Rotating Black Holes and Coriolis Effect

Xiaoning Wu,† Yi Yang,‡ Pei-Hung Yuan,* and Chia-Jui Chou†

1Institute of Mathematics, Academy of Mathematics and System Science, CAS, China
2Department of Electrophysics, National Chiao Tung University, Hsinchu, ROC
3Institute of Physics, National Chiao Tung University, Hsinchu, ROC

In this work, we consider the fluid/gravity correspondence for general rotating black holes. By using the Petrov-like boundary condition in near horizon limit, we study the correspondence between gravitational perturbation and fluid equation. We find that the dual fluid equation for rotating black holes contains a Coriolis force term, which is closely related to the angular velocity of the black hole horizon. This can be seen as a dual effect for the frame-dragging effect of rotating black hole under the holographic picture.

Introduction It is well-known that AdS/CFT correspondence is a great breakthrough on theoretical physics. This conjecture offers us a powerful tool to study properties of strongly coupled systems. An important application of this conjecture is the correspondence between gravity and fluid dynamics. Such correspondence was first observed by Policastro, Son and Starinets [1]. The main idea of the correspondence is that the infrared behaviour of the dual theories should be governed by hydrodynamics. Since the gravitational perturbation in bulk space-time could be identified with the perturbation of the dual field theory on the time-like boundary, there should be a natural relation between the long wavelength perturbed Einstein equation and the hydrodynamical equation. By considering long wavelength model of perturbation on black brane solutions, Son et. al. established such correspondence and calculated the associated shear viscosity of the dual fluid. During the last decade, this topic has attracted great attention of researchers. Many interesting fluid phenomenon have been realized holographically, e.g. turbulence [2] and Hall viscosity [3]. Since the gravity/fluid duality is a quite natural generalization of AdS/CFT correspondence, it is reasonable to believe that such correspondence should hold for general stationary black holes. Unfortunately, the original method in [1] requires the long wavelength condition, so that it can only be used to deal with black brane cases. Especially, the method can not be applied to rotating black holes. In 2011, Strominger and his colleagues proposed a new idea to realize the correspondence [4, 5]. They found that, by imposing a Petrov-like boundary condition in suitable circumstances, the Einstein equation reduces to the Navier-Stokes equation in one lower dimension. Mathematically, this method is much simpler than the original one and can be generalized to the cases of more general black holes. The fluid/gravity correspondence for general non-rotating black holes has been established in [6]. In this paper, we consider the case of general rotating black holes.

On the gravity side, rotating black holes have an important phenomenon, the frame-dragging effect [7]. Near horizon, the stationary observer will be forced to rotate with the black hole because of the distorted space-time geometry. In fact, such effect exists for any rotating massive objects and has been observed by the GPB experiment [8]. An interesting question is what the dual effect for the frame-dragging on the field theory side is. In this paper, we will study the gravity/fluid duality and focus on the physical effect on the dual fluid caused by the rotation of black holes. We discover that the dual fluid equation is an incompressible Navier-Stokes equation with Coriolis force. Our result implies that the holographic dual effect of frame-dragging is just the Coriolis force, at least at the hydrodynamical limit level.

In this paper, we first consider some basic properties of background black hole configurations. To describe a general stationary black hole, we use the theory of isolated horizon which is developed by Ashtekar and other authors [9]. Then we consider the fluid/gravity correspondence for rotating black holes by the Petrov-like boundary condition method. Finally, We conclude our results.

Asymptotic behaviour of metric near horizon In order to study the gravity/fluid correspondence in general cases, one needs to consider the properties of general stationary black hole configuration. To do this, Ashtekar’s isolated horizon theory is a suitable tool [9]. This theory was developed by Ashtekar and other authors for four dimensional at the end of last country. It can be shown that any stationary black hole horizon satisfies the definition of isolated horizon. Later, Lewandowski and his colleagues generalized it to general dimensions [10].

Definition 1 (Isolated Horizon in (p + 2)-dimensional space-time)

Let \((M, g)\) be a \((p + 2)\)-dim Einstein manifold with or without a cosmological constant. \(H\) is a \((p + 1)\)-dim null hypersurface in \(M\) and \(l\) is the null normal of \(H\). \(H\) is called an isolated horizon in \(M\) if

\((1)\) there exists an embedding \(S \times [0, 1] \rightarrow M\), \(H\) is the image of this map, \(S\) is a \(p\)-dimensional connected manifold and for every maximal null curve in \(H\) there
exists \( x \in S \) such that the curve is the image of \( x \times [0,1] \);   
(2). the expansion of \( l \) vanishes everywhere on \( \mathcal{H} \);  
(3). \( R_{\text{nl}}l^a|_{\mathcal{H}} = 0 \);  
(4). let \( \mathcal{D} \) denote the induced connection on \( \mathcal{H} \), \( [\mathcal{L}_l, \mathcal{D}] = 0 \) holds on \( \mathcal{H} \).

With the help of Killing equation, Rachardaury equation and Einstein equation, it is easy to show that any stationary black hole horizon satisfies above definition. In the rest part of this paper, we will focus on the gravitational perturbation near isolated horizon. As has been discussed in \([4]\), one can establish a Bondi-like coordinate \( \{t, r, x^i\} \) \( (i = 1, 2, \cdots, p) \) in a neighbourhood of isolated horizon. Further more, there is a special set of null tetrad \( \{l, n, E_I\} \) \( (I = 1, 2, \cdots, p) \), namely Bondi tetrad, which can be expressed by the Bondi-like coordinates as:

\[
\begin{align*}
    n &= \partial_r, \\
    l &= \partial_t + U \partial_r + X^i \partial_i, \\
    E_I &= W_I \partial_r + e_I^j \partial_j, & I, i = 1, 2, \cdots, p,
\end{align*}
\]

where \( (U, X^i, W_I, e_I^j) \) are functions of \( (t, r, x^i) \). As discussed in \([11]\), the Bondi gauge implies \( \hat{U} = X^i = \hat{W}_I = 0 \). According to Ref. \([4]\), all hatted quantities are the initial data of isolated horizon \( \mathcal{H} \). In Bondi coordinates, the general form of metric in the neighbourhood of horizon should be

\[
(g^\mu^\nu) = 
\begin{pmatrix}
0 & 1 & 0 \\
1 & 2U + W_I W_J X^i + W^I e_I^j & 0 \\
0 & X^j + W_I e_I^j & e_I^j e_I^j
\end{pmatrix},
\]

where the Cartan structure equations tell us that the behaviour of the unknown functions in metric are

\[
U = \hat{e} r + \frac{1}{2} (\hat{R}_{\text{nl}I} + 2 |\hat{\pi}|^2) r^2 + O(r^3),
\]

\[
W_I = -\hat{\pi}_I r + \frac{1}{2} (\hat{R}_{\text{nl}I} + 2 \hat{\theta}_{IJ} \hat{\pi}_J) r^2 + O(r^3),
\]

\[
X^i = -\hat{\pi}_I r + \frac{1}{2} (\hat{R}_{\text{nl}I} \hat{e}_I^j + 2 \hat{\theta}_{IK} \hat{e}_I^j) r^2 + O(r^3),
\]

\[
e_I^j = \hat{e}_I^j - \hat{\theta}_{IJ} \hat{e}_J^j r + \frac{1}{2} (\hat{R}_{\text{nl}I} \hat{e}_I^j + 2 \hat{\theta}_{IK} \hat{e}_I^j) r^2 + O(r^3),
\]

where \( \hat{e}_I^j \) is the tetrad on the section of horizon, \( \hat{\theta}_{IJ} := \langle E_I, \nabla J n \rangle \) and \( \hat{e} := \langle n, \nabla I n \rangle \) is the surface gravity of horizon, \( \pi_I := \langle E_I, \nabla I n \rangle \) is just the rotational 1-form potential in \([4]\), which is related to the angular momentum of horizon. \( \pi_I \neq 0 \) implies that the black hole is rotating. In previous works, all black holes considered are non-rotating \(^1\). The main concern of this paper is to study the effect induced by the non-zero \( \pi_I \).

Brown-York tensor of time-like boundary near horizon: When one studies the fluid/gravity correspondence by using Strominger’s Petrov-like boundary condition method \([4, 5]\), the asymptotic behaviour of extrinsic curvature of the time-like boundary is crucial. The first reason is that, based on the gauge/geometry dictionary, the radius of time-like boundary is related to the energy scale of the dual field theory on the boundary. So the radius approaching the horizon implies the low energy limit in the dual field theory\(^2\). The second reason is that the Brown-York tensor corresponds to the energy-momentum tensor of the dual field theory. It is well known that the dynamical equation of fluid comes from the conservation law of the energy-momentum tensor, so we need to know the asymptotic behaviour of Brown-York tensor. Such behaviour can be obtained by direct calculation based on the asymptotic results of metric in last section. We summarize our approach as follows: introduce a rescaling parameter \( \lambda \) and consider a time-like boundary \( r = r_c \) in the neighbourhood of the horizon, then define a new temporal coordinate \( \tau = 2 \varepsilon \lambda^2 t \) and \( r_c = 2 \varepsilon \lambda^2 \) as well. After taking \( \lambda \rightarrow 0 \) limitation, it turns out the near horizon and non-relativistic limit in the meanwhile.

\[
K^I = \frac{1}{2\lambda} + \beta \lambda + O(\lambda^3),
\]

\[
K^I = -\pi_I \lambda + \psi_I \lambda^3 + O(\lambda^5),
\]

\[
K^I = \xi_I^3 \lambda + O(\lambda^3),
\]

\[
K = \frac{1}{2\lambda} + (\beta + \xi) \lambda + O(\lambda^3),
\]

where

\[
\beta = - \frac{1}{4} (7 \hat{R}_{\text{nl}I} - 3 |\hat{\pi}|^2),
\]

\[
\psi_I = - \frac{1}{2} \nabla_i (\hat{R}_{\text{nl}I} - |\hat{\pi}|^2) - 4 \hat{\pi}^J \nabla[\hat{\pi}]^j
\]

\[
+ 2 \hat{\theta}^I_{JK} \hat{\pi}_J \hat{\pi}_K - \hat{\theta}^I_{JK} \hat{\pi}_J \hat{\pi}_K,
\]

\[
\xi_I^3 = - 2 \hat{\theta}^I_{JK} \nabla_j \hat{\pi}_K + 2 \hat{\pi}_J \hat{\pi}_K + 2 \hat{\theta}^I_{JK} \hat{\pi}_J \hat{\pi}_K,
\]

and \( \xi = \xi_I^3 \) also \( \nabla \) is induced derivative on the section of horizon. The asymptotic behaviour of associated Brown-York tensor are

\[
t^I = \xi_I^3 \lambda + O(\lambda^3),
\]

\[
\hat{t}_I^I = \hat{\pi}_I + O(\lambda^3),
\]

\[
t^I_r = O(\lambda^3),
\]

\[
t^I_j = \frac{1}{2\lambda} \delta^I_j + [(\beta + \xi) \delta^I_j - \xi^I_j] \lambda + O(\lambda^3),
\]

\[
t = \frac{1}{2} \lambda + [p (\beta + \xi)] \lambda + O(\lambda^3).
\]

\(^1\) When we finish this paper, there appears another paper \([13]\) discussing Kerr/fluid Duality.

\(^2\) Such limit also has been used to consider other topics about black hole which is related with AdS/CFT correspondence, such as Kerr/CFT correspondence \([14]\).
Petrov-like boundary condition and gravitational perturbation

Following gravity/liquid correspondence, one needs to consider the gravitational perturbation in bulk spacetime. A basic requirement for perturbation is to satisfy regular condition at horizon. In previous work, people solve the perturbation equation concretely to insure the regularity. For general isolated horizon, this method fails to work. One needs other method to insure the regularity of perturbation. Thanks for the results on initial-boundary value problem \[\text{[12]},\] one can insure such regularity by imposing suitable boundary condition. One of possible choice is Strominger’s Petrov-like boundary condition \[\text{[13,14]}\]. The Petrov-like boundary condition requires the perturbed Weyl curvature satisfy following condition on time-like boundary:

\[\hat{C}_{\mu
u} = 0,\]

where \(\xi\) is the out-pointed null normal of the time-like boundary. As the original paper by Strominger et. al., the perturbation is introduced in terms of Brown-York tensor:

\[t^a_b = t^{a(B)}_b + \sum_k t^{a(k)}_b \lambda^k,\]

where \(t^{a(B)}_b\) is the Brown-York tensor for back ground space-time, \(t^{a(k)}_b\) are gravitational perturbations and \(\lambda \ll 1\) is the perturbation parameter. Under such rescaling, the perturbed Petrov-like boundary condition implies that

\[t^a_j(1) = -2g^{ik}\hat{\nabla}_j(t^{a(1)} \hat{\pi})_k + 2t^{a(1)}(t^a_j + \hat{\pi}_j) + \frac{t^{a(1)}}{2}(\hat{\nabla}_i \lambda^j + \xi^j - \hat{R}^j_i + 2\frac{p^2 - 3}{p^2(p + 1)}\Lambda \dot{\lambda}^j);\]

Compare with the result of \[\text{[15]},\] it is clear that this equation reduce to the non-rotating condition if \(\hat{\pi}_I = 0\).

The dual Navier-Stokes equation

With preparation in previous sections, we are able to study the holographic dual of gravitational perturbation. The basic AdS/CFT dictionary tells us that the Brown-York tensor corresponds to the energy-momentum tensor of dual field theory. On the field theory side, the hydrodynamic limit of the conservation law of energy-momentum tensor should give the fluid equation. On the gravity side, the conservation equation of Brown-York tensor is just the Codazzi equation on the time-like boundary. Thus the hydrodynamic limit indeed corresponds to the near horizon limit. So what one has to consider the near horizon limit of the Codazzi equation, \(\hat{D}_a t^{a_B}_b = 0\), where \(\hat{D}\) is the induced derivative on time-like boundary. Since the inner geometry on time-like boundary is fixed, the perturbed Codazzi equation are

\[0 = \hat{D}_a t^{a(B)}_b + \hat{D}_a t^{a(1)}_b \lambda + O(\lambda^2).\]

Considering the \(\tau\) component of Codazzi equation, the first nontrivial equation is in the \(O(\lambda^{-1})\),

\[\hat{\nabla}_i (\hat{g}^ij t^a_j(1)) = 0.\]

For \(i\) components of Codazzi equation, under the near horizon limit, the first nontrivial equation is in the \(O(\lambda)\),

\[0 = \partial_t t^a_i(1) - 2\tau_j(1)\hat{\nabla}_j(t^a_i(1) + \hat{\pi}_j) - 2\tau^a_j(1)\hat{\nabla}_j(\xi^i - t^a_i(1)) + \hat{\nabla}_i \left(\beta + \xi - 4(\hat{R}_{\text{null}} - |\hat{\pi}|^2)\right).\]

Combined with Eq. (9) and Eq. (11) and used the concrete expression of \(\beta\) and \(\xi\) in Eq. (5), this equation becomes

\[0 = \partial_t t^a_i(1) + \hat{\nabla}_i (\tau^a_i) + 2\tau^a_j(1)\hat{\nabla}_j(\tau^a_i(1) + \hat{\pi}_j) - \hat{\nabla}_j(2\hat{R}_{\text{null}} + 4\hat{\nabla}_j(\hat{\pi})^2 - 5|\hat{\pi}|^2),\]

where \(\hat{R}^j_i\) is the Ricci curvature of the section metric \(\hat{g}_{ij}\).

Now let’s identify the geometric quantities with hydrodynamic quantities based on gauge/gravity dictionary. Since \(t^a_b\) corresponds to the energy-momentum tensor in the dual field theory, it should be identified with the fluid energy-momentum tensor under the hydrodynamic limit. Thus we take the standard identification \[\text{[15]},\]

\[t^a_i(1) \rightarrow \frac{1}{2}\pi_i v^a, \quad \frac{t^{a(1)}}{2} \rightarrow \frac{P}{2},\]

where \(P\) is the pressure and \(v_i\) is the velocity in the dual fluid. With this identification, Eq. (11) tells us that the dual fluid is incompressible, i.e. \(\hat{\nabla}_i v^a_i = 0\), and the fluid equation can be finally written as,

\[0 = \partial_v \pi_i + \hat{\nabla}_i P + v^a \hat{\nabla}_j v_a - \hat{\nabla}^a v_a - \hat{R}^a_i v_j - 4v^a \hat{\nabla}_j \hat{R}^a_i \hat{\pi}_j - 2\nabla^2 \hat{\pi}_i - 2\nabla_j \hat{R}^j_i - \frac{5}{\hat{R}_{\text{null}}}\hat{\nabla}_j \hat{\pi}_j - 4v^a \hat{\nabla}_j \hat{\pi}_j,\]

which can be realized as the forced incompressible Navier-Stokes equations. The first line in Eq. (15) are standard terms of Navier-Stokes equation in curved space-time. The second and third lines in Eq. (15) are total divergence of some quantities which is only dependent on back ground geometry and can be realized as external forces. The only unusual term is the fourth line in Eq. (15). An interesting recognizing is that this term has the form of Coriolis force. According to Eq. (19) and gauge/gravity dictionary, the vector \(\hat{\pi}_a\) is the velocity of the reference frame, and \(d\hat{\pi}_a\) is just the angular velocity. In order to see this, we consider the Gauss equation. Under near horizon limit, the perturbed Gauss equation gives

\[t^a_j(1) = -2g^{ij}t^a_i(1) - 2\hat{\pi}_I \hat{\pi}^j(1) - \xi + \hat{R}_i \hat{\pi}^j - \frac{1}{2}|\hat{\pi}|^2.\]
Based on the AdS/CFT dictionary, $t^\tau$ corresponds to the energy density of the dual fluid. Obviously, the first term can be recognized as the non-relativistic kinematic energy. This is agrees with that the Navier-Stokes equation describes the non-relativistic dynamics of fluid.

If the horizon section metric $\hat{g}_{ij}$ in a 5-dimension space-time is flat (based on characteristic initial value problem [11], such solution exists, at least locally.), we can identify that

$$\Omega = \hat{\nabla} \times \hat{\pi}.$$  \hspace{1cm} (17)

Then Eq.\((15)\) becomes

$$\partial_\tau v + v \cdot \hat{\nabla} v + \hat{\nabla} P - \hat{\nabla}^2 v + 2\Omega \times v + f = 0$$  \hspace{1cm} (18)

where the external force terms are:

$$f = -2\hat{\nabla}^2 \hat{\pi} - 2\hat{\nabla}(2\hat{C}^\text{nlm} + 4\hat{\nabla} \cdot \hat{\pi} - 5|\hat{\pi}|^2).$$  \hspace{1cm} (19)

Eq.\((18)\) is the standard incompressible Navier-Stokes equation in a flat 4-dim space-time with Coriolis force which is induced by the reference frame. From this correspondence one can see that $\hat{\pi}_a$ should be realized as the velocity of the reference frame. We thus show that the fluid/gravity correspondence can be established for general stationary black holes including rotation. Eq.\((15)\) is Navier-Stokes equation in a non-inertial frame.

**Conclusion** In this paper we studied the fluid/gravity correspondence for a general rotating black hole. We considered a rotating black hole with an isolated horizon, which is more general than an usual stationary horizon since only the geometry inside the horizon is required to be stationary in this case. Further calculation has shown that the fluid/gravity correspondence will be fail if one give up the isolated condition. We showed that the fluid/gravity correspondence can also be established for rotating black holes, especially Kerr black hole can be seen as a special case of our result. Further more, a more interesting result is that the dual fluid equation on rotating horizon contains a Coriolis force term. The associated angular velocity is determined by a rotational 1-form which is closely related to the angular velocity of horizon. Near a rotating black hole horizon there is a famous effect called frame-dragging [7], i.e. the stationary observer will be forced to rotating with the horizon. We thus proposed that the Coriolis effect should be the holographic dual of the frame-dragging effect in a rotating black hole. 

**Acknowledgments** This work is partially supported by the National Science Council (NSC 101-2112-M-009-005) and National Center for Theoretical Science, Taiwan. X. Wu is supported by the National Natural Science Foundation of China (Grant Nos. 11075206 and 11175245).

[1] G. Policastro, D. T. Son and A. O. Starinets, Phys. Rev. Lett. 87 (2001) 081601 ;
G. Policastro, D. T. Son and A. O. Starinets, JHEP 0209 (2002) 043.
[2] A. Adams, P.M. Chesler and H. Liu, Science 341 (2013) 368.
[3] O. Saremi and D. T. Son, J. High Energy Phys. 04 (2012) 091; H. Liu, H. Ooguri, B. Stoica and N. Yunes, Phys. Rev. Lett. 110 (2013) 211601.
[4] I. Bredberg, C. Keeler, V. Lysov and A. Strominger, JHEP 1103 (2011) 141.
[5] V. Lysov and A. Strominger, “From Petrov-Einstein to Navier-Stokes”, arXiv:1104.5502 [hep-th]
[6] Xiaoning Wu, Yi Ling, Yu Tian, Chengyong Zhang, Class. Quant. Grav. 30 (2013) 145012.
[7] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation*, W. H. Freeman and Company, 1973.
[8] C. W. F. Everitt et al. Phys. Rev. Lett. 106 (2011) 221101.
[9] A. Ashtekar and B. Krishnan, Living Rev. Relativity 7 (2004) 10; A. Ashtekar, C. Beetle and L. Lewandowski, Class. Quantum Grav. 19 (2002) 1195.
[10] M. Korzynski, J. Lewandowski and T. Pawlowski, Class. Quant. Grav. 22 (2005) 2001.
[11] H. Friedrich, Proc. Roy. Soc. Lond. A 378 (1981) 169-184, 401-421.
[12] H. Friedrich and G. Nagy, Commun. Math. Phys. 201 (1999) 619;
H. O. Kreiss, O. Reula, O. Sarbach and J. Winicour, Commun. Math. Phys., 289 (2009) 1099;
O. Sarbach and M. Tiglio, Living Rev. Relativity, 15 (2012) 9. [Online Article]: http://www.livingreviews.org/lrr-2012-9
[13] Ippei Fujisawa, Ryuichi Nakayama, “Kerr/Fluid Duality and Caustics of Null Geodesics on a Horizon”, arXiv : 1511.08002[hep-th].
[14] I. Bredberg, C. Keeler, V. Lysov and A. Strominger, Nucl. Phys. Proc. Suppl. 216 (2011) 194.