Dynamic Critical Scaling of the Holographic Spin Fluctuations

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Criticality with strong coupling is described by a theory in the vicinity of a non-Gaussian fixed point. The holographic duality conjectures that a theory at a non-Gaussian fixed point with strong coupling is dual to a gravitational theory. In this paper, we present a holographic theory in treating the strongly coupled critical spin fluctuations in quasi-2-dimension. We show that a universal frequency over temperature scaling law is a rather general property of the critical ac spin susceptibility at strongly coupled limit. Explicit results for the dynamic scaling of spin susceptibility are obtained in large-N and large ’t Hooft limit. We argue that such critical scaling are in good agreement with a number of experiments, some of which can not be explained by any perturbative spin-density-wave theory. Our results strongly suggest that the anomalous behavior of non-Fermi liquids in materials is closely related to the spin fluctuations described through the non-Gaussian fixed point. The exotic properties of non-Fermi liquids can be viewed as the Fermi liquids coupling to strongly coupled critical spin fluctuations.

I. INTRODUCTION

The subjets of spin fluctuations have attracted wide attentions [1,2], due to the intense interests in understanding the exotic properties of the heavy fermion (HF) metals and the normal state of non-conventional superconductors. The HF metals are materials in which the effective mass of the charge carriers is often hundreds or even thousands of times greater than the mass of bare electrons. It is now clear that the large effective mass arises from the hybridization between the conducting electrons and the magnetic moments or spins of the parent compounds (Kondo effect) [3]. The phase diagram of cuprate high-Tc superconductors looks similar to that of the HF metals, except that there is a pseudogap regime separating a weakly doped anti-ferromagnetic and superconducting phase around optimal doping.

Spin fluctuations play an important role not only in its anti-ferromagnetic regimes, but also in the superconducting dome, qualitatively, d-wave Cooper pairs can be formed by exchanging spin fluctuations between electrons [4].

It is by now established that the parent compounds of these two materials, at least at weakly doping, are well described through spin dynamics by using a spin-1/2 lattice Heisenberg model with nearest-neighbor interaction. As gradually chemical doped, such materials carrying constituent magnetic moments/spin fluctuations have been observed to develop quantum criticality when their transition temperature is driven to nearly zero [5], known as the Quantum Critical Point (QCP). The spin degrees of freedoms are present at all temperatures down to the QCP [6].

As the spin fluctuations become critical, the system exhibits strongly coupled dynamics due to the destruction of Kondo screening by the critical fluctuations. In this case, the critical spin fluctuation can not trivially explained by a weakly coupled spin-density-wave [4]. The properties of the material dramatically change by the critical fluctuations and form the so-called non-Fermi liquids or ‘strange metal’ phase, the anomalous behaviors of which depart the standard Landau’s Fermi-liquid theory that successfully describing most metals, for a review see [7]. The non-Fermi liquid behavior is closely related to the strongly coupled new state of matter in the Quantum Critical Regime (QCR), which is the finite temperature extension of a QCP. Understanding the behaviors of non-Fermi liquids is a major challenge in condensed matter physics. However, the mechanisms are still highly controversial, being almost due to the lack of exact theoretical computations on such strongly coupled critical systems.

The strongly coupled critical spin fluctuation as an example of quantum critical states is a tremendous difficult problem ever studied, see e.g. [8], and an important goal of theoretical studies in understanding non-Fermi liquids. There are trials in treating the system preserving traditional notion of quasi-particles, e.g. the Fermion condensation quantum phase transition theory, see review [8]. However, restrictly speaking, there is no well-defined quasi-particles/waves any more and even no Landau’s local order parameter description in most cases [10,12], which makes the system completely beyond the scope of using traditional method and needs completely new idea.

If one notes that near the critical point the correlation length is much larger than the microscopic length scale, the details of its microscopic structures is unimportant, so the fluctuations near the criticality behaves universal, in addition to the strongly coupled dynamics, it is more promising to conjecture that the strongly coupled critical system should be described by AdS/CFT correspondence or holographic duality discovered in string theory [13,14]. There are examples and evidences show that the holographic approach is much heuristic and even trustable, one successful prediction [15,16] was that the ratio of the shear viscosity to the entropy density of the strongly coupled quark-gluon plasma near the critical transition temperature, which agrees well with the measurements of it produced from relativistic heavy ion collision [17]. Different from the conventional renormalization group analysis of critical behavior, which relies largely on the existence of a Gaussian fixed point and the validity of perturbative calculation
near the fixed point. The strongly coupled Conformal Field Theory (CFT) without doubt describes a non-Gaussian fixed point or critical point of the system beyond the perturbation theory. Our discussion of the critical dynamic scaling will be based on such non-Gaussian fixed point, which are suggested in this paper response for the anomalous behavior of non-Fermi liquids.

This holographic approach has been actively applied to wide varieties of condensed matter problems in recent years, see reviews [18, 19], e.g. holographic superconductor [20–21] and superfluidity [22–23], holographic strange metals [24–26] and non-Fermi liquid [27–29], semi-holographic Fermi liquid [30–31], and quantum criticality [32–35]. The holographic superconductor or superfluidity implements a second order phase transition or mildly crossover by condensation of strongly coupled charged scalar fields (and more complicated higher integer spin fields generalizations [37, 38]) because of the intrinsic holographic instability [39]. In the works on holographic strange metals, strongly coupled Fermionic fields with spin-1/2 are studied by holographic approach, which realizes a non-Fermi liquid behavior with linear temperature resistivity. The semi-holographic Fermi liquid theory simplifies and generalizes the idea from holographic strange metals, introducing a hybridization between free fields and strongly coupled critical modes at infrared described by AdS$_2$ near the horizon of black hole.

The purpose of the paper is to examine the critical dynamic scaling behavior of the correlation function of strongly coupled critical spin fluctuations by holographic approach, the calculations lead to variety of testable predictions, and we try to connect our calculations to experimental observations at least qualitatively. A discussion for the correlation function of dual currents in 2+1 dimensional can be found in [40]. However, in this paper, we find the 3+1 dimensional holographic model is a more realistic model, because (i) the 3+1D holographic spin system is naturally a quasi-2-dimensional (quasi-2D) system similar to the real 2D-layer of a compound, as a consequence, the spin susceptibility is only a scaling of frequency over temperature ($\omega/T$) independent to wave vector in the quasi-2D plane; (ii) the scaling of spin ac susceptibility $\chi \sim (\omega/T)^\alpha$ takes a value $\alpha = 1$ at low frequency hydrodynamic limit, and $\alpha = 2/3$ at high frequency relativistic limit which can not be explained by any spin-density-wave QCP theory; (iv) the dc spin susceptibility approaches a constant independent to temperature; (v) these scaling behaviors are very crucial in understanding the exotic properties of non-Fermi liquid, and we show that our results are in good agreement with a number of experiments. The strongly coupled critical spin fluctuations in our paper play similar role with the infrared critical modes discussed in the semi-holographic Fermi liquid theory [30, 31]. The bulk of the paper is devoted to making these statements more precise.

The paper is organized as follows. We present the general arguments of holographic spin dynamics in Sec.II. In Sec.III we formulate the technical detail of the approach and evaluate the scaling exponent of spin susceptibility in three analytically tractable limits. In Sec.IV we discuss the comparison of our results with experimental phenomenon in two aspects, including qualitatively explaining the scaling law of spin susceptibility at criticality and linear-temperature resistivity in strange metals. Finally, we conclude in Sec.V.

II. HOLOGRAPHIC SPIN DYNAMICS

A. Non-Gaussian Fixed Point

A theoretical model for considering the spin dynamics doped with conduction electrons is described by the Kondo lattice model,

$$\mathcal{H} = \sum_{(i,j),s} t_{ij} c_{is}^\dagger c_{js} + J_K \sum_{i,s,s',s''} c_{is}^\dagger c_{i,s',s''} \cdot S_i + \sum_{(i,j)} I_{ij} S_i \cdot S_j,$$

(1)

where $c_i$ is the electron operator and $S_i$ is the on site spin operator at lattice site $i$. The first term is the hopping of the conduction electrons between neighbor site $i$ and $j$. At each lattice site, the local spin $S_i$ interacts via an exchange coupling $J_K$ with the spin of conduction electrons sitting at the site. The local spin $S_i$ interacts with the nearest neighbor spin $S_j$ by the Ruderman-Kittel-Kasuya-Yosida (RKKY) via exchange interaction $I_{ij}$.

The effective Hamiltonian for the on site local spin can be obtained by integrating out the fermionic degrees of freedom, i.e. the conduction electrons $c_{i,s}^\dagger, c_{i,s}$, and then all the effects of the conduction electron are absorbed into the the new effective exchange interaction $I_{ij}$, we get the effective spin dynamics described by the Heisenberg Hamiltonian,

$$\mathcal{H}_{eff} = \sum_{(ij)} J_{ij} S_i \cdot S_j.$$

(2)

The action of such effective Heisenberg model in the vicinity of critical point can be reduced to an equivalent
continuum field theory, written in terms of stagger order parameter,

\[ S_0[n] = \int d\tau dr \left[ (\nabla_r n)^2 + (\partial_r n)^2 + a n^2 + b (n^2)^2 \right], \tag{3} \]

in which

\[ n(r) = e^{iQ \cdot r} S_i \tag{4} \]

is the anti-ferromagnetic order parameter and \( Q \) is wavevector with each component at \( Q = \pi \).

The criticality of the model in Eq. (1) or Eq. (3) could develop via the competition between the fermionic and bosonic environments interacting with the local spin: the quenching of the local spin through its Kondo coupling to the conduction electron bath and the coupling between local spin and fluctuating magnetic field generated by the local spins at all other sites. At the QCP, the effects of the fermionic and bosonic baths come into balance, and the bosonic coupling succeeds in preventing the local spins from completely quenching, leading to a singular spin susceptibility.

The mechanism that develops criticality reflecting in the effective action Eq. (3) can be manifested in a more transparent way. Since the fermionic bath have been integrated out in the effective action, the criticality appears directly from the bosonic action. A rough observation in Eq. (3) shows that there exists two types of fundamentally different fixed points or critical points in the system. The first one is trivial, when \( a \to 0, b \to 0 \), which represents a trivial Gaussian fixed point without interaction, corresponding to the quenched free spin wave. It is worth noting that the Wilson-Fisher fixed point found at low dimension \( d<4 \) also resides in its perturbative character which is applicable only when the coupling \( b \) is sufficiently small, and hence the spin fluctuations also behave as quasi-particles. However, according to the AdS/CFT duality, it is conjectured that there exists a non-Gaussian fixed point when the interaction strength \( b \) is non-zero and takes large value. It corresponds to a strong tendency to polarize the local spin along the direction of the fluctuating magnetic field generated by the effective spin environment and hence represents a strong self-coupling between spin fluctuations. A comprehensive interpretation is as follows: the coupling with fermionic bath leads to a negative contribution to the typical energy scale characterized by parameter \( a \), while the bosonic coupling gives it a positive contribution, so when the coupling takes certain large value, the typical energy scale vanishes and the system manifests as an interaction criticality.

**B. Holographic Description of the Non-Gaussian Fixed Point**

These two types of fixed points relating to weak and strong coupling strength belong to different universality classes with different critical exponents. The dynamic critical exponents of the strongly coupled non-Gaussian fixed point is the major interest of the paper, since what we concern in this paper is the materials in the QCR exhibiting strongly coupled dynamics.

In the vicinity of the strongly coupled non-Gaussian fixed point, the theory is beyond the standard weak coupling technique and notoriously difficult to solve, the action Eq. (3) is then no longer directly useful for analytic perturbative calculations. However, the critical nature of the strongly coupled non-Gaussian fixed point indicates that it is a strongly coupled Conformal Fields Theory (CFT) which is conjectured to be dual to a string/M theory in an AdS space at large spin component limit. That is to say, the theory described by the holographic approach is a theory in the vicinity of the non-Gaussian fixed point. As a consequence, there must exists a holographic description to the large-N critical spin system at strongly coupled limit.

The most fundamental question of the critical spin fluctuation concerns the two-point retarded correlation function of spins, i.e. the spin susceptibility, which measures in response to weak applied external magnetic fields. It requires introduction of weak external magnetic fields \( B \) into the model by coupling the spin to the \( B \) fields

\[ S = S_0 + \int d\tau dr S \cdot B. \tag{5} \]

In this paper, we will study the spin susceptibility defined by

\[ \chi_{ij}(\omega, q) = -i \int d\tau d\theta(t) \langle [S_i(t, r), S_j(0, 0)] \rangle e^{-i\omega t + iq \cdot r}. \tag{6} \]

Obviously, if the system is anti-ferromagnetic as the action Eq. (3) refers to, the correlation function of the order parameter \( n \) must be anti-ferromagnetic. We can also define a spin susceptibility by the anti-ferromagnetic order
parameter. By using the Eq.(4), we have the relation between our definition Eq.(6) and the correlator in terms of the order parameters by
\[
\chi_{ij}(\omega, q)\delta(Q) = -i \int dt \theta(t) \langle [n_i(t, \mathbf{r}), n_j(0, 0)] \rangle e^{-i\omega t + i\mathbf{q} \cdot \mathbf{r}}.
\] (7)
in which the delta-function peaking at Q exhibits the expectation value is taken in an anti-ferromagnetic system.

The susceptibility defined in Eq.(6) can be directly obtained by evaluating the functional derivative of the partition function with respect to the external weak magnetic fields, according to the linear response theory,
\[
\chi_{ij} = 2 \left( \frac{\delta^2 \ln Z}{\delta B_i \delta B_j} \right),
\] (8)
where \(V\) is the volume of the spin system. Applying the AdS/CFT duality, the partition function of the strongly coupled critical spin fluctuations can be calculated from the gravitational side, in large-N and large \('t\) Hooft coupling limit the gravitational theory reduces to a classical gravity, the classical action gives a dominant contribution to the partition function by the saddle point approximation,
\[
Z[B] = \langle e^{-\int_{\partial M} S \cdot B_0} \rangle_{NG} = e^{-S_{cl}[B]},
\] (9)
where the subscript “NG” denotes the expectation value is taken at the non-Gaussian strongly coupled fixed point, \(\partial M\) stands for the boundary of the AdS space, i.e. our flat spacetime, \(B_0\) is the boundary value of the bulk magnetic fields \(B\). \(S_{cl}\) is the classical action for the magnetic fields propagating in an asymptotic AdS metric,
\[
S_{cl}[F] = -\frac{1}{4g_{YM}^2} \int d^{d+1}x \sqrt{-g} F_{IJ} F^{IJ},
\] (10)
where \(F_{IJ}\) is the field strength, and \(g_{YM}^2 = 16\pi^2 R/N^2\) the gauge coupling. So we have holographic version of spin susceptibility
\[
\chi_{ij} = 2 \left( \frac{\delta^2 S_{cl}}{\delta B_0 \delta B_{0j}} \right).
\] (11)

The strongly coupled spin fluctuations lived on the boundary of the AdS space share features of the critical theory of physical interest, there are no well-defined notion of quasi-particle, it is insensitive to microscopic details and displays universal behavior, the surprising success of the holographic approach at low energies also exhibit universality in its predictions. The holographic spin system captures the key features of the strongly coupled critical spin fluctuations. Historically, scaling plays a central role in the studying of criticality, so in the following section, we will turn to the calculation of the scaling behavior of the holographic spin fluctuations.

III. SCALING OF THE SPIN SUSCEPTIBILITY

A. The Calculation Framework

A method for calculating this quantity of R-current in the dual gravitational description was formulated in Ref [41]. In this paper, we consider a spin system in quasi-2 dimension, a proper framework for such system is not 2+1 but a 3+1 dimensional system in Minkovski spacetime \((M_4)\) as a direct consequence of the transverse nature of magnetic fields, which will be shown in the following discussion in more detail.

The large-N and large \('t\) Hooft coupling limit of the theory in \(M_4\) at finite temperature \(T\) corresponds to a gravitational background with a 4+1-dimensional asymptotically AdS metric embedding a Schwarzschild blackhole
\[
ds^2 = \left( \frac{\pi T R}{u} \right)^2 \left[ -f(u) dt^2 + dx^2 + dy^2 + dz^2 \right] + \frac{R^2}{4u^2 f(u)} du^2,
\] (12)
where \(f(u) = 1 - u^2\), \(u\) values from 0 (boundary) to 1 (horizon with Hawking temperature \(T\)), and \(R\) is the curvature radius of the AdS space. According to the duality prescription, the two-point function of spins in the holographic spin system is calculated by analyzing linear correspondence of magnetic fields propagating in the 4+1-dimensional AdS-Schwarzschild gravitational background. The perturbations of the magnetic fields obey the Maxwell’s equation
\[
\partial_I \left( \sqrt{-g} g^{IJ} g^{KL} F_{JK} \right) = 0,
\] (13)
where $g_{ij}$ is the metric of the background. The magnetic fields defined as $B_i = \frac{1}{4} \epsilon_{ijk} F_{jk}$ is a pseudo-vector propagating in the bulk space. The Fourier transformation of it by its boundary coordinates is given by

$$B_i(u, t, z) = \int \frac{d\omega dq_z}{(2\pi)^2} e^{iq_z z - i\omega t} B_i(u, \omega, q_z),$$  \hfill (14)

in which we have lied the wave vector $q$ along the $z$-direction for simplicity. By using the Bianchi identity for $F_{i,j}$ to relate magnetic $B$ fields to the electric $E$ fields, we obtain the wave equation

$$B''_T + \frac{j''}{j} B'_T + \frac{\tilde{\omega}^2 - \tilde{q}^2}{2f} B_T = 0,$$  \hfill (15)

in which $B_T = B_x, B_y$ denotes the transverse components of the magnetic fields, we also define dimensionless frequency and wave vector

$$\tilde{\omega} = \frac{\omega}{2\pi T}, \quad \tilde{q} = \frac{|q_z|}{2\pi T},$$  \hfill (16)

and the primes stand for derivatives with respect to $u$.

Note that the magnetic fields are always perpendicular to the direction of the wave vector (in the $z$-direction) due to their transverse nature, so all the dual physical effects of the spin system are constrained in the $x$ - $y$ quasi-2-dimensional plane, which means the spin fluctuations (duals to the transverse magnetic fields) are in fact constrained in 2D layer, but it is “quasi-” since it depends on external parameters in the $z$ dimension. For a fixed value of the external parameters, the system can be viewed as pure 2D. Now let us turn to the Maxwell action of the quasi-2D system in terms of magnetic fields

$$S_{el}[B] = -\frac{N^2 T^2}{16} V_2 \int du f(u) \int d\omega dq_z \left[ \frac{1}{(2\pi)^2 q_z^2} [B''_z(u, \tilde{\omega}, \tilde{q}) + B''_y(u, \tilde{\omega}, \tilde{q}) + \ldots] \right]$$  \hfill (17)

where $V_2$ is the area of this quasi-2D $x$ - $y$ plane.

Applying the prescription proposed by Son and Starinets [41], using Eq.(11), one finds the transverse spin susceptibility of the isotropic quasi-2D system, ($\chi_{zz} = 0$, $\chi_{xy} = \chi_{yx} = 0$, $\chi_{xx} = \chi_{yy} = \chi$)

$$\chi = c \Phi(\tilde{\omega}, \tilde{q}),$$  \hfill (18)

in which $c$ is a dimensionless coefficient and the universal scaling function $\Phi(\tilde{\omega}, \tilde{q})$ governed by the conformal nature of the holographic calculations

$$c = \frac{N^2}{32\pi^2 q_z^2}, \quad \Phi(\tilde{\omega}, \tilde{q}) = \lim_{u \to 0} \frac{B'_T(u, \tilde{\omega}, \tilde{q})}{B_T(u, \tilde{\omega}, \tilde{q})}.$$  \hfill (19)

Obviously, the $\chi$ is momentum-independent in the quasi-2D system ($\tilde{q}$ can be viewed as an external parameter or integration constant determined from initial condition). It is very natural that $\tilde{q}$ takes certain fixed value when $\tilde{\omega}$ is non-vanished in ac susceptibility, unless uniform and static limit is required to carefully take in dc susceptibility. So in the following discussions, we are only interested in the frequency over temperature dependent, i.e. $\Phi(\tilde{\omega})$.

One of the main purpose of the paper is to describe the critical scaling behavior of the universal function $\Phi(\tilde{\omega})$ and/or spin susceptibility, particularly in three exactly tractable asymptotics: (i) The low frequency/high temperature hydrodynamic limit ($\tilde{\omega} \ll 1$). (ii) The high frequency/low temperature limit in which we have assumed that the modes at high frequency are highly relativistic with linear dispersion ($1 \ll \tilde{\omega} = \tilde{q}$). (iii) The uniform and static limit dc susceptibility ($\tilde{q}, \tilde{\omega} \to 0$).

**B. Low Frequency Hydrodynamic Limit ($\tilde{\omega} \ll T$)**

The low frequency/high temperature asymptotics is a straightforward application of the perturbation theory to the wave equations Eq.(15). The solution obeying the incoming wave boundary condition at the horizon ($u = 1$) is controlled by a singular prefactor $(1 - u)^{-i\tilde{\omega}/2}$. Then the solution can be given perturbatively by using $\tilde{\omega}$ and $\tilde{q}$ as small expansion parameters

$$B_T(u) = C(1 - u)^{-i\tilde{\omega}/2} \left[ 1 + \frac{i\tilde{\omega}}{2} \ln \frac{1 + u}{2} + \frac{\tilde{q}^2}{2} \left( \frac{\pi^2}{2} - Li_2(-u) + \ln u \ln(1 + u) + Li_2(1 - u) \right) + O(\tilde{\omega}^2, \tilde{q}^4, \tilde{\omega}\tilde{q}^2) \right],$$  \hfill (20)

where $Li_n(x)$ is the polylogarithmic function.
where the renormalization constant $C(\tilde{\omega}, \tilde{q})$ is determined by the boundary condition $\lim_{u \to 0} B_T(u) = B_T^0$,

$$C(\tilde{\omega}, \tilde{q}) = \frac{8B_T^0}{8 - 4\tilde{\omega}\ln 2 + \pi^2\tilde{q}^2}. \quad (21)$$

we get the derivative of the B fields on the boundary $(u \to 0)$,

$$\lim_{u \to 0} B'_T = i\tilde{\omega}B_T^0. \quad (22)$$

At lowest order, the result is momentum independent, so the spin susceptibility has a simple scaling behavior $\Phi_h(\tilde{\omega}) = i\tilde{\omega}$, this can be seen from Eq.(18) and Eq.(19),

$$\chi_h(\tilde{\omega}) = ci\tilde{\omega}, \quad (\tilde{\omega} \ll 1). \quad (23)$$

We will see in the next section that this $\alpha = 1$ dynamic scaling law at low frequency/high temperature limit is very important in understanding the behaviors of a quantum critical regime, which is considered as a finite temperature extension of a quantum critical point.

### C. High Frequency Relativistic Limit ($T \ll \omega = q$)

To investigate the quantum critical point at near zero temperature, we need to study the high frequency/low temperature limit. This limit of the solution requires a careful WKB analysis. In this subsection, we assume that in this high frequency limit, the modes are relativistic with linear dispersion $\omega = q$, so the Eq.(15) becomes

$$B''_T + \frac{f'}{f}B'_T + \frac{u\tilde{\omega}^2}{f^2}B_T = 0. \quad (24)$$

For $\tilde{\omega} \gg 1$, by using Langer-Olver’s method [42, 43], we are able to obtain uniform asymptotics expansions to the solution (a version of the WKB approximation). If we introduce a new variable,

$$B_T = \frac{1}{\sqrt{-f(u)}} \phi, \quad (25)$$

then the wave equation is rewritten as

$$\phi'' = -\frac{u\omega^2}{(1-u^2)^2} \phi. \quad (26)$$

For large values of $\tilde{\omega}$, the solution has the formal expansions in terms of Airy function, which is chosen to obey the incoming wave boundary condition,

$$B_T(u) \sim C \left[ \frac{-u}{(1-u^2)^2 \zeta(-u)} \right]^{-1/4} \text{Ai} \left( \tilde{\omega}^{2/3} \zeta(-u) \right) + ... \quad (27)$$

where $\text{Ai}(z)$ is the Airy function, and

$$\zeta(x) = \frac{3^{2/3}}{2^{1/3}} \left( i\pi - 2 \arctan \sqrt{x} + \ln \frac{\sqrt{x} + 1}{\sqrt{x} - 1} \right). \quad (28)$$

The renormalization constant $C(\tilde{\omega})$ is

$$C(\tilde{\omega}) = 2\sqrt{\pi}e^{\pi/4} \tilde{\omega}^{-1/6}2^{-\omega-\tilde{\omega}/2}e^{i\pi\tilde{\omega}/4}B_T^0. \quad (29)$$

Similarly, by using Eq.(19), a fractional exponent $\alpha = 2/3$ is finally obtained, i.e. $\Phi_r(\tilde{\omega}) \propto \tilde{\omega}^{2/3}$, and

$$\chi_r(\tilde{\omega}) \sim c \frac{3^{1/3} \Gamma \left( \frac{4}{3} \right)}{\Gamma \left( \frac{1}{3} \right)} (-\tilde{\omega})^{2/3}, \quad (\tilde{\omega} \gg 1). \quad (30)$$

Different from the low frequency limit where the scaling is momentum-independence, in fact, numerical study manifests that the scaling function $\Phi(\tilde{\omega}, \tilde{q})$ is sensitive to $\tilde{q}$ in the high frequency limit. At first sight the $\alpha = 2/3$ scaling we obtained here is achieved by taking fine-tuned linear dispersion in Eq.(24), but it shows that the dispersion of vector mode at high frequency tends to linear [44], so $\omega = q$ at high frequency limit is expected as a promising pre-assumption for most real cases.
D. DC Susceptibility or Uniform Static Limit ($q, \omega \to 0$)

In principle the uniform ($q \to 0$) and static ($\omega \to 0$) limit can be taken straightforwardly in the low frequency hydrodynamic regime,

$$\chi_s = \lim_{\omega, q \to 0} \chi_h(\tilde{\omega}, \tilde{q}). \quad (31)$$

However, the $\tilde{q}$ dependence of the prefactor $c$ requires a more careful treatment, since $\tilde{q}$ of the modes will also tend to vanish as $\tilde{\omega}$ goes to zero because of the dispersion relation. The limit exists and be finite, if we note that $\tilde{\omega}$ and $\tilde{q}^2$ are of the same order in this hydrodynamic regime, and more precisely, there is a diffusive pole $i\tilde{\omega} = \tilde{q}^2 \left( i\omega = D\tilde{q}^2 \right.$ with $D = 1/2\pi T)$ governs the low energy dispersion for the longitudinal vector modes who share the same wave vector and frequency with the transverse vector modes in hydrodynamic regime. Therefore, we obtain a universal real constant susceptibility [45], independent with temperature,

$$\chi_s = \frac{N^2}{32\pi^2} \lim_{\omega, q \to 0} \frac{i\tilde{\omega}}{\tilde{q}^2} = \frac{N^2}{32\pi^2}. \quad (32)$$

The non-zero dimensionless constant dc spin susceptibility at high temperature for strongly coupled critical spin fluctuations is a non-trivial prediction from holographic theory at the critical regime. The universality of the strongly coupled critical spin susceptibility measuring the critical spin transport is a direct consequence of the “perfect fluid” behavior of a critical matter, which may be analogous to the universal value of the shear viscosity over entropy density of the strongly coupled quark-gluon plasma around critical temperature observed in heavy-ion-collision.

IV. CONNECTION TO EXPERIMENTS

The spin susceptibility we obtained from holographic large-N calculation is sensitive to the total number of degrees of freedom N, and hence it is not a good quantity to compare with measurements, but in this paper we propose two ways to connect our calculations to experimental facts: (i) the scaling exponent is universal and N-independent at lowest order in the holographic calculations, so it can be directly compared to experiments. (ii) The itinerant electrons coupling to the spin fluctuations give rise to a contributions $1/N^2$ in large-N theory, recalling that the order of spin susceptibility Eq. (19) is $N^2$, this reflects that the resistivity and self-energy of the N-component electrons due to exchange the spin fluctuations are of order $N^0$, and hence they are manifested in insensitive to the large-N technique.

A. Scaling of Critical Spin Susceptibility in Experiments

The results of scaling behavior for strongly coupled critical spin fluctuations lead to a variety of direct measurement of critical spin susceptibility by neutron-scattering, NMR, and magnetometry measurements. The low frequency hydrodynamic limit of the system can be measured in the phase diagram at large range of temperatures close to the doping/fields induced criticality, i.e. the QCR. The Eq. (23) suggests that, when frequency of the ac magnetic fields is fixed, the spin susceptibility decreases as the temperature increases. The simple $\omega/T$ scaling of spin susceptibility at low frequency and/or high temperature proposed in Eq. (23) is observed in La$_2$-Sr$_x$CuO$_4$ compound (at critical doping $x \approx 0.04 \pm 0.01$) in the pseudogap regime (which is conjectured as a QCR) [46], and it is also suggested as a phenomenological description in normal state of Cu-O high-Tc superconductor [47].

$$\text{Im} \chi \sim \frac{\omega}{T}, \quad (\omega \ll T). \quad (33)$$

When temperature is driven to small value, at the critical doping or QCP, the fluctuations become strong and almost quantum critical. In this case, we are interested in the $\chi \sim T^{-\alpha}$ type of scaling measured at fixed frequency, which is considered associated with the proximity to a QCP. We find the spin susceptibility diverges governed by a fractional scaling exponent $\alpha = 2/3$ in the high frequency or low temperature regime. This prediction of scaling from Eq. (23) is supported from the experimental observation in e.g. YbRh$_2$(Si$_{1-x}$Ge$_x$)$_2$ ($x \approx 0.05$) when reached a QCP. Although $\chi(T)$ tends to saturation below Kondo temperature, approaching to QCP from $0.3K$ to $10K$, it can be approximated by a power-law divergence [43],

$$(\chi(T) - \text{const}) \sim T^{-0.6}, \quad (T \sim 0.3K \div 10K), \quad (34)$$
where \( \text{const} = 0.215 \times 10^{-6} \text{m}^3 \text{mol}^{-1} \) is a small temperature-independent contribution. We can see at high \( T \), \( \chi \to \text{const} \). In our framework the constant may be interpreted as a constant lower bound of static spin susceptibility as suggested in Eq. (52).

The fractional scaling law of temperature has also been seen in the kagome lattice anti-ferromagnetic herbertsmithite ZnCu3(OH)6Cl2 at fixed frequency of weak ac magnetic fields, which is thought of possibly displaying quantum critical behavior and the proximity to a critical spin liquid ground state. For both real and imaginary part of susceptibility, from 0K to 35K, the divergence scaling behaves as\[\chi(T) \sim T^{-(0.66 \pm 0.02)}, \quad (T \sim 0K \div 35K).\] (35)

Other neutron scattering experiments on CeCu2−xAu6 near the critical doping \( x_c \approx 0.1 \) also gives an simple \( \omega/T \) scaling and a similar fractional critical exponent, fitting \( \chi \sim T^{-\alpha} \) with \( \alpha \approx 0.75 \) closed to our result [50].

Within the experimental resolution, it is a good agreement in our calculation to these measurements. Note that such fractional scaling can not be explained by any perturbative spin-density-wave QCP theories, and hence be a good support for our holographic treatment of the critical spin fluctuations. The author also recognize that there are other candidates, e.g. non-perturbative phenomenological approach [51] and dynamical mean field theory [52] yielding the similar fractional exponent.

### B. Linear-T Resistivity in Normal State of Cuprates

The normal state or strange metal phase of the cuprate is regarded as the central dogma in the theory of high-Tc superconductivity. Different from the \( R \sim T^2 \) behavior of resistivity due to phonon scattering in usual Fermi liquid metals, in cuprate layers, a linear-T resistivity \( R \sim T \) near optimal doping (as critical doping) over a wide range of temperature is a generic property of the normal state. The fact that the constant of proportionality seems similar for different cuprates may be a hint suggesting such behavior is closely related to the universality of critical phenomenon. We will show that the scaling behavior of spin susceptibility at low frequency/high temperature limit is very important in understanding the anomalous behavior, if we replace the phonons by the critical spin fluctuations.

\[ \chi(T) \sim T^{\alpha}, \quad \alpha \approx 0.75 \] (36)

in which the density of state is approximately a constant density in a unit volume at the Fermi surface \( \rho_F \). We extend the validity of \( B = -2\text{Im} \chi = -2\epsilon \omega \) to all range of frequency in our integral, since here it is treated in the high-T strange metal phase. A straightforward calculation then gives

\[ \text{Im} \Sigma(\epsilon, T) = J_K^2 N_0^2 \rho_F c T \Gamma \left( \frac{\epsilon}{T} \right), \] (39)
where $\mathcal{F}(\epsilon/T)$ is given by

$$\mathcal{F}(\frac{\epsilon}{T}) = \frac{\pi^2}{4} + \frac{\epsilon}{T} \ln 2 + ... \quad (\epsilon \ll T) \quad (40)$$

which approaches a constant for small $\epsilon/T$, we have

$$\text{Im} \Sigma(\epsilon, T) = \lambda T, \quad (\epsilon \ll T) \quad (41)$$

where $\lambda = \frac{1}{128} J^2 K \rho F / \tilde{q}^2$ is a dimensionless effective coupling constant of order $N^0$. By using the Kramers-Kronig relations again we obtain the real part and the full self-energy from the imaginary part

$$\Sigma(\epsilon, T) = -\frac{2}{\pi} \lambda \left( \epsilon \ln \left| \frac{x}{\epsilon_c} \right| - \left( \frac{\pi}{2} \right) \right), \quad (42)$$

where $x = \text{max}(\epsilon, T)$, $\epsilon_c$ is a ultraviolet cutoff scale. This form of self-energy has been proposed in the Marginal-Fermi-Liquid [47] and fits well with the ARPES measurement of cuprates in its high-T strange metal phase [53].

According to the Optical Theorem, the imaginary part of the self-energy is related to the scattering amplitude between the free electron and the critical spin fluctuations. Therefore, in the quasi-2D plane, we find the electric dc resistivity behaves linear in temperature,

$$R = \lambda T. \quad (43)$$

It is a generic property observed in normal state, strange metal and many non-Fermi liquids materials in 2D conductive layer, which are expected to exhibit QCR behavior related to the optimal doping, rather a $T^2$ behavior in Fermi liquids. In this sense, we could conclude that the non-Fermi liquids could arise from coupling a Fermi liquids to a strongly coupled critical spin fluctuations.

V. CONCLUSIONS

We conclude this paper by recollecting some highlights to the problem of the strongly coupled critical spin fluctuations. The dynamics of spin fluctuations is very important in understanding the behavior of normal state of the heavy fermion metal and high-Tc superconductor. The strongly coupled spin fluctuations develop quantum criticality and the notion of quasi-particle is no longer valid. Such critical system is described by continuum field theory in the vicinity of a non-Gaussian fixed point with strong coupling. The treatment of the system is beyond the traditional perturbative technique. We present a holographic theory to the system by following the AdS/CFT correspondence which conjectures that the strongly coupled critical system is dual to a gravitational theory in AdS space at large-N limit. We calculate the spin susceptibility in such holographic system at large 't Hooft coupling, and find that (i) the holographic spin system is quasi-2-dimensional; (ii) the $\omega/T$ scaling of spin susceptibility independent to wave vector is a general property of holographic spin fluctuations in the quasi-2-dimensional system; (iii) the scaling behavior of ac spin susceptibility is universal, $\chi \sim \omega/T$ at low frequency hydrodynamic limit and $\chi \sim (\omega/T)^{2/3}$ at high frequency relativistic limit, which can not be explained by traditional perturbative spin-density-wave QCP theories; (iv) the dc spin susceptibility approaches a constant $N^2/32\pi^2$ independent to temperature; (v) these scaling law shown in the susceptibility agree well with a number of experimental measurements, in which the test materials are tuned to nearly critical doping where quantum critical point/quantum critical regime appears; (vi) The $\text{Im} \chi \sim \omega/T$ scaling at low frequency/high temperature limit gives rise to a linear-temperature resistivity in strange metals and/or normal state of high-Tc superconductor. We argue that the non-Fermi liquid can arise from coupling the Fermi liquid to strongly coupled critical spin fluctuations.

Acknowledgments

The author would like to thank S.Sachdev, S.Hartnoll and H.Liu for helpful communications.

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