TASEP FLUCTUATIONS WITH SOFT-SHOCK INITIAL DATA

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Abstract. We consider the totally asymmetric simple exclusion process (TASEP) with soft-shock initial density, which is a step function increasing in the direction of flow and the step size chosen small to admit KPZ scaling.

We prove that the fluctuations of a particle at the macroscopic position of the shock converge to the maximum of two independent GOE Tracy-Widom random variables, establishing a conjecture of Ferrari and Nejjar. Furthermore, we show that the joint fluctuations of particles near the shock are described by the maximum of two lines with height shifts given by these two independent random variables. The microscopic position of the shock is then easily seen to be the difference of these two random variables.

Our proofs rely on determinantal formulae and a novel factorization of the associated kernels.

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1. Introduction

The continuous time totally asymmetric simple exclusion process (TASEP) is an interacting particle system on $\mathbb{Z}$. At time 0, there is a given initial configuration of particles such that every site of $\mathbb{Z}$ occupies at most one particle. The dynamics are as follows. A particle jumps randomly to its neighboring site to the right provided that it is empty. The jumps of a particle are independent of the others’ and performed with exponential waiting times having mean 1.

The particles are initially numbered by integers in increasing order from right to left. Denote by $X_t(n)$ the position of particle number $n$ at time $t$. The numbering of particles given the initial configuration $X_0(\cdot)$ is set such that

$$\cdots < X_0(3) < X_0(2) < X_0(1) < 0 \leq X_0(0) < X_0(-1) < X_0(-2) < \cdots.$$  

When there is a rightmost particle, $X_0(n) = +\infty$ for every $n$ after that particle, and similarly, $X_0(n) = -\infty$ for every $n$ preceding the leftmost particle. So for example, if the initial configuration occupies all sites at the negative integers then $X_0(1) = -1, X_0(2) = -2, X_0(3) = -3$ and so on, while $X_0(0) = X_0(-1) = \cdots = +\infty$.

The book [16] provides a detailed construction and basic features of the model. TASEP can also be seen as a randomly growing 1-dimensional interface whose gradient is the particle density. It belongs to the Kardar-Parisi-Zhang (KPZ) universality class. The surveys [5, 21] discuss the relationship of TASEP to KPZ.

Despite its simplicity, TASEP displays many of the interesting behaviour of non-equilibrium statistical mechanics. Consider a deterministic initial configuration such that the macroscopic particle density is $\rho_-$ to the left of the origin and $\rho_+$ to the right:

$$\rho_\pm = \lim_{t\to\infty} \frac{\# \{ \text{particles in the interval } [0, \pm t] \text{ at time } 0 \}}{t}.$$  

For instance, particles may be arranged periodically in large enough blocks to attain such a profile. On the macroscopic scale the evolution of the particle density is a solution to Burgers’ equation [24, 26]. More precisely, the limit

$$u(T, x) = \lim_{t \to \infty} \Pr \{ \text{there is a particle at site } \lfloor xt \rfloor \text{ at time } Tt \}$$

exists and is an entropic solution of Burgers’ equation:

$$(1.2) \quad \partial_t u + \partial_x (u(1-u)) = 0, \quad u(0, x) = \rho_- 1_{\{x<0\}} + \rho_+ 1_{\{x>0\}}.$$  

When $\rho_- < \rho_+$, there is a traffic jam in the system because particles to the left of the origin, moving at macroscopic speed $1 - \rho_-$, run into particles to the right of the origin moving at a slower speed of $1 - \rho_+$. In this case the relevant solution of
(1.2) is given by the travelling front
\[ u(T, x) = u(0, x - \nu T), \quad \text{where} \quad \nu = 1 - \rho_+ - \rho_. \]
The number \( \nu \) is the speed of the traffic jam. This is the shock in Burgers’ equation. It is of much interest to study the microscopic features of the shock, ergo, the fluctuations of TASEP with an initial particle configuration as above.

**Figure 1.** The travelling shock-front \( u(T, x) \) that solves (1.2).

A proxy for the location of the shock is the particle at macroscopic position \( \nu t \).

For large times, the number\(^1\) of said particle is
\[ n_{t}^{\text{shk}} = (\rho_- - \rho_+) t. \]
Its position fluctuates randomly to the order \( t^{1/3} \) and so one would like to calculate, for every \( a \in \mathbb{R} \),
\[
\lim_{t \to \infty} \Pr \left[ X_t(n_{t}^{\text{shk}}) \geq \nu t - at^{1/3} \right].
\]

1.1. **The soft shock.** This paper considers a softening of the shock where the parameters \( \rho_{\pm} \) are scaled as
\[
\rho_{\pm} = \frac{1 \pm \beta(t/2)^{-1/3}}{2}, \quad \beta \in \mathbb{R} \quad \text{and} \quad t \geq 2|\beta|^3.
\]
In the soft shock scenario, the TASEP is run until time \( t \) with the choice of \( \rho_{\pm} \) as in (1.3) and \( t \) being the parameter within \( \rho_{\pm} \). One then considers the fluctuation of \( X(n_{t}^{\text{shk}}) \) in the double limit of \( t \to \infty \) followed by \( \beta \to \infty \). It is governed by the GOE Tracy-Widom-squared law, as follows.

**Theorem 1.** Consider TASEP with a deterministic initial configuration of particles having macroscopic density as in (1.1) and \( \rho_{\pm} \) scaled as in (1.3). Then,
\[
\lim_{\beta \to \infty} \lim_{t \to \infty} \Pr \left[ X_t(n_{t}^{\text{shk}}) \geq -at^{1/3} \right] = F_1(2a)^2,
\]
where \( F_1 \) is the distribution function of the GOE Tracy-Widom law.

\(^1\)Rounding particle numbers to nearest integers is omitted throughout the paper.
The soft shock is introduced in [12] and Theorem 1 proves the conjecture therein as we explain in Section 1.6. The scaling (1.3) is an analogue of the one for the Baik-Ben Arous-Péché transition from random matrix theory within the context of last passage percolation; see [12]. The large time limit of the soft shock gives processes with interesting features as we remark below.

An advantage of the soft shock is that it allows one to describe the limiting law of \( X_t(n_{t}^{shk}) \) in the first limit transition, \( t \to \infty \), in terms of explicit determinantal formulae. This is then used to study the second limit transition of \( \beta \to \infty \) and to understand the mechanisms of shock fluctuations. This is how Theorem 1 will be proved: first, by deriving the large \( t \) limit of the joint fluctuations of particles that are near the shock and, second, by showing that the large \( \beta \) limit of the resulting stochastic process becomes an appropriately combined maximum of two independent GOE Tracy-Widom random variables. This general result is stated in Theorem 3 below. Fluctuations of the hard shock are expected to be given by the same law. Also, the methods of this paper should generalize to prove GOE Tracy-Widom cubed, quadrupled and so on limiting laws at the merger of soft shocks when the initial density has two jumps, three jumps, etc.

1.2. Large time limit of the soft shock. In the case of soft shock the particle numbered \( n_{t}^{shk} \) has non-trivial correlations with other particles that are within a distance of order \( t^{2/3} \) of its position. The positions of these particles also fluctuate on a scale of order \( t^{1/3} \). As such, consider particles having numbers

\[
n(t, x) = n_{t}^{shk} - x(t/2)^{2/3} = \frac{t}{4} - \frac{\beta^2}{2} (t/2)^{1/3} - x(t/2)^{2/3},
\]

for \( x \in \mathbb{R} \), which at time \( t \) have macroscopic positions

\[
m(t, x) = \frac{x(t/2)^{2/3}}{\rho_-} = \frac{2x}{1 - \beta(t/2)^{-1/3} (t/2)^{2/3}}.
\]

The first limit transition derives the law in the large \( t \) limit of the process

\[
x \mapsto \frac{X_t(n(t, x)) - m(t, x)}{-(t/2)^{1/3}}.
\]

**Theorem 2.** Given real numbers \( x_1 < x_2 < \cdots < x_m \) and \( a_1, \ldots, a_m \), as \( t \to \infty \), the probability

\[
\Pr \left[ X_t(n(t, x_i)) \geq m(t, x_i) - a_i(t/2)^{1/3}, 1 \leq i \leq m \right]
\]

converges to the probability

\[
\Pr \left[ h(1, x_i; 2\beta|y|) \leq \beta^2 - 2\beta x_i + a_i, 1 \leq i \leq m \right],
\]
where \( h(1, x; 2\beta | y|) \) is a random function of the variable \( x \). The multi-point distribution functions of \( h(1, x; 2\beta | y|) \) are given in terms of Fredholm determinants:

\[
\Pr \left[ h(1, x_i; 2\beta | y|) \leq a_i, \ 1 \leq i \leq m \right] = \det \left( I - e^{-x_m \partial^2} K_\beta e^{x_m \partial^2} \chi_{a_1} e^{(x_2-x_1)\partial^2} \chi_{a_2} \cdots e^{(x_m-x_{m-1})\partial^2} \chi_{a_m} \right)_{L^2(\mathbb{R})},
\]

where \( \chi_{a}(u) = 1 \{ u \leq a \} \) is projection onto \( L^2(-\infty, a) \) and \( K_\beta \) is an explicit operator.

\( K_\beta \) is defined separately in Section 2 since its introduction requires crucial terminology and concepts.

Complicated though the determinant in Theorem 2 may appear, observe the one-point distribution functions of \( h(1, x; 2\beta | y|) \) are given by the Fredholm determinant of operators \( e^{-x\partial^2} K_\beta e^{x\partial^2} \) over the spaces \( L^2(a, \infty) \). These will turn out simpler and play a crucial role in the proofs.

The reason we call the limit process \( h(1, x; 2\beta | y|) \) is that it is the height function at time 1 of the KPZ fixed point with initial data \( h_0(y) = 2\beta | y| \), as introduced in [17]. The KPZ fixed point refers to the asymptotic scaling invariant Markov process for the KPZ universality class, starting from general initial data. Although the KPZ fixed point motivates our paper to an extent, the kernels in this case were actually known previously in [12], and so the results used from [17] are somewhat auxiliary.

1.3. Transitioning into the shock. The main result of the paper is the large \( \beta \) limit law of the process \( h(1, x; 2\beta | y|) \).

**Theorem 3.** As \( \beta \to \infty \), the process

\[
x \mapsto h(1, (2\beta)^{-1} x; 2\beta | y|) - \beta^2
\]

converges in the sense of finite dimensional laws to the process

\[
(1.5) \quad x \mapsto \max \{ 2^{-2/3} X_{TW_1} - x, 2^{-2/3} X'_{TW_1} + x \},
\]

where \( X_{TW_1} \) and \( X'_{TW_1} \) are two independent GOE Tracy-Widom random variables.

In other words, as \( \beta \to \infty \),

\[
\Pr \left[ h(1, (2\beta)^{-1} x_i; 2\beta | y|) \leq a_i + \beta^2, \ 1 \leq i \leq m \right] \to \prod_{\ell=1,2} F_1 \left( 2^{2/3} \min \{ a_i + (-1)^\ell x_i \} \right).
\]

Stated in terms of the TASEP soft shock, Theorem 3 asserts that in the double limit of \( t \to \infty \) followed by \( \beta \to \infty \) the process

\[
x \mapsto \frac{X_t(\{n(t, (2\beta)^{-1} x)\}) - \beta^{-1} x(t/2)^{2/3}}{(t/2)^{1/3}}
\]

where \( X_t(\{n(t, (2\beta)^{-1} x)\}) \) is a random function of the variable \( x \). The multi-point distribution functions of \( h(1, x; 2\beta | y|) \) are given in terms of Fredholm determinants:
converges in law to the process (1.5). Process (1.5) may be thought of as the
asymptotic “shock process” of TASEP with initial density (1.1).

\[
\text{Flat region: Airy}_1 \text{ joint fluctuations on scales of } t^{1/3} \text{ for height and } t^{2/3} \text{ for space.}
\]

\[
\text{Shock region: Joint fluctuations given by the process (1.5) on scales of } t^{1/3} \text{ for both height and space.}
\]

Figure 2. Fluctuations of TASEP that arise from initial density (1.1) when \( \rho_\pm \) are given by (1.3).

**Remark on position of the shock.** The process (1.5) can be expressed as
\[
|X - X_T W| + Y,
\]
where \( X = (X_{TW} - X'_{TW})/2^{5/3} \) and \( Y = (X_{TW} + X'_{TW})/2^{5/3} \). The *microscopic position* of the shock is then at the minimizer of this function, which is
\[
X = (X_{TW} - X'_{TW})/2^{5/3}.
\]

1.4. **Remarks on the soft shock process.** “Soft shock” is bit of a misnomer since the shock manifests for large values of \( \beta \) whereas the process in Theorem 2 has interesting features for negative values of \( \beta \) as well. What we have is a family of processes interpolating from the Airy\(_2\) process at \( \beta = -\infty \) to the process (1.5) at \( \beta = +\infty \). This becomes easy to see from the framework of the aforementioned KPZ fixed point as the mapping from initial data \( h_0 \mapsto h(1, x; h_0) \) is continuous, so long as \( h_0 \) is upper semicontinuous with values in \([-\infty, \infty)\) and bounded from above by a linear function.

When \( \beta = 0 \), \( h(1, x; 0) \) is the Airy\(_1\) process corresponding to flat initial data \( h_0 \equiv 0 \). As \( \beta \to -\infty \), the function \( 2\beta|y| \) converges to \( -\infty \mathbf{1}_{\{y \neq 0\}} \), which is called the narrow wedge \((-\infty \times 0 = 0\))

Consequently, \( h(1, x; 2\beta|y|) \) converges to \( h(1, x; \text{narrow wedge}) \), which has the law of the Airy\(_2\) process minus a parabola. Its distribution at the point \( x = 0 \) is the GUE Tracy-Widom law.

The soft shock also interpolates between two Airy\(_1\) processes at \( x = -\infty \) and \( x = \infty \). Indeed, affine and translation symmetries of the KPZ fixed point [17, Prop. 4.7] imply that for constants \( c \) and \( u \),
\[
h(1, x + u; h_0(y)) - c(x + u) - \frac{c^2}{4} \overset{\text{law}}{=} h(1, x; h_0(y + u + \frac{c}{2}) - c(y + u + \frac{c}{2})).
\]
Thus, \( h(1, x \pm L; 2\beta|y| - \beta^2 \mp 2\beta(x \pm L) \) has the same law as \( h(1, x; 4\beta(x \pm (\beta + L)) \) \). The latter processes converge to \( h(1, x; 0) \) as \( L \to \infty \).
One also obtains the Airy$_{2 \rightarrow 1}$ process which is the law of $h(1, x; -\infty \cdot 1_{\{y<0\}})$. The initial data is called half-flat. Indeed, $h(1, x + \beta; 2\beta|y|) - \beta^2 + 2\beta(x + \beta)$ has the law of $h(1, x; 4\beta y_-)$ and $4\beta y_-$ converges to the half-flat function as $\beta \to -\infty$.

1.5. **An overview of the proof.** It is well known that the correlation functions of TASEP, which provide the probability of particles being at specific sites, are determinantal; see for instance [3, 5] and references therein. Our proofs rely on such determinantal formulae.

Let us summarize how the GOE TW-squared law arises in Theorem 1. The operator $K_\beta$ associated to the law of $h(1, 0; 2\beta|y|)$ can be approximately factorized in the form

$$I - K_\beta = (I - M_\beta K_0 M_\beta^{-1})(I - M_\beta^{-1} K_0 M_\beta) + \text{Err}_\beta.$$ 

Err$_\beta$ is an error term that is vanishingly small in the appropriate trace norm as $\beta \to \infty$. This effectively allows one to consider the Fredholm determinant of the product. This approximately becomes the product of determinants over the stipulated $L^2$ space. The conjugations by $M_\beta$ can then be removed. This results in the GOE TW-squared law in the large $\beta$ limit.

Observe that if one conjugates away $M_\beta$ from one of the factors in the above representation then the other factor is conjugated by $M_\beta^2$, and the resulting operator, $M_\beta^{\pm 2} K_0 M_\beta^{\mp 2}$, does not in fact converge as $\beta \to \infty$. This was a challenge faced by previous attempts.

1.6. **Review of literature.** The study of the TASEP shock has a rich history and the reader may find nice discussions in [8, 9] and the references therein. We provide an overview of prior works most directly related to this paper.

In [4] the authors find determinantal formulae for TASEP with particles having varying speeds, which allows them to study shock fluctuations with Bernoulli-random initial data. The fluctuations there are Gaussian to the order of $t^{1/2}$. The paper [1] has related results for Bernoulli initial data. Deterministic shock-like initial data is studied in [11, 13] by connecting TASEP to last passage percolation. The authors prove that shock fluctuations for various setups are governed by the maximum of various Tracy-Widom random variables, although they are unable to treat the basic case of the step initial density [1, 1].

The soft shock is introduced in [12] in a setup where the particles move at two different speeds instead of being spread with the two densities $\rho_{\pm}$. The authors prove the analogue of our Theorem 2. They present determinantal kernels for the multi-point distributions in terms of contour integrals, and one may verify that their kernel matches ours. They also conjecture our Theorem 1. A beautiful illustration
of the convergence of \(X_t(n_{shk})\) to the GOE TW-squared law is shown in [12, Figure 1]. The paper [18] also considers a scenario like the soft shock but with narrow-wedge-like initial data.

Finally, [10] proves that the asymptotic position of the second class particle with the step initial density (1.1) is the difference of two independent GOE Tracy-Widom random variables. One should think of the second class particle as a random walk in the potential well given by the TASEP height process, and, as expected, it sits at the minimum of the process (1.5), which is \((X_{TW_1} - X'_{TW_1})/2^{5/3}\).

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2. The soft shock operator

The operator \(K_\beta\) associated to the soft shock is defined in terms of operators

\[
S_x = \exp\{x\partial^2 + \frac{1}{3}\partial^3\}, \quad \text{for} \quad x \in \mathbb{R},
\]

acting on functions \(f \in L^2(a, \infty)\), for any fixed \(a > -\infty\). Note that the operator \(\exp\{x\partial^2\}\), which corresponds to the heat kernel, is ill-defined for \(x < 0\) but \(S_x\) is well-defined due to the presence of the third derivative operator. In terms of integral kernels,

\[
S_x(u, v) = e^{\frac{2}{3}x^3 + x(v-u)}\text{Ai}(v - u + x^2),
\]

where \(\text{Ai}(z)\) is the Airy function defined as

\[
\text{Ai}(z) = \frac{1}{2\pi i} \oint \text{d}we^{\frac{3}{2}w^3 - zw},
\]

and \(\oint\) is a contour consisting of two rays going from \(e^{-i\pi/3}\infty\) to \(e^{i\pi/3}\infty\) through 0. The operator \(S_0\) will often be denoted \(S\). We will frequently use the fact that

\[
S^*S = SS^* = I.
\]

We now introduce the important hitting operator. Let \(B(y)\), for \(y \geq 0\), denote a Brownian motion with diffusion coefficient 2. Let \(h : [0, \infty) \to [-\infty, \infty)\) be upper semicontinuous with at most linear growth in the sense that \(h(y) \leq C(1 + |y|)\) for some constant \(C\). Let

\[
\tau = \inf\{y \geq 0 : B(y) \leq h(y)\}.
\]
Define the operator $S_x^{\text{hypo}(h)}$ in terms of its integral kernel as
\begin{equation}
S_x^{\text{hypo}(h)}(u,v) = \mathbb{E} \left[ S_{x-\tau}(B(\tau),v)1_{\tau<\infty} \mid B(0) = u \right].
\end{equation}

If $u \leq h(0)$ then $S_x^{\text{hypo}(h)} = S_x$. If $h$ is continuous and $u > h(0)$ then
\[ S_x^{\text{hypo}(h)}(u,v) = \mathbb{E} \left[ S_{x-\tau}(h(\tau),v)1_{\tau<\infty} \mid B(0) = u \right]. \]

Consider also the projection operators onto $L^2(a,\infty)$ and $L^2(-\infty,a)$, respectively:
\begin{equation}
\chi_a(u,v) = 1_{\{ u=v, u>a \}} \quad \text{and} \quad \bar{\chi}_a = 1 - \chi_a.
\end{equation}

The hitting operator is defined as follows. Consider an upper semicontinuous function $h : \mathbb{R} \to [-\infty, \infty)$ that has at most linear growth. The hitting operator associated to $h$ requires choosing a split point $x \in \mathbb{R}$. Then consider the functions
\[ h_x^\pm(y) = h(x \pm y) \quad \text{for} \quad y \geq 0. \]

The hitting operator is
\begin{equation}
K^{\text{hypo}(h)} = I - \left( S_x - S_x^{\text{hypo}(h_x)} \right)^* \chi_{h(x)} \left( S_x - S_x^{\text{hypo}(h_x^+)} \right).
\end{equation}

It is a crucial property of the hitting operator that it does not depend on the choice of split point $x$ (see [23] for a proof).

The operator $K_\beta$ is the hitting operator associated to $h_\beta(y) = 2\beta |y|$. It is natural (and crucial for the large $\beta$ asymptotics) to take the split point at $x = 0$, which utilizes the fact that $h_\beta$ has different slopes on the two sides of the split point. So denoting $h_\beta^+(y) = 2\beta y$ for $y \geq 0$,
\begin{equation}
I - K_\beta = \left( S - S^{\text{hypo}(h_\beta^+)} \right)^* \chi_{0} \left( S - S^{\text{hypo}(h_\beta^+)} \right).
\end{equation}

Since $S^* S = I$, $K^{\text{hypo}(h_\beta)}$ can be expressed as
\[ S^* \bar{\chi}_0 S + S^* \chi_0 S^{\text{hypo}(h_\beta^+)} + (S^{\text{hypo}(h_\beta^+)}* \chi_0 S) - (S^{\text{hypo}(h_\beta^+)}* \chi_0 S^{\text{hypo}(h_\beta^+)}). \]

Each of these terms have a presence of the operator $\exp\{ \pm \partial^3/3 \}$ on both sides. This ensures that operator $\exp\{ x \partial^2 \}$ can be applied legally around $K^{\text{hypo}(h_\beta)}$ for every $x \in \mathbb{R}$, and so the operator inside the determinant from the statement of Theorem 2 is well-defined.

For general initial data $h_0$, the multi-point distribution functions of $h(1,x;h_0)$ are given as follows. Given reals $x_1 < \ldots < x_m$ and $a_1, \ldots, a_m$,
\begin{equation}
\Pr[h(1,x;h_0) \leq a_i; \; 1 \leq i \leq m] = \det \left( I - e^{-x_m \partial^2} K^{\text{hypo}(h_0)} e^{x_m \partial^2} \left( I - e^{(x_1-x_m)\partial^2} \bar{\chi}_{a_1} e^{(x_2-x_1)\partial^2} \bar{\chi}_{a_2} \ldots e^{(x_m-x_{m-1})\partial^2} \bar{\chi}_{a_m} \right) \right)_{L^2(\mathbb{R})}. 
\end{equation}
The determinantal expression for the multi-point distribution function is the 'path integral' version from [17]. There is an alternative 'extended kernel' version. The hitting operator is also introduced in [23] in a modified form and precursors appear in [2, 6, 20, 22].

3. First limit transition: proof of Theorem 2

Let us introduce a parameter $\varepsilon > 0$ and write

$$t = 2\varepsilon^{-3/2}.$$ 

Then $m(t, x) = 2x\varepsilon^{-1} + 2\beta x \varepsilon^{-1/2} + O(\beta^2)$. The events of interest are

$$X_{2\varepsilon^{-3/2}} \left( \frac{1}{2} \varepsilon^{-3/2} - x \varepsilon^{-1} - \frac{\beta^2}{2} \varepsilon^{-1/2} \right) \geq 2x\varepsilon^{-1} + (2\beta x - a)\varepsilon^{-1/2} + O(\beta^2).$$

In order to prove Theorem 2 one must derive the limiting joint probabilities of such events as $\varepsilon \to 0$. Upon replacing $x$ with $x - (\beta^2/2)\varepsilon^{1/2}$ the event becomes

$$X_{2\varepsilon^{-3/2}} \left( \frac{1}{2} \varepsilon^{-3/2} - x \varepsilon^{-1} \right) \geq 2x\varepsilon^{-1} - (\beta^2 - 2\beta x + a)\varepsilon^{-1/2} + O(\beta^2).$$

We may express the event (3.1) in terms of the height function of TASEP. For TASEP with initial data $X_0$, let

$$X_t^{-1}(u) = \min \{ n \in \mathbb{Z} : X_t(n) \leq u \}.$$ 

The height function $h_t : \mathbb{Z} \to \mathbb{Z}$ at time $t$ is

$$h_t(z) = -2(X_t^{-1}(z - 1) - X_0^{-1}(-1)) - z.$$ 

The KPZ-rescaled height function is

$$h^{\varepsilon}(T, x) = \varepsilon^{1/2} \left[ h_{2T\varepsilon^{-3/2}}(2x\varepsilon^{-1}) + T\varepsilon^{-3/2} \right].$$ 

In terms of the KPZ-rescaled height function one has

$$h^{\varepsilon}(T, x) \leq a \iff X_{2T\varepsilon^{-3/2}} \left( \frac{T}{2} \varepsilon^{-3/2} - x \varepsilon^{-1} \right) \geq 2x\varepsilon^{-1} - a\varepsilon^{-1/2} + X_0(1).$$

In the $\varepsilon \to 0$ limit the probability of the event in (3.1) remains unaffected if the term $O(\beta^2)$ is ignored. Thus, one must show that the limiting multi-point probabilities

$$\lim_{\varepsilon \to 0} \Pr \left[ h^{\varepsilon}(1, x_i) \leq \beta^2 - 2\beta x_i + a_i, \ 1 \leq i \leq k \right]$$

are given by the formula from Theorem 2 (It is easy to see that $h^{\varepsilon}(0, y)$ converges uniformly to $h_0(y) = 2\beta |y|$.)

Here there are several approaches. In [12], a determinantal formula is derived for these multi-point probabilities for the soft-shock data in a related setup, where particles to the left of the origin have a different speed than those to the right. Using
their formula, it is not difficult to guess a determinantal formula for our setup and then check it using the the bi-orthogonalization procedure from [3, 25]. On the other hand, [17, Theorem 2.6] provides a formula for any initial data with a rightmost particle. One can cutoff the soft-shock initial data far to the right and take a limit as the cutoff is removed to get a determinantal formula for the multi-point probabilities which coincides with the guess. Then by direct asymptotic analysis of the associated determinantal kernels one arrives at Theorem 2. Since the limiting kernel is the same as from [12], we omit the details.

4. Second limit transition: proofs of Theorem 1 and 3

The proof of Theorem 1 is presented first in Section 4.1 followed by the proof of Theorem 3 in Section 4.2 as the latter builds on the former. We first define the GOE Tracy-Widom law, introduced in [28], in a suitable form. For the remainder of the paper it is assumed that \( \beta \geq 0 \).

**The GOE Tracy-Widom law.** The distribution function of the GOE Tracy-Widom law may be written as a Fredholm determinant [14]. Consider the operator \( A \) with integral kernel
\[
A(u,v) = 2^{-1/3} \text{Ai}(2^{-1/3}(u + v)).
\]
If \( R \) is the reflection operator:
\[
Rf(x) = f(-x),
\]
then \( A \) may be expressed as
\[
A = RS^2 = S^*RS.
\]
The above representation uses that \( S^2 = e^{2\partial^3/3} \) and, as an integral kernel,
\[
\exp\left\{\frac{t}{3}\partial^3\right\}(u,v) = t^{-1/3} \text{Ai}(t^{-1/3}(v - u)) \quad \text{for } t > 0.
\]
The relation \( \partial R = -R\partial \) implies the second equality. It will turn out that \( A \) is the operator \( K_0 \). The GOE Tracy-Widom distribution function is
\[
F_1(2^{2/3}a) = \Pr\left[X_{TW_1} \leq 2^{2/3}a\right] = \det(I - \chi_a A \chi_a)_{L^2(\mathbb{R})}.
\]

4.1. **Proof of Theorem 1.** Let \( M_\beta \) denote the multiplication operator:
\[
M_\beta f(x) = e^{\beta x} f(x).
\]
Note also that the translation operator \( f \mapsto f(x + \lambda) \) is given by \( e^{\lambda \partial} \).

**Lemma 4.1.** The following commutation relations hold between \( M_\beta, S, R \) and the translation operator.

1. \( M_\beta S = \exp\left\{\frac{1}{3}(\partial - \beta)^3\right\} M_\beta, \)
2. \( M_\beta \exp\{\lambda \partial^2\} = \exp\{\lambda(\partial - \beta)^2\} M_\beta, \)
(3) $M_{\beta} \exp\{\lambda \partial\} = \exp\{\lambda (\partial - \beta)\} M_{\beta},$

(4) $M_{\beta} R = RM_{-\beta}$ and $\exp\{\lambda \partial\} S = S \exp\{\lambda \partial\}.$

Proof. Relations (1) – (3) follow from the identity $\partial M_{\beta} = M_{\beta}(\partial + \beta).$ Relation (4) is clear. \hfill \Box

The following lemma is key to calculating the hitting operator associated to $h_{\beta}(y) = 2\beta|y|.$

Lemma 4.2 (Reflection lemma). Let $h_{\beta}^{+}(y) = 2\beta y$ for $y \geq 0$. Then the operator

$$S^{\text{hypo}(h_{\beta}^{+})} = \chi_0(M_{\beta} RM_{-\beta}) S + \chi_0 S.$$ 

Proof. Recall that $S^{\text{hypo}(h_{\beta}^{+})}(u, v) = S(u, v)$ if $u \leq h_{\beta}^{+}(0) = 0$. This contributes the term $\chi_0 S$. Now assume that $u > 0$ and let $\tau$ be the hitting time of a Brownian motion of diffusion coefficient 2, starting from $u$, to the hypograph of $h_{\beta}^{+}$.

Observe that $S_{-\tau}(2\beta t, v) = e^{-2t^{3/2} - t(v - 2\beta t)} \text{Ai}(v - 2\beta t + t^2)$. Recall that the Airy function has the following decay: there is a constant $C$ such that

$$|\text{Ai}(z)| \leq C \text{ if } z \leq 0 \text{ and } |\text{Ai}(z)| \leq Ce^{-\frac{4}{3}z^{3/2}} \text{ if } z > 0.$$ 

The above implies that $S_{-\tau}(2\beta t, v)$ decays sufficiently fast that one has

$$S^{\text{hypo}(h_{\beta}^{+})}(u, v) = \lim_{T \to \infty} \mathbb{E} \left[ S_{-\tau}(2\beta \tau, v) 1_{\{\tau \leq T\}} \mid B(0) = u \right].$$

For $t \leq T$, $S_{-\tau} = e^{(T-\tau)\partial^2} S_{-\tau}$ and one recognizes the integral kernel of $e^{(T-\tau)\partial^2}$ at the transition density of Brownian motion (with diffusion constant 2) to go from $B(t) = u$ to $B(T) = v$. As such, the strong Markov property implies

$$\mathbb{E} \left[ e^{(T-\tau)\partial^2} (2\beta \tau, v) \mid B(0) = u \right] = \Pr \left[ \tau \leq T, B(T) \in dv \mid B(0) = u \right] / dv,$$

where the expression on the right is the transition density of $B(0) = u$ to $B(T) = v$ while hitting the curve $h_{\beta}^{+}$. Denote this expression $S^\text{hit}_T(u, v)$. Thus,

$$\mathbb{E} \left[ S_{-\tau}(2\beta \tau, v) 1_{\{\tau \leq T\}} \mid B(0) = u \right] = S^\text{hit}_T \cdot S_{-T}(u, v).$$

Let $X(t) = B(t) - h_{\beta}^{+}(t)$, so that $S^\text{hit}_T$ is the transition density of $X$ to go from $X(0) = u$ to $X(T) = v - 2\beta T$ while hitting 0. By the Cameron-Martin Theorem, $X$ becomes Brownian motion with diffusion coefficient 2 on $[0, T]$, starting from $u$, after a change of measure by the density $\exp\{-\beta(B(T) - u) - \beta^2 T\}$. Consequently,

$$S^\text{hit}_T(u, v) = \mathbb{E} \left[ e^{-\beta(B(T)-u)-\beta^2 T} \cdot 1\{B \text{ hits 0 on } [0, T], B(T) \in d(v - 2\beta T)\} \mid B(0) = u \right] / dv$$

$$= e^{\beta(u-v) + \beta^2 T} \Pr \left[ B \text{ hits 0 on } [0, T], B(T) \in d(v - 2\beta T) \mid B(0) = u \right] / dv.$$
Since \( u > 0 \), if \( v - 2\beta T \leq 0 \) then the latter transition density is simply the transition density of \( B \) to go from \( B(0) = u \) to \( B(T) = v - 2\beta T \). If \( v - 2\beta T > 0 \), however, one reflects along the time axis the initial segment of \( B \) from time 0 till the hitting time to the point zero. The reflection principle then implies that the latter transition density is the transition density of \( B \) to go from \( B(0) = -u \) to \( B(T) = v - 2\beta T \). Therefore, for \( u > 0 \),

\[
e^{\beta(v-u)-2\beta T} S_{\text{hit}}^T(u,v) = e^{T\beta^2}(u,v - 2\beta T)\chi_{2\beta T}(v) + e^{T\beta^2}(-u,v - 2\beta T)\chi_{2\beta T}(v)
\]

\[
= e^{T\beta^2+2\beta T\partial} \cdot \chi_{2\beta T}(u,v) + R \cdot e^{T\beta^2+2\beta T\partial} \cdot \chi_{2\beta T}(u,v).
\]

Relation (2) of Lemma 4.1 gives \( e^{T(\partial+\bar{\beta})^2} M_{-\beta} = M_{-\beta} e^{T\partial^2} \). Consequently, writing \( \chi_{2\beta T} \) as \( 1 - \bar{x}_{2\beta T} \) and expressing everything in operator notation shows that

\[
\chi_0 S_{\text{hit}}^T = \chi_0 M_\beta R e^{T(\partial+\bar{\beta})^2} M_{-\beta} + \chi_0 M_\beta(I - R) e^{T(\partial+\bar{\beta})^2} \chi_{2\beta T} M_{-\beta}
\]

\[
= \chi_0 (M_\beta R M_{-\beta}) e^{T\beta^2} + \chi_0 M_\beta(I - R) M_{-\beta} e^{T\beta^2} \bar{x}_{2\beta T}.
\]

Multiplying by \( S_{-T} \) now gives

\[
\chi_0 S_{\text{hit}}^T \cdot S_{-T} = \chi_0 (M_\beta R M_{-\beta}) S + \chi_0 M_\beta(I - R) M_{-\beta} e^{T\beta^2} \bar{x}_{2\beta T} S_{-T}.
\]

The operators \( \chi_0 \) and \( M_{\pm \beta} \) are diagonal, \( R \) is an anti-diagonal, and none depend on \( T \). The lemma thus follows if for every choice of \( u \) and \( v \) the quantity

\[
e^{T\beta^2} \cdot \bar{x}_{2\beta T} \cdot S_{-T}(u,v) \rightarrow 0 \quad \text{as} \quad T \rightarrow \infty.
\]

Let \((I)\) denote this quantity. Using the integral kernels of \( e^{T\beta^2} \) and \( S_{-T} \) one infers that \((I)\) equals

\[
\int_{-\infty}^{0} dz \frac{1}{\sqrt{4\pi T}} \exp\{-\frac{(z+2\beta T - u)^2}{4T} - \frac{2}{3} T^3 + T(z+2\beta T - v)\} \text{Ai}(v-z-2\beta T + T^2).
\]

In order to evaluate \((I)\), write the Airy function in terms of its contour integral representation \((2.3)\) and switch the contour integration with the integration over variable \( z \) by Fubini. The integral over \( z \) is a Gaussian integral, which equals

\[
\int_{-\infty}^{0} dz \ e^{-\frac{(z+u)^2}{4T} + z(w+T-\beta)} = \sqrt{\frac{4\pi T}{w}} e^{w(u+T-\beta)+T(w+T-\beta)^2} \Phi(-\sqrt{2T}(w+T-\beta) - \frac{u}{\sqrt{2T}}).
\]

Here \( \Phi(w) = (2\pi)^{-1/2} \int_{-\infty}^{w} ds e^{-s^2/2} \), where \( w \) is a complex argument and the integral is over the horizontal contour \( s \mapsto w - s \), for \( s \geq 0 \), oriented from \(-\infty\) to \( w \). Substituting into the expression for \((I)\), simplifying, and changing variables \( w \mapsto w - T \) gives the following.

\[
I = \frac{1}{2\pi i} \int_{(+)T} dw \ e^{\frac{1}{2}w^3-(v-u)w} \Phi(-\sqrt{2T}(w-\beta) - \frac{u}{\sqrt{2T}}).
\]
The contour \( \langle +T \rangle \) may be shifted back to \( \langle \rangle \) without changing the integral. Then changing variables \( w \mapsto w + \beta \) and shifting the contour \( \langle - \beta \rangle \) back to \( \langle \rangle \) gives
\[
I = \frac{1}{2\pi i} \oint_{\langle \rangle} dw \, e^{\frac{i}{2}(w+\beta)^3 - (v-u)(w+\beta)} \Phi\left( -\sqrt{2T}w - \frac{u}{\sqrt{2T}} \right). 
\]

If the contour is arranged such that \( |\arg(w)| = \pi/5 \) then \( \Phi\left( -\sqrt{2T}w - \frac{u}{\sqrt{2T}} \right) \to 0 \) as \( T \to \infty \). This is because \( \Phi\left( -w \right) \to 0 \) as \( w \to \infty \) within the sector \( |\arg(w)| \leq \pi/4 - \varepsilon \) for any \( \varepsilon > 0 \) \([19, \text{Eq. 7.2.4}]\). Moreover, if \( |\arg(w)| \geq \pi/6 + \varepsilon \) along the contour then the exponential factor decays in modulus to the order \( \exp\{-\delta \Re(w)^3\} \) for some \( \delta > 0 \). Arranging the contour as such, the dominated convergence theorem implies that \( (I) \to 0 \) as \( T \to \infty \).

We may now observe that the hitting operator \( K_0 \) is simply the operator \( A \) associated to the GOE Tracy-Widom law. Employing the definition from (2.7), Lemma 4.2, and using the fact \( S^*S = I \), it follows that
\[
(4.5) \quad K_0 = I - (S - S^{\text{hypo}(h_0^+)})*\chi_0(S - S^{\text{hypo}(h_0^+)}) \\
= S^*[I - (I - R)\chi_0(I - R)]S.
\]
The relations \( R\chi_0 = \bar{\chi}_0R \) and \( R^2 = I \) imply
\[
I - (I - R)\chi_0(I - R) = \chi_0R + R\chi_0 = R,
\]
which establishes the claim.

Theorem 2 gives
\[
\Pr \left[ h(1, 0; 2\beta|y|) \leq \beta^2 + a \right] = \det (I - K_\beta)_{L^2(\beta^2+a, \infty)} \\
= \det \left( I - e^{\beta^2\partial}K_\beta e^{-\beta^2\partial} \right)_{L^2(a, \infty)}. 
\]

**Lemma 4.3** (Factorization lemma). The operator \( e^{\beta^2\partial}K_\beta e^{-\beta^2\partial} \) admits the following factored from.
\[
I - e^{\beta^2\partial}K_\beta e^{-\beta^2\partial} = (I - M_\beta(A + E_\beta)M_{-\beta})^* (I - M_\beta(A + E_\beta)M_{-\beta}),
\]
where \( E_\beta = S^*_\beta\bar{\chi}_0(I - R)S_\beta \) and \( A = S^*RS \).

**Proof.** Lemma 4.2 and the relation \( \chi_0 = \chi_0SS^*\chi_0 \) imply that
\[
(4.6) \quad I - K_\beta = [S^*\chi_0(S - \chi_0M_{2\beta}RS - \bar{\chi}_0S)]^* [S^*\chi_0(S - \chi_0M_{2\beta}RS - \bar{\chi}_0S)].
\]
Since $S^*S = I$ and $M_\beta$ commutes with the projection $\chi_0$, we may write
\[
S^*\chi_0(S - \chi_0 M_\beta R M_{-\beta} S - \bar{\chi}_0 S) = I - S^*\chi_0 S - S^*\chi_0 M_\beta R M_{-\beta} S
= I - S^*\chi_0 S + S^*\chi_0 M_\beta R M_{-\beta} S - S^* M_\beta R M_{-\beta} S
= I - S^* M_\beta R M_{-\beta} S - S^* M_\beta \bar{\chi}_0 (I - R) M_{-\beta} S.
\]

We now conjugate the above equation by the translation $e^{\beta \partial}$ and use relations (1) and (3) from Lemma 4.1 to bring $M_{\pm \beta}$ to the outside. The adjoint of relation (1) gives $S^* M_\beta = M_\beta \exp \left\{ - \frac{1}{3} (\beta + \partial)^3 \right\}$. Thus, for the term $S^* M_\beta R M_{-\beta} S$ one infers
\[
e^{\beta \partial} S^* M_\beta R M_{-\beta} S e^{-\beta \partial} = M_\beta \exp \left\{ \beta^2 (\partial + \beta) - \frac{1}{3} (\beta + \partial)^3 \right\} R \times
\times \exp \left\{ -\beta^2 (\partial + \beta) + \frac{1}{3} (\beta + \partial)^3 \right\} M_{-\beta}
= M_\beta \exp \left\{ -\beta \partial^2 - \frac{1}{3} \beta^3 \right\} R \exp \left\{ \beta \partial^2 + \frac{1}{3} \beta^3 \right\} M_{-\beta}
= M_\beta R \exp \left\{ -\beta \partial^2 + \frac{1}{3} \beta^3 + \beta \partial^2 + \frac{1}{3} \beta^3 \right\} M_{-\beta}
= M_\beta A M_\beta.
\]
The last equation used that $A = RS^2$. A key point above is that conjugation by the translation cancels the term involving $\partial$ in the expansion of $\pm \frac{1}{3} (\beta + \partial)^3$.

Analogously, one computes to see that
\[
e^{\beta \partial} S^* M_\beta \bar{\chi}_0 (I - R) M_{-\beta} S e^{-\beta \partial} = M_\beta \exp \left\{ -\beta \partial^2 - \frac{\partial^3}{3} \right\} \bar{\chi}_0 (I - R) \times
\times \exp \left\{ \beta \partial^2 + \frac{\partial^3}{3} \right\} M_{-\beta}
= M_\beta E_\beta M_{-\beta}.
\]

In conclusion,
\[
e^{\beta \partial} S^* \chi_0(S - \chi_0 M_\beta R S - \bar{\chi}_0 S) e^{-\beta \partial} = I - M_\beta (A + E_\beta) M_{-\beta}.
\]
The lemma follows from this relation and the expression [4.6] for $I - K_\beta$. \qed

Lemma 4.3 and (4.5) give
\[
\Pr \left[ h(1, 0) \leq \beta^2 + a \right] = \det \left( (I - M_\beta (A + E_\beta) M_{-\beta})^* (I - M_\beta (A + E_\beta) M_{-\beta}) \right)_{L^2(a, \infty)}.
\]

Decompose the product above in the form $(I - X)^* \chi_a (I - X) + (I - X)^* \bar{\chi}_a (I - X)$. The determinant of the first of these terms factorizes and upon conjugating out $M_\beta$
from each factor one gets
(4.7)
\[
\det \left( (I-M_{\beta}(A+E_{\beta})M_{-\beta})^* \chi_a (I-M_{\beta}(A+E_{\beta})M_{-\beta}) \right)_{L^2(a,\infty)} = \det \left( I-A-E_{\beta} \right)_{L^2(a,\infty)}^2.
\]

The proof of Theorem \[\square\] will be completed by showing that the second term in the decomposition, as well as the term \(E_{\beta}\), provide negligible error as \(\beta \to \infty\). This is the content of the following two lemmas. The argument makes use of some standard inequalities between the Fredholm determinant, trace norm, Hilbert-Schmidt norm and operator norm that may be found in the book [27].

**Lemma 4.4.** As \(\beta \to \infty\), \(\|E_{\beta}\|_{tr} \to 0\) on \(L^2(a,\infty)\). Consequently,
\[
\det \left( I - A - E_{\beta} \right)_{L^2(a,\infty)} \to \det \left( I - A \right)_{L^2(a,\infty)}.
\]

**Proof.** The representation of \(E_{\beta}\) over \(L^2(a,\infty)\) is \(\chi_a E_{\beta} \chi_a\). Therefore, \(\chi_a E_{\beta} \chi_a = E_{\beta}^1 - E_{\beta}^2\) with
\[
E_{\beta}^1 = \chi_a S^*_{-\beta} \tilde{\chi}_0 S_{\beta} \chi_a \quad \text{and} \quad E_{\beta}^2 = \chi_a S^*_{-\beta} \tilde{\chi}_0 R S_{\beta} \chi_a.
\]
If suffices to show that both of the operators above have vanishingly small trace norm as \(\beta \to \infty\).

Consider the operator \(E_{\beta}^1\). Employing the inequality \(\|T_1 T_2\|_{tr} \leq \|T_1\|_{HS} \|T_2\|_{HS}\) with \(T_1 = \chi_a S^*_{-\beta} \tilde{\chi}_0\) and \(T_2 = \tilde{\chi}_0 S_{\beta} \chi_a\) gives
\[
\|E_{\beta}^1\|_{tr} \leq \|\chi_a S^*_{-\beta} \tilde{\chi}_0\|_{HS} \|\tilde{\chi}_0 S_{\beta} \chi_a\|_{HS} = \|\tilde{\chi}_0 S_{-\beta} \chi_a\|_{HS} \|\tilde{\chi}_0 S_{\beta} \chi_a\|_{HS}.
\]
Since \(S_{\pm\beta}(u,v) = \exp\{\pm \beta \beta^3 \pm \beta (v-u)\} \text{Ai}(v-u+\beta^2)\), one has
\[
\|\tilde{\chi}_0 S_{-\beta} \chi_a\|_{HS}^2 \cdot \|\tilde{\chi}_0 S_{\beta} \chi_a\|_{HS}^2 = \int_0^\infty du \int_a^\infty dv \ e^{-2\beta(v+u)} \text{Ai}^2(v+u+\beta^2) \times \int_0^\infty du \int_a^\infty dv \ e^{2\beta(v+u)} \text{Ai}^2(v+u+\beta^2).
\]

We change variable \(v \mapsto v + a\) in both integrals above. Then, changing variables \(y := u + v\) and \(x := u - v\) in both integrals gives
\[
\|\tilde{\chi}_0 S_{-\beta} \chi_a\|_{HS}^2 \cdot \|\tilde{\chi}_0 S_{\beta} \chi_a\|_{HS}^2 = \int_0^\infty dy \ e^{-2\beta y} \text{Ai}^2(\beta^2 + y + a) \int_0^\infty dy \ e^{2\beta y} \text{Ai}^2(\beta^2 + y + a).
\]
Rescaling the variable of the first integral as \(y \mapsto y/\beta\), and of the second as \(y \mapsto \beta y/2\), shows that
(4.8) \[
\|\tilde{\chi}_0 S_{-\beta} \chi_a\|_{HS}^2 \cdot \|\tilde{\chi}_0 S_{\beta} \chi_a\|_{HS}^2 = \frac{\beta^2}{16} \int_0^\infty dy \ e^{-y} \text{Ai}^2(\beta^2 + a + (2\beta)^{-1} y) \times \int_0^\infty dy \ e^{\beta y} \text{Ai}^2(\beta^2 (1 + \frac{y}{2}) + a).
\]
Recall there is a constant $C$ such that $|\text{Ai}(z)| \leq C \exp\{-\frac{2}{3}z^{3/2}\}$ if $z \geq 0$ and $|\text{Ai}(z)| \leq C$ if $z < 0$. Since $a$ is fixed, suppose $\beta$ satisfies $\beta^2 + a \geq 1$, say. Then due to the aforementioned bound on the Airy function the contribution to the first of the two integrals above results from $y$ being of bounded order, $y \approx 1$. In particular, there is a constant $C_a$ such that for sufficiently large $\beta$ (in terms of $a$),

$$
\int_0^\infty dy \, y e^{-y} \text{Ai}^2(\beta^2 + a + (2\beta)^{-1}y) \leq C_a \text{Ai}^2(\beta^2).
$$

The magnitude of the second integral from (4.8) may also be determined from a critical point analysis by using the bound on the Airy function above. By abusing notation a bit, there is a constant $C_a$ such that for large enough $\beta$,

$$
e^{\beta^2 y} \text{Ai}^2\left(\beta^2 (1 + \frac{y}{2}) + a\right) \leq C_a \exp\left\{\beta^3 \left(y - \frac{4}{3}(1 + \frac{y}{2})^{3/2}\right)\right\}.
$$

The function $y - (4/3)(1 + (y/2))^{3/2}$ is uniquely maximized at $y = 0$ and its value there is $-\frac{4}{3}$. Therefore the second integral from (4.8) is of order $e^{-\frac{4}{3}\beta^3}$ as $\beta \to \infty$. Consequently, there is a (new) constant $C_a$ such that for sufficiently large $\beta$, \begin{equation}
||\bar{\chi}_0 S_{-\beta} \chi_a||_{\text{HS}} \cdot ||\bar{\chi}_0 S_{\beta} \chi_a||_{\text{HS}} \leq C_a \beta^3 \text{Ai}^2(\beta^2) e^{-\frac{4}{3}\beta^3}.
\end{equation}

This shows that $||E_{\beta}^1||_{\text{tr}} \to 0$ as $\beta \to \infty$ since $\text{Ai}(\beta^2)$ is of order $e^{-\frac{4}{3}\beta^3}$.

Now consider the operator $E_{\beta}^2$. Using the definitions one has that

$$
E_{\beta}^2(u + a, v + a) = 1_{\{u \geq 0, v \geq 0\}} e^{(v - u)} \int_0^\infty dz \, e^{-2\beta z} \text{Ai}(\beta^2 + a + u + z) \text{Ai}(\beta^2 + a + v - z).
$$

The trace norm of $E_{\beta}^2$ is the same as that of $(u, v) \mapsto E_{\beta}^2(u + a, v + a)$ since the latter is a conjugation of the former by the unitary operation of translation. So we consider the latter kernel.

When $\beta^2 + a \geq 1$, the major contribution to the integral above comes from $z$ being in a region around zero, $z \approx 0$, due to the rapid decay of the integrand in the variable $z$. Consequently, for large $\beta$ there is a constant $C_a$ such that

$$
|E_{\beta}^2(u + a, v + a)| \leq C_a 1_{\{u \geq 0, v \geq 0\}} e^{\beta(v - u)} \text{Ai}(u + \beta^2) \text{Ai}(v + \beta^2).
$$

The right hand side above decays rapidly in the variable $u$, namely, it is at most of order $e^{-\frac{2}{3}(\beta^2 + u^{3/2})}$ $1_{\{u \geq 0\}}$. Consider its rate of decay in the variable $v$.

The asymptotics of the Airy function show that for $v \geq 0$,

$$
|e^{\beta v} \text{Ai}(v + \beta^2)| \leq \exp\left\{\beta v - \frac{2}{3}(v + \beta^2)^{3/2} + \text{const}\right\}.
$$

The exponent above is uniquely maximized at $v = 0$ whereby it equals $-\frac{2}{3}\beta^3$. Moreover, for large values of $v$ the exponent is of order $\beta v - \frac{2}{3}v^{3/2} - \beta^2 v^{1/2} + \text{const}$, which
is seen from a Taylor expansion of $(1 + x)^{3/2}$ around $x = 0$. If $\beta \geq 1$, say, then the quantity $\beta v - \frac{2}{5}v^{3/2} - \beta^2 v^{1/2} \leq -\frac{5}{8}v^{1/2}$. This is because $\beta^2 + \frac{2}{5}v - \beta v^{1/2}$ is at least $\frac{5}{8} \beta^2$, the minimum being at $v = \frac{9}{16\beta^2}$. The upshot is that for $v \geq 0$,

$$|e^{\beta v} A_i(v + \beta^2)| \leq \exp \left\{ -\frac{2}{3} \beta^3 - \frac{5}{8} v^{1/2} + \text{const} \right\}.$$ 

All in all, it follows that there are constants $C_a$ and $\kappa > 0$ such that

$$|E_\beta^2(u + a, v + a)| \leq C_a \mathbf{1}_{\{u \geq 0, v \geq 0\}} e^{-\frac{4}{3} \beta^3 - \kappa (u^{3/2} + v^{1/2})}.$$ 

This implies that the trace norm of $E_\beta^2$ decays to the order $e^{-\frac{4}{3} \beta^3}$, as required. □

**Lemma 4.5.** As $\beta \to \infty$, the difference of determinants

$$\det \left( (I - M_\beta(A + E_\beta)M_{-\beta})^*(I - M_\beta(A + E_\beta)M_{-\beta}) \right)_{L^2(a, \infty)} - \det (I - A - E_\beta)^2_{L^2(a, \infty)}$$

tends to zero.

**Proof.** In the following argument all Fredholm determinants are over $L^2(a, \infty)$. Denote $X = A + E_\beta$. On the space $L^2(a, \infty)$,

$$(I - M_\beta XM_{-\beta})^* \tilde{\chi}_a (I - M_\beta XM_{-\beta}) = (M_\beta XM_{-\beta})^* \tilde{\chi}_a (M_\beta XM_{-\beta})$$

because $\tilde{\chi}_a$ annihilates the identity on $L^2(a, \infty)$. Consequently, on $L^2(a, \infty)$,

$$(I - M_\beta XM_{-\beta})^*(I - M_\beta XM_{-\beta}) = \chi_a (I - M_\beta XM_{-\beta})^* \chi_a (I - M_\beta XM_{-\beta}) \chi_a + \chi_a (M_\beta XM_{-\beta})^* \tilde{\chi}_a (M_\beta XM_{-\beta}) \chi_a.$$ 

The determinant of $Y$ is $\det (I - X)^2$.

Since $\chi_a$ and $\tilde{\chi}_a$ are projections and commute with $M_{\pm \beta}$,

$$Y = M_{-\beta} (\chi_a - \chi_a X^* \chi_a) \chi_a M_{2\beta} (\chi_a - \chi_a X \chi_a) M_{-\beta},$$

$$E = M_{-\beta} (M_\beta \tilde{\chi}_a X \chi_a)^* (M_\beta \tilde{\chi}_a X \chi_a) M_{-\beta}.$$ 

The operators $I - X$ and $I - X^*$ are invertible on $L^2(a, \infty)$ for sufficiently large $\beta$ because $I - A$ is invertible there (since $\det(I - A)_{L^2(a, \infty)} = F_1(2^{2/3} a > 0)$ and $E_\beta$ has vanishingly small trace norm as $\beta \to \infty$. In fact, this means that both the operator norm and the Fredholm determinant of $I - X$ are uniformly bounded away from 0 for sufficiently large $\beta$. This implies invertibility of $Y$ on $L^2(a, \infty)$, and one observes from the above expressions for $Y$ and $E$ that on this space (4.10)

$$M_{-\beta} Y^{-1} EM_\beta = (I - \chi_a X \chi_a)^{-1} \chi_a M_{-2\beta} (I - \chi_a X^* \chi_a)^{-1} (M_\beta \tilde{\chi}_a X \chi_a)^* (M_\beta \tilde{\chi}_a X \chi_a).$$
In order to compare the determinant of $Y+E$ with that of $Y$ one first conjugates both operators as $M_{-\beta}(Y+E)M_\beta$ and $M_{-\beta}EM_\beta$, and then employs the inequality

$$|\det(I-T) - 1| \leq ||T||_\text{tr} e^{||T||_\text{tr} + 1}$$

to deduce that

$$|\det(Y+E) - \det(Y)| \leq |\det(Y)||M_{-\beta}Y^{-1}EM_\beta||_\text{tr} e^{||M_{-\beta}Y^{-1}EM_\beta||_\text{tr} + 1}.$$  

The determinant of $Y$ remains bounded in $\beta$ by Lemma 4.4.

The trace norm of $M_{-\beta}Y^{-1}EM_\beta$ may be bounded using the inequalities $||T_1T_2||_\text{tr} \leq ||T_1||_{\text{op}} ||T_2||_\text{tr}$ and $||T_1T_2||_\text{tr} \leq ||T_1||_\text{tr} ||T_2||_{\text{op}}$. The second follows from the first by taking adjoints and noting that both the operator norm and trace norm are invariant under taking adjoints. Thus,

$$\begin{align*}
|M_{-\beta}Y^{-1}EM_\beta||_\text{tr} &\leq ||(I - \chi_aX\chi_a)^{-1}||_{\text{op}} \times \\
&\quad \times ||\chi_aM_{-2\beta}(I - \chi_aX^*\chi_a)^{-1}(M_\beta\bar{\chi}_aX\chi_a)^*(M_\beta\bar{\chi}_aX\chi_a)||_\text{tr} \\
&\leq ||(I - \chi_aX\chi_a)^{-1}||_{\text{op}} ||\chi_aM_{-2\beta}||_\text{tr} \times \\
&\quad \times ||(I - \chi_aX^*\chi_a)^{-1}(M_\beta\bar{\chi}_aX\chi_a)^*(M_\beta\bar{\chi}_aX\chi_a)||_{\text{op}} \\
&\leq ||(I - \chi_aX\chi_a)^{-1}||^2_{\text{op}} ||\chi_aM_{-2\beta}||_\text{tr} ||M_\beta\bar{\chi}_aX\chi_a||^2_{\text{op}}.
\end{align*}$$

The first operator norm in the last expression above remains bounded for large $\beta$ as remarked. Since $\chi_aM_{-2\beta}$ is a diagonal operator, it has trace norm

$$||\chi_aM_{-2\beta}||_\text{tr} = \int_0^\infty du e^{-2\beta u} = \frac{e^{-2\beta a}}{2\beta}.$$ 

The operator norm of $M_\beta\bar{\chi}_aX\chi_a$ is at most $||M_\beta\bar{\chi}_aA\chi_a||_{\text{op}} + ||M_\beta\bar{\chi}_aE_\beta\chi_a||_{\text{op}}$. Observe that

$$M_\beta\bar{\chi}_aA\chi_a(u+a,v+a) = e^{\beta a} \cdot \left(e^{\beta u} 2^{-1/3} \text{Ai}(2^{-1/3}(u+v+2a))1_{\{u<0,v\geq 0\}} \right).$$

The operator norm of the kernel inside the big parentheses is bounded in terms of $\beta$ because the kernel decays to the order $e^{-\frac{\sqrt{2}}{\beta}v^{3/2}}$ for large values of $v$ and to the order $e^{-\beta |u|}$ for negative values of $u$. Since the operator displayed above is a conjugation of $M_\beta\bar{\chi}_aA\chi_a$ by a translation, it follows that $||M_\beta\bar{\chi}_aA\chi_a||_{\text{op}} \leq C_a e^{\beta a}$ for some constant $C_a$. Similarly, $M_\beta\bar{\chi}_aE_\beta\chi_a = e^{\beta a} e^{\beta a} \left(M_\beta\bar{\chi}_0 e^{-\beta a} E_\beta e^{\beta a} \chi_0 \right) e^{-\beta a}$. The operator norm of what sits within the big parentheses is vanishingly small in terms of $\beta$ by a calculation entirely analogous to that of Lemma 4.4. So in all, $||M_\beta\bar{\chi}_aX\chi_a||_{\text{op}} \leq C_a e^{\beta a}$ for some constant $C_a$. 


Therefore, (4.11) implies that for some (new) constant $C_a$,

$$||M_{-\beta}Y^{-1}EM_{\beta}||_{tr} \leq \frac{C_a}{\beta}.$$ 

Thus $||M_{-\beta}Y^{-1}EM_{\beta}||_{tr}$ tends to 0 as required. \qed

Lemma 4.4 and Lemma 4.5 together conclude the proof of Theorem 1.

This section concludes by extending Theorem 1 to arbitrary one-point distributions of $h(1, x; 2\beta |y|)$, which will be utilized in the proof of Theorem 3 below.

**Proposition 4.1.** For every $a, x \in \mathbb{R}$, as $\beta \to \infty$ the probability

$$\Pr[h(1, (2\beta)^{-1}x; 2\beta |y|) - \beta^2 \leq a] \to F_1(2^{2/3}(a + x))F_1(2^{2/3}(a - x)).$$

*Proof. *By Theorem 2, the probability

$$\Pr[h(1, x, 2\beta |y|) - \beta^2 \leq a] = \det \left( I - e^{-x\partial^2}K_{\beta}e^{x\partial^2} \right)_{L^2(\{a+\beta^2, \infty\})} \to \det \left( I - e^{-x\partial^2 + \beta^2\partial}K_{\beta}e^{x\partial^2 - \beta^2\partial} \right)_{L^2(a, \infty)}.$$ 

Factorization Lemma 4.3 then gives

$$I - e^{-x\partial^2 + \beta^2\partial}K_{\beta}e^{x\partial^2 - \beta^2\partial} = \left( I - e^{x\partial^2}M_{\beta}XM_{-\beta}e^{-x\partial^2} \right)^* \left( I - e^{-x\partial^2}M_{\beta}XM_{-\beta}e^{x\partial^2} \right),$$

where $X = A + E_{\beta}$. Commutation relation (2) of Lemma 4.1 implies that

$$e^{\mp x\partial^2}M_{\beta}XM_{-\beta}e^{\pm x\partial^2} = M_{\beta}e^{\mp x(\partial + \beta)^2}Xe^{\pm x(\partial + \beta)^2}M_{-\beta} = M_{\beta}e^{\mp x(\partial^2 + 2\beta\partial)}Xe^{\pm x(\partial^2 + 2\beta\partial)}M_{-\beta}.$$ 

The relation $\partial R = -R\partial$ now implies the following identities.

$$e^{\mp x(\partial^2 + 2\beta\partial)}Ae^{\pm x(\partial^2 + 2\beta\partial)} = e^{\mp 2\beta x\partial}Ae^{\pm 2\beta x\partial},$$

$$e^{\mp x(\partial^2 + 2\beta\partial)}E_{\beta}e^{\pm x(\partial^2 + 2\beta\partial)} = S_{-\beta \mp x}^* e^{\mp 2\beta x\partial}I_0(I - R)e^{\pm 2\beta x\partial}S_{\beta \pm x} = S_{-\beta \mp x}^* \chi_{\mp 2\beta x}(I - Re^{\mp 4\beta x\partial})S_{\beta \pm x}.$$ 

Substituting in $(2\beta)^{-1}x$ for $x$ now shows that $\Pr[h(1, (2\beta)^{-1}x; 2\beta |y|) - \beta^2 \leq a]$ equals

$$\det \left( \left( I - M_{\beta}(e^{x\partial}Ae^{-x\partial} + E_{\beta,x})M_{-\beta} \right)^* \left( I - M_{\beta}(e^{-x\partial}Ae^{x\partial} + E_{\beta,-x})M_{-\beta} \right) \right)_{L^2(a, \infty)},$$

where $E_{\beta,x} = S_{-\beta \mp x/2\beta}^* \chi_{x}(I - Re^{2\beta x\partial})S_{\beta \mp x/2\beta}$. The proof now proceeds exactly as in the arguments of Lemma 4.4 and Lemma 4.5. The argument of Lemma 4.5 remains the same, and in place of Lemma 4.4 one needs to show that the trace norm of $E_{\beta, \pm x}$ over $L^2(a, \infty)$ converges to zero as $\beta \to \infty$. The proof of the latter is entirely analogous to the proof of Lemma 4.4. We do not repeat the calculations for brevity.
In conclusion, as $\beta \to \infty$, $\Pr \left[ h(1, (2\beta)^{-1}x; 2\beta|y|) - \beta^2 \leq a \right]$ tends to
\[
\det(I - e^{x\partial}Ae^{-x\partial})_{L^2(a,\infty)} \cdot \det(I - e^{-x\partial}Ae^{x\partial})_{L^2(a,\infty)} =
\det(I - A)_{L^2(a+x,\infty)} \cdot \det(I - A)_{L^2(a-x,\infty)} = F_1(2^{2/3}(a + x)) F_1(2^{2/3}(a - x)).
\]

\[\square\]

4.2. **Proof of Theorem 3.** We will use an argument by way of the variational principle for the law of the process $h(1, x; 2\beta|y|)$. An Airy sheet $A_2(x,y)$ is a random function of real variables $x$ and $y$ defined by the identity

$$A_2(x,y) = h(1,x; -\infty 1_{\{z \neq y\}}) + (x - y)^2.$$  

Here, $-\infty 1_{\{z \neq y\}}$ is the narrow wedge at $y$. The height functions above are all coupled by a “common noise”. This noise is naturally present in TASEP and the coupled height functions may be obtained as a joint KPZ scaling limit of TASEPs with different wedge initial data that all move under a common dynamic. See Section 4.5 of [17].

Actually, [17] proves existence of an Airy sheet (due to tightness) but not its uniqueness. Nevertheless, the following properties we use are common to every Airy sheet. Every Airy sheet is continuous, invariant under switching variables, and has the law of the Airy$_2$ process in each variable when the other is held fixed. Also, the following variational principle applies to any Airy sheet [17, Theorem 4.11] (see also [7]).

**Variational principle.** Let $h_0 : \mathbb{R} \to (-\infty, \infty)$ be a upper semicontinuous function with at most linear growth. Then the KPZ height function $x \mapsto h(1,x; h_0)$ (as defined in (2.8)) satisfies

$$h(1,x; h_0) \overset{\text{law}}{=} \sup_{y \in \mathbb{R}} \left\{ A_2(x,y) - (x - y)^2 + h_0(y) \right\}.$$

Variational formulae like these originate in [15] and are similar to the Lax-Oleinik formula for solutions of Hamilton-Jacobi equations; see [7, 26].

**Lemma 4.6.** An Airy sheet has the following modulus of continuity uniformly over $y$ and $x_1,x_2$ with $|x_1 - x_2| \leq 1$.

$$|A_2(x_1, y) - A_2(x_2, y)| \leq O_p(|x_1 - x_2|^{1/4}).$$

The notation $O_p()$ means a random quantity that is finite with probability one.

**Proof.** For every fixed $y$, $A_2(x,y)$ is an Airy$_2$ process in $x$, which satisfies the modulus of continuity estimate stated above by [17] Theorem 4.4. (The Airy$_2$ process is Hölder-$(1/2 - \varepsilon)$ almost surely.) Thus, the modulus of continuity estimate above
holds for every fixed \(y\). By a union bound it then holds uniformly over all rational values of \(y\). By continuity of an Airy sheet, it also holds uniformly over all \(y\). □

Using the variational principle and separating the supremum over \(y \leq 0\) from the supremum over \(y \geq 0\), one has that

\[
\begin{align*}
\mathcal{I} &= \sup_{y \leq 0} \mathcal{A}_2(x, y) - (x - y)^2 - 2\beta y, \\
\mathcal{II} &= \sup_{y \geq 0} \mathcal{A}_2(x, y) - (x - y)^2 + 2\beta y.
\end{align*}
\]

Rewrite (I) by changing variable \(y \mapsto y - \beta - x\) and (II) by changing variable \(y \mapsto y + \beta + x\). Then

\[
\begin{align*}
h(1, x; 2\beta \vert y \vert) &\xrightarrow{\text{law}} \sup_{y \in \mathbb{R}} \mathcal{A}_2(x, y) - (x - y)^2 - 2\beta y = \max \{I, II\}, \\
&\text{where}
\end{align*}
\]

\[
\begin{align*}
\mathcal{I} &= \sup_{y \leq -\frac{x}{2\beta}} \mathcal{A}_2 \left( \frac{x}{2\beta}, y - \beta + \frac{x}{2\beta} \right) - y^2, \\
\mathcal{II} &= \sup_{y \geq \frac{x}{2\beta}} \mathcal{A}_2 \left( \frac{x}{2\beta}, y + \beta + \frac{x}{2\beta} \right) - y^2.
\end{align*}
\]

Now consider \(X_1(x)\) for a fixed value of \(x\). As \(y \mapsto \mathcal{A}_2(x/2\beta, y)\) has the law of the Airy_2 process, by the modulus of continuity estimate of Lemma 4.6 (the roles of \(x\) and \(y\) are now switched) one infers that

\[
\begin{align*}
\sup_{y \in \left[ -\frac{x}{2\beta}, \beta + \frac{x}{2\beta} \right]} \mathcal{A}_2 \left( \frac{x}{2\beta}, y - \beta + \frac{x}{2\beta} \right) &= \mathcal{A}_2 \left( \frac{x}{2\beta}, \frac{x}{2\beta} \right) + O_p(\beta^{-1/4}),
\end{align*}
\]

As a result, the supremum of \(\mathcal{A}_2 \left( \frac{x}{2\beta}, y - \beta + \frac{x}{2\beta} \right) - y^2\) over \(y \leq \beta - \frac{x}{2\beta}\) may be replaced by its supremum over \(y \leq \beta\) with an additive error of order \(O_p(1)\) as \(\beta \to \infty\) since the supremum on the leftover interval is of order \(O_p(1) - \beta^2\). (The notation \(O_p(1)\) denotes a term that converges to zero in probability as \(\beta \to \infty\).)

Furthermore, due to the modulus of continuity estimate in Lemma 4.6, the latter supremum may be replaced by the supremum of the process \(y \mapsto \mathcal{A}_2(0, y - \beta)\) over \(y \leq \beta\) with an additional penalty of \(O_p(\beta^{-1/4})\). This is because replacing the \(x/2\beta\) by 0 in the above introduces an additive error of order \(O_p(\beta^{-1/4})\). As a result,

\[
X_1(x) = \sup_{y \leq \beta} \mathcal{A}_2(0, y - \beta) - y^2 + o_p(1) \quad \text{as } \beta \to \infty.
\]

This same argument implies that

\[
X_2(x) = \sup_{y \geq -\beta} \mathcal{A}_2(0, y + \beta) - y^2 + o_p(1) \quad \text{as } \beta \to \infty.
\]

Observe that the two suprema above are \(X_1(0)\) and \(X_2(0)\), respectively.
Since the above holds for every fixed \( x \), it follows from the variational principle that for any finite number of points \( x_1, \ldots, x_m \), the joint law of the \( m \)-dimensional vector \( x_i \mapsto h(1,(2\beta)^{-1}x_i,2\beta|y|) \) satisfies

\[
(4.13) \quad h(1,(2\beta)^{-1}x;2\beta|y|) - \beta^2 \xrightarrow{\text{law}} \max \{X_1(0) - x, X_2(0) + x\} + o_p(1)
\]
as \( \beta \to \infty \) and \( x \) ranges over \( x_1, \ldots, x_m \).

**Lemma 4.7.** The random variables \( X_1(0) \) and \( X_2(0) \) jointly converge in law to two independent GOE Tracy-Widom random variables \( 2^{-2/3}X_{TW_1} \) and \( 2^{-2/3}X'_{TW_1} \), respectively, as \( \beta \to \infty \).

**Proof.** It suffices to show that given \( s, s' \in \mathbb{R} \), as \( \beta \to \infty \),

\[
\Pr \left[ X_1(0) \leq s, X_2(0) \leq s' \right] \xrightarrow{} F_1(2^{2/3}s)F_1(2^{2/3}s').
\]

There are numbers \( a \) and \( x \) such that \( s = a + x \) and \( s' = a - x \). Observe that the event \( \{X_1(0) \leq s, X_2(0) \leq s'\} \) equals the event \( \{\max \{X_1(0) - x, X_2(0) + x\} \leq a\} \). Since \( X_1(x) = X_1(0) + o_p(1) \) and \( X_2(x) = X_2(0) + o_p(1) \) as \( \beta \to \infty \), it suffices to show that as \( \beta \to \infty \),

\[
\Pr \left[ \max \{X_1(x) - x, X_1(x) + x\} \leq a \right] \xrightarrow{} F_1(2^{2/3}(a + x))F_1(2^{2/3}(a - x)).
\]
The law of the maximum above is \( h(1,(2\beta)^{-1}x;2\beta|y|) - \beta^2 \) by the representation (4.12). The required convergence is then the statement of Proposition 4.1. \( \square \)

Lemma 4.7 and the representation (4.13) imply that as \( \beta \to \infty \),

\[
h(1,(2\beta)^{-1}x;2\beta|y|) - \beta^2 \xrightarrow{} \max \{2^{-2/3}X_{TW_1} - x, 2^{-2/3}X'_{TW_1} + x\}
\]
in the sense of finite dimensional laws because the map that takes \( x_1, \ldots, x_m \) to \( \max \{X - x_i, X' + x_i\} \), for \( i = 1, \ldots, m \), is continuous. \( \square \)

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