CONSTRUCTING ENDOMORPHISM RINGS OF LARGE FINITE GLOBAL DIMENSION

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Dedicated to the memory of Ragnar-Olaf Buchweitz

Abstract. In this paper we study endomorphism rings of finite global dimension over a ring associated to a numerical semigroup. We construct these endomorphism rings in two ways, called the lazy and greedy construction. The first main result of this paper shows that the lazy construction enables us to obtain endomorphism rings of arbitrarily large global dimension. The second main result of this paper shows that the greedy construction gives us endomorphism rings which always have global dimension two. As a consequence, for a fixed numerical semigroup, the difference of the maximal possible value and the minimal possible value of the global dimension of an endomorphism ring over that numerical semigroup can be arbitrarily large.

Contents

1. Introduction 1
   1.1. Convention and notation 2
2. Numerical Semigroups and Numerical Semigroup Rings 2
3. Projective and Simple Modules Over $\text{End}_R(M)$ 5
4. The Functor $\left\lceil \cdot \right\rceil$ 9
5. Family of Starting Rings 11
   5.1. Constructing Endomorphism Rings of Large Global Dimension 11
   5.2. Constructing Endomorphism Rings of Global Dimension Two 15
6. Acknowledgements 17
References 17

1. Introduction

The global dimension of a ring is one of the most fundamental invariants. It measures the complexity of the category of modules over a ring $R$ by looking at how far $R$-modules are being from projective. It plays important roles in algebra and geometry. For example, Auslander-Buchsbaum-Serre Theorem characterizes commutative regular local rings in terms of finiteness of global dimension.

In representation theory, it often plays important roles to construct a finitely generated module $M$ over a given ring $R$ such that the endomorphism algebra $\text{End}_R(M)$ has finite global dimension. A basic example appears in Auslander-Reiten theory: When $M$ is an additive generator of $R\text{Mod}$ (finitely generated left $R$-modules, one can replace left by right), then $\text{End}_R(M)$ has global dimension at most two (see [2] and [4]). This gives a bijection $R \to \text{End}_R(M)$ between representation-finite algebras and algebras with global dimension at most two and dominant dimension at least two. Another basic example due to Auslander...
shows that
\begin{equation}
\text{End}_R(M), \text{ where } M = \bigoplus_{i \geq 0} R/\text{rad}^i R,
\end{equation}
has finite global dimension for any finite dimensional algebra $R$ (see [1, 3, 4]).

These classical results have been extensively studied by several authors, and a number of important applications are known, e.g. Auslander’s representation dimension, Dlab-Ringel’s approach to quasi-hereditary algebras of Cline-Parshall-Scott, Rouquier’s dimensions of triangulated categories, cluster tilting in higher dimensional Auslander-Reiten theory, and non-commutative resolutions in algebraic geometry due to Van den Bergh and others. In Krull dimension one, there is a natural analog of the construction (1.1).

**Theorem 1.1.** Let
\begin{equation}
(R, m) = (R_1, m_1) \subseteq (R_2, m_2) \subseteq \ldots \subseteq (R_{l-1}, m_{l-1}) \subseteq (R_l, m_l)
\end{equation}
be a chain of local Noetherian rings, where for each $i$, $R_i$ is commutative, reduced, complete (with respect to its Jacobson radical), has Krull dimension one, and $R_l$ is regular. If $R_{i+1} \subseteq \text{End}_{R_i}(m_i)$ for $1 \leq i \leq l - 1$, then
\begin{equation}
E := \text{End}_R(M), \text{ where } M := \bigoplus_{i=1}^l R_i,
\end{equation}
has global dimension at most $l$.

**Proof.** See [9] example 2.2.3 and [10].

The ring $R = R_1$ is called the starting ring for the chain (1.2). In general, given a ring $R$ of Krull dimension one it is a hard problem to understand all the endomorphism rings $\text{End}_R(M)$ with finite global dimension, since there are a huge number of modules $M$ with $\text{End}_R(M)$ having finite global dimension. A more reasonable problem is to determine the set of all possible values of the global dimension of $\text{End}_R(M)$ in (1.3), which Ballard-Favero-Katzarkov call the global spectrum of $R$. If $R$ is a commutative, reduced, complete, local Noetherian ring with Krull dimension one, then its normalization is an endomorphism ring of finite global dimension, which has global dimension one (since it is regular). In particular, for such rings, one is always an element of the global spectrum of $R$.

The structure of this paper is as follows: In section 2 we give some of the necessary background on numerical semigroups and introduce some of the notations and definitions which will be used throughout the paper. In section 3 we analyse the projective and simple modules over our endomorphism rings. In section 4 we introduce the functor $\lceil \rceil$ and some of its properties. This functor plays a crucial role in the proofs of the main results in this paper. In section 5 we prove the two main results of this paper, first of which gives us endomorphism rings with arbitrarily large (but finite) global dimension (Theorems 5.4, 5.5), and the second being the construction of endomorphism rings which always have global dimension two (Theorem 5.9).

### 1.1. Convention and notation.

Unless otherwise stated all rings in this paper are commutative, Noetherian, and reduced. When we say a ring is complete we mean it is complete with respect to its Jacobson radical.

### 2. Numerical Semigroups and Numerical Semigroup Rings

Let $\mathbb{N}$ be the set of the positive integers and $\mathbb{N}_0$ be the set of the non-negative integers. A set $\mathcal{H} \subseteq \mathbb{N}_0$ is called a numerical semigroup if zero is an element of $\mathcal{H}$, it is closed under addition, and $\mathbb{N}_0 \setminus \mathcal{H}$ is a finite set. The Frobenius number of $\mathcal{H}$, denoted by $F(\mathcal{H})$, is the largest integer not in $\mathcal{H}$ (this is a finite number as $\mathbb{N}_0 \setminus \mathcal{H}$ is a finite set). Notice that $F(\mathcal{H}) = -1$ if and only if $\mathcal{H} = \mathbb{N}_0$, otherwise $F(\mathcal{H}) \geq 2$. We define $e(\mathcal{H}) = \min\{n \in \mathbb{N} : n \in \mathcal{H}\}$, called the multiplicity of $\mathcal{H}$, $\Gamma(\mathcal{H}) = \min\{n \in \mathbb{N} : n \leq F(\mathcal{H}) + 1 \text{ and } n \in \mathcal{H}\}$. 
Let $k$ be a field with characteristic zero. We define $R(\mathcal{H})$ to be the subring of $k[[t]]$ generated by $t^n$ over $k$ for all $n \in \mathcal{H}$. We call $R(\mathcal{H})$ the numerical semigroup ring associated to $\mathcal{H}$. Notice that the normalization $\tilde{R}(\mathcal{H})$ of $R(\mathcal{H})$ is the ring of formal power series $k[[t]]$. We set $F(R(\mathcal{H})) = F(\mathcal{H})$, $e(R(\mathcal{H})) = e(\mathcal{H})$ and $\Gamma(R(\mathcal{H})) = \Gamma(\mathcal{H})$. Unless otherwise stated we assume the elements in $\Gamma(\mathcal{H})$ are written in ascending order. Notice that $R(\mathcal{H})$ is completely determined by the set $\Gamma(\mathcal{H})$. Given a ring $R(\mathcal{H})$, the principal ideal generated by $t^\alpha$ in $R(\mathcal{H})$ is denoted by $t^\alpha R(\mathcal{H})$.

If $\Gamma(\mathcal{H}) = \{\beta_1, \beta_2, \ldots, \beta_r\}$, we write $R(\mathcal{H}) = \langle 0, \beta_1, \beta_2, \ldots, \beta_r \rangle$. Given a natural number $b$, if $a_1 b, a_2 b, \ldots, a_\beta b \in \Gamma(\mathcal{H})$ with $0 = a_1 < a_2 < \ldots < a_\beta$, we write $R(\mathcal{H}) = \langle x_1 b, \ldots, x_\beta b \rangle$, where the square consists of all the elements in $\Gamma(\mathcal{H})$ that are not multiples of $b$. This convention is naturally extended when there is more than one number with distinct multiples of it in $\Gamma(\mathcal{H})$. We can also use this convention for maximal ideals of a ring or any other ideal, or subring of $R(\mathcal{H})$.

For any numerical semigroup $\mathcal{H}$, $R(\mathcal{H})$ is a local, commutative, Noetherian, reduced, complete ring that has Krull dimension 1. Moreover, the normalization of $R(\mathcal{H})$, denoted by $\tilde{R}(\mathcal{H})$, is $k[[t]]$ (which is a regular ring), and the total quotient ring of $R(\mathcal{H})$ (obtained by inverting all non-zero divisors in $R(\mathcal{H})$), denoted by $\mathcal{T}(\mathcal{H})$, is $(k(t))$ (which is a field).

Given $A = \{\alpha_1, \alpha_2, \ldots, \alpha_r\} \subseteq \mathbb{N}$, we say that $A$ generates $\mathcal{H}$ if

$\mathcal{H} = \langle A \rangle := \{x_1 \alpha_1 + x_2 \alpha_2 + \ldots + x_s \alpha_r : x_i \in \mathbb{N}_0\}$.

We call $A$ a generating set for $\mathcal{H}$. The set $A$ is called a minimal generating set for $\mathcal{H}$ if no proper subset of $A$ is a generating set for $\mathcal{H}$. It is a standard fact that $\langle A \rangle$ forms a numerical semigroup if and only if $\gcd(A) = 1$, and every numerical semigroup arises this way. Furthermore, every numerical semigroup has a unique minimal generating set, and this set has finitely many elements (see [16] and [17]). If $\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ is a minimal generating set for the numerical semigroup $\mathcal{H}$, then

$R(\mathcal{H}) = \left\{ \sum_{i \geq 0} a_i t^i : a_i \in k, \ i \in \mathcal{H} \right\} := k[[t^{\alpha_1}, t^{\alpha_2}, \ldots, t^{\alpha_r}]]$.

Definition 2.1. Suppose $\mathcal{H}$ is a numerical semigroup with minimal generating set $\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$. Given a non-negative integer number $b$, we define $\mathcal{H}[[b]]$ to be the numerical semigroup generated by $\{\alpha_1, \alpha_2, \ldots, \alpha_r, b\}$, i.e., $\mathcal{H}[[b]] = \langle \alpha_1, \alpha_2, \ldots, \alpha_r, b \rangle$.

Example 2.2. Let $\mathcal{H} = \langle 5, 8, 17, 19 \rangle$ and $\mathcal{H}' = \mathcal{H}[[14]]$. Then,

$R(\mathcal{H}) = k[[t^5, t^8, t^{17}, t^{19}]] = \langle 0, 5, 8, 10, 13, 15 \rangle = \langle 5x, 8, 13 : x = 0, 1, 2, 3 \rangle$,

and $e(R(\mathcal{H})) = 5$, $\Gamma(R(\mathcal{H})) = \{5, 8, 10, 13, 15\}$, $F(R(\mathcal{H})) = 14$.

Moreover, $\mathcal{H}' = \langle 5, 8, 14, 17, 19 \rangle$, $R(\mathcal{H}') = k[[t^5, t^8, t^{14}, t^{17}, t^{19}]] = \langle 5x, 8, 13 : x = 0, 1, 2 \rangle$, and $e(R(\mathcal{H}')) = 5$, $\Gamma(R(\mathcal{H}')) = \{5, 8, 10, 13\}$, $F(R(\mathcal{H}')) = 12$.

Definition 2.3. We call $\mathcal{R}$ a numerical semigroup ring provided $\mathcal{R} = R(\mathcal{H})$ for some numerical semigroup $\mathcal{H}$.

Notice that $\mathcal{H}[[b]] = \mathcal{H}$ if and only if $b \in \mathcal{H}$. Suppose $\mathcal{H}$ is a numerical semigroup such that $F(\mathcal{H}) > -1$. Then, $R(\mathcal{H}) \neq \tilde{R}(\mathcal{H}) = k[[t]]$, and we have $R(\mathcal{H}) \subseteq \text{End}_{R(\mathcal{H})}(m) \subseteq \tilde{R}(\mathcal{H})$, where $m$ is the maximal ideal of $R(\mathcal{H})$ (see [6, 7, 18]). Set $\mathcal{R}_1 = R(\mathcal{H})$ and $m = m_1$. It is easy to see that $\text{End}_{\mathcal{R}_1}(m_1) = \mathcal{R}(\mathcal{K})$ for some numerical semigroup $\mathcal{K}$, where $\mathcal{H} \subseteq \mathcal{K}$. Pick a ring $\mathcal{R}_2$ such that $\mathcal{R}_1 \subseteq \mathcal{R}_2 \subseteq \text{End}_{\mathcal{R}_1}(m_1)$. Again, it is easy to see that $\mathcal{R}_2 = R(\mathcal{H}')$ for some numerical semigroup $\mathcal{H}'$, where $\mathcal{H} \subseteq \mathcal{H}' \subseteq \mathcal{K}$. If $\mathcal{R}_2 = k[[t]]$, then $\mathcal{R}_2 = \text{End}_{\mathcal{R}_1}(m_1) = k[[t]]$ in which case we define $M := \mathcal{R}_1 \oplus \mathcal{R}_2$, and $E := \text{End}_{\mathcal{R}_1}(M)$. If $\mathcal{R}_2 \neq k[[t]]$, repeat the process to obtain $\mathcal{R}_3$ such that $\mathcal{R}_2 \subseteq \mathcal{R}_3 \subseteq \text{End}_{\mathcal{R}_2}(m_2) \subseteq k[[t]]$, where $m_2$ is the maximal ideal of $\mathcal{R}_2$. If $\mathcal{R}_3 = k[[t]]$, define $M := \mathcal{R}_1 \oplus \mathcal{R}_2 \oplus \mathcal{R}_3$, and $E := \text{End}_{\mathcal{R}_1}(M)$. If $\mathcal{R}_3 \neq k[[t]]$, repeat the process to obtain $\mathcal{R}_4$, and continue in this fashion. Notice that all the rings in our chain are numerical semigroup rings associated
to some numerical semigroup, and thus are complete, local, commutative, Noetherian, reduced, and have Krull dimension 1. Moreover, since $R_1 \subseteq R_i$ for all $i$, we have $\text{End}_{R_i}(m_i) = \text{End}_{R_1}(m_i)$. Of course, it is possible that $R_1 = R_2 = R_3 = \ldots$. To avoid such chains we make the additional restriction that all the containments must be strict except for finitely many. Since $R_1$ is missing only finitely many powers of $t$, there exists an $l$ such that $R_l = \tilde{R}_1 = k[[t]]$, at which time we stop the chain. This leads us to the following definition.

**Definition 2.4.** Let $(R, m)$ be a complete local ring such that its normalization $\tilde{R}$ is regular and $R \neq \tilde{R}$. A radical chain starting from $R$ is a chain of complete local rings

$$\begin{equation}
(R, m) = (R_1, m_1) \subseteq (R_2, m_2) \subseteq \ldots \subseteq (R_{l-1}, m_{l-1}) \subseteq (R_l, m_l) = \tilde{R}_1
\end{equation}$$

such that for each $2 \leq i \leq l$, we have $R_i \subseteq \text{End}_{R_{i-1}}(m_{i-1}) = \text{End}_{R_1}(m_{i-1})$.

**Remark 1.** Notice that all rings are allowed to be repeated in the radical chain except for the normalization $R_l = \tilde{R}_1$.

**Remark 2.** By the paragraph preceding Definition 2.4, any numerical semigroup ring has a radical chain, and every ring in the radical chain is a numerical semigroup ring. Moreover, there are several radical chains with the same starting ring.

**Example 2.5.** Let $\mathcal{H} = \langle 4, 5, 6, 7 \rangle$ and $R_1 = R(\mathcal{H}) = k[[t^4, t^5, t^6, t^7]]$. Then,

$$R_1 \subseteq k[[t^2, t^3, t^5]] \subseteq k[[t]] \text{ and } R_1 \subseteq k[[t^2, t^3]] \subseteq k[[t]]$$

are both radical chains starting at $R_1$.

Suppose $R = R(\mathcal{H})$ for some numerical semigroup $\mathcal{H}$ and (2.1) is a radical chain starting at $R$. For a fixed $1 \leq i \leq l$, if $\Gamma(R_i) = \{\beta_1, \beta_2, \ldots, \beta_r\}$ written in ascending order, we define $R_{i,0} = R_i$, and

$$R_{i,j} = \text{lead}\{\beta_j, \beta_{j+1}, \ldots, \beta_r\} = R_i/(1, t^{\beta_1}, \ldots, t^{\beta_{j-1}}) \text{ for } 1 \leq j \leq r,$$

where 1 is the multiplicative identity in $k$, and $\{1, t^{\beta_1}, \ldots, t^{\beta_{j-1}}\} = \{a_0 + a_1 t^{\beta_1} + \ldots + a_{j-1} t^{\beta_{j-1}} : a_i \in k\}$. Observe that $R_{i,j}$ is an ideal of $R_i$ for $0 \leq j \leq r$, and $R_{i,1} = m_i$.

**Example 2.6.** Let $\mathcal{H} = \langle 5, 8, 17, 19 \rangle$ and $R_1 = R(\mathcal{H})$. Then, $\Gamma(R_1) = \{5, 8, 10, 13, 15\}$. In particular, $R_{1,0} = R_1$, $R_{1,1} = m_1$ (where $m_1$ is the maximal ideal of $R_1$), $R_{1,2}$ is the ideal generated by $t^n$ over $k$, where $n \in \mathcal{H}$ and $n \geq 8$, $R_{1,3}$ is the ideal generated by $t^n$ over $k$, where $n \in \mathcal{H}$ and $n \geq 10$, $R_{1,4}$ is ideal generated by $t^n$ over $k$, where $n \in \mathcal{H}$ and $n \geq 13$, and $R_{1,5}$ is the ideal generated by $t^n$ over $k$, where $n \in \mathcal{H}$ and $n \geq 15$.

We now construct two radical chains with both having the same starting ring. One of these constructions maximizes the length of the radical chain (called the “lazy” construction), while the other minimizes the length of the radical chain (called the “greedy” construction).

Given a numerical semigroup $\mathcal{H} \neq N_0$, let $R = R(\mathcal{H})$. Notice that $\mathcal{H}$ has a minimal generating set, say $\{\alpha_1, \alpha_2, \ldots, \alpha_s\}$ written in ascending order. So $\mathcal{H} = \langle \alpha_1, \alpha_2, \ldots, \alpha_s \rangle$, equivalently $R = k[[t^{\alpha_1}, t^{\alpha_2}, \ldots, t^{\alpha_s}]]$.

Given a non-negative integer $b$ with $b \neq \alpha_i$, we define $\mathcal{H}[b] = \langle \alpha_1, \alpha_2, \ldots, \alpha_s, b \rangle$. Since $\gcd(\alpha_1, \alpha_2, \ldots, \alpha_s, b) = 1$ implies that $\gcd(\alpha_1, \alpha_2, \ldots, \alpha_s, b) = 1$, the set $\mathcal{H}[b]$ is a numerical semigroup. We define $R[[t^b]] = R(\mathcal{H}[b])$, i.e., $R[[t^b]]$ is the numerical semigroup ring associated to $\mathcal{H}[b]$. It should be noted that $\mathcal{H} \subseteq \mathcal{H}[b]$, and equality holds if and only if $b \in \mathcal{H}$. Set $R = R_1$ and define $R_i = R_{i-1}[[t^{F(R_{i-1})}]]$ for $i \geq 2$. Since only finitely many powers of $t$ are missing from $R_i$, there exists an $l \geq 2$ such that $R_l = k[[t]]$. In particular, we have constructed the following radical chain of rings: $R_1 \subseteq R_2 \subseteq \cdots \subseteq R_l = k[[t]]$. By Theorem 1.1, gl. dim$(E) \leq l$. The radical chain of rings just constructed, the module $M$, and the ring $E$ are said to be constructed via the “lazy” construction.

To the other extreme, let $R_1$ be the same ring as in the previous paragraph and define $R_2 = \text{End}_{R_1}(m_1)$. Notice that $R_2$ is a numerical semigroup ring and $R_1 \subseteq R_2 \subseteq \tilde{R}_1 = k[[t]]$ (see [6, 7, 18]). If $R_2 = k[[t]]$, then stop. If not, let $R_3 = \text{End}_{R_1}(m_2)$ ($R_3$ is a numerical semigroup ring and $R_2 \subseteq R_3 \subseteq \tilde{R}_2 = \ldots$).
\( \hat{R}_1 = k[[t]] \). If \( R_3 = k[[t]] \), then stop. Otherwise, continue the process. Since only finitely many positive powers of \( t \) are missing from \( R_1 \), there exist a natural number \( r \) such that \( R_l = k[[t]] \). In particular, \( R_i = \text{End}_{R_1}(m_{i-1}) \) for \( 2 \leq i \leq l \). Since \( R_1 \) is a numerical semigroup ring, \( R_i \) is a numerical semigroup ring for each \( 1 \leq i \leq l \). The radical chain of rings \( R_1 \subseteq R_2 \subseteq ... \subseteq R_l = k[[t]] \), the module \( M \), and the ring \( E \) are said to be constructed via the “greedy” construction. By Theorem 1.1, \( \text{gl.dim}(E) \leq l \). This is the construction given in [10].

3. Projective and Simple Modules Over \( \text{End}_R(M) \)

We begin with a well known result.

**Theorem 3.1.** Let \( R \) be a complete local Noetherian commutative ring, and \( A \) be a \( R \)-algebra which is finitely generated as an \( R \)-module. Then \( \overline{A} = A/J(A) \) is a semi-simple Artinian ring, where \( J(A) \) is the Jacobson radical of \( A \). Suppose that \( 1 = e_1 + ... + e_n \) is a decomposition of \( 1 \in A \) into orthogonal primitive idempotents in \( A \). Then

\[
A = \bigoplus_{i=1}^{n} e_i A
\]

is a decomposition of \( A \) into indecomposable right ideals of \( A \) and

\[
\overline{A} = \bigoplus_{i=1}^{n} \pi_i \overline{A}
\]

is a decomposition of \( \overline{A} \) into minimal right ideals. Moreover, \( e_i A \cong e_j A \) if and only if \( \pi_i \overline{A} \cong \pi_j \overline{A} \) (see [14] Theorem 6.18, 6.21 and Corollary 6.22).

The preceding theorem says that the indecomposable summands of \( A \) are of the form \( P_i = e_i A \). By definition, the \( P_i \) are the indecomposable projective modules over \( A \). The modules \( S_i = P_i/J(A) \) are the simple modules over \( A \) (as well as over the semi-simple algebra \( \overline{A} \)) and \( P_i \rightarrow S_i \rightarrow 0 \) is a projective cover. We denote the map \( P_i \rightarrow S_i \) by \( \pi_i \) (the quotient/natural map). In particular, \( (P_i, \pi_i) \) is a projective cover for \( S_i \).

Recall that a finitely generated \( R \)-module \( M \) is *torsion-free* provided the natural map \( M \rightarrow M \otimes_R \overline{R} \) is injective, where \( \overline{R} \) is the total quotient ring of \( R \). Suppose \( R \) and \( S \) are local, Noetherian, commutative, reduced rings, that are also complete with respect to their Jacobson radicals, respectively, and have Krull dimension 1. We say that \( S \) is a *birational extension* of \( R \) provided \( R \subseteq S \) and \( S \) is a finitely generated \( R \)-module contained in the total quotient ring \( \overline{R} \) of \( R \). Notice that if \( S \) is a birational extension of \( R \), then every finitely generated torsion-free \( S \)-module is a finitely generated torsion-free \( R \)-module, but not vice versa. The following lemma follows by clearing denominators.

**Lemma 3.2.** Suppose \( S \) is a birational extension of \( R \). Let \( C \) and \( D \) be finitely generated torsion-free \( S \)-modules. Then \( \text{Hom}_R(C, D) = \text{Hom}_S(C, D) \). Furthermore, if \( M \) is a finitely generated torsion-free \( R \)-module, and \( f : C \rightarrow M \) is an \( R \)-linear map, then the image of \( f \) is an \( S \)-module.

For the remainder of this section, unless otherwise stated \( (R, m) = (R_1, m_1) \) is a numerical semigroup ring and \( R \neq k[[t]] \). Given a radical chain (2.1), Theorem 1.1 implies that \( \text{gl.dim}(E) \leq l \). We can represent \( E \) as an \( l \times l \) matrix. More specifically, \( E_{ij} = \text{Hom}_{R_1}(R_j, R_i) \). Given an integer \( 1 \leq a \leq l \), the ring \( R_a \) is a birational extension of \( R_1 \). Moreover, \( R_i \) and \( R_j \) are finitely generated torsion-free \( R_a \)-modules provided \( a \leq i, j \leq l \). In particular, Lemma 3.2 implies that \( \text{Hom}_{R_1}(R_j, R_i) = \text{Hom}_{R_a}(R_j, R_i) \) provided \( a \leq i, j \leq l \). Hence, \( E_{ij} = R_i \) for \( 1 \leq j \leq i \leq l \).

For a fixed \( 1 \leq i \leq l \), the ring \( R_i \) can appear multiple times in a radical chain. Suppose \( a \) is the smallest natural number such that \( R_i = R_a \). Let \( n_a \) be the number of times that \( R_i = R_a \) appears in the radical
chain. The (Jacobson) radical of $E$ denoted by $J(E)$ is the matrix with the following entries in its $i$-th row (see [18]):

$$(J(E))_{ij} = \begin{cases} 
R_i = R_a & \text{if } 1 \leq j \leq a - 1 \\
m_i = m_a & \text{if } a \leq j \leq a + n_a - 1 \\
E_{ij} = E_{aj} & \text{if } a + n_a \leq j \leq l
\end{cases}.$$ 

It follows that if all the rings in a radical chain are distinct, then

$$(J(E))_{ij} = \begin{cases} 
m_i & \text{if } i = j \\
E_{ij} & \text{otherwise}
\end{cases}.$$ 

Since $E$ is an associative ring with unity that is module finite over $R_1$ in its centre, the global dimension of $E$ is the supremum of the projective dimensions of the simple $E$-modules (see [5], Proposition 6.7 page 125 or [12], 7.1.14). In particular, to find the global dimension of $E$ it suffices to find the projective dimension of all the simple $E$-modules.

The ring $E$ has a decomposition $I_i = e_1 + e_2 + \ldots + e_i$ into orthogonal primitive idempotents, where $e_i$ is the $l \times l$ matrix with 1 in the $ii$-th entry and zero otherwise, and $I_i$ is the $l \times l$ identity matrix. Since $R_1$ is a complete local Noetherian commutative ring and $E$ is a finitely generated $R$-module, Theorem 3.1 implies that the right indecomposable projective modules of $E$ are the matrices $P_i = e_i E$. We sometimes identify $P_i$ with its non-zero row, that is, we think of $P_i$ as the $i$-th row of $E$, and write $P_i^*$. Furthermore, the simple $E$-modules are $S_i = P_i / J(E)$. The maps $\pi_i : P_i \to S_i = P_i / J(E)$ are the quotient/natural maps and $(P_i, \pi_i)$ is a projective cover for $S_i$. Given $R_i$ in a radical chain, if $a$ is the smallest natural number such that $R_i = R_a$, and $n_a$ is the number of times that $R_i$ occurs in our radical chain, then

$$(S_i)_{bc} = \begin{cases} 
k & \text{if } b = i \text{ and } a \leq c \leq a + n_a - 1 \\
0 & \text{otherwise}
\end{cases}.$$ 

In particular, if all the rings in a radical chain are distinct, then $S_i = e_i D_l$, where $D_l$ is the $l \times l$ diagonal matrix with diagonal entries $k$. Similar to the identification for projective modules, we sometimes identify $S_i$ with its non-zero row, and write $S_i^*$. When we identify $P_i$ with its non-zero row, we use the notation $(P_i^*)_j$ to denote the $j$-th entry of $P_i^*$. Similarly, we use the notation $(S_i^*)_j$ to denote the $j$-th entry of $S_i^*$. Notice that $P_i^*$ and $S_i^*$ are still right $E$-modules.

Suppose $X$ is an $E$-module which is represented by an $l \times l$ matrix. Then $X_i = e_i X$ is both an $R_i$-module and also a right $E$-module, and we write

$$X = \bigoplus_{i=1}^l X_i.$$ 

We sometimes identify $X_i$ with its non-zero row and write $X_i^*$. Notice that $X_i^*$ still remain an $E$-module under this identification. Suppose $X_{i_1}^*, X_{i_2}^*, \ldots, X_{i_a}^*$ are the non-zero rows of $X$, where $1 \leq i_1 < i_2 < \ldots < i_a \leq l$ and $1 \leq a \leq l$. We identify

$$X \text{ with } X^* = \left( \bigoplus_{j=1}^a X_{ij} \right)^* := \left( \begin{array}{c} X_{i_1}^* \\ X_{i_2}^* \\ \vdots \\ X_{i_a}^* \end{array} \right).$$ 

This identifies $X$ with its non-zero rows. Notice that $X^*$ is an $E$-module under this identification. A similar identification is used for maps. That is, if $f : X \to Y$, then $f^* : X^* \to Y^*$ is the map obtained by removing from $f$ all row(s) corresponding to the zero row(s) of $Y$ and all column(s) corresponding to the zero row(s) of $X$. It should be noted that the above identification is only used for $E$-modules that can be represented by an $l \times l$ matrix. There are $E$-modules which are represented by $n \times l$ matrices (where $n \in \mathbb{N}$).
Example 3.3. Suppose \( R_1 = R_2 = k[[t^2, t^4, t^5]] \), \( R_3 = R_4 = k[[t^2, t^3]] \), and \( R_5 = k[[t]] \). For the radical chain \( R_1 \subseteq R_2 \subseteq R_3 \subseteq R_4 \subseteq R_5 \),

\[
E = \begin{pmatrix}
R_1 & R_1 & m_1 & m_1 & m_1 \\
R_1 & R_1 & m_1 & m_1 & m_1 \\
R_3 & R_3 & R_3 & R_3 & m_3 \\
R_3 & R_3 & R_3 & R_3 & m_3 \\
R_5 & R_5 & R_5 & R_5 & R_5
\end{pmatrix}, \quad J(E) = \begin{pmatrix}
m_1 & m_1 & m_1 & m_1 & m_1 \\
m_1 & m_1 & m_1 & m_1 & m_1 \\
R_3 & R_3 & m_3 & m_3 & m_3 \\
R_3 & R_3 & m_3 & m_3 & m_3 \\
R_5 & R_5 & R_5 & R_5 & m_5
\end{pmatrix}.
\]

Moreover,

\[
P_1 = \begin{pmatrix}
R_1 & R_1 & m_1 & m_1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

is identified with \( P_1^* = (R_1 \ R_1 \ m_1 \ m_1) \),

\[
P_1 \oplus P_3 \oplus P_4 = \begin{pmatrix}
R_1 & R_1 & m_1 & m_1 & m_1 \\
0 & 0 & 0 & 0 & 0 \\
R_3 & R_3 & R_3 & R_3 & m_3 \\
R_3 & R_3 & R_3 & R_3 & m_3 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

which we identify with

\[
(P_1 \oplus P_3 \oplus P_4)^* = \begin{pmatrix}
R_1 & R_1 & m_1 & m_1 & m_1 \\
R_3 & R_3 & R_3 & R_3 & m_3 \\
R_3 & R_3 & R_3 & R_3 & m_3 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
P_1^* \\
P_3^* \\
P_4^*
\end{pmatrix}.
\]

The map

\[
f : P_1 \oplus P_3 \oplus P_4 \to P_4 \text{ given by } f = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
1 & t^2 & t^5 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

is identified with

\[
f^* : (P_1 \oplus P_3 \oplus P_4)^* \to P_4^* \text{ given by } f^* = \begin{pmatrix}1 & t^2 & t^5\end{pmatrix},
\]

and

\[
\pi_1 : P_1 \to S_1 = P_1/J(E) \text{ is identified with } \pi_1^* = P_1^* \to S_1^* = P_1^*/(e_1 E)^*,
\]

where \( \pi_1^* \) is the quotient map.

One should realize that if we are given \( f^* \) then we can recover \( f \) from it. For example, the map \( f^* : (P_1 \oplus P_3 \oplus P_4)^* \to P_4^* \) in Example 3.3 is given by \( f^* = \begin{pmatrix}1 & t^2 & t^5\end{pmatrix} \). Since \( f : P_1 \oplus P_3 \oplus P_4 \to P_4 \) and the second and fifth rows of its domain is all zeros, the second and fifth columns of \( f \) is all zeros. Moreover, since all the rows except for the fourth row of \( P_4 \) (co-domain of \( f \)) are zeros, all the rows except for the fourth row of \( f \) are zero. Now we simply put the inputs that \( f^* \) gives us in the non-zero entries left (in the exact order that they appear) to get \( f \) back.
Notice that for any \( 1 \leq i, j \leq l \), \( \text{Hom}_E(P_i, P_j) = \text{Hom}_E(e_i E, e_j E) \cong e_j E e_i \subseteq k[[t]] \). Therefore, any non-zero morphism \( P_i \to P_j \) is of the form \( ut^\alpha \) for some \( \alpha \in \mathbb{N}_0 \) and \( u \) a unit (an automorphism of \( P_j \)). Adjusting the morphism by multiplication by \( u^{-1} \), we can assume without loss of generality that the non-zero morphisms from \( P_i \) to \( P_j \) are multiplication with some \( t^\alpha \).

**Lemma 3.4.** Let \( R_1 \subseteq R_2 \subseteq \ldots \subseteq R_l \) be a radical chain, and let \( \{P_1, P_2, \ldots, P_l\} \) be the set consisting of the indecomposable projective modules. Suppose

\[
P \xleftarrow{f} \bigoplus_{i=1}^n Q_i, \text{ where } f^* = (t^{\alpha_1} t^{\alpha_2} \cdots t^{\alpha_n}), \ n \in \mathbb{N}, \ \alpha_i \in \mathbb{N}_0, \ \text{and } P, Q_i \in \{P_1, P_2, \ldots, P_l\}.
\]

Then, \( f^* \) is injective if and only if \( n = 1 \).

**Proof.** If \( n = 1 \), then \( f \) is obviously injective. Conversely, suppose \( n > 1 \). Recall that \( (P_i^*)_1 = R_i \) for each \( 1 \leq i \leq l \). Let \( R_a, R_b, \) and \( R_c \) be the first entries of \( P^*, Q_i^*, \) and \( Q_j^* \), respectively. Let \( w_1 = F(R_a) + 1, \ w_2 = \alpha_1 + F(R_b) + 1, \ u_3 = \alpha_2 + F(R_c) + 1 \), and set \( w = \max\{w_1, w_2, u_3\} \). Then, \( t^w \in R_a, t^{u_1 - \alpha_1} \in R_b, \) and \( t^{u_2 - \alpha_2} \in R_c \). Define \( A \) to be the matrix with entries \( A_{11} = t^{u_1 - \alpha_1}, \) and \( A_{ij} = 0 \) for all other \( i, j \), and \( B \) to be the matrix with entries \( B_{21} = t^{u_2 - \alpha_2}, \) and \( B_{ij} = 0 \) for all other \( i, j \). Then, \( A, B \in \left( \bigoplus_{i=1}^n Q_i \right)^* \), \( f^*(A) = f^*(B) \in P^* \), but \( A \neq B \). Hence, \( f \) is not injective. \( \square \)

The next proposition gives us a lower bound for the projective dimension of the simple modules.

**Proposition 3.5.** Let \( R_1 \subseteq R_2 \subseteq \ldots \subseteq R_l \) be a radical chain.

1. \( \text{pd}_E(S_i) \geq 1 \) for \( 1 \leq i \leq l \).
2. \( \text{pd}_E(S_i) \geq 2 \) for \( S_i \not\cong S_1 \).
3. \( \text{pd}_E(S_l) = 2 \).

**Proof.** (1) Since none of the \( S_i \) are projective \( E \)-modules, we have \( \text{pd}_E(S_i) \geq 1 \) for \( 1 \leq i \leq l \).

(2) Suppose \( \text{pd}_E(S_i) = 1 \) where \( S_i \not\cong S_1 \). Theorem 3.1 and Lemma 3.4 yield the following exact sequence;

\[
0 \leftarrow S_i^* \xleftarrow{\pi_i^*} P_i^* \xleftarrow{t^\alpha} P_j^* \leftarrow 0, \text{ where } \alpha \geq 0 \text{ and } 1 \leq j \leq l.
\]

Since \( S_i \not\cong S_1 \), the exact sequence (3.1) gives the following exact sequence:

\[
0 \leftarrow (S_i^*)_1 = 0 \xleftarrow{\xi_1^*} (P_i^*)_1 = R_i \xleftarrow{t^\alpha} (P_j^*)_1 = R_j \leftarrow 0.
\]

This implies that \( t^\alpha R_j = R_i \), thus \( \alpha = 0 \) (since \( k \) is a subset of \( R_l \)), which implies that \( t^\alpha \) is the identity map. In particular, \( R_i = R_j, P_i \cong P_j \), and \( P_i^* = P_j^* \). Furthermore, the sequence (3.1) and the fact that \( t^\alpha \) is the identity map give the following exact sequence:

\[
0 \leftarrow (S_i^*)_1 = k = R_i/m_i \xleftarrow{\eta} (P_i^*)_1 = R_i \xleftarrow{\text{id}} (P_j^*)_1 = (P_i^*)_1 = R_i \leftarrow 0
\]

where \( \eta \) is the natural (quotient) map. That is, \( m_i = R_i \), a contradiction.

(3) The minimal projective resolution of \( S_i^* \) is

\[
0 \leftarrow S_i^* \xleftarrow{\pi_i^*} P_i^* \xleftarrow{(1, t)} (P_{i-1} \oplus P_i)^* \xleftarrow{\lambda} P_i^* \leftarrow 0, \text{ where } \lambda = \left( \begin{array}{c} t^{e(R_i-1)} \\ -t^{e(R_i-1)-1} \end{array} \right).
\]

Hence, \( \text{pd}_E(S_l) = \text{pd}_E(S_i^*) = 2. \) \( \square \)

For the remainder of this paper, unless otherwise stated \( (R, m) = (R_1, m_1) \) is a numerical semigroup ring and \( R \neq k[[t]] \).
4. The Functor $[\ ]$

In this section we introduce a functor, denoted by $[\ ]$ and state some of its properties. This functor plays a crucial role in the proofs of the main results in this paper.

Definition 4.1. Let

$$(R, m) = (R_1, m_1) \subseteq (R_2, m_2) \subseteq (R_3, m_3) \subseteq \ldots \subseteq (R_{t-1}, m_{t-1}) \subseteq (R_t, m_t) = \tilde{R}_1 = k[[t]]$$

be a radical chain starting from $(R, m)$. Given a non-negative integer $a$, we define

$$E[a] = \text{End}_{R_1}(M[a]),$$

where $M[a] = \bigoplus_{i=1}^{l+a} T_i$ and $T_i = \begin{cases} R_1 & \text{if } 1 \leq i \leq a \\ R_{i-a} & \text{if } a+1 \leq i \leq l+a. \end{cases}$

We can represent $E[a]$ as an $(l + a) \times (l + a)$ matrix. This matrix has the following block form:

$$E[a] = \begin{pmatrix} A_{a \times a} & B_{a \times l} \\ C_{l \times a} & E \end{pmatrix},$$

where the subscripts give the dimension of each matrix, and the entries of each matrix are as follows: $A_{ij} = E_{11}$ for $1 \leq i, j \leq a$, $B_{ij} = E_{1j}$ for $1 \leq i \leq a$ and $1 \leq j \leq l$, and $C_{ij} = E_{il}$ for $1 \leq i \leq l$ and $1 \leq j \leq a$. Notice that $E[a]$ and $E$ are Morita-equivalent, so their module categories are essentially the same.

We now define a functor $[\ ]$ from the category of right $E$-modules (denoted by $\text{Mod}_E$) to the category of right $E[a]$-modules (denoted by $\text{Mod}_{E[a]}$). If $X$ is an $E$-module, then it can be represented as an $n \times l$ matrix. We define $X[a]$ to be the $(n + a) \times (l + a)$ matrix with the following block form:

$$X[a] = \begin{pmatrix} A_{a \times a} & B_{a \times l} \\ C_{n \times a} & X \end{pmatrix},$$

where $A_{a \times a} = X_{11}$ for $1 \leq i, j \leq a$, $B_{ij} = X_{1j}$ for $1 \leq i \leq a$ and $1 \leq j \leq l$, and $C_{ij} = X_{il}$ for $1 \leq i \leq n$ and $1 \leq j \leq a$. It follows that $X[a]$ is an $E[a]$-module. Moreover, $J(X[a]) = (J(X))[a]$ and $S_i[a] = P_i[a]/((J(E))[a])$.

As for the maps, we define $\pi_i[a] : P_i[a] \to S_i[a]$ to be the quotient map. In general, if $f : X \to Z$ is an $E$-morphism, where $X$ and $Z$ are $E$-modules, there are two possibilities. If $f$ is of the form $t^\alpha$ for some $\alpha \in \mathbb{N}_0$, we define $f[a] : X[a] \to Z[a]$ by $f[a] = t^\alpha$. If $f$ is represented by a matrix, let $f_i$ be the $i$-th row of $f$ and $(f_i)_j$ be the $j$-th entry of $f_i$. We define $f_i[a]$ to be the $1 \times (l + a)$ matrix with the following entries:

$$(f_i[a])_j = \begin{cases} (f_i)_1 & \text{if } 1 \leq j \leq a \\ (f_i)_j-a & \text{if } a+1 \leq j \leq l+a, \end{cases}$$

and define $f[a] : X[a] \to Z[a]$ to be the $(l + a) \times (l + a)$ matrix with the following entries:

$$(f[a])_{pq} = \begin{cases} (f_1[a])_q & \text{if } 1 \leq p \leq a \\ (f_{p-a}[a])_1 & \text{if } a+1 \leq p \leq l+a. \end{cases}$$

Observe that $X[0] = X$. Given integers $a, b \geq 0$, $(X[a])[b] = X[a+b] = (X[b])[a]$. Notice that $f[a] : X[a] \to Z[a]$ is an $E[a]$-morphism for any $E$-modules $X$ and $Z$ (similar result for $\pi_i[a]$). Hence, $[a] : E \to E[a]$ is a covariant functor.

Example 4.2. If $R_1 = k[[t^3, t^4, t^5]]$ and $R_2 = k[[t]]$, then

$$E = \begin{pmatrix} R_1 & m_1 \\ R_2 & R_2 \end{pmatrix}, \quad J(E) = \begin{pmatrix} m_1 & m_1 \\ R_2 & m_2 \end{pmatrix}, \quad E[1] = \begin{pmatrix} R_1 & R_1 & m_1 \\ R_1 & R_1 & m_1 \\ R_2 & R_2 & R_2 \end{pmatrix} = \begin{pmatrix} A_{1 \times 1} & B_{1 \times 2} \\ C_{2 \times 1} & E \end{pmatrix},$$

$$(J(E))[1] = \begin{pmatrix} m_1 & m_1 & m_1 \\ m_1 & m_1 & m_1 \\ R_2 & R_2 & m_2 \end{pmatrix} = J(E[1]),$$

where $m_1 = t^3R_2$ and $m_2 = tR_2$. 


The indecomposable projective $E$-modules are

$$P_1 = \begin{pmatrix} R_1 & m_1 \\ 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 \\ R_2 & R_2 \end{pmatrix}, \quad P_1[1] = \begin{pmatrix} R_1 & R_1 & m_1 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_2[1] = \begin{pmatrix} 0 & 0 & 0 \\ R_2 & R_2 & R_2 \end{pmatrix}.$$ 

The simple $E$-modules are

$$S_1 = P_1/J(E) = \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix}, \quad S_1[1] = \begin{pmatrix} k & k & 0 \\ k & k & 0 \end{pmatrix}, \quad S_2[1] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$ 

The indecomposable projective $E[1]$-modules are

$$Q_1 = \begin{pmatrix} R_1 & 0 & m_1 \\ 0 & 0 & 0 \end{pmatrix} \cong \begin{pmatrix} 0 & 0 & 0 \\ R_1 & R_1 & m_1 \end{pmatrix} = Q_2, \quad Q_3 = P_2[1].$$ 

The simple $E[1]$-modules are

$$U_1 = Q_1/(J(E[1])) = \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix} \cong \begin{pmatrix} 0 & 0 & 0 \\ k & k & 0 \end{pmatrix} = U_2, \quad U_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$ 

Notice that $P_1[1] = Q_1 \oplus Q_2$, $S_1[1] = U_1 \oplus U_2$, and $S_2[1] = U_3$. Applying $[1]$ to the map $P_1 \oplus P_2 = E \xrightarrow{f} P_2$, where $f = \begin{pmatrix} 0 & 0 \\ 1 & t \end{pmatrix}$, gives the map $f[1] : E[1] \rightarrow P_2[1]$ given by the matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & t \end{pmatrix}.$$ 

We identify $f[1]$ with $(f[1])^* : (E[1])^* = E[1] \rightarrow (P_2[1])^*$ given by $(1 \ 1 \ t)$.

Notice that one has to keep track of the operations in each entry when applying the functor $[\ ]$. In the preceding example, $(S_1[1])_{11} = k = R_1/m_1$, however, $(S_2[1])_{33} = k = R_2/m_2$. The following lemma is an immediate consequence of our definitions above and we record it here for future reference.

**Lemma 4.3.** (a) Let $X$ and $Y$ be $E$-modules. If $\text{Im} \left( X \xrightarrow{f} Y \right) \subseteq J(Y)$ then $\text{Im} \left( X[a] \xrightarrow{f[a]} Y[a] \right) \subseteq J(Y[a])$ for any $a \in \mathbb{N}_0$.

(b) Let $X$ and $Y$ be $E$-modules. If $X \xrightarrow{f} Y \rightarrow 0$ is exact then $X[a] \xrightarrow{f[a]} Y[a] \rightarrow 0$ is exact for all $a \in \mathbb{N}_0$.

(c) Let $X$, $Y$, and $Z$ be $E$-modules. If the sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is exact at $Y$, then $\text{Im}(f[a]) \subseteq \ker(g[a])$ for all $a \in \mathbb{N}_0$.

(d) Let $X$, $Y$, and $Z$ be $E$-modules. If the sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is exact at $Y$ and the first row of $Y$ is all zeros, then

$$\text{Im} \left( X[a] \xrightarrow{f[a]} Y[a] \xrightarrow{g[a]} Z[a] \right) \text{ is exact at } Y[a] \text{ for any } a \in \mathbb{N}_0.$$ 

(e) Direct sum and the functor $[\ ]$ commute. That is, given $E$-modules $Q_1, Q_2, \ldots, Q_n$,

$$\left( \bigoplus_{i=1}^n Q_i \right)[a] = \bigoplus_{i=1}^n (Q_i[a]).$$ 

(f) For any $E$-module $X$, $J(X[a]) = (J(X))[a]$. 
5. Family of Starting Rings

Fix an even integer \( n \geq 6 \), and pick an integer \( \frac{3n}{2} + 1 \leq a \leq 2n - 1 \). Define \( A^n_0(1) = \text{lead} \{0, n, \frac{3n}{2}\} \) (this ring only depends on \( n \)),

\[
A^n_0(i) = \text{lead} \left\{0, \frac{jn}{2}, a + 1 + (i - 2)\frac{n}{2}; j = 2, 3, \ldots, i + 1 \right\} \quad \text{for each natural number} \ i \geq 2,
\]

and \( \mathcal{F}(n, a) = \{A^n_0(i); i \in \mathbb{N}\} \). Notice that \( A^n_0(i) \) is a numerical semigroup ring for all \( i \in \mathbb{N} \), and \( F(A^n_0(i)) = F(A^n_0(i - 1)) + \frac{n}{2} \) for each natural number \( i \geq 3 \). When \( a \) and \( n \) are understood, we write \( A(i) \) for \( A^n_0(i) \). For each \( i \in \mathbb{N} \), we construct a radical chain starting from \( A(i) \):

\[
A(i) = A(i)_1 \subseteq A(i)_2 \subseteq \ldots \subseteq A(i)_{l_i} = k[[t]],
\]

and we call \( \mathcal{F}(n, a) \) a family of starting rings. We define

\[
E^i = \text{End}_{A(i)}(M^i), \quad \text{where} \quad M^i = \bigoplus_{j=1}^{l_i} A(i)_j.
\]

The indecomposable projective \( E^i \)-modules are denoted by \( P^i_1, P^i_2, \ldots, P^i_{l_i} \). Similarly, the simple \( E^i \)-modules are denoted by \( S^i_1, S^i_2, \ldots, S^i_{l_i} \). By Theorem 1.1 and Proposition 3.5, \( 2 \leq \text{gl.dim}(E^i) \leq l_i \).

Of course, different constructions of the radical chain (5.1) give rise to different \( E^i \). So for each \( i \), we must first decide which construction to apply to get the radical chain (5.1).

5.1. Constructing Endomorphism Rings of Large Global Dimension. Throughout this section, we assume the radical chain (5.1), the module \( M^i \), and the ring \( E^i \) are constructed via the lazy construction for each \( i \in \mathbb{N} \). Observe that \( A(1)_1 = A(2)_{a+1-\frac{3n}{2}} \), and \( A(i + 1)_{j+\frac{n}{2}-1} = A(i)_{j} \) for \( i \geq 2 \) and \( 1 \leq j \leq l_i \). The following proposition is a direct consequence of this observation.

**Proposition 5.1.** Using the notation introduced at the beginning of this section,

(a) \( l_1 = \frac{3n}{2} - 1 \), \( l_2 = a - 2 \), and \( l_{i+1} = l_i + \frac{n}{2} - 1 \) for \( i \geq 2 \).

(b) For all \( i \geq 2 \) and \( j = 1, 2, \ldots, l_i \),

\[
((P^{i+1}_{j+\frac{n}{2}-1})^*)_b = ((P^i_j)^*)_{b-\frac{n}{2}+1} \quad \text{for} \quad \frac{n}{2} \leq b \leq l_{i+1}.
\]

(c) For all \( i \geq 2 \) and \( 2 \leq j \leq l_i \), \( P^i_j \left[\frac{n}{2} - 1\right] = P^{i+1}_{j+\frac{n}{2}-1} \) and \( S^i_j \left[\frac{n}{2} - 1\right] = S^{i+1}_{j+\frac{n}{2}-1} \).

**Lemma 5.2.** \( \text{gl.dim}(E^1) = 2 \).

**Proof.** A quick calculation shows that the minimal projective resolutions of the simple \( E^1 \)-modules are as follows:

\[
0 \leftarrow S^1_j \xrightarrow{\pi^1_j} P^1_j \xleftarrow{\alpha^1_j} P^n_1 \leftarrow 0, \quad \text{where} \quad (\alpha^1_j)^* = t^n
\]

\[
0 \leftarrow S^1_j \xleftarrow{\pi^1_j} P^1_j \xleftarrow{\alpha^1_j} P^1_{n+j-1} \oplus P^1_{n+j-2} \leftarrow 0 \quad \text{for} \quad 2 \leq j \leq \frac{n}{2}
\]

\[
0 \leftarrow S^1_j \xleftarrow{\pi^1_j} P^1_j \xleftarrow{\gamma^1_j} P^1_{n+j-1} \oplus P^1_{n-1-j} \leftarrow 0 \quad \text{for} \quad \frac{n}{2} + 1 \leq j \leq \frac{3n}{2} - 1 = l_1,
\]

where \( (\alpha^1_j)^* = (1 \ t^n) \), \( (\beta^1_j)^* = \left(\begin{smallmatrix} t^n \\ -1 \end{smallmatrix}\right) \), \( (\gamma^1_j)^* = (1 \ t^{\frac{3n}{2}-j}) \), and \( (\lambda^1_j)^* = \left(\begin{smallmatrix} t^{\frac{3n}{2}-j+1} \\ -t \end{smallmatrix}\right) \). \( \square \)

The following notation will be very useful throughout the remainder of this paper.

**Notation.** Let \( \varepsilon = a + 1 - \frac{3n}{2} \), \( \varepsilon_1 = a + 1 - n \), \( \varepsilon_2 = a + 1 - \frac{n}{2} \), \( \zeta = (t^n \ t^{\frac{3n}{2}}) \), and

\[
\tau = \left(\begin{smallmatrix} t^{\frac{3n}{2}} & t^{2n} \\ -t^n & -t^{\frac{3n}{2}} \end{smallmatrix}\right) = \left(\begin{smallmatrix} \tau_1 \\ \tau_2 \end{smallmatrix}\right), \quad \phi = \left(\begin{smallmatrix} t^{\varepsilon_1} \\ -t^{\varepsilon_1} \end{smallmatrix}\right), \quad \eta = \left(\begin{smallmatrix} t^{\varepsilon_2} \\ -t^{\varepsilon_2} \end{smallmatrix}\right), \quad \sigma = \left(\begin{smallmatrix} \frac{3n}{2} \ t^n \\ -t^n \ -t^{\frac{3n}{2}} \end{smallmatrix}\right), \quad \mu = \left(\begin{smallmatrix} t^{\frac{n}{2}} \\ -1 \end{smallmatrix}\right)
\]
Lemma 5.3. (a) The minimal projective resolutions of $S^2_1$ is

$$0 \leftarrow S^2_1 \xleftarrow{\pi^2_1} P^2_1 \xleftarrow{f_1} P^2_{n-1} \oplus P^2_{\frac{n}{2}-1} \xleftarrow{f_2} P^2_2 \leftarrow 0,$$

where $f_1^* = \zeta$ and $f_2^* = \phi$. In particular, $\text{pd}_{E^2}(S^2_1) = 2$.

(b) If $q \geq 1$, then

$$0 \leftarrow (S^1_{3q+2})^* \xleftarrow{\pi^1_{3q+2}^*} (P^2_{1})^* \xleftarrow{\zeta} (P^3q+2_{n-1} \oplus P^3q+2_{\frac{n}{2}-2})^* \xleftarrow{\mu} N^{3q-1} \leftarrow 0$$

is an exact sequence, where $N^{3q-1}$ is any non-zero row of $(J(P^1_{3q-1}))[-\frac{3q}{2}-3]$ (they are all the same), and $\zeta((P^3q+2_{n-1} \oplus P^3q+2_{\frac{n}{2}-2})^*) = \ker((\pi^1_{3q+2}^*)^*) = (J(P^1_{3q+2})^*)^*$.

Proof. (a) Notice that $(A^2_n(2))_1 = \text{lead} \{0, n, \frac{3n}{2}, a+1\}$, $(A^2_n(2))_{n-1} = \text{lead} \{0, \varepsilon_1\}$, $(A^2_n(2))_{\frac{n}{2}-1} = \text{lead} \{0, \varepsilon\}$. A quick calculation shows that the sequence

$$0 \leftarrow (S^2_1)^* \xleftarrow{\pi^2_1^*} (P^2_1)^* \xleftarrow{\zeta} (P^2_{n-1} \oplus P^2_{\frac{n}{2}-1})^* \xleftarrow{\phi} (P^2_2)^* \leftarrow 0$$

is exact, and the result follows by Proposition 3.5.

(b) Proof is similar to the proof of part (a). \qed

Now we are in position to prove the first main result.

Theorem 5.4. If $q \geq 0$, then

$$0 \leftarrow S^1_{3q+2} \xleftarrow{d_0} W_0 \xleftarrow{d_1} W_1 \xleftarrow{d_2} W_2 \xleftarrow{d_3} \ldots \xleftarrow{d_{q+1}} W_{q+1} \xleftarrow{d_{q+2}} W_{q+2} \leftarrow 0$$

is a minimal projective resolution for $S^1_{3q+2}$, where

$$W_j = \begin{cases} 
P^3q+2_1 & \text{if } j = 0 \\
P^{3q+2}_{(n-1)+3(j-1)(\frac{n}{2}-1)} \oplus P^{3q+2}_{(n-1)+3(j-1)(\frac{n}{2}-1)+(\frac{n}{2}-1)} & \text{if } j = 1, 2, \ldots, q \\
P^{3q+2}_{(n-1)+3q(\frac{n}{2}-1)} \oplus P^{3q+2}_{(n-1)+3q(\frac{n}{2}-1)+\frac{n}{2}} & \text{if } j = q+1 \\
P^{3q+2}_{l_{3q+2}} & \text{if } j = q+2 
\end{cases}$$

$$d^*_j = \begin{cases} 
(\pi^1_{3q+2})^* & \text{if } j = 0 \\
\zeta & \text{if } j = 1 \\
\tau & \text{if } j = 2, \ldots, q+1 \\
\phi & \text{if } j = q+2 
\end{cases}$$

In particular, $\text{pd}_{E^{3q+2}}(S^1_{3q+2}) = q + 2$ for $q \in \mathbb{N}_0$. Therefore, $q + 2 \leq \text{gl. dim}(E^{3q+2}) \leq l_{3q+2}$ for $q \in \mathbb{N}_0$.

Proof. We proceed by induction on $q$. The case $q = 0$ is Lemma 5.3(a). Assume the result holds for $q - 1$ (with $q \geq 1$). By Lemma 5.3(b), the following sequence is exact

$$0 \leftarrow (S^1_{3q+2})^* \xleftarrow{(\pi^1_{3q+2})^*} (P^2_{1})^* \xleftarrow{\zeta} (P^3q+2_{n-1} \oplus P^3q+2_{\frac{n}{2}-2})^* \xleftarrow{\mu} N^{3q-1} \leftarrow 0$$

(5.2)
By induction, pd_{E^3,q-1}(S_1^{3q-1}) = (q - 1) + 2 = q + 1 (since S_1^{3(q-1)+2} = S_1^{3q-1}) and

\[ 0 \leftarrow S_1^{3q-1} \xleftarrow{f_0} L_0 \xleftarrow{f_1} L_1 \xleftarrow{f_2} L_2 \xleftarrow{f_q} L_{q+1} \leftarrow 0 \]

is a minimal projective resolution for S_1^{3q-1}, where

\[
L_j = \begin{cases} 
P_1^{3q-1} & \text{if } j = 1 \\
P_{(n-1)+3(j-1)}(\frac{n}{2} - 1) \oplus P_{(n-1)+3(j-1)}(\frac{n}{2} - 1) & \text{if } j = 1, 2, \ldots, q - 1 \\
P_{(n-1)+3(q-1)}(\frac{n}{2} - 1) \oplus P_{(n-1)+3(q-1)}(\frac{n}{2} - 1) + \tau & \text{if } j = q \\
P_{3q-1} & \text{if } j = q + 1 
\end{cases}
\]

\[
f_j^* = \begin{cases} 
\zeta & \text{if } j = 0 \\
\tau & \text{if } j = 1 \\
f_j & \text{for } j = 2, \ldots, q \\
\phi & \text{if } j = q + 1 
\end{cases}
\]

Since Im(f_1) = ker(f_0) = J(P_1^{3q-1}), the exact sequence in (5.3) yields the following exact sequence:

\[ 0 \leftarrow J(P_1^{3q-1}) \xleftarrow{f_1} L_1 \xleftarrow{f_2} \cdots \xleftarrow{f_q} L_q \xleftarrow{f_{q+1}} L_{q+1} \leftarrow 0 \]

(5.4)

Lemma 4.3 and the exact sequence (5.4) gives the following complex:

\[ 0 \leftarrow (J(P_1^{3q-1})) [ \frac{3n}{2} - 3 ] \xleftarrow{g_1} L_1 [ \frac{3n}{2} - 3 ] \xleftarrow{g_2} \cdots \xleftarrow{g_q} L_q [ \frac{3n}{2} - 3 ] \xleftarrow{g_{q+1}} L_{q+1} [ \frac{3n}{2} - 3 ] \leftarrow 0 \]

where \( g_i = f_i [ \frac{3n}{2} - 3 ] \) for \( i = 1, 2, \ldots, q + 1 \). For \( j = 2, 3, \ldots, q + 1 \), none of the indices (subscripts) of the projective modules appearing as a direct summand of \( L_j \) is one, so Lemma 4.3 implies that this complex is exact everywhere except possibly at \( L_1 [ \frac{3n}{2} - 3 ] \) (since \( L_1 = P_1^{3q-1} \)). However, this gives rise to the following exact sequence:

\[ 0 \leftarrow N^{3q-1} \xleftarrow{g_1^*} (L_1 [ \frac{3n}{2} - 3 ])^* \xleftarrow{g_2^*} \cdots \xleftarrow{g_q^*} (L_q [ \frac{3n}{2} - 3 ])^* \xleftarrow{g_{q+1}^*} (L_{q+1} [ \frac{3n}{2} - 3 ])^* \leftarrow 0 \]

(5.5)

where \( g_1^* = \zeta, g_j^* = \tau \) for \( j = 2, 3, \ldots, q \), and \( g_{q+1}^* = \phi \). Splicing exact sequences (5.2) and (5.5) yields the following exact sequence:

\[ 0 \leftarrow (S_1^{3q+2})^{(\frac{3n+2}{2})}* \xleftarrow{\zeta} (P_1^{3q+2})^* \xleftarrow{\tau} (P_1^{3q+2} \oplus P_{n-1}^{3q+2})^* \xleftarrow{\mu \zeta} (L_1 [ \frac{3n}{2} - 3 ])^* \xleftarrow{g_2^*} \cdots \xleftarrow{g_q^*} (L_{q+1} [ \frac{3n}{2} - 3 ])^* \xleftarrow{g_{q+1}^*} (L_q [ \frac{3n}{2} - 3 ])^* \leftarrow 0 \]

(5.6)
Let

\[ W_j = \begin{cases} 
P_1^{3q+2} & \text{if } j = 0 \\
\sum_{n=1}^{3q+2} P_{n-1}^{3q+2} + P_{2n-2} & \text{if } j = 1 \\
L_{j-1} \left[ \frac{3n}{2} - 3 \right] & \text{if } j = 2, \ldots, q + 2 
\end{cases} \]

and \( d_j^* = \begin{cases} 
(\pi_1^{3q+2})^* & \text{if } j = 0 \\
\zeta & \text{if } j = 1 \\
\tau & \text{if } j = 2 \\
g_j^* & \text{if } j = 3, \ldots, q + 2 
\end{cases} \)

Then, (5.6) becomes the following exact sequence:

\[ 0 \gets (S_1^{3q+2})^* \gets d_0^* W_0^* \gets d_1^* W_1^* \gets d_2^* W_2^* \cdots d_{q+1}^* W_{q+1}^* \gets 0 \]

(5.7)

For \( j = 2, \ldots, q \), Lemma 4.3 and Proposition 5.1 yields

\[ W_j = L_{j-1} \left[ \frac{3n}{2} - 3 \right] = \left( P_{n-1}^{3q-1} + P_{n-1+3(j-1)-1}^{3q-1} \right) \left[ \frac{3n}{2} - 3 \right] \]

\[ = P_{n-1+3(j-1)-1}^{3q-1} \left[ \frac{3n}{2} - 3 \right] \]

A similar computation shows that

\[ W_{q+1} = P_{(n-1)+3q(\frac{2}{3}-1)}^{3q+2} + P_{n-1+3q(\frac{2}{3}-1)+2} \] and \( W_{q+2} = P_{3q+2} \).

Hence,

\[ 0 \gets S_1^{3q+2} \gets d_0^* W_0 \gets d_1^* W_1 \gets d_2^* W_2 \cdots d_{q+1}^* W_{q+1} \gets 0 \]

(5.8)

is a projective resolution for \( S_1^{3q+2} \). By Theorem 3.1, \( 0 \gets S_1^{3q+2} \gets d_0^* W_0 \) is a projective cover for \( S_1^{3q+2} \). Moreover, by Lemma 5.3(b), \( \text{Im}(d_1) = \ker d_0 = J(W_0) = J(P_1^{3q+2}) \). Minimality of the exact sequence (5.3) implies that

\[ \text{Im} \left( L_{j-1} \xrightarrow{f_j} L_{j-2} \right) \subseteq J(L_{j-2}) \] for \( 3 \leq j \leq q + 2 \).

In particular, for \( 3 \leq j \leq q + 2 \),

\[ \text{Im}(d_j) = \text{Im}(g_{j-1}) = \text{Im} \left( f_{j-1} \left[ \frac{3n}{2} - 3 \right] \right) \subseteq J \left( L_{j-2} \left[ \frac{3n}{2} - 3 \right] \right) \text{ (Lemma 4.3)} \]

\[ = J(W_{j-1}) \]

Furthermore, a quick calculation shows that \( \text{Im}(d_2^*) = \ker d_1^* \subseteq \ker \zeta \subseteq (J(W_1))^* \). In particular, \( \text{Im}(d_2) \subseteq J(W_1) \). Hence, (5.8) is a minimal projective resolution for \( S_1^{3q+2} \), as desired. The second part is a consequence of what we just proved. \( \Box \)

The following theorem covers the cases when \( i \) is congruent to zero or 1 mod 3 (proofs are similar to the one given in Theorem 5.4).

**Theorem 5.5.** (a) If \( q \geq 1 \), then

\[ 0 \gets S_1^{3q} \gets d_0^* W_0 \gets d_1^* W_1 \gets d_2^* W_2 \cdots d_q^* W_q \gets d_{q+1}^* W_{q+1} \gets 0 \]
Lemma 5.6. Let

\[ W_j = \begin{cases} 
  P_{3q}^1 & \text{if } j = 0 \\
  P_{3q}^{(n-1)+3(j-1)(\frac{n}{2}-1)} \oplus P_{3q}^{(n-1)+3(j-1)(\frac{n}{2}-1)+(\frac{n}{2}-1)} & \text{if } j = 1, 2, \ldots, q \\
  P_{3q}^{l_{3q}} & \text{if } j = q + 1 
\end{cases} \]

\[ d_j^* = \begin{cases} 
  (\pi_{1q}^3)^* & \text{if } j = 0 \\
  \zeta & \text{if } j = 1 \\
  \tau & \text{if } j = 2, \ldots, q \\
  \eta & \text{if } j = q + 1 
\end{cases} \]

In particular, \( \text{pd}_{E_{3q}}(S_1^{3q}) = q + 1 \) for \( q \in \mathbb{N} \). Therefore, \( q + 1 \leq \text{gl.dim}(E_{3q}) \leq l_{3q} \) for \( q \in \mathbb{N} \).

(b) If \( q \geq 1 \), then

\[ 0 \leftarrow S_1^{3q+1} \xleftarrow{d_0} W_0 \xleftarrow{d_1} W_1 \xleftarrow{d_2} W_2 \xleftarrow{\cdots} W_q \xleftarrow{d_{q+1}} W_{q+1} \leftarrow 0 \]

is a minimal projective resolution for \( S_1^{3q+1} \), where

\[ W_j = \begin{cases} 
  P_{3q+1}^1 & \text{if } j = 0 \\
  P_{3q+1}^{(n-1)+3(j-1)(\frac{n}{2}-1)} \oplus P_{3q+1}^{(n-1)+3(j-1)(\frac{n}{2}-1)+(\frac{n}{2}-1)} & \text{if } j = 1, 2, \ldots, q \\
  P_{3q+1}^{l_{3q+1}-1} & \text{if } j = q + 1 
\end{cases} \]

\[ d_j^* = \begin{cases} 
  (\pi_{1q+1})^* & \text{if } j = 0 \\
  \zeta & \text{if } j = 1 \\
  \tau & \text{if } j = 2, \ldots, q \\
  \sigma & \text{if } j = q + 1 
\end{cases} \]

In particular, \( \text{pd}_{E_{3q+1}}(S_1^{3q+1}) = q + 1 \) for \( q \in \mathbb{N}_0 \). Therefore, \( q + 1 \leq \text{gl.dim}(E_{3q+1}) \leq l_{3q+1} \) for \( q \in \mathbb{N}_0 \).

5.2. Constructing Endomorphism Rings of Global Dimension Two. Throughout this section, we assume the radical chain \((5.1)\), the module \( M \), and the ring \( E \) are constructed via the greedy construction for each \( i \in \mathbb{N} \). The second main result of this paper is that \( \text{gl.dim}(E_i) = 2 \) for all \( i \in \mathbb{N} \) (Theorem 5.9).

To begin, we describe the rings in the radical chain \((5.1)\) when the chain is constructed via the greedy construction. Fix \( i \in \mathbb{N} \), let \( (A(i))_j = R_j \) and write \( R_1 = \text{lead}\{0, \beta_1^1, \beta_1^2, \ldots, \beta_r^1\} \) where \( \beta_1^j \in \Gamma(R_1) \), \( \beta_1^r = F(R_1) + 1 \). Then \( R_2 = \text{End}_{R_1}(m_1) = \text{lead}\{0, \beta_2^1 - \beta_1^1, \beta_2^2 - \beta_1^2, \ldots, \beta_r^1 - \beta_1^1\} \) where \( \beta_r^1 - \beta_1^1 = F(R_2) + 1 \). Let \( \beta_2^a = \beta_1^{a+1} - \beta_1^1 \) for \( a = 1, 2, \ldots, r - 1 \). In particular, \( R_2 = \text{lead}\{0, \beta_2^1, \beta_2^2, \ldots, \beta_r^{r-1}\} \). Similarly, \( R_3 = \text{End}_{R_1}(m_2) = \text{lead}\{0, \beta_3^1 - \beta_2^1, \beta_3^2 - \beta_2^2, \ldots, \beta_r^{r-1} - \beta_2^1\} \).

Lemma 5.6. Fix \( i \in \mathbb{N} \). For a radical chain \((5.1)\), the following holds.

(a) \( l_i = \mathcal{G}(A(i))_1 + 1 = i + 2 \). Moreover, \( \mathcal{G}(A(i))_{j+1} = \mathcal{G}(A(i))_j - 1 \) for \( j = 1, 2, \ldots, l_i - 1 \).
(b) As a matrix, the entries of $E^i$ are as follows:

$$((P_j^i)^*)_{b} = (E^i)_{j_b} = \begin{cases} A(i)_{j,0} = A(i)_j & \text{if } 1 \leq b \leq j \\ A(i)_{j,b-j} & \text{if } j + 1 \leq b \leq l_i \end{cases}$$

(c) $A(i)_{j,b} = t^{e(A(i))} A(i)_{j+1,b-1}$ for $1 \leq b \leq G(A(i)_j)$.

(d) $A(1)_1 = \text{lead}\{0, n, \frac{3n}{2}\}$, $A(i)_{l_i} = k[t]$, and for $i \geq 2$,

$$A(i)_1 = \text{lead}\left\{0, \frac{bn}{2}, a + 1 + (i - 2) \frac{n}{2} : b = 2, 3, \ldots, i + 1\right\}$$

$$A(i)_j = \text{lead}\left\{\frac{bn}{2}, a + 1 + (i - 2 - j) \frac{n}{2} : b = 0, 1, 2, \ldots, i - j + 1\right\} \text{ for } 2 \leq j \leq l_i - 2$$

$$A(i)_{l_i-1} = \text{lead}\left\{0, a + 1 - \frac{3n}{2}\right\}$$

(e) For $i \geq 3$ and $3 \leq j \leq l_i$ we have $A(i)_j = A(i - 1)_j - 1$. In particular, $P_{j-1}^{i-1}[1] = P_j^i$, $(J(P_{j-1}^{i-1}))[1] = J(P_j^i)$, and $S_{j-1}^{i-1}[1] = S_j^i$.

**Lemma 5.7.** For each $i \in \mathbb{N}$, the minimal projective resolution of $S_1^i$ is

$$0 \leftarrow S_1^i \leftarrow P_1^i \leftarrow P_2^i \leftarrow 0,$$

where $(f^i)^* = t^i$. In particular, $\text{pd}_{P^i}(S_1^i) = 1$ for all $i \in \mathbb{N}$.

**Proof.** Notice that $e(A(i)_1) = n$ for all $i \in \mathbb{N}$. By Lemma 5.6, $((P_i^i)^*)_{b} = A(i)_{1,b-1}$ for $1 \leq b \leq l_i$, and

$$(\ker(\pi_1^i))^*_{b} = \begin{cases} A(i)_{1,1} & \text{if } b = 1, 2 \\ A(i)_{1,b-1} & \text{if } 3 \leq b \leq l_i \end{cases}$$

$$= \begin{cases} t^n A(i)_{2,0} & \text{if } b = 1, 2 \\ t^n A(i)_{2,b-2} & \text{if } 3 \leq b \leq l_i \end{cases} = t^n (P_2^i)^*,$$

and the result follows. □

A simple calculation proves the following lemma.

**Lemma 5.8.** (a) $\text{gl. dim}(E^1) = \text{gl. dim}(E^2) = 2$.

(b) The minimal projective resolution of $S_2^i$, $S_3^i$, $S_{l_i-1}^i$, and $S_{l_i}^i$ are as follows:

For all $i \in \mathbb{N}$, $0 \leftarrow S_2^i \leftarrow \pi_2^i P_2^i \leftarrow f_1^i P_1^i \oplus P_3^i \leftarrow f_2^i P_2^i \leftarrow 0$, $(f_1^i)^* = (1 \ t^n)$, $(f_2^i)^* = \left(\begin{array}{c} t^n \\ -t^n \end{array}\right)$

For $i \geq 3$, $0 \leftarrow S_3^i \leftarrow \pi_3^i P_3^i \leftarrow f_3^i P_2^i \oplus P_4^i \leftarrow f_4^i P_3^i \leftarrow 0$, $(f_3^i)^* = (1 \ t^n)$, $(f_4^i)^* = \left(\begin{array}{c} t^n \\ -1 \end{array}\right)$

For $i \geq 2$, $0 \leftarrow S_{l_i-1}^i \leftarrow \pi_{l_i-1}^i P_{l_i-1}^i \leftarrow f_5^i P_{l_i-2}^i \oplus P_{l_i}^i \leftarrow f_6^i P_{l_i-1}^i \leftarrow 0$, $(f_5^i)^* = (1 \ t^n)$, $(f_6^i)^* = \left(\begin{array}{c} t^n \\ -t^n \end{array}\right)$

For $i \geq 2$, $0 \leftarrow S_{l_i}^i \leftarrow \pi_{l_i}^i P_{l_i}^i \leftarrow f_7^i P_{l_i-1}^i \oplus P_{l_i}^i \leftarrow f_8^i P_{l_i}^i \leftarrow 0$, $(f_7^i)^* = (1 \ t^n)$, $(f_8^i)^* = \left(\begin{array}{c} t^n \\ -t^n \end{array}\right)$

Now we prove the second main result of this paper.
Theorem 5.9. (a) For \( i \geq 3 \) and \( 3 \leq j \leq l_i - 2 \), the minimal projective resolutions of the simple \( S_j^i \) is:

\[
0 \leftarrow S_j^i \xleftarrow{\pi_j^i} P_j^i \xleftarrow{f_9^i} P_{j-1}^i \oplus P_{j+1}^i \xleftarrow{f_{10}^i} P_j^i \leftarrow 0, \quad (f_j^i)^* = (1 \ t^\frac{n}{2}), \quad (f_{10}^i)^* = \left( t^\frac{n}{2} \right)_1
\]

(b) \( \text{gl. dim}(E^i) = 2 \) for all \( i \in \mathbb{N} \).

Proof. (a) We proceed by induction on \( i \). For \( i = 3 \), \( l_i = 5 \), and Lemma 5.8 gives the desired result for \( S_3^3 \). Assume the result is true for \( i - 1 \geq 3 \). The minimal projective resolution of \( S_3^3 \) is given by Lemma 5.8. If \( 4 \leq j \leq l_i - 2 \), then \( 3 \leq j - 1 \leq l_i - 3 = l_{i-1} - 2 \), and the induction hypothesis gives the exact sequence

\[
0 \leftarrow S_{j-1}^{i-1} \xleftarrow{\pi_{j-1}^{i-1}} P_{j-1}^{i-1} \xleftarrow{f_9^{i-1}} P_{j-2}^{i-1} \oplus P_{j-1}^{i-1} \xleftarrow{f_{10}^{i-1}} P_{j-1}^{i-1} \leftarrow 0.
\]

Since all indices appearing in the preceding exact sequence are greater than one, applying \([1]\) to the preceding exact sequence and using Lemma 4.3 gives the following exact sequence:

\[
0 \leftarrow S_{j-1}^{i-1}[1] \xleftarrow{\pi_{j-1}^{i-1}[1]} P_{j-1}^{i-1}[1] \xleftarrow{f_9^{i-1}[1]} (P_{j-2}^{i-1} \oplus P_{j-1}^{i-1})[1] \xleftarrow{f_{10}^{i-1}[1]} P_{j-1}^{i-1} \leftarrow 0.
\]

By Lemma 5.6, the preceding exact sequence is

\[
0 \leftarrow S_j^i \xleftarrow{\pi_j^i} P_j^i \xleftarrow{f_9^i} P_{j-1}^i \oplus P_{j+1}^i \xleftarrow{f_{10}^i} P_j^i \leftarrow 0,
\]

and minimality follows from Proposition 3.5.

(b) This is a direct consequence of part (a) and Lemmas 5.7 and 5.8. \( \square \)

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