Remark on the boundedness of the Cauchy singular integral operator on variable Lebesgue spaces with radial oscillating weights

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To Professor Kokilashvili on his seventieth birthday

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Abstract. Recently V. Kokilashvili, N. Samko, and S. Samko have proved a sufficient condition for the boundedness of the Cauchy singular integral operator on variable Lebesgue spaces with radial oscillating weights over Carleson curves. This condition is formulated in terms of Matuszewska-Orlicz indices of weights. We prove a partial converse of their result.

1. Introduction and main result

Let \( \Gamma \) be a rectifiable curve in the complex plane. We equip \( \Gamma \) with Lebesgue length measure \( |d\tau| \). We say that a curve \( \Gamma \) is simple if it does not have self-intersections. In other words, \( \Gamma \) is said to be simple if it is homeomorphic either to a line segment or to a circle. In the latter case
situation we will say that \( \Gamma \) is a Jordan curve. The Cauchy singular integral of \( f \in L^1(\Gamma) \) is defined by

\[
(Sf)(t) := \frac{1}{\pi i} \int_{\Gamma} \frac{f(\tau)}{\tau - t} \, d\tau \quad (t \in \Gamma).
\]

This integral is understood in the principal value sense, that is,

\[
\int_{\Gamma} \frac{f(\tau)}{\tau - t} \, d\tau := \lim_{R \to 0} \int_{\Gamma \setminus \Gamma(t,R)} \frac{f(\tau)}{\tau - t} \, d\tau,
\]

where \( \Gamma(t,R) := \{ \tau \in \Gamma : |\tau - t| < R \} \) for \( R > 0 \). David [4] (see also [3, Theorem 4.17]) proved that the Cauchy singular integral generates the bounded operator \( S \) on the Lebesgue space \( L^p(\Gamma) \), \( 1 < p < \infty \), if and only if \( \Gamma \) is a Carleson (Ahlfors-David regular) curve, that is,

\[
\sup_{t \in \Gamma} \sup_{R > 0} \frac{\left| \Gamma(t,R) \right|}{R} < \infty,
\]

where for any measurable set \( \Omega \subset \Gamma \) the symbol \( |\Omega| \) denotes its measure. To have a better idea about Carleson curves, consider the following example.

Let \( \alpha > 0 \) and

\[
\Gamma := \{0\} \cup \{ \tau \in \mathbb{C} : \tau = x + ix^\alpha \sin(1/x), \ 0 < x \leq 1 \}.
\]

One can show (see [3, Example 1.3]) that \( \Gamma \) is not rectifiable for \( 0 < \alpha \leq 1 \), \( \Gamma \) is rectifiable but not Carleson for \( 1 < \alpha < 2 \), and \( \Gamma \) is a Carleson curve for \( \alpha \geq 2 \).

A measurable function \( w : \Gamma \to [0, \infty] \) is referred to as a weight function or simply a weight if \( 0 < w(\tau) < \infty \) for almost all \( \tau \in \Gamma \). Suppose \( p : \Gamma \to [1, \infty] \) is a measurable a.e. finite function. Denote by \( L^{p(\cdot)}(\Gamma, w) \) the set of all measurable complex-valued functions \( f \) on \( \Gamma \) such that

\[
\int_{\Gamma} |f(\tau)w(\tau)/\lambda|^{p(\tau)} \, d\tau < \infty
\]

for some \( \lambda = \lambda(f) > 0 \). This set becomes a Banach space when equipped with the Luxemburg-Nakano norm

\[
\|f\|_{p(\cdot), w} := \inf \left\{ \lambda > 0 : \int_{\Gamma} |f(\tau)w(\tau)/\lambda|^{p(\tau)} \, d\tau \leq 1 \right\}.
\]

If \( p \) is constant, then \( L^{p(\cdot)}(\Gamma, w) \) is nothing else but the weighted Lebesgue space. Therefore, it is natural to refer to \( L^{p(\cdot)}(\Gamma, w) \) as a weighted generalized Lebesgue space with variable exponent or simply as a weighted
variable Lebesgue space. This is a special case of Musielak-Orlicz spaces [19]
(see also [13]). Nakano [20] considered these spaces (without weights) as
examples of so-called modular spaces, and sometimes the spaces $L^{p(\cdot)}(\Gamma, w)$
are referred to as weighted Nakano spaces.

Following [12, Section 2.3], denote by $W$ the class of all continuous
functions $\varphi : [0, |\Gamma|] \to [0, \infty)$ such that $\varphi(0) = 0$, $\varphi(x) > 0$ if $0 < x \leq |\Gamma|$, and $\varphi$ is almost increasing, that is, there is a universal constant $C > 0$
such that $\varphi(x) \leq C \varphi(y)$ whenever $x \leq y$. Further, let $\mathbb{W}$ be the set of all functions $\varphi : [0, |\Gamma|] \to [0, \infty)$ such that $x^\alpha \varphi(x) \in W$ and $x^{\beta}/\varphi(x) \in W$
for some $\alpha, \beta \in \mathbb{R}$. Clearly, the functions $\varphi(x) = x^\gamma$ belong to $\mathbb{W}$ for all $\gamma \in \mathbb{R}$. For $\varphi \in \mathbb{W}$, put

$$\Phi_\varphi^0(x) := \limsup_{y \to 0} \frac{\varphi(xy)}{\varphi(y)}, \quad x \in (0, \infty).$$

Since $\varphi \in \mathbb{W}$, one can show that the limits

$$m(\varphi) := \lim_{x \to 0} \frac{\log \Phi_\varphi^0(x)}{\log x}, \quad M(\varphi) := \lim_{x \to \infty} \frac{\log \Phi_\varphi^0(x)}{\log x}$$

exist and $-\infty < m(\varphi) \leq M(\varphi) < +\infty$. These numbers were defined by
Matuszewska and Orlicz [17, 18] (see also [15] and [16, Chapter 11]). We
refer to $m(\varphi)$ (resp. $M(\varphi)$) as the lower (resp. upper) Matuszewska-Orlicz
index of $\varphi$. For $\varphi(x) = x^\gamma$ one has $m(\varphi) = M(\varphi) = \gamma$. Examples of
functions $\varphi \in \mathbb{W}$ with $m(\varphi) < M(\varphi)$ can be found, for instance, in [1], [16, p. 93], [21, Section 2].

Fix pairwise distinct points $t_1, \ldots, t_n \in \Gamma$ and functions $w_1, \ldots, w_n \in \mathbb{W}$.
Consider the following weight

$$(1.1) \quad w(t) := \prod_{k=1}^n w_k(|t - t_k|), \quad t \in \Gamma.$$  

Each function $w_k(|t - t_k|)$ is a radial oscillating weight. The weight (1.1) is a
continuous function on $\Gamma \setminus \{t_1, \ldots, t_n\}$. This is a natural generalization of
so-called Khvedelidze weights $w(t) = \prod_{k=1}^n |t - t_k|^\lambda_k$, where $\lambda_k \in \mathbb{R}$ (see,
e.g., [3, Section 2.2], [9], [10]). Recently V. Kokilashvili, N. Samko, and S.
Samko have proved the following (see [12, Theorem 4.3] and also [11] for similar results for maximal functions).

**Theorem 1.1** ([12, Theorem 4.3]). Suppose $\Gamma$ is a simple Carleson curve
and $p : \Gamma \to (1, \infty)$ is a continuous function satisfying

$$(1.2) \quad |p(\tau) - p(t)| \leq A_T/\log |\tau - t| \quad \text{whenever} \quad |\tau - t| \leq 1/2,$$
where \( A_\Gamma \) is a positive constant depending only on \( \Gamma \). Let \( w_1, \ldots, w_n \in \mathbb{W} \) and the weight \( w \) be given by (1.1). If
\[
0 < 1/p(t_k) + m(w_k), \quad 1/p(t_k) + M(w_k) < 1 \quad \text{for all} \quad k \in \{1, \ldots, n\},
\]
then the Cauchy singular integral operator \( S \) is bounded on \( L^{p(.)}(\Gamma, w) \).

For the weight \( w(t) = \prod_{k=1}^n |t - t_k|^\lambda_k \), (1.3) reads as \( 0 < 1/p(t_k) + \lambda_k < 1 \) for all \( k \in \{1, \ldots, n\} \). This condition is also necessary for the boundedness of \( S \) on the variable Lebesgue space \( L^{p(.)}(\Gamma, w) \) with the Khvedelidze weight \( w \) (see [10]).

The author have proved in [8] that for Jordan curves condition (1.3) is necessary for the boundedness of the operator \( S \).

**Theorem 1.2** ([8, Corollary 4.3]). Suppose \( \Gamma \) is a rectifiable Jordan curve and \( p : \Gamma \to (1, \infty) \) is a continuous function satisfying (1.2). Let \( w_1, \ldots, w_n \in \mathbb{W} \) and the weight \( w \) be given by (1.1). If the Cauchy singular integral operator \( S \) is bounded on \( L^{p(.)}(\Gamma, w) \), then \( \Gamma \) is a Carleson curve and (1.3) is fulfilled.

The proof of this result given in [8] essentially uses that \( \Gamma \) is closed. In this paper we embark on the situation of non-closed curves. Our main result is a partial converse of Theorem 1.1. It follows from our results [6, 8] based on further development of ideas from [3, Chap. 1–3].

**Theorem 1.3** (Main result). Let \( \Gamma \) be a rectifiable curve homeomorphic to a line segment and \( p : \Gamma \to (1, \infty) \) be a continuous function satisfying (1.2). Suppose \( w_1, \ldots, w_n \in \mathbb{W} \) and the weight \( w \) is given by (1.1). If the Cauchy singular integral operator \( S \) is bounded on \( L^{p(.)}(\Gamma, w) \), then \( \Gamma \) is a Carleson curve and
\[
0 \leq 1/p(t_k) + m(w_k), \quad 1/p(t_k) + M(w_k) \leq 1 \quad \text{for all} \quad k \in \{1, \ldots, n\}.
\]
Moreover, if there exists an \( \varepsilon_0 > 0 \) such that the Cauchy singular integral operator \( S \) is bounded on \( L^{p(.)}(\Gamma, w^{1+\varepsilon}) \) for all \( \varepsilon \in (-\varepsilon_0, \varepsilon_0) \), then
\[
0 < 1/p(t_k) + m(w_k), \quad 1/p(t_k) + M(w_k) < 1 \quad \text{for all} \quad k \in \{1, \ldots, n\}.
\]

For standard Lebesgue spaces, the boundedness of the operator \( S \) on \( L^p(\Gamma, w) \), \( 1 < p < \infty \), implies that \( S \) is also bounded on \( L^{p(1+\varepsilon)}(\Gamma, w^{1+\varepsilon}) \) for all \( \varepsilon \) in a sufficiently small neighborhood of zero (see [3, Theorems 2.31 and 4.15]). Hence if \( 1 < p < \infty \), \( \Gamma \) is a simple Carleson curve, \( w_1, \ldots, w_n \in \mathbb{W} \), and the weight \( w \) is given by (1.1), then \( S \) is bounded on the standard Lebesgue space \( L^p(\Gamma, w) \), \( 1 < p < \infty \), if and only if
\[
0 < 1/p + m(w_k), \quad 1/p + M(w_k) < 1 \quad \text{for all} \quad k \in \{1, \ldots, n\}.
\]
We believe that all weighted variable Lebesgue spaces have this stability property.

**Conjecture 1.4.** Let $\Gamma$ be a simple rectifiable curve, $p : \Gamma \to [1, \infty]$ be a measurable a.e. finite function, and $w : \Gamma \to [0, \infty]$ be a weight such that the Cauchy singular integral operator $S$ is bounded on $L^{p(\cdot)}(\Gamma, w)$. Then there is a number $\varepsilon_0 > 0$ such that $S$ is bounded on $L^{p(\cdot)}(\Gamma, w^{1+\varepsilon})$ for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$.

If this conjecture would be true, we were able to prove the complete converse of Theorem 1.1 for non-closed curves, too.

## 2. Proof

In this section we formulate several results from [3, 6, 8] and show that Theorem 1.3 easily follows from them.

### 2.1 Muckenhoupt type condition.

Suppose $\Gamma$ is a simple rectifiable curve and $p : \Gamma \to (1, \infty)$ is a continuous function. Since $\Gamma$ is compact, one has

$$1 < \min_{\tau \in \Gamma} p(\tau), \quad \max_{\tau \in \Gamma} p(\tau) < \infty$$

and the conjugate exponent

$$q(\tau) := p(\tau) / (p(\tau) - 1) \quad (\tau \in \Gamma)$$

is well defined and also bounded and bounded away from zero. We say that a weight $w : \Gamma \to [0, \infty]$ belongs to $A_{p(\cdot)}(\Gamma)$ if

$$\sup_{t \in \Gamma} \sup_{R > 0} \frac{1}{R} \|w \chi_{(t, R)}\|_{p(\cdot)} \|w^{-1} \chi_{(t, R)}\|_{q(\cdot)} < \infty.$$ 

If $p = \text{const} \in (1, \infty)$, then this class coincides with the well known Muckenhoupt class. From the Hölder inequality for $L^{p(\cdot)}(\Gamma)$ (see e.g. [19, Theorems 13.12 and 13.13] for Musielak-Orlicz spaces over arbitrary measure spaces and also [13, Theorem 2.1] for variable Lebesgue spaces over domains in $\mathbb{R}^n$) it follows that if $w \in A_{p(\cdot)}(\Gamma)$, then $\Gamma$ is a Carleson curve.

Since $L^{p(\cdot)}(\Gamma, w)$ is a Banach function space in the sense of [2, Definition 1.1], the next result follows from [6, Theorem 6.1] (stated in [6] for Jordan curves, however its proof remains the same for curves homeomorphic to line segments, see also [7, Theorem 3.2]).

**Theorem 2.1.** Let $\Gamma$ be a simple rectifiable curve and let $p : \Gamma \to (1, \infty)$ be a continuous function. If $w : \Gamma \to [0, \infty]$ is an arbitrary weight such that the operator $S$ is bounded on $L^{p(\cdot)}(\Gamma, w)$, then $w \in A_{p(\cdot)}(\Gamma)$. 

If \( p = \text{const} \in (1, \infty) \), then \( w \in A_p(\Gamma) \) is also sufficient for the boundedness of \( S \) on the weighted Lebesgue space \( L^p(\Gamma, w) \) (see e.g. [3, Theorem 4.15]).

**2.2 Submultiplicative functions.** Following [3, Section 1.4], we say a function \( \Phi : (0, \infty) \to (0, \infty) \) is regular if it is bounded in an open neighborhood of 1. A function \( \Phi : (0, \infty) \to (0, \infty) \) is said to be submultiplicative if

\[
\Phi(xy) \leq \Phi(x)\Phi(y) \quad \text{for all } x, y \in (0, \infty).
\]

It is easy to show that if \( \Phi \) is regular and submultiplicative, then \( \Phi \) is bounded away from zero in some open neighborhood of 1. Moreover, in this case \( \Phi(x) \) is finite for all \( x \in (0, \infty) \). Given a regular and submultiplicative function \( \Phi : (0, \infty) \to (0, \infty) \), one defines

\[
\alpha(\Phi) := \sup_{x \in (0, 1)} \frac{\log \Phi(x)}{\log x}, \quad \beta(\Phi) := \inf_{x \in (1, \infty)} \frac{\log \Phi(x)}{\log x}.
\]

Clearly, \(-\infty < \alpha(\Phi) \) and \( \beta(\Phi) < \infty \).

**Theorem 2.2** (see [3, Theorem 1.13] or [14, Chap. 2, Theorem 1.3]). If a function \( \Phi : (0, \infty) \to (0, \infty) \) is regular and submultiplicative, then

\[
\alpha(\Phi) = \lim_{x \to 0} \frac{\log \Phi(x)}{\log x}, \quad \beta(\Phi) = \lim_{x \to \infty} \frac{\log \Phi(x)}{\log x}
\]

and \(-\infty < \alpha(\Phi) \leq \beta(\Phi) < +\infty \).

The quantities \( \alpha(\Phi) \) and \( \beta(\Phi) \) are called the lower and upper indices of the regular and submultiplicative function \( \Phi \), respectively.

**2.3 Indices of powerlikeness.** Fix \( t \in \Gamma \) and put \( d_t := \max_{\tau \in \Gamma} |\tau - t| \).

Suppose \( w : \Gamma \to [0, \infty] \) is a weight such that \( \log w \in L^1(\Gamma(t, R)) \) for every \( R \in (0, d_t] \). Put

\[
H_{w,t}(R_1, R_2) := \frac{\exp\left(\frac{1}{|\Gamma(t, R_1)|} \int_{\Gamma(t, R_1)} \log w(\tau) d\tau\right)}{\exp\left(\frac{1}{|\Gamma(t, R_2)|} \int_{\Gamma(t, R_2)} \log w(\tau) d\tau\right)}, \quad R_1, R_2 \in (0, d_t],
\]

Consider the function

\[
(V^0_t w)(x) := \lim_{R \to 0} \sup_{R} H_{w,t}(xR, R), \quad x \in (0, \infty).
\]
Combining Lemmas 4.8–4.9 and Theorem 5.9 of [6] with Theorem 3.4, Lemma 3.5 of [3], we arrive at the following.

**Theorem 2.3.** Let \( \Gamma \) be a simple rectifiable curve, \( p : \Gamma \to (1, \infty) \) be a continuous function satisfying (1.2), and \( w : \Gamma \to [0, \infty] \) be a weight such that \( w \in A_p(\cdot)(\Gamma) \). Then, for every \( t \in \Gamma \), the function \( V_t^0 w \) is regular and submultiplicative and

\[
0 \leq 1/p(t) + \alpha(V_t^0 w), \quad 1/p(t) + \beta(V_t^0 w) \leq 1.
\]

The numbers \( \alpha(V_t^0 w) \) and \( \beta(V_t^0 w) \) are called the lower and upper indices of powerlikeness of \( w \) at \( t \in \Gamma \), respectively (see [3, Chap. 3]). This terminology can be explained by the simple fact that for the power weight \( w(\tau) := |\tau - t|^\lambda \) its indices of powerlikeness coincide and are equal to \( \lambda \).

**2.4 Matuszewska-Orlicz indices as indices of powerlikeness.** If \( \varrho \in \mathcal{W} \), then \( \Phi_0^0 \varrho \) is a regular and submultiplicative function and its indices are nothing else but the Matuszewska-Orlicz indices \( m(\varrho) \) and \( M(\varrho) \). The next result shows that for radial oscillating weights indices of powerlikeness and Matuszewska-Orlicz indices coincide.

**Theorem 2.4** (see [8, Theorem 2.8]). Suppose \( \Gamma \) is a simple Carleson curve. If \( w_1, \ldots, w_n \in \mathcal{W} \) and \( w(\tau) = \prod_{k=1}^n w_k(|\tau - t_k|) \), then for every \( t \in \Gamma \) the function \( V_t^0 w \) is regular and submultiplicative and

\[
\alpha(V_t^0 w) = m(w_k), \quad \beta(V_t^0 w) = M(w_k) \quad \text{for} \quad k \in \{1, \ldots, n\},
\]

\[
\alpha(V_t^0 w) = 0, \quad \beta(V_t^0 w) = 0 \quad \text{for} \quad t \in \Gamma \setminus \{t_1, \ldots, t_n\}.
\]

Note that in [8], Theorem 2.4 is proved for Jordan curves. But the proof does not use the assumption that \( \Gamma \) is closed. It works also for non-closed curves considered in this paper.

**2.5 Proof of Theorem 1.3.** Suppose \( S \) is bounded on \( L^{p(\cdot)}(\Gamma, w) \). From Theorem 2.1 it follows that \( w \in A_{p(\cdot)}(\Gamma) \). By Hölder's inequality this implies that \( \Gamma \) is a Carleson curve. Fix an arbitrary \( t \in \Gamma \). Then, in view of Theorems 2.2 and 2.3 the function \( V_t^0 w \) is regular and submultiplicative, so its indices are well defined and satisfy \( 0 \leq 1/p(t) + \alpha(V_t^0 w) \) and \( 1/p(t) + \beta(V_t^0 w) \leq 1 \). From these inequalities and Theorem 2.4 it follows that

\[
0 \leq 1/p(t_k) + m(w_k), \quad 1/p(t_k) + M(w_k) \leq 1
\]

for all \( k \in \{1, \ldots, n\} \).
If $S$ is bounded on all spaces $L^{p(\cdot)}(\Gamma, w^{1+\varepsilon})$ for all $\varepsilon$ in a neighborhood of zero, then as before

$$0 \leq 1/p(t_k) + m(w_k^{1+\varepsilon}), \quad 1/p(t_k) + M(w_k^{1+\varepsilon}) \leq 1$$

for every $k \in \{1, \ldots, n\}$. It is easy to see that $m(w_k^{1+\varepsilon}) = (1 + \varepsilon)m(w_k)$ and $M(w_k^{1+\varepsilon}) = (1 + \varepsilon)M(w_k)$. Therefore

$$0 \leq 1/p(t_k) + (1 + \varepsilon)m(w_k), \quad 1/p(t_k) + (1 + \varepsilon)M(w_k) \leq 1$$

for all $\varepsilon$ in a neighborhood of zero and for all $k \in \{1, \ldots, n\}$. These inequalities immediately imply that $0 < 1/p(t_k) + m(w_k)$ and $1/p(t_k) + M(w_k) < 1$ for all $k$. □

**Remark 2.5.** The presented proof involves the notion of indices of powerlikeness, which were invented to treat general Muckenhoupt weights (see [3]). Weights considered in the present paper are continuous except for a finite number of points. So, it would be rather interesting to find a direct proof of the fact that $w \in A_{p(\cdot)}(\Gamma)$ implies (2.1), which does not involve the indices of powerlikeness $\alpha(V_0^0 w)$ and $\beta(V_0^0 w)$.

**2.6 Final remarks.** In connection with Conjecture 1.4, we would like to note that for standard Lebesgue spaces $L^p(\Gamma, w)$ there are two different proofs of the stability of the boundedness of $S$ on $L^p(\Gamma, w^{1+\varepsilon})$ for small $\varepsilon$. Simonenko’s proof [22] is based on the stability of the Fredholm property of some singular integral operators related to the Riemann boundary value problem. Another proof is based on the self-improving property of Muckenhoupt weights (see e.g. [3, Theorem 2.31]). One may ask whether does $w \in A_{p(\cdot)}(\Gamma)$ imply $w^{1+\varepsilon} \in A_{p(\cdot)}(\Gamma)$ for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ with some fixed $\varepsilon > 0$? The positive answer would give a proof of the complete converse of (1.3). The author does not know any stability result for the boundedness of $S$ or a self-improving property for $w \in A_{p(\cdot)}(\Gamma)$.

After this paper had been submitted, P. Hästö and L. Diening [5] have found a necessary and sufficient condition for the boundedness of the classical Hardy-Littlewood maximal function on weighted variable Lebesgue spaces in the setting of $\mathbb{R}^n$. Note that they write a weight as a measure (outside of $|\cdot|^{p(\cdot)}$). Their condition is another generalization of the classical Muckenhoupt condition. In the setting of Carleson curves (and the weight written inside of $|\cdot|^{p(\cdot)}$), the Hästö-Diening condition takes the form (2.2)

$$\sup_{t \in \Gamma} \sup_{R > 0} \left( \frac{1}{R^{p'(t,R)}} \int_{\Gamma(t,R)} w(\tau)^{p(\tau)} d\tau \right) \|w(\cdot)^{-p(\cdot)} \chi_{\Gamma(t,R)}(\cdot)\|_{q(\cdot)/p(\cdot)} < \infty,$$
where
\[ p_{\Gamma(t,R)} := \left( \frac{1}{|\Gamma(t,R)|} \int_{\Gamma(t,R)} \frac{1}{p(\tau)} \, |d\tau| \right)^{-1}. \]

Let \( HD_{p(\cdot)}(\Gamma) \) denote the class of weights \( w : \Gamma \to [0,\infty) \) satisfying (2.2). Following the arguments contained in [5, Remark 3.10], one can show that
\[ A_{L^p(\cdot)}(\Gamma) \supset HD_{p(\cdot)}(\Gamma) \]
whenever \( p : \Gamma \to (1, \infty) \) satisfies the Dini-Lipschitz condition (1.2). We conjecture that the H"ast"o-Diening characterization remains true also for the operator \( S \) in the setting of Carleson curves.

**Conjecture 2.6.** Let \( \Gamma \) be a simple Carleson curve, \( w : \Gamma \to [0,\infty] \) be a weight, and \( p : \Gamma \to (1, \infty) \) be a continuous function satisfying the Dini-Lipschitz condition (1.2). The operator \( S \) is bounded on \( L^p(\cdot)(\Gamma, w) \) if and only if \( w \in HD_{p(\cdot)}(\Gamma) \).

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