Solvable systems of two coupled first-order ODEs with homogeneous cubic polynomial right-hand sides

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Abstract

The solution $x_n(t), n = 1, 2,$ of the initial-values problem is reported of the autonomous system of 2 coupled first-order ODEs with homogeneous cubic polynomial right-hand sides,

$$\dot{x}_n = c_{n1}(x_1)^3 + c_{n2}(x_1)^2 x_2 + c_{n3}x_1(x_2)^2 + c_{n4}(x_2)^3, \quad n = 1, 2,$$

when the 8 (time-independent) coefficients $c_{n\ell}$ are appropriately defined in terms of 7 arbitrary parameters, which then also identify the solution of this model. The inversion of these relations is also investigated, namely how to obtain, in terms of the 8 coefficients $c_{n\ell}$, the 7 parameters characterizing the solution of this model; and 2 constraints are explicitly identified which, if satisfied by the 8 parameters $c_{n\ell}$, guarantee the solvability by algebraic operations of this dynamical system. Also identified is a related, appropriately modified, class of (generally complex) systems, reading

$$\dot{x}_n = i\omega \ddot{x}_n + c_{1}(\dot{x}_1)^3 + c_{2}(\dot{x}_1)^2 \ddot{x}_2 + c_{3}x_1(\dot{x}_2)^2 + c_{4}(\dot{x}_2)^3, \quad n = 1, 2,$$

with $i\omega$ an arbitrary imaginary parameter, which feature the remarkable property to be isochronous, namely their generic solutions are—as functions of real time—completely periodic with a period which is, for each of these models, a fixed integer multiple of the basic period $T = 2\pi/|\omega|$.

1 Introduction and presentation of the main results

The system characterized by the following 2 nonlinearly-coupled Ordinary Differential Equations (ODEs) with homogeneous cubic polynomial right-hand sides,

$$\dot{x}_n = c_{n1}(x_1)^3 + c_{n2}(x_1)^2 x_2 + c_{n3}x_1(x_2)^2 + c_{n4}(x_2)^3, \quad n = 1, 2,$$  \hspace{1cm} (1)

is a prototypical example of autonomous dynamical systems.

**Notation 1-1.** The 2 (possibly complex) numbers $x_n \equiv x_n(t), n = 1, 2,$ are the dependent variables; $t$ is the independent variable ("time"; but the treatment remains valid when $t$ is considered a complex variable); superimposed dots denote $t$-differentiations; the 8 $t$-independent (possibly complex) coefficients $c_{n\ell}$ ($n = 1, 2; \ell = 1, 2, 3, 4$) can be a priori arbitrarily assigned; but a key result of this paper is to identify 2 algebraic constraints (see below Subsection 3.2 and Appendix C) which—if satisfied by the 8 coefficients $c_{n\ell}$—allows the solution of the initial-values problem of the system (1) by algebraic operations (see below Proposition 1-1).

**Remark 1-1.** It is plain that the system (1) is invariant under the following simultaneous rescaling of the 8 coefficients $c_{n\ell}$ and of the independent variable $t$:

$$x_n(t) \Rightarrow \eta x_n(\eta^2 t), \quad n = 1, 2,$$  \hspace{1cm} (2)
with \( \eta \) an arbitrary nonvanishing parameter.

It is also plain (see (1)) that, by rescaling each of the 2 dependent variables \( x_1(t) \) and \( x_2(t) \) by appropriate \textit{constant} parameters, 2 of the 8 parameters \( c_{\alpha\ell} \) may generally be replaced by arbitrary constants (for instance by unity; of course unless that coefficient vanishes to begin with); while the other coefficients are of course also appropriately rescaled. But of course these features are of no help to satisfy the 2 \textit{constraints} on the 8 coefficients \( c_{\alpha\ell} \)—shown below to be sufficient for the \textit{algebraic solvability} of the model (1)—which are \textit{invariant} under such rescalings (see below). In the following we generally assume that the parameters \( c_{\alpha\ell} \) have \textit{generic} values (except for satisfying the \textit{constraints} mentioned just above); the only instance violating this rule is the special case (see below Subsection 3.5) with \( c_{14} = c_{21} = 0 \) which is particularly relevant in \textit{applicative} contexts and deserve therefore a separate treatment. ■

\textbf{Remark 1-2.} It is plain that the system (1) is \textit{invariant} under the following exchange of variables and parameters:

\[
x_1(t) \Leftrightarrow x_2(t) ; \quad c_{11} \Leftrightarrow c_{24} , \quad c_{12} \Leftrightarrow c_{23} , \quad c_{13} \Leftrightarrow c_{22} , \quad c_{14} \Leftrightarrow c_{21}.
\]

This property shall be used occasionally below to decrease the number of displayed equations. ■

\textbf{Remark 1-3.} The preceding Remark 1-2 might motivate the \textit{merely notational} replacement—possibly preferable in some \textit{applicative} contexts—of the system (1) with

\[
\dot{\xi} = \alpha_1 \xi^3 + \alpha_2 \xi^2 \eta + \alpha_3 \xi \eta^2 + \alpha_4 \eta^3,
\]

\[
\dot{\eta} = \beta_1 \eta^3 + \beta_2 \eta^2 \xi + \beta_3 \xi \eta^2 + \beta_4 \xi^3,
\]

(4) corresponding to the change of notation \( x_1(t) \Rightarrow \xi(t) \), \( x_2(t) \Rightarrow \eta(t) \), \( c_{1\ell} \Rightarrow \alpha_\ell \), \( c_{2\ell} \Rightarrow \beta_\ell \), \( \ell = 1, 2, 3, 4 \) and implying that the invariance property (3) take the neater form \( \xi(t) \Leftrightarrow \eta(t) \), \( \alpha_\ell \Leftrightarrow \beta_\ell \), \( \ell = 1, 2, 3, 4 \). But throughout this paper we stick with the notation implied by the more common notion of the system (1). ■

The system (1) has been investigated over time in an enormous number of purely \textit{mathematical}, or mainly \textit{applicative}, contexts. The first approach generally focussed on \textit{qualitative} features of its solutions \( x_n(t) \) as functions of \textit{real} \( t \) ("time"): mainly on the identification of its equilibria (if any), and on the behaviors close to them and asymptotically at large times (\( t \rightarrow \infty \)); and, as functions of \textit{complex} \( t \), on the \textit{analyticity} properties of its \textit{general solution}. The second approach was motivated by \textit{applications} in various contexts (mainly the time evolution of interacting "populations" or of other quantifiable entities, such as concentrations of chemicals or financial entries, you name it); and it was often pursued via numerical computations. This literature is too large to allow any attempt to provide a list of references which would do justice to the multitude of relevant papers. Here we limit ourselves to identify only the relatively recent (open access) paper [1] containing several references, and the 2 quite recent papers [2] and [3] because they have motivated this research and because from them additional relevant references can be traced. But we also like to mention the fundamental paper [4] by René Garnier, who 60 years ago investigated this type of systems—in fact, starting from the more general class of system analogous to (1) but featuring \textit{a priori arbitrary} polynomials on their right-hand sides and identifying subcases of this general class (including subcases of eq. (1)) such that their \textit{general} solutions—as functions of the independent variable \( t \) considered as a \textit{complex} variable—are \textit{holomorphic} ("uniform" in Garnier’s language). The present paper may be seen as a generalization of (some aspects) of Garnier’s work, as it allows the identification of a much larger subclass of the system (1), those which are solvable by \textit{algebraic operations}; hence such that their \textit{general solutions} (rather explicitly identified below) only feature—as functions of the independent variable \( t \) considered as a \textit{complex} variable—a \textit{finite} number of branch points (i. e., singularities of type \( (t - t_s)^\nu \)). This much larger subclass is clearly interesting from a purely mathematical point of view, and even more so because—as detailed below, see Section 4—the simple (\textit{complex}) extension of the model (1) reading

\[
\dot{x}_n = i\omega x_n + c_{n1} (x_1)^3 + c_{n2} (x_1)^2 x_2 + c_{n3} x_1 (x_2)^2 + c_{n4} (x_2)^3 , \quad n = 1, 2
\]

(5)

(with \( i\omega \) an arbitrary additional, purely imaginary, parameter) features then—provided all the exponents \( \nu_s \) are \textit{rational} numbers (a restriction that can be easily imposed, see below)—the remarkable property of \textit{isochrony}, its \textit{generic} solutions being then \textit{completely periodic} with a period which is an \textit{integer multiple} of the basic period \( T = 2\pi/|\omega| \) (see for instance, [5]). ■

Let us now state our main finding.

\textbf{Proposition 1-1.} Assume that the 8 parameters \( c_{\alpha\ell} \) of the dynamical system (1) are given in terms of the 7, \textit{a priori arbitrary}, parameters \( a_1, a_2, b_1, b_2, \gamma_1, \gamma_2, \gamma_3 \) by the following 8 formulas:

\[
c_{11} = \left[ (a_1)^3 b_2 - a_2 K_1 \right]/c ,
\]

(6a)
\[
c_{21} = \frac{-(a_1)^3 b_1 + a_1 K_1}{c}, \quad (6b)
\]
\[
c_{12} = \frac{3(a_1)^2 a_2 b_2 - a_2 K_2}{c}, \quad (6c)
\]
\[
c_{22} = \frac{3(a_1)^2 a_2 b_1 + a_1 K_2}{c}, \quad (6d)
\]
\[
c_{13} = \frac{3a_1 (a_2)^2 b_2 - a_2 K_3}{c}, \quad (6e)
\]
\[
c_{23} = \frac{-3a_1 (a_2)^2 b_1 + a_1 K_3}{c}, \quad (6f)
\]
\[
c_{14} = \frac{(a_2)^3 b_2 - a_2 K_4}{c}, \quad (6g)
\]
\[
c_{24} = \frac{-(a_2)^3 b_1 + a_1 K_4}{c}, \quad (6h)
\]

where, above and hereafter,
\[
K_1 \equiv \gamma_1 a_1 (b_1)^2 + \gamma_2 (a_1)^2 b_1 + \gamma_3 (a_1)^3 + (b_1)^3, \quad (7a)
\]
\[
K_2 \equiv \gamma_1 \left[ a_2 (b_1)^2 + 2a_1 b_1 b_2 \right] + \gamma_2 \left[ (a_1)^2 b_2 + 2a_1 a_2 b_1 \right] + 3\gamma_3 (a_1)^2 a_2 + 3 (b_1)^2 b_2, \quad (7b)
\]
\[
K_3 \equiv \gamma_1 \left[ a_1 (b_2)^2 + 2a_2 b_1 b_2 \right] + \gamma_2 \left[ (a_2)^2 b_1 + 2a_1 a_2 b_2 \right] + 3\gamma_3 a_1 (a_2)^2 + 3b_1 (b_2)^2, \quad (7c)
\]
\[
K_4 \equiv \gamma_1 a_2 (b_2)^2 + \gamma_2 (a_2)^2 b_2 + \gamma_3 (a_2)^3 + (b_2)^3, \quad (7d)
\]
\[
c \equiv a_1 b_2 - a_2 b_1. \quad (7e)
\]

Then the initial-values problem for the system (1) is solved by the following formulas:
\[
x_1 (t) = \frac{b_2 y (t) - a_2 w (t)}{c}, \quad (8a)
\]
\[
x_2 (t) = -\frac{b_1 y (t) - a_1 w (t)}{c}, \quad (8b)
\]

with \(c\) defined—above and hereafter—by eq. (8c). Note that these formulas are easily inverted, reading then
\[
y (t) = a_1 x_1 (t) + a_2 x_2 (t), \quad w (t) = b_1 x_1 (t) + b_2 x_2 (t), \quad (8c)
\]

which clearly also imply
\[
y (0) = a_1 x_1 (0) + a_2 x_2 (0), \quad w (0) = b_1 x_1 (0) + b_2 x_2 (0). \quad (8d)
\]

While the 2 functions \(y (t)\) and \(w (t)\) are given in terms of the above parameters and of their initial values \(y (0)\) and \(w (0)\) (themselves given in terms of the initial data \(x_1 (0)\) and \(x_2 (0)\) by (8c)) by the following explicit formula for \(y (t)\),
\[
y (t) = y (0) \left\{ 1 - 2 [y (0)]^2 t \right\}^{-1/2}, \quad (9)
\]

and by the relation
\[
w (t) = u (t) y (t), \quad (10a)
\]

where the function \(u (t)\) is defined implicitly by the following formula:
\[
\left[ \frac{u (t) - u_1}{u (0) - u_1} \right]^{-2\lambda_1} \left[ \frac{u (t) - u_2}{u (0) - u_2} \right]^{-2\lambda_2} \left[ \frac{u (t) - u_3}{u (0) - u_3} \right]^{-2\lambda_3} = 1 - 2 [y (0)]^2 t. \quad (10b)
\]

Here of course
\[
u (0) = w (0) / y (0); \quad (10c)
\]
while the 6 parameters $u_j$ and $\gamma_j$ ($j = 1, 2, 3$) are defined—in terms of (only!) the 3 parameters $\gamma_1$, $\gamma_2$, $\gamma_3$—by the following cubic equation,

$$u^3 + \gamma_1 u^2 + (\gamma_2 - 1) u + \gamma_3 = (u - u_1)(u - u_2)(u - u_3),$$

where $u$ is an arbitrary variable. The 3 roots $u_j$ of this cubic polynomial are of course related to the 3 parameters $\gamma_j$ as follows,

$$\begin{align*}
\gamma_1 &= -(u_1 + u_2 + u_3), \\
\gamma_2 &= u_1 u_2 + u_2 u_3 + u_3 u_1 + 1, \\
\gamma_3 &= -u_1 u_2 u_3;
\end{align*}$$

and are explicitly given in terms of the 3 parameters $\gamma_j$ by the standard Cardano formulas. And they determine the 3 parameters $\lambda_j$ via the following 3 relations,

$$\begin{align*}
\lambda_1 + \lambda_2 + \lambda_3 &= 0, \\
\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 &= 0, \\
\lambda_1 u_2 u_3 + \lambda_2 u_3 u_1 + \lambda_3 u_1 u_2 &= 1,
\end{align*}$$

implying

$$\lambda_j = \left[ \prod_{\ell=1, \ell\neq j}^3 (u_j - u_\ell) \right]^{-1}, \quad j = 1, 2, 3. \quad \blacksquare$$

**Remark 1-4.** The relation (10b) implies that if the 3 parameters $\lambda_j$, $j = 1, 2, 3$, are 3 arbitrary real rational numbers,

$$\lambda_j = N_j/M_j, \quad j = 1, 2, 3$$

(with $N_j$ arbitrary integers and $M_j$ arbitrary positive integers), then $u(t)$ is an algebraic function of $t$, being then the root of the polynomial equation determining $u(t)$—namely the equation that obtains by first multiplying eq. (10b) by the factor(s) $[u(t) - u_1]^{2\lambda_j}$ with $\lambda_j > 0$ and then raising the resulting equation to an appropriate positive integer power. Then the solutions $x_1(t)$ and $x_2(t)$ are clearly as well algebraic functions of the time $t$. A remarkable consequence (isochrony) of this fact has already been mentioned above (see (5)) and is discussed in more detail below, see Section 4. $\blacksquare$

The fact that the 8 coefficients $c_{n\ell}$ ($n = 1, 2; \ell = 1, 2, 3, 4$) are expressed by the formulas (6) in terms of 7 a priori arbitrary parameters might suggest that these 8 coefficients $c_{n\ell}$ are required to satisfy a single condition. As it turns out, they are in fact required to satisfy 2 restrictions: and various explicit avatars of these constraints—sufficient to guarantee that the system (1) possess the solution described by Proposition 1-1—are provided below (see Subsection 3.2 and Appendix C). This is discussed below (after the proof of Proposition 1-1: see Section 2) in the context (see Section 3) of the related—quite important in applicative contexts—issue of the inversion of the relations (6): namely the task of expressing in terms of the 8 coefficients $c_{n\ell}$ ($n = 1, 2; \ell = 1, 2, 3, 4$) the 7 parameters $a_n$, $b_n$ and $\gamma_j$ ($n = 1, 2; j = 1, 2, 3$)—hence of obtaining the rather explicit solution of the system (1) provided by Proposition 1-1, as well as the 2 constraints on the 8 coefficients $c_{n\ell}$ which entail this possibility.

Finally, let us suggest that readers primarily interested in the utilization in applicative contexts of the solvable subclass of the system of ODEs (1) (and/or (5)) have immediately a look below at Section 5, to assess if, and how, the findings reported in this paper are indeed likely to be useful for their purposes; and they might also be interested to have a quick look at Subsection 3.5, focussed on the subcase of the system (1) with $c_{14} = c_{21} = 0$, which is particularly relevant in applicative contexts.

**Remark 1-5.** Hereafter we assume that none of the 4 parameters $a_1$, $a_2$, $b_1$, $b_2$ vanishes; since it is plain from Proposition 1-1—see in particular the eqs. (5)—that if anyone of these 4 parameters vanishes the findings reported in this paper become rather less interesting. $\blacksquare$

### 2 Proof of Proposition 1-1

The starting point of our treatment is the trivially solvable ODE

$$\dot{y} = y^3,$$
Next, we introduce the ODE
\[ \dot{w} = w^3 + \gamma_1 yw^2 + \gamma_2 y^2 w + \gamma_3 y^3, \]
and we set (see (10a))
\[ w(t) = u(t)y(t), \]
implying of course
\[ u(t) = w(t)/y(t), \]
hence
\[ \dot{w}(t) = \dot{u}(t)y(t) + u(t)\dot{y}(t), \]
and (via (14))
\[ \dot{u} = y^2 \left[ u^3 + \gamma_1 u^2 + (\gamma_2 - 1)u + \gamma_3 \right], \]
\[ \dot{u} \left[ u^3 + \gamma_1 u^2 + (\gamma_2 - 1)u + \gamma_3 \right]^{-1} = y^2, \]
\[ \dot{u} \left( \frac{\lambda_1}{u - u_1} + \frac{\lambda_2}{u - u_2} + \frac{\lambda_3}{u - u_3} \right) = \left[ y(0) \right]^2 \left\{ 1 - 2 \left[ y(0) \right]^2 t \right\}^{-1}, \]
where the 3 parameters \( \gamma_j \) are a priori arbitrary and the 6 parameters \( \lambda_j \) and \( u_j \) are clearly defined in terms of them by the relations (11) and (12).

The formula (10b) is then an immediate consequence of this ODE (16d).

Finally, it is easy to verify—via a straightforward if tedious computation—that the relations (8a), implying of course
\[ \dot{x}_1(t) = \left[ b_2 y(t) - a_2 \dot{w}(t) \right]/c, \quad \dot{x}_2(t) = \left[ -b_1 \dot{y}(t) + a_1 \dot{w}(t) \right]/c, \]
hence, via (14) and (15),
\[ \dot{x}_1 = \left[ b_2 y^3 - a_2 \left( w^3 + \gamma_1 w^2 y + \gamma_2 w y^2 + \gamma_3 y^3 \right) \right]/c, \]
\[ \dot{x}_2 = \left[ -b_1 y^3 + a_1 \left( w^3 + \gamma_1 w^2 y + \gamma_2 w y^2 + \gamma_3 y^3 \right) \right]/c, \]
coincide—via the relations (8a)—with the system (11), provided the 8 parameters \( c_{n\ell} \) in the right-hand sides of (11) are given by the formulas (6) (with (7)) in terms of the 7 parameters \( \gamma_1, \gamma_2, \gamma_3, a_1, a_2, b_1, b_2. \)

**Proposition 1-1** is thereby proven.

### 3 Inversion of the relations (6) and constraints on the parameters \( c_{n\ell} \)

The task accomplished in this Section 3 is nontrivial, due to the nonlinear character of the algebraic equations to be solved: it could not be performed by the blind use of computer manipulations programs such as Mathematica; and it is clear that there is not a unique way to manage it.

The results reported in this Section 3 are likely to be particularly relevant for researchers who are interested to use in applicative contexts the findings reported in this paper.

**Remark 3-1.** Note that below we often take advantage of the definition (1c) of \( c \). ■

#### 3.1 First step towards inverting the relations (6)

The following easy consequences of the relations (6) constitute a first step towards their inversion.

From (6a) and (6b),
\[ a_1 c_{11} + a_2 c_{21} = (a_1)^3; \]
from (6c) and (6d),
\[ a_1 c_{12} + a_2 c_{22} = 3 (a_1)^2 a_2; \]
from (6e) and (6f),
\[ a_1 c_{13} + a_2 c_{23} = 3a_1 (a_2)^2; \]
from (6g) and (6h),

\[ a_1 c_{14} + a_2 c_{24} = (a_2)^3 . \]  

(18d)

**Remark 3.1-1.** Clearly these 4 algebraic equations imply the following 2 neat consequences:

\[ \sum_{n=1}^{2} [a_n (c_{n1} + s c_{n2} + c_{n3} + s c_{n4})] = (a_1 + s a_2)^3 , \quad s = \pm . \]  

(19)

But we shall not make use of these particular relation below.

### 3.2 Second step: determination of the 2 parameters \( a_n \) in terms of the 8 coefficients \( c_{n\ell} \) and of 2 constraints on these 8 coefficients

It is plain that the 4 equations (18) featuring the 10 dependent variables \( a_n \) and \( c_{n\ell} \) \((n = 1, 2; \ell = 1, 2, 3, 4)\) are independent of each other (since they involve different variables); and since they involve only the 2 parameters \( a_1 \) and \( a_2 \) in addition to the 8 coefficients \( c_{n\ell} \) it is quite natural to infer that they determine these 2 parameters \( a_1 \) and \( a_2 \) in terms of the 8 coefficients \( c_{n\ell} \) and that they moreover imply that the 8 coefficients \( c_{n\ell} \) satisfy 2 constraints. In this Subsection 3.2.2 we indeed obtain various explicit expressions of the 2 parameters \( a_n \) in terms of the 8 coefficients \( c_{n\ell} \) and we also obtain various avatars of 2 constraints implied by the 4 eqs. (18) for these 8 coefficients (with more implied by the formulas reported in Appendices A, B and C).

The determination from the 4 eqs. (18) of the 2 parameters \( a_1 \) and \( a_2 \) in terms of the 8 coefficients \( c_{n\ell} \), and of 2 constraints on the 8 coefficients \( c_{n\ell} \), can be achieved via different routes. A convenient way to make some initial progress is by introducing the auxiliary quantity

\[ \alpha = a_2/a_1 . \]  

(20)

Then, by taking the ratios of each of the (last 3 of the 4 ) equations (18) over that written before it, one easily gets the following 3 quadratic equations for the quantity \( \alpha \):

\[ 3 c_{21} \alpha^2 + (3c_{11} - c_{22}) \alpha - c_{12} = 0 , \]  

(21a)

\[ c_{22} \alpha^2 + (c_{12} - c_{23}) \alpha - c_{13} = 0 , \]  

(21b)

\[ c_{23} \alpha^2 + (c_{13} - 3c_{24}) \alpha - 3c_{14} = 0 . \]  

(21c)

Next, multiply (21b) by \( 3c_{21} \) and subtract from it (21a) itself multiplied by \( c_{22} \); likewise multiply (21c) by \( c_{22} \) and subtract from it (21b) itself multiplied by \( c_{23} \); and finally multiply (21a) by \( c_{23} \) and subtract from it (21c) itself multiplied by \( 3c_{21} \). In this manner the following 3 explicit expressions of \( \alpha \) in terms of the 8 coefficients \( c_{n\ell} \) are easily obtained:

\[ \alpha = \frac{c_{12} c_{22} - 3c_{13} c_{21}}{3(c_{11} c_{22} - c_{12} c_{21} + c_{21} c_{23}) - (c_{22})^2} , \]  

(22a)

\[ \alpha = \frac{c_{13} c_{23} - 3c_{14} c_{22}}{c_{12} c_{23} - c_{13} c_{22} - (c_{23})^2 + 3c_{22} c_{24}} , \]  

(22b)

\[ \alpha = \frac{c_{12} c_{23} - 9c_{14} c_{21}}{3(c_{11} c_{23} - c_{13} c_{21}) + 9c_{21} c_{24} - c_{22} c_{23}} . \]  

(22c)

**Remark 3.2-1.** Of course these are only 3 specific expressions of \( \alpha \), arbitrarily selected out of a plurality of different—but as well valid—formulas which might be obtained by analogous developments. And note that this remark is as well relevant for several of the following developments.

The simultaneous validity of these 3 expressions of \( \alpha \) implies of course the following 3 relations among the parameters \( c_{n\ell} \):

\[ (c_{12} c_{22} - 3c_{13} c_{21}) \left[ c_{12} c_{23} - c_{13} c_{22} - (c_{23})^2 + 3c_{22} c_{24} \right] \]

\[ = \left[ c_{13} c_{23} - 3c_{14} c_{22} \right] \left[ 3(c_{11} c_{22} - c_{12} c_{21} + c_{21} c_{23}) - (c_{22})^2 \right] , \]  

(23a)
Clearly—as implied by their derivation from the relations (22)—any two of these 3 relations imply the third; so they provide at most 2 independent constraints on the 8 coefficients \( c_{1\ell} \). But in fact they only entail 1 constraint on the 8 coefficients \( c_{1\ell} \), as demonstrated by the following remarkable phenomenon: if any one of the 3 eqs. (23) is solved for any specific one of the 8 coefficients \( c_{1\ell} \), then the 3 results thereby obtained from these 3 equations—in terms of the other 7 coefficients \( c_{1\ell} \)—are identical. Since each of the 3 eqs. (23) is linear or quadratic in each of the 8 coefficients \( c_{1\ell} \), these operations can be explicitly performed by hand (although it is wise to also check the result via Mathematica). The corresponding formulas are reported in Appendix A. Each of the 8 formulas reported there provides the explicit expression of one of the 8 parameters \( c_{1\ell} \) in term of the other 7; so each of them provides a constraint on the 8 coefficients \( c_{1\ell} \); and a constraint is as well provided by each of the 3 eqs. (23), or by any reasonable combinations of all these formulas. Of course all these constraints are equivalent. Nevertheless their explicit exhibition—especially in the "solved" version detailed in Appendix A—is worthwhile because of its potential usefulness in applicable contexts.

Clearly any one of the 3 formulas (22) provides an expression of \( \alpha \) in terms of the coefficients \( c_{1\ell} \); these 3 expressions of \( \alpha \) are of course equivalent if the coefficients \( c_{1\ell} \) satisfy the constraint mentioned above (see (23) and Appendix A), as we hereafter assume. It is then an easy task to get various explicit expressions of the parameters \( a_1 \) and \( a_2 \) from the equations (18); for instance, by dividing (18a) by \( a_1 \) one gets

\[
(a_1)^2 = c_{11} + \alpha c_{21} ,
\]

and then, from (18b),

\[
a_2 = \frac{a_1 c_{12}}{3 (a_1)^2 - c_{22}} = \frac{a_1 c_{12}}{3 (c_{11} + \alpha c_{21}) - c_{22}} ,
\]

where the second equality is of course implied by (24a) (and of course in these formulas \( \alpha \) is given by any one of the 3 formulas (22)).

There still remains the task to determine the second constraint on the 8 coefficients implied by the 4 eqs. (18). Other explicit expressions of the 2 parameters \( a_1 \) and \( a_2 \)—or rather of their squares—in terms of the 8 coefficients \( c_{1\ell} \) are also provided below in this context.

Let us begin by performing the following 4 operations on the 4 eqs. (18). (1) We multiply the first of these 4 equations by \( c_{22} \) and subtract from the result the second of these 4 equations itself multiplied by \( c_{21} \). (2) We multiply the third of these 4 equations by \( c_{14} \) and subtract from the result the fourth of these 4 equations itself multiplied by \( c_{13} \). (3) We multiply the first of these 4 equations by \( c_{23} \) and subtract from the result the third of these 4 equations itself multiplied by \( c_{21} \). (4) We multiply the second of these 4 equations by \( c_{14} \) and subtract from the result the fourth of these 4 equations itself multiplied by \( c_{12} \). There thus obtain the following 4 equations:

\[
c_{11} c_{22} - c_{12} c_{21} = c_{22} (a_1)^2 - 3 c_{21} a_1 a_2 ,
\]
\[
c_{14} c_{23} - c_{13} c_{24} = -c_{13} (a_2)^2 + 3 c_{14} a_1 a_2 ,
\]
\[
c_{11} c_{23} - c_{13} c_{21} = c_{23} (a_1)^2 - 3 c_{21} (a_2)^2 ,
\]
\[
c_{14} c_{22} - c_{12} c_{24} = -c_{12} (a_2)^2 + 3 c_{14} (a_1)^2 .
\]

Next, we perform the following cycle of analogous operations on these eqs. (25) (which are clearly equivalent to the 4 eqs. (18)). (1) We multiply the first of these 4 equations by \( c_{14} \) and we add to the result the second of these 4 equations itself multiplied by \( c_{21} \). (2) We multiply the third of these 4 equations by \( c_{12} \) and we subtract from the result the fourth of these 4 equations itself multiplied by \( 3 c_{21} \). (3) We multiply the third of these 4 equations by \( 3 c_{14} \) and we subtract from the result the fourth of these 4 equations itself multiplied by \( c_{23} \). There thus obtain the following 3 equations:

\[
c_{14} (c_{11} c_{22} - c_{12} c_{21}) + c_{21} (c_{14} c_{23} - c_{13} c_{24}) = c_{14} c_{22} (a_1)^2 - c_{13} c_{21} (a_2)^2 ,
\]
\[
(c_{12} c_{23} - 9 c_{14} c_{21}) \left[ c_{12} c_{23} - c_{13} c_{22} - (c_{23})^2 + 3 c_{22} c_{24} \right] ,
\]
\[
(c_{12} c_{23} - 9 c_{14} c_{21}) \left[ 3 (c_{11} c_{22} - c_{12} c_{21} + c_{21} c_{23}) - (c_{22})^2 \right] ,
\]
\[
(c_{12} c_{22} - 3 c_{13} c_{21}) \left[ 3 (c_{11} c_{23} - c_{13} c_{21}) + 9 c_{21} c_{24} - c_{22} c_{23} \right] .
\]

(23b)
c_12 \left(c_{11}c_{23} - c_{13}c_{21}\right) - 3c_21 \left(c_{14}c_{22} - c_{12}c_{24}\right) = \left(c_{12}c_{23} - 9c_{14}c_{21}\right) \left(a_1\right)^2,  \quad (26b)  
3c_14 \left(c_{11}c_{23} - c_{13}c_{21}\right) - c_{23} \left(c_{14}c_{22} - c_{12}c_{24}\right) = \left(c_{12}c_{23} - 9c_{14}c_{21}\right) \left(a_2\right)^2.  \quad (26c)

Next, we multiply the second of these 3 equations (26) by c_{14}c_{22} and we subtract from the result the last of these 3 equations itself multiplied by c_{13}c_{21}. We thereby get the following equation:

\[ c_{14} \left(c_{11}c_{23} - c_{13}c_{21}\right) \left(c_{12}c_{22} - 3c_{13}c_{21}\right) + c_{21} \left(c_{14}c_{22} - c_{12}c_{24}\right) \left(c_{13}c_{23} - 3c_{14}c_{22}\right) = \left(c_{12}c_{23} - 9c_{14}c_{21}\right) \left[c_{14}c_{22} \left(a_1\right)^2 - c_{13}c_{21} \left(a_2\right)^2\right], \quad (27) \]

hence, by comparing the right-hand side of this equation with the right-hand side of eq. (26a), we get finally the following constraint involving the 8 coefficients c_{nℓ}:

\[ c_{14} \left(c_{11}c_{23} - c_{13}c_{21}\right) \left(c_{12}c_{22} - 3c_{13}c_{21}\right) + c_{21} \left(c_{14}c_{22} - c_{12}c_{24}\right) \left(c_{13}c_{23} - 3c_{14}c_{22}\right) = \left(c_{12}c_{23} - 9c_{14}c_{21}\right) \left[c_{14} \left(c_{11}c_{22} - c_{12}c_{21}\right) + c_{21} \left(c_{14}c_{23} - c_{13}c_{24}\right)\right]. \quad (28) \]

Remark 3.2-1. This constraint is invariant under the transformation (3). ■

But, as it happens, this is rather a new avatar of the constraint on the 8 coefficients c_{nℓ} already obtained above: it is again a homogeneous polynomial of fifth degree involving these 8 coefficients, but it is only linear in the 2 coefficients c_{11} and c_{24}, and quadratic in each of the other 6 coefficients. Indeed, it is easily seen that this constraint implies the following expression of the coefficient c_{11} in terms of the other 7 coefficients (and of course an analogous expression of the coefficient c_{24} in terms of the other 7 coefficients can then be obtained via the transformation (3)):

\[ c_{11} = \left\{-3c_{12}c_{22}c_{24} + 3c_{14} \left[3c_{12}c_{21} + (c_{22})^2\right] - c_{21} \left[(c_{12})^2 + 9c_{14}c_{21}\right]c_{23} + c_{12} (c_{23})^2 + c_{13} \left[c_{12}c_{22} - c_{22}c_{23} + 9c_{21}c_{24}\right] - 3 \left(c_{13}\right)^2 c_{21}\right\} / \left[3(3c_{14}c_{22} - c_{13}c_{23})\right]. \quad (29) \]

And it is easily seen that this formula is in fact identical to eq. (25a).

And the same happens for the expressions of the other 7 coefficients c_{nℓ} obtained in an analogous manner: see Appendix A.

Finally, to make progress towards the identification of the second constraint we note that the 2 eqs. (26b) and (26c) provide the following 2 expressions of (the squares of) a_1 and a_2 in terms of the 8 coefficients c_{nℓ}:

\[ (a_n)^2 = A_n \left(C\right), \quad n = 1, 2 ; \quad (30a) \]

\[ A_1 \left(C\right) = \left|c_{12} \left(c_{11}c_{23} - c_{13}c_{21}\right) - 3c_{21} \left(c_{14}c_{22} - c_{12}c_{24}\right)\right| / \left(c_{12}c_{23} - 9c_{14}c_{21}\right), \quad (30b) \]

\[ A_2 \left(C\right) = \left|3c_{14} \left(c_{11}c_{23} - c_{13}c_{21}\right) - c_{23} \left(c_{14}c_{22} - c_{12}c_{24}\right)\right| / \left(c_{12}c_{23} - 9c_{14}c_{21}\right). \quad (30c) \]

Here and below we denote by C the set of 8 coefficients c_{nℓ}.

Let us then return to the original 4 eqs. (18), and let us replace—in an obvious manner—the squares of the variables a_n by their expressions (30a). We thus obtain the following system of 4 homogeneous linear equations for the 2 dependent variables a_n:

\[ c_{11} - A_1 \left(C\right) a_1 + c_{21} a_2 = 0, \quad (31a) \]

\[ c_{12} a_1 + \left[c_{22} - 3A_1 \left(C\right)\right] a_2 = 0, \quad (31b) \]

\[ c_{13} - 3A_2 \left(C\right) a_1 + c_{23} a_2 = 0, \quad (31c) \]

\[ c_{14} a_1 + \left[c_{24} - A_2 \left(C\right)\right] a_2 = 0. \quad (31d) \]

By selecting any pair of these 4 equations we get 6 different systems of 2 homogeneous linear equations, which must be satisfied by the 2 (nonvanishing!) dependent variables a_1 and a_2. Hence the 6 determinants of the coefficients
of these 6 systems must vanish, providing thereby—in principle—as many constraints on the 8 coefficients $c_{n\ell}$. But 2 of the resulting relations yield an identically vanishing result ($0 = 0$), and the other 4 the same outcome (provided the 8 coefficients have generic values, none of them vanishing): yielding the following second constraint on the 8 coefficients $c_{n\ell}$:

$$[c_{12} (c_{13} - 3c_{24}) + 3c_{14} (c_{13} - 3c_{11} + c_{22})] [3c_{21} (c_{13} - 3c_{24}) + c_{23} (c_{22} - 3c_{11})] = (c_{12}c_{23} - 9c_{14}c_{21})^2.$$  \hspace{1cm} (32)

The fact that this second constraint is not equivalent to the first is demonstrated by solving it for each of the 8 coefficients $c_{n\ell}$ and by noting that the resulting expressions of the 8 coefficients $c_{n\ell}$—as reported in Appendix B—do not coincide with those reported in Appendix A.

The formulas reported in Appendices A and B become of course pairwise equivalent if the 8 coefficients $c_{n\ell}$ are required to satisfy the 2 constraints identified in this Subsection 3.2: both of them!

The way is now open—solving simultaneously both the 2 independent constraints satisfied by the 8 coefficients $c_{n\ell}$—to obtain explicit expressions of each pair of the 8 coefficients $c_{n\ell}$ in terms of the other 6: remarkably, in all cases the relevant formulas could be explicitly obtained (via Mathematica) although in some cases they are too complicated to be usefully displayed: see Appendix C. To overcome as much as possible this difficulty the following remark turned out to be useful.

Remark 3.2-2. The second constraint \((32)\) is a homogeneous polynomial equation of degree 4 in the 8 variables $c_{n\ell}$; and several of the equivalent versions of the first constraint are also homogeneous polynomial equations, in particular the 3 versions \((23)\) are each homogeneous polynomial equations of degree 4; but 2 of the equations displayed in Appendix A—those containing no square-roots: see \((58a)\) and \((58d)\)—are (de facto) homogeneous polynomial equations of degree (only!) 3.

3.3 Third step: determination of the 2 parameters $b_n$

The starting point for the next steps to invert the 8 relations \((31)\) are the following 4 relations, obtained by summing pairwise the 8 relations \((6)\) appropriately multiplied by $b_1$ and $b_2$:

$$b_1c_{11} + b_2c_{21} = K_1,$$ \hspace{1cm} (33a)

$$b_1c_{12} + b_2c_{22} = K_2,$$ \hspace{1cm} (33b)

$$b_1c_{13} + b_2c_{23} = K_3,$$ \hspace{1cm} (33c)

$$b_1c_{14} + b_2c_{24} = K_4,$$ \hspace{1cm} (33d)

of course with the 4 quantities $K_\ell$ defined by the identities \((7)\).

We now note that these 4 relations can be re-written, via \((7)\), as follows

$$\gamma_1a_1 (b_1)^2 + \gamma_2 (a_1)^2 b_1 + \gamma_3 (a_1)^3 = -(b_1)^3 + b_1c_{11} + b_2c_{21},$$ \hspace{1cm} (34a)

$$\gamma_1 [a_2 (b_1)^2 + 2a_1b_1b_2] + \gamma_2 [(a_2)^2 b_2 + 2a_1a_2b_1] + 3\gamma_3 (a_1)^2 a_2$$

$$= -3 (b_1)^2 b_2 + b_1c_{12} + b_2c_{22},$$ \hspace{1cm} (34b)

$$\gamma_1 [a_1 (b_2)^2 + 2a_2b_1b_2] + \gamma_2 [(a_2)^2 b_1 + 2a_1a_2b_2] + 3\gamma_3 a_1 (a_2)^2$$

$$= -3b_1 (b_2)^2 + b_1c_{13} + b_2c_{23},$$ \hspace{1cm} (34c)

$$\gamma_1 a_2 (b_2)^2 + \gamma_2 (a_2)^2 b_2 + \gamma_3 (a_2)^3 = -(b_2)^3 + b_1c_{14} + b_2c_{24}.$$ \hspace{1cm} (34d)

These are clearly 4 linear equations satisfied by the 3 parameters $\gamma_j$ ($j = 1, 2, 3$); so that these 3 parameters $\gamma_j$ can then be explicitly determined by solving the system formed by any 3 of these 4 equations, obtaining thereby expressions of these 3 parameters $\gamma_j$ in terms of the 8 coefficients $c_{n\ell}$ appearing in the right-hand sides of these equations \((34)\), of the 2 parameters $a_n$—themselves already determined in terms of the 8 coefficients $c_{n\ell}$ (see
Subsection 3.2)—and of the still undetermined 2 parameters $b_n$. Once this step has been completed, the unused one of the 4 equations (54) provides a single explicit constraint on the 2 parameters $b_n$, implying the determination of one of them in terms of the other, or of their ratio (see below). But this rather natural route leads to quite complicated final equations. An equivalent, more practical, route is described below.

**Remark 3.3-1.** It is plain—by summing these 4 relations (54), with the second and fourth multiplied by $s = \pm$—that they imply the 2 relations

$$\gamma_1 (a_1 + sa_2) (b_1 + sb_2)^2 + \gamma_2 (a_1 + sa_2)^2 (b_1 + sb_2) + \gamma_3 (a_1 + sa_2)^3$$

$$= -(b_1 + sb_2)^3 + b_1 (c_{11} + sc_{12} + c_{13} + sc_{14}) + b_2 (c_{21} + sc_{22} + c_{23} + sc_{24}) ,$$

$$s = \pm . \quad (35)$$

But again we shall not need to take advantage of this relation. ■

**Remark 3.3-2.** By multiplying the formula (54a) by $(a_2)^3$ and by then subtracting from the result the formula (54d) itself multiplied by $(a_1)^3$ one gets (after some easy simplifications) the formula

$$a_1 a_2 (a_1 b_2 - a_2 b_1) [(a_1 b_2 + a_2 b_1) \gamma_1 + a_1 a_2 \gamma_2]$$

$$= -(a_1 b_2)^3 + (a_2 b_1)^3$$

$$+ (a_1)^3 (b_1 c_{14} + b_2 c_{24}) - (a_2)^3 (b_1 c_{11} + b_2 c_{21}) ; \quad (36a)$$

likewise, by multiplying the eq. (54b) by $a_2$ and by then subtracting from the result the formula (54d) itself multiplied by $a_1$ one gets (after some easy simplifications) the formula

$$(a_1 b_2 - a_2 b_1) [(a_1 b_2 + a_2 b_1) \gamma_1 + a_1 a_2 \gamma_2]$$

$$= -3 (a_1 b_2 - a_2 b_1) b_1 b_2 + b_1 (a_1 c_{23} - a_2 c_{22})$$

$$- b_1 (a_1 c_{13} - a_2 c_{12}) . \quad (36b)$$

Next, we multiply the second of these 2 formulas by $a_1 a_2$ and then subtract the result from the first of these 2 formulas, getting thereby (after some easy simplifications) the formula

$$(a_1 b_2 - a_2 b_1)^3 = b_2 [(a_1)^3 c_{24} - (a_2)^3 c_{21} - a_1 a_2 (a_1 c_{23} - a_2 c_{22})]$$

$$+ b_1 [(a_1)^3 c_{14} - (a_2)^3 c_{11} - a_1 a_2 (a_1 c_{13} - a_2 c_{12})] . \quad (36c)$$

Finally it is convenient to re-write this equation via the following positions:

$$a_2 = \alpha a_1 , \quad c_{n\ell} = \eta \hat{c}_{n\ell} , \quad b_1 = \eta , \quad b_2 = \alpha \beta \eta . \quad (37)$$

Here of course $\alpha$ is the parameter already introduced in Subsection 3.2 (see (22)) and determined there in terms of the 8 coefficients $c_{n\ell}$; $\eta$ is a parameter characterizing a rescaling of the 8 coefficients $c_{n\ell}$—introduced here to make notational contact with the invariance property of the system (11) mentioned in Remark 1-1 (see (2)) which implies that this parameter can be eventually altogether eliminated—i. e., assigned an arbitrary nonvanishing value (for instance, just the value $\eta = 1$)—via a corresponding appropriate rescaling of the independent variable $t$; $\beta$ is the parameter that we determine immediately below in terms of the 8 coefficients $c_{n\ell}$; and the last 2 eqs. (37) determine of course the 2 parameters $b_1$ and $b_2$, thereby completing the task indicated by the title of this Subsection 3.3.

To determine the parameter $\beta$ in terms of the 8 parameters $c_{n\ell}$—or, equivalently, $\hat{c}_{n\ell}$ (see (37) and (2))—we insert in eq. (36c) the positions (37), getting thereby the following cubic equation for this parameter:

$$(\beta - 1)^3 = \beta (\hat{c}_{24} \alpha^{-2} - \hat{c}_{23} \alpha^{-1} + \hat{c}_{22} - \hat{c}_{21} \alpha^{-1})$$

$$+ \hat{c}_{13} \alpha^{-2} - \hat{c}_{12} \alpha^{-1} + \hat{c}_{11} \alpha^{-1} . \quad (38)$$

Explicit solutions of this equation can of course be provided via the standard Cardano formulas.
### 3.4 Fourth step: determination of the 3 parameters $\gamma_j$ ($j = 1, 2, 3$) in terms of the 8 coefficients $c_{n\ell}$

The explicit determination of the 3 parameters $\gamma_j$ ($j = 1, 2, 3$) is now in principle a standard task, amounting—as shown below—to the solution of any triad of the several linear equations satisfied by these quantities: see for instance the 4 eqs. (34) and the 3 eqs. (36), or appropriate linear combinations of these equations. Of course care must be taken to use 3 equations which are independent of each other. The expressions obtained in this manner in terms of the 8 coefficients $c_{n\ell}$—and of the parameters $a_n$ and $b_n$ themselves expressed in terms of the 8 coefficients $c_{n\ell}$ as explained in Subsections 3.2 and 3.3—are of course only valid provided the 8 coefficients $c_{n\ell}$ satisfy the explicit constraints determined above (see Subsection 3.2 and Appendices A, B and C); and they are equivalent, but they need not look identical.

The most straightforward procedure is to use 3 out of the 4 eqs. (34); there are then of course 4 possible choices.

**Remark 3.4-1.** Hereafter $c = a_1b_2 - a_2b_1$, see (34). ■

The first choice are the 3 eqs. (34a), (34b) and (34d). One then gets (if need be, with the help of Mathematica)

$$
\gamma_1 = \left\{ -3(a_2)^2 (b_1)^3 + 6a_1a_2 (b_1)^2 b_2 - 3(a_1)^2 b_1 (b_2)^2 \\
+3(a_2)^2 (b_1c_{11} + b_2c_{21}) - 2a_1a_2 (b_1c_{12} + b_2c_{22}) \\
+ (a_1)^2 (b_1c_{13} + b_2c_{23}) \right\} / (a_1c^2),
$$

(39a)

$$
\gamma_2 = \left\{ 3(a_2)^2 (b_1)^4 - 6a_1a_2 (b_1)^3 b_2 + 3(a_1b_2)^2 - 3(a_2)^2 b_1 (b_1c_{11} + b_2c_{21}) \\
+3a_1a_2 \left( b_1c_{12} - (b_2)^2 c_{21} + b_1b_2 (-c_{11} + c_{22}) \right) \\
+ (a_1)^2 \left\{ -2(b_1)^2 c_{13} + (b_2)^2 c_{22} + b_1b_2 (c_{12} - 2c_{23}) \right\} \right\} / (a_1c)^2,
$$

(39b)

$$
\gamma_3 = \left\{ - (a_2)^2 (b_1)^5 + 2a_1a_2 (b_1)^4 b_2 - (a_1)^2 (b_1)^3 (b_2)^2 \\
+ (a_2b_1)^2 (b_1c_{11} + b_2c_{21}) \\
- a_1a_2b_1 \left( b_1c_{12} - (b_2)^2 c_{21} + b_1b_2 (-c_{11} + c_{22}) \right) \\
+ (a_1)^2 \left( b_1c_{13} + (b_2)^3 c_{21} + b_1b_2 (c_{12} - 2c_{23}) \right) \\
- (b_2)^2 c_{22} (c_{12} - c_{23}) \right\} / (a_1c)^3.
$$

(39c)

The second choice are the 3 eqs. (34a), (34b) and (34d). One then gets (if need be, with the help of Mathematica)

$$
\gamma_1 = \left\{ -2(a_2b_1)^3 + 3a_1 (a_2b_1)^2 b_2 - (a_1b_2)^3 \\
+2(a_2)^3 (b_1c_{11} + b_2c_{21}) - a_1 (a_2)^2 (b_1c_{12} + b_2c_{22}) \\
+ (a_1)^3 (b_1c_{14} + b_2c_{24}) \right\} / (a_1a_2c^2),
$$

(40a)

$$
\gamma_2 = \left\{ (a_2)^3 (b_1)^4 + 2(a_1)^3 b_1 (b_2)^3 - 3(a_1)^2 a_2 (b_1b_2)^2 \\
- (a_2)^3 b_1 (b_1c_{11} + b_2c_{21}) + (a_1)^2 a_2b_2 (b_1c_{12} + b_2c_{22}) \\
+a_1 (a_2)^2 \left( b_1c_{12} - 3(b_2)^2 c_{21} + b_1b_2 (c_{11} + c_{22}) \right) \\
-2a_1^3 b_1 (b_1c_{14} + b_2c_{24}) \right\} / [(a_1)^2 a_2c^2],
$$

(40b)
The third choice are the 3 eqs. (34a), (34c) and (34d). One then gets (if need be, with the help of Mathematica)

\[
\gamma_3 = \left\{ - (a_2)^2 (b_1)^4 b_2 + 2a_1 a_2 (b_1)^3 (b_2)^2 - (a_1 b_1)^2 (b_2)^3 \\
+ (a_2)^2 b_1 b_2 (b_1 c_{11} + b_2 c_{21}) \\
+ a_1 a_2 b_2 \left[ -(b_1)^2 c_{12} + (b_2)^2 c_{21} + b_1 b_2 (c_{11} - c_{22}) \right] \\
+ (a_1 b_1)^2 (b_1 c_{14} + b_2 c_{24}) \right\} \bigg/ \left[ (a_1)^2 a_2 c^2 \right].
\] (40c)

The fourth choice are the 3 eqs. (34b), (34c) and (34d). One then gets (if need be, with the help of Mathematica)

\[
\gamma_3 = \left\{ - (a_2)^2 (b_1)^3 (b_2)^2 - (a_1)^2 b_1 (b_2)^4 + 2a_1 a_2 (b_1)^2 (b_2)^3 \\
+ (a_2 b_2)^2 (b_1 c_{11} + b_2 c_{21}) + (a_1)^2 b_1 b_2 (b_1 c_{14} + b_2 c_{24}) \\
+ a_1 a_2 b_1 \left[ (b_1)^2 c_{14} - (b_2)^2 c_{23} - b_1 b_2 (c_{13} - c_{24}) \right] \right\} \bigg/ \left[ (a_1 a_2)^2 c^2 \right].
\] (41c)
The formulas displayed above provide explicit expressions of the 3 parameters $\gamma_j$ in terms of the 8 coefficients $c_{n\ell}$, and also in terms of the parameters $a_n$ and $b_n$, the determination of which in terms of the 8 coefficients $c_{n\ell}$ has been detailed in the preceding subsections of this Section 3.

These 4 expressions of the 3 parameters $\gamma_j$ are of course equivalent provided the 8 coefficients $c_{n\ell}$ satisfy the 2 constraints determined above (see Subsection 3.2 and Appendices A, B and C).

The task of inverting the formulas (40) is thereby completed.

3.5 The special case with $c_{14} = c_{21} = 0$

In several applications it is unreasonable to assume that the variations over time of the quantity $x_n(t)$ be influenced by a cause which depends only on the value of the other variable: hence that in the right-hand sides of the ODE characterizing the change over time of the dependent variable $x_n(t)$ associated to these phenomena, terms independent from the values of this variable $x_n(t)$ be present. This fact motivates the special interest of the subclass of the dynamical systems (1) characterized by the vanishing of the 2 coefficients $c_{14}$ and $c_{21}$,

$$c_{14} = c_{21} = 0,$$  \hspace{1cm} (43a)

hence characterized by the following reduced version of the system (1):

$$\dot{x}_1 = x_1 \left[ g_{11} \left( x_1 \right)^2 + g_{12} x_1 x_2 + g_{13} \left( x_2 \right)^2 \right],$$

$$\dot{x}_2 = x_2 \left[ g_{21} \left( x_1 \right)^2 + g_{22} x_1 x_2 + g_{23} \left( x_2 \right)^2 \right],$$  \hspace{1cm} (43b)

where, for notational convenience, we set (in addition to (43a))

$$c_{1j} \equiv g_{1j}, \quad c_{2\ell} \equiv g_{2\ell}, \quad j = 1, 2, 3, \quad \ell = j + 1.$$  \hspace{1cm} (43c)

In this Subsection 3.5 we tersely treat this particular subcase which—for the reason mentioned above—is of special applicative relevance.

Remark 3.5-1. Clearly in this case the invariance property (3) is replaced by the following formulas:

$$x_1(t) \Leftrightarrow x_2(t); \quad g_{11} \Leftrightarrow g_{23}, \quad g_{12} \Leftrightarrow g_{22}, \quad g_{13} \Leftrightarrow g_{21}.$$  \hspace{1cm} (44)

While of course for the system (43b) the invariance property (2) is just as valid as for the system (1). ■

The most direct way to obtain the results relevant to this special case is to insert in the previous treatment the restriction (43a) and the notational change (43c).

To assess the impact of the restriction (43a) we set it—together with the notational change (43c)—in the 2 eqs. (44).

It is then easily seen that its insertion in the simpler eqs. (44) implies the restriction

$$g_{12} = g_{22} = 0,$$  \hspace{1cm} (45a)

causing the system (43b) to take the following reduced form:

$$\dot{x}_1 = x_1 \left[ g_{11} \left( x_1 \right)^2 + g_{13} \left( x_2 \right)^2 \right],$$

$$\dot{x}_2 = x_2 \left[ g_{21} \left( x_1 \right)^2 + g_{23} \left( x_2 \right)^2 \right];$$  \hspace{1cm} (45b)

and moreover implying—see (24b)—the vanishing of the parameter $a_2$,

$$a_2 = 0,$$  \hspace{1cm} (45c)

signifying that in this special case our solution technique gets applied to a system whose solvable character is rather trivial.

The situation is instead different if one considers the alternative case (44b). Then, via (43a) and (43c), one obtains the following 2 new constraints on the 6 coefficients $g_{n\ell}$:

$$-g_{12} \left[ 2 \left( g_{22} \right)^2 + g_{21} \left( g_{13} - 3 g_{23} \right) \right]$$

$$+ g_{22} \left[ \left( g_{12} \right)^2 + \left( g_{22} \right)^2 + 3 g_{21} \left( g_{13} - g_{23} \right) \right] + FR_4 = 0,$$  \hspace{1cm} (46a)
The nonlinear context implies that there are many equivalent versions of a second independent relation constraining the coefficients $g_{nj}$, the simultaneous validity of both of them being then required in order that the solution of the system (43b) be provided by the insertion in Proposition 1-1 of the identities (43a) and (45c). The identification of which version of these 2 constraints is by revisiting the treatment of the preceding Subsections of this Section 3, highlighting the modifications implied by the restriction (43c) together with the notational change (43c).

The 3 expressions (22) of the parameter $\alpha$ read then as follows:

$$\alpha = \frac{g_{12}}{3g_{11} - g_{21}}, \quad \alpha = \frac{g_{13}g_{22}}{g_{12}g_{22} - g_{13}g_{21} - (g_{22})^2 + 3g_{21}g_{23}}, \quad \alpha = \frac{g_{12}}{3g_{11} - g_{21}};$$

and (since the first and last of these 3 equations are clearly identical) they clearly yield the following single constraint on the 6 coefficients $g_{nj} \ (n = 1, 2; \ j = 1, 2, 3)$:

$$g_{12} \left[ g_{12}g_{22} - g_{13}g_{21} - (g_{22})^2 + 3g_{21}g_{23} \right] - g_{13}g_{22} (3g_{11} - g_{21}) = 0. \quad (48)$$

The formulas reported below are valid for generic values of the coefficients $g_{nj}$: alternative solutions requiring some of the coefficients $g_{nj}$ to vanish are generally not reported, unless they are exceptionally simple (moreover, in the last case reported below, see (52a), it is the only result obtained for that case).
Hereafter
\[
G_1 = \sqrt{(g_{12})^2 + 4g_{13}g_{21} - 2g_{12}g_{22} + (g_{22})^2},
\]
\[
G_{11/23} = \sqrt{g_{11}/g_{23}},
\]
\[
G_2 = \sqrt{(g_{12})^2 + 12g_{21}g_{23}}.
\] (50)

We first report the formulas expressing the 3 pairs that are transformed into themselves by the transformations (43):

\[
g_{11} = \left[ (g_{12})^2 + 2g_{13}g_{21} - g_{12}g_{22} + g_{12}G_1 \right] / (6g_{13}),
\] (51a)
\[
g_{23} = \left[ 2g_{13}g_{21} - g_{12}g_{22} + (g_{22})^2 + g_{12}G_1 \right] / (6g_{21});
\] (51b)
\[
g_{12} = -(3g_{11} - g_{21})/G_{11/23}, \quad g_{22} = (g_{13} - 3g_{23})G_{11/23};
\] (51c)
\[
g_{13} = 3g_{23} - g_{22}/G_{11/23}, \quad g_{23} = 3g_{11} - g_{12}G_{11/23}.
\] (51d)

Next, the formulas expressing the 6 pairs that are transformed into 6 other pairs by the transformations (44):

\[
g_{11} = g_{21}/3, \quad g_{12} = 0;
\]
\[
g_{11} = (g_{22})^2 g_{23} / (g_{13} - 3g_{23})^2,
\]
\[
g_{12} = \frac{(g_{13})^2 g_{21} - 6g_{13}g_{21}g_{23} - 3 \left[(g_{22})^2 g_{23} - 3g_{21} (g_{23})^2\right]}{g_{22} (g_{13} - 3g_{23})};
\] (52a)
\[
g_{11} = (g_{12})^2 + 6g_{21}g_{23} + g_{12}G_2 / 18g_{23}, \quad g_{13} = g_{12}g_{22} + 6g_{21}g_{23} - g_{22}G_2 / 2g_{21};
\] (52b)
\[
g_{11} = (g_{22})^2 g_{23} / (g_{13} - 3g_{23})^2, \quad g_{21} = g_{22} \left[ g_{12} (g_{13} - 3g_{23}) + 3g_{22}g_{23} \right] / (g_{13} - 3g_{23})^2;
\] (52c)
\[
g_{11} = 6g_{21}g_{23} + g_{12} (g_{13} + G_2) / 18g_{23}, \quad g_{22} = -(g_{13} - 3g_{23}) (g_{12} + G_2) / 6g_{23};
\] (52d)
\[
g_{12} = (3g_{11} - g_{21})/G_{11/23}, \quad g_{13} = 3g_{23} + g_{22}/G_{11/23};
\] (52e)
\[
g_{12} = 0, \quad g_{23} = 3g_{11}.
\] (52f)

Hereafter we assume of course that the 6 coefficients \(g_{nj}\) satisfy the 2 constraints, many versions of which are provided above; and in addition the 2 constraints (46). Hence all the formulas we display below (in this Subsection 3.5) are just representative avatars of many other equivalent expressions implied by the formulas reported above, relating the 6 coefficients \(g_{nj}\) to each others.

The solution of the initial-values problem for the system (43) is then provided by Proposition 1-1, complemented by the eqs. (45a) and (45c).

For the applications of these findings the "inverse problem" to express the 7 parameters \(a_n, b_n\) and \(\gamma_j\) in terms of the coefficients \(g_{nj}\) is of course also quite important.

A quite neat version of the formulas expressing the 2 parameters \(a_n\) in terms of the coefficients \(g_{nj}\) of the system (43) reads as follows:

\[
a_1 = \sqrt{g_{11}}, \quad a_2 = \sqrt{g_{23}}.
\] (53)

For the determination of the 2 parameters \(b_n\) we refer to the relevant treatment provided at the end of Subsection 3.3, which is applicable with the following modifications: the equations (47) read now

\[
a_2 = \alpha a_1, \quad g_{nj} = \eta g_{nj}, \quad b_1 = \eta, \quad b_2 = \alpha \beta \eta,
\] (54)

where now \(\alpha\) is given in terms of the coefficient \(g_{nj}\) by anyone of the formulas (47); \(\eta\) is a parameter characterizing now a rescaling of the 6 coefficients \(g_{nj}\) introduced here to make notational contact with the invariance property.
Explicit solutions of this equation can of course be provided via the standard Cardano formulas. (43c) of the 8 coefficients just above are provided by the formulas (39)-(42), of course after the replacement implied by the identities (43a) and (see (2)) of the system (43b) mentioned above (see Remark 3.5-1 and Remark 1-1), again implying that this parameter can be eventually altogether eliminated—i.e., assigned an arbitrary nonvanishing value (for instance, just the value \( \eta = 1 \))—via a corresponding appropriate rescaling of the independent variable \( t \); \( \beta \) is the parameter that we determine immediately below in terms of the 6 coefficients \( g_{nj} \); and the last 2 eqs. (54) determine of course the 2 parameters \( b_1 \) and \( b_2 \), thereby completing the task indicated by the title of this Subsection 3.3.

To determine the parameter \( \beta \) in terms of the 6 parameters \( g_{nj} \)—or, equivalently, \( \hat{g}_{nt} \) (see (54) and (2))—we insert in eq. (43c) the positions (51), of course with (43a) and (43c), getting thereby the following cubic equation for this parameter:

\[
(\beta - 1)^3 = \beta (\hat{g}_{23} \alpha^{-2} - \hat{g}_{22} \alpha^{-1} + \hat{g}_{21}) - \hat{g}_{13} \alpha^{-1} + \hat{g}_{12} - \hat{g}_{11} \alpha^{-1}.
\]

(55)

**Explicit** solutions of this equation can of course be provided via the standard Cardano formulas.

Finally, several equivalent formulas expressing the parameters \( \gamma_j \) and involving the parameters \( a_n \) and \( b_n \) defined just above are provided by the formulas (49)-(42), of course after the replacement implied by the identities (43a) and (43c) of the 8 coefficients \( c_{nt} \) with the 6 coefficients \( g_{nj} \) (or possibly \( \hat{g}_{nj} \), see (54)).

4 The isochronous extension

In this Section 4 attention is restricted to real values of the independent variable \( t \) ("time"); except for the discussion in Remark 4.3.

It is well known (see, if need be, [5] and references therein) that via the following simple change of dependent and independent variables,

\[
\tilde{x}_n (t) = \exp(i \omega t) x_n (\tau) , \quad \tau = [\exp(2i \omega t) - 1] / (2i \omega) , \quad n = 1, 2 ,
\]

(56a)

the (autonomous) system (1) gets transformed into the, also autonomous, system (5). This new system differs from the system (1) only due to the additional presence of the linear term \( i \omega \tilde{x}_n \) in the right-hand side of its 2 ODEs. It is of course just as solvable as the system (1), to which it gets reduced via the (easily invertible) change of variables (55a). One then notes (see, for instance, [5]) that, if the parameter \( i \omega \) is an arbitrary imaginary number (as we hereafter assume), then—see (55a)—as the (real) time variable \( t \) evolves from its initial value 0 towards +\( \infty \), the auxiliary complex variable \( \tau \) rotates (counterclockwise for \( \omega > 0 \), clockwise for \( \omega < 0 \)) on the circle \( C \) of radius \( 1/|2\omega| \) centered at the point \( 1/(2\omega) \) in the complex \( \tau \)-plane: a clearly periodic evolution, with period

\[
T = \pi / |\omega| .
\]

(56b)

Hence any analytic function \( f(\tau) \) of the complex variable \( \tau \) featuring no singularity inside (nor on) the circle \( C \) in the complex \( \tau \)-plane evolves periodically with that period \( T \) as function of time, namely as the function \( \tilde{f}(t) \equiv f(\tau(t)) \) of the real variable \( t \) ("time"); and it also evolves periodically in time—with a period which is then an integer multiple of \( T \)—if the function \( f(\tau) \) features, as analytic function of the complex variable \( \tau \), a finite number of rational branch points inside the circle \( C \) (except for nongeneric initial data such that one of these branch points falls exactly on the circle \( C \) in the complex \( \tau \)-plane); since then the argument \( \tau(t) \) of the function \( \tilde{f}(t) \) travels on a Riemann surface with a finite number of sheets, and while possibly visiting (some of) these sheets it always eventually retraces the same path on the Riemann surface. Hence the system (5)—considered as a function of real time \( t \)—has the remarkable feature to be isochronous whenever the corresponding system (1) belongs to the class identified in this paper, such that its solutions only feature a finite number of rational branch points (see Remark 1-4): the period of its solutions being then generally a finite integer multiple of the basic period \( T \), see (56b), which does not change for sufficiently small changes of the initial data \( \tilde{x}_n(0) = x_n(0) \) (see (56a)); it may discontinuously change when the initial data are modified so that the consequential shifts of the branch point positions in the complex \( \tau \)-plane causes one of them to enter or exit the circle \( C \). While the special solutions—characterized by the exceptional initial data \( \tilde{x}_1(0) \) and \( \tilde{x}_2(0) \) such that one of the branch points of the corresponding solutions \( \tilde{x}_n(t) \) falls exactly on the circle \( C \)—then feature—at some special value (or values) \( t_s \) of the time variable \( t \)—a coincidence of the 2 functions \( \tilde{x}_1(t) \) and \( \tilde{x}_2(t) \), namely a "collision" of the 2 complex points \( \tilde{x}_1(t) \) and \( \tilde{x}_2(t) \), the loss of their individual identities, a phenomenon whose occurrence is associated with the singularity of that particular solution of the system (5). But of course these phenomena only happen for nongeneric values of the initial data \( \tilde{x}_1(0) \) and \( \tilde{x}_2(0) \).
Remark 4-1. Actually the periods of the generic solutions $\tilde{x}_n(t)$ of the system (5)—whenever the corresponding solutions of the system (1) are algebraic—are integer multiples of the period $\tilde{T} = 2T = 2\pi/|\omega|$, due to the prefactor $\exp(i\omega t) = \exp(2irnt/\tilde{T})$ (where $s = \omega/|\omega|$ is the sign of $\omega$) in the change of variables from $x_n(\tau)$ to $\tilde{x}_n(t)$, see (56a) and (56b); while $x_n(\tau) = x_n(\tau(t))$ is clearly itself periodic in $t$ with period $T$ (hence as well with period $\tilde{T} = 2T$) if the functions $x_n(\tau)$ are holomorphic in $\tau$, and instead with a positive integer multiple of $T$ if the functions $x_n(\tau)$ are not holomorphic but only feature a finite number of rational branch points as functions of the complex variable $\tau$. ■

The possibility reviewed in this Section 4 (and see [5] for analogous treatments in more general contexts), to associate to a homogeneous dynamical system such as (1) a corresponding isochronous system such as (5), underlines the interest to identify classes of homogeneous dynamical systems the general solutions of which—when considered as analytic functions of complex time—only feature a finite number of rational branch points; and in some cases to even obtain explicitly the solutions of their initial-values problems.

Remark 4-2. Let us also emphasize the obvious fact that the isochronous systems obtained in this manner—such as (5)—entail a doubling of the (real) dependent variables, since the presence of the imaginary parameter $i\omega$, in their right-hand side entails the need to treat the dependent variables $\tilde{x}_n(t)$ as complex numbers, featuring both a real and an imaginary part,

$$\tilde{x}_n(t) \equiv \tilde{x}_{Rn}(t) + i\tilde{x}_{In}(t), \quad n = 1, 2; \tag{57}$$

hence the 2 ODEs (5) define a dynamical system involving de facto the 4 real dependent variables $\tilde{x}_{R1}(t)$, $\tilde{x}_{R2}(t)$, $\tilde{x}_{I1}(t)$, $\tilde{x}_{I2}(t)$; and of course an analogous doubling by complexification may conveniently be made for its coefficients $c_{nt} \equiv c_{Rnt} + ic_{In t}$ whenever planning to use it in an applicative context, which generally entails the use of real numbers. ■

Let us complete this Section 4 by reporting (see for instance [5] and references therein) the following well-known

Remark 4-3. It is plain that, for any arbitrary assignment of the 8 parameters $c_{nt}$, the data $x_1(t) = x_2(t) = 0$ identify an equilibrium configuration of the system (1); and moreover that if the initial data $x_1(0)$ and $x_2(0)$ are sufficiently small—say, $|x_1(0)| < \varepsilon$ and $|x_2(0)| < \varepsilon$ with $\varepsilon$ a sufficiently small parameter—then the corresponding solution $x_n(t)$ of the system (1) features the property to be holomorphic as a function of its (complex) argument $t$ in the neighborhood of the initial data—namely, there exists a parameter $\delta$ (possibly quite small, but strictly positive, $\delta > 0$) such that $x_n(t)$ are both holomorphic functions of the complex variable $t$ inside the disk $|t| < \delta$. This clearly implies that the general system (5)—for any arbitrary assignment of its 8 parameters $c_{nt}$ and a sufficiently small value of the (real, nonvanishing) parameter $\omega$—features the isochronicity property to be—as function of the real variable $t$ ("time")—completely periodic with period $\tilde{T} = 2T$ (see (56b)) for the open set of its initial data $\tilde{x}_1(0) = x_1(0) < \varepsilon$ and $\tilde{x}_2(0) = x_2(0) < \varepsilon$ such that $2|\omega|\delta < 1$. ■

And let us also mention that the isochronous extension described in this Section 4 is of course also applicable to the special case treated in Subsection 3.5: the relevant treatment is left as an easy exercise for the interested reader.

5 Conclusions and outlook

In this last section we tersely review the main findings of this paper, and then mention possible analogous developments.

The initial-values problem of the system (1) can be solved—for arbitrary initial data $x_1(0)$ and $x_2(0)$—provided the 8 a priori arbitrary coefficients $c_{nt}$ satisfy 2 constraints: as discussed in detail above, see in particular the treatment in Section 3 and the determination of 2 of the 8 coefficients $c_{nt}$ in terms of the other 6 implied by these 2 constraints, as reported in Appendix C. The solution is provided by Proposition 1-1. It is given rather explicitly there in terms of the 7 parameters $a_1$, $a_2$, $b_1$, $b_2$, $\gamma_1$, $\gamma_2$, $\gamma_3$, themselves given in terms of the 8 coefficients $c_{nt}$, as follows: the 2 parameters $a_1$ and $a_2$ are explicitly given, for instance, by eqs. (24) with $\alpha$ given by any one of the 3 formulas (22); the 2 parameters $b_1$, $b_2$ can be determined as explained at the end of Subsection 3.3; and the 3 parameters $\gamma_1$, $\gamma_2$, $\gamma_3$ are then defined by any one of the 4 sets of explicit formulas displayed in the last part of Subsection 3.4. The remaining ambiguities in these determinations are then eventually eliminated when the solution of the initial-values problem is expressed—see Proposition 1-1—in terms of the 2 initial data $x_1(0)$ and $x_2(0)$; and the solutions $x_1(t)$ and $x_2(t)$ are then identified by continuity in $t$ starting from the initial data $x_1(0)$ and $x_2(0)$. 

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In an analogous manner the initial-values problem can be solved for the related isochronous system of ODEs \([\mathbf{5}]\); while the conditions implying its isochrony are detailed in Remark 1-4 complemented by the relevant formulas of Proposition 1-1 and by the treatment of this case in Section 4 (including in particular Remarks 4-2 and 4-3).

Finally, let us mention that natural extensions of the findings reported in this paper can be sought by considering dynamical systems analogous to, but more general than, \([\mathbf{1}]\): in particular systems involving more than 2 dependent variables, and systems featuring in their right-hand sides higher-degree—and possibly nonhomogeneous—polynomials; as well as dynamical systems involving higher-order ODEs; or evolutions in discrete—rather than continuous—time.

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7 Appendix A

In this Appendix A the formulas—implied by the first constraint on the 8 coefficients \(c_{n\ell}\) obtained in Subsection 3.2: see for instance eq. \([23]\) or any one of the 3 equivalent eqs. \([23]\)—are reported, which express each of these 8 coefficients in terms of the other 7; only 4 of these expressions are actually displayed below, since the other 4 can be obtained from those displayed via the transformations \([3]\). These formulas provide of course 8 constraints on the 8 coefficients \(c_{n\ell}\). These constraints are all equivalent among themselves.

\[
c_{11} = \left\{ -3 (c_{13})^2 c_{21} - (c_{12})^2 c_{23} + 3 c_{14} \left[ (c_{22})^2 - 3 c_{21} c_{23} \right] + c_{13} \left[ c_{22} (c_{12} - c_{23}) + 9 c_{21} c_{24} \right] + c_{12} \left[ 9 c_{14} c_{21} + (c_{23})^2 - 3 c_{22} c_{24} \right] \right\} / \left[ 3 (3 c_{14} c_{22} - c_{13} c_{23}) \right], 
\]

\[
c_{12} = 9 c_{14} c_{21} + c_{13} c_{22} + (c_{23})^2 - 3 c_{22} c_{24} \pm \left\{ \left[ 9 c_{14} c_{21} + c_{13} c_{22} + (c_{23})^2 - 3 c_{22} c_{24} \right]^2 - 4 c_{23} \left[ (c_{13})^2 c_{21} + 9 c_{11} c_{14} c_{22} - 3 c_{14} (c_{23})^2 + 9 c_{14} c_{21} c_{23} + c_{13} \left( -3 c_{11} c_{23} + c_{22} c_{23} - 9 c_{21} c_{24} \right) \right] \right\}^{1/2} / (2 c_{23}),
\]

\[
c_{13} = 9 c_{21} c_{24} + c_{12} c_{22} + 3 c_{11} c_{23} - c_{22} c_{23} \pm \left\{ \left[ (9 c_{21} c_{24} + c_{12} c_{22} + 3 c_{11} c_{23} - c_{22} c_{23})^2 + 12 c_{21} \left[ (c_{22})^2 c_{14} - 9 c_{11} c_{14} c_{22} + c_{12} (c_{23})^2 - 3 c_{12} c_{22} c_{24} + 9 c_{12} c_{14} c_{21} - (c_{12})^2 c_{23} - 9 c_{14} c_{21} c_{23} \right] \right] \right\}^{1/2} / (6 c_{21}),
\]

\[
c_{14} = -3 (c_{13})^2 c_{21} - c_{12} \left[ c_{12} c_{23} + 3 c_{22} c_{24} - (c_{23})^2 \right] + c_{13} \left[ c_{12} c_{22} + 3 c_{11} c_{23} - c_{22} c_{23} + 9 c_{21} c_{24} \right] \right\} / \left\{ 3 \left[ (c_{11} c_{22} - c_{12} c_{21}) + 3 c_{21} c_{23} - (c_{22})^2 \right] \right\},
\]
8 Appendix B

In this Appendix B the formulas—implied by the second constraint on the 8 coefficients \( c_{n\ell} \) obtained in Subsection 3.2: see for instance eq. (59)—are reported, which express each of these 8 coefficients in terms of the other 7; only 4 of these expressions are actually displayed below, since the other 4 can be obtained from those displayed via the transformations \( (3) \). These formulas provide of course 8 constraints on the 8 coefficients \( c_{n\ell} \). These constraints are all equivalent among themselves.

\[
c_{11} = [6c_{14}c_{22}c_{23} + (c_{12}c_{23} + 9c_{14}c_{21})(c_{13} - 3c_{24}) + (c_{12}c_{23} - 9c_{14}c_{21}) R_1] / (18c_{14}c_{23}) ,
\]

\[
c_{12} = \begin{cases} 3(c_{13})^2 \cdot c_{21} + 18c_{21} (c_{14}c_{21} - c_{13}c_{24}) - c_{13}c_{25} (3c_{11} - c_{22}) + 3c_{12}c_{23} (3c_{11} - c_{22}) + 9c_{21}c_{24} \\ + c_{13}c_{25} (3c_{11} - c_{22}) - 3c_{13}c_{21} + 9c_{21}c_{24} R_1] / [2(c_{23})^2] \end{cases} ,
\]

\[
c_{13} = [c_{12}c_{23} (3c_{11} - c_{22}) + 9c_{21} (3c_{11}c_{14} - c_{14}c_{22} + 2c_{12}c_{24}) + (c_{12}c_{23} - 9c_{14}c_{21}) R_2] / (6c_{12}c_{21}) ,
\]

\[
c_{14} = \begin{cases} -3c_{13}c_{24} (c_{13}c_{21} - 3c_{21}c_{24}) + c_{23} \left[ (3c_{11} - c_{22})^2 + 6c_{12}c_{21} \right] \\ + [3c_{21} (c_{13} - 3c_{24}) - c_{23} (3c_{11} - c_{22})] R_2] / [54(c_{21})^2] \end{cases} ,
\]

with

\[
R_1 = \sqrt{(c_{13} - 3c_{24})^2 + 12c_{14}c_{23}} , \quad R_2 = \sqrt{(3c_{11} - c_{22})^2 + 12c_{12}c_{21}} .
\]

9 Appendix C

In this Appendix C we report the expressions of pairs of the 8 coefficients \( c_{n\ell} \) in term of the other 6 coefficients, as implied by the 2 independent constraints on these 8 coefficients obtained in Subsection 3.2 (see in particular Remark 3.2-2): which are themselves sufficient for the solvability of the model \( (12) \) as described by Proposition 1-1. These expressions—being the solutions of nonlinear algebraic equations—are generally not unique; in some cases the only multiplicities are those implied by the sign ambiguities intrinsic in the square-root or cubic root operations; in other cases the multiplicities are less trivial and alternative solutions are then displayed. We only report results for generic values of the coefficients \( c_{n\ell} \); in particular we omit to report special solutions with vanishing coefficients. In all cases we were able to obtain an explicit solution via Mathematica; but in some cases the formulas so obtained were so complicated that it made no sense to report them here: these cases are identified below with the symbol \( \Box \) (which might also indicate more than one such solution).

Remark C-1. Some reader might wonder about the usefulness of formulas featuring this symbol "?!!". The point is that we are utilizing here formulas featuring a lot of algebraic symbols (such as \( c_{n\ell} \)); in applicative context, many of these symbols may be replaced by numbers, and in such cases Mathematica—or other equivalent computer subroutines—is likely to yield more useful outcomes. 

There are of course 28 \( (= 8 \cdot 7/2) \) different pairs of the 8 coefficients \( c_{n\ell} \), but only 16 formulas are displayed below, since the remaining 12 formulas are then implied by those displayed via the transformations \( (3) \).
Hereafter the square-roots $R_1$ and $R_2$ are defined as at the end of Appendix B. The other 2 square-root functions featured by the following formulas are defined as follows:

$$R_3 = \left\{ \left[ (c_{11})^2 (9c_{14}c_{21} + c_{13}c_{22} - 3c_{22}c_{24}) + (c_{13}c_{21})^2 \right] \right\}^{1/2} ,$$

$$R_4 = \sqrt{(c_{12} - c_{23})^2 + 4c_{13}c_{22}} .$$

9.1 The 4 pairs that are transformed into themselves by the transformations \[ \text{(3)} \]

$$c_{11} = \frac{2c_{13} (c_{22})^2 + (c_{12} - c_{23}) (c_{12}c_{22} + 3c_{13}c_{21}) + R_4 (c_{12}c_{22} - 3c_{13}c_{21})}{6c_{13}c_{22}} ,$$

$$c_{24} = - \left\{ 6 (c_{13})^3 c_{21}c_{22} + (c_{13})^2 \{ 3c_{21} (c_{23})^2 - c_{12} \left[ 2 (c_{22})^2 + 3c_{21}c_{23} \right] \} \right\} + (c_{12} - c_{23}) \left[ 3c_{12}c_{14} (c_{22})^2 + c_{13}c_{22} (c_{12}c_{23} - 9c_{14}c_{21}) \right] + (c_{13}c_{23} - 3c_{14}c_{22}) (c_{12}c_{22} - 3c_{13}c_{21}) / [6c_{13}c_{22} (c_{12}c_{22} - 3c_{13}c_{21})] .$$

$$c_{12} = \frac{3c_{14}(3c_{11} - c_{22})}{c_{13} - c_{24}} , \quad c_{23} = !?! ;$$

$$c_{12} = !?! , \quad c_{23} = !?! ;$$

$$c_{13} = !?! , \quad c_{22} = !?! ;$$

$$c_{14} = \frac{c_{12} (c_{13} - 3c_{24})}{3 (3c_{11} - c_{22})} , \quad c_{21} = \frac{c_{24} (3c_{11} - c_{22})}{3 (c_{13} - 3c_{24})} ;$$

9.2 The 12 pairs that are transformed into 12 other pairs by the transformations \[ \text{(3)} \]

$$c_{11} = \frac{c_{22}c_{23} + 3c_{21} (c_{13} - 3c_{24})}{3c_{23}} , \quad c_{12} = \frac{9c_{14}c_{21}}{c_{23}} ;$$

$$c_{21} = \left\{ -(c_{13})^4 c_{23} + c_{13} (3c_{11} - c_{22}) \left\{ c_{23} \left[ (c_{23})^2 + c_{22} (c_{13} - 3c_{24}) \right] - FR_4 \right\} + (c_{12})^3 \left[ 3 (c_{23})^2 + c_{22} (c_{13} - 3c_{24}) \right] + c_{12} \left\{ -3 (c_{13})^2 c_{22} (c_{11} - c_{22}) + (c_{23})^4 \right\} - c_{23}FR_4 - 3c_{22} (c_{23})^2 c_{24} + c_{13} \left[ 9c_{22}c_{24} (c_{11} - c_{22}) - (c_{23})^2 (6c_{11} - 5c_{22}) \right] \right\} + (c_{12})^2 \left\{ -3 (c_{13})^3 + FR_4 + c_{23} \left[ 3c_{11}c_{13} - c_{22} (5c_{13} - 6c_{24}) \right] \right\} / \left[ 6F (c_{13})^2 \right] ,$$

$$F = c_{22} (c_{13} - 3c_{24}) - c_{23} (c_{12} - c_{23}) .$$
\[
c_{11} = \left\{ \left( c_{13} \right)^2 c_{23} (c_{13} - 6c_{24}) + 27 (c_{14})^2 c_{21} (c_{13} - 3c_{24}) \\
+ 9c_{23} \left[ c_{13} (c_{24})^2 + c_{14} c_{23} (c_{13} - c_{24}) \right] \\
- \left\{ c_{13} c_{23} (c_{13} - 3c_{24}) + 3c_{14} [ (c_{23})^2 - 9c_{14} c_{21} ] \right\} R_{1} \right\} / \left( 54 (c_{14})^2 c_{23} \right) ,
\]
\[
c_{12} = \left\{ c_{13} (c_{13} c_{23} + 3c_{14} c_{22} - 3c_{23} c_{24}) + 3c_{14} \left[ 2 (c_{23})^2 - 3c_{22} c_{24} \right] \\
+ (3c_{14} c_{22} - c_{13} c_{23}) R_{1} \right\} / (6c_{14} c_{23}) ;
\] 

(65b)

\[
c_{11} = \left\{ 9 (c_{14})^2 c_{22} \left[ 3c_{12} c_{21} + (c_{22})^2 - 6c_{21} c_{23} \right] \\
+ 3 (c_{12})^2 c_{23} (c_{23} c_{24} - c_{14} c_{22}) + 3c_{14} c_{22} (c_{23})^2 (c_{12} + c_{23}) \\
- 9c_{14} (c_{22})^2 c_{24} (c_{12} + c_{23}) - 3c_{12} c_{23} c_{24} [(c_{23})^2 - 3c_{22} c_{24}] \\
+ C_{13} [3c_{14} c_{22} (c_{12} c_{22} + 9c_{21} c_{24}) - c_{12} c_{23} (c_{12} c_{23} + 6c_{22} c_{24}) \\
+ 9c_{14} c_{21} (c_{23})^2 ] + (C_{13})^2 c_{22} (c_{12} c_{23} - 9c_{14} c_{21}) \right\} / \\
/ \left\{ 9c_{14} \left[ 3c_{22} (c_{14} c_{22} - c_{23} c_{24}) + (c_{23})^2 (c_{23} - c_{12}) \right] \right\} ,
\]
\[
c_{13} \equiv C_{13} = 2c_{24} + \left[ c_{12} c_{23} + 2^{1/3} A_{3}/A_{4} - 2^{-1/3} A_{4} \right] / (3c_{22}) ,
\]
\[
A_{2} = 27c_{14} c_{22} \left[ -9c_{14} (c_{22})^3 + 2 (c_{12})^2 c_{22} c_{23} - 3c_{12} c_{22} (c_{23})^2 \\
+ 3c_{22} (c_{23})^3 + 3 (c_{23})^2 c_{24} (c_{12} + c_{23}) - 9c_{12} c_{22} (c_{23})^2 c_{24} (c_{12} + 3c_{23}) \\
+ 27 (c_{22})^2 c_{23} (c_{24})^2 (c_{12} - 6c_{23}) + 54 (c_{22} c_{24})^3 - 3 (c_{12} c_{23})^3 \right] ,
\]
\[
A_{3} = 9c_{14} (c_{22})^2 (3c_{23} - c_{12}) - c_{12} c_{23} (c_{12} c_{23} + 3c_{22} c_{24}) \\
- 9c_{22} c_{24} [(c_{23})^2 + c_{22} c_{24}] ,
\]
\[
A_{4} = \left( A_{2} + \sqrt{(A_{2})^2 + 4 (A_{3})^3} \right) 1/3 ;
\]
\[
c_{11} = \frac{3c_{13} c_{21} + c_{22} c_{23} - 9c_{21} c_{24}}{3c_{23}} , \quad c_{14} = \frac{c_{12} c_{23}}{9c_{21}} ;
\] 

(67a)

\[
c_{11} = \left\{ - (c_{12})^3 c_{22} c_{23} + c_{13} \left[ 2 (c_{22})^2 - 3c_{21} c_{23} \right] \left[ c_{13} c_{22} + (c_{23})^2 - 3c_{22} c_{24} \right] \\
+ (c_{12})^2 \left\{ c_{13} \left[ (c_{22})^2 - 3c_{21} c_{23} \right] + c_{22} \left[ 2 (c_{23})^2 - 3c_{22} c_{24} \right] \right\} \\
+ c_{12} \left\{ 3 (c_{13})^2 c_{21} c_{22} - c_{22} c_{23} \left[ (c_{23})^2 - 3c_{22} c_{24} \right] \\
- 3c_{13} \left[ (c_{23})^2 - 2c_{21} c_{23} \right] + 3c_{21} c_{22} c_{24} \right\} \\
+ (3c_{13} c_{21} - c_{12} c_{22}) \left[ c_{13} c_{22} - c_{12} c_{23} + (c_{23})^2 - 3c_{22} c_{24} \right] R_{1} \right\} / \\
\left\{ 6c_{13} c_{22} \left[ c_{13} c_{22} - c_{12} c_{23} + (c_{23})^2 - 3c_{22} c_{24} \right] \right\} , 
\]
\[
c_{14} = \left\{ c_{12} c_{23} (c_{12} - 2c_{23}) + 3c_{22} \left[ c_{12} c_{24} + c_{23} (c_{13} - c_{24}) \right] - c_{12} c_{13} c_{22} \\
( (c_{23})^3 + \left[ c_{13} c_{22} - c_{12} c_{23} + (c_{23})^2 - 3c_{22} c_{24} \right] R_{4} \right\} / \left( 6 (c_{22})^2 \right) ;
\] 

(67b)

\[
c_{11} = \frac{c_{12} c_{13} + 3c_{14} c_{22} - 3c_{12} c_{24}}{9c_{14}} , \quad c_{21} = \frac{c_{12} c_{23}}{9c_{14}} ;
\] 

(68)
\[ c_{11} = \left[ 54 (c_{14})^2 c_{23} \right]^{-1} \left\{ (c_{13})^3 c_{23} - 6 (c_{13})^2 c_{23} c_{24} + 9 c_{13} \left[ 3 (c_{14})^2 c_{21} + c_{14} (c_{23})^2 + c_{23} (c_{24})^2 \right] - 9 c_{14} \left[ (c_{23})^2 c_{24} + 9 c_{14} c_{21} c_{24} \right] + \left[ -(c_{13})^2 c_{23} + 3 c_{13} c_{23} c_{24} - 3 c_{14} (c_{23})^2 + 27 (c_{14})^2 c_{21} \right] R_1 \} , \]

\[ c_{22} = \left[ 18 (c_{14})^2 \right]^{-1} \left\{ (c_{13})^3 + 3 c_{14} \left[ -c_{12} c_{13} + 3 c_{13} c_{23} + 3 c_{12} c_{24} - 3 c_{23} c_{24} \right] + 3 c_{13} c_{24} (3 c_{24} - 2 c_{13}) + [c_{13} (c_{13} - 3 c_{24}) - 3 c_{14} (c_{12} - c_{23})] R_1 \} ; \quad (69) \]

\[ c_{11} = \frac{c_{12} c_{13} + 3 (c_{14} c_{22} - c_{12} c_{24})}{9 c_{14}} , \quad c_{23} = \frac{9 c_{14} c_{21}}{c_{12}} ; \quad (70a) \]
\[ c_{11} = \left\{ 27 \left( c_{13} c_{14} \right)^2 c_{22} \left[ c_{13} \left( 3c_{13} c_{21} - c_{12} c_{22} - 9c_{21} c_{24} \right) 
+ 3c_{12} \left( c_{22} c_{24} - 3c_{14} c_{21} \right) \right] + 81 \left( c_{14} \right)^3 \left( c_{22} \right)^2 \cdot 
\left[ \left( c_{12} \right)^2 c_{13} + 3c_{14} \left( 3c_{13} c_{21} - c_{12} c_{22} \right) \right] \right\}^{-1} \cdot 
\left\{ 9 \left( c_{12} c_{13} c_{14} \right)^2 \left[ c_{12} \left( c_{22} \right)^2 + 27c_{21} c_{22} c_{24} - 27c_{14} \left( c_{21} \right)^2 - 9c_{13} c_{21} c_{22} \right] 
+ 27 \left( c_{13} c_{14} \right)^2 c_{21} \left\{ 3c_{12} \left[ \left( c_{13} \right)^2 c_{21} + c_{14} \left( c_{22} \right)^2 - 3c_{13} c_{21} c_{24} \right] 
+ c_{22} \left[ c_{12} \left( c_{22} \right)^2 + 3 \left( 9c_{14} c_{21} - c_{13} c_{22} \right) c_{24} \right] \right\} \right\} 
+ 81 \left( c_{14} \right)^3 \left( c_{22} \right)^2 \left[ c_{14} c_{22} \left( 3c_{13} c_{21} - c_{12} c_{22} \right) + \left( c_{12} \right)^2 \left( c_{22} c_{24} - 3c_{14} c_{21} \right) \right] 
+ 9 \left( c_{12} c_{13} c_{22} \right)^2 c_{14} c_{24} \left( 2c_{13} - 3c_{24} \right) 
+ 9c_{12} \left( c_{13} \right)^3 c_{14} c_{22} \left[ 2 \left( c_{13} \right)^2 c_{21} - c_{14} \left( c_{22} \right)^2 - 12c_{13} c_{21} c_{24} + 18c_{21} \left( c_{24} \right)^2 \right] 
+ 27 \left( c_{12} \right)^3 c_{13} \left( c_{14} \right)^2 c_{22} \left( 3c_{14} c_{21} - c_{22} c_{24} \right) 
+ 27 \left( c_{12} c_{13} \left( c_{14} \right)^2 c_{22} \left[ 27 \left( c_{14} c_{21} \right)^2 - c_{22} c_{24} \left( 18c_{14} c_{21} - c_{13} c_{22} \right) \right] 
- 3 \left( c_{13} \right)^3 c_{14} c_{22} \left[ c_{12} \left( c_{13} \right)^2 c_{13} c_{22} + 162 \left( c_{14} \right)^2 \right] 
+ C_{23} \left\{ 3 \left( c_{12} c_{13} \right)^2 c_{14} \left[ 6 \left( c_{13} \right)^2 c_{21} + 18c_{12} c_{14} c_{21} + 2c_{12} c_{13} c_{22} - 3c_{14} \left( c_{22} \right)^2 \right] 
+ 6 \left( 3c_{13} c_{21} - c_{12} c_{22} \right) c_{24} \right\} + \left( c_{12} \right)^2 \left( c_{13} \right)^3 c_{22} \left[ 6c_{13} c_{24} - 9 \left( c_{24} \right)^2 - \left( c_{13} \right)^2 \right] 
+ 27 \left( c_{14} c_{22} \right)^2 \left[ c_{12} \right)^3 c_{14} + 9c_{12} \left( c_{14} \right)^2 c_{21} + \left( c_{12} \right)^2 \left( 3c_{13} c_{14} \right) \right] 
+ 27 \left( c_{13} c_{14} \right)^2 c_{21} \left( 18c_{12} c_{14} c_{21} + 5c_{12} c_{13} c_{22} + 9c_{13} c_{21} c_{24} \right) 
- 243c_{13} \left( c_{14} \right)^3 c_{21} c_{22} \left[ \left( c_{12} \right)^2 + 3c_{14} c_{21} \right] 
- 9c_{12} c_{13} \left( c_{14} \right)^2 c_{22} \left[ \left( c_{13} \right)^3 + 27c_{13} c_{21} c_{24} \right] \right\} 
\left( C_{23} \right)^2 \left\{ \left( c_{12} c_{13} \right)^3 c_{13} - 3c_{24} \right\} + 9 \left( c_{12} c_{13} \right)^2 c_{14} \left[ c_{22} \left( 2c_{24} - c_{13} \right) - 12c_{14} c_{21} \right] 
+ 81c_{13} \left( c_{14} \right)^3 c_{21} \left[ 2c_{12} c_{22} - 3c_{13} c_{21} \right] - 54c_{12} \left( c_{13} \right)^3 c_{14} c_{21} c_{24} 
+ 3 \left( c_{12} \right)^2 c_{14} \left[ 6c_{12} c_{13} c_{14} c_{22} - 9 \left( c_{14} c_{22} \right)^2 - \left( c_{12} c_{13} \right)^2 \right] \right\} 
+ 3 \left( C_{23} \right)^3 \left\{ \left( c_{12} c_{13} \right)^2 \left( 2c_{12} c_{14} + c_{13} c_{24} \right) + 3c_{12} c_{13} \left( c_{14} \right)^2 \left( 6c_{13} c_{21} - c_{12} c_{22} \right) \right\} 
- 3 \left( C_{23} \right)^4 \left( c_{12} c_{13} \right)^2 c_{14} \right\} 
\right\}, 
\right. 
\left. c_{23} \equiv C_{23} = 2c_{12}/3 + \left[ 3c_{13} c_{24} - 2^{1/3} \left( E_2 / E_3 \right) + 2^{-1/3} E_3 / \left( 9c_{14} \right) \right], \right. 
\\ 
E_1 = 27 \left\{ \left( c_{12} c_{14} \right)^2 \left[ 6 \left( c_{13} \right)^2 - 2c_{12} c_{14} - 3c_{13} c_{24} \right] + 9 \left( c_{13} c_{14} \right)^2 c_{22} \left( 3c_{24} - c_{13} \right) 
+ 3c_{12} c_{14} c_{24} \left[ \left( c_{13} \right)^3 - 9 \left( c_{14} \right)^2 c_{22} \right] + 27 \left( c_{14} \right)^3 c_{22} \left( 3c_{14} c_{22} - c_{12} c_{13} \right) 
+ c_{13} \left( c_{24} \right)^2 \left[ 3c_{12} c_{13} c_{14} - 45 \left( c_{14} \right)^2 c_{22} + 2 \left( c_{13} \right)^2 c_{24} \right] \right\}, 
\\ 
E_2 = 9 \left\{ \left( c_{14} \right)^2 \left[ 9c_{22} \left( c_{13} - c_{24} \right) - \left( c_{12} \right)^2 \right] - \left( c_{13} c_{24} \right)^2 - c_{12} c_{13} c_{14} \left( c_{13} + c_{14} \right) \right\}, 
\\ 
E_3 = \left( E_1 + \sqrt{\left( E_1 \right)^2 + 4 \left( E_2 \right)^3} \right)^{1/3} \cdot 
\\ 
\left. c_{12} = \frac{9c_{14} c_{21}}{c_{23}}, \quad c_{13} = \frac{3c_{11} c_{21} - c_{22} c_{23} + 9c_{21} c_{24}}{3c_{21}}; \right. 
\right.$
\[ c_{12} = \frac{9c_{11}c_{14} - 3c_{14}c_{22}}{c_{13} - 3c_{24}} \]  
\[ c_{21} = \frac{3c_{11}c_{23} - c_{22}c_{23}}{3(c_{13} - 3c_{24})} \]  
\[ B_2 = 27c_{13}(c_{21})^2 \left[ -81c_{13}(c_{21})^2 + 54(c_{11})^2c_{22} - 27c_{11}(c_{22})^2 \right] + 9(c_{22})^3 + 27c_{21}c_{23}(c_{11} + c_{22}) - 2 \left( (c_{11}c_{22})^3 - (c_{21}c_{23})^3 \right) \]
\[ -3c_{11}c_{21}c_{22}c_{23} \left[ c_{11}c_{22} + (c_{22})^2 - c_{21}c_{23} \right] - 6(c_{21}c_{22}c_{23})^2, \]
\[ B_3 = -81c_{13}(c_{21})^2(c_{11} - c_{22}) - 9c_{21}c_{22}c_{23}(c_{11} + c_{22}) - 9 \left[ (c_{11}c_{22})^3 + (c_{21}c_{23})^2 \right], \]  
\[ B_4 = \left( B_2 + \sqrt{(B_2)^2 + 4(B_3)^3} \right)^{1/3} \]

\[ c_{12} = \frac{(c_{13} - 3c_{24})(c_{13}c_{23} + 3c_{14}c_{22}) + 6c_{14}(c_{23})^2 + (3c_{14}c_{22} - c_{13}c_{23})R_1}{(6c_{14}c_{23})} \]  
\[ c_{21} = \frac{((c_{13})^3(c_{13} - 9c_{24}) + 9(c_{14})^22(c_{23})^2 + 9c_{11}c_{24})}{3c_{13}c_{21}^2 + 3(c_{24})^3 + 5c_{14}c_{23}c_{24}} + \frac{[-(c_{13})^3 + 6(c_{13})^2c_{24} - 9c_{13}(c_{24})^2 - 6c_{13}c_{14}c_{23} + 9c_{14}(3c_{11}c_{14} + c_{23}c_{24})R_1]}{162(c_{14})^3} \]  

\[ c_{12} = \frac{9c_{14}c_{21}}{c_{23}} \]  
\[ c_{22} = \frac{3(c_{11}c_{21} - c_{13}c_{21} + 3c_{21}c_{24})}{c_{23}} \]  
\[ c_{13} = \frac{3c_{11}c_{23} - c_{22}c_{23}}{3c_{21}} \]  
\[ c_{14} = \frac{c_{12}c_{23}}{9c_{21}} \]
\[ c_{13} = \left\{ c_{22} \left[ 9 (c_{11})^2 + 9c_{12}c_{21} + (c_{22})^2 - 3c_{21}c_{23} \right] \\
- 3c_{11} \left[ 3c_{12}c_{21} + 2 (c_{22})^2 - c_{21}c_{23} \right] \\
+ [3c_{21} (c_{12} - c_{23}) + c_{22} (3c_{11} + c_{22})] R_2 \} / \left[ 18 (c_{21})^2 \right], \]
\[ c_{14} = \left\{ 27 (c_{11})^2 \left[ c_{12}c_{21} - c_{11}c_{22} + (c_{22})^2 \right] + 9c_{12}c_{21} (2c_{12}c_{21} - 5c_{11}c_{22}) \\
+ (c_{22})^2 \left[ 12c_{12}c_{21} - 9c_{11}c_{22} + (c_{22})^2 \right] + (c_{21})^2 c_{24} (81c_{11} - 27c_{22}) \\
- \left[ 9c_{21} (c_{11}c_{12} + 3c_{21}c_{24}) - c_{22} \left[ 6c_{12}c_{21} + (c_{22})^2 \right] \\
+ 3c_{11}c_{22} (2c_{22} - 3c_{11}) \right] R_2 \} / \left[ 162 (c_{21})^3 \right] ; \] (75b)

\[ c_{13} = \frac{3(3c_{11}c_{14} + c_{12}c_{24} - c_{14}c_{22})}{c_{12}}, \quad c_{21} = \frac{c_{12}c_{23}}{9c_{14}} ; \] (76a)

\[ c_{13} = !?! , \quad c_{21} = !?! ; \] (76b)

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