ANALYTICAL APPROXIMATIONS FOR CALCULATING THE ESCAPE AND ABSORPTION OF RADIATION IN CLUMPY DUSTY ENVIRONMENTS

FRANK VÁROSI
Raytheon Information Technology and Scientific Services, Code 685, NASA Goddard Space Flight Center, Greenbelt, MD 20771; varosi@gsfc.nasa.gov

AND

ELI DWEK
Laboratory for Astronomy and Solar Physics, Code 685, NASA Goddard Space Flight Center, Greenbelt, MD 20771; edwek@stars.gsfc.nasa.gov

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ABSTRACT

We present analytical approximations for calculating the scattering, absorption, and escape of nonionizing photons from a spherically symmetric two-phase clumpy medium, with either a central point source of isotropic radiation, a uniform distribution of isotropic emitters, or uniformly illuminated by external sources. The analytical approximations are based on the mega-grains model of two-phase clumpy media, as proposed by Hobson & Padman, combined with escape and absorption probability formulae for homogeneous media. The accuracy of the approximations is examined by comparison with three-dimensional Monte Carlo simulations of radiative transfer, including multiple scattering. Our studies show that the combined mega-grains and escape/absorption probability formulae provide a good approximation of the escaping and absorbed radiation fractions for a wide range of parameters characterizing the clumpiness and optical properties of the medium. A realistic test of the analytic approximations is performed by modeling the absorption of a starlike source of radiation by interstellar dust in a clumpy medium and by calculating the resulting equilibrium dust temperatures and infrared emission spectrum of both the clumps and the interclump medium. In particular, we find that the temperature of dust in clumps is lower than in the interclump medium if the clumps are optically thick at wavelengths at which most of the absorption occurs. Comparison with Monte Carlo simulations of radiative transfer in the same environment shows that the analytic model yields a good approximation of dust temperatures and the emerging UV–FIR spectrum of radiation for all three types of source distributions mentioned above. Our analytical model provides a numerically expedient way to estimate radiative transfer in a variety of interstellar conditions and can be applied to a wide range of astrophysical environments, from clumpy star-forming regions to starburst galaxies.

Subject headings: dust, extinction — ISM: structure — methods: analytical — radiative transfer

1. INTRODUCTION

Radiative transfer plays an important role in the spectral appearance of almost all astrophysical systems, ranging from isolated star-forming regions to protogalactic and galactic systems. At wavelengths longer than the Lyman limit, absorption and scattering by dust in the intervening medium is the major factor affecting the transfer of radiation. Most of the transfer of radiation occurs in the interstellar medium (ISM) of the host system of the emitting sources. For simplicity, models of radiative transfer often assume a homogenous distribution of dust and gas, although in reality the structure of the ISM is observed to be significantly more complex.

In our Galaxy, the ISM is known to be composed of at least five phases that are in approximate pressure equilibrium: cold dense molecular clouds, cold diffuse clouds, warm diffuse clouds, H II regions, and hot low-density cavities created by supernova remnants (Spitzer 1978; Cox 1995; McKee 1995; Knapp 1995). The ISM is observed to have clumpy structure even at very small scales (Marscher, Moore, & Bania 1993; Stutzki & Güsten 1990). Data from CO line emission indicate a distribution of molecular clouds having a power-law cloud mass spectrum (Sanders, Scoville, & Solomon 1985). Dickey & Garwood (1989) find the same cloud mass spectrum based on 21 cm emission-line data. Most likely the ISM has a spectrum of densities and temperatures with correlated multiscale spatial structure, as evidenced by sky surveys such as the Infrared Astronomical Satellite (IRAS) and 21 cm surveys. Analysis of IRAS 100 $\mu$m sky flux indicates that the diffuse H I clouds have a fractal distribution (Bazell & Desert 1988; Waller et al. 1997). Analysis of CO column densities and emission-line profiles further suggests that a fractal distribution of matter applies to the molecular component of the ISM as well (Falgarone 1995; Elmegreen & Falgarone 1996). The complex, possibly fractal, distribution of gas and dust is also supported by theoretical arguments (Pfenniger & Combes 1994; Rosen & Bregman 1995; Norman & Ferrara 1996; Elmegreen 1997). Thus the ISM is clearly inhomogeneous; simulations of radiative transfer should account for this fact.

The simplest model of an inhomogeneous medium is that consisting of two phases: dense clumps embedded in a less-dense interclump medium (ICM). Natta & Panagia (1984) developed simple analytic approximations for the effective optical depth of clumpy media with no scattering and an empty ICM. Radiative transfer with isotropic scattering in a two-phase clumpy medium (nonempty ICM) with plane-parallel geometry was investigated by Boisè (1990). He used a Markov process model of the medium to develop analytical approximations for the intensity of radiation. Comparison with three-dimensional Monte Carlo simula-
tions verified the accuracy of the approximations. Hobson & Scheuer (1993) developed analytic solutions of radiative transfer with isotropic scattering in N-phase clumpy media using a Markov process model, which they compared with Monte Carlo simulations for the cases of two or three phases. Analytic approximations for radiative transfer in two-phase clumpy media were also developed by Neufeld (1991) and Hobson & Padman (1993), and we discuss and utilize them in later sections of this paper. Witt & Gordon (1996) performed extensive Monte Carlo simulations of the transfer of radiation from a central point source in a spherical two-phase clumpy medium when the scattering is not isotropic, as is more typical of UV photons scattered by dust. Their simulations showed that the nonlinear variation of effective optical depth and effective albedo with respect to parameters characterizing the clumpy medium could lead to erroneous estimates of the dust albedo and opacity if one assumes a homogeneous medium when modeling what may actually be a clumpy medium. Gordon, Calzetti, & Witt (1997) applied the Monte Carlo model of Witt & Gordon to the study of dust in starburst galaxies, concluding that a shell of clumpy dust around a starburst provides the best model of spectral data. Radiative transfer in a clumpy environment was also investigated by Wolf, Fischer, & Pfau (1998), with results similar to those of Witt & Gordon.

All the simulations and studies mentioned above verify the general expectation that a medium is more transparent when it is clumpy. This can be demonstrated in the following generic example, which then leads to the concept of effective optical depth in an inhomogeneous medium. Consider $N$ randomly chosen parallel and equal length lines of sight through an inhomogeneous medium acting as a foreground screen. Let $\tau_i$ be the optical depth of the ith line of sight, defined as the product of column density and cross section of the dust, so that $e^{-\tau_i}$ is the transmission. If $N$ is large (e.g., the number of photons per second emitted by a galaxy), then we can compute the average transmission of all the lines of sight, which defines the effective optical depth:

$$\tau_{\text{eff}} \equiv -\ln\left[1 - \frac{1}{N} \sum_i \exp(-\tau_i)\right]. \quad (1)$$

On the other hand, the average of all the optical depths equals the optical depth of the homogeneous medium with equal dust mass:

$$\tau_{\text{hom}} = \frac{1}{N} \sum_i \tau_i. \quad (2)$$

Using standard calculus, one can prove the inequality

$$\frac{1}{N} \sum_i \exp(-\tau_i) > \exp\left(-\frac{1}{N} \sum_i \tau_i\right), \quad (3)$$

which states that the average transmission of the inhomogeneous medium is greater than that of the equivalent homogeneous medium, allowing relatively more photons to escape. The expression can be an equality only when $\tau_i = \tau_{\text{hom}}$ for all lines of sight, i.e., only if the medium is homogeneous. Applying the negative natural logarithm to the inequality (3) and using the definitions (1) and (2) gives

$$\tau_{\text{eff}} < \tau_{\text{hom}}. \quad (4)$$

so the effective optical depth of an inhomogeneous medium is less than that of a homogeneous medium with an equal mass of dust. When the dust also scatters photons, the above inequality can be considered to apply approximately to the scattered photons, so we again expect generically a greater transmission through inhomogeneous media than the equivalent homogeneous medium. In this paper we study the dependence of the nonlinear relationship between $\tau_{\text{eff}}$ and $\tau_{\text{hom}}$ on parameters characterizing a clumpy medium and also give approximations for how absorption of photons is apportioned in each phase of the medium.

One consequence of the fact that $\tau_{\text{eff}} < \tau_{\text{hom}}$ for an inhomogeneous medium is that a large mass of dust could be concealed in dense regions even though observations of low extinction infer a small mass of dust. Such dense clumps of dust could have a lower temperature than if the dust were distributed homogeneously, providing an alternative explanation of observations that show an inconsistency among dust temperature, absorbed luminosity, and known luminosity sources as found in recent infrared spectral observations of the Galactic center (Chan et al. 1997). For these types of modeling efforts, easily computable analytic approximations of radiative transfer in clumpy media are desirable since they would enable the rapid exploration of the effects of varying the parameters in an astrophysical model. The more time-consuming Monte Carlo simulations can be used to guide the development and test the accuracy of analytical approximations.

We have developed a general Monte Carlo radiative transfer (MCRT) code for simulating radiative transfer with multiple scattering in a three-dimensional inhomogeneous medium, utilizing some novel techniques. The model is restricted to nonionizing radiation; hence, the only absorption and scattering considered is due to dust. We repeated many of the Monte Carlo simulations of Witt & Gordon, and our results are in good agreement. The medium in their model is composed of cubic clumps randomly located on a body-centered cubic lattice. We instead concentrate our studies on two-phase media composed of spherical clumps because the radiative transfer properties of such media are more directly approximated by analytic formulas in the mega-grains model of Hobson & Padman (1993, hereafter HP93). Furthermore, we extend the research of HP93 by computing the fraction of photons absorbed in each phase of the medium and improving the approximations for scattering. The resulting formulae allow for convenient and rapid exploration of the effects of varying the parameters defining the environment on the escape and absorption of radiation. The cases of a central isotropic point source, uniformly distributed isotropic emitters, and uniformly illuminating external sources are studied for homogeneous and clumpy media, with both MCRT simulations and analytic approximations, over a wide range of dust optical depths, scattering albedos, and from isotropic to forward scattering. The validity and accuracy of the analytical approximations is tested by detailed comparison with MCRT simulations.

We have also investigated the transfer of radiation in a fractal distribution of dust density (Városi & Dwek 1997) using MCRT, where the fractal construction is based on methods given in Elmegreen (1997). Briefly, the variation of the escape of radiation in a fractal medium as a function of $\tau_{\text{hom}}$ is similar to that of clumpy media, and we find that the mega-grains model can be used to approximate the escaping fraction of photons if some additional assumptions are
imposed. However, the distribution of absorbed radiation is more complicated than in two-phase clumpy media since there is a spectrum of densities in the fractal case. Studies and approximations of radiative transfer in complex and fractal media will be presented elsewhere (Városi & Dwek 1999).

The paper is organized as follows. Section 2 describes the MCRT code and its application to two-phase clumpy media. In § 3 we present analytical approximations for the fraction of radiation escaping from or absorbed by homogeneous media in spherical geometry, for each of the three types of sources mentioned above. In § 4 we present the mega-grains model of two-phase clumpy media and also some improvements and extensions. The mega-grains approach is combined with the formulae for homogeneous media to get approximations for the escaping/absorbed fractions in clumpy media and also the fractions absorbed in clumps and the ICM. Section 5 gives a summary of all the equations in a convenient list. Then in § 6, we demonstrate the validity of the analytic approximations by comparison with MCRT simulations. In § 7 we present a realistic test of the analytical model by simulating the scattering, escape, and absorption of a starlike emission spectrum by a clumpy distribution of interstellar dust and then comparing the predicted equilibrium dust temperatures and emerging UV–FIR spectrum with the results of MCRT simulations. The analytical model is also used to explore a large region of parameter space characterizing the clumpy medium, and we explain why the dust temperature in clumps is lower than in the ICM. The summary and conclusions are presented in § 8.

2. MONTE CARLO SIMULATION OF RADIATIVE TRANSFER

2.1. General Simulation Methods

The Monte Carlo code we have developed simulates coherent scattering and absorption of photons by dust with any kind of spatial distribution and composition. The spatial distribution of the dust is specified by a continuous function \( \rho(x, y, z) \), the mass of dust per unit volume. In practice this quantity is defined on a discrete three-dimensional grid. For each wavelength simulated, the number of photons absorbed by the dust in each volume element (voxel) of the three-dimensional grid is saved, allowing computation of the dust temperatures and resulting infrared emission spectrum.

The grid resolution is limited only by the available computer memory: increasing the number of grid elements does not significantly affect the computation time. This is achieved by employing the Monte Carlo method of rejections (Neumann 1951) in selecting the random distances each photon travels between interactions (absorption or scattering) with the dust, as described in Lux & Koblinger (1995, pp. 40–41). The method proceeds as follows. Let \( r \equiv (x, y, z) \) be the initial position of the emitted photon, and \( \rho_{\text{max}} \) be the maximum dust mass density in the direction \( \hat{l} \) the photon is traveling. Assume temporarily that the density is uniform and equal to \( \rho_{\text{max}} \) along its path. In that case the probability that the photon will travel a distance \( s \) without having an interaction is \( \exp(-sk\rho_{\text{max}}) \), where \( k \) is the dust mass extinction coefficient. Applying the fundamental principle of the Monte Carlo method, one can choose a uniformly distributed random variable \( 0 < \zeta < 1 \) and set \( \zeta \) equal to \( 1 - \exp(-sk\rho_{\text{max}}) \), the probability that an interaction will occur, and then solve for \( s \), the random value for the interaction distance, to give

\[
S = -\frac{\ln(1 - \zeta)}{k\rho_{\text{max}}}. \tag{5}
\]

Note that \((k\rho_{\text{max}})^{-1}\) is the worst case mean free path. One then has to play a rejection game in order to determine if the supposed interaction at the new location, \( r' = r + \vec{s} \), is accepted as a real interaction, because the density is of course not everywhere equal to \( \rho_{\text{max}} \). We choose another uniformly distributed random number \( 0 < \delta < 1 \) and compare it to \( \rho(r')/\rho_{\text{max}} \), the ratio of the actual density at the event location \( r' \) to the maximum density along \( \hat{l} \). If \( \delta \leq \rho(r')/\rho_{\text{max}} \) the supposed interaction is accepted as real. In particular, if \( \rho(r')/\rho_{\text{max}} = 1 \), all values of the random variable \( \delta \) fall below this ratio and the interaction will be accepted as real. Thus \( \rho(r')/\rho_{\text{max}} \) is equal to the probability that an interaction is real. If it happens that \( \delta > \rho(r')/\rho_{\text{max}} \), the interaction is rejected and called a virtual interaction. After a virtual interaction the photon is allowed to travel in the same direction another random distance selected by equation (5), and the above steps are repeated until an interaction event is accepted as real or the photon escapes. After evolving many photons in this manner, the method effectively integrates the density along all directions of travel while performing the Monte Carlo simulation of photon interactions. The three-dimensional medium need be specified with only a point-density function, and the method is simpler than performing numerical integrations along each photon path to determine in which voxel the photon will interact with dust. For most simulations one can set \( \rho_{\text{max}} \) equal to the maximum density of the entire medium, thereby further simplifying the process. However if \( \rho_{\text{max}} \gg \rho_{\text{min}} \) and the high-density regions occupy a very small fraction of the volume, it is worthwhile to have an algorithm for determining the maximum density in a given direction; otherwise, the method may iterate through many virtual interactions before getting to a real interaction or escaping, using up more CPU time.

Our Monte Carlo code follows a large group of randomly emitted photons simultaneously, typically \( 10^3 \) at a time, through multiple scatterings and to eventual absorption or escape. Each photon is given an initial flux weight of unity, and this weight changes as the photon interacts with the dust. The simulation proceeds via iterations, as follows. First, the group of photons travel randomly selected distances (as described above) depending on the dust density encountered, and then photons either escape the medium (reducing the number in the group) or interact with dust. Photons that interact are considered to experience both scattering and absorption: they are reduced in flux weight by the dust scattering albedo \( \omega \equiv \sigma_{\text{sca}}/\sigma_{\text{ext}} \) (the ratio of scattering to extinction cross section), and a fraction \( 1 - \omega \) of the energy of each interacting photon is absorbed by the dust at the locations of interaction. The path of each scattered photon is deflected by a random angle, \( \theta_{\text{ext}} \), which is distributed according to the Henyey-Greenstein (HG) phase function (Henyey & Greenstein 1941), fully characterized by the parameter \( g = \langle \cos \theta_{\text{ext}} \rangle \), the average value of the cosine of the deflection angles, also called the asymmetry parameter. The random scattering deflection angles are selected by the method given in Witt (1977). The remaining photons having new directions are fed back into the begin-
ning of the iteration procedure. This cycle of interaction, absorption, and scattering is repeated until the net flux of photons remaining in the medium is a small fraction of the flux that has escaped (e.g., less than 5%).

The flux of photons remaining after the iterations are terminated is divided into escaping and absorbed fractions according to a new scheme that considers the history of the escaping photons during each iteration and the dust scattering albedo. Let \( N_k \) be the number of photons remaining after iteration \( k \), and let \( m \) be the total number of iterations performed. If the remaining fraction after each iteration, given by the ratio \( \beta_k = N_k/N_{k-1} \leq 1 \), is tending toward a constant value, then the fraction of the final \( N_m \) photons that would escape if the iterations were continued ad infinitum is approximately (see Appendix A)

\[
f_{\text{esc}} = \frac{1 - \beta_m}{1 - \alpha \beta_m}.
\]

Thus upon termination after \( m \) scatterings, we assume that a fraction \( f_{\text{esc}}^m \) of the remaining flux escapes and a fraction \( 1 - f_{\text{esc}}^m \) is absorbed at the current location. Testing of the formula has shown that even if the multiple scattering is terminated after a few iterations when the flux remaining is 20% of the escaped flux, the final escaping fraction of flux obtained after applying equation (6) is as accurate as if the iteration was continued until the remaining flux is less than 1% of the escaped flux, and the total computation time is reduced. Of course, when the iterations are prematurely terminated, the locations of the photon absorptions and exits are not quite as accurate as when the iterations are continued, but the error is always less than the ratio of the flux that is remaining to that which has already escaped. In any case, using equation (6) is more intelligent than considering the remaining flux to be all escaped, all absorbed, or split up using just the scattering albedo.

### 2.2. Radiative Transfer in Two-Phase Clumpy Media

The main parameters defining a two-phase clumpy medium are the volume filling factor of the clumps \( f_c \), the ratio of the clump to interclump medium (ICM) densities \( \alpha = \rho_c/\rho_{\text{icm}} \), the total dust mass \( M \), and the volume \( V \) of the medium. Typical ranges that may describe parts of the interstellar medium are 0.01 < \( f_c \) < 0.3 and 10 < \( \alpha \) < 10\(^3\) (Spitzer 1978; van Buren 1989; Murthy, Walker, & Henry 1992; Gaustad & van Buren 1993). The number of clumps and the clump sizes are secondary parameters, inversely related, which we shall discuss later. To obtain a formula for the ICM density, first define the density of the equivalent homogeneous medium to be \( \rho_{\text{hom}} = M/V \). Then since the total dust mass is simply the sum of mass in each phase, the homogeneous density can be expressed as

\[
\rho_{\text{hom}} = \frac{M + M_{\text{icm}}}{V} = \frac{V_c M_c + V_{\text{icm}} M_{\text{icm}}}{V V_c} = f_c \rho_c + (1-f_c) \rho_{\text{icm}} = (\rho_c - \rho_{\text{icm}}) f_c + \rho_{\text{icm}},
\]

where \( M_c \) and \( M_{\text{icm}} \) are the dust masses in the clumps and ICM, respectively, \( V_c \) and \( V_{\text{icm}} \) are the respective volumes, and so \( \rho_c = M_c/V_c \) is the density of dust in the clumps, \( \rho_{\text{icm}} = M_{\text{icm}}/V_{\text{icm}} \) the density in the ICM, and \( f_c = V_c/V \) the filling factor of the clumps. Equation (7) can be rearranged by substituting for \( \rho_c \) with \( \alpha \rho_{\text{hom}} \) to give an equation for the ICM density:

\[
\rho_{\text{icm}} = \frac{\rho_{\text{hom}}}{(\alpha - 1) f_c + 1}.
\]

As \( f_c \to 0 \), \( \rho_{\text{icm}} \to \rho_{\text{hom}} \) and \( \rho_c \to \alpha \rho_{\text{hom}} \) from below, whereas, if \( f_c \to 1 \), \( \rho_{\text{icm}} \to \rho_{\text{hom}}/\alpha \) and \( \rho_c \to \rho_{\text{hom}} \) from above.

One possible type of two-phase clumpy medium is that of cubic clumps randomly located on a body-centered cubic lattice. The construction is similar to that used in percolation theory, where the probability of lattice site occupation is equivalent to the filling factor and the clumps are then the occupied sites, and the cubic clumps do not overlap. Witt & Gordon (1996) studied radiative transfer in such a two-phase clumpy medium defined on a cubic lattice, and we repeated some of their Monte Carlo simulations giving a successful test of our methods. In this work we use spherical clumps defined on a high-resolution three-dimensional grid, allowing clump overlaps and the random locations of clumps to be chosen at a resolution higher than the clump size. In addition, the radiative transfer properties are then readily approximated by analytic formulas, as we demonstrate in §§ 4 and 6. The stochastic media in the analytical models of Boisse (1990) and Hobson & Scheuer (1993) do not impose any restrictions on the shapes of the clumps; however, the models are developed for the case of isotropic scattering.

To construct a two-phase medium with randomly located clumps that may overlap, we must calculate the number of clumps needed to achieve the desired filling factor. This can be derived from the volume of a clump, \( v_c \), the total volume of the medium, \( V \), and the filling factor, \( f_c \), but the possible occurrence of overlapping clumps complicates the calculation. Assume that the clumps are all identical. Define the volume fraction of a single clump as

\[
p = \frac{v_c}{V};
\]

then the probability that a random point is not in any clump is equal to \( (1-p)^{N_c} \), where \( N_c \) is the number of clumps, and therefore

\[
f_c = 1 - (1-p)^{N_c}
\]

(see Appendix B for more discussion). Equation (10) is easily solved for the total number of clumps,

\[
N_c = \frac{\ln (1-f_c)}{\ln (1-p)}.
\]

in terms of the total filling factor, \( f_c \), and the single clump filling factor, \( p \). Although the above equations apply to identical clumps of any shape, we shall consider only spherical clumps of radius \( r_c \), with \( v_c = 4 \pi r_c^3/3 \). Given \( f_c, r_c, \) and \( V \), the clumpy medium is constructed by calculating \( N_c \) using equations (9) and (11), then selecting \( N_c \) random points uniformly distributed in \( V \), placing a sphere of radius \( r_c \) and density \( \rho_c \) around each point, setting the density between the spheres to \( \rho_{\text{icm}} \) and finally discretizing the medium on a high-resolution three-dimensional grid. When the randomly located clumps do overlap, the densities are not summed, so the medium is always characterized by only two possible densities: that of the ICM and that of the clumps.
Monte Carlo simulations of radiative transfer were computed for three types of photon sources in a two-phase clumpy medium contained within a sphere of unit radius $R_S = 1$, in which the spherical clumps have radius $r_c = 0.05$ and are 100 times denser than the ICM ($\alpha = 100$). The three-dimensional rectangular grid used to represent the medium has resolution of $127^3$ voxels, so that the clump diameters are about 6 grid elements. Figures 1, 2, and 3 show our results for an array of clump filling factors and optical depths, keeping $r_c$ and $\alpha$ constant. Three million photons were followed in each Monte Carlo simulation. Each of the images is the map of photons absorbed in a two-dimensional slice (127 $\times$ 127 pixels) through the center of the sphere. The gray scale from black to white indicates minimum to maximum absorption, on a logarithmic scale. Figure 1 represents the case of a single central isotropic luminosity source, analogous to a star or a centrally condensed cluster of stars in an H II region. Figure 2 represents the case of isotropic photon emission distributed uniformly in the sphere, analogous to a uniform distribution of stars in a galaxy. Figure 3 represents the case of an external uniformly illuminating source of isotropic photons, analogous to cold molecular clouds illuminated by the diffuse interstellar radiation field (IRF). In all cases the dust is characterized by a scattering albedo of $\omega = 0.6$ and asymmetry parameter $g = 0.6$, which are typical values for UV photons scattering off dust grains (Gordon et al. 1994). Recall that $g = 0$ results in isotropic scattering and $g = 1$ gives forward scattering.

Moving vertically in the array of slices in the figures corresponds to increasing the volume filling factor $f_c$ of the clumps while keeping the total dust mass constant (in each column). Thus the number of clumps appearing in each slice increases and $\rho_{icm}$ decreases from bottom to top. Moving horizontally increases the equivalent homogeneous optical depth of the sphere,

$$\tau_{hom} = \kappa \rho_{hom} R_S,$$

which is the radial optical depth of extinction (absorption plus scattering) that would result if the dust were distributed uniformly throughout the sphere instead of being clumpy. Increasing $\tau_{hom}$ can be viewed as either increasing the dust abundance (increasing $\rho_{icm}$ and $\rho_c$) or changing the wavelength of the photons, either case resulting in more absorption.

For the case of a central source (Fig. 1), as $\tau_{hom}$ increases the ICM absorbs more photons and the clumps become opaque, creating the apparent shadows behind the clumps, when $f_c$ is small. However scattering by the dust causes photons to go behind the clumps and become absorbed, thus diminishing the effect of what would otherwise be completely dark shadows in the case of no scattering. As the clumps become opaque ($\tau_{hom} = 40$), absorption occurs more at the clump surfaces. When $f_c$ is increased (keeping the total dust mass constant), the effect of shadows merges into the appearance of an absorption cavity. Similar effects occur for the case of uniformly distributed sources (Fig. 2); however, they are not as dramatic as when the clumps are illuminated by a central source, since there are no shadows when the photons are emitted everywhere in the medium. At high $f_c$ the clumps dominate the medium and absorb most of the photons. In the case of a uniformly illuminating...
Fig. 2.—Same as Fig. 1 for dust heated by a uniform distribution of internal sources. Scaling is logarithmic from the minimum of 0.1 photons (black) to the maximum of 40 photons absorbed per voxel (white). For further description see § 2.2.

Fig. 3.—Same as Fig. 1 for dust heated by an external uniformly illuminating source. Scaling is logarithmic from the minimum of 0.1 photons (black) to the maximum of 80 photons absorbed per voxel (white). For further description see § 2.2.
of is due to absorption by the increasing optical depth of the clumps. As $\tau_{\text{hom}} \to \infty$ the clumps become opaque and any further increase in absorption is due to the ICM, and because it has a lower density the slope of the $\tau_s$ curves is lower. The solid, dotted, and dashed curves are produced by the analytic approximations presented in §§ 3 and 4, and the agreement with Monte Carlo results is discussed further in § 6.

3. Escape and Absorption in Homogeneous Media

Since Monte Carlo simulations can require a large amount of computer time, it is useful to have analytical approximations for the basic results of radiative transfer, such as the fraction of photons that escape a bounded medium. In this section we present such approximations for homogeneous spherical media with internal or external sources. The escape and absorption probability approximations are also compared with results of Monte Carlo simulations to assess their accuracy. These escape probability formulae are later applied to the case of clumpy media.

3.1. Central Isotropic Point Source

Consider an isotropic point source in the center of a spherical homogeneous medium. When the medium does not scatter photons, the escape probability is simply $e^{-\tau}$, where $\tau = \pi \rho R$ is the optical radius of the sphere, characterized by a radius $R$ and a mass density $\rho$, and where $\kappa$ is the absorption cross section of the dust per unit mass. If the medium also scatters photons, we can construct an analytical approximation for the effective optical radius of the sphere, $\tau_s$, defined as the negative natural logarithm of the escaping fraction of photons, as in equation (13). The approximation formula is based on limiting cases of the optical parameters. When the optical depth of extinction (absorption and scattering), $\tau = \tau_{\text{abs}} + \tau_{\text{scat}}$, is large and the scattering is isotropic ($\eta \sim 0$), then theoretical analysis (Rybicky & Lightman 1979, pp. 37–38) suggests that $\tau_s$ is approximately

$$\tau_s \approx \tau(1 - \omega)^{1/2} \quad (\tau \gg 1 \text{ and } \eta \sim 0),$$

where $\omega \equiv \tau_{\text{scat}}/\tau$ is the scattering albedo. When the optical radius is small ($\tau \ll 1$) and $\eta$ is any value, we expect that

$$\tau_s \approx \tau(1 - \omega) \quad (\tau \ll 1 \text{ and any } \eta),$$

which is also an exact formula in the case of purely forward scattering at any optical depth. We can interpolate between these extreme cases using the following formula:

$$\tau_s(\tau, \omega, \eta) \equiv \tau(1 - \omega) \chi(\tau, \eta),$$

where the interpolation exponent is given by

$$\chi(\tau, \eta) \equiv 1 - \frac{1}{2}(1 - e^{-\tau/2})(1 - \eta)^{1/2}.$$  

The probability of escaping from the homogeneous sphere is then defined as

$$P_{\text{esc}}(\tau, \omega, \eta) \equiv \exp\left[-\tau_s(\tau, \omega, \eta)\right],$$

where the superscript “$c$” indicates that this is for a central point source.

To check this approximation, the Monte Carlo code was used to compute a three-dimensional matrix of effective optical radii as a function of optical parameters in the ranges $0 < \tau < 13$ with $\Delta \tau \leq 1$, $0 \leq \omega < 1$ with $\Delta \omega = 0.1$, and $0 \leq \eta < 1$ with $\Delta \eta = 0.1$, following more than $10^6$ photons in each simulation. Figure 5 shows contours of the

external source (Fig. 3), when $\tau_{\text{hom}}$ and $f_c$ are large we find that most of the impacting photons are absorbed in the outer layers of the spherical region and the center is shielded from radiation. However, for small $f_c$ the center of the sphere is less shielded, and instead the clump centers are shielded, most photons being absorbed on the surfaces of the clumps. In general, the clumpy medium allows photons to penetrate farther into the spherical region than if the medium were homogeneous, and this fact was the motivation and a conclusion in the work of Boissé (1990), Hobson & Scheuer (1993), and HP93, for the case of plane-parallel slab geometry.

By observing the fraction of photons that escape in the case of a central point source, we can compute the effective optical depth of the medium at a given wavelength $\lambda$ as

$$\tau_s(\lambda) = -\ln \left[ \frac{L_{\text{esc}}(\lambda)}{L_0(\lambda)} \right],$$

where $L_0(\lambda)$ is the luminosity of the central source and $L_{\text{esc}}(\lambda)$ the escaping luminosity (for simplicity we shall hereafter drop any explicit dependence on $\lambda$). We use the notation $\tau_s$ instead of $\tau_{\text{eff}}$ because $\tau_{\text{eff}}$ is reserved for the case of no scattering (absorption plus scattering considered as interaction), whereas $\tau_s$ includes the effects of scattering and is therefore dependent on the geometry of the medium, which in this case is spherical. Equation (13) is analogous to equation (1), and it follows that $\tau_s < \tau_{\text{eff}} < \tau_{\text{hom}}$ for a clumpy medium. Figure 4 shows the behavior of $\tau_s$ as a function of $\tau_{\text{hom}}$ for three cases of $f_c$, with other parameters having the values mentioned above. The diamonds indicate results of Monte Carlo simulations when $f_c = 0.3$ (corresponding to the upper row in Fig. 1), the triangles show $f_c = 0.2$, and the squares show $f_c = 0.1$. There are two slopes in each curve of $\tau_s$ versus $\tau_{\text{hom}}$, corresponding to the two phases of the medium. The steeper slope at low values of $\tau_{\text{hom}}$ is due to absorption by the increasing optical depth of the clumps. As $\tau_{\text{hom}} \to \infty$ the clumps become opaque and any further increase in absorption is due to the ICM, and because it has a lower density the slope of the $\tau_s$ curves is lower. The solid, dotted, and dashed curves are produced by the analytic approximations presented in §§ 3 and 4, and the agreement with Monte Carlo results is discussed further in § 6.

3. Escape and Absorption in Homogeneous Media

Since Monte Carlo simulations can require a large amount of computer time, it is useful to have analytical approximations for the basic results of radiative transfer, such as the fraction of photons that escape a bounded medium. In this section we present such approximations for homogeneous spherical media with internal or external sources. The escape and absorption probability approximations are also compared with results of Monte Carlo simulations to assess their accuracy. These escape probability formulae are later applied to the case of clumpy media.

3.1. Central Isotropic Point Source

Consider an isotropic point source in the center of a spherical homogeneous medium. When the medium does not scatter photons, the escape probability is simply $e^{-\tau}$, where $\tau = \pi \rho R$ is the optical radius of the sphere, characterized by a radius $R$ and a mass density $\rho$, and where $\kappa$ is the absorption cross section of the dust per unit mass. If the medium also scatters photons, we can construct an analytical approximation for the effective optical radius of the sphere, $\tau_s$, defined as the negative natural logarithm of the escaping fraction of photons, as in equation (13). The approximation formula is based on limiting cases of the optical parameters. When the optical depth of extinction (absorption and scattering), $\tau = \tau_{\text{abs}} + \tau_{\text{scat}}$, is large and the scattering is isotropic ($\eta \sim 0$), then theoretical analysis (Rybicky & Lightman 1979, pp. 37–38) suggests that $\tau_s$ is approximately

$$\tau_s \approx \tau(1 - \omega)^{1/2} \quad (\tau \gg 1 \text{ and } \eta \sim 0),$$

where $\omega \equiv \tau_{\text{scat}}/\tau$ is the scattering albedo. When the optical radius is small ($\tau \ll 1$) and $\eta$ is any value, we expect that

$$\tau_s \approx \tau(1 - \omega) \quad (\tau \ll 1 \text{ and any } \eta),$$

which is also an exact formula in the case of purely forward scattering at any optical depth. We can interpolate between these extreme cases using the following formula:

$$\tau_s(\tau, \omega, \eta) \equiv \tau(1 - \omega) \chi(\tau, \eta),$$

where the interpolation exponent is given by

$$\chi(\tau, \eta) \equiv 1 - \frac{1}{2}(1 - e^{-\tau/2})(1 - \eta)^{1/2}.$$  

The probability of escaping from the homogeneous sphere is then defined as

$$P_{\text{esc}}(\tau, \omega, \eta) \equiv \exp\left[-\tau_s(\tau, \omega, \eta)\right],$$

where the superscript “$c$” indicates that this is for a central point source.

To check this approximation, the Monte Carlo code was used to compute a three-dimensional matrix of effective optical radii as a function of optical parameters in the ranges $0 < \tau < 13$ with $\Delta \tau \leq 1$, $0 \leq \omega < 1$ with $\Delta \omega = 0.1$, and $0 \leq \eta < 1$ with $\Delta \eta = 0.1$, following more than $10^6$ photons in each simulation. Figure 5 shows contours of the
agreement is perfect. The dotted contour lines indicate where equation (16) gives values less than the Monte Carlo results. The worst case of the approximation occurs when \( \omega = 0.6, g = 0, \) and \( \tau > 4, \) as Figures 5b and 5c show that \( \tau_d(\tau, \omega, g) \) underestimates the Monte Carlo results by a value exceeding unity. Also, Figures 5a and 5c show that equation (14) is not always a good approximation. However the difference is relatively small compared to the value of \( \tau_d \approx 10 \) at \( (\tau, \omega, g) = (13, 0.6, 0). \) In addition, when modeling radiative transfer through interstellar dust, isotropic scattering occurs at near-infrared (NIR) wavelengths for which the optical depth is smaller than at UV wavelengths where the scattering is more forward directed, and so the approximation is likely to be good at all wavelengths, as shown in Figure 5b.

### 3.2. Uniformly Distributed Emission

Consider a spherical homogeneous medium with a uniform distribution of isotropically emitting sources. When there is no scattering, Osterbrock (1989, pp. 385–386) derived an exact solution for the photon escape probability given by

\[ P_e(\tau) = \frac{3}{4\tau} \left[ 1 - \frac{1}{2\tau^2} + \left( \frac{1}{\tau} + \frac{1}{2\tau^2} \right)e^{-2\tau} \right], \quad (19) \]

where \( \tau = \rho k R \) is the optical radius of the sphere (\( R \) is the radius of the sphere, \( \rho \) is the density of the absorbers, and \( k \) is the absorption cross section). We find that \( P_e(\tau) \) agrees exactly with the escaping fraction of photons computed in Monte Carlo simulations. Using the optical depth resulting from both absorption and scattering, \( \tau_{\text{ext}} = \tau_{\text{abs}} + \tau_{\text{scat}} \), we can get the probability, \( P_e(\tau_{\text{ext}}) \), that photons will escape the spherical medium without any interactions (absorption or scattering), and we call it the extinction escape probability. Making the assumption that the scattered photons have the same spatial and angular distribution as the emission sources, the extinction escape probability can be applied recursively to arrive at the scattering escape probability (Lucy et al. 1991),

\[ P_{e_s}(\tau, \omega) \equiv \frac{P_e(\tau)}{1 - \omega(1 - P_e(\tau))}, \quad (20) \]

where \( \tau \equiv \tau_{\text{ext}} \) and \( \omega = \tau_{\text{scat}}/\tau_{\text{ext}} \) is the scattering albedo. This formula estimates the effects of scattering, namely, that scattered photons may eventually escape, thereby increasing the escape probability. The superscript “u” indicates that \( P_{e_s} \) is for uniformly distributed emitters in a sphere. Actually, \( P_e(\tau) \) in the formula can be any extinction escape probability for any geometry, but this paper will concentrate on the case of spherical geometry, for which equation (19) applies. The derivations of equations (19) and (20) are reviewed in Appendix C.

Equations (19) and (20), which we shall call the Osterbrock-Lucy escape probability (OLEP) formula, were tested extensively against Monte Carlo radiative transfer simulations with multiple scattering and found to be a reasonable approximation of the fraction of photons escaping from a spherical homogeneous medium. Since the scattering asymmetry parameter, \( g = \langle \cos \theta_{\text{scat}} \rangle \), does not enter into the formula for \( P_{e_s}(\tau, \omega) \), the value of \( g \) for which the OLEP formula agrees exactly with Monte Carlo simulations is found to depend on the extinction optical depth \( \tau \). The upper panel in Figure 6 shows contours of the percent

![Figure 5](image_url)
relative difference between the OLEP formula and escaping fraction of photons found by Monte Carlo calculations (following $10^6$ photons in each simulation), as a function of $g$ and $\tau$, when the albedo is $\omega = 0.7$. The values of $g$ for which the OLEP agrees with Monte Carlo results are indicated by the zero contour level drawn with a thick solid line. The dotted contours indicate where the escape probability formula underestimates the escaping fraction, and the thin solid contours indicate where it overestimates the escaping fraction. For optically thin situations ($\tau < 1$) the OLEP formula agrees well with the Monte Carlo simulations for all values of $g$, with the best agreement occurring when the scattering is nearly isotropic ($g \sim 0$). As the optical depth increases the agreement shifts toward more forward-scattering cases ($g \sim 1$) of the Monte Carlo simulations. The maximum relative difference in the range shown is 26% overestimation at $\tau = 20$, $\omega = 0.7$, and $g = 0$, where the escaping fraction is only 0.10. The behavior of the relative difference is similar for other values of the albedo and always decreases with albedo, to zero as $\omega \rightarrow 0$, since then $f_{esc}(\tau, \omega) \rightarrow P_{esc}(\tau)$. The value of $g$ for which $f_{esc}(\tau, \omega)$ agrees with Monte Carlo calculations (the zero difference contour) is found to be a function of optical depth only, independent of the albedo $\omega$, and so we designate this special value $g^*(\tau)$.

The lower panel in Figure 6 shows contours labeled with the escaping fraction resulting from the Monte Carlo calculations, and the thick solid line is again $g^*(\tau)$, the zero difference contour from the upper panel. We find that a reasonable empirical approximation for $g^*(\tau)$ is given by

$$g^*(\tau) \approx 1 - \frac{3.3}{\tau + 3.3},$$

as shown by the dashed line.

There is a simple explanation for why the best agreement between $f_{esc}(\tau, \omega)$ and Monte Carlo simulations occurs at a value of $g$ that approaches unity as $\tau \rightarrow \infty$. To explain this fact we rewrite equation (20) to read

$$f_{esc}(\tau, \omega) = \frac{1}{\omega + (1 - \omega)/P_{esc}(\tau)},$$

where $P_{esc}(\tau)$ is the escape probability from a sphere given by equation (19). As $\tau \rightarrow \infty$ the extinction escape probability approaches zero as $P_{esc}(\tau) \rightarrow 3/4\pi$. Applying this limiting behavior to equation (22) gives

$$f_{esc}(\tau, \omega) \sim \frac{1}{\omega + 4\pi(1 - \omega)/3} \sim \frac{3}{4\pi(1 - \omega)},$$

since the term $4\pi(1 - \omega)/3 \gg \omega$ as $\tau \rightarrow \infty$. Now we recognize that $(1 - \omega)$ is exactly the optical depth in the case of purely forward scattering, when $g = 1$, since then scattering acts like a reduction in absorption. The forward-scattering escape probability is then exactly $P_{esc}[(1 - \omega)]$, which has the same limiting behavior as equation (23) when $\tau \rightarrow \infty$:

$$P_{esc}[(1 - \omega)] \sim \frac{3}{4\pi(1 - \omega)}.$$

Therefore as $\tau \rightarrow \infty$ the OLEP approximation approaches from below the exact escape probability for purely forward scattering:

$$f_{esc}(\tau, \omega) \rightarrow P_{esc}[(1 - \omega)].$$

Since the left-hand side is valid at $g = g^*(\tau)$ and the right-hand side is valid at $g = 1$, the limiting behavior implies that $g^*(\tau) \rightarrow 1$ as $\tau \rightarrow \infty$, which is satisfied by the approximation equation (21).

In summary, $f_{esc}(\tau, \omega)$ overestimates the actual escape probability when $g < g^*(\tau)$ and underestimates it when $g > g^*(\tau)$. For values of $g$ in the range $g^*(\tau) < g < 1$, the escape probability approximation can be improved by linear interpolation in $g$ between the values $f_{esc}(\tau, \omega)$ at $g^*(\tau)$ and $P_{esc}[(1 - \omega)]$ at $g = 1$. If an analytical approximation for the escape probability when $g = -1$ (purely backward scattering) were known, then three-point parabolic interpolation could be used to obtain a very good approximation to the actual escape probability for all $g$, and hence all $(\tau, \omega, g)$. When modeling absorption and scat-
tering by dust, large optical depths usually occur at UV wavelengths for which the scattering by dust tends to be more forward oriented, having values of $g$ near $g^*(\tau)$, so the error incurred by using the OLEP approximation may be relatively small.

3.3. Finite Number of Uniformly Distributed Point Sources

The question arises as to whether the escape probability formula can be successfully applied to a discrete collection of randomly placed isotropic point sources ("stars") instead of a continuous uniform distribution of emitters, and in particular what the minimum number of stars for which the escape probability formula gives correct results is. To answer these questions we performed Monte Carlo simulations for cases of 2–80 randomly placed "stars" in a homogeneous sphere, and for each case we simulated 200 trials, where in each trial $10^5$ photons were followed to find the fraction that eventually escapes. The results are shown in Figure 7 for when the optical radius of the sphere is $\tau = 5$, for (1) the case of no scattering ($\omega = 0$) and (2) the case with scattering ($\omega = 0.6$ and $g = 0.6$). The horizontal lines in cases 1 and 2 are the escape probabilities $P_i(\tau)$ and $P_{\text{esc}}(\tau, \omega)$, respectively, given by equations (19) and (20). Each diamond shows the average escaping fraction of the Monte Carlo trials for a fixed number of stars, and the error bars are plus and minus one standard deviation from the average value. The distribution of escaping fractions from the Monte Carlo trials is approximately Gaussian. We find that the standard deviation of the trials goes as the inverse square root of the number of stars $n_s$, as expected for sums of random variables, and in particular $\sigma \sim 0.1(n_s)^{-0.5}$ for all $\tau > 1$. Since the escaping fraction in the case of no scattering behaves as $P_i(\tau) \sim 3/4\tau$ when $\tau \rightarrow \infty$, the expected relative deviation ($\sigma/P_i$) in the escaping fraction of photons from $n_s$ stars randomly located in a sphere of optical depth $\tau \gg 1$ will be approximately $0.133n_s^{0.5}$. For the case of absorption only, and less when scattering is also involved. Thus the number of stars must increase as the square of the optical depth to maintain the same expected relative deviations. The analytic escape probability agrees well with the average escaping fraction found for each group of Monte Carlo trials; it is just a question of what relative deviation is acceptable.

3.4. Uniformly Illuminating External Source

In this section we introduce approximations for the probability that externally emitted isotropic photons will interact with and get absorbed by a homogeneous medium contained in a sphere. The term interaction includes both absorption and scattering events. Given that a distribution of photons encounters the sphere at a random impact parameter, the probability of interaction at any location inside the sphere is

$$P_i(\tau) = 1 - \frac{1}{2\tau^2} + \left(\frac{1}{\tau} + \frac{1}{2\tau^2}\right)e^{-2\tau},$$

where $\tau = \rho R$ is the extinction optical radius of the sphere. This equation is derived by averaging the transmission over all possible impact parameters and computing the ratio of net absorption versus impacting flux, using the same techniques as for the derivation of equation (19) (see Appendix C). When $\tau \ll 1$, we have $P_i(\tau) \sim 4\tau/3$, which is the optical path length through a sphere averaged over all impact parameters. Equation (26) was utilized by Neufeld (1991) and Hobson & Padman (1993) in their models for clumpy media, as we shall discuss in § 4.1. Note that equations (26) and (19) are related by

$$P_i(\tau) = \frac{4\tau}{3} P_e(\tau),$$

giving a duality between absorption (interaction) of an external source and escape (noninteraction) of an internal source.

If the medium scatters photons, we would like to know what fraction of $P_i(\tau)$, the interacting fraction, will eventually escape because of multiple scattering events and thus determine what fraction of photons are actually absorbed. In the special case of purely forward scattering, there is an exact formula for the absorbed fraction of photons. When the scattering deflection angle is always zero, it is equivalent to no scattering with a reduced optical depth equal to $\tau(1-\omega)$, where $\omega$ is the scattering albedo. So the actual fraction of photons absorbed in the purely forward-scattering case is

$$P_{\text{abs}}^x(\tau, \omega; g = 1) \equiv P_\text{abs}[\tau(1-\omega)],$$

where the superscript " $x$ " indicates that this is for the case of an external source and $g$ is the scattering asymmetry parameter. As expected, equations (26) and (28) agree exactly with Monte Carlo simulations.

For non–forward-scattering cases, when $g < 1$, we can use the methods discussed in § 3.2 for the case of internal uniformly distributed emitters, namely the OLEP equations (19) and (20). Given that photons interact with the medium inside the sphere, the probability that they scatter is simply $\omega$. These scattering events can be regarded as reemitted photons, and, assuming that they are approximately uniformly distributed in the sphere, the scattered photons have

\[\text{Fig. 7.—Monte Carlo simulations of the fraction of photons that escape from a homogeneous sphere of dust after being emitted by a small number of point sources ("stars"). As indicated on the horizontal axes, (a) Absorption only ($\omega = 0$); (b) with scattering ($\omega = 0.6$ and $g = 0.6$), and the optical radius of the sphere is $\tau = 5$ in both cases. The diamonds and error bars are each the means and standard deviations of 200 Monte Carlo simulations. The solid horizontal lines are predictions by the OLEP formula. The figure is discussed in § 3.3.} \]
a probability $P_{\text{esc}}(\tau, \omega)$ of escaping (eq. [20]). The fraction of the interacting photons that actually get absorbed is then $1 - \omega P_{\text{esc}}(\tau, \omega)$. Thus an approximation for the fraction of photons absorbed is

$$P_{\text{abs}}(\tau, \omega) \equiv P(\tau)[1 - \omega P_{\text{esc}}(\tau, \omega)].$$  \hspace{1cm} (29)

Recall that $P_{\text{esc}}(\tau, \omega)$ is valid only for $g = g^*(\tau)$ as discussed in § 3.2, and so we expect $P_{\text{abs}}(\tau, \omega)$ to follow the same pattern. When $\tau_c$ is large, the penetration depth of photons decreases so they will not be uniformly distributed in the sphere, as assumed above, but this does not severely affect the application the approximation as we show next.

The upper panel in Figure 8 shows contours of the percent relative difference between $P_{\text{abs}}(\tau, \omega)$ and the absorbed fraction of photons found by Monte Carlo calculations, as a function of $g$ and $\tau$, for $\omega = 0.7$. The values of $g$ for which equation (29) agrees with Monte Carlo results are indicated by the zero contour level drawn with a thick solid line. The dotted contours indicate where the absorption probability equation (29) underestimates the absorbed fraction, and the thin solid contours indicate where it overestimates. The difference is small and independent of $g$ for optically thin situations ($\tau < 1$), and as the optical depth increases the agreement shifts toward more forward-scattering cases ($g \rightarrow 1$) of the Monte Carlo simulations. Not surprisingly, the behavior of the relative difference contours is identical to that in Figure 6 of § 3.2, since we are using $P_{\text{esc}}(\tau, \omega)$ in the formula. The relative difference decreases as $\omega \rightarrow 0$, since then $P_{\text{abs}}(\tau, \omega) \rightarrow P(\tau)$ and $P(\tau)$ is an exact formula.

The lower panel in Figure 8 shows contours labeled with values of the absorbed fraction obtained from Monte Carlo simulations of a uniformly illuminating isotropic external source. Comparison with Figure 6 shows that the behavior of the absorbed fraction of an external source is quite different from the absorbed fraction of an internal source (one minus the escaping fraction shown in lower panel of Fig. 6). Note that for $g < 1$, as $\tau \rightarrow \infty$ the absorbed fraction tends asymptotically toward a constant value, since in the opaque limit scattering occurs at the surface of the sphere. The thick solid line is again the contour of zero difference between $P_{\text{abs}}(\tau, \omega)$ and Monte Carlo simulations, and the dashed line is the approximation of $g^*(\tau)$ given by equation (21). This zero difference contour is found to be independent of $\omega$ and fit well by equation (21), also because $P_{\text{esc}}(\tau, \omega)$ is used in the approximation formula.

Let us now examine the behavior of the absorption probability as $\tau \rightarrow \infty$ to show in more detail why the best agreement with Monte Carlo simulations occurs at a value of $g$ that approaches unity. Using equations (27) and (20), and defining $P \equiv P_4(\tau)$ for convenience, equation (29) can be rewritten as

$$P_{\text{abs}}^x(\tau, \omega) = \frac{4\tau}{3} P \left[1 - \frac{\omega P}{1 - \omega(1 - P)}\right]$$

$$= \frac{4\tau}{3} P \left[1 - \frac{\omega}{1 - \omega(1 - P)}\right]$$

$$= \frac{4\tau(1 - \omega)}{3} \left[\frac{P}{1 - \omega(1 - P)}\right]$$

$$= \frac{4\tau(1 - \omega)}{3} P_{\text{esc}}(\tau, \omega),$$  \hspace{1cm} (30)

thus extending equation (27), the duality between the cases of internal and external sources, to the case of $\omega > 0$. Equation (28) can be rewritten as

$$P_{\text{abs}}^x(\tau, \omega; g = 1) = \frac{4\tau(1 - \omega)}{3} P_4[\tau(1 - \omega)]$$  \hspace{1cm} (31)

where we have used equation (27) again. The analysis given by equations (23), (24), and (25) proved that $P_{\text{esc}}(\tau, \omega) \rightarrow P_4[\tau(1 - \omega)]$ from below as $\tau \rightarrow \infty$, thereby proving that

$$P_{\text{abs}}^x(\tau, \omega; g = g^*) \rightarrow P_{\text{abs}}(\tau, \omega; g = 1)$$  \hspace{1cm} (32)

from below, implying that $g^*(\tau) \rightarrow 1$ as $\tau \rightarrow \infty$. 

![Figure 8](image-url)
4. ESCAPE AND ABSORPTION IN CLUMPY MEDIA

A viable approach to the problem of estimating the escaping and absorbed fractions of photons in an inhomogeneous medium is to find effective values for the absorption and scattering properties of the inhomogeneous medium and then use these effective values in the escape/absorption probability formulae that were developed for homogeneous media. The work of Hobson & Padman (1993) (HP93) provides the means to such an approach for two-phase clumpy media by a model they called the “mega-grains” approximation, which reduces the inhomogeneous radiative transfer problem to an effectively homogeneous one. In the following sections we review the mega-grains approximation and then discuss how it is utilized to give escape/absorption probability approximations for clumpy media. We also introduce some improvements to the mega-grains approximation of HP93. In addition, we present new formulae giving the approximate fraction of photons absorbed in each phase of the medium, which can then be used to estimate the dust temperature and infrared emission from the clumps and ICM separately.

4.1. The Mega-Grains Approximation

Natta & Panagia (1984) developed analytic approximations for the extinction by dust having various kinds of inhomogeneous distributions, in particular, a clumpy distribution with an empty interclump medium (ICM). Neufeld (1991) proposed the approach of treating spherical clumps in a two-phase medium as large grains and applied it to the special case of the scattering, absorption, and escape of Lyz radiation, involving both gas and dust. HP93 proposed a more general set of formulae called the mega-grains approximation, which also treats the spherical clumps as large grains but with absorption and scattering coefficients in direct analogy with dust grains. Their mega-grains model gives equations for the effective optical depth and effective scattering albedo of a two-phase clumpy medium, with a nonempty ICM. In this section we review the derivation of the mega-grains approximation and introduce an improved equation for the effective scattering albedo. We also develop a new approximate formula for the effective scattering asymmetry parameter of the clumpy medium. To distinguish between similar properties of the clumps (mega-grains) and the dust we will assign them the subscripts “c” and “d,” respectively. Therefore, ωc will be the macroscopic albedo of the clumps and ωd is the microscopic albedo of the dust. Also, an implicit dependence on wavelength is assumed in all the following.

4.1.1. Effective Optical Depth

Assume that a collection of spherical clumps, each of radius r, and having dust mass density ρd, are randomly distributed in an interclump medium (ICM) having a constant dust mass density of ρicm. Following HP93, define the clump optical radius

\[
\tau_c = (\rho_c - \rho_{icm}) \kappa r_c = (\alpha - 1) \kappa \rho_{icm} r_c ,
\]  

where \( \kappa \) is the total absorption plus scattering cross section of the dust grains per unit mass, and recall that \( \alpha = \rho_d / \rho_{icm} \). The reason for the subtraction of densities in this definition is to disregard the clumps in the limit of when the clump density approaches the density of dust in the ICM, since then there are effectively no clumps. Thus the mega-grains are considered to be just the overdensity, \( \rho_c - \rho_{icm} \), allowing for analysis as a separate component superimposed on a continuous medium of lower density. Now considering the clumps as large grains, the probability that a randomly emitted photon will encounter a clump is just \( \tau_c^2 \). Once a clump is encountered, the probability that the photon will interact (get absorbed or scatter) with dust in clump is \( P(\tau_c) \), given by equation (26), the interaction probability for a sphere of optical radius \( \tau_c \). Then we can define the interaction coefficient per unit length, \( \Lambda_{mg} \), in the medium for just the mega-grains as

\[
\Lambda_{mg} = n_c \tau_c^2 P(\tau_c) ,
\]

where \( n_c \) is the number of clumps per unit volume. Here the use of the interaction probability \( P(\tau_c) \) is analogous to the absorption and scattering efficiency coefficient of dust grains.

Define the effective interaction coefficient per unit length, \( \Lambda_{eff} \), in the two-phase clumpy medium by including the ICM:

\[
\Lambda_{eff} = \Lambda_{mg} + \kappa \rho_{icm} .
\]

This reduces to the interaction coefficient for a homogeneous medium, \( \kappa \rho_{hom} \), when \( \rho_c - \rho_{icm} = 1 \) or \( \tau_c = 0 \), i.e., when the medium is homogeneous (see eqs. [7] and [8]). For a clumpy medium in a plane-parallel slab of thickness \( L \), or sphere of radius \( R = L \), the effective optical depth is then

\[
\tau_{eff} = L \Lambda_{eff} = L \Lambda_{mg} + L \kappa \rho_{icm} ,
\]

where by “effective” we mean the optical depth corresponding to the average transmission of a large collection of randomly chosen paths through the slab, as expressed by equation (1). In § 4.2 we analyze in more detail the implementation of the mega-grains equation (34), developing an improved version, and discuss the behavior of \( \tau_{eff} \) as a function of all the parameters characterizing the clumpy medium. Equation (34) is compared with the approximation developed by Natta & Panagia (1984) in § 4.3.

4.1.2. Effective Scattering Albedo

The effective scattering albedo of the two-phase clumpy medium is logically defined as a weighted combination of the albedos for each phase in the medium:

\[
\omega_{eff} = \frac{\omega_c \Lambda_{mg} + \omega_d \kappa \rho_{icm}}{\Lambda_{eff}} ,
\]

where \( \omega_d \) is the albedo of the dust and \( \omega_c \) is the effective albedo of a clump, which we discuss below. First note that in the limit as the medium becomes homogeneous (\( \tau_c \to 0 \)) we have \( \Lambda_{mg} \to 0 \) and \( \Lambda_{eff} \to \kappa \rho_{icm} \), so then \( \omega_{eff} \to \omega_d \) as expected.

Given that interactions with dust in a clump have occurred for a group of photons, the effective albedo of the clump is the fraction of those photons that manage to eventually escape from the clump by means of multiple scattering. HP93 suggested an approximation for the clump albedo for which we shall use the symbol \( \omega_{clump} \):

\[
\omega_{clump} = \frac{\omega_d}{1 + (1 - \omega_d) 4 \tau_c / 3} ,
\]

where \( \tau_c \) is the optical radius of a clump as given by equation (33). As \( \tau_c \to 0 \) we have \( \omega_{clump} \to \omega_d \) as required. As
\[ \tau_e \to \infty \text{ then } \omega_e^{HP} \to 0, \text{ but this cannot be generally true since we expect always some backscatter from the surface layer of the clump as long as the dust-scattering asymmetry parameter } g_d < 1. \text{ Note that the approximation does not consider the distribution of scattering angles. In addition we shall show that, for small } \tau_e, \text{ equation (38) is below the minimum possible value for the clump albedo.} \\

Let us examine the special case of purely forward scattering, when the asymmetry parameter of the dust is \( g_d = 1 \). In this case there is an exact formula for the clump albedo. The forward-scattering case for photons randomly impacting a spherical clump was discussed previously in the presentation of equations (26) and (28), and the actual fraction of photons absorbed is \( P[\tau_e(1 - \omega_d)] \). The fraction that escapes is then \( P(\tau_e) - P[\tau_e(1 - \omega_d)] \), and the ratio of escaping fraction over interacting fraction is the clump albedo,

\[ \omega^1_e = \frac{P(\tau_e) - P[\tau_e(1 - \omega_d)]}{P(\tau_e)} \quad (39) \]

where the superscript “ 1 ” indicates that \( g_d = 1 \). As \( \tau_e \to 0 \), we have \( P(\tau_e) \to 4 \tau_e / 3 \), and then

\[ \omega^1_e \to \frac{4 \tau_e / 3 - 4 \tau_e (1 - \omega_d) / 3}{4 \tau_e / 3} = 1 - (1 - \omega_d) = \omega_d, \]

reaching the correct limit. On the other hand, if \( \tau_e \to \infty \), then \( \omega^1_e \to 0 \), which is expected in this case of purely forward scattering. We shall show that \( \omega^1_e \) is the minimum clump albedo over all \(-1 \leq \omega_d \leq 1\).

To determine the actual clump albedo and how it depends on dust scattering parameters, we performed Monte Carlo simulations of photons randomly impacting a spherical clump for many values of the optical parameters in the ranges of \( 0 < \tau_e \leq 25, 0 < \omega_d < 1 \), and \( 0 \leq \omega_d \leq 1 \). The fraction of photons that interact with the clump is found to be exactly given by equation (26), as expected. Of these interacting photons, the fraction that scatters and eventually escapes from the clump is the clump albedo \( \omega_e \). Figure 9 shows \( \omega_e \) versus \( \tau_e \) for three values of the dust albedo \( \omega_d = 0.3, 0.6, \) and 0.8 (apparent when \( \tau_e = 0 \)). The squares indicate the case of \( g_d = 0 \) (isotropic scattering), the large diamonds show \( g_d = 1 \) (forward scattering), and intermediate values of \( 0 < g_d < 1 \) at increments of 0.1 are plotted as small diamonds vertically connecting the squares and large diamonds. The dotted line is \( \omega^1_e \) given by equation (39), the dashed line is \( \omega_e^{HP} \) given by equation (38), and the solid line we shall introduce shortly. Clearly, the theoretical forward-scattering clump albedo, \( \omega^1_e \) (dotted line), agrees exactly with the Monte Carlo results for \( g_d = 1 \) (large diamonds), as expected, and is a lower bound on the clump albedo. From the dashed line it is apparent that \( \omega^1_e < \omega^0_e \) when \( \tau_e < 2 \), which violates the lower bound.

Another formula for the clump albedo can be derived using the OLEP formula (eqs. [19] and [20]) in the same manner as was discussed in the derivation of equation (29). Given that a photon interacts with the dust inside a clump, the probability that it scatters is just \( \omega_d \). These scattering events can be regarded as reemitted photons, and, assuming that they are approximately uniformly distributed in the clump, they have a probability \( \mathcal{P}^d_{\text{esc}}(\tau_e, \omega_d) \) of escaping, thus obtaining the clump albedo

\[ \omega^* = \omega_d \mathcal{P}^d_{\text{esc}}(\tau_e, \omega_d), \quad (40) \]

which gives the solid line shown in Figure 9. The escape probability equation (20) for \( \mathcal{P}^d_{\text{esc}}(\tau_e, \omega_d) \) can be rearranged to give

\[ \omega^* = \frac{\omega_d}{\omega_d + (1 - \omega_d)P(\tau_e)}, \quad (41) \]

where \( P(\tau_e) \) is the extinction escape probability, as given by equation (19). As \( \tau_e \to 0 \), we have \( P(\tau_e) \to 1 \) and so \( \omega^* \to \omega_d \), because after a single scattering a photon will most likely escape an optically thin clump. As \( \tau_e \to \infty \), we have \( P(\tau_e) \to 3 / 4 \tau_e \) and then \( \omega^* \to 0 \). However, \( \omega^* \) gives the actual clump albedo for a single value of \( g_d \) for each \( \tau_e \), and this value, \( \omega^* \), is approximately given by equation (21), since we are using the approximate escape probability \( \mathcal{P}^d_{\text{esc}}(\tau_e, \omega_d) \) (see Fig. 6). Therefore the fact that \( \omega^* \to 0 \) as \( \tau_e \to \infty \) is just a consequence of the fact that \( g^*(\tau_e) \to 1 \) as \( \tau_e \to \infty \), coupled with the fact that the exact equation (39) for the case of \( g_d = 1 \) gives the limit of zero for the albedo of increasingly opaque clumps.

The clump albedo depends more strongly on \( g_d \) as \( \tau_e \) increases (see Fig. 9), creating a large spread in the values of \( \omega_e \) between the cases of forward scattering (\( g_d = 1 \); diamonds) and isotropic scattering (\( g_d = 0 \); squares). This is because as \( \tau_e \) increases the average photon penetration depth to first scattering decreases, and then forward-scattered photons are likely to be absorbed in the clump, whereas the spherical geometry presents many opportunities for the escape of any non-forward scattered photon. Thus a higher probability of backscatter, corresponding to smaller values of \( g_d \), increases the probability that a photon will escape, giving a larger effective clump albedo. For any \( g_d < 1 \) our Monte Carlo simulations indicate that the albedo of increasingly opaque clumps (\( \tau_e > 20 \)) remains nonzero.

In summary, Figure 9 compares the theoretical approximations for the clump albedo developed in this paper, \( \omega^* \).
the medium: combination of the asymmetry parameters of each phase in allowed. We define the effective phase function asymmetry for the effective albedo of the clumpy medium can be followed. We define the effective phase function asymmetry parameter for the two-phase clumpy medium as a weighted combination of the asymmetry parameters of each phase in the medium:

\[ g_{\text{eff}} = g_d \Lambda_{\text{mg}} + g_c \kappa \rho_{\text{icm}}/\Lambda_{\text{eff}}, \]

where \( g_d \) is the asymmetry parameter of the dust and \( g_c \) is the clump scattering asymmetry parameter, which we discuss below. Note that \( g_{\text{eff}} \rightarrow g_d \) as the medium becomes homogeneous (\( \tau_c \rightarrow 0 \)), since then \( \Lambda_{\text{mg}} \rightarrow 0 \) and \( \Lambda_{\text{eff}} \rightarrow \kappa \rho_{\text{icm}} \).

Let us assume that a collection of photons enter a clump with parallel ray paths, are scattered by the dust in the clump, and eventually escape from the clump. How does the exiting angular distribution depend on the dust scattering properties and the optical depth of the clump? The results of our Monte Carlo simulations shown in Figure 10 answer this question. We computed \( g_e = \langle \cos \theta_{\text{exit}} \rangle \) for all simulations, where \( \theta_{\text{exit}} \) is the exiting angle respect to the entering parallel rays, and plot \( g_e \) versus the clump optical radius \( \tau_c \), for values of \( g_d = \{ -0.4, 0.0, 0.6, 0.9 \} \) (evident when \( \tau_c = 0 \), and albedos \( 0.1 \leq \omega_d \leq 0.9 \). The squares are for when \( \omega_d \)-the large diamonds for \( \omega_d = 0.9 \), and intermediate values of \( \omega_d \) in increments of 0.1 are plotted as small diamonds connecting the squares and large diamonds. The distribution of exit angles does not actually follow the HG phase function with \( g = g_e \), since the scattering is complicated by the spherical geometry; however, the HG phase function is a reasonable approximation. In the case of forward scattering (\( g_d > 0 \)), the clump scattering distribution becomes more isotropic as the optical radius \( \tau_c \) of the clump increases and as the dust albedo \( \omega_d \) increases. If \( g_d = 0 \) (isotropic scattering by dust), then as the clump becomes opaque the clump asymmetry parameter approaches a backscatter distribution of \( g_e = -\frac{1}{4} \) independent of the dust albedo. In addition, when \( |g_d| \) is near zero and \( \tau_c \) is large, it is apparent that \( g_e \) approaches \( -\frac{1}{4} \) as \( \omega_d \) increases. Our computations of \( g_e \) agree with the three cases computed and mentioned by Code & Whitney (1995).

The solid lines in Figure 10 are produced by the following empirical formula:

\[ g_e(\tau_c, \omega_d, g_d) = g_d - C \left( 1 - \frac{1 + e^{-B/g_d}}{1 + e^{C-(B/g_d)}}, \right) \]

where

\[ A = 1.5 + 4g_d^4 + 2\omega_d \sqrt{g_d} \exp(-5g_d), \]

\[ B = 2 - g_d(1-g_d) - 2\omega_d g_d, \]

\[ C = \frac{1}{3 - \sqrt{2g_d - 2\omega_d g_d(1-g_d)}}. \]

The formula is a good approximation of the Monte Carlo results when \( g_d > 0 \). For \( g_d \) slightly negative one can shift the \( g_d = 0 \) curve downward to get an approximation of \( g_d \), or just use \( g_e = g_d \) when \( g_d < -0.2 \). The exact value of \( g_e \) is of importance mainly for the case of a central point source, since the escaping fraction is then very sensitive to the distribution of angular scattering.

4.2. The Extended Mega-Grains Approximation

The mega-grains approximation as presented by Hobson & Padman is limited to small values of the clump filling factor. In addition, we found that for optically thin situations it predicts an effective optical depth that is slightly greater than the equivalent homogeneous optical depth, in violation of equation (3). In this section we develop an improved version of the mega-grains approximation that resolves these problems and extends the approximation to all values of the clump filling factor.

An important term in equation (34) is \( n_c \), the density of clumps, which we shall now discuss in detail since it is the key to our extension of the mega-grains approximation. Define the porosity, \( Q_c \), of a randomly located collection of identical clumps as the ratio of the total volume of clumps, including possible overlaps, to the volume \( V \) of the...
medium:

\[ Q_c = \frac{N_c v_c}{V}, \]  

(44)

where \( N_c \) is the total number of clumps and \( v_c \) is the volume of just one clump. Equation (44) is easily solved for the density of clumps,

\[ n_c = \frac{N_c}{V} = \frac{Q_c}{v_c} = \frac{3Q_c}{4\pi r_c^3}, \]

(45)
as a function of the porosity and the radius of a spherical clump \( r_c \). Substituting for \( n_c \) in equation (34) gives

\[ \Lambda_{mg} = \frac{3Q_c}{4r_c} P(\tau_c) \]  

(46)

for the interaction coefficient of the mega-grains, where \( \tau_c \) is the optical radius of a clump.

The filling factor \( f_c \) is related to the porosity, \( Q_c \), of the clumps by

\[ f_c = 1 - e^{-Q_c}. \]  

(47)
because the probability that a random point is not in a clump goes as \( e^{-Q_c} \) (this is a very good approximation to eq. [10] when \( v_c < V \); see Appendix B). When \( f_c \ll 1 \), then \( f_c \approx Q_c \); however, as \( f_c \to 1 \) we have \( Q_c \to -\ln (1 - f_c) \to \infty \), since then the clumps tend to overlap. Thus an obvious problem with equation (46) is that as \( f_c \to 1 \) we have \( Q_c \to \infty \) causing \( \Lambda_{mg} \to \infty \). Indeed, Hobson & Padman found that \( f_c < 0.3 \) is the useful range for the mega-grains approximation. The clump overlapping fraction is calculated as \( (Q_c - f_c)/Q_c \), and for \( f_c < 0.3 \) it is less than 16% of the volume of clumps, but as \( f_c > 0.3 \) the overlapping volume fraction increases rapidly toward 100% so that the clumps can no longer be treated as separate mega-grains.

There is another problem with the previous equation, which is more subtle but still important. Consider the behavior of equation (46) as \( \tau_c \to 0 \), which occurs when either \( r_c \to 0 \), \( \kappa \to 0 \), or \( \rho_{\text{hom}} \to 0 \) (holding \( f_c \) and \( \alpha = \rho_{c}/\rho_{\text{icm}} \) constant). In that case \( P(\tau_c) \sim 4\tau_c^3/3 \), which simplifies equation (46), and upon substituting for \( \tau_c \) with equation (33), the clump radius cancels, giving

\[ \Lambda_{mg} \approx \frac{Q_c}{r_c} \tau_c = (\rho_c - \rho_{\text{icm}})\kappa Q_c \]  

(48)

when the clump optical radii are very small. Substituting this into equation (35) gives

\[ \Lambda_{\text{eff}} = \Lambda_{mg} + \kappa\rho_{\text{icm}} \approx \kappa[(\rho_c - \rho_{\text{icm}})Q_c + \rho_{\text{icm}}] \]  

(49)

for the behavior of the effective interaction coefficient of the two-phase clumpy medium as \( \tau_c \to 0 \). The problem with this behavior is that since \( Q_c > f_c \), we get that

\[ \Lambda_{\text{eff}} > \kappa[(\rho_c - \rho_{\text{icm}})f_c + \rho_{\text{icm}}] = \kappa\rho_{\text{hom}}, \]  

(50)

using equation (7) for the final equality. Thus as the clump optical depths become small, \( \Lambda_{\text{eff}} \) exceeds the interaction coefficient of the equivalent homogeneous medium. If \( L \) is the geometrical thickness of the medium then equation (50) implies that

\[ \tau_{\text{eff}} = L\Lambda_{\text{eff}} > L\kappa\rho_{\text{hom}} = \tau_{\text{hom}}, \]  

(51)

contradicting equations (3) and (4), which state that a clumpy medium is more transparent than the equivalent homogeneous medium. The inequality in equations (50) and (51) becomes greater in error as \( f_c \) increases, since then \( Q_c \to \infty \).

An immediate solution to the problem is to substitute \( f_c \) in place of \( Q_c \) in equation (46), obtaining

\[ \Lambda_{mg} = \frac{3f_c}{4r_c} P(\tau_c), \]  

(52)

which gives the correct behavior of \( \Lambda_{\text{eff}} < \kappa\rho_{\text{hom}} \) as \( \tau_c \to 0 \) for values of the clump filling factor. However, now consider the behavior as \( \tau_c \to \infty \), occurring when \( \rho_{\text{hom}} \to \infty \) or \( \kappa \to 0 \) while holding \( f_c \), \( \alpha \), and \( r_c \) constant. Then \( P(\tau_c) \to 1 \) and \( \Lambda_{mg} \to 3f_c/4r_c \), as the clumps become opaque. For values of \( f_c > 0.1 \) the resulting effective interaction coefficient \( \Lambda_{\text{eff}} = 3f_c/4r_c + \kappa\rho_{\text{icm}} \) underestimates the actual value found from Monte Carlo simulations (see Fig. 11b and discussion below). In addition, as \( f_c \to 1 \) we should have \( \Lambda_{\text{eff}} \to \kappa\rho_{\text{hom}} \), but instead equation (52) gives \( \Lambda_{\text{eff}} \to 3/4r_c + \kappa\rho_{\text{icm}} \), which because of the dependence on \( r_c \) will in general not reach the correct limit.

To extend the mega-grains approximation to all filling factors and retain the correct behavior as the clumps become optically thin or opaque, we propose to renormalize the clump radii by the factor \((1 - f_c)^\gamma \), where \( 0 < \gamma \leq 1 \) is a parameter that can fine tune the behavior as the clumps become opaque. In Appendix D we discuss the motivation and a possible interpretation of this renormalization of the clump radii. Substituting \( r_c(1 - f_c)^\gamma \) for all instances of \( r_c \) in equation (52), we get

\[ \Lambda_{mg} = \frac{3f_c}{4r_c} P(\tau_c(1 - f_c)^\gamma), \]  

(53)

our new definition of the mega-grains interaction coefficient. Now as \( \tau_c \to 0 \) or \( f_c \to 1 \) we have that \( P(\tau_c(1 - f_c)^\gamma) \sim (4/3)\tau_c(1 - f_c)^\gamma \), so the behavior of equation (53) is

\[ \Lambda_{mg} \sim \frac{f_c}{r_c} \tau_c = (\rho_c - \rho_{\text{icm}})\kappa f_c \]  

(54)
since the \((1 - f_c)^\gamma \) factor cancels out. This gives the desired result of \( \Lambda_{\text{eff}} \to \kappa\rho_{\text{hom}} \) as \( \tau_c \to 0 \) or \( f_c \to 1 \) and keeps \( \Lambda_{\text{eff}} < \kappa\rho_{\text{hom}} \) for a clumpy medium, thereby extending the mega-grains approximation to the full range \( 0 \leq f_c \leq 1 \).

In the other extreme, when \( \tau_c \gg (1 - f_c)^{-\gamma} \) then \( P(\tau_c(1 - f_c)^\gamma) \sim 1 \), resulting in

\[ \Lambda_{mg} \sim \frac{3f_c}{4r_c(1 - f_c)^\gamma}. \]  

(55)

Comparing this with the previously discussed versions of the mega-grains approximation gives the following sequence of inequalities in the limit as the clumps become opaque:

\[ \frac{3f_c}{4r_c(1 - f_c)^\gamma} > \frac{3Q_c}{4r_c} > \frac{3f_c}{4r_c}, \]  

(56)
corresponding to equations (53), (46), and (52), respectively, with \( \gamma = 1 \) and \( f_c < 1 \). Varying the parameter \( \gamma \) allows for adjustment of the opaque clump limit behavior of equation (53) over the complete range in the above sequence of inequalities: as \( \gamma \to 0 \) the left-hand side of the inequality approaches the right-hand side. We find that the optimal
value of $\gamma$ in our extended mega-grains approximation depends on the arrangement of the source of photons with respect to the geometry of the clumpy medium: $\gamma \sim 0.5$ gives a good approximation for the situation of photons impacting a slab of clumps with a fixed angle with respect to the surface normal, whereas $\gamma \sim 0.75$ works better for a central source within a sphere, and $\gamma = 1$ works best for uniformly distributed sources. When $f_c \ll 1$ the variation of the extended mega-grains approximation is small with respect to the parameter $\gamma$, and of course vanishes as the filling factor goes to zero. We shall use $\gamma = 1$ unless indicated otherwise.

Figure 11 compares the three versions of the mega-grains approximation (MGA) to Monte Carlo simulations. Figure 11a shows the ratio $\tau_{\text{eff}}/\tau_{\text{hom}}$ versus $\tau_{\text{hom}}$ for the case of $f_c = 0.2$ and $\alpha = 100$, with $r_c = 0.05R$, where $R$ is the extent of the medium ($f_c$ and $\alpha$ are dimensionless). By definition, $\tau_{\text{eff}} = RA_{\text{eff}}$ and $\tau_{\text{hom}} = RK\rho_{\text{hom}}$. The upper horizontal axis is the clump optical depth, $\tau_c$, which is proportional to $\tau_{\text{hom}}$ via

$$\tau_c = \left(\frac{r_c}{R}\right)\tau_{\text{hom}} \left[\frac{(\alpha - 1)}{(\alpha - 1)f_c + 1}\right],$$

derived by combining equations (8) and (33). The dashed line in Figure 11a results from the use of equation (46), the straightforward implementation of HP93, and it clearly exceeds unity for $\tau_c < 0.2$ and $\tau_{\text{hom}} < 1$, violating the requirement that $\tau_{\text{eff}} < \tau_{\text{hom}}$. The overshoot of unity becomes worse for larger filling factors. The diamonds are results from Monte Carlo simulations of a central source in a sphere, and $\tau_{\text{hom}}$ as required; however, the disagreement with Monte Carlo results becomes worse as $\tau_{\text{hom}}$ increases. The solid line results from the use of equation (53), our new definition of the mega-grains interaction coefficient, also giving $\tau_{\text{eff}} < \tau_{\text{hom}}$ and agreeing well with the Monte Carlo results.

Figure 11b compares $\tau_{\text{eff}}$ resulting from the three versions of the MGA at large $\tau_{\text{hom}}$ (equivalent to large $\tau_c$) for the same clumpy medium. As before, the diamonds are results from the same Monte Carlo simulations. The dotted line is produced by using equation (52), giving

$$\tau_{\text{eff}} \rightarrow R(3f_c/4r_c)$$

as $\tau_c \rightarrow \infty$, which clearly disagrees with the Monte Carlo results. The dashed line is produced by using equation (46), giving

$$\tau_{\text{eff}} \rightarrow R(3f_c/4r_c + 1)$$

as $\tau_c \rightarrow \infty$, which is in better agreement with the Monte Carlo results. The solid line is produced by our extended MGA, using equation (53), and as $\tau_c \rightarrow \infty$

$$\tau_{\text{eff}} \rightarrow R\left[\frac{3f_c}{4r_c(1 - f_c)} + 1\right],$$

which provides the best approximation of the Monte Carlo results. There is a smooth transition in the slope of the $\tau_{\text{eff}}$ curves as $\tau_c$ increases, and when the clumps become optically thick ($\tau_c > 4$) their contribution to the absorption becomes a fixed quantity so the only further change is due to the ICM density. Thus all versions of the MGA become linear as $\tau_c \gg 1$ and $\tau_{\text{hom}} \rightarrow \infty$, basically having the slope

$$\frac{\partial \tau_{\text{eff}}}{\partial \tau_{\text{hom}}} = \frac{1}{\partial \rho_{\text{ICM}}} = \frac{1}{(\alpha - 1)f_c + 1},$$

which for the parameters used for Figure 11b is a value of about 1/21. The intercepts of the asymptotic lines with the vertical axis are given by $R$ times the first term in each of the above equations (58), (59), and (60), and these values are approximately the average number of clumps encountered along a random line of sight (see Appendix B).

Now consider the behavior of the MGA as the clump radii vanish and other parameters are held fixed. This is
shown in Figure 12 for the case of $f_c = 0.2$, $x = 100$, and $\tau_{\text{hom}} = 1$, where $\tau_{\text{eff}}$ is plotted versus $r_c$ and $\tau$. The dashed line produced by using equation (46) exceeds the value of $\tau_{\text{hom}} = 1$ (dash–triple-dotted line) when $r_c < 0.03$ ($\tau < 0.2$), because as $r_c \to 0$ we have $\tau \to 0$, and by equations (48) and (49)

$$\tau_{\text{eff}} \to R\kappa[(\rho_c - \rho_{\text{icm}})Q_c + \rho_{\text{icm}}] > \tau_{\text{hom}}.$$ 

Thus the same problematic behavior of $\tau_{\text{eff}} > \tau_{\text{hom}}$ that was shown in Figure 11a for $\tau_{\text{hom}} \to 0$ occurs as $r_c \to 0$ and $\tau_{\text{hom}}$ is held constant. The solid line is produced by our extended MGA, using equation (53), and clearly yields the expected behavior $\tau_{\text{eff}} < \tau_{\text{hom}}$ for the full range of clump radii, since then as $r_c \to 0$ we have the exact limit of

$$\tau_{\text{eff}} \to R\kappa[(\rho_c - \rho_{\text{icm}})f_c + \rho_{\text{icm}}] = \tau_{\text{hom}}$$

(see eq. [54]). The diamonds are results from Monte Carlo simulations, in good agreement with our extended MGA. The dotted line is produced by using equation (52), which has the correct limit as $r_c \to 0$ but tends to underestimate $\tau_{\text{eff}}$ at larger clump radii.

Figure 13 compares the extended MGA to the original version over the full range of clump filling factors, $0 \leq f_c \leq 1$, with $x = 100$ and $r_c = 0.05$ held constant, for cases $\tau_{\text{hom}} = 1$ and $\tau_{\text{hom}} = 10$, as indicated. In each case the horizontal axis is $f_c$ and the vertical axis is $\tau_{\text{eff}}$, the effective optical depth. The diamonds are the $\tau_{\text{eff}}$ resulting from Monte Carlo simulations of a central source in a sphere of unit radius containing a two-phase clumpy medium with clump filling factors in the range $0.01 \leq f_c \leq 0.8$, and with $x = 100$ and $r_c = 0.05$ held constant. Clearly the extended MGA (using eq. [53] in eq. [35]; solid lines) gives the best agreement with the Monte Carlo results and satisfies the condition $\tau_{\text{eff}} < \tau_{\text{hom}}$ for the full range of clump filling factors. The original version of the MGA (using eq. [46]), shown as the dashed lines, starts diverging to infinity for $f_c > 0.1$ in the case of $\tau_{\text{hom}} = 1$, and diverges to infinity for $f_c > 0.5$ in the case of $\tau_{\text{hom}} = 10$. The dotted line is produced using equation (52), which underestimates the value of $\tau_{\text{eff}}$ and fails to reach the correct limit of $\tau_{\text{eff}} \to \tau_{\text{hom}}$ as $f_c \to 1$, especially when $\tau_{\text{hom}}$ is large. The fluctuations in the Monte Carlo results are due to variations in particular realizations of clumps in a finite volume, and these fluctuations grow larger as $f_c \to 1$.

The behavior of $\tau_{\text{eff}}$ versus $f_c$ always exhibits a minimum at a single value of $f_c$ that has a complicated dependence on the other parameters characterizing the clumpy medium. To study this behavior, Figure 14 shows the prediction of $\tau_{\text{eff}}$ by the extended MGA (solid lines) and the contribution from each phase on the clumpy medium: the dashed lines are the ICM component, $\tau_{\text{icm}} = R\kappa \rho_{\text{icm}}$, and the dotted lines are the component of the effective optical depth due to clumps, $\tau_{\text{mg}} = R\kappa_{\text{mg}}$, so that $\tau_{\text{eff}} = \tau_{\text{mg}} + \tau_{\text{icm}}$. The horizontal axis is now $\log_{10} f_c$, and the cases of $\tau_{\text{hom}} = R\kappa \rho_{\text{hom}} = 1$ and 10 are shown, with $x = 100$ and $r_c = 0.05$ (same as Fig. 13). Some of the same Monte Carlo results are shown (diamonds) to verify the behavior. Starting at $f_c \sim 0$, $\tau_{\text{icm}}$ decreases rapidly as $f_c$ increases since $\rho_{\text{icm}}$, given by equation (8), is inversely related to $f_c$ and the total mass is held constant. In fact the shape of the $\tau_{\text{icm}} = R\kappa \rho_{\text{icm}}$ curve
depends only on $\alpha$ and $f_c$, so only the magnitude changes with the value of $\tau_{\text{hom}}$: the shape of the variation of $\tau_{\text{icm}}$ versus $f_c$ is exactly the same for the two cases of $\tau_{\text{hom}}$ in Figure 14. The opacity of the clumps is also inversely related to their filling factor, as seen in equation (57), and so $\tau_{\text{hom}}$ is near maximum when $f_c \sim 0$, and then $\tau_{\text{mg}} \sim R(4f_c/3r_c)$. Thus $\tau_{\text{mg}}$ increases linearly with $f_c$ until the clumps become optically thin and they fill up the medium, causing $\tau_{\text{mg}} \rightarrow \tau_{\text{hom}}$. The linear increase of $\tau_{\text{mg}}$ does not compensate for the rapid decrease of $\tau_{\text{icm}}$ with $f_c$, the net effect being that $\tau_{\text{eff}}$ decreases with increasing $f_c$ until the clumps start to fill the medium. Therefore $\tau_{\text{eff}}$ goes through a minimum as a function of $f_c$ when the number of lines of sight that pass through the very low-density ICM greatly exceeds the number of paths intersecting optically thick clumps. For $f_c \ll 1$, the magnitude of $\tau_{\text{mg}}$ is independent of $\tau_{\text{hom}}$, whereas the magnitude of $\tau_{\text{icm}}$ is directly proportional to $\tau_{\text{hom}}$, so the minimum of $\tau_{\text{eff}}$ occurs at a larger value of $f_c$ for the case of $\tau_{\text{hom}} = 10$ than for the case of $\tau_{\text{hom}} = 1$.

In general, the minimum of $\tau_{\text{eff}}$ occurs at nearly the same value of $f_c$ for which $\tau_{\text{mg}} = \tau_{\text{icm}}$, and analysis of the equations shows why this is true. Assuming that $f_c \ll 1$, then

$$\frac{\partial \tau_{\text{mg}}}{\partial f_c} \approx R \left( \frac{4}{3r_c} \right) \approx \frac{\tau_{\text{mg}}}{f_c} .$$

(62)

Assuming that $\alpha \gg 1$, then

$$\frac{\partial \tau_{\text{icm}}}{\partial f_c} = -R\kappa\rho_{\text{icm}} \frac{x-1}{(x-1)f_c + 1} \approx -\frac{\tau_{\text{icm}}}{f_c + (1/\alpha)} .$$

(63)

Now since $\tau_{\text{eff}} = \tau_{\text{mg}} + \tau_{\text{icm}}$ we have that

$$\frac{\partial \tau_{\text{eff}}}{\partial f_c} \approx \frac{\tau_{\text{mg}} - \tau_{\text{icm}}}{f_c + (1/\alpha)} .$$

(64)

The local minimum occurs when the partial derivative is zero, so

$$\frac{\partial \tau_{\text{eff}}}{\partial f_c} = 0 \iff \left( 1 + \frac{1}{\alpha f_c} \right) \tau_{\text{mg}} \approx \tau_{\text{icm}} ,$$

(65)

which is approximately the relationship between the minimum of $\tau_{\text{eff}}$ and the intersection of the $\tau_{\text{mg}}$ and $\tau_{\text{icm}}$ curves in Figure 14. Consequently, as a function of $f_c$ (with constant $\alpha \gg 1$), the total luminosity absorbed by a clumpy medium attains a minimum when the luminosities absorbed in the clumps and in the ICM are equal, and this is demonstrated over a large region of parameter space in §7.3.

The previous discussion of $\tau_{\text{eff}}$ has been for the case of no scattering, or for scattering and absorption together considered as interaction with dust. To model scattering in a clumpy medium we need the effective scattering albedo, $\omega_{\text{eff}}$, given by equation (37), which depends on $\omega_{\text{c}}$, the clump albedo given by equation (41). We find that the clump radius renormalization technique introduced above also needs to be applied to the clump albedo formula in order to produce the correct approximation for scattering in the clumpy medium as $f_c \rightarrow 1$. The new equation is then

$$\omega_{\text{c}} = \omega_{\text{d}} \frac{\theta_{\text{esc}}}{\theta_{\text{eff}}} (\tau_{\text{c}}(1-f_c)^2, \omega_{\text{d}}) ,$$

(66)

so $\omega_{\text{c}} \rightarrow \omega_{\text{d}}$ as $f_c \rightarrow 1$, since then the escape probability becomes unity. This leads to the correct behavior of $\omega_{\text{eff}} \rightarrow \omega_{\text{d}}$ for $f_c \rightarrow 1$. For the same reasons, $\tau_{\text{c}}$ in equation (43) is replaced by $\tau_{\text{c}}(1-f_c)^2$ to give the correct limits as $f_c \rightarrow 1$ for $g_c$, the clump asymmetry parameter, and $g_{\text{eff}}$, the effective scattering asymmetry parameter. All the equations are summarized in §5.

In summary, we have extended and improved the formulas for the effective optical depth and albedo of a clumpy medium and introduced an approximation for the effective scattering asymmetry parameter. The validity of the extended mega-grains formulae, with scattering and for other source distributions, is further demonstrated by comparison with Monte Carlo simulations in §6 and §7.

4.3. Comparison with Earlier Theory

Here we compare the extended mega-grains approximation (MGA) for the effective optical depth of just clumps with the theory of Natta & Panagia (1984, hereafter NP84), when $\alpha$ is large so the ICM can be ignored. Figure 15 compares the approximations and quantities involved as function of $\log_{10} f_c$, for the case of $\tau_{\text{hom}} = 2$, $\alpha = 1000$, and $r_c = 0.05$. The upper panel shows $\tau_{\text{c}}$ (solid line), the clump optical radii, and two extreme values bounding $F_e$, the average number of clumps encountered along a random line of sight:

$$F_e = R \left( \frac{30}{4r_c} \right) .$$

(67)
random lines of sight are perpendicular to the plane of a F medium of radius $R$ which stays finite, so that (see Appendix B). Assuming that the number of clumps encountered along a random line of sight is Poisson distributed with mean $\lambda$, derived an equation for the effective random line of sight is $\lambda / \sqrt{\lambda}$, or radial in the case of a spherical medium, where $r_\ast$ is the porosity given by equation (44), and the dotted line is

\[ F' = R \left( \frac{3F_c}{4r_\ast} \right), \tag{68} \]

which stays finite, so that $F' \leq F_c \leq F'$ (see Appendix B). $F_c$ is also called the covering factor of the clumps, and the random lines of sight are perpendicular to the plane of a slab of thickness $R$, or radial in the case of a spherical medium of radius $R$. In the lower panel the solid line is $\tau_{mg} = R\lambda_{mg}$ using equation (53), the extended MGA. Assuming that the number of clumps encountered along a random line of sight is Poisson distributed with mean number $F_c$, NP84 derived an equation for the effective optical depth:

\[ \tilde{\tau} = F_c (1 - e^{-4c/3}), \tag{69} \]

where $(4/3)c_e$ is the expected optical depth of a random path through a spherical clump. The result of using $F_c = F_e$ in equation (69) is shown by the dashed line in the lower panel of Figure 15, and $\tilde{\tau}$ agrees with $\tau_{mg}$ (solid line) for $f_c < 0.1$, corresponding to $F_c < 1$, but $\tilde{\tau}$ exceeds $\tau_{mg}$ when $f_c > 0.3$ and $F_c > 5$, diverging to infinity. If instead $F_c = F'$ is used in equation (69), then $\tilde{\tau}$ remains finite, as shown by the dotted line. Applying the clump radii renormalization gives

\[ \tilde{\tau} = F' \left[ \frac{1 - \exp \left( -4c (1 - f_c)/3 \right)}{1 - f_c} \right] \tag{70} \]

(dotted line), which is almost identical to $\tau_{mg}$. So the approach used by NP84 gives almost the same result for $\tau_{mg}$ as the extended MGA when the average optical depth of a sphere, $(4/3)c_e$, is used. The quantity $1 - e^{-4c/3}$ happens to be a good approximation of $P(e)$, with a maximum relative difference of 5% overestimation occurring at $\tau \approx 1.5$ when $P(e) \approx 0.8$.

4.4. Escape and Absorption Probabilities for Clumpy Media

The effective optical depth, $\tau_{eff}$, effective scattering albedo, $\omega_{eff}$, and effective asymmetry parameter, $g_{eff}$, given by the mega-grains approximation reduces the basic radiative transfer properties of a two-phase clumpy medium to an effectively homogeneous medium. Therefore we propose to use $\tau_{eff}$, $\omega_{eff}$, and $g_{eff}$, given by equations (53), (35), and (36), equations (37) and (40), and equations (42) and (43), respectively, in the escape and absorption probability formulae that were developed for homogeneous media to estimate the escaping or absorbed fractions of photons in clumpy media. In particular, for isotropic emission uniformly distributed within a sphere, we use $\beta_{esc}$ given by equations (19) and (20); for a central point source, we use $\beta_{esc}$ given by equations (16), (17), and (18); and, for a uniformly illuminating external source, we use $\beta_{esc}$ given by equation (29). The full set of equations is summarized in a convenient list in § 5.

We find these analytic approximations give results in reasonable agreement with Monte Carlo simulations, as demonstrated in § 6. As suggested by HP93, we also conjecture that, given an escape probability function for a homogeneous medium having any geometry and any distribution of sources, the escape probability for a two-phase clumpy medium of same geometry and source distribution can be reasonably approximated by substituting the effective optical parameters given by the mega-grains equations into the homogeneous escape probability function.

4.5. The Fractions Absorbed in Each Phase of a Clumpy Medium

The energy of the radiation absorbed by the dust is converted into heat and then reradiated as infrared (IR) photons. The spectrum of the IR emission from the dust is dependent on the dust temperature, which is most cases is determined by the equilibrium between the energy absorbed and emitted. In this paper, we shall not deal with very small dust grains, which can undergo temperature fluctuations, an effect that will cause a wider distribution in dust temperature and enhanced IR emission in the $\sim 3-20 \mu m$ range (e.g., Dwek et al. 1997). For the case of a homogeneous medium with uniformly distributed emitters, most of the dust will be at the same temperature (accept at the boundary of the medium where the temperature will be lower). When the uniformly distributed emission occurs in an inhomogeneous medium there will be a distribution of dust temperatures. In a two-phase clumpy medium, the clumps will most likely attain a different radiative equilibrium temperature than the ICM, since the clumps will absorb/radiate a different amount of energy. Given the total amount of energy absorbed by the dust we need to know what fraction
is absorbed in the clumps and what fraction in the ICM, to
determine the approximate equilibrium temperature of each
phase of the clumpy medium.

In the mega-grains approximation it is the overdensity of
the clumps that enters into the formulation, in this way
separating the clumps and the ICM while keeping the ICM
continuous. However, to determine the fraction of photons
absorbed by the clumps we must consider the clumps as
being completely separate from the ICM, therefore con-
sidering the full density of dust in the clumps, and so the
ICM is not viewed as uniform and continuous but as having
holes occupied by clumps. This approach will give the
correct absorbed fractions as \( \alpha = \rho_c/\rho_{icm} \to 1 \). Thus we
define the full clump optical depth as

\[
\tau_e^c \equiv \kappa \rho_c r_e = \alpha \kappa \rho_{icm} r_e ,
\]

(71)

where \( \kappa \) is the total absorption plus scattering coefficient
of the dust grains per unit mass and \( r_e \) is the clump radius. We
then replace \( \tau_e \) with \( \tau_e^c \) in all the mega-grains approxima-
tion equations, and the superscript “\( c \)” shall indicate that all of
the clump optical depth is being used in the following quanti-
ties; in particular, \( A^c_{mg} \) is the result of using \( \tau_e^c \) in equation
(34).

First we deal with the case of a point source in the center
of a spherical clumpy medium and assume that the point
source is not in any clump. Let \( R \) be the radius of the sphere and
define optical depths

\[
\tau_{mg}^0 = R A_{mg}^0 ,
\]

(72)

\[
\tau_{icm}^0 = R \kappa \rho_{icm} ,
\]

(73)

because of the mega-grains (subscript “\( mg \)” and ICM,
respectively, where the superscript “\( 0 \)” indicates that this is
for no scattering. To model the effects of scattering we
employ the analytical approximation for the effective
optical radius of a homogeneous sphere with scattering, as
given by equations (16) and (17), defining new optical depths
as

\[
\tau_{mg} = \tau_{mg}^0 (1 - \omega_s^c \tau_{mg}^c; \omega_d) ,
\]

(74)

\[
\tau_{icm} = \tau_{icm}^0 (1 - \omega_s^c \tau_{icm}^c; \omega_d) ,
\]

(75)

for the mega-grains and ICM, respectively, where the function
\( \chi(\tau, \omega) \) is given by equation (17). Note that we use clump
scattering properties \( \omega_s^c \) and \( \theta_s^c \) for \( \tau_{mg} \), and the usual dust
scattering albedo and asymmetry parameter, \( \omega_d \) and \( \theta_d \),
for the ICM optical depth. Of the total photons absorbed
by the medium, the fraction, \( A_{icm}^x \), that gets absorbed by the
ICM is then estimated to be

\[
A_{icm}^x = \frac{(1 - f_c) \tau_{icm}^x}{\tau_{mg} + (1 - f_c) \tau_{icm}^x} ,
\]

(76)

where the superscript “\( c \)” indicates a central point source.
The factor \( 1 - f_c \) is introduced to get an approximation that
gives the correct behavior as \( \alpha \to 1 \) or \( f_c \to 1 \), because the
volume occupied by the clumps should not be included in
the ICM. Now if \( \beta_{esc}^c(\tau) \) is the fraction of photons that escapes
the sphere then \( A_{icm}^c(1 - \beta_{esc}^c) \) is the fraction absorbed in
the ICM and \( (1 - A_{icm}^c)(1 - \beta_{esc}^c) \) is the fraction absorbed
by clumps.

In the case of an internal uniform distribution of emitters
we consider the emission occurring in the clumps and ICM
separately, as follows. For the fraction \( 1 - f_c \) of the photons
that are emitted in the ICM we predict that \( A_{icm}^c \) will be
absorbed by the ICM. For the fraction \( f_c \) that are emitted in
the clumps we predict that a fraction \( \beta_{esc}^c(\tau, \omega_d) \) (given by
eqs. [19] and [20]) of those photons will escape the clumps
and then a fraction \( A_{icm}^x(1 - f_c) \beta_{esc}^c(\tau, \omega_d) \) of them will be
absorbed by the ICM. The total fraction of emitted photons that will be
absorbed by the ICM is then

\[
A_{icm}^x = A_{icm}^c(1 - f_c) + f_c \beta_{esc}^c(\tau, \omega_d) ,
\]

(77)

where the superscript “\( u \)” indicates that this is for the case of
uniformly distributed emission of photons. This gives
\( A_{icm}^x < A_{icm}^c \) because part of the emission occurs inside
clumps that are more dense than the ICM. Defining \( F_{abs}^x(\tau_{eff}, \omega_{eff}) \)
the total fraction of photons absorbed in the medium then \( A_{icm}^x F_{abs}^x \) is the fraction absorbed in the ICM and \( (1 - A_{icm}^x F_{abs}^x) \)
the clumps.

To estimate absorbed fractions for the case of a uniformly
illuminating external source, we just take the average of the
values for central source and uniform source:

\[
A_{icm}^x = \frac{A_{icm}^c + A_{icm}^x}{2} .
\]

(78)

The estimate is motivated by Monte Carlo simulations that
indicate that \( A_{icm}^x < A_{icm}^c < A_{icm}^c \). If \( \beta_{esc}^c(\tau_{eff}, \omega_{eff}) \) is
the total fraction absorbed in the sphere, then \( A_{icm}^x \beta_{esc}^c \) is the
fraction absorbed absorbed in the ICM and \( (1 - A_{icm}^x \beta_{esc}^c) \)
the fraction absorbed by clumps. The absorbed fractions predicted
by the above equations are compared with the results of
Monte Carlo simulations in § 6.

5. SUMMARY OF EQUATIONS

5.1. Escape and Absorption Probabilities

In the following formulae, use \( (\tau_{hom}, \omega_d, g_d) \) for all
occurrences of \( (\tau, \omega, g) \) if the medium is homogeneous, or if
the medium is clumpy use \( (\tau_{eff}, \omega_{eff}, g_{eff}) \) given by the
mega-grains equations below.

Central point source.—

\[
\beta_{esc}^c(\tau, \omega, g) \equiv \exp \left[ -\tau_{g}(\tau, \omega, g) \right] ,
\]

(80)

\[
\tau_{g}(\tau, \omega, g) \equiv (1 - \omega) \chi(\tau, \omega) ,
\]

(81)

\[
\chi(\tau, \omega) \equiv 1 - \frac{1}{2}(1 - e^{-1/2})(1 - g)^{1/2} .
\]

(82)

Uniform distribution of emitters.—

\[
\beta_{esc}^c(\tau, \omega) \equiv \frac{P_c(\tau)}{1 - \omega[1 - P_c(\tau)]} ,
\]

(83)

\[
P_c(\tau) = \frac{3}{4\tau} P_c(\tau) ,
\]

(84)

\[
P(\tau) = 1 - \frac{1}{2\tau^2} + \left( \frac{1}{\tau} + \frac{1}{2\tau^2} \right) e^{-2\tau} .
\]

(85)

Uniformly illuminating external source.—

\[
\beta_{esc}^c(\tau, \omega) \equiv P_c(\tau)[1 - \omega \beta_{esc}^c(\tau, \omega)]
\]

\[
= \frac{4\tau(1 - \omega)}{3} \beta_{esc}^c(\tau, \omega) .
\]

(86)
Recall that $P_{\text{esc}}$ and $P_{\text{abs}}$ are valid (most accurate) when $g = g^*(r)$ (see eq. [21]), whereas $P_i(x)$ and $P_i(y)$ are exact formulae.

5.2. Mega-grains Approximation

**Input parameters:**
- $\kappa$ = dust mass extinction coefficient ,
- $\omega_d$ = scattering albedo of dust grains ,
- $g_d$ = scattering asymmetry parameter ,
- $\rho_{\text{hom}}$ = average mass density of the dust ,
- $f_c$ = filling factor of the clumps ,
- $\gamma$ = MGA tuning parameter, ($\sim 1$)
- $\alpha = \rho_{icm}/\rho_{\text{hom}}$
- $r_c$ = radius of each spherical clump ,
- $R_S$ = radius of spherical medium .

**Effective optical depth $\tau_{\text{eff}}$:—**

$$\rho_{icm} = \frac{\rho_{\text{hom}}}{(\alpha - 1)f_c + 1} \quad \text{(ICM density)} ,$$

$$\tau_c = (\alpha - 1)f_c r_c \kappa \quad \text{(clump optical radius)} ,$$

$$\Lambda_{mg} = \frac{3f_c}{4r_c} P_i[(1 - f_c)\gamma]$$

$$\Lambda_{\text{eff}} = \Lambda_{mg} + \kappa \rho_{icm}$$

$$\tau_{\text{eff}} = R_S \Lambda_{\text{eff}} ,$$

(84)

where the formula for $\Lambda_{mg}$ utilizes $P_i(x)$ given by equation (82) and $0 < \gamma \leq 1$. Use $\gamma = 1$ for uniform internal/external sources, $\gamma = 0.75$ for central source, and $\gamma = 0.5$ for photons impacting a slab.

**Effective albedo $\omega_{\text{eff}}$:—**

$$\omega_{\text{eff}} = \frac{\omega_d \Lambda_{mg} + \omega_d \kappa \rho_{icm}}{\Lambda_{\text{eff}}} ,$$

where the clump albedo is

$$\omega_c = \omega_d P_{\text{esc}}^{\mu}[\tau_c(1 - f_c)\gamma, \omega_d]$$

and $P_{\text{esc}}^{\mu}$ is the OLEP formula stated in equation (80) above.

**Effective scattering asymmetry parameter $g_{\text{eff}}$:—**

$$g_{\text{eff}} = \frac{g_c \Lambda_{mg} + g_c \kappa \rho_{icm}}{\Lambda_{\text{eff}}} ,$$

where the clump asymmetry parameter is

$$g_c = g_d - C \left( 1 - \frac{1}{1 + \exp\left( -B/A \right)} \right)$$

$$A \equiv 1.5 + 4g_d \sqrt{g_d} \exp\left( -5g_d \right)$$

$$B \equiv 2 - g_d(1 - g_d) - 2\omega_d g_d$$

$$C \equiv [3 - \sqrt{2g_d} - 2\omega_d g_d(1 - g_d)]^{-1} .$$

5.3. Fractions Absorbed in Each Phase

Let $P_{\text{abs}} = 1 - P_{\text{esc}}$ be the generic absorption probability. Then the fraction of photons absorbed in the ICM is $A_{icm} P_{\text{abs}}$, and the fraction absorbed by clumps is $(1 - A_{icm}) P_{\text{abs}}$, where $A_{icm}$ is given by one of the following formulas, corresponding to the type of source. See below for definitions of $\tau_{mg}$ and $\tau_{icm}$.

**Central point source.**

$$A_{icm} \equiv \frac{(1 - f_c)\rho_{icm}}{\tau_{mg} + (1 - f_c)\rho_{icm}} .$$

(85)

**Uniform distribution of emitters.**

$$A_{icm} \equiv A_{icm}^e [(1 - f_c) + f_c P_{\text{esc}}(\tau_c, \omega_d)] .$$

(86)

**Uniformly illuminating external source.**

$$A_{icm} \equiv \frac{A_{icm}^e + A_{icm}^o}{2} .$$

(87)

**Component optical depths.**

$$\tau_{\text{mg}} = \kappa \rho_{icm} r_c = a \kappa \rho_{icm} r_c ,$$

$$\tau_{\text{mg}}^0 = R_S \Lambda_{mg} ,$$

$$\tau_{icm}^0 = R_S \kappa \rho_{icm} .$$

The superscript "a" indicates that the full clump optical depth is being used; i.e., $\Lambda_{mg}$ is the result of using $\tau_{icm}$ in equation (84). The superscript "0" indicates that the quantities are for no scattering. To include the effects of scattering in spherical geometry, apply equation (79):

$$\tau_{mg} = \tau_{mg}^0 (1 - \omega_d)^{\xi_{\text{mg}}^0(\omega_d)} ,$$

$$\tau_{icm} = \tau_{icm}^0 (1 - \omega_d)^{\xi_{\text{icm}}^0(\omega_d)} .$$

5.4. The Case of $\alpha < 1$

If one desires to model randomly distributed spherical cavities (e.g., supernova remnants instead of clumps) that have a lower density than the ICM (so that $\alpha < 1$), the mega-grains equations can be applied by redefining

$$f_c' = 1 - f_c ,$$

$$\alpha' = \frac{1}{\alpha} ,$$

(88)

and using these new inverted values in the above equations to obtain $(\tau_{\text{eff}}, \omega_{\text{eff}}, g_{\text{eff}})$ for the medium with cavities. However, when computing the fractions absorbed in each phase of the medium, we do not apply the inversion transform of equation (88). The inverted mega-grains approximation is demonstrated in § 6.3.

6. COMPARISON OF ANALYTIC APPROXIMATIONS WITH MONTE CARLO SIMULATIONS

In § 4.2 we developed the extended mega-grains approximation (MGA) and compared it with Monte Carlo simulations, which verified the effective optical depth predicted by the approximations for the case of no scattering and when emission is from a central point source in a spherical clumpy medium. We shall use the acronym MCRT when...
referring to Monte Carlo simulations of radiative transfer and the acronym MGEP when referring to the analytic approximations consisting of the mega-grains (MG) approximations combined with escape/absorption probability (EP) formulae, which were summarized § 5. In this section we present more detailed comparisons of the MGEP model with MCRT simulations when scattering is also involved and for all three types of source distributions discussed in § 3. In the following we designate the cases of a central isotropic source “C,” uniformly distributed internal sources “U,” and uniformly illuminating external sources “X.”

We shall compare the MGEP approximations with MCRT results over a wide range of parameters that define a two-phase clumpy medium in spherical geometry, except that the clump radii will be held fixed at \( r_c = 0.05 \) relative to the radius of the sphere \( R_s = 1 \). A larger value would be less realistic and cause the number of clumps to be less than needed for good statistics in the MCRT model, whereas spherical clumps with radii smaller than \( r_c = 0.05 \) become difficult to represent accurately on a grid. All the Monte Carlo simulations were performed using a grid of 100^3 or 127^3 voxels to represent the clumpy medium and followed at least 10^6 photons for each result. The quantities in the comparisons will be shown as a function of equivalent homogeneous optical depth \( \tau_{hom} \), clump filling factor \( f_c \), or clump-to-ICM density ratio \( \kappa \) in the following three subsections.

### 6.1. Dependence on Optical Depth

Recall that the equivalent homogeneous optical depth of extinction, \( \tau_{hom} = \kappa \tau_{hom} \), can vary because of changing wavelength or dust mass (\( R_s \) is held constant) and so is the major parameter involved in modeling the transfer of a spectrum of radiation in a clumpy medium. Figure 4 showed the behavior of the effective optical depth \( \tau_s \) (eq. [13]) versus \( \tau_{hom} \) resulting from Monte Carlo simulations with scattering (\( \omega_d = 0.6 \) and \( g_d = 0.6 \)) for three cases of the filling factor, \( f_c = 0.1, 0.2, \) and \( 0.3 \) (squares, triangles, and diamonds, respectively) with \( \kappa = 100 \). The dashed, dotted, and solid lines are produced by the extended mega-grains theory combined with equation (79) to compute \( \tau_s \) as a function \( \tau_{hom} \) goes from clump dominated (steeper slope at low values of \( \tau_{hom} \)) to ICM dominated (less slope at large \( \tau_{hom} \)) as the clumps become opaque, and the occurrence of this transition can be estimated by equation (57). Unless otherwise indicated, \( \kappa = 100 \) in this section.

In Figure 16 we study the effect of changing the dust scattering parameters for the case of \( f_c = 0.2 \), with the upper panel showing the effective optical depth with or without scattering (\( \tau_s \) or \( \tau_{eff} \), respectively) for source type C and the lower panel showing the escaping fraction of source type U, all versus \( \tau_{hom} \). The case of no scattering is indicated with squares for MCRT results and the dot-dashed line are the MGEP model. The diamonds and solid lines show MCRT and MGEP results, respectively, for \( \omega_d = 0.8 \) and \( g_d = 0.6 \) (isotropic scattering); the triangles and long-dashed lines show MCRT and MGEP results, respectively, for \( \omega_d = 0.8 \) and \( g_d = 0.8 \) (almost forward scattering). The values of \( \tau_{eff} \) given by MGA (dot-dashed line) for no scattering are used in equation (79), along with values of \( \omega_{eff} \) and \( g_{eff} \) (not shown), to compute \( \tau_d(\tau_{eff}, \omega_{eff}, g_{eff}) \), the solid and long-dashed lines in the upper panel. The solid line in the lower panel is computed by using \( \tau_{eff} \) and \( \omega_{eff} \) in equation (80) for source type U. The MCRT results show that, as expected, forward-scattering dust lowers the effective optical depth and increases the escape probability as compared to isotropic scattering dust, especially for a central source of photons, and the analytical approximations of MGEP correctly predict this effect. In the case of a uniformly distributed source, the dependence of the escaping fraction on the dust scattering asymmetry parameter \( g_d \) is weak, and so it is not a problem that \( g_d \) does not enter into the MGEP equations for source type U. The analytical approximations of MGEP agree well with the numerical results of MCRT at low optical depths, whereas at high optical depths there are some differences, especially for source type C, which are due to specific realizations of the clumpy medium in MCRT simulations, but the agreement is still acceptable.

Figure 17 compares in the upper panel the absorbed fraction of photons from each type of source, and in the lower panel the fraction of the total absorbed photons that are absorbed in just the ICM, as function of \( \tau_{hom} \), with \( f_c = 0.1 \), and with scattering parameters \( \omega_d = 0.6 \) and \( g_d = 0.6 \). The stars, diamonds, and squares represent MCRT numerical
results for source types C, U, and X, respectively. The solid, dot-dashed, and dashed lines represent MGEP predictions for source types C, U, and X, respectively. The absorbed fractions are computed as

$$P_{\text{abs}} = \begin{cases} 
1 - \mathcal{P}_{\text{esc}}(\tau_{\text{eff}}, \alpha_{\text{eff}}, g_{\text{eff}}) & \text{[C; see eq. (79)]} \\
1 - \mathcal{P}_{\text{esc}}(\tau_{\text{eff}}, \omega_{\text{eff}}) & \text{[U; see eq. (80)]} \\
\mathcal{P}_{\text{abs}}(\tau_{\text{eff}}, \omega_{\text{eff}}) & \text{[X; see eq. (83)]}
\end{cases}$$

for each indicated source type. The fractions absorbed by the ICM, $A_{\text{icm}}$, are computed by equations (76), (77), and (78) for source types C, U, and X, respectively. The MGEP model agrees with the MCRT results, showing the same behavior for each source type. Figure 18 compares the same quantities for $f_c = 0.3$. The MGEP model predicts a total absorbed fraction (upper panel) for source type X greater than the MCRT results when $\tau_{\text{hom}} > 10$, probably because of limitations of the assumptions used in deriving equation (29) for $\mathcal{P}_{\text{abs}}$. For this case of $f_c = 0.3$ (and actually for $f_c > 0.1$) the MGEP model predicts values for $A_{\text{icm}}$ (lower panel) that are lower than the MCRT results. When $f_c < 0.1$ the absorbed ICM fractions predicted by MGEP are slightly higher than the MCRT results. Overall, the agreement is acceptable, and the MGEP model exhibits the same behavior as the MCRT simulations.

6.2. Dependence on Filling Factor

We next compare MGEP and MCRT results as a function of the clump filling factor $f_c$, holding other parameters fixed. Figures 19 and 20 compare MCRT and MGEP predictions of the fraction of emitted photons that are absorbed in the medium, $P_{\text{abs}}$, in the upper panels, and the fraction of the total absorbed photons that are absorbed in the ICM, $A_{\text{icm}}$, in the lower panels, as function of $\tau_{\text{hom}}$, and $f_c$, with scattering parameters $\omega_d = 0.6$ and $g_d = 0.6$. Figure 19 is for the case of $\tau_{\text{hom}} = 2$ and Figure 20 is for $\tau_{\text{hom}} = 10$ (note the different ranges for $P_{\text{abs}}$ axis). The stars, diamonds, and squares represent MCRT numerical results for source types C, U, and X, respectively. The solid, dot-dashed, and dashed lines represent MGEP numerical results for source types C, U, and X, respectively. The analytical MGEP theory agrees well with the MCRT numerical results, with the exception of the absorbed fraction of a uniformly illuminating external source with scattering. For that case of source type X, when $\tau_{\text{hom}}$ is large and $f_c \rightarrow 1$ the MGEP theory overestimates the absorbed fraction with a difference that increases with $f_c$. This is probably because the assumption of uniformly distributed photons after first scattering (see § 3.4).
Fig. 19.—Comparison of MGEP absorption probabilities to MCRT absorbed fractions as function of with $a = 100$, and $f_{c}$. Scattering parameters are $\omega_d = 0.6$ and $g_d = 0.6$. The stars, diamonds, and squares represent MCRT results for source types C, U, and X, respectively. The solid, dot-dashed, and dashed lines represent MGEP results for source types C, U, and X. The dotted line and the triangles represent MGEP and MCRT results, respectively, for the case of no scattering.

becomes invalid as $\tau_{\text{hom}} \rightarrow \infty$ since then the externally impacting photons do not penetrate the sphere.

The dotted line and the triangles in the lower panels represent MGEP and MCRT results, respectively, for the case of no scattering ($\omega_d = 0$). The effect of turning on scattering is to decrease the fraction of photons absorbed in the ICM, since photons scattered by the ICM then have more of a chance of getting absorbed in the denser clumps. The MGEP model tends to overestimate $A_{\text{icm}}$ at low clump filling factors and to underestimate $A_{\text{icm}}$ at high filling factors, with best agreement at about $f_{c} = 0.1$. However, the MGEP theory gives the same variation of $A_{\text{icm}}$ with $f_{c}$ as the MCRT simulations, and overall the agreement is acceptable.

6.3. Dependence on Density Ratio

Finally, we compare MGEP and MCRT as a function of the clump to ICM density ratio, $a = \rho_{c}/\rho_{\text{icm}}$, holding other parameters fixed. Figure 21 shows (top row) the effective optical depth ($\tau_{\text{eff}}$ or $\tau_{\text{c}}$) for source type C, (second row) the escaping fraction of photons ($\gamma_{\text{esc}}^{u}$) from source type U, the fractions absorbed by the ICM (third row) for source type C ($A_{\text{icm}}^{C}$) and (fourth row) for source type U ($A_{\text{icm}}^{U}$), all as a function of $a$, for the cases $f_{c} = 0.1$, 0.5, and 0.9 as indicated, with $\tau_{\text{hom}} = 10$. The squares and diamonds indicate MCRT results for no scattering ($\omega_d = 0$) and with scattering (for $\omega_d = 0.6$ and $g_d = 0.6$), respectively. The solid and dashed lines indicate MGEP predictions with and without scattering, respectively. The value $a = 1$ corresponds to a homogeneous medium, and then $\tau_{\text{eff}} = \tau_{\text{hom}}$ when $\omega_d = 0$. Values of $a < 1$ show the case of spherical cavities in a denser ICM. For the case of denser clumps ($a > 1$), most of the variation in $\tau_{\text{eff}}$, $\tau_{\text{c}}$, and $\gamma_{\text{esc}}^{u}$ occurs for $1 < a < 100$ because in this range the clumps go from being optically thin to thick, and for $a > 100$ the clumps are essentially opaque. Turning on scattering decreases the effective optical depth and allows more photons to escape, but this effect is diminished as $a \rightarrow \infty$ and is correctly predicted by the MGEP model. The reason for the diminished effects of scattering is that $\omega_{\text{eff}}$ and $g_{\text{eff}}$ (not shown) decrease by more than a factor of 2 as $a \rightarrow \infty$.

Applying the MGEP model to the case of spherical cavities ($a < 1$) is accomplished by the inversion transform of equation (88), which then regards the cavities as the ICM and the rest of the medium as clumps. This is not the intended application of the mega-grains approximation, since there are then actually no isolated clumps, but there is some indication that the MGEP model could be tailored to give a reasonable approximation of the escaping radiation.
When $f_c = 0.9$ we see that the MGEP model does not give correct predictions for $\alpha < 1$, the case of spherical cavities. This is understandable, because when the cavities occupy 90% of the volume, the high-density regions are just shells between the cavities, which are not well approximated by clumps having the same radius as the cavities (recall $r_c = 0.05$). Choosing a smaller value for $r_c$ would improve the MGEP approximation of the MCRT results. The MGEP model provides an acceptable approximation of the MCRT results for $\alpha \geq 1$. When $f_c = 0.5$ and as $\alpha \rightarrow \infty$ there is an increasing difference between MGEP and MCRT results in the case of a central source, but this can be attributed to the specific realization of the clump locations relative to the source: there is a 50% chance that the point source will be inside a clump (as in this case), causing an increase of the optical depth in the MCRT simulation. Other simulations for the same filling factor give MCRT results that are less than the MGEP predictions, so we expect that on average the MGEP predictions will be close to MCRT results.

The fractions absorbed by the ICM are shown in the third and bottom rows of Figure 21 for source types C and U, respectively. The intersection of the horizontal and vertical dotted lines indicate the nominal value of expected $A_{\text{icm}}^c$ as the medium becomes homogeneous, since then the fraction photons absorbed by the ICM should be $1 - f_c$ of the total absorbed photons. For the case of when the clumps are denser than the ICM ($\alpha > 1$), the MGEP model agrees well with the MCRT results when the source is uniformly distributed. When the source is at a central point and $f_c = 0.5$, the MGEP predictions are less than the MCRT results, and this can be attributed to the fact that the point source happens to be in a clump, as discussed above. Note that the MGEP model correctly predicts that turning on scattering causes fewer photons to be absorbed by the ICM and more
to be absorbed in clumps. When \( x < 1 \) the MGEP model predicts values lower than MCRT for \( f_e > 0.3 \), since the MGA was not actually formulated for this application, but the MGEP model does exhibit the correct behavior.

6.4. Body-centered Cubic Lattice of Clumps

The mega-grains approximation can be applied to the case of cubic clumps randomly located on a body-centered cubic (BCC) lattice with partial success. The BCC lattice was used by Witt & Gordon (1996) in their Monte Carlo simulations. For source type U, the MGEP model does predict the same escaping fraction computed by MCRT when \( r_e = 1/N = 0.05 \), where \( N = 20 \) is the number of grid element along each axis of the lattice. However, this value of \( r_e \) in the MGEP model gives erroneous predictions for the effective optical depth seen by a central source. The problem is related to the fact that the apparent projected area of a cubic clump depends on the viewing angle. To match the MGEP model with MCRT results for source type C, we need to use \( r_e \approx 2^{1/2}/N = 0.071 \) in the mega-grains equations, but then the predictions for source type U are wrong.

7. Modeling the Absorption of Stellar Radiation by Dust and the Infrared Emission

In this section we model the transfer of a spectrum of radiation in a two-phase clumpy medium, from emission by starlike sources to the absorption and scattering by dust and the resulting infrared emission from dust heated by the absorbed radiation. Both detailed Monte Carlo simulations and the analytical approximations developed in previous sections are used to model the transfer of radiation, and the emerging spectral energy distribution (SED) from the two methods of modeling are compared. We shall again use the acronyms MCRT for the Monte Carlo radiative transfer model and MGEP for the mega-grains approximations combined with escape/absorption probability formulae.

The dust is assumed to be composed of 40% graphite and 60% silicates by mass and to have the grain size distribution

\[
\zeta(a) \propto a^{-3.5}
\]

normalized over the following range of grain sizes:

\[
0.001 \mu m < a < 0.25 \mu m
\]

Using the optical constants from Draine (1985), we applied Mie theory to calculate the absorption and scattering efficiencies of dust grains, \( Q_{abs}(a, \lambda) \) and \( Q_{scat}(a, \lambda) \), respectively, as a function of grain size, \( a \), and photon wavelength, \( \lambda \). The scattering asymmetry parameters, \( g(a, \lambda) = \langle \cos \theta_{scat}(a, \lambda) \rangle \), were also calculated by averaging with respect to the distribution of scattering angles. Then the cross sections and asymmetry parameters were averaged with respect to the grain size distribution to get the mass absorption and scattering coefficients and average asymmetry parameters used in the models:

\[
\kappa_{abs}(\lambda) = \frac{\langle \pi a^2 Q_{abs}(a, \lambda) \rangle_a}{\langle m_p \rangle_a},
\]

\[
\kappa_{scat}(\lambda) = \frac{\langle \pi a^2 Q_{scat}(a, \lambda) \rangle_a}{\langle m_p \rangle_a},
\]

\[
g_{\lambda}(\lambda) = \frac{\langle g(a, \lambda) \pi a^2 Q_{scat}(a, \lambda) \rangle_a}{\langle \pi a^2 Q_{scat}(a, \lambda) \rangle_a},
\]

where the averaging operator is defined as

\[
\langle \cdot \rangle_a = \int_{a_{\min}}^{a_{\max}} (\cdot) \zeta(a) da
\]

and the average grain mass is

\[
\langle m_p \rangle_a = \langle \frac{4}{3} \pi a^3 \rho \rangle_a,
\]

where \( \rho \) is the mass density of a graphite or silicate grain. All the following simulations have the same total dust mass of 1.01 \( M_\odot \) contained in spherical region 1 pc in radius. This corresponds to a dust mass density of \( \rho_{\text{hom}} = 1.63 \times 10^{-23} \text{ g cm}^{-3} \) in the homogeneous case, equivalent to a gas density of 1000 \( \text{ cm}^{-3} \) with a dust-to-gas mass ratio of 0.007, giving a homogeneous optical depth of 1.67 in the \( V \)-band.

Before discussing the results of the MCRT and MGEP simulations, we illustrate how the degree of clumpiness in the distribution of dust is as important as the total dust mass and scattering albedo in affecting the transfer of radiation through the medium. We compare, as a function of wavelength, in Figure 22a the effective optical depth \( \tau_{\text{eff}}(\lambda) \), in Figure 22b the effective albedo \( \omega_{\text{eff}}(\lambda) \), and in Figure 22c the effective asymmetry parameter, \( g_{\text{eff}}(\lambda) \), of a spherical region of dust with different degrees of clumpiness, computed using the mega-grains equations listed in § 5, with \( R_s = 1 \text{ pc} \), \( \rho_{\text{hom}} = 1.63 \times 10^{-23} \text{ g cm}^{-3} \),

\[
\kappa(\lambda) = \kappa_{abs}(\lambda) + \kappa_{scat}(\lambda), \quad \omega_{\lambda}(\lambda) = \kappa_{scat}(\lambda)/\kappa(\lambda).
\]

The dotted curves are for the extreme case of \( f_e = 0.01 \) and \( \rho_{\text{clum}} = 100 \text{ cm}^{-3} \), giving \( \tau(\lambda) \) values of 1.67. The dashed curves are for the case \( f_e = 0.1 \) with \( \rho_{\text{clum}} = 100 \text{ cm}^{-3} \), and the solid curves are for the case \( f_e = 0.1 \) with \( \rho_{\text{clum}} = 100 \text{ cm}^{-3} \), giving \( \tau_{\text{eff}}(\lambda) \) values of 1.67. The dotted curves are for the extreme case of \( \rho_{\text{clum}} = 100 \text{ cm}^{-3} \), and the solid curves are for the case \( f_e = 0.1 \) with \( \rho_{\text{clum}} = 100 \text{ cm}^{-3} \), giving \( \tau_{\text{eff}}(\lambda) \) values of 1.67. It is evident that the effective radiative transfer properties of the dusty medium can be radically affected by the degree of clumpiness. The effect is greatest at shorter wavelengths where the dust absorption and scattering coefficient is sufficiently large to make the clumps opaque, causing \( \tau_{\text{eff}}(\lambda) \) to become almost flat and featureless. The small features remaining are due to the ICM, but the opaque clumps dominate the effective optical depth. An explanation for this "gray" behavior when \( \tau_{\text{eff}}(\lambda) \gg 1 \) is obtained by using equation (61):

\[
\frac{\partial \tau_{\text{eff}}}{\partial \lambda} = \frac{\partial \tau_{\text{hom}}}{\partial \lambda} = \frac{1}{\frac{\partial \omega_{\text{hom}}}{\partial \lambda}} = \frac{\partial \tau_{\text{hom}}}{\partial \lambda} \approx \frac{\partial \tau_{\text{hom}}}{\partial \lambda}.
\]

Thus, as \( f_e \to \infty \) the variation of \( \tau_{\text{eff}}(\lambda) \) for the clumpy medium becomes a small fraction of the variation of \( \tau_{\text{hom}}(\lambda) \) in the equivalent homogeneous medium. For longer wavelengths (infrared) there is no difference between clumpy and homogeneous media because the clumps are optically thin.

To test the MGEP model using MCRT, we simulated a two-phase clumpy medium with \( f_e = 0.1 \), \( \rho_{\text{clum}} = 100 \text{ cm}^{-3} \), and \( r_e = 0.05 \text{ pc} \), so that the effective optical depths, albedos, and asymmetry parameters used in all the MGEP calculations are given by the dashed curves in Figure 22. The MCRT computations always use the optical parameters given by the homogeneous case (solid curves) in Figure 22, simulating the details of the clumpy medium on a three-dimensional grid of 127³ voxels. The radiation source in all cases is a blackbody spectrum with \( T_s = 15,000 \text{ K} \) and \( L_s = 33,000 \text{ L}_\odot \), so the spectrum of the
The transfer of radiation is computed by MCRT at 40 wavelengths from $\lambda_{\text{min}} = 0.1 \, \mu\text{m}$ to $\lambda_{\text{max}} = 14 \, \mu\text{m}$, following $10^7$ photons at each wavelength. The standard three types of source geometries are studied: uniformly distributed internal emission (U), uniformly illuminating external source (X), and a central isotropic point source (C).

### 7.1. Absorbed Luminosities and the Distribution of Dust Temperatures

Since we model only nonionizing photons experiencing coherent scattering, for a given type of source the radiative transfer of a unit emitted flux can be simulated at each wavelength separately, giving an escape and absorption response function for the particular choice of source, geometry, and clumpiness. Then we multiply the chosen source spectrum with the escape/absorption response function to get the actual escaping SED and the luminosity absorbed by each component of the dust. In the case of Monte Carlo simulations (MCRT) we obtain the three-dimensional spatial distribution of the absorbed luminosity, whereas using the analytical approximations (MGEP) gives the luminosity absorbed by all the clumps and luminosity absorbed by the ICM. The equilibrium dust temperatures, $T_d$ for graphite or silicate, are computed by equating the absorbed and emitted luminosities:

$$\int_{\lambda_{\text{min}}}^{\lambda_{\text{max}}} S_{\lambda} P_{\text{abs}}(\lambda) d\lambda = 4\pi m_k \int_{0}^{\infty} \kappa_\lambda B_\lambda(T_d) d\lambda ,$$

where $P_{\text{abs}}(\lambda)$ generically represents the fraction of flux at wavelength $\lambda$ absorbed by graphite or silicate in a particular three-dimensional voxel of mass $m_k$ in the case of MCRT, or it is the fraction absorbed by graphite or silicate in either the ICM or the clumps of mass $m_k$ in the case of MGEP. The energy conservation equation (97) is solved iteratively for $T_d$ using Brent’s method of finding the zeros of a function (Press et al. 1992, pp. 242–254), giving a three-dimensional distribution of dust temperatures for MCRT, or the average temperatures of the dust in clumps and the ICM when using the MGEP model.

Figure 23 shows the distribution of equilibrium dust temperatures computed by the MCRT model for clumps (upper row) and in the ICM (lower row), for source types U, X, and C in each column from left to right. The solid lines are the probability densities of graphite dust temperatures, and the dotted lines represent silicate dust. For the cases of uniformly distributed internal and external sources the probability densities are Gaussian on the high-temperature side and of exponential form on the low-temperature side (note slightly different temperature scales). In contrast, the central source creates a power-law distribution of temperatures (the third column is a log-log plot) since dust next to the source is heated much more than dust farther from the center. In fact, the straight lines in Figure 23 are power laws: the solid line is $T_d^{-8}$ and appears to be parallel to the graphite temperature distribution, whereas $T_d^{-8.5}$ (dashed line) is parallel to the temperature distribution of silicates.

Table I compares the luminosities absorbed by each component (graphite or silicates) and phase (ICM or clumps) of the medium, as computed by the MCRT and MGEP models, again for the cases of uniform internal, uniform external, and central source. The MGEP results were obtained by computing the escaping and absorbed SED using the mega-grains equations and the escape/absorption probability formulae, combined with formulae for the fractions absorbed by the ICM and clumps given in § 4.5. Also compared in Table 1 are the resulting dust temperatures: averages of the distributions shown in Figure 23 for the case of MCRT simulations, and in the case of the MGEP model when the source is not a central point source, a single temperature for each component and phase computed using equation (97).

To compute the average temperature of the dust in the case of a central source we use a theoretical power-law distribution of dust temperatures as described in Appendix E, along with two assumptions: that the absorbed luminosity decays with radial distance from the source like an
inverse power law with index $\eta = 2.5$ ($\eta = 2$ is normal for optically thin case), and we use the maximum dust temperatures found by MCRT in the voxel containing the source. Normally the maximum temperature would be the dust sublimation temperature, however, the MCRT simulation is not set up to fully resolve the volume next to the source so the sublimation temperature is not reached in the simulation. Since we are comparing with MCRT results we chose to use those maximum temperatures. We then solve for minimum temperature of the power-law distribution that matches the emitted and absorbed luminosities, and this also yields the average dust temperature.

Table 1 shows that the MCRT and MGEP results are in close agreement. The uniformly illuminating external source experiences more absorption (67%) than the uniformly distributed internal emission (49%) and therefore

![Graph](image-url)
heats the dust to slightly higher temperatures. Approximately the same fractions of luminosity are absorbed in the central and external source cases, but the average dust temperature is lower in the central source case since most of the dust is far from the source. In all cases the clumps absorb about 7 times more energy than the ICM, and this is in part due to the fact that the clumps contain more than 10 times the mass of the ICM,

$$\frac{M_c}{M_{icm}} = \frac{\rho_c f_c V}{\rho_{icm}(1 - f_c)V} = \frac{\alpha f_c}{1 - f_c},$$  

(98)

and this ratio is greater than 10 for $f_c = 0.1$ and $\alpha = 100$. However, the larger mass can easily radiate the absorbed energy in the IR so that the temperature of dust in clumps is generally lower than in the ICM. In § 7.3 we study in more detail the reasons for the lower temperatures of dust in clumps.

7.2. The Emerging UV–FIR SED

Computation of the dust temperatures also gives the IR emission from dust in each voxel of the three-dimensional simulation by the MCRT model, or from each phase of the medium in the MGEP model:

$$F_\lambda = 4\pi m_\lambda \kappa(\lambda)B_\lambda(T_d)\rho_{icm}(1 - f_c)V = \frac{\alpha f_c}{1 - f_c},$$  

(99)

where $T_d$ is the temperature of dust in a voxel of mass $m_\lambda$ in the MCRT model, or where $T_d$ is the temperature of a dust in clumps/ICM of mass $m_\lambda$ in the MGEP model. Equation (99) is used by the MGEP model only for uniformly distributed internal or external sources. In the case of a central source with a power-law distribution, $\rho(T_d)$ of dust temperatures, the MGEP model essentially computes

$$F_\lambda = 4\pi m_\lambda \kappa(\lambda) \int_{T_{min}}^{T_{max}} B_\lambda(T)p(T)dT,$$  

(100)

an approximation that is described in more detail in Appendix E. Integrating the IR emission over the three-dimensional volume in the case of the MCRT approach, or just adding the IR emission from the clumps and ICM in the case of the MGEP approach, gives the total IR emission spectrum. Figure 24 shows the IR emission spectra from graphite (upper row) and silicates (lower row), with the diamonds and triangles representing emission from clumps and the ICM, respectively, as computed by MCRT, and the solid and dotted lines representing emission from the ICM and clumps as computed by MGEP. The source types U, X, and C, are presented in columns from left to right. The MGEP model is in close agreement with the MCRT results. The emission from the clumps is in general greater than that from the ICM because the absorbed luminosity is larger. An exception is the case of a central source where the heating of dust adjacent to the source to much higher temperatures causes more emission from the ICM at short IR wavelengths.

The emerging SED is the sum of the escaping radiation and the IR emission from heated dust:

$$\lambda E_\lambda = \lambda S_\lambda P_{esc}(\lambda) + \lambda F_\lambda,$$  

(101)

where $P_{esc}(\lambda)$ generically designates an escape probability that is computed either numerically (MCRT) or analytically.
(MGEP) and depends on the model and source type. Figure 25 compares the SED computed by the MCRT and MGEP models for each type of source. The diamonds are the MCRT results, the solid lines are the MGEP theory, and the dashed line is the original source spectrum $\lambda S_{\lambda}$. The source types U, X, and C, are presented in panels from top to bottom. The SED at short wavelengths is the source radiation that escapes from the clumpy dusty environment, whereas the SED at long wavelengths is emission from dust heated by the absorbed radiation. The agreement between the MGEP and MCRT models is excellent. The SEDs for the uniform internal and external sources have very similar appearances, except that in the case of an external source more of the source energy is absorbed (68% vs. 49%). The central and external source cases have the same fraction of energy absorbed by the dust, so that the short wavelength portion of the SEDs are the same. However, in the case of a central source the emission from hot dust near the source increases the SED around $10\, \mu m$. In all cases, the relative paucity of emission in the $\sim 3-20\, \mu m$ region is caused by the omission of stochastic heating in the calculations.

**Fig. 25**—Comparison of the emerging UV–FIR SED [$\lambda E_\lambda$ (ergs s$^{-1}$)] resulting from the MGEP model (solid lines) and the MCRT model (diamonds). The dashed line is the SED of the source types U, X, and C (top to bottom, respectively).

### 7.3. Exploring Parameter Space with the MGEP Model

Since the MGEP equations are computationally fast, we can very easily model the escape and absorption of radiation over a wide range of parameters characterizing the clumpy medium or the radiation source. Figures 26 and 27 show the results of MGEP model calculations over the range of clump filling factors, $0.001 < f_c < 1$, and density ratios, $1 < \alpha < 10^4$, with clump radii $r_c = 0.05$. The dust composition and total mass is the same as that used in the above MCRT to MGEP comparisons. The sources of radiation are isotropic and uniformly distributed within the sphere, again having a blackbody spectrum with temperature $T_s = 15,000$ K and total luminosity $L_s = 33,000\, L_\odot$. In all the panels the diamond and cross marks the point $(f_c, \alpha) = (100, 0.1)$ for which we presented detailed comparisons with MCRT, and the horizontal dotted lines indicate the limiting value of $f_c (=0.0015)$ below which there are less

**Fig. 26**—Results of the MGEP model over a wide range of clumpy medium parameters $(\alpha, f_c)$ for a uniformly distributed blackbody source having $T_s = 15,000$ K and $L_s = 33,000\, L_\odot$. **Upper panel:** Absorbed luminosity fraction. **Lower panel:** Ratio of ICM/clump luminosities. The diamond and cross mark the parameters for which detailed comparisons with MCRT were presented. In both panels, the dashed line is the locus of parameters where $M_{\text{icm}} = M_c$, and the thick solid line is the locus of parameters where $L_{\text{icm}} = L_c$. More details are given in §7.3 of the text.
than 10 clumps in the medium, therefore, only filling factors above the dotted line are considered to be statistically reliable.

Figure 26 shows (upper panel) the contours of the fraction of the source luminosity that is absorbed by the clumpy medium \( (L_{\text{abs}}/L_s) \), and (lower panel) the contours of the logarithm of the ratio of luminosity absorbed by the ICM to that absorbed by clumps \( \log (L_{\text{icm}}/L_s) \). The dashed line (in both panels), extending from \((\alpha, f_c) = (1,0.5)\) to \((1000, 0.001)\), is the curve

\[
f_M(\alpha) = \frac{1}{\alpha + 1},
\]

(giving the locus parameters for which the ICM and clumps have equal mass, that is, \( M_{\text{icm}} = M_c \). The equation for \( f_M(\alpha) \) is obtained from equation (98), which we restate here as

\[
M_c = \left( \frac{\frac{\alpha f_c}{1 - f_c}}{M_{\text{icm}}} \right).
\]

Note that if \( f_c > f_M(\alpha) \), then \( M_c > M_{\text{icm}} \). The thick solid line in all panels represents the locus of parameters \( f_M(\alpha) \) for which the ICM and clumps absorb the same amount of energy, that is, \( L_{\text{icm}} = L_c \). The dotted contours in the lower panel of Figure 26 indicate when the ICM absorbs less energy than the clumps. The solid contours in the lower panel of Figure 26 indicate that if \( f_c < f_M(\alpha) \) then \( L_c < L_{\text{icm}} \). Combining this with the fact that \( M_c > M_{\text{icm}} \) when \( f_c > f_M(\alpha) \) we have that for the parameter region between the dashed and thick curves

\[
f_M(\alpha) < f_c < f_M(\alpha) \Rightarrow \left\{ \begin{array}{l} M_c > M_{\text{icm}} \\ L_c < L_{\text{icm}} \end{array} \right.,
\]

which guarantees that the temperature of dust in clumps will be less than that in the ICM since, compared to the ICM, a smaller amount of energy is absorbed by a larger number of dust particles, which can radiate this energy at a lower temperature.

The upper panel of Figure 26 also shows that, for a given density contrast \( \alpha_0 \), the minimum absorbed luminosity \( (L_{\text{abs}}) \) corresponds to the filling factor \( f_c = f_M(\alpha_0) \) for which \( L_c = L_{\text{icm}} \). When \( \alpha \gg 1 \), the condition \( L_c = L_{\text{icm}} \) is achieved when there is approximate equality between the effective optical depth of the clumps and that of the ICM, that is, \( \tau_{\text{mg}}(\lambda_p) \approx \tau_{\text{icm}}(\lambda_p) \), where \( \lambda_p \) is the wavelength where most of the source energy is absorbed. By equation (65), the value of \( f_c \) for which \( \tau_{\text{mg}} \approx \tau_{\text{icm}} \) is nearly the same value for which \( \tau_{\text{eff}} \) attains a minimum as a function of \( f_c \) (see also Figs. 14, 19, and 20). Consequently, as \( \alpha \to \infty \), the steepest descent of \( L_{\text{abs}} \) is along the curve \( f_M(\alpha) \). A detailed derivation of this result and the following equations would require studying \( \tau_{\text{mg}} \) and \( \tau_{\text{icm}} \) as given by equations (74) and (75), respectively, but here we assume that the effects of scattering are of second order importance.

Let us study the curve \( f_M(\alpha) \) in more detail. As \( \alpha \to 1 \) the condition \( L_c = L_{\text{icm}} \) is achieved by having \( M_c = M_{\text{icm}} \), and so \( f_M(\alpha) \approx f_M(\alpha) \) when \( \alpha < 5 \), as shown by the merging of the dashed and thick curves in Figure 26. However, as \( \alpha \to \infty \) we see that \( f_M(\alpha) > f_M(\alpha) \). This behavior is due to the change in the effective optical depth of the clumps, \( \tau_{\text{mg}} \), since the clumps become optically thick as \( \alpha \to \infty \), so we derive an equation for \( f_M(\alpha) \) as follows. Starting with \( \tau_{\text{mg}}(\lambda_p) = \tau_{\text{icm}}(\lambda_p) \) and applying the mega-grains equations from § 5, we obtain

\[
\frac{3f_c}{4\tau_c(1 - f_c)^2} = \frac{\kappa(\lambda_p)\rho_{\text{hom}}}{(\alpha - 1)f_c + 1},
\]

assuming optically thick clumps. Since we propose that equation (105) occurs when \( L_c = L_{\text{icm}} \), we now use the symbol \( f_c \) in place of \( f_M(\alpha) \) for the solution. Then as \( \alpha \to \infty \) we
know that $f_L \ll 1$ but $\alpha f_L \gg 1$, so equation (105) becomes

$$3f_L \approx \frac{\kappa(\lambda_p)\rho_{\text{hom}}}{\alpha f_L}. \quad (106)$$

Solving for $f_L$ gives

$$f_L(x) \approx \sqrt{\frac{4r_c\kappa(\lambda_p)\rho_{\text{hom}}}{3\alpha}} \quad (107)$$

when $\alpha$ is large. Using $\kappa(\lambda_p)\rho_{\text{hom}} = 1.7$, which occurs when $\lambda_p \sim 0.55 \mu m$, $f_L(x) \propto \sqrt{\alpha^{-1/2}}$, which produces a perfect fit to the zero contour of $\log(L_{\text{clumps}}/L_c)$ in the lower panel of Figure 26 when $x > 30$ (not shown). The transition from the behavior $f_L(x) \approx f_M(x)$ to the behavior $f_L(x) \propto \alpha^{-1/2}$ occurs naturally around values of $\alpha$ for which equation (107) is greater than $f_M(x)$, indicating that the clumps are becoming optically thick. We find that the same value of $\kappa(\lambda_p)\rho_{\text{hom}} = 1.7$ in equation (107) gives perfect fits to the large $\alpha$ behavior of $f_M(x)$ for all $0.1 \leq r_c \leq 0.8$ (not shown).

Figure 27 shows contours of the temperature of graphite dust in the ICM ($T_{\text{icm}}$: top panel), in the clumps ($T_c$: middle panel), and the temperature ratio, $T_{\text{icm}}/T_c$ (bottom panel). The thick solid lines are the curves $f_M(x)$ for which $L_c = L_{\text{icm}}$ and the dashed lines are the curves $f_M(x)$ for which $M_c = M_{\text{icm}}$. The contours of $T_{\text{icm}}$ wrap around $f_M(x)$, whereas the contours of $T_c$ wrap around $f_M(x)$. As $\alpha \to \infty$ and $f_c \to 0$, we find that $T_c$ decreases whereas $T_{\text{icm}}$ increases slightly. From the relation $L \propto MT^{4-\beta}$, where $\beta$ is the emissivity index, we get that

$$T_{\text{icm}} \approx T_c \left[ \left( \frac{M_c}{M_{\text{icm}}} \right) \left( \frac{L_c}{L_{\text{icm}}} \right) \right]^{1/(4+\beta)} \quad (108)$$

and with the conditions from equation (104), this equation explains why the situation of $f_M(x) < f_c < f_M(x)$ leads to $T_c < T_{\text{icm}}$. We can also explain why the temperature ratio must increase as $\alpha \to \infty$. Substituting equation (107) into equation (103) gives

$$M_c/M_{\text{icm}} \approx C\sqrt{\alpha} \quad (109)$$

along the $f_M(x)$ curve, where $L_c = L_{\text{icm}}$ and where $C$ is a constant. Using this in equation (108) we get that along the $f_M(x)$ curve

$$T_{\text{icm}} \approx (C\sqrt{\alpha})^{1/(4+\beta)}. \quad (110)$$

Therefore $T_{\text{icm}}/T_c$ must increase as $\alpha \to \infty$. The net result is that $T_c$ decreases faster than $T_{\text{icm}}$ increases because $M = M_c + M_{\text{icm}}$ is a constant whereas $L_{\text{abs}} = L_c + L_{\text{icm}} = 2L_c$ is decreasing as $\alpha \to \infty$ and $f_c = f_M(x)$. Note that the spectrum of the emitted radiation is dominated by dust at temperature $T_c$ if $f_c > f_M(x)$, since then $L_c > L_{\text{icm}}$, or it is dominated by dust at temperature $T_{\text{icm}}$ if $f_c < f_M(x)$.

The temperature of silicates also exhibits the same variations but with lower values since silicates absorb only about 25% of what graphite absorbs for the chosen composition. The same type of pattern, shown here for specific values of $r_c$, $T_c$, $L_c$, and source type, is found for other values of $r_c$, $T_c$, $L_c$, and the central point source or uniformly illuminating external source types. The dust temperatures are lower as $L_c$ is decreased, and there is less variation in $T_{\text{icm}}/T_c$ as $T_c$ or $r_c$ is decreased. Cases of $T_c > 20,000$ K would require modeling the ionization of gas and the heating of dust by absorption of Ly$_\alpha$ photons.

8. SUMMARY AND CONCLUSIONS

We have introduced an analytical model for the escape and absorption of radiation in two-phase clumpy media, based on combining the mega-grains approximation of HP93 with escape/absorption probability formulae for homogeneous media, as summarized in § 5, and referred to by the acronym MGEP. Enhancements of the mega-grains approximation developed in this paper include: (1) improving and extending the formula for effective optical depth to all clump filling factors, (2) improving the formula for the effective albedo, (3) developing a new approximation for the effective scattering asymmetry parameter, and (4) developing new approximations for the fractions absorbed by each phase of the medium. We also developed a new approximation for the fraction of photons from a uniformly illuminating external source that are absorbed in a sphere of dust when scattering occurs.

The space of parameters describing a two-phase clumpy dusty environment is six dimensional: the clump filling factor, the clump to interclump medium (ICM) density ratio, and the clump radii together characterize the morphology of the medium, and the interaction (absorption plus scattering) coefficient, scattering albedo, and asymmetry parameter are the three optical parameters of the dust that vary with wavelength. The analytic MGEP model was compared with Monte Carlo simulations of radiative transfer (MCRT) over a subspace of the six-dimensional parameter space, and we found good agreement for most of the parameter values checked. More importantly, the qualitative behavior of the MGEP model agrees very well with MCRT simulations, giving us confidence in applying the MGEP model to parameter ranges that have not yet been checked by MCRT simulations. Three types of source distributions were studied: the central point source (C), uniformly distributed internal sources (U), and uniform illumination by external sources (X). Source type U is the least absorbed, whereas source type X is the most absorbed at low optical depths and source type C is the most absorbed at high optical depths.

The MGEP model was shown to predict very well the emerging SED in a realistic simulation of starlike sources heating a clumpy dusty medium, as compared to MCRT simulations. Furthermore, the MGEP method requires just seconds to compute a full-spectrum simulation, in comparison to the hours of computation required by the MCRT method.

From MGEP simulations over a wide range parameters characterizing the clumpy medium we find that for constant clump to ICM density ratio, $\alpha$, the total luminosity absorbed by the clumpy medium attains a minimum at a filling factor, $f_c$, for which the luminosity absorbed by clumps ($L_c$) and the ICM ($L_{\text{icm}}$) are equal. The curve of $f_c$ versus $\alpha$ for which $L_c = L_{\text{icm}}$ is found to be proportional to $\alpha^{-1/2}$, a consequence of the clumps becoming optically thick, whereas the curve of $f_c$ for which the clumps and ICM have equal mass is proportional to $\alpha^{-1/2}$, and these diverging behaviors cause the temperature of dust in clumps to decrease as $\alpha \to \infty$ and $f_c \to 0$. Physically, the dust in opaque clumps shields itself from radiation, thus reaching a
lower equilibrium temperature than dust in the ICM. The extra parameters gained by introducing clumpiness allows for modeling more unusual relationships between the luminosity absorbed by dust and the resulting dust temperatures.

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APPENDIX A

FINAL FLUX APPORTIONMENT IN MONTE CARLO

Let \( N_0 \) be the number of photons emitted as a group at the start of the Monte Carlo simulation, and suppose that after \( k \) iterations of traveling, escape, absorption, and scattering there are \( N_k \) photons remaining in the medium. Since after each scattering the flux weight of each photon is reduced by the albedo \( 1 - \omega \), the actual flux remaining after \( k \) scatterings is \( \omega^k N_k \). For clarity in the following derivation define the flux weight factor

\[
W_k \equiv \omega^k .
\]

After one more iteration of traveling, there are \( N_{k+1} \) photons remaining, each still having weight \( W_k \), so that the flux escaping during the \( k + 1 \) iteration is \( E_{k+1} = W_k(N_k - N_{k+1}) \). The remaining photons interact with the medium and are apportioned into absorbed and scattered fractions: the absorbed flux is \( A_{k+1} = (1 - \omega)W_k N_{k+1} \), and the flux remaining after scattering is \( \omega W_k N_{k+1} = W_{k+1} N_{k+1} \). Carrying this analysis forward by induction gives the general formula for the escaping flux after each iteration,

\[
E_{k+1} = W_k(N_k - N_{k+1}) ,
E_{k+2} = W_{k+1}(N_{k+1} - N_{k+2}) ,
\vdots
E_{k+n} = \omega^{k+n-1}(N_{k+n-1} - N_{k+n}) ,
\]

and the absorbed flux after each iteration,

\[
A_{k+1} = (1 - \omega)W_k N_{k+1} ,
A_{k+2} = (1 - \omega)W_{k+1} N_{k+2} ,
\vdots
A_{k+n} = (1 - \omega)\omega^{k+n-1} N_{k+n} .
\]

We seek expressions for the infinite sums of \( E_{k+n} \) and \( A_{k+n} \) over \( n > 0 \) in order to terminate the Monte Carlo simulation at the \( k \)th iteration and still have an accurate estimate of the escaping flux. Assume that for all \( n > 0 \)

\[
\frac{N_{k+n}}{N_{k+n-1}} = \beta ,
\]

so that after \( k \) iterations the fraction of photons remaining after each additional iteration is assumed to have reached a steady state. Then we have

\[
N_{k+n} = \beta^n N_k ,
N_{k+n-1} - N_{k+n} = (1 - \beta)\beta^{n-1} N_k .
\]

The sum of the escaping flux is then

\[
\sum_{n=1}^{\infty} E_{k+n} = \omega^k \sum_{n=1}^{\infty} \omega^{n-1}(N_{k+n-1} - N_{k+n})
= \omega^k N_k (1 - \beta) \sum_{n=1}^{\infty} (\omega \beta)^{n-1}
= \omega^k N_k (1 - \beta)/(1 - \omega \beta) .
\]

Since \( \omega^k N_k \) is the known amount of flux remaining after \( k \) iterations, the fraction that would eventually escape if the iterations were continued is then

\[
f_k^{\text{esc}} = \frac{1 - \beta}{1 - \omega \beta} ,
\]
assuming that equation (A4) is true. Similarly for the sum of the absorbed flux

\[ \sum_{n=1}^{\infty} A_{k+n} = (1 - \omega) \beta^k \sum_{n=1}^{\infty} \omega^{n-1} N_{k+n} \]

\[ = \omega^k N_k (1 - \omega) \sum_{n=1}^{\infty} \omega^{n-1} \beta^n \]

\[ = \omega^k N_k (1 - \omega) \beta / (1 - \omega \beta) , \quad \text{(A9)} \]

and it is clear that

\[ f_k^{\text{esc}} + f_k^{\text{abs}} = \frac{1 - \beta}{1 - \omega \beta} + \frac{(1 - \omega) \beta}{1 - \omega \beta} = 1 , \quad \text{(A10)} \]

giving a check of the derivations. Note that the simpler assumption of \( f_k^{\text{esc}} = \omega \) and \( f_k^{\text{abs}} = 1 - \omega \) corresponds to the case of \( \beta = 1 / (1 + \omega) \), which incorporates no knowledge of the particular radiative transfer situation and could not possible work for all types of sources and geometries. In contrast, the final iteration escaping fraction given by equation (A8) is based on a partial history of absorption and scattering and therefore can better estimate the true value.

To examine the effects of relaxing the assumption of equation (A4), define the remaining fraction at the \( k \)th iteration:

\[ \beta_k = \frac{N_k}{N_{k-1}} , \quad \text{(A11)} \]

the ratio of the number of photons remaining after the \( k \)th iteration over the number of photons in the medium before the \( k \)th iteration. In our experience with Monte Carlo simulations we find that the sequence \( \beta_k \) (\( k = 1, 2, 3, \ldots \)) is monotonically either increasing or decreasing (does not oscillate), and the direction of convergence depends on the source geometry and the value of \( \beta \), the phase function asymmetry parameter. In our Monte Carlo simulations with uniformly distributed internal emitters, the sequence \( \beta_k \) converges rapidly, and using \( \beta_k \) in place of \( \beta \) in equation (A8) gives an \( f_k^{\text{esc}} \) that predicts the total escaping flux quite well. The remaining fraction \( \beta_k \) is found to be an increasing sequence when \( q < g^*(\tau) \), where \( g^*(\tau) \) is approximately given by equation (21), since photons tend to get trapped in the medium when scattering is more isotropic. When \( q > g^*(\tau) \) then \( \beta_k \) is a decreasing sequence, since photons tend to escape after more scatterings if the angular scattering distribution is on average more forward. When \( q = g^*(\tau) \) we find that the sequence \( \beta_k \) is essentially constant. This dependence on the asymmetry parameter \( q \) is related to the validity of the escape probability formula for scattering discussed in §3.2 and further analyzed in Appendix C2. The behavior of the sequence \( \beta_k \) is essentially the same for a uniformly illuminating external source of photons. In the extreme case of a single central source, it takes more scatterings for \( \beta_k \) to converge and it is always a decreasing function of \( k \). Thus \( f_k^{\text{esc}} \) always slightly overestimates the escaping fraction for the case of a central source but never underestimates it. When the scattering is more forward the rate of decreasing \( \beta_k \) is faster during the first few scatterings, since in this case the likelihood that a photon will escape increases after each scattering more than when the scattering is isotropic.

The discovery of the above equations and behavior was facilitated by the use of dynamic arrays to represent a group of photons, a standard feature of the Interactive Data Language (IDL), a product of Research Systems, Incorporated (RSI).

**APPENDIX B**

**RANDOM CLUMPS IN A TWO-PHASE MEDIUM**

In §2.2 we introduced equation (11) for the number of identical random clumps needed to produce a given volume filling factor in a two-phase medium. Here we give a more detailed derivation of that equation. Let \( X \) be a randomly chosen point in the medium. Then the probability that a randomly placed clump will contain the point \( X \) is simply

\[ p = \frac{v_c}{V} , \quad \text{(B1)} \]

where \( v_c \) is the volume of a clump and \( V \) is the total volume of the medium. Continue to randomly place more clumps in the medium, without regard to overlaps, for a total of \( N_c \) clumps. Then the probability \( P(n) \) that the point \( X \) will be contained in \( n \) clumps is given by the binomial distribution:

\[ P(n) = \binom{N_c}{n} p^n (1 - p)^{N_c - n} . \quad \text{(B2)} \]

In particular, the probability that \( X \) is not in any clump is

\[ P(0) = (1 - p)^{N_c} , \quad \text{(B3)} \]

so the fraction of the volume \( V \) occupied by the clumps is

\[ f_c = 1 - P(0) = 1 - (1 - p)^{N_c} . \quad \text{(B4)} \]
Solving for $N_c$ gives the desired equation

$$N_c = \frac{\ln(1 - f_c)}{\ln(1 - p)}$$

for the number of identical randomly placed clumps that will have a total filling factor of $f_c$, when each clump has a filling factor $p$.

The expectation value for $n$ of the binomial distribution (eq. [B2]) is

$$E(n) = \sum_{n=0}^{N_c} n P(n) = N_c p = \frac{N_c v_c}{V} = Q_c,$$

equaling the porosity of the clumps, $Q_c$, which was introduced for the mega-grains approximation. As $p \to 0$ and $N_c \to \infty$, the binomial distribution is well approximated by the Poisson distribution having the same expectation (e.g., Bevington & Robinson 1992, pp. 23–28):

$$P(n) \approx \frac{Q_c^n e^{-Q_c}}{n!}.$$

For $n = 0$ this approximation gives

$$f_c = 1 - P(0) \approx 1 - e^{-Q_c},$$

providing another equation relating porosity and filling factor, which is exact for our purposes, since $p < 10^{-3}$ (e.g., when $r_c < 0.1 R_S$) and $N_c \gg 1$. When $f_c \ll 1$ then of course $f_c \approx Q_c$.

Another quantity of interest is the average number of clumps encountered along a randomly chosen line of sight, also called the covering factor, $F_c$. Consider a cylinder of radius $r_c$ and length $L$ centered on the line of sight. The volume of the intersection of the cylinder with the random clumps is $\pi r_c^2 L f_c$, and we can estimate the number of clumps in the intersection by dividing by the volume of a clump:

$$F_c \approx \frac{\pi r_c^2 L f_c}{(4/3) \pi r_c^3} = L \left(\frac{3 f_c}{4r_c}\right).$$

This is actually more like a lower bound estimate since the clumps may overlap. An upper bound is obtained by using the porosity in place of the filling factor:

$$F_c < L \left(\frac{3Q_c}{4r_c}\right).$$

**APPENDIX C**

**ESCAPE AND INTERACTION PROBABILITIES FOR A SPHERE**

**C1. NO SCATTERING**

Here we derive equations (26) and (19), which are exact formulas for the interaction and escape probabilities of external and internal sources, respectively, in a homogeneous sphere of dust, ignoring the effects of scattering. Thus, scattering and absorption are considered together as interactions causing extinction of photons along a line of sight, and by extinction escape probability we mean the probability of escaping without being scattered or absorbed. Consider a ray intersecting the sphere at an angle $\theta$ with respect to the surface normal vector, which we call the impact angle. The length of the chord created by the intersection is $2R \cos \theta$, where $R$ is the radius of the sphere. Defining $\tau \equiv \rho k R$, the optical radius, where $\kappa$ is the dust interaction coefficient and $\rho$ is the density of dust, the extinction optical depth of the chord is $2 \tau \cos \theta$, and this will be used in the derivations.

To derive the interaction probability, equation (26), for the case of a uniformly illuminating external source, consider a beam of photons traveling in parallel rays impacting a hemisphere. Upon computing the transmission of the beam we can get the interacting fraction of photons, and then by symmetry this gives the interaction probability for all possible beams impacting the sphere, i.e., a uniformly illuminating external source. Let $I_0$ be the intensity of each parallel ray in the impacting beam. Then the intensity emerging from the other side is reduced by the extinction, which depends on the impact angle as follows:

$$I_{\text{out}}(\tau, \theta) = I_0 \exp (-2 \tau \cos \theta).$$

(C1)
The total flux emerging without interaction is computed by integrating over all impact angles with respect to solid angle:

\[ F_{\text{out}}(\tau) = 2\pi I_0 \int_0^{\pi/2} e^{-2\tau \cos \theta} \cos \theta \sin \theta \, d\theta \]

\[ = 2\pi I_0 \int_0^1 \mu e^{-2\tau \mu} \, d\mu \]

\[ = \pi I_0 \left[ \frac{1}{2\tau^2} - \left( \frac{1}{\tau} + \frac{1}{2\tau^2} \right) e^{-2\tau} \right] , \quad (C2) \]

where we have used the substitution of variables \( \mu = \cos \theta \) and integration by parts. The flux that would emerge if the sphere was empty is computed by integrating with \( \mu = 0 \) in equation (C2), obtaining

\[ F_0 \equiv F_{\text{out}}(0) = 2\pi I_0 \int_0^1 \mu \, d\mu = \pi I_0 . \quad (C3) \]

So the fraction of photons that interact is

\[ P_i(\tau) = \frac{F_0 - F_{\text{out}}(\tau)}{F_0} = 1 - \frac{1}{2\tau^2} + \left( \frac{1}{\tau} + \frac{1}{2\tau^2} \right) e^{-2\tau} , \quad (C4) \]

giving the interaction probability for a uniformly illuminating external source. For the case of an optically thin sphere note that the extinction behaves as

\[ e^{-2\tau \mu} \to 1 - 2\tau \mu , \quad \text{when} \quad \tau \ll 1 , \quad (C5) \]

and so the emerging flux is

\[ F_{\text{out}}(\tau \sim 0) \approx 2\pi I_0 \int_0^1 \mu (1 - 2\tau \mu) \, d\mu = \pi I_0 \left( 1 - \frac{4\tau}{3} \right) , \quad (C6) \]

and as expected this gives

\[ P_i(\tau \sim 0) \approx \frac{4\tau}{3} . \quad (C7) \]

For the case of uniformly distributed internal emission, the probability of escaping without interactions, equation (19), can be derived in a similar fashion. Let \( \rho \) be the emission per unit volume per second. Then the noninteracting intensity emerging from a ray at an angle \( \theta \) with respect to the surface normal is

\[ I_{\text{out}}(\tau, \theta) = \frac{\rho \epsilon}{\rho \kappa} (1 - e^{-2\tau \cos \theta}) , \quad (C8) \]

obtained in the standard fashion by integrating the transfer equation with no scattering along the chord of optical length \( 2\tau \cos \theta \) through the sphere. The total noninteracting flux emerging in any given direction is computed by integrating with respect to solid angle, as in equation (C2), over all the parallel rays:

\[ F_{\text{out}}(\tau) = \frac{2\pi \rho \epsilon}{\rho \kappa} \int_0^{\pi/2} (1 - e^{-2\tau \cos \theta}) \cos \theta \sin \theta \, d\theta \]

\[ = \frac{2\pi \rho \epsilon}{\rho \kappa} \int_0^1 (1 - e^{-2\tau \mu}) \mu \, d\mu \]

\[ = \frac{\pi \rho \epsilon}{\rho \kappa} \left[ 1 - \frac{1}{2\tau^2} \left( 1 + \frac{1}{2\tau^2} \right) e^{-2\tau} \right] , \quad (C9) \]

obtaining a result similar to the interaction probability above, because of the same exponential term in the integral. If the medium had zero absorption and scattering, the intensity emerging from a ray is

\[ I_0(R, \theta) = 2\rho \epsilon R \cos \theta . \quad (C10) \]

The total flux emerging in any given direction is again computed by integrating with respect to solid angle over all the parallel rays:

\[ F_0(R) = \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \cos \theta \sin \theta I_0(R, \theta) = 4\pi \rho \epsilon R \int_0^1 \mu^2 \, d\mu = \frac{4\pi \rho \epsilon R}{3} . \quad (C11) \]
Therefore, the fraction of photons that escape the sphere is
\[
P_{esc}(\tau) = \frac{F_{out}(\tau)}{F_{in}(R)} = \frac{3}{4\tau} \left[ 1 - \frac{1}{2\tau^2} + \left( \frac{1}{\tau} + \frac{1}{2\tau^2} \right) e^{-2\tau} \right],
\] (C12)

using the fact that \( \tau = \rho \kappa R \), giving the extinction escape probability for a uniformly distributed internal source.

### C2. INCLUDING SCATTERING

Equation (20), which approximately includes the effects of scattering into any extinction escape probability for the case of uniformly distributed isotropic emitters in a bounded medium, was presented in Lucy et al. (1991), but no derivation was given. In this section we derive the equation and point out some interesting properties. The scattering albedo of the medium is the fraction of the extinction optical depth that is due to scattering: where \( \sum \) is the sum of absorption and scattering optical depths from the center to the boundary of the medium, e.g., the radius of a sphere. Assume that we are given the escape probability, \( P_{esc}(\tau) \), for absorption only, as a function of the optical depth of the bounded medium. We can immediately use to give the fraction of emitted photons that escape without interacting (not absorbed or scattered), by using \( \tau = \tau_{ext} \). For convenience we shall identify in the following derivation. Thus a fraction
\[
\frac{P_{esc}(\tau)}{P_{tot}(\tau)}
\]

with another factor of \( \omega(1-P) \). The column labeled “scattered” feeds back into the “interacting” column of the next row of the table. The induction presented in the table shows that a fraction
\[
P_n = \frac{P_{esc}(\tau)}{1 - \omega(1-P)}
\]

after substituting equation (C13). As mentioned before, the number of scatterings occurring for the absorbed fractions listed in the third column of Table 2 is actually \( n - 1 \), where \( n \) is the number of interactions given in the first column. Therefore, the

| Number | Interacting | Absorbed | Scattered | Escaping |
|--------|-------------|----------|-----------|----------|
| 1      | \(1 - P\)   | \(1 - \omega(1-P)\) | \(\omega(1-P)\) | \(P_{esc}(1-P)\) |
| 2      | \(\omega(1-P)^2\) | \(1 - \omega(1-P)\) | \(\omega(1-P)^2\) | \(P_{esc}(1-P)^2\) |
| 3      | \(\omega^3(1-P)^3\) | \(1 - \omega(1-P)^3\) | \(\omega^3(1-P)^3\) | \(P_{esc}(1-P)^3\) |
| \vdots| \vdots | \vdots | \vdots | \vdots |
| \(n\)  | \(\omega^{n-1}(1-P)^n\) | \(1 - \omega\omega^{n-1}(1-P)^n\) | \(\omega^{n-1}(1-P)^n\) | \(P_{esc}(1-P)^n\) |

Table 2 summarizes the analysis of the fractions in each state and extends it by induction. The first column is the number of interactions that have occurred. The column labeled “scattered” feeds back into the “interacting” column of the next row with another factor of \(1 - P\) to give the next interaction. Note that for the first interaction, the photons that are absorbed have experienced zero scatterings, so that the number of scatterings for fractions in the “interacting” and “absorbed” columns is one less than the interaction number. The induction presented in the table shows that a fraction \(P_{esc}(1-P)^n\) escapes after \(n\) scatterings. Summing these escaping fractions over an infinite number of scatterings gives the sought after formula for the total escaping fraction:
\[
\mathcal{P}_{esc}(\tau, \omega) = \frac{P}{1 - \omega(1-P)}
\] (C13)

Recall that the formula is valid (most accurate) at a single value of the scattering asymmetry parameter, which in the case of spherical geometry is approximately given by equation (21). The absorbed fractions listed in the third column of Table 2 can also be summed to get the total absorbed fraction:
\[
\mathcal{P}_{abs}(\tau, \omega) = \frac{(1 - \omega)(1-P)}{1 - \omega(1-P)}
\] (C14)

finding that \(\mathcal{P}_{esc}(\tau, \omega) + \mathcal{P}_{abs}(\tau, \omega) = 1\), which gives a check of the above derivations.

Let us define distributions, \(p_n(n)\) and \(p_s(n)\), for the probability that a photon will escape or get absorbed after \(n\) scatterings. To arrive at a probability distribution, the fraction \(P_{esc}(1-P)^n\) that escapes after \(n\) scatterings must be normalized by dividing by the total escaping fraction, obtaining
\[
p_s(n) = \frac{P_{esc}(1-P)^n}{\mathcal{P}_{esc}(\tau, \omega)} = [1 - \omega(1-P)]\omega^n(1-P)^n,
\] (C15)
probability distribution of absorption after \( n \) scatterings is

\[
p_d(n) = \frac{(1 - \omega)\omega^n(1 - P)^{n+1}}{P_{\text{abs}}(\tau, \omega)} = [1 - \omega(1 - P)]\omega^n(1 - P)^n,
\]

resulting in the fact that \( p_d(n) = p_d(n) \) for all \( n \). Of course this can be true only when \( \mathcal{P}_{\text{esc}}(\tau, \omega) \) is valid. Monte Carlo simulations verify that the escaping and absorbed probability distributions are equal only when equation (C13) agrees with the Monte Carlo results, which occurs only for a single value of the scattering asymmetry parameter \( g \), as discussed in §3.2 for the case of spherical geometry.

Defining \( A \equiv 1 - P \), the average number of scatterings that a photon experiences before escape or absorption is calculated as

\[
\langle n_{\text{scat}} \rangle = \sum_{n=0}^{\infty} np_d(n) = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} p_d(n) = \sum_{k=0}^{\infty} \left[ 1 - (1 - \omega A) \left( \frac{1 - \omega^{k+1} A^{k+1}}{1 - \omega A} \right) \right] = \sum_{k=1}^{\infty} \omega^k(1 - P)^k = \frac{1}{1 - \omega(1 - P)} - 1 = \frac{\mathcal{P}_{\text{esc}}(\tau, \omega)}{P_d(\tau)} - 1.
\]

Monte Carlo simulations for the case of uniform emission within a homogeneous sphere of absorbers and scatterers again verifies that when the asymmetry parameter takes on the particular value \( g = g^*(\tau) \), approximately given by equation (21), then the average number of scatterings experienced by photons is the same whether the final state is escape or absorption: \( \langle n_{\text{scat}} \rangle_{\text{esc}} = \langle n_{\text{scat}} \rangle_{\text{abs}} \). For other values of \( g \) we find that \( \langle n_{\text{scat}} \rangle_{\text{esc}} < \langle n_{\text{scat}} \rangle_{\text{abs}} \) when \( g < g^*(\tau) \), so absorption occurs after more scatterings than escape since there is more absorption than predicted by equation (C13), whereas \( \langle n_{\text{scat}} \rangle_{\text{esc}} > \langle n_{\text{scat}} \rangle_{\text{abs}} \) when \( g > g^*(\tau) \), since there are more escaping photons than predicted. We conjecture that the ratio of \( \langle n_{\text{scat}} \rangle_{\text{esc}} \) to \( \langle n_{\text{scat}} \rangle_{\text{abs}} \) can be affected only by \( g^*(\tau) \) because it is dependent only on geometry: the directions of scattering and proximity to the boundary of the medium. The albedo \( \omega \) gives probability scattering relative to absorption and so affects the magnitude of \( \langle n_{\text{scat}} \rangle_{\text{abs}} \), but it does not enter into the geometry of photon paths, possibly explaining why \( g^*(\tau) \) is independent of \( \omega \). Note that as the medium becomes optically thick (\( \tau \rightarrow \infty \)) then \( P \rightarrow 0 \) and then equation (C17) predicts that \( \langle n_{\text{scat}} \rangle \rightarrow \omega/(1 - \omega) \) from below, and so for \( g \leq g^*(\tau) \) we have that \( \langle n_{\text{scat}} \rangle_{\text{esc}} \leq \omega/(1 - \omega) \), which is effectively for all \( g \) since \( g^*(\tau) \rightarrow 1 \) as \( \tau \rightarrow \infty \).

APPENDIX D

ANALYSIS OF CLUMP OVERLAPS

To extend the mega-grains approximation to high filling factors, consider that as \( f_c \to 1 \) it is the increase of clump overlaps that makes the mega-grains model become unrealistic. Since we ignore the extra density occurring in overlaps, the effective radii of the clumps can be reduced to eliminate most of the overlapping and the number of clumps then must be increased to retain the same filling factor. The volume of clump overlaps is \( V(Q_c - f_c) \) and so the average overlap per clump is

\[
\frac{V(Q_c - f_c)}{N_c} = v_c \left( \frac{Q_c - f_c}{Q_c} \right),
\]

where we have applied equation (44) and \( v_c \) is the volume of just one clump. Suppose there is a pair of clumps that overlap in volume by the amount given in equation (D1), then to find the reduced radius at which the pair of clumps would not overlap we must solve

\[
v_c \frac{2}{3} \left( \frac{Q_c - f_c}{Q_c} \right) = \frac{\pi}{3} h^2(3r_c - h)
\]

for \( h \), the half-depth of the overlap, where \( r_c \) is the clump radius. The reduced radius that eliminates the overlap is then given by \( r_c - h \). We have solved equation (D2) numerically for the full range of filling factors \( 0 < f_c < 1 \) and find that \( r_c(1 - f_c) < r_c - h \) as \( f_c \to 1 \), thus eliminating the overlap. We propose to substitute \( r_c(1 - f_c)^3 \) for all instances of \( r_c \) in the mega-grains equations, where \( \gamma \) is an optional tuning parameter. In this new model, when \( \gamma = 1 \), the density of clumps is

\[
n_c = \frac{3f_c}{4 \pi r_c^3(1 - f_c)^3} > \frac{3Q_c}{4 \pi r_c^3}
\]

and is thus greater than originally defined, and we assume that the radius renormalization has effectively eliminated overlaps while preserving the filling factor.

APPENDIX E

DISTRIBUTION OF DUST TEMPERATURES AROUND A POINT SOURCE OF RADIATION

Given an isotropic point source of radiation in a spherically symmetric medium, the resulting distribution of radiation and heating of the dust is then a function of only the radial distance \( r \) from the point source. We will show that the probability
distribution of dust temperatures can be approximated by a power-law function if the dust density and absorbed luminosity versus \( r \) and the emitted luminosity versus dust temperature can be approximated by power-law functions. Letting \( N(r) \) be the number of dust grains in a sphere of radius \( r \), our approach is to obtain approximations for

\[
\frac{dN}{dT} = \frac{dN}{dr} \left( \frac{dT}{dr} \right)^{-1},
\]

which then gives the functional form for the distribution of temperatures.

Assume that the luminosity absorbed by the dust at distance \( r \) varies like an inverse power law with exponent \( \eta \),

\[
L_{\text{abs}}(r) \sim m(r)r^{-\eta},
\]

which in the optically thin case is nearly exact with \( \eta = 2 \), where \( m(r) \) is the mass of dust in a thin shell at radius \( r \). When the dust is optically thick we expect \( \eta > 2 \), because the absorbed luminosity decays more rapidly than any power law (\( \eta \) is then also a function of \( r \)). The luminosity emitted by the dust scales with the dust temperature, \( T(r) \), approximately as

\[
L_{\text{em}}[T(r)] \equiv 4\pi m(r) \int_0^\infty B_\nu[T(r)]\kappa_\nu \, dv \sim m(r)T(r)^{4+\beta},
\]

if the emissivity per unit mass of the dust scales with frequency like \( \kappa_\nu \sim \nu^\beta \), and usually \( 0 < \beta \leq 2 \). Equating the absorbed and emitted luminosities of the dust we get an approximate relationship between dust temperature and radial distance:

\[
T(r) \sim r^{-[(4+\beta)/\eta]},
\]

The rate of change in temperature with respect to radial distance then varies as

\[
\frac{dT}{dr} \sim r^{-[(1+(4+\beta)/\eta)]} \sim T^{[(1+(4+\beta)/\eta)]},
\]

where we have also inverted equation (E4) and substituted for \( r \).

Assume that the dust density is approximately a power law \( \rho(r) \sim r^{-\delta} \), so the number of dust grains, \( N(r) \), in a sphere of radius \( r \) also varies like a power-law function:

\[
N(r) = 4\pi \int_0^r \rho(s)^2 \, ds \sim r^{3-\delta}.
\]

Then

\[
\frac{dN}{dr} \sim r^{2-\delta} \sim T^{(\delta-2)((4+\beta)/\eta)},
\]

where we have again substituted for \( r \) using equation (E4). Combining equations (E5) and (E7) into equation (E1), we find

\[
\frac{dN}{dT} \sim T^{(\delta-2)((4+\beta)/\eta)-(1-(4+\beta)/\eta)} = T^{(\delta-3)((4+\beta)/\eta)-1}.
\]

From this scaling approximation, we presume that the distribution of dust temperatures follows a power law,

\[
p(T) = aT^\mu,
\]

with exponent

\[
\mu = (\delta - 3) \left( \frac{4+\beta}{\eta} \right) - 1,
\]

which is certainly negative as long as \( \delta \leq 3 \). The normalizing factor \( a \) is determined by requiring

\[
\int_{T_{\text{min}}}^{T_{\text{max}}} p(T) dT = 1,
\]

where \( T_{\text{max}} \) is usually the dust sublimation temperature (~1500 K) and \( T_{\text{min}} \) is the minimum dust temperature. Actually, the minimum dust temperature is the major free parameter in this theory and is determined by balancing the absorbed and emitted dust luminosities:

\[
\int_0^R L_{\text{abs}}(r)r^2 \, dr = M \int_{T_{\text{min}}}^{T_{\text{max}}} p(T)K(T) \, dT,
\]

where \( M \) is the total mass of dust contained in a sphere of radius \( R \), and where

\[
K(T) = \int_0^\infty B_\nu(T)\kappa_\nu \, dv
\]
is the Planck averaged emissivity at temperature $T$. Most of the emission comes from dust at temperatures near $T_{\text{min}}$ since those dust grains occupy most of the volume and the dust temperature decays rapidly with distance from the point source. Once $T_{\text{min}}$ is determined, the IR emission spectrum from the dust grains is given by

$$F_{\nu} = 4\pi M \kappa_{\nu} \int_{T_{\text{min}}}^{T_{\text{max}}} B_{\nu}(T)p(T)dT.$$  \hspace{1cm} (E14)

The behavior of the emissivity of graphite or silicates changes at wavelengths longer than about 10 $\mu$m, therefore we find that the power-law approximation of the emissivity averaged with a Planck function at a given temperature (proportional to the luminosity emitted by dust),

$$K(T) \sim T^{4 + \beta_i},$$  \hspace{1cm} (E15)

requires two exponents, $\beta_i$, with $i = 1$ for $T > T_b$ and $i = 2$ for $T \leq T_b$, where $T_b$ is called the emissivity break temperature, corresponding to the wavelength at which the behavior of the dust emissivity changes. From examination of the behavior of $K(T)$, the values of $\beta_i$ and $T_b$ are found to be approximately

$$\beta_i = \left\{ \begin{array}{l} (1.0, 2.0) \text{ (graphite)} \nonumber \\
(0.5, 2.0) \text{ (silicates)} \end{array} \right\},$$  \hspace{1cm} (E16)

and

$$T_b = \left\{ \begin{array}{l} 80 \text{ K (graphite)} \\
150 \text{ K (silicates)} \end{array} \right\}. $$  \hspace{1cm} (E17)

Consequently, the power-law distribution of temperatures around a central source will have an exponent depending on the dust temperature range:

$$\mu_i = (\delta - 3) \left( \frac{4 + \beta_i}{\eta} \right) - 1,$$  \hspace{1cm} (E18)

yielding power-law probability distributions

$$p_\nu(T) \equiv a_i T^{\mu_i},$$  \hspace{1cm} (E19)

where the constants $a_i$ are determined by requiring that $p_1(T_b) = p_2(T_b)$ and

$$\int_{T_b}^{T_{\text{max}}} p_1(T)dT + \int_{T_{\text{min}}}^{T_b} p_2(T)dT = 1,$$  \hspace{1cm} (E20)

yielding (defining $\gamma_i = \mu_i + 1$)

$$a_2 = a_1 T_b^{\gamma_2},$$

$$a_1 = \frac{\gamma_1 \gamma_2}{\gamma_2 T_{\text{max}}^{\gamma_1} + (\gamma_1 - \gamma_2)T_b^{\gamma_2} + \gamma_1 T_{\text{min}}^{\gamma_2} T_b^{\gamma_1 - \gamma_2}.}$$  \hspace{1cm} (E21)

This approximation for the temperature distribution and resulting IR emission spectrum were found to be in reasonable agreement with Monte Carlo simulations for the case of $\delta = 0$.

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