GRAVITATION AS FIELD AND CURVATURE

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Abstract

We argue that space-time properties are not absolute with respect to the used frame of reference as is to be expected according to ideas of relativity of space and time properties by Berkley - Leibnitz - Mach - Poincaré. From this point of view gravitation may manifests itself both as a field in Minkowski space-time and as space-time curvature. If the motion of test particles is described by the Thirring Lagrangian, then in the inertial frames of reference, where space-time is pseudo-Euclidean, gravitation manifests itself as a field. In reference frames, whose reference body is formed by point masses moving under the effect of the field, it appears as Riemannian curvature which in these frames is other than zero. For realization of the idea the author bimetric gravitation equations are considered. The spherically-symmetric solution of the equations in Minkowski space-time does not lead to the physical singularity in the center. The energy of the gravitational field of a point mass is finite. It follows from the properties of the gravitational force that there can exist stable compact supermassive configurations of Fermi-gas without an events horizon.
1 Introduction

The key reason preventing a correct inclusion of the Einstein theory of gravitation in the interactions unification is that gravity is identified with space-time curvature. It is also a cause of such unsolved problems of the theory as an operational definition of the observable variables, the energy-momentum tensor problem and gravity quantization. In the present paper starting from [14], [15] we consider a likely reason of gravity geometrization. We argue that the gravitation properties are not absolute with respect to the used frame of reference. In inertial frames of reference gravitation can be considered as a field in flat space-time, while in so called proper frames of reference it manifests itself as space-time curvature.

The author’s gravitation equations which realize this idea are considered in details. They do not contradict available experimental data. The physical consequences resulting from the equations differ very little from the ones in general relativity if the distances from the attracting mass are much larger than the Schwarzschild radius $r_s$. However, they are completely different at the distances equal to $r_s$ or less than that. A number of new physical consequences follow from the equations.

2 Primary Principles

The geometrical properties of space-time can be described only by means of measuring instruments. At the same time, the description of the properties of measuring instruments, strictly speaking, requires knowledge of space-time geometry. One of the implications of it is that geometrical properties of space and time have no experimentally verifiable significance by themselves but only within the aggregate "geometry + measuring instruments". We got aware of it owing to Poincaré [1]. It is a development of the idea by Berkley - Leibnitz - Mach about relativity of space-time properties which is an alternative to the well known Newtonian approach.

If we proceed from the conception of relativity of space-time, we assume that there is no way of quantitative description of physical phenomena other than attributing them to a certain frame of reference which in itself is a physical device for space and time measurements. But then the relativity of the geometrical properties of space and time mentioned above is nothing else but relativity of space-time geometry with respect to the frame of reference being used [1].

Thus, it should be assumed that the concept of the reference frame as a physical object, whose properties are given and are independent of the properties of space and time, is approximate, and only the aggregate "frame of reference + space-time geometry" has a sense.

The Einstein theory of gravitation demonstrates relativity of space-time with respect to distribution of matter. However, space-time relativity with respect to measurement instruments hitherto has not been realized in physical theory. An attempt to show that there is also space-time relativity to the used reference frames for the first time has been undertaken in [14], [15].

At present we do not know how the space-time geometry in inertial frames of reference (IFRs) is connected with the frames properties. Under the circumstances, we simply postulate (according to special relativity) that space - time in IFRs is pseudo-Euclidean. Next, we find a space-time metric differential form in noninertial frames of reference (NIFRs) from the viewpoint of an observer in the NIFR who proceeds from the relativity of space and time in the Berkley - Leibnitz - Mach - Poincaré (BLMP) sense. It is shown that there are reasons to believe that side by side with generally accepted viewpoint on motion in noninertial frames of reference as a relative motion there can also be another point of view. According to this viewpoint the metric differential form $ds$ in the NIFR is completely conditioned by the properties of the frame being used as is to be expected according to the idea of relativity of space and time in the BLMP sense.

3 The Metric Form $ds$ in NIFR.

By a noninertial frame of reference we mean the frame, whose body of reference is fo/rmed by the point masses moving in the IFR under the effect of a given force field.

It would be a mistake to identify "a priori" a transition from the IFR to the NIFR with the transformation of coordinates related to the frames. If we act in such a way, we already assume that the properties of the space-time in both frames are identical. However, for an observer in the NIFR, who proceeds from the relativity of space and time in the BLMP sense, space-time geometry is not given "a priori" and must be ascertained from the analysis of experimental data.

We shall suppose that the reference body (RB) of the IFR or NIFR is formed by the identical point masses $m_p$. If the observer is at rest in one of the frames, his world line will coincide with the world line of some point of the reference body. It is obvious to the observer in the IFR that the accelerations of the point masses forming the reference body are equal to zero. Of course, this fact occurs in relativistic sense too. That is, if the differential metric form of space-time in the IFR is denoted by $\mathbf{c}$ and $= \mathbf{c} \cdot \mathbf{c} = \mathbf{c}$ is the 4-velocity vector of the point masses forming the reference body, then the absolute derivative of the vector is equal to zero, i.e.

$$D = \mathbf{c} = 0$$

(1)

(We mean that an arbitrary coordinate system is used).

Does this fact occur for an observer in the NIFR ? That is, if the differential metric form of space-time in the NIFR is denoted by $ds$, does the 4-velocity vector $= \mathbf{c} = ds$...
of the point masses forming the reference body of this NIFR obey the equation

\[ D = ds = 0 \]  \hspace{1cm} (2)

\[ \text{? The answer depends on whether space and time are absolute in Newtonian sense or they are relative in the BLMP sense.} \]

If space and time are absolute, the point masses of the NIFR reference body for an observer in this NIFR are at relative rest. A notion of relative acceleration can be determined in a covariant way \[5\]. This value is equal to zero. However, eqs. \[6\], strictly speaking, are not satisfied.

If space and time are relative in the BLMP sense, then for observers in the IFR and NIFR the motion of the point masses forming the reference body, which are kinematically equivalent, must be dynamically equivalent too (both in the nonrelativistic and relativistic sense). That is, if from the viewpoint of the observer in the IFR, the point masses forming the NIFR RB are at rest (are not subject to the influence of forces either). In other words, if for an observer in the IFR the world lines of the IFR RB points are, according to eq. \[1\], the geodesic lines, then for the observer in the NIFR the world lines of the NIFR RB points are also the geodesic lines in his space-time, which can be expressed by eq. \[2\].

The differential equations of these world lines at the same time are the Lagrange equations of motion of the NIFR RB points. The Lagrange equations, describing the motion of the identical RB point masses in the IFR, can be obtained from the Lagrange action \[s\] by the principle of least action. Therefore, the equations of the geodesic lines can be obtained from the differential metric form \[ds = k ds\] where \[k\] is the constant, \[ds = L (\dot{x};dx) dt\] and \[L\] is the Lagrange function. The constant \[k = (\eta_{p};c)^{-1}\] as it follows from the analysis of the case when the frame of reference is inertial. It is equal to \[(\eta_{p};c)^{-1}\].

Thus, if we proceed from relativity of space and time in the BLMP sense, then the differential metric form of space-time in the NIFR can be expected to have the following form

\[ ds = \eta_{p}(c) \eta_{p}(x;dx) \]  \hspace{1cm} (3)

In this equation \[S\] is the Lagrangean action describing (in an IFR) the motion of the identical point masses \[m_{p}\] forming the NIFR reference body.

So, the properties of space-time in the NIFR are entirely determined by the properties of the used frame in accordance with the idea of relativity of space and time in the BLMP sense.

Consider two examples of the NIFR.

1. The reference body is formed by noninteracting electric charges moving in a constant homogeneous electric field \[E\]. The motion of the charges is described in Cartesian coordinates by the Lagrangian

\[ L = m_{p}c^{2} (\dot{1} + V^{2} = c^{2})^{1/2} + E \cdot \dot{x} \]  \hspace{1cm} (4)

where \[V\] is the speed of the charges.

According to eq. \[3\] the space-time metric differential form in this frame is given by

\[ ds = c \eta \cdot (w \cdot \dot{x})^{2} \]  \hspace{1cm} (5)

where \[\eta = (c^{2} \cdot dt^{2} - dx^{2} - dy^{2} - dz^{2})^{1/2}\] is the metric differential form of the pseudo-Euclidean space-time in the IFR and \[w = \varepsilon_{p} m_{p}\] is the acceleration of the charges.

2. The reference body consists of noninteracting electric charges in a constant homogeneous magnetic field \[\mathbf{H}\] directed along the axis \[z\]. The Lagrangian describing the motion of the particles can be written as follows \[6\]

\[ L = m_{p}c^{2} (1 + V^{2} = c^{2})^{1/2} \eta_{p}(x_{0} = 2) (w \cdot \dot{x}) \]  \hspace{1cm} (6)

The points of such a system rotate in the plane \[xy\] around the axis \[z\] with the angular frequency

\[ \omega = \dot{\theta} (l + (\theta \circ r = c^{2})^{1/2}) \]  \hspace{1cm} (7)

where \[r = (x^{2} + y^{2})^{1/2}\]. The linear velocities of the BR points tend to \(c\) when \[r \rightarrow l\].

For the given NIFR

\[ ds = c \eta + (\omega = 2\pi) (\dot{w} \cdot \dot{x}) \]  \hspace{1cm} (8)

In the above NIFR \[ds\] is of the form

\[ ds = F (x;dx) \]  \hspace{1cm} (9)

where \[F (x;dx) = c + \mathbf{F} (x) \cdot \dot{x} \] ; \(\mathbf{F}\) is a vector field and

\[ c = [ (x;dx) \cdot \dot{x}]^{1/2} \]  \hspace{1cm} (10)

is the differential metric form of pseudo-Euclidean space-time of the IFR in the used coordinate system. \(\mathbf{F}\) is a homogeneous function of the first degree in \(dx\). Therefore, generally speaking, the space-time in NIFR is Finslerian \[7\] with the sign-indefinite differential metric form \[8\].

4 Space and Time Measurements in NIFR

For the \(3 + 1\) decomposition of space-time in noninertial frames of reference to 3-space and time we proceed from a covariant method which goes back to Ulman, Komar, Dehnen and other authors \[8\]. An ideal clock is a local periodic process measuring the length of its own world line to a certain scale. For an observer in the NIFR the direction of time in the point \(x\) is given by the vector of the 4-velocity of the BR point.

The physical 3-space in each point is orthogonal to the vector \(\dot{x}\). The arbitrary vector \(V\) in the point \(x\) can be represented as follows:

\[ V = \dot{r} + \mathbf{r} \]  \hspace{1cm} (11)
where \( \vec{a} \) are the spatial components and \( f \) is the function of \( x \).

Suppose any space vector \( \vec{a} \) in the point \( x \) is orthogonal to the vector \( \vec{n} \) in the sense of the Finslerian metric \( \mathcal{F}_x \):

\[
\vec{n} = 0; \quad \mathcal{F}_x (\vec{n}; \vec{x}) = 0
\]

where \( n \) is the covariant components of the vector \( \vec{a} \), which are given by

\[
A = \frac{\partial \mathcal{F}(\vec{x}; \vec{n})}{\partial \vec{x}}
\]

Since \( \mathcal{F}(\vec{x}; \vec{n}) = 1 \) this vector is of the form

\[
\vec{a} = f = + f
\]

where \( dx = c \) is the 4-velocity of the reference body point in the IFR. By multiplying eq. (10) to the vector we find that

\[
\vec{a} = H
\]

where \( H = \delta_{ik} \) and \( \delta \) is the Kroneker delta.

Eq. (10) yields for the vector \( dx :\)

\[
dx = \frac{\partial \mathcal{F}_x (\vec{x}; \vec{n})}{\partial \vec{x}} + c \delta = \frac{dx}{dx} = \mathcal{F}_x (\vec{x}; \vec{n}) + c \delta
\]

where \( dx \) is the spatial components of the vector \( \delta = c \) and

\[
d = c \cdot dx
\]

is the time element between the events in the points \( x \) and \( x + dx \) in the NIFR.

The metric form (9) and the spatial projection of the vector \( dx \) lead to the following covariant form of the spatial element in the NIFR:

\[
dL = (\frac{dx}{dx}) + f \frac{dx}{dx}
\]

This covariant equation is the simplest and clearest in the coordinates system in which

\[
= \frac{dx}{dx} = \mathcal{F}_x (\vec{x}; \vec{n})
\]

Indeed, in eq. (17)

\[
d = dds = 0
\]

where \( = 1 = (0 + f) \) and \( 0 = (0) \)

The zero-component of the tensor \( H \) is

\[
H_{00} = \phi_{00} \quad 2 \quad \quad \quad (0 + f) + (1 = ds) (0 + f)^2
\]

Since

\[
d = ds = (1 + f) \quad 1 = (0) \quad 1
\]

and \( (0 + f) = 1 \), the value of \( H_{00} \) is equal to zero identically.

The components

\[
H_{ik} = \frac{\partial \mathcal{F}_x (\vec{x}; \vec{n})}{\partial \vec{x}} + f \frac{\partial \mathcal{F}_x (\vec{x}; \vec{n})}{\partial \vec{x}}
\]

where \( i = 0 \) is \( (0) \) is zero. The spatial tensor \( H_{ik} = \delta \) with accuracy up to \( V = c \) where \( V \) is the linear speed of the reference body points.

We have also

\[
\frac{f \cdot dx}{dx} = \frac{f \cdot dx}{dx}
\]

The zero-component of the vector \( \frac{f \cdot dx}{dx} \) is equal to zero identically and the spatial-components are equal to \( f \) with accuracy up to \( V = c \).

Thus, in the used coordinates system the spatial element in the NIFR with accuracy up to \( V = c \) is of the form

\[
dL = (\frac{dx}{dx}) + f \frac{dx}{dx} = dl (1 + f) \frac{dx}{dx}
\]

where \( dl = (\frac{dx}{dx}) + f \frac{dx}{dx} \) is the Euclidean spatial element and \( k \) is a unit vector of the direction with respect to \( dl \).

The phase shift in the interference of two coherent light beams on a rotating frame was observed by Sagnac [8]. For a relativistic explanation of the effect it is usually postulated, that space-time in any frames of reference is pseudo-Euclidean [10]. If the motion in NIFR is considered as the relative one in absolute pseudo-Euclidean space-time.

However, for an isolated observer (in a "black box") in the rotating frame, who proceeds from the notion of space and time relativity in the BLMP sense, the observed anisotropy in the time of light propagation (which from his viewpoint contradicts the experiments of Michelson - Morley type) is not a trivial effect. It must have some "internal" physical explanation.

A rigid disk rotating in the plane \( xy \) with angular velocity \( \omega \) around the axis \( z \) is approximately identical to the NIFR described in example 2 of Sec.3. It follows from eq. (26) that the spatial element in the NIFR is anisotropic. We will show that the speed of light in the noninertial frame of reference is anisotropic too.

A triple of space basis vectors, necessary to compare the direction of a given vector from viewpoints of the NIFR and IFR, are not defined above in each point of the NIFR.
However, it does not prevent us from comparing the lengths of the vectors. In particular, we can find a dependency between the speeds of a particle in the NIFR and IFR.

The speed of the motion of a particle in the NIFR is

\[ v^0 = (u \cdot u)^{1/2} + f \cdot u; \]

where \( u = dx/dt \) is 4-velocity of the particle.

The term under the square root is given by

\[ H \cdot u \cdot u = u \cdot u \cdot (v \cdot v)^2 = u \cdot u \cdot \sqrt{2} (dl/dt^2 + 2c^2 / c_0 = c); \]

where we have used the following equalities:

\[ dx = dt = c t; \]
\[ dx = c_0 = c; \]
\[ 0 = c_1 u \cdot dx \]

The first term in eq. (28) for a photon is equal to zero and we obtain with accuracy up to \( v = c \) that \( (u \cdot u)^{1/2} = c \).

In the same approximation, the term \( f \cdot u = c c_1 k_i \).

Thus, in the used coordinates system the speed of the photon in the NIFR with accuracy up to \( v = c \) is

\[ v^0_{ph} = c(1 + c c_1 k_i): \]

Consider a disk rotating with the constant angular velocity \( \omega \) around the \( z \) axis. Let \( x \) and \( \dot{z} \) be the coordinates, defined by the equations

\[ x = r \cos(\theta); y = r \sin(\theta); \dot{r} = + \dot{t}; \]

In the coordinate system \( (x; jz; \dot{t}) \) the space-time metrical differential form \( ds \) in the rotating frame is of the form

\[ ds = \dot{t} (2c) \cdot dl \cdot (2c_1 k_i) \cdot dx^2; \]

where \( dl \) is the pseudo-Euclidean metric form:

\[ dl = [l (r \ddot{r} = c^2) (dx^2)^2 (dx^2) (dx^2)] ^{1/2} \]

\[ = 2 \cdot (dx^2) \cdot dl \cdot dx^2 \cdot dx^2; \]

In this coordinates system eq. (38) is satisfied with accuracy at least up to \( v = c \). In virtue of equations (27) and (36) the time of the motion of light from the point \( x^i \) to \( x^i + dx^i \) is \( dl = \rho = c_1 \cdot dl(1 + 2c c_1 k_i) \). It follows from eq. (30) that in the used coordinates system

\[ f c_1 k_i = \frac{r \cdot d \ddot{r}}{2c_2}; \]

For this reason the difference in the time interval between light propagation around the rotating disk in a clockwise and counterclockwise direction is \( 4 \csc^2 = c^2 \), which gives the Sagnac phase shift \( \Delta \).

The Sagnac effect for the isolated observer in the rotating frame can be treated as caused by the Finslerian metric of space-time in noninertial frames of reference.

5 Inertial Forces

Let us show that the existence of the inertial forces in NIFR can be interpreted as the exhibition of the Finslerian connection of space-time in such frames.

According to our initial assumption in Section 3, the differential equations of motion in an IFR of the point masses, forming the reference body of the NIFR, are the geodesic lines of space-time in NIFR. These equations can be found from the variational principle \( ds = 0 \). The equations of the form

\[ d = ds + G (\xi; \xi) = 0; \]

where \( G \) is the 4-velocity of the point mass, the world line of which is \( x = (\xi, \xi) \), and

\[ G (\xi; \xi) = + B + d(1) = ds; \]

where

\[ B = \theta f = \theta x; \theta f = \theta x; \]

In the Finslerian space-time a number of connections can be defined according to eq. (27) [3]. In particular, this equation can be interpreted in the sense that in the NIFR space-time the absolute derivative of a vector field \( \xi \) along the world line \( x = (\xi, \xi) \) is of the form

\[ D = ds = \dot{t} = ds + G (\xi; dx = ds) \]

where

\[ G (\xi; dx = ds) = B + dx = ds + d(1) = ds; \]

Equations (39) define a connection of Laugvitz type [5] in space-time of the NIFR, which is nonlinear relative to \( dx \). The change in the vector due to an infinitesimal parallel transport is

\[ d = G (\xi; dx) \]

Consider the motion of a particle of the mass \( m_p \), in a NIFR, unaffected by forces of any kind in the laboratory (inertial) frame of reference. The differential equations of motion of such a particle can be found from the variational principle \( d = 0 \). Since \( ds = d \rightarrow \) the equations of motion are

\[ D = ds = B u; \]

As an example, consider the nonrelativistic disk rotating in the \( xy \) plane about the \( z \) axis with the angular velocity \( \omega \). The equations of motion (57) are

\[ d \cdot = dt + \omega \cdot r = 0; \]
where \( \mathbf{r} = f(x; y; z) \) and the coordinates origin coincides with the disk center. The absolute derivative (50) of a vector \( \mathbf{v} \) is given by
\[
D \mathbf{v} = \frac{d}{dt} \mathbf{v} = \mathbf{v} + \mathbf{a},
\]
and the equations of motion (46) of the considered particle in the NIFR are
\[
D \mathbf{v} = \frac{d}{dt} \mathbf{v} = \mathbf{v} - \mathbf{a},
\]
where \( \mathbf{v} = f(x; y; z) \).

Next, for the 4-velocity \( \mathbf{u} \) we have
\[
\mathbf{u} = \mathbf{v} + \mathbf{a} + \mathbf{u},
\]
where \( \mathbf{u} \) is the spatial velocity of the particle in the NIFR.

The right-hand side of eq. (50) is the ordinary expression for the Coriolis forces and
\[
(43) \text{ is considered sometimes in classical dynamics nominally (12) just for the derivation of the inertial forces in the NIFRs.}
\]

6 Relativity of Inertia

A clock, which is in a NIFR at rest, is unaffected by acceleration in space - time of the frame. The change in rate of the ideal clock is a real consequence of the difference between the space - time metrics in the IFR and NIFR. It is given by the factor \( \Delta \mathbf{t} = \Delta \mathbf{t} \), from the equation \( \Delta s = c \Delta t \). For the rotating disk of the radius \( R = 1 !^2 \mathbf{R}^2 = 2c^2 \) which gives rise to the observed red shift in the well known Pound - Rebka - Snider experiments.

We consider here another experimentally verifiable consequence of the above theory.

Let \( \mathbf{p} = m_p c \mathbf{x} = 0 \) be 4-momentum of a particle in the IFR. Using 3 + 1 decomposition of space-time in the NIFR we have
\[
\mathbf{p} = \mathbf{p} + E \mathbf{u},
\]
where \( E = \mathbf{p} \) is the energy of the particle in the NIFR.

Thus, the inertial mass \( m_p^0 \) of the particle in the NIFR is given by
\[
\begin{align*}
\mathbf{m}_p^0 &= \mathbf{Q} m_p, \\
\end{align*}
\]

The quantity \( m_p^0 \) coincides with the proportionality factor between the momentum and the velocity of the nonrelativistic particle in the NIFR.

Since \( \mathbf{Q} \) is the function of \( x \), the inertial mass in the NIFR is not a constant. For example, on the rotating disk we have
\[
\mathbf{m}_p^0 = m_p = \left( \begin{array}{c} 1 \\ \mathbf{x} \end{array} \right) \left( \begin{array}{c} 1 \\ \mathbf{x} \end{array} \right) \mathbf{Q} c^2 \right),
\]
where \( \mathbf{x} \) is the rotation angular velocity and \( \mathbf{r} \) is the distance of the body from the disk center.

The difference between the inertial mass \( m_p^0 \) of a body on the Earth equator and the mass \( m_p^{pol} \) of the same body on the pole is given by
\[
\mathbf{m}_p^{eq} m_p^{pol} = m_p^{eq} = 1 \mathbf{2} 10^2
\]

The dependence of the inertial mass of particles on the Earth longitude can be observed by the Møssbauer effect. Indeed, the change in the wave length at the Compton scattering on particles of the masses \( m_p \) is proportional to \( m_p^{-1} \). If this value is measured for gamma - quanta with the help of the Møssbauer effect at a fixed scattering angle, then after transporting the measuring device from the longitude \( \mathbf{1} \) to the longitude \( \mathbf{2} \) we have
\[
\left( \begin{array}{c} 1 \\ \mathbf{1} \end{array} \right) \left( \begin{array}{c} 1 \\ \mathbf{1} \end{array} \right) \left( \begin{array}{c} \cos^2 \mathbf{1} + \cos \mathbf{2} \left( \begin{array}{c} \mathbf{1} \\ \mathbf{2} \end{array} \right) \right); \right.
\]
where \( \mathbf{1} = 1 \mathbf{2} 10^2 \).
7 Gravitation in Inertial and Proper Reference Frames

Consider a frame of reference whose reference body is formed by identical material points \( m_p \) moving under the effect of the field \( F \). These frames will be called the proper frames of reference (PRF) of the given field. Any observer, located in the PFR at rest, moves in space-time of this frame along the geodesic line of his space-time. This implies the space-time metric differential form in the NIFR is given by eq. (58), where \( S \) is the action describing describing the motion of particles forming the reference body of the NIFR.

Now suppose \( \textbf{[17]} \) that in pseudo-Euclidean space-time gravitation can be described as a tensor field \( \epsilon(x) \), and the Lagrangian describing motion of a test particle with the mass \( m_p \) is of the form

\[
L = m_p c g(x) \times \times \times j^{1,2}; \tag{58}
\]

where \( x = dx =dt \) and \( g \) is the symmetric tensor whose components are the function of \( \cdot \).

According to \( \textbf{[3]} \) the space-time metric differential form in the PFR is given by

\[
\text{d}a^2 = g(x) \times \text{d}x \times \text{d}x \tag{59}
\]

Thus, the space-time in the PFR is a Riemannian with the curvature other than zero. Viewed by an observer in the IFR, the motion of the test particle forming the reference body of the PFR is affected by the force field \( F \). But the observer located in the PRF will not observe the force properties of the field since he moves in space-time of the PRF along the geodesic line. For him the presence of the field will be displayed in another way — as space-time curvature differing from zero in these frames, e.g. as a deviation of the world lines of the neighbouring points of the reference body.

For example, when studying the Earth’s gravity, a frame of reference fixed to the Earth can be considered as an inertial frame if the forces of inertia are ignored. An observer located in this frame can consider motion of the particles forming the PRF reference body in flat space-time on the basis of eq. (58) without running into contradiction with experiments. However, the observer in the PFR (in a comoving frame for free falling particles) does not find the Earth gravity as some force field. If he proceeds from the relativistic space-time, he believes that point particles, forming the reference body of his reference frame, are the point of his physical space. They do not affect a force field and their accelerations in his space-time are equal to zero. In spite of that, he observes a change in the relative distances of these particles. Such an experimental fact has apparently the only explanation as non-relativistic display of the deviations of the geodesic lines caused by space-time curvature. So, we observe an important fact that only in proper frames of reference we have an evidence for gravitation identification with space-time curvature.

Thus, we arrive at the following hypothesis. In inertial frames of reference, where space-time is pseudo-Euclidean, gravitation is a field \( F \). In the proper frames of reference of the field \( V_n \), where space-time is Riemannian, gravitation manifests itself as curvature of space-time and must be described completely by the geometrical properties of the letter.

If this possibility really takes place in nature, then it will remove an isolation of the geometrical gravitational theories from the theories of other fields.

Of course, eq. (58) refers to any classical field. For instance, space-time in the PRF of an electromagnetic field is Finslerian. However, since \( ds \) depends on the mass \( m_p \) and charge \( e \) of the point masses forming the reference body, this fact is not of great significance.

It should be noted that the geometrical theory of gravitation in the PFR is not identical to Einstein’s theory. Gravitational equations should be some kind of differential equations for the function \( g \), which are invariant under a certain set of gauge transformations of the potentials \( \phi \). Since \( g \) = \( g(x) \), the Einstein equations are the equations both for \( g \) and for \( \phi \). Under the transformation \( \phi \) the quantities \( g(x) \) undergo some transformations too, and, as a consequence, the equations of the test particle motion resulting from eq. (58) and the Einstein’s equations do not remain invariant.

The equations of motion resulting from eq. (58) are at the same time the equations of a geodesic line of the Riemannian space-time \( V_n \) of the dimensionality \( n \) with the metric tensor \( g \). That is why if the given gauge transformation \( \phi \) leaves the equations of motion invariant, then the corresponding transformation \( g \) preserves \( V \) of the Riemannian spaces leaving geodesic lines invariant, i.e. it is a geodesic (projective) mapping. Let us assume that not only eq. (58) but also the field equations contain only in the form \( g \), then it becomes clear that the gauge-invariance of the equations of motion will be ensured if the field equations are invariant with respect to geodesic mappings of the Riemannian space \( V_n \). Thus, if we start from eq. (58), then the gravitational field equations as well as the physical field characteristics must be invariant with respect to geodesic (projective) mappings of the Riemannian space-time \( V_n \) with the metric tensor \( g \).

Simplest equations of that kind are analyzed in the next Section.

8 The geodesic-invariant equations of gravitation

In accordance with the basic principles the space-time in the PRF of the gravitational field \( V_n \) is the Riemannian space-time of dimension \( n=4 \) whose metric tensor is defined up to geodesic (projective) mappings \( g \). The geodesic
transformations of the metric tensor \( g \) are given by Levi-Chivita equations [19], [20]:
\[
\begin{align*}
\begin{array}{c}
r \left( \gamma \right) = 2' \gamma + \gamma + \gamma \; ; \\
\end{array}
\end{align*}
\]
where
\[
\begin{align*}
\gamma &= \frac{1}{2(n+1)} \Theta \times \left[ \Theta \gamma \right] \; ; \\
r \left( \gamma \right) &= \Theta \gamma = \Theta \times \gamma = \gamma \; ; \\
g &= \det \gamma \; .
\end{align*}
\]

The Christoffel symbols and the curvature tensor also do not remain invariant. In particular, the Christoffel symbols are transformed as follows [20]:
\[
\begin{align*}
\Gamma^i_{jk} &= \gamma^i_{jk} + \gamma^i_j + \gamma^i_k \; ; \\
\end{align*}
\]
where \((\kappa)\) is a vector field.

However, some objects, which are invariant under geodesic (projective) mappings of the space \( V_n \), also can be defined. Just these gauge-invariant objects have a physical sense in the theory under consideration.

The simplest gauge-invariant object is the Thomas symbols [22]:
\[
\Gamma^i_{jk} = (n+1)^{i} + \; ;
\]
where \ are Thomas symbols in the \( E_n \):
\[
\begin{align*}
\Gamma^i_{jk} &= (n+1)^{i} + \; ; \\
\end{align*}
\]

This geodesic - invariant tensor will be named the strength tensor of a gravitational field. Note that the equality \( B = 0 \) is satisfied identically.

According to eq. (58) the differential equations of motion of the test particle are given by
\[
\begin{align*}
\frac{d^2 x}{d\sigma^2} + \frac{dx}{d\sigma} \frac{dx}{d\sigma} &= 0 \; ; \\
\end{align*}
\]
where \( \Gamma^i_{jk} \) are the Christoffel symbols in \( V_n \). Following [23] we will show how one can define a geodesic - invariant connection and curvature in the spaces under consideration.

Let us define a scalar parameter \( \rho \) on the geodesic lines, that remains unaltered by geodesic mapping of \( V_n \); by means of a differential equation
\[
\rho \frac{d\sigma}{d\rho} = 2 \; \; \frac{dx}{d\sigma} \frac{dx}{d\sigma} \; ;
\]
where
\[
\rho \frac{d\sigma}{d\rho} = \frac{d^2 x}{d\sigma^2} = 2 \; \; \frac{dx}{d\sigma} \frac{dx}{d\sigma} \; ;
\]
is a nonzero constant and \( \frac{dx}{d\sigma} \) are a given function of \( x \).

By eq. (66) the parameter \( \rho \) is defined as the function of \( \sigma \) up to linear fractional transformations.

Since the parameter \( \rho \) must be a scalar, the object \( \frac{dx}{d\sigma} \) is the components of the covariant tensor of rank 2.

Let \( \frac{dx}{d\sigma} \) and \( \frac{dx}{d\sigma} \) be the components of \( \frac{dx}{d\sigma} \) and the line element \( d\sigma \), respectively, after some geodesic mapping of space-time \( V_n \). Then the new geodesic equations are given by
\[
\frac{d^2 x}{d\sigma^2} + \frac{dx}{d\sigma} \frac{dx}{d\sigma} = 0 \; ;
\]
On the other hand, after a geodesic mapping eqs. (65) are
\[
\frac{d^2 x}{d\sigma^2} + \frac{dx}{d\sigma} \frac{dx}{d\sigma} = 0 \; ;
\]
where \( \frac{dx}{d\sigma} = \frac{dx}{d\sigma} \) and \( \frac{dx}{d\sigma} = \frac{dx}{d\sigma} \); and these equations must be identical to (69). With regards to (61), this yields
\[
\begin{align*}
\frac{dx}{d\sigma} &= \exp 2 \; \frac{dx}{d\sigma} \; ;
\end{align*}
\]
Then, by putting
\[
\frac{dx}{d\sigma} = 2 \; \frac{dx}{d\sigma} \; ;
\]
and by taking notice of
\[
\frac{dx}{d\sigma} = \left( \frac{dx}{d\sigma} \right)^2 \left[ \frac{dx}{d\sigma} \right] \; ;
\]
we find (since \( \rho \) must be the invariant under geodesic mappings) the equations of transformation of \( \frac{dx}{d\sigma} \) under geodesic mappings:
\[
\frac{dx}{d\sigma} = \frac{dx}{d\sigma} \frac{dx}{d\sigma} \; ;
\]
Now on every geodesic we define a gauge variable ("5 th coordinate") by substitution
\[
\frac{dx}{d\sigma} = \frac{1}{2} \log \frac{ds}{d\sigma} \; ;
\]
Let \( \frac{dx}{d\sigma} \) be another parameter on the geodesic and
\[
\frac{dx}{d\sigma} = \frac{1}{2} \log \frac{ds}{d\sigma} \; ;
\]
is the gauge variable corresponding to the transition \( \rho \). Then it follows from (70) that for any path
\[
\frac{dx}{d\sigma} = \frac{1}{2} \log \frac{ds}{d\sigma} + \frac{1}{2} \; \; \frac{dx}{d\sigma} + C \; ;
\]
where integration is performed from an arbitrary fixed point \( q \) on the geodesic along the curve and \( C \) is the arbitrary constant.

Using the relation
\[
f_{\nu;\mu g} = \frac{d\nu}{du} \frac{\partial}{\partial q} f_{\nu;\gamma g}; \quad (77)
\]
we have
\[
f_{\nu;\rho g} = 2 \frac{\partial}{\partial q} \frac{\partial}{\partial q}; \quad (78)
\]
On replacing in this equation and the geodesic equations \((74)\) the derivations \( ds = dp \), \( d^2 s = dp^2 ; \) \( d^3 s = dp^3 \) by \( x^2 \); \( dx^3 = dp \), \( dx^4 = dp^2 \) according to the definition of the gauge variable \((74)\) we find that the geodesic equation \((65)\) and the equation \((78)\) which yields the definition of the parameter \( p \) can be written as
\[
\frac{d^2 x}{dp^2} + \frac{dx}{dp} \frac{dx}{dp} + 2 \frac{dx^4}{dp} = 0; \quad (79)
\]
\[
\frac{d^2 x^4}{dp^2} + \frac{dx^3}{dp} \frac{dx^4}{dp} + 4 \frac{dx^3}{dp} \frac{dx^4}{dp} = 0; \quad (80)
\]
respectively. These equations can be united as an equation of a geodesic in 5-dimensional space-time
\[
\frac{d^2 x^A}{dp^2} + A_{B C} \frac{dx^B}{dp} \frac{dx^C}{dp} = 0 \quad (81)
\]
The capital indices run from \( 0 \) to \( 4 \) and \( A_{B C} \) is the Kronecker symbols.

Suppose the functions satisfy the conditions:
\[
\theta = 0; \quad \theta = 0; \quad = 0; \quad (82)
\]
If we consider the transformations
\[
\vec{x} = f(x^0; x^1; x^2; x^3); \quad (83)
\]
and
\[
\vec{x}^0 = x^0 + \theta; \quad \vec{x} + C; \quad (84)
\]
where \( \frac{dx}{\theta} \) is the exact differential of a function of \( x \), as admissible coordinate transformations in the \( n + 1 \) dimensional manifold \( M_5 \), then eqs. \((81)\) can be regarded as the differential equations of the geodesics in homogeneous coordinates of projective geometry. A theory of the projective connection in such a way has been considered by \( [21] \) and other authors in 1921-1937.

The object components \( A_{B C} \) are transformed as follows:
\[
\bar{A}_{BC} = \bar{A}_{BC} + \bar{A}_{BD} \frac{\partial}{\partial q} \bar{A}_{BC}; \quad (85)
\]
The components of \( A_{B C} \) are coefficients of the projective connection \((22)\).

The object components \( R_{B C D} \) are given by
\[
R_{B C D} = \frac{\partial}{\partial q} \bar{A}_{BC} - \frac{\partial}{\partial q} \bar{A}_{BD} \frac{\partial}{\partial q} \bar{A}_{BC}; \quad (86)
\]
is transformed as the tensor relative to the coordinate transformations in \( M_5 \). The components of the tensor \( R_{B C D} \) vanish identically save \( R \) and
\[
R = R + \frac{\partial}{\partial q} \bar{A}_{BC}; \quad (87)
\]
where \( R \) is the curvature tensor of the affine connection in \( V_4 \). It has the following properties
\[
R = 0; \quad (88)
\]
\[
R + R = 0; \quad (89)
\]
\[
R + R + R = 0; \quad (90)
\]
It follows from eq. \((77)\) that the contracted tensor is given by
\[
R = R + (n + 1) \theta; \quad (91)
\]
It does not change with geodesic mappings of \( V_4 \).

The equations
\[
R = 0 \quad (92)
\]
are the simplest geodesic - invariant generalization of the Einstein vacuum equations.

Depending on a choice of the object \( 0 \) and we obtain a specific variant of the theory. In this paper we will assume that \( = 1 \) and
\[
\frac{1}{(n + 1)} \Theta \bar{Q} = \Theta x; \quad \bar{Q} = \Theta x; \quad (93)
\]
where \( \bar{Q} = \bar{Q} : \) The object defined in this way has the required properties under geodesic mappings \((73)\). Equation \((72)\) can be written in the form
\[
B \Theta B = 0; \quad (94)
\]
where the semi-; colon denotes a covariant derivative in the Pseudo - Euclidean space-time \( E_4 \). These equations were proposed first in \((8)\) from another viewpoint.

Equations \((24)\) are the system of the differential equations for the geodesic- invariant tensor \( B \) (or for the functions \( q \) ) which are defined up to arbitrary geodesic mappings. The coordinate system is defined by the used measurement instruments and is given. The equations do not contain the functions explicitly.

The simplest way of obtaining equations for is to set
\[
\frac{\partial}{\partial q} \bar{Q} = 0; \quad (95)
\]
where \( \bar{\xi} \) is the covariant derivative in flat space-time. The identity \( B \) is satisfied as it is to be expected according to the definition of the tensor \( B \).

Then, at the gauge condition \( \bar{\xi} = 0 \) eq. \((94)\) are given by

\[
B = \bar{\xi} (n + 1) \bar{\xi} + \bar{\xi} \bar{\xi}; \quad (96)
\]
\begin{align*}
\frac{r}{r} &= 0 \quad (96) \\
\frac{r}{r} &= 0;
\end{align*}

where \( \Gamma \) is the covariant Daanber operator in pseudo-Euclidean space-time.

It is natural to suppose that with the presence of matter these equations are given by

\begin{align*}
\mathcal{G} &= \{ T + t \}; \quad (97) \\
\mathcal{G} &= 0;
\end{align*}

where \( \mathcal{G} = 8 \mathcal{G}; \quad t = \{ r \quad r \quad \mathcal{T} \} \) and \( \mathcal{T} \) is the matter tensor of the energy-momentum.

Obviously, the equality

\begin{equation}
\mathcal{G} \left( T + t \right) = 0 \quad (98)
\end{equation}

is valid. Therefore, the magnitude \( t \) can be interpreted as the energy-momentum tensor of a gravitational field.\(^2\)

\section{9 Spherically-Symmetric Gravitational Field.}

Let us find the spherically symmetric solution of eqs. (94) in an inertial frame of reference, where space-time is supposed to be pseudo-Euclidean.

Because of the gauge (geodesic) invariance, additional conditions can be imposed on the tensor \( g \). In particular \( [13] \), under the conditions

\begin{equation}
\mathcal{G} = = 0 \quad (99)
\end{equation}

eqs. (94) will be reduced to the Einstein vacuum equations \( R = 0 \), where \( R \) is the Ricci tensor. Let us choose a spherical coordinate system. Then, if the test particle Lagrangian is invariant under the mapping \( t \), the fundamental metric form of space-time \( \mathcal{V}_{4} \) can be written as

\begin{align*}
\mathcal{d}s^{2} &= A \left( \mathcal{d}x \right)^{2} + B \left( \mathcal{d}y + \mathcal{A} \mathcal{N}^{2} \right) \left( \mathcal{d}y \right)^{2} + C \left( \mathcal{d}k^{0} \right)^{2}; \quad (100)
\end{align*}

where \( A, B \) and \( C \) are the functions of the radial coordinate \( r \). Proceeding from the above-stated, we shall find the functions \( A, B \) and \( C \) as the solution of the equations system

\begin{equation}
\mathcal{R} = 0 \quad (101)
\end{equation}

and

\begin{equation}
\mathcal{Q} = 0; \quad (102)
\end{equation}

which satisfy the conditions:

\begin{equation}
\lim_{r \to 1} A = 1; \quad \lim_{r \to 1} (B \mathcal{A}^{2}) = 1; \quad \lim_{r \to 1} C = 1; \quad (103)
\end{equation}

\footnotetext{1}{It should be noted that, when we introduce it in some way, we cannot be sure apriori that the equation for \( y \) yields all solutions of the equations for \( B \). We may introduce a potential also in another way.}

The equations \( R_{11} = 0 \) and \( R_{00} = 0 \) can be written\(^2\) as

\begin{align*}
B L_{2} &= 2C L_{4} = 0 \quad (104) \\
B 0^{0} &= 2B L_{2} = 0; \quad (105)
\end{align*}

where

\begin{equation}
L_{1} = R_{1212} = B 0^{0} = 2 \left( B^{0} \right)^{2} = (4B) \quad B 0^{0} = (4A) \quad (106)
\end{equation}

and the differentiation of \( A, B \) and \( C \) with respect to \( r \) is denoted by the primes.

The non zero eqs. (102) yield

\begin{equation}
B 2A C = r^{4}; \quad (106)
\end{equation}

First, the combination of eqs. (104) and (105) yields

\begin{equation}
4C L_{1} + B 0^{0} = 0; \quad (107)
\end{equation}

\begin{equation}
2B 0^{0} + (B 0^{0})^{2} = 0; \quad (108)
\end{equation}

Equation (107) then becomes

\begin{equation}
2B 0^{0} + (B 0^{0})^{2} = 0; \quad (108)
\end{equation}

or

\begin{equation}
\mathcal{u}^{0} = \mathcal{u} = 4 = r; \quad (109)
\end{equation}

where \( \mathcal{u} = (B 0^{0}) \mathcal{B} \). By using (103) we find

\begin{equation}
B = (r^{3} + K^{3})^{2/3}; \quad (110)
\end{equation}

where \( K \) is a constant.

Next, from eq. (105) we find by using eqs. (103).

\begin{equation}
C = 1 \quad Q = B 1^{2/3}; \quad (111)
\end{equation}

where \( Q \) is a constant.

Finally, we can find the function \( A \) from eq. (106).

Thus, in the spherical symmetric coordinate system the following functions \( A, B \) and \( C \) are obtained:

\begin{equation}
A = (r 0^{0})^{2} (Q = f) \quad B = f^{2}; \quad C = 1 \quad Q = f; \quad (112)
\end{equation}

where

\begin{equation}
f = (r^{3} + K^{3})^{1/3}
\end{equation}

and \( f 0^{0} = \mathcal{C}e = \mathcal{d}x \).

The nonzero components of the tensor \( B \) are given by
\[
\begin{align*}
B_{rr} &= \frac{1}{2} A \left( -\frac{1}{2} A \right) ; \\
B_{r} &= \frac{1}{2} r A \left( -\frac{1}{2} A \right) ; \\
B_{rt} &= \frac{1}{2} \frac{C}{A} ; \\
B_{tt} &= \frac{1}{2} \frac{C}{A} ;
\end{align*}
\]
(113)

If \( K = 0 \); then \( f = r \) and the solution \((112)\) coincides with the Droste-Weyl solution of the Einstein equations which is known as the Schwarzshild one \([28]\). If \( K = Q \); it coincides with the originally Schwarzshild solution \([22]\). However, it is important to understand that from the point of view of the considered theory, solution \((112)\) is obtained in a given (spherical) coordinate system, defined in pseudo-Euclidean space-time, and that different values of the constant \( Q \) and \( K \) yield different solutions of equation \((124)\) in the same coordinate system.

The equations of the motion of a test particle resulting from Lagrangian \((88)\) is given by

\[ \dot{x} + \left( \begin{array}{c} c^2 \end{array} \right) \dot{x} = 0 ; \]
(114)

In the nonrelativistic limit \( \dot{x} \rightarrow \dot{r} = \dot{r}_0 \); where \( \dot{r}_0 = \frac{c}{c} = \frac{\dot{c}}{\dot{c}} = \frac{\dot{c}}{\dot{c}} \); Therefore, to obtain the Newton gravity law it should be supposed that at large \( r \) the function \( \dot{r} \) and \( Q = \dot{r} = 2GM \) \( M \) is classical Schwarzshild radius.

At a given constant \( Q \) allowable solutions are obtained by change of the arbitrary constant \( K \). In particular, if we setting \( K = 0 \) the fundamental form of space-time \( V \) coincides with the Droste-Weyl solution of the Einstein equations \([28]\) (it is commonly named the Schwarzshild solution) which have an horizon horizon at \( r = r_0 \):

\[ ds^2 = \left( \frac{dr}{\dot{r}} \right)^2 \right) \left[ \dot{r}^2 + \sin^2 \theta \right] + \left( \frac{d\theta}{\dot{\theta}} \right)^2 \]
(115)

If we setting \( K = Q \), the solution coincides with the original Schwarzshild solution \([22]\) which have no the horizon and singularity in the center:

\[ ds^2 = \left( \frac{dr}{\dot{r}} \right)^2 \right) \left[ \dot{r}^2 + \sin^2 \theta \right] + \left( \frac{d\theta}{\dot{\theta}} \right)^2 ; \]
(116)

where \( \dot{r} = \left( \dot{r}_0^2 + \dot{r}^2 \right)^{1/2} \).

These solutions are related to the same coordinate system and are different solutions of the gravitation equations under consideration.

\footnote{A discussion of the difference in these solutions are given in \([28]\).}

Of course, a reader can say: however, we can obtain the Droste-Weyl solution from the original Schwarzshild one by a coordinate transformation. Supposing it is so. (There is an alternative point of view \([?]\)). However, in this case we must transform also space and time intervals to the new co-ordinates which, in contrast to the spherical ones, have no sense of values measured by rulers and clock. After that, of course, we obtain the same physical results as in the spherically co-ordinate. (Like classical electrodynamics in arbitrary co-ordinates). Therefore, since at \( K = Q \) the solution of our equation have no the singularity in the center and events horizon, it does not contain theirs and in others coordinate systems.

We can argue that the constant \( Q = r_0 \). Indeed, consider the 00-component of eq. \((77)\). Let us set \( T = c = c_0 \); where \( c_0 \) is the matter density and \( u \) is the 4-velocity of matter points. At the small macroscopic velocities of the matter we can set \( u_0 = 1 \) and \( u_1 = 0 \). Therefore, the equation is of the form

\[ = \left( \frac{c^2 + c_0}{c^2 + c_0} \right) \]
(117)

where \( = 4G = c_0^4 \) and \( c_0 \) is the 00-component of the tensor

\[ t = 1 \frac{B}{B} ; \]
(118)

Let us find the energy of a gravitational field of the point mass \( M \) as the following integral in the pseudo-Euclidean space-time

\[ E = \frac{1}{2} \frac{G}{K} dV ; \]
(119)

resulting from the above solution, where \( dV \) is the volume element. In the Newtonian theory this integral is divergent. In our case we have:

\[ t_0 = 2 \frac{1}{2} \frac{B^0}{B_0} \frac{B_0}{B_0} \frac{c_0^4}{c^4} \frac{Q^2}{G} \frac{K}{K} ; \]
(120)

and, therefore, in the spherical coordinates, we obtain

\[ E = \frac{1}{2} \frac{G}{K} dV ; \]
(121)

where

\[ J = \frac{dV}{t^2} = \frac{4}{3K} B \left( \theta ; \theta = 3 \right) \]
(122)

and

\[ B \left( \theta , \theta \right) = \frac{1}{(1 + t^2)^2 + \theta^2} dt \]
(123)

is B-function. Using the equality

\[ B \left( \theta , \theta \right) = \frac{\left( \theta \right)}{\left( \theta + \theta \right)} ; \]
(124)

where is -function we obtain \( J = 4 \) \( = K \), and, therefore,

\[ E = \frac{Q}{K} \frac{c^2}{2G} ; \]
(125)
We arrive at the conclusion that at $K \neq 0$ the energy of the point mass is finite and at $K = Q$ the rest energy of the point particle in full is caused by its gravitational field:

$$E = M c^2.$$ 

The spacial components of the vector $P = t_0$ are equal to zero.

Due to these facts we assume in the present paper that $K = Q = r_0$ and consider solution (112) in the spherical coordinates system at the used gauge condition as a basis for the subsequent analysis.

### 10 Orbits of Non-Zero Mass Particles.

The equations of motion of a test particle of a non-zero mass in the spherically symmetric field resulting from eqs. (94) are given by [18]

$$\dot{r}^2 = (c^2 \mathcal{C} = \Lambda) \left[ \mathcal{C} = \mathcal{E} \right] \left( 1 + \frac{\mathcal{E}^2}{r^2} \right);$$  \hspace{1cm} (126)$$

$$\dot{\varphi} = c \mathcal{C} \mathcal{J} = \Theta \mathcal{E}$$  \hspace{1cm} (127)

where $(\varphi'; \mathcal{C})$ are the spherical coordinates (is supposed to be equal to $-2$), $x = c\mathcal{C} = \Theta = c \mathcal{C} = \Theta$, $E = E = m c^2$, $J = J = m c$, $E$ is the particle energy, $J$ is the angular momentum. Let $\mathcal{C} = 1 = \mathcal{E}$, where $\mathcal{C} = (1 + \mathcal{P})^{1/3}$ and $\mathcal{E} = x = \mathcal{E}_0$. Then the differential equation of the orbits, following from eqs. (126) and (127) can be written as

$$(\mathcal{C} \mathcal{E} \mathcal{J} = \Theta \mathcal{E} )^2 = G(\mathcal{E})$$  \hspace{1cm} (128)

where

$$G(\mathcal{E}) = \mathcal{C}^3 - \mathcal{E} + \mathcal{C} \mathcal{E} \mathcal{J} \mathcal{C} - \mathcal{E}^2.$$ 

eq. (128) differs from the orbit equations of general relativity [27] by the function $\mathcal{C}$ instead of the function $\mathcal{E}$. Therefore, the distinction in the orbits becomes apparent only at the distances $\mathcal{E}$ of the order of 1 or less than that.

Setting $x = 0$ in eq. (126) we obtain $\mathcal{E} = N(\mathcal{E})$, where

$$N(\mathcal{E}) = \left( \frac{1}{1 + \mathcal{E}^{2} \mathcal{J}} \right) \mathcal{E}^2$$  \hspace{1cm} (129)

is the effective potential [27]. Fig.1 shows the function $N = N(\mathcal{E})$.

The function $N(\mathcal{E})$ differs from the one in general relativity in two respects:

1. It is defined at every point of the interval $(0; 1)$.  
2. It tends to zero when $\mathcal{E} = 0$.

Possible orbit types can be shown by the horizontals $E = c \cos \Theta t$. Two types of the orbits have peculiarity in comparison with the Einstein equations. The horizontals placed above the maximum of the curve $N(\mathcal{E})$ show the particles orbits which begin in the field center and end in the infinity. In other words, for each value of $\mathcal{E}$ there exists such a value of $E$ for which the gravitational field cannot keep particles escaping from the center. The events horizon is absent. Fig.1 shows an example of the orbits at $\mathcal{E} = 1.99$ and $E = 1$.

The horizontals placed between Y-axis and the curve $N(\mathcal{E})$ can show particles orbit kept by the gravitational field near the field center.

It follows from eqs. (126) and (127) that the velocity of a test particle freely falling to the point mass $M$ tends to zero when $x = 0$. The time of the motion of the particle from some distance $x = x_0$ to $x = 0$ is infinitely large. We can say that the spherically symmetric solution has no physical singularity.

The points of the minimum of the function $N(\mathcal{E})$ show stable closed orbits, the points of the maximum show unstable ones. The minimum of the function $N(\mathcal{E})$ exists only at $\mathcal{E} > \mathcal{E}_c$ which corresponds to the value of the function $\mathcal{E}(\mathcal{E}) > 3$. Therefore, stable circular orbits exist only at $\mathcal{E} > \mathcal{E}_c$, where $\mathcal{E}_c = \frac{1}{3} \mathcal{E}_c$. The orbital speed of the particle with $x = \mathcal{E}_c$ is equal to $0$ at $x \leq \mathcal{E}_c$.
the location of the maximums tends to $\tilde{r} = 3\tilde{r}_g$. Therefore, the minimum radius of the unstable circular orbit is $\tilde{r}_{\infty} = 133 \tilde{r}_g$. In general relativity it is equal to $15 \tilde{r}_g$. The speed of the motion of a particle on this orbit is equal to $0.5c$. The binding energy $E = 0.6572$, just as it occurs in general relativity.

The rotation frequency $\gamma = \gamma$ of the circular orbit will be

$$\gamma = \left\{ (\tilde{r}/1) - (\tilde{r} / \tilde{r}_g) \right\} (\tilde{r} = \tilde{r}_g)$$

(130)

In a circular motion $\tilde{r}$ is the constant and, therefore, the function $N(\tilde{r})$ has the minimum. Consequently, from the equation $dN = d\tilde{r} = 0$ we find

$$\tilde{J}^2 = \tilde{r}^2 = (\tilde{r} + 3)$$

(131)

Using (126) we have at $r = 0$

$$\tilde{E}^2 = 2(\tilde{r}^2 - 1) = \left[ \tilde{r} (2\tilde{r} - 1) \right]$$

(132)

Equations (129) yield

$$\gamma = \gamma^{1+2} \sqrt{\tilde{\tilde{r}}^2}$$

(133)

Hence, the circular orbits have the rotation period

$$\gamma = 2J^2 (\tilde{r} + 3)$$

(3 rd Kepler law). In comparison with general relativity the change in $\gamma$ is 2% at $\gamma = 3$ and 20% at $\gamma = 133$.

Consider the apsidal motion. For ellipsoidal orbits the function $G(\tilde{r})$ has 3 real roots $\tilde{r}_1 < \tilde{r}_2 < \tilde{r}_3$. The apsidal motion per one period is

$$\gamma = 2J^2 (\tilde{r} + 3) ;$$

where

$$\tilde{J}^2 = \tilde{J}^2 = \tilde{r}^2 = 1$$

(134)

Consequently, (31)

$$\tilde{J}^2 = \tilde{J}^2 = \tilde{r}^2 (\tilde{r} + 3)$$

(135)

and

$$\tilde{J}^2 = \tilde{J}^2 = \tilde{r}^2 (\tilde{r} + 3)$$

(136)

Let us introduce (by analogy with general relativity) the following notations:

$$\tilde{r}_1 = (\tilde{r} + 2\tilde{r}_g) ; \tilde{r}_2 = (\tilde{r} + \tilde{r}_g) ; \tilde{r}_3 = \tilde{r} + \tilde{r}_g$$

(139)

where the parameter $p$ at $\tilde{r} > 0$ becomes the focal parameter $p$ divided by $\tilde{r}_g$.

At $\tilde{r} = 1$ the value of $\tilde{r}$ is $2\tilde{r}_g$. At $\tilde{r} = \tilde{r}_g$ and, therefore,

$$F (\tilde{r} = \tilde{r}_g) = (\tilde{r} + 2\tilde{r}_g) \tilde{r}^2 = 1$$

and

$$\tilde{F} (\tilde{r} = \tilde{r}_g) = (\tilde{r} + 2\tilde{r}_g) \tilde{r}^2 \tilde{r} = 1$$

(140)

Using eqs. (135) and (138) we find with accuracy up to $\gamma = \gamma$.

$$\gamma = 3 \tilde{r} + \gamma = (\tilde{r} + \gamma) (\tilde{r} - 16\tilde{r}_g + 54 + \cdots)$$

(141)

For the orbits of Mercury or a binary pulsar (such as PSR 1913 + 16) the value of $\tilde{r}$ differs very little from the value of $r_g = x$. Consequently, the values of $\tilde{r} = 2(\tilde{r} + \tilde{r}_g)$ and $c = (\tilde{r}_g + \tilde{r} + \tilde{r}_g)$ differ very little from the values of $\tilde{r} = \tilde{r}_g$ and $c$. Hence, their apsidal motion differs very little from the general relativity prediction. Even, for example, at $\tilde{r} = 10$ and $\tilde{r} = 0.5$, the difference in $\gamma$ is about 6 $10^{-\alpha}$.

11 Photon Orbits.

The equations of motion of a photon in the spherical symmetric field are given by (18)

$$r^2 = (\tilde{r} - \gamma) (\tilde{r} - 16\tilde{r}_g + 54 + \cdots)$$

(142)

where $\tilde{r} =$ an impact parameter.

The differential equation of the orbits can be written as

$$\tilde{F} = \tilde{r}^{1/2} = G_1(\tilde{r})$$

(143)

where

$$G_1(\tilde{r}) = \tilde{r}^{1/2} - \tilde{r}^{1/2} - \tilde{r}_g$$

and

$$\tilde{r} = \tilde{r}_g$$

Setting $\tilde{r} = 0$ in eq. (143) we obtain $\tilde{r} = (\tilde{r}_g + \tilde{r}_g)^{1/2}$ where $\tilde{r} = \tilde{r}_g$. Fig 3 shows $\tilde{r}$ as a function of $\tilde{r}$, i.e. the location of the orbits turning points.

![Figure 3: The impact parameter $\tilde{b}$ as the function of $\tilde{r}$](image_url)

The function $\tilde{b}(\tilde{r})$ is defined at all $\tilde{r} > 0$. The minimal value of $\tilde{b}$, i.e. $\tilde{b}_{\min}$, is equal to $2\tilde{r}_g$. It is reached at $\tilde{r}_{\infty} = 133$. The motion of photons can be shown by the horizontal line $\tilde{r} = \tilde{r}_g$. There are two types of orbits which have the peculiarities due to the lack of the event horizon. The curves placed below the minimum of the curve $\tilde{r}(\tilde{r})$
show that the attracting mass cannot keep a photon escaping from the center at the parameter \( \beta < 3 \). The orbits of this type also show the gravitational capture of the photon. The photon finishes at the field center, unlike in general relativity, where it ends on the Schwarzschild sphere.

The angle of the light deflection at the distances close to \( \tau = 1:33 \) is given by

\[
\theta = \ln \left( \frac{d}{2} \right) \equiv \frac{\ln \left( \frac{d}{2} \right)}{2};
\]

where \( d = \frac{2 (l \cos \theta - 3)}{3} \) and \( l \cos \theta = (1 + \tau^2) \sqrt{3} \). It differs very little from the one in general relativity in a weak field. (See also [53].)

12 Spherical-symmetric solution in general case

Properties of the gravitational field, by definition, display themselves exclusively owing to its influence on the motion of test particles. Let us show that the motion of test particles described by Lagrangian (59) is insensitive to the fact whether \( A; B; C \) are depending on time, or not.

Suppose, in the metric form (100) the coefficients \( A, B, C \) are the functions of \( x^0 \). The Christoffel symbols for this form are given by

\[
\begin{align*}
\Gamma_{tt} &= \frac{1}{2} A_t; \quad \Gamma_{tr} = \frac{1}{2} A_r; \quad \Gamma_{rr} = \frac{1}{2} A_t \frac{A_r}{A}; \\
\Gamma_{rt} &= \frac{1}{2} A_t; \quad \Gamma_{rr} = \frac{1}{2} A_t; \quad \Gamma_{rr} = \frac{1}{2} A_t \\
\Gamma_{rr} &= \frac{1}{2} B; \quad \Gamma_{rr} = \frac{1}{2} A_t \\
\Gamma_{r0} &= \frac{1}{2} B; \quad \Gamma_{r0} = \frac{1}{2} A_t \\
\Gamma_{t0} &= \frac{1}{2} B; \quad \Gamma_{t0} = \frac{1}{2} A_t \\
\Gamma_{00} &= \frac{1}{2} B; \quad \Gamma_{00} = \frac{1}{2} A_t
\end{align*}
\]

In these formulas we denote by prime the partial derivative with respect to \( x \) and by point the partial derivative with respect to \( x^0 \). Then the equations of the motion can be written in the form

\[
\begin{align*}
\frac{d^2 x^0}{ds^2} + \frac{A}{2C} \frac{dx^0}{ds} + \frac{A}{2A} \frac{dx^0}{ds} + \frac{B}{2A} \frac{dx^0}{ds} & = 0; \\
\frac{d^2 x^1}{ds^2} + \frac{B}{2C} \frac{dx^1}{ds} & = 0; \\
\frac{d^2 x^2}{ds^2} + \frac{C}{2C} \frac{dx^2}{ds} & = 0;
\end{align*}
\]

that follows from the identity

\[
B \sin^2 \left( \frac{d}{2} \right) \frac{d^2}{ds^2} = 1
\]
The first of these equations is of the form
$$\frac{d^2 x^0}{dt^2} + \frac{1}{C} \frac{dc}{ds} \frac{dx^0}{ds} = 0; \quad (154)$$
where \(dc=ds\) is the total derivative of the function \(C\). Taking into account (150) and (155), we obtain
$$A \frac{dx^0}{ds} = 0; \quad (155)$$
and, after that, by using relation (151), in the form
$$\frac{dA}{ds} A \frac{dx}{ds} + A \frac{dx^0}{ds} \frac{dA}{ds} \frac{dx}{ds} + \frac{B^0}{ds} \frac{dx^0}{ds} = 0; \quad (156)$$
where \(dB=ds\) is the total derivative of the function \(B\). Taking into account (150) and (155), we obtain
$$\frac{dA}{ds} A \frac{dx}{ds} + \frac{1}{B} \frac{d^2 B}{ds^2} \frac{1}{C} \frac{dx^0}{ds} = 0; \quad (157)$$
which yields the third integral of the motion:
$$A \frac{dx}{ds} \frac{dx}{ds} + \frac{1}{B} \frac{d^2 B}{ds^2} \frac{1}{C} = E; \quad (158)$$
where \(E\) is a constant.

Now it can be demonstrated that the same integrals of the motion can be obtained from Lagrangian (158) for the static field. At \(= \pm 2\) it can be written in the form
$$L = m_p c^2 - \frac{A^2}{L} - \frac{B^2}{L} - \frac{1}{C} = 0; \quad (160)$$
where \(A, B\) and \(C\) are the functions of \(x\) only. Since \(L\) does not depend on \(x^0\) and \(\dot{x}\), there are two integrals of the motion:
$$\frac{\partial L}{\partial x} + \frac{\partial L}{\partial \dot{x}} \frac{dx}{ds} = \text{Const}; \quad (161)$$
$$\frac{\partial L}{\partial \dot{x}} = \text{Const}; \quad (162)$$

The first of these equations is of the form
$$\frac{d^2 x^0}{dt^2} + \frac{1}{C} \frac{dc}{ds} \frac{dx^0}{ds} = 0; \quad (154)$$

But it follows from eq. (160) that
$$A \frac{dx^0}{ds} = \frac{C}{L} \frac{L}{m_p c^2} = \text{Const}; \quad (163)$$
Consequently, \(C=L = \text{Const}; \quad (155)\)

Thus, we arrive at equation (155).

The second integral of the motion can be found from equations (162) and (164) and coincides with (150).

To obtain the third integral of the motion we start from the identity
$$\frac{d^2 L}{ds^2} + \frac{1}{B} \frac{d^2 B}{ds^2} \frac{1}{C} = 0; \quad (165)$$
where \(0\) or the particle with the mass \(m_p \neq 0\) and \(= 0\) for the particles with the mass \(m_p = 0\); Substituting (163) \(d'=ds\) and \(dx^0=ds\) from eqs. (163) and (164) into this equation, we obtain the equality which coincides with eq. (154).

Since the properties of a gravitation field are defined by their influence on the motion of the particles, a solution of any correct gravitation equations for the spherical symmetric field, based on Lagrangian (160), must be static.

Using the above results, we can also find gravitational field inside a spherically - symmetric matter layer. In order to reach a coincidence of the motion equations of the test particles in the nonrelativistic limit with the Newtonian ones, the constant \(x_0\) in eq. (111) in that case must be set equal to zero. Therefore, the spherically - symmetric matter layer does not create the gravitational field inside itself.

### 13 Equilibrium Configurations with Large Masses

In this Section we consider one of the most interesting consequences of gravitation equations (143) - the possibility of the existence of compact configurations of degenerated Fermi- gas with very large masses.

The radial component of the gravity force affecting a test particle at rest in the spherically - symmetric field is given by (143)
$$F^r = m \frac{B^0 x_0}{r^2} = \frac{G m_p M}{r^2} \left( \frac{x_0}{(r^3 + x_0^2)^{1/3}} \right) \quad (166)$$

It follows from this figure that \(F^r\) reaches its maximum at the distance \(r\) of the order of \(x_0\) and tends to zero at \(r! 0\).
we find from eq. (168) that the masses of the equilibrium configurations can be obtained from the equation of state

$$\frac{dp}{dr} = \frac{G m (r)}{r^2} \left( \frac{r_g}{r} \right)^{4/3} \left[ \left( \frac{r}{r_g} \right)^{2/3} - 1 \right] \left( \frac{r}{r_g} \right)^{1/3} \right)\#$$

(167)

In this equation, \(p\) is the pressure, \(m = m (r)\) is the mass inside a sphere of radius \(r\), \(r_g = r_g (r)\) is the matter density at the distance \(r\) from the center, \(r_g\) is the function of \(m (r)\).

Suppose the equation of state is \(p = K \rho\), where \(K\) and \(\rho\) are constants. For numerical estimates, we shall use their values from (166).

For rough estimates, we set \(\rho = \text{const}\) and replace \(dp/dr\) by \(\rho = \rho_0\), where \(\rho_0\) is the average matter pressure and \(\rho_0\) is its radius. Under the circumstances, we obtain from eq. (166)

$$\frac{dp}{dr} = \frac{G m (r)}{2 r} \left[ \frac{r_g}{r} \right] \left( \frac{r}{r_g} \right)^{1/3} \right)\#$$

(168)

If \(r \leq r_g\), then the term \(\frac{r_g}{r} \leq \frac{r}{r_g}\) is negligible. Setting \(M = \rho_0 R^2\), we find the mass of equilibrium states as a function of \(r\):

$$M = \rho_0 R^2 \left( \frac{r_g}{r} \right)^{4/3} \left( \frac{r}{r_g} \right)^{1/3} \right)\#$$

(169)

It yields the maximum Chandrasekhar mass \(M = 0.634 \, g\) \(= G M^2\) at \(0\).

However, according to eq. (168), there are also equilibrium configurations at \(r < r_g\). In particular, at \(r < r_g\) we find from eq. (168) that the masses of the equilibrium configurations are given by

$$M = \rho_0 R^2 \left( \frac{r_g}{r} \right)^{4/3} \left( \frac{r}{r_g} \right)^{1/3} \right)\#$$

(170)

These are the configurations with very large masses. For example, the following equilibrium configurations can be found:

- the nonrelativistic electrons: \(= 10^6 \, g = c m^4\), \(M = 1.3 \, 10^6 \, g, R = 2.3 \, 10^6 \, cm\)
- the relativistic electrons: \(= 10^7 \, g = c m^4\), \(M = 2.3 \, 10^7 \, g, R = 1.3 \, 10^7 \, cm\)
- the nonrelativistic neutrons: \(= 10^4 \, g = c m^4\), \(M = 3.9 \, 10^4 \, g, R = 1.6 \, 10^7 \, cm\).

The reason of the two types of configurations existence can be seen from fig. 4, where for \(= 10^5 \, g = c m^4\), \(K = 5 \, 10^9\) and \(= 5 = 3\) the plots of right-hand and left-hand sides of eq. (168) against mass \(M\) are shown.

Next, we consider the analysis of the plots of right-hand and left-hand sides of eq. (168) against mass \(M\).

The following conclusions can be made after considering the plots of the above kind:

1. For each value of \(\leq m_{ax}\) there are two equilibrium states (with \(R > r_g\) and \(R < r_g\)).

2. There are no equilibrium configurations whose density is larger than a certain value \(m_{ax} \leq 10^6 \, g = c m^4\). (At the densities exceeding \(m_{ax}\), the curves do not intersect).

More accurate conclusions about the internal structure of the configuration can be obtained from the equation of the hydrostatic equilibrium obtained for the gravity force (164):

$$\frac{dp}{dr} = \frac{G m (r)}{r^2} \left[ \frac{r_g}{r} \right] \left( \frac{r}{r_g} \right)^{1/3} \right)\#$$

(171)

where \(m = m (r)\) and \(r_g = r_g (r)\). The equation of state is

$$p = \frac{\rho_b}{\rho_0} \frac{c^2}{r^2} \right)\#$$

(172)

where the baryons density \(\rho_b\) as the function of \(r\) is given by the approximation Harrison equation (11) which takes place from \(7\) to at least \(10^16 \, g = c m^4\):

$$n_b = A (1 + B \, 4^{16}) \right)\#$$

(173)

where \(A = 6.2228 \, 10^8\) and \(B = 7.7483 \, 10^{16}\) in CGS units.

In addition to the ordinary solution (i.e. configurations of the white dwarfs and neutron stars) there exist solutions...
with large masses. Fig. [6] shows an example of that kind of solutions for \( r \). It is a configuration with the mass \( 2.6 \, 10^{30} \text{ g} \) and the radius \( 0.378 \, \text{ R} \).

\[ E_{\text{gr}} = \frac{Z}{r} \text{dm}(r) \text{ m}(r); \quad (174) \]

where

\[ (r) = \frac{1}{r} \int_{r}^{Z} \frac{d}{dr} (\frac{r}{Z})^2 \left( 1 + f \right); \]

\[ r_s = 2G \text{ m}(r_e^2), \quad f = \left( r_s \text{ (r)}^3 + \text{ (r)}^3 \right)^{1/3}, \]

\[ m(r) = 4 \int_{0}^{r} \frac{d}{dr} (\frac{r}{Z})^2 (\frac{r}{Z})^2; \quad (175) \]

The function \((r)\) is given by

\[ = \left( l=r \right) \left( l \right) \exp \left( 0.7 \right) \frac{Z}{r}; \quad (176) \]

where \( F \) is the degenerated hypergeometric function. Approximately

\[ = \left( l=r \right) \left( l \right) \exp \left( 0.7 \right) \frac{Z}{r}; \quad (177) \]

Therefore, at \( p = \text{ const.} \), up to a constant of the order one

\[ E_{\text{gr}} = \frac{G \, M^2}{R} \left( l \right) \exp \left( 0.7 \right) \frac{Z}{r}; \quad (178) \]

The intrinsic energy \( E_{\text{int}} = \text{ u dm} \), where \( u \) is the energy per mass unit. For the used equation of state \( u = K \left( 1 \right)^{1/3} \). Thus, up to the constant of the order of one

\[ E = K \, M^3 \left( 1 \right)^{1/3} \left( l \right) \exp \left( Q \right) \frac{M^3}{r}; \quad (179) \]

where \( Q = 0.7 \, c^2 = 2G \). As an example, fig. [7] shows the plot of the function \( E \) for the nonrelativistic degenerated Fermi gas of the mass \( M = 2.5 \, 10^{30} \text{ g} \) in comparison with the neutron star of the mass \( M = 10^{33} \text{ g} \) in fig. [6].

Figure 6: The density distribution inside the object with the mass \( M = 3.9 \, 10^{30} \text{ g} \) and the radius \( R = 1.6 \, 10^{10} \text{ m} \).

Figure 7: The plot of the energy \( E \) vs. the density for the configuration with the mass \( M = 2.5 \, 10^{30} \text{ g} \).

Figure 8: The plot of the energy \( E \) vs. the density for the neutron star with mass \( 10^{33} \text{ g} \).

The analysis of such plots shows that the function \( E = E (r) \) has the minimum. Thus, the above equilibrium states of the large masses are stable.

More rigorous investigation that confirms this result was carried out in [?].

### 14 Conclusion

It follows from the above results that the equations under consideration do not contradict available experimental data obtained in the Solar system. In paper [34] these equations were tested by the binary pulsar PSR1913+16 and it was found out that the results are very close to the ones in general relativity. It is a consequence of the fact that the used distances from attracting masses are much larger than the Schwarzschild radius. At the conditions the function \( f (r) \) is very close to the radial distance \( r \). However, the physical consequences between these equations are completely different at the distances \( r \gg r_s \). The events horizon is absent. There can exist supermassive configurations of the degenerated Fermi-gas. Candidates to the objects of such a kind are the galactic centers ([34] [35]).
References

[1] H. Poincaré, Dernières pensées Flammarion Paris (1913)

[2] The Work of George Berkeley. Vol.2, p.21-113; Vol.4, p.11-30. London (1949)

[3] The Leibniz - Clarke - Correspondence. Manchester (1956)

[4] E. Mach, The Science of Mechanics. Open Court, La Salle, 1960)

[5] H. Dehnen, Wissensch. Zeitschr. der Fridrich - Schiller Universitat. Jena, Math. - Naturw. Reihe. H.1 Jahrg., 15, 15 (1966)

[6] L. Landau and E. Lifshitz, The Classical Theory of Field. Addison - Wesley, Massacusetts (1971)

[7] H. Rund, The differential geometry of Finsler space. Springer (1959)

[8] E. Post, Rev. Mod. Phys. 39, 475 (1967)

[9] J. Ananden, Phys. Rev. D. 24, 338 (1981)

[10] A. Ashtekar and A. Magnon, Journ. of Math. Phys. 16, 341 (1975)

[11] C. Shibata, H. Shimada, M. Azuma and H. Yasuda, Tensor, 31, 219 (1977)

[12] J. L. Syng, Classical dynamics. Springer-Verlag (1960)

[13] L. V. Verozub, Ukr. Phys. Journ. 26, 1598 (1981)

[14] L. V. Verozub, Ukr. Phys.Journ. 10, 131, 778, 1598 (1981)

[15] L. V. Verozub, Phys. Essays, 8, 518 (1995)

[16] V. I. Rodichev, In: Einstein Collection, 286 (1974)

[17] W. Thirring, Ann. Phys. 16, 96 (1961)

[18] L. V. Verozub, Phys. Lett. A, 156, 404 (1991)

[19] T. Levi-Chivita, Ann. di Mat., Ser.2, 255 (1886)

[20] N. S. Sinukov, Geodesic Mappings of Riemmanian Spaces. Moscow, 255 (1979)

[21] A. Z. Petrov, New Methods in General Relativity. Moscow (1966)

[22] L. Berwald, Ann. of Math., 37, 879 (1936)

[23] H. Weyl, Bull. Amer. Math. Soc., 35, 716 (1929)

[24] J. H. Whitehead, Ann. Math., 32, 327 (1931)

[25] O. Veblen, Proc. Nat. Acad. Sci., 14, 154 (1928)

[26] K. Schwarzschild, d. Berl. Akad., 189 (1916). [ Also in : "Albert Einstein and theory of gravitation". Moscow, 199 (1979)]

[27] S. Chandrasekhar, The Mathematical Theory of Black Holes. Oxford Univ. Press, New York (1983)

[28] L. Abrams, Phys.Rev., 20, 2474 (1979)

[29] L. V. Verozub and A. Y. Kochetov, Astr. Nachr. , 3, 143 (2001)

[30] L. V. Verozub and A. Y. Kochetov, Grav and Cosmol., 6, 246 (2000)

[31] P. F. Byrd and M. D. Fridman, Handbook of elliptic integrals for engineers and physicists. Berlin - Göttingen - Heidelberg (1954)

[32] A. Eckart and R. Genzel, MNRAS, 284, 576 (1997)

[33] R. Genzel, D. Hollenbach and C. H. Townes Rep. Progr. Phys., 57, 417 (1994)

[34] A. M. Ghez, B. L. Klein, M. Morris and E. E. Becklin, E-prepr. astro-ph/9807210

[35] R. P. van der Marel, T. de Zeeuw and H-W. Rix, Ap. J., 493, 613 (1998)

[36] L. Greenhill, M. Nakai, P. Diamond and M. Inoue, Nature, 348, 127 (1995)

[37] L. Ferrarese, H. C. Ford and W. Jaffe, Ap. J., 470, 444 (1996)

[38] R. Bender, J. Kormendy and W. Dehnen, Ap. J., 464, 123 (1996)

[39] L. V. Verozub, Astr. Nachr., 317, 107 (1996)

[40] S. Shapiro and S. Teukolsky, Black Holes, White Dwarfs, and Neutron Stars. Jone Wiley & Sons. (1983)

[41] B. K. Harrison, K. S. Thorn, M. Wakano, J. A. Wheeler, Gravitational Theory and Gravitational Collapse. Univ. of Chicago Press, Chicago, Illinois (1966)