On $H^2$ solutions and $z$-weak solutions of the 3D Primitive Equations

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Abstract

Global in time well-posedness of $H^2$ solutions and $z$-weak solutions of the 3D Primitive equations in a bounded cylindrical domain is proved. More specifically, uniform in time boundedness and bounded absorbing sets are obtained for both $H^2$ solutions and $z$-weak solutions, as well as uniqueness of the $z$-weak solution for the 3D Primitive equations. The result for $H^2$ solutions improves a recent one proved in [6]. The result for $z$-weak solution positively resolves the problem of global existence and uniqueness of $z$-weak solutions of the 3D primitive equations, which has been open since the work of [14].

Keywords: 3D viscous Primitive Equations, global existence, uniqueness, regularity.

MSC: 35B40, 35B30, 35Q35, 35Q86.

1 Introduction

Given a bounded domain $D \subset \mathbb{R}^2$ with smooth boundary $\partial D$, we consider the following system of viscous Primitive Equations (PEs) of Geophysical Fluid Dynamics in the cylinder $\Omega = D \times (-h, 0) \subset \mathbb{R}^3$, where $h$ is a positive constant, see e.g. [17] and the references therein:
Conservation of horizontal momentum:
\[
\frac{\partial v}{\partial t} + (v \cdot \nabla)v + w \frac{\partial v}{\partial z} + \nabla p + f v^\perp + L_1 v = 0;
\]

Hydrostatic balance:
\[
\partial_z p + \theta = 0;
\]

Continuity equation:
\[
\nabla \cdot v + \partial_z w = 0;
\]

Heat conduction:
\[
\frac{\partial \theta}{\partial t} + v \cdot \nabla \theta + w \frac{\partial \theta}{\partial z} + L_2 \theta = Q.
\]

The unknowns in the above system of 3D viscous PEs are the fluid velocity field \((v, w) = (v_1, v_2, w) \in \mathbb{R}^3\) with \(v = (v_1, v_2)\) and \(v^\perp = (-v_2, v_1)\) being horizontal, the temperature \(\theta\) and the pressure \(p\). The Coriolis rotation frequency \(f = f_0(\beta + y)\) in the \(\beta\)-plane approximation and the heat source \(Q\) are given. For the issue concerned in this article, \(Q\) is assumed to be independent of \(t\). In the above equations and in this article, \(\nabla\) and \(\Delta\) denote the horizontal gradient and Laplacian:
\[
\nabla := (\partial_x, \partial_y) \equiv (\partial_1, \partial_2), \quad \Delta := \partial_x^2 + \partial_y^2 \equiv \sum_{i=1}^{2} \partial_i^2.
\]

We also use the following notation:
\[
\nabla_3 := (\partial_x, \partial_y, \partial_z) \equiv (\partial_1, \partial_2, \partial_3).
\]

The viscosity and the heat diffusion operators \(L_1\) and \(L_2\) are given respectively as follows:
\[
L_i := -\nu_i \Delta - \mu_i \frac{\partial^2}{\partial z^2}, \quad i = 1, 2,
\]
where the positive constants \(\nu_1, \mu_1\) are the horizontal and vertical viscosity coefficients and the positive constants \(\nu_2, \mu_2\) are the horizontal and vertical heat diffusivity coefficients.
The boundary of $\Omega$ is partitioned into three parts: $\partial \Omega = \Gamma_u \cup \Gamma_b \cup \Gamma_s$, where

\[
\Gamma_u := \{(x, y, z) \in \overline{\Omega} : z = 0\}, \\
\Gamma_b := \{(x, y, z) \in \overline{\Omega} : z = -h\}, \\
\Gamma_s := \{(x, y, z) \in \overline{\Omega} : (x, y) \in \partial D\}.
\]

Consider the following boundary conditions of the PEs as in [2] and [4]:

on $\Gamma_u$:
\[
\frac{\partial v}{\partial z} = h\tau, \quad w = 0, \quad \frac{\partial \theta}{\partial z} = -\alpha(\theta - \Theta),
\]
on $\Gamma_b$:
\[
\frac{\partial v}{\partial z} = 0, \quad w = 0, \quad \frac{\partial \theta}{\partial z} = 0,
\]
on $\Gamma_s$:
\[
v \cdot n = 0, \quad \frac{\partial v}{\partial n} \times n = 0, \quad \frac{\partial \theta}{\partial n} = 0,
\]
where $\tau(x, y)$ and $\Theta(x, y)$ are respectively the wind stress and typical temperature distribution on the surface of the ocean, $n$ is the normal vector of $\Gamma_s$ and $\alpha$ is a non-negative constant. The above system of PEs will be solved with suitable initial conditions.

We assume that $Q$, $\tau$ and $\Theta$ are independent of time. Notice that results similar to those to be presented here for the autonomous case can still be obtained for the non-autonomous case with proper modifications. For the autonomous case, assuming some natural compatibility conditions on $\tau$ and $\Theta$, one can further set $\tau = 0$ and $\Theta = 0$ without losing generality. See [2] for a detailed discussion on this issue.

Setting $\tau = 0$, $\Theta = 0$ and using the fact that
\[
w(x, y, z, t) = -\int_{-h}^{z} \nabla \cdot v(x, y, \xi, t) d\xi,
\]
\[
p(x, y, z, t) = p_s(x, y, t) - \int_{-h}^{z} \theta(x, y, \xi, t) d\xi,
\]
one obtains the following equivalent formulation of the system of PEs:
\[
\frac{\partial v}{\partial t} + L_1 v + (v \cdot \nabla) v - \left(\int_{-h}^{z} \nabla \cdot v(x, y, \xi, t) d\xi\right) \frac{\partial v}{\partial z} \\
+ \nabla p_s(x, y, t) - \int_{-h}^{z} \nabla \theta(x, y, \xi, t) d\xi + fv^\perp = 0. \tag{1.1}
\]
\[ \frac{\partial \theta}{\partial t} + L_2 \theta + v \cdot \nabla \theta - \left( \int_{-h}^{z} \nabla \cdot v(x, y, \xi, t) d\xi \right) \frac{\partial \theta}{\partial z} = Q; \quad (1.2) \]

\[ \frac{\partial v}{\partial z} \bigg|_{z=0} = \frac{\partial v}{\partial z} \bigg|_{z=-h} = 0, \quad v \cdot n \bigg|_{\Gamma_s} = 0, \quad \frac{\partial v}{\partial n} \times n \bigg|_{\Gamma_s} = 0, \quad (1.3) \]

\[ \left( \frac{\partial \theta}{\partial z} + \alpha \theta \right) \bigg|_{z=0} = \frac{\partial \theta}{\partial z} \bigg|_{z=-h} = 0, \quad \frac{\partial \theta}{\partial n} \bigg|_{\Gamma_s} = 0, \quad (1.4) \]

\[ v(x, y, z, 0) = v_0(x, y, z), \quad \theta(x, y, z, 0) = \theta_0(x, y, z). \quad (1.5) \]

We remark that the expressions of \( w \) and \( p \) via integrating the continuity equation and the hydrostatic balance equation were already used in [11] dealing with the Primitive Equations for large scale oceans. See also [10] for a similar treatment of the Primitive Equations for atmosphere.

Notice that the effect of the salinity is omitted in the above 3D viscous PEs for brevity of presentation. However, our results in this article are still valid when the effect of salinity is included. For simplicity of discussion, we only consider the case of \( Q \in L^2 \) being independent of time and set the right-hand side of (1.1) as zero. This is not technically essential. If the right-hand side of (1.1) is replaced by a non-zero time-independent given external force \( R \in L^2(\Omega) \), the results of this paper are still valid. The case with time-dependent \( Q \) and \( R \) can be treated similarly with minor proper adjustments.

To the best of our knowledge, the mathematical framework of the viscous primitive equations for the large scale ocean was first formulated in [11]; the notions of weak and strong solutions were defined and existence of weak solutions was proved. Uniqueness of weak solutions is still unresolved yet. Existence of strong solutions \textit{local in time} and their uniqueness were obtained in [3] and [17]. Existence of strong solutions \textit{global in time} was proved independently in [2] and [8]. See also [9] for dealing with some other boundary conditions. In [4], existence of the global attractor for the strong solutions of the system is proved in the functional space of strong solutions. It is proved in [6] that the \( H^2 \) solutions are uniformly bounded with a bounded absorbing set in \( H^2 \) space under the assumption that \( \alpha = 0 \) and \( Q, Q_z \in L^2 \) and thus the global attractor of the strong solutions has finite Hausdorff and fractal dimensions. In [5], it is further proved that the
global attractor of the strong solutions has finite Hausdorff and fractal dimensions for any $\alpha \geq 0$ and $Q \in L^2$ via an approach different from that of [6], without using uniform boundedness of $H^2$ solutions.

This article focuses on the study of global existence of $H^2$ solutions and $z$-weak solutions of the system of 3D viscous PEs and the uniqueness of $z$-weak solutions. Both of the problem are resolved in the sense to be discussed next.

The global existence of the $H^2$ solutions was obtained in [12] for the 3D viscous PEs with periodic boundary conditions, under the condition that $Q \in H^1$. However, this approach is not applicable to the case of non-periodic boundary conditions. A different and more involved analysis was presented recently in [5], which proves the uniform boundedness and existence of a bounded absorbing set for the $H^2$ solutions of the 3D viscous PEs with the set of boundary conditions as given by (1.3) and (1.4) under even less demanding condition that $Q, Q_z \in L^2$. This analysis also applies to the case with periodic boundary conditions, thus improving the result of [12]. However, the result of [6] requires the conditions that $Q, Q_z \in L^2$ and that $\alpha = 0$. As the first main result of this article, we further eliminate these two extra conditions via a somewhat different approach. See Theorem 3.1 in Section 3. The main idea relies on a new a priori entimate for $(v_t, \theta_t)$ that was obtained recently in [5].

The notion of a $z$-weak solution, i.e. a weak solution $(v, \theta)$ such that

$$(v_z, \theta_z) \in L^\infty(0, T; H) \cap L^2(0, T; V),$$

was introduced in [13], which proved global existence and uniqueness of $z$-weak solutions in a 2D domain with periodic boundary conditions. This problem was later studied in [14] for a 3D domain with non-periodic boundary conditions. Global existence and uniqueness of $z$-weak solutions was proved in [14] under the extra condition that $(v_0, \theta_0) \in L^6$. It seems to have been an open problem since the work of [14] that whether or not global existence and uniqueness of $z$-weak solutions are still valid for the 3D case. As our second main result, we resolve this problem positively. See Theorem 4.1 in Section 4 and Theorem 5.1 in Section 5.
The rest of this article is organized as follows:

In Section 2, we give the notations, briefly review the background results and present the problems to be studied and recall some important facts crucial to later analysis. In Section 3, we prove our first main result, Theorem 3.1 on global existence and uniform boundedness of $H^2$ solutions for any $\alpha \geq 0$ and $Q \in L^2$. In Section 4, we prove Theorem 4.1 on global existence and uniform boundedness of $z$-weak solutions. In Section 5, we prove Theorem 5.1 on uniqueness of $z$-weak solutions.

2 Preliminaries

We recall that $D$ is a bounded smooth domain in $\mathbb{R}^2$ and $\Omega = D \times [0, -h]$, where $h$ is a positive constant. We denote by $L^p(\Omega)$ and $L^p(D)$ ($1 \leq p < +\infty$) the classic $L^p$ spaces with the norms:

$$
\| \phi \|_p = \begin{cases} 
\left( \int_{\Omega} |\phi(x, y, z)|^p dxdydz \right)^{\frac{1}{p}}, & \forall \phi \in L^p(\Omega); \\
\left( \int_{D} |\phi(x, y)|^p dxdy \right)^{\frac{1}{p}}, & \forall \phi \in L^p(D).
\end{cases}
$$

Denote by $H^m(\Omega)$ and $H^m(D)$ ($m \geq 1$) the classic Sobolev spaces for square-integrable functions with square-integrable derivatives up to order $m$. We do not distinguish the notations for vector and scalar function spaces, which are self-evident from the context. For simplicity, we may use $d\Omega$ to denote $dxdydz$ and $dD$ to denote $dxdy$, or we may simply omit them when there is no confusion. Using the Hölder inequality, it is easy to show that, for $\varphi \in L^p(\Omega)$,

$$
\| \varphi \|_{L^p(\Omega)} = h^{\frac{1}{p}} \| \varphi \|_{L^p(D)} \leq \| \varphi \|_p, \quad \forall p \in [1, +\infty],
$$

where $\overline{\varphi}$ is defined as the vertical average of $\varphi$:

$$
\overline{\varphi}(x, y) = h^{-1} \int_{-h}^{0} \varphi(x, y, z)dz.
$$

Define the function spaces $H$ and $V$ as follows:

$$
H := H_1 \times H_2 := \{ v \in L^2(\Omega)^2 \mid \nabla \cdot \varphi = 0, \quad \varphi \cdot n|_{\Gamma_s} = 0 \} \times L^2(\Omega),
$$

$$
V := V_1 \times V_2 := \{ v \in H^1(\Omega)^2 \mid \nabla \cdot \varphi = 0, \quad v \cdot n|_{\Gamma_s} = 0 \} \times H^1(\Omega).
$$
Define the bilinear forms: $a_i : V_i \times V_i \rightarrow \mathbb{R}, i = 1, 2$ as follows:

$$a_1(v, u) = \int_\Omega (\nu_1 \nabla v_1 \cdot \nabla u_1 + \nu_1 \nabla v_2 \cdot \nabla u_2 + \mu v_z \cdot u_z) \, d\Omega;$$

$$a_2(\theta, \eta) = \int_\Omega (\nu_2 \nabla \theta \cdot \nabla \eta + \mu \theta z \eta z) \, d\Omega + \alpha \int_{\Gamma_u} \theta \eta \, dx \, dy.$$

Let $V_i'(i = 1, 2)$ denote the dual space of $V_i$. We define the linear operators $A_i : V_i \rightarrow V_i'$, $i = 1, 2$ as follows:

$$\langle A_1 v, u \rangle = a_1(v, u), \quad \forall v, u \in V_1; \quad \langle A_2 \theta, \eta \rangle = a_2(\theta, \eta), \quad \forall \theta, \eta \in V_2,$$

where $\langle \cdot, \cdot \rangle$ is the corresponding scalar product between $V_i'$ and $V_i$. We also use $\langle \cdot, \cdot \rangle$ to denote the inner products in $H_1$ and $H_2$. Define:

$$D(A_i) = \{ \phi \in V_i, A_i \phi \in H_i \}, \quad i = 1, 2.$$

Since $A_i^{-1}$ is a self-adjoint compact operator in $H_i$, by the classic spectral theory, the power $A_i^s$ can be defined for any $s \in \mathbb{R}$. Then $D(A_i)' = D(A_i^{-1})$ is the dual space of $D(A_i)$ and $V_i = D(A_i^{\frac{1}{2}})$, $V_i' = D(A_i^{-\frac{1}{2}})$. Moreover,

$$D(A_i) \subset V_i \subset H_i \subset V_i' \subset D(A_i)'$$

where the embeddings above are all compact. Define the norm $\| \cdot \|_{V_i}$ by:

$$\| \cdot \|_{V_i}^2 = a_i(\cdot, \cdot) = \langle A_i^\frac{1}{2} \cdot, A_i^\frac{1}{2} \cdot \rangle, \quad i = 1, 2.$$

The Poincaré inequalities are valid. There is a constant $c > 0$, such that for any $\phi = (\phi_1, \phi_2) \in V_1$ and $\psi \in V_2$

$$c\| \phi \|_2 \leq \| \phi \|_{V_1}, \quad c\| \psi \|_2 \leq \| \psi \|_{V_2}.$$

Therefore, there exist constants $c > 0$ and $C > 0$ such that for any $\phi = (\phi_1, \phi_2) \in V_1$ and $\psi \in V_2$,

$$c\| \phi \|_1 \leq \| \phi \|_{H^1(\Omega)} \leq C\| \phi \|_{V_1}, \quad c\| \psi \|_2 \leq \| \psi \|_{H^1(\Omega)} \leq C\| \psi \|_{V_2}.$$

Notice that, in the above first inequality, we have written $\| \phi \|_{H^1(\Omega)}$ instead of $\| \phi \|_{H^1(\Omega)^2}$. We could also simply write $\| \phi \|_{H^1}$.
we do not distinguish the notations for vector and scalar function spaces which are self-evident from the context. In this article, we use $c$ and $C$ to denote generic positive constants, the values of which may vary from one place to another.

Recall the following definitions of weak and strong solutions:

**Definition 2.1** Suppose $Q \in L^2(\Omega)$, $(v_0, \theta_0) \in H$ and $T > 0$. The pair $(v, \theta)$ is called a weak solution of the 3D viscous PEs (1.1)-(1.5) on the time interval $[0,T]$ if it satisfies (1.1)-(1.2) in the weak sense, and also

$$(v, \theta) \in C([0,T]; H) \cap L^2(0,T; V), \quad \partial_t(v, \theta) \in L^1(0,T; V').$$

If $(v_0, \theta_0) \in H$ and $(\partial_z v_0, \partial_z \theta_0) \in H$, a weak solution $(v, \theta)$ is called a $z$-weak solution of (1.1)-(1.5) on the time interval $[0,T]$ if, in addition, it satisfies

$$(v_z, \theta_z) \in L^\infty([0,T]; H) \cap L^2(0,T; V).$$

Moreover, if $(v_0, \theta_0) \in V$, a weak solution $(v, \theta)$ is called a strong solution of (1.1)-(1.5) on the time interval $[0,T]$ if, in addition, it satisfies

$$(v, \theta) \in C([0,T]; V) \cap L^2(0,T; D(A_1) \times D(A_2)).$$

The following theorem on global existence and uniqueness for the strong solutions was proved in [2]. See also a related result in [8].

**Theorem 2.1** Suppose $Q \in H^1(\Omega)$. Then, for every $(v_0, \theta_0) \in V$ and $T > 0$, there exists a unique strong solution $(v, \theta)$ on $[0,T]$ to the system of 3D viscous PEs, which depends on the initial data continuously in $H$.

**Remark 2.1** It is easy to see from the proof of Theorem 2.1 given in [2] that the condition $Q \in H^1(\Omega)$ can be relaxed to $Q \in L^6(\Omega)$. Notice that there are gaps between Definition 2.1 and Theorem 2.1 for the condition on $Q$, for the continuity of the strong solution with respect to time and for the continuous dependence of the strong solution with respect to initial data.

We now recall the following result proven in [4] for the existence of global attractor $\mathcal{A}$ for the strong solutions of the 3D viscous PEs (1.1)-(1.5).
Theorem 2.2 Suppose that $Q \in L^2(\Omega)$ is independent of time. Then the solution operator $\{S(t)\}_{t \geq 0}$ of the 3D viscous PEs (1.1)-(1.5): $S(t)(v_0, \theta_0) = (v(t), \theta(t))$ defines a semigroup in the space $V$ for $t \in \mathbb{R}_+$. Moreover, the following statements are valid:

1. For any $(v_0, \theta_0) \in V$, $t \mapsto S(t)(v_0, \theta_0)$ is continuous from $\mathbb{R}_+$ into $V$.
2. For any $t > 0$, $S(t)$ is a continuous and compact map in $V$.
3. $\{S(t)\}_{t \geq 0}$ possesses a global attractor $A$ in the space $V$. The global attractor $A$ is compact and connected in $V$ and it is the minimal bounded attractor in $V$ in the sense of the set inclusion relation; $A$ attracts all bounded subsets of $V$ in the norm of $V$.

We recall also the following important result proved in [5]:

Theorem 2.3 Suppose $Q \in L^2(\Omega)$ and $\alpha \geq 0$.

For any $(v_0, \theta_0) \in V$ and $(\partial_tv(0), \partial_t\theta(0)) \in H$, there exists a unique solution $(v, \theta)$ of (1.1)-(1.5) such that

$$(\partial_tv, \partial_t\theta) \in L^\infty(0, \infty; H) \cap L^2(0, \infty; V).$$

Moreover, there exists a bounded absorbing ball for $(\partial_tv, \partial_t\theta)$ in space $H$.

The main goal of this article is to prove global in time uniform boundedness of the $H^2$ solutions and global existence and uniqueness of $z$-weak or the so-called vorticity solutions, we also prove existence of bounded absorbing set of $z$-weak solutions.

To end this section, we introduce two technically critical lemmas to be used in our later analysis. First, we recall a lemma which will be useful for the a priori estimates in Section 3. See [1], and also [4], for a proof.

Lemma 2.1 Suppose that $\nabla v, \varphi \in H^1(\Omega), \psi \in L^2(\Omega)$. Then, there exists a constant $C > 0$ independent of $v, \varphi, \psi$ and $h$, such that

$$\left| \left( \int_{-h}^h \nabla \cdot v(x,y,\xi) d\xi \right) \varphi, \psi \right| \leq C \|\nabla v\|_2^\frac{1}{2} \|\nabla \|_{H^1}^\frac{1}{2} \|\varphi\|_{H^1}^\frac{1}{2} \|\varphi\|_{H^1}^\frac{1}{2} \|\psi\|_2.$$
Next, we prove a different anisotropic estimate which will be crucial in the analysis of Section 4 and Section 5.

**Lemma 2.2** Suppose that \( \Omega = D \times [-h, 0] \) as given previously and \( \phi, \psi, \varphi, \nabla \phi, \nabla \psi, \varphi_z \in L^2(\Omega) \). Then, there exists a positive constant \( C \) which is independent of \( \phi, \psi \) and \( \varphi \), such that

\[
\int_{\Omega} |\phi \psi \varphi| \leq C \|\phi\|_{L^2(D)}^{\frac{1}{2}} \left( \|\phi\|_{L^2(D)} + \|\nabla \phi\|_{L^2(D)} \right)^{\frac{1}{2}} \|\psi\|_{L^2(D)}^{\frac{1}{2}} \left( \|\psi\|_{L^2(D)} + \|\nabla \psi\|_{L^2(D)} \right)^{\frac{1}{2}}
\]

\[
\times \|\varphi\|_{L^2(D)}^{\frac{1}{2}} \left( \|\varphi\|_{L^2(D)} + \|\varphi_z\|_{L^2(D)} \right)^{\frac{1}{2}}.
\]

**Proof:** By Cauchy-Schwartz inequality, we have

\[
\int_{\Omega} |\phi \psi \varphi| \leq \int_{D} \|\varphi\|_{L^\infty(D)} \int_{-h}^{0} |\phi \psi| \, dz \, dD
\]

\[
\leq \left( \int_{D} \|\varphi\|_{L^\infty(D)}^2 \right)^{\frac{1}{2}} \left[ \int_{D} \left( \int_{-h}^{0} |\phi \psi| \, dz \right)^2 \right]^{\frac{1}{2}}.
\]

By Agmon’s inequality and Cauchy-Schwartz inequality, we have

\[
\int_{D} \|\varphi\|_{L^\infty(D)} \leq C \int_{D} \left( \int_{-h}^{0} |\varphi|^2 \right)^{\frac{1}{4}} \left[ \int_{-h}^{0} \left( |\varphi| + |\varphi_z|^2 \right) \right]^{\frac{3}{4}}
\]

\[
\leq C \|\varphi\|_{L^2(D)}(\|\varphi\|_{L^2(D)} + \|\varphi_z\|_{L^2(D)}).
\]

By Minkowski’s inequality and Cauchy-Schwartz inequality, we have

\[
\left[ \int_{D} \left( \int_{-h}^{0} |\phi \psi| \, dz \right)^2 \right]^{\frac{1}{2}} \leq \int_{-h}^{0} \left( \int_{D} |\phi \psi|^2 \right)^{\frac{1}{2}} \, dz
\]

\[
\leq \int_{-h}^{0} \left( \int_{D} |\phi|^4 \right)^{\frac{1}{4}} \left( \int_{D} |\psi|^4 \right)^{\frac{1}{4}} \, dz
\]

\[
\leq C \int_{-h}^{0} \|\phi\|_{L^2(D)}^{\frac{1}{2}} \left( \|\phi\|_{L^2(D)} + \|\nabla \phi\|_{L^2(D)} \right)^{\frac{1}{2}}
\]

\[
\times \|\psi\|_{L^2(D)}^{\frac{1}{2}} \left( \|\psi\|_{L^2(D)} + \|\nabla \psi\|_{L^2(D)} \right)^{\frac{1}{2}} \, dz
\]

\[
\leq C \|\phi\|_{L^2(D)}^{\frac{1}{2}} \left( \|\phi\|_{L^2(D)} + \|\nabla \phi\|_{L^2(D)} \right)^{\frac{1}{2}} \|\psi\|_{L^2(D)}^{\frac{1}{2}} \left( \|\psi\|_{L^2(D)} + \|\nabla \psi\|_{L^2(D)} \right)^{\frac{1}{2}},
\]

where in the above second to the last step we have used Gagliardo-Nirenberg-Sobolev inequality. This finishes the proof.

\[\square\]
3 The bounded absorbing set in \( D(A_1) \times D(A_2) \)

The existence of solutions, global in time, in the space \( H^2 \) was first proved in [6] for the case with boundary conditions (1.3) and (1.4). Uniform boundedness of the solutions in the space \( D(A_1) \times D(A_2) \) and the existence of a bounded absorbing ball for the solutions in the space \( D(A_1) \times D(A_2) \) were obtained in [6] under the condition that \( \alpha = 0 \) and that \( Q, Q_z \in L^2 \). What will be established in the following Theorem 3.1 is that the above result is still valid for any \( \alpha \geq 0 \) and \( Q \in L^2 \), thus eliminating the extra conditions \( \alpha = 0 \) and \( Q_z \in L^2 \). The proof to be presented here will be different from the one given by [6].

Now, we state our first main result of this article in the following theorem:

**Theorem 3.1** Suppose \( Q \in L^2(\Omega) \). For any \((v_0, \theta_0) \in D(A_1) \times D(A_2)\), there exists a unique solution \((v, \theta)\) of (1.1)-(1.5) such that

\[
(v, \theta) \in L^\infty(0, +\infty; D(A_1) \times D(A_2)).
\]

Moreover, there exists a bounded absorbing ball for the solutions of (1.1)-(1.5) in \( D(A_1) \times D(A_2) \).

**Proof:**

Uniqueness of such solutions follows immediately from the uniqueness of strong solutions.

Notice that for any \((v_0, \theta_0) \in D(A_1) \times D(A_2)\), we have \((v_t(0), \theta_t(0)) \in H\). Therefore, by Theorem 2.3 we have

\[
(v_t, \theta_t) \in L^\infty(0, \infty; H) \cap L^2(0, \infty; V).
\]

Moreover, there exists a bounded absorbing ball for \((\partial_t v, \partial_t \theta)\) in space \( H \).

Now, we prove existence of a bounded absorbing set in \( \mathbb{R}_+ \) for both \( \|v\|_{H^2} \) and \( \|\theta\|_{H^2} \) and the uniform boundedness of \( \|v\|_{H^2} \) and \( \|\theta\|_{H^2} \) for \( t > 0 \).

**Step 1.** Take inner product of (1.1) with \(-\partial_t^2 v\) and use (1.3) to obtain

\[
\nu_1 \|\nabla v_z\|^2 + \nu_1 \|v_{zz}\|^2 = \left\langle v_t + (v \cdot \nabla)v + wv_z + \nabla p_z - \int_{-h}^{z} \nabla \theta + f v^\bot, v_{zz} \right\rangle. \tag{3.1}
\]
First, by boundary condition (1.3), we have

\[
\langle \nabla p_s, v_{zz} \rangle = \int_D \nabla p_s \int_{-h}^0 v_{zz} = \int_D \nabla p_s \left( v_z \big|_{-h}^0 \right) = 0,
\]

\[
- \langle \int_{-h}^z \nabla \theta, v_{zz} \rangle = - \int_D \left( \int_{-h}^z \nabla \theta \right) \cdot v_{z}^0_{-h} + \int_D \int_{-h}^0 \nabla \theta \cdot v_z = \langle \nabla \theta, v_z \rangle,
\]

and, due to the fact that \( f_z = 0 \),

\[
\langle f v^\perp, v_{zz} \rangle = \int_D f \int_{-h}^0 v^\perp \cdot v_{zz} = \int_D f v^\perp v_z^0_{-h} - \int f v_z^\perp v_z = 0.
\]

Second,

\[
\langle (v \cdot \nabla) v, v_{zz} \rangle = \int_D \left( (v \cdot \nabla) v \right) \cdot v_{z}^0_{-h} - \int_D \left( (v_z \cdot \nabla) v + (v \cdot \nabla) v_z \right) \cdot v_z
\]

\[
= - \int_D \left( (v_z \cdot \nabla) v \right) \cdot v_z - \int_D (v \cdot \nabla) v_z \cdot v_z
\]

\[
= - \int_D \left( (v_z \cdot \nabla) v \right) \cdot v_z + \frac{1}{2} \int_D (\nabla \cdot v) |v_z|^2,
\]

where in the last equality we have used the following computation:

\[
\int_D \left( (v_z \cdot \nabla) v \right) \cdot v_z = \int_D \sum_{i=1}^{2} \partial_i \left( \frac{|v_z|^2}{2} \right)
\]

\[
= \int_D \sum_{i=1}^{2} \frac{1}{2} \left[ \int_{\partial D} v \cdot n |v_z|^2 - \int_D (\nabla \cdot v) |v_z|^2 \right]
\]

\[
= - \frac{1}{2} \int_D (\nabla \cdot v) |v_z|^2.
\]

Third, since \( w|_{z=-h} = w|_{z=0} = 0 \), we have

\[
\langle w v_z, v_{zz} \rangle = \frac{1}{2} \int_\Omega w \left( |v_z|^2 \right) = \frac{1}{2} \int_D \left( w |v_z|^2 \right)_{z=-h}^0 - \int_{-h}^0 w_z |v_z|^2
\]

\[
= \frac{1}{2} \int_\Omega (\nabla \cdot v) |v_z|^2.
\]

Therefore,

\[
\langle (v \cdot \nabla) v, v_{zz} \rangle + \langle w v_z, v_{zz} \rangle = - \int_\Omega \left( (v_z \cdot \nabla) v \right) \cdot v_z + \int_\Omega (\nabla \cdot v) |v_z|^2
\]
Plugging the above computations into (3.1) and noticing that \( v_z|_{z=0} = 0 \), we obtain

\[
\nu_1 \| \nabla v_z \|_2^2 + \mu_1 \| v_{zz} \|_2^2 \\
\leq \| v_t \|_2 \| v_{zz} \|_2 + \| \nabla \theta \|_2 \| v_z \|_2 - \int_\Omega [(v_z \cdot \nabla) v] \cdot v_z + \int_\Omega (\nabla \cdot v) |v_z|^2 \\
\leq \| \nabla \theta \|_2 \| v_z \|_2 + C \| \nabla v \|_2 \| v_z \|_2^2 \\
\leq C_\varepsilon (\| v_t \|_2^3 + \varepsilon \| v_{zz} \|_2^3 + \| \nabla \theta \|_2 \| v_z \|_2^2 + C \| \nabla v \|_2 \| v_z \|_2 \| \nabla^3 v_z \|_2^3) \\
\leq C_\varepsilon (\| v_t \|_2^3 + \| \nabla v \|_2^3 \| v_z \|_2^3) + \| \nabla \theta \|_2 \| v_z \|_2 + 2\varepsilon \| \nabla^3 v_z \|_2^3 \\
= C_\varepsilon (\| v_t \|_2^3 + \| \nabla v \|_2^3 \| v_z \|_2^3) + \| \nabla \theta \|_2 \| v_z \|_2 + 2\varepsilon (\| \nabla v \|_2^3 + \| v_{zz} \|_2^3).
\]

Thus, choosing \( \varepsilon > 0 \) sufficiently small yields

\[
\| \nabla v_z \|_2^2 + \| v_{zz} \|_2^2 \leq C (\| v_t \|_2^3 + \| \nabla v \|_2^3 \| v_z \|_2^3 + \| \nabla \theta \|_2 \| v_z \|_2^2).
\]

This gives the uniform boundedness of \( \| v_z \|_{H^1} \) and a bounded absorbing set for \( v_z \) in \( H^1 \).

**Step 2.** Take the inner product of (1.1) with \(-\Delta v\) and use (1.3) to obtain

\[
\nu_1 \| \Delta v \|_2^2 + \mu_1 \| \nabla v_z \|_2^2 = \langle v_t, \Delta v \rangle + \langle (v \cdot \nabla)v, \Delta v \rangle + \langle w v_z, \Delta v \rangle \\
- \langle \int_{-h}^z \nabla \cdot \theta, \Delta v \rangle + \langle f v^+, \Delta v \rangle.
\]

Therefore,

\[
\| \Delta v \|_2^2 + \| \nabla v_z \|_2^2 \leq C (\| v_t \|_2^3 + \| (v \cdot \nabla)v \|_2^3 + \| w v_z \|_2^3 + \| \nabla \theta \|_2^3 + \| v \|_2^3).
\]

(3.2)

Notice that

\[
\| (v \cdot \nabla)v \|_2^3 \leq \| v \|_6^2 \| \nabla v \|_3^2 \leq C \| v \|_{H^1}^2 \| \nabla v \|_2 \| \Delta v \|_2 \\
\leq C_\varepsilon \| v \|_{H^1}^3 + \varepsilon \| \nabla v \|_{H^1}^2 \\
= C_\varepsilon \| v \|_{H^1}^3 + \varepsilon \| \nabla v \|_2^3 + \varepsilon (\| \Delta v \|_2^2 + \| \nabla v_z \|_2^2),
\]

and that by Lemma 2.1, we have

\[
\| w v_z \|_2^2 \leq C \| \nabla v \|_2 \| \nabla v \|_{H^1} \| v_z \|_2 \| v_z \|_{H^1} \\
\leq C_\varepsilon \| \nabla v \|_2^3 \| v_z \|_2 \| v_z \|_{H^1} + \varepsilon \| \nabla v \|_{H^1}^2 \\
= C_\varepsilon \| \nabla v \|_2^3 \| v_z \|_2 \| v_z \|_{H^1} + \varepsilon (\| \Delta v \|_2^2 + \| \nabla v_z \|_2^2).
\]
Choosing \( \varepsilon > 0 \) sufficiently small, we derive from (3.2) that
\[
\|\Delta v\|_2^2 + \|\nabla v\|_2^2 \leq C (\|v_t\|_2^2 + \|v\|_{H^1}^2 + \|\nabla v\|_2^2 + \|\Delta v\|_2^2 + \|\nabla v\|_2^2). \tag{3.3}
\]
Notice that there exists a bounded absorbing set in \( \mathbb{R}_+ \) for each term on the right-hand side of (3.3) and that each term on the right-hand side of (3.3) is uniformly bounded for \( t > 0 \). Therefore, the same is true for \( \|\Delta v\|_2 \), and we can conclude the same for \( \|v\|_{H^2} \).

**Step 3.** Take the inner product of (1.2) with \(-\partial^2 \theta\) and use (1.4) to obtain
\[
\nu_2 \|\nabla \theta_2\|_2^2 + \alpha \nu_2 \|\nabla \theta|_{z=0}\|_2^2 + \mu_2 \|\theta_{zz}\|_2^2 = \langle \theta_t + v \cdot \nabla \theta + w \theta + Q, \theta_{zz} \rangle. \tag{3.4}
\]
First,
\[
\langle v \cdot \nabla \theta, \theta_{zz} \rangle \leq \|v\|_{\infty} \|\nabla \theta\|_2 \|\theta_{zz}\|_2 \leq C \|v\|_{H^2} \|\nabla \theta\|_2 \|\theta_{zz}\|_2 \leq C \|v\|_{H^2}^2 \|\nabla \theta\|_2^2 + \varepsilon \|\theta_{zz}\|_2^2.
\]
Second, using boundary condition \( w|_{z=0} = w|_{z=-h} = 0 \), we have
\[
\langle w \theta_z, \theta_{zz} \rangle = \frac{1}{2} \int_D \int_{-h}^0 w((\theta_z)^2)_z = -\frac{1}{2} \int_\Omega w_z(\theta_z)^2 = \frac{1}{2} \int_\Omega (\nabla \cdot v)(\theta_z)^2.
\]
Thus, noticing that \( \theta_z|_{z=-h} = 0 \),
\[
\langle w \theta_z, \theta_{zz} \rangle \leq \frac{1}{2} \|\nabla v\|_2 \|\theta_z\|_2^2 \leq C \|\nabla v\|_2 \|\theta_z\|_2^2 \|\nabla^3 \theta_z\|_2^2 \leq C \|\nabla v\|_2 \|\theta_z\|_2^2 + \varepsilon \|\nabla^3 \theta_z\|_2^2.
\]
Plugging the above estimates into (3.4) and choosing sufficiently small \( \varepsilon > 0 \) then yields
\[
\|\nabla \theta_z\|_2^2 + \|\nabla \theta|_{z=0}\|_2^2 + \|\theta_{zz}\|_2^2 \leq C (\|\theta_t\|_2^2 + \|v\|_{H^2}^2 \|\nabla \theta\|_2^2 + \|\nabla v\|_2 \|\theta_z\|_2^2 + \|Q\|_2^2).
\]
This finishes the proof of the uniform boundedness of \( \|\theta_z\|_{H^1} \) and the existence of a bounded absorbing set of \( \theta_z \) in \( H^1 \).

**Step 4.** Taking the inner product of (1.2) with \(-\Delta \theta\) and using (1.4), we obtain
\[
\nu_2 \|\Delta \theta\|_2^2 + \mu_2 \|\nabla \theta_z\|_2^2 = \langle \theta_t, \Delta \theta \rangle + \langle v \cdot \nabla \theta, \Delta \theta \rangle + \langle w \theta_z, \Delta \theta \rangle \leq C (\|\theta_t\|_2 + \|v \cdot \nabla \theta\|_2 + \|w \theta_z\|_2) \|\Delta \theta\|_2.
\]
Therefore, by Lemma 2.1, we obtain
\[ \| \Delta \theta \|_2^2 + \| \nabla \theta \|_2^2 \]
\[ \leq C(\| \theta_t \|_2^2 + \| v \cdot \nabla \theta \|_2^2 + \| w \theta_z \|_2^2) \]
\[ \leq C(\| \theta_t \|_2^2 + \| v \|_{L^\infty}^2 \| \nabla \theta \|_2^2 + \| \nabla v \|_{H^1} \| \theta_z \|_2 \| \theta_{zz} \|_{H^1}) \]
\[ \leq C(\| \theta_t \|_2^2 + \| v \|_{H^2}^2 \| \nabla \theta \|_2^2 + \| \nabla v \|_2 \| \nabla v \|_{H^1} \| \theta_z \|_2 \| \theta_{zz} \|_{H^1}), \]
from which follows the existence of a bounded absorbing set for \( \| \theta \|_{H^2} \) in \( \mathbb{R}_+ \) and the uniform boundedness of \( \| \theta(t) \|_{H^2} \) for \( t > 0 \).

\[ \square \]

\textbf{ Remark 3.1} Following the ideas of [6], Theorem 3.1 can be used to give a different proof of the main result of [5] on the finiteness of Hausdorff and fractal dimensions of the global attractor \( A \) for the strong solutions.

\section{Global existence of \( z \)-weak Solutions}

Existence and uniqueness of \( z \)-weak solutions, global in time, was first proved in [14] for the case with boundary conditions (1.3) and (1.4), under an extra assumption \((v_0, \theta_0) \in L^6 \). As pointed out in [14] that this assumption can be replaced by a slightly weaker assumption that \((\tilde{v}_0, \tilde{\theta}_0) \in L^6 \), where the notion of \( \tilde{\phi} \) is defined as the difference of a function \( \phi \) on \( \Omega \) and its vertical average:
\[ \tilde{\phi}(x, y, z) := \phi(x, y, z) - \frac{1}{h} \int_{-h}^{0} \phi(x, y, z) \, dz. \]
It seems to have been an open problem since the work of [14] that whether or not existence and uniqueness of \( z \)-weak solutions are valid for 3D Primitive equations, i.e. whether or not the extra assumption \((\tilde{v}_0, \tilde{\theta}_0) \in L^6 \) can be eliminated or not. In this and next section, we resolve this problem with a positive confirmation. Our first result is the global existence of \( z \)-weak solutions as presented in the following Theorem 4.1. In next section, we deal with uniqueness of \( z \)-weak solutions.

Now, we state our second main result of this article in the following:
Theorem 4.1 Suppose $Q \in L^2(\Omega)$. For any $(v_0, \theta_0), (\partial_z v_0, \partial_z \theta_0) \in H$, there exists a weak solution $(v, \theta)$ of (1.1)-(1.5) such that

$$(v_z, \theta_z) \in L^\infty(0, +\infty; H) \cap L^2(0, \infty; V).$$

Moreover, there exists a bounded absorbing ball for $(v_z, \theta_z)$ in $H$.

Proof: We prove the theorem in three steps:

Step 1. Take inner product of (1.1) with $-\partial_z^2 v$ and use (1.3) to obtain

$$\frac{1}{2} \frac{d}{dt} \|v_z\|^2 + \nu_1 \|\nabla v_z\|^2 + \nu_1 \|v_{zz}\|^2$$

$$= \langle (v \cdot \nabla)v + wv_z + \nabla p_s - \int_{-h}^z \nabla \theta + f v^\perp, v_{zz} \rangle,$$

where we have used the computation:

$$-\int_{\Omega} v_t \cdot v_{zz} = -\int_D v_t \cdot v_z\big|_{z=-h} + \int_{\omega} v_{zt} \cdot v_z = \frac{1}{2} \frac{d}{dt} \|v_z\|^2.$$

Moreover, by the few equalities in the previous subsection, and notice that

$$-\int_{\Omega} \int_{-h}^z \nabla \theta \cdot v_{zz} = -\int_D \left( \int_{-h}^z \nabla \theta \right) \cdot v_z\big|_{z=-h} + \int_{\Omega} \nabla \theta \cdot v_z$$

$$= \int_0^\infty \int_{\partial D} \theta \frac{\partial v_z}{\partial n} - \int_{\Omega} \theta \nabla \cdot v_z = -\int_{\Omega} \theta \nabla \cdot v_z,$$

we have

$$\frac{1}{2} \frac{d}{dt} \|v_z\|^2 + \nu_1 \|\nabla v_z\|^2 + \nu_1 \|v_{zz}\|^2$$

$$= \int_{\Omega} \left[ |(\nabla \cdot v)v_z|^2 - ((v_z \cdot \nabla)v) \cdot v_z \right] - \int_{\Omega} \theta \nabla \cdot v_z \quad (4.2)$$

$$= \int_{\Omega} \left[ (v_z \cdot \nabla v^\perp) \cdot v_z \right] - \int_{\Omega} \theta \nabla \cdot v_z,$$

where

$$v^\perp = (v_2, -v_1), \quad \nabla^\perp = (\partial_2, -\partial_1).$$

By Lemma 2.2, we have

$$\int_{\Omega} \left[ (v_z \cdot \nabla v^\perp) \cdot v_z \right]$$

$$\leq C \|\nabla v\|_{1/2}(\|v\|_2 + \|\nabla v_z\|_2) \|v_z\|_2(\|v_z\|_2 + \|\nabla v_z\|_2)$$

$$\leq C(\|\nabla v\|_2 + \|\nabla v\|_{1/2} \|\nabla v_z\|_{1/2})^2(\|v_z\|_2 + \|v_z\|_2 \|\nabla v_z\|_2)$$
By Hölder’s inequality, we have
\[(\|\nabla v\|_2 + \|\nabla v\|_2^\frac{1}{2} \|\nabla v_z\|_2^\frac{3}{2}) (\|v_z\|_2^2 + \|v_z\|_2 \|\nabla v_z\|_2) \]
\[= \|\nabla v\|_2 \|v_z\|_2^2 + \|\nabla v\|_2^\frac{1}{2} \|\nabla v_z\|_2^\frac{3}{2} \|v_z\|_2^2 \]
\[+ \|\nabla v\|_2 \|v_z\|_2 \|\nabla v_z\|_2 + \|\nabla v\|_2^\frac{1}{2} \|\nabla v_z\|_2 \|v_z\|_2 \|\nabla v_z\|_2^\frac{3}{2} \]
\[\leq C \varepsilon (\|\nabla v\|_2^2 + 1) \|v_z\|_2^2 + \|\nabla v\|_2^\frac{1}{2} \|\nabla v_z\|_2 \|v_z\|_2 \|\nabla v_z\|_2^\frac{3}{2} \]
\[\leq C \varepsilon (\|\nabla v\|_2^2 + 1) \|v_z\|_2^2 + \varepsilon \|\nabla v_z\|_2^2 \]
and
\[\left| \int_\Omega \theta \nabla \cdot v_z \right| \leq C \varepsilon \|\theta\|_2^2 + \varepsilon \|\nabla v_z\|_2^2 \]
Plugging the above estimates into (4.2) and choosing \(\varepsilon > 0\) sufficiently small yields
\[\frac{d}{dt} \|v_z\|_2^2 + \|\nabla v_z\|_2^2 + \|v_{zz}\|_2^2 \leq C \left[(\|\nabla v\|_2^2 + 1) \|v_z\|_2^2 + 1 + \|\theta\|_2^2 \right] . \quad (4.3)\]
Therefore,
\[\frac{d}{dt} (\|v_z\|_2^2 + 1) \leq C \left[(\|\nabla v\|_2^2 + \|\theta\|_2^2 + 1) (\|v_z\|_2^2 + 1)^2 \right]. \]
Let \(z(t) = \|v_z\|_2^2 + 1\). Then, there exists a \(t_1 > 0\), such that
\[Cz(0) \int_0^{t_1} \left(\|\nabla v(\tau)\|_2^2 + \|\theta(\tau)\|_2^2 + 1\right) d\tau < 1, \]
and
\[z(t) \leq \frac{z(0)}{1 - z(0) \int_0^t C \left(\|\nabla v(\tau)\|_2^2 + \|\theta(\tau)\|_2^2 + 1\right) d\tau}, \quad \forall \, t \in [0, t_1]. \]
This proves uniform boundedness of \(\|v_z(t)\|_2\) for \(t \in [0, t_1]\). Then, by (4.3), we have uniform boundedness of
\[\int_0^t (\|\nabla v_z(\tau)\|_2^2 + \|v_{zz}(\tau)\|_2^2) \, d\tau, \quad \forall \, t \in [0, t_1]. \]

**Step 2.** Take inner product of (1.2) with \(- \theta_{zz}\) to obtain:
\[\frac{1}{2} \frac{d}{dt} \left(\|\theta_z\|_2^2 + \alpha \|\theta|_{z=0}\|_2^2\right) + v_2 \left(\|\nabla \theta_z\|_2^2 + \alpha \|\nabla \theta|_{z=0}\|_2^2\right) + \mu_2 \|\theta_{zz}\|_2^2 \]
\[= \langle Q, \theta_{zz} \rangle + \langle v \cdot \nabla \theta + w \theta_z, \theta_{zz} \rangle . \quad (4.4)\]
In deriving (4.4), we have used boundary conditions (1.3) and (1.4) in the following calculations:

\[
\begin{align*}
- \int_{\Omega} \theta_t \theta_{zz} &= - \int_{D} \theta_t \theta_z^0 |_{z=-h} + \int_{\Omega} \theta_z \theta_z = \frac{1}{2} \frac{d}{dt} \left( \| \theta_z \|_2^2 + \alpha \| \theta |_{z=0} \|_2^2 \right), \\
\int_{\Omega} \Delta \theta \cdot \theta_{zz} &= \int_{D} \Delta \theta \cdot \theta_z^0 |_{z=-h} - \int_{\Omega} \Delta \theta_z \cdot \theta_z = - \alpha \int_{D} \Delta \theta \cdot \theta |_{z=0} - \int_{\Omega} \Delta \theta_z \cdot \theta_z \\
&= - \alpha \int_{\partial D} (n \cdot \nabla \theta) |_{z=0} + \alpha \int_{\partial D} |\nabla \theta |_{z=0}|^2 \\
&- \int_{-h}^{0} \int_{\partial D} (n \cdot \nabla \theta_z) \theta_z + \int_{\Omega} |\nabla \theta_z|^2 \\
&= \alpha \| \nabla \theta |_{z=0} \|_2^2 + \| \nabla \theta_z \|_2^2.
\end{align*}
\]

Now we reformulate the right-hand side of equation (4.4).

First of all, since \( w |_{z=-h} = w |_{z=0} = 0 \), we have

\[
\langle w \theta_z, \theta_{zz} \rangle = \frac{1}{2} \int_{\Omega} w(\theta_z^2)_z \\
= \frac{1}{2} \left( \int_{D} w(\theta_z^2)^0 |_{z=-h} - \int_{\Omega} w_z(\theta_z^2) \right) = \frac{1}{2} \int_{\Omega} (\nabla \cdot v) \theta_z^2.
\]

Second,

\[
\langle (v \cdot \nabla) \theta, \theta_{zz} \rangle = \int_{D} (v \cdot \nabla \theta \theta_z^0 |_{z=-h} - \int_{\Omega} [(v_z \cdot \nabla \theta) \theta_z + (v \cdot \nabla \theta) \theta_z] \\
= - \alpha \int_{D} (v \cdot \nabla \theta) \theta |_{z=0} - \int_{\Omega} (v_z \cdot \nabla \theta) \theta_z - \frac{1}{2} \int_{D} v \cdot \nabla (\theta_z^2) \\
= \frac{\alpha}{2} \int_{D} \theta^2 \nabla v |_{z=0} - \int_{\Omega} (v_z \cdot \nabla \theta) \theta_z + \frac{1}{2} \int_{\Omega} (\nabla \cdot v) \theta_z^2.
\]

Thus,

\[
\langle v \cdot \nabla \theta + w \theta_z, \theta_{zz} \rangle = \alpha \frac{1}{2} \int_{D} \theta^2 \nabla v |_{z=0} - \int_{\Omega} (v_z \cdot \nabla \theta) \theta_z + \int_{\Omega} (\nabla \cdot v) \theta_z^2.
\]

Therefore,

\[
\frac{1}{2} \frac{d}{dt} \left( \| \theta_z \|_2^2 + \alpha \| \theta |_{z=0} \|_2^2 \right) + \nu_2 \left( \| \nabla \theta_z \|_2^2 + \alpha \| \nabla \theta |_{z=0} \|_2^2 \right) + \mu_2 \| \theta_{zz} \|_2^2 \\
= \langle Q, \theta_{zz} \rangle + \frac{\alpha}{2} \int_{D} \theta^2 \nabla v |_{z=0} - \int_{\Omega} (v_z \cdot \nabla \theta) \theta_z + \int_{\Omega} (\nabla \cdot v) \theta_z^2 \quad (4.5)
\]

\[= :I_1 + I_2 + I_3 + I_4.\]
Now, we estimate $I_1$’s term by term. First, $I_1$ can be easily estimated as following:

$$I_1 \leq C_\varepsilon \|Q\|_2^2 + \varepsilon \|\theta_{zz}\|_2^2.$$ 

Next, since $v|_{z=0} = v + \int_z^0 v_z \, d\xi$, we have,

$$h|v|_{z=0}|^2 \leq 2 \left[ \int_{z=-h}^0 |v|^2 + \int_{z=-h}^0 \left( \int_z^0 |v_z| \, d\xi \right)^2 \, dz \right]$$

$$\leq 2 \left[ \int_{z=-h}^0 |v|^2 + h \int_{z=-h}^0 |v_z|^2 \, dz \right]$$

Thus,

$$h \int_D |v|_{z=0}|^2 \leq 2 \left[ \int_{\Omega} |v|^2 + h \int_{\Omega} |v_z|^2 \right],$$

that is

$$\|v|_{z=0}\|_2^2 \leq \frac{2}{h} \|v\|_2^2 + 2 \|v_z\|_2^2.$$  (4.6)

Therefore, by (4.6) and Gagliardo-Nirenberg-Sobolev inequality, we have

$$I_2 \leq \frac{\alpha}{2} \| (\nabla \cdot v) |_{z=0}\|_2 \|\theta|_{z=0}\|_4^2$$

$$\leq C(\|\nabla \cdot v\|_2 + \|\nabla \cdot v_z\|_2) \|\theta|_{z=0}\|_2 (\|\theta|_{z=0}\|_2 + \|\nabla \theta|_{z=0}\|_2)$$

$$\leq C(\|\nabla v\|_2 + \|\nabla v_z\|_2) \|\theta|_{z=0}\|_2^2$$

$$+ C_\varepsilon (\|\nabla v\|_2 + \|\nabla v_z\|_2)^2 \|\theta|_{z=0}\|_2^2 + \varepsilon \|\nabla \theta|_{z=0}\|_2^2$$

$$\leq C_\varepsilon (1 + \|\nabla v\|_2^2 + \|\nabla v_z\|_2^2) \|\theta|_{z=0}\|_2^2 + \varepsilon \|\nabla \theta|_{z=0}\|_2^2.$$ 

By Lemma 2.2, we have

$$I_3 \leq C \|v_z\|_{\frac{5}{2}} (\|v_z\|_2 + \|\nabla v_z\|_2) \|\nabla \theta\|_{\frac{5}{2}} (\|\nabla \theta\|_2 + \|\nabla \theta_z\|_2) \times \|\theta_z\|_{\frac{5}{2}} (\|\theta_z\|_2 + \|\nabla \theta_z\|_2) \frac{1}{2}$$

$$= C (\|v_z\|_2 + \|v_z\|_{\frac{5}{2}} \|\nabla v_z\|_{\frac{5}{2}}) (\|\nabla \theta\|_2 + \|\nabla \theta_z\|_{\frac{5}{2}})$$

$$\times (\|\theta_z\|_2 + \|\theta_z\|_{\frac{5}{2}} \|\nabla \theta_z\|_{\frac{5}{2}}).$$
Notice that
\[
\begin{align*}
&\left(\|v_z\|_2 + \|v_z\|_2^{2} \|
\frac{1}{2} \nabla v_z \|
\frac{1}{2} \right) \left(\|\nabla \theta\|_2 + \|\nabla \theta\|_2^{2} \|
\frac{1}{2} \nabla \theta_z \|
\frac{1}{2} \right)
\end{align*}
\]
and, by H"older's inequality, we have
\[
J_1 \leq \frac{1}{2} \|v_z\|_2 \|\theta_z\|_2^{2} + \frac{1}{2} \|
\nabla \theta\|_2,
\]
\[
J_2 \leq C \|v_z\|_2 \|\theta_z\|_2^{2} + \|\nabla \theta\|_2^{2} + \varepsilon \|
\nabla \theta_z\|_2^{2},
\]
\[
J_3 \leq C \|v_z\|_2 \|\theta_z\|_2^{2} + \|\nabla \theta\|_2^{2} + \varepsilon \|
\nabla \theta_z\|_2^{2},
\]
\[
J_4 = \|v_z\|_2 \|\nabla \theta\|_2^{2} \|\theta_z\|_2^{2} \|\nabla \theta_z\|_2
\leq C \|v_z\|_2 \|\theta_z\|_2^{2} + \|\nabla \theta\|_2^{2} + \varepsilon \|
\nabla \theta_z\|_2^{2},
\]
\[
J_5 \leq \frac{1}{4} \left(\|v_z\|_2^{2} + \|\nabla v_z\|_2^{2} \right) \|\theta_z\|_2^{2} + \frac{1}{2} \|
\nabla \theta\|_2^{2},
\]
\[
J_6 \leq C \|v_z\|_2 \|\nabla v_z\|_2 \|\theta_z\|_2^{2} + \|\nabla \theta\|_2^{2} + \varepsilon \|
\nabla \theta_z\|_2^{2},
\]
\[
J_7 \leq C \left(\|v_z\|_2^{2} + \|\nabla v_z\|_2^{2} \right) \|\theta_z\|_2^{2} + \|\nabla \theta\|_2^{2} + \varepsilon \|
\nabla \theta_z\|_2^{2},
\]
\[
J_8 = \|v_z\|_2^{2} \|\nabla v_z\|_2^{2} \|\nabla \theta\|_2^{2} \|\theta_z\|_2^{2} \|\nabla \theta_z\|_2
\leq C \|v_z\|_2^{2} \|\nabla v_z\|_2^{2} \|\theta_z\|_2^{2} + \|\nabla \theta\|_2^{2} + \varepsilon \|
\nabla \theta_z\|_2^{2}.
\]
Thus, for $t \in [0, t_1]$,
\[
I_3 \leq C \|v\|_2^{2} (\|v\|_2 + \|\nabla \theta\|_2^{2} + \|\nabla \theta_z\|_2^{2}).
\]
By Lemma 2.2 again, we have
\[
I_4 \leq C \|\nabla v\|_2^{2} (\|\nabla v\|_2 + \|\nabla \theta\|_2^{2} \|\theta_z\|_2 + \|\nabla \theta_z\|_2)
\leq C (\|\nabla v\|_2 + \|\nabla \theta\|_2^{2} \|\nabla \theta_z\|_2^{2})(\|\theta_z\|_2^{2} + \|\theta_z\|_2\|\nabla \theta_z\|_2).
By Hölder’s inequality, we have
\[
\left( \| \nabla v \|_2 + \| \nabla v \|_2^2 \| \nabla v_z \|_2^{1/2} \right) (\| \theta_z \|_2^2 + \| \nabla \theta_z \|_2^2) \\
= \| \nabla v \|_2 \| \theta_z \|_2^2 + \| \nabla v \|_2^2 \| \nabla \theta_z \|_2^{1/2} \| \theta_z \|_2^{1/2} \\
+ \| \nabla v \|_2 \| \theta_z \|_2 \| \nabla \theta_z \|_2 + \| \nabla v \|_2^2 \| \nabla \theta_z \|_2^{1/2} \| \theta_z \|_2^{1/2} \| \nabla \theta_z \|_2 \leq C \epsilon (1 + \| \nabla v \|_2^2 + \| \nabla v_z \|_2^2) \| \theta_z \|_2^2 + \epsilon \| \nabla \theta_z \|_2^2.
\]

Thus,
\[
I_4 \leq C \epsilon (1 + \| \nabla v \|_2^2 + \| \nabla v_z \|_2^2) \| \theta_z \|_2^2 + C \epsilon \| \nabla \theta_z \|_2^2.
\]

Therefore, for \( t \in [0, t_1] \), we have
\[
\frac{1}{2} \frac{d}{dt} (\| \theta_z \|_2^2 + \alpha \| \theta \|_{z=0}^2) + \nu v (\| \nabla \theta_z \|_2^2 + \alpha \| \nabla \theta \|_{z=0}^2) + \mu \| \theta_z \|_2^2 \\
\leq C \epsilon \| Q \|_2^2 + C \epsilon (1 + \| \nabla v \|_2^2 + \| \nabla v_z \|_2^2) \| \theta \|_{z=0}^2 \\
+ C \epsilon (1 + \| \nabla v \|_2^2 + \| \nabla v_z \|_2^2) \| \theta_z \|_2^2 + C \epsilon \| \nabla \theta \|_2^2 \\
+ \epsilon \| \theta_z \|_2^2 + \epsilon \| \nabla \theta \|_{z=0}^2 + C \epsilon \| \nabla \theta_z \|_2^2.
\]

Therefore, choosing sufficiently small \( \epsilon > 0 \), we have, for \( t \in [0, t_1] \),
\[
\frac{d}{dt} (\| \theta_z \|_2^2 + \alpha \| \theta \|_{z=0}^2) + \nu v \| \nabla \theta_z \|_2^2 + \nu \alpha \| \nabla \theta \|_{z=0}^2 + \mu \| \theta_z \|_2^2 \\
\leq C (\| Q \|_2^2 + \| \nabla \theta \|_2^2) + C (1 + \| \nabla v \|_2^2 + \| \nabla v_z \|_2^2) (\| \theta_z \|_2^2 + \alpha \| \theta \|_{z=0}^2).
\]

Noticing that
\[
\int_0^{t_1} (1 + \| \nabla v \|_2^2 + \| \nabla v_z \|_2^2 + \| \nabla \theta \|_2^2) dt < + \infty,
\]
and (4.6), i.e.
\[
\| \theta \|_{z=0}^2 \leq \frac{2}{\nu} (\| \theta \|_2^2 + 2 \| \theta_z \|_2^2)
\]
by Gronwall’s inequality, we obtain from (4.7) uniform boundedness of
\[
\| \theta_z (t) \|_2^2 + \alpha \| \theta (t) \|_{z=0}^2
\]
for \( t \in [0, t_1] \).

**Step 3.** Now, we can use global existence, uniqueness and uniform boundedness of the strong solutions to obtain global existence and uniform
boundedness of \((v_z, \theta_z) \in L^2\). Indeed, since \((v, \theta)\) is a weak solution, there exists a \(t_0 \in (0, t_1]\) such that
\[
\|v(t_0)\|_{V_1} < +\infty, \quad \|\theta(t_0)\|_{V_2} < +\infty.
\]
Therefore, both \(\|v(t)\|_{V_1}\) and \(\|\theta(t)\|_{V_2}\) are uniformly bounded for \(t \in [t_0, \infty)\).

Moreover, there exists a bounded absorbing set for \((v, \theta)\) in \(V\). Therefore, both \(\|v_z(t)\|_2\) and \(\|\theta_z(t)\|_2\) are uniformly bounded for \(t \geq 0\) and there is a bounded absorbing set for \((v_z, \theta_z)\) in \(H\).

\section{Uniqueness of \(z\)-weak Solutions}

Now we prove uniqueness of \(z\)-weak solutions. Indeed, we will prove the following Lipschitz continuity of the solutions in \(H\) space.

\textbf{Theorem 5.1} Suppose \(Q \in L^2(\Omega)\), \((v_0, \theta_0), (\partial_z v_0, \partial_z \theta_0) \in H\), then exists a unique weak solution \((v, \theta)\) of \((1.1)-(1.5)\), which depends continuously on the initial data in \(H\). More generally, let \((v^{(i)}, \theta_i)\) be a weak solution of the system of Primitive Equations with initial data \((v^{(i)}(0), \theta_i(0)) \in H\) with \(i = 1, 2\). Suppose \(\|\partial_z v^{(1)}(0)\|_2\) and \(\|\partial_z \theta_1(0)\|_2\) are finite. Then, for any \(T > 0\), there exists a constant \(C > 0\) which may depends on \((v^{(1)}(0), \theta_1(0))\) and \(T\), but not on \((v^{(2)}, \theta_2)\), such that, for all \(t \in [0, T]\),
\[
\|(v^{(1)}(t) - v^{(2)}(t), \theta_1(t) - \theta_2(t))\|_H \leq C\|(v^{(1)}(0) - v^{(2)}(0), \theta_1(0) - \theta_2(0))\|_H.
\]

\textbf{Proof:}

Denote \(p_{s,i}\) as the corresponding pressure term in the equation of \(v^{(i)}\),
\[
w_i = -\int_{-h}^{2} \nabla \cdot v^{(i)}(x, y, \xi) \, d\xi.
\]
and
\[
u = v^{(1)} - v^{(2)}, \quad \omega = w_1 - w_2, \quad \zeta = \theta_1 - \theta_2, \quad q_s = p_{1,s} - p_{2,s}.
\]
Then
\[ u_t - \nu_1 \Delta u - \mu_1 u_{zz} + (v^{(2)} \cdot \nabla)u + w_2 u_z \\
+ (u \cdot \nabla) v^{(1)} + \omega v_z^{(1)} + \nabla q_s - \int_{-h}^{z} \nabla \zeta + f u_{\perp} = 0. \tag{5.1} \]
\[ \zeta_t - \nu_2 \Delta \zeta - \mu_2 \zeta_{zz} + v^{(2)} \cdot \nabla \zeta + w_2 \zeta_z + u \cdot \nabla \theta_1 + \omega \theta_{1,z} = 0. \tag{5.2} \]
Take inner product of (5.1) with \( u \) to obtain
\[ \frac{1}{2} \frac{d}{dt} \| u \|^2 + \nu_1 \| \nabla u \|^2 + \mu_1 \| u_z \|^2 \\
- \int_{\Omega} [(u \cdot \nabla) v^{(1)} + \omega v_z^{(1)} - \int_{-h}^{z} \nabla \zeta] \cdot u, \tag{5.3} \]
where we have used the following fact:
\[ \int_{\Omega} [(v^{(2)} \cdot \nabla) u + w_2 u_z] \cdot u = \int_{\Omega} \nabla q_s \cdot u = \int_{\Omega} f u_{\perp} \cdot u = 0. \]
Take inner product of (5.2) with \( \zeta \) to obtain
\[ \frac{1}{2} \frac{d}{dt} \| \zeta \|^2 + \nu_2 \| \nabla \zeta \|^2 + \mu_2 \| \zeta_z \|^2 + \alpha \mu_2 \| \zeta \|_{z=0}^2 \\
- \int_{\Omega} [(v \cdot \nabla) \theta_1 + \omega \theta_{1,z}] \zeta, \tag{5.4} \]
where we have used the following fact:
\[ \int_{\Omega} [(v^{(2)} \cdot \nabla) \zeta + w_2 \zeta_z] \zeta = 0. \]
First, it is easy to have that
\[ \int_{\Omega} \int_{-h}^{z} (\nabla \zeta) \cdot u = - \int_{\Omega} \int_{-h}^{z} \zeta (\nabla \cdot u) \leq C z h^2 \| \zeta \|^2_2 + \epsilon \| \nabla u \|^2_2. \]
Next, we use Lemma 2.2 and Hölder’s inequality to get the following estimate:
\[ \int_{\Omega} [(u \cdot \nabla) v^{(1)}] \cdot u \leq C \| u \|_2 \| \nabla v^{(1)} \|_2 \| \nabla u \|_2 + \| \nabla u \|^2_2 (\| \nabla v^{(1)} \|_2 + \| \nabla v_z^{(1)} \|_2) \frac{1}{2} \\
\leq C \| \nabla v^{(1)} \|_2 (\| \nabla v^{(1)} \|_2 + \| \nabla v_z^{(1)} \|_2) \| u \|^2_2 \\
+ \frac{\epsilon}{2} (\| u \|^2_2 + \| \nabla u \|^2_2) \| \nabla v^{(1)} \|^2_2 + \| \nabla v_z^{(1)} \|^2_2 \\
\leq C \| \nabla v^{(1)} \|^2_2 + \| \nabla v_z^{(1)} \|^2_2 \| u \|^2_2 + \epsilon \| \nabla u \|^2_2. \]
We use Lemma 2.2 and Hölder’s inequality again to obtain:

\[ \int_{\Omega} \omega v_z^{(1)} \cdot u \leq C \|\omega\|_2^{\frac{1}{2}} \|v_z^{(1)}\|_2^{\frac{1}{2}} \|u\|_2 \left( \|\omega\|_2 + \|\omega_z\|_2 \right)^{\frac{1}{2}} \]

\[ \times \left( \|v_z^{(1)}\|_2 + \|\nabla v_z^{(1)}\|_2 \right)^{\frac{1}{2}} \|u\|_2 \|\nabla u\|_2 \]

\[ = C \|v_z^{(1)}\|_2 \left( \|v_z^{(1)}\|_2 + \|\nabla v_z^{(1)}\|_2 \right)^{\frac{1}{2}} \|u\|_2 \|\nabla u\|_2 \]

\[ \leq C \varepsilon (1 + \|\omega z\|_2^{\frac{1}{2}} + \|\nabla z\|_2^{\frac{1}{2}}) \|\nabla u\|_2 \]

\[ \leq C \varepsilon (1 + \|\nabla v_z^{(1)}\|_2^{\frac{1}{2}}) \|u\|_2^2 + \varepsilon \|\nabla u\|_2^2, \]

where we have used uniform boundedness of \(\|v_z^{(1)}\|_2\).

Similar to the treatment of the above two nonlinear terms, we also have

\[ \int_{\Omega} (u \cdot \nabla \theta_1) \zeta \leq C \|u\|_2^{\frac{1}{2}} \|\nabla \theta_1\|_2^{\frac{1}{2}} \|\zeta\|_2 \left( \|\zeta\|_2 + \|\nabla \zeta\|_2 \right)^{\frac{1}{2}} \]

\[ \times \left( \|\nabla \theta_1\|_2 + \|\nabla \theta_1,z\|_2 \right)^{\frac{1}{2}} \|\zeta\|_2 \|\nabla u\|_2 \]

\[ \leq C \varepsilon \|\nabla \theta_1\|_2 \|\zeta\|_2 \left( \|\nabla \theta_1\|_2 + \|\nabla \theta_1,z\|_2 \right)^{\frac{1}{2}} \|\zeta\|_2 \|\nabla u\|_2 \]

\[ + \varepsilon \left( \|\nabla u\|_2^2 + \|\nabla \zeta\|_2^2 \right), \]

and

\[ \int_{\Omega} \omega \theta_{1,z} \zeta \leq C \|\omega\|_2^{\frac{1}{2}} \|\theta_{1,z}\|_2^{\frac{1}{2}} \|\zeta\|_2 \left( \|\omega\|_2 + \|\omega_z\|_2 \right)^{\frac{1}{2}} \]

\[ \times \left( \|\theta_{1,z}\|_2 + \|\nabla \theta_{1,z}\|_2 \right)^{\frac{1}{2}} \|\zeta\|_2 \|\nabla u\|_2 \]

\[ \leq C \varepsilon \|\theta_{1,z}\|_2 \|\zeta\|_2 \left( \|\theta_{1,z}\|_2 + \|\nabla \theta_{1,z}\|_2 \right)^{\frac{1}{2}} \|\zeta\|_2 \|\nabla u\|_2 \]

\[ \leq C \varepsilon \|\theta_{1,z}\|_2 \|\zeta\|_2 \left( \|\theta_{1,z}\|_2 + \|\nabla \theta_{1,z}\|_2 \right)^{\frac{1}{2}} \|\zeta\|_2 \|\nabla u\|_2 \]

\[ + \varepsilon \left( \|\nabla u\|_2^2 + \|\nabla \zeta\|_2^2 \right), \]

where we have used uniform boundedness of \(\|\theta_{1,z}\|_2\).
Plugging the above estimates into (5.3) and (5.4) and adding up the two inequalities yields
\[
\frac{1}{2} \frac{d}{dt} \left( \|u\|_2^2 + \|\zeta\|_2^2 \right) + \nu_1 \|\nabla u\|_2^2 + \mu_1 \|u_z\|_2^2 + \nu_2 \|\nabla \zeta\|_2^2 + \mu_2 (\|\zeta\|_2^2 + \alpha \|\zeta|_{z=0}\|_2^2)
\leq C_\varepsilon \left( 1 + \|\nabla v^{(1)}\|_2^2 + \|\nabla \theta_1\|_2^2 + \|\nabla \theta_1, z\|_2^2 \right)
\times (\|u\|_2^2 + \|\zeta\|_2^2) + 4\varepsilon (\|\nabla u\|_2^2 + \|\nabla \zeta\|_2^2).
\]

Choose \(\varepsilon > 0\) sufficiently small, we have
\[
\frac{d}{dt} (\|u\|_2^2 + \|\zeta\|_2^2) + \nu_1 \|\nabla u\|_2^2 + \mu_1 \|u_z\|_2^2 + \nu_2 \|\nabla \zeta\|_2^2 + \mu_2 (\|\zeta\|_2^2 + \alpha \|\zeta|_{z=0}\|_2^2)
\leq C_\varepsilon \left( 1 + \|\nabla v^{(1)}\|_2^2 + \|\nabla v_2^{(1)}\|_2^2 + \|\nabla \theta_1\|_2^2 + \|\nabla \theta_1, z\|_2^2 \right)
\times (\|u\|_2^2 + \|\zeta\|_2^2).
\]

Noticing that, for a \(z\)-weak solution \((v^{(1)}, \theta_1)\),
\[
\int_0^\infty (\|\nabla v_2^{(1)}(t)\|_2^2 + \|\nabla \theta_1(t)\|_2^2) \, dt < +\infty,
\]
Integrating (5.5) with respect to \(t\), after dropping the dissipation and diffusion terms, yields Lipschitz continuity of \((v, \theta)\) with respect to \((v_0, \theta_0)\) in space \(H\), thus the uniqueness of \(z\)-weak solutions.

\(\square\)

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References

[1] C. Cao and E.S. Titi, Global Well-posedness and Finite Dimensional Global Attractor for a 3-D Planetary Geostrophic Viscous Model, Comm. Pure Appl. Math. 56, (2003), 198-233.

[2] C. Cao and E.S. Titi, Global well-posedness of the three-dimensional primitive equations of large scale ocean and atmosphere dynamics, Ann. of Math. (2) 166 (2007), no. 1, 245-267.

[3] F. Guillén-Gonzále, N. Masmoudi and M.A. Rodríguez-Bellido, Anisotropic estimates and strong solutions of the Primitive Equations, Diff. Integral Eq. 14(2001), no. 1, 1381-1408.
[4] N. Ju, *The global attractor for the solutions to the 3D viscous primitive equations*, Discrete and Continuous Dynamical Systems, 17 (2007), no. 1, 159-179.

[5] N. Ju, *Finite Dimensionality of the Global Attractor for 3D Primitive Equations with Viscosity*, preprint submitted, 2014. See also arXiv:1507.05992.

[6] N. Ju and R. Temam, *Finite Dimensions of the Global Attractor for 3D Primitive Equations with Viscosity*, J. Nonlinear Sci. 25 (2015), no. 1, 131-155.

[7] G. Kobelkov, *Existence of a solution 'in the large' for the 3D large-scale ocean dynamics equations*, C. R. Math. Acad. Sc. Paris 343 (2006), no. 4, 283-286. 86A05.

[8] G. Kobelkov, *Existence of a solution 'in the large' for ocean dynamics equations*, J. Math. Fluid Mech. 9 (2007), no. 4 588-610.

[9] I. Kukavica and M. Ziane, *On the regularity of the primitive equations of the ocean*, Nonlinearity, 20, (2007), no. 12, 2739-2753.

[10] J. L. Lions, R. Temam and S. Wang, *New formulations of the primitive equations of atmosphere and applications*, Nonlinearity 5, (1992), 237-288.

[11] J. Lions, R. Temam and S. Wang, *On the equations of the large scale Ocean*, Nonlinearity 5, (1992), 1007-1053.

[12] M. Petcu, *On the three dimensional primitive equations*, Adv. Dif. Eq., 11 (2006), 1201-1226.

[13] M. Petcu, *On the backward uniqueness of primitive equations*, J. Math. Pures Appl., 87 (2007), 275-289.

[14] T. Tachim Medjo, *On the uniqueness of z-weak solutions of the three-dimensional primitive equations of the ocean*, Nonlinear Anal. Real World Appl. 11 (2010), no. 3, 1413-1421.

[15] R. Temam, *Infinite Dimensional Dynamical Systems in Mechanics and Physics* Spring-Verlag, New York, Applied Mathematical Sciences Series, vol. 68, 1988. Second augmented edition, 1997.

[16] R. Temam, *Navier-Stokes equations. Theory and numerical analysis*, reprint of 3rd edition, AMS 2001.

[17] R. Temam and M. Ziane, *Some mathematical problems in geophysical fluid dynamics*, Handbook of Mathematical Fluid Dynamics, vol 3, S. Friedlander and D. Serre Editors, Elsevier, 2004, 535-658.