A combinatorial interpretation of the $\kappa^*_g(n)$ coefficients

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Abstract

Studying the virtual Euler characteristic of the moduli space of curves, Harer and Zagier compute the generating function $C_g(z)$ of unicellular maps of genus $g$. They furthermore identify coefficients, $\kappa^*_g(n)$, which fully determine the series $C_g(z)$. The main result of this paper is a combinatorial interpretation of $\kappa^*_g(n)$. We show that these enumerate a class of unicellular maps, which correspond 1-to-2 to a specific type of trees, referred to as $O$-trees. $O$-trees are a variant of the $C$-decorated trees introduced by Chapuy, Féray and Fusy. We exhaustively enumerate the number $s_g(n)$ of shapes of genus $g$ with $n$ edges, which is a specific class of unicellular maps with vertex degree at least three. Furthermore we give combinatorial proofs for expressing the generating functions $C_g(z)$ and $S_g(z)$ for unicellular maps and shapes in terms of $\kappa^*_g(n)$, respectively. We then prove a two term recursion for $\kappa^*_g(n)$ and that for any fixed $g$, the sequence $\{\kappa_{g,t}\}_{t=0}^g$ is log-concave, where $\kappa_{g,t}(n) = \kappa_{g,t}$, for $n = 2g + t - 1$.

Keywords: unicellular map, fatgraph, O-tree, shape-polynomial, recursion

1. Introduction

A unicellular map is a connected graph embedded in a compact orientable surface, in such a way that its complement is homeomorphic to a polygon. Equivalently, a unicellular map of genus $g$ with $n$ edges can also be seen as gluing the edges of $2n$-gon into pairs to create an orientable surface of genus $g$. It is related to the general theory of map enumeration, the study of moduli spaces of curves [11], the character theory of symmetric group [22, 13], the computation of matrix integrals [15], and also considered in a variety of application contexts [18, 2]. The most well-known example of unicellular maps is arguably the class of plane trees, enumerated by the Catalan numbers (see for example [21]).

In [11] Harer and Zagier study the virtual Euler characteristic of the moduli space of curves. The number $\epsilon_g(n)$ counting the ways of gluing the edges of $2n$-gon in order to obtain an orientable surface of genus $g$, i.e., the number of unicellular maps of genus $g$ with $n$ edges turns out to play a crucial role in their computations and they discover the two term recursion

$$ (n + 1)\epsilon_g(n) = 2(2n - 1)\epsilon_g(n - 1) + (n - 1)(2n - 1)(2n - 3)\epsilon_{g-1}(n - 2). $$

Subsequently, they identify certain coefficients, $\kappa^*_g(n)$, which they describe to “give the best coding of the information contained in the [. . .] series”, $\epsilon_g(n)$, [11]. The key link is the following functional relation between the generating function $K^*_g(z)$ of $\kappa^*_g(n)$ and the generating function $C_g(z)$ of $\epsilon_g(n)$:

$$ C_g(z) = \frac{1}{\sqrt{1 - 4z}} K^*_g\left(\frac{z}{1 - 4z}\right), $$

where $K^*_g(z) = \sum_{n=2g}^{3g-1} \kappa^*_g(n)z^n$. 

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Fig. 1. $\kappa_{1,1}$ enumerates the unique unicellular maps of genus 1 with 2 edges, which correspond 1-to-2 to O-trees from $\mathcal{R}_{1,1,0}$.

The main result of this paper is a combinatorial interpretation of the $\kappa_g^*(n)$ coefficients discovered by Harer and Zagier. $\kappa_g^*(n)$ enumerates a class of unicellular maps, which correspond 1-to-2 to certain O-trees. O-trees are a variant of the C-decorated trees introduced in [7], see Theorem 2. Using O-trees, we exhaustively enumerate a specific class of unicellular maps with vertex degree at least three, called shapes. We give combinatorial proofs for expressing the generating functions $C_g(z)$ and $S_g(z)$ for unicellular maps and shapes in terms of $\kappa_g^*(n)$, respectively. In particular the $\kappa_g^*(n)$ are positive integers that satisfy an analogue of eq. (1.1)

$$(n+1)\kappa_g^*(n) = (n-1)(2n-1)(2n-3)\kappa_{g-1}^*(n-2) + 2(2n-1)(2n-3)(2n-5)\kappa_{g-1}^*(n-3),$$

see Corollary 2 and Theorem 3. Eq. (1.2) has been independently discovered by Chekhov et al. [1] using the matrix model. We furthermore prove in Proposition 8 that for any fixed $g$, the sequence $\{\kappa_{g,t}\}_{t=0}^g$ is log-concave, where $\kappa_g^*(n) = \kappa_{g,t}$, for $n = 2g + t - 1$. We conjecture that the sequences $\{\kappa_{g,t}\}_{t=0}^g$ and $\{s_g(n)\}_{n=2g}^{6g-2}$ are infinitely log-concave.

2. Background

A map $M$ of genus $g \geq 0$ is a connected graph $G$ embedded on a closed compact orientable surface $S$ of genus $g$, such that $S\setminus G$ is homeomorphic to a collection of polygons, which are called the faces of $M$. Loops and multiple edges are allowed. The (multi)graph
G is called the underlying graph of M and S its underlying surface. Maps are considered up to homeomorphisms between the underlying surfaces. A sector of M consists of two consecutive edges around a vertex. A rooted map is a map with a marked sector, called the root; the vertex incident to the root is called the root-vertex. In figures, we represent the root, by drawing a dashed edge attaching the root-vertex and a distinguished vertex, called the plant. By convention, the plant, plant-edge and its associated sector (around the plant) are not considered, when counting the number of vertices, edges or sectors. From now on, all maps are assumed to be rooted and accordingly the underlying graph of a rooted map is naturally vertex-rooted. A unicellular map is a map with a unique face. By Euler’s characteristic formula $|V| - |E| + |F| = 2 - 2g$, a unicellular map of genus g with n edges has $n + 1 - 2g$ vertices. A plane tree is a unicellular map of genus 0.

We next introduce O-trees, which are directly implied by the concept of C-decorated trees by Chapuy et al. [7].

An O-permutation is a permutation where all cycles have odd length. For each O-permutation $\sigma$ on n elements, the genus of $\sigma$ is defined as $(n - \ell(\sigma))/2$, where $\ell(\sigma)$ is the number of cycles of $\sigma$.

An O-tree with n edges is a pair $\alpha = (T, \sigma)$, where $T$ is a plane tree with n edges and $\sigma$ is an O-permutation on $n + 1$ elements, see Figure 2(a). The genus of $\alpha$ is defined to be the genus of $\sigma$. We canonically number the $n + 1$ vertices of $T$ from 1 to $n + 1$ according to a left-to-right, depth-first traversal. Hence $\sigma$ can be seen as a permutation of the vertices of $T$, see Figure 2(b).

The underlying graph $G(\alpha)$ of $\alpha$ is the (vertex-rooted) graph $G$ with n edges, that is obtained from $T$ by merging the vertices in each cycle of $\sigma$ (so that the vertices of $G$ correspond to the cycles of $\sigma$) into a single vertex, see Figure 2(c).

**Fig. 2.** An O-tree and its underlying graph.

**Definition 1.** For $n, g$ nonnegative integers, let $E_g(n)$ denote the set of unicellular maps of genus g with n edges, let $O_g(n)$ be the set of O-permutations of genus g on n elements and let $T_g(n)$ denote the set of O-trees of genus g with n edges, i.e., $T_g(n) = E_0(n) \times O_g(n + 1)$.

For two finite sets $\mathcal{A}$ and $\mathcal{B}$, let $\mathcal{A} \uplus \mathcal{B}$ denote their disjoint union and $k\mathcal{A}$ denote the set made of k disjoint copies of $\mathcal{A}$. We write $\mathcal{A} \simeq \mathcal{B}$ if there exists a bijection between $\mathcal{A}$ and $\mathcal{B}$.
Let us first recall a combinatorial result of [6]:

**Proposition 1** (Chapuy [6]). For \( k \geq 1 \), let \( E_g^{(2k+1)}(n) \) denote the set of maps from \( E_g(n) \) in which a set of \( 2k+1 \) vertices is distinguished. Then for \( g > 0 \) and \( n \geq 0 \),

\[
2g E_g(n) \simeq \bigcup_{k=1}^{g} E_{g-k}^{(2k+1)}(n).
\]

In addition, if \( M \) and \( (M', S') \) are in correspondence, then the underlying graph of \( M \) is obtained from the underlying graph of \( M' \) by merging the vertices in \( S' \) into a single vertex.

**Remark:** one key feature of this bijection is that it preserves the underlying graph of corresponding objects. By multiplying with the factor \( 2^{2g} \) (which still preserves the underlying graph), we obtain

\[
2g 2^{2g} E_g(n) \simeq \bigcup_{k=1}^{g} 2^{2k} \cdot E_{g-k}^{(2k+1)}(n).
\]

In analogy to the above decomposition of unicellular maps, there exists a recursive way to decompose O-permutations:

**Proposition 2** (Chapuy et al. [7]). For \( k \geq 1 \), let \( O_g^{(2k+1)}(n) \) be the set of O-permutations from \( O_g(n) \) having \( 2k+1 \) labeled cycles. Then for \( g > 0 \) and \( n \geq 0 \),

\[
2g O_g(n) \simeq \bigcup_{k=1}^{g} 2^{2k} \cdot O_{g-k}^{(2k+1)}(n).
\]

Furthermore, if \( \pi \) and \( (\pi', S') \) are in correspondence, then the cycles of \( \pi \) are obtained from the cycles of \( \pi' \) by merging labeled cycles in \( S' \) into a single cycle.

Along these lines we furthermore observe:

**Proposition 3** (Chapuy et al. [7]). For \( k \geq 1 \), denote by \( T_g^{(2k+1)}(n) \) the set of O-trees from \( T_g(n) \) in which a set of \( 2k+1 \) cycles is distinguished. Then for \( g > 0 \) and \( n \geq 0 \),

\[
2g T_g(n) \simeq \bigcup_{k=1}^{g} 2^{2k} \cdot T_{g-k}^{(2k+1)}(n).
\]

Furthermore, if \( \alpha \) and \( (\alpha', S') \) are in correspondence, then the underlying graph of \( \alpha \) is obtained from the underlying graph of \( \alpha' \) by merging the vertices corresponding to cycles from \( S' \) into a single vertex.

The proofs of Proposition 2 and Proposition 3 are reformulations of those of [7] in the context of C-permutations and C-decorated trees. For completeness we give them in the Appendix.

**Remark:** the bijection for O-permutations preserves the cycles, which implies that the bijection for O-trees preserves the underlying graph of corresponding objects.

Combining Proposition 1 and Proposition 3, we inductively derive a bijection preserving the underlying graphs.
Theorem 1 (Chapuy et al. [7]). For any non-negative integers \( n \) and \( g \), there exists a bijection

\[
2^{2g} \mathcal{E}_g(n) \simeq \mathcal{T}_g(n) = \mathcal{E}_0(n) \times \mathcal{O}_g(n + 1).
\]

In addition, the cycles of an \( O \)-tree naturally correspond to the vertices of the associated unicellular map, such that the respective underlying graphs are the same.

Remark: in [7], Chapuy et al. prove the existence of a 1-to-2\(^{n+1}\) correspondence between \( C \)-decorated trees and unicellular maps. The notion of \( C \)-permutation and \( C \)-decorated tree therein can be viewed as \( O \)-permutation and \( O \)-tree carrying a sign with each cycle, respectively. The reduction from \( C \)-decorated trees to \( O \)-trees allows us derive a 1-to-2\(^g\) correspondence between \( O \)-trees and unicellular maps. Furthermore all the results in [7] for \( C \)-decorated trees have an \( O \)-tree analogue.

The proof of Theorem 1 is a reformulation of that for \( C \)-decorated trees [7]. We give its proof in the Appendix.

3. Shapes

Definition 2. A shape is a unicellular map having vertices of degree \( \geq 3 \).

We adopt the convention that the plant-edge is taken into account when considering the degree of the root vertex.

Proposition 4. [19] Given a shape of genus \( g \) with \( n \) edges, we have \( 2g \leq n \leq 6g - 2 \).

Proof. By Euler’s characteristic formula, we have \( |V| = n + 1 - 2g \), where \( V \) denotes the vertex set of a shape of genus \( g \) with \( n \) edges. On the one hand, any shape contains at least one vertex, which implies \( |V| = n + 1 - 2g \geq 1 \), i.e., \( n \geq 2g \). On the other hand, each vertex \( v \) of a shape has \( \deg(v) \geq 3 \). Then we derive \( 2(n + 1) = \sum_{v \in V} \deg(v) + 1 \geq 3|V| + 1 = 3(n + 1 - 2g) + 1 \), that is, \( n \leq 6g - 2 \). (Here we consider the plant and the plant-edge.) \( \square \)

Let \( \mathcal{S}_g(n) \) denote the set of \( \mathcal{E}_g(n) \)-shapes, i.e., shapes of genus \( g \) with \( n \) edges. Let \( \mathcal{R}_g(n) \) denote the set of \( O \)-trees from \( \mathcal{T}_g(n) \) such that each vertex in the underlying graph of the \( O \)-tree contains only vertices of degree \( \geq 3 \), that is

\[
\mathcal{R}_g(n) = \{(T, \sigma) \in \mathcal{E}_0(n) \times \mathcal{O}_g(n + 1) | \text{ each vertex of } G(T, \sigma) \text{ has degree } \geq 3\}.
\]

Lemma 1. For \( g \geq 1 \) and \( 2g \leq n \leq 6g - 2 \), we have the bijection

\[
2^{2g} \mathcal{S}_g(n) \simeq \mathcal{R}_g(n).
\]

In addition, the cycles of an \( O \)-tree naturally correspond to the vertices of the associated unicellular map, in such a way that the respective underlying graphs are the same.

Note that a unicellular map is a shape if and only if each vertex in the underlying graph of the map has degree \( \geq 3 \). Therefore, Lemma 1 follows directly from Theorem 1 by restricting the bijection to the set \( \mathcal{S}_g(n) \) of shapes since the bijection therein preserves the underlying graph of corresponding objects.
Lemma \[\text{II}\] allows us to obtain deeper insight into shapes via O-permutations. To this end we consider the cycle-type of an O-permutation, i.e., a partition with parts of odd size. Given an O-permutation from $\mathcal{O}_g(n + 1)$, its cycle-type is a partition $\beta$ of $n + 1$ with $n + 1 - 2g$ odd parts. We assume that $\beta = 1^{n+1-2g-t}3m_1 \ldots (2j + 1)m_j$, with $t = m_1 + \cdots + m_j$. The partition $\beta = 1^{n+1-2g-t}3m_1 \ldots (2j + 1)m_j$ naturally corresponds to the partition $\gamma = 1^{m_1} \ldots j^{m_j}$ of $g$. The fact that $\gamma$ is a partition of $g$ follows from the identity $(n + 1 - 2g - t) + 3m_1 + \cdots (2j + 1)m_j = n + 1$. Here $t = m_1 + \cdots + m_j = \ell(\gamma)$ denotes the number of odd parts $> 1$ of $\beta$, i.e., the number of parts of $\gamma$. Let $k = n + 1 - 2g - t$ denote number of parts $= 1$ of $\beta$. It is clear that this a one-to-one correspondence. Therefore the cycle type $\beta$ of an O-permutation from $\mathcal{O}_g(n + 1)$ can be indexed by a partition $\gamma$ of $g$.

The number $a_\gamma(k)$ of O-permutations of $n + 1 = 2g + t + k$ elements with cycle-type equal to $\beta = 1^{k}3m_1 \ldots (2j + 1)m_j$ is given by

$$a_\gamma(k) = \frac{(2g + t + k)!}{k! \prod_{i} m_i!(2i + 1)^{m_i}},$$

where $\gamma = 1^{m_1} \ldots j^{m_j}$.

Let $\mathcal{O}_{g,t,k}$ denote the set of O-permutations of genus $g$ with $k$ cycles of length 1 and $t$ cycles of length $> 1$. Note that the number of elements of an O-permutation from $\mathcal{O}_{g,t,k}$ is $n + 1 = 2g + t + k$. Then we have the following two cases:

1. For $k = 0$, the cardinality $\mathcal{O}_{g,t,0}$, denoted by $a_{g,t}$, counts O-permutations of genus $g$ on $2g + t$ elements without cycles of length 1 (or cycle-type having the form $\beta = 1^{m_1} \ldots (2j + 1)^{m_j}$). Hence it is given by

$$a_{g,t} = \sum_{\gamma^g \ell(\gamma) = t} a_\gamma(0) = (2g + t)! \sum_{\gamma^g \ell(\gamma) = t} \frac{1}{\prod_{i} m_i!(2i + 1)^{m_i}}.$$ 

where $\gamma = 1^{m_1}2^{m_2} \ldots j^{m_j}$ runs over all partitions of $g$ with $t$ parts.

2. For arbitrary $k$, each O-permutation in $\mathcal{O}_{g,t,k}$ consists of an O-permutation from $\mathcal{O}_{g,t,0}$ together with $k$ cycles of length 1. Then the set $\mathcal{O}_{g,t,k}$ can be counted by first picking up $k$ elements from $2g + t + k$ elements and then choosing an O-permutation from $\mathcal{O}_{g,t,0}$. Therefore

$$|\mathcal{O}_{g,t,k}| = \binom{2g + t + k}{k} a_{g,t} = \frac{(2g + t + k)!}{k!} \sum_{\gamma^g \ell(\gamma) = t} \frac{1}{\prod_{i} m_i!(2i + 1)^{m_i}}.$$ 

By definition,

$$\mathcal{O}_g(n + 1) = \biguplus_{t+k=n+1-2g} \mathcal{O}_{g,t,k}.$$ 

Set $n + 1 = 2g + t + k$. Let $\mathcal{R}_{g,t,k}$ denote the set of O-trees from $\mathcal{R}_g(n)$ such that their associated O-permutation has $k$ cycles of length 1 and $t$ cycles of length $> 1$, i.e.,

$$\mathcal{R}_{g,t,k} = \{(T, \sigma) \in \mathcal{E}_0(n) \times \mathcal{O}_{g,t,k} \mid \text{each vertex of } G(T, \sigma) \text{ has degree } \geq 3\}.$$
Hence
\[ R_g(n) = \bigcup_{t+k=n+1-2g} R_{g,t,k}. \]

**Lemma 2.** For \( k = 0 \), we have
\[ R_{g,t,0} = \mathcal{E}_0(2g + t - 1) \times \mathcal{O}_{g,t,0}. \]
Therefore
\[ |R_{g,t,0}| = \text{Cat}(2g + t - 1) a_{g,t} = \frac{(2(2g + t - 1))!}{(2g + t - 1)!} \sum_{\gamma \vdash g \ell(\gamma) = t} \frac{1}{\prod_i m_i!(2i + 1)^{m_i}}, \]
where \( \text{Cat}(n) := \frac{(2n)!}{n!(n+1)!} \) is the \( n \)-th Catalan number and \( \gamma = m_1 m_2 \cdots m_j \) runs over all partitions of \( g \) with \( t \) parts.

**Proof.** By definition, \( R_{g,t,0} \subseteq \mathcal{E}_0(2g + t - 1) \times \mathcal{O}_{g,t,0} \). Given \((T, \sigma) \in \mathcal{E}_0(2g + t - 1) \times \mathcal{O}_{g,t,0}\), each cycle of O-permutation \( \sigma \) has length \( \geq 3 \). Then the underlying graph \( G(T, \sigma) \) of \((T, \sigma)\), obtained from \( T \) by merging into a single vertex the vertices in each cycle of \( \sigma \), must have all vertices with degree \( \geq 3 \). It implies that \((T, \sigma) \in R_{g,t,0}\). Hence \( R_{g,t,0} = \mathcal{E}_0(2g + t - 1) \times \mathcal{O}_{g,t,0} \). \( \square \)

To enumerate O-trees from \( R_{g,t,k} \) for arbitrary \( k \), we observe that they can be reduced to O-trees from \( R_{g,t,0} \). The key idea is to eliminate the vertices corresponding to \( 1 \)-cycles from an O-tree, thereby reducing to an O-tree without \( 1 \)-cycles, i.e., O-tree from \( R_{g,t,0} \). This elimination on O-trees is reminiscent of Rémy’s bijection [20] on plane trees, which is briefly recalled below.

Rémy’s bijection reduces a plane tree \( T \) with \( n \) edges and a labeled vertex to a plane tree \( T' \) with \( n - 1 \) edges and a sector labeled by \(+\) or \(-\) as follow

- if the labeled vertex is a leaf, \( T' \) is obtained from \( T \) by contracting the edge connecting the labeled vertex and its father. Label by \(+\) the sector associated with the labeled vertex,
- if the labeled vertex is a non-leaf, \( T' \) is obtained from \( T \) by contracting the edge connecting the labeled vertex and its leftmost child. Label by \(-\) the sector separating the leftmost subtree and the remaining subtree of the labeled vertex.

Therefore \((n + 1)\mathcal{E}_0(n) \simeq 2(2n - 1)\mathcal{E}_0(n - 1)\), see Figure 3. Rémy’s bijection has been applied in [7, 12, 16].

Given an O-tree \((T, \sigma) \in R_{g,t,k}\), its **traversal** is defined as that of its underlying plane tree (traveling around the boundary of \( T \) starting from the root-sector). A vertex \( v \) of \((T, \sigma)\) is called a 1-cycle if the corresponding element in \( \sigma \) is in a cycle of length 1. All sectors around the vertex \( v \) are ordered according to the traversal of \((T, \sigma)\). A sector \( \tau \) at \( v \) in \((T, \sigma)\) is called permissible if

- \( \tau \) is not the last sector around \( v \) according to the traversal of \( T \),
- if vertex \( v \) is 1-cycle, then \( \tau \) is not the first two sectors around \( v \) according to the traversal of \( T \).
**Proposition 5.** Any O-tree \((T, \sigma) \in R_{g,t,k}\) has exactly \((2g + t - k - 1)\) permissible sectors.

**Proof.** By definition of \(R_{g,t,k}\), each vertex in the underlying graph \(G(T, \sigma)\) has degree \(\geq 3\). Then each 1-cycle vertex has degree \(\geq 3\) in \(T\) since it has the same degree as its corresponding vertex in the underlying graph.

Accordingly, any 1-cycle vertex has no less than 3 sectors, whence its first two sectors never coincide with its last sector. Note that any \((T, \sigma) \in R_{g,t,k}\) has \((2g + t + k - 1)\) edges, \((2g + t + k - 1) + 1\) sectors and \(k\) 1-cycle vertices. Thus in \((T, \sigma)\), the set of permissible sectors is the set of all \((T, \sigma)\)-sectors excluding all last sectors of vertices and all the first two sectors of 1-cycle vertices. Hence the number of permissible \((T, \sigma)\)-sectors is given by \(2(2g + t + k - 1) + 1 - (2g + t + k) - 2k = 2g + t - k - 1\). \(\square\)

Let \(R_{g,t,k}^{(l)}\) denote the set of \(R_{g,t,k}\) O-trees with \(l\) permissible, labeled sectors. By Proposition 5, \(0 \leq l \leq 2g + t - k - 1\) and \(|R_{g,t,k}^{(l)}| = \binom{2g + t - k - 1}{l}|R_{g,t,k}|\).

Let \(R_{g,t,k}^v\) denote the set of \(R_{g,t,k}\) O-trees with one labeled 1-cycle vertex. Since each \(R_{g,t,k}\) O-tree has \(k\) 1-cycle vertices, we have \(|R_{g,t,k}^v| = k|R_{g,t,k}|\).

**Lemma 3.** For \(k \geq 1\), there exists a bijection between \(R_{g,t,k}^v\), the set of \(R_{g,t,k}\) O-trees with one labeled 1-cycle vertex and \(R_{g,t,k-1}^{(l)}\), the set of \(R_{g,t,k-1}\) O-trees with one permissible, labeled sector. Accordingly we have

\[k|R_{g,t,k}| = (2g + t - k)|R_{g,t,k-1}|.\]
Proof. Suppose we are given \((T, \sigma, v) \in \mathcal{R}_{g,t,k}^v\) and a 1-cycle vertex \(v\). \(v\) has degree \(\geq 3\) and is not a leaf.

We construct an O-tree \((T', \sigma', s)\) from \(\mathcal{R}_{g,t,k-1}^{(1)}\) as follows: we apply Rémý’s bijection to the plane tree \(T\) with respect to the non-leaf \(v\), i.e., contracting the edge connecting \(v\) and its leftmost child. We obtain a plane tree \(T'\) together with a labeled sector \(s\). The correspondence between vertices in \(T\) and those in \(T'\) gives us a canonical relabelling of elements of the permutation \(\sigma\) excluding the 1-cycle corresponding to \(v\). Let \(\sigma'\) denote the permutation obtained from \(\sigma\) by deleting 1-cycle corresponding to \(v\) and relabeling, see Figure 4.

We define the mapping

\[
\Pi: \mathcal{R}_{g,t,k}^v \rightarrow \mathcal{R}_{g,t,k-1}^{(1)}, \quad (T, \sigma, v) \mapsto (T', \sigma', s).
\]

First we show that \(\Pi\) is well-defined. By construction, \(T'\) has \(2g + t + k - 2\) edges and all cycles in \(\sigma'\) have odd length, i.e., \(\sigma'\) is an O-permutation. Further \(\sigma'\) has \(2g + t + k - 1\) elements, \(k - 1\) cycles of length 1 and \(t\) odd cycles of length \(> 1\). Hence \((T', \sigma') \in \mathcal{R}_{g,t,k-1}^{(1)}\). Let \(v'\) denote the vertex in \(T'\), to which sector \(s\) belongs to. \(s\) is not the last sector around \(v'\), since otherwise, by construction of Rémý’s bijection, the 1-cycle vertex \(v\) in \(T\) has degree at most two, a contradiction. If \(v'\) is a 1-cycle in \((T', \sigma')\), then the leftmost child of \(v\) is a 1-cycle in \((T, \sigma)\) and \(v_1\) has degree at least \(\geq 3\) in \(T\). By the way of contracting the edge connecting \(v\) and \(v_1\) and labeling the sector, \(s\) is then not one of the first two sectors around \(v'\). This shows that \(s\) is permissible, whence \((T', \sigma', s) \in \mathcal{R}_{g,t,k-1}^{(1)}\) and \(\Pi\) is well-defined.

To recover \((T, \sigma, v) \in \mathcal{R}_{g,t,k}^v\) from \((T', \sigma', s) \in \mathcal{R}_{g,t,k-1}^{(1)}\), we apply the inverse of Rémý’s bijection to \(T'\) with respect to the sector \(s\), which is labeled – and obtain a plane tree \(T\) with non-leaf vertex \(v\). By construction of Rémý’s bijection and the definition of permissible sector, \(v\) has degree \(\geq 3\). Set \(\sigma\) to be the permutation obtained from \(\sigma'\) by adding the 1-cycle corresponding to \(v\) and relabeling according to the correspondence.
between vertices in $T'$ and those in $T$, see Figure 4. It is clear that this is the inverse of $\Pi$, whence $\Pi$ is bijective. □

By applying Lemma 3 successively, we derive

$$|R_{g,t,k}| = \frac{2g + t - k}{k} |R_{g,t,k-1}|$$

$$= \frac{2g + t - k}{k} \cdot \frac{2g + t - k + 1}{k - 1} |R_{g,t,k-2}|$$

$$= \cdots$$

$$= \binom{2g + t - 1}{k} |R_{g,t,0}|.$$ 

Accordingly, Lemma 3 induces a bijection from $R_{g,t,k}$ to $R_{g,t,0}^{(k)}$:

**Lemma 4.** For any $k$, there exists a bijection from $R_{g,t,k}$ to $R_{g,t,0}^{(k)}$, the set of $R_{g,t,0}$ O-trees with $k$ permissible, labeled sectors:

$$|R_{g,t,k}| = \binom{2g + t - 1}{k} |R_{g,t,0}|.$$ 

**Remark:** given an $R_{g,t,k}$ O-tree, the number $k$ of 1-cycle vertices is bounded by $k \leq 2g + t - 1$.

In Figure 5, we show how to generate all $R_{1,1,1}$ and $R_{1,1,2}$ O-trees from $R_{1,1,0}$ O-trees.
Let
\[ \kappa_{g,t} = \frac{|R_{g,t,0}|}{2^{2g}} = \frac{(2(2g + t - 1))!}{2^{2g}(2g + t - 1)!} \sum_{\gamma \vdash g} \prod_{\ell(\gamma) = t} \frac{1}{m_{i}(2i + 1)^{m_{i}}}, \]
where \( \gamma = 1^{m_{1}}2^{m_{2}} \cdots j^{m_{j}} \) is a partition of \( g \) with \( t \) parts.

For \( g \geq 1 \), let \( s_{g}(n) \) be the number of shapes of genus \( g \) with \( n \) edges and \( S_{g}(z) \) denote the corresponding generating polynomial \( S_{g}(z) = \sum_{n=2g}^{6g-2} s_{g}(n)z^{n} \). Then

**Lemma 5.** [12] For any \( g \geq 1 \), the generating polynomial of shapes is given by
\[ S_{g}(z) = \sum_{t=1}^{g} \kappa_{g,t}z^{2g+t-1}(1 + z)^{2g+t-1}. \]

**Proof.** By Lemma 1 we have \( 2^{2g}S_{g}(n) \simeq R_{g}(n) = \biguplus_{t+k=n+1-2g}^{6g-2} R_{g,t,k} \) and furthermore
\[ 2^{2g} \biguplus_{n=2g}^{6g-2} S_{g}(n) \simeq \biguplus_{n=2g}^{6g-2} R_{g}(n) = \biguplus_{t=1}^{g} \biguplus_{k=0}^{2g+t-1} R_{g,t,k}. \]

Therefore, by Lemma 4,
\[
S_{g}(z) = \sum_{n=2g}^{6g-2} \left| S_{g}(n) \right| z^{n} = \frac{1}{2^{2g}} \sum_{t=1}^{g} \sum_{k=0}^{2g+t-1} \left| R_{g,t,k} \right| z^{2g+t+k-1} = \frac{1}{2^{2g}} \sum_{t=1}^{g} \sum_{k=0}^{2g+t-1} \binom{2g + t - 1}{k} \left| R_{g,t,0} \right| z^{2g+t+k-1} = \frac{1}{2^{2g}} \sum_{t=1}^{g} \left| R_{g,t,0} \right| z^{2g+t-1} \sum_{k=0}^{2g+t-1} \binom{2g + t - 1}{k} z^{k} = \frac{1}{2^{2g}} \sum_{t=1}^{g} \left| R_{g,t,0} \right| z^{2g+t-1} (1 + z)^{2g+t-1} = \sum_{t=1}^{g} \kappa_{g,t}z^{2g+t-1}(1 + z)^{2g+t-1}. \]

\[ Q.E.D. \]

**Corollary 1.** We have
\[ s_{g}(n) = \sum_{t=1}^{g} \kappa_{g,t} \binom{2g + t - 1}{n - (2g + t - 1)}, \]
where \( \binom{n}{k} = 0 \) if \( k < 0 \) or \( k > n \).
Proof. By Lemma \([5]\) we have
\[
\sum_{n=2g}^{6g-2} s_g(n) z^n = \sum_{t=1}^{g} \kappa_{g,t} z^{2g+t-1}(1+z)^{2g+t-1} = \sum_{t=1}^{g} \sum_{i=0}^{2g+t-1} \binom{2g + t - 1}{i} (1 + z)^{2g+t-1+i}.
\]
Set \(n = 2g + t - 1 + i\). By comparing both sides of the above identity, we obtain the corresponding formula for \(s_g(n)\).

**Corollary 2.** The number \(\kappa_{g,t}\) is a positive integer.

Proof. The positivity of \(\kappa_{g,t}\) is clear by definition. We proceed by induction on \(t\): assume that \(\kappa_{g,j}\) is an integer for \(j < t\) and set \(n = 2g + t - 1\). By eq. (3.1), we have
\[
s_g(2g + t - 1) = \sum_{j=1}^{t} \kappa_{g,j} \binom{2g + j - 1}{t - j},
\]
i.e.,
\[
\kappa_{g,t} = s_g(2g + t - 1) - \sum_{j=1}^{t-1} \kappa_{g,j} \binom{2g + j - 1}{t - j}.
\]
Since \(s_g(2g + t - 1)\) and \(\kappa_{g,j}\) are integers for \(j < t\), \(\kappa_{g,t}\) is an integer.

### 4. The coefficients \(\kappa^*_g(n)\)

Let \(\epsilon_g(n)\) denote the number of unicellular maps of genus \(g\) with \(n\) edges. In the following we derive an explicit formula for the generating function of unicellular maps of genus \(g\), which has the same coefficients \(\kappa_{g,t}\) as in the generating polynomial of shapes of genus \(g\) in Lemma \([5]\). This result has been observed in \([12]\) by a different construction.

**Lemma 6.** For any \(g \geq 1\), the generating function of unicellular maps of genus \(g\) is given by
\[
C_g(z) = \sum_{t=1}^{g} \frac{\kappa_{g,t} z^{2g+t-1}}{(1 - 4z)^{2g+t-\frac{1}{2}}}. 
\]

Proof. Note that \(O_g(n+1) = \bigcup_{t=1}^{g} O_{g,t,k}\) and \(|O_{g,t,k}| = \binom{n+1}{2g+t} a_{g,t}\), where \(k = n + 1 - 2g - t\). Thus \(|O_g(n+1)| = \sum_{t=1}^{g} \binom{n+1}{2g+t} a_{g,t}\). By Theorem \([1]\) \(2^{2g} \mathcal{E}_g(n) \simeq \mathcal{E}_0(n) \times O_g(n+1)\) and we have
\[
\epsilon_g(n) = \frac{1}{2^{2g}} \text{Cat}(n) \sum_{t=1}^{g} \binom{n+1}{2g+t} a_{g,t}
\]
\[
= \sum_{t=1}^{g} \frac{(2n)!}{2^{2g} n!(n + 1 - 2g - t)!(2g + t)!} a_{g,t}.
\]
Therefore using
\[
\sum_{n \geq r-1} \frac{(2n)!}{n!(n + 1 - r)!} z^n = \frac{(2(r - 1))!}{(r - 1)!} \frac{z^{r-1}}{(1 - 4z)^{r-\frac{1}{2}}},
\]

\]
we compute

\[ C_g(z) = \sum_{n \geq 2g} \varepsilon_g(n) z^n \]

\[ = \sum_{n \geq 2g} \sum_{t=1}^g \frac{(2n)!}{2^{2g} t! (n+1-2g-t)! (2g+t)! a_{g,t} z^n} \]

\[ = \sum_{t=1}^g \frac{a_{g,t}}{2^{2g} (2g+t)!} \sum_{n \geq 2g} \frac{(2n)!}{n! (n+1-2g-t)!} z^n \]

\[ = \sum_{t=1}^g \frac{a_{g,t}}{2^{2g} (2g+t)!} \cdot \frac{(2(2g+t-1))!}{(2g+t-1)!} \cdot \frac{z^{2g+t-1}}{(1-4z)^{2g+t-\frac{1}{2}}} \]

\[ = \sum_{t=1}^g \frac{\text{Cat}(2g+t-1) a_{g,t}}{2^{2g}} \frac{z^{2g+t-1}}{(1-4z)^{2g+t-\frac{1}{2}}} \]

\[ = \sum_{t=1}^g \frac{\kappa_{g,t} z^{2g+t-1}}{(1-4z)^{2g+t-\frac{1}{2}}}. \]

\[ \square \]

Let

\[ K^*_{g}(z) = \sum_{n=2g}^{3g-1} \kappa^*_g(n) z^n, \]

then \[ 11 \] shows that

\[ C_g(z) = \frac{1}{\sqrt{1-4z}} K^*_{g}\left(\frac{z}{1-4z}\right). \]

In view of Lemma \[ 11 \] and Lemma \[ 6 \] this provides the following combinatorial interpretation of \( \kappa^*_g(n) \):

**Theorem 2.** \( \kappa^*_g(n) = \kappa_{g,t} \), where \( n = 2g + t - 1 \) and \( \kappa^*_g(n) \) counts the shapes of genus \( g \), which correspond to \( \mathcal{R}_{g,t,0} \subset \mathcal{R}_g(n) \) via the bijection in Lemma \[ 4 \].

In \[ 2 \], \( C_g(z) \) has been shown to have the form

\[ C_g(z) = \frac{P_g(z)}{(1-4z)^{3g-\frac{1}{2}}}, \]

where \( P_g(z) \) is a polynomial with integer coefficients.

Combining this with Lemma \[ 8 \] we obtain an explicit formula for the polynomials \( P_g(z) \) in terms of \( \kappa_{g,t} \):

**Corollary 3.** For any \( g \geq 1 \), the polynomial \( P_g(z) \) is given by

\[ P_g(z) = \sum_{t=1}^g \kappa_{g,t} z^{2g+t-1}(1-4z)^{g-t}. \]
| $a_{g,t}$ | $t = 1$ | $2$ | $3$ | $4$ | $5$ |
|----------|--------|-----|-----|-----|-----|
| $g = 1$  | 2      |     |     |     |     |
| 2        | 24     | 40  |     |     |     |
| 3        | 720    | 2688| 2240|     |     |
| 4        | 40320  | 245376|443520|246400|
| 5        | 3628800| 31426560|90934272|107627520|44844800|

*Table 1.* $a_{g,t}$ of O-permutations of genus $g$ on $2g + t$ elements having no cycles of length 1 and $t$ cycles of length $> 1$.

Combining Lemma and Lemma we also derive the following functional relation between $C_g(z)$ and $S_g(z)$

**Corollary 4.** For $g \geq 1$, we have

$$C_g(z) = \frac{1 + zC_0(z)^2}{1 - zC_0(z)^2}S_g\left(\frac{zC_0(z)^2}{1 - zC_0(z)^2}\right),$$

where the generating function $C_0(z)$ of plane trees with $n$ edges is given by

$$C_0(z) = \sum_n \epsilon_0(n)z^n = \frac{1 - \sqrt{1 - 4z}}{2z}.$$

This functional relation can also be derived via symbolic methods. More precisely, we can construct a general unicellular map from a shape by first replacing each edge by a path and then attaching a plane tree to each sector.

We shall proceed by giving a bijective proof of a recurrence of $a_{g,t}$.

**Proposition 6.** For any $1 \leq t \leq g$, there exists a bijection

$$\mathcal{O}_{g,t,0} \simeq (2g + t - 1)(2g + t - 2)(\mathcal{O}_{g-1,t,0} + \mathcal{O}_{g-1,t-1,0}).$$

Therefore $a_{g,t}$ satisfies the recurrence

$$a_{g,t} = (2g + t - 1)(2g + t - 2)(a_{g-1,t} + a_{g-1,t-1}),$$

with $a_{1,1} = 2$ and $a_{g,t} = 0$ if $t < 1$ or $t > g$. The values $a_{g,t}$ for $g \leq 5$ are given in Table.

**Proof.** Set $n = 2g + t$. Let $\mathcal{F}$ and $\mathcal{G}$ denote the subsets of $\mathcal{O}_{g,t,0}$ where the cycle containing element $n$ has length 3 and greater than 3, respectively. For any $\sigma \in \mathcal{O}_{g,t,0}$, we have two scenarios

1. if $\sigma \in \mathcal{G}$, assume the cycle $c$ containing $n$ is of the form $(h, \ldots, i, j, n)$. Removing $j$ and $n$ from $c$, we obtain an O-permutation with $2g + t - 2$ elements and $t$ cycles, which, after natural relabeling, corresponds to an O-permutation $\sigma'$ contained in $\mathcal{O}_{g-1,t,0}$. There are $2g + t - 1$ ways to choose $j$ and $2g + t - 2$ ways to insert $j$ and $n$. Thus $\mathcal{G}$ is in bijection with $(2g + t - 1)(2g + t - 2)\mathcal{O}_{g-1,t,0}$.

2. if $\sigma \in \mathcal{F}$, then the cycle $c$ containing $n$ is of the form $(i, j, n)$. By deleting $c$ from $\sigma$, we obtain an O-permutation with $2g + t - 3$ elements with $t - 1$ cycles, which, after natural relabeling, corresponds to an O-permutation $\sigma'$ contained in $\mathcal{O}_{g-1,t-1,0}$. The number of ways to choose $i, j$ is $(2g + t - 1)(2g + t - 2)$, whence $\mathcal{F}$ is in bijection with $(2g + t - 1)(2g + t - 2)\mathcal{O}_{g-1,t-1,0}$. 

κ_{g,t} & t = 1 & 2 & 3 & 4 & 5 \\ \hline g = 1 & 1 & & & & \\ 2 & 21 & 105 & & & \\ 3 & 1485 & 18018 & 50050 & & \\ 4 & 225225 & 4660227 & 29099070 & 56581525 & \\ 5 & 59520825 & 1804142340 & 18472089636 & 78082504500 & 117123756750 \\ \hline

Table 2. The numbers κ_{g,t}.

Since \( O_{g,t,0} = G \sqcup F \), we have a bijection
\[
\beta: O_{g,t,0} \to (2g+t-1)(2g+t-2)(O_{g-1,t,0} + O_{g-1,t-1,0})
\]
and eq. (4.2) follows immediately. \qed

Remark: since \( O_{g,t,0} \)-elements can viewed as sets of cycles of odd lengths > 1, we can derive via symbolic methods [9]
\[
1 + \sum_{g \geq 1} \sum_{t=1}^{g} \frac{1}{(2g+t)!} a_{g,t} y^{2g+t} x^t = \left( \frac{1+y}{1-y} \right)^{\frac{1}{2}} \exp(-xy).
\]

We proceed by deriving a recurrence for \( \kappa_{g,t} \), which is analogous to the proof for Harer-Zagier recurrence (1.1) in [7].

**Theorem 3.** For any \( 1 \leq t \leq g \), there exists a bijection
\[
nR_{g,t,0} \simeq 2(2n-3) \cdot 2(2n-5) \left( (n-2)R_{g-1,t,0} + 2(2n-7)R_{g-1,t-1,0} \right),
\]
where \( n = 2g+t \). Therefore \( \kappa_{g,t} \) satisfies the recurrence
\[
(4.3) \quad (2g+t)\kappa_{g,t} = (2(2g+t)-3)(2(2g+t)-5)(2(2g+t)-7)\kappa_{g-1,t} + 2(2(2g+t)-7)\kappa_{g-1,t-1},
\]
where \( \kappa_{1,1} = 1 \) and \( \kappa_{g,t} = 0 \) if \( t < 1 \) or \( t > g \), see Table 2.

**Proof.** Let \( R_{g,t,0}^* \) denote the set of \( R_{g,t,0} \) O-trees with a labeled vertex, \( v \). Let \( J \) and \( K \) denote the subsets \( R_{g,t,0}^* \) where the cycle containing the labeled vertex has length 3 and length greater than 3, respectively.

For any \( (T, \sigma, v) \in R_{g,t,0}^* \), we have two scenarios

1. if \( (T, \sigma, v) \in K \), then the cycle \( c \) containing \( v \) is of the form \( (v', \ldots, v_1, v_2, v) \).

   Applying Rémy’s bijection twice to \( T \) with respect to \( v \) and \( v_2 \), we obtain the O-tree \( (T', \sigma') \) where \( T' \) has \( n-2 \) vertices and \( \sigma' \) is \( \sigma \)-induced by removing \( v \) and \( v_2 \) from \( c \) and subsequent relabeling \( \sigma \) according to \( T' \).

   The number of possible positions \( v_1 \) where we can insert \( v_2 \) and \( v \) back is \( n-2 \), whence \( K \) is in bijection with \( 2(2n-3) \cdot 2(2n-5) \cdot (n-2)R_{g-1,t,0} \).

2. if \( (T, \sigma, v) \in J \), then the cycle \( c \) containing \( v \) is of the form \( (v_1, v_2, v) \).

   Applying Rémy’s bijection three times to \( T \) with respect to \( v, v_1 \) and \( v_2 \), we obtain the O-tree \( (T', \sigma') \) where \( T' \) has \( n-3 \) vertices and the O-permutation \( \sigma' \) is induced...
by $\sigma$ by deleting $c = (v_1, v_2, v)$ and relabeling $\sigma$ according to $T'$. Therefore $\mathcal{J}$ is in bijection with $2(2n - 3) \cdot 2(2n - 5) \cdot 2(2n - 7)\mathcal{R}_{g-1,t-1,0}$.

Since $\mathcal{R}_{g,t,0}^* \simeq n\mathcal{R}_{g,t,0}$ and $\mathcal{R}_{g,t,0}^* = \mathcal{K} + \mathcal{J}$, we have a bijection $n\mathcal{R}_{g,t,0} \simeq 2(2n - 3) \cdot 2(2n - 5) \cdot (n-2)\mathcal{R}_{g-1,t,0} + 2(2n - 3) \cdot 2(2n - 5) \cdot 2(2n - 7)\mathcal{R}_{g-1,t-1,0}$, for any $1 \leq t \leq g$.

Since $|\mathcal{R}_{g,t,0}| = 2^{2g}\kappa_{g,t}$, it is clear that this bijection implies eq. (4.3). \qed

Remark:

$$1 + 2 \sum_{g \geq 1} \sum_{t=1}^{g} \frac{\kappa_{g,t}}{(2(2g + t) - 3)!!} y^{2g+t}x^t = \left(\frac{1 + y}{1 - y}\right)^x \exp(-2xy),$$

which implies eq. (4.3).

We next turn to log-concavity of $\{\kappa_{g,t}\}_{t=0}^g$.

**Definition 3.** A sequence $\{a_i\}_{i=0}^n$ of nonnegative real numbers is said to be unimodal if there exists an index $0 \leq m \leq n$, called the mode of the sequence, such that $a_0 \leq a_1 \leq \cdots \leq a_m-1 \leq a_m \geq a_{m+1} \geq \cdots \geq a_n$. The sequence is said to be logarithmically concave (or log-concave for short) if

$$a_i^2 \geq a_{i-1}a_{i+1}, \quad 1 \leq i \leq n - 1.$$

Clearly, log-concavity of a sequence with positive terms implies unimodality. Let us say that the sequence $\{a_i\}_{i=0}^n$ has no internal zeros if there do not exist integers $0 \leq i < j < k \leq n$ satisfying $a_i \neq 0, a_j = 0, a_k \neq 0$. Then, in fact, a nonnegative log-concave sequence with no internal zeros is unimodal. We call a polynomial $f(x) = \sum_{i=0}^n a_ix^i$ unimodal and log-concave if the sequence $\{a_i\}_{i=0}^n$ of its coefficients is unimodal and log-concave, respectively.

**Lemma 7.** Assume that the number $b_{g,t}$ satisfies the recurrence $b_{g,t} = p_{g,t}b_{g-1,t} + q_{g,t}b_{g-1,t-1}$, for all $g \geq 1$, where $b_{g,t}$, $p_{g,t}$, $q_{g,t}$ are all nonnegative. If

- $\{b_{1,t}\}_t$ is log-concave,
- $\{p_{g,t}\}_t$ and $\{q_{g,t}\}_t$ are log-concave for any $g \geq 1$,
- $p_{g,t-1}q_{g,t+1} + p_{g,t+1}q_{g,t-1} \leq 2p_{g,t}q_{g,t}$ for any $g \geq 1$,

then $\{b_{g,t}\}_t$ is log-concave for any $g \geq 1$.

**Proof.** We prove this by induction on $g$. By induction hypothesis, $b_{g-1,m}b_{g-1,n} \geq b_{g-1,m-1}b_{g-1,n+1}$ for any $m \leq n$. For $g$, we expand the product using the recurrence

$$b_{g,t}^2 = (p_{g,t}b_{g-1,t} + q_{g,t}b_{g-1,t-1})^2$$

$$= p_{g,t}^2b_{g-1,t}^2 + 2p_{g,t}q_{g,t}b_{g-1,t}b_{g-1,t-1} + q_{g,t}^2b_{g-1,t-1}^2$$

and

$$b_{g,t-1}b_{g,t+1} = (p_{g,t-1}b_{g-1,t-1} + q_{g,t-1}b_{g-1,t-2})(p_{g,t+1}b_{g-1,t+1} + q_{g,t+1}b_{g-1,t})$$

$$= p_{g,t-1}p_{g,t+1}b_{g-1,t-1}b_{g-1,t-2} + p_{g,t-1}q_{g,t+1}b_{g-1,t-1}b_{g-1,t} + q_{g,t-1}p_{g,t+1}b_{g-1,t-2}b_{g-1,t+1} + q_{g,t-1}q_{g,t+1}b_{g-1,t-2}b_{g-1,t}.$$
We now compare corresponding terms in the expansion. By assumption and induction hypothesis, it is clear that
\[
p_{g,t} b_{g-1,t}^2 \geq p_{g,t-1} p_{g,t+1} b_{g-1,t-1} b_{g-1,t+1},
\]
\[
q_{g,t} b_{g-1,t}^2 \geq q_{g,t-1} q_{g,t+1} b_{g-1,t-2} b_{g-1,t-1},
\]
Also we have
\[
2p_{g,t} q_{g,t} b_{g-1,t} b_{g-1,t-1} \geq (p_{g,t-1} q_{g,t+1} + q_{g,t-1} p_{g,t+1}) b_{g-1,t} b_{g-1,t-1}
\]
\[
\geq p_{g,t-1} q_{g,t+1} b_{g-1,t-1} b_{g-1,t} + q_{g,t-1} p_{g,t+1} b_{g-1,t-2} b_{g-1,t-1},
\]
whence the lemma.

\[\square\]

**Proposition 7.** For any fixed \( g \), the sequence \( \{a_{g,t}\}_{t=0}^{q} \) is log-concave.

\[\text{Proof.}\] We just need to verify the conditions in Lemma 7. Set \( p_{g,t} = q_{g,t} = (2g + t - 1)(2g + t - 2) \) for \( g \geq 1 \). It is clear that \( \{p_{g,t}\}_t \) and \( \{q_{g,t}\}_t \) are log-concave for any \( g \geq 1 \). Furthermore, \( p_{g,t-1} q_{g,t+1} + p_{g,t+1} q_{g,t-1} \leq 2p_{g,t} q_{g,t} \) for all \( g \geq 1 \), whence \( \{a_{g,t}\}_{t=0}^{q} \) is log-concave.

\[\square\]

**Proposition 8.** For any fixed \( g \), the sequence \( \{\kappa_{g,t}\}_{t=0}^{q} \) is log-concave.

\[\text{Proof.}\] Set \( p_{g,t} = (2(2g+t)-3)(2(2g+t)-5)(2g+t-2) \) and \( q_{g,t} = \frac{2(2(2g+t)-3)(2(2g+t)-5)(2g+t-7)}{2g+t} \) for \( g \geq 1 \). It is clear that \( \{p_{g,t}\}_t \) and \( \{q_{g,t}\}_t \) are log-concave for any \( g \geq 1 \) and \( p_{g,t-1} q_{g,t+1} + p_{g,t+1} q_{g,t-1} \leq 2p_{g,t} q_{g,t} \) for all \( g \geq 1 \). Therefore the sequence \( \{\kappa_{g,t}\}_{t=0}^{q} \) is log-concave.

\[\square\]

**Remark:** combining the inductive proof of Lemma 7 with the bijective proof of recurrences of \( a_{g,t} \) and \( \kappa_{g,t} \), we can construct an injection from \( O_{g,t,0} \times O_{g,t,0} \) into \( O_{g,t+1,0} \times O_{g,t-1,0} \) and from \( R_{g,t,0} \times R_{g,t,0} \) into \( R_{g,t+1,0} \times R_{g,t-1,0} \). This provides combinatorial proofs for the log-concavity of \( \{a_{g,t}\}_{t=0}^{q} \) and \( \{\kappa_{g,t}\}_{t=0}^{q} \).

5. Discussion

Define \( \mathcal{L} \) to be an operator acting on the sequence \( \{a_i\}_{i=0}^{n} \) as given by
\[
\mathcal{L}(\{a_i\}_{i=0}^{n}) = \{b_i\}_{i=0}^{n}
\]
where \( b_i = a_i^2 - a_{i-1} a_{i+1} \) for \( 0 \leq i \leq n \) under the convention that \( a_{-1} = a_{n+1} = 0 \). Clearly, the sequence \( \{a_i\}_{i=0}^{n} \) is log-concave if and only if the sequence \( \{b_i\}_{i=0}^{n} \) is nonnegative. Given a sequence \( \{a_i\}_{i=0}^{n} \), we say that it is \( k \)-fold log-concave, or \( k \)-log-concave, if \( \mathcal{L}^j(\{a_i\}_{i=0}^{n}) \) is a nonnegative sequence for \( 1 \leq j \leq k \). A sequence \( \{a_i\}_{i=0}^{n} \) is said to be infinitely log-concave if it is \( k \)-log-concave for all \( k \geq 1 \).

It is well-known that, by Newton’s inequality, if the polynomial \( \sum_{i=0}^{n} a_i x^i \) with positive coefficients has only real zeros, then the sequence \( \{a_i\}_{i=0}^{n} \) is unimodal and log-concave (see [10]). Such a sequence of positive numbers whose generating function has only real zeros is called a Pólya frequency sequence in the theory of total positivity (see [14] [4] [5]). Furthermore, we have

**Theorem 4.** If the polynomial \( f(x) = \sum_{i=0}^{n} a_i x^i \) has only real and non-positive zeros, then the sequence \( \{a_i\}_{i=0}^{n} \) is infinitely log-concave.
This is conjectured independently by Stanley, McNamara–Sagan [17] and Fisk [8], and proved by Brändén [3].

Let \( A_g(x) \) denote the generating polynomial of \( a_{g,t} \), i.e., \( A_g(x) = \sum_{t=0}^g a_{g,t}x^t \).

**Proposition 9.** For any fixed \( g \), polynomial \( A_g(x) \) has only real zeros located in \((-1, 0]\).

Therefore, the sequence \( \{a_{g,t}\}_{t=0}^g \) is infinitely log-concave.

**Proof.** Set \( B_g(x) = x^{2g}A_g(x) = \sum_{t=0}^g a_{g,t}x^{2g+t} \). It suffices to show that polynomial \( B_g(x) \) has only real zeros located in \((-1, 0]\).

Recurrence (4.2) of \( a_{g,t} \) is equivalent to

\[
B_g(x) = 2x^3B_{g-1}(x) + 2x^3(2x + 1)\frac{d}{dx}B_{g-1}(x) + x^4(x + 1)\frac{d^2}{dx^2}B_{g-1}(x),
\]

i.e.,

\[
(5.1) \quad B_g(x) = x^3 \frac{d^2}{dx^2}\left[x(x + 1)B_{g-1}(x)\right].
\]

By induction hypothesis, \( B_{g-1}(x) \) has all 3\( g-3 \) roots in \((-1, 0]\), \( 2g-1 \) of which are at 0. Then, applying Rolle’s theorem twice we obtain, for the second derivative in eq. (5.1), at least \( g-2 \) roots in \((-1, 0]\) and \( 2g-2 \) roots at 0. Hence, \( B_g(x) \) has all of its \( 3g \) roots inside \((-1, 0]\), \( 2g + 1 \) of which are at 0, and \( g-1 \) roots are in \((-1, 0]\). \( \square \)

Let \( K_g(x) \) denote the generating polynomial of \( \kappa_{g,t} \), i.e., \( K_g(x) = \sum_{t=0}^g \kappa_{g,t}x^t \).

**Conjecture 1.** For any fixed \( g \), polynomial \( K_g(x) \) has only real zeros located in \((-\frac{1}{4}, 0]\).

Therefore, the sequence \( \{\kappa_{g,t}\}_{t=0}^g \) is infinitely log-concave.

Similarly, set \( H_g(x) = x^{2g}K_g(x) = \sum_{t=0}^g \kappa_{g,t}x^{2g+t} \). It suffices to show that polynomial \( H_g(x) \) has only real zeros located in \((-\frac{1}{4}, 0]\).

Recurrence (4.3) of \( \kappa_{g,t} \) is equivalent to

\[
\frac{d}{dx}H_g(x) = -6x^2H_{g-1}(x) + 3x^2(12x + 1)\frac{d}{dx}H_{g-1}(x) + 12x^3(6x + 1)\frac{d^2}{dx^2}H_{g-1}(x) + 4x^4(4x + 1)\frac{d^3}{dx^3}H_{g-1}(x).
\]

By Lemma\( \blacksquare \) the generating polynomial of shapes is given by

\[
S_g(x) = \sum_{t=1}^g \kappa_{g,t}x^{2g+t-1}(1 + x)^{2g+t-1}
\]

\[
= x^{-1}(1 + x)^{-1}\sum_{t=1}^g \kappa_{g,t}x^{2g+t}(1 + x)^{2g+t}
\]

\[
= x^{-1}(1 + x)^{-1}H_g(x(x + 1)).
\]

Therefore Conjecture\( \square \) implies that the polynomial \( S_g(x) \) has also only real zeros.
Conjecture 2. For any fixed \( g \), the generating polynomial \( S_g(x) = \sum_{n=2g}^{6g-2} s_g(n) x^n \) of shapes given by has only real zeros. Therefore, the sequence \( \{s_g(n)\}_{n=2g}^{6g-2} \) is infinitely log-concave.

6. Appendix

6.1. Recursive decomposition of O-trees. In analogy to the decomposition of C-permutations and C-decorated trees \([7]\), we derive a recursive method to decompose O-permutations and O-trees. This decomposition can be viewed also as an analogue to the decomposition of unicellular maps \([6]\).

Given an O-permutation \( \pi \), we can represent \( \pi \) as an ordered list of its cycles, such that all cycles start with its minimal element and are ordered from left to right such that the minimal elements are in descending order. We call this representation the canonical form of \( \pi \).

Let \( S_n \) denote the set of permutations on \([n]\), i.e., sequences of integers. A sign sequence of length \( n \) is an \( n \)-tuple \((i_1, \ldots, i_n)\), where \( i_k = \pm \).

Lemma 8 (Chapuy et al. \([7]\)). There is a bijection between permutations on \([n]\) and pairs of an O-permutation on \([n]\) with \( n - 2g \) cycles and a sign sequence of length \( n - 2g - 1 \), for arbitrary \( 0 \leq g \leq k = \left\lfloor \frac{n-1}{2} \right\rfloor \), i.e.,

\[
\{S_n\} \simeq \bigcup_{g=0}^{k} \langle -, + \rangle^{n-2g-1} \times O_g(n).
\]

In particular, the O-permutation has one cycle if and only if the sequence has odd length and starts with its minimal element.

The bijection is illustrated in the following example:

\[
\begin{align*}
78326154 & \to 78|3|26|154 \to 78|3|26|^+(154) \to 78|36|-(2)|^+(154) \\
& \to 78|^+(6)|-(3)|-(2)|^+(154) \to (8)|^-|(7)|^+(6)|^-|(3)|^-|(2)|^+(154) \\
& \to (8)(7)(6)(3)(2)(154), (-, +, -) \to (8)(7)(6)(3)(2)(154), (-, +, -, +, +).
\end{align*}
\]

We adopt the convention that signed cycles are represented with the sign preceding the cycle as an exponent, such as \(-(12)\).

Proof (Chapuy et al. \([7]\)). Given a sequence \( S \in S_n \), decompose \( S = x_1x_2 \ldots x_n \) into blocks \( S_1S_2 \ldots S_l \) as follows: traverse the sequence \( S \) from left to right. Set \( s_1 = x_1 \) and \( s_i \) to be the first element smaller than all elements traversed before. This procedure generates blocks \( S_i \) that start with \( s_i \).

Then we define a process to deal with the blocks successively from right to left. At each step, we have two cases:

1. if the block \( B \) has odd length, turn \( B \) into the signed cycle \(^+(B)\);
2. if \( B \) has even length, move the second element \( x \) of \( B \) out of \( B \) and turn \( B \) into the signed cycle \(^-(B)\). If \( x \) is the minimum of the elements to the left of \( B \), set \( \{x\} \) to
be a singleton-block before \(-B\) and append \(x\) at the end of the block preceding \(B\), otherwise.

This right-to-left process ends up with the last block \(B\) having odd length and we produce \((B)\) as the last cycle. This process generates a sign sequence, \(I\), together with an O-permutation, \(\pi\).

By construction, \(\pi\) is represented in its canonical form and furthermore the number of signs generated by the process is one less than the number of cycles of the O-permutation. Accordingly the process defines the mapping

\[
\Phi: S_n \to \bigcup_{g=0}^{k} \{-, +\}^{n-2g-1} \times O_g(n), \quad (I, \pi) \mapsto S.
\]

Conversely, given an O-permutation \(\pi\) with \(n - 2g\) cycles and a sign sequence \(I\) of length \(n - 2g - 1\), write \(\pi\) in its canonical form. Assign each cycle except of the leftmost one with the corresponding sign from the sign sequence \(I\). Turn the leftmost unsigned cycle \((B)\) into the block \(B\). Then treat the signed cycles from left to right, starting with the second one, as follows: let \(\epsilon(B)\) be the signed cycle to be processed and let \(B'\) be the block to the left of \(\epsilon(B)\). Process \(\epsilon(B)\) into the block \(B\), by either just removing the sign if \(\epsilon = +\) or by removing the sign \(\epsilon = -\) and moving the last element of \(B'\) to the second position of \(B\). This generates an ordered list of blocks, which can be viewed as a sequence \(S\), i.e. we have

\[
\Psi: \bigcup_{g=0}^{k} \{-, +\}^{n-2g-1} \times O_g(n) \to S_n, \quad (I, \pi) \mapsto S.
\]

By construction, \(\Psi \circ \Phi = \text{id}\) and \(\Phi \circ \Psi = \text{id}\).

An element of an O-permutation is called non-minimal if it is not the minimum in its cycle. Non-minimal elements play the same role for O-permutations (and O-trees) as tri-sections for unicellular maps \([6]\). Indeed, an O-permutation of genus \(g\) has \(2g\) non-minimal elements (Lemma 3 in \([6]\)), and moreover we have Proposition 2 and Proposition 3, which are an analogue of Proposition 1.

**Proof of Proposition 2** (Chapuy et al. \([7]\)). For \(k \geq 1\), let \(O_g^*(n)\) be the set of O-permutations from \(O_g(n)\) having one labeled non-minimal element. Note that \(O_g^*(n) \simeq 2gO_g(n)\) since an O-permutation in \(O_g(n)\) has \(2g\) non-minimal elements.

Given \(\pi \in O_g^*(n)\), we write the cycle containing the labeled element \(i\) of \(\pi\) as a sequence beginning with \(i\) and apply bijection \(\Phi\) in Lemma 8. This gives a collection \(S'\) of \((2k+1)\) ≥ 3 cycles of odd length, together with a sign-sequence \(I\) of length \(2k\). Hence, replacing the cycle containing the labeled element \(i\) with these \((2k+1)\) cycles, we obtain an O-permutation \(\pi'\) of genus \(g - k\) with \(2k + 1\) labeled cycles.

We have thus shown that \(O_g^*(n) \simeq \bigcup_{k=1}^{g}(-, +)^{2k} \times O_{g-k}^{(2k+1)}(n) \simeq \bigcup_{k=1}^{g} 2^{2k}O_{g-k}^{(2k+1)}(n)\).

By construction of \(\Phi\), the cycles of \(\pi\) are obtained from the cycles of \(\pi'\) by merging labeled cycles in \(S'\) into a single cycle and the proposition follows.
Proof of Proposition 3 (Chapuy et al. [7]). We have by definition $T_g(n) = E_0(n) \times O_g(n + 1)$ and Proposition 2 guarantees $2g O_g(n) \simeq \biguplus_{k=1}^{g} \{-, +\}^{2k} \times \mathcal{O}_{g-k}^{(2k+1)}(n)$. Therefore we have

$$2gT_g(n) \simeq g \biguplus_{k=1}^{g} \{-, +\}^{2k} \times \mathcal{T}_{g-k}^{(2k+1)}(n).$$

The statement about the underlying graphs follows from the fact that the bijection $\Phi$ in Lemma 8 merges the labeled cycles into a unique cycle.

Proof of Theorem 1 (Chapuy et al. [7]). We fix $n$ and prove the theorem by induction on $g$. The case $g = 0$ is obvious, as there is only one $O$-permutation of size $(n + 1)$ and genus $0$, i.e., the identity permutation and both sides are the set of plane trees with $n$ edges.

Assume $g > 0$. The induction hypothesis ensures that for each $g' < g$, $2g' E_{g'}^{(2k+1)}(n) \simeq \mathcal{T}_{g'}^{(2k+1)}(n)$, where the underlying graphs of the corresponding objects are by construction the same. Thus we have

$$\biguplus_{k=1}^{g} 2^{2k} \cdot 2^{2(g-k)} E_{g-k}^{(2k+1)}(n) \simeq \biguplus_{k=1}^{g} 2^{2k} \cdot \mathcal{T}_{g-k}^{(2k+1)}(n).$$

Combining this with eq. (2.3) of Proposition 1 and eq. (2.3) of Proposition 3, we derive

$$2g 2^{2g} E_g(n) \simeq 2gT_g(n),$$

where the underlying graphs of corresponding objects are the same. Note that by construction of corresponding bijections in Propositions 1 and 3, the $2g$ factor never affects the underlying graphs of corresponding objects. Hence, we can extract from this $2g$-to-$2g$ correspondence a 1-to-1 correspondence, i.e., $2^{2g} E_g(n) \simeq T_g(n)$, which also preserves the underlying graphs of corresponding objects. The following diagram

$$
\begin{align*}
2g' E_{g'}(n) & \rightarrow T_{g'}(n) \\
2g' E_{g'}^{(2k+1)}(n) & \rightarrow T_{g'}^{(2k+1)}(n) \\
\biguplus_{k=1}^{g} 2^{2k} \cdot 2^{2(g-k)} E_{g-k}^{(2k+1)}(n) & \rightarrow \biguplus_{k=1}^{g} 2^{2k} \cdot T_{g-k}^{(2k+1)}(n) \\
2g 2^{2g} E_g(n) & \rightarrow 2gT_g(n) \\
2^{2g} E_g(n) & \rightarrow T_g(n)
\end{align*}
$$

depicts the construction of the bijection.
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