Electromagnetic Siegert states for periodic dielectric structures

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The formalism of Siegert states to describe the resonant scattering in quantum theory is extended to the resonant scattering of electromagnetic waves on periodic dielectric arrays. The excitation of electromagnetic Siegert states by an incident wave packet and their decay is studied. The formalism is applied to develop a theory of coupled electromagnetic resonances arising in the electromagnetic scattering problem for two such arrays separated by a distance $2h$ (or, generally, when the physical properties of the scattering array depend on a real coupling parameter $h$). Analytic properties of Siegert states as functions of the coupling parameter $h$ are established by the Regular Perturbation Theorem which is an extension the Kato-Rellich theorem to the present case. By means of this theorem, it is proved that if the scattering structure admits a bound state in the radiation continuum at a certain value of the coupling parameter $h$, then there always exist regions within the structure in which the near field can be amplified as much as desired by adjusting the value of $h$. This establishes a rather general mechanism to control and amplify optical nonlinear effects in periodically structured planar structures possessing a nonlinear dielectric susceptibility.
I. INTRODUCTION

To outline the scope of the problem studied, it is helpful first to glean over the basic concepts of quantum resonant scattering theory. Consider the scattering problem ̂HΨ = EΨ for a quantum mechanical system described by the Hamiltonian ̂H = −1/2 ∆ + V(r) where ∆ is the Laplace operator in space, E is the spectral parameter (the energy), and the scattering potential V(r) is assumed to be of a finite range, i.e., V(r) = 0 if r = |r| > r₀ for some r₀. To avoid excessive and unnecessary technicalities, the potential is assumed to be spherically symmetric, V(r) = V(r). Suppose further that only a spherically symmetric scattered wave is of interest (the s-wave). Then the scattering theory [1, 2] requires that in the asymptotic region r > r₀ an eigenfunction of the Hamiltonian ̂H corresponding to an eigenvalue E is the superposition of an incident wave Ψᵢ ~ e^{ikr} and an outgoing wave Ψₛ:

Ψ = Ψᵢ + Ψₛ, \( Ψₛ = S \frac{e^{ikr}}{r}, \quad k = |k| = \sqrt{2E} \)

where S is called the scattering amplitude. The system is said to have a scattering resonance if S has a pole in the complex energy plane:

S ∼ \frac{Γ_n}{E - \mathcal{E}_n + iΓ_n}

where \( \mathcal{E}_n \) is the resonance position (or the resonant energy) and \( Γ_n \) is the resonance width. In this case the scattering cross section as a function of E exhibits a Lorentzian profile

\[ σ(E) ∼ |S|^2 ∼ \frac{Γ_n^2}{(E - \mathcal{E}_n)^2 + Γ_n^2} \]

In 1939, A. Siegert showed [3] that positions and widths of resonances for a given quantum system can be determined by solving the eigenvalue problem under the outgoing wave boundary condition which in the simplest case of a spherically symmetric scattered wave (the s-wave) reads

\[ ̂HΨ_n = EΨ_n, \quad \left( ∂_r (rΨ_n) - ikrΨ_n \right)_{r ≥ r₀} = 0 \]

Under this boundary condition the Hamiltonian is no longer hermitian, hence, the eigenvalues are generally complex, \( E = \mathcal{E}_n - iΓ_n \). The eigenstates Ψₙ associated with eigenvalues \( \mathcal{E}_n - iΓ_n \) are now called Siegert states. The scattered wave is then proved to have the form

Ψₛ = \[ \sum_n a_n \frac{1}{E - \mathcal{E}_n + iΓ_n} \Psi_n + \Psi_a \]

where the complex amplitudes \( a_n = a_n[Ψ_i] \) are homogeneous linear functionals of the incident wave and \( Ψ_a \) is the so-called background or potential scattering which is analytic in E (see e.g., [4, 5]). If Ψᵢ is a wave packet with a narrow energy distribution centered at \( E = \mathcal{E}_c \) (e.g., a narrow Gaussian wave packet), then the amplitude \( a_n \) is significant, provided \( \mathcal{E}_c ≈ \mathcal{E}_n \). In other words, a Siegert state contributes to the scattering wave if the incident wave has a resonant energy \( \mathcal{E}_n \) determined by the real part of the corresponding Siegert eigenvalue.

Note that the scattered modes always have positive energy \( E = k^2/2 > 0 \). The range of E corresponding to scattered modes is called the radiation continuum. Suppose that \( V = V(r) \), for simplicity. It is easy to see that the Siegert states contain bound states of the system \( \mathcal{E}_n < 0 \) and \( Γ_n = 0 \) which decay exponentially in the asymptotic region (as \( k = i\sqrt{-2\mathcal{E}_n} \)) and, hence, are square integrable eigenstates of the Hamiltonian. The bound states have a discrete spectrum which lies below the radiation continuum (\( \mathcal{E}_n < 0 \)). If the asymptotic behavior of the potential \( V \) is relaxed by demanding that \( V(r) → 0 \) as \( r → ∞ \), then there may exist bound states (i.e., square integrable solutions of the stationary Schrödinger equation) with positive energies \( \mathcal{E}_n > 0 \), \( Γ_n = 0 \). They are called bound states in the radiation continuum or resonances with the vanishing width (to emphasize the fact \( \mathcal{E}_n > 0 \)). Their existence was first predicted by von Neumann and Wigner in 1929 [6]. It can further be shown that the amplitudes \( a_n \) vanish if \( Γ_n = 0 \). Thus, no bound state (either above or below the radiation continuum) of the system can be excited by an incident wave [1].

The physical significance of Siegert states with \( Γ_n > 0 \) can be understood through the initial value problem for the time dependent Schrödinger equation \( i∂_t Ψ = ̂HΨ \) in which a wave packet \( Ψ_i \), initially positioned in the asymptotic region \( r > r₀ \) (i.e., at the initial time \( t = 0 \) the support of \( Ψ_i \) lies in the asymptotic region), propagates into the scattering region \( r < r₀ \) and passes through it giving rise to scattered waves. In the time-dependent picture, the
amplitude of each Siegert state (with a sufficiently small $\Gamma_n$) that has been excited by the incident wave packet is shown to decay exponentially [1]:

$$\Psi_s(t) = \sum_n \Omega_n(t) \Psi_n + \Psi_a(t), \quad \Omega_n(t) \sim e^{-i\varepsilon_n t} e^{-\Gamma_n t}$$

If the incident wave packet passes through the scattering region faster than the decay time $\tau_n = 1/\Gamma_n$ of a Siegert state, then the outgoing wave can be observed, and it resembles to a stationary state with the energy $\varepsilon_n$. The more narrow the scattering resonance is, the longer lives the corresponding Siegert state. So, Siegert states with $\Gamma_n \ll 1$ may be interpreted as quasi-stationary states of the system that can be excited by an incident wave and live long after the scattering process is over.

Resonant scattering phenomena are also quite common in electromagnetic theory. In the past decade much attention has been devoted to experimental and theoretical studies of reflection and transmission properties of periodically perforated films and similar periodic planar structures in infrared and visible light. The transmission and reflection coefficients of such structures exhibit typical resonant peaks at a wavelength about the structure period [2] (for a review see [3, 4]). Numerical studies of the scattering problem for an electromagnetic wave packet impinging a periodic structure show that a portion of the electromagnetic energy of the wave packet is trapped by the structure and remains in it long after the wave packet has passed the structure [5, 12]. Furthermore, each ”trapped” electromagnetic mode decays slowly by emitting a monochromatic radiation whose wavelength is close to the resonant wavelength. Broad and narrow resonances correspond to short-lived and long-lived ”trapped” modes, respectively. It has been shown that ”trapped” modes may occur either due to the structure geometry or the dispersive properties of the structure material [5, 11].

If two planar periodic structures are separated by a distance $2h$, then the resonance positions and their widths depend on $h$ so that under some conditions, there are critical values $h = h_b$ at which the widths of some resonances approach zero as $h \to h_b$ [3, 13, 14]. Studies of specific examples show that when $h = h_b$ the structure has a stable wave guiding mode localized near the structure which cannot be excited by an incident radiation despite the fact that the spectral parameters of this mode lie in the spectral range of the scattered modes (the radiation continuum). In other words, such wave guiding modes are a new type of localized solutions of Maxwell’s equations which are analogous to quantum mechanical bounds states in the radiation continuum.

It has been observed that in the spectral range near these localized solutions, there is a local amplification of near-fields (as compared to the amplitude of the incident radiation) for periodic scattering structures. The amplification effect becomes stronger if the system exhibits more narrow resonances [3, 12, 14].

The similarities between resonant quantum and electromagnetic scattering phenomena are evident. It is therefore natural to develop a similar mathematical formalism of Siegert states to study the resonant electromagnetic scattering for periodic structures. Although a qualitative analogy between electromagnetic ”trapped” modes and quantum Siegert states has been drawn, no rigorous quantitative studies have been carried out. It appears that there are two substantial differences between the theories of quantum mechanics and electromagnetism which prevent one from a trivial extension of quantum mechanical Siegert states to electromagnetism.

First, the electric field $E$ propagating in a non-homogeneous medium with a dielectric susceptibility $\varepsilon$ satisfies the wave equation $\varepsilon \partial_t^2 E = \Delta E$. Consequently, the stationary analog of the Schrödinger equation, which is obtained when $E(t, r) = e^{-i\omega t} E_\omega(r)$, is:

$$\left[-\Delta + k^2(1 - \varepsilon(r))\right] E_\omega = k^2 E_\omega, \quad k^2 = \frac{\omega^2}{c^2}$$

This equation can still be viewed as an eigenvalue problem, but in contrast to the stationary Schrödinger equation, the ”potential” $V(r) = k^2(1 - \varepsilon(r))$ depends on the spectral parameter $k^2$. Furthermore, the dielectric susceptibility could also be a complex-valued function of $k$ if the medium is dispersive, i.e., $\varepsilon = \varepsilon(k, r)$.

Second, for a periodic scattering structure, the amplitude $E_\omega$ must obey Bloch’s periodicity condition which is not compatible with the asymptotic boundary condition for quantum mechanical Siegert states. The stated differences require a modification of the very definition of Siegert states in electromagnetic theory. This problem is investigated in the present study.

In Section III electromagnetic Siegert states are defined for periodic scattering structures by means of the theory of compact integral operators when the polarization of the electromagnetic wave is preserved, i.e., when the vector wave equation is reduced to the scalar one. A relation between the Siegert states and the electromagnetic resonant scattering of waves is established. In Section IV the exponential decay of electromagnetic Siegert states excited by an incident wave packet is investigated. This allows one to identify the constructed Siegert states with ”trapped” quasi-stationary electromagnetic modes observed in aforementioned numerical studies of periodic structures. In quantum mechanics, resonances of a system that consists of two coupled resonant systems depend on a coupling parameter $h$, and this
II. ELECTROMAGNETIC SIEGERT STATES

Consider a scattering problem for an electromagnetic plane wave impinging a structure made of a non-dispersive dielectric. The structure is described by a dielectric function \( \varepsilon(r) \) that defines the value of the dielectric constant at every point \( r \) occupied by the structure and it equals the dielectric constant of the surrounding medium otherwise. Without loss of generality, the surrounding medium is assumed to be the vacuum, i.e., \( \varepsilon = 1 \). The material is said to be non-dispersive if its dielectric constant does not depend on the frequency of the incident wave. The structure is assumed to have a translational symmetry along a particular direction. In this case the dielectric function is independent of one of the spatial coordinates, say, the \( y \) coordinate. If the incident wave is polarized along the \( y \)-axis (the electric field is parallel to the \( y \)-axis), then the scattering problem can be formulated as the scalar Maxwell’s equation:

\[
\frac{\varepsilon}{c^2} \frac{\partial^2 E}{\partial t^2} = \Delta E, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}
\]  

(1)

where \( c \) is the speed of light in the surrounding medium (the vacuum), and \( E \) is the electric field. For a planar structure, the function \( \varepsilon \) differs from 1 only within a strip \( (x, z) \in (\infty, \infty) \times (-a, a) \). A planar structure is periodic if \( \varepsilon(x + D_g, z) = \varepsilon(x, z) \) where \( D_g \) is the period. In what follows, the units of length are chosen so that \( D_g = 1 \). It is further assumed that the structure’s dielectric constant exceeds that of the surrounding medium, i.e., \( \varepsilon(r) \geq 1 \) for all \( r \), and \( \varepsilon \) is bounded. Two examples of such structures are shown in Fig. 1 (Panels (a) and (b)) depicting periodic arrays of infinite dielectric cylinders, all parallel to the \( y \)-axis. The function \( \varepsilon(r) \) is piecewise continuous. For general periodic structures, \( \varepsilon \) is invariant under the action of a suitable affine Coxeter group in the \( xy \)-plane. In this case, vector Maxwell’s equations should be used as the polarization of the incident wave is not preserved in the scattering process. This general problem will not be studied here.

If the incident wave has a fixed frequency, then the field \( E \) has a harmonic time dependence \( E(r, t) = E_\omega(r)e^{-i\omega t} \) in Eq. (1) where the amplitude \( E_\omega \) satisfies the equation:

\[
\Delta E_\omega(r) + k^2 \varepsilon(r) E_\omega(r) = 0, \quad k^2 = \frac{\omega^2}{c^2}
\]  

(2)

Let the \( x \), \( y \), and \( z \) axes be oriented by the unit vectors \( e_1 \), \( e_2 \), and \( e_3 \), respectively. In the scattering theory, \( E_\omega \) is sought as a superposition of an incident wave, and the corresponding scattered wave \( E_\omega^s \):

\[
E_\omega(r) = e^{ikr} + E_\omega^s(r), \quad k = k_x e_1 + k_z e_3
\]  

(3)

where \( k \) is the wave vector of the incident wave. The units of the field \( E \) are chosen so that the amplitude of the incident wave is 1. The scattered wave obeys the outgoing wave boundary conditions at the spatial infinity. The periodicity of the scattering structure requires that the amplitude \( E_\omega \) satisfies Bloch’s periodicity condition,

\[
E_\omega(x + 1, z) = e^{ikr} E_\omega(x, z)
\]  

(4)

Under the specified boundary conditions, the amplitude \( E_\omega \) satisfies the Lippmann-Schwinger integral equation:

\[
E_\omega(r) = \hat{H}[E_\omega](r) + e^{ikr}
\]  

(5)

where \( \hat{H} \) is the integral operator defined by,

\[
\hat{H}[E](r) = \frac{k^2}{4\pi} \int_{\mathbb{R}^2} (\varepsilon(r_0) - 1)G_k(r,r_0)E(r_0)dr_0
\]
where \(x, z\) the function \(H\) does not depend on the rectangle \(D\) which to some extent is arbitrary. Indeed, the integrals of Eq. (7) extend only over the support \(S_{\varepsilon}\) of the function \((x, z) \rightarrow (\varepsilon(x, z) - 1)\) in the strip \(S = [0, 1] \times (-\infty, \infty)\) of the \(x, z\)-plane. The rectangle \(D\) is only introduced to obtain a connected and compact region of integration, which is convenient for the subsequent analysis. In general, the support of the function \((x, z) \rightarrow (\varepsilon(x, z) - 1)\) is not necessarily connected as, for example, in the case of multiple scatterers in the strip \(S\) depicted in Fig. 2(b). The solution to the Lippmann-Schwinger integral equation (6) will be sought in the Hilbert space \(L^2(D)\). Such a solution then extends naturally to the full \(x, z\)-plane by Bloch’s periodicity condition (4), and the Lippmann-Schwinger integral equation.

It is not hard to see that the summation and integration can be interchanged in Eq. (7) even though the underlying series is only conditionally convergent. The Poisson summation formula is then applied to yield the following form for the integral operator \(\hat{H}\),

\[
\hat{H}[E](\mathbf{r}) = \int_D (\varepsilon(\mathbf{r}_0) - 1)H(k^2, \mathbf{r} - \mathbf{r}_0)E(\mathbf{r}_0)d\mathbf{r}_0, \quad E \in L^2(D)
\]  

(8a)

FIG. 1. Panel (a): An example of scattering structures considered in this work. Infinitely long parallel cylinders are placed parallel to a \(y\)-axis periodically along an \(x\)-axis in the vacuum \(\varepsilon_0 = 1\). The cylinders are characterized by a dielectric constant \(\varepsilon_c > 1\). The rectangle \(D\) in Eq. (6) is enclosed by the dashed line, and the \(z\)-axis.

Panel (b): Two periodic arrays such as the one on Panel (a) are placed parallel to each other at a distance \(2h\) between the axes of any two opposing cylinders.

Panel (c): The \(C_{k_x}\) plane. The cuts run vertically from the diffraction thresholds (indicated by empty circles on the real axis) into the lower half of the complex plane. The diffraction thresholds divide the real axis into a countable set of intervals denoted \(I_l, l \geq 0\). The interval \(I_0\) lies below the radiation continuum. The rest of the intervals partition the radiation continuum.
where the function $H$ is given by,

$$
H(\zeta; x, z) = \frac{i\zeta}{2} \sum_{m \in \mathbb{Z}} e^{i(xk_{x,m} + |\zeta - k^{2}_{x,m}|)} \sqrt{\zeta - k^{2}_{x,m}}, \quad k_{x,m} = k_{x} + 2\pi m, \quad \zeta \neq k^{2}_{x,m}, \quad m \in \mathbb{Z}
$$

The square roots are defined by choosing the branch cut of the logarithm along the negative imaginary axis, i.e., if $w$ is a nonzero complex number, then

$$
\log w = \ln |w| + i \arg w, \quad -\frac{\pi}{2} < \arg w < \frac{3\pi}{2}
$$

In the scattering theory, the branch points $k^{2}_{x,m}, m \in \mathbb{Z}$, are called the *diffraction thresholds*, and they will be referred to as such in what follows.

Note that for $w$ real, then $\text{Im}\{\sqrt{w}\} \geq 0$. Therefore, $\text{Im}\{\sqrt{\zeta - k^{2}_{x,m}}\} > 0$ for all but a finite number of diffraction thresholds $k^{2}_{x,m}$ such that $\zeta = k^{2} > k^{2}_{x,m}$. It follows that in the series of Eq. (8b) all but a finite number of terms decay exponentially as $|z| \to \infty$. Hence the said series converges uniformly on $D$. The uniform convergence on $D$ still holds when the range of the variable $\zeta = k^{2}$ is extended to the cut complex plane $\mathbb{C}_{k_x}$ obtained by excluding all the vertical half lines running from the diffraction thresholds into the lower half of the complex plane, i.e.,

$$
\mathbb{C}_{k_x} = \mathbb{C} - \bigcup_{m \in \mathbb{Z}} \{k^{2}_{x,m} - is, s \leq 0\}
$$

This is because in this case too, $\text{Im}\{\sqrt{\zeta - k^{2}_{x,m}}\} > 0$ except for a finite number of terms. It follows that $\zeta \mapsto H(\zeta; r)$ extends to an analytic function on $\mathbb{C}_{k_x}$. Figure 11(c) shows a sketch of the cut plane $\mathbb{C}_{k_x}$.

Since for all $\zeta \in \mathbb{C}_{k_x}$, the kernel $(r, r_0) \mapsto (\varepsilon(r_0) - 1)H(\zeta; r - r_0)$ is square integrable on $D \times D$, the integral operator $\tilde{H}(\zeta)$ is Hilbert-Schmidt in $L^{2}(D)$, the space of bounded linear operators on $L^{2}(D)$. In particular, $\tilde{H}(\zeta)$ is compact for each $\zeta \in \mathbb{C}_{k_x}$. Moreover, the operator valued function defined on $\mathbb{C}_{k_x}$ by $\zeta \mapsto \tilde{H}(\zeta)$ is analytic. By the analytic Fredholm theorem [10], it then follows that if the inverse operator $(1 - \tilde{H}(\zeta))^{-1}$ exists at some point $\zeta \in \mathbb{C}_{k_x}$, it must be meromorphic throughout $\mathbb{C}_{k_x}$. That the said inverse exists at some point is obvious. Indeed, as $\zeta \to 0$ in $\mathbb{C}_{k_x}$, the norm of the operator $\tilde{H}(\zeta)$, i.e., $||\tilde{H}(\zeta)|| = ||(\varepsilon(\cdot) - 1)\tilde{H}(\cdot)||_{L^{2}(D \times D)}$, also converges to zero. In particular for $\zeta$ near 0, $||\tilde{H}(\zeta)|| < 1$ and therefore $(1 - \tilde{H}(\zeta))^{-1}$ exists as a Neumann series. Thus, the generalized resolvent $\zeta \mapsto (1 - \tilde{H}(\zeta))^{-1}$ is meromorphic in $\mathbb{C}_{k_x}$.

The poles $\{\zeta_n\}_n$ of $(1 - \tilde{H}(\zeta))^{-1}$ are isolated and form a discrete set. The same analytic Fredholm theorem guarantees that at each of the poles $\zeta_n$, there exists a nonzero solution to the generalized eigenvalue problem

$$
\tilde{H}(\zeta_n)[E_n] = E_n
$$

The generalized eigenfunctions $E_n$ will be referred to as the *Siegert states*.

It will be assumed throughout the rest of the work that the Siegert states are nondegenerate, i.e., the poles $\zeta_n$ are all *simple*. The results to be derived in the rest of the work could easily be generalized to the case when the poles are not simple. Yet, to our knowledge, there has been no report of resonance poles of multiplicity greater than one in either the literature of electromagnetism or that of quantum mechanics. However, we do not have a rigorous proof that higher multiplicity poles cannot occur in the studied scattering problem. For the structures depicted in Fig. 11 perturbation theory (when the radius of cylinders is much less than the structures period) shows that all the poles are simple [13, 14].

The assumed nondegeneracy of the Siegert states $\{E_n\}_n$ implies that the residues $\{\tilde{H}_n\}_n$ of $(1 - \tilde{H}(\zeta))^{-1}$ at the poles $\{\zeta_n\}_n$ are rank one operators on $L^{2}(D)$. In other words, if $(f, g) = \int_{D} f(r_0)g(r_0)dr_0$ is the inner product on the Hilbert space $L^{2}(D)$, then

$$
\forall n, \exists \varphi_n \in L^{2}(D): \forall \psi \in L^{2}(D), \quad \tilde{H}_n[\psi] = \langle \varphi_n, \psi \rangle E_n
$$

Since the incident wave $E_i(k^2, r) = e^{ik \cdot r}$ is analytic in $\zeta = k^2$ on $\mathbb{C}_{k_x}$, it follows that the amplitude $E_\omega = [1 - \tilde{H}(\zeta)]^{-1}[E_i(\zeta, \cdot)]$ extends to a meromorphic function of the variable $\zeta = k^2$ on $\mathbb{C}_{k_x}$. Its partial fraction expansion reads

$$
E_\omega(r) = e^{ik \cdot r} + \sum_{n} \frac{a_n}{k^2 - k^2_n + \text{i} \epsilon_n} E_n(r) + E_n(k^2; r), \quad a_n = \langle \varphi_n, E_i(k^2; \cdot) \rangle_{k^2 = k^2_n - \text{i} \epsilon_n}
$$

(11)
where each pole $\zeta_n$ has been decomposed in its real and imaginary parts as $\zeta_n = k_n^2 - i \Gamma_n$ owing to the fact that the imaginary parts of the poles are nonpositive as will be shown shortly. The remainder $E_n$ is analytic in $k^2$ on the cut plane $\mathbb{C}_{k_x}$. It describes the so called background or potential scattering, while the Siegert states account for the resonant scattering.

Since the Siegert states are solutions to the homogeneous linear equation $\tilde{H}(\zeta)[E] = E$, they should be normalized in some way in order for the functions $\varphi_n$ in Eq.\((13)\) and the coefficients $a_n$ in Eq.\((11)\) to be uniquely defined. A natural normalization condition will be introduced in Section III A for Siegert states that arise as perturbations of $k$ satisfies Eq.\((2)\) for $k^2 = \zeta_n$, and therefore

$$E_n \Delta E_n - E_n \Delta E_n - 2i \Gamma_n \varepsilon |E_n|^2 = 0$$

By Green’s theorem, it follows that,

$$2i \Gamma_n \int_{D'} |E_n(\mathbf{r}_0)|^2 \varepsilon(\mathbf{r}_0) d\mathbf{r}_0 = \oint_{\partial D'} (E_n(\mathbf{r}_0) \nabla E_n(\mathbf{r}_0) - E_n(\mathbf{r}_0) \nabla E_n(\mathbf{r}_0)) \cdot d\mathbf{n}_0 \quad (12)$$

where $D' = [0, 1] \times [z_1, z_2]$ is any rectangle containing the rectangle $D$, and $\mathbf{n}_0$ is the outward normal. By Bloch’s periodicity condition, the integrals on the line segments $(x, z) \in [0, 1] \times [z_1, z_2]$ of $\partial D'$ are canceled out so that the integral on the right side is to be carried out only over the line segments $(x, z) \in [0, 1] \times \{z_1, z_2\}$ of $\partial D'$. The result can be expressed in terms of the scattering amplitudes $S_m^\pm$ associated with the asymptotic behavior of the Siegert state $E_n$:

$$E_n(\mathbf{r}) = \begin{cases} 
S_m^+ e^{i(xk_x,m - z\sqrt{\zeta_n - k_{x,m}^2})}, & z < z_- \\
S_m^- e^{i(xk_x,m + z\sqrt{\zeta_n - k_{x,m}^2})}, & z > z_+ 
\end{cases} \quad (13a)$$

Equations \((13a)\) follow immediately from equations \((8)\). In particular, the amplitudes $S_m^\pm$ are given by the formulas,

$$S_m^\pm = \frac{i \zeta_n}{2 \sqrt{\zeta_n - k_{x,m}^2}} \int_D \varepsilon(\mathbf{r}_0) e^{i(xa_{k_x,m} + z_0a\sqrt{\zeta_n - k_{x,m}^2})} d\mathbf{r}_0, \quad \mathbf{r}_0 = x_0 \mathbf{e}_1 + z_0 \mathbf{e}_3 \quad (13b)$$

Evaluating the right hand side of Eq.\((12)\) yields,

$$\Gamma_n = \sum_{m \in \mathbb{Z}} \text{Re} \left\{ \sqrt{\zeta_n - k_{x,m}^2} \left[ |S_m^+|^2 e^{-2z_1 \text{Im} \sqrt{\zeta_n - k_{x,m}^2}} + |S_m^-|^2 e^{2z_1 \text{Im} \sqrt{\zeta_n - k_{x,m}^2}} \right] \right\} \int_{D'} |E_n(\mathbf{r}_0)|^2 \varepsilon(\mathbf{r}_0) d\mathbf{r}_0 \quad (14)$$

The series in Eq.\((14)\) is then split into the sums over the two complementary index sets,

$$I^\text{op}(\zeta_n) = \{ m \in \mathbb{Z} \mid \text{Re} \{ \zeta_n \} \geq k_{x,m}^2 \}, \quad I^\text{cl}(\zeta_n) = \{ m \in \mathbb{Z} \mid \text{Re} \{ \zeta_n \} < k_{x,m}^2 \} \quad (15a)$$

In the scattering theory, when $\zeta = k^2$ is real, the sets $I^\text{op}(k^2)$ and $I^\text{cl}(k^2)$ label open and closed diffraction channels respectively, thus the superscripts “op” for open, and “cl” for closed.

The choice of the branch cut for the logarithm given in Eq.\((9)\) ensures that

$$\text{Im} \left\{ \sqrt{\zeta_n - k_{x,m}^2} \right\} > 0, \quad \forall m \in I^\text{cl}(\zeta_n) \quad (15b)$$

In particular, in the series of Eq.\((14)\), the contributions from terms whose indices lie in $I^\text{cl}(\zeta_n)$ decay exponentially as $z_2 \to +\infty$ and $z_1 \to -\infty$. In the said limits, the aforementioned series is therefore reduced to a finite sum over the set $I^\text{op}(\zeta_n)$:

$$\Gamma_n = \lim_{z_1 \to -\infty, \ z_2 \to +\infty} \sum_{m \in I^\text{op}(\zeta_n)} \text{Re} \left\{ \sqrt{\zeta_n - k_{x,m}^2} \left[ |S_m^+|^2 e^{-2z_2 \text{Im} \sqrt{\zeta_n - k_{x,m}^2}} + |S_m^-|^2 e^{2z_2 \text{Im} \sqrt{\zeta_n - k_{x,m}^2}} \right] \right\} \int_{D'} |E_n(\mathbf{r}_0)|^2 \varepsilon(\mathbf{r}_0) d\mathbf{r}_0 \quad (16)$$

Observing that $\text{Re} \left\{ \sqrt{\zeta_n - k_{x,m}^2} \right\} \geq 0$ for all $m \in I^\text{op}(\zeta_n)$, and since the dielectric function $\varepsilon$ is positive, it follows that $\Gamma_n \geq 0$, and therefore the imaginary part of the pole $\zeta_n$ is indeed necessarily negative or zero, i.e., all the poles $\zeta_n$ are in the lower half of the cut complex plane $\mathbb{C}_{k_x}$.
III. REGULAR PERTURBATION THEORY

Suppose that the dielectric function $\varepsilon$ depends on a real parameter $h$ which is called a coupling parameter. The simplest example of such a coupling is given by two periodic arrays of dielectric scatterers that are parallel and separated by the distance $2h$ (Fig. 11(b)). The arrays are embedded in a medium of dielectric susceptibility 1. Each array is characterized by a dielectric function $\varepsilon_i(x, z)$, $i = 1, 2$ such that $\varepsilon_i \geq 1$ on the scatterers, and $\varepsilon_i$ is of period 1 in the $x$-direction. The resulting dielectric function describing the scattering of light on the structure is then,

$$
\varepsilon(h; x, z) = 1 + (\varepsilon_1(x, z - h) - 1) + (\varepsilon_2(x, z + h) - 1)
$$

(16)

In the most general case, the dependence of the dielectric function $\varepsilon$ on the coupling parameter $h$ implies that the poles of the generalized resolvent $\zeta \mapsto [1 - \hat{H}(h, \zeta)]^{-1}$ also depend on $h$. If the coupling is sufficiently smooth as indicated in the theorem stated below, then the poles of the generalized resolvent as well as the corresponding Siegert states depend continuously on $h$.

Before stating the theorem, a clarification must be made on the boundary of the rectangle $D$ in Eq.(6). This rectangle was chosen to contain the support $S_z$ of the function $(x, z) \mapsto (\varepsilon(x, z) - 1)$ in the strip $S = [0, 1] \times (-\infty, \infty)$ of the $x, z$-plane. In the current situation, the dielectric function depends on the coupling parameter $h$, and therefore the support $S_z(h)$ in question could change with $h$ resulting in a different choice for the set $D$. This is the case, for instance, in the example of the two parallel arrays separated by the distance $2h$. If the two scatterers in the strip $S$ are taken further apart, the rectangle $D$ is stretched further accordingly. So, in what follows it will always be assumed that $h$ varies in an open possibly finite interval $J_0$ such that $\bigcup_{h \in J_0} S_z(h)$ is bounded, and the rectangle $D$ will be chosen to contain the latter set. The theorem on the regular perturbation of electromagnetic Siegert states is formulated as follows:

**Regular Perturbation Theorem.** Suppose that the map $(h, \zeta) \mapsto \hat{H}(h, \zeta)$ from $J_0 \times C_{k_x}$ to $L^2(D)$ is continuously Frechet differentiable. Further, suppose that for some $h_0 \in J_0$, the generalized resolvent $\zeta \mapsto [1 - \hat{H}(h_0, \zeta)]^{-1}$ has a simple pole $\zeta_n \in C_{k_x}$, and let $E_n$ be the corresponding Siegert state. Then there exists an open interval $J \subset J_0$ containing $h_0$, and a unique continuously differentiable function $h \mapsto \zeta_n(h)$, $h \in J$, such that $\zeta_n(h_0) = \zeta_n$, and for all $h \in J$, $\zeta_n(h)$ is a simple pole of the generalized resolvent $\zeta \mapsto [1 - \hat{H}(h, \zeta)]^{-1}$. If $E_n(h)$ is the Siegert state corresponding to the pole $\zeta_n(h)$, then the function $h \mapsto E_n(h)$ from $J$ to $L^2(D)$ is continuously differentiable.

This theorem is an extension of the Kato-Rellich theorem [17] to the present case. Due to its length and complexity, the proof is left to the Appendix. It should be noted, however, that if the functions $(x, z) \mapsto \varepsilon_i(x, z)$, $i = 1, 2$ in Eq.(16) are piecewise differentiable, then $(h, \zeta) \mapsto \hat{H}(h, \zeta)$ is Frechet differentiable so that the theorem does indeed hold for two parallel arrays separated by the distance $h$.

A. Perturbation of bound states in the radiation continuum

The diffraction thresholds $k_{x,m}^2$ on the real line depend on the $x$-component $k_x$ of the incident wave vector. They can be ordered independently of the parameter $k_x$ by defining a sequence $\zeta_{\pm m}^*(k_x) = (2\pi m \pm |k_x|)^2$, $m \in \{0, 1, 2, \ldots\}$, where $|k_x|$ is the argument of $e^{ik_x}$ in the interval $(-\pi, \pi]$. The sequence $\{\zeta_{\pm m}^*(k_x)\}_{m=0}^\infty$ coincides with the sequence of diffraction thresholds $\{(k_x + 2\pi m)^2\}_{m \in \mathbb{Z}}$, and for all $k_x$,

$$
\zeta_0^*(k_x) \leq \zeta_1^*(k_x) \leq \zeta_2^*(k_x) \leq \cdots
$$

The threshold $\zeta_0^*(k_x)$ is called the radiation continuum threshold, and the interval $I_0 = (-\infty, \zeta_0^*(k_x))$ is said to lie below the radiation continuum. This is because whenever $k^2 < \zeta_0^*(k_x)$, then the scattered amplitude $E_n^s$ in Eq.(4) necessarily decays exponentially in the spatial infinity, $|z| \to \infty$, as can be inferred from Eqs.(4), (5), and (8). Hence, no electromagnetic flux is carried to the spatial infinity in this spectral range. In contrast, if $k^2 > \zeta_0^*(k_x)$, the amplitude $E_n^s$ oscillates and represents outgoing (scattered) radiation that carries an electromagnetic flux to the spatial infinity. The spectral range above $\zeta_0^*(k_x)$ on a real line is therefore referred to as the radiation continuum. It is the disjoint union of the intervals,

$$
I_1 = (\zeta_0^*(k_x), \zeta_1^*(k_x)), \quad I_2 = (\zeta_1^*(k_x), \zeta_2^*(k_x)), \quad I_3 = (\zeta_2^*(k_x), \zeta_3^*(k_x)), \quad \ldots
$$

Note that when $|k_x|$ is 0 or $\pi$, some of the intervals $I_i$ are empty, owing to the fact that some of the diffraction thresholds fuse.
Returning to Siegert states, recall that these states are generalized eigenfunctions to the generalized eigenvalue problem \( \tilde{H}(\zeta_0)[E_n] = E_n \) on \( L^2(D) \) with complex eigenvalues \( \zeta_n \). These states are naturally extended to the whole \( x, z \)-plane by means of Eqs. (13). In general, if the generalized eigenvalue \( \zeta_n = k^2_n - i\Gamma_n \) lies below or in the interval \( I_0 \) in the \( \mathbb{C}_k \)-plane, then the corresponding Siegert state is square integrable on the strip \( S = [0, 1] \times (-\infty, \infty) \) of the \( x, z \)-plane. This is because for such states, the set \( \Gamma^p(\zeta_n) \) of Eq. (15a) is empty, and therefore by Eqs. (13a) and (15b), these states decay exponentially in the asymptotic region \( |z| \to \infty \). In particular, the Siegert states for which the pole \( \zeta_0 \) is real and less than the continuum threshold \( \zeta_0(k, \varepsilon) \) are the bound states below the radiation continuum of the system.

On the contrary, the Siegert states \( E_n \) whose corresponding generalized eigenvalues \( \zeta_n = k^2_n - i\Gamma_n \) lie below an interval \( I_1 \), \( l \geq 1 \) of the radiation continuum, i.e., \( k^2_n \in I_1 \) and \( \Gamma_n > 0 \), are not necessarily square integrable on the strip \( S \). This is because for these states, the set \( \Gamma^p(\zeta_n) \) is not empty, and consequently, the terms of the series in Eq. (13a) indexed by this set are unbounded. However, when the pole \( \zeta_n \) is in \( I_1 \), \( l \geq 1 \), i.e., \( \Gamma_n = 0 \), it will be shown shortly that the resulting Siegert state is square integrable on the strip \( S \) and therefore, such a Siegert state is a bound state in the radiation continuum. The rest of this subsection is devoted to the proof of this assertion.

To proceed, suppose that for some value \( h_b \) of the coupling constant \( h \), there exists a Siegert state \( E_b \) whose corresponding generalized eigenvalue \( \zeta_b \) is real and lies in an interval \( I_1 \), \( l \geq 1 \), of the radiation continuum. By the Regular Perturbation Theorem there exist continuously differentiable functions \( h \mapsto \zeta_n(h) \) and \( h \mapsto E_n(h) \) on an interval \( J \) containing \( h_b \) such that \( \zeta_n(h_b) = \zeta_b \), and \( E_n(h_b) = E_b \). Furthermore, \( E_n(h) \) is the Siegert state corresponding to the pole \( \zeta_n(h) \) of the generalized resolvent \( \zeta \mapsto \{1 - \tilde{H}(\zeta, \cdot)\}^{-1} \) for all \( h \in J \). Put \( \zeta_n(h) = k^2_n(h) - i\Gamma_n(h) \) as before. Without loss of generality, the interval \( J \) is assumed to be sufficiently small so that for all \( h \in J \backslash \{h_b\} \), \( \Gamma_n(h) \neq 0 \). Equation (11) for the Siegert state \( E_n(h) \) is then rewritten as

\[
\int_{D^2} |E_n(r_0)|^2 \varepsilon(h; r_0) d\mathbf{r}_0 = \frac{1}{\Gamma_n} \sum_{m \in \mathbb{Z}} \text{Re} \left\{ \sqrt{\zeta_n - k^2_{x,m}} \left( |S^+_{m}|^2 e^{-2\varepsilon_2 \text{Im} \{\sqrt{\zeta_n - k^2_{x,m}}\}} + |S^-_{m}|^2 e^{2\varepsilon_1 \text{Im} \{\sqrt{\zeta_n - k^2_{x,m}}\}} \right) \right\}
\]

where it is understood that the Siegert state \( E_n \), the pole \( \zeta_n \), and its imaginary part \( -i\Gamma_n \), as well as the amplitudes \( S^\pm_{m} \) are all functions of \( h \). Note that the values of \( \varepsilon_1 \) and \( \varepsilon_2 \) are assumed to be independent of the coupling parameter \( h \). This is because \( h \) varies in a small interval \( J \), and therefore the values of \( \varepsilon_\pm \) in Eq. (6), which now depend on \( h \) are bounded. The condition \( \varepsilon_1 \leq z_+(h) < z_-(h) \leq \varepsilon_2 \), \( \forall h \in J \), can therefore be realized by choosing \( \varepsilon_1 \) and \( \varepsilon_2 \) sufficiently large.

As \( h \to h_b \), then \( E_n \to E_b \) in \( L^2(D) \), and therefore the left hand side of Eq. (11) remains finite as the dielectric function \( \varepsilon(h; \cdot) \) is bounded. The limit of the right hand side as \( h \to h_b \) may also be calculated by first computing explicitly the complex square roots involved according to the logarithmic branch cut described in Eq. (9). Put \( \xi_m(h) = k^2_n(h) - k^2_{x,m}, m \in \mathbb{Z} \). Then

\[
\sqrt{\zeta_n(h) - k^2_{x,m}} = \begin{cases} \sqrt{\xi_m(h)^2 + \Gamma_n(h)^2 + \zeta_m(h)} - i\Gamma_n(h) \frac{\Gamma_n(h)}{\sqrt{2(\xi_m(h)^2 + \Gamma_n(h)^2 + \zeta_m(h))}} & \text{if } m \in \Gamma^p(\zeta_n(h)) \\ \sqrt{\xi_m(h)^2 + \Gamma_n(h)^2 + \zeta_m(h)} + i\Gamma_n(h) \frac{\Gamma_n(h)}{\sqrt{2(\xi_m(h)^2 + \Gamma_n(h)^2 + \zeta_m(h))}} & \text{if } m \in \Gamma^d(\zeta_n(h)) \end{cases}
\]

where \( \Gamma^p \) and \( \Gamma^d \) are the index sets of Eq. (15a). In particular, if \( m \in \Gamma^p(\zeta_n(h)) \), then \( \xi_m(h) \geq 0 \), whereas \( \xi_m(h) < 0 \) whenever \( m \in \Gamma^d(\zeta_n(h)) \). As \( h \to h_b \), \( \Gamma_n(h) \to 0 \), and,

\[
\int_{D^2} |E_b(r_0)|^2 \varepsilon(h_b; r_0) d\mathbf{r}_0 = \lim_{h \to h_b} \left[ \frac{1}{\Gamma_n(h)} \sum_{m \in \Gamma^p(\zeta_n)} \sqrt{\xi_m(h) \left( |S^+_m(h)|^2 + |S^-_m(h)|^2 \right)} - \sum_{m \in \Gamma^d(\zeta_n)} \frac{1}{2\sqrt{-\xi_m(h_b)}} \left( |S^+_m(h_b)|^2 e^{-2\varepsilon_2 \sqrt{-\xi_m(h_b)}} + |S^-_m(h_b)|^2 e^{2\varepsilon_1 \sqrt{-\xi_m(h_b)}} \right) \right] (17)
\]

In particular, for each \( m \in \Gamma^p(\zeta_b) \), the limits \( \lim_{h \to h_b} |S^\pm_m(h)|^2 \) must exist and must be finite. By continuity of the functions \( h \mapsto S^\pm_m(h) \) on \( J \), there exist complex numbers \( S^\pm_{m,b} \) such that

\[
\lim_{h \to h_b} \frac{S^\pm_m(h)}{\sqrt{\Gamma_n(h)}} = S^\pm_{m,b}, \quad \forall m \in \Gamma^p(\zeta_b) \quad (18)
\]
As \( z_1 \to -\infty \) and \( z_2 \to \infty \), the second series in Eq. (17) converges to zero. Therefore the function \( r \to E_b(r)\sqrt{\varepsilon(h_b; r)} \) is square integrable on the strip \( S = [0, 1] \times (-\infty, \infty) \) of the \( x, z \)-plane, and

\[
\int_S |E_b(r_0)|^2\varepsilon(h_b; r_0)dr_0 = \sum_{m \in \Omega^+ (\zeta_h)} \sqrt{\xi_m(h_b)} \left(|S_{m,b}^+|^2 + |S_{m,b}^-|^2\right)
\]

(19a)

Since, \( \varepsilon(h_b, \cdot) \geq 1 \), it follows that \( E_b \in L^2(S) \) as claimed. Thus, if \( E_b \) is a bound state in the radiation continuum, it can always be normalized by the condition

\[
\int_S |E_b(r_0)|^2\varepsilon(h_b; r_0)dr_0 = 1
\]

(19b)

This result justifies the term "bound state" introduced by analogy with quantum mechanics where bound states represent square integrable eigenfunctions of the Hamiltonian operator, i.e., an electromagnetic bound state is a localized solution of Maxwell’s equations.

Finally, by continuity \( E_n(h) \to E_b \) as \( h \to h_b \), and therefore the normalization condition (19b) determines uniquely the Siegert states \( E_n(h) \) along the curve \( h \to \zeta_n(h), h \in J \).

B. Near field amplification mechanism

Resonant phenomena in the scattered electromagnetic flux can be described by the formalism of Siegert states as given in Eq. (11). However, a complete description requires calculating the residues \( a_n \) in Eq. (11). Here these residues are calculated for Siegert states that arise as perturbations of bound states in the radiation continuum in the sense of Section IIIA, i.e., for the states \( E_n(h) \) where \( |h - h_b|/h_b \ll 1 \). In particular, it is shown that if the scattering structure has bound states in the radiation continuum, then there exist regions (the so called "hot spots") in which the field amplitude can be amplified as much as desired by taking the value of the coupling parameter \( h \) close enough to \( h_b \). The unbounded local growth of the field amplitude is essentially due to the linearity of Maxwell’s equations. It disappears when a non-linear dielectric susceptibility, required for large field amplitudes, is included into Maxwell’s equations (12). Nevertheless, such a local field amplification enhances quite substantially optical nonlinear effects in the scattering structure as shown in (13). The following analysis also suggests that the concept to enhance optically non-linear effects by using bound states in the radiation continuum is universal because the existence of "hot spots" is proved to be a characteristic feature of such scattering structures.

When estimating the coefficients \( a_n \), it is convenient first to give another equation for them that is alternative to that of Eq. (11). Next, this equation will be analyzed in the vicinity of a bound state in the radiation continuum. In particular, if a bound state in the radiation continuum \( E_b \) exists at the critical value \( h_b \) of the coupling parameter, and \( J \) is the interval produced by the Regular Perturbation Theorem, while \( E_n(h; r) \) is the Siegert state that arises as a continuous perturbation of the bound state \( E_b \) for \( h \in J \), then it will be proved that \( a_n(h) \sim \sqrt{\Gamma_n(h)} \) as \( h \to h_b \).

Recall that \( \zeta_n(h) = k_n^2(h) - i\Gamma_n(h) \) is the generalized eigenvalue at which the Siegert state \( E_n(h) \) exists.

To proceed, put \( \tilde{E}_\omega = \zeta_n a_n E_\omega \) where \( E_\omega \) is the amplitude in Eq. (11). Then by the decomposition (11), \( \tilde{E}_\omega \to E_n \) as \( \zeta \to \zeta_n \). From the system,

\[
\begin{cases}
\Delta \tilde{E}_\omega + \zeta \varepsilon(h) \tilde{E}_\omega = 0 \\
\Delta E_n + \zeta \varepsilon(h) E_n = 0
\end{cases}
\]

it is derived by Green’s theorem that,

\[
\int_{\partial D'} \left( E_n \nabla \tilde{E}_\omega - \tilde{E}_\omega \nabla E_n \right) \cdot d\mathbf{n}_0 + (\zeta - \zeta_n) \int_{D'} \tilde{E}_\omega E_n \varepsilon(h) d\mathbf{r}_0 = 0
\]

where \( D' \) is the same rectangle as in Eq. (12). Then, by splitting the amplitude \( \tilde{E}_\omega \) in terms of the incident and scattered waves, \( \tilde{E}_\omega(r) = \zeta_n a_n e^{i k \cdot r} + \tilde{E}_s^a(r) \), one infers that

\[
a_n(h) = -\frac{(\zeta - \zeta_n) \int_{\partial D'} (i k E_n - \nabla E_n) e^{i k \cdot r_0} \cdot d\mathbf{n} + (\zeta - \zeta_n) \int_{D'} \tilde{E}_\omega E_n \varepsilon(h) d\mathbf{r}_0}{\int_{\partial D'} (E_n \nabla \tilde{E}_\omega - \tilde{E}_\omega \nabla E_n) \cdot d\mathbf{n} + (\zeta - \zeta_n) \int_{D'} \tilde{E}_\omega^a E_n \varepsilon(h) d\mathbf{r}_0}
\]
where it is understood that $E_n$ and $\zeta_n$ are functions of $h$. Since $a_n$ is independent of $\zeta$, the desired expression for $a_n$ is obtained by taking the limit $\zeta \to \zeta_n$:

$$ a_n(h) = -\frac{\int_{\partial D'} (i k_n E_n - \nabla E_n) \cdot d\mathbf{n}}{\int_{\partial D'} \left(E_n \nabla \delta \xi E_n - \partial \xi E_n \nabla E_n\right) \cdot d\mathbf{n} \Big|_{\zeta=\zeta_n} + \int_{\partial D'} |E_n|^2 \varepsilon(h) d\mathbf{n}}, \quad k_n = k|_{\zeta=\zeta_n} \tag{20} $$

where for the first term in the denominator, l'Hôpital's rule has been applied. This formula would not generally be useful as the term $E_n^+$ involves the coefficients $a_n$ implicitly. However, if the Siegert state $E_n(h)$ is taken near a bound state in the radiation continuum, as assumed here, the first-order perturbative expression of $a_n$ only depends on the Siegert state $E_n(h)$.

Such a perturbative expression is obtained by analyzing each of the integrals in Eq. (20) separately. First, it is observed that as $h \to h_b$, then $E_n(h) \to E_b$. By letting $z_1 \to -\infty$, and $z_2 \to \infty$, it follows from the normalization of Eq. (19b) that the second integral in the denominator of Eq. (20) can be made arbitrarily close to 1 provided $h$ is sufficiently close to $h_b$. Hence, if the first integral of the said denominator can be shown to converge to zero in the limits considered, this would imply that in the leading order of perturbation theory, the denominator of Eq. (20) is to be carried out on the segments $(x, z) \in [0, 1] \times \{z_1, z_2\}$ of the boundary of $D'$. Now, as $h \to h_b$, the Siegert state $E_n(h)$ becomes a bound state $E_b$, and therefore, it decays exponentially in the spatial infinity $|z| \to \infty$. Similarly, the same exponential decay can be established for the terms involving the derivatives of $E_n^+$ in the limit $h \to h_b$. It then follows that in the limits $h \to h_b$, $z_1 \to -\infty$ and $z_2 \to \infty$, the integral in question vanishes. Thus the denominator of Eq. (20) remains indeed close to 1, provided $h$ is sufficiently close to $h_b$.

On the other hand, the numerator to Eq. (20) can be evaluated in terms of the amplitudes $S_n^\pm$ in Eqs. (18) to yield

$$ \int_{\partial D'} (i k_n E_n - \nabla E_n) \cdot d\mathbf{n} = \sqrt{\Gamma_n(h)} A_n(h, z_1, z_2) $$

where if $k_n = k_x e_1 + \sqrt{\zeta_n - k_x^2} e_3$, then

$$ A_n(h, z_1, z_2) = 2i \Re \left\{ \frac{\sqrt{\zeta_n(h) - k_x^2}}{\sqrt{\Gamma_n(h)}} \right\} S_0^+(h) e^{-2iz_1 \Im \left\{ \sqrt{\zeta_n(h) - k_x^2} \right\}} + 2i \Im \left\{ \sqrt{\zeta_n(h) - k_x^2} \right\} \frac{S_0^-(h)}{\sqrt{\Gamma_n(h)}} e^{2iz_1 \Re \left\{ \sqrt{\zeta_n(h) - k_x^2} \right\}} $$

The case $k_n = k_x e_1 - \sqrt{\zeta_n - k_x^2} e_3$ is similar. As $h \to h_b$, then $\zeta_n(h) \to \zeta_b$, and since $\zeta_b > k_x^2$, it follows that $\Im \left\{ \sqrt{\zeta_n(h) - k_x^2} \right\} \to 0$. Hence, in the said limit, $A_n(h, z_1, z_2) \to 2i S_0^+ \sqrt{\zeta_b - k_x^2}$ for $S_0^+$ given in Eq. (18). Thus, in general, $a_n$ can be written as

$$ a_n(h) = \bar{a}_n(h) \sqrt{\Gamma_n(h)} \tag{21} $$

where $\bar{a}_n(h)$ is bounded and has the property $\bar{a}_n(h) \to 2i S_0^+ \sqrt{\zeta_b - k_x^2}$ as $h \to h_b$.

The most important consequence of the structure of the coefficients $a_n$ near a bound state in the radiation continuum is a local amplification of the amplitude $E_n$ as compared to the amplitude of the incident radiation. Indeed, if the wavenumber $k$ of the incident radiation and the coupling parameter $h$ are tuned to satisfy the condition $k = k_n(h), \forall h \in J$, then the amplitude becomes

$$ E_n(r) = \frac{i}{\sqrt{\Gamma_n(h)}} \bar{a}_n(h) E_b(r) + \bar{E}_n(h; r) \tag{22} $$

where $h \mapsto ||\bar{E}_n(h; \cdot)||_{L^2(D)}$ is bounded on $J$. It follows that,

$$ ||E_n||_{L^2(D)} \geq \left| \frac{||\bar{a}_n(h)||}{\sqrt{\Gamma_n(h)}} ||E_b||_{L^2(D)} - ||\bar{E}_n||_{L^2(D)} \right| $$

As $h \to h_b$, then $E_n \to E_b$ in $L^2(D)$, and therefore $||E_n||_{L^2(D)} \to ||E_b||_{L^2(D)} \neq 0$. Hence if the constant $S_0^+$ in Eq. (21) is nonzero, then,

$$ \lim_{h \to h_b} ||E_n||_{L^2(D)} = \infty $$
and this can only happen if the amplitude $E_\omega$ diverges in some regions of the rectangle $D$. Even though the fact that $\tilde{a}_n(h)$ is always nonzero in the limit $h \to b_i$ (i.e., $S_{0,b}^0 \neq 0$) could not be verified for all kinds of couplings, there are many systems in which it holds true. For example, this is the case of the normal incidence ($k_x = 0$) for a symmetric double array depicted in Fig. 1(b) when the dielectric functions $\varepsilon_1$ and $\varepsilon_2$ in Eq. (10) are identical, and symmetric with respect to the reflection $(x, z) \mapsto (x, -z)$. In this particular case, the parity operator, $\hat{S}[E](x, z) = E(x, -z)$ on $L^2(D)$, and the Lippmann-Schwinger integral operator $\hat{H}$ commute so that Siegert states are always symmetric or skew symmetric with respect to the reflection $(x, z) \mapsto (x, -z)$ in the $x, z$-plane. In particular, the amplitudes $S_{m}^\pm$ given in Eqs. (13) are such that $S_{m}^+ = \pm S_{m}^-$ depending on whether the Siegert state they correspond to is symmetric or skew symmetric. It follows that if the bound state $E_b$ happens to lie in the interval $I_1$ of the $C_{k_x}$-plane (case of one open diffraction channel), then Eqs. (19) are reduced to

$$|S_{0,b}^+|^2 = |S_{0,b}^-|^2 = \frac{1}{2} \left( k_b^2 - k_x^2 \right).$$

so that the coefficient in question is indeed nonzero and the amplitude $E_\omega$ is amplified in the vicinity of the bound state in the radiation continuum $E_b$. Note that the amplification of the amplitude $E_\omega$ was also observed perturbatively in the case of a more general incidence angle (i.e., $k_x \neq 0$) when the bound state $E_b$ lies in the intervals $I_1$ and $I_2$ of the radiation continuum [14].

The aforementioned amplification can only happen in a region near or within the scattering structure. This can be understood by analyzing the first term of Eq. (22) from which the amplification should result. Outside the scattering region, the Siegert state is expressed in terms of its scattered amplitudes by Eq. (21a). As before, the series involved in this expression split over the index sets $I^p(\zeta_b)$ and $I^d(\zeta_b)$ (for $h$ near $h_n$, $I^p(\zeta_n(h)) = I^p(\zeta_b)$ and $I^d(\zeta_n(h)) = I^d(\zeta_b)$). On one hand, the terms whose indices lie in $I^d(\zeta_b)$ decay exponentially in the spatial infinity so that no amplification of the field can be obtained from them. On the other hand, the terms whose indices lie in $I^p(\zeta_b)$, disappear near the bound state in the radiation continuum as these terms are proportional to $\sqrt{\Gamma_n(h)}$ as indicated by Eq. (15). From the physical point of view, the noted local field amplification results from a constructive interference of scattered fields from each scatterer (an elementary cell) of the periodic structure. It is important to note that the amplification magnitude can be regulated by varying the coupling parameter $h$, which provides a natural physical mechanism to control optical nonlinear phenomena if the structure has a nonlinear dielectric susceptibility. This mechanism has been used to demonstrate that a double periodic array of dielectric cylinders (depicted in Fig. 1(b)) with a non-zero second-order nonlinear susceptibility can convert as much as 44% of the incident flux into the second harmonics when the distance between the arrays is properly adjusted. The smallest distance at which such a high conversion rate can be achieved is about a half of the wave length of the incident radiation [15].

### IV. DECRY OF ELECTROMAGNETIC SIEGERT STATES

A Siegert state can be excited by an incident radiation, e.g., by a wave packet whose spectral distribution peaks around a desired wave number $k_e \approx k_n$. When passing through the structure, some of the wave packet energy is trapped by the structure and remains in there for a long period of time, decaying slowly by emitting a monochromatic radiation. This is established in a fashion similar to that of the study of the decay of unstable states in quantum mechanics [1]. To illustrate the principle, only the case of normal incidence is considered here (i.e., $k_x = 0$). Other incidence angles can be treated in a similar manner.

The exact statement which will be proved is as follows. Suppose that a Siegert state $E_n$ exists at the pole $\zeta_n = k_n^2 - i\Gamma_n$ such that $k_n > 0$ and $\Gamma_n > 0$ (i.e., the Siegert state is not a bound state in the radiation continuum). Then this state will decay with time by emitting a monochromatic radiation at a wavenumber $\tilde{k}_n$, and with a life time $\tau_n$:

$$\tilde{k}_n = \sqrt{\frac{k_n^2 + k_\omega^2 + i\Gamma_n}{2}} \quad \Rightarrow \quad k_n, \quad \tau_n = \frac{2k_n}{\Delta n} \quad \Rightarrow \quad \frac{2k_n}{\Delta n}$$

(23)

As a starting point, consider an incident wave packet,

$$E_i(r, t) = \int_0^\infty A(k) \cos(k \cdot r - \omega t) dk, \quad \omega = ck$$

where $A(k)$ is the distribution of wavenumbers in the wave packet. Then the solution $E$ of the wave equation (1) reads

$$E(r, t) = \text{Re} \left\{ \int_0^\infty A(k)E_\omega(r)e^{-i\omega t} dk \right\}$$
where $E_\omega$ is the amplitude of Eq. (2). From the meromorphic expansion of $E_\omega$ in Eq. (11) it then follows that,

$$E(r, t) = \text{Re} \left\{ E_i(r, t) + \sum_n E_n(r) \Omega_n(t) + \int_0^\infty A(k)E_n(k^2, r)e^{-ikt}dk \right\}$$

where the time dependence $\Omega_n(t)$ of the Siegert state $E_n$ is,

$$\Omega_n(t) = \bar{a}_n \sqrt{\Gamma_n} \int_0^\infty \frac{A(k)}{k^2 - k_n^2 + i\Gamma_n}e^{-ikt}dk$$  \hspace{1cm} (24)

for $\bar{a}_n$ defined in Eq.(21). As $t \to \infty$, then $\Omega_n(t) \to 0$ as required by the Riemann-Lebesgue lemma. For a typical physical wave packet, the function $A(k)$ decays fast as $k \to \infty$ and is also analytic in the complex $k$–plane. These properties allows for evaluating the integral (24) by the standard means of the complex analysis. It is also worth noting that bound states in the radiation continuum cannot be excited by an incident radiation because $\Gamma_n = 0$.

To avoid excessive technicalities of the general case, a specific form of the distribution $A(k)$ is chosen to illustrate the procedure, which is sufficient to establish the main properties of the decay of Siegert states. The simplest form that is also most commonly used in physics is a Gaussian wave packet centered around a wavenumber $k_c$,

$$A(k) = \frac{e^{-\frac{(k-k_c)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}$$

In the limit $\sigma \to 0$, $A(k) \to \delta(k - k_c)$, and the monochromatic case is recovered. The analysis will be carried out for a Siegert state $E_n$ corresponding to a pole $\zeta_n = k_n^2 - i\Gamma_n$ such that $\Gamma_n > 0$ and $\Gamma_n > 0$. Siegert states with $\Gamma_n < 0$ are discussed at the end of this section.

The integrand in (24) is highly oscillatory for large $t$, the contour of integration must be deformed to a curve along which the phase is constant to obtain a fast converging integral, and thereby to determine the leading term. To this end, first, the change of variable $\xi = \frac{k-k_c + it\sigma^2}{\sigma\sqrt{2}}$ is made to obtain the following more amenable form for $\Omega_n$:

$$\Omega_n(t) = \bar{a}_n \sqrt{\Gamma_n} \frac{e^{-ikt}e^{-\xi^2}}{2\sigma^2 \sqrt{\pi}} \int_{C_0} e^{-\xi^2}d\xi, \quad \xi = \frac{-k_c + it\sigma^2 \pm \sqrt{\zeta_n}}{\sigma\sqrt{2}}$$  \hspace{1cm} (25)

where the contour of integration $C_0$ is a horizontal ray outgoing from the point $\xi_0 = \frac{-k_c + it\sigma^2}{\sigma\sqrt{2}}$ toward the infinity in the $\xi$-plane as shown in Fig. 2. The position of the poles $\xi_\pm$ indicated on the same figure follows from the fact that for $\Gamma_n > 0$, then $\text{Re}\{\sqrt{\zeta_n}\} > 0$ while $\text{Im}\{\sqrt{\zeta_n}\} < 0$. Put $(u, v) = (\text{Re} \xi, \text{Im} \xi)$. Since the function $\xi \mapsto -\xi^2$ decays exponentially in the region $\text{Re}\{\xi^2\} > 0$, the contour of integration in Eq. (25) can be deformed to a contour that consists of the constant phase curve $C_1 = \{(u, v) | uv = -\frac{k_c t}{2}, \ u \leq \text{Re}\xi_0\}$ extending from $\xi_0$ to $-\infty$ and another constant phase contour $C_2 : \text{Im}\{\xi\} = 0$ from $-\infty$ to $\infty$. Figure 2 shows the modified contour. By Cauchy’s theorem, $\Omega_n(t)$ becomes the sum of three terms:

$$\Omega_n(t) = \bar{a}_n (A_n(t) + B_n(t) + C_n(t))$$

where $A_n$ is the residual contribution at the pole $\xi_+$, while $B_n$ and $C_n$ are the contributions of the line integrals along the contours $C_1$ and $C_2$, respectively. It is proved shortly that it is the residual term $A_n$ that accounts for an exponential time decay of the Siegert state $E_n$. For a long period of time, this term dominates in the decay radiation of a Siegert state, and it is only after it has decayed considerably that the term $B_n$ becomes dominant. The term $C_n$ remains small in comparison to $A_n$ and $B_n$.

To begin, the residual term $A_n$ is evaluated to yield the formula,

$$A_n(t) = \pi i \frac{\sqrt{\Gamma_n}}{\sqrt{\zeta_n}} A(\sqrt{\zeta_n})e^{-ikt\sqrt{\zeta_n}}$$

In particular, the wavenumber $k_n$, and the life time $\tau_n$ of the Siegert state given in Eq.(23) follow from the formula for $\sqrt{\zeta_n}$ which is,

$$\sqrt{k_n^2 - i\Gamma_n} = \sqrt{\frac{k_n^2}{2} + \sqrt{k_n^4 + \Gamma_n^2}} - i\frac{\Gamma_n}{\sqrt{2\left(k_n^2 + \sqrt{k_n^4 + \Gamma_n^2}\right)}}$$
Note that for $\Gamma_n \ll 1$ (a narrow resonance), the amplitude of the Siegert state is proportional to $A(\sqrt{\xi_n}) \approx A(k_n)$. Hence, the Siegert state corresponding to the scattering resonance at $k = k_n$ is excited if the Gaussian wave packet has a sufficient amplitude at $k = k_n$, or, ideally, is centered at $k_n$ to achieve the maximal effect.

The terms $B_n$ and $C_n$ also decay necessarily in time. For $C_n$, the integral along $C_2$ will necessarily remain bounded in time so that the multiplicative factor in Eq.(25) is dominant, i.e.,

$$C_n(t) = O(e^{-\frac{1}{2}c^2t^2\sigma^2})$$

The same relation holds for $B_n$, provided that

$$ct \leq \frac{k_c}{\sigma^2}$$

When this condition is violated, the point $\xi_0$ moves into the region $\text{Re}\{\xi^2\} < 0$, and the exponential $e^{-\xi^2}$ starts to grow large on a part of the contour $C_1$. Nonetheless, the term $B_n(t)$ still decays to zero in time. This is because, as noted earlier, the Riemann-Lebesgue lemma ensures the decay of $\Omega_n(t)$ in time, and the terms $A_n$ and $C_n$ have been proved to converge to zero.

For the actual decay rate of $B_n$, it can be inferred from Laplace’s method for the asymptotic expansion of integrals that $B_n = O(1/t^\alpha)$ for some positive number $\alpha$, which is of no importance here however. Indeed, it is obvious that as $t$ increases, the term $B_n$ should be larger than the residue at $\xi_+$, since the arc of the curve $C_1$ in the region $\text{Re}\{\xi^2\} < 0$ grows longer, and therefore the exponential decay of the Siegert state can no longer be observed above the background described by $B_n$. The condition (26) shows that the smaller the width $\sigma$ of the wave packet is, the longer the exponential decay can be observed. For instance, one can ensure the observation of the decay radiation of the state $E_n$ by requiring that the half-life time occurs before the decay amplitude becomes smaller than that of the background, i.e.,

$$c\tau_n \leq \frac{k_c}{\sigma^2} \iff \ln \frac{2\sqrt{k_n^2 + \sqrt{k_n^4 + \Gamma_n^2}}}{k_c} \leq \frac{\Gamma_n}{\sigma^2}$$

In other words, the width $\sigma$ of the packet must be comparable to $\sqrt{\Gamma_n}$, if not smaller.

For Siegert states with $k_+^2 < 0$, the pole $\xi_+$ is no longer enclosed by the contours $C_0$, $C_1$, and $C_2$ if $t$ is sufficiently large. The pole moves to the left of the contour $C_1$ and the exponential time decay is absent, owing to the fact that the incident wave packet does not contain any radiation at wavenumbers close to $k_n$ for the Siegert state to be excited.

**Appendix: Proof of the Regular Perturbation Theorem**

Recall that $\forall (h, \zeta) \in J_0 \times \mathbb{C}_{k_z}$, $\tilde{H}(h, \zeta) \in \mathcal{L}\{L^2(D)\}$, and that the map

$$J_0 \times \mathbb{C}_{k_z} \rightarrow \mathcal{L}\{L^2(D)\}$$

$$(h, \zeta) \mapsto \tilde{H}(h, \zeta)$$
is continuously Frechet differentiable. For \((h, \zeta) \in J_0 \times \mathbb{C}_k\), let \(R_\lambda(\hat{H}(h, \zeta)) = [\lambda - \hat{H}(h, \zeta)]^{-1}, \lambda \in \mathbb{C}\), be the resolvent of \(\hat{H}(h, \zeta)\) when it exists, and let \(\rho(\hat{H}(h, \zeta)) = \{\lambda \in \mathbb{C} | R_\lambda(\hat{H}(h, \zeta))\) exists\} be the resolvent set of \(\hat{H}(h, \zeta)\). The proof of the Regular Perturbation Theorem is aided with the following lemma:

**Lemma.** Let \(S = \{(h, \zeta, \lambda) \in J_0 \times \mathbb{C}_k \times \mathbb{C} | \lambda \in \rho(\hat{H}(h, \zeta))\}\). The set \(S\) is open in \(J_0 \times \mathbb{C}_k \times \mathbb{C}\) and nonempty.

**Proof.** That \(S\) is nonempty follows from the results of Section \(\S\). Indeed, if \(\zeta\) is not a pole of \(\zeta \mapsto [1 - \hat{H}(h, \zeta)]^{-1}\), then \((h, \zeta, 1) \in S\). To show that \(S\) is open, let \((h_0, \zeta_0, \lambda_0) \in S\). Then one has to prove that there always exists a neighborhood of \((h_0, \zeta_0, \lambda_0)\) in \(J_0 \times \mathbb{C}_k \times \mathbb{C}\) that lies in \(S\).

Since \(\rho(\hat{H}(h_0, \zeta_0))\) is open in \(\mathbb{C}\), there exists \(\lambda_1 \in \rho(\hat{H}(h_0, \zeta_0)), \lambda_1 \neq \lambda_0\) so that \([\lambda_1 - \hat{H}(h_0, \zeta_0)]^{-1}\) exists. Let then \(\Psi\) be the function,

\[
\Psi : J_0 \times \mathbb{C}_k \rightarrow \mathcal{L}\{L^2(D)\}
\]

where \(\hat{H}(h, \zeta)\) is a compact operator on the Hilbert space \(\mathcal{L}\{L^2(D)\}\). Then one has to prove that there always exists a neighborhood of \((h_0, \zeta_0, \lambda_0)\) in \(J_0 \times \mathbb{C}_k \times \mathbb{C}\) that lies in \(S\).

Now by the first resolvent formula,

\[
R_{\lambda_0}(\hat{H}(h_0, \zeta_0)) - R_{\lambda_1}(\hat{H}(h_0, \zeta_0)) = (\lambda_1 - \lambda_0)R_{\lambda_0}(\hat{H}(h_0, \zeta_0))R_{\lambda_1}(\hat{H}(h_0, \zeta_0))
\]

so that,

\[
R_{\lambda_0}(\hat{H}(h_0, \zeta_0)) \left(1 - (\lambda_1 - \lambda_0)R_{\lambda_1}(\hat{H}(h_0, \zeta_0))\right) = R_{\lambda_1}(\hat{H}(h_0, \zeta_0))
\]

It follows that \(1 - (\lambda_1 - \lambda_0)R_{\lambda_1}(\hat{H}(h_0, \zeta_0))\) is invertible. Now let \(\Phi\) be the function,

\[
\Phi : U \times \mathbb{C} \rightarrow \mathcal{L}\{L^2(D)\}
\]

\[
((h, \zeta), \lambda) \mapsto 1 - (\lambda_1 - \lambda)R_{\lambda_1}(\hat{H}(h, \zeta))
\]

Then \(\Phi\) is continuous. Since \(\Phi(h_0, \zeta_0, \lambda_0)\) is invertible, and again, the set of all invertible operators in \(\mathcal{L}\{L^2(D)\}\) is open, it follows that there exists an open set \(W \subset U \times \mathbb{C}\) such that for all \((h, \zeta, \lambda) \in W\), the operator \(\Phi(h, \zeta, \lambda)\) is invertible.

Thus, for \((h, \zeta, \lambda) \in W\), both \(\lambda_1 - \hat{H}(h, \zeta)\) and \(1 - (\lambda_1 - \lambda)R_{\lambda_1}(\hat{H}(h, \zeta))\) are invertible. Their composition is therefore invertible, and this composition is,

\[
(\lambda_1 - \hat{H}(h, \zeta))(1 - (\lambda_1 - \lambda)R_{\lambda_1}(\hat{H}(h, \zeta))) = \lambda - \hat{H}(h, \zeta)
\]

It follows that \(W \subset S\). As \(W\) is a neighborhood of \((h_0, \zeta_0, \lambda_0)\) in \(J_0 \times \mathbb{C}_k \times \mathbb{C}\), the set \(S\) is open.

The proof of the Regular Perturbation Theorem is as follows.

**Proof.** Let \(\zeta_n\) be a simple pole of \(\zeta \mapsto [1 - \hat{H}(h_n, \zeta)]^{-1}\) for some fixed \(h_n \in J_0\), and let \(E_n\) be the corresponding Siegert state. Then 1 is an eigenvalue of \(\hat{H}(h_n, \zeta_n)\) corresponding to the eigenfunction \(E_n\) in the usual sense. The theorem amounts to proving the existence of a curve \(h \mapsto \zeta(h)\) in \(J_0 \times \mathbb{C}_k\), along which the family of operators \(\hat{H}(h, \zeta(h))\) still has 1 as an eigenvalue. The corresponding eigenfunctions will then be the Siegert states of the said operators along the curve in question.

The proof starts by providing a general formula for the eigenvalue \(\lambda_0(h, \zeta)\) of \(\hat{H}(h, \zeta)\) which is the perturbed value of the eigenvalue 1 when the point \((h_n, \zeta_n)\) is displaced to \((h, \zeta)\) in the \(J_0 \times \mathbb{C}_k\) space. Then by the Implicit Function Theorem, a curve \(h \mapsto \zeta(h)\) is found along which the eigenvalue \(\lambda(h, \zeta)\) remains 1, i.e., \(\lambda(h, \zeta(h)) = 1\).

As a starting point, note that as \(\hat{H}(h_n, \zeta_n)\) is a compact operator on the Hilbert space \(L^2(D)\), the Riesz-Schauder theorem implies that 1 is necessarily an isolated eigenvalue. Therefore, there exists \(\delta > 0\) such that 1 is the only eigenvalue of \(\hat{H}(h_n, \zeta_n)\) in the disk \(\{\lambda \in \mathbb{C} | |\lambda - 1| \leq \delta\}\). It follows that \(V_\delta = \{(h_n, \zeta_n, \lambda) | \lambda \in \mathbb{C}, |\lambda - 1| = \delta\} \subset S\) where \(S\) is the set of the previous lemma. Since \(S\) is open, and \(V_\delta\) is compact, there exists an open \(W\) in \(J_0 \times \mathbb{C}_k \times \mathbb{C}\) such that \(V_\delta \subset W \subset S\). Therefore there exists a connected neighborhood \(U\) of \((h_n, \zeta_n)\) in \(J_0 \times \mathbb{C}_k\), such that \(\lambda \in \rho(\hat{H}(h, \zeta))\) for all \((h, \zeta) \in U\), and for all \(\lambda \in \mathbb{C}\) with \(|\lambda - 1| = \delta\).
Put
\[
\hat{P}(h, \zeta) = \frac{1}{2\pi i} \int_{|\lambda - 1| = \delta} [\lambda - \hat{H}(h, \zeta)]^{-1} d\lambda, \quad (h, \zeta) \in U
\] (27)

Then \( \hat{P}(h, \zeta) \in \mathcal{L}(L^2(D)) \), and by Theorem XII.6 of [17], \( \hat{P}(h, \zeta) \) is a projection. As the map \((h, \zeta) \mapsto \hat{P}(h, \zeta) \) of \( U \) to \( \mathcal{L}(L^2(D)) \) is continuous, it follows that the dimension of the range of \( \hat{P}(h, \zeta) \) is constant throughout \( U \).

Since the eigenvalue 1 of \( \hat{H}(h_n, \zeta_n) \) is nondegenerate, it follows that the range of \( \hat{P}(h_n, \zeta_n) \) has dimension 1, and therefore the dimension of the range of \( \hat{P}(h, \zeta) \) is 1 throughout \( U \). By Theorem XII.6 of [17], it follows that for all \((h, \zeta) \in U\), there exists a unique eigenvalue \( \lambda_0(h, \zeta) \) of \( \hat{H}(h, \zeta) \) in \( \{ \lambda \in \mathbb{C} : |\lambda - 1| \leq \delta \} \), and \( \hat{P}(h, \zeta) \) is the projection on the corresponding eigenspace. In particular, \( \hat{P}(h, \zeta)[E_n] \) is in the eigenspace of \( \lambda_0(h, \zeta) \), and therefore,

\[
\hat{H}(h, \zeta) \left[ \hat{P}(h, \zeta)[E_n] \right] = \lambda_0(h, \zeta) \hat{P}(h, \zeta)[E_n], \quad (h, \zeta) \in U
\]

Hence,

\[
\lambda_0(h, \zeta) = \frac{\langle E_n, \hat{H}(h, \zeta) \hat{P}(h, \zeta)[E_n] \rangle}{\langle E_n, \hat{P}(h, \zeta)[E_n] \rangle}, \quad (h, \zeta) \in U
\]

where once again, \( \langle \cdot, \cdot \rangle \) is the inner product on \( L^2(D) \). This is the formula for the perturbed eigenvalue announced in the preamble of this proof.

Now, observe that since \((h, \zeta) \mapsto \hat{H}(h, \zeta) \) is continuously Frechet differentiable in \( U \), so is \((h, \zeta) \mapsto \hat{P}(h, \zeta) \). Therefore, \( \lambda_0 \) is continuously Frechet differentiable in \( U \). Now, \( \lambda_0(h_n, \zeta_n) = 1 \), and as it will be shown shortly, \( \partial_\zeta \lambda_0(h_n, \zeta_n) \neq 0 \). It follows by the Implicit Function Theorem for Banach spaces that there exists an open interval \( J \subset J_0 \) containing \( h_n \), and a unique continuously differentiable function \( h \mapsto \zeta_n(h) \), \( h \in J \), such that \( \zeta_n(h_n) = \zeta_n \) and \( \lambda_0(h, \zeta_n(h)) = 1 \) for all \( h \in J \). Thus the proof of the Regular perturbation theorem is complete, provided the property \( \partial_\lambda \lambda_0(h_n, \zeta_n) \neq 0 \) is established. The latter is achieved by studying the analyticity of the resolvent \( R_\lambda(\hat{H}(h_n, \zeta)) = [\lambda - \hat{H}(h_n, \zeta)]^{-1} \) in the variables \( \lambda \) and \( \zeta \) separately when \( h = h_n \) is fixed.

Let \( \lambda \in \mathbb{C} \) and \( \zeta \in \mathbb{C}_{k_\delta} \) such that \( |\lambda - 1| = \delta \), \((h_n, \zeta) \in U\), and \( \zeta \neq \zeta_n \). Then the resolvent \( R_\lambda(\hat{H}(h_n, \zeta)) = [\lambda - \hat{H}(h_n, \zeta)]^{-1} \) exists by definition of the neighborhood \( U \). Also, the resolvent \( R_1(\hat{H}(h_n, \zeta)) = [1 - \hat{H}(h_n, \zeta)]^{-1} \) exists and is meromorphic in \( \zeta \).

By the first resolvent formula,

\[
R_\lambda(\hat{H}(h_n, \zeta)) - R_1(\hat{H}(h_n, \zeta)) = (1 - \lambda)R_\lambda(\hat{H}(h_n, \zeta))R_1(\hat{H}(h_n, \zeta))
\] (28)

Substituting Eq. (28) into Eq. (27), it follows that,

\[
\hat{P}(h, \zeta) = \frac{1}{2\pi i} \int_{|\lambda - 1| = \delta} (1 - \lambda)R_\lambda(\hat{H}(h_n, \zeta))R_1(\hat{H}(h_n, \zeta))d\lambda
\] (29)

Now, the meromorphic expansion of \( \lambda \mapsto R_\lambda(\hat{H}(h_n, \zeta)) \) at \( \lambda_0(h_n, \zeta) \) is,

\[
R_\lambda(\hat{H}(h_n, \zeta)) = \frac{1}{\lambda - \lambda_0(h_n, \zeta)}\hat{P}(h_n, \zeta) + \tilde{R}_\lambda(h_n, \zeta)
\] (30)

where \( \lambda \mapsto \tilde{R}_\lambda(h_n, \zeta) \) is analytic in the disk \( \{ \lambda \in \mathbb{C} : |\lambda - 1| \leq \delta \} \). The substitution of Eq. (30) into Eq. (29) yields

\[
\hat{P}(h_n, \zeta) = (1 - \lambda_0(h_n, \zeta))\hat{P}(h_n, \zeta)R_1(\hat{H}(h_n, \zeta))
\] (31)

Now, at the pole \( \zeta_n \), the generalized resolvent \( \zeta \mapsto R_1(h_n, \zeta) \) has the meromorphic expansion:

\[
R_1(h_n, \zeta) = \frac{1}{\zeta - \zeta_n} \hat{H}_n + \tilde{R}(h_n, \zeta)
\] (32)

where \( \hat{H}_n \) is the residue of \( \zeta \mapsto R_1(h_n, \zeta) \) as given in Eq. (10), and \( \zeta \mapsto \tilde{R}(h_n, \zeta) \) is analytic in the vicinity of \( \zeta_n \).

Substituting Eq. (32) into Eq. (31) and taking the limit as \( \zeta \to \zeta_n \) yields the equation,

\[
\hat{P}(h_n, \zeta_n) = -\partial_\zeta \lambda_0(h_n, \zeta_n)\hat{P}(h_n, \zeta_n)\hat{H}_n
\] (33)
Next, the action of both the sides of this operator equality is evaluated on the state $E_n$. Since $\hat{P}(h_n, \zeta_n)$ projects on $E_n$, and in light of Eq. (10), it follows that,

$$E_n = - \langle \varphi_n, E_n \rangle \partial \zeta \lambda_0(h_n, \zeta_n) E_n$$

As the Siegert state $E_n$ is not identically zero, it follows that $\partial \zeta \lambda_0(h_n, \zeta_n) \neq 0$, and the proof of the theorem is complete.

A final noteworthy remark is that the projection $\hat{P}(h_n, \zeta_n)$ and the residue $\hat{H}_n$ are proportional, which is established by applying both the sides of Eq. (33) to an arbitrary state $F \in L^2(D)$:

$$\hat{P}(h_n, \zeta_n)[F] = - \partial \zeta \lambda_0(h_n, \zeta_n) \hat{H}_n[F]$$

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