CLOSURES OF STEINBERG FIBERS IN TWISTED WONDERFUL COMPACTIFICATIONS

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Abstract. By a case-free approach we give a precise description of the closure of a Steinberg fiber within a twisted wonderful compactification of a simple linear algebraic group. In the non-twisted case this description was earlier obtained by the first author.

1. Introduction

Let $G$ be a simple linear algebraic group over an algebraically closed field $k$. Let $B$ be a Borel subgroup of $G$ and let $T \subset B$ denote a maximal torus. Let $W$ denote the associated Weyl group and $I$ denote the associated set of simple roots. For a subset $J$ of $I$ we let $W_J$ denote the subgroup of $W$ generated by the simple roots in $J$.

The wonderful compactification $X$ of $G$ (see e.g. [DP], [Str]), is a smooth projective $(G \times G)$-variety containing $G$ as an open subset. The $G \times G$-orbits in $X$ are indexed by the subsets $J$ of $I$, and we fix certain base points $h_J$ for these orbits. Let $\sigma$ denote a diagram automorphism of $G$ and let $X_\sigma$ be the associated twisted wonderful compactification of $G$, i.e. as a variety $X_\sigma$ is just $X$ but the $G \times G$-action is twisted by $\sigma$ on the second coordinate. Let $h_{I,\sigma}$ denote the point in $X_\sigma$ identified with $h_{\sigma(J)}$ in $X$. Then the collection $h_{\sigma(J)}$, $J \subset I$, are representatives for the $G \times G$-orbits in $X_\sigma$. A $G$-stable piece in $X_\sigma$ is then a locally closed and smooth subvariety in $X$ of the form $Z^w_{J,\sigma} = \text{diag}(G)(Bw,1)h_{J,\sigma}$, where $w \in W_{\sigma(J)}$ is a minimal length coset representative of $W/W_{\sigma(J)}$ and $\text{diag}(G)$ denotes the diagonal in $G \times G$. We then have a decomposition $X_\sigma = \bigsqcup_{J \subset I, w \in W_{\sigma(J)}} Z^w_{J,\sigma}$ (see [L2, 12.3] and [H2, 1.12]).

The $G$-stable pieces were first introduced by Lusztig to study the $G$-orbits and parabolic character sheaves. However, his original definition was based on some inductive method. The (equivalent) definition that we used above is due to the first author in [H1]. What we need in this paper is that the dimension of $Z^w_{J,\sigma}$ is equal to $\dim(G) - l(w) - |I - J|$, where $l(w)$ is the length of $w$ and $|I - J|$ is the cardinality of the set $I - J$ (see [L2, 8.20]). More properties about the $G$-stable pieces can be found in [L2] and [H2]. The $G$-stable pieces were also used by Evens and Lu in [EL] to study the Poisson structure and symplectic leaves.

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Consider $G$ as a $G \times G$-variety by left and right translation and define $G_{\sigma}$, similar to the definition of $X_{\sigma}$, by twisting the $G$-structure of $G$ on the second factor by $\sigma$. A $\sigma$-conjugacy class in $G_{\sigma}$ is then a diag($G$)-orbit in $G_{\sigma}$. The set of elements in $G_{\sigma}$ whose semisimple part lies in a fixed $\sigma$-conjugacy class is then called a Steinberg fiber of $G_{\sigma}$.

In this paper we study the closure of Steinberg fibers within $X_{\sigma}$. In [L2], Lusztig gave an explicit description for the closure of the unipotent variety in the group compactification when $G = PGL_2$ or $PGL_3$. In [Sp2], Springer studied the closure of an arbitrary Steinberg fiber for any connected, simple algebraic group and obtained some partial results. Based on their results, the first author obtained an explicit description of the closure of Steinberg fibers in the non-twisted case. The result in [H1] was formulated using $G$-stable pieces and the proof was based on a case-by-case checking. The main purpose of this paper is to generalize the result of [H1] to the twisted case with a more conceptual (and easier) proof. More precisely, we prove,

**Theorem.** Let $F$ be a Steinberg fiber of $G_{\sigma}$ and $\bar{F}$ be its closure in $X_{\sigma}$. Then

$$\bar{F} - F = \bigsqcup_{J \subset I, \text{w} \in W^{w}(J)} \bigsqcup_{\text{supp}_{\sigma}(\text{w}) = I} Z_{J,\sigma}^{w},$$

where $\text{supp}_{\sigma}(\text{w})$ denotes the minimal $\sigma$-stable subset of $I$ such that $\text{w}$ is contained in $W^{\text{supp}_{\sigma}(\text{w})}$.

As a consequence, the boundary of the closure is independent of the choice of the Steinberg fiber. Likewise it may be shown that the boundary of the closure of $F$ within any equivariant embedding of $G$ is independent of the choice of $F$ (see [H]). As a by-product, we will also give an explicit description of the “nilpotent cones” on $X$.

2. Wonderful compactifications and $G$-stable pieces

2.1. Let $G$ denote a simple linear algebraic group over an algebraically closed field $k$. We consider $G$ as a $G \times G$-variety by left and right translation. Let $B$ be a Borel subgroup of $G$, $B^{-}$ be an opposite Borel subgroup and $T = B \cap B^{-}$. The unipotent radical of $B$ (resp. $B^{-}$) will be denoted by $U$ (resp. $U^{-}$). Let $R$ denote the set of roots defined by $T$ and let $R^{+}$ denote the set of positive roots defined by $B$. Let $(\alpha_{i})_{i \in I}$ be the set of simple roots. For $i \in I$, we denote by $\omega_{i}$ and $s_{i}$ the fundamental weight and the simple reflection corresponding to $\alpha_{i}$.

We denote by $W$ the Weyl group associated to $T$. For any subset $J$ of $I$, let $W_{J}$ be the subgroup of $W$ generated by $\{s_{j} \mid j \in J\}$ and $W^{J}$ be the set of minimal length coset representatives of $W/W_{J}$.

For $J \subset I$, let $P_{J} \supset B$ be the standard parabolic subgroup defined by $J$ and let $P_{J}^{-} \supset B^{-}$ be the parabolic subgroup opposite to $P_{J}$. Set $L_{J} = P_{J} \cap P_{J}^{-}$. Then $L_{J}$ is a Levi subgroup of $P_{J}$ and $P_{J}^{-}$. The
semisimple quotient of $L_J$ of adjoint type will be denoted by $G_J$. We denote by $\pi_{P J}$ (resp. $\pi_{P J}^-$) the projection of $P_J$ (resp. $P_J^-$) onto $G_J$.

2.2. Let $X$ denote the wonderful compactification of $G$ ([DP], [SU]). Then $X$ is an irreducible, smooth projective $(G \times G)$-variety with finitely many $G \times G$-orbits $Z_J$ indexed by the subsets $J$ of $I$. As a $(G \times G)$-variety the orbit $Z_J$ is uniquely isomorphic to the product $(G \times G) \times_{P_J \times P_J} G_J$, where $P_J^-$ acts on $G \times G$ by $(q,p) \cdot (g_1, g_2) = (g_1 q^{-1}, g_2 p^{-1})$ and on $G_J$ by $(q,p) \cdot z = \pi_{P_J^-(q)} z \pi_{P_J^-(p)}^{-1}$. Let $h_J$ be the image of $(1,1,1)$ in $Z_J$ under this isomorphism.

We denote by diag$(G)$ the image of the diagonal embedding of $G$ in $G \times G$. For $J \subseteq I$ and $w \in W_J$, set $Z_J^w = \text{diag}(G)(Bw,1)h_J$. Then $Z_J^w$ is a locally closed subvariety of $X$ and (see [L2, 12.3] and [H2, 1.12])

$$X = \bigsqcup_{J \subseteq I, w \in W_J} Z_J^w.$$

We call $Z_J^w$ a $G$-stable piece.

3. Twisted actions

3.1. An automorphism $\sigma$ of $G$ which stabilizes the Borel subgroup $B$ and the maximal torus $T$ will induce a permutation of $I$. When the order of $\sigma$ as an automorphism of $G$ coincides with the order of the associated permutation of $I$, we say that $\sigma$ is a diagram automorphism. From now on $\sigma$ will denote a diagram automorphism of $G$. We also denote by $\sigma$ the corresponding bijection on $I$ and $W$.

Let $G_\sigma$ be the $(G \times G)$-variety which as a variety is isomorphic to $G$ and where the $G \times G$ action is twisted by the morphism $G \times G \to G \times G$, $(g,h) \mapsto (g, \sigma(h))$ for $g, h \in G$. Then we define the wonderful compactification $X_\sigma$ of $G_\sigma$ to be the $G \times G$-variety which as a variety is isomorphic to the wonderful compactification $X$ of $G$ and where the $G \times G$ action is twisted in the same way as above. Notice that we may regard $G_\sigma$ as a connected component of the semidirect product $G \rtimes < \sigma >$. In this case, $X_\sigma$ is the completion of $G_\sigma$ considered in [L2, 12].

The $G \times G$-orbits in $X_\sigma$ coincide with the associated orbits in $X$ and we let $Z_{J,\sigma}$ denote the orbit coinciding with $Z_{\sigma(J)}$. Accordingly we let $h_{J,\sigma}$ denote the point in $Z_{J,\sigma}$ identified with the base point $h_{\sigma(J)}$ of $Z_{\sigma(J)}$. For $J \subseteq I$ and $w \in W^{\sigma(J)}$, set $Z_{J,\sigma}^w = \text{diag}(G)(Bw,1)h_{J,\sigma}$. Then

$$X_\sigma = \bigsqcup_{J \subseteq I} \bigsqcup_{w \in W^{\sigma(J)}} Z_{J,\sigma}^w.$$

We call $(Z_{J,\sigma}^w)_{J \subseteq I, w \in W^{\sigma(J)}}$ the $G$-stable pieces of $X_\sigma$ (see [L2, 12.3] and [H2, 1.12]).
3.2. The orbits of \( \text{diag}(G) \) on \( G_\sigma \) are called \( \sigma \)-conjugacy classes. Let \( G//_\sigma G \) be the affine variety whose algebra is the subalgebra \( k[G]^{G,\sigma} \) of functions in \( k[G] \) invariant under \( \sigma \)-conjugacy. The inclusion \( k[G]^{G,\sigma} \to k[G] \) induces a morphism \( \text{St} : G_\sigma \to G//_\sigma G \). If \( \sigma \) is trivial, then \( \text{St} \) is just the Steinberg morphism of \( G_\sigma \) and the fibers the Steinberg fibers of \( G_\sigma \).

An element \( g \in G_\sigma \) is \( \sigma \)-conjugate to an element in \( B \) [Ste2, Lem.7.3]. Write \( b = tu \) where \( t \in T \) and \( u \) is an element of the unipotent radical \( U \) of \( B \). It is then easily seen that there exists an element \( t_1 \in T \), such that \( t_1 t \sigma(t_1)^{-1} \in T^\sigma \). Hence, \( g \) is \( \sigma \)-conjugate to some element in \( T^\sigma U \), i.e. we may assume that \( t \in T^\sigma \). Notice, that \( t \) is contained in the closure of the \( \sigma \)-conjugacy class of \( tu \) and thus, by geometric invariant theory, we find \( \text{St}(tu) = \text{St}(t) \). Moreover, considering \( t \sigma \) as an element of the semisimple group \( G \times < \sigma > \) it follows that \( t \sigma \) is quasi-semisimple in the sense of [Ste2, Sect.9], i.e. the automorphism of \( G \) obtained by conjugation by \( t \sigma \) will fix a Borel subgroup and a maximal torus thereof. As a consequence, the \( \sigma \)-conjugacy class of \( t \) in \( G_\sigma \) is closed [Spa II.1.15(f)]. We conclude that any Steinberg fiber of \( G_\sigma \) is of the form \( \bigcup_{g \in G} g(tU) \sigma(g)^{-1} \) for some \( t \in T^\sigma \). In particular, any Steinberg fiber is irreducible.

3.3. Let \( G_{sc} \) be the connected, simply connected, linear algebraic group associated to \( G \), and let \( B_{sc} \) (resp. \( T_{sc} \)) denote the Borel subgroup (resp. maximal torus) of \( G_{sc} \) associated to \( B \) (resp. \( T \)). By [Ste2 9.16] the automorphism \( \sigma \) of \( G \) may be lifted to an automorphism of \( G_{sc} \), which we also denote by \( \sigma \). We then define the \( G_{sc} \times G_{sc} \)-variety \( G_{sc,\sigma} \) similar to the definition of \( G_\sigma \). We may also form the quotient \( G_{sc}//_\sigma G_{sc} \) and define Steinberg fibers in \( G_{sc,\sigma} \) similar to the considerations in 3.2 for \( G_\sigma \).

The automorphism \( \sigma \) of \( G_{sc} \) induces a natural action of \( \sigma \) on the set \( \Lambda \) of \( T_{sc} \)-characters, and we let \( \Lambda^\sigma_{sc} \) denote the set of \( \sigma \)-invariant dominant weights. Let \( \mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_l \) denote the \( \sigma \)-orbits in \( I \), and set \( \omega_{\mathcal{C}_j} = \sum_{\mathcal{C}_j} \omega_i \). Then the elements \( \omega_{\mathcal{C}_j}, j = 1, \ldots, l \), is a generating set of the semigroup \( \Lambda^\sigma_{sc} \).

3.4. To any dominant weight \( \lambda = \sum_i a_i \omega_i \) we associate the subset \( I(\lambda) = \{ i \in I \mid a_i \neq 0 \} \) of \( I \). For \( w \in W \), let \( \text{supp}(w) \subset I \) be the set of simple roots whose associated simple reflections occur in some (or equivalently, any) reduced decomposition of \( w \) (see [Bon Prop.IV.1.7]) and let \( \text{supp}_\sigma(w) = \bigcup_{k \geq 0} \sigma^k(\text{supp}(w)) \). We have the following characterization of \( \text{supp}(w) \).

**Lemma 3.5.** Let \( w \in W \) and \( i \in I \). Then \( w \omega_i \neq \omega_i \) if and only if \( i \in \text{supp}(w) \). Hence for a dominant weight \( \lambda \), \( w\lambda \neq \lambda \) if and only if \( I(\lambda) \cap \text{supp}(w) \neq \emptyset \).
Proof. If $i \notin \text{supp}(w)$, then $w\omega_i = \omega_i$. Now we fix a reduced expression $w = s_{i_1} \cdots s_{i_n}$. Assume that $i \in \text{supp}(w)$. We show that $w\omega_i \neq \omega_i$ by induction on $n$. If $i_n \neq i$, then we are done by induction in $n$. Hence, we may assume that $i_n = i$. But then $w\alpha_i$ is a negative root. Thus $1 = \langle \omega_i, \alpha_i^\vee \rangle = \langle w\omega_i, (w\alpha_i)^\vee \rangle$ and, in particular, we cannot have $w\omega_i = \omega_i$. □

3.6. For any dominant weight $\lambda \in \Lambda_+$ let $H(\lambda)$ denote the dual Weyl module for $G_{sc}$ with lowest weight $-\lambda$. We then define $\sigma H(\lambda)$ to be the $G_{sc}$-module which as a vector space is $H(\lambda)$ and with $G_{sc}$-action twisted by the automorphism $\sigma$ of $G_{sc}$. Notice that up to a nonzero constant there exists a unique $G_{sc}$-isomorphism $\sigma H(\lambda) \cong H(\sigma(\lambda))$. In particular, when $\lambda \in \Lambda_{\sigma}^+$ is $\sigma$-invariant there exists a $G_{sc}$-equivariant isomorphism $f_\lambda : H(\lambda) \rightarrow \sigma H(\lambda)$. Fix $f_\lambda$ such that the its restriction to the lowest weight space $k_{-\lambda}$ in $H(\lambda)$ is the identity map (here we use the identification of $\sigma H(\lambda)$ with $H(\lambda)$ as vector spaces).

4. The “nilpotent cone” of $\mathcal{X}$

4.1. For any dominant weight $\lambda$ there exists (see [DS, 3.9]) a $G \times G$-equivariant morphism

$$\rho_\lambda : \mathcal{X} \rightarrow \mathbb{P}(\text{End}(H(\lambda)))$$

which extends the morphism $G \rightarrow \mathbb{P}(\text{End}(H(\lambda)))$ defined by $g \mapsto g[\text{Id}_\lambda]$, where $[\text{Id}_\lambda]$ denotes the class representing the identity map on $H(\lambda)$ and $g$ acts by the left action. By the definition of $X_\sigma$ we obtain a $G \times G$-equivariant morphism

$$X_\sigma \rightarrow \mathbb{P}(\text{Hom}_k(\sigma H(\lambda), H(\lambda))).$$

When $\lambda \in \Lambda_\sigma^+$ we may apply $f_\lambda$ to obtain an induced map

$$\rho_{\lambda,\sigma} : X_\sigma \rightarrow \mathbb{P}(\text{End}(H(\lambda)))$$

which is $G \times G$-equivariant.

4.2. An element in $\mathbb{P}(\text{End}(H(\lambda)))$ is said to be nilpotent if it may be represented by a nilpotent endomorphism of $H(\lambda)$. For $\lambda \in \Lambda_{\sigma}^+$ we let

$$\mathcal{N}(\lambda)_{\sigma} = \{ z \in X_\sigma | \rho_{\lambda,\sigma}(z) \text{ is nilpotent} \},$$

and call $\mathcal{N}(\lambda)_{\sigma}$ the nilpotent cone of $X_\sigma$ associated to the dominant weight $\lambda$. In 4.4, we will give an explicit description of $\mathcal{N}(\lambda)_{\sigma}$.

4.3. Define $ht$ to be the height map on the root lattice, i.e., the linear map on the root lattice which maps all the simple roots to 1.

Now fix $\lambda \in \Lambda_+$. Choose a basis $v_1, \ldots, v_m$ for $H(\lambda)$ consisting of $T$-eigenvectors with eigenvalues $\lambda_1, \ldots, \lambda_m$ satisfying $ht(\lambda_j + \lambda) \geq ht(\lambda_i + \lambda)$ whenever $j \leq i$. Then $B$ is upper triangular with respect to this basis.

Let $A_J$ be a representative of $\rho_\lambda(h_J)$ in $\text{End}(H(\lambda))$. Then when $\lambda_j + \lambda$ is a linear combination of the simple roots in $J$ we have that
and let \( X \). If \( \lambda_j + \lambda \) is not a linear combination of the simple roots in \( J \) then \( A_j v_j = 0 \). Assuming that \( \lambda \) is \( \sigma \)-invariant we obtain, by the definitions in 4.1, a similar description for a representative \( A_{J,\sigma} \) of \( \rho_{\lambda,\sigma}(h_{J,\sigma}) \) : if \( \lambda_j + \lambda \) is a linear combination of the simple roots in \( J \) then we have that \( A_{J,\sigma} v_j \in \kappa f(\lambda_j) \); otherwise \( A_{J,\sigma} v_j = 0 \). Notice that we regard \( f(\lambda_j) \) as an element of \( H(\lambda) \) and, as such, \( f(\lambda_j) \) is a \( T \)-eigenvector of weight \( \sigma(\lambda_j) \).

We now obtain.

**Proposition 4.4.** Let \( \lambda \in \Lambda^\sigma_+ \), then

\[
\mathcal{N}(\lambda)_\sigma = \bigcup_{J \subset I} \bigcup_{w \in W^\sigma(J)} Z_{J,\sigma}^w.
\]

Proof. Let \( w \in W^\sigma(J) \). Assume that \( w\lambda \neq \lambda \). Note that if \( x \) is a nonnegative linear combination of the simple roots in \( J \) then \( \text{ht}(w\sigma(x)) \geq \text{ht}(x) \). Hence,

\[
\text{ht}(w\sigma(-\lambda + x) + \lambda) = \text{ht}(w\sigma(x)) + \text{ht}(-w\lambda + \lambda) > \text{ht}(x).
\]

Therefore, \( \rho_{\lambda,\sigma}((b, 1)h_{J,\sigma}) \) is represented by a strictly upper triangular matrix with respect to the chosen basis in 4.3 above. As a consequence for any \( b \in B \), \( \rho_{\lambda,\sigma}((bw, 1)h_{J,\sigma}) \) is also represented by a strictly upper triangular matrix. So \( (bw, 1)h_{J,\sigma} \subset \mathcal{N}(\lambda)_\sigma \). Since \( \mathcal{N}(\lambda)_\sigma \) is \( \operatorname{diag}(G) \)-stable it follows \( Z_{J,\sigma}^w = \operatorname{diag}(G)(bw, 1)h_{J,\sigma} \subset \mathcal{N}(\lambda)_\sigma \).

Now assume that \( w\lambda = \lambda \). Let \( b \in B \) and \( z = (bw, 1)h_{J,\sigma} \). Denote by \( A \) a representative of \( \rho_{\lambda,\sigma}(z) \) in \( \operatorname{End}(H(\lambda)) \). Let \( V \) be the subspace of \( H(\lambda) \) spanned by \( v_1, \ldots, v_{m-1} \). Then \( Av_m \in \kappa v_m + V \) and \( AV \subset V \). Hence, \( A^n v_m \neq 0 \) for all \( n \in \mathbb{N} \). Thus \( z \notin \mathcal{N}(\lambda)_\sigma \). \( \square \)

**Corollary 4.5.** Let \( \lambda, \mu \in \Lambda^\sigma_+ \), then

\[
\mathcal{N}(\lambda + \mu)_\sigma = \mathcal{N}(\lambda)_\sigma \cup \mathcal{N}(\mu)_\sigma.
\]

Proof. This follows from the relation \( I(\lambda + \mu) = I(\lambda) \cup I(\mu) \). \( \square \)

5. A compactification of \( G_{sc} \)

5.1. Consider the morphism \( \psi_i : G_{sc} \to \mathbb{P}(\operatorname{End}(H(\omega_i)) \oplus k) \) defined by \( g \mapsto [(g : \operatorname{Id}_{H(\omega_i)} 1)] \), where \( \operatorname{Id}_{H(\omega_i)} \) denotes the identity map on \( H(\omega_i) \) and \( g \) acts on \( \operatorname{End}(H(\omega_i)) \) by the left action. Let furthermore \( \pi : G_{sc} \to X \) denote the the natural \( G_{sc} \times G_{sc} \)-equivariant morphism and let \( X_{sc} \) denote the closure of the image of the product map

\[
(\pi, \prod_{i \in I} \psi_i) : G_{sc} \to X \times \prod_{i \in I} \mathbb{P}(\operatorname{End}(H(\omega_i)) \oplus k).
\]

Then \( X_{sc} \) is a projective \( G_{sc} \times G_{sc} \)-equivariant variety containing \( G_{sc} \) as an open subset. Unlike \( X \) the variety \( X_{sc} \) need not be smooth and in general it is not even normal (e.g. in type \( A_3 \)). By abuse of notation
we use the notation \( \pi \) and \( \psi_i, i \in I \), for the natural extensions of the corresponding maps to \( X_{sc} \).

**Lemma 5.2.** The projective morphism \( \pi : X_{sc} \to X \) defines a bijection between \( X_{sc} - G_{sc} \) and \( X - G \). In particular, \( \pi \) is a finite morphism.

Proof. Notice that the \( G_{sc} \times G_{sc} \)-invariant homogeneous polynomial function on \( \text{End}(H(\omega_i)) \oplus k \) defined by \( (f, a) \mapsto \det(f) - a^{\text{dim}(H(\omega_i))} \) vanishes at \( (\text{Id}_{\text{H}(\omega_i)}, 1) \). Thus

\[
X_{sc} \subset X \times \prod_{i \in I} \left( \mathbb{P}(\text{End}(H(\omega_i)) \oplus k) - \mathbb{P}(0 \oplus k) \right),
\]

and we may consider the following commutative diagram

\[
\begin{array}{ccc}
X_{sc} & \longrightarrow & X \times \prod_{i \in I} \left( \mathbb{P}(\text{End}(H(\omega_i)) \oplus k) - \mathbb{P}(0 \oplus k) \right), \\
\downarrow^{\pi} & & \downarrow \\
X & \longrightarrow & X \times \prod_{i \in I} \mathbb{P}(\text{End}(H(\omega_i)))
\end{array}
\]

where all the maps are the natural ones. Assume now that \( x \) is an element of the boundary \( X_{sc} - G_{sc} \). As the dimensions of \( G_{sc} \) and \( X_{sc} \) coincide the \( (G_{sc}, 1) \)-stabilizer of \( x \) has strictly positive dimension. In particular, the images \( \psi_i(x) = [(f_i, a_i)], i \in I \), have the same property. Thus, the endomorphism \( f_i \) is not invertible and thus \( a_i = 0 \). This proves that

\[
X_{sc} - G_{sc} \subset X \times \prod_{i \in I} \mathbb{P}(\text{End}(H(\omega_i))),
\]

and hence \( \pi \) maps \( X_{sc} - G_{sc} \) injectively to the boundary \( X - G \). As \( \pi \) is dominant and projective, and thus surjective, this proves the first assertion. Finally \( \pi \) is finite as it is projective and quasifinite. \( \square \)

**5.3.** For a dominant weight \( \lambda \in \Lambda_+ \) let \( \psi_\lambda : G_{sc} \to \mathbb{P}(\text{End}(H(\lambda)) \oplus k) \) be the morphisms defined by \( \psi_\lambda(g) = [(g \cdot \text{Id}_{\text{H}(\lambda)}, 1)] \), for \( g \in G_{sc} \). Then we let \( X^\lambda_{sc} \) denote the closure of the image of the map

\[
G_{sc} \to X_{sc} \times \mathbb{P}(\text{End}(H(\lambda)) \oplus k)
\]

defined as the product of the inclusion \( G_{sc} \subset X_{sc} \) and \( \psi_\lambda \). We claim

**Lemma 5.4.** The canonical morphism \( \pi^\lambda : X^\lambda_{sc} \to X_{sc} \) is an isomorphism.

Proof. Let \( X_0 \) be the complement of the union of the closures \( \overline{B_s i B^-} \), \( i \in I \), within \( X \), and let \( X'_0 = \overline{T} \cap X_0 \). Then, by [B-K, Prop.6.2.3(i)], the natural morphism

\[
U \times U^- \times X'_0 \to X_0,
\]

\[
(g, h, x) \mapsto (g, h) \cdot x,
\]
is an isomorphism of varieties. Let $X_{sc,0} = \pi^{-1}(X_0)$ and $X'_{sc,0}$ denote the (scheme theoretic) inverse image $\pi^{-1}(X'_0)$. As $\pi$ is $G_{sc} \times G_{sc}$-equivariant we obtain an induced isomorphism
\[ U \times U^- \times X'_{sc,0} \rightarrow X_{sc,0}. \]
In particular, $X'_{sc,0}$ is an irreducible closed subvariety of the open subvariety $X_{sc,0}$ containing $T_{sc}$ as an open subset. Thus, $X'_{sc,0}$ is contained in the closure of $T_{sc}$ within $X_{sc}$. Let $\pi_\lambda$ denote the composition of $\pi$ and $\pi^{sc}$. Defining $X_{sc,0}^\lambda = \pi^{-1}_\lambda(X_0)$ and $(X_{sc,0}^\lambda)' = \pi^{-1}_\lambda(X'_0)$ we, similarly, obtain an isomorphism
\[ U \times U^- \times (X_{sc,0}^\lambda)' \rightarrow X_{sc,0}^\lambda. \]
Notice that $X_{sc,0}^\lambda = \pi^\lambda(X_{sc,0})$. Moreover, the $G \times G$-translates of $X_0$ over $X$ [B-K Thm.6.1.8]. Thus, it suffices to show that the morphism $(X_{sc,0}^\lambda)' \rightarrow X'_{sc,0}$ induced by $\pi^\lambda$ is an isomorphism. This will follow if $\pi^\lambda$ induces an isomorphism between the closures of $T_{sc}$ in $X_{sc}$ and $X_{sc}^\lambda$. Determining the latter closures of $T_{sc}$ and checking that they are isomorphic is now an easy exercise. \[ \square \]

It follows that we may consider $\psi_\lambda$ as the extended morphism
\[ \psi_\lambda : X_{sc} \rightarrow \mathbb{P}(\text{End}(H(\lambda)) \oplus k), \]
which we will do in the following. As in the proof of Lemma 5.2 we may prove that
\[ \psi_\lambda(X_{sc}) \subset \left( \mathbb{P}(\text{End}(H(\lambda)) \oplus k) - \mathbb{P}(0 \oplus k) \right) \]
and that the induced map $X_{sc} \rightarrow \mathbb{P}(\text{End}(H(\lambda)))$ is compatible with $\pi : X_{sc} \rightarrow X$ and the map $\rho_\lambda : X \rightarrow \mathbb{P}(\text{End}(H(\lambda)))$.

**5.5.** The variety $X_{sc}$ is a compactification of $G_{sc}$ with the $G_{sc} \times G_{sc}$ action defined in the natural way. Let $X_{sc,\sigma}$ be the $G_{sc} \times G_{sc}$-variety which as a variety is isomorphic to $X_{sc}$ and where the $G_{sc} \times G_{sc}$-action is twisted by the morphism $G_{sc} \times G_{sc} \rightarrow G_{sc} \times G_{sc}$, $(g, h) \mapsto (g, \sigma(g))$ for $g, h \in G_{sc}$. Thus we may identify $G_{sc,\sigma}$ of 3.3 with an open subset of $X_{sc,\sigma}$. Notice that by Lemma 5.2 we may identify the boundaries of $X_{sc,\sigma}$ and $X_\sigma$ and we may therefore also regard $Z_{J,\sigma}^\psi$, for $J \neq I$, as subsets of $X_{sc,\sigma}$.

**6. Steinberg fibers and trace maps**

**6.1.** Let $\text{Tr}_i$ denote the trace function on $\text{End}(H(\omega_{C_i}))$. To each $a_i \in k$ we may associate a global section $(\text{Tr}_i, a_i)$ of the line bundle $O_i(1) := O_{\mathbb{P}(\text{End}(H(\omega_{C_i})) \oplus k)}(1)$ on $\mathbb{P}(\text{End}(H(\omega_{C_i})) \oplus k)$. The pull back of $(\text{Tr}_i, a_i)$ to $X_{sc,\sigma}$, by the morphism $\psi_{\omega_{C_i}}$, is then a global section $f_{i,a_i}^{\sigma}$ of a line bundle on $X_{sc,\sigma}$. In the following, we will study the common zero set
of the sections $f_{i,a_i}$ for varying $a_i \in k$. By choosing a trivialization of the pull back of $O_i(1)$ to $G_{sc,\sigma}$ we may think of $f_{i,a_i}^\sigma$ as a function on $G_{sc,\sigma}$ and, by abuse of notation, we also denote this function by $f_{i,a_i}^\sigma$. We assume that the trivialization is chosen such that $f_{i,a}^\sigma - f_{i,0}^\sigma = a$ as functions on $G_{sc}$ (then the trivialization is actually unique). Then $f_{i,a_i}^\sigma$ is invariant under $\sigma$-conjugation by $G_{sc}$ and thus $f_{i,a_i}^\sigma$ induces a morphism $\overline{f}_{i,a_i}^\sigma : G_{sc}/\sigma G_{sc} \to k$. We then claim

**Proposition 6.2.** The product morphism

$$(\overline{f}_{1,0}^\sigma, \overline{f}_{2,0}^\sigma, \ldots, \overline{f}_{l,0}^\sigma) : G_{sc}/\sigma G_{sc} \to A^l,$$

is an isomorphism.

Proof. Let $f_{i,a}^\sigma$ denote the restriction of $f_{i,a}^\sigma$ to $T_{sc}$. An easy calculation shows that $f_{i,a}^\sigma$ equals

$$-w_0C_i + \sum_{\lambda \in \Lambda^\sigma, \lambda \neq \sigma C_i \neq H(C_i)} q_i, \lambda \lambda.$$

Thus $f_{i,a}^\sigma$ is contained in the semigroup algebra $k[\Lambda^\sigma]$ generated by the $\sigma$-invariant weights of $\Lambda$. Moreover, $f_{i,a}^\sigma$ is invariant under the action of the group $W^\sigma$ of $\sigma$-invariant elements of $W$. Hence, $f_{i,a}^\sigma$ is an element of the polynomial ring $k[\Lambda^\sigma]W^\sigma$ in $l$ variables [Sp3, proof of Cor.2], generated (as an $k$-algebra) by the elements

$$\text{sym}(C_i) := \sum_{w \in W^\sigma} wC_i, \ i = 1, \ldots, l.$$

As in [Ste] proof of Lem.6.3] we conclude that also $f_{i,a}^\sigma$, $i = 1, \ldots, l$, generates $k[\Lambda^\sigma]W^\sigma$ as an $k$-algebra. Now apply [Sp3] Thm.1.

**Corollary 6.3.** The intersection of $Z(a_1, \ldots, a_l)$ with the boundary $X_{sc,\sigma} - G_{sc,\sigma}$ of $X_{sc,\sigma}$ is independent of $a_1, \ldots, a_l$. Moreover, the intersection $Z(a_1, \ldots, a_l) \cap G_{sc,\sigma}$ is a single Steinberg fiber.

Proof. As in the proof of Lemma 5.2 it follows that $x$ is an element of $X_{sc,\sigma} - G_{sc,\sigma}$ exactly when the image $\psi_{\omega_C}(x)$ is of the form $[(f, 0)]$ for all $i = 1, \ldots, l$. Thus, the section $f_{i,a_i}^\sigma$ coincides with $f_{i,0}^\sigma$ on the boundary of $X_{sc,\sigma}$. This proves the first statement. The latter statement follows by Proposition 6.2.

7. Proofs of the main results

**Lemma 7.1.** Let $J \subsetneq I$, $w \in W^{\sigma(J)}$ and $b \in B$. If $f_{i,0}^\sigma((bw, 1)\mathbf{h}_{J,\sigma}) = 0$, then either (1) $w\omega_{C_i} \neq \omega_{C_i}$ or (2) $C_i \subset J$ and $\omega_{C_i} = \alpha_j$ for all $j \in C_i$.

Proof. Assume that $w\omega_{C_i} = \omega_{C_i}$. Then the diagonal entry of a representative $A$ of $\rho_{\omega_{C_i},\sigma}((bw, 1)\mathbf{h}_{J,\sigma})$ associated to the lowest weight space is nonzero. In particular, the relation $f_{i,0}^\sigma((bw, 1)\mathbf{h}_{J,\sigma}) = 0$ cannot
Lemma 7.2. \( \omega \in C \)

Let \( K \subseteq J \) denote the set of \( j \in J \) such that \( a_j \neq 0 \). As \( x - \omega_C \) is a weight of \( H(\omega_C) \) we know that \( C_i \cap K \) is nonempty. Now \( \sum_{j \in K} a_j w \alpha(j) = \sum_{j \in K} a_j \alpha_j \) and thus \( \sum_{j \in K} a_j (ht(w \alpha(j)) - ht(\alpha(j))) = 0 \). As \( w \in W^\sigma(J) \) we conclude that \( ht(w \alpha(j)) \geq 1 \) and consequently \( w \alpha(j) \) is a simple root for all \( j \in K \). By the assumption \( w \omega_C = \omega_C \), we know that \( w \alpha(j) = \alpha(j) \) for each \( j \in C_i \cap K \). In particular, when \( j \in C_i \cap K \) then \( a_{\alpha(j)} = a_j \). Hence, \( C_i \cap K \) is invariant under \( \sigma \) and as \( C_i \) is a single \( \sigma \)-orbit we have \( C_i \cap K = C_i \). This ends the proof. \( \square \)

**Lemma 7.2.** Let \( J \subseteq I \). Then

\[
Z(a_1, \cdots, a_l) \cap Z_{J, \sigma} = \bigcup_{w \in W^\sigma(J)} \bigcup_{\supp_w(w) = I} Z^w_{J, \sigma}.
\]

**Proof.** By Corollary 6.3 it suffices to consider the case when all \( a_i \) are zero. By Proposition 4.4,

\[
\bigcup_{J \subseteq I} \bigcup_{w \in W^\sigma(I)} Z^w_{J, \sigma} = \cap_{i} N(\omega_C)_\sigma \subset Z(0, \cdots, 0).
\]

This proves one inclusion. For \( z \in Z(0, \cdots, 0) \cap Z_{J, \sigma} \), we have that \( z = (g, g)(bw, 1)h_{J, \sigma} \) for some \( g \in G, b \in B \) and \( w \in W^\sigma(J) \). Then \( f^\sigma_i((bw, 1)h_{J, \sigma}) = 0 \) for all \( i = 1, \ldots, l \). It suffices to prove that \( \supp_w(w) = I \).

If \( w = 1 \), then by Lemma 7.1, \( C_i \subseteq J \) for each \( \sigma \)-orbit \( C_i \). Thus \( I = J \), which contradicts our assumption. Now assume that \( w \neq 1 \) and that \( \supp_w(w) \neq I \). As \( G \) is assumed to be simple, there exist simple roots \( \alpha_i \) and \( \alpha_j \) with \( n = -\langle \alpha_j, \alpha_i^\vee \rangle \neq 0 \) satisfying that \( i \in \supp_w(w) \) and \( j \notin \supp_w(w) \). Let \( C_i \) and \( C_j \) denote the associated \( \sigma \)-orbits of \( \alpha_i \) and \( \alpha_j \). As \( \supp_w(w) \) is \( \sigma \)-stable it follows that \( C_i \subseteq \supp_w(w) \) and \( C_j \subseteq I - \supp_w(w) \).

Now there exists \( m \in \mathbb{N} \), such that \( \sigma^m(i) \in \supp(w) \) and thus \( w \omega_{\sigma^m(i)} \neq \omega_{\sigma^m(i)} \). Hence redefining, if necessary, \( \alpha_i \) and \( \alpha_j \), we may assume that \( w \omega_i \neq \omega_i \). Consider then the relation \( \alpha_j = 2\omega_j - n\omega_i - \lambda \) with \( \lambda \) denoting a dominant weight. Then Lemma 7.1, applied to \( C_j \), implies that \( w\alpha_j = \alpha_j \) and \( w\omega_j = \omega_j \) and thus \( w(n\omega_i + \lambda) = n\omega_i + \lambda \). As both \( \omega_i \) and \( \lambda \) are dominant we conclude that \( w\omega_i = \omega_i \) which is a contradiction. \( \square \)

Now we will prove the main theorem.

**Theorem 7.3.** Let \( F \) be a Steinberg fiber of \( G_\sigma \) and \( \bar{F} \) its closure in \( X_\sigma \). Then

\[
\bar{F} - F = \bigcup_{J \subseteq I} \bigcup_{w \in W^\sigma(I)} Z^w_{J, \sigma}.
\]
which also coincides with the set \( Z(a_1, \ldots, a_l) \cap (X_{sc, \sigma} - G_{sc, \sigma}) \) for all \( a_1, \ldots, a_l \).

Proof. Let \( C \) be an irreducible component of \( Z(a_1, \cdots, a_l) \). Then by Krull’s principal ideal theorem, \( \dim(C) \geq \dim(G_{sc}) - l \). By [L2 8.20],

\[
\dim(Z_{I,\sigma}^w) = \dim(G) - l(w) - |I - J| < \dim(G_{sc}) - l,
\]

for \( J \neq I \) and \( w \in W_{\sigma(J)} \) with \( \text{supp}_\sigma(w) = I \). Thus by Lemma 7.2,

\[
\dim(C \cap (X_{sc, \sigma} - G_{sc, \sigma})) < \dim(G_{sc}) - l \leq \dim(C).
\]

Hence \( C \cap G_{sc, \sigma} \) is dense in \( C \). But, by Corollary 6.3, the intersection \( Z(a_1, \cdots, a_l) \cap G_{sc, \sigma} \) is a single Steinberg fiber \( F(a_1, \cdots, a_n) \) which, as in 3.2, is irreducible. We conclude that \( C \) is contained in the closure of \( F(a_1, \cdots, a_l) \), and thus the closure of \( F(a_1, \cdots, a_l) \) is \( Z(a_1, \cdots, a_l) \). In particular, \( Z(a_1, \cdots, a_l) \) is irreducible.

Let \( F \) be a Steinberg fiber of \( G_{\sigma} \). Then \( F = \pi(F(a_1, \cdots, a_l)) \) for some \( a_1, \cdots, a_l \in k \). Hence \( \bar{F} = \pi(Z(a_1, \cdots, a_l)) \). The statement now follows from Lemma 7.2 and Lemma 5.2.

\[ \square \]

Remark. 1. We call an element \( w \in W \) a \( \sigma \)-twisted Coxeter element if \( l(w) = l \) and \( \text{supp}_\sigma(w) = I \). (The notation of twisted Coxeter elements was first introduced by Springer in [Sp1]. Our definition is slightly different from his). It follows easily from Theorem 7.3 that \( Z_{I,\sigma}^w \) are the irreducible components of \( \bar{F} - F \), where \( i \in I \) and \( w \) runs over all \( \sigma \)-twisted Coxeter elements that are contained in \( W^{I-\{\sigma(i)\}} \).

2. By the proof of Theorem 7.3 we may also deduce that the closure of a Steinberg fiber \( F \) within \( X_{sc, \sigma} \) coincides with \( Z(a_1, \ldots, a_l) \) for certain uniquely determined \( a_1, \ldots, a_l \) depending on \( F \). This result may be considered as an extension of Corollary 2 in [Sp3] to the compactification \( X_{sc, \sigma} \) of \( G_{sc, \sigma} \). More precisely, notice that the statement of [Sp3 Corollary 2] is equivalent to saying that a Steinberg fiber \( F \) of \( \bar{G}_{sc, \sigma} \) is the common zero set of the functions \( f_{i,a_i}^\sigma \) for uniquely determined \( a_1, \ldots, a_l \). Here we think of \( f_{i,a_i}^\sigma \) as regular functions on \( G_{sc, \sigma} \) as explained in 6.1. When generalizing to \( X_{sc, \sigma} \) the only difference is that we have to regard \( \bar{f}_{i,a_i}^\sigma \) as sections of certain line bundles on \( X_{sc, \sigma} \).

Similar to [H1 4.6], we have the following consequence.

Corollary 7.4. Assume that \( G_{\sigma} \) is defined and split over \( \mathbb{F}_q \), then for any Steinberg fiber \( F \) of \( G_{\sigma} \), the number of \( \mathbb{F}_q \)-rational points of \( \bar{F} - F \) is

\[
\left( \sum_{w \in W} q^{l(w)} \right) \left( \sum_{\text{supp}_\sigma(w) = I} q^{l(w_0w) + L(w_0w)} \right),
\]

where \( w_0 \) is the maximal element of \( W \) and for \( w \in W \), \( l(w) \) is its length and \( L(w) \) is the number of simple roots \( \alpha \) satisfying \( w\alpha < 0 \).
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