EVOLUTION ALGEBRA OF A BISEXUAL POPULATION

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Abstract. We introduce an (evolution) algebra identifying the coefficients of inheritance of a bisexual population as the structure constants of the algebra. The basic properties of the algebra are studied. We prove that this algebra is commutative (and hence flexible), not associative and not necessarily power associative. We show that the evolution algebra of the bisexual population is not a baric algebra, but a dibaric algebra and hence its square is baric. Moreover, we show that the algebra is a Banach algebra. The set of all derivations of the evolution algebra is described. We find necessary conditions for a state of the population to be a fixed point or a zero point of the evolution operator which corresponds to the evolution algebra. We also establish upper estimate of the limit points set for trajectories of the evolution operator. Using the necessary conditions we give a detailed analysis of a special case of the evolution algebra (bisexual population of which has a preference on type “1” of females and males). For such a special case we describe the full set of idempotent elements and the full set of absolute nilpotent elements.

1. Introduction

The action of genes is manifested statistically in sufficiently large communities of matching individuals (belonging to the same species). These communities are called populations [13]. The population exists not only in space but also in time, i.e. it has its own life cycle. The basis for this phenomenon is reproduction by mating. Mating in a population can be free or subject to certain restrictions.

The whole population in space and time comprises discrete generations $F_0, F_1, \ldots$. The generation $F_{n+1}$ is the set of individuals whose parents belong to the $F_n$ generation. A state of a population is a distribution of probabilities of the different types of organisms in every generation. Type partition is called differentiation. The simplest example is sex differentiation. In bisexual population any kind of differentiation must agree with the sex differentiation, i.e. all the organisms of one type must belong to the same sex. Thus, it is possible to speak of male and female types.

The evolution (or dynamics) of a population comprises a determined change of state in the next generations as a result of reproduction and selection. This evolution of a population can be studied by a dynamical system (iterations) of a quadratic stochastic operator.
The history of the quadratic stochastic operators can be traced back to the work of S. Bernstein [1]. During more than 85 years this theory developed and many papers were published (see e.g. [1,6,9,13,15,18]). In recent years it has again become of interest in connection with numerous applications to many branches of mathematics, biology and physics.

A quadratic stochastic operator (QSO), \( V \), has meaning of a free population evolution operator, which arises as follows: Consider a free population consisting of \( m \) species. Let \( x^0 = (x^0_1, \ldots, x^0_m) \) be the probability distribution of species in the initial generations, and \( P_{ij,k} \) the probability that individuals in the \( i \)th and \( j \)th species interbreed to produce an individual \( k \). Then the probability distribution \( x' = (x'_1, \ldots, x'_m) \) of the species in the first generation can be found by the total probability i.e.

\[
x'_k = \sum_{i,j=1}^{m} P_{ij,k} x^0_i x^0_j, \quad k = 1, \ldots, m
\]

where the cubic matrix \( P \equiv P(V) = (P_{ij,k})_{i,j,k=1}^{m} \) satisfies the following conditions

\[
P_{ij,k} \geq 0, \quad \sum_{k=1}^{m} P_{ij,k} = 1, \quad i, j \in \{1, \ldots, m\}.
\]

This means that the association \( x^0 \mapsto x' \) defines a map \( V \) of the simplex

\[
S^{m-1} = \{ x = (x_1, \ldots, x_m) \in \mathbb{R}^m : x_i \geq 0, \sum_{i=1}^{m} x_i = 1 \}
\]

into itself, called the evolution operator.

The population evolves by starting from an arbitrary state \( x^{(0)} \), then passing to the state \( x' = V(x^{(0)}) \), then to the state \( x'' = V(V(x^{(0)})) \), and so on.

There are many papers devoted to study of the evolution of the free population, i.e. to study of dynamical system generated by the QSO \( (1.1) \), see e.g. [6,9,13,15,18].

There exist several classes of non-associative algebras (baric, evolution, Bernstein, train, stochastic, etc.), whose investigation has provided a number of significant contributions to theoretical population genetics. Such classes have been defined different times by several authors, and all algebras belonging to these classes are generally called genetic. Etherington introduced the formal language of abstract algebra to the study of the genetics in his series of seminal papers [3–5]. In recent years many authors have tried to investigate the difficult problem of classification of these algebras. The most comprehensive references for the mathematical research done in this area are [13,14,19,20].
In [13] an evolution algebra $A$ associated to the free population is introduced and using this non-associative algebra many results are obtained in explicit form, e.g. the explicit description of stationary quadratic operators, and the explicit solutions of a nonlinear evolutionary equation in the absence of selection, as well as general theorems on convergence to equilibrium in the presence of selection.

In [19] a new type of evolution algebra is introduced. This algebra also describes some evolution laws of genetics and it is an algebra $E$ over a field $K$ with a countable natural basis $e_1, e_2, \ldots$ and multiplication given by $e_ie_i = \sum_j a_{ij}e_j$, $e_ie_j = 0$ if $i \neq j$. Therefore, $e_ie_i$ is viewed as “self-reproduction”.

In this paper we consider a bisexual population (BP) and define an evolution algebra (EA) using inheritance coefficients of the population. This algebra is a natural generalization of the algebra $A$ of free population. The evolution algebra of a bisexual population (EABP) is different from the EA defined in [19].

The paper is organized as follows. In Section 2 we give evolution operator of BP. Section 3 contains the definition of EABP. Section 4 is devoted to basic properties of the algebra and therein we prove that the EABP is commutative but not associative and not power associative. In Section 5 we show that the EABP is not a baric algebra. In Section 6 we prove that the EABP is a dibaric algebra, hence its square is a baric algebra. In Section 7 we prove that the EABP is a Banach algebra. In Section 8 we describe the set of all derivations of EABP. Section 9 is devoted to study dynamics of the evolution operator, which corresponds to the evolution algebra. We find necessary conditions for a state of the population to be a fixed point or a zero point of the evolution operator. We also establish upper estimate of the limit points set for trajectories of the evolution operator. In the last section we give a detailed analysis of a special case of the evolution algebra (bisexual population of which has a preference on type “1” of females and males). For such a special case we describe the full set of idempotent elements and the full set of absolute nilpotent elements.

2. Evolution operator of a BP

In this section following [13], we describe the evolution operator of a BP. Assuming that the population is bisexual we suppose that the set of females can be partitioned into finitely many different types indexed by $\{1, 2, \ldots, n\}$ and, similarly, that the male types are indexed by $\{1, 2, \ldots, \nu\}$. The number $n + \nu$ is called the dimension of the population. The population is described by its state vector $(x, y)$ in $S^{n-1} \times S^{\nu-1}$, the product of two unit simplexes in $\mathbb{R}^n$ and $\mathbb{R}^\nu$ respectively. Vectors $x$ and $y$ are the probability distributions of the females and males over the possible types:

$$x_i \geq 0, \quad \sum_{i=1}^n x_i = 1; \quad y_i \geq 0, \quad \sum_{i=1}^\nu y_i = 1.$$
Denote \( S = S^{n-1} \times S^{\nu-1} \). We call the partition into types hereditary if for each possible state \( z = (x, y) \in S \) describing the current generation, the state \( z' = (x', y') \in S \) is uniquely defined describing the next generation. This means that the association \( z \mapsto z' \) defines a map \( V : S \to S \) called the evolution operator.

For any point \( z^{(0)} \in S \) the sequence \( z^{(t)} = V(z^{(t-1)}), t = 1, 2, \ldots \) is called the trajectory of \( z^{(0)} \).

Let \( P_{ik,j}^{(f)} \) and \( P_{ik,l}^{(m)} \) be inheritance coefficients defined as the probability that a female offspring is type \( j \) and, respectively, that a male offspring is of type \( l \), when the parental pair is \( ik \) \((i, j = 1, \ldots, n; \text{and } k, l = 1, \ldots, \nu)\). We have

\[
(2.1) \quad P_{ik,j}^{(f)} \geq 0, \quad \sum_{j=1}^{n} P_{ik,j}^{(f)} = 1; \quad P_{ik,l}^{(m)} \geq 0, \quad \sum_{l=1}^{\nu} P_{ik,l}^{(m)} = 1.
\]

Let \( z' = (x', y') \) be the state of the offspring population at the birth stage. This is obtained from inheritance coefficients as

\[
(2.2) \quad x_j' = \sum_{i,k=1}^{n,\nu} P_{ik,j}^{(f)} x_{i} y_{k}; \quad y_{j}' = \sum_{i,k=1}^{n,\nu} P_{ik,l}^{(m)} x_{i} y_{k}.
\]

We see from (2.2) that for a BP the evolution operator is a quadratic mapping of \( S \) into itself. But for free population the operator is quadratic mapping of the simplex into itself given by (1.1).

### 3. Definition of the EABP

In this section we give an algebra structure on the vector space \( \mathbb{R}^{n+\nu} \) which is closely related to the map (2.2).

Consider \( \{e_1, \ldots, e_{n+\nu}\} \) the canonical basis on \( \mathbb{R}^{n+\nu} \) and divide the basis as \( e_i^{(f)} = e_i, \ i = 1, \ldots, n \) and \( e_i^{(m)} = e_{n+i}, \ i = 1, \ldots, \nu \).

Now introduce on \( \mathbb{R}^{n+\nu} \) a multiplication defined by

\[
(3.1) \quad e_i^{(f)} e_k^{(m)} = e_k^{(m)} e_i^{(f)} = \frac{1}{2} \left( \sum_{j=1}^{n} P_{ik,j}^{(f)} e_j^{(f)} + \sum_{l=1}^{\nu} P_{ik,l}^{(m)} e_l^{(m)} \right), \quad e_i^{(f)} e_j^{(f)} = 0, \ i, j = 1, \ldots, n; \quad e_k^{(m)} e_l^{(m)} = 0, \ k, l = 1, \ldots, \nu.
\]

Thus we identify the coefficients of bisexual inheritance as the structure constants of an algebra, i.e. a bilinear mapping of \( \mathbb{R}^{n+\nu} \times \mathbb{R}^{n+\nu} \) to \( \mathbb{R}^{n+\nu} \).

The general formula for the multiplication is the extension of (3.1) by bilinearity, i.e. for \( z, t \in \mathbb{R}^{n+\nu} \),

\[
z = (x, y) = \sum_{i=1}^{n} x_i e_i^{(f)} + \sum_{j=1}^{\nu} y_j e_j^{(m)}; \quad t = (u, v) = \sum_{i=1}^{n} u_i e_i^{(f)} + \sum_{j=1}^{\nu} v_j e_j^{(m)}
\]
using (3.1), we obtain

\[ zt = \frac{1}{2} \sum_{k=1}^{n} \left( \sum_{i=1}^{n} \sum_{j=1}^{\nu} P_{ij,k}^{(f)} (x_i v_j + u_i y_j) \right) e_k^{(f)} + \]
\[ \frac{1}{2} \sum_{\nu=1}^{\nu} \left( \sum_{i=1}^{n} \sum_{j=1}^{\nu} P_{ij,l}^{(m)} (x_i v_j + u_i y_j) \right) e_l^{(m)}. \]

From (3.2) and using (2.2), in the particular case that \( z = t \), i.e. \( x = u \) and \( y = v \), we obtain

\[ zz = z^2 = \sum_{k=1}^{n} \left( \sum_{i=1}^{n} \sum_{j=1}^{\nu} P_{ij,k}^{(f)} x_i y_j \right) e_k^{(f)} + \]
\[ \sum_{\nu=1}^{\nu} \left( \sum_{i=1}^{n} \sum_{j=1}^{\nu} P_{ij,l}^{(m)} x_i y_j \right) e_l^{(m)} = V(z) \]

for any \( z \in S \).

This algebraic interpretation is very useful. For example, a BP state \( z = (x, y) \) is an equilibrium (fixed point, \( V(z) = z \)) precisely when \( z \) is an idempotent element of the set \( S \).

If we write \( z^{[t]} \) for the power \((\cdots (z^2)^2 \cdots) (t \text{ times})\) with \( z^{[0]} \equiv z \) then the trajectory with initial state \( z \) is \( V^t(z) = z^{[t]} \).

The algebra \( B = B_V \) generated by the evolution operator \( V \) (see (2.2)) is called the evolution algebra of the bisexual population (EABP).

**Remark 3.1.** 1. If a population is free then the male and female types are identical and, in particular \( n = \nu \), the inheritance coefficients are the same for male and female offsprings, i.e.

\[ P_{ik,j} = P_{ik,j}^{(f)} = P_{ik,j}^{(m)}. \]

The evolution algebra \( A \) associated with the free population is introduced and studied in [13]. Note that this algebra is commutative when the condition of symmetry \( P_{ik,j} = P_{ki,j} \) is satisfied, but it is not in general associative. In the next section we show that algebra \( B \) of bisexual population is commutative without any symmetry condition. Hence the algebra \( A \) is a particular case of the algebra \( B \).

2. It is easy to see that the EA introduced in [19] is completely different from our EABP, i.e. of \( B \).

3. The algebra \( B \) is a natural generalization of a zygotic algebra for sex linked inheritance (see [13, 14, 20]).

### 4. Basic properties of the EABP

The following theorem gives basic properties of the EABP.

**Theorem 4.1.** 1) Algebra \( B \) is not associative, in general.

2) Algebra \( B \) is commutative, flexible.

3) \( B \) is not power-associative, in general.
Proof. 1) Take $e_i^{(f)}, e_j^{(m)}$ such that $P_{i,j,s}^{(f)} \neq 0$ for some $s$ and take $e_k^{(m)}$ such that $P_{sk,r}^{(f)} \neq 0$ for some $r$ then

$$(e_i^{(f)} e_j^{(m)}) e_k^{(m)} = \frac{1}{2} \sum_{q=1}^{n} P_{ij,q}^{(f)} e_q^{(m)} e_k^{(m)} =$$

$$\frac{1}{2} \left( P_{ij,r}^{(f)} e_r^{(m)} + \sum_{q=1}^{n} P_{ij,q}^{(f)} e_q^{(m)} e_k^{(m)} \right) =$$

$$\frac{1}{4} P_{ij,r}^{(f)} P_{sk,r}^{(f)} e_r^{(m)} + \text{non-negative terms} \neq 0.$$

But

$$e_i^{(f)} (e_j^{(m)} e_k^{(m)}) = 0, \text{ i.e. } (e_i^{(f)} e_j^{(m)}) e_k^{(m)} \neq e_i^{(f)} (e_j^{(m)} e_k^{(m)}).$$

2) Commutativity of $\mathcal{B}$ follows from formula (3.2). An algebra is called flexible if it satisfies $z(tz) = (zt)z$ for any $z, t$. It is easy to see that a commutative algebra is flexible.

3) To show that $\mathcal{B}$ is not a power-associative, in general, we shall construct an example of $z$ such that $(zz)(zz) \neq ((zz)z)z$. Consider $n = 1, \nu = 2$. In this case

$$P_{11,1}^{(f)} = P_{12,1}^{(f)} = 1, \quad P_{11,1}^{(m)} + P_{11,2}^{(m)} = 1, \quad P_{12,1}^{(m)} + P_{12,2}^{(m)} = 1.$$

Denote $a = P_{11,1}^{(m)}, \quad b = P_{12,1}^{(m)}$. Take $z = e_1^{(f)} + e_1^{(m)}$. Then we have

$$z^2 = e_1^{(f)} + e_1^{(m)} + (1-a) e_2^{(m)}.$$

(4.1) $z^2 z^2 = e_1^{(f)} + (a^2 + (1-a)b) e_1^{(m)} + (1-a) a - b + 1) e_2^{(m)}.$

$$z^2 z = e_1^{(f)} + \frac{1}{2} (a^2 + (1-a)b + a) e_1^{(m)} + \frac{1}{2} (1-a)(a - b + 2) e_2^{(m)}.$$

(4.2) $$(z^2 z) z = e_1^{(f)} + \frac{1}{2} (a(a-b)^2 + (a + b)(a - b + 2)) e_1^{(m)} + \frac{1}{4} (1-a)(3 + (a-b+1)^2) e_2^{(m)}.$$

Assume $a = P_{11,1}^{(m)} \neq 1$ and $a \neq b = P_{12,1}^{(m)}$. Then from (4.1) and (4.2) we get $(zz)(zz) \neq ((zz)z)z$. □

5. $\mathcal{B}$ is not a baric algebra

A character for an algebra $A$ is a nonzero multiplicative linear form on $A$, that is, a nonzero algebra homomorphism from $A$ to $\mathbb{R}$. Not every algebra admits a character. For example, an algebra with the zero multiplication has no character. A pair $(A, \sigma)$ consisting of an algebra $A$ and a character $\sigma$ on $A$ is called a baric algebra. In [13] for the EA of a free population it is proven that there is a character $\sigma(x) = \sum_i x_i$, therefore that algebra is baric. But the following theorem says that the EABP, i.e. $\mathcal{B}$ is not baric.
Theorem 5.1. The EABP, $\mathcal{B}$, has no a nonzero character.

Proof. Assume $\sigma(z) = \sum_{i=1}^{n} a_{i}x_{i} + \sum_{j=1}^{\nu} b_{j}y_{j}$, $z = (x, y) \in \mathcal{B}$ is a character. We shall prove that $\sigma(z) \equiv 0$. For $z = (x, y)$, $t = (u, v)$ we have

$$
\sigma(zt) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{\nu} \left( \sum_{p=1}^{n} a_{p}P_{ij,p}^{(f)} + \sum_{q=1}^{\nu} b_{q}P_{ij,q}^{(m)} \right) (x_{i}v_{j} + u_{i}y_{j});$
$$
$$
\sigma(z)\sigma(t) = \sum_{i=1}^{n} \sum_{j=1}^{\nu} a_{i}a_{j}x_{i}u_{j} + \sum_{i=1}^{n} \sum_{j=1}^{\nu} a_{i}b_{j}(x_{i}v_{j} + u_{i}y_{j}) + \sum_{i=1}^{\nu} \sum_{j=1}^{\nu} b_{i}b_{j}y_{j}v_{j}.$$

From $\sigma(zt) = \sigma(z)\sigma(t)$ we get

$$a_{i}a_{j} = 0 \text{ for any } i, j = 1, \ldots, n$$
$$b_{i}b_{j} = 0 \text{ for any } i, j = 1, \ldots, \nu$$

This system of equations has only the solution $a_{1} = \cdots = a_{n} = b_{1} = \cdots = b_{\nu} = 0$. □

Remark 5.2. In [13] the baric EA of a free population is classified as algebra induced by a linear operator; unit algebra; constant algebra; Bernstein (stationary) algebra; genetic algebra; train algebra, etc. But since the EABP is not baric there are not similar algebras for bisexual population.

6. $\mathcal{B}$ IS A DIBARIC ALGEBRA

By Theorem 5.1 the algebra $\mathcal{B}$ is not a baric algebra. To overcome such complication, Etherington [5] for a zygotic algebra of sex linked inheritance introduced the idea of treating the male and female components of a population separately. In [11] Holgate formalized this concept by introducing sex differentiation algebras and a generalization of baric algebras called dibaric algebras. In this section we shall prove that the algebra $\mathcal{B}$ is a dibaric algebra.

Definition 6.1. [14][20] Let $\mathfrak{A} = \langle w, m \rangle_{\mathbb{R}}$ denote a two dimensional commutative algebra over $\mathbb{R}$ with multiplicative table

$$w^{2} = m^{2} = 0, \quad wm = \frac{1}{2}(w + m).$$

Then $\mathfrak{A}$ is called the sex differentiation algebra.

As usual, a subalgebra $B$ of an algebra $A$ is a subspace which is closed under multiplication. A subspace $B$ is an ideal if it is closed under multiplication by all elements in $B$. For example, the square of the algebra:

$$A^{2} = \text{span}\{zt : z, t \in A\}$$

is an ideal.

It is clear that $\mathfrak{A}^{2} = \langle w + m \rangle_{\mathbb{R}}$ is an ideal of $\mathfrak{A}$ which is isomorphic to the field $\mathbb{R}$. Hence the algebra $\mathfrak{A}^{2}$ is a baric algebra. Now we can define Holgate’s generalization of a baric algebra.
Definition 6.2. An algebra is called dibaric if it admits a homomorphism onto the sex differentiation algebra $\mathfrak{A}$.

Theorem 6.3. The algebra $\mathcal{B}$ is dibaric.

Proof. Consider mapping $\varphi : \mathcal{B} \to \mathfrak{A}$ defined by

$$\varphi(e_i^{(f)}) = w, \quad i = 1, \ldots, n; \quad \varphi(e_k^{(m)}) = m, \quad i = 1, \ldots, \nu.$$  

For $z, t \in \mathbb{R}^{n+\nu}$,

$$z = (x, y) = \sum_{i=1}^{n} x_i e_i^{(f)} + \sum_{j=1}^{\nu} y_j e_j^{(m)}, \quad t = (u, v) = \sum_{i=1}^{n} u_i e_i^{(f)} + \sum_{j=1}^{\nu} v_j e_j^{(m)},$$

using linearity of $\varphi$, (6.1), (2.1) and (3.2) we get

$$\varphi(zt) = \sum_{i=1}^{n} \sum_{j=1}^{\nu} (x_i v_j + u_i y_j) \left( \frac{1}{2}(w + m) \right).$$

We also have

$$\varphi(z)\varphi(t) = \left( \sum_{i=1}^{n} x_i w + \sum_{j=1}^{\nu} y_j m \right) \left( \sum_{i=1}^{n} u_i w + \sum_{j=1}^{\nu} v_j m \right) =$$

$$\sum_{i=1}^{n} \sum_{j=1}^{\nu} (x_i v_j + u_i y_j) \left( \frac{1}{2}(w + m) \right) = \varphi(zt),$$

i.e. $\varphi$ is a homomorphism. For arbitrary $u = \alpha w + \beta m \in \mathfrak{A}$ it is easy to see that $\varphi(z) = u$ if $z = \sum_{i=1}^{n} x_i e_i^{(f)} + \sum_{j=1}^{\nu} y_j e_j^{(m)} \in \mathcal{B}$ with $\sum_{i=1}^{n} x_i = \alpha$ and $\sum_{j=1}^{\nu} y_j = \beta$. Therefore $\varphi$ is onto. \qed

Proposition 6.4. If an algebra $A$ is dibaric, then $A^2$ is baric.

As a corollary of this Proposition and Theorem 6.3 we have

Corollary 6.5. The subalgebra $\mathcal{B}^2$ is a baric algebra.

Since $\mathcal{B}^2$ is a baric algebra all types of particular algebras mentioned in Remark 5.2 can be defined and studied for the algebra $\mathcal{B}^2$.

7. $\mathcal{B}$ is a Banach algebra

Define a norm $\| \cdot \| : \mathcal{B} \to \mathbb{R}$ as follows

$$\|z\| = \sum_{i=1}^{n} |x_i| + \sum_{j=1}^{\nu} |y_j|,$$

where $z = \sum_{i=1}^{n} x_i e_i^{(f)} + \sum_{j=1}^{\nu} y_j e_j^{(m)} \in \mathcal{B}$.

For a fixed $a \in \mathcal{B}$ consider the operator $L_a : \mathcal{B} \to \mathcal{B}$, left (right) multiplication, defined as

$$L_a(z) = az \quad (= za).$$
It is easy to see that the set \( \{ L_{e_i^{(f)}}, L_{e_j^{(m)}}, i = 1, \ldots, n, j = 1, \ldots, \nu \} \) spans a linear space which is the set of all the operators of the left (right) multiplications.

**Proposition 7.1.** Operator \( L_a \) is a bounded linear operator for any \( a = \sum_{i=1}^{n} a_i e_i^{(f)} + \sum_{j=1}^{\nu} b_j e_j^{(m)} \in \mathcal{B} \).

**Proof.** For \( z = \sum_{i=1}^{n} x_i e_i^{(f)} + \sum_{j=1}^{\nu} y_j e_j^{(m)} \in \mathcal{B} \) we have

\[
L_a(z) = \sum_{p=1}^{n} \left( \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{\nu} P_{ij,p}^{(f)} (a_i x_j + x_i b_j) \right) e_p^{(f)} + \sum_{q=1}^{\nu} \left( \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{\nu} P_{ij,q}^{(m)} (a_i y_j + x_i b_j) \right) e_q^{(m)}.
\]

\[
\|L_a(z)\| = \frac{1}{2} \sum_{p=1}^{n} \left( \sum_{i=1}^{n} \sum_{j=1}^{\nu} |P_{ij,p}^{(f)} (a_i x_j + x_i b_j)| \right) + \frac{1}{2} \sum_{q=1}^{\nu} \left( \sum_{i=1}^{n} \sum_{j=1}^{\nu} |P_{ij,q}^{(m)} (a_i y_j + x_i b_j)| \right)
\]

Using (2.1) from the last inequality we get

\[
\|L_a(z)\| \leq \|a\| \|y\| + \|b\| \|x\| \leq \max\{\|a\|, \|b\|\} \|z\|,
\]

where

\[
\|a\| = \sum_{i=1}^{n} |a_i|, \quad \|b\| = \sum_{j=1}^{\nu} |b_j|, \quad \|x\| = \sum_{i=1}^{n} |x_i|, \quad \|y\| = \sum_{j=1}^{\nu} |y_j|.
\]

Thus \( L_a \) is bounded for any fixed \( a \in \mathcal{B} \). \( \quad \square \)

**Theorem 7.2.** The algebra \( \mathcal{B} \) is complete as a normed linear space. That is, \( \mathcal{B} \) is a Banach space.

**Proof.** If \( \{z^k\} \) converges then its all coordinates also converge and therefore limit of \( z^k \) also will be an element of \( \mathcal{B} \). This completes the proof. \( \quad \square \)

**Corollary 7.3.** The algebra \( \mathcal{B} \) is a non associative Banach algebra.
8. The derivations of $\mathcal{B}$

There are many papers devoted to the subject of derivations of genetic algebras (see e.g. [2, 10, 12, 19]). In [12] an explanation of the genetic meaning of a derivation of a genetic algebra is given. For any algebra, it is known that the space of its derivations is a Lie algebra. The Lie algebra of derivations of a given algebra is an important tool for studying its structure, particularly in the non-associative case; so this is a natural development.

In this section we describe the set of all derivations of $\mathcal{B}$. Let $D \in \text{Der}(\mathcal{B})$ and suppose

$$D(e_k^{(f)}) = \sum_{i=1}^{n} d_{ki}^{ff} e_i^{(f)} + \sum_{l=1}^{\nu} d_{kl}^{fm} e_l^{(m)}, \quad k = 1, \ldots, n.$$  

$$D(e_k^{(m)}) = \sum_{i=1}^{n} d_{ki}^{mf} e_i^{(f)} + \sum_{l=1}^{\nu} d_{kl}^{mm} e_l^{(m)}, \quad k = 1, \ldots, \nu.$$  

By the definition of derivation $D(zt) = D(z)t + zD(t)$, we have

$$D(e_k^{(f)} e_j^{(f)}) = D(e_k^{(f)}) e_j^{(f)} + e_k^{(f)} D(e_j^{(f)}) =$$

$$\left( \sum_{i=1}^{n} d_{ki}^{ff} e_i^{(f)} + \sum_{l=1}^{\nu} d_{kl}^{fm} e_l^{(m)} \right) e_j^{(f)} + e_k^{(f)} \left( \sum_{i=1}^{n} d_{ji}^{ff} e_i^{(f)} + \sum_{l=1}^{\nu} d_{jl}^{fm} e_l^{(m)} \right) =$$

$$\frac{1}{2} \sum_{i=1}^{n} \left( \sum_{l=1}^{\nu} \left( d_{kl}^{fm} P_{jli}^{(f)} + d_{jl}^{fm} P_{kl,i}^{(f)} \right) \right) e_i^{(f)} +$$

$$\frac{1}{2} \sum_{q=1}^{\nu} \left( \sum_{l=1}^{\nu} \left( d_{kl}^{fm} P_{jql}^{(m)} + d_{jl}^{fm} P_{kl,q}^{(m)} \right) \right) e_q^{(m)} = 0$$

Consequently,

$$\sum_{l=1}^{\nu} \left( d_{kl}^{fm} P_{jli}^{(f)} + d_{jl}^{fm} P_{kl,i}^{(f)} \right) = 0, \quad i, j, k = 1, \ldots, n;$$

$$\sum_{l=1}^{\nu} \left( d_{kl}^{fm} P_{jql}^{(m)} + d_{jl}^{fm} P_{kl,q}^{(m)} \right) = 0, \quad j, k = 1, \ldots, n; \quad q = 1, \ldots, \nu.$$  

Similarly, from $D(e_k^{(m)} e_j^{(m)}) = D(e_k^{(m)}) e_j^{(m)} + e_k^{(m)} D(e_j^{(m)}) = 0$ we obtain

$$\sum_{l=1}^{n} \left( d_{kl}^{mf} P_{jls}^{(f)} + d_{jl}^{mf} P_{ik,s}^{(f)} \right) = 0, \quad s = 1, \ldots, n; \quad j, k = 1, \ldots, \nu;$$

$$\sum_{l=1}^{n} \left( d_{kl}^{mf} P_{jql}^{(m)} + d_{jl}^{mf} P_{ik,q}^{(m)} \right) = 0, \quad j, k, q = 1, \ldots, \nu.$$  

(8.2)
The equality \( D(e_i^{(f)} e_j^{(m)}) = D(e_i^{(f)})e_j^{(m)} + e_i^{(f)} D(e_j^{(m)}) \) gives the following conditions

\[
\begin{align*}
\sum_{p=1}^{n} \left( d_{ip}^f P_{ij,s}^f - d_{ip}^f P_{ij,p}^f \right) + \\
\sum_{q=1}^{\nu} \left( d_{jq}^m P_{ij,s}^m - d_{jq}^m P_{ij,q}^m \right) = 0, \quad i, s = 1, \ldots, n; \quad j = 1, \ldots, \nu; \\
\sum_{p=1}^{n} \left( d_{ip}^f P_{ij,t}^f - d_{ip}^m P_{ij,p}^f \right) + \\
\sum_{q=1}^{\nu} \left( d_{jq}^m P_{ij,t}^m - d_{jq}^m P_{ij,q}^m \right) = 0, \quad i = 1, \ldots, n; \quad j, t = 1, \ldots, \nu.
\end{align*}
\]

(8.3)

Since for any \( z = (x, y) \in \mathcal{B} \) we have
\[
D(z) = \sum_{i=1}^{n} x_i D(e_i^{(f)}) + \sum_{j=1}^{\nu} y_j D(e_j^{(m)}),
\]
\( D \) is uniquely defined by the matrix \( \mathcal{D} = D(D) = \left( d_{ij}^{f}, d_{ip}^{f}, d_{ip}^{m}, d_{iq}^{m} \right)_{i,j=1,\ldots,n} \).

Hence,
\[
\text{Der}(\mathcal{B}) = \{ D : \mathcal{D}(D) \text{ satisfies } (8.1), (8.2), (8.3) \}.
\]

**Example.** Consider the case \( n = 1, \nu = 2 \). In this case \( P_{11,1}^{(f)} = P_{12,1}^{(f)} = 1 \).

Denote \( P_{11,1}^{(m)} = a, P_{11,2}^{(m)} = 1 - a, P_{12,1}^{(m)} = b, P_{12,2}^{(m)} = 1 - b \). Assume \( a = 1 - b, a \neq \frac{1}{2} \). Then using (8.1), (8.2), (8.3) we get
\[
D(e_1^{(f)}) = 0; \quad D(e_1^{(m)}) = \alpha \cdot (e_1^{(m)} - e_2^{(m)}); \quad D(e_2^{(m)}) = \alpha \cdot (-e_1^{(m)} + e_2^{(m)}),
\]
where \( \alpha \in \mathbb{R} \). Consequently, for arbitrary element \( z = (x, y_1, y_2) \in \mathbb{R}^{1+2} \) we have
\[
D(z) = \alpha \cdot (y_1 - y_2)(e_1^{(m)} - e_2^{(m)}).
\]
Thus if \( n = 1, \nu = 2 \) and \( a = 1 - b, a \neq \frac{1}{2} \) then
\[
\text{Der}(\mathcal{B}) = \left\{ D : D(z) = \alpha(y_1 - y_2)(e_1^{(m)} - e_2^{(m)}), \quad z = (x, y_1, y_2) \in \mathcal{B}, \alpha \in \mathbb{R} \right\}.
\]

9. **Dynamics of the operator (2.2)**

Extend the operator (2.2) on \( \mathbb{R}^{n+\nu} \), i.e. consider the operator \( V : \mathbb{R}^{n+\nu} \to \mathbb{R}^{n+\nu}, \ z = (x, y) \mapsto z' = (x', y') = V(z) \) defined as

\[
x'_j = \sum_{i,k=1}^{n,\nu} P_{ik,j}^{(f)} x_i y_k; \quad y'_j = \sum_{i,k=1}^{n,\nu} P_{ik,j}^{(m)} x_i y_k.
\]

(9.1)

Consider the following linear form \( X : \mathbb{R}^n \to \mathbb{R} ( Y : \mathbb{R}^{\nu} \to \mathbb{R} ) \) defined by

\[
X(x) = \sum_{i=1}^{n} x_i, \quad Y(y) = \sum_{k=1}^{\nu} y_k.
\]

(9.2)

Denote

\[
H_i = \{ z = (x, y) : X(x) = Y(y) = i \}, \quad i = 0, 1,
\]

the product of the \( i \)-hyperplanes in \( \mathbb{R}^n \) and \( \mathbb{R}^{\nu} \), respectively.
A point \( z = (x, y) \in \mathbb{R}^{n+\nu} \) is called a fixed point (resp. zero point) of \( V \) if \( V(z) = z \) (resp. \( V(z) = 0 \)).

**Proposition 9.1.**

(1) If \( z \) is a fixed point then \( z \in H_0 \cup H_1 \).

(2) If \( z \) is a zero point then \( z \in \{ z = (x, y) : X(x)Y(y) = 0 \} \).

**Proof.** From (9.1) and using (2.1) we get

\[
X(x') = \sum_{j=1}^n x_j' = \sum_{j,k=1}^{n,\nu} \left( \sum_{j=1}^n P_{ik,j}^{(f)} \right) x_i y_k = X(x)Y(y) \\
Y(y') = \sum_{\nu=1}^{\nu} y_j' = \sum_{\nu=1}^{\nu} \left( \sum_{\nu=1}^{\nu} P_{ik,l}^{(m)} \right) x_i y_k = X(x)Y(y).
\]

(1) If \( z \) is a fixed point then \( X(x') = X(x) \) and \( Y(y') = Y(y) \), this by (9.1) gives that \( X(x) = Y(y) \) and \( X(x) = (X(x))^2 \). Hence \( X(x) = Y(y) = 0 \) or 1.

(2) If \( z \) is a zero point then \( X(x') = Y(y') = 0 \), and (9.1) gives \( X(x)Y(y) = 0 \).

\( \square \)

Let \( z^{(0)} = (x^{(0)}, y^{(0)}) \in \mathbb{R}^{n+\nu} \) be an initial point. Its trajectory is defined by \( z^{(t)} = (x^{(t)}, y^{(t)}) = V(z^{(t-1)}) \), \( t = 1, 2, \ldots \). Denote by \( \omega(z^{(0)}) \) the set of limit points of the trajectory \( \{ z^{(t)} \}_{t=0}^{\infty} \). If \( \omega(z^{(0)}) \) consists of a single point, then the trajectory converges, and \( \omega(z^{(0)}) \) is a fixed point of the operator \( V \).

The following theorem gives an upper estimate of the set \( \omega(z^{(0)}) \)

**Theorem 9.2.** We have

\[
\omega(z^{(0)}) \subset \begin{cases} 
H_0 & \text{if } |X(x^{(0)})Y(y^{(0)})| < 1 \\
H_1 & \text{if } |X(x^{(0)})Y(y^{(0)})| = 1 \\
H_\infty & \text{if } |X(x^{(0)})Y(y^{(0)})| > 1,
\end{cases}
\]

where \( H_\infty = \{ z = (x, y) : X(x) = Y(y) = +\infty \} \).

**Proof.** Using (9.1) we get \( X(x^{(t)}) = Y(y^{(t)}) = X(x^{(t-1)})Y(y^{(t-1)}) \) for any \( t = 1, 2, \ldots \). Iterating this recurrent equation we obtain

\[
X(x^{(t)}) = Y(y^{(t)}) = \left( X(x^{(0)})Y(y^{(0)}) \right)^{2^{t-1}}.
\]

Consequently,

\[
\lim_{t \to \infty} X(x^{(t)}) = \lim_{t \to \infty} Y(y^{(t)}) = \begin{cases} 
0 & \text{if } |X(x^{(0)})Y(y^{(0)})| < 1 \\
1 & \text{if } |X(x^{(0)})Y(y^{(0)})| = 1 \\
+\infty & \text{if } |X(x^{(0)})Y(y^{(0)})| > 1.
\end{cases}
\]

\( \square \)
10. A SPECIAL CASE OF AN EABP

In this section we consider a special case of an EABP giving an additional condition on heredity coefficients (2.1), i.e. consider the coefficients as follows

\[
P^{(f)}_{ik,j} = \begin{cases} 
  a_{ij} & \text{if } k = 1 \\
  1 & \text{if } k \neq 1, j = 1 \\
  0 & \text{if } k \neq 1, j \neq 1.
\end{cases} \quad P^{(m)}_{ik,l} = \begin{cases} 
  b_{kl} & \text{if } i = 1 \\
  1 & \text{if } i \neq 1, l = 1 \\
  0 & \text{if } i \neq 1, l \neq 1.
\end{cases}
\]

The matrices \( A = (a_{ij}) \) and \( B = (b_{kl}) \) by (2.1) satisfy the following conditions

\[
a_{ij} \geq 0, \quad \sum_{j=1}^{n} a_{ij} = 1, \quad i = 1, \ldots, n; \quad b_{kl} \geq 0, \quad \sum_{l=1}^{\nu} b_{kl} = 1, \quad k = 1, \ldots, \nu,
\]

i.e. both matrices are stochastic.

**Remark 10.1.** The condition (10.1) is taken to simplify our computations; in this way we reduced both cubic matrices \( (P^{(f)}_{ik,j}), (P^{(m)}_{ik,l}) \) to quadratic matrices. But that condition has very clear biological treatment: the type “1” of females and the type “1” of males have preference, i.e. any type of female (male) can be born if its father (mother) has type “1”. If the father (mother) has type \( \neq 1 \) then only type “1” female (male) can be born.

Under condition (10.1) the multiplication (3.1) became as

\[
e^{(f)}_i e^{(m)}_k = \begin{cases} 
  \sum_{j=1}^{n} a_{ij} e^{(f)}_j + \sum_{l=1}^{\nu} b_{kl} e^{(m)}_l, & \text{if } i = 1, k = 1 \\
  \sum_{j=1}^{n} a_{ij} e^{(f)}_j + e^{(m)}_1, & \text{if } i \neq 1, k = 1 \\
  e^{(f)}_1 + \sum_{l=1}^{\nu} b_{kl} e^{(m)}_l, & \text{if } i = 1, k \neq 1 \\
  e^{(f)}_1 + e^{(m)}_1, & \text{if } i \neq 1, k \neq 1.
\end{cases}
\]

\[
e^{(f)}_i e^{(f)}_j = 0, \quad i,j = 1, \ldots, n; \quad e^{(m)}_k e^{(m)}_l = 0, \quad k,l = 1, \ldots, \nu.
\]

Operator (2.2) has the following form

\[
x'_i = \sum_{i=1}^{n} (a_{i1} y_1 + \sum_{k=2}^{\nu} y_k) x_i \\
x'_j = y_1 \sum_{i=1}^{n} a_{ij} x_i, \quad j \neq 1
\]

\[
y'_1 = \sum_{k=1}^{\nu} (b_{k1} x_1 + \sum_{i=2}^{n} x_i) y_k \\
y'_l = x_1 \sum_{k=1}^{\nu} b_{kl} y_k, \quad l \neq 1
\]

Denote by \( B_1 \) the EABP defined by the multiplication table (10.3).
10.2. Idempotent elements of $B_1$. A element $z \in B$ is called idempotent if $z^2 = z$; such points of an EABP are especially important, because they are the fixed points of the evolution map $V$, i.e. $V(z) = z$. We denote by $\mathcal{I}d(B)$ the idempotent elements of an algebra $B$. Clearly, $0 \in \mathcal{I}d(B)$ and this set is an algebraic variety. By Proposition 9.1 we have $\mathcal{I}d(B) \subset H_0 \cup H_1$. In this subsection we shall describe idempotent elements of $B_1$. First we describe the set of idempotent elements which belong in $H_0 \cap \mathcal{I}d(B_1)$ and after that we shall describe elements of $H_1 \cap \mathcal{I}d(B_1)$.

**Idempotents in $H_0$.**

Using (10.4) and the condition that $z \in H_0$, from the equation $V(z) = z^2 = z$ we obtain

\[
\begin{align*}
x_j &= y_1 \sum_{i=2}^n (a_{ij} - a_{1j}) x_i, \quad j = 2, \ldots, n; \\
y_l &= x_1 \sum_{k=2}^{\nu} (b_{kl} - b_{1l}) y_k, \quad l = 2, \ldots, \nu.
\end{align*}
\]

*Case $x_1 y_1 = 0$: If $x_1 = 0$ then $y_2 = \cdots = y_\nu = 0$, consequently, $y_1 = 0$. Similarly, if $y_1 = 0$ we get $x_1 = \cdots = x_n = 0$. Hence in case $x_1 y_1 = 0$ we have unique idempotent $z = 0$. 

*Case $x_1 y_1 \neq 0$: Consider matrices $C_y = (c_{ij})_{i,j=2,\ldots,n}$ and $D_x = (d_{kl})_{k,l=2,\ldots,\nu}$ such that

\[
c_{ij} = \begin{cases} 
(a_{ij} - a_{1j}) y & \text{if } i \neq j \\
(a_{ij} - a_{1j}) y - 1 & \text{if } i = j 
\end{cases}, \quad d_{kl} = \begin{cases} 
(b_{kl} - b_{1l}) x & \text{if } k \neq l \\
(b_{kl} - b_{1l}) x - 1 & \text{if } k = l 
\end{cases}
\]

Then equation (10.5) can be written as

\[
C_y x = 0, \quad D_x y = 0,
\]

where $x = (x_2, \ldots, x_n)$, $y = (y_2, \ldots, y_\nu)$.

Consider now $x_1 \in \mathbb{R} \setminus \{0\}$ as a parameter, then equation $D_x y = 0$ has a unique solution $y = 0$ if $\det(D_x) \neq 0$, which gives $y_1 = 0$, i.e. this is a contradiction to the assumption that $y_1 \neq 0$.

If $\det(D_x) = 0$ then we fix a solution $x_1 = x_1^* \neq 0$ of the equation $\det(D_{x_1}) = 0$. In this case there are infinitely many solutions $y^* = (y_2^*, \ldots, y_\nu^*)$ of $D_{x_1} y = 0$. Substituting the solution $y_1^* = -\sum_{k=2}^{\nu} y_k^* C_{y_1} x = 0$, we get

\[
x_j = y_1^* \sum_{i=2}^n (a_{ij} - a_{1j}) x_i = 0, \quad j = 2, \ldots, n.
\]

This system has a unique solution $x_2 = \cdots = x_n = 0$ if $\det(C_{y_1}^*) \neq 0$ but in this case we get $x_1 = 0$ which is a contradiction to the assumption that $x_1 \neq 0$. If $\det(C_{y_1}^*) = 0$ then we have infinitely many solutions $x^* = (x_2^*, \ldots, x_n^*)$.

Hence we have proved the following

**Proposition 10.3.** We have

\[
H_0 \cap \mathcal{I}d(B_1) = \{0\} \cup \{((x_1^*, \ldots, x_n^*), (y_1^*, \ldots, y_\nu^*)): C_{y_1}^* x^* = 0, \quad D_{x_1}^* y^* = 0, \quad \det(D_{x_1}^*) = \det(C_{y_1}^*) = 0\}.
\]
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Idempotents in $H_1$. Using (10.4) and the condition that $z \in H_1$, the equation $V(z) = z^2 = z$ can be written as

\[\begin{align*}
&x_1 = \sum_{i=1}^n ((a_{i1} - 1)y_i + 1)x_i; \\
x_j = y_1 \sum_{i=1}^n a_{ij}x_i, & j \neq 1.
\end{align*}\]

(10.8) \[\begin{align*}
y_1 = \sum_{k=1}^\nu ((b_{k1} - 1)x_1 + 1)y_k; \\
y_l = x_1 \sum_{k=1}^\nu b_{kl}y_k, & l \neq 1.
\end{align*}\]

(10.9)

Consider several cases.

Case $x_1 = y_1 = 0$. In this case we get $z = 0$ which is not in $H_1$.

Case $x_1 \neq 0, y_1 = 0$. In this case $x_1 = 1, x_j = 0, j = 2, \ldots, n$.

Substituting $x_1 = 1$ in the system of equations (10.9) we get

\[\begin{align*}
0 &= \sum_{\nu}^\nu b_{k1}y_k \\
y_l &= x_1 \sum_{\nu}^\nu b_{kl}y_k, & l = 2, \ldots, \nu.
\end{align*}\]

(10.10)

Denote $B_2 = (b'_{kl})_{k,l=2,\ldots,\nu}$, with

\[b'_{kl} = \begin{cases} b_{kl} - 1 & \text{if } k = l \\ b_{kl} & \text{if } k \neq l. \end{cases}\]

If $\det(B_2) \neq 0$ then $B_2y = 0$ gives $y_1 = \cdots = y_\nu = 0$ but this does not satisfy $\sum_{\nu}^\nu y_k = 1$. If $\det(B_2) = 0$ then $B_2y = 0$ has infinitely many solutions $y = (y_2, \ldots, y_\nu)$. We must take these solutions which satisfy $\sum_{\nu}^\nu b_{k1}y_k = 0$.

So in this case we have the following idempotent elements (which belong to $H_1$)

\[I_0 = \begin{cases} \{(1,0,\ldots,0),(0,y_2,\ldots,y_\nu)\} : B_2y = 0, \sum_{\nu}^\nu b_{k1}y_k = 0 \} & \text{if } \det(B_2) = 0 \\ \emptyset & \text{if } \det(B_2) \neq 0. \end{cases}\]

Case $x_1 = 0, y_1 \neq 0$. This case is similar to the previous case. Denote $A_2 = (a'_{ij})_{i,j=2,\ldots,n}$, with

\[a'_{ij} = \begin{cases} a_{ij} - 1 & \text{if } i = j \\ a_{ij} & \text{if } i \neq j. \end{cases}\]

Then we have the following idempotent elements (which belong to $H_1$):

\[I_1 = \begin{cases} \{((0,x_1,\ldots,x_n),(1,0,\ldots,0)) : A_2x = 0, \sum_{j=2}^n a_{j1}x_j = 0 \} & \text{if } \det(A_2) = 0 \\ \emptyset & \text{if } \det(A_2) \neq 0. \end{cases}\]

Case $x_1 \neq 0, y_1 \neq 0$. Here, to avoid many special cases we assume that $a_{11} \neq 1$. Take $y_1$ as a parameter and solve the system (10.8) which is equivalent to the following

\[\begin{align*}
\sum_{i=2}^n \left( \frac{a_{1j}((a_{11} - 1)y_1 + 1)}{1 - a_{11}} + a_{ij}y_1 \right) x_i &= x_j, & j = 2, \ldots, n.
\end{align*}\]

(10.11)
Denote $U_y = (u_{ij})_{i,j=2,\ldots,n}$ with

$$ u_{ij} = \begin{cases} \frac{a_{ij}(a_{11}-1)y+1}{1-a_{11}} + a_{ij}y & \text{if } i \neq j \\ \frac{a_{ij}(a_{11}-1)y+1}{1-a_{11}} + a_{ij}y - 1 & \text{if } i = j \end{cases} $$

**Subcase** $\det(U_y) \neq 0$. In this case we have $x_2 = \cdots = x_n = 0$, consequently, $x_1 = 1$. For $x_1 = 1$ from the equation (10.9) we get $B_1 y = 0$ where $B_1 = (b_{kl})_{k,l=1,\ldots,\nu}$, with

$$ b_{kl} = \begin{cases} b_{kl} - 1 & \text{if } k = l \\ b_{kl} & \text{if } k \neq l. \end{cases} $$

So in this case we have the following set of idempotent elements of $H_1$.

$$ I_2 = \begin{cases} \{(1,0,\ldots,0),(y_1,\ldots,y_\nu)\} \text{ if } \det(B_1) = 0 \\ \emptyset & \text{if } \det(B_1) \neq 0. \end{cases} $$

**Subcase** $\det(U_y) = 0$. In this case we fix a solution $y_1 = y_1^*$ of $\det(U_y) = 0$. We have infinitely many solutions $(x_2^*,\ldots,x_n^*)$. Denote $C_x = (c_{kl}^*)_{k,l=1,\ldots,\nu}$, with

$$ c_{kl}^* = \begin{cases} (b_{11} - 1)x & \text{if } k = l = 1; \\ (b_{k1} - 1)x + 1 & \text{if } k \neq 1, l = 1; \\ b_{kl}x - 1 & \text{if } k = l \neq 1; \\ b_{kl}x & \text{if } k \neq l, l = 2,\ldots,\nu. \end{cases} $$

Now one has the following set of idempotent elements

$$ I_3 = \{((x_1^*,\ldots,x_n^*),(y_1^*,\ldots,y_\nu^*)) : C_{x_1^*} y_{1}^* = 0, U_{y_{1}^*} x_{1}^* = 0, \det(U_{y_{1}^*}) = \det(C_{x_1^*}) = 0\}. $$

Thus we have the following

**Theorem 10.4.** If $a_{11} \neq 1$ then the full set of the idempotent elements which belong to $H_1$ is

$$ \mathcal{I}d(B_1) \cap H_1 = I_0 \cup I_1 \cup I_2 \cup I_3. $$

**Remark 10.5.** By Proposition 9.3 we can conclude that Proposition 10.3 and Theorem 10.4 give the full set $\mathcal{I}d(B_1)$. But conditions described in the Proposition and Theorem are complicated in general. In the sequel of this subsection we shall consider additional conditions on $(a_{ij})$, $(b_{ij})$ and under these conditions we explicitly describe the set $\mathcal{I}d(B_1)$.

Now we consider a particular case and describe the full set of idempotent elements of $B_1$.

If we consider the case

$$ a_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}, \quad b_{kl} = \begin{cases} 1, & k = l \\ 0, & k \neq l \end{cases} $$

then the following is true

\[ \text{...} \]
Proposition 10.6. If the condition (10.12) is satisfied then

\[ I d(B_1) = \{ 0 \} \cup \left\{ ((1, x_2, \ldots, x_n), (1, y_2, \ldots, y_\nu)) : \sum_{i=2}^{n} x_i = \sum_{k=2}^{\nu} y_k = 0 \right\} \]

\[ \cup \left\{ ((1, x_2, \ldots, x_n), (1, y_2, \ldots, y_\nu)) : \sum_{i=2}^{n} x_i = \sum_{k=2}^{\nu} y_k = -1 \right\} \]

\[ \cup \left\{ ((0, 1, \ldots, 0), (y_1, \ldots, y_\nu)) : \sum_{k=1}^{\nu} y_k = 1, y_1 \neq 1 \right\} \]

\[ \cup \left\{ ((x_1, \ldots, x_n), (1, 0, \ldots, 0)) : \sum_{i=1}^{n} x_i = 1, x_1 \neq 1 \right\} . \]

Proof. The equation \( z^2 = z \), for \( z = (x, y) \in \mathbb{R}^{n+\nu} \), using condition (10.12) can be written as

\[
\begin{align*}
(1 - y_1) x_1 &= \sum_{k=2}^{\nu} y_k \sum_{i=1}^{n} x_i, \\
(1 - y_1) x_j &= 0, \quad j = 2, \ldots, n \\
(1 - x_1) y_1 &= \sum_{i=2}^{n} x_i \sum_{k=1}^{\nu} y_k, \\
(1 - x_1) y_l &= 0, \quad l = 2, \ldots, \nu.
\end{align*}
\]

(10.13)

The simple analysis of the system (10.13) gives the set of all idempotents. \( \square \)

10.7. Absolute nilpotent elements of \( B_1 \). The element \( z \) is called an absolute nilpotent if \( z^2 = 0 \), i.e. \( z \) is zero-point of the evolution operator \( V \). For \( B_1 \) the equation \( z^2 = 0 \) is equivalent to the following system of quadratic equations

\[
\begin{align*}
\sum_{i=1}^{n} (a_{ij} y_1 + \sum_{k=2}^{\nu} y_k) x_i &= 0; \\
y_1 \sum_{i=1}^{n} a_{ij} x_i &= 0, \quad j = 2, \ldots, n; \\
\sum_{k=1}^{\nu} (b_{kl} x_1 + \sum_{i=2}^{n} x_i) y_k &= 0; \\
x_1 \sum_{k=1}^{\nu} b_{kl} y_k &= 0, \quad l = 2, \ldots, \nu.
\end{align*}
\]

(10.14)

By Proposition 9.1 we have \( \sum_{i=1}^{n} x_i \sum_{k=1}^{\nu} y_k = 0 \). There are the following three cases.

- Case \( \sum_{i=1}^{n} x_i = \sum_{k=1}^{\nu} y_k = 0 \). In this case, from the system of quadratic equations (10.14) we get

\[
\begin{align*}
y_1 \sum_{i=1}^{n} a_{ij} x_i &= 0, \quad j = 1, \ldots, n; \\
x_1 \sum_{k=1}^{\nu} b_{kl} y_k &= 0, \quad l = 1, \ldots, \nu.
\end{align*}
\]

(10.15)

Subcase \( y_1 = x_1 = 0 \). In this case one easily gets the following set of solutions

\[ \mathcal{N}_{00}^0 = \left\{ ((0, x_2, \ldots, x_n), (0, y_2, \ldots, y_\nu)) : \sum_{i=2}^{n} x_i = 0, \sum_{k=2}^{\nu} y_k = 0 \right\} . \]
Subcase $x_1 = 0, y_1 \neq 0$. Denote $A_0 = (a_{ij})_{i,j=2,\ldots,n}$. It is easy to see that the set of solutions is as follows
\[ \mathcal{N}_0^{i1} = \begin{cases} \mathcal{N}^1 & \text{if } \det(A_0) \neq 0; \\ \mathcal{N}^0 & \text{if } \det(A_0) = 0, \end{cases} \]
where
\[ \mathcal{N}^1 = \left\{ (0, \ldots, 0, (y_1, \ldots, y_\nu)) : y_1 \neq 0, \sum_{k=1}^\nu y_k = 0 \right\}, \]
\[ \mathcal{N}^0 = \left\{ ((0, x_2, \ldots, x_n), (y_1, \ldots, y_\nu)) : \sum_{i=2}^n x_i = \sum_{i=2}^n a_{i1} x_i = 0, \right. \\
\left. A_0 x = 0, y_1 \neq 0, \sum_{k=1}^\nu y_k = 0 \right\}. \]

Subcase $x_1 \neq 0, y_1 = 0$. Denote $B_0 = (b_{kl})_{k,l=2,\ldots,\nu}$. In this case the set of solutions is
\[ \mathcal{N}_0^{i0} = \begin{cases} \mathcal{N}^2 & \text{if } \det(B_0) \neq 0; \\ \mathcal{N}^3 & \text{if } \det(B_0) = 0, \end{cases} \]
where
\[ \mathcal{N}^2 = \left\{ ((x_1, \ldots, x_n), (0, \ldots, 0)) : x_1 \neq 0, \sum_{i=1}^n x_i = 0 \right\}, \]
\[ \mathcal{N}^3 = \left\{ ((x_1, \ldots, x_n), (0, y_2, \ldots, y_\nu)) : x_1 \neq 0, \sum_{i=1}^n x_i = 0, \right. \\
\left. B_0 y = 0, \sum_{k=2}^\nu y_k = \sum_{k=2}^\nu b_{k1} y_k = 0 \right\}. \]

Subcase $x_1 \neq 0, y_1 \neq 0$. In this case it is easy to get the following
\[ \mathcal{N}_1^{i1} = \begin{cases} \emptyset & \text{if } \det(A) \neq 0 \text{ or } \det(B) \neq 0; \\ \mathcal{N}^4 & \text{if } \det(A) = \det(B) = 0, \end{cases} \]
where
\[ \mathcal{N}^4 = \left\{ ((x_1, \ldots, x_n), (y_1, \ldots, y_\nu)) : Ax = 0, By = 0 \right\}. \]

- **Case** $\sum_{i=1}^n x_i = 0, \sum_{k=1}^\nu y_k \neq 0$. In this case the system of quadratic equations (10.14) can be written as
\[ \sum_{i=1}^n (a_{i1} y_1 + \sum_{k=2}^\nu y_k) x_i = 0 \\
y_1 \sum_{i=1}^n a_{ij} x_i = 0, \quad j = 2, \ldots, n \\
x_1 \sum_{k=1}^\nu (b_{k1} - 1) y_k = 0 \\
x_1 \sum_{k=1}^\nu b_{kl} y_k = 0, \quad l = 2, \ldots, \nu. \]
Subcase $y_1 = x_1 = 0$. In this case we have the following set of absolute nilpotents

$$\mathcal{N}^1_{00} = \left\{ ((0, x_2, \ldots, x_n), (0, y_2, \ldots, y_\nu)) : \sum_{i=2}^n x_i = 0, \sum_{k=2}^\nu y_k \neq 0 \right\}. $$

Subcase $x_1 = 0, y_1 \neq 0$. From (10.16) we get

$$\sum_{i=2}^n (a_{i1} y_1 + \sum_{k=2}^\nu y_k) x_i = 0$$

(10.17)

$$\sum_{i=1}^n a_{ij} x_i = 0, \quad j = 2, \ldots, n.$$ 

This has the following set of solutions

$$\mathcal{N}^1_{01} = \begin{cases} \mathcal{N}^5 & \text{if } \det(A_0) \neq 0; \\ \mathcal{N}^6 & \text{if } \det(A_0) = 0, \end{cases}$$

where

$$\mathcal{N}^5 = \left\{ ((0, \ldots, 0), (y_1, \ldots, y_\nu)) : y_1 \neq 0, \sum_{k=1}^\nu y_k \neq 0 \right\},$$

$$\mathcal{N}^6 = \left\{ ((0, x_2, \ldots, x_n), (y_1, \ldots, y_\nu)) : \sum_{i=2}^n x_i = \sum_{i=2}^n (a_{i1} y_1 + \sum_{k=2}^\nu y_k) x_i = 0, \right. $$$$ \left. A_0 x = 0, y_1 \neq 0, \sum_{k=1}^\nu y_k \neq 0 \right\}. $$

Subcase $x_1 \neq 0, y_1 = 0$. In this case from (10.16) we get

$$\sum_{k=2}^\nu (b_{k1} - 1) y_k = 0$$

(10.18)

$$\sum_{k=2}^\nu b_{kl} y_k = 0, \quad l = 2, \ldots, \nu.$$ 

If $\det(B_0) \neq 0$ we get $y_2 = \cdots = y_\nu = 0$, this is impossible since we have condition $\sum_{k=1}^\nu y_k \neq 0$. For $\det(B_0) = 0$ the set of absolute nilpotents will be

$$\mathcal{N}^1_{10} = \left\{ ((x_1, \ldots, x_n), (0, y_2, \ldots, y_\nu)) : x_1 \neq 0, \sum_{i=1}^n x_i = 0, \right. $$$$ \left. B_0 y = 0, \sum_{k=2}^\nu y_k \neq 0, \sum_{k=2}^\nu (b_{k1} - 1) y_k = 0 \right\}. $$

Subcase $x_1 \neq 0, y_1 \neq 0$. Denote $A = (a_{ij})_{i,j=1,\ldots,n}$, with

$$a_{ij} = \begin{cases} 1 & \text{if } j = 1; \\ a_{ij} & \text{if } j \neq 1. \end{cases}$$
and $\mathcal{B} = (b_{kl})_{k,l=1,\ldots,\nu}$, with

$$b_{kl} = \begin{cases} 
  b_{k1} - 1 & \text{if } l = 1 \\
  b_{kl} & \text{if } l \neq 1.
\end{cases}$$

In this case we have

$$\mathcal{N}_{11}^1 = \begin{cases} 
  \emptyset & \text{if } \det(A) \neq 0 \text{ or } \det(B) \neq 0; \\
  \mathcal{N}^\emptyset & \text{if } \det(A) = \det(B) = 0,
\end{cases}$$

where

$$\mathcal{N}^\emptyset = \{((x_1, \ldots, x_n), (y_1, \ldots, y_\nu)) : Ax = 0, By = 0, \\
\sum_{i=1}^{n} \left( a_{i1} y_1 + \sum_{k=2}^{\nu} y_k \right) x_i = 0, \sum_{k=1}^{\nu} y_k \neq 0 \}.$$  

• Case $\sum_{i=1}^{n} x_i \neq 0, \sum_{k=1}^{\nu} y_k = 0$. This case is similar to the previous case, to get the set of nilpotents, one has to change sets $\mathcal{N}_{ij}^1, i, j = 0, 1$ as follows. Replace $x$ and $y$, also rearrange parameters $a_{ij}$ with $b_{kl}$.

Denote resulting sets by $\mathcal{N}_{ij}^2, i, j = 0, 1$ respectively.

Thus we have proved the following theorem

**Theorem 10.8.** The full set $\mathcal{N}$ of absolute nilpotent elements of the algebra $\mathcal{B}_1$ is

$$\mathcal{N} = \bigcup_{i=0}^{2} \left( \mathcal{N}_{00}^i \cup \mathcal{N}_{01}^i \cup \mathcal{N}_{10}^i \cup \mathcal{N}_{11}^i \right).$$

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