SYSTEM OF EQUATIONS INVOLVING FRACTIONAL $p$-LAPLACIAN AND DOUBLY CRITICAL NONLINEARITIES

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Abstract. This paper deals with existence of solutions to the following fractional $p$-Laplacian system of equations
\[
\begin{aligned}
(-\Delta_p)^s u &= |u|^{p^*_s-2}u + \frac{\alpha}{p^*_s}|u|^{\alpha-2}u|v|^{\beta} \quad \text{in } \Omega, \\
(-\Delta_p)^s v &= |v|^{p^*_s-2}v + \frac{\beta}{p^*_s}|v|^{\beta-2}v|u|^{\alpha} \quad \text{in } \Omega,
\end{aligned}
\]
where $s \in (0, 1)$, $p \in (1, \infty)$ with $N > sp$, $\alpha, \beta > 1$ such that $\alpha + \beta = p^*_s := \frac{Np}{N-sp}$ and $\Omega = \mathbb{R}^N$ or smooth bounded domains in $\mathbb{R}^N$. For $\Omega = \mathbb{R}^N$ and $\gamma = 1$, we show that any ground state solution of the above system has the form $(\lambda U, \tau \lambda V)$ for certain $\tau > 0$ and $U, V$ are two positive ground state solutions of 

\[
(-\Delta_p)^s u = |u|^{p^*_s-2}u \quad \text{in } \mathbb{R}^N.
\]

1. Introduction

We consider the following fractional $p$-Laplacian system of equations in $\mathbb{R}^N$:
\[
\begin{aligned}
(-\Delta_p)^s u &= |u|^{p^*_s-2}u + \frac{\alpha}{p^*_s}|u|^{\alpha-2}u|v|^{\beta} \quad \text{in } \mathbb{R}^N, \\
(-\Delta_p)^s v &= |v|^{p^*_s-2}v + \frac{\beta}{p^*_s}|v|^{\beta-2}v|u|^{\alpha} \quad \text{in } \mathbb{R}^N, \\
u, v \in \dot{W}^{s,p}(\mathbb{R}^N),
\end{aligned}
\]

where $0 < s < 1$, $p \in (1, \infty)$, $N > sp$ and $\alpha, \beta > 1$ such that $\alpha + \beta = p^*_s := \frac{Np}{N-sp}$. Here $(-\Delta_p)^s$ denotes the fractional $p$-Laplace operator which can be defined for the Schwartz class functions $\mathcal{S}(\mathbb{R}^N)$ as follows
\[
(-\Delta_p)^s u(x) := \text{P.V.} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p^*_s-2} (u(x) - u(y))}{|x-y|^{N+sp}} \, dy, \quad x \in \mathbb{R}^N,
\]
where P.V. denotes the principle value sense. Consider the following homogeneous fractional Sobolev space

\[ \dot{W}^{s,p}(\mathbb{R}^N) := \left\{ u \in L^p_s(\mathbb{R}^N) : \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \, dx \, dy < \infty \right\}. \]

The space \( \dot{W}^{s,p}(\mathbb{R}^N) \) is a Banach space with the corresponding Gagliardo norm

\[ \|u\|_{\dot{W}^{s,p}(\mathbb{R}^N)} := \left( \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \, dx \, dy \right)^{\frac{1}{p}}. \]

For simplicity of the notation we write \( \|u\|_{\dot{W}^{s,p}(\mathbb{R}^N)} \) instead of \( \|u\|_{\dot{W}^{s,p}(\mathbb{R}^N)} \). In the vectorial case, as described in [BCMP21], the natural solution space for \( (S) \) is the product space \( X = \dot{W}^{s,p}(\mathbb{R}^N) \times \dot{W}^{s,p}(\mathbb{R}^N) \) with the norm

\[ \|(u, v)\|_X := \left( \|u\|_{\dot{W}^{s,p}(\mathbb{R}^N)}^2 + \|v\|_{\dot{W}^{s,p}(\mathbb{R}^N)}^2 \right)^{\frac{1}{2}}. \]

**Definition 1.1.** We say a pair \((u, v)\) \(\in X\) is a positive weak solution of the system \((S)\) if \(u, v > 0\) and for every \((\phi, \psi)\) \(\in X\) it holds

\[
\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y)) (\phi(x) - \phi(y))}{|x-y|^{N+sp}} \, dx \, dy + \iint_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^{p-2} (v(x) - v(y)) (\psi(x) - \psi(y))}{|x-y|^{N+sp}} \, dx \, dy
\]

\[= \int_{\mathbb{R}^N} |u|^{p_s-2} u\phi \, dx + \int_{\mathbb{R}^N} |v|^{p_s-2} v\psi \, dx + \frac{\alpha}{p_s} \int_{\mathbb{R}^N} |u|^{p_s-2} u\phi \, dx + \frac{\beta}{p_s} \int_{\mathbb{R}^N} |v|^{p_s-2} v\psi \, dx.\]

Define

\[
(S) = \mathcal{S}_{\alpha+\beta} := \inf_{u \in \dot{W}^{s,p}(\mathbb{R}^N), u \neq 0} \frac{\|u\|_{\dot{W}^{s,p}(\mathbb{R}^N)}^p}{\left( \int_{\mathbb{R}^N} |u|^{p_s} \, dx \right)^{\frac{p}{p_s}}}. \]

In the limit case \( p = 1 \), the sharp constant \( \mathcal{S} \) has been determined in [FS08, Theorem 4.1] (see also [BLP14, Theorem 4.10]). The relevant extremals are given by the characteristic functions of balls, exactly as in the local case. For \( p > 1 \), (1.1) is related to the study of the following nonlocal integro-differential equation

\[
(1.2) \begin{cases}
(-\Delta_p)^s u = \mathcal{S} u^{p_s-1} & \text{in } \mathbb{R}^N, \\
u > 0, & u \in \dot{W}^{s,p}(\mathbb{R}^N).
\end{cases}
\]

In the Hilbertian case \( p = 2 \), it is known by [CT04, Theorem 1.1], the best Sobolev constant \( \mathcal{S} \) is attained by the family of functions

\[
U_t(x) = t^{\frac{2s-N}{2}} \left( 1 + \frac{|x-x_0|}{t} \right)^{\frac{2s-N}{2}}, \quad x_0 \in \mathbb{R}^N, \quad t > 0.
\]

Moreover, the family \( U_t \) is the only set of minimizers for the best Sobolev constant (see [CLO06]). However, for \( p \neq 2 \), the minimizers of \( \mathcal{S} \) is not yet known and it is not known.
whether (1.1) has any unique minimizer or not. In [BMS16], Brasco, et. al. have conjectured that the optimizers of $S$ in (1.1) are given by

$$U_t(x) = C t^{\frac{p-N}{p}} \left(1 + \left(\frac{|x-x_0|}{t}\right)^{\frac{p-N}{p-1}}\right)^{\frac{p-N}{p}}, \quad x_0 \in \mathbb{R}^N, \ t > 0,$$

but it remains as an open question till date. However, in [BMS16, Theorem 1.1], it has been proved that if $U$ is any minimizer of $S$ then $U$ is of constant sign, radially symmetric and monotone function with

$$\lim_{|x| \to \infty} |x|^{\frac{N-2p}{p-1}} U(x) = U_\infty,$$

for some constant $U_\infty \in \mathbb{R} \setminus \{0\}$.

Peng et. al in [PPW16] studied system (S) for $p = 2$ and $s = 1$ and among the other results they proved uniqueness of least energy solution. In the local case $s = 1$, a variant of system (S) (with $p = 2$) appears in various context of mathematical physics e.g. in Bose-Einstein condensates theory, nonlinear wave-wave interaction in plasma physics, nonlinear optics, for details see [AA99, Bh94, PW13] and the references therein. System of elliptic $p-$Laplacian type equations with weakly coupled nonlinearities we also cite [FP18, GPZ17] and the references therein. In the nonlocal case, there are not so many papers, in which weakly coupled systems of equations have been studied. We refer to [CS16, CMSY15, FMPSZ16, HSZ16], where Dirichlet systems of equations in bounded domains have been treated. For the nonlocal systems of equations in the entire space $\mathbb{R}^N$, we cite [BCP20, FPS17, FPZ19] and the references therein.

For $p = 2$ and $s \in (0, 1)$ Bhakta, et al. in [BCMP21] studied the following system:

$$\begin{cases}
(-\Delta)^s u = \frac{\alpha}{2} |u|^{\alpha-2} u |v|^{\beta} + f(x) & \text{in } \mathbb{R}^N, \\
(-\Delta)^s v = \frac{\beta}{2} |v|^{\beta-2} v |u|^{\alpha} + g(x) & \text{in } \mathbb{R}^N, \\
u, \ v > 0 & \text{in } \mathbb{R}^N,
\end{cases}
$$

(1.3)

where $f, g$ belongs to the dual space of $\dot{W}^{s,2}(\mathbb{R}^N)$. Among other results the authors proved that when $f = 0 = g$, any ground state solution of (1.3) has the form $(Bw, Cw)$, where $C/B = \sqrt{\beta/\alpha}$ and $w$ is the unique solution of (1.2) (corresponding to $p = 2$).

Being inspired by the above works, in this paper we generalize some of the above results in the fractional-$p$-Laplacian case.

**Definition 1.2.** (i) We say a weak solution $(u, v)$ of (S) is of the synchronized form if $u = \lambda w$, $v = \mu w$ for some constants $\lambda, \mu$ and a common function $w \in \dot{W}^{s,p}(\mathbb{R}^N)$.

(ii) We say a weak solution $(u, v)$ of (S) is a ground state solution if $(u, v)$ is a minimizer of $S_{\alpha,\beta}$ (see below (1.4)).

Define,

$$S_{\alpha,\beta} := \inf_{(u,v) \neq 0} \frac{||u||_{W^{s,p}}^p + ||v||_{W^{s,p}}^p}{\left(\int_{\mathbb{R}^N} \left( |u|^{p_s^*} + |v|^{p_s^*} + |u|^{\alpha} |v|^{\beta} \right) dx \right)^{\frac{p}{p_s}}}, \quad \alpha, \beta, p, s \in (0, 1),$$

(1.4)
Suppose that \( (S) \) has a positive solution of the synchronized form \((\lambda U, \mu U)\) for some \( \lambda > 0, \mu > 0 \) and \( U \in \dot{W}^{s,p}(\mathbb{R}^N) \) is a ground state solution of (1.2). Then it holds
\[
\lambda p^*_s - p + \frac{\alpha}{p^*_s} \mu^\beta \lambda^{\alpha - p} = 1 = \mu p^*_s - p + \frac{\beta}{p^*_s} \mu^\beta \lambda^\alpha.
\]
Now setting \( \mu = \tau \lambda \), we get
\[
\lambda p^*_s - p = \frac{p^*_s}{\alpha \tau + p^*_s} \text{ and } \tau \text{ satisfies (1.5)}
\]
\[
p^*_s + \alpha \tau^\beta - \beta \tau^\beta - p^*_s \tau^p - p^*_s = 0.
\]
On the other hand, we find that, if \( \tau \) satisfies (1.5), then \((\lambda U, \tau \lambda U)\) solves \((S)\).

Therefore, the natural question arises: is all the ground state solutions of \((S)\) of the synchronized form \((\lambda U, \tau \lambda U)\)?

If the answer of the above question is affirmative then it will hold
\[
S_{\alpha, \beta} = \frac{1 + \tau^p}{(1 + \tau^\beta + \tau^p)^p/p^*_s} S.
\]
This inspires us to define the following function
\[
h(\tau) := \frac{1 + \tau^p}{(1 + \tau^\beta + \tau^p)^p/p^*_s}.
\]
Note that \( h(\tau_{\text{min}}) = \min_{\tau \geq 0} h(\tau) \leq 1 \).

Below we state the main results of this paper.

**Theorem 1.3.** Let \((u_0, v_0)\) be any positive ground state solution of \((S)\). If one of the following conditions hold
\[
\begin{align*}
\text{(i)} & \quad 1 < \beta < p, \\
\text{(ii)} & \quad \beta = p \text{ and } \alpha < p, \\
\text{(iii)} & \quad \beta > p \text{ and } \alpha < p,
\end{align*}
\]
then, there exists unique \( \tau_{\text{min}} > 0 \) satisfying
\[
h(\tau_{\text{min}}) = \min_{\tau \geq 0} h(\tau) < 1,
\]
where \( h \) is defined by (1.6). Moreover,
\[
(u_0, v_0) = (\lambda U, \tau_{\text{min}} \lambda V),
\]
where \( U, V \) are two positive ground state solutions of (1.2). Further, \( \lambda p^*_s - p = \frac{p^*_s}{\alpha \tau_{\text{min}} + \alpha \tau_{\text{min}}^p} \).

**Remark 1.4.** Since for \( p \neq 2 \), uniqueness of ground state solutions of (1.2) is not yet known, we are not able to conclude whether any ground state solution of \((S)\) is of the synchronized form i.e, of the form of \((\lambda U, \tau_{\text{min}} \lambda U)\) or not.

Next, we consider \((S)\) with a small perturbation \( \gamma > 0 \), namely we consider the system
\[
(\tilde{S}_\gamma) \quad \begin{cases}
(\tilde{-\Delta})^s u = |u|^{p^*_s - 2} u + \frac{\alpha \gamma}{p^*_s} |u|^{\alpha - 2} u |v|^\beta \quad \text{in } \mathbb{R}^N, \\
(\tilde{-\Delta})^s v = |v|^{p^*_s - 2} v + \frac{\beta \gamma}{p^*_s} |v|^{\beta - 2} v |u|^\alpha \quad \text{in } \mathbb{R}^N,
\end{cases}
\]
\( u, v \in \tilde{W}^{s,p}(\mathbb{R}^N) \).
and prove existence of positive solutions to \(\tilde{S}_\gamma\) in various range of \(\gamma\). The corresponding energy functional of the problem \(\tilde{S}_\gamma\), given by for \((u, v) \in X\)

\[
J(u, v) = \frac{1}{p} \left( \|u\|_{W^{s,p}}^p + \|v\|_{W^{s,p}}^p \right) - \frac{1}{p^*_s} \int_{\mathbb{R}^N} \left( |u|^{p^*_s} + |v|^{p^*_s} + \gamma |u|^{\alpha} |v|^{\beta} \right) dx.
\]

We define

\[
\mathcal{N} = \left\{ (u, v) \in X : u \neq 0, v \neq 0, \|u\|_{W^{s,p}}^p = \int_{\mathbb{R}^N} \left( |u|^{p^*_s} + \frac{\alpha \gamma}{p^*_s} |u|^{\alpha} |v|^{\beta} \right) dx, \quad \|v\|_{W^{s,p}}^p = \int_{\mathbb{R}^N} \left( |v|^{p^*_s} + \frac{\beta \gamma}{p^*_s} |u|^{\alpha} |v|^{\beta} \right) dx \right\}.
\]

It is easy to see that \(\mathcal{N} \neq \emptyset\) and that any nontrivial solution of \(\tilde{S}_\gamma\) is belongs to \(\mathcal{N}\). Set

\[
A := \inf_{(u,v) \in \mathcal{N}} J(u, v).
\]

Consider the nonlinear system of algebraic equations

\[
\begin{align*}
\frac{p^*_s}{p} &+ \frac{\alpha \gamma}{p^*_s} \frac{\alpha - p}{\beta - p} k^\beta = 1, \\
\frac{p^*_s}{p} &+ \frac{\beta \gamma}{p^*_s} \frac{\beta - p}{\alpha - p} k^\alpha = 1, \\
k &> 0.
\end{align*}
\]

**Theorem 1.5.** Assume that one of the following conditions hold:

i) If \(\frac{N}{2} < p < \frac{N}{\alpha}, \alpha, \beta > p\) and

\[
0 < \gamma \leq \frac{p^*_s (p^*_s - p)}{p} \min \left\{ \frac{1}{\alpha} \left( \frac{\alpha - p}{\beta - p} \right)^{\frac{\alpha - p}{\beta - p}}, \frac{1}{\beta} \left( \frac{\beta - p}{\alpha - p} \right)^{\frac{\alpha - p}{\beta - p}} \right\};
\]

ii) If \(\frac{2N}{N + 2s} < p < \frac{N}{2s}, \alpha, \beta < p\) and

\[
\gamma \geq \frac{p^*_s (p^*_s - p)}{p} \max \left\{ \frac{1}{\alpha} \left( \frac{p - \beta}{p - \alpha} \right)^{\frac{p - \beta}{p - \alpha}}, \frac{1}{\beta} \left( \frac{p - \alpha}{p - \beta} \right)^{\frac{p - \alpha}{p - \beta}} \right\}.
\]

Then the least energy \(A = \frac{N}{N} (k_0 + \ell_0) S^{N/2p}\) and \(A\) is attained by \(\left( k_1^{1/p} U, \ell_1^{1/p} U \right)\), where \(U\) is a minimizer of (1.1), \(k_0, \ell_0\) satisfies (1.9) and

\[
k_0 = \min \{ k : (k, \ell) \text{ satisfies (1.9)} \}.
\]

**Theorem 1.6.** Assume that \(\frac{2N}{N + 2s} < p < \frac{N}{2s}\) and \(\alpha, \beta < p\). There exists \(\gamma_1 > 0\) such that for any \(\gamma \in (0, \gamma_1)\) there exists a solution \((k(\gamma), \ell(\gamma))\) of (1.9) such that \((k(\gamma))^{1/p} U, (\ell(\gamma))^{1/p} U\) is a positive solution of system \(\tilde{S}_\gamma\) with \(\tilde{J}(k(\gamma))^{1/p} U, (\ell(\gamma))^{1/p} U \geq \tilde{A}\), where \(U\) is a minimizer of (1.1),

\[
\tilde{A} = \inf_{(u,v) \in \mathcal{N}} J(u, v).
\]
and

\[ \mathcal{N} = \left\{ (u, v) \in X \setminus \{0\} : \|u\|_{W^{s,p}}^p + \|v\|_{W^{s,p}}^p = \int_{\mathbb{R}^N} \left( |u|^p + |v|^p + \gamma |u|^\alpha |v|^\beta \right) dx \right\}. \]

**Theorem 1.7.** Assume that \( \frac{2N}{N + 2s} < p < \frac{N}{2s} \) and \( \alpha, \beta < p \). Then the following system of equations

\[
\begin{aligned}
(-\Delta_p)^s u &= |u|^{p^*_s - 2} u + \frac{\alpha}{p^*_s} |u|^{\alpha - 2} u \beta \text{ in } B_R(0), \\
(-\Delta_p)^s v &= |v|^{p^*_s - 2} v + \frac{\beta}{p^*_s} |v|^{\beta - 2} v \alpha \text{ in } B_R(0), \\
u, v &\in W_0^{s,p}(B_R(0)),
\end{aligned}
\]

admit a radial positive solution \((u_0, v_0)\).

The organization of the rest of the paper is as follows: In Section 2, we prove Theorem 1.3. Section 3 deals with proof of Theorem 1.5, 1.7 and 1.6 respectively.

## 2. Proof of Theorem 1.3

**Lemma 2.1.** Suppose \( \alpha, \beta > 1 \) such that \( \alpha + \beta = p^*_s \). Then

i) \( S_{\alpha, \beta} = h(\tau_{\min}) \mathcal{S} \).

ii) \( S_{\alpha, \beta} \) has minimizers \((U, \tau_{\min} U)\), where \( U \) is a ground state solution of \((1.2)\) and \( \tau_{\min} \) satisfies

\[ \tau^{p-1} \left( p^*_s + \alpha \tau^\beta - \beta \tau^{\beta - p} - p^*_s \tau^{p^*_s - p} \right) = 0. \]

**Proof.** Let \( \{(u_n, v_n)\} \) be a minimizing sequence in \( X \) for \( S_{\alpha, \beta} \). Choose \( \tau_n > 0 \) such that \( \|u_n\|_{L^{p^*_s}(\mathbb{R}^N)} = \tau_n \|u_n\|_{L^{p^*_s}(\mathbb{R}^N)} \). Now set, \( w_n = \frac{u_n}{\tau_n} \). Therefore, \( \|u_n\|_{L^{p^*_s}(\mathbb{R}^N)} = \|w_n\|_{L^{p^*_s}(\mathbb{R}^N)} \) and applying Young’s inequality,

\[ \int_{\mathbb{R}^N} |u_n|^\alpha |w_n|^\beta dx \leq \frac{\alpha}{p^*_s} \int_{\mathbb{R}^N} |u_n|^{p^*_s} dx + \frac{\beta}{p^*_s} \int_{\mathbb{R}^N} |w_n|^{p^*_s} dx = \int_{\mathbb{R}^N} |u_n|^{p^*_s} dx = \int_{\mathbb{R}^N} |w_n|^{p^*_s} dx. \]

Therefore,

\[ S_{\alpha, \beta} + o(1) = \frac{\|u_n\|_{W^{s,p}}^p + \|v_n\|_{W^{s,p}}^p}{\left( \int_{\mathbb{R}^N} \left( |u_n|^{p^*_s} + |v_n|^{p^*_s} + |u_n|^{\alpha} |v_n|^{\beta} \right) dx \right)^{\frac{1}{p^*_s}}} \]

\[ = \frac{\|u_n\|_{W^{s,p}}^p}{\left( \int_{\mathbb{R}^N} \left( |u_n|^{p^*_s} + \tau_n^{p^*_s} |u_n|^{p^*_s} + \tau_n^{\beta} |u_n|^{\alpha} |w_n|^{\beta} \right) dx \right)^{\frac{1}{p^*_s}}} + \frac{\tau_n^p \|u_n\|_{W^{s,p}}^p}{\left( \int_{\mathbb{R}^N} \left( |u_n|^{p^*_s} + \tau_n^{p^*_s} |u_n|^{p^*_s} + \tau_n^{\beta} |u_n|^{\alpha} |w_n|^{\beta} \right) dx \right)^{\frac{1}{p^*_s}}}. \]
Thus, as $n \to \infty$, we have $h(\tau_{\text{min}})S \leq S_{\alpha,\beta}$. For the reverse inequality, we choose $u = U$, $v = \tau_{\text{min}}U$ to get $h(\tau_{\text{min}})S \geq S_{\alpha,\beta}$. In Lemma 2.2, we will show that point $\tau_{\text{min}}$ exists. This proves i).

ii) Taking $(u,v) = (U,\tau_{\text{min}}U)$, a simple computation yields that

$$\frac{1}{1 + \tau_n^\beta + \tau_n^{p^*_n}} \left[ \frac{\|u_n\|_{W^{s,p}}^p}{\left( \int_{\mathbb{R}^N} |u_n|^{p^*_n} \, dx \right)^{\frac{p}{p^*_n}}} + \frac{\tau_n^p \|w_n\|_{W^{s,p}}^p}{\left( \int_{\mathbb{R}^N} |w_n|^{p^*_n} \, dx \right)^{\frac{p}{p^*_n}}} \right] \geq \frac{1}{1 + \tau_n^\beta + \tau_n^{p^*_n}} S \geq \min_{\tau > 0} h(\tau)S.$$ 

By using i), we infer that $(U,\tau_{\text{min}}U)$ is a minimizer of $S_{\alpha,\beta}$. Further, since $\tau_{\text{min}}$ is a critical point of $h$, computing $h'(\tau_{\text{min}}) = 0$ yields that $\tau_{\text{min}}$ satisfies

$$\tau^{p-1} \left( p_s^* + \alpha \tau^\beta - \beta \tau^{\beta - p} - p_s^* \tau^{\beta - p} \right) = 0.$$

We observe from (1.6) that $h(0) = 1$ and $\lim_{\tau \to \infty} h(\tau) = 1$. Therefore, to ensure the existence of $\tau_{\text{min}}$ (i.e., minimum point of $h$ does not escape at infinity), $\tau_{\text{min}}$ is uniquely defined and $\tau_{\text{min}} > 0$, we need to investigate the solvability of the following equation

\begin{equation}
(2.1)
g(\tau) := p_s^* + \alpha \tau^\beta - \beta \tau^{\beta - p} - p_s^* \tau^{\beta - p} = 0.
\end{equation}

Lemma 2.2. Let $\alpha, \beta > 1$ and $\alpha + \beta = p_s^*$. Then (2.1) always has at least one root $\tau > 0$ and for any root $\tau > 0$, the problem $(S)$ has positive solutions $(\lambda U, \mu U)$, where

$$\mu = \tau \lambda, \quad \lambda^{p^*_s - p} = \frac{p_s^*}{p_s^* + \alpha \tau^\beta}.$$ 

Moreover, if one of the following conditions hold

(i) $1 < \beta < p$,
(ii) $\beta = p$ and $\alpha < p$,
(iii) $\beta > p$ and $\alpha < p$,

then, $\tau_{\text{min}} > 0$ and $h(\tau_{\text{min}}) < 1$. In all other cases, $\tau_{\text{min}} = 0$.

Proof. Clearly, if $\tau > 0$ solves

\begin{align*}
\begin{cases}
(p_s^* + \alpha \tau^\beta) \lambda^{p^*_s - p} = p_s^*, \\
(p_s^* \tau^{p^*_s - p} + \beta \tau^{\beta - p}) \lambda^{p^*_s - p} = p_s^*,
\end{cases}
\end{align*}

then $(\lambda U, \mu U)$ with $\mu = \tau \lambda$ solves $(S)$. Thus to prove the required result, it is enough to show that (2.1) has positive roots $\tau$, which we discuss in the following cases.
**Case 1:** If $1 < \beta < p$.

Therefore, $\lim_{\tau \to 0^+} g(\tau) = -\infty$.

Now, if $\alpha \geq p$, then $g(1) = \alpha - \beta > 0$. Thus, there exists $\tau \in (0, 1)$ such that $g(\tau) = 0$.

If $1 < \alpha < p$, then we have $p^*_s - p < p^*_s - \alpha = \beta$, and consequently, $\lim_{\tau \to \infty} g(\tau) = \infty$. Thus there exists $\tau > 0$ such that $g(\tau) = 0$.

Also observe that, by direct computation we obtain

$$h'(\tau) = f(\tau)g(\tau), \quad \text{where} \quad f(\tau) = \frac{p\tau^{p-1}}{p^*_s(1 + \tau^\beta + \tau^{p^*_s})^{\frac{p^*_s}{p^*_s + 1}}}.$$ 

Thus, $f(\tau) \geq 0$ for all $\tau > 0$ and $f(0) = 0$. This together with the fact that $\lim_{\tau \to 0^+} g(\tau) = -\infty$ implies $h'(\tau) < 0$ in $\tau \in (0, \epsilon)$ for some $\epsilon > 0$. This means $h$ is a decreasing function near 0.

Combining this with the fact that $h(0) = 1$ and $\lim_{\tau \to \infty} h(\tau) = 1$, we conclude that there exists a point $\tau_{\text{min}} \in (0, \infty)$ such that $\min_{\tau \geq 0} h(\tau) = h(\tau_{\text{min}}) < 1$ and this holds for all $\alpha > 1$.

**Case 2:** If $\beta = p$.

In this case $g$ becomes $g(\tau) = \alpha(1 + \tau^p) - p^*_s\tau^\alpha$. Hence $g(0) = \alpha > 0$ and $g(1) = \alpha - p$.

(i) If $\alpha = p$, then $N = 2sp$ and $g(\tau) = p - \tau^p$. Thus there exists a unique root $\tau_1 = 1$ of $g$.

Also note that $h$ is increasing near 0. Hence $\tau_1$ is the maximum point of $h$ with $h(\tau_1) > 1$. In this case $\min_{\tau \geq 0} h(\tau) = h(\tau_{\text{min}}) = h(0)$.

(ii) If $1 < \alpha < p$, then we have $2sp < N < sp(p + 1)$. Observe that $g(0) = \alpha > 0$, $g(1) = \alpha - p < 0$ and $\lim_{\tau \to \infty} g(\tau) = +\infty$. Also note that, $g$ is decreasing in $\left(0, \left(p^*_s/p\right)^{1/(p-\alpha)}\right)$ and increasing in $\left(\left(p^*_s/p\right)^{1/(p-\alpha)}, \infty\right)$. Therefore, $g$ has exactly one critical point $\left(p^*_s/p\right)^{1/(p-\alpha)}$ and two roots $\tau_i$ ($i = 1, 2$) with $\tau_1 \in (0, \left(p^*_s/p\right)^{1/(p-\alpha)})$, $\tau_2 \in (\left(p^*_s/p\right)^{1/(p-\alpha)}, \infty)$. Further, note that in this case $h$ is increasing near 0 which leads that first positive critical point of $h$, i.e., $\tau_1$ is
the local maximum for $h$ and $h(\tau_1) > 1$. Further, as $\lim_{\tau \to \infty} h(\tau) = 1$, the second root of $g$
\,i.e., $\tau_2$ becomes the 2nd and last critical point of $h$ and $h(\tau_2) < 1$. Therefore, in this case
\,\tau_{\min} = \tau_2 > 0$ is the minimum point of $h$ with $h(\tau_{\min}) < 1$.

(iii) If $\alpha > p$, then $sp < N < 2sp$ and $g(0) > 0$. We see that $g$ is increasing in \((0, \frac{1}{p/p^*})\)
\,and decreasing in \((\frac{1}{p/p^*}, \infty)\). This together with the fact $\lim_{\tau \to \infty} g(\tau) = -\infty$ leads that
\,there exists a unique $\tau > 0$ such that $g(\tau) = 0$.

Since in this case, $h$ is increasing near 0, so at $\tau$, $h$ attains the maximum with $h(\tau) > 1$.
\,Hence, $h$ has no other critical point and therefore, $h(\tau_{\min}) = h(0)$.

**Case 3:** If $\beta > p$.
\,If $1 < \alpha \leq p$, then $g(1) = \alpha - \beta \leq 0$. Since $g(0) > 0$, there is a $\tau \in (0, 1]$ such that $g(\tau) = 0$.
\,If $\alpha > p$ and $\alpha > \beta$, then $g(1) > 0$ and $\lim_{\tau \to \infty} g(\tau) = -\infty$. Thus there exists $\tau \in (1, \infty)$ such
\,that $g(\tau) = 0$. If $\alpha > p$ and $\alpha \leq \beta$, then $g(1) \leq 0$. As $g(0) > 0$, thus there exists $\tau \in (0, 1]$ such
\,that $g(\tau) = 0$. Next we analyse $\tau_{\min}$ in case 3 in the following three subcases.

(i) $\beta > p$ and $\alpha > p$.
\,Observe that in this case we have

$$\beta < p^*_s - p \quad \text{and} \quad \alpha < p^*_s - p.$$

Hence, without loss of generality we can assume $\alpha \geq \beta$.
\,Claim 1: $g(\tau) > 0$ for $\tau \in [0, 1)$. Indeed, using (2.2), $\tau \in [0, 1)$ implies $\tau^\alpha, \tau^\beta > \tau^{p^*_s-p}$.
\,Therefore,

$$g(\tau) > p^*_s \alpha \tau^\beta - \beta \tau^{\beta-p} - p^*_s \tau^\beta = p^*_s + (\alpha - p^*_s) \tau^\beta - \beta \tau^{\beta-p}$$
$$> p^*_s + \alpha - p^*_s - \beta \tau^{\beta-p} \quad (\text{as} \, \alpha < p^*_s \, \text{and} \, \tau^\beta < 1)$$
$$= \alpha - \beta \tau^{\beta-p} > 0,$$

where in the last inequality we have used the fact that $\tau^{\beta-p} < 1 \implies \tau^{\beta-p} < \beta \leq \alpha$. This proves the claim 1.
Claim 2: \( g \) is monotonically decreasing for \( \tau \geq 1 \). Indeed, \( \tau \geq 1 \) implies \( \tau^\alpha \geq \tau^p \). Therefore, using (2.2) we have

\[
g'(\tau) = \tau^{\beta-p-1} [\alpha \beta \tau^p - p^*_s(p^*_s - p) \tau^\alpha - \beta (\beta - p)] \\
\leq \tau^{\beta-p-1} [(\alpha \beta - p^*_s(p^*_s - p)) \tau^\alpha - \beta (\beta - p)] \\
\leq \tau^{\beta-p-1} [(p^*_s - p)(\beta - p^*_s) \tau^\alpha - \beta (\beta - p)] \\
< 0.
\]

This proves claim 2. Also observe that \( g(1) \geq 0 \) and \( g(\tau) \to -\infty \) as \( \tau \to \infty \). Combining these facts along with claim 1 and 2 above proves that \( g \) has only one root say \( \tau \in (0, \infty) \), which in turn implies \( h \) has only one critical point \( \tau \in (0, \infty) \). Since \( \beta > p \) implies \( h \) is increasing near 0, so at \( \tau \), \( h \) attains the maximum with \( h(\tau) > 1 \). Combining this with \( \lim_{\tau \to \infty} h(\tau) = 1 \) proves that \( h(\tau_{\min}) = h(0) = 1 \), i.e, \( \tau_{\min} = 0 \).

(ii) \( \beta > p \) and \( \alpha < p \).

In this case \( g(0) > 0, g(1) < 0 \) and we claim \( g \) is strictly decreasing in \((0, 1)\). Indeed, \( \alpha < p \implies \tau^p < \tau^\alpha \) for \( \tau \in (0, 1) \). Also \( \beta > p \implies \alpha < p^*_s - p \). Therefore,

\[
g'(\tau) = \tau^{\beta-p-1} [\alpha \beta \tau^p - p^*_s(p^*_s - p) \tau^\alpha - \beta (\beta - p)] \\
< \tau^{\beta-p-1} [(p^*_s - p)(\beta - p^*_s) \tau^\alpha - \beta (\beta - p)] \\
< 0.
\]

Claim: \( g \) has only one critical point in \((1, \infty)\). Indeed,

\[
g'(\tau) = \tau^{\beta-p-1} g_1(\tau), \quad \text{where} \quad g_1(\tau) := \alpha \beta \tau^p - p^*_s(p^*_s - p) \tau^\alpha - \beta (\beta - p).
\]

So to prove that \( g \) has only critical point in \((1, \infty)\), it’s enough to show that \( g_1 \) has only one root in \((1, \infty)\). Observe that, \( g_1(0) < 0, \lim_{\tau \to \infty} g_1(\tau) = \infty \) and a straight forward computation yields that \( g_1 \) is a decreasing function in \((0, \frac{p^*_s(p^*_s - p)}{p^*_s})\) and \( g_1 \) is an increasing function in \((\frac{p^*_s(p^*_s - p)}{p^*_s}, \infty)\). Thus, \( g_1 \) has only one root. Hence the claim follows. Next, we observe that \( \alpha < p \implies \beta > p^*_s - p \) and therefore, \( \lim_{\tau \to \infty} g(\tau) = \infty \). Combining all the above observations and claim, it follows that \( g \) has only one critical point in \((0, \infty)\) and two roots \( \tau_1, \tau_2 \) with \( \tau_1 \in (0, 1) \) and \( \tau_2 \in (1, \infty) \). Hence, \( h \) has exactly two critical points \( \tau_1, \tau_2 \).
Since $h$ is increasing near $0$ leads to the conclusion that first positive critical point of $h$, i.e., $\tau_1$ is the local maximum for $h$ and $h(\tau_1) > 1$ and since, $\lim_{\tau \to \infty} h(\tau) = 1$, at the second critical point of $h$ i.e., at $\tau_2$ we have $h(\tau_2) < 1$. Therefore, in this case $\tau_{\min} = \tau_2 > 0$ is the minimum point of $h$ with $h(\tau_{\min}) < 1$.

(iii) $\beta > p$, $\alpha = p$.
In this case, $g(0) > 0$ and $\alpha = p \implies \beta = p^*_s - p$. Therefore,

$$g'(\tau) = \tau^{\beta - p - 1}\left[\alpha\beta\tau^p - p^*_s(p^*_s - p)\tau^\alpha - \beta(\beta - p)\right]$$

$$= \tau^{\beta - p - 1}\left[(\alpha - p^*_s)\beta\tau^\alpha - \beta(\beta - p)\right] < 0,$$

i.e., $g$ is a strictly decreasing function. Also, observe that $\lim_{\tau \to \infty} g(\tau) = -\infty$. Hence, $g$ has only one root in $(0, \infty)$, i.e., $h$ has only critical point $\tau$ in $(0, \infty)$. Since $\beta > p$ implies $h$ is increasing near $0$, so at $\tau$, $h$ attains the maximum with $h(\tau) > 1$. Combining this with $\lim_{\tau \to \infty} h(\tau) = 1$ proves that $h(\tau_{\min}) = h(0) = 1$, i.e, $\tau_{\min} = 0$.

In order to prove Theorem 1.3, next we introduce an auxiliary system of equations with a positive parameter $\eta$,

$$\begin{cases} (-\Delta_p)^s u = \eta |u|^{p^*_s - 2} u + \frac{\alpha}{p^*_s} |u|^\alpha |v|^\beta \text{ in } \mathbb{R}^N, \\
(-\Delta_p)^s v = |v|^{p^*_s - 2} v + \frac{\beta}{p^*_s} |v|^\beta |u|^\alpha \text{ in } \mathbb{R}^N, \\
u, v \in \dot{W}^{s,p}(\mathbb{R}^N). \end{cases}$$

We define the following minimization problem associated to $(\mathcal{S}_\eta)$:
Similarly for $\tau > 0$, we define
\[
f_\eta(\tau) := \frac{1 + \tau^p}{(\eta + \tau^\beta + \tau p^*_s)^{p/p^*_s}}, \quad f_\eta(\tau_{\min}) = \min_{\tau \geq 0} f_\eta(\tau).
\]

Proceeding as in the proof of Lemma 2.2, we find $\epsilon \in (0, 1)$ small such that $\tau_{\min}(\eta), \lambda^*(\eta), \mu^*(\eta)$ are unique for $\eta \in (1 - \epsilon, 1 + \epsilon)$ and $\tau_{\min}(\eta)$ satisfies
\[
\tau^{p-1} (\eta p^*_s + \alpha \tau^\beta - \beta \tau^{\beta-p} - p^*_s \tau^{p^*_s-p}) = 0.
\]

Moreover, $\tau_{\min}(\eta), \lambda^*(\eta), \mu^*(\eta)$ are $C^1$ for $\eta \in (1 - \epsilon, 1 + \epsilon)$ and $\epsilon > 0$ small. Indeed, if we denote
\[
F(\eta, \tau) = \eta p^*_s + \alpha \tau^\beta - \beta \tau^{\beta-p} - p^*_s \tau^{p^*_s-p}.
\]

Then,
\[
\frac{\partial F}{\partial \eta} = \tau^{\beta-p-1} \left[ \alpha \beta \tau^p - p^*_s (p^*_s - p) \tau^{\alpha} - \beta (\beta - p) \right].
\]

Since $\tau_{\min}$ is the minimum of $h$, direct computation yields $g(\tau_{\min}) = 0$, $g'(\tau_{\min}) > 0$. Therefore, $F(1, \tau_{\min}) = 0$, $\frac{\partial F}{\partial \eta}(1, \tau_{\min}) > 0$. Consequently, by implicit function theorem, we obtain that $\tau_{\min}(\eta), \lambda^*(\eta), \mu^*(\eta)$ are $C^1$ for $\eta \in (1 - \epsilon, 1 + \epsilon)$.

**Proof of Theorem 1.3:** Let $(u_0, v_0)$ is a ground state solution of (S). First, we claim that
\[
(2.3) \int_{\mathbb{R}^N} |u_0|^{p^*_s} dx = \lambda p^*_s \int_{\mathbb{R}^N} |U|^s dx.
\]

In order to prove this, we define the following min-max problem associated to $(S_\eta)$
\[
B(\eta) := \inf_{(u,v) \in X \setminus \{0\}} \max_{t > 0} E_\eta(tu, tv),
\]
where
\[
E_\eta(u, v) := \frac{1}{p} \left( \|u\|_{W^{s,p}}^{p} + \|v\|_{W^{s,p}}^{p} \right) - \frac{1}{p^*_s} \int_{\mathbb{R}^N} \left( \eta (u^+)^{p^*_s} + (v^+)^{p^*_s} + (u^+)^{\alpha} (v^+)^{\beta} \right) dx.
\]

Observe that there exists $t(\eta) > 0$ such that $\max_{t > 0} E_\eta(tu_0, tv_0) = E_\eta(t(\eta)u_0, t(\eta)v_0)$ and moreover, $t(\eta)$ satisfies $H(\eta, t(\eta)) = 0$, where $H(\eta, t) = b^{p^*_s-p} (\eta G + D) - C$ with
\[
C := \|u_0\|_{W^{s,p}}^{p} + \|v_0\|_{W^{s,p}}^{p}, \quad D := \int_{\mathbb{R}^N} \left( |v_0|^{p^*_s} + |u_0|^{\alpha} |v_0|^{\beta} \right) dx \quad \text{and} \quad G := \int_{\mathbb{R}^N} |u_0|^{p^*_s} dx.
\]

As $(u_0, v_0)$ is a least energy solution of (S), then
\[
H(1, 1) = 0, \quad \frac{\partial H}{\partial t}(1, 1) > 0 \quad \text{and} \quad H(\eta, t(\eta)) = 0.
\]
Thus, by the implicit function theorem, there exists $\epsilon > 0$ such that $t(\eta) : (1 - \epsilon, 1 + \epsilon) \to \mathbb{R}$ is $C^1$ and

$$t'(\eta) = -\frac{\partial H}{\partial \eta} \bigg|_{\eta=1-t} = -\frac{G}{(p_*^s - p)(G + D)}.$$ 

By Taylor expansion, we also have $t(\eta) = 1 + t'(1)(\eta - 1) + O \left(|\eta - 1|^2\right)$ and thus

$$t^p(\eta) = 1 + pt'(1)(\eta - 1) + O \left(|\eta - 1|^2\right).$$

Since, $H(1, 1) = 0$ implies $C = G + D$, and $H(\eta, t(\eta)) = 0$ implies $C = t(\eta)p_*^{s-p} (\eta G + D)$. Therefore by definition of $B(\eta)$ and above we obtain

$$B(\eta) \leq E_\eta(t(\eta)u_0, t(\eta)v_0) = \frac{t(\eta)^p}{p} C - \frac{t(\eta)p_*^s}{p_*^s} (\eta G + D) = \frac{t(\eta)^p}{p} C = \frac{t(\eta)^p}{p} B(1)$$

$$= B(1) - \frac{p GB(1)}{(p_*^s - p)(G + D)}(\eta - 1) + O \left(|\eta - 1|^2\right).$$

Now, let us compute $B(1)$ from the definition

$$B(1) = \inf_{(u,v) \in X} E_1(t_{\max}u, t_{\max}v), \quad \text{where } p_{\max}^{s-p} = \frac{\|u\|_{W^{s,p}}^p + \|v\|_{W^{s,p}}^p}{\int_{\mathbb{R}^N} \left(|u|^{p_*^s} + |v|^{p_*^s} + |u|^\alpha |v|^\beta\right) dx}$$

$$= \frac{s}{N} \inf_{(u,v) \in X} \left( \frac{\|u\|_{W^{s,p}}^p + \|v\|_{W^{s,p}}^p}{\left(\int_{\mathbb{R}^N} \left(|u|^{p_*^s} + |v|^{p_*^s} + |u|^\alpha |v|^\beta\right) dx\right)^{p_*^s/p^{s-p}}} \right)^{p_*^s/p^{s-p}}$$

$$= \frac{s}{N} \left( \frac{\|u_0\|_{W^{s,p}}^p + \|v_0\|_{W^{s,p}}^p}{\left(\int_{\mathbb{R}^N} \left(|u_0|^{p_*^s} + |v_0|^{p_*^s} + |u_0|^\alpha |v_0|^\beta\right) dx\right)^{p_*^s/p^{s-p}}} \right)^{p_*^s/p^{s-p}}$$

$$= \frac{s}{N} (G + D).$$

Using this in (2.4), we obtain

$$B(\eta) \leq B(1) - \frac{G}{p_*^s} (\eta - 1) + O \left(|\eta - 1|^2\right).$$

Therefore, we have

$$\frac{B(\eta) - B(1)}{\eta - 1} \begin{cases} \leq -\frac{G}{p_*^s} + O \left(|\eta - 1|\right) & \text{if } \eta > 1, \\ \geq -\frac{G}{p_*^s} + O \left(|\eta - 1|\right) & \text{if } \eta < 1. \end{cases}$$
This implies that
\[ B'(1) = -\frac{G}{p^*_s} = -\frac{1}{p^*_s} \int_{\mathbb{R}^N} |u_0|^{p^*_s} \, dx. \]  

Arguing similarly as in the proof of Lemma 2.1, it follows that \( S_{\eta,\alpha,\beta} \) is attained by \((tU, \tau(\eta)tU)\). Therefore,
\[ B(\eta) = \frac{s}{N} \left( \frac{1+\tau(\eta)^p}{(\eta+\tau(\eta)^\beta + \tau(\gamma)^p s^*)^{\frac{\alpha}{p}}} \right) \int_{\mathbb{R}^N} |U|^{p^*_s} \, dx = \frac{s}{N} \left( \frac{1+\tau(\eta)^p}{(\eta+\tau(\eta)^\beta + \tau(\gamma)^p s^*)^{\frac{\alpha}{p}}} \right) \int_{\mathbb{R}^N} |U|^{p^*_s} \, dx. \]

Then, from a simple computation it follows
\[ B'(\eta) = \frac{(1 + \tau(\eta)^p)^{\frac{\alpha}{p}}}{s^*_p (\eta + \tau(\eta)^\beta + \tau(\gamma)^p s^*)^{\frac{\alpha}{p}}} \times \left[ \tau'(\eta)\tau(\eta)^{p-1} \left( \eta s^* + \alpha\tau(\eta)^\beta - \beta\tau(\eta)^{\beta-p} - p^*_s \tau(\eta)^{p^*_s-p} \right) - 1 - \tau(\eta)^p \right] \int_{\mathbb{R}^N} |U|^{p^*_s} \, dx. \]

Note that for \( \eta = 1, \tau(1) \) satisfies the equation \( g(\tau) = 0 \), where \( g(\tau) \) is given by (2.1), thus we obtain \( \tau(1) = \tau_{\min} \). Consequently,
\[ B'(1) = -\frac{1}{p^*_s} \left( \frac{1 + \tau_{\min}^\beta}{1 + \tau_{\min}^\beta + \tau_{\min}^p} \right) \int_{\mathbb{R}^N} |U|^{p^*_s} \, dx = \frac{\lambda^{p^*_s}}{p^*_s} \int_{\mathbb{R}^N} |U|^{p^*_s} \, dx. \]

Combining (2.5) and (2.6) we conclude (2.3). By a similar argument as in the proof of (2.3), we show that
\[ \int_{\mathbb{R}^N} |v_0|^{p^*_s} \, dx = \tau_{\min}^{p^*_s} \lambda^{p^*_s} \int_{\mathbb{R}^N} |U|^{p^*_s} \, dx, \quad \int_{\mathbb{R}^N} |u_0|^{\beta} \, dx = \tau_{\min}^{\beta} \lambda^{p^*_s} \int_{\mathbb{R}^N} |U|^{p^*_s} \, dx. \]

Therefore, by (2.3) and (2.7), we obtain
\[ \int_{\mathbb{R}^N} |u_0|^{\alpha} |v_0|^{\beta} \, dx = \tau_{\min}^{\beta} \int_{\mathbb{R}^N} |v_0|^{p^*_s}, \quad \int_{\mathbb{R}^N} |u_0|^{\alpha} |v_0|^{\beta} \, dx = \tau_{\min}^{p^*_s - \beta} \int_{\mathbb{R}^N} |v_0|^{p^*_s} \, dx. \]

Again, since \((\lambda U, \mu U)\) solves the problem \((S)\), we get
\[ \lambda^{p^*_s - \beta} + \frac{\alpha}{p^*_s} \mu^{\beta} \lambda^{\alpha - p} = 1 = \mu^{p^*_s - \beta} + \frac{\beta}{p^*_s} \lambda^{\beta - p} \lambda^{\alpha}. \]

Now define \((u_1, v_1) := \left( \frac{u_0}{\lambda}, \frac{v_0}{\mu} \right)\). Using (2.3), (2.7) and (2.8) we have
\[ \|u_1\|_{W^{s,p}}^p = \lambda^{-p} \|u_0\|_{W^{s,p}}^p = \lambda^{-p} \int_{\mathbb{R}^N} \left( |u_0|^{p^*_s} + \frac{\alpha}{p^*_s} |u_0|^{\alpha} |v_0|^{\beta} \right) \, dx. \]
\[ = \lambda^{-p} \left( \lambda u^\ast + \frac{\alpha}{p^-} \mu^\beta \lambda \right) \int_{\mathbb{R}^N} |U|^p dx = \|U\|^p_{W^{s,p}}. \]

Similarly, we obtain \( \|v_1\|^p_{W^{s,p}} = \|U\|^p_{W^{s,p}} \). Therefore, we have

\[(2.9) \quad \|u_1\|^p_{W^{s,p}} = \|U\|^p_{W^{s,p}} = \|v_1\|^p_{W^{s,p}}. \]

Also, by (2.3)

\[(2.10) \quad \int_{\mathbb{R}^N} |u_1|^p dx = \int_{\mathbb{R}^N} |U|^p dx, \]

and by (2.7),

\[(2.11) \quad \int_{\mathbb{R}^N} |v_1|^p dx = \int_{\mathbb{R}^N} |U|^p dx. \]

Thus, from (2.9) and (2.10), we conclude \( u_1 \) achieves \( S \). Further, from (1.1), (2.9) and (2.11) implies \( v_1 \) also achieves \( S \) in (1.1). This completes the proof.

3. Proof of Theorem 1.5, 1.6, and 1.7

In this section we study the system \((\tilde{S}_\gamma)\) which we introduced in the introduction. For the reader’s convenience, we recall \((\tilde{S}_\gamma)\) below

\[
(\tilde{S}_\gamma) \quad \begin{cases} (-\Delta_p)^s u = |u|^{p_s^* - 2} u + \frac{\alpha}{p^-} |u|^\alpha |v|^\beta \quad \text{in } \mathbb{R}^N, \\ (-\Delta_p)^s v = |v|^{p_s^* - 2} v + \frac{\beta}{p^-} |v|^\beta |u|^\alpha \quad \text{in } \mathbb{R}^N, \\ u, v \in \tilde{W}^{s,p}(\mathbb{R}^N). \end{cases}
\]

We also recall that (see (1.7)) the energy functional associated to the above system is

\[ \mathcal{J}(u, v) = \frac{1}{p} \left( \|u\|^p_{W^{s,p}} + \|v\|^p_{W^{s,p}} \right) - \frac{1}{p^-} \int_{\mathbb{R}^N} \left( |u|^{p_s^*} + |v|^{p_s^*} + \gamma |u|^\alpha |v|^\beta \right) \, dx, \quad (u, v) \in X. \]

and the Nehari manifold (1.8)

\[
\mathcal{N} = \left\{ (u, v) \in X : u \neq 0, v \neq 0, \|u\|^p_{W^{s,p}} = \int_{\mathbb{R}^N} \left( |u|^{p_s^*} + \frac{\alpha \gamma}{p^-} |u|^\alpha |v|^\beta \right) \, dx, \quad \|v\|^p_{W^{s,p}} = \int_{\mathbb{R}^N} \left( |v|^{p_s^*} + \frac{\beta \gamma}{p^-} |u|^\alpha |v|^\beta \right) \, dx \right\}. 
\]

Therefore, it follows

\[ A = \inf_{(u,v)\in\mathcal{N}} \mathcal{J}(u, v) = \inf_{(u,v)\in\mathcal{N}} s \int_{\mathbb{R}^N} \left( |u|^{p_s^*} + |v|^{p_s^*} + \gamma |u|^\alpha |v|^\beta \right) \, dx. \]
**Proposition 3.1.** Assume that $c, d \in \mathbb{R}$ satisfy

\[
\begin{cases}
\frac{c^2 - p}{c^p} + \frac{\alpha \gamma}{p_s^2} \cdot \frac{\alpha - p}{d^p} \geq 1, \\
\frac{d^2 - p}{d^p} + \frac{\beta \gamma}{p_s^2} \cdot \frac{\beta - p}{c^p} \geq 1,
\end{cases}
\]

with \( c, d > 0 \).

If \( \frac{c}{d} < p < \frac{d}{c} \), \( \alpha, \beta > p \) and (1.10) holds then \( c + d \geq k + \ell \), where \( k, \ell \in \mathbb{R} \) satisfy (1.9).

**Proof.** We use the change of variables \( y = c + d, \ x = c/d, \ y_0 = k + \ell \), and \( x_0 = k/\ell \) into (3.1) and (1.9), we obtain

\[
y_p^* \geq \frac{(x + 1)^{\alpha \gamma}}{x^p} + \frac{\alpha \gamma}{p_s^2} x^p =: f_1(x), \quad y_0^* = f_1(x_0),
\]

\[
y_p^* \geq \frac{(x + 1)^{\beta \gamma}}{\beta \gamma} \frac{\beta - p}{x^p} + \frac{\beta \gamma}{p_s^2} x^p =: f_2(x), \quad y_0^* = f_2(x_0).
\]

Then, one has

\[
\begin{align*}
f_1'(x) &= \frac{\alpha \gamma (x + 1)^{\alpha \gamma}}{p_s^2 \left( x^p + \frac{\alpha \gamma}{p_s^2} x^p \right)^2} \left[ -p_s^* (p_s^* - p) \frac{\beta}{x^p} + \beta x - \alpha + p \right] \\
&= \frac{\alpha \gamma (x + 1)^{\alpha \gamma}}{p_s^2 \left( x^p + \frac{\alpha \gamma}{p_s^2} x^p \right)^2} g_1(x),
\end{align*}
\]

\[
\begin{align*}
f_2'(x) &= \frac{\beta \gamma (x + 1)^{\beta \gamma}}{p_s^2 \left( 1 + \frac{\beta \gamma}{p_s^2} x^p \right)^2} \left[ p_s^* (p_s^* - p) \frac{\beta}{\beta \gamma} + (\beta - p) x^p - \alpha \frac{\alpha - p}{x^p} \right] \\
&= \frac{\beta \gamma (x + 1)^{\beta \gamma}}{p_s^2 \left( 1 + \frac{\beta \gamma}{p_s^2} x^p \right)^2} g_2(x).
\end{align*}
\]

Hence, we obtain \( x_1 = \left( \frac{\alpha \gamma}{p_s^2 (p_s^* - p)} \right)^{\frac{\beta}{\beta - p}} \) from \( g_1'(x) = 0 \) and similarly, for \( g_2 \) we have \( x_2 = \frac{\alpha - p}{\beta - p} \).

Now using (1.10) we conclude that

\[
\begin{align*}
\max_{x > 0} g_1(x) &= g_1(x_1) = \left( \frac{p \alpha \gamma}{p_s^2 (p_s^* - p)} \right)^{\frac{p}{\beta - p}} (\beta - p) - (\alpha - p) \leq 0, \\
\min_{x > 0} g_2(x) &= g_2(x_2) = \frac{p_s^* (p_s^* - p)}{\beta \gamma} - p \left( \frac{\alpha - p}{\beta - p} \right)^{\frac{\alpha - p}{\beta - p}} \geq 0.
\end{align*}
\]
Therefore, we conclude that the function \( f_1 \) is decreasing in \((0, \infty)\) and on the other hand the function \( f_2 \) is increasing in \((0, \infty)\). Thus, we have

\[
\frac{p^*_p - p}{y} \geq \max\{f_1(x), f_2(x)\} \geq \min_{x > 0} \left( \max\{f_1(x), f_2(x)\} \right) = \min_{\{f_1=f_2\}} \left( \max\{f_1(x), f_2(x)\} \right) = y_0^p.
\]

Hence the result follows. \( \square \)

We define the functions

\[
\begin{align*}
F_1(k, \ell) := & \frac{p^*_p - p}{y} + \frac{\alpha}{p^*_p} k^{\frac{\alpha}{p^*_p}} \ell^\frac{\beta}{\alpha} - 1, \quad k > 0, \ \ell \geq 0, \\
F_2(k, \ell) := & \frac{p^*_p - p}{y} + \frac{\beta}{p^*_p} \ell^{\frac{\beta}{p^*_p}} k^\frac{\alpha}{p^*_p} - 1, \quad k \geq 0, \ \ell > 0, \\
\ell(k) := & \left( \frac{p^*_p}{\alpha \gamma} \right)^\frac{\beta}{\alpha} k^{\frac{\alpha}{p^*_p}} \left( 1 - k^{\frac{p^*_p - p}{p}} \right)^{\frac{\beta}{\gamma}}, \quad 0 < k \leq 1, \\
k(\ell) := & \left( \frac{p^*_p}{\alpha \gamma} \right)^\frac{\beta}{\alpha} \ell^{\frac{\beta}{p^*_p}} \left( 1 - \ell^{\frac{p^*_p - p}{p}} \right)^{\frac{\beta}{\gamma}}, \quad 0 < \ell \leq 1.
\end{align*}
\]

Then \( F_1(k, \ell(k)) = 0 \) and \( F_2(k(\ell), \ell) = 0 \).

**Lemma 3.2.** Assume that \( \frac{2N}{N + 2s} < p < \frac{N}{2s} \) and \( \alpha, \beta < p \). Then

\[
F_1(k, \ell) = 0, \quad F_2(k, \ell) = 0, \quad k, \ell > 0,
\]

has a solution \((k_0, \ell_0)\) such that \( F_2(k, \ell(k)) < 0 \) for all \( k \in (0, k_0) \), that is \((k_0, \ell_0)\) satisfies (1.12). Similarly, \((3.3)\) has a solution \((k_1, \ell_1)\) such that \( F_1(k(\ell), \ell) < 0 \) for all \( \ell \in (0, \ell_1) \) that is \((k_1, \ell_1)\) satisfies (1.9) and \( \ell_1 = \min\{\ell : (k, \ell) \text{ satisfies (1.9)}\} \).

**Proof.** The proof is exactly similar to [GPZ17, Lemma 3.2]. \( \square \)

**Lemma 3.3.** Assume that \( \frac{N}{N + 2s} < p < \frac{N}{2s} \); \( \alpha, \beta < p \) and (1.11) holds. Then \( k_0 + \ell_0 < 1 \), where \((k_0, \ell_0)\) is same as in Lemma 3.2 and

\[
F_1(k(\ell), \ell) < 0 \quad \forall \ \ell \in (0, \ell_0), \quad F_2(k, \ell(k)) < 0 \quad \forall \ k \in (0, k_0).
\]

**Proof.** Using (3.2), we obtain

\[
\ell'(k) = \left( \frac{p^*}{\alpha \gamma} \right)^\frac{\beta}{\alpha} k^{\frac{\alpha}{p^*}} p^* k^{\frac{\alpha}{p^*}} \left( 1 - k^{\frac{p^* - p}{p}} \right)^{\frac{\beta}{\gamma}} \left( \frac{p - \alpha}{\beta} - k^{\frac{p^* - p}{p}} \right),
\]

and then we have

\[
\ell''(k) = \frac{(p - \beta)(p^* - p)}{p^*} \left( \frac{p^*}{\alpha \gamma} \right)^\frac{\beta}{\alpha} k^{\frac{\alpha}{p^*}} \left( 1 - k^{\frac{p^* - p}{p}} \right)^{\frac{\beta}{\gamma}} \left( \frac{p - \alpha}{\beta} - k^{\frac{p^* - p}{p}} \right).
\]
$$\left(1 - k \frac{p^* - p}{p - \rho} \right) \left( \frac{p(p - \alpha)}{\beta(p - \rho)} - k \frac{p^* - p}{p - \rho} \right).$$

Note that $\ell'(1) = 0 = \ell' \left( \frac{p - \alpha}{\beta} \right)$ and $\ell'(k) > 0$ for $0 < k < \left( \frac{p - \alpha}{\beta} \right)$, whereas $\ell'(k) < 0$ for $\left( \frac{p - \alpha}{\beta} \right)^{p^* - p} < k < 1$. From $\ell''(k) = 0$, we obtain $\tilde{k} = \left( \frac{p(p - \alpha)}{\beta(2p - p^* - p)} \right)^{p^* - p}$. Then by (11), we obtain

$$\min_{k \in [0,1]} \ell'(k) = \min_{k \in \left[ \frac{p - \alpha}{\beta} \right]^{p^* - p}, 1} \ell'(k) = \ell'(\tilde{k}) = - \left( \frac{p^*_s (p^*_s - p)}{\alpha \gamma p^{\alpha - \beta}} \right)^{p^{\alpha - \beta}} \left( \frac{p - \beta}{p - \alpha} \right)^{p^{\alpha - \beta} - 1} \geq 1.$$

The remaining proof follows from [GPZ17, Lemma 3.3] by considering $\mu_1 = 1 = \mu_2$ in their proof. □

**Lemma 3.4.** Assume that $\frac{N}{N + 2s} < p < \frac{N}{2s}$; $\alpha$, $\beta < p$ and (11) holds. Then

$$\begin{cases}
    k + \ell \leq k_0 + \ell_0, \\
    F_1(k, \ell) \geq 0, \ F_2(k, \ell) \geq 0, \\
    k, \ell \geq 0 \ (k, \ell) \neq (0,0),
\end{cases}$$

has a unique solution $(k, \ell) = (k_0, \ell_0)$, where $F_1$, $F_2$ are given by (3.2).

**Proof.** The proof follows from [GPZ17, Proposition 3.4]. □

**Proof of Theorem 1.5:** Using (1.9), we have $\left( k_0^{1/p} U, \ell_0^{1/p} U \right) \in \mathcal{N}$ is a nontrivial solution of $(\tilde{S}_\gamma)$ and

$$A \leq \mathcal{J} \left( k_0^{1/p} U, \ell_0^{1/p} U \right) = \frac{s}{N}(k_0 + \ell_0)S^{\frac{N}{s}}.$$  

Now, suppose $\{(u_n, v_n)\} \in \mathcal{N}$ be a minimizing sequence for $A$ such that $\mathcal{J}(u_n, v_n) \to A$ as $n \to \infty$. Let $c_n = \|u_n\|_{L^{p^*}(\mathbb{R}^\gamma)}^p$ and $d_n = \|v_n\|_{L^p(\mathbb{R}^\gamma)}^p$. Then by Hölder inequality we have

$$Sc_n \leq \|u_n\|_{W^{s, p}}^p = \int_{\mathbb{R}^N} \left( |u_n|^{p^*_s} + \frac{\alpha \gamma}{p^*_s} |u_n|^{\alpha} |v_n|^{\beta} \right) dx \leq \frac{p^*}{p} c_n^\alpha + \frac{\alpha \gamma}{p^*_s} \frac{\alpha}{\alpha - \beta} \frac{\beta}{p^*} d_n^\beta.$$

This implies that

$$c_n \frac{p^* - \rho}{p - \rho} + \frac{\alpha \gamma}{p^*_s} \frac{\alpha}{\alpha - \beta} \frac{\beta}{p^*} d_n^\beta \geq \left( \frac{p^*}{p} c_n^\alpha + \frac{\alpha \gamma}{p^*_s} \frac{\alpha}{\alpha - \beta} \frac{\beta}{p^*} d_n^\beta \right) \geq 1$$

i.e., $\mathcal{F}_1(c_n, d_n) \geq 0$,

where $c_n = \frac{c_n}{S^{\frac{N}{s} - \rho}}$, $d_n = \frac{d_n}{S^{\frac{N}{s} - \rho}}$. Similarly, we get

$$Sd_n \leq \|v_n\|_{W^{s, p}}^p = \int_{\mathbb{R}^N} \left( |v_n|^{p^*_s} + \frac{\beta \gamma}{p^*_s} |u_n|^{\alpha} |v_n|^{\beta} \right) dx \leq \frac{p^*}{p} d_n^\alpha + \frac{\beta \gamma}{p^*_s} \frac{\alpha}{\alpha - \beta} \frac{\beta}{p^*} d_n^\beta.$$

and thus $\mathcal{F}_2(c_n, d_n) \geq 0$. Then for $\alpha, \beta > p$, by Proposition 3.1 we have $c_n + d_n \geq k + \ell = k_0 + \ell_0$, on the other hand for $\alpha, \beta < p$, by Lemma 3.4 we have $c_n + d_n = k_0 + \ell_0$. Hence,
\[ c_n + d_n \geq (k_0 + \ell_0)S_{\frac{N-\gamma}{\alpha}}. \]

Since \( \mathcal{J}(u_n, v_n) = \frac{s}{\mathcal{N}} \left( \|u_n\|_{W_{s,p}}^p + \|v\|_{W_{s,p}}^p \right) \), using (3.4)-(3.6) we have

\[
\mathcal{S}(c_n + d_n) \leq \frac{N}{s} \mathcal{J}(u_n, v_n) = \frac{N}{s} A + o(1) \leq (k_0 + \ell_0)S_{\frac{N}{p}} + o(1)
\]

This implies that

\[ c_n + d_n \leq (k_0 + \ell_0)S_{\frac{N-\gamma}{\alpha}} + o(1). \]

Combining (3.7) and (3.8), we obtain \( c_n + d_n \to (k_0 + \ell_0)S_{\frac{N-\gamma}{\alpha}} \) as \( n \to \infty \). Therefore,

\[
A = \lim_{n \to \infty} \mathcal{J}(u_n, v_n) \geq \frac{s}{\mathcal{N}} \lim_{n \to \infty} (c_n + d_n) = (k_0 + \ell_0)S_{\frac{N-\gamma}{p}}.
\]

Therefore,

\[
A = \frac{s}{\mathcal{N}}(k_0 + \ell_0)S_{\frac{N}{p}} = \mathcal{J}\left(k_0^{1/p}U, \ell_0^{1/p}U \right).
\]

This completes the proof of Theorem 1.5. \( \square \)

Next, we prove existence of solutions of (1.13), namely Theorem 1.7. For this, define

\[
X(B_R(0)) = W_0^{s,p}(B_R(0)) \times W_0^{s,p}(B_R(0)),
\]

where \( W_0^{s,p}(B_R(0)) = \{ u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ in } \mathbb{R}^N \setminus B_R(0) \} \) with the norm \( \| \cdot \|_{W^{s,p}} \), and

\[
\mathcal{N}(R) = \left\{ (u, v) \in X(B_R(0)) \setminus \{0, 0\} : \|u\|_{W^{s,p}}^p + \|v\|_{W^{s,p}}^p \right\} = \left\{ \int_{B_R(0)} \left( |u|^{p^*_s} + |v|^{p^*_s} + \gamma |u|^{\alpha} |v|^\beta \right) dx \right\},
\]

and set \( \bar{A}(R) := \inf_{(u, v) \in \mathcal{N}(R)} \mathcal{J}(u, v) \). We also define

\[
\mathcal{N} = \left\{ (u, v) \in X \setminus \{0\} : \|u\|_{W_{s,p}}^p + \|v\|_{W_{s,p}}^p = \int_{\mathbb{R}^N} \left( |u|^{p^*_s} + |v|^{p^*_s} + \gamma |u|^{\alpha} |v|^\beta \right) dx \right\}.
\]

Set \( \bar{A} := \inf_{(u, v) \in \mathcal{N}} \mathcal{J}(u, v) \). Since \( \mathcal{N} \subset \mathcal{N} \), it follows \( \bar{A} \leq A \) and by the fractional Sobolev embedding \( \bar{A} > 0 \).

For \( \epsilon \in (0, \min\{\alpha, \beta\} - 1) \), consider

\[
\left\{ \begin{array}{ll}
(-\Delta_p)^s u = |u|^{p^*_s - 2 - \epsilon} u + \frac{(\alpha - \epsilon)\gamma}{p^*_s - 2\epsilon} |u|^{\alpha - 2 - \epsilon} u^\beta - \epsilon & \text{in } B_R(0), \\
(-\Delta_p)^s v = |v|^{p^*_s - 2 - \epsilon} v + \frac{(\beta - \epsilon)\gamma}{p^*_s - 2\epsilon} |v|^{\beta - 2 - \epsilon} v^\alpha - \epsilon & \text{in } B_R(0), \\
u, v \in W_0^{s,p}(B_R(0)).
\end{array} \right.
\]

The corresponding energy functional of the system (3.9) is given by
Proof. Let \( (u,v) \) by the Hölder and Sobolev inequalities. By Young’s inequality,

\[
\mathcal{J}_\epsilon(u,v) := \frac{1}{p} \left( \|u\|_{\dot{W}^s,p}^p + \|v\|_{\dot{W}^s,p}^p \right) - \frac{1}{p_s - 2\epsilon} \int_{B_R(0)} \left( |u|^{p_s^*-2\epsilon} + |v|^{p_s^*-2\epsilon} + \gamma |u|^{\alpha-\epsilon} |v|^{\beta-\epsilon} \right) dx.
\]

Define

\[
\tilde{N}_\epsilon(R) := \left\{ (u,v) \in X(B_1(0)) \setminus \{(0,0)\} : G_\epsilon(u,v) := \|u\|_{\dot{W}^s,p}^p + \|v\|_{\dot{W}^s,p}^p - \int_{B_R(0)} \left( |u|^{p_s^*-2\epsilon} + |v|^{p_s^*-2\epsilon} + \gamma |u|^{\alpha-\epsilon} |v|^{\beta-\epsilon} \right) dx = 0 \right\},
\]

and set \( \tilde{A}_\epsilon(R) := \inf_{(u,v) \in \tilde{N}_\epsilon(R)} \mathcal{J}_\epsilon(u,v) \).

Lemma 3.5. For any \( \epsilon_0 \in (0, \min\{\alpha - 1, \beta - 1, (p_s^*-p)/2\}) \), there exists a constant \( C_{\epsilon_0} > 0 \) such that

\( \tilde{A}_\epsilon(R) \geq C_{\epsilon_0} \forall \epsilon \in (0, \epsilon_0] \).

Proof. Let \( (u,v) \in \tilde{N}_\epsilon(R) \). Then

\[
\mathcal{J}_\epsilon(u,v) = \left( \frac{1}{p} - \frac{1}{p_s^*-2\epsilon} \right) \left( \|u\|_{\dot{W}^s,p}^p + \|v\|_{\dot{W}^s,p}^p \right),
\]

so it suffices to show that \( \|u\|_{\dot{W}^s,p}^p + \|v\|_{\dot{W}^s,p}^p \) is bounded away from zero. We have

\[
\|u\|_{\dot{W}^s,p}^p + \|v\|_{\dot{W}^s,p}^p \leq \int_{B_R(0)} \left( |u|^{p_s^*-2\epsilon} + |v|^{p_s^*-2\epsilon} + \gamma |u|^{\alpha-\epsilon} |v|^{\beta-\epsilon} \right) dx
\]

\[
\leq |B_R(0)|^{2\epsilon/p_s^*} \left[ \left( \int_{B_R(0)} |u|^{p_s^*} dx \right)^{(p_s^*-2\epsilon)/p_s^*} + \left( \int_{B_R(0)} |v|^{p_s^*} dx \right)^{(p_s^*-2\epsilon)/p_s^*} \right] + \gamma \left( \int_{B_R(0)} |u|^{p_s^*} dx \right)^{(\alpha-\epsilon)/p_s^*} \left( \int_{B_R(0)} |v|^{p_s^*} dx \right)^{(\beta-\epsilon)/p_s^*}
\]

\[
\leq |B_R(0)|^{2\epsilon/p_s^*} S^{-(p_s^*-2\epsilon)/p} \left( \|u\|_{\dot{W}^s,p}^{p_s^*-2\epsilon} + \|v\|_{\dot{W}^s,p}^{p_s^*-2\epsilon} + \gamma \|u\|^{\alpha-\epsilon}_{\dot{W}^s,p} \|v\|^{\beta-\epsilon}_{\dot{W}^s,p} \right)
\]

by the Hölder and Sobolev inequalities. By Young’s inequality,

\[
\|u\|^{\alpha-\epsilon}_{\dot{W}^s,p} \|v\|^{\beta-\epsilon}_{\dot{W}^s,p} \leq \frac{\alpha - \epsilon}{p_s^* - 2\epsilon} \|u\|^{p_s^*-2\epsilon}_{\dot{W}^s,p} + \frac{\beta - \epsilon}{p_s^* - 2\epsilon} \|v\|^{p_s^*-2\epsilon}_{\dot{W}^s,p} \leq \|u\|^{p_s^*-2\epsilon}_{\dot{W}^s,p} + \|v\|^{p_s^*-2\epsilon}_{\dot{W}^s,p} .
\]

Therefore, (3.10) gives

\[
\|u\|_{\dot{W}^s,p}^p + \|v\|_{\dot{W}^s,p}^p \leq (1 + \gamma) |B_R(0)|^{2\epsilon/p_s^*} S^{-(p_s^*-2\epsilon)/p} \left( \|u\|_{\dot{W}^s,p}^{p_s^*-2\epsilon} + \|v\|_{\dot{W}^s,p}^{p_s^*-2\epsilon} \right).
\]

Since \( (p_s^*-2\epsilon)/p > 1 \),

\[
\|u\|_{\dot{W}^s,p}^{p_s^*-2\epsilon} + \|v\|_{\dot{W}^s,p}^{p_s^*-2\epsilon} \leq \left( \|u\|_{\dot{W}^s,p}^p + \|v\|_{\dot{W}^s,p}^p \right)^{(p_s^*-2\epsilon)/p},
\]
thus (3.11) gives
\[ \|u\|_{W^{s,p}}^p + \|v\|_{W^{s,p}}^p \geq \left( \frac{\mathcal{S}(p^*_s - 2\epsilon)/p}{(1 + \gamma) |B_R(0)|^{2\epsilon/p^*_s}} \right)^{p/(p^*_s - p - 2\epsilon)}. \]

The desired conclusion follows from this since \( p^*_s - p - 2\epsilon \geq p^*_s - p - 2\epsilon_0 > 0 \) and the function \( h(t) = \left( \frac{\mathcal{S}(p^*_s - 2t)/p}{(1 + \gamma) |B_R(0)|^{2\epsilon/p^*_s}} \right)^{p/(p^*_s - p - 2t)} \) is continuous and positive in \([0, \epsilon_0]\). □

**Lemma 3.6.** Assume that \( \frac{2N}{N + 2s} < p < \frac{N}{2s} \) and \( \alpha, \beta < p \). For \( \epsilon \in (0, \min\{\alpha, \beta\} - 1) \), it holds

\[ \tilde{A}_\epsilon(R) < \min \left\{ \inf_{(u,0) \in \mathcal{N}_\epsilon(R)} \mathcal{J}_\epsilon(u,0), \inf_{(0,v) \in \mathcal{N}_\epsilon(R)} \mathcal{J}_\epsilon(0,v) \right\}. \]

**Proof.** Clearly, \( 2 < p^*_s - 2\epsilon < p^*_s \) from \( \min\{\alpha, \beta\} \leq \frac{p^*}{2} \). Then we may assume that \( u_1 \) is a least energy solution of

\[ \begin{align*}
(-\Delta_p)^s u &= |u|^{p^*_s - 2\epsilon} u \quad \text{in} \ B_R(0), \\
u &\in W_0^{s,p}(B_R(0)).
\end{align*} \]

Set \( \mathcal{J}_\epsilon(u_1,0) = a_10 := \inf_{(u,0) \in \mathcal{N}_\epsilon(R)} \mathcal{J}_\epsilon(u,0), \quad \mathcal{J}_\epsilon(0,u_1) = a_01 := \inf_{(0,v) \in \mathcal{N}_\epsilon(R)} \mathcal{J}_\epsilon(0,v). \)

We claim that for any \( \sigma \in \mathbb{R} \), there exists a unique \( t(\sigma) > 0 \) such that \( (t(\sigma)^{1/p}u_1, t(\sigma)^{1/p}\sigma u_1) \in \mathcal{N}_\epsilon(R). \)

\[ t(\sigma)^{\frac{p^*_s - p - 2\epsilon}{p}} = \frac{\|u_1\|_{W^{s,p}}^p + |\sigma|^p \|u_1\|_{W^{s,p}}^p}{\int_{B_R(0)} \left( |u_1|^{p^*_s - 2\epsilon} + |\sigma u_1|^{p^*_s - 2\epsilon} + \gamma |u_1|^{\alpha - \epsilon} |\sigma u_1|^{\beta - \epsilon} \right) dx} \]

\[ = \frac{qa_{10} + qa_{01}|\sigma|^p}{qa_{10} + qa_{01}|\sigma|^{p^*_s - 2\epsilon} + |\sigma|^{\beta - \epsilon} \gamma \int_{B_R(0)} |u_1|^{p^*_s - \epsilon} dx}, \]

where \( q := \frac{p(p^*_s - 2\epsilon)}{p^*_s - p - 2\epsilon} \), i.e., \( \frac{1}{q} = \frac{1}{p} - \frac{1}{p^*_s - 2\epsilon} \). Note that \( t(0) = 1 \), we have

\[ \lim_{\sigma \to 0} \frac{t'(\sigma)}{|\sigma|^{\beta - 2 - \epsilon}} = -\frac{(\beta - \epsilon) \gamma \int_{B_R(0)} |u_1|^{p^*_s - \epsilon} dx}{a_{10}(p^*_s - 2\epsilon)} \]

This implies that as \( \sigma \to 0 \)

\[ t'(\sigma) = -\frac{(\beta - \epsilon) \gamma \int_{B_R(0)} |u_1|^{p^*_s - \epsilon} dx}{a_{10}(p^*_s - 2\epsilon)} |\sigma|^{\beta - 2 - \epsilon} \sigma(1 + o(1)). \]

Then

\[ t(\sigma) = 1 - \frac{\gamma \int_{B_R(0)} |u_1|^{p^*_s - \epsilon} dx}{a_{10}(p^*_s - 2\epsilon)} |\sigma|^{\beta - \epsilon}(1 + o(1)) \quad \text{as} \ \sigma \to 0, \]
and therefore
\[ t(\sigma) \frac{p_s^*-2\varepsilon}{p} = 1 - \frac{\gamma}{p^{a_{10}}} \int_{B_R(0)} |u_1|^{p_s^*-\varepsilon} \, dx |\sigma|^{\beta-\varepsilon}(1 + o(1)) \text{ as } \sigma \to 0. \]

We get for $|\sigma|$ small enough
\[ \tilde{A}_e(R) \leq J_e \left( (t(\sigma))^{1/p} u_1, (t(\sigma))^{1/p} \sigma u_1 \right) \]
\[ = \frac{1}{q} \left( qa_{10} + qa_0 |\sigma|^{p_s^*-2\varepsilon} + |\sigma|^{\beta-\varepsilon} \gamma \int_{B_R(0)} |u_1|^{p_s^*-\varepsilon} \, dx \right) t(\sigma)^{p_s^*-2\varepsilon} \frac{1}{p} \]
\[ = a_{10} - \frac{1}{p_s^*-2\varepsilon} |\sigma|^{\beta-\varepsilon} \gamma \int_{B_R(0)} |u_1|^{p_s^*-\varepsilon} \, dx + o(|\sigma|^{\beta-\varepsilon}) < a_{10}. \]

Similarly, we see that $\tilde{A}_e(R) < a_{01}$. This completes the proof. \qed

Note that, similarly to Lemma 3.6, we obtain
\begin{equation}
(3.12) \quad \tilde{A} = \min \{ \inf_{(u,0) \in \tilde{N}_e} J(u,0), \inf_{(0,v) \in \tilde{N}'} J(0,v) \} = \min \{ J(U,0), J(0,U) \} = \frac{8}{N} \mathcal{S}_{\tilde{N}}^\mathcal{N}. \end{equation}

**Proposition 3.7.** For $0 < \varepsilon < \min \{ \min \{ \alpha, \beta \} - 1, \frac{p_s^*-p}{2} \}$, the system (3.9) has a positive least energy solution $(u_\varepsilon, v_\varepsilon)$ with $u_\varepsilon, v_\varepsilon$ are both radially symmetric nonincreasing functions.

**Proof.** By Lemma 3.5, $\tilde{A}_e(R) > 0$. Let $(u, v) \in \tilde{N}_e(R)$ with $u, v \geq 0$. We denote $u^*, v^*$ be Schwartz symmetrization of $u, v$ respectively. Then by nonlocal Pólya-Szegö inequality (see, [AL89]) and properties of the Schwartz symmetrization, we obtain
\[ \|u^*\|_{W_{s,p}}^p + \|v^*\|_{W_{s,p}}^p \leq \int_{B_R(0)} \left( |u^*|^{p_s^*-2\varepsilon} + |v^*|^{p_s^*-2\varepsilon} + \gamma |u^*|^{\alpha-\varepsilon} |v^*|^{\beta-\varepsilon} \right) \, dx. \]

Also, note that $J_e \left( t_s^{1/p} u^*, t_s^{1/p} v^* \right) \leq J_e(u,v)$ for some $t_s \in (0,1]$ such that $\left( t_s^{1/p} u^*, t_s^{1/p} v^* \right) \in \tilde{N}_e(R)$. Hence we choose a minimizing sequence $\{ (u_n, v_n) \} \subset \tilde{N}_e(R)$ of $\tilde{A}_e$ such that $(u_n, v_n) = (u_n^*, v_n^*)$ for any $n$ and $J_e(u_n, v_n) \to \tilde{A}_e$ as $n \to \infty$. Thus, we get both the sequence $\{ u_n \}$, $\{ v_n \}$ are bounded in $W_0^{s,p}(B_R(0))$. Since, $W_0^{s,p}(B_R(0))$ is a reflexive Banach space, up to a subsequence, $u_n \to u_\varepsilon$, $v_n \to v_\varepsilon$ weakly in $W_0^{s,p}(B_R(0))$. Moreover, as $W_0^{s,p}(B_R(0)) \hookrightarrow L^{p_s^*-2\varepsilon}(B_R(0))$ is a compact embedding, it follows $u_n \to u_\varepsilon$, $v_n \to v_\varepsilon$ strongly in $L^{p_s^*-2\varepsilon}(B_R(0))$. Therefore,
\[ \int_{B_R(0)} \left( |u_\varepsilon|^{p_s^*-2\varepsilon} + |v_\varepsilon|^{p_s^*-2\varepsilon} + \gamma |u_\varepsilon|^{\alpha-\varepsilon} |v_\varepsilon|^{\beta-\varepsilon} \right) \, dx \]
\[ = \lim_{n \to \infty} \int_{B_R(0)} \left( |u_n|^{p_s^*-2\varepsilon} + |v_n|^{p_s^*-2\varepsilon} + \gamma |u_n|^{\alpha-\varepsilon} |v_n|^{\beta-\varepsilon} \right) \, dx \]
\[ = \frac{p(p_s^*-2\varepsilon)}{p_s^*-2\varepsilon - p} \lim_{n \to \infty} J_e(u_n, v_n) = \frac{p(p_s^*-2\varepsilon)}{p_s^*-2\varepsilon - p} \tilde{A}_e(R) > 0, \]
this yields that \((u_\epsilon, v_\epsilon) \neq (0,0)\) and also \(u_\epsilon, v_\epsilon\) are nonnegative radially symmetric decreasing. Using the weak lower semicontinuity property of the norm, we also have
\[
\|u_\epsilon\|_{W^{s,p}}^p + \|v_\epsilon\|_{W^{s,p}}^p \leq \lim_{n \to \infty} \left( \|u_n\|_{W^{s,p}}^p + \|v_n\|_{W^{s,p}}^p \right),
\]
and therefore
\[
\|u_\epsilon\|_{W^{s,p}}^p + \|v_\epsilon\|_{W^{s,p}}^p \leq \int_{B_R(0)} \left( |u_\epsilon|^{p_s - 2\epsilon} + |v_\epsilon|^{p_s - 2\epsilon} + \gamma |u_\epsilon|^{\alpha - \epsilon} |v_\epsilon|^{\beta - \epsilon} \right) dx.
\]
Therefore there exists \(t_\epsilon \in (0,1)\) such that \((t_\epsilon^{1/p} u_\epsilon, t_\epsilon^{1/p} v_\epsilon) \in \tilde{N}_\epsilon\) and hence
\[
\tilde{A}_\epsilon(R) \leq J_\epsilon \left( t_\epsilon^{1/p} u_\epsilon, t_\epsilon^{1/p} v_\epsilon \right) = \frac{t_\epsilon (p_s^* - 2\epsilon - p)}{p (p_s^* - 2\epsilon)} \left( \|u_\epsilon\|_{W^{s,p}}^p + \|v_\epsilon\|_{W^{s,p}}^p \right)
\leq \frac{p_s^* - 2\epsilon - p}{p (p_s^* - 2\epsilon)} \lim_{n \to \infty} \left( \|u_n\|_{W^{s,p}}^p + \|v_n\|_{W^{s,p}}^p \right) = \lim_{n \to \infty} J_\epsilon(u_n, v_n) = \tilde{A}_\epsilon(R),
\]
which yields that \(t_\epsilon = 1\), \((u_\epsilon, v_\epsilon) \in \tilde{N}_\epsilon(R), \tilde{A}_\epsilon(R) = J_\epsilon(u_\epsilon, v_\epsilon)\) and
\[
\|u_\epsilon\|_{W^{s,p}}^p + \|v_\epsilon\|_{W^{s,p}}^p = \lim_{n \to \infty} \left( \|u_n\|_{W^{s,p}}^p + \|v_n\|_{W^{s,p}}^p \right).
\]
This proved that \(u_n \to u_\epsilon, v_n \to v_\epsilon\ strongly\ in\ W_0^{s,p}(B_R(0))\). Now by Lagrange multiplier theorem, there exists \(\lambda \in \mathbb{R}\) such that
\[
J_\epsilon'(u_\epsilon, v_\epsilon) + \lambda G_\epsilon'(u_\epsilon, v_\epsilon) = 0.
\]
Again since \(J_\epsilon'(u_\epsilon, v_\epsilon) = G_\epsilon(u_\epsilon, v_\epsilon) = 0\) and
\[
G_\epsilon'(u_\epsilon, v_\epsilon)(u_\epsilon, v_\epsilon) = -(p_s^* - 2\epsilon - p) \int_{B_R(0)} \left( |u_\epsilon|^{p_s^* - 2\epsilon} + |v_\epsilon|^{p_s^* - 2\epsilon} + \gamma |u_\epsilon|^{\alpha - \epsilon} |v_\epsilon|^{\beta - \epsilon} \right) dx < 0,
\]
we get \(\lambda = 0\) and hence \(J_\epsilon'(u_\epsilon, v_\epsilon) = 0\). Since \(\tilde{A}_\epsilon(R) = J_\epsilon(u_\epsilon, v_\epsilon)\) and by Lemma 3.6, we have \(u_\epsilon, v_\epsilon \neq 0\). By maximum principle (see, [DPQ17, Lemma 3.3]) we conclude the desired result.

\[\square\]

**Lemma 3.8.** For any \((u, v) \in \tilde{N}\), there is a sequence \((u_n, v_n) \in \tilde{N} \cap (C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N))\) such that \((u_n, v_n) \to (u, v)\ in\ X\ as\ n \to \infty.\)

**Proof.** By density, there is a sequence \((\tilde{u}_n, \tilde{v}_n) \in C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N)\) such that \((\tilde{u}_n, \tilde{v}_n) \to (u, v)\ in\ X\ as\ n \to \infty.\) Let
\[
t_n = \left( \frac{\|\tilde{u}_n\|_{W^{s,p}}^p + \|\tilde{v}_n\|_{W^{s,p}}^p}{\int_{\mathbb{R}^N} \left( |\tilde{u}_n|^{p_s^*} + |\tilde{v}_n|^{p_s^*} + \gamma |\tilde{u}_n|^{\alpha} |\tilde{v}_n|^{\beta} \right) dx} \right)^{1/(p_s^* - p)}
\]
and note that \(t_n \to 1\) since \((u, v) \in \tilde{N}\). Then \((u_n, v_n) = (t_n \tilde{u}_n, t_n \tilde{v}_n) \in \tilde{N} \cap (C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N))\) and \((u_n, v_n) \to (u, v)\ in\ X.\) \[\square\]

**Lemma 3.9.** There is a minimizing sequence \((u_n, v_n) \in \tilde{N} \cap (C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N))\) for \(\tilde{A}.\)
Proof. Let \((\tilde{u}_n, \tilde{v}_n) \in \tilde{N}\) be a minimizing sequence for \(\tilde{A}\), i.e., \(J(\tilde{u}_n, \tilde{v}_n) \to \tilde{A}\). By the continuity of \(J\) and Lemma 3.8, there is a \((u_n, v_n) \in \tilde{N} \cap (C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N))\) such that

\[ |J(u_n, v_n) - J(\tilde{u}_n, \tilde{v}_n)| < \frac{1}{n}. \]

Then \(J(u_n, v_n) \to \tilde{A}\), so \((u_n, v_n) \in \tilde{N} \cap (C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N))\) is a minimizing sequence for \(\tilde{A}\).

**Proof of Theorem 1.7:** First, we prove that

\[(3.13) \quad \tilde{A}(R) = \tilde{A} \text{ for every } R > 0. \]

Let \(R_1 < R_2\), then \(\tilde{N}(R_1) \subset \tilde{N}(R_2)\) and hence by definition we have \(\tilde{A}(R_2) \leq \tilde{A}(R_1)\). To prove reverse inequality, let \((u, v) \in \tilde{N}(R_2)\) and define

\[(u_1(x), v_1(x)) := \left( \frac{R_2}{R_1} \right)^{\frac{N}{s-p}} \left( u \left( \frac{R_2}{R_1} x \right), v \left( \frac{R_2}{R_1} x \right) \right). \]

Clearly, \((u_1, v_1) \in \tilde{N}(R_1)\). Therefore, we get

\[\tilde{A}(R_1) \leq J(u_1, v_1) = J(u, v), \text{ for any } (u, v) \in \tilde{N}(R_2),\]

and this implies that \(\tilde{A}(R_1) \leq \tilde{A}(R_2)\). So, we obtain \(\tilde{A}(R_1) = \tilde{A}(R_2)\). Let \((u_n, v_n) \in \tilde{N}\) be a minimizing sequence of \(\tilde{A}\). In view of Lemma 3.9, we may assume that \(u_n, v_n \in W_0^{s,p}(B_{R_n}(0))\) for some \(R_n > 0\). Then, \((u_n, v_n) \in \tilde{N}(R_n)\) and

\[\tilde{A} = \lim_{n \to \infty} J(u_n, v_n) \geq \lim_{n \to \infty} \tilde{A}(R_n) = \tilde{A}(R),\]

and hence (3.13) holds.

Let \((u, v) \in \tilde{N}(R)\) be arbitrary, then there exists \(t_\epsilon > 0\) with \(t_\epsilon \to 1\) as \(\epsilon \to 0\) such that \(\left( t_\epsilon^{1/p} u, t_\epsilon^{1/p} v \right) \in \tilde{N}_\epsilon(R)\). Therefore, we have

\[\limsup_{\epsilon \to 0} \tilde{A}_\epsilon \leq \limsup_{\epsilon \to 0} J_\epsilon \left( t_\epsilon^{1/p} u, t_\epsilon^{1/p} v \right) = J(u, v).\]

Thus, using (3.13) we obtain

\[(3.14) \quad \limsup_{\epsilon \to 0} \tilde{A}_\epsilon(R) \leq \tilde{A}(R) = \tilde{A}. \]

By Proposition 3.7, let \((u_\epsilon, v_\epsilon)\) be a positive least energy solution of (3.9), which is radially symmetric nonincreasing. Then by Lemma 3.5, for any \(\epsilon_0 \in (0, \min\{\alpha - 1, \beta - 1, (p_0^s - p)/2\})\), there exists a constant \(C_{\epsilon_0} > 0\) such that

\[(3.15) \quad \tilde{A}_\epsilon(R) = \frac{p_0^s - p - 2\epsilon}{p(p_0^s - 2\epsilon)} \left( \|u_\epsilon\|_{W^{s,p}}^p + \|v_\epsilon\|_{W^{s,p}}^p \right) \geq C_{\epsilon_0} \quad \forall \epsilon \in (0, \epsilon_0].\]

Therefore, from (3.14) we get \(u_\epsilon, v_\epsilon \in W_0^{s,p}(B_R(0))\) are uniformly bounded. Thus, by reflexivity up to a subsequence \(u_\epsilon \to u_0\) and \(v_\epsilon \to v_0\) weakly in \(W_0^{s,p}(B_R(0))\) as \(\epsilon \to 0\). Since (3.9)
is a subcritical system in bounded domain, passing the limit $\epsilon \to 0$, it follows that $(u_0, v_0)$ is a solution of the following system

\[
\begin{cases}
(-\Delta_p)^s u = |u|^{p^*_s-2}u + \frac{\alpha}{p^*_s} |u|^{\alpha-2}u|v|^{\beta} & \text{in } B_R(0), \\
(-\Delta_p)^s v = |v|^{p^*_s-2}v + \frac{\beta}{p^*_s} |v|^{\beta-2}v|u|^{\alpha} & \text{in } B_R(0), \\
\end{cases}
\]

\[
u, \ v \in W_0^{s,p}(B_R(0)).
\]

Also note that $u_0, v_0$ are nonnegative and from (3.15) we see that $(u_0, v_0) \neq (0, 0)$. We may now assume that $u_0 \not\equiv 0$. Therefore, by strong maximum principle (see, [DPQ17]) we obtain $u_0 > 0$ in $B_R(0)$. Further, we claim, $v_0 \not\equiv 0$. If $v_0 \equiv 0$ then substituting $(u_0, v_0)$ in the above system of equation leads $u_0$ is a positive solution to $(-\Delta_p)^s u = |u|^{p^*_s-2}u$ in $B_R(0)$. Since $u_0 \in W_0^{s,p}(B_R(0))$, it follows

\[
(3.16) \quad J(u_0, 0) = \frac{1}{p} \|u_0\|_{W^{s,p}}^p - \frac{1}{p^*_s} \int_{B_R(0)} u_0^{p^*_s} dx = \frac{1}{p} \|u_0\|_{W^{s,p}}^p - \frac{1}{p^*_s} \int_{B_R(0)} u_0^{p^*_s} dx = \frac{s}{N} \|\tilde{u}_0\|_{W^{s,p}}^p.
\]

We also observe that $(u_0, 0), (0, u_0) \in \mathcal{N}$. Therefore, using (3.12), we have

\[
(3.17) \quad \bar{A} < \min\{ \inf_{(u_0) \in \mathcal{N}} J(u, 0), \inf_{(0, v) \in \mathcal{N}} J(0, v) \} \leq \min\{ J(u_0, 0), J(0, u_0) \} = J(u_0, 0).
\]

Combining (3.16) and (3.17) together yields

\[
(3.18) \quad \bar{A} < \frac{s}{N} \|u_0\|_{W^{s,p}}^p.
\]

Further, by (3.14) and the fact that $(u_\epsilon, v_\epsilon)$ is a positive least energy solution of (3.9), it follows

\[
\hat{A} \geq \limsup_{\epsilon \to 0} \bar{A}_\epsilon(R) = \limsup_{\epsilon \to 0} J_\epsilon(u_\epsilon, v_\epsilon)
\]

\[
= \limsup_{\epsilon \to 0} \left[ \frac{1}{p} \|u_\epsilon\|_{W^{s,p}}^p + \|v_\epsilon\|_{W^{s,p}}^p \right] - \frac{1}{p^*_s - 2\epsilon} \int_{B_R(0)} (w\epsilon^{p^*_s-2\epsilon} + v\epsilon^{p^*_s-2\epsilon} + \gamma \epsilon^{\alpha-2\epsilon} \epsilon^{\beta-2\epsilon}) dx
\]

\[
= \limsup_{\epsilon \to 0} \left( \frac{1}{p} - \frac{1}{p^*_s - 2\epsilon} \right) \left( \|u_\epsilon\|_{W^{s,p}}^p + \|v_\epsilon\|_{W^{s,p}}^p \right) \geq \frac{s}{N} (\|u_0\|_{W^{s,p}}^p + \|v_0\|_{W^{s,p}}^p)
\]

\[
= \frac{s}{N} \|u_0\|_{W^{s,p}}^p \geq \bar{A} \quad \text{(by (3.18))},
\]

which is a contradiction. Hence $v_0 \neq 0$ and again by strong maximum principle we obtain $v_0 > 0$ in $B_R(0)$. Moreover, as $(u_\epsilon, v_\epsilon)$ is radial and $u_\epsilon \to u$ a.e. and $v_\epsilon \to v$ a.e. (up to a subsequence), we also have $u_0, v_0$ are radial functions. Hence $(u_0, v_0)$ is a positive radial solution to (1.13).

**Proof of Theorem 1.6:** To prove the existence of $(k(\gamma), \ell(\gamma))$ for small $\gamma > 0$, recalling (3.2), we denote $F_i(k, \ell, \gamma)$ instead of $F_i(k, \ell)$, $i = 1, 2$ in this case. Let $k(0) = 1 = \ell(0)$, then $F_i(k(0), \ell(0), 0) = 0, i = 1, 2$. Clearly, we have

\[
\frac{\partial F_1}{\partial k}(k(0), \ell(0), 0) = \frac{\partial F_2}{\partial \ell}(k(0), \ell(0), 0) = \frac{p^*_s - p}{p} > 0
\]
and
\[ \frac{\partial F_1}{\partial \ell}(k(0), \ell(0), 0) = \frac{\partial F_2}{\partial k}(k(0), \ell(0), 0) = 0. \]

Therefore, the jacobian determinant is
\[ \det J_F(k(0), \ell(0)) = (p^* - p)^2 > 0, \]
where \( F := (F_1, F_2). \)

Therefore, by the implicit function theorem, \( k(\gamma), \ell(\gamma) \) are well defined functions and of class \( C^1 \) in \( (-\gamma_2, \gamma_2) \) for some \( \gamma_2 > 0 \) and \( F_i(k, \ell, \gamma) = 0 \) for \( \gamma \in (-\gamma_2, \gamma_2). \) Then \( (k(\gamma)^{1/p}U, \ell(\gamma)^{1/p}U) \) is a positive solution of \( (S_\gamma). \) Since \( \lim_{\gamma \to 0} (k(\gamma) + \ell(\gamma)) = 2. \) Thus there exists \( \gamma_1 \in (0, \gamma_2] \) such that \( k(\gamma) + \ell(\gamma) > 1 \) for all \( \gamma \in (0, \gamma_1). \) Therefore, by (3.12) we get
\[ J \left( k(\gamma)^{1/p}U, \ell(\gamma)^{1/p}U \right) = \frac{s}{N} (k(\gamma) + \ell(\gamma)) S^N > \frac{s}{N} S^N = \bar{A}. \]

This completes the proof. \( \square \)

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