Cheeger–Colding–Tian Theory for Conic Kähler–Einstein Metrics

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Abstract
In this paper we extend the Cheeger–Colding–Tian theory to the conic Kahler–Einstein metrics. In general, there are no smooth approximations of a family of conic Kahler–Einstein metrics with Ricci curvature uniformly bounded from below. So we have to deal with the technical issues to extend the original arguments.

Keywords Cheeger–Colding–Tian theory · Conic Kahler–Einstein metrics

Mathematics Subject Classification 53C25

1 Introduction
In a series of papers [6–8], Cheeger and Colding studied singular structures of spaces which arise as limits of sequences of Riemannian manifolds with Ricci curvature bounded below in the Gromov–Hausdorff topology. One of fundamental results they proved is the existence of tangent cones of the limit space [7], that is,

Theorem 1.1 [7] Let \((M_i, g_i; p_i)\) be a sequence of n-dimensional Riemannian manifolds satisfying

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Ric\(_{M_i}(g_i) \geq -(n - 1)A^2 g_i\) and \(\text{vol}_{g_i}(B_{p_i}(1)) \geq v > 0\).

Assume that \((M_i, g_i; p_i)\) converge to a metric space \((Y, d; p_\infty)\) in the pointed Gromov-Hausdorff topology. Then for any \(y \in Y\) and sequence \(\{r_j\}\) with \(r_j \to 0\), there is a subsequence, say \(\{\bar{r}_k = r_{j(k)}\}\), such that \((Y, \bar{r}_k^{-2}d; y)\) converges in the pointed Gromov-Hausdorff topology to a metric space \(T_y Y\) which is a metric cone over another metric space whose diameter is less than \(\pi\). Such a \(T_y Y\) is referred to as a tangent cone of \(Y\) at \(y\).

Note that the tangent cone \(T_y Y\) is not necessarily unique and may depend on the sequence \(\{r_j\}\). As an application of this theorem, Cheeger and Colding were able to introduce a stratification of singularities of the limit space \(Y\).

**Definition 1.2** Let \((Y, d; p_\infty)\) be the limit of \((M_i, g_i; p_i)\) as in Theorem 1.1. Denote by \(\mathcal{R}\) the set of points which has a tangent cone isometric to \(\mathbb{R}^n\) and \(S = Y \setminus \mathcal{R}\). For \(k \leq n - 1\), we say that \(y \in S_k\) if there exist no tangent cones at \(y\) which can split off a Euclidean space \(\mathbb{R}^l\) isometrically with \(l > k\).

Applying Theorem 1.1 to iterated tangent cones, Cheeger and Colding showed

**Theorem 1.3** [7] We have that \(S = \bigcup_{k=0}^{n-2} S_k\) and \(\dim S_k \leq k\), where \(\dim\) denotes the Hausdorff dimension.

Based on the above theorem on existence of tangent cones, Cheeger, Colding and Tian [9] give further constraints on singularities of the limit space \(Y\) under certain curvature condition for \((M_i, g_i)\) (also see Cheeger [5]).

The purpose of this paper is to extend the Cheeger–Colding Theory to the following class of metrics. This extension provides a technical tool for [15] in which we prove a version of the Yau–Tian–Donaldson conjecture for Fano varieties with certain singularity.

**Definition 1.4** A length space \((M^n, d)\) is called a \(n\)-dimensional Riemannian manifold with singularity if there exists \(S \subseteq M\) with \(\mathcal{H}^n(S) = 0\) such that the followings hold:

(i) \(\mathcal{R} = M \setminus S\) is a smooth manifold and convex; moreover, the distance function \(d\) is induced from a smooth metric \(g\) on \(\mathcal{R}\).

(ii) for any \(\epsilon > 0\), denoting \(T_\epsilon = \{x \mid \text{dist}(x, S) \leq \epsilon\}\), there is a cut-off function \(\gamma_\epsilon \in C^\infty_0(M \setminus S)\) and

\[
\gamma_\epsilon \equiv 1 \text{ in } M \setminus T_\epsilon(S), \quad \int_\mathcal{R} |\nabla \gamma_\epsilon|^2 \leq \epsilon.
\]

(iii) on any domain \(U \subseteq M\) given a continuous function \(b\) defined in a neighborhood of \(\bar{U}\) and a real number \(c\), there is a bounded function \(h\) which is locally Lipschitz in \(U\) and continuous in \(\bar{U} \cap \mathcal{R}\) such that

\[
\begin{align*}
\Delta h &= c \text{ in } \mathcal{R}, \\
h|_{\partial U} \cap \mathcal{R} &= b|_{\partial U} \cap \mathcal{R}.
\end{align*}
\]
We will study the limit space of the $n$-dimensional Riemannian manifolds with singularity whose Ricci curvature is bounded from below. Let $\mathcal{M}(V, D, n)$ be the set of $n$-dimensional Riemannian manifolds $(M, d)$ with singularities satisfying

$$\mathcal{H}^n(M) \geq V, \quad \text{diam}(M, d) \leq D, \quad \text{Ric}(g) \geq 0 \text{ in } \mathcal{R}.$$ 

Let $(M_i, d_i)$ be a sequence of manifolds in $\mathcal{M}(V, D, n)$ and $(M_i, d_i) \to (X, d)$. In this paper, we will prove

**Theorem 1.5** For any $x \in X$ and sequence $\{r_j\}$ with $r_j \to 0$, there is a subsequence, say $\{\bar{r}_k = r_j(k)\}$, such that $(X, \bar{r}_k^{-2}d; x)$ converge in the pointed Gromov-Hausdorff topology to a metric space $T_x X$ which is a metric cone. Such $T_x X$ is referred as a tangent cone of $X$ at $x$. Moreover, there is a decomposition of $X$ into $\mathcal{R} \cup S$ such that $S = S_{n-2}$ and $\dim S_k \leq k$, where $S_k$ is defined as above.

In [2], Bamler considered another class of singular spaces modeled on Ricci bounded space or Ricci flow. His definition of singular space is stronger. Theorem 1.5 could be also proved using the theory of RCD spaces developed by Ambrosio and others [1,13,14,16]. Our proof here follows the approach of Cheeger–Colding by adapting their arguments to the conic case.

Let $M$ be a Kähler manifold and $D = \sum_{i=1}^k D_i$ be a normal crossing divisor. A metric $\omega$ is called a conic Kähler metric with conic angle $2\pi\beta_i$ along $D_i$, where $\beta_i \in (0, 1)$, if it is a smooth Kähler metric outside $D$ and for each point $p \in D$ where $D$ is defined by the equation $z_1 \ldots z_d = 0$ for some local coordinates $z_1, \ldots, z_n$, $\omega$ satisfies

$$C^{-1}\omega_{\text{cone}} \leq \omega \leq C\omega_{\text{cone}},$$

where $C$ is a positive constant and $\omega_{\text{cone}}$ is the model cone metric with cone angles $2\pi\beta_i$ along $\{z_i = 0\}$, that is,

$$\omega_{\text{cone}} = \sum_{i=1}^d \sqrt{-1} \frac{dz_i \wedge d\bar{z}_i}{|z_i|^{2(1-\beta_i)}} + \sum_{k=d+1}^n \sqrt{-1} dz_i \wedge d\bar{z}_i.$$

A conic Kähler metric is called a conic Kähler–Einstein metric on $M$ if for some constant $t$, $\omega$ satisfies

$$\text{Ric}(\omega) = t\omega + 2\pi \sum_{i=1}^k (1 - \beta_i) [D_i],$$

where $[D_i]$ denotes the current defined by integrating $2n - 2$-forms along $D_i$.

For any $\delta > 0$ and $V > 0$, we denote by $\mathcal{M}(n, k, \delta, V)$ the set of all $n$-dimensional conic Kähler–Einstein metrics $(M, \omega)$ satisfying

$$t \in [\delta, \delta^{-1}] \text{ and } \int_M \omega^n \geq V.$$
We will show

\[ \mathcal{M}(n, k, \delta, V) \subseteq \mathcal{M}(V, \pi \sqrt{(2n-1)/\delta}, 2n), \]

and consequently, we have the following:

**Theorem 1.6** For any limit space \( X \) of conic Kähler–Einstein metrics in \( \mathcal{M}(n, k, \delta, V) \), tangent cones of \( X \) exist, that is, for any \( x \in X \) and sequence \( \{ r_j \} \) with \( r_j \to 0 \), there is a subsequence, say \( \{ \tilde{r}_k = r_{j(k)} \} \), such that \( (X, \tilde{r}_k^{-2} d; x) \) converge in the pointed Gromov–Hausdorff topology to a metric space \( T_x X \) which is a metric cone. Moreover, there is a decomposition of \( X \) into \( \mathcal{R} \cup S \) such that \( S = S_{2n-2} \), \( S_{2k+1} = S_{2k} \) and \( \dim S_{2k} \leq 2k \).

### 2 Distance Function Comparison

Let \((M, d)\) be an \( n \)-dimensional Riemannian manifold with singularity which satisfies

\[ \text{Ric}(g) \geq 0 \ \text{in} \ \mathcal{R}. \]

We will derive some basic estimates on \( M \). On \( \mathcal{R} \), we have the Bochner formula:

\[ \frac{1}{2} \Delta |\nabla f|^2 = |\text{Hess} \ f|^2 + \text{Ric} \ (\nabla f \cdot \nabla f) + \langle \nabla f, \nabla \Delta f \rangle. \quad (1) \]

From this and the convexity of the regular part, the Laplacian comparison is the same as the smooth metric.

**Lemma 2.1** For any \( p \in \mathcal{R} \), \( r(\cdot) = \text{dist}(p, \cdot) \) satisfies

\[ \Delta r \leq \frac{n - 1}{r} \quad (2) \]

in the sense of distribution in \( \mathcal{R} \).

As a consequence, we have

**Lemma 2.2** For any \( p \in M \), the volume ratio \( r^{-n} \ \text{vol}(B_p(r)) \) is monotone decreasing.

**Proof** First we assume \( p \in \mathcal{R} \). Since the singular set \( S \) has zero volume, by the Fubini theorem, the \((n-1)\)-dimensional Hausdorff measure of \((\partial B_p(s)) \cap S\) vanishes for almost all \( s \in [0, r] \). Then by using the convexity of \( \mathcal{R} \) and arguing as in the smooth case using Lemma 2.1, we can conclude

\[ r^{-n} \ \text{vol}(B_p(r)) \leq s^{-n} \ \text{vol}(B_p(s)) \quad \text{for any} \ s < r. \]
In general, when \( s \leq r \) is given, we choose a sequence of point \( p_i \in \mathcal{R} \) converging to \( p \), then we have
\[
r^{-n} \text{vol}(B_{p_i}(r)) \leq s^{-n} \text{vol}(B_{p_i}(s)).
\]
Taking the limit as \( i \) goes to \( \infty \), we get the required monotonicity. \( \square \)

Using the convexity of the regular part, we can also show

**Lemma 2.3** Let \( A_1, A_2 \) be two bounded subsets of \( M \) and \( W \) be another subset of \( M \) satisfying
\[
\bigcup_{y_1 \in A_1, y_2 \in A_2} \gamma_{y_1y_2} \subseteq W,
\]
where \( \gamma_{y_1y_2} \) denotes a minimal geodesic connecting \( y_1 \) to \( y_2 \) in \( M \). Put
\[
D = \sup\{ d(y_1, y_2) \mid y_1 \in A_1, y_2 \in A_2 \}.
\]
Then for any smooth function \( e \) on \( W \), it holds
\[
\begin{align*}
\int_{(A_1 \cap \mathcal{R}) \times (A_2 \cap \mathcal{R})} & \int_0^{d(y_1, y_2)} e(\gamma_{y_1y_2}(s)) \, ds \\
& \leq c(n) D \left( \text{vol}(A_1) + \text{Vol}(A_2) \right) \int_W e \, dv. \quad (3)
\end{align*}
\]

**Proof** Note that
\[
\int_{(A_1 \cap \mathcal{R}) \times (A_2 \cap \mathcal{R})} \int_0^{d(y_1, y_2)} e(\gamma_{y_1y_2}(s)) \, ds
\]
\[
= \int_{A_1 \cap \mathcal{R}} dy_1 \int_{A_2 \cap \mathcal{R}} \frac{d(y_1, y_2)}{2} e(\gamma_{y_1y_2}(s)) \, ds \, dy_2
\]
\[
+ \int_{A_2 \cap \mathcal{R}} dy_2 \int_{A_1 \cap \mathcal{R}} \frac{d(y_1, y_2)}{2} e(\gamma_{y_1y_2}(s)) \, ds \, dy_1.
\]
On the other hand, for a fixed \( y_1 \in A_1 \cap \mathcal{R} \), by using the monotonicity formula (2), we have
\[
\int_{A_2 \cap \mathcal{R}} \int_{\frac{d(y_1, y_2)}{2}}^{d(y_1, y_2)} e(\gamma_{y_1y_2}(s)) \, ds \, dy_2
\]
\[
= \int_{A_2 \cap \mathcal{R}} \int_0^r e(\gamma_{y_1y_2}(s)) A(r, \theta) \, dr \, d\theta \, ds
\]
\[
\leq c(n) \int_{A_2 \cap \mathcal{R}} \int_0^r e(\gamma_{y_1y_2}(s)) A(s, \theta) \, dr \, d\theta \, ds
\]
\[ \leq c(n) D \int_W e \, dv. \]

Similarly,
\[ \int_{A_1 \cap \mathcal{R}} \int_{d(y_1, y_2)} e(y_{y_1, y_2}(s)) \, ds \, dy_1 \]
\[ \leq c(n) D \int_W e \, dv. \]

Then (3) follows from the above two inequalities. \qed

For any three points \( x, y, z \), put
\[ \int_{\gamma_{x,y}} e = \int_0^{d(x, y)} e(y_{y_1, y_2}(s)) \, ds \]
and
\[ \int_{\Delta_{xyz}} e = \int_{w \in \gamma_{xy}} \int_{\gamma_{zw}} e. \]

Then, by applying Lemma 2.3 twice, we get

**Lemma 2.4** Let \( A_1, A_2, A_3 \) be three bounded subsets of \( M \) and \( W, Z \) be another two subsets of \( M \) satisfying

\[ \bigcup_{x \in A_1, y \in A_2} \gamma_{y_1, y_2} \subseteq W \quad \text{and} \quad \bigcup_{w \in W, z \in A_3} \gamma_{zw} \subseteq Z. \]

Then for any smooth function \( e \) on \( Z \), it holds
\[ \int_{(A_1 \cap \mathcal{R}) \times (A_2 \cap \mathcal{R}) \times (A_3 \cap \mathcal{R})} \left( \int_{\Delta_{xyz}} e \right) \, dv \]
\[ \leq c(n) \text{diam}(W) \text{diam}(Z) (\text{vol}(A_1) + \text{vol}(A_2)) \]
\[ \text{vol}(A_3) + \text{vol}(W) \int_Z e \, dv. \] (4)

**Lemma 2.5** Let \( u \) be a bounded function in a bounded domain \( \Omega \). Assume that \( u \) is harmonic in \( \Omega \cap \mathcal{R} \) and \( u \leq 0 \) on \( \partial \Omega \). Then \( u \leq 0 \) in \( \Omega \).

**Proof** At first, we deal with the special case when \( u = 0 \) on \( \partial \Omega \). Then we have
\[ \int_{\Omega \cap \mathcal{R}} |\nabla u|^2 \gamma_e^2 = -2 \int_{\Omega \cap \mathcal{R}} u \gamma_e \langle \nabla u, \nabla \gamma_e \rangle - \int_{\Omega \cap \mathcal{R}} u \gamma_e^2 \Delta u. \]
\[
\leq \frac{1}{4} \int_{\Omega \cap \mathcal{R}} |\nabla u|^2 \gamma^2_\epsilon + 4 \int_{\Omega \cap \mathcal{R}} |u|^2 |\nabla \gamma_\epsilon|^2.
\]

So we have
\[
\int_{\Omega \cap \mathcal{R}} |\nabla u|^2 \gamma^2_\epsilon \leq C \int_{\Omega \cap \mathcal{R}} |\nabla \gamma_\epsilon|^2.
\]

Taking \( \epsilon \to 0 \), we get
\[
\int_{\Omega \cap \mathcal{R}} |\nabla u|^2 = 0
\]

which implies that \( u \equiv 0 \) in \( \Omega \).

Now we consider the general case. If there is a point \( p \in \Omega \cap \mathcal{R} \) such that \( u(p) > 0 \), then \( \Omega' = \{ x \mid u(x) > 0 \} \) is a non-empty domain. Since \( u \) vanishes on the boundary of \( \Omega' \), we deduce from the above special case that \( u \equiv 0 \) on \( \Omega' \). It is a contradiction. The lemma is proved. \( \square \)

Note that by using the cut-off function \( \gamma_\epsilon \) in ii), we can show that integration by parts holds on \( M \).

**Lemma 2.6** Assume that \( \Omega \) is a bounded domain, \( \phi \in C^0(\Omega) \cap C^\infty(\Omega \cap \mathcal{R}) \) and \( u \in L^\infty(\Omega) \cap C^\infty(\Omega \cap \mathcal{R}) \) satisfying
\[
|u|_{L^\infty(\Omega)} + |\nabla \phi|_{C^0(\Omega \cap \mathcal{R})} + |\Delta \phi|_{C^0(\Omega \cap \mathcal{R})} \leq C.
\]

If
\[
\int_{\Omega \cap \mathcal{R}} \phi^2 |\nabla u|^2 < \infty,
\]
we have
\[
\lim_{\epsilon \to 0} \int_\Omega \phi \gamma^2_\epsilon \Delta u = \int_{\Omega \cap \mathcal{R}} u \Delta \phi.
\]

**Proof** Using integration by parts, we get
\[
\int_\Omega \phi \gamma^2_\epsilon \Delta u = \int_\Omega u \gamma^2_\epsilon \Delta \phi + 2 \int_\Omega (\nabla \phi, \nabla \gamma_\epsilon) u \gamma_\epsilon - 2 \int_\Omega (\nabla u, \nabla \gamma_\epsilon) \phi \gamma_\epsilon \tag{5}
\]

Since
\[
\left| \int_\Omega (\nabla \phi, \nabla \gamma_\epsilon) u \gamma_\epsilon \right| \leq C \int_\Omega |\nabla \gamma_\epsilon| \to 0
\]
and
\[ \left| \int_{\Omega} (\nabla u, \nabla \gamma_\epsilon) \phi \gamma_\epsilon \right| \leq \left( \int_{\Omega} |\nabla \gamma_\epsilon|^2 \int_{\Omega} \phi^2 |\nabla u|^2 \right)^{\frac{1}{2}}, \]
we get the result. \qed

The integration condition can be obtained by applying the Bochner formula.

Lemma 2.7 Assume that \( \Omega \) is a bounded domain, \( \phi \in C^0(\Omega) \cap C^\infty(\Omega \cap \mathcal{R}) \) non-negative and \( u \in L^\infty(\Omega) \cap C^\infty(\Omega \cap \mathcal{R}) \) satisfying
\[ |u|_{L^\infty(\Omega)} + |\nabla \phi|_{C^0(\Omega \cap \mathcal{R})} + |\Delta \phi|_{C^0(\Omega \cap \mathcal{R})} \leq C. \]
If \( \Delta u \geq c |\nabla u|^2 \) in \( \Omega \cap \mathcal{R} \) for some \( c > 0 \), we have
\[ \int_{\Omega \cap \mathcal{R}} \phi |\nabla u|^2 < \infty. \]

Proof From (5), we have
\[ \int_{\Omega} \phi \gamma_\epsilon^2 \Delta u \leq \int_{\Omega} u \gamma_\epsilon^2 \Delta \phi + C \int_{\Omega} |\nabla \gamma_\epsilon| + 2 \left( \int_{\Omega} |\nabla \gamma_\epsilon|^2 \int_{\Omega} \phi \gamma_\epsilon^2 |\nabla u|^2 \right)^{\frac{1}{2}}. \quad (6) \]
Since \( \Delta u \geq c |\nabla u|^2 \), we have
\[ c \int_{\Omega} \phi \gamma_\epsilon^2 |\nabla u|^2 \leq \int_{\Omega} \phi \gamma_\epsilon^2 \Delta u. \]
Then the required estimate follows from (6) and the Cauchy–Schwarz inequality. \qed

Now we use the Moser iteration to derive the gradient estimate for harmonic functions. See [3] for the gradient estimate of harmonic functions on RCD spaces.

Lemma 2.8 Let \( u > 0 \) be a harmonic function defined on the unit ball \( B_p(1) \), i.e.,
\[ \Delta u = 0, \text{ in } B_p(1) \cap \mathcal{R}. \]
Then
\[ |\nabla u|^2 \leq C(n) u^2, \text{ in } B_p(1/4) \cap \mathcal{R}. \quad (7) \]
Proof Putting \( v = \ln u \), we have
\[ \Delta v = \frac{\Delta u}{u} - \frac{|\nabla u|^2}{u^2} = -|\nabla v|^2. \]
Denote $Q = |\nabla v|^2$, by the Bochner formula, we have
\[
\frac{1}{2} \Delta Q = |\text{Hess } v|^2 - \langle \nabla v, \nabla Q \rangle + \text{Ric}(\nabla v, \nabla v) \geq \frac{(\Delta v)^2}{n} - Q^{\frac{1}{2}} |\nabla Q|.
\]
(8)

For any Lipschitz function $\phi$ supported in $B_p(1)$, we have
\[
\int_{B_p(1)} \phi^2 \gamma^2 \epsilon Q^{p-1} \Delta Q \geq \frac{2}{n} \int_{B_p(1)} \phi^2 \gamma^2 \epsilon Q^{p+1} - 2 \int_{B_p(1)} \phi^2 \gamma^2 \epsilon Q^{p-\frac{1}{2}} |\nabla Q|.
\]
Integrating by parts, we have
\[
2 \int_{B_p(1)} \phi^2 \gamma^2 \epsilon \left( Q^{p-\frac{1}{2}} |\nabla Q| - \frac{1}{n} Q^{p+1} \right) \\
\geq \int_{B_p(1)} \left( (p-1)\phi^2 \gamma^2 \epsilon Q^{p-2} |\nabla Q|^2 \\
+ \gamma^2 \epsilon Q^{p-1} \langle \nabla \phi^2, \nabla Q \rangle + \phi^2 Q^{p-1} \langle \nabla \gamma^2 \epsilon, \nabla Q \rangle \right).
\]
(9)

Since
\[
\int_{B_p(1)} \phi^2 Q^{p-1} |\nabla \gamma^2 \epsilon, \nabla Q| \leq \delta \int_{B_p(1)} \phi^2 \gamma^2 \epsilon Q^{p-1} |\nabla Q|^2 \\
+ \delta^{-1} \int_{B_p(1)} \phi^2 |\nabla \gamma^2 \epsilon|^2 Q^{p-1}, (\forall \delta > 0)
\]
and $Q, |\nabla \phi|$ are bounded, taking $\epsilon \to 0$ and then $\delta \to 0$, we get
\[
\frac{4(p-1)}{p^2} \int_{B_p(1) \cap R} \phi^2 |\nabla Q|^2 \leq \frac{4}{p} \int_{B_p(1) \cap R} \phi |\nabla \phi||\nabla Q|^2 \frac{p}{Q} \\
+ \frac{4}{p} \int_{B_p(1) \cap R} \phi^2 |\nabla Q|^2 \frac{n+1}{Q} \\
- \frac{2}{n} \int_{B_p(1) \cap R} \phi^2 Q^{p+1}.
\]
Consequently, we obtain
\[
\int_{B_p(1) \cap R} \phi^2 |\nabla Q|^2 \leq 9 \int_{B_p(1) \cap R} |\nabla \phi|^2 Q^n \\
+ 9 \int_{B_p(1) \cap R} \phi^2 Q^{p+1} - \frac{p}{2n} \int_{B_p(1) \cap R} \phi^2 Q^{p+1}.
\]
Then we have

\[
\int_{B_p(1) \cap \mathcal{R}} |\nabla (\phi Q^\frac{n}{p})|^2 \leq \int_{B_p(1) \cap \mathcal{R}} \left( 20 |\nabla \phi|^2 Q^p + 20 \phi^2 Q^{p+1} - \frac{p}{n} \phi^2 Q^{p+1} \right).
\]

(10)

So for \( p_1 = 40 \), we have

\[
\int_{B_{p_1}(1) \cap \mathcal{R}} |\nabla (\phi Q^{\frac{p_1}{2}})|^2 \leq 20 \int_{B_{p_1}(1) \cap \mathcal{R}} |\nabla \phi|^2 Q^{p_1} - 20 \int_{B_{p_1}(1) \cap \mathcal{R}} \phi^2 Q^{p_1+1}.
\]

(11)

Let \( \psi \) be a cut-off function supported in \( B_p(\frac{1}{2}) \) satisfying \( \psi \equiv 1 \) in \( B_p(1) \) and \( |\nabla \psi| \leq 4 \). Put \( \phi = \psi^{p_1+1} \), we have

\[
|\nabla \phi|^2 \leq 16(p_1 + 1)^2 \phi^{2p_1}.
\]

Combined with the Hölder inequality, we get

\[
\int_{B_{p_1}(1) \cap \mathcal{R}} |\nabla \phi|^2 Q^{p_1} \leq C(n) \int_{B_{p_1+1}(1) \cap \mathcal{R}} \phi^{\frac{2p_1}{p_1+1}} Q^{p_1}
\]

\[
\leq C(n) \left( \int_{B_{p_1}(1) \cap \mathcal{R}} \phi^2 Q^{p_1+1} \right)^{\frac{p_1}{p_1+1}} \left( vol(B_p(1)) \right)^{\frac{1}{p_1+1}}
\]

\[
\leq \frac{1}{2} \int_{B_{p_1}(1) \cap \mathcal{R}} \phi^2 Q^{p_1+1} + C(n) vol(B_p(1)).
\]

(12)

By the Hölder inequality, we have

\[
\int_{B_{p_1}(1) \cap \mathcal{R}} \phi^2 Q^{p_1} \leq \left( \int_{B_{p_1}(1) \cap \mathcal{R}} \phi^2 Q^{p_1+1} \right)^{\frac{p_1}{p_1+1}} \left( \int_{B_{p_1}(1) \cap \mathcal{R}} \phi^2 \right)^{\frac{1}{p_1+1}}
\]

\[
\leq \frac{1}{2} \int_{B_{p_1}(1) \cap \mathcal{R}} \phi^2 Q^{p_1+1} + C(n) vol(B_p(1)).
\]

(13)

By [11], the Sobolev inequality holds in \( \mathcal{R} \). Combined with (11), (12) and (13), we can apply the Sobolev inequality to obtain

\[
\left( \int_{vol(B_p(1))} \right)^{\frac{1}{p}} \leq \frac{C(n)}{vol(B_p(1))} \int_{B_{p_1}(1) \cap \mathcal{R}} \left( |\nabla (\phi Q^{\frac{n}{p}})|^2 + \phi^2 Q^{p_1} \right) \leq C(n).
\]

(14)

For \( p \geq 20 \), we deduce from (10)

\[
\int_{B_{p_1}(1) \cap \mathcal{R}} |\nabla (\phi Q^\frac{n}{p})|^2 \leq 20 \int_{B_{p_1}(1) \cap \mathcal{R}} |\nabla \phi|^2 Q^p.
\]

(15)
Using (14), (15) and Moser’s iteration, we get

$$|Q|_{L^\infty(B_p(\frac{1}{4}))} \leq C.$$  

\[\square\]

**Lemma 2.9** For $p \in \mathcal{R}$, there exists a cut-off function $\phi$ supported in $B_p(2)$ such that

(i) $\phi \equiv 1$, in $B_p(1)$; (ii) $|\nabla \phi|_{B_p(2) \cap \mathcal{R}}, |\Delta \phi|_{B_p(2) \cap \mathcal{R}} \leq C(n)$.

**Proof** We will use an argument from Theorem 6.33 in [6]. First we consider a solution of ODE,

$$G'' + \frac{2n - 1}{r} G' = 1,$$

on $[1, 2]$, (16)

with $G(1) = a$ and $G(2) = 0$. When $a \geq a(n)$, we have $G' < 0$. Then by Lemma 2.1, we have

$$\Delta G(d(p, \cdot)) \geq 1.$$

Let $w$ be a solution of equation,

$$\Delta w = \frac{1}{a}, \text{ in } B_p(2) \setminus \overline{B_p(1)},$$

with $w = 1$ on $\partial B_p(1)$ and $w = 0$ on $\partial B_p(2)$. Thus by Lemma 2.5, we get

$$w \geq \frac{G(d(\cdot, p))}{a}.$$

Secondly, denote $H = \frac{r^2}{4n}$. Then by Lemma 2.1, we have

$$\Delta H(d(x, \cdot)) \leq 1, \text{ for any fixed point } x.$$

Thus by the maximum principle, we get

$$w(y) - \frac{H(d(x, y))}{a} \leq \max \left\{ 1 - \frac{H(d(x, p) - 1)}{a}, 0 \right\}$$

for any $y$ in the annulus $A_p(1, 2) = B_p(2) \setminus \overline{B_p(1)}$. It follows

$$w(x) \leq \max \left\{ 1 - \frac{H(d(x, p) - 1)}{a}, 0 \right\}, \forall x \in A_p(1, 2).$$
Now we choose a number \( \eta(n) \) such that \( \frac{G(1+\eta)}{a} > 1 - \frac{H(1-\eta)}{a} \) and we define a function \( \psi(x) \) on \([0, 1]\) with bounded derivative up to second order, which satisfies

\[
\psi(x) = 1, \text{ if } x \geq \frac{G(1+\eta)}{a}
\]

and

\[
\psi(x) = 0, \text{ if } x \leq \max \left\{ 1 - \frac{H(1-\eta)}{a}, 0 \right\}.
\]

It is clear that \( \phi = \psi \circ w \) is constant near the boundary of \( A_p(1, 2) \). So we can extend \( \phi \) inside \( B_p(1) \) by setting \( \phi = 1 \). By Lemma 2.8 and \( w \) is bounded, one sees that \( |\nabla \phi| \) is bounded by a constant \( C(n, \Lambda, A) \) in \( B_2(p) \). Since

\[
\Delta \phi = \psi'' |\nabla w|^2 + \psi' \Delta w,
\]

we also derive that \( |\Delta \phi| \leq C(n) \). □

3 Splitting Theorem

Let \((M_i, p_i) \in M(V, D, n)\) be a sequence of Riemannian manifold with singularity and converge to \((X, x)\) in the pointed Gromov-Hausdorff sense. In this section, we will prove

**Proposition 3.1** If \( X \) contains a line, then there exists a length space \( Y \) such that

\[
X \cong Y \times \mathbb{R}.
\]

As in [6], the proof depends on the following lemmas.

**Lemma 3.2** Let \( M \) be a Riemannian manifold with singularity with \( \text{Ric}(g) \geq 0 \) in \( \mathcal{R} \). Suppose that there are three points \( p, q^+, q^- \in \mathcal{R} \) which satisfy

\[
d(p, q^+) + d(p, q^-) - d(q^+, q^-) < \epsilon \tag{17}
\]

and

\[
d(p, q^+) > R. \tag{18}
\]

Then for any \( q \in B_p(1) \), the following holds:

\[
E(q) := d(q, q^+) + d(q, q^-) - d(q^+, q^-) < \Psi \left( \epsilon, \frac{1}{R}; n \right),
\]

where the quantity \( \Psi \left( \epsilon, \frac{1}{R}; n \right) \) means that it goes to zero as \( \epsilon, \frac{1}{R} \) go to zero while \( n \) is fixed.
Proof By Lemma 2.1, we have $\Delta E(q) \leq \frac{4n-2}{R}$. Put

$$G_L(r) = \frac{r^2}{4n} + \frac{L^{2n}}{4n(n-1)}r^{2-2n} - \frac{L^2}{4(n-1)}. \quad (19)$$

$G_L$ satisfies

$$G'_L < 0, \quad G_L(L) = 0, \quad \Delta G_L(d(p, \cdot)) \geq 1.$$

We will prove

Claim 3.3 For any $0 < c < 1$,

$$E(q) \leq 2c + \frac{4n-2}{R}G_L(c) + \epsilon, \quad \text{if} \quad \frac{4n-2}{R}G_L(1) > \epsilon.$$

Suppose that the claim is not true. Then there exists point $q_0 \in B_p(1)$ such that for some $c$,

$$E(q_0) > 2c + \frac{4n-2}{R}G(c) + \epsilon.$$

We consider

$$u(x) = \frac{4n-2}{R}G_L(d(q_0, x)) - E(x)$$

in the annulus $A_{q_0}(c, L)$. Clearly,

$$\Delta u \geq 0.$$

Note that we may assume that $p \in A_{q_0}(c, 1)$. Otherwise we have $E(q_0) \leq E(p) + 2c$. On the other hand, it is easy to see that on the inner boundary $\partial B_{q_0}(c)$,

$$u(x) = \frac{4n-2}{R}G_L(c) - E(x) \leq \frac{4n-2}{R}G_L(c) - E(q_0) - 2c \leq -\epsilon,$$

and on the outer boundary $\partial B_{q_0}(L)$,

$$u(x) = -E(x) \leq 0.$$

Thus applying the maximum principle, we obtain $u(p) \leq 0$. However,

$$u(p) = \frac{4n-2}{R}G_L(d(p, q_0)) - E(p) \geq \frac{4n-2}{R}G_L(1) - \epsilon > 0,$$

which is impossible. Therefore, the claim is true.
Now if $R \epsilon \leq G_2(1)$, we choose $L = 2$ and $c = (\frac{1}{R})^{\frac{1}{2n-1}}$, we have

$$E(q) \leq \epsilon + c(n) \left( \frac{1}{R} \right)^{\frac{1}{2n-1}}.$$  

Otherwise we choose $G_L(1) = \epsilon R$, $G_L(c) = Rc$ and get

$$E(q) \leq \epsilon + c(n) \epsilon^{\frac{1}{2n-1}}.$$  

The lemma is proved. \hfill \Box

$b^+(x) = d(q^+, x) - d(q^+, p)$ and let $h^+$ be a harmonic function which satisfies

$$\Delta h^+ = 0, \text{ in } B_p(1) \cap \mathcal{R},$$

with $h^+ = b^+$ on $\partial B_p(1) \cap \mathcal{R}$. Then

**Lemma 3.4** Under the conditions in Lemma 3.2, we have

\begin{align*}
\| h^+ - b^+ \|_{L^\infty(B_p(1))} & \leq \Psi(1/R, \epsilon), \quad (20) \\
\frac{1}{\text{vol}(B_p(1))} \int_{B_p(1) \cap \mathcal{R}} |\nabla h^+ - \nabla b^+|^2 dv & \leq \Psi(1/R, \epsilon), \quad (21) \\
\frac{1}{\text{vol}(B_{p(\frac{1}{2})})} \int_{B_{p(\frac{1}{2})} \cap \mathcal{R}} |\text{Hess } h^+|^2 dv & \leq \Psi(1/R, \epsilon). \quad (22)
\end{align*}

**Proof** Choose a point $q$ in $\partial B_p(2) \cap \mathcal{R}$ and let $g = \phi(d(q, \cdot))$, where $\phi(r)$ is chosen as in (16). Then

$$\Delta g = \varphi' \Delta r + \varphi'' \geq \frac{2n-1}{r} \varphi' + \varphi'' = 1, \text{ in } B_p(1) \mathcal{R}.$$  

(23)

It follows that

$$\Delta(h^+ - b^+ + \Psi(1/R, \epsilon) g) > 0, \text{ in } B_p(1) \cap \mathcal{R}.$$  

Thus by the maximum principle 2.5, we get

$$h^+ - b^+ \leq \Psi(1/R, \epsilon).$$  

On the other hand, we have

$$\Delta(-b^- - h^+ + \Psi(1/R, \epsilon) g) > 0, \text{ in } B_p(1),$$

where $b^- = d(q^-, x) - d(p, q^-)$. Since $b^+ + b^-$ is small as long as $1/R$ and $\epsilon$ are small by Lemma 3.2, by the maximum principle, we also get

$$h^+ - b^+ \geq -(b^+ + b^-) - \Psi(1/R, \epsilon) \geq -\Psi(1/R, \epsilon).$$

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For the second estimate (21), taking the cut-off function $\gamma_\eta$ for the Riemannian manifold with singularity, we have

$$
\int_{B_p(1) \cap R} \gamma_\eta^2 |\nabla h^+ - \nabla b^+|^2 dv
$$

$$
= \int_{B_p(1) \cap R} \gamma_\eta^2 (h^+ - b^+) (\Delta b^+ - \Delta h^+) dv
$$

$$
+ 2 \int_{B_p(1) \cap R} (b^+ - h^+) (\nabla h^+ - \nabla b^+, \nabla \gamma_\eta) \gamma_\eta dv
$$

$$
\leq \int_{B_p(1) \cap R} \left( \Psi(1/R, \epsilon) \gamma_\eta^2 |\Delta b^+| + \frac{1}{2} \gamma_\eta^2 |\nabla h^+ - \nabla b^+|^2 dv + 2(b^+ - h^+)^2 |\nabla \gamma_\eta|^2 \right) dv.
$$

Thus

$$
\int_{B_p(1) \cap R} \gamma_\eta^2 |\nabla h^+ - \nabla b^+|^2 dv \leq \Psi(1/R, \epsilon) \int_{B_p(1) \cap R} \gamma_\eta^2 |\Delta b^+| dv + C \eta.
$$

Now we see

$$
\int_{B_p(1) \cap R} \gamma_\eta^2 |\Delta b^+| dv
$$

$$
\leq \left| \int_{B_p(1) \cap R} \gamma_\eta^2 \Delta b^+ dv \right| + 2 \sup_{B_p(1)} (\Delta b^+) \text{vol}(B_p(1))
$$

$$
\leq \int_{B_p(1) \cap R} d v (\gamma_\eta^2 \nabla b^+) dv + 2 \int_{B_p(1) \cap R} |\nabla \gamma_\eta| dv + C \text{vol}(B_p(1))
$$

$$
\leq \text{vol}(\partial B_p(1)) + C \text{vol}(B_p(1)) \leq C \text{vol}(B_p(1)).
$$

Here we used (2) at the last inequality. Then (21) follows by letting $\eta \to 0$.

To get (22), we choose a cut-off function $\varphi$ supported in $B_p(1)$ as constructed in Lemma 2.9. Since

$$
\frac{1}{2} \Delta (|\nabla h^+|^2 - |\nabla b^+|^2) = |\text{Hess } h^+|^2 + \text{Ric}_g(\nabla h^+, \nabla h^+) \geq |\text{Hess } h^+|^2,
$$

and $|\nabla h^+|$ is bounded in the support of $\varphi$ by Proposition 2.8, for $u = |\nabla h^+|^2 - |\nabla b^+|^2$, we have

$$
\Delta u \geq C |\nabla u|^2.
$$

By Lemmas 2.6 and 2.7, we have

$$
\lim_{\eta \to 0} \int_{B_p(1)} \varphi \gamma_\eta^2 \Delta u = \int_{B_p(1) \cap R} u \Delta \phi.
$$
By (24), we derive (22) from (21) immediately.

**Lemma 3.5** For any \( \eta > 0 \), there exists \( \delta = \delta(\eta) \) having the following property: let \( x, y, z \) be three points in \( B_p(1) \cap R \) with

\[
|h^+(y) - h^+(x) - d(x, y)| \leq \delta, |h^+(x) - h^+(z)| \leq \delta.
\]

\( \gamma(s)(s \in [0, c])) \) is the minimal geodesic curve connecting \( x, y \) and \( \gamma_s(t)(s \in [0, l(s)], l(s) = d(z, \gamma(s))) \) is a family of minimal geodesic curves connecting \( z \) and \( \gamma(s) \). Assume that

(i) \( |h^+ - b^+|_{C^0(B_p(1))} \leq \delta; \)

(ii) \( \int_0^c |\nabla h^+(\gamma(s)) - \nabla b^+(\gamma(s))| \leq \delta; \)

(iii) \( \int_0^c \int_0^{l(s)} |Hess h^+(\gamma_s(t))| dt ds < \delta. \)

Then

\[
|d(z, x)^2 + d(x, y)^2 - d(y, z)^2| < \eta. \tag{25}
\]

**Proof** Since the rectangular is convex, we can follow the proof of Lemma 9.16 in [5]. From \( |h^+(y) - h^+(x) - d(x, y)| \leq \delta \), by (i) we know \( |b^+(y) - b^+(x) - d(x, y)| \leq 3\delta \). Since \( b^+ \) is 1-Lipschitz, we know that

\[
s = b^+(\gamma(s)) - b^+(x) + \Psi(\delta).
\]

Combined with \( |h^+(x) - h^+(z)| \leq \delta \), we get

\[
\frac{1}{2} d(x, y)^2 = \int_0^c s ds
\]

\[
= \int_0^c (h^+(\gamma(s)) - h^+(x)) ds + \Psi(\delta)
\]

\[
= \int_0^c (h^+(\gamma_s(l(s))) - h^+(\gamma_s(0))) ds + \Psi(\delta)
\]

\[
= \int_0^{l(s)} \int_0^c \langle \nabla h(\gamma_s(t)), \gamma_s'(t) \rangle dr ds + \Psi(\delta).
\]

On the other hand,

\[
|\langle \nabla h(\gamma_s(t)), \gamma_s'(t) \rangle - \langle \nabla h(\gamma_s(l(s))), \gamma_s'(l(s)) \rangle|
\]

\[
= \left| \int_t^{l(s)} hess h(\gamma_s'(t), \gamma_s'(t)) d\tau \right|
\]

\[
\leq \int_0^{l(s)} |hess h(\gamma_s'(t), \gamma_s'(t))| dt.
\]
Hence from the condition (iii), we get
\[
\frac{1}{2} d(x, y)^2 = \int_0^a \int_0^a (\nabla h(\gamma_t(l(s))), \gamma'_t(l(s))) dr ds + \Psi(\delta)
\]
\[
= \int_0^a (\nabla h(\gamma_t(l(s))), \gamma'_t(l(s))) l(s) ds + \Psi(\delta)
\]
\[
= \int_0^a (\nabla h(\gamma(s)), \gamma'_t(l(s))) l(s) ds + \Psi(\delta). \tag{26}
\]

Since \(|b^+(y) - b^+(x) - d(x, y)| \leq 3\delta\), we have
\[
\int_0^c |\nabla b^+ - \gamma'(s)| ds \leq c \int_0^c |\nabla b^+ - \gamma'(s)|^2 ds
\]
\[
= 2c \left( c - \int_0^c (b^+(s))' ds \right)
\]
\[
= 2c[c - (d(q, y) - d(q, x))] \leq 6c\delta. \tag{27}
\]

Combined with (ii) we get
\[
\int_0^c |\nabla h^+(\gamma(s)) - \gamma'(s)|
\]
\[
= \int_0^c |\nabla h^+(\gamma(s)) - \nabla b^+(\gamma(s))| ds + \int_0^c |\nabla b^+(\gamma(s)) - \gamma'(s)| \leq 20\delta. \tag{28}
\]

Now by the first variation formula of geodesic curve, we see that
\[
l'(s) = \langle \gamma'_t(l(s)), \gamma'(s) \rangle.
\]

Then by (28), we obtain
\[
\int_0^a (\nabla h(\gamma(s)), \gamma'_t(l(s))) l(s) ds
\]
\[
= \int_0^a l'(s) l(s) ds + \Psi(\delta)
\]
\[
= \frac{1}{2} (d(y, z)^2 - d(z, x)^2) + \Psi(\delta).
\]

Therefore, combined with (26), we derive (25). \qed

**Lemma 3.6** For any \( \eta > 0 \), there exists \( \delta = \delta(\eta) > 0, \ R_0 = R_0(\eta), \ \epsilon_0 = \epsilon_0(\eta) \) having the following property: under the condition of Lemma 3.2, if \( R \geq R_0, \ \epsilon \leq \epsilon_0 \), then for any three points \( x, y, z \) satisfying
\[
|h^+(y) - h^+(z)| \leq \delta, \ |h^+(x) - h^+(y) - d(x, y)| \leq \delta,
\]
we have

\[ |d^2(x, y) + d^2(y, z) - d^2(x, z)| \leq \eta. \]

**Proof** By Lemma 2.4 and Lemma 3.4, we get for any small positive \( \eta_1 \)

\[
\int_{B_x(\eta_1) \times B_y(\eta_1) \times B_z(\eta_1)} |\text{Hess } h^+| \\
\leq C(n) \left( \frac{\text{vol}(B_x(\eta_1)) \text{vol}(B_p(1))}{\text{vol}(B_p(1))} \right) \int_{B_p(1)} |\text{Hess } h^+| \\
\leq C(n) \left( \frac{\text{vol}(B_x(\eta_1)) \text{vol}(B_p(1))}{\text{vol}(B_p(1))} \right) \Psi \left( \frac{1}{R}, \epsilon \right).
\]

So there are points \( x^* \in B_x(\eta_1), y^* \in B_y(\eta_1), z^* \in B_z(\eta_1), \) such that

\[
\int_0^d(x^*, y^*) ds \int_0^d(z^*, y(s)) dt \leq \left( \frac{\text{vol}(B_q(1))}{\text{vol}(B_q(\eta_1))} \right)^2 \Psi \left( \frac{1}{R}, \epsilon \right),
\]

where \( \gamma_s(t) \) is the minimal geodesic curves connecting \( \gamma(s) \) and \( z^* \). Now we have

\[ |h^+(y^*) - h^+(x^*) - d(x^*, y^*)| \leq 2C \eta_1 + 2 \eta_1 + \delta. \]

By Lemma 3.5, we know that there exist \( \delta_0 = \delta_0(\eta) > 0, R_0 = R_0(\eta), \epsilon_0 = \epsilon_0(\eta), \eta_0 = \eta_0(\eta) \leq \frac{\eta}{12} \) such that if \( \delta \leq \delta_0, R \geq R_0, \epsilon \leq \epsilon_0, \eta_1 \leq \eta_0, \) we have

\[ |d^2(x^*, y^*) + d^2(y^*, z^*) - d^2(x^*, z^*)| \leq \frac{\eta}{2}. \]

As a consequence, we get

\[ |d^2(x, y) + d^2(y, z) - d^2(x, z)| \leq \frac{\eta}{2} + 6 \eta_1 \leq \eta. \]

\( \square \)

**Lemma 3.7** Suppose that \( X \) is a length space and \( x^* \) is point in \( X \). Assume that there is a function \( h \) having the following two properties:

(i) \( h \) is 1-Lipschitz with \( h(x^*) = 0 \),

(ii) for any point \( x \in B_x^+(1) \) and \( t \in [-1, 1] \), there exist \( x_t \in X \) and a minimal geodesic \( \gamma_t \) connecting \( x \) and \( x_t \) such that

\[ h(x_t) = t, \ d(x, x_t) = |h(x) - t|. \]

(iii) for three points \( x, y, z \in B_x^+(1) \) with \( h(x) = h(y), |h(x) - h(z)| = d(x, z) \), we have

\[ d(y, z)^2 = d(x, z)^2 + d(x, y)^2. \]
Then there exists a metric space $Y$ such that

$$B_{s^*}(1) \cong B_{s^* \times 0}(1) \subset Y \times \mathbb{R}.$$ 

**Proof** Define $Y = h^{-1}(0)$ with the distance induced from $X$. For any $x \in B_{s^*}(1)$, by (ii) there is a point $x_0 \in Y$ such that $d(x, x_0) = |h(x)|$. We show that such point $x_0$ is unique. Assume that $x_0$ is another point, then by (iii) we have

$$|h(x)|^2 = d(x, x_0) = d(x, x_0')^2 + d(x_0, x_0')^2 = |h(x)|^2 + d(x_0, x_0')^2.$$ 

It implies that $x_0 = x_0'$. Now denote $x_0$ by $x$. For any two points $x, y \in B_{s^*}(1)$, assuming that $|h(y)| \geq |h(x)|$, we can choose a point with $h(z) = h(x)$ and $d(z, y) = h(y) - h(x)$. We are going to show that $\pi(y) = \pi(z)$. We divide into two cases. The first case is that $h(y), h(z)$ have the opposite signs. Denoting the minimal geodesic connecting $y$ and $z$ by $\gamma(s)$, there is a point $w$ on $\gamma(s)$ with $h(w) = 0$. By (i) we know that

$$|h(y) - h(z)| = d(y, z) = d(y, w) + d(w, z) \geq |h(y)| + |h(z)|.$$ 

So we have $d(y, w) = |h(y)|$ which implies that $w = \pi(y) = \pi(z)$. For the second case, denote the minimal geodesic connecting $y$ and $\pi(y)$ by $\gamma(s)$. There is a point $w$ on $\gamma(s)$ with $h(w) = h(x)$ and $d(y, w) = |h(y) - h(w)|$. By (iii) we know that $d(w, z) = 0$ which implies that $\pi(z) = \pi(w) = \pi(y)$. Now by (iii) we also see that

$$d(x, y)^2 = d(x, z)^2 + |h(y) - h(x)|^2 = d(\pi(x), \pi(y))^2 + |h(y) - h(x)|^2.$$ 

It follows that $x \to (\pi(x), h(x))$ is an isometry. The lemma is proved. \hfill $\square$

Now we are ready to prove Proposition 3.1.

**Proof of Proposition 3.1** Denote the line in $X$ by $\gamma(t)$ and $\gamma(0) = x^*$. Let $q_i^+, q_i^- \in M_i$ be the points converging to $\gamma(i), \gamma(-i)$, respectively, such that

$$d(p_i, q_i^+) + d(p_i, q_i^-) - d(q_i^+, q_i^-) \leq \epsilon_i \to 0.$$ 

Denote by $h_i^+$ the functions constructed in Lemma 3.4. $h_i^+$ converges to a limit function $h$. By Lemma 3.4, we know that

$$h(x) = \lim_{s \to +\infty} b_s(x),$$
where \( b_s(x) = d(x, \gamma(s)) - s \). Denote

\[
h^-(x) = \lim_{t \to -\infty} d(x, \gamma(s)) - s.
\]

By Lemma 3.2, we know that

\[
h + h^- = 0.
\]

Now we show that \( h \) satisfies the conditions in Lemma 3.7. (i) is obvious. For (ii), let \( x \in B_{x^+}(1) \) be any point. For \( t \in [-1, h(x)] \), we choose a point \( x^+_t \) on the minima geodesic connecting \( x \) and \( \gamma(s) \) with

\[
d(x, x^+_t) = h(x) - t.
\]

Then \( b(x^+_t) = b(x) + t - h(x) \). Taking \( s_i \to -\infty \), we have \( x^+_i \to z \). Then we have

\[
h(z) = t.
\]

For \( t \in [h(x), 1] \), we can use \( h^-(x) \) instead of \( h(x) \) to obtain the point \( z \). For (iii), let \( x, y, z \) be three points in \( B_{x^+}(1) \), with \( h(x) = h(y), |h(x) - h(z)| = d(x, z) \). There are points \( x_i, y_i, z_i \in B_{x^+}(1) \) converging to \( x, y, z \), respectively, such that

\[
|h^+_i(x) - h^+_i(y)| \to 0, |h^+_i(x) - h^+_i(z) - d(x, z)| \to 0.
\]

By Lemma 3.6, we know that

\[
|d(x_i, y_i)^2 + d(x_i, z_i)^2 - d(y_i, z_i)^2| \to 0,
\]

which means that

\[
d(x, y)^2 + d(y, z)^2 = d(y, z)^2.
\]

\[\square\]

4 Metric Cone

We define the following set of Riemannian manifold with singularity

\[
\mathcal{M}(v, n) = \{(M^n, p, g)|\text{Ric}(g) \geq 0 \text{ in } \mathcal{R}, \text{vol}(B_{p}(1)) \geq v > 0.\}.
\]

Let \( (M_i, p_i) \) converge to \((X, x)\) in the Gromov-Hausdorff sense. In this section, we prove that every tangent cone of \( X \) is a metric cone:
Proposition 4.1 Let $T_{x^*}X$ be a tangent cone at $x^* \in X$. Then there is a length space $Y$ such that

$$T_{x^*}X \cong C(Y).$$

The proof depends on the following lemmas. We start with some estimates of approximate harmonic functions. Let $(M^n, p, g) \in \mathcal{M}(v, n)$ and $q \in R \subseteq M$ and $h$ be a solution of the following equation:

$$\Delta h = n, \text{ in } B_q(b) \setminus B_q(a), \quad h|_{\partial B_q(b) \cap R} = \frac{b^2}{2} \quad \text{and} \quad h|_{\partial B_q(a) \cap R} = \frac{a^2}{2}. \quad (29)$$

Let $\tilde{h} = \frac{r(q, .)^2}{2}$.

Lemma 4.2 Suppose that

$$\frac{\text{vol}(\partial B_q(b))}{\text{vol}(\partial B_q(a))} \geq (1 - \omega) \frac{b^{n-1}}{a^{n-1}} \quad (30)$$

for some $\omega > 0$. Then

$$\frac{1}{\text{vol}(A_q(a, b))} \int_{A_q(a, b) \cap R} |\nabla \tilde{h} - \nabla h|^2 \, dv < \Psi(\omega; a, b). \quad (31)$$

Moreover,

$$\|h - \tilde{h}\|_{L^\infty(A_q(a', b'))} < \Psi(\omega; a, b, a', b'), \quad (32)$$

where $a < a' < b' < b$.

Proof Since

$$\Delta r \leq \frac{n - 1}{r} \quad \text{in} \ R,$$

we have

$$\Delta \tilde{h} = 1 + r \Delta r \leq n, \quad \text{in} \ A(a, b) \cap R. \quad (33)$$

Thus we get

$$\frac{1}{\text{vol}(A_q(a, b))} \int_{A_q(a, b) \cap R} \Delta \tilde{h} \, dv \leq n. \quad (34)$$

On the other hand, by the monotonicity formula (2), we have

$$\text{vol}(A_q(a, b)) \leq \frac{b^n - a^n}{na^{n-1} \text{vol}(\partial B_q(a))}. \quad (35)$$
It follows by (30),

$$\text{vol}(A_q(a, b)) \leq (1 - \omega)^{-1} \frac{b^n - a^n}{n b^{n-1}} \text{vol}(\partial B_q(b)).$$

Since

$$\int_{A_q(a, b) \cap R} \Delta \tilde{h} \, dv = b \text{vol}(\partial B_q(b)) - a \text{vol}(\partial B_q(a)),$$

we get

$$\frac{1}{\text{vol}(A_q(a, b))} \int_{A_q(a, b) \cap R} \Delta \tilde{h} \, dv \geq (1 - \omega) \frac{n b^{n-1}}{b^n - a^n} \left( b - a \frac{\text{vol}(\partial B_q(a))}{\text{vol}(\partial B_q(b))} \right).$$

Hence we derive immediately,

$$\frac{1}{\text{vol}(A_q(a, b))} \int_{A_q(a, b) \cap R} \Delta \tilde{h} \, dv \geq n + \Psi(\omega; a, b). \quad (35)$$

By (34) and (35), we have

$$\int_{A_q(a, b) \cap R} \left| \Delta \tilde{h} - n \right| \, dv < \text{vol}(A_q(a, b)) \Psi(\omega; a, b). \quad (36)$$

From

$$0 = \int_{A_q(a, b) \cap R} \gamma^2_\epsilon |\nabla (\tilde{h} - h)|^2 + 2 \int_{A_q(a, b) \cap R} (\tilde{h} - h) \gamma_\epsilon \langle \nabla \gamma_\epsilon, \nabla (\tilde{h} - h) \rangle + \int_{A_q(a, b) \cap R} (\tilde{h} - h) \gamma^2_\epsilon (\Delta \tilde{h} - \Delta h),$$

and

$$\int_{A_q(a, b) \cap R} (\tilde{h} - h) \gamma_\epsilon \langle \nabla \gamma_\epsilon, \nabla (\tilde{h} - h) \rangle \leq C \left( \int_{A_q(a, b) \cap R} |\nabla \gamma_\epsilon|^2 \int_{A_q(a, b) \cap R} \gamma^2_\epsilon |\nabla (\tilde{h} - h)|^2 \right)^{\frac{1}{2}},$$

we get

$$\int_{A_q(a, b) \cap R} \gamma^2_\epsilon |\nabla (\tilde{h} - h)|^2 \leq C \int_{A_q(a, b) \cap R} |\Delta \tilde{h} - n| + C \epsilon.$$
Then by (36), we obtain (31).

Applying the following Lemma 4.3 to the function $\tilde{h} - h$ together with the estimate (31), we see that

$$
\frac{1}{\operatorname{vol}(A_q(a, b))} \int_{A_q(a, b) \cap \mathcal{R}} |\tilde{h} - h|^2 \, dv < \Psi(\omega; a, b).
$$

Then for any point $x \in A_q(a', b') \cap \mathcal{R}$, there is a point $y \in B_x(\eta) \cap \mathcal{R}$ such that

$$
|\tilde{h}(y) - h(y)|^2 \leq \frac{\operatorname{vol}(A_q(a, b))}{\operatorname{vol}(B_x(\eta))} \frac{1}{\operatorname{vol}(A_q(a, b))} \int_{A_q(a, b)} |\tilde{h} - h|^2 \, dv
\leq \frac{C(\Lambda, b)}{\eta^n} \Psi(\omega; a, b).
$$

On the other hand, by Proposition 2.8, we have

$$
|\langle \tilde{h}(x) - h(x), \tilde{h}(y) - h(y) \rangle| \leq (\|\nabla h\|_{C^0(A_q(a' - \eta, b' + \eta))} + 1) \operatorname{dist}(x, y)
\leq C(a, b, a' - \eta, b' + \eta) \eta.
$$

Thus we derive

$$
|\tilde{h}(x) - h(x)|
\leq \frac{C(\Lambda, b)}{\eta^n} \Psi(\omega; a, b) + C(a, b, a' - \eta, b' + \eta) \eta.
$$

Choosing $\eta = \frac{1}{\Psi \eta}$, we prove (32). \qed

**Lemma 4.3** Let $f \in L^\infty(A_q(a, b))$ be a locally Lipschitz function in $A_q(a, b) \cap \mathcal{R}$ and $f|_{\partial A_q(a, b) \cap \mathcal{R}} = 0$, then there is a positive number $\lambda_1 \geq C(b, n)$ such that

$$
\lambda_1 \int_{A_q(a, b) \cap \mathcal{R}} f^2 \leq \int_{A_q(a, b) \cap \mathcal{R}} |\nabla f|^2.
$$

**Proof** As in the proof of Lemma 3.2, let $G_b$ be the function satisfying $\Delta G_b \geq 1$. We have

$$
\int_{A_q(a, b) \cap \mathcal{R}} f^2 \leq \int_{A_q(a, b) \cap \mathcal{R}} f^2 \Delta G_b.
$$

Let $\gamma_\eta$ be the cut-off function, then we have

$$
\int_{A_q(a, b) \cap \mathcal{R}} f^2 \gamma_\eta \Delta G_b = -\int_{A_q(a, b) \cap \mathcal{R}} 2 f \langle \nabla f, \nabla G_b \rangle \gamma_\eta - \int_{A_q(a, b) \cap \mathcal{R}} f^2 \langle \nabla G_b, \nabla \gamma_\eta \rangle.
$$
\[ \leq C(b, n) \left( \int_{A_q(a', b') \cap \mathcal{R}} f^2 \right)^{\frac{1}{2}} \left( \int_{A_q(a, b) \cap \mathcal{R}} |\nabla f|^2 \right)^{\frac{1}{2}} + C\eta. \]

Taking \( \eta \to 0 \), we get

\[ \int_{A_q(a, b) \cap \mathcal{R}} f^2 \Delta G_b \leq C(b, n) \int_{A_q(a, b) \cap \mathcal{R}} |\nabla f|^2. \]

It follows that

\[ \int_{A_q(a, b) \cap \mathcal{R}} f^2 \leq C(b, n) \int_{A_q(a, b) \cap \mathcal{R}} |\nabla f|^2. \]

\[ \square \]

Furthermore, we have

**Lemma 4.4** **Under the condition in Lemma 4.2, it holds**

\[ \frac{1}{\text{vol}(A_q(a', b'))} \int_{A_q(a', b') \cap \mathcal{R}} |\text{Hess } h - g|^2 dv < \Psi(\omega; a, b, a', b'), \]  

where \( a < a' < b' < b \).

**Proof** First observe that

\[ |\text{Hess } h - g|^2 = |\text{Hess } h|^2 + (n - 2\Delta h). \]

Let \( \varphi \) be a cut-off function of \( A_q(a, b) \) as constructed in Lemma 2.9 which satisfies,

1. \( \varphi \equiv 1, \ \text{in } A_q(a', b') \cap \mathcal{R}; \)
2. \( |\nabla \varphi|, |\Delta \varphi| \) is bounded in \( A_q(a, b) \cap \mathcal{R}. \)

Then

\[ \frac{1}{\text{vol}(A_q(a, b))} \int_{A_q(a, b) \cap \mathcal{R}} \varphi |\text{Hess } h - g|^2 dv \]

\[ = \frac{1}{\text{vol}(A_q(a, b))} \int_{A_q(a, b) \cap \mathcal{R}} \varphi |\text{Hess } h|^2 dv \]

\[ - \frac{1}{\text{vol}(A_q(a, b))} \int_{A_q(a, b) \cap \mathcal{R}} n \varphi dv. \]  

By the Bochner formula Lemmas 2.6 and 2.7, we have

\[ \frac{2}{\text{vol}(A_q(a, b))} \int_{A_q(a, b) \cap \mathcal{R}} \varphi |\text{Hess } h|^2 dv \]

\[ \square \] Springer
\[
\begin{align*}
&\leq \frac{1}{\text{vol}(A_q(a, b))} \int_{A_q(a, b) \cap \mathcal{R}} \varphi \Delta |\nabla h|^2 \, dv \\
&= \frac{1}{\text{vol}(A_q(a, b))} \int_{A_q(a, b) \cap \mathcal{R}} |\nabla h|^2 \Delta \varphi \, dv.
\end{align*}
\]  

By Lemmas 4.2 and 2.6, we have
\[
\begin{align*}
\frac{1}{\text{vol}(A_q(a, b))} \int_{A_q(a, b) \cap \mathcal{R}} |\nabla h|^2 \Delta \varphi \, dv \\
&\leq \frac{1}{\text{vol}(A_q(a, b))} \int_{A_q(a, b) \cap \mathcal{R}} |\nabla \tilde{h}|^2 \Delta \varphi + \Psi(\omega; a, b, a', b') \\
&= \frac{2}{\text{vol}(A_q(a, b))} \int_{A_q(a, b) \cap \mathcal{R}} \tilde{h} \Delta \varphi + \Psi(\omega; a, b, a', b') \\
&= \frac{2}{\text{vol}(A_q(a, b))} \int_{A_q(a, b) \cap \mathcal{R}} \phi \Delta \tilde{h} \, dv + \Psi(\omega; a, b, a', b').
\end{align*}
\]

It follows from (38), (39) and (40),
\[
\begin{align*}
\int_{A_q(a_1, b_1) \cap \mathcal{R}} \text{Hess } h - g^2 \, dv \\
&\leq \frac{C(a_1, b_1, a, b)}{\text{vol}(A_q(a, b))} \int_{A_q(a, b) \cap \mathcal{R}} \varphi \text{Hess } h - g^2 \, dv \\
&\leq \frac{C(a_1, b_1, a, b)}{\text{vol}(A_q(a, b))} \int_{A_q(a, b) \cap \mathcal{R}} \varphi (\Delta \tilde{h} - n) \, dv + \Psi(\omega; a, b, a', b') \\
&\leq \Psi(\omega; a, b, a', b').
\end{align*}
\]

Here we used (33) at last inequality. \(\square\)

**Lemma 4.5** Given \(b > a > 0\), for any \(\epsilon > 0\), there exits \(\delta > 0\) such that the following holds: let \(x, y \in A_q(a, b)\) be two points with \(d(x, y) \leq r(y) - r(x) + \delta\), where \(r(\cdot) = \text{dist}(q, \cdot)\), and \(\gamma(s)(0 \leq s \leq c)\) be the unique geodesic connecting \(x\) and \(y\), and \(\gamma_s(t)\) be a family of geodesic curves connecting \(z\) and \(\gamma(s)\). Suppose that

(i) \(|h - \tilde{h}|_{C^0(A_q(a, b))} \leq \delta;\)

(ii) \(\int_0^c |\nabla h(\gamma(s)) - \nabla \tilde{h}| \leq \delta;\)

(iii) \(\int_0^c \int_0^1 |\text{Hess } h - g| \, dr \, ds \leq \delta.\)

Then
\[
|d(z, y)^2 r(x) - d(x, z)^2 r(y) + r(z)^2 (r(y) - r(x)) - r(x)r(y)(r(y) - r(x))| < \epsilon.
\]  

(41)
Proof} From $d(x, y) \leq r(y) - r(x) + \delta$, we know that $|r(\gamma(s)) - (r(x) + s)| \leq \delta$. Then by (i) we get

$$|h(\gamma(s)) - \frac{(r(x) + s)^2}{2}| \leq C(b)\delta. \quad (42)$$

From

$$h(\gamma_\delta(l(s))) - h(z) = \int_0^{l(s)} \frac{d}{dt} h(\gamma_\delta(t)), \quad \frac{d}{dt} h(\gamma_\delta(t))|_{t=l(s)} - \frac{d}{dt} h(\gamma_\delta(t)) = \int_t^{l(s)} \text{Hess } h(\gamma_\delta'(\tau), \gamma_\delta'(\tau), \gamma_\delta'(\tau), \gamma_\delta'(\tau)),$$

we have

$$l(s)h'(\gamma_\delta(l(s))) = h(\gamma_\delta(l(s))) - h(z) - \frac{l^2(s)}{2} + \int_0^{l(s)} \int_t^{l(s)} \text{Hess } h(\gamma_\delta'(\tau), \gamma_\delta'(\tau)) - g(\gamma_\delta'(\tau), \gamma_\delta'(\tau))d\tau.$$

From (42), we get

$$l(s)h'(\gamma_\delta(l(s))) = \frac{(r(x) + s)^2 - r^2(z)}{2} + \frac{l^2(s)}{2} + \int_0^{l(s)} \int_t^{l(s)} \text{Hess } h(\gamma_\delta'(\tau), \gamma_\delta'(\tau)) - g(\gamma_\delta'(\tau), \gamma_\delta'(\tau))d\tau + \Psi(\delta).$$

Hence we derive

$$\int_0^c \left( \frac{2l(s)h'(\gamma_\delta'(l(s)))}{(s + r(x))^2} - \frac{l^2(s)}{(s + r(x))^2} \right) ds = a + \frac{r^2(z)}{r(x) + c} - \frac{r^2(z)}{r(x)},$$

$$+ \int_0^c \frac{ds}{(r(x) + s)^2} \int_0^{l(s)} \text{Hess } h(\gamma_\delta'(\tau), \gamma_\delta'(\tau)) - g(\gamma_\delta'(\tau), \gamma_\delta'(\tau))d\tau + \Psi(\delta).$$

By iii), we get

$$\int_0^a \left( \frac{2l(s)h'(\gamma_\delta'(l(s)))}{(s + r(x))^2} - \frac{l^2(s)}{(s + r(x))^2} \right) ds = c + \frac{r^2(z)}{r(x) + c} - \frac{r^2(z)}{r(x)} + \frac{r^2(z)}{r(x)^2} + \Psi(\delta). \quad (43)$$

Since $d(x, y) \leq r(y) - r(x) + \delta$, we have

$$\int_0^c |\nabla d(q, .) - \gamma'(s)|ds \leq c \int_0^c |\nabla d(q, .) - \gamma'(s)|^2ds = 2c \left( c - \int_0^c d'(q, \gamma(s))ds \right)$$

\( \square \) Springer
\[ = 2c[c - (d(q, y) - d(q, x)))] \leq 2c\delta. \] (44)

Combined with (ii) we get
\[
\int_0^c |\nabla h(\gamma(s)) - r(\gamma(s))\gamma'(s)| \leq 2c\delta.
\] (45)

By the first variation formula,
\[ l'(s) = \langle \gamma'_s(l(s)), \gamma'(s) \rangle, \]

Now from (45) we get
\[
\int_0^C \frac{(l^2(s))'}{s + r(x)} \, ds \\
= \int_0^C \left( \frac{2l(s)l'(s)}{s + r(x)} - \frac{l^2(s)}{(s + r(x))^2} \right) \, ds \\
= \int_0^C \left( \frac{2l(s)(s + r(x))\gamma'_s(l(s)), \gamma'(s)}{(s + r(x))^2} - \frac{l^2(s)}{(s + r(x))^2} \right) \, ds + \Psi(\delta) \\
= \int_0^a c \left( \frac{2l(s)h'(\gamma'_s(l(s)))}{(s + r(x))^2} - \frac{l^2(s)}{(s + r(x))^2} \right) \, ds + \Psi(\delta).
\]

Combined with (43), we get (41) immediately. \qed

**Lemma 4.6** Given \( b > \epsilon > 0 \), there exists \( \delta > 0 \) such that the following holds: assume that \( x, y \in A_q(\epsilon, b) \) with \( d(x, y) \leq r(y) - r(x) + \delta \) and \( h \) satisfying

(i) \[ |h - \tilde{h}|c^0(A_q(\epsilon, b)) \leq \delta; \]

(ii) \[ \int_{A_q(\xi, b)} |\nabla h - \nabla \tilde{h}| < \delta << 1; \]

(iii) \[ \int_{A_q(\xi, b)} |\text{Hess } h - g| dr ds < \delta << 1. \]

Then for any \( z \in A_q(\epsilon, b) \), we have
\[
|d(z, y)^2r(x) - d(x, z)^2r(y) + r(z)^2(r(y) - r(x)) - r(x)r(y)(r(y) - r(x))| < \epsilon.
\] (46)
Proof From Lemmas 2.4 and 3.4, we know that there exists \( x' \in B_x(\eta) \cap R, y' \in B_y(\eta) \cap R, z' \in B_z(\eta) \cap R, \) such that

\[
\int_{\gamma x' y'} |\nabla h(\gamma(s)) - \nabla \bar{h}| \chi_{A_q(\frac{x}{\bar{g}}, b)} \leq \delta; \\
\int_{\Delta x' y' z'} |\text{Hess} h - g| \chi_{A_q(\frac{x}{\bar{g}}, b)} \leq \delta
\]

If the geodesics \( \gamma_s(t) \) all lies in \( A_q(\frac{\epsilon}{\bar{g}}, b) \), the result follows from Lemma 4.5.

Now, we define

\[
s_0 = \sup \left\{ s | d(q, \gamma_s) \leq \frac{\epsilon}{8} \right\}.
\]

Denoting \( \gamma(s_0) \) by \( w \), we have

\[
|d(w, z') - (r(w) + r(z'))| \leq \frac{\epsilon}{4}.
\]

It follows that

\[
d(z', x') \geq d(z', w) - d(x', w) \geq r(w) + r(z') - \frac{\epsilon}{4} - s_0.
\]

Since \( |r(w) - r(x') - s_0| \leq \delta + \frac{1}{10} \epsilon \), we have

\[
d(z', x') \geq r(x') + r(z') - \frac{\epsilon}{2} - \delta.
\]

Applying Lemma 4.5 to \( w, y', z' \), we have

\[
|d(z', y')^2 r(w) - d(w, z')^2 r(y') + r(z')^2 (r(y') - r(w)) - r(w) r(y') (r(y') - r(w))| = \Psi(\delta |b|).
\]

The lemma is proved. \( \square \)

Lemma 4.7 For any \( b > \epsilon > 0 \), there is \( \omega = \omega(b, \epsilon) \) such that the following holds: if

\[
\frac{\text{vol}(\partial B_q(b))}{\text{vol}(\partial B_p(\epsilon))} \geq (1 - \omega) \frac{b^{n-1}}{\epsilon^{n-1}},
\]

and \( x, y, z \in A_q(\epsilon, b) \) satisfies \( d(x, y) \leq r(y) - r(x) + \omega \), then we have

\[
d(z, y)^2 r(x) - d(x, z)^2 r(y) + r(z)^2 (r(y) - r(x)) - r(x) r(y) (r(y) - r(x)) < \epsilon.
\]

Proof The Lemma follows from Lemmas 4.4 and 4.6. \( \square \)
Lemma 4.8 Given $a < c < b$. For any $\eta > 0$, there exists $\omega = \omega(a, b, c, \eta, n)$ such that the following is true: if

$$\frac{\text{vol}(\partial B_p(b))}{\text{vol}(\partial B_p(a))} \geq (1 - \omega) \frac{b^{n-1}}{a^{n-1}},$$

(47)

then for any point $q$ on $\partial B_p(c)$, there exists $q'$ on $\partial B_p(b)$ such that

$$d(q, q') \leq b - a + \eta.$$  

(48)

Proof We prove by contradiction. If there is point $q \in \partial B_p(c)$ such that

$$d(q, x) \geq b - c + \eta, (\forall x \in \partial B_p(b)),$$

we can choose another point $q_1 \in \mathcal{R} \cap B_q(\eta^3)$ such that

$$d(q, x) \geq b - c + \frac{\eta}{3}, (\forall x \in \partial B_p(b)).$$

Since the $\mathcal{R}$ is convex, we know that every minimal geodesic connecting $p$ and $x \in \partial B_p(b)$ has no intersection with $B_{q_1}(\frac{\eta^2}{6})$. Then there is some $\frac{n}{4} < r < \frac{n}{3}$ such that

$$\text{vol}B_{q_1}\left(\frac{\eta}{3}\right) \cap S_p(c + r) \geq c(n)\eta^{n-1}\text{vol}A_p(a, b).$$

(49)

Using the monotonicity formula (2), we get

$$\text{vol}S_p(b) \leq \text{vol}\left(S_p(a + r) \setminus B_{q_1}\left(\frac{\eta}{3}\right)\right) \frac{b^{n-1}}{(a + r)^{n-1}}$$

$$\leq (\text{vol}S_p(a + r) - c(n)\eta^{n-1}\text{vol}A_p(a, b)) \frac{b^{n-1}}{(a + r)^{n-1}},$$

(50)

which is a contradiction. □

Lemma 4.9 Suppose $X$ is a length space and $x^*$ is a point in $X$. Assume that for any $x \in B_{x^*}(1)$ there exists $y \in \partial B_{x^*}(1)$ and a minimal geodesic $\gamma(t)$ from $x^*$ and $y$ containing $x$. Moreover, we assume that for any four points $y_1, y_2, z_1, z_2$ with

$$d(x^*, z_i) = d(x^*, y_i) + d(y_i, z_i)(1 \leq i \leq 2),$$

we have

$$\frac{d^2(x^*, y_1) + d^2(x^*, y_2) - d^2(y_1, y_2)}{d(x^*, y_1)d(x^*, y_2)} = \frac{d^2(x^*, z_1) + d^2(x^*, z_2) - d^2(z_1, z_2)}{d(x^*, z_1)d(x^*, z_2)}.$$  

(51)
Then there exists a metric space $Y$ such that

$B_{x^*}(1) \cong B_o(1) \subset B_o(C(Y))$.

**Proof** Let $Y$ be the set of all minimizing geodesics $\gamma : [0, 1] \to X$ with $\gamma(0) = x^*$. Then we can check the isometry directly. \hfill \Box

To get the condition of almost volume, we use the following lemma, which is an easy consequence of volume comparison and non-collapsing condition.

**Lemma 4.10** Given $0 < a < b$, for any $\omega > 0$, there exists $N = N(b, v, \omega)$, such that for any sequence of $r_i (1 \leq i \leq N)$ satisfying $ar_i \geq br_{i+1}$ and $(M, p) \in \mathcal{M}(n, k, v)$, there is some $j \in [1, N]$ such that

$$\frac{\text{vol}(A_p(br_j))}{\text{vol}(A_p(ar_j))} \geq (1 - \omega) \frac{b^{n-1}}{a^{n-1}}.$$

**Proof of Proposition 4.1** Without loss of generality, we can assume that $x^* = x = \lim_{i \to \infty} p_i$. We need to verify the conditions in Lemma 4.9 for

$$(T_{x^*}X, g_x, x^*) = \lim_{i \to \infty} \left( X, \frac{d}{r_i^\alpha}, x^* \right).$$

Let $x \in B_{x^*}(1, \omega_{x^*})$ with $d(x^*, x) = 2a > 0$. For any $\epsilon > 0$, let $\omega = \omega(\epsilon)$ be the constant determined in Lemma 4.8. By Lemma 4.10, we know that there is a subsequence of $r_{j_i} \to 0$ such that

$$\frac{\text{vol}(A_p(r_{j_i}))}{\text{vol}(A_p(\alpha r_{j_i}))} \geq (1 - \omega) \frac{1}{a^{n-1}}.$$

So by Lemma 4.8, there is a point $q_i \in B_{p_i}(1)$ with $d(q_i, p_i) \leq 1 - a + \eta$. Denoting the limit of $q_i$ by $y_\epsilon$, we have $1 - a \leq d(x, y_\epsilon) \leq 1 - a + \epsilon$. Since $\epsilon$ is arbitrary, we will find a point $y$ with $d(x, y) = 1 - a$.

Moreover, applying Lemma 4.7 to $y_1, z_1, z_2$, and $y_1, y_2, z_2$, we know that

$$d^2(z_1, z_2)r(y_1) + d^2(x^*, z_2)(r(z_1) - r(y_1))$$

$$- d^2(y_1, z_2)r(z_1) = r(y_1)r(z_1)(r(z_1) - r(y_1))$$

and

$$d^2(y_1, z_2)r(y_2) + d^2(x^*, y_1)(r(z_2) - r(y_2))$$

$$- d^2(y_1, y_2)r(z_2) = r(y_2)r(z_2)(r(z_2) - r(y_2)).$$

Combined these two identity, we get (51). The proposition is proved. \hfill \Box
5 Volume Convergence

In this section, we will prove a local version of volume convergence as in [10]. Let $M$ be a Riemannian manifold with singularity and $\text{Ric}(g) \geq 0$ in $\mathcal{R}$.

**Proposition 5.1** Given $\epsilon > 0$, there exist $R = R(\epsilon, n) > 1$ and $\delta = \delta(\epsilon, n)$ such that if $B_p(R) \subset M$ satisfies

$$d_{GH}(B_p(R), B_0(R)) < \delta, \quad (52)$$

then we have

$$\text{vol}(B_p(1)) \geq (1 - \epsilon)\text{vol}(B_0(1)). \quad (53)$$

**Proof** We need to construct a Gromov–Hausdorff approximation map by using harmonic functions constructed in Sect. 2. Choose $n$ points $q_i$ in $B_p(R)$ which is close to $Re_i$ in $B_0(R)$, respectively. Let $l_i(q) = d(q, q_i) - d(q_i, p)$ and $h_i$ a solution of

$$\Delta h_i = 0, \text{ in } B_1(p),$$

with $h_i = l_i$ on $\partial B_1(p) \cap \mathcal{R}$. Then by Lemma 3.4, we have

$$\frac{1}{\text{vol}(B_p(1))} \int_{B_p(1)} |\text{Hess } h_i|^2 < \Psi(1/R, \delta). \quad (54)$$

By using an argument in [10], it follows

$$\frac{1}{\text{vol}(B_p(1))} \int_{B_p(1) \cap \mathcal{R}} |\langle \nabla h_i, \nabla h_j \rangle - \delta_{ij}| < \Psi(1/R, \delta). \quad (55)$$

Define a map by $h = (h_1, h_2, \ldots, h_n)$. It is easy to see that the map $h$ is a $\Psi(1/R, \delta)$ Gromov–Hausdorff approximation to $B_p(1)$ by using the estimate (20) in Lemma 3.4. Since $h$ maps $\partial B_p(1)$ nearby $\partial B_0(1)$ with distance less than $\Psi$, by a small modification to $h$ we may assume that

$$h : (B_p(1), \partial B_p(1)) \rightarrow (B_0(1 - \Psi), \partial B_0(1 - \Psi)).$$

Now we can use the same degree argument in [4] to show that the image of $h$ contains $B_0(1 - \Psi)$. By using Vitali covering lemma, there exists a point $x$ in $B_p(\frac{1}{8}) \cap \mathcal{R}$ such that for any $r$ less than $\frac{1}{8}$ it holds

$$\frac{1}{\text{vol}(B_x(r))} \int_{B_x(r) \cap \mathcal{R}} |\text{Hess } h_i| < \Psi \quad (55)$$
and
\[
\frac{1}{\text{vol} (B_x(r))} \int_{B_x(r) \cap R} |\langle \nabla h_i, \nabla h_j \rangle - \delta_{ij}| < \Psi. \tag{56}
\]

Let \( \eta = \Psi^{\frac{1}{m+1}} \). For any \( y \) with \( d(x, y) = r < \frac{1}{8} \), applying Lemma 3, from (55) we get,
\[
\int_{B_x(\eta r) \cap R} \int_{\gamma \cap w} \|Hess h_i(\gamma', \gamma')\| < r \left( \text{vol} (B_x(\eta r)) + \text{vol} (B_y(\eta r)) \right) \text{vol} (B_x(r)) \Psi.
\]

It follows that
\[
\int_{B_x(\eta r) \cap R} \left( Q(r, \eta) \int_{B_y(\eta r) \cap R} \int_{\gamma \cap w} \sum_{i=1}^n |Hess h_i(\gamma', \gamma')| + |\langle \nabla h_i, \nabla h_j \rangle - \delta_{ij}| \right) < \text{vol} (B_x(\eta r)) \Psi,
\]
where \( Q(r, \eta) = \frac{\text{vol} B_x(\eta r)}{r \left( \text{vol} (B_x(\eta r)) + \text{vol} (B_y(\eta r)) \right) \text{vol} B_x(r)} \). Consider
\[
Q(r, \eta) \int_{B_y(\eta r) \cap R} \int_{\gamma \cap w} \sum_{i=1}^n \text{Hess} |h_i(\gamma', \gamma')| + |\langle \nabla h_i, \nabla h_j \rangle - \delta_{ij}| \]
as a function of \( z \in B_x(\eta r) \). Then one sees that there exists a point \( x^* \in B_x(\eta r) \cap R \) such that
\[
|\langle \nabla h_i, \nabla h_j \rangle(x^*) - \delta_{ij}| < \Psi \tag{57}
\]
and
\[
\sum_{i=1}^n \int_{B_y(\eta r) \cap R} \int_{\gamma \cap w} \text{Hess} h_i(\gamma', \gamma') < r \text{vol} (B_x(r)) \eta^{-n} \Psi. \tag{58}
\]

Moreover by (58), we can find a point \( y^* \in B_y(\eta r) \) such that
\[
\sum_{i=1}^n \int_{\gamma \cap w} \text{Hess} h_i(\gamma', \gamma') < \eta r. \tag{59}
\]

By a direct calculation with help of (57) and (59), we get
\[
(h(x^*) - h(y^*))^2 = \left( 1 + \Psi^{\frac{1}{m+1}} \right) r^2. \tag{60}
\]
This shows that \( h(x) \neq h(y) \) for any \( y \) with \( d(y, x) \leq \frac{1}{8} \). On the other hand, for any \( y \) with \( d(y, x) \geq \frac{1}{8} \), it is clear that \( h(x) \neq h(y) \) since \( h \) is a \( \Psi \) Gromov–Hausdorff approximation. Thus we prove that the pre-image of \( h(x) \) is unique. Therefore the degree of \( h \) is 1, and consequently, \( B_0(1 - \Psi) \subset h(B_p(1)) \). The lemma is proved because the volume of \( B_p(1) \) is almost same to one of \( h(B_p(1)) \) by (54).

\[ \Box \]

### 6 Structure of Limit Spaces

#### 6.1 Real Case

Let \( (M_i, g_i) \) be a sequence of Riemannian manifolds with singularity in \( M(V, D, n) \) and \( (M_i, g_i) \to (X, d) \).

**Theorem 6.1** Every tangent cone of \( X \) is a metric cone. There is a decomposition of \( X \) into \( \mathcal{R} \cup S \) and \( S = S_{n-2} \). Moreover, we have \( \dim S_k \leq k \).

**Proof** For any \( (M, g) \in M(V, D, n) \), by the volume comparison, we have

\[
\frac{\text{vol}(B_p(1))}{V} \geq \frac{\text{vol}(B_p(1))}{\mathcal{H}^n(M)} \geq \frac{1}{D^n}.
\]

Now by Proposition 4.1, we know that every tangent cone is a metric cone. By the argument in [7], we get \( S = S_{n-2} \) and \( \dim S_k \leq k \).

From this theorem and Proposition 5.1, we have

**Proposition 6.2** Denote by \( \mathcal{H}^n \) the \( n \)-dimensional Hausdorff measure, then

\[
\lim_{i \to \infty} \mathcal{H}^n(M_i) = \mathcal{H}^n(X).
\]

#### 6.2 Kähler Case

Now let \( M^n \) be a Kähler manifold, and \( \omega \) be a conic Kähler–Einstein metric on \( M \):

\[
\text{Ric}(\omega) = t\omega + 2\pi \sum_{i=1}^{k} (1 - \beta^i)D_i,
\]

where \( t \) is a positive constant and \( D_i \) are simple normal crossing divisors in \( M \). For any \( \delta > 0, V > 0 \), denote by \( M(n, k, \delta, V) \) the following set:

\[
\left\{ (M^n, \omega) \mid \omega \text{ is a conic Kähler–Einstein metric with } t \in [\delta, \delta^{-1}], \int_M \omega^n \geq V \right\}.
\]

**Lemma 6.3** Any manifold \( M \) with a conic Kähler–Einstein metric \( \omega \) is a Riemannian manifold with singularity.
**Proof** The convexity in Definition 1.4 follows from Theorem 1.1 in [12]. The existence of cut-off function is standard. To get the solution of Dirichlet problem, we use the approximation of $\omega$ by smooth Kähler metrics with Ricci curvature bounded from below. By Proposition 1.1 in [12] or Theorem 2.1 in [18], there is a sequence of smooth Kähler metrics $\omega_i$ satisfying $\omega_i \to \omega$ smoothly outside $D = \bigcup_{i=1}^{k} D_i$ and $\text{Ric}(\omega_i) \geq -C \omega_i$ for some constant $C$ depending on $(M, \omega)$. Let $h_i$ be the solution of
\[
\begin{cases}
\Delta_{\omega_i} h_i = c, \\
h_i|_{\partial U} = b|_{\partial U}.
\end{cases}
\]
By Cheng–Yau’s gradient estimate, we know that for any $V \subset \subset U$, $h_i$ is uniformly Lipschitz on $V$. For $\Omega \subset \subset \overline{U} \cap R$, there exists a constant $C = C(\Omega)$ such that $|\nabla h_i|_{\Omega} \leq C$. So we can take the limit of $h_i$ to get $h$. □

**Theorem 6.4** For any limit space $X$ of conic Kähler–Einstein metrics in $\mathcal{M}(n, k, \delta, V)$, every tangent cone of $X$ is a metric cone. There is a decomposition of $X$ into $R \cup S$ and $S = S_{2n-2}$. Moreover, we have $S_{2k+1} = S_{2k}$ and $\dim S_{2k} \leq 2k$.

**Proof** Since $M \setminus D$ is convex, we know that $\text{diam}(M, \omega) \leq \sqrt{\frac{2n-1}{\delta}}$. By the above lemma, we know that
\[
\mathcal{M}(n, k, \delta, V) \subseteq \mathcal{M}\left(V, \sqrt{\frac{2n-1}{\delta}}, 2n\right).
\]
To prove that $S_{2k+1} = S_{2k}$, we use the proof of Theorem 9.1 in [9]. For the function $h^+$ constructed before Lemma 3.4, we say that $\nabla h^+$ is an almost splitting direction. By Lemma 6.5 below, we know that if $\nabla h^+$ is an almost splitting direction, $J \nabla h^+$ is also an almost splitting direction. So the splitting direction is almost $J$–invariant. It follows that $S_{2k+1} = S_{2k}$. □

**Lemma 6.5** Under the conditions of Lemma 4.2, for a vector field $X$ on $A_p(a, b)$ which satisfies
\[
|X|_{C^0(A_p(a, b))} \leq C, \quad \frac{1}{\text{vol}(A_p(a, b))} \int_{A_p(a, b)} |\nabla X|^2 dv \leq \delta, \quad (61)
\]
there exists a harmonic function $\theta$ defined in $A_p(a', b')$ such that
\[
\frac{1}{\text{vol}(A_p(a', b'))} \int_{A_p(a', b')} |\nabla \theta - X|^2 dv < \Psi(\omega, \delta; a, b, a', b'), \quad (62)
\]
and
\[
\frac{1}{\text{vol}A_p(a_1, b_1)} \int_{A_p(a_1, b_1)} |\text{Hess} \theta|^2 dv
\]
where $A_p(a_1, b_1)$ is an even smaller annulus in $A_p(a', b')$.

**Proof** Let $h$ be the harmonic function constructed in (29) and $\theta_1 = \langle X, \nabla h \rangle$. Then

\[
\nabla \theta_1 = \langle \nabla X, \nabla h \rangle + \langle X, \text{Hess} h \rangle.
\]

It follows

\[
\int_{A_p(a, b) \cap R} |\nabla \theta_1 - X|^2 \, dv \\
\leq 2 \int_{A_p(a, b) \cap R} ((\nabla X, \nabla h)^2 \, dv + \langle X, \text{Hess} h - g \rangle^2) \, dv.
\]

Thus by (61) and Lemma 4.4, we get

\[
\frac{1}{\text{vol}(A_p(a, b))} \int_{A_p(a, b) \cap R} |\nabla \theta_1 - X|^2 \, dv \leq \Psi. \tag{64}
\]

Let $\theta$ be a solution of equation,

\[
\Delta \theta = 0, \text{ in } A_p(a, b) \cap R, \tag{65}
\]

with $\theta = \theta_1$ on $\partial A_p(a, b) \cap R$. Then

\[
0 = \int_{A_p(a, b)} \text{div} \left( \gamma_\eta (\theta - \theta_1) X \right) \, dv \\
= \int_{A_p(a, b)} ((\theta - \theta_1) \langle \nabla \gamma_\eta, X \rangle + \langle \nabla \theta - \nabla \theta_1, X \rangle + (\theta - \theta_1) \text{div} X) \, dv.
\]

Since

\[
\int_{A_p(a, b)} (\theta - \theta_1) \langle \nabla \gamma_\eta, X \rangle \, dv \leq C \left( \int_{A_p(a, b)} |\nabla \gamma_\eta|^2 \, dv \right)^{1/2} \leq C \sqrt{\eta},
\]

taking $\eta \to 0$ we have

\[
\int_{A_p(a, b) \cap R} (\nabla \theta - \nabla \theta_1, X) \, dv \leq \Psi. \tag{66}
\]

On the other hand, from

\[
0 = \int_{A_p(a, b)} \text{div} \left( \gamma_\eta^2 (\theta_1 - \theta) \nabla \theta \right)
\]
\begin{align}
&= \int_{A_p(a,b)} \gamma^2_\eta (\nabla \theta_1 - \nabla \theta, \nabla \theta) + 2 \int_{A_p(a,b)} \gamma_\eta (\theta_1 - \theta) \langle \gamma_\eta, \nabla \theta \rangle \\
&\leq \int_{A_p(a,b)} \gamma^2_\eta (\nabla \theta_1 - \nabla \theta, \nabla \theta) + \sqrt{\eta} \int_{A_p(a,b)} \gamma^2_\eta |\nabla \theta|^2 d\nu \\
&\quad + \frac{C}{\sqrt{\eta}} \int_{A_p(a,b)} |\nabla \gamma_\eta|^2 d\nu,
\end{align}

we get

\begin{align}
(1 - \sqrt{\eta}) \int_{A_p(a,b)} \gamma^2_\eta |\nabla \theta|^2 d\nu &\leq \int_{A_p(a,b)} \gamma^2_\eta (\nabla \theta, \nabla \theta_1) d\nu + C \sqrt{\eta} \\
&\leq \frac{1}{2} \int_{A_p(a,b)} \gamma^2_\eta |\nabla \theta|^2 d\nu \\
&\quad + \frac{1}{2} \int_{A_p(a,b)} \gamma^2_\eta |\nabla \theta_1|^2 d\nu + C \sqrt{\eta}.
\end{align}

Taking \( \eta \to 0 \), we have

\[\int_{A_p(a,b)} |\nabla \theta|^2 d\nu \leq \int_{A_p(a_2,b_2)} |\nabla \theta_1|^2 d\nu < C.\]

Hence,

\[\int_{A_p(a,b)} |\nabla \theta - X|^2 d\nu\]

\[= \int_{A_p(a,b)} (|\nabla \theta|^2 + |X|^2 - 2\langle \nabla \theta, X \rangle) d\nu\]

\[= \int_{A_p(a,b)} (\langle \nabla \theta, \nabla \theta_1 \rangle + |X|^2 - 2\langle \nabla \theta, X \rangle) d\nu\]

\[= \int_{A_p(a,b)} (\langle \nabla \theta_1 - X, \nabla \theta \rangle + \langle X, X - \nabla \theta_1 \rangle + \langle X, \nabla \theta_1 - \nabla \theta \rangle) d\nu.\]

Therefore, combining (61) and (66), we derive (62) immediately.

To get (63), we choose a cut-off function \( \phi \) which is supported in \( A_p(a, b) \) with bounded gradient and Laplacian as in Lemma 2.9. Then by the Bochner identity, we have

\[\int_{A_p(a,b)} \frac{1}{2} \phi \Delta |\nabla \theta|^2 d\nu = \int_{A_p(a,b)} \phi (|\text{hess} \theta|^2 + \text{Ric}(\nabla \theta, \nabla \theta)) d\nu.\]

Since

\[\int_{A_p(a,b)} \frac{1}{2} \phi \Delta |X|^2 d\nu = -\int_{A_p(a,b)} \langle \nabla \phi, (X, \nabla X) \rangle d\nu,\]
we obtain
\[ \int_{A_p(a,b)} \phi (|\text{Hess} \theta|^2) dv \leq \int_{A_p(a,b)} \frac{1}{2} \phi \Delta (|\nabla \theta|^2 - |X|^2) dv + C \delta. \]

Therefore, by Lemma 2.6, we derive (63) from (62).

Let \((M_i, \omega_i)\) be a sequence of conic Kähler–Einstein metrics in \(\mathcal{M}(n, k, \delta, V)\):

\[ \text{Ric}(\omega_i) = t_i \omega_i + \sum_{j=1}^k (1 - \beta_i^j) D_j. \]

Assuming that there is \(0 < \epsilon, 0 < T < 1\) such that \(\beta_i^j \in [\epsilon, T]\), we can characterize the cone angle in the limit space.

**Proposition 6.6** If there is a tangent cone \(T_x X \simeq \mathbb{C} \bar{\beta} \times \mathbb{R}^{n-2}\), we have

\[ 1 - \bar{\beta} = \sum_{j=1}^k m_j (1 - \beta_{\infty}^j), \]

where \(\beta_{\infty}^j\) is the limit of \(\beta_i^j\), and \(m_j\) are some positive integers.

**Proof** We use the arguments in [17]. Assume that

\[ (T_x X, x, \omega_x) = \lim_{i \to \infty} \left( X, \frac{d}{r_i^2}, p_i \right). \]

Using the estimate in Sects. 3 and 4, as Theorem 2.37 in [9], there are \(\epsilon_i \to 0\) and maps \((\Phi_i, u_i) : B_{p_i}(\frac{\delta}{2}, r_i^{-2} \omega_i) \to B_0(\frac{\delta}{2})\) such that

\[ \int_{|z| \leq 1} |V(z) - 2\pi \gamma| dz \leq \epsilon_i, \]

where \(V(z)\) is the volume of \(\Sigma_{\epsilon} = \Phi^{-1}(z) \cap u_i^{-1}[0, 1]\). \(K_M^{-1}\) restricts to a line bundle on \(\Sigma_{\epsilon}\) whose curvature \(\Omega\) is

\[ \text{Ric}(r_i^{-2} \omega_i) = t_i \omega_i + 2\pi \sum (1 - \beta_i^j) D_j. \]

Let \(\pi : S \Sigma_{\epsilon} \to \Sigma_{\epsilon}\) be the unit circle bundle, then

\[ \pi^* \Omega = d\theta. \]

Since \(K_M^{-1}\) is topologically equivalent to \(T_{\Sigma_{\epsilon}}\), there is a section \(v\) of \(K_M^{-1}\) which is equal to the outward unit normal of \(\partial \Sigma_{\epsilon}\) along the boundary of \(\Sigma_{\epsilon}\) and has non-degenerate
zeroes outside $\Sigma_z \setminus \cup D_j^i$. Put

$$s = \frac{v}{\|v\|} : \Sigma_z \setminus (D_j^i \cup v^{-1}(0)) \to S \Sigma_z.$$  

By Stokes theorem, we have

$$\int_{\Sigma_z \setminus \cup D_j^i} \Omega = \int_{\partial \Sigma_z} s^* \theta - \sum_{v(p) = 0 \text{ or } p \in \cup D_j^i} \lim_{\delta \to 0} \int_{\partial B_\delta(p, r_i^{-2} \omega_i)} s^* \theta.$$  

It follows that

$$\bar{\beta} - \chi(\Sigma_z) + \sum m_j (1 - \beta_j^i) = o(1).$$  

Since $\chi(\Sigma_z) \leq 1$, we must have $\chi(\Sigma_z) = 1$, and

$$1 - \bar{\beta} = \sum m_j (1 - \beta_j^i) + o(1).$$  

Taking $i \to \infty$, we get

$$1 - \bar{\beta} = \sum m_j (1 - \beta_{\infty}^j).$$

\[\square\]

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