A Weyl geometric scalar field approach to the dark sector

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Abstract

This paper explores the dark sector (dark matter and dark energy) from the perspective of Weyl geometric scalar tensor theory (integrable Weyl geometry). In order to account for the galactic dynamics successfully modelled by MOND ("modified Newtonian dynamics"), the non-minimally coupled scalar field considered here has a Lagrangian with two non-conventional contributions in addition to a standard kinetic term: one is inspired by Bekenstein/Milgrom’s RAQUAL ("relativistic a-quadratic Lagrangian") from 1983, the other one by a second order term introduced in cosmological studies by Novello et al. in 1993. See, however, the error warning below. We consider the transition to the Einstein gravity on one hand and to scalar field cosmology in the FRW framework on the other. A bouncing cosmological model is tentatively discussed at the end.

Error warning

Equ. (86) is wrong; it does not take contributions to $\frac{\delta L^{\text{braz}}}{\delta \phi^\nu}$ into account, which are due to the covariant derivative of the (scale covariant) gradient of the scalar field $\phi$. Therefore the contribution of $L^{\text{braz}}$ to the energy tensor of the scalar field is wrong. This flaw is fatal for the derivations in the Milgrom regime; the whole argumentation of part 2 can no longer be upheld. It is here reproduced for documentary reasons only.

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Introduction

Shortly after the rise of broader interest in dark matter at the turn to the 1980s Bekenstein and Milgrom explored a Lagrangian approach for explaining the flat rotation curves of galaxies in modified gravity (MOND) [9, 8, 45]. They assumed a scalar field $\phi$ with an unconventional “a-quadratic” kinetic Lagrangian term (AQUAL) and indicated how, at least in principle, a relativistic version called RAQUAL can be formulated along these lines. The authors observed that this approach did not provide for gravitational refraction of light sufficiently exceeding that of baryonic matter. In consequence a series of other, more or less ad-hoc, modifications of Einstein gravity were proposed for addressing this problem. A generic feature of them is the idea of a cubic kinetic term (enveloped in a more general transition function to Einstein gravity) which reproduces the achievements of MOND at the galactic level. Usually other fields are added, which take care of the necessary gravitational light refraction. Among them there are tensor-vector scalar field theory (TeVeS), generalized Einstein-Aether theory (gEA), superfluid theory, conformal emergent gravity [9, 8, 53, 11, 34, 35, 62], to name the
best known ones; most recently a “new relativistic theory for modified Newtonian dynamics” (nrMOND) \[63\]. For expositions and comparisons of some of the approaches with this background see \[61, 26, 45\]. Modified gravity theories in the cosmological range have a different motivation; for reviews see \[19, 27, 16\].

In many of these approaches new hypothetical degrees of freedom of the gravitational field are introduced more or less artificially in order to emulate the effects which in the \(\Lambda\)CDM approach are considered as the effects of dark matter. The present proposal is more parsimonious. It shows that one scalar field and a well founded generalization of the metric from Riemannian to (integrable) Weyl geometry (IWG) suffice for explaining dark matter effects at the astrophysical level of galaxies (and probably also of galaxy clusters). The corresponding gravitational light bending is induced by a scalar field the kinetic term of which is complemented, besides a cubic (RAQUAL inspired) one, by a second order derivative term first proposed in \[46\] in a cosmological context. This term endows the scalar field with an energy-stress tensor which represents a kind of dark matter/energy \textit{sui generis}, while the scalar field modifies gravity similar to Jordan-Brans-Dicke theory (JBD) \[28\]. Here the latter is related to the (integrable) scale connection of Weyl geometry; the present approach will therefore be called a \textit{Weyl geometric “dark” scalar theory} (WdST).

The paper starts with a short discussion of how the conformal transformations in JBD theory can be expressed in terms of Weyl geometry. The scale covariant scalar field of JBD is then easily taken over to IWG. In this framework the different frames of JBD appear as the different scale gauges of one and the same integrable Weylian metric. This results in a slight but important shift in the perspective on free fall trajectories of test bodies and on the question which affine connection ought to be considered as “physical” (sec. 1).

If one wants to explain MOND-type galactic dynamics in the framework of a relativistic scalar tensor theory, one needs a more general Lagrangian than in JBD. Moreover, although Einstein gravity will continue to hold in large regions of the universe, at the galactic and cluster level, i.e., in very weak field regions, it has to be changed, just like Newton theory is being changed by MOND. There exist many, often quite different reasons for looking at alternatives to Einstein gravity at the cosmological level. It is no longer clear that one universal Lagrangian of a new “fundamental” theory will cover all these gravitational regimes. More cautiously the next section proposes to distinguish between different gravitational regimes \[2.1\]. The main part of the section develops a Weyl geometric scalar field approach to galactic and cluster dynamics, a region here called the \textit{Milgrom regime} of WdST.

After introducing the Lagrangian of the Milgrom regime at the beginning of sec. \[2.2\] the dynamical equations for the Riemannian component of
the Weylian metric (generalized Einstein equation) and the scalar field equation (Milgrom equation) are derived, with some technicalities shifted to the appendix. A short look at the energy-stress tensor of the scalar field and the contribution of the scalar field to the (coordinate) acceleration in the Einstein gauge follow. While an investigation of the static weak field approximation leads to a Newton-type approximation for the \((0,0)\)-component of the metric sourced by baryonic matter only, the spatial components are strongly influenced by the scalar field pressures. This has the effect that the gravitational light refraction stands in agreement with the modification of the particle acceleration by the scalar field (sec. 2.3). Another upshot is the observation that a special type of MOND dynamics arises in the flat space limit. The effects for gravitational light bending can be studied by relativistic corrections to the latter (sec. 2.4).

We then come back to the question of how to represent other gravitational regimes mentioned above in WdST (sec. 3). Although it is clear how to incorporate Einstein gravity as a special case of WdST mathematically (by trivializing the scalar field dynamics), the physical reasons for the transition remain open. Also open is the question of how to represent the cosmological regime in WdST, even under the idealising symmetry assumption of FRW geometry. A first exploration of the question, including a new bouncing cosmological model is given in sec. 3.3. It is followed by a résumé and a discussion of what has been achieved in the paper (sec. 4). Some technical derivations are dealt with in the appendix (sec. 5).

1 JBD and Weyl geometric scalar tensor theory

1.1 A reminder on JBD gravity

Jordan-Brans-Dicke theory (JBD) assumes a scalar field \( \Phi \) contributing to the gravitational Lagrangian (Hilbert term) in the well known way:

\[
L_J = \frac{1}{2} \Phi R - \frac{\omega}{\Phi} \partial_\nu \Phi \partial^\nu \Phi + L_m, \quad L_J = L_J \sqrt{|\det g|} \tag{1}
\]

Here \( R \) denotes the Riemannian scalar curvature of a metric \( g_J = (g_{\mu\nu}) \) with signature \((-+++)\) in the Jordan frame.

Dicke postulated that the basic laws of physics ought to be independent of the choice of measuring units [22, p. 2163]. The acceptance of this postulate would demand a theory which is invariant under (nonsingular) conformal transformations, i.e. scale transformations, among different frames. That was similar to Weyl’s proposal (1918) of a scale gauge geometry [70], if one includes conformal rescaling of fields.\footnote{Weyl scaling is based on conformal rescaling of the metric \( g(x) \to \tilde{g}(x) = \Omega(x)^2 g(x) \) i.e., length and time intervals are rescaled by the point-dependent factor \( \Omega(x) \). The speed of light \( c \) is scale invariant and the Planck constant \( \hbar \) is postulated as such, i.e.,...}
Weyl generalized the Riemannian metric to what was later called a Weylian metric. The latter may be expressed in different scale covariantly connected forms, so-called (scale-)gauges. They are characterized by pairs \((g, \varphi)\) consisting of a Riemannian metric \(g\) (of any signature) and a differential 1-form \(\varphi\), the Weylian scale connection (in the physical literature often called the “Weyl field”). Different gauges of the Weylian metric, \((g, \varphi)\) and \((\tilde{g}, \tilde{\varphi})\) are connected by the rescaling of the Riemannian components \(\tilde{g} = \Omega^2 g\) and an accompanying gauge transformation of the connection \(\tilde{\varphi} = \varphi - d \log \Omega\) \[13, 47, 21\].

JBD theory may be rephrased in terms of integrable Weyl geometry (IWG) \[52, 2, 57\]. Integrable means here that the scale connection \(\varphi\) of any gauge \((g, \varphi)\) of the Weylian metric is pure gauge,

\[
\varphi = -d \log \Omega, \tag{2}
\]

where \(\Omega\) is the rescaling function s.th. \(g = \Omega^2 g_J\). Of course, different scale gauges in IWG correspond to different frames in JBD. Here we count the scaling weight of the scalar field as \(w(\Phi) = -2\) according to \(w(g_{\mu\nu}) = 2\), which means that length quantities scale with weight 1.\(^3\) As the Jordan frame expresses the Weylian metric in Riemannian terms we speak of it as the Riemann gauge of the underlying IWG.

Empirical interpretation of observable quantities presupposes a particular choice of measuring units, which results in an appropriate gauge fixing. Observable quantities are thus expressed in a specified frame/gauge which we will call the measuring gauge. The measuring gauge of a scale covariant field \(X\) will be denoted by \(X_m\). Of course one may want to know the reason for such a choice, at best given by a “breaking mechanism” of the scale symmetry. At the moment we leave this question open, but come back to it in section 4 (\(\rightarrow\) Higgs gauge). Because of the preferred relation to Einstein gravity the scale gauge in which the scalar field is a constant, \(\phi_0\), it is called a scalar field (\(\phi\)-)gauge. If the constant is normed to the gravitational constant

\[
\phi_0^2 = (8 \pi G_N[c^{-4}])^{-1} = (8 \pi \kappa)^{-1}, \tag{3}
\]

\((G_N\) the Newton constant and \(\kappa = G_N c^{-4}\)) it is called the Einstein gauge, respectively frame\(^4\) a field \(X\) in this gauge will be denoted by \(X_E\). More gauges will be introduced in the following.

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\(\text{energy and mass scale by the factor } \Omega^{-1}\). Respecting the new SI conventions, it is clear how dimensional quantities and thus also fields have to be rescaled. This is an extension of Weyl’s proposal and will be called Weyl scaling. It is logically equivalent to the rescaling in high energy physics, although with inverted scale weights.

\(\text{For a short introduction see, e.g., \[58, \text{sec. 11.2}\] and the literature given there in footnote 7.}\)

\(\text{By obvious reasons elementary particle physics prefers inverted signs of the weight convention.}\)

\(\text{Conventions for (physical) dimensions here: } [x_{\mu}] = 1 \text{ (dimensionless coordinates), } [g_{\mu\nu}] = L^2, \text{ then } [\mathcal{E}] = L E^{-1} \text{ (L length, E energy etc.)}\)
The Levi-Civita connections of the metrics in different frames define inequivalent affine structures. So an important question in JBD is: Which affine connection governs the inertial structure of JBD gravity? (Here the range of “affine connections” is implicitly delimited by the Levi-Civita connections of all frames conformal to $g$.) The debate often reduces the question to the decision between the Jordan and Einstein frames [27, sec. 2.1], [28, sec. 3]. From the point of view of IWG one would pose the question differently, as we discuss in the next subsection (compare [52, 2]).

1.2 The point of view of integrable Weyl geometry (IWG)

Different from JBD theory, in Weyl geometry exists a uniquely determined affine connection compatible with the Weylian metric. Given any gauge $(g, \varphi)$ it can be written as

$$\Gamma(g, \varphi) = \Gamma(g) + \Gamma(\varphi),$$

with $\Gamma(g)$ the Levi-Civita connection of $g$ and

$$\Gamma(\varphi)_{\lambda\nu} = \delta^\mu_{\varphi^\lambda} \varphi^\mu_{\nu} + \delta^\mu_\lambda \varphi^\nu_{\mu} - g^{\nu\lambda} \varphi^\mu_{\mu},$$

the contribution of the scale connection. $\Gamma(g, \varphi)$ induces a well determined (unique) scale covariant differential $D_\mu$ of fields, and leads to a generalized Riemannian curvature of Weyl geometry $Riem(g, \varphi)$ which is no longer anti-symmetric in the first two entries. The resulting Ricci curvature $Ric(g, \varphi)$ is scale invariant, while the scalar curvature $R(g, \varphi)$ is scale covariant of weight $w(R) = -2$,

$$R(g, \varphi) = R(g) - (n - 1)(n - 2)\varphi^\nu_\nu \varphi^\nu - 2(n - 1)\nabla(g)\varphi^\nu,$$

where $R(g)$ denotes the Riemannian scalar curvature of $g$, $\nabla(g)$ the covariant derivative of $\Gamma(g)$, and $n$ is the dimension of the Weylian manifold, see [30, 73], also [1, sec. 15.2].

Of course the path structure of affine geodesics $\gamma(\lambda)$, the autoparallels with regard to $\Gamma$, is independent of the chosen gauge; only the parametrization may change. We chose it such that in any gauge for non-null geodesics we get $g(\dot{\gamma}, \ddot{\gamma}) = \pm 1$; i.e. we use scale dependent parametrizations of geodesics with weight $w(\dot{\gamma}) = -1$.

It is known that a general, non-integrable scale connection (a “Weyl field” $\varphi$ with $d\varphi \neq 0$, i.e., non-vanishing field strength), leads to a mass term close to the order of Planck energy [64, 23, 18]. Even if it were physical, it could be integrated out at the energy scales of classical field theory, which we consider here. This leaves only the scalar degree of freedom for the integrable case. The arising Weyl geometric scalar tensor theory (WST) can also be arrived at by a slight modification of JBD gravity; then the scale
connection $\varphi$ is integrable from the outset. It has no own dynamical degree of freedom of its own but “shares” it, so to speak with the scalar field.

Its Lagrangian has no dynamical term of the Weyl field and is of a form close to JBD:

$$L = L^{(H)} + L^{(\text{kin})} + L^{(V)} + L^{(m)}, \quad \mathcal{L} = L\sqrt{|\det g|},$$  \hspace{1cm} (7)

where all building blocks $\mathcal{L}^{(X)}$ are scale invariant. The gravitational part is a Hilbert term,

$$\mathcal{L}^{(H)} = \frac{1}{2} \phi^2 R(g, \varphi),$$  \hspace{1cm} (8)

with $R(g, \varphi)$ the scalar curvature of Weyl geometry (weight -2), $\phi$ a gravitational scalar field (weight -1) similar to JBD theory ($\Phi = \phi^2$). $L^{(\text{kin})}$ and $L^{(V)} = -V(\phi)$, respectively their densities, are the kinetic and potential terms of the scalar field; $L^{(m)}$ is the matter term brought into a scale covariant form (weight -4) (cf. sec. 3.3). The simplest choice for the potential is the quartic monomial

$$V(\phi) = \frac{\lambda_4}{4} \phi^4 = V_4(\phi);$$  \hspace{1cm} (9)

a more refined one (biquadratically coupled to the Higgs field) will be discussed in sec 4. It is known that a general, non-integrable scale connection leads to a mass term close to the order of Planck energy [64, 23, 18]. Even if it were physical, it could be “integrated out” at the energy scales of classical field theory considered here. This leaves only the scalar degree of freedom for the integrable case.

For a quadratic kinetic term $L^{(\text{kin})}$ of the form

$$L^{(\text{q-kin})} = -\frac{\alpha}{2} D_\mu \phi D^\nu \phi$$  \hspace{1cm} (10)

the Lagrangian (7) is essentially the one of JBD theory, written in scale covariant form. Remember that $D$ denotes the scale covariant derivative, $D_\mu \phi = \partial_\mu \phi - \varphi_\mu \phi$. For a scalar field with $D_\mu \phi = 0$ (vanishing scale covariant gradient) the Einstein gauge coincides with the Riemann gauge and IWG gravity reduces to Einstein gravity. Let us add that for the Weyl geometric dark scalar tensor theory studied here (WdST) we will add two more terms (18), (21) to $L^{(\text{kin})}$.

Equations which hold only in Einstein gauge (frame) will be denoted by $= E$, similarly for the other gauges. For example $L^{(H)} = \frac{1}{2} \phi^2 R$, with

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5The Palatini variation approach used in [50, 51] and other work of the Brazilian group implies the constraint “Einstein gauge = Riemann gauge”. The approach, there called Weyl integrable spacetime (WIST), is thus a very special case of IWG gravity. Even for $L^{(\text{kin})} = L^{(\text{q-kin})}$ it boils down to a scale covariant description of Einstein gravity, cf. [24, sec. 4.1.C], [31].
\[ \phi_0 = (8\pi\kappa)^{-1}, \text{ thus } \phi_0 \sim E_{Pl} \text{ (reduced Planck energy).} \]

Let us write the scalar field in Riemann gauge/Jordan frame in an exponential form

\[ \phi(x) = R \phi_R(x) = \phi_0 e^{-\sigma(x)}. \quad (11) \]

In Einstein gauge \( \phi_E \) is constant due to (2); the scale connection is

\[ \varphi_E = d\sigma, \quad \text{respectively} \quad \varphi_\mu = E \partial_\mu \sigma. \quad (12) \]

Thus the kinetic term reduces to \( L^{(q-\text{kin})} \equiv E - \frac{\alpha}{2} \phi_0^2 \sigma \partial^\nu \sigma - \Lambda \) and the Lagrangian to

\[ L \equiv (8\pi\kappa)^{-1} \left( \frac{1}{2} R(g, d\sigma) - \frac{\alpha}{2} \partial_\mu \sigma \partial^\nu \sigma - \Lambda \right) + L^{(m)}, \quad (13) \]

with \( \Lambda = \lambda_4 \phi_0^2 \), where \( \lambda_4 \) is “cosmologically” small, i.e. it contains a very small “hierarchy” factor, \( \lambda_4 = (\lambda \beta^{-1})^2 \), where \( \beta^{-1} \) bridges the gap between the Planck scale and the scale of cosmologically small quantities like \( \Lambda^\frac{1}{2} \) or \( a_0 h \), with \( a_0 \) the MOND constant introduced below (19).

At first glance this resembles a Lagrangian of a minimally coupled field with a cosmological constant, but it is not, as the Hilbert term is formed from the Weyl geometric scalar curvature \( R(g, d\sigma) \). The “coupling” of \( \sigma \) to the Hilbert term, contained in (8) has taken on a specific form. The energy momentum tensor \( \Theta = (8\pi\kappa)T^{(\sigma)} \) of the scalar field and the scalar field equation come out differently from the analogous Lagrangian in a Riemannian framework. From [64, 13, 23, 59] we know:

\[ \Theta_{\mu\nu} = E \left( (\alpha + 6) \partial_\mu \sigma \partial_\nu \sigma - \left[ \frac{\alpha}{2} + 3 \right] \partial_\lambda \sigma \partial^\lambda \sigma - \lambda \phi_0^2 \right) g_{\mu\nu} \quad (14) \]

\[ \nabla^2 g(\sigma) \sigma = \frac{8\pi\kappa}{3+\frac{\alpha}{2}} \left( \rho^{(bar)} - 3p^{(bar)} \right) \quad (15) \]

\( \nabla(g) \) denotes the covariant differentiation with respect to the Levi-Civita connection of \( g \) etc. The values 6 and 3 depend on the dimension \( n \) of the spacetime, here \( n = 4 \). With minimal coupling of \( \sigma \) (respectively \( \phi \)) no matter coupling would arise in (15).

In the JBD tradition an equivalent result is derived in several steps: variation in the Jordan frame, conformal transformation to the Einstein frame, and a field redefinition of the scalar field [28]. The result is equivalent to our eg. (13).

(14) shows a decomposition \( \Theta = \Theta^{(de)} + \Theta^{(dm)} \), in a term proportional to \( g \) which may be considered as “dark energy”, \( \Theta^{(de)} \), and another one, \( \Theta^{(dm)} \), which may be considered as dark matter-like, if one uses a very generous concept of “dark matter”;

\[ \Theta^{(dm)}_{\mu\nu} = (\alpha + 6) \partial_\mu \sigma \partial_\nu \sigma, \quad \Theta^{(de)} = -\tilde{\Lambda}(x)g, \]
where \( \hat{\Lambda}(x) \triangleq (\frac{\alpha}{2} + 3) \partial \lambda \sigma(x) \partial^3 \sigma(x) + \Lambda \) and \( \Lambda \) like in (13). A similar decomposition holds for any scalar field theory (by the way, also in the case of minimal coupling).

### 1.3 Free fall and light bending in JBD and IWG gravity

JBD and IWG gravity make different assumptions for the dynamics of test particles. 

a) **JBD gravity** assumes coupling of matter to the metric \( g \) of one of the frames, usually to the Jordan frame or to the Einstein one. Freely falling particles then follow the trajectories of the Levi-Civita connection \( \Gamma(g) \) of the respective frame.

b) **Weyl geometric scalar tensor gravity**, as it is understood here, assumes a matter Lagrangian \( \mathcal{L}^{(m)} \) in a scale invariant form. Freely falling matter particles then follow the geodesics of the invariant Weyl geometric affine connection \( \Gamma(g, \varphi) \) which includes terms in the scale connection. The latter express an additional acceleration which adds to the one induced by the Levi-Civita connection of the Riemannian component \( g \) (see sec. 1.4 and [59, secs. 2.5, 6.3]).

The electromagnetic field and null geodesics depends only on the conformal structure; the gravitational bending of light can be described in any gauge by the respective Riemannian component \( g \) of the Weylian metric; it does not depend on (“respond to”) the respective scale connection, while matter particles do. The option b) may thus give the impression that inertial motion of particles and light rays follow different “laws”. But this is not the case, both are governed by one and the same geodesic structure of the Weyl metric. This may give rise to problems in the weak field approximation, but it need not do so.

In sec. 2.3 it will be shown that the impact of the stress components of the scalar field energy tensor in the present approach are strong enough to induce a non-negligible contribution to gravitational light bending, which is consistent with the additional acceleration for particles due to the scale connection in the Einstein frame. For this result it is important to choose a more elaborate kinetic term for the scalar field than the standard one (10).

### 1.4 Coordinate acceleration in IWG

If the matter Lagrangian \( \mathcal{L}^{(m)} \) is written in a scale invariant form, the energy momentum tensor \( T^{(m)} \) scales with weight \( w(T^{(m)}) = -2 \). A Geroch-Jang type argument [29] shows that in such a framework test bodies move along timelike geodesics of integrable Weyl geometry [59, app. 6.3]. In

\[ \text{This is the case for the SM fields of high energy physics before introducing the matter term of the Higgs field. For classical matter fields the scale invariance of } \mathcal{L}^{(m)} \text{ can be introduced formally. Although this looks artificial at the first sight, it may just as well express a deeper truth resulting from decoherence of the quantum domain.} \]
the perspective of JBD this would amount to coupling of matter to the Jordan frame metric. From the point of view of IWG, however, *matter does not couple to the metric of a specific frame* but to the scale invariant affine connection of Weyl geometry.

For low velocity trajectories parametrized in proper time $\tau$ the coordinate acceleration of Riemannian geometry is

$$a^j = -\Gamma^j_{00}$$  \hspace{1cm} (16)

[17, p. 153f.] or [67, eg. 9.1.2]. This also holds in the Weyl geometric context. (4) shows that the coordinate acceleration has two contributions

$$a^j = a(g)^j + a(\varphi)^j \quad (j = 1, 2, 3).$$  \hspace{1cm} (17)

The first one, $a(g)^j = -\Gamma(g)^j_{00}$, is due to the Levi Civita of the Riemannian component. The second one derives from the scale connection, $a(\varphi)^j = -\Gamma(\varphi)^j_{00}$; it vanishes only in the Riemann gauge. In Einstein gauge the scale connection is determined by the scalar field $\phi$, respectively $\sigma$ ($\varphi = d\sigma$); we therefore denote the additional acceleration $a(\varphi)$ also by $a(\sigma)$. In any scale gauge $(g, \varphi)$ different from the Riemannian one ($\varphi \neq 0$), the scale connection distracts free fall away from the Levi-Civita trajectories of the respective Riemannian component $g$. This general property of IWG is important for the dark scalar field theory of galactic and cluster dynamics which we now turn to.

## 2 Weyl geometric dark scalar field theory (WdST)

### 2.1 Gravitational regimes of WdST

The experience with MOND and its relativistic generalizations [45, 63, 111 sec. 5] is here taken into account by assuming three different regimes distinguished by (the norm of) the gradient of the scalar field $|\nabla \phi|$, respectively $|\nabla \sigma|$. The gravitational dynamics is governed by different, but related Lagrangians:

(i) For $\nabla \sigma = 0$ we are in the ordinary gravity regime governed by the dynamics of Einstein gravity (EG). It has been studied for more than a century in detail and with great success. As a special case of IWG gravity it will here be considered as the *Einstein regime* of WdST.

(ii) For spacelike gradient and $|\nabla \sigma|$ close to the order of the MOND constant $a_0 \approx H_0 / 6$ (see 3.1) we are in an (ultra) weak gravity regime typically obtained at the level of outer galaxies and clusters. In WdST its dynamics is characterized by a scale covariant scalar field which
modifies Einstein gravity. The weak field approximation results in a specific type of MOND dynamics\footnote{The specific type is characterized by the MOND-typical “interpolation function” \( \nu(y) = 1 + y^{-z} \) valid in the Milgrom regime, see sec \[2.4\].} This is the motivation for calling it the *Milgrom regime* of WdST. In this regime the scalar field is governed by two untypical kinetic terms: a cubic kinetic term similar to the one studied by Bekenstein/Milgrom in the deep MOND case \[9\]; it leads to a scale covariant relativistic Milgrom equation for the scalar field (generalizing the non-linear Poisson equation of ordinary MOND). Moreover a second order kinetic term is assumed; it was first introduced in a cosmological context by Novello et al. \[46\]) and endows the scalar field with non-negligible energy momentum important for gravitational light refraction.

iii) For \( \nabla \sigma \) timelike (or for an extremely small norm in the spacelike case; cf. sec. \[3.1\]) we are in a regime in which, under the idealizing assumption of homogeneity/isotropy, large scale models of FRW type are informative. It will be called the *FRW regime* of cosmology. The present experience with the WdST approach indicates that a straight-forward extrapolation of the dynamics of the Milgrom regime to this scale is unrealistic (sec. \[3.3\]). Weyl geometric or conformal approaches to the FRW regime are shortly discussed, but the research in this respect is far from conclusive.

In MOND and some of its relativistic generalizations a free function \( f(X) \) with appropriate asymptotic, or long range behaviour is used for characterizing the transition between the MOND/Milgrom and the Newton/Einstein regimes \[8, 61\]. In the present approach the passage from one regime to the other will be described by means of smooth transition functions between the different Lagrangians (sec. \[3.1\]). Berezhiani/Khoury have proposed that the physical cause for the transition between the Einstein regime and the Milgrom regime may lie in a phase transition of a hypothetical substrate to a superfluid state \[10, 11\]. If successful, this approach may also lead to an explanation of the appearance of fractional powers in the kinetic term of the scalar field (see below); but at the moment this seems still unclear.

Here we start by investigating the dynamics in the Milgrom regime. A short discussion of the other regimes and some remarks on the transition follow in sec. \[3.1\].

2.2 Milgrom regime

Two additional Lagrangian terms

In the Milgrom regime a cubic kinetic term for the scalar field (fractional in the quadratic kinetic expression) is assumed like in most modified gravity...
approaches aiming at a relativistic generalization of MOND. It is essentially the cubic term to which RAQUAL reduces in the deep MOND regime \[45, 26\]. Here it is brought into a scale covariant form of weight \(-4\), in order to get a scale invariant Lagrangian density.\footnote{The original RAQUAL Lagrangian term for the potential \(\varphi\) is \((-8\pi\kappa)^{-1}a_0^2/\bar{\varphi}(\sum x_i^2)^{\frac{3}{2}}\) \footnote{eg. (2b)}. It acquires its deep MOND form for \(\bar{\varphi}(x) \to x^2\), which is cubic in \(\sum x_i^2 = x^2\).}

\[8\]

It will be called the cubic term of the kinetic Lagrangian

\[ L^{(\text{cub})} = -\frac{2}{3}\beta\varphi^{-2}|D\varphi|^3, \quad |D\varphi| = |D_{\nu}\varphi D^\nu\varphi|^\frac{1}{2} \]

\[\equiv -\frac{2}{3}(\beta^{-1}\phi_0)^{-1}\phi_0^2|\nabla\sigma|^3 \quad \text{(18)}\]

The first line gives the scale invariant expression of the Lagrangian density, the second one its specification in Einstein gauge. The Riemann gauge (Jordan frame) expression can easily be read off from the scale invariant expression because of \(D_\nu \equiv \partial_\nu\). In this section we assume that the measuring gauge is identical to the Einstein gauge, at least in sufficient approximation; for a refinement see sec. 3.3.

The dimensional quantity

\[ \beta^{-1}\phi_0 = a_0, \quad a_0 \approx 3.9 \cdot 10^{-19} \text{ s}^{-1} \quad \text{(19)} \]

is a new constant of nature of physical dimension \(T^{-1}\) (T time), standing in a well known relation to the (empirically determined) MOND acceleration

\[ a_0 [c] \approx 1.2 \cdot 10^{-10} \text{ ms}^{-2}. \]

It is often expressed as \(a_0 \approx H_0/6\) (\(H_0\) the Hubble parameter); but this is a (cosmologically) transient characterisation only. A more principled relation exists with the “cosmological constant” \(\Lambda\) (the Einstein gauged quartic potential term of the scalar field),

\[ \Lambda = 36\lambda a_0^2 \]

with \(\lambda\) at the order 1 (see below, eq. (24)).

\(a_0\) will be called the Milgrom constant; it corresponds to the smallest physically meaningful energy quantity of present physics,

\[ E_M = a_0 [\hbar] \approx 2.6 \cdot 10^{-34} \text{ eV} \quad \text{(Milgrom energy)}. \]

The quotient

\[ \beta = \frac{\phi_0}{a_0} = \frac{E_P}{E_M} \sim 10^{62} \quad \text{(20)} \]

\[\text{Be aware that Weyl geometric scale invariance (cf. fn.[1]) is different from Milgrom’s scale invariance used in the latter’s discussion of the deep MOND regime [44, 45].}\]
is the "penultimate" hierarchy factor between the (reduced) Planck energy $E_P$ and $E_M$. It will be called the cosmological hierarchy factor.

In order to improve on the RAQUAL approach we have to add an energy carrying contribution to the kinetic Lagrangian of the scalar field, here realized by the second (derivative) order term. It has been introduced by Novello/Oliveira et al. in 1993 for cosmological considerations [46] and will be called the Brazilian term:

$$L^{(\text{braz})} = -\gamma \phi D_\nu D^\nu \phi$$

$$= -\frac{\gamma}{8\pi G} \left( \nabla(g)_\nu \partial^{\nu} \sigma + \partial_\nu \sigma \partial^{\nu} \sigma \right)$$  \hspace{1cm} (21)

Novello et al. noticed that $L^{(\text{braz})}$ does not increase the order of the scalar field equation because the derivative second order term $\sqrt{|g|} \nabla(g)_\nu \partial^{\nu} \sigma = \partial_\nu (\sqrt{|g|} \partial^{\nu} \sigma)$ is a divergence. So it can be shifted to a boundary integral and does not contribute to the variation $\delta \phi$. The scalar field equation remains of order 2.

All in all, the Lagrangian in the Milgrom regime of WdST is

$$L_M = L^{(H)} + L^{(\phi \text{kin})} + L^{(V_4)} + L^{(\text{bar})},$$

where

$$L^{(\phi \text{kin})} = L^{(q-\text{kin})} + L^{(\text{cub})} + L^{(\text{braz})}.$$  \hspace{1cm} (22)

The building blocks of the kinetic term are given in (10), (18), (21)), $L^{(H)}$ and $L^{(V_4)}$ in (8, 9); (bar) designates baryonic matter. The parameters $\alpha$, $\beta$, $\gamma$, $\lambda_4$ are specified below. The Lagrangian of the Milgrom regime can be rewritten in Einstein gauge (with $\alpha = -6$) as

$$L_M \overset{E}{=} (8\pi \kappa)^{-1} \left( \frac{R(g)}{2} - \gamma \nabla(g)_\nu \partial^{\nu} \sigma - (\gamma + 3) \partial_\nu \sigma \partial^{\nu} \sigma \right)$$

$$- \frac{2}{3} a_0^{-1} |\nabla \sigma|^3 - \Lambda \right) + L^{(\text{bar})}. \hspace{1cm} (23)$$

In the Milgrom regime WdST differs from RAQUAL in two respects: (i) the nonminimal coupling of $\phi$ to gravity; in the context of IWG it makes the scalar field a part of the gravitational sector. (ii) The second order term $L^{(\text{cub})}$ of the kinetic part of Lagrangian. (i) implies an indirect coupling of the scalar field to matter in the derivation of the scalar field equation like in JBD, (ii) endows the scalar field with a non-negligible energy-stress tensor and leads to gravitational light bending in accordance with the inertial dynamics.

Using the notation $\Theta^{(\text{cub})} \overset{E}{=} 8\pi \kappa T^{(\text{cub})}$, the contribution of $L^{(\text{cub})}$ to energy-momentum-stress on the right hand side (rhs) of the Einstein equation (see below) can be expressed as (85)

$$\Theta^{(\text{cub})}_{\mu \nu} \overset{E}{=} 2a_0^{-1}(|\nabla \sigma| \partial_\mu \sigma \partial_\nu \sigma - \frac{1}{3} |\nabla \sigma|^3 g_{\mu \nu}).$$

13
It is cubic in $|\partial \sigma|$, which makes it *negligibly small* in the Milgrom regime. The contribution of the Brazilian term to the energy-stress tensor of the scalar field is according to eq. (86))

$$\Theta^{\text{braz}} = \frac{2}{E} \gamma (\nabla (g)_{\mu} \partial_{\nu} \sigma - 3 \partial_{\mu} \sigma \partial_{\nu} \sigma) - \gamma (\nabla (g)_{\nu} \partial^{\nu} \sigma - \partial_{\lambda} \sigma \partial^{\lambda} \sigma) g_{\mu \nu} ,$$

where $\nabla (g)$ denotes the covariant derivative with regard to the Levi-Civita connection of $g$, and $\nabla (g)^{2}$ the corresponding d’Alembert operator. The second order terms of (86) are dominant in the Milgrom regime and crucial for the energy content of the scalar field.

**Parameter choice**

As mentioned already, $\beta \sim 10^{62}$ is the hierarchy factor between the constants $E_{Pl}$ and $E_{M}$ (corresponding to $a_{0}$). We see below that for $\alpha = -6$ (and only with this choice) the scalar field equation in Einstein frame acquires the form of a covariant Milgrom equation generalizing the deep MOND equation. This is the reason for assuming conformal coupling for the scalar field in the Milgrom regime. Because of the Brazilian term this does not imply a vanishing trace of the matter tensor. The coefficient of the Brazilian kinetic term will be chosen as $\gamma = 4$. With this choice the light bending of the scalar field is consistent with the additional acceleration induced by the scalar field (see sec. 2.3). The coefficient $\lambda_{4}$ of the quartic potential determines the value of a cosmological constant $\Lambda = \frac{\lambda_{4}}{4} \phi_{0}^{2}$ in the Einstein frame. Its order of magnitude is “extremely small”, but can be related to a coefficient $\lambda \sim 1$ by use of the cosmological hierarchy factor (squared):

$$\Lambda = \frac{\lambda_{4}}{4} \phi_{0}^{2} \approx \lambda H_{0}^{2} \approx 36 \lambda a_{0}^{2} .$$

Then $\Lambda$ has the order of magnitude typical for a cosmological constant ($\lambda = 3 \Omega_{\Lambda}$); in the Milgrom regime it is negligible.

| $\alpha$ | $\beta$ | $\gamma$ | $\lambda$ | $\lambda_{4}$ |
|---------|---------|---------|---------|---------|
| $-6$    | $\sim 10^{62}$ | 4       | $\sim 1$ | $\lambda (12 \beta^{-1})^{2}$ |

**Dynamical equations**

In integrable Weyl geometric scalar tensor theory the *scale invariant Einstein equation* is of the form

$$G = \text{Ric} - \frac{R}{2} g = \phi^{-2} T^{(\text{bar})} + \Theta(\phi) ,$$

with

$$\Theta(\phi) = \Theta(\phi^{\text{kin}}) + \Theta(V) + \Theta(H) .$$

Variation $\delta g_{E}$ in the Einstein gauge leads to the same second order term.
$G$, $Ric$, $R$ are the Weyl geometric quantities, $\Theta^{(\phi)}$ denotes the total contribution of the scalar field to energy-momentum-stress (depending on the gravitational regime). $\Theta^{(H)}$ arises from the non-minimally coupling scalar field to the Hilbert term (see appendix, (31)):

$$\Theta^{(H)}_{\mu\nu} = \phi^{-2}(D_{(\mu}D_{\nu)}\phi^2 - D_\lambda D^\lambda \phi^2 g_{\mu\nu})$$

It is well known in JBD theory [16, section 6.2]. In Weyl geometric gravity [23, 65], in Einstein gauge it is

$$\Theta^{(H)}_{\mu\nu} \doteq 2\nabla(g)(\mu,\nabla_{\nu})\sigma + 2\nabla(g)^2 g_{\mu\nu}, \quad \text{with } g = g_E.$$ 

In the decomposition of the Einstein tensors, $G(g, \varphi) = G(g) + G(\varphi)$, some terms of $G(\varphi) = Ric(\varphi) - \frac{R(g)}{2}g$ cancel with some of the rhs. In Einstein gauge we have [59, section 6.2]

$$G(\varphi)_{\mu\nu} \doteq 2\partial_{\mu}\sigma\partial_\nu\sigma - 2\nabla(g)(\mu,\nabla_{\nu})\sigma + (\partial_\lambda\sigma\partial^\lambda\sigma + 2\nabla(g)^2\sigma)g_{\mu\nu}.$$  (27)

With the notation

$$\Theta(\sigma) := \Theta^{(\phi\text{kin})} + \Theta^{(H)} - G(\varphi) \quad \text{(28)}$$

the Einstein equation acquires a more familiar form (in the Einstein frame):

$$G(g) = Ric(g) - \frac{R(g)}{2}g \doteq 8\pi\times T^{(\text{bar})} + \Theta(\sigma) + \Theta^{(V)}, \quad \text{(29)}$$

Note that $\Theta(\sigma)$ is a shorthand (not even scale covariant) for an important part of the rhs of the Einstein equation (29) in which the scale connection contribution of the Einstein tensor $G(\varphi)$ has been sub tractively included. The (scale covariant) energy momentum of the scalar field is:

$$\Theta^{(\phi)} = \Theta^{(\phi\text{kin})} + \Theta^{(H)} + \Theta^{(V)}, \quad \Theta^{(\phi\text{kin})} = \Theta^{(q\text{kin})} + \Theta^{(\text{cub})} + \Theta^{(braz)} \quad \text{(30)}$$

For a quartic potential, $V = V_4$, the energy momentum in the Einstein gauge boils down to a cosmological constant term, $\Theta^{(V)} \doteq -\Lambda g_E$ with $\Lambda$ as in (24). It is negligible in the Milgrom regime. From (28) and the calculations (83) ff. in the appendix we find

$$\Theta(\sigma)_{\mu,\nu} \doteq 2\gamma(\nabla(g)(\mu,\nabla_{\nu})\sigma - \frac{1}{2}\nabla(g)^2\sigma g_{\mu\nu}) + O_2(\partial\sigma),$$

where $O_2(\partial\sigma)$ contains the quadratic and cubic terms in $\partial\sigma$ [12].

The second order derivative terms of $\Theta^{(H)}$ cancel, the remaining ones derive from

\[11\] In flat space it agrees with an (ad hoc) “improvement” of the energy momentum tensor introduced by Callan/Coleman/Jackiw in quantum theory [15].

\[12\] More in detail:

$$O_2(\partial\sigma)_{\mu\nu} \doteq \left(\alpha - \frac{6\gamma - 2}{2}\right)\partial_\sigma\partial_\nu\sigma + 2a_0^{-1}\nabla\sigma|\partial_\sigma\partial_\nu\sigma$$

$$- \left(\frac{\alpha + \gamma + 2}{2}\right)\partial_\lambda\sigma\partial^\lambda\sigma + \frac{2}{3}\epsilon_\sigma a_0^{-1}\nabla\sigma^3)g_{\mu\nu}$$

with $\nabla = \nabla(g_E), g = g_E$ and $\epsilon_\sigma$ like in (88).
$L^{(brax)}$ For clusters and galaxies $(\partial_x \sigma)^2$ is at least 5 orders of magnitude smaller than $|\partial_x^2 \sigma|$. Thus it is reasonable to approximate the rhs in the Milgrom regime by its second order derivative terms:

$$\Theta(\sigma)_{\mu,\nu} \approx 2\gamma \left( (\nabla(g))_{(\mu} \partial_{\nu)} \sigma - \frac{1}{2} \nabla(g)^2 \sigma g_{\mu\nu} \right) \tag{31}$$

In the following this expression will be assumed as characteristic also for the non-centrally symmetric case.

The scalar field equation in the Milgrom regime can be calculated like in [59]. Like in JBD theory one subtracts the traced Einstein equation from the variational equation of $\phi$; in this way the trace of the baryonic matter tensor enters the dynamical equation of the scalar field without presupposing a Lagrangian coupling of the latter to matter (see app. 5.2). With the denotation $M(\phi)$ for the lhs of the scalar field equation we get the scale covariant Milgrom equation

$$M(\phi) = -\frac{1}{2} (\beta^{-1} \phi) \phi^{-2} \left( trT^{(bar)} + 4L(V) - \phi \partial_\phi L(V) \right) , \tag{32}$$

with $M(\phi)$ the (scale covariant) Milgrom operator given in general form in the appendix (94). For $V = V_4$ and the Einstein gauge the Milgrom operator is simply

$$M_E(\phi) \doteq E \nabla(g_E)_{\lambda} (|\nabla \phi| \partial^\lambda \phi) . \tag{33}$$

For an ideal fluid with matter density $\rho^{(bar)}$ and pressure $p^{(bar)}$ the scalar field equation acquires the form of a covariant generalization of the non-linear Poisson equation known from deep MOND (95):

$$M_E(\phi) \doteq (4\pi \kappa) a_0 (\rho^{(bar)} - 3p^{(bar)}) \tag{34}$$

The lhs of eq (34) derives from $L^{(cub)}$; it only holds for $\alpha = -6$. In the following it is called covariant Milgrom equation (without the addition “scale covariant”).

### 2.3 Weak field approximation, centrally symmetric case

In the following section we presuppose the Einstein frame and use the equality sign = as an abbreviation for $\doteq$; similarly $\approx$ is used for approximations in Einstein frame where the context is clear.

Let us shed a glance at the weak field approximation in the static centrally symmetric case (for details see appendix 5.3). In isotropic conformal coordinates with $x_0 = t, x_1 = r, x_2, x_3$ the Weylian metric is given by

$$ds^2 \doteq -A(r) dt^2 + B(r) \left( dr^2 + r^2 d\Omega^2 \right) , \quad \varphi \doteq d\sigma(r) = \sigma'(r) dr ,$$

\[33\]In a scale covariantly rewritten pure RAQUAL they would not be present.
with $d\Omega^2 = dx_2^2 + (\sin x_2)^2 dx_3^2$.

In the weak field case $g \equiv \eta + h$ with the Minkowski metric
$\eta = \text{diag}(-1, 1, r^2, r^2 \sin^2 x_2)$ and $h = \text{diag}(h_{00}, h_{11}, h_{11} r^2, h_{11} r^2 \sin^2 x_2)$ we have $A = 1 - h_{00}, B = 1 + h_{11}$. Equality up to first order in $h, h', h'', \sigma', \sigma''$ will be denoted by $\equiv$.

The Milgrom equation (34) in the vacuum becomes

$$0 = r \sigma'' + \left(1 + r \frac{A'}{A}\right) \sigma' = r \sigma'' + \sigma'.$$

(35)

It is solved by

$$\sigma' = C_1 A^{-\frac{1}{2}} r^{-1};$$

(36)

up to first order it is

$$\sigma' = \frac{C_1}{1} r^{-1}, \quad \sigma = \frac{C_1}{1} \ln \frac{r}{r_0},$$

(37)

with any value for $r_0$.

An inspection of the first components of the Ricci tensor by the reduced Einstein equation (appendix 5.3) shows that the weak field Riemannian metric is given by

$$h_{00} = -2\Phi_N^{(\text{bar})}, \quad h_{11} = 2\Phi_N^{(\text{bar})} + \frac{\gamma}{2} \sigma$$

(38)

with $\Phi_N^{(\text{bar})}$ the Newton potential of baryonic matter.

Outside a central mass $M$ this is

$$ds^2 = -(1 - 2 \frac{M}{r}) dt^2 + \left(1 + 2 \frac{M}{r} + \frac{\gamma}{2} \sigma\right) \left(dr^2 + r^2 (d\Omega^2)\right).$$

(39)

In the light of empirical evidence for MOND the constant $C_1$ is

$$C_1 = \sqrt{a_1 M}, \quad \text{with} \quad a_1 = a_0 c^{-1};$$

then

$$\partial_t \sigma = \frac{\sqrt{a_1 M}}{r}.$$  

(40)

We see that that the first order approximation of the relativistic Milgrom equation is identical to the deep MOND potential. The relativistic correction for $A$ in (36) is

$$(1 - h_{00}(r))^{-\frac{1}{2}} = (1 - 2 \frac{M}{r})^{-\frac{1}{2}} = (1 + \frac{M}{2r}) < 1 + a_1 r,$$

and thus negligible in the Milgrom regime.
Energy tensor of the scalar field

We have to distinguished between the energy-stress tensor of the scalar field $\Theta^{(\phi)}$ (30) on the rhs of the scale invariant Einstein equation (25) and the net energy-stress expression $\Theta(\sigma)$ (28) on the rhs of the equation (29) (the one with the Riemannian part of the Einstein tensor $G(g)$ on the lhs). The first one, taken in the Einstein gauge $\Theta^{(\phi)}_E$, represents the physical energy-momentum and stress properties of the scalar field, while the latter is crucial for the calculation of the Riemannian part of the metric. In the Einstein gauge they are at first order (30, 28)

$$
\Theta^{(\phi)}_{\mu\nu} = \frac{1}{2} (2\gamma - 2) \nabla_{(\mu} \sigma_{\nu)} + (2 - \gamma) \nabla^2 \sigma g_{\mu\nu} \right) \left[ -\Lambda g_{\mu\nu} \right] \\
\Theta(\sigma)_{\mu\nu} = \frac{1}{2} 2\gamma \left( \nabla_{(\mu} \sigma_{\nu)} - \frac{1}{2} \nabla^2 \sigma g_{\mu\nu} \right) \left[ +\Lambda g_{\mu\nu} \right],
$$

with negligible cosmological contributions in the Milgrom regime. For the central symmetric case and $\gamma = 4$ the term proportional to $g$ (vacuum energy like) dominates. With

$$
\nabla^2 \sigma = \sigma'' + \frac{2}{r} C' = C_1 r^{-2}
$$

the energy tensor of the scalar field $\Theta^{(\phi)}$ has the form of a variable vacuum energy with energy density falling off quadratically, $(4\pi\kappa)^{-1} C_1 r^{-2}$, plus a superimposed negative pressure term in the radial direction. The energy expression $\Theta(\sigma)$ in the “Riemannianized” Einstein equation (29) has a similar character (positive energy density, modified vacuum energy tensor) with the peculiar property that its reduction (by subtracting its half-trace times $g$) leads to a vanishing energy component:

$$
(\Theta(\sigma) - \frac{1}{2} tr \Theta(\sigma) g)_{00} = 0
$$

This property holds generally, independent of central symmetry; it is of major importance for the weak field approximation.

Contribution of the scalar field to acceleration and light refraction

In the Milgrom regime with Einstein gauge the scalar field induces the contribution

$$
a(\phi)^j = \partial^j \sigma g_{00}
$$

(43)

to the total acceleration of slow motions (17), (5). In the weak field case

$$
a(\phi) = -\nabla \sigma,
$$

(44)
i.e., $\sigma$ functions as an acceleration potential of the scalar field.
For a spherical symmetric Riemannian metric parametrized by isotropic conformal radius the gravitational refraction index $n_{\text{grav}}$ is well known (cf. [25])

$$n_{\text{grav}} = \frac{1}{2} B^{\frac{1}{2}}.$$

(45)

Here we have $n_{\text{grav}} = (1 - h_{00})^{-\frac{1}{2}} (1 + h_{11})^{-\frac{1}{2}} \approx 1 + \frac{1}{2} (h_{00} + h_{11})$ and thus

$$n_{\text{grav}} = 1 - 2(\Phi_{N}^{(\text{bar})} + \frac{\gamma}{4} \sigma).$$

(46)

In this sense, $\frac{\gamma}{4} \sigma$ functions as the light bending potential of the scalar field. For $\gamma = 4$ it agrees with the acceleration potential, so that we have good reasons to consider it as the “scalar field potential”

$$\Phi^{(\phi)} = \sigma.$$

Expecting a similar agreement for less symmetrical constellations we choose

$$\gamma = 4$$

(47)

as default value for the parameter. As mentioned above the energy momentum of the scalar field is then dominated by the vacuum energy like term proportional to $g$. The effect may thus be described as “light bending by dark energy” similar to an observation studied by Zhang in [74] in a different context.

2.4 Milgrom approximation of WdST gravity

A relativistic weak field background for MOND

In this section we continue to work in the Einstein frame and presuppose a general weak field metric $(g, \varphi)$ in the Milgrom regime of WdST, $g = \eta + h$ and $\varphi = d\sigma$, $h = \text{diag}(h_{00}, h_{11}, h_{22}, h_{33})$ without specialization to central symmetry.

For centrally symmetric constellations eq. (35) indicates that the first order approximation of the Milgrom equation reduces to the flat space non-linear Poisson equation. This is true more generally:

$$\mathcal{M}_{E}(\phi) = \nabla(g)_{\lambda} (|\nabla(g)\sigma| \partial^{\lambda} \sigma) = \nabla_{\lambda} (|\nabla\sigma| \partial^{\lambda} \sigma), \quad g = \eta + h,$$

with $\nabla = \nabla(\eta)$ the flat space operator. Up to first order the scalar field equation of WdST in the Milgrom regime is equivalent to the deep MOND equation in flat space (with pressures added)

$$\nabla_{\lambda} (|\nabla\sigma| \partial^{\lambda} \sigma) = (4\pi a_{0}) \rho_{0}^{(\text{bar})} - 3p_{0}^{(\text{bar})}.$$

(48)

At first order the scale connection part of the Weylian metric is determined by a flat space equation. It decouples from the Riemannian part.
On the other hand, the weak field equation for the (00)-component of the Riemannian contribution of the Weylian metric (98),

\[ R_{00}(g) = -\frac{1}{2} \nabla(\eta)^2 h_{00} = 4\pi \kappa \rho^{(\text{bar})} + \Theta(\sigma)_{00} - \frac{1}{2} \text{tr} \Theta(\sigma) g_{00}, \]

is independent of the scalar field, because by (42) the rhs reduces to the baryonic source term (see app. 5.3):

\[ h_{00} = -2\Phi^{(\text{bar})} \]

The other components \( h_{jj} \) \((j = 1, 2, 3)\) depend on the specific conditions of the system under study. They induce relativistic first order corrections of the flat space metric of MOND, important, among others, for the course of light rays (see above for the centrally symmetric case).

**Determining \( \sigma \) from the Newton acceleration**

Given the approximate baryonic matter distribution modelled in flat space, the scalar field contribution to the acceleration \( a(\phi) \) can be calculated without any symmetry conditions in two steps. At first the Newton acceleration of the baryonic matter density \( a_N := a_N^{(\text{bar})} \) has to be calculated, respectively the Newton potential \( \Phi_N \). A straightforward (although a bit tedious) vector calculus calculation shows that for

\[ a(\phi) = \sqrt{a_0 |a_N|} \frac{a_N}{|a_N|}. \]  

(50)

the function \( \sigma \) defined by

\[ \nabla \sigma = -a(\phi) \]

(51)

satisfies (48). With other words, once the Newtonian acceleration field \( a_N \) is known, \( a(\phi) \) can be calculated by the algebraic field transformation (50), and \( \sigma \) is found by integration (51).

(51) can also be written as

\[ \nabla^2 \sigma = \nabla \left( \sqrt{a_0 |\nabla \Phi_N|} \right) \frac{\nabla \Phi_N}{|\nabla \Phi_N|} = \nabla \left( \sqrt{\frac{a_0}{|\nabla \Phi_N|}} \nabla \Phi_N \right) \]

In the literature a similar idea is used for dealing with the deep MOND case. In slightly different guise this is known under the name “quasi-linear” (QUMOND) approach [43, 26, p. 46ff]. In our context it makes sense under the conditions of the Milgrom regime, i.e., for \( \nabla \sigma \leq 10a_0 \) (with \( l = 2 \) in the sense of sec. 3.1 below).
Flat space (Milgrom) approximation of weak field WdST

Let us now turn to the flat space picture of the weak field dynamics in the Milgrom regime. Because of (48) the flat limit of equation (34) for pressure free baryonic matter is the deep MOND equation:

$$\nabla_\lambda (|\nabla \sigma| \partial^\lambda \sigma) = (4\pi \kappa) a_0 \rho^{(\text{bar})}$$  \hspace{1cm} (52)

The (00)-component of the Einstein equation (29) delivers Newtonian dynamics in the flat space approximation\(^{14}\) which can easily be read off from the first order dynamics of WdST (using the Weylian metric \((\eta + h, \varphi = d\sigma)\)).

The scalar field potential \(\sigma = \Phi^2\) turns into the deep MOND potential \(\Phi^d = \Phi^2\) of the given matter distribution. At first order approximation the gravitational acceleration of massive particles is composed of the Riemannian contribution which can be written as a Newtonian acceleration

$$a^j_N = -\partial_j \Phi_N$$

and of the additional acceleration (17) due to the scalar field

$$a(\phi)^j = \Gamma(\varphi)_{00} = -\partial_j \sigma = -\partial_j \Phi^{(\phi)}.$$  

The total acceleration is then

$$a_{\text{tot}} = -\nabla (\Phi_N + \Phi^{(\phi)}) = a_N + a(\phi).$$

This result is the same as in the “conformal emergent gravity” approach (CEG) by Hossenfelder and Mistele \(^{35}\), although both are derived from different principles. With (50) we get

$$a_{\text{tot}} = \frac{a_N}{|a_N|} \left(|a_N| + \sqrt{a_0 |a_N|}\right).$$  \hspace{1cm} (53)

Denoting by \(a\) the same acceleration represented in flat space we get

$$a = \tilde{\nu}\left(\frac{|a_N|}{a_0}\right) a_N, \quad \text{with} \quad \tilde{\nu}(y) = 1 + y^{-\frac{1}{2}}, \quad y = \frac{|a_N|}{a_0}.$$  

Of course this holds only under the condition \(a(\phi) = \sqrt{a_0 |a_N|} \leq 10^{l-1} a_0\), respectively \(|a_N| \leq 10^{2l-2} a_0\), inherited from the delimitation of the Milgrom regime (cf sec.3.1). The smooth transition function \(h(x; a, b)\) introduced

\(^{14}\)The Riemannian (Levi-Civita) contribution to the acceleration of low velocity matter particles \(a(g)^j = \Gamma(g)^j_{00}\) depends crucially on \(h_00\):

$$\Gamma(g)^{ij}_{00} = \frac{\partial_j h_{00}}{2(1 + h_{jj})} = \frac{\partial_j h_{00}}{2} = -\partial_j \Phi^{(\text{bar})}_{ij}$$

and can thus be read off from the Newton approximation like in Einstein gravity.
there, (58), has value 0 for $x \leq a$ and value 1 for $x \geq b$. With it we can express $a$ more generally as

$$a = \nu \left( \frac{|a_N|}{a_0} \right) a_N,$$

(54)

with

$$\nu(y) = 1 + h(y^{-1}; 10^{-2l}, 10^{2-2l}) y^{-\frac{1}{2}}.$$  

(55)

This function hallmarks two well defined acceleration regimes:

$$\nu(y) = \begin{cases} 
1 & \text{for } y \gg 10^{2l} \rightarrow a = a_N \text{ (Newton regime)} \\
1 + y^{-\frac{1}{2}} & \text{for } y \ll 10^{2l-2} \rightarrow a = a_{tot} \text{ (MOND (WdST))}
\end{cases}$$  

(56)

A plausible value for $l$ is $l = 2$.

Functions of similar type are used in the MOND approach as “interpolation functions” between the deep MOND regime ($|a| \approx \sqrt{a_0 |a_N|}$) and the Newton regime ($a = a_N$). They are understood as expressing possible quantitative specifications of “Milgrom’s law” which postulates a smooth monotonous transition of gravitational dynamics between the Newton acceleration and the deep MOND one. Such a $\nu$-function has to satisfy two approximation conditions [26, sec. 5.1]:

$$\nu(y) \approx \begin{cases} 
1 & \text{for } y \gg 1 \rightarrow a \approx a_N \text{ (Newton regime)} \\
y^{-\frac{1}{2}} & \text{for } y \ll 1 \rightarrow a \approx \sqrt{a_0 |a_N|} \text{ (deep MOND)}
\end{cases}$$

In our case both are satisfied; i.e. the present flat space approximation of WdST satisfies Milgrom’s law. In analogy to the Newton approximation of Einstein gravity it will be called the Milgrom approximation of WdST, which is the MOND variant specified by (55) (WdST-MOND).

**Some unique features of WdST-MOND**

The dynamics of WdST MOND has some unique features among the MOND models and also some crucial differences to the whole class:

(i) The whole “interpolation” range between the Newton regime and the deep MOND region consists of two parts. The larger one, with roughly $|a_N| < 10^{2l-2} a_0$ ($l = 1$ or 2), the “upper” one in terms of distance, is determined by the dynamics of WdST in the Milgrom regime, expressed by the interpolation function (55). This interpolation function is the same as in the study of Hossenfelder/Mistele ([35]), in which 2693 data points from 153 galaxies are analysed. The authors show that the mass discrepancy relation for these data is extremely well reproduced (“predicted”) by MOND with this interpolation function.

15For $10^{2l} > y > 10^{2l-2}$ the value of $a$ is a formal interpolation between the regimes.

16A second interpolation function, usually denoted $\mu(x)$, is used. It has the property $a_N = \mu \left( \frac{|a|}{a_0} \right) a$; see, e.g., [29, sec. 5.1].
(ii) Only in the domain where \(10^{2l-2}a_0 < |a_N| < 10^{2l}a_0\) (the "lower" part in terms of distance) enters the conventionally chosen transition function \(h(x; a, b)\). In usual MOND models the whole interpolation function is conventional (with constraints).

(iii) In the ordinary MOND approach the deep MOND acceleration may be interpreted in terms of a fictitious Newton potential which would induce the same additional acceleration, in the central symmetric case:

\[
\Phi^{(ph)} = \sqrt{a_1 M \ln r}.
\]  

(57)

It is sometimes ascribed to fictional "phantom matter" of unknown origin and ontological status with density \(\rho^{(ph)}\) in agreement with Newton dynamics, \(\nabla^2 \Phi^{(ph)} = 4\pi \kappa \rho^{(ph)}\). MOND protagonists often postulate gravitational light deflection to be in accordance with the phantom matter density, often even with a "relativistic" factor 2, assuming that a general relativistic extension of MOND will justify such an assumption [26, sec. 8]. This is consistent with the present empirical evidence of microlensing; but in classical MOND the physical reason for such an effect is completely unclear. Here we have shown that this expectation is satisfied (sec. 2.3).

(iv) The scalar field has an energy tensor \(T^{(\phi)} = (8\pi \kappa)^{-1} \Theta^{(\phi)}\) with a non-negligible energy density \(\rho^{(\phi)}\) and stress/pressure terms (41). It resembles a type of dark energy \textit{sui generis} which is not "phantom". It supports the view that there need not be a strict dichotomy between modified gravity and dark matter/energy explanations of galactic and cluster dynamics, cf. [40].

(v) The energy density of the scalar field \(\rho^{(\phi)}\) does not enter the \((0, 0)\)-component of the relativistic weak field approximation (42). In spite of this the scalar field energy-momentum tensor as a whole enhances gravitational light deflection due to the baryonic matter sources through its pressure components. In the central symmetric case, the scalar field potential adds up to the Newton potential in the calculation of the light refraction index. For \(\gamma = 4\) the role of the phantom density for gravitational light deflection in the radial direction even in its relativistic extension (by factor 2) conjectured by some authors of the MOND program, is justified [46].

(vi) We have seen that the scalar field builds up a halo of energy density around mass concentrations in the Milgrom regime, which can be calculated in the weak field approximation (2.4). We Expect a superposition of the halos of "dark" energy density in the neighbourhood of

\[\text{Similar justifications arise, of course, also in other relativistic extensions of MOND like, e.g., [63].}\]
single galaxies and the halo accumulated at the scale of a galaxy cluster, calculated in a weak field (MOND) approximation at this larger scale. This makes an important difference to models of cluster dynamics in the usual MOND approach. A heuristic evaluation of cluster data on the basis of a precursor approach to the present one indicates that no additional dark matter besides the energy tensor of the scalar field may be necessary to get the dynamics at cluster level right [54]. This conjecture ought to be checked by astronomers in more detail.

3 Other regimes

3.1 Transition between the regimes and the overall Lagrangian

The gravitational regimes introduced in sec. 2.1 can be distinguished by the gradient of the solution of eq. (34), even in cases where the solution is only formal. We characterize the delimitation of the regimes by threshold values for the gradient of scalar field \((y_1 > y_2 > y_3 > y_4)\) as follows:

| Einstein regime | Milgrom regime | FRW regime |
|-----------------|----------------|------------|
| \(-\nabla \sigma\) | \(-\nabla \sigma\) | \(-\nabla \sigma\) |
| \(-\nabla \sigma\) | \(-\nabla \sigma\) | \(-\nabla \sigma\) |

(1) *Einstein regime* for \(|\nabla \sigma| \geq y_1\)

(2) *Milgrom regime* for \(\nabla \sigma\) spacelike and \(y_2 \geq |\nabla \sigma| \geq y_3\)

(3) *FRW regime* for \(\nabla \sigma\) timelike and/or \(|\nabla \sigma| \leq y_4\)

with roughly \(y_1 = 10^{l+2} a_0\), \(y_2 = 10^l a_0\), \(y_3 = 10^{-k} a_0\), \(y_4 = 10^{-k'} a_0\), for some \(k < k'\). In the light of cluster dynamics \(l = 1\) or \(2\) seems reasonable. The regions with \(|\nabla \sigma|\) between \(y_1, y_2\) and \(y_3, y_4\) are transition zones between the regimes.

For the transition we use the standard *smoothing function* \(h(x; a, b)\) defined stepwise by

\[
f(x) = \begin{cases} 
e^{-\frac{1}{x}} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0, \end{cases}
\]

from which one builds

\[
g(x) := \frac{f(x)}{f(x) + f(1 - x)},
\]

and finally

\[
h(x; a, b) = g\left(\frac{x - a}{b - a}\right).
\]

(58)

It is constant with values 0 for \(x \leq a\) and 1 for \(x \geq b\), with a smooth transition in between (fig. 1).
The functions
\[ h_1(x) = h(|\nabla \sigma(x)|^{-1}, y_1^{-1}, y_2^{-1}), \quad h_2(x) = h(|\nabla \sigma(x)|^{-1}, y_3^{-1}, y_4^{-1}) \]
characterize the transition between the Einstein regime (where \(|\nabla \sigma(x)| \geq y_1\)) with \(h_1(x) = 0\) and the Milgrom regime + beyond (where \(|\nabla \sigma(x)| \leq y_2\)) with \(h_1(x) = 1\). For the transition between the Milgrom regime and the FRW regime \(h_2\) plays a similar role.

The total Lagrangian of the present model is composed of the Lagrangians \(L_E\) \((60)\), \(L_M\) \((22)\) and a hypothetical Ansatz \((71)\) for the cosmological regime \(L_{FRW}\), with the respective transition functions:
\[ L = (1-h_1)L_E + h_1(1-h_2)L_M + h_2L_{FRW} \tag{59} \]
In Einstein gauge the terms of the Lagrangian are given by \((61)\), \((23)\) and \((73)\) respectively.

The transitions between the regimes are represented in the total Lagrangian \((59)\) formally, i.e., without making a claim for their unknown dynamics. The physical reason for the first transition may be a destabilization of the Milgrom regime scalar field, if the gradient of \(|\nabla \sigma|\) becomes too strong. This is to be expected if the scalar field represents a “superfluid phase” of some medium (Koury/Berezhiani, similarly Hossenfelder/Mistele); see sec. 2.1. The Lagrangian in the FRW regime considered below \((71)\) is a toy model and has to be improved.

### 3.2 Einstein regime

If the transition between the Milgrom and Einstein regimes is due to a destabilization of a superfluid phase, which suppresses the gradient of the physical \(\sigma\), the scalar field becomes constant in the Riemann gauge \((\nabla \sigma = 0)\),
\[ \phi \equiv \frac{\phi_0}{R} \]
This implies Riemann gauge = Einstein gauge. With other words the integrable Weyl geometry becomes Riemannian and IWG gravity reduces to
Einstein gravity. This leads to Einstein gravity written scale covariantly in the geometric framework of IWG, i.e. basically the gravity theory (WIST) studied by the Brazilian group, although with a different Lagrangian principle behind it.\footnote{The Palatini variation principle used in WIST implies that the Riemann gauge is identical with the Einstein gauge [51, sec. II].}

In our approach (WdST) the “freezing” of the scalar field in the Riemann gauge can be be expressed by a Lagrange multiplier term with scale covariant multipliers $\lambda_\nu(x)$ (weight $-4$), which fixes the dynamical degree of freedom of the scalar field:

$$L^{(\phi, \lambda)} = \sum_\nu (\phi^3 D_\nu \phi - \lambda_\nu)$$

The Lagrange constraint is $D_\nu \phi = 0 \leftrightarrow \partial_\nu \phi = 0$. The scalar field is trivialized (constant in the Riemann gauge) which reduces the Weyl geometric scalar field theory to Einstein gravity. The general form of the Lagrangian for Einstein gravity in our framework is

$$L_E = L^{(H)} + \sum_\nu (\phi^3 D_\nu \phi - \lambda_\nu) + L^{(m, \bar{m})}.$$ \hspace{1cm} (60)

In Einstein (= Riemann) gauge it reduces to the well known form

$$L_E \equiv (16\pi \kappa)^{-1} R(g) + L^{(m, \bar{m})}$$ \hspace{1cm} (61)

3.3 FRW regime and a non-singular cosmological model

No extension of the Milgrom regime Lagrangian to the FRW context

We start with a short look at the aspects of cosmology which can be modelled by a homogeneous and isotropic spacetime of FRW. The Riemannian component of the Weylian metric can be written in any gauge in a standard FRW form. Here we concentrate on the Einstein gauge with Riemannian component $g_E$:

$$ds^2 \equiv -dt^2 + a(t)^2 \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega_k^2 \right)$$ \hspace{1cm} (62)

(where $d\Omega_k^2$ denotes the line element on the unit sphere of the 3-geometry with constant scalar curvature $k \in -1, 0, +1$). If we write the scalar field in Riemann gauge as $\phi_R \equiv \phi_0 e^{-\sigma(t)}$ like above, the scale connection in Einstein gauge is given by (12), i.e., $\varphi_E = \dot{\sigma}dt$.

The Einstein equation can be brought into the Friedmann form with contributions $\Theta(\sigma) + \Theta^{(V)}$ of the scalar field in (29) on the rhs:

$$\begin{align*}
(i) \quad \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} & \equiv \frac{8\pi \kappa}{3} \rho^{(\bar{m})} + \frac{\Theta(\sigma)_{\theta\theta}}{3} + \frac{\Lambda}{3} \\
(ii) \quad \frac{\ddot{a}}{a} & \equiv \frac{4\pi \kappa}{3} (\rho^{(\bar{m})} + 3p^{(\bar{m})}) - \frac{\Theta(\sigma)}{6} + \frac{\Lambda}{3}
\end{align*}$$ \hspace{1cm} (63)
The Lagrangian (22) of the Milgrom regime with parameters like in section 2, $\alpha = -6, \beta \neq 0, \gamma = 4$ and $k = 0$ is not compatible with the symmetry conditions of the FRW spacetime. This can be seen by a qualitative investigation of the system of differential equations using the method of [36], which is applicable to scalar tensor theories of different types [51], [27, p. 102ff.]. For a non-trivial scalar field, $\dot{\sigma} \neq 0$, the second Friedmann equation (63 (ii)) and the scalar field equation (32) can be translated into a system of two first order differential equations (a vector field $F$ on $\mathbb{R}^2$) in the new variables $y = \dot{a}, z = \dot{\sigma}$. The first Friedmann equation turns then into an algebraic (here cubic) relation $R(y, z) = 0$. The phase diagram of the system $F$ (fig. 2) shows that the flow lines of the present dynamical system are transversal to the algebraic curve, i.e., the Friedmann equations and the scalar field equation are incompatible. For the trivial case, $\dot{\sigma} \equiv 0$, the Riemann gauge and Einstein gauge coincide and an (anti-) de Sitter - Lanczos model, $a(t) = e^{c_1 t + c_2}$ solves the equations.

For $\gamma = 0$, however, and $\alpha = -6, \beta \neq 0, k = 0$ the situation is different. Here we find a flow-line of the vector field along the compact component of the cubic curve (fig. 3 left). The numerical solution of the original equations satisfies the first Friedmann equation only up to numerical errors, indicating the existence of an exact bouncing solution of the dynamical equations for $\gamma = 0$ (fig. 3 right). In any case, a continuation of the Milgrom regime Lagrangian to the cosmological scale does not make sense without a major modification.
Other conformal (Weyl symmetric) approaches to FRW cosmology

Let us have a short look at some scale covariant approaches to cosmology, JBD cosmology [27], the Brazilian version of Weyl geometric (WIST) cosmology [51] or, most recently, the “Weyl symmetric” cosmological models of Bars, Turok, Steinhart et al. with Higgs and gravitational scalar fields [3, 7, 4, 5], here called BST cosmology. This is, of course, a small selection from a much wider field (see, e.g., [20, 38, 39, 31, 32]).

JBD cosmology has been studied since its inception in the paper by Brans and Dicke [14], extensive studies of explicit solutions by Lorentz-Petzold [37], and many other papers. The more recent monographs [27, 16] give an impressive survey of explicit examples and interesting phase space studies for JBD-FRW models. The explicit solutions for the warp function $a(t)$ and the scalar field $\phi(t)$ discussed there are often algebraic with fractional exponents and are not promising for realistic cosmological models. On the other hand, of Tretyakova and (I.D.) Novikov et al. have shown that a realistic looking bouncing solution, including baryonic matter, exists for a negative JBD-parameter $\omega \ll -1$ and a cosmological “constant” $\Lambda$ in the Jordan frame [66].

The Brazilian WIST approach boils down to scale covariantly written Einstein gravity with a minimally coupled scalar field $\phi(x)$. The phase space discussion for cosmological models with different types of potential and the selected explicit solutions studied by Pucheu, Romero et al. [51] show basic features which distinguish WIST cosmology clearly from the JBD case. From the perspective of WdST gravity the minimal coupling of the scalar field, however, and the reduction to Einstein gauge = Riemann gauge let these results appear too special.

That is different for the research program of conformal cosmology pursued
by Bars, Steinhardt, Turok, sometimes with other authors. In this approach a consistent “Weyl symmetric” (their terminology) approach linking up to the standard model of elementary particle physics (SM) is pursued. Initially the authors started investigating two (or more) non-minimally coupled scalar fields \( \phi, s \) motivated by a string or even M-theoretic background in higher dimensions, which was reduced to its “shadow” in 4 dimensions \([3, 6, 7]\). Then they turned towards interpreting one of the scalar fields \( s \) as a real field expression of a scale covariant version of the Higgs field \( H \), essentially its “expectation value” \([3, 5]\),

\[
s(x) = |H(x)| = \left( H^\dagger(x)H(x) \right)^{\frac{1}{2}},
\]

(64)

where in unitary gauge

\[
H(x) = \begin{pmatrix} 0 \\ h(x) \end{pmatrix}, \quad H_0(x) = \begin{pmatrix} 0 \\ h_0(x) \end{pmatrix},
\]

with real valued \( h_0 \) characterizing the ground state, and \( h(x) = h_0(x) + \Delta h(x) \). All constituents are assumed to be scale covariant of weight \(-1\) (here translated into length weights rather than the mass weights preferred in high energy physics). Because the Lagrangian density \( \mathcal{L}_{SM} \) is already “nearly” scale invariant, with the only exception of the Higgs mass, this links up nicely with SM physics, in particular if one assumes a common biquadratic potential for \( \phi \) and \( s \)

\[
\mathcal{L}^{(V-biq)} = -\frac{\lambda_H}{4}(s^2 - (\omega \phi)^2)^2
\]

(65)

and adds a quartic self-coupling of \( \phi \)

\[
\mathcal{L}^{(V)} = \mathcal{L}^{(V-biq)} + \mathcal{L}^{(V_4)} = -\frac{\lambda_H}{4}(s^2 - (\omega \phi)^2)^2 - \frac{\lambda_4}{4} \phi^4.
\]

(66)

Also other authors have argued that such a modification leads to a scale covariant SM-Lagrangian (weight \(-4\)) with scale invariant density \([60, 42]\). Generalizing to “curved spacetime” this may motivate to assume a long-range scalar field \( s = |h| \), a kind of “real Higgs field”. Its ground state value is obtained in the potential minimum,

\[
s = h_0(x) = \omega \phi(x).
\]

(67)

The ground state \( h_0 \) and the Higgs mass \( m_H \) appearing in the “tachyonic” looking mass term,

\[
\frac{m_H^2}{2} H^\dagger(x)H(x) \quad \text{with} \quad m_H^2 = \lambda_H (\omega \phi)^2 \quad \text{from} \quad (65),
\]

are both scale covariant of weight \(-1\). In this perspective, the Higgs field gains mass through its coupling to the gravitational scalar field \( \phi \), while the mass of
the gauge bosons and the elementary fermions is due to their coupling to the Higgs field. The coefficients have to be chosen such that in Einstein gauge $h_0(x) \equiv v$, the electroweak (ew) energy scale. This shows that $\omega$ plays the role of a hierarchy factor between the ew energy and the (reduced) Planck energy,

$$\omega = \frac{v}{E_P} \sim 10^{-16},$$

(68)

$\lambda_H$ is the coefficient between $v$ and the mass/energy $m_H$ of the Higgs boson at the electroweak scale

$$\lambda_H = \frac{m_H}{v} \approx 0.51.$$

According to their background program, BST assume non-minimal coupling not only for the gravitational scalar field $\phi$ but also for the (real) Higgs field $s$,

$$L_{\text{BST}}^{(H)} = (\phi^2 - s^2)R,$$

where $R$ is the Riemannian scalar curvature. They therefore have to postulate conformal coupling of both scalar fields to the Hilbert term. In such a framework of Riemannian based conformal gravity the authors develop an intriguing study of FRW cosmological models with explicit solutions for whole model classes (although often with a potential different from (65)); they establish geodesic completion for most of their models, some of them bouncing, some even “cyclic” etc. [3, 7, 4, 5]. A clue of their study is the considerable change between different gauges. In their approach the Einstein gauge is characterized by $(\phi^2 - s^2) \equiv E \text{ const} = 8\pi \kappa^{-1}$, Higgs gauge by $s \equiv H \text{ const} = v$, the ew energy scale, etc. These investigations are a challenge and an incentive for Weyl geometric studies of cosmology.

**Weyl’s adaptation argument on the measuring gauge reconsidered**

As SM particles acquire mass by their coupling to the Higgs field their masses are proportional to $s(x)$ in this approach. They are constant in the Higgs gauge only. For example, the electron with effective mass $\sqrt{\mu_e}v$ in the SM$^{19}$ has the scale covariant mass $m_e^2 = \mu_e s^2$ $^{20}$ Assuming a scale invariant fine structure constant $\alpha$, the Rydberg constant $R_{\text{ryd}} = \frac{\alpha^2 c}{4\pi \hbar} m_e$ responsible for the spectral frequencies of atoms turns also into a scale covariant quantity of weight -1, with its conventional value in the Higgs gauge$^{21}$

$$R_{\text{ryd}} = \frac{\alpha^2 c}{4\pi \hbar} \sqrt{\mu_e} s \equiv \frac{\alpha^2 c}{4\pi \hbar} \sqrt{\mu_e} v$$

(69)

$^{19}\mu_e \approx 2.1 \cdot 10^{-6}$

$^{20}$Take care not to confound Weyl geometric scaling with the running of mass with the energy scale.

$^{21}$Like $m_e$, the fine structure constant $\alpha$ becomes dependent on the energy scale under field quantization. Here we consider the low energy effective value of $\alpha$ only, and demand its invariance under Weyl rescaling comparable to $c$ and $h$ (cf. footnote 20).
The energy eigenvalues of, e.g., the Balmer series in the hydrogen atom are governed by the Rydberg constant $R_{\text{ryd}}$ and scale with the Higgs field $s$,

$$E_n = -R_{\text{ryd}} \frac{1}{n^2} = -\frac{1}{n^2} \frac{\alpha^2 c}{4\pi\hbar} \sqrt{\mu_e s}, \quad n \in \mathbb{N}. \quad (70)$$

As soon as one considers a scale covariant setting, typical atomic time intervals (“clocks”) become proportional to the reciprocal expectation value of the Higgs field $s^{-1} = |H|^{-1}$ and are constant in the Higgs gauge. This boils down to an adaptation of atomic clocks to the Higgs field. It is similar to, but now better founded than, Weyl’s ad hoc claim of an adaptation of clocks to the (Weylian) scalar curvature of spacetime, proposed during his discussion with Einstein in 1918 (also repeated, among others, in [71, p. 298]). In our context the Higgs gauge is the one in which the ticking of atomic clocks at low energies is directly expressed; it is the measuring gauge of WdST. As long as the Higgs field and the gravitational scalar field are closely linked by the potential (65) the Einstein gauge can just as well be used as a reliable approximation to the measuring gauge (see sec. 2).

**Higgs portal and the gravity sector**

The assumption of conformal coupling of the (real) Higgs field $s$ is not compulsory in WdST; in the light of the standard kinetic term of the SM Higgs field it even seems implausible. In the present framework it would be no problem to implement a long range real valued scalar field $s$ with a standard kinetic term $L^{(\text{kin})} = -\frac{1}{2} D_\mu s D^\mu s$ and $\bar{\alpha} = 1$. Here we confine ourselves to the even simpler assumption that $s$ is a scalar function expressing the expectation value of the Higgs field (64) without a proper field dynamics at the classical level and thus without a kinetic term of its own. The gravitational scalar field $\phi$, on the other hand, couples to the Higgs sector by the common biquadratic potential (65). Spoken metaphorically the gravitational scalar field enters the Higgs portal and connects the gravity sector with the Higgs field [12].

For the cosmological regime we consider the WdST Lagrangian

$$L_{\text{FRW}} = L^{(H)} + L^{(q-\text{kin})} + L^{(V-\text{biq})} + L^V_4 + L^{(m)}, \quad (71)$$

with

$$\alpha = -6, \quad \beta = \gamma = 0. \quad (72)$$

With these parameters the Einstein gauged Lagrangian is

$$L_{\text{FRW}} \propto (8\pi\kappa)^{-1} \left( \frac{R(g)}{2} - 3\nabla(g)_\nu \nabla^\nu \sigma - \Lambda \right) - \frac{\lambda_H}{4} (s^2 - v^2)^2 + L^{(m)}; \quad (73)$$

The “net” dynamical equation for $\phi$ (after subtracting the trace of the Einstein equation) [93] reduces to

$$\phi \partial_\phi L^{(V)} - 4L^{(V)} = tr T^{(m)}. \quad (73)$$
The $V_4$ contributions to the potential (66) on the lhs cancel and only the ones derived from the biquadratic term remain. This leads to the algebraic relation

$$s^2((\omega\phi)^2 - s^2) = -\lambda_H^{-1} \, tr \, T^{(m)}.$$  (74)

In consequence the conformal coupling of the scalar field $\phi$ is here compatible with the presence of a non-vanishing trace of the matter tensor. This is a crucial difference to Einstein gravity. Solving the quadratic equation in $s^2$ with the positive sign we get

$$2s^2 = (\omega\phi)^2 + ((\omega\phi)^4 + 4\lambda_H \, tr \, T^{(m)})^{1/2}.$$  (75)

For pressure-less matter with $-tr \, T^{(m)}$ at the order of magnitude of the critical cosmological density $\rho_{\text{crit}} = \frac{3H_0^2}{8\pi \kappa}$ we have (in the Higgs gauge) $\rho_{\text{crit}}[\hbar^2 c^5] \sim 10^{-11} \, eV \ll v_{\text{ew}}^4$. With (75) this implies

$$\omega^2\phi^2 \approx s^2.$$  (76)

Then $s^2$ is in the potential minimum. In large part of the universe the Higgs gauge and the Einstein gauge are thus approximately the same.

The only exceptions are regions of extremely high matter density close to a cosmological singularity. In such extreme regions the Higgs and Einstein gauges may diverge drastically; then the Einstein gauge can no longer be considered as the metrical gauge. This has important consequences for the physics close to a cosmological singularity, which cannot be pursued further at this occasion.

Two toy models of WdST-cosmology

With the parameters (72) the rhs contribution to the Einstein equation (29) is $\Theta(\sigma)_{00} \equiv -6\dot{\sigma}^2$, $\Theta(\sigma)_{jj} \equiv -2\dot{\sigma}^2 g_{jj}$. For $k = 0$ the Friedmann equations become:

$$\left(\frac{\ddot{a}}{a}\right)^2 \equiv \frac{8\pi \kappa}{6} \rho^{(\text{bar})} - 2\dot{\sigma}^2 + \frac{\Lambda}{3}$$

$$\frac{\ddot{a}}{a} \equiv \frac{4\pi \kappa}{3} (\rho^{(\text{bar})} + 3p^{(\text{bar})}) + 2\dot{\sigma}^2 + \frac{\Lambda}{3}$$

(77)

The scalar field equation (74) is trivialized by the potential condition (76).

In the baryonic vacuum this set of equations has a bouncing solution:

$$a(t) \equiv \frac{c_0}{E} \left(\cosh \left(\frac{\Lambda}{3}(2t + c_1)^{1/2}\right) \right)^{1/2}$$

$$\frac{\dot{a}}{a} \equiv \frac{\sqrt{\frac{\Lambda}{6}}}{E} \left(\cosh \left(\frac{\Lambda}{3}(2t + 3c_1)^{1/2}\right) \right)^{-1}$$

(78)

22 A divergence between Higgs gauge and Einstein gauge may also arise in the BST approach.
One can check that the scalar field energy density \( \Theta_{00}(\phi) \) remains positive \((\geq 0)\), although \( \Theta(\sigma) \) is negative.

If we denote by \( t_0 = 0 \) the present time, by \( t_{\text{min}} \) the time of the minimum (“big compression”) \( c_0 \) and \( c_1 \) can be chosen such that \( a(t_0) = 1 \) and the maximal redshift observable at \( t_0 \) gets an arbitrarily prescribed value \( z_{\text{max}} \).

Clearly this is a purely formal (“toy”) example. The deceleration \( q = -\frac{\ddot{a}}{a\dot{a}^2} \) in this model is \( q(t_0) = -1 \), thus not compatible with the astronomically determined empirical value \( q_0 \approx -0.65 \).

An explicit solution of the equations (77) with baryonic matter cannot be given as easily. A numerical solution with initial conditions at \( t_0 = 0 \),

\[
\begin{align*}
  a(0) &= a_0, \quad \dot{a}(0) = H_0 a_0, \quad \ddot{a}(0) = -q_0 H_0^2 a_0, \quad (79) \\
  a_0 &= 1, \quad H_0 = 7.66 \cdot 10^{-11} \text{[yr}^{-1}] \quad (\sim 75 \text{km} h^{-1} \text{Mpc}^{-1}), \quad q_0 = 0.66 \text{ and for realistic parameters expressing the relative energy densities of matter } \Omega_m \text{ and of the } \Lambda \text{-term (arising from the quartic potential } V_4) \Omega_\Lambda, \\
  \Omega_m &= 0.23, \quad \Omega_\Lambda \approx 0.773, \quad (80)
\end{align*}
\]

shows an intriguing behaviour. It runs extremely close to the present standard \( \Lambda \text{CDM} \) model in the astronomically observable part of the universe until redshift about \( z \approx 10 \), but then bounces back avoiding a cosmological singularity (fig. 5). Note that we have \( \Omega_m + \Omega_\Lambda > 1 \) (with the overshoot equal to \( 2 \dot{\sigma}(0)^2 H_0^{-2} \)). The time of the minimal warp factor \( t_{\text{min}} = -13.7 \cdot 10^9 \text{y} \) is slightly “earlier” than the time of the big bang of the standard model; the maximal redshift for a present observer is \( z_{\text{max}} = z(t_{\text{min}}) \approx 100 \). The energy density of the scalar field remains positive with the exception of a cosmologically “small” time interval about the bounce.

A squeeze factor \( a_0/a(t_{\text{min}}) \approx 100 \) leads to a matter energy density and radiation temperature an order of magnitude below the one necessary for the

\[
\begin{align*}
  c_1 &= (3\Lambda)^{-\frac{1}{2}} \text{arcCosh } ((1 + z_{\text{max}})^2), \quad c_0 = (\cosh(\sqrt{3\Lambda} c_1))^{-\frac{1}{2}} \\
  \text{For } \Lambda = 3 \text{ the Hubble parameter becomes } H(t_0) = \frac{2}{3}(0) = 1. \text{ Then the time unit is the Hubble time } H_0^{-1}; \text{ for other values of } \Lambda \text{ the unit is changed.}
\end{align*}
\]
decoupling of photons \((T = 3.6 \cdot 10^3 \, K)\) assumed in ΛCDM at the surface of last scattering. If we want to explore seriously whether this model can serve as an alternative for understanding our real universe, a different origin of the cosmic microwave background (CMB) has to be found. As a first conjecture one might think of the integrated blue-shifted extra galactic background light emitted at any \(t < t_{\text{min}}\) and concentrating at the throat \((t_{\text{min}})\) (in case the latter thermalizes at the right temperature 273 K). But it will be difficult to judge whether the CMB can be understood as a redshifted picture of such a kind of (Olbers) background radiation at the throat. The answer depends among others, on a new picture of the formation and passing by of galaxies, independent of the present one which has to fight with the problem of bringing the high degree of isotropy of the CMB in agreement with the astronomically observed structures already a “short” time after the assumed big bang.

4 Résumé and discussion

4.1 Résumé

In this paper a generalization of Einstein gravity has been studied. It works in the scale symmetric framework of integrable Weyl geometry (IWG) and assumes a scale covariant scalar field \(\phi\) non-minimally coupled to gravity in addition to baryonic matter. At the moment the most striking effects of this approach can be identified in regions of very weak gravitational fields at the level of galaxies and galaxy clusters (here called Milgrom regime). In this regime the gradient of the scalar field adds to the acceleration of test bodies, while its energy-stress enhances gravitational light deflection (sec. 25 A detailed investigation of the intensity of the extra galactic background light in expanding or static, but not in contracting cosmological models can be found in 63.)
2.2. Similar to Khoury/Berezhiani’s hypothesis of a superfluid theory of dark matter ([10, 11]) the scalar field is assumed to be no longer active in regions where the gradient surpasses a certain threshold; this can be formally expressed by a Lagrangian multiplier term in the action (sec. 3.2). Then the scalar field is inert, i.e., it is constant in the Riemann gauge, and the gravitational dynamics reduces to Einstein’s theory (Einstein regime). The Lagrangian of the Milgrom regime is incompatible with the symmetry assumptions of Friedmann-Robertson-Walker geometry; it cannot be extended to the FRW-cosmological regime without change. In sec. 3.3 a a reduction of the Lagrangian in the Milgrom regime to a conformally coupled scalar field is studied. The latter communicates by a common potential term with a second scale covariant real-valued scalar field $s$ expressing the expectation value of the Higgs field, similar to the proposal by Bars, Turok and Steinhardt ([5, 60]). Realistic assumptions for the parameters and initial conditions lead to a cosmological model with bouncing behaviour. This may shed new light on open questions in present cosmology (sec. 3.3).

The focus of this paper lies, however, in the Milgrom regime in which central features of modified Newtonian dynamics can be derived on the galactic scale. A strong indication that the “Keplerian” laws of galaxy dynamics ([26]) hold for the WdST version of MOND (in the flat space limit) can be found in Hossenfelder/Mistele’s study of the mass acceleration-discrepancy ([35]) in the framework of CEG, which leads to the same interpolation function $\nu$ as ours (section 2).

The present approach shares some characteristics with Jordan-Brans-Dicke theory in the Einstein frame, but differs in two important respects: the particle dynamics follows the trajectories of the Weylian metric, not the ones of the Riemannian metric of the Einstein frame (sec. 1.3); moreover it uses a modified Lagrangian inspired by Bekenstein/Milgrom’s RAQUAL and an additional term of the Lagrangian with coefficient $\gamma$, first proposed in a paper by Novello et al. (sec. 2.2, eqs. (18, 21, 22)). The Einstein equation for the Riemannian part of the Weylian metric in the Einstein frame has a familiar form (29). The scalar field $\phi$ satisfies a differential equations which, also written down in the Einstein gauge, is a general relativistic generalization of the nonlinear Poisson equation for the deep MOND potential (34); therefore it has been named “Milgrom equation”.

The contribution $\Theta^{(\phi)}$ of the scalar field to the total energy momentum has peculiar properties. Most strikingly, the $(0,0)$-component of the semi-trace reduced expression $\Theta^{(\phi)} - \frac{1}{2} tr \Theta^{(\phi)} g$, important for the weak field approximation, vanishes. This leads to the Newton approximation being sourced by baryonic matter only ([12]). As a result the weak field modification $h_{00}$ of the Minkowski metric stands in the same relation to the Newton potential as in Einstein gravity, $h_{00} = -2\Phi_N$. In IWG the trajectories deviate, however, from the Levi-Civita connection of the weak field (Riemannian) metric; an additional contribution to the acceleration ([44]) comes
from the gradient of the scalar field (more precisely from the gradient of the “scalar field potential” \( \sigma \), i.e., the logarithm of the scalar field in the Riemann gauge \( \phi = \frac{1}{R} \log e^\sigma \)). For the weak field dynamics of particles we thus encounter a superposition of the Newton potential and the scalar field potential, \( \Phi_N + \sigma \). Thus the particle dynamics is governed by a superposition of the Newton acceleration and a contribution induced by the scalar field equal to the deep MOND acceleration (sec. 2.3). In the flat space limit this leads to a special case of MOND with “interpolation function” \( \nu(y) = 1 + y^{-\frac{1}{2}} \) in the Milgrom regime.

Next it has been shown that for centrally symmetric constellations the gravitational light refraction stands in agreement with the changed dynamics; i.e., the scalar field potential \( \sigma \) also adds up to the light bending potential of baryonic matter (46). This is an important result which distinguishes the present approach from the original RAQUAL proposal of Milgrom and Bekenstein. It depends crucially on the Brazilian term of the Lagrangian and holds only for \( \gamma = 4 \).

4.2 Discussion and outlook

The present approach is far from a fundamental theory behind MOND. After all it shows that already one scalar fields suffices for deriving the MOND dynamics in weak field constellations at the level of galaxies and clusters. Integrable Weyl geometry (with the scalar field governing the deviation of the Weyl geometric affine connection in the Einstein gauge from its Riemannian contribution) signifies an extremely moderate and convincing change of the Riemannian framework of Einstein gravity. No disformal deformation of the metric (TeVeS), no timelike unit field breaking the local Lorentz invariance at the general level (Einstein aether theory), or any other complicated and/or artificial gadget is necessary. All we need is a Weyl symmetric generalization of Riemannian geometry, which is of advantage for high energy physics anyhow and possibly an important feature for quantizing gravity [49, 30].

Like in other scalar-tensor theories the scalar field shows interesting qualitative features in addition to its more technical properties. Its energy-stress tensor \( \Theta(\phi) = \frac{8\pi G}{c^4} T(\phi) \) appearing on the rhs of the Einstein equation (29) decomposes into two parts, one proportional to the Riemannian metric \( g_E \) like for dark energy, \( \Theta(\text{de}) = \frac{\Lambda(\phi)}{c^4} g_E \) with variable coefficient \( \Lambda \), and another one, \( \Theta(\text{dm}) \), which looks like a “dark fluid” tensor,

\[
\Theta(\phi) = \frac{\Theta(\text{dm})}{c^4} + \Theta(\text{de})
\]

This decomposition should, however, not been taken literally. For the weak field approximation in the Milgrom regime and for \( \gamma = 4 \), e.g., we have found

\footnote{A similar result claimed on a flawed basis in [55] is herewith corrected.}
\[ \Theta^{(\phi)}_{\mu\nu} = 6 \nabla_{(\mu} \sigma^{\nu)} - (2 \nabla^2 \sigma + \Lambda) g_{\mu\nu}, \]

where the “cosmological” contribution \( \Theta^{(V)} = \Lambda g \) may be neglected. In a static weak field approximation this leads to a positive overall energy density of the scalar field \( \rho^{(\sigma)} = \frac{4 \pi \kappa}{E} - 1 \nabla (g_E)^2 \sigma \) which is due only to the “dark energy” contribution \( \Theta^{(de)} \), while the “dark fluid” has vanishing energy density. In the ordinary “dark” language, we might describe this as an energy tensor of the scalar field, the most important (energy carrying) part of which has the form of a generalized dark energy tensor with a superimposed “dark fluid”-like modification of the pressure components. Taken together \( \Theta^{(\phi)} \) results in a modification of Einstein gravity with all the effects usually attributed to particle dark matter. In this sense the scalar field \( \phi \) plays a double role in the present approach: it modifies gravity by its coupling to the Hilbert term and the induced Weylian scale connection in the Einstein gauge, and it contributes to the rhs of the Einstein equation like a peculiar combination of dark matter and dark energy. In the latter role it underpins the analysis in [10] in which an often assumed strict dichotomy between modified gravity and dark matter has been put into question.

Two final remarks on the cosmological level. (i) Also in this approach no direct connection between the Milgrom regime dynamics and cosmology could be established. After all an indirect link between \( a_0 \) and \( \Lambda \) arises from introducing a common cosmological hierarchy factor \( \beta \) \([20]\) and its implementation in the kinetic term of the scalar field \([18]\) and the scale invariant potential term \([9, 24]\).

(ii) It remains an open question, whether the example of the cosmological model \([77]\) (with initial conditions and parameters \([79, 80]\)) can develop into a realistic approach; but it seems worthwhile to check. It is not impossible that this, or a similar model built upon a more refined scalar field approach, may become a game changer for cosmological modelling leading to more awareness of the astronomically accessible part of the universe than in the present cosmological discourse.

5 Appendix

5.1 Energy-stress of the scalar field

In the variation of \( L^{(\phi)} \) the non-minimal coupling of the scalar field results in a term \( \Theta^{(\phi)} \) (in addition to \( Ric - \frac{\Lambda}{2} g \) \([23]\) eg. (2.17), \([16]\) eg. (3.5), \([59]\)
sec. 6.2:
\[ \Theta^{(H)}_{\mu\nu} = \phi^{-2}(D_\mu D_\nu \phi^2 - D_\lambda D^\lambda \phi^2 g_{\mu\nu}) \]
\[ = 2\phi^{-2}(D_\mu \phi D_\nu \phi + \phi D_\mu D_\nu \phi - (\phi D_\lambda D^\lambda \phi + D_\lambda \phi D^\lambda \phi) g_{\mu\nu}) \]
\[ \approx_E -2\nabla_\mu \partial_\nu \sigma + 2\nabla^2 \sigma g_{\mu\nu} \] \hspace{1cm} (81)

with \( \nabla = \nabla(g_\nu) \) in the whole appendix 5.1.

The contribution \( \Theta^{(\phi)} \) of the scalar field to the rhs of the Einstein equation (25) derived from the non-Hilbert terms of the Lagrangian contains the kinetic and the potential contributions,
\[ \Theta^{(\phi'H)} = \Theta^{(\phi'kin)} + \Theta^{(V)} , \] \hspace{1cm} (82)

where in the Milgrom regime with the Lagrangian (22)
\[ \Theta^{(\phi'kin)} = \Theta^{(q'kin)} + \Theta^{(cub)} + \Theta^{(braz)} . \] \hspace{1cm} (83)

\( \Theta^{(X)} \) denotes the contribution of \( X \) to the rhs of the Einstein equation:
\[ \Theta^{(X)} = \phi^{-2} T^{(X)} = -\phi^{-2} \left( \frac{2}{\sqrt{|g|}} \frac{\delta \mathcal{L}^{(X)}}{\delta g} \right) \]

where
\[ \epsilon_\sigma = \begin{cases} +1 & \text{for } \nabla \sigma \text{ spacelike} \\ -1 & \text{for } \nabla \sigma \text{ timelike or null} \end{cases} \] \hspace{1cm} (88)

For \( V = V_4 \) a cosmological constant arises in Einstein gauge (24):
\[ \Theta^{(V_4)}_{\mu\nu} \]
\[ \approx_E -\frac{\lambda H_0^2}{4} \phi^2 g_{\mu\nu} = -\Lambda g_{\mu\nu} , \quad \Lambda = \lambda H_0^2 \] \hspace{1cm} (89)

Tracing the (scale invariant) Einstein equation and multiplying with \(-\phi^2\) leads to
\[ -2L_H - tr T^{(bar)} - (2\gamma + 6) \phi D_\lambda D^\lambda \phi + (\alpha + 6) D_\lambda \phi D^\lambda \phi \]
\[ -L^{(cub)} - 4L^{(V)} = 0 . \] \hspace{1cm} (90)
5.2 The scalar field equation

The scale covariant variation with regard to φ, \( \frac{\delta L}{\delta \phi} = \frac{\partial L}{\partial \phi} - D_\lambda \frac{\partial L}{\partial (D_\lambda \phi)} \), contains the partial contributions [59] app. 6.2:

\[
\frac{\delta L (q-\text{kin})}{\delta \phi} = \alpha D_\lambda D^\lambda \phi \\
\frac{\delta L (\text{cub})}{\delta \phi} = 2\beta \phi^-2 D_\lambda \left( |D\phi| D^\lambda \phi \right) + 4\phi^{-1} L^{(\text{braz})}
\]

In the second line we encounter a scale covariant form of the non-linear modification of the d’Alembert operator typical for relativistic MOND theories.

For \( L^{(\text{braz})} \) it is recommendable to use the Einstein gauge. Because of

\[
\nabla^\lambda(g) \partial_\lambda \sigma = \frac{1}{\sqrt{|g|}} \partial_\lambda (\sqrt{|g|} \partial_\lambda \sigma)
\]

the second order derivative term of \( L^{(\text{braz})} \) in Einstein gauge is a divergence

\[-\frac{\gamma}{4} \phi^2_0 \partial_\lambda (\sqrt{|g|} \partial^\lambda \sigma) . \]

For the variation of \( \phi \) (with fixed \( g \)) its integral can be shifted to a boundary term outside the support of \( \delta \phi \) and does not contribute to the Euler-Lagrange equation of the scalar field. This not the case for the variation \( \delta g \). For the variation \( \delta \phi \) in Einstein gauge only the term \(-\frac{\gamma}{4} \phi^2_0 \partial_\lambda \sigma \partial^\lambda \sigma \) remains as the reduces Brazilian term. Its scale covariant version

\[ L^{\text{braz red}} = -\frac{\gamma}{4} \phi^2_0 D_\lambda \sigma D^\lambda \sigma \]

has the same form as \( L^{(q-\text{kin})} \) and leads to a second degree dynamical equation for \( \phi \) (respectively \( \sigma \)). In terms of \( \phi \):

\[
\frac{\delta L^{(\text{braz})}}{\delta \phi} = 2\gamma D_\lambda D^\lambda \phi . \tag{91}
\]

\( L^{(H)} \) and \( L^{(V_i)} \) are monomials in \( \phi \) with \( \frac{\delta \phi^k}{\delta \phi} = \frac{\partial \phi_k}{\partial \phi} = k \phi^{k-1} \).

After summing up and multiplying with \( \phi \) we arrive at the gross scalar field equation:

\[
2L_H + (\alpha - 2\gamma) \phi D_\lambda D^\lambda \phi + 4L^{(\text{braz})} + 2\beta \phi^{-1} D_\lambda \left( |D\phi| D^\lambda \phi \right) + \phi \partial_\phi L^{(V)} = 0 \tag{92}
\]

\[\text{This has been noted by the authors of [16].}\]
Addition of the traced Einstein equation (90) leads to the scale covariant, (net) scalar field equation (in arbitrary gauge):

\[ 2\beta \phi^{-1} D_\lambda \left( |D\phi|D^\lambda \phi \right) + (\alpha + 6) \left( D_\lambda \phi D^\lambda \phi + \phi D_\lambda D^\lambda \phi \right) + L^{(\text{braz})} + \phi \partial_\phi L(V) - 4L(V) = tr T^{(\text{bar})} \]  \hspace{1cm} (93)

For \( \beta = \gamma = 0 \) and vanishing or quartic potential, \( V = 0 \) or \( V_4 \), this implies the well known constraint \( tr T^{(\text{bar})} = 0 \) for conformal coupling (\( \alpha = -6 \)), not so however for different potentials, e.g. the biquadratic one used in sec. 3.3.

In the \textit{MG regime} (\( \alpha = -6 \)) (93) simplifies to

\[ 2\beta \phi^{-1} D_\lambda \left( |D\phi|D^\lambda \phi \right) + 3L^{(\text{braz})} = tr T^{(\text{bar})} + 4L(V) - \phi \partial_\phi L(V). \]

Multiplying by \( -\frac{1}{2} (\beta^{-1}\phi)^{-2} \) implies the \textit{scale covariant Milgrom equation} of the main text (32),

\[ \mathcal{M}(\phi) = -\frac{1}{2} \phi^{-2} (\beta^{-1}\phi) \left( tr T^{(\text{bar})} + 4L(V) - \phi \partial_\phi L(V) \right), \]

with the the \textit{scale covariant Milgrom operator} on the lhs:

\[ \mathcal{M}(\phi) = -\left( \phi^{-2} D_\lambda \left( |D\phi|D^\lambda \phi \right) - \phi^{-3} |D\phi| \right) \]  \hspace{1cm} (94)

In the Einstein gauge we find \( D_\lambda (|D\phi|D^\lambda \phi) \overset{E}{=} -\phi_0^2 D_\lambda (|\nabla\sigma|\partial^\lambda \sigma) \) and \( \phi \overset{E}{=} \phi_0 \). For a perfect fluid (energy density \( \rho_{\text{bar}} \) and pressure \( p_{\text{bar}} \)) and \( V = V_4 \):

\[ D_\lambda \left( |\nabla\sigma|\partial^\lambda \sigma \right) - \epsilon_\sigma |\nabla\sigma|^3 \overset{E}{=} \frac{1}{2} \phi_0^{-2} (\beta^{-1}\phi_0) \left( \rho^{(\text{bar})} - 3p^{(\text{bar})} \right). \]

The cubic term on the rhs cancels against the one in\footnote{\( |\nabla\sigma| = \epsilon_\sigma \partial_\sigma \partial^\sigma \) and therefore \( w(|\nabla\sigma|) = -2 \), so we get

\[ D_\lambda (|\nabla\sigma|\partial^\lambda \sigma) = \frac{s}{\lambda} (|\nabla\sigma|\partial^\lambda \sigma) + \frac{1}{\lambda} \epsilon_\sigma |\nabla\sigma|\partial^\sigma \sigma - 3\epsilon_\sigma |\nabla\sigma|\partial^\lambda \sigma = \frac{s}{\lambda} (|\nabla\sigma|\partial^\lambda \sigma) + \epsilon_\sigma |\nabla\sigma|\partial^\lambda \sigma \]

and

\[ \nabla (g_\lambda) \left( |\nabla\sigma|\partial^\lambda \sigma \right) \overset{E}{=} 4\pi \kappa_0 a_0 (\rho - 3p^{(\text{bar})}), \]  \hspace{1cm} (95)}

and the scalar field equation in the Milgrom regime simplifies to

\[ \nabla (g_\lambda) \left( |\nabla\sigma|\partial^\lambda \sigma \right) \overset{E}{=} 4\pi \kappa_0 a_0 (\rho - 3p^{(\text{bar})}), \]  \hspace{1cm} (95)

with the Einstein gauged \textit{covariant Milgrom operator} for a scalar field \( X \)

\[ \mathcal{M}_E(X) \overset{E}{=} \nabla (g_\lambda) \left( |\nabla X|\partial^\lambda X \right) \]  \hspace{1cm} (96)

on the lhs. For the flat metric and static fields this is the non-linear Laplace operator of classical MOND theory \( \nabla_j (|\nabla X|\partial^j X) \) (with the Euclidean \( \nabla \) operator).
5.3 The weak field approximation for \( g \) in the Milgrom regime

The weak field approximation in the Milgrom regime considers a metric \( g \equiv E \eta + h \) where only first order terms in \( h, h', h'', \sigma, \sigma'' \) are considered. The equality up to first order is denoted by \( \equiv \).

Here we concentrate on the (quasi-)static central symmetric case with conformal spherical coordinates and Weylian metric

\[
\frac{ds^2}{E} = -A(r) \, dt^2 + B(r) \, (dr^2 + r^2 d\Omega^2) , \quad \varphi \equiv \frac{d\sigma(r)}{E} = \sigma'(r) \, dr .
\]

Other cases like the case of cylindrical symmetry can be treated similarly. Here we have \( \eta = \text{diag}(-1, 1, r^2, r^2 \sin^2 x_2) \), \( h = \text{diag}(h_{00}, h_{11}, h_{11} r^2, h_{11} r^2 \sin^2 x_2) \) and \( A = 1 - h_{00}, B = 1 + h_{11} \), where “\( \equiv \)” stands for \( \equiv \).

Important are the following Levi-Civita connection coefficients

\[
\begin{align*}
\Gamma(g)^1_{00} &= \frac{A'}{2B} \frac{h_{00}'}{2}, \quad \Gamma(g)^1_{11} = \frac{B'}{2B} \frac{h_{11}'}{2}, \quad \\
\Gamma(g)^1_{22} &= -r(1 + \frac{B'}{2B}) \frac{1}{1} - r(1 + \frac{r}{2} h_{11}'), \quad \\
\Gamma(g)^2_{00} &= \frac{A'}{2AB} + \frac{2}{rB} + \frac{B'}{2B^2} \frac{f'}{r} = f'' + \left( -\frac{h_{00}'}{2} + \frac{h_{11}'}{2} + \frac{2}{r} f' \right), \quad \\
\Gamma(g)^2_{11} &= \frac{B'^2}{2B^2} + \frac{B'}{2B} - \frac{B'}{rB} - \frac{B''}{2B^2} + \frac{B''}{rB} \frac{1}{2}, \\
\Gamma(g)^2_{22} &= -r \left( \frac{A'}{2A} + \frac{r A'}{2AB} - \frac{B'^2}{4B^2} + \frac{3B'}{2B} + \frac{r B''}{2AB} \right) \frac{1}{2} \left( \frac{h_{00}'}{2} - \frac{3h_{11}'}{2r} - \frac{h_{11}''}{2} \right) r^2 .
\end{align*}
\]

the d’Alembert operators

\[
\begin{align*}
\nabla (g)^2 f &= \frac{f''}{B} + \left( \frac{A'}{2AB} + \frac{2}{rB} + \frac{B'}{2B^2} \right) f' \frac{1}{r} = f'' + \left( -\frac{h_{00}'}{2} + \frac{h_{11}'}{2} + \frac{2}{r} f' \right), \\
\nabla (\eta)^2 f &= f'' + \frac{2}{r} f', \\
\end{align*}
\]

and the Ricci tensor components:

\[
\begin{align*}
R_{00}(g) &= -\frac{A^2}{4AB} + \frac{A'B'}{4AB^2} + \frac{A'}{rB} + \frac{A''}{2B^2} \frac{1}{2} = -\frac{1}{2} \left( \frac{h_{00}''}{r} + \frac{2h_{00}'}{r} \right) \frac{1}{2} = -\frac{1}{2} \nabla (\eta)^2 h_{00}, \\
R_{11}(g) &= \frac{A^2}{4A^2} + \frac{A'B'}{4AB} + \frac{A''}{2AB} - \frac{B'^2}{2B^2} - \frac{B'}{rB} - \frac{B''}{2B^2} \frac{1}{2} = \frac{h_{00}''}{r} - \frac{h_{11}'}{r} - \frac{h_{11}''}{2}, \\
R_{22}(g) &= -r \left( \frac{A'}{2A} + \frac{r A'}{2AB} - \frac{B'^2}{4B^2} + \frac{3B'}{2B} + \frac{r B''}{2AB} \right) \frac{1}{2} \left( \frac{h_{00}'}{2} - \frac{3h_{11}'}{2r} - \frac{h_{11}''}{2} \right) r^2 .
\end{align*}
\]

Consider the half-trace (times \( g \)) reduced Einstein equation\(^{[29]} \)

\[
Ric(g) \equiv \frac{1}{E} (8\pi \kappa) (T^{(\text{bar})} - \frac{1}{2} tr T^{(\text{bar})} \, g) + \Theta(\sigma) - \frac{1}{2} tr \, \Theta(\sigma) \, g .
\]

The terms on the rhs will be called the reduced energy tensors of baryonic matter and of the scalar field:

\[
T^{(b, \text{red})} = T^{(\text{bar})} - \frac{1}{2} T^{(\text{bar})} \, g , \quad \Theta(\sigma)^{(\text{red})} = \Theta(\sigma) - \frac{1}{2} tr \, \Theta(\sigma) \, g .
\]

\(^{[29]} \)The contribution of \( \Theta^{(V_4)} \) is cosmologically small (\( \Lambda g \)) and therefore negligible.
For pressure-free baryonic matter with \( T^{(\text{bar})}_{00} = \rho^{(\text{bar})} \) the reduced energy tensor has components \( T^{(\text{red})}_{00} = \frac{1}{2} \rho^{(\text{bar})} \) and \( T^{(\text{bar})}_{jj} = \frac{1}{2} \rho^{(\text{bar})} g_{jj} \). For the scalar field the (semi-trace) reduction of (31) leads to

\[
\Theta(\sigma)^{(\text{red})}_{\mu\nu} = \frac{1}{2} \gamma \nabla(\eta)(\mu \partial_\nu)\sigma ,
\]

(97)

with components

\[
\Theta(\sigma)^{(\text{red})}_{00} = 0, \quad \Theta(\sigma)^{(\text{red})}_{11} = 2\gamma \sigma'' , \quad \Theta(\sigma)^{(\text{red})}_{22} = 2\gamma \sigma' .
\]

This shows that the scalar field does not contribute to the energy component of the reduced Einstein equation. Note that (97) is independent of the assumption of central symmetry and the energy component of \( \Theta(\sigma) \) vanishes in the static case.

We therefore get

\[
R_{00}(g) = -\frac{1}{2} \nabla(\eta)^2 h_{00} = 4\pi \kappa \rho^{(\text{bar})} ,
\]

(98)

exactly like in Einstein gravity independent of central symmetry. The usual identification

\[
h_{00} = -2\Phi^{(\text{bar})}_N
\]

(99)

leads then to the well-known Newton approximation.

Remember, however, that in our framework the point particles do not follow the Levi-Civita connection of \( g \) (approximated by the Newton acceleration of baryonic matter), but are subject to an additional acceleration derived from the scalar field potential \( \sigma \) (see sec. 1.4).

In the (11) and (22) components of the reduced Einstein equation the scalar field becomes visible:

\[
R_{11} = \frac{h_{00}'}{2} - \frac{h_{11}'}{r} - \frac{h''_{11}}{r} = 4\pi \kappa \rho^{(\text{bar})} + 2\gamma \sigma''
\]

\[
R_{22} = \left( \frac{h_{00}'}{2r} - \frac{3h_{11}'}{2r} - \frac{h_{11}'}{2r} \right) r^2 = (4\pi \kappa \rho^{(\text{bar})} + 2\gamma \sigma') r^2
\]

This leads to

\[
R_{11}(g) + \frac{2}{r^2} R_{22}(g) = \frac{1}{2} \nabla(\eta)^2 h_{00} - 2 \nabla(\eta)^2 h_{11} = 12\pi \kappa \rho^{(\text{bar})} + 2\gamma \nabla(\eta)^2 \sigma ,
\]

from which

\[
2 \nabla(\eta)^2 h_{11} = -4 \nabla(\eta)^2 \Phi^{(\text{bar})}_N - 2\gamma \nabla(\eta)^2 \sigma
\]

and finally

\[
h_{11} = -2 \left( \Phi^{(\text{bar})}_N + \frac{\gamma}{2} \sigma \right).
\]

(100)
With $\gamma = 4$ the gravitational refraction is identical to the relativistic refraction induced by a (Newtonian) potential $\Phi_{N}^{(\text{bar})} + \sigma$ and thus in agreement with the acceleration of test particles (see sec. 2.3).

The weak field approximation of the Riemannian metric in the central symmetric vacuum case with central mass $M$ is:

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 + \frac{2M}{r} - 4\sqrt{a_1M \log \frac{r}{r_0}})(dr^2 + r^2(d\Omega^2)) \quad (101)$$

It fits well to the Schwarzschild metric in the Einstein regime and can be glued to the latter by the smooth transition function (58).

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