HIGHER SYZYGIES OF ELLIPTIC RULED SURFACES

FRANCISCO JAVIER GALLEGOL *
AND
B. P. PURNAPRAJNA

10 June 1995

Introduction
1. Background material
2. An example and a general result
3. Some lemmas and commutative diagrams
4. Cohomology vanishings on ruled elliptic surfaces with invariant \( e = -1 \)
5. Cohomology vanishings on elliptic ruled surfaces with invariant \( e \geq 0 \)
6. Syzygies of elliptic ruled surfaces
7. Open questions and conjectures

INTRODUCTION

The purpose of this article is to study the minimal free resolution of homogeneous coordinate rings of elliptic ruled surfaces.

Let \( X \) be an irreducible projective variety and \( L \) a very ample line bundle on \( X \), whose complete linear series defines the morphism

\[
\phi_L : X \rightarrow \mathbb{P}(H^0(L))
\]

Let \( S = \bigoplus_{m=0}^{\infty} S^m H^0(X, L) \) and \( R(L) \bigoplus_{m=0}^{\infty} H^0(X, L^\otimes m) \). Since \( R(L) \) is a finitely generated graded module over \( S \), it has a minimal graded free resolution. We say that the line bundle \( L \) is normally generated if the natural maps

\[
S^m H^0(X, L) \rightarrow H^0(X, L^\otimes m)
\]

are surjective for all \( m \geq 2 \). If \( L \) is normally generated, then we say \( L \) satisfies property \( N_p \), if the matrices in the free resolution of \( R \) over \( S \) have linear entries until the \( p \)th stage.

In this article we prove the following result (Theorem 6.1): Let \( X \) be an elliptic ruled surface and let \( L = B_1 \otimes ... \otimes B_{p+1} \) be a line bundle on \( X \), such that each \( B_i \)

We would like to thank our advisor David Eisenbud for his help, patience and encouragement. We would also like to thank Robert Lazarsfeld and Mohan Kumar for their encouragement and advice. * Partially supported by DGICYT, PB93-0440-C03-01.
is base-point-free and ample. Then $L$ satisfies property $N_p$. As a corollary of this result we show that the adjoint bundle $\omega_X \otimes A_1 \otimes \cdots \otimes A_{2p+3}$ satisfies property $N_p$, for arbitrary ample line bundles $A_i$.

To put things in perspective, we would like to recall some well known results in this area. On the subject of adjoint linear series, Reider recently proved (c.f. [R]) that if $X$ is a surface over the complex numbers and $A$ is an ample line bundle, then $\omega_X \otimes A^{\otimes 4}$ is very ample. Mukai has conjectured that $\omega_X \otimes A^{\otimes p+4}$ satisfies property $N_p$. Some work in this direction has been done by David Butler in [Bu], where he studies the syzygies of adjoint linear series on ruled varieties. He proves that if the dimension of $X$ is $n$, then $\omega_X \otimes A^{\otimes 2n+1}$ is normally generated and $\omega_X \otimes A^{\otimes 2n+2np}$ satisfies property $N_p$. In particular if $X$ is a ruled surface Butler’s result says that $\omega_X \otimes A^{\otimes 5}$ is normally generated and $\omega_X \otimes A^{\otimes 4+4p}$ satisfies property $N_p$.

Of course a more general question would be: given a very ample line bundle $L$ on $X$, what is the largest $p$ such that property $N_p$ holds for $L$? A relevant result in this line is due to Yuko Homma (c.f. [Ho1] and [Ho2]), who has classified all line bundles which are normally generated on an elliptic ruled surface. Another result is obtained in [GP], where we characterize those line bundles which satisfy property $N_1$ on an elliptic ruled surface. Therefore, it is in the light of the above more general question that Theorem 6.1 should be regarded. Certainly the mentioned results provide information about the particular case of adjoint linear series. The result of Homma and our results in [GP] imply Mukai’s conjecture, in the case of elliptic ruled surfaces, for $p = 0$ and $p = 1$ respectively. Accordingly, as a corollary of Theorem 6.1 we improve the bound obtained by Butler by almost a factor of two in the case of elliptic ruled surfaces.

One of the tools we use in this article is the so-called Koszul cohomology, developed by Mark Green, which links the study of the vanishing of graded Betti numbers of minimal free resolutions to the study of cohomology vanishings of certain vector bundles. We also use a theorem of Castelnuovo on surjectivity of multiplication maps of global sections, generalized by Mumford, (c.f. Theorem 1.3). Then in Sections 2 and 3 we develop the machinery that will allow us to prove our main results. For the set-up of this machinery we use induction on the number of base-point-free divisors and on the dimension of the variety. Briefly, we will associate to a line bundle, which is a product of certain number of base-point-free line bundles, another bundle. We will show that the first cohomology group of the associated bundle vanishes. To achieve this goal for a variety $X$ of arbitrary dimension we restrict the line bundle to a divisor of $X$ and then use induction on the dimension. And to achieve the goal for an arbitrary number of base-point-free divisors in the mentioned product, we use induction on the number of base-point-free divisors.

The usefulness of these constructions is not limited to the example of elliptic ruled surfaces. In this article (see Theorem 2.2) we also obtain from them, results for all surfaces with geometric genus 0 (a class that includes Enriques surfaces). In fact our results can be summarized in the following principle: let $L$ be the tensor product of $p+1$ ample, base-point-free line bundles; If certain cohomology vanishings occur...
then $L$ should satisfy property $N_p$. This principle holds in wider generality: In two forthcoming articles ([GP1],[GP2]) we use the machinery developed here to show that the principle mentioned above holds for surfaces with $\kappa = 0$, Fano varieties of dimension $n$ with index bigger than or equal to $n-2$ and elliptic surfaces. We also show that the principle holds, with some minor modifications which depend on the dimension of the variety, for pluricanonical models of surfaces of general type and Calabi-Yau threefolds. The former answers an open question in [B].

1. Background material

Convention. Throughout this paper we will work over an algebraically closed field $k$.

One of the tools we will use is this beautiful cohomological characterization by Green of the property $N_p$ (c.f. [G], [GL], [L]). Let $L$ be a globally generated line bundle. We define the vector bundle $M_L$ as follows:

$$0 \to M_L \to H^0(L) \otimes \mathcal{O} \to L \to 0$$

(1.1)

 Lemma 1.2. Let $L$ be a normally generated, nonspecial line bundle on a surface $X$ with geometric genus $0$. Then, $L$ satisfies the property $N_p$ iff $H^1(\wedge^{p'+1} M_L \otimes L) = 0$ for all $1 \leq p' \leq p$.

Proof. See [GL] §1.

(1.2.1) If the characteristic of $k$ is strictly bigger than $p+1$, then the vanishings of $H^1(\wedge^{p'+1} M_L \otimes L)$ for all $0 \leq p' \leq p$ will follow from the vanishings of $H^1(M_L^{\otimes p'+1} \otimes L)$, because $\wedge^{p'+1} M_L \otimes L$ is a direct summand of $M_L^{\otimes p'+1} \otimes L$.

The other main tool we will use is a generalization by Mumford of a lemma of Castelnuovo:

Theorem 1.3. Let $L$ be a base-point-free line bundle on a variety $X$ and let $\mathcal{F}$ be a coherent sheaf on $X$. If $H^i(\mathcal{F} \otimes L^{-i}) = 0$, for all $i \geq 1$, then the multiplication map

$$H^0(\mathcal{F} \otimes L^{\otimes i}) \otimes H^0(L) \to H^0(\mathcal{F} \otimes L^{\otimes i+1})$$

is surjective, for all $i \geq 0$.

Proof. [Mu], p. 41, Theorem 2. Note that the assumption there of $L$ being ample is unnecessary.

It will be useful to have the following characterization of projective normality:

Lemma 1.4. Let $X$ be a surface with geometric genus $0$ and let $L$ be an ample, base-point-free line bundle. If $H^1(L) = 0$, then $L$ is normally generated iff $H^1(M_L \otimes L) = 0$.

Proof. See [GP], Lemma 1.4.
2. An example and a general result

In [GP] we gave a complete characterization of the divisors on an elliptic ruled surface satisfying the property $N_1$. In particular we showed ([GP], Corollary 4.4) that

\begin{equation}
(2.1) \quad B^\otimes 2 \text{ satisfies the property } N_1 \text{ if } B \text{ is ample and base-point-free.}
\end{equation}

To give an idea of how we will generalize to higher syzygies the results and techniques from [GP] we will focus in this section on the generalization of (2.1). We recall that the statement in (2.1) was shown to be true for a larger class of surfaces (see [GP], Proposition 2.1 and Corollary 2.8), namely, those with $p_g = 0$, if one requires $B$ to be nonspecial (the latter condition is automatically satisfied by ample base-point-free line bundles on elliptic ruled surfaces). Thus, we will prove the following

**Theorem 2.2.** Let $X$ be a surface with $p_g = 0$. Let $\text{char } k > p + 1$ or equal to 0. Let $B$ be a nonspecial, ample, and base-point-free line bundle. Then $B^\otimes p+1$ satisfies the property $N_p$, for all $p \geq 1$.

(2.2.1) The same statement is false for $p = 0$. Consider for instance an elliptic ruled surface of invariant $e = -1$, let $C_0$ be a minimal section of $X$ and let $B = \mathcal{O}_X(2C_0)$. $B$ is an ample, base-point-free line bundle (c.f. Propositions 3.4 and 3.5) but it is not very ample since its restriction to $C_0$ is not very ample.

Before we prove Theorem 2.2 we will require the following

**Lemma 2.3.** Let $X$ and $B$ be as in Theorem 2.2. If $p \geq 1$ and $p_1,p_2 \geq p$, then the cohomology groups $H^1(M^\otimes p+1 \otimes B^\otimes p_1 \otimes B^\otimes p_2)$ and $H^2(M^\otimes p+1 \otimes B^\otimes p_1 \otimes B^\otimes p_2)$ vanish.

Before we give the proof of Lemma 2.3 we make two observations

**Observation 2.4.** Let $X$ be a surface with geometric genus 0, let $B$ be a base point free line bundle and let $P$ be an effective line bundle such that $H^1(P) = H^1(B) = 0$. Then $H^1(B \otimes P) = 0$.

**Observation 2.5.** Let $X$ be a surface, let $P$ be an effective line bundle and $L$ a coherent sheaf. If $H^2(L) = 0$, then $H^2(L \otimes P) = 0$.

(2.6) Proof of Lemma 2.3. The proof is by induction on $p$. If $p = 1$ we have to prove that

\begin{align}
(2.6.1) \quad & H^1(M^\otimes 2 \otimes B^\otimes b) = 0 \\
(2.6.2) \quad & H^2(M^\otimes 2 \otimes B^\otimes b^{-1}) = 0
\end{align}

for all $a,b \geq 2$. Note that if $H^1(M^\otimes a \otimes B^\otimes b) = 0$, the vanishing in (2.6.1) is equivalent to the surjectivity of the following multiplication map:

$$H^0(M^\otimes a \otimes B^\otimes b) \otimes H^0(B^\otimes a) \rightarrow H^0(M^\otimes a \otimes B^\otimes a+b).$$
To show the surjectivity of $\alpha$ it suffices to show the surjectivity of
\[ H^0(M_{B^a} \otimes B^{b}) \otimes H^0(B)^a \to H^0(M_{B^{a+b}} \otimes B^{a+b}) \].

From all the above and from Theorem 1.3, it follows that in order to prove (2.6.1),
it is enough to show that
\[
\begin{align*}
(2.6.3) & \quad H^1(M_{B^a} \otimes B^{b'} - 1) = 0 \\
(2.6.4) & \quad H^2(M_{B^a} \otimes B^{b'' - 2}) = 0
\end{align*}
\]
for all $a, b' \geq 2$. From Observation 2.4 it follows that $H^1(B^{\otimes c}) = 0$ for all $c \geq 1$.
Therefore the vanishing (2.6.3) is equivalent to the surjectivity of the map
\[ H^0(B^a) \otimes H^0(B^{b'} - 1) \to H^0(B^{a+b' - 1}). \] (2.6.5)
Thus it suffices to show the surjectivity of
\[ H^0(B^{\otimes r}) \otimes H^0(B)^s \to H^0(B^{\otimes r+s}) \]
for all $r$ and $s$ such that $r \geq s$, $r \geq 2$ and $s \geq 1$. This follows at once from Theorem 1.3, since, by Observations 2.4 and 2.5, $H^1(B^{\otimes c}) = H^2(B^{\otimes d}) = 0$ for all $c \geq 1$ and all $d \geq 0$.

From the exact sequence 1.1, it follows that the vanishings of both $H^1(B^{\otimes a+b'-2})$ and $H^2(B^{\otimes b''-2})$ imply (2.6.4). Now we prove (2.6.2). From the exact sequence 1.1 it is enough to show that $H^1(M_{B^a} \otimes B^{a+b-1})$ and $H^2(M_{B^a} \otimes B^{a+b-1})$ vanish. These vanishings are special cases of (2.6.3) and (2.6.4).

Now assume that the result is true for $p - 1$. Then in particular $H^1(M_{B^a}^{p+1} \otimes B^{p+1}) = 0$ and therefore the vanishing of the group $H^1(M_{B^a}^{p+1} \otimes B^{p+1})$ is equivalent to the surjectivity of the following map:
\[ H^0(M_{B^a}^{p+1} \otimes B^{p+1}) \otimes H^0(B^{p+1}) \to H^0(M_{B^a}^{p+1} \otimes B^{p+1+p+2}). \]
The surjectivity of $\gamma$ follows from the surjectivity of
\[ H^0(M_{B^a}^{p+1} \otimes B^{p+2}) \otimes H^0(B^{p+1}) \to H^0(M_{B^a}^{p+1} \otimes B^{p+2}) \]
and this in turn follows from Theorem 1.3 and induction hypothesis.

To show that $H^2(M_{B^a}^{p+1} \otimes B^{p+2}) = 0$ it suffices, again by the exact sequence 1.1, to check that the groups $H^1(M_{B^a}^{p+1} \otimes B^{p+1+p+1})$ and $H^2(M_{B^a}^{p+1} \otimes B^{p+2})$ vanish. This is true by induction. \(\square\)

\[
(2.7) \textbf{Proof of Theorem 2.2.} \text{ By Lemmas 1.2 and 1.4 and by (1.2.1) it is enough to show that } H^1(M_{B^a}^{p'+1} \otimes B^{p'+1}) \text{ vanishes for all } 0 \leq p' \leq p. \text{ The vanishing follows when } 1 \leq p' \leq p \text{ as a particular case of Lemma 2.3. Since } H^1(B^{\otimes c}) = 0 \text{ for all } c \geq 1, \text{ the vanishing of } H^1(M_{B^a}^{p+1} \otimes B^{p+1}) \text{ is equivalent to the surjectivity of the multiplication map }
\]
\[ H^0(B^{\otimes p+1}) \otimes H^0(B^{\otimes p+1}) \to H^0(B^{\otimes 2p+2}). \]
and that is a special case of (2.6.5). \(\square\)
Corollary 2.7.1. Let $X$ be an Enriques surface over an algebraic closed field of characteristic 0. Let $B$ be an ample base-point-free line bundle. Then $B^\otimes p + 1$ satisfies the property $N_p$, for all $p \geq 1$.

Proof. Since $K_X \equiv 0$ and $B$ is ample, $\omega_X \otimes B$ is also ample and by Kodaira vanishing, $H^1(B) = 0$. Thus we can apply Theorem 2.2. 

In Theorem 2.2 we have dealt with line bundles which are powers of a base-point-free line bundle. Obviously not all the line bundles on a surface $X$ are of this form. Therefore we want now to study the syzygies of a wider variety of line bundles. For this purpose it is convenient to abstract and somehow generalize the formalism of Lemma 2.3. We will do so in the next lemma, which is key to the proof of Propositions 4.1, 4.2 and 5.1, on which the results of Section 6 are based.

Lemma 2.8. Let $X$ be a surface. Let $q_0$ be a positive integer and let $\mathcal{B}$ and $\mathcal{P}$ be two subsets of $\text{Pic}(X)$ satisfying the following properties:

2.8.1. All elements in $\mathcal{B}$ are base-point-free and if $B \in \mathcal{B}$ and $B \equiv B'$, then $B' \in \mathcal{B}$. The set $\mathcal{B}$ is contained in $\mathcal{P}$ and if $P^1$ and $P^2$ belong to $\mathcal{P}$, then $P^1 \otimes P^2$ belongs to $\mathcal{P}$.

2.8.2. For all $B_1, \ldots, B_{q_0+3} \in \mathcal{B}$, the line bundle $B_1^\otimes 2 \otimes B_2 \otimes \cdots \otimes B_{q_0+2} \otimes B_{q_0+3}^*$ belongs to $\mathcal{P}$.

2.8.3. For all $B_1, \ldots, B_{q_0+1}, B_1', \ldots, B_{q_0+2}, C_1, \ldots, C_n \in \mathcal{B}$ such that $B_i \equiv B'_i$ and for any line bundle $P \in \mathcal{P}$, the line bundles
\[
R_3 = B_1 \otimes \cdots \otimes B_{q_0+1} \otimes C_1 \otimes \cdots \otimes C_n \quad \text{and} \quad R'_3 = B'_1 \otimes \cdots \otimes B'_{q_0+1} \otimes P \otimes B'^*_{q_0+2}
\]
satisfy $H^2(M_{R_3}^\otimes q_0 + 1 \otimes R'_3) = 0$.

2.8.4. For all $B_1, \ldots, B_{q_0+1}, B_1', \ldots, B_{q_0+1}, C_1, \ldots, C_n \in \mathcal{B}$ such that $B_i \equiv B'_i$ and for any line bundle $P \in \mathcal{P}$, the line bundles
\[
R_4 = B_1 \otimes \cdots \otimes B_{q_0+1} \otimes C_1 \otimes \cdots \otimes C_n \quad \text{and} \quad R'_4 = B'_1 \otimes \cdots \otimes B'_{q_0+1} \otimes P
\]
satisfy $H^1(M_{R_4}^\otimes q_0 + 1 \otimes R'_4) = 0$.

Given $q \geq q_0$, let $T_1, \ldots, T_{q+1}, T'_1, \ldots, T'_{q+1}, S_1, \ldots, S_n \in \mathcal{B}$ such that $T_i \equiv T'_i$ and let $Q \in \mathcal{P}$. If $R = T_1 \otimes \cdots \otimes T_{q+1} \otimes S_1 \otimes \cdots \otimes S_n$ and $R' = T'_1 \otimes \cdots \otimes T'_{q+1} \otimes Q$, then
\[
H^1(M_{R}^\otimes q + 1 \otimes R') = 0.
\]

Proof. We prove the lemma using induction on $q$. If $q = q_0$ the result is just Condition 2.8.4. Now assume that the result is true for $q_0, \ldots, q-1$. After tensoring exact sequence 1.1 by $M_R^\otimes q \otimes R'$ and taking global sections we obtain:
\[
H^0(M_R^\otimes q \otimes R') \otimes H^0(R) \xrightarrow{\alpha} H^0(M_R^\otimes q \otimes R' \otimes R) \\
\rightarrow H^1(M_R^\otimes q + 1 \otimes R') \rightarrow H^1(M_R^\otimes q \otimes R') \otimes H^0(R)
\]
Using 2.8.1 and induction hypothesis on $q-1$ it follows that $H^1(M_R^{\otimes q} \otimes R')$ vanishes. Thus the surjectivity of $\alpha$ is equivalent to the vanishing of the group $H^1(M_R^{\otimes q+1} \otimes R')$. We argue like this: The surjectivity of $\alpha$ follows from the surjectivity of

$$H^0(M_R^{\otimes q} \otimes R') \otimes \bigotimes_{i=1}^{q+1} H^0(T_i) \otimes \bigotimes_{j=1}^{n} H^0(S_j) \xrightarrow{\beta} H^0(M_R^{\otimes q} \otimes R' \otimes R)$$

and to obtain the surjectivity of $\beta$, by Theorem 1.3, it is sufficient to check the following vanishings:

(2.8.5) $$H^1(M_R^{\otimes q} \otimes T_1^{\otimes 2} \otimes \cdots \otimes T_{i-1}^{\otimes 2} \otimes T_{i+1} \otimes \cdots \otimes T_{q+1} \otimes N \otimes Q) = 0$$ for all $1 \leq i \leq q+1$ and any $N$ nef

(2.8.6) $$H^2(M_R^{\otimes q} \otimes T_1^{\otimes 2} \otimes \cdots \otimes T_{i-1}^{\otimes 2} \otimes T_{i+1} \otimes \cdots \otimes T_{q+1} \otimes T_i^* \otimes N \otimes Q) = 0$$ for all $1 \leq i \leq q+1$ and any $N$ nef

(2.8.7) $$H^1(M_R^{\otimes q} \otimes T_1^{\otimes 2} \otimes \cdots \otimes T_{q+1}^{\otimes 2} \otimes S_1 \otimes \cdots \otimes S_{j-1} \otimes S_j^* \otimes N \otimes Q) = 0$$ for all $1 \leq j \leq n$ and any $N$ nef

(2.8.8) $$H^2(M_R^{\otimes q} \otimes T_1^{\otimes 2} \otimes \cdots \otimes T_{q+1}^{\otimes 2} \otimes S_1 \otimes \cdots \otimes S_{j-1} \otimes S_j^{-2} \otimes N \otimes Q) = 0$$ for all $1 \leq j \leq n$ and any $N$ nef

First we check (2.8.6). If $q = q_0+1$, (2.8.6) follows from Conditions 2.8.1 ($B \otimes N \in \mathfrak{B}$ for all $B \in \mathfrak{B}$) and 2.8.3. If $q \geq q_0 + 2$, using exact sequence 1.1 it suffices to check that

(2.8.9) $$H^1(M_R^{\otimes k} \otimes R \otimes T_1^{\otimes 2} \otimes \cdots \otimes T_{i-1}^{\otimes 2} \otimes T_{i+1} \otimes \cdots \otimes T_{q+1} \otimes T_i^* \otimes N \otimes Q) = 0$$ for all $1 \leq i \leq q+1$, for all $q_0 + 1 \leq k \leq q-1$ and any $N$ nef

(2.8.10) $$H^2(M_R^{\otimes q_0+1} \otimes T_1^{\otimes 2} \otimes \cdots \otimes T_{i-1}^{\otimes 2} \otimes T_{i+1} \otimes \cdots \otimes T_{q+1} \otimes T_i^* \otimes N \otimes Q) = 0$$ for all $1 \leq i \leq q+1$ and any $N$ nef

The vanishing in (2.8.10) follows from Condition 2.8.1 and Condition 2.8.3. We will postpone the proof of (2.8.9) for the moment. Now we check (2.8.8): If $q = q_0 + 1$, then (2.8.8) follows from Conditions 2.8.1, 2.8.2 and 2.8.3. If $q \geq q_0 + 2$, again using
Then, for all $F$ line bundle on $X$, line bundle on $X$ so does the tensor product of Lemma 3.1. Let three lemmas will be a key element in the proofs of Propositions 4.1, 4.2 and 5.1. They give us a way to prove that if a line bundle $L$ of $M$ the cohomology of $L$ Consider in addition to $M$ to relate the vanishing of the cohomology of a similar bundle on a divisor $Y$ of $X$, obtained by restricting $L_1$ and $L_2$ to $Y$. The second and third lemma deal roughly with the following situation: Consider in addition to $L_1$ and $L_2$, two “bigger” line bundles $L_1'$ and $L_2'$ (in the sense that $L_1' \otimes L_2'$ is an effective line bundle). We would like to relate the vanishing of the cohomology of $M_{\mathcal{L}_1'}^{\otimes p+1} \otimes L_2$ and $M_{\mathcal{L}_1}^{\otimes p+1} \otimes L_2$ to the vanishing of the cohomology of $M_{\mathcal{L}_1}^{\otimes p+1} \otimes L_2$. The usefulness of these kinds of results is quite clear. For example, they give us a way to prove that if a line bundle $L$ satisfies the property $N_p$, then so does the tensor product of $L$ with certain effective line bundles. Therefore these three lemmas will be a key element in the proofs of Propositions 4.1, 4.2 and 5.1.

**Lemma 3.1.** Let $X$ be a projective variety, let $q$ be a nonnegative integer and let $F_i$ be a base-point-free line bundle on $X$ for all $1 \leq i \leq q+1$. Let $Q$ be an effective line bundle on $X$ and let $q$ be a reduced and irreducible member of $|Q|$. Let $R$ be a line bundle on $X$ such that

3.1.1. $H^1(F_i \otimes Q^*) = 0$
3.1.2. $H^1(R \otimes \mathcal{O}_q) = 0$
3.1.3. $H^1(M(F_{i1} \otimes \mathcal{O}_q) \otimes \cdots \otimes M(F_{iq'+1} \otimes \mathcal{O}_q) \otimes R) = 0$ for all $0 \leq q' \leq q$

Then, for all $-1 \leq q'' \leq q$ and any subset $\{j_k\} \subseteq \{i\}$ with $\# \{j_k\} = q'' + 1$ and for all $0 \leq k' \leq q'' + 1$,

$$H^1(M_{F_{j_1}} \otimes \cdots \otimes M_{F_{j_{k'}}} \otimes M_{(F_{k'+1} \otimes \mathcal{O}_q)} \otimes \cdots \otimes M_{(F_{q''+1} \otimes \mathcal{O}_q)} \otimes R \otimes \mathcal{O}_q) = 0$$
Proof. We prove the result by induction on $q''$. For $q'' = -1$ the corresponding statement is nothing but Condition 3.1.2. Assume that the result is true for $q'' - 1$. In order to prove the result for $q''$ we will use induction on $k'$. If $k' = 0$, the statement is just Condition 3.1.3. Assume that the result is true for $k' - 1$. Because of Condition 3.1.1 we can write for $F_i$ this commutative diagram:

\[
\begin{array}{c}
0 \\
\downarrow \\
0 \rightarrow H^0(F_i \otimes Q^*) \otimes O_q \\
\downarrow \\
0 \rightarrow M_{F_i} \otimes O_q \\
\downarrow \\
0 \rightarrow M(F_i \otimes O_q) \\
\downarrow \\
0 \rightarrow H^0(F_i \otimes O_q) \otimes O_q \\
\downarrow \\
0 \rightarrow F_i \otimes O_q \\
\rightarrow 0
\end{array}
\]

Setting $i = j_{k'}$, tensoring the left hand side vertical exact sequence by

\[
M_{F_j} \otimes \cdots \otimes M_{F_{j_{k'-1}}} \otimes M(F_{j_{k'+1}} \otimes O_q) \otimes \cdots \otimes M(F_{j_{q''+1}} \otimes O_q) \otimes R \otimes O_q
\]

and taking global sections we obtain this sequence:

\[
H^0(F_{j_{k'}} \otimes Q^*) \otimes H^1(\bigotimes_{r=1}^{k'-1} M_{F_{j_{r}}} \otimes \bigotimes_{r=k'+1} M(F_{j_{r}} \otimes O_q) \otimes R \otimes O_q)
\]

\[
\rightarrow H^1(\bigotimes_{r=1}^{k'-1} M_{F_{j_{r}}} \otimes \bigotimes_{r=k'+1} M(F_{j_{r}} \otimes O_q) \otimes R \otimes O_q)
\]

\[
\rightarrow H^1(\bigotimes_{r=1}^{k'-1} M_{F_{j_{r}}} \otimes \bigotimes_{r=k'} M(F_{j_{r}} \otimes O_q) \otimes R \otimes O_q).
\]

The group

\[
H^1(M_{F_{j_1}} \otimes \cdots \otimes M_{F_{j_{k'-1}}} \otimes M(F_{j_{k'+1}} \otimes O_q) \otimes \cdots \otimes M(F_{j_{q''+1}} \otimes O_q) \otimes R \otimes O_q)
\]

vanishes by the induction hypothesis for $q'' - 1$ and

\[
H^1(M_{F_{j_1}} \otimes \cdots \otimes M_{F_{j_{k'-1}}} \otimes M(F_{j_{k'}} \otimes O_q) \otimes \cdots \otimes M(F_{j_{q''+1}} \otimes O_q) \otimes R \otimes O_q)
\]

vanishes by induction on $k'$ (we have assumed the result to be true for $q''$ and $k' - 1$). Therefore we obtain the vanishing of the group sitting in the middle of (3.1.4). □
Lemma 3.2. Let $X$ be a projective variety, let $q$ be a nonnegative integer and let $F_i$ be a base-point-free line bundle on $X$ for all $1 \leq i \leq q+1$. Let $Q$ be an effective line bundle on $X$ and let $q$ be a reduced and irreducible member of $|Q|$. Let $R$ be a line bundle on $X$ such that

3.2.1. $H^i(F_i \otimes Q^*) = 0$
3.2.2. $H^i(R \otimes Q \otimes \mathcal{O}_q) = 0$
3.2.3. $H^i(M(F_{i_1} \otimes \mathcal{O}_q) \otimes \cdots \otimes M(F_{i_{q+1}} \otimes \mathcal{O}_q) \otimes R \otimes Q) = 0$ for all $0 \leq q' \leq q$

If $H^i(M_{F_1} \otimes \cdots \otimes M_{F_{q+1}} \otimes R) = 0$, then

$$H^i(M_{F_1} \otimes \cdots \otimes M_{F_{q+1}} \otimes R \otimes Q) = 0.$$  

Proof. From the exact sequence

$$0 \rightarrow Q^* \rightarrow \mathcal{O} \rightarrow \mathcal{O}_q \rightarrow 0,$$

after tensoring by $M_{F_1} \otimes \cdots \otimes M_{F_{q+1}} \otimes R \otimes Q$ and taking global sections we obtain

$$H^i(M_{F_1} \otimes \cdots \otimes M_{F_{q+1}} \otimes R) \rightarrow H^i(M_{F_1} \otimes \cdots \otimes M_{F_{q+1}} \otimes R \otimes Q) \rightarrow H^i(M_{F_1} \otimes \cdots \otimes M_{F_{q+1}} \otimes R \otimes Q \otimes \mathcal{O}_q).$$

The group $H^i(M_{F_1} \otimes \cdots \otimes M_{F_{q+1}} \otimes R)$ vanishes by hypothesis. To obtain the vanishing of $H^i(M_{F_1} \otimes \cdots \otimes M_{F_{q+1}} \otimes R \otimes Q \otimes \mathcal{O}_q)$ we use Lemma 3.1 (the line bundle $R$ in Lemma 3.1 is now $R \otimes Q$ and we set $q'' = q$ and $k' = q + 1$). □

Lemma 3.3. Let $X$ be a variety. Let $F$, $Q$ and $R$ be line bundles on $X$ such that $F$ and $F \otimes Q$ are base-point-free and $Q$ is effective. Let $q$ be an effective divisor in $|Q|$, reduced and irreducible. Let $q$ be an integer. Assume that there exists an integer $q_0 \leq q$ such that for all $0 \leq l \leq q - q_0 - 1$, the following conditions are satisfied:

3.3.1. $H^i(F) = H^i(F \otimes Q^*) = 0$
3.3.2. $H^i(R \otimes Q^{-l} \otimes \mathcal{O}_q) = 0$
3.3.3. $H^i(M_{(F \otimes \mathcal{O}_q)}^{i+j} \otimes M_{(F \otimes Q \otimes \mathcal{O}_q)}^{j} \otimes R \otimes Q^{-l}) = 0$ for all $1 \leq i + j \leq q - l + 1$
3.3.4. $H^i(M_{F}^{q-l+1} \otimes R \otimes Q^{-l}) = 0$
3.3.5. $H^i(M_{F}^{q'} \otimes M_{F \otimes Q}^{q'} \otimes R \otimes Q^{-(q-q_0)}) = 0$ for all $\alpha' + \beta' q_0 + 1$.

Then, $H^i(M_{F}^{\alpha} \otimes M_{F \otimes Q}^{\beta} \otimes R \otimes Q^{-m}) = 0$ for all $m$ such that $0 \leq m \leq q - q_0$ and for all $\alpha, \beta$ nonnegative integers such that $\alpha + \beta - 1 = q - m$. In particular, $H^i(M_{F \otimes Q}^{q+1} \otimes R) = 0$.

Proof. We prove the lemma by induction on $q_0 \leq q' = q - m \leq q$. If $q - m = q_0$, the conclusion of the theorem is just Condition 3.3.5. Assume that the statement
is true for \( q' - 1 = q - m \). We will show that it also holds for \( q' = q - m \). Now consider \( \alpha \) and \( \beta \) such that \( \alpha + \beta - 1 = q' \). We use induction on \( \beta \). If \( \beta = 0 \), the statement is just Condition 3.3.4 considered for \( l = q - q' \). Assume that the theorem holds for \( \beta - 1 \) and we will prove that it holds also for \( \beta \). We will consider two commutative diagrams which yield two exact sequences relating the bundles \( M_F, M_{F \otimes Q} \) and \( M_{F \otimes Q \otimes \mathcal{O}_q} \) (we will set \( F \otimes Q \otimes \mathcal{O}_q = \mathcal{G} \), for notational convenience):

\[
\begin{array}{cccc}
0 & 0 & 0 \\
0 & M_F & \rightarrow & H^0(F) \otimes \mathcal{O}_X \\
0 & M_{F \otimes Q} & \rightarrow & H^0(F \otimes Q) \otimes \mathcal{O}_X \\
0 & K & \rightarrow & H^0(G) \otimes \mathcal{O}_X \\
0 & H^1(M_{F \otimes \alpha} \otimes M_{F \otimes Q} \otimes R \otimes Q^{-(q-q')}) & \rightarrow & H^1(M_{F \otimes \alpha} \otimes M_{F \otimes Q} \otimes R \otimes Q^{-(q-q')}) \\
0 & H^1(M_{F \otimes \alpha} \otimes M_{F \otimes Q} \otimes R \otimes Q^{-(q-q')}) & \rightarrow & H^1(M_{F \otimes \alpha} \otimes M_{F \otimes Q} \otimes R \otimes Q^{-(q-q')}) \\
0 & H^1(M_{F \otimes \alpha} \otimes M_{F \otimes Q} \otimes R \otimes Q^{-(q-q')}) & \rightarrow & H^1(M_{F \otimes \alpha} \otimes M_{F \otimes Q} \otimes R \otimes Q^{-(q-q')}) \\
0 & H^1(M_{F \otimes \alpha} \otimes M_{F \otimes Q} \otimes R \otimes Q^{-(q-q')}) & \rightarrow & H^1(M_{F \otimes \alpha} \otimes M_{F \otimes Q} \otimes R \otimes Q^{-(q-q')}) \\
\end{array}
\]

Note that the exactness at the bottom of the central vertical column of the first diagram follows from Condition 3.3.1. The two exact sequences we are interested in are the ones in the left hand side of each diagram. From the first one, after tensoring by \( M_{F \otimes \alpha} \otimes M_{F \otimes Q} \otimes R \otimes Q^{-(q-q')} \) and taking global sections, we obtain the sequence

\[
H^1(M_{F \otimes \alpha}^{\otimes +1} \otimes M_{F \otimes Q}^{\otimes +1} \otimes R \otimes Q^{-(q-q')}) \rightarrow H^1(M_{F \otimes \alpha} \otimes M_{F \otimes Q}^{\otimes +1} \otimes R \otimes Q^{-(q-q')}) \\
\rightarrow H^1(K \otimes M_{F \otimes \alpha} \otimes M_{F \otimes Q}^{\otimes +1} \otimes R \otimes Q^{-(q-q')}).
\]

The group \( H^1(M_{F \otimes \alpha}^{\otimes +1} \otimes M_{F \otimes Q}^{\otimes +1} \otimes R \otimes Q^{-(q-q')}) \) vanishes because, by induction on \( \beta \), we have assumed the result to be true for \( q' = q - m \) and \( \beta - 1 \). Therefore we need only to check that \( H^1(K \otimes M_{F \otimes \alpha} \otimes M_{F \otimes Q}^{\otimes +1} \otimes R \otimes Q^{-(q-q')}) \) vanishes. For that we use the left hand side exact sequence of the second diagram. After tensoring it
by $M_F^\otimes \otimes M_F^\otimes Q^{-1} \otimes R \otimes Q^{-(q-q')}$ and taking global sections we obtain

$$H^0(G) \otimes H^1(M_F^\otimes \otimes M_F^\otimes Q^{-1} \otimes R \otimes Q^{-(q-q')}) \rightarrow H^1(K \otimes M_F^\otimes \otimes M_F^\otimes Q^{-1} \otimes R \otimes Q^{-(q-q')}) \rightarrow H^1(M_G \otimes M_F^\otimes \otimes M_F^\otimes Q^{-1} \otimes R \otimes Q^{-(q-q')})$$

The group $H^1(M_F^\otimes \otimes M_F^\otimes Q^{-1} \otimes R \otimes Q^{-(q-q')})$ vanishes by induction hypothesis for $q' - 1 = q - m$. The vanishing of the cohomology group $H^1(M_G \otimes M_F^\otimes \otimes M_F^\otimes Q^{-1} \otimes R \otimes Q^{-(q-q')})$ is obtained from Lemma 3.1 using Conditions 3.3.1 to 3.3.3.

The vanishing of $H^1(M_F^{q+1} \otimes R)$ is obtained from the general statement by setting $m = 0$ and $\beta = q + 1$. □

The remaining lemmas and proposition are less general (they are stated for surfaces with geometric genus 0) and they basically yield a slightly more general version of the vanishing of cohomology obtained in [GP], Proposition 2.1, which we will need in the arguments of Sections 4 and 5 when we apply Lemma 2.8 (concretely when we check that Condition 2.8.4 is satisfied).

**Lemma 3.4.** Let $X$ be a surface with geometric genus 0, let $F_1$ and $F_2$ be two base-point-free, nonspecial line bundles and let $R = F_1 \otimes F_2$. Assume moreover that if $F_1' \equiv F_1$, then $F_1'$ is base-point-free and nonspecial. If $H^2(F_2 \otimes F_1') = 0$ for all $F_1' \equiv F_1$, then $H^1(M_R' \otimes F_1'^n) = 0$, for all $n \geq 1$ and all $F_1' \equiv F_1$.

**Proof.** Mimic word by word the proof of Lemma 2.5 of [GP] with $R$ playing here the role of $L$; $F_1$, $F_1'$ and $F_1''$, the role of $B_1$ and $F_2$, the role of $B_2$. □

**Lemma 3.5.** Let $X$ be a surface with geometric genus 0, let $F_1$ and $F_2$ be two base-point-free line bundles and let $R = F_1 \otimes F_2$. Assume that $R'$ is nonspecial for all $R' \equiv R$. Assume also that if $F_1' \equiv F_1$ and $F_2' \equiv F_2$, then $F_1'$ and $F_2'$ are base-point-free and they satisfy the conditions $H^1(F_1'^2) = H^1(F_2') = 0$ and $H^2(F_2' \otimes F_1'^*) = H^2(F_1'^2 \otimes F_2'^*) = 0$.

If $Q$ is any effective line bundle on $X$ such that either $H^1(Q) = 0$ or $Q \simeq O$, then $H^1(M_R \otimes R' \otimes Q) = 0$ for any $R' \equiv R$.

**Proof.** Mimic word by word the proof of Lemma 2.6 in [GP] with $R$ and $R'$ playing the role of $L$ and $F_1$ and $F_1'$, the role of $B_1$). □

**Proposition 3.6.** Let $X$ be a surface with geometric genus 0, let $F_1$ and $F_2$ be base-point-free line bundles and let $R = F_1 \otimes F_2$. Assume that if $F_1' \equiv F_1$ and $F_2' \equiv F_2$, $F_1'$ and $F_2'$ are base-point-free and nonspecial and that they satisfy $H^2(F_2' \otimes F_1'^*) = H^2(F_2' \otimes F_1'^*) = 0$.

Then $H^1(M_R^{q+1} \otimes R') = 0$ for $q = 0, 1$ and any $R' \equiv R$.

**Proof.** Mimic the proof of Proposition 2.1 in [GP], using now Lemmas 3.4 and 3.5 instead of Lemmas 2.5 and 2.6 of [GP] and with $R$ and $R'$ playing the role of $L$ and $F_i$ and $F_i'$, the role of $B_i$). □
4. Cohomology vanishings on ruled elliptic surfaces with invariant $e = -1$

In this section $X$ will denote an elliptic ruled surface with invariant $e = -1$. This means that $X = \mathbb{P}(E)$, where $E$ is a normalized vector bundle of rank 2 and degree 1 over a smooth elliptic curve $C$. We set $\mathcal{O}(e) = \wedge^2 E$ and $e = -\text{deg} e = -1$. We fix a minimal section $C_0$ such that $\mathcal{O}(C_0) = \mathcal{O}_{\mathbb{P}(E)}(1)$. The group $\text{Num}(X)$ is generated by $C_0$ and by the class of a fiber. We will denote by $f$ the class of a fiber of $X$. If $a$ is a divisor on $D$, $af$ will denote the pullback of $a$ to $X$ by the projection from $X$ to $D$. Sometimes when $\text{deg} a = 1$ we will write, by an abuse of notation, $f$ instead of $a$. The canonical divisor $K_X$ is linearly equivalent to $-2C_0 + ef$, and hence numerically equivalent to $-2C_0 + f$.

As we said at the beginning of Section 2, we want to obtain sufficient conditions for a line bundle $L$ to satisfy the property $N_p$. We will be considering $L$ to be a nonspecial, normally generated line bundle, hence (see Section 1, especially Lemma 1.2 and (1.2.1)) we are interested in knowing when the group $H^1(M^{\otimes p+1} \otimes L)$ vanishes. Sometimes the approach to a particular problem is simplified by considering instead a more general problem. So we do here: In this section and in the next, using the results from Sections 2 and 3, we obtain sufficient conditions on line bundles $L_1$ and $L_2$, so that the group $H^1(M^{\otimes p+1} \otimes L_1 \otimes L_2)$ vanishes. In this line, the following are the main results of this section:

**Proposition 4.1.** Let $B_1^1 = B_1^1 \otimes \cdots \otimes B_{p+1}^1$ and $B_2^2 = B_1^2 \otimes \cdots \otimes B_{p+1}^2$ be line bundles on $X$ such that $B_i^1 \equiv B_i^2$ and $B_i^j$ is in the numerical class of either $2C_0$ or $C_0 + f$. Let $P_1^1$ and $P_2^2$ be two effective line bundles on $X$ such that $P_1^1$ is in the numerical class of $aC_0 + bf$ for some $a, b \geq 0$. If $L_i = B_i^i \otimes P_i$, then

$$H^1(M^{\otimes p+1}_{L_1} \otimes L_2) = 0$$

**Proposition 4.2.** Let $p \geq 1$. Let $B_1^1 \otimes \cdots \otimes B_{p+1}^1$ and $B_2^2 = B_1^2 \otimes \cdots \otimes B_{p+1}^2$ be line bundles on $X$ such that $B_i^3$ is in the numerical class of $2C_0$. Let $P_1^1$ and $P_2^2$ be two effective line bundles on $X$. If $L_i = B_i^i \otimes P_i$, then

$$H^1(M^{\otimes p+1}_{L_1} \otimes L_2) = 0$$

To prove the above propositions we need to do some preliminary work. We start by recalling when a line bundle on $X$ is ample, when is base point free, when is effective and when its higher cohomology vanishes.

**Proposition 4.3 ([GP], Proposition 3.1; [Ho1], §2, [Ho2], Proposition 2.3).**

Let $L$ be a line bundle on $X$, numerically equivalent to $aC_0 + bf$. Then
Proposition 4.4 ([GP], Proposition 3.2).

4.4.1. There exist three effective line bundles in the numerical class of $2C_0 - f$. They are $\mathcal{O}(2C_0 + (e + \eta_i)f)$, where the $\eta_i$s are the nontrivial degree 0 divisors corresponding to the three nonzero torsion points in Pic$^0(C)$. The unique element in $|2C_0 + (e + \eta_i)f|$ is a smooth elliptic curve $Y_i$. 

4.4.2. For each $n > 0$, there are only four effective line bundles numerically equivalent to $n(2C_0 - f)$. They are $\mathcal{O}(2nC_0 + n(e + \eta_i)f)$ and $\mathcal{O}(2nC_0 + n\epsilon f)$. The only smooth (elliptic) curves (and indeed the only irreducible curves) in these numerical classes are general members in $|4C_0 + 2\epsilon f|$.

The number of linearly independent global sections of these line bundles are summarized in the following table:

| $a$   | $b$  | $h^0(L)$ | $h^1(L)$ | $h^2(L)$ |
|-------|------|----------|----------|----------|
| $a \geq 0$ | $b > -a/2$ | $>0$ | $0$ | $0$ |
|       | $b = -a/2$ | ? | ? | 0 |
|       | $b < -a/2$ | 0 | $>0$ | 0 |
| $a = -1$ | any $b$ | 0 | 0 | 0 |
| $a \leq -2$ | $b > -a/2$ | 0 | $>0$ | 0 |
|       | $b = -a/2$ | 0 | ? | ? |
|       | $b < -a/2$ | 0 | 0 | $>0$ |

(4.4.3) We will fix one of the smooth elliptic curves in the numerical class of $2C_0 - f$ and we will call it $E$.

Proposition 4.5 ([H], V.2.21.b; [GP], Proposition 3.5 and Remark 3.5.3). Let $L$ be a line bundle on $X$ in the numerical class of $aC_0 + bf$.

4.5.1 $L$ is ample iff $a > 0$ and $b > -\frac{1}{2}a$

4.5.2 $L$ is base point free if $a \geq 0$, $a + b \geq 2$ and $a + 2b \geq 2$.

4.5.3 $L$ is ample and base point free iff $a \geq 1$, $a + b \geq 2$ and $a + 2b \geq 2$.

We will need some lemmas dealing with the vanishing of the cohomology of certain bundles on curves:
Lemma 4.6. Let $p \geq -1$ and let $B_i$ be a line bundle on $\mathbb{P}^1$ for all $1 \leq i \leq p+1$, such that $b_i \deg(B_i) \geq 1$. Let $L$ be a line bundle on $\mathbb{P}^1$ such that $l = \deg(L) \geq p$. Then $H^1(M_{B_1} \otimes \cdots \otimes M_{B_{p+1}} \otimes L) = 0$.

Proof. Note in the first place that each $B_i$ is base-point-free so it makes sense to define $M_{B_i}$. The bundle $M_{B_i}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus b_i}$ (the sequence defining $M_{B_i}$ is the sheafification of $0 \to S^{b_i}(-1) \to S^{b_i+1} \to S(b_i) \to 0$ where $S$ denotes the homogeneous coordinate ring of $\mathbb{P}^1$). Hence using that

$$H^1(M_{B_1} \otimes \cdots \otimes M_{B_{p+1}} \otimes L) = H^1(\mathcal{O}(l-p-1)^{\oplus (\prod_{i=1}^{p+1} b_i)})$$

and that $l - p - 1 \geq -1$, we obtain the result. \qed

Lemma 4.7. Let $Y$ be a smooth elliptic curve. Let $p \geq -1$ and let $B_i$ be a line bundle on $Y$ for all $1 \leq i \leq p + 1$. Let $L$ be another line bundle on $Y$. Let $b_i = \deg(B_i) \geq 2$ and let $l = \deg(L)$. If $\sum_{i=1}^{p+1} \frac{b_i}{b_i - 1} < l$, then

$$H^1(M_{B_1} \otimes \cdots \otimes M_{B_{p+1}} \otimes L) = 0.$$

In particular, if $b_i \geq p + 3$ for all $1 \leq i \leq p + 1$ and $l \geq p + 2$, then

$$H^1(M_{B_1} \otimes \cdots \otimes M_{B_{p+1}} \otimes L) = 0.$$

Proof. Note first that the $B_i$s are base-point-free, since their degrees are greater or equal than 2; hence $M_{B_i}$ makes sense. Let $r_i = r(B_i) = h^0(B_i) - 1$. Then

$$\text{rk}(M_{B_1} \otimes \cdots \otimes M_{B_{p+1}} \otimes L) = r_1 \cdots r_{p+1} \quad \text{and}$$

$$\deg(M_{B_1} \otimes \cdots \otimes M_{B_{p+1}} \otimes L) = l \cdot r_1 \cdots r_{p+1} - \sum_{i=1}^{p+1} b_i \cdot r_1 \cdots \hat{r}_i \cdots r_{p+1}$$

and therefore

$$\mu(M_{B_1} \otimes \cdots \otimes M_{B_{p+1}} \otimes L) = l - \sum_{i=1}^{p+1} \frac{b_i}{r_i}$$

The bundle $M_{B_1} \otimes \cdots \otimes M_{B_{p+1}} \otimes L$ is semistable by [Bu], Theorem 1.2 and [Mi], Corollary 4.9 and §5. Therefore $H^1(M_{B_1} \otimes \cdots \otimes M_{B_{p+1}} \otimes L) = 0$ if $\mu(M_{B_1} \otimes \cdots \otimes M_{B_{p+1}} \otimes L) > 2g(Y) - 2$. Since $Y$ is elliptic, $r_i = b_i - 1$ and $2g(Y) - 2 = 0$ and the conclusion of the lemma is clear. \qed

Now we prove some lemmas which have in account the particular properties of elliptic ruled surfaces.
Lemma 4.8. Let \( L = B_1 \otimes B_2 \) and \( L' = B_3 \) be line bundles on \( X \) satisfying the following properties:

4.8.1 \( B_1 \equiv B_3 \)
4.8.2 \( B_1 \) is in the numerical class of \( 2C_0 \) or in the numerical class of \( C_0 + f \).
4.8.3 \( B_2 \) is the numerical class of \( 2C_0 \), in the numerical class of \( C_0 + f \) or in the numerical class of \( 2f \).

Let \( P \) and \( P' \) be effective. Then the map

\[
H^0(L \otimes P) \otimes H^0(L' \otimes P') \xrightarrow{\alpha} H^0(L \otimes L' \otimes P \otimes P')
\]

surjects.

Proof. We start by noting that the multiplication map

\[
H^0(L \otimes P) \otimes H^0(L') \xrightarrow{\beta} H^0(L \otimes L' \otimes P)
\]

surjects. This is a consequence of Theorem 1.3. Indeed. The line bundle \( L' \) is base-point-free by Proposition 4.5 and since \( L \otimes P \otimes L' \equiv B_2 \otimes P \) and \( L \otimes P \otimes L'^{-2} \equiv B_2 \otimes B_3^* \otimes P \), it follows from Proposition 4.3 that \( H^1(L \otimes P \otimes L'^*) = H^2(L \otimes P \otimes L'^{-2}) = 0 \). Since, by Proposition 4.3, \( H^1(L') = 0 \), the surjectivity of \( \beta \) is equivalent to the vanishing of the group \( H^1(M_{L \otimes P} \otimes L') \). Since, again by Proposition 4.3, \( H^1(L' \otimes P') = 0 \), the surjectivity of \( \alpha \) is equivalent to the vanishing of \( H^1(M_{L \otimes P} \otimes L' \otimes P') \). We use Lemma 3.2 to prove the latter vanishing. We can assume without loss of generality that \( P' \mathcal{O}(aC_0 + bf + cE) \). Thus we carry out induction on \((a,b,c)\). If \((a,b,c)(0,0,0)\) the content of the statement we want to prove is nothing but the vanishing of \( H^1(M_{L \otimes P} \otimes L') \), which we have just shown. Now we assume the result to be true for \((a-1,0,0)\) and we will prove that it is also true for \((a,0,0)\). For that we apply Lemma 3.2 to \( q = 0 \), \( F_1 = L \otimes P \), \( Q = \mathcal{O}(C_0) \), \( Q = C_0 \) and \( R = L' \otimes \mathcal{O}((a-1)C_0) \). We need to see that the conditions required by Lemma 3.2 are satisfied. For Condition 3.2.1 it is enough to check that \( H^1(L \otimes P \otimes \mathcal{O}(-C_0)) = 0 \) and this is true by Proposition 4.3. Using that \( \deg(L' \otimes \mathcal{O}(-C_0)) \geq 3 \) we see that Condition 3.2.2 is satisfied. Condition 3.2.3 follows from Lemma 4.7 because \( \deg(L' \otimes \mathcal{O}(-C_0)) \geq 3 \) and \( \deg(L \otimes P \otimes \mathcal{O}(-C_0)) \geq 4 \). The argument for the induction on \( b \) and \( c \) is analogous (in the case of \( b \) we use Lemma 4.6 instead of Lemma 4.7) to the one we have just made and we will not show it here.

Lemma 4.9. Let \( a, b \) be two integers such that \( a \geq 1 \), \( a + b \geq 4 \) and \( a + 2b \geq 4 \). Let \( L \) be a line bundle in the numerical class of \( aC_0 + bf \) and \( P \) a line bundle whose numerical class contains an effective representative. Then

\[
H^2(M_L \otimes P) = 0
\]

Proof. Note first that \( L \) is base-point-free (c.f. Proposition 4.5) and therefore it make sense to talk about \( M_L \). From exact sequence 1.1 we obtain these two exact

\[
\begin{align*}
&H^0(L \otimes P) \otimes H^0(L') \xrightarrow{\beta} H^0(L \otimes L' \otimes P) \\
&\xrightarrow{\alpha} H^0(L \otimes L' \otimes P' \otimes P) \xrightarrow{\gamma} H^0(L \otimes L' \otimes P' \otimes P) \\
&\xrightarrow{\delta} H^0(L \otimes L' \otimes P' \otimes P) \xrightarrow{\epsilon} H^0(L \otimes L' \otimes P' \otimes P)
\end{align*}
\]

\[
\begin{align*}
&\xrightarrow{\theta} H^0(L \otimes L' \otimes P' \otimes P) \\
&\xrightarrow{\varphi} H^0(L \otimes L' \otimes P' \otimes P) \xrightarrow{\psi} H^0(L \otimes L' \otimes P' \otimes P) \\
&\xrightarrow{\chi} H^0(L \otimes L' \otimes P' \otimes P) \\
&\xrightarrow{\delta} H^0(L \otimes L' \otimes P' \otimes P) \xrightarrow{\epsilon} H^0(L \otimes L' \otimes P' \otimes P)
\end{align*}
\]
sequences:

$$H^1(M_L \otimes L \otimes P) \to H^2(M_L^\otimes \otimes P) \to H^0(L) \otimes H^2(M_L \otimes P)$$

$$H^1(L \otimes P) \to H^2(M_L \otimes P) \to H^0(L) \otimes H^2(P)$$

The vanishing of $H^1(M_L \otimes L \otimes P)$ follows from Lemma 4.5 and Proposition 4.3. The vanishings of $H^1(L \otimes P)$ and $H^2(P)$ follow from Proposition 4.3, and hence we obtain the result. □

**Lemma 4.10.** Let $B$ be a line bundle in the numerical class of $2(p+1)C_0$ for some $p \geq 1$. Then one can choose a divisor $\mathfrak{d}$ of degree 1 on $C$ and $B_i$s in the numerical class of $2C_0$ for all $1 \leq i \leq p+1$ such that $B_i \otimes O(-\mathfrak{d})$ is effective for all $1 \leq i \leq p$, but $B \otimes O(-(p+1)\mathfrak{d})$ is not effective.

**Proof.** The line bundle $B$ is equal to $O(2(p+1)C_0 + a\mathfrak{d})$ for some degree 0 divisor $a$ on $C$. Choose $\mathfrak{d}$ satisfying $2(a + (p+1)(\mathfrak{d} - \mathfrak{e})) \neq 0$ and set, for all $1 \leq i \leq p$, $B_i$ equal to $O(2C_0 + (\mathfrak{d} - \mathfrak{e} - \eta)$ for some divisor $\eta$ on $C$ such that $2\eta \sim 0$ and $\eta \not\sim 0$. Then Proposition 4.4 implies that $B_i \otimes O(-\mathfrak{d})$ is effective for all $1 \leq i \leq p$ and that $B \otimes O(-(p+1)\mathfrak{d})$ is not effective. □

We are now ready to give the proof of Propositions 4.3 and 4.4:

(4.11) Proof of Proposition 4.1.

**Step 1.** $H^1(M_{B_1 \otimes B_2 \otimes \cdots \otimes B_n}^{\otimes p+1} \otimes L_2) = 0$

We will use Lemma 2.8. The set $\mathfrak{B}$ will consist of those line bundles on $X$ belonging either to the numerical class of $2C_0$ or to the numerical class of $C_0 + f$; the set $\mathfrak{B}$ will be the set of all effective line bundles of $X$ and $q_0$ will be equal to 1. Therefore if $\mathfrak{B}$ and $\mathfrak{B}$ satisfy the conditions of Lemma 2.8, we are done (simply take $n$ to be 0 and $Q = P^2$ in the conclusion of Lemma 2.8). Conditions 2.8.1 and 2.8.2 are satisfied, (c.f. Proposition 4.3 and Proposition 4.5). For Condition 2.8.3 note that $B_1 \otimes B_2 \otimes B_3^\ast$ is an effective line bundle for any $B_1$, $B_2$ and $B_3$ in $\mathfrak{B}$ (this follows again from Proposition 4.3). On the other hand, the line bundle $R_3$ defined in the statement of Lemma 2.8 satisfies the hypothesis for the line bundle $L$ in Lemma 4.9. Thus applying the mentioned lemma we are done. For Condition 2.8.4 we have to show that

$$H^1(M_{B_1 \otimes B_2 \otimes \cdots \otimes B_n}^{\otimes 2} \otimes B_1' \otimes B_2' \otimes P) = 0$$

for all $B_1, B_2, B_1', B_2', C_1, \ldots, C_n \in \mathfrak{B}$ and $P$ effective line bundle satisfying the condition $B_i \equiv B_i'$. Note that $C_1 \otimes \cdots \otimes C_n$ is numerically equivalent to $O(aC_0 + b\mathfrak{e})$ for some $a, b \geq 0$, so we will prove this more general fact instead: Let $P^1$ be an effective line bundle in the numerical class of $a_1C_0 + b_1\mathfrak{e}$ for some $a_1, b_1 \geq 0$ and let $P^2$ be another effective line bundle. Then

(4.11.1) $H^1(M_{B_1 \otimes B_2 \otimes P^1}^{\otimes 2} \otimes B_1' \otimes B_2' \otimes P^2) = 0$. 
First we show using Lemma 3.1 that $\text{H}^1(M_{B_1 \otimes B_2 \otimes P_1} \otimes B_1' \otimes B_2') = 0$. By Lemma 4.7 we may assume without loss of generality that

\[(4.11.2) \ P^1 = \mathcal{O}(a_1C_0 + b_1f), \text{ and if } B_1' \equiv B_2' \equiv \mathcal{O}(2C_0), \text{ the line bundle } B_1' \otimes \mathcal{O}(-f) \text{ is effective and the line bundle } B_1' \otimes B_2' \otimes \mathcal{O}(-2f) \text{ is not effective.} \]

In particular, $\text{H}^1(B_1' \otimes B_2' \otimes \mathcal{O}(-2f)) = 0$.

We use induction on $(a_1, b_1)$. If $(a_1, b_1) = (0, 0)$, the result follows from Proposition 3.6. Now assume that

$$
\text{H}^1(M_{B_1 \otimes B_2 \otimes \mathcal{O}((a_1 - 1)C_0)} \otimes B_1' \otimes B_2') = 0
$$

for $a_1 \geq 1$. We apply Lemma 3.3 to $F = B_1 \otimes B_2 \otimes \mathcal{O}((a_1 - 1)C_0)$, $Q = \mathcal{O}(C_0)$, $q = C_0$, $R = B_1' \otimes B_2'$, $q = 1$, $q_0 = -1$, $\alpha = 0$ and $m = 0$. Condition 3.3.1 is satisfied by Proposition 4.3. Condition 3.3.2 is satisfied because $\deg(B_1' \otimes B_2' \otimes \mathcal{O}_C(-lC_0)) \geq 3$ for $l = 0, 1$. We check that Condition 3.3.3 is satisfied by using Lemma 4.7, noting that

$$\deg(B_1 \otimes B_2 \otimes \mathcal{O}_C(a_1C_0)) > \deg(B_1 \otimes B_2 \otimes \mathcal{O}_C((a_1 - 1)C_0)) \geq 4$$

and that

$$\deg(B_1' \otimes B_2' \otimes \mathcal{O}_C(-lC_0)) \geq 3.$$  

Condition 3.3.4 requires that

$$\text{H}^1(M_{B_1 \otimes B_2 \otimes \mathcal{O}((a_1 - 1)C_0)} \otimes B_1' \otimes B_2' \otimes \mathcal{O}(-C_0)) = 0,$$

which is a consequence of Lemma 4.8 and Proposition 4.3, and that

$$\text{H}^1(M_{B_1 \otimes B_2 \otimes \mathcal{O}((a_1 - 1)C_0)} \otimes B_1' \otimes B_2') = 0$$

which is true by the induction hypothesis on $a_1 - 1$. Condition 3.3.5 requires the vanishing of $\text{H}^1(B_1' \otimes B_2' \otimes \mathcal{O}(-2C_0)) = 0$ which follows from Proposition 4.3. Now we carry out induction on $b_1$. If $b_1 = 0$, the required statement has just been proven. Assume that the result is true for $b_1 - 1 (b_1 \geq 1)$. We will use again Lemma 3.3 setting $F = B_1 \otimes B_2 \otimes \mathcal{O}(a_1C_0 + (b_1 - 1)f)$, $R = B_1' \otimes B_2'$, $Q = \mathcal{O}(f)$, $q = f$, $q = 1$ and $q_0 = -1$. Condition 3.3.1 is satisfied because of Proposition 4.3. Condition 3.3.2 is satisfied because $\deg(B_1' \otimes B_2' \otimes \mathcal{O}_f(-lf)) \geq 2$. Condition 3.3.3 follows from Lemma 4.6, since

$$\deg(B_1 \otimes B_2 \otimes \mathcal{O}_f(a_1C_0 + b_1f)) \geq 2$$

$$\deg(B_1 \otimes B_2 \otimes \mathcal{O}_f(a_1C_0 + (b_1 - 1)f)) \geq 2$$

and

$$\deg(B_1' \otimes B_2' \otimes \mathcal{O}_f(-lf)) \geq 2.$$  

Condition 3.3.4 follows by induction hypothesis on $b_1 - 1$ and from Lemma 4.8 and (4.11.2). Condition 3.3.5 follows from Proposition 4.3 and (4.11.2).
To finish the proof of (4.11.1) we apply Lemma 3.2 inductively (as done for instance in the proof of Lemma 4.8) setting the line bundles $F_1$ and $F_2$ both equal to $B_1 \otimes B_2 \otimes P^1$, $Q$ equal to $\mathcal{O}(C_0)$, $\mathcal{O}(f)$ or $\mathcal{O}(E)$, $q$ equal to $C_0$, $f$, or $E$ and $q = 2$.

**Step 2.** $H^1(M_{L_1}^{\otimes p+1} \otimes L_2) = 0$.

Again by Lemma 4.10 we may assume without loss of generality the following:

(4.11.3) $L_1 = B_1^1 \otimes \cdots \otimes B_{p+1}^1 \otimes \mathcal{O}(a_1 C_0 + b_1 f)$ and if all the $B_i^2$'s are in the numerical class of $2C_0$, then $B_1^2 \otimes \mathcal{O}(-f)$ is effective for all $2 \leq i \leq p + 1$ but $B_1^2 \otimes \cdots \otimes B_{p+1}^2 \otimes \mathcal{O}(-p f)$ is not effective, for all $p \geq 1$. In particular $H^1(B_1^2 \otimes \cdots \otimes B_{p+1}^2 \otimes \mathcal{O}(-p f)) = 0$.

We will prove that

(4.11.4) $H^1(M_{B_i^1 \otimes \cdots \otimes B_{p+1}^i}^{\otimes p+1} \otimes B_1^2 \otimes \cdots \otimes B_{p+1}^2 \otimes P^2) = 0$

for all $1 \leq p \leq p$. We use induction on $p'$. If $p' = 1$ we must prove that

$H^1(M_{B_1^2 \otimes B_2^2}^{\otimes 2} \otimes B_1^2 \otimes B_2^2 \otimes P^2) = 0$.

This is the content of (4.11.1).

Now we assume that (4.11.4) holds for $1, \ldots, p' - 1$ ($p' \geq 2$) and we prove that it holds also for $p'$. Again we make induction on $(a_1, b_1)$. If $(a_1, b_1) = (0, 0)$ the statement was proven in Step 1. Assume that the result is true for $(a_1 - 1, 0)$. We apply Lemma 3.3 to $F = B_1^1 \otimes \mathcal{O}((a_1 - 1) C_0)$, $R = L_2$, $q = C_0$, $p = p'$ and $q_0 = -1$. Condition 3.3.1 is satisfied by Proposition 4.3. Condition 3.3.2 follows from the fact that $\deg(L_2 \otimes \mathcal{O}_{C_0}(-l C_0)) \geq p' + 2 \geq 4$ for all $0 \leq l \leq p'$. Condition 3.3.3 follows from Lemma 4.7 and from the fact that $\deg(L_2 \otimes \mathcal{O}_{C_0}(-l C_0)) \geq p' + 2$ and

$\deg(B_1^1 \otimes \mathcal{O}_{C_0}(a_1 C_0)) \geq \deg(B_1^1 \otimes \mathcal{O}_{C_0}((a_1 - 1) C_0)) \geq 2p' + 2 \geq p' + 3$.

Condition 3.3.4 requires the vanishing of

$H^1(M_{B_1^1 \otimes \mathcal{O}((a_1 - 1) C_0)}^{\otimes p''+1} \otimes L_2 \otimes \mathcal{O}(-l C_0))$ (4.11.5)

for $l = p' - p''$ and $0 \leq p'' \leq p'$. If $p'' = p'$, the vanishing of (4.8.5) is simply the induction hypothesis for $a_1 - 1$. If $1 \leq p'' \leq p' - 1$, the vanishing of (4.11.5) follows from the induction hypothesis on $1, \ldots, p' - 1$. Indeed. The line bundle $B_1^1 \otimes \mathcal{O}((a_1 - 1) C_0)$ can be written as the tensor product of $B_1^1 \otimes \cdots \otimes B_{p''+1}^1$ with an effective line bundle numerically equivalent to $a C_0 + b f$ for some $a, b \geq 0$. The line bundle $L_2 \otimes \mathcal{O}(-l C_0)$ can be written as the tensor product of $B_2^2 \otimes \cdots \otimes B_{p''+1}^2$ with an effective line bundle, since $B_2^2 \otimes \mathcal{O}(-C_0)$ is effective. If $p'' = 0$, the vanishing of (4.11.5) follows from Lemma 4.8 and Proposition 4.3. Condition 3.3.5 requires the vanishing of $H^1(L_2 \otimes \mathcal{O}(-(p' + 1) C_0))$ which follows from Proposition 4.3.
The induction argument on \( b_1 \) is similar to the one on \( a_1 \) and we will only highlight here the differences and the delicate points. We make again iterated use of Lemma 3.3. Condition 3.3.3 follows from Lemma 4.3. Condition 3.3.4 is obtained as before (assumption (4.11.3) assures us that \( B^2_i \otimes \mathcal{O}(-f) \) is effective for all \( 1 \leq i \leq p' \)). Condition 3.3.5 is obtained from Propositions 4.3 and 4.4 and assumption (4.11.3). \( \square \)

(4.12) Proof of Proposition 4.2. Without loss of generality we may assume that \( P^1 \) is isomorphic to \( \mathcal{O}(a_1C_0 + b_1f + c_1E) \). We prove the result by induction on \( p \). First we prove it for \( p = 1 \). We will use induction on \( c_1 \). If \( c_1 = 0 \) the result follows from Proposition 4.1. Assume that the result is true for \( (a_1, b_1, c_1 - 1) \) and \( c_1 \geq 1 \). We apply Lemma 3.3 to \( FB^1 \otimes \mathcal{O}(a_1C_0 + b_1f + (c_1 - 1)E) \), \( q = E \), \( R = L_2 \), \( q = 1 \) and \( q_0 = -1 \). We have to check that the conditions of Lemma 3.3 are satisfied. Condition 3.3.1 follows from Proposition 4.3. Condition 3.3.2 follows from the fact that \( \deg(L_2 \otimes \mathcal{O}_E(-lE)) = \deg(L_2 \otimes \mathcal{O}_E) > 0 \). Condition 3.3.3 follows from Lemma 4.7 using the fact that \( \deg(L_2 \otimes \mathcal{O}_E(-lE)) \geq 4 \) and \( \deg(B^1 \otimes \mathcal{O}(a_1C_0 + b_1f + (c_1 - 1)E)) \geq 4 \). Condition 3.3.4 requires the vanishing of \( H^1(M_{B^1} \otimes \mathcal{O}(a_1C_0 + b_1f + (c_1 - 1)E) \otimes L_2 \otimes \mathcal{O}(-E)) \) which follows from Lemma 4.8 and the vanishing of \( H^1(M_{B^1} \otimes \mathcal{O}(a_1C_0 + b_1f + (c_1 - 1)E) \otimes L_2) \) which follows from the induction hypothesis on \( c_1 = 1 \). Condition 3.3.5 follows from Proposition 4.3.

Now let us assume the result to be true for \( 1, \ldots, p - 1 \). To prove the result for \( p \geq 2 \) we will again use induction on \( c_1 \). If \( c_1 = 0 \) the result follows from Proposition 4.11. Assume that the result is true for \( (a_1, b_1, c_1 - 1) \) and \( c_1 \geq 1 \). We apply Lemma 3.3 to \( FB^1 \otimes \mathcal{O}(a_1C_0 + b_1f + (c_1 - 1)E) \), \( q = E \), \( R = L_2 \), \( q = p \) and \( q_0 = -1 \). We see now that the conditions of Lemma 3.3 are satisfied. Condition 3.3.1 follows from Proposition 4.3. Condition 3.3.2 follows from the fact that \( \deg(L_2 \otimes \mathcal{O}_E(-lE)) = \deg(L_2 \otimes \mathcal{O}_E) > 0 \). Condition 3.3.3 follows from Lemma 4.7 using the fact that

\[
\begin{align*}
\deg(L_2 \otimes \mathcal{O}_E(-lE)) & \geq 2p + 2 > p + 2 \\
\deg(B^1 \otimes \mathcal{O}_E(a_1C_0 + b_1f + (c_1 - 1)E)) & \geq 2p + 2 \geq p + 3 .
\end{align*}
\]

Condition 3.3.4 requires the vanishing of

\[
H^1(M_{B^1} \otimes \mathcal{O}(a_1C_0 + b_1f + (c_1 - 1)E) \otimes L_2 \otimes \mathcal{O}(-lE))
\]

for all \( 0 \leq p' \leq p \) and \( l = p - p' \). If \( p' = p \) the vanishing follows from the induction hypothesis on \( c_1 = 1 \). If \( 1 \leq p' \leq p - 1 \) the vanishing follows from the induction hypothesis on \( 1, \ldots, p - 1 \). Finally, if \( p' = 0 \) the vanishing follows from Lemma 4.8 and Proposition 4.3. Condition 3.3.5 requires the vanishing of \( H^1(L_2 \otimes \mathcal{O}(-(p + 1)E)) \) which follows from Proposition 4.3. \( \square \)

5. Cohomology vanishings on elliptic ruled surfaces with invariant \( e \geq 0 \)

In this section we duplicate for an elliptic ruled surface of invariant \( e \geq 0 \) the work done in the previous one for an elliptic ruled surface with invariant \( e = -1 \).
Thus \( X \) will denote throughout this section an elliptic ruled surface with invariant \( e \geq 0 \). Again \( C_0 \) will be a minimal section of \( X \). We will denote by \( f \) the class of a fiber of \( X \). If \( a \) is a divisor on \( D \), \( af \) will denote the pullback of \( a \) to \( X \) by the projection from \( X \) to \( D \). Sometimes if \( \deg a = 1 \) we will write, by an abuse of notation, \( f \) instead of \( af \). The canonical divisor \( K_X \) is linearly equivalent to \(-2C_0 + ef\), and hence numerically equivalent to \(-2C_0 - ef\).

Our main result in this section is

**Proposition 5.1.** Let \( B^1 = B^1_1 \otimes \cdots \otimes B^1_{p+1} \) and \( B^2 = B^2_1 \otimes \cdots \otimes B^2_{p+1} \) be line bundles such that \( B^i_j \) is in the numerical class of \( C_0 + (e+2)f \). Let \( P^1 \) and \( P^2 \) be two effective line bundles on \( X \) such that \( P^j \) is in the numerical class of \( a_j(C_0 + ef) + b_jf \) for some \( a_j, b_j \geq 0 \). If \( L_i = B^i \otimes P^i \), then

\[
H^1(M^{\otimes p+1}_L \otimes L_2) = 0
\]

Before we prove Proposition 5.1 we need to recall some properties of the line bundles on \( X \).

**Proposition 5.2 ([GP] Proposition 3.1 or [Ho1], §2).**

Let \( L \) be a line bundle on \( X \), numerically equivalent to \( aC_0 + bf \). Then

\[
\begin{array}{|c|c|c|c|}
\hline
a & b & h^0(L) & h^2(L) \\
\hline
a \geq 0 & b > 0 & > 0 & 0 \\
& b = 0 & ? & 0 \\
& b < 0 & 0 & 0 \\
\hline
a = -1 & any b & 0 & 0 \\
\hline
a \leq -2 & b > -e & 0 & 0 \\
& b = -e & 0 & ? \\
& b < -e & 0 & > 0 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|}
\hline
a & b & h^1(L) \\
\hline
a \geq 0 & b > ae & 0 \\
& b = ae & ? \\
& b < ae & > 0 \\
\hline
a = -1 & any b & 0 \\
\hline
a \leq -2 & b > e(a+1) & 0 \\
& b = e(a+1) & ? \\
& b < e(a+1) & > 0 \\
\hline
\end{array}
\]
Proposition 5.3 (c.f. [GP] proposition 3.3). The general member of $|C_0 - ef|$ is a smooth elliptic curve and those are the only smooth curves in the numerical class of $C_0 + ef$.

(5.3.1) We will fix once and for all a smooth elliptic curve $F$ in the numerical class of $C_0 + ef$.

Proposition 5.4 ([H], V.2.21.b; [GP] Proposition 3.5 and Remark 3.5.4). Let $L$ be line bundle on $X$ in the numerical class of $aC_0 + bf$.

5.4.1 $L$ is ample iff $a > 0$ and $b > ae$
5.4.2 $L$ is base point free if $a \geq 0$ and $b - ae \geq 2$.
5.4.3 $L$ is ample and base point free iff $a \geq 1$ and $b - ae \geq 2$.

We need another two lemmas in order to prove Proposition 5.1

Lemma 5.5. Let $L = B_1 \otimes B_2$ and $L' = B_3$ be line bundles on $X$ satisfying the following properties:
5.5.1 $B_1 \equiv B_3$
5.5.2 $B_i$ is in the numerical class of $C_0 + (e + 2)f$.

Let $P$ and $P'$ be effective line bundles in the numerical classes of $a(C_0 + ef) + bf$ and of $a'(C_0 + ef) + b'f$ respectively for some $a, b, a', b' \geq 0$. Then the map

$$H^0(L \otimes P) \otimes H^0(L' \otimes P') \xrightarrow{\alpha} H^0(L \otimes L' \otimes P \otimes P')$$

surjects.

Proof. Analogous to the proof of Lemma 4.8. □

Lemma 5.6. Let $a$, $b$ be two integers such that $a \geq 1$ and $b - ae \geq 4$. Let $L$ be a line bundle in the numerical class of $aC_0 + bf$ and $P$ a line bundle in the numerical class of $a'(C_0 + ef) + b'f$ for some $a', b' \geq 0$. Then

$$H^2(M_L^{22} \otimes P) = 0$$

Proof. Analogous to the proof of Lemma 4.9. □

(5.7) Proof of Proposition 5.1.

Step 1. $H^1(M_{B_1}^{p+1} \otimes L_2) = 0$

We will use Lemma 2.8. The set $\mathfrak{B}$ will be the numerical class of $C_0 + (e+2)f$, the set $\mathfrak{P}$ will consists of all line bundles numerically equivalent to $a(C_0 + ef) + bf$ for some $a, b \geq 0$ and $p_0$ will be equal to 1. Therefore if $\mathfrak{B}$ and $\mathfrak{P}$ satisfy the conditions of Lemma 2.8, we are done (simply take $n$ to be 0 and $P = P^2$ in the conclusion of the Lemma 2.8). Conditions 2.8.1 and 2.8.2 are satisfied, (c.f. Proposition 5.2 and 5.4). For Condition 2.8.3 note that $B_1 \otimes B_2 \otimes B_3^*$ belongs to $\mathfrak{P}$ for any $B_1, B_2$ and $B_3$ in $\mathfrak{B}$. In the other hand, the line bundle $L_3$ defined in Lemma 2.8 satisfies
the hypothesis for the line bundle \( L \) in Lemma 5.6. Thus applying the mentioned lemma we are done. For Condition 2.8.4 we have to show that

\[
H^1(M_{B_1 \otimes B_2 \otimes C_1 \otimes \cdots \otimes C_n} \otimes B'_1 \otimes B'_2 \otimes P) = 0
\]

for all \( B_1, B_2, B'_1, B'_2, C_1, \ldots, C_n \in \mathfrak{B} \) and \( P \in \mathfrak{P} \). Note that by 2.8.2 \( C_1 \otimes \cdots \otimes C_n \) belongs to \( \mathfrak{P} \). Thus we will prove this more general result:

\[
(5.7.1) \quad H^1(M_{B_1 \otimes B_2 \otimes P_1} \otimes B'_1 \otimes B'_2 \otimes P^2) = 0
\]

for any \( P_1 \) and \( P^2 \) in \( \mathfrak{P} \).

First we show using Lemma 3.3 that

\[
(5.7.2) \quad H^1(M_{B_1 \otimes B_2 \otimes P_1} \otimes B'_1 \otimes B'_2) = 0.
\]

We may assume without loss of generality that \( P_1 = \mathcal{O}(a_1F + b_1f) \). We use induction on \((a_1, b_1)\). If \((a_1, b_1) = (0,0)\), the result follows from Lemma 3.6. Assume that

\[
H^1(M_{B_1 \otimes B_2 \otimes \mathcal{O}((a_1-1)F)} \otimes B'_1 \otimes B'_2) = 0
\]

We apply Lemma 3.3 to \( B = B_1 \otimes B_2 \otimes \mathcal{O}((a_1-1)F), P = \mathcal{O}(C_0 + \epsilon f), p = F, L = B'_1 \otimes B'_2, p_1 = 1, p_0 = -1, a = 0 \) and \( m = 0 \). Condition 3.3.1 is satisfied by Proposition 5.2. Condition 3.3.2 is satisfied because \( \deg(B_1 \otimes B_2 \otimes \mathcal{O}_F(-lF)) \geq e + 4 \geq 4 \). We check that Condition 3.3.3 is satisfied by using Lemma 4.7, noting that

\[
\deg(B_1 \otimes B_2 \otimes \mathcal{O}_F(a_1F)) > \deg(B_1 \otimes B_2 \otimes \mathcal{O}_F((a_1-1)F)) \geq (2+a_1-1)e+4 \geq 4 \geq 4-l
\]

and that

\[
\deg(B_1 \otimes B_2 \otimes \mathcal{O}_F(-lF)) \geq 4 \geq 3-l.
\]

Condition 3.3.4 requires that \( H^1(M_{B_1 \otimes B_2 \otimes \mathcal{O}((a_1-1)F)} \otimes B'_1 \otimes B'_2 \otimes \mathcal{O}(-F)) = 0 \), which is a consequence of Lemma 5.5, and that

\[
H^1(M_{B_1 \otimes B_2 \otimes \mathcal{O}((a_1-1)F)} \otimes B'_1 \otimes B'_2) = 0
\]

which is true by the induction hypothesis on \( a_1 - 1 \). Condition 3.3.5 requires the vanishing of \( H^1(B'_1 \otimes B'_2 \otimes \mathcal{O}(-2F)) \) which follows from Proposition 5.2.

To finish the proof of (5.7.2) we do induction on \( b_1 \). If \( b_1 = 0 \), the required statement has just been proven. Assume that the result is true for \( b_1 - 1 \) (\( b_1 \geq 1 \)). We will use again Lemma 3.3 setting \( B = B_1^1 \otimes B_2^1 \otimes \mathcal{O}(a_1(C_0 - \epsilon f) + (b_1 - 1)f), L = B'_1^1 \otimes B'_2^1, P = \mathcal{O}(f), p = f, p_1 = 1 \) and \( p_0 = -1 \). Condition 3.3.1 is satisfied because of Proposition 5.2. Condition 3.3.2 is satisfied because \( \deg(L_2 \otimes \mathcal{O}_f(-lf)) \geq a_2 + 2 \geq 2 \). Condition 3.3.3 follows from Lemma 5.6, since

\[
\deg(B_1^1 \otimes B_2^1 \otimes \mathcal{O}_f(a_1(C_0 + \epsilon f) + (b_1 - 2)f)) \geq a_1 + 2 \geq 2
\]

and
deg(L_2 \otimes \mathcal{O}_f(-lf)) \geq a_2 + 2 \geq 2.

Condition 3.3.4 follows by induction hypothesis on b_1 - 1 and from Lemma 5.5. Condition 3.3.5 follows from Proposition 5.2.

To finish the proof of (5.7.1) we apply Lemma 3.2 inductively (as done for instance in the proof of Lemma 4.8) setting the line bundles B_1 and B_2 in the statement of Lemma 3.2 both equal to B_1 \otimes B_2 \otimes P^1, P equal to \mathcal{O}(F) or \mathcal{O}(f), p equal to F or f and p = 2.

Step 2. H^1(M_2^{p+1} \otimes L_2) = 0. We may assume without loss of generality that L_1 = B_1^1 \otimes \cdots \otimes B_{p+1}^2 \otimes \mathcal{O}(a_1(C_0 - ef) + b_1f). Thus we want to prove

\begin{equation}
(5.7.3) \quad H^1(M_2^{p+1} \otimes B_1^1 \otimes \cdots \otimes B_{p+1}^2 \otimes \mathcal{O}(a_1(C_0 - ef) + b_1f) \otimes L_2) = 0.
\end{equation}

We will use induction on p, starting at p = 1. If p = 1 (5.7.3) follows from (5.7.1).

Now we assume that (5.7.3) holds for 1, \ldots, p - 1 for p \geq 2 and we will prove that it holds also for p. Again we do induction on (a_1, b_1). If (a_1, b_1) = (0, 0) the statement was proven in Step 1. Assume the result is true for (a_1 - 1, 0). We apply Lemma 3.3 to B = B_1^1 \otimes \mathcal{O}((a_1 - 1)(C_0 + ef)), L = L_2, p = F and p_0 = -1. Condition 3.3.1 is satisfied by Proposition 5.2. Condition 3.3.2 follows from the fact that deg(L_2 \otimes \mathcal{O}_F(-lf)) \geq (p + 1)(e + 2) + (a_1 - 1)e \geq 2p + 2 > 0. Condition 3.3.3 follows from Lemma 4.7 and from the fact that deg(L_2 \otimes \mathcal{O}_F(-lf)) \geq 2p + 2 > p + 2 and

\begin{equation}
\text{deg}(B^1 \otimes \mathcal{O}_F(aF)) \geq \text{deg}(B^1 \otimes \mathcal{O}_F((a_1 - 1)F)) \geq 2p + 2 \geq p + 3.
\end{equation}

Condition 3.3.4 requires the vanishing of

\begin{equation}
(5.7.4) \quad H^1(M_{p+1}^{B_1^1 \otimes \mathcal{O}((a_1 - 1)F)} \otimes L_2 \otimes \mathcal{O}(-lf))
\end{equation}

for l = p - p' and 0 \leq p' \leq p. If p = p', (5.7.4) is simply the induction hypothesis for a_1 - 1. If 1 \leq p' \leq p - 1, (5.7.4) is nothing but the induction hypothesis on 1, \ldots, p - 1. Indeed. The line bundle L_2 \otimes \mathcal{O}(-lf) can be written as the tensor product of B_1^2 \otimes \cdots \otimes B_{p+1}^2 with an effective line bundle in the numerical class of a_2(C_0 + ef) + (b_2 + 2(p - p'))f. If p' = 0, (5.7.4) follows from Lemma 5.5. Condition 3.3.5 requires the vanishing of H^1(L_2 \otimes \mathcal{O}(-(p + 1)f)) which follows from Proposition 5.2.

The induction argument on b_1 is similar to the one on a_1. \(\square\)

6. Syzygies of elliptic ruled surfaces

In this section we assume that \text{char}(k) > p + 1 or equal to 0. We will use the results obtained in Sections 4 and 5 to prove the following
Theorem 6.1. Let $X$ be an elliptic ruled surface and let $p \geq 1$. Let $a$, $b$ be integers and let $L$ be a line bundle in the numerical class of $aC_0 + bf$.

6.1.1 If $e = e(X) = -1$ and $a \geq p + 1$, $a + b \geq 2p + 2$ and $a + 2b \geq 2p + 2$, then $L$ satisfies the property $N_p$.

6.1.2 If $e = e(X) \geq 0$ and $a \geq p + 1$, $b - ae \geq 2p + 2$, then $L$ satisfies the property $N_p$.

(6.1.3) Note that if $p = 1$, we recover from Theorem 6.1, the “if” part of Theorem 4.2 of [GP], except for the case when $a = 1$.

(6.1.4) Proof of Theorem 6.1. The line bundle $L$ is normally generated (see [Ho1] and [Ho2]; see also [GP], Lemma 2.6 and Theorem 4.2). Hence by Lemma 1.2 and 1.2.1 (this is the reason why we need the hypothesis on the characteristic of $k$), it is enough to show that

\[ H^1(M_L^{\otimes k+1} \otimes L) = 0 \text{ for all } 1 \leq k \leq p. \]

If $e = -1$, $L$ can be written for all $1 \leq k \leq p$ either as $B_1 \otimes \cdots \otimes B_{k+1} \otimes P$, where $B_i$ is in the numerical class of $2C_0$ or of $C_0 + f$ and $P$ is effective in the numerical class of $aC_0 + bf$ for some $a, b \geq 0$ or as $B_1 \otimes \cdots \otimes B_{k+1} \otimes P$, where $B_i$ is in the numerical class of $2C_0$ and $P$ is effective. Thus by Proposition 4.1 and Proposition 4.2 we obtain the result.

If $e \geq 0$, $L$ can be written for all $1 \leq k \leq p$ as $B_1 \otimes \cdots \otimes B_{k+1} \otimes P$, where $B_i$ is the numerical class of $C_0 + (e + 2)f$ and $P$ is effective in the numerical class of $a(C_0 + ef) + bf$ for some $a, b \geq 0$. Thus by Proposition 5.1 we obtain the result. □

As a corollary of Theorem 6.1 we obtain the following result on adjoint linear series, which is a generalization to higher syzygies of Corollary 4.6 of [GP]. Note however that we obtain there a sharper bound in the case $e \geq 1, p = 1$.

Corollary 6.2. Let $X$ be an elliptic ruled surface and let $p \geq 1$. Let $A_i$ be an ample line bundle on $X$ for all $1 \leq i \leq q$. If $q \geq 2p + 2 - \min(e(X), p - 1)$, then $\omega_X \otimes A_1 \otimes \cdots \otimes A_q$ satisfies the property $N_p$.

Proof. Let $A_i$ be in the numerical class of $a_iC_0 + b_if$ and $\omega_X \otimes A_1 \otimes \cdots \otimes A_q$ in the numerical class of $aC_0 + bf$. If $e = -1$, $A_i$ is ample iff $a_i \geq 1$ and $a_i + 2b_i \geq 1$ (c.f. Proposition 4.5). In particular we also have that if $A_i$ is ample, then $a_i + b_i \geq 1$. Since $\omega_X$ is numerically equivalent to $-2C_0 + f$ it follows that

\[
\begin{align*}
    a &\geq q - 2 \geq 2p + 1 > p + 1, \\
    a + b &\geq q - 1 \geq 2p + 2 \quad \text{and} \\
    a + 2b &\geq q \geq 2p + 3.
\end{align*}
\]

Hence by Theorem 6.1, $\omega_X \otimes A_1 \otimes \cdots \otimes A_q$ satisfies the property $N_p$. 
If $e \geq 0$ $A_i$ is ample iff $a \geq 1$ and $b_i - a_i e \geq 1$ (c.f. Proposition 5.4). Since $\omega_X$ is numerically equivalent to $-2C_0 - ef$ it follows that $a \geq q - 2$ and $b - ae \geq q + e$. By hypothesis, $q \geq 2p - 2 - e$ and $q \geq p + 3$; hence $a \geq p + 1$ and $b - ae \geq 2p + 2$ and by Theorem 6.1, $\omega_X \otimes A_1 \otimes \cdots \otimes A_q$ satisfies the property $N_p$. □

We also obtain this generalization of Corollary 4.4 of [GP]:

**Corollary 6.3.** Let $X$ be as above and let $p \geq 1$. Let $B_i$ be an ample and base-point-free line bundle on $X$ for all $1 \leq i \leq q$. If $q \geq p + 1$, then $B_1 \otimes \cdots \otimes B_q$ satisfies the property $N_p$.

**Proof.** Let $B_i$ be in the numerical class of $a_i C_0 + b_i f$ and $B_1 \otimes \cdots \otimes B_q$ in the numerical class of $aC_0 + bf$. If $e = -1$, by Proposition 4.5, $B_i$ is ample and base-point-free iff $a_i \geq 1$, $a_i + b_i \geq 2$ and $a_i + 2b_i \geq 2$. Thus we obtain that

\[
\begin{align*}
a & \geq q \geq p + 1 \\
q & \geq p + 1 \\
a + b & \geq 2q \geq 2p + 2 \\
a + 2b & \geq 2q \geq 2p + 2
\end{align*}
\]

Hence, by Theorem 6.1, $B_1 \otimes \cdots \otimes B_q$ satisfies the property $N_p$.

If $e \geq 0$, $B_i$ is ample and base-point-free iff $a_i \geq 1$ and $b_i - a_i e \geq 2$ (c.f. Proposition 5.4). Thus $a \geq q \geq p + 1$ and $b - ae \geq 2q \geq 2p + 2$. Hence, by Theorem 6.1, $B_1 \otimes \cdots \otimes B_q$ satisfies the property $N_p$. □

**Corollary 6.4.** Let $X$ as above and let $p \geq 1$. Let $A_i$ be an ample line bundle on $X$ for all $1 \leq i \leq q$. If $q \geq 2p + 2$, then $A_1 \otimes \cdots \otimes A_q$ satisfies the property $N_p$.

**Proof.** If suffices to note that if $A$ and $A'$ are ample line bundles on $X$, then $A \otimes A'$ is ample and base-point-free (this follows from Propositions 4.5 and 5.4). Then we apply Corollary 6.3. □

(6.5) Note that the assumption on the characteristic was made because we wanted to be able to consider $\bigwedge^{p+1} M_L \otimes L^{\otimes p' + 1}$ as a direct summand of $M_L^{\otimes p' + 1} \otimes L^{\otimes p' + 1}$, for all $1 \leq p' \leq p$. That way we obtained from the vanishings of $H^1(\bigwedge^{p' + 1} M_L \otimes L^{\otimes p' + 1})$, the vanishings of $H^1(\bigwedge^{p + 1} M_L \otimes L^{\otimes p + 1})$, for all $1 \leq p' \leq p$. These were the vanishings required by Lemma 1.2 in order that $L$ satisfied the property $N_p$. However, in particular situations, those conditions required in Lemma 1.2 can be relaxed. Precisely, if $L$ is a normally generated line bundle such that $H^i(L^{\otimes 2 - i}) = 0$ and $p$ is less or equal than the codimension of $X$ inside $P^N = P(H^0(L))$, then $L$ satisfies the property $N_p$ iff the group $H^1(\bigwedge^{p + 1} M_L \otimes L^{\otimes p + 1})$ vanishes (c.f. [GL], Lemma 1.10). We claim that the above condition on $p$ and the codimension is satisfied under the conditions of Theorem 6.1.

Indeed. If $L$ belongs to the numerical class of $aC_0 + bf$, using Riemann-Roch one easily obtains that $h^0(L) = \frac{1}{2}(a(b - 1) + (a + 2)) - a(a + 2)e$. Thus, if $e = -1$, we want to see that

\[
\frac{(a + 1)(a + 2b)}{2} - 3 = \text{cod}(X, P^N) \geq p .
\]
The latter inequality follows from the numerical conditions satisfied by \((a, b)\):

\[
\frac{(a + 1)(a + 2b)}{2} - 3 \geq (p + 2)(p + 1) - 3 \geq p , \quad \text{for all } p \geq 1.
\]

If, on the other hand, \(e \geq 0\), using again the numerical conditions satisfied by \((a, b)\), we see that

\[
\text{cod}(X, \mathbb{P}^N) = \frac{(a + 2)(b - ae) + a(b - 1)}{2} - 3
\]

\[
\geq \frac{(p + 3)(2p + 2) + 3(p + 1)}{2} - 3 \geq p
\]

for all \(p \geq 1\).

Hence the results of this section hold in slightly greater generality, namely, they hold when \(\text{char}(k)\) does not divide \(p + 1\).

7. Open questions and conjectures

We foresee two directions in which these results on syzygies of elliptic ruled surfaces could be improved:

7.1. In Section 4 of [GP] we prove that the product of two base-point-free divisors (not necessarily both of them ample) satisfies the property \(N_1\) iff it is ample. Therefore one may ask whether a similar statement is true for any \(p \geq 1\), i.e., whether the product \(L\) of \(p + 1\) base-point-free divisors (not necessarily all of them ample) satisfies the property \(N_p\) whenever \(L\) is ample. This is expressed graphically for the case \(e(X) = -1\) in Figure 1 and for the case \(e \geq 0\) in Figure 2. In these figures the integral points of the coordinate plane represent the classes of \(\text{Num}(X)\)) and the shadowed regions contain the divisors which could satisfy the property \(N_p\).

(7.2) When \(e(X) = -1\), Homma proved (see [Ho2]) that if \(L\) is a line bundle in the numerical class of \(aC_0 + bf\), then \(L\) satisfies the property \(N_0\) iff \(a \geq 1\), \(a + b \geq 3\), and \(a + 2b \geq 3\). We prove in Theorem 4.2 of [GP] that \(L\) satisfies the property \(N_1\) iff \(a \geq 1\), \(a + b \geq 4\), and \(a + 2b \geq 4\). Hence one could ask whether \(L\) satisfies the property \(N_2\) if \(a \geq 1\), \(a + b \geq 5\), and \(a + 2b \geq 5\). Evidence suggesting an affirmative answer is the fact that the free resolution of \(R(L)\) is linear until the second stage if \(L\) is in the numerical class of \(5f\) and if \(L\) is certain line bundle in the numerical class of \(C_0 + 4f\) and in the class of \(2C_0 + 3f\) (these two cases were checked using the computer program Macaulay). Analogously, one expects similar statements for \(p \geq 3\) and also for the case \(e \geq 0\). We make the following

**Conjecture 7.3.** Let \(X\) be an elliptic ruled surface and let \(L\) be a line bundle on \(X\) in the numerical class \(aC_0 + bf\).

If \(e = -1\), \(L\) satisfies the property \(N_p\) iff \(a \geq 1\), \(a + b \geq p + 3\), and \(a + 2b \geq p + 3\).
If \( e(X) \geq 0 \), \( L \) satisfies the property \( N_p \) iff \( a \geq 1 \) and \( b - ae \geq p + 3 \).

In Figure 3 we show, for the case \( e = -1 \) the lines (dashed) joining the numerical classes of those line bundles which are conjectured to be optimal “\( N_p \) line bundles”. If this conjecture is true, \( \omega_X \otimes A^{p+4} \) will satisfy the property \( N_p \). Hence Conjecture 7.3 implies Mukai’s conjecture in the case of elliptic ruled surfaces. It also implies an affirmative answer for Question 7.1.

(7.4) Observe the analogy of Conjecture 7.3 and Green’s Theorem for curves, which says that \( L \) satisfies the property \( N_p \) if \( \deg(L) \geq 2g + p + 1 \). There the difference between two consecutive bounds is 1, i.e., the minimal degree for an ample line bundle on a curve. Going back to elliptic ruled surfaces, the “difference” between the line joining the conjectured optimal “\( N_p \) line bundles” and the line joining the conjectured optimal “\( N_{p+1} \) line bundles” is \( C_0 \), which is the “minimal” ample divisor.
Figure 1

$N_p$ (by theorem 6.1)

Figure 2
Figure 3

REFERENCES

[B] E. Bombieri, Canonical models of surfaces of general type, IHES, 42 (1973)
[Bu] D. Butler, Normal generation of vector bundles over a curve, J. Differential
Geometry 39 (1994) 1-34.
[G] M. Green, Koszul cohomology and the geometry of projective varieties, J.
Differential Geometry 19 (1984) 125-171.
[GL] M. Green & R. Lazarsfeld, Some results on the syzygies of finite sets and
algebraic curves, Compositio Math. 67 (1989) 301-314.
[GP] F.J. Gallego & B.P. Purnaprajna, Normal Presentation on Elliptic Ruled
Surfaces (1995) Preprint.
[GP1] F.J. Gallego & B.P. Purnaprajna, Syzygies of K3 surfaces and Fano varieties
(1995) Preprint.
[GP2] F.J. Gallego & B.P. Purnaprajna, Syzygies of surfaces and Calabi-Yau three-
folds (1995) In preparation.
[H] R. Hartshorne Algebraic Geometry, Springer, Berlin, 1977.
[Ho1] Y. Homma, Projective normality and the defining equations of ample invert-
able sheaves on elliptic ruled surfaces with $e \geq 0$, Natural Science Report,
Ochanomizu Univ. 31 (1980) 61-73.
[Ho2] ____, *Projective normality and the defining equations of an elliptic ruled surface with negative invariant*, Natural Science Report, Ochanomizu Univ. **33** (1982) 17-26.

[L] R. Lazarsfeld, *A sampling of vector bundles techniques in the study of linear series*, Lectures on Riemann Surfaces, World Scientific Press, Singapore, 1989, 500-559.

[Mi] Y. Miyaoka, *The Chern class and Kodaira dimension of a minimal variety*, Algebraic Geometry –Sendai 1985, Advanced Studies in Pure Math., Vol. 10, North-Holland, Amsterdam, 449-476.

[Mu] D. Mumford, *Varieties defined by quadratic equations*, Corso CIME in Questions on Algebraic Varieties, Rome, 1970, 30-100.

[R] I. Reider, *Vector bundles of rank 2 and linear systems on an algebraic surface*, Ann. of Math. (2) **127** (1988) 309-316.

F.J. Gallego, Dpto. de Algebra, Facultad de Matemáticas, U.C.M., 28040 Madrid SPAIN

E-mail address: gallego@sunali.mat.ucm.es

B.P. Purnaprajna, Dept. of Mathematics, Brandeis University, Waltham MA 02254-9110 USA.

E-mail address: purna@max.math.brandeis.edu