On the problem of differentiation of hyperelliptic functions

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Abstract
We describe a construction that leads to an explicit solution of the problem of differentiation of hyperelliptic functions. A classical genus $g = 1$ example of such a solution is the result of Frobenius and Stickelberger (J Reine Angew Math 92:311–337, 1882). Our method follows the works Buchstaber (Proc Steklov Inst Math 294:176–200, 2016) and Bunkova (Eur J Math 4(1):93–112, 2018) that led to constructions of explicit solutions of the problem for genus $g = 2$ and $g = 3$.

Keywords Abelian functions · Elliptic functions · Jacobians · Hyperelliptic curves · Hyperelliptic functions · Lie algebra of derivations · Polynomial vector fields

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1 Introduction
We consider meromorphic functions $f$ in $\mathbb{C}^g$. A vector $\omega \in \mathbb{C}^g$ is called a period for $f$ if $f(z + \omega) = f(z)$ for any $z \in \mathbb{C}^g$. If the periods of $f$ form a lattice $\Gamma$ of rank $2g$ in $\mathbb{C}^g$, then $f$ is called an Abelian function. We say that an Abelian function is a meromorphic function on the complex torus $T^g = \mathbb{C}^g/\Gamma$. We denote the coordinates in $\mathbb{C}^g$ by $(z_1, z_3, \ldots, z_{2g-1})$.

Let us consider hyperelliptic curves of genus $g$ in the model $\mathcal{V}_\lambda = \{(X, Y) \in \mathbb{C}^2 : Y^2 = X^{2g+1} + \lambda_4 X^{2g-1} + \lambda_6 X^{2g-2} + \cdots + \lambda_{4g} X + \lambda_{4g+2}\}.$

Such a curve depends on the parameters $\lambda = (\lambda_4, \lambda_6, \ldots, \lambda_{4g}, \lambda_{4g+2}) \in \mathbb{C}^{2g}$.

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Denote by $\mathcal{B} \subset \mathbb{C}^{2g}$ the subspace of parameters such that $\mathcal{V}_\lambda$ is non-singular for $\lambda \in \mathcal{B}$. We have $\mathcal{B} = \mathbb{C}^{2g} \setminus \Sigma$ where $\Sigma$ is the discriminant hypersurface.

A hyperelliptic function of genus $g$ (see [3,7]) is a meromorphic function in $\mathbb{C}^g \times \mathcal{B}$, such that for each $\lambda \in \mathcal{B}$ its restriction to $\mathbb{C}^g \times \lambda$ is Abelian with $T^g$ the Jacobian $\mathcal{J}_\lambda$ of $\mathcal{V}_\lambda$. We denote the field of hyperelliptic functions of genus $g$ by $\mathcal{F}$. See [7] for its properties.

Let $\mathcal{U}$ be the space of the fiber bundle $\pi : \mathcal{U} \to \mathcal{B}$ with fiber over $\lambda \in \mathcal{B}$ the Jacobian $\mathcal{J}_\lambda$ of the curve $\mathcal{V}_\lambda$. Thus, a hyperelliptic function is a meromorphic function on $\mathcal{U}$. According to the Dubrovin–Novikov theorem [13], there is a birational isomorphism between $\mathcal{U}$ and the complex linear space $\mathbb{C}^{3g}$.

**Problem 1.1** ([7]) For each $g$ describe the Lie algebra $\text{Der} \mathcal{F}$ of differentiations of $\mathcal{F}$, that is find $3g$ independent differential operators $\mathcal{L}$ such that $\mathcal{L} \mathcal{F} \subset \mathcal{F}$.

In the case $g = 1$ the solution to this problem is classical [14]. A method for solving it in general was presented in [9,10]. A good overview of this approach is given in [7]. It turned out that it is hard to follow this method to obtain explicit answers.

Explicit solutions to this problem for $g = 2$ and $g = 3$ were first found in [3] and [12]. These works allow us to present a general method that is useful for any genus. Here we describe the general construction of this method.

We use the theory of hyperelliptic Kleinian functions (see [2,4–6], and [15] for elliptic functions). Take the coordinates $(z, \lambda) = (z_1, z_2, \ldots, z_{2g-1}, \lambda_4, \lambda_6, \ldots, \lambda_{4g}, \lambda_{4g+2})$ in $\mathbb{C}^g \times \mathcal{B} \subset \mathbb{C}^{3g}$. Let $\sigma(z, \lambda)$ be the hyperelliptic sigma function (or elliptic sigma function in the genus $g = 1$ case). We denote $\partial_k = \frac{\partial}{\partial z_k}$. Following [3,7,12], we use the notation

\[
\xi_k = \partial_k \ln \sigma(z, \lambda), \quad \wp_{i; k_1, \ldots, k_n} = -\partial_i^{1} \partial_{k_1} \cdots \partial_{k_n} \ln \sigma(z, \lambda),
\]

where $n \geq 0$, $i + n \geq 2$, $k_s \in \{1, 3, \ldots, 2g - 1\}$. In the case $n = 0$ we will skip the semicolon. Note that our notation for the variables $z_k$ differs from the one in [4–6] as $u_i = z_{2g+1-i}$. The functions $\wp_{i; k_1, \ldots, k_n}$ provide us with examples of hyperelliptic functions.

A key to our approach to the problem is the following theorem.

**Theorem 1.2** ([5]) For $i, k \in \{1, 3, \ldots, 2g - 1\}$ we have the relations

\[
\wp_{3; i} = 6 \wp_2 \wp_{1; i} + 6 \wp_{1; i+2} - 2 \wp_{0; 3, i} + 2 \lambda_4 \delta_{i, 1},
\]

\[
\wp_{2; i} \wp_{2; k} = 4 \left( \wp_2 \wp_{1; i} \wp_{1; k} + \wp_{1; k} \wp_{1; i+2} + \wp_{1; i} \wp_{1; k+2} + \wp_{0; k+2, i+2} \right) - 2 \left( \wp_{1; i} \wp_{0; 3, k} + \wp_{1; k} \wp_{0; 3, i} + \wp_{0; k, i+4} + \wp_{0; i, k+4} \right) + 2 \lambda_4 (\delta_{i, 1} \wp_{1; k} + \delta_{k, 1} \wp_{1; i}) + 2 \lambda_{i+4, k+4} (2 \delta_{i, k} + \delta_{k, i-2} + \delta_{i, k-2}).
\]

**Proof** In [5] we have formulas (4.1) and (4.8). Using the notation (1) we get (2) from (4.1) and (3) from (4.8). □
2 The problem for polynomial vector fields

The work [8] constructs the theory of polynomial Lie algebras. Here we describe its connection with Problem 1.1.

Consider the complex space $\mathbb{C}^{3g}$ with coordinates $x = (x_{i,j})$, where $i \in \{1, 2, 3\}$, $j \in \{1, 3, \ldots, 2g - 1\}$. We define the map $\varphi : \mathcal{U} \rightarrow \mathbb{C}^{3g}$ by

$$\varphi : (z, \lambda) \mapsto (x_{i,j}) = (\varphi_{i,j}(z, \lambda)).$$

This map has the following property, observed by Buchstaber (see [3]).

**Theorem 2.1** The functions $\varphi^*(x_{i,j})$ give a set of generators of $\mathcal{F}$.

**Proof** Let us show that the functions $\varphi_{i,j}(z, \lambda)$, where $i \in \{1, 2, 3\}$, $j \in \{1, 3, \ldots, 2g - 1\}$, give a set of generators of $\mathcal{F}$. We use a fundamental result from the theory of hyperelliptic Abelian functions (see [6, Chapter 5]): Any hyperelliptic function can be presented as a rational function in $\varphi_{1,k}$ and $\varphi_{2,k}$, where $k \in \{1, 3, \ldots, 2g - 1\}$. Theorem 1.2 gives a set of relations between the derivatives of these functions.

Now by [12, Corollary 5.2], the functions $(\varphi^*(x_{i,j}), \varphi^*(w_{k,l}), \varphi^*(\lambda_{s}))$ in the notation of this corollary give a set of generators of $\mathcal{F}$. By [12, Theorem 5.3] we obtain the claim of Theorem 2.1. \qed

Another property of $\varphi$ follows from [12, Corollary 5.5]. For each $g$ there is a polynomial map $p : \mathbb{C}^{3g} \rightarrow \mathbb{C}^{2g}$, such that we get the diagram

$$\begin{array}{ccc}
\mathcal{U} & \xrightarrow{\varphi} & \mathbb{C}^{3g} \\
\pi \downarrow & & \downarrow p \\
\mathcal{B} & \hookrightarrow & \mathbb{C}^{2g}.
\end{array}$$

Here $\mathcal{B} \subset \mathbb{C}^{2g}$ is the inclusion like in Sect. 1, with coordinates $\lambda$ in $\mathbb{C}^{2g}$.

We note that the proof of [12, Theorem 5.3] gives a construction to obtain the polynomial maps $p$ explicitly. Examples of these maps for $g = 1, 2, 3$ are given in [12]. The work [7, Theorem 3.2] claims that these polynomial maps are of degree at most 3.

We refer the reader to [8] for the theory of polynomial Lie algebras. Denote the ring of polynomials in $\lambda \in \mathbb{C}^{2g}$ by $\mathcal{P}$. Let us consider the polynomial map $p : \mathbb{C}^{3g} \rightarrow \mathbb{C}^{2g}$. A vector field $\mathcal{L}$ in $\mathbb{C}^{3g}$ will be called projectable for $p$ if there exists a vector field $L$ in $\mathbb{C}^{2g}$ such that

$$\mathcal{L}(p^*f) = p^*L(f) \quad \text{for any } f \in \mathcal{P}.$$  

The vector field $L$ will be called the pushforward of $\mathcal{L}$. A corollary of this definition is that for a projectable vector field $\mathcal{L}$ we have $\mathcal{L}(p^*\mathcal{P}) \subset p^*\mathcal{P}$.  

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Problem 2.2 ([12, Problem 6.1]) Find $3g$ polynomial vector fields in $\mathbb{C}^{3g}$ projectable for $p : \mathbb{C}^{3g} \to \mathbb{C}^{2g}$ and independent at any point in $p^{-1}(\mathcal{B})$. Construct their polynomial Lie algebra.

The connection of this problem to Problem 1.1 is straightforward. Given a solution to Problem 2.2 for each of the $3g$ vector fields $\mathcal{L}_k$ with pushforwards $L_k$, we will restore the vector fields $\mathcal{L}_k$ projectable for $\pi$ with pushforwards $L_k$ and such that $\mathcal{L}_k(\phi^s x_{i,j}) = \phi^s \mathcal{L}_k(x_{i,j})$ for the coordinate functions $x_{i,j}$ in $\mathbb{C}^{3g}$. As $\phi^s x_{i,j}$ are the generators of $\mathcal{F}$ and $\mathcal{L}_k(x_{i,j})$ is a polynomial in $x_{i,j}$, this gives $\mathcal{L}_k(\phi^s x_{i,j}) \in \mathcal{F}$ and $\mathcal{L}_k \in \text{Der } \mathcal{F}$.

The plan to solve Problem 2.2 is the following. For each $g$:

- Find the “odd polynomial vector fields”, i.e., the $g$ independent polynomial vector fields $\mathcal{L}_1, \mathcal{L}_3, \ldots, \mathcal{L}_{2g-1}$ projectable for $p$ with zero pushforward.
- Define $2g$ independent polynomial vector fields $L_0, L_2, L_4, \ldots, L_{4g-2}$ in $\mathcal{B}$.
- Find the “even polynomial vector fields”, i.e., the $2g$ polynomial vector fields $\mathcal{L}_0, \mathcal{L}_2, \mathcal{L}_4, \ldots, \mathcal{L}_{4g-2}$ projectable for $p$ with pushforwards $L_0, L_2, L_4, \ldots, L_{4g-2}$.
- Construct their polynomial Lie algebra.

We will do these steps in the following sections. Namely, in Sect. 3 we deal with the first step, while Sect. 4 gives an approach to the third step for some classical vector fields taken in the second step. It is based on a condition for the polynomial Lie algebra in the last step. In Sect. 5 we give the explicit solutions to Problem 1.1 that can be constructed by this method (see [12]).

3 Odd polynomial vector fields

Lemma 3.1 ([12, Lemmas 6.2 and 6.3]) We have

$$\mathcal{L}_1 = \sum_j x_{2,j} \frac{\partial}{\partial x_{1,j}} + x_{3,j} \frac{\partial}{\partial x_{2,j}} + 4(2x_2x_{2,j} + x_3x_{1,j} + x_{2,j+2}) \frac{\partial}{\partial x_{3,j}}$$

where $x_{2,2g+1} = 0$. For $s = 3, 5, \ldots, 2g - 1$ we have

$$\mathcal{L}_s = x_{2,s} \frac{\partial}{\partial x_2} + x_{3,s} \frac{\partial}{\partial x_3} + \mathcal{L}_1(x_{3,s}) \frac{\partial}{\partial x_4} + \sum_{k=1}^{g-1} z_{1,s,2k+1} \frac{\partial}{\partial x_{1,2k+1}}$$

$$+ \mathcal{L}_1(z_{1,s,2k+1}) \frac{\partial}{\partial x_{2,2k+1}} + \mathcal{L}_1(\mathcal{L}_1(z_{1,s,2k+1})) \frac{\partial}{\partial x_{3,2k+1}}$$

for some $y_{1,s,2k+1} = \mathcal{L}_s(x_{1,2k+1})$.

This lemma determines the odd polynomial vector fields, given the value $\mathcal{L}_s(x_{1,2k+1})$. For this value we use the construction of Korteweg–de Vries hierarchy [4, Section 4.4].

The Korteweg–de Vries equation

$$u_t = 6uu_x - u_{xxx}$$
for \( x = z_1, -4t = z_3, \Phi_2 = \frac{1}{2} u, \Phi_4 = -\frac{3}{2} \Phi_2^2 + \frac{1}{4} \partial_1 \Phi_2 \) takes the form
\[
\partial_3 \Phi_2 = \partial_1 \Phi_4.
\]
It is the first equation of the Korteweg–de Vries hierarchy, which is an infinite system of differential equations
\[
\partial_{2k-1} \Phi_2 = \partial_1 \Phi_{2k}, \quad k = 2, 3, 4, \ldots,
\]
where
\[
\partial_1 \Phi_{2k+2} = \mathcal{R} \partial_1 \Phi_{2k} \quad \text{and} \quad \mathcal{R} = \frac{1}{4} \partial_1^2 - 2 \Phi_2 - \Phi'_2 \partial_1^{-1}.
\]

**Theorem 3.2** ([4, Theorem 4.12]) *The function \( u = 2\wp_2(z) \) is a g-gap solution of the Korteweg–de Vries system.*

This gives us a system of equations
\[
\mathcal{L}_s(x_2) = \mathcal{L}_1 \Phi_s(x_2)
\]
with differential polynomials \( \Phi_s \). Thus in Lemma 3.1 we have
\[
\mathcal{L}_s(x_{1,2k+1}) = \mathcal{L}_s(\Phi_{2k}(x_2)).
\]
This determines \( y_{1,2k+1} \).

### 4 Even polynomial vector fields

First we define the polynomial vector fields in \( \mathcal{B} \). Recall \( \mathcal{B} = \mathbb{C}^{2g} \setminus \Sigma \) where \( \Sigma \) is the discriminant hypersurface. For the vector fields \( L_0, L_2, L_4, \ldots, L_{4g-2} \) in \( \mathcal{B} \) we take the vector fields tangent to \( \Sigma \), that are obtained from the convolution of invariants of the group \( A_{2g} \), see the construction by Fuchs in [1, Section 4]. See also [8,11].

We consider \( \mathbb{C}^{2g} \) with coordinates \( (\lambda_4, \lambda_6, \ldots, \lambda_{4g}, \lambda_{4g+2}) \) and set \( \lambda_s = 0 \) for every \( s \notin \{4, 6, \ldots, 4g, 4g+2\} \). For \( k, m \in \{1, 2, \ldots, 2g\}, k \leq m \) set
\[
T_{2k,2m} = 2(k + m) \lambda_{2k+2m} + \sum_{s=2}^{k-1} 2(k + m - 2s) \lambda_{2s} \lambda_{2k+2m-2s} - \frac{2k(2g - m + 1)}{2g + 1} \lambda_{2k} \lambda_{2m},
\]
and for \( k > m \) set \( T_{2k,2m} = T_{2m,2k} \). For \( k = 0, 1, 2, \ldots, 2g - 1 \) we have the vector fields
\[
L_{2k} = \sum_{s=2}^{2g+1} T_{2k+2,2s-2} \frac{\partial}{\partial \lambda_{2s}}. \tag{4}
\]
The expressions (4) give polynomial vector fields tangent to the discriminant hypersurface.

Now we need to find polynomial vector fields $\mathcal{L}_{2k}$ projectable for $p$ with pushforwards $L_{2k}$. The vector field $\mathcal{L}_0$ is the Euler vector field on $\mathbb{C}^{3g}$, we have

$$\mathcal{L}_0 = \sum_j (j + 1) x_{1,j} \frac{\partial}{\partial x_{1,j}} + (j + 2) x_{2,j} \frac{\partial}{\partial x_{2,j}} + (j + 3) x_{3,j} \frac{\partial}{\partial x_{3,j}}.$$ 

All the other vector fields are determined using the condition on the polynomial Lie algebra

$$\begin{pmatrix}
[L_1, L_0] & [L_1, L_2] & [L_1, L_4] & [L_1, L_6] & \cdots & [L_1, L_{4g-4}] & [L_1, L_{4g-2}]
\end{pmatrix} =
\begin{pmatrix}
-1 & 0 & 0 & \cdots & 0
x_{1,1} & -1 & 0 & \cdots & 0
x_{1,3} & x_{1,1} & -1 & \cdots & 0
x_{1,5} & x_{1,3} & x_{1,1} & \cdots & 0
\cdots & \cdots & \cdots & \cdots & \cdots
0 & 0 & \cdots & x_{1,2g-3} & x_{1,2g-1}
0 & 0 & \cdots & 0 & x_{1,2g-1}
\end{pmatrix} \cdot
\begin{pmatrix}
L_1
L_3
\vdots
L_{2g-1}
\end{pmatrix}.$$ 

A demonstration of this method for genus $g = 4$ will follow in our upcoming works.

### 5 Explicit solutions of the problem of differentiation of hyperelliptic functions

#### 5.1 Genus 1

See [14]. The generators of the $\mathcal{F}$-module $	ext{Der} \mathcal{F}$ are

$$\mathcal{L}_0 = L_0 - \zeta_1 \partial_1, \quad \mathcal{L}_1 = \partial_1, \quad \mathcal{L}_2 = L_2 - \zeta_1 \partial_1.$$ 

Their Lie algebra is $[\mathcal{L}_0, \mathcal{L}_1] = \mathcal{L}_1, [\mathcal{L}_0, \mathcal{L}_2] = 2 \mathcal{L}_2, [\mathcal{L}_1, \mathcal{L}_2] = \varphi_2 \mathcal{L}_1$.

#### 5.2 Genus 2

The generators of the $\mathcal{F}$-module $	ext{Der} \mathcal{F}$ are (see [3, Theorem 29]):

$$\mathcal{L}_0 = L_0 - \zeta_1 \partial_1 - 3 \zeta_3 \partial_3, \quad \mathcal{L}_2 = L_2 + \left( - \zeta_1 + \frac{4}{5} \lambda_4 \zeta_3 \right) \partial_1 - \zeta_1 \partial_3,$$

$$\mathcal{L}_1 = \partial_1, \quad \mathcal{L}_4 = L_4 + \left( - \zeta_3 + \frac{6}{5} \lambda_6 \zeta_3 \right) \partial_1 - (\zeta_1 + \lambda_4 \zeta_3) \partial_3,$$

$$\mathcal{L}_3 = \partial_3, \quad \mathcal{L}_6 = L_6 + \frac{3}{5} \lambda_8 \zeta_3 \partial_1 - \zeta_3 \partial_3.$$ 

Their Lie algebra can be found in [3, Theorem 32].
5.3 Genus 3

The generators of the $\mathcal{F}$-module $\text{Der} \mathcal{F}$ are (see [12, Theorem 10.1]):

\[
\begin{align*}
\mathcal{L}_1 &= \partial_1, \quad \mathcal{L}_3 = \partial_3, \quad \mathcal{L}_5 = \partial_5, \\
\mathcal{L}_0 &= L_0 - z_1 \partial_1 - 3z_3 \partial_3 - 5z_5 \partial_5, \\
\mathcal{L}_2 &= L_2 - \left( \xi_1 - \frac{8}{7} \lambda_4 z_3 \right) \partial_1 - \left( z_1 - \frac{4}{7} \lambda_4 z_5 \right) \partial_3 - 3z_3 \partial_5, \\
\mathcal{L}_4 &= L_4 - \left( \xi_3 - \frac{12}{7} \lambda_6 z_3 \right) \partial_1 - \left( \xi_1 + \lambda_4 z_3 - \frac{6}{7} \lambda_6 z_5 \right) \partial_3 - (z_1 + 3\lambda_4 z_5) \partial_5, \\
\mathcal{L}_6 &= L_6 - \left( \xi_5 - \frac{9}{7} \lambda_8 z_3 \right) \partial_1 - \left( \xi_3 - \frac{8}{7} \lambda_8 z_5 \right) \partial_3 - (\xi_1 + \lambda_4 z_3 + 2\lambda_6 z_5) \partial_5, \\
\mathcal{L}_8 &= L_8 + \left( \frac{6}{7} \lambda_{10} z_3 - \lambda_{12} z_5 \right) \partial_1 - \left( \xi_5 - \frac{10}{7} \lambda_{10} z_5 \right) \partial_3 - (\xi_3 + \lambda_8 z_5) \partial_5, \\
\mathcal{L}_{10} &= L_{10} + \left( \frac{3}{7} \lambda_{12} z_3 - 2\lambda_{14} z_5 \right) \partial_1 + \frac{5}{7} \lambda_{12} z_5 \partial_3 - \xi_5 \partial_5.
\end{align*}
\]

Their Lie algebra can be found in [12, Corollary 10.2].

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