Expansion in the Width: the Case of Vortices *

by

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Abstract
We construct an approximate solution of field equations in the Abelian Higgs model which describes motion of a curved vortex. The solution is found to the first order in the inverse mass of the Higgs field with the help of the Hilbert-Chapman-Enskog method. Consistency conditions for the approximate solution are obtained with the help of a classical Ward identity. We find that the Higgs field of the curved vortex of the topological charge $n \geq 2$ in general does not have single $n$-th order zero. There are two zeros: one is of the (n-1)-th order and it follows a Nambu-Goto type trajectory, the other one is of the first order and its trajectory in general is not of the Nambu-Goto type. For $|n| = 1$ the single zero in general does not lie on Nambu-Goto type trajectory.

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1 Introduction

In the recent paper [1] we have shown how to construct approximate domain wall solution with the help of Hilbert-Chapman-Enskog method [2] for solving equations with singular perturbations. The solution has the form of a perturbative series in positive powers of the width of the domain wall. Crucial role is played by certain consistency conditions for the perturbative expansion. Their appearence is related to existence of zero modes. In order to satisfy the consistency conditions we have had to allow for the possibility that the so called co-moving coordinate system is tied to an auxilliary membrane instead to zeros of the scalar field. Our approach differs essentially from earlier attempts to use the expansion in the width for domain walls.

In the present paper we would like to apply this same method to obtain a solution in the Abelian Higgs model describing a generic non-selfinteracting vortex. By the non-selfinteracting vortex we mean a vortex such that no its parts which are distant along the vortex are spatially close to each other. This excludes, e.g., spikes and selfintersections. It turns out that the method works quite well also in the case of vortices. The calculations we report in the present paper are parallel to the ones in [1]. Therefore, we do not repeat here general motivation and comments about the course of calculations. In [1] one can also find a more extensive list of references including also papers on dynamics of vortices.

The vortex in the Abelian Higgs model is the classic example of a vortex. It is called local because its fields are different from their vacuum values only in a close vicinity of a single line – far from this line the fields are exponentially close to the vacuum values. Thus, there are no long range forces acting on pieces of the vortex. For this reason the dynamics of such a vortex is probably simpler than dynamics of a global vortex with its long range Goldstone field. This is the reason we have chosen to apply the new method first to the local vortex. Unfortunately, the Abelian Higgs model is not simple on the mathematical side. It involves six real fields ($Re\Phi, Im\Phi, A_\mu$) which obey a non-linear set of equations. Moreover, except for the special case of Bogomol’nyi limit there are present two length scales, given by the inverses of masses of the scalar and vector fields. Even in the simplest case of a straight-linear, static vortex the exact analytical form of the corresponding solution is not known. Therefore, one should be prepared for rather cumbersome calculations.
The plan of our paper is the following. In the next Section we introduce the co-moving coordinate system and we write the field equations in the new coordinates. Most of this material is not new. We quote it here for convenience of the reader and in order to fix our notation. Moreover, we would like to emphasize the fact that the co-moving coordinate system apriori is not tied to position of zeros of the Higgs field. The approximate position of the zeros are determined at the end, from the constructed approximate solution of the field equations. In Section 3 we find the zeroth order term in the expansion in the width of the vortex. By the width we mean the inverse mass of the Higgs field. We also check the consistency conditions appropriate for the zeroth order and characteristic for the Hilbert-Chapman-Enskog method. Section 4 is devoted to the first order solution and the relevant consistency conditions. Section 5 contains final remarks. In Appendix A we derive a classical Ward identity which helps to calculate troublesome integrals encountered in Section 4. In Appendix B we give a list of constants appearing in Section 4.

2 Equations for the scalar and vector fields in the co-moving coordinates

We take the Lagrangian of the Abelian Higgs model in the following form

\[ L = D^\mu \Phi^* D_\mu \Phi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{\lambda}{4} (|\Phi|^2 - \frac{M^2}{\lambda})^2, \]  

(1)

where

\[ D_\mu \Phi = \partial_\mu \Phi + iqA_\mu \Phi, \quad D_\mu \Phi^* = (D_\mu \Phi)^* = (D_\mu)^*(\Phi)^*. \]

The corresponding field equations are

\[ D^\mu D_\mu \Phi + \frac{\lambda}{2} \Phi \left(|\Phi|^2 - \frac{M^2}{\lambda}\right) = 0, \]  

(2)

\[ \partial^\mu F_{\mu\nu} = iq(\Phi^* D_\nu \Phi - \Phi D_\nu \Phi^*). \]  

(3)

The vacuum fields are characterised by the following conditions

\[ |\Phi| = \Phi_0 \equiv \frac{M}{\sqrt{\lambda}}, \quad D_\mu \Phi = 0. \]  

(4)
The masses of the Higgs and the vector particle are equal to \( m_H^2 = M^2 \), \( m_A^2 = \frac{2q^2}{\lambda} m_H^2 \), respectively. We consider the case \( \frac{2q^2}{\lambda} \leq 1 \) which corresponds to superconductors of the second kind.

In the case of single vortex considered in this paper, the fields \( \Phi, A_\mu \) at each instant of time have the vacuum values in the whole space \( \mathbb{R}^3 \) except for a vicinity of a line \( C \) where the energy density is significantly different from zero. We do not assume that the Higgs field vanishes precisely on this line. The world-sheet of the line \( C \) in Minkowski space-time we shall denote by \( \Sigma \). The line \( C \) we will call the string, in anticipation of the result of the next Section, where we find that in the framework of \( 1/M \) expansion \( C \) has to obey an equation identical in its form with Nambu-Goto equation for a relativistic string. The string is used to define the coordinate system co-moving with the vortex. To this end, we parametrize \( \Sigma \) with two parameters \( \tau, \sigma \)

\[
[Y^\mu(\tau, \sigma)] \in \Sigma.
\]

The parameter \( \tau \) is by assumption time-like, i.e. \( Y^\mu_\tau Y^\nu_\tau > 0 \), while \( \sigma \) is the space-like one, i.e. \( Y^\mu_\sigma Y^\nu_\sigma < 0 \). In the following we use a more compact notation: \( (\tau, \sigma) = (u^a), \) \( a = 0, 3 \), with \( a = 0 \) (\( a = 3 \)) corresponding to \( \tau \) (\( \sigma \)). The co-moving coordinates are defined in a vicinity of the string world-sheet \( \Sigma \) by the following formula

\[
x^\mu = Y^\mu(u^a) + \xi_i n^\mu_i(u^a), \tag{5}
\]

where \( x^\mu \) are Cartesian, laboratory frame coordinates in Minkowski space-time; \( i = 1, 2 \); \( n_i(u^a) \) are two orthonormal space-like four-vectors, orthogonal to \( \Sigma \) at the point \([Y^\mu(u^a)]\) in the covariant sense, i.e.

\[
Y^\rho_\alpha(u) n_i_{\rho\mu}(u) = 0, \; n_i^\mu(u)n_k_{\mu\nu}(u) = -\delta_{ik}. \tag{6}
\]

The conditions (6) do not fix the four-vectors \( n_i \) uniquely – there is still a freedom of local (i.e. \( u^a \)-dependent) SO(2) rotations acting on the indices \( i, k \). The coordinates \( \xi^1, \xi^2 \) vanish on the string. We shall denote the full set of co-moving coordinates by \((\zeta^\alpha)\): \( (\zeta^\alpha) = (u^a, \xi^i), \; \alpha=0,3,1,2 \). The co-moving coordinate system has been used in numerous papers, see, e.g., [3, 4, 5, 6], with the difference that in those papers it was attached to a line on which the Higgs field was assumed to vanish.
The next step is to write Eqs.(2),(3) in the co-moving coordinates. It is convenient to introduce several mathematical objects:
- the metric $g_{ab}$ on $\Sigma$,
  \[ g_{ab} = Y_{\mu}^{a}Y_{\nu}^{b}; \]
- the extrinsic curvature coefficients,
  \[ K_{ab}^{i} = -n_{i}^{\mu}Y_{\mu,ab}; \]
- the torsion coefficients
  \[ \omega_{a} = \frac{1}{2}\epsilon^{ik}n_{i}n_{k,a}, \]
where $\epsilon^{ik}$ is the 2-dimensional totally antisymmetric symbol ($\epsilon^{12} = 1$). The torsion and extrinsic curvature coefficients depend on the choice of the four-vectors $n_{i}$. In general it is not possible to adjust them in such a manner that the torsion vanishes. This can be seen from Ricci equation
\[
\partial_{a}\omega_{b} - \partial_{b}\omega_{a} = \epsilon^{ij}K_{ic}^{a}K_{bc}^{j},
\]
where $K_{a}^{ic} = K_{ab}^{i}g^{bc}$. Vanishing of the torsion would imply that the extrinsic curvatures at a given point could not be arbitrary.

The Minkowski space-time metric tensor in the new coordinates has the following form
\[
\left[ G_{\alpha\beta} \right] = \left[ \begin{array}{cc} G_{ab} & G_{ak} \\ G_{ia} & G_{ik} \end{array} \right],
\]
where
\[
G_{ab} = g_{ab} + 2K_{ab}^{i}\xi^{i} + K_{a}^{ic}K_{bc}^{j}\xi^{i}\xi^{j} - \delta_{i}^{i}\xi^{i}\omega_{a}\omega_{b},
\]
\[
G_{ai} = G_{ia} = \epsilon^{ik}\xi^{k}\omega_{a},
\]
\[
G_{ik} = -\delta_{ik}.
\]
It follows from these formulae that
\[
\sqrt{-G} = \sqrt{-g} h(u^{a},\xi^{i}),
\]
where $G = det[G_{\alpha\beta}]$, $g = det[g_{ab}]$, and
\[
h(u^{a},\xi^{i}) = 1 + K_{a}^{ia}\xi^{i} + \frac{1}{2}(K_{a}^{ia}K_{b}^{jb} - K_{a}^{ib}K_{b}^{ja})\xi^{i}\xi^{j}. \quad (8)
\]
We shall also need components of the inverse metric tensor \( G^{\alpha\beta} \). They are
given by the following formulae:

\[
G^{ab} = h^{-2}[g^{ab}(1 + \xi^i K^{ic}_c)^2 - 2\xi^i K^{iab}(1 + \xi^k K^{kc}_c) + \xi^i \xi^j K^{ia}_c K^{jbc}],
\]
\[
G^{ai} = \epsilon^i_k \xi^k \omega_b G^{ab},
\]
\[
G^{ik} = -\delta^{ik} + (\delta^{ik} \xi^l - \xi^i \xi^k) \omega_a \omega_b G^{ab}.
\]

We shall write the Higgs and Maxwell equations (2),(3) in the co-moving coordinates using rescaled variables:

\[
\xi^i = \frac{\sqrt{2}}{M} s^j, \quad \Phi(u^a, \xi^i) = \Phi_0 \phi(u^a, s^i).
\]

Also the vector field \( A_\mu \) is transformed to the new coordinates. It has the following components

\[
(A_\alpha) = (A_a, A_i),
\]

where \( a = 0, 3, \ i = 1, 2 \), and

\[
A_a = \epsilon^\mu_a(u, \xi) A_\mu, \quad A_i = \frac{\sqrt{2}}{M} n^\mu_i(u) A_\mu.
\]

Here

\[
\epsilon^\mu_a = \frac{\partial x^\mu}{\partial u^a} = Y^\mu_{a} + \xi^i K^i_{ab} Y^\mu_{b} + \epsilon^{ij} \omega_a \xi^i n^\mu_j.
\]

Notice that the formula defining the components \( A_i \) contains the factor \( \sqrt{2}/M \), so that \( A_i \) are dimensionless, similarly as \( s^i \) and \( \phi \). We do not rescale the coordinates \( u^a \) and the components \( A_a \). The variables \( \xi^i \) present in \( G^{\alpha\beta} \) and \( \sqrt{-G} \) are also rescaled by the same factor. We shall use the following notation for the U(1) gauge covariant derivatives

\[
D_a = \frac{\partial}{\partial u^a} + i q A_a, \quad D_i = \frac{\partial}{\partial s^i} + i q A_i.
\]

Using the rescaled co-moving coordinates and the rescaled fields we can write Eq.(2) in the following form

\[
D_i D_i \phi + \phi - |\phi|^2 \phi = -\frac{\partial h}{h} D_i \phi
\]
The function \( h \) is given by formula (8); \( \partial_i \) denotes \( \partial/\partial s^i \).

The Maxwell equation (3) in the co-moving coordinates has the form

\[
\partial_\alpha (\sqrt{-G} G^{\alpha \beta} F_{\beta \gamma} ) = iq \sqrt{-G} (\Phi^* D_\gamma \Phi - \Phi D_\gamma \Phi^*),
\]

where

\[
F_{\beta \gamma} = \frac{\partial A_\gamma}{\partial \zeta^\beta} - \frac{\partial A_\beta}{\partial \zeta^\gamma}.
\]

It is convenient to split Eq.(13) into two equations and to pass to the rescaled variables. In the case \( \gamma = k = 1, 2 \) Eq.(13) gives

\[
\lambda \partial_i F_{ik} + 2iq (\phi^* D_k \phi - \phi D_k \phi^*) = -\lambda \frac{\partial h}{h} F_{ik}
\]

+ \[
\frac{2\lambda}{M^2} \sqrt{-gh} \left( \partial_a + \epsilon^t s^t \omega_a \partial_t \right) \left[ \sqrt{-gh} G^{ab} (F_{bk} + \epsilon^s r^s \omega_b F_{sk}) \right],
\]

where

\[
F_{ik} = \partial_i A_k - \partial_k A_i, \quad F_{bk} = \partial_b A_k - \partial_k A_b.
\]

For \( \gamma = c = 0, 3 \) we obtain from (13)

\[
\lambda \partial_i F_{ic} + 2iq (\phi^* D_c \phi - \phi D_c \phi^*) = -\lambda \frac{\partial h}{h} F_{ic}
\]

+ \[
\frac{2\lambda}{M^2} \sqrt{-gh} \left( \partial_a + \epsilon^k s^k \omega_a \partial_t \right) \left[ \sqrt{-gh} G^{ab} (F_{bc} + \epsilon^t s^t \omega_b F_{tc}) \right].
\]

Equations (12), (14), (15) have the form convenient for constructing the perturbative expansion in powers of \( 1/M \).

The reason for the splitting of the Maxwell equations into the two groups is that the components \( A_i \) play different dynamical role in the motion of the vortex than \( A_c \). The components \( A_i \), determined basically from Eq.(14), do not vanish for any vortex and they are present even in the zeroth order approximation. The components \( A_c \) are determined from Eq.(15). They vanish in the zeroth and first order approximations for all vortices, and for some vortices to all orders (e.g. for a straight-linear vortex). Moreover, we shall see that in the \( 1/M \) expansion we first find the n-th order contributions to \( \phi \) and \( A_i \) and the n-th order contribution to \( A_c \) is calculated afterwards.
The co-moving coordinate system in general does not cover the whole space-time. We assume that the range of its validity includes the physically most interesting region – outside of it the fields are exponentially close to the vacuum. The range of validity of the co-moving coordinates can be determined from the condition $h(u^a, \xi^i) > 0$. We will not discuss here this purely mathematical point.

3 Expansion in the width – the zeroth order solution and zero modes

The present Section is devoted to a preliminary discussion of the expansion in the width for the vortex solution constructed with the help of Hilbert-Chapman-Enskog method. The zeroth order solution we obtain coincides with the one known from earlier applications of the expansion in the width to vortices, [3]. In the next Section we calculate the first order contribution.

We seek approximate solutions of Eqs. (12), (14), (15) in the form of power series in $1/M$:

$$\phi = \phi^{(0)} + \frac{1}{M}\phi^{(1)} + \frac{1}{M^2}\phi^{(2)} + ...$$  \hspace{1cm} (16)  

$$A_i = A_i^{(0)} + \frac{1}{M}A_i^{(1)} + \frac{1}{M^2}A_i^{(2)} + ...$$  \hspace{1cm} (17)  

$$A_a = A_a^{(0)} + \frac{1}{M}A_a^{(1)} + \frac{1}{M^2}A_a^{(2)} + ...$$  \hspace{1cm} (18)

We also assume that these fields do not have any component oscillating in $\tau$ with a frequency proportional to $M$. This assumption ensures that when acting with the derivative $\partial_\tau$ on the n-th order contribution we obtain still the n-th order contribution. To illustrate the point: the $\tau$-derivative of the n-th order function $M^{-n}\cos M\tau$ belongs to the (n-1)-th order. This assumption has a physical content – it eliminates a whole class of solutions, in particular the ones describing radiation of the Higgs and vector particles. In order to include such solutions one would have to construct an extended perturbative scheme, e.g., in analogy to considerations presented in [4], where such oscillating component was taken into account in the case of domain walls in the framework of a polynomial approximation.

In the perturbative calculations we would like to preserve the local U(1) gauge invariance of the Abelian Higgs model. To this end, we use the trick
borrowed from background field techniques in field theory: we assume that
the gauge transformations act only on $A^{(0)}_\alpha$, while $A^{(n)}_\alpha$ with $n > 0$ are gauge
invariant. Then, the total vector field $A_\alpha$ has the right transformation law.
As for the Higgs field, each contribution $\phi^{(n)}$ is multiplied by the same $(\zeta^{(n)})$-
dependent phase factor. Then, in the gauge covariant derivatives it is suffi-
cient to use the zeroth order vector field $A^{(0)}_\alpha$, i.e.

$$D^{(0)}_a \phi = \left( \frac{\partial}{\partial u^a} + iqA^{(0)}_a \right) \phi, \quad D^{(0)}_k \phi = \left( \frac{\partial}{\partial s^k} + iqA^{(0)}_k \right) \phi.$$  

As explained in detail in paper [1], the expansion in $1/M$ is an example of
singular perturbations. In order to calculate the perturbative contributions
correctly we use the Hilbert-Chapman-Enskog method. In this method, apart
from obtaining the solution in a given order it is essential to look also at the
equations in the higher orders – from them one obtains consistency conditions
for the solution in the given order. The consistency condition for the zeroth
order solution follows from the equations in the first order. The equations
in the zeroth and first orders are obtained by inserting the expansions (16-
18) into Eqs.(12),(14),(15) and collecting the terms of the order $(1/M)^0$ and
$(1/M)^1$.

In the zeroth order, Eqs.(12),(14),(15) are reduced to the following equations:

$$D^{(0)}_i (D^{(0)}_i \phi^{(0)}) + \phi^{(0)} - |\phi^{(0)}|^2 \phi^{(0)} = 0, \quad (19)$$

$$\lambda \partial_i F^{(0)}_{ik} = -2iq[\phi^{*(0)}(D^{(0)}_k \phi^{(0)}) - \phi^{(0)}(D^{(0)}_k \phi^{(0)})^*], \quad (20)$$

$$\lambda \partial_i F^{(0)}_{ic} = -2iq[\phi^{*(0)}(D^{(0)}_c \phi^{(0)}) - \phi^{(0)}(D^{(0)}_c \phi^{(0)})^*]. \quad (21)$$

Equations (19),(20) coincide with equations for the static Abrikosov-Nielsen-
Olesen (A-N-O) vortex [3].

Let us quote here basic facts about the A-N-O vortex. We shall use them
in the next Section. The axially symmetric Ansatz for the A-N-O vortex has
the form

$$\phi^{(0)} = e^{in\theta} F(s), \quad A^{(0)}_i = e^{ik} \frac{s^k}{s} \left( n \frac{q}{qs} - H(s) \right), \quad (22)$$

where $s = \sqrt{(s^1)^2 + (s^2)^2}$, $\theta$ is the azimuthal angle in the $(s^1, s^2)$-plane, and
$n$ is the topological charge of the vortex. This Ansatz reduces Eqs.(19),(20)
to the following equations

$$F'' + \frac{1}{s} F' - q^2 H^2 F + F - F^3 = 0, \quad (23)$$
\[ B'(s) = \frac{4q^2}{\lambda} F^2 H, \tag{24} \]

where

\[ B = H' + \frac{H}{s}, \]

and ' denotes the derivative \( d/ds \). These equations are supplemented by the boundary conditions for the functions \( F, H \):

\[ F(0) = 0, \quad F(\infty) = 1, \tag{25} \]

\[ \lim_{s \to 0} sH(s) = \frac{n}{q}, \quad H(\infty) = 0. \tag{26} \]

We will call the functions \( F, H \) obeying Eqs.(23), (24) with the boundary conditions (25), (26) the A-N-O functions, and denote them by \( F_0(s), H_0(s) \). They exponentially vanish for large \( s \). The magnetic field of the vortex is equal to

\[ F_{ik}^{(0)} = e^{ik} \left( \frac{H_0}{s} + H_0' \right) \equiv e^{ik} B_0(s). \tag{27} \]

Explicit, exact analytical form of the A-N-O functions is not known. Nevertheless, they are very well investigated, see, e.g. [10].

For \( \phi^{(0)}, A_c^{(0)} \) given by the A-N-O solution, equation (21) is reduced to

\[ \triangle A_c^{(0)} - \frac{4q^2}{\lambda} F_0^2(s) A_c^{(0)} = 0, \tag{28} \]

where \( \triangle \equiv \partial_i \partial_i \) is the 2-dimensional Laplacian. The operator on the l.h.s. of this equation does not have non-zero regular, localised solutions. This can be easily seen by interpreting the operator \( -\triangle + \frac{4q^2}{\lambda} F_0^2 \) as a Schrödinger operator with the potential \( \frac{4q^2}{\lambda} F_0^2 \). Because this potential is non-negative, there are no bound states with zero eigenvalues. Therefore,

\[ A_c^{(0)} = 0. \tag{29} \]

Notice that Eq.(28) coincides with Eq.(24) for \( H(s) \). The fact that the solution \( H_0(s) \) of Eq.(24) is non-zero is not in contradiction with the conclusion (29) because \( H_0 \) is singular at \( s = 0 \) (see (26)), while \( A_c^{(0)} \) is required to be regular for all \( s \) and exponentially vanishing for large \( s \). These regularity conditions follow from the requirement that the fields \( F_{\alpha\beta} \) should be regular and exponentially localised.
Notice that by adopting the concrete form of the zeroth order solution we implicitly fix the gauge for two local gauge symmetries. The first local gauge group is formed by the \( (x^\mu) \)-dependent "electromagnetic" U(1) transformations. The other one consists of \( (u^a) \)-dependent SO(2) rotations in the \( (\xi^1, \xi^2) \)-plane, equivalent to SO(2) rotations of the basis \( (n_1, n_2) \) in the plane orthogonal to the world-sheet \( \Sigma \) of the string. The third local gauge symmetry, namely invariance with respect to reparametrisations of the world-sheet of the string remains explicitly preserved. In the present paper we prefer to work with the concrete gauge fixing for the former two gauge symmetries. Actually, one can provide a formulation of the Abelian Higgs model in the co-moving coordinates which is explicitly invariant under the three local gauge symmetries, \[11\].

Now we will turn to the consistency conditions for the zeroth order solution. To this end we have to derive equations in the first order. It is convenient to introduce a compact notation for the fields and the field equations. Thus,

\[
\Psi = \begin{pmatrix} \phi \\ \phi^* \\ A_k \end{pmatrix}.
\] (30)

The compact notation for the left hand side of the set of equations formed by Eq.(12), its complex conjugate counterpart, and Eq.(14) is \( \mathcal{F}(\Psi) \). Then, the zeroth order equations (19), (20) can be written as

\[
\mathcal{F}(\Psi^{(0)}) = 0,
\] (31)

\[
\Psi^{(0)} = \begin{pmatrix} \phi^{(0)} \\ \phi^{*\text{(0)}} \\ A_k^{(0)} \end{pmatrix}.
\] (32)

In this notation we regard \( \phi \) and \( \phi^* \) as two independent fields obeying Eq.(12) and its complex conjugate counterpart, correspondingly. This is equivalent to considering the real and imaginary parts of the field \( \phi \) and Eq.(12). Notice that we have left aside Eq.(15) – it will be solved later. This is possible because \( A_a \) is not present on the l.h.s. of Eqs.(12),(14).

Let us now consider a small correction

\[
\delta \Psi = \begin{pmatrix} \delta \phi \\ \delta \phi^* \\ \delta A_k \end{pmatrix}
\] (33)
to the zeroth order solution $\Psi^{(0)}$. Because of Eq.(31) we have

$$\mathcal{F}(\Psi^{(0)} + \delta \Psi) = \hat{L} \delta \Psi + \mathcal{O}(\delta \Psi).$$

(34)

Here $\hat{L}$ is a linear operator obtained by expanding the l.h.s. of Eqs.(12),(14) around $\phi^{(0)}$, $A^{(0)}_k$. Including also $\phi^*$ and the complex conjugate of Eq.(12) we find that

$$\hat{L} = \begin{bmatrix}
D^{(0)}_i D^{(0)}_i + 1 - 2|\phi^{(0)}|^2 & -(\phi^{(0)})^2 \\
-(\phi^{(0)})^2 & D^{*(0)}_i D^{*(0)}_i + 1 - 2|\phi^{(0)}|^2 \\
2iq[\phi^{*(0)} D^{(0)}_k - (D^{(0)}_k \phi^{(0)})^\ast] & -2iq[\phi^{(0)} D^{*(0)}_k - (D^{(0)}_k \phi^{(0)})^\ast] \\
2iq(D^{(0)}_l \phi^{(0)}) + iq\phi^{(0)} \partial_l & -2iq(D^{(0)}_l \phi^{(0)})^\ast - iq\phi^{*(0)} \partial_l \\
(\lambda \partial_l \partial_l - 4q^2 |\phi^{(0)}|^2)\delta^{kl} - \lambda \partial_k \partial_l 
\end{bmatrix}.$$

(35)

The equation for the first order correction to $\Psi^{(0)}$, i.e. for

$$\Psi^{(1)} = \begin{pmatrix}
\phi^{(1)} \\
\phi^{*(1)} \\
A^{(1)}_k
\end{pmatrix},$$

(36)

easily follows from formula (34) and Eqs.(12),(14). It has the following form

$$\hat{L} \Psi^{(1)} = -\sqrt{2} \begin{pmatrix}
D^{(0)}_i \phi^{(0)} \\
(D^{(0)}_i \phi^{(0)})^\ast \\
\lambda F^{(0)}_{ik}
\end{pmatrix} K^{ia}_a,$$

(37)

where the r.h.s. has been obtained by expanding the r.h.s. of Eqs.(12), (14). Equation (37) will be solved in the next Section.

From Eq.(37) we obtain the consistency conditions for the zeroth order solution. The crucial observation is that there are left eigenvectors of operator $\hat{L}$ with the corresponding eigenvalue equal to zero, i.e. normalizable, non-zero $\Psi^{l.z.m.}$, such that

$$\int ds^1 ds^2 (\Psi^{l.z.m.})^* T \hat{L} \Psi = 0$$

(38)

for all $\Psi$ vanishing sufficiently quickly for large $s$. In condition (38) $T$ denotes the matrix transposition, and the abbreviation l.z.m. stands for ”left zero
mode". It is easy to check that the operator $\hat{L}$ is Hermitean with respect to the scalar product given by

$$<\Psi_1|\Psi_2> = \int ds^1 ds^2 \Psi'^T \Psi,$$

in the space of $\Psi$’s which have the second component equal to the complex conjugation of the first one. (Therefore, it is a Hilbert space over real numbers and not over complex numbers.) Actually, the Higgs self-coupling constant $\lambda$ has been placed in front of the $F_{\alpha\beta}$ field in Eqs.(14),(15) just in order to obtain $\hat{L}$ Hermitean with respect to the simple scalar product (39). In the case of Hermitean operator the l.z.m.’s coincide with right zero modes, which are defined as normalizable (non-zero) solutions of the equation

$$\hat{L}\Psi^{r.z.m.} = 0.$$  \hspace{1cm} (40)

The zero modes can be easily found with the help of formula (34) by using a particular $\delta\Psi$ corresponding to symmetries of Eqs.(19),(20). These equations are regarded here as equations for functions on the plane $(s^1, s^2)$. Because of invariance of Eqs.(19),(20) with respect to translations in the $(s^1, s^2)$-plane, one can generate from the solution $\Psi^{(0)}$ infinitely many other solutions by applying the translations. We shall denote these new solutions by $T_a\Psi^{(0)}$. Thus,

$$T_a\Psi^{(0)} = \begin{pmatrix} \phi^{(0)}(s^i + a^i) \\ \phi^{* (0)}(s^i + a^i) \\ A_k^{(0)}(s^i + a^i) \end{pmatrix},$$

where $a = (a^i)$ is a constant vector defining the translation. It is clear that

$$\mathcal{F}(T_a\Psi^{(0)}) = 0.$$  

Taking infinitesimal $a$, we obtain from formula (34) that

$$\hat{L}\Psi_t = 0,$$

where

$$\Psi_t = \frac{\partial(T_a\Psi^{(0)})}{\partial a^l} |_{a=0} = \begin{pmatrix} \frac{\partial\phi^{(0)}}{\partial s^l} \\ \frac{\partial\phi^{* (0)}}{\partial s^l} \\ \frac{\partial A_k^{(0)}}{\partial s^l} \end{pmatrix}.$$
These $\Psi_l$ can not be used as the zero modes because they are not normalizable. The correct zero modes are obtained by combining the translation with a gauge transformation in the spirit of improved variations of Jackiw and Manton, \[12\]. Instead of $T_a \Psi^{(0)}$ we take

$$\tilde{T}_a \Psi^{(0)} = \begin{pmatrix}
e^{i\chi(a; s^k)} \phi^{(0)}(s^i + a^i) \\
e^{-i\chi(a; s^k)} \phi^{(0)}(s^i + a^i) \\
A_k^{(0)}(s^i + a^i) - \frac{1}{q} \partial_k \chi(a; s^i)
\end{pmatrix},$$

where $\chi(a; s^i) = qa^i A_1^{(0)}(s^i)$. It is clear that also $\tilde{T}_a \Psi^{(0)}$ is a solution of Eqs.\((19),(20)\). As the zero modes we take

$$\Theta_l = \frac{\partial (\tilde{T}_a \Psi^{(0)})}{\partial a^l} \bigg|_{a=0} = \begin{pmatrix}
D_l^{(0)} \\
(D_l^{(0)} \phi^{(0)})^* \\
F_{lk}^{(0)}
\end{pmatrix}.$$ 

where $l = 1, 2$. They are normalizable and orthogonal – one can easily check that

$$\langle \Theta_k | \Theta_l \rangle = \delta^{kl} \int ds^1 ds^2 (B_0^2 + F_0^{*2} + q^2 H_0^2 F_0^2),$$

where $F_0, H_0$, and $B_0$ are the A-N-O functions.

There also exists a solution of Eq.\((40)\) related to the U(1) gauge invariance. It has the form

$$\Psi_g = \begin{pmatrix} i\phi^{(0)} \\ -i\phi^{*(0)} \\ 0 \end{pmatrix}.$$ 

Another possible solution, corresponding to rotations around the origin in the $(s^1, s^2)$-plane is in fact identical with $\Psi_g$ because in the case of the A-N-O vortex the rotations are equivalent to global U(1) gauge transformations. Notice however that $\Psi_g$ is not normalizable with respect to the scalar product \((39)\). Therefore, it can not be accepted as the zero mode.

The consistency conditions follow from Eq.\((37)\) by integrating it with the zero modes $\Theta_l$, like in formula \((38)\). The r.h.s. of Eq.\((37)\) in general is not a linear combination of the zero modes because of the factor $\lambda$ in front of $F_{lk}^{(0)}$. Nevertheless, the integrations are simple and we obtain the following conditions

$$K_a^{la} = 0,$$ 

(43)
where \( l = 1, 2 \) and \( a = 0, 3 \).

The conditions (43) are equivalent to Nambu-Goto equation for a relativistic string whose world-sheet is given by the surface \( \Sigma \) in Minkowski space-time.

To summarize, the zeroth order solution is given by formulae (22) with \( F(s), H(s) \) equal to the A-N-O functions \( F_0(s), H_0(s) \). The components \( A_{a}^{(0)} \) vanish. The variables \( s = M\xi^i \) are related to the Cartesian coordinates by formula (5). For the functions \( Y^\mu(u^a) \) in that formula we can take any solution of Nambu-Goto equation (43). In the zeroth order approximation the string coincides with the zeros of the Higgs field.

4 The first order corrections

The first order corrections to \( \phi \) and \( A_k \) obey Eq.(37) with vanishing the r.h.s., due to the consistency conditions (43). This equation is satisfied by linear combinations of the zero modes and \( \Psi_g \) with coefficients which do not dependent on \( s^i \), i.e.

\[
\Psi^{(1)} = C^l(u^a)\Theta_l(s^i) + C^3(u^a)\Psi_g(s^i),
\]

where \( l = 1, 2 \). Now we have included \( \Psi_g \) because in Eq.(37) there are no integrals. The functions \( C^l, C^3 \) are real, otherwise the second component of \( \Psi^{(1)} \) would not be equal to the complex conjugation of the first one. In the following we shall show that the functions \( C^l, C^3 \) are not arbitrary.

We also have to solve the first order equation obtained from Eq.(15) for \( A_b \). With the Nambu-Goto equations (43) and the zeroth order results taken into account, it has the following form

\[
\Delta A^{(1)}_b - \frac{4q^2}{\lambda} F_0^2(s)A^{(1)}_b = 0.
\]

We already know from the discussion following Eq.(28) that the only regular, localised solution of Eq.(45) is the trivial one,

\[
A^{(1)}_b = 0.
\]

By the analogy with the case of domain walls [1] we expect consistency conditions for the first order solution coming from the second and third order
equations. These conditions have the form of equations which must be obeyed by the functions \( C^l, C^3 \). Therefore, we have to consider the perturbative equations in the second and third orders. As for still higher orders, we expect that the corresponding consistency conditions can be "saturated" by new arbitrary functions, introduced in each order as the general solution of the homogeneous equation \( \hat{L}\Psi^{(n)} = 0 \). All such solutions have the form (44) with the functions \( C^l, C^3 \) replaced by a new set of functions in each order. They have to be added to a solution of the full perturbative equation in the given order to obtain the general solution in the n-th order. The full perturbative equations in the second and third orders are given below, see Eqs.(47), (52).

The equation in the second order has the following form

\[
\hat{L}\Psi^{(2)} = \begin{pmatrix}
  f^{(2)} \\
  f^{*\prime\prime}(2) \\
  a_k^{(2)}
\end{pmatrix},
\]

where

\[
f^{(2)} = 2\phi^{(0)}|\phi^{(1)}|^2 + (\phi^{(1)})^2\phi^{*(0)} + q^2A_{i}^{(1)}A_{i}^{(1)}\phi^{(0)}
-2iqA_{i}^{(1)}D_{i}^{(0)}\phi^{(1)} - iq(\partial_iA_{i}^{(1)})\phi^{(1)} + 2K_{a}^{ib}K_{b}^{ja}s_jD_{i}^{(0)}\phi^{(0)}
+ \frac{2}{\sqrt{-g}}(\partial_a + \epsilon^{id}s^l\omega_aD_{i}^{(0)})(\sqrt{-gg}g^{ab}\epsilon^{kr}s^r\omega_bD_{k}^{(0)}\phi^{(0)}),
\]

and

\[
a_k^{(2)} = -2iq(\phi^{*(1)}D_{k}^{(0)}\phi^{(1)} - \phi^{(1)}D_{k}^{(0)}\phi^{*(1)})
+2q^2A_{k}^{(1)}(\phi^{*(1)}\phi^{(0)} + \phi^{(1)}\phi^{*(0)}) + 2\lambda K_{a}^{ib}K_{b}^{ja}s_jF_{ik}^{(0)}
+ \frac{2\lambda}{\sqrt{-g}}(\partial_a + \epsilon^{id}s^l\omega_a\partial_i)(\sqrt{-gg}g^{ab}\epsilon^{tr}s^r\omega_bF_{ik}^{(0)}).
\]

We also have the following equation for \( A_{c}^{(2)} \) (obtained by expanding Eq.(15))

\[
\Delta A_{c}^{(2)} - \frac{4q^2}{\lambda}F_0^2(s)A_{c}^{(2)} = \partial_c(\partial_iA_{i}^{(2)})
- \frac{2q}{\lambda}(\phi^{*(0)}\partial_c\phi^{(2)} - \phi^{(0)}\partial_c\phi^{*(2)} + \phi^{*(1)}\partial_c\phi^{(1)} - \phi^{(1)}\partial_c\phi^{*(1)}).
\]

In the formulae for \( f^{(2)}, a_k^{(2)} \), and on the r.h.s. of Eq.(50) the zeroth order results and the Nambu-Goto equation (43) have already been taken into account.
From Eq.(47) one can in principle determine the second order correction $Ψ^{(2)}$. In practice we do not hope to obtain an explicit formulae for $Ψ^{(2)}$ – we do not even know the explicit form of the zeroth order functions $φ^{(0)}$, $A_k^{(0)}$ which are present in the operator $\tilde{L}$. On the other hand, the functions $φ^{(0)}$, $A_k^{(0)}$ are known numerically, and the coefficients and the r.h.s. in Eq.(47) are regular. Therefore, we expect that it will be possible to investigate the solutions of this equation by numerical methods. This task we will leave for a separate work. In the present paper we would like to focus on the description of the perturbative scheme and on the first order correction $Ψ^{(1)}$.

As explained in Section 3, the consistency conditions are obtained by integrating the both sides of the perturbative equation, i.e. Eq.(47) in the present case, multiplied by the zero modes $Θ_l$:

$$0 = \int ds_1 ds_2 [(D_l^{(0)} φ^{(0)})^* f^{(2)} + \text{c.c.} + F_{lk}^{(0)} a_k^{(2)}]$$

(c.c. stands for "complex conjugate"). Most contributions to the integral vanish because the integrands are odd functions of the variables $s^i$ and the integration range is from $-\infty$ to $\infty$. The non-vanishing terms on the r.h.s. lead to the following condition

$$0 = -d_0 q \epsilon^{lk} C^l(u^a) C^3(u^a),$$

where $l, k = 1, 2$, and $d_0$ is a non-vanishing constant given in the Appendix B.

We shall see that in general $C^l$ are not equal to zero. Therefore, we put

$$C^3 = 0. \tag{51}$$

Equation (50) does not yield any restrictions on the functions $C^l, C^3$ because the operator on the l.h.s. of it (the same as in Eq.(28)) does not have zero eigenvalues.

Further consistency conditions follow from the equations in the third order. These conditions have the form of non-linear, intercoupled, inhomogeneous wave equations for $C^l(u^a)$ regarded as fields on the world-sheet $\Sigma$ of the Nambu-Goto string. Unfortunately, now the calculations become rather cumbersome – one could hardly expect simple formulae in the third order approximation in the model with six coupled fields. The equation for the
third order correction $\Psi^{(3)}$ has the form

$$\hat{L}\Psi^{(3)} = \begin{pmatrix} \hat{f}^{(3)} \\ \hat{f}^{*^{(3)}} \\ a_k^{(3)} \end{pmatrix},$$

(52)

where

$$\hat{f}^{(3)} = \hat{f}^{(3)}_1 + \hat{f}^{(3)}_2, \quad a_k^{(3)} = a_{1k}^{(3)} + a_{2k}^{(3)}.$$  

Here $\hat{f}^{(3)}_1, a_{1k}^{(3)}$ ( $\hat{f}^{(3)}_2, a_{2k}^{(3)}$ ) denote the terms obtained by expanding the l.h.s.'s ( the r.h.s.'s ) of Eqs.(12), (14), respectively. We find that

$$f^{(3)}_1 = 2\phi^{(0)}(1)\phi^{(2)} + 2(\phi^{*^{(1)}}\phi^{(0)} + \phi^{*^{(0)}}\phi^{(1)})\phi^{(2)} + \phi^{(1)}|\phi^{(1)}|^2$$

$$-2i\gamma \hat{A}^{(1)}_i \hat{D}^{(0)}_i \phi^{(1)} - 2i\gamma \hat{A}^{(2)}_i (\hat{D}^{(0)}_i \phi^{(1)} + i\gamma \hat{A}^{(1)}_i \phi^{(0)})$$

$$-i\gamma (\partial_i \hat{A}^{(1)}_i)\phi^{(2)} - i\gamma (\partial_i \hat{A}^{(2)}_i)\phi^{(1)},$$

(53)

$$a_{1k}^{(3)} = -2i\gamma [\phi^{*^{(2)}} \hat{D}^{(0)}_k \phi^{(1)} - \phi^{(2)}(\hat{D}^{(0)}_k \phi^{(1)})^*]$$

$$+\phi^{*^{(1)}} \hat{D}^{(0)}_k \phi^{(2)} - \phi^{(1)}(\hat{D}^{(0)}_k \phi^{(2)})^*]$$

$$+4\gamma A^{(1)}_k (\phi^{*^{(2)}} \phi^{(0)} + \phi^{(2)}\phi^{*^{(0)}}) + 4\gamma A^{(2)}_k (\phi^{*^{(1)}} \phi^{(0)} + \phi^{(1)}\phi^{*^{(0)}}),$$

(54)

and

$$f^{(3)}_2 = +2K^{ib} K^{ja}_b s^j (\hat{D}^{(0)}_i \phi^{(1)} + i\gamma \hat{A}^{(1)}_i \phi^{(0)})$$

$$+2i\gamma e^s t s^r \omega^a c s^b g^{ab}(\hat{D}^{(0)}_k \phi^{(1)})\hat{A}^{(1)}_i$$

$$+\frac{2}{\sqrt{-g}}(\partial_a + e^s t s^\omega D^{(0)}_a) [\sqrt{-g} g^{ab}(\partial_b \phi^{(1)} + e^s t s^r \omega^b (\hat{D}^{(0)}_k \phi^{(1)} + i\gamma \hat{A}^{(1)}_i \phi^{(0)})])$$

$$-\frac{4\gamma}{\sqrt{-g}}(\partial_a + e^s t s^\omega D^{(0)}_a) (\sqrt{-g} \hat{D}^{(0)}_k \phi^{(1)} [\sqrt{-g} K^{pab} \omega_b s^p e^s t s^r D^{(0)}_k \phi^{(0)})],$$

(55)

$$a_{2k}^{(3)} = 2\lambda K^{ib}_a K^{ja}_b s^j F^{(1)}_{tk}$$

$$+\frac{2\lambda}{\sqrt{-g}}(\partial_a + e^s t s^\omega \partial_a) [\sqrt{-g} g^{ab}(F^{(1)}_{bk} + e^s t s^r \omega^b F^{(1)}_{tk})$$

$$-2\sqrt{2}\frac{\sqrt{-g} K^{pab} \omega_b s^p e^s t s^r F^{(0)}_{tk}}{\gamma}].$$

(56)
In these formulae we have taken into account the Nambu-Goto equation as well as the fact that $A_{b}^{(1)}$ vanishes.

In the third order we also have an equation for $A_{c}^{(3)}$. We will not present it here because it does not lead to consistency conditions – the operator on the l.h.s. of it is the same as in Eqs. (28), (45).

The next step is to integrate over $s^1, s^2$ the r.h.s. of Eq. (52) multiplied by the zero modes. At first look this might seem a difficult task because the second order corrections $\phi^{(2)}, A_k^{(2)}$ are given only implicitly by Eq. (47). The other functions, including the zero modes, are explicitly expressed by the A-N-O functions $F_0(s), H_0(s)$. Luckily, it turns out that the $1/M$ expansion has the special property that the troublesome integrals can be expressed by integrals involving only the zeroth and first order corrections, which are explicitly given in terms of the A-N-O functions. Derivation of this relation is based on the translational and U(1) gauge invariances of the model, and it strongly reminds derivations of Ward identities for Green functions in quantum field theory. In the quantum case, instead of the integration over $s^1, s^2$ there is a functional integral over fields. For this reason, we will call the identity presented below the classical Ward identity.

All the terms with $\phi^{(2)}, \phi^{*^{(2)}}, A_k^{(2)}$ present on the r.h.s. of Eq. (52) are obtained by expanding the l.h.s.’s of Eqs. (12), (14). They are collected as $f_{l}^{(3)}$ and $a_{lk}^{(3)}$. The crucial observation is that they follow from the expression

$$-\hat{L}(\Psi^{(0)} + \frac{1}{M} \Psi^{(1)})\Psi^{(2)}$$

by expanding it with respect to $1/M$ and keeping the term linear in $1/M$ (with $1/M$ dropped out). The operator $\hat{L}(\Psi^{(0)} + \frac{1}{M} \Psi^{(1)})$ is given by formula (35) with $\Psi^{(0)}$ replaced by $\Psi^{(0)} + \frac{1}{M} \Psi^{(1)}$. The integrals we are discussing have the form

$$\mathcal{I}_l \equiv -\int ds^1 ds^2 \Theta^l_{\Psi} \left. \frac{\partial \hat{L}(\Psi^{(0)} + \Psi^{(1)}/M)}{\partial (1/M)} \right|_{1/M=0} \Psi^{(2)},$$

where $l = 1, 2$ and $\Psi^{(1)}$ is given by formula (44) with $C^3 = 0$. In the Appendix A we prove that

$$\mathcal{I}_l = \int ds^1 ds^2 \mathcal{C}^l_{r} \left( \frac{D^{(0)}_{r} D^{(0)}_{l} \phi^{(0)}}{\partial \phi_{lk}} \right) \hat{L}(\Psi^{(0)}) \Psi^{(2)}. \quad (57)$$
The expression $\hat{L}(\Psi^{(0)})\Psi^{(2)}$ on the r.h.s. of the classical Ward identity (57) is eliminated with the help of Eq.(47). In this manner $\mathcal{I}_l$ is given by integrals involving the zeroth order solution and the first order corrections only.

Further steps in obtaining the consistency conditions are nothing more than laborious calculating of the integrals present in the initial form of these conditions which is

$$0 = \mathcal{I}_l + \int ds^1 ds^2 [(D_l^{(0)} \phi^{(0)})^* f^{(3)}_2 + \text{c.c.} + F_{lk}^{(0)} a_{2k}^{(3)}].$$

The result we have obtained has the form of the following equation for $C^l$

$$\Box^{(3)} C^l - 2 \omega_a g^{ab} \epsilon^{lk} \partial_b C^k - \omega_b \omega_b g^{ab} C^l - \frac{1}{\sqrt{-g}} \partial_a (\sqrt{-g} g^{ab} \omega_b) \epsilon^{lk} C^k$$

$$- K^a_k K^b_j C^j + \frac{d_2}{d_1} \omega_a g^{ab} (\epsilon^{lk} \partial_b C^k + \omega_b C^l) - \frac{d_3}{d_1} (C^k C^k) C^l = (58)$$

$$\frac{d_4}{d_1} [\epsilon^{lk} \frac{1}{\sqrt{-g}} \partial_a (\sqrt{-g} g^{ab} \omega_b) + \omega_a \omega_b K^{ab} + \frac{d_5}{d_1} \omega_a \omega_b K^{ab}],$$

where

$$\Box^{(3)} C^l = \frac{1}{\sqrt{-g}} \partial_a (\sqrt{-g} g^{ab} \partial_b C^l).$$

The constants $d_i$, $i=1,...5$, are listed in Appendix B. The fact that the coefficients in Eq.(58) are correlated is due to the implicit local SO(2) invariance mentioned below formula (29).

Let us summarize our results. The obtained to the first order in $1/M$ vortex solution has the following form (in the co-moving coordinates)

$$\phi = e^{in\theta} [F_0(s) + \frac{1}{M} C^l(u) (\frac{s^i}{s} F'_0(s) - i q \epsilon^{ik} \frac{s^k}{s} H_0(s) F_0(s))],$$

$$A_i = \epsilon^{ik} \frac{s^k}{s} (\frac{n}{qs} - H_0(s)) - \frac{1}{M} C^k(u) \epsilon^{ik} B_0(s),$$

$$A_b = 0,$$

where $C^k(u)$ obey Eq.(58) and $B_0 = H'_0 + H_0/s$. The problem how to fix initial data for equation (58) is discussed below.
From formulae (59),(60) one can find approximate position of the zeros of the Higgs and vector fields. Using the fact that for small $s^i$ 

$$\phi^{(0)} \simeq c_0(s^1 + is^2)^{|n|},$$

where $c_0$ is a constant, we find that the n-th order zero of $\phi^{(0)}$ at $s^1 = s^2 = 0$ is split by the first order correction into (n-1)-th order zero at $s^1 = s^2 = 0$ and a first order zero at

$$s_H^i \approx -\frac{n}{M}C^i.$$ (62)

For $|n| = 1$ only the zero (62) is present. Thus, the (n-1)-th zero lies on the string and therefore follows a Nambu-Goto type trajectory. The other zero is shifted from the Nambu-Goto trajectory if the $C^i$ functions do not vanish.

As for the zeros of the $A_k$ field (k=1,2), we find that the both components vanish at the point

$$s_A^i \approx \frac{2}{M}C^i.$$ (63)

which is in general different from the origin and $s_H^i$. This first order zero is not so important because its position can be shifted by the local U(1) gauge transformations dependent on $s^1, s^2$.

The final problem we have to deal with is to fix the initial data for Eq.(58) from which we calculate the functions $C^i$, and also for the Nambu-Goto equation (43) from which we find the functions $Y^\mu(u^a)$. The input is the initial data for the vortex fields in the laboratory frame coordinates. The string is only an auxiliary mathematical object. Therefore, we can choose its initial position and velocity in a convenient manner.

For $|n| = 1$ the most natural choice is that at the initial $\tau = \tau_0$ the position and velocity of the string coincide with the ones of the line of zeros of the Higgs field. This implies that at $\tau = \tau_0$

$$C^i = 0, \quad \partial_\tau C^i = 0.$$ (64)

The values of the $C^i$ functions at later time depend on the r.h.s. of Eq.(58). Only if it vanishes for all $\tau$ then $C^i$ remain equal to zero for all $\tau$. Notice that for non-vanishing of r.h.s. of Eq.(58) it is necessary that the torsion coefficients $\omega_a$ do not vanish identically.

For $|n| > 1$ the initial data can be fixed again by looking at the zeros of the initial Higgs field. In general, it is possible that the zeros are split.
already at the initial time and that they have nonvanishing relative velocity. For \( |n| \geq 2 \) the split zeros are of different order – the initial position and velocity of the string are determined from the \((n-1)\)-th order zero, while the initial values of the functions \( C^i, \partial_\tau C^i \) are fixed by the initial position and velocity of the line on which lie the first order zeros. For \( |n| = 2 \) we can take any of the two zeros to determine the initial data for the Nambu-Goto string. Then, the other one will give the initial data for \( C^i \). The two choices are equivalent because Eq.\( (58) \) is symmetric with respect to the change of sign of the functions \( C^i \).

In all cases the functions \( C^i(u^a) \) give deviations of the trajectory of the first order zero of the Higgs field from the Nambu-Goto trajectory.

## 5 Remarks

In the present paper we have adapted the Hilbert-Chapman-Enskog method to the problem of evolution of the vortex. The method, previously applied to domain walls \[\text{[1]}\], seems to work quite well also in the case of vortices, eventhough in the present case the calculations are more complicated.

There are important differences between the aproach proposed in our paper and the earlier applications of the expansion in the width to dynamics of vortices in Minkowski space-time, \[\text{[3, 4, 5, 6]}\]. Apart from using the Hilbert-Chapman-Enskog method in which the consistency conditions play the prominent role, also new is the fact that the co-moving coordinate system does not have to be tied to the zeros of the Higgs field.

Already the first order correction \( \Psi^{(1)} \) reveals interesting phenomena in the vortex dynamics, like the splitting of the zeros of the Higgs field, or the possibility that for the \( |n| = 1 \) vortex the zeros do not follow the Nambu-Goto type trajectory. It would be also interesting to know the second order correction, but it can be calculated probably only by means of numerical methods.

Detailed discussion of properties of the obtained vortex solution, as well as concrete examples of evolution of the vortex obtained by solving Eqs.\( (43), (58) \) we will present in a separate paper, because the present one already seems to be rather voluminous.

Let us end the present paper with a general remark that in our approach to the vortex dynamics there is a quite interesting interplay of the 4-dimensional
and 2-dimensional field theoretical models. First, the zeroth order solution as well as the zero modes are determined from Eqs.(19),(20) which can be regarded as equations of the 2-dimensional Euclidean Abelian Higgs model. Second, the functions $C^l(u)$ can be regarded as fields on the 2-dimensional Nambu-Goto manifold $\Sigma$. This non-trivial background manifold is to be determined from the Nambu-Goto equation (43). We expect that in higher orders new sets of functions on $\Sigma$ will also appear. In this manner, the original description of the vortex by the 4-dimensional fields $\Phi, A_\mu$ would be transformed into another description in terms of the Nambu-Goto string and the fields on its world-sheet.

APPENDIX A. The derivation of the classical Ward identity (57)

We start from the identity

$$0 = \int ds^1 ds^2 (\tilde{T}_a \Theta_k)^\dagger \hat{L}(\tilde{T}_a \Psi^{(0)}) \tilde{T}_a \Psi^{(2)}, \quad (A1)$$

valid for any $a$ independent of $s^1, s^2$. The operator $\tilde{T}_a$ is given by formula (41). The identity (A1) is just Eq.(38) written in terms of the translated and gauge transformed fields. It is valid for any $\Psi^{(2)}$ with the proviso that the integral is convergent, but here $\Psi^{(2)}$ is of course the second order correction. We take the particular $a$ equal to

$$a^l = \frac{C^l(u)}{M},$$

where $C^l$ are the functions appearing in the first order solution (44). Let us recall that $C^3 = 0$. Next, we expand in $1/M$:

$$\tilde{T}_a \Psi^{(0)} \cong \Psi^{(0)} + \frac{1}{M} \Psi^{(1)},$$

$$\tilde{T}_a \Theta_k \cong \Theta_k + \frac{1}{M} C^l \left[ \begin{array}{ccc} D_l^{(0)} & 0 & 0 \\ 0 & D_l^{(0)} & 0 \\ 0 & 0 & \frac{\partial}{\partial s^l} \end{array} \right] \Theta_k,$$
\[ \tilde{T}_a \Psi^{(2)} = \Psi^{(2)} + \frac{1}{M} \delta \Psi^{(2)} \]

(the precise form of \( \delta \Psi^{(2)} \) is not needed). Substituting these formulae on the r.h.s. of identity (A1) we obtain up the first order in \( 1/M \)

\[
0 = \int ds^1 ds^2 \left[ (\Theta_k)^{\dagger} \hat{L}(\Psi^{(0)}) \Psi^{(2)} + \frac{1}{M} (\Theta_k)^{\dagger} \hat{L}(\Psi^{(0)}) \delta \Psi^{(2)} \right] + \frac{1}{M} \int ds^1 ds^2 C \left[ \begin{bmatrix} D^{(0)}_l & 0 & 0 \\ 0 & D^{*(0)}_l & 0 \\ 0 & 0 & \frac{\partial}{\partial s} \end{bmatrix} \Theta_k \right] \hat{L}(\Psi^{(0)}) \Psi^{(2)}
\]

\[ - \frac{1}{M} I_k + \mathcal{O}(\frac{1}{M^2}). \]

The first integral on the r.h.s. vanishes because \( \Theta_k \) is the zero mode, and we obtain the identity (57).

**APPENDIX B. The constants appearing in Section 4**

In order to obtain the formulae listed below we have used Eqs.(23),(24), the boundary conditions (25),(26) and we have applied multiple integration by parts.

\[
d_0 = 2\pi \int_0^\infty ds s B_0 (F_0^2 - F_0^4 + F_0'^2 + q^2 H_0^2 F_0^2),
\]

\[
d_1 = \int_0^\infty ds s (\lambda B_0^2 + F_0'^2 + q^2 H_0^2 F_0^2),
\]

\[
d_2 = \frac{9}{2} \lambda \int_0^\infty ds s B_0^2,
\]

\[
d_3 = \frac{1}{4} \int_0^\infty ds \left[ \frac{3}{2} F_0'^4 + \left( \frac{3}{2} - \frac{4q^2}{\lambda} \right) qF_0 H_0 F_0^4 + \left( 1 - \frac{4q^2}{\lambda} \right) qF_0 H_0 F_0^2 \right] + (2qB_0)^2 (F_0'^2 + q^2 H_0^2 F_0^2) + \frac{2q^4}{\lambda} H_0^2 F_0^4 (1 - F_0^2) + \frac{2q^4}{\lambda} H_0 F_0^3 F_0' \left( \frac{4}{s} H_0 + B_0 \right),
\]

23
\[ d_4 = \sqrt{2} \int_0^\infty ds s^3 (\lambda B_0^2 + 2q^2 H_0^2 F_0^2), \]

\[ d_5 = d_4 - 3\sqrt{2}\lambda \int_0^\infty ds s^3 B_0^2. \]

References

[1] H. Arodz, preprint TPJU-2/95 (hep-th/9502018).

[2] See, e.g., N.G. van Kampen, Stochastic Processes in Physics and Chemistry. North-Holland Publ. Comp.; Amsterdam, 1987. Chapt.8, §7.

[3] D. Förster, Nucl. Phys. B81, 84 (1974).

[4] N. Turok and K. Maeda, Phys. Lett. B202, 376 (1988).

[5] R. Gregory, Phys. Lett. B206, 199 (1988).

[6] R. Gregory, D. Haws and D. Garfinkle, Phys. Rev. D42, 343 (1990).

[7] H. Arodz and A. L. Larsen, Phys. Rev. D49, 4154 (1994);

[8] A. A. Abrikosov, Zh. Eksp. Teor. Fiz. 32, 1442 (1957).

[9] H. B. Nielsen and P. Olesen, Nucl. Phys. B61, 45 (1973).

[10] See, e.g., A. Jaffe and C. Taubes, Vortices and Monopoles. Birkhauser; Boston, Basel, Stuttgart, 1980.

[11] H. Arodz, T. Dobrowolski and P. Wegrzyn, in preparation.

[12] R. Jackiw and N. Manton, Ann. Phys. (N.Y.) 127, 257 (1980).