Mass-transfer instability of ground-states for Hamiltonian Schrödinger systems

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Abstract

We study generic semilinear Schrödinger systems which may be written in Hamiltonian form. In the presence of a single gauge invariance, the components of a solution may exchange mass between them while preserving the total mass. We exploit this feature to unravel new orbital instability results for ground-states. More precisely, we first derive a general instability criterion and then apply it to some well-known models arising in several physical contexts. In particular, this mass-transfer instability allows us to exhibit $L^2$-subcritical unstable ground-states.

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1 Introduction

In this paper, we consider general semilinear Schrödinger systems in pseudo-Hamiltonian form

$$u_t = JH'(u)$$

where $u = (u_1, \ldots, u_m)$, $u_j : \mathbb{R}^d \rightarrow \mathbb{C}$, $J = \text{diag}(\frac{1}{\lambda_j})$ with $\lambda_j \in \mathbb{R}$, $\lambda_j \neq 0$ and the Hamiltonian $H$ is of the form

$$H(u) = \frac{1}{2} \sum_j | \nabla u_j |^2 + \frac{1}{2} N(u), \quad N(u) = \sum_k n_k(u), \quad n_k : \mathbb{C}^m \rightarrow \mathbb{R} \text{ homogeneous of degree } \alpha_k.$$

There are many physically relevant models which may be written in this form. Such applications arise in several contexts, such as plasma physics, nonlinear optics or Bose-Einstein condensates, among others. The mathematical theory regarding the scalar case is, by now, well-established, covering local well-posedness, existence and stability of bound-states, global existence vs. finite time blow-up and scattering theory. In the last twenty years, the study of Schrödinger systems has become a very active field of research: on the one hand, the vector case presents a larger array of interesting physical models; on the other hand, one may observe new dynamical features that were not available in the scalar case. However, we believe that these new features have yet to be thoroughly explored.

We shall focus on the stability properties of bound-states, that is, solutions of the form

$$u(t) = (e^{i\omega_1 t} Q_1, \ldots, e^{i\omega_m t} Q_m).$$

In order for system (1) to admit bound-state solutions, we will require the existence of a gauge invariance:

$$H(u_1 e^{i\omega_1 t}, \ldots, u_m e^{i\omega_m t}) = H(u_1, \ldots, u_m), \quad \text{for all } u \in (H^1(\mathbb{R}^d))^m, \quad t \in \mathbb{R}.$$ (2)

This condition is usually verified in any physically relevant model, since it is equivalent (see Appendix A) to the conservation of the total mass

$$M(u(t)) = \frac{1}{2} \sum_j \lambda_j \omega_j | u_j(t) |^2, \quad \lambda_j \omega_j > 0.$$
A direct computation shows that (the profiles of) bound-states, \( Q = (Q_1, \ldots, Q_m) \), are precisely the critical points of the action functional
\[
S(u) = M(u) + H(u).
\]

As in the scalar case, a special attention should be given to the bound-states with minimal action among all bound-states, the so-called ground-states. Indeed, it turns out that these solutions determine many dynamical properties of the full evolution problem. However, it is important to observe that this definition of ground-state is not very useful from a mathematical point of view, since it provides no information on the behavior of the action functional (even locally). For this reason, an important effort has to be made in order to show that ground-states are the solutions to some specific minimization problems (usually minimizing the action on a codimension one manifold in \((H^1(\mathbb{R}^d))^m\)). Only then may one derive the numerous interesting properties regarding these solutions. In the general pseudo-Hamiltonian form, it is quite non-trivial to determine a suitable minimization problem: it depends on the specific power of the nonlinearities, their signs and also on the spatial dimension \( d \). Since our goal is not to prove such a variational characterization, we shall define the set of minimal bound-states as

\[
\mathcal{B}_0 = \{ Q \neq 0 \text{ bound-state } : Q \text{ is a local minimum of } S \text{ over a manifold } V \subset (H^1(\mathbb{R}^d))^m \text{ of codimension } 1 \},
\]

and study their instability. Since, in all known cases, minimal bound-states and ground-states coincide, we feel that this definition is in no way harmful to the validity of our work.

When studying the stability of minimal bound-states, it is essential to take into account the gauge and translation invariances. In fact, some simple arguments (see, for example, [2] Section 8.3]) show that these invariances always induce an unstable behavior. Therefore, one should weaken the notion of stability: a bound-state is said to be \emph{orbitally stable} if, for any given initial data sufficiently close to it, the corresponding solution remains close \emph{modulo gauge and translation invariances}.

For the nonlinear Schrödinger equation on \( \mathbb{R}^d \)
\[
iu_t + \Delta u + |u|^{p-1} u = 0,
\]
(3)
it is well-known that the ground-state is orbitally stable if and only if \( p < 4/d \) (corresponding to the \( L^2 \)-subcritical case): on the one hand, if \( p \geq 4/d \), one may use a Virial-type argument to show that finite-time blow-up occurs for some initial datum arbitrarily close to the ground-state; on the other hand, if \( p < 4/d \), the ground-state can be shown to be (up to phase and translation) the minimizer of the action on a surface of constant total mass. The fact that both mass and action are preserved by the dynamical flow of (3) then implies the orbital stability. Evidently, for very particular systems of type (3), the same dichotomy can be verified and the dynamical properties near the ground-state are the same as in the scalar case. Recalling our goal to find new dynamical behavior, instead of trying to disclose the optimal conditions that ensure this precise threshold, we will analyze other properties, \emph{intrinsic to the vector-valued case}, that may induce instability.

In the seminal papers [7], [8], the authors present very generic conditions that allow a complete characterization of the stability properties of ground-states. However, they assume a very precise knowledge of the linearized equation around the ground-state, specifically in what concerns the number of negative and null eigenvalues. In our context, the fine study of the linearized operator is quite challenging, especially due to the presence of multiple nonlinear terms and couplings. On the other hand, it was noticed in [8] that the minimality of the ground-state \( Q \) on the manifold
\[
\mathcal{V} = \left\{ u \in H^1(\mathbb{R}^d) : \int |u|^{p+1} = \int |Q|^{p+1} \right\},
\]
along with the existence of an unstable direction, is enough to prove orbital instability. Heuristically, the minimality condition implies that the number of negative eigenvalues is either zero or one, thus reducing the stability problem to the existence of a negative direction. As we shall prove, this observation may be further extended to generic manifolds of codimension 1. Consequently, orbital instability will follow from the existence of a negative direction.

In this work, we study the conditions under which the curve
\[
\Gamma(t) = \left( \gamma_1(t) \lambda^\frac{4}{d}(t) Q_1(\lambda(t)x), \ldots, \gamma_m(t) \lambda^\frac{4}{d}(t) Q_m(\lambda(t)x) \right), \quad \Gamma(0) = Q, \quad M(\Gamma(t)) = M(Q)
\]

2
provides a direction for instability. The scaling factor $\lambda$ is connected to the Virial argument, while the coefficients $\gamma_j : [0, 1] \to \mathbb{R}$ provide a way to exchange mass between components in such a way that the total mass is preserved. Consequently, any instability result obtained through the analysis of this curve shall be referred to as a mass-transfer instability. In a previous work [1], we exploited this mechanism in a very concrete situation. Here, our goal is to derive a general criterion which may easily be applied to several semilinear Schrödinger systems at once.

At this point, it is important to notice that, if some other gauge invariance is present, then the choice of $\gamma_j$ is further restricted. In particular, if one has an invariance for each individual component, then the individual masses are conserved, thus preventing the mass-transfer mechanism (all $\gamma_j$ must be constant). Therefore, our results will be applied to systems presenting a single gauge invariance.

Before we state our main results, we introduce a few notations and assumptions. To abbreviate, we write $u = (e^{i\omega_1 t}u_1, \ldots, e^{i\omega_m t}u_m)$ and $\omega = (\omega_1 u_1, \ldots, \omega_m u_m)$. Define

$$\beta_{j,k} = \text{homogeneity degree of } n_k \text{ with respect to the } j^{th} \text{ component},$$

and assuming, without loss of generality, that the $m$-th component of the minimal bound-state $Q$ is nonzero, we denote

$$k_j = \frac{\lambda_j\omega_j}{\lambda_m\omega_m}\frac{|Q_j|^2}{|Q_m|^2}.$$

The first assumption concerns the initial value problem

$$u_t = JH'(u), \quad u(0) = u_0. \quad (4)$$

**Assumption 1** (Local well-posedness). For $u_0 \in (H^1(\mathbb{R}^d))^m$, there exists $T = T(\|u_0\|_{(H^1(\mathbb{R}^d))^m})$ and a unique $u \in C([0,T], (H^1(\mathbb{R}^d))^m)$ solution of (4). Moreover, we suppose that

$$H' \in C((H^1(\mathbb{R}^d))^m; (H^{-1}(\mathbb{R}^d))^m).$$

In the vector-valued case, it may happen that the orbit of a bound-state is not closed. Our method does not cover this possibility. It is an interesting open problem to analyze orbital stability in this case. In the examples given below, the following assumption will be a consequence of the fact that $\omega_j \in \mathbb{Q}$ for all $j$.

**Assumption 2** (Periodicity). The orbit $\{e^{i\omega t}Q\}_{t \in \mathbb{R}}$ is closed.

Finally, we require some regularity for the minimal bound-states, which may be verified using classical elliptic regularity bootstrap arguments.

**Assumption 3** (Regularity). The bound-state $Q$ satisfies $S''(Q) : (H^1(\mathbb{R}^d))^m \to (H^{-1}(\mathbb{R}^d))^m$ and $x \cdot \nabla Q \in (H^1(\mathbb{R}^d))^m$.

**Theorem 1.1.** Consider a real minimal bound-state $Q$ of (1) and the symmetric matrix $A \in \mathcal{M}_{m \times m}$ given by

$$a_{0,0} = \frac{d}{2} \sum_k \left( \frac{\alpha_k}{2} - 1 \right) \frac{d\alpha_k}{2} - 2 \int n_k(Q)$$

$$a_{0,j} = \sum_k \left( \frac{d\alpha_k}{4} - \frac{d}{2} - 1 \right) (\beta_{j,k} - k_j) \int n_k(Q)$$

$$a_{j,j} = \sum_k \left( \frac{1}{2} k_j^2 \beta_{m,k}(\beta_{m,k} - 2) + \frac{1}{2} \beta_{j,k}(\beta_{j,k} - 2) - k_j \beta_{j,k} \right) \int n_k(Q)$$

$$a_{j,j} = \sum_k \left( \frac{1}{2} k_j k_{m,k}(\beta_{m,k} - 2) + \beta_{j,k} \beta_{m,k} - k_j \beta_{m,k} + \beta_{j,k} \right) \int n_k(Q)$$

for $1 \leq j < m$ and $j \neq j_0$. If $A$ admits one negative eigenvalue then $Q$ is orbitally unstable.

**Remark 1.** Even though it is not trivial to derive generic sufficient conditions for the existence of a negative eigenvalue for $A$, Theorem 1.1 may be applied quite easily to any particular system, as we illustrate below. In fact, since this criterion is amenable to perturbations, there is no need to know the exact values of $\sum n_k(Q)$. Therefore, in the cases where the exact derivation of a formula for the minimal bound-state is challenging, one may use a numerical approximation, thus allowing for computer-assisted proofs of orbital instability.

3
Before we proceed to study concrete examples, we apply Theorem 1.1 to derive some simple instability results. The first can be obtained directly from the analysis of the scaling parameter $\lambda$ and thus it is just a generalization of the scalar $L^2$-supercritical instability.

**Proposition 1.2** ($L^2$-supercritical instability). Given $p > 2 + 4/d$, suppose that $N$ admits the following decomposition:

$$N(u) = N_2(u) + N_{<p}(u) + N_p(u) + N_{>p}(u),$$

where

1. $N_2$ is quadratic;
2. $N_{<p}$ is the sum of homogeneities smaller than $p$ and is nonnegative;
3. $N_p$ has homogeneity equal to $p$;
4. $N_{>p}$ is the sum of homogeneities larger than $p$ and is nonpositive.

Then any real minimal bound-state is orbitally unstable.

We now focus on the $L^2$-critical case:

$$N(u) = N_2(u) + N_p(u),$$

where $N_2$ is quadratic and $N_p$ is homogeneous of degree $p = 2 + 4/d$.

**Proposition 1.3** ($L^2$-critical instability I). Suppose that

$$N_2(u) = \sum_{j=1}^{m} c_j \int |u_j|^2, \quad c_j \in \mathbb{R}.$$ 

If $c_1 \lambda_m \omega_m \neq c_m \lambda_1 \omega_1$, then all real minimal bound-states $Q$ with $Q_1, Q_m \neq 0$ are orbitally unstable.

**Proposition 1.4** ($L^2$-critical instability II). Suppose that

$$N_2(u) = \lambda \text{Re} \int u_1 \bar{u}_m, \quad \lambda \neq 0.$$ 

If a real minimal bound-state $Q$ satisfies $\lambda_1 \omega_1 \sqrt{Q_1^2} \neq \lambda_m \omega_m \sqrt{Q_m^2}$ and $\sqrt{Q_1 Q_m} \neq 0$, then $Q$ is orbitally unstable.

**Remark 2.** The above results provide criteria depending on the first and last components of the bound-state. Evidently, since one may choose the order of the components, this causes no loss in generality. Moreover, the scaling parameter by itself is not sufficient to conclude instability, which means that these results are truly intrinsic to the vector-valued case.

Finally, by continuity, we realize that instability of minimal bound-states may even occur in $L^2$-subcritical cases:

**Corollary 1.5** (Instability in almost $L^2$-critical cases). Suppose that

$$N(u) = N_2(u) + N_p(u),$$

where $N_p$ is homogeneous of degree $p$ and $N_2$ is as in Proposition 1.3. Assume that there exists a continuous curve of real minimal bound-states $p \mapsto Q(p)$ around $p_0 = 2 + 4/d$. If the hypotheses of Proposition 1.3 are verified for $p_0$, then all bound-states $Q(p), |p - p_0| \ll 1$, are orbitally unstable. The result remains valid if one replaces Proposition 1.3 by Proposition 1.4.

For illustrative purposes, we will apply these results to the following concrete models:
Example 1 (Quadratic Schrödinger system I).

\[
\begin{align*}
    iu_t + \Delta u + \overline{v} v &= 0, \\
    i\sigma v_t + \Delta v - \beta v + \frac{1}{2} u^2 &= 0, \quad (x, t) \in \mathbb{R}^d \times \mathbb{R}.
\end{align*}
\]  

(5)

This model governs the resonant interaction between waves propagating in a \( \chi^{(2)} \) dispersive medium in several physical contexts, such as magneto-hydrodynamics or nonlinear optics (see for instance [11], [12], [13]). In [3], the authors prove the existence of ground-state solutions of the form \( e^{i\omega t} Q_1(x), e^{i\omega t} Q_2(x) \) for this system by minimizing the action over the manifold

\[
V_\lambda = \left\{ (u, v) \in (H^1(\mathbb{R}^d))^2 : \text{Re} \int u^2 \overline{\bar{v}} = \lambda \right\},
\]

for some specific \( \lambda \in \mathbb{R} \). Moreover, they show orbital stability in the \( L^2 \)-subcritical dimension \( d = 2 \) when \( \beta = 0 \).

We may recover the instability results that we derived in [4] for the \( L^2 \)-(super)critical cases: the ground-state \( Q = (Q_1, Q_2) \) is orbitally unstable if

1. \( d \geq 5 \) (Corollary 1.2);
2. \( d = 4 \) and \( \beta \neq 0 \) (Proposition 1.3).

Also, in the synchronous case \( \beta = \omega(1 - 2\sigma) \), the ground-state can be computed explicitly. Indeed, one has

\[
Q = Q_{sync} = \left( \sqrt{\frac{2}{3} q}, \frac{1}{\sqrt{3}} q \right),
\]

where \( q \) is the ground-state of

\[-\omega q + \Delta q + \frac{1}{\sqrt{3}} q^2 = 0.\]

In this situation, in the subcritical cases \( d \leq 3 \), Theorem 1.1 yields that for \( \sigma \gg 1 \), \( Q_{sync} \) is orbitally unstable:

Proposition 1.6. Let \( d \leq 3 \) and \( \beta = \omega(1 - 2\sigma) \). Define \( \sigma_0 \) as the positive root of

\[3(4 - d)(1 + 4\sigma) - (1 - 2\sigma)^2 = 0.\]

Then, for \( \sigma > \sigma_0 \), \( Q_{sync} \) is orbitally unstable by the flow of (5).

To the best of our knowledge, there are no examples in the literature of such \( L^2 \)-subcritical unstable ground-states for Schrödinger-type coupled systems. We will exhibit another one in the next example:

Example 2 (Quadratic Schrödinger system II).

\[
\begin{align*}
    3iu_t + \Delta u - \beta_1 u &= -v w, \\
    2iv_t + \Delta v - \beta v &= -\frac{1}{2} w^2 - u \overline{\bar{w}}, \\
    iw_t + \Delta w - w &= -u \overline{\bar{w}} - u \overline{\bar{w}}, \quad (x, t) \in \mathbb{R}^d \times \mathbb{R}.
\end{align*}
\]  

(6)

This system is a generalization of the previous one for three-wave interactions in the framework of optical fiber systems. (see [10]). Both quadratic systems arise when second order nonlinear processes, such as second harmonic generation, are taken into account. Under some conditions on \( \beta, \beta_1 \), the existence of ground-states may be achieved through the minimization of the action on

\[
V_\lambda = \left\{ (u, v) \in (H^1(\mathbb{R}^d))^2 : \text{Re} \int (2u \overline{\bar{w}} + w^2 \overline{\bar{v}}) = \lambda \right\},
\]

for some specific \( \lambda \in \mathbb{R} \).
for a well-chosen $\lambda \in \mathbb{R}$.

Again, in the synchronous case $\beta_1 = -7$ and $\beta = -2$, it is possible to compute explicitly the ground-state $Q_{\text{sync}} := (aq, bq, cq)$, where $(a, b, c) \in \mathbb{S}^2$ and $q$ is the ground-state of

$$-2q + \Delta q + \frac{1}{6}(\sqrt{3} + \sqrt{5})q^2 = 0.$$  

In this framework, Theorem 1.1 yields the following result:

**Proposition 1.7.** Let $d = 1$, $\beta_1 = -7$ and $\beta = -2$. Then the ground-state $Q_{\text{sync}}$ is orbitally unstable by the flow of $\text{sync}$.

Notice that our method does not seem to allow any conclusions in the subcritical dimensions $d = 2$ and $d = 3$.

**Remark 3.** Interestingly enough, in 15, the author showed that $Q_{\text{sync}}$ is spectrally stable in dimension $d = 1$, i.e., that the linearized operator around the ground-state does not have any negative eigenvalues. The spectral stability of the ground-state only implies that there are no exponentially diverging solutions in its neighborhood (and thus it is a weaker notion than that of orbital stability). Therefore there is no contradiction with Proposition 1.7.

**Example 3** (Cubic Schrödinger system I).

$$\begin{cases} iu_t + \Delta u - u + \left(\frac{1}{9}|u|^2 + 2|w|^2\right)u + \frac{1}{3}w^2w = 0 \\ i\sigma w_t + \Delta w - \mu w + (9|w|^2 + 2|u|^2)w + \frac{1}{9}w^3 = 0, \ (x, t) \in \mathbb{R}^d \times \mathbb{R}. \end{cases}$$

This system, derived by Sammut et al. in 16, models the resonant interaction between a monochromatic beam of frequency $\omega$ propagating in a Kerr $\chi^{(3)}$ material and its third harmonic. The third-harmonic generation leads to features typical of $\chi^{(2)}$ media. In 14, the existence of a ground-state $Q$ for $\sigma, \mu > 0$ and $\omega > \max\{-1, -\mu/3\sigma\}$ proved by minimizing the action over the Nehari manifold

$$\mathcal{V} = \{(u, v) \in (H^1(\mathbb{R}^d))^2 \setminus \{(0, 0)\} : \langle S'(u, v), (u, v) \rangle_{H^{-1} \times H^1} = 0\}.$$  

As in the quadratic cases, we conclude that the ground-state solution is orbitally unstable if:

1. $d \geq 3$ (Corollary 1.2),
2. $d = 2$ and $\mu \neq 3\sigma$ (Proposition 1.8).

**Example 4** (Cubic Schrödinger system II).

$$\begin{cases} iu_t + \Delta u + \lambda v + k_{11}|u|^2u + k_{12}|v|^2u = 0 \\ iv_t + \Delta v + \lambda u + k_{12}|v|^2u + k_{22}|v|^2v = 0, \ (x, t) \in \mathbb{R}^2 \times \mathbb{R}. \end{cases}$$

This system models a two-component Bose-Einstein condensate irradiated by an external electromagnetic field with no trapping potential and with Rabi frequency $\lambda$ (11, 9). The existence of bound-states $(u, v) = e^{i\omega t}(P, Q)$, $\omega > \lambda$, can be achieved by minimizing the action on the manifold

$$\mathcal{V}_\lambda = \left\{(u, v) \in (H^1(\mathbb{R}^2))^2 : \int \left(k_{11}|u|^4 + 2k_{12}|u|^2|v|^2 + k_{22}|v|^4\right) = \lambda \right\}, \ \lambda > 0.$$  

If $k_{12} > 0$, a standard application of the Schwarz symmetrization reveals that $P, Q$ are real and radially decreasing. Since neither $P$ nor $Q$ can be zero, $\int PQ \neq 0$. Applying Proposition 1.4 we derive the following:

**Proposition 1.8.** Set $\lambda \neq 0$ and $k_{12} > 0$. If the ground-state $(P, Q)$ satisfies $\int P^2 \neq \int Q^2$, then it is orbitally unstable.
As it should be clear from the above examples, the existence of linear terms may induce unstable behavior through the mass-transfer instability. When there is a gauge invariance for each individual component, these linear terms may be absorbed using a simple change of variables. In the cases considered, however, the presence of a single, complete gauge invariance prevents this procedure.

2 Weak instability of ground-states

We recall the action functional

\[ S_p^u_q : \mathcal{M}_{p}^u_q \]

and the set of minimal bound-states

\[ B_0 = \{ Q \neq 0 \text{ bound-state} : Q \text{ is a local minimum of } S \text{ over a manifold } \mathcal{V} \subset (H^1(\mathbb{R}^d))^m \text{ of codimension 1} \} . \]

Remark 4. Observe that \( Q \) is a bound-state if and only if \( S'(Q) = 0 \). Indeed, taking \( u = e^{i\omega t} Q = (e^{i\omega_1 t} Q_1(x), \ldots, e^{i\omega_m t} Q_m(x)) \),

\[ u_t = JH'(u) \Leftrightarrow \forall j, i\omega_j e^{i\omega_j t} Q_j = \frac{1}{i\lambda_j} \frac{\partial H}{\partial u_j}(Q) \Leftrightarrow \forall j, \frac{\partial}{\partial u_j}(M + H)(Q) = 0 \Leftrightarrow S'(Q) = 0. \]

As previously explained, one needs to study the stability properties modulo the invariance

\[ Q \mapsto f(\theta, y)Q := (e^{i\omega_1 \theta} Q_1(\cdot + y), \ldots, e^{i\omega_m \theta} Q_m(\cdot + y)) , \quad \theta \in \mathbb{R} , \ y \in \mathbb{R}^d. \]

We therefore define the orbit of a bound-state \( Q \)

\[ O_Q = \{ f(\theta, y)Q , (\theta, y) \in \mathbb{R} \times \mathbb{R}^d \}. \]

Since the total mass is invariant under both the dynamical flow and the gauge and translation invariances, a great part of the analysis shall be performed on

\[ \mathcal{M}_Q = \{ u \in (L^2(\mathbb{R}^d))^m : M(u) = M(Q) \}. \]

To simplify the exposition, we will drop the subscript and write \( \mathcal{M} = \mathcal{M}_Q \).

Finally, we say that a bound-state \( Q \) is orbitally unstable if there exist solutions of \( \Box \) with initial data near \( Q \) which move away from the orbit of \( Q \). More precisely:

**Definition 1** (Orbital instability). A bound-state \( Q \) is said to be orbitally unstable by the flow of \( \Box \) if there exist \( \epsilon > 0 \) and a sequence \( (u_0)_k \rightarrow Q \) in \( (H^1(\mathbb{R}^d))^m \) such that the solution \( u_k \) of \( \Box \) with initial data \( (u_0)_k \)

\[ T_k := \sup\{ t : d(u_k(t), O_Q) < \epsilon \} < +\infty. \]

2.1 An orbital instability condition

In this paragraph, we prove the following orbital instability condition:

**Theorem 2.1.** Let \( Q \in B_0 \). If there exists \( \Psi \in T_Q \mathcal{M} \) such that

- \( J^{-1} \Psi \) is \( L^2 \)-orthogonal to \( J^{-1} \omega Q \) and to \( \partial_{x_j} Q \) for all \( j = 1, \ldots, d \);
- For all \( j = 1, \ldots, d \), \( J^{-1} \omega Q \) and \( \partial_{x_j} Q \) are linearly independent;
- \( \langle S''(Q) \Psi, \Psi \rangle_{H^{-1} \times H^1} < 0 \),

then \( Q \) is orbitally unstable.
We follow the main ideas for the scalar case presented in [17], in the context of the nonlinear Schrödinger equation. We extend the arguments of Shatah and Strauss to more general semilinear Schrödinger systems and for bound-states in $B_0$.

Throughout this section, $Q \in B_0$ will be fixed. We define the $L^2$-orthogonal hyperplane to the orbit as

$$L = \{ w \in (L^2(\mathbb{R}^d))^m : w \perp_{L^2} J^{-1}\omega Q \text{ and } \forall j, w \perp_{L^2} \partial_{x_j} Q \}$$

and define the neighborhoods

$$L_\delta = \{ w \in (H^1(\mathbb{R}^d))^m \cap (Q + L) : \|w - Q\|_{(H^1(\mathbb{R}^d))^m} < \delta \} \text{ and } \mathcal{O}_{Q,\delta} = \{ f(\theta, y)L_\delta, (\theta, y) \in \mathbb{R}^{d+1} \}.$$

Lemma 2.2. There exists $\delta > 0$ such that $\forall (\theta, y) \in (0, 2\pi/\tilde{\omega}) \times \mathbb{R}^d,$

$$L_\delta \cap f(\theta, y)L_\delta = \emptyset.$$ 

Here, $2\pi/\tilde{\omega}$ is the minimal period of $u = e^{i\omega t}Q$. As a consequence, $\mathcal{O}_{Q,\delta}$ is an open set in $(H^1(\mathbb{R}^d))^m$.

Remark 5. The existence of $\tilde{\omega}$ is ensured by Assumption 2.

Proof. Let $\delta > 0$ and $s_0 \equiv 0 \mod 2\pi/\tilde{\omega}$. We set

$$F(\theta, y, z, w) = z - f(s, y)w, \quad (\theta, y, z, w) \in \mathbb{R} \times \mathbb{R}^d \times L_\delta \times L_\delta.$$ 

Observe that $F(0, 0, Q, Q) = 0$ and that the Jacobian matrix of $F$ at $(0, 0, Q, Q)$ with respect to $s, w$ and $z$ is given by

\[
\begin{bmatrix}
J^{-1}\omega Q & 0 & 0 & \ldots & 0 \\
0 & \partial_{x_j} Q & 0 & \ldots & 0 \\
0 & 0 & \ldots & \ldots & 0 \\
0 & 0 & \ldots & \partial_{x_n} Q & 0 \\
0 & 0 & 0 & \ldots & Q
\end{bmatrix}.
\]

Since $Q \in \langle J^{-1}\omega Q, \partial_{x_1} Q, \ldots, \partial_{x_n} Q \rangle$ and $J^{-1}\omega Q, \partial_{x_1} Q, \ldots, \partial_{x_n} Q$ are linearly independent, this matrix is invertible. Hence, by the Implicit Function Theorem, for $\delta$ small enough and $w \in L_\delta$, there exists a unique $(\theta, y, w)$, small in $(\mathbb{Z}/(2\pi/\tilde{\omega})\mathbb{Z}) \times \mathbb{R}^d \times L_\delta$, such that $z = f(\theta, y)w$. The choice is obviously $(\theta, y, w) = (0, 0, z)$ and the Lemma is proved for $\delta$ small enough and $(\theta, y)$ in a neighbourhood of $(0, 0) \in (\mathbb{Z}/(2\pi/\tilde{\omega})\mathbb{Z}) \times \mathbb{R}$.

Now, by contradiction, let us assume the existence of sequences $\delta_n \to 0$, $(\theta_n, y_n) \in [0, 2\pi/\tilde{\omega}] \times \mathbb{R}^d$ and $(w_n)_{n \in \mathbb{N}} \in L_{\delta_n}$ such that $z_n = f(\theta_n, y_n)w_n \in L_{\delta_n}$. By definition of $L_{\delta_n}, w_n \to Q$ and $z_n \to Q$ in $H^1(\mathbb{R}^d)$. Also, since $(\theta_n)_{n \in \mathbb{N}}$ is bounded, we may assume that $\theta_n \to \theta_0 \in [0, 2\pi/\tilde{\omega}]$.

If $(y_n)_{n \in \mathbb{N}}$ is also bounded, then $y_n \to y_0$ up to a subsequence. Since $L_{\delta_n} \cap f(\theta, y)L_{\delta_n} = \emptyset$ for small $\theta$, $y$ and $\delta$, $(\theta_0, y_0) \notin [0, 2\pi/\tilde{\omega}] \times \{0\}$. Also,

$$\|Q - z_n\|_{L^2} = \|Q - f(\theta_n, y_n)w_n\|_{L^2} \to \|f(-\theta_n, -y_n)Q - w_n\|_{L^2} \to \|f(-\theta_0, y_0)Q - Q\|_{L^2} = 0.$$ 

Hence $Q = f(\theta_0, y_0)Q$, which contradicts the fact that $(\theta_0, y_0) \notin [0, 2\pi/\tilde{\omega}] \times \{0\}$.

On the other hand, if $(y_n)_{n \in \mathbb{N}}$ is unbounded, $|y_n| \to +\infty$ up to a subsequence. Then,

$$\|Q - z_n\|_{L^2}^2 = \|Q\|_{L^2}^2 + \|w_n\|_{L^2}^2 - 2 \int f(-\theta_n, -y_n)Qw_n$$

$$= \|Q\|_{L^2}^2 + \|w_n\|_{L^2}^2 - 2 \int f(-\theta_n, -y_n)Q(w_n - Q) - 2 \int f(-\theta_n, -y_n)QQ.$$ 

It is clear that $\|w_n\|_{L^2} \to \|Q\|_{L^2}$ and $\int f(-\theta_n, -y_n)Q(w_n - Q) \to 0$. Furthermore,

$$\int f(-\theta_n, -y_n)QQ = e^{-i\theta_n\omega} \int Q(x - y_n)Q(x)dx \to 0$$

since $Q \in (L^2(\mathbb{R}^d))^m$. The contradiction now follows from (7).
We now consider a smooth path $\Gamma : t \in [0, \epsilon] \rightarrow \Gamma(t) \in \mathcal{M}$ with $\Gamma(0) = \mathbf{Q}$ and $\Gamma'(0) = \mathbf{w}$. We define the projection of the orbital neighborhood onto the orthogonal neighborhood $G : \Omega_{Q, \delta} \rightarrow L_\delta$ as

$$G(w) = \tilde{w} \in L_\delta, \text{ where } w = f(\theta, y)\tilde{w}, \text{ for some } (\theta, y) \in [0, 2\pi/\omega) \times \mathbb{R}^d.$$Notice that $G$ is well-defined: if $w = f(\theta_1, y_1)w_1 = f(\theta_2, y_2)w_2$ with $w_1, w_2 \in L_\delta$, then $w_1 = f(\theta_2 - \theta_1, y_2 - y_1)w_2$ and $(\theta_1, y_1) = (\theta_2, y_2)$ by Lemma 3.2.

We set

$$A(w) = \langle J^{-1} \mathbf{w}, G(w) \rangle_{L^2}, \quad \text{for } w \in \Omega_{Q, \delta}.$$Notice that, by definition of $L$,

$$(L^2(\mathbb{R}^d))^m = L \oplus \langle J^{-1} \omega \mathbf{Q} \rangle \oplus \langle \partial_{\phi} \mathbf{Q} \rangle.$$Since $A$ is invariant by the action of $f(s, y)$, it is constant along $\mathbf{Q} + \langle J^{-1} \omega \mathbf{Q} \rangle = \mathbf{Q} + \langle \partial_{\phi} f(\theta, y) \mathbf{Q} \rangle_{(0,0)}$ and $\mathbf{Q} + \langle \partial_x \mathbf{Q} \rangle = \mathbf{Q} + \langle \partial_y f(\theta, y) \mathbf{Q} \rangle_{(0,0)}$. Hence $A'(\mathbf{Q}) = A|_{L}^{(\mathbf{Q})}$, and, for all $\gamma \in L$,

$$\langle A'(\mathbf{Q}), \gamma \rangle_{L^2} = \frac{d}{dt}A(\mathbf{Q} + \gamma t)|_{t=0} = \frac{d}{dt}\langle J^{-1} \mathbf{w}, \mathbf{Q} + \gamma t \rangle_{L^2}|_{t=0} = \langle J^{-1} \mathbf{w}, \gamma \rangle_{L^2}.$$Recalling that $J^{-1} \mathbf{w} \in L$, we conclude that $A'(\mathbf{Q}) = J^{-1} \mathbf{w}$. Moreover, since, by Assumption 3, $\mathbf{w} \in (H^1(\mathbb{R}^d))^m$, one may easily check that $A \in C^1((H^1(\mathbb{R}^d))^m)$.

Consider, for $v \in \Omega_{Q, \delta}$, the flow $z = \Lambda(t)v$ generated by the pseudo-Hamiltonian system

$$\begin{cases}
    z_t = JA'(z) \\
    z(0) = v
\end{cases}.$$**Remark 6.** The mass $M$ is conserved by the flow $\Lambda$. Indeed, notice that for all $u \in \Omega_{Q, \delta}$,

$$0 = \frac{d}{ds}A(e^{i\omega s}u)|_{s=0} = \langle A'(u), i\omega u \rangle_{L^2} = \langle A'(u), -JM'(u) \rangle_{L^2}$$

and so, taking $u = \Lambda(t)v$,

$$\frac{d}{dt}M(\Lambda(t)v) = \langle M'(\Lambda(t)v), JA'(\Lambda(t)v) \rangle_{L^2} = 0.$$Set

$$P : w \in \Omega_{Q, \delta} \rightarrow P(w) = \langle H'(w), JA'(w) \rangle_{H^{-1} \times H^{1}}.$$Due to Assumption 1, $P$ is well-defined and is continuous. We have the following result:

**Lemma 2.3.** For $w \in \Omega_{Q, \delta}, \delta$ small enough, and for small $t \neq 0$,

$$H(\Lambda(t)v) - H(v) < tP(v).$$

**Proof.** The Taylor expansion of $H(\Lambda(t)\mathbf{Q})$ at $t = 0$ reads

$$H(\Lambda(t)\mathbf{Q}) = H(\mathbf{Q}) + t\langle H'(\mathbf{Q}), JA'(\mathbf{Q}) \rangle_{L^2} + \frac{1}{2}t^2\langle H''(\mathbf{Q})A'(\mathbf{Q}), A'(\mathbf{Q}) \rangle_{L^2} + \langle H'(\mathbf{Q}), \Lambda''(0) \rangle_{L^2} + o(t^2)$$

$$= H(\mathbf{Q}) + t\langle H'(\mathbf{Q}), \Psi \rangle_{L^2} + \frac{1}{2}t^2\langle H''(\mathbf{Q})\Psi, \Psi \rangle_{L^2} + \langle H'(\mathbf{Q}), \Lambda''(0) \rangle_{L^2} + o(t^2). \quad (8)$$

Since $M$ is conserved by $\Lambda$,

$$\langle M'(\mathbf{Q}), \Psi \rangle_{L^2} = \langle M''(\mathbf{Q})\Psi, \Psi \rangle_{L^2} + \langle M'(\mathbf{Q}), \Lambda''(0) \rangle_{L^2} = 0.$$
Adding these terms to (5) and recalling that $S'(Q) = 0$,

$$H(\Lambda(t)Q) = H(Q) + tP(Q) + \frac{1}{2} t^2 \langle S''(Q)\Psi, \Psi\rangle_{L^2} + o(t^2).$$

Therefore, for some $C > 0$,

$$H(\Lambda(t)Q) - H(Q) - tP(Q) < -Ct^2, \quad t \text{ small}.$$

By continuity,

$$H(\Lambda(t)u) - H(u) - tP(u) \leq -Ct^2, \quad u \in O_{Q,\delta}, \quad t \text{ small}$$

and the Lemma is proved. \(\square\)

Let $V$ be the codimension one manifold on which $Q$ minimizes the action $S$. Observe that $J^{-1}\Psi$ is transverse to the manifold $V$: on one hand, $\langle S''(Q)J^{-1}\Psi, J^{-1}\Psi\rangle_{L^2} = \langle S''(Q)\Psi, \Psi\rangle_{L^2} < 0$; on the other hand, $Q$ is a minimum of the action over $V$, and therefore, considering the projection $J^{-1}\hat{\Psi}$ of $J^{-1}\Psi$ on the tangent space $T_QV$, $\langle S''(Q)J^{-1}\hat{\Psi}, J^{-1}\hat{\Psi}\rangle_{L^2} \geq 0$.

Set $K = O_{Q,\delta} \setminus V$ and fix $u \in K \cap M$. We claim that there exists $\epsilon > 0$ and $t_u \in ]-\epsilon, \epsilon[$ with

$$H(Q) < H(u) + t_u P(u). \quad (9)$$

Indeed, as a consequence of the transversality of $A'(Q) = J^{-1}\Psi$ with respect to $V$, the Implicit Function Theorem implies that, for $\epsilon > 0$ small and $v \in O_{Q,\delta}$, there exists $t_v \in ]-\epsilon, \epsilon[$ such that $\Lambda(t_v)v \in V$. Since $u \in K$, then necessarily $t_u \neq 0$. It follows from Lemma 2.3 that

$$H(\Lambda(t_u)u) < H(u) + t_u P(u).$$

The minimality of $Q$, together with $M(\Lambda(t_u)u) = M(u) = M(Q)$, implies that

$$H(Q) + M(Q) = S(Q) \leq S(\Lambda(t_u)u) = H(\Lambda(t_u)u) = H(\Lambda(t_u)u) + M(Q).$$

The claim now follows from the two previous inequalities.

**Proof of Theorem 2.7.** Consider $S_+ := \{u \in O_{Q,\delta} : H(u) < H(Q), M(u) = M(Q), P(u) > 0\}$. Since $Q$ is a local minimum of $S$ over $V$, $V \cap S_+ = \emptyset$, meaning that $S_+ \subset K \cap M$. If $u_0 \in S_+$, by the conservation of $M$ and $H$ and in view of (9),

$$0 < H(Q) - H(u_0(t)) < t_{u_0(t)} P(u(t)), \quad t_u \neq 0.$$

Hence $P(u(t)) \neq 0$ and, by continuity, $P(u(t)) > 0$, that is, $u \in S_+$. Hence $S_+$ is conserved by the flow generated by (1).

We may now conclude has in (17): for $u_0 \in S_+$ with $\|u_0 - Q\|_{L^2(Q)} = \delta$,\n
$$\frac{d}{dt} A(u(t)) = \langle u_t, A'(u(t))\rangle_{H^{-1} \times H^1} = \langle JH'(u(t)), J^{-1} u_t\rangle_{H^{-1} \times H^1} = \langle JH'(u(t)), A'(u(t))\rangle_{H^{-1} \times H^1} = P(u(t)) = \frac{H(Q) - H(u_0)}{t_u} > \frac{H(Q) - H(u_0)}{\epsilon} > 0,$$

one has

$$\lim_{t \to +\infty} |A(u(t))| = +\infty.$$

The contradiction follows from $|A(u(t))| \leq \|\Psi\|_{L^2}[G(u(t))]_{L^2} \leq \|\Psi\|_{L^2}(\|Q\|_{H^1} + \delta)$. \(\square\)
2.2 Construction of an unstable direction - Proof of Theorem [1.1]

In what follows, let us consider a real bound-state \( Q \in B_0 \) and a smooth path \( \Gamma : [0, \epsilon] \to \mathcal{M} \) given by

\[
\Gamma(t) = \left( \gamma_1(t)\lambda^2Q_1(\lambda(t)x), \ldots, \gamma_m(t)\lambda^2Q_m(\lambda(t)x) \right),
\]

with \( \Gamma(0) = Q \) (i.e. \( \lambda(0) = \gamma_j(0) = 1 \)). Our goal is to suitably choose \( \lambda, \gamma_j \) so that \( \Psi = \Gamma'(0) \) satisfies the conditions of Theorem 2.1. Now:

- By Assumption 3, \( \Psi \in (L^2(\mathbb{R}^d))^m \) and \( \langle S''(Q)\Psi, \Psi \rangle_{H^{-1} \times H^1} \) is well-defined.

- The condition \( \Gamma(t) \in \mathcal{M} \), which implies that \( \Psi \) is tangent to \( \mathcal{M} \) at \( Q \), is equivalent to \( \frac{d}{dt} M(\Gamma(t)) = 0 \), that is,

\[
\sum_j \gamma_j(t)\gamma_j'(t)\lambda_j\omega_j \int Q_j^2 = 0. \tag{10}
\]

- The fact that \( J^{-1}\Psi \) has complex components immediately implies that

\[
\langle J^{-1}\Psi, \partial_x \Psi \rangle_{L^2} = 0.
\]

Furthermore, \( \langle J^{-1}\Psi, i\omega Q \rangle_{L^2} = \langle \Psi, \nabla M(Q) \rangle_{L^2} = 0 \).

- Again, since \( Q \) is a real bound-state, \( J^{-1}\omega Q \) is orthogonal to \( \partial_x Q \).

Hence, we only need to see that \( \langle S''(Q)\Psi, \Psi \rangle_{H^{-1} \times H^1} < 0 \). We have

\[
\frac{d}{dt} S(\Gamma(t)) = \langle S'(\Gamma(t)), \Gamma'(t) \rangle
\]

and

\[
\frac{d^2}{dt^2} S(\Gamma(t)) = \langle S'(\Gamma(t)), \Gamma''(t) \rangle + \langle S''(\Gamma'(t)), \Gamma'(t) \rangle.
\]

Evaluating the above equality at \( t = 0 \), since \( Q = \Gamma(0) \) is a bound-state, we get

\[
\langle S''(\Psi), \Psi \rangle < 0 \iff \frac{d^2}{dt^2} S(\Gamma(t))|_{t=0} < 0 \iff \frac{d^2}{dt^2} H(\Gamma(t))|_{t=0} < 0
\]

(recall that \( \Gamma \subset \mathcal{M} \)). We now compute the Hamiltonian \( H \) along the path \( \Gamma \):

\[
H(\Gamma(t)) = \frac{1}{2} \sum_j \gamma_j^2(t)\lambda_j^2 \int |\nabla Q_j|^2 + \frac{1}{2} \sum_k \lambda_j^2 \frac{d}{dt} \left( \int n_k(\gamma_1(t)Q_1, \ldots, \gamma_m(t)Q_m) \right) = \frac{1}{2} \lambda^2(t) \sum_j \gamma_j^2(t) \int |\nabla Q_j|^2 + \frac{1}{2} \sum_k \left( \prod_j \gamma_j^{\beta_j,k}(t) \right) \lambda_j^{d_j-\gamma_j-k-1} \int n_k(Q).
\]

Differentiating with respect to \( t \), we obtain

\[
\frac{d}{dt} H(\Gamma(t)) = \lambda'(t)A(t) + \sum_j \gamma_j'(t)B_j(t)
\]

with

\[
A(t) = \lambda(t) \sum_j \gamma_j^2(t) \int |\nabla Q_j|^2 + \frac{d}{dt} \sum_k \left( \frac{\alpha_k}{2} - 1 \right) \lambda_j^{d_j-\gamma_j-k-1} \left( \prod_j \gamma_j^{\beta_j,k}(t) \right) \int n_k(Q)
\]

and

\[
B_j(t) = \lambda^2(t) \gamma_j(t) \int |\nabla Q_j|^2 + \frac{1}{2} \sum_k \lambda_j^{d_j-\gamma_j-k-1} \int n_k(Q).
\]
Since $Q \neq 0$, we may assume, without loss of generality, that $Q_m \neq 0$. Condition (10) then yields

$$\gamma'_m(t) = -\frac{1}{\gamma_m(t)} \sum_{j \neq m} k_j \gamma_j(t) \gamma'_j(t).$$

Thus,

$$\frac{d}{dt} H(\Gamma(t)) = \lambda'(t) A(t) + \sum_{j < m} \gamma'_j(t) \left( B_j(t) - \frac{1}{\gamma_m(t)} k_j \gamma_j(t) B_m(t) \right).$$

Now, observe that since $Q$ is a bound-state, $\frac{d}{dt} H(\Gamma(t)) \big|_{t=0} = 0$ independently of the choices of $\lambda, \gamma_1, \ldots, \gamma_{m-1}$. Hence,

$$A(0) = 0 \quad \text{and} \quad B_j(0) - k_j B_m(0) = 0,$$

and we easily deduce the following $m$ independent equalities regarding the bound-state $Q$:

**Proposition 2.4** (First integrals). Let $Q = (Q_1, \ldots, Q_m)$ a real bound-state of system (11), with $Q_m \neq 0$. Define

$$K = \sum_j k_j = \frac{1}{\lambda_m \omega_m} Q_m^2 M(Q).$$

Then, for any $1 \leq j \leq m$,

$$\int |\nabla Q_j|^2 = \frac{1}{2} \sum_k \left[ -\frac{k_j}{K} \left( 2 \frac{\alpha_k}{2} - 1 \right) \sum_l (k_l \beta_{m,k} - \beta_{l,k}) + k_j \beta_{m,k} - \beta_{j,k} \right] \int n_k(Q) =: \sum_j \sigma_{j,k} \int n_k(Q).$$

In particular,

$$\sum_j \int |\nabla Q_j|^2 = \sum_{j,k} \sigma_{j,k} \int n_k(Q) = -\frac{d}{2} \sum_k \left( \frac{\alpha_k}{2} - 1 \right) \int n_k(Q).$$

**Remark 2.5.** It is important to observe that

$$\sigma_{j,k} - k_j \sigma_{m,k} = -\frac{1}{2} (\beta_{j,k} - k_j \beta_{m,k}).$$

We now compute the second derivative at $t = 0$:

$$\frac{d^2}{dt^2} H(\Gamma(t)) \bigg|_{t=0} = \lambda'' A + \lambda' A' + \sum_{j < m} \gamma''_j \left( B_j - \frac{1}{\gamma_m} k_j \gamma_j B_m \right) + \sum_{j < m} \gamma'_j \left( B_j - \frac{1}{\gamma_m} k_j \gamma_j B_m \right)'.$$

From (11), the first and third terms are zero, and so

$$\frac{d^2}{dt^2} H(\Gamma(t)) \bigg|_{t=0} = \left[ \lambda' A' + \sum_{j < m} \gamma'_j \left( B'_j - k_j \gamma'_j B_m - k_j B'_m + k_j \gamma'_m B_m \right) \right] \bigg|_{t=0}.$$

Thus the second derivative is a quadratic form applied to $(\lambda'(0), \gamma'_1(0), \ldots, \gamma'_{m-1}(0))$. To simplify the following exposition, we write $\lambda'(0)$ (resp. $\gamma'_j(0)$) instead of $\lambda'(0)$ (resp. $\gamma'_j(0)$).

$$A'(0) = \sum_k \left[ \sum_j (\lambda' + 2 \gamma_j) \sigma_{j,k} + \frac{d}{2} \left( \frac{\alpha_k}{2} - 1 \right) \left( \frac{d\alpha_k}{2} - d - 1 \right) \lambda' + \frac{d}{2} \sum_{j} \left( \frac{\alpha_k}{2} - 1 \right) \beta_{j,k} \gamma'_j \right] \int n_k(Q).$$
For a fixed \( j_0 < m \),
\[
B'_{j_0}(0) = \sum_k \left( 2\lambda' + \gamma'_{j_0} \right) \sigma_{j_0,k} + \frac{d}{2} \left( \frac{\alpha_k}{2} - 1 \right) \beta_{j_0,k} \lambda' + \frac{1}{2} \beta_{j_0,k} (\sum_{j \neq j_0} \beta_{j,k}) \gamma'_{j_0} + \frac{1}{2} \beta_{j_0,k} \left( \sum_{j \neq j_0,m} \beta_{j,k} \gamma'_{j} \right) \int n_k(Q)
\]
\[
= \sum_k \left( 2\lambda' + \gamma'_{j_0} \right) \sigma_{j_0,k} + \frac{d}{2} \left( \frac{\alpha_k}{2} - 1 \right) \beta_{j_0,k} \lambda' + \frac{1}{2} \beta_{j_0,k} (\beta_{j_0,k} - 1) \gamma'_{j_0} + \frac{1}{2} \beta_{j_0,k} \left( \sum_{j \neq j_0,m} \beta_{j,k} \beta_{m,k} k_{j} \gamma'_{j} \right) \int n_k(Q)
\]

and
\[
B'_{m}(0) = \left[ 2\lambda' + \sum_{j < m} k_j \gamma'_{j} \right] \sigma_{m,k} + \frac{d}{2} \left( \frac{\alpha_k}{2} - 1 \right) \beta_{m,k} \lambda' + \frac{1}{2} \beta_{m,k} \sum_{j < m} (\beta_{j,k} - k_j (\beta_{m,k} - 1)) \gamma'_{j} \int n_k(Q)
\]

Writing the quadratic form in terms of a symmetric matrix \( A \), we now collect the various entries:

- \((\lambda')^2\):
  \[
a_{0,0} = \sum_k \left[ \sigma_{j,k} + \frac{d}{2} \left( \frac{\alpha_k}{2} - 1 \right) \left( \frac{\alpha_k}{2} - d - 1 \right) \right] \int n_k(Q) = \sum_k \frac{d}{2} \left( \frac{\alpha_k}{2} - 1 \right) \left( \frac{\alpha_k}{2} - d - 2 \right) \int n_k(Q)
\]

- \(\lambda' \gamma_{j_0}\)\:
  \[
a_{0,j_0} = \sum_k \left[ 2(\sigma_{j,k} - k_j \sigma_{m,k}) + \frac{d}{2} \left( \frac{\alpha_k}{2} - 1 \right) (\beta_{j,k} - k_j \beta_{m,k}) \right] \int n_k(Q)
\]

- \((\gamma'_{j_0})^2\):
  \[
a_{j_0,j_0} = \sum_k \left[ \sigma_{j_0,k} + \frac{1}{2} (\beta_{j_0,k} - 1) \beta_{j_0,k} - \frac{1}{2} k_{j_0,k} \beta_{m,k} \beta_{j_0,k} - k_{j_0} (\sigma_{m,k} + \frac{1}{2} \beta_{m,k}) \right.
  + \frac{k_{j_0}^2}{2} \beta_{m,k} + \frac{1}{2} k_{j_0,k}^2 \beta_{m,k} (\beta_{m,k} - 1) - \frac{1}{2} k_{j_0} \beta_{m,k} \beta_{j_0,k} - \frac{1}{2} \beta_{m,k} k_{j_0}^2 \left] \int n_k(Q)
\]

- \(\gamma_{j_0} \gamma_{j_1}, j_0 \neq j_1\):
  \[
a_{j_0,j_1} = \sum_k \left[ \frac{1}{2} \beta_{j_0,k} \beta_{j_1,k} - \frac{1}{2} k_{j_1} \beta_{m,k} \beta_{j_0,k} + k_{j_0,k} \sigma_{m,k} + \frac{1}{2} k_{j_0,k} \beta_{j_1,k} \right.
  - \frac{1}{2} k_{j_0} \beta_{m,k} \beta_{j_1,k} - k_{j_0,k} \beta_{m,k} - \frac{1}{2} k_{j_0} k_{j_1} \beta_{m,k} \left] \int n_k(Q)
\]

Hence the symmetric form is represented by the matrix \( A \) given in the statement of Theorem \ref{thm:1}. Since, by assumption, \( A \) has a negative eigenvalue, there exists a nontrivial choice of \((\lambda'(0), \gamma'_1(0), \ldots, \gamma'_{m-1}(0))\) such that
\[
\langle S'(Q) \Gamma'(0), \Gamma'(0) \rangle = \frac{d}{dt} H(t) \bigg|_{t=0} < 0.
\]

This concludes the proof of Theorem \ref{thm:1}. \qed
Proof of Proposition 1.2. Just notice that

\[ a_{0,0} = \frac{d}{2} \sum_k \left( \frac{\alpha_k}{2} - 1 \right) \left( \frac{d\alpha_k}{2} - d - 2 \right) \int n_k(Q) \]

\[ \leq \frac{d}{2} \left( \frac{dp}{2} - d - 2 \right) \sum_k \left( \frac{\alpha_k}{2} - 1 \right) \int n_k(Q) = -\left( \frac{dp}{2} - d - 2 \right) \sum_j \|Q_j\|^2 < 0 \]

from Proposition 2.4. \qed

Proof of Proposition 1.3. The specific homogeneities of \( N \) imply that \( a_{0,0} = 0 \). Furthermore, \( a_{0,1} = -2 \left( c_1 \int Q_1^2 - k_1 c_m \int Q_m^2 \right) = -2 \left( c_1 - \frac{c_m \lambda \omega}{\lambda m \omega_m} \right) \int Q_1^2 \neq 0. \)

Hence the principal minor

\[ A_1 = \begin{bmatrix} a_{0,0} & a_{0,1} \\ a_{0,1} & a_{1,1} \end{bmatrix} \]

has negative determinant and \( A \) cannot be semi-positive definite. \qed

Proof of Proposition 1.4. As in the previous proof, it suffices to check that \( a_{0,0} = 0 \) and \( a_{0,1} \neq 0 \), which is a direct consequence of the hypothesis. \qed

3 Applications

Before we begin to study the examples stated in the Introduction, it will be useful to recall some facts regarding synchronous systems. Suppose that \( N(u) = N_2(u) + N_p(u) \),

\[ N_2(u) = \sum_{j=1}^m c_j \int |u_j|^2, \quad N_p(u) = -\int f(u) \]

with \( f : \mathbb{C}^m \to \mathbb{R} \), homogeneous of degree \( p \), such that \( f(X) \leq f(|X|) \), for all \( X \in \mathbb{C}^m \). By synchronicity, we mean that

\[ c_j + \lambda_j \omega_j = c + \lambda \omega > 0, \quad j = 1, \ldots, m. \]

Define

\[ f_{\text{max}} = \sup_{X \in S^{m-1}} f(X). \]

\[ \mathcal{X} = \{ X \in S^{m-1} : f(X) = f_{\text{max}} \}. \]

and consider the scalar equation

\[ -(c + \lambda \omega)u + \Delta u + au^{p-1} = 0. \]  \( (13) \)

Proposition 3.1. If \( X_0 \in \mathbb{R}^m \) is a critical point of \( f \) on the sphere and \( u \) is a solution of (13) with \( a = pf(X_0)/2 \), then \( X_0u \) is a bound-state of (1). Furthermore, the set of ground-states is given by

\[ G = \{ Xq : X \in \mathcal{X}, \quad q \text{ ground-state of (13)} \text{ with } a = pf_{\text{max}}/2 \}. \]

Proof. Observe that, for some \( \gamma \in \mathbb{R} \), \( \nabla f(X_0) = \gamma X_0. \) Since \( f \) is homogeneous of degree \( p \),

\[ pf(X_0) = \nabla f(X_0) \cdot X_0 = \gamma |X_0|^2 = \gamma. \]

One may now check that \( X_0u \) satisfies the elliptic system for the bound-states. The characterization of \( G \) follows the exact same argument as in [5]. \qed
3.1 Quadratic Schrödinger system I

We recall (5):

\[
\begin{align*}
& iu_t + \Delta u + \overline{u}v = 0 \\
& i\sigma v_t + \Delta v - \beta v + \frac{1}{2}u^2 = 0, \quad (x, t) \in \mathbb{R}^d \times \mathbb{R}.
\end{align*}
\]

Here,

\[
N(u, v) = \beta \int |v|^2 - \text{Re} \int |\overline{u}v|,
\]

\[n_1 = \beta |v|^2 \quad (\beta_{1,1} = 0, \beta_{2,1} = 2), \quad n_2 = -\text{Re} \beta \overline{u}^2v \quad (\beta_{1,2} = 2, \beta_{2,2} = 2)\]

and

\[
M(u, v) = \int |u|^2 + 2\sigma \int |v|^2, \quad k_1 = \frac{\int Q_1^2}{2\sigma \int Q_2^2}.
\]

The associated matrix defined in Theorem 1.1 is

\[
A = \begin{bmatrix}
\frac{d}{8}(4 - d) \int Q_1^2 Q_2 & 2k_1\beta \int Q_2^2 + \frac{1}{4}(d - 4)(k - 2) \int Q_1^2 Q_2 \\
2k_1\beta \int Q_2^2 + \frac{1}{4}(d - 4)(k - 2) \int Q_1^2 Q_2 & \frac{k_1}{2} (k_1 + 4) \int Q_1^2 Q_2
\end{bmatrix}
\]

Proof of Proposition 1.6. For a given frequency \( \omega > 0 \), one may look for bound-states of the form \((u, v) = (e^{i\omega t}P, e^{2i\sigma \omega t}Q)\). The corresponding stationary system is

\[
\begin{align*}
& -\omega P + \Delta P + \overline{P}Q = 0 \\
& -2\omega \sigma + \Delta Q + \frac{1}{2}P^2 = 0
\end{align*}
\]

Observe that, if \( \beta = \omega (1 - 2\sigma) \), the system is synchronous. Thus, by Proposition 3.1, the ground-state can be computed explicitly:

\[
(P, Q) = \left( \sqrt{\frac{2}{3}}q, \frac{1}{\sqrt{3}}q \right),
\]

where \( q \) is the ground-state of

\[-\omega q + \Delta q + \frac{1}{\sqrt{3}}q^2 = 0.\]

Consequently,

\[
k_1 = \frac{1}{\sigma}, \quad k_1\beta = \frac{1 - 2\sigma}{\sigma} - \omega, \quad \int Q_1^2 Q_2 = \frac{2}{3\sqrt{3}} \int q^3, \quad \int Q_2^2 = \frac{1}{3} \int q^2.
\]

Furthermore, one can see (cf. [2 Corollary 8.1.3]) that

\[
\int q^2 = \frac{6 - 4}{6\sqrt{3}\omega} \int q^3.
\]

Since \( a_{0,0} > 0 \) in all \( L^2 \)-subcritical cases, the condition for instability reduces to \( \det(A) < 0 \):

\[
\frac{d(4 - d)(1 + 4\sigma)}{108\sigma^2} - \left( \frac{(1 - 2\sigma)(6 - d)}{9\sqrt{3}\sigma} + \frac{(d - 4)(1 - 2\sigma)}{6\sqrt{3}\sigma} \right)^2 < 0, \text{ that is, } (2\sigma - 1)^2 > 3(4 - d).
\]

Since \( \sigma > 0 \), we observe orbital instability when \( \sigma > \sigma_0 \).

\[\Box\]
3.2 Quadratic Schrödinger system II

\[
\begin{align*}
3iu_t + \Delta u - \beta_1 u &= -vw \\
2iv_{x} + \Delta v - \beta v &= -\frac{1}{2}w^2 - u\overline{w} \\
iw_t + \Delta w - w &= -\overline{u}w - u\overline{w}, \quad (x, t) \in \mathbb{R} \times \mathbb{R}.
\end{align*}
\]

In this case, \((\omega_1, \omega_2, \omega_3) = (3, 2, 1), (\lambda_1, \lambda_2, \lambda_3) = (3, 2, 1)\) and

\[
N(u, v, w) = \beta_1 \int |u|^2 + \beta \int |v|^2 + \int |w|^2 - Re \int \overline{w}^2 v - 2 \int \overline{w}vw.
\]

The synchronicity is obtained for \(\beta_1 = -7\) and \(\beta = -2\), and, in this situation,

\[
N_2(u, v, w) = -7 \int |u|^2 - 2 \int |v|^2 + \int |w|^2 \text{ and } N_3(u, v, w) = -Re \int \overline{w}^2 v - 2 \int \overline{w}vw.
\]

**Proof of Proposition** We apply Proposition setting \(f(x, y, z) = yz^2 + 2xyz\), elementary computations show that the maximum is \(f_{max} = (\sqrt{3} + \sqrt{3})/9\) and that it is achieved at

\[
(a, b, c) = \left(\sqrt{\frac{1}{15}(5 - \sqrt{5})}, \frac{1}{\sqrt{3}}, \sqrt{\frac{1}{15}(5 + \sqrt{5})}\right) \text{ and } (-a, b, -c).
\]

Hence the set of ground-states is given by

\[
G = \{(aQ, bQ, cQ), (-aQ, bQ, -cQ)\} \text{ modulo translations and rotations,
}
\]

where \(Q\) satisfies

\[-2Q + \Delta Q + 3f_{max}Q^2 = 0.\]

For either ground-state, the entries of matrix \(A\) read

\[
a_{0, 0} = \frac{3}{4}f_{max} \int Q^3; \quad a_{1, 1} = abc(k_1 + 1)^2 \int Q^3; \quad a_{2, 2} = bc(2k_2 + \frac{1}{2})c + (k_2 + 1)^2a \int Q^3,
\]

\[
a_{0, 1} = (14a^2 + 2k_1c^2) \int Q^2 - \frac{3}{4}(2k_1bc^2 + (k_1 - 1)abc) \int Q^3
\]

\[
a_{0, 2} = (4b^2 + 2k_2c^2) \int Q^2 - \frac{3}{4}(2k_2 - 1)bc^2 + 2(k_2 - 1)abc) \int Q^3
\]

\[
a_{1, 2} = \left(bc^2k_1 + abc(k_1 + k_2 + k_1k_2 - 1)\right) \int Q^3.
\]

Once again, by Corollary 8.1.3,

\[
\int Q^2 = \frac{5}{4}f_{max} \int Q^3.
\]

Finally, since \(k_1 = \frac{9a^2}{c^2}\) and \(k_2 = \frac{4b^2}{c^2}\), putting \(a_{i,j} = \tilde{a}_{i,j} \int Q^3\),

\[
\tilde{a}_{0, 0} = \frac{3}{4}f_{max}
\]

\[
\tilde{a}_{1, 1} = \frac{c^3}{ab(9a^2 + c^2)^2}
\]

\[
\tilde{a}_{2, 2} = \frac{f_{max} + b(4b^2 + c^2)^2}{c^3}
\]

\[
\tilde{a}_{0, 1} = \frac{115}{16}f_{max} - \frac{15}{16}f_{max} \left(18a^2b + \frac{ab(9a^2 + c^2)}{c^2}\right)
\]

\[
\tilde{a}_{0, 2} = \frac{15b^2f_{max} - \frac{3}{4}(8b^2 - bc^2 + 2ab(4b^2 - c^2))}{c^3}
\]

\[
\tilde{a}_{1, 2} = \frac{9a^2b + \frac{ab}{c}(9a^2 + 4b^2 - c^2 + \frac{36a^2b^5}{c^2})}{c^3}
\]

With these values, we get \(\text{det}(A) < 0\), and therefore both ground-states are orbitally unstable. \(\square\)
In this section, we formally deduce some identities regarding (1) using the Hamiltonian structure. Suppose that one wishes to understand the evolution of a functional $G$ through the trajectories of (1):

$$\frac{d}{dt} G(p) = \langle G'(u(t)), u_t \rangle = \langle G'(u(t)), JH'(u(t)) \rangle = P(u(t)).$$

Consider the Hamiltonian system generated by $G$,

$$v_t = JG'(v).$$

(14)

and prescribe an initial condition $v_0$. Then

$$\frac{d}{dt} H(v) = \langle H'(v(t)), v_t \rangle = \langle H'(v(t)), JG'(v(t)) \rangle = -\langle G'(v(t)), JH'(v(t)) \rangle = -P(v(t)).$$

Taking $t = 0$, we obtain an alternative definition for $P$:

$$P(v_0) = \frac{d}{dt} H(v) \bigg|_{t=0}.$$

Therefore, the variation of $G$ along the trajectories generated by $H$ is symmetric to the variation of $H$ along the trajectories generated by $G$ at the same state. This duality corresponds to the symmetry of the Poisson bracket in Hamiltonian mechanics. The advantage of this formulation is that the dynamical system (14) is usually explicitly solvable and the computation becomes trivial.

**Proposition A.1** (Conservation of mass and energy). Regarding the flow generated by (1),

1. the Hamiltonian $H$ is conserved;
2. the mass

$$M(u(t)) = \frac{1}{2} \sum_j \int \lambda_j \omega_j |u_j(t)|^2$$

is conserved.

**Proof.** The conservation of $H$ is trivial: one takes $G = H$ and obtains $P = -P$. For the conservation of mass, observe that the dynamical system generated by $M$ is

$$(u_j)_t = -i \omega_j v_j, \quad j = 1, \ldots, m.$$ 

The solution of the IVP $v(0) = v_0$ is

$$v_j(t) = e^{-i \omega_j t} v_0 \quad j = 1, \ldots, m.$$ 

The invariance (2) implies that $H$ is constant along these trajectories, and thus $P = 0$. \qed

**Proposition A.2** (Virial identities). Define

$$V(u) = \frac{1}{2} \sum_j \int \lambda_j \omega_j |x|^2 |u_j|^2 dx.$$ 

Define the space

$$\Sigma = \{ u \in (H^1(\mathbb{R}^d))^m : V(u) < \infty \} ,$$

equipped with the natural norm. Given $u_0 \in \Sigma$, the corresponding solution $u$ of (1) belongs to $C([0, T(u_0)), \Sigma)$ and

$$\frac{d}{dt} V(u(t)) = 2 \sum_j \int \omega_j u_j(t) x \cdot \nabla u_j(t) dx.$$ 

(15)

Furthermore, if $\omega_j/\lambda_j = \omega_0/\lambda_0$, 

$$\frac{d^2}{dt^2} V(u(t)) = \frac{8 \omega_0}{\lambda_0} H(u_0) - \frac{\omega_0}{\lambda_0} \sum_k (4 - N(\alpha_k - 2)) \int n_k(u(t)).$$ 

(16)
Proof. We present only the formal computation. The rigorous justification of \( u \in C([0, T(u_0)), \Sigma) \) follows from the same arguments as for the single (NLS) equation (see, for example, [2, Chapter 6]). The solution of the initial value problem
\[
v_t = JV'(v), \quad v(0) = v_0
\]
is
\[
v_j(t) = e^{-i\omega_j|\cdot|^2t}(v_0)_j, \quad j = 1, \ldots, m.
\]
The invariance [2] implies that
\[
\frac{d}{dt} H(v(t)) = \frac{1}{2} \sum_j \int |\nabla (v_0)_j - 2i\omega_j x (v_0)_j|^2 dx
\]
The identity [15] follows from the evaluation at \( t = 0 \).

For the second derivative, one must solve the auxiliary dynamical system associated with the right-hand side of (15),
\[
(v_j)_t = \frac{4\omega_j}{\lambda_j} x \cdot \nabla v_j - \frac{2d\omega_j}{\lambda_j} v_j, \quad v_j(0) = (v_0)_j, \quad j = 1, \ldots, m.
\]
Since \( \omega_j/\lambda_j = \omega_0/\lambda_0 \), the explicit solution is given by
\[
v_j(t) = e^{-\frac{2d\omega_0}{\lambda_0}t}(v_0)_j(e^{-\frac{4\omega_0}{\lambda_0}t}x), \quad j = 1, \ldots, m.
\]
Hence, using a change of variables,
\[
H(v(t)) = \frac{1}{2} \sum_j \int e^{-\frac{2d\omega_0}{\lambda_0}t}|\nabla (v_0)_j|^2 dx + \frac{1}{2} \sum_k \int e^{(4-2\alpha_k)}\frac{4\omega_0}{\lambda_0}n_k(v_0) dx
\]
Differentiating in time and taking \( t = 0 \), one arrives at the identity [16].

Remark 7. If \( u = e^{i\omega t}Q \) is a bound-state, then one obtains the Pohozaev identity
\[
0 = \frac{\lambda_0}{\omega_0} \frac{d^2V(u(t))}{dt^2} = H(Q) - \sum_k (4 - d(\alpha_k - 2)) \int n_k(Q).
\]
If \( \alpha_k = 2 + 4/d \) for all \( k \), one sees that \( H(Q) = 0 \).

As a direct consequence from the Virial identities, one has the standard blow-up result using Glassey’s argument:

**Proposition A.3** (Blow-up). Suppose that \( \omega_j/\lambda_j = \omega_0/\lambda_0 \) and that

- \( n_k \leq 0 \) for \( \alpha_k > 2 + 4/d \);
- \( n_k \geq 0 \) for \( \alpha_k < 2 + 4/d \).

Then any initial data \( u_0 \in \Sigma \) with negative energy gives rise to a solution which blows-up in finite-time. In particular, if \( \alpha_k = 2 + 4/d \) for all \( k \), any bound-state is orbitally unstable.

Remark 8. Even in the scalar case, the instability of bound-states through a blow-up argument does not work when there are terms in the Hamiltonian which are not \( L^2 \)-critical. In the \( L^2 \)-supercritical case, one may still obtain instability for ground-states, by proving that they are the minimizers of a certain minimization problem. For general nonlinearities and/or bound-states, instability by blow-up remains a challenging open problem.
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