DEFINING RELATIONS OF LOW DEGREE OF INVARIANTS OF TWO $4 \times 4$ MATRICES

VESSELIN DRENSKY AND ROBERTO LA SCALA

Abstract. The trace algebra $C_{nd}$ over a field of characteristic 0 is generated by all traces of products of $d$ generic $n \times n$ matrices, $n, d \geq 2$. Minimal sets of generators of $C_{nd}$ are known for $n = 2$ and 3 for any $d$ and for $n = 4$ and 5 and $d = 2$. The explicit defining relations between the generators are found for $n = 2$ and any $d$ and for $n = 3, d = 2$ only. Defining relations of minimal degree for $n = 3$ and any $d$ are also known. The minimal degree of the defining relations of any homogeneous minimal generating set of $C_{42}$ is equal to 12. Starting with the generating set given recently by Drensky and Sadikova, we have determined all relations of degree $\leq 14$. For this purpose we have developed further algorithms based on representation theory of the general linear group and easy computer calculations with standard functions of Maple.

Introduction

Let $K$ be any field of characteristic 0. All vector spaces, tensor products, algebras considered in this paper are over $K$. Let $X_i = \left( x_{pq}^{(i)} \right)$, $p, q = 1, \ldots, n$, $i = 1, \ldots, d$, be $d$ generic $n \times n$ matrices. We consider the pure (or commutative) trace algebra $C_{nd}$ generated by all traces of products $\text{tr}(X_{i_1} \cdots X_{i_k})$. It coincides with the algebra of invariants of the general linear group $GL_n = GL_n(K)$ acting by simultaneous conjugation on $d$ matrices of size $n \times n$. The algebra $C_{nd}$ is finitely generated. An upper bound for the degree of the trace monomials sufficient to generate $C_{nd}$ is given in terms of the Nagata-Higman theorem in the theory of PI-algebras. The defining relations of $C_{nd}$ are described by the Razmyslov-Procesi theory [R, P] in the

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language of ideals of the group algebras of symmetric groups. For a background on the algebras of matrix invariants see e.g. [F, DF] and for computational aspects of the theory see [D2].

Explicit minimal sets of generators of $C_{nd}$ are known for $n = 2$ and 3 for any $d$, and $n = 4$ and 5 for $d = 2$ only. The exact upper bound of the degree $k \leq N(n)$ of the trace polynomials $\text{tr}(X_{i_1} \cdots X_{i_k})$ sufficient to generate $C_{nd}$ is $N(2) = 3$, $N(3) = 6$, and $N(4) = 10$. Even less is known for the defining relations between these minimal sets of generators. For details on the explicit form of the defining relations for $n = 2, d \geq 2$ see e.g. [DF]. For $n = 3, d = 2$, a minimal generating set of $C_{32}$ consisting of 11 trace monomials of degree $\leq 6$ was found by Teranishi [T1]. He also calculated the Hilbert (or Poincaré) series of $C_{32}$. It follows from his description that, with respect to these generators, $C_{32}$ has a single defining relation of degree 12. The explicit form of the relation was found by Nakamoto [N], over $\mathbb{Z}$, with respect to a slightly different system of generators. Abeasis and Pittaluga [AP] found a system of generators of $C_{3d}$, for any $d \geq 2$, in terms of representation theory of the symmetric and general linear groups, in the spirit of its usage in theory of PI-algebras. Aslaksen, Drensky and Sadikova [ADS] gave the defining relation of $C_{32}$ with respect to the generators from [AP].

For $n = 3$ and $d > 2$ the defining relations of $C_{3d}$ seem to be very complicated. Recently, Benenti and Drensky [BD] have shown that for all $d > 2$ the minimal degree of the defining relations of $C_{3d}$ is equal to 7 and have found explicitly these relations with respect to the generators from [AP]. For $d = 3$ they have given also the relations of degree 8, using additional information from the Hilbert series of $C_{33}$ calculated by Berele and Stembridge [BS]. Independently, the defining relations of the algebra $C_{33}$ have been studied in the recent master thesis of Hoge [H]. Using representation theory of general linear groups and computer calculations with Maple, as in [ADS] and [BD], he developed a general algorithm and found the relations of degree 7 and some of the relations of degree 8.

For $C_{42}$, a set of generators was found by Teranishi [T1, T2] and a minimal set by Drensky and Sadikova [DS], in terms of the approach in [AP]. Djoković [Dj] gave another minimal set of 32 generators of $C_{42}$ consisting of trace monomials only (he found also a minimal set of 173 generators of $C_{52}$). Any homogeneous minimal generating set $\{u_i \mid i = 1, \ldots, 32\}$ of $C_{42}$ consists of $g_i$ elements of degree $i = 1, 2, \ldots, 10$,
where
\[ g_1 = 2, \quad g_2 = 3, \quad g_3 = 4, \quad g_4 = 6, \quad g_5 = 2, \]
\[ g_6 = 4, \quad g_7 = 2, \quad g_8 = 4, \quad g_9 = 4, \quad g_{10} = 1. \]

Hence \( C_{42} \) is isomorphic to the factor algebra \( K[y_1, \ldots, y_{32}] / I \). Defining \( \deg(y_i) = \deg(u_i) \), the ideal \( I \) is homogeneous. The comparison of the Hilbert series of \( C_{42} \) calculated by Teranishi [T2] (with some typos) and corrected by Berele and Stembridge [BS], with the Hilbert series of \( K[y_1, \ldots, y_{32}] \) gives that any homogeneous minimal system of generators of the ideal \( I \) contains no elements of degree \( \leq 11 \) and 5 elements of degree 12, see [DS]. The purpose of the present paper is to find the explicit form of the defining relations of minimal degree for \( C_{42} \), with respect to the generating set in [DS]. We have performed similar computations also for higher degrees, up to 14. The proofs are based on representation theory of \( GL_2 \) combined with computer calculations with Maple and develop further ideas of [ADS, DS]. In particular, we have found a way to write the defining relations in a compact form. Our methods are quite general and can be successfully used for further investigation of generic trace algebras and other algebras close to them.

Having in hand some defining relations of \( C_{42} \), we face the problem what is their meaning. We suggest the following point of view. It is known that the algebra \( C_{nd} \) is Cohen-Macaulay. It has a homogeneous system of parameters \( u_1, \ldots, u_p \) which are algebraically independent and \( C_{nd} \) is a finitely generated free \( K[u_1, \ldots, u_p] \)-module. Here \( p = (d-1)n^2 + 1 \) is the Krull dimension of \( C_{nd} \). In our case the homogeneous system of parameters of \( C_{42} \) consists of 17 of the 32 generators \( u_i \) of \( C_{42} \), say \( u_1, \ldots, u_{17} \), and the free \( K[u_1, \ldots, u_{17}] \)-module \( C_{42} \) is freely generated by a finite set of products

\[ \{ u_1^{a_1} \cdots u_{32}^{a_{32}} \mid (a_{18}, \ldots a_{32}) \in A \} \]

for some set of indices \( A \). The form of the relations of low degree which we have found agrees with the fact that every product \( u_1^{b_{18}} \cdots u_{32}^{b_{32}} \) can be written as a linear combination of the elements from (2) with coefficients from \( K[u_1, \ldots, u_{17}] \) and gives some restrictions on the indices \( (a_{18}, \ldots a_{32}) \).

1. Preliminaries

Till the end of the paper we fix \( n = 4 \) and \( d = 2 \) and denote by \( X, Y \) the two generic \( 4 \times 4 \) matrices. We shall denote \( C_{42} \) by \( C \). It is a standard trick to replace the generic matrices with generic traceless
matrices. We express \( X \) and \( Y \) in the form

\[
X = \frac{1}{4} \text{tr}(X)e + x, \quad Y = \frac{1}{4} \text{tr}(Y)e + y,
\]

where \( e \) is the identity \( 4 \times 4 \) matrix and \( x, y \) are generic traceless matrices. Then

\[
(3) \quad C \cong K[\text{tr}(X), \text{tr}(Y)] \otimes C_0,
\]

where the algebra \( C_0 \) is generated by the traces of products \( \text{tr}(z_1 \cdots z_k) \), \( z_i = x, y, \ 2 \leq k \leq 10 \). Hence the problem for the generators and the defining relations of \( C \) can be replaced by a similar problem for \( C_0 \).

As in the case of “ordinary” generic matrices, up to similarity we may replace \( x \) by a generic traceless diagonal matrix. Although not essential, this results in a simplification from a computational point of view. In fact, one of the worst drawbacks when computing with traces of polynomials in generic matrices is that these are commutative polynomials with a very high number of monomials. Then, without loss of generality we can fix the two generic traceless matrices as

\[
x = \begin{pmatrix}
x_{11} & 0 & 0 & 0 \\
0 & x_{22} & 0 & 0 \\
0 & 0 & x_{33} & 0 \\
0 & 0 & 0 & -(x_{11} + x_{22} + x_{33})
\end{pmatrix},
\]

\[
y = \begin{pmatrix}
y_{11} & y_{12} & y_{13} & y_{44} \\
y_{21} & y_{22} & y_{23} & y_{24} \\
y_{31} & y_{32} & y_{33} & y_{34} \\
y_{41} & y_{42} & y_{43} & -(y_{11} + y_{22} + y_{33})
\end{pmatrix},
\]

We summarize now the necessary background on representation theory of general linear groups \( GL_d \). We shall state everything for \( d = 2 \) only. See [M, W] for details on polynomial representations of \( GL_d \) and [D1] for their applications to PI-algebras. The group \( GL_2 = GL_2(K) \) acts in a canonical way on the vector space with basis \( \{x, y\} \) and this action induces a diagonal action on the free associative algebra \( K\langle x, y \rangle \):

\[
g(z_1 \cdots z_k) = g(z_1) \cdots g(z_k), \quad z_i = x, y, \quad g \in GL_2.
\]

The action of \( GL_2 \) on \( K\langle x, y \rangle \) induces an action on the algebras \( C \) and \( C_0 \). For \( C_0 \) it is given by

\[
g(\text{tr}(z_1 \cdots z_k)) = \text{tr}(g(z_1) \cdots g(z_k)), \quad z_i = x, y, \quad g \in GL_2.
\]

The \( GL_2 \)-module \( K\langle x, y \rangle \) is a direct sum of irreducible polynomial modules, described in terms of partitions \( \lambda = (\lambda_1, \lambda_2) \). We denote by \( W(\lambda) \) the corresponding \( GL_2 \)-module.
The $GL_2$-submodules and factor modules $W$ of $K\langle x, y \rangle$ inherit its natural bigrading which counts the entries of $x$ and $y$ in each monomial. We denote by $W^{(p, q)}$ the corresponding homogeneous component of degree $p$ and $q$ in $x$ and $y$, respectively. The formal power series

$$H(W, t, u) = \sum_{p, q \geq 0} \dim(W^{(p, q)})t^p u^q$$

is called the Hilbert series of $W$. The Hilbert series of $W(\lambda)$ is the Schur function $S_\lambda(t, u)$ which, in the case of two variables, has the simple form

$$S_\lambda(t, u) = (tu)^{\lambda_2} (t^{\lambda_1-\lambda_2} + t^{\lambda_1-\lambda_2-1}u + \ldots + tu^{\lambda_1-\lambda_2-1} + u^{\lambda_1-\lambda_2})$$.

The Hilbert series of $W$ plays the role of its character. The module $W(\lambda)$ participates in $W$ with multiplicity $m(\lambda)$, i.e.,

$$W = \bigoplus(W(\lambda))^{\oplus m(\lambda)}, \quad m(\lambda) \in \mathbb{N} \cup \{0\},$$

if and only if

$$H(W, t, u) = \sum m(\lambda)S_\lambda(t, u).$$

Let $C_0^+ = \omega(C_0)$ be the augmentation ideal of $C_0$. It consists of all trace polynomials $f(x, y) \in C_0$ without constant terms, i.e., satisfying the condition $f(0, 0) = 0$. Any minimal system of generators of $C_0^+$ lying in $C_0^+$ forms a basis of the vector space $C_0^+$ modulo $(C_0^+)^2$. Abeasis and Pittaluga [AP] suggested to fix the minimal system of generators of $C_{nd}$ in such a way that it spans a $GL_2$-module $G$. Then $C_{nd}$ is a homomorphic image of the symmetric algebra $K[G] = \text{Sym}(G)$ and the defining relations correspond to the generators of the kernel of the natural homomorphism $K[G] \to C_{nd}$. Drensky and Sadikova [DS] found that the minimal $GL_2$-module of generators of $C_{42}$ is decomposed as

$$G = W(1, 0) \oplus W(2, 0) \oplus W(3, 0) \oplus W(4, 0) \oplus W(2, 2)$$

$$\oplus W(3, 2) \oplus W(4, 2) \oplus W(3, 3) \oplus W(4, 3)$$

$$\oplus W(5, 3) \oplus W(4, 4) \oplus W(6, 3) \oplus W(5, 5).$$

Hence the minimal generating $GL_2$-module $G_0$ of $C_0$ is the direct sum of those modules in (5) which are different from $W(1, 0)$. For $\lambda = (\lambda_1, \lambda_2) \neq (5, 5)$, one may choose as a generator of $W(\lambda_1, \lambda_2) \subset G_0$ the canonical element

$$w_\lambda(x, y) = \text{tr}((xy - yx)^{\lambda_2} x^{\lambda_1}y^{\lambda_2})$$.
A generator of $W(5, 5)$ may be chosen as
\[ w_{(5,5)}(x, y) = \text{tr}((xy - yx)^3(x^2y^2 - xy^2x - yx^2y + y^2x^2)). \]

In [DS] it corresponds to the standard tableau

\[
\begin{bmatrix}
1 & 3 & 5 & 7 & 8 \\
2 & 4 & 6 & 9 & 10
\end{bmatrix}
\]

Since $C_0 \cong K[G_0]/J$ for an ideal $J$ which is also graded, the difference of the Hilbert series of $K[G_0]$ and $C_0$ gives the Hilbert series of $J$. By [DS], the Hilbert series of $J$ is
\[ H(J, t, u) = H(C_0, t, u) - H(K[G_0], t, u) = (S_{(7,5)}(t, u) + 2S_{(6,6)}(t, u)) \]
\[ + (S_{(8,5)}(t, u) + 2S_{(7,6)}(t, u)) + (2S_{(9,5)}(t, u) + 6S_{(8,6)}(t, u) + 2S_{(7,7)}(t, u)) \]
\[ + (2S_{(10,5)}(t, u) + 9S_{(9,6)}(t, u) + 7S_{(8,7)}(t, u)) + \ldots \]

Hence, the $GL_2$-modules $R_{12}, R_{13},$ and $R_{14}$ of the defining relations of degree 12, 13, and 14 are, respectively,
\[ R_{12} = W(7, 5) \oplus 2W(6, 6), \]
\[ R_{13} = W(8, 5) \oplus 2W(7, 6), \]
\[ R_{14} = 2W(9, 5) \oplus 6W(8, 6) \oplus 2W(7, 7). \]

Any submodule $W(\lambda) = W(\lambda_1, \lambda_2)$ of $K\langle x, y \rangle$ is generated by a unique, up to a multiplicative constant, homogeneous element $w(\lambda)(x, y)$ of degree $\lambda_1$ and $\lambda_2$ in $x$ and $y$, respectively, called the “highest weight vector” of $W(\lambda)$. It is characterized in the following way, see [DEP], [ADF] and [K] for the version which we need. We state it for two variables only. Recall that a linear operator $\delta$ on an algebra $R$ is called a derivation if $\delta(uv) = \delta(u)v + u\delta(v)$ for all $u, v \in R$. We define a derivation $\Delta$ of $K\langle x, y \rangle$ by putting
\[ \Delta(x) = 0, \quad \Delta(y) = x \]
and a linear operator $h \in GL_2$ by $h(x) = x$, $h(y) = x + y$, i.e.,
\[ h = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \]

The $GL_2$-submodules and factor modules of $K\langle x, y \rangle$ are invariant under the action of $\Delta$ and we can extend $\Delta$ also to tensor products, symmetric algebras, and other constructions with such modules. For example, if $W_1, W_2 \subset K\langle x, y \rangle$, we define $\Delta$ on the tensor product $W_1 \otimes W_2$ by $\Delta(w_1 \otimes w_2) = \Delta(w_1) \otimes w_2 + w_1 \otimes \Delta(w_2)$, $w_i \in W_i$. 
Lemma 1.1. ([ADF, DEP, K], see also [BD]) Let $\Delta$ and $h$ be defined as in (9) and (10), respectively. The homogeneous polynomial $w_\lambda(x, y) \in K \langle x, y \rangle$ of degree $(\lambda_1, \lambda_2)$ is a highest weight vector for some $W(\lambda_1, \lambda_2)$ if and only if $\Delta(w_\lambda(x, y)) = 0$ or, equivalently, $h(w_\lambda(x, y)) = w_\lambda(x, y)$.

If $W_i \subset K \langle x, y \rangle$, $i = 1, \ldots, k$, are $k$ isomorphic copies of $W(\lambda)$ and $w_i \in W_i$ are highest weight vectors, then $w_1, \ldots, w_k$ span a vector subspace $V = Kw_1 + \cdots + Kw_k$ of $K\langle x, y \rangle$ with the following property. The nonzero elements of $V$ are highest weight vectors of submodules $W(\lambda)$ of the sum $W_1 + \cdots + W_k$ and every highest weight vector can be obtained in such a way. The sum $W_1 + \cdots + W_k$ is direct if and only if $w_1, \ldots, w_k$ are linearly independent. The following statement is a direct consequence of Lemma 1.1.

Corollary 1.2. If $W(\lambda)$, $\lambda = (\lambda_1, \lambda_2)$, participates with multiplicity $m(\lambda)$ in the $GL_2$-submodule $W$ of $K\langle x, y \rangle$, then the vector space of the highest weight vectors $w_\lambda(x, y)$ is an $m(\lambda)$-dimensional subspace of the homogeneous component $W^{(\lambda_1, \lambda_2)}$ of $W$. Any basis $\{w_1, \ldots, w_{m(\lambda)}\}$ of this subspace generates the direct sum $(W(\lambda))^{\otimes m(\lambda)} \subset W$ as $GL_2$-submodule.

2. Algorithms

For our concrete computations we need the explicit form of the highest weight vectors in the symmetric algebra $K[G_0]$, where $G_0 = G/W(1,0)$ generates $C_0$ and $G$ is given in (5). In [ADS, BD, DS] a similar problem was solved by careful study of the symmetric tensor powers $K[W(\lambda)]$ and their tensor products, based on the Littlewood-Richardson rule (or, for $d = 2$, on its partial case, the Young rule) and symmetric tensor powers on the level of [M, Th]. In the present paper we use a simplified approach and work directly in the symmetric algebra $K[G_0]$. (After we had finished the computations we learned that a similar simplification was used independently by Hoge [H].) We define the derivation $\Delta_1$ of $K\langle x, y \rangle$ by

\begin{equation}
\Delta_1(x) = y, \quad \Delta_1(y) = 0
\end{equation}

and a linear operator $h_1 \in GL_2$ by $h_1(x) = x + y$, $h_1(y) = y$, i.e.,

\begin{equation}
h_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.
\end{equation}

As in the case of $\Delta$ from (9) we extend the action of $\Delta_1$ on $GL_2$-modules related with $K\langle x, y \rangle$. The following lemma gives an algorithm which finds a basis of $W(\lambda)$. 
Lemma 2.1. If \( \lambda = (a + b, b) \) and \( w(x, y) \in W(\lambda) \subset K[x, y] \) is a highest weight vector, then the set
\[
\left\{ \frac{\Delta_1(w)}{a}, \frac{\Delta_2^0(w)}{a(a - 1)}, \ldots, \frac{\Delta_a^0(w)}{a(a - 1) \cdots 2 \cdots 1} \right\}
\]
is a basis of the module \( W(\lambda) \). Here \( \Delta_1 \) is the derivation defined in (11).

Proof. It is well known that, starting with a highest weight vector \( w \in W(\lambda) \), the homogeneous components of \( h_1(w) \) form a basis of \( W(\lambda) \), where \( h_1 \in GL_2 \) is from (12). Now the proof follows from the fact that, up to a multiplicative constant, \( \Delta_k^1(w) \) is equal to the homogeneous component of degree \( (a + b - k, b + k) \) of \( \varepsilon_1(w) \), where \( \varepsilon_1 = \exp(\Delta_1) = 1 + \frac{\Delta_1}{1!} + \frac{\Delta_2^1}{2!} + \cdots \) is the related exponential automorphism of the locally nilpotent derivation \( \Delta_1 \), and \( h_1 = \exp(\Delta_1) \). \( \square \)

Example 2.2. (i) The \( GL_2 \)-module \( F(a) \) of the forms of degree \( a \) in the polynomial algebra \( K[x, y] \) in two variables \( x, y \) is isomorphic to \( W(a, 0) \) and \( w = x^a \) is its highest weight vector. Since \( \Delta_1(y) = 0 \), we obtain
\[
\Delta_1(w) = ax^{a-1}y, \quad \Delta_2^1(w) = a(a - 1)x^{a-2}y^2, \ldots,
\Delta_1^{a-1}(w) = a(a-1) \cdots 2:xy^{a-1}, \quad \Delta_2^a(w) = a(a-1) \cdots 2 \cdot 1 \cdot y^a, \quad \Delta_1^{a+1}(w) = 0.
\]
Hence Lemma 2.1 gives the basis of \( F(a) \)
\[
\{ x^a, x^{a-1}y, \ldots, xy^{a-1}, y^a \}.
\]

(ii) Consider the submodules of \( G_0 \) in (5). The basis of \( W(4, 0) \) consists of the highest weight vector
\[
w = \text{tr}(x^4),
\]
\[
\frac{\Delta_1(w)}{4} = \frac{1}{4} \text{tr}(yx^3 + xyx^2 + x^2y + x^3y) = \text{tr}(x^3y),
\]
\[
\frac{\Delta_2^1(w)}{4 \cdot 3} = \frac{1}{3} \text{tr}((yx^2 + xy + x^2y)y) = \frac{1}{3}(2\text{tr}(x^2y^2) + \text{tr}(xyxy)),
\]
\[
\frac{\Delta_3^1(w)}{4 \cdot 3 \cdot 2} = \text{tr}(xy^3),
\]
\[
\frac{\Delta_4^1(w)}{4 \cdot 3 \cdot 2 \cdot 1} = \text{tr}(y^4).
\]
The basis of \( W(5, 3) \subset G_0 \) consists of
\[
w = \text{tr}((xy - yx)^3x^2),
\]
\[
\frac{\Delta_1(w)}{2} = \frac{1}{2} \text{tr}((xy - yx)^3(yx + xy)),
\]
\[
\Delta_2^2(w) = \frac{1}{2} \text{tr}((xy - yx)^3y^2).
\]

Note that we make use of the fact that the trace of a product does not change under a cyclic permutation of its factors.

Applying Corollary 1.2 we obtain the following algorithm which is in the base of our further computations.

**Algorithm 2.3.**

**Input.** A partition \( \lambda = (\lambda_1, \lambda_2) \) and a system of highest weight vectors \( w_i \in W_i, \ i = 1, \ldots, k, \) where each \( W_i \) is an irreducible \( GL_2 \)-submodule of \( K[x, y] \).

**Output.** A basis of the vector space of highest weight vectors \( w_\lambda(x, y) \) in the symmetric algebra \( K[W] \) of the direct sum \( W = W_1 \oplus \cdots \oplus W_k. \)

**Step 1.** Applying Lemma 2.1, find homogeneous bases \( \{u_{i_0}, \ldots, u_{i_\alpha} \} \) of the modules \( W_i, \ i = 1, \ldots, k. \)

**Step 2.** In \( K[W] \), form all products

\[
w_p = \prod_{i=1}^{k} \prod_{j=0}^{a_i} u_{ij}^{r_{ij}}, \quad p = 1, \ldots, P,
\]

\[
v_q = \prod_{i=1}^{k} \prod_{j=0}^{a_i} u_{ij}^{s_{ij}}, \quad q = 1, \ldots, Q,
\]

which are of degree \( (\lambda_1, \lambda_2) \) and \( (\lambda_1 + 1, \lambda_2 - 1) \), respectively. Present each \( \Delta(w_p) \) in the form

\[
\Delta(w_p) = \sum_{q=1}^{Q} \alpha_{qp} v_q, \quad \alpha_{qp} \in K.
\]

**Step 3.** Consider the element

\[
w = \sum_{p=1}^{P} \xi_p w_p,
\]

with unknown coefficients \( \xi_p \in K. \) Calculate

\[
\Delta(w) = \sum_{q=1}^{Q} \left( \sum_{p=1}^{P} \alpha_{qp} \xi_p \right) v_q.
\]

**Step 4.** Solve the homogeneous linear system

\[
\sum_{p=1}^{P} \alpha_{qp} \xi_p = 0, \quad q = 1, \ldots, Q,
\]
whose equations are obtained from the equation $\Delta(w) = 0$.

**Step 5.** Any basis

$$\{ \Xi_r = (\xi_{1}^{(r)}, \ldots, \xi_{p}^{(r)}) \mid r = 1, \ldots, s \}$$

of the vector space of solutions of the system gives rise to a basis of the space of highest weight vectors.

**Remark 2.4.** Instead of solving one big system (16), we may solve several systems of smaller size. Let $W_i = W(\nu^{(i)})$ for some partition $\nu^{(i)}$. For each $m_1, \ldots, m_k$ such that

$$\sum_{i=1}^{k} m_i |\nu_i| = |\lambda|$$

the vector space $V(m_1, \ldots, m_k)$ spanned on those elements $w_p$ from (14) with

$$\sum_{j=0}^{a_i} r_{ij} = m_i, \quad i = 1, \ldots, k,$$

is a $GL_2$-submodule of $K[W]$. Since

$$K[W] = K[W_1 \oplus \cdots \oplus W_k] \cong K[W_1] \otimes \cdots \otimes K[W_k],$$

we derive that

$$V(m_1, \ldots, m_k) \cong W_1^\otimes m_1 \otimes \cdots \otimes W_k^\otimes m_k,$$

where $W_i^\otimes m_i$ is the $m_i$-th symmetrized tensor power of $W_i$. The sum of all $V(m_1, \ldots, m_k)$ is direct, and we may choose a basis of the vector space of the $\lambda$-highest weight vectors in $K[W]$ as the union of the corresponding bases in $V(m_1, \ldots, m_k)$. If $k > 1$, the homogeneous linear systems corresponding to $V(m_1, \ldots, m_k)$ are simpler than the whole system (16) for most of the $\lambda$.

Obvious modifications of Algorithm 2.3 give the highest weight vectors in other situations. For example, let $W_1$ and $W_2$ have homogeneous bases $\{u_0, u_1, \ldots, u_p\}$ and $\{v_0, v_1, \ldots, v_q\}$, respectively. If we want to find the highest weight vectors in the tensor product $W_1 \otimes W_2$, we have to solve the homogeneous linear system obtained from the equation

$$\Delta \left( \sum \xi_{ij} u_i \otimes v_j \right) = \sum \xi_{ij} (\Delta(u_i) \otimes v_j + u_i \otimes \Delta(v_j)) = 0,$$

where the sum is on all $i, j$ such that $u_i \otimes v_j$ is homogeneous of degree $(\lambda_1, \lambda_2)$. 


For each $W$ assume that $W$ is isomorphic to the tensor product $\text{det}^b$ in (6) and (7). Rewriting $\lambda$ one-dimensional $\text{GL}(19)$ in (5). We put:

$$H(W, t, u) = \sum_{i=1}^k S_{\nu, \lambda}(t, u) = \sum a_{bc} t^b u^c, \quad a_{bc} \in \mathbb{N} \cup \{0\}.$$  

Hence the Hilbert series of $K[W]$ is

$$H(K[W], t, u) = \prod_{b, c} \frac{1}{(1 - t^b u^c)^a_{bc}} = \sum_{p, q} h(p, q) t^p u^q, \quad h(p, q) \in \mathbb{N} \cup \{0\}.$$  

By (11) the Schur function $S_{\mu}(t, u)$ contains the summand $t^{\lambda_1} u^{\lambda_2}$ if and only if $\mu_1 + \mu_2 = \lambda_1 + \lambda_2$ and $\mu_1 \geq \lambda_1$. This easily implies that the multiplicity of $W(\lambda)$ is given by the formula

(18) $$m(\lambda) = h(\lambda_1, \lambda_2) - h(\lambda_1 + 1, \lambda_2 - 1).$$

Similarly, if we want to find the multiplicity of $W(\lambda)$ in the tensor product $W_1 \otimes \cdots \otimes W_k$, $W_i = W(\nu^{(i)})$, $i = 1, \ldots, k$, we have to present the product of the corresponding Schur functions in the form

$$\prod_{i=1}^k S_{\nu, \lambda}(t, u) = \sum_{p, q} h(p, q) t^p u^q, \quad h(p, q) \in \mathbb{N} \cup \{0\},$$

and to obtain the multiplicity of $W(\lambda)$ by the formula (18).

We want now to give a compact form for the highest weight vectors of the tensor products $V(m_1, \ldots, m_k)$ defined in (17). We fix an order on the summands $W_i$ in the decomposition of $G_0 = G/W(1, 0)$ given in (3). We put:

$$W_1 = W(2, 0), \quad W_2 = W(3, 0), \quad W_3 = W(4, 0),$$

$$W_4 = W(2, 2), \quad W_5 = W(3, 2), \quad W_6 = W(4, 2),$$

(19) $$W_7 = W(3, 3), \quad W_8 = W(4, 3), \quad W_9 = W(5, 3),$$

$$W_{10} = W(4, 4), \quad W_{11} = W(6, 3), \quad W_{12} = W(5, 5).$$

For each $W_i = W(\lambda)$ we fix a highest weight vector $w_i = w_\lambda(x, y)$ given in (6) and (7). Rewriting $\lambda = (\lambda_1, \lambda_2)$ in the form $\lambda = (a + b, b)$ we assume that $W_i = W(a_i + b_i, b_i)$. The $\text{GL}_2$-module $W(a_i + b_i, b_i)$ is isomorphic to the tensor product $\text{det}^b \otimes W(a_i, 0)$, where $\text{det}^b$ is the one-dimensional $\text{GL}_2$-module with $\text{GL}_2$-action defined by

$$g(v) = (\det(g))^b \cdot v, \quad g \in \text{GL}_2, v \in \text{det}^b.$$
The module $W(a_i, 0)$ has a natural realization as the module $F(a_i)$ of the forms of degree $a_i$ in two variables $x_i, y_i$. We fix a nonzero element of $\det^{b_i}$ and denote it by $t_i^{b_i}$. Omitting the symbol $\otimes$ for the tensor product, $\det^{b_i} \otimes F(a_i)$ has a basis
\[
\{ t_i^{b_i} x_i^{a_i}, t_i^{b_i} y_i^{a_i-1}, \ldots, t_i^{b_i} x_i y_i^{a_i-1}, t_i^{b_i} y_i^{a_i} \}
\]
with action of $GL_2$ defined by
\[
g( t_i^{b_i} x_i^{a_i} y_i^{a_i-1} ) = (\det(g))^{b_i} (g(x_i))^{\lambda_i} (g(y_i))^{\lambda_j}, \quad g \in GL_2.
\]
Using the highest weight vector $w_i$ of $W(a_i + b_i, b_i)$ from (6) and (7), we fix the $GL_2$-module isomorphism
\[
(20) \quad \varphi_i : \det^{b_i} \otimes F(a_i) \to W_i = W(a_i + b_i, b_i)
\]
which sends $t_i^{b_i} x_i^{a_i}$ to $w_i(x, y)$. The concrete form of the image of $t_i^{b_i} x_i^{a_i} y_i^{a_i-1}$ in $W(a_i + b_i, b_i)$ can be obtained applying Lemma 2.1. For the derivation $\Delta_1$ from (11),
\[
\Delta_1(\det^{b_i}) = 0,
\]
\[
\frac{1}{j} \Delta_1(x_i^{j} y_i^{a_i-j}) = x_i^{j-1} y_i^{a_i+1-j}
\]
and we define recursively
\[
\varphi_i(t_i^{b_i} x_i^{a_i}) = w_i(x, y),
\]
\[
\varphi_i(t_i^{b_i} x_i^{a_i-1} y_i^{a_i+1-j}) = \frac{1}{j} \Delta_1(\varphi_i(t_i^{b_i} x_i^{j} y_i^{a_i-j})), \quad j = a_i, a_i - 1, \ldots, 2, 1.
\]
For example, if $\lambda = (6, 3)$, then $W(\lambda) = W_{11}$ in the notation of (19),
\[
\varphi_{11}(t_{11}^{3} x_{11}^{3}) = w_{11}(x, y) = \text{tr}([x, y]^3 x^3),
\]
\[
\varphi_{11}(t_{11}^{3} x_{11}^{2} y_{11}) = \frac{1}{3} \Delta_1(w_{11}) = \frac{1}{3} \text{tr}([x, y]^3 (yx^2 + xyx + x^2 y)),
\]
\[
\varphi_{11}(t_{11}^{3} x_{11} y_{11}) = \frac{1}{3} \text{tr}([x, y]^3 (y^2 x + yxy + xy^2)),
\]
\[
\varphi_{11}(t_{11}^{3} y_{11}^{3}) = \text{tr}([x, y]^3 y^3).
\]
Now we extend the $GL_2$-module isomorphisms $\varphi_i$ to the symmetric algebras. Let
\[
(21) \quad \Phi : K \left[ \bigoplus_{i=1}^{12} \det^{b_i} \otimes F(a_i) \right] \to K[G_0]
\]
be defined by
\[
\Phi \left( \prod_{i=1}^{12} \prod_{j=0}^{a_i} (t_i^{b_i} x_i^{j} y_i^{a_i-j} c_{i,j} \right) = \prod_{i=1}^{12} \prod_{j=0}^{a_i} (\varphi_i(t_i^{b_i} x_i^{j} y_i^{a_i-j})) c_{i,j}, \quad c_{i,j} \geq 0.
\]
In order to avoid the confusion and to distinguish e.g. \((x_1y_1)^2 = (x_1y_1) \otimes (x_1y_1)\) and \((x_2^2(y_2^2) = x_2^2 \otimes y_2^2\) in \(F(2)^{\otimes 2}\), in the summands where \(\sum_{j=0}^{\alpha_i} C_{ij} > 1\), we shall denote the elements \(x_i^j y_i^{a_i-j}\) by \(z_i^{(j,a_i-j)}\). Hence, instead of \((x_1y_1)^2 = (x_1y_1) \otimes (x_1y_1)\) and \(x_1^2 y_1^2 = x_1^2 \otimes y_1^2\) we shall write \((z_1^{(1,1)})^2\) and \((z_1^{(2,0)})(z_1^{(0,2)})\), respectively. There is no confusion using \(t_i\) because

\[
W(a_i + b_i, b_i)^{\otimes m_i} \cong \det_{b_i}^{m_i} \otimes F(a_i)^{\otimes m_i}.
\]

For example, using the notation of (17) and (19) one has

\[
V(1, 2, 1, 0, \ldots, 0) = W_1 \otimes W_2^{\otimes 2} \otimes W_3
\]

\[
= W(2, 0) \otimes W(3, 0)^{\otimes 2} \otimes W(4, 0),
\]

\[
\varphi_1(y_1^2) = \text{tr}(y^2),
\]

\[
\varphi_2(x_2^3) = \varphi_2(z_2^{(3,0)}) = \text{tr}(x^3), \quad \varphi_2(x_2y_2^2) = \varphi_2(z_2^{(1,2)}) = \text{tr}(xy^2),
\]

\[
\varphi_3(x_3^3y_3) = \text{tr}(x^3y),
\]

\[
\Phi(y_1^2(z_2^{(3,0)})(z_2^{(1,2)})x_3^3y_3) = \text{tr}(y^2)\text{tr}(x^3)\text{tr}(xy^2)\text{tr}(x^3y).
\]

For

\[
V(2, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0) = W_1^{\otimes 2} \otimes W_2 \otimes W_5
\]

\[
= (W(2, 0)^{\otimes 2} \otimes W(3, 0) \otimes W(3, 2),
\]

\[
\varphi_1(x_1y_1) = \varphi_1(z_1^{(1,1)}) = \text{tr}(xy),
\]

\[
\varphi_1(y_1^2) = \varphi_1(z_1^{(0,2)}) = \text{tr}(y^2),
\]

\[
\varphi_2(x_2y_2^2) = \text{tr}(xy^2),
\]

\[
\varphi_5(t_5^2 x_5) = \text{tr}([x, y]^2 x),
\]

\[
\Phi((z_1^{(1,1)})(z_1^{(0,2)})x_2y_2^2t_5^2x_5) = \text{tr}(xy)\text{tr}(y^2)\text{tr}(xy^2)\text{tr}([x, y]^2 x).
\]

For

\[
V(2, 0, 2, 0, \ldots, 0) = W_1^{\otimes 2} \otimes W_4^{\otimes 2} = W(2, 0)^{\otimes 2} \otimes W(2, 2)^{\otimes 2},
\]

\[
\varphi_1((z_1^{(2,0)})(z_1^{(0,2)})) = \text{tr}(x^2)\text{tr}(y^2),
\]

\[
\varphi_4(t_4^2) = \text{tr}^2([x, y]^2),
\]

\[
\Phi((z_1^{(2,0)})(z_1^{(0,2)})t_4^2) = \text{tr}(x^2)\text{tr}(y^2)\text{tr}^2([x, y]^2).
\]
3. Computations and Results

We shall explain now the computations for degree 12. From (8) we see that it is sufficient to consider the cases \( \lambda = (7, 5) \) and \( \lambda = (6, 6) \) only. First, we use Algorithm 2.3 to find the highest weight vectors \( w_\lambda(x, y) \in K[G_0] \). Applying Step 1 of the algorithm we find bases of the submodules \( W_1, \ldots, W_{12} \) of \( G_0 \). By Step 2, we form all products (14) and (15) of degree \( (\lambda_1, \lambda_2) \) and \( (\lambda_1 + 1, \lambda_2 - 1) \), respectively.

For \( \lambda = (7, 5) \) we obtain that \( P = 155 \) and \( Q = 119 \), i.e., there are 155 products (14) of degree \( (7, 5) \) and 119 products (15) of degree \( (8, 4) \). Applying Steps 3 and 4 we compute that the system (16) has \( s = 36 \) linearly independent solutions which give rise to 36 linearly independent highest weight vectors. Hence \( W(7, 5) \) participates with multiplicity 36 in \( K[G_0] \). We call these 36 highest weight vectors \( w_1, \ldots, w_{36} \).

For \( \lambda = (6, 6) \) the corresponding data are \( P = 185 \), \( Q = 155 \) and the number of the linear independent highest weight vectors \( w_{(6,6)}(x, y) \in K[G_0] \) is \( s = 30 \).

The next step of the computations is to find the highest weight vectors of the \( GL_2 \)-modules \( W(\lambda) \subset R_{12} \) of the defining relations of degree 12. For \( \lambda = (7, 5) \) we proceed in the following way. We form the trace polynomial in \( K[G_0] \)

\[
(22) \quad w = \sum_{i=1}^{36} \zeta_i w_i
\]

where \( w_i \) are the 36 linearly independent highest weight vectors corresponding to the submodules \( W(7, 5) \) of \( K[G_0] \) and \( \zeta_i \) are unknown coefficients. Then we evaluate \( w \) on the generic traceless 4 \( \times \) 4 matrices \( x \) and \( y \) and obtain

\[
(23) \quad w(x, y) = \sum_{i=1}^{36} \zeta_i w_i(x, y) = \sum_{p,q=1}^{4} \sum_{i=1}^{36} \zeta_i w_i^{(p,q)}(x, y)e_{pq},
\]

where the \( (p, q) \)-entry \( w_i^{(p,q)}(x, y) \) of \( w_i(x, y) \) is a homogeneous polynomial of degree 12 in the entries \( x_{aa}, y_{b_1b_2} \) of \( x \) and \( y \). We require that \( w(x, y) = 0 \) which is equivalent to

\[
w^{(p,q)}(x, y) = \sum_{i=1}^{36} \zeta_i w_i^{(p,q)}(x, y) = 0, \quad p, q = 1, 2, 3, 4.
\]

We rewrite the relations \( w^{(p,q)}(x, y) = 0 \) in the form

\[
w^{(p,q)}(x, y) = \sum_{c,d} \alpha_{cd}^{(p,q)}(\zeta_1, \ldots, \zeta_{36}) \prod_a x_{aa}^{c_a} \prod_{b_1,b_2} y_{b_1b_2}^{d_{b_1b_2}} = 0.
\]
Since the coefficients $\alpha_{cd}^{(p,q)}$ are equal to 0, we obtain a homogeneous linear system

$$\alpha_{cd}^{(p,q)}(\zeta_1, \ldots, \zeta_{36}) = 0$$

with unknowns $\zeta_1, \ldots, \zeta_{36}$. The solutions of the system give rise to the highest weight vectors which generate the submodules $W(7,5)$ of the $GL_2$-module $R_{12}$ of defining relations of degree 12. The result of the computations is that the system has a unique nonzero solution which we shall give explicitly soon.

Similar computations for $\lambda = (6,6)$ give that the system, which corresponds to (24) in this case, has two linearly independent solutions. One of them is relatively simple:

$$v'_{(6,6)} = 3u_1(x, y) + 4u_2(x, y) + 6u_3(x, y) = 0,$$

where

$$u_1 = \text{tr}(x^2)\text{tr}(y^3)\text{tr}([x, y]^3x) - \text{tr}(y^2)\text{tr}(xy^2)\text{tr}([x, y]^3y) - 2\text{tr}(xy)\text{tr}(xy^2)\text{tr}([x, y]^3x) + 2\text{tr}(xy)\text{tr}(x^2y)\text{tr}([x, y]^3y) + \text{tr}(y^2)\text{tr}(x^2y)\text{tr}([x, y]^3x) - \text{tr}(y^2)\text{tr}(x^3)\text{tr}([x, y]^3y),$$

$$u_2 = -\text{tr}(y^3)\text{tr}([x, y]^3x^3) + 3\text{tr}(xy^2)\text{tr}([x, y]^3(yx^2 + xyx + x^2y)) - 3\text{tr}(x^2y)\text{tr}([x, y]^3(y^2x + yxy + xy^2)) + \text{tr}(x^3)\text{tr}([x, y]^3y^3),$$

$$u_3 = -\text{tr}([x, y]^2x)\text{tr}([x, y]^3y) + \text{tr}([x, y]^2y)\text{tr}([x, y]^3x).$$

Applying the isomorphism $\Phi$ from (21) we rewrite $u_1, u_2, u_3$ as

$$u_1 = \Phi(- (x_1y_2 - y_1x_2)^2(x_2y_8 - y_2x_8)t_8^3),$$

$$u_2 = \Phi((x_2y_{11} - y_2x_{11})^3t_{11}^3),$$

$$u_3 = \Phi(-(x_5y_8 - y_5x_8)t_{58}^2t_8^3),$$

Hence (25) has the form

$$v'_{(6,6)} = \Phi(- (x_1y_2 - y_1x_2)^2(x_2y_8 - y_2x_8)t_8^3 + 4(x_2y_{11} - y_2x_{11})^3t_{11}^2 - 6(x_5y_8 - y_5x_8)t_{58}^2t_8^3) = 0.$$
In the same notation the only solution for the case \( \lambda = (7, 5) \) is

\[
(27)
\]

\[
\begin{align*}
v_{(7,5)} &= \Phi(-6z_1^{(2,0)}(z_1^{(2,0)}z_1^{(0,2)}) - (z_1^{(1,1)})^2)t_7^3 \\
&- 4((z_1^{(2,0)}z_1^{(0,2)} - 6(z_1^{(1,1)})^2)x_9^2 + 10z_1^{(2,0)}z_1^{(1,1)}x_9y_9 - 5(z_1^{(2,0)})^2y_9^2)t_9^3 \\
&+ 4(x_1y_2 - y_1x_2)x_2((x_1y_2 + y_1x_2)x_8 - 2x_1x_2y_8)t_8^3 \\
&+ 16(x_1y_3 - y_1x_3)^2x_3^3t_7^3 - x_1^2t_4^2t_7^3 + 8x_1^2t_1^5 \\
&+ 28(z_2^{(3,0)}z_2^{(1,2)} - (z_2^{(2,1)})^2)t_7^3 - 48x_2x_{11}(x_2y_{11} - y_2x_{11})^2t_{11}^3 \\
&- 48x_3^2(x_3y_9 - y_3x_9)^2t_9^3 - 16x_9^2t_4^2t_9^3 \\
&- 24x_5x_8t_5^2t_8^3 + 4x_6^2t_6^2t_7^3) = 0
\end{align*}
\]

and the second relation for \( \lambda = (6, 6) \) is
\[ e^\alpha = \Phi(-108(z_2^{(2,0)}z_1^{(0,2)} - (z_1^{(1,1)})^2)^3 + 216(z_1^{(2,0)}z_1^{(0,2)} - (z_1^{(1,1)})^2)(x_3^2z_1^{(0,2)} - 2x_3yz_0x_1^{(1,1)} + y_3^2z_1^{(2,0)})^2 - 180(z_1^{(2,0)}z_1^{(0,2)} - (z_1^{(1,1)})^2)^3 t_4^2 - 12(54z_1^{(2,0)}(z_1^{(1,1)})^2(z_2^{(1,2)})^2 + 12(z_1^{(2,0)})^2z_1^{(0,2)}z_2^{(2,1)}z_3^{(0,3)} + 30z_1^{(2,0)}(z_1^{(1,1)})z_2^{(2,1)}z_3^{(0,3)} - 42(z_1^{(2,0)})^2z_1^{(1,1)}z_2^{(1,2)}z_3^{(0,3)} - 72z_1^{(2,0)}(z_1^{(1,1)}z_1^{(0,2)}z_2^{(2,1)}z_3^{(1,2)} + 9(z_1^{(2,0)})^2z_1^{(2,1)}(z_2^{(1,2)})^2 - 12z_1^{(2,0)}(z_1^{(1,1)}z_1^{(0,2)}z_2^{(3,0)}(0,3) - 42z_1^{(1,1)}(z_1^{(0,2)}z_2^{(3,0)}z_2^{(2,1)}) + 54(z_1^{(1,1)})^2z_1^{(0,2)}(z_2^{(2,1)})^2 - 54(z_1^{(1,1)})^2z_1^{(2,1)}z_2^{(2,1)})^2 + 30(z_1^{(2,0)}(z_1^{(1,1)})^2z_1^{(0,2)}z_2^{(3,0)}(0,3) + 7(z_1^{(2,0)}(z_1^{(0,2)}z_2^{(3,0)}z_3^{(0,3)} + 9z_1^{(2,0)}(z_1^{(0,2)})^2z_2^{(2,1)}z_3^{(0,3)} - 2(z_1^{(1,1)})^2z_3^{(0,3)} + 12z_1^{(2,0)}(z_1^{(0,2)}z_2^{(3,0)}z_2^{(1,2)} + 216(z_1^{(2,0)}z_1^{(0,2)} - (z_1^{(1,1)})^2)(z_1^{(0,2)}x_6y_6 + 2z_1^{(2,1)}x_6y_6 + z_1^{(2,0)}y_6^2)t_0^2 + 432(z_1^{(0,2)}x_7^2 - 2z_1^{(1,1)}x_6y_2 + z_1^{(2,0)}y_2^2)(x_2x_5 + z_1^{(1,1)}(x_2y_5 + x_5y_2) - z_1^{(2,0)}y_2x_5) + 432(-2(z_1^{(2,0)})^2z_1^{(1,1)}(z_1^{(3,1)}z_2^{(0,4)} - z_3^{(2,2)}z_1^{(1,3})) - 4z_1^{(2,0)}z_1^{(1,1)}z_3^{(3,0)}z_2^{(2,2)} + 7(z_1^{(2,0)})^3(z_3^{(0,3)}z_1^{(1,3)}) - z_1^{(0,2)}z_1^{(0,2)}z_2^{(2,2)}z_3^{(0,3)}) + (z_1^{(1,1)})^2(-5(z_1^{(2,0)})^2 + z_3^{(4,0)}z_1^{(0,4)} + 4z_3^{(3,1)}z_1^{(1,3)}) - 4z_1^{(1,1)}z_1^{(0,2)}z_2^{(4,0)}z_1^{(0,4)} + z_1^{(1,1)}z_3^{(3,1)}z_1^{(2,2)} + 2(z_1^{(0,2)}z_2^{(4,0)}z_2^{(2,2)} - z_3^{(3,1)})^2)) + 216(z_1^{(0,2)}x_4^2 - 2z_1^{(1,1)}x_3y_3 + 2z_1^{(2,0)}y_3^2x_4^2 + 33(z_1^{(2,0)}z_1^{(0,2)} - (z_1^{(1,1)})^2)t_4^2 + 36(-z_2^{(0,3)}x_3^3 + 3z_1^{(1,2)}x_3y_3 - 3z_2^{(2,1)}x_3y_3 + z_1^{(3,0)}y_3^3)(-x_1^2x_3z_2^{(0,3)} + x_1(x_1y_3 + 2x_1y_3)z_2^{(1,2)} - y_1(y_1x_3 + 2x_1y_3)z_2^{(2,1)} + y_1y_3z_2^{(2,1)}) + 45(x_1^2(z_2^{(2,1)}z_1^{(0,3)})^2 - (z_1^{(1,2)})^2) + 9(x_1y_6 - y_1x_6)^2t_4^2 - 108(x_1y_6 - y_1x_6)^2t_4^2 - 9(4(z_2^{(2,1)})^3z_1^{(0,3)} - 6z_2^{(3,0)}z_2^{(1,2)}z_1^{(0,3)} + (z_2^{(3,0)})^2(z_2^{(0,3)})^2 + 4z_2^{(3,0)}(z_1^{(2,1)})^2 - 3(z_1^{(2,1)})^2(z_1^{(0,2)})^2 - 144(x_2y_3 - y_2x_3)^3(x_3y_5 - y_3x_5)t_4^2 - 108((z_2^{(2,1)})^2z_2^{(0,3)} - (z_2^{(1,2)})^2z_2^{(0,3)} - z_2^{(2,1)}z_2^{(1,2)}x_6y_6 + (z_2^{(3,0)}z_2^{(1,2)} - (z_2^{(2,1)})^2y_6)t_6^2 + 432(-z_2^{(3,1)}z_2^{(0,4)} + 2z_2^{(3,1)}z_2^{(2,2)} + z_2^{(4,0)}(z_3^{(0,4)} - z_3^{(4,0)}z_2^{(0,4)}) - z_2^{(3,1)}z_3^{(3,1)}z_2^{(2,1)}z_3^{(0,4)} + 3(z_2^{(2,2)})^2t_4^2 - 36(z_2^{(4,0)}z_2^{(0,4)} - 4z_2^{(3,1)}z_3^{(3,1)} + 3(z_2^{(2,2)})^2t_4^2 + 14t_4^2 - 36t_4^2t_4^2 - 108(z_6^{(2,0)}z_6^{(0,2)} - (z_6^{(1,1)})^2t_4^2 + 24t_4^2 = 0.\]
We state the results as a theorem.

**Theorem 3.1.** The defining relations of degree 12 of the pure trace algebra generated by two traceless $4 \times 4$ generic matrices form a $GL_2$-module isomorphic to $W(7, 5) \oplus 2W(6, 6)$. The corresponding highest weight vectors

$$v_{(7,5)} = 0, \quad v'_{(6,6)} = 0, \quad v''_{(6,6)} = 0$$

are given in (27), (26), and (28).

The computational results for degree 13 and 14 are quite long expressions and we shall discuss them in the next section. Here for each $\lambda$ corresponding to a defining relation in (8) we give only the numbers $P$ and $Q$ of the vectors (14) and (15), the number $s$ of linearly independent solutions of the system (16) and the multiplicity $r$ of $W(\lambda)$ from (8):

$$
\begin{align*}
\lambda &= (8, 5) : \quad P = 203, \quad Q = 136, \quad s = 67, \quad r = 1, \\
\lambda &= (7, 6) : \quad P = 252, \quad Q = 203, \quad s = 49, \quad r = 2, \\
\lambda &= (9, 5) : \quad P = 284, \quad Q = 188, \quad s = 96, \quad r = 2, \\
\lambda &= (8, 6) : \quad P = 390, \quad Q = 284, \quad s = 106, \quad r = 6, \\
\lambda &= (7, 7) : \quad P = 418, \quad Q = 390, \quad s = 28, \quad r = 2.
\end{align*}
$$

(29)

4. Conclusions

The homogeneous system of parameters of $C_{42}$ found by Teranishi [T1, T2] contains all traces of degree $\leq 4$ and two elements of degree $(4, 2)$ and $(2, 4)$, respectively. If in the system of Teranishi we remove $\text{tr}(X), \text{tr}(Y)$ and replace $X, Y$ with $x, y$, respectively, we obtain a homogeneous system of parameters of the algebra $C_0$ generated by the generic traceless $4 \times 4$ matrices $x$ and $y$. Following [DS], we can choose for a homogeneous system of parameters of $C_0$ any 13 trace polynomials which form a $K$-basis of

$$W(2, 0) \oplus W(3, 0) \oplus W(4, 0) \oplus W(2, 2) = W_1 \oplus W_2 \oplus W_3 \oplus W_4 \subset C_0$$

and two more trace polynomials

$$\text{tr}((xy - yx)^2 x^2), \text{tr}((xy - yx)^2 y^2) \in W(4, 2) = W_6 \subset C_0.$$
Hence in (2) we can choose for $u_{18}, \ldots, u_{32}$ any basis of the direct sum from (19)

$$W_5 \oplus W_7 \oplus W_8 \oplus W_9 \oplus W_{10} \oplus W_{11} \oplus W_{12}$$

and the only homogeneous polynomial of degree $(3, 3)$ in $W(4, 2) = W_6$.

Using Lemma 2.1 we construct homogeneous bases $\{u_{i0}, \ldots, u_{ia_i}\}$ of the modules $W_i = W(a_i + b_i, b_i)$ from (19). In this notation, we fix the homogeneous system of parameters of $C_0$ consisting of

$$(30) \quad \{u_{ij}, u_{60}, u_{62} \mid i = 1, 2, 3, 4, j = 0, 1, \ldots, a_i\}$$

and complete it to a system of generators of $C_0$ by

$$(31) \quad \{u_{ij}, u_{61} \mid i = 5, 7, 8, 9, 10, 11, 12, j = 0, 1, \ldots, a_i\}.$$ 

It is easy to see that the finitely generated free $S$-module $C_0$, where

$$(32) \quad S = K[u_{ij} \mid i = 1, 2, 3, 4, j = 0, 1, \ldots, a_i, i = 6, j = 0, 2],$$ 

has a basis of the form

$$(33) \quad B = \left\{ \prod_{i,j} u_{ij}^{b_{ij}} \mid i = 5, 6, \ldots, 12 \right\},$$

where the $u_{ij}$’s are from (31) and the $b_{ij}$’s belong to some set of indices. Hence every product of elements from (31) can be presented as a linear combination of elements in $B$ from (33) with coefficients in $S$. We shall use the defining relations of degree 12, 13, and 14, to give some restriction on the integers $b_{ij}$.

We start with the highest weight vectors $v_{(7,5)}$, $v'_{(6,6)}$, $v''_{(6,6)}$ from (26), (27), and (28). The trace polynomial $v_{(7,5)}$ is of the form

$$(34) \quad v_{(7,5)} = -24u_{50}u_{80} + 4u_{60}u_{70} + \cdots ,$$

where $\cdots$ stays for the linear combination of products of the generators (31) with coefficients which are polynomials in $S$, do not depend on $u_{60}, u_{62}$, and are without constant term (i.e., from the augmentation ideal $\omega(S)$ of $S$). By Lemma 2.1 the $GL_2$-module generated by $v_{(7,5)}$ has a basis

$$\left\{ v_{(7,5)}, \frac{1}{2}\Delta_1(v_{(7,5)}), \frac{1}{2}\Delta_1^2(v_{(7,5)}) \right\}.$$ 

Direct computations show that

$$\Delta_1(v_{(7,5)}) = -24(\Delta_1(u_{50})u_{80} + u_{50}\Delta_1(u_{80}))$$

$$+ 4(\Delta_1(u_{60})u_{70} + u_{60}\Delta_1(u_{70})) + \cdots ,$$

and, since $\Delta_1(u_{70}) = 0$,

$$(35) \quad \frac{1}{2}\Delta_1(v_{(7,5)}) = -24(u_{51}u_{80} + u_{50}u_{81}) + 4u_{61}u_{70} + \cdots .$$
Similarly

\[ \frac{1}{2} \Delta_2^2(v_{(7,5)}) = -48u_{51}u_{81} + 4u_{62}u_{70} + \cdots . \]

The equations (34), (35), and (36) imply that

\[ \frac{1}{2} \Delta_1(v_{(7,5)}) = -24u_{50}u_{80}, \]

\[ \frac{1}{2} \Delta_1^2(v_{(7,5)}) = -24(u_{51}u_{80} + u_{50}u_{81}) + 4u_{61}u_{70}, \]

\[ \frac{1}{2} \Delta_2^1(v_{(7,5)}) = -48u_{51}u_{81} \]

modulo \( \omega(S)B \). In the same way, \( v'_{(6,6)}, v''_{(6,6)} \) generate one-dimensional \( GL_2 \)-modules isomorphic to \( W(6,6) \) and can be written in the form

\[ v'_{(6,6)} = -6(u_{50}u_{81} - u_{51}u_{80}), \]

\[ v''_{(6,6)} = 108u_{61}^2 + 24u_{70}^2 \]

modulo \( \omega(S)B \). We order the trace polynomials from (31) by

\[ u_{50} > u_{51} > u_{70} > u_{80} > u_{81} > u_{61} > u_{90} > \cdots > u_{12,0}, \]

i.e., \( u_{i_1j_1} > u_{i_2j_2} \) if \( i_1 < i_2 \) or \( i_1 = i_2, j_1 < j_2 \), except the case \( u_{70} > u_{80} > u_{81} > u_{61} \). Then we extend the order lexicographically on the products of (31). Hence, (37) and (38) give five relations such that, modulo \( \omega(S)B \), their leading monomials are

\[ u_{50}u_{80}, \quad u_{50}u_{81}, \quad u_{51}u_{80}, \quad u_{51}u_{81}, \quad u_{70}^2. \]

The defining relations of degree 13 and 14 which have been found in the same way as the defining relations of degree 12 show that the corresponding highest weight vectors are of the form

\[ v_{(8,5)} = u_{50}u_{90} - u_{60}u_{80} + \cdots , \]

\[ v'_{(7,6)} = 4(u_{50}u_{91} - u_{51}u_{90}) + (u_{60}u_{81} - u_{61}u_{80}) + \cdots , \]

\[ v''_{(7,6)} = u_{50}u_{10,0} - 2u_{70}u_{80} + \cdots . \]
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\[ v'_{(9,5)} = 2u_{50}u_{11,0} - u_{60}u_{90} + \cdots, \]
\[ v''_{(9,5)} = \cdots, \]
\[ v'_{(8,6)} = u_{60}u_{10,0} - 2u_{70}u_{90} + \cdots, \]
\[ v''_{(8,6)} = u_{50}u_{11,1} - u_{51}u_{11,0} + u_{60}u_{91} - u_{61}u_{90} + \cdots, \]
\[ v''_{(8,6)} = -7u_{60}u_{10,0} + 12u_{70}u_{90} + 12u^2_{80} + \cdots, \]
\[ v^{(4)}_{(8,6)} = \cdots, \]
\[ v^{(5)}_{(8,6)} = \cdots, \]
\[ v^6_{(8,6)} = \cdots, \]
\[ v'_{(7,7)} = -6(u_{60}u_{92} - 2u_{61}u_{91} + u_{62}u_{90}) + u_{70}u_{10,0} + \cdots, \]
\[ v''_{(7,7)} = \cdots, \]

with the same meaning of \( \cdots \) as above. Applying several times the derivation \( \Delta_1 \) on the highest weight vectors from (41) and (42) we obtain that they generate irreducible \( GL_2 \)-modules with bases which, modulo \( \omega(S)B \), have leading monomials of the form

\[ u_{50}u_{90}, \quad u_{50}u_{91}, \quad u_{51}u_{90}, \quad u_{50}u_{92}, \quad u_{51}u_{91}, \quad u_{51}u_{92}, \]
\[ u_{50}u_{10,0}, \quad u_{51}u_{10,0}. \]

The leading monomials in the case of degree 14 are

\[ u_{50}u_{11,0}, \quad u_{50}u_{11,1}, \quad u_{51}u_{11,0}, \quad u_{50}u_{11,2}, \]
\[ u_{51}u_{11,1}, \quad u_{50}u_{11,3}, \quad u_{51}u_{11,2}, \quad u_{51}u_{11,3}, \]
\[ u_{70}u_{90}, \quad u_{70}u_{91}, \quad u_{70}u_{92}, \]
\[ u_{70}u_{10,0}, \quad u_{50}^2, \quad u_{80}^2u_{81}, \quad u_{81}^2. \]

**Theorem 4.1.** Let us fix the homogeneous system of parameters (30) of \( C_0 \), complete it to a system of generators by (31), and let \( S \) be defined in (32). The finitely generated free \( S \)-module \( C_0 \) has a basis of the form (33) such the products in \( B \) do not contain factors \( u_{i_1j_1}u_{i_2j_2} \) from the lists given in (40), (43), (44).

**Proof.** We order the elements (31) by (39). The leading monomials of the defining relations of degree 12, 13, and 14 are given in (40), (43), and (44). If the free generating set contains a monomial from these
lists, we can replace it by a linear combination of monomials which are lower in the lexicographic order and monomials from $\omega(S)C_0$.

\begin{remark}
The generating function of the leading monomials from (40), (43), and (44) is equal to $L(t, u) = L_{12} + L_{13} + L_{14}$, where
\begin{align*}
L_{12}(t, u) &= t^7u^5 + 3t^6u^6 + t^5u^7 = S(7, 5)(t, u) + 2S(6, 6)(t, u), \\
L_{13}(t, u) &= t^8u^5 + 3t^7u^6 + 3t^6u^7 + t^5u^8 = S(8, 5)(t, u) + 2S(7, 6)(t, u), \\
L_{14}(t, u) &= t^9u^5 + 4t^8u^6 + 5t^7u^7 + 4t^6u^8 + t^5u^9 \\
&= S(9, 5)(t, u) + 3S(8, 6)(t, u) + S(7, 7)(t, u).
\end{align*}
Comparing with the Hilbert series $H(R_i, t, u)$ of the defining relations $R_i$ of degree $i = 12, 13, 14$, from (8), respectively, we see that $L_{12}(t, u) = H(R_{12}, t, u)$, $L_{13}(t, u) = H(R_{13}, t, u)$, and
\begin{align*}
H(R_{14}, t, u) - L_{14}(t, u) &= S(9, 5)(t, u) + 3S(8, 6)(t, u) + S(7, 7)(t, u).
\end{align*}
The explanation is the following. We multiply the trace polynomials $v_{(7, 5)}, \frac{1}{2}\Delta_1(v_{(7, 5)}), \frac{1}{2}\Delta_2(v_{(7, 5)}), v'_{(6, 6)}, v''_{(6, 6)}$ of degree 12 by the polynomials $u_{10}, u_{11} \in W_1 = W(2, 0) \subset G_0$ and obtain linearly independent relations of degree 14 with generating function which turns to be equal to the difference $H(R_{14}, t, u) - L_{14}(t, u)$. Hence the new relations of degree 14, which cannot be obtained from relations of lower degree, form a $GL_2$-module isomorphic to
\begin{align*}
W(9, 5) \oplus 3W(8, 6) \oplus W(7, 7).
\end{align*}

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Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Acad. G. Bonchev Str., Block 8, 1113 Sofia, Bulgaria
E-mail address: drensky@math.bas.bg

Dipartimento di Matematica, Università di Bari, Via E. Orabona 4, 70125 Bari, Italia
E-mail address: lascala@dm.uniba.it