SUMMARY We initiate the study of Ramsey numbers of trails. Let \( k \geq 2 \) be a positive integer. The Ramsey number of trails with \( k \) vertices is defined as the smallest number \( n \) such that for every trail \( H \) with \( n \) vertices, \( H \) or the complete \( \overline{H} \) contains a trail with \( k \) vertices. We prove that the Ramsey number of trails with \( k \) vertices is at most \( k \), and at least \( 2 \sqrt{k} + o(1) \). This improves the trivial upper bound of \( \lfloor 3k/2 \rfloor - 1 \). We note here that a trivial upper bound is \( \lfloor 3k/2 \rfloor - 1 \), which will be sketched in the next section.

1. Introduction

Ramsey theory is one of the topics in discrete mathematics that has been studied over the years [1], [4]. For graphs, the Ramsey number was first studied for complete graphs, and later it was studied for other classes of graphs such as paths, cycles, and trees; a good source of references is given by Radziszowski [5]. Applications of Ramsey theory ranges from pure mathematics such as number theory and harmonic analysis to computer science such as approximation algorithms and complexity theory [3], [6].

For graphs \( G \) and \( G' \), the Ramsey number of the pair \((G, G')\) is the smallest number \( n \) such that for every graph \( H \) with \( n \) vertices, \( H \) contains a copy of \( G \) or the complement \( \overline{H} \) contains a copy of \( G' \). It is known that for every pair \((G, G')\) of finite graphs, the Ramsey number of \((G, G')\) exists, and the determination of the Ramsey number is the ultimate goal. However, even for complete graphs, the exact Ramsey number is not known: When \( G = G' = K_5 \) we only know that the Ramsey number lies between 43 and 48 [5].

In this paper, we initiate the study of Ramsey numbers for trails. Unlike paths, trails may have a repetition of vertices. To study the Ramsey number of trails, we first fix the number of vertices in a trail. Let \( k \) and \( \ell \) be integers. Then, the Ramsey number of trails with \( k \) vertices and \( \ell \) vertices is defined as the smallest number \( n \) such that for every trail \( H \) with \( n \) vertices, \( H \) contains a trail with \( k \) vertices or \( \overline{H} \) contains a trail with \( \ell \) vertices.

The ultimate goal is to determine the Ramsey number of trails. Unfortunately, we are unable to provide a definite answer. Nonetheless, we give a progress toward the ultimate goal. We concentrate on the diagonal case, i.e., the case where \( k = \ell \). Our main theorems give an improved upper bound of \( k \), and also a lower bound of roughly \( 2 \sqrt{k} \). We note here that a trivial upper bound is \( \lfloor 3k/2 \rfloor - 1 \), which will be sketched in the next section.

2. Preliminaries

In this paper, all graphs are finite, simple and undirected. A graph \( G \) is defined as a pair \((V, E)\) of a finite set \( V \) and \( E \subseteq \{\{u, v\} \mid u, v \in V, u \neq v\} \), where \( V \) is the set of vertices of \( G \) and \( E \) is the set of edges of \( G \). The degree of a vertex \( v \in V \) is the number of edges incident to \( v \), i.e., \( |\{e \in E \mid v \in e\}| \).

A graph \( G' = (V', E') \) is a subgraph of a graph \( G = (V, E) \) if \( V' \subseteq V \), \( E' \subseteq E \) and \( u, v \in V' \) for every \( e = \{u, v\} \in E' \). For a graph \( G = (V, E) \), the complement of \( G \), denoted by \( \overline{G} \), is a graph with vertex set \( V \) and edge set \( \overline{E} = \{\{u, v\} \mid u, v \in V, u \neq v, \{u, v\} \notin E\} \). Namely, \( \overline{G} = (V, \overline{E}) \). A pair \((G, H)\) of graphs is called complementary if \( H = \overline{G} \).

A graph is complete if each pair of vertices is joined by an edge. The complete graph with \( n \) vertices is denoted by \( K_n \). A graph \( P = (V, E) \) is a path if \( V = \{v_1, v_2, \ldots, v_n\} \), and \( E = \{\{v_i, v_{i+1}\} \mid 1 \leq i \leq n - 1\} \). The path with \( n \) vertices is denoted by \( P_n \).

A walk is a sequence \( v_1 e_1 v_2 \ldots e_{k-1} v_k \) of vertices \( v_i \) and edges \( e_i \) such that for \( 1 \leq i \leq k \), the edge \( e_i = \{v_i, v_{i+1}\} \).

Here, \( k \) is the number of vertices of the walk, \( v_1 \) and \( v_k \) are called endpoints of the walk. A trail is a walk in which all the edges are different from each other. A trail that satisfies \( v_1 = v_k \) is called a circuit. A graph is connected if it has a trail from any vertex to any other vertex.

Let \( G \) be a connected graph. An Eulerian circuit of \( G \) is a circuit of \( G \) that passes every edge exactly once. If \( G \) has an Eulerian circuit, then \( G \) is called Eulerian. An Eulerian trail of \( G \) is a trail of \( G \) that passes every edge exactly once. If \( G \) has an Eulerian trail but no Eulerian circuit, then \( G \) is called a semi-Eulerian. It is well-known and easy to prove that a connected graph \( G \) is Eulerian if and only if the degree of every vertex of \( G \) is even, and \( G \) is semi-Eulerian if and only if the number of odd-degree vertices is two.

For \( k \geq 1 \), we denote by \( T_k \) the set of connected graphs that have an Eulerian circuit or an Eulerian trail with \( k \) vertices (see Fig.1). Note that in our definitions, vertices in trails and circuits are counted multiple times if they are passed multiple times. Therefore, some graphs in \( T_k \) may have less than \( k \) vertices.

Let \( C \) and \( C' \) be two graph classes, i.e., possibly infinite
sets of graphs. Then, the Ramsey number of \( C \) and \( C' \) is the smallest number \( n \) such that for every graph \( H \) with \( n \) vertices, \( H \) contains a graph in \( C \) or \( \overline{H} \) contains a graph in \( C' \). The Ramsey number of \( C \) and \( C' \) is denoted by \( R(C, C') \). If \( C \) and \( C' \) are singletons (i.e., contain only one graph as \( C = \{ G \} \) and \( C' = \{ G' \} \)), then the Ramsey number of \( C \) and \( C' \) is denoted by \( R(G, G') \).

Gerencsér and Gyárfás [2] determined the exact value of the Ramsey number of paths, as in the following theorem.

**Lemma 1 ([2]):** Let \( k \geq \ell \geq 2 \). Then, \( R(P_k, P_\ell) = k + \lfloor \ell/2 \rfloor - 1 \).

Since \( P_k \) belongs to \( T_k \), \( R(T_k, T_k) \leq R(P_k, P_k) = k + \lfloor k/2 \rfloor - 1 = \lfloor 3k/2 \rfloor - 1 \). This is the trivial upper bound mentioned in the previous section.

To gain the first impression, we have conducted a computer search of the Ramsey number \( R(T_k, T_k) \) for small values of \( k \). This has been performed with the following procedure. For \( 2 \leq n \leq 7 \), we generate all graphs \( G \) with \( n \) vertices. For each such \( G \), we calculate \( t(G) \), which is defined as the number of vertices in the longest trail in \( G \) or \( \overline{G} \). Then, we determine \( \text{value}(n) \), which is defined as the minimum value of \( t(G) \) for all \( G \) with \( n \) vertices. If \( k \) satisfies \( \text{value}(n - 1) < k \leq \text{value}(n) \), then we know that \( R(T_k, T_k) \) is equal to \( n \).

The result of the computer search is summarized in Table 1. We may observe that the upper bound of \( \lfloor 3k/2 \rfloor - 1 \) should be improved.

### 3. Main Theorem: Lower Bound

We begin with a lower bound of \( R(T_k, T_k) \).

**Theorem:** Let \( k \) be a positive integer. Then,

\[
R(T_k, T_k) \geq \begin{cases} 
  k & \text{if } k \leq 6, \\
  1 + \sqrt{16k - 7} \over 2 & \text{if } k \geq 7.
\end{cases}
\]

The rest of the section is devoted to the proof of Theorem 1. We first consider the case when \( k \leq 6 \).

Let \( k = 2 \). Then, there is no graph of \( T_2 \) in the complete graph \( K_1 \). Therefore, \( R(T_2, T_2) \geq 2 \).

Let \( k = 3 \). Then, the complete graph \( K_3 \) has only one edge. So, there is no graph of \( T_3 \) in \( K_2 \). Thus, \( R(T_3, T_3) \geq 3 \).

Let \( k = 4 \). Consider the complementary pair of graphs with three vertices as shown in Fig. 2. Since those two graphs have at most two edges, no element of \( T_4 \) is contained in either graph. Therefore, \( R(T_4, T_4) \geq 4 \).

Let \( k = 5 \). Consider the complementary pair of graphs with four vertices as shown in Fig. 3. Since those two graphs have three edges, no element of \( T_5 \) is contained in either graph. Hence, \( R(T_5, T_5) \geq 5 \).

Let \( k = 6 \). Consider the complementary pair of graphs with five vertices as shown in Fig. 4. Those graphs have five edges. Hence, for an element of \( T_6 \) to be contained in either of the two graphs, one of the two graphs must be Eulerian or semi-Eulerian. However, each graph has four odd-degree vertices. Thus, these graphs are neither Eulerian nor semi-Eulerian, and have no elements of \( T_6 \).

Next, we consider the case where \( k \geq 7 \). To complete the proof, we use the following two lemmas.

**Lemma 2:** The number of vertices of a complete graph with \( m \geq 0 \) edges is \((1 + \sqrt{1 + 8m})/2)\).

**Proof:** Let \( n \geq 1 \) be the number of vertices of a complete graph with \( m \) edges. In this case, \( m = n(n - 1) \). Solving for \( n \geq 1 \), we have \( n = (1 + \sqrt{1 + 8m})/2)\).

**Lemma 3:** Let \( k \geq 7 \) and let \( n \) be the number of vertices of a complete graph with at most \( 2k - 2 \) edges. Then, there exists a subgraph \( G = (V, E) \) of \( K_n \) such that \( G \) and \( \overline{G} \) have no element of \( T_k \).

Before proving Lemma 3, we finish the proof of Theorem 1 using Lemmas 2 and 3.

**Proof of Theorem 1 when \( k \geq 7 \):** Let \( k \geq 7 \) and let \( n \) be the number of vertices of a complete graph with at most \( 2k - 1 \) edges. Then, \( R(T_k, T_k) \geq n \) from Lemma 3. Therefore, by
Lemma 2

\[ R(\mathcal{T}_k, \mathcal{T}_k) \geq \frac{1 + \sqrt{1 + 8(2k - 1)}}{2} = \frac{1 + \sqrt{16k - 7}}{2}. \]

Thus, it suffices to prove Lemma 3.

**Proof of Lemma 3.** We distinguish the cases \(|E| \leq 2k - 4, \ |E| = 2k - 3\), and \(|E| = 2k - 2\).

> Case 1: \(|E| \leq 2k - 4\).

Choose \(G = (V, E_1)\) as any subgraph with \(|E_1| = \lfloor |E|/2 \rfloor\). Then, since \(|E_1| = \lfloor |E|/2 \rfloor \leq k - 2, |E - E_1| = \lfloor |E|/2 \rfloor \leq k - 2\), it follows that \(G\) and \(G\) have no element of \(\mathcal{T}_k\).

> Case 2: \(|E| = 2k - 3\).

Consider a complete graph \(K_n\) with \(2k - 3\) edges. Then,

\[ n = \frac{1 + \sqrt{1 + 8(2k - 3)}}{2} = \frac{1 + \sqrt{16k - 23}}{2} > 5 \]

from Lemma 2. Let \(v_1, \ldots, v_n\) be the vertices of \(K_n\), and let \(E_c = \{\{v_i, v_{i+1}\} \mid i = 1, 2, \ldots, n - 1\} \cup \{v_n, v_1\}\). Then, the graph \(C = (V, E_c)\) is a cycle contained in \(K_n\). We construct an Eulerian or a semi-Eulerian graph \(S\) with \(k + 1\) edges that contains \(C\).

Before constructing such \(S\), we observe that this is enough for our purpose. Since \(S\) contains \(C\) and \(n \geq 6\), there exist two edges \(e_1, e_2\) of \(C\) such that \(S' = S - \{e_1, e_2\}\) has exactly four odd-degree vertices. Thus, \(S'\) is neither Eulerian nor semi-Eulerian. Since \(S'\) has only \(k - 1\) edges, \(S'\) includes no element of \(\mathcal{T}_k\). Further, \(S\) has only \(k - 2\) edges, and \(S'\) includes no element of \(\mathcal{T}_k\).

To find a subgraph \(S\) with the desired properties, we further distinguish two cases according to the parity of \(n\).

> Case 2-1: \(n\) is odd.

Let \(G = (V, E - E_c)\). Then, \(G\) is Eulerian since the degree of each vertex of \(G\) is even and \(G\) is connected. Thus, \(G\) contains a trail with \(|E| - n + 1 = 2k - 2 - n \geq 2k - 2\) vertices. Let \(T\) be a subgraph of \(G\) obtained by the first \(k + 1 - n\) edges of such a trail, and let \(S = C \cup T\). Then, \(S\) is Eulerian or semi-Eulerian with \(k + 1\) edges.

> Case 2-2: \(n\) is even.

Let \(G = (V, E - E_c - \{\{v_i, v_{i+2}\} \mid i = 1, 2, \ldots, n/2\})\). Then, \(G\) is Eulerian since the degree of each vertex of \(G\) is even and \(G\) is connected.

Thus, \(G\) contains a trail with \(|E| - 3n/2 + 1 = 2k - 2 - 3n/2 \geq k + 2 - n\) vertices since \(n > 5\) and \(n(n-1)/2 = 2k - 3\). Let \(T\) be a subgraph of \(G\) obtained by the first \(k + 1 - n\) edges of such a trail, and let \(S = C \cup T\). Then, \(S\) is Eulerian or semi-Eulerian with \(k + 1\) edges.

> Case 3: \(|E| = 2k - 2\).

This case is analogous to Case 2 where \(|E| = 2k - 3\). Note that for a complete graph \(K_n\) with \(2k - 2\) edges. We have

\[ n = (1 + \sqrt{1 + 8(2k - 2)})/2 = (1 + \sqrt{16k - 15})/2 > 5 \]

from Lemma 2.

We have to take care of the argument after constructing \(S\) because \(S'\) has \(k - 1\) edges and we need a different argument to show that \(S'\) includes no element of \(\mathcal{T}_k\). Remind that \(S'\) contains \(k - 1\) edges from a trail of \(G\) and the edges of \(C\).

We distinguish two cases according to the parity of \(n\). First, let \(n\) be odd. Then, the degree of every vertex of \(K_n\) is even. Since \(S'\) has four odd-degree vertices, \(S'\) has four odd-degree vertices, too. Thus, \(S'\) is neither Eulerian nor semi-Eulerian. Since \(S'\) has only \(k - 1\) edges, \(S'\) includes no element of \(\mathcal{T}_k\).

Next, let \(n\) be even. We first observe that \(n \geq 8\). We already know that \(n \geq 6\), but if \(n = 6\), then the number of edges of \(K_n\) is 15, which is not of the form \(2k - 2\); this is impossible. Therefore, \(S'\) has at least four odd-degree vertices since \(S'\) has \(n - 4\) even-degree vertices and \(n - 4 \geq 4\). Thus, \(S'\) is neither Eulerian nor semi-Eulerian. Since \(S'\) has only \(k - 1\) edges, \(S'\) includes no element of \(\mathcal{T}_k\).

\[ \square \]

4. Main Theorem: Upper Bound

We already observed that \(R(\mathcal{T}_k, \mathcal{T}_k) \leq \lfloor 3k/2 \rfloor - 1\) as a trivial upper bound. Now, we improve the upper bound in the next theorem.

**Theorem 2:** For every integer \(k \geq 2\), it holds that \(R(\mathcal{T}_k, \mathcal{T}_k) \leq k\).

To this end, for any graph \(G\) with \(k\) vertices, we prove either \(G\) or its complement \(\overline{G}\) contains a trail with \(k\) vertices.

We begin with the following lemma which will be used in the proof of the theorem.

**Lemma 4:** Let \(G = (V, E)\) be a bipartite graph with partite sets \(A\) and \(B\), i.e., \(A \cup B = V, A \cap B = \emptyset\) and each edge of \(G\) joins a vertex of \(A\) and a vertex of \(B\). If \(|A| = 3\) and the degree of every vertex of \(B\) is two, then \(G\) contains a trail such that both endpoints belong to \(A\) and the number of edges is \(|2|B|\).

**Proof.** Denote the three elements of \(A\) by \(a_1, a_2, a_3\). We distinguish the following two cases according to the existence of an isolated vertex (i.e., a vertex of degree zero) in \(A\).

> Case 1: \(A\) has an isolated vertex.

Without loss of generality, assume that \(a_3\) is an isolated vertex. Since each vertex in \(B\) has degree two, it is adjacent to \(a_1\) and \(a_2\). Hence, the bipartite graph \(G' = G - a_3\) is connected. Furthermore, the number of odd-degree vertices in \(G'\) is zero or two since the degree of \(a_1\) and \(a_2\) is \(|B|\), and the degree of every vertex in \(B\) is two. Thus, \(G'\) is Eulerian or semi-Eulerian and has \(2|B|\) edges. When \(G'\) is Eulerian, \(G'\) contains a trail with \(2|B|\) edges such that both endpoints coincide with \(a_1\). When \(G'\) is semi-Eulerian, \(G'\) contains a trail with \(2|B|\) edges such that one endpoint is \(a_1\) and the other endpoint is \(a_2\).
Case 2: A has no isolated vertex.

Without loss of generality, assume that there exists a vertex \( b \in B \) adjacent to \( a_1 \) and \( a_2 \). Since there is no isolated vertex, there is a vertex \( b' \in B \) adjacent to \( a_3 \). As the degree of \( b' \) is two, \( b' \) is adjacent to either \( a_1 \) or \( a_2 \). Therefore, the three vertices of \( a_1, a_2 \) and \( a_3 \) are connected by paths. This implies that \( G \) is connected since every vertex in \( B \) is adjacent to one of \( a_1, a_2 \) and \( a_3 \). Since \( G \) is bipartite and the degree of every vertex in \( B \) is two, the sum of the degrees of \( a_1, a_2 \) and \( a_3 \) is even. If there are an odd number of odd-degree vertices in \( A \), then the sum of the degrees of \( a_1, a_2 \) and \( a_3 \) is odd, contradicting the fact that the sum of the degrees of \( a_1, a_2 \) and \( a_3 \) is even. Therefore, the number of odd-degree vertices is zero or two.

When there is no odd-degree vertex, then \( G \) is Eulerian, and contains a trail with 2\(|B|\) edges such that both endpoints coincide with \( a_1 \). When there are two odd-degree vertices, let them be \( a_1 \) and \( a_2 \). Then, \( G \) is semi-Eulerian, and contains a trail with 2\(|B|\) edges such that one endpoint is \( a_s \) and the other endpoint is \( a_t \).

We are now ready for the proof of Theorem 2.

Proof of Theorem 2. The proof uses the induction on \( k \). When \( k \leq 10 \), \( R(T_k, T_k) \leq k \) holds from Table 1.

Now, fix an arbitrary integer \( k \geq 11 \) and suppose that the statement is true for \( k' < k \). Consider a graph \( G = (V, E) \) with \( k \) vertices. For a subgraph \( G' \) with \( k-1 \) vertices of \( G \), by induction hypothesis, either \( G' \) or \( \overline{G'} \) contains a trail \( S \) with \( k-1 \) vertices. If \( G' \) contains \( S \), then \( G \) contains \( S \) because \( G' \) is a subgraph of \( G \). If \( \overline{G'} \) contains \( S \), then \( G \) contains \( S \) because \( \overline{G'} \) is a subgraph of \( \overline{G} \). Therefore, either \( G \) or \( \overline{G} \) contains \( S \). Without loss of generality, suppose \( G \) contains \( S \). Let \( S = u_1e_1u_2e_2\ldots e_{k-2}u_{k-1} \) where \( e_i = \{u_i, u_{i+1}\} \) for all \( i \in \{1, 2, \ldots, k-2\} \), \( U = \{u_1, u_2, \ldots, u_{k-1}\} \) be the set of vertices in \( S \), and \( W = V - U \). Note that the size of \( U \) can be smaller than \( k-1 \) since some vertices can be identical.

If there exists a vertex \( w \in W \) such that \( \{u_1, w\} \in E \), then \( G \) contains the trail \( w \{u_1, w\} \) in \( k \) vertices. Similarly, if there exists a vertex \( w \in W \) such that \( \{u_{k-1}, w\} \in E \), then \( G \) contains the trail \( S \{u_{k-1}, w\} \) with \( k \) vertices. If there is a vertex \( u \in S \) such that \( \{u, u_1\} \in E \) is not included in \( S \), then \( G \) contains the trail \( u \{u, u_1\} \) with \( k \) vertices. Similarly, if there is a vertex \( u \in U \) such that \( \{u, u_{k-1}\} \in E \) is not included in \( S \), then \( G \) contains the trail \( S \{u_{k-1}, u\} \) with \( k \) vertices. In all of these cases, \( G \) contains a trail with \( k \) vertices and we are done.

Hence, we only need to consider the cases where the following two conditions are satisfied.

**Condition 1.** For every \( w \in W \), \( \{u_1, w\} \not\in E \) and \( \{u_{k-1}, w\} \not\in E \). That is, \( \{u_1, w\} \in \overline{E} \) and \( \{u_{k-1}, w\} \in \overline{E} \).

**Condition 2.** For every \( u \in U \), if \( \{u, u_1\} \) is not included in \( S \), then \( \{u, u_1\} \not\in E \). That is, \( \{u, u_1\} \in \overline{E} \). If \( \{u, u_{k-1}\} \) is not included in \( S \), then \( \{u, u_{k-1}\} \not\in E \). That is, \( \{u, u_{k-1}\} \in \overline{E} \).

We distinguish the cases according to the “shape” of \( S \).
or semi-Eulerian. Since $G'$ has $2|W|$ edges, $G'$ contains a trail $T'$ with $2|W|+1 \geq |U|+|W|+1 = |U|+(k-|U|)+1 = k+1$ vertices. Since $G'$ is a subgraph of $\overline{G}$, we conclude that $\overline{G}$ contains $T$.

Case 3: $S$ is not a path or a circuit.

Since $S$ is not a path, $|U| \leq k-2$. Since $S$ is not a circuit, $u_1 \neq u_{k-1}$. We distinguish cases according to the size of $U$.

Case 3-1: $|U| = k-2$.

Since $S$ is a trail with $k-1$ vertices and $|U| = k-2$, there is only one vertex $x$ that is used more than once in $S$. If $x \neq u_1$ and $x \neq u_{k-1}$, then there are at most two vertices adjacent to either $u_1$ or $u_{k-1}$ in $G$. If $x = u_1$ or $u_{k-1}$, then there are at most four vertices adjacent to either $u_1$ or $u_{k-1}$ in $G$.

Let $U'$ be the set of elements of $U - \{u_1, u_{k-1}\}$ that are not adjacent to either $u_1$ or $u_{k-1}$ in $G$. Every vertex $u' \in U'$ satisfies $\{u', u_1\} \notin E$ and $\{u', u_{k-1}\} \notin E$, i.e., $\{u_1, u_{k-1}\} \in E$ and $\{u', u_1\}, \{u', u_{k-1}\} \in E$. Further, $|U'| \geq |U| - \{u_1, u_{k-1}\} = k-4 = |U|-2-4 = k-8$. From Condition 1, every vertex $w \in W$, we have $\{u_1, w\} \in \overline{E}$ and $\{u_{k-1}, w\} \in \overline{E}$. Let $V' = U' \cup W' \cup \{u_1, u_{k-1}\}$. $E' = \{(w, u_1), (w, u_{k-1}) \mid w \in W\} \cup \{(u_1, u'), (u_{k-1}, u') \mid u' \in U'\} \subseteq \overline{E}$ and consider the subgraph $G' = (V', E')$ of $\overline{G}$. Since every vertex in $U' \cup W$ is adjacent to $u_1$ and $u_{k-1}$, $G'$ is connected. The degree of every vertex in $U' \cup W$ is two, and the degree of $u_1$ and $u_{k-1}$ is both $|U'| + |W|$. Hence, the number of odd-degree vertices in $G'$ is zero or two. Therefore, $G'$ is Eulerian or semi-Eulerian. Since $G'$ has $2(|U'|+|W|)$ edges, $G'$ contains a trail $T$ with $2(|U'|+|W|)+1 \geq (2k-8)+2+1 = 2k-11 \geq k$ vertices. Since $G'$ is a subgraph of $\overline{G}$, we conclude that $\overline{G}$ contains $T$.

Case 3-2: $|U| \leq \lfloor k/2 \rfloor$.

From Condition 1, for each vertex $w \in W$, we have $\{u_1, w\} \in \overline{E}$ and $\{u_{k-1}, w\} \in \overline{E}$. Let $V' = U' \cup \{u_1, u_{k-1}\}$, $E' = \{(w, u_1), (w, u_{k-1}) \mid w \in W\} \subseteq \overline{E}$ and consider the subgraph $G' = (V', E')$ of $\overline{G}$. Since every vertex in $W$ is adjacent to $u_1$ and $u_{k-1}$, $G'$ is connected. The degree of every vertex in $W$ is two, and the degree of $u_1$ and $u_{k-1}$ is both $|W|$. Hence, the number of odd-degree vertices in $G'$ is zero or two. Therefore, $G'$ is Eulerian or semi-Eulerian. Since $G'$ has $2|W|$ edges, $G'$ contains a trail $T$ with $2|W|+1 \geq 2 \lfloor k/2 \rfloor +1 \geq k+1$ vertices. Since $G'$ is a subgraph of $\overline{G}$, we conclude that $\overline{G}$ contains $T$.

Case 3-3: $\lfloor k/2 \rfloor < |U| \leq k-3$.

By the induction hypothesis, in $G$ or $\overline{G}$, there exists a trail $T$ with $|W| \geq 3$ vertices such that every vertex in $T$ is an element of $W$. Let $T = w_1e'_1w_2e'_2 \ldots e'_{|W|-1}w_{|W|}$ with $e'_i \notin \{w_1, w_{i+1}\}, i \in \{1, 2, \ldots, |W|-1\}$ and $W'$ be the set of vertices used in $T$.

We further distinguish two cases according to the containment of $T$ in $G$ or $\overline{G}$.

Case 3-3-1: $T$ is included in $G$.

Assume that there exists a vertex $u_i \in U$ adjacent to two vertices $w_x, w_y \in W'$ where $w_x \neq w_y, x < y$. Then, we have a trail $S' = u_1e_1u_2e_2 \ldots u_iw_xw_y \ldots w_y \{w_y, u_i\}u_i e_1 \ldots e_{k-2}u_{k-1}$. The number of vertices of $S'$ is at least $k$. Thus, we only need to consider the case where, for every vertex $u \in U$, there exists at most one element of $W'$ adjacent to $u$ in $G$.

Let $w'_1, w'_2$ and $w'_3$ be any three different vertices of $W'$. For every vertex $u \in U$, there exists at most one element of $W'$ adjacent to $u$ in $G$. Then, $u$ is adjacent to at least two vertices of $w_1, w_2$ and $w_3$ in $\overline{G}$. Therefore, $\overline{G}$ has the following bipartite graph $G'$ as a subgraph:

- The partite sets of $G'$ are $A = \{w'_1, w'_2, w'_3\}$ and $B = U$;
- The degree of each vertex in $B$ is two.

From Lemma 4, $G'$ has a trail $X$ with $2|B|$ edges, i.e., $2|B|+1 = 2|U|+1 > 2 \lfloor k/2 \rfloor +1 \geq k$ vertices. Since $G'$ is a subgraph of $\overline{G}$, $\overline{G}$ also has $X$.

Case 3-3-2: $T$ is included in $\overline{G}$.

Let $w'_1, w'_2$ and $w'_3$ be any three different vertices of $W'$. First, assume that in $G$ there exist three vertices $u_1, u_y, u_2 \in U - \{u_1, u_{k-1}\}$ that are adjacent to at least two of the vertices $w'_1, w'_2$ and $w'_3$. Then, one of the graphs in Fig. 5 always appears as a subgraph of $G$. In both cases, there exists a cycle $C$ that has a vertex in $U$. Therefore, we have a trail $S = u_1e_1u_2e_2 \ldots e_{x-1}C e_x \ldots e_{k-2}u_{k-1}$, and the number of vertices of $S$ is at least $k$.

Second, assume there are at most two elements of $U - \{u_1, u_{k-1}\}$ that are adjacent to at least two vertices of $w'_1, w'_2$ and $w'_3$ in $G$. Let $c$ and $d$ be those two vertices of $U - \{u_1, u_{k-1}\}$. Then, there is a subbipartite graph $G'$ in $\overline{G}$:

- The partite sets of $G'$ are $A = \{w'_1, w'_2, w'_3\}$ and $B = U - \{u_1, u_{k-1}, c, d\}$;
- The degree of each vertex in $B$ is two.

From Lemma 4, $G'$ has a trail $X$ with $2|B|+1 = 2(|U|-4)+1$ vertices. Let $s, t \in W'$ be the endpoints of $X$. We now construct a trail $T'$ with $|W|$ vertices such that it only consists of the vertices and edges used in $T$, and does not start at $t$. If $w_1 \neq t$, then we have $T' = T$. If $w_1 = t$ and $T$ is a circuit, then we have $T' = w_2e'_2 \ldots e'_{|W|-1}w_{|W|}e_1w_1$ since $w_1 = w_{|W|}$ and $w_\neq w_1$. If $w_1 = t$ and $T$ is not a circuit, then we have $T' = w_{|W|}e'_{|W|-1}w_{|W|-2} \ldots e'_1w_1$ since $w_1 \neq w_{|W|}$. Therefore, $T'$ can be constructed.
Since $T$ is included in $\overline{G}$, $T'$ is also included in $\overline{G}$. Let $w, x$ be the endpoints of $T'$. Then, we have a trail $Y = X\{t, u_1\}u_1\{u_1, w\}T'\{x, u_{k-1}\}u_{k-1}$ with $2|U| + |W| - 5$ edges. Hence, $Y$ is a trail with $2|U| + |W| - 4 = 2|U| + (k - |U|) - 4 = k + |U| - 4 > k + \lfloor k/2 \rfloor - 4 \geq k$ vertices.

5. Conclusion

From Theorems 1 and 2, we conclude that $R(T_k, T_k) = k$ when $k \leq 6$ and $2\sqrt{k} + \Theta(1) \leq R(T_k, T_k) \leq k$ when $k \geq 7$.

Future work is to find stricter upper and lower bounds. Another challenge is to find upper and lower bounds of $R(T_k, T_\ell)$ for any $k$ and $\ell$.

Acknowledgments

The author thanks Yoshio Okamoto for his dedication to this work, and anonymous referees for their helpful suggestions.

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