case, this can be understood along the following line of arguments. In the theory described by the action (2.20), one can define the effective gravitational constant,

$$G_{\text{eff}} = G(1 - \frac{4\pi}{3}G\phi^2)^{-1}. \quad (4.65)$$

It is easy to show that $G_{\text{eff}} \to \infty$ as $\tau \to 0$ (compare equations (3.30), (3.34), (3.35) and (4.65)). On the other hand, we saw that the throat is "visible" to distant observers as a massive body (the "light bending" effect, etc.). As the throat becomes singular at $\tau = 0$, the observers will feel its "tail" via the Newtonian potential. If the "mass parameter" $m = 0$, the (single) cosmological horizon is regular. In the latter case, the solution describes a
homogeneous isotropic universe with a "big bang" (or "big crash") singularity.

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