NEW LATTICE POINT ASYMPTOTICS FOR PRODUCTS OF UPPER HALF PLANES

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Abstract. Let $\Gamma$ be an irreducible lattice in $\text{PSL}_2(\mathbb{R})^d$ ($d \in \mathbb{N}$) and $z$ a point in the $d$-fold direct product of the upper half plane. We study the discrete set of componentwise distances $D(\Gamma, z) \subset \mathbb{R}^d$ defined in (1). We prove asymptotic results on the number of $\gamma \in \Gamma$ such that $d(z, \gamma z)$ is contained in strips expanding in some directions and also in expanding hypercubes. The results on the counting in expanding strips are new. The results on expanding hypercubes improve the existing error terms ([6]) and generalize the Selberg error term for $d = 1$.

We give an asymptotic formula for the number of lattice points $\gamma z$ such that the hyperbolic distance in each of the factors satisfies $d((\gamma z)_j, z_j) \leq T$. The error term, as $T \to \infty$ generalizes the error term given by Selberg for $d = 1$, also we describe how the counting function depends on $z$. We also prove asymptotic results when the distance satisfies $A_j \leq d((\gamma z)_j, z_j) < B_j$, with fixed $A_j < B_j$ in some factors, while in the remaining factors $0 \leq d((\gamma z)_j, z_j) \leq T$ is satisfied.

Contents

1. Introduction 2
2. Lie groups and discrete subgroups 6
3. A priori estimates 7
4. The Selberg transform and spectral estimates 10
4.1. The Selberg transform 10
4.2. Spectral decomposition 12
4.3. Spectral measure 13
5. Proof of the lattice points theorems 14
5.1. The main parameters 14
5.2. Test functions and auxiliary parameters 15
5.3. Terms in the asymptotic formula 16
5.4. The explicit term 18
5.5. Sum over the spectrum 19
5.6. Difference between sums with sharp and smooth bounds 23
5.7. Asymptotic estimate, case $E = \emptyset$ 24
5.8. Asymptotic estimate, case $E \neq \emptyset$ 25
6. Estimates of Selberg transforms 28
6.1. Integral representations 28

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1. Introduction

Let $\mathbb{H} = \{ x+iy \in \mathbb{C} : y > 0 \}$ be the upper halfplane equipped with the hyperbolic metric $d : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ and its invariant measure induced by $\frac{dx \, dy}{y^2}$. The group of orientation preserving isometries of this metric space is $\text{PSL}_2(\mathbb{R})$. Let now $d$ be a natural number and consider the semisimple Lie group $\text{PSL}_2(\mathbb{R})^d$ as acting on its corresponding symmetric space $\mathbb{H}^d$. We write $z = (z_1, \ldots, z_d)$ for the coordinates $z_1, \ldots, z_d \in \mathbb{H}$ of a point $z \in \mathbb{H}^d$. Let us consider the vector valued distance function

\begin{equation}
(1) \quad d(z, u) := (d(z_1, u_1), \ldots, d(z_d, u_d)) \in \mathbb{R}^d
\end{equation}

for points $z = (z_1, \ldots, z_d), u = (u_1, \ldots, u_d) \in \mathbb{H}^d$. The canonical invariant distance of $z$, $u$ is then the euclidean norm of $d(z, u)$. But other choices of norms (like the maximum norm) also induce $\text{PSL}_2(\mathbb{R})^d$-invariant metrics on $\mathbb{H}^d$.

Let $\Gamma \subset \text{PSL}_2(\mathbb{R})^d$ be an irreducible lattice. A lattice in $\text{PSL}_2(\mathbb{R})^d$ is a discrete subgroup $\Gamma \subset \text{SL}_2(\mathbb{R})^d$ of finite covolume, that is, the volume of the quotient $\Gamma \backslash \mathbb{H}^d$ in the canonical measure is finite. The lattice $\Gamma \subset \text{SL}_2(\mathbb{R})^d$ is called irreducible if all projections of $\Gamma$ to non-trivial subproducts of $\text{PSL}_2(\mathbb{R})^d$ are dense. A main example is the Hilbert modular group $\text{PSL}_2(O)$ for the ring of integers $O$ of a totally real number field $F$ of degree $d$ over $\mathbb{Q}$, embedded in the product $\text{PSL}_2(\mathbb{R})^d$ by the $d$ embeddings of $F$ into $\mathbb{R}$. The embedded group $\text{PSL}_2(O)$ and all its subgroups of finite index are irreducible lattices in $\text{PSL}_2(\mathbb{R})^d$. They are not cocompact, which means that the quotient $\Gamma \backslash \mathbb{H}^d$ is not compact. In case $d \geq 2$ every irreducible lattice in $\text{PSL}_2(\mathbb{R})^d$ which is not cocompact contains a subgroup of finite index which is $\text{PSL}_2(\mathbb{R})^d$-conjugate to a subgroup of finite index in one of the $\text{PSL}_2(O)$. Irreducible cocompact lattices in $\text{PSL}_2(\mathbb{R})^d$ are constructed from quaternion algebras over totally real number fields $F$. In case $d \geq 2$ these are up to conjugacy the only examples, by Margulis’ arithmeticity theorem. See Section 2 for more details.

Let $\Gamma \subset \text{PSL}_2(\mathbb{R})^d$ be an irreducible lattice and let $z \in \mathbb{H}^d$ be fixed. Consider the set of vector valued distances

\begin{equation}
(2) \quad D(\Gamma, z) := \{ d(z, \gamma z) \in \mathbb{R}^d : \gamma \in \Gamma \}.
\end{equation}

This clearly is an infinite discrete subset of $\mathbb{R}^d$. But what more can be said? In this paper we shall prove results which describe the distribution of the points of $D(\Gamma, z)$ in various regions like strips or expanding polyhedra in $\mathbb{R}^d$.

To give a precise formulation of our main results, we need to discuss some aspects of the spectral theory of $L^2(\Gamma \backslash \mathbb{H}^d)$. This Hilbert space has an infinite dimensional subspace $L^2(\Gamma \backslash \mathbb{H}^d)$ with an orthonormal basis $\{ \psi_\ell \}$ ($\ell \in \mathbb{N} \cup \{0\}$) of joint
eigenvectors of the Laplace operators $\Delta_j = -y_j^2 \partial_{y_j}^2 - y_j^2 \partial_{x_j}^2$ ($j = 1, \ldots, d$) in the factors. Among the eigenfunctions is the constant function $\psi_0(z) = (\text{vol}(\Gamma \backslash \mathbb{H}^d))^{-1/2}$ for which the eigenvalues of all $\Delta_j$ are all equal to 0. The corresponding multi-eigenvalues $\lambda_\ell$ have finite multiplicities and form a discrete set in $[0, \infty)^d$. For $\ell \geq 1$ one knows that $\lambda_{\ell,j} > 0$ for all $j = 1, \ldots, d$. If $\Gamma$ is cocompact, then $L^2,\text{disc}(\Gamma \backslash \mathbb{H}^d)$ is all of $L^2(\Gamma \backslash \mathbb{H}^d)$. Otherwise the elements of the orthogonal complement of $L^2,\text{disc}(\Gamma \backslash \mathbb{H}^d)$ can be described as sums of integrals of Eisenstein series.

We call a multi-eigenvalue $\lambda_\ell$ exceptional if $0 < \lambda_{\ell,j} < \frac{1}{\ell}$ for some coordinate $j \in \{1, \ldots, d\}$. If $d \geq 2$, there may be infinitely many exceptional eigenvalues, since there is no bound on the other coordinates. If $0 < \lambda_{\ell,j} < \frac{1}{\ell}$ for all $j$ we call $\lambda_\ell$ totally exceptional. There are at most finitely many totally exceptional eigenvalues.

For a further discussion we use the parametrization $\lambda = \frac{1}{\ell} - \tau^2$ by the spectral parameter $\tau$. In $L^2(\Gamma \backslash \mathbb{H}^d)$ all eigenvalues of local Laplace operators are in $[0, \infty)$, so we can choose $\tau \in i[0, \infty) \cup [0, \frac{1}{2}]$. Thus we have $\tau_{0,j} = \frac{1}{\ell}$ for all $j$, and $\Re \tau_{\ell,j} < \frac{1}{\ell}$ for all $j$.

For a congruence subgroup $\Gamma$ of a Hilbert modular group it has been shown by Kim and Shahidi, [13], that $\Re \tau_{\ell,j} \leq \frac{1}{\ell}$ for all $\ell \geq 1$ and all $j$. For this situation a conjecture called after Selberg says that $\Re \tau_{\ell,j} = 0$ for all $\ell \geq 1$ and all $j$. Below we will discuss other results concerning $\Re \tau_{\ell,j}, \ell \geq 1$. For the formulation of our results we summarize the information concerning exceptional eigenvalues in the quantity

$$\hat{\tau} = \hat{\tau}(\Gamma) := \sup_{\ell \geq 1, 1 \leq j \leq d} \Re \tau_{\ell,j}.$$  

This, by definition, is an element of $[0, \frac{1}{2}]$. Since there may be infinitely many $\lambda_\ell$, the value $\frac{1}{2}$ might occur in (3), although here we omit $\ell = 0$. If $\hat{\tau} < \frac{1}{2}$ one says that $\Gamma \backslash \mathbb{H}^d$ has a strong spectral gap. In our later arguments we use that $\hat{\tau} < \frac{1}{2}$, a fact proved in [12] by Kelmer and Sarnak for cocompact $\Gamma$. If $\Gamma$ is not cocompact and $d \geq 2$ then $\Gamma$ contains a subgroup of finite index which is conjugate to a congruence subgroup of a Hilbert modular group $\text{SL}_2(O)$, for which the results of Kim and Shahidi, [13], imply that $\hat{\tau} \leq \frac{1}{2}$. Here we use the important fact, proved in [26] that every subgroup of finite index in $\text{SL}_2(O)$ is a congruence subgroup.

Let now $E \subset \{1, \ldots, d\}$ be a non-empty subset and let $I := \{I_j : j \in E\}$ be a set of bounded intervals $I_j := [A_j, B_j) \subset [0, \infty)$. Define for $T > 0$

$$S(E, I; T) := \{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_j \in I_j \text{ for } j \in E, 0 \leq x_j \leq T \text{ for } j \notin E\}.$$  

We think of $S(E, I; T)$ as a strip of increasing height $T$ in $\mathbb{R}^d$. Given $z \in \mathbb{H}^d$ we introduce the counting quantity

$$N_E(z; T) := \# \{ \gamma \in \Gamma : d(z, \gamma z) \in S(E, I; T)\}.$$  

We show

**Theorem 1.1.** Let $\Gamma$ be an irreducible lattice in $\text{PSL}_2(\mathbb{R})^d$, with $d \geq 2$. Let $E \subset \{1, \ldots, d\}$ be a subset with $e := \# E \geq 1$. Define $Q := \{1, \ldots, d\} \setminus E$ and assume
Let finite intervals \( \{A_j, B_j\} \subset [0, \infty) \) be given for \( j \in E \), the quantity \( N_E(z; T) \) in (5) satisfies as \( T \to \infty \):

\[
N_E(z; T) = \frac{\pi^d}{\text{vol}(\Gamma \backslash \mathbb{H}^d)} e^{\tau T} \prod_{j \in E} \left( \cosh B_j - \cosh A_j \right)
\]

\[
+ \begin{cases}
\text{O}_{\Gamma, E}(n(z) \exp \left( \frac{d+1}{d+2} qT \right)) & \text{if } \hat{\tau} \leq \frac{q}{2(d+2)}, \\
\text{O}_{\Gamma, E}(n(z) \exp \left( \frac{1+2\hat{\tau}+e}{2+e} qT \right)) & \text{if } \frac{q}{2(d+2)} \leq \hat{\tau} < \frac{1}{2}.
\end{cases}
\]

This is a specialization of Theorem 5.5 where we allow somewhat more general conditions on the components of the vector valued distances. The \( E \) in \( \text{O}_{\Gamma, E} \) implies an implicit dependence on the intervals \( \{A_j, B_j\} \) with \( j \in E \). In Theorem 1.1 we have \( d \geq 2 \) and \( 1 \leq q = \#Q \leq d - 1 \). As explained above \( \hat{\tau} < \frac{1}{2} \) holds and it depends on the relative sizes of \( d \) and \( q \) which of the error terms is applicable.

We shall describe now another result pertaining to the more standard lattice point problems. We consider the asymptotic distribution of the orbit points \( \gamma z \) (\( \gamma \in \Gamma \)) for a given point \( z \in X \) and a discontinuous group of motions \( \Gamma \) acting on a symmetric space \( X \). In the case when \( X = \mathbb{H} \) is the upper half plane many authors have contributed to this problem, for instance [10], [20] and [11]. The best result concerning error terms is due to Selberg (see the Bombay and Göttingen lectures in [25]). It gives

\[
\# \{ \gamma \in \Gamma : d(\gamma z, z) \leq T \} = \frac{\pi^d}{\text{vol}(\Gamma \backslash \mathbb{H}^d)} e^{\tau T}
\]

\[
+ \sum_{\ell} \pi^{1/2} |\psi_\ell(z)|^2 \frac{\Gamma(\tau_\ell)}{\Gamma(\tau_\ell + 3/2)} e^{(1/2+\tau_\ell)T} + O(e^{3T}) \quad (T \to \infty).
\]

The functions \( \psi_\ell \) form an finite orthonormal system (possibly empty) of eigenfunctions with eigenvalue \( \frac{1}{4} - \tau_\ell^2 \) of the hyperbolic Laplace operator acting on \( L^2(\Gamma \backslash \mathbb{H}) \) with \( 0 < \tau_\ell < \frac{1}{4} \). This result holds for all cofinite discrete subgroups of \( \text{PSL}_2(\mathbb{R}) \), cocompact or not.

Let \( \Gamma \) now be an irreducible lattice in \( \text{PSL}_2(\mathbb{R})^d \), with \( d \in \mathbb{N} \). For \( z \in \mathbb{H}^d \) we define

\[
N(z; T) := \# \left( \text{D}(\Gamma, z) \cap \{ x \in \mathbb{R}^d : \max(x) \leq T \} \right)
\]

where \( \max(x) \) is the maximum of the absolute values of the coordinates of the vector \( x \in \mathbb{R}^d \) and \( \text{D}(\Gamma, z) \) is defined in (2). We show

**Theorem 1.2.** Let \( \Gamma \) be an irreducible lattice in \( \text{PSL}_2(\mathbb{R})^d \), with \( d \in \mathbb{N} \) and let \( z \in \mathbb{H}^d \) be given. With \( \hat{\tau} = \hat{\tau}(\Gamma) \) as in (3), and with the quantity \( n(z) \) as defined in (16), the counting function \( N(z; T) \) has the following asymptotic behavior as \( T \to \infty \):

- If \( 0 \leq \hat{\tau}(\Gamma) \leq \frac{d}{2(d+2)} \) (large spectral gap), then

\[
N(z; T) = \frac{\pi^d}{\text{vol}(\Gamma \backslash \mathbb{H}^d)} e^{dT} + O \left( n(z) \exp \left( \frac{d+1}{d+2} dT \right) \right).
\]
If $\frac{d}{2(d+2)} \leq \frac{1}{\tilde{\tau}(\Gamma)} \leq \frac{1}{2}$ (small spectral gap), then

$$N(z; T) = \frac{\pi^d}{\text{vol}(\Gamma\backslash \mathbb{H}^d)} e^{dT} + \sum_{\ell \geq 1, \forall j \in (0,1/2)} |\psi_{\ell}(z)|^2 \prod_{j=1}^d \left( \frac{\sqrt{\pi} \Gamma(\tau_{\ell,j})}{\Gamma(3/2 + \tau_{\ell,j})} e^{(1/2 + \tau_{\ell,j})T} \right)$$

$$+ O_T\left(n(z) \exp\left(\frac{2d + 2(d-1)\tilde{\tau}}{3} T\right)\right).$$

Theorem 1.2 is a special case of Theorem 5.4, where we allow a more general counting quantity than $N(z; T)$. The function $n(z)$ in (16) is positive on $\Gamma\backslash \mathbb{H}^d$ and grows when $z$ approaches a cusp.

We note that we do not try to put one distance function on $\mathbb{H}^d$, but work with the vector of the distances in the factors. Partly, this is because in this way it is easier to apply the spectral theory. Partly, it reflects the fact that there is not one distance function on $\mathbb{H}^d$ that is preserved by the action of $\text{PSL}_2(\mathbb{R})^d$, but infinitely many.

For a small spectral gap totally exceptional eigenfunctions appear explicitly in the asymptotic estimate. Of course, some of these exceptional contributions may happen to be absorbed by the error term. For large spectral gaps, further improvement of our knowledge of $\tilde{\tau}(\Gamma)$ does not improve the quality of the error term in our asymptotic formula. This holds in particular for the congruence case, in which we know that $\tilde{\tau} \leq \frac{1}{2}$. The main term is always larger than the error term, even if we would have $\tilde{\tau}(\Gamma) = \frac{1}{2}$ (no spectral gap).

The case $d = 1$ in Theorem 1.2 concerns lattice point counting for groups acting on the upper half plane. The best known error term $O(e^{\tilde{\tau} T})$ coincides with the error term in Theorem 1.2 for $d = 1$. The papers [7], [1], [15], [16] and [2] treat lattice point counting for other symmetric spaces of rank one. The situation in this paper, with rank $d$, falls within the scope of [5] and [6], in which Gorodnik and Nevo consider counting of lattice points over quite general families of sets in quotients of more general Lie groups. Their error terms for $\Gamma\backslash \mathbb{H}^d$ are weaker than those in Theorem 1.2. They get $O(\exp(\tilde{\tau} T))$ in the case $d = 1$ and $\Gamma$ not cocompact, and $O(\exp(\frac{d+1}{4(d+2)} dT))$ for the Hilbert modular case and $d \geq 2$, which should be compared with Selberg’s bound $O(\tilde{\tau} T)$ for $d = 1$, and with $O(\exp(\frac{d+1}{4(d+2)} dT))$ in Theorem 1.2 for general $d$. We emphasize that the class of counting problems considered by Gorodnik and Nevo is much larger than ours. They consider quite general families $t \mapsto G_t$ of regions in much more general groups than $\text{PSL}_2(\mathbb{R})^d$, that have to grow in all directions. They use an ergodic method that can be applied in all these cases, without using more spectral information than the size of the spectral gap. The counting in Theorem 1.1 is over regions that are constant in some coordinate directions, and hence do not satisfy the conditions in [5].

In the proofs we apply the spectral theory of automorphic forms. We give the main proof in §5. The approach is sketched in the introduction to §4 and in Subsections 5.1 and 5.2. The idea is that the sums $N(z; T)$ and $N_E(z; T)$ are replaced by smooth approximations. This smoothness ensures that the new quantities have a spectral decomposition that converges pointwise. In this spectral expansion we single out the terms corresponding to the constant functions and to totally exceptional...
eigenfunctions, if these are present. These give the main terms in the asymptotic expansion. The remaining part of the spectral decomposition is estimated using the estimate of the spectral measure in Theorem 4.2. We use an approach similar to one in [11] that makes explicit the dependence on the point \( z \in \mathbb{S}^d \).

For the handling of the spectral decomposition the Selberg transform discussed in §4.1 is essential. In §6 we prove the estimates and other facts that we need. In the proof of the main theorems we also use some estimates of the counting function (Lemmas 3.1 and 3.2) obtained without the use of spectral theory. A main role in the proof is played by an estimate of the spectral function given in Theorem 4.2, proved in §7.2.

2. Lie groups and discrete subgroups

Let \( G \) be the Lie group \( \text{PSL}_2(\mathbb{R})^d \) for some integer \( d \geq 1 \). The group \( G \) acts on the product \( \mathbb{S}^d \) of upper half planes by fractional linear transformations in each factor. We will use the letter \( j \) to index these factors. \( G \) leaves invariant the vector valued distance function

\[
d(z, w) = (d_j(z_j, w_j))_{j \in \{1, \ldots, d\}},
\]

where \( d_j \) is the hyperbolic distance in the \( j \)-th factor. By \( \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right] \) we denote the class in \( G \) represented by \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{R}) \).

We consider an irreducible lattice \( \Gamma \subset G \), as described in Definition 5.20 and Corollary 5.21 of [22]. So, for each of the genuine subproducts \( H \) of \( \text{PSL}_2(\mathbb{R})^d \) the projection \( \Gamma \to H \) has dense image. In particular \( \Gamma \backslash \mathbb{S}^d \) has finite volume, and the projection to each of the factors is injective on \( \Gamma \). (See also Corollary 5.23 in loc. cit.)

Hilbert modular groups \( \text{PSL}_2(O) \) and their subgroups of finite index, mentioned in the introduction, are examples. Cases for which \( \Gamma \backslash G \) is compact can be derived from quaternion algebras \( \mathcal{H} \) over a totally real number field \( F \) for which there is a non-empty set \( S \) of infinite places \( j \) for which the tensor product \( F_j \otimes_F \mathcal{H} \), with the completion \( F_j \), is a division algebra. Suppose that \( \#S = d > 0 \). Let \( \mathcal{H}_O \) be an order in \( \mathcal{H} \). Then the elements of reduced norm 1 in \( \mathcal{H}_O \) have as their image in \( \prod_{j \in S} \text{PSL}_2(F_j) \) a cocompact discrete subgroup satisfying the assumptions above.

In the case \( d = 1 \) most of the subgroups with finite index in \( \text{PSL}_2(\mathbb{Z}) \) are not the image of a congruence subgroup of \( \text{SL}_2(\mathbb{Z}) \). Moreover there are irreducible discrete subgroups of \( \text{PSL}_2(\mathbb{R}) \) that are not commensurable to \( \text{PSL}_2(\mathbb{Z}) \). For \( d \geq 2 \), Margulis has shown that all irreducible discrete subgroups of \( \text{PSL}_2(\mathbb{R})^d \) are arithmetic, i.e., commensurable to a Hilbert modular group or to a unit group of a quaternion algebra. (See Theorem 1.11 in Chap. IX of [18], or the discussion in §7 of [23].) Serre, [26], has shown that all subgroups of finite index in \( \text{SL}_2(O) \) are congruence subgroups. So all non-cocompact irreducible lattices contain a conjugate of a congruence subgroup as a subgroup with finite index.
3. A priori estimates

From here on we follow the usual practice of not working directly with the hyperbolic distance \( d \) on \( \mathcal{H} \), but with
\[
  u(z, z') = \frac{|z - z'|^2}{4yy'} = (\sinh \frac{1}{2}d(z, z'))^2,
\]
(9)
\[
d(z, z') = 2 \log \left( \sqrt{u(z, z')} + \sqrt{u(z, z') + 1} \right).
\]

For \( U, V \in [0, \infty)^d \) such that \( U_j < V_j \) for all \( j \), we consider the counting quantity
\[
  N(U, V; z) = \# \{ \gamma \in \Gamma : U_j < u((\gamma z)_j, z_j) < V_j \text{ for all } j \}.
\]
(10)

To relate this to the quantities \( N(z; T) \) and \( N_{E}(z; T) \) used in the introduction we will use that \( d \downarrow 0 \) corresponds to \( u \downarrow 0 \) in such a way that
\[
  u = \frac{d^2}{4} + O(d^4), \quad d = 2 \sqrt{u} + O(u^{3/2}),
\]
and that \( u \to \infty \) corresponds to \( d \to \infty \) in such a way that
\[
  u = \frac{1}{4} e^d + O(1), \quad d = \log u + \log 4 + O(u^{-1}).
\]
(12)

We will need a starting point for the estimation of \( N(U, V; z) \). We give two estimates for the counting function. The first is based on a simple volume argument. The second is important for the dependence of our results on the geometry of \( \Gamma \backslash \mathcal{H}^d \). As \( z \in \mathcal{H}^d \) approaches a cusp there are more and more \( \gamma \in \Gamma \) for which \( \gamma z \) is near \( z \).

**Lemma 3.1.** For \( z \in \mathcal{H}^d \) and \( U, V \in [0, \infty)^d \) such that \( U_j < V_j \) for all \( j \), we have
\[
  N(U, V; z) \ll_{\Gamma, z} \prod_j (V_j - U_j + 1).
\]

**Proof.** For \( w \in \mathcal{H}^d \) and \( \delta > 0 \) we put
\[
  B(w, \delta) = \{ v \in \mathcal{H}^d : \forall j \ u(w_j, v_j) < \delta \}.
\]

Let \( z \in \mathcal{H}^d \) be given. The subgroup \( \Gamma_z \) of \( \Gamma \) fixing \( z \) is finite. See, e.g., Remark 2.14 in [4]. By the discontinuity of the action there is \( \delta > 0 \) such that \( B(z, \delta) \cap B(\gamma z, \delta) = \emptyset \) for all \( \gamma \in \Gamma \setminus \Gamma_z \). The \( \Gamma \)-invariance of \( u \) implies that \( B(\gamma_1 z, \delta) \cap B(\gamma_2 z, \delta) = \emptyset \) for all \( \gamma_1, \gamma_2 \in \Gamma \) for which \( \gamma_1 z \neq \gamma_2 z \).

For \( P, Q \in [0, \infty)^d \) denote by \( A(P, Q) \) the multi-annulus
\[
  \{ v \in \mathcal{H}^d : \forall j \ P_j \leq u(v_j, z_j) < Q_j \}.
\]

The pairwise disjoint sets \( B(\gamma z, \delta) \) with \( \gamma z \in \Gamma_z \cap A(U, V) \) are contained in a slightly larger multi-annulus \( A(U(\delta), V(\delta)) \) with \( U(\delta)_j = U_j - O(1) \) and \( V(\delta)_j = V_j + O(1) \). See [5]. Thus we have
\[
  \#(\Gamma_z \cap A(U, V)) \leq \text{vol}(A(U(\delta), V(\delta)))/\text{vol}(B(z, \delta)) \ll_\delta \prod_j (V_j - U_j + 1).
\]
(The volume computation is easiest in a distance coordinate \( u = u(z, i) \) and an angular coordinate \( \phi \). Then \( du \) on \( \mathfrak{S} \) is given by \( 4 \, du \, d\phi \). See (1.17) in \[11\].) Since \( \mathcal{N}(U, V; z) = \# \Gamma_z \cdot \#(\Gamma z \cap A(U, V)) \), this proves the lemma.

Next we aim at an estimate of \( \mathcal{N}(z, w; 0, V) \) when all \( V_j \) are small. In this estimate, the dependence on \( z \) will be explicit. In order to do this, we take into account some facts concerning the geometry of the action of \( \Gamma \) on \( \mathfrak{S}^d \). (The approach is motivated by that in the proof of Corollary 2.12 in \[11\].)

The assumptions on \( \Gamma \) imply that we can find a fundamental domain \( \mathfrak{F} \) that is compact in the case of cocompact \( \Gamma \), and is contained in a union of Siegel domains otherwise:

\[
\mathfrak{F} \subset \bigcup_k g_k \mathfrak{D}_\kappa,
\]

\[
\mathfrak{D}(X_\kappa, Y_\kappa, V_\kappa) = \left\{ z : x_j \in [-X_\kappa, X_\kappa] \text{ for all } j, \ y_1 y_2 \cdots y_d \geq Y_\kappa, \ V_\kappa^{-1} \leq \frac{y_j}{y_{j+1}} \leq V_\kappa \text{ for } 1 \leq j \leq d - 1 \right\}
\]

where \( \kappa \) runs through a finite set of representatives \( \kappa = g_\kappa \infty \) of the \( \Gamma \)-classes of cusps, with \( g_\kappa \in G, X_\kappa, Y_\kappa > 0 \) and \( V_\kappa > 1 \). (See \[4\], Chap. I, §2.) Enlarging \( Y_\kappa \) decreases \( \mathfrak{D}(X_\kappa, Y_\kappa, V_\kappa) \). There exists \( A > 0 \) such that the \( g_\kappa \mathfrak{D}_\kappa(X_\kappa, A, V_\kappa) \) are disjoint. For each \( B \geq A \) there is a compact set \( C_B \) such that

\[
\mathfrak{F} \subset C_B \cup \bigcup_k g_k \mathfrak{D}(X_\kappa, B, V_\kappa).
\]

We fix a fundamental domain and a disjoint decomposition of it induced by (13), and define \( \Gamma \)-invariant functions \( \eta_1, \ldots, \eta_d \) on \( \mathfrak{D} \) determined by the requirement that for \( z \in \mathfrak{F} \):

\[
\eta_j(z) = \begin{cases} 
1 & \text{if } \Gamma \text{ is cocompact, or if } z \in C_A, \\
\text{Im}(g_\kappa^{-1} z_j) & \text{if } z \in g_\kappa \mathfrak{D}(X_\kappa, A, V_\kappa). 
\end{cases}
\]

The product of the \( \eta_j(z) \) measures how far up in a cusp sector the point \( z \) is situated. Note that the \( \eta_j \) may be discontinuous, but are bounded away from 0.

For \( T \in (0, \infty)^d \) we put

\[
n_j(T_j, z) = \max(1, \eta_j(z)/T_j), \quad n(T, z) = \prod_j n_j(T_j, z).
\]

We take

\[
n(z) = n(1, z) = \prod_j \max\left(1, \eta_j(z)\right),
\]

with \( 1 = (1, 1, \ldots, 1) \in \mathbb{R}^d \). The quantity \( n(z) \) occurs in the error terms in the final estimates in Theorems \[5.4\] and \[5.5\] describing the dependence on \( z \in \mathfrak{S}^d \).

**Lemma 3.2.** For all sufficiently small \( \delta_1 > 0, \ldots, \delta_d > 0 \), we have for \( 0 \in \mathbb{R}^d \) and \( \delta = (\delta_j)_j \):

\[
\mathcal{N}(z; 0, \delta) \ll_{\Gamma} n(\delta^{-1/2}, z),
\]
where \( \delta^{-1/2} = (\delta_j^{-1/2})_j \).

**Proof.** It suffices to consider \( z \) in a fundamental domain \( \mathcal{F} \) chosen as indicated above. As long as \( z \) stays in a compact region, the value of \( N(z; 0, \delta) \) is at most the maximal order of totally elliptic elements of \( \Gamma \) provided we take the \( \delta_j > 0 \) sufficiently small. In the previous proof we have seen that this maximal order is bounded for each \( \Gamma \). This proves the lemma for cocompact \( \Gamma \).

For other \( \Gamma \) we fix \( B > 2A \). If \( z \in C_B \), with \( C_B \) as in (15), we have \( N(z; 0, \delta) = \text{O}_1(1) \) for all sufficiently small \( \delta \). Suppose now that \( z \in g_\kappa \Xi(X_\kappa, B, V_\kappa) \). If the \( \delta_j \) are sufficiently small, then all \( \gamma \in \Gamma \) such that \( u((\gamma z)_j, z_j) \leq \delta_j \) lie in \( \Gamma \cap P_\kappa \), where \( P_\kappa \) is the parabolic subgroup fixing \( \kappa \).

For the remaining computations, we can assume that \( \kappa = \infty \) and \( g_\kappa = 1 \). Denote by \( N = \left[ \begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix} \right] \in G \) the unipotent of \( P_\infty = \left[ \begin{smallmatrix} t & x \\ 0 & 1 \end{smallmatrix} \right] \in G \). Elements in \( N_\Gamma = N \cap \Gamma \subset \Gamma_\infty = \Gamma \cap \Gamma \) have the form \( \left[ \begin{smallmatrix} 1 & \omega \\ 0 & 1 \end{smallmatrix} \right] \) with \( \omega \) running through a lattice \( \Lambda \subset \mathbb{R}^d \). The quotient \( \Gamma_\infty/N_\Gamma \) is represented by elements of the form \( \left[ \begin{smallmatrix} \varepsilon & \alpha/\varepsilon \\ 0 & 1/\varepsilon \end{smallmatrix} \right] \), where \( \alpha \mod \Lambda \) is determined by \( \varepsilon \), and where \( \varepsilon \) runs through a discrete subgroup of \( (\mathbb{R}^*)^d \) such that \( \varepsilon^d \Lambda = \Lambda \), and such that the \( (\log |\varepsilon_1|, \ldots, \log |\varepsilon_d|) \) run through a lattice in the hyperplane \( \sum_j x_j = 0 \) in \( \mathbb{R}^d \).

We need a bound for the number of \( \gamma \in \Gamma_\infty \) with

\[
u(\vartheta_j^2(z_j + \alpha_j), z_j) = \frac{(\vartheta_j^2 - 1)^2 y_j^2 + ((\vartheta_j^2 - 1)x_j + \alpha_j)^2}{4\vartheta_j^2 y_j^2} \leq \delta_j,
\]

for \( \delta_j \in (0, 1) \) for all \( j \). Hence the following quantities have to be non-negative:

\[
4\delta_j \vartheta_j^2 y_j^2 - (\vartheta_j^2 - 1)^2 y_j^2
\]

This implies

\[
\log(1 + 2\delta_j - 2 \sqrt{\delta_j + \delta_j^2}) \leq 2 \log |\varepsilon_j| \leq \log(1 + 2\delta_j + 2 \sqrt{\delta_j + \delta_j^2}).
\]

Since \( \log |\varepsilon| \) runs through a lattice in a hyperplane in \( \mathbb{R}^d \), this leaves \( \text{O}(1) \) possibilities for the choice of \( \varepsilon \). Taking the maximum of the quantity in (17), we find for all \( j \):

\[
|(\vartheta_j^2 - 1)x_j + \alpha_j| \leq 2 \sqrt{\delta_j + \delta_j^2} y_j.
\]

Since \( \alpha \) runs through a coset modulo the lattice \( \Lambda \), this gives at most

\[
\text{O}
\left(\prod_j (1 + \sqrt{\delta_j} y_j)\right) \ll \prod_j n_j(\delta_j^{-1/2}, z)
\]

possibilities for the choice of \( \omega \).

For \( z \in g_\kappa \Xi(X_\kappa, B, V_\kappa) \) replace \( y_j \) by \( \text{Im } g_\kappa^{-1}z_j \). Together with the bound \( \text{O}(1) \) for \( z \in C_B \), we get the statement in the lemma. \( \Box \)
4. The Selberg transform and spectral estimates

If $k_1, \ldots, k_d$ are bounded functions on $[0, \infty)$ with compact support, then the sum
\begin{equation}
K(z, w) = \sum_{\gamma \in \Gamma} \prod_j k_j(u((\gamma z)_j), w_j)
\end{equation}
converges absolutely, and defines a function on $(\Gamma \backslash \mathbb{H}^d) \times (\Gamma \backslash \mathbb{H}^d)$. If we take each $k_j$ equal to the characteristic function of the interval $[U_j, V_j)$, then
\[ K(z, z) = N(U, V; z) . \]

It will turn out preferable to use smooth $k_j$, so we will take for the $k_j$ approximations of those characteristic functions. In this case $K(z, z)$ is only an approximation of $N(U, V; z)$, but its spectral expansion as an element of $L^2(\Gamma \backslash \mathbb{H}^d)$ converges pointwise, and we can write
\begin{equation}
K(z, z) = K_{\text{expl}}(z, z) + K'(z, z)
\end{equation}
for each $z \in \mathbb{H}^d$, where $K_{\text{expl}}(z, z)$ is the contribution to the spectral expansion of a finite number of $\psi_\ell$ (among them $\psi_0$), and where $K'(z, z)$ is the remainder. The main idea is that $K_{\text{expl}}(z, z)$ will yield the explicit terms in the asymptotic expansion of $N(U, V; z)$, and that estimates of the difference $K(z, z) - N(U, V; z)$ and of $K'(z, z)$ will contribute to the error term.

To carry this out, we have to see how the spectral expansion depends on the functions $k_j$. That leads us to a study of the Selberg transform (§4.1). We also have to know what is the size of the contributions of various parts of the spectrum (§4.3). In this section we state the results that we need, and refer for most proofs to §6 and §7.

4.1. The Selberg transform. We can do most of the work on $\mathbb{H}^d$ factor by factor. So we work first on $\mathbb{H}$.

Functions $k$ on $[0, \infty)$ yield kernel operators on functions $f$ on $\mathbb{H}$:
\begin{equation}
L_k f(z) = \int_{\mathbb{H}} k(u(z, w)) f(w) d\mu(w),
\end{equation}
where $d\mu(w) = \frac{d \text{Re} u \, d \text{Im} u}{(1 + u)^2}$ is the invariant measure associated to the Riemannian metric on $\mathbb{H}$. We take $k \in C_0^\infty(0, \infty)$. (This implies in particular that all derivatives are well defined and continuous at $u = 0$.) We assume that the function $f$ is continuous. That suffices for the convergence in (19).

The Selberg transform associates to the function $k \in C_0^\infty(0, \infty)$ an even holomorphic function $h$ on $\mathbb{C}$, given by the following three steps:

\begin{align*}
g(p) &= \int_{-p}^\infty k(u) \frac{du}{\sqrt{u - p}}, \text{ for } p \geq 0, \\
g(r) &= 2q \left( \sinh(\frac{r}{2}) \right)^2, \text{ for } r \in \mathbb{R}, \\
h(\tau) &= \int_{-\infty}^\infty e^{\pi \tau} g(r) dr, \text{ for } \tau \in \mathbb{C}.
\end{align*}
See, e.g., [11], p. 33, but note that Iwaniec uses $ir$ as the variable in $h$. (See also [23].) The relation can be described in one step:

\begin{equation}
\label{eq:21}
h(\tau) = \int_{\mathbb{H}} k(u(z,i)) y^{1/2 - \tau} d\mu(z),
\end{equation}

which can be made more explicit by use of a hypergeometric function

\begin{equation}
\label{eq:22}
h(\tau) = 4\pi \int_0^\infty k(u) F_1 \left( \frac{1}{2} + \tau, \frac{1}{2} - \tau; 1; u \right) du.
\end{equation}

See (1.62') and the proof of Theorem 1.16 in [11]. In fact, $h$ is the spherical transform of $k$. See, e.g., [14], Chap. V, §4. We have in particular

\begin{equation}
\label{eq:23}
h(\frac{1}{2}) = 4\pi \int_0^\infty k(u) du.
\end{equation}

The Selberg transform has the important property that if $\Delta f = (\frac{1}{4} - \tau^2) f$, then

\begin{equation}
\label{eq:24}
L_k f = h(t) f
\end{equation}

(Theorem 1.16 in [11]).

Next we consider $h_1, \ldots, h_d \in C_\infty(0, \infty)$, and form the kernel function

\begin{equation}
\label{eq:25}
k(z, w) = \prod_j k_j(u(z_j, w_j))
\end{equation}

on $\mathbb{S}^d \times \mathbb{S}^d$. Thus we have the operator

\begin{equation}
\label{eq:26}
L_k f(z) = \int_{\mathbb{S}^d} k(z, w) f(w) d\mu(w),
\end{equation}

with $d\mu = \prod_j d\mu_j$ the product of the invariant measures. This converges absolutely if $f$ is continuous on $\mathbb{S}^d$. If moreover we have $\Delta_j f = (\frac{1}{4} - \tau_j^2) f$ for the local Laplace operators $\Delta_j = -y_j^2 \frac{\partial^2}{\partial x_j^2} - y_j^2 \frac{\partial^2}{\partial y_j^2}$, then

\begin{equation}
\label{eq:27}
\Delta_j (L_k f) = h_j(\tau_j) f \quad \text{for each } j,
\end{equation}

where $h_j$ is the Selberg transform of $h_j$.

By Lemma [3.1] the sum

\begin{equation}
\label{eq:28}
K(z, w) := \sum_{\gamma \in \Gamma} k(\gamma z, w)
\end{equation}

converges absolutely, and defines a function in $C^\infty((\Gamma \setminus \mathbb{S}^d) \times (\Gamma \setminus \mathbb{S}^d))$ that satisfies

\begin{equation}
\label{eq:29}
K(z, w) = O_\varepsilon(1).
\end{equation}

The boundedness of $K(z, w)$ is uniform for $z$ varying in compact sets. If $f$ is square integrable on $\Gamma \setminus \mathbb{S}^d$ for the invariant measure $d\mu$ then

\begin{equation}
\label{eq:30}
K_k f(z) = \int_{\Gamma \setminus \mathbb{S}^d} K(z, w) f(w) d\mu(w)
\end{equation}

converges absolutely, and defines an operator

$K_k : L^2(\Gamma \setminus \mathbb{S}^d) \rightarrow C^\infty(\Gamma \setminus \mathbb{S}^d)$.
where \( f \mapsto \mathcal{K}_s f(z) \) is continuous on \( L^2(\Gamma \backslash \mathbb{H}^d) \) for each \( z \in \mathbb{H}^d \).

### 4.2. Spectral decomposition.

A consequence of the irreducibility assumption for the lattice \( \Gamma \) is that the spectral theory \( L^2(\Gamma \backslash \mathbb{H}^d) \) is well known. The Hilbert space \( L^2(\Gamma \backslash \mathbb{H}^d) = L^2(\Gamma \backslash \mathbb{H}^d, \mu_\Gamma) \) has a spectral decomposition in terms of automorphic forms. In the cocompact case, each element can be written in \( L^2 \)-sense as

\[
\sum_{\ell \geq 0} a_\ell \psi_\ell, \tag{31}
\]

where the \( \psi_\ell \) form a complete orthonormal system in \( L^2(\Gamma \backslash \mathbb{H}^d) \) of simultaneous eigenfunctions of the \( \Delta_j \):

\[
\Delta_j \psi_\ell = (\frac{j}{4} - \tau_{\ell,j}^2) \psi_\ell, \tag{32}
\]

with \( \tau_{\ell,j} \in i[0, \infty) \cup (0, \frac{1}{2}] \). Among these eigenfunctions we choose \( \psi_0 = \frac{1}{\sqrt{\text{vol}(\Gamma \backslash \mathbb{H}^d)}} \), a constant function; hence \( \tau_{0,j} = \frac{1}{2} \) for all \( j \). For each \( \ell \geq 1 \) we know that \( \tau_{\ell,j} \in i[0, \infty) \cup (0, \frac{1}{2}) \). The \( a_\ell \) form a sequence in the Hilbert space \( \ell^2 \).

If \( \Gamma \) has cusps, there is a subspace \( L^2_{\text{discr}}(\Gamma \backslash \mathbb{H}^d) \) with the same structure as in the cocompact case. It always contains the constant function \( \psi_0 \). If \( d = 1 \) there may be finitely many \( \ell \geq 1 \) for which \( \psi_\ell \) is a residue of an Eisenstein series and at most countably many \( \psi_\ell \) that are cusp forms. The orthogonal complement \( L^2_{\text{cont}}(\Gamma \backslash \mathbb{H}^d) \) is a sum of direct integrals. Elements of this space can be written in \( L^2 \)-sense in the form

\[
\sum_\kappa 2c_\kappa \sum_{\mu \in L_\kappa} \int_0^\infty b_{\mu,\kappa}(t) E(\kappa; s, it) dt.
\]

Here \( \kappa \) runs over representatives of the finitely many cuspidal \( \Gamma \)-orbits, \( c_\kappa \) are positive constants, \( L_\kappa \) is a lattice in the hyperplane \( \sum_j x_j = 0 \) in \( \mathbb{R}^d \), and \( E(\kappa; s, it) \) is an Eisenstein series, satisfying

\[
\Delta_j E(\kappa; s, it) = \left( \frac{j}{4} + (s + i\mu j)^2 \right) E(\kappa; s, it)
\]

for each \( j \). For \( f \in L^2(\Gamma \backslash \mathbb{H}^d) \) we have \( a_\ell = (f, \psi_\ell) \). If \( f \) is bounded and sufficiently smooth, then \( b_{\mu,\kappa} \) is given by integration against \( E(\kappa; it, i\mu) \).

The quantity \( \hat{\tau}(\Gamma) = \sup_{\ell \geq 1, s \in \mathbb{R}^d} \text{Re } \tau_{\ell,j} \) in equation (3) is related to the quantity \( \tau(\Gamma \backslash G) \in [2, \infty) \) in [12], with \( G = \text{PSL}_2(\mathbb{R})^d \), by

\[
p(\Gamma \backslash G) \geq \frac{1}{\frac{1}{2} - \hat{\tau}} \quad \text{or equivalently} \quad \tau \leq \frac{1}{2} - \frac{1}{p(\Gamma \backslash G)}.
\]

So \( \hat{\tau} = \frac{1}{2} \) would imply \( p(\Gamma \backslash G) = \infty \) (no strong spectral gap), and \( p(\Gamma \backslash G) = 2 \) implies \( \hat{\tau} = 0 \) (no exceptional eigenvalues at all). We have to be careful to use inequalities in (34). Kelmer and Sarnak take all irreducible representations of \( G = \text{PSL}_2(\mathbb{R})^d \) in \( L^2_{\text{discr}}(\Gamma \backslash G) \) into account. Such a representation is visible in \( L^2_{\text{discr}}(\Gamma \backslash \mathbb{H}^d) \) only if all \( d \) components of the representation have a non-trivial \( \text{PSO}(2) \)-invariant vector. We recall that in the congruence case (including all non-cocompact \( \Gamma \) if \( d \geq 2 \)) we have \( \hat{\tau}(\Gamma) \leq \frac{1}{2} \). For all cocompact \( \Gamma \) we have \( \hat{\tau}(\Gamma) < \frac{1}{2} \).
We return to the kernel function $K$ in (28). By (27) and the invariance of the kernel $k(z, w)$, we have for fixed $z \in \mathcal{H}^d$:

$$\int_{\Gamma \setminus \mathcal{H}^d} K(z, w) \overline{\psi_\ell(w)} \, d\mu(w) = \int_{\mathcal{H}^d} k(z, w) \overline{\psi_\ell(w)} \, d\mu(w) = h(\tau_\ell) \psi_\ell(z),$$

with

$$h(\tau) = \prod_j h_j(\tau_j).$$

Therefore the scalar product of $K(z, \cdot)$ with $\psi_\ell$ makes sense. If $\Gamma$ is not cocompact, we find in a similar way that the coefficients $b_{k, \mu}(t)$ in (33) are given by

$$\prod_j h_j(it + i\mu_j) \overline{E(\kappa; it, i\mu; z)} E(\kappa; it, i\mu; w) \, dt.$$ 

In the cocompact case, we understand the sum over $\kappa$ to be absent.

This spectral expansion converges in the Hilbert space $L^2(\Gamma \setminus \mathcal{H}^d)$. To use it to investigate the counting function $N(U, V; z)$ in the way indicated in the introduction of this section, we need it to make sense pointwise.

**Theorem 4.1.** Let $f \in C^2(\Gamma \setminus \mathcal{H}^d)$ be bounded, and suppose that the derivatives $\Delta^a_1 \Delta^a_2 \cdots \Delta^a_d f$ are bounded for all choices of $a_j \in \{0, 1, 2\}$. Then the spectral expansion of $f$ converges absolutely and uniformly on compacta.

In particular, if the $k_j$ in (25) are in $C^\infty_c[0, \infty)$ for all $j$, then the expansion

$$K(z, w) = \sum_\ell h(\tau_\ell) \overline{\psi_\ell(z)} \psi_\ell(w)$$

$$+ \sum_\kappa 2c_\kappa \sum_{\mu \in L_\kappa} \int_0^\infty h(it + i\mu) \overline{E(\kappa; it, i\mu; z)} E(\kappa; it, i\mu; w) \, dt.$$

converges absolutely for each choice $z, w \in \mathcal{H}^d$.

This result is more or less standard. We will sketch a proof in §7.1.

**4.3. Spectral measure.** As indicated in the introduction of this section, we will need to know how the various parts of the spectral set

$$i \mathbb{R} \cup \left(0, \frac{1}{2}\right)^d$$

contribute to the spectral expansion of $K(z, \cdot)$. We write $i \mathbb{R}$ instead of $i[0, \infty)$, since in the term in the spectral expansion (37) corresponding to the continuous spectrum there are quantities $i(t + \mu_j)$, which in some cases are in $i(-\infty, 0)$.
For \( X \in [1, \infty)^d \) we put

\[
Y(X) = \prod_j \left( (0, \frac{1}{2}] \cup i(-X_j, X_j) \right),
\]

and define

\[
S(X; z, w) = \sum_{\ell, t \in Y(X)} \psi_\ell(z)\psi_\ell(w)
+ \sum_\kappa \sum_{\mu \in L_\kappa} \int_{t \geq 0, (t + \mu_j) \in Y(X)} E(\kappa; it, i\mu; z) E(\kappa; it, i\mu; w) \, dt.
\]

This is a smooth function on \((\Gamma \backslash \mathcal{S}^d) \times (\Gamma \backslash \mathcal{S}^d)\). (The sum over \( \ell \) is finite. The region of integration is finite for all \((\kappa, \mu)\), and empty for almost all \((\kappa, \mu)\).)

The following estimate of the spectral function \( S(X; z, z) \) will play an important role in \( \S 5 \) in the proof of our main results:

**Theorem 4.2.** For \( X \in [1, \infty)^d \) and \( z \in \mathcal{S}^d \):

\[
S(X; z, z) \ll \Gamma X^{2} X_{2}^{2} \cdots X_{d}^{2} n(X, z).
\]

The quantity \( n(X, z) \) has been defined in \( (16) \). It makes explicit the dependence of the spectral measure on the point \( z \in \mathcal{S} \). This constitutes a difference with \( [2] \), where we used a result of Hörmander to estimate the spectral measure uniformly for \( z \) in compact sets, obtaining an asymptotic formula for the lattice point counting function on symmetric spaces of rank one that was uniform for \( z \) varying in compact sets only.

We prove Theorem \( 4.2 \) in \( \S 7.2 \). The proof uses the a priori estimate of the lattice point counting function in Lemma \( 3.2 \).

5. **Proof of the lattice points theorems**

This section is the heart of this paper, where we carry out the plan sketched in the introduction of \( \S 4 \). We consider the asymptotic behavior of the quantity \( N(U, V; z) \) defined in \( (10) \), with \( U, V \in [0, \infty)^d \), \( U_j < V_j \) for all \( j \). In Theorem \( 5.4 \) which is slightly more general than Theorem \( 1.2 \) in the introduction, all \( V_j \) tend to \( \infty \), whereas in Theorem \( 5.5 \) generalization of Theorem \( 1.1 \) some intervals \([U_j, V_j)\) stay fixed. For most of the section we handle the proofs simultaneously.

5.1. **The main parameters.** We partition the set \( \{1, \ldots, d\} \) into two disjoint subsets \( Q \) and \( E \), with the requirement that \( Q \) is non-empty.

For each \( j \in Q \) we let \( V_j \geq 1 \) tend to \( \infty \), and choose \( U_j = 0 \) or \( U_j = \frac{1}{2} V_j \). (This choice may depend on the place \( j \in Q \).) The simplest is to take all \( V_j \) with \( j \in Q \) equal to each other. We wish also to include the case that \( V_j = T^{a_j} \) with positive exponent \( a_j \), where \( T \) tends to infinity. However, we do not let the \( V_j \) run apart too much, by fixing a parameter \( \hat{q} \) satisfying

\[
\hat{q} \geq \#Q, \quad \text{and require } \min_{j \in Q} V_j^{\hat{q}} = \prod_{j \in Q} V_j.
\]
Thus, if all $V_j$ with $j \in Q$ are equal to each other, then $\hat{q} = \#Q$. For each $j \in E$ we keep the non-empty interval $[U_j, V_j)$ fixed.

These are the parameters used in the Theorems 5.4 and 5.5. They constitute the "main parameters" in Table 1.

### Table 1. Overview of the parameters in §5

| **Main parameters** |  |
|---------------------|-----------------|
| $Q$ | a non-empty subset of $\{1, \ldots, d\}$ |
| $E$ | the complement $\{1, \ldots, d\} \setminus Q$ |
| $V_j$ $j \in Q$ | $V_j \geq 2$, $V_j \to \infty$, |
| $V_j$ $j \in E$ | $V_j > 0$ fixed |
| $U_j$ $j \in Q$ | $Q_j = 0$ (fixed), or $Q_j = \frac{1}{2}V_j \to \infty$ |
| $U_j$ $j \in E$ | $U_j \in [0, V_j)$ fixed |
| $\hat{q}$ | $\hat{q} \geq \#Q$ fixed, $\hat{q} \log V_{\min} = \sum_{j \in Q} \log V_j$ |
| $V_{\min}$ | $V_{\min} = \min_{j \in Q} V_j \geq 2$ |

| **Auxiliary parameters** |  |
|--------------------------|-----------------|
| $\vartheta$ | $\vartheta \in (0, 1)$ |
| $Y_E$ | $Y_E \in (0, 1]$, $Y_E \downarrow 0$ |
| $Y_j$ $j \in Q$ | $Y_j = V_j^\vartheta$, $Y_j \geq 1$, $Y_j \to \infty$ |
| $Y_j$ $j \in E$ | $Y_j = Y_E \downarrow 0$ |
| $c$ | $0 < c < \frac{1}{2}$, if $\hat{c} > 0$ then $c < \hat{c}$ |

#### 5.2. Test functions and auxiliary parameters.

In the introduction of §4 we have sketched our plan to prove the main results in §5.7 and 5.8. We take the approximations $h_j \in C_c^\infty(0, \infty)$ of the characteristic functions $[U_j, V_j)$ in the following way:

$$0 \leq k \leq 1, \quad k_j^{(l)} = O_l(Y_j^l) \text{ for } l \in \mathbb{N},$$

$$k_j = 1 \text{ on } \begin{cases} [U_j + Y_j, V_j - Y_j] & \text{if } U_j > 0, \\ [0, V_j - Y_j] & \text{if } U_j = 0, \end{cases}$$

$$k_j = 0 \text{ on } \begin{cases} [0, U_j - Y_j] \cup [V_j + Y_j, \infty) & \text{if } U_j > 0, \\ [V_j + Y_j, \infty) & \text{if } U_j = 0. \end{cases}$$

![Graph](image-url)
The parameters $Y_j$ control how quickly $k_j$ changes from 0 to 1 and back. We require

$$Y_j \leq \frac{V_j - U_j}{2} \quad \text{if } U_j > 0, \quad Y_j \leq \frac{1}{2} V_j \quad \text{if } U_j = 0.$$  

If one fixes a smooth function $\omega \in C^\infty(\mathbb{R})$ that increases from 0 to 1 on an interval contained in $(0, 1)$, then $h_j(u) = \omega((u - U_j)/Y_j)$ on $[U_j, U_j + Y_j]$ satisfies on this interval the condition on the derivatives, and goes from 0 to 1. On $[V_j - Y_j, V_j]$ we proceed similarly. This gives a choice such that $k_j = 0$ outside $[U_j, V_j]$. We can equally well arrange that $k_j = 1$ on $[U_j, V_j]$.

The $Y_j$ are new parameters. They play a role in the proof, not in the theorems. At the end of the proof we try to choose them optimally. To avoid having to keep track of too many auxiliary parameters, we assume from the start that $Y_j = V_j^\theta$ for $j \in Q$, with $\theta \in (0, 1)$ a single auxiliary parameter. So the $Y_j$ are large parameters for $j \in Q$.

We let the $Y_j$ with $j \in E$ tend to zero. It seems that we do not lose much if we take all these parameters equal to a quantity $Y_E$ tending to 0, for which we require that $2Y_E \leq V_j - U_j$ for all $j \in E$, and $2Y_E \leq U_j$ for all $j \in E$ with $U_j > 0$.

The estimates of Selberg transforms in §4 depend on a small positive parameter $c \in (0, 1)$. In Lemma 6.3(c) there is also a small positive parameter $\delta$. We choose $\delta$ such that $\delta < U_j$ for all $j \in E$ with $U_j > 0$. The dependence on $\delta$ of the implicit constants in the estimates is absorbed in the dependence of the choice of the intervals $[U_j, V_j]$ with $j \in E$. The positive constant $c$ will turn up in exponents in some estimates. We take $c < \hat{\tau}$ if $\hat{\tau} > 0$. (We recall that $\hat{\tau}$ measures the spectral gap, which is maximal if $\hat{\tau} = 0$.)

### 5.3. Terms in the asymptotic formula

With the test functions in (40) we form the $\Gamma$-invariant kernel

$$K(z, w) = \sum_{y \in \Gamma} k(u(z, w)),$$

as indicated in the introduction of §4. The diagonal value $K(z, z)$ gives an approximation of $N(U, V; z)$. By $h(\tau) = \prod_j h_j(\tau_j)$ we denote the product of the Selberg transforms of the $k_j$.

From the absolutely convergent spectral decomposition in (37) we single out

$$K^{\text{expl}}(z, z) := h(\tau_0) \frac{1}{\text{vol}(\Gamma \backslash \mathbb{H}^d)} + \sum_{\ell \geq 1, \forall j} h(\tau_\ell) |\psi_\ell(z)|^2.$$

Here we have made the choice to take not only the contribution of the constant functions, but also all of the terms corresponding to totally exceptional eigenvalues. This term $K^{\text{expl}}(z, z)$ will lead to the explicit term in our asymptotic expansions.
As the explicit term in the final results we use

\[
\mathcal{E}(U, V; z) = \sum_{\ell \geq 0, \, \tau_{\ell,j} \in (0, \frac{1}{2}]} |\psi_{\ell}(z)|^2 \prod_{j \in \mathcal{E}} \eta(U_j, V_j; \tau_{\ell,j}) \\
\cdot \prod_{j \in Q} \frac{\sqrt{\pi} 2^{1+2\tau_{\ell,j}} \Gamma(\tau_{\ell,j})}{\Gamma(\frac{3}{2} + \tau_{\ell,j})} (V_j^{\frac{1}{2} + \tau_{\ell,j}} - U_j^{\frac{1}{2} + \tau_{\ell,j}}),
\]

where

\[
\eta(a, b; \tau) = \int_{z \in \mathcal{A}, \, a < d(z) < b} y^{\frac{1}{2} + \tau} \, d\mu(z)
\]
is the Selberg transform of the characteristic function of \([a, b]\). So we will need to estimate the difference between \(\mathcal{E}(U, V; z)\) and \(K_{\text{expl}}(z, z)\). We note that there might not exist totally exceptional eigenvalues for the group \(\Gamma\). In that case \(\mathcal{E}(U, V; z)\) is equal to the term for \(\ell = 0\):

\[
\frac{1}{\text{vol}(\Gamma \backslash \mathcal{A}^d)} (4\pi)^d \prod_{j} (V_j - U_j).
\]

(See (88) in Lemma 6.1)

The remaining part of the spectral decomposition is split up according to subsets \(Z(n)\) of the space of spectral parameters. For \(n \in \mathbb{N}^d\) we put

\[
Z(n) = \left\{ \tau \in (i[0, \infty) \cup (0, \frac{1}{2}))^d : \right. \\
\tau_j \in i[-n_j, 1-n_j] \cup i[n_j - 1, n_j) \text{ if } n_j > 1, \\
\tau_j \in (0, \frac{1}{2}) \cup i(-1, 1) \text{ if } n_j = 1 \left. \right\}.
\]

For \(n \in \mathbb{N}^d, \, n \neq 1 = (1, 1, \ldots, 1)\) we define

\[
K_n(z, z) = \sum_{\ell \geq 1, \, \tau_{\ell,j} \in Z(n)} h(\tau_{\ell}) |\psi_{\ell}(z)|^2 \\
+ \sum_{k} 2c_k \sum_{\mu \in L_d} \int_{i[0, \infty) \cup i(\infty, 0) \subseteq Z(n)} h(it + i\mu) |E(\kappa; it, i\mu; z)|^2 \, dt.
\]

For \(n = 1\) we modify the term from the discrete spectrum by requiring not only \(\tau_{\ell} \in Z(1)\), but also \(\tau_{\ell,j} \in i[0, \infty)\) for some \(j\). (The totally exceptional terms go into \(\mathcal{E}(U, V; z)\).) With these definitions, we have

\[
K(z, z) - K_{\text{expl}}(z, z) = \sum_{n \in \mathbb{N}^d} K_n(z, z).
\]

It will be hard work to estimate this sum.

Finally, we also will have to estimate the difference between \(K(z, z)\) and the counting quantity \(N(U, V; z)\). Table 2 gives an overview of the estimates to be carried out.
The explicit term. The explicit term $E(U, V; z)$ in (44) is a finite sum. Each of its terms contains as a factor, for $j \in E$, the Selberg transform $\eta(U_j, V_j; \tau_{\ell,j})$ of the characteristic function of $[U_j, V_j]$, and, for $j \in Q$, the approximation of this Selberg transform given in (90) in Lemma 6.1. That approximation is uniform on intervals $(50)$ that approximation is uniform on intervals $(50)$ for each $c > 0$. Here we want to apply it with $\tau$ equal to the coordinates $\tau_{\ell,j}$ of the totally exceptional eigenvalues. These coordinates form a finite subset of $(0, \frac{1}{2})$. We take $c \in (0, \frac{1}{2})$ smaller than the minimum of these finitely many $\tau_{\ell,j}$, and then apply (90) uniformly. Thus, this parameter $c$ depends on the group $\Gamma$, and will lead to an implicit dependence of the error terms on $\Gamma$.

**Lemma 5.1.** The explicit term satisfies

$$E(U, V; z) - K_{\text{expl}}(z, z) \ll_{\Gamma, E} n(z) \left( V_{\min}^{\theta-1} + Y_E \right) \prod_{j \in Q} V_j. $$

The factor $n(z)$ has been defined in (16). See Table 7 for $Y_E$, $V_{\min}$ and $\theta$. The $E$ in $\ll_{\Gamma, E}$ indicates an implicit dependence on all $U_j$ and $V_j$ with $j \in E$.

**Proof.** For each of the finitely many $\tau = \tau_{\ell}$ occurring in $E(U, V; z)$ and $K_{\text{expl}}(z, z)$, we have by Lemma 6.3 a):

$$\prod_j \eta(U_j, V_j; \tau) - \prod_j h_j(\tau) \ll \sum_j (\eta(U_j, V_j; \tau) - h_j(\tau)) \sum_{l \neq j} \eta(U_j, V_j; \tau).$$

We apply Lemmas 6.1 and 6.2 with $c \in (0, \frac{1}{2})$ chosen so that $c < \tau_{\ell,j}$ for all $j$ for all $\ell$ occurring in the explicit term. We find, uniformly for $c \leq \tau_j \leq \frac{1}{2}$:

$$j \in Q : \eta(U_j, V_j; \tau_j) - h_j(\tau_j) \ll_c V_j^{T_{j}^{\tau_j - \frac{1}{2}}} = V_j^{T_{j}^{\tau_j + \theta - \frac{1}{2}}} \quad \text{Lemma 6.2 c),}$$

$$\eta(U_j, V_j; \tau_j) \ll E Y_j \quad \text{Lemma 6.2 c),}$$

Note that we leave implicit the influence of the fixed quantities $U_j$ and $V_j$, but keep their difference $Y_j$ explicit. The difference in (51) is estimated by the following quantity, uniformly in the $\tau = \tau_{\ell}$ under consideration:

$$\ll_{\Gamma, E} \sum_{j \in Q} V_j^{T_{j}^{\tau_j + \theta - \frac{1}{2}}} \cdot \prod_{l \in Q \setminus j} V_l^{T_{l}^{\tau_l + \frac{1}{2}}} \cdot O(1) + \sum_{j \in E} Y_j \cdot \prod_{l \in Q} V_l^{T_{l}^{\tau_l + \frac{1}{2}}} \cdot O(1)$$

**Table 2.** Overview of the error term estimates.
\[
\ll \left( \sum_{j \in Q} V_j^{\theta^2} + Y_E \right) \prod_{i \in Q} V_i^{\tau + \frac{1}{2}} \ll (V_{\min}^{\theta^2} + Y_E) \prod_{j \in Q} V_j,
\]

where we have used \( \tau_j \leq \frac{1}{2} \) in the last step.

We still have to estimate the finitely many \( \psi_\ell(z) \). If \( \psi_\ell \) is a cusp form or if \( \ell = 0 \), then \( |\psi_\ell(z)| = O(1) \). If \( \psi_\ell \), with \( \ell \geq 1 \), arises from a residue of an Eisenstein series, it satisfies \( \psi_\ell(g_\kappa z) = O(N(y)^{\frac{1}{2} - \rho_\ell}) \) as \( N(y) = \prod_j y_j \to \infty \) for all cusps \( \kappa \), for some \( \rho_\ell \in (0, \frac{1}{2}) \). Hence \( |\psi_\ell(z)| \ll n(z)^{1/2} \). Since the explicit term and \( K_\kappa(z, z) \) run over finitely many \( \ell \), this estimate can be used uniformly, thus giving the lemma. Note that here arises another implicit dependence of the error terms on the group \( \Gamma \). \( \square \)

5.5. Sum over the spectrum. We turn to the estimation of \( \sum_{n \in \mathbb{N}} K_n(z, z) \), as defined in \([48]\), with the given modification for \( n = 1 \). We will use that

\[
K_n(z, z) \leq M(n) S_n,
\]

where, with \( Z(n) \) as defined in \([47]\),

\[
M(n) = \sup_{\tau \in Z(n)} |h(\tau)|,
\]

\[
S_n = \sum_{\ell \geq 1, \tau \in Z(n)} |\psi_\ell(z)|^2
\]

\[
+ \sum_k 2c_\kappa \sum_{\mu \in \mathcal{E}} \int_{E(\kappa; i\mu; z)} |E(\kappa; i\mu; z)|^2 dt.
\]

Lemma 5.2. For each sufficiently small \( c \in (0, \frac{1}{2}) \) the quantity \( M(n) \) has for each \( n \in \mathbb{N}^d \) and each \( l \in \mathbb{N}^d \) an estimate

\[
M(n) \ll_{E, l, c} \prod_j f_{l, j}(n_j),
\]

where for all \( l \in \mathbb{N}^d \)

\[
f_{l, j}(1) = \begin{cases} V_j^{l+\frac{1}{2}} & \text{if } j \in Q \text{ and } \tau > 0, \\
V_j^{l+\frac{1}{2}} & \text{if } j \in Q \text{ and } \tau = 0, \\
1 & \text{if } j \in E,
\end{cases}
\]

\[
f_{l, j}(n) = \begin{cases} n^{-l+\frac{1}{2}} Y_j^{l-1} V_j^{l+\frac{1}{2}} & \text{if } j \in Q \text{ and } n \geq 2, \\
Y_j^{l-1} & \text{if } j \in E \text{ and } n \geq 2.
\end{cases}
\]

If \( E = \emptyset \) and \( \tau > 0 \), we have the slightly better estimate

\[
M(1) \ll_c V_{\min}^{\tau-\frac{1}{2}} \prod_j f_{l, j}(1).
\]

We recall that \( \tau \in [0, \frac{1}{2}] \) is the supremum of the real parts \( \Re \tau_{\ell, j} \), \( \ell \geq 1 \), of the spectral parameters. It measures the spectral gap.
Proof. If $\hat{\tau} > 0$, we take $c \in (0, \hat{\tau})$. We use the estimates in (56) of the Selberg transforms $h_j$ of the $k_j$ with this value.

If $n = 1$ we have to consider $\tau_j \in (0, \hat{\tau}) \cup (i[-1, 1])$. For $\tau \in [c, \frac{1}{2})$, which can occur only if $\hat{\tau} > 0$, we use Lemma 6.2 c) and (50) in Lemma 6.1 to get

$$h_j(\tau_j) \ll_c V_j^{\tau_j + \frac{1}{2}} + Y_j m,$$

with

$$m = \max(V_j^{\tau_j - \frac{1}{2}}, U_j^{\frac{1}{2}}) \text{ if } U_j > 0, \quad m = \max(V_j^{\tau_j}, V_j^{\tau_j - \frac{1}{2}}) \text{ if } U_j = 0.$$

If $j \in Q$ we get a bound by $O(V_j^{\tau_j + \frac{1}{2}}) = O(V_j^{\tau_j + \frac{1}{2}})$. For $j \in E$ the dependence on $U_j$ and $V_j$ is left implicit, so we can use the bound 1. If $|\tau_j| \leq c$, Lemma 6.3 b) gives the bound $O_c(V_j^{\tau_j + \frac{1}{2}})$ for $j \in Q$, and $O_{c,E}$ if $j \in E$. For $\tau_j \in i\mathbb{R}, c \leq |\tau_j| \leq 1$, we use Lemma 6.3 c) with $l = 1$. For $j \in E$ we take care to choose the $\delta$ in Lemma 6.3 such that $U_j \geq \delta$ if $U_j > 0$. We use that $k' \ll Y^{-1}$ to find $O(V_j^{\tau_j})$ if $j \in Q$ and $O(1)$ if $j \in E$.

If $n \geq 2$ we use Lemma 6.3 c) and the condition $k_j^{(i)} = O(Y^{-1})$ to obtain the bounds by $f_{j}(n)$.

In the case of $M(1)$ we have the additional information that $\tau_j \in i[-1, 1]$ for at least one $j$. If $E = \emptyset$ this leads to the estimate in (56).

The problem with $S_n$ in (54) is that we do not have a direct estimate for it. All we have is Theorem 4.2 which gives

$$(57) \quad \sum_{n \in \mathbb{N}^d, V_{j_m} \leq n_j} S_m \ll \Gamma n(n, z) \prod \frac{n_j}{j} \prod \max(n_j^2, \eta_j(z) n_j).$$

(See (15) for $n(n, z, \ldots)$ So we need to carry out a $d$-dimensional partial summation.

**Lemma 5.3.** Let $z \in \mathbb{D}$. For $c \in (0, \frac{1}{2})$ as in the previous lemma, we have if $\hat{\tau} > 0$

$$(58) \quad \sum_{n \in \mathbb{N}^d} K_n(z, z) \ll_{\Gamma, E, c} n(z) \cdot \begin{cases} Y_E^{\frac{1}{2}+\#E} \prod_{j \in Q} V_j^{1-\frac{1}{2} \theta} & \text{if } \theta \leq 1 - 2\hat{\tau}, \\ Y_E^{\frac{1}{2}+\#E} \prod_{j \in Q} V_j^{\frac{1}{2}+\frac{1}{2} \theta} & \text{if } \theta \geq 1 - 2\hat{\tau} \text{ and } E \neq \emptyset, \\ V_{c-\frac{1}{2}}^{\frac{1}{2}-\frac{1}{2} \theta} \prod_{j=1}^d V_j^{1+\frac{1}{2}} & \text{if } 1 - 2\hat{\tau} \leq \theta \leq 1 - 2c \text{ and } E = \emptyset, \\ V_{c-\frac{1}{2}}^{\frac{1}{2}-\frac{1}{2} \theta} \prod_{j=1}^d V_j^{1+\frac{1}{2}} & \text{if } \theta \geq 1 - 2c \text{ and } E = \emptyset, \end{cases}$$

and if $\hat{\tau} = 0$

$$(59) \quad \sum_{n \in \mathbb{N}^d} K_n(z, z) \ll_{\Gamma, E, c} n(z) \cdot \begin{cases} Y_E^{\frac{1}{2}+\#E} \prod_{j \in Q} V_j^{1-\frac{1}{2} \theta} & \text{if } \theta \leq 1 - 2c, \\ Y_E^{\frac{1}{2}+\#E} \prod_{j \in Q} V_j^{\frac{1}{2}+\frac{1}{2} \theta} & \text{if } \theta \geq 1 - 2c. \end{cases}$$
Proof. We have $S_n = \sum_{H \in \{1, \ldots, d\}} (-1)^{#H} S(Y(n - 1_H); z, z)$, where $1_H \in \mathbb{N}^d$ has coordinate 1 if $j \in H$ and coordinate 0 otherwise. We understand that $S(Y(m); z, z)$ is zero if one of the coordinates of $m$ vanishes.

To estimate $\sum_n K_n(z, z)$ it suffices to consider

$$\sum_{m \in \mathbb{N}^d} S(Y(m); z, z) \sum_{H \subseteq \{1, \ldots, d\}} (-1)^{#H} M(m + 1_H).$$

For $S(Y(m); z, z)$ we have an estimate with product structure. Lemma 5.2 estimates $M(n)$ also by a product over the places, with the exception of $n = 1$ in some cases. To handle that exception we take subsums $T_F$ of (60) characterized by $m_j \geq 2$ if and only if $j \in F$, with $F$ running over the nonempty subsets of $\{1, \ldots, d\}$.

$$T_F = \sum_{m \in \mathbb{N}^d, m_j \geq 2 \forall j \in F} \max(m_j^2, \eta_j(z)) m_j$$

$$\cdot \sum_{H \subseteq \{1, \ldots, d\}} (-1)^{#H} \prod_{j \in H} f_{l,j}(m_j + 1) \prod_{j \notin H} f_{l,j}(m_j)$$

$$= \prod_{j \in F} \max(m_j^2, \eta_j(z)) m_j (f_{l,j}(m_j + 1) - f_{l,j}(m_j + 1))$$

$$\cdot \prod_{j \notin F} \max(1, \eta_j(z)) (f_{l,j}(1) - f_{l,j}(2)).$$

For each $j$ and each $m_j$, we will choose a value of $l$ in $f_{l,j}(m_j)$. Since the implicit constant in the estimate depends on $l$, we have to choose the $l$ from a finite set in $\mathbb{N}$. For “small values” of $m_j \geq 2$ we take $l = 1$, and $l = 3$ for “large” values. We will determine the boundary between small and large in a moment. If $m_j = 1$, $f_{l,j}(1)$ does not depend on $l$.

For $j \in F$, the sum

$$\sum_{m_j \geq 2} \max(m_j^2, \eta_j(z)) m_j (f_{l,j}(m_j + 1) - f_{l,j}(m_j + 1))$$

has two critical values of $m_j$. The first occurs near $m_j = \eta_j(z)$. The other one, which we denote by $A_j$ occurs where $f_{l,j}(m_j) = f_{l,j}(m_j)$. We take $l = 3$ for $m \geq A_j + 1$ and $l = 1$ for $2 \leq m \leq A_j$, with $A_j = \lfloor V_j Y_j \rfloor = [V_j^{1-\theta}]$ if $j \in Q$ and $A_j = [V_E^{-1}]$ for $j \in E$. The choices of the parameters in 5.1 and 5.2 is such that $A_j$ is a large quantity in all cases.

With the notations $P_1 = V_j^{1/2}$, $P_2 = V_j^{1-\theta}$ if $j \in Q$, and $P_1 = 1$, $P_2 = V_E^{-1}$ if $j \in E$, the sum for $j \in F$ is equal to

$$\sum_{2 \leq m \leq A_j - 1} \max(m^2, \eta_j(z)) m (m^{-1/2} - (m + 1)^{-1/2}) P_1$$

$$+ \max(A_j^2, \eta_j(z)) A_j (A_j^{-1/2} P_1 - (A_j + 1)^{-1/2} P_2)$$

$$+ \sum_{m \geq A_j + 1} \max(m^2, \eta_j(z)) m (m^{-1/2} - (m + 1)^{-1/2}) P_2.$$

(62)
If $\eta_j(z) \geq A_j$, the first of the sums is estimated by

$$\eta_j(z) \sum_{2 \leq m \leq A_j - 1} m^{-\frac{2}{7}} P_1 \ll \eta_j(z) P_1.$$  

If $\eta_j(z) < A_j$ we split up the first sum, and obtain

$$\sum_{2 \leq m < \eta_j(z)} m^{-\frac{2}{7}} P_1 + \sum_{\eta_j(z) < m \leq A_j - 1} m^{-\frac{2}{7}} P_1 \ll \eta_j(z) P_1 + A_j^{-\frac{2}{7}} P_1.$$  

For the last sum we obtain if $\eta_j(z) \geq A_j$

$$\sum_{A_j + 1 \leq m < \eta_j(z)} \eta_j(z) m^{-\frac{2}{7}} P_2 + \sum_{m \geq \eta_j(z)} m^{-\frac{2}{7}} P_2 \ll \eta_j(z) A_j^{-\frac{2}{7}} P_2 + \eta_j(z)^{-\frac{2}{7}} P_2,$$

and if $\eta_j(z) < A_j$

$$\sum_{m \geq A_j + 1} m^{-\frac{2}{7}} P_2 \ll A_j^{-\frac{2}{7}} P_2.$$  

The transitional middle term in (62) can be estimated by

$$\max(A_j, \eta_j(z)) A_j^{-\frac{2}{7}} P_1.$$  

In total we get for $\eta_j(z) \geq A_j$

$$\ll \eta_j(z)(P_1 + A_j^{-\frac{2}{7}} P_2) + \eta_j(z)^{-\frac{2}{7}} P_2 + \eta_j(z) A_j^{-\frac{2}{7}} P_1$$

and for $\eta_j(z) < A_j$

$$\ll \eta_j(z) P_1 + A_j^\frac{1}{2} P_1 + A_j^{-\frac{2}{7}} P_2 + A_j^{-\frac{2}{7}} P_1$$

In the case $\eta_j(z) \geq A_j$ we use that $\eta_j(z)^{-\frac{2}{7}} V_j^{\frac{5}{2} - \theta} \leq V_j^{1 - \frac{1}{2} \theta}$ in the case $j \in Q$ and $\eta_j(z)^{-\frac{2}{7}} Y_E^2 \leq Y_E^{-\frac{1}{2} \theta}$ for $j \in E$, to get the following bound for the quantity in (62):

$$\ll \begin{cases} \eta_j(z) V_j^{\frac{1}{2}} + V_j^{1 - \frac{1}{2} \theta} & \text{if } j \in Q, \\ \eta_j(z) + Y_E^{-\frac{1}{2}} & \text{if } j \in E. \end{cases}$$

(63)

So this estimates the factors with $j \in F$ in (61).
We estimate the factors for $j \not\in F$ by $n_j(z)f_j(1)$. By Lemma 5.2 we get

$$T_F \ll_{E,c} \prod_{j \in F \cap Q} n_j(V_j^{1-\frac{1}{2}\theta},z) V_j^{1-\frac{1}{2}\theta} \prod_{j \in Q \setminus F} V_j^{\max(c,\hat{c})+\frac{1}{2}} \prod_{j \in E \setminus F} n_j(V_j^{-\frac{1}{2}},z) V_j^{-\frac{1}{2}} \prod_{j \in E} 1. \tag{64}$$

Since we have already $n(z)$ in the error term in Lemma 5.1, it seems sensible to replace $n_j(\cdot, z)$ by $n_j(1, z)$ in these estimates. So we put $n(z)$ in front, and remove the $n_j(\cdot, \cdot)$ from the products.

The next step is to determine which non-empty $F \subset \{1, \ldots, d\}$ has the maximal contribution. The factors for $j \in E$ are maximal if $j \in F$. So we consider $F \supset E$. The factors for $j \in Q$ are maximal for $j \not\in F$ if $\hat{c} \geq 1 - 2 \max(c, \hat{c})$, and maximal for $j \in F$ otherwise. If $E = \emptyset$, we have to put one place in $F$ anyhow, which gives the maximal contribution if $V_j = V_{\min}$. We find the following maximal value:

$$S(Y(1); z, z)M(1) \ll n(z) \begin{cases} \prod_{j \in Q} V_j^{\max(c, \hat{c})+\frac{1}{2}} & \text{if } \hat{c} \geq 1 - 2 \max(c, \hat{c}) \text{ and } E \neq \emptyset, \\ \prod_{j \in Q \setminus F} V_j^{\max(c, \hat{c})+\frac{1}{2}} & \text{if } \hat{c} \geq 1 - 2 \max(c, \hat{c}) \text{ and } \#Q = d, \\ \prod_{j \in Q \cap F} V_j^{-\frac{1}{2}\theta} & \text{if } \hat{c} \leq 1 - 2 \max(c, \hat{c}). \end{cases} \tag{65}$$

In the latter case, the maximum is attained for $F = \{1, \ldots, d\}$, and in the former case for $F = E$, if $E \neq \emptyset$. If $E = \emptyset$, one place has to be in $F$, and $j$ such that $V_j = V_{\min}$ gives the maximal value.

Finally we have to consider the term with $m = 1$ in (60). It is estimated by

$$S(Y(1); z, z)M(1) \ll n(z) \begin{cases} \prod_{j \in Q} V_j^{\max(c, \hat{c})+\frac{1}{2}} & \text{if } E \neq \emptyset, \\ \prod_{j \in Q} V_j^{\max(c, \hat{c})+\frac{1}{2}} & \text{if } E = \emptyset. \end{cases} \tag{66}$$

If $E \neq \emptyset$ or if $\hat{c} \leq 1 - 2 \max(c, \hat{c})$ this is absorbed in the term that we have already obtained. In the case $E = \emptyset$ and $\hat{c} \geq 1 - 2 \max(c, \hat{c})$ we have to compare the factors $V_j^{\max(c, \hat{c})-\frac{1}{2}\theta}$ and $V_j^{-\max(c, \hat{c})}$. If $\hat{c} = 0$, we have $\max(c, \hat{c}) = c$, in which case the latter factor, $V_j^0 = 1$, is the largest. In the remaining case there is another transition point at $\hat{c} = 1 - 2c$.

This leads to the statements in the lemma. We resist the temptation to simplify the lemma by choosing $c < \frac{1}{2}\theta$. That would cause a dependence of the implicit constant in the estimates on the auxiliary parameter $\theta$. \hfill \Box$

5.6. Difference between sums with sharp and smooth bounds. We have obtained $K(z, z) = E(U, V; z) + Err_1 + Err_2$ for the sum $K(z, z)$ in (42), the explicit term $E(U, V; z)$ in (44), with error terms $Err_1$ estimated in Lemma 5.1 and $Err_2$ in Lemma 5.3. The sum $K(z, z)$ depends on the choice of the local test functions $k_j$ as indicated in (40). In particular the estimate is valid for the sum $K^+(z, z)$ based on test functions with $k_j^+ = 1$ on $[U_j, V_j]$ for all $j$, and also for the sum $K^-(z, z)$ built with $\text{Supp}(k_j^-) \subset [U_j, V_k]$. Since the characteristic function $\chi$ of $\prod_j [U_j, V_j]$
satisfies $\prod_j k_j^r \leq \chi \leq \prod_j k_j^{r^+}$, we have $K^-(z, z) \leq N(U, V; z) \leq K^+(z, z)$. Thus we have also

$$\tag{67} N(U, V; z) = E(U, V; z) + Err_1 + Err_2,$$

with error terms satisfying the estimates in Lemmas 5.1 and 5.3.

5.7. Asymptotic estimate, case $E = 0$. First we choose the auxiliary parameters in the case $E = 0$. This leads to the asymptotic result in Theorem 5.4 of which Theorem 1.2 is a special case.

The auxiliary parameter $\vartheta \in (0, 1)$ has to be adapted to the $V_j$ to get the minimal value of the bound

$$\leq_{\vartheta,c} n(z)V_{\min}^{\vartheta-1} \prod_j V_j$$

$$+ n(z) \begin{cases} \prod_j V_j^{1-\frac{\vartheta}{\vartheta^+}} & \text{if } \vartheta \leq 1 - 2 \max(c, \hat{\vartheta}), \\ \prod_j V_j^{\frac{\vartheta}{\vartheta^+} + \max(c, \hat{\vartheta})} & \text{if } \vartheta \geq 1 - 2 \max(c, \hat{\vartheta}). \end{cases}$$

The parameter $c \in (0, \frac{1}{2})$ is allowed to depend on $\Gamma$, but not on the $V_j$. We have assumed that $0 < c < \hat{\vartheta}$ if $\hat{\vartheta} > 0$. The $V_j$ influence the estimate by their product $\prod_j V_j$, which tends to $\infty$. We have prescribed that the minimal $V_j$ is coupled to the product by $V_{\min}^{\hat{\vartheta}} = \prod_j V_j$, with $\hat{\vartheta} \geq d$. (See (39).) Expressing the logarithm of the quantity to consider in terms of $\log V_{\min}$, we arrive at the following quantity to minimize:

$$\begin{cases} \max(\vartheta - 1 + \hat{\vartheta}, \hat{\vartheta} - \frac{1}{2} \hat{\vartheta} \vartheta) & \text{if } \vartheta \leq 1 - 2 \max(c, \hat{\vartheta}), \\ \max(\vartheta - 1 + \hat{\vartheta}, \frac{\vartheta}{2} + (\hat{\vartheta} - 1)\hat{\vartheta} + \max(\frac{1}{2} \vartheta, c)) & \text{if } \vartheta \geq 1 - 2 \hat{\vartheta}, \hat{\vartheta} > 0, \\ \max(\vartheta - 1 + \hat{\vartheta}, \hat{\vartheta}(\frac{\vartheta}{2} + c) + \max(\frac{1}{2} \vartheta, c)) & \text{if } \vartheta \geq 1 - 2c, \hat{\vartheta} = 0. \end{cases}$$

We choose $0 < c < \frac{1}{10}$ in addition to the requirement that $c < \hat{\vartheta}$ if $\hat{\vartheta} > 0$.

If $\hat{\vartheta} = 0$ the value $\vartheta_1$ for which $\vartheta_1 - 1 + \hat{\vartheta} = \hat{\vartheta} - \frac{1}{2} \hat{\vartheta} \vartheta_1$ is $\vartheta_1 = \frac{1}{\vartheta^2 + 2}$. Since $\hat{\vartheta} \geq d \geq 1$, we have $\vartheta_1 \leq \frac{\vartheta}{2} < 1 - 2c$. Hence this is the optimal choice.

For $\hat{\vartheta} > 0$ we have $\vartheta_1 = \frac{2}{\vartheta^2 + 2}$ and $\vartheta_2 = 1 - \frac{2}{\vartheta^2 + 2} \vartheta_2$, for the intersections of the graph of $\vartheta \mapsto \vartheta - 1 + \hat{\vartheta}$ with respectively, $\vartheta \mapsto \hat{\vartheta} - \frac{1}{2} \hat{\vartheta} \vartheta$ and $\vartheta \mapsto \frac{1}{2} \hat{\vartheta} + (\hat{\vartheta} - 1)\hat{\vartheta} + \frac{1}{2} \hat{\vartheta}$. If $\hat{\vartheta} \leq \frac{2}{2(\vartheta^2 + 2)}$, then $\vartheta_1 \leq 1 - 2 \hat{\vartheta}$ gives the optimal choice. Otherwise, $\vartheta_2 \geq 1 - 2 \hat{\vartheta}$ is optimal, since it is between $1 - 2 \hat{\vartheta}$ and $1 - 2c$.

This leads to the following optimal bound of the quantity in (69):

$$\tag{70} \frac{\vartheta_1 + 1}{\vartheta_2 + 2} \text{ if } 0 \leq \hat{\vartheta} \leq \frac{\vartheta_1}{2(\vartheta_2 + 2)}, \quad \frac{\vartheta_2(\hat{\vartheta} + 1)}{\vartheta_2 + 2} - \frac{2 \hat{\vartheta}}{3} \text{ if } \hat{\vartheta} \geq \frac{\vartheta_1}{2(\vartheta_2 + 2)}.$$

Now we have chosen $c$ depending only on quantities determined by $\Gamma$, and we have arrived at the following estimate:

$$\tag{71} N(U, V; z) - \mathcal{E}(U, V; z) \leq_{\vartheta,c} n(z) \begin{cases} \prod_j V_j^{\hat{\vartheta} + 1} & \text{if } \hat{\vartheta} \leq \frac{\vartheta}{2(\vartheta^2 + 2)}, \\ \prod_j V_j^{\frac{2\hat{\vartheta}}{\vartheta_2(\hat{\vartheta} + 1) - \frac{2\hat{\vartheta}}{3}}} & \text{if } \hat{\vartheta} \geq \frac{\vartheta}{2(\vartheta^2 + 2)}. \end{cases}$$
If there are totally exceptional eigenvalues, the explicit sum \( \mathcal{E}(U, V; z) \) in (44) contains the corresponding terms, which are of the size \( O(n(z) \prod_j V_j^{\frac{1}{2} + \tilde{\tau}}) \). Since \( \tau_{\ell, j} \leq \tilde{\tau} \) for all exceptional coordinates, these terms are swallowed by the error term obtained for the case \( \tilde{\tau} \leq \frac{\hat{q}}{2(q+2)} \).

We have thus obtained the following asymptotic result for the counting function:

**Theorem 5.4.** Let \( \Gamma \) be an irreducible lattice in \( \text{PSL}_2(\mathbb{R})^d \) with \( d \in \mathbb{N} \). Let \( \tilde{\tau} \) be the quantity in (3), measuring the spectral gap. Denote by \( V_j \geq 1 \), \( 1 \leq j \leq d \), large quantities subject to the condition \( \min_j V_j^d = \prod_j V_j \) for a fixed number \( \hat{q} \geq d \). Choose \( U_j = 0 \) or \( U_j = V_j \) for each \( j = 1, \ldots, d \).

Let \( z \in \mathbb{S}^d \). The number \( N(U, V; z) \) of \( \gamma \in \Gamma \) such that \( U_j \leq u((\gamma z)_j, z_j) \leq V_j \) for all \( j \), with \( u(\cdot, \cdot) \) as in (9), satisfies

\[
N(U, V; z) = \frac{(4\pi)^d}{\text{vol}(\Gamma \setminus \mathbb{S}^d)} \prod_{j=1}^d (V_j - U_j) + O_{\Gamma}(n(z) \prod_{j=1}^d V_j^{(\hat{q}+1)/(\hat{q}+2)})
\]

if \( \tilde{\tau} \leq \frac{\hat{q}}{2(q+2)} \), and

\[
N(U, V; z) = \frac{(4\pi)^d}{\text{vol}(\Gamma \setminus \mathbb{S}^d)} \prod_{j=1}^d (V_j - U_j) + \sum_{\ell \geq 1, \forall \tau_j \in (0, \frac{1}{\ell})} |\psi_\ell(z)|^2
\]

\[
\cdot \prod_{j=1}^d \frac{\sqrt{\pi} \cdot 2^{1+2\tau_{\ell, j}} \Gamma(\tau_{\ell, j})}{\Gamma(\frac{1}{2} + \tau_{\ell, j})} (V_j^{\frac{1}{2} + \tau_{\ell, j}} - U_j^{\frac{1}{2} + \tau_{\ell, j}})
\]

\[
+ O_{\Gamma}(n(z) \prod_{j=1}^d V_j^{2(\tilde{\tau}+1)-2\tilde{\tau}/3\hat{q}})
\]

if \( \tilde{\tau} \geq \frac{\hat{q}}{2(q+2)} \). The factor \( n(z) \) is as in (16).

Note that even if there is no spectral gap (\( \tilde{\tau} = \frac{1}{2} \)) the error term is still smaller than the main term.

In the special case when all \( V_j \) are equal to the same quantity \( V \) we have \( \hat{q} = d \). The relation (9) implies that the condition \( d((\gamma z)_j, z_j) \leq T \) is equivalent to \( u((\gamma z)_j, z_j) \leq \frac{1}{\ell}e^T (1 + O(e^{-T})) \). Thus, we obtain Theorem 1.2

5.8. **Asymptotic estimate, case \( E \neq \emptyset \).** We turn to the case when both parts of the partition \( \{1, \ldots, d\} = Q \sqcup E \) are non-empty.

We have obtained the following estimate for the error terms:

\[
\ll_{\Gamma, E} n(z) (V_\theta - 1 + Y_{\#E}) \prod_{j \in Q} V_j + n(z) Y_E^{\frac{1}{2} + \#E} \prod_{j \in Q} V_j^{\max(1 - \frac{1}{2} + \frac{1}{2} + \tilde{\tau}, \frac{1}{2} + \tilde{\tau} + c)}.
\]

We try to choose \( \theta \) and \( Y_E \) optimally, depending on the \( V_j, j \in Q \). The parameter \( c \in (0, \frac{1}{2}) \) satisfies \( c < \tilde{\tau} \) if \( \tilde{\tau} > 0 \), and can be further adapted to the situation, but is not allowed to depend on the \( V_j \) with \( j \in Q \).
We take $x = \log V_{\min}$ as the large variable. Then $\prod_{j \in Q} V_j = \hat{e}^{\hat{q}x}$, with $\hat{q} \geq q$ fixed. We assume that $Y_E = V_{\min}^{-\eta}$ with $\eta > 0$ is a sensible choice. To simplify the formulas we work for the moment with the notations $e = |E|$, $m = \max(c, \hat{\tau})$. So $0 < m \leq \frac{1}{2}$. We try to choose $\vartheta \in [0, 1]$ and $\eta \geq 0$ such that the following quantity is minimal:

$$M(\eta, \vartheta) = \max\left(\vartheta - 1 + \hat{q}, \hat{q} - \eta, \frac{e}{2}\eta + \hat{q}\left(-\frac{\vartheta}{2}\right), \frac{e}{2}\eta + \hat{q}\left(1 - \frac{\vartheta}{2}\right)\right).$$

We allow for the moment $\vartheta$ and $\eta$ to assume boundary values. If we end up with an optimal choice on the boundary, we will see how to handle the problem of satisfying the conditions in §5.2.

The lines $\vartheta + \eta = 1$ and $\vartheta = 1 - 2m$ give four subsets of the region in which $(\eta, \vartheta)$ varies:

|   |   |   |   |   |   |
|---|---|---|---|---|---|
|  \vartheta  |  1-2m | A | B | C | D |
|   |   |   |   |   |   |
|  \eta        |   |   |   |   |   |

We have

$$M(\eta, \vartheta) = \begin{cases} 
\max(\hat{q} - \eta, \frac{e}{2}\eta + \hat{q}(1 + m)) & \text{on } A, \\
\max(\hat{q} - 1 + \vartheta, \frac{e}{2}\eta + \hat{q}(1 + m)) & \text{on } B, \\
\max(\hat{q} - \eta, \frac{e}{2}\eta + \hat{q}(1 - \frac{\vartheta}{2})) & \text{on } C, \\
\max(\hat{q} - 1 + \vartheta, \frac{e}{2}\eta + \hat{q}(1 - \frac{\vartheta}{2})) & \text{on } D. 
\end{cases}$$

On $B$ and $D$ these expressions for $M(\eta, \vartheta)$ contain $\eta$ only once. So in the search for optimal values we can take $\eta$ minimal in these cases. This brings them into the cases $C$ and $D$, respectively. In region $C$, the variable $\vartheta$ occurs only once. So it makes sense to take it optimal, i.e., $\vartheta = 1 - 2m$ if $0 \leq \eta \leq 2m$ and $\vartheta = 1 - \eta$ if $2m \leq \eta \leq 1$. This reduces our search for the optimum to the lines $E$ and $F$ in the following figure.

|   |   |   |   |   |
|---|---|---|---|---|
|   |   |   |   |   |
|  \vartheta  |   |   |   |   |
|   |   |   |   |   |
|  \eta        |   |   |   |   |

Thus, we are left with a one-dimensional problem: Find the minimum for $0 \leq \eta \leq 1$ of the maximum of the two functions $\alpha(\eta) = \hat{q} - \eta$ and

$$\beta(\eta) = \begin{cases} 
\frac{e}{2}\eta + \hat{q}(1 + m) & \text{for } 0 \leq \eta \leq 2m, \\
\frac{e}{2}(e + \hat{q}) + \frac{\hat{q}}{2} & \text{for } 2m \leq \eta \leq 1.
\end{cases}$$
Since \((\frac{1}{2} + m)\hat{q} \leq \hat{q} \leq \hat{q} + \frac{\eta}{2}\), the graphs of \(\alpha\) and \(\beta\) intersect for some value of \(\eta\) in \([0, 1]\). This gives the value of the optimum we look for. This leads to the optimal value

\[
M(\frac{\hat{q}}{\hat{q} + \frac{\eta}{2}}) = \frac{\hat{q} + \frac{\eta}{2}}{\hat{q} + \eta + 2} \quad \text{if } m \leq \frac{\hat{q}}{2(\hat{q} + \eta + 2)},
\]

\[
M(\frac{\eta + 2m}{\eta + 2}) = \frac{\hat{q} + \frac{\eta}{2}}{\hat{q} + \eta + 2} \quad \text{if } m \geq \frac{\hat{q}}{2(\hat{q} + \eta + 2)}.
\]

We note that if \(m = \max(x, \hat{r}) = \frac{1}{2}\) (no spectral gap), the optimal value is at the boundary point \((0, 1)\), which we did not want to use. However, this optimal value is \(\hat{q} \geq q\), hence it provides an error term that swallows the main term. So we assume that \(\hat{r} < \frac{1}{2}\) from this point on.

If \(\hat{r} > 0\), we can just replace \(m\) by \(\hat{r}\) in the results. If \(\tau = 0\) we take \(0 < c < \frac{\#Q}{2(d + e)}\). This depends on \(d\), hence on \(\Gamma\), and on the partition in \(Q\) and \(E\). We now have

\[
\frac{\hat{q}}{2(\hat{q} + \eta + 2)} = \frac{\hat{q}}{2} - \frac{2 + e}{2(\hat{q} + 2 + e)} \geq \frac{\hat{q} + \frac{\eta}{2}}{2(2 + e + \#Q)} = \frac{\#Q}{2(d + e)} > c = m.
\]

So we can apply the estimate for \(m \leq \frac{\hat{q}}{2(\hat{q} + \eta + 2)}\).

Thus, we obtain the following estimates for the error terms:

\[
O_{\Gamma,E}(n(z) \prod_{j \in Q} V_j^{(\hat{q} + 1 + \#E)/(\hat{q} + 2 + \#E)}) \quad \text{if } \hat{r} \leq \frac{\hat{q}}{2(\hat{q} + 2 + \#E)},
\]

\[
O_{\Gamma,E}(n(z) \prod_{j \in Q} V_j^{(1 + 2\hat{r} + \#E)/(2 + \#E)}) \quad \text{if } \frac{\hat{q} + \frac{\eta}{2}}{2(\hat{q} + 2 + \#E)} \leq \hat{r} < \frac{1}{2}.
\]

Each exceptional term in \(E(U, V; z)\) in \((44)\) contributes at most \(\prod_{j \in Q} V_j^{1 + \#E}\) and is absorbed by the error term in \((76)\). We are left with the term corresponding to the constant function. See \((88)\) in Lemma \((6.1)\) for its simple form.

Thus we have obtained the following asymptotic result:

**Theorem 5.5.** Let \(\Gamma\) be an irreducible lattice in \(\text{PSL}_2(\mathbb{R})\) with \(d \geq 2\). Suppose that the quantity \(\hat{r}\) in \((3)\) satisfies \(\hat{r}(\Gamma) < \frac{1}{2}\). We partition the set \(\{1, \ldots, d\}\) into two disjoint non-empty subsets \(Q\) and \(E\). For each \(j \in E\) we fix a bounded interval \([U_j, V_j]\) \(\subset (0, \infty)\). For each \(j \in Q\), let \(V_j \to \infty\), under the assumption that \(\min_{j \in Q} V_j^\beta = \prod_{j \in Q} V_j\) for some fixed real number \(\beta \geq \#Q\). Also, choose \(U_j = 0\) or \(U_j = \frac{1}{2} V_j\) for \(j \in Q\).
Let $z \in \mathcal{S}$. The number $N(U, V; z)$ in (10) satisfies

$$N(U, V; z) = \frac{(4\pi)^d}{\text{vol}(\Gamma \setminus \mathcal{S}^d)} \prod_{j=1}^d (V_j - U_j)$$

(77)

$$\begin{cases}
\Omega_{\Gamma, E}(n(z) \prod_{j \in Q} V_j^{(\hat{q}+1+\#E)/(\hat{q}+2+\#E)}) & \text{if } \hat{q} \leq \frac{\hat{q}}{2(\hat{q}+2+\#E)}, \\
\Omega_{\Gamma, E}(n(z) \prod_{j \in Q} V_j^{(1+2+\#E)/(2+\#E)}) & \text{if } \frac{\hat{q}}{2(\hat{q}+2+\#E)} \leq \hat{q} < \frac{1}{2}.
\end{cases}$$

The implicit constants in the estimates depend on the discrete group $\Gamma$, on the partition $[1, \ldots, d] = Q \sqcup E$ and on the choice of the intervals $(U_j, V_j)$ for $j \in E$.

We note that the presence of totally exceptional eigenvalues for $\Gamma$ has no explicit influence on this asymptotic formula. The size of the spectral gap does influence the quality of the error term only if it is larger than $\frac{\hat{q}}{2(\hat{q}+2+\#E)}$.

Like we did in the previous subsection we take all $V_j$ with $j \in Q$ equal to $V$, and all $U_j$ for $j \in Q$ equal to $0$. With the relation in (19) between $u((yz_j, z_j) \in [U_j, V_j)$ and the hyperbolic distance $d((yz_j, z_j) \in [A_j, B_j)$, the main term takes the form

$$\frac{4\theta^d}{\text{vol}(\Gamma \setminus \mathcal{S}^d)} \int_{\mathcal{S}^d} \prod_{j \in E} e^{B_j + e^{-B_j} - e^{A_j} - e^{-A_j}} \frac{du}{4},$$

in the notations of Theorem [11], which leads to the main term in the asymptotic formula in that theorem. For the error terms we use that for equal $V_j$ for $j \in Q$, the parameter $\hat{q}$ is equal to $\#Q$.

6. Estimates of Selberg Transforms

Here we collect and prove the estimates of Selberg transforms that we have used. This can be done factor by factor. So in this section we work on $\mathcal{S}$, we do not use an index $j$ and we denote real numbers by $U$ and $V$.

6.1. Integral representations. The results we need are given in Lemmas [6.1] [6.2] and [6.3]. To derive these lemmas we start with an arbitrary measurable compactly supported function $k$ on $[0, \infty)$ with values in $[0, 1]$. By the definitions in 4.1 we have the following integral representation for the Selberg transform $h$ of $k$:

$$h(\tau) = 2 \int_{r=-\infty}^{\infty} e^{\tau r} \int_{u=\sinh^2 r/2}^{\infty} k(u) \frac{du}{\sqrt{u - \sinh^2 r/2}} dr$$

(78)

$$= 4 \int_0^{\infty} \cosh \tau r \int_{u=\sinh^2 r/2}^{\infty} k(u) \frac{du}{\sqrt{u - \sinh^2 r/2}} dr.$$

The inner integral gives a non-negative compactly supported function. So $h(0) \geq |h(it)|$ for $t \in \mathbb{R}$, which gives

$$|h(it)| \leq h(0) \quad (t \in \mathbb{R}), \quad \tau \mapsto h(\tau) \text{ is increasing on } [0, \frac{1}{\tau}].$$

(79)

(For the latter we use that $\tau \mapsto \cosh \tau r$ is increasing.)
We interchange the order in the double integration, and obtain

\[
(80) \quad h(\tau) = 2 \int_{u=0}^{\infty} k(u) \int_{r \in \mathbb{R}, \sinh^2 r/2 \leq u} e^{\tau (u - \sinh^2 r/2)^{-1/2}} \, dr \, du.
\]

Following the approach in [20], we define \( x = x(u) \geq 1 \) by \( 2 + 4u = x + x^{-1} \), hence \( x(u) = 1 + 2u + 2 \sqrt{u + u^2} \). The condition \( \sinh^2 \frac{r}{2} \leq u \) amounts to \( |r| \leq \log x \). With the substitution \( e^\tau = x^{-1}(1 + y(x^2 - 1)) \) the inner integral equals

\[
\int_{-\log x}^{\log x} e^{\tau ((x + x^{-1} - e^\tau - e^{-\tau})/4)^{-1/2}} \, dr
\]

\[
= 2 \int_{-1}^{1} (1 + y(x^2 - 1))^{\Re \tau} \, dy
\]

\[
= 2 \pi x^{\Re \tau} \, \text{E}_{1}(1, x^2 - 1),
\]

by the standard integral representation of the hypergeometric series in [3], §2.1.3, (10). Let us work under the standing assumption that \( 0 \leq \Re \tau \leq \frac{1}{2} \).

From (81) we obtain the bound \( O(x^{\Re \tau - \frac{3}{2}}) \) for the inner integral in (80). Hence if \( \text{Supp}(k) \subset [A, B] \), then

\[
(83) \quad h(\tau) \ll \int_{x(A)}^{x(B)} x^{2 \Re \tau - \frac{3}{2}} \, dx \ll \frac{x(B)^{\Re \tau + \frac{1}{2}} - x(A)^{\Re \tau + \frac{1}{2}}}{\Re \tau + \frac{1}{2}}
\]

\[
= (B - A) \left( 1 + 2T + 2 \sqrt{T^2 + T} \right)^{\Re \tau - \frac{1}{2}} \quad \text{with} \ A \leq T \leq B
\]

\[
\ll \begin{cases} 
(B - A)(1 + A^{\Re \tau - \frac{1}{2}}) & \text{if } A > 0, \\
1 + B^{\Re \tau + \frac{1}{2}} & \text{if } A = 0 \text{ and } B > 0.
\end{cases}
\]

Proceeding with (82) we get

\[
(84) \quad h(t) = \pi \int_{1}^{\infty} k\left( (x + x^{-1} - 2)/4 \right) x^{\frac{1}{2} - \tau} (x^2 - 1)
\]

\[
\cdot \, \text{E}_{1}\left( \frac{1}{2} - \tau, \frac{1}{2}; 1 - x^2 \right) \, dx
\]

\[
(85) \quad = \pi \int_{1}^{2} k\left( (x + x^{-1} - 2)/4 \right) x^{\frac{1}{2} - \tau} (x^2 - 1)
\]

\[
\cdot \, \text{E}_{1}\left( \frac{1}{2} - \tau, \frac{1}{2}; 1 - x^2 \right) \, dx
\]

\[
+ \sqrt{\pi} \int_{2}^{\infty} k\left( (x + x^{-1} - 2)/4 \right) x^{\frac{1}{2} - \tau} (x^2 - 1)^{\frac{1}{2}}
\]

\[
\cdot \sum_{\pm} \frac{\Gamma(\pm \tau)}{\Gamma(\frac{1}{2} \pm \tau)} x^{\pm \tau} \, \text{E}_{1}\left( \frac{1}{2} \pm \tau; 1 - x^2 \right) \, dx.
\]

(For (85) we have used a Kummer relation, [3], §2.9, (34), (10), (13)).

Let us use (85) in the case when \( \text{Supp}(k) \subset [A, B] \). For the part of \([A, B]\) corresponding to a subinterval \( x \in [1, 2] \) we have the estimates in (83). We now estimate
the integral over an interval \([A, B]\) with \(x(A) \geq 2\) by
\[
\sum_{\pm} \left| \frac{\Gamma(\pm \tau)}{\Gamma(\frac{1}{2} \pm \tau)} \right| \int_{x(A)}^{x(B)} x^{-\frac{1}{2} \pm \text{Re} \tau} \left| \text{E}_\Gamma \left( \frac{1}{2}, \frac{1}{2}; 1 \mp \Re \tau; \frac{1}{1-x} \right) \right| dx.
\]
Uniformly for \(\tau\) in a compact set \(T\) we find an estimate by
\[
(86) \quad \sum_{\pm} \left| \frac{\Gamma(\pm \tau)}{\Gamma(\frac{1}{2} \pm \tau)} \right| (B - A)^{\pm \text{Re} \tau - \frac{1}{2}}.
\]
Note that this estimate is bad if \(\tau\) is near 0. To handle a neighborhood of \(\tau = 0\), we consider the second integral in (83) as a holomorphic function of the complex variable \(\tau\). It is holomorphic at \(\tau = 0\), since the contributions of \(\Gamma(\tau)\) and \(\Gamma(-\tau)\) cancel each other. If \(\text{Supp}(k)\) is contained in \([A, B]\) with \(x(A) \geq 2\), we find for \(|\tau| = c\) with a small \(c > 0\) by the reasoning that led to the estimate in (86) a bound
\[
O_c((B - A)A^{c-\frac{1}{2}}).
\]
By holomorphy, this bound extends to \(|\tau| \leq c\). Thus, if \(\text{Supp}(k) \subset \left[ A, B \right]\) with \(x(A) \geq 2\), and if \(|\tau| \leq c\), then
\[
h(\tau) \ll_c (B - A)A^{c-\frac{1}{2}}.
\]
For \(\tau = it \in i\mathbb{R}\) with \(x \geq 2\) we use that
\[
\left| \text{E}_\Gamma \left( \frac{1}{2}, \frac{1}{2}; 1 \pm it; \frac{1}{1-x} \right) \right| \leq 2\text{E}_\Gamma \left( \frac{1}{2}, \frac{1}{2}; 1; \frac{1}{1-x} \right).
\]
By Stirling’s formula we get a bound \((1 + |t|)^{-1/2} (B - A)^{1/2}\) uniformly on \(|t| \geq c\).

In Table 3 we have combined these results. We will use these estimates repeatedly in the proofs of the following lemmas. In some cases, we shall return to the integral representations.

| \(i\) | \(c \leq \tau \leq \frac{1}{2}\) | \(|\tau| \leq c\) | \(\tau = it\) in \(i\mathbb{R} \setminus (-c, c)\) |
|---|---|---|---|
| a) | \(1 \leq A < B\) | \((B - A)A^{\tau - \frac{1}{2}}\) | \(|\tau|^{-1/2}(B - A)A^{\tau - \frac{1}{2}}\) |
| b) | \(0 < A < B \leq 1\) | \((B - A)A^{1/2}\) | |
| c) | \(0 = A < B \leq 1\) | \(B^{\tau}\) | |
| d) | \(0 < A < B\) | \((B - A)\cdot \max(A^{-\tau}, A^{-\tau})\) | \((B - A)A^{-\frac{1}{2}}\) |
| e) | \(0 = A < B\) | \(1 + B^{\tau + \frac{1}{2}}\) | \(1 + |\tau|^{-\frac{1}{2}}B\) |

Table 3. Bounds for \(h(\tau)\) under the assumption that \(\text{Supp}(k) \subset [A, B]\). These bounds depend implicitly on \(c \in (0, \frac{1}{2})\).
6.2. **Lemmas for Selberg transforms.** First we consider the Selberg transform \( \eta(U, V; \tau) \) in (85) of the characteristic function \( \chi \) of a bounded interval \([U, V] \subset [0, \infty)\).

**Lemma 6.1.**

\[ \tau \mapsto \eta(U, V; \tau) \text{ is positive and increasing on } [0, \frac{1}{2}] , \]
\[ \eta(U, V; \frac{1}{2}) = 4\pi (V - U) , \]
\[ |\eta(U, V; it)| \leq \eta(U, V; 0) \quad (t \in \mathbb{R}) . \]

Moreover, if \( c \in (0, \frac{1}{2}) \) is fixed, then we have uniformly for \( c \leq \tau \leq \frac{1}{2} \) the estimate

\[ \eta(U, V; \tau) = \sqrt{\frac{2\pi}{\tau}} \frac{2^{2\tau+1} \Gamma(\tau)}{\Gamma(\frac{3}{2} + \tau)} (V^{\tau+\frac{1}{2}} - U^{\tau+\frac{1}{2}}) + O_c(V^{-\tau+\frac{1}{2}}) \quad (V \to \infty) . \]

If the difference \( V - U \) is small, then the \( O \)-term may be larger than the explicit term in (90).

**Proof.** We apply the computations in 6.1 to the characteristic function \( \chi \) of \([U, V]\). In (79) we find (87) and (89). Taking \( \tau = \frac{1}{2} \) in (81) gives (88).

For (90) we need an asymptotic formula, not an estimate. We use (85). If \( x(U) \geq 2 \), we need only the integral over \([2, \infty)\). Writing \( \zeta \left( \frac{1}{2} - \tau, \frac{1}{2}; 1 - x^2 \right) = 1 + O(x^{-2}) \) as \( x \to \infty \), we get, uniformly for \( \tau \in \left[c, \frac{1}{2}\right] \):

\[ \sum_{\pm} \pi \frac{\Gamma(\pm \tau)}{\Gamma(\frac{3}{2} \pm \tau)} \int_{x(U)}^{x(V)} \left( 1 + O_c(x^{-2}) \right) dx \]
\[ = \sum_{\pm} \pi \frac{\Gamma(\pm \tau)}{\Gamma(\frac{3}{2} \pm \tau)} (x(V)^{\frac{1}{2} \pm \tau} - x(U)^{\frac{1}{2} \pm \tau} + O_c(x(U)^{\pm \tau - \frac{1}{2}})) . \]

For \( T \geq 1 \) we have \( x(T) = 4T + O(1) \). So the main term with \( \pm = + \) gives the explicit term in (90). The other terms give \( O(U^{\tau+\frac{1}{2}}) + O(V^{\frac{1}{2} - \tau}) = O(V^{\frac{1}{2} - \tau}) \).

If \( x(U) \leq 2 \) we get from \( x \in [2, \infty) \) the contribution

\[ \pi \frac{\Gamma(1+\tau)}{\Gamma(\frac{3}{2} + \tau)} (V^{\frac{1}{2} + \frac{1}{2} - O(1)}) + O(V^{\frac{1}{2} - \tau}) . \]

We add to it \( O(a(2) - U) = O(1) \) from \( i) d \) in Table 3 and obtain (90) in this case as well. \( \square \)

Next we consider functions approximating the characteristic function of \([U, V]\), satisfying the following conditions.

\[ k \in C^\infty_0(0, \infty) , \quad 0 \leq k \leq 1 , \]
\[ \exists_{T > 0} \text{ such that } 2Y \leq U \text{ if } U > 0, 2Y \leq V - U, \text{ and} \]
\[ k = 1 \text{ on } \left\{ \begin{array}{ll}
[U + Y, V - Y] & \text{if } U > 0 , \\
[0, V - Y] & \text{if } U = 0 , 
\end{array} \right.
\]
\[ k = 0 \text{ on } \left\{ \begin{array}{ll}
[0, U - Y] \cup [V + Y, \infty) & \text{if } U > 0 , \\
[V + Y, \infty) & \text{if } U = 0 .
\end{array} \right. \]
Lemma 6.2. The Selberg transform $h$ in (20) of a function $k \in C^\infty[0,\infty)$ satisfying the conditions (21) has the following properties:

a) $4\pi(V - U - 2Y) \leq h\left(\frac{\tau}{2}\right) \leq 4\pi(V - U + 2Y)$.

b) If $V < 1$, then
$$\left|h(\tau) - h\left(\frac{1}{2}\right)\right| \ll V^{3/2} \left|\frac{1}{2} - \tau\right|$$
for all $\tau$ with $0 \leq \text{Re} \, \tau \leq \frac{1}{2}$.

c) For each $c \in (0, \frac{1}{2})$ the difference with the Selberg transform $\eta(U,V; \tau)$ of the characteristic function of $[U,V]$ satisfies the estimate
$$\eta(U,V; \tau) - h(\tau) \ll c \begin{cases} Y \max(V^{\frac{1}{2}} - 1, U^{-\frac{1}{2}}) & \text{if } U > 0, \\ Y \max(V^{-\frac{1}{2}}, V^{\frac{1}{2}} - 1) & \text{if } U = 0, \end{cases}$$
uniformly in $\tau \in [c, \frac{1}{2}]$.

Proof. Part a) follows by a comparison of $k$ with the characteristic functions of the intervals $[U + Y, V - Y]$ and $[U, V]$, and an application of (88) in Lemma 6.1.

For b) we use (21), and note that $y^s - 1 = s(y - 1)(1 + \xi(y - 1))^{s-1}$ for some $\xi = \xi_{x,y} \in (0,1)$ to obtain
$$\left|h(\tau) - h\left(\frac{1}{2}\right)\right| \leq \left|\frac{1}{2} - \tau\right| \int_{\delta} k(u(z,i)) |y - 1| (1 + \xi(y - 1))^{-\frac{1}{2} - \text{Re} \, \tau} \, d\mu(z).$$

For small $V$ the values of $y$ that occur in the integral are between $1 - O(\sqrt{V})$ and $1 + O(\sqrt{V})$. Thus the integral is bounded by $O(\sqrt{V}) \int_{\delta} k(u(z,i)) \, d\mu(z)$, which gives b).

For c) we apply the estimate i) in Table 3 to a function with support in the union of the intervals $[U - Y, U + Y]$ and $[V - Y, V + Y]$. For $U > 0$ we find:

$$Y \max(U^{\frac{1}{2}} - 1, U^{-\frac{1}{2}}) + Y \max((V - Y)^{\frac{1}{2}} - 1, (V - Y)^{-\frac{1}{2}}) \ll Y \max(V^{\frac{1}{2}} - 1, U^{-\frac{1}{2}}).$$

If $U = 0$ we have only the contribution of $[V - Y, V + Y]$. \hfill $\square$

Lemma 6.3. The Selberg transform $h$ of a function $k \in C^\infty[0,\infty)$ satisfying the conditions (21) has the following properties:

a) $h(\tau) = \eta(U,V; \tau) + Y$ for $\tau \in [0, \frac{1}{2}]$.

b) Let $c \in (0, \frac{1}{2})$. Then we have, uniformly for $\tau \in [-c, \frac{1}{2}]$:

$$h(\tau) \ll c \begin{cases} (V - U)^{U^{\frac{1}{2}}} & \text{if } 1 \leq U \leq V, \\ V^{c + \frac{1}{2}} & \text{if } U = 0, \ V \geq 1, \end{cases}$$

and, without dependence on $c$:

$$h(\tau) \ll V.$$ 

c) Let $c \in (0, \frac{1}{2})$ and take $l \in \mathbb{N}$. Then we have for each $\delta > 0$, uniformly for $t \in \mathbb{R} \setminus (-c, c)$:

$$h(it) \ll c,l \begin{cases} Y \|k^{(l)}\|_\infty \max(V^{l - \frac{1}{2}} - 1, U^{-\frac{l}{2}}) |t|^{-l - \frac{1}{2}} & \text{if } \delta \leq U < V, \\ Y \|k^{(l)}\|_\infty \max(V^{-\frac{l}{2}} - 1, V^{l - \frac{1}{2}}) |t|^{-l - \frac{1}{2}} & \text{if } U = 0, \ V > \delta. \end{cases}$$
We note that we have stated what we need, not the best estimate one might prove by separating more cases. In the proof we will see that in c) we have to avoid intervals with $U \in (0, \delta)$, to be able to apply an asymptotic estimate for hypergeometric functions.

**Proof.** Part a) is a direct consequence of (21), the inequalities $0 \leq k(u) \leq \chi(u)$, where $\chi$ is the characteristic function of $[U, V)$, and Lemma 6.1.

The first estimate in b) can be read off from Table 3. The bound $h(\tau) \ll V$ follows from a) and (88) in Lemma 6.1.

For d) we modify the discussion in §6.1. By the smoothness of $k$ we find in (20)

$$q(p) = \frac{(-1)^l \sqrt{\pi}}{\Gamma(l + \frac{1}{2})} \int_p^\infty k^{(l)}(u - p)^{l - \frac{3}{2}} du$$

for each $l \in \mathbb{N}$. Proceeding as in (78), (80)–(82), (84)–(85), we obtain

$$h(\tau) = \frac{\sqrt{\pi}(-1)^l}{2^l \Gamma(l + \frac{1}{2})} \int_1^\infty k^{(l)}((x + x^{-1} - 2)/4) \int_0^1 x^{\frac{1}{2} - \tau} \cdot (x^2 - 1)^{2l+1} \, dy \, dx$$

$$\cdot (y(1-y))^{l-\frac{1}{2}} (y(1-y))^{l-\frac{1}{2}} \, dy \, dx$$

$$= \frac{\pi(-1)^l}{24l!} \int_1^\infty k^{(l)}((x + x^{-1} - 2)/4) \int_0^1 x^{\frac{1}{2} - \tau} \cdot (x^2 - 1)^{2l+1} \, dy \, dx$$

$$\cdot 2F_1(l + \frac{1}{2} - \tau, l + \frac{1}{2}; 2l + 1; 1 - x^2) \, dx$$

and

$$\cdot \frac{\sqrt{\pi}(-1)^l}{2^l} \int_2^\infty k^{(l)}((x + x^{-1} - 2)/4) \sum_{\pm} \frac{\Gamma(\pm \tau)}{\Gamma(l + \frac{1}{2} \pm \tau)} x^{\frac{1}{2} \pm \tau - l}$$

$$\cdot (x^2 - 1)^{l+\frac{1}{2}} 2F_1\left(l - \frac{1}{2}, l + \frac{1}{2}; 1 + \tau; (1 - x^2)^{-1}\right) \, dx .$$

We apply this for $\tau = \pm \in \mathbb{R}$ with $|\tau| \geq c$. Consider first an interval $[U_1, V_1]$ with $1 < x(U_1) < x(V_1) \leq 2$. Then we use the first integral in (92) to get a bound

$$\ll_l \|k^{(l)}\|_\infty \int_{x(U_1)}^{x(V_1)} |2F_1(l + \frac{1}{2} + it, l + \frac{1}{2}; 2l + 1; 1 - x^{-2})| \, dx .$$

For $x \geq 1 + \delta > 1$ we have by formulas (14) and (15) in §2.3.2 of [3]

$$2F_1\left(l + \frac{1}{2}, l + 1; l + \frac{1}{2} + it; 2l + 1; 1 - x^{-2}\right) \ll_{l, \delta} |t|^{-\frac{1}{2} - l} .$$

This gives a bound

$$O_l,\delta(\|k^{(l)}\|_\infty (x(V_1) - x(U_1)))$$

$$\ll \|k^{(l)}\|_\infty (V_1 - U_1)\left(1 + \frac{1}{\sqrt{V_1 + V_1^2 + U_1 + U_1^2}}\right)$$

$$\ll \|k^{(l)}\|_\infty (V_1 - U_1) V_1^{-\frac{1}{2}} .$$
We stress that the use of $\delta$ is critical for the application of the asymptotic behavior from loc. cit. If we allow $x$ to get down to 1 the implicit constant blows up.

For an interval $[U_2, V_2]$ with $x(U_2) \geq 2$ we can use the second integral in (94). The hypergeometric series shows that

$$\left| \frac{\Gamma(1/2)}{\Gamma(1)} \right| \leq 2F_1 \left( \frac{1}{2} + l, \frac{1}{2} + l; 1 \right) (1 - x^2)^{-1},$$

which is $O(x)$ for $x \geq 2$. By Stirling’s formula we get an estimate by

$$O \left( \frac{\Gamma(x)}{\Gamma(x - 1)} \right) \ll \int_{x(U_2)}^{x(V_2)} x^{1/2 + i} \, dx \ll \int_{x(U_2)}^{x(V_2)} x^{1/2 + i} \, dx \ll \int_{x(U_2)}^{x(V_2)} x^{1/2 + i} \, dx.$$

If $U = 0$ we have only to estimate the integral over $[V - Y, V + Y]$. This gives the bound $\frac{\Gamma(x)}{\Gamma(x - 1)} \max(V^{-\frac{1}{2}}, V^{1/2})$. If $U \geq \delta$, we get from the interval $[U - Y, U + Y]$ the bound $\frac{\Gamma(x)}{\Gamma(x - 1)} \max(U^{-\frac{1}{2}}, U^{1/2})$. Together with the bound for the interval $[V - Y, V + Y]$ we get the other bound in c) of the lemma.

\[ \square \]

7. Spectral theory

7.1. Spectral expansion. The pointwise convergence of the spectral expansion of sufficiently differentiable elements of $L^2(\Gamma \backslash \mathbb{H}^d)$ in Theorem 4.1 is similar to well known facts for the case $d = 1$ (e.g., Theorems 4.7 and 7.4 in [11]), and for rank-one Lie groups (Lemma 2.2 in [19]). We sketch how to obtain the pointwise convergence in Theorem 4.1 in the present context.

We consider first the mechanism of the Selberg transform on $\mathbb{H}$, given in §4.1. We replace $k \in C_0^\infty(0, \infty)$ by

$$r_s(u) = \frac{\Gamma(2s)}{4\pi^2} u^{1/2} F_1(s, s; 2s; -1/u),$$

with $\Re s > 1$. It has a logarithmic singularity at $u = 0$ and its support is not compact. Nevertheless, it determines a kernel function $(z, w) \mapsto r_s(u(z, w))$ on $\mathbb{H}$ such that the corresponding convolution operator $R_s : f \mapsto R_s f$ is well defined for bounded $f \in C_0^\infty(\mathbb{H})$. It is in fact the free space resolvent on $\mathbb{H}$, and satisfies

$$R_s(\Delta - s^2) f = f.$$ 

See §1.9 of [11]. If $a, s \in \mathbb{C}$ both have real part larger than 1, the difference $r_s - r_{s,a}$ has no singularity at $u = 0$, and has the Selberg transform

$$h_{s,a}(t) = \frac{s - s^2 - a + a^2}{(t^2 + (s - \frac{1}{2})^2)(t^2 + (a - \frac{1}{2})^2)}$$

for $|t| < \Re s - \frac{1}{2}$. Moreover, the resolvent equation gives on bounded functions in $C_0^\infty(\mathbb{H})$ such that $\Delta f$ and $\Delta^2 f$ are also bounded:

$$L_{s,a}(\Delta - s^2)(\Delta - a + a^2) f = (R_s - R_a)(\Delta - s^2)(\Delta - a + a^2) f = (s - s^2 - a + a^2) f.$$
Taking \( s, a \in \mathbb{C}^d \) with \( \text{Re} \ s_j > 1, \text{Re} a_j > 1, s_j \neq a_j \) for all \( j \), we form

\[
k_{s,a}(u(z, w)) = \prod_j r_{s_j,a_j}(u(z_j, w_j)) = \prod_j (r_{s_j}(u(z_j, w_j)) - r_{a_j}(u(z_j, w_j)) ,
\]

and obtain a kernel operator \( \mathcal{K}_{s,a} \) on \( \Gamma \setminus \mathbb{H}^d \) given by the kernel function

\[
R_{s,a}(z, w) = \sum_{y \in T} k_{s,a}(u(yz, w)) .
\]

We apply this operator to differentiable bounded functions \( f \) on \( \Gamma \setminus \mathbb{H}^d \) for which the derivatives \( \Delta_1^{b_1} \cdots \Delta_d^{b_d} f \) are bounded for all choices \( b_j \in \{0, 1, 2\} \).

\[
f = \prod_j \frac{1}{(s_j - s_j^2 - a_1 + a_1^2) \cdots (s_j - s_j^2 - a_d + a_d^2)} \mathcal{K}_{s,a} f_1 ,
\]

\[
f_1 = (\Delta_1 - s_1 + s_1^2) \cdots (\Delta_d - s_d + s_d^2)(\Delta_1 - a_1 + a_1^2) \cdots (\Delta_d - a_d + a_d^2) f .
\]

Now we note that the values \( \mathcal{K}_{s,a} f(z) \) are given by a scalar product in \( L^2(\Gamma \setminus \mathbb{H}^d) \):

\[
\mathcal{K}_{s,a} f(z) = \langle f_1, R_{s,a}(z, \cdot) \rangle = \langle f_1, R_{s,a}(z, \cdot) \rangle .
\]

Taking the scalar product is a continuous operation \( L^2(\Gamma \setminus \mathbb{H}^d) \rightarrow \mathbb{C} \). Thus, using (86) and (97) we conclude that the \( L^2 \)-expansion of \( f_1 \) is transformed in a pointwise expansion:

\[
\mathcal{K} f_1(z) = \sum_{\ell} h_{s,a}(t_\ell) \psi_\ell(z) a_{\ell}^{(1)}
\]

\[
+ \sum_k 2c_k \sum_{\mu \in \mathbb{Z}^d} \int_0^\infty \overline{h_{s,a}(t + \mu)} E(\kappa; it, i\mu; z) b_{\mu,k}^{(1)}(t) dt ;
\]

\[
h_{s,a}(t) = \prod_j \frac{s_j - s_j^2 - a_j + a_j^2}{(t_j^2 + (s_j - \frac{1}{2})^2)(t_j^2 + (a_j - \frac{1}{2})^2)} ,
\]

where \( a_{\ell}^{(1)} = \langle f_1, \psi_\ell \rangle \) and \( b_{\mu,k}^{(1)}(t) = \int_{\Gamma \setminus \mathbb{H}^d} f_1(z)E(\kappa; it, i\mu; z) d\mu(z) \). Since \( R_{s,a}(z, w) \) is bounded for \( z \in \mathbb{H}^d \) uniformly in \( z \) in compact sets, the convergence of the expansion (99) is also uniform on compact sets.

For two times differentiable functions in \( L^2(\Gamma \setminus \mathbb{H}^d) \) application of \( \Delta_j \) changes \( a_{\ell} \) in the spectral expansion into \( (\frac{1}{4} - \mu_{\ell,j}^2) a_{\ell} \), and \( b_{\mu,k}(t) \) into \( (\frac{1}{4} + (t + \mu_j)^2) b_{\mu,k}(t) \). Taking this into account, the pointwise spectral expansion of \( f \) takes the form

\[
f(z) = \sum_{\ell} \psi_\ell(z) a_{\ell} + \sum_k 2c_k \sum_{\mu \in \mathbb{Z}^d} \int_0^\infty E(\kappa; it, i\mu; z) b_{\mu,k}(t) dt .
\]

Now we turn to the sum in (23) defining \( K(z, w) \). The sum is locally finite in \( w \), uniform for \( z \) in a fixed compact set, and defines a smooth bounded differentiable function \( f : w \mapsto K(z, w) \) with compact support modulo \( \Gamma \). Its derivatives are bounded, uniform for \( z \) in compact sets, hence its spectral expansion in \( w \) converges pointwise.
7.2. **Spectral measure.** To prove Theorem 4.2 we use the estimate of the counting function in Lemma 3.2 and apply the mechanism of the Selberg transform.

We take \( k_j \in C^\infty[0, \infty) \) that satisfy \( 0 \leq k_j \leq 1 \), \( k_j = 1 \) on \([0, \eta_j]\) and \( k_j = 0 \) on \([\delta_j, \infty)\) for quantities \( \eta_j = (\eta_j)_j \) and \( \delta_j = (\delta_j)_j \) with \( 0 < \eta_j < \delta_j < \frac{1}{2} \) to be chosen later. We form \( k \) and \( K \) as in (25) and (28). We have seen in (29) that \( K(z, \cdot) \in L^2(\Gamma \backslash \mathbb{H}^d) \). We shall give two inequalities in which the norm \( \|K(z, \cdot)\|_2 \) occurs.

We have

\[
\|K(z, \cdot)\|_2^2 = \int_{\Gamma \backslash \mathbb{H}^d} |K(z, w)|^2 \, d\mu(w) \\
= \sum_{\gamma, \delta} \int \gamma z, w, k(\delta z, w) \, d\mu(w) \\
= \sum_{\gamma, \delta} \int \gamma^-1 z, \delta^-1 w, k(z, \delta^-1 w) \, d\mu(w) \\
= \sum_{\gamma} \int \gamma z, w, k(z, w) \, d\mu(w).
\]

The second factor restricts the domain of integration to \( w \) with \( u(z_j, w_j) \leq \delta_j \) for all \( j \), and the first factor to \( w \) with \( u((\gamma z)_j, w_j) \leq \delta_j \) for all \( j \). For the hyperbolic distances this means that \( d((\gamma z)_j, z_j) \leq 2 \nu_j \), where \( \nu_j \) corresponds to \( \delta_j \) according to the relation (9). For small values we have \( \delta_j \sim \frac{1}{4} \nu_j^2 \). Hence \( u(\gamma j z_j, z_j) \leq \tilde{\delta}_j \) with \( \tilde{\delta}_j \sim 4 \delta_j \) as \( \delta_j \downarrow 0 \). Hence, with \( \tilde{\delta} = (\tilde{\delta}_j)_j \):

\[
\|K(z, \cdot)\|_2^2 \leq N(z; 0, \tilde{\delta}) \int \tilde{\delta} z, w, d\mu(w) = N(z; 0, \tilde{\delta}) \prod_j h(\frac{1}{4}).
\]

Note that Lemma 6.2a) implies that \( \prod_j h(\frac{1}{4}) \) is a positive quantity between \( (4\pi)^d \prod_j (\delta_j - \eta_j) \) and \( (4\pi)^d \prod_j \delta_j \).

Let \( X \in [1, \infty)^d \). We recall that in (38) we have given a bounded subset of the spectral set depending on \( X \). Theorem 4.1 implies that

\[
\|K(z, \cdot)\|_2^2 = \sum_{\ell} |h(t_\ell)|^2 |\psi_\ell(z)|^2 \\
+ \sum_{k} 2c_k \sum_{\mu \in L_k} \int_0^\infty |h(t + \mu)|^2 |E(\kappa; it, i\mu; z)|^2 \, dt \\
\geq \min\{|h(t)|^2 : t \in Y(X)\} \left( \sum_{\ell, \mu \in Y(X)} |\psi_\ell(z)|^2 \\
+ \sum_{k} 2c_k \sum_{\mu \in L_k} \int_{t \geq 0, (t + i\mu_j) \in Y(X)} |E(\kappa; it, i\mu; z)|^2 \, dt \right). \tag{102}
\]
To get a hold on a lower bound of \( h \) on \( Y(X) \), we use Lemma 6.2 b). For \( \tau \in Y(X) \) it gives
\[
|h(j(\tau)) - h(j(\frac{1}{2})))| \ll \delta_j^{3/2} X_j,
\]
\[
|h(j(\tau))| \geq h(j(\frac{1}{2})) - O(\delta_j^{3/2} X_j) = 4\pi(\delta_j - \eta_j) - O(\delta_j^{3/2} X_j).
\]

We take \( \eta_j = \frac{1}{2} \delta_j \), and \( \delta_j = \varepsilon X_j^{-2} \) with \( \varepsilon > 0 \) sufficiently small to have \( |h(j(\tau))| \geq \delta_j \). This gives
\[
\min\{|h(\tau)|^2 : \tau \in Y(X)\} \geq \frac{1}{(4\pi)^d} \prod_j h(j(\frac{1}{2}))^2.
\]

Thus we obtain from (102) the inequality
\[
(103) \quad \|K(z, \cdot)\|_{L^2}^2 \geq c S(X; z) \prod_j h(j(\frac{1}{2}))^2,
\]
for some positive constant \( c_1 \), which does not depend on \( \Gamma \). If the \( X_j \) are sufficiently large, the \( \delta_j \) and the \( \tilde{\delta}_j \) are sufficiently small to apply Lemma 3.2. By (103) and (101) we get
\[
S(X; z) \leq \frac{1}{c_1} \prod_j h(j(\frac{1}{2}))^2 \frac{N(z; 0, \tilde{\delta})}{\prod_j h(j(\frac{1}{2}))^2} \leq \prod_j \delta_j^{\frac{d-1}{2}} n(\tilde{\delta}^{-1/2}, z) \ll n(X, z) \prod_j X_j^2.
\]

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