Projection Operator in Adaptive Systems

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Abstract

The projection algorithm is frequently used in adaptive control and this note presents a detailed analysis of its properties.

1 Introduction

These notes started in [2] as a personal communication from Eugene to colleagues in the field of adaptive control and summarized results from [5, 3, 1, 4]. Properties of the projection operator are explored in detail in the following section.

2 Properties of Convex Sets and Functions

Definition 1. A set $E \subset \mathbb{R}^k$ is convex if

$$\lambda x + (1 - \lambda)y \in E$$

whenever $x \in E$, $y \in E$, and $0 \leq \lambda \leq 1$

Remark. Essentially, a convex set has the following property. For any two points $x, y \in E$ where $E$ is convex, all the points on the connecting line from $x$ to $y$ are also in $E$.

Definition 2. A function $f : \mathbb{R}^k \to \mathbb{R}$ is convex if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

$\forall 0 \leq \lambda \leq 1$. 

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Lemma 3. Let $f(\theta) : \mathbb{R}^k \to \mathbb{R}$ be a convex function. Then for any constant $\delta > 0$ the subset $\Omega_\delta = \{ \theta \in \mathbb{R}^k | f(\theta) \leq \delta \}$ is convex.

Proof. Let $\theta_1, \theta_2 \in \Omega_\delta$. Then $f(\theta_1) \leq \delta$ and $f(\theta_2) \leq \delta$. Since $f(x)$ is convex then for any $0 \leq \lambda \leq 1$

$$f\left(\lambda \theta_1 + (1 - \lambda) \theta_2\right) \leq \lambda f(\theta_1) + (1 - \lambda) f(\theta_2) \leq \lambda \delta + (1 - \lambda) \delta = \delta$$

$\therefore f(\theta) \leq \delta$, thus, $\theta \in \Omega_\delta$.

Lemma 4. Let $f(\theta) : \mathbb{R}^k \to \mathbb{R}$ be a continuously differentiable convex function. Choose a constant $\delta > 0$ and consider $\Omega_\delta = \{ \theta \in \mathbb{R}^k | f(\theta) \leq \delta \} \subset \mathbb{R}$. Let $\theta^\ast$ be an interior point of $\Omega_\delta$, i.e. $f(\theta^\ast) < \delta$. Choose $\theta_b$ as a boundary point so that $f(\theta_b) = \delta$. Then the following holds:

$$ (\theta^\ast - \theta_b)^T \nabla f(\theta_b) \leq 0 \quad (1) $$

where $\nabla f(\theta_b) = \left( \frac{\partial f(\theta)}{\partial \theta_1} \cdots \frac{\partial f(\theta)}{\partial \theta_k} \right)^T$ evaluated at $\theta_b$.

Proof. $f(\theta)$ is convex $\therefore$

$$ f(\lambda \theta^\ast + (1 - \lambda) \theta_b) \leq \lambda f(\theta^\ast) + (1 - \lambda) f(\theta_b) $$

equivalently,

$$ f(\theta_b + \lambda (\theta^\ast - \theta_b)) \leq f(\theta_b) + \lambda (f(\theta^\ast) - f(\theta_b)) $$

For any $0 < \lambda \leq 1$:

$$ \frac{f(\theta_b + \lambda (\theta^\ast - \theta_b)) - f(\theta_b)}{\lambda} \leq f(\theta^\ast) - f(\theta_b) \leq \delta - \delta = 0 $$

and taking the limit as $\lambda \to 0$ yields (1). \qed

3 Projection

Definition 5. The Projection Operator for two vectors $\theta, y \in \mathbb{R}^k$ is now introduced as

$$ \text{Proj}(\theta, y, f) = \begin{cases} y - \frac{\nabla f(\theta)(\nabla f(\theta))^T}{\| \nabla f(\theta) \|^2} y f(\theta) & \text{if } f(\theta) > 0 \land y^T \nabla f(\theta) > 0 \\ y & \text{otherwise.} \end{cases} \quad (2) $$

where $f : \mathbb{R}^k \to \mathbb{R}$ is a convex function and $\nabla f(\theta) = \left( \frac{\partial f(\theta)}{\partial \theta_1} \cdots \frac{\partial f(\theta)}{\partial \theta_k} \right)^T$. Note that the following are notationally equivalent $\text{Proj}(\theta, y) = \text{Proj}(\theta, y, f)$ when the exact structure of the convex function $f$ is of no importance.
Remark. A geometrical interpretation of (2) follows. Define a convex set \( \Omega_0 \) as
\[
\Omega_0 \triangleq \{ \theta \in \mathbb{R}^k | f(\theta) \leq 0 \} \tag{3}
\]
and let \( \Omega_1 \) represent another convex set such that
\[
\Omega_1 \triangleq \{ \theta \in \mathbb{R}^k | f(\theta) \leq 1 \} \tag{4}
\]
From (3) and (4) \( \Omega_0 \subset \Omega_1 \). From the definition of the projection operator in (7) \( \theta \) is not modified when \( \theta \in \Omega_0 \). Let
\[
\Omega_A \triangleq \Omega_1 \setminus \Omega_0 = \{ \theta | 0 < f(\theta) \leq 1 \}
\]
represent an annulus region. Within \( \Omega_A \) the projection algorithm subtracts a scaled component of \( y \) that is normal to boundary \( \{ \theta | f(\theta) = \lambda \} \). When \( \lambda = 0 \), the scaled normal component is 0, and when \( \lambda = 1 \), the component of \( y \) that is normal to the boundary \( \Omega_1 \) is entirely subtracted from \( y \), so that \( \text{Proj}(\theta, y, f) \) is tangent to the boundary \( \{ \theta | f(\theta) = 1 \} \). This discussion is visualized in Figure [1].

![Figure 1: Visualization of Projection Operator in \( \mathbb{R}^2 \).](image)

Remark. Note that \((\nabla f(\theta))^T \text{Proj}(\theta, y) = 0 \forall \theta \) when \( f(\theta) = 1 \) and that the general structure of the algorithm is as follows
\[
\text{Proj}(\theta, y) = y - \alpha(t) \nabla f(\theta) \tag{5}
\]
for some time varying \( \alpha \) when the modification is triggered. Multiplying the left hand side of the equation by \((\nabla f(\theta))^T \) and solving for \( \alpha \) one finds that
\[
\alpha(t) = (\nabla f(\theta))^T \nabla f(\theta)^{-1} (\nabla f(\theta))^T y \tag{6}
\]
and thus the algorithm takes the form
\[
\text{Proj}(\theta, y) = y - \nabla f(\theta) (\nabla f(\theta))^T \nabla f(\theta)^{-1} (\nabla f(\theta))^T y f(\theta) \tag{7}
\]
where the modification is active. Notice that the \( f(\theta) \) has been added to the definition, making (7) continuous.

**Lemma 6.** One important property of the projection operator follows. Given \( \theta^* \in \Omega_0 \),
\[
(\theta - \theta^*)^T (\text{Proj}(\theta, y, f) - y) \leq 0. \tag{8}
\]
Proof. Note that

\[(\theta - \theta^*)(\text{Proj}(\theta, y, f) - y) = (\theta^* - \theta)^T(y - \text{Proj}(\theta, y, f))\]

If \(f(\theta) > 0 \land y^T\nabla f(\theta) > 0\), then

\[ (\theta^* - \theta)^T \left( y - \left( y - \frac{\nabla f(\theta)(\nabla f(\theta))^T}{\|\nabla f(\theta)\|^2} y f(\theta) \right) \right) \]

and using Lemma 4

\[ \frac{(\theta^* - \theta)^T \nabla f(\theta) (\nabla f(\theta))^T y}{\|\nabla f(\theta)\|^2} \begin{cases} \leq 0 & \text{if } f(\theta) \leq 0 \\ > 0 & \text{if } f(\theta) > 0 \end{cases} \]

otherwise \(\text{Proj}(\theta, y, f) = y\). \hfill \Box

**Definition 7 (Projection Operator).** The general form of the projection operator is the \(n \times m\) matrix extension to the vector definition above.

\[
\text{Proj}(\Theta, Y, F) = [\text{Proj}(\theta_1, y_1, f_1) \ldots \text{Proj}(\theta_m, y_m, y_m)]
\]

where \(\Theta = [\theta_1 \ldots \theta_m] \in \mathbb{R}^{n \times m}, Y = [y_1 \ldots y_m] \in \mathbb{R}^{n \times m}, \) and \(F = [f_1(\theta_1) \ldots f_m(\theta_m)]^T \in \mathbb{R}^{m \times 1}\). Recalling (2)

\[
\text{Proj}(\theta_j, y_j, f_j) = \begin{cases} y_j - \frac{\nabla f_j(\theta_j)(\nabla f_j(\theta_j))^T}{\|\nabla f_j(\theta_j)\|^2} y_j f_j(\theta_j) & \text{if } f_j(\theta_j) > 0 \land y_j^T \nabla f_j(\theta_j) > 0 \\ y_j & \text{otherwise} \end{cases}
\]

\(j = 1 \text{ to } m\).

**Lemma 8.** Let \(F = [f_1 \ldots f_m]^T \in \mathbb{R}^{m \times 1}\) be a convex vector function and \(\hat{\Theta} = [\hat{\theta}_1 \ldots \hat{\theta}_m], \Theta = [\theta_1 \ldots \theta_m], Y = [y_1 \ldots y_m] \) where \(\hat{\Theta}, \Theta, Y \in \mathbb{R}^{n \times m}\) then,

\[
\text{trace} \left\{ (\hat{\Theta} - \Theta)^T (\text{Proj}(\hat{\Theta}, Y, F) - Y) \right\} \leq 0.
\]

**Proof.** Using (8),

\[
\text{trace} \left\{ (\hat{\Theta} - \Theta)^T (\text{Proj}(\hat{\Theta}, Y, F) - Y) \right\} = \sum_{j=1}^{m} (\hat{\theta}_j - \theta_j)^T (\text{Proj}(\hat{\theta}_j, y_j, f_j) - y_j) \leq 0 \Box
\]

The application of the projection algorithm in adaptive control is explored below.

**Lemma 9.** If an initial value problem, i.e. adaptive control algorithm with adaptive law and initial conditions, is defined by:

1. \(\dot{\theta} = \text{Proj}(\theta, y, f)\)
2. \(\theta(t = 0) = \theta_0 \in \Omega_1 = \{\theta \in \mathbb{R}^k | f(\theta) \leq 1\}\)
3. $f(\theta): \mathbb{R}^k \to \mathbb{R}$ is convex

Then $\theta(t) \in \Omega_1 \forall t \geq 0$.

**Proof.** Taking the time derivative of the convex function

$$\dot{f}(\theta) = (\nabla f(\theta))^T \dot{\theta} = (\nabla f(\theta))^T \text{Proj}(\theta, y, f)$$

(9)

Substitution of (9) into (2) leads to

$$\dot{f}(\theta) = (\nabla f(\theta))^T \text{Proj}(\theta, y, f)$$

$$= \begin{cases} (\nabla f(\theta))^T y (1 - f(\theta)) & \text{if } f(\theta) > 0 \wedge y^T \nabla f(\theta) > 0 \\ (\nabla f(\theta))^T y & \text{if } f(\theta) \leq 0 \vee y^T \nabla f(\theta) \leq 0 \end{cases}$$

therefore

$$\dot{f}(\theta) = \begin{cases} > 0 & \text{if } 0 < f(\theta) < 1 \wedge y^T \nabla f(\theta) > 0 \\ = 0 & \text{if } f(\theta) = 1 \wedge y^T \nabla f(\theta) > 0 \\ < 0 & \text{if } f(\theta) \leq 0 \vee y^T \nabla f(\theta) \leq 0 \end{cases}$$

Thus $f(\theta_0) \leq 1 \Rightarrow f(\theta) \leq 1 \forall t \geq 0$. □

**Remark.** Given $\theta_0 \in \Omega_0$, $\theta$ may increase up to the boundary where $f(\theta) = 1$. However, $\theta$ never leaves the convex set $\Omega_1$.

**Example 10** (Projection Algorithm in Adaptive Control Law). Let $\Theta(t): \mathbb{R}^+ \to \mathbb{R}^{m \times n}$ represent a time varying feedback gain in a dynamical system. This feedback gain is implemented as:

$$u = \Theta(t)^T x$$

where $u \in \mathbb{R}^n$ represents the control input and $x \in \mathbb{R}^m$ the state vector. The time varying feedback gain is adjusted using the following adaptive law

$$\dot{\Theta} = \text{Proj}(\Theta, -xe^T PB, F)$$

where $e \in \mathbb{R}^m$ is an error signal in the state vector space, $P \in \mathbb{R}^{m \times m}$ is a square matrix derived from a Lyapunov relationship and $B \in \mathbb{R}^{m \times n}$ is the input Jacobian for the LTI system to be controlled and $F(\Theta) = [f_1(\theta_1) \ldots f_m(\theta_m)]^T$. The projection algorithm operates with the family of convex functions

$$f(\theta; \vartheta, \varepsilon) = \frac{||\theta||^2 - \vartheta^2}{2\varepsilon \vartheta + \varepsilon^2}.$$  

Then, the components of the convex vector function $F$ are chosen as

$$f_i(\theta_i) = f(\theta_i; \vartheta_i, \varepsilon_i).$$

(10)

Each $i$–th component of $F$ is associated with two constant scalar quantities $\vartheta_i$ and $\varepsilon_i$. From (10), $f_i(\theta_i) = 0$ when $||\theta_i|| = \vartheta_i$, and $f_i(\theta_i) = 1$ when $||\theta_i|| = \vartheta_i + \varepsilon_i$. If the initial condition for $\Theta$ is such that $\Theta(t = 0) \in \Theta_0 = [\theta_{0,1} \ldots \theta_{0,m}]$ where $\{\theta_{0,i} | f_i(\theta_i) \leq 0 \}$ is a subset of $\{\theta_{0,i} | f_i(\theta_i) \leq 0 \}$, then each $\theta_i$ satisfies all three conditions for Lemma 9. Thus $||\theta_i(t)|| \leq \vartheta_i + \varepsilon_i \forall t \geq 0.$
4 $\Gamma$–Projection

**Definition 11.** A variant of the projection algorithm, $\Gamma$–projection, updates the parameter along a symmetric positive definite gain $\Gamma$ as defined below

$$
\text{Proj}_\Gamma(\theta, y, f) = \begin{cases} 
\Gamma y - \Gamma \frac{\nabla f(\theta)(\nabla f(\theta))^T}{(\nabla f(\theta))^T \Gamma \nabla f(\theta)} \Gamma y f(\theta) & \text{if } f(\theta) > 0 \wedge y^T \Gamma \nabla f(\theta) > 0 \\
\Gamma y & \text{otherwise}
\end{cases}
$$

(11)

This method was first introduced in [1].

**Lemma 12.** Given $\theta^* \in \Omega_0$,

$$(\theta - \theta^*)^T (\Gamma^{-1} \text{Proj}_\Gamma(\theta, y, f) - y) \leq 0.
$$

(12)

Proof. If $f(\theta) > 0 \wedge y^T \Gamma \nabla f(\theta) > 0$, then

$$(\theta^* - \theta)^T \left( y - \Gamma^{-1} \left( \Gamma y - \Gamma \frac{\nabla f(\theta)(\nabla f(\theta))^T}{(\nabla f(\theta))^T \Gamma \nabla f(\theta)} \Gamma y f(\theta) \right) \right)
$$

and using Lemma [4]

$$
\frac{(\theta^* - \theta)^T \nabla f(\theta) (\nabla f(\theta))^T \Gamma y}{(\nabla f(\theta))^T \Gamma \nabla f(\theta)} \geq 0 \
\frac{f(\theta)}{\geq 0} \leq 0
$$

otherwise $\text{Proj}_\Gamma(\theta, y, f) = \Gamma y$. \qed

**References**

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