Linkage of Sets of Cyclic Algebras

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Abstract

Let $p$ be a prime integer and $F$ the function field in two algebraically independent variables over a smaller field $F_0$. We prove that if $\text{char}(F_0) = p \geq 3$ then there exist $p^2 - 1$ cyclic algebras of degree $p$ over $F$ that have no maximal subfield in common, and if $\text{char}(F_0) = 0$ then there exist $p^2$ cyclic algebras of degree $p$ over $F$ that have no maximal subfield in common.

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1. Introduction

A cyclic algebra of prime degree $p$ over a field $F$ takes the form

$$(\alpha, \beta)_{p,F} = F(x, y : x^p = \alpha, y^p = \beta, yxy^{-1} = px),$$

for some $\alpha, \beta \in F^\times$ when $\text{char}(F) \neq p$ and $F$ contains a primitive $p$th root of unity $\rho$. This algebra is a division algebra if $\alpha \notin (F^\times)^p$ and $\beta$ is not a norm in the field extension $F[\sqrt[p]{\alpha}] / F$, and otherwise it is the matrix algebra $M_p(F)$. When $\text{char}(F) = p$, a cyclic algebra of degree $p$ over $F$ takes the form

$$(\alpha, \beta)_{p,F} = F(x, y : x^p - x = \alpha, y^p = \beta, yxy^{-1} = x + 1),$$

for some $\alpha \in F$ and $\beta \in F^\times$. This algebra is a division algebra if $\alpha \notin \varphi(F) = \{\lambda^p - \lambda : \lambda \in F\}$ and $\beta$ is not a norm in the field extension $F[x : x^p - x = \alpha] / F$, and otherwise it is the matrix algebra $M_p(F)$. These algebras won their significance for being the generators of $\mu \text{Br}(F)$ (see [12] and [10, Chapter 9]). These algebras are called “quaternion algebras” when $p = 2$.

We say that cyclic algebras $A_1, \ldots, A_\ell$ of degree $p$ over $F$ are linked if they share a common maximal subfield. We say that $\mu \text{Br}(F)$ is $\ell$-linked if every $\ell$ cyclic algebras of degree $p$ over $F$ are linked.

The linkage properties of such algebras demonstrate a deeper phenomenon yet to be fully understood: clearly if $A$ and $B$ are linked then $A \otimes B$ is not a division algebra,
but for quaternion algebras the converse holds true as well. This means that \(z Br(F)\) is 2-linked if and only if its symbol length is \(\leq 1\) (i.e., every class is represented by a single quaternion algebra). Moreover, if \(z Br(F)\) is 2-linked then the \(u\)-invariant of \(F\) is either 0,1,2,4 or 8 (\([8]\) and \([5]\)), and for nonreal fields \(F\), \(z Br(F)\) is 3-linked if and only if \(u(F) \leq 4\) (see \([2]\) and \([3]\)).

For local fields \(F\), \(u Br(F)\) is clearly \(\ell\)-linked for any \(\ell\). It follows from the local-global principle (e.g., see \([7]\) and \([13]\) Proposition 15) that for global fields \(F\), \(u Br(F)\) is \(\ell\)-linked for any \(\ell\) too. A question was raised (\([2]\)) on whether function fields \(F = Br(F_0(\alpha, \beta))\) in two algebraically independent variables over algebraically closed fields \(F_0\) satisfy this property. It was answered in the negative for quaternion algebras (\([7]\) for \(char(F) = 0\) and \([3]\) for \(char(F) = 2\)), showing that for such fields \(z Br(F)\) is not 4-linked.

In the current paper, we extend this observation to cyclic algebras of odd prime degree \(p\) over \(F = Br(F_0(\alpha, \beta))\), showing that when \(char(F_0) = p\), the group \(p Br(F)\) is not \((p^2 - 1)\)-linked, and when \(char(F_0) = 0\), the group \(p Br(F)\) is not \(p^2\)-linked.

2. Characteristic \(p\)

**Lemma 2.1.** Let \(A = \{\alpha, \beta\}_{p,F}\) be a cyclic algebra of degree \(p\) generated by \(x\) and \(y\) over a field \(F\) of \(char(F) = p\), and write \(Tr : A \rightarrow F\) for its reduced trace map. Then for any \(\lambda = \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} c_{ij} x^i y^j \in A\), \(Tr(\lambda) = -c_{p-1,0}\).

**Proof.** For each \(j \in \{1, \ldots, p - 1\}\), every element \(v \in F(x) y^j\) satisfies \(v^p \in F\), and so \(Tr(v) = 0\). The problem therefore reduces to calculating \(Tr_{K/F}(x')\) where \(K = F(x)\) with \(x\) a root of the irreducible polynomial \(X^p - X - \alpha\) in \(F[X]\). For this, let \(L_{x'} : K \rightarrow K\) be the \(F\)-linear transformation given by multiplication by \(x'\). For \(i \in \{1, 2, \ldots, p - 1\}\), the matrix \([L_{x'}]\) of \(L_{x'}\) relative to the \(F\)-basis \(\{1, x, x^2, \ldots, x^{p-1}\}\) of \(K\) has the following form: a diagonal of \(1\)-s starting at the \((i + 1, 1)\)-entry (i.e., row \(i + 1\) and column \(1\)), a diagonal of \(\alpha\)-s starting at the \((1, p - i + 1)\)-entry, a \(1\) directly below each \(\alpha\), and all other entries are \(0\)-s. Thus, \(Tr_{K/F}(x') = Tr[L_{x'}] = 0\) for \(i \in \{1, 2, \ldots, p - 2\}\), while \(Tr[L_{x'}(x^{p-1})] = Tr[L_{x'}] = p - 1\).

**Remark 2.2.** The last statement appeared in \([4]\) Remark 2.2, but we provided here a simpler proof which was suggested by an anonymous colleague. Note that the trace argument works in a more general setting, in any characteristic and for roots \(x\) of any irreducible polynomial \(X^n - X - \alpha\) for any natural number \(n\).

**Theorem 2.3.** Let \(p\) be an odd prime, \(F_0\) a field of \(char(F_0) = p\) and \(F = Br(F_0(\alpha, \beta))\) the function field in two algebraically independent variables \(\alpha\) and \(\beta\) over \(F_0\). Then there exist \(p^2 - 1\) cyclic algebras of degree \(p\) over \(F\) that share no maximal subfield.

**Proof.** Note that \(F\) is endowed with the right-to-left \((\alpha^{-1}, \beta^{-1})\)-adic valuation, which we denote by \(v\). This is in fact the restriction to the standard rank 2 valuation on \(F_0(\alpha^{-1})(\beta^{-1})\). Write \(\Gamma_F\) for the value group of \(F\) with respect to \(v\). Note \(\Gamma_F = \mathbb{Z} \times \mathbb{Z}\). For each \((i, j) \in I = \{0, 1, \ldots, p - 1\}\), write \(A_{i,j} = \begin{cases} (\alpha^j \beta^{-1}, \beta)_{p,F} & i \neq 0 \\ (\beta^i, \alpha)_{p,F} & i = 0. \end{cases}\)
Since the values of $\alpha'\beta'$ and $\beta$ when $i \neq 0$ are negative and $\mathbb{F}_p$-independent in $\Gamma_F/p\Gamma_F$, the valuation $v$ extends to $A_{i,j}$ and $A_{i,j}$ is totally ramified over $F$ with value group $\Gamma_{A_{i,j}} = \frac{1}{p}\mathbb{Z} \times \frac{1}{p}\mathbb{Z}$ (see [14]). For a similar argument, the valuation extends also to $A_{0,j}$, when $j \neq 0$, and $A_{0,j}$ is totally ramified over $F$ with value group $\Gamma_{A_{0,j}} = \frac{1}{p}\mathbb{Z} \times \frac{1}{p}\mathbb{Z}$ (see also [4, Remark 3.2]). Write $V_{i,j}$ for the subspace of trace zero elements of $A_{i,j}$. It follows from Lemma 2.1 that $v(V_{i,j})/\Gamma_F \neq (\frac{1}{p}, \frac{2}{p})$, because writing $x$ and $y$ for the standard generators of $A_{i,j}$, $V_{i,j} = \text{Span}_{\mathbb{F}_p}(x^iy^j : (k, \ell) \in \{0, 1, \ldots, p-1\} \times \{(p-1, 0)\})$, the values of the $x^iy^j$'s are distinct modulo $\Gamma_F$ and none is congruent to $(\frac{1}{p}, \frac{2}{p})$. Therefore the intersection of all the $v(V_{i,j})$s modulo $\Gamma_F$ is trivial, which means that $\bigcap_{i,j \neq 0} v(V_{i,j}) = \Gamma_F$.

Now, suppose the contrary, that the algebras above share a maximal subfield $K$. Since $K$ is a subfield of each $A_{i,j}$, it is totally ramified over $F$. Write $W$ for its subspace of elements of trace 0. Then $\dim_F W$ is at least $p - 1$. Since $W \subseteq \bigcap_{i,j \neq 0} V_{i,j}$, the values of all the nonzero elements in $W$ are in $\Gamma_F$. Recall that $p > 2$. We can therefore choose two elements $w_1$ and $w_2$ in $K$ whose values are $\mathbb{F}_p$-independent in $\Gamma_K/\Gamma_F$. As a result, they are also linearly independent, and there is a nonzero linear combination of theirs $w_3 \in Fw_1 + Fw_2$ which lives in $W$. Hence, the value of $w_3$ is either $v(w_1)$ or $v(w_2)$. In either case, $v(w_3)$ is not in $\Gamma_F$, despite the fact that $w_3 \in W$, contradiction. Consequently, the algebras $A_{i,j}$ have no maximal subfield in common. \hfill $\Box$

**Remark 2.4.** Theorem 2.3 holds true also if one replaces $F_0(\alpha, \beta)$ with the field of iterated Laurent series $F_0((\alpha^{-1})(\beta^{-1}))$ in two variables over $F_0$ of char($F_0$) = $p$. This demonstrates another difference between the behaviour of Laurent series in the good characteristic and the bad characteristic, because over an algebraically closed field $F_0$ of char($F_0$) = 0, the group $\text{Br}(F)$ for $F = F_0((\alpha^{-1})(\beta^{-1}))$ is generated by a single division cyclic algebra of degree $p$ and thus $\text{Br}(F)$ is $\ell$-linked for any $\ell$.

### 3. Characteristic 0

For the proof of the main result, we need the following observation about inseparable field extensions:

**Lemma 3.1.** Let $E$ be a field of char($E$) = $p > 0$, and $\alpha \in E \setminus E^p$.

1. Then the $E^p$-vector spaces $V_i = \text{Span}_{E^p}((\alpha - i)^k : k \in \{1, \ldots, p-1\})$ for $i \in \{0, \ldots, p-1\}$ satisfy $\bigcap_{i=0}^{p-1} V_i = \{0\}$.

2. Furthermore, if there is another element $\beta \in E \setminus E^p(\alpha)$, then the vector spaces $W_{i,j}$ given by $W_{i,0} = \text{Span}_{E^p}((\alpha - i)^m\beta^n : m, n \in \{0, \ldots, p-1\}, (m, n) \neq (0, 0))$ and $W_{i,j} = \text{Span}_{E^p}(\alpha^m(\alpha\beta - j)^n : m, n \in \{0, \ldots, p-1\}, (m, n) \neq (0, 0))$ for $i \in \{0, \ldots, p-1\}$ and $j \in \{1, \ldots, p-1\}$, satisfy $\bigcap_{i,j=0}^{p-1} W_{i,j} = \{0\}$.

**Proof.** The first statement follows from the fact that an element $v = c_0 + c_1\alpha + \cdots + c_{p-1}\alpha^{p-1} \in E^p(\alpha)$ is in $V_i$ if and only if $c_0 + c_1i + c_2i^2 + \cdots + c_{p-1}i^{p-1} = 0$.

If we assume that $v \in \bigcap_{i=0}^{p-1} V_i$, then the polynomial $c_0 + c_1X + \cdots + c_{p-1}X^{p-1} \in E^p[X]$ has at least $p$ distinct roots in $E^p$ (which are the elements of the subfield $\mathbb{F}_p$), it must be the zero polynomial, i.e., $c_0 = c_1 = \cdots = c_{p-1} = 0$.  

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Theorem 3.2. Let $F$ and the intersection on the right-hand side is clearly trivial.

Proof. Since the intersection

$$W_{i,0} = \operatorname{Span}_{E^{(a)}}\{\beta^k : k \in \{1, \ldots, p - 1\}\} \oplus \operatorname{Span}_{E^{(a)}}\{(\alpha - i)^k : k \in \{1, \ldots, p - 1\}\} = \operatorname{Span}_{E^{(a)}}\{\beta^k : k \in \{1, \ldots, p - 1\}\} \oplus \operatorname{Span}_{E^{(a)}}\{(\alpha - i)^k : k \in \{1, \ldots, p - 1\}\}. $$

It follows from the first statement that

$$\bigcap_{i=0}^{p-1} W_{i,0} = \operatorname{Span}_{E^{(a)}}\{\beta^k : k \in \{1, \ldots, p - 1\}\}. $$

Now, for any $i \in \{0, \ldots, p - 1\}$, the space $W_{0,0}$ can also be written as

$$W_{0,0} = \operatorname{Span}_{E^{(a)}}\{\alpha^k : k \in \{1, \ldots, p - 1\}\} \oplus \operatorname{Span}_{E^{(a)}}\{(\alpha^i - 0)^k : k \in \{1, \ldots, p - 1\}\}, $$

and for any $j \in \{1, \ldots, p - 1\}$ we can write the space $W_{i,j}$ as

$$W_{i,j} = \operatorname{Span}_{E^{(a)}}\{\alpha^k : k \in \{1, \ldots, p - 1\}\} \oplus \operatorname{Span}_{E^{(a)}}\{(\alpha^i - j)^k : k \in \{1, \ldots, p - 1\}\} = \operatorname{Span}_{E^{(a)}}\{\alpha^k : k \in \{1, \ldots, p - 1\}\} \oplus \operatorname{Span}_{E^{(a)}}\{(\alpha^i - j)^k : k \in \{1, \ldots, p - 1\}\}. $$

It follows from the first statement that for any $i \in \{0, \ldots, p - 1\}$,

$$W_{0,0} \cap \left(\bigcap_{j=1}^{p-1} W_{i,j}\right) = \operatorname{Span}_{E^{(a)}}\{\alpha^k : k \in \{1, \ldots, p - 1\}\}. $$

Since the intersection $\bigcap_{(i,j) \in \{0, \ldots, p-1\}^2} W_{i,j}$ can be written as

$$\bigcap_{(i,j) \in \{0, \ldots, p-1\}^2} W_{i,j} = \left(\bigcap_{i=0}^{p-1} W_{i,0}\right) \cap \left(\bigcap_{i=1}^{p-1} \left( W_{0,0} \cap \left(\bigcap_{j=1}^{p-1} W_{i,j}\right)\right)\right), $$

we conclude that

$$\bigcap_{(i,j) \in \{0, \ldots, p-1\}^2} W_{i,j} = \operatorname{Span}_{E^{(a)}}\{\beta^k : k \in \{1, \ldots, p - 1\}\} \cap \left(\bigcap_{i=1}^{p-1} \left( \operatorname{Span}_{E^{(a)}}\{\alpha^k : k \in \{1, \ldots, p - 1\}\}\right)\right), $$

and the intersection on the right-hand side is clearly trivial. \qed

**Theorem 3.2.** Let $F_0$ be a field of char$(F_0) = 0$ and $F = F_0(\alpha, \beta)$ the function field in two algebraically independent variables over $F_0$. Then there exist $p^2$ cyclic algebras of degree $p$ over $F$ that have no maximal subfield in common.

**Proof.** Consider the algebras $A_{i,j} = (\gamma_{i,j}, \delta_{i,j})_{p,F}$ for $i, j \in \{0, \ldots, p - 1\}$ where $(\gamma_{i,j}, \delta_{i,j})$ are given (as elements of $F \times F$) by the formula

$$(\gamma_{i,j}, \delta_{i,j}) = \begin{cases} (\alpha - i, \beta) & (i, j) \in \{0, \ldots, p - 1\} \times \{0\} \\ (\alpha^i - j, \alpha) & (i, j) \in \{0, \ldots, p - 1\} \times \{1, \ldots, p - 1\} \end{cases}$$
In the rest of the proof we can assume that $F_0$ is algebraically closed. If it is not, we can extend scalars to $F_0^{alg}(\alpha, \beta)$. If the algebras do not have a common maximal subfield under this restriction, they did not have any common maximal subfield from the beginning. Denote by $V_{i,j}$ the subspace of $A_{i,j}$ of elements of trace zero. Every maximal subfield is generated by an element of trace zero, and therefore in order for the algebras to have a common maximal subfield, they must possess nonzero elements of trace zero of the same reduced norm. Write $\varphi_{i,j}$ for the restriction of the reduced norm to $V_{i,j}$, and thus a necessary condition for the algebras to share a maximal subfield is that the forms $\varphi_{i,j}$ for $i, j \in \{0, \ldots, p - 1\}$ represent a common nonzero value. Now, the $p$-adic valuation extends from $\mathbb{Q}$ to $F_0$ with residue field $k$, and thus to $F$ with residue field $E = k(\alpha, \beta)$. (See [9, Chapter 3] for details.) Since the value group of $F_0$ is divisible, if the forms represent a common nonzero value, we can suppose the equality $\varphi_{1,1}(v_1,1) = \cdots = \varphi_{p-1,p-1}(v_{p-1,p-1})$ is obtained for elements $v_{1,1}, \ldots, v_{p-1,p-1}$ of minimal value 0. If such a solution to the system above exists, then it gives rise to a solution to the system

$$\varphi_{1,1}(v_{1,1}) = \cdots = \varphi_{p-1,p-1}(v_{p-1,p-1})$$

and their value in $k(\alpha, \beta)$ is nonzero as the residue of an element of value zero.

The valuation extends from $F$ to the algebras $A_{i,j}$, which are unramified, and their residue algebras are $k(\sqrt[p]{\alpha}, \sqrt[p]{\beta})$. (This follows from [13, Proposition 3.38] and the fact that $k(\sqrt[p]{\alpha}, \sqrt[p]{\beta})$ is a degree $p^2$ field extension of $k$.) Fixing generators $x$ and $y$ for $A_{i,j}$, the image of the reduced norm of an element $t = \sum_{m=0}^{p} \sum_{n=0}^{p} c_{m,n} x^m y^n$ of value zero in the residue field $k(\alpha, \beta)$ is $\overline{t}^p$ (by [13, Lemma 11.16]), which is $\sum_{m=0}^{p} \sum_{n=0}^{p} \overline{c_{m,n}} x^m y^n$. If $\text{Tr}(t) = 0$ then $c_{0,0} = 0$. Thus, the solution to the system [1] gives rise to a nontrivial intersection of the $E^p$-vector spaces $W_{i,j} = \text{Span}\{\overline{\gamma}^m \delta^n : m, n \in \{0, \ldots, p - 1\}, (m,n) \neq (0,0)\}$ for $i, j \in \{0, \ldots, p - 1\}$. However, they intersect trivially by Lemma [3,1] Hence, the algebras $A_{i,j}$ for $i, j \in \{0, \ldots, p - 1\}$ share no maximal subfield.

4. In the opposite direction

It is important to point out what is known about the linkage of $p \cdot \text{Br}(F)$ for function fields $F = F_0(\alpha, \beta)$ over algebraically closed fields $F_0$:

- When $p = 2$, $2 \cdot \text{Br}(F)$ is 3-linked in any characteristic, so the story is complete in this case, for previous papers have shown that it need not be 4-linked.

- When $p = 3$, $3 \cdot \text{Br}(F)$ is 2-linked by an easy argument mentioned in [1] based on [11]. Here we show that it need not be 8-linked in characteristic 3, or 9-linked in characteristic 0. Between 2 and 8 or 9 there is still a significant gap.

- There are no results in this direction for $p > 3$ to the author’s knowledge. There are results on the related period-index problem but that does not settle the problem yet.

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References

[1] M. Artin. Brauer-Severi varieties. In *Brauer groups in ring theory and algebraic geometry (Wilrijk, 1981)*, volume 917 of *Lecture Notes in Math.*, pages 194–210. Springer, Berlin-New York, 1982.

[2] K. J. Becher. Triple linkage. *Ann. K-Theory*, 3(3):369–378, 2018.

[3] A. Chapman. Linkage of sets of quaternion algebras in characteristic 2. *Comm. Algebra*. to appear.

[4] A. Chapman and M. Chapman. Symbol $p$-algebras of prime degree and their $p$-central subspaces. *Arch. Math. (Basel)*, 108(3):241–249, 2017.

[5] A. Chapman and A. Dolphin. Differential forms, linked fields, and the u-invariant. *Arch. Math. (Basel)*, 109(2):133–142, 2017.

[6] A. Chapman, A. Dolphin, and D. B. Leep. Triple linkage of quadratic Pfister forms. *Manuscripta Math.*, 157(3-4):435–443, 2018.

[7] A. Chapman and J.-P. Tignol. Linkage of Pfister forms over $\mathbb{C}(x_1, \ldots, x_n)$. *Ann. K-Theory*, 4(3):521–524, 2019.

[8] R. Elman and T. Y. Lam. Quadratic forms and the $u$-invariant. II. *Invent. Math.*, 21:125–137, 1973.

[9] A. J. Engler and A. Prestel. *Valued fields*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2005.

[10] P. Gille and T. Szamuely. *Central simple algebras and Galois cohomology*, volume 101 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2006.

[11] S. Lang. On quasi algebraic closure. *Ann. of Math. (2)*, 55:373–390, 1952.

[12] A. S. Merkur’ev and A. A. Suslin. $K$-cohomology of Severi-Brauer varieties and the norm residue homomorphism. *Izv. Akad. Nauk SSSR Ser. Mat.*, 46(5):1011–1046, 1135–1136, 1982.

[13] A. S. Sivatski. Linked triples of quaternion algebras. *Pacific J. Math.*, 268(2):465–476, 2014.

[14] J.-P. Tignol. Classification of wild cyclic field extensions and division algebras of prime degree over a Henselian field. In *Proceedings of the International Conference on Algebra, Part 2 (Novosibirsk, 1989)*, volume 131 of *Contemp. Math.*, pages 491–508. Amer. Math. Soc., Providence, RI, 1992.

[15] J.-P. Tignol and A. R. Wadsworth. *Value Functions on Simple Algebras, and Associated Graded Rings*. Springer Monographs in Mathematics. Springer, 2015.