Semi-analytical solutions to consolidation for unsaturated soils by vertical drains under arbitrary time-dependent loading

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Abstract. This paper presents general semi-analytical solutions to equal strain consolidation for unsaturated soil layer by vertical drains, and the well resistance is considered. Firstly, the coupled governing equations are transformed into ordinary differential equations (ODEs) by Laplace transform and introducing two new variables, and combined with boundary conditions to obtain solutions. On this basis, the semi-analytical solutions in the time domain are achieved by performing inverse Laplace transform with Crump’s method. Secondly, the correctness of the semi-analytical solutions are verified by comparison with the existing analytical solutions of radial consolidation for unsaturated soils under constant loading. Finally, the effect of well resistance factor, permeability coefficient ratio $k_a/k_w$ and load parameter on the average degree of consolidation was analyzed under exponential loading.

1. Introduction

At present, the traditional calculation method of saturated soils is usually used to predict the consolidation settlement of unsaturated soils subgrade. But in fact, the existence range of real saturated soil is limited. In the subgrade near the surface, due to the evaporation and transpiration of water, the subgrade is not fully saturated, which is especially common in arid and semi-arid areas. Therefore, there has been an essential need to study the consolidation characteristics of unsaturated soils near the ground.

In recent years, research on the problems related to the consolidation for unsaturated soils has made some progress. The typical one is the solution of the one-dimensional unsaturated soil consolidation problem proposed with the double-stress variable theory by Fredlund and Hanson [1]. On the basis of this theory, for the consolidation of one dimensional (1D) unsaturated soil, Qin et al. [2] obtained semi analytical solutions by Laplace transform and Cayley Hamilton method. Shan et al. [3] studied the analytical solution for different boundary conditions by separating variables, while Zhou et al. [4] introduced the analytical solution under different initial conditions and boundary conditions by difference integral method (DQM). In addition, Wang et al. [5, 6] made further improvement on the consolidation problem of unsaturated soil, and presented the semi analytical solutions of one-dimensional (1D) and two-dimensional (2D) consolidation problem of unsaturated soil. For the application of the axisymmetric consolidation model in unsaturated soil, Qin et al.[7] first proposed a semi analytical solution for the drainage consolidation of unsaturated soil through an ideal sand well. Then, Zhou et al. [8] solved this problem by differential quadrature method (DQM). In addition, Ho et al. [9] and Zhou et al. [10] used equal strain assumption to get the solution of consolidation for unsaturated soil by vertical drains. In practical engineering, the load usually changes with time, but there is no research on the consolidation of unsaturated soil layer by vertical drain under arbitrary time-dependent loading.
In this paper, on the basis of Fredlund’s one-dimensional consolidation theory of unsaturated soils, the semi-analytical solutions to equal strain consolidation for unsaturated soils by vertical drains considering well resistance under arbitrary time-dependent loading are derived. The Laplace transform technique and the method of introducing two new variables are used to solve the proposed solution. Then, the dissipation of the average excess pore pressure is compared with the existing analytical solution [10]. And the exponential loading is selected to study the influence of well resistance factor, permeability coefficient ratio $k_a/k_w$ and load parameter on the average degree of consolidation.

2. Mathematical model

2.1 Governing Equations

Figure 1 shows the details of consolidation modeling to unsaturated soil by a vertical drain considering well resistance. Dimensions of the unsaturated soil subgrade system include the thickness of unsaturated soil layer $H$ (m), the radius of vertical drain $r_w$ (m) and the radius of influence zone $r_e$ (m). The air and water permeability coefficients of unsaturated soil in the radial direction are $k_a$ and $k_w$ (m/s), respectively. $k_{aw}$ and $k_{ww}$ are the air and water permeability coefficients of the vertical drain in the vertical direction (m/s), respectively. In addition, the top and bottom boundaries (i.e. $z = 0$, $z = H$), and the outer boundary of influence zone (i.e. $r = r_e$) are both impermeable to air and water. $q(t)$ is a loading that varies with time in the vertical direction on the top surface of the foundation (kPa).

![Figure 1. Consolidation modeling of unsaturated soils by vertical drains considering well resistance](image)

The research hypothesis are similar to those proposed by Zhou et al.[10] They are listed as follows:

1. The unsaturated soil is homogeneous.
2. The air flux conforms Fick’s law, while the water flux refers to Darcy’s law. In addition, both flow of those are continuous and independent.
3. The soil grain and water are incompressible.
4. The coefficients of permeability with regard to air and water, and volume change for the soil remain constant throughout the consolidation process.
5. The strain occurring during consolidation is small strain.

Note: the parameters in assumption (4) throughout the consolidation process of unsaturated soils are functions of stress state. However, if these parameters are nonlinear changed during consolidation, it is impossible to get the semi-analytical solution. According to the relevant studies on unsaturated soils[1-10], it is assumed that these parameters remain constant during consolidation process.

Referring to the studies given by Zhou et al. [10] the governing equations are expressed as follows:

$$\frac{\partial \bar{u}}{\partial t} = -C_s \frac{\partial \bar{u}_w}{\partial t} - C_{ve} \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) + C_{ve} \frac{\partial q}{\partial t}$$  \hspace{1cm} (1a)
\[
\frac{\partial \bar{u}_a}{\partial t} = -C_a \frac{\partial \bar{u}_a}{\partial t} - C^a \left( \frac{\partial^2 u_a}{\partial r^2} + \frac{1}{r} \frac{\partial u_a}{\partial r} \right) + C^q \frac{\partial q}{\partial t} \tag{1b}
\]

where \(\bar{u}_a\) and \(\bar{p}_w\) are the average excess pore-air and pore-water pressures (kPa), respectively. They can be expressed as follows:

\[
\bar{u}_a = \frac{1}{\pi} \left( r_c^2 - r_w^2 \right) \int_{r_w}^{r_c} u_a 2\pi r dr
\]

\[
\bar{p}_w = \frac{1}{\pi} \left( r_c^2 - r_w^2 \right) \int_{r_w}^{r_c} u_w 2\pi r dr
\]

where

\[
C_a = \frac{m^a_a}{m^a_k - m^a_e - u_{atm} n_0 (1 - s_0) (\bar{u}_a)^2}, \quad C^a = \frac{kRT}{\mu \bar{u}_a M \left( m^a_k - m^a_e - u_{atm} n_0 (1 - s_0) (\bar{u}_a)^2 \right)}, \quad C^q = \frac{k_w}{\gamma_w m^w_k}, \quad C^w = \frac{m^w_k}{m^w_e}
\]

\(\bar{u}_a + u_{atm}\) is the absolute initial average excess pore-air pressure (kPa), and \(u_{atm}\) is the atmospheric pressure. \(\bar{u}_a^0\) is the absolute initial average excess pore-air pressure (kPa). \(u_a\) and \(u_w\) mean excess pore-air and pore-water pressures (kPa) in the influence zone, respectively. \(m^a_k\) and \(m^e_k\) are the coefficients of air and water volume change with regard to a change in the net normal stress (\(q - \bar{u}_a\)) (kPa\(^1\)), respectively. \(m^a_k\) and \(m^e_k\) are the coefficients of air and water volume change with regard to a change in the matrix suction (\(\bar{u}_a - \bar{u}_w\)) (kPa\(^1\)), respectively. \(S_0\) and \(n_0\) are the initial degree of saturation and initial porosity, respectively. \(M\) is the molecular mass of air (0.02895 kg•mol\(^{-1}\)). \(R\) is the molar gas constant (8.314 J•mol\(^{-1}\)•K\(^{-1}\)), and \(T (t_0 + 273)\) is the absolute temperature. \(n_0\) is the ambient temperature (20 °C). \(g\) is the gravitational acceleration (9.81 m/s\(^2\)), and \(\gamma_w\) is the water unit weight (9.81 kN•m\(^{-3}\)).

2.2 Initial and boundary conditions

In this paper, it is assumed that the initial excess pore-air and pore-water pressures are equally scattered with the depth \(z\). Thus, they are given as follows:

\[
u_a^0 (r, z, t = 0) = u_a^0
\]

\[
u_w^0 (r, z, t = 0) = u_w^0
\]

As shown in figure 1, the boundary conditions are expressed as followed:

\[
u_{ax} (r = r_c, z, t) = u_{ax} (r = r_c, z, t) = 0
\]

\[
u_a (r = r_w, z, t) = u_{aw} (z, t)
\]

\[
u_w (r = r_w, z, t) = u_{ww} (z, t)
\]

\[
u_{aw} (r = r_w, z, t) = \frac{r_w k_{aw}}{2k_a} u_{aw, t} (z, t) = 0
\]

\[
u_{aw} (r = r_w, z, t) = \frac{r_w k_{aw}}{2k_w} u_{ww, t} (z, t) = 0
\]

\[
u_{aw} (z = 0, t) = u_{aw} (z = 0, t) = 0
\]

\[
u_{aw} (z = H, t) = u_{aw} (z = H, t) = 0
\]

3. Derivation of semi-analytical solutions

3.1 General semi-analytical solutions
The governing Equations (1) can be integrated with regard to \( r \) and the boundary condition presented in Equation (4) is considered. Thus, the following Equations can be acquired:

\[
\begin{align}
\frac{\partial u_a}{\partial r} &= -\frac{1}{2}\left(\frac{r_e^2 - r^2}{r}\right) \frac{1}{C_w} \left( -C_a \frac{\partial u_a}{\partial t} - \frac{\partial u_a}{\partial t} + C_a \frac{dq}{dt} \right) \quad (9a) \\
\frac{\partial u_w}{\partial r} &= -\frac{1}{2}\left(\frac{r_e^2 - r^2}{r}\right) \frac{1}{C_w} \left( -C_w \frac{\partial u_w}{\partial t} - \frac{\partial u_w}{\partial t} + C_w \frac{dq}{dt} \right) \quad (9b)
\end{align}
\]

Then, an integration of Equations (9) with regard to \( r \) while considering the boundary conditions presented in Equations (5) gives:

\[
\begin{align}
u_a &= -\frac{1}{2}\left(\frac{r_e^2}{r}\right) \ln \left(\frac{r}{r_w}\right) - \frac{r^2 - (r_w)^2}{2} \frac{1}{C_w} \left( -C_a \frac{\partial u_a}{\partial t} - \frac{\partial u_a}{\partial t} + C_a \frac{dq}{dt} \right) + u_{aw} \quad (10a) \\
u_w &= -\frac{1}{2}\left(\frac{r_e^2}{r}\right) \ln \left(\frac{r}{r_w}\right) - \frac{r^2 - (r_w)^2}{2} \frac{1}{C_w} \left( -C_w \frac{\partial u_w}{\partial t} - \frac{\partial u_w}{\partial t} + C_w \frac{dq}{dt} \right) + u_{ww} \quad (10b)
\end{align}
\]

Thereafter, substituting Equations (10) into Equations (2) results in:

\[
\begin{align}
\bar{u}_a &= A_a \frac{\partial u_a}{\partial t} + C_A (A_a \frac{\partial u_a}{\partial t} - C_A A_a \frac{dq}{dt} + u_{aw}) \quad (11a) \\
\bar{u}_w &= W_w \frac{\partial u_w}{\partial t} + C_w W_w \left( \frac{\partial u_w}{\partial t} - C_q W_q \frac{dq}{dt} + u_{ww} \right) \quad (11b)
\end{align}
\]

where \( A_a, W_w \) and \( F \) are the constants of integration.

\[
A_a = \frac{(r_e^2 \ F)}{2C_w}, \quad W_w = \frac{(r_e^2 \ F)}{2C_w}, \quad F = \frac{N^2}{N^2 - 1} \left( \ln N + \frac{1}{N^2} - \frac{1}{4N^4} - \frac{3}{4} \right).
\]

\( N \) is the ratio of influence radius to vertical drain radius, \( N = r_e r_w \).

The Laplace transform is used to decouple Equations (11). Accordingly, the average excess pore-air and pore-water pressures in the Laplace transform domain are obtained as follows:

\[
\bar{u}_a = \frac{1}{D} \left[ ((s W_w - 1) \bar{u}_{aw} - s C_a A_a \bar{u}_a + B_a) \right] \quad (12a) \\
\bar{u}_w = \frac{1}{D} \left[ ((s A_a - 1) \bar{u}_{aw} - s C_w W_w \bar{u}_w + B_w) \right] \quad (12b)
\]

where \( B_a = (s C_a A_a W_w - s A_a W_w + A_a) \bar{u}_0^a + C_A A_a \bar{u}_0^a + (s C_a A_a C_q W_w - s C_q A_a W_w + C_q A_q) (s q - q_b) \),

\( B_w = (s C_a A_a W_w - s A_a W_w + W_w) \bar{u}_0^w + C_w W_w \bar{u}_0^w + (s C_q A_a C_q W_w - s C_q A_q W_w + C_q W_q) (s q - q_b) \),

\( D = s^2 C_A A_a C_w W_w - s^2 A_w W_w + s A_a + s W_w - 1 \). \( \bar{u}_a, \bar{u}_w, \bar{q} \) are the result of the Laplace transform of \( u_a, u_w, q(t) \), respectively. And \( \bar{u}_0^a, \bar{u}_0^w, q_b \) are initial value of \( u_a, u_w, q(t) \), respectively.

The boundary condition Equations (6) is substituted into Equations (9), then substituted into Equations (10), and the following Equations is obtained through Laplace transform:

\[
\begin{align}
\frac{d^2 \bar{u}_aw}{dz^2} &= \rho_a (\bar{u}_a - \bar{u}_{aw}) \quad (13a) \\
\frac{d^2 \bar{u}_ww}{dz^2} &= \rho_w (\bar{u}_w - \bar{u}_{ww}) \quad (13b)
\end{align}
\]

where \( G_a, G_w \) are the well resistance factor with regard to air and water, respectively. \( \rho_a, \rho_w \) are two constant parameters, they are defined as:

\[
\rho_a = \frac{-8(N^2 - 1) G_a}{H^2 F} , \quad \rho_w = \frac{-8(N^2 - 1) G_w}{H^2 F}, \quad G_a = \frac{k_a}{k_w} \left( \frac{H}{d_w} \right)^2, \quad G_w = \frac{k_w}{k_w} \left( \frac{H}{d_w} \right)^2.
\]
\( d_1 \) is the diameter of the vertical drain, \( d_w = 2r_w \).

Equation (12) is combined with Equation (13) to obtain the ordinary differential equations (ODEs) of variable \( z \) as follows:

\[
\begin{align*}
\frac{d^2 \tilde{u}_w}{dz^2} & = \Omega_1 \tilde{u}_w + \Omega_2 \tilde{u}_w + \Omega_3 \\
\frac{d^2 \tilde{u}_w}{dz^2} & = \gamma_a \tilde{u}_w + \gamma_d \tilde{u}_w + \gamma_q
\end{align*}
\]

(14a)

(14b)

where

\[
\begin{align*}
\Omega_1 & = \rho_s (s \rho_w - 1 - D) D^{-1}, \quad \Omega_2 = -s \rho_w \rho_m \rho_d D^{-1}, \quad \Omega_3 = \rho_d D^{-1}, \\
\gamma_a & = \rho_a (s \rho_w - 1 - D) D^{-1}, \quad \gamma_d = -s \rho_w \rho_a D^{-1}, \quad \gamma_q = \rho_q D^{-1}
\end{align*}
\]

The above equation (14) is decoupled by introducing two new variables proposed by Qin et al. [7]

\[
\begin{align*}
\frac{d^2 \tilde{\phi}_1}{dz^2} - Q_1 \tilde{\phi}_1 - (\Omega_1 + \gamma_a q_{21}) & = 0 \\
\frac{d^2 \tilde{\phi}_2}{dz^2} - Q_2 \tilde{\phi}_2 - (\Omega_2 q_{12} + \gamma_q) & = 0
\end{align*}
\]

(15a)

(15b)

where

\[
\begin{align*}
q_{21} & = -\frac{\Omega_2}{\Omega_1 - \gamma_a}, \quad q_{12} = -\frac{\gamma_a}{\Omega_2 - \Omega_1}, \quad Q_{1,2} = \frac{1}{2} \left\{ \Omega_1 + \gamma_a + \left[ (\Omega_1 - \gamma_a)^2 + 4\Omega_2 \gamma_a \right]^{1/2} \right\}
\end{align*}
\]

\( \tilde{\phi}_1 \) and \( \tilde{\phi}_2 \) are intermediate variables, \( \tilde{\phi}_1 = \tilde{u}_w + \tilde{u}_w q_{21}, \quad \tilde{\phi}_2 = \tilde{u}_w q_{12} + \tilde{u}_w \).

By combining the boundary conditions (7), (8) after Laplace transform with the Equations (15), the solution can be obtained as follows:

\[
\begin{align*}
\tilde{\phi}_1 & = \xi_1 (\lambda_1 - 1) \\
\tilde{\phi}_2 & = \xi_2 (\lambda_2 - 1)
\end{align*}
\]

(16a)

(16b)

where

\[
\begin{align*}
\xi_1 & = \frac{\Omega_1 + \gamma_a q_{21}}{Q_1}, \quad \lambda_1 = \cosh \left[ H (Q_1)^{1/2} - z (Q_1)^{1/2} \right] \text{sech} \left[ H (Q_1)^{1/2} \right], \\
\xi_2 & = \frac{\gamma_a q_{12} + \gamma_q}{Q_2}, \quad \lambda_2 = \cosh \left[ H (Q_2)^{1/2} - z (Q_2)^{1/2} \right] \text{sech} \left[ H (Q_2)^{1/2} \right].
\end{align*}
\]

Therefore, the average excess pore-air and pore-water pressure can be expressed as:

\[
\begin{align*}
\bar{\tilde{u}} & = \frac{1}{\rho_a} \left[ \xi_1 (\lambda_1 - 1) \right] + \frac{1}{\rho_a} \left[ \xi_2 (\lambda_1 - 1) \right] \\
\tilde{u}_w & = \frac{1}{\rho_q} \left[ \xi_2 (\lambda_2 - 1) \right] + \frac{1}{\rho_q} \left[ \xi_2 (\lambda_2 - 1) \right]
\end{align*}
\]

(17a)

(17b)

Referring to Fredlund and Hasan’s [1] method of using stress state variables to express the volumetric strain of soil unit, the average degree of consolidation in the Laplace transform domain can be expressed as:

\[
\tilde{U} = \left[ \frac{\tilde{e}_s (s)}{\tilde{e}_s (\infty)} \right] = \frac{(m_1 - m_{1s})(\tilde{u}_s - \bar{u}_s / s) - m_2 (\tilde{u}_s - \bar{u}_s / s) - m_1 (\tilde{q} - q_o / s) + m_1 \tilde{u}_s}{(m_1 - m_{1s}) \tilde{u}_s + m_2 \tilde{u}_s + m_1 (q_s - q_o)}
\]

(18)

where \( \tilde{U} \), \( \tilde{e}_s (s) \), \( \tilde{e}_s (\infty) \) are the average degree of consolidation degree, the volumetric strain and the final volumetric strain in the Laplace transform domain, respectively. \( m_{1s} \) \( (m_{1s} = m_{1s} + m_{ws}) \) and \( m_2 \) \( (m_2 = m_2 + m_{w2}) \) are the coefficients of soil element volume change with regard to the net normal stress \( (q - \tilde{u}_s) \) (kPa) and the matric suction \( (\tilde{u}_s - \bar{u}_s) \) (kPa), respectively. \( q_s \) is the final value of \( q(t) \).
The general semi-analytical solutions can be obtained by the inverse Laplace transform of Equations (17) and (18) with Crump’s method. Details of the Crump’s method can refer to Wang et al. [5] The common time-dependent loading after Laplace transform \( \tilde{q} \) can be referred to Xu et al. [11]

3.2 Verification
In order to verify the reliability of the proposed semi-analytical solution, a special consolidation case under constant loading \((q=100 \text{ kPa})\) is considered, and compared with this of the analytical solution obtained by Zhou et al. [10] The following unsaturated soil examples are used for the verification and the analysis of consolidation behavior and influence factors later. It is worth noting that the calculation method of initial value of excess pore pressure under constant loading can refer to Ho et al. [9]

\[
r_w = 0.2 \text{ m}, \ r_c = 1.8 \text{ m (i.e. } N = 9 \text{ )}, \ H = 10 \text{ m}, \ S_0 = 80\%, \ n_0 = 50\%,
\]
\[
k_w = 1 \times 10^{-10} \text{ m/s}, \ m_{w0} = -2 \times 10^4 \text{ kPa}^{-1}, \ m_{a0} = -5 \times 10^{-5} \text{ kPa}^{-1}, \ m_{w}^* = -2.5 \times 10^4 \text{ kPa}^{-1},
\]
\[
m_a^* = 1 \times 10^{-4} \text{ kPa}^{-1}, \ m_a^* = -2 \times 10^{-4} \text{ kPa}^{-1}, \ m_a = 1 \times 10^{-4} \text{ kPa}^{-1}, \ q_0 = 100 \text{ kPa} (u_a^0=20 \text{ kPa}, u_w^0=40 \text{ kPa}).
\]

Figure 2. Variations of (a) average excess pore-air pressure and (b) average excess pore-water pressure with different ratio of \(k_a/k_w\)

Figure 2(a) and (b) are respectively the dissipation curves of average excess pore-air pressure and pore-water pressure with different permeability coefficient ratio of \(k_a/k_w\) on the equal strain assumption. During this analysis, it is assumed that the permeability of water is constant (i.e. \(k_w = 10^{-10} \text{ m/s}\)), and the well resistance factor with respect to air and water are equal (i.e. \(G_a = G_w\)). As shown in figure 2, the dissipation curve of the average excess pore pressure obtained from the semi analytical solution under different permeability coefficient ratio of \(k_a/k_w\) in this paper has an almost-identical dissipation trend with this achieved from the analytical solution by Zhou et al. [10] Thus, it can be concluded that the current semi analytical solution is reliable.

4. Case study and discussions
In the case study, it mainly discusses the influence of permeability coefficient ratio \(k_a/k_w\), well resistance factor with regard to air and water (i.e. \(G_a, G_w\)) and load parameter \(\lambda\) on the consolidation of unsaturated soils by vertical drains under exponential loading. The exponential loading is a simplified mathematical expression of construction loading, and it can be expressed as:

\[
q(t) = q_0 + bq_0\left(1 - e^{-bt}\right)
\]

where \(q_0\) is the initial surcharge (100 kPa), \(b\) is the load constant and \(b = 1\) in this paper. \(\lambda\) is the loading parameter, which controls the loading rate of exponential loading. Hence, \(\tilde{q} = 2q_0s^{-1} - q_0\left(\lambda + s\right)^{-1}\).

Figure 3(a), (b) shows the effects of well resistance factor with regard to air and water (i.e. \(G_a, G_w\)) on the average degree of consolidation under exponential loading. The adopted parameters for the analysis include \(r_w = 0.2 \text{ m}, N = 9, k_a/k_w = 10, k_w = 10^{-10} \text{ m/s and } \lambda = 5 \times 10^4 \text{ s}^{-1}\). Figure 3(a) presents the
The degree of consolidation increases with the increase of load parameter $\lambda$ (i.e. the faster the loading rate). It is interesting to note that, as shown in figure 3(a), the change curve of average degree of consolidation is obviously divided into two stages, and when $G_w$ is constant, the change of $G_a$ only affects the first stage. When other conditions are the same, the time needed to reach the same degree of consolidation increases with the increase of $G_a$. After entering the second stage, the curve tends to be consistent. This can be explained by the fact that after entering the second stage, the excess pore-air pressure has completely dissipated. This phenomenon is consistent with Zhou et al. [10] in the case of constant loading.

Figure 3(b), on the other hand, illustrates the average degree of consolidation varying with time for different well resistance factor $G$ with $G_a = G_w = G$. Where $G = 0$ means to ignore the influence of well resistance. When comparing it with other curves, it can be known that when the well resistance factor reaches a certain value, its influence on the average degree of consolidation is obvious. Additionally, the time needed to reach the same degree of consolidation decreases with the decrease of the well resistance factor. Moreover, when the well resistance factor $G$ is less than 0.1, the influence of the well resistance can be neglected.

Figure 4(a), (b) depicts the average degree of consolidation varying with permeability coefficient ratio of $k_a/k_w$ and loading parameter $\lambda$, respectively. The adopted parameters are $r_a = 0.2m, N = 9, G_a = G_w = G = 0.5$ and $k_w = 10^{-10}$ m/s. The variation of the average degree of consolidation with time in figure 4(a) is similar to that of figure 3(a). That is because the $k_w$ value is assumed to be constant of consolidation only occurs in the first stage, and the consolidation becomes faster as $k_a$ increases. The effect of loading rate on average degree of consolidation is shown in figure 4(b). The consolidation process is accelerated with the increase of load parameter $\lambda$ (i.e. the faster the loading rate). It is intere-
sting to observe that when the load parameter $\lambda$ is greater than $5 \times 10^4$ s$^{-1}$, a new stage appears in the first stage when the excess pore pressure is not completely dissipated. The reason for this phenomenon may be that when the time reaches $10^4$ s, the load of $\lambda = 5 \times 10^4$ s$^{-1}$ has reached the maximum value and remains constant. The load of $\lambda = 5 \times 10^5$ s$^{-1}$ remains at the stage of load increase. The increase in load accelerates the rate of consolidation.

5. Conclusions

Based on the equal strain assumption, well resistance and arbitrary time-dependent loading are considered in this paper to get the semi-analytical solutions of consolidation for unsaturated soils by vertical drains. The reliability of the obtained solution is verified by comparing with the analytical solution of radial consolidation of unsaturated soil considering well resistance under instantaneous loading conditions. Then, the influence of well resistance factors, permeability coefficient ratio $k_a/k_w$, and load parameter on the average degree of consolidation are analyzed under exponential loading. In addition, the general semi-analytical solutions can be used to analyze the consolidation problem with more complex loads in practical engineering.

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