Comparison of mass renormalization schemes for simple model systems

E V Stefanovich\textsuperscript{1}, R E Wagner\textsuperscript{2}, Q Su\textsuperscript{2} and R Grobe\textsuperscript{2}

\textsuperscript{1} 2255 Showers Drive, Unit 153, Mountain View, CA 94040, USA
\textsuperscript{2} Intense Laser Physics Theory Unit and Department of Physics, Illinois State University, Normal, IL 61790-4560, USA

Received 13 September 2012, in final form 14 October 2012
Accepted for publication 31 October 2012
Published 7 February 2013
Online at stacks.iop.org/LP/23/035302

Abstract
We discuss an alternative method to mass renormalize a quantum field Hamiltonian based on the requirement that the vacuum and single-particle sectors are not self-scattering. We illustrate the feasibility of this method for the concrete example of bosonic quadratic and quartic interactions. The results are compared with those obtained by the standard renormalization technique based on a spectral analysis of the usual field-based propagators. We also discuss a potential method for renormalizing a quantum field theory in a non-perturbative way using computational methods.

1. Introduction

The ultimate goal of our work is to develop a methodology that would permit us to study the quantum field theoretical interaction of electrons, positrons and photons with full space–time resolution. However, there are numerous conceptual and also computational challenges that need to be addressed first. Presently, all experimentally verified predictions of quantum electrodynamics, such as transition rates and cross-sections, are based on the matrix elements of the scattering operator \[1, 2\]. This approach relates the properties of asymptotic incoming and outgoing scattering states and it is difficult to see how it could be used to also provide temporal and spatial information about the dynamics inside the interaction region. The latter could be read off the time-dependent field theoretical state \[|\Psi(t)\rangle\]. While the QED Hamiltonian would describe the entire time evolution of \[|\Psi(t)\rangle\], its energy eigenvalues and eigenvectors cannot be used immediately to construct the time evolution. This problem exists on top of the conceptual difficulty of converting the information contained in \[|\Psi(t)\rangle\] to meaningful physical quantities that should at least in principle be observable. In order to overcome the singularities of the original Hamiltonian, it needs to be renormalized first \[3, 4\]. This step can usually be performed perturbatively, but there are serious theoretical questions about the convergence of the complete perturbative series even after each term is renormalized, although this issue does not prevent practical application of perturbative methods in QED.

Many textbooks discuss renormalization techniques in an abstract way and concrete examples that apply these techniques for specific Hamiltonian systems are rare \[5–9\]. If the system is renormalizable, then the free (bare) parameters (such as the mass, charge or coupling constant) in the Hamiltonian can be adjusted in order to modify the spectrum of the Hamiltonian to fit experimental measurements of the physical parameters. This can be expressed equivalently by the construction of appropriate additional interactions (counterterms) to the Hamiltonian, where the coefficients of the counterterms are related to the difference between the bare and physical values of the parameters.

The motivation for this work is three-fold. While the general abstract principles of renormalizing a system on a Hamiltonian level are standard in most texts on quantum field theory, there are only a very limited number of specific examples showing how these abstract concepts are applied to concrete systems. Secondly, we introduce a new way of constructing counterterms so that the \(S\)-matrix operator satisfies a no-self-scattering condition for the vacuum and single-particle states. Finally, we introduce a new, albeit purely computational method to numerically renormalize a Hamiltonian. This method is non-perturbative and could be used as a yardstick to determine the accuracy of perturbative methods.
This work is focused mainly on the mass renormalization of simple model quantum field theories in one spatial dimension. Extension of our approaches to the coupling constant renormalization is deferred to future studies. The work is structured as follows. In section 2 we describe the two model Hamiltonians used in this work and comment critically on the relationship between manifest covariance and relativistic invariance as defined by the Poincaré relationships. In section 3 we first briefly review the usual renormalization technique and introduce a new alternative technique that is based on imposed self-scattering conditions. It requires the determination of some of the S-matrix elements, and the associated self-scattering conditions for some free energy eigenstates lead to the addition of counterterms to the original Hamiltonian. The remainder of the work is devoted to providing concrete examples for this new renormalization method, illustrating the commonalities with, and differences from the usual approach. In section 4 we compare both approaches for the quadratic interaction, which while being seemingly trivial, has exact solutions making it an ideal illustration of the two approaches to renormalization. In section 5 we discuss both the more general quartic interactions and the new role of single- and two-particle masses for renormalization. Here we compare estimates based on perturbation theory and the usual Feynman technique with numerical results obtained from a direct diagonalization of the Hamiltonian in a truncated Hilbert space. We also apply the alternative S-matrix based approach to this system. In section 6 we provide a brief summary and an extended critical discussion of the results, its relevance for QED, open questions and future work.

2. The model Hamiltonians and their Poincaré invariance

To construct a quantum field theory, one can equivalently take either the field operators or the particle creation and annihilation operators as the more fundamental quantities. Below we adopt a particle-based approach, similar to that used in Weinberg’s book [10], which seems to us to be the closest to an ideal axiomatic-like method in the sense that it relies on the smallest number of heuristic assumptions. We restrict our discussion here to a scalar boson in one spatial direction, but generalizations to bosonic and fermionic systems such as QED in higher dimensions should be straightforward. The unperturbed, free Hamiltonian is given by

$$H_0 = \int dp \, \omega(p; m) \hat{a}(p) \hat{a}^\dagger(p)$$

(2.1)

where the usual bosonic annihilation and creation operators fulfil the commutator relation \( [\hat{a}(p_1), \hat{a}^\dagger(p_2)]_\pm = \delta(p_1 - p_2)\), the energy is given by \( \omega(p; m) = \sqrt{m^2 c^4 + c^2 p^2} \), and \( m \) is the bare mass of the boson. Here and below we use \( \hbar = 1 \) au. We denote with \( |0; m \rangle \) and \( \hat{a}^\dagger(p) |0; m \rangle \) the bare eigenvectors of \( H_0 \) with energies \( 0 \) and \( \omega(p; m) \). This Hamiltonian \( H_0 \) is relativistically invariant, and we show in the appendix A that by using the operators \( \hat{a}^\dagger(p) \) and \( \hat{a}(p) \) one can explicitly construct the two translation operators for the position and velocity \( P \) and \( K_0 \) such that the three Poincaré relationships \( [H_0, P] = 0, [K_0, P] = -i H_0/c^2 \) and \( [K_0, H_0] = -i P \) are fulfilled.

Weinberg’s method of constructing an interaction that leads to a Lorentz invariant scattering matrix requires two steps. First, he constructs the free field operator from the particle creation and annihilation operators in the \( H_0 \)-based interaction picture

$$\hat{\phi}_0(z, t) = (2\pi)^{-1/2} e^{i \lambda t} \int d^3 p \, \omega(p)^{-1/2} \hat{a}(p) \exp(ipoz - io ant)$$

$$+ \hat{a}^\dagger(p) \exp(-ipoz + io ant).$$

(2.2)

It was constructed in such a way that it transforms under the Lorentz transformation as a scalar, \( \exp[-iK_{0e}t] \hat{\phi}_0(z, t) \) \( e^{iK_{0e}t} \hat{\phi}_0(L^{-1}z, t) \) describes the inverse of the usual Lorentz transformation defined as \( L(a, b) \equiv (a \cosh \theta - b/c \sinh \theta, b \cosh \theta - a/c \sinh \theta) \) with the rapidity parameter \( \theta \equiv \tanh^{-1}(v/c) \). Manifest covariance means that applying the boost operator to the field can be performed by simply replacing the parameters \( z \) and \( t \) by the Lorentz transformed ones. In other words, the boosted field has the same functional form as the unboosted one, except it is now a function of \( z' \) and \( t' \) instead of \( z \) and \( t \).

As a second step we construct a general manifestly covariant Hamiltonian density in the interaction picture as a polynomial of this field and its derivatives with respect to the parameters \( z \) and \( t \). More specifically, in this work we will examine the quadratic and quartic interactions \( H = H_0 + \lambda \int d^3 z \hat{\phi}(z) \hat{\phi}(z) \) for \( N = 2 \) and 4. Even though we have not been able to prove its Poincaré invariance, we will also not discuss below the normal-ordered form \( V_{N4} \equiv N \int d^3 z \hat{\phi}_0(z) \hat{\phi}_0(z) \), where \( N \) denotes normal ordering of creation and annihilation operators.

3. General approaches to renormalization

It turns out that due to the quadratic or quartic interaction, the parameters \( m \) and \( \lambda \) are different from the observable mass and the real coupling (or charge) of the physical particles. These two observables must be correctly matched to the experimental values for a theory that contains interactions. In order to repair the theory, one has to first find how the parameters \( m \) and \( \lambda \) are related to the physical observables. To make a better comparison, we first briefly review the basic ideas for the usual field- and propagator-based approach to renormalization and then introduce in section 3.2 our proposed scattering matrix based approach.

3.1. Lagrangian- and field-based approach to renormalization

As one of the key equations in quantum field theory, the LSZ reduction formula relates the scattering amplitude between the asymptotic initial and final states to the time-ordered vacuum expectation value of products of the interacting field operator [11, 12]. Its validity requires that (i) the vacuum expectation value of the field vanishes \( \langle VAC | \hat{\phi}(0) | VAC \rangle = 0 \), and that (ii) the probability amplitude for the field to
create or annihilate a one-particle energy eigenstate of $H$ with momentum $P$ takes the same value as it does in a free theory, $(P|\hat{\psi}(0)|\text{VAC}) = (p|\hat{\psi}(0)|0)$. Here $|\text{VAC}\rangle$ is defined as the vacuum state of the interacting Hamiltonian, which is also the state of lowest energy. $|P\rangle$ and $|p\rangle$ are sharp momentum eigenstates of the full ($H$) and free ($H_0$) Hamiltonians, respectively. If we are to use the Feynman rules to calculate the $S$-matrix elements, we should first renormalize the fields to satisfy these two requirements [13]. The first condition can be accomplished by adding a constant to the field $\hat{\phi} \rightarrow \hat{\phi} + \nu$ for some vacuum expectation value $\nu$, and the second condition is accomplished by a multiplicative constant, $\hat{\phi} \rightarrow \sqrt{Z}\hat{\phi}$, as discussed below.

As a result, the renormalized field is one whose associated Green’s function has the same behavior near its pole as for a free field, and the renormalized mass ($M$) is defined by the position of the pole. In this traditional method it is customary to use the Lagrangian framework and try

$$L = 1/2(\partial_\mu \hat{\phi}\partial^\mu\hat{\phi}) - 1/2(\partial_\mu \hat{\phi}_0\partial^\mu\hat{\phi}_0) - 1/2 m^2 c^2 \phi_0^2 - \lambda \phi_0^n. \quad (3.1)$$

Here it is important to note that the free parameters $m$ and $\lambda$ are not the physical mass or coupling. In order to re-express the Lagrangian in terms of the physical quantities, we add and subtract terms from the Lagrangian such that we can write the same Lagrangian density of equation (3.1) as:

$$L = (1/2)\partial_\mu \hat{\phi}\partial^\mu\hat{\phi} - (1/2)\partial_\mu \hat{\phi}_0\partial^\mu\hat{\phi}_0 - \lambda \phi_0^n + \text{Counter}(m, \lambda) \quad (3.2)$$

where the form of the counterterm Counter$(m, \lambda)$ is

$$\text{Counter}(m, \lambda) = (1/2)\partial_\mu \hat{\phi}_0\partial^\mu\hat{\phi}_0 - \lambda \phi_0^n.$$

The counterterm contains both $\hat{\phi}_0$ and $\hat{\phi}_r$. To simplify the term Counter$(m, \lambda)$ an additional and possibly restrictive condition is used, namely, it is assumed that $\hat{\phi}_0$ and $\hat{\phi}_r$ differ by a constant of proportionality, $\hat{\phi}_r = Z(m, \lambda)^{1/2}\hat{\phi}_0$. This condition is imposed in order to satisfy the relation described above that the field operator creates a single-particle state, $(p|\hat{\phi}(0)|0) = (P|\hat{\psi}(0)|\text{VAC})$. Usually it is then assumed that instead of merely finding $m$ and $\lambda$ we now have to find the values of three parameters. The bonus, however, is that we can now eliminate $\hat{\phi}_0$ from our counterterm and everything depends only on one field. The counterterms then take the form Counter$(m, \lambda) = (Z - 1)(1/2)\partial_\mu \hat{\phi}\partial^\mu\hat{\phi} - \lambda \phi_0^n.$

In contrast to the spectrally based method discussed above, in the $S$-matrix based method proposed here we equate the bare parameters from the very beginning with the physical ones [15–17]. We want to assign some physical reality to our original states $|0\rangle$ and $\hat{\phi}(p)|0\rangle$. As a result, the usual annihilation and creation operators are associated with the corresponding physical states. In this method, the counterterms in the Hamiltonian need to be constructed in such a way that the no-self-scattering property of the states $|0\rangle$ and $\hat{\phi}(q)|0\rangle$ is guaranteed. Here, no-self-scattering means that the $S$-matrix carries these states into themselves,

$$S|0\rangle = |0\rangle \quad (3.4a)$$

$$\hat{S}\hat{a}^\dagger(p_1)|0\rangle \equiv |0\rangle \quad (3.4b)$$
or, in terms of matrix elements \( \langle 0|S|0 \rangle = 1 \) and \( \langle q'|S(q) \rangle = \delta(q - q') \), where the state \( |q \rangle \) is given by \( |q \rangle \equiv \hat{a}^\dagger(q)|0 \rangle \).

In this approach, the central quantity is the scattering operator, defined as \( S = \lim_{t \to -\infty, t_0 \to -\infty} S(t, t_0) \), where

\[
S(t, t_0) \equiv 1 - i \int dt_1 V(t_1) - i \int dt_1 V(t_1) \int dt_2 V(t_2) + \cdots
\]

where \( V(t) \equiv \exp[iH(t-t_0)]V \exp[-iH(t-t_0)] \) and the lower integration limit is \( t_0 \). Here we are still in the Schrödinger picture and the states are time-independent. It turns out that this scattering operator allows one to effectively split off the interaction-based portion from the full time evolution operator, \( \exp[iH(t-t_0)] = \exp[-iH_0(t-t_0)]S(t, t_0) \). This allows a consecutive propagation for any state first by \( S(t, t_0) \), which can describe transitions between bare eigenstates of \( H_0 \), and then consecutively the free evolution associated solely with \( H_0 \).

A physical \( S \)-operator must be cluster separable so that \( S \)-matrix elements representing separate processes at large distances from each other do not contain any dependences between each other. Furthermore, it is assumed that the distances from each other do not contain any dependences.

4. First example: the quadratic interaction

In order to illustrate the \( S \)-matrix based method proposed here and compare its predictions with that of the standard field-based approach, we use the simplest quantum field theory that predicts the undesirable self-scattering in the bare vacuum as well as the bare single-particle sector. The non-renormalized Hamiltonian is given by

\[
H(m, \lambda) = \int dp \omega(p; m)\hat{a}^\dagger(p; m)\hat{a}(p; m) + \lambda \int dz (\hat{q}_0(z))^2.
\]

Note that the Hamiltonian is expressed in terms of the operators \( \hat{a}(p; m) \) that create a particle with bare mass \( m \). We denoted with \( |0; m \rangle \) the bare vacuum and with \( |q; m \rangle \equiv \hat{a}^\dagger(q; m)|0; m \rangle \) the bare single-particle state with momentum \( q \). This Hamiltonian is simple enough that one could immediately compute its energy spectrum exactly. The vacuum’s expected energy is given by \( E_{\text{vac}} = \delta(0) \int dp \left[ \omega(p, m) - \omega(p; m) \right]^2 + 2\lambda/c^2/[4\omega(p, m)] \), where the effective mass is \( m_* \equiv \left[ 1 + 2\lambda/c^2/(m_*^2) \right]^{1/2}/m \). While the first term \( \int dp \left[ \omega(p, m) - \omega(p; m) \right]^2 \) is finite for any \( \lambda \), which is contained in \( m_* \), the integral over the second term proportional to \( \int dp/[\omega(p, m)] \) diverges logarithmically. For a comparison which will be needed below we should mention the perturbative expression of the vacuum energy \( E_{\text{vac}} = \delta(0)\lambda/(2c^2) \int dp \omega(p; m) + O(\lambda^2) \). The exact single-particle energy is given by \( E_{\text{vac}} + \omega(p, m) \). As we are focused here on illustrating the general method, for which the exact spectrum is usually unknown, we will not use this knowledge about \( E_{\text{vac}}, \omega(p, m) \) and \( m_* \) below.

4.1. Spectral approach for the single-particle mass

In this approach, we try to relate the (free) bare parameters \( m \) and \( \lambda \) to the known value of the physical mass, denoted by \( M \). There are several methods to do so, ranging from Feynman diagrams to numerical diagonalization to simple perturbation theory in the bare parameters. At this level, the standard renormalization treatment stops and uses these computed bare parameters for \( S \)-matrix calculations. However, in order to compare this traditional approach with our alternative approach in which a new Hamiltonian (called \( H_{\text{phys}} \)) is obtained, as a second step we need to rewrite the Hamiltonian (4.1) in terms of the physical particle operators \( \hat{a}(p; M) \). This rewriting will allow us to identify a new interaction in the original Hamiltonian. Using simple first-order perturbation theory in \( \lambda \), we find \( \langle 0; m|H|0; m \rangle = \lambda c^2/2|0 \rangle \int dp \omega(p)^{-1} \) and \( \langle q; m|H|q; m \rangle = \lambda c^2/2|\delta(0) \int dp \omega(p)^{-1} + \omega(p; m) + \lambda c^2/2\omega(p)^{-1} \rangle \). It should be noted that the correction to the vacuum energy is infinite, associated with the logarithmic divergence of the integral \( \int dp \omega(p)^{-1} \). The second infinite factor, \( \delta(0) \), becomes \( L/(2\pi) \) in a finite box of total length \( L \). This factor is expected for a state that is translationally invariant and represents a vacuum energy density.

We subtract the vacuum energy \( \langle 0; m|H|0; m \rangle \) first and then define the \( O(\lambda) \) effective mass as \( M_c \equiv \langle q; m|H|q; m \rangle|_{q=0} = \omega(q; m) + \lambda c^2\omega(q)^{-1} \rangle|_{q=0} = mc^2 + \lambda m^{-1} \). This equation determines the correct value of the bare mass, which we denote by \( m_1 \). If we invert this solution up to \( O(\lambda^4) \), we find that the bare mass has to take the specific value \( m_1 \) given by

\[
m_1 = M - \lambda/(M_c^2) + O(\lambda^2).
\]

Now, as a second step, we rewrite the same Hamiltonian \( H(m_1, \lambda) \) in terms of more meaningful operators that create a particle of mass \( M \). In general, the corresponding operators that create or annihilate a particle with either mass \( m_1 \) and energy \( \omega_1(p) \equiv [m_1^2 + c^2p^2]^{1/2} \) or mass \( M \) and energy \( \Omega(p) \equiv [M^2 + c^2p^2]^{1/2} \) are related by the Bogoliubov–Valatin transformation [18]:

\[
\tilde{a}(p; m_1) = (\Omega + \omega_1)(4\Omega\omega_1)^{-1/2}a(p; M) + (\omega_1 - \Omega)(4\Omega\omega_1)^{-1/2}\tilde{a}(-p; M).
\]

If we apply this equation to our situation (4.2) and stay within \( O(\lambda^4) \), we find \( \tilde{a}(p; m_1) = \hat{a}^\dagger(p; M) - \lambda c^2/(2\Omega^2)\tilde{a}(p; M) \). Then replacing the bare operators \( \hat{a}(p; m_1) \) by the more
physical ones $\hat{a}(p; M)$, which we denote by $\hat{A}(p) \equiv \hat{a}(p; M)$, the original Hamiltonian becomes

$$H(m_1, \lambda) = \int dp \, \omega(\hat{A}^\dagger(p)\hat{A}(p) - \lambda c^2/(2\Omega(p)^2))$$

$$\times \left[ \hat{A}^\dagger(p)\hat{A}^\dagger(-p) + \hat{A}(p)\hat{A}(-p) \right] + \lambda c^2/2$$

$$\times \int dp \, \Omega(p)^{-1} \left[ \hat{A}^\dagger(p)\hat{A}^\dagger(-p) + \hat{A}(p)\hat{A}(-p) \right].$$

We have expanded the original interaction $\lambda \int dz (\hat{q}(z))^2$ immediately in terms of the $A(p)$ and $A^\dagger(p)$. The common prefactor before the term $A^\dagger(p)A(p)$ is given by $\omega_1 + \lambda c^2\Omega^{-1}$, which is then equivalent to $\Omega + O(\lambda^2)$. Furthermore, if we also use $-\lambda c^2\omega_1/(2\Omega^2) + \lambda(c^2/2)\Omega^{-1} = 0 + O(\lambda^2)$, several other terms cancel out. As a result, the Hamiltonian can then be expressed as

$$H(m_1, \lambda) = \int dp \, \Omega(p)^{-1} \left[ \hat{A}^\dagger(p)\hat{A}(p) + \lambda c^2/2 \right] \Omega(p)^{-1} \delta(0).$$

In this new representation the Hamiltonian has the required eigenvalues up to $O(\lambda^2)$: energies of single-particle states are $\omega(p; M) + E_{\text{vac}}$, i.e., those of free particles with mass $M$ shifted by the constant vacuum energy $E_{\text{vac}} = \lambda(c^2/2) \int dp \, \Omega(p)^{-1} \delta(0)$. The second term in equation (4.5) can be interpreted as a new interaction, which for the quadratic interaction happens to simply be an infinite constant, but for more complicated interactions it can be a function of the creation and annihilation operators.

### 4.2. Alternative $S$-matrix based approach to mass renormalization

In contrast to equation (4.1), in this approach we assume from the very beginning that the non-renormalized Hamiltonian and the quantum field $\hat{q}(z)$ are expressed in the correct physical mass $M$ and the ‘correct’ operators $\hat{A}(p)$:

$$H = \int dp \, \omega(p; M) \hat{A}^\dagger(p)\hat{A}(p) + \lambda \int dz (\hat{q}(z))^2.$$  

The interaction part of this Hamiltonian, however, cannot be correct as it violates our required zero-scattering conditions for the vacuum and the single-particle states indicated in equation (3.4). In order to ‘repair’ this Hamiltonian we add some yet undetermined counterterm interactions $H_{\text{phys}} = H + V_{\text{counter}}$ so that scattering matrix elements computed with $H_{\text{phys}}$ vanish for the vacuum and single-particle states. The space of those relevant energy densities that would lead to a quadratic Poincaré-invariant interaction can be spanned by the square of the field operator $(\hat{q}(z))^2$ and its covariant derivative $(\partial_{\mu}\hat{q}(z))^2 - (\partial^\mu\hat{q}(z))^2$, so we can try to repair our Hamiltonian with the ansatz

$$H_{\text{phys}} = \int dp \, \omega(p; M) \hat{A}^\dagger(p)\hat{A}(p) + \lambda \int dz (\hat{q}(z))^2$$

$$\times \left[ B(\hat{q}(z))^2 + C(\partial_\mu\hat{q}(z))^2 - C(\partial^\mu\hat{q}(z))^2 \right].$$

where we hope that we can determine the two unknown coefficients $B$ and $C$ such that the corresponding first-order scattering matrix elements simultaneously fulfill $\langle 0|S_1|0 \rangle = 0$ and $\langle q|S_1|q \rangle = 0$, where $S_1$ is the $O(\lambda)$ contribution to the $S$-matrix in (3.5), $S_1 = -i \int_{-\infty}^{\infty} dt \, V(t)$. Using $\Xi = \int dp/(\omega(p))$ we obtain from these two conditions

$$\lambda \Xi + B \Xi + CM^2c^2\Xi = 0 \quad \text{(4.8a)}$$

$$\lambda[\Xi + 1/(\delta(0))\delta(0)] + B[\Xi + 1/(\delta(0))\delta(0)] = 0. \quad \text{(4.8b)}$$

The solutions are $B = -\lambda - M^2c^2C$ and $C$ is arbitrary. Note that both coefficients are at most linear in $\lambda$. If we insert these coefficients into the physical Hamiltonian, we obtain

$$H_{\text{phys}} = \int dp \, \omega(p; M) \hat{A}^\dagger(p)\hat{A}(p) - C \int dp \, \omega(p; M)$$

$$\times \left( \hat{A}^\dagger(p)\hat{A}^\dagger(-p) + \hat{A}(p)\hat{A}(-p) \right).$$

In other words, the incorrect interaction $\lambda \int dz (\hat{q}(z))^2$ needs to be effectively replaced by the physical interaction $\int dp \, \omega(p; M) \hat{A}^\dagger(p)\hat{A}(p) - \hat{A}(p)\hat{A}(-p)$. We note that while the repaired Hamiltonian equation (4.5) (according to the conventional method) contains an infinite constant in addition to the free Hamiltonian term, the result of this section, equation (4.9), contains an interaction of the form $\hat{A}^\dagger(p)\hat{A}^\dagger(-p) + \hat{A}(p)\hat{A}(-p)$. The free parameter $C$ characterizing the strength of this interaction could then be determined by additional conditions on the system. In this case, it turns out that at second order in perturbation theory $C$ must be set equal to 0, or else it leads to interactions that are not cluster separable.

We remark that while the quadratic interaction predicts a non-trivial self-scattering of zero- and one-particle states, due to its simple structure it has no connected scattering processes between two-particle states. In order to illustrate the alternative renormalization technique for a system with non-trivial two-particle scattering, we study in the next section the more complicated quartic interaction.

### 5. Second example: the quartic interactions

As a model system we choose here the $\hat{q}^4$-interaction [19–22], given by the Hamiltonian

$$H(m, \lambda) = \int dp \, \omega(p; m) \hat{a}^\dagger(p; m)\hat{a}(p; m)$$

$$+ \lambda \int dz (\hat{q}(z))^4. \quad (5.1)$$

While it can describe many experimentally observable features of critical phenomena, we use it here as a simple but concrete model system to study renormalization [23]. It is our hope that our findings are general enough to also be applicable to the more complicated QED Hamiltonian. In order to have easier access to numerical data we again restrict the spatial dimensions to one.
If we return to the Schrödinger picture representation for the auxiliary field $\hat{\phi}(z)$, the interaction takes the form:

$$\int dz \hat{\phi}(z)^4 = N \left[ \int dz \hat{\phi}(z)^4 + (2\pi)(4\pi)^{-1/2} c^4 \right]$$

$$\times \int dp / \omega(p) \int dp_1 \int dp_2 / \omega(p_1)$$

$$\times \omega(p_2)^{1/2} \left[ 6 \hat{a}(p_1) \hat{a}(p_2) \delta(p_1 + p_2) + 12 \hat{a}^\dagger(p_1) \hat{a}(p_2) \delta(p_2 - p_1) + 6 \hat{a}^\dagger(p_1) \right]$$

$$\times \left[ (4\pi)^{-1/2} c^4 \right] \int dp / \omega(p)^2 \delta(0)$$

(5.2)

where for notational convenience the five quartic terms associated with the normal-ordered products are abbreviated with

$$N \left[ \int dz \hat{\phi}(z)^4 \right] = (2\pi)(4\pi)^{-1/2} c^4 \int dp_1 \int dp_2$$

$$\times \int dp_3 \int dp_4 / \omega(p_1) \omega(p_2)$$

$$\times \omega(p_3) \omega(p_4)^{-1} \left[ \hat{a}(p_1) \hat{a}(p_2) \right]$$

$$\times \hat{a}(p_3) \hat{a}(p_4) \delta(p_1 + p_2) + p_3 + p_4$$

$$+ 4 \hat{a}^\dagger(p_1) \hat{a}(p_2) \delta(p_2 - p_1)$$

$$\times \delta(-p_1 - p_2 + p_3 + p_4)$$

$$+ 6 \hat{a}^\dagger(p_1) \hat{a}^\dagger(p_2) \hat{a}(p_3)$$

$$\times \left[ (4\pi)^{-1/2} c^4 \right] \left[ \hat{a}(p_1) \hat{a}(p_2) \right]$$

$$\times \delta(-p_1 - p_2 - p_3 - p_4)$$

(5.3)

Furthermore, if we replace the infinite upper and lower integration limits with the cutoff momentum $\Lambda$, the integral $\int dp / \omega(p)$ in equation (5.2) can be evaluated as $(2c)\sinh^{-1}(\Lambda/mc) \equiv \Xi$. As it will be relevant for the discussion below, we point out that the interaction can also be expressed as

$$\int dz \hat{\phi}(z)^4 = N \left[ \int dz \hat{\phi}(z)^4 + 6(4\pi)^{-1/2} c^4 \Xi \right]$$

$$\times \left[ (4\pi)^{-1/2} c^4 \Xi^2 \delta(0) \right]$$

$$= N \left[ \int dz \hat{\phi}(z)^4 + 6(4\pi)^{-1/2} c^4 \Xi \right]$$

$$\times \left[ \int dz \hat{\phi}(z)^4 + 3(2\pi) \right]$$

$$\times \left[ (4\pi)^{-1/2} c^4 \Xi^2 \delta(0) \right].$$

(5.4)

Here we have used an interesting equality that shows that by adding a term $\int dz \hat{\phi}(z)^2$ to $\int dz \hat{\phi}(z)^4$ the resulting expression contains only normal-ordered terms:

$$\int dz \hat{\phi}(z)^4 - 3(4\pi)^{-1/2} c^2 \Xi \int dz \hat{\phi}(z)^2$$

$$= N \left[ \int dz \hat{\phi}(z)^4 + 3(4\pi)^{-1/2} c^2 \Xi \right]$$

$$\times N \left[ \int dz \hat{\phi}(z)^2 \right].$$

(5.5)

5.1. The role of single- and two-particle masses

Let us assume that as a result of a measurement of our system, we know that the mass of a single particle has the specific value denoted by $M_1$. Furthermore, let us assume that due to another measurement (via some cross-section, e.g.) we also know that the interaction strength between two particles is such that the lowest energy of a state with two particles takes the specific value of $M_2^2$. For the special case of no interaction ($\lambda = 0$), we would trivially find that the two-particle energy eigenstate is just the direct product of two single-particle states, $\hat{a}^\dagger(p_1) \hat{a}^\dagger(p_2)\langle \text{vac} \rangle$ with the eigenvalue $\omega(p_1) + \omega(p_2)$. In this special (non-interacting) case, the lowest energy eigenvalue $M_2^2$ would be equal to $2M_1^2$, associated with $p_1 = p_2 = 0$. In other words, the effective two-particle mass is simply the sum of two single masses. However, if the two particles interact with each other in an attractive way then the lowest possible energy can be less than $2M_1^2$. The true size of $M_2^2$ is therefore a measure for the strength of the interaction. The ‘two-particle mass’ $M_2$ is related to the charge (or coupling strength) whose size determines the magnitude of the interaction. If the original Hamiltonian $H_0 + \lambda \int dz \hat{\phi}(z)^4$ was a correct description of our system, we would have to reproduce the two desired values $M_1^2$ and $M_2^2$ as its lowest eigenvalues associated with the single- and two-particle sector. Furthermore, as its vacuum state is supposed to be free of any particles, we would also expect its lowest energy eigenvalue to be zero. However, $H$ has none of these desired spectral properties in its present form.

A closer inspection of the Hamiltonian $H = H_0 + \lambda \int dz \hat{\phi}(z)^4$ shows that it can be re-expressed as the sum of normal-ordered products plus an infinite constant which we denote by $\Theta \equiv \lambda \lambda \lambda (2\pi)^{-1/2} c^4 \Xi^2 \delta(0)$, which depends on the bare mass $m$ through $\Xi = \int dp / \omega(p)$. For a system in a finite box of length $L$, the $\delta(0)$ should be replaced by $\delta(0) = L/(2\pi)$. We will return to this in our numerical discussion of section 5.4 below. In order to renormalize our Hamiltonian by finding the parameters $m$ and $\lambda$, we first subtract the constant $\Theta$ from it. This subtraction changes the spectrum in a trivial way and leaves all eigenvectors the same, giving the new interaction $V = \lambda \int dz \hat{\phi}(z)^4 - \Theta = \lambda \lambda \lambda [\int dz \hat{\phi}(z)^4 + 6 \lambda \lambda \lambda (4\pi)^{-1/2} c^2 \Xi \int dz \hat{\phi}(z)^2]$ the necessary vanishing vacuum expectation value, as discussed in appendix A. This interaction $V$ now consists entirely of normal ordered products.

In order to repair (renormalize) the new Hamiltonian $H_0 + V$, we can compute its spectrum and adjust the two free parameters $m$ and $\lambda$ until the spectrum matches the two renormalization conditions, here given by $\min(|E(1)|) =$
5.1.1. Single-particle mass using Rayleigh–Schrödinger perturbation theory. In usual (non-degenerate) Rayleigh–Schrödinger perturbation theory [24] the energies $\Omega$ of the Hamiltonian $H = H_0 + V$ can be computed as the sum $\Omega = \Sigma_{n\omega}$. Here, the general form of the lowest two terms is given by $\omega^{(0)} = \langle 0|V|\omega^{(0)}\rangle / \langle 0|\omega^{(0)}|0\rangle = \langle 0|\omega^{(0)}|\omega^{(0)}\rangle^{-1}\Sigma_{\beta}\beta(\beta^{(0)}|V|\omega^{(0)})|^2(\omega^{(0)} \beta^{(0)} \rangle^{-1}$. Here the summation (integration) $\Sigma_{\beta}$ extends over all eigenstates $|\beta^{(0)}\rangle$ of $H_0$ with energy $\beta^{(0)}$, except the state $|0^{(0)}\rangle$.

If we restrict our scheme to only the lowest order in $\lambda$, the two renormalization conditions simplify to (1) $\omega(p)_{p=0} = p|V|p|/\delta(0)|_{p=0} = M_1c^2$ and (2) $2\omega(p)_{p=0} + (p,p)V,p,p/\delta(0)|_{p=0} = M_2c^2$. In other words, we simply have to compute the three diagonal elements of the potential and invert them to find $m$ and $\lambda$ as a function of the given values $M_1$ and $M_2$.

To incorporate the first requirement we have to compute the energy of the diagonal element for the single-particle state $p|V|p$, which amounts to

$$\langle p|V|p \rangle = \lambda(p) \int dz \phi_0(z)^4 - \Theta(p)$$

$$= \lambda(p)(2\pi)(4\pi\lambda)^{-1/2} \Xi \int dp_1 \int dp_2 \omega(p_1) \times \omega(p_2) \frac{1}{12} \lambda \lambda_1 \lambda_2 \lambda_3 \lambda_4 \lambda_5 \lambda_6$$

$$\lambda = \lambda(2\pi)(4\pi\lambda)^{-1/2} \Xi \omega(p)^{-1}2 + O(\lambda^2),$$

which is a transcendental equation for $m$ as a function of the required mass $M_1$. Consistent with our perturbative approach, we can solve this equation up to order $O(\lambda^2)$ and find for our renormalized mass $m$

$$m = M_1 - \lambda 3(\pi mc)^{-1} \sinh^{-1}(\Lambda/mc) + O(\lambda^2)$$

In the limit of $\Lambda \to \infty$, we can replace $\sinh^{-1}$ by its asymptotic behavior $\sinh^{-1}(\Lambda/Mc) \to \ln[2\Lambda/Mc] + (1/4)(\Lambda/Mc)^{-2}$. The first-order renormalized mass can then be used to compute the second-order correction to the mass. The second-order contribution to the mass shift can be calculated from the second-order perturbative correction to the energy

$$E^{(2)} = \delta(0)^{-1} \int d\beta |\beta|V|p|^2 / \omega(p)$$

where $\int d\beta$ is the integration over all possible intermediate states and the factor of $\delta(0)$ arises from the normalization of the single-particle state, $p|p| = \delta(0)$. The two terms in the interaction $V$ which modify the energy of a single-particle state $|p\rangle$ at second order in $\lambda$ are proportional to $\hat{a}^\dagger \hat{a}^\dagger \hat{a}^\dagger \hat{a}$, corresponding to $|\beta\rangle$ being a 3-particle state, and $\hat{a}^\dagger \hat{a}^\dagger \hat{a}^\dagger \hat{a}$, where $|\beta\rangle$ will be a 5-particle state. We will define the two corresponding second-order energy corrections as $E^{(2)}_3(p)$ and $E^{(2)}_5(p)$. The details are worked out in appendix B. The final contribution to the single-particle energy $E^{(2)}(p)$ is

$$E^{(2)}(p) = E^{(2)}_3(p) + E^{(2)}_5(p) - E^{(2)}_{\text{vac}}$$

$$= 3/(2\pi^2) c^2 \lambda^2 \int dp_1 dp_2 \omega(p_1) \omega(p_2) \times \omega(p-p_1-p_2) \omega(p_1) - \omega(p_2) - \omega(p-p_1+p_2) + 3/(2\pi^2)$$

$$\lambda^2 \lambda_1 \lambda_2 \lambda_3 \lambda_4 \lambda_5 \lambda_6$$

$$\omega(p-p_1-p_2) \omega(p_1)^{-1} \omega(p_2) - \omega(p_2) - \omega(p-p_1+p_2)$$

$$E^{(2)}_3(p)$$

$$E^{(2)}_5(p)$$

where $E^{(2)}_{\text{vac}}$ is the energy of the vacuum state itself, which must be subtracted from the single-particle state’s energy in order to isolate that part of the state’s energy which corresponds to the shift in the particle’s mass due to the interaction. The correction to the mass can be evaluated by evaluating $E^{(2)}(p = 0)$ numerically; for instance, for a bare mass of $m = 1$, this mass correction is $-\lambda^2 \times 0.75$. This result agrees with the result that will be calculated from Feynman graphs in section 5.1.2.

In order to make contact with computational methods, to be presented in section 5.1.4, we can discretize the momentum by confining the system to a finite box with periodic boundary conditions [25]. For a box of length $L$ the momentum modes of the system are discrete with spacing $\Delta = 2\pi/L$. We can therefore replace all momentum integrals $\int dp$ by $\sum p$. The creation and annihilation operators must be normalized by $\sqrt{\Delta}\hat{a}_p \equiv \sqrt{\Delta}\hat{a}\alpha(p) = 0$. We therefore take the subscript $p$ to be an integer and the physical momentum corresponding to integer momentum $p$ is given by $p\Delta$. This normalization results in the normalized creation and annihilation operators satisfying the commutation relations $[\hat{a}_p, \hat{a}^\dagger_{p'}] = \delta_{pp'}$, where $\delta_{pp'}$ is the Kronecker delta.

With these substitutions, the second-order perturbative correction to the energy of a single-particle state from the term $\hat{a}^\dagger \hat{a}^\dagger \hat{a}^\dagger \hat{a}$ for this discretized system is

$$E^{(2)}_3(p) = \lambda^2/(2\pi^2) c^2 \lambda^2 \Xi \int dp_1 dp_2 \omega(p_1) \omega(p_2) \times \omega(p-p_1-p_2) \omega(p_1) - \omega(p_2) - \omega(p-p_1+p_2)$$

$$E^{(2)}_5(p) = \lambda^2/(2\pi^2) c^2 \lambda^2 \Xi \int dp_1 dp_2 \omega(p_1) \omega(p_2) \times \omega(p-p_1-p_2) \omega(p_1) - \omega(p_2) - \omega(p-p_1+p_2)$$

The second term in this equation is equal to the second-order correction term of the vacuum state $|0\rangle$ for a discrete system.
and is therefore defined by the transcendental equation $M^2 = m^2 - \Sigma(M^2)$. To $O(\lambda^2)$, this can be written as $M^2 = m^2 - \Sigma(m^2)$. The integrals in equation (5.15) can be evaluated numerically at $k^2 = m^2$, and for a bare mass of $m = 1$ the result is $\Sigma ((m = 1)^2) \approx -\lambda^2 \times 1.51$. The renormalized mass is therefore $M = \sqrt{(m^2 - \Sigma(m^2))} = m\sqrt{(1 - \Sigma(m^2))} \approx m - 1/2 \Sigma(m^2)/m \approx 1 - \lambda^2 \times 0.75$, in agreement with the mass correction calculated in appendix B.

5.1.3. Two-particle mass using perturbation theory. To incorporate the binding energy renormalization condition we have to compute the energy of the diagonal element for the two-particle state $\langle p, p | V | p, p \rangle$, which contains two terms associated with the interactions proportional to $\hat{a}^\dagger \hat{a}$ and $\hat{a}^\dagger \hat{a}^\dagger \hat{a}$

\[ \langle p, p | V | p, p \rangle = \lambda \langle p, p | \int d\zeta \hat{\phi}_0(\zeta)^2 - \Theta(p, p) \right] = \lambda \langle p, p | (2\pi)^{-1/2} c^4 \right] \int dp_1 \]

\[ \times \int dp_2 \left[ \omega(p_1)\omega(p_2) \right] \times \hat{a}(p_2)\delta(p_2 - p_1) \right] \int dp_1 \]

\[ + \lambda \langle p, p | (2\pi)^{-1/2} c^4 \right] \int dp_1 \]

\[ \times \int dp_2 \int dp_3 \int dp_4 \omega(p_1)\omega(p_2) \right] \int dp_1 \]

\[ + \omega(p_3)\omega(p_4) \right] \times \hat{a}(p_3)\hat{a}(p_4)\delta(p_3 + p_4 \right] \int dp_1 \]

\[ - p_1 - p_2)\right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \r...
it is equal to normal-ordered products of operators, which have zero vacuum expectation value. With the addition of this counterterm, the two-particle mass becomes

\[ M_2 c^2 = 2\omega(0) + (\lambda + C) 6\sqrt{2}/2(4\pi)^{-1} \]
\[ \times \epsilon^4 [22\pi m^{-1}c^{-2} + m^{-2}c^{-4}]. \quad (5.18) \]

The value of \( C \) must be chosen to be an infinite constant, such that the two-particle mass becomes finite and equal to the known, measured value.

5.1.4. Complete spectrum using direct computational methods. In this approach we try to find an efficiently truncated set of basis states that permits us to determine the energy eigenvalues directly by diagonalizing the corresponding Hamiltonian matrix numerically \[25, 26\]. If we assume that the system is inside a box of length \( L \) with periodic boundary conditions, we can discretize the spatial axis into \((2N + 1)\) grid points. As a result, the discretized momenta take the form \( p_j = j(2\pi)/L \), where \( j = 0, \pm 1, \pm 2, \pm N \) and the largest momentum is \( \Lambda = N(2\pi)/L \). We have used the discretized versions of the energy eigenstates of \( H_0 \) as basis states.

After discretization, the system is still infinite dimensional as there can be an arbitrary number of bosons in each state. Thus, the system must be further restricted by setting an upper limit \( N_0 \) to the total number of bosons. Due to momentum conservation each momentum block can be diagonalized separately, and the largest momentum block (generally the zero momentum subspace) can, due to memory limitations, be at most a \( 10^5 \times 10^5 \) Hamiltonian matrix.

Diagonalization can be further simplified, due to the \( \phi \rightarrow -\phi \) symmetry, such that states with an even number of bosons do not couple with states that have odd boson numbers, which allows each of these subspaces to be considered separately. With recently developed computational techniques based on dynamically allocated pointer variables and suitable memory swapping, one can now simulate the fully coupled quantum field theoretical fermion–boson interaction in less than 50 CPU-hours of computing time.

If the maximum allowed momentum mode is \( N = 10 \) and the number of allowed bosons is \( N_0 = 5 \), the zero momentum subspace with an odd number of bosons is a \( 1456 \times 1456 \) matrix which must be diagonalized. For coupling constant \( \lambda = 0.01 \), which is well inside the perturbative regime, and for box length \( L = 0.02 \), the lowest lying eigenvalue is 18 778.843 in atomic units, where \( c = 137.036 \) au, which when compared to the bare energy of a boson of zero momentum \( c^2 = 18 778.865 \), yields an energy correction \( \Delta E = -0.022 \). This result is to be compared with the results for discretized perturbation theory, equations (5.11)–(5.13). Equation (5.11) gives \( E^{(2)}_{\text{Dis}}(k = 0) = -0.012 \) for these parameters, while from equation (5.12) we have \( E^{(2)}_{\text{Dis}}(k = 0) = -0.010 \). So, the total energy correction of \(-0.022\) is identical to the numerical result. We should note that equation (5.12) suggests that the perturbative correction to the energy of the single-particle state contains a term identical to the energy correction of the vacuum state. This change to the vacuum energy should be subtracted from the state’s energy, and whatever energy correction remains is the true change to the particle’s rest energy. For these parameters, the vacuum energy correction is \( E^{(2)}_{\text{vac Dis}} = -0.003 \), so the actual change to the particle’s rest energy is \( \Delta m c^2 = -0.019 \).

This numerical method of solving for the spectrum of a quantum field theory can in principle be used to renormalize such a theory non-perturbatively. The restriction of the number of momentum modes to some maximum \( N \), which was necessary in order to make the number of modes finite so that the theory becomes numerically tractable, also automatically regulates the integrals by playing the role of a momentum cutoff. The bare parameters can then be adjusted until the observables, such as the rest energy \( E = mc^2 \), match their physical values. These bare parameters would depend on both the box length and the maximum momentum mode \( N \), since the momentum cutoff is given by the momentum of the highest mode, which is \( \Lambda = N(2\pi)/L \). This procedure could be carried out iteratively, and since the numerical method is non-perturbative the values of the bare parameters derived from this procedure would be exact and non-perturbative.

5.2. Alternative S-matrix based approach to renormalization

In this alternative renormalization approach, we have to construct the counterterms \( V_{\text{counter}} \) for the Hamiltonian

\[ H = \int dp \Omega(p)\hat{A}^\dagger(p)\hat{A}(p) + \lambda \int dz \hat{\phi}^4(z), \]

where \( \Omega(p) = [M^2 c^4 + \epsilon^2 p^2]^{1/2} \), such that the no-self-scattering conditions of equations (3.4) are fulfilled. In full generality we start here with the ansatz

\[ V_{\text{counter}} = B \int dz \hat{\phi}(z)^2 + C \int dz \hat{\phi}^2(z) + F \int dz D(\phi) \]
\[ + E \int dz D(\phi)^2 \] (5.19)

where \( D(\phi) = (\partial_\mu \hat{\phi}(z))^2 - (\partial_\mu \hat{\phi}(z))^2 \). The four unknown coefficients \( B, C, F, E \) will be determined by trying to satisfy the vacuum \( \langle 0|S|0 \rangle = 1 \) and one-particle \( \langle p'|S|p \rangle = \delta(p - p') \) no-scattering conditions from section 3.2.

The perturbative expansion (3.5) of the \( S \) matrix is given by \( S = \sum_{n=0}^\infty S_n \), where \( S_n = (-i)^n/n! \int \cdots \int dt_1 dt_2 \cdots dt_n \)
\[ T[V(t_1)V(t_2)\cdots V(t_n)] \]
Note that we have introduced the Dyson expansion form, which is equivalent to the expansion equation (3.5), but because all integrals extend from \( -\infty \) to \( \infty \), it requires the introduction of the time-ordering operator \( T \).

We compute the matrix elements for the vacuum first by inserting the mode decomposition of the field equation (2.2) into the corresponding equations. After a lengthy calculation we obtain the rather simple results:

\[ \int dz\langle 0|\hat{\phi}(z, t)|^2|0 \rangle = c^2/2\delta(0)\Xi \] (5.20a)
\[ \int dz\langle 0|\hat{\phi}(z, t)|^4|0 \rangle = 3(8\pi)^{-1}c^4\Xi^2\delta(0) \] (5.20b)
\[ \int dz\langle 0|(\partial_\mu \hat{\phi}(z))^2 - (\partial_\mu \hat{\phi}(z))^2|0 \rangle = (m^2 c^4/2)\Xi\delta(0) \] (5.20c)
\[ \int d\zeta(0)\langle(\hat{\alpha}_0\psi(\zeta)^2 - (\hat{\alpha}_0^2\psi(\zeta)^2)\rangle(0) \]
\[ = (8\pi)^{-1}[m^2\xi^2\delta(0) + (4\pi)^{-1}\left\{ \int dp_1 \omega(p_1) \right\}^2 + \left\{ \int dp c^2p^2/\omega(p) \right\}^2 \delta(0)) \]
\[ = (8\pi)^{-1}[m^2\xi^2\delta(0) + (4\pi)^{-1}\left\{ \int dp_1 \omega(p_1) \right\}^2 + \left\{ \int dp c^2p^2/\omega(p) \right\}^2 \delta(0)) \]

In appendix C we show that the corresponding values for the single-particle sector can be obtained directly from these expressions if we use the two general relationships:

\[ \langle q|\hat{\psi}(z, t)^*\rangle = \langle 0|\hat{\psi}(z, t)^*\rangle \delta(0) \]
\[ + N(N - 1)\hat{c}^2(4\pi)^{-1}\omega(q)\langle 0|\hat{\psi}(z, t)^*\rangle \delta(0) \]
\[ = \langle 0|\hat{\psi}(z, t)^*\rangle \delta(0) \]
\[ + 2N(4\pi)^{-1}c^2m^2/\omega(q) \]
\[ \times \langle 0|\hat{\psi}(z, t)^*\rangle \delta(0) \]
\[ + (4\pi)^{-1}\omega(q)\langle 0|\hat{\psi}(z, t)^*\rangle \delta(0) \]
\[ = (8\pi)^{-1}[m^2\xi^2\delta(0) + (4\pi)^{-1}\left\{ \int dp_1 \omega(p_1) \right\}^2 + \left\{ \int dp c^2p^2/\omega(p) \right\}^2 \delta(0)) \]

Applying these relationships to our operators, we obtain

\[ \int d\zeta\langle q|\hat{\psi}(z, t)^*\rangle = c^2/2\delta(0)\xi(0) \]
\[ + c^2\omega(0)^{-1}\delta(0) \]
\[ \int d\zeta\langle q|\hat{\psi}(z, t)^*\rangle = 3(8\pi)^{-1}c^4\xi^2\delta(0) \]
\[ + 3(2\pi)^{-1}c^4\omega(0)^{-1}\delta(0) \xi(0) \]
\[ \int d\zeta\langle q|\hat{\psi}(z, t)^*\rangle = (m^2\xi^2/2)\xi(0) \delta(0) + c^2m^2/\omega(q)\delta(0) \]
\[ \times (8\pi)^{-1}[m^2\xi^2\delta(0) + (4\pi)^{-1}\left\{ \int dp_1 \omega(p_1) \right\}^2 + \left\{ \int dp c^2p^2/\omega(p) \right\}^2 \delta(0)) \]
\[ \times \delta(0)\delta(0) + (4\pi)^{-1}\left\{ \int dp_1 \omega(p_1) \right\}^2 + \left\{ \int dp c^2p^2/\omega(p) \right\}^2 \delta(0) \]
\[ \times \delta(0)\delta(0) + (4\pi)^{-1}\left\{ \int dp_1 \omega(p_1) \right\}^2 + \left\{ \int dp c^2p^2/\omega(p) \right\}^2 \delta(0) \]

We may solve these equations for B and E in terms of the other unknown coefficients C and F. Doing so yields

\[ B = -Fm^2c^2 - 3/(4\pi)(C + \lambda)c^2\xi + (6C + \lambda)c^2\xi \]
\[ \times \left( m^4\xi^2/(8\pi) + 1/(4\pi) \left\{ \int dp \omega(p) \right\}^2 \right) \]

associated with the vacuum is

\[ 0 = (0|S_1|0) = -i \int d\zeta \left[ Bc^2/2\xi(0) + (C + \lambda)3(8\pi)^{-1} \times c^4\xi^2\delta(0) + F(m^2\xi^2/2)\xi(0) + E \left( 8\pi \right)^{-1} \times [m^2\xi^2\delta(0) + (4\pi)^{-1}\left\{ \int dp_1 \omega(p_1) \right\}^2 + \left\{ \int dp c^2p^2/\omega(p) \right\}^2 \delta(0)) \right] \]

while the equation coming from the no-self-scattering condition applied to the single-particle states is

\[ 0 = \langle q|S_1|q \rangle = -i \int d\zeta B[c^2/2\xi(0)\xi(0) + c^2(4\pi)^{-1}\omega(q)\delta(0) \delta(0) + 2(4\pi)^{-1}c^2m^2/\omega(q) \delta(0) \xi(0) \]
\[ \times \delta(0)\delta(0) + (4\pi)^{-1}\left\{ \int dp_1 \omega(p_1) \right\}^2 + \left\{ \int dp c^2p^2/\omega(p) \right\}^2 \delta(0) \]
\[ \times \delta(0)\delta(0) + (4\pi)^{-1}\left\{ \int dp_1 \omega(p_1) \right\}^2 + \left\{ \int dp c^2p^2/\omega(p) \right\}^2 \delta(0) \]

We may solve these equations for B and E in terms of the other unknown coefficients C and F. Doing so yields

\[ B = -Fm^2c^2 - 3/(4\pi)(C + \lambda)c^2\xi + (6C + \lambda)c^2\xi \]
\[ \times \left( m^4\xi^2/(8\pi) + 1/(4\pi) \left\{ \int dp \omega(p) \right\}^2 \right) \]
Laser Phys. 23 (2013) 035302

E V Stefanovich et al.

\[ + c^4 \left( \int \frac{d\mathbf{p}}{\omega(p)} \right)^2 \left( m^4 c^8 \Xi^2 + 4q^2 c^4 \Xi \right) \]
\[ \times \int \frac{d\mathbf{p}}{\omega(p)} - 2c^4 \left( \int \frac{d\mathbf{p}}{\omega(p)} \right)^2 \]
\[ + 4 \Xi \omega(q)^2 \int \frac{d\mathbf{p}}{\omega(p)} \]  
\[ E = -3(C + \lambda)c^4 \Xi^2 (m^4 c^8 \Xi^2 + 4q^2 c^4 \Xi) \int \frac{d\mathbf{p}}{\omega(p)} \]
\[ - 2c^4 \left( \int \frac{d\mathbf{p}}{\omega(p)} \right)^2 + 4 \Xi \omega(q)^2 \int \frac{d\mathbf{p}}{\omega(p)} )^{-1}. \]  
\[ (5.26a) \]

The remaining coefficients of the counterterms \( C \) and \( F \) must be determined by other physical requirements. One such requirement is that when these counterterms are applied to higher orders in perturbation theory the \( S \)-matrix remains cluster separable. Also, the measured coupling constant will set further constraints on these coefficients and thereby determine the counterterms.

6. Discussion and open questions

In the standard field theoretical treatment of renormalization, the field theory is renormalized based on the requirement that the mass and coupling match their measured values and the field strength is renormalized by requiring that the field operator creates single-particle states, as assumed by the LSZ reduction formula. But the renormalization program can also be carried out by applying a no-self-scattering condition on the \( S \)-matrix, so that the vacuum and single-particle states are stable and do not scatter with themselves. Both of these techniques were applied to both \( \bar{\psi}(z) \) and \( \bar{\psi}(z)^4 \) interactions in this work.

In the standard treatment of renormalization, there are two renormalization conditions for each particle species in the theory: mass renormalization and wavefunction renormalization. In the \( S \)-matrix renormalization presented here, these two conditions are seemingly replaced by the sole condition that single-particle states should not be self-scattering. This raises interesting questions as to whether this renormalization condition is complete, that is, whether it contains enough conditions to fully determine the parameters of the theory. It is interesting to note that we had to employ the additional condition of cluster separability; whether cluster separability is sufficient to fix all possible parameters at all higher orders in perturbation theory remains an open question.

Further work must also be done to include coupling constant renormalization into this method. The issue was briefly studied here by examining the two-particle mass, which is the energy of the lowest lying two-particle eigenstate divided by \( c^2 \). Such a quantity may make an adequate assessment of the coupling strength for use in the numerical techniques presented here, since our numerical method involves the diagonalization of the Hamiltonian, which directly gives the energies of all states, including two-particle states. However, the two-particle mass of a bound state is obviously not a good measurement to use for renormalization within the context of \( S \)-matrices, as bound states are not easily handled in \( S \)-matrices and are fundamentally non-perturbative.

Quantum field theories can also be renormalized using numerical techniques. The numerical discretization automatically regulates loop integrals, and the bare mass and coupling could be adjusted iteratively until they match the desired physical parameters. Since the numerical method employed here is a non-perturbative solution to the \( \bar{\psi}(z)^4 \) theory, such a renormalization scheme promises to provide a non-perturbative technique for the regularization and renormalization of any quantum field theory, including QED. Such a non-perturbative solution to QFT has not been fully realized, partly due to computational limitations and partly due to conceptual difficulties, but we hope that the current study may be a starting point for this line of research.

Although the techniques presented here were applied to the simple, concrete examples of \( \bar{\psi}(z)^2 \) and \( \bar{\psi}(z)^4 \) theories in one spatial dimension, such methods may be applicable to theories of higher spin in higher dimension, including the Yukawa theory or QED. In QED, the no-self-scattering conditions could be applied to three states: the vacuum, the single-photon state, and the single-electron state. Other physical conditions such as cluster separability provide further constraints on the parameters used in the counterterms.

There remain many open questions, which must be reserved for future works. One such question is whether the \( S \)-matrix renormalization method presented here can be shown to work at all orders of perturbation theory: ideally, one would like to have a proof that the theory is renormalizable order-by-order at all orders of the coupling constant. A further question remains as to how to apply numerical techniques to a gauge theory due to the gauge invariance. In particular, numerical methods naturally lend themselves to a maximum momentum cutoff regulator, but such regulators do not preserve the gauge-invariance of the theory, which may introduce complications that must be resolved.

Acknowledgments

We enjoyed several helpful discussions with S Meuren, Professors C C Gerry, A Di Piazza, K Z Hatsagotsyan, C Müller and Y T Li. QS and RG acknowledge the kind hospitality of their host institutions (Chinese Academy of Sciences, Beijing and MPIK Heidelberg) during their sabbatical leaves. This work has been supported by the NSF and the NSFC (#11128409).

Appendix A

A system is relativistically invariant if we are able to construct the Hamiltonian \( H \), the total momentum \( P \), and the boost operator \( K \) in such a way that they fulfil the three Poincaré commutator relationships given by

\[ [H, P] = 0 \]  
\[ [K, P] = -iH/c^2 \]  
\[ [K, H] = -iP. \]  
\[ (A.1a) \]
\[ (A.1b) \]
\[ (A.1c) \]
For the special case of a non-interacting system, the Hamiltonian is given by \( H_0 = \int dp \omega(p) \hat{a}^\dagger(p) \hat{a}(p) \) and the total momentum operator by \( P_0 = \int dp \hat{p} \hat{a}^\dagger(p) \hat{a}(p) \). The generator for velocity takes a more complicated form, \( K_0 = i \int dp \frac{p}{2} \omega(p) \hat{a}^\dagger(p) \hat{a}(p) + \omega(p)/c^2 \hat{a}^\dagger(p) \hat{p} \hat{a}(p) \). This operator \( K_0 \) can also be represented in an equivalent form using the position operator \( Z = \int dz \hat{z} \hat{a}^\dagger(z) \hat{a}(z) \), where \( \hat{a}(z) \equiv (2\pi)^{-1/2} \int dp \hat{a}(p) \exp[ipz] \) is the Fourier transform of \( \hat{a}(p) \). This position operator fulfills \( [Z, P] = i \) and we can use it to express the (interaction-free) boost operator as \( K_0 = -(ZH + HZ)/(2c^2) \). We note that the proof for \( [K, H] = -iP \) requires us to assume that \( \omega(p) \hat{a}^\dagger(p) \hat{a}(p) \int_0^\infty = 0 \), i.e., that states with infinitely large momentum should not be excited.

Any operator, when viewed from another coordinate system which is shifted by a distance \( s \), a time \( \tau \), or a velocity \( v \) (associated with the rapidity parameter \( \theta = \text{tanh}(v/c) \)), can be obtained from its original form by a unitary transformation. The quantum field operator \( \hat{\psi}(z, t) \) in equation (2.2) is constructed to have the remarkably simple transformation property

\[
\exp[-iP\theta]\hat{\psi}(z, t)\exp[iP\theta] = \hat{\psi}(z - s, t - \tau)
\]

(A.2a)
\[
\exp[iH\theta]\hat{\psi}(z, t)\exp[-iH\theta] = \hat{\psi}(z, t - \tau)
\]

(A.2b)
\[
\exp[-iK\theta]\hat{\psi}(z, t)\exp[iK\theta] = \hat{\psi}(L_{-\theta}[z, t]) \tag{A.2c}
\]

where the linear operator \( L_{-\theta}[z, t] \) describes the inverse of the usual Lorentz transformation, which is defined as \( L_{0}(a, b) \equiv (a \cosh \theta - b/c \sinh \theta, b \cosh \theta - a/c \sinh \theta) \).

For the interacting case of the \( \hat{\psi}(z) \) theory, we prove here that the new triplet, given by

\[
H = H_0 + \lambda \int dz \hat{\psi}(z)^N, \tag{A.3a}
\]
\[
P = P_0 \tag{A.3b}
\]
\[
K = K_0 + (\lambda/c^2) \int dz \hat{\psi}(z)^N \tag{A.3c}
\]
also fulfills the Poincaré Lie algebra for any integer \( N \) except \( N = 0 \). The proof is straightforward if we use the invariant transformation properties of the quantum field operator \( \hat{\psi}(z) \) described in equations (A.2).

The first Poincaré relationship equation (A.1a) can be proved as follows: \([H, P] = [H_0, P_0] + [\lambda, \int dz \hat{\psi}(z)^N, P_0] = [\lambda, \int dz \hat{\psi}(z)^N, P_0] \). To simplify the commutators further, we can use the general relationship between a general propagator \( \exp[-iGa] \) and its associated generator \( G \) for arbitrary operators \( O \) according to \( \partial/O[a]_{a=0} = \partial/\exp[GA] \exp[GA]_{a=0} = [O, G(a = 0)] \). We obtain

\[
\left[ \lambda, \int dz \hat{\psi}(z)^N, P_0 \right] = \lim(s \to 0) \partial/\partial s \left\{ \exp[-iP_0]\lambda \int dz \hat{\psi}(z)^N \exp[iP_0s] \right\}
\]
\[
= \lim(s \to 0) \lambda \partial/\partial s \int dz \hat{\psi}(z + s)^N
\]
\[
= \lim(s \to 0) \lambda \partial/\partial s \int dz \hat{\psi}(z)^N
\]
\[
= 0. \tag{A.4}
\]

Thus the first required relationship \([H, P] = 0 \) is fulfilled. The proof for the second relationship \([K, P] = -iH/c^2 \) is very similar. We have \([K, P] = [K_0, P_0] = [K_0, P_0] + [(\lambda/c^2) \int dz \hat{\psi}(z)^N, P_0] \). For the second commutator we write:

\[
\begin{align*}
\left[ \lambda/c^2, \int dz \hat{\psi}(z)^N, P_0 \right] &= \lim(s \to 0)i\partial/\partial s \left\{ \exp[iP_0s](\lambda/c^2) \right\} \\
&= i \lim(s \to 0) \int dz \hat{\psi}(z + s)^N \right. \\
&= i \lim(s \to 0) \int dz \hat{\psi}(z - s)^N \right. \\
&= -i\lambda \int dz \hat{\psi}(z)^N/c^2. \tag{A.5}
\end{align*}
\]

Here it is important to note that equation (A.5) is not valid for \( N = 0 \) as the derivative of a constant operator \( \int dz \hat{\psi}(z)^0 \) (even when unitarily transformed) is zero. As we can use for the first term \( [K_0, P_0] = -iH_0/c^2 \), we have therefore shown the validity of equation (A.1b), i.e. \([K, P] = -iH/c^2 \).

We now prove the third relationship equation (A.1c), which is \([K, H] = -iP \). Again, inserting the expressions above from equations (A.3), we obtain the four terms \([K, H] = [K_0, H_0] + [K_0, \lambda \int dz \hat{\psi}(z)^N] + [(\lambda/c^2) \int dz \hat{\psi}(z)^N, H_0] + [(\lambda/c^2) \int dz \hat{\psi}(z)^N, \lambda \int dz \hat{\psi}(z)^N] \). The first term is \(-iP_0 \) while the fourth term vanishes. The remaining second and third terms are

\[
\begin{align*}
&\left[ K_0, \lambda \int dz \hat{\psi}(z, t = 0)^N \right] = \lim(\theta \to 0)i\partial/\partial(\cosh \theta) \left\{ \exp[-iK_0\theta] \right\} \\
&\times \int dz \hat{\psi}(z, t = 0)^N \exp[iK_0\theta] \right. \\
&= \left\{ \lambda \int dz \hat{\psi}(z \cosh \theta, z/c \sinh \theta)^N \right.
\end{align*}
\]

(A.6)

If we use \( \exp[-iK_0\theta] \hat{\psi}(z, t = 0) \exp[iK_0\theta] = \hat{\psi}(z \cosh \theta, z/c \sinh \theta) \), the commutator becomes

\[
\begin{align*}
&\left[ K_0, \lambda \int dz \hat{\psi}(z, t = 0)^N \right] = i \lim(\theta \to 0) \partial/\partial (\cosh \theta) \\
&\times \left\{ \lambda \int dz \hat{\psi}(z \cosh \theta, z/c \sinh \theta)^N \right.
\end{align*}
\]

(A.7)
For the third commutator, we apply the same reasoning
\[
\left[ \frac{\lambda}{c^2} \int dz \hat{z} \hat{\phi}(z)^N, H_0 \right] = \lim(t \to 0) - i \frac{\partial}{\partial t} \left\{ \exp[-iH_0 t] \left( \frac{\lambda}{c^2} \right) \right\} \times \int dz \hat{z} \hat{\phi}(z)^N \exp[iH_0 t]
\]
\[= \lim(t \to 0) - i \frac{\partial}{\partial t} \left\{ \frac{\lambda}{c^2} \int dz \hat{z} \hat{\phi}(z, t)^N \right\} \]
\[= -i \frac{\lambda}{c^2} \int dz \frac{\partial}{\partial t} \hat{\phi}(z, t)^N \bigg|_{t=0} \quad \text{(A.8)}
\]
As the sum of the two terms in equations (A.7) and (A.8) cancel, we have the final result \([K, H] = -iP_0 = -iP\).

Finally, we point out that in any relativistic theory describing particles of mass \(m\), the spectrum of the Hamiltonian operator \(H\) must satisfy two conditions: the spectrum should be non-negative, and it should contain 0 as a non-degenerate eigenvalue. These conditions follow from the existence of the non-particle (vacuum) state \(|\text{vac}\rangle\) which carries the trivial 1-dimensional irreducible representation of the Poincaré group, i.e., \(H|\text{vac}\rangle = P|\text{vac}\rangle = K|\text{vac}\rangle = 0.\) If any Hamiltonian \(H\) is a legitimate Hamiltonian satisfying these conditions, then for any non-zero constant \(N\) the operator \(H' = H + N\) does not satisfy these conditions and is not a legitimate Hamiltonian. First, note that any eigenvector of \(H\) is automatically an eigenvector of \(H'\). Therefore, the spectrum of \(H'\) can be obtained from the spectrum of \(H\) by simply shifting it by the constant \(N\). If \(N\) were positive, then the spectrum of \(H'\) would violate the condition that \(0\) should be an eigenvalue, and if \(N\) were negative, then the spectrum violates the condition that the spectrum should be non-negative.

Even though it might seem only as a trivial energy shift, we have not been able to construct the corresponding velocity boost generator \(K\) which would make the Hamiltonian \(H_0 + N\) with \(N\) a constant, fulfill the Poincaré relationships, which is strictly required for relativistic invariance. While this infinite constant could be important for phenomena in systems with finite extension (such as the Casimir effect, e.g.), a system with a boundary would not be translationally invariant and we presently do not know if it is possible to generalize the Poincaré relationships to account for the effective interactions that are usually approximated mathematically by boundary conditions.

**Appendix B**

The two terms in the interaction \(V\) which modify the energy of a single-particle state \(|p\rangle\) at second order in \(\lambda\) are proportional to \(a^\dagger a^\dagger a a\), corresponding to an intermediate 3-particle state, and \(a^\dagger a a^\dagger a\), which now the intermediate state will be a 5-particle state. We will define the two corresponding second-order energy corrections as \(E^{(2)}_3\) and \(E^{(2)}_5\). For the term \(E^{(2)}_3\), the matrix element to be calculated is
\[
\langle p_1 p_2 p_3 | V | p \rangle = \langle p_1 p_2 p_3 | (2\pi)^{-1/2} \int dk_1 \int dk_2 \left[ \int dk_3 \int dk_4 \delta(-k_1 - k_2 - k_3 + k_4) \times 4 \hat{a}_1^\dagger (k_1) \hat{a}_2^\dagger (k_2) \hat{a}_3^\dagger (k_3) \hat{a}(k_4) \right] \]
\[
\times [\omega(p_1) \omega(p_2) \omega(p_3)] f_{123} \int dk_1 \int dk_2 \int dk_3 \delta(-k_1 - k_2 - k_3 + p) \times [\omega(p_1) \omega(p_2) \omega(p_3) \omega(p)]^{1/2} (\omega(p) - (\omega(p_1) + \omega(p_2) + \omega(p_3))).
\]

The factor of 3! in the final line arises from the number of ways in which the \(\hat{a}(p_i)\) can contract on the \(\hat{a}^\dagger (k_i)\), and \(f_{123} = f(p_1, p_2, p_3)\) is the combinatorial factor which arises from the operators \(\hat{a}^\dagger (p_i)\) acting on the bare vacuum state \(|\text{vac}\rangle\); \(f_{123}\) is equal to 1 if all three momenta differ, it is equal to \(\sqrt{2}/2\) if two momenta are equal to each other (but not all three), and it is equal to \(\sqrt{6}/6\) if all three momenta are identical.

This matrix element is to be integrated over all possible 3-particle intermediate states \(|p_1 p_2 p_3\rangle\) with the integration range is restricted to \(p_1 \leq p_2 \leq p_3\) to avoid double counting. The integration range can be extended to include all momenta if we also divide by the symmetry factor \(M_{123} = M(p_1, p_2, p_3)\), which counts the number of times the triplet \(p_1, p_2, p_3\) appears in the integration (or summation, in the case of a finite box). This symmetry factor works out to be \(M_{123} = 3! f_{123}^2\). It might appear that a combinatorial factor such as \(f_{123}\) is irrelevant inside an integral, since it requires two or more momenta to be identical for it to be different from 1, which is of measure 0. However, if the integral is divergent, a measure 0 contribution to an integral may yield a finite contribution. While the integral \(E^{(2)}_3\) is finite, the integral \(E^{(2)}_5\) diverges because it contains a term equal to the vacuum’s energy correction. This vacuum energy correction is to be subtracted from all states in the Hilbert space, and in order to do this correctly finite corrections to these divergent integrals must be treated very carefully. The energy correction corresponding to 3-particle states is therefore
\[
E^{(2)}_3 = \delta(0) - 3! \int dp_1 dp_2 dp_3 M_{123}^{-1} \lambda f_{123}^{1/2} \int \omega(p_1) \omega(p_2) \omega(p_3) \omega(p)\]
\[
\times [\omega(p_1) \omega(p_2) \omega(p_3) \omega(p)]^{1/2} (\omega(p) - (\omega(p_1) + \omega(p_2) + \omega(p_3)))
\]
where the integration range is restricted to $p$, and the other cancels the factor $L^2 \lambda_4$. The intermediate states particles contracting on $\pi_4 = (2\pi)^{-1/2} c^4 \lambda^2 f_{12345}^{-1}$.

The factor of $\delta(-p_1 - p_2 - p_3 + p)$ is squared; one delta function removes an integral, and the other cancels the factor of $\delta(0)^{-1}$ which comes from the normalization of the state $|p\rangle$, yielding a finite correction to the energy. The computation of $E^{(2)}_5$ proceeds similarly. The matrix element to be computed is

$$\langle p_1 p_2 p_3 p_4 p_5 | V | p \rangle = \langle p_1 p_2 p_3 p_4 p_5 | (2\pi)^{-1} c^4 \lambda f_{12345}^{-1} \delta(0)^{-1} \delta(-p_1 - p_2 - p_3 + p) \rangle^{(p) = \omega(p_1) \omega(p_2) \omega(p_3) \omega(p_4) \omega(p_5)}$$

$$= \int \frac{d\mathbf{k}_1}{(2\pi)^{3/2}} \int \frac{d\mathbf{k}_2}{(2\pi)^{3/2}} \int \frac{d\mathbf{k}_3}{(2\pi)^{3/2}} \int \frac{d\mathbf{k}_4}{(2\pi)^{3/2}} \delta(k_1 + k_2 + k_3 + k_4) \times \left[ \omega(k_1) \omega(k_2) \omega(k_3) \omega(k_4) \right]^{-1/2}$$

$$\times \hat{a}^\dagger(k_2) \hat{a}(k_3) \hat{a}^\dagger(k_4) \hat{a}(p)$$

$$= \langle 0 | (\hat{a}^\dagger p_1 \hat{a}(p_2) \hat{a}(p_3) \hat{a}(p_4) \hat{a}(p_5) f_{12345}^{-1} - (2\pi)^{-1} c^4 \lambda f_{12345}^{-1} \delta(-p_5 + p) \rangle^{(p) = \omega(p_1) \omega(p_2) \omega(p_3) \omega(p_4) \omega(p_5)}$$

$$\times \int \frac{d\mathbf{k}_1}{(2\pi)^{3/2}} \int \frac{d\mathbf{k}_2}{(2\pi)^{3/2}} \int \frac{d\mathbf{k}_3}{(2\pi)^{3/2}} \int \frac{d\mathbf{k}_4}{(2\pi)^{3/2}} \delta(k_1 + k_2 + k_3 + k_4) \times \left[ \omega(k_1) \omega(k_2) \omega(k_3) \omega(k_4) \right]^{-1/2}$$

$$\times \hat{a}^\dagger(k_2) \hat{a}^\dagger(k_3) \hat{a}^\dagger(k_4) \hat{a}(p)$$

$$= \delta(0)^{-1} \int \frac{d\mathbf{p}_1}{(2\pi)^{5/2}} \int \frac{d\mathbf{p}_2}{(2\pi)^{5/2}} \int \frac{d\mathbf{p}_3}{(2\pi)^{5/2}} \int \frac{d\mathbf{p}_4}{(2\pi)^{5/2}} \int \frac{d\mathbf{p}_5}{(2\pi)^{5/2}} \delta(p_1 + p_2 + p_3 + p_4 + p_5) \times \delta(p_5 - p) \delta(p_1 + p_2 + p_3 + p_4)$$

$$\times \omega(p_1) \omega(p_2) \omega(p_3) \omega(p_4) \omega(p_5) \delta(p_1 + p_2 + p_3 + p_4 + p_5)$$

$$+ 4 \text{ permutations} \times \omega(p) - \omega(p) \rangle^{(p) = \omega(p_1) \omega(p_2) \omega(p_3) \omega(p_4) \omega(p_5)}.$$
where the $\omega(p_1)\omega(p_2)\omega(p_3)$ is subtracted, is equal to equation (B.2) plus 20 times the correction to the energy, if the vacuum term equation (B.5) differs only by a permutation of integration variables. The total Laser Phys. cancels the vacuum computing the matrix elements of the scattering operator for section 5.1.2. To prove the useful relationships equations (C.1) and (C.3). In below we state to know their values for single-particle states. Below we prove the useful relationships equations (C.1) and (C.3). In first-order perturbation theory, the calculation of the scattering matrix elements requires us to compute the expectation values of powers of the operators $\hat{\phi}(z, t)$ and $D(\hat{\phi}) \equiv \langle \hat{\phi}(z, t)^2 \rangle - (\hat{\phi}(z, t))^2$.

We will first prove the general relationship

$$
\langle q_1| \hat{\phi}^N |q_2 \rangle = \langle 0| \hat{\phi}^N |0 \rangle \delta(q_1 - q_2) + (N - 1)N! \langle 0| \hat{\phi}^N |0 \rangle Y(q_1) Y^*(q_2) \tag{C.1}
$$

where the c-number $Y(q) \equiv [\hat{a}(q), \hat{\phi}(z, t)] = c(4\pi T)^{-1/2} \omega(q)^{-1/2} \exp[-iqz + i\omega(q)t]$. Note that for the special case $q \equiv q_1 = q_2$ this reduces to

$$
\langle q| \hat{\phi}^N |q \rangle = \langle 0| \hat{\phi}^N |0 \rangle \delta(0) + (N - 1)N! \langle 0| \hat{\phi}^N |0 \rangle Y(q) Y^*(q) \tag{C.2}
$$

There are two steps involved in the derivation. First we commute $\hat{a}(q_1)$ in $\langle 0| \hat{a}(q_1) \hat{\phi}^N |q_2 \rangle$ to the right, obtaining $\langle 0| \hat{a}(q_1) \hat{\phi}^N |q_2 \rangle = \langle 0| \hat{\phi}^N \hat{a}(q_1) |q_2 \rangle + \langle 0| \hat{a}(q_1) \hat{\phi}^N |q_2 \rangle$. The first term simplifies to $\langle 0| \hat{\phi}^N |0 \rangle \delta(q_1 - q_2)$. As the commutator $[\hat{a}(q_1), \hat{\phi}^N] = Y(q_1)$ is a c-number and therefore commutes with $\hat{\phi}^N$, we can simplify the term $\langle 0| \hat{a}(q_1) \hat{\phi}^N |q_2 \rangle = NY(q_1)\hat{\phi}^{N-1}$. This can easily be seen from the solution to the recursive relationship $[\hat{a}(q_1), \hat{\phi}^N] = NY(q_1)\hat{\phi}^{N-1}$. So our two terms simplify to $\langle 0| \hat{a}(q_1) \hat{\phi}^N |q_2 \rangle = \langle 0| \hat{\phi}^N |0 \rangle \delta(q_1 - q_2) + NY(q_1)(0\hat{\phi}^{N-1}|q_2)\rangle$.

As a second step we commute $\hat{a}(q_2)$ in $\langle 0| \hat{\phi}^{N-1} |q_2 \rangle = \langle 0| \hat{\phi}^{N-1} \hat{a}(q_2) |0 \rangle$ to the leftmost position, leading to $\langle 0| \hat{\phi}^{N-1} \hat{a}(q_2) |0 \rangle = \langle 0| \hat{\phi}^{N-1} |q_2 \rangle \delta(q_1 - q_2) + \langle 0| \hat{\phi}^{N-1} |q_2 \rangle \delta(q_1 - q_2)$, where the second term vanishes. Following the same sort of recursion relationship argument as above, we find that $[\hat{\phi}^{N-1}, \hat{a}(q_2)] = (N - 2)Y(q_2)\hat{\phi}^{N-1}$. Such that the final correction term becomes $NY(q_1)(0\hat{\phi}^{N-1}|q_2\rangle = (N - 1)(N - 2)Y(q_1) Y^*(q_2)|0\hat{\phi}^{N-2}|0 \rangle$ and the proof of equation (C.1) is complete.

A second relationship between the vacuum and single-particle state expectation value is given by

$$
\langle q_1| \hat{\phi}^N |q_2 \rangle = \langle 0| \hat{\phi}^N |0 \rangle \delta(q_1 - q_2) + 2N[T(q_1) T^*(q_2) - Z(q_1) Z^*(q_2) \hat{a}(q_2) \hat{\phi} + Z(q_1) \hat{a}(q_2) \hat{\phi} T(q_2) T^*(q_2) - Z(q_1) Z^*(q_2) \hat{a}(q_2) \hat{\phi}] \langle 0| \hat{\phi}^{N-2} |0 \rangle \tag{C.3}
$$

Here we have defined the operators $\hat{A}(q) \equiv [\hat{a}(q), \hat{\phi}(z, t)] = 2[T(q) \hat{a}(q) \hat{\phi} - \hat{a}(q) \hat{\phi} T(q)]$ and the two c-numbers $Z(q) \equiv [\hat{a}(q), \hat{\phi}(z, t)] = \partial \hat{\phi} \hat{a}(q) + \partial \hat{a}(q) \hat{\phi}$. For the interesting special case $q_1 = q_2 \equiv q$ this simplifies to

$$
\langle q| \hat{\phi}^N |q \rangle = \langle 0| \hat{\phi}^N |0 \rangle \delta(0) + N(2\pi)^{-1/2} \omega(q) \langle 0| \hat{\phi}^N |0 \rangle \delta(0) \tag{C.4}
$$

To prove this relationship, we follow a similar two-step approach as above. First, we again commute $\hat{a}(q_1)$ in $\langle 0| \hat{a}(q_1) \hat{\phi}^N |q_2 \rangle$ to the right, obtaining $\langle 0| \hat{a}(q_1) \hat{\phi}^N |q_2 \rangle = \langle 0| \hat{\phi}^N \hat{a}(q_1) |q_2 \rangle + \langle 0| \hat{a}(q_1) \hat{\phi}^N |q_2 \rangle$. The first term simplifies to $\langle 0| \hat{\phi}^N |0 \rangle \delta(q_1 - q_2)$. As the commutator $[\hat{a}(q_1), \hat{\phi}^N] = Y(q_1)$ is an operator that commutes with $\hat{\phi}(z, t)$, $[\hat{a}(q_1), \hat{\phi}(z, t)] = 0$, we can simplify the term $\langle 0| \hat{a}(q_1) \hat{\phi}^N |q_2 \rangle = NY(q_1)\hat{\phi}^{N-1}$. This can be easily seen from the solution to the recursive relationship $[\hat{a}(q_1), \hat{\phi}^N] = NY(q_1)\hat{\phi}^{N-1}$.
$[\hat{a}(q_1), D(\hat{\psi})]D(\hat{\psi})^{N-1}$. So the second term simplifies to
$\langle 0| [\hat{a}(q_1), D(\hat{\psi})]^{N-1}|q_2\rangle = N\langle 0| \hat{A}(q_1)D(\hat{\psi})^{N-1}|q_2\rangle$.

The second step is to commute $\hat{a}^\dagger(q_2)$ in $N\langle 0| \hat{A}(q_1)D(\hat{\psi})^{N-1}\hat{a}^\dagger(q_2)|0\rangle$ to the leftmost position, leading to

$$N\langle 0| \hat{A}(q_1)D(\hat{\psi})^{N-1}\hat{a}^\dagger(q_2)|0\rangle = \langle 0| \hat{A}(q_1)[D(\hat{\psi})^{N-1}\hat{a}^\dagger(q_2)]|0\rangle.$$ (C.5)

Using the commutator $[D(\hat{\psi})^{N-1}, \hat{a}^\dagger(q_2)] = (N-1)\hat{A}^\dagger(q_2)D(\hat{\psi})^{N-2}$, this simplifies to

$$N\langle 0| \hat{A}(q_1)\hat{A}^\dagger(q_2)D(\hat{\psi})^{N-2}|0\rangle = N\langle 0| \hat{A}(q_1), \hat{a}^\dagger(q_2)|0\rangle = \langle 0| \hat{A}(q_1)\hat{A}^\dagger(q_2)|0\rangle.$$ (C.6)

The commutator $[\hat{A}(q_1), \hat{a}^\dagger(q_2)] = 2[T(q_1)\hat{a}^\dagger(q_2) - Z(q_1)\hat{a}^\dagger(q_2)]$ simplifies to $2[T(q_1)\hat{a}^\dagger(q_2) - Z(q_1)\hat{a}^\dagger(q_2)]$. If we use the definitions of $T(q_2)$ and $Z(q_2)$, this commutator simplifies to

$$(2\pi)^{-1}\langle \omega(q_1) / \omega(q_2) \rangle^{1/2}$$

$$+ c^2 q_1 q_2(\omega(q_1) / \omega(q_2))^{1/2} \exp[-i q_1 z]$$

$$+ i \omega(q_1) r \exp[i q_2 z - i \omega(q_2) r].$$

Furthermore, the term $\hat{A}(q_1)\hat{A}^\dagger(q_2)$ is a short-hand notation for the operator $\hat{A}(q_1)\hat{A}^\dagger(q_2) = 4[T(q_1)\hat{a}^\dagger(q_2) - Z(q_1)\hat{a}^\dagger(q_2)]$. We therefore obtain the final result as indicated by equation (C.3):

$$\langle q_1| D(\hat{\psi})^N|q_2\rangle = \langle 0| D(\hat{\psi})^N|0\rangle \delta(q_1 - q_2) + N2(T(q_1)T^*(q_2) - Z(q_1)Z^*(q_2))$$

$$- (Z(q_1)Z^*(q_2))|0\rangle [D(\hat{\psi})^{N-1}|0\rangle + 4N(N-1)|0\rangle \times [T(q_1)\hat{a}^\dagger(q_2) - Z(q_1)\hat{a}^\dagger(q_2)]T^*(q_2)\hat{a}^\dagger(q_2)$$

$$- Z^*(q_2)\hat{a}^\dagger(q_2)]D(\hat{\psi})^{N-2}|0\rangle.$$ (C.7)

This expression simplifies considerably if we assume $q_1 = q_2 \equiv q$. In this case, the second expression $T(q_1)T^*(q_2) - Z(q_1)Z^*(q_2)$ reduces to $4(\pi)^{-1}(\omega(q) - c^2 q^2 / \omega(q))$ reduces to the factor $(4\pi)^{-1}m^2 c^4 / \omega(q)$. Furthermore, the product of the two operators $[T(q_1)\hat{a}^\dagger(q_2) - Z(q_1)\hat{a}^\dagger(q_2)]T^*(q_2)\hat{a}^\dagger(q_2) - Z(q_2)\hat{a}^\dagger(q_2)$ reduces to $(4\pi)^{-1}(\omega(q)[\hat{a}^\dagger(q_2) + c q / \omega(q)]\hat{a}^\dagger(q_2)]^2$. If we insert these expressions into equation (C.7) we obtain equation (C.4).

References

[1] Bialynicki-Birula I and Bialynicka-Birula Z 1975 Quantum Electrodynamics (Oxford: Pergamon)
[2] Gerry C C and Knight P L 2004 Introductory Quantum Optics (Oxford: Cambridge University Press)
[3] Collins J 2003 Renormalization (Cambridge: Cambridge University Press)
[4] Delamotte B 2004 Am. J. Phys. 72 170
[5] Bogoliubov N N and Shirkov D V 1959 The Theory of Quantized Fields (New York: Interscience)
[6] Schweber S S 1962 An Introduction to Relativistic Quantum Field Theory (New York: Harper & Row)
[7] Roman P 1969 Introduction to Quantum Field Theory (New York: Wiley)
[8] Itzykson C and Zuber J 1980 Quantum Field Theory (New York: McGraw-Hill)
[9] Ryder L H 1985 Quantum Field Theory (Cambridge: Cambridge University Press)
[10] Weinberg S 1995 The Quantum Theory of Fields vol 1 (Cambridge: Cambridge University Press)
[11] Peskin M E and Schroeder D V 1995 An Introduction to Quantum Field Theory (Boulder: Westview Press)
[12] Srednicki M 2007 Quantum Field Theory (Cambridge: Cambridge University Press)
[13] Mandl F and Shaw G 2006 Quantum Field Theory (Chichester: Wiley)
[14] Takayanagi K 1991 Phys. Rev. A 44 59
[15] Stefanovich E V 2001 Ann. Phys. 292 139
[16] Stefanovich E V 2005 Renormalization and dressing in quantum field theory arXiv:hep-th/0503076
[17] Stefanovich E V 2004 Relativistic quantum dynamics: a non-traditional perspective on space, time, particles, fields, and action-at-a-distance, arXiv:physics/0504062v15 [physics.gen-ph]
[18] Efimov G V 1989 Int. J. Mod. Phys. A 4 4977
[19] Glimm J and Jaffe A 1968 Phys. Rev. 176 1945
[20] Jaffe A M and Powers R T 1968 Commun. Math. Phys. 7 218
[21] Glimm J and Jaffe A 1970 Ann. Math. 91 362
[22] Wagner R E, Acosta S, Glasgow S A, Su Q and Grobe R 2012 J. Phys. A: Math. Theor. 45 275303
[23] Kaku M 1993 Quantum Field Theory (Oxford: Oxford University Press)
[24] Messiah A 1966 Quantum Mechanics vol II (Amsterdam: Noth-Holland)
[25] Cheng T, Gospodarczyk E R, Su Q and Grobe R 2010 Ann. Phys. 325 265
[26] Montvay I and Münster G 1994 Quantum Fields on a Lattice (Cambridge: Cambridge University Press)