ON SERRE DUALITY FOR COMPACT HOMOLOGICALLY SMOOTH DG ALGEBRAS

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To Leonid L’vovich Vaksman on his 55th birthday, with gratitude

1. Introduction

Let $X$ be a smooth projective variety over a perfect field $k$. It is a classical fact that the bounded derived category $\mathcal{D}^b(\text{Coh} \, X)$ of the category of coherent sheaves on $X$ is equivalent (as a triangulated category) to the derived category $\mathcal{D}_{\text{per}}(A)$ of perfect modules over a DG (=differential graded) algebra $A$ (see [4, 23] and references therein). The equivalence $\mathcal{D}^b(\text{Coh} \, X) \simeq \mathcal{D}_{\text{per}}(A)$ implies some properties of $A$. First of all, the total cohomology of $A$ has to be finite-dimensional (such DG algebras are called compact [11, 19]). Smoothness of $X$ boils down to $A$ being perfect as an $A$-bimodule [1] (such DG algebras are called homologically smooth [11, 19]; see also [24]).

In view of the above observation, it is natural to expect that compact homologically smooth DG algebras possess many properties of smooth projective varieties. For example, if some invariant of smooth projective varieties depends on the derived category $\mathcal{D}^b(\text{Coh} \, X)$, rather then on $X$ itself, then one can try to define and study the corresponding invariant for compact homologically smooth DG algebras. The aim of this approach is two-fold. On one hand, one can try to obtain ‘geometry-free’ proofs of some classical results which may prove useful for the purposes of noncommutative algebraic geometry. On the other hand, some of the classical results (e.g. collapsing of the Hodge-to-de Rham spectral sequence), if true in this noncommutative setting, would have some interesting and important applications (e.g. to topological conformal field theories [7, 19]).

In this paper, we deal with an algebraic counterpart of the Hodge cohomology groups $H^p(X, \Omega^q_X)$ which is the Hochschild homology $\text{HH}_n(A)$ of a DG algebra $A$.

The Hochschild homology was originally defined for ordinary associative algebras. It has been generalized in different directions: there is a definition of the Hochschild homology of any DG algebra or any scheme which agree with that of ordinary algebras. It is a consequence of deep results of [15, 16, 17] that the equivalence $\mathcal{D}^b(\text{Coh} \, X) \simeq \mathcal{D}_{\text{per}}(A)$ implies an isomorphism of the Hochschild homology groups $\text{HH}_n(X) \cong \text{HH}_n(A)$. It turns out that

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1It seems that a rigorous proof of this statement is yet to be written.
for a smooth projective variety $X$, one has an isomorphism $\HH^n(X) = \oplus_i H^{i-n}(X, \Omega^i_X)$. Thus, $\HH_n(A)$ is a ‘right’ replacement of the Hodge cohomology in the general setting.

The first fundamental property of the Hodge cohomology of smooth projective varieties is finite-dimensionality: $\sum_{p,q} \dim H^p(X, \Omega^q_X) < \infty$. Therefore, it is natural to expect that $\sum_n \dim \HH_n(A) < \infty$ for an arbitrary compact homologically smooth DG algebra $A$. Furthermore, by the classical Serre duality, there exists a non-degenerate pairing $H^p(X, \Omega^q_X) \times H^s(X, \Omega^t_X) \to k$ provided $p+s=q+t=\dim X$. Again, we may hope that there exists a non-degenerate pairing $\HH_n(A) \times \HH_{-n}(A) \to k$ on the algebraic side. The aim of the paper is to prove the above two assertions (see Theorem 4.6). Both results, we believe, are well-known to the experts (cf. [19]). For the case of associative algebras, they can be derived from [25]. A similar “categorical” approach to Serre duality in the geometric setting can be found in [20, 21, 5].

We would like to point out one interesting corollary of the existence of the pairing on $\HH_\bullet(A)$. Namely, if $A$ is a compact homologically smooth DG algebra concentrated in non-positive degrees then its Hochschild homology is concentrated in degree 0. Indeed, it follows from the explicit form of the bar-resolution of a DG algebra $A$ (see, for example, [9]) that $\HH_n(A) = 0$ for $n > 0$ provided $A$ is concentrated in non-positive degrees. This, together with the existence of a non-degenerate pairing $\HH_n(A) \times \HH_{-n}(A) \to k$, implies the result. This corollary gives, for example, an alternative proof of the main result of [6] which describes the Hochschild homology of some quiver algebras with relations.

There is yet another application of the corollary. It is related to an analog of the aforementioned collapsing of the Hodge-to-de Rham spectral sequence. By the classical Hodge theory, the de Rham differential vanishes on the Hodge cohomology of a smooth projective variety. An algebraic counterpart of this fact is the following statement: Connes’ differential $B : \HH_\bullet(A) \to \HH_{\bullet-1}(A)$ vanishes whenever $A$ is compact and homologically smooth. This statement is the well known Noncommutative Hodge-to-de Rham degeneration conjecture formulated by M. Kontsevich and Y. Soibelman several years ago (actually, the conjecture is a stronger statement than just vanishing of $B$ on the Hochschild homology; its precise formulation can be found in [19]). Recently, D. Kaledin [13] proved this conjecture in the special (although very difficult) case of DG algebras concentrated in non-negative degrees. We would like to notice that the above corollary of our main result implies the conjecture in the much easier case of homologically smooth compact DG algebras concentrated in non-positive degrees. Indeed, the Hochschild homology of such a DG algebra is concentrated in degree 0 and $B$ is bound to vanish.
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Notation
Throughout the paper, we work over a fixed field $k$. All vector spaces, algebras, linear categories are defined over $k$. All the definitions regarding DG algebras and DG categories we are going to use can be found in [14].

If $A$ is a DG algebra
$$A = \bigoplus_{n \in \mathbb{Z}} A^n, \quad d = d_A : A^n \to A^{n+1}$$
then $A^{\text{op}}$ stands for the opposite DG algebra. We denote the DG algebra $A^{\text{op}} \otimes A$ with the standard DG algebra structure by $A^e$.

The DG category of right DG $A$-modules is denoted by $\text{Mod}(A)$. We will write $n.a$ for the action of an element $a \in A$ on an element $n$ of a right DG $A$-module $N$. The homotopy and the derived categories of $\text{Mod}(A)$ are denoted by $\text{Ho}(A)$ and $D(A)$, respectively.

Except for few cases, we work exclusively with right DG modules over DG algebras. Therefore, we don’t reserve a notation for the category of left DG modules. We use the fact that right $A^{\text{op}}$-modules are canonically left $A$-modules. For example, we consider the tensor product $N \otimes_A M$ of a right $A$-module $N$ and a right $A^{\text{op}}$-module $M$.

Let us recall the notation for some standard derived functors. Let $A$ and $B$ be two DG algebras. Take $N \in \text{Mod}(A)$ and $X \in \text{Mod}(A \otimes B)$. The vector space $\text{Hom}_{\text{Mod}(A)}(N, X)$ is canonically a right DG $B$-module:

\begin{equation}
(1.1) \quad f.b(n) = (-1)^{|b||n|} f(n).b.
\end{equation}

The assignment $(N, X) \mapsto \text{Hom}_{\text{Mod}(A)}(N, X)$ is a DG bifunctor. It gives rise to a triangulated bifunctor $\text{Ho}(A)^{\text{op}} \times \text{Ho}(A \otimes B) \to \text{Ho}(B)$ denoted by $\text{Hom}_A(-, -)$. Fix $X$. The corresponding derived functor $D(A)^{\text{op}} \to D(B)$ is denoted by $\text{RHom}_A(-, X)$. It is defined by means of appropriate projective resolutions [14]. If $B = k$ then $H^0(\text{RHom}_A(N, X)) \simeq \text{Hom}_{D(A)}(N, X)$.

Let $C$ be yet another DG algebra. Given $N \in \text{Mod}(C \otimes A)$ and $M \in \text{Mod}(A^{\text{op}} \otimes B)$, consider $N \otimes_A M \in \text{Mod}(C \otimes B)$. The assignment $(N, M) \mapsto N \otimes_A M$ is a DG bifunctor. It induces a triangulated bifunctor $\text{Ho}(C \otimes A) \times \text{Ho}(A^{\text{op}} \otimes B) \to \text{Ho}(C \otimes B)$ which we
denote by \(-\hat{\otimes}_A\). The total derived functor \(D(C \otimes A) \times D(A^{\text{op}} \otimes B) \to D(C \otimes B)\) is denoted by \(- \otimes_A^L\). It is also defined by means of projective resolutions.

2. **Perfect modules and the notion of smoothness**

We start by recalling the construction of the DG category \(\text{Tw}(A)\) of twisted modules over \(A\) (see [2]).

Let us view \(A\) as a DG category with one object. The first step is to enlarge \(A\) to a new DG category \(ZA\) by adding formal shifts of the object. Namely, the objects of \(ZA\) are enumerated by integers and denoted by \(A[n], n \in \mathbb{Z}\). The space of morphisms \(\text{Hom}_{ZA}(A[n], A[m])\) coincides with \(\text{Hom}_A(A, A)[m-n] \simeq A[m-n]\) as a \(\mathbb{Z}\)-graded vector space; the differential \(d_{ZA}\) on \(\text{Hom}_{ZA}(A[n], A[m])\) is given by \(d_{ZA} = (-1)^m d_A\).

The objects of the category \(\text{Tw}(A)\) are pairs \((\bigoplus A[r_j], \alpha)\), where \(r_1, r_2, \ldots r_n\) are integers, \(\alpha = (\alpha_{ij})\) is a strictly upper triangular \(n \times n\)-matrix of morphisms \(\alpha_{ij} \in \text{Hom}^1_{ZA}(A[r_j], A[r_i]) \simeq A^{1+r_i-r_j}\) satisfying the Maurer-Cartan equation

\[
d_{ZA}(\alpha) + \alpha \cdot \alpha = 0.
\]

The space \(\text{Hom}_{\text{Tw}(A)}((\bigoplus_{j=1}^n A[r_j], \alpha), (\bigoplus_{i=1}^m A[s_i], \beta))\) is a \(\mathbb{Z}\)-graded space whose \(p\)-th component consists of \(m \times n\)-matrices \(f = (f_{ij})\) of morphisms \(f_{ij} \in \text{Hom}^p_{ZA}(A[r_j], A[s_i]) = A^{p+s_i-r_j}\). The differential \(d_{\text{Tw}(A)}\) is defined as follows

\[
d_{\text{Tw}(A)}(f) = d_{ZA}(f) + \beta \cdot f - (-1)^{|f|} f \cdot \alpha.
\]

The direct sum on objects of the category \(\text{Tw}(A)\) is defined in the obvious way:

\[
(\bigoplus A[r_j], \alpha) \bigoplus (\bigoplus A[s_i], \beta) = (\bigoplus A[r_j] \oplus A[s_i], \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}).
\]

To make \(\text{Tw}(A)\) additive, we add also the zero object \(0\) satisfying the usual axioms. \(A\) is embedded into \(\text{Tw}(A)\) as a full DG subcategory via the identification \(A = (A, 0)\).

The homotopy category \(\text{Ho}(\text{Tw}(A))\) is equipped with a canonical triangulated structure as follows. The shift functor on \(\text{Tw}(A)\) is given by

\[
(\bigoplus A[r_j], \alpha)[1] = (\bigoplus A[r_j + 1], -\alpha)
\]

(the shift acts trivially on morphisms). Suppose \(f \in \text{Hom}^0_{\text{Tw}(A)}((\bigoplus A[r_j], \alpha), (\bigoplus A[s_i], \beta))\) is closed, i.e. \(d_{\text{Tw}(A)}(f) = 0\). The cone of \(f\), \(\text{Cone}(f)\), is the object of \(\text{Tw}(A)\) defined by

\[
\text{Cone}(f) = (\bigoplus A[s_i] \oplus A[r_j + 1], \begin{pmatrix} \beta & f \\ 0 & -\alpha \end{pmatrix}).
\]
By definition, a triangle in $\text{Ho}(\text{Tw}(A))$ is distinguished if it is isomorphic to a triangle of the form

$$X \xrightarrow{f} Y \xrightarrow{j} \text{Cone}(f) \xrightarrow{p} X[1]$$

where $f$ is a degree 0 closed morphism and $j$ and $p$ are defined in the obvious way.

The category $\text{Tw}(A)$ can be embedded into the DG category $\text{Mod}(A)$ by means of the so called Yoneda embedding $[2]$. Let us describe the image of $\bigoplus A[r_j]$ under $Y$. We can replace the formal direct sum by the usual direct sum and view $X = \bigoplus A[r_j]$ as a usual graded space. The matrix $\alpha$ allows us to equip $X$ with the structure of a DG module by $dX = d + \alpha$. Here $d$ acts on $A[r_j]$ as the differential of $A$ multiplied by $(-1)^r j$. Finally, the right $A$-module structure is given by the coordinate-wise multiplication from the right:

$$[a_1, a_2, \ldots], a = [a_1 a, a_2 a, \ldots], \quad a \in A, \quad a_j \in A[r_j].$$

The functor $Y$ induces an embedding $\text{Ho}(\text{Tw}(A)) \to \text{Ho}(A)$ which allows us to view $\text{Ho}(\text{Tw}(A))$ as a triangulated subcategory of $\text{Ho}(A)$. Let $\text{Ho}_{\text{per}}(A)$ be the smallest full triangulated subcategory of $\text{Ho}(A)$ containing $\text{Ho}(\text{Tw}(A))$ and closed under taking isomorphisms and direct summands. It embeds into a larger triangulated subcategory $\overline{\text{Ho}_{\text{per}}}(A) \subset \text{Ho}(A)$ defined as follows: $N \in \overline{\text{Ho}_{\text{per}}}(A)$ if there exists a distinguished triangle in $\text{Ho}(A)$ of the form

$$pN \rightarrow N \rightarrow aN \rightarrow pN[1]$$

where $pN \in \text{Ho}_{\text{per}}(A)$ and $aN$ is acyclic. In other words, if we denote the triangulated subcategory of acyclic modules by $\text{Ho}_{\text{ac}}(A)$ then

$$\overline{\text{Ho}_{\text{per}}}(A) = \text{Ho}_{\text{per}}(A) \ast \text{Ho}_{\text{ac}}(A)$$

(see $[1]$ for a description of the operation $\ast$). We define the triangulated subcategory $\text{D}_{\text{per}}(A) \subset \text{D}(A)$ of perfect modules to be the image of $\overline{\text{Ho}_{\text{per}}}(A)$ under the localization $\text{Ho}(A) \to \text{D}(A)$.

The above chain of definitions can be rephrased as follows: a perfect DG $A$-module is a module which can be resolved by a direct summand of a twisted module.

Let us recall several useful results about perfect modules which we are going to use in the sequel.

A DG $A$-module $P$ is called homotopically projective if $\text{Hom}_{\text{Ho}(A)}(P, N) = 0$ for any acyclic module $N$ (such modules are also called $K$-projective $[1]$). It is known that objects of $\text{Ho}_{\text{per}}(A)$ are homotopically projective modules $[12, \S 13]$. This observation allows us to apply some well known facts about homotopically projective modules to perfect modules. For example, one has $[1]$ Corollary 10.12.4.4:
Proposition 2.1. If $P$ is homotopically projective then $P \otimes_A N$ is acyclic whenever $N$ is an acyclic left DG $A$-module.

Here is one more application. It is a classical fact that the restriction of the localization $\text{Ho}(A) \to \text{D}(A)$ onto the homotopy subcategory of homotopically projective modules is an equivalence (the quasi-inverse functor sends modules to their homotopically projective resolutions). Therefore $\text{D}_{\text{per}}(A)$ is equivalent to $\text{Ho}_{\text{per}}(A)$.

Finally, we are ready to formulate the main definition of this paper: A DG algebra $A$ is said to be homologically smooth if $A \in \text{D}_{\text{per}}(A^e)$ [19, 11].

Observe that if $A$ is homologically smooth then so is $A^{\text{op}}$. Also, if $A, B$ are homologically smooth then their tensor product is a homologically smooth DG algebra\footnote{There is another important version of the notion of smoothness in non-commutative geometry, namely, the formal smoothness (or quasi-freeness) introduced in [8]. Notice that the tensor product of two formally smooth algebras is not a formally smooth algebra in general.}. In particular, $A^e$ is homologically smooth whenever $A$ is. We will use these facts later.

3. Smoothness and saturatedness

The goal of this section is to prove the following

Theorem 3.1. If $A$ is compact (i.e. $\sum_n \dim H^n(A) < \infty$) and homologically smooth then the triangulated category $\text{D}_{\text{per}}(A)$ is saturated.

Let us recall the definition of a saturated triangulated category [3]. Let $\mathcal{T}$ be a triangulated category. It is said to be Ext-finite if $\sum_n \dim \text{Hom}_\mathcal{T}(X, Y[n]) < \infty$ for all $X, Y \in \mathcal{T}$. $\mathcal{T}$ is right (resp. left) saturated if it is Ext-finite and any contravariant (resp. covariant) cohomological functor $h : \mathcal{T} \to \text{Vect}(k)$ of finite type (i.e. such that $\sum_n \dim h(X[n]) < \infty$ for any $X$) is representable. $\mathcal{T}$ is saturated if it is both right and left saturated.

We will prove the above Theorem for the equivalent category $\text{Ho}_{\text{per}}(A)$. Let us start with

Lemma 3.2. If $A$ is compact then $\text{Ho}_{\text{per}}(A)$ is Ext-finite.

**Proof.** Obviously it is enough to check that $\text{Ho}(\text{Tw}(A))$ is Ext-finite.

When $X = A[n]$ and $Y = A[m]$, $\sum_n \dim \text{Hom}_{\text{Ho}(\text{Tw}(A))}(X, Y[n]) < \infty$ is a straightforward consequence of the compactness of $A$. The general case can be reduced to this special case as follows.

Since $\text{Hom}_{\text{Ho}(\text{Tw}(A))}(-, -)$ is exact with respect to both arguments, it suffices to prove that any object $X = (\bigoplus_{j=1}^n A[r_j], \alpha)$ of $\text{Tw}(A)$ is the cone of a degree 0 morphism between two objects of length at most $n - 1$. Let $\beta = (\alpha_{ij})_{i,j=1,\ldots,n-1}$ and $f = (\alpha_{in})_{i=1,\ldots,n-1}$. Then
it is easy to see that $X = \text{Cone}(f)$, where $f$ is viewed as a morphism $(A[r_n - 1], 0) \to (\bigoplus_{j=1}^{n-1} A[r_j], \beta)$. The Lemma is proved.

We know that $\text{Ho}_{\text{per}}(A)$ is equivalent to $\text{Ho}_{\text{per}}(A^{\text{op}})^{\text{op}}$ (see the proof of Proposition [A.1]). If we can prove that $\text{Ho}_{\text{per}}(A)$ is right saturated then it would imply that $\text{Ho}_{\text{per}}(A^{\text{op}})$ is left saturated. Since we can interchange $A$ and $A^{\text{op}}$ in this argument ($A^{\text{op}}$ is also compact and homologically smooth!), we see that to prove Theorem 3.1 it would be enough to show that $\text{Ho}_{\text{per}}(A)$ is right saturated.

One has the following result:

**Theorem 3.3.** [4, Theorem 1.3] Assume $\mathcal{T}$ is an Ext-finite and Karoubian triangulated category. If it has a strong generator then it is right saturated.

Let us explain the last statement. If $\mathcal{T}$ is a triangulated category and $E \in \mathcal{T}$, define $(E)^n_\mathcal{T}$ to be the full subcategory of objects that can be obtained from $E$ by taking shifts, finite direct sums, direct summands, and at most $n$ cones (details can be found in [4, §2.1]). An object $E \in \mathcal{T}$ is called a strong generator if, for some $n$, $(E)^n_\mathcal{T}$ is equivalent to $\mathcal{T}$.

The category $\text{Ho}_{\text{per}}(A)$ is Karoubian by its definition. Let us show that, for a compact and homologically smooth $A$, $\text{Ho}_{\text{per}}(A)$ is strongly generated. We are going to use the idea of the proof of Theorem 3.1.4 from [4, §3.4].

Since $A$ is homologically smooth, there exists a quasi-isomorphism $pA \to A$, where $pA \in \text{Ho}_{\text{per}}(A^{e})$ is a direct summand of some twisted module $(\bigoplus_{j=1}^{n} A^{e}[r_j], \alpha)$. This means that $pA \in (A^{e})^{\text{Ho}_{\text{per}}(A^{e})}_n$. Take $N \in \text{Ho}_{\text{per}}(A)$. Since $N$ is homotopically projective, $N \simeq N \otimes_A A \simeq N \otimes_A pA$ (the latter isomorphism follows from Proposition 2.1). Therefore

$$N \simeq N \otimes_A pA \in N \otimes_A (A^{e})^{\text{Ho}_{\text{per}}(A^{e})}_n \subset (N \otimes_A A^{e})^{\text{Ho}_{\text{per}}(A^{e})}_n.$$  

(The latter inclusion is due to the fact that the functor $N \otimes_A - : \text{Ho}_{\text{per}}(A^{e}) \to \text{Ho}_{\text{per}}(A)$ is triangulated.) Observe that $N \otimes_A A^{e} \simeq N \otimes_k A$ in $\text{Ho}(A)$. Since each DG $k$-module is homotopically equivalent to its total cohomology $H(N) = \bigoplus_{n \in \mathbb{Z}} H^n(N)$, we have $N \otimes_k A \simeq H(N) \otimes_k A$ in $\text{Ho}(A)$. $H(N) \otimes_k A$ is a free module. It remains to show that $\dim H(N) < \infty$.

The proof of this fact repeats the proof of Lemma 3.2 since the cohomology functor is cohomological. Theorem 3.1 is proved.

Let us point out a useful corollary of the above computation:

**Proposition 3.4.** Let $A$ be a compact and homologically smooth DG algebra. Then $N \in D_{\text{per}}(A)$ iff $\dim H(N) < \infty$.  

That \( N \in \mathcal{D}_{\text{per}}(A) \) implies \( \dim H(N) < \infty \) is explained in the end of the above proof. For the converse statement, repeat the above computation: if \( N \) is any module and \( pN \) its homotopically projective resolution then \( pN \) is homotopically equivalent to a direct summand of some \( n \)-fold extension of the module \( H(pN) \otimes_k A \).

4. Serre duality on \( \mathcal{D}_{\text{per}}(A) \) and its applications

Recall the definition of a Serre functor \([3]\). Let \( \mathcal{T} \) be a \( k \)-linear Ext-finite triangulated category. A Serre functor \( S : \mathcal{T} \to \mathcal{T} \) is defined as a covariant auto-equivalence of \( \mathcal{T} \) such that there exists an isomorphism of bifunctors

\[
\text{Hom}_\mathcal{T}(X, Y)^* \simeq \text{Hom}_\mathcal{T}(Y, S(X)).
\]

If such a functor exists, it is unique up to an isomorphism.

We want to describe a Serre functor on the category \( \mathcal{D}_{\text{per}}(A) \), where \( A \) is compact and homologically smooth. The answer is known in the case of ordinary associative algebras (see, for example, \([10, \S 21]\)); we just show that the same construction works in our setting. Then we will compute the inverse functor and prove the main result of the paper, namely, the existence of a non-degenerate pairing on the Hochschild homology of a compact homologically smooth DG algebra.

We notice that existence of a Serre functor follows from Theorem 3.1 and the following

**Theorem 4.1.** \([3, \S 3.5]\) If \( \mathcal{T} \) is a saturated triangulated category then it has a Serre functor.

Let \( S_A \) stand for the so called Nakayama functor on \( \mathcal{D}(A) \):

\[ S_A : N \mapsto (N^\vee)^* \]

(see \([A.2], [A.4]\)).

**Theorem 4.2.** If \( A \) is compact and homologically smooth then the functor \( S_A \) preserves the subcategory \( \mathcal{D}_{\text{per}}(A) \subset \mathcal{D}(A) \) and induces a Serre functor on it.

Let us explain why \( S_A \) preserves \( \mathcal{D}_{\text{per}}(A) \). By Proposition 3.4 it suffices to prove that \( S_A(N) \) has finite dimensional total cohomology whenever \( N \) is perfect. As we know (Proposition A.1), \( N^\vee \in \mathcal{D}_{\text{per}}(A^{\text{op}}) \). Then, again by Proposition 3.4 the cohomology of the latter module is finite dimensional, whence the result.

What we are going to show is that \( S_A \) is a right Serre functor (i.e. it satisfies (4.1) but is not necessarily an equivalence). Then Theorem 4.2 will follow from the existence of a Serre functor on \( \mathcal{D}_{\text{per}}(A) \) and the fact that any two right Serre functors are isomorphic \([22, \S 1.1]\).
Let \( N, M \in \mathcal{D}_{\text{per}}(A) \). By (A.5), \( \text{RHom}_A(N, M)^* \simeq (M \otimes^L_A N^\vee)^* \). By (A.6), \( (M \otimes^L_A N^\vee)^* \simeq \text{RHom}_A(M, (N^\vee)^*) \). Theorem 4.2 is proved.

It is natural to ask whether the functor \( S_A \) can be written in the form \( - \otimes A^e \) for some right DG \( A^e \)-module \( X \). To answer this question, consider \( A^* \). It carries a canonical right DG \( A^e \)-module structure coming from the natural left DG \( A^e \)-module structure on \( A^e \):

\[
(a' \otimes a'')a = (-1)^{|a'||a''|} a''a'^* \tag{4.2}
\]

Then, by Proposition A.3

**Theorem 4.3.** \( S_A \) is isomorphic to \( - \otimes^L_A A^* \).

Let us compute the inverse \( S_A^{-1} \). Consider the right DG \( A^e \)-module

\[
A^l = \text{RHom}_{(A^e)^e}(A, (A^e)^e).
\]

Here we are using the left DG \( A^e \)-module structure on \( A^e \) defined above (or, more precisely, the corresponding right DG \( (A^e)^e \)-module structure).

**Theorem 4.4.** \( - \otimes^L_A A^l \) is inverse to \( S_A \).

To show this, it suffices to prove that the two functors form an adjoint pair:

\[
\text{Hom}_{\mathcal{D}(A)}(N, S_A(M)) = \text{Hom}_{\mathcal{D}(A)}(N \otimes^L_A A^l, M), \quad N, M \in \mathcal{D}_{\text{per}}(A)
\]

which is equivalent to \( (\text{Hom}_{\mathcal{D}(A)}(M, N))^* = \text{Hom}_{\mathcal{D}(A)}(N \otimes^L_A A^l, M) \). By (A.6) and (A.1),

\[
\text{RHom}_A(N \otimes^L_A A^l, M) \simeq (N \otimes^L_A A^l \otimes^L_A M^*)^* \simeq ((N \otimes_k M^*) \otimes^L_A A^l)^*.
\]

Proposition A.2 implies \( ((N \otimes_k M^*) \otimes^L_A A^l)^* \simeq \text{RHom}_{A^e}(A, N \otimes_k M^*)^* \) and (A.3) finishes the proof.

Thus, we have

**Theorem 4.5.** \( A^* \) and \( A^l \) are mutually inverse invertible bimodules, i.e. we have isomorphisms

\[
A^* \otimes^L_A A^l \simeq A \simeq A^l \otimes^L_A A^*
\]

in \( \mathcal{D}(A^e) \).

Now we are ready to prove our main result. Recall [17] that the Hochschild homology groups \( \text{HH}_n(A) \) are defined as follows

\[
\text{HH}_n(A) = H^n(A \otimes^L_{A^e} A).
\]

The tensor product on the right hand side is defined via the action (4.2).
Theorem 4.6. Suppose $A$ is compact and homologically smooth. Then $\sum_n \dim HH_n(A) < \infty$ and there exists a canonical non-degenerate pairing

$$HH_n(A) \times HH_{-n}(A) \rightarrow k.$$ 

Since $A$ is compact and homologically smooth, so is $(A^e)^{\text{op}}$. Proposition 3.4 assures that $A$ is a perfect right DG $(A^e)^{\text{op}}$-module (with respect to the action (4.2)). Therefore, by Proposition A.1, there is a canonical isomorphism $A \simeq \text{RHom}_{A^e}(A^l, A^e)$. By Proposition A.2

$$A \otimes_{A^e}^L A \simeq A \otimes_{A^e}^L \text{RHom}_{A^e}(A^l, A^e) \simeq \text{RHom}_{A^e}(A^l, A).$$

In particular, $\sum_n \dim HH_n(A) < \infty$ since $A^l, A$ are perfect DG $A^e$-modules (see Lemma 3.2). Finally, by Theorem 4.5 and (A.6)

$$\text{RHom}_{A^e}(A^l, A) \simeq \text{RHom}_{A^e}(A^l \otimes_A^L A^*, A \otimes_{A^l}^L A^*) \simeq \text{RHom}_{A^e}(A, A^*) \simeq (A \otimes_{A^e}^L A)^*$$

and therefore, for any $n$, we have a canonical non-degenerate pairing

$$H^n(A \otimes_{A^e}^L A) \times H^{-n}(A \otimes_{A^e}^L A) \rightarrow k.$$ 

Appendix A. Some canonical isomorphisms

In this Appendix, we give an account of all canonical isomorphisms used in the paper.

Let $A$ be a DG algebra and $N \in \text{Mod}(A), M \in \text{Mod}(A^{\text{op}})$ arbitrary modules. The tensor product $N \otimes_k M$ is canonically a right DG $A^e$-module. Fix a module $X \in \text{Mod}(A^e)$. One has an obvious isomorphism

(A.1) \hspace{1cm} N \otimes_A^L X \otimes_A^L M \simeq (N \otimes_k M) \otimes_{A^e}^L X.

Let $N$ be a DG $k$-module. Define

(A.2) \hspace{1cm} N^* = \text{Hom}_{\text{Mod}(k)}(N, k).

Here $k$ stands for the DG $k$-module whose 0-th component is $k$ and other components are 0. If $N$ is a right DG module over a DG algebra then $N^*$ inherits a canonical structure of a right DG module over the opposite DG algebra. Observe that $N^{**} \simeq N$ on the derived level whenever $N$ has finite dimensional total cohomology.

Let $N, M \in \text{Mod}(A)$. Both the tensor product $N \otimes_k M^*$ and $A$ are right DG $A^e$-modules. One has an isomorphism

(A.3) \hspace{1cm} \text{Hom}_{D(A^e)}(A, N \otimes_k M^*) \simeq \text{Hom}_{D(A)}(M, N).
Let us now list some canonical isomorphisms involving perfect $A$-modules. All of them are well known (see, for example, [10, §21] for a review of the case of ordinary associative algebras); therefore we give only sketches of proofs.

Take $N \in \text{Mod}(A)$ and set
\[(A.4) \quad N^\vee = \text{Hom}_{\text{Mod}(A)}(N, A).\]
Thus, $N^\vee \in \text{Mod}(A^{\text{op}})$.

**Proposition A.1.** The functor $N \mapsto N^\vee$ preserves perfect modules and induces an equivalence $D_{\text{per}}(A) \to D_{\text{per}}(A^{\text{op}})^{\text{op}}$. More precisely, there is a canonical isomorphism $N \simeq (N^\vee)^\vee$.

Indeed, the functor $N \mapsto \text{Hom}_{\text{Mod}(A)}(N, A)$ is easily seen to preserve twisted modules (the space $\text{Hom}_{\text{Tw}(A)}((\bigoplus_{j=1}^n A[r_j], \alpha), (A, 0))$ consists of $n \times 1$-matrices of elements of $A(= A^{\text{op}})$, and the differential and the $A$-action are exactly of the same form as those in $\text{Tw}(A^{\text{op}})$). It clearly descents to a functor from $\text{Ho}_{\text{per}}(A)$ to $\text{Ho}_{\text{per}}(A^{\text{op}})^{\text{op}}$ since $\text{Hom}_A(\cdot, A)$ sends direct summands to direct summands. Furthermore, one has an obvious canonical map
\[N \to \text{Hom}_{A^{\text{op}}}(\text{Hom}_A(N, A), A^{\text{op}}).\]
It is easy to see that this map is an isomorphism when $N$ is a twisted module. If $N$ is a direct summand of a twisted module $P$, we have
\[P \simeq \text{Hom}_{A^{\text{op}}}(\text{Hom}_A(P, A), A^{\text{op}}), \quad N \hookrightarrow \text{Hom}_{A^{\text{op}}}(\text{Hom}_A(N, A), A^{\text{op}})\]
which implies $N \simeq \text{Hom}_{A^{\text{op}}}(\text{Hom}_A(N, A), A^{\text{op}})$. To finish the proof, it remains to observe that, for an arbitrary perfect module $N$, $(N^\vee)^\vee \simeq \text{Hom}_{A^{\text{op}}}(\text{Hom}_A(pN, A), A^{\text{op}})$, where $pN$ is a resolution of $N$.

**Proposition A.2.** If $N$ is perfect and $M$ is arbitrary then there is a canonical isomorphism
\[(A.5) \quad \text{RHom}_A(N, M) \simeq M \otimes_A N^\vee.\]

To prove this Proposition, consider the map of DG modules
\[F_{N,M} : M \otimes_A \text{Hom}_{\text{Mod}(A)}(N, A) \to \text{Hom}_{\text{Mod}(A)}(N, M)\]
given by $F_{N,M}(m \otimes f)(n) = mf(n)$, where $m \otimes f \in M \otimes_A \text{Hom}_{\text{Mod}(A)}(N, A)$ and $n \in N$. It induces a map
\[\bar{F}_{N,M} : M \otimes_A \text{Hom}_A(N, A) \to \text{Hom}_A(N, M).\]
If $N$ is a twisted module then a straightforward computation shows that $F_{N,M}$ is bijective, and so is $\bar{F}_{N,M}$. If $N$ is a direct summand of a twisted module $P$ then one can show that the image of $M \otimes_A \text{Hom}_A(N, A)$ under $\bar{F}_{P,M}$ is inside of $\text{Hom}_A(N, M) \subset \text{Hom}_A(P, M)$.

Finally, we want to mention the following

**Proposition A.3.** Suppose $N$ is a perfect right DG $A$-module and $M$ is an arbitrary right DG $A \otimes B$-module, where $B$ is yet another DG algebra. Then there is a natural isomorphism of right DG $B^{\text{op}}$-modules

$$(A.6) \quad (\text{RHom}_A(N, M))^* \simeq N \otimes_A^L M^*.$$  

The proof of this proposition is completely analogous to the proof of the preceding one. This time the isomorphism comes from the canonical map

$$G_{N,M} : N \otimes_A M^* \to (\text{Hom}_{\text{Mod}(A)}(N, M))^*$$

given by $G_{N,M}(n \otimes \nu)(f) = (-1)^{|n||\nu|+|f|}\nu(f(n))$, where $n \otimes \nu \in N \otimes_A M^*$ and $f \in \text{Hom}_{\text{Mod}(A)}(N, M)$. If $N$ is a direct summand of a twisted $A$-module then $G_{N,M}$ is obviously bijective.

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