THE BOUNDED COMPLEX OF A UNIFORM AFFINE ORIENTED MATROID IS A BALL

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Abstract. Zaslavsky [8] conjectures that the bounded complex of a simple hyperplane arrangement is homeomorphic to a ball. We prove this conjecture for the more general uniform affine oriented matroids.

1. Introduction

A hyperplane arrangement in a Euclidean space partitions the space into faces. The bounded faces are those bounded in the usual metric sense. The collection of all bounded faces is a polyhedral complex called the bounded complex of the hyperplane arrangement. More generally, affine oriented matroids have a topological model as arrangements of pseudo-hyperplanes, each obtained from a flat hyperplane by tame topological deformation. Since we can still talk about faces and metric in a pseudo-hyperplane arrangement, the bounded complex of an affine oriented matroid can be defined as the regular cell complex consisting of all bounded faces. Seemingly a simple object, the topology of the bounded complex is still not completely understood.

Study on the bounded complex goes back to Zaslavsky’s 1975 paper [8], in which he proves several face counting formulas for hyperplane arrangements. One of the formula is for the number of bounded regions of a hyperplane arrangement. The formula can be proven using Möbius inversion provided that one knows that the Euler characteristic of the bounded complex is one. This prompts Zaslavsky [8] to conjecture that the bounded complex is contractible. This conjecture was proven in Ziegler [9] for hyperplane arrangements, and in Björner and Ziegler [1] for affine oriented matroids. In fact, Zaslavsky [8] also conjectures that the bounded complex is star-convex, that is, there is a “center point” from which all other points in the bounded complex can be seen. However this is false (see, e.g., [2, Exercise 4.29]). Here I should remark that some of Zaslavsky’s formulas were independently discovered by Las Vergnas [4]. The reader may consult [2, §4.6] for an account of relevant formulas and their histories.

Zaslavsky [8] also proves that the bounded complex of a hyperplane arrangement is pure, that is, all the maximal bounded faces have the same dimension. However his proof does not generalize to affine oriented matroids. In a recent paper [3] it is shown that the bounded complex of an affine oriented matroid is pure by using covector axioms. It is also shown in [3] that the bounded complex is collapsible. A collapsible complex is contractible, but not vice versa. The collapsibility of the bounded complex will play a crucial role in this paper.
In general the bounded complex of a hyperplane arrangement is not necessarily a (closed) ball. For instance, let us consider the four lines defined by equations $x = 0$, $y = 0$, $x+y = 1$ and $x+y = -1$ respectively in a plane. The bounded complex of this line arrangement consists of two triangles joined at a vertex. However if the hyperplanes are in general position (such an arrangement is called simple), then it is intuitively plausible that the bounded complex should be a ball. This was conjectured to be the case in Zaslavsky [8] (see also Stanley [7]). The main objective of this paper is to prove this conjecture. The rough idea of the proof is as follows. It is known that a collapsible piecewise-linear (PL) manifold is a PL ball (however a contractible PL manifold is not necessarily a ball, see Mazur [5] for one of the first such examples). Therefore it is sufficient to show that the bounded complex is a PL manifold since it is known to be collapsible. This will be accomplished by showing that the link of a vertex in the order complex $\Delta(L^{++})$ is either a PL ball or a PL sphere. The proof works for the more general uniform affine oriented matroids.

The paper is organized as follows. Notations and preliminary facts about affine oriented matroids are reviewed in Section 2. Section 3 outlines the proof of Zaslavsky’s conjecture. In Section 4, the link of a vertex in the order complex $\Delta(L^{++})$ gets a detailed study, and the proof of the conjecture is completed. Finally Section 5 summarizes some further open questions.

2. AFFINE ORIENTED MATROIDS

Let us start with a quick review of the necessary definitions and terminology for oriented matroids. We mostly follow Section 4.1 of [2]. The only difference is the notation for the support of a sign vector.

Let $E$ be a finite set and consider the sign vectors $X, Y \in \{+, -, 0\}^E$. The support of a vector $X$ is $\text{sp}(X) = \{e \in E : X_e \neq 0\}$; its zero set is $z(X) = E \setminus \text{sp}(X) = \{e \in E : X_e = 0\}$.

The opposite of a vector $X$ is $-X$, defined by

$$(-X)_e = \begin{cases} - & \text{if } X_e = +; \\ + & \text{if } X_e = -; \\ 0 & \text{if } X_e = 0. \end{cases}$$

The zero vector is 0, with $0_e = 0$ for all $e \in E$. The composition of two vectors $X$ and $Y$ is $X \circ Y$, defined by

$$(X \circ Y)_e = \begin{cases} X_e & \text{if } X_e \neq 0; \\ Y_e & \text{otherwise.} \end{cases}$$

The separation set of $X$ and $Y$ is $S(X, Y) = \{e \in E : X_e = -Y_e \neq 0\}$. Notice that $X \circ Y = Y \circ X$ if and only if $S(X, Y) = \emptyset$, in which case we say that $X$ and $Y$ are conformal. We are now ready for the definition of oriented matroids in terms of covectors (see [2 4.1.1]).

**Definition 2.1** (Covector Axioms). An oriented matroid is a pair $(E, \mathcal{L})$, where $E$ is a finite set and $\mathcal{L} \subseteq \{+, -, 0\}^E$ is the set of covectors satisfying:
(L0) 0 \in \mathcal{L};
(L1) X \in \mathcal{L} implies that \(-X \in \mathcal{L};
(L2) X, Y \in \mathcal{L} implies that \(X \circ Y \in \mathcal{L};
(L3) if X, Y \in \mathcal{L} and e \in S(X, Y) then there exists Z \in \mathcal{L} such that \(Z_e = 0\) and \(Z_f = (X \circ Y)_f = (Y \circ X)_f\) for all \(f \notin S(X, Y)\).

Let \(\leq\) be the partial order on the set \(+, -\, 0\) defined by \(0 < +\) and \(0 < -\), with + and − incomparable. This induces a product partial order on \(+, -\, 0\). Thus \(Y \leq X\) if and only if \(Y_e \in \{0, X_e\}\) for all \(e \in E\). As a subset of \(+, -\, 0\) the set of covectors \(\mathcal{L}\) has an induced partial order with bottom element 0. Let \(\mathcal{L}\) denote the poset \(\mathcal{L}\) with a top element \(\hat{1}\) adjoined. Then \(\mathcal{L}\) is a lattice called the big face lattice of \((E, \mathcal{L})\). The join in \(\mathcal{L}\) of \(X\) and \(Y\) equals \(X \circ Y = Y \circ X\) if \(S(X, Y) = \emptyset\), and equals \(\hat{1}\) otherwise.

An affine oriented matroid is a triple \((E, \mathcal{L}, g)\), where \((E, \mathcal{L})\) is an oriented matroid and \(g \in E\) is a distinguished element which is not a loop. Recall that \(g\) is a loop if \(X_g = 0\) for all \(X \in \mathcal{L}\). We now define the bounded complex as in Definition 4.5.1 of [2]. For an affine oriented matroid \((E, \mathcal{L}, g)\) let

\[
\mathcal{L}^+ = \{X \in \mathcal{L} : X_g = +\} \quad \text{and} \quad \hat{\mathcal{L}}^+ = \mathcal{L}^+ \cup \{0, \hat{1}\}.
\]

With the induced order as a subset of \(\hat{\mathcal{L}}\), we call \(\hat{\mathcal{L}}^+\) the affine face lattice of \((E, \mathcal{L}, g)\). The bounded complex of \((E, \mathcal{L}, g)\) is

\[
\mathcal{L}^{++} = \{X \in \mathcal{L}^+ : \mathcal{L} \leq X \subseteq \hat{\mathcal{L}}^+\}.
\]

Let \(T\) denote the set of maximal covectors (called topes) of \(\mathcal{L}\). Let \(B \in T\). Then the tope poset \(T(\mathcal{L}, B)\) is a partial order on the set \(T\) defined by \(T \leq T'\) if and only if \(S(B, T) \subseteq S(B, T')\). The following lemma is easy to deduce from this definition (see Corollary 4.2.11 of [2]).

Lemma 2.2. Let \(T < T'\) in \(T(\mathcal{L}, B)\). Then the interval \([T, T']\) has the same structure in \(T(\mathcal{L}, T)\) as in \(T(\mathcal{L}, B)\).

The following is a theorem of Lawrence (see Proposition 4.3.2 of [2]).

Proposition 2.3. Every linear extension of \(T(\mathcal{L}, B)\) is a shelling. Therefore \(\mathcal{L} \setminus \{0\}\) is the face poset (with the empty face excluded) of a shellable regular cell decomposition of an \((r - 1)\)-sphere, where \(r\) is the rank of the underlying matroid.

We also need the following more general result (see Corollary 4.3.7 of [2]).

Proposition 2.4. Let \((X, Y)\) be an interval in \(\mathcal{L}\). Then \((X, Y)\) is isomorphic to the face poset of a shellable regular cell decomposition of the \((\text{rank}(Y) - \text{rank}(X) - 2)\)-sphere.

\(\mathcal{L}^+\) is an order filter in \(\mathcal{L}\). It is pure of length \(r - 1\) where \(r\) is the rank of the underlying matroid, i.e., every maximal chain is of the same length \(r - 1\) (see Proposition 4.5.3 of [2]). In particular, the minimal covectors in \(\mathcal{L}^+\) are also atoms in \(\mathcal{L}\). It is now easy to see that \(\mathcal{L}^{++}\) is never empty: if \(X\) is a minimal covector in \(\mathcal{L}^+\) then \(X\) is an atom of \(\mathcal{L}\), therefore \(X \in \mathcal{L}^{++}\). In the realizable case this corresponds to the fact that the bounded complex of an essential hyperplane arrangement is
nonempty. The poset $\mathcal{L}^{++}$ is an order ideal of $\mathcal{L} \setminus \{0\}$, hence the face poset of a subcomplex of the $(r-1)$-sphere. For other basic facts about $\mathcal{L}$, $\mathcal{L}^+$ and $\mathcal{L}^{++}$ we refer the reader to Chapter 4 of [2].

In this paper, the same symbol will often be used to denote a regular cell complex and its face poset if no confusion can arise. In particular we use $\mathcal{L}^{++}$ to denote both the poset and the underlying cell complex. Some useful results about the bounded complex from [3] are collected in the following proposition.

**Proposition 2.5 ([3]).** Let $(E, \mathcal{L}, g)$ be an affine oriented matroid.

1. $\mathcal{L}^{++}$ is pure.
2. All the maximal covectors in the bounded complex $\mathcal{L}^{++}$ have the same support, say, $E_1$. Let $\mathcal{L}_{1}^{++}$ denote the bounded complex of $(E_1, \mathcal{L}_1, g)$, where $\mathcal{L}_1$ is the deletion $\mathcal{L} \setminus (E - E_1) = \{X|_{E_1} : X \in \mathcal{L}\}$. Then $\mathcal{L}_{1}^{++} \cong \mathcal{L}^{++}$.

**Proof.** Part (1) is Corollary 3.3 of [3]. Part (2) follows from Theorem 3.2(2) and Theorem 5.1 of [3].

A matroid of rank $r$ is **uniform** if every $r$-element subset of the ground set $E$ is a basis. An oriented matroid $(E, \mathcal{L})$ is **uniform** if its underlying matroid is uniform. Similarly an affine oriented matroid $(E, \mathcal{L}, g)$ is **uniform** if $(E, \mathcal{L})$ is uniform. The realizable uniform affine oriented matroids correspond to exactly the simple hyperplane arrangements. The following proposition characterizes uniform matroids in terms of several different systems of axioms. We omit its straightforward proof.

**Proposition 2.6.** Let $M$ be a matroid of rank $r$ on the ground set $E$. Then $M$ is uniform if and only if it satisfies one of the following equivalent conditions:

1. Every $r$-element subset of $E$ is a basis.
2. $I \subseteq E$ is independent if and only if $|I| \leq r$.
3. $F \subseteq E$ is a flat if and only if either $F = E$ or $|F| \leq r - 1$.
4. For any subset $A \subseteq E$,
   \[
   \text{rank}(A) = \begin{cases} 
   |A|, & \text{if } |A| \leq r - 1; \\
   r, & \text{otherwise}.
   \end{cases}
   \]

For an oriented matroid $(E, \mathcal{L})$, recall that the set $L = \{z(X) : X \in \mathcal{L}\}$ is the collection of flats of the underlying matroid. The map $z : \mathcal{L} \to L$ is a cover-preserving, order-reversing surjection of $\mathcal{L}$ onto the geometric lattice $L$. Therefore we have the following characterization of a uniform oriented matroid.

**Corollary 2.7.** An oriented matroid $(E, \mathcal{L})$ of rank $r$ is uniform if and only if it satisfies one of the following equivalent conditions:

1. For every subset $F \subseteq E$ with $|F| \leq r - 1$, there exists a covector $X \in \mathcal{L}$ with $z(X) = F$.
2. $\text{rank}(X) = r - |z(X)|$ for all $X \in \mathcal{L} \setminus \{0\}$.

### 3. Outline of the proof

Let us review some necessary terminology from PL topology. Recall that a simplicial complex $K$ is a PL $d$-ball if $K$ and the standard $d$-simplex have isomorphic
subdivisions. A simplicial complex \( K \) is a PL \((d−1)\)-sphere if \( K \) and the boundary of the standard \( d \)-simplex have isomorphic subdivisions. A simplicial complex \( K \) is a PL \( d \)-manifold if the link of every vertex is either a PL \((d−1)\)-sphere or a PL \((d−1)\)-ball.

Recall that the order complex of a poset \( P \) is the abstract simplicial complex whose vertices are the elements of \( P \) and whose faces are the chains \( x_0 < x_1 < \cdots < x_k \) in \( P \). The geometric realization of the order complex will also be denoted by \( \Delta(P) \), or even just by \( P \) if no confusion can arise.

Let \( \Gamma \) be a regular cell complex with face poset \( \mathcal{F}(\Gamma) \). Then the order complex \( \Delta(\mathcal{F}(\Gamma)) \) is homeomorphic to \( \Gamma \) (see Proposition 4.7.8 of [2]). It is the first barycentric subdivision of \( \Gamma \). The PL definitions are now applicable to regular cell complexes. A regular cell complex \( \Gamma \) is a PL \( d \)-ball if and only if its simplicial subdivision \( \Delta(\mathcal{F}(\Gamma)) \) is a PL \( d \)-ball, and similarly for PL spheres and PL manifolds (see Lemma 4.7.25 of [2]).

The main objective of this paper is to prove the following theorem.

**Theorem 3.1.** Let \((E, L, g)\) be a uniform affine oriented matroid. Then its bounded complex \( L^{++} \) is a PL ball.

The tool of collapsing will be used to prove the theorem. Let \( \Gamma \) be a regular cell complex, and suppose that \( \sigma \in \Gamma \) is a proper face of exactly one face \( \tau \in \Gamma \). Then the complex \( \Gamma' = \Gamma \setminus \{\sigma, \tau\} \) is obtained from \( \Gamma \) by an elementary collapse. Note that the condition on \( \sigma \) and \( \tau \) implies that \( \tau \) is a maximal face of \( \Gamma \) and \( \sigma \) is a maximal proper face of \( \tau \). If \( \Gamma \) can be reduced to a single point by a sequence of elementary collapses, then \( \Gamma \) is collapsible. The following is Theorem 6.11 of [3].

**Theorem 3.2 (3).** Let \((E, L, g)\) be an affine oriented matroid. Then its bounded complex \( L^{++} \) is collapsible.

**Remark 3.3.** The collapsing in PL topology is more restrictive than our above definition. To distinguish between the two let us define PL collapsing here. We follow the notation in [3]. Suppose that \( X \supset Y \) are locally conical sets (called polyhedra in [3]), \( B^n \) is a PL \( n \)-ball, and \( B^{n−1} \) is a PL \((n−1)\)-ball contained in the boundary of \( B^n \). If \( X = Y \cup B^n \) and \( Y \cap B^n = B^{n−1} \), then we say that there is an elementary PL collapse of \( X \) on \( Y \). We say that \( X \) is PL collapsible if \( X \) reduces to a point via a sequence of elementary PL collapses.

We want to show that \( L^{++} \) is PL collapsible. For this we need the following proposition (see Proposition 4.7.26 of [2]).

**Proposition 3.4.** Let \( \Gamma \) be a regular cell decomposition of the \( d \)-sphere. If \( \Gamma \) is shellable then \( \Gamma \) is a PL sphere. If \( \Gamma \) is a PL sphere then every closed cell in \( \Gamma \) is a PL ball.

**Corollary 3.5.** \( L^{++} \) is PL collapsible.

**Proof.** Regular cell complexes are locally conical sets since they admit simplicial subdivisions. The cells in \( L^{++} \) are all PL balls by Proposition 2.3 and Proposition 3.4. Therefore the bounded complex \( L^{++} \) is in fact PL collapsible. \( \square \)
We need the following fact from PL topology. For a proof see Corollary 3.28 of [6].

**Theorem 3.6.** If a PL manifold is PL collapsible, then it is a PL ball.

To prove the main Theorem 3.1, it remains to show that the following is true.

**Lemma 3.7.** Let \((E, \mathcal{L}, g)\) be a uniform affine oriented matroid. Then its bounded complex \(\mathcal{L}^+\) is a PL manifold.

This lemma will be proven in the next section, by showing that the link of a vertex in \(\Delta(\mathcal{L}^+)\) is either a PL ball or a PL sphere.

**4. The links in \(\Delta(\mathcal{L}^+)\)**

Let \((E, \mathcal{L}, g)\) be an affine oriented matroid (not necessarily uniform). Let \(X \in \mathcal{L}^+\) be a covector in its bounded complex. For simplicity we will not distinguish \(\{X\}\) and \(X\). Then \(X\) is a vertex in the order complex \(\Delta(\mathcal{L}^+)\). The link of \(X\) in \(\Delta(\mathcal{L}^+)\) will be denoted by

\[
\text{link}(X, \Delta(\mathcal{L}^+)) := \{\sigma \in \Delta(\mathcal{L}^+) : X \notin \sigma \text{ and } \{X\} \cup \sigma \in \Delta(\mathcal{L}^+)\}.
\]

Recall that the *join* of two simplicial complexes \(K_1\) and \(K_2\) on disjoint vertex sets is

\[
K_1 \ast K_2 := \{\sigma_1 \cup \sigma_2 : \sigma_1 \in K_1, \sigma_2 \in K_2\}.
\]

Let \(P\) be a finite poset and \(x \in P\). Then it is easy to see that

\[
\text{link}(x, \Delta(P)) = \Delta(P_{\prec x}) \ast \Delta(P_{\succ x}),
\]

where \(P_{\prec x} = \{y \in P : y < x\}\) and similarly for \(P_{\succ x}\). It follows that

\[
\text{link}(X, \Delta(\mathcal{L}^+)) = \Delta(\mathcal{L}^+_{\prec X}) \ast \Delta(\mathcal{L}^+_{\succ X}).
\]

Let us first consider \(\Delta(\mathcal{L}^+_{\prec X})\). Recall that \(\mathcal{L}^+\) is an order ideal in \(\mathcal{L} \setminus \{0\}\). Hence \(\mathcal{L}^+_{\prec X}\) is the same as the interval \((0, X]\) in \(\mathcal{L}\). By Proposition 2.4 \((0, X]\) is isomorphic to the face poset of a shellable regular cell decomposition of a sphere of dimension \(\text{rank}(X) - 2\). Therefore \(\mathcal{L}^+_{\prec X}\) is a PL sphere by Proposition 3.4, and so is \(\Delta(\mathcal{L}^+_{\prec X})\). Note that when \(X\) is of rank one, \(\Delta(\mathcal{L}^+_{\prec X})\) is the complex \(\{\emptyset\}\) which we consider as a sphere of dimension \(-1\).

Next let us consider \(\Delta(\mathcal{L}^+_{\succ X})\). First we make some reductions. By Proposition 2.3 \((2)\) we may assume that \(\mathcal{L}^+\) is full dimensional, so that the maximal covectors in \(\mathcal{L}^+\) are topes of \(\mathcal{L}\). If \(\mathcal{L}^+_{\succ X} = \emptyset\) then \(\Delta(\mathcal{L}^+_{\succ X})\) is a sphere of dimension \(-1\). If \(\mathcal{L}^+_{\succ X} = \mathcal{L}_{\succ X}\) then \(\Delta(\mathcal{L}^+_{\succ X})\) is the first barycentric subdivision of \(\mathcal{L}_{\succ X}\) which is a shellable regular cell decomposition of a sphere, hence \(\Delta(\mathcal{L}^+_{\succ X})\) is a PL sphere.

In what follows we assume that \(\emptyset \subseteq \mathcal{L}^+_{\succ X} \subseteq \mathcal{L}_{\succ X}\).

From now on we shall use the fact that \((E, \mathcal{L}, g)\) is uniform. The following several lemmas are not true for general affine oriented matroids. They are the crux of our proof of the conjecture.
Lemma 4.1. Let \((E, \mathcal{L})\) be a uniform oriented matroid and \(X \in \mathcal{L} \setminus \{0\}\). Then the map
\[
d : \mathcal{L}_{\geq X} \to \{+, -, 0\}^{z(X)}
\]
\[
Y \mapsto Y \setminus \text{sp}(X)
\]
is an isomorphism of posets.

Proof. Let \(r\) be the rank of \((E, \mathcal{L})\). By Corollary 2.7, \(|z(X)| \leq r - 1\) since \(X \neq 0\). Again by Corollary 2.7, for every \(e \in z(X)\) there exists a covector \(Y \in \mathcal{L}\) with \(z(Y) = z(X) - \{e\}\). It follows that for every \(e \in z(X)\) there is a covector \(Z \in \mathcal{L} \setminus \text{sp}(X)\) with \(\text{sp}(Z) = \{e\}\). The covector axioms then imply that \(\mathcal{L} \setminus \text{sp}(X) = \{+, -, 0\}^{z(X)}\). Finally note that \(\mathcal{L}_{\geq X}\) is isomorphic to the deletion \(\mathcal{L} \setminus \text{sp}(X)\) via the map \(Y \mapsto Y \setminus \text{sp}(X)\).

Since the same symbol is often used to denote a face poset and its underlying regular cell complex in this paper, when a face poset \(P\) is said to be simplicial or shellable it is meant that the underlying complex is simplicial or shellable.

Corollary 4.2. Let \((E, \mathcal{L})\) be a uniform oriented matroid and \(X \in \mathcal{L} \setminus \{0\}\). Then \(\mathcal{L}_{\geq X}\) is simplicial.

Proof. By Lemma 4.1, \(\mathcal{L}_{\geq X}\) is isomorphic to the face poset of the boundary of a cross polytope, which is simplicial.

Recall that without loss of generality \(\mathcal{L}^{++}\) is assumed to be full dimensional. We may also assume that \(|E| > 1\) to exclude the trivial case \(E = \{g\}\). Under these assumptions, for every \(X \in \mathcal{L}^+\), the deletion \(X \setminus g \neq 0\). Otherwise, if \(X \setminus g = 0\), then it is easy to show that \(\mathcal{L}^{++} = \{X\}\). This contradicts the full-dimensionality of \(\mathcal{L}^{++}\).

Corollary 4.3. If \(X \in \mathcal{L}^+, \) then \(X \setminus g \in \mathcal{L} / g\) if and only if \(X \notin \mathcal{L}^{++}\).

Proof. Let \(Z \in \{+, -, 0\}^E\) be the sign vector defined by \(Z_g = 0\) and \(Z_e = X_e\) otherwise. Note that \(Z \neq 0\) since \(X \setminus g \neq 0\).

If \(X \notin \mathcal{L}^{++}\), then there exists \(0 < Y < X\) such that \(Y_g = 0\). Note that \(Y \leq Z\) as sign vectors. Applying Lemma 4.1 to \(\mathcal{L}_{\geq Y}\), we see that \(Z\) is a covector. Therefore \(X \setminus g \in \mathcal{L} / g\). Conversely, if \(X \setminus g \in \mathcal{L} / g\), then \(Z \in \mathcal{L}\). It follows that \(X \notin \mathcal{L}^{++}\) since \(0 < Z < X\) and \(Z_g = 0\).

For \(X \in \mathcal{L}^{++}\) let \(\mathcal{C}_X\) denote the set of topes that are in \(\mathcal{L}_{\geq X} = \mathcal{L}_{\geq X}^{+}\) but not in \(\mathcal{L}^{++}\). Let \(\mathcal{D}_X\) denote the set of topes \(T\) of the contraction \(\mathcal{L} / g\) with the following property: \(T \in \mathcal{D}_X\) if and only if \(T_e = X_e\) for all \(e \in \text{sp}(X \setminus g)\). Equivalently, \(T \in \mathcal{D}_X\) if and only if \(T \geq X / g\). We should, however, note that \(X \setminus g \notin \mathcal{L} / g\) in this case.

Lemma 4.4. A tope \(T \in \mathcal{L}_{\geq X}\) is in \(\mathcal{C}_X\) if and only if the deletion \(T \setminus g\) is a tope in \(\mathcal{D}_X\). Moreover the map \(r : \mathcal{C}_X \to \mathcal{D}_X\) defined by \(r(T) = T \setminus g\) is a bijection.

Proof. The first statement follows from Corollary 4.3. To prove the second statement, let \(T \in \mathcal{D}_X\). Define \(h(T) \in \{+, -, 0\}^E\) by \(h(T)_g = X_g = +\) and \(h(T)_e = T_e\) for all \(e \in E \setminus \{g\}\). Define \(i(T) \in \{+, -, 0\}^E\) by \(i(T)_g = 0\) and \(i(T)_e = T_e\) for all \(e \in E \setminus \{g\}\). Then \(i(T) \in \mathcal{L}\) since \(T \in \mathcal{L} / g\). Therefore \(h(T) = i(T) \circ X \in \mathcal{L}^+.\)
Moreover $h(T) \notin \mathcal{L}^{++}$ since $0 < i(T) < h(T)$ in $\mathcal{L}$. Hence $h(T) \in \mathcal{C}_X$. Note that $h$ is the inverse map of $r$, showing that $r$ is bijective. □

Let $[D_X] = \{Y \in \mathcal{L}/g : 0 < Y \leq T \text{ for some } T \in D_X\}$. Then $[D_X]$ is a subcomplex of the shellable sphere $\mathcal{L}/g$.

Lemma 4.5. $[D_X]$ is shellable.

Proof. Fix a tope $B \in D_X$. For a tope $T' \in D_X$ we have $T'_e = B_e = X_e$ for all $e \in \text{sp}(X \setminus g)$. If $T \in \mathcal{L}/g$ and $T \leq T'$ in the tope poset $T(\mathcal{L}/g)$, then $S(B,T) \subseteq S(B,T')$. Hence $T_e = T'_e = B_e = X_e$ for all $e \in \text{sp}(X \setminus g)$, so $T \in D_X$. Therefore $D_X$ is an initial segment of the tope poset, so there is a linear extension of the tope poset in which $D_X$ is an initial segment. Hence $D_X$ is an initial segment of a shell of the tope poset in $\mathcal{L}/g$ by Proposition 2.3 proving that $[D_X]$ is shellable. □

Let $[C_X] = \{Y \in \mathcal{L}_{>X} : Y \leq T \text{ for some } T \in C_X\}$. It turns out that a shell on $D_X$ induces a shell on $C_X$. The shellability of a regular cell complex is defined recursively (see Definitions 4.7.14 and 4.7.17 of [2]). However in the case of a simplicial complex, the definition of a shell is much simpler. Fortunately for us, the poset $[C_X]$ is an order ideal in $\mathcal{L}_{>X}$ and hence the face poset of a simplicial complex by Lemma 4.2.

First let us formulate the definition of a shellable simplicial complex in terms of its face poset. Let $P$ be the face poset of a pure $d$-dimensional simplicial complex. Let $\hat{P} = P \cup \{0, \hat{1}\}$ denote the (augmented) face lattice by adding new elements such that $0 < p < \hat{1}$ for all $p \in P$. A linear ordering $c_1, c_2, \ldots, c_t$ of the coatoms of $\hat{P}$ is a shelling if for all $1 \leq i \leq j \leq t$ there exists $1 \leq k < j$ such that $c_i \wedge c_j \leq c_k \wedge c_j \wedge c_j < c_j$, where $< \wedge$ is the covering relation in the poset. Note that if $P$ is the face poset of a regular cell complex whose augmented face poset is a lattice (as in the case of $[D_X]$), then the above condition on the linear order of coatoms is necessary, although not sufficient, for being a shelling.

Lemma 4.6. $[C_X]$ is shellable.

Proof. Fix a tope $B \in D_X$. Let $d_1(= B), d_2, \ldots, d_t$ be the shell of $D_X$ obtained from an initial segment of a linear extension of the tope poset $T(\mathcal{L}/g,B)$. Let $c_i = h(d_i) \in \mathcal{C}$ for all $1 \leq i \leq t$, where $h$ is as defined in the proof of Lemma 4.3. We want to show that $c_1, c_2, \ldots, c_t$ is a shell of $[C_X]$. As in the above definition of shell we shall work with the augmented face lattices $[\hat{C}_X]$ and $[\hat{D}_X]$. We identify $0$ in $[\hat{C}_X]$ with $X$, and $0$ in $[\hat{D}_X]$ with $0$ in $\mathcal{L}/g$. Let $1 \leq i < j \leq t$, we want to find $1 \leq k < j$ such that $c_i \wedge c_j \leq c_k \wedge c_j < c_j$.

Case 1: If $c_i \wedge c_j \notin \mathcal{L}^{++}$, then $(c_i \wedge c_j)/g \in \mathcal{L}/g$ by Corollary 4.3 Therefore $(c_i \wedge c_j)/g \leq d_i \wedge d_j$. By the shell of $D_X$ there exists $1 \leq k < j$ such that $d_i \wedge d_j \leq d_k \wedge d_j < d_j$. Since $(c_i \wedge c_j)/g \leq d_i \wedge d_j \leq d_k$, we get $c_i \wedge c_j \leq c_k$. Hence $c_i \wedge c_j \leq c_k \wedge c_j$. Since $\mathcal{L}/g$ is uniform we have $|z(d_k \wedge d_j)| = 1$. It follows that $|z(c_k \wedge c_j)| = 1$ and hence $c_k \wedge c_j < c_j$.

Case 2: If $c_i \wedge c_j \in \mathcal{L}^{++}$, then consider $D_{c_i \wedge c_j}$ which is a subset of $D_X$. The interval $[d_i, d_j]$ has the same structure in $T(\mathcal{L}/g, d_i)$ and $T(\mathcal{L}/g, B)$ by Lemma 2.2. Note that $[d_i, d_j] \subseteq D_{c_i \wedge c_j} \subseteq D_X$ since $D_{c_i \wedge c_j}$ is an order ideal in the tope poset.
There is a linear extension of \( T(L/g, d_i) \) (hence a shelling of \( L/g \)) such that

1. \([d_i, d_j]\) is an initial segment;
2. the linear order of the elements in \([d_i, d_j]\) is the same as the restriction of the shelling of \( D_X \) on \([d_i, d_j]\).

Therefore there exists \( d_k \in [d_i, d_j] \) (so \( i \leq k < j \)) such that \( d_k \wedge d_j \succ d_j \). Once again, since \( L/g \) is uniform we have \(|z(d_k \wedge d_j)| = 1\). It follows that \(|z(c_k \wedge c_j)| = 1\) and hence \( c_k \wedge c_j \prec c_j \). Finally \( d_k \in D_{c_i \wedge c_j} \) implies that \( c_i \wedge c_j \leq c_k \), so \( c_i \wedge c_j \leq c_k \wedge c_j \).

\[ \square \]

**Corollary 4.7.** \([C_X]\) is a PL ball.

**Proof.** Since \([C_X]\) is shellable, it is either a PL ball or a PL sphere. Since \( L^{++} \) is nonempty, \([C_X]\) is a proper subset of \( L_X \) which is a PL sphere of the same dimension. Therefore \([C_X]\) has to be a PL ball. \[ \square \]

The following lemma is known as Newman’s Theorem (see Theorem 4.7.21(iii) of [2]).

**Lemma 4.8.** The closure of the complement of a PL \( d \)-ball embedded in a PL \( d \)-sphere is itself a PL \( d \)-ball.

\([C_X]\) is a full dimensional PL ball embedded in the PL sphere \( L_X \). The closure of its complement is exactly \( L^{++} \). It follows that \( L^{++} \) is a PL ball, and so is \( \Delta(L^{++}) \).

We have seen that \( \Delta(L^{++}) \) is a PL sphere and \( \Delta(L^{++}) \) is either a PL sphere or a PL ball, so

\[
\text{link}(X, \Delta(L^{++})) = \Delta(L^{++}) \ast \Delta(L^{++})
\]

is either a PL sphere or a PL ball by the following lemma (see Proposition 2.23 of [2]).

**Lemma 4.9.** The join of two PL spheres is a PL sphere. The join of a PL sphere and a PL ball is a PL ball.

By Proposition 2.3(1) \( L^{++} \) is pure of dimension \( r - 1 \), so \( \text{link}(X, \Delta(L^{++})) \) is either a PL \((r - 2)\)-sphere or a PL \((r - 2)\)-ball. We conclude that \( \Delta(L^{++}) \) is a PL \((r - 1)\)-manifold, and so is \( L^{++} \). Since \( L^{++} \) is known to be PL collapsible, it is in fact a PL \((r - 1)\)-ball. The proof of our main Theorem 3.1 is now complete.

5. **Final remarks**

Zaslavsky [8] in fact also conjectures that the bounded complex of a hyperplane arrangement is a ball as long as there is no parallelism among the hyperplanes and intersections. Generalizing to affine oriented matroids, we get the following conjecture.

**Conjecture 5.1.** Let \((E, L, g)\) be an affine oriented matroid. If the contraction \( L/g \) is uniform, then the bounded complex \( L^{++} \) is a ball.

Another open question in [2] asks whether the bounded complex of a simplicial affine oriented matroid is a ball. We phrase it as a conjecture.
**Conjecture 5.2.** Let \((E, \mathcal{L}, g)\) be a simplicial affine oriented matroid. Then the bounded complex \(\mathcal{L}^{++}\) is a ball.

More generally one can ask for which kind of affine oriented matroids the bounded complex is a ball. A related question is for which kind of affine oriented matroids the bounded complex is shellable. If the bounded complex is shellable then it must be a ball, but not vice versa. It is not known whether the bounded complex of a uniform affine oriented matroid is shellable.

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