Energetic Stability of the Solutions
of the Einstein Field Equations for
Spherically Symmetric Liquid Shells

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Abstract

We interpret the exact solutions previously obtained for spherically symmetric shells
of liquid fluid in General Relativity in terms of the energies involved. We show that
a certain parameter that was introduced into the solutions by the interface boundary
conditions is related to the binding energies of the gravitational systems. We then use
this fact in order to discuss the energetic stability of those solutions.

We include in the stability discussion the well-known interior Schwarzschild solution
for a liquid sphere, which can be obtained as a specific limit of the solutions that we
previously obtained for the liquid shells. We show that this solution turns out to be a
maximally unstable one, from the energetic point of view discussed here.

We also perform a numerical exploration of the energetic stability criterion of the
solutions, and show that indeed there is a particular subset of the solutions which are
energetically stable. All these solutions have the form of shells with non-vanishing
internal radii. This reduces the original three-parameter family of solutions to a two-
parameter family of energetically stable solutions.

1 Introduction

The issue of the energy in General Relativity is a difficult one, and its discussion in specific
examples quite often becomes involved and obscure. The difficulties start at the very foun-
dations of the theory, with the impossibility of defining an energy-momentum tensor density
for the gravitational field itself, a problem which apparently is related to the impossibility
of localizing the energy of the gravitational field in the general case [1]. However, a recently
discovered new class of static and time-independent exact solutions provides us with an
opportunity to discuss the subject in a clear, precise and complete manner. It leads to a
simple and clear characterization of all the energies involved in this class of solutions, as
well as a characterization of the relations among them.

In a previous paper [2] we established the solution of the Einstein field equations for the
case of a spherically symmetric shell of liquid fluid located between the radial positions $r_1$

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and $r_2$ of the Schwarzschild system of coordinates. The matter distribution is characterized by the radii $r_1$ and $r_2$, by its total asymptotic gravitational mass $M$, associated to the Schwarzschild radius $r_M$, and by a matter energy density $\rho_0$ which is constant with the radial Schwarzschild coordinate $r$. In this work we will use the time-like signature $(+,−,−,−)$, following [1]. In terms of the coefficients of the metric, for an invariant interval given in terms of the Schwarzschild coordinates $(t, r, \theta, \phi)$ by

$$ds^2 = e^{2\nu(r)}c^2 dt^2 - e^{2\lambda(r)}dr^2 - r^2 [d\theta^2 + \sin^2(\theta)d\phi^2],$$

(1)

where $\exp[\nu(r)]$ and $\exp[\lambda(r)]$ are two positive functions of only $r$, as was explained in [2] the Einstein field equations reduce to the set of three first-order differential equations

$$\begin{align*}
\left\{1 - 2 \left[r\lambda'(r)\right]\right\} e^{-2\lambda(r)} &= 1 - \kappa r^2 \rho(r), \\
\left\{1 + 2 \left[r\nu'(r)\right]\right\} e^{-2\lambda(r)} &= 1 + \kappa r^2 P(r), \\
[r(\rho(r) + P(r))\nu'(r) &= -P'(r),
\end{align*}$$

(2-4)

where $\rho(r)$ is the energy density of the matter, $P(r)$ is the isotropic pressure, $\kappa = 8\pi G/c^4$, $G$ is the universal gravitational constant and $c$ is the speed of light. In these equations the primes indicate differentiation with respect to $r$. Given these equations, as presented in [2] the complete solution for $\lambda(r)$ is given by

$$\lambda(r) = \begin{cases} 
-\frac{1}{2} \ln \left(\frac{r + r_\mu}{r}\right) & \text{for } 0 \leq r \leq r_1, \\
-\frac{1}{2} \ln \left[\frac{\kappa \rho_0 \left(r_2^3 - r_\mu^3\right) + 3(r - r_M)}{3r}\right] & \text{for } r_1 \leq r \leq r_2, \\
-\frac{1}{2} \ln \left(\frac{r - r_M}{r}\right) & \text{for } r_2 \leq r < \infty,
\end{cases}$$

(5)

where $r_M = 2GM/c^2$, while for $\nu(r)$ we have

$$\nu(r) = \begin{cases} 
\frac{1}{2} \ln \left(\frac{1 - r_M/r_2}{1 + r_\mu/r_1}\right) + \frac{1}{2} \ln \left(\frac{r + r_\mu}{r}\right) & \text{for } 0 \leq r \leq r_1, \\
\frac{1}{2} \ln \left(\frac{r_2 - r_M}{r_2}\right) + \ln[z(r)] & \text{for } r_1 \leq r \leq r_2, \\
\frac{1}{2} \ln \left(\frac{r - r_M}{r}\right) & \text{for } r_2 \leq r < \infty,
\end{cases}$$

(6)

and finally the pressure within the shell, that is, for $r_1 \leq r \leq r_2$, is given by

$$P(r) = \rho_0 \frac{1 - z(r)}{z(r)}. $$

(7)

This solution is valid under the condition that $r_2 > r_M$. In all these expressions we have that $r_\mu$ is given in terms of the parameters characterizing the system by

$$r_\mu = \frac{\kappa \rho_0}{3} \left(\frac{r_2^3}{r_1^3} - 1\right) - r_M,$$

(8)

that $\rho_0$ is determined algebraically in terms of $r_1$, $r_2$ and $r_M$ as the solution of the algebraic equation

$$P(r_1) = 0.$$
\( \sqrt{\frac{r_2}{3 (r_2 - r_M)}} = \sqrt{\kappa \rho_0 \left( \frac{r_1}{r_2} \right)^3} + 3 \left( r_1 - r_M \right) + \frac{3}{2} \int_{r_1}^{r_2} dr \frac{\kappa \rho_0 r^{5/2}}{\left[ \kappa \rho_0 \left( \frac{r_3}{r_2} - r^3 \right) + 3 (r - r_M) \right]^{3/2}} \). \tag{9}

and that the real function \( z(r) \) is determined in terms of a non-trivial elliptic real integral by the equation

\[
z(r) = \frac{\sqrt{\kappa \rho_0 \left( \frac{r_3}{r_2} - r^3 \right) + 3 (r - r_M)}}{r} \times \left\{ \frac{r_2}{3 (r_2 - r_M)} + \frac{3}{2} \int_{r_2}^{r} ds \frac{\kappa \rho_0 s^{5/2}}{\left[ \kappa \rho_0 \left( \frac{r_3}{r_2} - s^3 \right) + 3 (s - r_M) \right]^{3/2}} \right\}. \tag{10}\]

The relation shown in Equation (8) is a direct consequence of the field equations and of the interface boundary conditions associated to them. In [2] we proved that, so long as the pressure of the liquid is positive, we must have \( r_\mu > 0 \). In fact, the hypotheses of that proof can be weakened to require only that the pressure be strictly positive at a single point. This strictly positive value of \( r_\mu \) implies that the solution has a singularity at the origin. However, that singularity is not associated to an infinite concentration of matter, but rather, as explained in [2], to zero energy density at that point. Also, the solution introduces into the system the new physical parameter \( r_\mu \) with dimensions of length, which can be associated to a mass parameter \( \mu \) in the same way that \( M \) is associated to \( r_M \).

This paper is organized as follows: in Section 2 we review the interior Schwarzschild solution; in Section 3 we establish the interpretation of the shell solutions in terms of the energies involved, and extend that analysis to the interior Schwarzschild solution; in Section 4 we perform a small numerical exploration of the energetic stability of the shell solutions; in Section 5 we establish integral formulas for all the energies involved; and in Section 6 we state our conclusions.

2 The Interior Schwarzschild Solution

It is an interesting and somewhat remarkable fact that the well-known interior Schwarzschild solution [3, 4] can be obtained from our solution for a shell, even though the interior Schwarzschild solution has no singularity at the origin, while our solution always has that singularity. Curiously enough, we must start by assuming that \( r_\mu = 0 \), even though we proved in [2] that one must have \( r_\mu > 0 \) in the shell solutions. The subtle point here is that the proof given in [2] relies on the existence of a shell with \( r_1 > 0 \), while in the case of the interior Schwarzschild solution we will have to use \( r_1 = 0 \), so that the shell becomes a filled sphere. If we start by first putting \( r_\mu = 0 \) and then putting \( r_1 = 0 \) in Equation (8), we are led to the relation

\[
\kappa \rho_0 = \frac{3 r_M}{r_2^3}, \tag{11}\]

so that we may substitute \( \kappa \rho_0 \) in terms of \( r_M \) and the radius \( r_2 \) of the resulting sphere. Following the usual notation for the interior Schwarzschild solution, we now define a parameter \( R \), with dimensions of length, such that \( R^2 = r_2^3/r_M \), in terms of which we have

\[
\kappa \rho_0 = \frac{3}{R^2}. \tag{12}\]

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Note that the required condition that $r_2 > r_M$ is translated here as the condition that $R > r_2$. Making this substitution we have for $\lambda(r)$ inside the resulting sphere, directly from the line in Equation (5) for the case of the matter region, in the case in which $r_\mu = 0$ and $r_1 = 0$,

$$\lambda_i(r) = -\frac{1}{2} \ln \left[ 1 - \left( \frac{r}{R} \right)^2 \right],$$

which implies that for the radial metric coefficient we have

$$e^{-\lambda_i(r)} = \sqrt{1 - \left( \frac{r}{R} \right)^2}. \quad (13)$$

In order to obtain $\nu(r)$ inside the sphere we must first work out the function $z(r)$. Making the substitution of $\kappa \rho_0$ in terms of $R$ in the result for $z(r)$ given in Equation (10) we get

$$z(r) = \sqrt{1 - \left( \frac{r}{R} \right)^2} \left[ \sqrt{\frac{r_2}{r_2 - r_M}} + \frac{3}{2} \int_{r_2}^{r} ds \frac{s/R^2}{(1 - s^2/R^2)^{3/2}} \right]. \quad (15)$$

Is is now easy to see that in this case the remaining integral can be done, and we get

$$z(r) = \frac{3}{2} - \frac{1}{2} \sqrt{\frac{r_2}{r_2 - r_M}} \left[ 1 - \left( \frac{r}{R} \right)^2 \right]. \quad (16)$$

Using again the definition of $R$, which implies that we have $r_M/r_2 = (r_2/R)^2$, we may write this as

$$z(r) = \frac{3}{2} - \frac{1}{2} \sqrt{\frac{1 - (r/R)^2}{1 - (r_2/R)^2}}. \quad (17)$$

Note that we have $z(r_2) = 1$, so that the boundary condition for $z(r)$ at $r_2$ is still satisfied. From this we may now obtain all the remaining results for the interior Schwarzschild solution. From the line in Equation (5) for the case of the matter region, in the case in which $r_\mu = 0$ and $r_1 = 0$, we get for $\nu(r)$ in the interior of the sphere

$$\nu_i(r) = \frac{1}{2} \ln \left[ 1 - \left( \frac{r_2}{R} \right)^2 \right] + \ln \left[ \frac{3}{2} - \frac{1}{2} \sqrt{\frac{1 - (r/R)^2}{1 - (r_2/R)^2}} \right],$$

which implies that for the temporal metric coefficient we have

$$e^{\nu_i(r)} = \frac{3}{2} \sqrt{1 - \left( \frac{r_2}{R} \right)^2} - \frac{1}{2} \sqrt{1 - \left( \frac{r}{R} \right)^2}. \quad (18)$$

Finally, from Equation (7), in the case in which $r_\mu = 0$ and $r_1 = 0$, we get for the pressure $P(r)$ within the sphere

$$P(r) = \rho_0 \frac{\sqrt{1 - (r/R)^2} - \sqrt{1 - (r_2/R)^2}}{3 \sqrt{1 - (r_2/R)^2} - \sqrt{1 - (r/R)^2}}. \quad (20)$$

These are indeed the correct results for the case of the interior Schwarzschild solution. Note that all the arguments of the logarithms and of the square roots are positive due to the conditions that $R > r_2 > r$. Note also that in the $r_1 \to 0$ limit the lines in Equations (5) and (6) for the case of the inner vacuum region become irrelevant, since this region reduces
to a single point. On the other hand, the lines for the case of the outer vacuum region do not change at all.

It is therefore apparent that the \( r_1 \to 0 \) limit of our solution does reproduce the interior Schwarzschild solution, so long as we adopt the value zero for \( r_\mu \). Our interpretation of these facts is that the \( r_1 \to 0 \) limit to the interior Schwarzschild solution is a non-uniform one, in which we have to leave out one point, the origin. In the \( r_1 \to 0 \) limit the singularity of the shell solutions becomes a strictly point-like one, and therefore a removable one, by a simple continuity criterion. This is certainly the case for the energy density \( \rho(r) \), which in the limit is non-zero everywhere around the origin but at a single point, the origin itself. The same is true for the pressure \( P(r) \), which in the limit is also non-zero around the origin but at the origin itself. Similar situations hold for \( \lambda(r) \) and \( \nu(r) \), as is not difficult to see numerically. It seems that all these functions converge in the \( r_1 \to 0 \) limit to functions with a point-like removable discontinuity at the origin.

### 3 Interpretation in Terms of Energy

Our first task here is to establish the physical interpretation of the new parameter \( r_\mu \).

For this purpose, recalling that \( \kappa = 8\pi G/c^4 \), that \( r_M = 2GM/c^2 \), and defining a mass parameter \( \mu \) such that \( r_\mu = 2G\mu/c^2 \), we write Equation (8) as

\[
\mu c^2 = \frac{4\pi}{3}(r_2^3 - r_1^3)\rho_0 - Mc^2.
\]

In this equation the second term on the right-hand side, according to the usual interpretation in General Relativity, is the total energy of the bound system of particles that constitutes the shell, including both the energy of the matter and the energy of the gravitational field. The term on the left-hand side is also an energy, whose nature we wish to establish. The first term on the right-hand side is an energy as well, namely the product of the constant energy density \( \rho_0 \) of the matter by the coordinate volume of the shell. Observe that this is not the proper volume of the shell, but just its coordinate volume, as measured in terms of the Schwarzschild system of coordinates, as if space was flat inside and around the shell. In general, for a non-constant \( \rho(r) \), this energy would be given by the integral

\[
M_u c^2 = \int_{r_1}^{r_2} dr \int_0^\pi d\theta \int_0^{2\pi} d\phi r^2 \sin(\theta) \rho(r),
\]

where \( M_u \) is a mass parameter which we associate to it. This expression reduces to the first term of the right-hand side of Equation (21) in the case in which \( \rho(r) = \rho_0 \) is a constant.

Let us make a geometrical transformation on this integral in order to establish its meaning. We start by recalling that \( \rho(r) \) is an energy density, so that \( \rho(r)\delta V \) is the energy of the matter within an infinitesimal coordinate volume \( \delta V \) around a point at the radial position \( r \), as seen by a stationary local observer at that point. This may include the rest energies of certain numbers of particles of various types, a certain amount of thermal energy, and anything else that constitutes the matter.

Let us now consider an arbitrary Riemann partition of this integral, consisting of a finite number of cells \( \delta V_n \) with coordinate volume and linear coordinate dimensions below certain maximum values, where \( n \in \{1, \ldots, N\} \). By definition the sum of all these volume elements is equal to the coordinate volume of the shell, and each volume element is at the spatial position \( \vec{r}_n \). The energy \( M_u c^2 \) can therefore be written as the integration limit of the Riemann sum,
\[ M_u c^2 = \lim_{{N \to \infty}} \sum_{{n=1}}^{{N}} \rho(r_n)\delta V_n. \]  

We now consider the mathematical transformation in which we map each volume element at \( \vec{r}_n \) onto an identical volume element at the coordinate position \( \alpha \vec{r}_n \), for some large positive real number \( \alpha \), without changing the coordinate volume of the volume elements. The result is a new set of volume elements, all at large distances from each other, whose sum is still equal to the coordinate volume of the shell.

After taking them to large distances in this fashion, we now put within each one of these new volume elements exactly the same amount of mass and energy that we have in the corresponding coordinate volume elements of the shell. This means putting into each volume element \( \delta V_n \) at infinity the same numbers of the same types of particles, as well as the same amount of thermal energy, as seen by a stationary local observer at that position. In other words, we associate to each volume element at infinity the same value of the energy density \( \rho(r_n) \) that we had for the corresponding volume element of the shell, where \( r_n = |\vec{r}_n| \) and \( \vec{r}_n \) is the original position of the volume element of the shell. Note that no physical transport of the energy elements of the shell is meant here, so that there are no physical transformations involved.

For large values of \( \alpha \) these elements of mass and energy are all at large distances from each other, so as to render the gravitational interactions among them negligible. In the \( \alpha \to \infty \) limit all the gravitational interactions among volume elements go to zero. Besides, in the integration limit each element of mass and energy so constructed tends to zero, so that the gravitational self-interactions within each volume element also become negligible. However, independently of either limit, by construction the total coordinate volume of the elements of volume at infinity remains equal to the coordinate volume of the shell. Therefore, the corresponding sum of all elements of energy \( \rho(r_n)\delta V_n \) is the same as the integral given in Equation (22). Note that this whole argument is general, in the sense that it is not limited to the case in which \( \rho(r) = \rho_0 \) is a constant.

Now, at radial infinity spacetime is flat, so that the coordinate volume of each volume element \( \delta V_n \) coincides with its proper volume, and hence the energy element \( \rho(r_n)\delta V_n \) is the total energy of that element of matter, so that the sum of all the energy elements is the total energy of the matter at infinity. In other words, the integral given in Equation (22) gives us the total energy of the system at infinity. Hence we will name the quantity \( M_u c^2 \) the total energy of the unbound system. This is the total energy of the system when all gravitational interactions have been eliminated by increasing without limit the distances among its elements. This is in both analogy and contrast with the quantity \( Mc^2 \), which is the total energy of the bound system, after all its parts have been brought together to form the shell. In our case here, since \( \rho(r) = \rho_0 \) is a constant, the total energy of the unbound system is just the product of \( \rho_0 \) by the coordinate volume of the shell, as shown in Equation (21).

That same Equation (21) now gives us the interpretation of the parameter \( r_\mu \) and of the corresponding mass parameter \( \mu \). The energy parameter \( \mu c^2 \) is the difference between the total energy of the unbound system and the total energy of the bound system,

\[ \mu c^2 = M_u c^2 - Mc^2, \]  

and is therefore the binding energy of the system. It is the amount of energy that must be given to the system in order to disperse its elements to infinity, thus eliminating all the gravitational bindings between those elements. It is also the amount of energy that must be
dissipated by the system during the process of its assembly into the bound system, stating from the unbound system at infinity. The theorem we proved in [2], in the $\rho(r) = \rho_0$ case that we have here, namely that we must have $r_\mu > 0$, is equivalent to the statement that the bound system must have a finite and non-zero binding energy. This is therefore closely related to the attractive nature of the gravitational interaction between particles.

3.1 Energetic Stability

This interpretation of the parameters involved leads right away to the idea that we may define a notion of energetic stability of the solutions obtained, in the general spirit of the principle of virtual work. Given certain constraints regarding some of the parameters of the solutions, we may obtain the parameter $r_\mu$ as a function of the remaining parameters of the system. Within this class of solutions, if there are two with different values of $r_\mu$, which is proportional to the binding energy $\mu c^2$, then in principle the constrained system will tend to go from the one with the smaller value of $r_\mu$ to the one with the larger value, given the existence of a permissible path between the two solutions.

Let us exemplify this with our current system, in a way that is physically illustrative. Our system contains four parameters, namely $r_1, r_2, r_M$ and $\rho_0$, of which only three are independent. As was explained in [2], these four parameters are related by the condition in Equation (9). Given any three of the parameters, that equation can be used to determine the fourth in terms of those three. Let us assume that we are given fixed values of both $M$ and $\rho_0$, thus determining the local properties of the matter and the total amount of energy of the bound system. This is equivalent to fixing $r_M$ and $\rho_0$, and therefore the result of solving Equation (9) is to establish $r_1$ as a function of $r_2$. We therefore are left with a collection of solutions parametrized by a single real parameter, the external radius $r_2$. We may then determine $r_\mu(r_2)$ and verify whether this function has a single local maximum at a certain value of $r_2$. This then identifies that particular solution which is stable, or that has the largest binding energy, among all others, given the constraints described.

Another approach, slightly more indirect, but perhaps simpler and more physically compelling, would be to keep constant the local parameter $\rho_0$ and the energy $M u c^2$ of the unbound system. This fixes the local properties of the matter and the total energy of the unbound system we start with, and we may then ask which is the solution that corresponds to the most tightly bound system that can be assembled from that unbound system. Since the energy of the unbound system is the product of $\rho_0$ by the coordinate volume of the shell, keeping fixed both $\rho_0$ and $M u$ corresponds to keeping fixed that coordinate volume, which is given by

$$V_0 = \frac{4\pi}{3} \left(r_2^3 - r_1^3\right).$$

This immediately determines $r_2$ as a simple function $r_2(r_1)$ of $r_1$. Then solving Equation (9) results in $r_M$ being given as a function $r_M(r_1)$ of $r_1$ for the fixed value of $\rho_0$ and the fixed coordinate volume $V_0$. This corresponds to the energy of the bound system with internal radius $r_1$, for the given fixed values of $\rho_0$ and $V_0$. The minimum of this function gives us the value of $r_1$ that corresponds to the most tightly bound system that can be assembled from a given unbound system. Other solutions in the same family, with other values of $r_1$, will tend to decay into this one, given a permissible decay path between the two solutions involved. We will execute this program numerically in Section 4.

We saw that in the case of the interior Schwarzschild solution we have the value zero for $r_\mu$. This implies that the resulting solution has zero binding energy, and that its energy is the same as the energy of the corresponding unbound system, which is a very strange
and even bizarre situation indeed. This means that the resulting solution is not only ener-
getically unstable, but that it is in fact maximally energetically unstable, since the bound
system cannot possibly have more energy than the unbound system. Given a permissible
path, in principle one would be able to disassemble the matter distribution of the interior
Schwarzschild solution, taking every element of matter do infinity, without giving any en-
ergy at all to the system. This may be the reason why this solution has never proved to be
a very useful one.

4 Numerical Exploration of the Binding Energy

Here we will explore numerically the issues of the binding energy and of the energetic
stability of the solutions. In this exploration we will keep fixed the local energy density
parameter $\rho_0$, as well as the total energy $M_u c^2$ of the unbound system. Our objective
will be then to determine the existence and the parameters of the maximally bound shell
solution. We will do this by calculating the energy $M c^2$ of the bound system and showing
that it has a point of minimum as a function of $r_1$. Since we keep fixed the parameter $\rho_0$,
and since the energy of the unbound system is given by $M_u c^2 = \rho_0 V_0$, this implies that we
also keep fixed the coordinate volume $V_0$ of the shell,

$$V_0 = \frac{4\pi}{3} \left( r_2^3 - r_1^3 \right),$$

which immediately establishes $r_2$ as a given function of $r_1$,

$$r_2(r_1) = \left( r_1^3 + \frac{3V_0}{4\pi} \right)^{1/3}. \quad (27)$$

Therefore, of the three free parameters of our solutions, which can be taken to be $r_1$, $r_2$ and
$\rho_0$, one is being kept fixed and another is a given function, so that we are left with only one
free parameters, which we will take to be $r_1$. Under these circumstances we have that $r_M$, and
therefore both the mass $M$ and the energy $M c^2$ of the bound system, are functions of
$r_1$, with values that are left to be determined numerically.

In order to perform the numerical work it is convenient to first rescale the variables,
creating a set of equivalent dimensionless variables. Since under these conditions $\kappa \rho_0$ is a
constant which has dimensions of inverse square length, we will define a constant $r_0$ with
dimensions of length by

$$r_0 = \frac{1}{\sqrt{\kappa \rho_0}} \quad (28)$$

Having now the known constant $r_0$, we define the set of dimensionless parameters given by

$$\xi_1 = \frac{r_1}{r_0},$$
$$\xi_2 = \frac{r_2}{r_0},$$
$$\xi_M = \frac{r_M}{r_0},$$
$$\vartheta_0 = \frac{3V_0}{4\pi r_0^3}, \quad (29)$$

where $\vartheta_0$ is the ratio between the coordinate volume $V_0$ of the shell and the volume of an
Euclidean sphere of radius $r_0$. The expression in Equation (27) giving $r_2$ as a function of
$r_1$ is now translated as
Figure 1: Graph of the energy of the bound system as a function of $\xi_1$, for a fixed energy of the unbound system, given by $\vartheta_0 = 2$, and with $\xi_1$ in $[1, 5]$.

$$\xi_2(\xi_1) = \left(\vartheta_0 + \xi_1^3\right)^{1/3}. \quad (30)$$

Note, for subsequent use, that this can also be written as $\xi_2^3 - \xi_1^3 = \vartheta_0$. The relation which we must now use in order to determine $\xi_M$ is that given in Equation (9), which upon rescalings by $r_0$ can be written as

$$\sqrt{\frac{\xi_1}{3 (\xi_2 - \xi_M)}} = \sqrt{\frac{\xi_1^3}{\xi_2^3 - \xi_1^3 + 3 (\xi_1 - \xi_M)}} + \frac{3}{2} \int_{\xi_1}^{\xi_2} d\xi \frac{\xi_2^{5/2}}{[\xi_2^3 - \xi_1^3 + 3 (\xi - \xi_M)]^{3/2}}. \quad (31)$$

where we changed variables in the integral from $r$ to $\xi = r/r_0$. Substituting for $\vartheta_0$ where possible we have the following non-trivial algebraic equation that determines $\xi_M$ and therefore $r_M$,

$$\sqrt{\frac{\xi_1}{\vartheta_0 + 3 (\xi_1 - \xi_M)}} - \sqrt{\frac{\xi_2}{3 (\xi_2 - \xi_M)}} + \frac{3}{2} \int_{\xi_1}^{\xi_2} d\xi \frac{\xi^{5/2}}{[\xi_2^3 - \xi_1^3 + 3 (\xi - \xi_M)]^{3/2}} = 0. \quad (32)$$

Our objective here is to solve this equation in order to get $\xi_M(\xi_1)$, given a fixed value of $\vartheta_0$ and with $\xi_2$ given by Equation (30). Note that, due to the homogeneous scalings leading from the dimensionfull quantities to the dimensionless ones, shown in Equation (29), each solution of this equation is valid for any value of $\rho_0$, which no longer appears explicitly.
The same is true of the graphs to be generated using this equation. Given a value of \( \vartheta_0 \), the corresponding graph represents the results for all the possible values of \( \rho_0 \).

There are two main numerical tasks here, the calculation of the integral and the resolution of this algebraic equation for \( \xi_M \). The integral can be readily and efficiently calculated by a cubic interpolation method, using the values of the integrand and of its derivative at the two ends of each integration interval. So long as we can return the value of the integral without too much trouble, Equation (32) can be readily and efficiently solved by an exponential sandwich (or bisection) method \[5\]. There are two readily available and robust initial upper and lower bounds for the value of \( \xi_M \), the minimum possible lower bound being zero, and the maximum possible upper bound being the energy of the unbound system, since we must have that \( M c^2 < M_u c^2 \), which in terms of the dimensionless parameters translates as \( \vartheta_0 / 3 \) for the upper bound of \( \xi_M \). We may therefore start the process with a lower bound \( \xi_{M\ominus} = 0 \) and an upper bound \( \xi_{M\oplus} = \vartheta_0 / 3 \) for \( \xi_M \). In practice, the efficiency of this algorithm may be highly dependent on the use of a tighter pair of bounds.

A few examples of the functions obtained in this way can be seen in Figures 3 through 6 which show \( \xi_M \) as a function of \( \xi_1 \), for fixed values of the energy of the unbound system, that is, for fixed values of \( \vartheta_0 \). Each graph consists of 81 data points. In order to ensure good numerical precision we used \( 10^6 \) integration intervals in the domain \([\xi_1, \xi_2]\). The exponential sandwich was iterated until a relative precision of the order of \( 10^{-12} \) was reached. The four graphs shown were generated on a high-end PC in approximately 25 hours, 15 hours, 62 hours and 154 hours, respectively. As one can see, the graphs clearly display minima of \( \xi_M \), which are located at certain values of \( \xi_1 \). At these minima the pairs of values \((\xi_1, \xi_2)\) are given approximately, in each case, by \((2.79, 2.87), (2.72, 2.93), (2.60, 3.02)\) and \((2.35, 3.21)\).
respectively. There is freely available an open-source program that can be used to perform these calculations [6].

The minima of these functions give us the value of $\xi_1$ that corresponds to the most tightly bound system that can be assembled from the given unbound system in each case. With the given values of $\rho_0$ and $M_u c^2$, in each case this establishes the value of $r_1$ for the most tightly bound and therefore energetically stable solution, and hence determines the values of $r_2$, $r_M$ and of all the functions describing both the spacetime geometry and the matter for that stable solution. The limiting value of $\xi_M$ when $\xi_1 \to 0$, not shown in these graphs, corresponds to the interior Schwarzschild solution and thus to the energy of the unbound system in each case, which in terms of the variables shown in the graphs is given by $\vartheta_0/3$. The $\xi_1 \to \infty$ limit to the other side rises fairly slowly and does not seem to approach this same value asymptotically, a situation that is probably due to the fact that an infinitesimally thin shell at infinity still has some binding energy, as compared to the corresponding set of isolated infinitesimal point masses.

5 Integral Expressions for the Energies

In addition to the integral expression giving $M_u$, shown in Equation (22), which upon integration over the angles can be written as

$$M_u c^2 = 4\pi \int_{r_1}^{r_2} dr \, r^2 \rho(r),$$

---

Figure 3: Graph of the energy of the bound system as a function of $\xi_1$, for a fixed energy of the unbound system, given by $\vartheta_0 = 10$, and with $\xi_1$ in $[1, 5]$. 
it is possible to express both $M$ and $\mu$ as integrals of $\rho(r)$ over coordinate volumes, in a way similar to what is usually done for $M$ in the literature \[7, 8\]. In order to do this in a simple and organized way, we first change variables from $\lambda(r)$ to $\beta(r)$, which is defined to be such that

$$e^{2\lambda(r)} = \frac{r}{r - r_M \beta(r)},$$

(34)

which then implies that we have for the corresponding derivatives

$$2r \lambda'(r) = -r_M \frac{\beta(r) - r \beta'(r)}{r - r_M \beta(r)}.$$

(35)

Note that $\beta(r) = 0$ corresponds to $\lambda(r) = 0$ and therefore to $\exp[2\lambda(r)] = 1$ for the radial coefficient of the metric. Substituting these expressions in the component field equation shown in Equation 2, a very simple relation giving the derivative of $\beta(r)$ in terms of $\rho(r)$ results,

$$\beta'(r) = \frac{\kappa r^2 \rho(r)}{r_M}.$$

(36)

Therefore, wherever $\rho(r) = 0$, we have that $\beta(r)$ is a constant. Note that these facts are general, in the sense that they are not limited to the case in which $\rho(r)$ is constant within the matter region. It then follows from Equation 5 that we have that $\beta(r) = 1 > 0$ in the outer vacuum region, and that we have that $\beta(r) = -r_M / r_M < 0$ in the inner vacuum region. Since $\beta(r)$ is a continuous function that goes from negative to positive values, it
follows that there is a radial position \( r_z \) within the matter region where \( \beta(r_z) = 0 \), regardless of whether or not \( \rho(r) \) is constant within the shell. At this particular radial position we also have that \( \lambda(r_z) = 0 \). Let us consider now the integral of the energy density over the coordinate volume from \( r_z \) to \( r_2 \). Using Equation (36) we get

\[
4\pi \int_{r_z}^{r_2} dr \, r^2 \rho(r) = 4\pi \frac{r_M}{\kappa} \int_{r_z}^{r_2} dr \beta'(r). \tag{37}
\]

One can now see that the integral is trivial, and since we have that \( \beta(r_z) = 0 \) and that \( \beta(r_2) = 1 \), we get

\[
Mc^2 = 4\pi \int_{r_z}^{r_2} dr \, r^2 \rho(r), \tag{38}
\]

where we have replaced \( \kappa \) and \( r_M \) by their values in terms of \( M \) and \( c \). We have therefore an expression for the energy of the bound system in terms of a coordinate volume integral of the energy density. Note however that the integral does not run over the whole matter region, since it starts at \( r_z \) rather than at \( r_1 \). In a similar way, if we consider the integral from \( r_1 \) to \( r_z \), we get

\[
4\pi \int_{r_1}^{r_z} dr \, r^2 \rho(r) = 4\pi \frac{r_M}{\kappa} \int_{r_1}^{r_z} dr \beta'(r). \tag{39}
\]

Once again one can see that the integral is trivial, and since we have that \( \beta(r_z) = 0 \) and that \( \beta(r_1) = -r_{\mu}/r_M \), we now get

\[
\mu c^2 = 4\pi \int_{r_1}^{r_z} dr \, r^2 \rho(r), \tag{40}
\]

where we have replaced \( \kappa \) and \( r_{\mu} \) by their values in terms of \( \mu \) and \( c \). We have therefore an expression for the binding energy in terms of a coordinate volume integral of the energy density. If we consider Equation (33) together with the two equations derived above, we once again obtain the result that

\[
M_{\mu}c^2 = \mu c^2 + Mc^2. \tag{41}
\]

We see therefore that the point \( r_z \) where \( \beta(r_z) = 0 \) and therefore \( \lambda(r_z) = 0 \) plays a particular role when it comes to the determination of the energies involved. Note that all this is true for any function \( \rho(r) \) within the matter region. For our specific case here, we find from Equation (1) that we have within the matter region

\[
\beta(r) = 1 - \frac{\kappa \rho_0}{3r_M} \left( r_2^3 - r^3 \right) = 1 - \frac{1}{Mc^2} \frac{4\pi \rho_0}{3} \left( r_2^3 - r^3 \right), \tag{42}
\]

so that in this case we have for \( r_z \)

\[
r_z = \left( r_2^3 - \frac{3r_M}{\kappa \rho_0} \right)^{1/3}. \tag{43}
\]

Note that, although all these integrals are written in terms of the energy density \( \rho(r) \) of the matter, the energy \( Mc^2 \) is not the energy \( M_{\mu}c^2 \) of just the matter within the bound system. That would be given by the integral with the full Jacobian factor \( \sqrt{-g} \), where \( g \) is the determinant of \( g_{\mu\nu} \), which in our case here results in
\[ M_m c^2 = 4\pi \int_{r_1}^{r_2} dr \ r^2 e^{\lambda(r) + \nu(r)} \rho(r). \]  
\text{(44)}

As a partial consistency check, it is not difficult to verify that this is always smaller than \( M_u c^2 \), due to the fact that the exponent \( \lambda(r) + \nu(r) \) is always negative within the matter region. In order to show this we just take the difference between Equations \( \text{(3)} \) and \( \text{(2)} \), thus obtaining

\[ [\lambda(r) + \nu(r)]' = \frac{\kappa}{2} e^{2\lambda(r)} r [\rho(r) + P(r)]. \]
\text{(45)}

Since all quantities appearing on the right-hand side are positive or zero, we may conclude that the derivative of the exponent is non-negative. However, we have that \( \lambda(r_2) + \nu(r_2) = 0 \), since this exponent is identically zero within the outer vacuum region. It follows that

\[ \lambda(r) + \nu(r) < 0, \]
\text{(46)}

and therefore that

\[ e^{\lambda(r) + \nu(r)} < 1, \]
\text{(47)}

throughout the whole matter region, with the exception of a single point where the exponential is equal to one. Therefore, it follows for the integrals that

\[ 4\pi \int_{r_1}^{r_2} dr \ r^2 e^{\lambda(r) + \nu(r)} \rho(r) < 4\pi \int_{r_1}^{r_2} dr \ r^2 \rho(r), \]
\text{(48)}

and therefore that \( M_m c^2 < M_u c^2 \). The difference \( M_c^2 - M_m c^2 \) is the part of the energy of the bound system which is not the energy of the matter itself, but rather the energy stored in the gravitational field. In general, in order to determine this difference, \( M_m c^2 \) has to be calculated numerically.

6 Conclusions

In this paper we have established the energetic interpretation of the exact solutions obtained in a previous paper for spherically symmetric shells of liquid fluid \[2\]. All the energies involved were precisely characterized, including the total energies of the unbound systems, the total energies of the bound systems, the binding energies and the energies stored in the gravitational field. This led to a characterization of the stability of the bound systems in terms of their binding energies. We have identified a two-parameter family of energetically stable solutions, within the original three-parameter family of solutions. In a few cases the stable solutions were identified numerically.

The energetic interpretation was extended to the case of the interior Schwarzschild solution for a sphere \[3,4\], which can be obtained as a particular limit of the shell solutions, and which turns out to be a maximally unstable one. This means that there is a strong tendency of the solution for a filled sphere to spontaneously generate an internal vacuum region and thus become a shell solution. This is clearly connected to the repulsive character of the gravitational field around the origin, pushing matter and energy away from that origin, as was discussed and characterized in the previous paper \[2\].

Integral expressions for all the energies involved were presented, as integrals of the matter energy density over various coordinate volumes. Some of these expressions hold more generally than just in the case of constant energy density \( \rho(r) = \rho_0 \) that we are
directly dealing with here. A particular radial position \( r_z \) within the matter region, at which we have \( \lambda(r_z) = 0 \) and therefore \( \text{exp} [\lambda(r_z)] = 1 \) for the radial coefficient of the metric, was identified as playing a special role in relation to the integral expressions for the various energies. This is the single finite radial position where the three-dimensional space is neither stretched nor contracted.

The crucial development leading to all this was the introduction of the parameter \( r_\mu \) in the previous paper, which was shown there to be necessarily strictly positive in that case, for the correct resolution of the differential equations and the corresponding interface boundary conditions, as implied by the Einstein field equations. The apparently traditional routine of choosing \( r_\mu = 0 \) in order to eliminate the singularity at the origin not only is often incompatible with the correct resolution of the differential system, but is tantamount to selecting a solution which has no binding energy at all and is therefore maximally unstable from the energetic point of view. Both from the purely mathematical point of view and from the physical point of view, this is more often than not the incorrect choice, which we are simply not at liberty to make.

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