A practical method in calculating one loop quantum fluctuations to the energy of the non-topological soliton

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I have used a practical method to calculate the one-loop quantum correction to the energy of the non-topological soliton in Friedberg-Lee model. The quantum effects which come from the quarks of the Dirac sea scattering with the soliton bag are calculated by a summation of the discrete and continuum energy spectrum of the Dirac equation in the background field of soliton. The phase shift of the continuum spectrum is numerically calculated in an efficient way and all the divergences are removed by the same renormalization procedure.

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Non-topological soliton models which are effective models inspired from the underlying QCD theory are phenomenologically successful in describing the low energy nuclear physics. However, the main calculation methods in these models are based on mean field approximation, in other words treating the fields classically [1–4]. The quantum corrections in the background fields of spatially non-trivial configurations are very difficult to calculate. This is partly due to the fact that these calculations are nonlocal. During the past decades different calculation methods and approximate schemes have been developed on this problem [5–12]. As the calculation of quantum corrections of solitons is much more complex than those usual calculations of quantum loop corrections of trivial background fields, most studies on this problem are based on the derivative expansion method [3,4]. The renormalization in this method is a very nontrivial task. One remarkable calculation method was that developed by Farhi, Graham, Haagensen and Jaffe [12]. It is a systematic and efficient scheme for calculating the quantum corrections about static field configuration in renormalizable field theories, in which all the divergences are removed by the same renormalization procedure. As originally this method was applied in the Higgs like models and the main interest was focused on studying solitons in the standard electroweak models [13,14], there are no applications of this method, as far as I know, in strong interaction hadronic models, like the Friedberg-Lee(FL) model, the linear sigma model and other QCD effective models. In recent years topological solitons in strong interaction QCD theory have drawn lots of attentions [15,16]. One needs an efficient method to calculate the quantum correction of the soliton in effective QCD theories [17]. So in this paper as the first small step I want to introduce this method to calculate the one loop quantum fluctuation of the non-topological soliton in the FL model. In this method one makes the energy level summation by calculating the discrete and continuous energy spectrum and the continuum contribution is determined through evaluating scattering phase shift in a concise way. The renormalization of the field configuration energy could be done in a manner consistent with on-shell mass and coupling constant renormalization in the perturbative sector. Comparing to the precedent calculation technics in the literatures this method is more efficient and practical.

Consider the Lagrangian of the FL model,

$$\mathcal{L} = \bar{\psi}(i\gamma_{\mu}\partial^{\mu} - g\sigma)\psi + \frac{1}{2}(\partial_{\mu}\sigma)(\partial^{\mu}\sigma) - U(\sigma),$$

(1)

where

$$U(\sigma) = \frac{1}{2!}a\sigma^2 + \frac{1}{3!}b\sigma^3 + \frac{1}{4!}c\sigma^4 + B.$$  

(2)

ψ represents the quark field, and σ denotes the phenomenological scalar field. a, b, c, and g are the constants which are generally fitted in producing the properties of hadrons properly. B is the bag constant. In the background of a nontrivial σ field there might be some bound state levels with energy 0 < E_n < m which can allocate the quarks lowering the energy of the whole system at the expense of creating such nontrivial configuration of σ field. This is the non-topological soliton solution in the FL model. Generally speaking the spherical configuration of the σ field will take the following form,

$$\sigma(r) = \sigma_v - \frac{\sigma_0}{1 + e^{(r-R)/r_v}},$$

(3)

where the second term on the r.h.s of the equation is a Woods-Saxon potential well with depth σ_0. Inside a sphere of radius R the σ field almost vanish, while outside the well it takes its asymptotic vacuum value σ_v. The valence quarks are bounded in the well and form a classical soliton. By fitting the hadron properties the model parameters could be fixed but not uniquely. There are some flexibilities in choosing the parameters. For baryons if one takes N = 3 and chooses one set of values of parameters as a = 17.7 fm^{-2}, b = -1457.4 fm^{-1}, c = 20000, g = 12.16 [2], one obtains a classical soliton energy E_cl ≈ 6.4 fm^{-1} ≈ 1262 MeV.

Next I will study the quantum correction of the classical soliton. In principle the quantum corrections in FL model should include loop corrections from both quark fields and the σ field. However since the σ field is only
a phenomenological field describing the long-range collective effects of QCD, the loop corrections coming from the sigma field will be ignored. So I just consider one loop fluctuations from the quark field in a static nontrivial configuration of $\sigma$ field background. In this case the one loop effective action after integrating out the quark field is given by

$$S_{\text{eff}} = S_{\text{ct}} + S_{\text{ct}} - i \log \det D$$

(4)

where $S_{\text{ct}}$ is the classical part, $S_{\text{ct}}$ is the counterterm part and $D$ is the Dirac operator which general form is $D = i\gamma_\mu \partial^\mu - g\sigma(r)$. The total energy could be derived by $E_{\text{tot}} = -S_{\text{eff}} / \int dt$ and the result is

$$E_{\text{tot}} = E_{\text{ct}} + E_{\text{ct}} + 4\psi_{\text{vac}}$$

(5)

where $E_{\text{ct}}$ is the necessary renormalization counterterm and $E_{\text{vac}}$ is the vacuum correction as a result of the energy level summation from both discrete and continuum spectrum. The whole energy spectrum is determined by the following stationary dirac equation

$$[-i\alpha \cdot \slashed{\nabla} + \beta g\sigma(r)]\psi = E\psi.$$  

(6)

One could solve the Dirac equation and get the continuous energy spectrum $E(k) = \sqrt{k^2 + m^2}$ where $m = g\sigma_v$ and some possible discrete energy spectrum $0 < E_n < m$, thus the energy level sum over discrete and continuous spectrum is

$$E_{\text{vac}} = -\sum_n E_n - \sum_l (2l + 1) \int dk \rho_l(k)E(k),$$

(7)

where $\rho_l(k)$ is the density of states in momentum space with the angular momentum quantum number $l$ and $(2l + 1)$ is the degenerate factor of the angular momentum projection. The density of states $\rho_l(k)$ will relate to the scattering phase shift $\delta_l(k)$ in the following way [12]

$$\rho_l(k) = \rho_l^{\text{free}}(k) + \frac{1}{\pi} \frac{\delta_l(k)}{dk},$$

(8)

where $\rho_l^{\text{free}}(k)$ is the density of states when the background $\sigma$ field is trivial. In our case this part will be subtracted from the density of states since I only consider the quantum corrections of the nontrivial background $\sigma$ field.

The main difficulties come from the calculations of the scattering phase shift $\delta_l(k)$ and the renormalization. To eliminate the divergence of the integral over continuum spectrum in equation (7) the phase shift needs to be rendered by a Born approximation according to the stand method in quantum mechanics. In one loop calculation only the first and second Born approximation should be subtracted from the phase shift. Therefore the subtracted phase shift is defined as

$$\delta_l(k) \equiv \delta_l(k) - \delta_l^{(1)}(k) - \delta_l^{(2)}(k),$$

(9)

in which $\delta_l^{(1)}(k)$ and $\delta_l^{(2)}(k)$ are the first and second Born approximations to $\delta_l(k)$. These phase shifts can be determined by solving the equation (6). In order to solve it one need to decompose the quark field into

$$\psi(r) = \frac{1}{r} \left( \frac{F(r)}{i\sigma \cdot \hat{r}G(r)} \right) y_{\kappa m},$$

(10)

where $y_{\kappa m} \equiv y_{\kappa m}^1$ is the two-component Pauli spinor harmonic, $\kappa$ is the Dirac quantum number $\kappa = -(l+1)$ and $\hat{r}$ is the spatial unit vector. Substitute it into the equation (6) one obtains two coupled first order radial equations of upper component $F$ and lower component $G$. These two equations can be decoupled to two second order differential equations about $F$ and $G$. One can use either of them to evaluate the phase shift. The equation of upper component $F$ is

$$F'' - \frac{g\sigma'}{E + g\sigma} F' - \left[ \frac{\kappa - g\sigma'}{r} \right] F + \left[ \frac{\kappa(k + 1)}{r^2} - (E^2 - g^2 \sigma^2) \right] F = 0,$$

(11)

where the prime denotes the differentiation with respect to $r$. In the following I will use this equation to calculate the phase shift. When $r >> R$ the asymptotic form of equation (11) is

$$F'' - \left[ \frac{\kappa(k + 1)}{r^2} - k^2 \right] F = 0,$$

(12)

where $k^2 = E^2 - g^2 \sigma^2$. The solutions will be spherical Hankel functions. At the same time for equation (11) the solution should satisfy that $F(r) \to 0$ as $r \to 0$. Thus one could introduce two linearly independent solutions to equation (11) as

$$F_l^{(1)}(r) = e^{i\beta_l(k,r)r}h_l^{(1)}(kr),$$

(13)

$$F_l^{(2)}(r) = e^{-i\beta_l(k,r)r}h_l^{(2)}(kr),$$

(14)

where $h_l^{(1)}(kr)$ and $h_l^{(2)}(kr)$ are the Hankel functions of the first and second kinds and $h_l^{(1)*}(kr) = h_l^{(1)}(kr)$. The function $\beta_l(k,r)$ should satisfy $\beta_l(k,r) \to 0$ as $r \to \infty$. Then the scattering solution is

$$F_l(r) = F_l^{(2)}(r) + e^{i\beta_l(k)}F_l^{(1)}(r),$$

(15)

and obeys $F_l(0) = 0$, which leads to the result of the scattering phase shift

$$\delta_l(k) = -2\text{Re}\beta_l(k,0),$$

(16)

where Re means the real part. By substituting $F_l^{(1)}$ into equation (11) one could obtain the equation of $\beta_l$

$$i\beta_l''r h_l + 2i\beta_l'(h_l + rh_l') - \beta_l^2 r h_l - \frac{g\sigma'}{E + g\sigma} (i\beta_l' r h_l + h_l$$

$$+ r h_l') - \left[ \frac{\kappa - g\sigma'}{E + g\sigma} + g^2 (\sigma^2 - \sigma_v^2) \right] h_l = 0.$$  

(17)
In the fixed background soliton field of $\sigma(r)$ this equation could be numerically solved to obtain the phase shift $\delta_l(k)$. To get the Born approximation to the phase shift one should expand $\beta_l$ in powers of $g$ as

$$\beta_l = g\beta_1 + g^2\beta_2 + \cdots.$$ \hspace{1cm} (18)

Substituting the expansion into equation (17) and neglecting the higher order terms $O(g^3)$ one can obtain a set of coupled differential equations about $\beta_1$ and $\beta_2$ as

$$i\beta''_1 rh_1 + (2i\beta'_1 - \frac{\sigma'}{E})(h_1 + rh'_1) - \frac{\kappa \sigma'}{E} h_1 = 0, \hspace{1cm} (19)$$

$$i\beta''_2 rh_2 - \beta'_1 rh_1 - \frac{\sigma'}{E} \beta'_2 rh_1 + (2i\beta'_2 + \frac{\sigma' \sigma}{E^2})(h_1 + rh'_1) + \left[ \frac{\kappa \sigma'}{r E^2} - (\sigma^2 - \sigma^2_\sigma) \right] rh_1 = 0. \hspace{1cm} (20)$$

These equations could be numerically solved to obtain the first and second Born approximations of the phase shift namely $\delta^{(1)}_l$ and $\delta^{(2)}_l$ as

$$\delta^{(1)}_l = -2g \text{Re}\beta_1(k, r = 0), \hspace{1cm} \delta^{(2)}_l = -2g^2 \text{Re}\beta_2(k, r = 0). \hspace{1cm} (21)$$

Finally the subtracted phase shift $\bar{\delta}_l$ can be determined by equation $\text{(9)}$.

In Ref. 13 it is shown that these subtractions are added back into the energy by using their explicit diagrammatic representation in terms of divergent diagrams as one and two insertions of $\sigma$ field to the fermion loop, which will be renormalized by the counterterms $E_{\text{ct}}$ and yield a finite contribution denoted by $\Gamma_2$. Thus one has the renormalized one loop quantum correction energy

$$E_{\text{ren}} = -\sum_n E_n - \sum_l (2l + 1) \int \frac{d^4k}{(2\pi)^4} \frac{1}{\pi} \frac{d\delta_l(k)}{dk} E(k) + \Gamma_2. \hspace{1cm} (22)$$

The detail calculation of the renormalized energy part $\Gamma_2$ is in the following. The only divergent Feynman diagram which needs to be evaluated could be expressed in the following form

$$i\Pi(q^2) = -g^2 \int \frac{d^4p}{(2\pi)^4} \text{Tr} \left[S(p + q)S(p)\right] \hspace{1cm} (23)$$

Where $S(p)$ is the fermion propagator. By the standard dimensional regulation one obtains the regulated result

$$\Pi(q^2) = \frac{g^2}{4\pi^2} \left\{ \frac{1}{\mu^2} (q^2 + 2m^2) + \frac{1}{6} q^2 + m^2 \hspace{1cm} \right\} - \int_0^1 dx \left[ 3x(1 - x)q^2 + m^2 \right] \ln \frac{D}{\mu^2}, \hspace{1cm} (24)$$

where $D = x(1 - x)q^2 + m^2$. The divergent part can be renormalized by a on-shell mass renormalization which means

$$\Pi_{\text{ren}}(q^2) |_{q^2 = -m^2} = 0, \hspace{1cm} \frac{d\Pi_{\text{ren}}(q^2)}{dq^2} \bigg|_{q^2 = -m^2} = 0. \hspace{1cm} (25)$$

where $m_\sigma$ is taken as the usual sigma meson mass $m_\sigma = 550\text{MeV}$. The divergent parts are removed by the counter terms. Then the renormalized result is

$$\Pi_{\text{ren}}(q^2) = \frac{g^2}{4\pi^2} \left\{ \int_0^1 dx \left[ 3x(1 - x)q^2 + m^2 \right] \frac{\ln m^2 + x(1 - x)q^2}{m^2 - x(1 - x)m^2_\sigma} + (q^2 + m^2_\sigma) \right\} \int_0^1 dx (1 - x) \frac{3x(1 - x)m^2_\sigma - m^2}{m^2 - x(1 - x)m^2_\sigma}. \hspace{1cm} (26)$$

In the following calculation $I$ will change the four momentum to the three momentum as $q = |\vec{q}|$ by setting $q_0 = 0$. Now the finite energy term $\Gamma_2$ can be evaluated as

$$\Gamma_2 = \int_0^\infty \frac{q^2 dq}{2\pi^2} \Pi_{\text{ren}}(q^2) \tilde{\sigma}(q)^2, \hspace{1cm} (27)$$

where $\tilde{\sigma}(q)$ is the Fourier transform of $\sigma(\vec{r})$ which result is

$$\tilde{\sigma}(q) = \frac{4\pi^2 r_0 \sigma_0}{q \sinh(\pi q r_0)} R \cos(q R) - \frac{\pi r_0}{\tanh(\pi q r_0)} \sin(q R). \hspace{1cm} (28)$$

Notice that the homogeneous background field $\sigma_v$ is subtracted from $\sigma(r)$ because it will generate the energy from the scattering effect to the homogeneous vacuum background which is infinite and not relevant to our physical result. Substituting the above result of $\tilde{\sigma}(q)$ into equation (27) together with the result of $\Pi_{\text{ren}}(q^2)$ from equation (26), the renormalized energy part $\Gamma_2$ can be numerically calculated.

The numerical result of the phase shift $\delta_l(k)$ is presented in figure (I(a)). It could be seen that the amplitudes of the phase shift decrease with $l$ increasing. However at $k \to \infty$ the phase shift decreases with momentum increasing in a logarithmic way and approaches zero very slowly. By subtracting the Born approximation $\delta^{(1)}_l$ and $\delta^{(2)}_l$ one obtain the subtracted phase shift $\bar{\delta}_l(k)$. In figure (I(b)) the numerical result of $\bar{\delta}_l(k)$ is presented. When $k \to \infty$ they all approach zero exponentially which makes the integral over momentum finite in equation (22). Here the energy term associated with different angular momentum $l$ can be defined as

$$E^{(l)} = \int \frac{1}{\pi} \frac{d\bar{\delta}_l(k)}{dk} E(k), \hspace{1cm} (29)$$

The numerical results of them are in the following

$$E^{(0)} = -4.26 \text{fm}^{-1}, \hspace{1cm} E^{(1)} = -0.53 \text{fm}^{-1}, \hspace{1cm} E^{(2)} = -0.3 \text{fm}^{-1}, \hspace{1cm} E^{(3)} = -0.19 \text{fm}^{-1}, \hspace{1cm} \ldots \hspace{1cm} (30)$$

Additionally there is only one bound state which energy is $E_1 = 1.6 \text{fm}^{-1}$. However the energy level of this bound state has already been occupied by the three valence quarks, so this bound state does not contribute to the correction energy. The vacuum correction only comes from summing the continuum energy spectrum of the
scattering states. Considering the summation of the energy over \( l \) from \( l = 0 \) to \( l = 3 \) together with \( \Gamma_2 \) which is \( \Gamma_2 = -0.42 \, fm^{-1} \) the renormalized quantum correction energy will be \( E_{\text{vac}}^{\text{ren}} \sim 8.26 \, fm^{-1} \) which magnitude has already exceeded the value of the classical energy which is \( E_{\text{cl}} \approx 6.4 \, fm^{-1} \), not to mention the energy terms with \( l > 3 \). Thus the one loop quantum correction is quite large and will not support the existence of the soliton in FL model in this context.

In this paper I have used a practical method to study the quantum fluctuations of the non-topological solitons in FL model. One could see that it is efficient and all the divergences have been removed by the same renormalization procedure. At the present level I just focus the study at the zero temperature case. However an interesting issue is the quantum correction of soliton at finite temperature. In that case one can even study the quantum fluctuations of solitons during the deconfinement phase transition in FL model. These calculations are doable and will be a separate work deserving thorough discussion in the next step. Another interesting issue is that the practical method could be further extended to the chiral soliton model and other soliton models based on QCD theory. In recent years the topological solitons in QCD, like instantons and dyons, have been actively studied by Shuryak and Zahed \[14\,15]. These nontrivial vacuum structures of QCD are believed to be able to produce both confinement and chiral symmetry breaking. Most of the calculations on instantons and dyons are semi-classical. The quantum fluctuations are also important in these systems, especially during the phase transition. However the calculations of the quantum fluctuations of the QCD dyon ensemble at finite temperature will be a very challenging task. All these issues are under consideration and will be studied in the future work.

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