LOCAL STATISTICAL PROPERTIES OF SCHMIDT EIGENVALUES OF BIPARTITE ENTANGLEMENT FOR A RANDOM PURE STATE

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Abstract. Consider the model of bipartite entanglement for a random pure state emerging in quantum information and quantum chaos, corresponding to the fixed trace Laguerre unitary ensemble (LUE) in Random Matrix Theory. We focus on correlation functions of Schmidt eigenvalues for the model and prove universal limits of the correlation functions in the bulk and also at the soft and hard edges of the spectrum, as these for the LUE. Further we consider the bounded trace LUE and obtain the same universal limits.

1. Introduction and main results

Quantum entanglement has recently been studied extensively [23, 18, 32, 25, 13, 21, 16, 3] due to its central role in quantum information and quantum computation, which is treated as an indispensable resource [20]. The entanglement of random pure quantum states is of much interest in the context of bipartite entanglement, and statistical properties of such random states are relevant to quantum chaotic systems, see [16, 13] and references therein.

In the present paper, we consider a bipartite quantum system (a system with its surrounding environment). Given a composite system $A \otimes B$ of an $(NM)$-dimensional Hilbert space $\mathcal{H}^{(NM)} = \mathcal{H}_A^{(N)} \otimes \mathcal{H}_B^{(M)}$, let $|e_A^{\alpha}\rangle_{i=1}$ and $|e_B^{\beta}\rangle_{j=1}$ be two complete orthogonal basis states for the subsystems $A$ and $B$, respectively. Without loss of generality, we assume $N \leq M$. Any quantum state $|\Phi\rangle \in A \otimes B$ can be expanded as a linear combination

$$\sum_{i,j} W_{ij} |e_A^{\alpha}\rangle |e_B^{\beta}\rangle$$

where these coefficients $X_{ij} \in \mathbb{C}$ form a rectangular $N \times M$ complex matrix $X = [X_{ij}]$. The composite state $|\Phi\rangle$ is fully unentangled (separable) if $|\Phi\rangle$ can be written as a direct product of two states $|\Phi^A\rangle \in A$ and $|\Phi^B\rangle \in B$, i.e., $|\Phi\rangle = |\Phi^A\rangle \otimes |\Phi^B\rangle$, otherwise referred to as an entangled state [30]. The composite state $|\Phi\rangle$ is normalized pure state system if the density matrix of $|\Phi\rangle$ is given by $\rho = |\Phi\rangle \langle \Phi|$ satisfying $\text{tr} [\rho] = 1$. The reduced density matrix of the subsystem $A$ by tracing over the states of the subsystem $B$ is defined [3, 16] by

$$\rho_A = \text{tr}_B [\rho] = \sum_{i,j} W_{ij} |e_A^{\alpha}\rangle \langle e_A^{\alpha}|,$$

where $W_{ij}$ are the entries of $N \times N$ square matrix $W = XX^\dagger$ with $\text{tr} W = 1$ due to the normalized restriction that $\text{tr} [\rho] = 1$. Analogously, $\rho_B = \text{tr}_A [\rho]$. It is not
difficult to prove that the reduced density matrices $\rho_A$ and $\rho_B$ have the same set of non-negative eigenvalues $x_1, x_2, \ldots, x_N$ (we call them Schmidt eigenvalues of the quantum state $|\Phi\rangle$) satisfying $\sum_{i=1}^{N} x_i = 1$. Let $v_i^A$ denote the eigenvector of the square matrix $W$ corresponding to the eigenvalue $x_i$. Then $\rho_A$ can be expressed as $\rho_A = \sum_{i=1}^{N} x_i |v_i^A\rangle \langle v_i^A|$. A similar representation holds for $\rho_B$. The composite state $|\Phi\rangle$ has a well-known Schmidt spectral decomposition 4

$$|\Phi\rangle = \sum_{i=1}^{N} \sqrt{x_i} |v_i^A\rangle \otimes |v_i^B\rangle.$$  

A pure state is random if these coefficients $X_{ij}$ are random. The simplest and most common random state is to choose $X_{ij}$ as independent and identically distributed Gaussian variables 16. However, the set of complex Wishart matrices invariant under every unitary transformation but without any other constraint is referred as complex Wishart ensemble or Laguerre unitary ensemble 17. Its joint probability density function (p.d.f.) of $N$ unordered eigenvalue $x_1, x_2, \ldots, x_N$ of complex Wishart matrix $W$ is written as

$$P_{LU}^{s}(x_1, \ldots, x_N) = \frac{1}{Z_s} \prod_{1 \leq i < j \leq N} |x_i - x_j|^2 \prod_{i=1}^{N} x_i^\alpha e^{-x_i/s},$$  

where $s > 0$ and $\alpha = M - N \geq 0$. In this present paper, we extend the scope of the index $\alpha$ and $s$ to $\alpha > -1$ and $\text{Re} s > 0$, respectively. The partition function reads

$$Z_s = s^N \prod_{j=1}^{N} \Gamma(1 + j) \Gamma(\alpha + j),$$  

calculated in the book of Mehta 17. On the other hand, in case of a random pure state $|\Phi\rangle$, all the eigenvalues of $W = XX^\dagger$ are not quite same as these of complex Wishart matrix due to the additional constraint that $\text{tr} \rho_A = \text{tr} W = 1$. Thus, the eigenvalues of the reduced density matrix $\rho_A$ are distributed according to (1.4) in addition to the constraint $\sum_{i=1}^{N} x_i = 1$. More precisely,

$$P_{LU}^{s,r}(x_1, \ldots, x_N) = \frac{1}{Z_{s}^r} \delta \left( \sum_{i=1}^{N} x_i - r \right) \prod_{1 \leq i < j \leq N} |x_i - x_j|^2 \prod_{i=1}^{N} x_i^\alpha,$$

where $\delta(x)$ denotes the Dirac measure, $r = 1$ and $\alpha = M - N$. We refer this ensemble as fixed trace Laguerre unitary ensemble (FTLUE), following the classic book by Mehta 17 where he referred to fix trace and bounded trace Gaussian ensembles as restricted trace ensembles (this class of ensembles has been generalized in II). We will extend the scope of the index $r$ to $\text{Re} r > 0$. It follows from (1.5) that the partition function $Z_{s}^r$ equals

$$Z_{s}^r = r^N \Gamma(1 + j) \Gamma(\alpha + j) \prod_{j=1}^{N} \Gamma(\alpha + j),$$  

When $M = N$, the joint p.d.f. of (1.6) in 22, 25 is referred to the ensemble of random density matrices with respect to the Hilbert-Schmidt metric in the set $D_N$ of all density matrices of size $N$. It is worthy of stressing that another ensemble of random density matrices with respect to the Bures metric is quite distinguished, because its features support the claim that without any prior knowledge on a certain
density matrix, the optimal way to mimic it is to generate it at random with respect to the Bures measure [25].

The study of the eigenvalues of the reduced density matrix $\rho_A = W$ is crucial for understanding and utilizing entanglement. In principle, all information about the spectral properties of the subsystem $A$, including its degree of entanglement, can be encoded in the p.d.f. of (1.6). For example, one classic measure of entanglement is the von Neumann entropy defined by

$$S = -\text{tr} \rho_A \ln \rho_A = -\sum_{i=1}^{N} x_i \ln x_i,$$

which is a random variable. The average entropy $\langle S \rangle$ is close to $\ln N - N/2M$ for large $N$ when $M \geq N$ [23]. Besides, some known results on the FTLUE which are the same in the limit as these of the LUE include: the global density [25, 3], the largest eigenvalue distribution [21], the smallest eigenvalue distribution when $M = N$ [16].

In this paper, we focus on the so-called correlation functions of the FTLUE. We also consider another closely relevant ensemble: bounded trace LUE (BTLUE), whose joint p.d.f. for the eigenvalues is given by

$$P^\phi_r(x_1, \ldots, x_N) = \frac{1}{Z^\phi_r} \prod_{i=1}^{N} \theta(r - \sum_{i=1}^{N} x_i) \prod_{1 \leq i < j \leq N} |x_i - x_j|^2 \prod_{i=1}^{N} x_i^\alpha,$$

where $\theta(x)$ denotes the Heaviside step function, i.e., $\theta(x) = 1$ for $x \geq 0$, otherwise $\theta(x) = 0$. Note that the FTLUE or BTLUE bears the same relationship to the LUE that the micro-canonical ensembles to the canonical ensembles in statistical mechanics. Section 27 in [17], Mehta posed the “equivalence of ensembles” problem whether all local statistical properties of the eigenvalues between fixed trace and unconstrained random matrix ensembles are identical, and further speculated that working out the eigenvalues spacing distribution for bounded trace ensembles is much more difficult. Although universal local results have been obtained for very broad classes of canonical random matrix ensembles [9, 8, 7, 28], only very few results on the local limit behavior of the correlation functions for the restricted ensembles (no orthogonal polynomial techniques are available!). Recently, some progress has been made for fixed trace Gaussian ensembles [2], [12, 11], [15].

Before we state our main results, let us first recall that the definitions of correlation function and some universal results on the LUE. The $n$-point correlation function $R_n^{LUE,s}(x_1, \ldots, x_n)$ of the LUE is defined as

$$R_n^{LUE,s}(x_1, \ldots, x_n) = \frac{N!}{(N-n)!} \int_{\mathbb{R}^{N-n}} P_N^{LUE,s}(x_1, \ldots, x_N) dx_{n+1} \cdots dx_N.$$

Analogously, the $n$-point correlation function of the FTLUE or BTLUE is defined as

$$R_n^{\phi,r}(x_1, \ldots, x_n) = \frac{N!}{(N-n)!} \int_{R^{N-n}} P_N^{\phi,r}(x_1, \ldots, x_N) dx_{n+1} \cdots dx_N,$$

where $\phi$ denotes $\delta$ or $\theta$. In particular, when $n = 1$, $R_1^{LUE,s}$ is called the level density or the density of states. A classical result for the LUE says that

$$\lim_{N \to \infty} \frac{1}{N} R_1^{LUE,s}(x) = \psi(x) = \frac{2}{\pi} \sqrt{\frac{1-x}{x}} 1_{(0,1)}(x),$$

where the symbol $1_{(0,1)}(x)$ denotes the characteristic function of the set $(0, 1]$, and $\psi(x)$ is the Marchenko-Pastur law [19]. However, for $n \geq 2$, the study of a finer
asymptotics near a point of the spectrum shows \cite{22}; in the bulk, i.e., \( u \in (0, 1) \),
\[
\lim_{N \to \infty} \frac{1}{(N \psi(u))^n} R^{LUE}_{n} \left( u + \frac{t_1}{N \psi(u)}, \ldots, u + \frac{t_n}{N \psi(u)} \right) = \det[K_{\sin}(t_i, t_j)]_{i,j=1}^n
\]
where \( K_{\sin}(t_i, t_j) = \frac{\sin(\pi(t_i - t_j))}{\pi(t_i - t_j)} \) is the so-called sine kernel; at the soft edge,
\[
\lim_{N \to \infty} \frac{1}{(2N^{2/3})^n} R^{LUE}_{n} \left( 1 + \frac{t_1}{(2N^{2/3})}, \ldots, 1 + \frac{t_n}{(2N^{2/3})} \right) = \det[K_{\text{Airy}}(t_j, t_k)]_{j,k=1}^n
\]
where \( K_{\text{Airy}}(u, v) = \frac{\text{Ai}'(u) \text{Ai}(v) - \text{Ai}'(v) \text{Ai}(u)}{u - v} \)
and the Airy function \( \text{Ai}(x) \) satisfies the equation \( \text{Ai}''(x) = x \text{Ai}(x) \); at the hard edge,
\[
\lim_{N \to \infty} \frac{1}{(16N^2)^n} R^{LUE}_{n} \left( \frac{t_1}{16N^2}, \ldots, \frac{t_n}{16N^2} \right) = \det(K_{J_{\phi,\theta}}(t_i, t_j))_{i,j=1}^n,
\]
where
\[
J_{\phi,\theta}(u, v) = \frac{J_{\phi}(\sqrt{u}) \sqrt{v} J_{\phi}'(\sqrt{v}) - J_{\phi}(\sqrt{v}) \sqrt{u} J_{\phi}'(\sqrt{u})}{2(u - v)}
\]
and \( J_{\alpha}(z) \) denotes the Bessel function of the index \( \alpha \).
On the other hand, the limit global density of the FTLUE (see \cite{25}) and the BTLUE is also the Marchenko-Pastur law, i.e.,
\[
\lim_{N \to \infty} \frac{1}{N} R^{\phi, \theta}_{n}(x) = \psi(x)
\]
where \( \phi \) denotes \( \delta \) or \( \theta \). In the case of the BTLUE, we will prove the claimed result in Sect. 5. Considering universality in the bulk, at the soft and hard edges of the spectrum of the restricted trace LUE, we have the same local limit behavior as that for the LUE.

**Theorem 1.** Let \( R^{\phi, \theta}_{n} \) be the \( n \)-point correlation function of eigenvalues of bipartite entanglement for a random pure state, defined by \( \text{(1.10)} \). The following asymptotic properties hold.

(i) The bulk of the spectrum: for every \( u \in (0, 1) \) and \( t_i \in \mathbb{R}, 1 \leq i \leq n \),
\[
\lim_{N \to \infty} \frac{1}{(N \psi(u))^n} R^{\phi, \theta}_{n}(u + \frac{t_1}{N \psi(u)}, \ldots, u + \frac{t_n}{N \psi(u)}) = \det[K_{\sin}(t_i, t_j)]_{i,j=1}^n
\]
uniformly for \( t_1, \ldots, t_n \) in compact subsets of \( \mathbb{R} \) and for \( u \) in a compact subset of \( (0, 1) \).

(ii) The soft edge of the spectrum: for any \( f \in \mathbb{C}_c(\mathbb{R}^n) \), the set of all continuous functions on \( \mathbb{R}^n \) with compact support,
\[
\lim_{N \to \infty} \frac{1}{(2N^{2/3})^n} \int_{\mathbb{R}^n} f(t_1, \ldots, t_n) R^{\phi, \theta}_{n}(1 + \frac{t_1}{(2N^{2/3})}, \ldots, 1 + \frac{t_n}{(2N^{2/3})}) \, dt^{n} = \int_{\mathbb{R}^n} f(t_1, \ldots, t_n) \det[K_{\text{Airy}}(t_j, t_k)]_{j,k=1}^n \, dt^{n}.
\]

(iii) The hard edge of the spectrum:
\[
\lim_{N \to \infty} \frac{1}{(16N^2)^n} R^{\phi, \theta}_{n}(\frac{t_1}{16N^2}, \ldots, \frac{t_n}{16N^2}) = \det(K_{J_{\phi,\theta}}(t_i, t_j))_{i,j=1}^n
\]
uniformly for \( t_1, \ldots, t_n \) in bounded subsets of \( (0, \infty) \).
Theorem 2. Let $R^\theta_n$ be the $n$-point correlation function of the bounded trace LUE, defined by (1.1). Let $f \in C_c(R^n)$, the set of all continuous functions on $\mathbb{R}^n$ with compact support, the following asymptotic properties hold.

(i) The bulk of the spectrum: for every $x \in (0, 1)$,
\[
\lim_{n \to \infty} \frac{1}{(N\psi(x))^n} \int_{\mathbb{R}^n} f(t_1, \ldots, t_n) R_n^{\theta, N+\alpha} \left( x + \frac{t_1}{N\psi(x)}, \ldots, x + \frac{t_n}{N\psi(x)} \right) d^n t
= \int_{\mathbb{R}^n} f(t_1, \ldots, t_n) \det[K_{\sin}(t_i, t_j)]_{i,j=1}^n d^n t.
\]

(ii) The soft edge of the spectrum:
\[
\lim_{n \to \infty} \frac{1}{((2N)^{2/3})^n} \int_{\mathbb{R}^n} f(t_1, \ldots, t_n) R_n^{\theta, N+\alpha} \left( 1 + \frac{t_1}{(2N)^{2/3}}, \ldots, 1 + \frac{t_n}{(2N)^{2/3}} \right) d^n t
= \int_{\mathbb{R}^n} f(t_1, \ldots, t_n) \det[K_{\text{Airy}}(t_j, t_k)]_{j,k=1}^n d^n t.
\]

(iii) The hard edge of the spectrum:
\[
\lim_{n \to \infty} \frac{1}{(16N^2)^n} \int_{\mathbb{R}^n} f(t_1, \ldots, t_n) R_n^{\theta, N+\alpha} \left( \frac{t_1}{16N^2}, \ldots, \frac{t_n}{16N^2} \right) d^n t
= \int_{\mathbb{R}^n} f(t_1, \ldots, t_n) \det[K_{J\alpha}(t_i, t_j)]_{i,j=1}^n d^n t.
\]

To the best of our knowledge, this is the first result about the local properties of correlation functions for the bounded trace ensembles. Theorems 1 and 2 give an affirmative answer to Mehta’s “equivalence of ensembles” problem in the case of Laguerre unitary ensemble. In fact, our method can deal with some more general ensembles which will be considered in a forthcoming paper.

The plan of the remaining part of our paper is the following. Sections 2, 3 and 4 are devoted to the proof of Theorem 1. Section 5 deals with Theorem 2. In Sect. 2, the asymptotic behavior of $R_n^{\theta, N+\alpha}$ in the bulk of the spectrum is given based on the rigorous estimates of the correlation function $R_n^{LUE, \sigma}$ in the complex plane, inspired by [11]. Some of the results in [28] play an important role on our proof. In sect. 3, by using the similar method introduced by the authors in [15], the universality at the soft edge of the spectrum is proved. In sect. 4, based on a heuristic idea in [2] where universality at zero is considered for fixed trace Gauss-type ensembles, the asymptotic behavior at the hard edge is derived. In the last section, a “sharp” concentration phenomenon is observed, then local statistical properties of the eigenvalues between the fixed and bounded LUEs can be proved to be identical in the limit. So we extend the results from Theorem 1 to Theorem 2.

2. PROOF OF THEOREM 1 THE BULK OF THE SPECTRUM

2.1. The relation between $R_n^{LUE, \sigma}$ and $R_n^{\theta, \sigma}$. For every $\vartheta \in \mathbb{R}$, through this section, we will denote by $(\cdot)^\vartheta$ the function
\[
(\cdot)^\vartheta : \mathbb{C} \setminus (-\infty, 1] \to \mathbb{C} : z \mapsto \exp \vartheta \log z,
\]
where log denotes the principle branch of the logarithm. The constants $C(\mu), C_1(\mu), C_2(\mu), \Omega(\mu)$, depending on the parameter $\mu$, may change from one line to another line.
Let $s > 0$, and let for every $k \geq 0$, $\tilde{h}_k(x, s)$ be a polynomial of degree $k$ with positive leading coefficient such that for $\tilde{h}_k(x, s)$,

\begin{equation}
(2.2) \quad \int_0^\infty \tilde{h}_k(x, s)\tilde{h}_j(x, s)x^\alpha e^{-x/s}dx = \delta_{k,j}, \quad j, k = 0, 1, \ldots .
\end{equation}

The generalized Laguerre polynomials $(\tilde{L}^\alpha_j(x, s))_{j \geq 0}$ with the positive leading coefficients $\alpha$ are defined by

\begin{equation}
(2.3) \quad \sum_{j=0}^{\infty} \tilde{L}^\alpha_j(x, s)x^j = (1 + sw)^{-\alpha - 1} \exp(-x/s),
\end{equation}

and for $i, j = 0, 1, \ldots$, satisfy the relation

\begin{equation}
(2.4) \quad \int_0^\infty e^{-x/s}x^\alpha \tilde{L}^\alpha_i(x, s)\tilde{L}^\alpha_j(x, s)dx = s^{\alpha + 1 + i + j} \frac{\Gamma(i + \alpha + 1)}{\Gamma(i + 1)} \delta_{i,j}.
\end{equation}

Hence

\begin{equation}
(2.5) \quad \tilde{h}_i(x, s) = s^{-i+\alpha+1} \left( \frac{\Gamma(i + \alpha + 1)}{\Gamma(i + 1)} \right)^{-1/2} \tilde{L}^\alpha_i(x, s).
\end{equation}

The functions defined by

\begin{equation}
(2.6) \quad \varphi_i(x, s) = x^{\alpha/2}e^{-x/(2s)}\tilde{h}_i(x, s), \quad i = 0, 1, \ldots,
\end{equation}

form an orthogonal sequence of functions in the Hilbert space $L^2(0, \infty)$. Next let us consider the standardized Laguerre polynomials \cite{24} with the positive leading coefficient with respect to the weight $x^\alpha e^{-x}$ by the relation

\begin{equation}
(2.7) \quad L^\alpha_i(x) = \tilde{L}^\alpha_i(x, 1), \quad \tilde{h}_i(x, 1) = \tilde{h}_i(x, 1), \quad \varphi_i(x) = \varphi_i(x, 1),
\end{equation}

which satisfy

\begin{equation}
(2.8) \quad L^\alpha_i(x, s) = s^i L^\alpha_i(xs^{-1}), \quad \tilde{h}_i(x, s) = s^{-(\alpha+1)/2}h_i(x s^{-1}), \quad \varphi_i(x, s) = s^{-1/2}\varphi_i(x s^{-1}).
\end{equation}

The following three recurrence formula for $L^\alpha_j(x, s)$ holds:

\begin{equation}
(2.9) \quad jL^\alpha_j(x, s) = (xs^{-1} - 2j - \alpha + 1)sL^\alpha_{j-1}(x, s) - s^2L^\alpha_{j-2}(x, s)(j + \alpha - 1)
\end{equation}

where $j = 2, 3, \ldots$. The $n$-point correlation function $R^L_{nE,s}$ of the LUE could be expressed as

\begin{equation}
(2.10) \quad R^L_{nE,s}(x_1, \ldots, x_n) = \det(\tilde{K}_N(x_i, x_j, s))_{i,j=1}^n,
\end{equation}

where

\begin{equation}
(2.11) \quad \tilde{K}_N(x, y, s) = \sum_{k=0}^{N-1} \varphi_k(x, s)\varphi_k(y, s).
\end{equation}

Let

\begin{equation}
(2.12) \quad K_N(x, y) \doteq \tilde{K}_N(x, y, 1),
\end{equation}

from (2.2), we get

\begin{equation}
(2.13) \quad \tilde{K}_N(x, y, s) = s^{-1}K_N(xs^{-1}, ys^{-1}).
\end{equation}
The Christoffel-Darboux formula for kernels $K_N$ and $\tilde{K}_N$ reads

\begin{equation}
K_N(x, y) = \sqrt{N(N + \alpha)} \frac{\varphi_N(x)\varphi_{N-1}(y) - \varphi_N(y)\varphi_{N-1}(x)}{x - y}
\end{equation}

and

\begin{equation}
\tilde{K}_N(x, y, s) = \sqrt{N(N + \alpha)} \frac{\tilde{\varphi}_N(x, s)\tilde{\varphi}_{N-1}(y, s) - \tilde{\varphi}_N(y, s)\tilde{\varphi}_{N-1}(x, s)}{x - y}
\end{equation}

Note that the reproducing kernel $K_N(x, y)$ has the following integral representation (Eq.(4.2), [29], or Eq.(3.6), [14])

\begin{equation}
K_N(x, y) = \frac{\sqrt{N(N + \alpha)}}{2} \int_0^{+\infty} S_1(x + z)S_2(y + z) + S_1(y + z)S_2(x + z)dz
\end{equation}

where

\begin{equation}
S_1(x) = \sqrt{N} \frac{\varphi_N(x)}{x} + \sqrt{N + \alpha} \frac{\varphi_{N-1}(x)}{x},
\end{equation}

\begin{equation}
S_2(x) = \sqrt{N + \alpha} \frac{\varphi_N(x)}{x} + \sqrt{N} \frac{\varphi_{N-1}(x)}{x},
\end{equation}

and so, for every $s > 0$, the following relation

\begin{equation}
K_N\left(\frac{x}{s}, \frac{y}{s}\right) = \frac{\sqrt{N(N + \alpha)}}{2} \int_0^{+\infty} S_1\left(\frac{x}{s} + z\right)S_2\left(\frac{y}{s} + z\right) + S_1\left(\frac{y}{s} + z\right)S_2\left(\frac{x}{s} + z\right)dz
\end{equation}

holds. The relations (2.8), (2.9), (2.13) and our extension of the function $(\cdot)^\theta$ to $\mathbb{C}\setminus[-\infty, 1]$ allow us to continue $L_i(x, s)$, $\tilde{L}_i(x, s)$ and $K_N(x, y, s)$ to this domain analytically in the parameter $s$. The relations (2.8)-(2.15) remain valid under these continuations. So does (2.19) whenever this integral is well-defined.

Next, for $R^{L,E,s}_n$ and $R^{b,r}_n$, we will prove that one can be expressed by the other. Let us stress that the similar relation will be frequently used in the proofs of Theorems 1 and 2.

**Proposition 3.** Let $R^{L,E,s}_n$ and $R^{b,r}_n$ be the n-point correlation functions, defined by (1.9) and (1.10) respectively, then we have the following integral equation

\begin{equation}
R^{L,E,s}_n(x_1, \ldots, x_n) = \int_0^\infty R^{b,r}_n(x_1, \ldots, x_n) \gamma\left(\frac{u}{s}\right)s^{-1}du,
\end{equation}

where $\gamma(x)$ is defined by

\begin{equation}
\gamma(x) \doteq \frac{1}{\Gamma(N(N + \alpha))} e^{-x} x^{N^2 + N\alpha - 1} 1_{[0, \infty)}(x).
\end{equation}

**Proof.** By (1.9), for any $f \in L^\infty(\mathbb{R}^n)$,

\begin{align*}
\int_{\mathbb{R}^n} f(x_1, \ldots, x_n)R^{L,E,s}_n(x_1, \ldots, x_n)dx_1 \ldots dx_n &= \frac{1}{Z_s(N-n)!} \int_0^\infty \int_\triangle \int_{\mathbb{R}^n} f(uy_1, \ldots, uy_N) \prod_{1 \leq i < j \leq N} |y_i - y_j|^2 \prod_{i=1}^N y_i^\alpha e^{-\frac{1}{2} u y_i^2} u^{N^2 + N\alpha - 1}dy_1 \ldots dy_N du.
\end{align*}

Here we make the change of variables: $x_i = uy_i, 1 \leq i \leq N$, and $y_i$ belongs to the standard N-simplex $\triangle = \{(y_1, \ldots, y_N) | \sum_{i=1}^N y_i = 1, y_i \geq 0\}$. By (1.10), the
Here we have used Proposition 3. □

The right-hand side of the above equality equals
\[
\frac{Z_1}{Z_s} \int_{\mathbb{R}^n} f(y_1, \ldots, y_n) R_{n}^{\delta,1}(y_1, \ldots, y_n) e^{-\frac{u}{\mu}} u^{N^2 + N\alpha - 1} dy_1 \ldots dy_n du
\]
\[= \frac{Z_1}{Z_s} \int_{\mathbb{R}^n} f(y_1, \ldots, y_n) R_{n}^{\delta,1} \left( \frac{y_1}{u}, \ldots, \frac{y_n}{u} \right) \frac{1}{u^n} e^{-\frac{u}{\mu}} u^{N^2 + N\alpha - 1} dy_1 \ldots dy_n du
\]
\[= \frac{Z_1}{Z_s} \int_{\mathbb{R}^n} f(x_1, \ldots, x_n) \int_{0}^{\infty} R_{n}^{\delta,u}(x_1, \ldots, x_n) e^{-\frac{u}{\mu}} u^{N^2 + N\alpha - 1} du dx_1 \ldots dx_n.
\]

Here we have used the following fact that
\[
(2.22) \quad R_{n}^{\delta,r}(x_1, \ldots, x_n) = R_{n}^{\delta,1}(x_1 r^{-1}, \ldots, x_n r^{-1}) r^{-n}.
\]

Thus we prove that
\[
(2.23) \quad R_{n}^{LUE,s}(x_1, \ldots, x_n) = \frac{Z_1}{Z_s} \int_{0}^{\infty} R_{n}^{\delta,u}(x_1, \ldots, x_n) e^{-\frac{u}{\mu}} u^{N^2 + N\alpha - 1} du,
\]

It follows from (1.5) and (1.7) that
\[
(2.24) \quad Z_s = \Gamma(N(N + \alpha)) s^{N(N + \alpha)} Z_1^\frac{1}{\alpha}.
\]

This proves this proposition. □

Note that the relations Eq. (2.20) and Eq. (2.22) also hold when extending the parameter \( s \) and \( r \) to the domain \( \text{Re} s, \text{Re} r > 0 \). By Proposition 2, set \( u = (N + \alpha + v)/4, s = 1/4N \), we have
\[
(2.25) \quad R_{n}^{LUE,\frac{1}{\alpha}} = \int_{0}^{\infty} R_{n}^{\delta,u,N + \alpha + v} N \gamma(N(N + \alpha + v)) dv.
\]

We will state the following lemma, which plays a central role in our proof of universality in the bulk of the spectrum.

**Lemma 4.**

(2.26) \[
\int_{0}^{\infty} R_{n}^{\delta,u,N + \alpha + v} N \gamma(N(N + \alpha + v)) \exp(-iyuv) dv = \phi_N(y)R_{n}^{LUE,\frac{1}{\alpha} \cdot \frac{1}{N}},
\]

where
\[
(2.27) \quad \phi_N(y) = e^{iy(N + \alpha)} \left( 1 + \frac{iy}{N} \right)^{-(N^2 + N\alpha)}.
\]

Note that the function \( \phi_N(\cdot) \) is the characteristic function of \( N \gamma(N(N + \cdot)) \).

**Proof.** Set \( (N + \alpha + v)/4 = u \), the left-hand side of Eq. (2.26) equals
\[
\int_{0}^{\infty} R_{n}^{\delta,u,4N \gamma(4Nu)} \exp(-iy(4u - N - \alpha)) du
\]
\[= \frac{e^{iy(N + \alpha)}}{\Gamma(N^2 + N\alpha)} \int_{0}^{\infty} R_{n}^{\delta,u,4N} \left( 4Nu \right)^{N^2 + N\alpha - 1} e^{-4Nu} \exp(-iy4u) du
\]
\[= \int_{0}^{\infty} R_{n}^{\delta,u} e^{-4Nu(1 + \frac{iy}{N})} \left( 4Nu(1 + \frac{iy}{N}) \right)^{N^2 + N\alpha - 1} 4N(1 + \frac{iy}{N}) du
\]
\[\times \frac{e^{iy(N + \alpha)}}{\Gamma(N^2 + N\alpha)} \left( 1 + \frac{iy}{N} \right)^{-(N^2 + N\alpha)}
\]
\[= e^{iy(N + \alpha)} \left( 1 + \frac{iy}{N} \right)^{-(N^2 + N\alpha)} R_{n}^{LUE,\frac{1}{\alpha} \cdot \frac{1}{N}} = \phi_N(y)R_{n}^{LUE,\frac{1}{\alpha} \cdot \frac{1}{N}}.
\]

Here we have used Proposition 3. □
Here $\beta$ exists a positive number and zero respectively, depicted in Figure 1 (See Figures 1 and 5, [28]). Hence there holds uniformly for all $x$ such that

$$
R_n^{LUE, s} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \phi_N(y)R_n^{LUE,E_0(N+\alpha)} dy.
$$

2.2. Estimate of $R_n^{LUE, s}$ in the complex plane. For the convenience of the reader, we will review some basic results in [28]. The function (Eq.(3.38), [28])

$$
\hat{\psi}(z) = \frac{2}{i\pi} \frac{(z - 1)^{1/2}}{z^{1/2}}, \text{ for } z \in \mathbb{C}\{0, 1\}
$$

with principle branches of powers denotes analytic continuation of the standard Marčenko-Pastur law in the domain $\mathbb{C}\{0, 1\}$. The following two formulas:

$$
g(z) = \int_0^1 \log(z - y)\psi(y)dy, \text{ for } z \in \mathbb{C}\{(-\infty, 1]\}
$$

and

$$
\xi(z) = -i\pi \int_0^z \hat{\psi}(y)dy, \text{ for } z \in \mathbb{C}\{(-\infty, 1]\}
$$

come from Eq.(3.30) and Eq.(3.40) respectively in [28]. The uniqueness of analytic function (Eq.(5.2), [28]) shows that

$$
2\xi(z) = 2g(z) - 2z - l, \text{ for } z \in \mathbb{C}\{(-\infty, 1]\}.
$$

Here $l$ is given by Proposition 3.12 in [28].

For $0 < \theta < 1/2$ and $\beta > 0$, the sets $S_{\theta, \beta}$ and $\overline{S}_{\theta, \beta}$ are defined by

$$
S_{\theta, \beta} = \{z \in \mathbb{C}| \theta < \text{Re}(z) < 1 - \theta, |\text{Im} z| < \beta\},
$$

$$
\overline{S}_{\theta, \beta} = \{z \in \mathbb{C}| \theta \leq \text{Re}(z) \leq 1 - \theta, |\text{Im} z| \leq \beta\}.
$$

Lemma 6. Let $\Lambda(H) = 1 + iH$. For every $0 < \theta < 1/2$, there exists a positive number $H_0(\theta) > 0$ such that for kernel $\tilde{K}_N(x, y, 1/4N)$ denoted by (2.11), and for $A > 0$ the relation

$$
\lim_{N \to \infty \frac{1}{N\psi(x)} \tilde{K}_N \left( (x + \frac{u}{N\psi(x)})\Lambda(H), (x + \frac{v}{N\psi(x)})\Lambda(H), 1/4N \right) = \frac{\sin \left( \pi(u - v)\Lambda(H) \right)}{\pi(u - v)\Lambda(H)}
$$

holds uniformly for all $x \in [0, 1 - \theta]$, $0 \leq H \leq H_0(\theta)$ and $|u|, |v| \leq A$.

Proof. For any fixed $0 < \theta < 1/2$, taking $\delta = \theta/4$ (see Figure 5 in the section 3.8, [28]) such that

$$
[\theta/2, 1 - \theta/2] \cap \partial U_\delta = \emptyset, [\theta/2, 1 - \theta/2] \cap \partial \tilde{U}_\delta = \emptyset.
$$

Here $\partial U_\delta$ and $\partial \tilde{U}_\delta$ denote the boundary of the two disks $U_\delta$ and $\tilde{U}_\delta$ around unity and zero respectively, depicted in Figure 1 (See Figures 1 and 5, [28]). Hence there exists a positive number $\beta(2\delta) > 0$ such that this set $S_{\theta/2, \beta(2\delta)}$ is the subset of the set $A_\delta \cup B_\delta \cup (\delta, 1 - \delta)$, described in Figure 1. Let

$$
w_1 = (x + \frac{u}{N\psi(x)})\Lambda(H), w_2 = (x + \frac{v}{N\psi(x)})\Lambda(H).
$$

It is not difficult to check that for all $x \in [\theta, 1 - \theta]$ and $|u|, |v| \leq A$, there exists $N_0$, when $N \geq N_0$, $\text{Re} w_1, \text{Re} w_2 \in [\theta/2, 1 - \theta/2]$. Furthermore, there exists a
Here we have used Eq. (2.32). By (2.33), the following asymptotic behavior

\[ n_\xi (w_2) - n_\xi (w_1) = -N \pi i \int_{(x + \pi e^{\pi i}) \Lambda (H)}^{(x + \pi e^{-\pi i}) \Lambda (H)} \hat{\psi} (y) dy = \pi i \frac{\hat{\psi} (x \Lambda (H))}{\hat{\psi} (x)} \psi (x) \Lambda (H) (u - v) + O \left( \frac{1}{N} \right) \]

holds uniformly when \( x, u, v, H \) satisfy the assumptions of this lemma. By (2.36), we conclude the proof of this lemma.
Next, first we will obtain an upper bound about orthogonal polynomial \( h_N(z) \) in the complex plane. As a consequence, basing on the integral representation Eq. (2.19), we can derive the upper bound estimate of the reproducing kernel \( \tilde{K}_N(z,\cdot, s) \) (See Lemma 8).

**Lemma 7.** For every \( \mu > 0 \), there exist constants \( C(\mu) \) and \( \Omega(u) \) such that the two inequalities

\[
|h_N(4Nz)| \leq C(\mu)N^{-\frac{\alpha+1}{2}}\Omega(\mu)|z|^{N+\frac{1}{2}}
\]

and

\[
|h_{N-1}(4Nz)| \leq C(\mu)N^{-\frac{\alpha+1}{2}}\Omega(\mu)^{N-1}|z|^{N-\frac{1}{2}}
\]

hold for every \( N \) and every \( z \) satisfying \( \text{Re} \ z \geq 0, \text{Im} \ z \geq \mu \).

**Proof.** Theorem 2.4 in [28] shows that for \( \text{Re} \ z \geq 0 \) and \( \text{Im} \ z \geq \mu \),

\[
h_N(4Nz) = (4Nz)^{-\frac{\alpha}{2}}e^{2Nz}\sqrt{\frac{1}{2N\pi}}
\]

\[
(\Psi(z))^{\frac{\alpha+1}{2}}\frac{2\pi}{2z^{1/4}(z-1)^{1/4}}\exp(-\pi i N \int_1^z \Psi(s)ds)(1 + O(\frac{1}{N}))
\]

where the error term is uniform, and

\[
\Psi(z) = 2z - 1 + 2z^{1/2}(z-1)^{1/2}, \text{ for } z \in \mathbb{C}[0,1].
\]

Note that the function \( g(z) \) satisfies

\[
g'(z) = 2(1 - \frac{(z-1)^{1/2}}{z^{1/2}})
\]

with the initial condition (See (3.26), (3.28) and (3.35), [28])

\[
e^{N g(z)} = z^N + O(z^{N-1}).
\]

A direct computation shows that

\[
g(z) = 2z - 1 - 2z^{1/2}(z-1)^{1/2} - 2\ln(z^{1/2} - (z-1)^{1/2}) - 2\ln 2.
\]

Note that \( l = -2 - 4\ln 2 \) (Remark 2.3, [28]). It follows from (2.31) and (2.32) that

\[
h_N(4Nz) = (4Nz)^{-\frac{\alpha}{2}}\sqrt{\frac{1}{2N\pi}}\frac{2\pi}{2z^{1/4}(z-1)^{1/4}}(\Psi(z))^{\frac{\alpha+1}{2}}e^{2N(z-z^{1/2}(z-1)^{1/2})}
\]

For a given analytic branch, \((z^{1/2})^2 = z, (z-1) = ((z-1)^{1/2})^2\). Set \( w = z^{1/2}, \) which is a univalent analytic function in \( \mathbb{C}\setminus(-\infty, 1) \). Let \( x(w) \) be the root of this equation: \( x + \frac{1}{x} = 2w \) satisfying \( |x| < 1 \). Hence

\[
x(w) = w - \sqrt{w^2 - 1},
\]

where the function \( \sqrt{w^2 - 1} \) is a univalent analytic function in \( \mathbb{C}\setminus(-\infty, 1] \) satisfying the relation \( (\sqrt{w^2 - 1})^2 > 0 \) for \( w > 1 \). For every \( 0 < r < 1 \), the equation \( |x(w)| = r \) defines the ellipse

\[
\frac{(\text{Re} w)^2}{(\frac{1}{r} + r)^2} + \frac{(\text{Im} w)^2}{(\frac{1}{r} - r)^2} = 1
\]

and \( |x(w)| \leq r \) denotes the outside of this ellipse \( E_r \). Thus for \( w \in E_r \),

\[
|x(w)| = |w - \sqrt{w^2 - 1}| \leq r < 1,
\]
and so
\begin{equation}
(2.47) \quad \text{Re}(w - \sqrt{w^2 - 1})^2 \leq r^2.
\end{equation}

By Eq. (2.44), for \( w \in E_r \) we get
\begin{equation}
(2.48) \quad \min_{|x(w)|=|w|} |w| = \frac{1}{2}(\frac{1}{r} - r).
\end{equation}

Hence,
\begin{equation}
(2.49) \quad \frac{1}{|w - \sqrt{w^2 - 1}|} = \frac{1}{|x(w)|} = |2w - x(w)|
\end{equation}

\begin{equation}
(2.50) \quad < 1 + 2|w| = |w|(2 + \frac{1}{|w|}) \leq |w|(2 + \frac{2}{(\frac{1}{r} - r)}).
\end{equation}

Further, we have
\begin{equation}
(2.51) \quad |w^2 - 1| = |w - 1||w + 1| = \frac{1}{2}(x + \frac{1}{x}) - 1||\frac{1}{2}(x + \frac{1}{x}) + 1|
\end{equation}

\begin{equation}
(2.52) \quad \leq C_1(r)N^{-\frac{N-1}{2}}L(r)^NL(|z|N^{\frac{1}{2}}).
\end{equation}

Applying (2.46) - (2.51) to (2.48), we find that for every fixed \( 0 < r < 1 \), the function \(|h_N(4Nz)| |c|\) could be controlled by
\begin{align*}
|h_N(4Nz)| &= |(4Nz)^{-\frac{1}{2}}|\sqrt{\frac{1}{2N\pi(z - 1)^{1/4}}} |(z^{1/2} - (z - 1)^{1/2})| |e^{N(z^{1/2} - (z - 1)^{1/2})^2}|
\end{align*}

\begin{align*}
&\leq C_1(r)N^{-\frac{N-1}{2}}L(r)^NL(|z|N^{\frac{1}{2}}). \quad (2.53)
\end{align*}

Let \( z_N = \frac{N-1}{N}z \). Note that by the convexity of \( C \backslash E_r \), \( z_N \in E_r \) if \( z \in E_r \). Thus we have
\begin{align*}
|h_{N-1}(4(N-1)z_N)| &= |h_{N-1}(4(N-1)z_N)| \leq C_1(r)(N-1)^{-\frac{N-1}{2}}L(r)^NL(|z_N|N^{\frac{1}{2}})
\end{align*}

\begin{equation}
(2.54) \quad \leq C_2(r)N^{-\frac{N-1}{2}}L(r)^NL(|z_N|N^{\frac{1}{2}}).
\end{equation}

Here we have used the following fact that
\begin{equation}
\lim_{N \to \infty} \left(\frac{N-1}{N}\right)^{N^{\frac{1}{2}}} = e^{-1}.
\end{equation}

According to the assumption about \( z \) of this lemma, it follows from \( w = |z|^{1/2}e^{i\arg z} \) that \( \text{Re} w \geq 0, \text{Im} w \geq 0 \). Note that \( w^2 = (\text{Re} w)^2 - (\text{Im} w)^2 + 2i\text{Re} w \text{Im} w \). The assumption \( \text{Re} z \geq 0 \) implies that \( \text{Re} w \geq \text{Im} w \). For every \( \mu > 0 \), let
\begin{equation}
r = \sqrt{\frac{\mu + 2}{2}} - \sqrt{\frac{\mu}{2}},
\end{equation}

then \( 0 < r < 1 \) is a solution of the equation \( (\frac{1}{r} - r)^2 = 2\mu \). Hence \( \text{Im} z \geq \mu \) implies that
\begin{equation}
2(\text{Re} w)^2 \geq 2 \text{Re} w \text{Im} w = \text{Im} w^2 = \text{Im} z \geq \frac{1}{2}(\frac{1}{r} - r)^2,
\end{equation}

and
\begin{equation}
\text{Re}(w - \sqrt{w^2 - 1})^2 \leq r^2.
\end{equation}
so \( \text{Re} w \geq \frac{1}{2} (\frac{1}{r} - r) \), which establishes this fact that \( w \in E_r \). It follows from Eq. (2.52) and Eq. (2.53) that for every \( \mu > 0 \), there exist some constants \( C(\mu) \) and \( \Omega(\mu) \) such that

\[
|h_N(4Nz)| \leq C(\mu)N^{-\frac{r+3}{2}} \Omega(\mu)^N |z|^{N+\frac{1}{2}},
\]

\[
|h_N(-1(4Nz)| \leq C(\mu)N^{-\frac{r+3}{2}} \Omega(\mu)^{N-1} |z|^{N-\frac{1}{2}}.
\]

Lemma 8. For every \( \mu > 0 \), there exist \( C(\mu) \) and \( \Omega(\mu) \) such that

\[
\frac{1}{N}|\hat{K}_N(z_1, z_2, \frac{1}{4N})| \leq C(\mu)\Omega(\mu)^{2N-2}(2P+1)|\text{Im} z_1|^{P^\frac{3}{2}}|\text{Im} z_1|^{P^\frac{1}{2}}
\]

holds for all complex numbers \( z_i, i = 1, 2 \) satisfying \( \text{Re} z_i > 0, \text{Im} z_i \geq \mu, \) and \( N \geq \frac{1}{4r} \). Here \( P = N + \left[ \frac{3}{2} \right] + 1 \), \( [x] \) denotes the integer part of a real number \( x \).

Proof. By (2.19), one finds that

\[
\frac{1}{N}\hat{K}_N(z_1, z_2, \frac{1}{4N}) = 4K_N(4Nz_1, 4Nz_2) = 4 \frac{\sqrt{N(N+\alpha)}}{2} 4N
\]

\[
\times \int_0^{+\infty} S_1(4N(z_1 + \theta)) S_2(4N(z_2 + \theta)) + S_1(4N(z_2 + \theta)) S_2(4N(z_1 + \theta)) d\theta.
\]

It follows from Eq. (2.6), Eq. (2.17) and Lemma 7 that

\[
|S_1(4N(z_1 + \theta))| \leq \sqrt{N^2} \frac{|\varphi_N(4N(z_1 + \theta))|}{|4N(z_1 + \theta)|} + \sqrt{N^2} \frac{|\varphi_{N-1}(4N(z_1 + \theta))|}{|4N(z_1 + \theta)|}
\]

\[
\leq \sqrt{N^2} \frac{|4N(z_1 + \theta)|^{2N - 2|\text{Re} z_1 + \theta|} C(\mu)N^{-\frac{2N-1}{2}} \Omega(\mu)^N |z_1 + \theta|^{N^\frac{1}{2}}}{|4N(z_1 + \theta)|}
\]

\[
+ \sqrt{N^2} \frac{|4N(z_1 + \theta)|^{2N - 2|\text{Re} z_1 + \theta|} C(\mu)N^{-\frac{2N-1}{2}} \Omega(\mu)^{N-1} |z_1 + \theta|^{N-\frac{1}{2}}}{|4N(z_1 + \theta)|}.
\]

Here we have used this fact that \( \theta \geq 0, \text{Re}(z_1 + \theta) > 0 \) and \( \text{Im}(z_1 + \theta) \geq \mu \). A direct calculation tells us that

\[
|S_1(4N(z_1 + \theta))| \leq 4^{\alpha^2 - 1} N^{-1} C(\mu) \Omega(\mu)^N |z_1 + \theta|^{N+\frac{\alpha+1}{2}} e^{-2N \text{Re}(z_1 + \theta)}
\]

\[
+ 4^{\alpha^2 - 1} N^{-3/2} \sqrt{N+\alpha} C(\mu) \Omega(\mu)^N |z_1 + \theta|^{N+\frac{\alpha+1}{2}} e^{-2N \text{Re}(z_1 + \theta)}
\]

\[
= 4^{\alpha^2 - 1} N^{-1} C(\mu) \Omega(\mu)^{N-1} e^{-2N \text{Re}(z_1 + \theta)} |z_1 + \theta|^{N+\frac{\alpha+1}{2}} (\Omega(\mu) + \frac{1}{|z_1 + \theta|} \sqrt{\frac{N + \alpha}{N}})
\]

\[
\leq N^{-1} C(\mu) \Omega(\mu)^{N-1} e^{-2N \text{Re}(z_1 + \theta)} |z_1 + \theta|^{N+\frac{\alpha+1}{2}}.
\]

Note that \( |z_1 + \theta|^{-1} \leq 1/\mu \). Analogously, we have

\[
|S_2(4N(z_2 + \theta))| \leq N^{-1} C(\mu) \Omega(\mu)^{N-1} e^{-2N \text{Re}(z_2 + \theta)} |z_2 + \theta|^{N+\frac{\alpha-1}{2}}.
\]
Applying the two estimates of $S_1$ and $S_2$ to (2.57), we find that

$$\frac{1}{N} |\tilde{K}_N(z_1, z_2, \frac{1}{4N})| \leq C(\mu) \Omega(\mu)^{2N-2} \int_0^{+\infty} e^{-2N \text{Re}(z_1+\theta)} e^{-2N \text{Re}(z_2+\theta)} |z_1 + \theta|^{N + \frac{\alpha - 1}{2}} |z_2 + \theta|^{N + \frac{\alpha - 1}{2}} d\theta$$

(2.59)

$$\leq C(\mu) \Omega(\mu)^{2N-2} \left( \int_0^{+\infty} e^{-4N \text{Re}(z_1+\theta)} |z_1 + \theta|^{2(N + \frac{\alpha - 1}{2})} d\theta \right)^{\frac{1}{2}}$$

$$\times \left( \int_0^{+\infty} e^{-4N \text{Re}(z_2+\theta)} |z_2 + \theta|^{2(N + \frac{\alpha - 1}{2})} d\theta \right)^{\frac{1}{2}}.$$

Let $\{ \frac{\alpha - 1}{2} \} = \frac{\alpha - 1}{2} - [\frac{\alpha - 1}{2}]$. For every complex number $z$ satisfying $\text{Re} z \geq 0$, $\text{Im} z \geq \mu$, we obtain

$$\int_0^{+\infty} e^{-4N \text{Re}(z+\theta)} |z + \theta|^{2(N + \frac{\alpha - 1}{2})} d\theta \leq \mu^{2(\frac{\alpha - 1}{2})-1} \int_0^{+\infty} e^{-4N \text{Re}(z+\theta)} |z + \theta|^{2P} d\theta$$

$$= \mu^{2(\frac{\alpha - 1}{2})-1} \int_{\text{Re} z}^{+\infty} e^{-4N \text{Re} \eta} |\eta + i \text{Im} z|^{2P} d\eta$$

$$= \mu^{2(\frac{\alpha - 1}{2})-1} \int_{\text{Re} z}^{+\infty} e^{-4N \text{Re} \eta} |\eta^2 + (\text{Im} z)^2|^{P} d\eta$$

$$\leq \mu^{2(\frac{\alpha - 1}{2})-1} \int_0^{+\infty} e^{-4N \text{Re} \eta} |\eta^2 + (\text{Im} z)^2|^{P} d\eta$$

$$= \mu^{2(\frac{\alpha - 1}{2})-1} (\text{Im} z)^{2P+1} \int_0^{+\infty} e^{-4N \text{Re} \eta} |\eta|^{2P} d\eta$$

(2.60)

Here we have used the inequality: $\eta^2 + 1 \leq (\eta + 1)^2$ for $\eta \geq 0$. We also notice that (P.17, [10])

$$\frac{1}{\Gamma(m+1)} \int_c^{+\infty} x^m e^{-x} dx = e^{-c} \left( 1 + \frac{c}{1!} + \frac{c^2}{2!} + \cdots + \frac{c^m}{m!} \right), \text{ for } m \in \mathbb{N}, \ c > 0.$$

Then we have

$$e^{4N \mu} \left( \frac{1}{4N \mu} \right)^{2P+1} \int_{4N \mu}^{+\infty} e^{-y} y^{2P} dy = e^{4N \mu} \left( \frac{1}{4N \mu} \right)^{2P+1} \Gamma(2P+1) \int_{4N \mu}^{+\infty} e^{-y} y^{2P} \Gamma(2P+1) dy$$

$$= e^{4N \mu} \left( \frac{1}{4N \mu} \right)^{2P+1} \Gamma(2P+1) \left( e^{-4N \mu} + \frac{4N \mu}{1!} + \frac{(4N \mu)^2}{2!} + \cdots + \frac{(4N \mu)^{2P}}{(2P)!} \right)$$

(2.61)

$$\leq \frac{1}{4N \mu} \Gamma(2P+1)e.$$
Here we have used the inequality: $4N\mu \geq 1$. It follows from (2.59), (2.60) and (2.61) that

$$\frac{1}{N} |\tilde{K}_N(z_1, z_2, \frac{1}{4N})| \leq \frac{1}{\sqrt{N}} C(\mu) \Omega(\mu)^{2N-2} \mu^{(2N-1)-1} \frac{e}{4N\mu} \Gamma(2P + 1) (\text{Im } z_1)^{P+\frac{1}{2}} (\text{Im } z_2)^{P+\frac{1}{2}}.$$

Hence we complete the proof of this lemma.

**Corollary 9.** Let $R_{n}^{LUE,\ast}$ be $n$-point correlation function of the LUE, defined by (1.9), for all real $u, t_i, i = 1, \ldots, n$, if $N$ is sufficiently large, the following inequality

$$\left| \frac{1}{N^n} R_{n}^{LUE,\ast \ast} \left( (u + \frac{t_i}{N\psi(u)}) \Lambda(H), \ldots, (u + \frac{t_n}{N\psi(u)}) \Lambda(H) \right) \right| \leq n! C(\theta)^n \Omega(\theta)^{(2N-2)n} \Gamma(2P + 1)^n H^{2P+1} (1 - \frac{\theta}{2})^{2P+1}.$$

holds in all $u \in [\theta, 1 - \theta], |t_i| \leq A$ and $H \geq H_0(\theta)$. Here $0 < \theta < 1, A > 0$ and $H_0(\theta)$ as in Lemma 8.

**Proof.** For any $u \in [\theta, 1 - \theta]$, there exists $N'$ such that $|t_i/N\psi(u)| \leq \theta/2$ for all $N > N'$. Hence we have

$$\text{Re}((u + \frac{t_i}{N\psi(u)}) \Lambda(H)) \geq \theta/2 > 0$$

and

$$\text{Im}((u + \frac{t_i}{N\psi(u)}) \Lambda(H)) \geq H \frac{\theta}{2} \geq H_0(\theta) \frac{\theta}{2}.$$

It follows from Lemma 8 that

$$\frac{1}{N} |\tilde{K}_N((u + \frac{t_i}{N\psi(u)}) \Lambda(H), (u + \frac{t_j}{N\psi(u)}) \Lambda(H), \frac{1}{4N})| \leq C(\theta) \Omega(\theta)^{(2N-2)\Gamma(2P + 1)} |H(u + \frac{t_i}{N\psi(u)})|^{P+\frac{1}{2}} |H(u + \frac{t_j}{N\psi(u)})|^{P+\frac{1}{2}} \leq C(\theta) \Omega(\theta)^{(2N-2)\Gamma(2P + 1)} H^{2P+1} (1 - \frac{\theta}{2})^{2P+1}.$$

By the definition of determinant and Eq. (2.10), we have

$$\left| \frac{1}{N^n} R_{n}^{LUE,\ast \ast} \left( (u + \frac{t_i}{N\psi(u)}) \Lambda(H), \ldots, (u + \frac{t_n}{N\psi(u)}) \Lambda(H) \right) \right| = \frac{1}{N^n} |\det(\tilde{K}_N((u + \frac{t_i}{N\psi(u)}) \Lambda(H), (u + \frac{t_j}{N\psi(u)}), \frac{1}{4N}))_{i,j=1}^n|$$

$$\leq n! \max_{|t_i|, |t_j| \leq A} \left| \frac{1}{N} \tilde{K}_N((u + \frac{t_i}{N\psi(u)}) \Lambda(H), (u + \frac{t_j}{N\psi(u)}), \frac{1}{4N}) \right|^n \leq n! C(\theta)^n \Omega(\theta)^{(2N-2)n} \Gamma(2P + 1)^n H^{2P+1} (1 - \frac{\theta}{2})^{2P+1} n.$$

We complete the proof of this corollary. □
2.3. Proof of Theorem 1 in the bulk of the spectrum. Now, we turn to the proof of Theorem 1 in the bulk.

Proof. From the Stirling’s formula \( \Gamma(z) = \sqrt{2\pi} \exp(-z)z^{z-1/2}(1 + O(\frac{1}{z})) \) if \( |z| \to \infty, |\arg(z)| < \pi \), we know that

\[
\lim_{N \to \infty} N \gamma(N(N + \alpha + v)) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\alpha^2}{2}}.
\]

Particularly taking \( v = 0 \), then \( N \gamma(N(N + \alpha)) \to 1/\sqrt{2\pi} \) as \( N \to \infty \). A straightforward calculation proves that

\[
\lim_{N \to \infty} \int_{-\infty}^{\infty} \phi_N(y)dy = \sqrt{2\pi}.
\]

By Eq. (2.28), it suffices to show that

\[
\lim_{N \to \infty} \frac{1}{(N\psi(u))^n} \int_{-\infty}^{+\infty} \phi_N(y)R_n^{LUE, \frac{1}{\sqrt{N(1+y/N)}}}(u + \frac{t_1}{N\psi(u)}, \ldots, u + \frac{t_n}{N\psi(u)})dy = \sqrt{2\pi} \det \left( \frac{\sin(\pi(t_i - t_j))}{\pi(t_i - t_j)} \right)_{i,j=1}^{n}
\]

holds uniformly in \( u, t_1, \ldots, t_n \) satisfying the assumptions of this theorem. Note that

\[
\phi_N(-y) = \phi_N(y), \quad R_n^{LUE, \frac{1}{\sqrt{N(1+y/N)}}} = R_n^{LUE, \frac{1}{\sqrt{N(1+y/N)}}},
\]

and

\[
\lim_{N \to \infty} \int_{0}^{\infty} \text{Re} \phi_N(y)dy = \frac{\sqrt{2\pi}}{2}.
\]

It is enough to show that

\[
\lim_{N \to \infty} \frac{1}{(N\psi(u))^n} \int_{0}^{+\infty} \text{Re}[\phi_N(y)R_n^{LUE, \frac{1}{\sqrt{N(1+y/N)}}}(u + \frac{t_1}{N\psi(u)}, \ldots, u + \frac{t_n}{N\psi(u)})dy = \lim_{N \to \infty} \int_{0}^{\infty} \text{Re} \phi_N(y)dy \det \left( \frac{\sin(\pi(t_i - t_j))}{\pi(t_i - t_j)} \right)_{i,j=1}^{n}.
\]

Making the change of variables: \( y/N = H \), we find

\[
\frac{1}{(N\psi(u))^n} \int_{0}^{+\infty} \text{Re}[\phi_N(y)R_n^{LUE, \frac{1}{\sqrt{N(1+y/N)}}}(u + \frac{t_1}{N\psi(u)}, \ldots, u + \frac{t_n}{N\psi(u)})dy = \frac{N}{(N\psi(u))^n} \int_{0}^{+\infty} dH
\]

\[
\text{Re} \left[ \phi_N(NH)R_n^{LUE, \frac{1}{\sqrt{N(1+y/N)}}}(u + \frac{t_1}{N\psi(u)}\Lambda(H), \ldots, u + \frac{t_n}{N\psi(u)}\Lambda(H) \right) \Lambda(H)^n
\]

\[
= \frac{N}{(N\psi(u))^n} \left( \int_{0}^{\bar{H}} + \int_{\bar{H}}^{+\infty} \right) = I_1 + I_2
\]

where \( \bar{H} \in (0, H_{0}(\theta)) \) fixed and \( H_{0}(\theta) \) as in Lemma 5. Here we have used the following fact that

\[
R_n^{LUE, s}(x_1, \ldots, x_n) = R_n^{LUE, \frac{1}{s}}(x_1\sigma^{-1}, \ldots, x_n\sigma^{-1})\sigma^{-n}
\]
holds for any \( \operatorname{Re} \sigma > 0 \) and \( s > 0 \). On the other hand, let

\[
\int_0^\infty \operatorname{Re} \phi_N(y) dy \left| \det \left( \frac{\sin(\pi(t_i - t_j))}{\pi(t_i - t_j)} \right)_{i,j=1}^n \right| \\
= \left( \int_0^H + \int_H^{+\infty} \right) \operatorname{Re} \phi_N(NH) dH \left| \det \left( \frac{\sin(\pi(t_i - t_j))}{\pi(t_i - t_j)} \right)_{i,j=1}^n \right| = J_1 + J_2.
\]

Next we will prove that when \( N \to \infty \),

\[ (2.65) \quad I_1 - J_1 \to 0, \quad I_2 \to 0, \quad J_2 \to 0. \]

By Lemma 5, for every \( 0 < \theta < 1 \), \( A > 0 \) the following relation

\[
\lim_{N \to \infty} \frac{1}{(N\psi(u))^n} R_n^{LUE, \frac{1}{N}} \left( u + \frac{t_1}{N\psi(u)} \right) \Lambda(H), \ldots, \left( u + \frac{t_n}{N\psi(u)} \right) \Lambda(H) \nonumber \]

\[
= \det \left( \frac{\sin \pi(t_i - t_j) \Lambda(H) \psi(u \Lambda(H))}{\pi(t_i - t_j) \Lambda(H)} \right)_{i,j=1}^n \nonumber \]

\[ \equiv \tilde{K}(u, H) \]

holds uniformly in all \( u \in [\theta, 1 - \theta] \), \( |t_i| \leq A \) and \( H \in [0, \tilde{H}] \). The uniform continuity of the function \( \tilde{K}(u, H) \Lambda(H)^n \) with respect to \( H \) means that for every \( \epsilon \), there exists an \( H(\epsilon) \leq H_0(\theta) \) such that

\[ (2.66) \quad |\tilde{K}(u, H) \Lambda(H)^n - \tilde{K}(u, 0)| < \epsilon \]

holds for all \( 0 \leq H \leq H(\epsilon) \). We choose \( \tilde{H} = H(\epsilon) \). Note that

\[ (2.67) \quad |\phi_N(y)| = \exp\left( -\frac{N(N + \alpha)}{2} \ln(1 + \frac{y^2}{N^2}) \right), \]

and so

\[ (2.68) \quad N \int_0^\tilde{H} |\operatorname{Re} \phi_N(NH)| dH \leq 2 \int_0^{+\infty} \exp\left( -\frac{y^2}{2} \right) dy. \]

Hence, for large \( N \) the difference \( I_1 - J_1 \) can be controlled by

\[
|I_1 - J_1| \leq N \int_0^H |\operatorname{Re} \phi_N(NH) \left( \frac{1}{(N\psi(u))^n} R_n^{LUE, \frac{1}{N}} - \tilde{K}(u, H) \right) \Lambda(H)^n| dH \\
+ N \int_0^{+\infty} |\operatorname{Re} \left( \phi_N(NH) \tilde{K}(u, H) \Lambda(H)^n - \tilde{K}(u, 0) \right)| dH \\
\leq \epsilon A \int_0^{+\infty} \exp\left( -\frac{y^2}{2} \right) dy.
\]

The uniform boundedness of the function \( \tilde{K}(u, 0) \) with respect to \( u \in [\theta, 1 - \theta] \) implies that as \( N \to \infty \), the following relation

\[
|J_2| \leq |\tilde{K}(u, 0)| N \int_0^{+\infty} |\operatorname{Re} \phi_N(NH)| dH \leq |\tilde{K}(u, 0)| 2 \int_0^{+\infty} \exp\left( -\frac{y^2}{2} \right) dy \to 0
\]
holds uniformly in all \( u \). It follows from Corollary \( 3 \) that
\[
|I_2| \leq \frac{N}{V(u)^n} n! C(\theta)^n \Omega(\theta)^{(2N-2)n} \Gamma(2P + 1)^n (1 - \theta^2)^{(2P + 1)n} \\
\times \int_{\mathbb{H}}^{+\infty} \text{Re} \left[ \phi_N(NH)\Lambda(H)^n \right] |H|^{(2P + 1)n} dH \\
\leq \frac{N}{V(u)^n} n! C(\theta)^n \Omega(\theta)^{(2N-2)n} (1 - \theta^2)^{(2P + 1)n} \\
\times \Gamma(2P + 1)^n \int_{\mathbb{H}}^{+\infty} e^{-\frac{N\theta^N}{2} \ln(1+H^2)(1 + H^2)^{\frac{1}{2}}} H^{(2P + 1)n} dH \\
\leq e^{C''} \Gamma(2P + 1)^n \int_{\mathbb{H}}^{+\infty} e^{-\frac{N\theta^N}{2} \ln(1+H^2)(1 + H^2)^{n(P+1)} \frac{1}{1 + H^2}} dH \\
\leq e^{C''} \Gamma(2P + 1)^n e^{-\frac{N\theta^N}{2} \ln(1+H^2)(1 + H^2)^{n(P+1)-1}} \int_{0}^{+\infty} \frac{1}{1 + H^2} dH.
\]
Here the constant \( C'' \) depends on \( \theta \) and \( n \). On the other hand, we notice that
\[
\Gamma(2P + 1)^n = \sqrt{2\pi} \exp(n(2P + 1/2) \ln(2P + 1) - 2P - 1) \sim O(e^{n2N\ln(2N)}),
\]
thus \( I_2 \to 0 \) as \( N \to \infty \). We complete the proof of this theorem in the bulk. \( \square \)

3. PROOF OF THEOREM \( 1 \) THE SOFT EDGE OF THE SPECTRUM

From Eq. (2.20), we get
\[
(\cdot)_{\Omega}(x_1, x_2, \cdots, x_n) = \int_{0}^{\infty} R_n(x_1, x_2, \cdots, x_n) \frac{1}{u^N} \gamma(N\alpha) N \alpha du,
\]
where \( N\alpha = N(N + \alpha) \). Next, We prove a more refined asymptotic result than Eq. (2.10).

**Lemma 10.** Let \( \{b_N\} \) be a sequence such that \( b_N \to 0 \) but \( Nb_N/\sqrt{\ln N} \to \infty \) as \( N \to \infty \), then we have
\[
\int_{0}^{\infty} \gamma(N\alpha) N \alpha du = \int_{u_-}^{u_+} \gamma(4Nu) 4N du + O(e^{-\frac{1}{2}(Nb_N)^2(1+o(1))}),
\]
where \( u_\pm = \frac{N}{4}(1 \pm b_N) \).

**Proof.** Divide the left hand side of the Eq. (3.2) into three parts
\[
\int_{0}^{\infty} \gamma(4Nu) 4N du = \int_{0}^{1-b_N} \int_{1+b_N}^{1+b_N} \int_{1-b_N}^{1+b_N} \gamma(N(N + \alpha)u) N(N + \alpha) du.
\]
First consider \( \int_{0}^{u_-} \gamma(4Nu) 4N du \). By \( \gamma'(4Nu) = 0 \), we get the maximum point
\[
\max = \frac{(N + \alpha)}{4} > u_-
\]
for sufficiently large \( N \). Note that the function \( \gamma(4Nu) \) is monotonically increasing when \( u \in (0, u_-) \), thus
\[
\int_{0}^{u_-} \gamma(4Nu) 4N du \leq \gamma(N\alpha(1 - b_N)) N\alpha(1 - b_N).
\]
It follows from (2.21) that
\[
\gamma(N_\alpha(1-b_N))N_\alpha(1-b_N) = \frac{(1-b_N)^{N_\alpha} e^{-N_\alpha(1-b_N)}}{N_\alpha^{-N_\alpha} \Gamma(N_\alpha)}.
\]

Using Stirling’s formula, we get
\[
\ln[\gamma(N_\alpha(1-b_N))N_\alpha(1-b_N)] = N_\alpha \ln(1-b_N) + \frac{1}{2} \ln N_\alpha + O(1)
\]
\[
= -N_\alpha b_N^2 + N(N + \alpha) O(b_N^3) + \frac{1}{2} \ln N_\alpha + O(1)
\]
\[
= -N_\alpha b_N^2 \left[ 1 + O(b_N) + \frac{1}{N_\alpha b_N^2} \ln N_\alpha + O\left( \frac{1}{N_\alpha b_N^2} \right) \right]
\]
\[
= -N_\alpha b_N^2 \left[ 1 + 2 \ln \frac{N}{N_\alpha b_N^2} + o(1) + O\left( \frac{1}{\ln N} \right) \right] = -\frac{1}{2} N b_N^2 (1 + o(1)).
\]

In the last two equalities we used \( b_N \to 0 \) and \( N b_N / \sqrt{\ln N} \to \infty \) as \( N \to \infty \).

Next, we estimate \( \int_{u_+}^{\infty} u^2 \gamma(4Nu)4N \, du \). Notice that the unique maximum point of \( u^2 \gamma(4Nu) \) satisfies
\[
\tilde{u}_{\text{max}} = \frac{N_\alpha + 1}{4N} < u_+
\]
for sufficiently large \( N \), and \( u^2 \gamma(4Nu) \) is monotonically decreasing when \( u \in [u_+ , \infty) \), thus
\[
\int_{u_+}^{\infty} \gamma(4Nu)4N \, du = \int_{u_+}^{\infty} u^{-2}(u^2 \gamma(4Nu)4N) \, du
\]
\[
\leq u_+^2 \gamma(4Nu_+)4N \int_{u_+}^{\infty} u^{-2} \, du = 4Nu_+ \gamma(4Nu_+) = \gamma(N_\alpha(1+b_N))N_\alpha(1+b_N).
\]

Similarly, we can get
\[
\ln[\gamma(N_\alpha(1+b_N))N_\alpha(1+b_N)] = -\frac{1}{2} N^2 b_N^2 (1 + o(1)).
\]

Combining (3.7) and (3.8), this completes the proof. \( \Box \)

Remark 11. In Lemma 10, let us take \( b_N = N^{-\kappa}, \kappa \in (0,1) \). It is a well-known fact that the scaling at the soft edge of the spectrum is proportional to \( N^{-2/3} \), thus we can choose \( \kappa > 2/3 \) and give a very close approximation of correlation functions near the radial sharp cutoff point. Then using known results about the unconstrained ensembles, we obtain Airy kernel for the fixed trace ensembles. Such arguments can also deal with Bessel kernel at the hard edge. However, it seems to be insufficient for proving universality in the bulk. The main difficulty is that the “rate” index \( \kappa \) has been rather sharp, in the sense that it cannot be replaced with a larger number than 1.
Proof: (Theorem 1: the soft edge)
For \( f \in C_c(\mathbb{R}^n) \), From (3.11) one finds that

\[
(3.9) \quad \frac{1}{(2N)^{2/3}} \int_{\mathbb{R}^n} f(t_1, \ldots, t_n) R_n^{L_{J_E, \pi_N}} \left(1 + \frac{t_1}{(2N)^{2/3}}, \ldots, 1 + \frac{t_n}{(2N)^{2/3}}\right) \, dt
= \frac{1}{(2N)^{2/3}} \int_{\mathbb{R}^n} f(t_1, \ldots, t_n) \int_0^{\infty} R_n^{\delta, N_{\alpha}} \left(u^{-1}(1 + \frac{t_1}{(2N)^{2/3}}), \ldots, u^{-1}(1 + \frac{t_n}{(2N)^{2/3}})\right) \frac{1}{u^{\gamma}} \gamma(N_{\alpha}u)N_{\alpha} \, du \, dt
\]

Next we will prove that \( I \) \( \leq \parallel f \parallel \parallel R_n^{\delta, N_{\alpha}} \parallel \int_{\mathbb{R}^n} \gamma(N_{\alpha}u)N_{\alpha} \, du \)

\[
(3.10) \quad t_i = (2N)^{2/3}(u_i - 1) + uy_i, i = 1, \ldots, n.
\]

Choose \( b_N = N^{-\gamma} \) for fixed \( \gamma \in (\frac{2}{3}, 1) \). By Lemma 10 we get

\[
|I| \leq \frac{\|f\|_{\infty}}{(2N)^{2/3}} \int_{\mathbb{R}^n} \int_0^{1-b_N} \gamma(N_{\alpha}u)N_{\alpha} R_n^{\delta, N_{\alpha}} \left(1 + \frac{y_1}{(2N)^{2/3}}, \ldots, 1 + \frac{y_n}{(2N)^{2/3}}\right) \, du \, d^n y
\]

\[
\leq \frac{\|f\|_{\infty}}{(2N)^{2/3}} \int_{\mathbb{R}^n} R_n^{\delta, N_{\alpha}} \left(1 + \frac{y_1}{(2N)^{2/3}}, \ldots, 1 + \frac{y_n}{(2N)^{2/3}}\right) d^n y \int_0^{1-b_N} \gamma(N_{\alpha}u)N_{\alpha} \, du
\]

\[
= \|f\|_{\infty} \frac{N!}{(N-n)!} O(e^{-1/2(Nb_N^2)(1+o(1))}) = o(1).
\]

Similarly, again by Lemma 10, we have \( |I_1| = o(1) \). Thus

\[
(3.11) \quad (3.9) = I_2 + o(1).
\]

Let

\[
I_2 = \frac{1}{(2N)^{2/3}} \int_{\mathbb{R}^n} \int_{1-b_N}^{1+b_N} f(y_1, \ldots, y_n)
\]

\[
\times R_n^{\delta, N_{\alpha}} \left(1 + \frac{y_1}{(2N)^{2/3}}, \ldots, 1 + \frac{y_n}{(2N)^{2/3}}\right) \gamma(N_{\alpha}u)N_{\alpha} \, du \, d^n y
\]

\[
= \frac{1}{(2N)^{2/3}} \int_{\mathbb{R}^n} f(y_1, \ldots, y_n) R_n^{\delta, N_{\alpha}} \left(1 + \frac{y_1}{(2N)^{2/3}}, \ldots, 1 + \frac{y_n}{(2N)^{2/3}}\right) d^n y
\]

\[
\times \int_{1-b_N}^{1+b_N} \gamma(N_{\alpha}u)N_{\alpha} \, du.
\]

Next we will prove that

\[
(3.12) \quad \lim_{N \to \infty} |I_2 - I_2'| = 0.
\]
Since \( f \in C_c(\mathbb{R}^n) \) and
\[
(3.13) \quad (2N)^{2/3}(u - 1) + uy_i \xrightarrow{N \to \infty} y_i, \quad i = 1, \ldots, n,
\]
we can choose a ball \( B_R \) of the radius \( R \) in \( \mathbb{R}^n \) centered at zero such that \( \text{supp}(f) \subset B_R \) and
\[
\{(2N)^{2/3}(u - 1) + uy_1, \ldots, (2N)^{2/3}(u - 1) + uy_n\} \subset \text{supp}(f), \quad 1 - b_R \leq u \leq 1 + b_R \subset B_R.
\]
From \( f \in C_c(\mathbb{R}^n) \), given \( \epsilon > 0 \), there exists some \( \delta(\epsilon) > 0 \) such that \( |f(x_1, \ldots, x_n) - f(y_1, \ldots, y_n)| < \epsilon \), whenever \( ||(x_1, \ldots, x_n) - (y_1, \ldots, y_n)|| < \delta(\epsilon) \). On the other hand, there exist \( N_0 \) independent of \( (y_1, \ldots, y_n) \) in \( B_R \) such that
\[
((2N)^{2/3}(u - 1) + uy_1, \ldots, (2N)^{2/3}(u - 1) + uy_n) - (y_1, \ldots, y_n) \leq \sqrt{n}|u - 1|(R + (2N)^{2/3}) \leq \sqrt{n}N^{-\gamma}(R + (2N)^{2/3}) \leq \delta(\epsilon)
\]
for \( N > N_0 \). Therefore, \( \forall (y_1, \ldots, y_n) \in \text{supp}(f) \)
\[
|f((2N)^{2/3}(u - 1) + uy_1, \ldots, (2N)^{2/3}(u - 1) + uy_n) - f(y_1, \ldots, y_n)| < \epsilon.
\]
Furthermore, we get
\[
|I_2 - I_2'| \leq \frac{\epsilon}{((2N)^{2/3})^n} \int_{B_R} \int_{1 - b_R}^{1 + b_R} \frac{1}{((2N)^{2/3})^n} R_u^{\delta + \alpha} (1 + \frac{y_1}{(2N)^{2/3}}, \ldots, 1 + \frac{y_n}{(2N)^{2/3}})^\gamma (N_u N_d du d^n y = \epsilon \int_{1 - b_R}^{1 + b_R} \gamma (N_u N_d du \int_{B_R} R_u^{\delta + \alpha} (1 + \frac{y_1}{(2N)^{2/3}}, \ldots, 1 + \frac{y_n}{(2N)^{2/3}}) d^n y \leq \epsilon (1 + o(1))C_R.
\]
Here \( C_R \) is a constant and we have used Lemma 12 below. Hence the relation (3.12) holds. Combining (3.9) and (3.11), we have (3.9) = I_2 + o(1) for sufficiently large \( N \), more precisely,
\[
\frac{1}{((2N)^{2/3})^n} \int_{\mathbb{R}^n} f(t_1, \ldots, t_n) R_{\delta + \alpha} \left(1 + \frac{t_1}{(2N)^{2/3}}, \ldots, 1 + \frac{t_n}{(2N)^{2/3}}\right) d^n t = \frac{1}{((2N)^{2/3})^n} \int_{\mathbb{R}^n} f(y_1, \ldots, y_n) R_{\delta + \alpha} \left(1 + \frac{y_1}{(2N)^{2/3}}, \ldots, 1 + \frac{y_n}{(2N)^{2/3}}\right) d^n y (1 + o(1)) + o(1),
\]
this proves the anticipated result. \( \square \)

**Lemma 12.** For any fixed \( R > 0 \), let \( B_R \) be the ball of the radius \( R \) in \( \mathbb{R}^n \) centered at zero. There exists some constant \( C_R \) such that
\[
(3.15) \quad \frac{1}{((2N)^{2/3})^n} \int_{B_R} R_{\delta + \alpha} \left(1 + \frac{y_1}{(2N)^{2/3}}, \ldots, 1 + \frac{y_n}{(2N)^{2/3}}\right) d^n y \leq C_R.
\]

**Proof.** Given \( 0 < \delta < R \), there exists \( N_0(R, \delta) \) such that when \( N > N_0(R, \delta) \) and \( (y_1, \ldots, y_n) \in B_R \),
\[
(3.16) \quad \sum_{i=1}^{n} ((2N)^{2/3}(u - 1) + uy_i)^2 < (R + \delta)^2
\]
where \( u \in [1 - N^{-\gamma}, 1 + N^{-\gamma}] \). Let \( \eta \in (0, 1) \) be a real number and \( \phi(t) \) be a smooth decreasing function on \([0, \infty)\) such that \( \phi(t) = 1 \) for \( t \in [0, R + \delta) \).
and $\phi(t) = 0$ for $t \geq (1 + \eta)(R + \delta)$. Set $\varphi(x_1, \ldots, x_n) = \phi(||(x_1, \ldots, x_n)||)$ for $(x_1, \ldots, x_n) \in \mathbb{R}^n$. For $N > N_0$, we have

\[
(3.17) \quad \frac{1}{((2N)^{2/3})^n} \int_{B_R} R_n^{\delta,N} \gamma\left((1 + \frac{y_1}{(2N)^{2/3}}, \ldots, 1 + \frac{y_n}{(2N)^{2/3}}) d^ny \right.
\]

Multiplying by $\gamma(N_\alpha u)N_\alpha$ and then integrating with respect to $u$ on $[1 - b_N, 1 + b_N]$, one obtains

\[
\int_{1-b_N}^{1+b_N} \gamma(N_\alpha u)N_\alpha du \frac{1}{((2N)^{2/3})^n} \int_{B_R} R_n^{\delta,N}(1 + \frac{y_1}{(2N)^{2/3}}, \ldots, 1 + \frac{y_n}{(2N)^{2/3}}) d^ny \leq \frac{1}{((2N)^{2/3})^n} \int_{R^n} \varphi((2N)^{2/3}(u - 1) + uy_1, \ldots, (2N)^{2/3}(u - 1) + uy_n) \gamma(N_\alpha u)N_\alpha R_n^{\delta,N} (1 + \frac{y_1}{(2N)^{2/3}}, \ldots, 1 + \frac{y_n}{(2N)^{2/3}}) du d^ny \left.\right)
\]

\[
= \frac{1}{((2N)^{2/3})^n} \int_{R^n} \varphi(t_1, \ldots, t_n) R_n^{LUE, \frac{N_\alpha}{\alpha}} (1 + \frac{t_1}{(2N)^{2/3}}, \ldots, 1 + \frac{t_n}{(2N)^{2/3}}) d^nt + o(1).
\]

Here we make use of Eq. (3.11). This completes the proof of this lemma. \[\square\]

4. PROOF OF THEOREM [1] THE HARD EDGE OF THE SPECTRUM

First we prove that the $n$-point correlation function $R_n^{\delta,r}$ could be expressed as the inverse Laplace transform of $R_n^{LUE, \frac{N_\alpha}{\alpha}}$, which is slightly different from Eq. (2.26).

Proposition 13. Let $R_n^{LUE, \frac{N_\alpha}{\alpha}}$ and $R_n^{\delta,r}$ be the $n$-point correlation functions of eigenvalues for the LUE and FTLUE, respectively. Then we have the relation

\[
R_n^{\delta,r}(x_1, \ldots, x_n) = \frac{\Gamma(N_\alpha)}{\pi^{N_\alpha-1}(4N)^n} \mathcal{L}^{-1}[t^{-(N_\alpha - n)} R_n^{LUE, \frac{N_\alpha}{\alpha}}(\frac{t}{4N}, x_1, \ldots, \frac{t}{4N}, x_n)](r)
\]

where $\mathcal{L}^{-1}[h(t)](x)$ is the inverse Laplace transform of a function $h(t)$, and $N_\alpha = N(N + \alpha)$.

Proof. For $h \in L(\mathbb{R}^N)$, let $< h(\cdot) >$ and $< h(\cdot) >_\delta$ denote that the ensemble average is taken in the LUE and the FTLUE, respectively. Consider the integral

\[
I[h] = \int_{\mathbb{R}^N} h(x_1, \ldots, x_N) \delta(r - \sum_{i=1}^{N} x_i) \prod_{i=1}^{N} x_i^\alpha \prod_{j<k} |x_j - x_k|^2 d^N x.
\]

Making the change of variables: $x_j = 4N y_j, j = 1, \ldots, N$, we have

\[
I[h] = (4N)^{N_\alpha} \int_{\mathbb{R}^N} h(4Nx_1, \ldots, 4Nx_N) \delta(r - 4N \sum_{i=1}^{N} x_i) \prod_{i=1}^{N} x_i^\alpha \prod_{j<k} |x_j - x_k|^2 d^N x.
\]
Multiply both sides by $e^{-tr}$ and integrate on $r$ from 0 to $\infty$, we get
\[
\int_0^\infty e^{-tr}I[h]dr = (4N)^N \int_{\mathbb{R}^N} h(4Nx_1,\ldots,4Nx_N) \\
\times \exp \left(-t4N \sum_{i=1}^N x_i \prod_{i=1}^N x_i^\alpha \prod_{j<k} |x_j - x_k|^2 d^Nx \right) \\
= \frac{(4N)^N}{t} \int_{\mathbb{R}^N} h(\frac{4N}{t} x_1,\ldots,\frac{4N}{t} x_N) \exp \left(-4N \sum_{i=1}^N x_i \prod_{i=1}^N x_i^\alpha \prod_{j<k} |x_j - x_k|^2 d^Nx \right) \\
= Z_{1/4N} \left(\frac{4N}{t})^N h(\frac{4N}{t}) \right). 
\]

Here we have made the change of variables $x_j = t^{-1} y_j$, $j = 1,\ldots,N$, where $t^{-1}$ denotes the principal branch of the power for complex variable $t$. Using the inverse Laplace transform, we have
\[
I[h] = Z_{1/4N} \mathcal{L}^{-1}[\left(\frac{4N}{t})^N h(\frac{4N}{t}) \right)](r). 
\]

Notice that
\[
\mathcal{L}^{-1}[t^{-\gamma}](x) = \frac{x^{\gamma-1}}{\Gamma(\gamma)} \theta(x), \quad \text{Re}(\gamma) > 0. 
\]

The ensemble average $\langle h \rangle_\delta$ reads
\[
\langle h \rangle_\delta = \frac{I[h]}{I[1]} = \frac{\Gamma(N_\alpha)}{r_{N_\alpha-1} \mathcal{L}^{-1}[t^{-N_\alpha} h(\frac{4N}{t}) \rangle]}(r). 
\]

In particular, taking
\[
h(x_1,\ldots,x_N) = \sum_{1 \leq i_1 < \cdots < i_n \leq N} f(x_{i_1},\ldots,x_{i_n}),
\]
then we find that
\[
\int_{\mathbb{R}^n} f(x_1,\ldots,x_n) R_n^{LUE}(x_1,\ldots,x_n) d^n x \\
= \frac{\Gamma(N_\alpha)}{r_{N_\alpha-1} (4N)^n} \int_{-i\infty+0^+}^{-i\infty+0^+} dt \ e^{r t} \Gamma(-N_\alpha) \\
\times \int_{\mathbb{R}^n} f(\frac{4N}{t} x_1,\ldots,\frac{4N}{t} x_n) R_n^{LUE, \frac{4N}{t}}(x_1,\ldots,x_n) d^n x \\
= \frac{\Gamma(N_\alpha)}{r_{N_\alpha-1} (4N)^n} \int_{-i\infty+0^+}^{-i\infty+0^+} dt \ e^{r t} \Gamma(-N_\alpha-n) \\
\times \int_{\mathbb{R}^n} f(x_1,\ldots,x_n) R_n^{LUE, \frac{4N}{t}}(\frac{t}{4N} x_1,\ldots,\frac{t}{4N} x_n) d^n x \\
= \frac{\Gamma(N_\alpha)}{r_{N_\alpha-1} (4N)^n} \int_{\mathbb{R}^n} f(x_1,\ldots,x_n) d^n x \frac{1}{2\pi i} \int_{-i\infty+0^+}^{-i\infty+0^+} e^{r t} \Gamma(-N_\alpha-n) \\
\times R_n^{LUE, \frac{4N}{t}}(\frac{t}{4N} x_1,\ldots,\frac{t}{4N} x_n) d t.
\]
Since $R_n^{LUE, N+\alpha}$ and $R_n^{\delta, r}$ are continuous, we get

$$R_n^{\delta, r}(x_1, \ldots, x_n) = \frac{\Gamma(N_n)}{r^{N_n-1}(4N)^n} \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{rt-(N_n-n)} R_n^{LUE, \frac{1}{4N}} \left( \frac{t}{4N} x_1, \ldots, \frac{t}{4N} x_n \right) dt$$

$$= \frac{\Gamma(N_n)}{r^{N_n-1}(4N)^n} \mathcal{L}^{-1} [ t^{-(N_n-n)} ] R_n^{LUE, \frac{1}{4N}} \left( \frac{t}{4N} x_1, \ldots, \frac{t}{4N} x_n \right) \{ r \}.$$

This completes the proof. □

Proof: (Theorem 1 the hard edge)

Now we make use of the fact that $n$-point correlation function $R_n^{LUE, \frac{1}{4N}} (x_1, \ldots, x_n)$ can be expanded as follows:

$$R_n^{LUE, \frac{1}{4N}} (x_1, \ldots, x_n) = \prod_{i=1}^n x_i^{\alpha} \exp \left( -4N \sum_{i=1}^n x_i \right) \sum_{l_1, \ldots, l_n=0}^{2N-2} c^{(N)}_{\{l_1, \ldots, l_n\}} x_1^{l_1} \cdots x_n^{l_n}$$

$$= \exp \left( -4N \sum_{i=1}^n x_i \right) \sum_{l_1, \ldots, l_n=0}^{2N-2} c^{(N)}_{\{l_1, \ldots, l_n\}} x_1^{l_1} \cdots x_n^{l_n},$$

where $l_i = l_i + \alpha$, $i = 1, \ldots, n$. It follows from Proposition 13 and Eq. (4.1) that

$$R_n^{\delta, N+\alpha} (x_1, \ldots, x_n)$$

$$= \frac{4^n N_n - 1}{(N+\alpha)^n (4N)^n} \sum_{l_1, \ldots, l_n=0}^{2N-2} c^{(N)}_{\{l_1, \ldots, l_n\}} x_1^{l_1} \cdots x_n^{l_n} \mathcal{L}^{-1} [ t^{-(N_n-n-\sum l_i)} ] \left( \frac{N+\alpha}{4} - \sum_{i=1}^n x_i \right)$$

$$= \theta \left( \frac{N+\alpha}{4} \right) \sum_{l_1, \ldots, l_n=0}^{2N-2} \left( \frac{N+\alpha}{4} - \sum_{i=1}^n x_i \right) N_n - \sum l_i - 1 \cdot \frac{4^n N_n - 1}{(N+\alpha)^n (4N)^n} \sum_{l_1, \ldots, l_n=0}^{2N-2} c^{(N)}_{\{l_1, \ldots, l_n\}} x_1^{l_1} \cdots x_n^{l_n}.$$

With rescaling $x_i = \frac{l_i}{N\gamma}$, we will deal with the different factors in (4.2). First, the $\theta$-function term

$$\theta \left( \frac{N+\alpha}{4} \right) \frac{1}{N^2} \sum_{i=1}^n \frac{l_i}{16} \rightarrow 1.$$
as $N \to \infty$. Since $\sum l'_i \leq n(2N - 2 + \alpha)$, the first term in the sum yields

\[
\sum_{l'_i} \frac{1}{N^2(N + \alpha)} \sum_{i=1}^{n} t_i \frac{N_\alpha - n - \sum l'_i}{n} \tag{4.4}
\]

\[
= 1 - \left(N_\alpha - n - \sum l'_i - 1\right) \frac{4}{N^2(N + \alpha)} \sum_{i=1}^{n} t_i + \frac{1}{N^2} C_N(t_1, \cdots, t_n) \tag{4.5}
\]

Here $C_N(t_1, \cdots, t_n)$ is uniformly bounded in all $t_1 \in (0, A], \cdots, t_n \in (0, A]$ for given $A > 0$.

Before evaluating the factor containing Gamma functions, we introduce a lemma about ratio of two Gamma functions, due to Tricomi and Erdélyi [31], see also Copson’s book [5].

**Lemma 14.** Let $a > 0$, for sufficiently large $x$ the following expansion holds:

\[
\frac{\Gamma(x)}{x^a \Gamma(x - a)} = \sum_{s=0}^{\infty} \frac{a(a-1)\cdots(a-s+1)}{s! x^s} B_s^{(a+1)}(0). \tag{4.6}
\]

Here $B_s^{(a)}(x)$ is Nörlund’s generalized Bernoulli polynomial in $x$ of degree $s$, and $B_s^{(a)}(0)$ is also a polynomial in $a$ of degree $s$.

Note that there are only finite terms in the sum of the right-hand side of Eq.(4.6) if $\alpha$ is an integer.

For the convenience, we write

\[
L_s(a) = a(a-1)\cdots(a-s+1)B_s^{(a+1)}(0).
\]

It is a polynomial in $a$ of degree $2s$. By Lemma[14] the factor containing Gamma-functions reads:

\[
\frac{\Gamma(N_\alpha)}{(N_\alpha)\sum l'_i + n \Gamma(N_\alpha - n - \sum l'_i)} = \sum_{s=0}^{\infty} \frac{1}{s!(N_\alpha)^s} L_s(n + \sum l'_i). \tag{4.7}
\]

Since $B_0^{(a)}(x) = 1$, we have $L_0(n + \sum l'_i) = 1$. However, if $\sum l'_i \sim N$, the terms on the right-hand side of Eq.(4.7)

\[
\frac{1}{(N_\alpha)^s} L_s(n + \sum l'_i) \tag{4.8}
\]

approach to a finite non-zero number as $N \to \infty$, not sub-leading. With the aid of multiplication and partial differential operators, we can rewrite

\[
L_s(n + \sum l'_i) t_1^{l'_1} \cdots t_n^{l'_n} = L_s(\sum_{i=1}^{n} \partial_{t_i} t_i) t_1^{l'_1} \cdots t_n^{l'_n}. \tag{4.9}
\]
Combing (4.3), (4.4), (4.7) and (4.9), it follows from (4.2) that

\[
\frac{1}{(16N^2)^n} R_{n}^{LUE, \xi}(t_1/16N^2, \ldots, t_n/16N^2) = \frac{1}{(16N^2)^n} \sum_{l_1, \ldots, l_n=0}^{2N-2} \left(1 - \left(\frac{1}{N} - n - \sum_{i=1}^{l_i} - \frac{1}{2}\right) \frac{4}{N^2(N + \alpha)} \sum_{i=1}^{n} \frac{t_i}{16}\right) \\
+ \frac{1}{N^2} C_N(t_1, \ldots, t_N) \sum_{s=0}^{\infty} L_s(N + \sum_{i=1}^{l_i} \frac{1}{t_i}) \frac{\partial}{\partial l_i} c_{(l_1, \ldots, l_n)} \left(\frac{t_1}{16N^2}\right)^{l_1} \cdots \left(\frac{t_n}{16N^2}\right)^{l_n} \\
\tag{4.11}
\]

\[
\frac{1}{(16N^2)^n} R_{n}^{LUE, \xi}(t_1/16N^2, \ldots, t_n/16N^2) = \frac{4(N_n - 1)}{N^2(N + \alpha)} \sum_{i=1}^{n} \frac{t_i}{16} + \frac{4}{N^2(N + \alpha)} \sum_{i=1}^{n} \frac{t_i}{16} \\
\times \frac{1}{(16N^2)^n} \sum_{l_1, \ldots, l_n=0}^{2N-2} \left(\sum_{i=1}^{n} \frac{\partial}{\partial t_i} t_i \right) \sum_{s=0}^{\infty} L_s(N + \sum_{i=1}^{l_i} \frac{1}{t_i}) \frac{\partial}{\partial l_i} c_{(l_1, \ldots, l_n)} \left(\frac{t_1}{16N^2}\right)^{l_1} \cdots \left(\frac{t_n}{16N^2}\right)^{l_n} \\
+ \frac{1}{(16N^2)^n} \sum_{l_1, \ldots, l_n=0}^{2N-2} \frac{1}{N^2} C_N(t_1, \ldots, t_n) \sum_{s=0}^{\infty} L_s(N + \sum_{i=1}^{l_i} \frac{1}{t_i}) \frac{\partial}{\partial l_i} c_{(l_1, \ldots, l_n)} \left(\frac{t_1}{16N^2}\right)^{l_1} \cdots \left(\frac{t_n}{16N^2}\right)^{l_n} \\
\times c_{(l_1, \ldots, l_n)} \left(\frac{t_1}{16N^2}\right)^{l_1} \cdots \left(\frac{t_n}{16N^2}\right)^{l_n} = I + I_2 + I_3 + I_4 \\
\tag{4.10}
\]

where

\[
I = \sum_{s=0}^{\infty} L_s(N + \sum_{i=1}^{l_i} \frac{1}{t_i}) \frac{1}{(16N^2)^n} \sum_{l_1, \ldots, l_n=0}^{2N-2} c_{(l_1, \ldots, l_n)} \left(\frac{t_1}{16N^2}\right)^{l_1} \cdots \left(\frac{t_n}{16N^2}\right)^{l_n}.
\]

On the other hand, we know from the expansion (4.1) for the correlation function of the LUE that

\[
\lim_{N \to \infty} \frac{1}{(16N^2)^n} R_{n}^{LUE, \xi}(t_1/16N^2, \ldots, t_n/16N^2) = \lim_{N \to \infty} \frac{1}{(16N^2)^n} \exp \left(-\sum_{i=1}^{n} \frac{t_i}{4N} \sum_{l_1, \ldots, l_n=0}^{2N-2} c_{(l_1, \ldots, l_n)} \left(\frac{t_1}{16N^2}\right)^{l_1} \cdots \left(\frac{t_n}{16N^2}\right)^{l_n} \right) \\
= \lim_{N \to \infty} \frac{1}{(16N^2)^n} \sum_{l_1, \ldots, l_n=0}^{2N-2} c_{(l_1, \ldots, l_n)} \left(\frac{t_1}{16N^2}\right)^{l_1} \cdots \left(\frac{t_n}{16N^2}\right)^{l_n} \\
= \det[K_{\xi}(t, t)]_{i,j=1}^{n},
\]

uniformly for \(t_1, \ldots, t_n\) in bounded subsets of \((0, \infty)\). Comparing (4.11) and (4.12), the terms where \(s > 0\) in the sum of (4.11) vanish. Hence we get

\[
\lim_{N \to \infty} I = \det[K_{\xi}(t_i, t_j)]_{i,j=1}^{n}.
\]
Thus we have that $I_2 \to 0$ as $N \to \infty$. Note that
\[
\lim_{N \to \infty} I_3 = \lim_{N \to \infty} \frac{4}{N^2(N + \alpha)} \left( \sum_{i=1}^{n} \frac{t_i}{16} \sum_{s=0}^{\infty} \frac{(\sum_{i=1}^{n} \partial_i t_i)L_s(\sum_{i=1}^{n} \partial_i t_i)}{s!(N^{\alpha})^s} \right) \\
\times \frac{1}{(16N^2)^n} \sum_{t_1, \ldots, t_n=0}^{2N-2} c^{(N)}_{\{t_1, \ldots, t_n\}} (\frac{t_1}{16N^2})^{t'_{1}} \ldots (\frac{t_n}{16N^2})^{t'_{n}} = 0
\]

We also notice the following fact: for any sequences $\{a_i\}_{i=1}^{N}$ and $\{b_i^{(N)}\}_{i=1}^{N}$, if $a_1 + \cdots + a_N \to C$ as $N \to \infty$, and $b_i^{(N)} = O(\frac{1}{N^2})$, then
\[
|a_1 b_1^{(N)} + a_2 b_2^{(N)} + \cdots + b_N^{(N)} a_N| \leq \sum_{i=1}^{N} |S_i - S_{i-1}| |b_i^{(N)}| = O\left(\frac{1}{N}\right)
\]

where $S_i = a_1 + a_2 + \cdots + a_i$. Thus we have
\[
|I_4| \leq C n(2N - 2 + \alpha)O(\frac{1}{N^2}) = O(\frac{1}{N}),
\]
uniformly for $t_1, \ldots, t_n$ in bounded subsets of $(0, \infty)$. Here we have used the fact of (4.13).

This completes the proof. □

5. PROOF OF THEOREM 2

First we give a representation of correlation functions for the BTLUE in terms of these for the FTLUE.

**Proposition 15.** Let $R_n^{\theta,r}$ and $R_n^{\delta,r}$ be the $n$-point correlation functions for the BTLUE and FTLUE respectively, then we have the following relation
\[
R_n^{\theta,r}(x_1, \ldots, x_n) = \int_{0}^{1} N_{\alpha} u^{N_{\alpha} - 1} \frac{1}{u^n} R_n^{\delta,r}(\frac{x_1}{u}, \ldots, \frac{x_n}{u}) du,
\]
where $N_{\alpha} = N(N + \alpha)$.

**Proof.** It suffices to prove
\[
R_n^{\theta,r}(x_1, \ldots, x_n) = \int_{0}^{1} N_{\alpha} u^{N_{\alpha} - 1} \frac{1}{u^n} R_n^{\delta,r}(\frac{x_1}{u}, \ldots, \frac{x_n}{u}) du.
\]

For every $u > 0$, let
\[
\Delta_N(u) = \{(x_1, \ldots, x_N) \mid \sum_{j=1}^{N} x_j = u, x_j \geq 0, j = 1, \ldots, N\}
\]
be a simplex in $\mathbb{R}^N$, which carries the volume element induced by the standard Euclidean metric on $\mathbb{R}^N$, denoted by $u^{N-1}d\sigma_N$. For $h \in L^\infty(\mathbb{R}^N)$, let $< h(\cdot) >_{\theta}$ and $< h(\cdot) >_{\delta}$ denote the ensemble average taken in the BTLUE and the FTLUE,
respectively. From (1.6) and (1.8), we have

\[ <h(\cdot)>_\theta = \frac{1}{Z_\theta} \int_0^r u^{N-1} du \int_{\Delta_N(u)} h(x_1, \ldots, x_N) \prod_{1 \leq i < j \leq N} |x_i - x_j|^2 \prod_{i=1}^N x_i^\sigma d\sigma_N \]

\[ = \frac{1}{Z_\theta} \int_0^r u^{N-1} du \int_{\Delta_N(1)} h(u x_1, \ldots, u x_N) \prod_{1 \leq i < j \leq N} |x_i - x_j|^2 \prod_{i=1}^N x_i^\sigma d\sigma_N \]

and

\[ <h(a \cdot)>_\delta = \frac{1}{Z_\delta} \int_{\Delta_N(r)} r^{N-1} h(ax_1, \ldots, ax_N) \prod_{1 \leq i < j \leq N} |x_i - x_j|^2 \prod_{i=1}^N x_i^\sigma d\sigma_N \]

\[ = \frac{1}{Z_\delta} \int_{\Delta_N(1)} r^{N-1} h(ar x_1, \ldots, ar x_N) \prod_{1 \leq i < j \leq N} |x_i - x_j|^2 \prod_{i=1}^N x_i^\sigma d\sigma_N. \]

Choose \( a = \frac{r}{\sigma} \), we get

\[ (5.4) \quad <h(\cdot)>_\theta = \frac{Z_\theta}{Z_\delta} \int_0^r \left( \frac{u}{r} \right)^{N-1} <h(u \cdot)>_\delta d\, u. \]

Setting \( h = 1 \), we get the ratio of the partition functions \( Z_\theta^r = Z_\theta^r \). Substituting this ratio, we then obtain

\[ (5.5) \quad <h(\cdot)>_\theta = \int_0^r \frac{N_\alpha}{r^{N_\alpha}} u^{N_\alpha-1} <h(u \cdot)>_\delta d\, u. \]

In particular, taking

\[ (5.6) \quad h(x_1, \ldots, x_N) = \sum_{1 \leq i_1 < \cdots < i_n \leq N} f(x_{i_1}, \ldots, x_{i_n}), \]

we have

\[ \int_{\mathbb{R}^n} f(x_1, \ldots, x_n) R_n^{\theta}(x_1, \ldots, x_n) \, d^n x = \int_{\mathbb{R}^n} \frac{N_\alpha}{r^{N_\alpha}} u^{N_\alpha-1} \, d\, u \int_{\mathbb{R}^n} f(u x_1, \ldots, u x_n) R_n^{\delta}(x_1, \ldots, x_n) \, d^n x \]

\[ = \int_{\mathbb{R}^n} f(x_1, \ldots, x_n) \, d^n x \int_0^r \frac{N_\alpha}{r^{N_\alpha}} u^{N_\alpha-1} \left( \frac{r}{u} \right)^n \, d\, u R_n^{\delta}(x_1, \ldots, x_n) \, d^n x. \]

Since \( R_n^{\theta} \) and \( R_n^{\delta} \) are both continuous, we complete the proof. \( \square \)

Next, we notice a “sharp” concentration phenomenon along the radial coordinate between correlation functions of the BTLUE and FTLUE. Although its proof is simple, the following lemma plays a crucial role in dealing with local statistical properties of the eigenvalues between the fixed and bounded ensembles.

**Lemma 16.** Let \( \{b_N\} \) be a sequence such that \( b_N \to 0 \) but \( N^2 b_N \to \infty \) as \( N \to \infty \), then we have

\[ (5.7) \quad \int_{0}^{1} N_\alpha u^{N_\alpha-1} \, du = \int_{u}^{1} N_\alpha u^{N_\alpha-1} \, du + e^{-N^2 b_N (1+o(1))}, \]
where \( u^- = 1 - b_N \).

Proof.

\[
\int_0^{u^=} N_\alpha u^{N_\alpha - 1} du = (1 - b_N)^{N_\alpha} = e^{N_\alpha \ln(1 - b_N)} = e^{N_\alpha \left(-b_N + O(b_N^2)\right)} = e^{-N^2 b_N (1 + o(1))}.
\]

This completes the proof. \( \square \)

Remark 17. In Lemma 16, let us take \( b_N = N^{-\kappa}, \kappa \in (0, 2) \). Since the “rate” index \( \kappa \) can be chosen larger than 1 while the scaling in the bulk is proportional to \( N^{-1} \) and at the soft edge of the spectrum is proportional to \( N^{-2/3} \), in principle we can prove all local statistical properties of the eigenvalues between the fixed and bounded trace ensembles are identical in the limit. Such arguments apply to the equivalence of ensembles between the fixed trace and bounded trace ensembles with monomial potentials, where we exploit some homogeneity of the monomial potentials.

Before we prove Theorem 2, let us prove the claimed result in Sect. 1: limit global density for the BTLUE is also Marchenko-Pastur law.

**Theorem 18.** Let \( R^{\delta, \alpha}_{1} \) be the 1-point correlation function of the BTLUE, for any \( f \in L^\infty(\mathbb{R}) \cap C_{\text{lip}}(\mathbb{R}) \) where the set \( C_{\text{lip}}(\mathbb{R}) \) denotes all Lipschitz continuous functions on \( \mathbb{R} \), we have

\[
\lim_{N \to \infty} \int f(x) \frac{1}{N} R^{\delta, \alpha}_{1} (x) dx = \int f(x) \psi(x) dx
\]

where \( \psi(x) \) is the Marchenko-Pastur law.

Proof. By Proposition 15,

\[
\int f(x) \frac{1}{N} R^{\delta, \alpha}_{1} (x) dx = \int \int_0^1 N_\alpha u^{N_\alpha - 1} f(x) \frac{1}{N} u^{\delta, \alpha}_{1} \left( \frac{x}{u} \right) du dx
\]

\[
= \int \int_0^1 N_\alpha u^{N_\alpha - 1} f(ux) \frac{1}{N} R^{\delta, \alpha}_{1} \left( \frac{x}{u} \right) du dx
\]

\[
= \int \left( \int_0^{u^-} + \int_{u^=}^1 \right) N_\alpha u^{N_\alpha - 1} f(ux) \frac{1}{N} R^{\delta, \alpha}_{1} \left( \frac{x}{u} \right) du dx = I_1 + I_2.
\]

By Lemma 16,

\[
I_1 \leq \|f\|_\infty \int \int_0^{u^-} N_\alpha u^{N_\alpha - 1} \frac{1}{N} R^{\delta, \alpha}_{1} \left( \frac{x}{u} \right) du dx
\]

\[
= \|f\|_\infty \int_0^{u^-} N_\alpha u^{N_\alpha - 1} du \int R^{\delta, \alpha}_{1} \left( \frac{x}{u} \right) dx
\]

\[
= \|f\|_\infty e^{-N^2 b_N (1 + o(1))} \to 0
\]

as \( N \to \infty \). On the other hand,

\[
I_2 = \int \int_{u^-}^{1} N_\alpha u^{N_\alpha - 1} \left( (f(ux) - f(x)) + f(x) \right) \frac{1}{N} R^{\delta, \alpha}_{1} \left( \frac{x}{u} \right) du dx \approx I_{21} + I_{22}.
\]
Since \( f \in C_{lip}(\mathbb{R}) \), we have
\[
I_{21} \leq (1 - u_-) L \int_{u_-}^{1} N_{\alpha} u^{N_{\alpha} - 1} du \int_{\mathbb{R}} |x| \frac{1}{N} R_{1}^{\alpha + \alpha_{1}}(x) \, dx \to 0
\]
for some constant \( L \). Here we have used the fact
\[
\lim_{N \to \infty} \int_{\mathbb{R}} |x| \frac{1}{N} R_{1}^{\alpha + \alpha_{1}}(x) \, dx = \int_{\mathbb{R}} |x| \psi(x) \, dx.
\]
Again by Lemma 16,
\[
\lim_{N \to \infty} I_{22} = \int_{\mathbb{R}} f(x) \psi(x) \, dx.
\]
This completes the proof. \( \square \)

Now we turn to the proof of Theorem 2.

**Proof.** The proof is very similar to that of the soft edge of the spectrum in Theorem 1 we only point out some different places in the bulk case.

In Lemma 16 choose \( b_{N} = N^{-\kappa} \), \( \kappa \in (1, 2) \). The change of variables corresponding to (3.10) reads:
\[
t_{i} = (u - 1) N x \psi(x) + u y_{i}, i = 1, \ldots, n
\]
where fixed \( x \in (0, 1) \). The condition that \( b_{N} = N^{-\kappa} \), \( \kappa \in (1, 2) \) ensures \( (1 - u) N \leq N^{-\kappa + 1} \to 0 \) as \( N \to \infty \) for \( u \in [u_{-}, 1] \). On the other hand, by Theorem 1, the following fact similar to Lemma 12 is obvious: for any fixed \( R > 0 \),
\[
\frac{1}{(N \psi(x))^{n}} \int_{B_{R}} R_{n}^{\alpha + \alpha_{1}}(x + \frac{y_{1}}{N \psi(x)}, \ldots, x + \frac{y_{n}}{N \psi(x)}) \, d^{n}y \leq C_{R}.
\]
Here \( B_{R} \) is the ball of the radius \( R \) in \( \mathbb{R}^{n} \) centered at zero, and \( C_{R} \) is a constant.

Using Proposition 15 and Theorem 1 we complete the proof after a similar procedure. \( \square \)

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