On the Fibonacci universality classes in nonlinear fluctuating hydrodynamics

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Abstract We present a lattice gas model that without fine tuning of parameters is expected to exhibit the so far elusive modified Kardar-Parisi-Zhang (KPZ) universality class. To this end, we review briefly how non-linear fluctuating hydrodynamics in one dimension predicts that all dynamical universality classes in its range of applicability belong to an infinite discrete family which we call Fibonacci family since their dynamical exponents are the Kepler ratios $z_i = F_{i+1}/F_i$ of neighbouring Fibonacci numbers $F_i$, including diffusion ($z_2 = 2$), KPZ ($z_3 = 3/2$), and the limiting ratio which is the golden mean $z_\infty = (1 + \sqrt{5})/2$. Then we revisit the case of two conservation laws to which the modified KPZ model belongs. We also derive criteria on the macroscopic currents to lead to other non-KPZ universality classes.

1 Introduction

It is well-known that in one dimension transport in stationary states is usually anomalous even when the microscopic interactions are short-ranged and noise is uncorrelated [15]. This property manifests itself in transport coefficients that diverge, usually algebraically, with system size, in contrast to normal, i.e., diffusive transport where the transport coefficients are material-dependent constants. Also the spatio-temporal fluctuations of the densities $\rho_\alpha(x,t)$ of globally conserved quantities $N_\alpha$ (such as mass, energy, etc.) are frequently characterized by non-diffusive scaling properties with dynamical exponents $z_\alpha \neq 2$, including the celebrated Kardar-Parisi-Zhang (KPZ) universality class where $z = 3/2$ [11, 27], and another universality class with $z = 3/2$ [1, 2, 17, 25].

If the large-scale behaviour of the fluctuations is dominated by the long wavelength modes of the conserved fields then the theory of nonlinear fluctuating hy-
Nonlinear fluctuating hydrodynamics (NLFH) predicts that, in a comoving frame with collective velocity $v_\alpha$, the normalized dynamical structure functions, i.e., the stationary correlations $S_\alpha(x,t) := \langle \phi_\alpha(x,t) \phi_\alpha(0,0) \rangle$ of the centered normal modes $\phi_\alpha(x,t) = \sum_\beta R_{\alpha \beta}(\rho_\beta(x,t) - \rho_\beta)$, have a scaling limit of the form $S_\alpha(x,t) = t^{-1/z_\alpha} f_\alpha((x - v_\alpha t)/(\lambda_\alpha t)^{z_\alpha})$. Here $f_\alpha(\cdot)$ is a universal scaling function that does not depend on the microscopic details of the interaction. Non-universal are the scale factors $\lambda_\alpha$ as well as the collective velocities and the coefficients $R_{\alpha \beta}$ that both depend on the stationary densities $\rho_\alpha = \langle \rho_\alpha(x,t) \rangle$ and the stationary currents $j_\alpha(\rho_1, \ldots, \rho_n)$ associated with the conserved quantities. For diffusion the scaling function is a Gaussian, while for the KPZ universality class one has the Prähofer-Spohn function \[21\].

Thus the main quantities of interest in the study of spatio-temporal fluctuations in one space dimension are the dynamical exponents $z_\alpha$ and the scaling functions $f_\alpha(\cdot)$. They determine the dynamical universality class that a given microscopic model belongs to. Remarkably, it was found \[19\] that the possible dynamical exponents are either sequences of the Kepler ratios $z_i = F_{i+1}/F_i$ of neighbouring Fibonacci numbers $F_i$ beginning with $z = 2 = 2/1$ or with $z = 3/2$, or the golden mean $z = (1 + \sqrt{5})/2$ which is the limiting value $i \to \infty$ of the Kepler sequence. Also the corresponding scaling functions have been determined, with one exception, which is the so-called modified Kardar-Parisi-Zhang universality class \[26\] for which, however, until now no generic microscopic model has been proposed.

In the following we briefly review the reasoning that leads to these predictions. We follow mainly the arguments put forward in Refs. \[25\] and \[20\] which lead, via mode coupling theory, to the conclusion that the dynamical universality class of a mode can be deduced from the above-mentioned macroscopic stationary current-density relation $j_\alpha(\rho_1, \ldots, \rho_n)$ through the so-called mode-coupling matrices (Sec. 2). Then we revisit the case of two conservation laws studied in some detail already in \[18\] and \[26\]. In Sec. 3 we construct a microscopic lattice gas model that, without fine-tuning of parameters, is predicted to be in the modified Kardar-Parisi-Zhang universality class. Finally, in Sec. 4, we present in a “consumer-friendly” fashion the criteria on the currents $j_\alpha(\rho_1, \ldots, \rho_n)$ under which only non-KPZ universality classes appear in systems with two conservation laws.

2 Nonlinear fluctuating hydrodynamics

2.1 Notation and general properties of fluctuations

In order to fix ideas we consider discrete microscopic models that evolve in continuous time $t$ and that have $n$ locally conserved quantities. By this we mean the following. Let $S$ be some set, $\Lambda$ denote a contiguous set of integers, and $\eta_k \in S$ with $k \in \Lambda$ be the local state variable. The index $k$ denotes a lattice site or a particle in a chain, depending on the type of model one has in mind. A microscopic configuration
The compressibility matrix $K_{\alpha \beta}$ at time $t$ is thus given by $\eta(t) = \{ \eta_k(t) : k \in \Lambda \}$ \textsuperscript{1} The generator of the dynamics is denoted by $\mathcal{L}$. The translation operator is denoted by $\mathcal{T}$ and defined by the property $\mathcal{T}(\eta_k) = \eta_{k+1}$, and similar for functions of the local state variables. We assume the dynamics to be translation invariant, i.e., $\mathcal{T}\mathcal{L} = \mathcal{L}\mathcal{T}$, with the identification $\eta_k \equiv \eta_{k+L}$ if $\Lambda$ is the integer torus.

In order to introduce conservation laws consider a cylinder function $\xi^\alpha_k(\eta)$ where $\alpha \in \{1, \ldots, n\}$ and $j^\alpha_k(\eta) \equiv \mathcal{T}^k(\xi^\alpha_k(\eta))$. We shall assume that (i) the $\xi^\alpha_k(\eta)$ satisfy the discrete continuity equations

$$\mathcal{L}(\xi^\alpha_k(\eta)) = j^\alpha_k(\eta) - j^\alpha_{k-1}(\eta)$$

for all $\alpha$, (ii) only the $\xi^\alpha_k(\eta)$ have this property, and (iii) that also the so-called microscopic currents $j^\alpha_k(\eta)$ are cylinder functions. We shall drop the dependence of the conserved quantities $\xi^\alpha_k$ and currents $j^\alpha_k$ on the configuration $\eta$.

Following \textsuperscript{10} we postulate that there exists a family of translation invariant grand-canonical measures parametrized by fugacities $\phi^\alpha$ which are translation invariant and invariant under the dynamics generated by $\mathcal{L}$. Expectations under this measure are denoted by $\langle \cdot \rangle$. In particular, we introduce the stationary conserved densities $\rho^\alpha := \langle \xi^\alpha_k \rangle$ and the stationary currents $j^\alpha_k := \langle j^\alpha_k \rangle$. The first and second derivatives of the currents $j^\alpha_k$ w.r.t. the densities $\rho^\alpha$, $\rho^\beta$ are denoted by $j^\alpha_\beta$ and $j^\alpha_\beta\gamma$.

The discrete fluctuation fields at time $t$ are the centered random variables $\bar{\rho}^\alpha_k(t) := \xi^\alpha_k(t) - \rho^\alpha$. The dynamical structure matrix $\bar{S}_k(t)$ is defined by the matrix elements

$$\bar{S}_k^{\alpha\beta}(t) := \langle \bar{\rho}^\alpha_k(t) \bar{\rho}^\beta_0 (0) \rangle. \text{ (2)}$$

The compressibility matrix $K_{\alpha \beta} = \sum_k \bar{S}_k(t)$ is the covariance matrix of the conserved quantities which is independent of time due to the conservation laws \textsuperscript{11}. The current Jacobian $J$ is the matrix with matrix elements $J_{\alpha \beta} = j^\alpha_\beta$ and the Hessians $H_{\alpha \beta \gamma}^\alpha = j^\alpha_\beta j^\alpha_\gamma$. The $n$-dimensional unit matrix is denoted by $\mathbb{1}$. Transposition of a matrix is denoted by the superscript $T$.

We make the mild (but essential !) assumptions on the invariant measure that for all $\alpha, \beta$ one has $K_{\alpha \beta} < \infty$ and $\lim_{n \to \infty} n(\bar{\rho}^\alpha_k \bar{\rho}^\beta_n) = 0$. These hypotheses imply the Onsager-type current symmetry \textsuperscript{10}

$$\frac{\partial j^\beta}{\partial \phi^\alpha} = \frac{\partial j^\alpha}{\partial \phi^\beta} \text{ (3)}$$

for all $\alpha, \beta$. \textsuperscript{2} In particular, the chain rule implies \textsuperscript{25}

$$JK = (JK)^T. \text{ (4)}$$

\textsuperscript{1} When the time $t$ is irrelevant we drop the dependence on $t$.

\textsuperscript{2} It seems to have gone unnoticed that, quite remarkably, this symmetry relates a purely static property of the invariant measure (the covariances $K_{\alpha \beta}$) with the microscopic dynamics which give rise to the currents $j^\alpha$. This restricts severely the possible microscopic dynamics for which a given measure can be invariant.
The current symmetry ensures that the current Jacobian has real eigenvalues which we denote by $v_\alpha$. Counterexamples with non-decaying correlations are models that exhibit phase separation [3, 12, 22]. It should be noted that the assumption of locality of the conserved quantity and the associated current as well as the finite number $n$ of conservation laws also rules out models with infinitely many and non-local conservation laws. Nevertheless, some of the phenomenology of such models seems to be similar to the finite and local case [14].

Throughout this work we shall assume complete absence of degeneracy of the eigenvalues $v_\alpha$. Then one can always write $V := RJR^{-1} = \text{diag}(v_1, \ldots, v_n)$. By convention we choose the normalization $RKR^T = 1$. (5)

The eigenmodes are defined to be the transformed fluctuation fields $\phi_\alpha^\beta(t) := \sum_\beta R_{\alpha \beta} \bar{\rho}_\beta(t)$. They give rise to the normal form of the dynamical structure matrix $S_k(t) := R \bar{S}_k(t) R^T$ (6) which satisfies $\sum_k S_k(t) = 1$ for all $t$. The conservation law, translation invariance and the mild decay of stationary correlations as assumed for the invariant measure yields the exact relation $d/dt \sum_k S_k(t) = V$ for all $t$.

We point out that the dynamical structure functions $S^\alpha\beta_k(t)$ have an alternative meaning as describing the relaxation of a microscopic perturbation of the invariant measure at the origin $k = 0$ [18]. As the perturbation evolves in time, it separates into distinct density peaks, one for each mode $\alpha$. The eigenvalues $v_\alpha$ are the center-of-mass velocities of these perturbations. The variance w.r.t. $k$ of the diagonal structure function $S^\alpha\alpha_k(t)$ describes, on lattice scale, the spatial spreading of mode $\alpha$.

2.2 Nonlinear fluctuating hydrodynamics

In the hydrodynamic limit where the “lattice” spacing tends to zero we denote the scaling limits of the density by $\rho_\alpha(x,t)$, $\bar{\rho}_\alpha(x,t)$, and $\phi_\alpha(x,t)$ resp. Under Eulerian scaling one expects from the law of large numbers and local stationarity [13] that the discrete continuity equation (1) gives rise to the system of conservation laws $\partial_t \rho_\alpha(x,t) + \partial_x j_\alpha(x,t) = 0$ where the currents $j_\alpha(x,t)$ depend on $x$ and $t$ only through the local densities $\rho_\alpha(x,t)$ via the stationary current-density relation. Thus one can write $j_\alpha(x,t) = j_a(\rho_1(x,t), \ldots, \rho_n(x,t))$ which gives

$$\partial_t \rho_\alpha(x,t) + \sum_\beta J^\alpha\beta(x,t) \partial_x \rho_\beta(x,t) = 0.$$  (7)

Non-degeneracy of $J$ implies that this nonlinear system of conservation laws is strictly hyperbolic. Obviously, the constant functions $\rho_\alpha(x,t) = \rho_\alpha$ form a translation invariant stationary solution.
In order to study fluctuations one expands around a fixed stationary solution and adds to the current a phenomenological diffusion term with diffusion matrix $D(p_1, \ldots, p_n)$ and Gaussian white noise $\tilde{\zeta}(x,t)$ with an amplitude that is usually taken to satisfy the fluctuation-dissipation theorem. Renormalization group arguments suggest that only terms up to second order in the density expansion are relevant. Third order terms may lead to logarithmic corrections to the fluctuations, but only if the second-order term vanishes. All higher order terms vanish in the scaling limit of large $x$ and large $t$ [5]. Thus, omitting arguments, one arrives at the non-linear fluctuating hydrodynamics equation [25]

$$\partial_t \tilde{\rho}_\alpha(x,t) + \partial_x \left[ \sum_\beta J_{\alpha\beta} \tilde{\rho}_\beta(x,t) + \frac{1}{2} \sum_{\beta, \gamma} \tilde{\rho}_\beta(x,t) H_{\beta\gamma}^{\alpha} \tilde{\rho}_\gamma(x,t) + \tilde{\zeta}_\alpha(x,t) \right] = 0 \tag{8}$$

with linear current operator $J_{\alpha\beta} = J_{\alpha\beta} - D_{\alpha\beta} \partial_t$.

In terms of the eigenmodes one has

$$\partial_t \phi_\alpha(x,t) + \partial_x \left[ \sum_\beta \tilde{V}_{\alpha\beta} \partial_x \phi_\beta(x,t) + \sum_{\beta, \gamma} \phi_\beta(x,t) G_{\beta\gamma}^{\alpha} \phi_\gamma(x,t) + \tilde{\zeta}_\alpha(x,t) \right] = 0 \tag{9}$$

with $\tilde{V}_{\alpha\beta} = v_\alpha \delta_{\alpha\beta} - D_{\alpha\beta} \partial_t$, where $D = DR^{-1}$, the symmetric mode coupling matrices

$$G^{\alpha} = \frac{1}{2} \sum_\chi R_{\alpha\chi} (R^{-1})^T H^\chi R^{-1}, \tag{10}$$

and transformed noise $\tilde{\zeta}_\alpha(x,t) = \sum_\lambda R_{\alpha\lambda} \tilde{\zeta}_\lambda(x,t)$ with covariance $\langle \tilde{\zeta}_\alpha(x,t) \tilde{\zeta}_\beta(x',t') \rangle = 2D_{\alpha\beta} \delta(x-x') \delta(t-t')$. One recognizes in [9] a system of coupled noisy Burgers equations which with the substitution $\tilde{\rho}_\alpha(x,t) = \partial_t \tilde{h}_\alpha(x,t)$ turns into a system of coupled KPZ equations [6, 7, 9, 24].

### 2.3 Mode coupling theory

Following [25] one writes the stochastic pde [9] in discretized form $\phi_\alpha(x,t) \rightarrow \phi_\alpha(n,t)$ with $n \in \mathbb{Z}$ in terms of a generator $L = L_0 + L_1$ where $L_0$ represents the linear part involving $\tilde{V}$ and $L_1$ represents the non-linear part involving the mode-coupling matrices $G^{\alpha}$. This yields $S_{\alpha\beta}(n,t) = \langle \phi_\beta(0,0) e^{L_1 \phi_\alpha(n,0)} \rangle$ and therefore

$$\frac{d}{dt} S_{\alpha\beta}(n,t) = \langle \phi_\beta(0,0) e^{L_1 \phi_\alpha(n,0)} \rangle + \langle \phi_\beta(0,0) \frac{d}{dt} e^{L_1 \phi_\alpha(n,0)} \rangle. \tag{11}$$

The discretization of the generator is chosen such that a product of mean-zero Gaussian measures for the $\phi_\alpha(n) \equiv \phi_\alpha(n,0)$ is invariant under the stochastic evolution.

We insert the identity $e^{L_1} = e^{L_1'} + \int_0^1 ds e^{L_0(s-t)} L_1 e^{L_0 s}$ into the second term on the r.h.s. of (11). The first contribution involving only the linear evolution vanishes since
by closer inspection one realizes that one is left with the expectation of cubic terms which are zero. The second contribution involves higher order correlators which due to the Gaussian measure can be factorized into pair correlations using the Wick rule. Finally, one replaces the bare evolution \(e^{L_0(t-s)}\) by the interacting evolution \(e^{L(t-s)}\) and takes the continuum limit. One arrives at the mode coupling equation [25]

\[
\partial_t S_{\alpha \beta}(x,t) = -v_\alpha \partial_x S_{\alpha \beta}(x,t) + \sum_{\gamma} D_{\alpha \gamma} \partial_x^2 S_{\gamma \beta}(x,t) + \int_0^t ds \int_{-\infty}^{\infty} dy \partial_y^2 \sum_{\gamma} M_{\alpha \gamma}(y,s) S_{\gamma \beta}(x-y, t-s)
\]

with the memory term

\[
M_{\alpha \gamma}(y,s) = 2 \sum_{\mu \nu} G_{\alpha \mu \nu} G_{\gamma \mu \nu} S_{\mu \mu}(y,s) S_{\nu \nu}(y,s).
\]

Next we recall that in the strictly hyperbolic case all modes drift with different velocities. Hence after time \(t\) their centers of mass are a distance of order \(t\) apart. On the other hand, the broadening of the peaks is expected to grow sublinearly. Hence, eventually the offdiagonal terms \(S_{\alpha \beta}(x, t)\) die out and can be neglected. With \(S_{\alpha}(x, t) \equiv S_{\alpha \alpha}(x, t)\) the mode-coupling equations thus reduce to

\[
\partial_t S_{\alpha}(x,t) = -v_\alpha \partial_x S_{\alpha}(x,t) + D_{\alpha \alpha} \partial_x^2 S_{\alpha}(x,t) + \int_0^t ds \int_{-\infty}^{\infty} dy \partial_y^2 M_{\alpha \alpha}(y,s) S_{\alpha}(x-y, t-s)
\]

with the memory term

\[
M_{\alpha \alpha}(y,s) = 2 \sum_{\mu \nu} \left( G_{\mu \nu} \right)^2 S_{\mu}(y,s) S_{\nu}(y,s).
\]

### 2.4 Fibonacci universality classes

It was found in [19, 20] that the mode coupling equations [14] have scaling solutions which can be obtained in explicit form after Fourier and Laplace transformation. In order to classify the solutions we recall the recursive definition of the Fibonacci numbers \(F_{n+1} = F_n + F_{n-1}\) with \(F_1 = F_2 = 1\) and introduce the set \(\mathbb{F}_\alpha := \{ \beta : G_{\beta \beta}^\alpha \neq 0 \}\) of modes \(\beta\) that give rise to a non-linear term in the time-evolution of mode \(\alpha\).

**Theorem 1 (Refs. [19, 20]).** Let \(u = pt^{1/z_\alpha}\) with dynamical exponent \(z_\alpha > 1\). Then

1. In Fourier representation \(\hat{S}_\alpha(p,t) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx e^{-i px} S_{\alpha}(x,t)\) the mode coupling equation [14] with memory term [15] has for finite \(|u|\) the scaling solution

\[
\lim_{t \to \infty} e^{i p t} \hat{S}_\alpha(p,t) = \hat{f}_\alpha(u)
\]

This theorem provides a classification of the solutions of the mode coupling equations, which can be understood in terms of Fibonacci numbers.
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where

- **Case 1:** For modes $\alpha$ such that $I_\alpha = \emptyset$ one has $z_\alpha = 2$ and $\hat{f}_\alpha(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} u^2}$ (diffusive universality class).

- **Case 2:** For modes $\alpha$ such that $I_\alpha \neq \emptyset$ and $\alpha \notin I_\alpha$ the dynamical exponents satisfy the nonlinear recursion $z_\alpha = \min_{\beta \in I_\alpha} \left( 1 + \frac{1}{z_\beta} \right)$ and the scaling function is given by $\hat{f}_\alpha(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} E_\alpha |u|^z_\alpha \left( 1 - i A_\alpha \tan \left( \pi z_\alpha^2 u / |u| \right) \right)}$ with explicit real constants $E_\alpha > 0$, $A_\alpha \in [-1, 1]$ given in [20] (Lévy universality class).

- **Case 3:** If $\alpha \in I_\alpha$ then $z_\alpha = 3/2$. (a) If there is no diffusive mode $\beta \in I_\alpha$, then $\hat{f}_\alpha(u) = \hat{f}_{\text{MCT}}^{\text{KPZ}}(u)$ given in [20] (KPZ universality class). (b) If there is at least one diffusive mode $\beta \in I_\alpha$, then $\hat{f}_\alpha(u) = \hat{f}_{\text{MCT}}^{\text{mKPZ}}(u)$ given in [20] (modified KPZ universality class).

(2) The non-linear recursion for the dynamical exponents in case 2 has as unique solution the sequence of Kepler ratios of Fibonacci numbers $z_i = \frac{F_{i+1}}{F_i}$, starting from $z_3 = 3/2$ (if at least one diffusive mode $\beta \in I_\alpha$), or $z_4 = 5/3$ (if no diffusive mode but at least one KPZ mode $\beta \in I_\alpha$) or else the golden mean $z_i = \frac{1 + \sqrt{5}}{2}$ for all modes $i$.

Remark 1. The subballistic scaling $z_\alpha > 1$ is motivated by the locality of interactions, conservation laws and currents. Since all dynamical exponents that can appear are Kepler ratios of neighbouring Fibonacci numbers we call the whole family of universality classes comprising diffusion, Lévy, KPZ and modified KPZ the Fibonacci universality classes.

The main ingredients in the proof of item (a) are strict hyperbolicity and power counting of the leading singularities in the Fourier-Laplace representation of the mode-coupling equation. Item (b) follows from the recursion of the Fibonacci numbers by a judiciously chosen ordering of the modes belonging to case 2.

We stress that the theorem deals with the function $S_\alpha(x,t)$ satisfying the mode coupling equations (14). There is no general rigorous result how this function relates to the true scaling limit of the dynamical structure function $S_\alpha^{\text{true}}(t)$. However, in the diffusive case 1 one expects the Gaussian scaling function to be the true scaling limit, up to possible logarithmic corrections. For specific models there are numerical [17,18,19] and mathematically rigorous results [1,2] that suggest that the true scaling form in case 2 is indeed generally a Lévy distribution. However, the coefficients $A_\alpha, E_\alpha$ arising from the mode-coupling equations are not believed to correspond to the true values. The scaling limit of $S_\alpha(x,t)$ has a closed expression in Fourier-Laplace representation also in case 3. However, for the case of a single conservation law (the usual KPZ universality class) it is known that this scaling function, studied in detail in [4,8], is not exact but rather given by the Prähofer-Spohn scaling function [21]. Correspondingly, one does not expect the scaling forms $\hat{f}_{\text{MCT}}^{\text{KPZ}}(u)$ and $\hat{f}_{\text{MCT}}^{\text{mKPZ}}(u)$ solving the mode coupling equations (14) to exact either.

With these provisos Theorem 1 shows that, according to the mode coupling theory reviewed above, the dynamical universality classes of all modes are fully de-
terminated by whether or not the diagonal elements of the mode coupling matrices vanish. This in turn is fully determined by the stationary current-density density relation. Thus, as long as mode-coupling theory is valid, one can read off from this macroscopic stationary quantity alone the dynamical universality classes of all modes.

3 Two-lane lattice gas for the modified KPZ universality class

The modified KPZ universality class \[26\] arises for \[G_{11}^2 = G_{12}^2 = 0 \quad \text{and} \quad G_{22}^2 \neq 0\], see case 3b) in Theorem 1. However, so far no microscopic model with this property has been proposed. Here we present a two-lane lattice gas that belongs to case 3b) in Theorem 1 for all values of the conserved densities without fine-tuning of model parameters. This model consists of two coupled one-dimensional lattice gases without self-interaction, but where the jump rates between sites \(k\) and \(k+1\) of the particles of one gas depend on the number of particles of the other gas on the same pair of sites. It is convenient to describe this a model as a two-lane lattice gas model in the spirit of the multi-lane exclusion processes of Popkov and Salerno \[16\], but without requiring exclusion.

We denote the number of particles on site \(k\) on lane \(i\) as particles of type \(n_i^k\). Thus the local state variable is the pair \(\eta_k = (n_1^k, n_2^k) \in \mathbb{N}_0^2\). We introduce the parameters

\[
\begin{align*}
  r_i &= \frac{1}{2} (w_i + f_i), \quad 
  \ell_i &= \frac{1}{2} (w_i - f_i), \quad 
  g^\pm &= \frac{1}{2} (\alpha \pm \gamma) 
\end{align*}
\]

with strictly positive constants \(w_i, \alpha > 0\), and \(|f_i| \leq w_i, \gamma | \leq \alpha\). We also define the mean pair occupation numbers

\[
\bar{n}_k^i = \frac{1}{2} (n_k^i + n_{k+1}^i). 
\]

Informally, the stochastic dynamics is then defined as follows. A particle on lane 1 jumps independently of all other particles on lane 1 after an exponential waiting time from site \(k\) to site \(k+1\) of lane 1 with rate \(r^1(\bar{n}^2_k) = r_1 + g^+ \bar{n}^2_k\) and from site \(k+1\) to \(k\) with rate \(\ell^1(\bar{n}^2_k) = \ell_1 + g^- \bar{n}^2_k\). Likewise, particles on lane 2 jump with rates \(r^2(\bar{n}^1_k) = r_2 + g^+ \bar{n}^1_k\) and from site \(k+1\) to \(k\) with rate \(\ell^2(\bar{n}^1_k) = \ell_2 + g^- \bar{n}^1_k\). Thus \(w_i\) are the “bare” jump rates (i.e., in the absence of interaction), \(f_i\) are the bare jump biases, \(\alpha\) is the interaction strength and \(\gamma\) is the interaction asymmetry.

With the updated state variable

\[
\eta_l^{i,k,k'} = \begin{cases} 
  n_k^l - 1 & \text{if } l = k \\
  n_k^l + 1 & \text{if } l = k' \\
  n_k^l & \text{else}
\end{cases}
\]

the generators for independent random walkers read
\[ \mathcal{L}_i = \sum_k \left[ \ell_i \left( f(\eta^{i,k+1}) - f(\eta) \right) + \ell_i \left( f(\eta^{i,k+1}) - f(\eta) \right) \right]. \]  \tag{20} 

The interaction between the lanes is given by the generator
\[ \mathcal{L}_i (f(\eta)) = \sum_k \left\{ \tilde{\eta}_k^2 \left[ g^+ f(\eta^{i,k+1}) + g^- f(\eta^{i,k+1}) \right] - (g^+ + g^-) f(\eta) \right\} \]  \tag{21} 

The generator of the full interacting process is then
\[ \mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_1. \]  \tag{22} 

The interaction between the lanes does not change the invariant measure of the non-interacting part. Thus one arrives at

**Proposition 1.** For parameters \( \rho_{1,2} \geq 0 \) the product measure with factorized Poisson marginals
\[ \mu(\eta_k) = \frac{\rho_1^{n_1} e^{-\rho_1} \rho_2^{n_2} e^{-\rho_2}}{(n_1)! (n_2)!} \]  \tag{23} 

for the occupation at site \( k \) is a translation invariant measure of the process defined by the generator \( \mathcal{L}_1 \).

**Proof.** We prove the proposition for the finite torus \( \Lambda = \{1, \ldots, L\} \) with \( L \) sites in the quantum operator formalism \[23,24\]. Since \( \mathcal{L}_1 \) is invariant for the non-interacting part \( \mathcal{L}_1 + \mathcal{L}_2 \) of the generator one needs to prove only invariance under \( \mathcal{L}_1 \). For \( n \in \mathbb{N}_0 \) let \( |n\rangle \) be the infinite-dimensional vector with components \( \langle n \rangle_i = \delta_i \) and define the tensor vectors \( |n^1, n^2\rangle = |n^1\rangle \otimes |n^2\rangle \). Furthermore, define the matrices \( a^\pm \) and \( \hat{n} \) by \( a^+ |n\rangle = |n+1\rangle \), \( a^- |n\rangle = n|n-1\rangle \), \( \hat{n} |n\rangle = n|n\rangle \), \( 1|n\rangle = n|n\rangle \) and also the tensor products \( a^1 \otimes a^2 = a^1 \otimes a^2, \hat{n}^1 = \hat{n} \otimes 1, \hat{n}^2 = 1 \otimes \hat{n}, 1_{12} = 1 \otimes 1 \) and \( X_k = 1_{12}^{L-k} \otimes X \otimes 1_{12}^{L-k} \) where \( X \) is any of the two-fold tensor products just defined. The matrix form \( \hat{H}_i \) of the generator \( \mathcal{L}_1 \) is then given by \( \hat{H}_i = \sum_k (g^1_k + g^2_k) \) with

\[ g^1_k = -\frac{1}{2} \left\{ (\hat{n}_k^2 + \hat{n}_{k+1}^2) \left[ g^+ \left( a_{k+1}^{1,+} - a_k^{1,+} \right) + g^- \left( a_{k+1}^{1,-} a_k^{1,-} - \hat{n}_{k+1}^1 \right) \right] \right\} \]  \tag{24} 
\[ g^2_k = -\frac{1}{2} \left\{ (\hat{n}_k^1 + \hat{n}_{k+1}^1) \left[ g^+ \left( a_{k+1}^{2,+} - a_k^{2,+} \right) + g^- \left( a_{k+1}^{2,-} a_k^{2,-} - \hat{n}_{k+1}^2 \right) \right] \right\}. \]  \tag{25} 

Let \( |\rho\rangle = e^{-\rho} \sum_{n=0}^{\infty} \rho^n / (n!) |n\rangle \) and \( |\rho_1, \rho_2\rangle = (|\rho_1\rangle \otimes |\rho_2\rangle)^{\otimes L} \) for \( \rho_{1,2} \geq 0 \). One has \( a_k^{\alpha,+} |\rho_1, \rho_2\rangle = \hat{n}_k^\alpha / \rho_\alpha \) and \( a_k^{\alpha,-} |\rho_1, \rho_2\rangle = \rho_\alpha |\rho_1, \rho_2\rangle \). \[a_k^{\alpha,+} |\rho_1, \rho_2\rangle = (\hat{n}_k^\alpha - \hat{n}_{k+1}^\alpha) |\rho_1, \rho_2\rangle \]  \tag{26} 
\[a_k^{\alpha,-} |\rho_1, \rho_2\rangle = (\hat{n}_k^\alpha - \hat{n}_{k+1}^\alpha) |\rho_1, \rho_2\rangle \]  \tag{27}
and therefore
\[ g^1_k |\rho_1, \rho_2\rangle = -\frac{\gamma}{2} (\hat{n}^2_k + \hat{n}^3_{k+1}) (\hat{n}^1_{k+1} - \hat{n}^1_k) |\rho_1, \rho_2\rangle \]  
\[ g^2_k |\rho_1, \rho_2\rangle = -\frac{\gamma}{2} (\hat{n}^1_k + \hat{n}^1_{k+1}) (\hat{n}^2_{k+1} - \hat{n}^2_k) |\rho_1, \rho_2\rangle. \]  
(28)  
(29)

The telescopic property of the lattice sum then yields \( H_I |\rho_1, \rho_2\rangle = 0. \)

In Proposition 1 the parameters \( \rho_i = \langle n^i_k \rangle \) are the conserved stationary densities and one finds immediately the compressibility matrix \( K \) with matrix elements \( K_{\alpha \beta} = \rho_\alpha \delta_{\alpha \beta} \). From the definition (22) of the generator one computes the microscopic currents defined up to an irrelevant constant by the discrete continuity equation (1). The factorization of the invariant measure then yields the stationary current-density relation
\[ j^1(\rho_1, \rho_2) = \rho_1 (f_1 + \gamma \rho_2), \quad j^2(\rho_1, \rho_2) = \rho_2 (f_2 + \gamma \rho_1). \]  
(30)

Remarkably, the stationary currents depend only on the hopping biases \( f_1, f_2 \) and the interaction asymmetry \( \gamma \), not on the interaction strength \( \alpha \).

The main result is the following.

**Theorem 2.** Define the quantities
\[ \Delta := \sqrt{(f_1 - f_2 + \gamma (\rho_2 - \rho_1))^2 + 4\gamma^2 \rho_1 \rho_2} \]  
\[ \xi := \frac{\Delta - (f_2 - f_1 + \gamma (\rho_1 - \rho_2))}{2\gamma \sqrt{\rho_1 \rho_2}}, \quad \gamma := \frac{f_2 - f_1 + \gamma (\rho_1 - \rho_2)}{2\gamma \sqrt{\rho_1 \rho_2}}. \]  
(31)  
(32)

For all bare hopping rates \( w_{1,2}, f_{1,2} \), all strictly positive densities \( \rho_{1,2} \) and all non-zero interaction parameters \( \alpha, \gamma \), the current Jacobian of the process defined by (22) with invariant measure given in Proposition 1 is non-degenerate and has eigenvalues
\[ v_{\pm} = \frac{1}{2} (f_1 + f_2 + \gamma (\rho_1 + \rho_2) \pm \Delta). \]  
(33)

The mode coupling matrices (10) where mode 1 (2) has collective velocity \( v_+ (v_-) \) are given by
\[ G^\alpha = -g^\alpha \begin{pmatrix} -1 & y \\ y & 1 \end{pmatrix} \]  
(34)

with
\[ g_1 = \rho_1 \sqrt{\frac{\gamma \xi}{\Delta^3}} (\gamma (\rho_1 + \rho_2) + \Delta + f_2 - f_1) \]  
\[ g_2 = \rho_2 \sqrt{\frac{\gamma \xi}{\Delta^3}} (\gamma (\rho_1 + \rho_2) - (\Delta + f_2 - f_1)). \]  
(35)  
(36)

\(^3\) The product measure (23) remains invariant also for different interaction strength \( \alpha_1 \neq \alpha_2 \) which leaves the currents unchanged. However, equal interaction asymmetry is required.
Remark 2. For $\gamma = 0$ one has $g_1 = g_2 = 0$ for all $f_1, f_2, \rho_1, \rho_2$ so that both modes are diffusive with drift velocities $f_{1,2}$. For interaction asymmetry $\gamma > 0$ and strictly positive densities the drift velocities of the two modes are different even for equal individual bare hopping asymmetries $f_1 = f_2$.

Remark 3. For $\gamma > 0$ and strictly positive densities one has $g_1 \neq 0$ for all $f_1, f_2$. On the other hand, $g_2 \neq 0$ if and only if $f_1 \neq f_2$. Thus according to case 3 of Theorem 1 one expects that for any $\gamma > 0$ and strictly positive densities $\rho_1, \rho_2$ both modes are KPZ if $f_1 \neq f_2$ whereas for equal asymmetries $f_1 = f_2$ mode 1 is modified KPZ and mode 2 is diffusive, without fine-tuning of parameters.

Remark 4. The offdiagonal elements of the mode coupling matrices vanish for $j_1 = j_2$, which is equivalent to $f_1 - f_2 = \gamma (\rho_1 - \rho_2)$.

Proof. The proof of Theorem 2 is computational. From the current-density relation \ref{eq:30} one obtains the current Jacobian
\[ J = \begin{pmatrix} f_1 + \gamma \rho_2 & \gamma \rho_1 \\ \gamma \rho_2 & f_2 + \gamma \rho_1 \end{pmatrix} \] \hfill \tag{37}

Solving the eigenvalue equation proves \ref{eq:31}. With
\[ u = \sqrt{\frac{\rho_1}{\rho_2}} \] \hfill \tag{38}

the diagonalizing matrix defined by
\[ V := RJR^{-1} = \begin{pmatrix} v_+ & 0 \\ 0 & v_- \end{pmatrix} \] \hfill \tag{39}

is computed to be
\[ R = \begin{pmatrix} x_+ & x_+ \xi^{-1} u \\ -x_- \xi^{-1} u & x_- \end{pmatrix} \] \hfill \tag{40}

with $\det(R) = x_+ x_- (1 + \xi^{-2})$ and with free parameters $x_\pm$.

The parameters $x_\pm$ are fixed by the normalization condition \ref{eq:5}. One finds
\[ x_+ = \frac{1}{\sqrt{\rho_1 (1 + \xi^{-2})}} = \sqrt{\frac{\gamma_\xi}{\rho_1}} \\ x_- = \frac{1}{\sqrt{\rho_2 (1 + \xi^{-2})}} = \sqrt{\frac{\gamma_\xi}{\rho_2}} \] \hfill \tag{41}

For the Hessians one finds
\[ H^1 = H^2 = \gamma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = : H \] \hfill \tag{42}

and therefore
\[ (R^{-1})^T H R^{-1} = 2 \gamma \rho_1 \rho_2 \frac{\gamma}{\delta} \begin{pmatrix} 1 & -y \\ -y & -1 \end{pmatrix} \] \hfill \tag{43}

The definition \ref{eq:10} then yields \ref{eq:34}. \qed
4 Criterion for Lévy universality classes for systems with two conservation laws

In the case of two conservation laws we denote the universality classes of the two modes by a pair \((\cdot,\cdot)\) where the possible entries are \(D\) for diffusion \((z = 2)\), \(2L\) for the Lévy universality class with \(z = 3/2\) and \(GM\) for the Lévy universality class with the golden mean \(z = (1 + \sqrt{5})/2\).

4.1 Diagonalization of the current Jacobian

Let

\[
J = \begin{pmatrix} j_1^1 & j_1^2 \\ j_2^1 & j_2^2 \end{pmatrix}
\] (44)

be a current Jacobian. The two eigenvalues of \(J\) are

\[
v_{\pm} = \frac{1}{2} \left( j_1^1 + j_2^2 \pm \Delta \right)
\] (45)

with

\[
\Delta = \sqrt{(j_1^1 - j_2^2)^2 + 4j_1^2 j_2^1}.
\] (46)

We consider only the strictly hyperbolic case \(\Delta > 0\). The diagonalizer \(R\) with the property (39) reads

\[
R = \begin{pmatrix} x_+ & \frac{2j_1^2}{\Delta + (j_1^1 - j_2^2)} \\ -x_- & \frac{2j_2^1}{\Delta + (j_1^1 - j_2^2)} \end{pmatrix}
\] (47)

with constants \(x_{\pm}\) satisfying \(x_+ x_- \neq 0\) and to be chosen such that \(R\) has well-defined limits \(j_2^1 \to 0\) or \(j_2^2 \to 0\).

4.2 Non-KPZ universality classes

According to cases 1 and 2 in Theorem 1 mode coupling theory predicts two non-KPZ universality classes for \(G_{11}^1 = G_{22}^2 = 0\) and specifically

- \((DD)\) if and only if \(G_{22}^1 = G_{11}^2 = 0\)
- \((D, 2L)\) if and only if \(G_{22}^1 = 0, G_{11}^2 \neq 0\)
- \((2L, D)\) if and only if \(G_{22}^1 \neq 0, G_{11}^2 = 0\)
- \((GM, GM)\) if and only if \(G_{22}^1 \neq 0, G_{11}^2 = 0\).
The diagonal elements of the mode coupling matrices have been computed explicitly in [18], albeit in a form that does not directly express them in terms of the current-density relation. Moreover, the expressions in [18] depend on the normalization factors $x_{\pm}$ in [47] fixed by (5), which, however, is irrelevant with regard to whether a diagonal element is zero or not and therefore irrelevant to the question which universality class one expects. A more “user-friendly” form that expresses the conditions on the various allowed universality classes directly in terms of the current derivatives is the following result.

**Theorem 3.** Let $J$ be a current Jacobian and $G^\alpha$ be mode coupling matrices as defined in (10). Then one has the generic non-KPZ conditions $G_{12}^1 = G_{21}^2 = 0$ if and only if

\begin{align}
J_1^2 (2j_{12}^2 + j_{22}^2) + j_2^1 j_{11}^2 - j_1^1 (j_{22}^2 - j_{11}^2) &= 0 \\
J_2^2 (2j_{22}^1 + j_{11}^1) + j_1^2 j_{22}^1 + j_{22}^2 (j_{11}^2 - j_{11}^1) &= 0
\end{align}

and the specific conditions

\begin{align}
(D,D) \iff j_1^2 j_{12}^2 + j_2^1 j_{11}^1 &= j_1^2 j_{22}^1 + j_2^1 j_{11}^1 = 0 \\
\end{align}

for two diffusive modes,

\begin{align}
(D,\frac{1}{2}L) \iff (j_1^2)^2 j_{12}^2 + j_2^1 j_{11}^1 &= \frac{1}{2} (j_1^2 - j_2^2 - \delta) (j_1^2 j_{22}^1 + j_2^1 j_{11}^1) \\
(\frac{1}{2}L,D) \iff (j_1^2)^2 j_{12}^2 + j_2^1 j_{11}^1 &= \frac{1}{2} (j_1^2 - j_2^2 + \delta) (j_1^2 j_{22}^1 + j_2^1 j_{11}^1)
\end{align}

for the mixed case with one diffusive and one $3/2$-Lévy mode, and

\begin{align}
(GM,GM) \iff (j_1^2)^2 j_{12}^2 + j_2^1 j_{11}^1 &\neq \frac{1}{2} (j_1^2 - j_2^2 \pm \delta) (j_1^2 j_{22}^1 + j_2^1 j_{11}^1)
\end{align}

for two golden mean Lévy modes.

**Proof.** We invert (10) to find $2R^T G^\alpha R = R_{\alpha 1} H^1 + R_{\alpha 2} H^2$. Requiring the the generic non-KPZ conditions $G_{12}^1 = G_{21}^2 = 0$ and using that the mode coupling matrices and the Hessians are symmetric leads to six independent equations involving the current derivatives. In term of the parameters

\begin{align}
u = \sqrt{\frac{j_2^1}{j_1^2}}, \quad \xi = \frac{\Delta - (j_2^2 - j_1^1)}{2 \sqrt{j_2^1 j_1^2}}, \quad y = \frac{j_2^2 - j_1^1}{2 \sqrt{j_2^1 j_1^2}}
\end{align}

they read

\begin{align}
\xi u^{-1} j_{11}^1 + j_{11}^2 &= 2G_{22}^1 \frac{(x-\Delta \xi)^2}{x^2 + u^2 \xi^{-1}} \left( \frac{u^{-1}}{\xi} \right)^2 - 4G_{12} x u^{-2}
\end{align}
\[ \xi u^{-1} j_{12}^1 + j_{12}^2 = -2G_{12}^{(1)} \frac{(x_\perp)^2}{x_+ u \xi^{-1}} \frac{u^{-1}}{\xi} - 4G_{12}^{(1)} x_+ u\xi^{-1} \frac{1 - \xi^2}{2\xi} \] 
\[ \xi u^{-1} j_{22}^1 + j_{22}^2 = 2G_{12}^{(1)} \frac{(x_\perp)^2}{x_+ u \xi^{-1}} + 4G_{12}^{(1)} x_- \] 
\[ \frac{u^{-1}}{\xi} j_{12}^1 - j_{12}^1 = -2G_{11}^{(1)} \frac{(x_+ u \xi^{-1})^2}{x_-} \left( \xi u^{-1} \right)^2 + 4G_{12}^{(1)} x_+ u\xi^{-1} \frac{1}{u^2} \] 
\[ \frac{u^{-1}}{\xi} j_{12}^2 - j_{12}^2 = -2G_{11}^{(1)} \frac{(x_+ u \xi^{-1})^2}{x_-} \xi u^{-1} + 4G_{12}^{(1)} x_+ u\xi^{-1} \frac{1}{u^2} \frac{1 - \xi^2}{2\xi} \] 
\[ \frac{u^{-1}}{\xi} j_{22}^1 - j_{22}^1 = -2G_{11}^{(1)} \frac{(x_+ u \xi^{-1})^2}{x_-} - 4G_{12}^{(1)} x_+ u\xi^{-1} \] 

Since \(G_{12}^{(1)}\) and \(G_{12}^{(2)}\) are arbitrary, we can introduce arbitrary new constants
\[ A = -\frac{4G_{12}^{(1)} x_- + 4G_{12}^{(2)} x_+ u\xi^{-1}}{\xi^{-1} + \xi}, \quad B = -\frac{4G_{12}^{(1)} x_- \xi^{-1} - 4G_{12}^{(2)} x_+ u\xi^{-1} \xi}{\xi^{-1} + \xi} \]
so that the six non-KPZ equations become
\[ j_{11}^1 = u^{-1} \left( \frac{2G_{12}^{(1)} \frac{(x_\perp)^2}{x_+ u \xi^{-1}} \xi^{-2} - 2G_{11}^{(1)} \frac{(x_+ u \xi^{-1})^2}{x_-} \xi^2}{\xi^{-2} + \xi} + A \right) \] 
\[ j_{12}^1 = \frac{-2G_{12}^{(1)} \frac{(x_\perp)^2}{x_+ u \xi^{-1}}}{\xi^{-2} + \xi} \xi^{-1} - 2G_{11}^{(1)} \frac{(x_+ u \xi^{-1})^2}{x_-} \xi^2 + A \frac{1 - \xi^2}{2\xi} \] 
\[ j_{22}^1 = u \left( \frac{2G_{12}^{(1)} \frac{(x_\perp)^2}{x_+ u \xi^{-1}} - 2G_{11}^{(1)} \frac{(x_+ u \xi^{-1})^2}{x_-}}{\xi^{-2} + \xi} \right) - A \] 
\[ j_{11}^2 = u^{-2} \left( \frac{2G_{12}^{(1)} \frac{(x_\perp)^2}{x_+ u \xi^{-1}} \xi^{-3} + 2G_{11}^{(1)} \frac{(x_+ u \xi^{-1})^2}{x_-} \xi^3}{\xi^3 + \xi^{-1}} + B \right) \] 
\[ j_{12}^2 = u^{-1} \left( \frac{-2G_{12}^{(1)} \frac{(x_\perp)^2}{x_+ u \xi^{-1}} \xi^{-2} - 2G_{11}^{(1)} \frac{(x_+ u \xi^{-1})^2}{x_-} \xi^2}{\xi^2 + \xi^{-1}} \right) + B \frac{1 - \xi^2}{2\xi} \] 
\[ j_{22}^2 = \frac{2G_{12}^{(1)} \frac{(x_\perp)^2}{x_+ u \xi^{-1}} \xi^{-1} + 2G_{11}^{(1)} \frac{(x_+ u \xi^{-1})^2}{x_-} \xi}{\xi^2 + \xi^{-1}} - B. \]

Now we use the fact that \(\xi^\pm 1 = \sqrt{1 + y^2} \mp y\) to write
\[ 1 = \xi^{\pm 1} = \pm 2y, \xi^\pm 3 = \xi^\pm 1 \mp 2y. \]
Defining
\[ C = 2 \left( \frac{G_1^2(x_u^2) - G_1^2(x_u u^2)}{\xi + \xi^{-1}} \right), \quad D = 2 \left( \frac{G_2^2(x_u^2) + G_2^2(x_u u^2)}{\xi + \xi^{-1}} \right) \] (69)
this yields
\[ \frac{2G_1^2(x_u^2) - 2G_1^2(x_u u^2)}{\xi + \xi^{-1}} = C - 2yD \] (70)
\[ \frac{2G_2^2(x_u^2) - 2G_2^2(x_u u^2)}{\xi + \xi^{-1}} = D + 2yC. \] (71)

Thus the six non-KPZ conditions can be recast in the form
\[ j_{11}^1 = u^{-1}(C + A) \] (72)
\[ j_{12}^1 = -D + yA \] (73)
\[ j_{22}^1 = u(C - 2yD - A) \] (74)
\[ j_{11}^2 = u^{-2}(D + 2yC + B) \] (75)
\[ j_{12}^2 = u^{-1}(-C + yB) \] (76)
\[ j_{22}^2 = D - B \] (77)

Next we choose the arbitrary functions as
\[ A = u j_{11}^1 - C, \quad B = D - j_{22}^2 \] (78)
to obtain
\[ j_{12}^1 = -(D + yC) + yu j_{11}^1 \] (79)
\[ j_{22}^1 = 2u(C - yD) - u^{2} j_{11}^1 \] (80)
\[ j_{11}^2 = 2u^{-2}(D + yC) - u^{-2} j_{22}^2 \] (81)
\[ j_{12}^2 = -u^{-1}(C - yD) - yu^{-1} j_{22}^2. \] (82)

With the short-hands
\[ F_1 = 2u(C - yD), \quad F_2 = 2(D + yC) \] (83)
these equations take the form
\[ 2j_{12}^1 + j_{22}^2 + u^2 j_{11}^2 - 2uy j_{11}^1 = 0 \] (84)
\[ 2j_{11}^2 + j_{12}^1 + u^{-2} j_{22}^2 + 2u^{-1} y j_{22}^2 = 0 \] (85)
\[ j_{22}^2 + u^2 j_{11}^1 = F_1 \] (86)
\[ j_{22}^2 + u^2 j_{11}^1 = F_2 \] (87)
Next we observe

\[
F_1 = u \left( 2G_{22}^1 \frac{(x_-)^2}{x_+ u \xi^{-1}} - 2G_{11}^2 \frac{(x_+ u \xi^{-1})^2}{x_-} \right) \tag{88}
\]

\[
F_2 = 2G_{22}^1 \frac{(x_-)^2}{x_+ u \xi^{-1}} \xi^{-2} + 2G_{11}^2 \frac{(x_+ u \xi^{-1})^2}{x_-} \xi^2. \tag{89}
\]

Thus, by setting the respective diagonal elements $G_{\alpha\beta\beta}$ to zero,

\[
(D, D) : G_{22}^1 = G_{11}^2 = 0 \Rightarrow F_1 = F_2 = 0 \tag{90}
\]

\[
(D, \frac{3}{2}L) : G_{22}^1 = 0, G_{11}^2 \neq 0 \Rightarrow F_1 = -\xi^{-1} uF_2 \neq 0 \tag{91}
\]

\[
(\frac{3}{2}L, D) : G_{22}^1 \neq 0, G_{11}^2 = 0 \Rightarrow F_1 = \xi uF_2 \neq 0 \tag{92}
\]

\[
(GM, GM) : G_{22}^1 \neq 0, G_{11}^2 \neq 0 \Rightarrow F_1 \neq \pm u\xi^\pm 1 F_2. \tag{93}
\]

In terms of the derivatives one has

\[
uy = \frac{j_2^2 - j_1^1}{2j_1^1}, \quad u^{-1}y = \frac{j_2^2 - j_1^1}{2j_2^2}
\]

\[
u\xi = \frac{\delta - (j_2^2 - j_1^1)}{2j_2^1}, \quad u\xi^{-1} = \frac{\delta + (j_2^2 - j_1^1)}{2j_1^1}
\]

which yields the conditions (48) - (53) as stated in the theorem. Conversely, one proves that the required diagonal elements vanish by assuming the conditions to be valid and using the definition (10) of the mode coupling matrices. □

References

1. C. Bernardin and P. Gonçalves, Anomalous fluctuations for a perturbed Hamiltonian system with exponential interactions. Commun. Math. Phys. 325, 291–332 (2014).

2. C. Bernardin, P. Gonçalves, and M. Jara, 3/4-fractional superdiffusion in a system of harmonic oscillators perturbed by a conservative noise, Arch. Rational Mech. Anal. 220, 505–542 (2016).

3. S. Chakraborty, S. Pal, S. Chatterjee and M. Burma, Large compact clusters and fast dynamics in coupled nonequilibrium systems, Phys. Rev. E 93, 050102(R) (2016).

4. P. Colaiori and M.A. Moore, Numerical Solution of the Mode-Coupling Equations for the Kardar-Parisi-Zhang Equation in One Dimension Phys. Rev. E 65, 017105 (2001).

5. P. Devillard and H. Spohn, Universality class of interface growth with reflection symmetry. J. Stat. Phys. 66, 1089–1099 (1992).

6. D. Ertaş and M. Kardar, Dynamic relaxation of drifting polymers: A phenomenological approach, Phys. Rev.E 48, 1228–1245 (1993).

7. P.L. Ferrari, T. Sasamoto and H. Spohn, Coupled Kardar-Parisi-Zhang equations in one dimension. J. Stat. Phys. 153, 377–399 (2013).
8. E. Frey, U.C. Täuber, T. Hwa, Mode-coupling and renormalization group results for the noisy Burgers equation. Phys. Rev. E 53, 4424–4438 (1996).
9. T. Funaki, Infinitesimal invariance for the coupled KPZ equations, Memoriai Marc Yor – Séminaire de Probabilités XLVII, Lect. Notes Math., 2137, 37–47. (Springer, Switzerland, 2015).
10. R. Grisi and G.M. Schütz, Current symmetries for particle systems with several conservation laws, J. Stat. Phys. 145, 1499–1512 (2011).
11. T. Halpin-Healy, K.A. Takeuchi, A KPZ Cocktail-Shaken, not Stirred..., J. Stat. Phys. 160(4), 794–814 (2015).
12. Y. Kafri, E. Levine, D. Mukamel, G.M. Schütz, and R.D. Willmann Phase-separation transition in one-dimensional driven models, Phys. Rev. E 68, 035101(R) (2003).
13. C. Kipnis and C. Landim, Scaling limits of interacting particle systems in: Grundlehren der mathematischen Wissenschaften Vol. 320 (Springer, Berlin, 1999).
14. A. Kundu, and A. Dhar, Equilibrium dynamical correlations in the Toda chain and other integrable models, Phys. Rev. E 94, 062130 (2016).
15. S. Lepri (ed.), Thermal Transport in Low Dimensions: From Statistical Physics to Nanoscale Heat Transfer, Lecture Notes in Physics 921, (Springer, Switzerland, 2016).
16. V. Popkov and M. Salerno, Hydrodynamic limit of multichain driven diffusive models. Phys. Rev. E 69, 046103 (2004).
17. V. Popkov, J. Schmidt, and G.M. Schütz, Superdiffusive modes in two-species driven diffusive systems. Phys. Rev. Lett. 112, 200602 (2014).
18. V. Popkov, J. Schmidt, and G.M. Schütz, Universality classes in two-component driven diffusive systems. J. Stat. Phys. 160, 835–860 (2015).
19. V. Popkov, A. Schadschneider, J. Schmidt, and G.M. Schütz, Fibonacci family of dynamical universality classes, Proc. Natl. Acad. Science (USA) 112(41) 12645–12650 (2015).
20. V. Popkov, A. Schadschneider, J. Schmidt, G.M. Schütz, Exact scaling solution of the mode coupling equations for non-linear fluctuating hydrodynamics in one dimension, J. Stat. Mech. 093211 (2016).
21. M. Prähofer and H. Spohn, Exact scaling function for one-dimensional stationary KPZ growth, J. Stat. Phys. 115, 255–279 (2004).
22. S. Ramaswamy, M. Barma, D. Das, and A. Basu, Phase Diagram of a Two-Species Lattice Model with a Linear Instability, Phase Transit. 75, 363–375 (2002).
23. G.M. Schütz, Exactly solvable models for many-body systems far from equilibrium, in: Phase Transitions and Critical Phenomena. Vol. 19, C. Domb and J. Lebowitz (eds.), Academic Press, London (2001).
24. G.M. Schütz and B. Wehefritz-Kaufmann, Kardar-Parisi-Zhang modes in d-dimensional directed polymers, Phys. Rev. E 96, 032119 (2017).
25. H. Spohn, Nonlinear Fluctuating hydrodynamics for anharmonic chains, J. Stat. Phys. 154, 1191–1227 (2014).
26. H. Spohn and G. Stolz, Nonlinear fluctuating hydrodynamics in one dimension: The case of two conserved fields, J. Stat. Phys. 160 861–884 (2015).
27. H. Spohn, The Kardar-Parisi-Zhang equation - a statistical physics perspective, Les Houches Summer School July 2015 session CIV “Stochastic processes and random matrices”, edited by Gregory Schehr, Alexander Aftalion, Yan V. Fyodorov, Neil O’Connell, and Leticia F. Cugliandolo, Oxford University Press, Oxford (2017).
28. A. Sudbury and P. Lloyd, Quantum operators in classical probability theory. II: The concept of duality in interacting particle systems, Ann. Probab. 23(4), 1816–1830 (1995).