STATISTICS OF PARAMETRICALLY EXCITED
PHOTON-ADDED COHERENT STATE

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Abstract

Photon distribution function, means and dispersions are found explicitly for the nonclassical state of light which is created from the photon–added coherent state $|\alpha, m\rangle$ due to a time–dependence of the frequency of the electromagnetic field oscillator. Generating function for factorial momenta is obtained. The Wigner function and Q–function are constructed explicitly for the excited photon–added coherent state of light. Influence of added photons on known oscillations of photon distribution function for squeezed light is demonstrated.
1 Introduction

There are several types of nonclassical states of one–mode light, statistics of which has been studied last decade. They include squeezed light [1], [2], [3], correlated light [4], even and odd coherent states [5], [6] or Schrödinger cat states [7]. All these nonclassical states demonstrate either sub–poissonian or super–poissonian statistics of photons. The statistics of the coherent photons [8], [9], [10] is described by the Poisson distribution function. The coherent states of light are considered as the classical states.

The squeezed light has the property of the reducing the dispersion in one of the photon quadrature components. The photon distribution function of squeezed light has wavy behaviour for large squeezing [11], [12]. The quadrature components of the squeezed light satisfy the condition of the minimization of the Heisenberg uncertainty relation [13].

The correlated states of light [4] satisfy the condition of the minimization of Schrödinger uncertainty relation [14], [15]. The photon distribution function of the correlated light has the oscillatory behaviour [16], [17] depending on the correlation coefficient of the quadrature components which is extra physical parameter characterizing the state of the electromagnetic field oscillator. The property of oscillations of the photon distribution function is presented also for two–mode light [18], [19], [20].

In Ref. [21] the state $|\alpha, m\rangle$, which is obtained by the multiple action of the creation operator on the usual coherent state has been studied. The photon–added coherent state $|\alpha, m\rangle$ depends on extra discrete parameter, number of excited photons, which influences the statistics of photons. The statistics of this light including distribution function, means and dispersions of quadrature components, Wigner function and $Q$–function were found explicitly. This state of light is interesting because it may be created due to interaction of one–mode light with the two–level system [21]. The photon–coherent states of light $|\alpha, m\rangle$ have the specific property since they do not contain the contribution of states with fixed photon number $|n\rangle$ if the number $n$ is less than $m$. Thus the distribution function for photon–added states $p(n)$ is equal to zero up to the number $n = m$.

The photon–added coherent states should be distinguished from the states that are obtained by the action of the displacement operator on the number state. These states have been studied in [22], [23], [24]. For these states the photon distribution function may be nonzero for $n < m$. One of mechanisms producing the squeezing and correlation of the photon quadrature components is the parametric excitation of the electromagnetic field oscillator [25]. The parametric excitation may change the statistics of the photon–added coherent states.

The aim of this work is to study the photon statistics of the photon added states subjected to the parametric excitation. We obtain the Wigner and $Q$–function of the parametrically excited photon added states and study the deformations of photon distribution function of squeezed and correlated light under the influence of the extra physical parameter—the quantity of added photons. We will show that the characteristic property of the distribution for photon–added coherent states to be equal to zero up to a certain number of photons is destroyed by the parametric excitation of the electromagnetic field oscillator.
2 State $|\alpha, m\rangle$ of Stationary Oscillator

Let us consider the state defined by the formula

$$|\alpha, m\rangle = N \exp\left(-\frac{|\alpha|^2}{2}\right) \frac{\partial^m}{\partial \alpha^m} \left( \exp\left(\frac{|\alpha|^2}{2}\right) |\alpha\rangle \right),$$

(1)

where $N = (m!L_m(-|\alpha|^2))^{-1/2}$ is a normalization constant, $|\alpha\rangle$ is a coherent state and $m$ is a positive integer. The state $|\alpha\rangle$ can be defined as

$$|\alpha\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right) \exp(\alpha \hat{a}^+)|0\rangle,$$

(2)

where $|0\rangle$ is a vacuum state and $\hat{a}, \hat{a}^+$ are photon creation and annihilation operators. Using this series we obtain

$$|\alpha, m\rangle = \hat{a}^m |\alpha\rangle \frac{1}{(m!L_m(-|\alpha|^2))^{1/2}}.$$

(3)

This state coincides with the state $|\alpha, m\rangle$ that has been studied in [21]. With the aid of (1) and using the formula for the Hermite polynomial

$$H_m(\xi) = (-1)^m e^{\xi^2} \frac{d^m}{d\xi^m} e^{-\xi^2},$$

(4)

we can easily write the evident expression for wave function of the state $|\alpha, m\rangle$ in the coordinate representation

$$\langle q|\alpha, m\rangle = \frac{2^{-m} H_m(q - \alpha / \sqrt{2}) \langle q | \alpha\rangle}{(m!L_m(-|\alpha|^2))^{1/2}}.$$

(5)

The states $|\alpha, m\rangle$ should be distinguished from the states $|n, \alpha\rangle$ that are defined as

$$|n, \alpha\rangle = D(\alpha)|n\rangle.$$

These states are related to the shifted harmonic oscillators. The properties and statistics of these states were studied in [22, 23, 24]. It easy to notice that the state $|n, \alpha\rangle$ is the superposition of the states $|\alpha, m\rangle$

$$|n, \alpha\rangle = \frac{1}{\sqrt{n!}} \sum_{m=0}^{n} C_n^m (-\alpha^*)^{n-m} (m!L_m(-|\alpha|^2))^{1/2} |\alpha, m\rangle.$$

(6)

The reciprocal expression can be derived as well

$$|\alpha, m\rangle = (m!L_m(-|\alpha|^2))^{-1/2} \sum_{n=0}^{m} C_n^m (\alpha^*)^{m-n} \sqrt{n!} |n, \alpha\rangle.$$

The photon distribution function for the state $|\alpha, m\rangle$ has been found in [21]

$$p(n) = \frac{n!|\alpha|^{2(n-m)} \exp(-|\alpha|^2)}{[(n-m)!]^2 L_m(-|\alpha|^2)m!}. $$

(6)
We could calculate the generating function for this distribution in the form
\[
G(\lambda) = \frac{e^{(\lambda-1)|\alpha|^2} \lambda^m L_m(-\lambda|\alpha|^2)}{L_m(-|\alpha|^2)},
\]
which is associated with the expression (3) by the formula
\[
G(z) = \sum_{n=0}^{\infty} p(n) z^n.
\]

Below this expression will be generalized for the nonstationary case.

3 Time–Dependent Photon–Added Coherent States

We introduce the state \(|\alpha, m, t\rangle\) which is analogous to the state \(|\alpha, m\rangle\) of stationary oscillator, but the new state satisfies the Schrödinger equation with the Hamiltonian
\[
\hat{H} = \frac{\hat{p}^2}{2} + \Omega^2(t) \hat{q}^2/2.
\]
The initial value for the frequency \(\Omega(t)\) is taken to be \(\Omega(0) = 1\). We choose the initial state of the parametric oscillator to be photon–added state \(|\alpha, m, 0\rangle = |\alpha, m\rangle\), and we will study its evolution. In terms of evolution operator \(\hat{U}(t)\) of the oscillator it means that
\[
|\alpha, m, t\rangle = \hat{U}(t)|\alpha, m, 0\rangle = \hat{U}(t)|\alpha, m\rangle, \quad \hat{U}(0) = \hat{1}.
\]
The evolution operator \(\hat{U}(t)\) is unitary one,
\[
\hat{U}(t) \hat{U}^+(t) = \hat{1}.
\]
Thus we can write
\[
|\alpha, m, t\rangle \sim \hat{U}(t) \hat{a}^+ m \hat{U}^+(t) |\alpha\rangle = \hat{A}^+ m(t) |\alpha, t\rangle,
\]
where \(|\alpha, t\rangle\) is a time–dependent coherent state \([26]\), and
\[
\hat{A}^+(t) = \hat{U}(t) \hat{a}^+ \hat{U}^+(t)
\]
is the integral of motion of the parametric oscillator. From (3) and (11) we can derive that \(\hat{A}^+(t)\) satisfies the initial condition
\[
\hat{A}^+(0) = \hat{a}^+.
\]
By using hermitian conjugation of (12) and (13) we get another integral of motion
\[
\hat{A}(t) = \hat{U}(t) \hat{a} \hat{U}^+(t), \quad \hat{A}(0) = \hat{a}.
\]
Due to unitarity property of the evolution operator for hermitian Hamiltonian, the operators \(\hat{A}(t)\) and \(\hat{A}^+(t)\) satisfy to the same commutation relation as \(a\) and \(a^+\)
\[
[\hat{A}(t), \hat{A}^+(t)] = \hat{1}.
\]
Thus the algebra defined by the operators \( \hat{A}(t) \), \( \hat{A}^+(t) \), \( \hat{1} \) is the same Heisenberg–Weyl algebra as in the case of operators \( \hat{a} \), \( \hat{a}^+ \), \( \hat{1} \). Using the definition of invariants we get

\[
\hat{A}(t) = \frac{i}{\sqrt{2}} (\varepsilon(t) \hat{p} - \dot{\varepsilon}(t) \hat{q}), \quad \hat{A}^+(t) = -\frac{i}{\sqrt{2}} (\varepsilon^*(t) \hat{p} - \dot{\varepsilon}^*(t) \hat{q})
\]

where \( c \)-number function \( \varepsilon(t) \) satisfies the equation

\[
\ddot{\varepsilon}(t) + \Omega^2(t) \varepsilon(t) = 0,
\]

with the initial conditions \( \varepsilon(0) = 1, \ \dot{\varepsilon}(0) = i \), which means that the Wronskian is

\[
\varepsilon \dot{\varepsilon} - \varepsilon^* \dot{\varepsilon}^* = -2i.
\]

Thus, the solutions to Schrödinger equation for the arbitrary time–dependence of the frequency are expressed in terms of the solution \( \varepsilon(t) \) of the classical equation of motion \( (17) \).

Let us write down the evident expression for the state \( |\alpha, m, t \rangle \) in the coordinate representation

\[
\langle q | \alpha, m, t \rangle = \frac{\left( \frac{\varepsilon^*}{2\varepsilon} \right)^{\frac{m}{2}} H_m \left( \frac{q}{|\varepsilon|} - \sqrt{\frac{\varepsilon^*}{2\varepsilon}} \alpha \right)}{m! L_m(-|\alpha|^2)^{1/2}} \langle q | \alpha, t \rangle,
\]

where \( \langle q | \alpha, t \rangle \) is the time-dependent coherent state

\[
\langle q | \alpha, t \rangle = \pi^{-1/4} e^{-1/2} \exp\left( \frac{i\varepsilon q^2}{2\varepsilon} + \sqrt{2\alpha q} \frac{\varepsilon^* - |\alpha|^2}{2\varepsilon} - \frac{|\alpha|^2}{2} \right).
\]

We see that the wave function of parametrically excited photon–added coherent state is expressed in terms of Hermite polynomials.

### 4 First and Second Quadrature Component Moments

It is easy to derive from \( (16) \) the expressions for photon quadrature components in terms of the integral of motion

\[
\hat{p} = \frac{1}{\sqrt{2}} (A(t) \dot{\varepsilon}^*(t) + A^+(t) \dot{\varepsilon}(t)), \quad \hat{q} = \frac{1}{\sqrt{2}} (A(t) \varepsilon^*(t) + A^+(t) \varepsilon(t)).
\]

So, the problem of finding the moments of the quadratures is reduced to finding the average values of products of the operators \( A \) and \( A^+ \) in the states \( |\alpha, m, t \rangle \). Using the results \( (21) \) we get

\[
\langle \hat{q} \rangle = \langle \alpha, m, t | \hat{q} | \alpha, m, t \rangle = \frac{L_m^{(1)}(-|\alpha|^2)}{\sqrt{2} L_m(-|\alpha|^2)} (\varepsilon^* \alpha + \varepsilon \alpha^*),
\]

\[
\langle \hat{p} \rangle = \frac{L_m^{(1)}(-|\alpha|^2)}{\sqrt{2} L_m(-|\alpha|^2)} (\dot{\varepsilon}^* \alpha + \dot{\varepsilon} \alpha^*),
\]

\[
\langle \hat{q}^2 \rangle = \frac{(\varepsilon^2 \alpha^2 + \varepsilon^2 \alpha^*^2) L_m^{(2)}(-|\alpha|^2) + 2|\varepsilon|^2 (m + 1) L_{m+1}(-|\alpha|^2) - |\varepsilon|^2 L_m(-|\alpha|^2)}{2 L_m(-|\alpha|^2)},
\]

\[
\langle \hat{p}^2 \rangle = \frac{(\varepsilon^* \alpha^2 + \varepsilon \alpha^*^2) L_m^{(2)}(-|\alpha|^2) + 2|\varepsilon|^2 (m + 1) L_{m+1}(-|\alpha|^2) - |\varepsilon|^2 L_m(-|\alpha|^2)}{2 L_m(-|\alpha|^2)}.
\]
\[\langle \hat{p}^2 \rangle = \frac{(\hat{\varepsilon}^2 \alpha^2 + \hat{\varepsilon}^2 \alpha^* \alpha^*)}{2 L_m(-|\alpha|^2)} L_m^2(-|\alpha|^2) + 2|\hat{\varepsilon}|^2(m+1) L_{m+1}^2(-|\alpha|^2) - |\hat{\varepsilon}|^2 L_m(-|\alpha|^2). \quad (25)\]

With the help of these expressions it is easy to write dispersions of the quadratures \(\hat{q}\) and \(\hat{p}\)
\[\sigma_q^2(t) = \langle \hat{q}^2 \rangle - \langle \hat{q} \rangle^2, \quad \sigma_p^2(t) = \langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2. \quad (26)\]

All the moments and dispersions are the functions of time as well as of the amplitude \(|\alpha|\) and the phase of \(\alpha\). In the stationary case the results coincide with the results obtained in Ref. [21].

5 Mean and Dispersion of Photon Number

Let us express the creation and annihilation operators of photons in terms of the integrals of motion. We get
\[a = \frac{1}{2}(A(\varepsilon^* + i\hat{\varepsilon}^*) + A^+(\varepsilon + i\hat{\varepsilon})), \quad a^+ = \frac{1}{2}(A^+(\varepsilon - i\hat{\varepsilon}) + A(\varepsilon^* - i\hat{\varepsilon}^*)). \quad (27)\]

The average number of photons in the state \(|\alpha, m, t\rangle\) is
\[\langle N \rangle = \langle \alpha, m, t|a^+a|\alpha, m, t\rangle = (|\varepsilon|^2 + |\hat{\varepsilon}|^2)(m+1) L_{m+1}(-|\alpha|^2) \]
\[- \frac{1}{4}(|\varepsilon|^2 + |\hat{\varepsilon}|^2 - \frac{L_{m+1}^2(-|\alpha|^2)}{4 L_m(-|\alpha|^2)}(\alpha^2|\varepsilon|^2 + \hat{\varepsilon}^2) + \alpha^2(\varepsilon^2 + \hat{\varepsilon}^2)\), \quad (28)\]

and the average of number of photons squared is given by the formula
\[\langle (a^+a)^2 \rangle = \alpha^4(\varepsilon^2 + \hat{\varepsilon}^2)^2 + \alpha^4(\varepsilon^2 + \hat{\varepsilon}^2)^2 \frac{L_{m+1}^2(-|\alpha|^2)}{4 L_m(-|\alpha|^2)} \]
\[+ (|\varepsilon|^2 + |\hat{\varepsilon}|^2)(m+1)(\alpha^2|\varepsilon|^2 + \hat{\varepsilon}^2) + \alpha^2(\varepsilon^2 + \hat{\varepsilon}^2)) \frac{L_{m+1}^2(-|\alpha|^2)}{4 L_m(-|\alpha|^2)} \]
\[+ (m+1)(m+2)(|\varepsilon|^2 + \hat{\varepsilon}^2)^2 + 2(|\varepsilon|^2 + |\hat{\varepsilon}|^2)^2 \frac{L_{m+2}(-|\alpha|^2)}{8 L_m(-|\alpha|^2)} \]
\[+ (m+1)(2(|\varepsilon|^2 + |\hat{\varepsilon}|^2)(|\varepsilon|^2 + |\hat{\varepsilon}|^2 + 1) + |\varepsilon|^2 + \hat{\varepsilon}^2)^2 \frac{L_{m+1}(-|\alpha|^2)}{4 L_m(-|\alpha|^2)} \]
\[- (3(|\varepsilon|^2 + |\hat{\varepsilon}|^2 + 2)(\alpha^2|\varepsilon|^2 + \hat{\varepsilon}^2) + \alpha^2(\varepsilon^2 + \hat{\varepsilon}^2)) \frac{L_{m}^2(-|\alpha|^2)}{8 L_m(-|\alpha|^2)} \]
\[+ \frac{1}{16}((|\varepsilon|^2 + |\hat{\varepsilon}|^2 + 2)^2 + 2|\varepsilon|^2 + \hat{\varepsilon}^2)^2). \quad (29)\]

With the help of these expressions one can write down the expression for the function which describes the dispersion of the number of photons [27]
\[Q(\alpha, m) = \frac{\langle (a^+a)^2 \rangle - \langle a^+a \rangle^2}{\langle a^+a \rangle}. \quad (30)\]

It’s worth noticing also that the average values of products of any number of operators \(A^n\) and \(A^+m\) (where \(m\) and \(n\) are the arbitrary integers) could be evaluated from the commutation relations.
6 Time–Dependent Photon Distribution Function

Next we study the photon distribution of the field in the state \(|\alpha, m, t\rangle\). By definition we could write for this distribution function the expression

\[
p(n, t) = |\langle n|\alpha, m, t \rangle|^2.
\]

To calculate the integral \(\langle n|\alpha, m, t \rangle\) we will use the evident expression for the states \(|n\rangle\) and \(|\alpha, m, t\rangle\) in the coordinate representation, in which the Hermite polynomials are taken in the following form

\[
H_m(\xi) = \lim_{\beta \to 0} \frac{\partial^m}{\partial \beta^m} \exp(-\beta^2 + 2\beta \xi).
\]  

(31)

Since \(\beta\) is an independent parameter to find the projection of the state \(|\alpha, m, t\rangle\) on the vector \(|n\rangle\) means to calculate the Gaussian integral. Using the Hermite polynomials of two variables, that are defined by means of the generating function (see, for example [25]),

\[
\exp(-\frac{1}{2} \hat{a} \hat{a}^\dagger + \hat{a}^\dagger \hat{a} y) = \sum_{n,m=0}^{\infty} \frac{a^n a_m^m}{n!m!} H^{(R)}_{nm}(y_1, y_2).
\]  

(32)

we get the distribution function in the form

\[
p(n, t) = \frac{2|H^{(R)}_{mn}(-i\varepsilon^* \alpha/|\varepsilon|, 0)|^2}{n!m!(|\varepsilon|^2 + |\varepsilon|^2 + 2)^1/2} L_m(-|\alpha|^2)^{1/2} \times \exp\left(-|\alpha|^2 - \text{Re}\left(\frac{\varepsilon^{*2} + \varepsilon^{*2}}{|\varepsilon|^2 + |\varepsilon|^2 + 2} \alpha^2\right)\right),
\]  

(33)

where \(H^{(R)}_{mn}(\xi, \eta)\) is a Hermite polynomial of two variables defined by the \(2 \times 2\)–matrix \(R\) with the following matrix elements

\[
R_{11} = \frac{\varepsilon i \varepsilon^* - \varepsilon}{\varepsilon^* \varepsilon - i \varepsilon}, \quad R_{22} = \frac{\varepsilon + i \varepsilon^*}{\varepsilon - i \varepsilon},
\]

\[
R_{12} = R_{21} = \frac{2\varepsilon}{(\varepsilon - i \varepsilon)|\varepsilon|}.
\]

In the stationary case the matrix \(R\) reduces to

\[
R \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

and if taking into account that [25]

\[
H^{(00)}_{mn}(\xi, \eta) = (-1)^{\mu_{mn}} H_{\mu_{mn}}^{(0)} \xi^{(n - m + |n - m|)/2} \eta^{(m - n + |m - n|)/2} L_{\mu_{mn}}^{m-n}(\xi \eta),
\]

where

\[
\mu_{mn} = \text{min}(m, n),
\]

we reproduce the result of Ref. [21] for stationary case.
Now we write down the generating function, which is defined as

\[ G(z) = \sum_{n=0}^{\infty} p(n, t) z^n. \]  

(34)

Calculating the expression for the state \(|\alpha, m, t\rangle\) (see above) we derived also a relationship between the states \(|\alpha, t\rangle\) and \(|\alpha, m, t\rangle\). Therefore, it is easy to obtain the relation between the projections of these states on the state \(|n\rangle\)

\[ \langle n|\alpha, m, t\rangle = (m! L_m(-|\alpha|^2))^{-1/2} \frac{\partial^m}{\partial \alpha^m} \exp\left(\frac{|\alpha|^2}{2}\right) \langle n|\alpha, t\rangle. \]

Let us introduce the distribution function for \(m = 0\).

\[ p_0(n, t) = \langle n|\alpha, t\rangle^2. \]  

(35)

Then the relation between the functions \(p(n, t)\) and \(p_0(n, t)\) is given by the formula

\[ p(n, t) = (m! L_m(-|\alpha|^2))^{-1} e^{-|\alpha|^2} \frac{\partial^m}{\partial \alpha^m} \frac{\partial^m}{\partial \alpha^*^m} \exp\left(\frac{|\alpha|^2}{2}\right) p_0(n, t). \]  

(36)

The same relation exists for the generating functions as well. Due to this to find \(G(z, t)\) it is enough to compute the function

\[ G^0(z, t) = \sum_{n=0}^{\infty} p_0(n, t) z^n. \]  

(37)

The function \(p_0\) is proportional to the \(|H_n|^2\), therefore, we need to obtain the relation for summing the modules square of the Hermite polynomials. To obtain it we consider the product of two generating functions for the Hermite polynomials

\[ \exp(-x^2 + 2x\xi) \exp(-y^2 + 2y\xi^*) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{x^n}{n!} \frac{y^m}{m!} H_n(\xi) H_m(\xi^*), \]  

(38)

making a substitution of variables

\[ x = \sqrt{\lambda} \chi, \quad y = \sqrt{\lambda} \chi^*, \]

where \(\lambda\) is a real parameter. Then multiplying both sides of (38) by the factor \(\exp(-\chi \chi^*)\) and calculating the Gaussian integral with the account of the relation

\[ \frac{i}{2\pi} \int \chi^* \chi^m \exp(-\chi \chi^*) d\chi d\chi^* = n! \delta_{nm}, \]

we obtain the expression for \(G^0(z, t)\), where the relation between \(\lambda\) and \(z\) is given by the following formula

\[ \lambda = \frac{z |\varepsilon + i\varepsilon|}{2 |\varepsilon - i\varepsilon|}. \]
Using (36) we find the expression for the generating function

\[ G(z, t) = \sqrt{\sigma_0} m! L_m(-|\alpha|^2) H^{(J)}_{mm}(\alpha, \alpha^*) \exp \left[ (z\sigma_0 - 1)|\alpha|^2 - \text{Re}(J_{11}\alpha^2) \right]. \]  

(39)

where \( J \) is a 2 \( \times \) 2-matrix with the following matrix elements

\[ J_{11} = \frac{2\varepsilon* - [\varepsilon* + i\dot{\varepsilon}* + (\varepsilon* - i\dot{\varepsilon}*)z^2]\sigma_0}{2\varepsilon}, \]

\[ J_{12} = J_{21} = -z\sigma_0, \]

\[ \sigma_0 = \frac{4}{(|\varepsilon|^2 + |\dot{\varepsilon}|^2)(1 - z^2) + 2(1 + z^2)}. \]

One can check that \( G(1, t) = 1 \). From the definition (34) it follows that

\[ \langle a^+ a \rangle = \lim_{z \to 1} \frac{\partial G(z, t)}{\partial z}. \]

Taking the limit \( z \to 1 \) we see that this result coincides with the expression (28) for \( \langle a^+ a \rangle \) obtained above.

### 7 Quasiprobability Distributions for Field in \( |\alpha, m, t\rangle \)-State

Next we study the \( Q \)-function which is the diagonal matrix elements of the density operator in coherent states basis (see, for example, [28]). Thus, the \( Q \)-function is defined by formula

\[ Q(z) = \langle z|\alpha, m, t\rangle\langle\alpha, m, t|z\rangle. \]

(40)

Let us calculate the matrix element \( \langle z|\alpha, m, t\rangle \). As in the case of finding the distribution function the Hermite polynomials can be expressed through the generating function. The calculation gives

\[ Q(z) = \frac{\exp(-|z|^2 - |\alpha|^2)2|\vartheta|^{2m}|H_m(\varsigma)|^2}{L_m(-|\alpha|^2)m!(|\varepsilon|^2 + |\dot{\varepsilon}|^2 + 2)^{1/2}} \times \exp \left( \text{Re} \left( \frac{\varepsilon + i\dot{\varepsilon}}{\varepsilon - i\dot{\varepsilon}} \right) z^* 2 z^* \alpha - (\varepsilon^* - i\dot{\varepsilon}^*)\alpha^2 \right), \]  

(41)

where

\[ \vartheta = \left( \frac{\varepsilon^* - i\dot{\varepsilon}^*}{2(\varepsilon - i\dot{\varepsilon})} \right)^{1/2}, \quad \varsigma = \vartheta \left( \frac{2z^*}{\varepsilon^* - i\dot{\varepsilon}^*} - \alpha \right). \]

A transition to the stationary case is obvious. Preexponential factor in the formula (41) is related to \( m \) added photons and it changes the Gaussian form of the \( Q \)-function of the coherent state described by the exponent. Thus, the \( Q \)-function of the photon–added coherent state has the shape of deformed Gaussian with extra maxima and minima.
The Wigner function of an arbitrarily state, determined by the density matrix \( \rho(q, q') \), is defined by the relation

\[
W(q, p) = \int_{-\infty}^{+\infty} \rho(q + \frac{r}{2}, q - \frac{r}{2}) \exp(-ipr) dr.
\]

For the pure state \( |\alpha, m, t\rangle \) we have

\[
\rho(q, q') = \langle q|\alpha, m, t\rangle \langle \alpha, m, t|q'\rangle.
\]

Therefore, to calculate the integral (42) we make the same computation as for the integral \( \langle n|\alpha, m, t\rangle \) found above. For two-dimensional Hermite polynomial having zero diagonal matrix elements of the matrix \( R \) the following relation holds

\[
H_{nn}^{(00)}(\xi, \eta) = (-1)^n n! L_n(\xi \eta), \quad R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

Due to this we can represent the Wigner function in the form

\[
W(q, p) = \frac{(-1)^m 2L_m(|\gamma|^2)}{L_m(|\alpha|^2)} \exp(-|i\xi p - i\xi x - \sqrt{2}\alpha|),
\]

where \( \gamma \) is defined by

\[
\gamma = ip|\varepsilon|\sqrt{2} - i\sqrt{2} \frac{\varepsilon^*}{|\varepsilon|} - \frac{\alpha \varepsilon^*}{|\varepsilon|}.
\]

The physical parameters \( m \) describing extra photons produces the factor \( L_m(|\gamma|^2) \) in the formula (43) and this factor changes the Gaussian shape of the Wigner function of the coherent state. Since Laguerre polynomial has the wavy behaviour the Wigner function of photon–added state has local maxima and minima. Thus, we obtained the phase space distributions for the photon–added states of the parametric oscillator.

8 Numerical Evaluation of Means and Dispersions for Specific Time-Dependence of Frequency

In this Section we consider a specific time-dependence of the oscillator frequency \( \Omega^2(t) \) in form of "the saw with one tooth", i.e.,

\[
\Omega^2(t) = 1, \quad \text{if } t < 0 \text{ or } t \geq T,
\]

and

\[
\Omega^2(t) = 1 + kt, \quad \text{if } 0 < t < T.
\]

The time-dependence is determined by two parameters \( T \) and \( k \). Function \( \varepsilon(t) \) for such a "saw" is given by the formula

\[
\varepsilon(t) = e^{it}, \quad \text{if } t < 0,
\]

\[
\varepsilon(t) = \sqrt{1 + kt(C_1 J_{1/3}(\frac{2(1 + kt)^{3/2}}{3k})) + C_2 Y_{1/3}(\frac{2(1 + kt)^{3/2}}{3k})}, \quad \text{if } 0 < t < T,
\]
where $C_1$ and $C_2$ are defined by

$$C_1 = \frac{Y_{4/3}(\frac{2}{3k}) - Y_{1/3}(\frac{2}{3k})(k - i)}{J_{1/3}(\frac{2}{3k})Y_{4/3}(\frac{2}{3k}) - J_{4/3}(\frac{2}{3k})Y_{1/3}(\frac{2}{3k})},$$

$$C_2 = \frac{J_{1/3}(\frac{2}{3k})(k - i) - J_{4/3}(\frac{2}{3k})}{J_{1/3}(\frac{2}{3k})Y_{4/3}(\frac{2}{3k}) - J_{4/3}(\frac{2}{3k})Y_{1/3}(\frac{2}{3k})},$$

and if $t \geq T$, then $\varepsilon(t)$-function is given by

$$\varepsilon = D_1 e^{it} + D_2 e^{-it},$$

where $D_1$ and $D_2$ are given by

$$D_1 = \frac{e^{-iT}}{2}(\varepsilon(T) - i\dot{\varepsilon}(T)),$$

$$D_2 = \frac{e^{iT}}{2}(\varepsilon(T) + i\dot{\varepsilon}(T)).$$

At the fixed perturbation parameters $k$ and $T$ the dispersions of quadratures $\hat{q}$ and $\hat{p}$ will be periodical functions of time when $t > T$. The minimal value of dispersion squared multiplied by the factor 2 is the squeezing coefficient $s$

$$s = \frac{1}{2\sigma_q^2}.$$

We studied numerically the dependence of squeezing on $k, \alpha, m$ at the fixed $T = 1$. In Fig. 1 there are presented the results of the numerical calculation for parameter $\alpha = 10$, which corresponds to the multiphoton field. Different curves correspond to different $m$. We can see that they are very close. Their difference can be seen better at $k \sim 30$. In this range the values of $m = 1, 5, 10, 50$ correspond to the curves, if one counts from top to bottom. The dependence of squeezing on $k$ is an oscillating function. When $k \approx 60$ we observe the local minimum of squeezing, which is equal approximately to 0.02. In the following local minimum the squeezing is even stronger.

The parameter $\alpha = 1$ corresponds to the Fig. 2. In this case the field state is intermediate between classical and quantum one. We can see that the dependence on $m$ is stronger. In the range of $k$ from 30 to 80, the upper curve corresponds to $m = 5$, and the lower one—to $m = 1$.

In Fig. 3 there are presented the plots for quantum state of the field $\alpha = 0.1$. The dependence on $m$ becomes even stronger. The curves do not overlap as it was in Figs. 1 and 2. The number $m = 1$ corresponds to the upper curve and $m = 5$ to the lower one.

So we can conclude that with the decrease of $|\alpha|$ the field becomes more sensitive to photon pumping.

Now we discuss the behaviour of average number of photons and the dispersions of a number of photons. When $t > T$ the Hamiltonian becomes stationary and therefore the operator of a number of photons $a^+a$ is the integral of motion. The average values of $\langle \hat{N} \rangle$, $\langle \hat{N}^2 \rangle$ and therefore the dispersion of a number of photons do not depend on time when
The dependences of the ratio of the dispersion of a number of photons squared and mean photon number \( \langle N \rangle \) (which is denoted \( Q(\alpha, m) \)) on \( k \) at the fixed \( T = 1, t > T, \alpha = 1, m = 5 \) is presented in Fig. 4. As in the case of squeezing they have the tendency of the increasing role of \( m \) when \( |\alpha| \) decreases. If we compare Fig. 4 with Fig. 2 we can notice that when squeezing increases an average number of photons and the dispersion of it in the field also increases.

Let us discuss the results of the distribution function investigation. For the stationary case \( t > T \) the distribution \( p(n) \) does not depend on time and the most interesting is its dependence on \( n \) and \( k \). In Fig. 5 \( p(n) \) is plotted for \( m = 2, \alpha = 1 \). We chose the range of \( k \) from 50 to 80. According to Fig. 2 it is the range in which strong squeezing is achieved. We can see here the known oscillations of \( p(n) \) [11], which proves nonclassical nature of the field. At smaller squeezing these oscillations are smaller, too. The dependence of the photon distribution function on \( k \) behaves as smoothly oscillating (Fig. 5). While varying \( k \), the behaviour of maxima and minima of \( p(n) \) is changing. This is seen in Fig. 5 when \( n \) varies from 15 to 20.

In Fig. 6 the dependence of the distribution function on \( \alpha \) is illustrated. Here we chose \( k = 63, T = 1, m = 2, t > T \) and \( \alpha \) is real. At \( \alpha = 0 \) we get the distribution function of a squeezed Fock state \( |m, t \rangle \) which looks in the coordinate representation as

\[
\langle q|m, t \rangle = \left( \frac{\hat{\epsilon}^*}{2\varepsilon} \right)^{m/2} (m!\varepsilon\sqrt{\pi})^{-1/2} \exp \left( \frac{i\hat{\epsilon}^* q^2}{2\varepsilon} \right) \frac{H_m}{\varepsilon} \left( \frac{q}{|\varepsilon|} \right).
\]

While \( \alpha \) is changing in the quantum range from 0 to 1, the distribution oscillations can be easily noticed. The increase of \( \alpha \) results in the classical limit of a large number of photons. It produced a strong delocalization of the distribution function of the parameter \( n \). Thus, the oscillations become smaller and invisible in the graph.

The distribution function at \( k = 63, T = 1, \alpha = 1, t > T \) for \( m = 1 \) and \( m = 2 \) is shown in Fig. 7. For \( m = 1 \) at \( n > 40 \) oscillations disappear while for \( m = 2 \) they exist at large values of \( n \).

Generally speaking, the problem of dependence of the distribution function for the squeezed and, simultaneously, correlated state on the parameter \( m \) requires special investigation. We can just suppose that the trend of the dependence on \( m \) will be different at different extent of squeezing and different correlation between quadrature components.

In Fig. 8 there is \( Q \)-function at \( T = 1, k = 10, \alpha = 1, m = 5, t = 10 \). If \( \alpha \) is increased, the form of a \( Q \)-function peak will resemble the form shown at fig.8. However, a shape and a position will be different. Approximately, the same trend is observed while increasing \( m \). At the values of \( \alpha \) corresponding to the quantum case the form of peak becomes much more sensitive to changes of parameters. It is illustrated in Fig. 9, which represents the \( Q \)-function at \( \alpha = 0.1, T = 1, k = 1, m = 10, t = 10 \).

The Wigner function is more sensitive to changes of \( m \). In Fig. 10a there is Wigner function at \( T = 1, k = 10, \alpha = 2, m = 9, t = 10 \). The Wigner function of the coherent state is the Gaussian function with one maximum. On the other hand, for the photon–added states such shape is deformed and several local maxima are observable. Fig.10b shows the same plot that Fig.10a but from the direction of \( p \) axis. In this figure it is clearly seen that the function \( W(q, p) \) can be negative.
In Figs. 11 and 12 the Wigner function is plotted at the same values of parameters $T = t = 1$, $\alpha = 2$, $m = 1$ but at different values of $k$, 10, and 60, respectively. At $k = 10$ the dispersion of $q$ is greater than at $k = 60$ but the dispersion of $p$ is less. That is why the peak of the Wigner function in Fig. 11 is significantly narrower along $q$ and wider along $p$, than in Fig. 12.

Thus we demonstrated that parametrical excitation of the electromagnetic field oscillator changes the statistics of photon–added coherent states since the squeezing and correlation of quadratures produced due to nonstationarity of the oscillator influences the dispersions and means of photon numbers in these states.
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FIGURE CAPTIONS

**Fig. 1.** Squeezing of $q$ as a function of $k$ for $T = 1$, $\alpha = 10$, and $m = 1$, 5, 10, 50.

**Fig. 2.** Squeezing of $q$ as a function of $k$ for $T = 1$, $\alpha = 1$, and $m = 1$, 5.

**Fig. 3.** Squeezing of $q$ as a function of $k$ for $T = 1$, $\alpha = 0.1$, and $m = 1$, 5.

**Fig. 4.** The dispersion of the number of photons squared and divided by the average number of photons $Q(\alpha, m)$ for $T = 1$, $t > T$, $\alpha = 1$, and $m = 1$, 5, 10.

**Fig. 5.** Photon number distribution $p(n, t)$ as a function of $n$ and $k$ for $T = 1$, $t > T$, $\alpha = 1$, and $m = 2$.

**Fig. 6.** Photon number distribution $p(n, t)$ as a function of $n$ and $\alpha$ for $T = 1$, $t > T$, $k = 63$, and $m = 2$.

**Fig. 7.** Photon number distribution $p(n, t)$ for $T = 1$, $t > T$, $k = 63$, $\alpha = 1$, and $m = 1$, 2.

**Fig. 8.** $Q$–function $Q(z)$ for $T = 1$, $t = 10$, $k = 10$, $\alpha = 1$, and $m = 5$.

**Fig. 9.** $Q$–function $Q(z)$ for $T = 1$, $t = 10$, $k = 10$, $\alpha = 0.1$, and $m = 10$.

**Fig. 10a (b).** Wigner function $W(q, p)$ for $T = 1$, $t = 10$, $k = 10$, $\alpha = 2$, and $m = 9$.

**Fig. 11.** Wigner function $W(q, p)$ for $T = 1$, $t = 1$, $k = 10$, $\alpha = 2$, and $m = 1$.

**Fig. 12.** Wigner function $W(q, p)$ for $T = 1$, $t = 1$, $k = 60$, $\alpha = 2$, and $m = 1$. 
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