Meson structure on the light-front III
The Hamiltonian, heavy quarkonia, spin and orbit mixing

Edward Shuryak* and Ismail Zahed†
Center for Nuclear Theory, Department of Physics and Astronomy,
Stony Brook University, Stony Brook, New York 11794–3800, USA

This is the third paper on hadronic light front wave functions (LFWFs). We derive a light front Hamiltonian from first principles, using the key features of the QCD vacuum at low resolution. In the first approximation, it gives transverse oscillator and longitudinal harmonic modes, and yields the correct Regge trajectories. For heavy quarkonia, we compare its spectrum to that obtained from the usual Schroedinger equation in the rest frame. We use the same approach for light quarks, and investigate the role of confinement and chiral symmetry breaking in the quark-antiquark sector. We then study spin-spin and spin-orbit mixing, resulting in e.g. quadrupole moments of vector mesons. For the light mesons, we show how to extend the famed 't Hooft interaction to the light front, which solves the U(1) problem and helps produce a light pion. We use the ensuing light front wavefunctions, to derive the pertinent parton distribution functions, parton amplitudes and low energy constants.

I. INTRODUCTION

The physics of hadrons is firmly based in Quantum Chromodynamics, a theory over half a century old. One might think that by now this subject has reached a solid degree of maturity with most issues settled. Yet persisting tension remains between the non-perturbative aspects of the theory and empirical measurements using inclusive and exclusive processes.

More specifically, first principle approaches – lattice and semi-classics – are focused on the ground state properties of the QCD vacuum, using Euclidean time formulation. Hadrons are then studied via certain correlation functions (a brief review will be given in the next subsection). However, a significant part of the experimental information – parton distribution functions (PDFs) used in deep inelastic inclusive processes, and distribution amplitudes (DAs) used for exclusive processes – are defined in the light front kinematics, and therefore are not directly accessible by the Euclidean formulation. Only recently, the first attempt to formulate the appropriate kinematical limits [1], and use the lattice for calculating the PDFs [2, 3] were carried out with some success.

Bringing the two sides of hadronic physics together is not just a technical issue related with kinematics. Even the main pillars of the theory – confinement and chiral symmetry breaking – become contentious. In particular, 60 years ago Nambu and Jona-Lasinio (NJL) [4] have explained that pions are light because they are near-massless vacuum waves due to the spontaneous breaking of chiral symmetry. The mechanism creating the vacuum quark condensate and the ensuing organization using chiral perturbation theory, have since been discussed and confirmed in countless papers.

More importantly, the QCD vacuum in the mesoscopic limit, reveals a multitude of multi-quark correlations captured by universal spectral fluctuations in the Zero Mode Zone (ZMZ) [5]. They are analogous to the universal conductance fluctuations around Fermi surfaces in dirty metals [6]. We regard these mesoscopic fluctuations as strong evidence, in support of the topological origin of the spontaneous breaking of chiral symmetry in QCD. Most of the current hadronic models fail to reproduce these fluctuations.

And yet, parton dynamics is still treated as if the vacuum is “empty” and quarks are treated...
as massless. There are even suggestions that on the light front, there are no condensates\cite{7,8}. The pion was also suggested to be massless due to other reasons\cite{9}. Recently these arguments were revisited\cite{10}, and “quasi-PDF” have been calculated on the lattice\cite{11} (and references therein), obviously with all nonperturbative effects included.

Still, there remains a significant gap between light-front observables used and hadronic spectroscopy (as well as atomic and nuclear ones): the former focuses on certain matrix elements (DAs, PDFs, TMDs, etc), rather than the wave functions, or the underlying Hamiltonian. This approach is entirely driven by information deduced from experiment.

Indeed, one can calculate various inclusive and exclusive reactions using DAs. But their number is in principle infinite, as there are unlimited number of operators. (The standard approach is to include only those with the smallest values of their twist, dimension minus spin, thereby limiting the discussion to the large $Q^2$ domain.) The normalization of the DAs is done via a number of empirical constants like $f_\pi,f_\rho$.

Hadronic spectroscopy, as in many other similar fields, goes in the opposite direction, from the underlying theory to effective Hamiltonians, to wave functions, to matrix elements. Indeed, one Hamiltonian produces many eigenstates, with well defined wave functions, naturally normalized and mutually orthogonal. From them any number of matrix elements of interest can be calculated.

The light-front wave functions were classified in well-known papers such as\cite{12}, but hardly used. Only for the pion – a very special particle, a Nambu-Goldstone mode – there is determination of both its components, from model-dependent Bethe-Salpeter equations\cite{13,14}, and from quasi-DAs in the instanton vacuum\cite{15}.

Model Hamiltonians were invented, but not related to the underlying physics. The spin-dependent forces – so important in spectroscopy – have not been included. In\cite{16} we reviewed their perturbative and non-perturbative aspects in the rest frame, and in\cite{17} we showed how to extend the non-perturbative contributions to the light front.

In this paper, a comprehensive derivation of the perturbative spin contributions will be given using Wilson lines on the light front. When combined with the non-perturbative contributions from\cite{17}, it provides a first principle Hamiltonian on the light front. The spectroscopic implications of this Hamiltonian for heavy and light mesons will be investigated.

The organization of the paper is as follows: in section section II-III we give a first principle derivation of the light front Hamiltonian, through an analytical continuation of pertinent Wilson loops from Euclidean to Minkowski signature. The derivation includes both the perturbative and non-perturbative gluonic contributions in the QCD vacuum at low resolution. In section IV we limit the light front Hamiltonian to the contributions stemming from confinement and Coulomb, and analyze their role on heavy quarkonia, with Upsilonium as an example. In section V we briefly review how parity is defined on the light front, and how it is used to organize the light front wave-functions for mesons. In section VI we consider the mixing induced by the tensor contribution to the light front Hamiltonian, onto heavy quarkonia. We show that the quadrupole moment of Upsilonium on the light front is about comparable to the one extracted from other approaches both at rest and also on the light front. In section VII, we consider the additional mixing induced by spin-orbit coupling on the light front, and apply in this case to the light meson spectrum. In section VIII we show how the subtle zero-modes associated to tunneling through instantons in Euclidean signature are lifted to the light front, using the LSZ reduction in coordinate space. We use it to derive the famed t’Hooft interaction on the light front. In section IX we use our light front wavefunctions to derive the parton distributions functions and amplitudes of heavy and light mesons, and their pertinent low energy constants. The extraction of the mesonic form factors, is also briefly dis-
cussed. Our conclusions are in section X. A number of Appendices are added to complement some of the results in the text.

FIG. 1. Wilson loop for a $\bar{Q}Q$ meson on the light-front.

II. PERTURBATIVE LIGHT-FRONT HAMILTONIAN, VIA ANALYTIC CONTINUATION FROM EUCLIDEAN AMPLITUDES

In the infinite momentum frame, a meson state composed of a quark and antiquark $Q\bar{Q} \equiv Q_1Q_2$ is characterized by the closed Wilson loop or a dipole, sloped along the light cone with rapidity $\chi$ as shown in Fig. 1. The same Wilson loop follows from the Euclidean Wilson loop at an angle $\theta$ by analytical continuation $\theta \rightarrow -i\chi$, as we discussed in the second paper of this series [17]. This construction follows the original suggestion for quark-quark scattering in [18], and its extension to dipole-dipole scattering in the QCD vacuum [19, 20], many years ago. The same construction was used in the holographic context, to address hadron-hadron scattering in the Regge limit [21–23].

With this in mind, the result is the squared meson mass operator, or light front Hamiltonian $H_{LF}$

$$H_{LF} \approx \frac{k^2}{x^\perp} + M^2 + 2M(V_{Cg}(\xi_x) + V_C(\xi_x)) + V_{SD}(\xi_x, b_{\perp}) + V_{TH}(\xi_x, b_{\perp}) \quad (1)$$

The non-perturbative contributions in (1) were discussed in [17], along with the ordering ambiguities. The perturbative contributions will be derived below. On the light front, the invariant distance $\xi_x$ is

$$\xi_x = \left( \frac{|id/dx|}{M} \right)^2 + b_{\perp}^2 \right)^{\frac{1}{2}} \quad (2)$$

with longitudinal distance $\gamma b_3 = id/dx/M$, the conjugate of Bjorken-$x$ or $x = k^3/P^3$. The explicit $\gamma$-factor compensates the Lorentz contraction along the 3-direction.

A. Wilson lines dressed by spin variables

The perturbative contribution to the central potential on the light front, induced by a one-gluon exchange with an effective mass $m_G$ in the RIV, can be constructed using the general technique of a sloped Wilson loop as we detailed in [17]. In particular, the one-gluon interaction with spin effects, follows by dressing the Wilson loop or holonomies in Fig. 1, with explicit spin factors.
\[ \langle \text{TrP} \left[ \exp \left( +g \int d\tau_1 (i\dot{x}(\tau_1) \cdot A(x(\tau_1)) + \frac{1}{4} \sigma_{1\mu\nu} F_{\mu\nu}(x(\tau_1)) \right) \right. \right. \]
\[
\times \left. \exp \left( -g \int d\tau_2 (i\dot{x}(\tau_2) \cdot A(x(\tau_2)) + \frac{1}{4} \sigma_{2\mu\nu} F_{\mu\nu}(x(\tau_2)) \right) \right] \rangle \] (3)

with \( \sigma_{\mu\nu} = \frac{i}{2} [\gamma_\mu, \gamma_\nu] \), and \( \sigma_{\mu\nu} = \eta_{\alpha\mu\nu} \sigma^\alpha \) using 't Hooft symbol. The averaging is understood using the QCD action. We have made explicit the gauge coupling \( g \), for a perturbative treatment to follow.

For massive quarks traveling on straight trajectories, the affine time \( \tau \) relates to the conventional time \( t \) through
\[
\mu = \frac{dt}{d\tau} = \frac{m_Q}{\sqrt{1 + x^2}} \rightarrow \gamma m_Q \] (4)
in Euclidean signature. We note that the holonomies tracing out the Wilson loop in Fig. 1 are unaffected by the exchange holonomies, in contrast to the spin contributions which get rescaled by \( 1/\mu \). This will be exploited below.

\[ g^2 T_1^A T_2^B \int dt_1 \int dt_2 \left( \cos^2 \theta (A_1^A(t_1) A_2^B(t_2)) + \sin^2 \theta (A_3^A(t_1) A_3^B(t_2)) + 2 \sin \theta \cos \theta (A_4^A(t_1) A_4^B(t_2)) \right) \] (6)

with the gluon correlator in Feynman gauge
\[
\langle A_\mu^A(t_1) A_\nu^B(t_2) \rangle = \frac{1}{2\pi^2} \frac{\delta^{AB} \delta_{\mu\nu}}{|x(t_1) - x(t_2)|^2} \] (7)

Inserting (7) into (6) and changing variables \( T_E = t_1 + t_2 \) and \( \tau = t_1 - t_2 \), yield
\[
g^2 T_1^A T_2^A \int \frac{dT_E}{2} \int \frac{dt_2}{2} \frac{1}{t_2^2 + \cos^2 \theta b_3^2 + b_4^2} \]
\[
= \frac{g^2 T_1^A T_2^A}{4\pi} \frac{T_E}{\sqrt{b_3^2 \cos^2 \theta + b_4^2}} \] (8)
The analytical continuation \( \theta \rightarrow -i\chi \) and

\[ T_E \rightarrow iT_M \] of (8), re-exponentiates to the Coulomb contribution
\[
\exp \left[ -i\gamma T_M \left( -\frac{g^2 T_1^A T_2^A}{4\pi} \frac{1}{\sqrt{\gamma^2 b_3^2 + b_4^2}} \right) \right] \] (9)

with \( \gamma T_M \) the dilatated time along the light-like Wilson loop. The Coulomb contribution to the light front \( QQ \) Hamiltonian \( P_{\gamma}^2 \) follows, leading the squared invariant mass as

**B. One-gluon exchange and the Coulomb interaction**

The Coulomb interaction between a \( QQ \equiv Q_1 Q_2 \) pair attached to the Wilson lines, can be obtained in perturbation theory by expanding the holonomies, and averaging the \( AA \) correlator in leading order. For that, we parametrize the world-lines by
\[
x_\mu(t_1) = (0,0,\sin \theta t_1, \cos \theta t_1) \]
\[
x_\mu(t_2) = (b_1, b_2, \sin \theta t_2 + b_3, \cos \theta t_2) \] (5)
The perturbative one-gluon contribution from (3) reads

\[ T_E \rightarrow iT_M \] of (8), re-exponentiates to the Coulomb contribution
\[
\exp \left[ -i\gamma T_M \left( -\frac{g^2 T_1^A T_2^A}{4\pi} \frac{1}{\sqrt{\gamma^2 b_3^2 + b_4^2}} \right) \right] \] (9)

with \( \gamma T_M \) the dilatated time along the light-like Wilson loop. The Coulomb contribution to the light front \( QQ \) Hamiltonian \( P_{\gamma}^2 \) follows, leading the squared invariant mass as
\[ 2P^+ P^-_{cg} = 2P^+ \left( - \frac{g^2 T_1^A T_2^A}{4\pi} \frac{1}{\sqrt{\gamma^2 b_3^2 + b_1^2}} \right) \]

\[ \rightarrow 2M \left( - \frac{g^2 T_1^A T_2^A}{4\pi} \frac{1}{\xi_x} \right) = 2MV_{cg}(\xi_x) \quad (10) \]

with \( P^+/M = \gamma \), and \( \gamma b_3 \to id/dx/M \) the conjugate of Bjorken-\( x \).

In the random instanton vacuum (RIV), the perturbative gluons acquire a momentum dependence mass from their rescattering through the instanton-anti-instanton ensemble [24]

\[ m_G(k\rho) = m_G \left( k\rho K_1(k\rho) \right) \]

\[ m_G \rho \approx 2 \left( \frac{6\kappa}{N_C - 1} \right)^{1/2} \approx 0.55 \quad (11) \]

using the estimate \( \kappa = \pi^2 \rho^4 n_{t+1} \) in the rightmost result. With this in mind, (10) is now

\[ V_{cg}(\xi_x) = - \frac{g^2 T_1^A T_2^A}{2\pi^2} \frac{1}{\xi_x} \int_0^\infty dx \sin x \frac{dx}{x^2 + (\xi_x m_G(x\rho/\xi_x))^2} \rightarrow - \frac{g^2 T_1^A T_2^A}{4\pi} e^{-m_G \xi_x} \quad (12) \]

with the right-most result following for a constant gluon mass.

### C. Spin-spin interaction

The perturbative spin-spin interaction follows from the cross term in (3)

\[ \frac{g^2}{16} \int d\tau_1 d\tau_2 \langle \sigma_{1\mu} F_{\mu\nu}(x(\tau_1)) \sigma_{2\alpha} F_{\alpha\beta}(x(\tau_2)) \rangle \quad (13) \]

Note that the perturbative electric field is purely imaginary in Euclidean signature, leading mostly to phases and not potentials in the long time limit. Also, the Dirac representation \( \sigma_{4i} \) is off-diagonal, an indication that the electric contribution mixes particles and anti-particles, which is excluded by the use of straight Wilson lines on the light front. With this in mind and using (4), we can reduce (13) to

\[ - \frac{g^2}{4\mu^2} \int dt_1 dt_2 (\sigma_{1i} F_{ij}(x(t_1)) \sigma_{2k} F_{kl}(x(t_2))) \]

\[ = - \frac{g^2}{4\mu^2} \sigma_1^i \sigma_2^j \int dt_1 dt_2 (B_a(x(t_1)) B_b(x(t_2))) \quad (14) \]

\[ V_{SS}(\xi_x) = 2M \left[ \frac{\sigma_1^i \cdot \sigma_2^j}{4m_Q_1 m_Q_2} \left( \nabla_{\perp} V_{cg}(\xi_x) \right) \right] \]

\[ = 2M \nabla_{SS}(\xi_x, b_{\perp}) \quad (17) \]
D. Spin-orbit interaction

1. Cross spin-orbit

The cross spin-orbit interaction is readily obtained from the 12 + 21 cross terms

\[ -\frac{g^2 T_A^B T_B^A}{2}\sigma_a^a \]

\[ \times \int d\tau_1 d\tau_2 \dot{x}i(\tau_1)\langle A_A^a(x(\tau_1))B_B^a(x(\tau_2)) \rangle \]

\[ +1 \leftrightarrow 2 \] (18)

which can be reduced to

\[ -ig^2 T_A^B T_B^A \sin \theta \]

\[ \times \int dt_1 dt_2 \langle A_A^a(x(t_1))B_B^a(x(t_2)) \rangle \]

\[ +1 \leftrightarrow 2 \] (19)

with \( s_{1,2} = \text{sgn}(v_{1,2})^3 \) the signum of the 3-velocity of particle 1,2 (a more refined definition will be given below). After carrying the integrations, and the analytical continuations, the spin-orbit contribution to the squared mass is

\[ H_{SL,12} = 2M \left[ \left( \frac{\sigma_2 \cdot (b_{12} \times s_1 \hat{3})}{2m_{Q_2}} - \frac{\sigma_1 \cdot (b_{21} \times s_2 \hat{3})}{2m_{Q_1}} \right) \left( \frac{1}{\xi_\theta} \mathcal{V}_{Cg}(\xi_\theta) \right) \right] \] (20)

in general, with \( b_{21} = -b_{12} \equiv b_\perp \).

2. Standard spin-orbit

The standard self spin-orbit interaction with Thomas precession is more subtle. To unravel it, we note that the insertion of a single spin contribution along the path-ordered Wilson loop amounts to expanding the spin factors in (3) to first order, and retaining the holonomies to all orders in \( \frac{1}{\theta} \), namely

\[ \frac{1}{4\mu} \sigma_{1\mu} \int dt_1 \langle gF_{a\nu}(x(t_1)) \mathbf{1}_\theta \rangle + 1 \leftrightarrow 2 \] (21)

with the path ordered color-spin trace subsumed. Here \( \mathbf{1}_\theta \) refers to the slated Wilson loop in Fig. 1 without the spin dressing. We now decompose

\[ F_{\mu\nu} = v_{1\mu}v_{1\alpha}F_{a\nu} + F_{a\mu}^{\perp} = F_{\mu\nu}^{||} + F_{\mu\nu}^{\perp} \] (22)

into a contribution parallel to \( v_1 = \dot{x}_1 \) and a contribution orthogonal to \( v_1 \). The contribution parallel to the worldline when inserted in (21) can be undone by the identity (see Eq.71 in [17])

\[ \int dt_1 \langle gv_{\alpha}F_{a\nu}(x(t_1)) \mathbf{1}_\theta \rangle \]

\[ = -i\partial_{1\nu}(\mathbf{1}_\theta) \equiv -i\partial_{1\nu} e^{-T_\theta \mathcal{V}_{C}(\xi_\theta)} \] (23)

with \( \mathcal{V}_{C}(\xi_\theta) \approx \mathcal{V}_{Cg}(\xi_\theta) \) the central Coulomb potential in perturbation theory. The longitudinal contribution to (21)

\[ -\frac{1}{4\mu} \sigma_{1\mu} v_{1\mu} i\partial_{1\nu} e^{-T_\theta \mathcal{V}_{C}(\xi_\theta)} + 1 \leftrightarrow 2 \] (24)

is gauge-invariant. After carrying the analytical continuation, (24) contributes both a real and imaginary part. The latter is an irrelevant phase factor in Euclidean signature. The real part contributes to the direct mass squared operator as
in leading order in perturbation theory. This is the standard spin-orbit contribution with the correct Thomson correction on the light front, familiar from atomic physics in the rest frame. The total perturbative spin contribution on the light front, is the sum of (17), (20) and (25),

\[
H_{LS,11} = 2M \left[ \left( \frac{\sigma_1 \cdot (b_{12} \times s_1 \hat{3})}{4m_{Q1}} - \frac{\sigma_2 \cdot (b_{21} \times s_2 \hat{3})}{4m_{Q2}} \right) \left( \frac{1}{\xi_x} \nabla V_C(\xi_x) \right) \right]
\]

(25)

III. INSTANTON CONTRIBUTIONS TO WILSON LINE AMPLITUDES

A. Central potential

The central potential-operator induced by instantons is given by

\[
V_C(\xi_x) = \left( \frac{4\kappa}{N_c \rho} \right) H(\xi_x)
\]

(28)

with the integral operator

\[
H(\xi_x) = \int_0^{\infty} y^2 dy \int_{-1}^{1} dt \times \left[ 1 - \cos \left( \frac{\pi y}{\sqrt{y^2 + 1}} \right) \cos \left( \pi \left( \frac{y^2 + \xi_x t^2 + 2\xi_x y t}{y^2 + \xi_x t^2 + 2\xi_x y t + 1} \right)^{\frac{1}{2}} \right) \right. \\
\left. - \frac{y + \xi_x t}{(y^2 + \xi_x t^2 + 2\xi_x y t)^{\frac{1}{2}}} \sin \left( \frac{\pi y}{\sqrt{y^2 + 1}} \right) \sin \left( \pi \left( \frac{y^2 + \xi_x t^2 + 2\xi_x y t}{y^2 + \xi_x t^2 + 2\xi_x y t + 1} \right)^{\frac{1}{2}} \right) \right]
\]

(29)

with the dimensionless invariant distance on the light front \( \xi_x = \xi_x / \rho \). \( H(\xi_x) \) admits the short distance limit

\[
H(\xi_x) \approx + \left( \frac{\pi^3}{48} - \frac{\pi^3}{3} J_1(2\pi) \right) \xi_x^2 \\
+ \left( - \frac{\pi^3(438 + 7\pi^2)}{30720} + \frac{J_2(2\pi)}{80} \right) \xi_x^4
\]

(30)
and large distance limit
\[ H(\xi_x) \approx -\frac{2\pi^2}{3} \left( \pi J_0(\pi) + J_1(\pi) \right) + \frac{C}{\xi_x} \]
with \( p \ll 1 \) and \( C > 0 \). The large asymptotic is to be subtracted in the definition of the potential. This will be subsumed throughout. In the dense instanton vacuum discussed in [17], the central potential (28) is almost linear at intermediate distances \( 0.2 - 0.5 \text{ fm} \). At larger distances, the linearly confining potential with string tension \( \sigma_T \) takes over, in good agreement with most lattice simulations.

**B. Spin dependent potentials**

On the light front, the spin-dependent interactions captured by \( V_{SD} \) and due to the non-zero modes in (1), have been discussed in general in [17], with the results

\[
V_{SD}(\xi_x, b_\perp) = + \left[ \frac{\sigma_1 \cdot (b_{12} \times s_1 \hat{3})}{4m_{Q_1}} - \frac{\sigma_2 \cdot (b_{21} \times s_2 \hat{3})}{4m_{Q_2}} \right] \frac{1}{\xi_x} V'_C(\xi_x) \\
+ \left[ \frac{\sigma_1 \cdot (b_{12} \times s_1 \hat{3})}{2m_{Q_1}} - \frac{\sigma_2 \cdot (b_{21} \times s_2 \hat{3})}{2m_{Q_2}} \right] \frac{1}{\xi_x} V'_1(\xi_x) \\
+ \left[ \frac{\sigma_2 \cdot (b_{12} \times s_1 \hat{3})}{2m_{Q_2}} - \frac{\sigma_1 \cdot (b_{21} \times s_2 \hat{3})}{2m_{Q_1}} \right] \frac{1}{\xi_x} V'_2(\xi_x) \\
+ \left[ \frac{1}{4m_{Q_1}m_{Q_2}} \left( \sigma_{1\perp} \cdot \hat{b}_{21} \sigma_{2\perp} \cdot \hat{b}_{21} - \frac{1}{2} \sigma_{1\perp} \cdot \sigma_{2\perp} \right) \right] V''(\xi_x)
\] (32)

with again \( b_{21} = -b_{12} \equiv b_\perp \). All potentials follow from the central instanton potential \( V_C(\xi_x) \), thanks to self-duality

\[
V_1(\xi_x) = V_2(\xi_x) - V_C(\xi_x) = -\frac{1}{2} V_C(\xi_x) \\
V_2(\xi_x) = + \frac{1}{2} V_C(\xi_x) \\
V_3(\xi_x) = + \frac{b_\perp^2}{\xi_x^2} V''_C(\xi_x)
\] (33)

As a result, the first and second contribution in (32) cancel out, leaving only the cross spin orbit plus tensor contributions in the instanton vacuum,

\[
H_{LS} = 2M \left( \frac{I_{1\perp} \cdot S_{2\perp}}{m_{Q_1}m_{Q_2}} - \frac{I_{2\perp} \cdot S_{1\perp}}{m_{Q_1}m_{Q_2}} \right) \frac{1}{\xi_x} V'_C(\xi_x) + \frac{1}{m_{Q_1}m_{Q_2}} \left( S_{1\perp} \cdot \hat{b}_{1\perp} S_{2\perp} \hat{b}_{2\perp} - \frac{1}{2} S_{1\perp} \cdot S_{2\perp} \right) \frac{b_\perp^2}{\xi_x} V''_C(\xi_x)
\] (34)
IV. HEAVY QUARKONIA ON THE LIGHT FRONT

In our second paper [17] we introduced “the basic problem” of meson structure, of two constituent quarks connected by a classical relativistic string, which was then studied using both a semiclassical approach, and a relativistic Klein-Gordon equation. Our main focus there was on the correspondence between the conventional treatment in the rest frame, versus the analysis on the light front using the Hamiltonian we derived. Of course, frame-invariant quantities –masses in particular – obtained in both ways must agree. We specifically investigated the linear rise of the Regge trajectories, with the principal quantum number $M^2_n \sim n$ (not angular momentum).

In this paper we will carry out a larger set of studies, including not only the confining string, but also various other terms in the Hamiltonian. In particular, the perturbative (Coulomb) term, and most importantly, the terms containing spin and orbital momentum variables. In doing so, it is also natural to widen the set of applications. Therefore here we start with heavy quarkonia, before returning to the light quark systems.

In the quarkonia settings, we can use the large quark mass as an extra parameter, to discriminate between distinct physical contributions. Remarkably, on the light front all meson problems, from bottomonia to pions, can be studied in essentially the same setting, just by changing the mass value.

A. Excited states of bottomonium via the Schroedinger equation in the rest frame

Let us start by focusing first on heavy quarkonia. Such an approach is more convenient in this work, devoted to the mixing between states with different spin and orbital momenta. In heavy quarkonia these relativistic effects are naturally suppressed by the nonrelativistic motion of heavy quarks (or, in other terms, their small magnetic moments $\mu \sim g/m_Q$).

The first question to be addressed is: how well the heavy quarkonia states can be represented by linear-linear Regge trajectories? In Fig.2 we show the experimental masses of the $(nS), n = 0 − 5$ Upsilon, compared to the standard results from the Schroedinger equation, with the Cornell potential (black triangles) and with only its linear part $V_{conf} = \sigma_T r$ (blue circles). The first observation is that using a linear potential alone (blue circles), we find a nearly-linear Regge trajectory. This observation will be important in the next subsection, as it shows that even for heavy bottomonia, the light-front Hamiltonian can be approximated by an oscillator with good accuracy. Note however, that the slope of the straight line, is here completely different from the $1/\alpha'$ slope of a similar trajectory for light mesons (e.g. for $\omega$ mesons we used in [16]).

The second observation is that the expected contribution from the spin-dependent potential $V_{SS}$ (responsible for splitting between squares and triangles), is positive and decreases with $n$. The former is due to the positivity of the spin factor $\vec{S}_1 \cdot \vec{S}_2 = 1/4$, and the second to the fact that $V_{SS}(r)$ is rather short range, in comparison to the size of the lowest Upsilon, but much smaller than the sizes of the excited ones. Another way to anticipate the accuracy of an oscillator approximation in the light front description (discussed in [17] and using $\omega_3$ mesons with $L = 2$), is to study the mass dependence of bottomonium on its orbital momentum $L$. In Fig.3 we show the calculated 18 squared masses for $n = 0−5$ (left-to-right) and $L = 0, 1, 2$ (bottom-to-top). While the Coulomb potential was included, it affects mostly and only $n = 0, L = 0$ Upsilon. The Regge trajectories for nonzero angular momentum $L = 1, 2$, show better linear dependence on $n$ than $L = 0$. The corresponding wave function vanishes at the origin $r = 0$, and is less affected by short-range Coulomb and spin-dependent forces.

We further note, that at larger $n$ (right side
FIG. 2. $M_{n+1}^2$ (GeV)$^2$ versus $n + 1$, $n = 0, \ldots, 5$, for the six $S$ zero orbital momentum ($L=0$) states of bottomonium. The red squares correspond to the experimentally observed Upsilons. The black triangles show the masses obtained from the Schrödinger equation, with the Cornell potential (linear and Coulomb potentials, no spin forces). The blue circles show the masses if the Coulomb potential is switched off, and only the linear potential is used. The straight line is shown for comparison.

B. Bottomonium on the light front

In [17], we described how we may include the linear confining term in $H_{LF}$ (instanton induced at intermediate distances), and make it more user friendly, by eliminating the square root using the well-known einbein $e = 1/a$ trick, i.e.

$2 M V_C(a,b,x,b_{\perp}) \\ \approx \sigma_T \left( |\frac{1}{a} dL/dx|^2 + b b_{\perp}^2 + a \right)$ (35)

Here $a, b$ are variational parameters. The minimization with respect to $a$ is assumed, followed by the substitution $b \rightarrow M^2 \approx (2m_Q)^2$ for heavy mesons, and most light ones. (For the pion, this last substitution is not valid, as we have shown in [17]).

For a numerical analysis of (35), we used in [17] a basis set of functions composed of a 2-dimensional transverse oscillator, times longitudinal states $sin(\pi n x)$ with odd $n$, as we briefly review in Appendix A. More specifically, the light front Hamiltonian can be re-arranged as follows

$H_{LF} = H_0 + \tilde{V} + V_{perp} + V_{spin}$ (36)
with the spin-part including both the perturbative and non-perturbative instanton contributions. As we noted earlier, in the dense instanton vacuum, the central part is hardly differentiable from the linear confining potential at intermediate distances.

The first contribution $H_0$

$$H_0 = \frac{\sigma_T}{a} \left( -\frac{\partial^2}{\partial x^2} - b \frac{\partial^2}{\partial \vec{k}_\perp^2} \right) + \sigma_T a + 4(m_Q^2 + k_\perp^2)$$  \hspace{1cm} (37)

is diagonal in the functional basis used [17]. In this form, we make use of the momentum representation, with $\vec{k}_\perp$ as variable. Similarly, one can use the coordinate representation with $\vec{b}_\perp$ as a variable, and $\vec{k}_\perp = i\partial/\partial \vec{b}_\perp$. The latter choice is much more convenient when discussing states with nonzero angular momenta, in relation to the azimuthal angle coordinate $\phi$ in the transverse plane (see more on that in Appendix).

The second contribution $\tilde{V}$

$$\tilde{V}(x, \vec{k}_\perp) \equiv (m_Q^2 + k_\perp^2) \left( \frac{1}{x^2} - 4 \right)$$  \hspace{1cm} (38)

has nonzero matrix elements $\langle n_1 | V(x, \vec{k}_\perp) | n_2 \rangle$ for all $n_1, n_2$ pairs. The perturbative part $V_{\text{pert}}$ for heavy quarks is the Coulomb term, with running coupling and other radiative corrections. Finally, the last term $V_{\text{spin}}$ contains matrices in spin variables and in orbital momenta, which we will consider later.

We truncate the basis set to a $12 \times 12$ matrix, and diagonalize $H_0 + \tilde{V}$, to find its eigenvalues as a function of the remaining parameter $a$. The results for the three lowest states $n = 1, 2, 3$ are shown in Fig. 4 (top). We see that while the minima in $a$ exist, they are not at the same value. Thus the dilemma: one can either select different $a$ for different states, and then somehow re-orthogonalize them, or one can use some “optimal” value of $a$ common for all states, and then be sure that all states are orthogonal. Since the dependence on $a$ is rather flat, we opt for the second approach and use $a = 25$.

The calculated masses (shifted by a constant “mass renormalization”, to make $n = 0, m = 0$ states the same) are shown in Fig. 4. The bottom part shows good agreement between the masses obtained solving the Shroedinger equation in the rest frame (blue circles), and the masses following from the light-front frame (red-triangles). The slope is correct, and is de-
The splittings in orbital momentum are of the same scale, but not identical. This is expected, as we compare the 2-dimensional \( m \)-states on the light front, with the 3-dimensional \( L \)-states in the center of mass frame.

The irregularity between the third and fourth set of states, is due to our use of a modest basis set, with only three radial functions (altogether 12 functions if one counts them with 4 longitudinal harmonics). This can be eliminated using a larger set.

As a final note in this section, we recall that the chief goal of these calculations is to generate the pertinent LFWFs for all these states, from the light front Hamiltonian \( H_{LF} \). The details about the setting and some of these wavefunctions can be found in Appendices A,B,C.

C. Matrix elements of the Coulomb term and further operators on the LF

We start with a calculation of the contribution of the Coulomb force, which demonstrate how to deal with any function of transverse coordinate \( F(r_\perp) \). Of course, the standard way is to transform all functions into the coordinate representation.

Recall that our LFWFs are defined using a transverse oscillator in the momentum representation, and so one possible strategy is to trade

\[ r_\perp \rightarrow i \frac{\partial}{\partial k_\perp} \]

as in the confining potential. It can work for other polynomial functions \( F(r_\perp) \) or their Taylor expansions. Unfortunately, for a transverse Coulomb potential

\[ V_\perp = -\frac{C_C}{r_\perp} \]  

this strategy does not work. A straightforward solution is to Fourier transform the LFWFs to coordinate representation. Note that the LFWFs are of the form

\[ \psi(p_\perp, x) = \sum_n \phi_n(p_\perp) \sin(\pi nx) , \]

as a result, the integration over \( x \) in the matrix element \( \langle \psi | F(r_\perp) | \psi \rangle \) removes terms with \( n_1 \neq n_2 \), and reduces to

\[ \langle \psi | F(r_\perp) | \psi \rangle = \sum_n \int d^2 r_\perp |\tilde{\phi}_n(r_\perp)|^2 F(r_\perp) \]  

where tilde stands for the Fourier transform.

In particular, the lowest state in our basis has a simple Gaussian form \( \phi_1(p_\perp) \sim \exp(-Ap_\perp^2) \), and its Fourier transform is also a Gaussian \( \tilde{\phi}_1(r_\perp) \sim \exp(-r_\perp^2/4A) \). The average Coulomb contribution to the squared mass \( M_\Upsilon^2 \) is then found to be

\[ -4M_\Upsilon \langle \psi_1 | \frac{C_C}{r_\perp} | \psi_1 \rangle \]

\[ = -4M_\Upsilon C_C \sqrt{\frac{\pi}{2A}} \approx -15 \text{ GeV}^2 \]  

which approximately agrees with \( \Delta M_\Upsilon^2 \approx -17 \text{ GeV}^2 \) obtained from the Schröedinger equation in the CM frame (and shown in Fig.2).

With growing \( \Upsilon \) number their sizes grow, which reduces the Coulomb contribution.

In general, the operators to be averaged (e.g. spin-dependent potentials) depend on the invariant distance \( \xi_x \), which includes the longitudinal distance with the derivative \( id/dx \). In this case the matrix element should be calculated using the eigenvalue \( l \) decomposition of this derivative operator. For example, by approximating the ground state LFWF as

\[ \phi_0(x, p_\perp) \approx \left( \frac{2\alpha}{\pi} \right)^{1/2} e^{-\alpha p_\perp^2/2} \sum_{odd \ l} \varphi_l \sin(l\pi x) \]  

with a simple Fourier transform \( p_\perp \rightarrow b_\perp \)

\[ \tilde{\phi}_0(x, b_\perp) \approx \left( \frac{2}{\pi \alpha} \right)^{1/2} e^{-b_\perp^2/2\alpha} \sum_{odd \ l} \varphi_l \sin(l\pi x) \]  

(43)
one can use it to evaluate matrix elements of a
the potential depending on this invariant \( V(\xi) \) as follows

\[
\langle \phi_0 | V(\xi_x) | \phi_o \rangle = \int db_\perp \frac{e^{-b_\perp^2/\alpha}}{\pi \alpha} \sum_{odd l} |\phi_l|^2 V\left( \left( \frac{l\pi}{M \rho} \right)^2 + \frac{b_\perp^2}{\rho^2} \right)^{1/2}
\]

The same procedure applies for the excited states. (44) shows that the natural transverse
cutoff in \( b_\perp \sim \pi/M \sim \pi/2m_Q \) for the heavy states.

V. SPIN AND ORBITAL MOMENTUM
MIXING OF THE LFWFS

In so far, we only discussed the LFWFs diag-
ontal in longitudinal orbital momentum \( L_z = m \).
In general, this is not a conserved quantity,
but for heavy quarkonia it is approximately
conserved, as the spin and orbital momentum-
dependent effects are suppressed by large quark
masses. As we will proceed to light quark states, this approach would become invalid, and
spin-spin and spin-orbit mixing is mandatory.

A. Parity on the light front

There are some obvious differences between
the description in the rest frame, and on the
light front. For instance, there are different
symmetries: 3-dimensional angular momenta
\( \vec{S}, \vec{L}, \vec{J} \) are reduced to their 2-dimensional
transverse parts, with spin \( \vec{S} \) and orbital momentum \( \vec{L} \) projected onto longitudinal momentum \( \vec{P} \). The projection of \( \vec{J} \) is denoted by
“meson helicity” \( \Lambda \). Obviously, hadron states
with different \( \Lambda \) values, are treated differently: say \( \rho(\Lambda = 0) \) and \( \rho(\Lambda = \pm 1) \) have different wave functions (even more than one: see be-
low). While masses, magnetic and quadrupole moments, etc should turned out to be the same,
the 3-dimensional rotation is some complicated
transformation, involving all components of the
wave functions, and we will not attempt to ex-
plitly use it.

(This situation is of course not new. For
example, different isospin components are also
treated differently in the rest frame. The \( \bar{d}u \)
charged states are described by a potential,
while the \( \bar{d}d, \bar{u}u \) follow from annihilation. In
the isospin symmetric limit, the same mass for \( \pi^+ \) and \( \pi^0 \) needs to be explicitly demonstrated.)

Another important difference between the
rest frame and the light front frame notations,
relates to the different definitions of parity. The
usual P-parity is the sign change of all 3 spatial
coordinates, or mirror reflection. On the light
front, one would like to keep the main beam
direction (of \( \vec{P} \)) intact, so \( P \) is supplemented
by an additional rotation, by \( \pi \) around some
transverse axes: this operator is called \( \hat{Y} \). The
state’s helicity \( \Lambda \) changes sign, so its action for
\( \Lambda \neq 0 \) is given by the so called Jacob-Wick re-
lation

\[
\hat{Y}|\vec{P},\Lambda> = (-)^{S-\Lambda} \eta|\vec{P},-\Lambda>
\]

where \( \eta \) is the intrinsic parity of the state, neg-
ative for quark-antiquark states, positive for
quark states, and negative for an antiquark
plus gluon. Since parton’s momenta are gener-
ically not in the direction of \( \vec{P} \), \( k_\perp \neq 0 \), one
should remember that only one component of \( k_\perp \) changes sign under \( \hat{Y} \). For \( \Lambda = 0 \), \( \hat{Y} \) turns
the state to itself, so for these states one can
define \( Y \)-parity. The changed definition of parity
completely changes the parity mixing rules,
and respectively the number of light front wave
functions, as described in detail in [25].
Yet the light front wave functions in the helicity basis, have different rules. The classification is not done via the total $S_1, S_2, L$: only their $z$-components (meson directed) are used. They satisfy the obvious constraint

$$\Lambda = S_1^z + S_2^z + L^z$$

In the following we will drop the $z$ upper-script. The $\Lambda = 0$ states are eigenstates of $\hat{Y}$, minus for pions and plus for rho mesons: those have two wave functions (see section C) unlike $\Lambda = \pm 1$ states.

In [17] we focused on the spin-dependent forces, with $\hat{S}_1, \hat{S}_2, \hat{S}, \hat{L}$ and tensor. Now we will have their analogues in beam projections, which we refer to by the same labels, without vectors. Some of them are non-diagonal. For instance, the tensor force can mix $|S_1 = S_2 = \frac{1}{2}, L = 0\rangle$ with $|S_1 = S_2 = -\frac{1}{2}, L = 2\rangle$.

$$H_\Lambda = (\psi_{\Lambda-1}, \psi_{\Lambda}, \psi_{\Lambda+1}) \begin{pmatrix} V_{diag} & V_{\pm 1} & V_{\pm 2} \\ V_{\pm 1} & V_{diag} & V_{\pm 1} \\ V_{\pm 2} & V_{\pm 1} & V_{diag} \end{pmatrix} \begin{pmatrix} \psi_{\Lambda-1} \\ \psi_{\Lambda} \\ \psi_{\Lambda+1} \end{pmatrix}$$ (46)

The spin-orbit $V_{SL}$ interaction changes $L$ by $\pm 1$, and the tensor interaction $V_T$ changes $L$ by $\pm 2$. So, in general, any meson has three wave functions, mixed by spin- and $L_z = m$-flipping forces.

For the important case of $\Lambda = 0$ – the pseudoscalar ($\eta, \ldots \pi$) and vector ($\Upsilon, \ldots \rho$) – is now diagonal. Thus these mesons have different par-tions.

$$|P\rangle = \int [1|d[2] \frac{\delta_{ij} N_c}{\sqrt{N_c}} \left[ \psi^P_0 (x, k_\perp) (Q_{i1}(1)Q^{\dagger}_{j1}(2) - Q_{i1}(1)Q^{\dagger}_{j1}(2)) + \psi^P_{+1}(x, \vec{k}_\perp)Q^{\dagger}_{i1}(1)Q^{\dagger}_{j1}(2) + \psi^P_{+1}(x, \vec{k}_\perp)Q^{\dagger}_{i1}(1)Q^{\dagger}_{j1}(2) \right] 0 \rangle$$ (47)

$$|V\rangle = \int [1|d[2] \frac{\delta_{ij} N_c}{\sqrt{N_c}} \left[ \psi^V_0 (x, k_\perp) (Q_{i1}(1)Q^{\dagger}_{j1}(2) + Q_{i1}(1)Q^{\dagger}_{j1}(2)) + \psi^V_{+1}(x, \vec{k}_\perp)Q^{\dagger}_{i1}(1)Q^{\dagger}_{j1}(2) - \psi^V_{+1}(x, \vec{k}_\perp)Q^{\dagger}_{i1}(1)Q^{\dagger}_{j1}(2) \right] 0 \rangle$$ (48)

with $N_c = 3$. The subscripts 0 and $\pm 1$ on the wave-functions, refer to $L_z$, the $z$-projections of

\[ B. \text{ General form of LFWFs for mesons with different } \Lambda \]

To proceed with the spin effects, we need to get the full spin-orbit structure of the LFWFs. In order to explain what we mean, consider a meson with total helicity $\Lambda$ (with longitudinal projection $J_z$). The total of two quark spins $\vec{S} = \vec{S}_1 + \vec{S}_2$ can be $S = 0$ or $S = 1$: in the former case $\Lambda = L_z$, and in the latter there are three cases: $L_z = \Lambda - 1, L_z = \Lambda, L_z = \Lambda + 1$. These 4 states (like e.g. $\chi, \eta_b$) are in general mixed by spin-dependent forces. Schematically for the last three states, the mixing is captured by a 3x3 matrix (the index of $\psi$ is the longitudinal projection of orbital momentum, or $M_L$)
the orbital momentum. Note that compared to the notations in [25], there are no explicit factors of $k_1^\pm = k_1 \pm ik_2$ here because they naturally belong to our wave functions, consistently defined not only for $m = L_z = 1$, but for any $m$ value.

The $\Lambda = 0$ state of the vector mesons are called “transversely polarized”. The two other polarizations, with $\Lambda = \pm 1$ are “longitudinally” polarized. They are a bit more complicated, with three components each with different wave-functions, corresponding to $L = 2, 1, 0$.

The invariant measure in (47-48) refers to the on-shell covariant one, with overall momentum conservation

$$d[1]d[2] = \frac{dx}{\sqrt{4x\bar{x}}} \frac{dp_\perp}{(2\pi)^3}$$

(49)

Here $x, \bar{x}$ are the fraction of longitudinal momenta carried by particle-1 and the antiparticle-2, or $x = p_1^+/P^+$ and $\bar{x} = p_2^+/P^+$ with $x + \bar{x} = 1$. The creation and annihilation operators in (47-48) obey the anti-commutation rules

$$[Q_\alpha(k_1), Q^{\dagger}_\beta(k_2)]_+ = \delta_{\alpha\beta}2k_1^+(2\pi)^3 \delta(k_1^+ - k_2^+) \delta(k_{1\perp} - k_{2\perp})$$

for equal light-front time, so that the $|P,V\rangle$ states are covariantly normalized on the light front, e.g.

$$\langle P|P'\rangle = 2P^+(2\pi)^3 \delta(P^+ - P'^+) \delta(P_{\perp} - P'_{\perp})$$

(50)

It is readily checked that the light front wavefunctions in (47-48) are normalized by

$$\int \frac{d^2k_\perp dx}{(2\pi)^3} \left(|\psi_0|^2 + |\psi_1|^2 + |\psi_{-1}|^2\right) = 1$$

(51)

Below we show that $\psi_0$ refers to the twist-2 and $\psi_{\pm 1}$ to the (tensor) twist-3 contribution to the mesonic distribution amplitude.

VI. QUADRUPOLE MOMENT OF VECTOR MESONS AND $m \pm 2$

“TENSOR” MIXING

To explain why the effects mixing spin and orbital momenta are important, let us take the classic example of the quadrupole moment. In the rest frame, these phenomena are well known in nuclear physics, for example the deuteron $d = pn$ state has total $J = 1$ and, in nonrelativistic notation, it is a mixture of $L = 0, J = 1$ and $L = 2, J = 1$ states induced by the tensor force.

In the light front formulation, the rotational symmetry turns to a hidden symmetry, with apparent distinctions between longitudinal and transverse coordinates. Therefore, the LFWM mixing related to the quadrupole moment, takes two different forms:

1. for $\Lambda = 1$ it is mixing of $\Psi_{0,0}, \Psi_{0,2}$;

2. for $\Lambda = 0$ it is mixing of $\Psi_{0,-1}, \Psi_{0,1}$.

Note that the indices here are the quantum numbers $n$ and $m$.

A. S-D mixing in the rest frame

To assess the S-D mixing for Upsilon in the center of mass frame, we need to first consider the splitting due to the repulsive centrifugal potential $(6/r^2)$ originating from the free Laplacian plus the Cornell potential, with the result

$$E_2 - E_0 = 0.46669 - (-0.47682) = 0.943 GeV$$

which should is still subject to corrections by spin-dependent forces). This value is to be compared to the empirical mass difference

$$M_{\Upsilon_2} - M_{\eta_b} = 10.2325 - 9.3987 = 0.834 (GeV)$$

The S-D mixing requires two states with the same $J = 1$, which are constructed in a standard way, via Clebsch-Gordon coefficients
\[ \psi_0^{M_j=1} = \psi_0(r) Y_0^0 \chi_1^1 \]
\[ \psi_2^{M_j=1} = \psi_2(r) \left( \sqrt{\frac{3}{5}} Y_2^2 \chi_1^{-1} - \sqrt{\frac{3}{10}} Y_1^0 \chi_1^0 + \sqrt{\frac{1}{10}} Y_0^0 \chi_1^1 \right) \]  

Here $Y_\ell^m(\theta, \phi)$ are spherical harmonics, and $\chi_S^{M_s}$ are states of total spin composed of $Q$ and $\bar{Q}$. To proceed, we use the standard notation for the tensor force $V_T(r)S_{12}$, and the non-diagonal matrix element with the angular integral

\[ \int d\Omega(Y_0^0 \chi_1^1) S_{12} \]
\[ \times \left( \sqrt{\frac{3}{5}} Y_2^2 \chi_1^{-1} - \sqrt{\frac{3}{10}} Y_1^0 \chi_1^0 + \sqrt{\frac{1}{10}} Y_0^0 \chi_1^1 \right) = \sqrt{8} \]

As a result, the quadrupole moment is given by an integral

\[ \langle Q \rangle_T = \frac{\epsilon_{02}}{\sqrt{2}} \int dr r^4 \Psi_{00}(r) \Psi_{02}(r) \approx \epsilon_{02}(0.14 \text{ GeV}^{-2}) \]  

where the admixing amplitude of the D state is $\epsilon_{02}$. If we assume it to be small, it is then given by the perturbative matrix element of the tensor mixing operator, sandwiched between states calculated using the Cornell potential

\[ \epsilon_{02} = \frac{\sqrt{8} \int dr r^2 \Psi_{00}(r) V_{+2}(r) \Psi_{02}(r) }{E_2 - E_0} \]

For an estimate, we may use the perturbative contribution with

\[ V_T(r) = \frac{4}{3} \frac{\alpha_s}{r^3} \]  

\[ \text{B. Quadrupole moment of Upsilon meson from the light front Hamiltonian} \]

In this subsection we still consider the case of $\Upsilon$, a vector meson made of $b\bar{b}$ quark pair. In the rest frame we just discussed, the cases of transversely polarized $\Lambda = 0$ and longitudinally polarized $\Lambda = \pm 1$ are related by the $O(3)$ rotational symmetry. The matrix elements of the various tensor operators over the corresponding states, are tied by the Wigner-Eckart theorem, and given by Clebsch-Gordon coefficients times a “reduced” (rotationally invariant) matrix element independent of the meson orientation.

In the light front Hamiltonian $H_{LF}$ in (36), the spin-tensor potential for heavy quarkonia comes from the last term with the instanton-induced effects (32). The spin operator can be re-written in a more transparent way as

\[ \left( \sigma_1 \cdot \hat{b}_2 \sigma_2 \cdot \hat{b}_1 - \frac{1}{2} \sigma_1 \cdot \sigma_2 \right) \]
\[ = \frac{1}{4} \left( \sigma_1 - \sigma_2 - e^{2i\phi} \sigma_1 + \sigma_2 + e^{-2i\phi} \right) \]

The dependence on the azimuthal angle $\phi$ reflects on the mixing between the $m$ and $m \pm 2$ states. We recall that the instanton contribution to the central potential was discussed in detail in [17], for the “dense instanton ensemble” with diluteness parameter $\kappa$ set to one. The plot of the central potential $V_C(\xi_x)$ and its second derivative $V''_C(\xi_x)$ are given in Fig.5. Note the change in sign at $b_1 \sim 1.5 \text{ GeV}^{-1}$, a distance comparable to the size of Upsilon.

With our usual approximation for heavy quarkonia, $M \approx 2m_Q$ and $\xi_x \approx b_1$, its contribution to the mixing part of $H_{LF}$ takes the form

\[ \langle n_1 m | V_{\pm 2} | n_2, m \pm 2 \rangle \]
\[ = \int d^2 b_1 d\xi \Psi_{n_1 m}^* V_T(b_1) e^{-2i\phi} \Psi_{n_2 m+2} \]

\[ \text{(58)} \]

We show the factors depending on $\phi$ explicitly, but omit the spin operators.
FIG. 5. The central part of the instanton induced potential $V_C(\xi_x)$ versus the distance $\xi_x = r$ (top), and its second derivative $V''_C(\xi_x)$ (bottom). See text.

To simplify the wave functions, let us for now ignore $\tilde{V}$ in the Hamiltonian, which means using instead of $\Psi_{0,0}, \Psi_{0,2}$ LFWFs, the functions $\psi_{0,0}, \psi_{0,2}$ of the oscillator basis, (A3) and (C3). Recall that in the coordinate representation for the $\bar{b}b$ mesons, the size parameter $\beta \approx 0.62 \text{GeV}$. The result is

$$\langle 00 | V_{\pm 2} | 0, 2 \rangle \approx -0.011 \text{GeV}^2$$  \hspace{1cm} (59)

Note that if the size integral is split into the contributions stemming from the small plus large radial intervals, i.e. $[0, 1.5 \text{GeV}^{-1}]$ plus $[1.5 \text{GeV}^{-1}, \infty]$, we find 0.007 and -0.018, respectively.

When (59) is divided by the difference of the mass squared for the two mixing states

$$\Delta M^2 = M^2_{\Upsilon_2} - M^2_{\eta_b}$$

$$= 10.2325^2 - 9.3987^2 \approx 16.4 \text{GeV}^2$$

we get our estimates of the mixing parameter $\epsilon_{02} \approx 0.00064$. As a result, the estimate for the Upsilon quadrupole moment is then

$$Q_{\Upsilon} \approx 2\epsilon_{02} \int d^2b_\perp dx \Psi_{0,0} \Psi_{0,2} b^2_\perp \approx -0.0095 \text{GeV}^2$$

which means that the usually quoted combination is

$$Q_{\Upsilon} M^2_{\Upsilon} \approx -0.87$$ \hspace{1cm} (61)

As we will see in the next subsection, it is right in the ballpark of other determination. Unfortunately, this result is relatively uncertain since it comes from significant cancellations of small and large ranges in the $b_\perp$-mixing integral. Deformations of the instantons – e.g. in instanton-antiinstanton “molecule” configuration described by streamline or thimble configuration – would change this number. Putting this observation into a positive direction, we may conclude that the quadrupole moments of mesons, are sensitive to the exact nature of the nonperturbative vacuum fluctuations.

We recall that in [16], we extracted the matrix element of the tensor force from the masses of the $P$-wave states of mesons with different quark species, ranging from the heavy $\chi_b$ to the light $K, \pi$. We noted that this matrix element changes sign, in going from heavy to light mesons. This observation is consistent with the calculation of this section. This issue clearly deserves further studies.

C. Quadrupole moments of vector mesons from lattice and other approaches

There have been several lattice measurements of the quadrupole moments of vector
mesons, and in all fairness we will not be able to cover them in this comparative study. In a recent lattice study of vector mesons composed of light, strange and charmed quarks with $V = \rho, K^*, \rho_s, \rho_c$ (the latters carry artificial charge assignments), it was numerically found that $Q_V M_V^2 \approx -0.3$ [26], which is comparable to an earlier study with $Q_V M_V^2 \approx -0.23(2)$ [27]. When extrapolated to bottomium, the recent lattice result gives $Q_\Upsilon \approx -0.003\,GeV^2$, with a mixing parameter $\epsilon_{02} \approx 0.02$ from our analysis.

Adhikari et al [28] have used their version of light-front Hamiltonian and wave functions, and calculated form-factors for the lowest states of charmonia and bottomonia. From their Table V, we see that their value for Upsilon is larger $Q_\Upsilon M_\Upsilon^2 = -0.731(9)$. In sum, the spread of these numbers is about a factor of 3, so the magnitude of the quadrupole moment of Upsilon remains relatively uncertain.

VII. GENERAL SPIN AND ORBIT MIXING FOR LIGHT QUARKS

In so far, we have considered mixing between the $m = 0$ and $m = 2$ components of the quarkonia wave function by the tensor force $V_{zz}$. Similarly, we can include spin-orbit force $V_{z1}$ and spin-spin forces, generating the whole $3 \times 3$ mixing matrix. However, since we know that all mixing is suppressed by powers of the heavy quark mass, we can treat these mixings perturbatively and additively, as we did above.

Instead, we now switch to the more involved case of light quarks, where the mixing is not expected to be suppressed. Of course, this is well known from the spectroscopy in the rest frame: heavy quarkonia are nonrelativistic, while light quark systems are not. In the rest frame, it is difficult to compare these two limiting cases of the meson spectroscopy. Fortunately, in the light front the comparison is possible, as light and heavy quarks are treated democratically.

The light front Hamiltonian has the same form for both cases, with only few parameters due for change. The only special case is the pion as a Goldstone mode, that we will address in the next section (see also our qualitative analysis in [17]).

The “basic problem” of two constituent quarks connected by a confining string was already considered in [17]. There we did not yet have mixing of states with different (transverse) orbital momentum $m$, and considered only the set of functions with $m = 0$. The basis functions with $m \neq 0$ are discussed in Appendix C, including the transition from the momentum to the coordinate representation.

The general form of the mixing matrix $H_A$ for a meson with helicity $\Lambda = J_z$, was already given in (46). The derivation of the perturbative and instanton-induced spin- and orbital-m-changing effects were given in our recent analysis in [17] and above. Our current task is to evaluate the corresponding matrix elements.

The diagonal in the $m$-part or $V_{\text{diag}}$, consists of two parts, the one coming from the spinless $H_0 + V$ Hamiltonian, and the one from the spin part. There is no need to describe in details the former. In brief, the values of the squared masses for the $m = 0, 1, 2$ states were obtained after its diagonalization. Using the states detailed in the Appendix, we get for the longitudinally polarized vector $\Lambda = 1$ case the following diagonal elements

$$H_1^{00} = 2.229, \quad H_1^{11} = 2.833, \quad H_1^{22} = 3.434\,(GeV^2)$$

(62)

Three spin states $S = 1, S_z = 1, 0, -1$ are

$$|\uparrow\uparrow\rangle, (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)/\sqrt{2}, |\downarrow\downarrow\rangle$$

respectively. The spin-spin forces are proportional to the same value $\langle S_1 \cdot S_2 \rangle = 1/4$, and the corresponding matrix elements of the perturbative (26), and instanton-induced (34) potentials with $|\Psi_{\text{on}}|^2$.

The near-diagonal $V_{z1}$ part of the mixing matrix is due to the spin-orbit forces, from the perturbative (26) and instanton-induced effects (34). The former is proportional to $-\hat{S}_1 \cdot \hat{L}_2 + \hat{S}_2 \cdot \hat{L}_1$, in which $\hat{L}_2 = -\hat{L}_1$ so the two terms are added into the total spin
\[ \begin{align*}
\hat{S}. \text{ We only consider the non-diagonal operators } S^+L^- + S^-L^+ \text{ that flip the spin and } m \text{ by } \pm 1. \text{ The perturbative potential is } & \sim 1/\xi^3. \\
\text{If we ignore the longitudinal distance and use } \xi^2 \approx \vec{b}^2 \perp, \text{ we may worry of the convergence of the integrand at the origin. In the transition between } m = 0 \text{ and } m = 1, \text{ one indeed finds a logarithmic divergence} \\
\int d^2b_\perp \Psi_{00} \frac{1}{b^3} \Psi_{01} \\
\sim \int db_\perp b_\perp * \frac{1}{b^4} * b_\perp & \sim \log(b_{min})
\end{align*} \]

since at small \( b_\perp, \Psi_{00} \sim b_\perp^0, \Psi_{01} \sim b_\perp \). This logarithmic divergence is cut-off by the small longitudinal distance of about \( \pi/M \approx \pi/2m_Q \).

In contrast, in the transition between \( m = 1 \) and \( m = 2 \) the integral has \( \Psi_{02} \sim b_\perp^2 \) instead of \( \Psi_{00} \), so it is convergent.

The instanton-induced spin-orbit contributes to the light front Hamiltonian \( H_{LF} \), with the corresponding potential \( V'_C(\xi_x) \) shown in Fig.6. It is regular at the origin, but with a relatively small range of about an instanton size \( \sim 1.5 GeV^{-1} \sim 0.3 \text{ fm} \).

The \( V_{\pm 2} \) or tensor forces were discussed in details above, for the case of the transition between the \( m = 0 \) and \( m = 2 \) states. The corresponding instanton-induced potential is shown in the lower Fig.5.

After the evaluation of all matrix elements, we obtain the following mixing matrix

\[ H_{\Lambda=1} = \begin{pmatrix}
M_0^2 + C^{00} + SS_C^{00} + SS_{inst}^{00}, & SL_C^{01} + SL_{inst}^{01}, & M_1^2 + C^{11} + SS_C^{11} + SS_{inst}^{11} & T_{inst}^{02} \\
SL_C^{01} + SL_{inst}^{01}, & M_1^2 & SL_C^{12} + SL_{inst}^{12}, & M_2^2 + C^{22} + SS_C^{22} + SS_{inst}^{22}
\end{pmatrix}
\]

All entries in (64) are explained in Appendix D.

The mixing changes the squared masses of the three \( \Lambda = 1 \) states as follows

\[ \{M_0^2, M_1^2, M_2^2\} = \{2.229, 2.833, 3.434\} \rightarrow \{M_0^2, M_1^2, M_2^2\} = \{1.940, 2.779, 3.307\} \]

The largest change is, as expected, a downward shift of the ground state.
\[ \Psi_a = 0.922252\psi_0 - 0.377037\psi_1 + 0.0854116\psi_2 \]
\[ \Psi_b = 0.381066\psi_0 + 0.923825\psi_1 - 0.0365531\psi_2 \]
\[ \Psi_c = -0.0651235\psi_0 + 0.0662586\psi_1 + 0.995675\psi_2 \]

There is significant 0-1 mixing due to spin-orbit interactions.

**VIII. INCLUDING 'T HOOFT EFFECTIVE LAGRANGIAN**

The zero-mode contributions due to tunneling are captured by the 't Hooft determinantal interaction, in the rest frame. The 3-flavor determinantal interaction reduced to 2-flavor reads [16, 17]

\[
V_{TH}(1, 2) = -\frac{1}{4}\hat{\kappa}_2|A_{2N}\left(1 - \tau_1 \cdot \tau_2\right)\
\times \left(1 - 16B_{2N}S_1 \cdot S_2\right) \delta(\vec{x}_{12})
\]

(65)

with the pair spatial distance \(\vec{x}_{12} = \vec{x}_1 - \vec{x}_2\), in the ultra-local approximation for the instanton. However, the fermionic zero modes ride a non-local instanton tunneling process, and its interpretation in a boosted frame, requires analytical continuation.

To derive the analogue of (65) on the light front, we need to show how to analytically continue the fermionic tunneling process to the light front, where the in-out quarks are nearly on mass-shell. We need an LSZ reduction scheme in *Euclidean signature*, that extends to the light front in Minkowski signature. For that, we follow the proposal suggested in [29].

**A. LSZ reduction in the rest frame**

For two flavors, the LSZ reduced 't Hooft vertex between on-shell light quarks in the rest frame is

\[
\langle \chi_R^\dagger(k_2)\Phi_0^\dagger(-k_2)\frac{1}{i\rho_q}\Phi_0^\dagger(k_1)i\hat{k}_1\chi_L(k_1) \rangle \times
\]
\[
\left[\chi_R^\dagger(k_2)\Phi_0^\dagger(-k_2)\frac{1}{i\rho_q}\Phi_0^\dagger(k_1)i\hat{k}_1\chi_L(k_1) \right]_U
\]

The averaging in (66) is over the \(SU(N_c)\) color moduli \(U\). Here each factor is on mass shell using the long time limit in Euclidean signature. More specifically, for the in-coming left-handed and on-shell \(\chi_L(k_1)\) going through an instanton, we define

\[
\Phi_0^\dagger(k_1)i\hat{k}_1\chi_L(k_1) e^{-|\vec{k}_1|/|T|} = \lim_{|T| \to \infty} \int d^3y \left(\frac{\rho_q^2}{\pi} U^1 ey \chi_L(k_1)\right) e^{-i\vec{k}_1 \cdot \vec{y}}(66)
\]

with \(\Pi_y = 1 + \rho^2/y^2\), and \(y = (-T, \vec{y})\). In the large *Euclidean* time limit \(\Pi_y \to 1\), and the \(y\)-integration reduces to

\[
\lim_{|T| \to \infty} \int d^3y \frac{-T^1 - i\vec{\sigma} \cdot \vec{y}}{(T^2 + \vec{y}^2)\Pi_y^2} e^{-i\vec{k}_1 \cdot \vec{y}}
\]

(67)

Note the appearance of the mass-shell condition \(E_1 = |\vec{k}_1|\) in the exponent, supporting the LSZ reduction in Euclidean signature. As a result, (66) simplifies

\[
\Phi_0^\dagger(k_1)i\hat{k}_1\chi_L(k_1) = (\pi\rho_q^2)\left[U^+ e(1 - \vec{\sigma} \cdot \vec{k}_1) \chi_L(k_1)\right] = (2\pi\rho_q^2)\left[U^+ e \chi_L(k_1)\right]
\]

(68)

Since \(\chi_L(k_1)\) is left-handed in the small current mass \(m_q\) limit, the right-most result follows. A repeat of this analysis, yields (66) in the form
B. LSZ reduction on the light front

To carry the preceding analysis to the light front, we first carry the analogue of the LSZ reduction along \( y_+ = \cos \theta y_4 + \sin \theta y_3 \) for large \( y_+ \) in Euclidean signature, integrate over the remaining orthogonal directions \( y_- = \sin \theta y_4 - \cos \theta y_3 \) and \( y_\perp \), and then analytically continue \( \theta \to -i \chi \). More specifically, the analogue of (67) is now

\[
\lim_{y_+ \to \infty} \int dy_-dy_\perp e^{-ik_{1+}y_- - ik_{1-}y_\perp} \times \left( \frac{y_+ (\cos \theta - i \sin \theta \sigma_3) + y_- (\sin \theta + i \cos \theta \sigma_3) - i \sigma_\perp \cdot y_\perp}{(y_+^2 + y_-^2 + y_\perp^2)^{3/2}} \right)
\]

\[
= \pi^2 \left( (\cos \theta \mathbf{1} - i \sin \theta \sigma_3) - \frac{ik_{1+}}{k} (\sin \theta \mathbf{1} + i \cos \theta \sigma_3) - i \sigma_\perp \cdot y_\perp \right) e^{-k|y_+|}
\]

in the large \( y_+ \) limit, with

\[
k_{1+} = \cos \theta k_4 + \sin \theta k_3 \\
k_{1-} = \sin \theta k_4 - \cos \theta k_3 \\
k = (k_{1-}^2 + k_{1+}^2)^{3/2}
\]

The analytical continuation \( \theta \to -i \chi, y_4 \to iy_0 \) and \( k_4 \to -ik_0 \) yield (70) in the form

\[
2\gamma \pi^2 (1 - \sigma_3) e^{-k|y_+|} \to \pi^2 (1 - \sigma_3) e^{-k|y_+|}
\]

In retrospect, (72) shows that (69) extends to the light front in the form

\[
\left( \frac{4\pi^2 \rho^3}{im_q} \right)^2 \left\langle \left[ \chi_R^\dagger(k_2)eU \right] [U^\dagger \epsilon \chi_L(k_1)] \times [\chi_R^\dagger(k_1) eU] [U^\dagger \epsilon \chi_L(k_2)] \right\rangle_U
\]

with the \( L, R \) polarizations solely along the 3-direction. In other words, on the light front helicity and chirality are identified: a left-handed quark with spin down, flips to a right-handed quark with spin up as it tunnels through an instanton on the light front. The opposite flip takes place through an anti-instanton.
C. Zero mode induced ('t Hooft Lagrangian) interaction on the light front

(73) does not carry any form factor in leading order, by the LSZ reduction. Its contribution to the invariant meson squared mass is

\[- \int \frac{dk^+}{4\pi k^+} \frac{dk^+}{4\pi k^+} \frac{dk_\perp}{(2\pi)^2} \frac{dk_\perp}{(2\pi)^2} \times (2\pi)^3 2P^+ \delta \left( k^+_1 + k^+_2 - k^+_3 - k^+_4 \right) \delta \left( k_{1\perp} + k_{2\perp} - k_{3\perp} - k_{4\perp} \right) \]

\[
\times \left( \frac{4\pi^2}{i m_q} \right)^2 \left\langle \left[ \chi_{3R}^+(k_2) e U \right] \left[ U^\dagger \epsilon \chi_{3L}(k_1) \right] \times \left[ \chi_{3R}^+(k_2) e U \right] \left[ U^\dagger \epsilon \chi_{3L}(k_1) \right] + L \leftrightarrow R \right\rangle_U
\]

(74)

with the anti-instanton contribution added. Recall that in the rest frame, an effective form factor of the form (Euclidean signature)

\[
\left( M(k_1) M(k_2) M(k_3) M(k_4) \right)^{\frac{1}{2}} \rightarrow \left( M(k_{1\perp}) M(k_{2\perp}) M(k_{3\perp}) M(k_{4\perp}) \right)^{\frac{1}{2}}
\]

(75)

is induced, with \( M(k) \) the running constituent mass. (75) regulates the momentum transfers, since the in-out quark pair is not on mass-shell. We expect the rightmost form factor in (75) to carry to the light front when the strict mass-shell limit is lifted as noted in [15, 30].

The color averaging in (74) is similar to the color averaging carried in the rest frame (see Eq. 75 in [16]). The result for the bracket in the mesonic channel is

\[
\left\langle \left[ \overline{U}(k_2, s_2) U(k_1, s_1) \overline{V}(k_4, s_4) V(k_3, s_3) + \overline{U}(k_2, s_2) \gamma^5 U(k_1, s_1) \overline{V}(k_4, s_4) \gamma^5 V(k_3, s_3) \right. \right. \]

\[
\left. \left. - \overline{U}(k_2, s_2) \gamma^a U(k_1, s_1) \overline{V}(k_4, s_4) \gamma^a V(k_3, s_3) - \overline{U}(k_2, s_2) \gamma^a \gamma^5 \gamma^a U(k_1, s_1) \overline{V}(k_4, s_4) \gamma^a \gamma^5 \gamma^a V(k_3, s_3) \right) \left. \right. \]

\[
\left. \left. - 4B_{2N} \left( \overline{U}(k_2, s_2) \sigma^a U(k_1, s_1) \overline{V}(k_4, s_4) \sigma^a V(k_3, s_3) + \overline{U}(k_2, s_2) \gamma^a \gamma^5 \gamma^a U(k_1, s_1) \overline{V}(k_4, s_4) \gamma^a \gamma^5 \gamma^a V(k_3, s_3) \right) \right. \left. \right. \]

\[
\left. \left. - \overline{U}(k_2, s_2) \sigma^a \gamma^b U(k_1, s_1) \overline{V}(k_4, s_4) \sigma^a \gamma^b V(k_3, s_3) - \overline{U}(k_2, s_2) \gamma^a \gamma^5 \gamma^a \gamma^b U(k_1, s_1) \overline{V}(k_4, s_4) \gamma^a \gamma^5 \gamma^a \gamma^b V(k_3, s_3) \right) \right\rangle
\]

(76)

with \( U(k, s), V(k, s) \) the quark and the antiquark LF spinors respectively (see Appendix F). Note that the same interaction holds for

\[(\overline{U} U)(\overline{V} V) \rightarrow (\overline{U} V)(\overline{V} U)\]

through Fierz re-arrangements.

Besides the standard flavor-determinantal character of the squared mass operator (74) after color averaging, its chief effect is to flip the helicity/chirality/spin of the in-out nearly on-shell light-front quark pair, in the chiral limit. For three light flavors \( u, d, s \) in QCD, the arguments are similar but the strange quark loops in the sea as \( \langle \bar{s}s \rangle < 0 \). The reduction to two flavors \( u, d \) is structurally identical to (74), with only the overall coupling modified and sign switched.
On the LF, each of the contributions in (76) is spin-valued. If we denote by \([s_2s_1]\) the entries with \(s_1\) for the initial spin and \(s_2\) for the final spin, then the matrix valued forms are

\[
\begin{align*}
\mathcal{U}_L(k_2, s_2)\mathcal{U}_R(k_1, s_1) &= +\sqrt{k_1^+k_2^+} \begin{pmatrix}
\frac{m_{Q_1}}{k_1^+} & 0 \\
\frac{k_1^+ - k_2^+}{k_1^+ k_2^+} & \frac{m_{Q_1}}{k_2^+}
\end{pmatrix} = +\sqrt{x_1 x_2} \begin{pmatrix}
\frac{m_{Q_1}}{x_1} & 0 \\
\frac{k_1^+ - k_2^+}{x_1 x_2} & \frac{m_{Q_1}}{x_2}
\end{pmatrix}, \\
\mathcal{U}_R(k_2, s_2)\mathcal{U}_L(k_1, s_1) &= +\sqrt{k_1^+k_2^+} \begin{pmatrix}
\frac{m_{Q_2}}{k_2^+} & 0 \\
\frac{k_2^+ - k_1^+}{k_2^+ k_1^+} & \frac{m_{Q_2}}{k_1^+}
\end{pmatrix} = +\sqrt{x_1 x_2} \begin{pmatrix}
\frac{m_{Q_2}}{x_2} & 0 \\
\frac{k_2^+ - k_1^+}{x_1 x_2} & \frac{m_{Q_2}}{x_1}
\end{pmatrix}, \\
\mathcal{V}_L(k_1, s_1)\mathcal{V}_R(k_2, s_2) &= -\sqrt{k_1^+k_2^+} \begin{pmatrix}
\frac{m_{Q_2}}{k_2^+} & 0 \\
\frac{k_2^+ - k_1^+}{k_2^+ k_1^+} & \frac{m_{Q_2}}{k_1^+}
\end{pmatrix} = -\sqrt{x_1 x_2} \begin{pmatrix}
\frac{m_{Q_2}}{x_2} & 0 \\
\frac{k_2^+ - k_1^+}{x_1 x_2} & \frac{m_{Q_2}}{x_1}
\end{pmatrix}, \\
\mathcal{V}_R(k_1, s_1)\mathcal{V}_L(k_2, s_2) &= -\sqrt{k_1^+k_2^+} \begin{pmatrix}
\frac{m_{Q_1}}{k_1^+} & 0 \\
\frac{k_1^+ - k_2^+}{k_1^+ k_2^+} & \frac{m_{Q_1}}{k_2^+}
\end{pmatrix} = -\sqrt{x_1 x_2} \begin{pmatrix}
\frac{m_{Q_1}}{x_1} & 0 \\
\frac{k_1^+ - k_2^+}{x_1 x_2} & \frac{m_{Q_1}}{x_2}
\end{pmatrix}
\end{align*}
\]

(77)

with \(k_{1,2}^+ = x_{1,2} P^+\). For eikonalized longitudinal momenta \(k_1^+ \approx k_2^+\) commensurate with our use of the Wilson lines, (77) simplify

\[
\begin{align*}
\mathcal{U}_L(k_2, s_2)\mathcal{U}_R(k_1, s_1) &\rightarrow \begin{pmatrix} m_{Q_1} & 0 \\ \Delta_R & m_{Q_1} \end{pmatrix} = m_{Q_1} 1 + \Delta_R \sigma^- \\
\mathcal{U}_R(k_2, s_2)\mathcal{U}_L(k_1, s_1) &\rightarrow \begin{pmatrix} m_{Q_1} & -\Delta_R \\ 0 & m_{Q_1} \end{pmatrix} = m_{Q_1} 1 - \Delta_R \sigma^+ \\
\mathcal{V}_L(k_1, s_1)\mathcal{V}_R(k_2, s_2) &\rightarrow \begin{pmatrix} -m_{Q_2} & 0 \\ \Delta_R & -m_{Q_2} \end{pmatrix} = -m_{Q_2} 1 + \Delta_R \sigma^- \\
\mathcal{V}_R(k_1, s_1)\mathcal{V}_L(k_2, s_2) &\rightarrow \begin{pmatrix} -m_{Q_2} & 0 \\ -\Delta_R & -m_{Q_2} \end{pmatrix} = -m_{Q_2} 1 - \Delta_R \sigma^+
\end{align*}
\]

(78)

with the momentum transfer \(\Delta^\mu = k_1^\mu - k_2^\mu\), \(\Delta_L = \Delta^1 - i\Delta^2 = \Delta_R^\ast\), and \(\sigma^\pm = \frac{1}{2}(\sigma^1 \pm i\sigma^2)\). Inserting (78) into (76) yields the local determinantal 2-body interaction potential (76) on the light front as

\[
V_{FH}^i \simeq -|\tilde{\kappa}_2| A_{2N} \frac{1}{4} (1 - \tau_1 \cdot \tau_2) \\
\times \left[ 4m_{Q_1} m_{Q_2} 1_1 1_2 - 2(\sigma_{1 \perp} \cdot i\nabla_\perp m_{Q_2} 1_2 - m_{Q_1} 1_1 \sigma_{2 \perp} \cdot i\nabla_\perp) + \sigma_{1 \perp} \cdot \sigma_{2 \perp} \nabla_\perp^2 \right] \delta(P^+ z^-) \delta(x_\perp)
\]

(79)

in the U(1) or \(\eta'\) channel, where only the leading \(1/N_c\) contribution is shown. We have set

\[
A_{2N} = \frac{2N_c - 1}{2N_c (N_c^2 - 1)} \quad \tilde{\kappa}_2 = 3G_{\text{Hooft}} \langle \bar{s}s \rangle < 0 \quad G_{\text{Hooft}} = \frac{n_f + 1}{2} (4\pi^2 \rho^3) \int_{f=u,d,s} \frac{1}{m_f^3} 
\]

(80)

with \(\sigma_{1 \perp}\) the \(\perp\)-Pauli matrices for particle \(i = 1, 2\). The flavor permutation \(P_{12}^f\) is manifest in (76) as carried in (80)

\[
\frac{1}{4} (1 - \tau_1 \cdot \tau_2) = \frac{1}{2} (1 - P_{12}^f)
\]

This is just the projector on the flavor singlet states. This is at the origin of the famed 't Hooft determinantal interaction, which helps solve the U(1) problem for the 3-flavor case. (Note that the interaction is repulsive in the un-projected 3-flavor U(1) channel).
Both spin contributions in (79) flip the spin of the incoming quark pair from L-down to R-up in the instanton contribution, and vice-versa in the anti-instanton contribution. In the ultra-local approximation, we may trade $\nabla^2_\perp \rightarrow 1/\rho^2$, hence the instanton induced spin-spin interaction in the 2-flavor singlet channel,

$$-|\tilde{\kappa}_2|A_{2N}\frac{1}{2}(1 - \tau_1 \cdot \tau_2) S_{1\perp} \cdot S_{2\perp} \delta(P^+ z^-) \delta(x_\perp)$$

(81)

The nature of the instanton induced interactions in the other meson channels with different spin-flavor, is manifest in the individual contributions in the Fierzed form (76), with no contribution to the vector and pseudo-vector channels. For instance, in the pion channel, the light front interaction is

$$|\tilde{\kappa}_2|A_{2N}\frac{1}{4}(1 - \tau_1 \cdot \tau_2) S_{1\perp} \cdot S_{2\perp} \delta(P^+ z^-) \delta(x_\perp)$$

(82)

which is attractive in the isospin-triplet and spin-singlet state, key for a massless pion. This point has been addressed in the literature many times before, including briefly in this series [17] (note that in the latter the interaction was assumed local in the invariant distance $\xi$ for simplicity). If we were to treat this interaction perturbatively, it may appear that we should fit the magnitude of the 4-fermion coupling constant to put the total pion mass to zero. However, this is not correct. This effect is dominant and leading in the pion channel, and should not be treated perturbatively, as we showed qualitatively in [17]. For any large enough coupling (and thus instanton density) it breaks chiral symmetry and (in the chiral limit) produces exactly massless Nambu-Goldstone modes, the pions. Yet to show that quantitatively, we need to re-derive the essentials of chiral symmetry breaking on the light front, a task we will discuss later in these series.

IX. VARIOUS OBSERVABLES

A. Parton distribution functions

The parton distributions functions or PDFs count the parton content of a given hadronic state. They are matrix elements of various operators sandwiched between pertinent light front wavefunctions. For the hadronic states (47-48) limited to the lowest 2-particle wavefunction contributions with net helicity zero, the pseudo-scalar and vector PDFs are given by

$$q_{P,V}(x) = \int \frac{d^2 k_\perp}{(2\pi)^2} (|\psi^0_{P,V}|^2 + |\psi^1_{P,V}|^2 + |\psi^{-1}_{P,V}|^2)$$

(83)

B. Distribution amplitudes

The distribution amplitudes DAs are defined as matrix elements of certain nonlocal operators on the light cone, sandwiched between the vacuum and pertinent hadronic states. They capture the longitudinal momentum, and transverse location of a parton in the hadronic state. DAs are widely used in the theory of hard exclusive reactions, such as the hadronic form-factors in elastic scattering, and heavy meson semi-leptonic decays. At high momentum transfer, factorization allows a split of the scattering amplitude into a “hard block operator”, sandwiched between two DAs. The moments of various DAs have been calculated on the lattice, for a reviews see [31].
The DAs are classified by the twist (dimension minus spin) of the operator involved. To give an example, since the early 1980’s, most exclusive reactions involving the pions are based on the following three DAs

$$
\int_{-\infty}^{+\infty} \frac{p^+ d^z}{2\pi} e^{ixp\cdot z} \langle 0 \left| \bar{d}(0)[0, z]u(0) \right| \pi^+(p) \rangle = \left( + \frac{if_\pi}{4} \gamma^5 \right) \left( \bar{\psi}\varphi^A_{\pi^+}(x) - \chi_\pi \varphi^P_{\pi^+}(x) - i\chi_\pi \sigma_{\mu\nu} \frac{p^\mu n^\nu}{p\cdot n} \frac{\varphi^T_{\pi^+}(x)}{6} \right)_{\alpha\beta} \tag{85}
$$

with $n^\nu$ a light-like vector in the $z$-direction. Note that although these matrix elements are nonlocal, the integral is just one-dimensional, taken along the light cone coordinate. The symbol $[x, y]$ is the shorthand notation for the gauge link between two points on the light front, and $\sigma_{\mu\nu} = \frac{i}{2} [\gamma_\mu, \gamma_\nu]$. In this example the first term contains momentum, while the other two do not. Therefore the axial A-DA is of leading twist, while the two others $P, T$-DA are subleading (next twist) at large momentum $p \to \infty$.

The three functions $\varphi^i(x)$ have indices $i = A, P, T$ standing for axial, pseudoscalar and tensor gamma matrices in the operator. They are all normalized to 1. Their explicit definition follows from (85) by inversion

$$
\begin{align*}
\varphi^A_{\pi^+}(x) &= \frac{1}{if_\pi} \int_{-\infty}^{+\infty} \frac{dz}{2\pi} e^{ixp\cdot z} \langle 0 \left| \bar{d}(0)\gamma^+ \gamma_5 [0, z]u(z) \right| \pi^+(p) \rangle \\
\varphi^P_{\pi^+}(x) &= \frac{1}{f_\pi \chi_\pi} \int_{-\infty}^{+\infty} \frac{d^z}{2\pi} e^{ixp\cdot z} \langle 0 \left| \bar{d}(0)i\gamma_5 [0, z]u(z) \right| \pi^+(p) \rangle \\
\varphi^T_{\pi^+}(x) &= \frac{-6}{f_\pi \chi_\pi p\cdot n} \int_{-\infty}^{+\infty} \frac{p^+ dz}{2\pi} e^{ixp\cdot z} \langle 0 \left| \bar{d}(0)\sigma_{\mu\nu} \gamma_5 [0, z]u(z) \right| \pi^+(p) \rangle \tag{86}
\end{align*}
$$

The constants DAs are normalized to 1. The axial pion DA is normalized by the weak pion decay constant $f_\pi \approx 133$ MeV, and the pseudoscalar and tensor pion DAs are normalized by $\chi_\pi$ which follows from the chiral algebra. These chiral parameters are a measure of the pion wavefunction at the origin of the transverse plane (zero transverse distance $b_\perp = 0$). The DAs are only a function of the longitudinal momentum $x$.

**Pion axial DA:**

To relate the pion axial DA in (86) to the pseudoscalar light front wavefunction with helicity-0 in (47), we note the identity

$$
\tau^+ \left( \frac{1}{4} \bar{\psi} \gamma^5 \psi \right)_{\alpha\beta} = \
\frac{1}{2} \frac{1}{\sqrt{4\pi x \bar{x}}} (u_{\alpha\uparrow}(1) \bar{d}_{\beta\downarrow}(2) - u_{\alpha\downarrow}(1) \bar{d}_{\beta\uparrow}(2)) \tag{87}
$$

in the light front limit. With this in mind, the leading twist-2 or axial distribution amplitude for $P = \pi^+$ (or any pseudoscalar) in (86), matches the $m = 0$ contribution in (47),

$$
\varphi^P_{\pi^+}(x) = \frac{2\sqrt{N_c}}{f_P} \int \frac{dk_\perp}{(2\pi)^2} \psi_0^P(x, k_\perp) \tag{88}
$$
FIG. 7. Distribution amplitudes for $\bar{b}b$ (upper) and a "generic" light $\bar{q}q$ meson (lower) as a function of $x$, for the three lowest states $n = 0, 1, 2$. For bottomium, the difference between the three curves is too small to be visible. For the light meson, the differences are visible. With increasing $n$, the DAs become narrower and higher at $x = \frac{1}{2}$.

By the same reasoning, the corresponding contribution for the vectors, matches the $m = 0$ contribution in (48)

$$\varphi_V(x) = \frac{2\sqrt{N_c}}{f_V} \int \frac{dk_\perp}{(2\pi)^3} \psi_0^V(x, k_\perp)$$  \hspace{1cm} (89)

**Pion tensor DA:**

To relate the tensor pion DA amplitude in (86) to the pseudoscalar light front wavefunction with helicity-1 in (47), we note that the dominant tensor matrix element on the light cone reads

$$\langle 0 | \bar{d}(0)i\sigma^+i\gamma_5 u(z^-, z_\perp) | \pi^+(p) \rangle = 2p^+ \frac{\partial}{\partial z_\perp} \psi_1^T(z^-, z_\perp)$$  \hspace{1cm} (90)

with $(x = k^+/p^+)$

$$\psi_1^T(x, k_\perp) = \int dz^-dz_\perp e^{i(k^+z^- - k_\perp \cdot z_\perp)} \psi_1^P(z^-, z_\perp)$$  \hspace{1cm} (91)

Note the relation

$$\psi_{\pm 1}^P(x, k_\perp) = k_\perp^{\pm} \psi_1^P(x, k_\perp)$$

with $k_\perp^{\pm} = k_x \pm ik_y$. A comparison with (86) shows that the twist-3 pion distribution amplitude in $\varphi_T(x, k_\perp)$ matches the $m = 1$ contribution in (47) through

$$\frac{\partial}{\partial k_\perp} \varphi_T(x, k_\perp) = \frac{6}{f_\pi \chi_\pi} k_\perp^i \psi_1^P(x, k_\perp)$$  \hspace{1cm} (92)

**Bottomium DA:**

The generic LFWFs depend on all three variables $x, b_\perp$. The DAs are LFWF’s at the origin in the coordinate representation, or the overall integral

$$\Psi_{nm}(b_\perp = 0, x) = \int \frac{d^2k_\perp}{(2\pi)^2} \Psi_{nm}(k_\perp, x)$$  \hspace{1cm} (93)

in the momentum representation. They are normalized so that the integral over $x$ is one. In Fig. 7 we show the DAs for $n = 0, 1, 2$ LFWFs, for $\bar{b}b$ and generic $\bar{q}q$ mesons.

Note first, that the heavier the meson, the slower the quark motion, with the DAs concentrated near $x = \frac{1}{2}$. Note also, that the bottomonia states with different orbital momenta $m$, have practically the same DAs. Note finally, that the oscillations in the DAs for bottomonia, are an artifact of the small basis set of functions.
in $x$, we used. Before normalization, the lowest bottomium DA reads

$$ DA(n = 0, m = 0) \sim (2.19 \sin(\pi x) - 1.79 \sin(3\pi x) + 1.16 \sin(5\pi x) - 0.44 \sin(7\pi x)). $$

Note that the signs of the harmonics alternate, with a net effect being a suppression of the DA near the edges $x \to 0, 1$.

Decay constants:
These matrix elements are best interpreted in the chiral basis. For the pseudoscalar pion, the leading twist-2 DA $\varphi_\pi(x)$ is chirally-diagonal, but the subleading twist-3 are chirally non-diagonal. Fortunately, a matrix related the chiral basis to the spin basis has been already defined in the previous section.

The weak pion decay constant follows

$$ (0 | \bar{d}(0) \gamma^\mu (1 - \gamma^5) u(0) | \pi^+(p)) = -\text{Tr} \left( \gamma^\mu (1 - \gamma^5) \left( \frac{i}{4} \gamma^5 p \right) \right) 2 \int \frac{dk_\perp}{(2\pi)^3} \frac{dx}{\sqrt{N_c}} \psi_0^P(x, k_\perp) \equiv i f_\pi p^\mu $$

(94)

Chiral constant:
Isospin symmetry and charge conjugation force $\varphi_\pi(x) = \varphi_\pi(\bar{x})$. As pointed out initially in [32], there are two twist-3 and chirally non-diagonal independent DA $\varphi_\pi^P(x)$ and $\varphi_\pi^T(x)$, characterizing the pseudoscalar and tensor strength in the pion respectively. The latter are tied by the current identity

$$ \partial^\nu (\bar{d}(0)\sigma_{\mu\nu} \gamma_5 u(z)) = -\partial_\mu (\bar{d}(0)\gamma_5 u(z)) + m \bar{d}(0) \gamma_\mu \gamma_5 u(z) $$

and share the same couplings. The value of the dimensionful coupling constant $\chi_\pi$ can be fixed by the divergence of the axial-vector current and the PCAC relation

$$ (m_u + m_d) \langle 0 | \bar{d}(0) i \gamma^5 u(0) | \pi^+(p) \rangle = - (m_u + m_d) \text{Tr} \left( i \gamma^5 \left( \frac{i f_\pi}{4} \gamma^5 \chi_\pi \right) \right) \int_0^1 dx \varphi_\pi^P(x) = (m_u + m_d) f_\pi \chi_\pi $$

(99)

with $\varphi_\pi^P(x)$ normalized to 1. Using the Gell-Mann-Oakes-Renner relation

$$ f_\pi^2 m_\pi^2 = -2(m_u + m_d) \langle \bar{q} q \rangle $$

(100)

with $| \langle \bar{q} q \rangle | \approx (240 \text{ MeV})^3$, which yields

$$ \chi_\pi = \frac{m_\pi^2}{(m_u + m_d)} $$

(101)
C. Form-factors

Elastic scattering is the simplest exclusive process, and the corresponding mesonic form-factors have been studied extensively theoretically and experimentally, for about five decades. Here we do not have space for a review of their history, let us just say that early asymptotic predictions at large $Q^2$, based on one-gluon exchange, are not yet met, neither in experiment or on the lattice. (The “semi-hard” domain $Q^2 \sim \text{few GeV}^2$ is dominated by nonperturbative effects. In particular, in our previous paper [33], we have calculated the instanton-induced contributions to the hard block, and showed that they are comparable to the perturbative amplitudes. When combined, they reproduce the experimental/lattice data, provided that the diluteness parameter is not small $\pi^2 n_I + \bar{n}_I \rho^4 = O(1)$.)

On the light front, the form factors follow from the Drell-Yan-West construction using the good current $J^+ = J^0 + J^3$ [34, 35]. The analogue of the Breit-frame with fixed energy in the rest frame, is the Drell-Yan frame in the light front frame, with fixed longitudinal momentum $P^+ = P'^+$, for $P' = P + q$ with space-like squared momentum transfer $q^2 = -Q^2 = -q_{\perp}^2$. The key feature of this choice of current and frame, is that the vacuum production and annihilation diagrams are suppressed, and parton number is conserved in-out in the form-factor viewed as a process $\gamma^* + P \rightarrow P'$.

For instance, the Drell-Yan-West form-factor for charged pseudo-scalars is helicity preserving with [34–36]

$$F_P(Q^2) = \langle P', 0, 0 | J^+(0) | P, 0, 0 \rangle$$

$$= \int_0^1 dx \int \frac{dk_{\perp}}{(2\pi)^3} \psi^*_0(x, k_{\perp} + \bar{x} q_{\perp}) \psi_0^*(x, k_{\perp})$$

in the 2-particle approximation (low resolution). Similar expressions for the charged vectors can be derived. The thorough analyses of these form factors will be given in a sequel.

X. CONCLUSIONS

The chief aim of this series of papers is to derive the light-front Hamiltonian $H_{LF}$ and corresponding wave functions from first principles, using the theoretical and empirical information we have at the moment. The important distinctions between our approach and other versions of $H_{LF}$ in the literature are among others: (i) our $H_{LF}$ is not a model, meaning we do not invent its form, but derive it using certain approximations; (ii) therefore we do not fit any of the parameters to the experimental data, rather we use the standard values for the quark masses, the string tension $\sigma_T$, the perturbative coupling $\alpha_s$ and the instanton ensemble parameters; (iii) we do not consider just one or two lowest states in each quark channel, but look for as many states as feasible, by relating the results to Regge behavior; (iv) central to our derivation, is the QCD vacuum as we know it, at low resolution.

In a wider perspective, these works continue to bridge light front physics with the development in Euclidean space-time. The latter – lattice QCD simulations and semiclassical ensembles of instantons – have elucidated a rich vacuum structure dominated by inhomogeneous and topologically nontrivial gauge configurations. These configurations explain why chiral symmetry is spontaneously broken, and account for the emergence of mass through running quark effective masses.

A massless left-handed quark tunneling through an instanton emerges as a right-handed massless quark as a topological zero mode, a remarkable manifestation of the axial anomaly. This phenomenon is the essence of the dynamical breaking of chiral symmetry, which yields a running constituent mass. The collectivization of these zero modes, is well understood from detailed numerical studies of instanton ensembles carried in the 1990’s, and produces an octet of massless Nambu-Goldstone modes. The QCD vacuum possesses the “Zero Mode Zone” with no gap near zero Dirac eigenvalue.
In a way, we may say that the QCD vacuum is 'metallic', with the scalar and vector mesons as weakly correlated 'excitons'.

However, chiral symmetry breaking is not the only instanton-induced effect. Apart from well-isolated instantons producing near-zero Dirac modes, there are also more numerous fluctuations which can be described as instanton-antiinstanton molecules. In the first paper of the series [16], we showed that by including them in Wilson line correlators, we can account for a significant part of the central inter-quark potential, as well as the spin-dependent potentials in heavy quarkonia.

The light front observables at large normalization scale $\mu^2$ paint a picture of hadrons containing multiple gluons, and a rich quark-antiquark sea. Yet at small resolution, of the order of $\mu \sim 1/\rho \approx 600 \text{ MeV}$, we expect this picture to morph into the spectroscopic quark model, dominated by the minimal (2 for mesons, 3 for baryons) configurations. This is especially obvious for heavy quarkonia, which is the focus in this paper.

"Potentials" independent and dependent on spin variables, are defined via certain nonlocal observables, such as the well known Wilson lines. We know how to evaluate them in the Euclidean version of the QCD vacuum, on the lattice or when it can be regarded as a correlated ensemble of certain topological solitons, such as instantons and anti-instantons.

Bridging Euclidean vacuum with the light front, is one of the major challenge for theory today. In the second paper [17], we outlined a method to do so, by performing calculations in Euclidean space-time, and then analytically continuing the results (not the field configurations themselves). Wilson loop correlators are mapped on the light front, in terms of a slope angle $\theta$ in Euclidean signature, that is continued to $-i\chi$ (the rapidity) in Minkowski signature. This proposal was made long ago, and tested both at the perturbative [18], and non-perturbative [19, 20] levels.

In a way, this proposal is different, although similar in spirit, to the large momentum effective approach [1], where Euclidean equal-time correlators are made to asymptote their light cone counterparts. The large momentum limit in this case, is analogous to the large rapidity $\theta \rightarrow -i\chi$ continuation in our case.

Finally, in this paper we explore the fortunate fact that on the light front, all hadrons – from bottomonia to the pions – can be approached from the same LF Hamiltonian. In the first approximation, it is just a transverse oscillator and longitudinal harmonic functions. In the next approximation, the non-factorizable part $\tilde{V}$ is included. For the lowest states, its influence is not too drastic.

The next approximation brings in the Coulomb, perturbative and instanton-induced mixing, in spin and angular momenta. For heavy quarkonia these effects are suppressed by large quark masses. For light quark states, these effects are also not large, except for the ground states. The reason is that these mixing effects are short-range. Narrow Wilson loops are mostly sensitive to the topologically active instanton and anti-instanton gauge configurations and chiral symmetry breaking. Wide Wilson loops are mostly sensitive to the flux disordering gauge vortices and confinement. Contrary to common lore, the QCD vacuum is dominant on the light front too, and is central for the emergence of mass, confinement and spin mixing.

The idea to derive all LF observables – DAs, PDFs, GPDs, Form-factors – from a common model Hamiltonian, was put forth by Brodsky and his collaborators [37], and by Vary and his collaborators [38, 39], with the intent to combine results of various experiments into a common framework. Their Hamiltonians were largely guessed, using insights from holographic QCD models.

Our approach is different. It is based on a derivation of a light front Hamiltonian, from well established features of the perturbative and nonperturbative QCD vacuum fields. We also used a wider set of states and quark masses, and showed that the excited states are much closer to the generic Regge trajectories. Indeed,
for the excited states, the confining string becomes the only relevant physical effect, as explained in our introductory remarks in [16].

Looking at all works on $H_{LF}$, we believe that this is just the beginning of a successful modeling of hadrons on the light front, that factors in the wealth of information from the QCD vacuum. While our analyses deal solely with mesons, their generalization to baryons is relatively straightforward. Also, one can analyze multi-quark wavefunctions of mesons (tetraquarks) and baryons (pentaquarks), the anti-quark sea etc.

In many ways, the theoretical construction and tools presented and used in these serial analyses, provide a common framework for both hadronic and nuclear physics.

Acknowledgements
This work is supported by the Office of Science, U.S. Department of Energy under Contract No. DE-FG-88ER40388.

Appendix A: Functional basis using a 2d oscillator Hamiltonian

Following our analysis in [17], we use the eigenfunction basis of the “2d oscillator” imbedded in part of the Hamiltonian $H_0$ in (37). We start by recalling its generic properties, and then use it either in the momentum or coordinate representation, whichever is more convenient.

The generic Hamiltonian is

$$H_{osc} = \frac{\vec{p}^2}{2\mu} + \rho^2 \frac{\mu \omega^2}{2} \quad (A1)$$

where $\vec{\rho}$ is a 2d coordinate. One way to generate all wave functions is to use two 1d oscillator notations, but a more convenient one is to use polar coordinates.

so the Hamiltonian matrix consists of separate sectors of $N = 3 \times 4$ size.

The basic LF Hamiltonian includes a diagonal $H_0$ and a nonfactorizable part

$$\tilde{V} = (M^2 + \rho^2_\perp) \left( \frac{1}{x(1-x)} - 4 \right) \quad (A2)$$

which can be calculated either in momentum or coordinate representations.

Note that the wave functions we use here, are all normalized using

$$\int_0^1 dx \int d^2\rho_\perp \psi_{nm}^2(x, \rho_\perp) = 1,$$

which is natural in coordinate space. When used in momentum space, pertinent powers of $1/(2\pi)^3$ will be added whenever needed.

The wave functions which are independent of the azimuthal angle (angular momentum zero) will be used mostly in the momentum representation, with $\vec{\rho}$ as the transverse momentum. The orthonormal wave functions are

$$\{\psi_{n0}\} = e^{-\beta^2 \rho^2/2} \sqrt{\frac{1}{\pi \beta}} \times \{1, (1 - \beta^2 \rho^2), (1 - 2\beta^2 \rho^2 + \beta^4 \rho^4/2), \ldots\} \quad (A3)$$

with the $\beta$-parameter given in terms of the Hamiltonian parameters $\beta = (4a/b/\sigma T)^{1/4}$.

The harmonic set of orthonormal functions for longitudinal momentum fraction $x$, the set of functions $\chi_l(x)$, are labeled by odd integer $l = 1, 3, 5,\ldots$

$$\chi_l(x) = \sqrt{2} \times \{\sin(\pi x), \sin(3\pi x), \sin(5\pi x), \sin(7\pi x), \ldots\} \quad (A4)$$

The products of these two sets, define the set of states we used in our (angle-independent) calculations, in the momentum representation.

The part of the Hamiltonian matrix used is thus limited by three maximal values of indices $n, m, l$, so the total size of of the matrix is $N \times N$, with $N = n_{max} m_{max} l_{max}$. Obviously,
the calculations significantly slow down with increase $N$, and so for this exploratory paper we use a rather modest value of $N = 3 \times 3 \times 4$. Furthermore, before we account for mixing of states with different orbital momenta, $m$ is conserved.

**Appendix B: From bottomonium to generic light mesons, on the light front**

$$M^2 \approx \{360., 341., 328., 169., 161., 153., 127., 121., 114., 113., 107., 101.\} \text{ (GeV}^2\text{)}$$

The LFWF of the ground state is approximately factorized into

$$\Psi_{00} = e^{-1.302\rho^2} \left( -0.915 \sin (\pi x) + 0.749 \sin (3\pi x) - 0.485 \sin (5\pi x) + 0.183 \sin (7\pi x) \right) \quad (B1)$$

where we have omitted all terms with coefficients smaller than 0.01. Note that the ground state is then the product of just a Gaussian in transverse momentum, times certain functions of $x$. However, the next eigenstates are not that simple. The LFWFs for the next two states with $n = 1, 2$ (and still independent on $\phi$ or for $m = 0$) are

$$\Psi_{10} = e^{-1.302\rho^2} \left[ (0.905 - 4.726\rho^2 + 3.85\rho^4) \sin (\pi x) + (-0.741 + 3.876\rho^2 - 2.533\rho^4) \sin (3\pi x) \right.$$

$$+ (0.480 - 2.521\rho^2 + 1.651\rho^4) \sin (5\pi x) - (0.182 + 0.957\rho^2 - 0.628\rho^4) \sin (7\pi x) \right] \quad (B2)$$

$$\Psi_{20} = e^{-1.302\rho^2} \left[ (0.905 - 4.726\rho^2 + 3.85\rho^4) \sin (\pi x) + (-0.741 + 3.876\rho^2 - 2.533\rho^4) \sin (3\pi x) \right.$$

$$+ (0.480 - 2.521\rho^2 + 1.651\rho^4) \sin (5\pi x) - (0.182 + 0.957\rho^2 - 0.628\rho^4) \sin (7\pi x) \right] \quad (B3)$$

Here $\rho^2 = p_{\perp}^2$ (GeV$^2$) and all coefficients are also in GeV units with appropriate powers. The $\Psi_{n0}$ functions are eigenstates of $H$, and should not be confused with the basis set $\psi_{nm\ell}$ introduced above.

Their integrals of $\Psi_{nm}$ over $p_{\perp}$ give the DAs discussed in section IX. Their $x$-dependence are very similar. Their $p_{\perp}$-dependence is shown in Fig.8. In contrast to the $x$-dependence, the $p_T$-dependences are very different, as each curve reflects on the proper number of $n$ zeros.

The same construction for generic light quarks with $m_q = 0.35\text{GeV}$, was discussed in [17]. Here, we slightly modify the setting by selecting the variational minima at $a = 4$. The $p_{\perp}$ dependence of the first three states is shown in Fig.8.

For reference, the squared masses of the 12 lowest eigenvalues are
and the lowest wave function is

$$\Psi_{00} = e^{-7.14286}\rho^2 ((-3.12914 + 1.86018\rho^2) - 0.72094\rho^4)\sin(\pi x)$$
$$+ (0.16657 + 0.636644\rho^2 - 0.490831\rho^4)\sin[3\pi x]$$
$$+ 0.0267555\sin[5\pi x] + 0.111208\rho^2\sin[5\pi x] - 0.152454\rho^4\sin[5\pi x]$$
$$+ 0.00919945\sin[7\pi x] + 0.0402368\rho^2\sin[7\pi x] - 0.0577019\rho^4\sin[7\pi x]) \quad (B4)$$

FIG. 8. Three lowest LFWFs with $n = 0, 1, 2, m = 0$ as a function of $p_\perp$ (GeV) at $x = \frac{1}{2}$, for bottomonium (left) and a typical light meson (right). The number of zeros are commensurate with $n$.

Appendix C: Wave functions with non-zero angular momentum $L_z = m$

The functions with nonzero orbital momentum $m$, are generated by the corresponding right (plus) creation operators

$$a^+_R = \frac{1}{2} e^{i\phi} (\beta\rho - \frac{1}{\beta} \frac{\partial}{\partial \rho} - i \frac{\partial}{\beta \rho \partial \phi})$$
$$a^+_L = \frac{1}{2} e^{-i\phi} (\beta\rho - \frac{1}{\beta} \frac{\partial}{\partial \rho} + i \frac{\partial}{\beta \rho \partial \phi}) \quad (C1)$$

and the needed extra factors for proper wave function normalization $1/\sqrt{n_R!n_L!}$ depending on the numbers of right- and left-rotating “quanta”.

In particular, we use the following orthonor-
mal sets of functions $\psi_{nm}$ depending on $\rho$ and azimuthal angle $\phi$, with principle quantum number $n = 0, 1, 2...$ and angular momentum $m = 0, 1, 2...$

\[
\{\psi_0\} = e^{-\beta^2 \rho^2/2 + i\phi} \sqrt{\frac{1}{\pi}} \beta^2 \rho \times \{1, (-2 + \beta^2 \rho^2)/\sqrt{2}, (6 - 6\beta^2 \rho^2 + \beta^4 \rho^4)/2\sqrt{3}, ...\} \quad (C2)
\]

\[
\{\psi_2\} = e^{-\beta^2 \rho^2/2 + 2i\phi} \sqrt{\frac{1}{2\pi}} \beta^3 \rho^2 \times \{1, (3 - \beta^2 \rho^2)/\sqrt{3}, (12 - 8\beta^2 \rho^2 + \beta^4 \rho^4)\sqrt{2}/4\sqrt{3}, ...\} \quad (C3)
\]

When matrix elements of some potentials are evaluated, it is more natural to switch to the coordinate representation. One way to do it is to rederive an oscillatory basis in which $\rho^2 = r^2_\perp$ and $\beta^2$ is interpreted as a Laplacian containing angular (centrifugal) term. In this case, the parameter $\beta$ is inverted. Simpler is to go to coordinate representation by 2d Fourier transform. While doing so, it is convenient to return to Cartesian coordinates, e.g. $r e^{\pm i\phi} \rightarrow p_x \pm ip_y$, which after double Fourier transform produces factors $x \pm iy$. With slight abuse of notation, we write the latter combination as $r e^{\pm i\phi}$, although the angles $\phi$ in momentum and coordinate representations do not have the same meaning. Also note that in coordinate representation one may better use the inverted scale parameter

$$\beta_p \rightarrow \beta_r = \left(\frac{4\alpha}{b\sigma_T}\right)^{-\frac{1}{4}}$$

Recall that $a$ is to be determined from the mass minimization, $b = M_{mes}^2 \approx (2m_Q)^2$, and the string tension is standard $\sigma_T = (0.4 \text{ GeV})^2$. Note also that, as expected, the wave functions at small distances are $\sim \rho^m$.

Now we return to the momentum representation, and diagonalize $H_0 + \hat{V}$ for $m = 1, 2$. For the bottomonium parameters the lists of the six lowest squared masses are

\[
M_{0, \pm 1}^2 = \{131.1, 124.4, 117.8, 116.6, 110.4, 104.2\}
\]

\[
M_{0, \pm 2}^2 = \{134.4, 127.7, 121.1, 119.7, 113.5, 107.3\}
\]

\[
\Psi_{01} = e^{-1.30\rho^2} \rho e^{\pm i\phi} [(1.48 - 0.015\rho^2) \sin(\pi x) + (-1.21 + 0.011\rho^2) \sin(3\pi x) + 0.787\sin(5\pi x) - 0.298\sin(7\pi x)], \quad (C4)
\]

\[
\Psi_{02} = e^{-1.30\rho^2} \rho^2 e^{\pm 2i\phi} [1.67\sin(\pi x) - 1.37\sin(3\pi x) + 0.89\sin(5\pi x) - 0.34\sin(7\pi x)] \quad (C5)
\]

\[
\Psi_{03} = e^{+1.30\rho^2} \rho e^{\pm i\phi} [(1.48 - 0.015\rho^2) \sin(\pi x) + (-1.21 + 0.011\rho^2) \sin(3\pi x) + 0.787\sin(5\pi x) - 0.298\sin(7\pi x)], \quad (C6)
\]

\[
\Psi_{04} = e^{+1.30\rho^2} \rho^2 e^{\pm 2i\phi} [1.67\sin(\pi x) - 1.37\sin(3\pi x) + 0.89\sin(5\pi x) - 0.34\sin(7\pi x)]
\]

\[
\Psi_{05} = e^{+1.30\rho^2} \rho e^{\pm i\phi} [(1.48 - 0.015\rho^2) \sin(\pi x) + (-1.21 + 0.011\rho^2) \sin(3\pi x) + 0.787\sin(5\pi x) - 0.298\sin(7\pi x)], \quad (C7)
\]

\[
\Psi_{06} = e^{+1.30\rho^2} \rho^2 e^{\pm 2i\phi} [1.67\sin(\pi x) - 1.37\sin(3\pi x) + 0.89\sin(5\pi x) - 0.34\sin(7\pi x)]
\]

Appendix D: Mixing matrix elements

In (64) we defined the $3 \times 3$ mixing matrix between states with different azimuthal quantum numbers $m = 0, 1, 2$, for a meson with fixed helicity $\Lambda = 1$. For simplicity, we did not use the states $\Psi_{nm}(x, \vec{k}_\perp)$ determined in the previous
section, but rather the basic and simple oscillatory states $\psi_{nm}(k_\perp)$. These states carry the azmuthal dependence through $e^{in\phi}$, and the $x$-dependence through $\sqrt{2}\sin(n\pi x)$, which are standard and not explicitly shown.

In this simplified basis, the Coulomb interaction

$$2MV_C = 2M_{\text{mes}} \left(-\frac{4\alpha S}{3\rho}\right)$$

is diagonal

$$C_{mm} = \int_0^\infty |\psi_{0m}|^2 V_C(\rho) 2\pi \rho d\rho \quad (D1)$$

For $m = \{0, 1, 2\}$, the entries are explicitly

$$C_{mm} = M_{\text{mes}} \sqrt{\pi \alpha S \beta \{-8/3, -4/3, -1\}}$$

Note that here, we use the coordinate representation and the oscillator parameter $\beta$, the inverse of the oscillator parameter in the momentum representation.

The perturbative spin-spin interactions is $\hat{S}_1 \hat{S}_2 = \frac{\lambda}{2}$, for the states with total spin $S = 1$, so that $\hat{S}_{1\perp} \hat{S}_{2\perp} = \frac{2\lambda}{3}$. Its associated transverse Coulomb potential $\nabla^2_{\perp}/\rho$ is regulated at short distances, through

$$V_{\text{pert}}^{SS} = \frac{2}{3} \left(\frac{2M_{\text{mes}}}{m_q^2}\right) \left(-\frac{4\alpha S}{3}\right) \nabla^2 \frac{1}{\sqrt{\rho^2 + \epsilon^2}}$$

$$= -\frac{4M_{\text{mes}} \alpha S}{9m_q^2} \frac{-2\epsilon^2 + \rho^2}{(\epsilon^2 + \rho^2)^{3/2}} \quad (D2)$$

In the limit $\epsilon \to 0$, it reduces to

$$S_{\text{pert}}^{SS} = \frac{2M_{\text{mes}} \sqrt{\pi \alpha S \beta^3}}{m_q^2} \{4/9, -2/9, -1/18\}$$

for $m = \{0, 1, 2\}$

The induced instanton spin-spin, spin-orbit and tensor forces cannot be carried analytically. We evaluated them numerically, using the potentials shown in Figs.5,6, as explained in the text. The numerics are carried for light quarks with mass $m_q = 0.35 \text{ GeV}$, and $M_{\text{mes}} = 2m_q$. The matrix elements are integrals of these potentials times the pertinent wave functions $\psi_{0m}, m = 0, 1, 2$ in coordinate space. The results are given in the second line of (64).

Appendix E: Spin, helicity and chirality spinors

The light-front wave functions in [12] are built in terms of the spin and angular momentum projected along the $z$-direction, which is the hadron momentum $\vec{P}$ direction. In the case of mesons, there are two sets of spin variables $S_Q \tilde{Q} = S_{1,2}$ and a single orbital momentum $L$.

Let the direction of the quark momentum be described by standard polar angles $\theta, \phi$, with $p_\perp = p \sin(\theta)$ etc. In this case the spin up and down basis (with standard Dirac matrices) is

$$|\text{spin } \uparrow\rangle = \sqrt{\frac{E + m}{2m}} \times \begin{pmatrix} 1 \\ 0 \\ \frac{p \cos(\theta)}{E + m} \sin(\theta) e^{i\phi} \end{pmatrix}$$

$$|\text{spin } \downarrow\rangle = \sqrt{\frac{E + m}{2m}} \times \begin{pmatrix} 0 \\ 1 \\ \frac{p \cos(\theta)}{E + m} \sin(\theta) e^{-i\phi} \end{pmatrix}$$

(E2)

The helicity $\lambda = \vec{S} \cdot \vec{k}$ defines a different basis, because the spin projection is defined not along the in-coming $z$-axis, but along the quark momentum. Of course, quarks in the hadron have nonzero transverse components to it, $|p_\perp| = p \sin(\theta)$. The nonrelativistic 2-component spinors with $\lambda = \pm 1$ are obtained by rotation

$$h_+ = (\cos(\theta/2), e^{i\phi} \sin(\theta/2))$$

$$h_- = (-\sin(\theta/2), e^{i\phi} \cos(\theta/2))$$

and, after a boost, the corresponding Dirac spinors are

$$|h_+\rangle = \sqrt{\frac{E + m}{2m}} \times \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) e^{i\phi} \end{pmatrix}$$

$$|h_-\rangle = \sqrt{\frac{E + m}{2m}} \times \begin{pmatrix} \sin(\theta/2) \\ \cos(\theta/2) e^{i\phi} \end{pmatrix}$$

(E4)
\[ |h^-\rangle = \sqrt{\frac{E + m}{2m}} \begin{bmatrix} -\sin(\theta/2) \\ \cos(\theta/2)e^{-i\phi} \\ \frac{p}{E + m}\sin(\theta/2) \\ \frac{2m}{E + m}\cos(\theta/2)e^{-i\phi} \end{bmatrix} \] (E5)

We will also use the chiral basis, which is obtained from the helicity basis by taking the ultrarelativistic limit \((m \to 0, p/(E + m) \to 1)\) inside the spinor

\[ |c+\rangle = \sqrt{\frac{E + m}{2m}} \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta)e^{i\phi} \\ \cos(\theta/2) \\ \sin(\theta/2)e^{i\phi} \end{bmatrix} \] (E6)

\[ |c-\rangle = \sqrt{\frac{E + m}{2m}} \begin{bmatrix} -\sin(\theta/2) \\ \cos(\theta/2)e^{-i\phi} \\ \sin(\theta/2) \\ -\cos(\theta)e^{-i\phi} \end{bmatrix} \] (E7)

so that they become eigenvectors of the chiral projectors \(P_{\pm} = (1 \pm \gamma_5)/2\).

Having specified these spinors, one can define the matrices rotating one set to the other. In particular, the transition between the spin and helicity states, takes the simple form

\[
\begin{align*}
\langle s \uparrow | h^+ \rangle &= \cos(\theta/2), & \langle s \downarrow | h^+ \rangle &= \sin(\theta/2)e^{i\phi} \\
\langle s \uparrow | h^- \rangle &= -\sin(\theta/2), & \langle s \downarrow | h^- \rangle &= \cos(\theta/2)e^{i\phi}
\end{align*}
\] (E8)

which – in the ultrarelativistic limit – is the same as the matrix between the spin-basis and chirality-basis.

### Appendix F: LF Dirac spinors

The LF Dirac spinors used to derive (77) are for the L-quark spinor with mass \(m_{Q_1}\)

\[
U_L(k, \uparrow) = \frac{1}{(\sqrt{2}k^+)^{\frac{3}{2}}} \begin{bmatrix} m_{Q_1} \\ -k_R \end{bmatrix}
\]

\[
U_L(k, \downarrow) = \frac{1}{(\sqrt{2}k^+)^{\frac{3}{2}}} \begin{bmatrix} -k_L \\ \sqrt{2}k^+ + \frac{1}{2}m_{Q_1} \end{bmatrix}
\]
and the R-quark spinor with the same mass

\[
U_R(k, \uparrow) = \frac{1}{(\sqrt{2}k^+)^{\frac{3}{2}}} \begin{bmatrix} \sqrt{2}k^+ + \frac{1}{2}m_{Q_1} \\ k_R \end{bmatrix}
\]

\[
U_R(k, \downarrow) = \frac{1}{(\sqrt{2}k^+)^{\frac{3}{2}}} \begin{bmatrix} \frac{m_{Q_1}}{k_R} \\ m_{Q_1} \end{bmatrix}
\]

For the L-antiquark spinor with mass \(m_{Q_2}\), we have

\[
V_L(k, \uparrow) = \frac{1}{(\sqrt{2}k^+)^{\frac{3}{2}}} \begin{bmatrix} -k_L \\ \sqrt{2}k^+ + \frac{1}{2}m_{Q_2} \end{bmatrix}
\]

\[
V_L(k, \downarrow) = \frac{1}{(\sqrt{2}k^+)^{\frac{3}{2}}} \begin{bmatrix} -m_{Q_2} \\ k_R \end{bmatrix}
\]

and for the R-antiquark with the same mass

\[
V_R(k, \uparrow) = \frac{1}{(\sqrt{2}k^+)^{\frac{3}{2}}} \begin{bmatrix} -k_L \\ -m_{Q_2} \end{bmatrix}
\]

\[
V_R(k, \downarrow) = \frac{1}{(\sqrt{2}k^+)^{\frac{3}{2}}} \begin{bmatrix} \sqrt{2}k^+ + \frac{1}{2}m_{Q_2} \\ k_R \end{bmatrix}
\]

---

[1] X. Ji, Phys. Rev. Lett. 110, 262002 (2013), arXiv:1305.1539 [hep-ph].

[2] J.-H. Zhang, J.-W. Chen, X. Ji, L. Jin, and H.-W. Lin, Phys. Rev. D 95, 094514 (2017), arXiv:1702.00008 [hep-lat].

[3] C. Alexandrou, K. Cichy, M. Constantinou, K. Jansen, A. Scapellato, and F. Steffens, Phys. Rev. Lett. 121, 112001 (2018), arXiv:1702.00008 [hep-lat].
[4] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961), [hep-lat].

[5] J. J. M. Verbaarschot and I. Zahed, Phys. Rev. Lett. **70**, 3852 (1993), arXiv:hep-th/9303012.

[6] G. Montambaux, Phys. Rev. B **55**, 12833 (1997), arXiv:cond-mat/9703003.

[7] S. J. Brodsky and R. Shrock, Proc. Nat. Acad. Sci. **108**, 45 (2011), arXiv:0905.1151 [hep-th].

[8] S. J. Brodsky, C. D. Roberts, R. Shrock, and P. C. Tandy, Phys. Rev. C **85**, 065202 (2012), arXiv:1202.2376 [nucl-th].

[9] G. Montambaux, Phys. Rev. B **55**, 12833 (1997), arXiv:cond-mat/9703003.

[10] G. F. De T´eramond and S. J. Brodsky, (2021), arXiv:2103.16628 [hep-ph].

[11] X. Ji, Nucl. Phys. B, 115181 (2020), arXiv:2003.04478 [hep-ph].

[12] X. Ji, Y.-S. Liu, Y. Liu, J.-H. Zhang, and Y. Zhao, Rev. Mod. Phys. **93**, 035005 (2021), arXiv:2004.03543 [nucl-th].

[13] C. Shi and I. C. Cloet, Phys. Rev. Lett. **122**, 082001 (2019), arXiv:1806.04799 [nucl-th].

[14] A. Kock and I. Zahed, (2021), arXiv:2110.06989 [hep-ph].

[15] E. Shuryak and I. Zahed, (2021), arXiv:2110.15927 [hep-ph].

[16] E. Shuryak and I. Zahed, (2021), arXiv:2111.01775 [hep-ph].

[17] E. Meggiolaro, Eur. Phys. J. C **4**, 101 (1998), arXiv:hep-th/9702186.

[18] V. Shuryak and I. Zahed, Phys. Rev. D **62**, 085015 (2000). arXiv:hep-ph/0005152.

[19] M. Giordano and E. Meggiolaro, eCONF **C0906083**, 31 (2009), arXiv:0909.3710 [hep-ph].

[20] R. A. Janik and R. B. Peschanski, Nucl. Phys. B **586**, 163 (2000), arXiv:hep-th/0003059.

[21] G. Basar, D. E. Kharzeev, H.-U. Yee, and I. Zahed, Phys. Rev. D **85**, 105005 (2012), arXiv:1202.0831 [hep-th].

[22] E. Shuryak and I. Zahed, Annals Phys. **396**, 1 (2018), arXiv:1707.01885 [hep-ph].

[23] M. Musakhanov and U. Yakhshtev, in *9th International Conference on New Frontiers in Physics* (2021) arXiv:2103.16628 [hep-ph].

[24] X.-d. Ji, J.-P. Ma, and F. Yuan, Eur. Phys. J. C **33**, 75 (2004), arXiv:hep-ph/0304107.

[25] Y.-Z. Xu, D. Binosi, Z.-F. Cui, B.-L. Li, C. D. Roberts, S.-S. Xu, and H. S. Zong, Phys. Rev. D **100**, 114038 (2019), arXiv:1911.05199 [nucl-th].

[26] J. J. M. Verbaarschot and I. Zahed, (2021), arXiv:2103.16628 [hep-ph].

[27] L. Adhikari, Y. Li, M. Li, and J. P. Vary, Phys. Rev. C **99**, 035208 (2019), arXiv:1809.06475 [hep-ph].

[28] X. Ji, Y.-S. Liu, Y. Liu, J.-H. Zhang, and Y. Zhao, Rev. Mod. Phys. **93**, 035005 (2021), arXiv:2004.03543 [nucl-th].

[29] B. Geshkenbein and M. Terentev, Phys. Lett. B **117**, 243 (1982).

[30] S. D. Drell and T.-M. Yan, Phys. Rev. Lett. **24**, 181 (1970).

[31] E. Meggiolaro, Eur. Phys. J. C **4**, 101 (1998), arXiv:hep-th/9702186.

[32] V. V. Pietro, A. Schäfer, R. W. Schiel, and A. Sternbeck, PoS **QCDEV2015**, 009 (2015), arXiv:1510.07429 [hep-lat].

[33] G. B. West, Phys. Rev. Lett. **24**, 1206 (1970).

[34] G. P. Lepage and S. J. Brodsky, Phys. Rev. D **22**, 2157 (1980).

[35] S. J. Brodsky, G. F. de Teramond, H. G. Dosch, and J. Erlich, Phys. Rept. **584**, 1 (2015), arXiv:1407.8131 [hep-ph].

[36] Y. Li, P. Maris, X. Zhao, and J. P. Vary, Phys. Lett. B **758**, 118 (2016), arXiv:1509.07212 [hep-ph].
[39] C. Mondal, J. Lan, K. Fu, S. Xu, Z. Hu, X. Zhao, and J. P. Vary, in *50th International Symposium on Multiparticle Dynamics* (2021) arXiv:2109.12921 [hep-ph].