The best sextic approximation of hyperbola with order twelve

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ABSTRACT

In this article, the best uniform approximation for the hyperbola of degree 6 that has approximation order 12 is found. The associated error function vanishes 12 times and equioscillates 13 times. For an arc of the hyperbola, the error is bounded by $2.4 \times 10^{-4}$. We explain the details of the derivation and show how to apply the method. The method is simple and this encourages and motivates people working in CG and CAD to apply it in their works.

1. INTRODUCTION

Bézier curves were invented by Pierre Bézier and Paul de Casteljau in the 1960’s. The primary application was in the automobile industry, but today they are widely used in many computer applications and codes to sketch curves. Approximation using polynomials of low degree is favourable, and in many cases is a must issue. Many CAD systems are limited to using only parametric polynomial curves of low degree with low errors. The big error causes two major disadvantages: high accumulated error, and slow and costly software. So, it is favourable to approximate using low degree polynomials. Parabolas are represented exactly using parametric curves of degree 2. In [1] methods of best uniform approximation of a circular arc of degree 6 with order 12 are accomplished. These approximations are optimal and have 13 equioscillations. In [2], a method of best uniform approximation of a hyperbola of degrees 2 with order 4 is accomplished. These results were motivated by the results of approximating curves with high accuracy [3-6].

In this paper, we find approximation for the hyperbola of degree 6 and we get the order 12. Bézier curve techniques are used to represent the approximations of the hyperbola using parametric polynomials of degree 6 that have the least uniform error. The method is simple and practical. To achieve this approximation, proper arrangements and symmetries of the hyperbola are applied to determine the Bézier points that define the approximation of the hyperbola, and therefore, makes a CAD system more efficient and minimizes the cost. Using suitable translation and scaling, the hyperbola can be written in the basic forms: $y^2 - x^2 = 1$ and $x^2 - y^2 = 1$. Every form has two branches. So, there are four branches. Geometrically, all of these branches

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are identical. Therefore, it is sufficient to represent one branch and the other three branches can be represented using rotation of this branch. So, we consider the upper branch of the hyperbola \( y^2 - x^2 = 1 \), see Figure 1. It can be written in parametric form \( c : t \mapsto (\sinh(t), \cosh(t)) \), \( t \in \mathbb{R} \). We are interested in finding out the longest arc of the hyperbola that can be approximated and that the error function is the Chebyshev polynomial, see [2]. It is impossible to exactly represent a hyperbola with a polynomial curve [7-11]. It can be represented exactly using rational Bézier curves, a polynomial parametric form is preferred in many applications. The ability to represent a primitive hyperbola is a must issue especially in computer graphics and data and image processing. Thus, there is a demand to find a parametrically defined polynomial curve \( p_6 : t \mapsto (x_6(t), y_6(t)) \), \( 0 \leq t \leq 1 \), where \( x_6(t), y_6(t) \) are polynomials of degree 6. The \( p_6 \) has to approximate \( c \) within tolerable error. In this paper, degree 6 parametric curves are considered, and it is shown that the error is very small and the results are competitive.

![Figure 1. The Hyperbola \( y^2 - x^2 = 1 \)](image)

The Euclidean error function is used to measure the error between \( p_6 \) and \( c \). The error is defined by:

\[
E(t) := \sqrt{y_6^2(t) - x_6^2(t) - 1}.
\]  

(1)

\( E(t) \) is replaced by the following error function:

\[
e(t) := y_6^2(t) - x_6^2(t) - 1.
\]

(2)

The approximation problem is formulated as follows. **The approximation problem** is to find \( p_6 : t \mapsto (x_6(t), y_6(t)) \), \( 0 \leq t \leq 1 \), where \( x_6(t), y_6(t) \) are polynomials of degree 6, that approximates \( c \) by satisfying the following three conditions:

(a) \( p_6 \) minimizes \( \max_{t \in [0,1]} |e(t)| \),

(b) \( p_6 \) approximates \( c \) with order twelve [12],

(c) \( e(t) \) equioscillates thirteen times over \([0, 1]\).

The solution to this problem is shown in section 3. This solution is presented in Figure 2 and Figure 3; the corresponding error is shown in Figure 4. More related results can be found in [13-15] and the references therein.

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2. **RESEARCH METHOD**

Let \( p_6(t) = (x_6(t), y_6(t)) \) be a sextic polynomial parametric representation of the curve \( c \). In CAGD, curves are presented using the Bézier form, see Figure 5 for possible choice of the Bézier points.

The Bézier curve \( p_6(t) \) has the following form:

\[
p_6(t) = \sum_{i=0}^{6} p_i B^6_i(t) =: \begin{pmatrix} x_6(t) \\ y_6(t) \end{pmatrix}, \quad 0 \leq t \leq 1,
\]

where \( p_0, p_1, p_2, p_3, p_4, p_5, p_6 \) are the Bézier points, and the Bernstein polynomial basis of degree 6 is defined by:

\[
B^6_i(t) = \binom{6}{i} (1 - t)^{6-i} t^i, \quad i = 0, \ldots, 6, \quad t \in [0, 1].
\]
The Bézier polynomial curve $y$ where

Proof: From equation (5), we get

Thus, the point $p_k$ is kicked. Thus, the point $p_k$ should have the form $p_k = (\alpha, \beta)$. The point $p_1 = (-\gamma, \delta)$, so, to obey the symmetry, the point $p_5$ should have the form $p_5 = (\gamma, \delta)$. The point $p_2 = (-\xi, \psi)$, so, to obey the symmetry, the point $p_4$ should have the form $p_4 = (\xi, \psi)$. There is one remaining point; if this point lies in either halve of the plane around the $y$-axis, then the symmetry of the hyperbola is kicked. Thus, the point $p_3$ must lie on the $y$-axis and has the form $p_3 = (0, \omega)$. Therefore, the proper choice for the Bézier points is

$$p_0 = \left( -\frac{\alpha}{\beta} \right), \; p_1 = \left( -\frac{\gamma}{\delta} \right), \; p_2 = \left( -\frac{\xi}{\psi} \right), \; p_3 = \left( 0, \frac{\omega}{\omega} \right), \; p_4 = \left( \frac{\xi}{\psi} \right), \; p_5 = \left( \frac{\gamma}{\delta} \right), \; p_6 = \left( \frac{\alpha}{\beta} \right).$$

(4)

The Bézier polynomial curve $p_0(t)$ in (3) is given in the form

$$\begin{align*}
x_0(t) &= \left( -\alpha B^6_0(t) - \gamma B^6_1(t) - \xi B^6_2(t) + \gamma B^6_3(t) + \alpha B^6_4(t) \right), \\
y_0(t) &= \left( \beta B^6_0(t) + \delta B^6_1(t) + \psi B^6_2(t) + \omega B^6_3(t) + \beta B^6_4(t) \right),
\end{align*}$$

(5)

where $0 \leq t \leq 1$. The seven parameters $\alpha, \beta, \gamma, \delta, \xi, \psi, \omega$ are used to have the polynomial approximation $p_0$ comply with the conditions of the approximation problem; this is done in the following section.

3. RESULTS AND ANALYSIS

The values of $\alpha, \beta, \gamma, \delta, \xi, \psi, \omega$ that minimize the uniform error and satisfy the conditions of the approximation problem are given in the following theorem. A symbolic programming language is used to get the values of the parameters in (4) and are rounded in decimal form.

Theorem 1: The Bézier curve in (5) with the Bézier points in (4), where

$$\alpha = 21.396696163346007, \; \beta = 21.420062908119476, \; \gamma = -0.3481294343024657,$$
$$\delta = -0.3015887594987887, \; \xi = 0.5937616113806532, \; \psi = 0.705589524349383,$$
$$\omega = 0.1813438330271954$$

(6)

satisfies the three conditions of the Approximation Problem. More precisely, the error functions satisfy:

$$-\frac{1}{2^{11}} \leq e(t) \leq \frac{1}{2^{11}}, \quad -\frac{1}{2^{11}(2-\epsilon)} \leq E(t) \leq \frac{1}{2^{11}(2+\epsilon)}, \quad \text{where} \; \epsilon = \max_{0 \leq t \leq 1} |E(t)|.$$  

(7)

Proof: From equation (5), we get

$$\begin{align*}
x_0(t) &= \alpha (B^6_0(t) - B^6_1(t)) + \gamma (B^6_1(t) - B^6_2(t)) + \xi (B^6_2(t) - B^6_3(t)) + \beta (B^6_0(t) + B^6_1(t)) + \delta (B^6_1(t) + B^6_2(t)) + \psi (B^6_2(t) + B^6_3(t)) + \omega B^6_3(t), \\
y_0(t) &= \beta (B^6_0(t) + B^6_1(t)) + \delta (B^6_1(t) + B^6_2(t)) + \psi (B^6_2(t) + B^6_3(t)) + \omega B^6_3(t).
\end{align*}$$

Substituting $x_0(t)$ and $y_0(t)$ into the error function $e(t)$ in (2) and rewriting the result in terms of powers of $t$ we get the following equality:

$$e(t) = t^{12} (4\beta^2 - 48\beta\delta + 144\delta^2 + 120\beta\xi - 720\delta\xi + 900\xi^2 - 80\beta\omega + 480\delta\omega - 1200\xi\omega + 400\omega^2) \quad + t^{11} (-243\beta^2 + 288\beta\delta - 864\delta^2 - 720\beta\xi + 4320\delta\xi - 5400\xi^2 + 480\beta\omega - 2880\delta\omega + 7200\xi\omega - 2400\omega^2) \quad + t^{10} (-36\alpha^2 + 288\alpha\gamma - 576\gamma^2 - 360\alpha\xi + 1440\gamma^2 - 900\xi^2 + 96\beta^2 - 1032\beta\delta + 2736\delta^2 + 2400\beta\psi - 12600\psi\omega + 7800\psi^2 - 1560\beta\omega + 8160\delta\omega - 18600\psi\omega + 600\omega^2) \quad + t^9 (1800\alpha^2 - 1440\alpha\gamma + 2880\gamma^2 + 1800\alpha\xi - 7200\xi\omega + 4500\xi^2 + 260\beta^2 + 2520\beta\delta - 576\delta^2 \quad - 2520\beta\psi - 12600\psi\omega + 7800\psi^2 - 1560\beta\omega + 8160\delta\omega - 18600\psi\omega + 600\omega^2).$$

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The approximation intersects the hyperbola and the approximating Bézier curve, Figure 4 shows the corresponding error, and Figure 6 shows the Euclidean error. We substitute the values of α, β, γ, δ, ξ, ψ and ω from (6) into the last equation to get

\[ e(t) = 4096 t^{12} - 24576 t^{11} + 64512 t^{10} - 97280 t^9 + 93024 t^8 - 58752 t^7 + 24752 t^6 
   - 6864 t^5 + \frac{19305}{16} t^4 - \frac{1001}{8} t^3 + \frac{429}{64} t^2 - \frac{9}{64} t^1 + \frac{1}{2048}, \quad t \in [0, 1]. \]

Making the substitution \( t = \frac{u + \frac{1}{2}}{1 + \frac{1}{2}} \) reduces the error function to the following form

\[ e(u) = \frac{1}{2048} - 9 \frac{u^2 + 105}{256} u^4 - 7 \frac{u^6 + 27}{8} u^8 + 3 t^{10} + t^{12}, \quad u \in [-1, 1]. \]

The last form of \( e(u) \) coincides with the monic Chebyshev polynomial \( \tilde{T}_{12}(u) \), \( u \in [-1, 1] \), which is the unique polynomial of degree 12 that equioscillates 13 times between \( \pm \frac{1}{2} \) for all \( u \in [-1, 1] \) and has the least deviation [12]. This shows that \( p_6 \) satisfies the conditions of the approximation problem. The error formula for \( E(t) \) can be proved using its relation to the error function \( e(t) \). This proves Theorem 1.

It is clear that the approximation is the best uniform approximation from Figure 4. Figure 2 and Figure 3 show the hyperbola and the approximating Bézier curve, Figure 4 shows the corresponding error, and Figure 6 shows the Euclidean error.

4. CONCLUSION

In this paper, the best uniform approximation of the hyperbola with parametrically defined polynomial curves of degree 6 are explicitly given. The error function equioscillates 13 times; the approximation order is 12. The method of construction demonstrates the efficiency and simplicity of the approximation method. The approximation intersects the hyperbola 12 times with maximum error \( 2.4 \times 10^{-4} \). Reflecting the upper branch of the hyperbola around the \( x \)-axis gives the lower branch. The hyperbola is shown in Figure 7. Note that the points \( p_1(p_6) \) and \( p_2(p_6) \) in both branches are very close to each other and can not be distinguished from each other. The results in this paper can be used to improve the results obtained in [16-23] see also the results in [24, 25].
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REFERENCES

[1] A. Rababah, "Best sextic approximation of circular arcs with thirteen equioscillations," Proceedings of the Jangjeon Mathematical Society, vol. 21, Issue 1, pp. 111–123, 2018. DOI: http://doi.org/10.17777/pjms2018.21.1.111

[2] A. Rababah, "The Best Quadratic Approximation of Hyperbola with Order Four," Journal of Inequalities and Special Functions, vol. 7, Issue 3, pp. 1–11, 2016.

[3] M. Floater, "High order approximation of conic sections by quadratic splines," Computer Aided Geometric Design, vol. 12, pp. 617–637, 1995.

[4] C. de Boor, et al., "High accuracy geometric Hermite interpolation," Computer Aided Geometric Design, vol. 4, pp. 269–278, 1988.

[5] A. Rababah, "Taylor theorem for planar curves," Proceedings of the American Mathematical Society, vol. 119, Issue 3, pp. 803–810, 1993.

[6] A. Rababah, "High accuracy Hermite approximation for space curves in $\mathbb{R}^d$," Journal of Mathematical Analysis and Applications, vol. 325, Issue 2, pp. 920–931, 2007.

[7] K. Höllig and J. Hörner, "Approximation and Modeling with B-Splines," SIAM, Titles in Applied Mathematics, vol. 132, 2013.

[8] H. Prautzsch, et al., "Bézier and B-Spline Techniques," Springer 2002.
[9] P. Bézier, "The mathematical basis of the UNISURF CAD system," Butterworth-Heinemann Newton, MA, USA, ISBN 0-408-22175-5, 1986.

[10] Carl de Boor, A Practical Guide to Splines, Series Title: Applied Mathematical Sciences, Vol 27, Springer-Verlag New York, 1978.

[11] Gerald Farin, Curves and surfaces for computer-aided geometric design: a practical guide. Elsevier, 2014.

[12] J. Rice, "The approximation of functions," linear theory, Addison-Wesley, vol. 1, 1964.

[13] Y. J. Ahn and C. Hoffmann, "Circle approximation using LN Bézier curves of even degree and its application," Journal of Mathematical Analysis and Applications, vol. 410 Issue 1, pp. 257–266, February 2014.

[14] S. W. Kim and Y. J. Ahn, "Circle approximation by quartic $G^2$ spline using alternation of error function," J. KSIAM, vol. 17, Issue 5, pp. 171–179, 2013.

[15] A. Rababah, et al., "Multiple Degree Reduction and Elevation of Bézier Curves Using Jacobi-Bernstein Basis Transformations," Numerical Functional Analysis and Optimization, vol. 28, Issue 9-10, pp. 1179–1196, 2007.

[16] S. Alabed, "Performance Analysis of Differential Beamforming in Decentralized Networks," International Journal of Electrical and Computer Engineering (IJCE), vol. 8 Issue 3, pp. 1692-1700, 2018. DOI: http://doi.org/10.11591/ijece.v8i3.pp1692-1700

[17] C. Suwannapong and C. Khunboa, "An Approximation Delay between Consecutive Requests for Congestion Control in Unicast CoAP-based Group Communication," International Journal of Electrical and Computer Engineering (IJCE), vol. 9, Issue 3, June 2019. DOI: http://doi.org/10.11591/ijece.v9i3.

[18] Bhargav Bhatkalkar, Abhishek Joshi, Srikanth Prabhu, Sulatha Bhandary, Automated fundus image quality assessment and segmentation of optic disc using convolutional neural networks. International Journal of Electrical and Computer Engineering (IJCE) Vol.10, No.1, 2020, pp. 816 827, ISSN: 2088-8708, DOI: 10.11591/ijece.v10i1.pp816-827.

[19] Rodiah, Sarifuddin Madenda, Diana Tri Susetianigtias, Dewi Agushinta Rahayu, EtySutanty, Optic Disc and Macula Localization from Retinal Optical Coherence Tomography and Fundus Image, International Journal of Electrical and Computer Engineering (IJCE) Vol.8, No.6, 2018, pp. 5050-5060, ISSN: 2088-8708, DOI: 10.11591/ijece.v8i6.pp5050-5060.

[20] G. Daway, Hana H. kareem, Ahmed Rafid Hashim, Pupil Detection Based on Color Difference and Circular Hough TransformHazim, International Journal of Electrical and Computer Engineering (IJCE) Vol.8, No.5, 2018, pp. 3278-3284, ISSN: 2088-8708, DOI: 10.11591/ijece.v8i5.pp3278-3284.

[21] A. Rababah, L-2 degree reduction of triangular Bézier surfaces with common tangent planes at vertices. International Journal of Computational Geometry & Applications, Vol. 15, No. 5 (2005) 477-490. http://www.worldscinet.com/ijcga/15/1505/S0218195905015050.html

[22] A. Rababah, Distances with rational triangular Bézier surfaces. Applied Mathematics and Computation 160, (2005) 379-386. http://www.elsevier.com/locate/amc

[23] A. Rababah and Moath Jaradat, Approximating offset Curves using Bézier curves with high accuracy, International Journal of Electrical and Computer Engineering (IJCE) Vol.10, No.2, 2020 (PART I).

[24] A. Rababah, "Approximation von Kurven mit Polynomen und Splines," Ph. Dissertation, Stuttgart Universität, West Germany, 1992.

[25] A. Rababah, The best quintic Chebyshev approximation of circular arcs of order ten. International Journal of Electrical and Computer Engineering (IJCE)Vol. 9, No. 5, October 2019, pp. 3779-3785, ISSN: 2088-8708, DOI: 10.11591/ijece.v9i5.pp3779-3785.
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Abedallah Rababah is a professor of mathematics at United Arab Emirates University and is on leave from Jordan University of Science and Technology. He is working in the field of Computer Aided Geometric Design, abbreviated CAGD. In particular, his research is on degree raising and reduction of Bézier curves and surfaces with geometric boundary conditions, Bernstein polynomials, and their duality. He is known for his research in describing approximation methods that significantly improve the standard rates obtained by classical (local Taylor, Hermite) methods. He proved the following conjecture for a particular set of curves of nonzero measure: Conjecture: A smooth regular planar curve can, in general, be approximated by a polynomial curve of degree n with order 2n. The method exploited the freedom in the choice of the parametrization and achieved the order 4n/3, rather than n + 1. Generalizations were also proved for space curves. Professor Rababah is also doing research in the fields of classical approximation theory, orthogonal polynomials, Jacobi-weighted orthogonal polynomials on triangular domains, and best uniform approximations. Since 1992, he has been teaching at German, Jordanian, American, Canadian, and Emirates’ universities. He is active in the editorial boards of many journals in mathematics and computer science. Further info can be found on his homepage at ResearchGate: https://www.researchgate.net/profile/Abedallah_Rababah or at http://www.just.edu.jo/eportfolio/Pages/Default.aspx?email=rababah

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