ON THE BOUNDED GENERATION OF ARITHMETIC SL$_2$

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Abstract. Let $K$ be a number field and $\mathcal{O}$ be the ring of $S$-integers in $K$. Morgan, Rapinchuk, and Sury have proved that if the group of units $\mathcal{O}^\times$ is infinite, then every matrix in $\text{SL}_2(\mathcal{O})$ is a product of at most 9 elementary matrices. We prove that under the additional hypothesis that $K$ has at least one real embedding or $S$ contains a finite place we can get a product of at most 8 elementary matrices. If we assume a suitable Generalized Riemann Hypothesis, then every matrix in $\text{SL}_2(\mathcal{O})$ is the product of at most 5 elementary matrices if $K$ has at least one real embedding, the product of at most 6 elementary matrices if $S$ contains a finite place, and the product of at most 7 elementary matrices in general.

1. Introduction

Let $K$ be a number field and $S$ be a finite set of primes of $K$ containing the archimedean valuations. Denote by $\mathcal{O} = \mathcal{O}_S$ the ring of $S$-integers in $K$:

$$\mathcal{O} = \mathcal{O}_S = \{ x \in K^\times \mid v(x) \geq 0 \text{ for all } v \notin S \}.$$ 

For $x \in \mathcal{O}$ we define the upper triangular matrix $U(x)$ and the lower triangular matrix $L(x)$ by

$$U(x) := \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad L(x) := \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}. \quad (1)$$

The elementary matrices over $\mathcal{O}$ are the matrices $U(x), L(x)$ for $x \in \mathcal{O}$.

Consider the case where $K$ is the field of rational numbers $\mathbb{Q}$. Taking $\mathcal{O} = \mathbb{Z}$ we have that every $A \in \text{SL}_2(\mathbb{Z})$ is a product of elementary matrices, but the number required is unbounded. However, if we take $\mathcal{O} = \mathbb{Z}[1/p]$ for $p$ prime, the situation is different. Every matrix $A \in \text{SL}_2(\mathbb{Z}[1/p])$ is a product of at at most 5 elementary matrices as was proved by Vsemirnov [Vse14, Theorem 1.1].

The key difference between the $\mathbb{Z}$ and $\mathbb{Z}[1/p]$ for this bounded generation question for $\text{SL}_2$ is their units: $\mathbb{Z}^\times = \langle \pm 1 \rangle$ is finite whereas $\mathbb{Z}[1/p]^\times$ is infinite. Vaserstein [Vas72] proved that if $\mathcal{O}$ has infinitely many units, then $\text{SL}_2(\mathcal{O})$ is generated by elementary matrices. Morgan, Rapinchuk, and Sury [MRS18, Theorem 1.1] recently proved an explicit general result on bounded generation:

**Theorem 1.1** (Morgan, Rapinchuk, and Sury). Assume that the group of units $\mathcal{O}^\times$ is infinite. Then every matrix in $\text{SL}_2(\mathcal{O})$ can be written as a product of at most 9 elementary matrices with the first one lower triangular.

The lower triangular assertion follows from their proof: see [MRS18, Eq. (21) and following].

Here we prove two theorems on a matrix $A \in \text{SL}_2(\mathcal{O})$:

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Theorem 1.2. Suppose that $S$ contains a finite place or suppose that the group of units $\mathcal{O}^\times$ is infinite and $K$ has at least one real embedding. Then $A \in \text{SL}_2(\mathcal{O})$ can be written as the product of at most 8 elementary matrices with the first one lower triangular.

Theorem 1.3. Assume that the group of units $\mathcal{O}^\times$ is infinite and assume the Generalized Riemann Hypothesis [3.7]. Then $A \in \text{SL}_2(\mathcal{O})$ can be written as the product of at most 5 elementary matrices if $K$ has at least one real embedding, the product of at most 6 elementary matrices if $S$ contains a finite place, and the product of at most 7 elementary matrices in general with the first one lower triangular in each case.

We give diophantine applications of Theorems 1.2 and 1.3 in [JZ19]. These applications require us to know that the first matrix in our factorization into elementary matrices can be taken to be lower triangular. Hence we keep track of this here, whereas it is not a concern in [MRS18].

2. Theorem 1.2

2.1. Reducing the first row of a matrix $A \in \text{SL}_2(\mathcal{O})$. Following [MRS18, Section 4], let

\[ \mathcal{R}(\mathcal{O}) = \{(a, b) \in \mathcal{O}^2 \mid a\mathcal{O} + b\mathcal{O} = \mathcal{O}\}. \]  

The $(a, b) \in \mathcal{R}(\mathcal{O})$ are precisely the first rows of matrices in $\text{SL}_2(\mathcal{O})$. The effect on the first row of a matrix $A = \begin{bmatrix} a & b \\ * & * \end{bmatrix} \in \text{SL}_2(\mathcal{O})$ from right multiplying by an elementary matrix as in (1) is

\[
AL(x) = \begin{bmatrix} a & b \\ * & * \end{bmatrix} \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} = \begin{bmatrix} a + bx & b \\ * & * \end{bmatrix},
\]

\[
AU(x) = \begin{bmatrix} a & b \\ * & * \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & ax + b \\ * & * \end{bmatrix},
\]

for $x \in \mathcal{O}$.

The following succinct notation using only the first rows of matrices is convenient:

Definition 2.1. For $x \in \mathcal{O}$ and $(a, b) \in \mathcal{R}(\mathcal{O})$, set $(a, b)\ell(x) = (a + bx, b)$ and $(a, b)u(x) = (a, ax + b)$.

If there exist $x_1, \ldots, x_k \in \mathcal{O}$ with

\[
(c, d) = \begin{cases} 
(a, b)\ell(x_1)u(x_2) \cdots \ell(x_k) & \text{k odd} \\
(a, b)\ell(x_1)u(x_2) \cdots u(x_k) & \text{k even}
\end{cases}
\]

for $(a, b), (c, d) \in \mathcal{R}(\mathcal{O})$, write $(a, b) \overset{k,\ell}{\Rightarrow} (c, d)$. Similarly, if there exist $x_1, \ldots, x_k \in \mathcal{O}$ with

\[
(c, d) = \begin{cases} 
(a, b)u(x_1)\ell(x_2) \cdots u(x_k) & \text{k odd} \\
(a, b)u(x_1)\ell(x_2) \cdots \ell(x_k) & \text{k even}
\end{cases}
\]

for $(a, b), (c, d) \in \mathcal{R}(\mathcal{O})$, write $(a, b) \overset{k,u}{\Rightarrow} (c, d)$. As in [MRS18, Section 4], write $(a, b) \overset{k}{\Rightarrow} (c, d)$ if $(a, b) \overset{k,\ell}{\Rightarrow} (c, d)$ or $(a, b) \overset{k,u}{\Rightarrow} (c, d)$. 

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2.2. The Proof of Theorem 1.2. First we need the following Lemma 2.3 which requires a definition.

Definition 2.2. [MRS18, Section 3.1.] A prime \( q \) of the number field \( K \) lying above the rational prime \( q \) is \( \mathbb{Q} \)-split if \( q > 2 \) and \( K_q \cong \mathbb{Q}_q \).

If \( K \) is a number field, \( \mu = \mu(K) \) is the number of roots of unity in \( K \).

Lemma 2.3. (cf. [MRS18, Lemma 4.4].) Suppose \( K \) has at least one real embedding or \( S \) contains a finite place and \( (a, b) \in R(O) \). Let \( \mu = \mu(K) \). Then there exists \( a' \in O \) and infinitely many \( \mathbb{Q} \)-split prime principal ideals \( q \) of \( O \) with a generator \( \mathfrak{q} \) such that for any \( m \equiv 1 \mod \phi(a'O) \) we have \((a, b) \xrightarrow{3\mathfrak{q}} (a', \mathfrak{q}^m) \).

Proof. Let \( v \) be either a real place of \( K \) or a finite place in \( S \). To simplify subsequent notation we use the convention that the valuation of an element \( \alpha \in K \) with respect to a real place \( v \) is given by \( \text{val}_v(\alpha) = 1 \) if \( \alpha \) is negative with respect to \( v \) and 0 otherwise.

Let \( b' \in O \) be a prime relatively prime to \( \mu \), congruent to \( b \mod a \), and such that \( \text{val}_v(b') = 1 \). Such a \( b' \) exists by Dirichlet’s theorem. This is clear in the archimedean case. In the nonarchimedean case, this can be done by finding an ideal \( b \subset O_K \) in the same ideal class as \( I_{-1}^{-1} \) with \( \text{b} \equiv b \mod (a'O) \). Note that \( (a, b) \xrightarrow{1\mathfrak{q}} (a, b') \).

For a prime \( w \) of \( K \), denote by \( \left( \frac{a'}{w} \right)_\mu \) the power residue symbol of degree \( \mu \) at \( w \) (cf. [BMS67, p. 85]). Find a prime \( a' \) of \( O \) congruent to a mod \( b' \) such that

1. \( (\frac{a'}{v_i})_\mu = 1 \) for all places \( v_i \) in \( S \) or dividing \( \mu \) except \( v \) and
2. \( (\frac{a'}{v})_\mu = (\frac{b'}{v})_\mu^{-1} \).

Note that

1. In the nonarchimedean case, \( b' \) are all congruence conditions modulo sufficiently high powers of \( I_{v_i}O_K \).
2. \( b' \) is a condition modulo a power of the product of the ideals above \( v \) and \( b' \). That it is nonempty follows from the fact that the map given by \( \left( \frac{a'}{v} \right)_\mu \) is surjective if \( \text{val}_v(b') \equiv 1 \mod (\mu) \).

We can see that such an \( a' \) exists from Dirichlet’s theorem. Note that \( a' \) and \( b' \) are relatively prime and \( (a, b) \xrightarrow{1\mathfrak{q}} (a', b') \).

Observe that by the reciprocity law \( \left( \frac{a'}{\mathfrak{q}O} \right)_\mu = 1 \). This implies that \( b' \equiv x^\mu \mod (a'O) \) for some residue \( x \) using [BMS67, (A.16)]; cf. [MRS18, p. 18]. By the generalization of Dirichlet’s theorem to \( \mathbb{Q} \)-split primes, see [MRS18, Theorem 3.3] there are infinitely many odd, degree-1 principal prime ideals \( q \) with a generator \( \mathfrak{q} \equiv x \mod (a'O) \). Then for all these \( \mathfrak{q} \) and for all \( m \equiv 1 \mod \phi(a'O) \) we have \((a', b') \xrightarrow{1\mathfrak{q}} (a', \mathfrak{q}^m) \). Hence we are done.

Proof of Theorem 1.2. Suppose \( S \) contains a finite place or \( \#O^\times = \infty \) and \( K \) has at least one real embedding. Let \( A = \left[ \begin{smallmatrix} a & c \\ b & d \end{smallmatrix} \right] \in \text{SL}_2(O) \). Proceed as in the proof of [MRS18, Section 4] only use Lemma 2.3 instead of [MRS18, Lemma 4.4]. Thus we don’t need to use [MRS18, Lemma 4.3] and we end up showing \((a, b) \xrightarrow{7} (1, 0) \) instead of \((a, b) \xrightarrow{8} (1, 0) \) as in [MRS18, Eq. (21)]. Hence \( A \) is the product of 8 elementary matrices beginning with a lower triangular matrix.
3. **Theorem 1.3**

### 3.1. Division Chains.

**Definition 3.1.** (cf. [CW75, Section 2].) Let \((a, b) \in \mathcal{R}(\mathcal{O})\) as in (2). A **division chain** of length \(k\) starting with \((a, b)\) is a sequence of equations

\[
\begin{align*}
  a &= q_1 b + r_1 \\
  b &= q_2 r_1 + r_2 \\
  &\vdots \\
  r_{k-3} &= q_{k-1} r_{k-2} + r_{k-1} \\
  r_{k-2} &= q_k r_{k-1} + r_k
\end{align*}
\]

with \(q_i \in \mathcal{O}, 1 \leq i \leq k\). The division chain is **terminating** if \(r_k = 0\). Notice that since \(a\) and \(b\) are relatively prime, in the terminating case \(r_{k-1}\) must be a unit.

**Remark 3.2.** The division chains of Definition 3.1 are closely related to the row reductions of Definition 2.1. The division chain in (6) of length \(k\) starting with \((a, b) \in \mathcal{R}(\mathcal{O})\) is equivalent to

\[
(a, b) \xrightarrow{k,\ell} \begin{cases} (r_{k-1}, r_k) & \text{if } k \text{ is even} \\ (r_k, r_{k-1}) & \text{if } k \text{ is odd} \end{cases}
\]

The following lemma is elementary:

**Lemma 3.3.** We have \(b \equiv v \mod a\) for \(v \in \mathcal{O}^\times\) if and only if there exists a terminating division chain of length 2 starting with \((b, a)\).

### 3.2. Terminating division chains of length 2.

Consider the matrix

\[
A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(\mathcal{O}).
\] (7)

Assume in this subsection that there is a terminating division chain of length 2 starting with \((b, a)\). Therefore, by Lemma 3.3 we have \(b \equiv v \mod a\), or \(b - v = ax\) for \(x \in \mathcal{O}\), with a unit \(v \in \mathcal{O}^\times\).

**Proposition 3.4.**

\[
AU(-x)L(v^{-1}(1 - a))U(-v) = L(w)
\]

for some \(w \in \mathcal{O}\).

**Proof.** Multiplying matrices verifies that

\[
AU(-x)L(v^{-1}(1 - a))U(-v) =: B = \begin{bmatrix} 1 & 0 \\ * & * \end{bmatrix}.
\]

But the entry \(B_{22}\) must be 1 since \(B \in \text{SL}_2(\mathcal{O})\). Hence \(B = L(w)\) for some \(w \in \mathcal{O}\). \(\Box\)

**Theorem 3.5.** Let \(A\) be as in (7) assume there is a terminating division chain of length 2 starting with \((b, a)\). Then \(A\) can be written as product of at most 4 elementary matrices with the first one lower triangular.
Proof. From Proposition 3.4 we have
\[ A = L(w)U(-v)^{-1}L(v^{-1}(1-a))^{-1}U(-x)^{-1}. \] (8)
But for any \( s \in \mathcal{O} \) we have \( U(s)^{-1} = U(-s) \) and \( L(s)^{-1} = L(-s) \). Hence (8) becomes
\[ A = L(w)U(v)L(v^{-1}(a-1))U(x). \]
\[ \square \]

3.3. General Matrices in SL\(_2(\mathcal{O})\).

**Theorem 3.6.** Let \( A \) be as in (7). If there exists a terminating division chain of length \( k > 1 \) starting at
\[
\begin{cases}
(a, b) & \text{if } k \text{ is odd} \\
(b, a) & \text{if } k \text{ is even},
\end{cases}
\]
then \( A \) can be written as the product of at most \( k + 2 \) elementary matrices with the first one lower triangular.

**Proof.** We proceed by induction on \( k \). The \( k = 2 \) case is Theorem 3.5.

Suppose \( k \) is odd. Then by the definition of a terminating division chain there exist \( s \in \mathcal{O} \) such that
\[ a - r = bs \]
and \((b, r)\) has a terminating division chain of length \( k - 1 \). Then
\[ AL(-y) = \begin{bmatrix} r & b \\ * & * \end{bmatrix} \]
is the product of \( k + 1 \) elementary matrices with the first one lower triangular by the induction hypothesis.

The \( k \) even case is handled similarly only switch the roles of \( a \) and \( b \) as well as multiply by \( U(-y) \) instead of \( L(-y) \). \[ \square \]

Note that this construction is similar to that used in [CW75] Corollary 2.3 except ours is more efficient, so we end up with \( k + 2 \) rather than the \( k + 4 \) elementary matrices produced by the construction of [CW75] p. 496–498. This accounts for why our numbers are two smaller than theirs.

3.4. The Generalized Riemann Hypothesis and the Proof of Theorem 1.3. The relevant Riemann hypothesis is most clearly stated in [Len77, Theorem 3.1].

**Riemann Hypothesis 3.7.** The \( \zeta \)-function of \( K(\zeta_n, \sqrt{\mathcal{O}}^\times) \) satisfies the Riemann hypothesis for all integers \( n > 0 \).

**Proof of Theorem 1.3** Let \( \mathcal{O} \) be the \( S \)-integers in \( K \) and \((a, b) \in \mathcal{R}(\mathcal{O}) \) as in (2). Assume Hypothesis 3.7. Then by [CW75, Theorem 2.2] there is a terminating division chain of length 5 starting with \((a, b)\). If \( S \) contains at least one finite prime, then there is a terminating division chain of length 4 starting with \((a, b)\) by [CW75, Theorem 2.9], attributed to Lenstra. If \( K \) has a real place, then [CW75, Theorem 2.14] shows that there is a terminating division chain of length 3 starting with \((a, b)\). Now apply Theorem 3.6. \[ \square \]
Morgan, Rapinchuk, and Sury [MRS18, Proposition 5.1] show that if \( p > 7 \) is a prime, then not every matrix in \( \text{SL}_2(\mathbb{Z}[1/p]) \) is a product of 4 elementary matrices. Hence the bound of 5 elementary matrices if \( K \) has a real embedding in Theorem 1.3 assuming Hypothesis 3.7 would be strict.

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