THE GENERALISED PÓLYA CONJECTURE FOR THE DIRICHLET EIGENVALUES

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Abstract. In this paper, we prove the Generalized Pólya conjecture for the Dirichlet eigenvalues. In other words, we show that

$$\lambda_k(\alpha) \geq \frac{(2\pi)^\alpha k^{\alpha/n}}{(\omega_n \cdot \text{vol}(\Omega))^{\alpha/n}} \cdot \omega_n \cdot \text{vol}(\Omega)$$

where \(\lambda_k(\alpha)\) is the \(k\)-th Dirichlet eigenvalue for the fractional Laplacian \((-\Delta)^{\alpha/2}\) with \(\alpha \in (0, 2]\) in a bounded domain \(\Omega \subset \mathbb{R}^n\).

1. Introduction

A symmetric \(\alpha\)-stable process \(X = \{X_t, t \geq 0, \mathbb{P}_x, x \in \mathbb{R}^n\}\) of order \(\alpha \in (0, 2]\) in \(\mathbb{R}^n\) is a Lévy process such that

$$E_x[e^{i\xi \cdot (X_t - X_0)}] = e^{-t|\xi|^\alpha}$$

for every \(x \in \mathbb{R}^n\) and \(\xi \in \mathbb{R}^n\).

The infinitesimal generator of a symmetric \(\alpha\)-stable process \(X\) in \(\mathbb{R}^n\) is the fractional Laplacian \((-\Delta)^{\alpha/2}\), which is a prototype of nonlocal operators. The fractional Laplacian can be written in the form

$$(-\Delta)^{\alpha/2} u(x) = c \lim_{\epsilon \to 0} \int_{\{y \in \mathbb{R}^n | |x - y| > \epsilon\}} \frac{u(x) - u(y)}{|x - y|^{n+\alpha}} dy$$

for some constant \(c = c(n, \alpha)\) (see [7]). For every bounded domain \(\Omega \subset \mathbb{R}^n\) with \(n \geq 2\), we denote by \(X^\Omega\) the subprocess of \(X\) killed upon leaving \(\Omega\). The infinitesimal generator of \(X^\Omega\) is the Dirichlet fractional Laplacian \((-\Delta)^{\alpha/2}|_\Omega\) (the fractional Laplacian with zero exterior condition). It is known (see [7]) that \(X^\Omega\) has a transition density \(p(t, x, y)\) with respect to the Lebesgue measure that is jointly Hölder continuous. With the aid of the Fourier transform, fractional Laplacians restricted to \(\Omega\) can also be defined to be a pseudodifferential operator as follows

$$(-\Delta)^{\alpha/2}|_\Omega u := \mathcal{F}^{-1}[|\xi|^\alpha \mathcal{F}[u\chi_\Omega]]$$

1991 Mathematics Subject Classification. 35P15, 58C40, 58J50, 65N25.
Key words and phrases. Dirichlet eigenvalue, Generalised Pólya conjecture.
Here \( x \to \chi_\Omega(x) \) stands for the characteristic function defined to be 1 when \( x \in \Omega \) and 0 when \( x \notin \Omega \), and \( \mathcal{F}[u] \) denotes the Fourier transform of a function \( u : \mathbb{R}^n \to \mathbb{R} \) and is defined by
\[
\mathcal{F}[u](\xi) = \hat{u}(\xi) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i(x,\xi)} u(x) \, dx.
\]
We refer the reader to the book [9] or articles [13] for the proof of the equivalence between (1.2) and (1.3).

Two important examples of symmetric \( \alpha \)-stable processes are Brownian motion, which is obtained by setting \( \alpha = 2 \), and the Cauchy process, which is obtained by setting \( \alpha = 1 \). In addition, the transition density in the case of the Brownian motion is given by
\[
p_2(t, x, y) = \frac{1}{(4\pi t)^{n/2}} e^{-|x-y|^2/(4t)}, \quad t > 0, \ x, y \in \mathbb{R}^n,
\]
and the transition density in the case of the Cauchy process is represented by
\[
p_1(t, x, y) = \frac{c_n}{(t^2 + |x-y|^2)^{(n+2)/2}}, \quad t > 0, \ x, y \in \mathbb{R}^n
\]
where \( c_n = \pi^{-\frac{n+1}{2}} \Gamma \left( \frac{n+1}{2} \right) \) is the semiclassical constant that appears in the Weyl’s law for the eigenvalues of the Laplacian. Therefore, the infinitesimal generator of the Brownian motion for \( \alpha = 2 \) and the Cauchy process with the corresponding killing condition on \( \partial \Omega \) is \((-\Delta)^{\alpha/2}|_\Omega\). For more interesting results involving stable processes and Cauchy processes, please refer to the papers [18], [3], [4], [5], [6], [7] and references therein.

We consider the following Dirichlet fractional Laplacian eigenvalue problem:
\[
(1.4) \left\{ \begin{array}{ll}
(-\Delta)^{\alpha/2} u = \lambda u & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega.
\end{array} \right.
\]
As is well-known, for fixed \( \alpha \in (0, 2] \), the spectrum of the \((-\Delta)^{\alpha/2}|_\Omega\) with Dirichlet boundary condition is discrete and consists of a sequence \( \{\lambda_k(\alpha)\}_{k=1}^\infty \) of eigenvalues (with finite multiplicity) written in increasing order according to their multiplicity (see, for example, [6]):
\[
0 < \lambda_1(\alpha) < \lambda_2(\alpha) \leq \cdots \leq \lambda_k(\alpha) \leq \cdots \nearrow +\infty.
\]
Weyl’s asymptotic formula for the fractional Laplacian with Dirichlet boundary condition (see [2] or [11]) says that
\[
(1.5) \quad \lambda_k \sim \frac{(2\pi)^\alpha k^{\alpha/n}}{(\omega_n \cdot \text{vol}(\Omega))^{\alpha/n}} \quad \text{as } k \to \infty,
\]
where \( \omega_n = \frac{n^{n/2}}{\Gamma(1+\frac{n}{2})} \) is the volume of the unit ball in \( \mathbb{R}^n \) and \( \text{vol}(\Omega) \) is the volume of \( \Omega \). In 1961, G. Pólya [12] (see also [11]) conjectured that for \( \alpha = 2 \), the following inequality holds
\[
\lambda_k \geq \frac{(2\pi)^2 k^{2/n}}{(\omega_n \cdot \text{vol}(\Omega))^{2/n}}, \quad \text{for } k = 1, 2, 3, \cdots.
\]

In this paper, by considering an equivalent eigenvalue problem of a multiple fractional Laplacian \((-\Delta)^{\alpha/2}|_\Omega)^m\) with Dirichlet boundary conditions and by applying Li-Yau’s technique (see [10]) to such operator, and finally by letting \( m \to +\infty \), we prove the generalized Pólya conjecture for the Dirichlet eigenvalues. Our main result is the following:

**Theorem 1.1.** Let \( \Omega \subset \mathbb{R}^n \) (\( n \geq 2 \)) be a bounded domain, and let \( 0 < \lambda_1(\alpha) < \lambda_2(\alpha) \leq \lambda_3(\alpha) \leq \cdots \leq \lambda_k(\alpha) \leq \cdots \) be the Dirichlet eigenvalues of the fractional Laplacian \((-\Delta)^{\alpha/2}|_\Omega\)
with \( \alpha \in (0.2] \). Then
\[
\lambda_k(\alpha) \geq \frac{(2\pi)^{\alpha} k^{\alpha/n}}{(\omega_n \cdot \text{vol}(\Omega))^{\alpha/n}}, \quad \text{for } k = 1, 2, 3, \ldots.
\]

**Remark 1.2** In particular, by taking \( \alpha = 2 \) in (1.8) we have
\[
\lambda_k \geq (2\pi)^2 \left( \frac{1}{\omega_n \cdot \text{vol}(\Omega)} \right)^{2/n} k^{2/n}, \quad \text{for } k = 1, 2, 3, \ldots,
\]
i.e., the Pólya conjecture is true for the Dirichlet eigenvalues (see also \( \text{[8]} \)).

## 2. Proof of main theorem

**Lemma 2.1.** Let \( f : \mathbb{R}^n \to [0, \infty) \) be a real valued nonnegative function in \( L^\infty(\mathbb{R}^n) \). Assume that there exists a number \( M > 0 \) such that
\[
\int_{\mathbb{R}^n} |\xi|^{m\alpha} f(\xi) d\xi \leq M.
\]
Then, \( f \in L^1(\mathbb{R}^n) \) and
\[
\|f\|_{L^1(\mathbb{R}^n)} \leq \left( \frac{\|f\|_{L^\infty(\mathbb{R}^n)}}{\Gamma(1 + n/2)} \right)^{m\alpha} \left( \frac{n + m\alpha}{n} M \right)^{\frac{n}{n+m\alpha}}.
\]

**Proof.** The proof is similar to that of \( \text{[18]} \) just by replacing \( \alpha \) by \( m\alpha \). \( \Box \)

**Proof of theorem 1.1.** (i) We first assume that \( \Omega \) is a bounded domain with smooth boundary. In this case, the eigenvalue problem (1.4) is equivalent to the following eigenvalue problem of the multiple fractional Laplacian:
\[
\begin{cases}
((-\Delta)^{\alpha/2})_\Omega^m u = (\lambda(\alpha))^m u & \text{in } \Omega, \\
((-\Delta)^{\alpha/2})_\Omega^{l-1} u = 0 & \text{on } \partial \Omega, \ l = 1, 2, \ldots, m,
\end{cases}
\]
where \( m = 1, 2, 3, \ldots \). In other words, if \( u_k \) is the eigenfunction corresponding to the \( k \)-th Dirichlet eigenvalue \( \lambda_k(\alpha) \), then \( (\lambda_k(\alpha))^m \) (respectively, eigenfunction \( u_k \)) must be the \( k \)-th eigenvalue (respectively, eigenfunction \( u_k \)) of problem (2.3). The converse is also true.

Let \( \{u_j\}_{j=1}^\infty \) be the set of orthonormal eigenfunctions in \( L^2(\Omega) \) corresponding to the Dirichlet fractional Laplacian eigenvalues \( \{\lambda_j(\alpha)\}_{j=1}^\infty \). We extend the eigenfunction \( u_j \) such that \( u_j \) is defined in \( \mathbb{R}^n \) by letting \( u_j(x) = 0 \) when \( x \in \mathbb{R}^n \setminus \Omega \). By using Plancherel’s theorem, the set of Fourier transform \( \{\hat{u}_j\}_{j=1}^\infty \) of eigenfunctions \( \{u_j\}_{j=1}^\infty \) also forms an orthonormal set in \( L^2(\mathbb{R}^n) \).

Set
\[
U_k(\xi) := \sum_{j=1}^k |\hat{u}_j(\xi)|^2 = \sum_{j=1}^k \left| \frac{1}{(2\pi)^{n/2}} \int_{\Omega} e^{-i(x, \xi)} u_j(x) dx \right|^2.
\]
Notice that the integral is taken over \( \Omega \) instead of \( \mathbb{R}^n \) because the support of \( u_j \) is \( \Omega \). Interchanging the sum and integral and using \( \|\hat{u}_j\|_2 = 1 \), we get
\[
\int_{\mathbb{R}^n} U_k(\xi) d\xi = k.
\]
Note that for each fixed \( \xi \in \mathbb{R}^n \),
\[
e^{-i(x,\xi)} = \sum_{j=1}^{\infty} \left( \int_{\Omega} e^{-i(x,\xi)} u_j(x) dx \right) u_j(x)
\]
in the sense of the \( L^2 \)-norm. By the Bessel inequality of the Fourier series, we get an upper bound for \( U_k(\xi) \):
\[
U_k(\xi) = \sum_{j=1}^{k} \left| \frac{1}{(2\pi)^{n/2}} \int_{\Omega} e^{-i(x,\xi)} u_j(x) dx \right|^2 \leq \frac{1}{(2\pi)^n} \int_{\Omega} |e^{-i(x,\xi)}|^2 dx = \frac{\text{vol}(\Omega)}{(2\pi)^n}.
\]
Since the support of \( u_j \) is \( \Omega \), we have that
\[
(\lambda_j(\alpha))^m = \langle u_j, \left(-\Delta\right)^{\alpha/2} \mid _{\Omega}^m \rangle = \langle u_j, \mathcal{F}^{-1} |[|\xi|^m \mathcal{F}[u_j]|] \rangle = \int_{\mathbb{R}^n} |\xi|^m |\hat{u}_j(\xi)|^2 d\xi.
\]
Put
\[
f(\xi) := U_k(\xi) = \sum_{j=1}^{k} |\hat{u}_j(\xi)|^2, \quad \xi \in \mathbb{R}^n.
\]
It is easy to see that
\[
k = \int_{\mathbb{R}^n} f(\xi) d\xi \quad \text{and} \quad \sum_{j=1}^{k} (\lambda_j(\alpha))^m = \int_{\mathbb{R}^n} |\xi|^m f(\xi) d\xi = M.
\]
Applying (2.2) of Lemma 2.1, we obtain that
\[
k \leq \left( \frac{\pi^{\frac{n}{2}} \|f\|_{L^\infty(\mathbb{R}^n)}}{\Gamma(1 + n/2)} \right)^{\frac{1}{n + m\alpha}} \frac{n + m\alpha}{n} \left( \sum_{j=1}^{k} (\lambda_j(\alpha))^m \right)^{\frac{n}{n + m\alpha}},
\]
i.e.,
\[
\sum_{j=1}^{k} (\lambda_j(\alpha))^m \geq \frac{n}{n + m\alpha} \left( \frac{\pi^{\frac{n}{2}} \|f\|_{L^\infty(\mathbb{R}^n)}}{\pi^{\frac{n}{2}} \text{vol}(\Omega)} \right)^{\frac{m\alpha}{n}} k^{1 + \frac{m\alpha}{n}}.
\]
According (2.5), we find that for any \( \xi \in \mathbb{R}^n \),
\[
\|f\|_{L^\infty(\mathbb{R}^n)} = \max_{\xi \in \mathbb{R}^n} U_k(\xi) \leq \frac{\text{vol}(\Omega)}{(2\pi)^n}.
\]
Inserting this into (2.7) we get
\[
\sum_{j=1}^{k} (\lambda_j(\alpha))^m \geq \frac{n}{n + m\alpha} \left( \frac{(2\pi)^n \Gamma(1 + \frac{n}{2})}{\pi^{\frac{n}{2}} \text{vol}(\Omega)} \right)^{\frac{m\alpha}{n}} k^{1 + \frac{m\alpha}{n}}.
\]
Because of \( (\lambda_j(\alpha))^m \leq (\lambda_k(\alpha))^m \) for all \( 1 \leq j \leq k \), it follows from (2.8) that
\[
k (\lambda_k(\alpha))^m \geq \frac{n}{n + m\alpha} \left( \frac{(2\pi)^n \Gamma(1 + \frac{n}{2})}{\pi^{\frac{n}{2}} \text{vol}(\Omega)} \right)^{\frac{m\alpha}{n}} k^{1 + \frac{m\alpha}{n}}.
\]
Consequently, we obtain
\begin{equation}
\lambda_k(\alpha) \geq \left( \frac{n}{n + m\alpha} \right)^{1/m} \left( \frac{(2\pi)^n \Gamma(1 + \frac{n}{2})}{\pi^{\frac{n}{2}} \vol(\Omega)} \right)^{\frac{1}{k^\alpha}} \frac{\alpha}{k}.
\end{equation}

Note that \( \lim_{m \to +\infty} \left( \frac{n}{n + m\alpha} \right)^{1/m} = 1 \). From (2.9) we find by letting \( m \to +\infty \) that for each \( \alpha \in (0, 2] \),
\begin{equation}
\lambda_k(\alpha) \geq \left( \frac{2\pi}{\omega_n \cdot \vol(\Omega)} \right)^{\alpha/n} = 1.\end{equation}

(ii) For any bounded domain, we can choose a sequence \( \{\Omega_l\}_{l=1}^\infty \) of bounded domains with smooth boundaries such that \( \Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_p \subset \cdots \subset \Omega \) and \( \Omega_p \) converges to \( \Omega \) as \( p \to +\infty \). By this property and inequality (2.10), we conclude that the inequality (1.7) still holds for arbitrary bounded domain.

Acknowledgments
This research was supported by SRF for ROCS, SEM (No. 2004307D01) and NNSF of China (11171023/A010801).

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