A MODULI SPACE FOR SUPERSINGULAR ENRIQUES SURFACES

KAI BEHRENS

Abstract. We construct a moduli space of adequately marked Enriques surfaces that have a supersingular K3 cover over fields of characteristic $p \geq 3$. We show that this moduli space exists as a quasi-separated algebraic space locally of finite type over $\mathbb{F}_p$. Moreover, there exists a period map from this moduli space to a period scheme and we obtain a Torelli theorem for supersingular Enriques surfaces.

Introduction

Over the complex numbers there exists a Torelli theorem for K3 surfaces in terms of Hodge cohomology. Moreover, for K3 surfaces together with a polarization of fixed even degree there is a coarse moduli space which is a quasi-projective variety of dimension 19 over $\mathbb{C}$ [Pvv71], [BR75].

Over a field of characteristic $p \neq 2$, Enriques surfaces are precisely the quotients of K3 surfaces by fixed point free involutions. Using this connection between Enriques surfaces and K3 surfaces, Namikawa proved a Torelli theorem for complex Enriques surfaces and showed that there is a 10-dimensional quasi-projective variety which is a coarse moduli space for complex Enriques surfaces [Nam85]. If $Y$ is an Enriques surface, then its Neron-Severi group $\text{NS}(Y)$ is isomorphic to the lattice $\Gamma' = \Gamma \oplus \mathbb{Z}/2\mathbb{Z}$ with $\Gamma = U_2 \oplus E_8(-1)$. By the Torelli theorem for complex K3 surfaces, fixed point free involutions of a K3 surface $X$ can then be characterized in terms of certain embeddings $\Gamma(2) \hookrightarrow \text{NS}(X)$.

Now we turn to characteristic $p > 2$. For Enriques surfaces that are quotients of ordinary K3 surfaces over perfect fields of positive characteristic, that is K3 surfaces $X$ with $h^1(X) = 1$, Laface and Tirabassi recently proved a Torelli theorem [LT19].

For supersingular K3 surfaces over an algebraically closed field of characteristic at least 3, crystalline cohomology plays a role similar to the role of Hodge cohomology in characteristic zero. Ogus proved a Torelli theorem for supersingular K3 surfaces [Ogu83] which shows that supersingular K3 surfaces are determined by their corresponding K3 crystals. For a K3 lattice $N$, an $N$-marking of a supersingular K3 surface $X$ is an embedding of lattices $\gamma: N \hookrightarrow \text{NS}(X)$. Supersingular K3 surfaces are stratified by the Artin invariant $\sigma$, where $-p^{2\sigma}$ is the discriminant of $\text{NS}(X)$. We always have $1 \leq \sigma \leq 10$ [Art74].

A version of Ogus’ Torelli theorem states that for families of $N$-marked supersingular K3 surfaces of Artin invariant at most $\sigma$ there exists a fine moduli space $S_\sigma$ which is a smooth scheme of dimension $\sigma - 1$, locally of finite type, but not separated. There is an étale surjective period map $\pi_\sigma: S_\sigma \rightarrow \mathcal{M}_\sigma$ from $S_\sigma$ to a period scheme $\mathcal{M}_\sigma$. The latter is smooth and projective of dimension $\sigma - 1$ and is a moduli space for marked K3 crystals. The functors represented by $S_\sigma$ and $\mathcal{M}_\sigma$ have interpretations in terms of so-called characteristic subspaces of $pN'/pN$. If $X$ is a supersingular K3 surface over an algebraically closed field of characteristic $p \geq 3$ and $\iota: X \rightarrow X$ is a fixed point free involution, we write $G = \langle \iota \rangle$ for the cyclic group of order 2 which is generated by $\iota$.

Definition. A quotient of surfaces $X \rightarrow X/G = Y$ defined by such a pair $(X, \iota)$ is called a supersingular Enriques surface $Y$. The Artin invariant of a supersingular Enriques surface $Y$ is the Artin invariant of the supersingular K3 surface $X$ that universally covers $Y$.

In this article we construct a fine moduli space for marked supersingular Enriques surfaces. More precisely, writing $\mathcal{A}_{\mathbb{F}_p}$ for the category of algebraic spaces over $\mathbb{F}_p$, $N_\sigma$ for a fixed K3...
lattice of Artin invariant $\sigma$ and $\Gamma' = \Gamma \oplus \mathbb{Z}/2\mathbb{Z}$ as above, we study the functor

$$\mathcal{E}_\sigma: \mathcal{A}^{\text{gp}}_{\mathbb{F}_p} \rightarrow \text{(Sets)}$$

$$S \mapsto \left\{ \begin{array}{l}
\text{Isomorphism classes of families of } \Gamma'\text{-marked supersingular Enriques surfaces } (f: \mathcal{Y} \rightarrow S, \tilde{\gamma}: \Gamma' \rightarrow \text{Pic}_{\mathcal{Y}/S}) \\
\text{such that the canonical K3 cover } \mathcal{X} \rightarrow \mathcal{Y} \\
\text{admits an } N_{\sigma}\text{-marking}
\end{array} \right\}. $$

Using the supersingular Torelli theorem, we attack this moduli problem by starting with the moduli space for $N$-marked supersingular K3 surfaces. Similar to the construction in the complex case by Namikawa [Nam85], we regard Enriques surfaces as equivalence classes of certain embeddings of $\Gamma(2)$ into the Néron-Severi lattice of a K3 surface. Over the complex numbers this means that Namikawa obtains the moduli space of Enriques surfaces by taking a certain open subscheme of the moduli space of K3 surfaces and then taking the quotient by a group action.

However, the supersingular case is more complicated than the situation over the complex numbers. One of the main problems we face is the fact that in our situation we are, morally speaking, dealing with several moduli spaces $\mathcal{S}_i$ nested in each other, with group actions on these subspaces. We use different techniques from [TT16] and [Ryd13] concerning pushouts of algebraic spaces and quotients of algebraic spaces by group actions, and finally obtain the following result.

**Theorem.** The functor $\mathcal{E}_\sigma$ is represented by a quasi-separated algebraic space $\mathcal{E}_\sigma$ which is locally of finite type over $\mathbb{F}_p$, and there exists a separated $\mathbb{F}_p$-scheme $\mathcal{Q}_\sigma$ of finite type and AF, and a canonical étale surjective morphism $\pi^E_\sigma: \mathcal{E}_\sigma \rightarrow \mathcal{Q}_\sigma$.

Here, a scheme $X$ is called AF, if every finite subset of $X$ is contained in an affine open subscheme of $X$.

The geometry of the space $\mathcal{E}_\sigma$ is complicated in general, but we have some results on the number of its connected and irreducible components. In short, these numbers depend on properties of the lattice $N_{\sigma}$ and we refer to Section 6 for details.

Since the scheme $\mathcal{Q}_\sigma$ in the theorem above was constructed from the scheme $\mathcal{M}_\sigma$, we also obtain a Torelli theorem for Enriques quotients of supersingular K3 surfaces.

**Theorem.** Let $Y_1$ and $Y_2$ be supersingular Enriques surfaces. Then $Y_1$ and $Y_2$ are isomorphic if and only if $\pi^E_\sigma(Y_1) = \pi^E_\sigma(Y_2)$ for some $\sigma \leq 5$.

The period map $\pi^E_\sigma$ is defined in the following way: the scheme $\mathcal{Q}_\sigma$ represents the functor that associates to a smooth scheme $S$ the set of isomorphism classes of families of K3 crystals $H$ over $S$ together with maps $\gamma: \Gamma(2) \hookrightarrow T_H \hookrightarrow H$ that are compatible with intersection forms and such that there exists a factorization $\gamma: \Gamma(2) \hookrightarrow N_{\sigma} \hookrightarrow T_H \hookrightarrow H$ without $(-2)$-vectors in the orthogonal complement $\gamma(\Gamma(2))^\perp \subset N_{\sigma}$. For a supersingular Enriques surface $Y$ we can choose a $\Gamma$-marking $\gamma: \Gamma \hookrightarrow \text{NS}(Y)$ and this induces a point $\pi^E_\sigma(Y, \gamma) \in \mathcal{Q}_\sigma$. We show that $\pi_\sigma(Y, \gamma)$ is independent of the choice of $\gamma$ and set $\pi^E_\sigma(Y) = \pi^E_\sigma(Y, \gamma)$. This construction justifies calling $\pi^E_\sigma(Y)$ the period of $Y$ and we call $\mathcal{Q}_\sigma$ the period space of supersingular Enriques surfaces of Artin invariant at most $\sigma$.

It remains to mention characteristic $p=2$. Here, there are three types of Enriques surfaces and a moduli space in this case has two components [BM76, Lie15]. For the component corresponding to simply-connected Enriques surfaces, Ekedahl, Hyland and Shepherd-Barron [EHSB12] constructed a period map and established a Torelli theorem. In their work, however, the K3-like cover is not smooth and the covering is not étale, which is why the theory has a slightly different flavor.

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1. Prerequisites and notation

In this section we fix some notation and recall known results on supersingular K3 surfaces.

Let $k$ be a perfect field of characteristic $p \geq 3$. A K3 surface $X$ over $k$ is called supersingular if and only if $\text{rk}(\text{NS}(X)) = 22$. This definition of supersingularity is due to Shioda. There is a second definition for supersingularity due to Artin. Namely, a K3 surface $X$ over $k$ is called Artin supersingular if and only if its formal Brauer group $\Phi^X_k$ is of infinite height. Over perfect fields of characteristic at least 5, any K3 surface is Artin supersingular if and only if it is Shioda supersingular $[\text{Man14}]$. By a lattice $((L, \langle \cdot, \cdot \rangle))$ we mean a free $\mathbb{Z}$-module $L$ of finite rank together with a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle : L \times L \to \mathbb{Z}$.

Most of the following content is due to Ogus $[\text{Ogu79}, \text{Ogu83}]$. A strong inspiration for our treatment in this section and a good source for the interested reader is $[\text{Lie16}]$.

1.1. K3 crystals. For the definition of $F$-crystals and their slopes we refer to $[\text{Kat79}$, Chapter I.1$]$. Given a supersingular K3 surface $X$, it turns out that a lot of information is encoded in its second crystalline cohomology. We say that $H^2_{\text{crys}}(X/W)$ is a supersingular K3 crystal of rank 22 in the sense of the following definition, due to Ogus $[\text{Ogu79}]$.

**Definition 1.1.** Let $k$ be a perfect field of positive characteristic $p$ and let $W = W(k)$ be its Witt ring with lift of Frobenius $\sigma : W \to W$. A supersingular K3 crystal of rank $n$ over $k$ is a free $W$-module $H$ of rank $n$ together with an injective $\sigma$-linear map $\varphi : H \to H,$

i.e. $\varphi$ is a morphism of abelian groups and $\varphi(a \cdot m) = \sigma(a) \cdot \varphi(m)$ for all $a \in W$ and $m \in H$, and a symmetric bilinear form

$\langle -, - \rangle : H \times H \to W,$

such that

1. $p^2 H \subseteq \text{im}(\varphi),$
2. the map $\varphi \otimes_W k$ is of rank 1,
3. $\langle -, - \rangle$ is a perfect pairing,
4. $\langle \varphi(x), \varphi(y) \rangle = p^2 \sigma(\langle x, y \rangle),$ and
5. the $F$-crystal $(H, \varphi)$ is purely of slope 1.

The Tate module $T_H$ of a K3 crystal $H$ is the $\mathbb{Z}_p$-module

$$T_H := \{ x \in H \mid \varphi(x) = px \}.$$  

One can show that if $H = H^2_{\text{crys}}(X/W)$ is the second crystalline cohomology of a supersingular K3 surface $X$ and $c_1 : \text{Pic}(X) \to H^2_{\text{crys}}(X/W)$ is the first crystalline Chern class map, we have $c_1(\text{Pic}(X)) \subseteq T_H$. If $X$ is defined over a finite field, the Tate conjecture is known, see $[\text{Cha13}]$ $[\text{MP15}]$, and it follows that we even have the equality $c_1(\text{NS}(X)) \otimes \mathbb{Z}_p = T_H$. The following proposition on the structure of the Tate module of a supersingular K3 crystal is due to Ogus $[\text{Ogu79}]$.

**Proposition 1.2.** Let $(H, \varphi, \langle -, - \rangle)$ be a supersingular K3 crystal and let $T_H$ be its Tate module. Then $\text{rk}_W H = \text{rk}_{p^2} T_H$ and the bilinear form $(H, \langle -, - \rangle)$ induces a non-degenerate form $T_H \times T_H \to \mathbb{Z}_p$ via restriction to $T_H$ which is not perfect. More precisely, we find

1. $\text{ord}_p(T_H) = 2\sigma$ for some positive integer $\sigma,$
2. $(T_H, \langle -, - \rangle)$ is determined up to isometry by $\sigma,$
3. $\text{rk}_W H \geq 2\sigma$ and
there exists an orthogonal decomposition
\[(T_H, \langle -, - \rangle) \cong (T_0, p\langle -, - \rangle) \perp (T_1, \langle -, - \rangle),\]
where \(T_0\) and \(T_1\) are \(\mathbb{Z}_p\)-lattices with perfect bilinear forms and of ranks \(\text{rk} T_0 = 2\sigma\) and \(\text{rk} T_1 = \text{rk}_W H - 2\sigma\).

The positive integer \(\sigma\) is called the Artin invariant of the K3 crystal \(H\) [Ogu79]. When \(H\) is the second crystalline cohomology of a supersingular K3 surface \(X\), we have \(1 \leq \sigma(H) \leq 10\).

1.2. K3 lattices. The previous subsection indicates that the Néron-Severi lattice \(\text{NS}(X)\) of a supersingular K3 surface \(X\) plays an important role in the study of supersingular K3 surfaces via the first Chern class map. We say that \(\text{NS}(X)\) is a supersingular K3 lattice in the sense of the following definition due to Ogus [Ogu79].

**Definition 1.3.** A supersingular K3 lattice is an even lattice \((N, \langle -, - \rangle)\) of rank 22 such that
\begin{enumerate}
\item the discriminant \(d(N \otimes \mathbb{Q})\) is \(-1\) in \(\mathbb{Q}^*/\mathbb{Q}^{*2}\),
\item the signature of \(N\) is \((1, 21)\), and
\item the lattice \(N\) is \(p\)-elementary for some prime number \(p\).
\end{enumerate}

When \(N\) is the Néron-Severi lattice of a supersingular K3 surface \(X\), then the prime number \(p\) in the previous definition turns out to be the characteristic of the base field. One can show that if \(N\) is a supersingular K3 lattice, then its discriminant is of the form \(d(N) = -p^{2\sigma}\) for some integer \(\sigma\) such that \(1 \leq \sigma \leq 10\). The integer \(\sigma\) is called the Artin invariant of the lattice \(N\). If \(X\) is a supersingular K3 surface, we find that \(\sigma(\text{NS}(X)) = \sigma(H_{\text{crys}}^2(X/W))\). The following theorem is due to Rudakov and Shafarevich [RS81, Section 1].

**Theorem 1.4.** The Artin invariant \(\sigma\) determines a supersingular K3 lattice up to isometry.

1.3. Characteristic subspaces and K3 crystals. In this subsection we introduce characteristic subspaces. These objects yield another way to describe K3 crystals, a little closer to classic linear algebra in flavor. For this subsection we fix a prime \(p > 2\) and a perfect field \(k\) with Frobenius \(F: k \to k, x \mapsto x^p\).

**Definition 1.5.** Let \(\sigma\) be a non-negative integer and let \(V\) be a \(2\sigma\)-dimensional \(\mathbb{F}_p\)-vector space together with a non-degenerate and non-neutral quadratic form \(\langle -, \rangle: V \times V \to \mathbb{F}_p\).

The condition that \(\langle -, \rangle\) is non-neutral means that there exists no \(\sigma\)-dimensional isotropic subspace of \(V\). Set \(\varphi := \text{id}_V \otimes F: V \otimes_{\mathbb{F}_p} k \to V \otimes_{\mathbb{F}_p} k\). A \(k\)-subspace \(G \subset V \otimes_{\mathbb{F}_p} k\) is called characteristic if
\begin{enumerate}
\item \(G\) is a totally isotropic subspace of dimension \(\sigma\), and
\item \(G + \varphi(G)\) is of dimension \(\sigma + 1\).
\end{enumerate}

A strictly characteristic subspace is a characteristic subspace \(G\) such that
\[V \otimes_{\mathbb{F}_p} k = \sum_{i=0}^{\infty} \varphi^i(G)\]
holds true.

We can now introduce the categories
\[\text{K3}(k) := \{ \text{Supersingular K3 crystals with only isomorphisms as morphisms} \}\]
and
\[ C_3(k) := \begin{cases} \text{Pairs } (T, G), \text{ where } T \text{ is a supersingular} \\ \text{K3 lattice over } \hat{\mathbb{Z}}_p, \text{ and } G \subseteq T_0 \otimes_{\mathbb{Z}} k \\ \text{is a strictly characteristic subspace} \\ \text{with only isomorphisms as morphisms} \end{cases} \]

It turns out that over an algebraically closed field these two categories are equivalent.

**Theorem 1.6.** ([Ogu79] Theorem 3.20) Let \( k \) be an algebraically closed field of characteristic \( p > 0 \). Then the functor
\[ K_3(k) \to C_3(k), \]
\[ (H, \varphi, (\cdot, \cdot)) \mapsto \left(T_H, \ker \left(T_H \otimes_{\mathbb{Z}_p} k \to H \otimes_{\mathbb{Z}_p} k \right) \subset T_0 \otimes_{\mathbb{Z}_p} k \right) \]
defines an equivalence of categories.

If we denote by \( C_3(k)_\sigma \), the subcategory of \( C_3(k) \) consisting of objects \((T, G)\) where \( T \) is a supersingular K3 lattice of Artin invariant \( \sigma \), there is a coarse moduli space.

**Theorem 1.7.** ([Ogu79] Theorem 3.21) Let \( k \) be an algebraically closed field of characteristic \( p > 0 \). We denote by \( p_n \) the cyclic group of \( n \)-th roots of unity. There exists a canonical bijection
\[ (C_3(k)_\sigma, \sim) \to \mathbb{A}^{p-1}_k(1)/\mu_{p^{n+1}}(k). \]

The previous theorem concerns characteristic subspaces defined on closed points with algebraically closed residue field. Next, we consider families of characteristic subspaces.

**Definition 1.8.** Let \( \sigma \) be a non-negative integer and let \((V, \langle \cdot, \cdot \rangle)\) be a \(2\sigma\)-dimensional \(\mathbb{F}_p\)-vector space together with a non-neutral quadratic form. If \( A \) is an \(\mathbb{F}_p\)-algebra, a direct summand \( G \subset V \otimes_{\mathbb{F}_p} A \) is called a \textit{geneatrix} if \( \text{rk}(G) = \sigma \) and \( \langle \cdot, \cdot \rangle \) vanishes when restricted to \( G \). A \textit{characteristic geneatrix} is a geneatrix \( G \) such that \( G + FA(G) \) is a direct summand of rank \( \sigma + 1 \) in \( V \otimes_{\mathbb{F}_p} A \). We write \( \mathcal{M}_V \) for the set of characteristic geneatrices in \( V \otimes_{\mathbb{F}_p} A \).

It turns out that there exists a moduli space for characteristic geneatrices.

**Proposition 1.9.** ([Ogu79] Proposition 4.6) The functor
\[ (\mathbb{F}_p\text{-algebras})^{op} \to (\text{Sets}), \]
\[ A \mapsto \mathcal{M}_V \]
is representable by an \(\mathbb{F}_p\)-scheme \(\mathcal{M}_V \) which is smooth, projective and of dimension \( \sigma - 1 \).

If \( N \) is a supersingular K3 lattice with Artin invariant \( \sigma \), then \( N_0 = pN/\langle pN \rangle \) is a \(2\sigma\)-dimensional \(\mathbb{F}_p\)-vector space together with a non-degenerate and non-neutral quadratic form induced from the bilinear form on \( N \).

**Definition 1.10.** We set \( \mathcal{M}_\sigma := \mathcal{M}_N \) and call this scheme the \textit{moduli space of \( N \)-rigidified K3 crystals}.

1.4. **Ample cones.** Next, we will need to enlarge \( \mathcal{M}_\sigma \) by equipping \( N \)-rigidified K3 crystals with ample cones. For the rest of this section we fix a prime \( p \geq 3 \).

**Definition 1.11.** Let \( N \) be a supersingular K3 lattice. The set \( \Delta_N := \{ l \in N \mid l^2 = -2 \} \) is called the set of \textit{roots of } \( N \). The Weyl group \( W_N \) of \( N \) is the subgroup of the orthogonal group \( O(N) \) generated by all automorphisms of the form \( s_l : x \mapsto x + \langle x, l \rangle l \) with \( l \in \Delta_N \). We denote by \( \pm W_N \) the subgroup of \( O(N) \) generated by \( W_N \) and \( \pm \text{id} \). Further, we define
\[ V_N := \{ x \in N \otimes \mathbb{R} \mid x^2 > 0 \text{ and } \langle x, l \rangle \neq 0 \text{ for all } l \in \Delta_N \}. \]
The set \( V_N \) is an open subset of \( N \otimes \mathbb{R} \) and each of its connected components meets \( N \). The connected components of \( V_N \) are called the \textit{ample cones of } \( N \) and we denote by \( C_N \) the set of ample cones of \( N \).
Remark 1.12. The group $\pm W_N$ operates simply and transitively on $C_N$ \cite{Ogu83}.

Definition 1.13. Let $N$ be a supersingular K3 lattice of Artin invariant $\sigma$ and let $S$ be an algebraic space over $\mathbb{F}_p$. For a characteristic geneatrix $G \in \mathcal{M}_{\sigma}(S)$ and any point $s \in S$ we define

$$\lambda(s) := N_0 \cap G(s),$$

$$N(s) := \{ x \in N \otimes \mathbb{Q} \mid px \in N \text{ and } p\mathfrak{a} \in \lambda(s) \},$$

$$\Delta(s) := \{ l \in N(s) \mid l^2 = -2 \}.$$

An ample cone for $G$ is an element $\alpha \in \prod_{s \in S} C_N(s)$ such that $\alpha(s) \subseteq \alpha(t)$ whenever $s \in \{t\}$.

2. Moduli spaces of $N_\sigma$-marked supersingular K3 surfaces

This section discusses the moduli spaces for lattice-marked K3 surfaces that were introduced in \cite{Ogu83}.

We fix a prime $p \geq 3$ and for each integer $\sigma$ with $1 \leq \sigma \leq 10$ a representative $N_\sigma$ for the unique isomorphism class of K3 lattices with $\sigma(N_\sigma) = \sigma$. A family of supersingular K3 surfaces is a smooth and proper morphism $f : \mathcal{X} \rightarrow S$ of algebraic spaces over $\mathbb{F}_p$ such that for each field $k$ and each $k$-valued point $\text{Spec } k \rightarrow S$ the fiber $\mathcal{X}_k \rightarrow \text{Spec } k$ is a projective supersingular K3 surface. By \cite[Theorem 3.1.1]{Riz06} the relative Picard functor $\text{Pic}_{\mathcal{X}/S}$ is representable by a separated algebraic space $\text{Pic}_{\mathcal{X}/S}$ over $S$. An $N_\sigma$-marking of a family of supersingular K3 surfaces $f : \mathcal{X} \rightarrow S$ is a morphism $\psi : N_\sigma \rightarrow \text{Pic}_{\mathcal{X}/S}$ of group objects in the category of algebraic spaces that is compatible with intersection forms. There is an obvious notion of morphisms of families $N_\sigma$-marked K3 surfaces. From now on we will write $\mathcal{A}^o_{\mathbb{F}_p}$ for the category of algebraic spaces over $\mathbb{F}_p$. We consider the following moduli problem

$$\mathcal{S}_\sigma : \mathcal{A}^o_{\mathbb{F}_p} \rightarrow (\text{Sets})$$

$$S \mapsto \{ \text{Isomorphism classes of families of } N_\sigma \text{-marked supersingular K3 surfaces } (f : \mathcal{X} \rightarrow S, \psi : N_\sigma \rightarrow \text{Pic}_{\mathcal{X}/S}) \}.$$

It is a classical result of Ogus that the functor $\mathcal{S}_\sigma$ is representable by an $\mathbb{F}_p$-scheme $\mathcal{S}_\sigma$ that is smooth of dimension $\sigma - 1$ and locally of finite type over $\mathbb{F}_p$ \cite{Ogu83}. Further, $\mathcal{S}_\sigma$ satisfies the existence part of the valuative criterion for properness. However, $\mathcal{S}_\sigma$ is in general neither quasi-compact nor separated.

Via the period map the functor $\mathcal{S}_\sigma$ is canonically isomorphic to a functor $\mathcal{P}_\sigma$ \cite{Ogu83} which is defined to be

$$\mathcal{P}_\sigma : \mathcal{A}^o_{\mathbb{F}_p} \rightarrow (\text{Sets})$$

$$S \mapsto \{ \text{characteristic geneatrices } K \subseteq N_\sigma \otimes_{\mathbb{F}_p} O_S \text{ together with an ample cone} \}.$$

Ogus originally proved that the period morphism $\pi : \mathcal{S}_\sigma \rightarrow \mathcal{P}_\sigma$ is an isomorphism over fields of characteristic at least 5, but Bragg and Lieblich recently showed that Ogus’ results also hold true in characteristic 3 \cite[Section 5.1]{BL18}.

If we consider the functor

$$\mathcal{M}_\sigma : \mathcal{A}^o_{\mathbb{F}_p} \rightarrow (\text{Sets})$$

$$S \mapsto \{ \text{characteristic geneatrices } G \subseteq N_\sigma \otimes_{\mathbb{F}_p} O_S \} ;$$

then there is a canonical surjection of functors $\pi_\sigma : \mathcal{S}_\sigma \rightarrow \mathcal{M}_\sigma$, which is given by forgetting the choice of an ample cone. The functor $\mathcal{M}_{\sigma'}$ is representable by a smooth connected projective scheme $\mathcal{M}_{\sigma'}$ of dimension $\sigma - 1$ and the morphism of schemes $\pi_{\sigma'}$ is étale. For further details on the functor $\mathcal{M}_\sigma$ we refer the interested reader to \cite{Ogu79} and for further details on the functor $\mathcal{S}_\sigma$ we refer to \cite{Ogu83}.

Now let $\sigma' < \sigma$ be positive integers with $\sigma \leq 10$. In our construction of the moduli space of marked Enriques surfaces we will use an inductive argument. Therefore, we begin with an observation on the relation between the schemes $\mathcal{S}_\sigma$ and $\mathcal{S}_{\sigma'}$. There exists an embedding of
lattices \( j: N_{\sigma} \hookrightarrow N_{\sigma'} \) which makes \( N_{\sigma'} \) into an overlattice of \( N_{\sigma} \). We say that two such embeddings \( j \) and \( j' \) are isomorphic embeddings if there exists an automorphism \( \alpha: N_{\sigma'} \to N_{\sigma'} \) such that \( \alpha \circ j = j' \).

By [Nik80 Proposition 1.4.1] there are only finitely many isomorphism classes of such embeddings \( j: N_{\sigma} \hookrightarrow N_{\sigma'} \). For each isomorphism class we choose a representative \( j \) and denote by \( R_{\sigma', \sigma} \) the set of these representatives. An embedding \( j: N_{\sigma} \hookrightarrow N_{\sigma'} \) induces a morphism of \( F_p \)-schemes

\[
\Phi_j: S_{\sigma'} \to S_{\sigma}
\]

by mapping

\[
(f: X \to S, \psi: N_{\sigma'} \to \text{Pic}_X/S) \mapsto (f: X \to S, \psi \circ j: N_{\sigma} \to \text{Pic}_X/S)
\]
on \( S \)-valued points. Similarly, we also obtain a morphism \( \Psi_j: M_{\sigma'} \to M_{\sigma} \). It follows from [Ogu79 Remark 4.8] that the \( \Psi_j \) are closed immersions. Analogously, we see that the finite union \( M_{\sigma'} = \bigcup_{j \in R_{\sigma', \sigma}} \Psi_j(M_{\sigma'}) \) is the closed subscheme in \( M_{\sigma} \) corresponding to characteristic subspaces \( G \) of \( N_{\sigma} \) with Artin invariant \( \sigma(G) \leq \sigma' \). We now want to show that the morphisms \( \Phi_j \) are also closed immersions.

**Lemma 2.1.** The commutative diagrams

\[
\begin{array}{ccc}
S_{\sigma'} & \xrightarrow{\Phi_j} & S_{\sigma} \\
\downarrow{\pi_{\sigma'}} & & \downarrow{\pi_{\sigma}} \\
M_{\sigma'} & \xrightarrow{\Psi_j} & M_{\sigma}
\end{array}
\]

are cartesian.

**Proof.** It is easy to see that the \( \Phi_j \) are monomorphisms of functors. So we only need to check the existence part in the definition of fiber products. To this end, we claim that there is an equality \( \Phi_j(S_{\sigma'}) = \pi_{\sigma}^{-1}(\Psi_j(M_{\sigma'})) \). Indeed, the inclusion \( \Phi_j(S_{\sigma'}) \subseteq \pi_{\sigma}^{-1}(\Psi_j(M_{\sigma'})) \) is clear by definition and we easily see that the two subschemes have the same underlying topological space, that is, we have an equality of sets \( \{ x \in \pi_{\sigma}^{-1}(\Psi_j(M_{\sigma'})) \} = \{ x \in \Phi_j(S_{\sigma'}) \} \). The scheme \( \pi_{\sigma}^{-1}(\Psi_j(M_{\sigma'})) \) is reduced since \( \pi_{\sigma} \) is an étale morphism and \( \Psi_j(M_{\sigma'}) \) is reduced. Hence, we obtain the desired equality of subschemes.

Thus, given an \( F_p \)-scheme \( S \) and \( S \)-valued points \( y \in M_{\sigma'}(S) \) and \( z \in S_{\sigma}(S) \) such that \( \Psi_j(y) = \pi_{\sigma}(z) \), we find that \( z \in \Phi_j(S_{\sigma'}(S)) \). If we let \( x \) be the preimage of \( z \) under \( \Phi_j(S) \), then \( \Phi_j(x) = z \) and \( \pi_{\sigma}(x) = y \) which shows the claim.

**Proposition 2.2.** The morphisms of functors \( \Phi_j: S_{\sigma'} \to S_{\sigma} \) are closed immersions of schemes and the subfunctor \( S''_{\sigma'} \hookrightarrow S_{\sigma} \) which is defined to be

\[
S''_{\sigma'}: \mathcal{A}_{\mathbb{Z}_p}^{\text{op}} \to (\text{Sets})
\]

\[
S \mapsto \left\{ \begin{array}{l}
\text{Isomorphism classes of families of } N_{\sigma}-\text{marked} \\
\text{supersingular K3 surfaces } (f: X \to S, \psi: N_{\sigma} \to \text{Pic}_X/S) \\
\text{such that each fiber } \chi_x \text{ has } \sigma(\chi_x) \leq \sigma' \end{array} \right\}
\]

is representable by the closed subscheme \( S''_{\sigma'} = \bigcup_{j \in R_{\sigma', \sigma}} \Phi_j(S_{\sigma'}) \subseteq S_{\sigma} \).

**Proof.** We already mentioned that the morphisms \( \Psi_j \) are closed immersions, and thus Lemma 2.1 implies that the morphisms \( \Phi_j \) are closed immersions as well. The assertion on the functor represented by the union \( \bigcup_{j \in R_{\sigma', \sigma}} \Phi_j(S_{\sigma'}) \) is a consequence of the equality

\[
\bigcup_{j \in R_{\sigma', \sigma}} \Phi_j(S_{\sigma'}) = \pi_{\sigma}^{-1} \left( \bigcup_{j \in R_{\sigma', \sigma}} \Psi_j(M_{\sigma'}) \right),
\]

which follows from the proof of Lemma 2.1.
3. Auxiliary functors and moduli spaces

In this section we will introduce some auxiliary functors which we will then use to construct the main functor in the subsequent section.

In the following, we let $\sigma \leq 10$ be a positive integer. We consider the lattice $\Gamma = U_2 \oplus E_8(-1)$, which is up to isomorphism the unique unimodular, even lattice of signature $(1,9)$. The Picard group of any Enriques surface is isomorphic to $\Gamma \oplus \mathbb{Z}/2\mathbb{Z}$. Our idea is as follows: if $Y$ is an Enriques surface with a supersingular covering $\mathbb{Q}$ such that $\Gamma(2)^{\perp}$ contains an ample divisor and such that there is no $(2)$-vector in $\Gamma(2)^{\perp}$ such that there is a $(2)$-vector in $\Gamma(2)^{\perp} \subseteq \text{NS}(X)$, then we talk about quotients $X \to Y'$ of $X$ by an involution that maybe has a non-trivial fixed point locus.

We will therefore define various functors of $\Gamma(2)$-marked $K3$ surfaces and in Section 5 we then show that the main functor $E_{\sigma}$ of $\Gamma(2)$-marked $K3$ surfaces in Section 4 is isomorphic to a functor of $\Gamma$-marked Enriques surfaces.

By Corollary 2.4 in [Jan15], there exists a primitive embedding of lattices $\gamma: \Gamma(2) \hookrightarrow N_{\sigma}$ such that $\gamma(\Gamma(2))^{\perp} \subseteq N_{\sigma}$ contains no vector of self-intersection number $-2$ if and only if $\sigma \leq 5$, and further there are only finitely many isomorphism classes $[\gamma: \Gamma(2) \hookrightarrow N_{\sigma}]$ of such embeddings. We fix for each such isomorphism class a representative $\gamma$ and denote by $R_{\sigma}$ the finite set formed by these elements. For $\sigma > 5$ we have $R_{\sigma} = \emptyset$. For an embedding $\gamma \in R_{\sigma}$ we consider the subfunctor $\mathcal{S}_{\sigma} \subseteq \mathcal{S}_{\sigma}$ which is defined to be

$$
\mathcal{S}_{\sigma}^\prime: \mathcal{A}_{p,\sigma}^\text{op} \longrightarrow \text{(Sets)}
$$

\begin{align*}
S & \mapsto \begin{cases} 
\text{Isomorphism classes of families of } N_{\sigma}\text{-marked} \\
\text{supersingular } K3 \text{ surfaces } \quad (f: \mathcal{X} \to \mathcal{S}, \psi: N_{\sigma} \to \text{Pic}_{\mathcal{X}/S}) \\
\text{such that for each geometric fiber } s \in S \\
\text{the sublattice } \gamma_s(\Gamma(2)) \hookrightarrow \text{NS}(\mathcal{X}_s) \\
\text{contains an ample line bundle}
\end{cases}
\end{align*}

It follows from the following lattice theoretic lemma that the induced embedding of lattices $\gamma_s: \Gamma(2) \hookrightarrow \text{NS}(\mathcal{X}_s)$ is primitive even on the locus where the $N_{\sigma}$-marking $\psi: N_{\sigma} \to \text{Pic}_{\mathcal{X}/S}$ is not an isomorphism.

**Lemma 3.1.** Let $\gamma: \Gamma(2) \hookrightarrow N_{\sigma}$ be a primitive embedding and let $j: N_{\sigma} \hookrightarrow N_{\sigma-1}$ be an embedding of $K3$ lattices. Then the composition $j \circ \gamma: \Gamma(2) \hookrightarrow N_{\sigma-1}$ is a primitive embedding.

**Proof.** We write $\Gamma(2)^{\text{sat}}$ for the saturation of $\Gamma(2)$ in $N_{\sigma-1}$. Then we have an inclusion $2 \cdot \Gamma(2)^{\text{sat}} \subseteq \Gamma(2)$, because the lattice $\Gamma(2)$ is 2-elementary. On the other hand, we find that $N_{\sigma} + \Gamma(2)^{\text{sat}}$ is an overlattice of $N_{\sigma}$ with $2 \cdot (N_{\sigma} + \Gamma(2)^{\text{sat}}) \subseteq N_{\sigma}$. Since the lattice $N_{\sigma}$ is $p$-elementary and we have $p \neq 2$, it follows that $N_{\sigma} + \Gamma(2)^{\text{sat}} = N_{\sigma}$. Thus we have an equality $\Gamma(2) = \Gamma(2)^{\text{sat}}$. \hfill $\square$

For the rest of the discussion, we will always assume an embedding of $\Gamma(2)$ into some $\sigma$ to be primitive. The next thing we are interested in, is the representability of the functor $\mathcal{S}_{\sigma}$ for some fixed $\gamma \in R_{\sigma}$. The following result is probably known to experts, but we report it for convenience to the reader.

**Proposition 3.2.** The functor $\mathcal{S}_{\gamma}$ is an open subfunctor of $\mathcal{S}_{\sigma}$.

**Proof.** By definition, we have to show that for any $\mathcal{F}_p$-scheme $S$ and any isomorphism class $x = (f: X \to S, \psi: N_{\sigma} \hookrightarrow \text{Pic}_{X/S}) \in \mathcal{S}_{\gamma}(S)$ the locus $S_x \subseteq S$ such that $\gamma_x(\Gamma(2))$ contains an ample line bundle for all geometric points $s \in S_x$ is an open subscheme of $S$.

Given an $\mathcal{F}_p$-scheme $S$ and an $S$-valued point $x = (f: X \to S, \psi: N_{\sigma} \hookrightarrow \text{Pic}_{X/S}) \in \mathcal{S}_{\gamma}(S)$, using Lemma 3.1, we obtain a unique involution $\sigma^\gamma_\ast: \text{Pic}_{X/S} \to \text{Pic}_{X/S}$ which is induced from $\sigma_\ast: \Gamma(2) \to \text{id}\Gamma(2)$ and $\sigma^\gamma_\ast: \Gamma(2)^{\perp} = -\text{id}\Gamma(2)^{\perp}$. By [Ogu83] Proposition 2.1.1. and the argument in [Jan13] Lemma 4.3., the automorphism $\sigma^\gamma_\ast$ is induced...
from an automorphism of $S$-algebraic spaces $\iota_{\gamma}: \mathcal{X} \to \mathcal{X}$ if and only if $\gamma(\Gamma(2)) \hookrightarrow \text{Pic}_{\mathcal{X}/S}$ intersects the ample cone in $\text{NS}(\mathcal{X})$ for all points $s \in S$.

Now, if there is no point $s \in S$ such that $\gamma_{\gamma}(\Gamma(2)) \hookrightarrow \text{NS}(\mathcal{X})$ contains an ample line bundle, then $S_{e} = \emptyset$ is the empty scheme, which is an open subscheme of $S$. Else, let $s \in S$ be a point such that $\gamma_{s}(\Gamma(2)) \hookrightarrow \text{NS}(\mathcal{X})$ contains an ample line bundle. Let $\mathcal{O}_{S,s}$ be the local ring of $S$ at $s$, then $(\text{Fun}_{X} : s) \in \text{Aut}(\mathcal{O}_{S,s})$ is also an element of $\text{NS}(\mathcal{X})$ by the discussion in [Ogu83] pages 373-374]. If $\{U_{i}\}_{i \in I}$ is the directed system of all open subschemes of $S$ such that $s \in U_{i}$, then $\text{Spec} \mathcal{O}_{S,s} = \text{lim}U_{i}$ and we consider the commutative diagram

$$
\begin{array}{ccc}
\text{colim} \left( \text{Aut}_{U_{i}}(\mathcal{X}_{U_{i}}) \right) & \longrightarrow & \text{colim} \left( \text{Aut}(\text{Pic}_{\mathcal{X}_{U_{i}}/U_{i}}) \right) \\
\mathbb{Z} & \longrightarrow & \mathbb{Z}
\end{array}
$$

The morphisms $\mathcal{X} \to S$ and $\text{Pic}_{\mathcal{X}/S} \to S$ are locally of finite presentation, and it follows from [Sta19] Proposition 31.6.1] that the vertical arrows in the diagram are isomorphisms. Further, the horizontal arrows are injective by the Torelli theorem [Ogu83] and the fact that filtered colimits of sets preserve injections. Since the automorphism $\iota_{\gamma}^{\prime}|_{\text{Spec} \mathcal{O}_{S,s}}$ is induced from an automorphism $\iota \in \text{Aut}_{\text{Spec} \mathcal{O}_{S,s}}(\mathcal{X}_{\text{Spec} \mathcal{O}_{S,s}})$ it follows that there exists an open neighborhood $U(s)$ of $s$ such that $\iota_{\gamma}^{\prime}|_{U(s)}$ is induced from an automorphism $\iota \in \text{Aut}(\text{Pic}_{\mathcal{X}_{U_{i}}/U(s)})$.

Thus, the sublattice $\gamma_{\gamma}(\Gamma(2)) \hookrightarrow \text{NS}(\mathcal{X})$ contains an ample line bundle for all $s \in U(s)$, and the set of all $s \in S$ such that $\gamma_{\gamma}(\Gamma(2)) \hookrightarrow \text{NS}(\mathcal{X})$ contains an ample line bundle. Let $A$ be the set of all $s \in S$ such that $\gamma_{\gamma}(\Gamma(2)) \hookrightarrow \text{NS}(\mathcal{X})$ contains an ample line bundle.

We next want to be able to forget about the choice of a basis for $N_{\sigma}$ in the definition of $S_{\gamma}^{\prime}$. To do so, we consider the functor $\tilde{S}_{\gamma}^{\prime}: S_{\gamma}^{\prime} \to \text{Sets}$. To do so, we consider the function $\tilde{S}_{\gamma}^{\prime}: \mathcal{X}_{\mathbb{F}_{p}}^{op} \to \text{Sets}$

$$
\begin{array}{c}
S \mapsto \left\{ \text{Isomorphism classes of families of supersingular K3 surfaces } f: \mathcal{X} \to S \text{ together with a sublattice } \\
\mathbb{R} \subseteq \text{Pic}_{\mathcal{X}/S} \text{ and an embedding } \gamma^{\prime}: \Gamma(2) \hookrightarrow \mathbb{R} \text{ such that } (\gamma: \Gamma(2) \hookrightarrow N_{\sigma}) \cong (\gamma^{\prime}: \Gamma(2) \hookrightarrow \mathbb{R}) \text{ and such that for each geometric fiber } s \in S \\
\text{the sublattice } \gamma^{\prime}_{s}(\Gamma(2)) \hookrightarrow \text{NS}(\mathcal{X}) \text{ contains an ample line bundle} \right\}.
\end{array}
$$

We are again interested in the representability of the functor $\tilde{S}_{\gamma}^{\prime}$. We will see in the proof of the following proposition that $\tilde{S}_{\gamma}^{\prime}$ is in fact a quotient of $S_{\gamma}^{\prime}$ by a finite group action.

**Proposition 3.4.** The functor $\tilde{S}_{\gamma}^{\prime}$ is representable by a quasi-separated algebraic space $\tilde{S}_{\gamma}^{\prime}$ which is locally of finite type over $\mathbb{F}_{p}$ and there exists a canonical finite surjective morphism of algebraic spaces $q: S_{\gamma}^{\prime} \to \tilde{S}_{\gamma}^{\prime}$.

**Proof.** Consider the group $O(N_{\sigma}, \gamma) = \{ \varphi \in O(N_{\sigma}) \mid \varphi \circ \gamma = \gamma \circ \varphi \}$ of isometries of $N_{\sigma}$ that preserve the embedding $\gamma$. The group $O(N_{\sigma}, \gamma)$ is a subgroup of $O(\gamma(\Gamma(2))^{\perp})$, and the
latter group is finite because the lattice $\gamma(\Gamma(2))^\perp$ is negative definite. Hence it follows that $O(N_\sigma, \gamma)$ is a finite group. There is a group action of $O(N_\sigma, \gamma)$ on the functor $S'_\gamma$ which is given on $S$-valued points for connected schemes $S$ via

$$\varphi \cdot (f : X \to S, \psi : N_\sigma \to Pic_{X/S}) = (f : X \to S, \psi \circ \varphi : N_\sigma \to Pic_{X/S}).$$

The rest of the proof is separated into two steps.

**Step 1:** There is a canonical isomorphism of functors $F : S'_\gamma/\gamma(O(N_\sigma, \gamma)) \to S'_\gamma$.

There is a canonical morphism of functors $F' : S'_\gamma \to S'_\gamma$, which is given on $S$-valued points via

$$(f : X \to S, \psi : N_\sigma \to Pic_{X/S}) \mapsto (f : X \to S, \psi(N_\sigma) \subseteq Pic_{X/S}, \psi \circ \gamma : \Gamma(2) \leftrightarrow \psi(N_\sigma)).$$

This morphism is invariant under the action of $O(N_\sigma, \gamma)$ on $S'_\gamma$, and therefore it descends to a morphism of functors $F : S'_\gamma/O(N_\sigma, \gamma) \to S'_\gamma$. We want to show that $F$ is an isomorphism of functors by checking that for any $F_p$-scheme $S$ the induced map of sets $F(S)$ is a bijection.

a) **Surjectivity:** It suffices to show that the map $F'(S) : S'_\gamma(S) \to S'_\gamma(S)$ is surjective. To this end, we consider an element $s = (f, R, \gamma') \in S'_\gamma(S)$ and we choose an isomorphism of lattice embeddings $\psi : (\gamma : \Gamma(2) \leftrightarrow N_\sigma) \mapsto (\gamma' : \Gamma(2) \leftrightarrow R)$. Then the pair $s' = (f, \psi) \in S'_\gamma(S)$ is a preimage of $s$ under $F'$.

b) **Injectivity:** For an element $s = (f, R, \gamma') \in S'_\gamma(S)$ we have to show that any two preimages $s'$ and $s''$ in $S'_\gamma(S)$ only differ by some isometry $\varphi \in O(N_\sigma, \gamma)$. To this end, we write $s' = (f, \psi')$ and $s'' = (f, \psi'')$. We find that $\psi' \circ \varphi \in O(N_\sigma, \gamma)$ and we obtain the equality $(\psi' \circ \varphi) : s' = s''$. This concludes Step 1.

**Step 2:** The functor $S'_\gamma/O(N_\sigma, \gamma)$ is representable by an algebraic space $S'_\gamma/O(N_\sigma, \gamma)$ which is quasi-separated and locally of finite type over $F_p$ and the corresponding quotient morphism $q : S'_\gamma \rightarrow S'_\gamma/O(N_\sigma, \gamma)$ is finite.

Analogously to the action of the group $O(N_\sigma, \gamma)$ on $S'_\gamma$ we obtain an action of $O(N_\sigma, \gamma)$ on the scheme $M_\sigma$. Using the period map $S'_\sigma \rightarrow \mathcal{P}_\sigma$, it is clear that the $O(N_\sigma, \gamma)$-action on the open subscheme $S'_\sigma$ of $S_\sigma$ is the pullback of the $(O(N_\sigma, \gamma)$-action on $M_\sigma$ under the morphism $\pi'_\gamma : S'_\gamma \rightarrow M_\sigma$.

Next, we claim that the morphism $\pi'_\gamma$ is fixed-point reflecting in the sense of [Ryd13, Definition 2.2] by [Ryd13, Corollary 5.4]. Further, the quotient morphism $M_\sigma \rightarrow M_\sigma/O(N_\sigma, \gamma)$ is finite and $M_\sigma/O(N_\sigma, \gamma) \rightarrow \text{Spec} \ F_p$ is proper and of finite type by [Ryd13, Proposition 4.7]. By [Ryd13, Theorem 3.15] the quotient $M_\sigma \rightarrow M_\sigma/O(N_\sigma, \gamma)$ satisfies the descent condition in the sense of [Ryd13, Definition 3.6] and it follows that the quotient $q : S'_\gamma \rightarrow S'_\gamma/O(N_\sigma, \gamma)$ exists as an algebraic space and is topological quotient, the morphism $q$ is finite and the morphism $S'_\gamma/O(N_\sigma, \gamma) \rightarrow M_\sigma/O(N_\sigma, \gamma)$ is étale. \hfill $\square$
Remark 3.5. We do not expect $\tilde{S}_\gamma$ to be a scheme in general. A sufficient and necessary condition for $\tilde{S}_\gamma$ to be a scheme is that every orbit of the $O(N_\sigma, \gamma)$-action on $S'_\gamma$ is contained in an affine open subscheme of $S'_\gamma$ [Ryd13, Theorem 4.4]. Since $S'_\gamma$ is non-separated, we generally expect this condition to fail.

However, it turns out that the corresponding quotient of $M_\sigma$ which lies under $\tilde{S}_\gamma$ is still a scheme.

Proposition 3.6. There exist a projective $\mathbb{P}_p$-scheme $\tilde{M}'_\gamma$ and a canonical étale surjective morphism of algebraic spaces $\tilde{\pi}'_\gamma: \tilde{S}'_\gamma \to \tilde{M}'_\gamma$.

Proof. We can take the quotient $\tilde{M}'_\gamma = M_\sigma/O(N_\sigma, \gamma)$. This quotient is indeed a scheme because $M_\sigma$ is projective and in particular it has the property from Remark 3.5. Further, the scheme $M_\sigma/O(N_\sigma, \gamma)$ is projective by [Ryd13, Proposition 4.7.]. The other assertions have already been shown in the proof of Proposition 3.4. \hfill □

We will use the scheme $\tilde{M}'_\gamma$ later to construct the period scheme of supersingular Enriques surfaces.

We will now consider the subfunctor $\tilde{S}''_\gamma$ of $\tilde{S}'_\gamma$ that only allows $\Gamma(2)$-markings without vectors of self-intersection $-2$ in the complement, which is defined to be

$$\tilde{S}''_\gamma: \mathcal{A}_p^{sp} \to \text{(Sets)}$$

where

$$S \mapsto \left\{ \begin{array}{ll}
\text{Isomorphism classes of families of supersingular} \\
\text{K3 surfaces } f: X \to S \text{ together with a sublattice } \\
\mathcal{R} \subseteq \text{Pic}_X/S \text{ and an embedding } \gamma': \Gamma(2) \hookrightarrow \mathcal{R} \\
\text{such that } (\gamma: \Gamma(2) \hookrightarrow N_\sigma) \cong (\gamma': \Gamma(2) \hookrightarrow \mathcal{R}) \text{ and } \\
\text{such that for each geometric fiber } s \in S \\
\text{the sublattice } \gamma'_s(\Gamma(2)) \hookrightarrow \text{NS}(X_s) \\
\text{contains an ample line bundle} \\
\text{and } \gamma'_s(\Gamma(2))'^{-1} \hookrightarrow \text{NS}(X_s) \text{ contains no } (-2)\text{-vector} \\
\end{array} \right\}.$$

The points of $\tilde{S}''_\gamma$ should be seen as quotients of supersingular K3 surfaces by a fixed point free involution. For an explanation we refer to the proof of Theorem 4.1. in [Jan13]. We are again interested in the representability of the functor $\tilde{S}''_\gamma$.

Proposition 3.7. The functor $\tilde{S}''_\gamma$ is representable by an open algebraic subspace $\tilde{S}''_\gamma$ of $\tilde{S}'_\gamma$.

Proof. We consider the set $R'$ of representatives of all isomorphism classes of embeddings $j: N_\sigma \hookrightarrow N_\sigma'$ such that $j(\gamma(\Gamma(2)))^{-1} \subseteq N_\sigma'$ contains a (-2)-vector. Then the set $R'$ is a subset of the finite set $\bigcup_{\sigma' < \sigma} R_{\sigma', \sigma}$. For each $j$, the algebraic subspace $q(\Phi_j(S_{\sigma'} \cap S'_\gamma) \subseteq \tilde{S}'_\gamma$ is closed, and it is clear that the open algebraic subspace

$$\tilde{S}''_\gamma = \tilde{S}'_\gamma \setminus \left( \bigcup_{j \in R'} q(\Phi_j(S_{\sigma'})) \cap S'_\gamma \right)$$

represents the functor $\tilde{S}''_\gamma$. \hfill □

We also find an open subscheme of $\tilde{M}'_\gamma$ that lies under $\tilde{S}''_\gamma$.

Proposition 3.8. There exist a quasi-projective $\mathbb{P}_p$-scheme $\tilde{M}''_\gamma$ and a canonical étale surjective morphism of algebraic spaces $\tilde{\pi}''_\gamma: \tilde{S}''_\gamma \to \tilde{M}''_\gamma$.

Proof. The morphism $\tilde{\pi}''_\gamma: \tilde{S}''_\gamma \to \tilde{M}''_\gamma$ is universally open. Hence we may take $\tilde{M}''_\gamma$ to be the image of $\tilde{S}''_\gamma$ under $\tilde{\pi}''_\gamma$ and $\tilde{\pi}''_\gamma$ to be the restriction of $\tilde{\pi}''_\gamma$ to $\tilde{S}''_\gamma$. \hfill □
Proof. We prove that \( f \) has finite discrete fibers, because the fibers of \( g \circ f \) surject onto the fibers of \( g \). Further, the morphism \( g \) is quasi-compact \([\text{Sta19} \text{ Lemma 61.23.6}]\). It is clear that \( g \) has finite discrete fibers, because the fibers of \( g \circ f \) surject onto the fibers of \( g \). Thus, the fibers of \( g \circ f \) are finite. We leave the proper case to the reader. Since \( Y \) and \( Z \) are locally of finite type, the morphism \( g \) is locally of finite type \([\text{Nam85} \text{ Theorem 1.14}]\). We removed a divisor or the empty set in each sub moduli space \( S_{\sigma'} \subseteq S_{\sigma} \).

4. Moduli spaces of \( \Gamma(2) \)-marked supersingular K3 surfaces

Next, we want to get rid of having to make a choice of a sublattice \( \mathcal{R} \) in \( \text{Pic}_{X/S} \). The idea is, that on an open dense subset of the moduli space \( \tilde{S}_{\sigma'} \) we do not have a choice anyways, and the closed complement of this open subspace can be contracted to the corresponding moduli space for Artin invariant \( \sigma - 1 \) by forgetting about the sublattice \( \mathcal{R} \).

We now introduce the functor

\[
\tilde{E}_{\sigma} : \mathcal{A}^{op}_{\mathbb{F}_p} \longrightarrow (\text{Sets})
\]

\[
S \mapsto \left\{ \begin{array}{l}
\text{Isomorphism classes of families of supersingular K3 surfaces } f : X \rightarrow S \text{ that admit an } N_{\sigma} \text{-marking} \\
\text{together with an embedding } \gamma : \Gamma(2) \hookrightarrow \text{Pic}_{X/S} \\
\text{such that for each geometric fiber } s \in S \\
\text{the sublattice } \gamma_{s}(\Gamma(2)) \hookrightarrow \text{NS}(X_s) \\
\text{contains an ample line bundle} \\
\text{and } \gamma_{s}(\Gamma(2)) \hookrightarrow \text{NS}(X_s) \text{ contains no } -2 \text{-vector}
\end{array} \right\}.
\]

We are again interested in an object \( \tilde{E}_{\sigma} \) which represents the functor \( \tilde{E}_{\sigma} \). The discussion will use an inductive argument, so we start by discussing the case \( \sigma = 1 \).

Proposition 4.1. The functor \( \tilde{E}_1 \) is representable by a zero-dimensional quasi-separated algebraic space \( \tilde{E}_1 \) locally of finite type over \( \mathbb{F}_p \) which has finitely many connected components and each of these components is irreducible.

Proof. For each \( \gamma \in R_1 \) there is a canonical morphism of functors \( \tilde{E}''_{\gamma} \rightarrow \tilde{E}_1 \) which is given on \( S \)-valued points by forgetting about the choice of a sublattice \( \mathcal{R} \subseteq \text{Pic}_{X/S} \). Since any such sublattice \( \mathcal{R} \subseteq \text{Pic}_{X/S} \) is actually already equal to \( \text{Pic}_{X/S} \), we see that this morphism is injective on \( S \)-valued points and it follows that \( \coprod_{\gamma \in R_1} \tilde{E}''_{\gamma} \rightarrow \tilde{E}_1 \) is an isomorphism of functors. Hence, the functor \( \tilde{E}_1 \) is represented by the algebraic space \( \coprod_{\gamma \in R_1} \tilde{E}''_{\gamma} \).

Remark 4.2. More precisely, since \( S_1 \) is isomorphic to a disjoint union of finitely many copies of \( \text{Spec} \mathbb{F}_{p^2} \) and \( \tilde{E}''_{\gamma} \) is just an open subscheme of a quotient of an open subscheme of \( S_\gamma \), we easily see that \( \tilde{E}_1 \) is just a disjoint union of finitely many copies of \( \text{Spec} \mathbb{F}_{p^2} \) as well.

We will need the following lemma which might be well-known, but we did not find it in the literature in full generality. That is, we do not require any assumptions on being a scheme, being noetherian or separatedness.

Lemma 4.3. Let \( X, Y \) and \( Z \) be algebraic spaces that are locally of finite type over a base scheme \( S \) together with \( S \)-morphisms \( f : X \rightarrow Y \) and \( g : Y \rightarrow Z \) such that \( g \circ f \) is proper (respectively finite) and \( f \) is proper (respectively finite) and surjective. Then \( g \) is proper (respectively finite).

Proof. We prove that \( g \) is finite when \( f \) and \( g \circ f \) are finite. We leave the proper case to the reader. Since \( Y \) and \( Z \) are locally of finite type, the morphism \( g \) is locally of finite type \([\text{Sta19} \text{ Lemma 61.23.6}]\). It is clear that \( g \) has finite discrete fibers, because the fibers of \( g \circ f \) surject onto the fibers of \( g \). Further, the morphism \( g \) is quasi-compact \([\text{Sta19} \text{ Lemma} \text{61.23.6}]\).
It follows that \( g \) is quasi-finite. Further, if \( T \to Z \) is any morphism and \( Q \subseteq Y_T \) is a closed subscheme, then the subscheme \( g_T(Q) = g_T \circ f_T(f_T^{-1}(Q)) \) is closed. This shows that \( g \) is universally closed. Further, the fact that \( g \) is affine follows from a version of Chevalley’s theorem [Ryd15, Theorem 8.1]. All these properties together imply that \( g \) is finite. \( \square \)

Since every family of supersingular K3 surfaces that admits an \( N_{\sigma-1} \)-marking also admits an \( N_{\sigma} \)-marking, the functor \( \mathcal{E}_{\sigma-1} \) is a subfunctor of \( \mathcal{E}_{\sigma} \). For each positive integer \( \sigma \leq 10 \) there is a canonical morphism of functors

\[
p_{\sigma} : \prod_{\gamma \in R_{\sigma}} \tilde{\mathcal{S}}_{\gamma} \to \tilde{\mathcal{E}}_{\sigma}
\]

which is given on \( S \)-valued points by forgetting about the sublattice \( \mathcal{R} \subseteq \text{Pic}_{X/S} \). Then the preimage of the subfunctor \( \mathcal{E}_{\sigma-1} \rightarrow \tilde{\mathcal{E}}_{\sigma} \) under \( p_{\sigma} \) is given by the closed algebraic subspace

\[
p_{\sigma}^{-1}(\mathcal{E}_{\sigma-1}) = \prod_{\gamma \in R_{\sigma}} \left( \left( \bigcup_{j \in R_{\sigma-1, \sigma}} q(\Phi_j(S_{\sigma-1}) \cap S_j') \right) \setminus \left( \bigcup_{j' \in R'} q(\Phi_{j'}(S_{\sigma}) \cap S_{j'}) \right) \right)
\]

of \( \prod_{\gamma \in R_{\sigma}} \tilde{\mathcal{S}}_{\gamma} \).

**Definition 4.4.** For \( \gamma \in R_{\sigma} \) and \( j \in R_{\sigma-1, \sigma} \), we write \( W_j^\gamma \) for the locally closed subspace of \( S_{\sigma} \) defined to be

\[
W_j^\gamma = (\Phi_j(S_{\sigma-1}) \cap S_j') \setminus \left( \bigcup_{j' \in R'} \Phi_{j'}(S_{\sigma}) \cap S_{j'} \right).
\]

**Remark 4.5.** The image of \( W_j^\gamma \) under \( q : \mathcal{S}_{\gamma}' \rightarrow \tilde{\mathcal{S}}_{\gamma} \) is contained in \( \tilde{\mathcal{S}}_{\gamma}' \). In fact, we have the equality \( \bigcup_{\gamma \in R_{\sigma}, j \in R_{\sigma-1, \sigma}} q(W_j^\gamma) = p_{\sigma}^{-1}(\mathcal{E}_{\sigma-1}) \). Moreover, since \( W_j^\gamma \) is a closed subspace of \( S_j' \), it follows from Proposition 3.4 that the morphism \( q_{|W_j^\gamma} : W_j^\gamma \rightarrow \tilde{\mathcal{S}}_{\gamma}' \) is finite.

Further, since \( \Phi_j(S_{\sigma-1}) \cap S_j' \) is canonically isomorphic to the open subscheme \( S_{j_{\gamma, \gamma}}' \) of \( S_{\sigma-1} \), we also have a natural finite morphism \( q : W_j^\gamma \rightarrow S_{j_{\gamma, \gamma}}' \).

**Lemma 4.6.** Assume that \( \mathcal{E}_{\sigma-1} \) is representable by an algebraic space \( \tilde{\mathcal{E}}_{\sigma-1} \) that is locally of finite type over \( \mathbb{F}_p \) and that the canonical morphism \( \prod_{\gamma \in R_{\sigma}, j \in R_{\sigma-1, \sigma}} W_j^\gamma \rightarrow \tilde{\mathcal{E}}_{\sigma-1} \) is finite. Then the restriction of \( p_{\sigma} \) to \( p_{\sigma}^{-1}(\mathcal{E}_{\sigma-1}) \) is a finite morphism.

**Proof.** This is a direct consequence of Lemma 4.3 and the previous remark. \( \square \)

**Theorem 4.7.** Let \( \sigma \leq 10 \) be a positive integer.

1. The functor \( \mathcal{E}_{\sigma} \) is representable by an algebraic space \( \tilde{\mathcal{E}}_{\sigma} \) which is locally of finite type over \( \mathbb{F}_p \) and quasi-separated.

2. For each isomorphism class of primitive embeddings \( \gamma : \Gamma(2) \hookrightarrow N_{\sigma+1} \) such that there is no \((-2)\)-vector in \( \gamma(\Gamma(2))^\perp \subset N_{\sigma+1} \) and each embedding of lattices \( j : N_{\sigma+1} \hookrightarrow N_{\sigma} \) such that there is no \((-2)\)-vector in \( j(\gamma(\Gamma(2)))^\perp \subset N_{\sigma} \), there is a canonical finite morphism \( W_j^\gamma \rightarrow \tilde{\mathcal{E}}_{\sigma} \).

**Proof.** We do induction over \( \sigma \). For \( \sigma = 1 \), the theorem follows from Proposition 4.1 and its proof.

We will now assume that the theorem holds for \( \sigma - 1 \). We consider the pushout diagram

\[
p_{\sigma}^{-1}(\mathcal{E}_{\sigma-1}) \xrightarrow{\epsilon} \prod_{\gamma \in R_{\sigma}} \tilde{\mathcal{S}}_{\gamma} \xrightarrow{\Gamma} \tilde{\mathcal{E}}_{\sigma-1} \xrightarrow{\mathcal{P}}
\]

By Lemma 4.6, the morphism \( p_{\sigma} : p_{\sigma}^{-1}(\mathcal{E}_{\sigma-1}) \rightarrow \tilde{\mathcal{E}}_{\sigma-1} \) is finite, hence the Ferrand pushout datum \( \mathcal{E}_{\sigma-1} \xrightarrow{\epsilon} p_{\sigma}^{-1}(\mathcal{E}_{\sigma-1}) \xrightarrow{\Gamma} \prod_{\gamma \in R_{\sigma}} \tilde{\mathcal{S}}_{\gamma} \) is effective by [TT16, Theorem 6.2] and the
pushout $\mathcal{P}$ exists as an algebraic space over $\mathbb{F}_p$. Further, the morphism of algebraic spaces $\coprod_{x \in \mathbb{F}_p} \tilde{S}'_\gamma \to \mathcal{P}$ is finite by [TT16, Theorem 6.6] and $\mathcal{P} \to \text{Spec} \mathbb{F}_p$ is quasi-separated by [TT16, Theorem 6.8].

We obtain from [TT16, Theorem 4.8] that the topological space underlying $\mathcal{P}$ is just the pushout in the category of topological spaces, there exists a natural isomorphism of algebraic spaces $p^{-1}_*(\tilde{E}_{\sigma-1}) \cong \tilde{E}_{\sigma-1} \times_{\mathcal{P}} \coprod_{x \in \mathbb{F}_p} \tilde{S}'_\gamma$, the morphism $\tilde{E}_{\sigma-1} \to \mathcal{P}$ is a closed immersion of algebraic spaces, the morphism $\left( \coprod_{x \in \mathbb{F}_p} \tilde{S}'_\gamma \right) \setminus \left( p^{-1}_*(\tilde{E}_{\sigma-1}) \right) = U \to \mathcal{P}$ is an open immersion of algebraic spaces and we have an equality of sets $|\mathcal{P}| = |\tilde{E}_{\sigma-1}| \sqcup |U|$. The finite morphism $\tilde{E}_{\sigma-1} \sqcup \coprod_{x \in \mathbb{F}_p} \tilde{S}'_\gamma \to \mathcal{P}$ is surjective as a map of topological spaces, and it follows from [AM69, Proposition 7.8] that $\mathcal{P}$ is locally of finite type over $\mathbb{F}_p$.

We now show that the algebraic space $\mathcal{P}$ represents the functor $\tilde{E}_\sigma$ and that the morphism of algebraic spaces $\coprod_{x \in \mathbb{F}_p} \tilde{S}'_\gamma \to \mathcal{P}$ represents the canonical morphism $\coprod_{x \in \mathbb{F}_p} \tilde{S}'_\gamma \to \tilde{E}_\sigma$.

**Step 1**: We define a morphism of presheaves $F: \tilde{E}_\sigma \to \mathcal{P}$.

If $S$ is an irreducible and reduced $\mathbb{F}_p$-scheme, we define the map $F(S): \tilde{E}_\sigma(S) \to \mathcal{P}(S)$ in the following way. If $x = (f: X \to S, \gamma): \Gamma(2) \to \text{Pic}_{X/S}$ is such that for every $s \in S$ the fiber $X_s$ has Artin invariant $\sigma(\text{NS}(X_s)) \leq \sigma - 1$, then $x$ is an element of the subset $\tilde{E}_{\sigma-1}(S) \subset \tilde{E}_\sigma(S)$. In this case, we set $F(S)(x)$ to be the image of $x$ under the canonical map $\tilde{E}_{\sigma-1}(S) \to \mathcal{P}(S)$. Note, that by the commutativity of the pushout diagram, if $x$ lies in the image of $p_*\gamma$, we equivalently could have chosen a preimage $x'$ of $x$ in $\tilde{S}_\gamma(S)$ for some $\gamma'$ and set $F(S)(x')$ to be the image of $x'$ under the canonical map $\tilde{S}'_\gamma(S) \to \mathcal{P}(S)$.

If, on the other hand, $x$ is such that there exists an $s \in S$ with $\sigma(\text{NS}(X_s)) = \sigma$, then the subset $U \subseteq S$ where $X_s$ has Artin invariant $\sigma$ is open. We choose an arbitrary lift $x' = (f, R', \gamma')$ of $x$ to $\tilde{S}_\gamma(S)$. We claim that this lift is unique. Indeed, let $x'' = (f, R'', \gamma'')$ be another such lift. We take preimages $\tilde{x}' = (f, \psi')$ and $\tilde{x}'' = (f, \psi'')$ in $\tilde{S}'_\gamma(S)$ and after applying an automorphism of $N_\sigma$ that preserves the embedding $\gamma': \Gamma(2) \to N_\sigma$, we may assume that $\psi'_U = \psi''_U$. But by [Riz06, Theorem 3.1.1] the morphism of algebraic spaces $\text{Pic}_{X/S} \to S$ is separated and it therefore follows that $\psi' = \psi''$. Thus, we have an isomorphism $x' \cong x''$.

We set $F(S)(x)$ to be the image of $x' \in \tilde{S}_\gamma(S)$ under the canonical map $\tilde{S}'_\gamma(S) \to \mathcal{P}(S)$. It is clear from the construction that the class of maps $F(S)$ yields a morphism of functors.

**Step 2**: We define a morphism of presheaves $G: \mathcal{P} \to \tilde{E}_\sigma$ which is an inverse to $F$.

Using the induction hypothesis, we write

$$X_{\tilde{E}_{\sigma-1}} \to \tilde{E}_{\sigma-1}$$

and

$$X_{\coprod_{x \in \mathbb{F}_p} \tilde{S}'_\gamma} \to \coprod_{x \in \mathbb{F}_p} \tilde{S}'_\gamma$$

for the universal elements of the functors $\tilde{E}_{\sigma-1}$ and $\coprod_{x \in \mathbb{F}_p} \tilde{S}'_\gamma$. Since the scheme corresponding to an $S$-valued point of $\coprod_{x \in \mathbb{F}_p} \tilde{S}'_\gamma$ maps to the same scheme corresponding to an $S$-valued point of $\tilde{E}_{\sigma-1}$ under $p_\sigma$, and we are only forgetting about additional structure, there exists a unique isomorphism

$$p_\sigma^* X_{\tilde{E}_{\sigma-1}} \cong \gamma^* X_{\coprod_{x \in \mathbb{F}_p} \tilde{S}'_\gamma}.$$ 

We choose a representative for this pullback of algebraic spaces and denote it by $X_{p_\sigma^{-1}(\tilde{E}_{\sigma-1})}$.

We find that the Ferrand pushout datum $X_{\tilde{E}_{\sigma-1}} \leftarrow X_{p_\sigma^{-1}(\tilde{E}_{\sigma-1})} \to X_{\coprod_{x \in \mathbb{F}_p} \tilde{S}'_\gamma}$ is effective using the same argument as above and we choose a pushout $X_\mathcal{P}$ for this datum. The canonical morphism

$$\left( X_{\tilde{E}_{\sigma-1}} \leftarrow X_{p_\sigma^{-1}(\tilde{E}_{\sigma-1})} \to X_{\coprod_{x \in \mathbb{F}_p} \tilde{S}'_\gamma} \right) \to \left( \tilde{E}_{\sigma-1} \leftarrow \tilde{E}_{\sigma-1} \leftarrow \coprod_{x \in \mathbb{F}_p} \tilde{S}'_\gamma \right) \to \coprod_{x \in \mathbb{F}_p} \tilde{S}'_\gamma$$
is a flat morphism of pushout data in the sense of [TT16, Chapter 2.2]. Hence, the induced morphism $\mathcal{X}_P \to \mathcal{P}$ is smooth by [TT16, Theorem 6.3.2,(ii)] and proper by Lemma 4.3. Moreover, the morphism

$$X_{\mathcal{E}_{\sigma}} \rightarrow \tilde{\mathcal{E}}_{\sigma}$$

is just the pullback of $X_P$ along the morphism

$$\prod_{\gamma \in R_{\sigma}} \tilde{S}_{\gamma}'' \rightarrow \mathcal{P}$$

and the morphism

$$X_{\mathcal{E}_{\sigma-1}} \rightarrow \tilde{\mathcal{E}}_{\sigma-1}$$

is just the pullback of $X_P$ along

$$\tilde{\mathcal{E}}_{\sigma-1} \rightarrow \mathcal{P}$$

by [TT16, Theorem 6.3.2,(i)]. Since $\mathcal{P}$ is set-theoretically covered by $\prod_{\gamma \in R_{\sigma}} \tilde{S}_{\gamma}''$ and $\tilde{\mathcal{E}}_{\sigma-1}$, and the geometric fibers of these algebraic spaces are projective supersingular K3 surfaces, it follows that the geometric fibers of $X_P \to \mathcal{P}$ are projective supersingular K3 surfaces as well. Hence $X_P \to \mathcal{P}$ is a family of supersingular K3 surfaces. The construction of the relative Picard functor is compatible with base change. Therefore, we obtain a morphism of algebraic group spaces compatible with intersection forms

$$\text{Pic}_{\prod_{\gamma \in R_{\sigma}} \tilde{S}_{\gamma}'' / \mathcal{P}} \rightarrow \text{Pic}_{X_P / \mathcal{P}}$$

which induces a $\Gamma(2)$-marking $\gamma : \Gamma(2) \hookrightarrow \text{Pic}_{X_P / \mathcal{P}}$. If $S$ is an $\mathbb{F}_p$-scheme and $y : S \to \mathcal{P}$ is a morphism of $\mathbb{F}_p$-schemes, we define $G(S)(y) \in \tilde{\mathcal{E}}_{\sigma}(S)$ to be the pullback of $X_P$ under $y$.

A straightforward computation shows that the morphisms $F$ and $G$ are mutually inverse to each other.

Since we have shown that the canonical morphism $\prod_{\gamma \in R_{\sigma}} \tilde{S}_{\gamma}'' \rightarrow \mathcal{E}_{\sigma}$ is finite, it follows from Remark 4.5 that for each $\gamma \in R_{\sigma+1}$ and $j \in R_{\sigma,\sigma+1}$ the canonical morphism $W^j_{\gamma} \rightarrow \tilde{\mathcal{E}}_{\sigma}$ is finite.

Again, there exists a nice scheme for which $\tilde{\mathcal{E}}_{\sigma}$ is an étale cover. However, this scheme may not be quasi-projective anymore and we introduce the following slightly weaker finiteness property.

**Definition 4.8.** [Ryd13, Definition B.1] A scheme $X$ is called an AF scheme if for every finite subset $\{x_i\}$ of $X$ there exists an affine open subscheme $U$ in $X$ such that $\{x_i\}$ is contained in $U$.

**Remark 4.9.** Any quasi-projective scheme over a field $k$ is AF. Further, if $X$ is an AF scheme and $G$ is a finite group acting on $X$, then the quotient $X/G$ always exists as a scheme, see Remark 3.5.

**Remark 4.10.** To our knowledge, the term AF scheme was first used in [Ryd13]. However, schemes with this property have been studied before [Art+63, Exp. V], [Art71, §4], [Fer03]. For more facts on AF schemes see [Ryd13, Appendix B].

**Proposition 4.11.** There exists a separated $\mathbb{F}_p$-scheme $Q_{\sigma}$ which is of finite type and AF, and a canonical étale surjective morphism $\tilde{\mathcal{E}}_{\sigma} \rightarrow Q_{\sigma}$.

**Proof.** For $\sigma = 1$ we can take the quasi-projective scheme $Q_{\sigma} = \prod_{\gamma \in R_1} \tilde{M}_{\gamma}''$. This proves the assertion in this case.
We now do induction over $\sigma$ and assume that the assertion is true for $\sigma - 1$. The pushout diagram of $\mathbb{F}_p$-algebraic spaces

$$
p_{\sigma}^{-1}(\tilde{E}_{\sigma-1}) \quad \xrightarrow{i} \quad \coprod_{\gamma \in R_\sigma} \tilde{S}_\gamma^\prime
\quad \xrightarrow{\gamma} \quad \tilde{E}_{\sigma-1}
\xrightarrow{p_{\sigma}} \quad \tilde{E}_{\sigma}
$$

induces a pushout diagram of separated $\mathbb{F}_p$-schemes of finite type and AF

$$
p_{\sigma}^{-1}(Q_{\sigma-1}) \quad \xrightarrow{i} \quad \coprod_{\gamma \in R_\sigma} \tilde{M}_\gamma^\prime
\quad \xrightarrow{\gamma} \quad Q_{\sigma-1}
\xrightarrow{p_{\sigma}} \quad Q_{\sigma}
$$

together with an étale and surjective morphism of pushout data

$$
\left( \tilde{E}_{\sigma-1} \leftarrow p_{\sigma}^{-1}(\tilde{E}_{\sigma-1}) \to \coprod_{\gamma \in R_\sigma} \tilde{S}_\gamma^\prime \right) \quad \xrightarrow{\gamma} \quad \left( Q_{\sigma-1} \leftarrow p_{\sigma}^{-1}(Q_{\sigma-1}) \to \coprod_{\gamma \in R_\sigma} \tilde{M}_\gamma^\prime \right).
$$

It follows from the previous discussion of the schemes $M_\sigma$ that $i$ is a closed immersion and that we may inductively assume that $p_{\sigma}$ is finite. By [Fer03, Théorème 5.4.], the pushout $Q_\sigma$ exists as an AF scheme and the induced morphism $Q_{\sigma-1} \coprod_{\gamma \in R_\sigma} \tilde{M}_\gamma^\prime \to Q_\sigma$ is finite surjective. Since $\coprod_{\gamma \in R_\sigma} \tilde{M}_\gamma^\prime$ is of finite type over $\mathbb{F}_p$ and we may inductively assume that $Q_{\sigma-1}$ is of finite type over $\mathbb{F}_p$ as well, it follows that $Q_\sigma$ is of finite type over $\mathbb{F}_p$. That $Q_\sigma$ is separated follows from [TT16, Theorem 6.8.] and by [TT16 Theorem 6.4.] the induced morphism $\tilde{E}_{\sigma} \to Q_\sigma$ is étale and surjective.

\[ \square \]

**Remark 4.12.** We will prove in Section 7 that the scheme $Q_\sigma$ constructed in the proof of Proposition 4.11 is a coarse moduli scheme for supersingular Enriques surfaces.

### 5. From $\Gamma(2)$-marked K3 surfaces to $\Gamma^\prime$-marked Enriques surfaces

Although we want to construct a moduli space for Enriques surfaces, we have only discussed K3 surfaces so far. In this section we establish the connection between $\Gamma(2)$-marked supersingular K3 surfaces and $\Gamma^\prime$-marked Enriques surfaces that are quotients of supersingular K3 surfaces.

**Definition 5.1.** If $X$ is a supersingular K3 surface and $\iota: X \to X$ is a fixed point free involution, we write $G = \langle \iota \rangle$ for the cyclic group of order 2 which is generated by $\iota$. A quotient of surfaces $X \to X/G = Y$ defined by such a pair $(X, \iota)$ is called a **supersingular Enriques surface** $Y$. The **Artin invariant** of a supersingular Enriques surface $Y$ is the Artin invariant of the supersingular K3 surface $X$ that universally covers $Y$. A family of **supersingular Enriques surfaces** is a smooth and proper morphism of algebraic spaces $f: \mathcal{Y} \to S$ over $\mathbb{F}_p$ such that for each field $k$ and each $s: \text{Spec} \ k \to S$ the fiber $f_s: \mathcal{Y}_s \to \text{Spec} \ k$ is a supersingular Enriques surface.

Recall from Section 3 that we defined $\Gamma$ to be the lattice $\Gamma = U_2 \oplus E_8(-1)$. If $Y$ is a supersingular Enriques surface, then there exists an isomorphism of lattices $\text{Pic}(Y) \cong \Gamma \oplus \mathbb{Z}/2\mathbb{Z}$ and we denote the latter lattice by $\Gamma^\prime$. In arbitrary characteristic, by [Lie15 Proposition 4.4], if $Y \to S$ is a family of supersingular Enriques surfaces, then the torsion part $\text{Pic}_{Y/S}^T$ of the Picard scheme is a finite flat group scheme of length 2 over $S$. In particular, when $p \geq 3$ we have an equality of sheaves of groups $\text{Pic}_{Y/S}^T \cong \mathbb{Z}/2\mathbb{Z}$ with generator $\omega_{Y/S}$. Further, in arbitrary characteristic, the quotient $\text{Pic}_{Y/S}/\text{Pic}_{Y/S}^T$ is a locally constant sheaf of torsion-free finitely generated abelian groups. In characteristic $p \geq 3$ this implies that there exists an étale covering $\{U_i \to S\}_{i \in I}$ such that we have an isomorphism $\text{Pic}_{U_i/S} \cong \Gamma \oplus \mathbb{Z}/2\mathbb{Z}$ for each $i \in I$. 

Definition 5.2. A $\Gamma$-marking of a family $f: \mathcal{Y} \to S$ of supersingular Enriques surfaces is the choice of a morphism $\tilde{\gamma}: \mathcal{Y} \to \text{Pic}_{\mathcal{Y}/S}$ of group objects in the category of algebraic spaces compatible with the intersection forms. Analogously we define the notion of a $\Gamma'$-marking. There are obvious notions of morphisms of families of marked supersingular Enriques surfaces.

As before, we will in the following always assume that $p \neq 2$. We first show that for any family of $\Gamma'$-marked supersingular Enriques surfaces there exists a canonical family of supersingular K3 surfaces which covers it.

Proposition 5.3 (and Definition). Given a family of $\Gamma'$-marked supersingular Enriques surfaces $(f: \mathcal{Y} \to S, \tilde{\gamma}: \Gamma' \to \text{Pic}_{\mathcal{Y}/S})$ there exists a family of supersingular K3 surfaces $\mathcal{X} \to S$ together with a morphism $\mathcal{X} \to \mathcal{Y}$ which makes $\mathcal{X}$ into a $\mathbb{Z}/2\mathbb{Z}$-torsor over $\mathcal{Y}$. Further, this family carries a canonical $\Gamma(2)$-marking $\gamma: \Gamma(2) \to \text{Pic}_{\mathcal{X}/S}$ induced from the $\Gamma'$-marking on $\mathcal{Y}$ and the tuple $(f: \mathcal{X} \to S, \gamma: \Gamma(2) \to \text{Pic}_{\mathcal{X}/S})$ is unique up to isomorphism. We call $\mathcal{X} \to \mathcal{Y}$ the canonical K3 cover of $\mathcal{Y}$.

Proof. Note that we always assume characteristic $p \neq 2$, thus for the Cartier dual we get the equality $\mathbb{Z}/2\mathbb{Z}^D = \mathbb{Z}/2\mathbb{Z}$. Let $(f: \mathcal{Y} \to S, \tilde{\psi}: \Gamma' \to \text{Pic}_{\mathcal{Y}/S})$ be a family of $\Gamma'$-marked supersingular Enriques surfaces. There is a unique isomorphism $\mathbb{Z}/2\mathbb{Z} \xrightarrow{\sim} \text{Pic}_{\mathcal{Y}/S}$ which corresponds to the unique $\mathbb{Z}/2\mathbb{Z}$-torsor $\mathcal{X} \to \mathcal{Y}$ of algebraic spaces over $S$, cf. [Ray70 Proposition 6.2.1].

The morphism $\mathcal{X} \to \mathcal{Y}$ is finite and \'etale, thus it follows that $\mathcal{X} \to S$ is proper and smooth. Further, every fiber $\mathcal{X}_s \to \mathcal{Y}_s$ is just the universal K3 cover of the Enriques surface $\mathcal{Y}_s$ and it follows that $\mathcal{X} \to S$ is a family of supersingular K3 surfaces.

Pullback of line bundles induces a morphism $\text{Pic}_{\mathcal{Y}/S} \to \text{Pic}_{\mathcal{X}/S}$ of group objects in the category of algebraic spaces over $S$, and because the morphism $\mathcal{X} \to \mathcal{Y}$ is unramified and 2-to-1, the intersection form under this morphism gets multiplied by 2. In other words, after twisting the intersection form of $\text{Pic}_{\mathcal{Y}/S}$ by the factor 2, we obtain a morphism $\text{Pic}_{\mathcal{Y}/S}(2) \to \text{Pic}_{\mathcal{X}/S}$ of group objects in the category of algebraic spaces over $S$ compatible with intersection forms. Now precomposing with the marking $\psi_0(2): \Gamma(2) \to \text{Pic}_{\mathcal{Y}/S}(2)$ yields an embedding $\gamma: \Gamma(2) \to \text{Pic}_{\mathcal{X}/S}$.

Next, we show that any $\Gamma$-marking on a family of supersingular Enriques surfaces extends in a unique way to a $\Gamma'$-marking.

Lemma 5.4. Let $S$ be an algebraic space over $\mathbb{F}_p$. The forgetful functor

\[ \{ \text{Families of } \Gamma' \text{-marked supersingular Enriques surfaces } \} \to \{ \text{Families of } \Gamma \text{-marked supersingular Enriques surfaces } \} \]

is an equivalence of categories.

Proof. The automorphism group of the constant group scheme $\mathbb{Z}/2\mathbb{Z}$ is trivial. Thus, every $\Gamma$-marking extends \'etale locally in a unique way to a $\Gamma'$-marking and by uniqueness to a global $\Gamma'$-marking.

We now consider the functor

\[ \mathcal{E}_\sigma: \mathcal{A}^{\mathcal{Y}/S}_\sigma \to \{ \text{Sets} \} \]

\[ S \mapsto \{ \text{Isomorphism classes of families of } \Gamma' \text{-marked supersingular Enriques surfaces } \}

\[ (f: \mathcal{Y} \to S, \tilde{\gamma}: \Gamma' \to \text{Pic}_{\mathcal{Y}/S}) \]

\[ \text{such that the canonical K3 cover } \mathcal{X} \to \mathcal{Y} \]

\[ \text{admits an } N_\sigma \text{-marking} \} \].

We are interested in the representability of the moduli functor $\mathcal{E}_\sigma$. In the following proposition we show that the functor $\mathcal{E}_\sigma$ is isomorphic to the functor $\tilde{\mathcal{E}}_\sigma$ from Section 4.

Proposition 5.5. There exists an isomorphism of functors $\text{cov}: \mathcal{E}_\sigma \to \tilde{\mathcal{E}}_\sigma$. 

Proof. We first define the morphism \( 	ext{cov}: \mathcal{E}_\sigma \to \tilde{\mathcal{E}}_\sigma \). To this end, we consider a family of \( \Gamma' \)-marked supersingular Enriques surfaces \( y = (f: \mathcal{Y} \to S, \gamma: \Gamma(2) \to \text{Pic}_{\mathcal{X}/S}) \) which has the canonical K3 cover \( (f: \mathcal{X} \to S, \gamma: \Gamma(2) \to \text{Pic}_{\mathcal{X}/S}) \). If \( s: \text{Spec} \mathbb{F} \to S \) is a geometric point, then the orthogonal complement of \( \gamma_s(\Gamma(2)) \) in \( \text{NS}(\mathcal{X}_s) \) contains no \((-2)\)-vector. Since the fiber \( \mathcal{Y}_s \) is projective, it has an ample divisor. Pullback along finite morphisms preserves ampleness of divisors, so the sublattice \( \gamma_s(\Gamma(2)) \to \text{NS}(\mathcal{X}_s) \) also contains an ample divisor. We can thus define \( \text{cov}(S)(y) = (f: \mathcal{X} \to S, \gamma: \Gamma(2) \to \text{Pic}_{\mathcal{X}/S}) \) and this clearly yields a morphism of functors.

We will now define another morphism of functors \( \text{quot}: \tilde{\mathcal{E}}_\sigma \to \mathcal{E}_\sigma \) such that the morphisms \( \text{cov} \) and \( \text{quot} \) are mutually inverse to each other. To this end, we let \( S \) be a scheme and let \( x = (f: \mathcal{X} \to S, \gamma: \Gamma(2) \to \text{Pic}_{\mathcal{X}/S}) \in \mathcal{E}_\sigma(S) \). We consider the involution \( \iota_x: \mathcal{X} \to \mathcal{X} \) from the proof of Proposition 3.2. Then \( \iota_x \) induces a free \( \langle \iota_x \rangle \)-action on \( \mathcal{X} \) and thus the quotient \( \mathcal{Y} = \mathcal{X}/\langle \iota_x \rangle \) exists as an algebraic space over \( S \) and the morphism \( \mathcal{X} \to \mathcal{Y} \) makes \( \mathcal{X} \) into a \( \mathbb{Z}/2\mathbb{Z} \)-torsor over \( \mathcal{Y} \). Thus, for every \( s \in S \), \( \mathcal{X}_s \) is a \( \mathbb{Z}/2\mathbb{Z} \)-torsor over \( \mathcal{X}_s \) and it follows that \( \mathcal{Y}_s \) is a supersingular Enriques surface for each \( s \in S \). Further, the canonical morphism \( \text{Pic}_{\mathcal{Y}/S} \to \text{Pic}_{\mathcal{X}/S} \) induces an isomorphism \( \psi: \text{Pic}_{\mathcal{Y}/S}(2) \to \gamma(\Gamma(2)) \). We define \( \tilde{\gamma}: \Gamma' \to \text{Pic}_{\mathcal{Y}/S} \) to be the unique \( \Gamma' \)-marking of \( \text{Pic}_{\mathcal{Y}/S} \) which is induced from \( \psi^{-1} \) using Lemma 5.4. Now setting \( \text{quot}(S)(x) = (f: \mathcal{Y} \to S, \tilde{\gamma}: \Gamma' \to \text{Pic}_{\mathcal{Y}/S}) \) yields the desired inverse.

The following theorem, which is one of the main results in this work, can be seen as a supersingular version of the results on complex Enriques surfaces in [Nam85] or as a version for Enriques surfaces of the results on supersingular K3 surfaces in [Ogu83].

Theorem 5.6. The functor \( \mathcal{E}_\sigma \) is represented by a quasi-separated algebraic space \( \mathcal{E}_\sigma \) which is locally of finite type over \( \mathbb{F}_p \) and there exists a separated \( \mathbb{F}_p \)-scheme \( Q_\sigma \) of finite type and \( AF \), and a canonical étale surjective morphism \( \pi^{E}_{\sigma}: \mathcal{E}_\sigma \to Q_\sigma \).

Proof. This follows directly from Theorem 4.7, Proposition 4.11 and Proposition 5.5.

Remark 5.7. It follows from [Jan13] Proposition 3.5] that for any \( \sigma \geq 5 \) we have a canonical isomorphism \( \mathcal{E}_\sigma \overset{\sim}{\longrightarrow} \mathcal{E}_5 \).

The previous remark motivates the following definition.

Definition 5.8. We call \( \mathcal{E}_5 \) the modal space of \( \Gamma' \)-marked supersingular Enriques surfaces and \( Q_5 \) the period space of \( \Gamma' \)-marked supersingular Enriques surfaces.

Remark 5.9. From the constructions it follows directly that, similar to the case of marked supersingular K3 surfaces, there are canonical stratifications \( \mathcal{E}_1 \hookrightarrow \mathcal{E}_2 \hookrightarrow \mathcal{E}_3 \hookrightarrow \mathcal{E}_4 \hookrightarrow \mathcal{E}_5 \) and \( Q_1 \hookrightarrow Q_2 \hookrightarrow Q_3 \hookrightarrow Q_4 \hookrightarrow Q_5 \) via closed immersions. However, the latter are not sections to fibrations of the form \( Q_\sigma \to Q_{\sigma-1} \). The main difference to the situation for marked supersingular K3 surfaces, and therefore the reason why such a fibration does not exist, is the following. While the embedding \( M_{\sigma-1} \hookrightarrow M_\sigma \) depends on the choice of an embedding \( j: N_\sigma \to N_{\sigma-1} \), the embedding \( Q_{\sigma-1} \hookrightarrow Q_\sigma \) corresponds to the union over all images of such embeddings \( M_{\sigma-1} \hookrightarrow M_\sigma \), but the inclusion \( \bigcup_{j \in K_{\sigma-1}} \Phi(j(\mathcal{M}_{\sigma-1})) \hookrightarrow M_\sigma \) does not have an inverse.

Remark 5.10. The period spaces \( Q_\sigma \) come with canonical compactifications which we denote \( Q_\sigma^1 \). Namely, we consider the functor

\[
\tilde{\mathcal{E}}_\sigma^1: \mathcal{A}^{\text{gp}}_{\mathbb{G}_m} \longrightarrow \text{(Sets)}
\]

\[
S \longmapsto \left\{ \text{Isomorphism classes of families of supersingular } \begin{array}{ll}
\mathcal{E}_\sigma \hookrightarrow \text{K3 surfaces } f: \mathcal{X} \to S \text{ admitting an } N_\sigma\text{-marking } \\
together \text{with an embedding } \gamma: \Gamma(2) \to \text{Pic}_{\mathcal{X}/S} \\
such that for each geometric fiber } s \in S \\
\text{the sublattice } \gamma_s(\Gamma(2)) \hookrightarrow \text{NS}(\mathcal{X}_s) \\
\text{contains an ample line bundle} \right\}.
\]
By an argument analogous to the proof of Theorem \[4.47\] it follows that the functor \(\tilde{\mathcal{E}}^1\) is representable by a quasi-separated algebraic space \(\tilde{E}\) which is locally of finite type over \(\mathbb{F}_p\). Further, there exists a proper \(\mathbb{F}_p\)-scheme \(\mathcal{Q}_1^\dagger\) and a canonical étale surjective morphism \(\tilde{E}^1 \rightarrow \mathcal{Q}_1^\dagger\) by an argument analogous to the one in the proof of Proposition \[4.11\].

The scheme \(\mathcal{Q}_1^\dagger\) is indeed proper because inductively there exists a finite surjection of the proper \(\mathbb{F}_p\)-scheme \(\mathcal{Q}_{\sigma-1}^\dagger \amalg \bigcup_{\gamma \in R_{\sigma}} \tilde{\mathcal{M}}_\gamma\) onto \(\mathcal{Q}_1^\dagger\). The canonical morphism of schemes \(\mathcal{Q}_{\sigma} \rightarrow \mathcal{Q}_1^\dagger\) is open immersion and a subscheme of the closed locus \(\mathcal{Q}_1^\dagger \backslash \mathcal{Q}_{\sigma}\) corresponds to quotients of K3 surfaces by involutions that fix a divisor. This is an analogue to the so-called Coble locus in the characteristic zero setting, see [DK13].

6. Some remarks about the geometry of the moduli space \(\mathcal{E}_\sigma\)

The geometry of \(\mathcal{E}_\sigma\) is quite complicated. However, it is clear that the algebraic space \(\mathcal{E}_\sigma\) is reduced, but in general it will not be connected, since already in the case \(\sigma = 1\) it has multiple connected components.

Moreover, we can not expect the connected components of \(\mathcal{E}_\sigma\) to be irreducible, since they are glued together from the algebraic spaces \(\tilde{S}_\gamma'^\dagger\) with \(\gamma \in R_{\sigma}\) and we can not expect the irreducible components to be smooth: a priori the action of \(O(N_{\sigma}, \gamma)\) on \(S_{\gamma'}\) which we took the quotient by is not free and we do not expect it to factorize over a free action.

Further, when taking the pushout in the proof of Theorem \[4.47\] we expect more singularities to show up. However, there are some simple general observations on the geometry of the algebraic space \(\mathcal{E}_\sigma\).

We will first introduce a subfunctor \(\tilde{\mathcal{E}}'_\sigma\) of \(\tilde{\mathcal{E}}_\sigma\) to help us understand the geometry of the algebraic space \(\tilde{\mathcal{E}}_\sigma \cong \tilde{\mathcal{E}}\). We define

\[
\tilde{\mathcal{E}}'_\sigma : \mathcal{A}^{op}_{\mathbb{F}_p} \rightarrow \text{(Sets)}
\]

\[
S \mapsto \begin{cases}
\text{Isomorphism classes of families of supersingular K3 surfaces } f : \mathcal{X} \rightarrow S \\
\text{together with a marking } \gamma : \Gamma(2) \hookrightarrow \text{Pic}_{\mathcal{X}/S} \\
\text{such that there exists an embedding } \\
\psi : N_{\sigma} \hookrightarrow \text{Pic}_{\mathcal{X}/S} \text{ with } \gamma(\Gamma(2)) \subset N_{\sigma} \text{ and } \\
\text{such that for each geometric fiber } s \in S \\
\text{the sublattice } \gamma_s(\Gamma(2)) \hookrightarrow \text{NS}(\mathcal{X}_s) \\
\text{contains an ample line bundle and } \\
\gamma_s(\Gamma(2))^{-1} \hookrightarrow \text{NS}(\mathcal{X}_s) \text{ contains no } (-2)\text{-vector}
\end{cases}
\]

The proof of the following proposition goes similarly to the proof of Theorem \[4.47\]. We therefore only highlight the main differences in the proof.

**Proposition 6.1.** The functor \(\tilde{\mathcal{E}}'_\sigma\) is representable by a closed algebraic subspace \(\tilde{\mathcal{E}}'_\sigma\) of \(\tilde{\mathcal{E}}_\sigma\).

**Proof.** We do induction over \(\sigma\). The case \(\sigma = 1\) is clear, because in this case we have \(\tilde{\mathcal{E}}'_1 \cong \tilde{\mathcal{E}}'_1\).

We write \(\tilde{\mathcal{E}}'^{\alpha}_{\sigma-1}\) for the subfunctor of \(\tilde{\mathcal{E}}_{\sigma-1}\) which is defined to be as follows: the \(S\)-valued points of \(\tilde{\mathcal{E}}'^{\alpha}_{\sigma-1}\) are the families \(f : \mathcal{X} \rightarrow S\) in \(\tilde{\mathcal{E}}'_{\sigma-1}(S)\) that admit markings of the form \(\gamma : \Gamma(2) \hookrightarrow N_{\sigma-1} \hookrightarrow \text{Pic}_{\mathcal{X}/S}\) such that there is a factorization \(\gamma : \Gamma(2) \rightarrow N_{\sigma} \hookrightarrow N_{\sigma-1} \hookrightarrow \text{Pic}_{\mathcal{X}/S}\).

Then \(\tilde{\mathcal{E}}'^{\alpha}_{\sigma-1} \subseteq \tilde{\mathcal{E}}'_{\sigma-1}\) is a closed subfunctor, since \(\tilde{\mathcal{E}}'^{\alpha}_{\sigma-1}\) is representable by the image of the finite morphism \(p_\sigma : p_\sigma^{-1}(\tilde{\mathcal{E}}_{\sigma-1}) \rightarrow \tilde{\mathcal{E}}_{\sigma-1}\). We consider the pushout diagram

\[
p_\sigma^{-1}(\tilde{\mathcal{E}}'^{\alpha}_{\sigma-1}) \xrightarrow{\epsilon} \coprod_{\gamma \in R_{\sigma}} \tilde{\mathcal{E}}'^{\alpha}_\gamma \\
p_\sigma \downarrow \quad \quad \downarrow \quad \quad \downarrow \\
\tilde{\mathcal{E}}'^{\alpha}_{\sigma-1} \quad \quad \tilde{\mathcal{E}}_{\sigma-1} \quad \quad \mathcal{P}.
\]
We note that $p_\sigma: p_\sigma^{-1}(E_{\sigma-1}^\prime) \to E_{\sigma-1}^\prime$ is finite surjective and therefore also $\bigsqcup_{\gamma \in R_\sigma} S'^\gamma_\gamma \to P$ is finite surjective. Analogously to the proof of Theorem 4.7 we can show that $P$ exists as an algebraic space and represents the functor $E_{\sigma-1}^\prime$. Thus, we set $E_{\sigma-1}^\prime = P$. Since $E_{\sigma-1}^\prime$ is closed in $E_{\sigma-1}$ it follows from the construction of the algebraic space $E_\sigma$ that $E_{\sigma-1}^\prime$ is a closed subspace of $E_\sigma$. □

Again, the functor $E_{\sigma}^\prime$ has a description in terms of Enriques surfaces. Namely, we define $E_{\sigma}^\prime: \mathcal{S} \to (\text{Sets})$

$$S \mapsto \left\{ \begin{array}{l} \text{Isomorphism classes of families of } \Gamma\text{-marked} \\ \text{supersingular Enriques surfaces } (f: \mathcal{Y} \to S, \gamma: \Gamma \to \text{Pic}_S) \\ \text{such that the canonical K3 cover } X \to \mathcal{Y} \text{ admits} \\ \text{an } N_\sigma\text{-marking such that the induced map} \\ \Gamma(2) \to \text{Pic}_X/S \text{ factorizes through } N_\sigma \end{array} \right\}.$$ 

The proof of the following proposition goes completely analogously to the proof of Proposition 5.5 and we therefore leave it to the reader.

**Proposition 6.2.** There exists an isomorphism of functors $\text{cov}: E_{\sigma}^\prime \to E_{\sigma}^\prime$.

We will write $E_{\sigma}^\prime$ for the algebraic space representing the functor $E_{\sigma}^\prime$. Coming back to the discussion of the geometry of the space $E_\sigma$, we note that the space $E_\sigma$ is of dimension $\sigma - 1$, but its irreducible components might in general not be equidimensional. The upshot of constructing the functor $E_{\sigma}^\prime$ lies in the following result.

**Proposition 6.3.** For any $\sigma' \leq \sigma$, the algebraic space $E_{\sigma'}$, is a closed subspace of $E_{\sigma}$ and we have the equality

$$\bigcup_{\sigma' \leq \sigma} E_{\sigma'} = E_{\sigma}.$$ 

Further, $E_{\sigma}$ is the maximal closed subspace in $E_{\sigma}$ with the property that all of its irreducible components are of dimension $\sigma - 1$.

**Proof.** The first statement follows from the construction of the space $E_{\sigma}$ via induction over $\sigma$ and the second statement follows directly from the construction of $E_{\sigma'}$ and $E_{\sigma}$ and the fact that the morphism $\bigsqcup_{\gamma \in R_\sigma} S'^\gamma_\gamma \to E_{\sigma'}$ is a finite surjection. □

**Remark 6.4.** We do not know if the functors $E_{\sigma}$ and $E_{\sigma}^\prime$ are unequal in general. This boils down to asking whether there exist embeddings $\Gamma(2) \hookrightarrow N_{\sigma-1}$ that do not factorize over an embedding $j: N_{\sigma} \hookrightarrow N_{\sigma-1}$. However, we suspect that such embeddings may exist and that for $\sigma > 1$ we should have $E_{\sigma} \neq E_{\sigma}^\prime$.

There exists a scheme lying under $E_{\sigma}$ in analogy to Proposition 4.11.

**Proposition 6.5.** There exists a separated $\mathbb{F}_p$-scheme $Q_{\sigma}^\prime$, which is a closed subscheme of $Q_\sigma$, and a canonical étale surjective morphism $E_{\sigma}^\prime \to Q_{\sigma}^\prime$.

**Proof.** The proof goes analogously to the proof of Proposition 4.11 by replacing $Q_{\sigma-1}$ with the image of $p_\sigma^{-1}(Q_{\sigma-1})$ in $Q_{\sigma-1}$ in the pushout construction. □

The following proposition is an analogue to Proposition 6.3.

**Proposition 6.6.** For any $\sigma' \leq \sigma$, the scheme $Q_{\sigma'}^\prime$ is a closed subscheme of $Q_{\sigma}$ and we have an equality

$$\bigcup_{\sigma' \leq \sigma} Q_{\sigma'}^\prime = Q_{\sigma}.$$ 

Further, $Q_{\sigma}^\prime$ is the maximal closed subscheme in $Q_{\sigma}$ whose irreducible components are all of dimension $\sigma - 1$. 

In the following, we give some results on the geometry of the spaces $E'_\sigma$ and $Q'_\sigma$. It follows from Proposition 6.3 and Proposition 6.6 that the geometry of these spaces is intimately related to the geometry of the spaces $E_\sigma$ and $Q_\sigma$.

**Definition 6.7.** We write $\varepsilon_\sigma$ for the number of irreducible components of $E'_\sigma$.

**Remark 6.8.** We recall from Section 2 that the $\mathbb{F}_p$-scheme $S_\sigma$ is smooth. In particular each of its connected components is irreducible. From its description as the moduli space of characteristic subspaces together with ample cones it is clear that $S_\sigma$ only has finitely many connected components.

**Proposition 6.9.** The morphism $p_\sigma: \coprod_{\gamma \in R_\sigma} \tilde{S}'_\gamma \to E'_\sigma$ induces a bijection between the sets of irreducible components of $\coprod_{\gamma \in R_\sigma} \tilde{S}'_\gamma$ and $E'_\sigma$. If we write $\tau_\sigma$ for the number of connected components of $S_\sigma$, we obtain the inequality

$$\varepsilon_\sigma \leq \tau_\sigma \cdot |R_\sigma|.$$  

**Proof.** For $\gamma \in R_\sigma$, each irreducible component of the algebraic space $\tilde{S}'_\gamma$ over $\mathbb{F}_p$ is of dimension $\sigma - 1$. Since there exists a dense open subspace $U \subseteq \coprod_{\gamma \in R_\sigma} \tilde{S}'_\gamma$ such that the restriction $p_\sigma|_U: U \to E'_\sigma$ is an open immersion, it follows that if $E_1, E_2 \subseteq \coprod_{\gamma \in R_\sigma} \tilde{S}'_\gamma$ are two different irreducible components, then the intersection $p_\sigma(E_1) \cap p_\sigma(E_2)$ is at least of codimension 1. Thus, the morphism $p_\sigma$ induces a bijection between the sets of irreducible components of $\coprod_{\gamma \in R_\sigma} \tilde{S}'_\gamma$ and $E'_\sigma$. The inequality follows from the fact that the open subscheme $S'_\gamma \subset S_\sigma$ surjects onto $\tilde{S}'_\gamma$ and each connected component of $S_\sigma$ is irreducible. \hfill \Box

**Proposition 6.10.** There is an equality

$$\# \{\text{irreducible components of } Q'_\sigma \} = |R_\sigma|.$$

**Proof.** This follows since the schemes $\tilde{M}'_\gamma$ are irreducible and there is a dense open subscheme of $\coprod_{\gamma \in R_\sigma} \tilde{M}'_\gamma$ which is isomorphic to a dense open subscheme of $Q'_\sigma$. \hfill \Box

**Definition 6.11.** On the set $R_\sigma$ of isomorphism classes $[\gamma: \Gamma(2) \hookrightarrow N_\sigma]$ of embeddings of lattices we define an equivalence relation via

$$[\gamma: \Gamma(2) \hookrightarrow N_\sigma] \sim [\gamma': \Gamma(2) \hookrightarrow N_\sigma]$$

if and only if there exists a positive integer $\sigma' \leq \sigma$ and embeddings $j: N_\sigma \hookrightarrow N_{\sigma'}$ and $j': N_{\sigma'} \hookrightarrow N_\sigma$, such that the sublattice $j(\gamma(\Gamma(2)))/j'(\gamma(\Gamma(2)))) \subset N_{\sigma'}$ contains no $(\sim 2)$-vectors and such that there is an equality

$$[j \circ \gamma: \Gamma(2) \hookrightarrow N_{\sigma'}] = [j' \circ \gamma': \Gamma(2) \hookrightarrow N_{\sigma'}]$$

of elements in $R_{\sigma'}$.

Using this equivalence relation we obtain the following results.

**Proposition 6.12.** There is an equality

$$\# \{\text{connected components of } Q'_\sigma \} = |R_\sigma/ \sim|.$$  

**Proof.** It follows from the construction in the proof of Proposition 6.6 that under the surjection of schemes $\coprod_{\gamma \in R_\sigma} \tilde{M}'_\gamma \to Q'_\sigma$ two connected components $\tilde{M}'_{\gamma_1}$ and $\tilde{M}'_{\gamma_2}$ map to the same connected component of $Q'_\sigma$ if and only if $\gamma_1 \sim \gamma_2$. \hfill \Box

**Proposition 6.13.** We write $\tau_\sigma$ for the number of connected components of $S_\sigma$ and $\varepsilon'_\sigma$ for the number of connected components of $E'_\sigma$. There is an inequality

$$\varepsilon'_\sigma \leq \tau_\sigma \cdot |R_\sigma/ \sim|.$$
Proof. We consider the surjection of algebraic spaces $\coprod_{\gamma \in R_{\sigma}} \tilde{S}_{\gamma}^\prime \to \mathcal{E}_{\sigma}^\prime$. For each $\gamma \in R_{\sigma}$, the algebraic space $\tilde{S}_{\gamma}^\prime$ has at most $\tau_{\sigma}$ many connected components. If $\gamma_1 \sim \gamma_2$, say with $[j_1 \circ \gamma_1] = [j_2 \circ \gamma_2]$, then $\tilde{S}_{j_1 \circ \gamma_1}^\prime \cong \tilde{S}_{j_2 \circ \gamma_2}^\prime$ is a subspace of both $\tilde{S}_{\gamma_1}^\prime$ and $\tilde{S}_{\gamma_2}^\prime$ which touches each of the connected components of the $\tilde{S}_{\gamma_i}^\prime$. Thus, the image of $\tilde{S}_{\gamma_i}^\prime$ in $\mathcal{E}_{\sigma}^\prime$ has at most $\tau_{\sigma}$ many connected components and this implies the statement of the proposition. □

Proposition 6.14. We denote by $\alpha_{\sigma}$ the number of isomorphism classes $[\gamma : \Gamma(2) \hookrightarrow N_{\sigma}]$ in $R_{\sigma}$ such that that for each positive integer $\sigma' < \sigma$ and each embedding of lattices $j : N_{\sigma} \hookrightarrow N_{\sigma'}$, there is a $-2$-vector in the sublattice $j(\gamma(\Gamma(2)))^\perp \subset N_{\sigma'}$. Then we have an inequality

$$\alpha_{\sigma} \leq \varepsilon_{\sigma} \leq \tau_{\sigma} \cdot (\alpha_{\sigma} + \varepsilon_{\sigma-1}).$$

Proof. The lower bound is a very weak estimate: if $\gamma$ is such that for each positive integer $\sigma' < \sigma$ and each $j : N_{\sigma} \hookrightarrow N_{\sigma'}$, there is a $(-2)$-vector in the sublattice $j(\gamma(\Gamma(2)))^\perp \subset N_{\sigma'}$, then $[\gamma]$ is the only element in its equivalence class of $\sim$. Hence, the image of $\tilde{S}_{\gamma}^\prime$ in $\mathcal{E}_{\sigma}^\prime$ is disjoint from the image of any $\tilde{S}_{\gamma'}^\prime$ in $\mathcal{E}_{\sigma}^\prime$ for all $\gamma' \neq \gamma$.

For the upper bound, we remark that each $\gamma \in R_{\sigma}$ is either as above, or there exists a positive integer $\sigma' < \sigma$ and an element $\gamma' \in R_{\sigma'}$ such that the images of $\tilde{S}_{\gamma'}^\prime$ in $\mathcal{E}_{\sigma}^\prime$, $\tilde{S}_{\gamma}^\prime$ and $\tilde{S}_{\gamma'}^\prime$ intersect non-trivially. □

Analogously to the compactification $\mathcal{E}_{\sigma}^1$ of $\mathcal{E}_{\sigma}$, we can construct a compactification $\mathcal{E}_{\sigma}^{\tilde{1}}$ of $\mathcal{E}_{\sigma}^\prime$. In analogy to Proposition 6.13 we have the following proposition.

Proposition 6.15. For any $\sigma' \leq \sigma$, the algebraic space $\mathcal{E}_{\sigma}^{\tilde{1}}$ is a closed subspace of $\mathcal{E}_{\sigma}^1$ and we have the equality

$$\bigcup_{\sigma' \leq \sigma} \mathcal{E}_{\sigma}^{\tilde{1}} = \mathcal{E}_{\sigma}^1.$$ 

Further, $\mathcal{E}_{\sigma}^{\tilde{1}}$ is the maximal closed subspace in $\mathcal{E}_{\sigma}^1$ with the property that all of its irreducible components are of dimension $\sigma - 1$.

We leave the proof to the reader and obtain the following result.

Proposition 6.16. There are inequalities

$$\# \{\text{connected components of } \mathcal{E}_{\sigma}^{\tilde{1}} \} \leq \# \{\text{connected components of } \mathcal{E}_{\sigma}^{\tilde{1}} \}$$

and

$$\# \{\text{irreducible components of } \mathcal{E}_{\sigma}^1 \} \leq \# \{\text{irreducible components of } \mathcal{E}_{\sigma}^{\tilde{1}} \}.$$ 

Proof. The proof of the first inequality goes analogously to the proof of the upper bound in the previous proposition. The second inequality is clear since $\mathcal{E}_{\sigma}^1$ is an open algebraic subspace in $\mathcal{E}_{\sigma}^{\tilde{1}}$. □

7. Torelli theorems for supersingular Enriques surfaces

The algebraic spaces $\mathcal{E}_{\sigma}$ are fine moduli spaces for $\Gamma'$-marked supersingular Enriques surfaces with Artin invariant at most $\sigma$, but their geometry is very complicated. However, it turns out that the much nicer schemes $Q_{\sigma}$ from Proposition 4.11 are coarse moduli spaces for this moduli problem. The next proposition is a direct consequence of the Torelli theorem for supersingular K3 surfaces [Ogu83] and does not use any of our prior results.

Proposition 7.1. Let $Y$ and $Y'$ be supersingular Enriques surfaces over an algebraically closed field $k$ of characteristic $p \geq 3$ which have universal K3 covers $X$ and $X'$ respectively. Let $\tilde{\phi} : \text{NS}(Y) \to \text{NS}(Y')$ be a morphism of lattices that maps the ample cone of $Y$ to the ample cone of $Y'$ and such that the induced morphism of lattices $\phi : \text{NS}(X) \to \text{NS}(X')$ extends via the first Chern map to an isomorphism $H^2_{\text{crys}}(X/W) \cong H^2_{\text{crys}}(X'/W)$. Then $\phi$ is induced from an isomorphism $\Phi : Y \to Y'$ of supersingular Enriques surfaces.
Proof. This follows immediately from a version of the Torelli theorem for supersingular K3 surfaces \cite{Ogu83}, cf. Theorem II] and the fact that pullback along finite morphisms preserves ampleness of divisors.

We next want to show that the schemes \( Q_{\sigma} \) are coarse moduli spaces for Enriques surfaces in the sense that their points parametrize isomorphism classes of Enriques surfaces without having to choose any kind of marking.

**Definition 7.2.** Recall from Theorem 5.6 that there is a canonical étale surjective morphism \( \pi^E: \mathcal{E}_\sigma \to Q_{\sigma} \). If \( Y \) is a supersingular Enriques surface of Artin invariant \( \sigma' \leq \sigma \) over an algebraically closed field \( k \) of characteristic \( p \geq 3 \), we define the period \( \pi^E_{\sigma} \) of \( Y \) in \( Q_{\sigma} \) to be \( \pi^E_{\sigma}(Y) = \pi^E_{\sigma}(Y, \gamma) \), where \( \gamma \) is any \( \Gamma \)-marking of \( Y \).

The following proposition shows that \( \pi^E_{\sigma} \) is well-defined and does not depend on the chosen marking.

**Proposition 7.3.** Let \( k \) be an algebraically closed field of characteristic \( p \geq 3 \), let \( \sigma \leq 5 \) be a positive integer and let \( Y \) be a supersingular Enriques surface of Artin invariant at most \( \sigma \) over \( k \). For any choice of markings \( \tilde{\gamma}_1: \Gamma \to \text{NS}(Y) \) and \( \tilde{\gamma}_2: \Gamma \to \text{NS}(Y) \) we have an equality \( \pi^E_{\sigma}(Y, \tilde{\gamma}_1) = \pi^E_{\sigma}(Y, \tilde{\gamma}_2) \). In other words, the period of \( Y \) in \( Q_{\sigma} \) is independent of the choice of a marking.

**Proof.** From the construction of \( Q_{\sigma} \) in Proposition 4.11 and the discussion in \cite{Ogu79} §4 and §5 it follows that the scheme \( Q_{\sigma} \) represents the functor that associates to a smooth scheme \( S \) the set of isomorphism classes of families of K3 crystals \( H \) over \( S \) together with maps \( \gamma: \Gamma(2) \to T_H \) that are compatible with intersection forms and such that there exists a factorization \( \gamma: \Gamma(2) \to N_\sigma \to T_H \) without \((-2)\)-vectors in the orthogonal complement \( \gamma(\Gamma(2))^\perp \subset N_\sigma \).

Now, let \( Y \) be a supersingular Enriques surface which has the universal K3 covering \( X \to Y \) and we let \( \tilde{\gamma}_1: \Gamma \to \text{NS}(Y) \) and \( \tilde{\gamma}_2: \Gamma \to \text{NS}(Y) \) be two choices of markings. We consider the period points
\[
\pi^E_{\sigma}(Y, \tilde{\gamma}_1) = [\gamma_1: \Gamma(2) \hookrightarrow T_{H^2_{\text{crys}}(X/W)} \hookrightarrow H^2_{\text{crys}}(X/W)]
\]
and
\[
\pi^E_{\sigma}(Y, \tilde{\gamma}_2) = [\gamma_2: \Gamma(2) \hookrightarrow T_{H^2_{\text{crys}}(X/W)} \hookrightarrow H^2_{\text{crys}}(X/W)].
\]
We have that \( \text{disc}(\Gamma(2)) = -2^{10} \), therefore \( \gamma_1(\Gamma(2)) \otimes W = \gamma_2(\Gamma(2)) \otimes W \subset H^2_{\text{crys}}(X/W) \) is a unimodular \( W \)-sublattice, and we can write \( H^2_{\text{crys}}(X/W) = K \oplus L \) for some sublattice \( L \subset H^2_{\text{crys}}(X/W) \) and \( K = \gamma_1(\Gamma(2)) \otimes W \). Since the sublattice \( K \) is contained in \( T_{H^2_{\text{crys}}(X/W)} \), it follows that \( K \) is closed under the Frobenius action on \( H^2_{\text{crys}}(X/W) \) and therefore its orthogonal complement \( L = K^\perp \) is also closed under this action. Thus, the automorphism of the K3 crystal \( H^2_{\text{crys}}(X/W) \) given by \( (\gamma_2 \circ \gamma_1^{-1}, \text{id}_L): K \oplus L \to K \oplus L \) induces an isomorphism
\[
(\gamma_1: \Gamma(2) \hookrightarrow T_{H^2_{\text{crys}}(X/W)} \hookrightarrow H^2_{\text{crys}}(X/W)) \cong (\gamma_2: \Gamma(2) \hookrightarrow T_{H^2_{\text{crys}}(X/W)} \hookrightarrow H^2_{\text{crys}}(X/W))
\]
of \( \Gamma(2) \)-structures on \( H^2_{\text{crys}}(X/W) \) and it follows that \( \pi^E_{\sigma}(Y, \tilde{\gamma}_1) = \pi^E_{\sigma}(Y, \tilde{\gamma}_2) \). □

**Theorem 7.4.** Let \( Y_1 \) and \( Y_2 \) be supersingular Enriques surfaces. Then \( Y_1 \) and \( Y_2 \) are isomorphic if and only if \( \pi^E_{\sigma}(Y_1) = \pi^E_{\sigma}(Y_2) \) for some \( \sigma \leq 5 \).

**Proof.** It follows from Proposition 7.3 that writing \( \pi^E_{\sigma}(Y) \) makes sense since the period of \( Y \) does not depend on the choice of a marking. We also directly obtain the ‘only if’ part of the theorem as a consequence of Proposition 7.3. We now let \( Y_1 \) and \( Y_2 \) be supersingular Enriques surfaces with the same period point and let \( X_1 \to Y_1 \) and \( X_2 \to Y_2 \) be their canonical K3 covers. We choose two markings \( \gamma_1: \Gamma \to \text{NS}(Y_1) \) and \( \gamma_2: \Gamma \to \text{NS}(Y_2) \). These induce \( \Gamma(2) \)-markings \( \gamma_1: \Gamma(2) \hookrightarrow \text{NS}(X_1) \) and \( \gamma_2: \Gamma(2) \hookrightarrow \text{NS}(X_2) \), and we may choose extensions of the morphisms \( \gamma_i \) that are \( N_\sigma \)-markings \( \eta_1: N_{\sigma} \to \text{NS}(X_1) \) and \( \eta_2: N_{\sigma} \to \text{NS}(X_2) \). From
the construction of $Q_\sigma$ in Proposition 4.11, it follows that the markings $\gamma_1: \Gamma(2) \hookrightarrow \text{NS}(X_1)$ and $\gamma_2: \Gamma(2) \hookrightarrow \text{NS}(X_2)$ are isomorphic embeddings, say $[\gamma_1] = [\gamma_2] = [\gamma] \in R_\sigma$, and after applying some isometry $\varphi \in O(N_\sigma, \gamma)$ we may assume that the marked K3 surfaces $(X_1, \eta_1)$ and $(X_2, \eta_2)$ have the same period in $M_\sigma$. Hence, there exists an isomorphism of K3 crystals $\psi: H^2_{\text{crys}}(X_1) \longrightarrow H^2_{\text{crys}}(X_2)$ and a commutative diagram

$$
\begin{array}{cccccc}
\Gamma(2) & \gamma & N_\sigma & \eta_1 & \text{NS}(X_1) & H^2_{\text{crys}}(X_1) \\
\downarrow \text{id} & & \downarrow \text{id} & & \downarrow \psi & \\
\Gamma(2) & \gamma & N_\sigma & \eta_2 & \text{NS}(X_2) & H^2_{\text{crys}}(X_2).
\end{array}
$$

By a version of the Torelli theorem [Ogu83, cf. Theorem II] the isomorphism $\psi$ is induced by some isomorphism of K3 surfaces $\Psi: X_1 \rightarrow X_2$. Since $\psi(\gamma_1(\Gamma(2))) = \gamma_2(\Gamma(2))$, if $i_1: X_1 \rightarrow X_1$ and $i_2: X_2 \rightarrow X_2$ are the involutions induced by the $\gamma_i$, we have that $\Psi \circ i_1 = i_2 \circ \Psi$ and it follows that the morphism $\Psi$ descends to an isomorphism of the Enriques quotients $\tilde{\Psi}: Y_1 \rightarrow Y_2$. \qed
## Notation and list of symbols

For reference, here is a list of some of the objects we introduce. In general, if $F$ is a representable functor, we write $F$ for the object representing this functor.

| Symbol | Description |
|--------|-------------|
| $\mathcal{A}_p$ | category of algebraic spaces over $\mathbb{F}_p$ |
| $\mathcal{E}_\sigma$ | families in $\mathcal{E}_\sigma$ s.t. induced $\Gamma(2)$-marking of covering K3 factorizes over an $N_\sigma$-marking |
| $\mathcal{E}_\sigma'$ | families of $\Gamma'$-marked ssg. Enriques surfaces s.t. K3 cover admits $N_\sigma$-marking; isomorphic to $\tilde{\mathcal{E}}_\sigma$ |
| $\mathcal{E}_\sigma''$ | families in $\mathcal{E}_\sigma$ s.t. $\Gamma(2)$-marking factorizes over an $N_\sigma$-marking |
| $\mathcal{E}_\sigma$ | families of ssg. K3 surfaces admitting an $N_\sigma$-marking, with a $\Gamma(2)$-marking that has an ample bundle and no $-2$-vector in complement |
| $\tilde{\mathcal{E}}_\sigma$ | compactification of $\mathcal{E}_\sigma$ |
| $\Gamma$ | lattice $U_2 \oplus E_8(-1)$ |
| $\Gamma'$ | Neron Severi lattice of Enriques surface, $\Gamma \oplus \mathbb{Z}/2\mathbb{Z}$ |
| $\mathcal{M}_\sigma$ | period space of K3 crystals with Artin invariant $\leq \sigma$; characteristic generatrices of $N_\sigma$ |
| $\mathcal{M}_\sigma'$ | characteristic generatrices of Artin invariant $\sigma'$ in $N_\sigma$ |
| $\tilde{\mathcal{M}}_\sigma'$ | scheme lying under $\tilde{\mathcal{S}}_\sigma'$ |
| $\mathcal{M}_\gamma$ | scheme lying under $\mathcal{E}_\sigma$ |
| $\mathcal{N}_\sigma$ | K3 lattice of Artin invariant $\sigma$ |
| $\mathcal{Q}_\sigma$ | scheme lying under $\mathcal{E}_\sigma$; coarse moduli space of supersingular Enriques surfaces |
| $\tilde{\mathcal{Q}}_\sigma$ | compactification of $\mathcal{Q}_\sigma$ |
| $\mathcal{R}_\sigma$ | set of embeddings $\gamma: \Gamma(2) \hookrightarrow N_\sigma$ |
| $\mathcal{R}_\sigma'$ | set of embeddings $N_\sigma \hookrightarrow N_\sigma'$ |
| $\mathcal{S}_\sigma$ | families in $\mathcal{S}_\sigma$ s.t. induced $\Gamma(2)$-marking has an ample bundle |
| $\mathcal{S}_\sigma'$ | families of $N_\sigma$-marked ssg. K3 surfaces |
| $\mathcal{S}_\sigma''$ | families in $\mathcal{S}_\sigma$ of Artin invariant $\sigma'$ |
| $\mathcal{S}_\gamma'$ | families in $\mathcal{S}_\gamma'$ s.t. induced $\Gamma(2)$-marking has no $-2$-vector in complement |
| $\tilde{\mathcal{S}}_\gamma$ | families of ssg. K3 surfaces with a sublattice $\mathcal{R}$ in Pic s.t. induced $\Gamma(2)$-marking has an ample bundle |

## References

For reference, here is a list of some of the objects we introduce. In general, if $F$ is a representable functor, we write $F$ for the object representing this functor.

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