A UNIQUE PAIR OF TRIANGLES

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ABSTRACT. A rational triangle is a triangle with sides of rational lengths. In this short note, we prove that there exists a unique pair of a rational right triangle and a rational isosceles triangle which have the same perimeter and the same area. In the proof, we determine the set of rational points on a certain hyperelliptic curve by a standard but sophisticated argument which is based on the 2-descent on its Jacobian variety and Coleman’s theory of p-adic abelian integrals.

1. Main theorem and its proof

A rational (resp. integral) triangle is a triangle with sides of rational (resp. integral) lengths. Such a triangle has arithmetic interest: For instance, every rational right triangle has sides of lengths \((k(1 + x^2), k(1 - x^2), 2kx)\) with positive rational numbers \(k, x > 0\). We may check this fact from the Pythagorean theorem and the uniqueness of the prime decomposition, which are most elementary theorems in geometry and arithmetic respectively.

From the point of view of arithmetic, perimeter and area are fundamental invariants of a rational triangle. Therefore, it is natural to try to classify rational triangles by their perimeters and/or areas. Indeed, there are several works on construction of infinitely many pairs of rational triangles which have the same perimeters and the same areas (see e.g. [Bre06], [vL07] and references there).

A primitive triangle is an integral triangle such that the greatest common divisor of the lengths of its sides is 1. We can prove that there exists no pair of a primitive right triangle and a primitive isosceles triangle which have the same perimeter and the same area. We give an elementary proof of this fact in Appendix. How many such pairs are there in the category of rational triangles? In this short note, we give the complete answer to this question, that is, there exists only one such pair of triangles.

Theorem 1.1. Up to similitude, there exists a unique pair of a rational right triangle and a rational isosceles triangle which have the same perimeter and the same area. The unique pair consists of the right triangle with sides of lengths \((377, 135, 352)\) and the isosceles triangle with sides of lengths \((366, 366, 132)\).

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Since the proof for the primitive case is quite elementary, it may be surprising that our proof of Theorem 1.1 given below depends on a sophisticated theory of modern arithmetic geometry of hyperelliptic curves (of genus 2), which contains the 2-descent argument on the Jacobian variety of a hyperelliptic curve ([FPS97], [Sto01]) and Coleman’s theory of $p$-adic abelian integrals ([Col85], [Col85b]). In particular, the following theorem of Chabauty and Coleman is a key ingredient of our proof:

**Theorem 1.2** ([Cha41], [Col85b, Corollary 4a], [MP12, Theorem 5.3 (b)]). Let $C$ be a curve of genus $g \geq 2$ over $\mathbb{Q}$, and $J$ be its Jacobian variety. Let $p$ be a prime number. Suppose that the rank of $J(\mathbb{Q})$ is smaller than $g$, and $p > 2g$ and $C$ has good reduction at $p$. Then, we have

$$\#C(\mathbb{Q}) \leq \#C(\mathbb{F}_p) + (2g - 2).$$

Although there are several works on rational triangles, the only previous work which uses such a theory is [ZP17], where the authors prove that there exists no pair of an integral isosceles triangle and a certain integral rhombus which have the same perimeters and the same integral areas.

**Proof of Theorem 1.1.** Assume that there exists such a pair of triangles. First, note that since every rational isosceles triangle with a rational area has a rational height, it is a union of the two copies of a rational right triangle. Moreover, by rescaling both of the given triangles, we may assume that the given triangles have sides of lengths

1. $(k(1 + x^2), k(1 - x^2), 2kx)$ and $(1 + u^2, 1 + u^2, 4u)$, or
2. $(k(1 + x^2), k(1 - x^2), 2kx)$ and $(1 + u^2, 1 + u^2, 2(1 - u^2))$

respectively with positive rational numbers $x, u, k$.

In case (1), we have a simultaneous equation

$$\begin{cases}
k + kx = 1 + 2u + u^2 \\
k^2x(1 - x^2) = 2u(1 - u^2).
\end{cases}$$

Since $x, u, k > 0$, this is equivalent to

$$\begin{cases}
k(1 + x) = w^2 \\
(w^2 - k)w(2k - w^2) = 2k(w - 1)(w - 2),
\end{cases}$$

where we set $w = u + 1$. Since the former equation has a unique solution $(x, w, k)$ for every solution $(w, k)$ with $w > 1, k > 0$ of the latter, this simultaneous equation is equivalent to the single latter equation under the condition $w > 1, k > 0$. Moreover, it is equivalent to

$$-w^5 + 3kw^3 - 2k^2w = 2kw^2 - 6kw + 4k,$$

i.e.,

$$2wk^2 + (-3w^3 + 2w^2 - 6w + 4)k + w^5 = 0.$$ 

Since $k$ is a rational number, the discriminant of the left hand side as a polynomial of $k$ is a square integer, say, $r^2$. Therefore, we obtain

$$r^2 = (-3w^3 + 2w^2 - 6w + 4)^2 - 8w^6,$$

which defines an affine curve. We denote its non-singular compactification by $C_1$, which is a hyperelliptic curve.

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1Here, a curve over $\mathbb{Q}$ means a smooth projective geometrically integral scheme of dimension 1 over $\text{Spec}(\mathbb{Q})$.

2Cf. the proof of Theorem 1.3.
It is easy to check that $C_1$ has at least ten rational points, that is, $(w, r) = (0, \pm 4), (1, \pm 1), (2, \pm 8), (12, \pm 868)$ and two points at infinity. All of these points do not give triangles. On the other hand, we can prove that the Mordell-Weil rank of the Jacobian variety of $C_1$ over $\mathbb{Q}$ is at most 1. Indeed, we can check it by Magma Calculator (cf. [BCP97]). Here are the inputs and output, where the algorithm is based on [Sto01].

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> R<w>:=PolynomialRing(Rationals());
> C:=HyperellipticCurve((-3*w^3+2*w^2-6*w+4)^2-8*w^6);
> J:=Jacobian(C);
> RankBound(J);
1
```

Finally, we may easily check that $C_1$ has good reduction at 5 and $\#C_1(\mathbb{F}_5) = 8$. Therefore, by applying Theorem 1.2, we obtain that $\#C_1(\mathbb{Q}) \leq 10$, i.e., there exists no other rational point than the above ten rational points.

Next, we consider case (2). We have a simultaneous equation

$$\begin{cases} k + kx = 2 \\ k^2(x(1 - x^2)) = 2u(1 - u^2). \end{cases}$$

Since $x, u, k > 0$, this is equivalent to

$$\begin{cases} k(1 + x) = 2 \\ (2 - k)(2k - 2) = ku(1 - u^2). \end{cases}$$

Since the former equation has a unique solution $(x, u, k)$ for every solution $(u, k)$ with $0 < k < 2$ of the latter, this simultaneous equation is equivalent to the single latter equation under the condition $0 < k < 2$. Moreover, it is equivalent to

$$2k^2 - (u^3 - u + 6)k + 4 = 0.$$

Since $k$ is a rational number, the discriminant of the left hand side as a polynomial of $k$ is a square integer, say, $s^2$. Therefore, we obtain

$$s^2 = (u^3 - u + 6)^2 - 32,$$

which defines an affine curve. We denote its non-singular compactification by $C_2$, which is a hyperelliptic curve. It is easy to check that $C_2$ has at least ten rational points, that is, $(u, s) = (0, \pm 2), (1, \pm 2), (1, \pm 2), (5/6, \pm 217/216)$ and two points at infinity. The former six points do not give triangles. Moreover, the equalities $(u, s) = (5/6, 217/216)$ and $(5/6, -217/216)$ imply that $(k, x, u) = (27/16, 5/27, 5/6)$ and $(32/27, 11/16, 5/6)$ respectively, both of which give the unique pair in the statement up to similitude. Finally, by the same argument as in case (1), we may also check that $\#C_2(\mathbb{Q}) \leq 10$, i.e., there exists no other rational point than the above ten rational points. This completes the proof. □

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3 Indeed, the equalities $(w, r) = (0, \pm 4), (1, \pm 1)$ imply that the bottom length $4u$ of the isosceles triangle is less than or equal to zero, and $(w, r) = (2, \pm 8), (12, \pm 868)$ imply that the height $1 - u^2$ of the isosceles triangle is less than or equal to zero.

4 In fact, it has exactly rank 1.

5 http://magma.maths.usyd.edu.au/calc

6 See also the Magma handbook: Example CrvHyp_sha_visibility (H131E34) in https://magma.maths.usyd.edu.au/magma/handbook/text/1501.

7 In fact, the two curves $C_1$ and $C_2$ are isomorphic to each other via the isomorphism induced by a birational map given by $(u, s) = (1 - 2/w, 2r/w^3)$.

8 Indeed, the equalities $(u, s) = (0, \pm 2)$ imply that the height $2u$ of the isosceles triangle is zero, and $(u, s) = (1, \pm 2), (-1, \pm 2)$ imply that the bottom length $2(1 - u^2)$ of the isosceles triangle is zero.
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Appendix: No pair of a primitive right triangle and an isosceles triangle

**Theorem 1.3.** There exists no pair of a primitive right triangle and a primitive isosceles triangle which have the same perimeter and the same area.

**Proof.** We prove this statement by contradiction. Assume that there exists such a pair of triangles. As we have done in the proof of Theorem 1.1, we may assume that the given triangles have sides of lengths

\[(1) \ (x^2 + y^2, x^2 - y^2, 2xy) \text{ and } (u^2 + v^2, u^2 - v^2, 4uv), \]

\[(2) \ (x^2 + y^2, x^2 - y^2, 2xy) \text{ and } (u^2 + v^2, u^2 - v^2, 2(u^2 - v^2))\]

respectively with positive integers \(x, y, u, v\) such that \(x\) and \(y\) (resp. \(u\) and \(v\)) are coprime, and exactly one of \(x\) and \(y\) (resp. \(u\) and \(v\)) are even.

In the case (1), we have a simultaneous equation

\[
\begin{cases}
2x^2 + 2xy = 2u^2 + 4uv + 2v^2 \\
xy(x^2 - y^2) = 2uv(u^2 - v^2).
\end{cases}
\]

Since \(x, x + y, u + v > 0\), this is equivalent to

\[
\begin{cases}
x(x + y) = (u + v)^2 \\
y(x - y) = \frac{2uv(u-v)}{u+v}.
\end{cases}
\]

Here, note that since \(y(x - y)\) is a positive integer, \(2uv(u - v)/(u + v)\) is also a positive integer. On the other hand, since either \(u\) or \(v\) is even, we see that \(u + v\) is an odd positive integer. Moreover, since \(u\) and \(v\) are coprime, \(u + v\) and \(uv\) are also coprime. Since \(2uv(u - v)/(u + v)\) is a positive integer, this implies that \((u - v)/(u + v)\) is a positive integer, which is impossible whenever \(u, v > 0\).

The proof for case (2) is the same. This completes the proof. □

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