Abstract

We study the problem of agnostically learning homogeneous halfspaces in the distribution-specific PAC model. For a broad family of structured distributions, including log-concave distributions, we show that non-convex SGD efficiently converges to a solution with misclassification error $O(\text{opt}) + \epsilon$, where opt is the misclassification error of the best-fitting halfspace. In sharp contrast, we show that optimizing any convex surrogate inherently leads to misclassification error of $\omega(\text{opt})$, even under Gaussian marginals.

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1 Introduction

1.1 Background and Motivation

Learning in the presence of noisy data is a central challenge in machine learning. In this work, we study the efficient learnability of halfspaces when a fraction of the training labels is adversarially corrupted. As our main contribution, we show that non-convex SGD efficiently learns homogeneous halfspaces in the presence of adversarial label noise with respect to a broad family of well-behaved distributions, including log-concave distributions. Before we state our contributions, we provide some background and motivation for this work.

A (homogeneous) halfspace is any function $f : \mathbb{R}^d \to \{\pm 1\}$ of the form $f(x) = \text{sign}(\langle w, x \rangle)$, where the vector $w \in \mathbb{R}^d$ is called the weight vector of $f$, and the function $\text{sign} : \mathbb{R} \to \{\pm 1\}$ is defined as $\text{sign}(t) = 1$ if $t \geq 0$ and $\text{sign}(t) = -1$ otherwise. Halfspaces are arguably the most fundamental concept class and have been studied since the beginning of machine learning, starting with the Perceptron algorithm \cite{Ros58, Nov62}. In the realizable setting, halfspaces are efficiently learnable in the distribution-independent PAC model \cite{Val84} via linear programming (see, e.g., \cite{MT94}). On the other hand, in the agnostic model \cite{Hau92, KSS94}, even weak distribution-independent learning is computationally intractable \cite{GR06, FGKP06, Dan16}. The distribution-specific agnostic (or adversarial label noise) setting – where the label noise is adversarial but we have some prior knowledge about the structure of the marginal distribution on examples – lies in between these two extremes. In this setting, computationally efficient noise-tolerant learning algorithms are known \cite{KKMS08, KLS09a, ABL17, Dan15, DKS18} under various distributional assumptions. We start by defining the distribution-specific agnostic model.

**Definition 1.1 (Distribution-Specific PAC Learning with Adversarial Label Noise).** Given i.i.d. labeled examples $(x, y)$ from a distribution $D$ on $\mathbb{R}^d \times \{\pm 1\}$, such that the marginal distribution $D_x$ is promised to belong in a known family $\mathcal{F}$ but the labels $y$ can be arbitrary, the goal of the learner is to output a hypothesis $h$ with small misclassification error $\text{err}^D_{0-1}(h) \overset{\text{def}}{=} \Pr_{(x,y) \sim D}[h(x) \neq y]$, compared to $\text{opt} \overset{\text{def}}{=} \inf_{g \in C} \text{err}^D_{0-1}(g)$, where $C$ is the target concept class.

\cite{KKMS08} gave an algorithm that learns halfspaces in this model with error $\text{opt} + \epsilon$ under any isotropic log-concave distribution, with sample complexity and runtime $d^{m(1/\epsilon)}$, for an appropriate function $m$, which is at least polynomial. Moreover, there is evidence that any algorithm that achieves error $\text{opt} + \epsilon$ requires time exponential in $1/\epsilon$, even under Gaussian marginals \cite{DGZ20, GGK20}. Specifically, recent work \cite{DKZ20, GGK20} obtained Statistical Query (SQ) lower bounds of $d^{\text{poly}(1/\epsilon)}$ for this problem.

A line of work \cite{KLS09a, ABL17, Dan15, DKS18} focused on obtaining $\text{poly}(d, 1/\epsilon)$ time algorithms with near-optimal error guarantees. Specifically, \cite{ABL17} gave a polynomial time constant-factor approximation algorithm – i.e., an algorithm with misclassification error of $C \cdot \text{opt} + \epsilon$, for some universal constant $C > 1$ – for homogeneous halfspaces under any isotropic log-concave distribution. More recent work \cite{DKS18} gave an algorithm achieving this error bound for arbitrary halfspaces under Gaussian marginals. The algorithms of \cite{ABL17, DKS18} rely on an iterative localization technique and are quite sophisticated. Moreover, while their complexity is polynomial, they do not appear to be practical. The motivation for this work is the design of simple and practical algorithms for this problem with the same near-optimal error guarantees as these prior works.
1.2 Our Contributions

Our main result is that SGD on a non-convex surrogate of the zero-one loss solves the problem of learning a homogeneous halfspace with adversarial label noise when the underlying marginal distribution on the examples is well-behaved. As we already mentioned, prior work [ABL17, DKS18] uses more complex methods and custom algorithms that run in multiple phases using multiple passes over the samples. In contrast, we take a direct optimization approach and define a single loss function over the space of halfspaces whose approximate stationary points are near-optimal solutions. This implies that any optimization method that is guaranteed to converge to stationary points, for example SGD, will yield a halfspace with error $O(\text{opt}) + \epsilon$.

Our loss function is a smooth version of the 0-1 loss using a sigmoid function. In our case, we use the logistic function $S_\sigma(t) = 1/(1 + e^{-t/\sigma})$. Our overall objective is:

$$L_\sigma(w) = E_{(x,y) \sim D}[S_\sigma(-y \langle w, x \rangle)]$$

and we optimize it over the unit sphere $\|w\|_2 = 1$. We show that, for a broad class of distributions, any stationary point of this loss function corresponds to a halfspace with near-optimal error. In more detail, we require that the distribution on the examples is sufficiently well-behaved (Definition 1.2) satisfying natural (anti-)concentration properties.

In [DKTZ20], it was shown that the (approximate) stationary points of the objective of Equation (1) are (approximately) optimal halfspaces under Massart noise, which is a milder noise assumption than adversarial label noise. Interestingly, our results suggest that optimizing this objective is a unified approach for learning halfspaces under label noise, as we show that it works even in the more challenging adversarial noise setting.

**Definition 1.2 (Well-behaved distributions).** Let $U, R > 0$ be absolute constants and $t : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-negative function. An isotropic (i.e., zero mean and identity covariance) distribution $D_x$ on $\mathbb{R}^d$ is called well-behaved if for any projection $(D_x)_V$ of $D_x$ onto a 2-dimensional subspace $V$ the corresponding pdf $\gamma_V$ on $\mathbb{R}^2$ satisfies the following properties:

1. $\gamma_V(x) \geq 1/U$, for all $x \in V$ such that $\|x\|_2 \leq R$ (anti-anti-concentration).
2. For all $x \in V$, we have $\gamma_V(x) \leq t(\|x\|_2)$ and also $\sup_{x \in V} t(\|x\|_2) \leq U$, $\int_V t(\|x\|_2)dx \leq U$, $\int_V\|x\|_2 t(\|x\|_2)dx \leq U$ (anti-concentration and concentration).

Our class of distributions contains well-known distribution classes such as Gaussian and log-concave. In addition to distributions with strong concentration properties, our results also handle distributions with very weak concentration such as heavy-tailed distributions. In particular, we handle distributions whose density function decays only polynomially with the distance from the origin, see Table 1.

We use the non-convex objective of Equation (1) and SGD to obtain our main algorithmic result.

**Theorem 1.3.** Let $D$ be a distribution on $\mathbb{R}^d \times \{\pm 1\}$ such that the marginal $D_x$ on $\mathbb{R}^d$ is well-behaved. Then SGD on the objective (1) has the following performance guarantee: For any $\epsilon > 0$, it draws $m = \tilde{O}(d/\epsilon^4)$ labeled examples from $D$, uses $O(m)$ gradient evaluations, and outputs a hypothesis halfspace with misclassification error $O(\text{opt}) + \epsilon$ with probability at least 99%.

Theorem 1.3 gives a simple and practical learning algorithm for halfspaces with adversarial label noise with respect to a broad family of marginal distributions.
A natural question is whether the non-convexity of our surrogate loss (1) is required. In many practical settings, convex surrogates of the 0/1 loss such as Hinge or ReLU loss are used, see [BJM06] for more choices. In general, given a convex and increasing loss $\ell(\cdot)$ the following objective is defined.

$$
C(w) = \mathbb{E}_{(x,y) \sim D}[\ell(-y \langle x, w \rangle)].
$$

(2)

One such convex optimization problem closely related to our non-convex formulation is logistic regression. In that case, the convex surrogate is simply $\ell(t) = \log(S_\sigma(t))$ (compare with Equation (1)).

To complement our positive result, we show that convex surrogates are insufficient for the task at hand. In particular, for any convex surrogate objective, one will obtain a halfspace with error $\omega(\text{opt})$. In more detail, we construct a single noisy distribution whose $x$-marginal is well-behaved such that optimizing any convex objective over this distribution will yield a halfspace with error $\omega(\text{opt})$. We establish a fine-grained result showing that the misclassification error of convex objectives degrades as the distributions become more heavy tailed, see Table 1.

**Theorem 1.4.** Let $D_x$ be the standard normal distribution on $\mathbb{R}^d$. There exists a distribution $D$ on $\mathbb{R}^d \times \{\pm 1\}$ such that for every convex and non-decreasing loss $\ell(\cdot)$ the objective $C(w) = \mathbb{E}_{(x,y) \sim D}[\ell(-y \langle x, w \rangle)]$ is minimized at some halfspace $h$ with misclassification error $\Omega(\text{opt} \sqrt{\log(1/\text{opt})})$. Moreover, if the marginal $D_x$ is allowed to be log-concave (resp. $s$-heavy tailed, $s > 2$) the error of any minimizer is $\Omega(\text{opt} \log(1/\text{opt}))$ (resp. $\Omega(\text{opt}^{1-1/s})$).

In fact, our lower bound result shows a strong statement about convex surrogates: Even under the nicest distribution possible, i.e., a Gaussian, there is some simple label noise (flipping the labels of points far from the origin) that does not depend on the convex loss $\ell(\cdot)$ such that no convex objective can achieve $O(\text{opt})$ error. This suggests that the shortcoming of convex objectives is not due to pathological cases and complicated noise distributions that are designed to fool each specific loss function, but is rather inherent.

**Table 1:** Common well-behaved distribution families with their corresponding parameters $U, R, t(\cdot)$, see Definition 1.2. The last two columns show the best possible error achievable by convex objectives and our non-convex objective of Eq.(1).

| Distribution          | $U, R$ | $t(x)$ | Any Convex Loss | Our Loss, Eq.(1) |
|-----------------------|--------|--------|-----------------|------------------|
| Gaussian              | $\Theta(1)$ | $e^{-\Omega(\|x\|_2^2)}$ | $\Omega(\text{opt} \sqrt{\log(1/\text{opt})})$ [Thm 1.4] | $O(\text{opt})$ [Thm 1.3] |
| Log-Concave           | $\Theta(1)$ | $e^{-\Omega(\|x\|_2)}$ | $\Omega(\text{opt} \log(1/\text{opt}))$ [Thm 1.4] | $O(\text{opt})$ [Thm 1.3] |
| $s$-Heavy Tailed, $s > 2$ | $\Theta(1)$ | $\frac{O(1)}{(\|x\|_2 + 1)^{2+s}}$ | $\Omega(\text{opt}^{1-1/s})$ [Thm 1.4] | $O(\text{opt})$ [Thm 1.3] |

**1.3 Overview of Techniques**

Our approach is inspired by the recent work [DKTZ20], where the authors use the same loss function for learning halfspaces under the (weaker) Massart noise model. Under similar distributional assumptions to the ones we consider here, [DKTZ20] shows that the gradient of the loss function points towards the parameters of the optimal halfspace. A major difference between the two settings is that under Massart noise there exists a unique optimal halfspace and is identifiable. In the agnostic setting, there may be multiple halfspaces achieving optimal error. However, as we show,
for the class of distributions we consider, all these solutions lie on a small cone, see Claim 3.4 establishing that the angle between any two halfspaces is small. Our algorithm aims to move towards the cone with every gradient step.

To achieve this, we must carefully set the parameter $\sigma$ of the objective. Smaller values of $\sigma$ amplify the contribution to the gradient of points closer to the current guess and enable using local information to obtain good gradients. This localization approach is necessary and is commonly used to efficiently learn halfspaces under structured distributions [ABL17, DKS18]. In the Massart model, the authors of [DKTZ20] show that for the loss function of Equation (1) any sufficiently small value for $\sigma$ suffices to obtain a gradient pointing towards the optimal halfspace. This is not true in the agnostic setting that we consider here. In particular, choosing small values of $\sigma$ may put a lot of weight on points close to the halfspace that may all be noisy. To prove our structural result, we show that there exists an appropriate setting of a not-too-small $\sigma$ that will guarantee convergence to a solution with $O(\text{opt})$ error. This is our main structural result, Lemma 3.2.

Our lower bound hinges on the fact that such a trade-off can only be achieved using non-convex loss functions. In particular, our lower bound construction leverages the structure of convex objectives to design a noisy distribution where any convex objective results in misclassification error $\omega(\text{opt})$. In more detail, we exploit the fact that all optimal halfspaces lie in a small cone, and show that there exists a fixed noise distribution such that all convex loss functions have non-zero gradients inside this cone.

1.4 Related Work

Here we provide a detailed summary of the most relevant prior work with a focus on poly($d/\epsilon$) time algorithms. [KLS09b] studied the problem of learning homogeneous halfspaces in the adversarial label noise model, when the marginal distribution on the examples is isotropic log-concave, and gave a polynomial-time algorithm with error guarantee $\tilde{O}(\text{opt}^{1/3}) + \epsilon$. This error bound was improved by [ABL17] who gave an efficient localization-based algorithm that learns to accuracy $O(\text{opt}) + \epsilon$ for isotropic log-concave distributions. [DKS18] gave a localization-based algorithm that learns arbitrary halfspaces with error $O(\text{opt}) + \epsilon$ for Gaussian marginals. [BZ17] extended the algorithms of [ABL17] to the class of $s$-concave distributions, for $s > -\Omega(1/d)$. Inspired by the localization approach, [YZ17] gave a perceptron-like learning algorithm that succeeds under the uniform distribution on the sphere. The algorithm of [YZ17] takes $O(d/\epsilon)$ samples, runs in time $\tilde{O}(d^2/\epsilon)$, and achieves error of $\tilde{O}(\log d \cdot \text{opt}) + \epsilon$ — scaling logarithmically with the dimension $d$. We also note that [DKTZ20] established a structural result regarding the sufficiency of stationary points for learning homogeneous halfspaces with Massart noise. Finally, we draw an analogy with recent work [DGK+20] which established that convex surrogates suffice to obtain error $O(\text{opt}) + \epsilon$ for the related problem of agnostically learning ReLUs under well-behaved distributions. This positive result for ReLUs stands in sharp contrast to the case of sign activations studied in this paper (as follows from our lower bound result). An interesting direction is to explore the effect of non-convexity for other common activation functions.

2 Preliminaries and Notation

For $n \in \mathbb{Z}_+$, let $[n] \overset{\text{def}}{=} \{1, \ldots, n\}$. We will use small boldface characters for vectors. For $x \in \mathbb{R}^d$ and $i \in [d]$, $x_i$ denotes the $i$-th coordinate of $x$, and $\|x\|_2 \overset{\text{def}}{=} (\sum_{i=1}^d x_i^2)^{1/2}$ denotes the $\ell_2$-norm of $x$. We will use $(x, y)$ for the inner product of $x, y \in \mathbb{R}^d$ and $\theta(x, y)$ for the angle between $x, y$. We will also denote $1_A$ to be the characteristic function of the set $A$, i.e., $1_A(x) = 1$ if $x \in A$.
and \(I_A(x) = 0\) if \(x \not\in A\). Let \(e_i\) be the \(i\)-th standard basis vector in \(\mathbb{R}^d\). Let \(\text{proj}_U(x)\) be the projection of \(x\) onto subspace \(U \subset \mathbb{R}^d\). Let \(\mathbb{E}[X]\) denote the expectation of random variable \(X\) and \(\mathbb{P}[\mathcal{E}]\) the probability of event \(\mathcal{E}\). We consider the binary classification setting where labeled examples \((x, y)\) are drawn i.i.d. from a distribution \(\mathcal{D}\) on \(\mathbb{R}^d \times \{\pm 1\}\). We denote by \(\mathcal{D}_x\) the marginal of \(\mathcal{D}\) on \(x\). The misclassification error of a hypothesis \(h : \mathbb{R}^d \rightarrow \{\pm 1\}\) (with respect to \(\mathcal{D}\)) is \(\text{err}^{D}_{0-1}(h) \overset{\text{def}}{=} \mathbb{P}_{(x, y) \sim \mathcal{D}}[h(x) \neq y]\). The zero-one error between two functions \(f, h\) (with respect to \(\mathcal{D}_x\)) is \(\text{err}^{\mathcal{D}_x}_{0-1}(f, h) \overset{\text{def}}{=} \mathbb{P}_{x \sim \mathcal{D}_x}[f(x) \neq h(x)]\).

## 3 Non-Convex SGD Learns Halfspaces with Adversarial Noise

In this section, we prove our main algorithmic result, whose formal version we restate here.

**Theorem 3.1.** Let \(\mathcal{D}\) be a distribution on \(\mathbb{R}^d \times \{\pm 1\}\) such that the marginal \(\mathcal{D}_x\) on \(\mathbb{R}^d\) is well-behaved. There is an algorithm with the following performance guarantee: For any \(\epsilon > 0\), it draws \(m = \tilde{O}(d \log(1/\delta)/\epsilon^4)\) labeled examples from \(\mathcal{D}\), uses \(O(m)\) gradient evaluations, and outputs a hypothesis vector \(\hat{w}\) that satisfies \(\text{err}^{\mathcal{D}}_{0-1}(\hat{w}) \leq O(\text{opt}) + \epsilon\) with probability at least \(1 - \delta\), where \(\text{opt}\) is the minimum classification error achieved by halfspaces.

The crucial component in the proof of Theorem 3.1 is the following structural lemma, Lemma 3.2. We show that by carefully choosing the parameter \(\sigma > 0\) of the non-convex surrogate loss \(S_\sigma\) of Equation (1), we get that any approximate stationary point of this objective will be close to some optimal halfspace. Instead of optimizing over the unit sphere, we can normalize our objective \(\mathcal{L}_\sigma\) defined in Equation (1), as follows

\[
\mathcal{L}_\sigma(w) = \mathbb{E}_{(x, y) \sim \mathcal{D}} \left[ S_\sigma \left( -y \frac{\langle w, x \rangle}{\|w\|_2} \right) \right],
\]

where \(S_\sigma(t) = \frac{1}{1 + e^{-\sigma t}}\) is the logistic function with growth rate \(1/\sigma\). We prove the following:

**Lemma 3.2 (Stationary points of \(\mathcal{L}_\sigma\) suffice).** Let \(\mathcal{D}_x\) be a well-behaved distribution on \(\mathbb{R}^d\) and let \(w^*\) be a halfspace achieving optimal classification error \(\text{opt}\). Fix \(\sigma > 0\) and let \(\theta = (4\sqrt{2}\pi U/R)\cdot \sigma\). If \(\text{opt} \leq R^4/(2^{15} U^3)\cdot \sigma\), then for every \(\tilde{w}\) such that \(\theta(\tilde{w}, w^*) \in (\theta, \theta - \pi)\) it holds \(\|\nabla_{\tilde{w}} \mathcal{L}_\sigma(\tilde{w})\|_2 \geq \frac{R^2}{\text{opt}}\).

**Proof.** To simplify notation, we will write \(h(w, x) = \frac{\langle w, x \rangle}{\|w\|_2}\). Note that \(\nabla_{\tilde{w}} h(w, x) = \frac{x}{\|w\|_2^2} - \langle w, x \rangle \frac{w}{\|w\|_2}\). We define the “noisy” region \(S\), as follows \(S = \{x \in \mathbb{R}^d : y \neq \text{sign}(\langle w^*, x \rangle)\}\). The gradient of the objective \(\mathcal{L}_\sigma(w)\) is

\[
\nabla_w \mathcal{L}_\sigma(w) = \mathbb{E}_{(x, y) \sim \mathcal{D}} \left[ -S_\sigma'( \langle w, x \rangle ) \nabla_w h(w, x) y \right] = \mathbb{E}_{(x, y) \sim \mathcal{D}} \left[ -S_\sigma'( \|h(w, x)\|) \nabla_w h(w, x) \right] = \mathbb{E}_{x \sim \mathcal{D}_x} \left[ -S_\sigma'( \|h(w, x)\|) \nabla_w h(w, x) (1 - 2 \cdot \text{1}_S(x) \text{sign}(\langle w^*, x \rangle)) \right].
\]

Let \(V = \text{span}(w^*, w)\). Since projections can only decrease the norm of a vector, we have \(\|\nabla_w \mathcal{L}_\sigma(w)\|_2 \geq \|\text{proj}_V \nabla_w \mathcal{L}_\sigma(w)\|_2\). Without loss of generality, we may assume that \(\tilde{w} = e_2\) and \(w^* = -\sin \theta \cdot e_2\).
\[ e_1 + \cos \theta \cdot e_2. \] Then, we have \( \text{proj}_V(h(w, x)) = (x_1, 0). \) Using the above and the triangle inequality, we obtain

\[
\| \nabla_w L_\sigma(w) \|_2 \geq \left\| \frac{E_{x \sim D_x}[-S'_\sigma(|h(w, x)|) (x_1, 0) \sign((w^*, x))]}{I_1} \right\|_2
- 2 \left\| \frac{E_{x \sim D_x}[-1 \cdot S(x)S'_\sigma(|h(w, x)|) (x_1, 0) \sign((w^*, x))]}{I_2} \right\|_2.
\]

Let \( R, U \) be absolute constants from the Definition 1.2. We will first bound from above the term \( I_2 \), i.e., the contribution of the noisy points to the gradient. Using the fact that \( S'_\sigma(|t|) \leq e^{-|t|/\sigma}/\sigma \) we obtain

\[
I_2 \leq \frac{E_{x \sim D_x} \left[ e^{-|x_2|/\sigma} / \sigma \right]}{\sigma} |x_1| \cdot 1_S(x) \leq \sqrt{\frac{E_{x \sim D_x} [1_S(x)]}{\sigma \cdot E_{x \sim D_x} [-e^{-|x_2|/\sigma} / \sigma^2 \cdot x_1^2]} \left[ e^{-2|x_2|/\sigma} / \sigma \cdot x_1^2 \right]} \leq \sqrt{\frac{\text{opt}}{\sigma}} \cdot \sqrt{\frac{E_{x \sim D_x} [1_S(x)]}{\sigma \cdot E_{x \sim D_x} [-e^{-2|x_2|/\sigma} / \sigma \cdot x_1^2]} \left[ e^{-2|x_2|/\sigma} / \sigma \cdot x_1^2 \right]},
\]

where the first inequality follows from the Cauchy-Schwarz inequality and for the second we used the fact that the set \( S \) has probability at most opt. To finish the bound, notice that the remaining expectation depends only on \( x_1, x_2 \) and therefore we can use the upper bound \( t(\cdot) \) on the density function. Using polar coordinates we obtain

\[
E_{x \sim (D_x)^V} \left[ e^{-2|x_2|/\sigma} / \sigma \cdot x_1^2 \right] \leq 4 \int_0^\infty \int_0^{\pi/2} \int_0^3 \frac{r^3}{\sigma} \cos^2(\phi)e^{-2r \sin(\phi)/\sigma} t(r)d\phi dr
\]

\[
\leq 2 \int_0^\infty r^2 t(r) \int_0^{\pi/2} \frac{2r}{\sigma} \cos(\phi)e^{-2r \sin(\phi)/\sigma} d\phi dr
\]

\[
= 2 \int_0^\infty r^2 t(r)(1 - e^{-2r/\sigma})dr \leq 2 \int_0^\infty r^2 t(r)dr \leq 2U,
\]

where for the last inequality we used the fact that \( 1 - e^{-2r/\sigma} \leq 1 \). We thus have \( I_2 \leq \sqrt{2U \text{opt}/\sigma}. \)

We now bound \( I_1 \) from below. Observe that since inner products with \( w^*, w \) are preserved when we project \( x \) to \( V \), we have \( I_1 = \left| E_{x \sim (D_x)^V} [S'_\sigma(|x_2|)|x_1|1_G(x)] \right| \). Now, if we define \( G = \{ (x_1, x_2) \in \mathbb{R}^2 : x_1 \sign((w^*, x)) > 0 \} \), using the triangle inequality we have

\[
I_1 \geq E_{x \sim (D_x)^V} \left[ S'_\sigma(|x_2|)|x_1|1_G(x) \right] - E_{x \sim (D_x)^V} \left[ S'_\sigma(|x_2|)|x_1|1_{G^c}(x) \right].
\]

Moreover, using the fact that \( e^{-|t|/\sigma}/(4\sigma) \geq S'_\sigma(|t|) \leq e^{-|t|/\sigma}/\sigma \) we get

\[
I_1 \geq \frac{1}{4} E_{x \sim (D_x)^V} \left[ |x_1|1_G(x)e^{-|x_2|/\sigma}/\sigma \right] - E_{x \sim (D_x)^V} \left[ |x_1|1_{G^c}(x)e^{-|x_2|/\sigma}/\sigma \right].
\]

We can now bound each term separately using the fact that the distribution \( D_x \) is well-behaved. Assume first that \( \theta(w^*, \bar{w}) = \theta \in (0, \pi/2) \). Then we can express the region \( G \) in polar coordinates as \( G = \{ (r, \phi) : \phi \in (0, \theta) \cup (\pi/2, \pi + \theta) \cup (3\pi/2, 2\pi) \} \).
We denote by $\gamma(x, y)$ the density of the 2-dimensional projection on $V$ of the marginal distribution $D_x$. Since the integral is non-negative, we can bound from below the contribution of region $G$ on the gradient by integrating over $\phi \in (\pi/2, \pi)$. Specifically, we have:

$$
\mathbb{E}_{x \sim (D_x)V} \left[ \frac{e^{-|x_2|/\sigma}}{\sigma} |x_1| \mathbb{I}_{G^c}(x) \right] \geq \int_0^\infty \int_0^{\pi/2} \gamma(r \cos \phi, r \sin \phi) r^2 \cos \phi \frac{e^{-r \sin \phi}}{\sigma} d\phi dr
$$

$$
= \int_0^\infty \int_0^{\pi/2} \gamma(r \cos \phi, r \sin \phi) r^2 \cos \phi \frac{e^{-r \sin \phi}}{\sigma} d\phi dr
$$

$$
\geq \frac{1}{U} \int_0^R r^2 dr \int_0^{\pi/2} \cos \phi \frac{e^{-R \sin \phi}}{\sigma} d\phi
$$

$$
= \frac{1}{3U} R^2 \left( 1 - e^{-R/\sigma} \right) \geq \frac{1}{4U} R^2, \tag{5}
$$

where for the second inequality we used the lower bound $1/U$ on the density function $\gamma(x, y)$ (see Definition 1.2) and for the last inequality we used that $\sigma \leq \frac{R}{8}$ and that $1 - e^{-R/8} \geq 3/4$.

We next bound from above the contribution of the gradient in region $G^c$. Note that $G^c = \{(r, \phi) : \phi \in B_\theta = (\pi/2 - \theta, \pi/2) \cup (3\pi/2 - \theta, 3\pi/2)\}$. Hence, we can write:

$$
\mathbb{E}_{x \sim (D_x)V} \left[ \frac{e^{-|x_2|/\sigma}}{\sigma} |x_1| \mathbb{I}_{G^c}(x) \right] = \frac{1}{\sigma} \int_0^\infty \int_{\phi \in B_\theta} \gamma(r \cos \phi, r \sin \phi) r^2 \cos \phi \frac{e^{-r \sin \phi}}{\sigma} d\phi dr
$$

$$
\leq \frac{2U}{\sigma} \int_0^\infty \int_{\theta}^{\pi/2} r^2 \cos \phi \frac{e^{-r \sin \phi}}{\sigma} d\phi dr
$$

$$
= \frac{2U \sigma^2 \cos^2 \theta}{\sin^2 \theta}, \tag{6}
$$

where the inequality follows from the upper bound $U$ on the density $\gamma(x, y)$ (see Definition 1.2). Putting everything in (4), we obtain $I_1 \geq R^2/(16U) - 2U \sigma^2 / \sin^2 \theta$. Notice now that the case where $\theta(\tilde{w}, w^*) \in (\pi/2, \pi - \theta)$ follows similarly. Finally, in the case where $\theta = \pi/2$, the region $G^c$ is empty, and we again get the same lower bound on the gradient. Let $A > 0$, and set $\theta = A \cdot \sigma < \pi/2$, and let $\tau = \text{opt}/\sigma$. Since $\sin(t) \geq 2t/\pi$ for every $t \in [0, \pi/2]$, we have

$$
I_1 - 2I_2 \geq \frac{R^2}{16U} - \frac{\pi^2 U}{2A^2} - 2\sqrt{2U \tau}.
$$

For $\tau \leq \frac{R^4}{2U^2}$ and $A \geq 4\sqrt{2U} / R$, it holds $I_1 - 2I_2 \geq R^2/(32U)$.

Using Lemma 3.2 we get our main algorithmic result. Our algorithm proceeds by Projected Stochastic Gradient Descent (PSGD), with projection on the $\ell_2$-unit sphere, to find an approximate stationary point of our non-convex surrogate loss. Since $L_\sigma(w)$ is non-smooth for vectors $w$ close to $0$, at each step, we project the update on the unit sphere to avoid the region where the smoothness parameter is high. We are going to use the following result about the convergence of non-convex, smooth SGD on the unit sphere.

Lemma 3.3 (Lemma 4.2 and 4.3 of [DKTZ20]). Let $L_\sigma(w)$ be as in Equation (1). After $T$ iterations, where $T = \Theta(d \log(1/\delta)/(\sigma^4 \rho^4))$, the output $(w^{(1)}, \ldots, w^{(T)})$ of Algorithm 1 satisfies $\min_{i=1,...,T} \|\nabla w L_\sigma(w^{(i)})\|_2 \leq \rho$, with probability at least $1 - \delta$. 

7
Algorithm 1 PSGD for $f(w) = E_{Z \sim \mathcal{D}}[g(z, w)]$

1: procedure PSGD($f, T, \beta$) \> $f(w) = E_{Z \sim \mathcal{D}}[g(z, w)]$: loss, $T$: number of steps, $\beta$: step size.
2: \> $w(0) \leftarrow e_1$
3: \> for $i = 1, \ldots, T$ do
4: \> \> Sample $z(i)$ from $\mathcal{D}$.
5: \> \> $v(i) \leftarrow w(i-1) - \beta \nabla_w g(z(i), w(i-1))$
6: \> \> $w(i) \leftarrow v(i)/\|v(i)\|_2$
7: \> return $(w(1), \ldots, w(T))$.

In order to relate the misclassification error of a candidate halfspace with the angle that it forms with an optimal halfspace, we are going to use the following claim that states that the disagreement error between two halfspaces is $\Theta(\theta(u, v))$ under well-behaved distributions.

Claim 3.4. Let $\mathcal{D}_x$ be a distribution on $\mathbb{R}^d$. Let $f \in \arg\min_{g \in \mathcal{C}} \text{err}_{0-1}^P(g)$, where $\mathcal{C}$ is the class of halfspaces, then for any $u \in \mathbb{R}^d$, it holds that $\text{err}_{0-1}^P(h_u, f) - \text{err}_{0-1}^P(f) \leq \text{err}_{0-1}^P(h_u) \leq \text{err}_{0-1}^P(h_u, f)$. Moreover, if the distribution $\mathcal{D}_x$ is well-behaved, then $\text{err}_{0-1}^P(h_u, \nu) = \Theta(\theta(u, \nu))$.

Proof. Let $S = \{x \in \mathbb{R}^d : y \neq f(x)\}$, then we have

$$\text{err}_{0-1}^P(h_u, f) = \int_S 1\{h_u(x) \neq y\} \gamma(x)dx + \int_S 1\{h_u(x) = y\} \gamma(x)dx$$

$$= \int_{\mathbb{R}^d} 1\{h_u(x) \neq y\} \gamma(x)dx + 2 \int_S 1\{h_u(x) = y\} \gamma(x)dx - \int_S \gamma(x)dx$$

$$= \text{err}_{0-1}^P(h_u) + 2 \int_S 1\{h_u(x) = y\} \gamma(x)dx - \text{err}_{0-1}^P(f).$$

Using that $\int_S 1\{h_u(x) = y\} \gamma(x)dx \geq 0$, the result follows. To prove that $\text{err}_{0-1}^P(h_u, f) - \text{err}_{0-1}^P(f) \leq \text{err}_{0-1}^P(h_u)$, we work as follows

$$\text{err}_{0-1}^P(h_u, f) = \int_S 1\{h_u(x) \neq y\} \gamma(x)dx + \int_S 1\{h_u(x) = y\} \gamma(x)dx$$

$$= \int_{\mathbb{R}^d} 1\{h_u(x) \neq y\} \gamma(x)dx + \int_S \gamma(x)dx - 2 \int_S 1\{h_u(x) \neq y\} \gamma(x)dx$$

$$= \text{err}_{0-1}^P(h_u) + \text{err}_{0-1}^P(f) - 2 \int_S 1\{h_u(x) \neq y\} \gamma(x)dx.$$

To finish the proof, note that $\int_S 1\{h_u(x) \neq y\} \gamma(x)dx \geq 0$. \qed

Now assuming that we know the value of opt, we can readily use SGD and obtain a halfspace with small classification error. The following lemma, which relies on Claim 3.4, shows that SGD will output a list of candidate vectors, one of which will have error opt + $O(\sigma)$. For our structural result to work, we need opt $\leq C\sigma$ which gives the $O(\text{opt})$ error overall. Recall that for all well-behaved distributions the parameters $U, R$ are absolute constants.

Lemma 3.5. Let $\mathcal{D}$ be a distribution on $\mathbb{R}^d \times \{\pm 1\}$ such that the marginal $\mathcal{D}_x$ on $\mathbb{R}^d$ is well-behaved. Algorithm 1 has the following performance guarantee: If opt $\leq C \cdot \sigma$ where $C = \frac{R^4}{2\sigma^2 U}$, it draws $m = \text{poly}(U/R) \cdot \frac{\log(1/\delta)}{\sigma}$ labeled examples from $\mathcal{D}$, uses $O(m)$ gradient evaluations, and outputs a hypothesis list of vectors $L$, such that there exists a vector $w \in L$ that satisfies $\text{err}_{0-1}^P(h_w) \leq \text{opt} + O(\sigma)$ with probability at least $1 - \delta$, where opt is the minimum classification error achieved by halfspaces.
Proof. Let $R, U$ be the absolute constants from the Definition 1.2. If we set $\rho = \frac{R}{2^{5d}}$, by Claim 3.4, to guarantee $\text{err}^D_{0,1}(h_{\tilde{w}}, f) \leq \sigma$ it suffices to show that the angle $\theta(\tilde{w}, w^*) \leq \mathcal{O}(\sigma) =: \theta_0$. Using (the contrapositive of) Lemma 3.2, if the norm squared of the gradient of some vector $w \in \mathbb{S}^{d-1}$ is smaller than $\rho$, then $w$ is close to either $w^*$ or $-w^*$ – that is, $\theta(w, w^*) \leq \theta_0$ or $\theta(w, -w^*) \leq \theta_0$. Therefore, it suffices to find a point $w$ with gradient $\|\nabla_w \mathcal{L}_\sigma(w)\|_2 \leq \rho$. From Lemma 3.3, after $T = \mathcal{O}(\frac{d}{\sigma^4} \log(1/\delta))$ steps, the norm of the gradient of some vector in the list $(w(0), \ldots, w(T))$ will be at most $\rho$ with probability $1 - \delta$. Therefore, the required number of iterations is $T = \text{poly}(U/R) \cdot d \log(1/\delta)$. Note that one of the hypotheses in the list that is returned by Algorithm 1 is $\sigma$-close to the true $w^*$. From Claim 3.4, we have that there exists a $\tilde{w} \in L$ such that $\text{err}^D_{0,1}(h_{\tilde{w}}) \leq \text{opt} + \mathcal{O}(\sigma)$. \hfill \IEEEQEDsmall

We now give the proof of our main theorem, Theorem 1.3.

Proof of Theorem 1.3. Let $R, U$ be the absolute constants from Definition 1.2. and let $C = 2^{15}U^3/R^4$. We will do binary search to find the correct value of $\sigma$ using a grid of size $\mathcal{O}(1/\epsilon)$. In particular, we consider $\sigma \in \{C\epsilon, (C + 1)\epsilon, \ldots, C\}$. We now analyze our binary search over this grid. We have three cases. We first assume that $\epsilon \leq \text{opt} \leq C$. Let $L_k$ be the list of candidates output by Algorithm 1 for $\sigma = k\epsilon$. Note that there is a value of $k$ such that opt $< C\sigma$ and opt $> C\sigma - \epsilon$. Then we have that there exists $\tilde{w} \in L_k$ such that $\text{err}_{0,1}(h_{\tilde{w}}) \leq \text{opt} + \mathcal{O}(\sigma) = \text{opt} + \epsilon$. To find the right value of $k$, we do binary search in the $(1/\epsilon)$-sized grid of possible values and check each time if we obtained a weight vector that decreased the overall error. Thus, we will overall construct $\text{poly}(R/U) \cdot \log(1/\epsilon)$ lists. Finally, to evaluate all the vectors from the list, we need a small number of samples from the distribution $D$ to obtain the best among them, i.e., the one that minimizes the zero-one loss. The maximum size of each list of candidates is $\text{poly}(U/R) \cdot d \log(1/\epsilon^2)$. Therefore, from Hoeffding’s inequality, it follows that $\mathcal{O}(\log(d/(\epsilon\delta))/\epsilon^2)$ samples are sufficient to guarantee that the excess error of the chosen hypothesis is at most $\epsilon$ with probability at least $1 - \delta$. Similarly, in the case where $\text{opt} \leq \epsilon$ we have that for $\sigma = C\epsilon$, by running Algorithm 1, we obtain a list $L_1$ of candidates. From Lemma 3.5, we get that there is a vector $\tilde{w} \in L_1$, such that $\text{err}_{0,1}(h_{\tilde{w}}) \leq \text{opt} + \mathcal{O}(\sigma) \leq \mathcal{O}(\epsilon)$. If $\text{opt} \geq C$ then any halfspace will have error $\text{err}_{0,1}(h_{\tilde{w}}) \leq \mathcal{O}(R/U) = \mathcal{O}(\text{opt})$. We conclude that the total number of samples will be $\mathcal{O}(d \log(1/\delta)/\epsilon^4)$. This completes the proof. \hfill \IEEEQEDsmall

4 Convex Objectives Do Not Work

In this section, we show that optimizing convex surrogates of the zero-one loss cannot get error $\mathcal{O}(\text{opt}) + \epsilon$. We first recall the agnostic PAC learning setting that we assume here. Given a distribution $D_x$ on $\mathbb{R}^d$ and a halfspace $w^*$, we can define a noiseless instance $D$ on $\mathbb{R}^d \times \{\pm 1\}$ by setting the label of each point $x$ to $y = \text{sign}(\langle w^*, x \rangle)$. In this setting, $w^*$ achieves 0 classification error. To get a distribution where $w^*$ achieves error opt $> 0$, we can simply flip the labels of an opt fraction of points $x$. In this section, we show that optimizing convex surrogates of the zero-one loss cannot get error $\mathcal{O}(\text{opt}) + \epsilon$, even under Gaussian marginals. Recall that we consider objectives of the form

$$C(w) = \mathbb{E}_{x,y \sim D}[\ell(-y \langle x, w \rangle)],$$

where $\ell(\cdot)$ is a convex loss function. Notice that by considering the population version of the objective in Equation (2), we essentially rule out the possibility of sampling errors to be the reason that the minimizer of the convex objective did not achieve low classification error. With standard tools from empirical processes, one can readily get the same result for the empirical objective

$$C_n(w) = \frac{1}{n} \sum_{i=1}^n \ell(-y_i \langle x_i, w \rangle),$$
Figure 1: The green region depicts all points with +1 label and the red region depicts points with −1 label. We have \( y = -\text{sign}(\langle w^*, x \rangle) \) for all points in \( S \setminus C \), this corresponds to the hatched region. We have \( \theta(w^*, w) = \theta_1 \) and \( \theta(\tilde{w}, w) = \theta_2 \).

\[
(1/N) \sum_{i=1}^{N} \ell((-1)^i \langle x^{(i)}, w \rangle) \text{ assuming that the sample size } N \text{ is sufficiently large. We now restate the main result of this section that allows us to show Theorem 1.4.}
\]

**Theorem 4.1.** Fix \( Z > 0, \theta \in (0, \pi/8) \), and let \( D_x \) be a radially symmetric distribution on \( \mathbb{R}^2 \) such that

1. For all \( t > 0 \) it holds \( \Pr_{x \sim D_x}[\|x\|_2 \geq t] > 0 \).

2. \( \mathbb{E}_{x \sim D_x}[1\{\|x\|_2 \geq Z\} \|x\|_2] > 24\theta \mathbb{E}_{x \sim D_x}[\|x\|_2] \).

Then there exists a distribution \( D \) on \( \mathbb{R}^2 \times \{\pm 1\} \) and a halfspace \( w^* \) such that \( \text{err}^D_{\theta - 1}(w^*) \leq \Pr_{x \sim D_x}[\|x\|_2 \geq Z] \), the \( x \)-marginal of \( D \) is \( D_x \), and for every convex, non-decreasing, non-constant loss \( \ell(\cdot) \) and every \( w \) such that \( \theta(w, w^*) < \theta \) it holds \( \nabla_w C(w) \neq 0 \), where \( C \) is defined in Eq. (2).

**Proof.** We start by constructing the noisy distribution \( D \). Fix any unit vector \( w^* \) and let \( \tilde{w} \)  be a vector such that \( \theta(w^*, \tilde{w}) = \theta_2 \), where \( 2\theta \leq \theta_2 \leq \pi/4 \). Denote by \( \tilde{w}^\perp \) the vector that is perpendicular with \( \tilde{w} \) and satisfies \( \langle w^*, \tilde{w}^\perp \rangle \geq 0 \). We now define the regions \( C, S \) that will help us define the parts of the distribution where we will introduce noise by flipping the \( y \)-labels, see also Figure 1.

\[
C = \left\{ x : \langle w^*, x \rangle \langle \tilde{w}, x \rangle \geq 0 \text{ and } \langle \tilde{w}^\perp, x \rangle \leq 0 \right\} \quad \text{ and } \quad S = \left\{ x : \|x\|_2 \geq Z \right\}.
\]

We are now ready to define our noisy distribution \( D \): *we flip the labels of all points in the set \( S \setminus C \).* Observe that \( \text{err}^D_{\theta - 1}(w^*) \leq \Pr_{x \sim D_x}[\|x\|_2 \geq Z] \). Take any \( w \) such that \( \theta_1 = \theta(w, w^*) \leq \theta \). We are going to bound from below the norm of the gradient of \( C \) at \( w \). The gradient of \( C(w) \) is

\[
\nabla_w C(w) = \mathbb{E}_{(x,y) \sim D}[-yx \ell'(-y \langle x, w \rangle)].
\]

Without loss of generality, we may assume that \( w = \rho e_2 \), where \( \rho = \|w\|_2 > 0 \). We have that the first coordinate of the gradient is

\[
\langle \nabla_w C(w), e_1 \rangle = \mathbb{E}_{(x,y) \sim D}[-yx_1 \ell'(-y \rho x_2)]. \tag{8}
\]
In what follows, we are going to use polar coordinates \((r, \phi)\) with the standard relation to Cartesian \((x_1, x_2) = (r \cos \phi, r \sin \phi)\). Now assume that we want to compute the contribution of a specific region \(A = \{r \in [r_1, r_2], \phi \in [\phi_1, \phi_2]\}\) to the gradient of Equation (8). We denote the 2-dimensional density of the radially symmetric distribution \(D_x\) as \(\gamma(r)\). We have

\[
E_{(x, y) \sim D}[-y x_1 \ell'(-y x_2) 1_A(x)] = \int_{r_1}^{r_2} r \gamma(r) \int_{\phi_1}^{\phi_2} -y r \cos \phi \ell'(-y r \sin \phi) d\phi dr \\
= \frac{1}{\rho} \int_{r_1}^{r_2} r \gamma(r) \int_{\phi_1}^{\phi_2} (\ell(-y r \sin \phi))' d\phi dr = \frac{1}{\rho} \int_{r_1}^{r_2} r \gamma(r)(\ell(-y r \sin \phi_2) - \ell(-y r \sin \phi_1)) dr. \tag{9}
\]

Without loss of generality, we consider the two cases shown in Figure 1. We start with the first case, where \(w\) lies between \(w^*\) and \(\bar{w}\). We first compute the contribution to the gradient in \(S^c\), i.e., the points where \(\gamma = \text{sign}((w^*, x))\). Since the distribution is radially symmetric, we have \(E_{(x, y) \sim D}[-y x_1 \ell'(-y x_2) 1_{S^c}(x)] = 2 E_{(x, y) \sim D}[-y x_1 \ell'(-y x_2) 1_{R_1}(x)]\), where \(R_1 = \{r \in [0, Z], \phi \in [\theta_1, \pi + \theta_1]\}\). From Equation (9), we obtain that

\[
I_{S^c} = \frac{\rho}{2} \int_0^Z r \gamma(r)(\ell(\rho r \sin \theta_1) - \ell(-\rho r \sin \theta_1)) dr.
\]

Observe that since \(\ell(\cdot)\) is non-decreasing we have \(I_{S^c} \geq 0\). Next we compute the contribution of region \(S\) to the gradient. Recall that \(S\) contains \(S \setminus C\), i.e., the region we flipped the labels, \(y = -\text{sign}((w^*, x))\), see Figure 1. Using again the fact that the distribution is radially symmetric and Equation (8) for the region \(R_2 = \{r \in [Z, +\infty), \phi \in [\pi/2 - \theta_2, 3\pi/2 - \theta_2]\}\), we obtain

\[
I_S = \frac{\rho}{2} \int_0^Z r \gamma(r)(\ell(\rho r \cos \theta_2) - \ell(-\rho r \cos \theta_2)) dr.
\]

Similarly to the previous case, the fact that \(\ell(\cdot)\) is non-decreasing implies that \(I_S \leq 0\).

Now we are going to use the facts that \(\ell(\cdot)\) is convex and non-decreasing. Since both \(\theta_1, \theta_2 \leq \pi/4\), we have that \(\cos \theta_2 \geq \sin \theta_1\) and therefore, from the convexity of \(\ell(\cdot)\), we obtain

\[
\frac{\ell(\rho r \sin \theta_1)) - \ell(-\rho r \sin \theta_1)}{2 \rho r \sin \theta_1} \leq \frac{\ell(\rho r \cos \theta_2) - \ell(-\rho r \sin \theta_1)}{\rho r \cos \theta_2 + \rho r \sin \theta_1}.
\]

Since \(\ell(\cdot)\) is also non-decreasing, we have that \(\ell(\rho r \cos \theta_2) - \ell(-\rho r \sin \theta_1) \leq \ell(\rho r \cos \theta_2) - \ell(-\rho r \cos \theta_2)\) and therefore,

\[
\ell(\rho r \sin \theta_1)) - \ell(-\rho r \sin \theta_1) \leq \frac{2 \sin \theta_1}{\cos \theta_2 + \sin \theta_1}(\ell(\rho r \cos \theta_2) - \ell(-\rho r \cos \theta_2)) .
\]

To simplify notation, we define the functions \(\bar{\ell}(r) = \ell(\rho r \cos \theta_2)\) and \(h(r) = \bar{\ell}(r) - \bar{\ell}(-r)\). Observe that \(\bar{\ell}(\cdot)\) enjoys exactly the same properties as \(\ell(\cdot)\), that is \(\bar{\ell}(\cdot)\) is convex, non-decreasing, and non-constant. Moreover, observe that \(h(r)\) is non-negative and non-decreasing. Using the above inequalities, we obtain that

\[
\rho \langle \nabla_w C(w), e_1 \rangle = \rho (I_S + I_{S^c}) \leq \frac{4 \sin \theta_1}{\cos \theta_2 + \sin \theta_1} \int_{I_2}^Z r \gamma(r) h(r) dr - 2 \int_{I_1}^Z r \gamma(r) h(r) dr . \tag{10}
\]
We will now show that instead of dealing with every convex and increasing \( \ell(\cdot) \), we can restrict our attention to simple piecewise-linear convex and increasing functions. First, we observe that without loss of generality we may assume that the convex function \( \ell(r) \) is constant for all \( r \leq -Z \), since that part only increases \( I_1 \). To construct \( s(\cdot) \), we use the supporting lines of \( \ell(\cdot) \) at \(-Z\) and \( 0 \), and the secant line from \( 0 \) to \( Z \). We will first assume that \( \ell(Z) > 0 \). Let \( a_0 \) be a subgradient of \( \ell(\cdot) \) at \( 0 \). Then the secant from \( 0 \) to \( Z \) is some line \( a_1 r - a_0 Z_0 \) for some \( a_1 \in [a_0, \ell'(Z)] \). Then, for every convex and non-decreasing \( \ell(\cdot) \), the following piecewise-linear function \( s(r) \) makes the ratio \( I_1/I_2 \) smaller. In what follows, \( Z_0 \in [-Z, 0] \) is the intersection point of the supporting line \( a_0 r - a_0 Z_0 \) and the constant supporting line at \(-Z\).

\[
s(r) = b + \begin{cases} 
0, & r \leq Z_0 \\
(a_1 + a_0)r, & 0 \leq r \leq -Z_0, \\
(a_1 - a_0)Z_0, & 0 < r \\
\end{cases}
\]

We have

\[
h(r) = \begin{cases} 
(a_1 + a_0)r, & 0 \leq r \leq -Z_0, \\
(a_1 - a_0)Z_0 - Z_0, & -Z_0 < r \\
\end{cases}
\]

\[
I_1 = a_1 \int_{Z}^{\infty} r^2 \gamma(r) dr - a_0 Z_0 \int_{Z}^{\infty} r \gamma(r) dr \geq a_1 \int_{Z}^{\infty} r^2 \gamma(r) dr.
\]

\[
I_2 = (a_1 + a_0) \int_{0}^{-Z_0} r^2 \gamma(r) dr + a_1 \int_{-Z_0}^{Z} r^2 \gamma(r) dr - a_0 Z_0 \int_{-Z_0}^{Z} r \gamma(r) dr
\leq 2(a_1 + a_0) \int_{0}^{Z} r^2 \gamma(r) dr \leq 4a_1 \int_{0}^{Z} r \gamma(r) dr.
\]

Using the above bounds in Equation (10), we obtain

\[
\langle \nabla_w \mathcal{C}(w), e_1 \rangle \leq \frac{2a_1}{\rho} \left( \frac{8 \sin \theta_1}{\cos \theta_2 + \sin \theta_1} \int_{0}^{Z} r^2 \gamma(r) dr - \int_{Z}^{\infty} r^2 \gamma(r) dr \right).
\]

Removing the positive quantity \( \sin \theta_1 \) of the denominator and replacing \( \theta_1 \) by its upper bound \( \theta \), we obtain the claimed bound. Since \( \cos \theta_2 \) is decreasing in \([0, \pi/2] \), we may choose \( \theta = 2\theta \). Our final bound is then

\[
\langle \nabla_w \mathcal{C}(w), e_1 \rangle \leq \frac{2a_1}{\rho} \left( 8 \tan(2\theta) \int_{0}^{Z} r^2 \gamma(r) dr - \int_{Z}^{\infty} r^2 \gamma(r) dr \right)
\leq \frac{2a_1}{\rho} \left( 24\theta \mathbf{E}_{x \sim \mathcal{D}_x} [||x||_2] - \mathbf{E}_{x \sim \mathcal{D}_x} [I\{||x||_2 > Z \} \cdot ||x||_2] \right),
\]

where for the last inequality we used the fact that \( \tan(2\theta) \leq 3\theta \) for all \( \theta \in [0, \pi/8] \). In the case where \( \ell'(\rho Z \cos \theta_2) = 0 \), the above bound vanishes. We fist assume that this is not the case. Then, using Assumption 2 of our theorem, we obtain that \( \langle \nabla_w \mathcal{C}(w), e_1 \rangle \neq 0 \) and therefore \( \nabla_w \mathcal{C}(w) \neq 0 \).

In the case where \( \ell'(\rho Z \cos \theta_2) = 0 \), we observe that \( I_{S^c} \) vanishes. To finish the proof, we need to bound from above and away from zero the integral \( I_S \). Since \( \ell(\cdot) \) is non-constant, there exists a point \( Z' > Z \) with \( \ell(Z) > 0 \). Convexity of \( \ell(\cdot) \) implies \( h(r) \geq \ell(Z) r \). Using this fact, we get

\[
I_S \leq -\ell(Z') \int_{Z'}^{\infty} r^2 \gamma(r) dr.
\]
Using Assumption 1 of our theorem, we again get that $\nabla_w C(w) \neq 0$.

Next we handle the case where the candidate $w$ lies out of the cone formed by $w^*$ and $\bar{w}$. In that case, similarly to before, we compute the contribution to the gradient of the noisy samples $S$ and the non-noisy $S^c$.

$$I_{S^c} = \mathbb{E}_{(x,y) \sim D} [ -y x_1 \ell'(y x_2) 1_{S^c}(x) ] = \frac{2}{\rho} \int_{0}^{Z} r \gamma(r)(\ell(-\rho r \sin \theta_1) - \ell(\rho r \sin \theta_1)) dr .$$

and

$$I_{S} = \mathbb{E}_{(x,y) \sim D} [ -y x_1 \ell'(y x_2) 1_{S}(x) ] = \frac{2}{\rho} \int_{Z}^{\infty} r \gamma(r)(\ell(-\rho r \cos \theta_2) - \ell(\rho r \cos \theta_2)) dr .$$

In contrast to the previous case, we now observe that since $\ell(\cdot)$ is non-decreasing, both $I_S$ and $I_{S^c}$ have the same sign, i.e., they are both non-positive. From Assumption 1, and the fact that $\ell(\cdot)$ is non-constant, we obtain that $I_S + I_{S^c} < 0$, which in turn implies that $\nabla_w C(w) \neq 0$.

We are now ready to give the proof of Theorem 1.4, which we restate below for convenience.

**Theorem 1.4.** Let $D_x$ be the standard normal distribution on $\mathbb{R}^d$. There exists a distribution $D$ on $\mathbb{R}^d \times \{\pm 1\}$ such that for every convex, non-decreasing loss $\ell(\cdot)$, the objective $C(w) = \mathbb{E}_{x,y \sim D}[\ell(-y \langle x, w \rangle)]$ is minimized at some halfspace $h$ with error $\text{err}^D_{0-1}(h) = \Omega(\text{opt} \log(1/\text{opt})).$

Moreover, there exists a log-concave marginal $D_x$ (resp. $s$-heavy tailed marginal) such that $\text{err}^D_{0-1}(h) = \Omega(\text{opt} \log(1/\text{opt}))$ (resp. $\text{err}^D_{0-1}(h) = \Omega(\text{opt}^{1-1/s})$).

**Proof.** Since all the examples that we are going to consider will be radially invariant distributions, we note that the “disagreement” error of two halfspaces with normal vectors $v, u$ is $\theta(v, u)/\pi$. From Claim 3.4, we have that the classification error of any candidate $w$ is lower bounded by $\theta(w, w^*)/\pi - \text{opt}$. We will construct a distribution $D$ such that there is some $w^*$ that achieves error opt, but at the same time $C(w)$ is minimized at some halfspace such that $\theta(w, w^*) = \omega(\text{opt})$. This means that the minimizer of $C$ has classification error $\omega(\text{opt})$.

We assume first that $D_x$ is the standard normal and without loss of generality work in two dimensions. Recall that the density function in this case is radially invariant, i.e., $\gamma(x_1, x_2) = \frac{1}{2\pi} e^{-|x|^2/2}$. If $\ell$ is a constant function, any halfspace would minimize it and therefore, this case is trivial. Clearly, Assumption 1 of Theorem 4.1 holds in this case. We now show that we can pick $Z > 0$ such that the probability of all points with flipped label is $O(\text{opt})$ and make Assumption 2 of Theorem 4.1 true. Since the distribution is Gaussian, we have that for $Z = \Theta(\sqrt{\log(1/\text{opt})})$ it holds

$$\Pr[\|x\|_2 \geq Z] \leq \text{opt}.$$ 

Since the distribution is isotropic, we have $\mathbb{E}_{x \sim D_x}[\|x\|_2^2] = 1$. Moreover, we have that

$$\mathbb{E}_{x \sim D_x}[\mathbb{1}\{\|x\|_2 \geq Z\} \|x\|_2] = \int_{Z}^{\infty} r^2 e^{-r^2/2} dr \geq e^{-Z^2/2} Z = \Theta(\text{opt} \sqrt{\log(1/\text{opt})}) .$$

Now we can fix some $\theta = \Omega(\text{opt} \sqrt{\log(1/\text{opt})}) < \pi/8$ and observe that Assumption 2 of Theorem 4.1 is satisfied. Therefore, we have that for any halfspace with normal vector $w$ with $\theta(w, w^*) \leq \theta = \Omega(\text{opt} \sqrt{\log(1/\text{opt})})$ it holds that $\nabla_w C(w) \neq 0$, and therefore it cannot be a minimizer of $C(w)$.

For the log-concave marginals the argument is similar. We work again in two dimensions and pick $\gamma(x) = \frac{6}{\pi} e^{-2\sqrt{3}\|x\|_2}$. This distribution is isotropic log-concave. We have that for $Z = \Theta(\log(1/\text{opt}))$ it holds that $\Pr[\|x\|_2 \geq Z] \leq \text{opt}.$ Moreover, we have $\mathbb{E}_{x \sim D_x}[\mathbb{1}\{\|x\|_2 \geq Z\} \|x\|_2] = \Omega(\text{opt} \log(1/\text{opt}))$. Now we can fix some $\theta = \Omega(\text{opt} \log(1/\text{opt}) < \pi/8$ and observe that Assumption 2 of Theorem 4.1 is satisfied. Therefore, we have that for any halfspace with normal vector $w$ with
\[ \theta(w, w^*) \leq \theta = \Omega(\text{opt}\log(1/\text{opt})) \] it holds that \( \nabla_w C(w) \neq 0 \), and as a result it cannot be a minimizer of \( C(w) \).

For the heavy tailed marginals, the argument is similar. We work again in two dimensions, and for any \( s > 2 \) we pick
\[
\gamma(x) = \frac{b_s}{\left(\frac{\|x\|_2}{a_s} + 1\right)^{2+s}},
\]
where the constants \( a_s, b_s \) depend only on \( s > 2 \) and are appropriately picked so that the distribution is isotropic. Using polar coordinates, we have
\[
\Pr[\|x\|_2 \geq Z] = 2\pi \int_{Z}^{\infty} \frac{r b_s}{\left(\frac{r}{a_s} + 1\right)^{2+s}} dr = \frac{2\pi b_s}{s(1+s)} \frac{a_s + (s + 1)Z}{(a_s + Z)^{1+s}}.
\]
Therefore, for \( Z = \Theta((1/\text{opt})^{1/s}) \) it holds that \( \Pr[\|x\|_2 \geq Z] \leq \text{opt} \). Moreover, we have
\[
E_{x \sim D_x} [\mathbb{1}\{\|x\|_2 \geq Z\} \|x\|_2] = 2\pi \int_{Z}^{\infty} \frac{r^2 b_s}{\left(\frac{r}{a_s} + 1\right)^{2+s}} dr = \frac{b_s}{s(s-1)} \frac{2a_s^2 + 2a_s(s+1)Z + s(s+1)Z^2}{(a_s + Z)^{s+1}}.
\]
Therefore, for \( Z = \Theta((1/\text{opt})^{1/s}) \) it holds \( E_{x \sim D_x} [\mathbb{1}\{\|x\|_2 \geq Z\} \|x\|_2] = \Omega(\text{opt}^{1-1/s}) \). We can now fix some \( \theta = \Omega(\text{opt}^{1-1/s}) < \pi/8 \) and observe that Assumption 2 of Theorem 4.1 is satisfied. Therefore, we have that for any halfspace with normal vector \( w \) with \( \theta(w, w^*) \leq \theta = \Omega(\text{opt}^{1-1/s}) \) it holds that \( \nabla_w C(w) \neq 0 \), and as a result it cannot be a minimizer of \( C(w) \).

References

[ABL17] P. Awasthi, M. F. Balcan, and P. M. Long. The power of localization for efficiently learning linear separators with noise. J. ACM, 63(6):50:1–50:27, 2017.

[BJM06] P. L. Bartlett, M. I. Jordan, and J. D. Meauliffe. Convexity, classification, and risk bounds. Journal of the American Statistical Association, 101(473):138–156, 2006.

[BZ17] M.-F. Balcan and H. Zhang. Sample and computationally efficient learning algorithms under s-concave distributions. In Advances in Neural Information Processing Systems, pages 4796–4805, 2017.

[Dan15] A. Daniely. A PTAS for agnostically learning halfspaces. In Proceedings of The 28th Conference on Learning Theory, COLT 2015, pages 484–502, 2015.

[Dan16] A. Daniely. Complexity theoretic limitations on learning halfspaces. In Proceedings of the 48th Annual Symposium on Theory of Computing, STOC 2016, pages 105–117, 2016.

[DGK+20] I. Diakonikolas, S. Goel, S. Karmalkar, A. Klivans, and M. Sohlanolkotabi. Approximation schemes for relu regression. In COLT 2020, to appear, 2020. Available at https://arxiv.org/abs/2005.12844.

[DKS18] I. Diakonikolas, D. M. Kane, and A. Stewart. Learning geometric concepts with nasty noise. In Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing, STOC 2018, pages 1061–1073, 2018.
[DKTZ20] I. Diakonikolas, V. Kontonis, C. Tzamos, and N. Zarifis. Learning halfspaces with massart noise under structured distributions. arXiv, February 2020. Available at https://arxiv.org/abs/2002.05632. To appear in COLT 2020.

[DKZ20] I. Diakonikolas, D. M. Kane, and N. Zarifis. Near-optimal sq lower bounds foragnostically learning halfspaces and rehus under gaussian marginals. Manuscript, 2020.

[FGKP06] V. Feldman, P. Gopalan, S. Khot, and A. Ponnuswami. New results for learning noisy parities and halfspaces. In Proc. FOCS, pages 563–576, 2006.

[GGK20] S. Goel, A. Gollakota, and A. Klivans. Statistical-query lower bounds via functional gradients. Manuscript, 2020.

[GR06] V. Guruswami and P. Raghavendr. Hardness of learning halfspaces with noise. In Proc. 47th IEEE Symposium on Foundations of Computer Science (FOCS), pages 543–552. IEEE Computer Society, 2006.

[Hau92] D. Haussler. Decision theoretic generalizations of the PAC model for neural net and other learning applications. Information and Computation, 100:78–150, 1992.

[KKMS08] A. Kalai, A. Klivans, Y. Mansour, and R. Servedio. Agnostically learning halfspaces. SIAM Journal on Computing, 37(6):1777–1805, 2008.

[KLS09a] A. Klivans, P. Long, and R. Servedio. Learning halfspaces with malicious noise. To appear in Proc. 17th Internat. Colloq. on Algorithms, Languages and Programming (ICALP), 2009.

[KLS09b] A. Klivans, P. Long, and R. Servedio. Learning Halfspaces with Malicious Noise. Journal of Machine Learning Research, 10:2715–2740, 2009.

[KSS94] M. Kearns, R. Schapire, and L. Sellie. Toward Efficient Agnostic Learning. Machine Learning, 17(2/3):115–141, 1994.

[MT94] W. Maass and G. Turan. How fast can a threshold gate learn? In S. Hanson, G. Drastal, and R. Rivest, editors, Computational Learning Theory and Natural Learning Systems, pages 381–414. MIT Press, 1994.

[Nov62] A. Novikoff. On convergence proofs on perceptrons. In Proceedings of the Symposium on Mathematical Theory of Automata, volume XII, pages 615–622, 1962.

[Ros58] F. Rosenblatt. The Perceptron: a probabilistic model for information storage and organization in the brain. Psychological Review, 65:386–407, 1958.

[Val84] L. G. Valiant. A theory of the learnable. In Proc. 16th Annual ACM Symposium on Theory of Computing (STOC), pages 436–445. ACM Press, 1984.

[YZ17] S. Yan and C. Zhang. Revisiting perceptron: Efficient and label-optimal learning of halfspaces. In Advances in Neural Information Processing Systems 30: Annual Conference on Neural Information Processing Systems 2017, pages 1056–1066, 2017.