Continuous space-time symmetries in a lattice field theory

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For purposes of regularization as well as numerical simulation, the discretization of Lorentz invariant continuum field theories on a space-time lattice is often convenient. In general this discretization destroys the rotational or Lorentz-frame independence of the theory, which is only recovered in the continuum limit. The Baxter 8-vertex model may be regarded as a particular discretization of a self-interacting massive Dirac fermion field theory in two dimensions (the massive Thirring model). Here it is shown that, in the 8-vertex/massive Thirring model, the Lorentz-frame independence of the continuum theory remains undisturbed on the lattice. The only effect of the discretization is to compactify the manifold of Lorentz frames. The relationship between this lattice Lorentz symmetry and the Yang-Baxter relations is discussed.

1 Introduction

The early work of Jim McGuire on the delta-function gas was the first analysis to focus on the essential feature of consistent factorization of 3-body scattering amplitudes into products of 2-body amplitudes. Subsequent developments of integrable systems technology have generalized and enshrined this idea in the form of a generic algebraic structure, the Yang-Baxter relations, which is common to a wide variety of solvable models. The recognition that the transfer matrix methods used by Baxter to solve the 8-vertex model arose from precisely the same algebraic structure (this time as a statement about vertex Boltzmann weights instead of scattering amplitudes) led to the quantum inverse scattering method, an elegant unification of Yang-Baxter relations, the Bethe ansatz, and classical soliton methods. In spite of all these developments, the physical nature of the symmetries imposed by the Yang-Baxter relations is still somewhat obscure. In this paper, I would like to discuss a viewpoint on integrable models and Yang-Baxter relations which arises from a lattice construct called a corner transfer matrix (CTM), invented and developed by Rodney Baxter in the late 1970's. I will briefly review some arguments which expose a connection between the CTM and Yang-Baxter relations of the 8-vertex model and the question of how space-time symmetries (or their remnants) are realized in a lattice field theory. Simply stated, the corner transfer matrix is

\textsuperscript{a}To appear in Statistical Physics on the Eve of The Twenty-First Century, a festschrift for James B. McGuire.
the lattice generalization of a Lorentz boost (or Euclidean rotation) operator. The miraculous properties of the CTM (which follow from the Yang-Baxter relations) are a manifestation of the fact that the lattice theory supports an exact, continuous analog of the Lorentz symmetry of the continuum theory. The obvious question of how a continuous space-time symmetry can survive in any sense on the lattice is answered by the introduction of the elliptic function or “lattice rapidity” parametrization of momentum space.

To understand the nature of lattice rapidity, let us first recall the role of rapidity in a Lorentz invariant continuum theory in one spatial dimension. The energy-momentum dispersion relation $\omega(p) = \sqrt{p^2 + m^2}$ for a relativistic particle of mass $m$, may be parametrized by introducing a rapidity variable $\alpha$,

$$p = m \sinh \alpha, \quad \omega = m \cosh \alpha$$

(1)

The rapidity variable “uniformizes” momentum space, in the sense that the relativistic phase space volume element becomes

$$\frac{dp}{2\omega(p)} = \frac{1}{2} d\alpha$$

(2)

Rapidity also uniformizes momentum space from a dynamical point of view. Specifically, Lorentz invariance implies that invariant scattering amplitudes depend only on relative rapidity variables. The scattering amplitudes are unchanged by a uniform shift of all the rapidity variables in the scattering state, which is equivalent to a change of the observer’s Lorentz frame. As I discuss here, the 8-vertex model and XYZ spin chain have the amazing property that all of this structure carries over undisturbed to the the lattice theory. The only change introduced by the lattice is that momentum space is compactified in the real rapidity direction. The uniformity of rapidity space associated with Lorentz invariance remains unchanged. For example, two-body phase shifts and Bethe ansatz kernels depend only on the relative rapidity of the colliding spin waves (leading to the “difference kernel” form of the Bethe ansatz equations). Note that the periodicity of the continuum rapidity parametrization in the imaginary direction $\alpha \to \alpha + 2\pi i$ corresponds to periodicity under Euclidean rotations by $2\pi$. In the lattice theory, the rapidity parametrization is given in terms of doubly periodic elliptic functions, combining the periodicity under Euclidean rotations (imaginary rapidity) with the lattice periodicity of boosts (real rapidity) corresponding to momentum shifts by $2\pi/a$ where $a = $ lattice spacing. To see this structure in a simple context, in Section 3 I will look specifically at the single particle eigenmode operators of the XY spin...
chain,
\[
H = -\frac{1}{2} \sum_j \left[ \sigma_j^x \sigma_{j+1}^x + k \sigma_j^y \sigma_{j+1}^y \right]
\]  
(3)
which corresponds to the free fermion case of the Thirring model. The eigen-modes of \( H \) can be parametrized in terms of lattice rapidity variables labelling the momenta of the spin waves. These variables are exactly analogous to the continuum rapidity in a relativistic theory. The corner transfer matrix, when applied to a spin-chain eigenstate, produces another eigenstate with a shifted rapidity variable. This is completely analogous to the Lorentz boost operator in a continuum theory, which sweeps through the set of states of different total momentum corresponding to the same physical state in different Lorentz frames. In the continuum case, the states can be classified in terms of irreducible representations of the 2-dimensional Poincare algebra,
\[
[\hat{K}, \hat{H}] = \hat{P}, \quad [\hat{K}, \hat{P}] = \hat{H}, \quad [\hat{H}, \hat{P}] = 0
\]  
(4)
where \( \hat{H} \), \( \hat{P} \), and \( \hat{K} \) are the Hamiltonian, the total momentum operator, and the boost generator, respectively. In the 8-vertex model, the Poincare algebra is replaced by the lattice boost generator \( K \) (which is given by the log of the CTM, see below) and an infinite tower of commuting conserved quantities \( H_n \)
\[
[H_n, H_m] = 0
\]  
(5)
which generalize the role of \( \hat{H} \) and \( \hat{P} \). Instead of closing as in the continuum algebra, repeated commutation by \( K \) walks up the tower of conserved quantities,
\[
[K, H_n] = H_{n+1}
\]  
(6)
The lowest member of this hierarchy \( H_1 \equiv H \) is the nearest-neighbor \( XYZ \) spin chain Hamiltonian, with the higher conserved operators \( H_n \) involving interactions ranging up to \( n \)th nearest neighbor.

It was shown long ago by Alan Luther\(^{11}\) that the continuum theory obtained by approaching the critical point of the 8-vertex model is equivalent to the massive Thirring model, a self-interacting, relativistic Dirac fermion theory. Perforce we may view the 8-vertex model as the result of taking the continuum massive Thirring model and “putting it on a lattice,” i.e. in some way discretizing the fermionic variables of that theory. The direct connection between the vertex Boltzmann weights and the two-dimensional action of the Dirac fermion theory would be of great interest, but this connection has not yet been fully clarified. The relationship between the two models is most easily discussed in the transfer matrix or Hamiltonian framework. It can
be shown that the spin-chain Hamiltonian may be directly transformed into a lattice Dirac Hamiltonian as discussed in Section 4. The comparison of the spin-chain eigenmodes with those of the continuum Dirac Hamiltonian clearly exhibits the structure imposed by the lattice Lorentz invariance embodied in the CTM formalism.

The direct transcription of the lattice spin Hamiltonian to a lattice Dirac Hamiltonian allows us to address the interesting question of exactly how the spin chain constitutes a discretization of the Dirac field. The spin-chain fermion operators \( c_x^j, c_y^j \) on site \( j \) are obtained via a Jordan-Wiger transformation of the local Pauli spin matrices,

\[
c_x^j = \sigma_x^j \prod_{l<j} \sigma_z^l, \quad c_y^j = \sigma_y^j \prod_{l<j} \sigma_z^l
\]  

(7)

This gives us two real fermion operators on each site. A lattice Dirac field consists of a two-component complex Dirac spinor, i.e. four real fermion operators on each site. In a particular representation of Dirac matrices (with \( \gamma^1 \) diagonal), the \( x \) and \( y \) labels of the spin-chain fermions \( c_x^j \) and \( c_y^j \) correspond to upper and lower components of the Dirac spinor. However, the real and imaginary parts of a single Dirac spinor component correspond to nearest neighbor combinations of spin-chain operators. (See Section 4.) As a result, the vector charge symmetry of the Dirac field, corresponding to local phase rotations of the complex spinor components, is expressed in terms of spin-chain operators by a non-local mixing of nearest neighbor pairs. The associated conserved charge is thus not locally defined on the spin lattice. (In fact, the conserved vector charge on the lattice is the kink number associated with the Bethe ansatz for this model, and introduced via Baxter’s SOS transformation.) On the other hand, the chiral symmetry of the theory in the limit of zero fermion mass is related to a very simple and obvious symmetry of the spin chain. The massless fermion theory corresponds to a spin-spin interaction which is isotropic in the \( \sigma_x - \sigma_y \) plane, i.e. has equal coefficients for the \( \sigma_x^j \sigma_x^{j+1} \) and \( \sigma_y^j \sigma_y^{j+1} \) terms. The global chiral symmetry which arises in the massless fermion theory corresponds to the symmetry of the isotropic spin chain under global rotations in the \( \sigma_x - \sigma_y \) plane. Some time ago, I argued that the chiral and Lorentz properties of the Heisenberg spin chain and 8-vertex model made this a particularly interesting model for studying properties of chiral lattice fermions. It can be shown that the free lattice Dirac Hamiltonian obtained from the \( XY \) spin-chain is a Wilson-Dirac Hamiltonian operator with Wilson parameter \( r = 1 \) and hopping parameter \( \frac{1}{2} k \), and that, for the massless case \( k = 1 \), it satisfies a one-dimensional version of the Ginsparg-Wilson relation:

\[
\{ \gamma^5, D \} = D \gamma^5 D
\]  

(8)
where $D = \gamma_0 H$, and $H$ is the lattice Dirac Hamiltonian. Furthermore, it can be shown that this particular Wilson-Dirac operator is unique in that it is the only lattice Dirac operator in one-dimension involving a finite number of nearest-neighbor hopping terms which satisfies the Ginsparg-Wilson relations. This may suggest a connection between vertex models and the form of lattice chiral symmetry that is embodied in the Ginsparg-Wilson relations. This problem is currently under investigation.

2 Lattice anisotropy, commuting transfer matrices, and Lorentz frame independence

To introduce the discussion of lattice Lorentz invariance, let me describe in generic terms how this Lorentz invariance is related to the Yang-Baxter relations. In the row-to-row transfer matrix (or quantum inverse) formalism, a vertex is represented by a $2 \times 2$ matrix of spin operators $V_j(\alpha)$ where each element of this matrix contains Pauli matrices acting on spin $j$. The vertex is a function of a rapidity-like anisotropy parameter $\alpha$ (sometimes called the spectral parameter because of its role in the quantum inverse method), which determines the Boltzmann weights via Baxter’s elliptic function parametrization. (The other two parameters in the vertex weights are essentially the mass and coupling of the Thirring model and are treated as fixed constants.) The transfer matrix $T(\alpha)$ is then given by a row of vertices of the form

$$T(\alpha) = \text{Tr} \left( \prod_{j=-L}^{j=L} V_j(\alpha) \right)$$

where the trace and product are over the $2 \times 2$ matrix space (horizontal arrows of the vertex model). Although I have implicitly assumed spatial periodic boundary conditions by taking the trace in (9), in what follows, I will effectively assume that the chain of spins stretches from $-\infty$ to $\infty$ and ignore the subtleties associated with boundary terms. (In the CTM framework, issues associated with the limit of infinite spatial volume are replaced by issues of analytic continuation to complex momentum or rapidity (c.f. (4)).) The Yang-Baxter relations for this model are trilinear algebraic relations among vertices $V_j(\alpha)$. It is easy to show that the Yang-Baxter relations, in the limit where two of the rapidity parameters involved are nearly equal, reduce to a simple statement giving the commutator of the nearest-neighbor spin-chain Hamiltonian term $H_{j,j+1}$ and the product of the two vertices at sites $j$ and $j+1$, namely

$$[H_{j,j+1}, V_j(\alpha)V_{j+1}(\alpha)] = V_j(\alpha) \left( \frac{\partial}{\partial \alpha} V_{j+1}(\alpha) \right) - \left( \frac{\partial}{\partial \alpha} V_j(\alpha) \right) V_{j+1}(\alpha)$$
In the following discussion, the properties of two important lattice operators constructed from $\mathcal{H}_{j,j+1}$ are relevant. One is the standard Heisenberg spin chain Hamiltonian,

$$H = \sum_j \mathcal{H}_{j,j+1} \tag{11}$$

Most of the following discussion applies to the general symmetric 8-vertex model, equivalently, the fully anisotropic XYZ spin-chain, for which

$$\mathcal{H}_{j,j+1} = -\frac{1}{2} \left[ \sigma_j^x \sigma_{j+1}^x + k \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z \right] \tag{12}$$

In Section 3, I will analyze the free fermion case $\Delta = 0$ more completely. The other operator essential to the discussion is the lattice boost generator, which is given by the first moment of the same Hamiltonian density,

$$K = \sum_j j \mathcal{H}_{j,j+1} \tag{13}$$

(Note that this is completely analogous to the continuum boost generator which is given, at $t = 0$, by the first moment of the Hamiltonian density, $K = \int x \mathcal{H}(x) \, dx$.) An important property of $K$ is that, because the $j = 0$ term in the sum (13) vanishes, it separates into two commuting operators which act separately on the left and right half-chain,

$$K = K^> + K^< \tag{14}$$

As Baxter showed in his original work, the corner transfer matrix in the infinite volume limit is, up to an overall constant, exactly the exponential of the operator $K^>$ or $K^<$ (sweeping out the left- and right-hand corners of the lattice, respectively).

From the Yang-Baxter commutator (10) it follows that $K$ has the following commutator with the transfer matrix:

$$[K, T(\alpha)] = \frac{\partial}{\partial \alpha} T(\alpha) \tag{15}$$

Thus $K$ generates a shift of the rapidity (anisotropy) parameter,

$$e^{i\beta K} T(\alpha) e^{-i\beta K} = T(\alpha + \beta) \tag{16}$$

In a sense, we can interpret $T(\alpha)$ for a particular value of $\alpha$ as the transfer matrix in a particular Lorentz frame. The boost operator $K$ generates Lorentz transformations from one frame to another by shifting the rapidity of
the Lorentz frame. Among other things, this explains why transfer matrices with different values of $\alpha$ commute with each other (the famous result that led Baxter to his solution of the model),

$$[T(\alpha), T(\alpha')] = 0$$  (17)

This follows from the fact that any two observers of the same theory in two different frames will construct the same set of Hamiltonian or transfer matrix eigenstates. Thus, $T(\alpha)$ and $T(\alpha')$ are simultaneously diagonalizable. In the continuum, this follows from the fact that $H$ and $P$ commute (because a boost simply mixes $H$ and $P$). On the lattice, it requires an infinite number of commuting conserved quantities. If the rapidity parameter which appears in the vertex weights is taken to be imaginary, it is most appropriately interpreted as a lattice anisotropy parameter which determines the relative scale between the space and time directions. The Heisenberg spin-chain Hamiltonian is the first nontrivial term in an expansion of $\log T(\alpha)$ in powers of $\alpha$. Here $\alpha \rightarrow 0$ corresponds to the limit in which the lattice spacing in the time direction goes to zero. In this sense, the Heisenberg spin-chain is the time-continuum Hamiltonian formulation of the two-dimensional 8-vertex model.

3 Lorentz transformation of lattice spin waves and Dirac fermions

To exhibit some of the simpler consequences of lattice Lorentz invariance, I will look at the eigenmode operators for the $XY$ Hamiltonian. For comparison, we first review the Lorentz transformation properties of eigenmodes of the free massive continuum Dirac Hamiltonian. In a Hamiltonian framework, the most direct statement of Lorentz invariance is in terms of the action of the boost operator on the eigenstates of the Hamiltonian. It is this formulation that generalizes most directly to the lattice theory. We will work in Minkowski space, taking $(\gamma^0)^2 = 1$, $(\gamma^1)^2 = -1$, and $\gamma^5 = \gamma^0 \gamma^1$. The continuum Dirac Hamiltonian is

$$\tilde{H} = \frac{1}{2} \int dx \left[ \tilde{\psi} \left( -i \gamma^1 \partial_1 + m \right) \psi + \text{h.c.} \right]$$  (18)

The comparison with the $XY$ spin chain is facilitated by choosing a particular representation of gamma matrices for which $\gamma^1$ is diagonal. Specifically, I will take $\gamma^0 = \sigma^z$, $\gamma^1 = i \sigma^z$, and $\gamma^5 = \sigma^y$, where the $\sigma$’s are the standard Pauli matrices. In this basis, we can derive the Hamiltonian equations for the free Dirac field in terms of the fourier transformed field,

$$\tilde{\psi}(p) = \int dx \ e^{-ipx} \psi(x)$$  (19)
The spinor components satisfy
\[
\begin{align*}
\left[ \tilde{\psi}_1(p), \tilde{\mathcal{H}} \right] &= (m - ip) \tilde{\psi}_2(p) \\
\left[ \tilde{\psi}_2(p), \tilde{\mathcal{H}} \right] &= (m + ip) \tilde{\psi}_1(p)
\end{align*}
\] (20)

Defining
\[
\begin{align*}
b_1(p) &= (m + ip)^{1/2} \tilde{\psi}_1(p) \\
b_2(p) &= (m - ip)^{1/2} \tilde{\psi}_2(p)
\end{align*}
\] (21)

the Hamiltonian equations reduce to
\[
\begin{align*}
\left[ b_1(p), \tilde{\mathcal{H}} \right] &= \omega(p) b_2(p) \\
\left[ b_2(p), \tilde{\mathcal{H}} \right] &= \omega(p) b_1(p)
\end{align*}
\] (22)

where
\[
\omega(p) = (p^2 + m^2)^{1/2}
\] (23)

Thus, the eigenmodes of \( \tilde{\mathcal{H}} \) are
\[
b^\pm(p) = b_1(p) \mp b_2(p)
\] (24)

which satisfy
\[
\left[ \tilde{\mathcal{H}}, b^\pm(p) \right] = \pm \omega(p) b^\pm(p)
\] (25)

Since \( b^+(p) \) and \( b^-(p) \) carry the same fermionic charge but have opposite energy, they must be interpreted as particle creation and antiparticle annihilation operators, respectively.

Next, consider the action of the continuum Lorentz boost operator
\[
\tilde{\mathcal{K}} = \frac{1}{2} \int dx \ x \left[ \bar{\psi} \left( -i \gamma^1 \partial_1 + m \right) \psi + \text{h.c.} \right]
\] (26)

The commutators of \( \tilde{\mathcal{K}} \) with the Dirac field may be computed in a similar way to the Hamiltonian, with the additional factor of \( x \) in the integrand of (26) giving rise to a derivative with respect to momentum. The commutators with \( \tilde{\mathcal{K}} \) are particularly simple when expressed in terms of the operators defined in (21),
\[
\begin{align*}
\left[ b_1(p), \tilde{\mathcal{K}} \right] &= i \omega(p) \frac{\partial}{\partial p} b_2(p) \\
\left[ b_2(p), \tilde{\mathcal{K}} \right] &= i \omega(p) \frac{\partial}{\partial p} b_1(p)
\end{align*}
\] (27)
Introducing the continuum rapidity, where $p = m \sinh \alpha$, we see that the boost operator induces a uniform shift of the rapidity of Hamiltonian eigenmodes,

$$[b^\pm, \tilde{K}] = i \frac{\partial}{\partial \alpha} b^\pm$$  \hspace{1cm} (29)

For comparison with the corresponding lattice results, it is useful to note that the square root factors appearing in the expressions for the eigenmodes are entire functions of rapidity

$$(m \pm ip)^\pm = \sqrt{2m} \cosh \left(\frac{\alpha}{2} \pm i\frac{\pi}{4}\right)$$  \hspace{1cm} (30)

Thus, we may write the eigenmodes entirely in terms of rapidity

$$b^\pm = \sqrt{2m} \left[ \cosh \left(\frac{\alpha}{2} + i\frac{\pi}{4}\right) \tilde{\psi}_1(p(\alpha)) \mp \sinh \left(\frac{\alpha}{2} + i\frac{\pi}{4}\right) \tilde{\psi}_2(p(\alpha)) \right]$$  \hspace{1cm} (31)

Written in terms of the fermionized spin operators (7), the Hamiltonian becomes

$$H = -\frac{i}{2} \sum_j \left[ c^x_{j+1} c^y_j + kc^x_j c^y_{j+1} \right]$$  \hspace{1cm} (32)

The procedure for diagonalizing this operator is well-known and consists of a Fourier transform to momentum space followed by a Bogoliubov transformation. Define the momentum-space fermion operators

$$a^x,y(z) = \sum_j z^j c^x,y_j$$  \hspace{1cm} (33)

It is easy to show that

$$[H, a^x(z)] = \frac{i}{2} (z + k z^{-1}) a^y(z)$$  \hspace{1cm} (34)$$[H, a^y(z)] = -\frac{i}{2} (z^{-1} + k z) a^x(z)$$

Thus, if we define

$$B^x(z) = (1 + k z^2)^{1/2} a^x(z)$$  \hspace{1cm} (35)$$B^y(z) = z (1 + k z^{-2})^{1/2} a^y(z)$$  \hspace{1cm} (36)

then

$$[H, B^x(z)] = i \omega(z) B^y(z)$$  \hspace{1cm} (37)$$[H, B^y(z)] = -i \omega(z) B^x(z)$$  \hspace{1cm} (38)
where we have defined the single-particle energy by

\[ \omega(z) = \frac{1}{2} (1 + k z^2)^{\frac{1}{2}} (1 + k z^{-2})^{\frac{1}{2}} \] (39)

The Hamiltonian eigenmodes are given by

\[ B^\pm(z) = B_x(z) \pm i B_y(z) \] (40)

which satisfy

\[ [H, B^\pm(z)] = \pm \omega(z) B^\pm(z) \] (41)

Thus, \( B^+(z) \) \((B^-(z))\) is a single particle creation (annihilation) operator for a particle of energy \( \omega(z) \). With a particular choice of branch cuts for the square roots in (39), the energy function \( \omega(z) \) is continuous and positive definite on the unit circle \(|z| = 1\). The minima of \( \omega(z) \) are at \( z = \pm i \), so we define the single-particle momentum \( p \) to be given by \( z = i e^{ip} \). (Here and elsewhere I take \( a = 1 \), expressing quantities like \( p \) in lattice units.) In the continuum limit \( p \to 0, k \to 1 \), \( \omega(z) \) reduces to the continuum relativistic energy,

\[ \omega(z) \to \sqrt{m^2 + p^2} \] (42)

where the fermion mass is

\[ ma = \frac{1}{2k} - \frac{1}{2} \] (43)

The commutators of the lattice boost operator \( K \) with the spin chain fermion operators \( a_x(z) \) and \( a_y(z) \) are easily computed in a manner similar to the Hamiltonian commutators. The action of the boost operator is very simply expressed in terms of the Hamiltonian eigenmode operators. In terms of the spin chain fermions, the boost generator is

\[ K = -\frac{i}{2} \sum_j \left[ c^x_{j+1} c^y_j + kc^x_j c^y_{j+1} \right] \] (44)

By direct commutation one obtains

\[ [K, B_x(z)] = iz\omega(z) \frac{\partial}{\partial z} B_y(z) \] (45)

\[ [K, B_y(z)] = -iz\omega(z) \frac{\partial}{\partial z} B_x(z) \] (46)

so that the eigenmode operators satisfy

\[ [K, B^{\pm}(z)] = z\omega(z) \frac{\partial}{\partial z} B^{\pm}(z) \] (47)
These commutators provide the essential statement of Lorentz transformations on a lattice. Just as in the continuum, we may uniformize momentum space by defining a rapidity $\alpha$, which is determined up to an overall constant by the differential relation

$$d\alpha = \frac{dp}{\omega(p)} = -i \frac{dz}{z\omega(z)}$$  \hspace{1cm} (48)$$

The solution to this is an elliptic function,

$$z(\alpha) = i \sqrt{k} \text{sn} \frac{1}{2}(\alpha - \alpha_0)$$ \hspace{1cm} (49)

where $\text{sn}$ is a Jacobian elliptic function of modulus $k$. The choice of $\alpha_0$ will be made so that the lattice rapidity $\alpha$ corresponds to the ordinary rapidity in the continuum limit. This is accomplished by taking

$$\alpha_0 = 2\hat{K} + i\hat{K}'$$ \hspace{1cm} (50)

where $\hat{K}$ and $\hat{K}'$ are the complete elliptic integrals of modulus $k$ and $k' = \sqrt{1 - k^2}$, respectively. ($\hat{K}$ and $\hat{K}'$ are the real and imaginary elliptic quarter-periods.)

Note that the Bogoliubov factors which appear in the Hamiltonian eigenmode operators (c.f. Eq. 35) are also simply expressed in terms of elliptic functions:

$$1 + k z^2 \frac{1}{2} = \text{dn} \frac{1}{2}(\alpha - \alpha_0)$$ \hspace{1cm} (51)$$

$$z(1 + k z^{-2}) \frac{1}{2} = \sqrt{k} \text{cn} \frac{1}{2}(\alpha - \alpha_0)$$ \hspace{1cm} (52)$$

These two expressions are the analog of the continuum expressions (30). Thus, we can write the lattice eigenmodes as functions of rapidity

$$B^+ = \text{dn} \frac{1}{2}(\alpha - \alpha_0) \ a_x (z(\alpha)) + i\sqrt{k} \text{cn} \frac{1}{2}(\alpha - \alpha_0) \ a_y (z(\alpha))$$ \hspace{1cm} (53)$$

It is a simple exercise to show that the coefficients of $a_x$ and $a_y$ reduce to those of $\tilde{\psi}_1$ and $\tilde{\psi}_2$ in the continuum eigenmode (31). In taking the continuum limit ($k \to 1$) of the elliptic functions, the fact that the shift $\alpha_0$ goes to infinity in this limit must be noted. A useful identity to use before taking the continuum limit is

$$\sqrt{k} \text{sn}(\beta - \hat{K} - i\hat{K}') = \left( \frac{\text{cn} \beta \text{dn} \beta + i(1 - k)\text{sn} \beta}{\text{cn} \beta \text{dn} \beta - i(1 - k)\text{sn} \beta} \right)^{\frac{1}{2}}$$ \hspace{1cm} (54)$$

along with similar identities for $\text{dn}$ and $\text{cn}$. To summarize, the lattice boost operator $\hat{K}$ given by the first moment of the spin chain Hamiltonian has exactly
the same effect on the lattice spin wave eigenmodes $B^\pm$ that the continuum Dirac boost operator has on the eigenmodes of the Dirac Hamiltonian. That is, it generates a shift of the rapidity variable. This is the central manifestation of lattice Lorentz invariance in the free fermion theory. I will not discuss the interacting case $\Delta \neq 0$ in any detail, and some issues remain to be investigated for this case. However, it is clear from corner transfer matrix and Bethe ansatz results that the Lorentz invariance applies to this case as well. For example, the two-body phase shift which appears in the Bethe ansatz for the interacting case is Lorentz invariant in the sense that it depends only on the relative rapidity of the two colliding spin waves. For zero mass, the interacting case reduces to the $XXZ$ spin chain. The boost properties of the eigenmodes for this case have been investigated in detail\[10\].

4 Spin-chain fermions and Dirac fermions

The comparison of spin chain eigenmodes $B^\pm$ and Dirac eigenmodes $b^\pm$ in the last Section would constitute a complete identification of the two theories if the spin chain operators $a_x(z)$ and $a_y(z)$ reduced to the Dirac fermion operators $\tilde{\psi}_1(p)$ and $\tilde{\psi}_2(p)$ in the continuum limit. However, the correspondence is not that simple, because of the fact that $a_x(z)$ and $a_y(z)$ are the Fourier transforms of real lattice fermions $c^x_j$ and $c^y_j$, and satisfy

\[(a_{x,y}(z))^\dagger = a_{x,y}(z^*)\] (55)

This contrasts with the Dirac field which is complex. The states created by $b^+$ and $(b^-)^\dagger$ carry opposite vector charge, so that the particle spectrum consists of two distinct species (particle and antiparticle). At first site, the reality of $a_x$ and $a_y$ would seem to indicate that the spin chain contains only one species, and therefore cannot constitute a Dirac fermion. The resolution of this question, and the appearence of vector charge in the spin chain, depends on the well-known doubling of the spectrum associated with lattice fermions.

The energy function $\omega(z)$ is an even function of $z = ie^{ip}$ and has two distinct low-energy regions in the continuum limit, at $z = \pm i$. To see how this doubled spectrum gets converted into the charge of the fermion, let’s look at the spin chain eigenmode as a function of $z$,

\[B(z) \equiv B^+(z) = C(z)a_x(z) + izC(z^{-1})a_y(z)\] (56)

where

\[C(z) \equiv (1 + kz^2)^\frac{1}{2}\] (57)

Because of the degeneracy under $z \rightarrow -z$, we may ”reduce the Brillouin zone” by defining eigenmodes which are even functions of $z$ and allowing $p$ to go from
\(-\frac{\pi}{2}\) to \(\frac{\pi}{2}\) instead of from \(-\pi\) to \(\pi\). Define the eigenmodes in the reduced zone by

\[
B_1(z) = B(z) + B(-z) \quad (58)
\]
\[
B_2(z) = z^{-1} (B(z) - B(-z)) \quad (59)
\]

Note that

\[
B_1(z) = C(z) a^e_x(z) + iz C(z^{-1}) a^o_y(z) \quad (60)
\]
\[
B_2(z) = z^{-1} C(z) a^o_x(z) + i C(z^{-1}) a^e_y(z) \quad (61)
\]

where \(a^e,o_x\) are Fourier transforms over even and odd sublattices,

\[
a^e,o_{x,y}(z) = a_{x,y}(z) \pm a_{x,y}(-z) \quad (62)
\]

Since \(B_1(z)\) and \(B_2(z)\) in the reduced zone are two independent creation operators with degenerate energy, we may construct a symmetry transformation which mixes \(B_1\) and \(B_2\). The corresponding positively and negatively charged eigenmodes are \(B_1 \pm iB_2\). From this construction, we can identify the local lattice Dirac fields, which are given by

\[
\tilde{\Psi}_1(z) = \frac{1}{\sqrt{2}} (a^e_x(z) + iz^{-1} a^o_x(z)) \quad (63)
\]
\[
\tilde{\Psi}_2(z) = \frac{1}{\sqrt{2}} (za^o_y(z) + ia^e_y(z)) \quad (64)
\]

Equivalently, we may define the lattice Dirac fermion field \(\Psi_j^{1,2}\) at site \(j\) (on a lattice with twice the lattice spacing of the original spin chain lattice) by

\[
\Psi_j^1 = \frac{1}{\sqrt{2}} (c^e_{2j} + ic^e_{2j+1}) \quad (65)
\]
\[
\Psi_j^2 = \frac{1}{\sqrt{2}} (c^y_{2j} - ic^y_{2j-1}) \quad (66)
\]

The vector fermion charge in terms of spin chain operators is thus,

\[
Q = \sum_j \left[ \Psi_j^{1\dagger} \Psi_j^1 + \Psi_j^{2\dagger} \Psi_j^2 \right] = \sum_{j\text{ even}} \left[ c^e_{2j} c^e_{2j+1} + c^y_{2j-1} c^y_{2j} \right] \quad (67)
\]

It is easy to show explicitly that \(Q\) commutes with the \(XY\) Hamiltonian,

\[
[Q, H] = 0 \quad (68)
\]
It is interesting to note that this conserved charge has appeared in the literature on the 8-vertex model. In Baxter’s original papers on the Bethe ansatz for the 8-vertex model, he introduced an SOS formulation of the model and an associated conserved “kink” number. (The existence of this conserved number of kinks is crucial for the formulation of the Bethe ansatz.) This SOS transformation was studied for the free fermion case by Jones. There it was shown that the conserved kink number associated with Baxter’s SOS transformation is precisely the operator $Q$ defined by (67).

Using the identification (65-66), the spin chain Hamiltonian may be directly transformed to a Wilson-Dirac Hamiltonian with Wilson parameter $r = 1$ and hopping parameter $\frac{1}{2}$,

$$H = \sum_j \left[ \Psi_j^{1+} \Psi_j^2 + \Psi_j^{1+} \Psi_{j+1}^2 + h.c \right]$$

(69)

$$= \sum_j \left[ \bar{\Psi}_j \Psi_j + \frac{1}{2} k \left( \Psi_j \left( 1 + i \gamma^1 \right) \Psi_{j+1} + \bar{\Psi}_j \left( 1 - i \gamma^1 \right) \Psi_{j-1} \right) \right]$$

(70)

5 Conclusion

The use of a space-time lattice to regularize relativistic quantum field theories is now commonplace in both theoretical and numerical investigations. For many nonperturbative questions, the lattice formulation is the only well-defined cutoff scheme available. Although there are typically an infinite number of different lattice theories that correspond to a given continuum theory, in practice one generally tries to retain in the lattice theory as much of the symmetry of the continuum theory as possible. The incorporation of exact global and vector-like gauge symmetries on the lattice is usually not problematic, but fundamental difficulties are encountered in the formulation of fermions interacting with chiral gauge fields, as in the electroweak Standard Model. Much effort has been devoted to understanding the chiral lattice fermion problem, and there have been recent promising developments. In a continuum field theory, the chiral structure of a fermion field may be defined kinematically in terms of its Lorentz transformation properties (i.e. the chiral components of the field are irreducible under proper Lorentz transformations). In this paper, I have shown that a two-dimensional Dirac fermion theory may be discretized as a vertex model, and that this particular discretization retains the full Lorentz symmetry of the continuum theory and merely compactifies the manifold of Lorentz frames. Further investigation of the chiral structure of this model may reveal some useful insights.
Acknowledgments

I am grateful to I. Horvath for discussion of these and related issues. This work was supported in part by the Department of Energy under grant DE-FG02-97ER41027.

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