Asymptotic Measures and Links in Simplicial Complexes

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Abstract
We introduce canonical measures on a locally finite simplicial complex $K$ and study their asymptotic behavior under infinitely many barycentric subdivisions. We prove that the simplices of each dimension in $\operatorname{Sd}^d(K)$ equidistribute in $|K|$ with respect to the Lebesgue measure as $d$ grows to $+\infty$ and then prove that their asymptotic link and dual block are universal.

Keywords
Simplicial complex · Barycentric subdivisions · Face vector · Face polynomial · Link of a simplex · Dual block · Measure

Mathematics Subject Classification 52C99 · 28C15 · 28A33

1 Introduction
Let $K$ be a finite $n$-dimensional simplicial complex and $\operatorname{Sd}^d(K)$, $d \geq 0$, be its $d$-th barycentric subdivision, see [9]. We denote by $f_p(K)$, $p \in \{0, \ldots, n\}$, the $p$-th face number of $K$, that is the number of $p$-dimensional simplices of $K$ and by $q_K(T)$ its face polynomial $\sum_{p=0}^{n} f_p(K) T^p$ (following the notation of [4,5]). The asymptotic of $f_p^d(K) := f_p(\operatorname{Sd}^d(K))$ has been studied in [4,5] and it is equivalent to $q_{p,n} f_n(K)(n + 1)!^d$ as $d$ grows to $+\infty$, for some positive universal constant $q_{p,n}$. We provide an expression for these constants in Theorem 1.3 but they remain puzzling. Our aim is to introduce canonical measures on the underlying topological space.
|K|, namely a Lebesgue measure $d\text{vol}_K$, see Definition 3.1, and a family of discrete measures $(\gamma^p, K)_{0 \leq p \leq n}$ which we call the $p$-skeleton measures of $K$, see Definition 3.2. Since $|\text{Sd}^d(K)| = |K|$ for every $d \geq 0$, we get by considering iterated barycentric subdivisions of $K$ a sequence of measures on $|K|$ and study their asymptotic behavior, see Theorem 1.4, enriching the asymptotic study of [4,5]. Theorem 1.4 implies that the simplices of given dimension in $\text{Sd}^d(K)$ equidistribute in $|K|$ with respect to the Lebesgue measure as $d$ grows to $+\infty$. We then prove that their asymptotic link and dual block are universal, that is they only depend on the dimensions of $K$ and on the simplices, see Theorems 1.5 and 1.6.

It has been proved in [4] that the roots of the limit face polynomial $q_n^\infty(T) := \sum_{p=0}^n q_{p,n} T^p$ are all simple and real in $[-1, 0]$, and in [5] that this polynomial is symmetric with respect to the involution $T \mapsto -T - 1$, see Theorem 2.4. We first observe that this symmetry actually follows from a general symmetry phenomenon obtained by I. G. Macdonald in [8] which can be formulated as follows (see Sect. 2.1 for a proof). We set $R_K(T) := Tq_K(T) - \chi(K)T$, where $\chi(K)$ denotes the Euler characteristic of $K$.

**Theorem 1.1** ([8, Thm. 2.1]) Let $K$ be a triangulated compact homology $n$-manifold. Then, $R_K(-1 - T) = (-1)^{n+1} R_K(T)$.

Recall that a homology $n$-manifold is a topological space $X$ such that for every $x \in X$, the relative homology $H_\ast(X, X \setminus \{x\}; \mathbb{Z})$ is isomorphic to $H_\ast(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; \mathbb{Z})$. Any smooth or topological manifold is thus a homology manifold and Poincaré duality holds true in such compact homology manifolds, see [9].

We then deduce the following theorem (see Corollary 2.2 and Theorem 2.5), the first part of which is a corollary of Theorem 1.1 that has been independently (not as a corollary of Theorem 1.1) observed by Akita [1].

**Theorem 1.2** Let $K$ be a compact triangulated homology manifold of even dimension. Then $\chi(K) = q_K\left(-\frac{1}{2}\right) = \sum_{p=0}^{\dim K} (-1)^{p} \frac{f_p(K)}{2^p}$.

Moreover, $t = -1$ in odd dimensions and $t = -1$ together with $t = -\frac{1}{2}$ in even dimensions are the only complex values $t$ for which $q_K(T)$ equals $\chi(K)$ for every compact triangulated homology manifold of the given dimension.

Having spheres in mind for instance, Theorems 1.1 and 1.2 exhibit a striking behavior of simplicial structures compared to cellular structures. In [10], we also provide a probabilistic proof of the first part of Theorem 1.2 and revisit it in Corollary 3.1 of [11].

The limit face polynomial $q_n^\infty(T)$ remains mysterious, but we have been able to prove Theorem 1.3 which provides an expression for $(q_{p,n})_{0 \leq p \leq n}$.

Let $\Lambda := (\lambda_{i,j})_{i,j \geq 1}$ be the infinite lower triangular matrix whose entries $\lambda_{i,j}$ are the numbers of interior $(j - 1)$-faces on the subdivided standard simplex $\text{Sd}(\Delta_{i-1})$ and let $\Lambda_n := (\lambda_{i,j})_{1 \leq i,j \leq n+1}$, see Fig. 1. These numbers $\lambda_{i,j}$ coincide with $j! S(i, j)$, where $S(i, j)$ denotes the Stirling numbers of the second kind, see Lemma 2.6. The diagonal entries of $\Lambda$ are thus $\lambda_{i,i} = i!$. We set as a convention $\lambda_{0,0} := 1$ and $\lambda_{i,0} := 0$ whenever $l > 0$. It follows from [5] that the vector $(q_{p,n})_{0 \leq p \leq n}$ is the eigenvector of the transpose matrix $\Lambda^T$ associated to the eigenvalue $(n + 1)!$ and normalized in such
a way that $q_{n,n} = 1$ and $q_{p,n} = 0$ if $p > n$. We provide a geometric proof of this fact in Corollary 4.2 and prove the following.

**Theorem 1.3** For every $0 \leq p < n$,

$$q_{p,n} = \sum_{(p_1, ..., p_j) \in \mathcal{P}_{p,n}} \frac{\lambda_{n+1,p_j} \cdot \cdot \cdot \lambda_{p_2,p_1}}{(\lambda_{n+1,n+1} - \lambda_{p_j,p_j}) \cdots (\lambda_{n+1,n+1} - \lambda_{p_1,p_1})},$$

where $\mathcal{P}_{p,n} := \{(p_1, ..., p_j) \in \mathbb{N}^j | j \geq 1 \text{ and } p + 1 = p_1 < \cdots < p_j < n + 1\}$.

For every $0 \leq p \leq n$, we define the $p$-skeleton measure $\gamma_{p,K}$ of $K$ to be $\sum_{\sigma \in K[p]} \delta_{\hat{\sigma}}$, where $\delta_{\hat{\sigma}}$ denotes the Dirac measure on the barycenter $\hat{\sigma}$ of $\sigma$ and $K[p]$ the set of $p$-dimensional simplices of $K$, see Definition 3.2. Likewise, for every $d \geq 0$, we set $\gamma_{p,K}^d := \frac{1}{(n+1)!} \gamma_{p,Sd(K)}$, which provides a canonical sequence of Radon measures on the underlying topological space $|K|$, see Sect. 3. The latter is also equipped with the Lebesgue measure $d\text{vol}_K := \sum_{\sigma \in K[n]} (f_{\sigma})_\ast d\text{vol}_{\Delta_n}$, where $f_{\sigma} : \Delta_n \to \sigma$ denotes any simplicial isomorphism between the standard $n$-simplex $\Delta_n$ and $\sigma$, and $d\text{vol}_{\Delta_n}$ denotes the Lebesgue measure normalized in such a way that $\Delta_n$ has volume 1, see Definition 3.1. We prove the following equidistrubition result, see Sect. 3.

**Theorem 1.4** For every $n$-dimensional locally finite simplicial complex $K$ and every $0 \leq p \leq n$, the sequence of measures $(\gamma_{p,K}^d)_{d \in \mathbb{N}}$ weakly converges to $q_{p,n}d\text{vol}_K$ as $d$ grows to $+\infty$.

By weak convergence, we mean that for every continuous function $\varphi$ with compact support in $|K|$, $\int_{|K|} \varphi \, d\gamma_{p,K}^d \xrightarrow{d \to +\infty} q_{p,n} \int_{|K|} \varphi \, d\text{vol}_K$. Note that when $K$ is finite, Theorem 1.4 recovers the asymptotic of $f_{p}^d(K)$ as $d$ grows to $+\infty$, by integration of the constant function 1.

Recall that the *link* $\text{LK}(\sigma, K)$ of a simplex $\sigma$ in $K$ is by definition the set of simplices $\tau \in K$ such that $\sigma$ and $\tau$ are disjoint and both are faces of a simplex in $K$. Likewise, the *block dual* to $\sigma$ is the set $D(\sigma) := \{[\hat{\sigma}_0, \cdots, \hat{\sigma}_p] \in \text{Sd}(K) | p \in \{0, \ldots, n\} \text{ and } \sigma_0 = \sigma\}$, see [9]. Recall also that the simplices of $\text{Sd}(K)$ are by definition of the form $[\hat{\sigma}_0, \cdots, \hat{\sigma}_p]$, where $\sigma_0 < \cdots < \sigma_p$ are simplices of $K$ with $<$ meaning being a proper
face. The dual blocks form a partition of $S_d(K)$, see [9], and the links $L_k(\sigma, K)$ encode in a sense the local complexity of $K$ near $\sigma$. We finally prove the following asymptotic behavior of this complexity, see Sect. 4.

**Theorem 1.5** For every $n$-dimensional locally finite simplicial complex $K$ and every $0 \leq p < n$, the sequence of measures $(q_{L_k(\sigma, S_d(K))}(T) d\gamma^d_{p,K}(\sigma))_{d \in \mathbb{N}}$ (with values in $\mathbb{R}_{n-p-1}(T)$) weakly converges to $(\sum_{l=0}^{n-p-1} q_{p+l-1,n} f_p(\Delta_{p+l+1}) T^l) d\text{vol}_K$ as $d$ grows to $+\infty$.

The measure $q_{L_k(\sigma, S_d(K))}(T) d\gamma^d_{p,K}(\sigma)$ appearing in Theorem 1.5 is defined as $\frac{1}{(n+1)!d} \sum_{\sigma \in S_d(K) \mid p} q_{L_k(\sigma, S_d(K))}(T) \delta_\sigma$. Likewise, we denote by $q_{D(\sigma)}(T) d\gamma^d_{p,K}(\sigma)$ the measure $\frac{1}{(n+1)!d} \sum_{\sigma \in S_d(K) \mid p} q_{D(\sigma)}(T) \delta_\sigma$.

**Theorem 1.6** For every $n$-dimensional locally finite simplicial complex $K$ and every $0 \leq p \leq n$, the sequence of measures $(q_{D(\sigma)}(T) d\gamma^d_{p,K}(\sigma))_{d \in \mathbb{N}}$ weakly converges to $\sum_{l=0}^{n-p} (\sum_{h=0}^{n-p} q_{p+h,n} f_p(\Delta_{p+h}) \lambda_{h,l}) T^l d\text{vol}_K$ as $d$ grows to $+\infty$.

Theorems 1.5 and 1.6 imply that the asymptotic complexity of the link and the dual block of a simplex is almost everywhere constant with respect to $d\text{vol}_K$ and that this constant complexity is universal, depending only on $p$ and $n$ but not on $K$. It is also local, in the sense that it is already obtained starting with $K = \Delta_n$. This universality and local property have thus the same flavor as the ones appearing in Weyl’s asymptotic law or in the asymptotic study of the Bergman kernel for instance.

In [10], we study the asymptotic topology of a random subcomplex in a finite simplicial complex $K$ and its iterated barycentric subdivisions. It turns out that the Betti numbers of such a subcomplex get controlled by the measures given in Theorem 1.6.

### 2 The Face Polynomial of a Simplicial Complex

#### 2.1 The Symmetry Property

Let $K$ be a finite $n$-dimensional simplicial complex. We set $R_K(T) := T q_K(T) - \chi(K) T$, where $q_K(T) := \sum_{p=0}^{n} f_p(K) T^p$ and $\chi(K)$ is the Euler characteristic of $K$, so that $R_K(0) = R_K(-1) = 0$.

**Example 2.1** 1. If $K = \partial \Delta_{n+1}$, then $T q_K(T) = (1 + T)^{n+2} - 1 - T^{n+2}$.
2. If $K = S^0 \ast \cdots \ast S^0$ is the $n$-th iterated suspension of the 0-dimensional sphere, then

$$R_K(T) = T q_K(T) - T \chi(K) = \begin{cases} (2T + 1)((2T + 1)^n - 1) & \text{if } n \text{ is even}, \\ (2T + 1)^{n+1} - 1 & \text{if } n \text{ is odd}. \end{cases}$$

Recall that if $K$ is a triangulated compact homology $n$-manifold, its face numbers satisfy the following Dehn–Sommerville relations ([7], see also for example [6]):
\( \forall 0 \leq p \leq n, \ f_p(K) = \sum_{i=p}^{n} (-1)^{i+n} \binom{i+1}{p+1} f_i(K). \)

The Dehn–Sommerville relations imply that \( R_K(T) \) satisfy the striking symmetry property observed by Macdonald \[8\] which we recalled in Theorem 1.1. Let us now give a proof of the latter for the reader’s convenience.

**Proof of Theorem 1.1** Observe that

\[
R_K(-1 - T) = \sum_{p=0}^{n} f_p(K)(-1 - T)^{p+1} + \chi(K)(1 + T)
= \sum_{p=0}^{n} f_p(K)(-1)^{p+1} \sum_{q=0}^{p+1} \binom{p + 1}{q} T^q + \chi(K)(1 + T)
= \sum_{p=0}^{n} f_p(K)(-1)^{p+1} \sum_{q=0}^{p} \binom{p + 1}{q + 1} T^{q+1} + \chi(K)T
= \sum_{q=0}^{n} T^{q+1} \sum_{p=q}^{n} \binom{p + 1}{q + 1} f_p(K)(-1)^{p+1} + \chi(K)T.
\]

Then, the Dehn–Sommerville relations imply

\[
R_K(-1 - T) = -\sum_{q=0}^{n} T^{q+1}(-1)^{n+1} f_q(K) + \chi(K)T
= (-1)^{n+1} R_K(T) + (1 + (-1)^{n+1})\chi(K)T.
\]

Now, if \( n \) is even, \( 1 + (-1)^{n+1} = 0 \) while if \( n \) is odd, \( \chi(K) = 0 \) by Poincaré duality with \( \mathbb{Z}/2\mathbb{Z} \) coefficients, see \[9\]. Therefore in both cases, we get \( R_K(-1 - T) = (-1)^{n+1} R_K(T) \).

**Corollary 2.2** Let \( K \) be a triangulated compact homology \( n \)-manifold.

1. If \( n \) is even, then \( q_K\left(-\frac{1}{2}\right) = \chi(K) \).
2. If \( n \) is odd, the polynomial \( T q_K(T) \) is preserved by the involution \( T \mapsto -1 - T \).
3. If \( \chi(K) \leq 0 \), the real roots of \( R_K(T) = T q_K(T) - \chi(K)T \) lie on the interval \([-1, 0]\).

**Proof** When \( n \) is even, \( R_K \) has an odd number of real roots, invariant under the involution \( T \mapsto -1 - T \) whose unique fixed point is \(-\frac{1}{2}\). Theorem 1.1 thus implies that \( R_K\left(-\frac{1}{2}\right) = 0 \). Hence the first part. When \( n \) is odd, \( \chi(K) = 0 \) by Poincaré duality so that \( R_K(T) = T q_K(T) \) and the second part. Finally, if \( \chi(K) \leq 0 \), the coefficients of the polynomial \( R_K(T) \) are all positive, so that its real roots are all negative. It thus follows from Theorem 1.1 that they lie on the interval \([-1, 0]\). \( \square \)
**Remark 2.3** The first part of Corollary 2.2 was independently (not as a corollary of Theorem 1.1) observed by T. Akita [1]. In [10], we provide a probabilistic proof of it and revisit it in [11]. The third part of Corollary 2.2 always holds true when $n$ is odd, since then $\chi(K) = 0$.

The first part of Corollary 2.2 raises the following question: given some dimension $n$, what are the universal parameters $t$ such that $q_K(t) = \chi(K)$ for every compact triangulated homology $n$-manifolds? We checked that $t = -1$ in odd dimensions and $t = -1/2$ in even dimensions are the only ones, see Theorem 2.5.

### 2.2 The Asymptotic Face Polynomial

Let $K$ be a finite $n$-dimensional simplicial complex. We denote by $f(K)$ its *face vector* $(f_0(K), f_1(K), \ldots, f_n(K))$, that is the vector formed by its face numbers. Now, for every $d > 0$, we set $f_d^p(K) := f_p(Sd^d(K))$, where $Sd^d(K)$ denotes the $d$-th barycentric subdivision of $K$. How does the face vector change under barycentric subdivisions and what is the asymptotic behavior of $f_d^p(K) := (f_0^d(K), f_1^d(K), \ldots, f_n^d(K))$?

These questions have been treated in [4,5], leading to the following.

**Theorem 2.4 ([4,5])** For every $0 \leq p \leq n$, there exist $q_{p,n} > 0$ such that for every $n$-dimensional finite simplicial complex $K$, $\lim_{d \to +\infty} f_d^p(K) = q_{p,n}$. Moreover, the $n+1$ roots of the polynomial $Tq_n^\infty(T)$ are simple, belong to the interval $[-1, 0]$ and are symmetric with respect to the involution $T \mapsto -T - 1 \in \mathbb{R}$ whenever $n > 0$, where $q_n^\infty(T) := \sum_{p=0}^{n} q_{p,n} T^p$.

The symmetry property of $Tq_n^\infty(T)$ follows from Theorem 1.1 and the first part of Theorem 2.4, since the Euler characteristic remains unchanged under subdivisions. This symmetry has been observed in [5] (with a different proof). It implies that $q_n^\infty(-1) = 0$ and that $q_n^\infty(-\frac{1}{2}) = 0$ whenever $n$ is even, as the number of roots of $Tq_n^\infty(T)$ is then odd and $-\frac{1}{2}$ is the unique fixed point of the involution.

**Theorem 2.5** The reals $t = -1$ if $n$ is odd and $t = -1/2$ together with $t = -1/2$ if $n$ is even are the only complex values for which the face polynomial $q_K(T) := \sum_{p=0}^{\dim K} f_p(K) T^p$ equals $\chi(K)$ for every compact triangulated homology $n$-manifold $K$.

**Proof** Let us equip the $n$-dimensional sphere with the triangulation given by the boundary of the $(n + 1)$-simplex $\Delta_{n+1}$. Then, for every $0 \leq p \leq n$, $f_p(S^n) = \binom{n+2}{p+1}$ and $q_{S^n}(T) = \frac{1}{T} \left( (1+T)^{n+2} - 1 - T^{n+2} \right)$. Now, the polynomial $q_{S^n}(T) - \chi(S^n)$ has only one real root if $n$ is odd and two real roots if $n$ is even. Indeed, differentiating the polynomial $Tq_{S^n}(T) - \chi(S^n) T$ once if $n$ is odd and twice if $n$ is even, we get, up to a factor, $(1+T)^{n+1} - T^{n+1}$ or respectively $(1+T)^n - T^n$ which vanishes only for $t = -\frac{1}{2}$ on the real line. From Rolle’s theorem we deduce that 0 and $-1$ (respectively $0, -\frac{1}{2}, -1$) are the only real roots of $Tq_{S^n}(T) - \chi(S^n) T$ when $n$ is odd (respectively, when $n$ is even).
Finally, if \( t_0 \in \mathbb{C} \) is such that \( q_K(t_0) = \chi(K) \) for all triangulated manifolds of a given dimension \( n \), then in particular, \( R_{Sd}(t_0) = 0 \) for every \( d > 0 \). Dividing by \( f_n(K)(n+1)^d \) and passing to the limit, we deduce that \( q_n^\infty(t_0) = 0 \). But from Theorem 2.4 we know that the roots of \( Tq^\infty(T) \) are all real, hence the result. \( \square \)

Let \( S(i, j) \) be the Stirling number of the second kind, that is the number of partitions of a set of \( i \) elements into \( j \) disjoint non-empty subsets. Explicit values for these numbers are given on page 8 of [12] (see also [3] and compare with Fig. 1).

**Lemma 2.6** For every \( 1 \leq j \leq i \), \( \lambda_{i,j} = j! S(i, j) \). Also, \( \lambda_{i,i} = \sum_{p=1}^{i-1} \binom{i}{p} \lambda_{p,i-1} \) where \( \binom{i}{p} \) denotes the binomial coefficient. In particular, \( \lambda_{i,i} = i! \).

**Proof** The standard simplex \( \Delta_{i-1} \) has \( i \) vertices. An ordered partition \( I_1 \cup \cdots \cup I_j \) of this set \( I \) of vertices defines a \( (j-1) \)-simplex \( \{\hat{0}, \ldots, \hat{j}\} \) of \( Sd(\Delta_{i-1}) \), where \( \hat{0} \) is the barycenter of the simplex with vertices \( I_1 \), \( \hat{1} \) is the barycenter of the simplex with vertices \( I_1 \cup I_2 \), and \( \hat{j} \) is the barycenter of the simplex with vertices \( I = I_1 \cup \cdots \cup I_j \). This correspondence between the interior simplices of \( Sd(\Delta_{i-1}) \) and the set of ordered partitions of the set of vertices of \( \Delta_{i-1} \) is one-to-one. Thus for every \( 1 \leq j \leq i \), \( \lambda_{i,j} = j! S(i, j) \).

Now, the interior \((j-1)\)-faces of \( Sd(\Delta_{i-1}) \) are cones over the \((j-2)\)-faces of the boundary of \( Sd(\Delta_{i-1}) \). The latter are interior to some \((p-1)\)-simplex of \( \partial \Delta_{i-1} \), \( j-1 \leq p \leq i - 1 \). The result follows from the fact that for every \( 1 \leq p \leq i - 1 \), \( \partial \Delta_{i-1} \) has \( \binom{i}{p} \) many \((p-1)\)-dimensional faces while each such face contains \( \lambda_{p,i-1} \) many \((j-2)\)-dimensional faces of \( Sd(\Delta_{i-1}) \) in its interior. \( \square \)

The first part of Theorem 2.4 is basically deduced in [4,5] from the following observation: for every \( n \)-dimensional finite simplicial complex \( K \), the face vector \( f(K) \) is deduced from the face vector \( f(Sd(K)) \) by multiplication on the right by \( \Lambda_n \), that is \( f(Sd(K)) = f(K) \Lambda_n \), while the matrix \( \Lambda_n \) is diagonalizable with eigenvalues given by Lemma 2.6.

We deduce from [4,5] that the vector \( (q_{p,n})_{0 \leq p \leq n} \) is the eigenvector of \( \Lambda_n^T \) associated to the eigenvalue \( \lambda_{n+1,n+1} = (n+1)! \) normalized by the relation \( q_{0,n} = 1 \). A geometric proof of this fact will be given in Sect. 4, see Corollary 4.2. This observation makes it possible to compute \( q_{p,n} \) in terms of the coefficients \( \lambda_{i,j} \), proving Theorem 1.3.

**Proof of Theorem 1.3** Having in mind that \( \Lambda_n \) is a lower triangular matrix and by Lemma 2.6, \( (n+1)! = \lambda_{n+1,n+1} \). The equation \( \Lambda_n^T(q_{p,n}) = (n+1)! (q_{p,n}) \) results in the following system. For all \( 0 \leq p < n \),

\[
q_{p,n} = \sum_{k=0}^{n-p-1} \frac{\lambda_{n+1-k,p+1}q_{n-k,n}}{\lambda_{n+1,n+1} - \lambda_{p+1,p+1}}.
\]

The solution of this system is obtained by induction on \( r = n - p \) by setting \( q_{n,n} = 1 \). The result follows from the fact that the partitions \((p_1, \ldots, p_j)\) of integers between \( p + 1 \) and \( n + 1 \) such that \( p + 1 = p_1 < \cdots < p_j < n + 1 \) are obtained (except the one with single term \( p_1 = p + 1 \)) from those \( p + 1 + s = p_1' < \cdots < p_j' < n + 1 \) for all \( 1 \leq s \leq r \) by setting \( p_1 = p + 1 \) and \( p_{i+1} = p_i' \) for \( i \in \{1, \ldots, j\} \). \( \square \)
Remark 2.7  The numbers $\lambda_{i,j}$ are classically computed, see Theorem 1 of [3] and the references therein. Namely, for every $1 \leq j \leq i$, $\lambda_{i,j} = \sum_{p=0}^{j} \binom{j}{p} (-1)^{i-p} p^i$. (This result appeared as Lemma 6.1 of [5]. The left hand side in Lemma 6.1 of [5] should read $\lambda_{i-1,j-1}$ and our $\lambda_{i,j}$ corresponds to $\lambda_{i-1,j-1}$ in [5].)

For every $j \geq 1$, let $L_j(T) := \frac{1}{j!} \prod_{i=0}^{j-1} (T - i) \in \mathbb{R}[T]$ be the $j$-th Lagrange polynomial, so that $L_j(p) = 0$ if $0 \leq p < j$ and $L_j(p) = \binom{j}{p}$ if $p \geq j$. We deduce the following interpretation of the transpose matrix $\Lambda^t$.

Corollary 2.8  For every $j \geq 1$, $T_j = \sum_{i=1}^{j} \lambda_{j,i} L_i(T)$.

Corollary 2.8 means that $\Lambda^t$ is the matrix of the vectors $(T_j)_{j \geq 0}$ in the basis $(L_i)_{i \geq 0}$ of the space $\mathbb{R}[T]$ of real polynomials in one variable, setting $T^0 = L_0 = 1$. This point of view was actually the way Stirling defined his numbers, see [12, p. 8] and also [3].

Proof  Let $i \geq 1$. Then, for every $l \geq i$,

$$\sum_{p=0}^{l} \binom{l}{p} (-1)^{l-p} L_i(p) = \sum_{p=i}^{l} \binom{l}{p} (-1)^{l-p} \binom{p}{i}$$

$$= \binom{l}{i} \sum_{p=i}^{l} \binom{l-i}{l-p} (-1)^{l-p}$$

$$= (-1)^{l-i} \binom{l}{i} \sum_{q=0}^{l-i} \binom{l-i}{q} (-1)^q$$

$$= \delta_{li},$$

where $\delta_{li} = 0$ if $l \neq i$ and $\delta_{li} = 1$ otherwise. This result also holds true for $l \in \{0, \ldots, i-1\}$. We deduce that for $0 \leq l \leq j$,

$$\sum_{p=0}^{l} \binom{l}{p} (-1)^{l-p} \left( \sum_{i=0}^{j} \lambda_{j,i} L_i(p) \right) = \lambda_{j,l}.$$  

The result now follows from the fact that $\lambda_{i,j} = \sum_{p=0}^{j} \binom{j}{p} (-1)^{j-p} p^i$, see Remark 2.7 and that a degree $j$ polynomial is uniquely determined by its values on the $j+1$ integers $\{0, \ldots, j\}$, since the above linear combinations for $l \in \{0, \ldots, j\}$ define an invertible triangular matrix.  

3 Canonical Measures on a Simplicial Complex

Let us equip the standard $n$-dimensional simplex $\Delta_n$ with the Lebesgue measure $d\text{vol}_{\Delta_n}$ inherited by some affine embedding of $\Delta_n$ in a Euclidian $n$-dimensional space.
\(E\) in such a way that the total measure of \(\Delta_n\) is 1. This measure does not depend on the embedding \(\Delta_n \hookrightarrow E\) for two such embeddings differ by an affine isomorphism which has constant Jacobian 1.

**Definition 3.1** For every \(n\)-dimensional locally finite simplicial complex \(K\), we denote by \(d\text{vol}_K\) the Lebesgue measure \(\sum_{\sigma \in K[n]} (f_{\sigma})_*(d\text{vol}_{\Delta_n})\) of \(|K|\), where \(K[n]\) denotes the set of \(n\)-dimensional simplices of \(K\) and \(f_{\sigma} : \Delta_n \to \sigma\) any simplicial isomorphism.

The topological space \(|K|\) is equipped with its \(\sigma\)-algebra of Borel subsets, the smallest \(\sigma\)-algebra containing its open subsets. It is Hausdorff and locally compact and the measure \(d\text{vol}_K\) given by Definition 3.1 turns out to be a positive Radon measure with respect to the topology \(|K|\), see for instance Definition 7.1.1 of [2]. If \(K\) is a finite \(n\)-dimensional simplicial complex, the total measure of \(|K|\) is thus \(f_n(K)\) and its \((n-1)\)-skeleton has vanishing measure.

The topological space \(|K|\) inherits in addition the following family of (canonical) discrete measures.

**Definition 3.2** For every \(p \in \mathbb{N}\), the \(p\)-skeleton measure of a locally finite simplicial complex \(K\) is the measure \(\gamma_{p,K} := \sum_{\sigma \in K[p]} \delta_{\hat{\sigma}}\), where \(\delta_{\hat{\sigma}}\) denotes the Dirac measure on the barycenter \(\hat{\sigma}\) of \(\sigma\).

Recall that the Dirac measure \(\delta_{\hat{\sigma}}\) is the measure such that \(\int_{|K|} \varphi \delta_{\hat{\sigma}} = \varphi(\hat{\sigma})\) for every continuous function \(\varphi\) on \(|K|\). If \(K\) is finite, the total measure \(\int_{|K|} 1 d\gamma_{p,K}\) thus equals \(f_p(K)\). For every \(d \geq 0\), we denote the \(p\)-skeleton measure of the \(d\)-th barycentric subdivision \(S_d(K)\) of \(K\) by

\[
\gamma_{p,K}^d := \frac{1}{(n+1)!d} \sum_{\sigma \in S_d(K)[p]} \delta_{\hat{\sigma}}.
\]

Our aim is now to prove Theorem 1.4. Let \(p \in \{0, \ldots, n\}\). For every \(l, d \geq 0\), we set

\[
\theta_{p,d}^l := \frac{1}{(n+1)!d} \sum_{\sigma \in S_d(\Delta_n)[p]} (f_{\sigma})_*(\gamma_{p,\Delta_n}^d) - \gamma_{p,\Delta_n}^{l+d},
\]

where \(f_{\sigma} : \Delta_n \to \sigma\) denotes any simplicial isomorphism. We need first the following lemma.

**Lemma 3.3** For every \(p \in \{0, \ldots, n\}\) and every \(l, d \geq 0\), the total measure of \(\theta_{p,d}^l\) converges to zero as \(d\) grows to \(+\infty\).

**Proof** In a subdivided \(n\)-simplex \(S_d(\Delta_n)\), every \(p\)-simplex \(\tau\) is a face of at least one \(n\)-simplex and the number of such \(n\)-simplices is by definition \(f_{n-p-1}(\text{Lk}(\tau, S_d(\Delta_n)))\) where \(\text{Lk}(\tau, S_d(\Delta_n))\) denotes the link of the simplex \(\tau\) in the simplicial complex \(S_d(\Delta_n)\). Since \(S_d^{l+d}(\Delta_n) = S_d^d(S_d(\Delta_n))\), we deduce that for every \(d \geq 0\),
\[
\gamma_{p, \Delta_n}^{l+d} = \frac{1}{(n+1)!^l} \sum_{\sigma \in \text{Sd}^l(\Delta_n)^{[n]} \setminus \{\emptyset\}} (f_{\sigma})_*(\gamma_{p, \Delta_n}^{d}) \\
- \frac{1}{(n+1)!^{l+d}} \sum_{\tau \in \text{Sd}^l(\Delta_n)^{(n-1)}} (f_{n-\dim \tau-1}(Lk(\tau, \text{Sd}^l(\Delta_n)))) - 1) \sum_{\alpha \in \text{Sd}^d(\tau)[\rho]} \delta_\hat{\alpha},
\]

where \(\text{Sd}^l(\Delta_n)^{(n-1)}\) denotes the \((n-1)\)-skeleton of \(\text{Sd}^l(\Delta_n)\). Thus, \(\theta_{p}^{l}(d)\) is equal to

\[
\frac{1}{(n+1)!^{l+d}} \sum_{\tau \in \text{Sd}^l(\Delta_n)^{(n-1)}} (f_{n-\dim \tau-1}(Lk(\tau, \text{Sd}^l(\Delta_n)))) - 1) \sum_{\alpha \in \text{Sd}^d(\tau)[\rho]} \delta_\hat{\alpha}.
\]

And so the total mass \(\int_{\Delta_n} 1 d\theta_{p}^{l}(d)\) of \(\theta_{p}^{l}(d)\) is less than or equal to

\[
\left(\frac{1}{(n+1)!^l} \sup_{\tau} \left( f_{n-\dim \tau-1}(Lk(\tau, \text{Sd}^l(\Delta_n)))) - 1 \right) \times \# \text{Sd}^l(\Delta_n)^{(n-1)} \right) \sup_{\tau} f_{p}^{d}(\tau) \frac{\text{Sd}^{l}(\Delta_n)}{(n+1)!^{l+d}}.
\]

Since \(\dim \tau < n\), we know from Theorem 2.4 that \(\frac{\sup_{\tau} f_{p}^{d}(\tau)}{(n+1)!^{l+d}} \xrightarrow{d \to +\infty} 0\). Hence the result. \(\Box\)

**Proof of Theorem 1.4** Let us first assume that \(K = \Delta_n\) and let \(\varphi \in C^0(\Delta_n)\) be a continuous function on \(\Delta_n\). We set, for every \(l, d \geq 0\), \(R_{l,d} := \int_{\Delta_n} \varphi d\gamma_{p, \Delta_n}^{l+d} - q_{p,n} \int_{\Delta_n} \varphi d\text{vol}_{\Delta_n}\) and deduce from Lemma 3.3

\[
R_{l,d} = \frac{1}{(n+1)!^l} \sum_{\sigma \in \text{Sd}^l(\Delta_n)^{[n]} \setminus \{\emptyset\}} \left( \int_{\Delta_n} f_{\sigma}^* \varphi d\gamma_{p, \Delta_n}^{d} - q_{p,n} \int_{\Delta_n} f_{\sigma}^* \varphi d\text{vol}_{\Delta_n} \right) \\
- \int_{\Delta_n} \varphi d\theta_{p}^{l}(d),
\]

since by definition \((f_{\sigma})_*d\text{vol}_{\Delta_n} = (n+1)!^l d\text{vol}_{\Delta_n}|_{\sigma}\). Thus,

\[
R_{l,d} = \frac{1}{(n+1)!^l} \sum_{\sigma \in \text{Sd}^l(\Delta_n)^{[n]} \setminus \{\emptyset\}} \left( \int_{\Delta_n} (f_{\sigma}^* \varphi - \varphi(\hat{\sigma})) d\gamma_{p, \Delta_n}^{d} \right) \\
+ \left( f_{p}(\text{Sd}^d(\Delta_n)) \frac{1}{(n+1)!^{d}} - q_{p,n} \right) \frac{1}{(n+1)!^l} \sum_{\sigma \in \text{Sd}^l(\Delta_n)^{[n]} \setminus \{\emptyset\}} \varphi(\hat{\sigma}) - \int_{\Delta_n} \varphi d\theta_{p}^{l}(d).
\]

Now, since \(\varphi\) is continuous, \(\sup_{\sigma \in \text{Sd}^l(\Delta_n)^{[n]} \setminus \{\emptyset\}} (\sup_{\sigma} |\varphi - \varphi(\hat{\sigma})|)\) converges to 0 as \(l\) grows to \(+\infty\), while \(\frac{1}{(n+1)!^l} \sum_{\sigma \in \text{Sd}^l(\Delta_n)^{[n]} \setminus \{\emptyset\}} \varphi(\hat{\sigma})\) remains bounded by \(\sup_{\Delta_n} |\varphi|\). Likewise
by Theorem 2.4, \( f_p^d(\Delta_n) \) converges to \( q_{p,n} \) as \( d \) grows to \( +\infty \), while by Lemma 3.3, 
\[
\int_{\Delta_n} 1 \, d\theta_p^l(d) \to 0.
\]
By letting \( d \) grow to \( +\infty \) and then \( l \) grow to \( +\infty \), we deduce that \( R_{l,d} \) can be as small as we want for \( l, d \) large enough. This proves the result for \( K = \Delta_n \).

Now, if \( K \) is a locally finite \( n \)-dimensional simplicial complex, we deduce the result by summing over all \( n \)-dimensional simplices of \( K \), since from Theorem 2.4, the measure of the \( (n-1) \)-skeleton of \( K \) with respect to \( \gamma_p^d \) converges to 0 as \( d \) grows to \( +\infty \).

Note that by integration of the constant function 1, Theorem 1.4 implies that for a finite simplicial complex \( K \), 
\[
\frac{f_p^d(K)}{(n+1)!d} \to q_{p,n},
\]
recovering the first part of Theorem 2.4. Also, since \( q_{n,n} = 1 \), it implies that \( \gamma_{n,K}^d \to d\text{vol}_K \). This actually quickly follows from Riemann integration, since for every \( \varphi \in C^0(\Delta_n) \),
\[
\int_{\Delta_n} \varphi \, d\text{vol}_{\Delta_n} = \lim_{d \to +\infty} \frac{1}{(n+1)!d} \sum_{\sigma \in \text{Sd}^d(\Delta_n)[n]} \varphi(\hat{\sigma})
\]
\[
= \lim_{d \to +\infty} \int_{|\Delta_n|} \varphi \, d\gamma_{n,\Delta_n}^d.
\]

Let us give another point of view on this fact. For every \( \sigma \in \text{Sd}(\Delta_n)[n] \), let us choose once and for all a simplicial isomorphism \( f_\sigma : \Delta_n \to \sigma \). We consider then the space \( \Omega \) of sequences \( i \in \mathbb{N}^* \mapsto \sigma_i \in \text{Sd}(\Delta_n)[n] \) of \( n \)-simplices in \( \text{Sd}(\Delta_n) \) and equip the latter with the normalized \( n \)-skeleton measure 
\[
\frac{1}{(n+1)!} \sum_{\sigma \in \text{Sd}(\Delta_n)[n]} \delta_{\hat{\sigma}}.
\]
The space \( \Omega \), as a product space \( \Omega = \text{Map}(\mathbb{N}^*, \text{Sd}(\Delta_n)[n]) = (\text{Sd}(\Delta_n)[n])^{\mathbb{N}^*} \), thus inherits the induced product measure which we denote by \( \omega \).

**Definition 3.4** For every \( d \in \mathbb{N}^* \), the \( d \)-th subdivision map of \( \Delta_n \) is the map
\[
\Phi_d : \Omega \times \Delta_n \to \Delta_n
\]
\[
((\sigma_i)_{i \in \mathbb{N}^*}, x) \mapsto f_{\sigma_1} \circ \ldots \circ f_{\sigma_d}(x).
\]

Moreover, the subdivision map of \( \Delta_n \) is the limit
\[
\Phi : \Omega \times \Delta_n \to \Delta_n
\]
\[
((\sigma_i)_{i \in \mathbb{N}^*}, x) \mapsto \lim_{d \to +\infty} f_{\sigma_1} \circ \ldots \circ f_{\sigma_d}(x).
\]

**Theorem 3.5** For every \( n \in \mathbb{N}^* \), the subdivision map \( \Phi \) of \( \Delta_n \) is well defined, continuous, surjective and contracts the second factor \( \Delta_n \). Moreover,
\[
d\text{vol}_{\Delta_n} = \Phi_*(\omega \times d\text{vol}_{\Delta_n}) = \Phi_*(\omega \times \delta_{\Delta_n}) = \lim_{d \to +\infty} \gamma_{n,\Delta_n}^d.
\]
(This result may be compared to the general Borel isomorphism theorem.) In Theorem 3.5, by “contracts the second factor”, we mean that for every \((\sigma_i)_{i \in \mathbb{N}^*} \in \Omega\) and every \(x, y \in \Delta_n\), \(\Phi((\sigma_i)_{i \in \mathbb{N}^*}, x) = \Phi((\sigma_i)_{i \in \mathbb{N}^*}, y)\).

**Proof** For every \((\sigma_i)_{i \in \mathbb{N}^*} \in \Omega\), the sequence of compact subsets \(\text{Im}(f_{\sigma_1} \circ \cdots \circ f_{\sigma_d})\) decreases as \(d\) grows to \(+\infty\). These subsets are \(n\)-simplices of the barycentric subdivision \(\text{Sd}^d(\Delta_n)\) so that their diameters converge to zero. We deduce the first part of Theorem 3.5.

Now, from the first part \(\Phi\) contracts the second factor and is measurable, so the pushforward \(\Phi_*(\omega \times \mu)\) does not depend on the probability measure \(\mu\) on \(\Delta_n\). In particular, \(\Phi_*(\omega \times d\text{vol}_{\Delta_n}) = \Phi_*(\omega \times \delta_{\Delta_n})\). Now, we have by definition \((\Phi_d)_*(\omega \times d\text{vol}_{\Delta_n}) = \frac{1}{(n+1)!} \sum_{\tau \in \text{Sd}^d(\Delta_n)} (f_{\tau})_* (d\text{vol}_{\Delta_n}),\) where \(f_{\tau}\) is the corresponding simplicial isomorphism \(f_{\sigma_1} \circ \cdots \circ f_{\sigma_d}\) between \(\Delta_n\) and \(\tau\), so that \((\Phi_d)_*(\omega \times d\text{vol}_{\Delta_n}) = d\text{vol}_{\Delta_n}\) for every \(d\) since \((f_{\tau})_* (d\text{vol}_{\Delta_n}) = (n+1)! d\text{vol}_{\Delta_n}\). Likewise, \((\Phi_d)_*(\omega \times \delta_{\Delta_n}) = \frac{1}{(n+1)!} \sum_{\tau \in \text{Sd}^d(\Delta_n)} (f_{\tau})_* (\delta_{\Delta_n}) = \gamma_{n, \Delta_n}^d\) by definition. Since the sequence \((\Phi_d)_{d \in \mathbb{N}^*}\) of continuous maps converges pointwise to \(\Phi\), we deduce from Lebesgue’s dominated convergence theorem that for every probability measure \(\mu\) on \(\Delta_n\), the sequence \((\Phi_d)_*(\omega \times \mu)\) weakly converges to \(\Phi_*(\omega \times \mu)\).

Recall that by definition, the Dirac measure \(\delta_{\Delta_n}\) in Theorem 3.5 coincides with the measure \(\gamma_{n, \Delta_n}\). For \(p < n\), we get

**Theorem 3.6** For every \(0 \leq p \leq n\), \(f_p(\Delta_n) d\text{vol}_{\Delta_n} = \Phi_*(\omega \times \gamma_{p, \Delta_n})\). Moreover the sequence of measures \(\left\{ \frac{1}{(n+1)!} \sum_{\sigma \in \text{Sd}^d(\Delta_n)|^{[p]}} f_{n-p-1}(\text{Lk}(\sigma, \text{Sd}^d(\Delta_n))) \delta_{\hat{\sigma}} \right\}_{d \in \mathbb{N}}\) weakly converges to this measure \(f_p(\Delta_n) d\text{vol}_{\Delta_n}\) as \(d\) grows to \(+\infty\).

We will denote the measure \(\frac{1}{(n+1)!} \sum_{\sigma \in \text{Sd}^d(\Delta_n)|^{[p]}} f_{n-p-1}(\text{Lk}(\sigma, \text{Sd}^d(\Delta_n))) \delta_{\hat{\sigma}}\) appearing in Theorem 3.6 by \(f_{n-p-1}(\text{Lk}(\sigma, \text{Sd}^d(\Delta_n))) d\gamma_{p, \Delta_n}^d(\sigma)\).

Before starting to prove the above theorem, let us recall that \(f_p(\Delta_n) = \binom{n+1}{p+1}\) and that by definition \(f_{-1}(\text{Lk}(\sigma, \text{Sd}^d(\Delta_n))) = 1\).

**Proof** From Theorem 3.5, \(\Phi\) contracts the second factor. Since the mass of \(\gamma_{p, \Delta_n}\) equals \(f_p(\Delta_n)\) by definition, we deduce the first equality. Now, as in the proof of Theorem 3.5, we deduce from Lebesgue’s dominated convergence theorem that the sequence \((\Phi_d)_*(\omega \times \gamma_{p, \Delta_n})\) weakly converges to \(\Phi_*(\omega \times \gamma_{p, \Delta_n})\). It remains thus to compute \((\Phi_d)_*(\omega \times \gamma_{p, \Delta_n})\). By definition \((\Phi_d)_*(\omega \times \gamma_{p, \Delta_n}) = \frac{1}{(n+1)!} \sum_{\sigma \in \text{Sd}^d(\Delta_n)|^{[p]}} (f_{\tau})_* (\gamma_{p, \Delta_n}),\) where \(f_{\tau}\) is the corresponding simplicial isomorphism \(f_{\sigma_1} \circ \cdots \circ f_{\sigma_d}\) between \(\Delta_n\) and \(\tau\). In this sum, we see that each \(p\)-simplex of \(\text{Sd}^d(\Delta_n)\) receives as many Dirac measures as the number of \(n\)-simplices adjacent to it. The number of \(n\)-simplices adjacent to \(\sigma \in \text{Sd}^d(\Delta_n)|^{[p]}\) is by definition \(f_{n-p-1}(\text{Lk}(\sigma, \text{Sd}^d(\Delta_n)))\). We deduce

\[
(\Phi_d)_*(\omega \times \gamma_{p, \Delta_n}) = \frac{1}{(n+1)!} \sum_{\sigma \in \text{Sd}^d(\Delta_n)|^{[p]}} f_{n-p-1}(\text{Lk}(\sigma, \text{Sd}^d(\Delta_n))) \delta_{\hat{\sigma}} = f_{n-p-1}(\text{Lk}(\sigma, \text{Sd}^d(\Delta_n))) d\gamma_{p, \Delta_n}^d(\sigma).
\]
Corollary 3.7 For every $n$-dimensional locally finite simplicial complex $K$ and every $0 \leq p \leq n$, the sequence of measures $\left( f_{n-p-1}(\text{Lk}(\sigma, \text{Sd}^d(K))) \right) d\gamma_{p,K}^d = \sum_{\sigma \in K^{[n]}} \gamma_{p,\sigma}^d - \sum_{\tau \in K^{(n-1)}} (f_{n-\dim \tau-1}(\text{Lk}(\tau, K)) - 1) \left( \frac{(\dim \tau + 1)!}{(n+1)!} \right) \gamma_{p,\tau}^d$

Proof By definition

$$\gamma_{p,K}^d = \sum_{\sigma \in K^{[n]}} \gamma_{p,\sigma}^d - \sum_{\tau \in K^{(n-1)}} (f_{n-\dim \tau-1}(\text{Lk}(\tau, K)) - 1) \left( \frac{(\dim \tau + 1)!}{(n+1)!} \right) \gamma_{p,\tau}^d$$

since for every $\tau \in K^{(n-1)}$ and every $\sigma \in K^{[n]}$ such that $\tau < \sigma$, $\gamma_{p,\sigma}|_{\tau} = \left( \frac{(\dim \tau + 1)!}{(n+1)!} \right) \gamma_{p,\tau}^d$ by definition and $\tau$ is a face of exactly $f_{n-\dim \tau-1}(\text{Lk}(\tau, K))$ such $\sigma$’s. The result thus follows from Theorems 1.4 and 3.6.

4 Limit Density of Links in a Simplicial Complex

Corollary 3.7 computes the limit density as $d$ grows to $+\infty$ of the top face numbers of the links of $p$-dimensional simplices in $\text{Sd}^d(K)$, $p \in \{0, \ldots, n\}$. We are now going to prove Theorem 1.5, which extends this result to all the face numbers of these links.

Proof of Theorem 1.5 Let $\varphi \in C_c^0(|K|)$ be a continuous function with compact support on $|K|$. For every $0 \leq l \leq n - p - 1$, let us introduce the set

$$\mathcal{I}_l := \{(\sigma, \tau) \in \text{Sd}^d(K)^{[p]} \times \text{Sd}^d(K)^{[p+l+1]} | \sigma < \tau\}. \quad (1)$$

It is equipped with the projections $p_1 : (\sigma, \tau) \in \mathcal{I}_l \mapsto \sigma \in \text{Sd}^d(K)^{[p]}$ and $p_2 : (\sigma, \tau) \in \mathcal{I}_l \mapsto \tau \in \text{Sd}^d(K)^{[p+l+1]}$. We observe that for every $\sigma \in \text{Sd}^d(K)^{[p]}$, $\# p_1^{-1}(\sigma) = f_l(\text{Lk}(\sigma, \text{Sd}^d(K)))$ while for every $\tau \in \text{Sd}^d(K)^{[p+l+1]}$, $p_2^{-1}(\tau)$ is in bijection with $\tau^{[p]}$ (given by $p_1$). Let us set

$$\varphi_1 : (\sigma, \tau) \in \mathcal{I}_l \mapsto \varphi(\hat{\sigma}) \in \mathbb{R},$$

$$\varphi_2 : (\sigma, \tau) \in \mathcal{I}_l \mapsto \varphi(\hat{\tau}) \in \mathbb{R},$$

$$\gamma_l := \frac{1}{(n+1)!d} \sum_{(\sigma, \tau) \in \mathcal{I}_l} \delta(\sigma, \tau). \quad (2)$$

Then, we deduce

$$\int_{|K|} \varphi f_l(\text{Lk}(\sigma, \text{Sd}^d(K))) \, d\gamma_{p,K}^d(\sigma) = \int_{\mathcal{I}_l} \varphi_1 \, d\gamma_l$$

$$= \int_{\mathcal{I}_l} \varphi_2 \, d\gamma_l + \int_{\mathcal{I}_l} (\varphi_1 - \varphi_2) \, d\gamma_l$$

$$= \int_{\text{Sd}^d(K)^{[p+l+1]}} (p_2)_*(\varphi_2 \, d\gamma_l) + \int_{\mathcal{I}_l} (\varphi_1 - \varphi_2) \, d\gamma_l$$

$$= \int_{|K|} \varphi f_p(\tau) \, d\gamma_{p+l+1,K}(\tau) + \int_{\mathcal{I}_l} (\varphi_1 - \varphi_2) \, d\gamma_l.$$
From Theorem 1.4, the first term \( \int_{K} \varphi f_p(\tau) d \gamma_{p+l+1,K}(\tau) \) in the right hand side converges to \( q_{p+l+1,n} \int_{K} \varphi d \text{vol}_K \) as \( d \) grows to \(+\infty\) while the second term \( \int_{I_0} (\varphi - \varphi_2) d \gamma_l \) converges to zero. Indeed, \( \varphi \) is continuous with compact support and the diameter of \( \tau \in \text{Sd}^d(\mathcal{K}) \) uniformly converges to zero on this compact subset as \( d \) grows to \(+\infty\). Thus, the supremum of \( (\varphi - \varphi_2) \) converges to zero as \( d \) grows \(+\infty\). On the other hand, the total mass of \( \gamma_l \) remains bounded, since

\[
\int_{I_0} 1 \gamma_l = \int \text{Sd}^d(\mathcal{K})^{p+l+1} (p_2)_{*} (d \gamma_l) = f_p(\Delta_{p+l+1}) \int_{K} \gamma_{p+l+1,K}^d
\]

and the latter is bounded from Theorem 1.4. The result follows by definition of \( q_{L,k(\text{Sd}^d(\mathcal{K}))}(T) \).

The \((n-1)\)-skeleton of \( \mathcal{K} \) has vanishing measure with respect to \( d \text{vol}_K \) while for every \( \sigma \in \text{Sd}^d(\mathcal{K})^{[n]} \) interior to an \( n \)-simplex, its link is a homology \((n-p-1)\)-sphere (Theorem 63.2 of [9]). After evaluation at \( T = -1 \) and integration of the constant function 1, Theorem 1.5 thus provides the following asymptotic Dehn–Sommerville relations:

\[
\sum_{l=p}^{n} q_{l,n} \binom{l+1}{p+1} (-1)^{n+l} = q_{p,n}.
\]

Now, recall that the dual block \( D(\sigma) \) of a simplex \( \sigma \in \mathcal{K} \) is the union of all open simplices \( [\hat{\sigma}_0, \ldots, \hat{\sigma}_p] \) of \( \text{Sd}(\mathcal{K}) \) such that \( \sigma_0 = \sigma \), see [9]. The closure \( \overline{D}(\sigma) \) of \( D(\sigma) \) is called the closed block dual to \( \sigma \) and following [9] we set \( \hat{D}(\sigma) : = \overline{D}(\sigma) \setminus D(\sigma) \).

We are now ready to prove Theorem 1.6.

**Proof of Theorem 1.6** By definition, the dual block \( D(\sigma) \) has only one face in dimension 0, namely \( \hat{\sigma} \), so that for the coefficient \( l = 0 \), the result follows from Theorem 1.4. Let us now assume that \( 0 < l \leq n - p \) and choose \( \varphi \in C^0_c([K]) \). We set

\[
\mathcal{J}_l := \{ (\sigma, \theta) \in \text{Sd}^d(\mathcal{K})^{[n]} \times \text{Sd}^{d+1}(\mathcal{K})^{[l-1]} \mid \theta \in \hat{D}(\sigma) \}.
\]

Let \( p_1 : (\sigma, \theta) \in \mathcal{J}_l \mapsto \sigma \in \text{Sd}^d(\mathcal{K})^{[n]} \). Then, for every \( \sigma \in \text{Sd}^d(\mathcal{K})^{[n]} \), \# \( p_1^{-1}(\sigma) = f_l(D(\sigma)) \), since \( p_1^{-1}(\sigma) \) is in bijection with \( \hat{D}(\sigma) \) and by taking the cone over \( \hat{\sigma} \) we get an isomorphism \( \tau \in \hat{D}(\sigma) \mapsto \hat{\sigma} \ast \tau \in \hat{D}(\sigma) \setminus \hat{\sigma} \) where \( \ast \) denotes the join operation. (Recall that if \( \tau = [e_0, \ldots, e_k] \) the join \( \hat{\sigma} \ast \tau \) is \( [\hat{\sigma}, e_0, \ldots, e_k] \).)

Likewise by definition, every simplex \( \theta \in \hat{D}(\sigma)^{[l-1]} \) reads \( \theta = [\hat{\tau}_0, \ldots, \hat{\tau}_{l-1}] \) where \( \sigma < \tau_0 < \cdots < \tau_{l-1} \) are simplices of \( \text{Sd}^d(\mathcal{K}) \) (see Theorem 64.1 of [9]). We deduce a map
\[ \pi : \mathcal{J}_{l} \rightarrow \bigsqcup_{h=l-1}^{n-p-1} \mathcal{I}_{h} \]

\[ (\sigma, [\hat{0}, \ldots, \hat{t}_{l-1}]) \mapsto (\sigma, \tau_{l-1}) \]

where \( \mathcal{I}_{h} \) is the set defined in (1).

We then set \( p_{2} : (\sigma, \tau) \in \bigsqcup_{h=l-1}^{n-p-1} \mathcal{I}_{h} \mapsto \tau \in \text{Sd}^{d}(K) \setminus \text{Sd}^{d}(K)^{(p+l-1)} \). As in the proof of Theorem 1.5, for every \( \tau \in \text{Sd}^{d}(K) \setminus \text{Sd}^{d}(K)^{(p+l-1)} \), \( p_{2}^{-1}(\tau) \) is in bijection with \( \tau^{[p]} \) and \( \pi^{-1}(\sigma, \tau) \) with the set of interior \((l - 1)\)-dimensional simplices of \( \text{Sd}(\text{Lk}(\sigma, \tau)) \), so that \#\( \pi^{-1}((\sigma, \tau)) = \lambda_{h+1,l} \) if \( \dim \tau = p + h + 1 \). Let us set

\[ \tilde{\varphi}_{1} : (\sigma, \tau) \in \mathcal{J}_{l} \mapsto \varphi(\hat{\sigma}) \in \mathbb{R} \text{ and } \tilde{\gamma}_{l} := \frac{1}{(n+1)!} \sum_{(\sigma, \theta) \in \mathcal{J}_{l}} \delta(\sigma, \theta). \]

Then, we deduce

\[ \int_{|\mathcal{K}|} \varphi f_{l}(D(\sigma)) \, d\gamma^{d}_{p,K}(\sigma) = \int_{\mathcal{J}_{l}} \tilde{\varphi}_{1} \, d\tilde{\gamma}_{l} = \sum_{h=l-1}^{n-p-1} \lambda_{h+1,l} \int_{\mathcal{I}_{h}} \varphi_{1} \, d\gamma_{h} \]

by pushing forward \( \tilde{\varphi}_{1} \, d\tilde{\gamma}_{l} \) onto \( \bigsqcup_{h=l-1}^{n-p-1} \mathcal{I}_{h} \) with \( \pi \), where \( \varphi_{1} \) and \( \gamma_{h} \) are defined by (2).

Now, we have established in the proof of Theorem 1.5 that as \( d \) grows to \(+\infty\), \( \int_{\mathcal{I}_{h}} \varphi_{1} \, d\gamma_{h} \) converges to \( f_{p}(\Delta_{p+h+1})q_{p+h+1} \int_{K} \varphi(\text{vol}K) \). We deduce that

\[ f_{l}(D(\sigma))d\gamma^{d}_{p,K}(\sigma) \]

weakly converges to \( (\sum_{h=l}^{n-p} \lambda_{h,1} f_{p}(\Delta_{p+h})q_{p+h,n}) \text{vol}K \). Hence the result.

\[ \square \]

**Remark 4.1** In [10], we study the expected topology of a random subcomplex in a finite simplicial complex \( K \) and its barycentric subdivisions. The Betti numbers of such a subcomplex turn out to be asymptotically controlled by the measure given by Theorem 1.6.

Let us now finally observe that Theorem 1.6 provides a geometric proof of the following (compare Theorem A of [5]).

**Corollary 4.2** The vector \( (q_{p,n})_{0 \leq p \leq n} \) is the eigenvector of \( \Lambda_{n}^{d} \) associated to the eigenvalue \((n + 1)!\), normalized by the relation \( q_{n,n} = 1 \).

**Proof** By [9, Thm. 64.1], we know that the dual blocks of a complex \( K \) are disjoint and that their union is \( |K| \). We deduce that for every \( d \in \mathbb{N}^{*} \),

\[ \frac{1}{(n + 1)!d} q_{\text{Sd}^{d+1}(\Delta_{n})}(T) = \sum_{p=0}^{n} \int_{\Delta_{n}} q_{D(\sigma)}(T) \, d\gamma^{d}_{p,\Delta_{n}}(\sigma). \]

By letting \( d \) grow to \(+\infty\), we now deduce from Theorem 1.6, applied to \( K = \Delta_{n} \) and after integration of 1, that

\[ \square \]
\[(n+1)! \sum_{p=0}^{n} q_{p,n} T^p = \sum_{p=0}^{n} \left( \sum_{l=0}^{n-p} T^l \left( \sum_{h=p+l}^{n} q_{h,n} f_p(\Delta_h) \lambda_{h-p,l} \right) \right) = \sum_{l=0}^{n} T^l \left( \sum_{h=l}^{n} q_{h,n} \sum_{p=0}^{h-l} f_p(\Delta_h) \lambda_{h-p,l} \right). \]

Now, \(\sum_{p=0}^{h-l} f_p(\Delta_h) \lambda_{h-p,l} = \sum_{p=l}^{h} (\binom{h+1}{p}) \lambda_{p,l} = \lambda_{h+1,l+1} \) from Lemma 2.6. Hence, for every \(p \in \{0, \ldots, n\}, \ (n+1)! q_{p,n} = \sum_{h=l}^{n} q_{h,n} \lambda_{h+1,l+1}. \)

\[\square\]

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