THE FUNDAMENTAL SOLUTION OF AN ELLIPTIC EQUATION WITH SINGULAR DRIFT

VLADIMIR MAZ’YA AND ROBERT MCOWEN

Abstract. For $n \geq 3$, we study the existence and asymptotic properties of the fundamental solution for elliptic operators in nondivergence form, $\mathcal{L}(x, \partial_x) = a_{ij}(x) \partial_i \partial_j u + b_k(x) \partial_k u$, where the $a_{ij}$ have modulus of continuity $\omega(r)$ satisfying the square-Dini condition and the $b_k$ are allowed mild singularities of order $r^{-1} \omega(r)$. A singular integral is introduced that controls the existence of the fundamental solution. We give examples that show the singular drift $b_k \partial_k$ may act as a perturbation that does not dramatically change the fundamental solution of $\mathcal{L}^0 = a_{ij} \partial_i \partial_j$, or it could change an operator $\mathcal{L}^0$ that does not have a fundamental solution to one that does.

0. Introduction

Consider a 2nd-order operator in nondivergence form with a 1st-order term

$$\mathcal{L}(x, \partial_x)u = a_{ij}(x) \partial_i \partial_j u + b_k(x) \partial_k u, \quad x \in U,$n

where the coefficients $a_{ij} = a_{ji}$ and $b_k$ are measurable, real-valued functions in an open set $U \subset \mathbb{R}^n$, $n \geq 3$; here and throughout the paper we use the summation convention for repeated indices. We assume $\mathcal{L}(x, \partial_x)$ is pointwise elliptic, i.e. the matrix $a_{ij}(x)$ is positive definite for each $x \in U$. Our objective is to study the existence and asymptotics of its fundamental solution, i.e. a function $F(x, y)$ satisfying $F(x, \cdot) \in L^1_{\text{loc}}(U)$ for each $x \in U$ and

$$- \mathcal{L}(x, \partial_x)F(x, y) = \delta(x - y) \quad \text{for} \ x, y \in U,$n

in a distributional sense that needs to be made clear. In the classical case that the coefficients are sufficiently smooth (e.g. Hölder continuous), the existence of a fundamental solution and its asymptotic properties are well-known: cf. Miranda [17].

In [11] we studied this problem for $\mathcal{L}^0(x, \partial_x) = a_{ij}(x) \partial_i \partial_j$ when the modulus of continuity $\omega(r)$ for the $a_{ij}$ satisfies the square-Dini condition

$$\int_0^1 \frac{\omega^2(r)}{r} \, dr < \infty.$n

Notice that this assumption is weaker than $\lambda$-Hölder continuity since $[3]$ is satisfied by $\omega(r) = r^\lambda$ for $0 < \lambda < 1$; $[3]$ is also weaker than the Dini-condition, $\int_0^1 r^{-1} \omega(r) \, dr < \infty$. The hypotheses and conclusions of [11] are most easily stated if we fix $y = 0$ and let $U = B_\varepsilon = \{ x \in \mathbb{R}^n : |x| < \varepsilon \}$ for $\varepsilon$ sufficiently small. By an affine change of coordinates, we may arrange $a_{ij}(0) = \delta_{ij}$, so we may assume

$$\sup_{|x| = r} |a_{ij}(x) - \delta_{ij}| \leq \omega(r) \quad \text{for} \ 0 < r < \varepsilon.$$n

Date: August 31, 2022.

2020 Mathematics Subject Classification. 35A08 (Primary), 35B40, 35J15 (Secondary).

Key words and phrases. Elliptic equations, nondivergence form, square-Dini condition, singular drift, fundamental solution, asymptotics.
It was found in [11] that the existence of a solution of
\begin{equation}
- \mathcal{L}^\alpha(x, \partial_x)F(x) = \delta(x) \quad \text{in } B_\varepsilon,
\end{equation}
for \( \varepsilon \) sufficiently small depends on the behavior of the integral
\begin{equation}
I^\alpha(r) = \frac{1}{|S^{n-1}|} \int_{r<|z|<\varepsilon} \left( \text{tr}(A_z) - \nu \frac{\langle A_z z, z \rangle}{|z|^2} \right) \frac{dz}{|z|^n}
\end{equation}
as \( r \to 0 \); here \( A_x \) denotes the matrix \( (a_{ij}) \) and \( \langle A_z z, z \rangle = a_{ij}(z)z_iz_j \). If \( I^\alpha(0) = \lim_{r \to 0} I^\alpha(r) \) exists and is finite, then we showed there is a solution of (5). However, if \( I^\alpha(r) \to -\infty \) as \( r \to 0 \), then we found a solution \( F(x) \) of
\begin{equation}
\mathcal{L}^\alpha(x, \partial_x)F(x) = 0 \quad \text{in } B_\varepsilon,
\end{equation}
which has a singularity at \( x = 0 \), but \( F(x) = o(|x|^{2-n}) \) as \( |x| \to 0 \) and \( F(x) \) is not a solution of (5); this violates the “extended maximum principle” of [5].

In the present paper, we generalize the results of [11] by including the 1st-order term \( b_k \partial_k \), which is called a drift term: cf. [2], [11]. If the \( b_k \) are bounded in \( U \), they will not effect the existence of the fundamental solution, so we will allow the drift to be singular at \( x = 0 \), but satisfy the condition:
\begin{equation}
\sup_{|x|=r} |b_k(x)| \leq c \omega(r) \quad \text{for } 0 < r < \varepsilon.
\end{equation}
(Here and throughout this paper, \( c \) denotes a generic constant.) We again construct an unbounded solution \( Z(x) \) of
\begin{equation}
\mathcal{L}(x, \partial_x)Z(x) = 0 \quad \text{for } x \in B_\varepsilon \setminus \{0\},
\end{equation}
and then check to see whether \( Z \) has the appropriate singular behavior as \( |x| \to 0 \) so that, in a distributional sense,
\begin{equation}
- \mathcal{L}(x, \partial_x)Z(x) = C_0 \delta(x) \quad \text{for some constant } C_0.
\end{equation}
If so, then setting \( F(x) := C_0^{-1} Z(x) \) defines a solution of
\begin{equation}
- \mathcal{L}(x, \partial_x)F(x) = \delta(x) \quad \text{for } x \in B_\varepsilon.
\end{equation}

To determine whether this can be done, the quantity (6) is generalized to
\begin{equation}
I(r) = \frac{1}{|S^{n-1}|} \int_{r<|z|<\varepsilon} \left( \text{tr}(A_z) - \nu \frac{\langle A_z z, z \rangle}{|z|^2} + \langle B_z, z \rangle \right) \frac{dz}{|z|^n},
\end{equation}
where \( B_z \) denotes the vector with components \( b_k(z) \) and \( \langle B_z, z \rangle = b_k(z)z_k \). In general, \( I(r) \) may have a singularity at \( r = 0 \), but it is weaker than logarithmic: for any \( \lambda > 0 \) there exists \( c_\lambda > 0 \) such that
\begin{equation}
|I(r)| \leq \lambda |\log r| + c_\lambda \quad \text{for } 0 < r < \varepsilon.
\end{equation}

Before stating the main results of this paper, we need some notation and an additional assumption. For an open set \( U \), \( 1 < p < \infty \), and integer \( m = 0, 1, 2 \), let \( W^{m,p}_{\text{loc}}(U) \) denote the Sobolev space of functions whose derivatives up to order \( m \) are locally \( L^p \)-integrable in \( U \). The \( L^p \)-mean of \( w \in L^p_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) \) on the annulus \( A_r := \{ x : r < |x| < 2r \} \) is defined by
\begin{equation}
M_p(w, r) := \left( \int_{A_r} |w(x)|^p \, dx \right)^{1/p}.
\end{equation}
Similarly, we define
\begin{equation}
M_{1,p}(w, r) = r M_p(Dw, r) + M_p(w, r) \quad \text{for } w \in W^{1,p}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}),
\end{equation}
For $p = \infty$ we can analogously define $M_\infty(w, r)$, $M_{1, \infty}(w, r)$ and $M_{2, \infty}(w, r)$. The $L^p$-mean over annuli centered at $y \neq 0$ will be denoted by $M_p(w, r; y)$ and similarly for $M_{1,p}(w, r; y)$ and $M_{2,p}(w, r; y)$. We also define

$$M_{2,p}(w, r) = r^2 M_p(D^2 w, r) + M_{1,p}(w, r) \quad \text{for } w \in W^{2,p}_\text{loc}(\mathbb{R}^n \setminus \{0\}).$$

Under the assumptions of Theorem 1 and Theorem 2.

\[ (14c) \]

where $\omega_m$ over annuli centered at $y = 0$ will be denoted by $M_p(w, r; y)$ and similarly for $M_{1,p}(w, r; y)$ and $M_{2,p}(w, r; y)$. We also define

$$\sigma(r) = \int_0^r \omega^2(\rho) \frac{d\rho}{\rho},$$

which satisfies $\sigma(r) \rightarrow 0$ as $r \rightarrow 0$ because of (3). Finally, as a modulus of continuity we want $\omega(r)$ to be nondecreasing for $0 < r < 1$, but we also assume for some $\kappa \in (0, 1)$ that

$$\omega(r) r^{-1+\kappa} \quad \text{is nonincreasing on } 0 < r < 1.$$

This is natural since we are interested in moduli of continuity that vanish slower than $r \rightarrow 0$.

Our first result generalizes Theorem 1 in [11], which only applied to $L^\infty(x, \partial_x)$. Theorem 1. For $n \geq 3$ and $p \in (1, \infty)$, assume $a_{ij} = a_{ji}$ satisfy (1) while the $b_k$ satisfy (8). Then for $\varepsilon > 0$ sufficiently small, there is a solution of (3) in the form

\[ (17a) \]

where $I(r)$ is given by (12), $\zeta(r)$ satisfies

$$M_{2,p}(\zeta, r) \leq c \max(\omega(r), \sigma(r)),$$

and $v(x)$ satisfies

$$M_{2,p}(v, r) \leq cr^{-2-n}e^{I(r)}\omega(r).$$

Moreover, for any $u \in W^{2,p}_\text{loc}((B_\varepsilon \setminus \{0\}))$ that is a strong solution of $\mathcal{L}(x, \partial_x)u = 0$ in $B_\varepsilon \setminus \{0\}$ subject to the growth condition

$$M_{2,p}(u, r) \leq cr^{1-n+\varepsilon_0} \text{ where } \varepsilon_0 > 0,$$

there exist constants $c_0, c_1$ (depending on $u$) such that

$$u(x) = c_0 Z(x) + c_1 + w(x),$$

where $M_{2,p}(w, r) \leq cr^{1-\varepsilon_1}$ for any $\varepsilon_1 > 0$.

We will prove this result in Section II but let us here observe that $I(r)$ satisfies $|I'(r)| \leq cr^{-1} \omega(r)$, so we may integrate by parts and take $p > n$ to conclude

$$Z(x) = \frac{|x|^{2-n}e^{I(|x|)}}{n-2} (1 + \xi(x)),$$

where $M_{1,\infty}(\xi, r) \leq c \max(\omega(r), \sigma(r))$. This shows that $I(r)$ controls how closely $Z$ adheres to the fundamental solution of the Laplacian.

Our second result generalizes Theorem 2 in [11] and shows that the existence and finiteness of the limit $I(0) = \lim_{r \rightarrow 0} I(r)$ determines whether $Z$ solves (10) for some constant $C_0$.

Theorem 2. Under the assumptions of Theorem 1 and $\varepsilon > 0$ sufficiently small:

(i) If $I(0) = \lim_{r \rightarrow 0} I(r)$ exists and is finite, then we can solve (10) in $B_\varepsilon$ with $C_0 = |S^{n-1}| e^{I(0)}$.

(ii) If $I(r) \rightarrow -\infty$ as $r \rightarrow 0$, then solving (10) in $B_\varepsilon$ yields $C_0 = 0$, and so $Z$ solves $-\mathcal{L}(x, \partial_x)Z(x) = 0$ in $B_\varepsilon$, despite its singularity at $x = 0$. 
We will prove this result in Section 2 but let us observe that in case (i) we have found a solution of (11) of the form

$$F(x) = \frac{|x|^2 - n}{(n - 2)|S^{n-1}|} (1 + o(|x|)) \quad \text{as } |x| \to 0,$$

so the comparison with the fundamental solution of the Laplacian is even more explicit. On the other hand, if $I(r) \to +\infty$ as $r \to 0$, then the singular solution $Z(x)$ grows more rapidly as $|x| \to 0$ than the fundamental solution for the Laplacian, and we are not able to solve (10).

Let us consider a simple example to illustrate the effect of the drift term on the fundamental solution.

Example 1. Let $a_{ij} = \delta_{ij}$ so that our operator (11) becomes

$$\mathcal{L}(x, \partial_x) = \Delta + b_k(x) \partial_x,$$

and the quantity $I(r)$ in (12) reduces to

$$I(r) = \frac{1}{|S^{n-1}|} \int_{r < |x| < \varepsilon} \frac{\langle B_z, z \rangle}{|z|^n} \, dz.$$

If $\omega(r)$ satisfies the Dini condition, then from condition (3) we easily conclude that $I(0)$ exists and is finite; but this finite limit may exist without the Dini condition: e.g. we could take $b_k(x) = x_k \sin(|x|^{-1}) \omega(|x|)/|x|^2$. In any case, provided $I(0)$ is finite, we have a solution $F(x)$ of (10) that is comparable to the fundamental solution of the Laplacian.

In [2], Cranston and Zhao consider operators of the form $\mathcal{L} = \frac{1}{2} \Delta + b \cdot \nabla$ with vector field $b$. Assuming $U$ is a bounded Lipschitz domain and $b(x)$ satisfies the conditions

$$\limsup_{r \to 0} \sup_{x \in U} \int_{|x - y| < r} \frac{|b(y)|^2}{|x - y|^{n-2}} \, dy = 0 \quad \text{and} \quad \limsup_{r \to 0} \sup_{x \in U} \int_{|x - y| < r} \frac{|b(y)|}{|x - y|^{n-1}} \, dy,$$

they conclude that the Green’s function $G(x, y)$ for $\mathcal{L}$ in $U$ exists and is comparable to the Green’s function $G_0(x, y)$ for $\mathcal{L}_0 = \frac{1}{2} \Delta$, i.e.

$$c^{-1} G_0(x, y) \leq G(x, y) \leq c G_0(x, y) \quad \text{for } x, y \in U, \quad x \neq y.$$

If $U$ contains the origin, then setting $F(x) := G(x, 0)$ defines a solution of (11), so let us compare our hypotheses and conclusions with those of [2]. If $b$ has a singularity at $x = 0$ of the form $|b(x)| = |x|^{-1} \omega(|x|)$, the conditions in (24a) require

$$\int_0^r \frac{\omega^2(\rho)}{\rho} \, d\rho < \infty \quad \text{and} \quad \int_0^r \frac{\omega(\rho)}{\rho} \, d\rho < \infty.$$

The first of these is the square-Dini condition (3) that we have required, while the second is the Dini condition that we have not required: to conclude that the fundamental solution exists at $x = 0$, we only require the function $I(r)$ given in (23) to have a finite limit $I(0) = \lim_{r \to 0} I(r)$. Moreover, (20) is a sharper estimate than (24b).

In Example 1, the drift term plays the role of a perturbation which, if not too large, does not affect the existence of the fundamental solution. We now consider an example where singular drift can convert an operator $\mathcal{L}^o$ that does not have a fundamental solution to one for which a fundamental solution exists!

Example 2. Consider $\mathcal{L}^o = a_{ij} \partial_i \partial_j$ with coefficients

$$a_{ij}(x) = \delta_{ij} + g(r) \frac{x_i x_j}{|x|^2},$$

where
where \(|g(r)| \leq \omega(r)|\) with \(\omega\) satisfying (3). Coefficients of the form \(\frac{1}{r}\) were first considered by Gilbarg & Serrin \([5]\), and have proven useful in both nondivergence and divergence form equations (cf. \([11]\), \([12]\), \([13]\)). We can use (6) to calculate

\[(26)\]
\[I^o(r) = (1 - n) \int_r^1 g(\rho) \frac{d\rho}{\rho}.\]

Note that \(I^o(0) = \lim_{r \to 0} I^o(r)\) exists and is finite when \(\omega\) satisfies the Dini condition, but this finite limit may exist without Dini continuity: e.g. \(g(r) = \sin(r^{-1})\omega(r)\).

Now let us assume \(g(r) > 0\) and \(\omega\) does not satisfy the Dini condition. Then \(I^o(r) \to -\infty\) as \(r \to 0\), so the fundamental solution for \(\mathcal{L}^o\) does not exist at \(y = 0\). However, if we add the first-order coefficients

\[(27)\]
\[b_k(x) = (n - 1) \frac{x_k}{|x|^2} [g(|x|) + g^2(|x|)]\]

to obtain \(\mathcal{L}\) as in (1), then we can use (12) to calculate

\[(28)\]
\[I(r) = (n - 1) \int_r^1 \frac{g^2(\rho)}{\rho} d\rho.\]

Since \(g^2(r) \leq \omega^2(r)\) and \(\omega\) satisfies (3), we see that \(I(r)\) has a finite limit as \(r \to 0\), and so we can solve (10) to conclude the fundamental solution exists at \(y = 0\).

As previously stated, we will prove Theorem 1 in Section 1 and Theorem 2 in Section 2, but in those sections we will also state and prove Corollaries 1 and 2 respectively, which show how the formulas for the singular solution \(Z\) change when \(y \neq 0\) and \(a_{ij}(y) \neq \delta_{ij}\). To formulate these results, we need to generalize (11): for a given \(y \in U\), choose \(\varepsilon\) so that \(0 < \varepsilon < \text{dist}(y, \partial U)\) and assume

\[(29)\]
\[\sup_{|x - y| = r} \|A_x - A_y\| \leq \omega(r)\quad \text{for } 0 < r < \varepsilon.\]

We continue to assume the \(b_k\) are only singular at \(x = 0\) and satisfy (8). We also generalize the function \(I(r)\) in (12) as

\[(30a)\]
\[I_y(r) = \frac{1}{|S^{n-1}|} \int_{r < |z - y| < \varepsilon} H(z, y) \frac{dz}{|z - y|^n},\]

where the integrand \(H(z, y)\) is

\[(30b)\]
\[\text{tr}(A_z A_y^{-1}) - n \frac{\langle A_z A_y^{-1/2} (z - y), A_y^{-1/2} (z - y) \rangle}{|z - y|^2} + \langle B_z A_y^{-1/2}, (z - y) \rangle,\]

with \(A_y^{-1}\) denoting the inverse matrix. If \(y = 0\) and \(A_0 = I\), then (30) coincides with (12); if \(y \neq 0\) and we also stipulate \(0 < \varepsilon < |y|\), then \(B_z\) is bounded on \(|z - y| < \varepsilon\), so the last term in (30b) will not play a role in whether the limit \(I_y(0)\) exists and is finite. Now we state the main result of this paper.

**Theorem 3.** Suppose \(\mathcal{L}(x, \partial_x)\) as in (1) is an elliptic operator in a bounded open set \(U \subset \mathbb{R}^n\), \(n \geq 3\), where the coefficients \(a_{ij} = a_{ji}\) are continuous functions with modulus of continuity \(\omega(r)\) satisfying (3). Suppose \(U\) contains the origin and the \(b_i\) satisfy (8) but otherwise are bounded in \(U\). For each \(y \in U\) assume that the limit \(I_y(0) = \lim_{r \to 0} I_y(r)\) exists and is finite. Then \(\mathcal{L}(x, \partial_x)\) has a fundamental solution \(F(x, y)\) in \(U\) and it has the asymptotic behavior

\[(31)\]
\[F(x, y) = \frac{\langle A_y^{-1}(x - y), (x - y)^2 \rangle}{(n - 2)|S^{n-1}| \sqrt{\det A_y}} (1 + o(1))\quad \text{as } x \to y.\]
This is proved in Section 3. Note that the leading asymptotic in (31) is familiar as the “Levi function” that occurs in the classical case (cf. [17]). As a consequence of Theorem 3, we see that the singular drift may affect the existence of the fundamental solution but does not play a role in its asymptotic behavior as \( x \to y \).

Many of the arguments in this paper also appear in [11], but we have repeated them here for the convenience of the reader. From a more general perspective, the asymptotic analysis used here is related to that developed in [9].

Let us compare our results (in [11] and this paper) with recent work estimating the singularity of the Green’s function for \( L_0(x, \partial_x) = a_{ij}(x) \partial_i \partial_j \) when the coefficients \( a_{ij} \) satisfy the Dini mean oscillation (DMO) condition: cf. [7], [8], [4]. If \( U \) is a bounded \( C^{1,1} \)-domain and the \( a_{ij} = a_{ji} \) are continuous functions on \( \overline{U} \), then it is well-known that the Green’s function \( G(x, y) \) exists. If the \( a_{ij} \) satisfy the DMO condition in \( U \), then [4] shows that, for any \( x_0 \in U \), the following limit holds:

\[
(32) \lim_{x \to x_0} \frac{|x - x_0|^{2-n}}{2} |G(x, x_0) - G_{x_0}(x, x_0)| = 0,
\]

where \( G_{x_0} \) denotes that Green’s function for the constant coefficient operator \( L_0(x, \partial_x) \) in \( U \). Since the Green’s function is a particular fundamental solution and since (32) is equivalent to (31), it is natural to compare the hypotheses of the two results. As shown in [3], there are coefficients that are DMO but do not satisfy our square-Dini condition (3). On the other hand, there are coefficients of the form (25) which satisfy (3) but are not DMO. In fact, as shown in [13], with

\[
(33) g(r) = \sin(|\log r|) |\log r|^{-\gamma},
\]

the coefficients in (25) are DMO only for \( \gamma > 1 \), but they satisfy (3) for \( \gamma > 1/2 \) and the limit \( I(0) \) exists and is finite for all \( \gamma > 0 \). This example with \( 1/2 < \delta \leq 1 \) shows that the results of [4] do not cover our results for \( L_0(x, \partial_x) \), let alone the operator (1) with singular drift.

1. Construction of the singular solution \( Z \)

In this section, we will not only prove Theorem 1, but we will state and prove Corollary 1, which shows how the formulas change when we no longer assume \( y = 0 \) and \( a_{ij}(y) = \delta_{ij} \). Instead of constructing \( Z(x) \) in a small ball, we replace the condition that \( \omega(\rho) \) satisfies (3) with

\[
(34) \sigma(1) = \int_0^1 \omega^2(\rho) \frac{d\rho}{\rho} < \mu^2
\]

where \( \mu > 0 \) is sufficiently small, and then show existence in the unit ball \( B_1 \). In fact, with \( \kappa \in (0, 1) \) as in (16), this also implies

\[
(35) \omega(r) < c_\kappa \mu \quad \text{for} \quad 0 < r \leq 1,
\]

since

\[
\mu^2 > \int_0^r \omega^2(\rho) \frac{d\rho}{\rho} \geq \omega^2(r) r^{-2+2\kappa} \int_0^r \rho^{1-2\kappa} d\rho = \frac{\omega^2(r)}{2(1 - \kappa)}.
\]

Moreover, it will be useful to consider \( L \) on all of \( \mathbb{R}^n \), so we assume

\[
(36) a_{ij}(x) = \delta_{ij} \quad \text{and} \quad b_k(x) = 0 \quad \text{for} \quad |x| > 1,
\]

and construct a solution of \( LZ = 0 \) on \( \mathbb{R}^n \setminus \{0\} \).

Proof of Theorem 1. As in [11], we use spherical means: for a function \( f(x) \) we denote its mean value over the sphere \( |x| = r \) by \( \overline{f}(r) \):

\[
\overline{f}(r) = \int_{S^{n-1}} f(r\theta) \, d\theta,
\]
Z(x) = h(|x|) + v(x), where h(r) := Z(r),
so that
\( \nabla(r) = 0. \)
If we take the spherical mean of the equation \( \mathcal{L}(h + v) = 0, \) we obtain
\[
(38a) \quad \alpha(r) h'' + \left[ \frac{\alpha_n(r) - \alpha(r) + \beta(r)}{r} \right] h' + \overline{\mathcal{L}v}(r) = 0,
\]
where
\[
(38b) \quad \alpha(r) := \int_{S^{n-1}} a_{ij}(r \theta) \partial_i \partial_j \theta \, d\theta, \quad \alpha_n(r) := \int_{S^{n-1}} a_{ii}(r \theta) \, d\theta,
\]
and
\[
(38c) \quad \beta(r) := r \int_{S^{n-1}} b_k(r \theta) \partial_k \, d\theta.
\]
In terms of these, instead of (12) we can write
\[
(39) \quad I(r) = \int_r^1 (\alpha_n(s) - n \alpha(s) + \beta(s)) \frac{ds}{s}.
\]
Using (11), we see that
\[
(40a) \quad \alpha(r) = 1 + O(\omega(r)) \quad \text{and} \quad \alpha_n(r) = n + O(\omega(r)) \quad \text{as } r \to 0,
\]
and using (8) we see that
\[
(40b) \quad |\beta(r)| \leq \omega(r).
\]
Hence the integrand in (39) is bounded by \( c \omega(s) s^{-1}. \) Since \( \omega(r) \) need not satisfy the Dini condition, we do not know whether \( I(r) \) has a finite limit as \( r \to 0; \) this is, of course, the significance of Theorem 2.

Turning to the \( \overline{\mathcal{L}v}(r) \) term, if we write
\[
a_{ij} \partial_i \partial_j v = \bar{a}_{ij} \partial_i \partial_j v + \Delta v\]
where \( \bar{a}_{ij} := (a_{ij} - \delta_{ij}) \) and use \( \Delta v = \Delta \nabla = 0, \) we see that
\[
\overline{\mathcal{L}v}(r) = \frac{a_{ij} \partial_i \partial_j v(r) + b_k \partial_k v(r)}.\]
Notice that \( |\overline{\mathcal{L}v}(r)| \leq c \omega(r) \left( |D^2v| + r^{-1} |Dv| \right) \) for \( 0 < r < 1 \) and \( \overline{\mathcal{L}v}(r) = 0 \) for \( r > 1. \) Since \( \alpha(r) \to 1 \) as \( r \to 0 \) and \( \alpha(r) = 1 \) for \( r > 1, \) we may assume \( \alpha(r) \geq \delta > 0 \) for \( 0 < r < \infty. \) Hence we may divide (38a) by \( \alpha(r) \) and replace \( h' \) by \( g \) to obtain
\[
(41a) \quad g' + \left[ \frac{n - 1 + R(r)}{r} \right] g = B[v](r),
\]
where
\[
(41b) \quad R(r) = \frac{\alpha_n(r) + \beta(r)}{\alpha(r)} - n
\]
satisfies \( |R(r)| \leq c \omega(r) \) as \( r \to 0 \) and \( R(r) = 0 \) for \( r > 1, \) and \( B[v](r) \) satisfies
\[
(41c) \quad |B[v](r)| \leq c \omega(r) \left( |D^2v(r)| + r^{-1} |Dv(r)| \right) \quad \text{for } 0 < r < 1\]
and \( B[v](r) = 0 \) for \( r > 1. \) Moreover, the monotonicity of \( \omega(r) \) and (10) imply
\[
(41d) \quad \sup_{r < \rho < 2r} \omega(\rho) \leq c \omega(r),
\]
where \( S^{n-1} \) is the unit sphere, the slashed integral denotes mean value, \( r = |x|, \theta = x/|x| \in S^{n-1}, \) and \( d\theta \) denotes standard surface measure on \( S^{n-1}. \) Let us write
\[
(37a) \quad Z(x) = h(|x|) + v(x), \quad \text{where } h(r) := Z(r),
\]
so that
\[
(37b) \quad \nabla(r) = 0.
\]
so we consequently obtain

\[(41c)\quad r^2 M_p(B[v],r) \leq c \omega(r) M_{2p}(v,r).\]

Solving (41a) involves the integrating factor $r^{n-1}E_-(r)$, where we introduce

\[(42)\quad E_\pm(r) = \exp\left[ \pm \int_r^\infty R(s) \frac{ds}{s} \right] = \exp\left[ \pm \int_r^1 R(s) \frac{ds}{s} \right] = \frac{1}{E_\mp(r)}.\]

Notice that $E_-(r)E_+(\rho) = \exp(\int_\rho^r R(s) s^{-1} ds)$ so by (35) we have

\[(43a)\quad \left( \frac{\rho}{r} \right)^{c_\omega \mu} \leq \exp\left( \pm \int_\rho^r R(s) \frac{ds}{s} \right) \leq \left( \frac{r}{\rho} \right)^{c_\omega \mu} \quad \text{for } 0 < \rho < r \leq 1.\]

In particular, we have

\[(43b)\quad c_1 E_\pm(r) \leq E_\pm(\rho) \leq c_2 E_\pm(r) \quad \text{for } r < \rho < 2r,\]

and for any $f \in L^p_\text{loc}(\mathbb{R}^n \setminus \{0\})$ and $\nu \in \mathbb{R}$ we have

\[(43c)\quad M_p(|x|^\nu E_\pm(|x|) f(x),r) \leq c r^{\nu} E_\pm(r) M_p(f,r).\]

While $E_+(r)$ is used to solve (41a), we observe that it is equivalent to $e^{I(r)}$. In fact,

\[(44a)\quad E_+(r) = A e^{I(r)} (1 + \tau(r))\]

where $|R(s)(1 - \alpha(s))| \leq c \omega^2(s)$ implies

\[(44b)\quad A = \exp\left[ \int_0^1 R(s)(1 - \alpha(s)) s^{-1} ds \right] \quad \text{is finite and positive,}\]

and

\[(44c)\quad \tau(r) = \exp\left[ - \int_0^r R(s)[1 - \alpha(s)] \frac{ds}{s} \right] - 1 \quad \text{satisfies } |\tau(r)| \leq c \sigma(r).\]

Hence, for some constants $c_1, c_2$ we have

\[(45)\quad c_1 E_+(r) \leq e^{I(r)} \leq c_2 E_+(r).\]

We consider (41a) as an ODE that depends on $v \in Y$, where $Y$ is the Banach space of functions $v \in W^{2p}_\text{loc}(\mathbb{R}^n \setminus \{0\})$ for which

\[(46)\quad \|v\|_Y := \sup_{0<\rho<1} \frac{M_{2p}(v,r) r^{n-2}}{\omega(r) e^{I(r)}} + \sup_{r>1} \frac{M_{2p}(v,r) r^{n-1}}{\mu} < \infty.\]

If we let $\phi(r) = r^{n-1} E_-(r) g(r)$, then (41a) implies $\phi'(r) = r^{n-1} E_-(r) B[v](r)$. This can be integrated to find

\[(47)\quad \phi(r) = \phi(0) + \int_0^r \rho^{n-1} E_-(\rho) B[v](\rho) d\rho,\]

where $\phi(0)$ is an arbitrary constant. Of course, for (47) to be valid we need to know that $r^{n-1} E_-(r) B[v](r)$ is integrable at $r = 0$. In fact, we will show below that for $v \in Y$ we have

\[(48)\quad \int_0^r \rho^{n-1} E_-(\rho) |B[v](\rho)| d\rho \leq c \mu^2 \quad \text{for all } r > 0,\]

where $c$ may be taken uniformly for all $v \in Y$ with $\|v\|_Y \leq 1$. Hence (47) is valid and we conclude

\[(49a)\quad g(r) = h'(r) = r^{1-n} E_+(r) \left[ \phi(0) + \int_0^r \rho^{n-1} E_-(\rho) B[v](\rho) d\rho \right].\]
and
\[ (49b) \quad h''(r) = \frac{1 - n - R(r)}{r^n} E_+(r) \left[ \phi(0) + \int_0^r \rho^{n-1} E_-(\rho) B[v](\rho) \, d\rho \right] + B[v](r). \]

To verify (49), we observe that \( v \in Y \) implies \( M_{2,p}(v, r) \leq c \omega(r) r^{2-n} E_+(r) \), so we can use Proposition 1 in Appendix A.

(52)\[ v \]

Then we obtain
\[ \int_0^r \rho^{n-1} E_-(\rho) |B[v](\rho)| \, d\rho \leq c \int_0^r \rho^{n-1} \left( |D^2 v(\rho)| + \rho^{-1} |Dv(\rho)| \right) \, d\rho \]
\[ \leq c E_-(r) \omega(r) \int_{r < |x| < 2r} (|D^2 v(x)| + |x|^{-1} |Dv(x)|) \, dx \]
\[ \leq c E_-(r) \omega(r) r^{n-2} M_{2,p}(v, r) \leq c \omega^2(r). \]

Now if we write
\[ \int_0^r \rho^{n-1} E_-(\rho) |B[v](\rho)| \, d\rho = \sum_{j=0}^{r} \int_{j/2j+1}^{r/2j+1} \rho^{n-1} E_-(\rho) |B[v](\rho)| \, d\rho, \]
then we obtain
\[ \int_0^r \rho^{n-1} E_-(\rho) |B[v](\rho)| \, d\rho \leq c \sum_{j=0}^{r} \omega^2 \left( \frac{r}{j+1} \right) \]
\[ \leq c \int_0^r \omega^2(\rho) \frac{d\rho}{\rho} = c \sigma(r) < c \mu^2. \]

This confirms (49) with \( c \) uniform for \( \|v\|_Y \leq 1 \).

We also have a PDE for \( v \) that depends upon \( h \). From \( \mathcal{L}Z - ZZ = 0 \) we find:
\[ -\Delta v = \bar{a}_{ij} \partial_i \partial_j h - \bar{a}_{ij} \partial_i \partial_j v + \bar{a}_{ij} \partial_i \partial_j v - \bar{a}_{ij} \partial_i \partial_j v \]
\[ + b_k \partial_k h - b_k \partial_k h + + b_k \partial_k h - b_k \partial_k h. \]

For a given \( v \in Y \), we solve (41a) for \( h \) and use (49a) and (49b) to write
\[ b_k \partial_k h = r^{-n} E_+(r) \left[ \phi(0) + \int_0^r \rho^{n-1} E_-(\rho) B[v](\rho) \, d\rho \right] \psi_1 \]
and
\[ \bar{a}_{ij} \partial_i \partial_j h = r^{-n} E_+(r) \left[ \phi(0) + \int_0^r \rho^{n-1} E_-(\rho) B[v](\rho) \, d\rho \right] \psi_2 + B[v \bar{a}_{ij} \theta_i \theta_j], \]
where
\[ \psi_1(r \theta) = r b_k(r \theta) \theta_k \quad \text{and} \quad \psi_2(r \theta) = \bar{a}_{ii}(r \theta) - (n + R(r)) \bar{a}_{ij}(r \theta) \theta_i \theta_j \]
also satisfy \( |\psi_i(r \theta)| \leq c \omega(r) \) for \( i = 1, 2 \). Plugging this into (51), we obtain an equation of the form \(-\Delta v = F[v]\). We want to apply \( K \), convolution by the fundamental solution of the Laplacian, to solve this, but there could be a problem: \( F[v] \) may not be integrable at \( x = 0 \). However, from (51) we see that \( F[v] = 0 \) and multiplying \( F[v](x) \) by \( |x| \) makes it integrable, so we can use Proposition 1 in Appendix A.

Applying \( K \) to both sides of (51), we obtain an equation for \( v \) alone:
\[ v + S_1 v + S_2 v = \phi(0) w, \]
where
\[ w(x) = K_{y \to x} \left( |y|^{-n} E_+(|y|) \left[ \psi(y) - \bar{\psi}(|y|) \right] \right) \]
Hence $M_{(45)}$ to replace $M_{(55b)}$ and similarly (55a) $\omega$ in Appendix A to satisfy $10 \VLADIMIR\ MAZ'YA\ AND\ ROBERT\ MCOwen$

and we have let $\psi := \psi_1 + \psi_2$. To find $v \in Y$ satisfying (53) we need to show $w \in Y$ and $S_i : Y \to Y$ is small operator norm for $i = 1, 2$.

To show $w \in Y$, we must estimate $M_{2,p}(w, r)$ for $0 < r < 1$ and for $r > 1$. We apply Proposition 1 in Appendix A to $f(x) = |x|^{-n}E_+(|x|)(\psi(x) - \psi(|x|))$, which vanishes for $|x| > 1$, to conclude

$M_{2,p}(w, r) \leq c \left( r^{1-n} \int_0^r E_+(\rho)\omega(\rho) d\rho + r \int_r^1 E_+(\rho)\omega(\rho)r^{-n} d\rho \right)$ for $0 < r < 1$.

We can use (53a) and the monotonicity of $\omega(r)$ to estimate

(55a) $\int_0^r E_+(\rho)\omega(\rho) d\rho \leq E_+(r)\omega(r) r^{c_\mu} \int_0^r \rho^{-c_\mu} d\rho = crE_+(r)\omega(r)$

and similarly

(55b) $\int_r^1 E_+(\rho)\omega(\rho) \rho^{-n} d\rho \leq cr^{1-n}E_+(r)\omega(r)$.

Using these in (54), we obtain

$M_{2,p}(w, r)^{n-2} \leq c_\omega(r)E_+(r)$ for $0 < r < 1$, and we can invoke (53a) to replace $E_+(r)$ by $e^{r(r)}$ as required in the norm for $Y$. Meanwhile, for $r > 1$ we use (53a) and $E_+(\rho) \leq \rho^{-c_\mu}$ for $0 < \rho < 1$ to conclude

$M_{2,p}(w, r) \leq c_\mu r^{1-n}$.

Hence $M_{2,p}(w, r)^{n-1} \leq c_\mu$ for $r > 1$. These estimates confirm that $w \in Y$.

Next let us show that $S_1 : Y \to Y$ with small operator norm. We assume $\|v\|_Y \leq 1$ and we want to estimate $M_{2,p}(S_1 v, r)$ separately for $0 < r < 1$ and $r > 1$. Using (48) we see that the function

$f_1(y) = |y|^{-n}E_+(|y|) \int_0^{|y|} r^{n-1}E_-(\rho)B[v](\rho) d\rho (\psi(y) - \psi(|y|))$

satisfies $M_p(f_1, r) \leq c_\mu^2 E_+(r)\omega(r)r^{-n}$ for $0 < r < 1$ and $M_p(f_1, r) = 0$ for $r > 1$. Since $S_1v = -Kf_1$, we can apply Proposition 1 in Appendix 1 to obtain

$M_{2,p}(S_1 v, r) \leq c_\mu^2 \left( r^{1-n} \int_0^r E_+(\rho)\omega(\rho) d\rho + r \int_r^1 E_+(\rho)\omega(\rho)r^{-n} d\rho \right)$.

Using (55a) and (55b), we conclude $M_{2,p}(S_1 v, r)^{n-2} \leq c_\mu^2 \omega(r)E_+(r)$ for $0 < r < 1$. On the other hand, for $r > 1$, Proposition 1 in Appendix 1 implies

$M_{2,p}(S_1 v, r) \leq c_\mu^3 r^{1-n} \int_0^1 E_+(\rho)\omega(\rho) d\rho \leq c_\mu^3 r^{1-n}$,

so $M_{2,p}(S_1 v, r)^{n-1} \leq c_\mu^3$ for $r > 1$. Combining these estimates, we see that $S_1 : Y \to Y$ has small operator norm.

Finally, we show that $S_2 : Y \to Y$ with small operator norm. Again we assume $\|v\|_Y \leq 1$ and estimate $M_{2,p}(S_2 v, r)$ separately for $0 < r < 1$ and $r > 1$. Notice that the function

$f_2 = B[v] (\bar{a}_{ij}^\theta \theta_j - \tilde{a}_{ij}^\theta \theta_j)$

satisfies

$M_p(f_2, r) \leq \omega(r)M_p(B[v], r) \leq c_\omega^3(r)E_+(r)r^{-n}$ for $0 < r < 1$, and

$M_p(f_2, r) \leq c_\mu^3 E_+(r)r^{-n}$ for $r > 1$. Since $\omega(r) \leq \omega(1)$, we use (55a) to estimate

$M_{2,p}(f_2, r) \leq c_\mu^3 r^{1-n} \int_0^r E_+(\rho)\omega(\rho) d\rho \leq c_\mu^3 r^{1-n}$,

so $M_{2,p}(f_2, r)^{n-1} \leq c_\mu^3$ for $r > 1$. Combining these estimates, we see that $S_2 : Y \to Y$ has small operator norm.
where \( c \) is independent of \( v \), and \( M_p(f_2, r) = 0 \) for \( r > 1 \). Similarly, the function

\[
f_3 = \tilde{a}_{ij} \partial_i \partial_j v - a_{ij} \partial_i \partial_j v
\]

satisfies

\[
M_p(f_3, r) \leq \omega(r) M_p(D^2 v, r) \leq \omega^2(r) E_+(r) r^{-n}
\]

for \( 0 < r < 1 \), and \( M_p(f_3, r) = 0 \) for \( r > 1 \). For \( 0 < r < 1 \), we apply Proposition 1 in Appendix 1 to \( S_2 v = -K(f_2 + f_3) \) to conclude

\[
M_{2,p}(S_2 v, r) \leq \frac{c}{r^{1-n}} \int_0^r \omega^2(\rho) E_+(\rho) d\rho + \frac{r}{r} \int_0^1 \omega^2(\rho) E_+(\rho) \rho^{-n} d\rho.
\]

Using (55), (55a), and (55b), we conclude that \( M_{2,p}(S_2 v, r) r^{n-2} \leq c \mu \omega(r) E_+(r) \) for \( 0 < r < 1 \). Meanwhile, for \( r > 1 \), we use (55) and (55a) to estimate

\[
M_{2,p}(S_2 v, r) \leq c r^{1-n} \int_0^1 \omega^2(\rho) E_+(\rho) d\rho \leq c \mu^2 r^{1-n} \int_0^1 \rho^{-c} \mu d\rho \leq c \mu^2 r^{-1-n}.
\]

Consequently, \( M_{2,p}(S_2 v, r) r^{n-1}/\mu \leq c \mu \). These estimates show that \( S_2 : Y \rightarrow Y \) has small operator norm.

Since \( S_1 + S_2 \) has small operator norm on \( Y \), we conclude that (55) has a unique solution \( v \in Y \), depending on the choice of the constant \( c_\star = \phi(0) \). But once \( c_\star \) and \( v \) are known, we find \( g(r) \) from (49a) and integrate to find \( h(r) \):

\[
h(r) = \int_r^\infty s^{1-n} E_+(s) \left[ c_\star + \int_0^s \rho^{n-1} E_-(\rho) B[v](\rho) d\rho \right] ds + c_2
\]

where \( c_2 \) is an arbitrary constant. To show that the solution \( Z(x) = h(|x|) + v(x) \) is of the form (17), we choose \( c_\star \) to enable us to replace \( E_+(s) \) by \( e^{I(s)} \); recalling (44a) we see that we should take \( c_\star = A^{-1} \) and write \( h(r) = h_0(r) + h_1(r) + c \) where

\[
h_0(r) = \int_r^1 s^{1-n} e^{I(s)} ds
\]

and (recalling (44c))

\[
h_1(r) = \int_r^1 s^{1-n} e^{I(s)} \tau(s) ds + \int_r^1 s^{1-n} E_+(s) \int_0^s \rho^{n-1} E_-(\rho) B[v](\rho) d\rho ds.
\]

Integrating by parts and using \( |I'(s)| \leq c s^{-1} \omega(s) \) we can show

\[
\left| h_0(r) - \frac{r^{2-n}}{n-2} e^{I(r)} \right| \leq c r^{2-n} e^{I(r)} \omega(r).
\]

Using \( |\tau(s)| \leq c \sigma(s) \) and \( 50 \), we can also estimate

\[
|h_1(r)| \leq c r^{2-n} e^{I(r)} \max(\omega(r), \sigma(r)) \quad \text{for} \quad 0 < r < 1.
\]

If we define

\[
\zeta(r) = \frac{h_1(r)}{h_0(r)} \quad \text{for} \quad 0 < r < 1,
\]

then we can easily estimate \( |\zeta(r)|, |r \zeta'(r)| \leq c \max(\omega(r), \sigma(r)) \). To estimate \( \zeta'' \), let us write \( h_0 \zeta'' = h_1'' - h_0'' \zeta - 2 h_0' \zeta' \) where

\[
h_0''(r) = (n - 1) r^{-n} e^{I(r)} - r^{1-n} e^{I(r)} I'(r)
\]
and
\[h''_1(r) = r^{-n}e^{I(r)}[(n-1)\tau(r) - rI'(r)\tau(r) - r\tau'(r)]\]
\[(59b) + r^{-n}E_+(r)(n-1 + R(r))\int_0^\rho n^{-1} R(\rho) B[v](\rho) d\rho - B[v](r).\]

We can estimate $h''_1$ and $h'_0$ pointwise, but $h''_1$ needs to be estimated in $M_p$. However, using (41c) and $v \in Y$, we may conclude $M_p(r^2\chi^u, r) \leq c \max(\omega(r), \sigma(r))$. Combining with the estimates of the lower-order derivatives, we have shown (17c). Since (17c) follows from $v \in Y$, we have proved the first part of Theorem 1.

Let us now turn to the second part of Theorem 1. Suppose $u \in W^{2,p}_{\delta_0}(B_1 \setminus \{0\})$ satisfies $Lu = 0$ in $B_1 \setminus \{0\}$ and $M_{2,p}(u, r) \leq cr^{1-n+\varepsilon_0}$ for some $\varepsilon_0 > 0$; we want to show $u$ is of the form (19). We will use properties of the bounded linear map
\[(60) \mathcal{L} : W^{2,p}_{\delta_0, \delta_1}(\mathbb{R}^n_o) \rightarrow L^p_{\delta_0+2, \delta_1+2}(\mathbb{R}^n_o),\]
where $\mathbb{R}^n_o = \mathbb{R}^n \setminus \{0\}$ and $W^{2,p}_{\delta_0, \delta_1}(\mathbb{R}^n_o), L^p_{\delta_0+2, \delta_1+2}(\mathbb{R}^n_o)$ are the weighted Sobolev spaces that are defined in Appendix B. Since we are interested in the behavior of functions at the origin, we fix $\delta_1 \in (-n/p, -2+n/p')$ and allow $\delta_0$ to vary. Note that (60) is a perturbation of (82), and the norm of the difference $\mathcal{L} - \Delta$ depends on the magnitude of
\[\sup_{|v|<1} (|a_{ij}(x) - \delta_{ij}| + |x| b_k(x)).\]

Thus, provided we take $\mu$ in (34) sufficiently small, we can arrange that (60) and (82) are not only Fredholm for exactly the same values of $\delta_0$ and $\delta_1$, but the nullity and deficiency of the two maps depend on the magnitude of
\[\sup_{|v|<1} (|a_{ij}(x) - \delta_{ij}| + |x| b_k(x)).\]

Introduce a cutoff function $\chi \in C^\infty_0(B_1)$ with $\chi = 1$ on $B_{1/2}$, then $M_{2,p}(\chi u, r) \leq cr^{1-n+\varepsilon_0}$ implies $\chi u \in W^{2,p}_{\delta_0}(B_1)$ provided $\delta_0 \geq 1$. Let us choose $\delta_0^* \in (-n/p + 1 + n/p')$ and let $f = \mathcal{L}(\chi u)$. Since $f = 0$ for $|x| < 1/2$ and for $|x| > 1$, we have $f \in L^p_{\delta_0+2, \delta_1+2}(\mathbb{R}^n_o)$ for all $\delta_0$ so let us choose $\delta_0 \in (-n/p, -2+n/p')$. By (i) we can find $v = \mathcal{L}^{-1}f \in W^{2,p}_{\delta_0}(\mathbb{R}^n_o)$. Then $\chi u - v \in W^{2,p}_{\delta_0, \delta_1}(\mathbb{R}^n_o)$ satisfies $\mathcal{L}(\chi u - v) = 0$. By (ii), $\mathcal{L} : W^{2,p}_{\delta_0, \delta_1}(\mathbb{R}^n_o) \rightarrow L^p_{\delta_0+2, \delta_1+2}(\mathbb{R}^n_o)$ has nullity 1. Since $\mathcal{L} \neq 0$, the nullspace must be spanned by $Z$ and so $\chi u - v = c_0 Z$ for some constant $c_0$.

It only remains to show that $v = c_1 + w$ where $M_{2,p}(w, r) \leq cr^{1-\varepsilon_1}$ for any $\varepsilon_1 > 0$. Let us pick $\delta_0^- \in (-1 - n/p, -n/p)$ so that (iii), the map
\[(61) \mathcal{L} : W^{2,p}_{\delta_0^-}(\mathbb{R}^n_o) \rightarrow L^p_{\delta_0^-+2, \delta_1+2}(\mathbb{R}^n_o),\]
is injective with deficiency 1. Let $\zeta$ be a linear functional on $L^p_{\delta_0^-+2, \delta_1+2}(\mathbb{R}^n_o)$ that vanishes on the image of (61). Note that $\mathcal{L} \chi = 0$ for $|x| < 1/2$ and for $|x| > 1$, so $\mathcal{L} \chi \in L^p_{\delta_0^-+2, \delta_1+2}(\mathbb{R}^n_o)$. But $\chi \not\in W^{2,p}_{\delta_0^-, \delta_1}(\mathbb{R}^n_o)$ since $\delta_0^- < -n/p$, so $\mathcal{L} \chi$ is not in the image of (61), and hence $\zeta[\mathcal{L}(\chi u)] \neq 0$. This enables us to find $c_1$ so that
\[\zeta[\mathcal{L}(\chi u - c_1 \chi)] = 0, \quad \text{i.e., } \mathcal{L}(\chi u - c_1 \chi) \in \text{image of (61)}, \quad \text{so } \mathcal{L} w = \mathcal{L}(\chi u - c_1 \chi) \text{ for some } w \in W^{2,p}_{\delta_0^-, \delta_1}(\mathbb{R}^n_o) \subset W^{2,p}_{\delta_0^-}(\mathbb{R}^n_o).\]

Since $\delta_0 \in (-n/p, -2+n/p')$, by the isomorphism (i) we have $w = \chi u - c_1 \chi$. In other words, for $|x| < 1/2$ we have $v = c_1 + w$, where $w \in W^{2,p}_{\delta_0^-}(\mathbb{R}^n_o)$. 

For any $\varepsilon_1 \in (0, 1)$ we can let $\delta_0 = \varepsilon_1 - 1 - n/p$ and conclude $M_{2,p}(w, r) \leq cr^{1-\varepsilon_1}$, as stated in Theorem 1. This completes the proof. □

Now let us combine Theorem 1 with a change of variables to treat a general $y \in \mathbb{R}^n$ and we do not assume $a_{ij}(y) = \delta_{ij}$. We let $B_\varepsilon(y) = \{ x : |x - y| < \varepsilon \}$ and want to construct a singular solution of

$$L(x, \partial_x)Z_y(x) = 0 \quad \text{for } x \in B_\varepsilon(y) \setminus \{ y \},$$

provided $\varepsilon$ is sufficiently small. Since $A_y$ is symmetric and positive definite, we can define the symmetric matrix $A_y^{-1/2}$. This enables us to define the function $I_y(r)$ as in (60).

**Corollary 1.** For $n \geq 3$, $p \in (1, \infty)$, and fixed $y \in U$, assume that $A_y$ is positive definite and the coefficients $a_{ij}$ satisfy (29). If $y = 0$, we assume (3) but otherwise the $b_k$ are bounded on $U$. Then, for $\varepsilon$ sufficiently small, there exists a solution $Z_y$ of (62) in the form

$$(63)\begin{align*}
Z_y(x) &= h_y(|A_y^{-1/2}(x - y)|) + v(x), \\
h_y(r) &= \int_r^\varepsilon s^{1-n}e^{I_y(s)}ds \left(1 + \zeta_y(r)\right)
\end{align*}$$

with $I_y(r)$ given by (60) and $\zeta_y$ satisfying

$$(63)\begin{align*}
M_{2,p}(\zeta_y, r) &\leq c \max(\omega(r), \sigma(r)), \\
v &\text{ satisfying } M_{2,p}(v, r; y) \leq c r^{2-n}e^{I_y(r)}\omega(r).
\end{align*}$$

Moreover, for any $u \in W^{2,p}_{\text{loc}}(\overline{B_\varepsilon(y)} \setminus \{ y \})$ that is a strong solution of $L(x, \partial_x)u = 0$ in $\overline{B_\varepsilon(y)} \setminus \{ y \}$ and subject to the growth condition

$$M_{2,p}(u, r; y) \leq cr^{1-n+\varepsilon_0} \quad \text{where } \varepsilon_0 > 0,$$

there exist constants $c_0, c_1$ such that

$$u(x) = c_0 Z_y(x) + c_1 + w(x),$$

where $M_{2,p}(u, r; y) \leq cr^{1-\varepsilon_1}$ for any $\varepsilon_1 > 0$.

If we use integration by parts, we can write the solution of Corollary 1 as

$$Z_y(x) = \frac{(A^{-1}_y(x - y), (x - y))^{2-n}}{(n-2)} e^{I_y(\sqrt{(A^{-1}_y(x - y), (x - y))})}(1 + \xi_y(x))$$

where $M_{1,\infty}(\xi_y, r; y) \leq c \max(\omega(r), \sigma(r))$ for $0 < r < \varepsilon$. This generalizes (20).

**Proof of Corollary 1.** First we consider $y = 0$. Since $A_0$ is positive definite, we can define a symmetric matrix by $J = A_0^{-1/2}$, so $JA_0J = I$. If we introduce a change of variables by $\tilde{x} = Jx$ and new coefficients $\tilde{a}_{ij}(\tilde{x})$ and $\tilde{b}_k(\tilde{x})$ by

$$\tilde{A}_x = JA_xJ \quad \text{and} \quad \tilde{B}_2 = B_xJ,$$

then $\tilde{A}_0 = I$ and

$$L(x, \partial_x) = a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + b_k \frac{\partial}{\partial x_k} = \tilde{a}_{ij} \frac{\partial^2}{\partial \tilde{x}_i \partial \tilde{x}_j} + \tilde{b}_k \frac{\partial}{\partial \tilde{x}_k} = \tilde{L}(\tilde{x}, \partial_{\tilde{x}}).$$
Hence we may apply Theorem 1 in the variables \( \tilde{x} \) to conclude the existence of a solution \( \tilde{Z} \) of
\[
\mathcal{L}(\tilde{x}, \partial_{\tilde{x}}) \tilde{Z}(\tilde{x}) = 0 \quad \text{for} \quad 0 < |\tilde{x}| < \varepsilon \quad \text{in the form} \quad \tilde{Z}(\tilde{x}) + \tilde{h}(|\tilde{x}|) + \tilde{v}(\tilde{x})
\]
where
\[
\tilde{h}(r) = \int_r^\varepsilon s^{1-n} e^{I(s)} \, ds \quad (1 + \zeta(r))
\]
with \( M_{2,\rho}(\zeta, r) \leq c \max(\omega(r), \sigma(r)), \) \( M_{2,\rho}(\tilde{v}, r) \leq c r^{2-n} e^{I(r)} \omega(r), \) and
\[
I(r) = \frac{1}{|S^{n-1}|} \int_{r < |z| < \varepsilon} \left( \text{tr}(\tilde{A}_z) - n \frac{\langle \tilde{A}_z, z \rangle}{|z|^2} + \langle \tilde{B}_z, z \rangle \right) \, \frac{dz}{|z|^n}. \]

In terms of the original variables, we see that \( Z(x) = \tilde{h}(|Jx|) + \tilde{v}(Jx) \) satisfies \( \mathcal{L}(x, \partial_x)Z = 0 \) for
\( 0 < |Jx| < \varepsilon, \) hence for \( 0 < |x| < \varepsilon_1 \) with \( \varepsilon_1 \) sufficiently small.

Finally, if \( y \) is a general point in \( U, \) then we use the change of variables \( \tilde{x} = J(x - y) \) with
\[ J = A_y^{1/2} \] and let \( \tilde{A}_\tilde{x} = (\partial_{ij}(\tilde{x})) = JA_xJ. \) Since \( \tilde{x} = 0 \) corresponds to \( x = y, \) we have \( \tilde{a}_{ij}(0) = \delta_{ij}, \)
so we can apply Theorem 1 to \( \mathcal{L}(\tilde{x}, \partial_{\tilde{x}}) = \mathcal{L}(x, \partial_x) \) to obtain the solution \( \tilde{h}(|\tilde{x}|) + \tilde{v}(\tilde{x}). \) We obtain the solution of (67) as
\[
Z_y(x) = \tilde{h}(|J(x - y)|) + \tilde{v}(J(x - y)),
\]
where \( \tilde{h}(r) \) involves the above \( I(r). \) To transform \( I(r) \) to the original variables, replace \( \tilde{A}_z \) by
\( A_z \) and every occurrence of \( \tilde{z} \) by \( x - y; \) we find \( I \) is of the desired form (69). Moreover, since
\( \tilde{v} \) satisfies \( M_{2,\rho}(\tilde{v}, r) \leq c r^{2-n} e^{I(r)} \omega(r), \) we find that \( v(x) = \tilde{v}(J(x - y)) \) satisfies \( M_{2,\rho}(v, r; y) \leq c r^{2-n} e^{I(r)} \omega(r), \) as desired. \( \square \)

2. Finding the constant \( C_y \) so that \(-\mathcal{L}Z(x) = C_y \delta(x - y)\)

In this section we first prove Theorem 2, then state and prove Corollary 2, which shows how the formulas change when we no longer assume \( y = 0 \) and \( a_{ij}(y) = \delta_{ij}. \) As in the proof of Theorem 1, we shall assume (34) holds for \( Z_x = 0 \) and only involve \( v \) by \( \delta(x), \) i.e. satisfy (65) for some constant \( C_0. \) This was done for \( \mathcal{L}_0 \) in (11), and many of the arguments here are the same, so we shall be brief; for more details, consult (11).

From Theorem 1, we obtain estimates on \( \partial Z \) and \( \partial^2 Z \) that show \( \mathcal{L}(x, \partial_x)Z(x) \) can be regularized at \( x = 0 \) as a distribution \( \mathcal{F}_0 \) on \( C^0_0(U), \) the Hölder continuous functions with compact support in \( U. \) So if we can define \( \mathcal{L}(x, \partial_x)Z(x) \) as a distribution \( \mathcal{F}, \) then it must be supported at \( x = 0 \) and only involve \( \delta(x), \) not derivatives of \( \delta(x), \) i.e. satisfy (65) for some constant \( C_0. \)

The difficulty in defining \( \mathcal{L}(x, \partial_x)Z(x) \) as a distribution comes from the lack of regularity of the coefficients, especially \( a_{ij}. \) In particular, there is no difficulty in defining 2nd-order distributional derivatives of \( Z \) by
\[
\langle \partial_i \partial_j Z, \phi \rangle = -\int_U \partial_i Z(x) \partial_j \phi(x) \, dx \quad \text{for} \quad \phi \in C^0_0(U),
\]
since the integral on the right converges. So let us try to define the distribution \( \mathcal{L}Z \) by
\[
\langle \mathcal{L}Z, \phi \rangle = \int_U \left( (a_{ij} - \delta_{ij}) \partial_i \partial_j Z \phi - \partial_i Z \partial_j \phi + b_k \partial_k Z \phi \right) \, dx \quad \text{for} \quad \phi \in C^0_0(U).
\]
This is an improper integral due to the singularities in \( \partial_i \partial_j Z \) and \( b_k \partial_k Z \) at \( x = 0, \) but if the integral converges then we conclude (65) holds and we can compute \( C_0 \) from
\[
- C_0 = \lim_{\varepsilon \to 0} \int_U \left( (a_{ij} - \delta_{ij}) \partial_i \partial_j Z \phi_x - \partial_i Z \partial_j \phi_x + b_k \partial_k Z \phi_x \right) \, dx,
\]
where $\phi_{\varepsilon}(|x|) = \chi(|x|/\varepsilon)$ with $\chi(r)$ being a smooth cutoff function that is 1 for $0 < r < 1/4$ and vanishes for $r > 1/2$. (We may assume $\phi(x) = \phi(|x|)$ since we can write $\phi(x) = \phi_0(|x|) + \phi_1(x)$ where $|\phi_1(x)| + |x| |\nabla \phi_1(x)| \leq c |x|$ for $|x| < 1$, which shows that $\langle L \mu, \phi_1 \rangle$ is well-defined as an integral and contributes nothing to $C_0$.)

**Proof of Theorem 2.** Recall from the proof of Theorem 1 the decomposition $Z(x) = h(|x|) + v(x)$ in (56). We first show that $v$ makes no contribution to determining the value of $C_0$. Since $v \in Y$, we have $M_{2,p}(v, r) \leq c r^{2-n} \omega(r) e^{I(r)}$. But $I(r)$ is bounded above in both cases (i) and (ii), so we have

\begin{equation}
M_{2,p}(v, r) \leq c r^{2-n} \omega(r) \quad \text{for } 0 < r < 1.
\end{equation}

Then, as $\varepsilon \to 0$, we have\footnote{In the following, integrals involving $\int_{|x|<\varepsilon}$ should be interpreted as improper: $\lim_{\eta \to 0} \int_{\eta<|x|<\varepsilon}$.}

\[
\int_{|x|<\varepsilon} |(a_{ij} - \delta_{ij}) \partial_i \partial_j v \phi_{\varepsilon}| \, dx \leq c \int_0^\varepsilon \omega^2(r) \frac{dr}{r} = c \sigma(\varepsilon) \to 0,
\]

\[
\int_{|x|<\varepsilon} |\partial_i v \partial_i \phi_{\varepsilon}| \, dx \leq c \varepsilon^{-1} \int_0^\varepsilon \omega(r) \, dr \leq c \omega(\varepsilon) \to 0
\]

and

\[
\int_{|x|<\varepsilon} |b_k \partial_k v \phi_{\varepsilon}| \, dx \leq c \int_0^\varepsilon \omega^2(r) \frac{dr}{r} = c \sigma(\varepsilon) \to 0.
\]

So $v$ makes no contribution to $C_0$.

Now we consider $h(r)$. In fact, from (56) we can write $h(r) = h_0(r) + h_1(r) + c$, where $c$ is an arbitrary constant and

\[
h_0(r) = c_s \int_r^1 s^{1-n} E_+(s) \, ds,
\]

\[
h_1(r) = \int_r^1 s^{1-n} E_+(s) \int_0^s \rho^{n-1} E_-(\rho) B[v](\rho) \, d\rho \, ds,
\]

with $E_+$ defined in (42) and $c_s$ chosen as in the proof of Theorem 1 so that $c_s E_+(0) = e^{I(0)}$. (Note that this decomposition of $h(r)$ is slightly different from (57).) Let us show that $h_1$ and $c$ do not contribute to $C_0$. We compute

\[
\partial_t h_1 = -x_i r^{-n} E_+(r) \int_0^r \rho^{n-1} E_-(\rho) B[v](\rho) \, d\rho,
\]

\[
\partial_i \partial_j h_1 = -r^{-n} E_+(r) \left( \delta_{ij} - (n + R(r)) \frac{x_i x_j}{r^2} \right) \int_0^r \rho^{n-1} E_-(\rho) B[v](\rho) \, d\rho - \frac{x_i x_j}{r^2} B[v](r),
\]

and hence

\[
\int_{|x|<\varepsilon} [(a_{ij} - \delta_{ij}) \partial_i \partial_j h_1 \phi_{\varepsilon} + b_k \partial_k h_1 \phi_{\varepsilon} - \partial_t h_1 \partial_t \phi_{\varepsilon}] \, dx
\]

\[
= \int_0^\varepsilon \left[ \frac{R(r)}{r} E_+(r) \chi \left( \frac{r}{\varepsilon} \right) + E_+ \frac{d}{dr} \chi \left( \frac{r}{\varepsilon} \right) \right] \int_0^r \rho^{n-1} E_-(\rho) B[v](\rho) \, d\rho \, dr
\]

\[
- \int_0^\varepsilon (\alpha(r) - 1) r^{n-1} B[v](r) \chi \left( \frac{r}{\varepsilon} \right) \, dr
\]

\[
+ \varepsilon^{-1} \int_0^\varepsilon E_+(r) \int_0^r \rho^{n-1} E_-(\rho) B[v](\rho) \, d\rho \chi \left( \frac{r}{\varepsilon} \right) \, dr.
\]
Let us denote these three terms $I_1(e), I_2(e), I_3(e)$ and estimate them separately. First,

$$I_1(e) = \int_0^e \frac{d}{dr} \left[ E_+(r) \chi \left( \frac{r}{\varepsilon} \right) \int_0^{\rho^{-1} E_-(\rho) B(v)(\rho) \, d\rho \right) dr - \int_0^e r^{n-1} \chi \left( \frac{r}{\varepsilon} \right) B[v](r) \, dr.$$

$$= E_+(e) \int_0^e r^{n-1} E_-(r) B[v](r) \, dr - \int_0^e r^{n-1} B[v](r) \chi \left( \frac{r}{\varepsilon} \right) \, dr.$$

But recall from (50) that $\int_0^e r^{n-1} E_-(r) B[v](r) \, dr \leq c \sigma(e)$ and $I(r)$ is bounded implies $E_+(r)$ are bounded above and below by positive constants, so $|I_1(e)| \leq c \sigma(e) \to 0$ as $\varepsilon \to 0$. This also implies $|I_2(e)| \to 0$ as $\varepsilon \to 0$. As for $I_3(e)$, we also have

$$|I_3(e)| \leq c \varepsilon^{-1} \int_0^\varepsilon \chi(r) \, dr \leq c \sigma(e) \to 0 \text{ as } \varepsilon \to 0.$$

We conclude that $h_1$ does not contribute to $C_0$.

Finally, we consider $h_0$. We can calculate $\partial_1 h_0 = -c_r r^{-n} E_+(r) x_i$, and

$$\partial_1 \partial_j h_0 = -c_r r^{-n} E_+(r) \left( \delta_{ij} - n \frac{a_{ij} x_i x_j}{r^2} - \frac{x_i x_j}{r^2} R(r) \right).$$

It is easy to verify that

$$(a_{ij} - \delta_{ij}) \partial_i \partial_j h_0 = -c_r r^{-n} E_+(r) \left( a_{ii} - n \frac{a_{ij} x_i x_j}{r^2} - \left( \frac{a_{ij} x_i x_j}{r^2} - 1 \right) R(r) \right).$$

Notice that

$$\int_{|x|<\varepsilon} r^{-n} E_+(r) \left( a_{ii} - n \frac{a_{ij} x_i x_j}{r^2} - \left( \frac{a_{ij} x_i x_j}{r^2} - 1 \right) R(r) \right) \phi_e(|x|) \, dx$$

$$= |S^{n-1}| \int_0^\varepsilon E_+(r) \frac{\alpha_\alpha(r) - n \alpha(r) - (\alpha(r) - 1) R(r)}{r} \chi \left( \frac{r}{\varepsilon} \right) \, dr$$

$$= |S^{n-1}| \int_0^\varepsilon E_+(r) \frac{R(r) - \beta(r)}{r} \chi \left( \frac{r}{\varepsilon} \right) \, dr,$$

since $\alpha(r) R(r) = \alpha_\alpha(r) + \beta(r) - n \alpha(r)$. Similarly, we can verify

$$\int_{|x|<\varepsilon} b_k \partial_k h_0 \phi_e \, dx = -c_r |S^{n-1}| \int_0^\varepsilon E_+(r) \frac{\beta(r)}{r} \chi \left( \frac{r}{\varepsilon} \right) \, dr,$$

so

$$\langle L h_0, \phi_e \rangle = c_r |S^{n-1}| \int_0^\varepsilon \left( -E_+(r) \frac{R(r)}{r} \chi \left( \frac{r}{\varepsilon} \right) + E_+(r) \frac{\beta(r)}{r} \chi \left( \frac{r}{\varepsilon} \right) \right) \, dr.$$

Hence

$$\langle L h_0, \phi_e \rangle = -c_r |S^{n-1}| \int_0^\varepsilon \frac{d}{dr} \left[ E_+(r) \chi \left( \frac{r}{\varepsilon} \right) \right] \, dr = c_r |S^{n-1}| E_+(0).$$

Letting $\varepsilon \to 0$, we conclude $Z$ satisfies (65) with $C_0 = |S^{n-1}| e^{f(0)}$. \hfill \square

Now let us consider a general fixed $y \in U$ and try to solve

$$(71) \quad -\mathcal{L}(x, \partial_x) Z_y(x) = C_y \delta(x - y) \quad \text{for } x \in B_1(y)$$

for some constant $C_y$. We replace (66) with

$$(72) \quad \langle LZ_y, \phi \rangle = \int_U \left( (a_{ij} - a_{ij}(y)) \partial_i \partial_j Z_y \phi - a_{ij}(y) \partial_i Z_y \partial_j \phi + b_k \partial_k Z_y \phi \right) \, dx.$$

**Corollary 2.** Suppose the conditions of Corollary 1 hold and $Z_y$ is the singular solution found there with $I_y(r)$ given by (60a).
(i) If \( I_y(0) = \lim_{r \to 0} I_y(r) \) exists and is finite, then we can solve (21) in \( B_1(y) \) with \( C_y = |S^{n-1}| \sqrt{\det A_y} e^{I_y(0)} \).

(ii) If \( I_y(r) \to -\infty \) as \( r \to 0 \), then solving (10) in \( B_1(y) \) yields \( C_0 = 0 \), and so \( Z \) solves 
\[-\mathcal{L}(x, \partial_x)Z(x) = 0 \] 
in \( B_0(y) \), despite its singularity at \( x = y \).

If \( I_y(0) \) exists and is finite, we see from (64) that the solution \( F_y \) of 
\[-\mathcal{L}(x, \partial_x)F_y(x) = \delta(x-y) \] 
in \( B_0(y) \) has the asymptotic behavior

\[
F_y(x) = \frac{\langle A_y^{-1}(x-y), (x-y) \rangle^{2-n}}{(n-2)|S^{n-1}| \sqrt{\det A_y}} (1 + o(1)) \quad \text{as } x \to y.
\]

This generalizes (21) and establishes (31).

**Proof of Corollary 2.** We only need to show that

\[
\langle -\mathcal{L}Z_y, \phi \rangle = |S^{n-1}| \langle (\det A_y)^{-1/2} e^{I_y(0)} \phi \rangle
\]

for some \( \phi \in C_0^\infty(B_{\varepsilon_x}(y)) \). Let us recall the change of coordinates used in the proof of Theorem 1, namely \( \tilde{x} = J(x-y) \) where \( J = A_y^{-1/2} \), and let \( \tilde{\phi} = \phi(x) \). Then

\[
-\int \mathcal{L}(x, \partial_x)Z_y(x) \phi(x) \, dx = -(\det A_y)^{1/2} \int \tilde{\mathcal{L}}(\tilde{x}, \partial_{\tilde{x}}) \tilde{Z}_y(\tilde{x}) \tilde{\phi}(\tilde{x}) \, d\tilde{x}.
\]

But Theorem 2 implies \( -\langle \tilde{\mathcal{L}} \tilde{Z}_0, \tilde{\phi} \rangle = |S^{n-1}| e^{I_y(0)} \tilde{\phi}(0) \). Since \( \tilde{\phi}(0) = \phi(y) \), we obtain the desired result. \( \square \)

### 3. Constructing the fundamental solution

**Proof of Theorem 3.** For each \( y \in U \), denote the \( \varepsilon \) in Corollary 1 by \( \varepsilon_y \), and use Corollary 2 to calculate \( C_y \), which is positive since \( I_y(0) \) is finite. We conclude that 
\[-\mathcal{L}(x, \partial_x)Z_y(x)/C_y = \delta(x-y) \] 
for all \( x, y \in U \) with \( |x-y| < \varepsilon \). For fixed \( y \in U \), let \( \eta_y(r) \) be a smooth cutoff function satisfying \( \eta_y(r) = 1 \) for sufficiently small \( r \) and define

\[
F(x, y) = \eta_y(|x-y|)Z_y(x)/C_y + v(x, y),
\]

where \( v(x, y) \) is to be determined. But if we apply \( -\mathcal{L}(x, \partial_x) \) we obtain

\[
-\mathcal{L}(x, \partial_x)F(x, y) = \delta(x-y) + \psi(x, y) - \mathcal{L}(x, \partial_x)v(x, y),
\]

where \( \psi(\cdot, y) \in L^p(U) \) for \( 1 < p < \infty \) and vanishes near \( y \). We may assume that \( U \) has a smooth boundary (else we can embed it in such a bounded domain and extend the coefficients, see [16]), so we may find \( v(x, y) \) by solving the Dirichlet problem (for fixed \( y \in U \)):

\[
-\mathcal{L}(x, \partial_x)v(x, y) = \psi(x, y) \quad \text{for } x \in U,
\]

\[
v(x, y) = 0 \quad \text{for } x \in \partial U.
\]

When \( b_k \in L^\infty(U) \), it is well-known (cf. Theorem 9.15 in [6]) that (75) has a unique solution \( v(\cdot, y) \in W^{2,p}(U) \cap W^{1,p}_0(U) \) for \( 1 < p < \infty \); associated with this unique solvability is the à priori inequality \( \|v\|_{W^{2,p}(U)} \leq C \|v\|_{W^{1,p}_0(U)} \) for \( v \in W^{2,p}(U) \cap W^{1,p}_0(U) \). But our assumption [6] enables us to write \( \mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 \), where \( \mathcal{L}_1 \) has bounded coefficients in \( U \) and \( \mathcal{L}_2 = \tilde{b}_k \partial_k \) has coefficients supported in a ball \( B_{\varepsilon} \). For any \( \varepsilon > 0 \), we can take \( \sigma \) sufficiently small that \( \sup_{x \in B_{\varepsilon}} |b_k(x)| \leq \varepsilon |x|^{-1} \). For \( 1 < p < n \), by Hardy’s inequality we have \( \|\mathcal{L}_2 w\|_{L^p} \leq \varepsilon \|w\|_{W^{2,p}} \) for \( w \in W^{2,p} \) with support in \( B_{\varepsilon} \). If we take \( \varepsilon \) sufficiently small, we can arrange that the à priori inequality \( \|v\|_{W^{2,p}(U)} \leq C \|v\|_{W^{1,p}_0(U)} \) holds for \( v \in W^{2,p}(U) \cap W^{1,p}_0(U) \). We conclude that (75) admits a unique solution \( v(\cdot, y) \in W^{2,p}(U) \cap W^{1,p}_0(U) \) for \( 1 < p < n \). Taking \( p \in (n/2, n) \), we see that \( v(x, y) \) is continuous in \( x \in U \).
To confirm that $F(x,y)$ satisfies (31), we know from (74) and (64) that
\[
F(x,y) = \frac{(A_n^{-1}(x-y), (x-y))_{L^2}}{(n-2)|S^{n-1} \sqrt{\det A_y}|} e^{t_y \sqrt{(A_n^{-1}(x-y), (x-y))}} - t_y(0)(1 + \epsilon_y(x))
\]
where $M_{1,\infty}(\xi_y, r; y) \leq c \max(\omega(r), \sigma(r))$ for $0 < r < \varepsilon$. Since the exponential term tends to 1 as $|x - y| \to 0$, we obtain (31). □

**Appendix A. $L^p$-Mean Estimates for Convolutions**

In [11], we proved $L^p$-mean estimates for distribution solutions of
\[
\Delta u = f \quad \text{in} \quad \mathbb{R}^n \setminus \{0\},
\]
when $f$ has zero spherical mean, i.e. $\mathcal{F}(r) = 0$. Let $K$ denote convolution by the fundamental solution $\Gamma$ of the Laplacian. The following appears as Corollary 1 in Section 1 of [11].

**Proposition 1.** Suppose $n \geq 2$, $p \in (1, \infty)$, and $f \in L^p_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ satisfies
\[
\mathcal{F}(r) = 0 \quad \text{and} \quad \int_{|x|<1} |x| |f(x)| \, dx + \int_{|x|>1} |x|^{1-n} |f(x)| \, dx < \infty.
\]
Then $u = Kf = \Gamma * f$ is a distribution solution of (76) that satisfies
\[
M_{2,p}(Kf, r) \leq c \left( r^{1-n} \int_0^r M_p(r, \rho) \, d\rho + r \int_r^\infty M_p(f, \rho) \, d\rho \right).
\]

**Appendix B. The Laplacian on Weighted Sobolev Spaces**

Many authors have studied the mapping properties of elliptic operators such as the Laplacian on weighted Sobolev spaces on $\mathbb{R}^n$ and other noncompact manifolds with conical or cylindrical ends: cf. [10], [14], [15]. We will recall some of these results for punctured Euclidean space $\mathbb{R}^n_0 := \mathbb{R}^n \setminus \{0\}$. Since we are mostly concerned in this paper with singularities at $x = 0$, let us first investigate the weighted $L^p$-norm on the punctured unit ball $B_0 := B_1 \setminus \{0\}$. For $\delta \in \mathbb{R}$ and $1 < p < \infty$, define the Banach space $L^p_\delta(B_0)$ by the norm
\[
\|u\|_{L^p_\delta(B_0)} := \int_{|x|<1} |x|^{\delta p} |u(x)|^p \, dx.
\]
For example, the constants are in $L^p_\delta(B_0)$ if and only if $\delta > -n/p$. To compare (79) with $L^p$-means, it is easy to see (cf. [11]) that $M(u, r) \leq c r^\varepsilon$ for $0 < r < 1$ implies that $u \in L^p_\delta(B_0)$ provided $\varepsilon + \delta > -n/p$, and conversely $u \in L^p_\delta(B_0)$ implies $M_p(u, r) \leq c_\varepsilon r^\varepsilon$ for $0 < r < 1$ if $\varepsilon = -\delta - n/p$.

Now we introduce the weighted $L^p$-norm for functions on $\mathbb{R}^n_0$ with separate weights at the origin and at infinity. For $\delta_0, \delta_1 \in \mathbb{R}$, define
\[
\|u\|_{L^p_{\delta_0,\delta_1}(\mathbb{R}^n_0)} := \int_{|x|<1} |x|^\delta |u(x)|^p \, dx + \int_{|x|>1} |x|^\delta |u(x)|^p \, dx.
\]
We then define the weighted Sobolev space $W^{2,p}_{\delta_0,\delta_1}(\mathbb{R}^n_0)$ to be functions $u \in W^{2,p}_{\text{loc}}(\mathbb{R}^n_0)$ for which
\[
\|u\|_{W^{2,p}_{\delta_0,\delta_1}(\mathbb{R}^n_0)} := \sum_{|\alpha| \leq 2} \| |x|^{\alpha} \partial^\alpha u \|_{L^p_{\delta_1,\delta_2}}
\]
is finite. It is clear that
\[
\Delta : W^{2,p}_{\delta_0,\delta_1}(\mathbb{R}^n_0) \to L^p_{\delta_0+2,\delta_1+2}(\mathbb{R}^n_0)
\]
is a bounded linear operator, and (using the analysis of [10], [14], [15], for example) it can be shown that (82) is an isomorphism for $-n/p < \delta_0, \delta_1 < -2 + n/p'$, where $p' = p/(p-1)$. (Since $n \geq 3$, such $\delta_0, \delta_1$ exist.) Moreover, provided $\delta_0, \delta_1$ do not take the values $-n/p - k$ or $-2 + n/p' + k$ where $k$ is a nonnegative integer, then (82) is a Fredholm operator whose nullspace and cokernel are easily described in terms of harmonic polynomials. Since we are principally interested in the behavior of functions at the origin, we will fix

$$-n/p < \delta_1 < -2 + n/p'$$

and allow $\delta_0$ to vary. We only require a small range of values for $\delta_0$.

**Proposition 2.** Assume $1 < p < \infty$ and (83). Then the map (82) is

(a) an isomorphism for $-n/p < \delta_0 < -2 + n/p'$;
(b) surjective with nullspace spanned by $|x|^{2-n}$ if $-2 + n/p' < \delta_0 < -1 + n/p'$;
(c) injective with cokernel spanned by $1$ if $-1 - n/p < \delta_0 < -n/p$.

**Acknowledgement:** This paper has been supported by the RUDN University Strategic Academic Leadership Program.

**References**

[1] R. Alvarado, D. Brigham, V. Maz’ya, M. Mitrea, E. Ziadé, *Sharp geometric maximum principles for semieliptic operators with singular drift*, Math. Res. Lett. **18** (2011), 10001-10007.

[2] M. Cranston, Z. Zhao, *Conditional transformation of drift formula and potential theory for $1/2 \Delta + b(\cdot) \cdot \nabla$*, Comm. Math. Phys. **112** (1987), 613-625.

[3] H. Dong, S. Kim, *On C1, C2, and weak type (1,1) estimates for linear elliptic operators*, Comm. Partial Differential Equations, **42** (2017), 417-435.

[4] H. Dong, S. Kim, S. Lee, *Note on Green’s functions of non-divergence elliptic operators with continuous coefficients*, arXiv:2201.03764.

[5] D. Gilbarg, J. Serrin, *On isolated singularities of solutions of second-order elliptic equations*, J. Analyse Math. **4** (1955/56), 309-340.

[6] D. Gilbarg, N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd ed, Springer-Verlag, New York, 1983.

[7] S. Hwang, S. Kim, *Green’s function for second order elliptic equations in non-divergence form*, Potential Anal. **52** (2020), no. 1, 27–39.

[8] S. Kim, S. Lee, *Estimates for Green’s functions of elliptic equations in non-divergence form with continuous coefficients*, Ann. Appl. Math. **37** (2021), no. 2, 111–130.

[9] V.A. Kozlov, V.G. Maz’ya, *Differential Equations with Operator Coefficients*, Springer-Verlag, New York, 1999.

[10] R. Lockhart, R. McOwen, *Elliptic differential operators on noncompact manifolds*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (4) **12** (1985), 409-447.

[11] V.G. Maz’ya, R. McOwen, *On the fundamental solution of an elliptic equation in nondivergence form*, AMS Translations: special volume dedicated to Nina Uraltseva 229 (2010), 145-172.

[12] V.G. Maz’ya, R. McOwen, *Differentiability of solutions to second-order elliptic equations via dynamical systems*, J. Differential Equations, **250** (2010), 1137-1168.

[13] V.G. Maz’ya, R. McOwen, *Gilbarg-Serrin equation and Lipschitz regularity*, J. Differential Equations **312** (2022), 45-64.

[14] V. Maz’ya, B. Plamenevski, *Estimates in $L_p$ and Hölder classes and the Miranda-Agmon maximum principle solutions of elliptic boundary problems in domains with singular points on the boundary* (Russian), Math. Nachr. **81** (1978), 25-82.

[15] R. McOwen, *The behavior of the Laplacian on weighted Sobolev spaces*, Comm. Pure Appl. Math. **32** (1979), 783-795.

[16] V. Mil’man, *Extension of functions that preserve the modulus of continuity*, Math. Notes **61** (1997), 193-200.

[17] C. Miranda, *Partial Differential Equations of Elliptic Type*, Springer-Verlag, Berlin, 1970.
