The long $n$-exact sequence theorem in $n$-abelian categories

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Abstract

In this note, we show that the long $n$-exact sequence theorem holds in any $n$-abelian category. As an application, the $n$-abelian version of Wakamatu’s Lemma is given.

Key words: $n$-abelian category; $n$-derived functor; $n$-Baer sum; long $n$-exact sequence theorem.

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1 Introduction

Let $0 \to A \to B \to C \to 0$ be an short exact sequence of injectively cogenerated abelian category $A$. Then there exists a long exact sequence of cohomology associated to this short exact sequence. For example, assume that $F$ is a covariant left exact functor from $A$ to an abelian category $B$, it is well known that there exists a long exact sequence of cohomology

$$0 \to FA \to FB \to FC \to R^1F(A) \to R^1F(B) \to R^1F(C) \to \cdots$$

where $R^iF$ is the $i$-th right derived functor. In particular, there exists a long exact sequence of abelian groups

$$0 \to A(X,A) \to A(X,B) \to A(X,C) \to \text{Ext}^1_A(X,A) \to \text{Ext}^1_A(X,B) \to \text{Ext}^1_A(X,C) \to \cdots$$

For projectively generated abelian category, we also have similar conclusions above.

In any abelian category $A$, we can define $\text{Ext}^1_A(A,B)$ (even if it has no projectives and injectives) to be the class of equivalence classes of extensions under Baer sum by using pushout and pullback. Baer’s description of $\text{Ext}^1_A(A,B)$ as extension $E(A,B)$ has been generalized to a description of $\text{Ext}^m_A(A,B)$ for all $m \geq 1$ by Yoneda etc. Elements of $\text{Ext}^m_A(A,B)$ are certain equivalence classes of $m$-fold exact sequences and can also be viewed as a generalization of Baer sum, see [2, 15, 20, 17] [18, 19, 18]. The exactness of the long exact sequence without the use of projectives or injectives was proved by Schanuel [16] VII Proposition 2.2 and Theorem 5.1.

In both higher dimensional Auslander-Reiten theory and higher homological algebra, $n$-cluster tilting subcategories of abelian and triangulated categories play a fundamental role [6, 7, 8]. This was a starting point for developing the theory of $(n + 2)$-angulated categories and $n$-abelian categories in the sense of of Geiss-Keller-Oppermann [5] and Jasso [10]. Jasso [10] proved that any $n$-cluster tilting subcategory of an

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abelian category is $n$-abelian, and any small projectively generated can be embedded into the abelian category. Building on work of Jasso, Kvanne [12] showed that any projectively generated $n$-abelian category is equivalent to an $n$-cluster tilting subcategory of an abelian category with enough projectives. Ebrahimi and Nasr-Isfahani [3] proved that any $n$-abelian category can be embedded into an injectively cogenerated abelian category, which is a higher dimensional version of the Freyd-Mitchell embedding theorem for $n$-abelian categories.

The first author [14] studied the homological theory in $n$-abelian categories, defined $n$-exact functor, $n$-resolution functor and $n$-derived functor. The notion of $n$-derived functor $n\text{Ext}^i(-,-)$ of Hom-functor $\text{Hom}(-,-)$ can be viewed as a generalization of $\text{Ext}^i(-,-)$. In an $n$-abelian category, $n\text{Ext}^i(B,A)$ denote the equivalence classes of $i$-fold $n$-exact sequences starting at $A$ and ending with $B$. Let $\mathcal{A}$ be a small $n$-abelian category with enough injectives, and $F$ be a covariant left $n$-exact functor from $\mathcal{A}$ to an abelian category $\mathcal{B}$, there exists a long exact sequence

$$
0 \to FX^0 \to FX^1 \to \cdots \to FX^{n+1} \xrightarrow{\partial_n} nR^{1}F(X^0) \to nR^{1}F(X^1) \to \cdots \to nR^{1}F(X^{n+1})
$$

for any $n$-exact sequence $X^0 \to X^1 \to \cdots \to X^{n+1}$. It is called the long $n$-exact sequence theorem, where $nR^{1}F$ is the $i$-th left $n$-derived functor.

In this note, we consider the higher dimensional analogue of the long exact sequence theorem of $n$-extentions for $n$-abelian categories without the use of projectives or injectives. Our main result is the following.

**Theorem 1.1.** (see Theorem 3.3 for details) Let $\mathcal{A}$ be an $n$-abelian category and $X : X^0 \xrightarrow{\alpha^0} X^1 \xrightarrow{\alpha^1} \cdots \xrightarrow{\alpha^n} X^{n+1}$ an $n$-exact sequence in $\mathcal{A}$. Then the sequences of abelian groups

$$
0 \to \mathcal{A}(A, X^0) \to \cdots \to \mathcal{A}(A, X^{n+1}) \xrightarrow{X^*} n\text{Ext}^i_{\mathcal{A}}(A, X^0) \xrightarrow{\langle\alpha^0\rangle^{i+1}} \cdots \xrightarrow{\langle\alpha^n\rangle^{i+1}} n\text{Ext}^i_{\mathcal{A}}(A, X^{n+1})
$$

and

$$
0 \to \mathcal{A}(X^{n+1}, A) \to \cdots \to \mathcal{A}(X^0, A) \xrightarrow{X^*} n\text{Ext}^i_{\mathcal{A}}(X^{n+1}, A) \xrightarrow{\langle\alpha^0\rangle^{i+1}} \cdots \xrightarrow{\langle\alpha^n\rangle^{i+1}} n\text{Ext}^i_{\mathcal{A}}(X^0, A)
$$

are exact for any $A \in \mathcal{A}$.

As an application, we show that the $n$-abelian version of Wakamatsu’s Lemma, see Theorem 4.2.

This article is organized as follows. In Section 2, we recall the definitions of $n$-abelian categories, $n$-pushout diagram, cohomology and $n$-extension group, and give some properties which are needed in the sequel. In Section 3, we prove the long $n$-exact sequence theorem in any $n$-abelian category. In Section 4, we show that $n$-abelian version of Wakamatsu’s Lemma.
2 Definitions and preliminaries

2.1 \( n \)-abelian categories

Let \( n \) be a positive integer and \( C \) an additive category. We denote the category of complexes in \( C \) by \( \text{Ch}(C) \). For convenience, we denote by \( \text{Ch}^n(C) \) the full subcategory of \( \text{Ch}(C) \) given by all complexes

\[
X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1}
\]

which are concentrated in degrees 0, 1, \ldots, \( n+1 \). We denote \( \mathcal{C}(X,Y) \) the set of morphisms from \( X \) to \( Y \) in \( C \) if \( X,Y \in C \).

Let \( d^0 \in \mathcal{C}(X^0,X^1) \). Then an \( n \)-cokernel of \( d^0 \) is a sequence of morphisms

\[(d^1,\ldots,d^n) : X^1 \xrightarrow{d^1} X^2 \xrightarrow{d^2} \cdots \xrightarrow{d^n} X^{n+1}\]

such that for all \( 1 \leq k \leq n-1 \) the morphism \( d^k \) is a weak cokernel of \( d^{k-1} \), and \( d^n \) is moreover a cokernel of \( d^{n-1} \). In this case, we say the sequence (2.1)

is right \( n \)-exact \([13, \text{Definition 2.4}]\). The concepts of \( n \)-kernel and left \( n \)-exact are defined dually. If \( n \geq 2 \), the \( n \)-cokernels and \( n \)-kernels are not unique in general, but their are unique up to isomorphism in the homotopy category of \( C \), see \([10]\).

A sequence (2.1) is called an \( n \)-exact sequence if it is both right \( n \)-exact and left \( n \)-exact.

As a generalization of classical abelian categories, Jasso introduced a notion of \( n \)-abelian categories in \([10]\).

**Definition 2.1.** \([10, \text{Definition 3.1}]\) An additive category \( \mathcal{A} \) is called \( n \)-abelian category if it satisfies the following axioms:

(A0) The category \( \mathcal{A} \) is idempotent complete.

(A1) Every morphism in \( \mathcal{A} \) has an \( n \)-kernel and an \( n \)-cokernel.

(A2) For every monomorphism \( f^0: X^0 \rightarrow X^1 \) in \( \mathcal{A} \), and for every \( n \)-cokernel \((f^1,f^2,\ldots,f^n)\), the following sequence is \( n \)-exact:

\[
X^0 \xrightarrow{f^0} X^1 \xrightarrow{f^1} \cdots \xrightarrow{f^{n-1}} X^n \xrightarrow{f^n} X^{n+1}.
\]

(A2\textsuperscript{op}) For every epimorphism \( g^n: X^n \rightarrow X^{n+1} \) in \( \mathcal{A} \), and for every \( n \)-kernel \((g^0,g^1,\ldots,g^{n-1})\), the following sequence is \( n \)-exact:

\[
X^0 \xrightarrow{g^0} X^1 \xrightarrow{g^1} \cdots \xrightarrow{g^{n-1}} X^n \xrightarrow{g^n} X^{n+1}.
\]

Note that 1-abelian categories are precisely abelian categories in the usual sense since abelian categories are idempotent complete.

Recall that a subcategory \( \mathcal{D} \) of an abelian category \( \mathcal{A} \) is cogenerating if for every object \( X \in \mathcal{A} \) there exists an object \( Y \in \mathcal{D} \) and a monomorphism \( X \rightarrow Y \). The concept of generating subcategory is defined dually. Let \( \mathcal{A} \) be an \( n \)-abelian category. We say that \( \mathcal{A} \) is projectively generated if for every object \( A \in \mathcal{A} \)
there exists a projective object \( P \in \mathcal{A} \) and an epimorphism \( P \to A \). The notion of \textit{injectively cogenerated} \( n \)-abelian category is defined dually.

Let \( \mathcal{A} \) be an abelian category and \( \mathcal{D} \) a generating-cogenerating full subcategory of \( \mathcal{A} \). \( \mathcal{D} \) is called an \textit{n-cluster-tilting subcategory} of \( \mathcal{A} \) if \( \mathcal{D} \) is functorially finite in \( \mathcal{A} \) and

\[
\mathcal{D} = \{ X \in \mathcal{A} \mid \forall i \in \{1, \ldots, n-1\} \quad \text{Ext}^i_{\mathcal{A}}(X, \mathcal{D}) = 0 \} = \{ X \in \mathcal{A} \mid \forall i \in \{1, \ldots, n-1\} \quad \text{Ext}^i_{\mathcal{D}}(\mathcal{D}, X) = 0 \}.
\]

Note that \( \mathcal{A} \) itself is the unique 1-cluster-tilting subcategory of \( \mathcal{A} \).

\textbf{Lemma 2.2.} \cite{10} Theorem 3.16 and \cite{12} Theorem 1.3] Let \( \mathcal{A} \) be an abelian category and \( \mathcal{D} \) an \textit{n-cluster tilting subcategory} of \( \mathcal{A} \). Then \( \mathcal{D} \) is an \( n \)-abelian category. Conversely, every small projectively generated (or injectively cogenerated) \( n \)-abelian category can be embedded in an abelian category as an \( n \)-cluster tilting subcategory.

There are some examples of projectively generated (or injectively cogenerated) \( n \)-abelian category which are not \( n \)-cluster tilting subcategories of the given abelian categories.

\textbf{Example 2.3.} Let \( \Lambda_n \) be a finite-dimensional algebra given by the quiver

\[
n \xrightarrow{a_n} n - 1 \xrightarrow{a_{n-1}} \cdots \xrightarrow{a_2} 1 \xrightarrow{a_1} 0
\]

with relations \( a_ia_{i+1} = 0 \) for any \( 1 \leq i \leq n - 1 \), it is a basic absolutely \( (n+1) \)-complete \( n \)-Auslander algebra. Then the Auslander-Reiten quiver of \( \Lambda_n \) is the following, where \( P_i \) and \( S_i \) are projective and simple modules associated with the vertex \( i \), respectively.

\[
\begin{array}{ccccccc}
P_0 & \xleftarrow{P_1} & S_0 & \rightarrow & S_1 & \rightarrow & S_2 & \rightarrow & \cdots & \rightarrow & S_{n-2} & \rightarrow & S_{n-1} & \rightarrow & P_n & \rightarrow & S_n
\end{array}
\]

\( \text{add}(\Lambda_n \oplus S_n) \) is the unique \( n \)-cluster tilting subcategory of \( \text{mod} \Lambda_n \), and \( \text{End}_{\Lambda_n}^{\text{op}}(\Lambda_n \oplus S_n) \simeq \Lambda_{n+1} \). One can easily check that there are isomorphisms of algebras \( \text{End}_{\Lambda_n}^{\text{op}}(\bigoplus_{0 \leq i \leq j} P_i) \simeq \Lambda_j \) for any \( 0 \leq j \leq n \), this implies that \( \text{add}(\bigoplus_{0 \leq i \leq j} P_i) \) is a \( j \)-abelian category for any \( 1 \leq j \leq n \), but it is not an \( n \)-cluster tilting subcategory of \( \text{mod} \Lambda_n \).

\section{2.2 \( n \)-pullback diagrams and \( n \)-pushout diagrams}

Let \( \mathcal{C} \) be an additive category and \( f : X \to Y \) a morphism in \( \text{Ch}^{n-1}(\mathcal{C}) \). We have the following commutative diagram:

\[
\begin{array}{cccccccc}
X & \xrightarrow{f} & X^0 & \xrightarrow{d_0^1} & X^1 & \xrightarrow{d_1^2} & \cdots & \xrightarrow{d_{n-2}^{n-1}} & X^{n-1} & \xrightarrow{d_{n-1}^{n}} & X^n \\
\downarrow{f} & & \downarrow{f^0} & & \downarrow{f^1} & & \cdots & & \downarrow{f^{n-2}} & & \downarrow{f^{n-1}} & & \downarrow{f^n} \\
Y & \xrightarrow{g} & Y^0 & \xrightarrow{d_0^1} & Y^1 & \xrightarrow{d_1^2} & \cdots & \xrightarrow{d_{n-2}^{n-1}} & Y^{n-1} & \xrightarrow{d_{n-1}^{n}} & Y^n
\end{array}
\]

The \textit{mapping cone} \( C = C(f) \in \text{Ch}^{n}(\mathcal{C}) \) is

\[
X^0 \xrightarrow{d_0^1} X^1 \oplus Y^0 \xrightarrow{d_0^1} \cdots \xrightarrow{d_{n-2}^{n-1}} X^n \oplus Y^{n-1} \xrightarrow{d_{n-1}^{n}} Y^n
\]
where
\[
d_C^k = \left( \begin{array}{cc} -d_{X}^{k+1} & 0 \\
 f_{Y}^{k+1} & d_{Y}^{k} \end{array} \right) : X^{k+1} \oplus Y^{k} \to X^{k+2} \oplus Y^{k+1}
\]
for each \( k \in \{0, 1, \ldots, n-2\} \), \( d_C^{-1} = \left( \begin{array}{c} -d_{X}^{0} \\
 f_{Y}^{0} \end{array} \right) \) and \( d_C^{-n-1} = (f^n, d^n_{X}^{-1}) \).

(1) The diagram \( \text{(2.2)} \) is called an \text{n-pullback diagram of } \( Y \text{ along } f^n \) if \( C(f) \) is left \( n \)-exact;
(2) The diagram \( \text{(2.2)} \) is called an \text{n-pushout diagram of } \( X \text{ along } f^0 \) if \( C(f) \) is right \( n \)-exact;
(3) The diagram \( \text{(2.2)} \) is called an \text{n-bicartesian(or n-exact diagram) } \) if \( C(f) \) is \( n \)-exact.

Jasso in [10] proved that the \( n \)-pullback (resp. pushout) exists for any \( X \in \text{Ch}^{n-1}(C) \) (resp. \( Y \)) along \( f_0 \) (resp. \( f^n \)), and is unique up to homotopy equivalence.

Lemma 2.4. [14] Proposition 2.7] Let \( C \) be an additive category which satisfies axioms (A0) and (A1), and \( X \in \text{Ch}^n(C) \) be a right \( n \)-exact sequence, for any \( f^0 : X^0 \to Y^0 \) be a morphism. Then, for any \( n \)-pushout diagram of \( (d^n_{X}, \ldots, d^n_{X}^{-1}) \) along \( f^0 \)
\[
\begin{array}{cccccccc}
X^0 & d^n_{X} & X^1 & \cdots & X^{n-1} & d^{n-1}_{X} & X^n & d^n_{X} \\
\downarrow f^0 & \downarrow 1 & \downarrow 1 & \cdots & \downarrow f^{n-1} & \downarrow 1 & \downarrow f^n & \\
Y^0 & d^n_{Y} & Y^1 & \cdots & Y^{n-1} & d^{n-1}_{Y} & Y^n & \cdots \to X^{n+1} 
\end{array}
\]
there exists a cokernel \( d^n_{X} : Y^n \to X^{n+1} \) of \( d^{n-1}_{X} \) such that the diagram is commutative and the bottom row is right \( n \)-exact. Moreover, if \( d^0_{X} \) is a monomorphism, both rows are \( n \)-exact sequences.

2.3 \text{(co)Homology of } n \text{-abelian categories}

Now we recall the right (resp. left) \( n \)-derived functors of covariant or contravariant left (resp. right) \( n \)-exact functors.

Let \( A \) be an \( n \)-abelian category, \( B \) an abelian category, and \( F : A \to B \) a covariant additive functor.

(i) \( F \) is called \text{left } n\text{-exact} if for any left \( n \)-exact sequence \( X^0 \overset{d^0}{\to} X^1 \overset{d^1}{\to} \cdots \overset{d^{n-1}}{\to} X^n \overset{d^n}{\to} X^{n+1} \) in \( A \),
\[
0 \to FX^0 \to FX^1 \to \cdots \to FX^n \to FX^{n+1} \text{ is an exact sequence of } B.
\]

(ii) \( F \) is called \text{right } n\text{-exact} if for any right \( n \)-exact sequence \( X^0 \overset{d^0}{\to} X^1 \overset{d^1}{\to} \cdots \overset{d^{n-1}}{\to} X^n \overset{d^n}{\to} X^{n+1} \) in \( A \),
\[
FX^0 \to FX^1 \to \cdots \to FX^n \to FX^{n+1} \to 0 \text{ is an exact sequence of } B.
\]

(iii) \( F \) is called \text{n-exact} if for any \( n \)-exact sequence \( X^0 \overset{d^0}{\to} X^1 \overset{d^1}{\to} \cdots \overset{d^{n-1}}{\to} X^n \overset{d^n}{\to} X^{n+1} \) in \( A \),
\[
0 \to FX^0 \to FX^1 \to \cdots \to FX^n \to FX^{n+1} \to 0 \text{ is an exact sequence of } B.
\]
The notions of covariant (contravariant) additive left (right) exact functors are defined dually. For example, the hom-functors \( A(X, -) \) (resp. \( A(-, X) \)) is covariant (resp. contravariant) left \( n \)-exact by the definition of \( n \)-kernel (resp. \( n \)-cokernel).

Lemma 2.5. [4] Proposition 3.2] Let \( A \) be an \( n \)-abelian category and \( B \) an abelian category. A covariant functor \( F : A \to B \) is an \( n \)-exact functor if and only if it is both left and right \( n \)-exact functor.

Let \( P \) (resp. \( I \)) be the category of projective (resp. injectives) objects in an abelian category \( A \). We say that \( A \) has enough projectives if for every object \( M \in A \) there exist \( P_1, P_2, \ldots, P_n \in P \) and an \( n \)-exact
sequence $N \to P_n \to \cdots \to P_1 \to M$. Write $\Omega_n M = N$, and we have $n$-exact sequences

$$\cdots, \Omega_n^i M \xrightarrow{j_i} P_n \xrightarrow{d_n} \cdots \to P_{(i-1)n+1} \xrightarrow{\pi_{i-1}} \Omega_n^{i-1} M, \quad \cdots, \quad \Omega_n M \xrightarrow{j_1} P_n \xrightarrow{d_n} \cdots \to P_1 \xrightarrow{\pi_0} M$$

Then we can “splice” these $n$-exact sequences and write $d_{in+1} = j_i \pi_i$, thus we have a sequence

$$\cdots \to P_{3n} \xrightarrow{d_{3n}} \cdots \to P_{2n+1} \xrightarrow{d_{2n+1}} P_{2n} \xrightarrow{d_n} P_{n+1} \xrightarrow{d_n} \cdots \to P_n \xrightarrow{d_n} \cdots \to P_1 \xrightarrow{d_1} M \quad (2.3)$$

which was called a projective $n$-resolution of $M$, simply write $P_\bullet \xrightarrow{d_\bullet} M$. We call $\Omega_n^k M$ the $k$-th $n$-syzygy of $M$ for $k \geq 0$. The notions of injective $n$-resolution, $k$-th $n$-cosyzygy $\Omega_n^{-k} M$ of $M$ are defined dually.

For objects $M, N \in A$, we denote by $\mathcal{P}(M, N)$ (resp. $\mathcal{I}(M, N)$) the subgroup of $\mathcal{A}(M, N)$ of morphisms which factor through a projective (resp. injective) object. The projective (resp. injective) stable category of $A$, denoted by $\mathcal{A}$ (resp. $\mathcal{A}$), is the category with the same objects as $A$ and with morphisms groups defined by

$$\mathcal{A}(M, N) := A(M, N)/\mathcal{P}(M, N), \quad (resp. \mathcal{A}(M, N) := A(M, N)/\mathcal{I}(M, N)).$$

**Remark 2.6.** Similar to an abelian category, $\Omega_n^k$ is not a functor in general, and one can easily use Comparison Lemma to show that $\Omega_n^k : A \to \mathcal{A}$ is functor for $k > 0$, and defines a functor $\Omega_n^k : \mathcal{A} \to \mathcal{A}$ is functor for $k < 0$. Moreover if $A$ is moreover a Frobenious $n$-abelian category, $\Omega_n^k$ and $\Omega_n^{-k}$ are inverse equivalences.

Let $F : A \to B$ be a contravariant left $n$-exact functor. The right $n$-derived functors $nR^{i} F$ for $i \geq 0$ as follows, for any $M \in A$, choose a projective $n$-resolution $P_\bullet \xrightarrow{d_\bullet} M$ as (2.3) and define

$$nR^{i} F(M) := H_{in+1} (FP_\bullet) := \ker Fd_{in+2}/\operatorname{im} Fd_{in+1} \quad for \quad i = 0, 1, \cdots.$$ Note that $nR^{0} F(M) \simeq FM$, $nR^{i} F(-)$ is an additive functor from $A$ to $B$ and $nR^{i} F(P) = 0$ for all projective object $P$, $i > 0$. The notions of right (left) $n$-derived functors of covariant or contravariant left (right) exact functors are defined dually. In particular, for the contravariant (resp. covariant) left $n$-exact functor $A(-, B)$ (resp. $A(A, -)$),

$$\text{Ext}^{i}_{A}(-, B) := nR^{i} A(-, B) \quad (\text{resp.} \text{Ext}^{i}_{A}(A, -) := nR^{i} A(A, -)).$$

If $A$ is an $n$-cluster tilting subcategory of a projectively generated(or injectively cogenerated) abelian category $\mathcal{D}$. Then $\text{Ext}^{i}_{A}(A, B) \simeq \operatorname{Ext}^{i}_{\mathcal{D}}(A, B)$. $\operatorname{Ext}^{i}_{\mathcal{D}}(A, B) = 0 \forall A, B \in A, m \geq 0, 1 \leq i \leq n - 1$, and $P$ is a projective object if and only if if $\text{Ext}^{i}_{A}(P, B) = 0$ for $i > 0$ if and only if if $\text{Ext}^{i}_{A}(P, B) = 0$.

**Theorem 2.7.** \cite{[4]} Long n-exact Sequence Theorem 4.5] Let $A$ be a small $n$-abelian category with enough injectives (or $A$ is projectively generated), and $X : X^0 \xrightarrow{\alpha_0} X^1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} X^{n+1}$ an n-exact sequence in $A$. Then for any object $A \in A$, we have exact sequences

$$0 \to A(A, X^0) \to \cdots \to A(A, X^{n+1}) \xrightarrow{\partial \alpha^{n+1}} nE^i_A(A, X^0) \xrightarrow{(\alpha^{n})^{i-1}} \cdots \xrightarrow{(\alpha^{1})^{i-1}} nE^1_A(A, X^n) \xrightarrow{X^{n+1}} A(A, X^{n+1})$$


and
\[
0 \to \mathcal{A}(X^{n+1}, A) \to \cdots \to \mathcal{A}(X^0, A) \xrightarrow{X_{n-1}} n\mathcal{E}_A^*(X^{n+1}, A) \xrightarrow{(a_n)_*,1} \cdots \xrightarrow{(a_0)_*,1} n\mathcal{E}_A^*(X^0, A) \xrightarrow{X_{n-1}} \cdots
\]

2.4 \ n\text{-extension}

Let $\mathcal{A}$ be an $n$-abelian category and $A, B \in \mathcal{A}$. An $n\text{-extension}$ $X$ of $A$ by $B$ is an $n$-exact sequence $B \to X^1 \to \cdots \to X^n \to A$ in $\mathcal{A}$. And two $n\text{-extensions}$ $X, X'$ of $A$ by $B$ are equivalent if there is a commutative diagram

\[
\begin{array}{ccc}
X : & B & \xrightarrow{d_X} X^1 & \to \cdots & \to X^{n-1} & \to X^n & \to A \\
& \downarrow & \downarrow f^1 & & \downarrow f^{n-1} & & \downarrow f^n & \\
X' : & B & \xrightarrow{d_{X'}} Y^1 & \to \cdots & \to Y^{n-1} & \to Y^n & \to A
\end{array}
\]

We shall simply write $0$ for the $n$-exact sequence $0 \to 0 \to \cdots \to 0$ if no confusion appears. The following lemma shows that the equivalent is an equivalence relation.

Lemma 2.8. [10] Proposition 4.10] Let $\mathcal{A}$ be an $n$-abelian category. If there exists an equivalence of $n$-exact sequences $f : X \to X'$, then there exists an equivalence of $n$-exact sequences $g : X' \to X$ such that $f$ and $g$ are mutually inverse isomorphisms in the homotopy category of $\mathcal{A}$.

We write the equivalence class of an $n$-extension $X$ by $[X]$, and we define

\[
n\mathcal{E}(A, B) = \{ [X] \mid X \text{ is an } n\text{-extension of } A \text{ by } B \}, \text{ and } X \equiv X' \text{ if } [X] = [X'].
\]

$X$ is contractible if it is equivalent to an $n$-exact sequence $X'$ with $d_{X'}^0$, is a split monomorphism [10] Proposition 2.6, and if and only if there exists an equivalence

\[
\begin{array}{ccc}
X : & B & \xrightarrow{d_X} X^1 & \to X^2 & \to \cdots & \to X^{n-1} & \to X^n & \to A \\
& \downarrow & \downarrow f^1 & & \downarrow & & \downarrow & \\
O_{A,B} : & B & \xrightarrow{1_n} B & \to 0 & \to \cdots & \to 0 & \to A & \xrightarrow{1_A} A
\end{array}
\]

When there is danger of no confusion we shall write $O$ in place of $O_{A,B}$.

Proposition 2.9. [11] Proposition 5.3] Let $\mathcal{A}$ be an $n$-ableian category, $X : B \to B^1 \to \cdots \to B^n \to A$ an $n$-extension of $A$ by $B$ and $f^0 : B \to C$ a morphism in $\mathcal{A}$. Taking an $n$-pushout along $f^0$, there is a morphism between $n$-exact sequences

\[
\begin{array}{ccc}
X : & B & \xrightarrow{f} B^1 & \to \cdots & \to B^n & \to A \\
& \downarrow & \downarrow f^0 & & \downarrow f^n & & \downarrow f^n & \\
X_{po} : & C & \xrightarrow{f^0} T^1 & \to \cdots & \to T^n & \to A
\end{array}
\]

Then, $[X_{po}]$ is unique determined by $[X]$ and $f^0$.

Notation 2.10. By Proposition 2.9 we write $f \cdot X = X_{po}$, and $[f \cdot X]$ does not depend on the choice of $n$-pushout of $X$ along $f$ and nor on the choice of representative element of $[X]$, thus we can write $f \cdot [X] := [f \cdot X]$. The notions $X \cdot g$ and $[X] \cdot g := [X \cdot g]$ for $g : A' \to A$ are defined dually.
Proposition 2.11. [14 Proposition 5.7] Let $\mathcal{A}$ be an $n$-abelian category, $X : B \to X^1 \to \cdots \to X^n \to A$ an $n$-extension of $A$ by $B$. Then, for any morphisms $B \xrightarrow{f} B' \to B''$, and any morphism $A'' \xrightarrow{g'} A' \xrightarrow{g} A$, we have

(i) $(f'f) \cdot [X] = f' \cdot [f \cdot X] = [f' \cdot (f \cdot X)]$.

(ii) $[X] \cdot (gg') = [X \cdot g] \cdot g' = [(X \cdot g) \cdot g']$.

(iii) $f \cdot [X] \cdot g := (f \cdot [X]) \cdot g = f \cdot ([X] \cdot g)$.

(iv) $0 \cdot [X] = [O]$, $[X] \cdot 0 = [O]$.

The following lemma shows that any $n$-pushout (resp. $n$-pullback) along the first (resp. last) morphism of an $n$-exact sequence yields a contractible $n$-exact sequence, is a generalization of [15 Lemma 1.7].

Lemma 2.12. Let $\mathcal{C}$ be an additive category, and $X : X^0 \xrightarrow{\alpha^0} X^1 \xrightarrow{\alpha^1} \cdots \xrightarrow{\alpha^n} X^{n+1}$ be an $n$-exact sequence, then

(i) $f : T \to X^{n+1}$ can be factored through $\alpha^n$ if and only if $[X] \cdot f = [O]$.

(ii) $g : X^0 \to T$ can be factored through $\alpha^0$ if and only if $g \cdot [X] = [O]$.

In particular, $\alpha^0 \cdot X$ and $X \cdot \alpha^n$ are contractible.

Proof. We only prove (i). The proof of (ii) is similar. The morphism $f : T \to X^{n+1}$ factors through $\alpha^n$ if and only if there exist a morphism $f' : T \to X^n$ such that $f = \alpha^n f'$, if and only if the following diagram commutative

$$
\begin{array}{ccc}
X \cdot f : & X^0 & \xrightarrow{\alpha^0} X^1 & \xrightarrow{\alpha^1} \cdots & \xrightarrow{\alpha^n} X^{n+1} \\
[\alpha] & \xrightarrow{[\alpha]} & \xrightarrow{[\alpha]} & \cdots & \xrightarrow{[\alpha]} \\
X : & X^0 & \xrightarrow{\alpha^0} X^1 & \xrightarrow{\alpha^1} \cdots & \xrightarrow{\alpha^n} X^{n+1} \\
\end{array}
$$

2.5 $m$-fold $n$-extension groups

Let $\mathcal{A}$ be an $n$-abelian category, and

$$
X_2 : C \to E_{2n} \to \cdots \to E_{n+1} \to B, \quad \text{and} \quad X_1 : B \to E_n \to \cdots \to E_1 \to A
$$

are two $n$-exact sequences of $\mathcal{A}$. Then we can splicce the two $n$-exact sequences, and obtain a complex in $\text{Ch}^{2n}(\mathcal{A})$

$$
X_2 \circ X_1 : C \to E_{2n} \to \cdots \to E_{n+1} \to E_n \to \cdots \to E_1 \to A,
$$

generally, we call the sequence

$$
S : A_m \to E_{mn} \xrightarrow{d_m} \cdots \xrightarrow{E_{j} d_{j}} E_1 \xrightarrow{d_1} A_0 \tag{2.4}
$$

an $m$-fold $n$-exact sequence starting at $A_m$ and ending at $A_0$ if $d_{m+1}$ can be written as a composite $E_{m+1} \to A_i \to E_i$ for each $i \in \{1, 2, \cdots, m - 1\}$ such that $X_j : A_j \to E_{jn} \to \cdots \to E_{(j-1)n+1} \to A_{j-1}$
are n-exact sequences for \( j \in \{1, 2, \cdots, m\} \). Conventionally, we write \( S = X_m \circ X_{m-1} \circ \cdots \circ X_1 \). Similarly, we write

\[
S_d \circ S_{d-1} \circ \cdots \circ S_1
\]

for \( n \)-fold n-exact sequences \( S_i \).

**Remark 2.13.** It is easily checked that \( d_m \) is a weak cokernel of \( d_{m+1} \), \( d_{m+2} \) is a weak kernel of \( d_{m+1} \), but \( S = X_m \circ X_{m-1} \circ \cdots \circ X_1 \) may be false a mn-exact sequence for \( m \geq 2 \), and it is easy to see that \( S \) is an mn-exact sequence if and only if \( X_1, X_2, \cdots, X_m \) are contractible.

**Definition 2.14.** Let \( \mathcal{A} \) be an n-abelian category, \( m \) be a positive integer. An \( m \)-fold n-extension of \( A_0 \) by \( A_m \) is an \( m \)-fold n-extension \( 2.4 \) in \( \mathcal{A} \). The \( m \)-fold n-extension \( S \) of \( A_0 \) by \( A_m \) is similar to the \( m \)-fold n-extension \( S' \) if there is a commutative diagram

\[
\begin{array}{cccccc}
S & \overset{f}{\longrightarrow} & A_m & \longrightarrow & E_{mn} & \longrightarrow \cdots \longrightarrow & E_1 & \longrightarrow & A_0 \\
\downarrow & & \downarrow & & \downarrow f_{mn} & & \downarrow f_1 & & \downarrow & \\
S' & \overset{g}{\longrightarrow} & A_m & \longrightarrow & E'_{mn} & \longrightarrow \cdots \longrightarrow & E'_1 & \longrightarrow & A_0
\end{array}
\]

The \( m \)-fold n-extension sequence \( S \) is equivalent to \( S' \) and write \( S \equiv_m S' \) if there exists a finite sequence of \( m \)-fold n-exact sequences \( S_0, S_1, \cdots, S_r \) such that \( S = S_0, S' = S_r \) and \( S_i \rightarrow S_{i+1} \) or \( S_i \leftarrow S_{i+1} \) for \( i = 0, 1, \cdots, r - 1 \), and the minimal number \( r \) is called the length of walk from \( S \) to \( S' \), write \( r = \ell_{S,S'} \).

Specially, we write \( \equiv \) for \( \equiv_1 \).

It is easy to see that “\( \equiv_m \)” is an equivalence relation. We call \( S \) a contractible \( m \)-fold n-extension sequence if \( S \) equivalent to \( O_{A_0,A_m} : X_m \circ 0 \circ \cdots \circ 0 \circ X_1 \) with \( X_m : A_m \stackrel{1_{A_m}}{\longrightarrow} A_m \rightarrow 0 \rightarrow \cdots \rightarrow 0, X_1 : 0 \rightarrow \cdots \rightarrow 0 \rightarrow A_0 \stackrel{1_{A_0}}{\longrightarrow} A_0 \) for \( m \geq 2 \). When there is danger of no confusion we shall write \( O \) in place of \( O_{A_0,A_m} \).

We denote the equivalence class of an \( m \)-fold n-exact sequence \( S \) by \([S]\), and we define

\[
nE^m(A, B) = \{[S] | S \text{ is an } m\text{-fold } n\text{-extension of } A \text{ by } B\}.
\]

Especially, \( nE^0(A, B) = \mathcal{A}(A, B) \). \( nE^m(A, B) \) may not be a set, and if \( \mathcal{A} \) is small, then \( nE^m(A, B) \) will be a set. And it can be shown that \( nE^m(A, B) \) is a set if \( \mathcal{A} \) has projectives or injectives (see Theorem 2.17), similarly if \( \mathcal{A} \) has a generator or a cogenerator. In the following, we may assume that the underlying class of ordinary abelian group need not be a set.

Similar to Notation 2.10 we write

\[
\begin{align*}
\alpha \cdot [S] & := [\alpha \cdot S] = [\alpha \cdot (X_m \circ X_{m-1} \circ \cdots \circ X_1)] := [(\alpha \cdot X_m) \circ X_{m-1} \circ \cdots \circ X_1], \\
[S] \cdot \gamma & := [S \cdot \gamma] = [(X_m \circ X_{m-1} \circ \cdots \circ X_1) \cdot \gamma] := [X_m \circ \cdots \circ X_2 \circ (X_1 \cdot \gamma)].
\end{align*}
\]

Sometimes, we write \( \alpha \circ S \) for \( \alpha \cdot S \) when no confusion appears.

Similar to Proposition 2.11 we have

**Proposition 2.15.** [14] Proposition 5.1] Let \( \mathcal{A} \) be an n-abelian category, \( S \) an \( m \)-fold n-extension of \( A \) by \( B \).

(i) If \( S : B \overset{i}{\rightarrow} E_{mn} \overset{d_{mn}}{\longrightarrow} \cdots \overset{d_2}{\longrightarrow} E_1 \overset{\pi}{\rightarrow} A \), then \( i \cdot [S] = [O] \) and \( [S] \cdot \pi = [O] \).
The morphisms $X_{2.6}$

In the rest of this section, we simply and dually, which defined for any equivalence classes $[S], [S'] \in nE^m(A, B)$, and this makes $nE^m(A, B)$ an abelian group.

**Theorem 2.16.** [13] Theorem 5.14 Let $A$ be an $n$-abelian category, $A, B$ two objects of $A$. Then $nE^m(A, B)$ is an abelian group under $n$-Baer sum: $[S] + [S'] = [(\nabla(S \oplus S'))_{\Delta}]$, with zero element being the class of contractible $m$-fold $n$-extensions $[O]$, the inverse of any $[S]$ is the $m$-fold $n$-extension $(-1_B) \cdot [S]$.

For any $[R] \in nE^k(C, A)$, $[T] \in nE^m(B, M)$, $k, l \geq 0$, we have

$$( [S] + [S'] ) \circ [R] = [S] \circ [R] + [S'] \circ [R], \quad [T] \circ ([S] + [S']) = [T] \circ [S] + [T] \circ [S'].$$

Given a morphism $f : B \to C$, $m > 0$, define

$$f^{*,m} = nE^m(A, f) : nE^m(A, B) \to nE^m(A, C), \quad \text{by } f^{*,m}([S]) = f \cdot [S],$$

and dually,

$$f_{*,m} = nE^m(f, A) : nE^m(C, A) \to nE^m(B, A), \quad \text{by } f_{*,m}([S]) = [S] \cdot f.$$

By Proposition 2.15 and Theorem 2.16 we see that $nE^m(-, -)$ is an additive bifunctor from $A$ to abelian group category $G$, contravariant in the first variable and covariant in the second. Then we have the pairing

$$nE^m(B, C) \times nE^1(A, B) \to nE^{m+1}(A, C),$$

which defined for $m, l \geq 1$ by splicing sequences, and defined for $m$ or $l = 0$ by the $n$-pushout or $n$-pullback, this pairing is bilinear by Theorem 2.16. Let $A \in A$, and $X : X^0 \xrightarrow{\alpha^0} X^1 \xrightarrow{\alpha^1} \cdots \xrightarrow{\alpha^n} X^{n+1}$ be an $n$-exact sequence, we can define group homomorphisms

$$X^* : A(X^0, X^{n+1}) \to nE^1(A, X^0), \quad \text{by } X^* : f \mapsto [X] \cdot f$$

$$(\alpha^j)^*: nE^1(A, X^j) \to nE^1(A, X^{j+1}), \quad \text{by } (\alpha^j)^*: [S] \mapsto \alpha^j \cdot [S], \quad \text{for } j = 0, \ldots, n, i \geq 1$$

$$X^{*,i} : nE^i(A, X^{n+1}) \to nE^i(A, X^0), \quad \text{by } X^{*,i} : [S] \mapsto [X] \circ [S], \quad \text{for } i \geq 1$$

and dually

$$X_* : A(X^0, A) \to nE^1(X^{n+1}, A), \quad \text{by } X_* : f \mapsto f \cdot [X]$$

$$(\alpha^j)_*: nE^1(X^{j+1}, A) \to nE^1(X^j, A), \quad \text{by } (\alpha^j)_*: [S] \mapsto \alpha^j \cdot [S], \quad \text{for } j = 0, \ldots, n, i \geq 1$$

$$X_{*,i} : nE^i(X^0, A) \to nE^i(X^{n+1}, A), \quad \text{by } X_{*,i} : [S] \mapsto [S] \circ [X], \quad \text{for } i \geq 1.$$

The morphisms $X^*, X_*$ are called the $i$-th $n$-connecting morphisms. In particular, $X^*, X_*$ are called the $n$-connecting morphisms.

### 2.6 $m$-fold $n$-extensions and $n\text{Ext}^m$

In the rest of this section, we simply $\mathcal{A}(f, M)$ by $f^*$ for any morphism $f$. 
Let \( S : B = A_n \rightarrow E_{mn} \overset{\alpha_{mn}}{\rightarrow} \cdots \overset{\alpha_2}{\rightarrow} E_1 \overset{\alpha_1}{\rightarrow} A_0 = A \) be an \( m \)-fold \( n \)-extension of \( A \) by \( B \) with \( m \geq 1 \). Taking an \( n \)-resolution \( P_\bullet \rightarrow A \) of \( A \), then we have a commutative diagram lifting \( 1_A \)

\[
P_\bullet \rightarrow A : \quad \cdots \quad \xrightarrow{d_{mn+2}} P_{mn+1} \xrightarrow{d_{mn+1}} P_{mn} \xrightarrow{d_{mn}} \cdots \xrightarrow{d_1} P_1 \xrightarrow{d^0} A
\]

\[
S : \quad \cdots \quad \xrightarrow{f_{mn+2}} P_{mn+1} \xrightarrow{f_{mn+1}} P_{mn} \xrightarrow{f_{mn}} \cdots \xrightarrow{f_1} P_1 \xrightarrow{f^0} A
\]

We have \( f_{mn+1}d_{mn+2} = 0 \). The following theorem shows that we can define the Ext group \( \text{nExt}^m_A(A, B) \) even without projective objects or injective objects.

**Theorem 2.17.** Let \( A \) be a projectively generated \( n \)-abelian category, \( A, B \) two objects of \( A \). Then, there is a functorial isomorphism of abelian groups \( \Theta_m : nE^n(A, B) \rightarrow \text{nExt}^m_A(A, B) \), in which the class of contractible \( n \)-fold \( n \)-extensions correspond to the element \( 0 \in \text{nExt}^m_A(A, B) \).

**Proof.** Applying \( A(-, B) \) to the above diagram, define

\[
\Theta_m : nE^n(A, B) \rightarrow \text{nExt}^m_A(A, B) \quad \text{by} \quad [S] \mapsto f_{mn+1} + \text{Im} A(d^*_{mn+1}, B).
\]

In [14, Theorem 5.17], we have proved that \( \Theta_m \) is an isomorphism of abelian groups. To show that \( \Theta_m : nE^n(-, -) \rightarrow \text{nExt}^m_A(-, -) \) is a functorial morphism, we must show that for any morphism \( f : B \rightarrow C \) and any \( A \), the following diagrams commute:

\[
\begin{array}{ccc}
nE^n(A, B) & \xrightarrow{f^*} & nE^n(A, C) \\
\text{nExt}^m(A, B) & \xrightarrow{\Theta_m} & \text{nExt}^m(A, C),
\end{array}
\begin{array}{ccc}
nE^n(C, A) & \xrightarrow{f^*} & nE^n(B, A) \\
\text{nExt}^m(C, A) & \xrightarrow{\Theta_m} & \text{nExt}^m(B, A).
\end{array}
\]

Let \([S] \in nE^n(A, B)\). Then

\[
\text{nExt}^m(A, f)\Theta_m([S]) = \text{nExt}^m(A, f)(A(f_{mn+1}, B)(1_B) + \text{Im} A(d^*_{mn+1}, B)) = ff_{mn+1} + \text{Im} A(d^*_{mn+1}, C) = \Theta_m(f \cdot [S]) = \Theta_m f^*[S].
\]

This shows that the left diagram is commutative, and dually the right diagram is also commutative. \( \square \)

### 3 Long \( n \)-exact sequence theorem

In order to prove the main result, we need the following two lemmas.

**Lemma 3.1.** Let \( X : X^0 \overset{\alpha_0}{\rightarrow} X^1 \overset{\alpha_1}{\rightarrow} \cdots \overset{\alpha_n}{\rightarrow} X^{n+1} \) be a left \( n \)-exact sequence in \( n \)-abelian category \( A \). Then there exists a morphism \( X^n \overset{u}{\rightarrow} Y \) such that \( X^0 \overset{\alpha_0}{\rightarrow} X^1 \overset{\alpha_1}{\rightarrow} \cdots \overset{\alpha_n}{\rightarrow} X^n \overset{u}{\rightarrow} Y \) is \( n \)-exact.

**Proof.** By [14, Theorem 4.3], the \( n \)-exact functor \( \tilde{H} : A \rightarrow \mathcal{L}_2(A, G) \) induced an exact sequence (Lemma 2.5)

\[
A(X^{n+1}, -) \rightarrow A(X^n, -) \rightarrow \cdots \rightarrow A(X^1, -) \rightarrow A(X^0, -) \rightarrow 0,
\]

where \( \mathcal{L}_2(A, G) \) is the category of absolutely pure mono functors from \( A \) to abelian group category \( G \). Then
for any \( M \in \mathcal{A} \),
\[
\mathcal{A}(X^{n+1}, M) \to \mathcal{A}(X^n, M) \to \cdots \to \mathcal{A}(X^1, M) \to \mathcal{A}(X^0, M)
\]
is an exact sequence of abelian groups, this shows that \( \alpha^i \) is a weak cokernel of \( \alpha^{i-1} \), for \( i = 1, 2, \ldots, n \). By Proposition 3.7, \( \alpha^{i-1} \) admits a cokernel \( X^n \xrightarrow{u} Y \) in \( \mathcal{A} \), therefore \( X^n \xrightarrow{\alpha^n} X^1 \xrightarrow{\alpha^1} \cdots \xrightarrow{\alpha^{n-1}} X^n \xrightarrow{u} Y \) is \( n \)-exact.

\[\square\]

**Lemma 3.2.** Let \( E \) and \( F \) be \( r \)-fold, \( s \)-fold \( n \)-extensions, respectively \((r, s \geq 1)\), such that the splicing \( E \circ F \) is defined. Then the following are equivalent.

(i) \( E \circ F \equiv_{r+s} O \).

(ii) There is an \( r \)-fold \( n \)-extension \( G \) and a morphism \( \varphi \) such that \( E \equiv_r G \cdot \varphi \) and \( \varphi \cdot F \equiv_s O \).

(iii) There is an \( s \)-fold \( n \)-extension \( H \) and a morphism \( \psi \) such that \( F \equiv_s \psi \cdot H \) and \( E \cdot \psi \equiv_r O \).

**Proof.** Our proof is an adaptation of the proof of\([10]\) Lemma 4.1 if \( r + s \geq 3 \).

It follows immediately from Proposition\([2.15]\) that \((iii) \Rightarrow (i)\) and \((ii) \Rightarrow (i)\). We first prove \((iii) \Rightarrow (ii)\) in the case \( r = s = 1 \), and by duality \((ii) \Rightarrow (iii)\).

Let \( F \equiv_1 \psi \cdot H \) and \( E \cdot \psi \equiv_1 O \). By Proposition\([2.8]\) we can assume that \( F = \psi \cdot H \) and \( E \cdot \psi \) is contractible. By Lemma\([2.12]\) and Proposition\([2.8]\) there exists a morphism \( \rho \) such that \( \lambda_E \rho = \psi \), where \( \lambda_E \) is the rightmost morphism of \( E \). Then \( F = \lambda_E \rho \cdot H = \lambda_E \cdot (\rho \cdot H) \). Since \( E \cdot \lambda_E \equiv_1 O \) by Lemma\([2.4]\) we may assume \( \psi = \lambda_E \). Taking good \( n \)-pushout in the sense of\([10]\) Definition-Proposition 2.14], such that \( f^2, \ldots, f^n \) are split monomorphism, and we have commutative diagram

\[
\begin{array}{ccc}
C & \xrightarrow{\alpha^n} & C \\
| & \downarrow{\alpha^0} & \downarrow{\alpha^0} \\
E^1 & \xrightarrow{\alpha^1} & E^1 \\
| & \downarrow{\alpha^1} & \downarrow{\alpha^1} \\
\vdots & \vdots & \vdots \\
E^{n-1} & \xrightarrow{\alpha^{n-2}} & E^{n-1} \\
\downarrow{\alpha^{n-1}} & \downarrow{u \alpha^{n-1}} & \downarrow{\lambda_E} \\
E^n & \xrightarrow{\lambda_E} & H^n \\
| & \Downarrow{f^1} & \Downarrow{f^1} \\
B & \xrightarrow{\mu \phi} & F^n \\
F = \lambda_E \cdot H : & & \xrightarrow{-} \\
& F^{n-1} & \xrightarrow{f^{n-1}} \xrightarrow{f^n} A
\end{array}
\]

One can easily show that \( \alpha^{n-2} \) is a weak kernel of \( u \alpha^{n-1} \). The morphism \( u \alpha^{n-1} \) is a weak kernel of \( f^1 \). Indeed, let \( t : M \to H^1 \) be a morphism such that \( f^1 t = 0 \). \( d^1 t = 0 \) since \( f^2 \) is a monomorphism. By the \( n \)-exactness of the mapping cone of \( n \)-pushout \( H \to \lambda_E \cdot H \), there exists \( s : M \to E^n \) such that \( u s = t, \lambda_E s = 0 \). Then there exists \( k : M \to E^{n-1} \) such that \( \alpha^{n-1} k = s \). It follows that

\[ t = u s = u \alpha^{n-1} k . \]
This shows that the second column is left $n$-exact, by Lemma \ref{lem:exactness}. $f^1$ is a weak cokernel of $u_\alpha n^{-1}$ and there is an $n$-exact sequence

$$G : C \xrightarrow{\alpha^0} E^1 \xrightarrow{\alpha^1} \cdots \xrightarrow{\alpha^{n-2}} E^{n-1} \xrightarrow{\alpha^{n-1}} H^1 \xrightarrow{\psi} Y.$$  

Then there exists a morphism $w : F^1 \rightarrow Y$ such that $u = w f^1$, and

$$E = G \cdot (w_\mu F), (w_\mu F) \cdot F = w \cdot (\mu_\psi F) \equiv_1 w \cdot O \equiv_1 O.$$  

Next we show that (i) $\Rightarrow$ (ii) and (iii), in the case $r = s = 1$. We do it by induction on the length $\ell_{E\circ F,O}$ of walk from $E \circ F$ to $O$. If $\ell_{E\circ F,O} = 0$, that is, if $E \circ F = O$, then this is trivial. Otherwise, we may have $E = E' \cdot \eta$ where $\ell_{E' \cdot \eta F,O} < \ell_{E\circ F,O}$, or we may have $F = \gamma \cdot F'$ where $\ell_{F' \cdot \eta F,O} < \ell_{E\circ F,O}$. In the former case we have by induction $E' = G \cdot \varphi'$ where $\varphi' = (\eta \cdot F) = O$. Hence $E = E' \cdot \eta = G \cdot (\varphi' \cdot \eta)$, and $(\varphi \cdot \eta) \cdot F = O$, so that we may take $\varphi = \varphi' \cdot \eta$ to see that (ii) holds. In the latter case we see by duality that (iii) holds.

We proceed now by induction on $r + s$. We assume (i), (ii), (iii) are equivalent when $r + s < m (m > 2)$, and take $r + s = m$. We first show that (ii) $\Rightarrow$ (iii). We have $E \equiv_r G \cdot \varphi$ where $\varphi \cdot F \equiv_s O$.

If $s > 1$, write $F = F_1 \circ F_{s-1}$, where $F_1, F_{s-1}$ are 1 and $(s - 1)$-fold $n$-extensions respectively. Then $(\varphi \cdot F_1) \circ F_{s-1} \equiv_s O$, so by induction we can write

$$F_{s-1} \equiv_{s-1} \eta \cdot H_{s-1}, (\varphi \cdot F_1) \cdot \eta \equiv_1 O_1.$$  

Hence, $E \circ (F_1 \cdot \eta) \equiv_{r+1} G \cdot \varphi \circ (F_1 \cdot \eta) \equiv_{r+1} O$, so again by induction we can find a morphism $\psi$ such that $F_1 \cdot \eta = \psi \cdot H_1, E \cdot \psi \equiv_r O$. Then we have

$$F = F_1 \circ F_{s-1} \equiv_s F_1 \circ (\eta \cdot H_{s-1}) = (\psi \cdot H_1) \circ H_{s-1}$$  

and therefore we can take $H = H_1 \circ H_{s-1}$.

If $s = 1$, then $r > 1$, so we can write $G \cdot \varphi = (G_{r-1} \circ G_1) \circ \varphi$. From the case $r = s = 1$, we then have a morphism $\phi$ such that $F_1 = \psi \cdot H$ and $G_1 \cdot (\varphi \psi) \equiv_1 O_1$. Then $E \cdot \psi = G_{r-1} \circ (G_1 \cdot \varphi) \cdot \psi \equiv_r O$. This shows (ii) $\Rightarrow$ (iii), and dually (iii) $\Rightarrow$ (ii).

We now have only to show that (i) $\Rightarrow$ (ii) and (iii). We do it by induction on the length $\ell_{E\circ F,O}$ of walk from $E \circ F$ to $O$. Again, this is trivial if $\ell_{E\circ F,O} = 0$. Otherwise, we may have $E = E' \cdot \eta$ where $\ell_{E' \cdot \eta F,O} < \ell_{E\circ F,O}$, or we may have $F = \gamma \cdot F'$ where $\ell_{F' \cdot \eta F,O} < \ell_{E\circ F,O}$. In the former case we have by induction we can write $E' \equiv_r G \cdot \varphi$ where $\varphi \cdot F \equiv_s O$. Then $E' = E' \circ \eta \equiv_r G \cdot (\varphi' \cdot \eta)$, so that we may take $\varphi = \varphi' \cdot \eta$. In the latter case we see by duality that (iii) holds.

Now we state and prove our main result in this section.

**Theorem 3.3.** Let $A$ be an $n$-abelian category and $X: X^0 \xrightarrow{\alpha^0} X^1 \xrightarrow{\alpha^1} \cdots \xrightarrow{\alpha^n} X^{n+1}$ an $n$-exact sequence in $A$. Then the sequences of abelian groups

$$
\begin{array}{c}
0 \rightarrow A(A,X^0) \xrightarrow{X^*} \cdots \rightarrow A(A,X^{n+1}) \xrightarrow{X^*} nE_A^1(A,X^0) \xrightarrow{(\alpha^0)_{r+1}} \cdots \xrightarrow{(\alpha^n)_{r+1}} nE_A^1(A,X^{n+1}) \\
X^{r+1} \rightarrow \cdots \xrightarrow{X^{r+1}} nE_A^1(A,X^0) \xrightarrow{(\alpha^0)_{r+1}} \cdots \xrightarrow{(\alpha^n)_{r+1}} nE_A^1(A,X^{n+1}) \xrightarrow{X^{r+1}} \cdots
\end{array}
$$

\[\]
and

\[ 0 \to \mathcal{A}(X^{n+1}, A) \to \cdots \to \mathcal{A}(X^0, A) \xrightarrow{X^1} nE_{\mathcal{A}}(X^{n+1}, A) \xrightarrow{(\alpha^2)_{1,0}} \cdots \xrightarrow{(\alpha_n)_{1,0}} nE_{\mathcal{A}}(X^0, A) \]

are exact for any \( A \in \mathcal{A} \). This Theorem is called Long \( n \)-exact Sequence Theorem.

**Proof.** We only prove the first part of this theorem, and the second part proves similarly.

By Lemma 2.12 Proposition 2.11 Proposition 2.15 Theorem 2.16 \( X^*, X^*, (\alpha^1)^*, i \) are well defined group homomorphisms, and the sequence \( \triangleleft \) is a complex.

**Step 1.** It is trivial that the sequence \( \triangleleft \) is exact at \( \mathcal{A}(A, X^j) \) for \( j = 0, 1, \cdots, n \). Suppose that \( g \in \mathcal{A}(A, X^{n+1}) \) is a morphism such that \( X^*(g) = [X] \cdot g \) contractible in \( n\mathcal{E}(A, X^0) \). By Lemma 2.12 there exists a morphism \( t : A \to X^n \) such that \( g = \alpha^n t = \mathcal{A}(A, \alpha^n)t \), this proves that \( \text{Ker}X^* \subset \text{Im} \mathcal{A}(A, \alpha^n) \) and the sequence \( \triangleleft \) is exact at \( \mathcal{A}(A, X^{n+1}) \).

**Step 2.** We show that the sequence \( \triangleleft \) is exact at \( n\mathcal{E}(A, X^0) \). Suppose that \( E : X^0 \xrightarrow{d^0} Y^1 \to \cdots \to Y^n \to A \) is an \( n \)-extension of \( n\mathcal{E}(A, X^0) \) such that \( (\alpha^n)^* \cdot ([E]) = \alpha^0 \cdot [E] \) is contractible. By Lemma 2.12 there exists a morphism \( f^1 : Y^1 \to X^1 \) such that \( f^1d^0 = \alpha^0 \). Since \( d^{i+1} \) is a weak cokernel of \( d^i \), there is a commutative diagram

\[
\begin{array}{c}
E : & X^0 \xrightarrow{d^0} Y^1 \xrightarrow{d^1} Y^2 \xrightarrow{\cdots} Y^n \xrightarrow{d^n} A \\
\downarrow & \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
X : & X^0 \xrightarrow{\alpha^0} X^1 \xrightarrow{\alpha^1} X^2 \xrightarrow{\cdots} X^n \xrightarrow{\alpha^n} X^{n+1}
\end{array}
\]

(3.5)

This proves that \([E] = [X] \cdot f = X^*(f)\). Thus the sequence \( \triangleleft \) is exact at \( n\mathcal{E}(A, X^0) \).

**Step 3.** We show that the sequence \( \triangleleft \) is exact at \( n\mathcal{E}(A, X^1) \). Suppose that \( (\alpha^1)^* \cdot ([E]) = \alpha^1 \cdot [E] \) is contractible. Then we have a commutative diagram

\[
\begin{array}{c}
E : & X^0 \xrightarrow{u} T \xrightarrow{d^1f^{-1}t} Y^1 \xrightarrow{d^2} \cdots \xrightarrow{d^n} Y^n \xrightarrow{d^n} A \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
X^1 \xrightarrow{d^0} Y^1 \xrightarrow{d^1} Y^2 \xrightarrow{\cdots} Y^n \xrightarrow{d^n} A \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\alpha^1 \cdot E : & X^2 \xrightarrow{0} \cdots \xrightarrow{0} A \\
\downarrow \quad \downarrow \\
\alpha^2 \\
\downarrow \\
X^{n+1} \xrightarrow{\alpha^n} X^{n+1}
\end{array}
\]

(3.6)

where \( f^{-1} \) is a weak kernel of \( f^1 \), and then \( u \) is a monomorphism.

It is easily seen that \( f^1 \) is a weak kernel of \( \alpha^2 \), so we can choose \( f^0 \) as a kernel of \( f^1 \) by [10 Dual of Proposition 3.7], this shows that the second column of this diagram is an \( n \)-exact sequence.

We now show that \( d^1f^0 \) is a weak kernel of \( d^2 \). Let \( t : M \to Y^2 \) be a morphism such that \( d^2t = 0 \). Then there exist a morphism \( t^1 : M \to Y^1 \) such that \( d^1t^1 = t \). But \( \alpha^2f^1t^1 = 0 \), then there exists a morphism \( t^2 : M \to X^1 \) such that \( \alpha^1t^2 = f^1t^1 \) since \( \alpha^1 \) is a weak kernel of \( \alpha^2 \). It follows that
\( f^1(d^0t^2 - t^1) = \alpha^1t^2 - f^1t^1 = 0 \). So there exists a morphism \( s : M \to T \) such that \( f^0s = d^0t^2 - t^1 \) since \( f^0 \) is a weak kernel of \( f^1 \). Hence we have
\[
t = d^1t^1 = d^1(d^0t^2 - f^0s) = d^1f^0(-s).
\]

We claim that \( u \) is a weak kernel of \( d^1f^0 \) and then \( u \) is a kernel of \( d^1f^0 \) since \( u \) is monomorphism.

Indeed, let \( h : M \to T \) be a morphism such that \( d^1f^0h = 0 \), then there exist a morphism \( h^1 : M \to X^1 \) such that \( d^0h^1 = f^0h \). But \( \alpha^1h^1 = f^1d^0h^1 = f^1f^0h = 0 \), then there exists a morphism \( h^2 : M \to X^0 \) such that \( \alpha^0h^2 = h^1 \) since \( \alpha^0 \) is a kernel of \( \alpha^1 \). It follows that \( uh^2 = h \) since \( f^0(uh^2 - h) = d^0\alpha^0h^2 - f^0h = d^0h^1 - f^0h = 0 \) and \( f^0 \) is a monomorphism.

This means that the first row is an \( n \)-extension, write \( T \), and \( [E] = \alpha^0 \cdot [T] = (\alpha^0)^{\cdot 1}([T]) \). This proves that the sequence \( \bullet \) is exact at \( \text{ne}^1(A, X^1) \).

**Step 4.** We show that the sequence \( \bullet \) is exact at \( \text{ne}^1(A, X^i) \), for \( i = 2, 3, \ldots, n - 1 \). Suppose that \( (\alpha^i)^{\cdot 1}([E]) = \alpha^i \cdot [E] \) is contractible. Then we have a commutative diagram
\[
\begin{array}{ccccccccc}
X^i & \xrightarrow{u^i} & T^{i-1} & \xrightarrow{d^if^{i-1}} & Y^2 & \xrightarrow{d^2} & \cdots & \xrightarrow{d^{n-1}} & Y^n & \xrightarrow{d^n} & A \\
E : & \downarrow{\alpha^{i-1}} & \downarrow{f^{i-1}} & & \downarrow{d^2} & & \cdots & & \downarrow{d^n} & & \\
X^i & \xrightarrow{d^0} & Y^1 & \xrightarrow{d^1} & Y^2 & \xrightarrow{d^2} & \cdots & \xrightarrow{d^{n-1}} & Y^n & \xrightarrow{d^n} & A \\
\alpha^i \cdot E : & \downarrow{\alpha^i} & \downarrow{f^i} & & \downarrow{d^2} & & \cdots & & \downarrow{d^n} & & \\
X^{i+1} & \xrightarrow{d^{i+1}} & X^{i+1} & \xrightarrow{d^1} & 0 & \xrightarrow{d^2} & \cdots & \xrightarrow{d^{n-1}} & 0 & \xrightarrow{d^n} & A \\
\downarrow{\alpha^{i+1}} & \downarrow{d^{i+1}} & & & & & & & & & \\
X^{i+2} & \xrightarrow{d^{i+2}} & X^{i+2} & & & & & & & & \\
\end{array}
\]
where \( f^{i-1} \) is a weak kernel of \( f^i \).

Similar to Step 3, \( f^i \) is a weak kernel of \( \alpha^{i+1} \), \( d^1f^{i-1} \) is a weak kernel of \( d^2 \). Generally, \( u^{i-1} \) is not a weak kernel of \( d^1f^{i-1} \). By \[10\] Proposition 3.7, \( d^1f^{i-1} \) admits a kernel \( u : T \to T^{i-1} \), and there exist an unique morphism \( \alpha : T \to X^i \) such that \( d^0\alpha = f^{i-1}u \). We claim that \( \alpha \) is a weak kernel of \( \alpha_i \). Indeed, let \( t : M \to X^i \) be a morphism such that \( \alpha^i t = 0 \), then there exist a morphism \( t^1 : M \to T^{i-1} \) such that \( d^0t = f^{i-1}t^1 \). By \[10\] Proposition 4.8 and the definition of \( n \)-pushout, we have commutative diagram
\[
M \xrightarrow{(-\cdot t)} C : T \xrightarrow{(-u, \alpha)^T} T^i-1 \oplus X^i \xrightarrow{d^0_C} Y^2 \oplus Y^1 \xrightarrow{d^2} \cdots \xrightarrow{d^{n-1}} Y^n \oplus Y^{n-1} \xrightarrow{d^n} Y^n
\]
where \( d^0_C = \begin{pmatrix} -d^1f^{i-1} & 0 \\ f^{i-1} & d^0 \end{pmatrix} \), and \( C \) is an \( n \)-exact sequence since \( u \) is a monomorphism. Then there exists an unique morphism \( s : M \to T \) such that \( \alpha s = t \) since \( (-u, \alpha)^T \) is a kernel of \( d^0_C \).

\( \alpha^i \cdot \alpha^{i-1} \) are all of weak kernel of \( \alpha^i \), then there exists a morphism \( v : T \to X^i \) such that \( \alpha = \alpha^{i-1}v \). Write \( n \)-extension \( T \to T^{i-1} \to Y^2 \to \cdots \to Y^n \) by \( E \), we have
\[
E = \alpha \cdot E = (\alpha^{i-1}v) \cdot E = \alpha^{i-1} \cdot (v \cdot E).
\]
This proves that the sequence \( \bullet \) is exact at \( \text{ne}^1(A, X^i) \), \( i = 2, 3, \ldots, n - 1 \).

**Step 5.** The sequence \( \bullet \) is exact at \( \text{ne}^1(A, X^n) \). Suppose that \( (\alpha^n)^{\cdot 1}([E]) = \alpha^n \cdot [E] \) is contractible.
The fact that $\alpha^n \cdot E$ is contractible which gives a commutative diagram

\[
\begin{array}{cccccccc}
X^{n-1} & \xrightarrow{u^{n-1}} & T^{n-1} & \xrightarrow{d^{n-1}} & Y^2 & \xrightarrow{d^2} & \cdots & \xrightarrow{d^2} & Y^n & \xrightarrow{d^n} & A \\
\downarrow{\alpha^{n-1}} & & \downarrow{f^{n-1}} & & \| & & \| & & \| & & |
\end{array}
\]

where $f^{n-1}$ is a weak kernel of $f^n$.

If $d^1 f^{n-1}$ is a weak kernel of $d^2$, similar to Step 4, the sequence $\bullet$ is exact at $E(X^n)$. If $d^1 f^{n-1}$ is not a weak kernel of $d^2$, we can replace $T^{n-1}$ by $T^{n-1} \oplus Y^1$, $u^{n-1}$ by $(u^{n-1}, 0)^T$, $f^{n-1}$ by $(f^{n-1}, 1)$.

Then, $(f^{n-1}, 1)$ is also a weak kernel of $f^n$, and $(d^1 f^{n-1}, d^1)$ is a weak kernel of $d^2$. So, we can choose $T^{n-1} f^{n-1} = Y^1$ such that $d^1 f^{n-1}$ is a weak kernel of $d^2$.

**Step 6.** The sequence $\bullet$ is exact at $nE(A, X^{n+1})$, $i \geq 1$. If $X^{*\cdot i}([S]) = [X \circ [S] = [X \circ S] = [O]$, then by Lemma 52 we have $S \equiv_i \psi \cdot F$ where $X \cdot \psi$ is contractible, by Lemma 212 we can write $\psi = \alpha^n \psi'$, we have

$$[S] = (\alpha^n \psi') \cdot [F] = (\alpha^n)^* \cdot ([\psi' \cdot F]).$$

This proves the sequence $\bullet$ is exact at $nE(A, X^{n+1}), i \geq 1$.

**Step 7.** The sequence $\bullet$ is exact at $nE(A, X^n), i \geq 2$. Suppose that $(\alpha^0)^* \cdot ([S]) = \alpha^0 \cdot [S] = [O]$, we write $S = S_1 \circ S_{i-1}$, where $S_1$ and $S_{i-1}$ are 1 and $(i-1)$-fold $n$-extensions respectively.

$$\alpha^0 \cdot S = (\alpha^0 \cdot S_1) \circ S_{i-1},$$

Applying Step 6 to $X = \alpha^0 \cdot S_1$, we have $S_{i-1} \equiv_i \lambda_{\circ \circ . S_i} S'_{i-1}$, where $\lambda_{\circ \circ . S_i}$ is the rightmost morphism of $\alpha^0 \cdot S_1$ and $S'_{i-1}$ is an $(i-1)$-fold $n$-extension. Hence, we have

$$S = S_1 \circ S_{i-1} \equiv_i (S_1 \cdot \lambda_{\circ \circ . S_i}) \circ S'_{i-1}.$$

Observe that

$$(\alpha^0)^* \cdot ([S_1 \cdot \lambda_{\circ \circ . S_i}]) = \alpha^0 \cdot [S_1 \cdot \lambda_{\circ \circ . S_i}] = [(\alpha^0 \cdot S_1) \cdot \lambda_{\circ \circ . S_i}] = [O],$$

by Step 2, there exists a morphism $\varphi \in A(A, X^{n+1})$ such that $S_1 \cdot \lambda_{\circ \circ . S_i} = X \cdot \varphi$, therefore

$$[S] = [(S_1 \cdot \lambda_{\circ \circ . S_i}) \circ S'_{i-1}] = [(X \cdot \varphi \circ S'_{i-1})] = [X \cdot (\varphi \circ S'_{i-1})] = X^{*\cdot i \cdot \varphi} \circ S'_{i-1}.$$

This proves the sequence $\bullet$ is exact at $nE(A, X^n), i \geq 2$.

**Step 8.** The sequence $\bullet$ is exact at $nE(A, X^j)$, for $i \geq 2$, $j = 1, 2, \cdots, n$. Suppose that $(\alpha^j)^* \cdot ([S]) = \alpha^j \cdot [S] = [O]$, we write $S = S_1 \circ S_{i-1}$, where $S_1$ and $S_{i-1}$ are 1 and $(i-1)$-fold $n$-extensions respectively.

$$\alpha^j \cdot S = (\alpha^j \cdot S_1) \circ S_{i-1},$$

Applying Step 6 to $X = \alpha^j \cdot S_1$, we have $S_{i-1} \equiv_i \lambda_{\circ \circ . S_i} S'_{i-1}$, where $\lambda_{\circ \circ . S_i}$ is the rightmost morphism of $\alpha^j \cdot S_1$ and $S'_{i-1}$ is an $(i-1)$-fold $n$-extension. Hence, we have

$$S = S_1 \circ S_{i-1} \equiv_i (S_1 \cdot \lambda_{\circ \circ . S_i}) \circ S'_{i-1}.$$
Observe that
$$(\alpha^j)^{-1}([S_1 \cdot \lambda_{\alpha^j} \cdot S_1]) = \alpha^j \cdot [S_1 \cdot \lambda_{\alpha^j} \cdot S_1] = [(\alpha^j \cdot S_1) \cdot \lambda_{\alpha^j} \cdot S_1] = [O],$$
by Step 3, 4, and 5, there exists an $n$-extension $[E] \in n\mathcal{E}(A, X^{j-1})$ such that $S_1 \cdot \lambda_{\alpha^j} \cdot S_1 = \alpha^j \cdot [E]$, therefore
$$[S] = [(S_1 \cdot \lambda_{\alpha^j} \cdot S_1) \circ S_{i-1}] = [(\alpha^j \cdot E) \circ S_{i-1}] = [\alpha^j \cdot (E \circ S_{i-1})] = (\alpha^j)^{-1} \cdot [E \circ S_{i-1}].$$
This proves the sequence $\clubsuit$ is exact at $n\mathcal{E}(A, X^j)$, for $i \geq 2, j = 1, 2, \cdots, n$. \hfill \qed

\section{Wakamatsu’s Lemma}

In this section, we give an application about the long $n$-exact sequence theorem. The following observation is useful.

\textbf{Remark 4.1.} If $\mathcal{X}$ is a generating subcategory of an $n$-abelian category $\mathcal{A}$, for all $A \in \mathcal{A}$ each right $\mathcal{X}$-approximation of $A$ is an epimorphism. If $\mathcal{X}$ is a cogenerating subcategory of an $n$-abelian category $\mathcal{A}$, for all $A \in \mathcal{A}$ each left $\mathcal{X}$-approximation of $A$ is a monomorphism.

Recall that a subcategory $\mathcal{X}$ of an abelian category $\mathcal{A}$ is called extension closed if each $n$-exact sequence in $\mathcal{A}$ of the form $X : X^0 \rightarrow A^1 \rightarrow A^2 \rightarrow \cdots \rightarrow A^n \rightarrow X^{n+1}$ with $X^0, X^{n+1} \in \mathcal{X}$ is equivalent to an $n$-exact sequence in $\mathcal{A}$ $X' : X^0 \rightarrow X^1 \rightarrow X^2 \rightarrow \cdots \rightarrow X^n \rightarrow X^{n+1}$ with $X^1, X^2, \cdots, X^n \in \mathcal{X}$.

The following is an $n$-abelian version of Wakamatsu’s Lemma.

\textbf{Theorem 4.2.} Let $\mathcal{X}$ be a full extension closed generating-cogenerating subcategory of an $n$-abelian category $\mathcal{A}$.

1. If $\phi : X_C \rightarrow C$ is a minimal right $\mathcal{X}$-approximation, an $n$-kernel of $\phi$ is a sequence

$$(a_1, a_2, \cdots, a_n) : 0 \rightarrow A^1 \xrightarrow{a_1} A^2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} A^n \xrightarrow{a_n} X_C$$

then
$$0 \rightarrow n\mathcal{E}^1(\mathcal{X}, A^1) \rightarrow n\mathcal{E}^1(\mathcal{X}, A^2) \rightarrow \cdots \rightarrow n\mathcal{E}^1(\mathcal{X}, A^{n-1}) \rightarrow n\mathcal{E}^1(\mathcal{X}, A^n) \rightarrow 0$$

is exact.

2. If $\varphi : C \rightarrow X_C$ is a minimal left $\mathcal{X}$-approximation, an $n$-cokernel of $\phi$ is a sequence

$$(b_1, b_2, \cdots, b_n) : X^C \xrightarrow{b_1} B^1 \xrightarrow{b_2} B^2 \xrightarrow{b_3} \cdots \xrightarrow{b_n} B^n \rightarrow 0$$

then
$$0 \rightarrow n\mathcal{E}^1(B^n, \mathcal{X}) \rightarrow n\mathcal{E}^1(B^{n-1}, \mathcal{X}) \rightarrow \cdots \rightarrow n\mathcal{E}^1(B^2, \mathcal{X}) \rightarrow n\mathcal{E}^1(B^1, \mathcal{X}) \rightarrow 0$$

is exact.

\textbf{Proof.} We prove only (1), since the proof of (2) is dual.

(1) Since $\mathcal{X}$ is a generating subcategory of $\mathcal{A}$, by Remark 4.1 we have that $\phi$ is an epimorphism. Thus the following sequence
$$0 \rightarrow A^1 \rightarrow A^2 \rightarrow \cdots \rightarrow A^n \rightarrow X_C \xrightarrow{\phi} C \rightarrow 0$$

is $n$-exact. For any $X \in \mathcal{X}$, by the Long $n$-exact Sequence Theorem \cite{5,6}, we have an exact sequence of groups

$$
\mathcal{A}(X, X_C) \xrightarrow{\phi} \mathcal{A}(X, C) \rightarrow nE^1(X, A^1) \xrightarrow{\delta} nE^1(X, A^2) \rightarrow \cdots \xrightarrow{\xi} nE^1(X, X_C) \xrightarrow{\phi^*} nE^1(X, C).
$$

Since $\phi$ is right $\mathcal{X}$-approximation of $C$, we have that $\phi_*$ is an epimorphism. It follows that $\delta$ is a monomorphism. We need to show that $\epsilon$ is an epimorphism. For this it is enough to show that $\phi^*$ is a monomorphism. For any $[E] \in nE^1(X, X_C)$ such that $\phi^*[E] = [O]$, we have the following commutative diagram

$$
\begin{array}{ccccccccc}
E : & & X_C & \xrightarrow{f} & M^1 & \rightarrow & M^2 & \rightarrow & \cdots & \rightarrow & M^{n-1} & \rightarrow & M^n & \rightarrow & X \\
\phi \cdot E & & C & \xrightarrow{\phi} & C & \xrightarrow{f^1} & 0 & \rightarrow & \cdots & \rightarrow & 0 & \rightarrow & X & \rightarrow & X
\end{array}
$$

Since $\mathcal{X}$ is extension closed and $X_C, X \in \mathcal{X}$, we can choose $M^1$ from $\mathcal{X}$. Since $\phi$ is right $\mathcal{X}$-approximation, there exists a morphism $t: M^1 \rightarrow X_C$ such that $\phi t = f^1$. It follows that

$$
\phi tf = f^1 f = \phi.
$$

Since $\phi$ is right minimal, we have that $tf$ is an isomorphism implies that $f$ is a split monomorphism. Hence, $[E] = [O]$. This shows that $\phi^*$ is a monomorphism.

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