Quantum phase transition in the $\mathbb{Z}_3$ Kitaev-Potts model

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Competition between the topological order induced by the $\mathbb{Z}_3$ Kitaev model and the local order induced by the 3-state Potts model is studied. We show that the low energy sector of the Kitaev-Potts model is mapped to the Potts model in the presence of transverse magnetic field. Our study relies on two high-order series expansion based on continuous unitary transformations in the limits of small- and large-Potts couplings as well as mean-field approximation. Our analysis reveals that the topological phase of the $\mathbb{Z}_3$ Kitaev model breaks down to the Potts model through a first order phase transition. We capture the phase transition by analysis of the ground state energy, one-quasiparticle gap and geometric measure of entanglement.

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I. INTRODUCTION

Quantum computers are much more powerful than the classical ones. However, a practical realization of such machines is still a big challenge ahead. Qubit as the essential part of a quantum computer is a very fragile entity and sensitive to errors arising from interaction with environment. During the past years, several error correcting methods have been proposed. Unfortunately, error correcting methods are themselves error prone which cause other complexities. In fact, the error threshold below which fault tolerant quantum computation is possible, is very low. In order to overcome such problems, it has recently been proposed to store information in physical systems which are intrinsically robust against errors. Systems with topologically ordered ground states are promising playgrounds for the so-called fault tolerant quantum computation.

The toric code by Kitaev is the first model introduced for the purpose of topological quantum computation. In this model, information is stored in the topologically degenerate ground states of the system and the computation is performed by braiding the quasiparticles (QP) of the model. Here one can use the gapped ground state and the robust nature of the topological phase of the model to protect information against local errors. The only perturbations that cause logical error are those with length equal to the system size.

In his seminal paper, Kitaev showed that universal quantum computation is not possible with an Abelian group and in order to perform all universal gates of quantum computation, one needs to resort to the non-Abelian case. However, as physical realizations of the non-Abelian models are difficult, one can try to extend the Abelian models, i.e. those based on $\mathbb{Z}_d$ Abelian group, by adding non-topological operations. It has been shown that when $d$ is prime, by adding some non-topological operations such as measurement, one can perform universal quantum computation.

Aside from the ability of $d$-level Abelian models to perform universal quantum computation in the above sense, their robustness and stability against external perturbations is still a crucial question which has to be investigated. Plenty of recent studies have been devoted to responding to such questions. It has been shown that the $\mathbb{Z}_3$ Kitaev model is more robust against temperature than the $\mathbb{Z}_2$ one. The robustness of the model has also been studied in transverse magnetic field and it has been shown that the topological order transforms to the magnetic order via a first order phase transition. The competition between topological order and ferromagnetic order has also been studied in Ref., where it was shown by a mean field analysis and estimation of Wilson loop that topological order breaks down when the strength of Ising interaction exceeds a critical value. Here we want to extend these studies by studying the competition of $\mathbb{Z}_3$ topological order induced by the $\mathbb{Z}_3$ Kitaev model and the local order induced by the 3-state Potts model. The reason for restricting to 3-state Potts model instead of the general d-state Potts model is that this model has an equidistant spectrum which is needed for the Perturbative Continuous Unitary Transformations (PCUT), as we shall see.

The outline of the paper is as follows: In Sec. II and III, we briefly review the $\mathbb{Z}_d$ Kitaev and Potts models and their essential features needed for our study in this paper, respectively. In Sec. IV, by defining a new basis, we show that the $\mathbb{Z}_3$ Kitaev model in the presence of the 3-state Potts model is mapped to the Potts model in transverse magnetic field. We start our analysis of the mapped model in Sec. V by using mean field approximation and Perturbative Continuous Unitary Transformations method. By applying the PCUT method to the small Potts coupling limits, we calculate the 1-QP gap and also the Geometric Measure of Entanglement. We further compute the ground-state energy in both small and large Potts couplings and capture the phase transition and the breakdown of the topological phase of the Kitaev model by analysis of the ground state energy and its derivatives, as well as the Geometric Measure of Entanglement and energy gap. Finally, Sec. VI is devoted to the conclusion.
FIG. 1. The conventions for defining vertex and plaquette operators. $A_s := \prod_{i \in s} \sigma_{x,i}^{\pm 1}$, where the $+$ ($-$) corresponds to links with inward (outward) direction and $B_p := \prod_{i \in p} \sigma_{z,i}^{\pm 1}$, where the $+$ ($-$) corresponds to counterclockwise (clockwise) direction of the links around plaquettes.

II. $Z_d$ KITAEV MODEL

$Z_d$ Kitaev model is the generalization of the Kitaev model from $Z_2$ to $Z_d$ group. The model can be defined on any two-dimensional lattice. In the present work, we consider the model on a square lattice on which the periodic boundary conditions are imposed on both sides and the manifold forms a torus. The lattice has $N$ plaquettes, $N$ vertices and $2N$ edges. The qudits live on the edges of the lattice. Hamiltonian of the model consists of two kinds of operators, i.e. the plaquette and vertex operators. The operators are define based on the generalized Pauli operators (acting on a qudit) as: 

$$\sigma_{x,i}^0 = |i + 1 \text{ mod } d\rangle$$

and  

$$\sigma_{z,i}^0 = \omega^{|i\rangle}, \omega = e^{2\pi i/d}.$$ 

These operators are not Hermitian and they don’t square to $I$ except for $d$=2. They further obey the following commutation relation: $\sigma_x \sigma_z = \omega \sigma_z \sigma_x$. The Hamiltonian of the model is given by the sum of the plaquette and vertex operators as:

$$H_{\text{Kitaev}} := -J \sum_s (A_s + A_s^\dagger) - K \sum_p (B_p + B_p^\dagger),$$

where $s$ and $p$ denote the stars (vertices) and plaquettes respectively. In order for the model to be exactly solvable, we define the $A_s$’s and $B_p$’s such that they commute with each other. To this end we ascribe an arbitrary direction to each edge and define $A_s$ and $B_p$ operators as follows:

- $A_s := \prod_{i \in s} \sigma_{x,i}^{\pm 1}$, if the link’s direction is inward, $\sigma_x$ is applied, otherwise $\sigma_x^{-1}$ is,

- $B_p := \prod_{i \in p} \sigma_{z,i}^{\pm 1}$, by starting from a link and moving counterclockwise, if each link’s direction is the same as moving’s direction $\sigma_z$ is applied, otherwise $\sigma_z^{-1}$ is. (Fig. 1)

These conventions lead to the commutativity of all $A_s$’s and $B_p$’s. Let us note that these arbitrary orientations do not have a physical significance, i.e. they lead to unitary equivalent models. That is if we change the orientations on some links ($L$) in an arbitrary manner, we will end up at a model ($\tilde{H}$) which is iso-spectral with the original one ($H$). This unitary operator should transform $\sigma_x$ to $\sigma_x^\dagger$ and $\sigma_z$ to $\sigma_z^\dagger$ in the Hamiltonian:

$$\tilde{H} = UHU^\dagger, \quad U = \bigotimes_{i \in L} u_i,$$

where

$$u_i \sigma_{x,i} u_i^\dagger = \sigma_{x,i}^\dagger, \quad u_i \sigma_{z,i} u_i^\dagger = \sigma_{z,i}^\dagger$$

It is straightforward to show that $u$ is equivalent to:

$$u = \sum_k |d-k\rangle \langle k|.$$ 

Note that there are $2N$ stabilizers in the Hamiltonian, but only $2N-2$ of them are independent, because of these two constraints on the torus:

$$\prod_s A_s = \prod_p B_p = I.$$ 

So there are $d^2$ degenerate ground states. The ground state is the state that is stabilized by all of the star and plaquette operators simultaneously and is equal to:

$$|\tilde{0}\rangle := \prod_s (1 + A_s + A_s^\dagger + A_s^2 + \ldots + A_s^{d-1})|0\rangle_{\otimes 2N}. \quad (6)$$

In order to construct the other $d^2-1$ degenerate ground states, we define the following four string operators:

- $T_{x,1} = \prod_{i \in C_1} \sigma_{x,i}^\dagger$, by starting from a link on the loop $C_1$ and moving on it, if each link’s direction is the same as moving’s direction $\sigma_z$ is applied, otherwise $\sigma_z^{-1}$ is,

- $T_{x,2} = \prod_{i \in C_2} \sigma_{x,i}^\dagger$, by starting from a link on the loop $C_2$ and moving on it, if each link’s direction is the same as moving’s direction $\sigma_x$ is applied, otherwise $\sigma_x^{-1}$ is,

- $T_{x,1} = \prod_{i \in \tilde{C}_1} \sigma_{x,i}^\dagger$, if the links’ direction is the same as moving’s direction on $C_1$, $\sigma_x$ is applied, otherwise $\sigma_x^{-1}$ is,

- $T_{x,2} = \prod_{i \in \tilde{C}_2} \sigma_{x,i}^\dagger$, if the links’ direction is the same as moving’s direction on $C_2$, $\sigma_x$ is applied, otherwise $\sigma_x^{-1}$ is,

where $C_1$, $C_2$, $\tilde{C}_1$ and $\tilde{C}_2$ are shown in Fig. 2. All the degenerate ground states are given as:

$$|\tilde{i,j}\rangle = T_{x,1} T_{x,2}^\dagger |\tilde{0}\rangle, \quad i, j = 0, 1, 2, ..., d - 1. \quad (7)$$

One can further check that:

$$T_{x,1} |\tilde{i,j}\rangle = \omega^i |\tilde{i,j}\rangle, \quad T_{x,2} |\tilde{i,j}\rangle = \omega^j |\tilde{i,j}\rangle. \quad (8)$$
III. POTT S MODEL

Considering any lattice of interest, the classical Potts model is defined by the following Hamiltonian:

\[ H_{\text{Potts}} = -\sum_{i,j} \delta_{s_i, s_j}, \] (9)

where the sum runs over the nearest neighbor sites of the lattice \((i,j)\) and \(s_i\) takes \(d\) different values. One can take these values to be \(d\) different roots of unity. For \(d=2\), \(s_i = \{1, -1\}\), which reduces the Hamiltonian (9) to the renowned Ising model. This simplification is a consequence of the following definition for the delta function: \(\delta_{s_i, s_j} = \frac{1}{2}(1 + s_i s_j)\). Such a definition can be extended to \(d\)-level and the analogue formula for the Potts model reads:

\[ \delta_{s_i, s_j} = \frac{1}{2d} \sum_{r=0}^{d-1} ((s_i^r s_j^*) + (s_i^* s_j)^r), \] (10)

where \(s_j^*\) is the complex conjugate of \(s_j\).

The quantum Potts model consists of \(d\)-level spins (qudits). The nearest neighbor spins interact with each other by the following Hamiltonian:

\[ H_{\text{Potts}} = -\frac{1}{2d} \sum_{(i,j)} \sum_{r=0}^{d-1} ((\sigma_{z,i}^r \sigma_{z,j}^{r*}) + (\sigma_{z,i}^{r*} \sigma_{z,j})^r). \] (11)

The ground state of the system is the state of all spins, polarized in the same direction. So there are \(d\) degenerate ground states as follows:

\[ |\tilde{i}\rangle := |i\rangle^\otimes L, \quad i = 0, 1, ..., d - 1, \] (12)

where \(L\) is the number of spins.

IV. KITA E V-POTTS MODEL FOR QU TRITS

Our aim is to study the phase transition of the Kitaev model in presence of the Potts interaction. As our series expansion technique, i.e. the PCUT method is only applicable to those models with equidistant spectrum, we confine our study to the \(d = 3\) or qutrits and show that \(Z_3\) Kitaev and 3-state Potts models have equidistant spectrum. The full Hamiltonian of the \(Z_3\) Kitaev model perturbed by Potts interaction is given by:

\[
H = H_{\text{Kitaev}} + \lambda H_{\text{Potts}}
= -J \sum_s (A_s + A_s^\dagger) - K \sum_p (B_p + B_p^\dagger)
- \frac{\lambda}{6} \sum_{(i,j)} \sum_{r=0}^{2} ((\sigma_{z,i}^r \sigma_{z,j}^{r*}) + (\sigma_{z,i}^{r*} \sigma_{z,j})^r) - 2KN. \quad (13)
\]

where \(\lambda\) is the perturbation parameter and also a measure of the strength of the Potts interaction. The perturbed Kitaev Hamiltonian is no longer exactly solvable, this is due to the fact that the \(\sigma_z\) operators in the Potts model do not commute with vertex operators of the Kitaev model. However, the plaquette operators still commute with the full Hamiltonian (13). The ground state of the Hamiltonian (13) is therefore in the sector in which \(B_p = 1\) for all plaquette operators. In this sector the Kitaev-Potts’s Hamiltonian reduces to the following form:

\[
H = -J \sum_s (A_s + A_s^\dagger)
- \frac{\lambda}{6} \sum_{(i,j)} \sum_{r=0}^{2} ((\sigma_{z,i}^r \sigma_{z,j}^{r*}) + (\sigma_{z,i}^{r*} \sigma_{z,j})^r) - 2KN. \quad (14)
\]

In order to tackle the Hamiltonian (14), we first define the following new basis and rewrite the full Hamiltonian in this new basis:

\[ |r\rangle = |r_1, r_2, ..., r_N\rangle := \prod_i A_i^r |0\rangle^\otimes 2N, \quad r_i = 0, 1, 2. \quad (15)\]

One can readily check that \(B_p |r\rangle = |r\rangle\). We can therefore interpret Eq. (15) as the basis for the sector in which \(B_p = 1\) for all \(p\). Next, we determine the action of the different terms in the Hamiltonian (14) on this new basis:

\[ A_s |r\rangle = A_s \prod_i A_i^{r_s} |0\rangle^\otimes 2N = |r_1, r_2, ..., r_s + 1, ..., r_N\rangle, \quad (16)\]

so \(A_s\) in this basis acts like generalized Pauli operator, \(\sigma_x\), which we denote by \(\tilde{X}\). Clearly, the action of \(A_i^{r_s}\) can be regarded as \(\tilde{X}^s\). In order to recast the Potts interaction in the new basis, we attach an arrow to each link. The direction of the arrows are illustrated in Fig. 2. The action of a Potts interaction term like \(\sigma_{z,a} \sigma_{z,b}^\dagger\) in the new
basis, is denoted by:

\[
\sigma_{z,a}^\dagger \sigma_{z,b}^\dagger |r\rangle = \sigma_{z,a}^\dagger \sigma_{z,b}^\dagger A_{1}^{\dagger} \cdots A_{3}^{\dagger} \prod_{i \neq 1,2,3} A_{i}^{\dagger} |0\rangle \otimes 2^{N}(17)
\]

which means that the Potts interaction term, \( \sigma_{z,a} \sigma_{z,b} \), commutes with all of the vertex operators except \( A_{1} \) and \( A_{3} \), actually it acts like \( \hat{Z}_{i} \hat{Z}_{3} \) in this new basis:

\[
\sigma_{z,a} \sigma_{z,b} = \hat{Z}_{i} \hat{Z}_{3}.
\]

It is straightforward to check that the following relations further hold in the new basis:

\[
\begin{align*}
\sigma_{z,a}^\dagger \sigma_{z,b}^\dagger & = \hat{Z}_{i}^\dagger \hat{Z}_{3}, \\
\sigma_{z,c}^\dagger \sigma_{z,d}^\dagger & = \hat{Z}_{1}^\dagger \hat{Z}_{3}, \\
\sigma_{z,e}^\dagger \sigma_{z,d}^\dagger & = \hat{Z}_{1}^\dagger \hat{Z}_{3}.
\end{align*}
\]

Therefore the \( \mathbb{Z}_{3} \) Kitaev-Potts model in the new basis is given by:

\[
\hat{H} = \hat{H}_{A} + \hat{H}_{B} - 2KN,
\]

where

\[
\begin{align*}
\hat{H}_{A} & = -J \sum_{i \in A}(\hat{X}_{i} + \hat{X}_{i}^\dagger) - \lambda \sum_{(i,j) \in A} \sum_{r=0}^{2}(\hat{Z}_{i}^r \hat{Z}_{j}^r + \hat{Z}_{j}^r \hat{Z}_{i}^r), \\
\hat{H}_{B} & = -J \sum_{i \in B}(\hat{X}_{i} + \hat{X}_{i}^\dagger) - \lambda \sum_{(i,j) \in B} \sum_{r=0}^{2}(\hat{Z}_{i}^r \hat{Z}_{j}^r + \hat{Z}_{j}^r \hat{Z}_{i}^r).
\end{align*}
\]

Hamiltonian (20) is nothing but the sum of two Potts models in a transverse magnetic field, wherein the Potts interactions act on nearest neighbour vertices in the two de-coupled sublattices shown in Fig. 3.

The two de-coupled Hamiltonians are totally the same. In the following sections we present our results for \( \hat{H}_{A} \), The results can be extended to the full Hamiltonian (20), without loss of generality. From now on to the end of the paper by \( H \), we mean \( H_{A} \), unless stated otherwise.

V. METHODS

In this section, we present the solution to the mapped model by applying the Mean field approximation and Perturbative Continuous Unitary Transformations method to the Hamiltonian (21).

A. Mean Field approximation

Suppose we are interested in finding the ground state energy of a given Hamiltonian. We therefore need to minimize \( \langle \Psi | H | \Psi \rangle \) over all \( |\Psi\rangle \)'s in the Hilbert space. One approximation (Mean field approximation) would be that we do the minimization only over product states. But the Hamiltonian has translational symmetry so we can search for the minimum energy only over states of the form \( |\Psi\rangle = |\Phi\rangle \otimes n \), which have translational symmetry. For the mapped Hamiltonian \( n \) is the number of vertices in each sublattice, \( n = \frac{N}{2} \).

\[
E = \min_{{|\Phi\rangle}} \otimes n \langle \Phi | H | \Phi \rangle \otimes n, \tag{23}
\]

where \( |\Phi\rangle \) is a general one-qutrit state. For the mapped Hamiltonian, we have:

\[
H = -J \sum_{i}(\hat{X}_{i} + \hat{X}_{i}^\dagger) - \frac{\lambda}{3} \sum_{(i,j)} \sum_{r=0}^{2}(\hat{Z}_{i}^r \hat{Z}_{j}^r + \hat{Z}_{j}^r \hat{Z}_{i}^r),
\]

\[
E_{0} = -Jn\left(\langle \hat{X} \rangle + \langle \hat{X}^\dagger \rangle \right) - 4n \lambda \sum_{r=0}^{2}(\langle \hat{Z}^r \rangle \langle \hat{Z}^{-r} \rangle). \tag{24}
\]

Taking \( |\Phi\rangle = \sum_{i=0}^{a_{i}|i\rangle} \), we should minimize:

\[
E_{0} = -Jn\left(\sum_{i}a_{i}a_{i+1} + a_{i}^*a_{i+1}\right) - 4n \lambda \sum_{i} |a_{i}|^4. \tag{25}
\]

It’s easy to check that the above equation recasts into:

\[
E_{0} = -Jn\left(|a_{0} + a_{1} + a_{2}|^2 - 1 \right) - 4n \lambda \left(|a_{0}|^4 + |a_{1}|^4 + |a_{2}|^4\right). \tag{26}
\]

that has a permutation symmetry by changing the \( a_{i} \)'s indices from \( (0,1,2) \) to \( (\sigma(1), \sigma(2), \sigma(3)) \). If the state that minimize this expression respects the full symmetry,
the derivation of energy at \( x_c \approx 0.115 \) signals the first order quantum phase transition.

By minimizing \( \varepsilon_0 \), the ground state energy per site reduces to:

\[
\varepsilon_0 = -2J - \frac{4}{3}\lambda. \tag{27}
\]

Setting \( J = 1 \), Eq. (27) is analytic for any value of \( \lambda \) and one does not observe any phase transition. However, if some of the symmetries are broken and the ground state remains invariant under only one permutation, it will be in the form of \( |\Phi\rangle = a_0|0\rangle + a_1|1\rangle + a_2|2\rangle \). By setting \( a_0 = \sin \theta, a_1 = \frac{1}{\sqrt{2}} e^{i\alpha} \cos \theta \), the ground state energy per site reduces to:

\[
\varepsilon_0 = -J(\cos^2 \theta + \sqrt{2} \sin 2\theta \cos \alpha) - 4\lambda(\sin^4 \theta + \frac{1}{2} \cos^4 \theta). \tag{28}
\]

By minimizing \( \varepsilon_0 \), the ground state as a function of perturbation parameter \( x = \frac{2\lambda}{J} \) (this explicit form of \( x \) is chosen because it makes the comparison between mean-field and PCUT results easier) is given as follows:

\[
|\Phi\rangle = \begin{cases} 
\frac{1}{\sqrt{2}}(|0\rangle + |2\rangle) & \text{for } x < 0.115 \\
\sin \theta|0\rangle + \frac{\cos \theta}{\sqrt{2}}|1\rangle + \frac{\cos \theta}{\sqrt{2}}|2\rangle & \text{for } x > 0.115
\end{cases} \tag{29}
\]

where,

\[
x = \frac{\sin 2\theta_x - 2\sqrt{2} \cos 2\theta_x}{36 \sin^2 \theta_x (\sin^2 \theta_x - \frac{1}{2} \cos^2 \theta_x)}. \tag{30}
\]

The above relation shows that the nature of the ground state changes at a critical point \( x_c = 0.115 \), which can be a signal of the phase transition. Finally we analyse the case where the symmetry is fully broken and the ground state has the general form as \( |\Phi\rangle = \sum_{i=0}^{2} a_i|i\rangle \). In this case, we minimize the energy numerically. Figure 4 demonstrates the mean-field ground state energy per site as a function of \( x = \frac{2\lambda}{J} \). As we can see, there is a small kink in the ground state energy curve (a sharp jump in its derivative) at \( x_c \approx 0.115 \). So the results show that the ground state actually respects some permutation symmetries. This jump signals a first order quantum phase transition. In order to investigate the critical point more accurately, we resort to a more accurate approximation technique, i.e. the Perturbative Continuous Unitary Transformation method\(^{25-26}\) which is the subject of the next subsection.

### B. Perturbative Continuous Unitary Transformations

In this section, we briefly review the Perturbative Continuous Unitary Transformation (PCUT) method and apply it to the small- and large-coupling limits of the problem.

The Continuous Unitary Transformation (CUT) method which was first introduced by Wegner\(^{23}\) in the framework of condensed matter theory, is basically used to rather diagonalize or block-diagonalize a given Hamiltonian by applying an infinite number of unitary operators to the initial Hamiltonian in a continuous fashion as:

\[ H(\ell) = U(\ell)HU(\ell), \tag{31} \]

where \( \ell \) is the continuous flow parameter such that \( H = H(\ell = 0) \) and \( H_{\text{eff}} = H(\ell = \infty) \) is the (block-) diagonal Hamiltonian. The Hamiltonian is transformed by a unitary operator which its evolution is governed by:

\[ \partial_\ell U(\ell) = -U(\ell)\eta(\ell). \tag{32} \]

In which \( \eta(\ell) \) is the anti-Hermitian generator of the unitary transformation \( U(\ell) \). Combining Eq. (31) and (32) together one can show that the initial Hamiltonian flows in the form of a differential commutator equation:

\[ \partial_\ell H(\ell) = [\eta(\ell), H(\ell)]. \tag{33} \]

The method therefore requires the choice of a suitable generator for the unitary operators to obtain the desirable form of \( H_{\text{eff}} \). Uhrig and Knetter introduced the quasiparticle (QP) conserving generator which is very well suited for our purpose. We refer the interested reader to Refs\(^{24,25}\) for detailed discussions on QP conserving generator.

The perturbative version of the CUT method (PCUT), can be applied to the Hamiltonians of the form \( H = Q + AV \) where the first part of the Hamiltonian, \( Q \), is diagonal with an equidistant spectrum bounded from below and the second part can be treated as a perturbation (\( \lambda \) is the expansion parameter). The method further requires that the perturbing part can be written in the form \( V = \sum_{n=-N}^{N} T_n \), where \( T_n \) increments (decrements,
if \( n < 0 \) the number of excitations (quasiparticles) by \( n \) such that \( \{Q, T_n\} = n T_n \) \(^{24}\). Transforming the initial problem by using the QP conserving generator, the effective Hamiltonian is brought to the form that conserves the number of quasiparticles, \( [H_{\text{eff}}, Q] = 0 \). The energy spectrum of the system can therefore be extracted perturbatively by acting the \( H_{\text{eff}} \) on the ground state and the multi-particle sectors of the Hilbert space.

1. Small-coupling limit (\( \lambda \ll J \))

In the following we discuss the procedure of applying the PCUT method to the small-coupling limit (\( \lambda \ll J \)) of the mapped Kitaev-Potts model, i.e. the Potts model in transverse magnetic field on the A or B sublattices of Fig. 3:

\[
H = -J \sum_{i \in A} (\hat{X}_i + \hat{X}_i^\dagger) - \frac{\lambda}{3} \sum_{(i,j) \in A} \left( \sum_{r=0}^2 \left( (\hat{Z}_i \hat{Z}_j)^r + (\hat{Z}_j \hat{Z}_i)^r \right) \right),
\]

(34)
The first term in Eq. (34) is an effective field term which is diagonal. Denoting the local vacuum of each site by \( |0\rangle \), the elementary excitations of the model for \( d = 3 \) are two separate spin flips labeled by \( |1\rangle \) and \( |2\rangle \) which correspond to the eigenstates of \( X \) operator with \( \omega \) and \( \omega^{-1} \) eigenvalues, respectively. Let us note that either of the excitations cost an energy of \( 3J \). The elementary excitations are energetically indistinguishable. So the first term for \( d = 3 \) has an equidistant spectrum and can be regarded as \( Q \) for implementation of the PCUT, and the PCUT results obtained in 1-QP sector of the Hilbert space such as 1-QP gap are degenerate for both of the excitations.

The Potts interaction at the right side of Eq. (34) can be treated as a perturbation \( V \) which for \( d = 3 \) is denoted by:

\[
V = -\frac{2\lambda}{3} \sum_{(i,j)} \left( (\hat{Z}_i \hat{Z}_j^\dagger) + (\hat{Z}_j \hat{Z}_i^\dagger) \right) + C,
\]

(35)
where \( C \) is a constant. The perturbing part consists of two-body interactions and can change the number of excitations over the ground state of the effective field term \( Q \) by \( n = \{0, \pm1, \pm2\} \) when it acts on the bonds of the square lattice. Therefore the Hamiltonian (34) can be written as:

\[
H = Q - x(T_2 + T_1 + T_0 + T_{-1} + T_{-2}),
\]

(36)
where

\[
Q = \sum_i -\frac{(\hat{X}_i + \hat{X}_i^\dagger)}{3} + 2I,
\]

(37)
is the quasiparticle counting operator (\( I \) is the identity operator), \( x = \frac{2\lambda J}{3} \) is the expansion parameter and \( T_n \) operators are given by:

\[
T_{+2} = \sum_{(i,j)} |12\rangle_{i,j} \langle 00| + |21\rangle_{i,j} \langle 00|,
\]

(38)
\[
T_+ = \sum_{(i,j)} |22\rangle_{i,j} \langle 01| + |22\rangle_{i,j} \langle 10| + |11\rangle_{i,j} \langle 02| + |11\rangle_{i,j} \langle 20|,
\]

\[
T_0 = \sum_{(i,j)} |12\rangle_{i,j} \langle 21| + |01\rangle_{i,j} \langle 10| + |02\rangle_{i,j} \langle 20| + |21\rangle_{i,j} \langle 12| + |10\rangle_{i,j} \langle 01| + |20\rangle_{i,j} \langle 02|.
\]

From the hermiticity condition \( T_n^\dagger = T_{-n} \), and absorbing the expansion parameter \( -x \) in the definition of \( T_n \) operators, Eq. (36) is recast into:

\[
H = Q + \sum_{n=-2}^{n=2} T_n.
\]

(39)
Under the CUTs, the above Hamiltonian is continuously transformed with the flow parameter \( \ell \) as:

\[
H(\ell) = Q + \sum_{n=-2}^{n=2} T_n(\ell),
\]

(40)
where we wish to reach a situation where \( T_n(\ell = \infty) = 0 \) for all \( n \neq 0 \). In order to fulfill this demand, we choose the quasiparticle conserving generator in the following form\(^{24,25}\):

\[
\eta(\ell) = T_{+2}(\ell) + T_{+1}(\ell) - T_{-1}(\ell) - T_{-2}(\ell).
\]

(41)
With this choice of generator, the flow Eq. (33) can be written as:

\[
\partial_\ell T_0(\ell) = 2[T_{+2}(\ell), T_{-2}(\ell)] + 2[T_{+1}(\ell), T_{-1}(\ell)],
\]

(42)
\[
\partial_\ell T_{+2}(\ell) = -2T_{+2}(\ell) + [T_{+2}(\ell), T_0(\ell)],
\]

\[
\partial_\ell T_{+1}(\ell) = -T_{+1}(\ell) + 2[T_{+2}(\ell), T_{-1}(\ell)] + [T_{+1}(\ell), T_0(\ell)].
\]

(43)
Let us stress that the Hamiltonian remains hermitian under the unitary transformation. We can therefore calculate \( T_{-2}(\ell) \) and \( T_{-1}(\ell) \) from the hermiticity condition. Solving the flow equation is still a very cumbersome task because there are an infinite number of terms in \( T_n(\ell) \). However, we can tackle the problem by performing a perturbative expansion of the flow equation. We can therefore introduce the expansion of the \( T_n \) operators as\(^{26}\):

\[
T_n(\ell) = \sum_{i=1}^{\infty} T_n^{(i)}(\ell),
\]

(43)
where \( i \) is the order of perturbation. Using this relation, the perturbative expansion of the flow equation is written
as:

\[
\partial_{\ell} T_0^{(k)}(\ell) = 2 \sum_{j=1}^{k-1} [T_{2j}^{(j)}(\ell), T_{2j-2}^{(k-j)}(\ell)] + 2 \sum_{j=1}^{k-1} [T_{2j+2}^{(j)}(\ell), T_{2j-2}^{(k-j)}(\ell)],
\]

\[
\partial_{\ell} T_{+1}^{(k)}(\ell) = -2T_{+1}^{(k)}(\ell) + \sum_{j=1}^{k-1} [T_{+2j}^{(j)}(\ell), T_{0}^{(k-j)}(\ell)],
\]

\[
\partial_{\ell} T_{+1}^{(k)}(\ell) = -T_{+1}^{(k)}(\ell) + 2 \sum_{j=1}^{k-1} [T_{+2j}^{(j)}(\ell), T_{-1}^{(k-j)}(\ell)] + \sum_{j=1}^{k-1} [T_{+1}^{(j)}(\ell), T_{0}^{(k-j)}(\ell)].
\]

Solving this set of equations for the initial condition \(T_n^{(\ell)}(\ell = 0) = \delta_{1,n} T_n\) and then taking the limit of \((\ell \to \infty)\), we can obtain \(H_{\text{eff}}\). The effective QP conserving Hamiltonian for the small-coupling limit up to order 3 in perturbation parameter is obtained as:

\[
H_{\text{eff}}^{(3)} = Q - xT_0 + x^2[T_1, T_{-1}] + \frac{x^2}{2} [T_2, T_{-2}] - \frac{x^3}{8} [T_2, [T_0, T_{-2}]] + [T_2, [T_0, T_{-2}]] - \frac{x^3}{2} [T_1, [T_1, T_{-2}]] + [T_2, [T_{-1}, T_{-1}]] - \frac{x^3}{2} [T_1, [T_0, T_{-1}]] + [T_1, [T_0, T_{-1}]].
\]

The ground state energy and 1-QP gap can be obtained by acting the effective hamiltonian on 0P and 1P sector of the Q. We have calculated the ground state energy and 1-QP energy gap of the system in the small-coupling limit up to order 8 in perturbation theory:

\[
\epsilon_0^{\text{sc}} = -\frac{2}{3} - 2x^2 - x_0^3 - \frac{17x_0^4}{2} - \frac{847x_0^5}{36} - \frac{18407x_0^6}{144} (46)
\]

\[
\Delta^{\text{sc}} = 1 - 4x^0 - 10x_0^2 - 5x_0^3 - \frac{1895x_0^4}{6} + \frac{14107x_0^5}{18} - \frac{3572759x_0^6}{216} + \frac{2656267x_0^7}{324} - \frac{85919559673x_0^8}{77760}. (47)
\]

Fig. 6 illustrates the 1-QP gap of the system in the small-coupling limit as a function of \(x = \frac{2}{3}\). The bare and extrapolated series are well converged. Closure of the 1-QP gap occurs at \(x_c \approx 0.129\). However, one should note that the knowledge of the gap is not solely sufficient to determine the first- or second-order nature of the phase transition\(^{38}\) and one has to analyze the ground state energy of the system and its derivatives\(^{33}\) to capture the phase transition correctly. We therefore postpone further discussion on the phase transition to Sec. V B 3 after we calculate the ground state energy series in large-coupling limits.

2. Large-coupling limit (\(\lambda \gg J\))

Following our discussion, we now apply the PCUT method to the large-coupling limit of the problem (\(\lambda \gg J\)). For \(J = 0\), the Hamiltonian is 3-state Potts model which is ferromagnetically ordered and has an equidistant spectrum thus satisfying the first condition for the PCUT method. The excitations of the model further correspond to the anti-ferromagnetic bonds of the square lattice. When \(J \neq 0\), the effective field term in Eq. (34) can be considered as a perturbation which changes the number of excitations by \(n = \{0, \pm 1, \pm 2, \pm 3, \pm 4\}\). We...
can therefore write the Hamiltonian of the model in the large-coupling limit in terms of $T_n$ operators as:

$$H = Q - h \sum_{n=-4}^{4} T_n,$$

where $Q$ is the quasiparticle counting operator defined as:

$$Q = \frac{2I - \sum_{\langle i,j \rangle} (\hat{Z}_i \hat{Z}_j^\dagger) + (\hat{Z}_j \hat{Z}_i^\dagger)}{3},$$

and $h = J/2\lambda$ is the expansion parameter. Using the PCUT method, we have calculated the ground state energy per site, $\varepsilon_0$, in the large-coupling limit up to order 8 in perturbation parameter by acting the $H_{\text{eff}}$ on the 0-QP sector of the Hilbert space:

$$\varepsilon_0^c = -\frac{8}{3} \frac{h^2}{2} - \frac{h^3}{8} - \frac{19 h^4}{672} - \frac{3 h^5}{128} - \frac{1277 h^6}{846720}$$

$$= \frac{1052987 h^7}{121927680} - \frac{458808396457 h^8}{62768369664000}$$

As we have already mentioned in the previous section, one can obtain a better understanding about the nature of the phase transition by analysis of the ground state energy results. We will therefore provide strong evidences for the first-order phase transition in the $\mathbb{Z}_3$ Kitaev-Pots model in the next section.

### 3. Analyzing series expansion results

To investigate the nature of the phase transition, we study the ground state energy per site of the system in the whole range of the expansion parameter by merging the small- and large-coupling results. Setting $\lambda = \sin \theta$ and $J = \cos \theta$, we can join the small- and large-coupling results to obtain a complete picture. Fig. 7 demonstrates the ground state energy per site $\varepsilon_0$ as a function of $\theta$. The small- and large-coupling series cross each other at $\theta_c \approx 0.52$, giving rise to a kink in the $\varepsilon_0$ curve which is fully consistent with the first-order phase transition. The location of the kink is essentially the same for different Padé extrapolants and has strong agreement with the closure of the 1-QP gap ($x_c = \frac{2}{5} \tan \theta_c \approx 0.129$).

Using the Feynman-Hellman theorem, we have calculated the first derivative of $\varepsilon_0$ which is equivalent to the magnetization in statistical mechanics. Figs. 8 and 9 depict first and second derivatives of $\varepsilon_0$ for different $\theta$ values. The sharp jumps in the first derivative is a clear signature of the first-order phase transition. Let us further note that the first order nature of the phase transition in the $\mathbb{Z}_3$ Pots model in transverse magnetic field has already been confirmed using series expansion combined with infinite projected entangled-pair state (IPEPS) method in Ref.\textsuperscript{22}. The jump in the second derivative further confirms the location of the transition point which is exactly the same as that of the ground-state energy for the bare series and different Padé extrapolants.

### C. Geometric Measure of Entanglement

In this section we calculate a measure of multipartite entanglement, Geometric Measure of Entanglement (GME)\textsuperscript{29}, as another tool for capturing the phase transition. GME is a measure of multipartite entanglement in quantum many body systems which measures the distance, Hilbert Schmidt distance, between a given state, $|\Psi(x)\rangle$, and the closest product state, $|P\rangle$, as follows:

$$\text{GME} = - \log_2 \left( \max_{|P\rangle} |\langle P|\Psi(x)\rangle|^2 \right),$$

(51)
where the maximization is over all product states. This implies that the more entangled the states are, more distance from the set of product states, which is a convex set, they will have. In order to calculate the GME, we have to calculate the ground state of the mapped Hamiltonian, $|\Psi(x)\rangle$, by PCUT procedure up to a specific order of perturbation. The PCUT method transforms the initial Hamiltonian by a unitary transformation ($U(\infty)$) to a block-diagonal form ($H_{\text{eff}}$), the basis of the Hilbert space is rotated, such that $H_{\text{eff}}$ commutes with $Q$. So they have the same eigenstates. It’s also known that $H_{\text{eff}}$’s ground state is the vacuum state of $Q$ with no excitations ($|\tilde{0}\rangle$). The $H_{\text{eff}}$ and the initial Hamiltonian, $H$, are unitary equivalent and their ground states are related to each other as follows:

$$|\Psi(x)\rangle = U(\infty)|\tilde{0}\rangle.$$  

(52)

In order to determine $U(\ell)$, we expand Eq. (32) as a function of the perturbation parameter $x$:

$$\partial_{\ell}U^{(k)}(\ell) = -\sum_{j=0}^{k-1}U^{(j)}(\ell)\eta^{(k-j)}(\ell),$$  

(53)

where

$$\eta^{(k-j)}(\ell) = T_{+2}^{(k-j)}(\ell) + T_{+1}^{(k-j)}(\ell) - T_{-1}^{(k-j)}(\ell) - T_{-2}^{(k-j)}(\ell),$$  

(54)

and $k$ is the order of perturbation. By using Eqs. (44), and the initial condition $U^{(0)}(0) = 1$, one can solve the above equation perturbatively up to second order in $x$.

Taking the limit $\ell \to \infty$, the $U(\infty)$ is found to be:

$$U = 1 + \frac{x}{2}(T_{+2} - T_{-2}) + x(T_{+1} - T_{-1})$$

$$+ \frac{x^2}{2}(T_{+1} - T_{-1})^2 + \frac{x^2}{8}(T_{+2} - T_{-2})^2$$

$$+ \frac{x^2}{4}[T_{0}, (T_{+2} + T_{-2})] + x^2[T_{0}, (T_{+1} + T_{-1})]$$

$$+ \frac{x^2}{6}(T_{+1}T_{+2} + T_{-1}T_{-2} + 2T_{+2}T_{+1} + 2T_{-2}T_{-1})$$

$$+ \frac{x^2}{2}(|T_{1}, T_{-2} + |T_{-1}, T_{2}| - T_{-2}T_{1} - T_{2}T_{-1}).$$  

(55)

The ground state of the initial Hamiltonian $H$ is therefore given by:

$$|\Psi(x)\rangle = U(0) = (1 - \frac{4nx^2}{8})|\tilde{0}\rangle + \frac{x}{2} \sum_{2n} |1 2\rangle$$

$$+ \frac{x^2}{4} \left( \sum_{2n} |1 2\rangle + 2 \sum_{8n} |0 2\rangle + 2 \sum_{4n} |1 0\rangle \right)$$

$$+ \frac{x^2}{6} \left( 2 \sum_{4n} |2 0\rangle + 2 \sum_{4n} |1 1\rangle + 2 \sum_{2n} |2 2\rangle + 2 \sum_{2n} |1 0\rangle \right)$$

$$+ \frac{x^2}{8} \left( \sum_{8n(2n-7)} |1 2\rangle \right).$$

(56)

This state is a superposition of some product states, these product states are eigenstates of $Q$. The first term, $|\tilde{0}\rangle$ refers to the ground state of $Q$ with no excitations, which is the state of all spins in the $X$ eigenstate with $+1$ eigenvalue, $|\tilde{0}\rangle$. $|1 2\rangle$ is the state of all the spins in...
\( |0\rangle\) state, except for the two of them, which are nearest neighbour, and have the states of \( |1\rangle\) and \( |2\rangle\). The summation must be done over all the states of this kind, and one should note that in a lattice with \( n \) vertices the number of this kind of states are \( 2 \) times the number of bonds, i.e. \( 2 \times 2n = 4n \). Furthermore, \( |1 0 0\rangle \) and \( |1 0 2\rangle \) denote the state with all spins in \( |0\rangle \) state, except for the two of them, which are next nearest neighbours and have the states of \( |1\rangle \) and \( |2\rangle \). The last term, \( |1 2 1\rangle \), also refers to the state with all spins in \( |0\rangle \) state except for the four of them. The state of these four spins consist of two clusters of \( |1 2\rangle \), where these two clusters can be separate or not.

\( |\Psi(x)\rangle \) is exact up to the second order in perturbation parameter \( x \) and is normalized. We have also calculated the ground state of the initial Hamiltonian up to the fourth order, numerically and calculated the GME. Since the ground state has translational symmetry, the closest product state also preserves this symmetry. We can therefore perform the maximization only over the states in the form \( |P\rangle = |\phi\rangle^{\otimes n} \), where \( |\phi\rangle = \cos(\theta) |0\rangle + e^{-i\alpha} \sin(\theta) \sin(\varphi) |1\rangle + e^{-i\beta} \sin(\theta) \cos(\varphi) |2\rangle \). In other word, the maximization is only over 4 parameters which makes the numerical calculation of the GME possible. Fig. 10 illustrates the geometric measure of entanglement as a function of \( x \). As we can see, the convexity of the GME changes sign from positive to negative close to a critical point, \( x_c \approx 0.16 \). The sharp jump in the derivative of GME captures the phase transition more clearly, since it’s known that for two-dimensional systems discontinuity in the derivative of multipartite entanglement leads to a quantum phase transition. The critical point is very close to the PCUT results, \( x_c \approx 0.129 \). The difference might be the result of the different orders of perturbation.

VI. CONCLUSION

The \( \mathbb{Z}_d \) Kitaev model is a system with a topologically ordered ground state which is well suited for the purpose of universal quantum computation without resorting to non-Abelian groups. It is therefore of great interest to study the stability and robustness of the topological phase of the model in the presence of external perturbations. In this paper, we studied the competition between the topological order induced by the \( \mathbb{Z}_3 \) Kitaev model and the local order induced by the 3-state Potts model on the square lattice. We showed that the Kitaev model in the presence of the Potts interaction is mapped to Potts model in a transverse magnetic field. Using the high-order series expansion based on the continuous unitary transformations in the small- and large- Potts interactions, we showed that the topological phase breaks down to a non-topological phase with local order parameter through a first-order quantum phase transition at \( x_c = \frac{2\lambda}{\beta} \approx 0.129 \). Our results were further in good agreement with the mean-field approximation results at \( x_c \approx 0.115 \). Computing the Geometric Measure of Entanglement also shows that the derivative of GME has a sharp jump very close to the critical point and also, the convexity of GME changes sign from positive to negative in this point.

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