Threshold resonance and controlled filtering in quantum star graphs

ONDŘEJ TUREK and TAKSU CHEON\textsuperscript{(a)}

Laboratory of Physics, Kochi University of Technology - Tosa Yamada, Kochi 782-8502, Japan

received 19 March 2012; accepted in final form 15 May 2012
published online 13 June 2012

PACS 03.65.-w – Quantum mechanics
PACS 73.63.Nm – Quantum wires

Abstract – We design two simple quantum devices applicable as an adjustable quantum spectral filter and as a flux controller. Their function is based upon the threshold resonance in a Fülöp-Tsutsui–type star graph with an external potential added on one of the lines. The adjustment of the potential strength directly controls the quantum flow from the input to the output line. This is the first example to date in which the quantum flux control is achieved by addition of an external field not on the channel itself, but on other lines connected to the channel at a vertex.

The phenomenon described in this paper should be observable not only in quantum systems, but also in systems in which classical waves are used, for example in optical fibre networks, microwave guides, or optical laser systems.

When a quantum particle with mechanical energy $E$ living on a star graph comes in the vertex from the $j$-th line, it is scattered at the vertex into all the lines. The $i$-th component of the final-state wave function equals

$$
\psi_{ij}(x) = \begin{cases} 
\frac{1}{\sqrt{k_j}} e^{-ik_jx} + S_{jj} \frac{1}{\sqrt{k_j}} e^{ik_jx}, & \text{for } i = j, \\
S_{ij} \frac{1}{\sqrt{k_i}} e^{ik_ix}, & \text{for } i \neq j, 
\end{cases}
$$

(1)

where $S_{ij}$ are the scattering amplitudes, $k_i$ are the momenta on the corresponding lines, and the coefficients $1/\sqrt{k_i}$ are involved for proper normalization. For any $i$, the momentum $k_i$ is equal to $k_i = \sqrt{E - U_i}$, where $U_i$ is the potential on the $i$-th line. The matrix $S = \{S_{ij}\}$ is the scattering matrix of the graph. For a normalized wave function coming in from the $j$-th line, $S_{ij}$ is interpreted as the complex amplitude of transmission into the $i$-th line (for $i \neq j$), whereas $S_{jj}$ represents the complex amplitude of reflection. The matrix $S$ depends, besides on the internal properties of the vertex, also on $E$ and $U_1, U_2, \ldots, U_n$.

\textsuperscript{(a)}E-mail: taksu.cheon@kochi-tech.ac.jp
To derive a formula for $S$, let us define the matrices $M = \{\psi_{ij}(0)\}$ and $M' = \{\psi'_{ij}(0)\}$. With regard to (1), these holds

$$
M = K^{-1} + K^{-1}S, \\
M' = iK^2(-K^{-1} + K^{-1}S),
$$

(2)

where $K = \{\sqrt{k}\delta_{ij}\}$. Any wave function $\Psi(x) = (\psi_1(x), \ldots, \psi_n(x))^T$ (the superscript $T$ denotes the transposition) on the graph obeys the boundary condition determining the vertex, which is usually written (also called "scale invariant") singular coupling (cf. [6–8]) with the explicit notation

$$
\psi_j(x) = \psi_j'(x), \ldots, \psi_n(x)\),
$$

for $j = 1, \ldots, n$, where $A, B \in \mathbb{C}^{n,n}$, cf. [5]. In particular, the boundary condition must be satisfied by the final-state wave function $\psi_{ij}(x) = (\psi_{1j}(x), \ldots, \psi_{nj}(x))^T$ determined in (1) for all $j$, hence $AM + BM' = 0$, which together with (2) leads to the sought expression for $S$:

$$
S = -(AK^{-1} + iBK)^{-1}(AK^{-1} - iBK).
$$

(3)

The squared moduli of the elements of $S$ have the following interpretation: $|S_{ij}|^2$ for $j \neq i$ represents the probability of transmission from the $i$-th to the $j$-th line, $|S_{jj}|^2$ is the probability of reflection on the $j$-th line.

Now consider an $n = 3$ star graph with a Fülöp-Tsutsui (also called "scale invariant") singular coupling (cf. [6–8]) in its vertex. For the sake of convenience, the coupling will be described by a boundary condition written in the so-called $ST$-form ($BK' = -A\Psi$ with specially structured $A, B, \Psi$; see [9] and [10]) with the explicit notation

$$
\begin{pmatrix}
1 & a & b \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\psi'_j(0) \\
| \psi'_j(0) \\
| \psi'_j(0)
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & 1 \\
a & 0 & 0 \\
-b & 1 & 0
\end{pmatrix}
\begin{pmatrix}
\psi_1(0) \\
| \psi_2(0) \\
| \psi_3(0)
\end{pmatrix}.
$$

(4)

The graph is schematically illustrated in fig. 1. The roles of individual lines are the following:

- Line 1 is the input. Particles of various energies are coming in the vertex along this line.
- Line 2 is the output. Particles passed through the vertex are gathered on this line.
- Line 3 is the controlling line. We assume that this line is subjected to a constant (but adjustable) external potential $U$.

A quantum particle with energy $E = k^2$ coming in the vertex from the input line 1 is scattered at the vertex into all the lines. The scattering amplitudes can be calculated by substituting the matrices $A, B$ from the boundary condition (4) together with

$$
k_1 = k_2 = k, \quad k_3 = \sqrt{k^2 - U}
$$

(5)

into equation (3). We obtain

$$
S_{21}(k; U) = \frac{2a}{1 + a^2 + b^2\sqrt{1 - \frac{U}{k^2}}},
$$

(6)

$$
S_{11}(k; U) = \frac{1 - a^2 - b^2\sqrt{1 - \frac{U}{k^2}}}{1 + a^2 + b^2\sqrt{1 - \frac{U}{k^2}}},
$$

(7)

$$
S_{31}(k; U) = \frac{2b\left(1 - \frac{U}{k^2}\right)^{\frac{3}{2}}}{1 + a^2 + b^2\sqrt{1 - \frac{U}{k^2}}}.
$$

(8)

The Heaviside step function $\Theta(k - \sqrt{U})$ is added in equation (8) to make the expression valid for all energies $k^2$, including $k^2 < U$. It represents asymptotically no transmission to the line 3 below the threshold momentum $k_{\text{th}} = \sqrt{U}$.

We are interested in the probability of transmission into the output line 2, which we denote by $\mathcal{P}(k; U)$, and particularly in its $k$-dependence. Since $\mathcal{P}(k; U) = |S_{21}(k; U)|^2$, we have from (6)

$$
\mathcal{P}(k; U) = \frac{4a^2}{(1 + a^2 + b^2\sqrt{1 - \frac{U}{k^2}})^2}, \quad \text{for } k \geq \sqrt{U},
$$

$$
\mathcal{P}(k; U) = \frac{4a^2}{(1 + a^2 + b^2(U/k^2 - 1))}, \quad \text{for } k \leq \sqrt{U}.
$$

(9)

We observe that for a given constant potential on the line 3, $\mathcal{P}(k; U)$ as a function of $k$ grows in the interval $(0, \sqrt{U})$, attains its maximum at $k = \sqrt{U}$, and decreases in the interval $(\sqrt{U}, \infty)$. In particular, there holds

$$
\lim_{k \to 0} \mathcal{P}(k; U) = 0,
$$

(10)

$$
\mathcal{P}(\sqrt{U}; U) = \left(\frac{2a}{1 + a^2}\right)^2,
$$

(11)

$$
\lim_{k \to \infty} \mathcal{P}(k; U) = \left(\frac{2a}{1 + a^2 + b^2}\right)^2.
$$

(12)

If the parameters $a, b$ satisfy $b \gg a \geq 1$, the function $\mathcal{P}(k; U)$ has a sharp peak at $k = \sqrt{U}$. Equation (11) implies that the peak is highest possible (attaining 1) for $a = 1$. We conclude: If $b \gg a = 1$, the system has high input–output transmission probability for particles having momenta $k \approx \sqrt{U}$ and the transmission is perfect for $k = \sqrt{U}$, whereas there is just a very small transmission probability for other values of $k$. The situation is numerically illustrated in fig. 2. The quantum graph schematically depicted in fig. 1 can therefore be used as an adjustable spectral filter, controllable by the potential put on the controlling line 3.

50005-p2
potential set to $U$. The transmission probability $\mathcal{P}(k; U)$ as a function of $k$ with the value of the potential set to $U = 1$ is plotted in the top figure. The lower figure shows the reflection probability $|S_{11}(k; U)|^2$ and the probability of transmission to the controlling line $|S_{31}(k; U)|^2$.

The bandwidth $W$ of the filter, i.e., the width of the interval of energies $k^2$ for which $\mathcal{P}(k; U) > 1/2$, depends on $U$ and $b$, and for $b \gg 1$ it is approximately given as $W \approx 4.7U/b^4$. Let us remark that the resonance at the threshold momentum $k_{th} = \sqrt{U}$ is related to the pole of the scattering matrix which is located on the positive real axis at $k_{pol} = \frac{1}{\sqrt{b^2 - (1 + a^2)^2}} \sqrt{U}$ on the unphysical Riemann surface which is connected to the physical Riemann surface at the cut that runs between $k = \pm \sqrt{U}$.

In order to develop another quantum device, let us consider an $n = 4$ star graph as schematically illustrated in fig. 3. The meaning of the first three lines will be the same as in the previous model: $1 = $ input, $2 = $ output, $3 = $ controlling line subjected to a constant external potential $U$. The line No. 4 is a drain and is included in the model for technical reasons: our considerations showed that the device we wish to construct is mathematically feasible using a vertex of degree $n = 3$. The vertex coupling is again assumed to be of the Fülöp-Tsutsui-type, and its properties are determined by the boundary condition written for convenience in the $ST$-form

$$
\begin{pmatrix}
1 & 0 & a & a \\
0 & 1 & -a & -a \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\psi_1(0) \\
\psi_2(0) \\
\psi_3(0) \\
\psi_4(0)
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-a & -a & 1 & 0 \\
-a & a & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\psi_1(10) \\
\psi_2(10) \\
\psi_3(10) \\
\psi_4(10)
\end{pmatrix}
$$

(13)

with $a \in \mathbb{R}$. The block $\begin{pmatrix} a & a \\ b & c \end{pmatrix}$ is a special choice that ensued from our analysis; generally, the $ST$-form admits $\begin{pmatrix} a & a \\ b & c \end{pmatrix}$ for any $a, b, c, d \in \mathbb{C}$, cf. [9] or [10].

For a particle with energy $E = k^2$ coming in the vertex from the input line 1, we have

$$k_1 = k_2 = k, \quad k_3 = \sqrt{k^2 - U}, \quad k_4 = k,$$

and the scattering amplitudes can be calculated as

$$S_{21}(k; U) = \frac{2a^2}{(1 + 2a^2)} \frac{1 - \sqrt{1 - \frac{U}{k^2}}}{1 + 2a^2 \sqrt{1 - \frac{U}{k^2}}}$$

(15)

and

$$S_{11}(k; U) = \frac{1 - 4a^4}{(1 + 2a^2)} \frac{\sqrt{1 - \frac{U}{k^2}}}{1 + 2a^2 \sqrt{1 - \frac{U}{k^2}}}$$

(16)

$$S_{31}(k; U) = \frac{2a}{1 + 2a^2} \sqrt{1 - \frac{U}{k^2}}$$

(17)

$$S_{41}(k; U) = \frac{2a}{1 + 2a^2}$$

(18)

We again denote the transmission probability input $\rightarrow$ output by $\mathcal{P}(k; U) = |S_{21}(k; U)|^2$. There holds

$$\mathcal{P}(k; U) = \begin{cases}
4a^4 U/k^2 & \text{for } k \leq \sqrt{U}, \\
(1 + 2a^2)^2 (1 - 4a^4 + 4a^4 U/k^2)^2 & \text{for } k \geq \sqrt{U},
\end{cases}$$

(19)

hence

$$\lim_{k \to 0} \mathcal{P}(k; U) = \frac{1}{1 + 2a^2},$$

(20)

$$\lim_{k \to \infty} \mathcal{P}(k; U) = \frac{4a^4}{(1 + 2a^2)^2}$$

(21)

If $U$ is fixed, $\mathcal{P}(k; U)$ as a function of $k$ quickly falls off to zero at $k > \sqrt{U}$. A typical behaviour is illustrated in a numerical example in fig. 4. The peak at the threshold momentum $k_{th} = \sqrt{U}$, appearing for $a > 1/\sqrt{2}$, is again related to the pole in the unphysical Riemann plane at $k_{pol} = \frac{2a^2}{\sqrt{4a^4 + 1}} \sqrt{U}$.
There is a value of the parameter $a$ that deserves a particular attention, namely $a = 1/\sqrt{2}$. For this choice of $a$ the peak disappears and the function $P(k; U)$ becomes constant in the whole interval $(0, \sqrt{U})$:

$$P(k; U) = \begin{cases} \frac{1}{4}, & \text{for } k \leq \sqrt{U}, \\ \frac{1}{4} \left( \frac{1-\sqrt{1-U/k^2}}{1+\sqrt{1-U/k^2}} \right)^2, & \text{for } k > \sqrt{U}, \end{cases} \quad (23)$$

see fig. 5. The device then behaves as a spectral filter with a flat passband that transmits (with the probability of $1/4$) quantum particles with momenta $k \in [0, \sqrt{U}]$ to the output, whereas particles with higher momenta are diverted to other lines, mainly to 3 and 4. The process is directly controlled by the external potential $U$.

Since increasing $U$ opens the channel $1 \rightarrow 2$ for more particles, the device can be regarded as a quantum sluice-gate, applicable as a quantum flux controller. When there are many particles described by the momentum distribution $\rho(k)$ on the line 1, the flux $J$ to the line 2 is given by $J(U) = \int dk \rho(k) k P(k; U)$. Assuming a Fermi distribution with Fermi momentum $k_F$ larger than our range of operation of $\sqrt{U}$, we can set $\rho(k) = \rho = \text{const}$. With the approximation $P(k; U) \approx \frac{1}{2} \Theta(\sqrt{U} - k)$, we obtain $J(U) = \frac{1}{8} \rho U$, which indicates the linear flux control.

The sluice-gate built from an $n = 4$ star graph has one more operation mode. If the line No. 4 (the drain) is subjected to another external field $V$, $0 < V < U$, the channel $1 \rightarrow 2$ opens for particles with $k \in [\sqrt{V}, \sqrt{U}]$ and mostly closes for particles with $k$ outside this interval. The gate then works as a fully tunable band spectral filter. However, in contrast to the standard $V = 0$ operation mode, the filter with $V > 0$ does not have a flat passband.

It should be emphasized that the studied controllable filter devices using the threshold resonance became possible only with “exotic” Fülöp-Tsutsui-type couplings in the vertices. Standard vertex couplings (the free and the $\delta$-coupling) would not work this way. It is therefore essential, for the proposed designs to be viable, that the required Fülöp-Tsutsui vertices can be created One of
the possible ways might be to tailor them from standard couplings, which have a simple physical interpretation themselves [11]. Using the procedure devised in [9] and [10], any Fülöp-Tsutsui coupling given by a boundary condition with real matrices $A, B$ can be approximately constructed by assembling a few $\delta$-couplings. The solution for our case is shown in fig. 6.

It can be proven that the results showing the controllable filtering remain essentially the same even when we replace the constant potential on the whole controlling line by a potential separated from the vertex by a short gap, or a potential supported by a long enough finite segment. This is understood from the physical point of view because the filtering behavior of the system is based essentially on the blockade of the natural flow pattern by the added potential. The technical details, along with an extension of our model, are to be treated in full in a forthcoming publication.

***

We thank Prof. E. F. Redish for helpful suggestions. This research was supported by the Japan Ministry of Education, Culture, Sports, Science and Technology under the Grant No. 24540412.

REFERENCES

[1] Albeverio S., Gesztesy F., Hoegh-Krohn R. and Holden H., Solvable Models in Quantum Mechanics, 2nd edition with appendix by Exner P. (AMS Chelsea, RI) 2005.

[2] Exner P., Keating J.P., Kuchment P., Sunada T. and Tepljaev A. (Editors), Analysis on Graphs and Applications, AMS Proceedings of Symposia in Pure Mathematics Ser., Vol. 77 (AMS, Providence, RI) 2008, and references therein.

[3] Lawniczak M., Bauch S., Hul O. and Sirko L., Phys. Rev. E, 81 (2010) 046204.

[4] Cheon T., Exner P. and Turek O., J. Phys. Soc. Jpn., 78 (2009) 124004.

[5] Kostrykin V. and Schrader R., J. Phys. A: Math. Gen., 32 (1999) 595.

[6] Fülöp T. and Tsutsui I., Phys. Lett. A, 264 (2000) 366.

[7] Naimark K and Solomyak M., Proc. London Math. Soc., 80 (2000) 690.

[8] Sobolev A. V. and Solomyak M., Rev. Math. Phys., 14 (2002) 421.

[9] Cheon T., Exner P. and Turek O., Ann. Phys. (N.Y.), 325 (2010) 548.

[10] Cheon T. and Turek O., Phys. Lett. A, 374 (2010) 4212.

[11] Exner P., Lett. Math. Phys., 38 (1996) 313.