Biquaternions algebra is complex expansion of quaternions algebra, which was elaborated by W.R.Hamilton in the middle of XIX century [1]. It is very useful and very suitable apparatus for description of many physical processes. In last decennial these algebras become be actively used in research of different problems of electrodynamics [2-4], quantum mechanics [5-9], fields theory [10-13] and others.

This algebra gives possibility to write many well-known systems of differential equations in theoretical physics in form of one biquaternionic equation in bigradiental form. Concept bigradient is theoretical generalization of gradient concept, differential operator which characterizes a direction and intensity of exchange of scalar field. By analogy bigradient characterizes a direction and intensity of exchange of biquaternionic fields which are used for description of complex scalar-vector physical fields.

Here we elaborate the differential algebra of biquaternions on Minkowski space for construction of generalize solutions of bigradiental biquaternionic equations, which are equivalent to the hyperbolic systems of differential equations of 8 order. We consider biquaternionic wave equation (biwave equation) and obtain on the base of Fourie transform its fundamental and generalized solutions. Shock waves are researched and conditions on their fronts are determined. Invariance of biwave equation for the groups of orthogonal transforms, Lorentz and Poincaré transforms are explored and relativistic formulas are constructed.The new scalar equations for potentials of bigradiental fields (KGFSh-equation)) is obtained, which contains Klein-Gordon-Fokk and Shrédinger operators, and its generalized solutions are defined. The biquaternionic representation of Maxwell and Dirac equations are considered, their fundamental and generalized solutions are obtained, which describe nonstationary, harmonic and static scalar-vector electromagnetic fields, spinors and spinors fields.

Properties of superposition of bigradiental operators, which is very easy calculated in biquaternions algebra, and methods of generalized functions theory make the process of solutions construction simple and fine, as you’ll see hereinafter.

1. Biquaternions algebra

We introduce some notions and indications, which will be used here. Let \( e_1, e_2, e_3 \) are basis vectors of cartesian coordinate system in \( R^3 \), \( e_0 = 1 \); \( F \) is three dimensional vector with complex components: \( F = F_1 e_1 + F_2 e_2 + F_3 e_3 \), \( F_j, f \in \mathbb{C} \) are complex numbers; \( \varepsilon_{jkl} \) is Leavy-Chivitta pseudotensor, \( \delta_{jk} \) is Kronecker symbol.

We consider the linear space of hypercomplex numbers (biquaternions) \( B = \{ F = f + F \} \):

\[
aF + bG = a(f + F) + b(g + G) = (af + bg) + (aF + bG), \quad \forall a, b \in \mathbb{C},
\]

with operation of quaternionic multiplication [1]:

\[
F \circ G = (f + F) \circ (g + G) = (fg - (F, G)) + (fG + gF + [F, G]).
\]
Here and hereinafter we designate as \( (F, G) = \sum_{j=1}^{3} F_j G_j \) the scalar product of vectors \( F \) and \( G \); their vector product is \([F, G] = \sum_{j,k,l=1}^{3} \varepsilon_{jkl} F_j G_k e_l\).

Biquaternions algebra is associative:
\[
F \circ G \circ H = (F \circ G) \circ H = F \circ (G \circ H),
\]
but not commutative:
\[
< F, G >= F \circ G - G \circ F = 2[F, G].
\]
It may be proved on the base of these properties of Leavy-Chivita pseudotensor and basic elements:
\[
\varepsilon_{jkl} = \varepsilon_{ijk} = \varepsilon_{kjl}, \quad \varepsilon_{jkl} = -\varepsilon_{jlk} = -\varepsilon_{kjl}; \quad \varepsilon_{jkl} \varepsilon_{mnl} = \delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km},
\]
\[
e_0 \circ e_0 = e_0, \quad e_0 \circ e_j = e_j, \quad e_j \circ e_k = -\delta_{jk} + \varepsilon_{jkl} e_l,
\]
\[
(e_j \circ e_k) \circ e_m = -\varepsilon_{jkm} - \delta_{jk} e_m - \delta_{km} e_j + \delta_{mj} e_k = e_j \circ (e_k \circ e_m), \quad j, k, l, m, n = 1, 2, 3,
\]
(over similar indexes in product there are summation from 1 to 3, like to tensor convolution).

From the properties of commutator (3) it follows

**Jacobi identity**
\[
<< F, G >, H > + << H, F >, G > + << G, H >, F >> =
\]
\[
= 4[[F, G], H] + 4[[H, F], G] + 4[[G, H], F] =
\]
\[
= -4(F(H, G) - G(H, F) + H(G, F) - F(G, H) + G(F, H) - H(F, G)) = 0.
\]
I.e. biquaternions algebra is **Lie algebra**.

From (3) we see, that the product of two biquaternions is commutative if one of them is a scalar or if their vector parts are parallel.

**Some definitions.**

Bq. \( F^- = f - F \) is called **mutual** for \( F = f + F \).

Bq. \( \bar{F} = \bar{f} + \bar{F} \) is named **complex-conjugate** to \( F \) (overline marks the complex conjugate number). If \( \bar{F} = F \circ \bar{F} = F \circ F = 1 \), then we name \( F \) **unitary**.

\( F^* = \bar{f} - \bar{F} \) is termed **conjugate** to \( F \). If \( F^* = F \), then it is **selfconjugated**.

Selfconjugated biquaternion has the form: \( F = f + iF \), where \( f \) and \( F \) are real.

**Scalar product** of \( F_1, F_2 \) is the bilinear operation \( (F_1, F_2) = f_1 f_2 + (F_1, F_2) \).

**Norm of** \( F \) is nonnegative scalar value
\[
\|F\| = \sqrt{(F, \bar{F})} = \sqrt{f \cdot \bar{f} + (F, \bar{F})} = \sqrt{|f|^2 + \|F\|^2}.
\]
If \( F \) is real biquaternion (quat**ernion**) then \( F^* = F^- \) and
\[
\|F\|^2 = F^* \circ F = F \circ F^*.
\]

**Pseudonorm** of \( F \) is the value
\[
\langle\langle F\rangle\rangle = \sqrt{f \cdot \bar{f} - (F, \bar{F})} = \sqrt{|f|^2 - \|F\|^2}, \quad Re\langle\langle F\rangle\rangle \geq 0.
\]
It's easy to see, if \( F \) is selfconjugated then \( \bar{F} = F^\perp \) and
\[
\bar{F} \circ F = F \circ \bar{F} = \langle \langle F \rangle \rangle^2.
\] (7)

If \( F \circ G = 1 \) then \( G \) is right inverse for \( F \) and it is termed as \( G = F^{-1} \), thereafter \( F \) is left inverse for \( G \) and it is termed as \( -1^G G \).

Simply the equalities are proved:
\[
(F \circ G)^* = G^* \circ F^*, \quad (F \circ G)^{-1} = G^{-1} \circ F^{-1}.
\] (8)

Also simply the next lemma and theorem are proved.

**Lemma 1.1.** If \( (F, F) \neq 0 \), then both inverse biquaternions exist and they are equal:
\[
F^{-1} = -1^F F = F^{-1}/(F, F).
\] (9)

If \( (F, F) = 0 \) then inverse Bq. does not exist.

**Theorem 1.1.** By known \( F \) and \( B \) biquaternionic linear (bilinear) equations of a kind
\[
F \circ G = B \quad \text{and} \quad G \circ F = B
\]
have unique solution \( G = F^{-1} \circ B \) and \( G = B \circ F^{-1} \) appropriately, if and only if \( (F, F) \neq 0 \).

Proof of this theorem follows from the lemma 1.1.

If \( (F, F) = 0 \), then existence of the decision depends on \( B \) (see [14]).

We introduce here the very useful Bq.
\[
\Xi = W + iP = \frac{1}{2} F \circ F^* = (|f|^2 + \|F\|^2)/2 + i \left( \text{Im} \left( \overline{f}F \right) + \left[ \text{Re} \, F, \text{Im} \, F \right] \right).
\]
\[
\langle \langle \Xi \rangle \rangle^2 = W^2 - \|P\|^2 = \Xi \circ \Xi^\perp.
\]

Here \( W \) and \( P \) are real values. It describes the energy-impulse density of scalar-vector field in problems of theoretical physics.

2. Lorentz and Poincaré transformations on Minkowski space

Let consider biquaternions on Minkowski space \( M = \{(\tau, x) : \tau \in R^1, x \in R^3\} \) and groups of linear transformations on it. We quoternize \( M \) by using complex-conjugate Bqs.:
\[
Z = \tau + ix, \quad \bar{Z} = Z^\perp = \tau - ix, \quad \tau \in R^1, \quad x \in R^3.
\]
They are selfconjugated: \( Z = Z^* \), \( \bar{Z} = \bar{Z}^* \), and have equal norms and pseudonorms:
\[
\|Z\|^2 = \|\bar{Z}\|^2 = \tau^2 + \|x\|^2 = (Z, \bar{Z}), \quad \langle \langle Z \rangle \rangle^2 = \langle \langle \bar{Z} \rangle \rangle^2 = \tau^2 - \|x\|^2 = Z \circ \bar{Z}
\] (10)

and
\[
Z^{-1} = \frac{\bar{Z}}{\langle \langle Z \rangle \rangle^2}, \quad \bar{Z}^{-1} = \frac{Z}{\langle \langle Z \rangle \rangle^2}.
\]

Hence on light cone (\( |\tau| = \|x\| \)) inverse Bq. for \( Z, \bar{Z} \) does not exist.

**Orthogonal transformation.** Let consider conjugate Bqs.:
\[
U(\varphi, e) = \cos \varphi + e \sin \varphi, \quad U^* = U^\perp = \cos \varphi - e \sin \varphi, \quad \|e\| = 1, \quad \varphi \in R^1.
It’s easy to see:

\[ \| U \| = \| U^* \| = 1, \quad U \circ U^* = U^- \circ U = 1. \]  \hspace{1cm} (11)

**Lemma 2.1.** Conjugate biquaternions \( U(\varphi, e), U^*(\varphi, e), \varphi \in R^1 \), define the group of orthogonal transformation on \( M \), which are orthogonal on vector part of \( Bq \). \( Z' = U \circ Z \circ U^*, \ Z = U^* \circ Z' \circ U. \)

This transformation is the rotation of space \( R^3 \) around vector \( e \) through angle \( 2\varphi \). \( \hspace{1cm} \)

**Proof.** By calculating lemmas formulae we get conservation of scalar part and specified rotating of vector part:

\[ \tau' = \tau, \quad x' = e(e, x) + (x - e(e, x))\cos 2\varphi + [e, x] \sin 2\varphi. \]

Under (10)-(11) the pseudonorm \( \langle\langle Z'\rangle\rangle^2 = U \circ Z \circ U^* \circ U \circ Z' \circ U \rangle = \langle\langle Z\rangle\rangle^2 \). Since \( \tau = \tau' \), the norm of vector \( Z \) hold true: \( \| Z \| = \| Z' \|. \)

Let consider superposition of two orthogonal transformation:

\[ U_1 \circ U_2 = U_3 = u_3 + U_3 = (\cos \varphi_1 + e_1 \sin \varphi_1) \circ (\cos \varphi_2 + e_2 \sin \varphi_2) = \]

\[ \cos \varphi_1 \cos \varphi_2 - (e_1, e_2) \sin \varphi_1 \sin \varphi_2 + e_2 \sin \varphi_1 \cos \varphi_2 + \]

\[ + [e_1, e_2] \sin \varphi_2 \sin \varphi_2. \]

Since

\[ U_3^{-1} = (U_1 \circ U_2)^{-1} = U_2^{-1} \circ U_1^{-1} = U_2^* \circ U_1^*, \]

\[ U_3^* = (U_1 \circ U_2)^* = U_2^* \circ U_1^* = U_2 \circ U_1^- = U_3^-, \]

we get:

\[ U_3 \circ U_3^- = (U_1 \circ U_2) \circ (U_1 \circ U_2)^- = U_1 \circ U_2 \circ U_2^- \circ U_1^- = 1 \]

Hence \( U_3 \) is also orthogonal transformation:

\[ U_3 = \cos \varphi_3 + e_3 \sin \varphi_3, \quad \varphi_3 = \arccos(u_3), \quad e_3 = \frac{U_3}{\| U_3 \|}. \]

Lemma has been proved.

**Lorentz transformations.** Let consider mutual selfconjugated Bgs.

\[ L(\theta, e) = \cosh \theta + i e \sinh \theta, \quad L^- = \cosh \theta - i e \sinh \theta, \quad \theta \in R^1, \quad \| e \| = 1 \]

(here hyperbolic sine and cosine are used). It’s easy to see that they are unitary:

\[ L \circ L^- = \cosh \theta^2 - \sinh \theta^2 = 1. \]  \hspace{1cm} (12)

Next lemmas are proved by use of simply calculations [14].

**Lemma 2.2.** Lorentz transformation has such biquaternionic representation:

\[ Z' = L \circ Z \circ L, \quad Z = L^- \circ Z' \circ L^-, \quad \langle\langle Z'\rangle\rangle^2 = \langle\langle Z\rangle\rangle^2. \]  \hspace{1cm} (13)
Also easy to show that pseudonorm conserves on the associativity and unitarians (12):

\[ \langle\langle Z'\rangle\rangle^2 = L \circ Z \circ L \circ L^- \circ Z \circ L^- = \langle\langle Z\rangle\rangle^2. \]

If to enter designations:

\[ ch2\theta = (1 - v^2)^{-1/2}, \quad sh2\theta = v(1 - v^2)^{-1/2}, \quad |v| < 1, \]

After calculating we get known formulas for scalar and vector parts of \( Z' \) and \( Z \) [13]: relativistic formulas

\[
\tau' = \frac{\tau + v(e, x)}{\sqrt{1 - v^2}}, \quad x' = (x - e(e, x)) + e\left(\frac{e(e, x)}{\sqrt{1 - v^2}} + v\tau\right),
\]

\[
\tau = \frac{\tau' - v(e, x)}{\sqrt{1 - v^2}}, \quad x = (x' - e(e, x')) + e\left(\frac{e(e', x')}{\sqrt{1 - v^2}} - v\tau'\right),
\]

It corresponds to motion of coordinates system \( \{X_1, X_2, X_3\} \) in direction of vector \( e \) with velocity \( v \).

Superposition of two Lorentz transformations with equal \( e_j \) possess with group properties:

\[
L_1 \circ L_2 = L_3 = l_3 + L_3 = (ch\varphi_1 + iesh\theta_1) \circ (ch\varphi_2 + iesh\theta_2) =
\]

\[
= (ch\varphi_1 ch\varphi_2 + sh\theta_1 sh\theta_2) + i e (sh\theta_2 ch\varphi_1 + sh\theta_1 ch\varphi_2) =
\]

\[
= ch(\varphi_1 + \varphi_2) + esh(\theta_1 + \theta_2)
\]

But superposition of two Lorentz transformations with nonparallel \( e_1 \) and \( e_2 \) does not constitute the Lorentz transformation:

\[
L_1 \circ L_2 = L_3 = l_3 + L_3 = (ch\varphi_1 + i e_1 sh\theta_1) \circ (ch\varphi_2 + i e_2 sh\theta_2) =
\]

\[
= (ch\varphi_1 ch\varphi_2 + (e_1, e_2) sh\theta_1 sh\theta_2) + i [e_2 sh\theta_2 ch\varphi_1 + e_1 sh\theta_1 ch\varphi_2] - [e_1, e_2] sh\theta_2 sh\theta_1
\]

What see, this Bq. contains real value in vector part.

**Poincaré transformation.** Note that

\[
L(\theta, e) = U(-i\theta, e).
\]

Using superpositions of these two transformations we get general form of the transformation, which may be named Poincaré transformation.

**Definition.** Poincaré transformation on \( M \) is the linear transformation of the type:

\[
Z' = P \circ Z \circ P^*, \quad Z = P^- \circ Z' \circ P^* - ,
\]

\[
P = U \circ L = cos(\varphi + i\theta) + esin(\varphi + i\theta), \quad P^* = L^* \circ U^* = cos(\varphi - i\theta) - esin(\varphi - i\theta),
\]

which conserves the pseudonorm: \( \langle\langle Z\rangle\rangle = \langle\langle Z'\rangle\rangle \).

It’s easy to prove because \( P \circ P^* = P^* \circ P^* - = 1 \). By such transformations light cone \( (\tau = \|x\|) \) is invariant set as \( \langle\langle Z\rangle\rangle = \langle\langle Z'\rangle\rangle = 0 \).

Superposition of two Poincaré transformations with equal \( e_j \) also possess with group properties. It’s easy to show:

\[
P_1 \circ P_2 = L_1 \circ U_1 \circ L_2 \circ U_2 = L_1 \circ L_2 \circ U_1 \circ U_2 = L_3 \circ U_3 = P_3.
\]
But superposition of two Poincaré transformations with nonequal $e_j$ doesn’t possess such properties.

3. The space of generalized biquaternions $\mathcal{B}'(M)$

We will consider on $M$ the functional space of Bqs. $\mathcal{B}(M) = \{ F = f(\tau, x) + F(\tau, x) \}$, where $f$ is complex function and $F$ is three dimensional vector-function with complex components $F_j, j = 1, 2, 3$. The partial derivative from Bq. over $\tau$ or $x_j$ we designate so:

$$\partial_\tau F = \partial_\tau f + \partial_\tau F, \quad \partial_j F = \frac{\partial f}{\partial x_j} + \frac{\partial F}{\partial x_j}, \quad j = 1, 2, 3.$$

Let introduce two biquaternions spaces, basic one is $\mathcal{B}(M) = \{ \Phi = \varphi(\tau, x) + \Phi(\tau, x) \}$, $\varphi \in D(R^4)$, $\Phi_j \in D(R^4)$, $j = 1, 2, 3$, where $D(R^4)$ is the space of finite infinitely differentiable functions on $R^4$, and conjugate space $\mathcal{B}'(M) = \{ \hat{F} = \hat{f} + \hat{F} \}$ of linear continues functionals on $\mathcal{B}(M)$:

$$\langle \hat{F}, \Phi \rangle = \langle \hat{f}, \varphi \rangle + \sum_{j=1}^{3} \langle \hat{F}_j, \Phi_j \rangle, \quad \forall \Phi \in \mathcal{B}(M),$$

which be named the space of generalized biquaternions and we mark such Bq. with cap over it.

Any regular Bq. $F$ corresponds to the functional, which can be presented in integral form:

$$\langle \hat{F}, \Phi \rangle = \int_{R^4} (F(\tau, x), \Phi(\tau, x)) d\tau dx_1 dx_2 dx_3, \quad \forall \Phi \in \mathcal{B}(M).$$

Bq. is singular if its action on $\mathcal{B}(M)$ can not be presented in such form. Example of singular Bq. is singular functions from $D'(R^4)$ because $D'(R^4) \subset \mathcal{B}'(M)$. Using them the more complex biquaternions may be constructed.

In particular, in the mathematical physics problems the next singular functions are often used as simple layers. There generalization on $\hat{B}(M)$ are generalize Bqs. $F \delta_S$, which define the functionals of the type:

$$\langle F \delta_S, \Phi \rangle = \int_S (F(\tau, x), \Phi(\tau, x)) dS, \quad \forall \Phi \in \mathcal{B}(M).$$

Here is the surface integral on a surface $S \subset R^4$, which dimension may be equal 1,2,3. Defined and integrable on $S$ Bq. $F$ we name the density of simple layer as is customary.

Differentiation. Analogically as in the theory of generalized function [15] using the definition of the partial derivatives of generalized Bq.:

$$\langle \partial_j \hat{F}, \Phi \rangle = -\langle \hat{F}, \partial_j \Phi \rangle \quad \text{for} \quad \forall \Phi \in \mathcal{B}(M),$$

the derivatives of singular Bqs. and the derivatives of more higher order can be constructed.

From the properties of differentiation of regular functions with finite gaps on some 3-dimensional surface $S \subset R^4$ there is

$$\partial_j \hat{F} = \partial_j F + n_j[F]_S \delta_S, \quad j = \tau, 1, 2, 3 \quad (14)$$
Here $\partial_j F$ is classic derivative, $n = \{n_\tau, n_1, n_2, n_3\}$ is unit normal to $S$. In square brackets the gap of $F$ on $S$ stands,

$$[F(\tau, x)]_S = \lim_{\varepsilon \to +0} \{F(\tau + \varepsilon n_\tau, x + \varepsilon n) - F(\tau - \varepsilon n_\tau, x - \varepsilon n)\}, \quad (\tau, x) \in S.$$ 

**Definition.** Bq. $F$ is **generalize solution** of differential equation $D(\partial_\tau, \partial_x)F = G$ if

$$(D\hat{\Phi}, \Phi) = (\hat{G}, \Phi) \quad \text{for } \forall \Phi \in B(M)$$

(here $D$ is linear differential operator). If $\hat{F}$ is quasi everywhere differentiable regular function $F$ then it is classic solution.

**Convolution** of two biquaternions is the bilinear operation:

$$A(\tau, x) * B(\tau, x) = a * b - \sum_{i,j,l=1}^{3} (A_j * B_j) + (a * A_j) e_j + (b * B_j) e_j + \varepsilon_{ijl} (A_i * B_j) e_l,$$

where usual convolutions of generalized functions [15] stand in brackets on the right. It’s easy to see, here two operations of biquaternionic multiplication and convolution are united.

In virtue of properties of convolution differentiation [15] we get the formula for derivative of Bqs. convolution:

$$\partial_j(A * B) = (\partial_j A) * B = A * \partial_j B, \quad \partial_j = \partial_\tau, \partial_1, \partial_2, \partial_3.$$  \hspace{1cm} (15)

It gives possibility to use more convenient form by its calculating.

**Fourier transformation.** Generalized Fourier transformation (FTr.) of $\hat{G}$ is $\tilde{G}$ which satisfies the equality:

$$(\hat{\Phi}, \Phi) = (\tilde{\Phi}, \tilde{\Phi}) \quad \text{for } \forall \Phi \in B(M).$$

Here $\tilde{\Phi}$ is classic FTr. of $\Phi$:

$$F[\Phi(\tau, x)] = \tilde{\Phi}(\omega, \xi) = \int_{\mathbb{R}^4} \Phi(\tau, x) \exp(i\tau \omega + i(\xi, x)) d\tau dx_1 dx_2 dx_3, \quad \Phi \in B(M),$$

which always exists in virtue of properties $B(M)$. If Bq. is regular and has classic FTr. then it is also generalized FTr.

Also it’s easy to prove (similarly in [15]), that

$$F[A(\tau, x) * B(\tau, x)] = \tilde{A}(\omega, \xi) \circ \tilde{B}(\omega, \xi).$$ \hspace{1cm} (16)

Using properties of FTr. of generalized functions we could get many similar properties of Fourier transformation for biquaternions, but we don’t do this here.

4. **Bigradients. Poincaré and Lorentz transformations**

Let consider special cases of differential operators which are representative for mathematical physics equations. But we will analyze them on $B'(M)$.

We introduce operators, named **mutual complex gradients**:

$$\nabla^+ = \partial_\tau + i\nabla, \quad \nabla^- = \partial_\tau - i\nabla,$$ \hspace{1cm} (18)
where $\nabla = \text{grad} = (\partial_1, \partial_2, \partial_3)$. We name them shortly bigradients.

In the sense of given definitions their symbols are complex conjugate and selfconjugated: $(\nabla^-)^* = \nabla^-$, $(\nabla^+)^* = \nabla^+$. Their action is defined in according to biquaternions algebra

$$\nabla^\pm \mathbf{F} = (\partial_\tau \pm i \nabla) \circ \mathbf{f} + (\partial_\tau \mp i \nabla, \mathbf{F}) \pm \partial_\tau \mathbf{F} \pm i \nabla \mathbf{f} \pm i [\nabla, \mathbf{F}]$$

(correspond to signs) or in conventional record

$$\nabla^\pm \mathbf{F} = (\partial_\tau \mathbf{f} \mp i \text{div} \mathbf{F}) \pm \partial_\tau \mathbf{F} \pm i \text{grad} \mathbf{f} \pm i \text{rot} \mathbf{F}.$$  

It’s easy to test that wave operator ( $\Box$ ) is presented in the form of superposition of mutual bigradients:

$$\Box = \frac{\partial^2}{\partial \tau^2} - \Delta = \nabla^- \circ \nabla^+ = \nabla^+ \circ \nabla^-,$$  

(19)

where $\Delta = \sum_{j=1}^{3} \partial_j \partial_j$ is Laplace operator. Using this property, we can construct particular solutions of differential biquaternionic equations on $B'(M)$ of a type:

$$\nabla^\pm \mathbf{K}(\tau, x) = \mathbf{G}(\tau, x),$$  

(20)

which be named biwave equation. Solutions of this equations be named $\pm$ bipotentials of $\mathbf{G}$.

By Poincaré transformations bigradients and biwave equations transform according consecutive affirmations.

**Theorem 4.1.** By Poincaré transformations $(\mathbf{Z} \to \mathbf{Z}' = \mathbf{P} \circ \mathbf{Z} \circ \mathbf{P}^*)$ biwave equation transforms to biwave equation:

$$\mathbf{D}' \mathbf{K}' = \mathbf{G}'$$

$$\mathbf{D} = \nabla^\pm = \partial_\tau \pm i \text{grad}_x, \quad \mathbf{D}' = \mathbf{P}^- \circ \mathbf{D} \circ \mathbf{P} = \partial_\tau' \pm i \text{grad}_{x'}.$$  

(relativistic formulas)

$$\mathbf{K}' = \mathbf{P}^- \circ \mathbf{K} \circ \mathbf{P}, \quad \mathbf{G}' = \mathbf{P}^- \circ \mathbf{G} \circ \mathbf{P}.$$  

**Proof.** See [14] and equalities

$$\mathbf{D}' \mathbf{K}' = (\mathbf{P}^- \circ \mathbf{D} \circ \mathbf{P}) \circ (\mathbf{P}^- \circ \mathbf{K} \circ \mathbf{P}) = \mathbf{P}^- \circ \mathbf{D} \circ \mathbf{K} \circ \mathbf{P} = \mathbf{P}^- \circ \mathbf{G} \circ \mathbf{P} = \mathbf{G}'.$$

Lets go to solving biwave equations.

5. Generalized solutions of biwave equations. Shock waves

**Theorem 5.1.** Generalized solution of biwave equation (20) can be presented in the form:

$$\mathbf{K} = \nabla^\pm \mathbf{G} \ast \mathbf{\psi} + \mathbf{K}_0.$$  

(21)

where $\mathbf{\psi}(\tau, x)$ is simple layer on light cone $\tau = \|x\|$:

$$\mathbf{\psi} = (4\pi \|x\|)^{-1}\delta(\tau - \|x\|),$$  

(22)

which is the fundamental solution of wave equation:

$$\Box \mathbf{\psi} = \delta(\tau)\delta(x).$$  

(23)
Bq. \( K_0(\tau, x) \) is a solution of homogeneous biwave equation (by \( \hat{G} = 0 \)). It can be presented as
\[
K_0 = \nabla^\pm \{ G_0 * \psi_0(\tau, x) \},
\]
where \( \psi_0(\tau, x) \) is the a solution of homogeneous wave equation:
\[
\Box \psi_0 = 0,
\]
\[
\psi_0(\tau, x) = \int_{\mathbb{R}^3} \phi(\xi) \exp \left( i \left( (\xi, x) \pm \|\xi\| \tau \right) \right) dV(\xi), \quad \forall \phi(\xi) \in L_1(\mathbb{R}^3),
\]
\( G_0(\tau, x) \) is arbitrary Bq., capable of this convolution with \( \psi_0 \), or it can be presented as a sum of such type solutions.

Proof. Using associativity and properties (15), (23) for first summ and in (21) we get
\[
\nabla^\pm \hat{K} = \nabla^\pm \nabla^\mp \left( \hat{G} * \psi \right) = \Box \left( \hat{G} * \psi \right) = \hat{G} * \Box \psi = \hat{G} * \delta(\tau) \delta(x) = \hat{G}.
\]
Last equality in virtue of the property of \( \delta \)-function [15]. By analogy for the second summand:
\[
\nabla^\pm K_0 = \nabla^\pm \nabla^\mp \{ C_0 * \psi_0(\tau, x) \} = \Box \{ C_0 * \psi_0(\tau, x) \} = \{ \Box \psi_0(\tau, x) \} * C_0 = 0.
\]
Inversely, if \( K_0 \) is the solution of homogeneous biwave equation, then every its components is the solution of Eq. (26). Thus and so it can be presented in the form (25) or decomposed in sum of forth such bipotentials for every components generated with different solutions of Eq. (26). In virtue of linearity their sum in (21) is a solution of Eq. (20).

The form of solutions of wave equation (26), including fundamental one (22), is well known [15].

**Fundamental solution of biwave equation** we obtain if suppose \( G = \delta(\tau) \delta(x) \). Then by use formula (21) we have its presentation:
\[
\Psi(\tau, x) = \partial_\tau \psi \pm i \text{grad} \psi.
\]
It gives possibility to construct solutions of biwave equation in form of the convolution:
\[
\hat{K} = \hat{G} * \Psi + K_0
\]
for arbitrary right part of Eq.(20), accepting such convolution.

**Shock waves.** Let consider generalized solutions of biwave equation. Whereas it is hyperbolic, there are solutions which are non differentiable on characteristic surfaces \( (S) \) where
\[
n_\tau^2 = \|n\|^2
\]
\((n_\tau, n_1, n_2, n_3)\) is unit normal to \( S \) in \( M \), \( n = (n_1, n_2, n_3) \), \( \|n\|^2 = n_1^2 + n_2^2 + n_3^2 \). This is a cone of characteristic normals of wave equation.

Let define the conditions on the gaps of derivatives for regular solutions of biwave equation. By using formula (14) we get:
\[
\nabla^+ \hat{K} = \nabla^+ \hat{K} + \{ n_\tau [k]_S - i (n, [K]_S) + n_\tau [K]_S + in [k]_S + i[n, [K]_S] \} \delta_S(\tau, x) =
\]
\[
= \nabla^+ \hat{K} + (n_\tau + in) \circ [K]_S \delta_S(\tau, x) = 0
\]
Because $\nabla^+ K = 0$ then

$$n_\tau[k]_S - i(n, [K]_S) = 0, \quad n_\tau[K]_S + in[k]_S + i[n, [K]_S] = 0. \quad (29)$$

Such solution is named shock wave. In $R^3$ it has mobile wave front $S_t$ with normal $(n_1, n_2, n_3)$, which propagates in virtue (28) with speed

$$c = -\frac{n_\tau}{\|n\|} = 1. \quad (30)$$

This result is formulated in this theorem.

**Theorem 5.2.** Shock waves satisfy to the conditions on its fronts:

$$[K]_S = i\mathbf{m} \circ [K]_S,$$

where $\mathbf{m} = n/\|n\|$ is a unit wave vector directed along the speed of its expansion in $R^3$.

**Proof.** If to divide (29) per $\|n\|$ subject to (30), we get

$$[k]_S = -i(\mathbf{m}, [K]_S), \quad [K]_S = i\mathbf{m}[k]_S + i[\mathbf{m}, [K]_S] \quad (31)$$

Pass to its biquaternionic record we get the formula of the theorem.

First equation (31) characterizes longitudinal shock waves.

If to substitute it to second equation, we get formula for tangent component of vector $K$ to wave front:

$$[K]_S - \mathbf{m}(\mathbf{m}, [K]_S) = i[\mathbf{m}, [K]_S]. \quad (32)$$

They give the connection between gaps of real and imaginary parts of Bq.

**Generalized Kirchhoff formula for biwave equation** Now we solve Cauchy problem for biwave equation. Let *initial condition* is given:

$$K(0, x) = K^0(x). \quad (33)$$

We have to construct the solution of Eq.(20), satisfying to these data.

We use for this *generalized functions method*. Let introduce regular functions $\hat{G} = H(\tau)G(\tau, x)$, where $H(\tau)$ is Heaviside function. If $K^0$ is regular then

$$\nabla^\pm \hat{K} = \hat{G} + \delta(\tau)K^0(x). \quad (34)$$

Therefore the solution is

$$K(\tau, x) = \nabla^+ \{H(\tau)G \ast \psi\} + G(0, x) \ast \psi + \nabla^+ \{K^0(\tau) \ast \psi\}, \quad (35)$$

where sign $\ast$ implies a convolution only over $x$. For regular $\hat{G}$ it has integral representation:

$$4\pi \hat{K}(\tau, x) = -\nabla^+ \left\{ \int_{\tau \leq r} G(\tau - r, y) \frac{dV(y)}{r} + \tau^{-1} \int_{r = \tau} K^0(y) dS(y) \right\} - \tau^{-1} \int_{r = \tau} G(0, y) dS(y). \quad (36)$$
Here and further \( r = \| y - x \| \), \( dV(y) = dy_1dy_2dy_3 \), \( dS(y) \) is differential of spheres surface \( r = \tau \) in point \( y \). This formula is generalization of well known Kirchhoff formula for solution of wave equation [15].

6. Biquaternionic form of Maxwell equations and its modification

As example of applications of biwave equation theory let consider Maxwell equation for electromagnetic field. It contains 8 differential Eqs. (2 vectorial and 2 scalar Eqs.).

In space of biquaternions it has the form of biwave equation [2,4,16]:

\[
\nabla^+ A + \Theta = 0. \tag{37}
\]

Biquaternions of intensity \( A \) and charge - current \( \Theta \) of EM-field are

\[
A = 0 + A = \sqrt{\varepsilon} E(\tau, x) + i\sqrt{\mu} H(\tau, x), \quad \Theta = i\rho(\tau, x) + J(\tau, x), \tag{38}
\]

where \( E \) is electric field intensity, \( H \) is magnetic field intensity; a density \( \rho \) and pulse \( J \) are denominated with use of density of electric charge \( \rho^E \) and electric current density \( j^E \) by formula:

\[
\rho = \rho^E / \sqrt{\varepsilon} = \sqrt{\varepsilon} \text{div} E, \quad J = \sqrt{\mu} j^E,
\]

\( \varepsilon, \mu \) are the constants of electric conduction and magnetic conductivity; \( \tau = ct, \) \( t \) is time, \( c = 1 / \sqrt{\varepsilon\mu} \) is light speed.

Scalar and vector part of Eq. (37) gives us Hamilton form of Maxwell equations [17,18]:

\[
\rho = \text{div} A, \quad \partial_\tau A + i \text{rot} A + J = 0. \tag{39}
\]

Real and imaginary part of Eqs.(20) gives us the system of 8 Eqs. of classical form.

If to take mutual bigradient from Eq.(37), we get charge conservation law in scalar part and wave equation for intensities of EM-field in vector part:

\[
\partial_\tau \rho + \text{div} J = 0, \quad \Box A = i \text{rot} J - \text{grad} \rho - \partial_\tau J.
\]

Bq. energy-pulse of \( A \)-field

\[
\Xi = W + iP = \frac{1}{2} A \circ A^* \tag{40}
\]

contains energy density of EM-field

\[
W = 0.5 \| A \|^2 = \frac{1}{2} \left( \varepsilon \| E \|^2 + \mu \| H \|^2 \right),
\]

and Pointing vector

\[
P = \frac{1}{2} [\bar{A}, A] = c^{-1} E \times H.
\]

Relativistic formulas for Maxwell equations for Poincaré transformation have the form:

\[
\nabla^+ A' = \Theta', \quad \text{where} \quad A' = P^- \circ A \circ P, \quad \Theta' = P^- \circ \Theta \circ P.
\]

Shock EM-waves. From theorem 5.2 we get conditions on fronts of shock EM-waves:

\[
[A]_S = i \text{m} \circ [A]_S. \tag{41}
\]
Its scalar and vector parts be of the form:

\[(\mathbf{m}, [A]_S) = 0, \quad [A]_S = i[\mathbf{m}, [A]_S].\]

First condition shows that shock EM-waves are transversal. From second condition we have

\[\sqrt{\varepsilon} [E]_S = \sqrt{\mu} ([H]_S, \mathbf{m}), \quad \sqrt{\mu} [H]_S = \sqrt{\varepsilon} [\mathbf{m}, [E]_S].\]

Using vectors of electric displacement and magnetic induction

\[D = \varepsilon E, \quad B = \mu H,\]

we state result.

**Theorem 6.1.** On the fronts of shock EM-waves the gaps of intensities of EM-field satisfy to conditions: \([E]_S = c [[B]_S, \mathbf{m}], \quad [H]_S = c [\mathbf{m}, [D]_S].\) That is equivalent to conditions: \([D]_S = c^{-1} [[H]_S, \mathbf{m}], \quad [B]_S = c^{-1} [\mathbf{m}, [E]_S].\)

**Generalized Kirchhoff formula for Maxwell equation** be of the form:

\[4\pi \mathbf{A} = \nabla^{-} \left\{ \int_{r \leq \tau} \Theta(\tau - r, y) \frac{dV(y)}{r} + \tau^{-1} \int_{r = \tau} \mathbf{A}^{0}(y) dS(y) \right\} + \tau^{-1} \int_{r = \tau} \Theta(0, y) dS(y). \quad (42)\]

Integral representation for \(E, H\) can be get from here but they will be composite form.

**Modified Maxwell equations.** In Hamilton form of Maxwell Eqs. (34) vector Eq. defines currents, scalar Eq. is definition of a charge. In bigradient of intensity of EM-field the scalar part is equal to 0. Corollary of it is law of charge conservation. I.e. Maxwell equations describes closed systems of electric charges and currents and generated by them EM-field.

For open system the Maxwell Eqs. must be modified with use of scalar field \(a(\tau, x)\) in Bq. \(\mathbf{A} = a(\tau, x) + A(\tau, x).\) Then from Eq. (32) follows its modified form:

\[\partial_\tau A + i \text{rot} A + J = \text{grad} a, \quad \rho = \text{div} A - \partial_\tau a. \quad (43)\]

If \(\rho\) and \(J\) are known this system for determination \(a\) and \(A\) is closed. Scalar field \(a(\tau, x)\) may be called resistance-attraction field [11,12].

**Comment.** Biquaternionic form of Maxwell equation (32) has deep physical sense: electric charges and currents are physical appearance of EM-field bigradient, and intensities of EM-field are the bipotential for its charge-current.

### 7. Biquaternionic presentation of Dirac operators and their properties

Biwave equation (20) can be written in the matrix form:

\[\sum_{j=0}^{3} D^\pm_{m,j} b_j = g_m, \quad m, j = 0, 1, 2, 3, \quad (44)\]

where \(b_0 = b, \quad g_0 = g, \quad b_j = B_j, \quad g_j = G_j, \quad j = 1, 2, 3; \) and \(D^\pm_{m,j}\) are the components of matrix \(D^\pm\) (corresponding to sign), which have the form:

\[D^+ = D = \begin{pmatrix}
\partial_\tau & -i\partial_1 & -i\partial_2 & -i\partial_3 \\
i\partial_1 & \partial_\tau & -i\partial_3 & i\partial_2 \\
i\partial_2 & i\partial_3 & \partial_\tau & -i\partial_1 \\
i\partial_3 & -i\partial_2 & i\partial_1 & \partial_\tau
\end{pmatrix}, \quad D^- = D = \begin{pmatrix}
\partial_\tau & i\partial_1 & i\partial_2 & i\partial_3 \\
-i\partial_1 & \partial_\tau & i\partial_3 & -i\partial_2 \\
-i\partial_2 & -i\partial_3 & \partial_\tau & i\partial_1 \\
-i\partial_3 & i\partial_2 & -i\partial_1 & \partial_\tau
\end{pmatrix} \quad (45)\]
Easy to test that their product (operators superposition) satisfy to relation:

\[ \sum_{j=0}^{3} D_{mj}D_{jl} = \delta_{ml}\Box, \quad j, m, l = 0, 1, 2, 3. \quad (46) \]

We’ll show that matrixes (45) are the differential matrix operators of Dirac, which possess by such properties \[13,17\]. For this we afford them in matrix form:

\[ D^+ = \sum_{j=0}^{3} D^j \partial_j, \quad (47) \]

where, as follow from (45), matrixes \( D^j \) have the components:

\[
D^0 = I, \quad D^1 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}, \quad D^2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad D^3 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}
\]

here \( I \) is unit matrix. As you see, this is 4-dimensional Dirac matrixes, composed from 2-dimensional Pauli matrix:

\[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \pm i & 0 \\ 0 & \mp i \end{pmatrix}, \quad \begin{pmatrix} 0 & \pm i \\ \pm i & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \pm i \\ \mp i & 0 \end{pmatrix}.
\]

Therefore Dirac matrix \( D^\pm \) has \textit{biquaternionic presentation} \( \nabla^\pm \).

The differential operators:

\[
D^+_m = \nabla^+ + m, \quad D^-_m = \nabla^- + m
\]

constitute biquaternionic presentation of matrix Dirac operators: \( D^\pm + mI \). By \( m = 0 \) it is biquaternionic form of Maxwell operator. By this cause biquaternionic equation of form:

\[
D^\pm_m B \equiv (\nabla^\pm + m) \circ B = F, \quad m \in \mathcal{C}, \quad (48)
\]

is the biquaternionic presentation of generalized Maxwell-Dirac equation (\textit{MD-equation}).

After simply calculation we get this lemma.

**Lemma 7.1.**

\[
D^+_m \circ D^-_m = D^-_m \circ D^+_m = \Box + m^2 + 2m\partial_\tau.
\]

For imaginary \( m = i\rho \)

\[
D^+_{i\rho} \circ D^-_{i\rho} = \Box - \rho^2 + 2i\rho\partial_\tau. \quad (49)
\]

It is interesting that right part (49) contains Klein-Gordon-Fokk operator \( \Box - \rho^2 \), and Srödinger operator \( \Delta + 2i\rho\partial_\tau \). By this cause we will name the equation

\[
\Box u + 2m\partial_\tau u + m^2u = f(\tau, x) \quad (50)
\]

\textit{Klein-Gordon-Fokk-Srödinger} equation or shortly \textit{KGFSh-equation}.

As you’ll see further, addition of \( 2m\partial_\tau \) to KGF-operator essentially simplify construction of solutions of Maxwell and Dirac equations and their representation.
8. Generalized solutions of Maxwell-Dirac equation (48).

KGFSSh-equation and scalar potentials

Using lemma 7.1 and theory of generalized functions it is easy to construct generalized solutions of MD-equation (48).

**Theorem 8.1.** Generalized solutions of MD-equation (48) have the form:

\[ B = B^0 + D_m^\pm (\psi^* F) = B^0 + \Psi^* F, \]

where \( B^0(\tau, x) \) is a solution of homogeneous equation:

\[ D_m^\pm \circ B^0 = 0, \]

\( \Psi \) is its fundamental solution:

\[ \Psi(\tau, x) = \partial_\tau \psi + m\psi \pm i \text{grad} \psi, \]

\( \psi(\tau, x) \) is fundamental solution of KGFSSh-equation (50).

**Proof.** In virtue of linearity of equation, it is sufficiently to prove this assertion for the second summand in formula (51). If to substitute it to Eq. (48), by use of properties (15) and \( \delta \)-function, we get:

\[ D_m^\pm D_m^\pm (\psi^* F) = F^* (\Box \psi + 2m\partial_\tau \psi + m^2 \psi) = F^* \delta(\tau)\delta(x) = F. \]

Ex facto, whatever solution of Eq. (48) can be presented in form (51).

Let formulate this theorem for imaginary \( m = i\rho \).

**Theorem 8.2.** Generalized solutions of MD-equation:

\[ (\nabla^\pm + i\rho) \circ B = F \]

can be presented in the form:

\[ B = B_0 + (\nabla^\pm + i\rho) \circ (F^* \psi), \]

where \( \psi(\tau, x) \) is fundamental solution of KGFSSh-equation:

\[ \Box \psi - \rho^2 \psi + 2i\rho \partial_\tau \psi = \delta(\tau)\delta(x), \]

\( B_0(\tau, x) \) is a solution of Dirac equation:

\[ (\nabla^\pm + i\rho) \circ B_0 = 0. \]

From these theorems it follows that solutions of KGFSSh-equation determine solutions of MD-equation.

**Theorem 8.3** Generalized solutions of KGFSSh-equation (50) are presented in the form:

\[ u = f^* \psi + u_0, \]

where \( \psi \) is its fundamental solutions:

\[ \psi = \frac{1}{4\pi \|x\|} \left( a e^{-m\|x\|} \delta(\tau - \|x\|) + (1 - a) \delta(\tau + \|x\|) e^{m\|x\|} \right) + \psi_0, \quad \forall a \in \mathcal{C}, \]
\( \delta(\tau \pm \|x\|) \) is simple fiber on light cone \( \|x\| = \mp \tau \), \( u_0(\tau, x) \) is a solution of homogeneous Eq., which exists only for imaginary \( m = i \rho \) and

\[
u_0(\tau, x) = e^{-i \rho \tau} \int_{\mathbb{R}^3} \phi(\xi) \exp (i ((\xi, x) \pm \|\xi\| \tau)) \, dV(\xi), \quad \forall \phi(\xi) \in L_1(\mathbb{R}^3). \tag{57}
\]

**Proof.** From equation for fundamental solution:

\[
\Box \psi + m^2 \psi + 2m \partial_\tau \psi = \delta(\tau)\delta(x), \tag{58}
\]

follows Helmholtz equation:

\[
\{ \Delta - k^2 \} F_\tau[\psi] + \delta(x) = 0, \quad k = i \omega - m.
\]

fundamental solutions of which are well known:

\[
F_\tau[\psi] = \frac{1}{4 \pi \|x\|} \left( ae^{(i \omega \rho \pm \|x\|)} + (1 - a) e^{-(i \omega \rho \pm \|x\|)} \right), \quad \forall a \in \mathcal{C}.
\]

From here by use of properties of direct and inverse Fourier transformation the first formula of theorem follows.

Support of first summand is expanding over time sphere of radius \( \tau \) (\( \tau > 0 \)), and for second one it is converging over time sphere of radius \( |\tau| \) (\( \tau < 0 \)).

If \( m \) is imaginary (\( m = i \rho \)) and support of solution over time is \( \tau > 0 \), then

\[
\psi = \frac{e^{-i \rho \|x\|}}{4 \pi \|x\|} \delta(\tau - \|x\|). \tag{59}
\]

It’s interesting that here density of simple fiber in light cone is fundamental solutions of Helmholtz Eq. with wave number \( \rho \).

**Solutions of homogeneous KGFSh-equation.** Let consider

\[
\Box u + m^2 u + 2m \partial_\tau u = 0 \tag{60}
\]

which in Fourier transform space satisfy to algebraic Eq.

\[
\left( \|\xi\|^2 - (\omega + im)^2 \right) u^*(\omega, \xi) = 0. \tag{61}
\]

If \( \text{Re} \, m \neq 0 \), then \( \|\xi\|^2 - (\omega + im)^2 \neq 0 \) by \( \forall \{\xi, \omega\} \in \mathbb{R}^4 \). In this case the equation have only trivial solution: \( u^* = 0 \).

Provided \( m = i \rho \) this Eq. has infinitely set of singular solutions of the type:

\[
u^*(\omega, \xi) = \phi(\omega, \xi) \delta \left( \|\xi\|^2 - (\omega - \rho)^2 \right), \tag{62}
\]

where \( \phi(\omega, \xi) \) is a density of simple fiber. It is arbitrary given and integrable function on cones \( \|\xi\| = |\omega - \rho| \). By calculating of originals, we get:

\[
u(\tau, x) = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} d\omega \int_{\|\xi\| = |\omega - \rho|} \phi(\omega, \xi) \exp (-i(\xi, x) - i\omega \tau) \, dS(\xi) =
\]
\[ e^{-i\rho \tau} \int_{\mathbb{R}^3} \left\{ \phi(\rho + \|\xi\|, \xi) e^{-i\|\xi\|\tau} - \phi(\rho - \|\xi\|, \xi) e^{i\|\xi\|\tau} \right\} \exp(-i(\xi, x)) d\xi_1 d\xi_2 d\xi_3. \]

In virtue of arbitrary \( \phi \), we get formula (57). The theorem is proved.

**Harmonic vibrations.** Let construct solutions of MD-equation (48) in case of harmonic vibration with frequency \( \omega \):

\[ B = B(x) e^{i\omega \tau}, \quad F = F(x) e^{i\omega \tau}. \]

Then we have the equation for complex amplitude of type:

\[ \nabla_{(\omega + \rho)}^\pm B(x) \equiv (\omega + \rho + \nabla) \circ B(x) = F(x). \tag{61} \]

which be named *gradiental*. It easy to calculate

\[ \nabla_\omega \circ \nabla_{-\omega} = -(\omega^2 + \Delta). \]

By using this property it easy to prove the theorem.

**Theorem 8.4.** Generalized solutions of gradiental equation (61) can be formulary:

\[ B = \nabla_{\omega + \rho}^\pm (\chi \ast F) + \text{Sp}^{(\omega + \rho)}, \]

where

\[ \chi = -\frac{ae^{-ik\|x\|}}{4\pi \|x\|} + \frac{(a-1)e^{ik\|x\|}}{4\pi \|x\|}, \quad k = |\omega + \rho| \neq 0, \quad \forall a \in \mathbb{C}, \tag{62} \]

\[ \text{Sp}^{(\omega + \rho)} = \nabla_{\omega + \rho}^\pm (\chi_0 \ast C(x)), \]

\( \chi_0(x) \) is a solution of Helmholtz equation with wave number \( k \):

\[ \chi_0(x) = \int_{\|e\|=1} p(e) e^{-ik(e,x)} dS(e), \]

\( p(e) \) is \( \forall \) integrable on unit sphere function, \( C(x) \) is \( \forall \ Bq., \) for which this convolution exists.

If \( \omega > 0 \) first summand in (62) describes radiated spherical wave, second one – converging spherical wave. The wave number \( k \) defines the length of these waves \( \lambda = 2\pi/k \), which (for positive \( \rho \)) decreases by increasing \( \rho \). But if \( \rho < 0 \) then \( \lambda \) in interval \( \{-\omega, 0\} \) increases by increasing \( |\rho| \) and by \( \rho = -\omega \) spherical waves vanished.

**Statics.** This theorem also defines solutions of static MD-equation (when \( \omega = 0 \)).

**Theorem 8.5.** Generalized solutions of static MD-equation can be formulary:

\[ B = \nabla_{\omega}^\pm (\chi \ast F) + \text{Sp}^{(\rho)}, \]

where

\[ \chi = -\frac{ae^{-i\rho x}}{4\pi \|x\|} + \frac{(a-1)e^{i\rho x}}{4\pi \|x\|}, \quad \forall a \in \mathbb{C}, \]

\[ \text{Sp}^{(\rho)} = \nabla_{\rho}^\pm (\chi_0 \ast C(x)), \]
\( \chi_0(x) \) is a solution of of Helmholtz equation with wave number \(|\rho|\):

\[
\chi_0(x) = \int_{\|e\|=1} p(e) e^{-i|\rho|(e,x)} dS(e),
\]

\( p(e) \) is \( \forall \) integrable on unit sphere function, \( C(x) \) is \( \forall \) Bq., for which this convolution exists.

We remark that Bq. \( C \in B'(M) \). For regular B this formulas gives classic solutions of MD-equations.

If \( \rho = 0 \) from these two theorems we get solutions of modified Maxwell equations in cases of harmonic vibrations and statics.

9. Spinors and spinors fields

In quantum mechanics solutions of Dirac equation:

\[
(\nabla^\pm + i\rho) \circ \Psi = 0, \quad \text{Re}\rho = 0.
\]

are called spinors \([13,17,19]\).

**Theorem 9.1.** Spinors may be formulary:

\[
\Psi = D_{i\rho}^\pm \circ (\psi_0 \ast C(\tau,x)),
\]

where \( \psi_0 \) is a solution of homogeneous KGFS\(h\)-equation, or can be presented as the sum of such form solutions, \( \forall C \in B'(M) \) for which this convolution exists.

**Proof.** By substituting formula (64) into Eq.(63) we get:

\[
D_m^\pm \Psi = D_m^\pm (\psi_0 \ast C) = (\Box \psi_0 + 2m\partial_\tau \psi_0 + m^2\psi_0) \ast C = 0, \quad m = i\rho.
\]

Inversely, if spinor (64) is a solution of (63), then

\[
(\Box + 2m\partial_\tau + m^2) \circ \Psi = D_m^\pm D_m^\mp \Psi = D_m^\mp 0 = 0.
\]

Therefore scalar part and components of vector part \( \Psi \) are the solutions of KGFS\(h\)-equation and \( \Psi \) may be formulary as the sum of solutions of type (64).

In particular case, when \( C = 1 \), we have spinor of scalar \( \psi_0 \)-field :

\[
\Psi^\pm_0 = (\nabla^\mp + m) \circ \psi_0 = m\psi_0 + \partial_\tau \psi_0 \mp i\text{grad}\psi_0,
\]

Consequently \( C\)-field spinors , generated by potential \( \psi_0 \), may by formulary:

\[
\Psi^\mp = \Psi^\mp_0 \ast C(\tau,x).
\]

**Scalar harmonic potentials.** Let see formula (57) where there are two plane harmonic wave:

\[
\varphi^\pm_\xi(\tau,x) = \exp (i((\xi,x) - \rho\tau \pm \|\xi\|\tau)).
\]

They also are solutions of homogeneous KGFS\(h\)-equation. Wave vector \( \xi \) defines direction of wave motion. The length of these waves is equal to \( \lambda = 2\pi/\|\xi\| \), their frequencies are \( \omega = |\rho \pm \|\xi\|| \), periods \( T = 2\pi/|\rho \pm \|\xi\|| \). In depend of sign, one of them has the supersonic
phase speed \((V > 1)\), other one has the subsonic phase speed \((V < 1)\), as \(V = 1 \pm \rho \xi / \| \xi \|\). \(V \to 1 \pm 0\) by \(\| \xi \| \to \infty\), but \(\omega \to \infty\). By \(\| \xi \| \to |\rho|\) velocities \(V \to 1; 0\) and frequencies \(\omega \to \frac{\xi}{\rho}; \infty\) (corresponding to sign).

Generated by these waves spinors

\[
(\nabla^\pm + i \rho) \circ \varphi^\pm_\xi (\tau, x) = \pm (i \| \xi \| + \xi) \varphi^\pm_\xi.
\]

have the form, norm and pseudonorm:

\[
Sp^\pm_\xi = \exp \left( \frac{i ((\xi, x) - \rho \tau \pm \| \xi \| \tau)}{\sqrt{2}} \right) \left( i + \frac{\xi}{\| \xi \|} \right),
\]

\[
\| Sp^\pm_\xi \| = 1, \quad \langle \langle Sp^\pm_\xi \rangle \rangle = 0.
\]

We name them \textit{elementary \(\xi\)-oriented harmonic spinors}. Its energy-impulse is equal to

\[
\Xi = Sp^\mp_\xi \circ (Sp^\pm_\xi)^* = 1 - i \frac{\xi}{\| \xi \|} \Rightarrow \| \Xi \| = 2, \quad \langle \langle \Xi \rangle \rangle = 0.
\]

\textbf{C-field \(\xi\)-oriented harmonic spinor} is the spinor:

\[
Cp_\xi = C(\tau, x) \ast Sp^\pm_\xi (\tau, x).
\] (66)

From formula (64) it follows that \textit{C-field harmonic spinors} may be formulary as

\[
Cp = C(\tau, x) \ast \int_{R^3} Sp^\pm_\xi (\tau, x) \phi(\xi) d\xi_1 d\xi_2 d\xi_3, \quad \phi(\xi) \in L_1(R^3),
\]

\textit{Elementary harmonic \((\omega + \rho)\)-spinors} are defined as:

\[
\Psi_0^{(\omega + \rho)}(x, e) = \frac{1}{k \sqrt{2}} (\nabla + \omega + \rho) \circ e^{-i k (e, x)} = \frac{1}{k \sqrt{2}} (\omega + \rho - i k e) e^{-i k (e, x)}
\] (68)

It easy to test that

\[
\| \Psi_0^{(\omega + \rho)} \| = 1, \quad \langle \langle \Psi_0^{(\omega + \rho)} \rangle \rangle = 0.
\] (69)

Here \(e\) is the direction of the spinor, \(k = |\omega + \rho|\) is its wave number. Energy-impulse of \(\Psi_0^{(\omega + \rho)}\) are equal to

\[
\Xi = \Psi_0^{(\omega + \rho)} \circ \{ \Psi_0^{(\omega + \rho)} \}^* = 1 - i e \text{sign}(\omega + \rho)
\]

\textbf{Theorem 9.1.} \textit{G-field harmonic spinors} can be formulary

\[
Gp^{(\omega + \rho)}(x, e) = G(x) \ast \Psi_0^{(\omega + \rho)}(x, e) \quad \text{\((e\)-oriented spinors field\)}
\]

or

\[
Gp^{(\omega + \rho)}(x) = G(x) \ast \int_{|e|=1} p(e) \Psi_0^{(\omega + \rho)}(x, e) dS(e) \quad \text{\((nonoriented \ spinors \ field)\)}
\]

\(\forall p(e) \in L_1(\{e \in R^3 : |e| = 1\})\).

\textit{Static spinors} are obtained by \(\omega = 0\).
**Conclusion.** The solutions of biquaternionic form of Maxwell-Dirac equations are here received in class of generalized function. That allows to build the decisions as for regular biquaternionic functions, so and at presence of singular sources in its right part. It’s possible to use at building of biquaternionic theories of the elementary particles. At calculation of spinors fields, it is possible to flip differentiation on components $C$-field, when this suitable. $C$-field too can be singular generalized function.

Bigradients, biwave equations and their decisions were used by author earlier for building of one models of electro-gravymagnetic fields and interactions [11,12,20]. It’s possible to find much other useful applications of the differential algebra of biquaternions, what is offered to interested reader.

**Key words:** biquaternion, bigradient, biwave equation, Maxwell-Dirac equations, generalized solutions, shock waves, spinors, harmonic spinors, spinors field

**References**

1. W.R. Hamilton. *On a new Species of Imaginary Quantities connected with a theory of Quaternions*, Proceedings of the Royal Irish Academy, 2 (Nov 13, 1843), 424-434.

2. J.D. Edmonds Jr. *Eight Maxwell equations as one quaternionic*, Amer. J. Phys., 46 (1978), No. 4, 430.

3. G.L. Shpilker. *Hypercomplex solutions of Maxwell equations*, Report of USSR Academy of Sciences, 272 (1983), № 6, 1359-1363.

4. M. Acevedo M., J. Lopez-Bonilla and M. Sanchez-Meraz. *Quaternions, Maxwell Equations and Lorentz Transformations*, Apeiron, 12 (2005), No. 4, 371.

5. S.L. Adler. *Quaternionic quantum mechanics and quantum fields*, New York: Oxford University Press, 1995.

6. P. Rotelli. *The Dirac equation on the quaternionic field*, Mod. Phys. Lett. A 4 (1989), 933-940.

7. A.J. Davies, *Quaternion Dirac equation*, Phys. Rev. D 41 (1990), 2628-2630.

8. D. Finkelstein, J. M. Jauch, S. Schiminovich, D. Speiser. *Foundations of quaternion quantum mechanics*, J. Math. Phys., 3 (1992), 207-220.

9. S. De Leo, W. A. Rodrigues Jr. *Quaternionic electron theory: geometry, algebra and Dirac’s spinors*, Int. J. Theor. Phys., 37 (1998), 1707-1720.

10. V.V. Kassandrov. *Biquaternion electrodynamics and Weyl-Cartan geometry of spacetime*, Gravitation and cosmology, 1 (1995), 3, 216-222.

11. L.A.Alexeyeva. *Equations of interaction of A-fields and Newton laws*, Intelligence of National Academy of Sciences of Rep.Kazakhstan, Physical and mathematical series, No.3 (2004), 45-53.

12. V. V. Kravchenko. *On force-free magnetic fields: quaternionic approach*. Mathematical Methods in the Applied Sciences, 2005, v. 28, No. 4, 379-386.

13. N.V.Bogoljubov, A.A.Logunov, A.A.Oksak, I.T.Todorov. *General principles to quantum theory of the field*. Moscow:Science, 1987. 616 p.

14. L.A.Alexeyeva. *Differential algebra of biquaternions. 1. Lorentz transformations*, Mathematical journal, 10 (2010), No.1, 33-41.

15. V.S.Vladimirov. *Generalized functions in mathematical physics*, Moscow:Science, 1976, 512 p.

16. L.A.Alexeyeva. *Quaternions of Hamilton form of Maxwell equations*, Mathematical journal, 3 (2003), No.4, 20-24.
17. A.I.Ahiezer, D.B.Berestezkij. *Quantum electrodynamics*, Moscow:Science, 1981. 320 p.

18. L.A.Alexeyeva. *Hamilton form of Maxwell equations and its generalized solutions*, Differential equations,39 (2003), No.6, 769-776.

19. *Mathematical encyclopedia*, Moscow:Science.2 (1982).

20 Alexeyeva L.A. *Newton’s laws for a biquaternionic model of the electro-gravimagnetic fields, charges, currents, and their interactions*, Ashdin publishing, Journal of Physical Mathematics. 2009. Vol.1. Article ID S090604

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