LOCALLY FINITE GROUPS OF FINITE CENTRALISER
DIMENSION

ALEXANDRE BOROVIK AND ULLA KARHUMÄKI

Abstract. We describe structure of locally finite groups of finite centraliser
dimension.

1. Introduction

This paper describes structure of locally finite groups of finite centraliser
dimension; the result will be used by the second author in [6].

Theorem 1. Let $G$ be a locally finite group of finite centraliser dimension $c$. Then
$G$ has a normal series

$$1 \leq S \leq L \leq G,$$

where

(a) $S$ is solvable of solvability degree bounded by a function of $c$.
(b) $\mathcal{T} = L/S$ is a direct product $\mathcal{T} = \mathcal{T}_1 \times \cdots \times \mathcal{T}_m$ of finitely many non-abelian
simple groups.
(c) Each $\mathcal{T}_i$ is either finite, or a Chevalley group, or a twisted analogue of a
Chevalley group, or one of the non-algebraic twisted groups of Lie type $^{2}B_2$,
$^{2}F_4$, or $^{2}G_2$, over a locally finite field.
(d) The factor group $G/L$ is finite.

Our proof uses the Classification of Finite Simple Groups.

2. Proof of Theorem 1

Our theorem absorbs a number of known results, and we shall handle the proof
issue by issue.

We work with the group $G$ which satisfies the assumptions of Theorem 1.

2.1. Control of sections. If $G$ is a group and $H \leq G$ and $K \triangleleft H$, then the factor
group $H/K$ is called a section of $G$.

One immediate difficulty encountered in any study of groups of finite centraliser
dimension is that the descending chain condition for centralisers is inherited by
subgroups of $G$, but not by factor groups or sections. This happens even in the
class of periodic nilpotent groups of finite centraliser dimension (and they are of
course locally finite), an example can be found in [1 Section 4].

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Academy of Science and Letters.
So we need a sufficiently strong property which holds in every locally finite group of finite centraliser dimension and is inherited by its sections. Luckily, this property is provided by the following result.

**Fact 1** (Khukhro [7]). Periodic locally solvable groups of finite centralizer dimension are solvable and have derived length bounded by function of centralizer dimension.

Let us call a group $G$ constrained if derived lengths of its solvable subgroups are bounded. In view of Fact 1, periodic groups of finite centralizer dimension are constrained.

**Lemma 1.** Let $G$ be a locally finite group of finite centraliser dimension and $H = H/K$ its section. Then $H$ is constrained.

**Proof.** By Fact 1 we may assume that every solvable subgroup in $G$ has derived length at most $d$. Since $H$ is locally finite, it suffices to prove that an arbitrary finite solvable subgroup $S$ in $H$ has derived length at most $d$. Pick representatives $s_1, \ldots, s_n$ of cosets of $S$ of $K$ and generate by them a subgroup $R$; it is a finite subgroup, and by the well-known Frattini Argument for finite groups, $R$ contains a subgroup $P$ such that $P(R \cap K) = R$ and $P \cap K$ is nilpotent. Thus, $P$ is a solvable subgroup of $G$, hence has derived length at most $d$; but $PK/K = RK/K = S$, hence the derived length of $S$ is also at most $d$. □

### 2.2. Simple sections in locally finite groups of finite centraliser dimension.

The following result by Brian Hartley (based on the Classification of Finite Simple Groups) allows us to identify constrained simple locally finite groups.

**Fact 2** (Hartley [5]). Let $L$ be an infinite simple locally finite group. If some finite group is not involved in $L$, then $L$ is a Chevalley group, or a twisted version of a Chevalley group, or one of the non-algebraic twisted groups of Lie type $2B_2$, $2F_4$, or $2G_2$ over a locally finite field.

We shall call groups in conclusion of Fact 2 simple groups of Lie type.

As an immediate corollary, we have the following theorem.

**Theorem 2.** If $L$ is an infinite simple section of a locally finite group of finite centraliser dimension, then $L$ is of Lie type.

It is a partial generalisation of the result by Simon Thomas (1983) (which is also based on the Classification of Finite Simple Groups).

**Fact 3.** (Thomas [9]) An infinite simple locally finite group which satisfies the minimal condition on centralizers is of Lie type over a locally finite field.

### 2.3. Quasisimple simple locally finite groups of Lie type.

Recall that a group $H$ is called quasisimple if $H = [H, H]$ and $H/Z(H)$ is a non-abelian simple group.

**Proposition 1.** If $H$ is a quasisimple locally finite group and $H/Z(H)$ is a simple group of Lie type then $|Z(H)|$ is finite and bounded by a constant depending only on $H/Z(H)$.

**Proof.** The group of Lie type $\overline{H} = H/Z(H)$ is defined over a locally finite field, say $F$, and taking groups of points in $\overline{H}$ of finite subfields of $F$, we can construct a sequence of subgroups

$$1 < H_1 < H_2 < \ldots$$
such that their images $\overline{H_i}$ in $\overline{H}$ are simple groups of the same Lie type as $\overline{H}$, $H_i = [H_i, H_i]$ for all $i = 1, 2, \ldots$ and $\overline{H} = \bigcup_{i=1}^{\infty} \overline{H_i}$. Let $Z = Z(H)$. The subgroups $Z(H_i) = H_i \cap Z$ are factor groups of the Schur multipliers of groups of Lie type $\overline{H_i}$ and have orders of bounded size (see [4, §6.1]). Therefore the sequence of groups

$$1 \leq Z(H_1) \leq Z(H_2) \leq \ldots$$

stabilises at some finite group $Z_*$ of bounded order. If $Z$ is infinite, there is an element $z \in Z \setminus Z_*$ which is written as a product of some commutators from $H$:

$$z = [h_1, h_2] \cdots [h_{2k-1}, h_{2k}]$$

with elements $h_1, h_2, \ldots, h_{2k}$ contained in one of the subgroups $H_m$; but then $z \in [H_m, H_m] = H_m$ and therefore belongs to $Z(H_m) \leq Z_*$ — a contradiction. □

2.4. Composition factors and composition series. Now we have to address another difficulty: the concept of a composition factor is somewhat vague in the case of locally finite groups because the classical Jordan-Hölder Theorem for composition series of finite groups is no longer true; a spectacular example is given by Brian Hartley [5, pp. 2–3]. Moreover, there are countably infinite locally finite simple groups for which the following holds: $G$ possesses a series

$$\ldots G_{-2} \triangleleft G_{-1} \triangleleft G_0 \triangleleft G_1 \triangleleft G_2 \ldots$$

of proper subgroups of $G$ where the factors are finite and $G = \bigcup_{i=1}^{\infty} G_i$ (ibid. Theorem 1.27) and [3]).

For that reason we formulate the theorem by Buturlakin and Vasiliev [3] in a weaker form more suitable for our use.

Fact 4. (Buturlakin and Vasiliev [3]) Let $G$ be a locally finite group of centraliser dimension $c$ and

$$1 = G_0 \triangleleft G_1 \triangleleft G_2 \triangleleft \cdots \triangleleft G_l = G$$

a finite subnormal series in $G$. Then the number of distinct non-solvable factors $G_i/G_{i-1}$, $i \geq 1$, is at most $5c$.

2.5. Proof of Theorem [1]: solvable radical and the layer. We start building the normal series

$$1 \leq L \leq G$$

of Theorem [1].

For a constrained group $H$, we denote by $R(H)$ the maximal locally solvable normal subgroup of $H$; note that $R(H)$ is soluble and $H/R(H)$ has no non-trivial soluble normal subgroups. We also denote by $Q(H)$ the minimal normal subgroup of $H$ with soluble factor $H/Q(H)$; since $H$ is constrained, $Q(H)$ exists and coincides with the last term of the derived series of $H$. Also, $Q(H)$ has no non-trivial soluble factor groups. We shall call $H$ truncated if $R(H) = 1$ and $Q(H) = H$.

Observe also that $R(H)$ and $Q(H)$ are characteristic subgroups of $H$.

We start with the normal series

$$1 = G_0 \triangleleft G_1 = G,$$

and refine and re-build it and get subsequent series

$$1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_l = G,$$

appropriately changing numeration at every step, in accordance with the following rules:
• For every factor $G_i/G_{i-1}$ that is not truncated, we insert subgroups
  \[ G_{i-1} \trianglelefteq G_j \trianglelefteq G_k \trianglelefteq G_i, \]
  where $G_j$ in the full preimage of $R(G_i/G_{i-1})$ and $G_k$ is the full preimage of $Q(G_i/G_{i-1})$.
• If $G_i/G_{i-1}$ and $G_j/G_{j-1}$, $i < j - 1$, are two non-trivial truncated factors and there are no truncated factors $G_k/G_{k-1}$ for $i < k < j$, then $G_{j-1}/G_i$ is solvable and we can remove from the series its members $G_{i+1}, \ldots, G_{j-2}$.
• If $G_i/G_{i-1}$ is a truncated factor and there is a normal subgroup $G_j \trianglelefteq G$ fitting into $G_{i-1} < G_j < G_i$, we insert it in the series and repeat the process from the beginning.

In view of Fact \[4\] the process terminates after finitely many steps, producing a finite series of normal subgroups where every factor is solvable or truncated without non-trivial proper characteristic subgroups. Again applying Fact \[4\] we see that these truncated factors are finite direct sums of isomorphic non-abelian simple groups which are either finite or of Lie type.

Now we can coherently define $S = R(G)$ and $L$ as the full pre-image in $G$ of the layer $L(G/S)$, that is, the product of all simple subnormal subgroups (components) of $G/S$. Therefore $L = L/S$ is a direct product $L = \prod_{i=1}^{m} L_i$ of finitely many non-abelian simple groups. This proves Theorem \[3\] as soon as we do the point (d) of the Theorem \[3\] finiteness of $G/L$ – see next Sections.

2.6. Action of $G$ on $G/S$. We retain notation from the previous Section. Set $\overline{G} = G/S$.

Our first observation is that $C_{\overline{G}}(\overline{L}) \cap \overline{L} = 1$ and $C_{\overline{G}}(\overline{L}) \trianglelefteq \overline{G}$. If $C_{\overline{G}}(\overline{L}) \neq 1$, then, applying analysis of the previous Section to the full preimage of $C_{\overline{G}}(\overline{L})$ in $G$, we see that the group $C_{\overline{G}}(\overline{L})$ has subnormal non-abelian simple subgroups which are subnormal in $\overline{G}$ but do not belong to $\overline{L}$, which contradicts to the way $\overline{L}$ was constructed. Hence $C_{\overline{G}}(\overline{L}) = 1$.

Now the group $G$, in its action on $\overline{L}$ by conjugation, permutes simple subgroups $\overline{L}_i$; the kernel of this permutation action, say $G^\circ$, is a normal subgroup of finite index in $G$. Without loss of generality, we can assume that $G^\circ = G$ and each $\overline{L}_i \trianglelefteq \overline{G}$.

Let now $\overline{M} = \overline{L}_1 \times \cdots \times \overline{L}_k$ be the product of all infinite components of $\overline{G}$; if $\overline{M} = 1$, then $\overline{L}$ and hence $\overline{G}$ are finite, thus the point (d) holds and Theorem \[3\] is proven.

So we can assume that $\overline{M} \neq 1$. Denote by $\overline{N} = \overline{L}_{k+1} \times \cdots \times \overline{L}_m$ the product of all finite components of $\overline{G}$. If $\overline{N} \neq 1$, then $C_{\overline{G}}(\overline{N})$ is the kernel of the action of $\overline{G}$ on $\overline{N}$ by conjugation and therefore has finite index in $\overline{G}$. Again, we can assume without loss of generality that $C_{\overline{G}}(\overline{N}) = \overline{G}$ and $\overline{N} = 1$, that is, all components of $\overline{G}$ are infinite simple groups of Lie type over (infinite) locally finite fields.

2.7. The factor group $G/L$ is abelian-by-finite. We turn our attention to the action of $\overline{G}$ on $\overline{L}$.

It is well-known that every automorphism of a group of Lie type over a locally finite field $F$, say $X = X(F)$, is a product of inner, diagonal, graph, and field automorphisms. If $\text{Out} X = \text{Aut} X / \text{Inn} X$ is the group of outer automorphisms of $X$, then images in $\text{Out} X$ of diagonal and graph automorphisms of $X$ generate a finite subgroup. Also, the image $\Gamma$ in $\text{Out} X$ of the group of field automorphisms is naturally isomorphic to the group $\text{Aut} F$. It is well known that $\text{Aut} F$ is a factor
group of $\hat{\mathbb{Z}}$, the profinite completion of the additive group of integers (the latter is the Galois group of the algebraic closure of a finite prime field).

We have a natural embedding

$$\overline{G}/\overline{L} \hookrightarrow \prod_{i=1}^{m} \text{Out} \overline{L}_i.$$  

We see now that if $\overline{G}/\overline{L}$ is infinite, it contains an abelian subgroup $\Delta$ of finite index which either centralises, or acts by field automorphisms on components $T_i$ – in the sense that elements from the preimage $D$ induce on $L_i$ automorphisms that are products of inner and field automorphisms.

Let $D$ be the full preimage of $\overline{D}$ in $G$, then $D$ has a finite index in $G$. So for the rest of the proof we can assume, without loss of generality, that $G/L$ is abelian and outer automorphism induced from $D$ on Lie type subgroups $L_i$ are field automorphisms.

2.8. Frattini Argument. At this point it becomes essential to find a more efficient way around the fact that the descending chain condition for centralisers in general is not preserved under taking factor groups.

The general situation is the following: we have a group $G$ of finite centraliser dimension and a subgroup $K \triangleleft G$; we wish to derive some information about $\hat{G} = G/K$ without being able to prove directly that $\hat{G}$ has finite centraliser dimension. The idea is to calculate instead in an appropriate partial complement $M \leq G$ to $K$, that is, a subgroup such that $G = MK$ and $M$ is sufficiently small for easier deduction of the desired facts about $G/K \simeq M/(M \cap K)$.

The classical way to construct partial complements is the Frattini Argument. For a set of prime numbers $\pi$, a periodic group $H$ is called a $\pi$-group (correspondingly, $\pi'$-group), if prime factors of orders of elements from $H$ belong (correspondingly, do not belong) to $\pi$. A Sylow $\pi$-subgroup of a group $H$ is a maximal $\pi$-subgroup of $H$.

**Lemma 2.** (Frattini Argument) Let $H$ be a locally finite group and $K \lhd H$ a normal subgroup. Assume that the Sylow $\pi$-subgroups in $K$ are conjugate in $K$ and $P$ is one of them. Then $H = KN_H(P)$.

**Proof.** Proof is exactly the same as the well-known proof in the finite case. □

The value of the Frattini Argument is obvious because of following important fact.

**Fact 5.** (Bryant and Hartley [2, Theorem 1.6]) In a locally finite group $H$ with descending chain condition for centralisers and for all set of primes $\pi$, Sylow $\pi$-subgroups are conjugate.

We shall start now using the Frattini argument as a tool for carving out from $G$ subgroups where we have better control of centralisers.

2.9. Toward finiteness of $G/L$. Let us now look at the group $\Delta < \overline{G}/\overline{L}$ constructed in Section 2.7 and its full preimage $\overline{D}$ in $\overline{G}$.

To prove that $\overline{G}/\overline{L}$ is finite it will suffice to prove that $\Delta$ is finite. So let us assume that $\Delta$ is infinite and work towards a contradiction. Among components $\overline{L}_i \triangleleft \overline{L}$, we pick one, say $\overline{K}$, such that $\overline{D}$ induces in its action on $\overline{K}$ an infinite group
of outer automorphisms. The group \( \overline{K} \) is of Lie type over a locally finite field \( F \); let \( p \) be the of characteristic of \( F \). The natural homomorphism
\[
\rho : \overline{D} \to \overline{K} \times \Gamma, \quad \text{where} \quad \Gamma = \text{Aut} F;
\]
take \( E = \text{Im} \rho \cap \Gamma \).

The group \( \Gamma \) is the continuous image of \( \hat{\mathbb{Z}} \) (the profinite completion of \( \mathbb{Z} \), the Galois group of the algebraic closure of the prime field \( \mathbb{F}_p \)), and \( E \), as a locally finite subgroup of \( \Gamma \), is locally cyclic and is a direct sum of finite cyclic groups of pairwise coprime prime power orders; let \( \epsilon_1, \epsilon_2, \ldots \), be generators of the cyclic direct summands of \( E \), then we have an infinite decreasing sequence of fields
\[
C_F(\langle \epsilon_1 \rangle) > C_F(\langle \epsilon_1, \epsilon_2 \rangle) > C_F(\langle \epsilon_1, \epsilon_2, \epsilon_3 \rangle) > \ldots,
\]
and correspondingly an infinite descending chain of centralisers
\[
C_{\overline{K}}(\langle \epsilon_1 \rangle) > C_{\overline{K}}(\langle \epsilon_1, \epsilon_2 \rangle) > C_{\overline{K}}(\langle \epsilon_1, \epsilon_2, \epsilon_3 \rangle) > \ldots.
\]
This would produce a contradiction if we knew that \( \overline{\mathbb{K}} \) had a descending chain condition for centralisers; but we don’t, so we have to make some further surgery on the group \( G \), and, in particular, cut \( E \) to a manageable size.

Pick in \( E \) elements \( \alpha_1, \ldots, \alpha_n \), of pairwise different prime orders \( p_1, \ldots, p_n \), none of which is \( p \), and none divides the order of the center of any quasisimple extension of \( \mathbb{K} \) (see Proposition [11]), making sure that \( n \) is bigger than the centraliser dimension \( c \) of \( G \). Take their product \( \alpha = \alpha_1 \alpha_2 \cdots \alpha_n \) and its preimage \( \bar{\alpha} \) in \( \overline{G} \); let \( \alpha \) be some coset representative of \( \bar{\alpha} \) and \( A = \langle \alpha \rangle \) is the cyclic group generated by \( \alpha \); replacing \( \alpha \) by another coset representative, we can ensure that prime divisors of \( |A| \) are exactly \( p_1, \ldots, p_n \).

Let now \( K \) be the full preimage of \( \overline{K} \); we can replace, without loss of generality, \( G \) by \( KA \); the solvable radical \( S \) could slightly grow up by absorbing part of \( A \), but this does not affect our considerations; we still have the property that, for appropriate powers \( a_1, \ldots, a_n \) of \( \alpha \), we have a descending chain of centralisers
\[
C_{\overline{K}}(\langle a_1 \rangle) > C_{\overline{K}}(\langle a_1, a_2 \rangle) > \cdots > C_{\overline{K}}(\langle a_1, a_2, \ldots, a_n \rangle).
\]

2.10. Trimming the solvable radical. Let \( U \) be a Sylow \( p \)-subgroup of \( S \), where \( p \) is the characteristic of \( F \), the underlying field of \( \overline{K} \). Using Frattini Argument, we replace \( G \) by \( N_G(U) \) and assume, without loss of generality, that \( U \triangleleft G \). Now we take a Sylow \( p' \)-subgroup \( Q \) in \( S \) and, applying the Frattini Argument again, replace \( G \) by \( N_G(Q) \), thus assuming, without loss of generality, that \( Q \triangleleft G \). Now \( S = U \times Q \).

Let now \( P \) be a Sylow \( p \)-subgroup in \( K \), then its image \( \overline{P} \) in \( \overline{K} \) is a maximal unipotent subgroup in the group \( \overline{K} \) of Lie type over an infinite field of characteristic \( p \); hence \( \overline{P} \) contains an infinite elementary abelian subgroup \( \overline{V} \) (the center of any root subgroup can be used for that purpose). We denote by \( V \) a Sylow \( p \)-subgroup in the full preimage of \( \overline{V} \) in \( K \), then \( V \) acts on \( Q \) by conjugation and \( U \) belongs to the kernel of this action. The group \( VQ \) has finite centraliser dimension, and therefore has uniformly bounded lengths of chains of centralisers of subsets from \( V \) in \( Q \). At this point we can invoke the following fact.

Fact 6. (Khukhro [[7, Lemma 3]]) If an elementary abelian \( p \)-group \( W \) of order \( p^n \) acts faithfully on a finite nilpotent \( p' \)-group \( Q \), then there exists a series of subgroups
\[
W = W_0 > W_1 > W_2 > \cdots > W_n = 1
\]
such that
\[ C_Q(W_0) < C_Q(W_1) < \cdots < C_Q(W_n). \]

Indeed, the assumption that \( Q \) is finite and nilpotent can be easily replaced by “locally finite and soluble” using induction on the derived length of \( Q \). After that, we see that \( Y = C_V(Q) \) has finite index in \( V \) and therefore the image \( \overline{Y} \) of \( Y \) in \( \overline{K} \) is infinite. Let \( H = C_K(Q) \); obviously, \( H \leq K \) and contains \( \overline{Y} \); but \( \overline{K} \) is simple, hence \( \overline{K} = H \) and contains the image \( \overline{Y} \) of \( W \) in \( G \). But \( \overline{Y} \) is an infinite subgroup of \( \overline{K} \), hence \( \overline{K} \leq \overline{H} \). So, without loss of generality, we can replace \( K \) by \( H = C_K(Q) \), and then replace \( G \) by \( K \).

Now \( Q \leq Z(K) \). At the last step of our trimming procedure, we replace \( K \) by the intersection of its derived series, making sure that \( K = [K, K] \), and then again replace \( G \) by \( K \).

Consider \( \hat{K} = K/U \) and let \( \hat{Q} \) be the image of \( Q \) in \( \hat{K} \), then \( \hat{Q} \leq Z(\hat{K}) \). The centers of locally finite quasisimple groups of Lie type are finite by Proposition 1, hence \( \hat{Q} \) and \( Q \) are finite. In the resulting normal series
\[ 1 \trianglelefteq U \trianglelefteq S \trianglelefteq K \trianglelefteq G \]
the factor \( S/U \) is a finite abelian group in view of Proposition 1. In particular, only finitely many distinct prime numbers divide orders of elements in \( S \).

2.11. End of proof. Now we can return to the chain of centralisers
\[ C_{\overline{K}}(\langle a_1 \rangle) > C_{\overline{K}}(\langle a_1, a_2 \rangle) > \cdots > C_{\overline{K}}(\langle a_1, a_2, \ldots, a_n \rangle) \]
constructed in Section 2.9.

Recall that \( A \) was constructed in a way that \( a_i \in A \) have orders coprime to orders of elements in \( U \) and in \( S/U \). These restrictions have been forced on \( A \) with the aim of lifting centralisers of subgroups \( B \leq A \) in \( \overline{K} \) to centralisers in \( K \), that is, proving that \( C_{\overline{K}}(B) \) is the image of \( C_K(B) \) in \( \overline{K} \). For that, we need a simple tool from finite group theory.

The following fact is well-known and appears to be part of folklore; we give only a brief sketch of a proof.

Fact 7. Let \( B \) be a finite cyclic \( \pi \)-group of automorphisms of a finite group \( H \) and let \( R \) be an \( B \)-invariant normal \( \pi' \)-subgroup of \( H \). Then \( C_{H/R}(B) \) is the image of \( C_H(B) \) in \( H/R \).

Proof. A standard induction by two parameters:
- The number of prime divisors in \( |B| \), and
- the minimal length of a series of characteristic subgroups in \( R \) whose factors are groups of prime power order

reduces the proof to the case when \( B \) is a \( q \)-group and \( R \) is a \( r \)-group for primes \( r \neq q \). Let \( |R| = r^m \) and and \( hR \) be a coset in \( C_{H/R}(B) \). Orbits of \( B \) on \( hR \) have sizes of the form \( q^k, k = 0, 1, 2, \ldots \). If \( B \) does not have a fixed point in \( hR \), then the cardinality of \( hR \) is divisible by \( q \), which is impossible, since it equals to \( |R| = r^m \). Hence every coset \( hR \in C_{H/R}(B) \) has a point fixed by \( B \), which completes the proof.

Now we can expand, in a routine way, Fact 7 to locally finite groups.
Lemma 3. Let $B$ be a finite cyclic $\pi$-group of automorphisms of a locally finite group $K$ and let $S$ be an $B$-invariant normal solvable $\pi'$-subgroup of $H$. Assume, in addition, that orders of elements from $S$ are divisible only by finitely many different prime numbers. Then $C_{K/S}(B)$ is the image of $C_K(B)$ in $K/P$.

Now, after lifting centralisers in the chain
\[ C_K(\langle a_1 \rangle) > C_K(\langle a_1, a_2 \rangle) > \cdots > C_K(\langle a_1, a_2, \ldots, a_n \rangle), \]
from $K$ to $K$, we have the chain of centralisers
\[ C_K(\langle a_1 \rangle) > C_K(\langle a_1, a_2 \rangle) > \cdots > C_K(\langle a_1, a_2, \ldots, a_n \rangle), \]
of the length exceeding the centraliser dimension of $G$.
This contradiction completes the proof of Theorem 1. □ □

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School of Mathematics, University of Manchester, UK; alexandre ≫ at ≪ borovik.net
School of Mathematics, University of Manchester, UK; ukarhumaki ≫ at ≪ gmail.com