The Phase of Neutrino Oscillations

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Abstract

Using an analogy with the well-known double-slit experiment, we show that the standard phase of neutrino oscillations is correct, refuting recent claims of a factor of two correction. We also improve the wave packet treatment of neutrino oscillations taking into account explicitly the finite coherence time of the detection process.

The experimental and theoretical investigation of neutrino oscillations is presently one of the most active fields of research in high-energy physics. For the interpretation of experimental data it is important to have a clear and correct theoretical formulation of the neutrino oscillation mechanism.\(^1\) In particular, the phase \(\Phi_{kj}\) of neutrino oscillations in vacuum due to the interference of the contributions of two massive neutrinos \(v_k\) and \(v_j\) with masses \(m_k\) and \(m_j\), respectively, depends on the mass-squared differences \(\Delta m^2_{kj} \equiv m_k^2 - m_j^2\). A correct theoretical expression of the oscillation phase \(\Phi_{kj}\) is important in order to extract from the experimental data correct information on the value of \(\Delta m^2_{kj}\).

The standard expression for the oscillation phase \(\Phi_{kj}\) in the relativistic approximation is

\[
\Phi_{kj} = -\frac{\Delta m^2_{kj} L}{2E},
\]

where \(E\) is the neutrino energy and \(L\) is the distance between the neutrino source and detector.

It has been argued by some authors [3,4] that the expression (1) is wrong by a factor of two on the basis of the following reasoning. Neutrino experiments measure neutrino oscillations as a function of the source-detector distance \(L\). Different massive neutrinos propagate with different velocities

\[
v_k = \frac{p_k}{E_k},
\]

where \(E_k\) and \(p_k\) are, respectively, the energy and momentum of the neutrino with mass \(m_k\), related by the relativistic dispersion relation

\[
E_k^2 = p_k^2 + m_k^2.
\]

Hence, the phases of the different massive neutrinos wave functions after a propagation distance \(L\) should take into account the different times of propagation of different massive neutrinos:

\[
\Phi_k = p_k L - E_k t_k.
\]

The different propagation times are given by

\[
t_k = \frac{L}{v_k} = \frac{E_k}{p_k} L, \quad \text{Eq. (5)}
\]

which lead, in the relativistic approximation, to the phase difference

\[
\Delta \Phi_{kj} = \Phi_k - \Phi_j = \frac{-\Delta m^2_{kj} L}{E}.
\]

This phase difference is twice the standard one in Eq. (1). A similar disagreement by a factor of two has been claimed to exist in the case of kaon oscillations in Refs [5–7].

Let us notice that in Eq. (4) we have considered the possibility of different energies and momenta for different massive neutrino wave functions. Indeed, it has been shown in Ref. [8] that Lorentz invariance implies that in general in oscillation experiments different massive neutrinos have different energies and different momenta. The energy \(E\) in Eqs (1) and (6) is the energy of the massive neutrinos in the massless limit, i.e. neglecting the differences of energy due to the masses.

The authors of Refs [9–11] claimed that a correct way to obtain the standard oscillation phase is to assume the same energy for the different massive neutrino wave functions. This is an unphysical assumption, as discussed in Ref. [8]. Moreover, as already noticed in Ref. [3], it is not true that the disagreement of a factor of two disappears assuming the same energy for the different massive neutrino wave functions, as clearly shown by the above derivation of Eq. (6), in which the energies of the different massive neutrino wave functions could have been taken to be equal. Indeed, even if the different massive neutrino wave functions have the same energy, the time contribution \(-Et_k + Et_j\) to the phase difference \(\Delta \Phi_{kj}\) does not disappear, because \(t_k \neq t_j\). This contribution has been missed in Refs [9–11].

In order to test the validity of the reasoning of the authors of Refs [3,4], let us apply their method to the well-known double-slit interference experiment depicted in Fig. 1. The particles emitted by the source \(S\) and detected could be photons or electrons, or others. The screen \(A\) has two holes through which the particles can reach the screen \(B\) on which a detector registers the arrival of the particles at a distance \(x\) from the center of the screen.

In the standard approach, the phases of the two waves at \(x\) at the time \(t\) are given by

\[
\Phi_k = p_k x - E t = p(r_A + r_{Bk}) - E t, \quad \text{Eq. (7)}
\]

for \(k = 1, 2\). Hence, the phase difference is

\[
\Delta \Phi = \Phi_2 - \Phi_1 = p \Delta r, \quad \text{Eq. (8)}
\]

\(^1\)See Ref. [1] for a review of the theory of neutrino oscillations and Ref. [2] for an exhaustive list of references.
with
\[ \Delta \tau = r_2 - r_1 = r_{B2} - r_{B1} \approx \frac{2a}{d}. \] (9)

where the approximation holds for \( x, a \ll d \). Since \( p = 2\pi /\lambda \), where \( \lambda \) is the wavelength, the maxima of interference are given by the usual well known formula \( x = n\lambda d/2a \), with \( n = 0, 1, \ldots \) which has been confirmed by many experiments without any doubt.

Let us apply now the reasoning of the authors of Refs [3,4]. If one takes into account the velocity of the particles, the two paths have different propagation times
\[ t_k = \frac{r_k}{v}, \] (10)

where \( v \) is the velocity,
\[ v = \frac{p}{E}. \] (11)

In this approach, the phases of the two waves at \( x \) are given by
\[ \Phi_k = pr_k - E t_k. \] (12)

The phase difference at \( x \) turns out to be
\[ \Delta \Phi = \Phi_2 - \Phi_1 = \left( p - \frac{E}{v} \right) \Delta \tau \simeq -2a \frac{E^2}{d} x \frac{m^2}{p}, \] (13)

where the approximation holds for \( x, a \ll d \). This phase difference is very different from the correct one in Eq. (8), and it even vanishes for massless particles (as photons).

Therefore, the reasoning presented in Refs [3,4] is wrong.

The mistake in the claim formulated in Refs [3,4] is due to a wrong use of the group velocity in the phase, which depends on the phase velocity. The group velocity (given in Eq. (2) for neutrinos and in Eq. (11) for the double-slit experiment) is the velocity of the factor that modulates the amplitude of a wave packet describing a localized particle. In the double-slit experiment there are two wave packets which propagate along the two paths in Fig. 1. The envelopes of these wave packets take different times to cover the two different distances from the source \( S \) to the point \( x \) on the screen. But this has no effect on the phases.

Only the amplitude of the final wave function is determined by the modulating factors of the amplitudes of the two wave packets. The different arrival times of the envelopes of the wave packets reduces the overlap of the two wave packets, leading to a decoherence effect.

Let us illustrate these concepts in a one-dimensional formalism using a Gaussian momentum distribution with width \( \sigma_p \), centered at the momentum \( p \):
\[ \psi(p') = \left( \frac{\sqrt{2\pi} \sigma_p}{2} \right)^{-1/2} \exp \left[ -\frac{(p' - p)^2}{4\sigma_p^2} \right]. \] (14)

For \( \sigma_p \ll p \), the corresponding wave packet in space-time is
\[ \psi(r, t) = \left( \frac{\sqrt{2\pi} \sigma_r}{2} \right)^{-1/2} \exp \left[ -\frac{(r - vt)^2}{4\sigma_r^2} \right], \] (15)

where \( \sigma_r = 1/2\sigma_p, E = \sqrt{p^2 + m^2} \), and \( v = p/E \) is the velocity of the envelope
\[ \left( \frac{\sqrt{2\pi} \sigma_r}{2} \right)^{-1/2} \exp \left[ -\frac{(r - vt)^2}{4\sigma_r^2} \right] \] (16)

of the wave packet. The real part of the wave packet (15) at \( t = 0 \) is depicted by the solid line in Fig. 2, with the envelope (16) represented by the dashed line. It is important to understand that the wave packet in the approximation (15) is a monochromatic wave whose amplitude is modulated by the envelope factor (16), that has no effect on the phase factor
\[ \exp[ipr - iEt] \] (17)

of the wave packet.

Fig. 2. Real part of the wave packet (15) at \( t = 0 \). The dashed line represents the envelope (16).
The amplitudes of each of the two wave packets traveling the two paths in the double-slit experiment in Fig. 1 at the distance \( x \) from the center of screen \( B \) at the time \( t \) is given by

\[
\psi(r_k, t) = \frac{1}{\sqrt{2}} \left( 2\pi \sigma_r \right)^{-1/2} \exp \left[ i \sqrt{\frac{1}{2} (r_k - v t)^2} \right] e^{-i E_k t - \frac{(r_k - v t)^2}{4\sigma_r^2}},
\]

for \( k = 1, 2 \), with \( r_k = r_k^i + r_k^b \). The probability to detect the particle at the point \( x \) at the time \( t \) is

\[
P(x, t) \propto |\psi(r_1, t) + \psi(r_2, t)|^2,
\]

\[
\approx \frac{1}{2 \sqrt{2\pi \sigma_r}} \left\{ \exp \left[ -\frac{(r_1 - v t)^2}{2\sigma_r^2} \right] + \exp \left[ -\frac{(r_2 - v t)^2}{2\sigma_r^2} \right] + 2 \cos(p \Delta r) \exp \left[ -\frac{(r_1 - v t)^2 + (r_2 - v t)^2}{4\sigma_r^2} \right] \right\},
\]

where we have omitted an appropriate normalization factor, that must be inserted in order to normalize the probability over the screen.

From Eq. (19) it is clear that the probability to detect the particle at \( x \) is appreciable only when one of the two wave packets cover the detector. Interference of the two wave functions is possible if both wave packets cover the detector simultaneously. However, usually the detector operates for a long time without recording the precise instant of detection. In this case, the probability of detection at \( x \) is given by the time average of \( P(x, t) \) in Eq. (19):

\[
P(x) \propto \frac{1}{2} \left\{ 1 + \cos(p \Delta r) \exp \left[ -\frac{(r_1 - v_{1D} t)^2}{2\sigma_{1D}^2} \right] + \exp \left[ -\frac{(r_2 - v_{1D} t)^2}{2\sigma_{1D}^2} \right] + 2 \cos(p \Delta r) \exp \left[ -\frac{(r_1 - v_{1D} t)^2 + (r_2 - v_{1D} t)^2}{4\sigma_{1D}^2} \right] \right\}.
\]

This equation has the same structure as Eq. (19), with the width \( \sigma_r \) of the wave packets replaced by \( \sigma_r \). Obviously, the average over the unmeasured detection time \( t_D \) leads to an expression for the probability to detect the particle at \( x \) given by Eq. (20) with \( \sigma_r \) replaced by \( \sigma_r \), which takes into account also the coherence time of the detection process.

Therefore, we have seen that the coherence time of the detection process, which allows a coherent absorption of wave packets arriving at the detector at different times, do not have any effect on the phase of the interference between different wave functions, refuting the arguments presented in Refs [3,4]. The resulting prescription in calculations performed in the plane wave approximation (i.e. neglecting the wave packet character of wave functions that describe localized particles) is to calculate the interference of different wave functions at the same time and the same space point [13,14].

The physical explanation of the independence of the phase of the interference between different wave functions from their arrival time at the detector has been presented in Ref. [15]. It consists in taking into account that there is interference only if the different wave functions are detected coherently. Coherence means that there is a precise phase difference of the detection process between the arrival times of the different wave functions that must be taken into
account. When the second wave packet arrives, the detection process is already excited with a frequency due to its interaction with the first wave packet and its phase is determined by the time difference between the arrivals of the two wave packets. It is easy to see [15] that the phase of the detection process cancels exactly the phase difference of the two wave functions due to the different detecting times, leading to the practical prescription to calculate the interference of different wave functions at the same time and the same space point.

Let us now apply the wave packet formalism to neutrino oscillations, following the lines presented in Refs [16,17]. The Gaussian wave packets that describe massive neutrinos in one spatial dimension \(x\) along the source-detector direction are

\[
\psi_k(x, t) = \left( \frac{2\pi \sigma_{x,p}}{i\hbar} \right)^{-1/2} \exp \left[ \frac{i p_k x - i E_k t - \left( \frac{x - v_k t}{4\sigma_{x,p}} \right)^2}{4\sigma_{x,p}} \right],
\]

(26)

where \(\sigma_{x,p}\) is the width of the wave packets, that is determined by the production process, \(E_k\) and \(p_k\) are, respectively, the energy and momentum of the neutrino with mass \(m_k\), related by the relativistic dispersion relation (3), and \(v_k\) is the group velocity given in Eq. (2).

Taking into account the possibility that the detection process has a finite coherence time interval \(\sigma_{x,D}\) and a finite spatial coherence width \(\sigma_{x,D}\), we describe the detection process of the massive neutrino \(v_k\) with the Gaussian wave packet

\[
\psi_{Dk}(x, t, L, T) \propto \exp \left[ \frac{ip_k(x - L) - iE_k(t - T)}{4\sigma_{x,D}^2} - \frac{(x - L)^2}{4\sigma_{x,D}^2} - \frac{(t - T)^2}{4\sigma_{x,D}^2} \right],
\]

(27)

where \(L\) is the source-detector distance and \(T\) is the time elapsed between neutrino production and detection. Notice that with respect to the wave packet (21), in Eq. (27) we have considered also a finite spatial coherence width of the detection process that was neglected in the discussion of the double-slit experiment. On the other hand, with respect to the wave packet treatment of neutrino oscillations presented in Ref. [17] we have added the explicit terms that take into account the finite coherence time interval \(\sigma_{x,D}\) of the detection process.

The frequency and wave number of the detection process are the same as those of the detected massive neutrino wave function, which excites the corresponding degree of freedom of the detection process. In the framework of quantum field theory the detection process is described in terms of the leptonic weak charged current

\[
J^\nu(x) = \sum_{2\to\ell, \mu, \tau} \sum_k \bar{\nu}_k(x) j^\nu(1 - \gamma^5) U_{\ell \nu_k}(x),
\]

(28)

where \(U\) is the lepton mixing matrix (see [18–21]). Only the field \(\nu_k(x)\) is excited by the arrival of the wave function of the corresponding massive neutrino \(v_k\), and the frequency and wave number of the field excitations are the same as those of the incoming wave function.

The amplitude of detection of each massive neutrino wave packet is given by the overlap of the incoming neutrino wave packet with the detection process wave packet:

\[
A_k(L, T) \propto \int dx \, dt \psi_{Dk}(x, t, L, T) \psi_k(x, t).
\]

(29)

Performing the integration over \(x\) and \(t\) we obtain

\[
A_k(L, T) \propto \exp \left[ i p_k L - i E_k t - \frac{L - v_k t}{4\sigma_{x,P}^2} \right].
\]

(30)

with

\[
\sigma_{x,k}^2 \equiv \sigma_{x,p}^2 + \sigma_{x,D}^2 + v_k^2 \sigma_{\delta}^2.
\]

(31)

This result is simple and important. It shows clearly that the coherent absorption of each massive neutrino wave function occurs in a space-time interval of width \(\sigma_{x,k}\) around the coordinates \(L, T\) of the detection process. The size of this space-time interval is determined by the width of the neutrino wave packet and the coherent spatial and temporal widths of the detection process. The largest width dominates. The contribution of the temporal width of the detection process is weighted by the velocity of the incoming neutrino wave packet for obvious reasons. Since this velocity depends on the mass of the neutrinos, the width \(\sigma_{x,k}\) depends on the index \(k\) labeling massive neutrinos. However, for relativistic neutrinos the contribution of neutrino mass to \(\sigma_{x,k}\) is negligible and we can safely approximate

\[
\sigma_{x,k}^2 \approx \sigma_{x,p}^2 + v_k^2 \sigma_{x}^2.
\]

(32)

Notice that, on the other hand, the contribution of neutrino mass to the other terms in Eq. (30) cannot be neglected because it is amplified by the macroscopic quantities \(L\) and \(T\). In the following we use the relativistic approximation, which implies that

\[
E_k \approx E + \frac{m_k^2}{2E}, \quad p_k \approx E - (1 - \gamma) \frac{m_k^2}{2E}, \quad v_k \approx 1 - \frac{m_k^2}{2E},
\]

(33)

where \(E\) is the neutrino energy in the limit of zero mass. The first order correction to the momentum and energy due to neutrino mass \(m_k\) is proportional to \(m_k^2\) because of the relativistic dispersion relation (3), and must be divided by the energy \(E\) for dimensional reasons. The coefficient \(\gamma\) depends on the production process, but we will see that it has no effect on the phase of neutrino oscillations, as already shown in Refs [15–17].

The probability of flavor transitions is given by (see [18–21])

\[
P_{\nu_\ell \to \nu_k}(L, T) \propto \sum_k \left| U_{\ell \nu_k} A_k(L, T) U_{\nu k} \right|^2,
\]

(34)

where \(U\) is the lepton mixing matrix. Performing the average of \(P_{\nu_\ell \to \nu_k}(L, T)\) over the unmeasured neutrino propagation time \(T\), we finally obtain

\[
P_{\nu_\ell \to \nu_k}(L) = \sum_k \left| U_{\nu_k} U_{\nu k} \right|^2 + 2 \text{Re} \sum_{k \neq j} U_{\nu_k} U_{\nu j} U_{\nu k} U_{\nu j} F_{kj}
\]

\[
\times \exp \left[ -i \frac{\Delta m_{\nu_k}^2 L}{2E_k} \left( \frac{L}{L_{\text{coh}}^k} \right)^2 \right],
\]

(35)
where
\[ L_{\text{coh}}^{\text{co}} = \frac{4\sqrt{2}E^2}{|\Delta m_{\text{kg}}^2|} \sigma_x \]  
are the coherence lengths and
\[ F_{ij} = \exp \left[ -2\pi^2 \xi^2 \left( \frac{\sigma_x}{L_{\text{coh}}^{\text{co}}} \right)^2 \right], \]
with the oscillation lengths
\[ L_{\text{osc}}^{\text{co}} = \frac{4\pi E}{|\Delta m_{\text{kg}}^2|}. \]

The coherence length \( L_{\text{coh}}^{\text{co}} \) is the distance beyond which the contributions of the massive neutrinos \( v_k \) and \( v_j \) do not interfere any more because the separation between their wave packets is larger than the size of the wave packets and the spatial and temporal coherence widths of the detection process. The coefficient \( F_{ij} \) suppresses the interference of \( v_k \) and \( v_j \) if \( \sigma_x \gtrsim L_{\text{coh}}^{\text{co}} \), i.e. if the production or the detection process is not localized in a space-time region much smaller than the oscillation length. This obvious constraint for the observation of neutrino oscillations was discussed for the first time in Ref. [22] and is satisfied in all experiments. Therefore, in practice one can safely approximate \( F_{ij} \approx 1 \).

From Eq. (35) one can see that a proper wave packet treatment of neutrino oscillations that takes into account the different propagation times of massive neutrino wave packets leads to the standard oscillation phase (1), refuting the disagreement of a factor of two claimed in Refs [3,4]. Neutrino oscillations are due to the different phase velocities of different massive neutrinos, which produces interference. This interference is obviously the same whether it is calculated in the plane wave approach or in the wave packet treatment. The different conclusion reached in Refs [3,4] is due to a wrong use of the group velocity in the phase.

In conclusion, using an analogy with the well-known double-slit experiment we have shown that the standard phase (1) of neutrino oscillations is correct, refuting the claim of a factor of two correction presented in Refs [3,4]. We have also improved the wave packet treatment of neutrino oscillations presented in Refs [16, 17], taking into account explicitly the finite coherence time of the detection process.

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