Decentralized and Personalized Federated Learning

Abdurakhmon Sadiev\textsuperscript{1}, Darina Dvinskikh\textsuperscript{1,3,4}, Aleksandr Beznosikov\textsuperscript{1,2}, and Alexander Gasnikov\textsuperscript{1,2,4}

\textsuperscript{1} Moscow Institute of Physics and Technology, Russia
\textsuperscript{2} HSE University, Russia
\textsuperscript{3} Weierstrass Institute for Applied Analysis and Stochastics, Germany
\textsuperscript{4} Institute for Information Transmission Problems RAS, Russia

Abstract. In this paper, we consider the personalized federated learning problem minimizing the average of strongly convex functions. We propose an approach which allows to solve the problem on a decentralized network by introducing a penalty function built upon a communication matrix of decentralized communications over a network and the application of the Sliding algorithm \cite{10}. The practical efficiency of the proposed approach is supported by the numerical experiments.

Keywords: Federated Learning, Distributed Optimization

1 Introduction

Over the past few years, there has been a great interest in minimizing the average of convex functions (local losses)

$$\min_{x \in \mathbb{R}^d} \frac{1}{n} \sum_{k=1}^{n} f_k(x).$$ \hspace{1cm} (1)

The interest is due to this problem arises in many machine learning and statistical applications, e.g., empirical risk minimization, maximum likelihood estimation and etc. To overcome the computational issues typically arising while solving such kind of problems due to requirement of enormous calculations, we address to federated learning. Federated learning \cite{8,12,9} shares the global task between computing nodes which collaboratively solve the global problem. This allows to process the problem faster and without additional collecting the data if it comes from multiple source. The standard federated learning minimizes the separable function $\frac{1}{n} \sum_{k=1}^{n} f_k(x_k)$ under the constraint $x_1 = x_2 = \cdots = x_n$. The personalized federated learning (see \cite{6,5,7}) allows the local arguments $x_k$’s ($x_k \in \mathbb{R}^d$) be mutually different by penalizing for their discrepancy and seeks to solve the following unconstrained optimization problem

$$\min_{x = [x_1, \ldots, x_n] \in \mathbb{R}^{nd}} \frac{1}{n} \sum_{k=1}^{n} f_k(x_k) + \frac{\lambda}{2} r(x),$$ \hspace{1cm} (2)

* Supported by organization x.
where $r(x)$ is a convex penalty function, $\lambda \geq 0$. The simplest choice of the penalty function is quadratic penalization [5, 6, 7]: $r(x) = \frac{1}{n} \sum_{k=1}^{n} \|x_i - \bar{x}\|^2$, where $\bar{x}$ is the average of $x_1, x_2, \ldots, x_n$. In this work, we refuse the centralized communication imposing by the quadratic penalization [5, 6, 7] and provide another regularization with allow decentralized communications [11, 4]. Moreover, we propose a more transparent and universal approach based on the Sliding algorithm [10] in comparison with approach of [5, 6, 7] but of the same rates of convergence.

To do so, we consider a distributed system solving which is given by a communication matrix $W$. Then, as in the work [1], the federated learning problem (2) can be formulated as following problem

$$\min_{x=[x_1, \ldots, x_n] \in \mathbb{R}^d} F(x) \triangleq \frac{1}{n} \sum_{k=1}^{n} f_k(x_k) + \frac{\lambda}{2} \langle x, Wx \rangle.$$  \hspace{1cm} (3)

The penalization for the dissimilarity is fulfilled by the fact $Wx = 0$ if and only if $x_1 = \cdots = x_n$ (the Perron–Frobenius theorem). Further, we comment on the critical values of penalizing parameter $\lambda$:

* If $\lambda = 0$, the problem (3) turns to the problem with probably different arguments $x_k$’s minimizing the local functions $f_k$’s

$$\min_{x=[x_1, \ldots, x_n] \in \mathbb{R}^d} \frac{1}{n} \sum_{k=1}^{n} f_k(x_k).$$

* When $\lambda \to +\infty$, the problem 3 turns to the distributed problem with equal local argument $x_k$’s

$$\min_{x_1=\cdots=x_n \in \mathbb{R}^d} \frac{1}{n} \sum_{k=1}^{n} f_k(x_k).$$

The communication matrix $W$ in (3) is defined as the Kronecker product of matrix $\hat{W}$ (to be defined further) and the identity matrix $I_d$ to take into the consideration that all $x_k \in \mathbb{R}^d (k = 1, \ldots, n)$: $W = \hat{W} \otimes I_d$. The gossip matrix $\hat{W}$ satisfies the following assumptions (see [13]):

1. $\hat{W}$ is symmetric positive semi-definite;
2. The kernel $\hat{W}$ consists of vector $1 = (1, \ldots, 1)^T$;
3. $\hat{W}$ is defined on the edges of the communication network: $\hat{w}_{i,j} \neq 0$ if and only if $i = j$ or $(i, j) \in E$.

The simplest choice of $\hat{W}$ is the Laplace matrix.

In this paper, we restrict ourselves by considering only smooth and strongly convex functions $f_k$’s in (3). That is, the functions $f_k$’s satisfy the following assumption.

Assumption LT (Lipschitz and Strongly convex assumption) Function $f(x)$ is
1. $L$-smooth w.r.t $\ell_2$-norm $\| \cdot \|_2$, i.e. for all $x_1, x_2 \in \mathbb{R}$ we have
   \[ \| \nabla f(x_2) - \nabla f(x_1) \|_2 \leq L \| x_2 - x_1 \|_2. \]

2. strongly-convex w.r.t. $\ell_2$-norm $\| \cdot \|_2$, i.e. for all $x_1, x_2 \in \mathbb{R}$ we have
   \[ f(x_2) - f(x_1) \geq \langle \nabla f(x_1), x_2 - x_1 \rangle + \frac{\mu}{2} \| x_2 - x_1 \|_2^2. \]

2 Constraint Gap

This section demonstrates that we control the accuracy of consensus constraints implementation by the choice of penalizing parameter $\lambda$. To do so, we consider a decentralized formulation of the problem minimizing the average of convex functions can be formulated as follows

\[
\min_{\sqrt{W}x = 0, \; x = [x_1, \ldots, x_n] \in \mathbb{R}^n} f(x) \triangleq \frac{1}{n} \sum_{k=1}^{n} f_k(x_k), \tag{4}
\]

where we used $x_1 = x_2 = \cdots = x_m$ if and only if $Wx = 0$ and $\sqrt{W}x = 0$. It can be easily seen that an exact solution $x^*$ of (4) is an exact solution of problem of interest (1).

**Theorem 1.** Let $x^N$ be an $\varepsilon$-solution to problem 3, i.e
   \[ F(x^N) - F(x^*) \leq \varepsilon, \]
   where $x^*$ is an exact solution of problem 3. Then for $x^N$ and problem (4), we have
   \[ f(x^N) - f(x^*) \leq \varepsilon, \quad \| \sqrt{W}x^N \|_2 \leq 2 \frac{R_y}{\lambda} + \sqrt{\frac{2\varepsilon}{\lambda}}, \tag{5} \]
   where $R_y^2 = \| y^* \|_2^2$, $y^*$ is the Lagrangian multiplier for the constraint $\sqrt{W}x = 0$ in (4).

In particular, if we choose $\lambda = R_y^2/2\varepsilon$, we have
   \[ f(x^N) - f(x^*) \leq \varepsilon, \quad \| \sqrt{W}x^N \|_2 \leq 2\varepsilon/R_y. \]

**Proof.** By the definition of $x^N = [x^N_1, \ldots, x^N_m]$, we have
   \[ F(x^N) - F(x^*) = f(x^N) - f(x^*) + \frac{\lambda}{2} \| \sqrt{W}x^N \|_2^2 \leq \varepsilon. \]

Using the definition of the Lagrange multiplier $y^*$, we estimate $f(x^N) - f(x^*)$:
   \[
   f(x^*) = f(x^*) - \langle y^*, \sqrt{W}x^* \rangle = \min_{x \in \mathbb{R}^n} \left\{ f(x) - \langle y^*, \sqrt{W}x \rangle \right\} \\
   \leq f(x^N) - \langle y^*, \sqrt{W}x^N \rangle.
   \]
Then, using the Cauchy-Schwarz inequality, we have
\[-R_y \|\sqrt{W}x^N\|_2 \leq \langle y^*, \sqrt{W}x^N \rangle \leq f(x^N) - f(x^*),\]
where \(R_y = \|y^*\|_2\), for an estimate of the value, see [2]. Hence, we get
\[-R_y \|\sqrt{W}x^N\|_2 + \frac{\lambda}{2} \|\sqrt{W}x^N\|_2^2 \leq \varepsilon.\]
Finally, we have
\[\|\sqrt{W}x^N\|_2 \leq 2 \frac{R_y}{\lambda} + \sqrt{\frac{2\varepsilon}{\lambda}}.\]
Thus, by controlling the choice of the penalizing parameter \(\lambda\), we control the accuracy of consensus constraints implementation. \(\square\)

2.1 Accelerated Meta-Algorithm

We consider problem (3) as composite optimization problem. The authors of [3] proposed Accelerated-Meta-Algorithm (see Algorithm 1 of this paper) to solve the following composite optimization problem

\[
\min_{x \in \mathbb{R}^d} \{F(x) \triangleq f(x) + g(x)\},
\]
where \(g : \mathbb{R}^d \rightarrow \mathbb{R}\) and \(f : \mathbb{R}^d \rightarrow \mathbb{R}\) are convex functions.

We assume that the function \(f\) is \(L_p(f)\)-Lipschitz of order \(p \geq 1\), i.e.
\[
\|D^p f(x_2) - D^p f(x_1)\| \leq L_p(f)\|x_2 - x_1\|, \quad \forall x_1, x_2 \in \mathbb{R}^d,
\]
where \(D^p f\) is \(p\)-th order derivative of function \(f\). Similarly to the article [3], we denote the Taylor decomposition of a function \(f\) of order \(p\):
\[
\Omega_p(f, x; y) = f(x) + \sum_{k=1}^{p} \frac{1}{k!} D^k f(x)[y - x]^k
\]

Now, let we assume that the function \(F\) is \(\mu\)-strongly convex, i.e.
\[
F(y) \geq F(x) + \langle \nabla F(x), y - x \rangle + \frac{\mu}{2} \|y - x\|^2, \quad \forall x, y \in \mathbb{R}^d \mu > 0.
\]
Via this assumption, the function \(F = f + g\) will be \(\mu\)-strongly convex, and for \(\mu\)-strongly convex composite optimization problem (6), we can use the Restarted Accelerated Meta-Algorithm [3], this algorithm listed as Algorithm 2, below.

For Algorithm 2, the following result was proved in [3].

Theorem 2 (see works [3,14]). Let \(z_N\) is the output of Algorithm 2 after \(N\) iterations. If \(H \geq (p + 1)L_p(f)\), then to achieve \(F(z_N) - F(x^*) \leq \varepsilon\), we need to make
\[
N = \mathcal{O} \left( \ln(\varepsilon^{-1}) \left( \frac{HR_y^{-1} - 1}{\mu} \right)^{\frac{2}{p+1}} \right)
\]
Algorithm 1 Accelerated Meta-Algorithm (AM) [3] for problem (6). AM($z_0, f, g, H, K$).

1: **Input**: $p \in \mathbb{N}$, number of iterations $K$, starting point $z_0$, parameter $H > 0$.
2: $A_0 = 0$, $y_0 = z_0$.
3: **for** $k = 0, \ldots, K - 1$ **do**
4: Find $\lambda_{k+1}$ and $y_{k+1} \in \mathbb{R}^d$, such that
\[
\frac{1}{2} \leq \lambda_{k+1} H \frac{\|y_{k+1} - w_k\|^{p-1}}{p!} \leq \frac{p}{p + 1},
\]
where
\[
y_{k+1} = \mathop{\text{argmin}}_{y \in \mathbb{R}^d} \left\{ \Omega_p(f, w_k; y) + g(y) + \frac{H}{(p+1)!} \|y - w_k\|^{p+1} \right\},
\]
\[
a_{k+1} = \frac{\lambda_{k+1} + \sqrt{\lambda_{k+1}^2 + 4 \lambda_{k+1} A_k}}{2},
\]
\[
A_{k+1} = A_k + a_{k+1},
\]
\[
w_k = \frac{A_k - y_k + a_{k+1} z_k}{A_{k+1}}.
\]
5: $z_{k+1} := z_k - a_{k+1} \nabla f(y_{k+1}) - a_{k+1} \nabla g(y_{k+1})$.
6: **end for**
7: **Output**: AM($z_0, K$) := $y_K$.

Algorithm 2 Restarted Accelerated Meta-Algorithm [3], for strongly convex composite optimization problem. RAM($z_0, f, g, p, \mu, H, s$).

1: **Input**: starting point $z_0$, $H > 0, \mu > 0$, number of restarts $s \geq 1$, $R_0$ s.t. $\|z_0 - x^*\|^2 \leq R_0^2$, where $y_*$ is the solution of the problem (6).
2: **for** $k = 0, \ldots, s - 1$ **do**
3: $R_k = R_0 \cdot 2^{-k}$,
4: $N_k = \max\left\{ \left( \frac{(p+1) \cdot \left( \frac{8 H^p}{\mu \cdot p!} R_k^{p-1} \right)^{2(p+1)}}{\mu \cdot p! R_k^{p+1}} \right), 1 \right\}$,
5: $z_{k+1} := \text{AM}(z_k, N_k)$ (the output of Algorithm 1, with starting point $z_k$ and $N_k$ iterations),
6: **end for**
7: **Output**: $z_s$.

calculations of (8), where $R_0 = \|y_0 - y_*\|$, $y_*$ is the solution of problem (6).

In particular, if $p = 2$, also we have the accuracy of solution to problem 8, satisfies to the following inequality
\[
\delta \leq \frac{\varepsilon \mu}{864^2 (L(f) + L(g) + H)^2},
\]
where $L(f), L(g)$ are constants of smoothness of function $f$ and $g$ respectively.
3 Main Results. Near-Optimal Algorithm

In this section, we present the main convergence theorem. Our approach consists of solving problem 3 as composite optimization problem 6. Therefore, we solve the problem 3 by Algorithm 2. Now we state the convergence theorem for this case.

Theorem 3. Let each functions $f_k(\cdot)$ satisfy the Assumption LT and let $\delta > 0$ be accuracy of solution auxiliary problem 8. Then to achieve $\varepsilon$-solution to problem 3 solving by Algorithm 8 it needs the number of communications

$$N^{\text{comm}} = \mathcal{O} \left( \min \left\{ \sqrt{\frac{\lambda \lambda_{\max}(W)}{\mu}}, \sqrt{\frac{L}{\mu}} \right\} \log \frac{\|z_0 - x^*\|^2}{\varepsilon} \log \frac{1}{\delta} \right),$$

where $\chi = \frac{\lambda_{\max}(W)}{\lambda_{\min}(W)}$, and the number of local oracle calls (gradient of each functions $f_k(\cdot)$)

$$N^{\text{loc}} = \mathcal{O} \left( \sqrt{\frac{L}{\mu}} \log \frac{\|z_0 - x^*\|^2}{\varepsilon} \log \frac{1}{\delta} \right),$$

where we take $\delta$ as follows

$$\delta = \frac{\varepsilon \mu}{864^2 (L + \lambda \lambda_{\max}(W) + H)^2}$$

Proof. For analysis we consider auxiliary problem 8 of Algorithm 1 with $p = 1$:

$$y_{k+1} = \arg\min_{y \in \mathbb{R}^d} \left\{ f(w_k) + \langle \nabla f(w_k), y - w_k \rangle + g(y) + \frac{H}{2} \|y - w_k\|^2 \right\} \quad (11)$$

Case 1 $\lambda \lambda_{\max}(W) \geq L$ Then we take $f(x)$ like sum component, $g(x)$ like $\frac{1}{2} \langle x, Wx \rangle$ and we know that number of calls of gradient of $f$:

$$N_{\nabla f_k} = \mathcal{O} \left( \sqrt{\frac{L}{\mu}} \log \frac{\|z_0 - x^*\|^2}{\varepsilon} \right) \quad (12)$$

Now we look carefully at auxiliary problem. We know Ker $W$ is not empty. And function $g(x)$ takes zero on this subspace Ker $W$. Then we can divide our problem on two subproblem: minimization of quadratic form with matrix $H \cdot I$ on Ker $W$ and minimization of quadratic form with matrix $\lambda W + H \cdot I$ on $(\text{Ker} W)^\perp$. Complexity of the first problem is equal to $\mathcal{O}(1)$. Complexity of the second problem is equal to

$$\mathcal{O} \left( \sqrt{\frac{H + \lambda \lambda_{\max}(W)}{\max\{H, \lambda \lambda_{\min}(W)\}}} \log \frac{1}{\delta} \right),$$

where $\delta$ is accuracy of solution to the auxiliary problem 8. Then we can say number of calls of gradient of $g$:

$$N_{Wx} = \mathcal{O} \left( \sqrt{\frac{L}{\mu}} \sqrt{\frac{H + \lambda \lambda_{\max}(W)}{\max\{H, \lambda \lambda_{\min}(W)\}}} \log \frac{\|z_0 - x^*\|^2}{\varepsilon} \log \frac{1}{\delta} \right). \quad (13)$$
Calculate the following
\[ \sqrt{\frac{H + \lambda_{\max}(W)}{\max\{H, \lambda_{\min}(W)\}}} = \min \left\{ \sqrt{\frac{H + \lambda_{\max}(W)}{H}}, \sqrt{\frac{H + \lambda_{\max}(W)}{\lambda_{\min}(W)}} \right\} \]

Taking \( H \) be equal to \( L \), we get
\[ N_{Wx} = \mathcal{O} \left( \min \left\{ \sqrt{\frac{\lambda_{\max}(W)}{\mu}}, \sqrt{\frac{L \lambda_{\max}(W)}{\mu \lambda_{\min}(W)}} \right\} \log \frac{z_0 - x^*}{\varepsilon} \frac{1}{\log \delta} \right) \] (14)

**Case 2** \( \lambda_{\max}(W) < L \) Then we take \( g(x) \) like sum component, \( f(x) \) like \( \frac{\lambda}{2} \langle x, Wx \rangle \) and we know that number of calls of gradient of \( f \):
\[ N_{Wx} = \mathcal{O} \left( \sqrt{\frac{\lambda_{\max}(W)}{\mu}} \log \frac{z_0 - x^*}{\varepsilon} \frac{1}{\log \delta} \right) \] (15)

we know that number of calls of gradient of \( g \):
\[ N_{\nabla f_k} = \mathcal{O} \left( \sqrt{\frac{L}{\mu}} \log \frac{z_0 - x^*}{\varepsilon} \frac{1}{\log \delta} \right) , \]
where \( \delta \) is accuracy of solution to the auxiliary problem 8. Taking \( H \) be equal to \( \lambda_{\max}(W) \), we get
\[ N_{\nabla f_k} = \mathcal{O} \left( \sqrt{\frac{L}{\mu}} \log \frac{z_0 - x^*}{\varepsilon} \frac{1}{\log \delta} \right) \]

Overall, we have
\[ N_{Wx} = \mathcal{O} \left( \min \left\{ \sqrt{\frac{\lambda_{\max}(W)}{\mu}}, \sqrt{\frac{L \lambda_{\max}(W)}{\mu \lambda_{\min}(W)}} \right\} \log \frac{z_0 - x^*}{\varepsilon} \frac{1}{\log \delta} \right) \] (16)
and
\[ N_{\nabla f_k} = \mathcal{O} \left( \sqrt{\frac{L}{\mu}} \log \frac{z_0 - x^*}{\varepsilon} \frac{1}{\log \delta} \right) \] (17)

Now we consider \( \delta \) accuracy of auxiliary problem 8. According to Theorem 2, we can take \( \delta \) like this
\[ \delta = \frac{\varepsilon \mu}{864^2(L + \lambda_{\max}(W) + H)^2} , \]
because function \( f(x) \) is \( L \)-smooth and \( \frac{\lambda}{2} \langle x, Wx \rangle \) is \( \lambda_{\max}(W) \)-smooth.
Remark 1. It is curious to note that in case of totally connected communication network we have that $\chi = 1$, $\lambda_{\text{max}}(W) = 1$ and our method converges with the following rates:

$$N^{\text{comm}} = \tilde{O} \left( \min \left\{ \sqrt{\frac{\lambda}{\mu}}, \sqrt{\frac{L}{\mu}} \right\} \right), \quad N^{\text{loc}} = \tilde{O} \left( \sqrt{\frac{L}{\mu}} \right).$$

Moreover, this convergence bounds coincide with lower bounds for such optimization problem (see work [5]).

4 Experiments

In this section, we demonstrate the efficiency of Algorithm 2 for problem 3 (Figure 1). We consider the following functions $f_k$.

$$f_k(x) = \log (1 + \exp (y_k \langle a_k, x \rangle)), \quad (18)$$

where $y_k$ takes only 1 and $-1$ value with equal probability, $a_k$ is a column of matrix $A$ which is generated randomly. The experiment are performed on the cycle graph which has the following Laplasian matrix

$$W = \begin{pmatrix}
1 & -1 & & \\
-1 & 2 & -1 & \\
& -1 & 2 & -1 & \\
& & -1 & 2 & -1 & \\
& & & & \cdots & \\
& & & & & -1 & 2 & -1 \\
& & & & & & -1 & 1
\end{pmatrix}.$$
Fig. 1: Convergence of Algorithm 2 with different parameter $\lambda$.

References

1. Aleksandr Beznosikov, Vadim Sushko, Abdurakhmon Sadiev, and Alexander Gasnikov. Decentralized personalized federated min-max problems. arXiv preprint arXiv:2106.07289, 2021.
2. Alexander Gasnikov. Universal gradient descent, 2020.
3. Alexander Gasnikov, Darina Dvinskikh, Pavel Dvurechensky, Dmitry Kamzolov, Vladimir Matykhin, Dmitry Pasechnyuk, Nazarii Tupitsa, and Alexei Chernov. Accelerated meta-algorithm for convex optimization. arXiv preprint arXiv:2004.08691, 2020.
4. Eduard Gorbunov, Darina Dvinskikh, and Alexander Gasnikov. Optimal decentralized distributed algorithms for stochastic convex optimization. arXiv preprint arXiv:1911.07363, 2019.
5. Filip Hanzely, Slavomír Hanzely, Samuel Horváth, and Peter Richtárik. Lower bounds and optimal algorithms for personalized federated learning. arXiv preprint arXiv:2010.02372, 2020.
6. Filip Hanzely and Peter Richtárik. Federated learning of a mixture of global and local models. arXiv preprint arXiv:2002.05516, 2020.
7. Filip Hanzely, Boxin Zhao, and Mladen Kolar. Personalized federated learning: A unified framework and universal optimization techniques, 2021.
8. Jakub Konečný, H Brendan McMahan, Daniel Ramage, and Peter Richtárik. Federated optimization: Distributed machine learning for on-device intelligence. arXiv preprint arXiv:1610.02527, 2016.
9. Viraj Kulkarni, Milind Kulkarni, and Aniruddha Pant. Survey of personalization techniques for federated learning. In 2020 Fourth World Conference on Smart Trends in Systems, Security and Sustainability (WorldS4), pages 794–797. IEEE, 2020.
10. Guanghui Lan. Gradient sliding for composite optimization. *Mathematical Programming*, 159(1):201–235, 2016.

11. Huan Li, Cong Fang, Wotao Yin, and Zhouchen Lin. Decentralized accelerated gradient methods with increasing penalty parameters. *IEEE Transactions on Signal Processing*, 68:4855–4870, 2020.

12. Brendan McMahan, Eider Moore, Daniel Ramage, Seth Hampson, and Blaise Agueray Arcas. Communication-efficient learning of deep networks from decentralized data. In *Artificial Intelligence and Statistics*, pages 1273–1282. PMLR, 2017.

13. Kevin Scaman, Francis Bach, Sébastien Bubeck, Yin Tat Lee, and Laurent Massoulié. Optimal algorithms for smooth and strongly convex distributed optimization in networks. *arXiv preprint arXiv:1702.08704*, 2017.

14. Vladislav Tominin, Yaroslav Tominin, Ekaterina Borodich, Dmitry Kovalev, Alexander Gasnikov, and Pavel Dvurechensky. On accelerated methods for saddle-point problems with composite structure. *arXiv preprint arXiv:2103.09344*, 2021.