HOMOGENIZATION VIA SPRINKLING

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Abstract. We show that a superposition of an $\epsilon$-Bernoulli bond percolation and any everywhere percolating subgraph of $\mathbb{Z}^d$, $d \geq 2$, results in a connected subgraph, which after a renormalization dominates supercritical Bernoulli percolation. This result, which confirms a conjecture from [BHS00], is mainly motivated by obtaining finite volume characterizations of uniqueness for general percolation processes.

1. Introduction

Consider a deterministic subset $X$ of the edges of the standard $d$-dimensional lattice $\mathbb{Z}^d$, $d \geq 2$. Assume that $X$ is percolating everywhere, meaning that every vertex of $\mathbb{Z}^d$ is in an infinite connected component of the graph $(\mathbb{Z}^d, X)$. Example of such graphs are foliation by lines or spanning forests. Consider then the random set of edges $Y = X \cup \omega$, obtained by adding to $X$ the open edges $\omega$ of a Bernoulli percolation with density $\epsilon > 0$. We prove that for every choice of $X$ and every $\epsilon > 0$, the graph $Y$ is almost surely connected and has large scale geometry similar to that of supercritical Bernoulli percolation. In [BHS00], this result was already proved in dimension $d = 2$, and conjectured for higher dimensions. The proof of [BHS00] for $d = 2$ relies strongly on planar duality, and cannot extend to higher dimensions. In this paper, we develop new robust methods that allow us to extend the result of [BHS00] to any dimension $d \geq 2$. This is the content of Theorem 1 below. The main step in our proof is of independent interest (see Lemma 1.1). We obtain a finite-volume characterization for the uniqueness of the infinite connected component in $Y$. More precisely, we show that with high probability, all the points in the ball of radius $n$ are connected by a path of $Y$ which lies inside the ball of radius $2n$. This finite-size criterion approach to uniqueness is in the same spirit of the original proof of uniqueness for Bernoulli percolation (see [AKN87] and the recent work of [Cer13]).

As a consequence of the finite-size criterion mentioned above, we show that a renormalized version of $Y$ dominates highly supercritical percolation. In particular, $Y$ percolates in sufficiently thick slabs and in half-spaces. This result is analogous to the Grimmett-Marstrand theorem [GM90], and we expect most of the properties of supercritical percolation to hold for $Y$ (see [Gri09], Chapters 7 and 8). The Grimmett-Marstrand theorem is a fundamental and powerful tool in supercritical Bernoulli percolation, but its proof does not provide directly quantitative estimates and relies on the specific symmetries of $\mathbb{Z}^d$ (see [MT13] for an extension to some graphs with less symmetries than the canonical $\mathbb{Z}^d$).

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We hope that the method in the present paper could be useful to obtain quantitative and robust proofs of the Grimmett-Marstrand theorem.

We do not require any symmetry hypothesis on the set $X$. Thus, the percolation process $Y$ is not necessarily invariant under the symmetries of $\mathbb{Z}^d$. Therefore, the uniqueness of the infinite cluster cannot be derived from the Burton-Keane theorem [BK89]. In this sense, our result can be seen as a generalization of the Burton-Keane theorem. Recently, Teixeira [Tei14] considered general percolation processes with high marginals on graphs with polynomial volume growth. Under some additional assumptions but without requiring symmetries or invariance, he obtains uniqueness of the infinite cluster. Since $Y$ does not necessarily have high marginals, our uniqueness result is not implied by Teixeira’s work.

We also obtain that the critical value for Bernoulli percolation on $Y$ satisfies $p_c(Y) < 1$. This is related to the initial motivation in [BHS00] for introducing the random set $Y$. Proving that $p_c(Y) < 1$ can be seen as an intermediate question in order to understand under which conditions a random infinite subgraph $G$ of $\mathbb{Z}^d$ has $p_c(G) < 1$. Finding such conditions is very challenging, and related to famous open problems in percolation, e.g. the absence of infinite cluster at criticality for Bernoulli percolation (see [BHS00] for more details). We give related questions in Section 1.2.

1.1. Main results. We prove the following theorem, which confirms a conjecture of Benjamini, Häggström and Schramm [BHS00]. (If needed, see Section 1.4 for notation and definitions.)

**Theorem 1.** Let $X$ be a fixed everywhere percolating subgraph of $\mathbb{Z}^d$, and let $Y = X \cup \omega$ be obtained from $X$ by adding an $\varepsilon$-percolation $\omega$. For any $\varepsilon > 0$, the following hold.

(i) The subgraph $Y$ is connected a.s.
(ii) The critical parameter for Bernoulli percolation on $Y$ satisfies $p_c(Y) < 1$ a.s.
(iii) The subgraph $Y$ percolates in the upper half-space a.s.
(iv) There exists $L = L(\varepsilon, d)$ such that $Y$ percolates in the slab $\mathbb{Z}^2 \times \{0, \ldots, L\}$ a.s.
(v) For every fixed $p < 1$, a renormalized version of $Y$ stochastically dominates a $p$-Bernoulli percolation.

The precise signification of Item (v) is the following. Define the percolation process $Y^{(n)}$ on $\mathbb{Z}^d$ by declaring an edge $e = \{x, y\} \in \mathbb{E}^d$ open if the vertex $2nx$ is $Y$-connected to $2ny$ inside $n(x + y) + \Lambda_{2n}$. Item (v) occurs if for every $p < 1$, there exists $n \geq 1$ such that the process $Y^{(n)}$ dominates stochastically a $p$-Bernoulli percolation (see [LSS97] for more details on stochastic domination).

**Remarks.**

a. As in [BHS00], a straightforward extension of our proof shows that an analogue of Theorem 1 holds if we only assume that $X$ is densely percolating. A subgraph $X$ is said to be densely percolating if there exists $R$ such that every box $2R, z + \Lambda_R$, $z \in \mathbb{Z}^2$, intersects an infinite component of $X$. In this framework, Item (i) needs to be replaced
by uniqueness of the infinite cluster, and the definition of renormalization in Item (v) has
to be slightly adapted.

b. Once we know that a rescaled version of $Y$ dominates supercritical Bernoulli perco-
lation, one could use the known results for Bernoulli percolation to obtain other properties
of $Y$ (see [Gri99], Chapters 7 and 8).

1.2. Questions. Let us begin by rewriting the question of [BHS00] that motivates the
problem studied here.

Question 1. Is there an invariant, finite energy percolation $X$ on $\mathbb{Z}^d$, which a.s. perco-
lates and satisfies $p_c(X) = 1$?

An invariant percolation is a probability measure on the percolation configuration that
is invariant under the symmetries of $\mathbb{Z}^d$. The finite energy property was considered in [NS81]. In the sense of Lyons and Schramm [LS99], it corresponds to insertion and
deletion tolerance: given an edge $e$, the conditional probability that $e$ is present (resp. absent) given the status of all the other edges is positive. Benjamini, Häggström and
Schramm [BHS00] showed that Question 1 has a positive answer if we replace the finite
energy condition by the insertion tolerance. They construct an insertion tolerant invariant
process $X$ (obtained by adding and $\varepsilon$-percolation to a well-chosen invariant percolation),
that percolates but satisfies $p_c(X) = 1$.

Adding an $\varepsilon$-percolation to a percolation process is an easy way to build insertion
tolerant processes. Of course, one may add a more general process instead. Our proof
of Theorem 1 uses strongly that our process was constructed by adding an $\varepsilon$-percolation.
With Question 1 in mind, it would be interesting to understand the effect of adding a
more general process. We suggest the following question.

Question 2. Let $X$ be a fixed everywhere percolating subgraph of $\mathbb{Z}^d$. Let $\eta$ be a perco-
lation process such that $\eta \neq \emptyset$ almost surely. Assume that $\eta$ is ergodic with respect to the
translations and invariant with respect to the whole automorphisms of the grid. Which
properties among (i), (ii), (iii), (iv) and (v) are satisfied by $Y := X \cup \eta$?

Let us give a particular case. In $\mathbb{Z}^3$, consider a superposition of two independent
ergodic invariant spanning forests. It is a.s. connected?

Another natural generalization of the problem treated in this paper is to consider
graphs other than the hypercubic lattice. In [BLPS01] it is shown that non-amenable
graphs, admit a spanning forest that stays disconnected after adding the edges of an
$\varepsilon$-percolation, for $\varepsilon$ sufficiently small. A positive answer to the following question would
show that, in the context of transitive graphs with $p_c < 1$, the existence of an everywhere
percolating disconnected subgraph that remains disconnected after adding an $\varepsilon$-sprinkling
is equivalent to non-amenability.

Question 3. Let $G$ be a transitive amenable graph with critical value for Bernoulli per-
colation satisfying $p_c(G) < 1$. Let $X$ be an everywhere percolating subgraph of $G$, and
let $Y = X \cup \omega$ be obtained from $X$ by adding an $\varepsilon$-percolation $\omega$. Is $Y$ connected almost
surely?
As an intermediate step toward Question 3, one can start first with transitive graphs of polynomial volume growth (a framework in which our methods are more likely to be adapted).

Another perspective is to study the simple random walk on a superposition of an everywhere percolating subgraph of the lattice and an independent sprinkling (for example, verify diffusivity in this framework). Adding the sprinkling can be viewed as homogenisation, this suggests to study spaces of harmonic functions on this environment (in the spirit of [BDKTY11]).

1.3. Organization of the paper. For the rest of the paper, we fix the values of \( d \geq 2 \) and \( \varepsilon > 0 \).

Constants. In the proof, we will introduce constants, denoted by \( C_0, C_1, \ldots \). By convention, the constants are elements of \((0, \infty)\), they may depend on \( d \) and \( \varepsilon \), but never depend on any other parameter of the model. In particular, they never depend on the chosen everywhere percolating subgraph \( X \), or the size of the box \( n \).

In Section 2, we study the effect of an \( \varepsilon \)-percolation on a finite highly connected graph. More precisely we consider a graph \( G \) with \( O(N^{2d}) \) vertices that is \( N \)-connected (\( G \) is connected and remains connected if we erase any set of \( N \) edges). We show that an \( \varepsilon \)-percolation on such a graph is connected with high probability.

The main new ideas are presented in Section 3, where we prove the following lemma.

**Lemma 1.1.** Let \( X \) be a fixed everywhere percolating subset of edges of \( \mathbb{Z}^d \). Let \( Y = X \cup \omega \) be obtained from \( X \) by adding an \( \varepsilon \)-percolation \( \omega \). For every \( n \geq 1 \), we have

\[
P \left[ \text{For all } x, y \in \Lambda_n, x \text{ is } Y \text{-connected to } y \text{ inside } \Lambda_{2n} \right] \geq 1 - C_3 e^{-C_1 \sqrt{n}} \tag{1}
\]

Let us give the strategy used to prove Lemma 1.1. The restriction of an everywhere percolating subgraph of \( \mathbb{Z}^d \) to the finite box \( \Lambda_n \) gives a partition of \( \Lambda_n \) into finite clusters. If we forget about possible small clusters at the boundary of \( \Lambda_n \) and assume that all the finite clusters partitioning \( \Lambda_n \) intersect the box \( \Lambda_{n-\sqrt{n}} \), then contracting these clusters result in a highly connected graph. Indeed, a non-trivial union of clusters must “cross” the annulus \( \Lambda_n \setminus \Lambda_{n-\sqrt{n}} \), which implies that its boundary must have size at least \( \sqrt{n} \). We can thus apply the result of Section 2. The main difficulty is to treat carefully the small clusters at the boundary of \( \Lambda_n \).

In Section 4, we deduce Theorem 1 from Lemma 1.1. That section uses standard renormalization and stochastic domination tools, which were already used by Benjamini, Häggström and Schramm [BHS00] to treat the case \( d = 2 \).

1.4. Notation, definitions. Let \((\mathbb{Z}^d, E^d)\), \( d \geq 2 \), be the standard \( d \)-dimensional hypercubic lattice. For \( r, R > 0 \) (not necessarily integer), we write \( \Lambda_r := [-r, r]^d \cap \mathbb{Z}^d \) and \( \Lambda_{r,R} := \Lambda_R \setminus \Lambda_r \). For \( z \in \mathbb{Z}^d \), we denote by \( \Lambda_r(z) := z + \Lambda_r \) the translate of \( \Lambda_r \) by vector \( z \).

In this paper, we call subgraph of \( \mathbb{Z}^d \) a set of edges \( X \subset E^d \), and identify it to the graph \((\mathbb{Z}^d, X)\). (Notice that the vertex set of every subgraph of \( \mathbb{Z}^d \) considered here will always be the entirety of \( \mathbb{Z}^d \).) We say that two vertices \( x, y \in \mathbb{Z}^d \) are \( X \)-connected, if there exists a sequence of disjoint vertices \( x_1, \ldots, x_\ell \in \mathbb{Z}^d \), such that \( x_1 = x \), \( x_\ell = y \), and for every
1 ≤ i < ℓ the edge \{x_i, x_{i+1}\} belongs to X. We say that x and y are X-connected inside S ⊂ Z^d if, in addition, all the x_i’s belong to S.

A subgraph X is said to be everywhere percolating if all its connected components are infinite. In other words, a subgraph is everywhere percolating if for every vertex x in Z^d, x is X-connected to infinity. We say that X percolates in S ⊂ Z^d if the graph induced by X on S contains an infinite connected component.

We call p-Bernoulli percolation (or simply p-percolation) the random subgraph ω of Z^d, constructed as follows: each edge of E^d is examined independently of the other and is declared to be an element of ω with probability p.

2. PERCOLATION ON A HIGHLY CONNECTED FINITE GRAPH

In this section, we show that an ε-percolation on any N-connected finite graph with O(N^2d) vertices connects all the vertices with large probability. Let us begin with the definitions needed to state the result. A finite multigraph is given by a pair G = (V, E): V is a finite set of vertices, and E is a multiset of pairs of unordered vertices (multiple edges and loops are allowed). We work with multigraphs rather than standard graphs because we will consider multigraphs obtained from other graphs by contraction, and we want to keep track of the multiple edges created by the contraction procedure. A multigraph G = (V, E) is said to be N-connected \(^1\) if it is connected, and removing any set of N edges does not disconnect it. In other words, for any subset F ⊂ E such that |F| ≤ N, the graph (V, E \ F) is connected. An ε-percolation on G is defined equivalently as on Z^d: it is a random set of edges ω ⊂ E such that the events \{e ∈ ω\} are independent, each of them having probability equal to ε.

**Proposition 2.** Let N ∈ N. Let G = (V, E) be an N-connected multigraph with |V| ≤ N^2d. Let ω ⊂ E be an ε-percolation on G. Then

\[ P[The \ graph \ (V, \omega) \ is \ connected] \geq 1 - C_2e^{-C_1N}. \]

We begin with the following lemma, which says that adding an (ε/4d)-percolation on a subgraph of G either connects all the vertices or shrinks the number of connected components by a factor smaller than 1/√N (with large probability). We will then prove the proposition by applying this lemma 4d times. Given X ⊂ E, we write K(X) for the number of connected components in the graph (V, X).

**Lemma 2.1.** Let ω_1 be an (ε/4d)-percolation on G. For every X ⊂ E, we have

\[ P[K(X \cup ω_1) > 1 \lor \frac{K(X)}{\sqrt{N}}] \leq C_0e^{-C_1N}, \]  

\[ \text{(2)} \]

**Proof.** We can assume that K(X) > 1 (if K(X) = 1, then Equation (2) is trivially true). We say that a set of vertices S ⊂ V generated by X, if it can be exactly written as a disjoint union

\[ S = S_1 \cup \cdots \cup S_m, \]

\[ \text{(3)} \]

\(^1\)In graph theory, the terminology “N-edge-connected” is used in this case to distinguish with vertex-connectivity.
where \( S_1, \ldots, S_m \) are disjoint connected components of \( X \). For such a set \( S \), we define 
\[
m(S) := m
\]

as the number of connected components in the decomposition \( (3) \). Notice that every non-empty set generated by \( X \) satisfies 
\[
1 \leq m(S) \leq K(X).
\]

The case \( m(S) = 1 \) corresponds to \( S \) being a single connected component of \((V, X)\), and \( m(S) = K(X) \) corresponds to \( S = V \).

If \( K(X \cup \omega_1) > 1 \lor K(X)/\sqrt{N} \), then there must exist a connected component \( S \subseteq V \) of \( X \cup \omega_1 \) that satisfies 
\[
1 \leq m(S) < \sqrt{N}.
\]

By the union bound, we obtain
\[
\mathbb{P} \left[ K(X \cup \omega_1) > 1 \lor \frac{K(X)}{\sqrt{N}} \right] \leq \sum_{S \subseteq V} \mathbb{P} \left[ \text{S is a connected component of } X \cup \omega_1 \right]. \tag{4}
\]

Now, if a non-empty set \( S \subseteq V \) is a connected component of \( X \cup \omega_1 \), then all the edges connecting a vertex of \( S \) to a vertex in \( V \setminus S \) must be \( \omega_1 \)-closed. Since the graph \( G \) is \( N \)-connected there are at least \( N \) such edges, and we obtain
\[
\mathbb{P} \left[ \text{S is a connected component of } X \cup \omega_1 \right] \leq (1 - \epsilon/4d)^N. \tag{5}
\]

Plugging Equation \( (5) \) in \( (4) \), we find
\[
\mathbb{P} \left[ K(X \cup \omega_1) > 1 \lor \frac{K(X)}{\sqrt{N}} \right] \leq \{S : 1 \leq m(S) < \sqrt{N}\} \leq (1 - \epsilon/4d)^N
\]
\[
\leq \sum_{1 \leq k < \sqrt{N}} \binom{K(X)}{k} (1 - \epsilon/4d)^N
\]
\[
\leq \sqrt{N} N^{2d/\sqrt{N}} (1 - \epsilon/4d)^N
\]
\[
\leq C_0 e^{-C_1 N}.
\]

In the third line we use that \( K(X) \leq |V| \leq N^{2d} \) to bound the binomial coefficient by \( N^{2d/\sqrt{N}} \).

Proof of Proposition 2. Let \( \omega_1, \ldots, \omega_{4d} \) be \( 4d \) independent \((\epsilon/4d)\)-percolations on \( G \), set \( \eta_0 = \emptyset \) and \( \eta_k := \omega_1 \cup \ldots \cup \omega_k \). By Lemma 2.1 we have, for all \( 1 \leq k \leq 4d \),
\[
\mathbb{P} \left[ K(\eta_k) > 1 \lor (K(\eta_{k-1})/\sqrt{N}) \right] \leq C_0 e^{-C_1 N}.
\]

Since \( K(\eta_0) = |V| \leq N^{2d} \), we find by induction
\[
\mathbb{P} \left[ K(\eta_k) > N^{2d-k/2} \right] \leq kC_0 e^{-C_1 N}.
\]

Setting \( k = 4d \) in the equation above, we obtain that \( \eta_{4d} \) is connected with probability larger than \( 1 - C_2 e^{-C_1 N} \). This concludes the proof, since \( \eta_{4d} \) is stochastically dominated by an \( \epsilon \)-percolation.
Let $X$ be an everywhere percolating subgraph of $\mathbb{Z}^d$, let $n \geq 1$. We define $C_{8dn}(X)$ as the set of connected components for the graph induced by $X$ on the box $\Lambda_{8dn}$ (two vertices are in the same connected component if they are $X$-connected inside $\Lambda_{8dn}$). Fix $n_0$ such that, for every $n \geq n_0$, the size the boundary of $\Lambda_{8dn}$ is smaller than $n^d$. (We call boundary of $\Lambda_n$ the set $\Lambda_n \setminus \Lambda_{n-1}$). Notice that $n_0$ depends only on the dimension $d$.

Since any element of $C_{8dn}(X)$ contains at least a vertex at the boundary of $\Lambda_{8dn}$, we also have, for every $n \geq n_0$,

$$|C_{8dn}(X)| \leq n^d. \tag{6}$$

For $0 < a \leq b \leq 8dn$, define $U_{a,b}(X)$ as the number of sets $C \in C_{8dn}(X)$ such that $C \cap \Lambda_a = \emptyset$ and $C \cap \Lambda_b \neq \emptyset$. In other words,

$$U_{a,b}(X) = \text{Card}\{C \in \Lambda_{8dn} : a < d(0,C) \leq b\},$$

where $d(0,C)$ denotes the $L^\infty$-distance between the origin and the set $C$. Define $U_{0,b}$ as the number of sets $C \in C_{8dn}(X)$ such that $C \cap \Lambda_b \neq \emptyset$. That is

$$U_{0,b}(X) = \text{Card}\{C \in \Lambda_{8dn} : d(0,C) \leq b\}.$$

Notice that $U_{a,b}(X)$ depends on $n$, but we keep this dependence implicit to lighten the notation.

**Lemma 3.1.** Fix an everywhere percolating subgraph $X$ of $\mathbb{Z}^d$. Let $n_0 \leq n \leq m \leq m + \sqrt{n} \leq 8dn$. Let $\omega$ be an $\varepsilon$-percolation restricted to $A_{m,m+2\sqrt{n}}$, then

$$\mathbb{P}\left[U_{0,m}(X \cup \omega) > 1 \lor U_{m,m+2\sqrt{n}}(X)\right] \leq C_2 e^{-C_1 \sqrt{n}}.$$

(In the statement of the lemma, the $\varepsilon$-percolation restricted to $A_{m,m+2\sqrt{n}}$ is defined by $\omega = \eta \cap A_{m,m+2\sqrt{n}}$, where $\eta$ is an $\varepsilon$-percolation in $\mathbb{Z}^d$.)

**Proof.** Let us first assume $U_{m,m+\sqrt{n}}(X) = 0$. In this case, the proof is easier and we directly show that

$$\mathbb{P}\left[U_{0,m}(X \lor \omega') > 1\right] \leq C_2 e^{-C_1 \sqrt{n}}, \tag{7}$$

where $\omega' = \omega \cap A_{m,m+\sqrt{n}}$ is the restriction of $\omega$ to the annulus $A_{m,m+\sqrt{n}}$.

Let $G$ be the graph obtained from $\Lambda_{m+\sqrt{n}}$ by the following contraction procedure.

1. Start with $\Lambda_{m+\sqrt{n}}$, with the standard graph structure induced by $\mathbb{Z}^d$
2. Examine the elements of $C_{8dn}(X)$ one after the other. For every $C \in C_{8dn}(X)$, contract all the points $x \in C \cap \Lambda_{m+\sqrt{n}}$ into one vertex.

Since $U_{m,m+\sqrt{n}}(X) = 0$, the graph $G$ has exactly $U_{0,m}(X)$ vertices, and $U_{0,m}(X \lor \omega')$ corresponds the number of connected components resulting from an $\varepsilon$-percolation on $G$. Therefore, Equation (7) follows from Proposition 2 applied to $G$ with $N = \sqrt{n}$. Indeed, the graph $G$ has at most $(\sqrt{n})^{2d}$ vertices by (4), and is $\sqrt{n}$-connected. To see that $G$ is $\sqrt{n}$-connected, observe that any non-trivial union of elements of $C_{8dn}(X)$ that intersect $\Lambda_{m+\sqrt{n}}$ must “cross” the annulus $A_{m,m+\sqrt{n}}$ (this follows from the hypothesis $U_{m,m+\sqrt{n}}(X) = 0$).
We now turn to the case $U_{m,m+\sqrt{n}}(X) > 0$, in which we show
\[
P \left[ U_{0,m}(X \cup \omega) \leq U_{m,m+2\sqrt{n}}(X) \right] \geq 1 - C_2 e^{-C_1 \sqrt{n}}. \tag{8}
\]
The strategy is very similar to the one used in the case $U_{m,m+\sqrt{n}}(X) = 0$, except that
here we need to consider carefully the clusters that intersect the annulus $A_{m,m+2\sqrt{n}}$ but
not the box $\Lambda_m$. More precisely, we partition the elements of $C_{8dn}(X)$ intersecting the box
$\Lambda_{m+2\sqrt{n}}$, into the following two types:

- the bulk-clusters, defined as the elements $C \in C_{8dn}(X)$ such that $C \cap \Lambda_m \neq \emptyset$;
- the boundary-clusters, defined as the elements $C \in C_{8dn}(X)$ such that $C \cap \Lambda_m = \emptyset$.

Notice that
\[
U_{0,m+2\sqrt{n}}(X) = U_{0,m}(X) + U_{m,m+2\sqrt{n}}(X),
\]
where $U_{0,m}(X)$ counts the bulk-clusters, and $U_{m,m+2\sqrt{n}}(X)$ counts the boundary-clusters.

Define $\tilde{C}$ as the union of all the boundary-clusters. The hypothesis $U_{m,m+\sqrt{n}}(X) > 0$
implies that at least one boundary-cluster intersects the annulus $A_{m,m+\sqrt{n}}$. Thus, we have
\[
\tilde{C} \cap \Lambda_{m+\sqrt{n}} \neq \emptyset.
\]
We now construct a $\sqrt{n}$-connected graph $G$ by the following contraction procedure.

1. Start with $\Lambda_{m+2\sqrt{n}}$, with the graph structure induced by $\mathbb{Z}^d$.
2. For every bulk-cluster $C \in C_{8dn}(X)$, contract all the points $x \in C \cap \Lambda_{m+2\sqrt{n}}$ into
   one vertex.
3. Contract all the vertices $x$ that belong to $\tilde{C} \cap \Lambda_{m+2\sqrt{n}}$ into one vertex.

To see that the graph is $\sqrt{n}$-connected, observe that all the bulk-clusters and $\tilde{C}$ cross
the annulus $A_{m+\sqrt{n},m+2\sqrt{n}}$. Proposition 2 applied with $N = \sqrt{n}$, ensures that an $\varepsilon$-
percolation on $G$ connects all its vertices with probability larger than $1 - C_2 e^{-C_1 \sqrt{n}}$. This
proves Equation (8), because the $\varepsilon$-percolation $\omega$ can be interpreted as an $\varepsilon$-percolation
on $G$, except that the $U_{m,m+2\sqrt{n}}(X)$ boundary-clusters were “artificially” merged into one
point in the construction of $G$.

\begin{lemma}
\textbf{Lemma 3.2.} Fix an everywhere percolating subgraph $X$ of $\mathbb{Z}^d$. Let $n_0 \leq n \leq m \leq m + 2n \leq 8dn$. Let $\omega$ be an $\varepsilon$-percolation restricted to the annulus $A_{m,m+2n}$; then
\[
P \left[ U_{0,m}(X \cup \omega) > 1 \vee \frac{U_{0,m+2n}(X)}{\sqrt{n}} \right] \leq C_2 e^{-C_1 \sqrt{n}}. \tag{9}
\]
\end{lemma}

\begin{proof}
First observe that
\[
U_{0,m}(X) + \sum_{i=0}^{[\sqrt{n}]-1} U_{m+2\sqrt{n},m+2(i+1)\sqrt{n}}(X) \leq U_{0,m+2n}(X).
\]
Among the $1 + [\sqrt{n}]$ terms summed on the left hand side, at least one of them must be
smaller than or equal to $U_{m,m+2n}(X)/\sqrt{n}$. If $U_{0,m}(X) \leq U_{m,m+2n}(X)/\sqrt{n}$, then Equation (9)
is trivially true. Otherwise, one can fix $i$ such that
$U_{m+2i\sqrt{n},m+2(i+1)\sqrt{n}}(X) \leq U_{m,m+2n}(X)/\sqrt{n}$. By Lemma 3.1 we have
\[
U_{0,m+2n}(X \cup \omega) > 1 \vee \frac{U_{0,m+2n}(X)}{\sqrt{n}} \leq C_2 e^{-C_1 \sqrt{n}}.
\]
Then, use the inequality $U_{0,m}(X \cup \omega) \leq U_{0,m+2i\sqrt{\pi}}(X \cup \omega)$ to conclude the proof.

Proof of Lemma 1.1. Since $n_0$ depends only on $d$, it is sufficient to prove Equation (11) in Lemma 1.1 for $n \geq n_0$ (recall that $n_0$ was defined at the beginning of Section 3). We wish to apply Lemma 3.2 recursively in the $2d$ disjoint annuli

$$A_{(8d-2)n,8dn}, A_{(8d-4)n,(8d-2)n}, \ldots, A_{4dn,(4d+2)n}.$$  

For $i = 1, \ldots, 2d$, set $m(i) = (8d-2i)n$, and $\omega_i = \omega \cap \Lambda_{m(i),8dn}$. By Lemma 3.2, we have for all $i < 2d$

$$\mathbb{P} \left[ U_{0,m(i+1)}(X \cup \omega_{i+1}) > 1 \vee \frac{U_{0,m(i)}(X \cup \omega_i)}{\sqrt{n}} \right] \leq C_2 e^{-C_1 \sqrt{n}}. \quad (10)$$

By Equation (6), we have $U_{0,8dn}(X) \leq n^d$ for all $n \geq n_0$. This implies that

$$\mathbb{P} [U_{0,4dn}(X \cup \omega) = 1] \geq \mathbb{P} \left[ \text{For all } i, U_{0,m(i+1)}(X \cup \omega_{i+1}) \leq 1 \vee \frac{U_{0,m(i)}(X \cup \omega_i)}{\sqrt{n}} \right] \geq 1 - 2dC_2 e^{-C_1 \sqrt{n}},$$

where the last line follows from Equation (10). This proves that with probability larger than $1 - C_3 e^{-C_1 \sqrt{n}}$, all the vertices of $\Lambda_{4n}$ are $(X \cup \omega)$-connected inside $\Lambda_{8n}$. Lemma 1.1 follows straightforwardly. □

4. PROOF OF THEOREM 1

Let $X$ be a fixed everywhere percolating subgraph of $\mathbb{Z}^d$, and let $Y = X \cup \omega$ be obtained from $X$ by adding an $\varepsilon$-percolation $\omega$. In this section, we will show how to derive Theorem 1 from the following estimate, stated in Lemma 1.1:

$$\mathbb{P} [\text{For all } x, y \in \Lambda_n, x \text{ is } Y\text{-connected to } y \text{ inside } \Lambda_{2n}] \geq 1 - C_3 e^{-C_1 \sqrt{n}}$$

Recall that the constants $C_1, C_3$ do not depend on the underlying everywhere percolating graph $X$. Thus, by considering translates of $X$, we show that for every $z \in \mathbb{Z}^d$,

$$\mathbb{P} [\text{For all } x, y \in \Lambda_n(z), x \text{ is } Y\text{-connected to } y \text{ inside } \Lambda_{2n}(z)] \geq 1 - C_3 e^{-C_1 \sqrt{n}} \quad (11)$$

4.1. Proof of Item (v). Let $p < 1$. By Corollary 1.4 in [LS97], one can pick $p' < 1$ such that any 3-dependent $^2$ percolation process $Z$ on $\mathbb{Z}^d$, satisfying for every edge $e \in E^d$

$$\mathbb{P} [e \in Z] > p'$$

dominates stochastically a $p$-Bernoulli percolation.

Recall that the process $Y^{(n)}$ is defined by setting $\{x, y\} \in Y^{(n)}$ if $2nx$ is $Y$-connected to $2ny$ inside $\Lambda_{4n}(nx + ny)$. One can easily verify that for every $n \geq 1$, the process $Y^{(n)}$ is 3-dependent. Thus, in order to show that for some $n$, $Y^{(n)}$ dominates a $p$-Bernoulli percolation, one only need to prove that for every edge $\{x, y\} \in E^d$,

$$\mathbb{P} [2nx \text{ is } Y\text{-connected to } 2ny \text{ inside } \Lambda_{2n}(nx + ny)] > p'.$$ \quad (12)

$^2$A percolation process $Z$ is said to be 3-dependent if, given two sets of edges $A$ and $B$ such that any edge of $A$ is at $L^\infty$ distance at least 3 from any edge of $B$, the processes $Z \cap A$ and $Z \cap B$ are independent.
Since both $2nx$ and $2ny$ belong to $\Lambda_n(nx + ny)$, we get that Equation (12) holds for $n$ large enough, by applying Equation (11) with $z = nx + ny$.

4.2. Proof of Items (iii) and (iv). Let $p > p_c(Z^2)$. By Item (v), one can pick $n$ such that $Y^{(n)}$ dominates a $p$-Bernoulli percolation on $Z^d$. It is known that a $p$-Bernoulli percolation percolates in the half-plane $\{2, 3, \ldots\} \times \mathbb{Z} \times \{0, \ldots, 2\} \times \{0, \ldots, 2\}$ (see e.g. [Kes82]). By stochastic domination, the same holds for $Y^{(n)}$. This implies that $Y$ percolates in the half space $N \times Z^d$.

4.3. Proof of Item (iii). We begin as in the proof of Item (v). By stochastic domination arguments, one can fix $p' < 1$ such that any $3$-dependent percolation process $Z$ with $\Pr[e \in Z] > p'$ percolates in $Z^d$. Let $p'' \in (p', 1)$. By Equation (11), one can fix $n$ such that for every edge $\{x, y\} \in E^d$,

$$
\Pr[2nx \text{ is } Y\text{-connected to } 2ny \text{ inside } n(x + y) + \Lambda_{2n}] > p''.
$$

Let $\eta_q$ be a $q$-Bernoulli percolation process in $Z^d$, independent of $\omega$, and set $Y_q = Y \cap \eta_q$. This way, $Y_q$ corresponds exactly to a $q$-percolation on $Y$. Choose $q < 1$ such that all the edges of $\Lambda_{2n}$ belong to $\eta_q$ with probability larger than $1 - (p'' - p')$. This way, for fixed $z$, the processes $Y_q$ and $Y$ differs in the box $z + \Lambda_{2n}$ with probability smaller than $(p'' - p')$. Thus, for every edge $\{x, y\} \in E^d$, we have

$$
\Pr[2nx \text{ is } Y_q \text{-connected to } 2ny \text{ inside } n(x + y) + \Lambda_{2n}] \\
\geq \Pr[2nx \text{ is } Y\text{-connected to } 2ny \text{ inside } n(x + y) + \Lambda_{2n}] - (p'' - p') \\
> p'
$$

By the same stochastic argument that we already used several times, this is enough to guarantee that the process $Y_q$ percolates in $Z^d$. This proves that $p_c(Y) \leq q < 1$ almost surely.

4.4. Proof of Item (ii). Let $U_n$ be the event that all the pairs of points in $\Lambda_n$ are $Y$-connected in $Z^d$. By Lemma 1.1, we have

$$
\sum \Pr[U_n] < \infty.
$$

Thus, by the Borel-Cantelli Lemma, we have $\Pr[\bigcup_{n \geq n} U_n] = 1$, which implies that $Y$ is almost surely connected.

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