Optimality gap of constant-order policies decays exponentially in the lead time for lost sales models

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Inventory models with lost sales and large lead times have traditionally been considered intractable, due to the curse of dimensionality which arises from having to track the set of orders placed but not yet received (i.e. pipeline vector). Recently, Goldberg et al. (2012) laid the foundations for a new approach to solving these models, by proving that as the lead time grows large (with the other problem parameters fixed), a simple constant-order policy (proposed earlier by Reiman (2004)) is asymptotically optimal. This was quite surprising, as it is exactly this setting (i.e. large lead times) that was previously believed intractable. However, the bounds proven there are impractical, requiring the lead time to be very large before the constant-order policy becomes nearly optimal, e.g. requiring a lead time which is $\Omega(\epsilon^{-2})$ to ensure a $(1 + \epsilon)$-approximation guarantee, and involving a massive prefactor. The authors note that numerical experiments of Zipkin (2008b) suggest that the constant-order policy performs quite well even for small lead times, and pose closing this gap (thus making the results practical) as an open problem.

In this work, we make significant progress towards resolving this open problem and closing this gap. In particular, for the infinite-horizon variant of the finite-horizon problem considered by Goldberg et al. (2012), we prove that the optimality gap of the same constant-order policy actually converges exponentially fast to zero, i.e. we prove that a lead time which is $O(\log(\epsilon^{-1}))$ suffices to ensure a $(1 + \epsilon)$-approximation guarantee. We demonstrate that the corresponding rate of exponential decay is at least as fast as the exponential rate of convergence of the expected waiting time in a related single-server queue to its steady-state value. We also derive simple and explicit bounds for the optimality gap. For the special case of exponentially distributed demand, we further compute all expressions appearing in our bound in closed form, and numerically evaluate them, demonstrating good performance for a wide range of parameter values. Our main proof technique combines convexity arguments with ideas from queueing theory.

Key words: inventory, lost-sales, constant-order policy, lead time, asymptotic optimality, steady-state.
1. Introduction

It is well-known that there is a fundamental dichotomy in the theory of inventory models, depending on the fate of unmet demand. If unmet demand remains in the system and can be met at a later time, we say the system exhibits backlogged demand; if unmet demand is lost to the system, we say the system exhibits lost sales. Which of these assumptions is appropriate depends heavily on the application of interest. For example, in many retail applications one must manage an inventory in a competitive environment, i.e. demand can in principle be met by a competing supplier, making lost sales a more appropriate assumption. Indeed, as pointed out in Bijvank and Vis (2011), recent studies have shown that retailers across many sectors lose over 75% of the potential demand which they cannot satisfy immediately, and we refer the interested reader to Gruen, Corsten and Bharadway (2002), and Verhoef and Sloot (2006) for further details.

A second important feature of many inventory models, intimately related to the above dichotomy, is that of positive lead times, i.e. settings in which there is a multi-period delay between when an order for more inventory is placed and when that order is received. In principle, this feature leads to an enlarged state-space (growing linearly with the lead time), to track all orders already placed but not yet received, i.e. the pipeline vector. It is a classical result, indeed one of the foundational results of the field, that models with backlogged demand remain tractable even in the presence of positive lead times. Namely, it can be proven that a so-called base-stock (i.e. order-up-to) policy, based only on the total inventory position (i.e. sum of the current inventory and all orders in the pipeline vector), is optimal in this setting (cf. Scarf (1960), Iglehart (1963), Veinott (1966)). Intuitively, this follows from the fact that when demand is backlogged, inventory is a linear function of orders placed and past demands, along with certain convexity arguments. However, it is known that such simple policies are no longer optimal for models with lost sales and positive lead times (cf. Karlin and Scarf (1958)). For over fifty years, inventory models with lost sales and positive lead
times were generally considered intractable, as the primary solution method (dynamic programming) suffered from the curse of dimensionality as the lead time grew. As noted in Bijvank and Vis (2011), this has led to many researchers using models with backlogging as approximations for settings in which a lost sales assumption is more appropriate, which may lead to very suboptimal solutions.

Although the optimal policy for lost-sales models with positive lead times remains poorly understood, the model has been studied now for over fifty years (cf. Gaver (1959), Morse (1959), Yaspan (1961), Morton (1969), Morton (1971), Pressman (1977), Nahmias (1979), Downs, Metters and Semple (2001), Johansen (2001), Janakiraman and Roundy (2004), Johansen and Thorstenson (2008)), and we refer to Bijvank and Vis (2011) for a comprehensive review. Due to the difficulty of computing the optimal policy, there has been considerable focus on understanding structural properties of an optimal policy, and analyzing heuristics. In particular, convexity results were obtained in Karlin and Scarf (1958), Morton (1969), and Zipkin (2008a), and used to bound the optimal ordering quantity. Janakiraman, Seshadri and Shanthikumar (2007) compared the optimal costs between the backlogged and lost sales systems with identical problem parameters, and showed that the lost sales system always had a lower cost. Huh et al. (2009) further proved that the base-stock policy was asymptotically optimal as the lost-sales penalty became large compared to the holding cost, and similar results were derived in Lu, Squillante and Yao (2012). In a breakthrough work, Levi, Janakiraman and Nagarajan (2008) proposed the family of so-called dual-balancing policies, motivated by previous work on other relevant models (cf. Levi et al. (2008a), Levi et al. (2008b)), and proved that the cost incurred by such a policy was always within a factor of 2 of optimal. Another recent line of research, based on carefully truncating and rounding the relevant dynamic programs, yields efficient approximation algorithms for any fixed lead time, with run-time polynomial in the inverse of the desired approximation error but possibly growing exponentially in the other problem inputs (e.g. the lead time) (cf. Halman et al. (2009), Halman, Orlin and Simchi-Levi (2012), Chen, Dawande and Janakiraman (2014a)). Despite this progress, the aforementioned work leaves open the problem of deriving efficient algorithms with arbitrarily small error, when the lead time is large.
A very simple and natural policy, which will be the subject of our own investigations, is the so-called constant-order policy, which places the same order in every period, independent of the state of the system. Perhaps surprisingly, Reiman (2004) proved that for lost sales inventory models with positive lead times, sometimes the best constant-order policy outperforms the more sophisticated base-stock policy, and performed a detailed analysis under a certain asymptotic scaling. This phenomena was further illuminated by the computational study of Zipkin (2008b), which confirmed that in several scenarios the constant-order policy performed favorably. In all of their experiments, the constant-order policy always incurred an expected cost at most twice that incurred by the optimal policy; in 62.5% of the cases, it incurred a cost at most 1.33 times that incurred by the optimal policy; and in 38% of the cases, it incurred a cost at most 1.12 times that incurred by the optimal policy.

These observations were recently given a solid theoretical foundation by Goldberg et al. (2012), who proved that for lost sales inventory models with positive lead times, as the lead time grows with all other parameters remaining fixed, the best constant-order policy is in fact asymptotically optimal. This is quite surprising, as the policy is so simple, and performs nearly optimally exactly in the setting which had stumped researchers for over fifty years. However, the bounds proven there are impractical, requiring the lead time to be very large before the constant-order policy becomes nearly optimal, e.g. requiring a lead time which is $\Omega(\epsilon^{-2})$ to ensure a $(1 + \epsilon)$-approximation guarantee, and involving a massive prefactor. The authors note that the numerical experiments of Zipkin (2008b) suggest that the constant-order policy performs quite well even for small lead times, and pose closing this gap (thus making the results practical) as an open problem. The authors also point out that if one could prove that the constant-order policy performs well even for small to moderate lead times, this would open the door for the creation of practical hybrid algorithms, which solve large dynamic programs when the lead time is small, and gradually transition to more naive algorithms for larger lead times.
1.1. Our contributions

In this work, we make significant progress towards resolving this open problem and closing this gap. In particular, for the infinite-horizon variant of the finite-horizon problem considered by Goldberg et al. (2012), we prove that the optimality gap of the same constant-order policy actually converges \textit{exponentially fast} to zero, i.e. we prove that a lead time which is $O\left(\log(e^{-1})\right)$ suffices to ensure a $(1 + \epsilon)$-approximation guarantee. We demonstrate that the corresponding rate of exponential decay is at least as fast as the exponential rate of convergence of the expected waiting time in a related single-server queue to its steady-state value, which we prove to be monotone in the ratio of the lost-sales penalty to the holding cost. We also derive simple and explicit bounds for the optimality gap. For the special case of exponentially distributed demand, we further compute all expressions appearing in our bound in closed form, and numerically evaluate them, demonstrating good performance for a wide range of parameter values. Our main proof technique combines convexity arguments with ideas from queueing theory, and is simpler than the coupling argument of Goldberg et al. (2012).

1.2. Outline of paper

The rest of the paper is organized as follows. We formulate our problem, and introduce several elementary properties of the stationary inventory process, in Section 2.1 We describe the constant-order policy in Section 2.2 and review the results of Goldberg et al. (2012) in Section 2.3 We state our main results in Section 2.4 We provide a more detailed analysis (both analytical and numerical) for the special case of exponentially distributed demand in Section 2.4.1 and discuss the monotonicity of our bounds in the ratio of the lost-sales penalty to the holding cost in Section 2.4.2 The proof of our main results are given in Section 3 Finally, we summarize our main results and propose directions for future research in Section 4 A technical appendix is provided in Section 5

2. Main results

2.1. Model description, problem statement, and assumptions

In this section, we formally define our lost-sales inventory optimization problem, noting that a finite-horizon variant of the same model was considered in Goldberg et al. (2012). Let $\{D_t, t \geq 1\}$
be a sequence of independent and identically distributed (i.i.d.) demand realizations, distributed as the non-negative random variable (r.v.) $D$ with distribution $D$, which we assume to have finite mean, and (to rule out certain degenerate cases) to have strictly positive variance. Let $T$ be the time horizon, $L$ be the deterministic lead time, and $h, c (> 0)$ be the unit holding cost and lost-sales penalty respectively. In addition, let $I_t$ denote the on-hand inventory, and $x_t = (x_{1,t}, \ldots, x_{L,t})$ denote the pipeline vector of orders placed but not yet delivered, at the beginning of time period $t$, where $x_{i,t}$ is the order to be received in period $i + t - 1$. The ordered sequence of events in period $t$ is then as follows.

- A new amount of inventory $x_{1,t}$ is delivered and added to the on-hand inventory;
- A new order is placed;
- The demand $D_t$ is realized;
- Costs for period $t$ are incurred, and the on-hand inventory and pipeline vector are updated.

Note that the on-hand inventory is updated according to $I_{t+1} = \max(0, I_t + x_{1,t} - D_t)$, and the pipeline vector is updated such that $x_{1,t}$ is removed, $x_{i,t+1}$ is set equal to $x_{i+1,t}$ for $i \in [1, L - 1]$, and $x_{L,t+1}$ is set equal to the new order placed. We require that this new order $x_{L,t+1}$ be a (possibly random) function of realized demands, inventory levels, ordering quantities, and pipeline vectors, as well as the problem primitives $h, c, T, L, D$ and current time $t$, but cannot depend on future demands. We call the corresponding family of policies admissible, and denote this family by $\Pi$.

Define $C_t$ to be the sum of the holding cost and lost-sales penalty in time period $t$:

$$C_t \triangleq h(I_t + x_{1,t} - D_t)^+ + c(I_t + x_{1,t} - D_t)^-, \quad \text{where } x^+ \triangleq \max(x, 0), \quad x^- \triangleq \max(-x, 0).$$

For simplicity, we suppose that the problem initial conditions are to start with the all zeros pipeline vector, i.e. $x_1 = 0$, and zero inventory, i.e. $I_1 = 0$. We note that our problem will differ from that considered in Goldberg et al. (2012) in a single important way: we will consider the corresponding infinite-horizon problem, while Goldberg et al. (2012) considered the finite-horizon problem. Namely, for a policy $\pi$, let $C(\pi)$ denote the long-run average cost incurred:

$$C(\pi) \triangleq \limsup_{T \to \infty} \frac{\sum_{t=1}^T \mathbb{E}[C_t]}{T}.$$
The corresponding infinite-horizon (i.e. long-run average cost) lost-sales inventory optimization problem is given by

$$\text{OPT}(L) \triangleq \inf_{\pi \in \Pi} C(\pi).$$  \hspace{1cm} (1)

Recall that a stationary policy is one that places orders only based on the current state information (i.e., the on-hand inventory and pipeline vector), as well as the problem primitives $h, c, L, D$, but not the current time period $t$ or time horizon $T$. Under a stationary policy, the evolution of the on-hand inventory and pipeline vector evolves as a discrete time, finite-dimensional Markov Chain. It follows from the results of Huh, Janakiraman and Nagarajan (2011) that: an optimal policy for Problem 1 exists (i.e. is not simply approached), and furthermore that there always exists at least one such optimal policy which is stationary, so restricting oneself to the family of stationary policies is without loss of generality (w.l.o.g.). We will further assume that of these stationary optimal policies, there exists at least one such policy $\pi^*$ whose corresponding induced Markov chain converges in distribution to a unique stationary measure when initialized with $x_1 = 0$ and $I_1 = 0$. We will also assume that under this policy $\pi^*$ (with the given initialization), $E[x_t], E[I_t], E[C_t]$ are finite for all $t$, and converge to the corresponding expected values under the given stationary measure (which we also assume to be finite), i.e. $L^1$ convergence. We refer to the set of such policies as convergent. Such a convergence is to be expected from the basic theory of Markov chains, and we refer the interested reader to Asmussen (2003) and Meyne and Tweedie (2009) for further details. We note that this is especially so, in light of the results of Zipkin (2008a), which demonstrate that there exists an optimal stationary policy which, under the given initialization, with probability (w.p.) 1 belongs to a fixed compact set for all time (i.e. the ordering quantities and inventory levels are uniformly bounded over time as functions of $h, c, L, D$ only). For any such stationary and convergent policy $\pi$, let $(I^\pi, \chi^\pi)$ denote a vector distributed as the stationary measure of the corresponding Markov chain, with $I^\pi$ corresponding to the stationary inventory level, and $\chi^\pi$ corresponding to the stationary pipeline vector.
2.2. Constant-order policy

In this section, we formally define the constant-order policy, and characterize the best constant-order policy. As a notational convenience, let us define all empty sums to equal zero, let \( \mathbf{1} \) denote the vector with all entries equal to unity, \( e \) denote Euler’s number, \( \log(x) \) denote the natural logarithm of \( x \), \( \frac{1}{\infty} \) denote 0, \( \frac{1}{0} \) denote \( \infty \), \( \log(\infty) \) denote \( \infty \), and \( \mathbb{I}(A) \) denote the indicator of the event \( A \).

For any \( r \in [0, \mathbb{E}(D)] \), the constant-order policy \( \pi_r \) is the policy that places the constant order \( r \) in every period. It is well-known (cf. Goldberg et al. (2012)) that the corresponding steady-state on-hand inventory level, which we denote by \( I_r^\infty \), has the same distribution as the steady-state waiting time in the corresponding \( GI/GI/1 \) queue with interarrival distribution \( D \) and processing time distribution the constant \( r \). For two r.v.s \( X, Y \), let \( X \sim Y \) denote equivalence in distribution between \( X \) and \( Y \). In that case, it is well-known (cf. Asmussen (2003)) that

\[
I_r^\infty \sim \sup_{j \geq 0} \left( jr - \sum_{i=1}^{j} D_i \right). \tag{2}
\]

We note that in Goldberg et al. (2012), the authors considered a slightly modified constant-order policy which ordered \( I_r^\infty + r \) in the first period and \( r \) in all subsequent periods, to make the corresponding sequence of inventory levels stationary. As both policies have the same steady-state distribution, for our purposes this distinction is irrelevant.

We now formalize the notion of the best constant-order policy, and begin by briefly reviewing several well-known properties of the stationary inventory level under any stationary and convergent policy \( \pi \) (not just the constant-order policy). As the inventory update dynamics imply that \( I^\pi \sim (I^\pi + \chi^\pi - D)^+ \), with \( D \) independent of \( I^\pi \) and \( \chi^\pi \), it follows that \( \mathbb{E}[I^\pi] = \mathbb{E}[(I^\pi + \chi^\pi - D)^+] \). A straightforward algebraic manipulation further demonstrates that

\[
\mathbb{E}[(I^\pi + \chi^\pi - D)^-] = \mathbb{E}[D] - \mathbb{E}[\chi^\pi]. \tag{3}
\]

Combining the above, we conclude that

\[
\mathbb{E}[\chi^\pi] \leq \mathbb{E}[D], \tag{4}
\]
and

\[ C(\pi) = hE[I^\pi] + cE[D] - cE[I^\pi]. \tag{5} \]

Customizing (2) - (5) to the constant-order policy, we conclude that for any \( r \in [0, E[D]) \),

\[ C(\pi_r) = hE \left[ \sup_{j \geq 0} \left( jr - \sum_{i=1}^{j} D_i \right) \right] + cE[D] - cr, \tag{6} \]

and note that \( E[I^{\pi_r}] < \infty \) for all \( r \in [0, E[D]) \) (cf. Asmussen (2003)). As it is easily verified that \( C(\pi_r) \) is thus a convex function of \( r \) on \( [0, E[D]) \), to find the best possible constant-order policy, it suffices to select the \( r \) minimizing this one-dimensional convex function over the compact set \( [0, E[D]) \). We note that the existence of at least one such optimal \( r \) follows from the well-known properties of convex optimization over a compact set, and that the set of all such optimal solutions must be bounded away from \( E[D] \), since by assumption \( h > 0 \), and \( \lim_{t \to E[D]} E[I^\pi_t] = \infty \) (since \( D \) has strictly positive variance, cf. Asmussen (2003)). Let \( r_\infty \in \arg \min_{0 \leq r \leq E[D]} C(\pi_r) \) denote the infimum of this set of optimal ordering quantities, in which case the best constant-order policy will refer to \( \pi_{r_\infty} \).

### 2.3. Review of results of Goldberg et al. (2012)

In this section, we formally review the results of Goldberg et al. (2012), and begin by introducing some additional notations. Let \( Q \) denote the \( \frac{h}{c+h} \) quantile of the demand distribution, i.e. \( Q \overset{\Delta}{=} \inf \{ s \in \mathbb{R}^+ : P(D > s) \leq \frac{h}{c+h} \} \). We note that \( Q \) is the optimal inventory level for the corresponding single-stage newsvendor problem, i.e. for any policy \( \pi \) and any time \( t \), \( E[C_t^\pi] \geq g \overset{\Delta}{=} hE[(Q - D)^+] + cE[(D - Q)^+] \) (cf. Zipkin (2000)). Equivalently, \( g = \text{OPT}(0) \), the long-run optimal cost when there is zero leadtime, as in this case one can always order-up to this optimal level \( Q \). That \( g \in (0, \infty) \) follows from the assumption that \( D \) has finite mean and is not deterministic. Also, let \( \sigma \) denote the standard deviation of \( D \), \( \zeta \overset{\Delta}{=} \mathbb{E}[|D - \mathbb{E}[D]|^3] \sigma^{-3} \) denote the so-called skewness of \( D \),

\[ m \overset{\Delta}{=} \left[ \frac{26(3\zeta + c(h\sigma)^{-1}\mathbb{E}[D] + 1)}{2} \right]^2, \]

and

\[ y(\epsilon) \overset{\Delta}{=} \max \left( 2^{14}h(Q + 2^4\mathbb{E}[D])(\mathbb{E}[D] + \mathbb{E}[D^2])^3 \sigma^{-6} m^3 g^{-1} \epsilon^{-1}, \left( 12 c g^{-1} (2 c h^{-1})^2 + 3 \right)^2 \epsilon^{-2} \right). \]
Then the main result of Goldberg et al. (2012) is as follows. We only state the implications of those results for the infinite-horizon problem, as that is our focus in this paper and to do otherwise would require several additional definitions and notations, but do note that the results of Goldberg et al. (2012) also apply to the finite-horizon setting.

**Theorem 1.** Suppose $E[D^3] < \infty$. Then for all $\epsilon \in (0, 1)$ and $L \geq y(\epsilon)$, $\frac{C(\pi_r^\infty)}{\text{OPT}(L)} \leq 1 + \epsilon$.

Theorem 1 represented significant progress in our understanding of lost sales models with large lead times, as it proved that the simple constant-order policy performs well exactly when the problem becomes challenging to solve by dynamic programming and other means, i.e. when $L$ becomes large. However, as discussed in Goldberg et al. (2012), the explicit bounds of Theorem 1 require $L$ to be so large as to make the results impractical. In addition to the massive prefactor, they require $L$ to be $\Omega(\epsilon^{-2})$ to achieve a $(1 + \epsilon)$-approximation, which requires e.g. a lead time on the order of 400 to be within 5% of optimal. As pointed out in Goldberg et al. (2012), this massive prefactor and unfavorable scaling with $\epsilon$ leave much to be desired, and are a far cry from the good numerical performance of the constant-order policy even for small lead times demonstrated in Zipkin (2008b). Goldberg et al. (2012) pose tightening these bounds and closing this gap as a significant open question, as doing so would represent a large step towards making the bounds practical, e.g. proving that when $L$ is small one can solve a large dynamic program, and when $L$ becomes even moderately large one can use simple policies such as the constant-order policy.

### 2.4. Main results

In this section, we present our main results, demonstrating that the optimality gap of the best constant-order policy decays exponentially in the lead time. For $\theta \geq 0$, let us define

$$\phi(\theta) \triangleq \exp(\theta r^\infty) \mathbb{E}[\exp(-\theta D)] \quad , \quad \gamma \triangleq \inf_{\theta \geq 0} \phi(\theta),$$

and $\vartheta \in \arg \min_{\theta \geq 0} \phi(\theta)$ denote the supremum of the set of minimizers of $\phi(\theta)$, where we define $\vartheta$ to equal $\infty$ if the above infimum is not actually attained. Note that $\phi(\theta)$ is a continuous and convex function of $\theta$ on $(0, \infty)$, and right-continuous function of $\theta$ at 0. In addition, it follows
from Folland (1999) Theorem 2.27 that $\phi(\theta)$ is right-differentiable at zero, with derivative equal to $r_\infty - \mathbb{E}[D]$, assuming only that $\mathbb{E}[D] < \infty$ (along with our default assumption of non-negativity). As $r_\infty < \mathbb{E}[D]$, we conclude from the definition of derivative and a straightforward contradiction argument that $\vartheta > 0$ (i.e. $\vartheta$ is strictly positive), and $\gamma \in [0,1)$ (i.e. $\gamma$ is strictly less than 1). It follows from the celebrated Cramér’s Theorem, and more generally the theory of large deviations, that up to exponential order (and under appropriate technical assumptions), $\Pr(kr_\infty \geq \sum_{i=1}^k D_i)$ decays like $\gamma^k$ as $k \to \infty$ (cf. Deuschel and Stroock (1989)). Furthermore, as we will explore in detail later in the proof of our main result, $\gamma$ corresponds (again up to exponential order, under appropriate assumptions) to the rate at which the expected waiting time in an initially empty single-server queue, with inter-arrival distribution $D$ and processing time distribution (the constant) $r_\infty$, converges to its steady-state value (cf. Kingman (1962)). Then our main result is as follows.

**Theorem 2 (Exponential convergence of constant-order policy to optimality).** For all $L \geq 1$,

$$\frac{C(\pi r_\infty)}{\text{OPT}(L)} \leq 1 + h((1 - \gamma)g)^{-1} \left(\mathbb{E}[D] - r_\infty + (e\vartheta(L+1))^{-1}\right)^{L+1}. \quad (7)$$

Our results prove that for the corresponding infinite-horizon problem, the optimality gap of the constant-order policy converges exponentially fast to zero. In particular, a lead time which is $O(\log(\epsilon^{-1}))$ suffices to ensure a $(1 + \epsilon)$-approximation guarantee. This contrasts with the bounds of Goldberg et al. (2012), which had an inverse polynomial dependence on $\epsilon$. Furthermore, our explicit bounds are much tighter than those of Goldberg et al. (2012), and we only require a finite first moment, i.e. our results also hold for heavy-tailed distributions. This takes a large step towards answering several open questions posed in Goldberg et al. (2012) with regards to deriving bounds tight enough to be useful in practice. In particular, our bounds suggest that for small values of $L$, one can solve a large dynamic program to derive the optimal policy (whose size may be exponential in $L$), while for larger values of $L$ one can simply use the constant-order policy. We again note that since our results only hold for the infinite-horizon problem, and will use critically certain stationarity properties that only hold in this regime, our results are not directly comparable to those of Goldberg et al. (2012), whose bounds also hold for finite-horizon problems. Closing this gap, and proving tighter bounds for the finite-horizon problem, remains an interesting open question.
2.4.1. Example: exponentially distributed demand

In this section, for the special case of exponentially distributed demand, we further compute all expressions appearing in our bound in closed form, and numerically evaluate them, demonstrating good performance for a wide range of parameter values. Thus suppose demand is exponentially distributed with rate $\lambda$, i.e. mean $\lambda^{-1}$. In this case, it is well-known that for $r \in [0, \mathbb{E}[D])$, $\mathbb{E}[I_\infty]$ is the expected steady-state waiting time in a corresponding $M/D/1$ queue, and equals $\frac{r^2\lambda}{2(1-r\lambda)}$ (cf. Haigh (2013)). For $c, h > 0$, let

$$\tau_{c,h} \triangleq \sqrt{\frac{h}{2c+h}}, \quad \gamma_{c,h} \triangleq (1 - \tau_{c,h}) \exp(\tau_{c,h}).$$

It may be easily demonstrated that $\gamma_{c,h} \in (0,1)$. In that case, when demand is exponentially distributed, Theorem 2 is equivalent to the following bound, for which we provide a complete derivation in the appendix.

**Corollary 1 (Case of exponentially distributed demand).** Suppose $D$ is exponentially distributed with rate $\lambda$. Then $C(\pi_\infty) = \lambda^{-1}(\sqrt{h(2c+h)} - h)$, and for all $L \geq 1$,

$$\frac{C(\pi_\infty)}{OPT(L)} \leq 1 + \left(\tau_{c,h} + (\tau_{c,h}^{-1} - 1)(e(L+1))^{-1}\right)\left((1 - \gamma_{c,h}) \log(1 + ch^{-1})\right)^{-1}\gamma_{c,h}^{L+1}. \quad (8)$$

Note that (8) does not depend on $\lambda$, which follows from the scaling properties of the exponential distribution. We now numerically evaluate (8) under different lost-demand penalty and lead time scenarios, with the holding cost fixed to 1, and present the results in Table 1. For each $c$, we also give the value of the best constant-order policy, $C(\pi_\infty)$, further assuming $\mathbb{E}[D] = 1$ (i.e. $\lambda = 1$).
We note that when $\frac{c}{h}$ is small (i.e. less than or equal to 1), our bounds demonstrate an excellent performance by the constant-order policy even for lead times as small as 10. When $\frac{c}{h}$ is moderate (i.e. less than or equal to 9), our bounds demonstrate a similarly good performance for lead times on the order of 70. Even when $\frac{c}{h}$ is very large, our bounds still imply non-trivial performance guarantees, e.g. the constant-order policy is always within a factor of 4 of optimal when $c = 39$ and $L = 100$. Combining (8) with our explicit evaluation of $C(\pi_{r_\infty})$ yields tight bounds on $\text{OPT}(L)$ whenever (8) is close to 1. For example, for $c = h = 1$, our bounds imply that $\text{OPT}(10) \in [.63,.73]$.

Given the complexity of computing $\text{OPT}(L)$ exactly using dynamic programming as $L$ grows (cf. Zipkin (2008b), Chen, Dawande and Janakiraman (2014a)), and the very large values of $L$ required for the earlier results of Goldberg et al. (2012) to apply, we believe that our bounds provide the first window into the behavior of $\text{OPT}(L)$ for moderate, but realistic, values of $L, c, h$.

### 2.4.2. Impact of the ratio $\frac{c}{h}$ on our bounds

In this section, we discuss the dependence of our demonstrated exponential rate of convergence $\gamma$ on the ratio of the lost-sales penalty to the holding cost. Indeed, the numerical results of Section 2.4.1 suggest a degradation in performance as $\frac{c}{h}$ grows, and we now formalize this. In particular, we show that $\gamma$ is non-decreasing in $\frac{c}{h}$. Note
that by a simple scaling argument, for any fixed demand distribution \( D \), \( r_\infty \) (and thus \( \gamma \)) is a function of \( \frac{c}{h} \) only, as opposed to the particular values of \( c, h \). To make this dependence explicit, let \( \gamma(\rho) \) denote the value of \( \gamma \) when \( \frac{c}{h} = \rho \) (with the dependence on \( D \) implicit).

**Lemma 1.** Under the same assumptions as Theorem 3, for any fixed demand distribution \( D \), \( \gamma(\rho) \) is non-decreasing in \( \rho \).

We include a proof of Lemma 1 in the appendix. This result suggests that the optimality gap of the constant-order policy may be larger when \( \frac{c}{h} \) is large. Interestingly, this is exactly the regime in which Huh et al. (2009) proved that order-up-to policies are nearly optimal. More formally understanding this connection remains an interesting open question.

### 3. Proof of Theorem 2

In this section, we prove Theorem 2. Recall that \( \pi^* \) denotes some fixed stationary and convergent policy which is optimal for Problem 1, where the existence of such a policy follows from our assumptions. Let \((I^*, \chi^*)\) denote a vector distributed as the stationary measure of the corresponding Markov chain, and \( \{D_i, i \geq 1\} \) an i.i.d. sequence of demands, distributed as \( D \), independent of \((I^*, \chi^*)\). Let \( \delta_{i,j} = 1 \) if \( i = j \), and 0 otherwise. It follows from stationarity, the inventory dynamics, and a straightforward induction that \( I^* \sim \max_{j=0, \ldots, L} \left( \sum_{i=1}^j (\chi^*_{L+1-i} - D_{L+1-i} + \delta_{j, L} I^*) \right) \). Thus

\[
E[I^*] = E \left[ \max_{j=0, \ldots, L} \left( \sum_{i=1}^j (\chi^*_{L+1-i} - D_{L+1-i} + \delta_{j, L} I^*) \right) \right].
\]

The crux of our argument consists of two simple observations. First, we conclude the following from stationarity and the manner in which the pipeline vector is updated.

**Observation 1** \( \chi^*_i \) has the same distribution for all \( i \in [1, L] \), and \( E[\chi^*_i] = E[\chi^*_1] \) for all \( i \in [1, L] \).

Second, we note that the right-hand side of (9) is a jointly convex function of \( \chi^* \) and \( I^* \), which will allow us to apply the multi-variate Jensen’s inequality (cf. Dudley (2002)). In particular, for fixed \( d \in \mathbb{R}^L \), let us define \( f_d(\chi_1, \ldots, \chi_L, I) \triangleq \max_{j=0, \ldots, L} \left( \sum_{i=1}^j (\chi_{L+1-i} - d_{L+1-i} + \delta_{j, L} I) \right) \).
Observation 2 For each fixed $d \in \mathbb{R}^L$, $f_d(\chi_1, \ldots, \chi_L, I)$ is a jointly convex function of $(\chi_1, \ldots, \chi_L, I)$ over $\mathbb{R}^{L+1}$. Combining with Observation 1, the multi-variate Jensen’s inequality, and the i.i.d. property of $\{D_i, i \geq 1\}$, we conclude that

$$E[I^*] \geq E \left[ \max_{j=0, \ldots, L} \left( jE[\chi_1] - \sum_{i=1}^{j} D_i + \delta_{j,L} E[I^*] \right) \right]. \quad (10)$$

We note that the observation of such a convexity in terms of the on-hand inventory and pipeline vector is not new. Indeed, the so called $L$-natural-convexity of the relevant cost-to-functions has been studied extensively (cf. Karlin and Scarf (1958), Morton (1969), Zipkin (2008a), Chen, Dawande and Janakiraman (2014a)), and used to obtain both structural results and algorithms. In contrast, here we use convexity to relate the expected inventory under an optimal policy to the expected inventory under a particular constant-order policy, intuitively that which orders $E[\chi_1]$ in every period. Very similar ideas and arguments have appeared previously in the queueing theory literature, to demonstrate the extremality (with regards to expected waiting times) of certain queueing systems with constant service (or inter-arrival) times (cf. Humblet (1982), Hajek (1983)). We also note that although a related idea appears in the proofs of Goldberg et al. (2012), the fact that they do not work in the stationary regime results in their obtaining much weaker results, since outside of stationarity one can no longer assume that all pipeline vector components have the same mean.

Before proceeding, let us define several additional notations. In particular, for $r \in [0, E[D]]$ and $L \geq 1$, let

$$I_r^L \triangleq \max_{j=0, \ldots, L} \left( j r - \sum_{i=1}^{j} D_i \right), \quad C_L(r) \triangleq hE[I_r^L] + cE[D] - cr,$$

and $r_L \in \arg \min_{r \leq E[D]} C_L(r)$ denote the infimum of the set of minimizers of $C_L(r)$. We note that $I_r^L$ is distributed as the waiting time of the $L$-th customer in the corresponding GI/GI/1 queue (initially empty) with interarrival distribution $D$ and processing time $r$. We also note that for $r \in [0, E[D])$, $I_r^\infty$ is the weak limit, as $L \to \infty$, of $I_r^L$. Similarly, $C(\pi_r) = \lim_{L \to \infty} C_L(r)$, and $C_L(r)$ is monotone increasing in $L$.

We now combine (10) with (5), non-negativity, and definitions to bound the optimality gap of the constant-order policy.
\textbf{Lemma 2.} $\text{OPT}(L) \geq C_L(\gamma_L)$, and

$$C(\gamma_L) - \text{OPT}(L) \leq h \left( \mathbb{E}\left[I_{\infty}^L\right] - \mathbb{E}\left[I_{L}^L\right] \right) + h \left( \mathbb{E}\left[I_{\infty}^L\right] - \mathbb{E}\left[I_{L}^L\right] \right) - c(r_{\infty} - r_L). \quad (11)$$

\textbf{Proof} Combining (11) with the nonnegativity of $\mathbb{E}[\gamma^*]$, we conclude that $\mathbb{E}[\gamma^*] \geq \mathbb{E}\left[I_{L}^\epsilon\right]$. Thus by (6), $\text{OPT}(L) \geq h\mathbb{E}\left[I_{L}^\epsilon\right] + c\mathbb{E}[D] - c\mathbb{E}[\epsilon]$. Combining with (11) and the definition of $r_L$, we conclude that $\text{OPT}(L) \geq C_L(\gamma_L)$. It then follows from (6) that

$$C(\gamma_L) - \text{OPT}(L) \leq \left( h\mathbb{E}\left[I_{\infty}^L\right] + c\mathbb{E}[D] - cr_{\infty} \right) - \left( h\mathbb{E}\left[I_{L}^L\right] + c\mathbb{E}[D] - cr_L \right)$$

$$= h \left( \mathbb{E}\left[I_{\infty}^L\right] - \mathbb{E}\left[I_{L}^L\right] \right) + h \left( \mathbb{E}\left[I_{\infty}^L\right] - \mathbb{E}\left[I_{L}^L\right] \right) - c(r_{\infty} - r_L),$$

completing the proof. \qed

We proceed by bounding the terms appearing in the right-hand side of (11) separately. We begin by recalling a classical result of Kingman (1962), which uses the celebrated Spitzer’s identity to bound the difference between the expected waiting time of the $L$th job to arrive to a single-server queue (initially empty), and the steady-state expected waiting time. As this difference is exactly $\mathbb{E}\left[I_{\infty}^L\right] - \mathbb{E}[I_L]$, the result will allow us to bound the relevant term of (11). We state Kingman’s results as customized to our own setting, notations, and assumptions.

\textbf{Lemma 3 (Theorems 1, 4, Kingman (1962))}. For all $r \in [0, \mathbb{E}[D]]$ and $L \geq 1$,

$$\mathbb{E}[I_L^r] = \sum_{n=1}^{L} \frac{1}{n} \mathbb{E} \left[ \left( nr - \sum_{i=1}^{n} D_i \right)^+ \right].$$

If in addition $r < \mathbb{E}[D]$, then

$$\mathbb{E}[I_{\infty}^r] = \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{E} \left[ \left( nr - \sum_{i=1}^{n} D_i \right)^+ \right].$$

Also,

$$\mathbb{E}\left[I_{\infty}^r\right] - \mathbb{E}\left[I_{L}^r\right] \leq (1 - \gamma)e^\theta(L + 1)^{-1} \gamma^{L+1}.$$
Lemma 4. \( r_\infty \leq r_L \) for all \( L \geq 1 \).

Proof Suppose for contradiction that there exists \( L \in [1, \infty) \) such that \( r_L < r_\infty \). Note that in this case, both \( r_L, r_\infty < E[D] \), and thus by Lemma 3 both \( E[I^r_\infty], E[I^r_L] < \infty \). From definitions and the associated respective optimality of \( r_L, r_\infty \), we conclude that

\[
\begin{align*}
    hE[I^r_\infty] + cE[D] - cr_\infty &\leq hE[I^r_L] + cE[D] - cr_L, \\
    hE[I^r_L] + cE[D] - cr_L &\leq hE[I^r_\infty] + cE[D] - cr_\infty.
\end{align*}
\]

Summing these two inequalities together implies that

\[
E[I^r_\infty] + E[I^r_L] \leq E[I^r_L] + E[I^r_\infty],
\]

which, by Lemma 3 is equivalent to

\[
\sum_{n=L+1}^{\infty} \frac{1}{n} E \left[ \left( nr_\infty - \sum_{i=1}^{n} D_i \right)^+ \right] \leq \sum_{n=L+1}^{\infty} \frac{1}{n} E \left[ \left( nr_L - \sum_{i=1}^{n} D_i \right)^+ \right] < \infty. \tag{12}
\]

Here we note that the desired result intuitively follows from (12) and the monotonicity of the relevant functions, i.e. the fact that \( x > y \) implies \( E \left[ (nx - \sum_{i=1}^{n} D_i)^+ \right] \geq E \left[ (ny - \sum_{i=1}^{n} D_i)^+ \right] \). However, we must rule out certain subtle problems that could potentially arise from the function \( E \left[ (nr - \sum_{i=1}^{n} D_i)^+ \right] \) not being strictly monotonic in \( r \), and proceed as follows. Definitions, non-negativity, and the fact that \( r_L < r_\infty \), together imply that

\[
\left( nr_\infty - \sum_{i=1}^{n} D_i \right)^+ = \mathbb{I} \left( nr_\infty \geq \sum_{i=1}^{n} D_i \right) \left( nr_\infty - \sum_{i=1}^{n} D_i \right) \geq \mathbb{I} \left( nr_L \geq \sum_{i=1}^{n} D_i \right) \left( nr_\infty - \sum_{i=1}^{n} D_i \right).
\]

Combining with (12), we conclude that

\[
\sum_{n=L+1}^{\infty} \frac{1}{n} E \left[ \mathbb{I} \left( nr_L \geq \sum_{i=1}^{n} D_i \right) \left( nr_\infty - \sum_{i=1}^{n} D_i \right) \right] \leq \sum_{n=L+1}^{\infty} \frac{1}{n} E \left[ \mathbb{I} \left( nr_L \geq \sum_{i=1}^{n} D_i \right) \left( nr_L - \sum_{i=1}^{n} D_i \right) \right] < \infty.
\]

It follows that

\[
\sum_{n=L+1}^{\infty} \frac{1}{n} E \left[ \mathbb{I} \left( nr_L \geq \sum_{i=1}^{n} D_i \right) (nr_\infty - nr_L) \right] \leq 0.
\]
However, since by assumption $nr_\infty - nr_L > 0$, it follows from non-negativity that

$$
\sum_{n=L+1}^{\infty} \mathbb{E} \left[ \mathbb{I} \left( nr_L \geq \sum_{i=1}^{n} D_i \right) \right] = 0,
$$

and thus $\mathbb{P}(nr_L \geq \sum_{i=1}^{n} D_i) = 0$ for all $n \geq L + 1$. Further noting that $\mathbb{P}(nr_L \geq \sum_{i=1}^{n} D_i) \geq \mathbb{P}^n(r_L \geq D_1)$, we conclude that $\mathbb{P}(r_L \geq D_1) = 0$. It follows that $\mathbb{E}[I_{r_L}^\infty] = \mathbb{E}[I_L^r] = 0$, and thus by (5), $C(\pi_{r_L}) = C_L(r_L)$. Combining with Lemma 2, which implies that $C_L(r_L) \leq C(\pi_{r_\infty})$, and the optimality of $r_\infty$, we conclude that $r_L \in \text{arg min}_{0 \leq r \leq \mathbb{E}(D)} C(\pi_r)$. However, as $r_\infty$ is by definition the infimum of $\text{arg min}_{0 \leq r \leq \mathbb{E}(D)} C(\pi_r)$, the fact that $r_L < r_\infty$ thus yields a contradiction, completing the proof. \(\square\)

Before proceeding, we also derive a certain critical inequality, which we will use to show that the term $h(\mathbb{E}[I_L^\infty] - \mathbb{E}[I_L^r])$ and the term $c(r_\infty - r_L)$ essentially “cancel out”. This inequality follows from the first-order optimality conditions of the convex optimization problem associated with $r_\infty$, but requires some care, as the relevant functions are potentially non-differentiable, and the desired statement in principle involves an interchange of expectation and differentiation.

**Lemma 5.** $\sum_{n=1}^{\infty} \mathbb{P}(nr_\infty \geq \sum_{i=1}^{n} D_i) \geq \frac{1}{n}$.

**Proof** Since $r_\infty < \mathbb{E}(D)$, there exists $\delta > 0$ such that $r_\infty + \epsilon < \mathbb{E}(D)$ for all $\epsilon \in [0, \delta]$. Let us fix any such $\epsilon > 0$. The definition and associated optimality of $r_\infty$ implies that $\mathbb{E}[\pi_{r_\infty}] \leq \mathbb{E}[\pi_{r_\infty} + \epsilon]$. Combining with Lemma (5) and (6), we conclude that

$$
h \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{E} \left[ \mathbb{I} \left( nr_\infty \geq \sum_{i=1}^{n} D_i \right) \left( nr_\infty - \sum_{i=1}^{n} D_i \right) \right]
$$

is at most

$$
h \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{E} \left[ \mathbb{I} \left( n(r_\infty + \epsilon) \geq \sum_{i=1}^{n} D_i \right) \left( n(r_\infty + \epsilon) - \sum_{i=1}^{n} D_i \right) \right] - ce.
$$

Combining with the fact that

$$
\mathbb{I} \left( nr_\infty \geq \sum_{i=1}^{n} D_i \right) \left( nr_\infty - \sum_{i=1}^{n} D_i \right) \geq \mathbb{I} \left( n(r_\infty + \epsilon) \geq \sum_{i=1}^{n} D_i \right) \left( n(r_\infty + \epsilon) - \sum_{i=1}^{n} D_i \right),
$$

it follows that

$$
h \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{E} \left[ \mathbb{I} \left( n(r_\infty + \epsilon) \geq \sum_{i=1}^{n} D_i \right) \left( n(r_\infty + \epsilon) - \sum_{i=1}^{n} D_i \right) \right]$$
is at most
\[ h \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{E} \left[ \mathbb{I} \left( n(r_\infty + \epsilon) \geq \sum_{i=1}^{n} D_i \right) \left( n(r_\infty + \epsilon) - \sum_{i=1}^{n} D_i \right) \right] - c \epsilon. \]

Equivalently (as all relevant sums are finite)
\[ \sum_{n=1}^{\infty} \mathbb{P} \left( n(r_\infty + \epsilon) \geq \sum_{i=1}^{n} D_i \right) \geq \frac{c}{h}. \]

As this holds for all sufficiently small \( \epsilon \), the only remaining step is to demonstrate validity at \( \epsilon = 0 \).

By monotonicity, for each fixed \( n \) and all \( \epsilon \in [0, \delta] \),
\[ \mathbb{P} \left( n(r_\infty + \epsilon) \geq \sum_{i=1}^{n} D_i \right) \leq \mathbb{P} \left( n(r_\infty + \delta) \geq \sum_{i=1}^{n} D_i \right). \]

Furthermore, since \( r_\infty + \delta < \mathbb{E}[D] \), for any fixed \( \nu > 0 \), there exists \( M_\nu < \infty \) (depending only on \( \nu, D, r_\infty, \delta \)) such that
\[ \sum_{n=M_\nu+1}^{\infty} \mathbb{P} \left( n(r_\infty + \delta) \geq \sum_{i=1}^{n} D_i \right) \leq \nu. \]

Indeed, the above follows from a standard argument (the details of which we omit) in which each term is bounded using Chernoff’s inequality, and the terms are summed as an infinite series (cf. Deuschel and Stroock (1989)). Combining the above, we conclude that for all \( \nu > 0 \), and \( \epsilon \in (0, \delta] \),
\[ \sum_{n=1}^{M_\nu} \mathbb{P} \left( n(r_\infty + \epsilon) \geq \sum_{i=1}^{n} D_i \right) \geq \frac{c}{h} - \nu. \]

As \( \mathbb{P}(nx \geq \sum_{i=1}^{n} D_i) \) is a right-continuous function of \( x \) (by the right-continuity of cumulative distribution functions), it follows that \( \sum_{n=1}^{M_\nu} \mathbb{P}(nx \geq \sum_{i=1}^{n} D_i) \) is similarly right-continuous in \( x \). Right-continuity at \( \epsilon = 0 \) follows, and we conclude that
\[ \sum_{n=1}^{M_\nu} \mathbb{P} \left( nr_\infty \geq \sum_{i=1}^{n} D_i \right) \geq \frac{c}{h} - \nu. \]

As this holds for all \( \nu \), letting \( \nu \downarrow 0 \) completes the proof. \( \square \)

With Lemmas 3, 4, and 5 in hand, we now complete the proof of our main result, Theorem 2.

**Proof of Theorem 2**
Recall that the remaining term on the right-hand side of (11) which we are yet to bound is
\[ h \left( \mathbb{E}[I_L^\infty] - \mathbb{E}[I_L^L] \right) - c (r_\infty - r_L). \]

(13)
We first bound \( E[I_L^\infty] - E[I_L^L] \), which by Lemma 3 equals
\[
\sum_{n=1}^{L} \frac{1}{n^2} \left( \left( nr^\infty - \sum_{i=1}^{n} D_i \right) I( nr^\infty \geq \sum_{i=1}^{n} D_i ) \right) - \sum_{n=1}^{L} \frac{1}{n} \left[ \left( nr_L - \sum_{i=1}^{n} D_i \right) I( nr_L \geq \sum_{i=1}^{n} D_i ) \right].
\]
Combining with Lemma 4 (i.e. the fact that \( r_L \geq r_\infty \)), which implies that
\[
\left( nr_L - \sum_{i=1}^{n} D_i \right) I( nr_L \geq \sum_{i=1}^{n} D_i ) \geq \left( nr_L - \sum_{i=1}^{n} D_i \right) I( nr^\infty \geq \sum_{i=1}^{n} D_i ),
\]
we conclude that \( E[I_L^\infty] - E[I_L^L] \) is at most
\[
\sum_{n=1}^{L} \frac{1}{n^2} \left[ \left( nr^\infty - \sum_{i=1}^{n} D_i \right) I( nr^\infty \geq \sum_{i=1}^{n} D_i ) \right] - \sum_{n=1}^{L} \frac{1}{n} \left[ \left( nr_L - \sum_{i=1}^{n} D_i \right) I( nr^\infty \geq \sum_{i=1}^{n} D_i ) \right],
\]
which itself equals
\[
- (r_L - r^\infty) \sum_{n=1}^{L} \mathbb{P}( nr^\infty \geq \sum_{i=1}^{n} D_i ) .
\]
It follows that \((13)\) is at most
\[
(r_L - r^\infty) \left( c - h \sum_{n=1}^{L} \mathbb{P}( nr^\infty \geq \sum_{i=1}^{n} D_i ) \right).
\] (14)
Note that Lemma 5 implies that
\[
\sum_{n=L+1}^{\infty} \mathbb{P}( nr^\infty \geq \sum_{i=1}^{n} D_i ) \geq c - h \sum_{n=1}^{L} \mathbb{P}( nr^\infty \geq \sum_{i=1}^{n} D_i ),
\] (15)
Combining (14) and (15), we conclude that \((13)\) is at most
\[
(r_L - r^\infty) h \sum_{n=L+1}^{\infty} \mathbb{P}( nr^\infty \geq \sum_{i=1}^{n} D_i ) .
\]
It follows from the well-known Chernoff’s inequality (cf. Deuschel and Stroock (1989)) that
\[
\mathbb{P}( nr^\infty \geq \sum_{i=1}^{n} D_i ) \leq \gamma^n.
\]
By summing the associated geometric series, and combining with the fact that by definition \( r_L \leq E[D] \), we conclude that \((13)\) is at most
\[
h(E[D] - r^\infty)(1 - \gamma)^{-1} \gamma^{L+1}.
\]
Combining the above bound for \((13)\) with Lemma 4, plugging into \((11)\), and applying the fact that \( \text{OPT}(L) \geq g \), completes the proof. \(\square\)
4. Conclusion

In this paper, we proved that for a family of challenging inventory models (i.e. lost sales models with large lead times), the optimality gap of the simple constant-order policy converges exponentially fast to zero as the lead time grows with the other problem parameters held fixed, and derived effective explicit bounds for this optimality gap. This takes a large step towards answering several open questions of Goldberg et al. (2012), who recently proved the asymptotic optimality of the constant-order policy in this setting, but whose bounds on the rate of convergence were impractical, involving a massive prefactor and an inverse polynomial dependence on the relevant error term.

We also demonstrated that the corresponding rate of exponential decay is at least as fast as the exponential rate of convergence of the expected waiting time in a certain single-server queue to its steady-state value, which we proved to be monotone in the ratio of the lost-sales penalty to the holding cost. For the special case of exponentially distributed demand, we further computed all expressions appearing in our bound in closed form, and numerically evaluated these bounds, demonstrating good performance for a wide range of parameter values.

This work leaves many interesting directions for future research. First, it would be interesting to investigate the tightness of our exponential bound, e.g. to determine whether our exponential rate captures the true exponential rate of convergence of the optimality gap of the constant-order policy. Although one can come up with pathological examples for which this is not true, e.g. discrete demand distributions with probability at least $\frac{c}{c+h}$ at 0 (for which $Q = 0$, $\pi_0$ is optimal amongst all policies for all $L \geq 0$, yet $\gamma > 0$), we conjecture that under mild assumptions $\gamma$ indeed captures the true rate of convergence of the optimality gap. On a related note, it is an open question to bridge the gap between our own results, which hold only in the stationary regime, and the results of Goldberg et al. (2012), which also hold for the corresponding finite-horizon problem. Although (as noted by the authors) the arguments of Goldberg et al. (2012) could likely be rederived using convexity-type (as opposed to coupling) arguments, resolving the fundamental question of whether (and precisely in what sense) the optimality gap decays exponentially quickly for finite-horizon
problems seems to entail overcoming several challenging problems related to quantifying the “rate of convergence to stationarity” of optimal policies for lost sales inventory models, i.e. both the rate at which the finite-horizon problem “converges” to the infinite-horizon problem, and the rate at which the Markov chains associated with stationary optimal policies for the infinite-horizon problem converge to their steady-state behavior. Although recent progress has been made on several related questions (cf. Huh, Janakiraman and Nagarajan (2011)), a complete resolution seems beyond the reach of current techniques.

Second, it is an interesting open challenge to analyze the performance of more sophisticated policies for lost sales inventory models, e.g. affine policies, which should exhibit even better performance. For example, it is an open question whether the optimality gap of a more sophisticated (but still simple and efficient) policy can be made to decay (exponentially) faster as the lead time grows. Another interesting question involves the formal construction and analysis of “hybrid” algorithms, which e.g. solve large dynamic programs when $L$ is small and transition to using simpler policies when $L$ is large, or use base-stock policies when $\frac{c}{h}$ is large (relative to $L$) and a constant-order policy when $\frac{c}{h}$ is small. More generally, it is an open challenge to classify the family of all algorithms which are asymptotically optimal as the lead time grows, and to better understand the relative performance of these policies. For example, it is an open question whether the order-up-to policy of Huh et al. (2009), or the dual-balancing policy of Levi, Janakiraman and Nagarajan (2008), exhibit such asymptotic optimality.

Third, it would be interesting to prove that a similar phenomena occurs in more general inventory settings. As a first step, one could extend our results to models in which demand is not i.i.d., models with a fixed ordering cost, and models with integrality constraints. In these settings, although one would expect that the constant-order policy may no longer be asymptotically optimal, one can ask what analogous simple policy should have the asymptotic optimality property. We also suspect that our methodology may be able to prove that simple policies work well for considerably more complex inventory models. For example, it is an open question whether the so-called tailored base-surge policy, which combines the constant-order policy with a base-stock policy, is asymptotically
optimal for the more sophisticated dual-sourcing inventory model, a natural generalization of the lost sales model considered here (cf. Janakiraman, Seshadri and Sheopuri (2014)).

On a final note, our results and methodology (combined with that of Goldberg et al. (2012)) provide a fundamentally new approach to lost sales inventory models with positive lead times. We believe that our approach, combined with other recent developments in inventory theory (e.g. the efficient solution of related dynamic programs), represents a considerable step towards making these models solvable in practice. Such progress may ultimately help to free researchers from having to use backlogged demand inventory models as approximations to lost-sales inventory models, even when such an approximation is not appropriate, which has been recognized as a major problem in the inventory theory literature (cf. Bijvank and Vis (2011)).

5. Appendix

Proof of Corollary 7 Suppose $D$ is exponentially distributed with rate $\lambda$. It is well-known that in this case, for all $r \in [0, \lambda^{-1})$, $\mathbb{E}[I^r_\infty] = \frac{r^2\lambda}{2(1-r\lambda)}$ (cf. Haigh (2013)). It follows from (6) that for all $r \in [0, \lambda^{-1})$,

$$C(\pi_r) = h\mathbb{E}[I^r_\infty] + c\mathbb{E}[D] - cr = h\frac{r^2\lambda}{2(1-r\lambda)} + c\lambda^{-1} - cr.$$ 

Recall that $C(\pi_r)$ is a convex function of $r$ on $[0, \lambda^{-1})$, and note that

$$\frac{d}{dr}C(\pi_r) = \frac{h}{2}((\lambda r - 1)^{-2} - 1) - c. \quad (16)$$

As it is easily verified that the right-hand side of (16) strictly increases from $-c$ to $\infty$ on $[0, \lambda^{-1})$, it follows that $r_\infty$ must be the unique solution to the equation $\frac{d}{dr}C(\pi_r) = 0$ on $[0, \lambda^{-1})$. It then follows from a straightforward calculation (the details of which we omit) that

$$r_\infty = \lambda^{-1}(1 - \tau_{c,h}), \quad C(\pi_{r_\infty}) = \lambda^{-1}(\sqrt{h(2c + h)} - h).$$

As $\mathbb{E}[\exp(-\theta D)] = \lambda(\lambda + \theta)^{-1}$ for all $\theta \geq 0$, we conclude that

$$\phi(\theta) = \exp(\lambda^{-1}(1 - \tau_{c,h})\theta)\lambda(\lambda + \theta)^{-1}.$$
As above, it follows from another straightforward calculation (the details of which we omit) that \( \vartheta \) equals the unique solution to \( \frac{d}{d \theta} \phi(\theta) = 0 \) on \([0, \infty)\), and thus

\[
\vartheta = \tau_{c,h} \lambda (1 - \tau_{c,h})^{-1}, \quad \gamma = \gamma_{c,h} = (1 - \tau_{c,h}) \exp(\tau_{c,h}).
\]

Finally, let us compute \( Q \) and \( g \). Noting that \( Q \) is the \( c^{-1} + h^{-1} \) quantile of the demand distribution, i.e., \( 1 - \exp(-\lambda Q) = \frac{c}{c+h} \), implies \( Q = \lambda^{-1} \log(1 + ch^{-1}) \). It follows that

\[
g = \int_{0}^{Q} (Q-x) \lambda \exp(-\lambda x) \, dx + c \int_{Q}^{\infty} (x-Q) \lambda \exp(-\lambda x) \, dx
\]

\[
= (h+c) \int_{0}^{Q} (Q-x) \lambda \exp(-\lambda x) \, dx + c (E[D] - Q)
\]

\[
= (h+c) \left( Q - \lambda^{-1} + \lambda^{-1} \exp(-\lambda Q) \right) + c \left( \lambda^{-1} - Q \right)
\]

\[
= h \lambda^{-1} \log \left( 1 + ch^{-1} \right).
\]

Combining the above with a straightforward calculation (the details of which we omit) completes the proof. \( \square \)

**Proof of Lemma 7** Suppose \( \frac{c_1}{h_1} < \frac{c_2}{h_2} \), and let \( r^i_\infty \in \arg\min_{0 \leq r \leq E[D]} \{ h_i E[I^r_\infty] + c_i E[D] - c_i r \}, i = 1,2 \). From the respective optimality of \( r^1_\infty, r^2_\infty \), we conclude that

\[
E[I^r^1_\infty] + \frac{c_1}{h_1} E[D] - \frac{c_1}{h_1} r^1_\infty \leq E[I^r^2_\infty] + \frac{c_2}{h_2} E[D] - \frac{c_2}{h_2} r^2_\infty,
\]

\[
E[I^r^2_\infty] + \frac{c_2}{h_2} E[D] - \frac{c_2}{h_2} r^2_\infty \leq E[I^r^1_\infty] + \frac{c_1}{h_1} E[D] - \frac{c_1}{h_1} r^1_\infty.
\]

Summing these two inequalities together implies

\[
\left( \frac{c_2}{h_2} - \frac{c_1}{h_1} \right) (r^2_\infty - r^1_\infty) \geq 0.
\]

It follows that \( r^2_\infty \geq r^1_\infty \), and for all \( \theta \geq 0 \),

\[
\exp(\theta r^2_\infty) E[\exp(-\theta D)] \geq \exp(\theta r^1_\infty) E[\exp(-\theta D)].
\]

Combining with the definition of \( \gamma \) completes the proof. \( \square \)

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