LOWER BOUNDS FOR THE PRINCIPAL GENUS OF DEFINITE BINARY QUADRATIC FORMS

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ABSTRACT. We apply Tatzuawa’s version of Siegel’s theorem to derive two lower bounds on the size of the principal genus of positive definite binary quadratic forms.

Introduction. Suppose \(-D < 0\) is a fundamental discriminant. By genus theory we have an exact sequence for the class group \(\mathcal{C}(−D)\) of positive definite binary quadratic forms:

\[
P(−D) \overset{\text{def}}{=} \mathcal{C}(−D)^2 \rightarrow \mathcal{C}(−D) \rightarrow \mathcal{C}(−D)/\mathcal{C}(−D)^2 \cong (\mathbb{Z}/2)^{g−1},
\]

where \(D\) is divisible by \(g\) primary discriminants (i.e., \(D\) has \(g\) distinct prime factors). Let \(p(−D)\) denote the cardinality of the principal genus \(P(−D)\). The genera of forms are the cosets of \(\mathcal{C}(−D)\) modulo the principal genus, and thus \(p(−D)\) is the number of classes of forms in each genus. The study of this invariant of the class group is as old as the study of the class number \(h(−D)\) itself. Indeed, Gauss wrote in [3, Art. 303] . . . Further, the series of [discriminants] corresponding to the same given classification (i.e. the given number of both genera and classes) always seems to terminate with a finite number . . . However, rigorous proofs of these observations seem to be very difficult.

Theorems about \(h(−D)\) have usually been closely followed with an analogous result for \(p(−D)\). When Heilbronn [4] showed that \(h(−D) \rightarrow \infty\) as \(D \rightarrow \infty\), Chowla [1] showed that \(p(−D) \rightarrow \infty\) as \(D \rightarrow \infty\). An elegant proof of Chowla’s theorem is given by Narkiewicz in [8, Prop 8.8 p. 458].

Similarly, the Heilbronn-Linfoot result [3] that \(h(−D) > 1\) if \(D > 163\), with at most one possible exception was matched by Weinberger’s result [14] that \(p(−D) > 1\) if \(D > 5460\) with at most one possible
exception. On the other hand, Oesterlé’s exposition of the Goldfeld-
Gross-Zagier bound for $h(-D)$ already contains the observation that
the result was not strong enough to give any information about $p(-D)$.

In [13] Tatzuawa proved a version of Siegel’s theorem: for every $\varepsilon$
there is an explicit constant $C(\varepsilon)$ so that

$$h(-D) > C(\varepsilon)D^{1/2-\varepsilon}$$

with at most one exceptional discriminant $-D$. This result has never
been adapted to the study of the principal genus. It is easily done;
the proofs are not difficult so it is worthwhile filling this gap in the
literature. We present two versions. The first version contains a tran-
scendental function (the Lambert $W$ function discussed below). The
second version gives, for each $n \geq 4$, a bound which involves only ele-
mentary functions. For each fixed $n$ the second version is stronger on
an interval $I = I(n)$ of $D$, but the first is stronger as $D \to \infty$. The
second version has the added advantage that it is easily computable.
(N.B. The constants in Tatzuawa’s result have been improved in [6] and
[7]; these could be applied at the expense of slightly more complicated
statements.)

**Notation.** We will always assume that $g \geq 2$, for if $g = 1$ then $-D = -4, -8$, or $-q$ with $q \equiv 3 \mod 4$ a prime. In this last case $p(-q) = h(-q)$ and Tatzuawa’s theorem [13] applies directly.

**First Version**

**Lemma 1.** If $g \geq 2$,

$$\log(D) > g \log(g).$$

**Proof.** Factor $D$ as $q_1, \ldots, q_g$ where the $q_i$ are (absolute values) of pri-
mary discriminants, i.e. 4, 8, or odd primes. Let $p_i$ denote the $i$th
prime number, so we have

$$(1) \quad \log(D) = \sum_{i=1}^{g} \log(q_i) \geq \sum_{i=1}^{g} \log(p_i) \overset{\text{def}}{=} \theta(p_g).$$

By [11] (3.16) and (3.11), we know that Chebyshev’s function $\theta$ satisfies
$\theta(x) > x(1 - 1/\log(x))$ if $x > 41$, and that

$$p_g > g(\log(g) + \log(\log(g)) - 3/2).$$

After substituting $x = p_g$ and a little calculation, this gives $\theta(p_g) >
$$g \log(g)$ as long as $p_g > 41$, i.e. $g > 13$. For $g = 2, \ldots, 13$, one can
easily verify the inequality directly. \qed
Let $W(x)$ denote the Lambert $W$-function, that is, the inverse function of $f(w) = w \exp(w)$ (see [2], [10, p. 146 and p. 348, ex 209]). For $x \geq 0$ it is positive, increasing, and concave down. The Lambert $W$-function is also sometimes called the product log, and is implemented as `ProductLog` in Mathematica.

**Theorem 1.** If $0 < \varepsilon < 1/2$ and $D > \max(\exp(1/\varepsilon), \exp(11.2))$, then with at most one exception

$$p(-D) > \frac{1.31}{\pi} \varepsilon D^{1/2-\varepsilon-\log(2)/W(\log(D))}.$$ 

**Proof.** Tatuzawa’s theorem [13], says that with at most one exception

$$\frac{\pi \cdot h(-D)}{\sqrt{D}} = L(1, \chi_D) > .655 \varepsilon D^{-\varepsilon},$$

thus

$$p(-D) = \frac{2h(-D)}{2^g} > \frac{1.31 \varepsilon \cdot D^{1/2-\varepsilon}}{\pi \cdot 2^g}.$$ 

The relation $\log(D) > g \log(g)$ is equivalent to

$$\log(D) > \exp(\log(g)) \log(g),$$

Thus applying the increasing function $W$ gives, by definition of $W$

$$W(\log(D)) > \log(g),$$

and applying the exponential gives

$$\exp(W(\log(D))) > g.$$ 

The left hand side above is equal to $\log(D)/W(\log(D))$ by the definition of $W$. Thus

$$-\log(D)/W(\log(D)) < -g,$$

$$D^{-\log(2)/W(\log(D))} = 2^{-\log(D)/W(\log(D))} < 2^{-g},$$

and the Theorem follows. □

**Remark.** Our estimate arises from the bound $\log(D) > g \log(g)$, which is nearly optimal. That is, for every $g$, there exists a fundamental discriminant (although not necessarily negative) of the form

$$D_g \overset{\text{def}}{=} \pm 3 \cdot 4 \cdot 5 \cdot 7 \ldots p_g,$$

and

$$\log |D_g| = \theta(p_g) + \log(2).$$

From the Prime Number Theorem we know $\theta(p_g) \sim p_g$, so

$$\log |D_g| \sim p_g + \log(2)$$
while [1] 3.13 shows $p_g < g(\log(g) + \log(\log(g)))$ for $g \geq 6$.

**Second version**

**Theorem 2.** Let $n \geq 4$ be any natural number. If $0 < \varepsilon < 1/2$ and $D > \max(\exp(1/\varepsilon), \exp(11.2))$, then with at most one exception

$$p(-D) > \frac{1.31\varepsilon}{\pi} \cdot \frac{D^{1/2-\varepsilon-1/n}}{f(n)},$$

where

$$f(n) = \exp \left[ (\pi(2^n) - 1/n) \log 2 - \theta(2^n)/n \right];$$

here $\pi$ is the prime counting function and $\theta$ is the Chebyshev function.

**Proof.** First observe

$$f(n) = \frac{2\pi(2^n)}{2^{1/n} \prod_{\text{primes } p < 2^n} p^{1/n}}.$$

From Tatuzawa’s Theorem [2], it suffices to show $2^g \leq f(n)D^{1/n}$. Suppose first that $D$ is not $\equiv 0 \pmod{8}$.

Let $S = \{4, \text{ odd primes } < 2^n\}$, so $|S| = \pi(2^n)$. Factor $D$ as $q_1 \cdots q_g$ where $q_i$ are (absolute values) of coprime primary discriminants, that is, 4 or odd primes, and satisfy $q_i < q_j$ for $i < j$. Then, for some $0 \leq m \leq g$, we have $q_1, \ldots, q_m \in S$ and $q_{m+1}, \ldots, q_g \not\in S$, and thus $2^n < q_i$ for $i = m+1, \ldots, g$. This implies

$$2^{gn} = 2^n \cdots 2^n \geq 2^{mn} q_{m+1}q_{m+2} \cdots q_g$$

$$= \frac{2^{mn}}{q_1 \cdots q_m} \frac{D}{\prod_{q \in S} q} \cdot D$$

as we have included in the denominator the remaining elements of $S$ (each of which is $\leq 2^n$). The above is

$$= \frac{2^{\pi(2^n)-n}}{2^{1/n} \prod_{\text{primes } p < 2^n} p} \cdot D = f(n)^n \cdot D.$$

This proves the theorem when $D$ is not $\equiv 0 \pmod{8}$. In the remaining case, apply the above argument to $D' = D/2$; so

$$2^{gn} \leq f(n)^n D' < f(n)^n D.$$

□
Examples. If $0 < \varepsilon < 1/2$ and $D > \max(\exp(1/\varepsilon), \exp(11.2))$, then with at most one exception, Theorem 2 implies

\[
p(-D) > 0.10199 \cdot \varepsilon \cdot D^{1/4-\varepsilon} \quad (n = 4)
\]
\[
p(-D) > 0.0426 \cdot \varepsilon \cdot D^{5/10-\varepsilon} \quad (n = 5)
\]
\[
p(-D) > 0.01249 \cdot \varepsilon \cdot D^{1/3-\varepsilon} \quad (n = 6)
\]
\[
p(-D) > 0.00188 \cdot \varepsilon \cdot D^{5/14-\varepsilon} \quad (n = 7)
\]

Comparison of the two theorems

How do the two theorems compare? Canceling the terms which are the same in both, we seek inequalities relating

\[
D^{-\log 2/W(\log D)} \quad \text{v.} \quad \frac{D^{-1/n}}{f(n)}.
\]

Theorem 3. For every $n$, there is a range of $D$ where the bound from Theorem 2 is better than the bound from Theorem 1. However, for any fixed $n$ the bound from Theorem 1 is eventually better as $D$ increases.

For fixed $n$, the first statement of Theorem 3 is equivalent to proving

\[
D^{\log(2)/W(\log(D))} - 1/n \geq f(n)
\]

on a non-empty compact interval of the $D$ axis. Taking logarithms, it suffices to show,

Lemma 2. Let $n \geq 4$. Then

\[
x \left( \frac{\log 2}{W(x)} - \frac{1}{n} \right) \geq \log f(n)
\]

on some non-empty compact interval of positive real numbers $x$.

Proof. Let $g(n, x) = x (\log 2/W(x) - 1/n)$. Then

\[
\frac{\partial g}{\partial x} = \frac{\log 2}{W(x)} - \frac{1}{n} \quad \text{and} \quad \frac{\partial^2 g}{\partial x^2} = \frac{-\log 2 \cdot W(x)}{x(W(x) + 1)^3}.
\]

This shows $g$ is concave down on the positive real numbers and has a maximum at

\[
x = 2^n(n \log 2 - 1)/e.
\]

Because of the concavity, all we need to do is show that $g(n, x) > \log f(n)$ at some $x$. The maximum point is slightly ugly so instead we let $x_0 = 2^n n \log 2/e$. 
Using $W(x) \sim \log x - \log \log x$, a short calculation shows

$$g(n, x_0) \sim \frac{1}{e} \cdot \frac{2^n}{n}.$$ 

By [12, 5.7]), a lower bound on Chebyshev’s function is

$$\theta(t) > t \left(1 - \frac{1}{40 \log t}\right), \quad t > 678407.$$ 

(Since we will take $t = 2^n$ this requires $n > 19$ which is not much of a restriction.) By [11, (3.4)], an upper bound on the prime counting function is

$$\pi(t) < \frac{t}{\log t - 3/2}, \quad t > e^{3/2}.$$ 

Hence $-\theta(2^n) < 2^n \left(1/(40n \log 2) - 1\right)$ and so

$$\log f(n) = \left(\frac{\pi(2^n) - 1}{n}\right) \log 2 - \frac{\theta(2^n)}{n}$$

$$< \left(\frac{2^n}{n \log 2 - 3/2} - 1\right) \log 2 + \frac{2^n}{n} \left(\frac{1}{40n \log 2} - 1\right)$$

$$\sim \frac{61}{40 \log 2} \cdot \frac{2^n}{n^2}.$$ 

Comparing the two asymptotic bounds for $g$ and $\log f$ respectively we see that

$$\frac{1}{e} \cdot \frac{2^n}{n} > \frac{61}{40 \log 2} \cdot \frac{2^n}{n^2},$$

for $n \geq 6$; small $n$ are treated by direct computation. 

Figure 1 shows a log-log plot of the two lower bounds, omitting the contribution of the constants which are the same in both and the terms involving $\varepsilon$. That is, Theorem 2 gives for each $n$ a lower bound $b(D)$ of the form

$$b(D) = C(n)\varepsilon D^{1/2-1/n-\varepsilon},$$

so

$$\log(b(D)) = (1/2 - 1/n - \varepsilon) \log(D) + \log(C(n)) + \log(\varepsilon).$$

Observe that for fixed $n$ and $\varepsilon$, this is linear in $\log(D)$, with the slope an increasing function of the parameter $n$. What is plotted is actually $(1/2 - 1/n) \log(D) + \log(C(n))$ as a function of $\log(D)$, and analogously for Theorem 1. In red, green, and blue are plotted the lower bounds from Theorem 2 for $n = 4$, 5, and 6 respectively. In black is plotted the lower bound from Theorem 1.

\[1\] The details of the asymptotics have been omitted for conciseness.
**Examples.** The choice $\varepsilon = 1/\log(5.6 \cdot 10^{10})$ in Theorem 1 shows that $p(-D) > 1$ for $D > 5.6 \cdot 10^{10}$ with at most one exception. (For comparison, Weinberger [14, Lemma 4] needed $D > 2 \cdot 10^{11}$ to get this lower bound.) And, $\varepsilon = 1/\log(3.5 \cdot 10^{14})$ in Theorem 1 gives $p(-D) > 10$ for $D > 3.5 \cdot 10^{14}$ with at most one exception. Finally, $n = 6$ and $\varepsilon = 1/\log(4.8 \cdot 10^{17})$ in Theorem 2 gives $p(-D) > 100$ for $D > 4.8 \cdot 10^{17}$ with at most one exception.

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