A DETERMINISTIC MODEL FOR SIMULTANEOUS PLAY GAMES:
THE CHEATING ROBOT AND INSIDER INFORMATION

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Abstract

Combinatorial games are two-player games of pure strategy where the players, usually called Left and Right, move alternately. In this paper, we introduce simultaneous-play combinatorial games except that Right has extra information: he knows what move Left is about to play and can react in time to modify his move. Right is ‘cheating’ and we assume that Left is aware of this. This knowledge makes for deterministic, not probabilistic, strategies.

The basic theory and properties are developed, including showing that there is an equivalence relation and partial order on the games. Whilst there are no inverses in the class of all games, we show that there is a sub-class, simple hot games, in which the ‘integers’ have inverses. In this sub-class, the optimal strategies are obtained by the solutions to a minimum-weight matching problem on a graph whose number of vertices equals the number of summands in the disjunctive sum. This is further refined, in a version of SIMULTANEOUS TOPPLING DOMINOS, by reducing the number of edges in the underlying graph to be linear in the number of vertices.

Keywords: combinatorial game theory, economic game theory, simultaneous play games, deterministic model, weighted matchings

1 Introduction

‘Insider knowledge’ can be represented in several ways. We present it in a game situation and develop a theory with interesting properties and many unsolved questions. It can be viewed as the confluence of two disparate research directions: cheating in zero-sum games and a search for a deterministic theory for simultaneous combinatorial games.

The name Cheating-Robot, originates in [13] and was inspired by the Japanese robot that always wins Rock-Paper-Scissors against humans (see [14], [17]). It ‘cheats’ by having

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reflexes fast enough to see what the human hand is forming and then responds correctly. An equivalent formulation for humans versus humans with inside information is the following. Each player has three cards: Rock, Paper, Scissors. The players choose a card from their hand and reveal them simultaneously. However, one player’s cards, call her Left, are marked and the opponent, call him Right, can see which card she is about to play and chooses his card appropriately.

Combinatorial games, as presented in [1, 4, 18], are two-player games of pure strategy where the players move alternately. These include checkers, chess, and go. Our other motivation, is to analyze combinatorial games with simultaneous moves. The players are Left (female) and Right (male). As is, the resulting theory has probabilistic strategies, see [13] for more on this approach. Since letting Right know what Left is about to play is tantamount to Right ‘responding’ to Left’s moves, the strategies for both players are deterministic.

In extending an alternating play combinatorial game to simultaneous play, our guiding philosophy is twofold: (i) the players must know when they have a move and (ii) if a player does not have a move they cannot win.

We will return to the impact of the second part, i.e., ‘who wins?’, in Definition 4. In alternating play CGT, the winning condition, ‘last player to move loses’ (misère) [18] and ‘greater score’ [15] have also led to interesting theories but with fewer algebraic properties. We leave those for another time.

The first part of the philosophy impacts the structure of the games. For each combination of Left and Right moves there must exist a simultaneous move. A convenient way to represent the simultaneous options is using a matrix where the rows and columns are indexed by the Left, respectively, Right moves. This is similar to zero-sum games except now the ‘payoffs’ are positions in a combinatorial game.

We assume that in the simultaneous game, both Left and Right have the same moves as in the alternating game. The first hurdle is to decide what happens if the two moves result in an illegal position in the underlying, alternating game. In [13], it was shown that this must be covered explicitly in the amended ruleset for the simultaneous game. See the end of this section for the ruleset for bw simultaneous toppling dominoes, a simultaneous combinatorial game which we use throughout the paper. For other examples, see [13].

Why isn’t the Cheating-Robot theory trivial? In any zero-sum matrix game if one player knows the move of the other before playing this is a huge advantage. This advantage is not so clear when the entries of the matrix are positions from combinatorial games since each simultaneous move results in another matrix, or finishes the game. Also, Singmaster [19] proves that almost all combinatorial games are first player wins. These competing advantages do indeed lead to a theory with interesting questions and hope for the human player.
Our paper will proceed as follows. We develop the basic theory for all short (finite number of positions and no repeated positions) Cheating-Robot games in Section 2. We begin by presenting the formal definition of Cheating-Robot games. The matrix representation allows for the domination concepts from zero-sum games to be generalized to Cheating-Robot games (see Theorems 5, 6, 7). Theorem 1 proves there are only three outcome classes: Left win, Right win, and Draw.

The disjunctive sum induces an equivalence relation and a partial order on the games (see Theorem 2). In normal-play alternating combinatorial game theory (CGT) games, the disjunctive sum induces an abelian group, the ‘0’ being the game with no moves, and there are many games equivalent to 0. The only Cheating-Robot game equivalent to 0 is 0 itself, Corollary 1. This is similar to misère games (see [18]). The game where only one player has $n$ consecutive moves, ending at the 0 game, clearly should be called $n$ or $-n$. Unfortunately, these do not add like integers since, in this theory, $1 - 1 \neq 0$. Several properties of integers do hold when the sign of the integers in the sum are the same, Theorem 9. As in the theory of all short misère games, the negative of a game $G$ is not its additive inverse. In fact, in Lemma 2 we prove that if $G \neq 0$ then $0 \succ G - G$.

In Section 3, we investigate a sub-class of Cheating-Robot games in which both players want to move. Within this sub-class, for an integer $n$, it is now true that $n - n \equiv 0$. Moreover, Left only needs a simple search to find her optimal move, and Right has to solve a quadratic-time optimization problem. In Section 3.1 is a case study of BW SIMULTANEOUS TOPPLING DOMINOES to demonstrate the theory. The structure of this ruleset allows for an improvement in the running time of the algorithm to find the optimal robot strategy.

**Ruleset for BW simultaneous toppling dominoes**

The rules are based on the physics of a line of dominoes. For properties of the underlying CGT game and the extension to other colour dominoes, see [12].

- **Board:** Rows composed of black (B) and white (W) dominoes. Dominoes are spaced so that a toppling domino can only topple an adjacent domino. If a row has an empty place then the sub-rows to the left and to the right are regarded as separate rows. Moves are only applied to a single row.

- **Moves:**
  - Left: she topples a black domino either to the left or the right.
  - Right: he topples a white domino either to the left or the right.
  - Effect of the move: A toppling domino topples the adjacent standing domino, in that direction, which in turn topples the next adjacent standing domino and so on. This series of toppling stops at an empty space or when the next domino either has toppled or is toppling. A domino can be ‘toppled’ by the dominoes on both sides at the same time. At the end of the move, all toppled dominoes are removed.
The next examples illustrate the rules, with no pretense of these being good moves. Let ‘_’ indicate an empty spot and $\overrightarrow{K}$ or $\overleftarrow{K}$ indicate the direction in which the player topples the piece $K$. Possible moves are

$$\begin{align*}
B\overrightarrow{BW\overrightarrow{WB}} &\rightarrow B\_\_\_\_ \\
B\overleftarrow{BW\overrightarrow{WB}} &\rightarrow B\_\_WB \\
\overleftarrow{BBWW\overrightarrow{WB}} &\rightarrow _BWW\_ \\
B\overrightarrow{W_BWW} &\rightarrow B\_W\_W \\
\overrightarrow{B_BWW} &\rightarrow B\_WW\_ \\
\overrightarrow{BWW} &\rightarrow _BWW\_ \\
B\_\_W_B\overrightarrow{WW} &\rightarrow B\_W\_W
\end{align*}$$

In combinatorial game theory, we want to consider several games being played at the same time. In BW SIMULTANEOUS TOPPLING DOMINOES these would be the different rows. On a player’s turn, a player chooses one game and moves in it (which is denoted by ‘+’). It is possible, as in the fourth example, that both players do not play in the same row. The rules must indicate what happens in this case. For us, those moves will be identical with the underlying, alternating, CGT game.

**Question 1.** Consider the following BW SIMULTANEOUS TOPPLING DOMINOES game with ten rows of dominoes, where one row is represented by $(r, s)$ which means there are $r$ black dominoes followed by $s$ white dominoes:

$$(56, 7) + (37, 11) + (24, 15) + (20, 17) + (18, 18) + (16, 29) + (16, 14) + (12, 18) + (8, 19) + (8, 33)$$

*If you were Left, would you want to play and, if so, how would you play?*

## 2 Rules and Approaches to the Cheating-Robot Play Convention

One important aspect of (alternating) combinatorial game theory is that the games frequently decompose into components. Play does not have to alternate in the components, only in the whole game. The approach, as in [1, 4, 18], is to analyze the components first and use this information to reduce the analysis needed for the total game. We carry that over to the Cheating-Robot model. Because of the various ways in which simultaneous moves can be defined, care is needed when defining the concepts that are seemingly clear when generalizing from alternating combinatorial game theory. As is defined for alternating combinatorial games, a simultaneous combinatorial game $G$ is *short* if it is has finitely many sub-positions and there is no sequence of moves starting at $G$ that repeats a position (loop-free).

Our philosophy that both players must know when they have a move, demands that if both Left and Right have a move then there is a simultaneous move corresponding to both. Moreover, if there is a simultaneous move then there must be a Left and a Right move that give rise to it.
Definition 1. [13] Given a set of game positions $\Omega$, a simultaneous ruleset over $\Omega$ consists of three functions $L, R, S : \Omega \rightarrow 2^\Omega$. For $G \in \Omega$, $L(G)$ is the set of Left options, which we will denote as $G^L$, $R(G)$ is the set of Right options, denoted $G^R$, and $S(G)$ is the set of simultaneous options, denoted $G^S$. Moreover, $G^S$, is given by a matrix $M(G)$ where rows are indexed by the elements of $G^L$ and the columns by the elements of $G^R$. An entry $G^S_{i,j}$ will be the result of Left playing option $i$ and Right playing option $j$ in $G$. A game $G$ is called a terminal position if $G^S = \emptyset$.

Hence if Left has $m$ options and Right has $n$ options we have, 

$$M(G) = \begin{bmatrix}
G^R_1 & G^R_2 & \ldots & G^R_n \\
G^S_{1,1} & G^S_{1,2} & \ldots & G^S_{1,n} \\
G^S_{2,1} & G^S_{2,2} & \ldots & G^S_{2,n} \\
\vdots & \vdots & \ddots & \vdots \\
G^S_{m,1} & G^S_{m,2} & \ldots & G^S_{m,n}
\end{bmatrix}.$$ 

Consider the generic (non-cheating, simultaneous) game represented by the matrix in Figure 1. Looking at the extensive-form game (Figure 1) from Right’s perspective, since players are moving at the same time, Right does not know which node, (1) or (2), he is at when making his decision. In economic game theory terminology (see [3]), Right sees the encircled nodes as the same information set, as he does not know which one Left will choose. Consider this game but under the Cheating-Robot model. In Figure 2, the extensive-form is adjusted to account for Right’s knowledge of where Left will move. Since Right has more information within the Cheating-Robot model, his information set has split; they are now singletons. He has perfect information and can use this to his advantage. This also warrants changing the matrix to be a column vector where Right chooses a best response to Left’s choice.

In normal play alternating CGT, $G^L$ and $G^R$ are typically viewed as sets. However, in our context, it is more appropriate to view them as lists considering the correspondence between the Left, Right, and simultaneous options.
Definition 2. [Cheating-Robot] Given a set of game positions $\Omega$, a Cheating-Robot ruleset over $\Omega$ is a simultaneous ruleset except $M(G)$ is replaced by a column matrix $M_{cr}(G)$ whose rows are indexed by $G^L$ and whose $(i,1)$ entry is $G^{S_i} = [G^{S_i,1}, G^{S_i,2}, \ldots, G^{S_i,n}]$.

Although it is possible to have games with infinite options, this paper only considers finite games.

Definition 3. Let $\text{CR}$ be the set of short Cheating-Robot games.

A Cheating-Robot game is defined by its simultaneous options, but it is important to know the individual Left and Right moves. For this reason, we write $G = \{G^L \mid G^S \mid G^R\}$. For example, let $G$ be the bw simultaneous toppling dominoes position $G = \text{BWB}$. The Left and Right options for $G$ are $G^L = [\text{WB}, \ldots, \ldots, \text{BW}]$ and $G^R = [\ldots, \text{B}, \ldots]$, respectively. The simultaneous moves are summarized in Figure 3 and are pictured in Figure 4.

![Figure 3: $M_{cr}(G)$, where $G = \text{BWB}$.](image)

There are three outcomes.

Definition 4. [Winning Convention] Let $G$ be a terminal game. Left wins $G$ if $G^L \neq \emptyset$ and $G^S = G^R = \emptyset$; Right wins if $G^R \neq \emptyset$ and $G^S = G^L = \emptyset$; it is a draw if $G^L = G^S = G^R = \emptyset$.

We extend this to: $G \in \mathcal{L}$, or $o_{cr}(G) = \mathcal{L}$, if Left can force a win; $G \in \mathcal{R}$, or $o_{cr}(G) = \mathcal{R}$, if Right can force a win; otherwise $G \in \mathcal{D}$, or $o_{cr}(G) = \mathcal{D}$, the game is a draw, where $o_{cr}(G)$ is the outcome of $G$ under the Cheating-Robot model.
Figure 4: First level of the CR game tree of the **bw simultaneous toppling dominoes** position \( G = \text{BWB} \). The simultaneous option representation requires both a Left and a Right option. Right’s responses for each of Left’s choices indicate the resulting position after toppling a domino to the left (left slanted options) and toppling a domino to the right (right slanted options).

It will also be useful to include all the options in the game tree of \( G \). The *birthday* of \( G \), denoted \( b(G) \), is the depth of the game tree and forms the basis for the technique of *inducting on the options*. The game with no options is called 0, i.e., \( 0 = \{ \emptyset | \emptyset | \emptyset \} \). For brevity we replace \( \emptyset \) with \( \cdot \), i.e., \( 0 = \{ \cdot | \cdot | \cdot \} \). The outcome of a game can be evaluated by backtracking from the leaves of the game tree. This gives the Fundamental Theorem of Cheating-Robot games.

**Theorem 1.** Let \( G \in \text{CR} \). Then \( o_{cr}(G) \in \{ \mathcal{L}, \mathcal{D}, \mathcal{R} \} \).

**Proof.** The game \( \{ \cdot | \cdot | \cdot \} \) is in \( \mathcal{D} \). This forms the basis for induction. If \( G \neq 0 \) and \( G^S = \emptyset \) then either \( G^L \neq \emptyset \) and, by definition, \( G \in \mathcal{L} \), or \( G^R \neq \emptyset \) and \( G \in \mathcal{R} \).

We may now assume that \( G^S \neq \emptyset \). If, in each \( G^S_i \) there is an option in \( \mathcal{R} \) then Right can force a win, i.e., \( G \in \mathcal{R} \). If, in each \( G^S_i \) that does not contain an option in \( \mathcal{R} \), there is an option in \( \mathcal{D} \) then \( G \in \mathcal{D} \) since Right can force a draw and Left cannot do any better. If in each \( G^S_i \) there are only options in \( \mathcal{L} \) then \( G \in \mathcal{L} \) since Left can force a win. The result now follows by induction.

Just as in alternate play CGT, Cheating-Robot games decompose. The components form the *disjunctive sum*. The Left and Right options are now required for the formal definition.
Definition 5. The disjunctive sum of $G$ and $H$ is given by

$$G + H = \{(G + H)^L | (G + H)^S | (G + H)^R\} = \{G^L + H, G^H + H^L, G^H + H^S, H, G + H^S | G^R + H, G + H^R\}.$$  

For example, in the **bw simultaneous toppling dominoes** positions $BW + W$ means both Left and Right get to choose one component and move in it. We leave in duplications to recall that the entries correspond to specific options for the players.

$$(BW + W)^L = [-W + W, - + W]$$

$$(BW + W)^R = [\_ - + W, B_ - + W, BW + -, BW + -]$$

$$(BW + W)^S = [\_ - + W, \_ - + W, \_W + -, \_W + -, \_ + W, \_ + W, \_- + -, \_- + -]$$

Two observations should be made at this point. Firstly, in the disjunctive sum of two games, $G + H$, $G$ and $H$ can both be terminal but that does not imply that $G + H$ is terminal. Secondly, it is tempting to regard the lists of options as sets because then duplications would be eliminated. However, in any analysis the connection between the Left and the Right options and the resulting simultaneous options must be maintained.

As in the other theories of CGT, the disjunctive sum leads directly to an equivalence relation and to a partial order. Intuitively, the games $G$ and $H$ are equivalent if, in all disjunctive sums, replacing $G$ by $H$ does not affect the outcome. In other words, both players are indifferent to including $G$ or $H$.

Definition 6. Let $G, H \in CR$. The games $G$ and $H$ are equivalent, $G \equiv H$, if $o_{cr}(G + X) = o_{cr}(H + X)$ for all $X \in CR$; and $G$ is greater or equal to $H$, $G \succeq H$, if $o_{cr}(G + X) \geq o_{cr}(H + X)$ for all $X \in CR$.

A standard approach to proving $G \succeq H$ is to show that if Right can win $G + X$ then he can win $H + X$. To prove $G \equiv H$ it suffices to show that $G \succeq H$ and $H \succeq G$.

Theorem 2. The relation $\equiv$ is an equivalence relation on $CR$. The relation $\succeq$ is a partial order on $CR/Eq$.

The proofs are standard and follow from the properties of the outcome function. We only present a sketch of the proofs.

Proof. For any position $G$, let $G \in CR$, clearly $G \equiv G$ and $G \succeq G$.

Let $G, H, K \in CR$. Given $X \in CR$, if $o_{cr}(G + X) = o_{cr}(H + X)$ and $o_{cr}(H + X) = o_{cr}(K + X)$ then $o_{cr}(G + X) = o_{cr}(K + X)$. Thus $G \equiv K$ and it follows that $\equiv$ is an equivalence relation.

Similarly, given $X \in CR$, if $o_{cr}(G + X) \geq o_{cr}(H + X)$ and $o_{cr}(H + X) \geq o_{cr}(K + X)$ then $o_{cr}(G + X) \geq o_{cr}(K + X)$. Thus $G \succeq K$ and it follows that $\succeq$ is a partial order. \qed
Note that in the next result there is equality.

**Theorem 3.** Let $G, H, K \in \mathbb{CR}$. We have $G + H = H + G$ and $(G + H) + K = G + (H + K)$.

The proofs that $+$ is both commutative and associative are also standard and somewhat tedious so we omit them.

The usual desired order properties also hold.

**Theorem 4.** Let $A, B, C, D \in \mathbb{CR}$.

1. If $A \succeq B$ then $A + C \succeq B + C$;
2. If $A \succeq B$ and $C \succeq D$ then $A + C \succeq B + D$.

**Proof.** By definition, if $A \succeq B$ then for all $Y \in \mathbb{CR}$, $o_{cr}(A + Y) \geq o_{cr}(B + Y)$. For any $X \in \mathbb{CR}$, that $o_{cr}(A + C + X) \geq o_{cr}(B + C + X)$ holds follows from letting $Y = C + X$.

We have $o_{cr}(A + C + X) \geq o_{cr}(B + C + X)$. In a similar fashion, since $o_{cr}(C + Y) \geq o_{cr}(D + Y)$, letting $Y = B + X$, gives $o_{cr}(C + B + X) \geq o_{cr}(D + B + X)$. Combining the inequalities gives $o_{cr}(A + C + X) \geq o_{cr}(B + D + X)$. □

The partial order can be used to identify poor options and, since we are assuming best play, these can then be eliminated, as shown in the next three results. The first two are, respectively, what Left can eliminate and what Right can eliminate. The last result is a technical lemma: given a list of options from a single Left move, Right can replace all of the dominated options by his best option. The correspondence between the Left and Right options and the simultaneous options would no longer exist if dominated options had been eliminated.

**Theorem 5.** [Row elimination] For a game $G \in \mathbb{CR}$ and suppose that for some $i, j$,

1. $G^{L_i} \succeq G^{L_j}$; and
2. for all $r$ there exists $s$ with $G^{S_{i,r}} \succeq G^{S_{j,s}}$.

Let $G'$ be the same game as $G$ except the Left option $j$ has been removed then $G \equiv G'$.

**Proof.** In $G$, Left has the same plus more options than in $G'$ thus $G \succeq G'$.

Now we show $G' \succeq G$. Suppose Right can win (draw) $G' + X$. In $G + X$, Right mimics the winning (drawing) moves from $G' + X$. The exception is if Left moves to $G^{L_j} + X$. Right now considers Left moving from $G + X$ to $G^{L_i} + X$. If the winning Right move is to $G^{L_i} + X^R$ then in $G^{L_j} + X$ Right moves to $G^{L_j} + X^R$. Since $G^{L_i} \succeq G^{L_j}$, Right wins $G^{L_j} + X^R$.

If instead, a winning move is to $G^{S_{i,r}} + X$ then there is some $s$ with $G^{S_{i,r}} \succeq G^{S_{j,s}}$ and Right moves from $G^{L_j} + X$ to $G^{S_{j,s}} + X$ and wins. Thus $G' \succeq G$ and $G' \equiv G$. □
**Theorem 6.** [Column' elimination] For a game \( G = \{ G^L \mid G^S \mid G^R \} \), with \( G^L = \{ L_1, L_2, \ldots, L_m \} \), \( G^R = \{ R_1, R_2, \ldots, R_n \} \), and \( G^S_i = [G^{S_i,1}, G^{S_i,2}, \ldots, G^{S_i,n}] \), let \( H \) be the game obtained from \( G \) by deleting the \( R_n \) option.

That is, \( H^L = G^L \), \( H^R = \{ R_1, R_2, \ldots, R_{n-1} \} \), and \( H^{S_i} = G^{S_i} \setminus G^{S_i,n} \). In \( G \), if for every \( i \), there exists \( j_i \neq 1 \) such that \( G^{S_i,1} \geq G^{S_i,j_i} \), then \( G \equiv H \).

**Proof.** Since Right has more options in \( G \) than in \( H \) and Left has all the same options in both games, we have \( o_{cr}(H) \geq o_{cr}(G) \).

For the reverse implication, suppose that Right can win \( G + X \) we claim that Right can win \( H + X \). There is only one case where the moves in \( H + X \) are not the mimic of those in \( G + X \). That is, consider for a Left move \( H^{L_i} + X \), this is a Left move in \( G + X \), and a winning response is to \( G^{S_i,n} + X \). In \( H^{L_i} + X \), Right moves to \( G^{S_i,j_i} + X \). By assumption, \( G^{S_i,j_i} \leq G^{S_i,n} \), so since Right can win \( G^{S_i,n} + X \), Right can win \( G^{S_i,j_i} + X = H^{S_i,j_i} + X \). That is, Right can win \( H + X \). \( \square \)

**Theorem 7.** [Right’s dominated moves] For a game \( G \), let \( G^S_i = [G^{S_i,1}, G^{S_i,2}, \ldots, G^{S_i,n}] \) and suppose \( G^{S_i,1} \geq G^{S_i,2} \). Let \( G' \) be the game where \( M_{cr}(G') = M_{cr}(G) \) except \( G^{S_i,1} \) is replaced by \( G^{S_i,2} \). Then \( G \equiv G' \).

**Proof.** In \( G \), Right has the same plus more options than in \( G' \) thus \( G' \geq G \). For the reverse implication, suppose that Right can win (draw) \( G + X \). In \( G' + X \), Right mimics the winning (drawing) moves from \( G + X \). The exception is if Left moves to \( G^{L_i} \) and Right’s winning move in \( G + X \) is to \( G^{S_i,1} + X \). In \( G' + X \), Right now responds to \( G^{S_i,2} + X \) and since \( G^{S_i,1} \geq G^{S_i,2} \) Right wins \( G^{S_i,2} + X \). Thus \( G \equiv G' \). \( \square \)

Zero is appropriately named and note that there is equality not just equivalence in the next result.

**Lemma 1.** \( 0 + G = G \)

**Proof.** By definition of disjunctive sum we have

\[
0 + G = \{ (0 + G)^L \mid (0 + G)^S \mid (0 + G)^R \}
\]

but since 0 has no options this leaves

\[
0 + G = \{ 0 + G^L \mid 0 + G^S \mid 0 + G^R \}
\]

\[
= \{ G^L \mid G^S \mid G^R \},
\]

and the last equality follows by induction. \( \square \)

**Theorem 8.** Let \( G \) be a \( \mathbb{C} \mathbb{R} \) game. \( G \succ 0 \) if and only if \( G^L \neq \emptyset \) and \( G^R = \emptyset \).
Proof. Suppose that $G \succ 0$. By Definition 6 this means $o_{cr}(G + X) \geq o_{cr}(X)$ for all $X \in \mathbb{CR}$. Suppose Right doesn’t have a move in $G$. Since the game with no options is equal to zero and $G \succ 0$ then $G^L \neq \emptyset$.

Suppose $G$ has a Right move. Let $b(G) = n$ and let $X = \{-n - 1 \mid 0 \mid -n - 1\}$, and $o_{cr}(X) = D$. Consider Left’s moves:

- (i) Left plays in $G$, Right plays in $X$ and wins (since he plays through his $n+1$ moves and Left has at most $n$ moves); or
- (ii) Left plays in $X$, Right plays in $G$ and $b(G^R) < n$ and Right has $n + 1$ moves in $X^L$.

Now suppose $G^L \neq \emptyset$ and $G^R = \emptyset$. We need to show that if Right can win $G + X$ then Right can win $X$ for all $X \in \mathbb{CR}$. Suppose Right can win $G + X$ and Left plays in $X$ to $G + X^L$. Since $G^R = \emptyset$, Right responds to $G + X^S$. Since Right can win this game $G + X^S$, he can win $X^S$ by induction.$\square$

From the forward implication of Theorem 8 if $G^R \neq \emptyset$ then $G \neq 0$ and if $G^R = \emptyset$ and $G^L \neq \emptyset$ then again $G \neq 0$. That is, the equivalence class of 0 contains only 0. A further implication is that in $G + H$ if one of the option sets is none empty then $G + H \neq 0$.

Corollary 1. If $G \equiv 0$, then $G = 0$. Moreover, if $G + H \equiv 0$ then $G = H = 0$.

In CGT, the intuitive notion of the negative in normal play or ‘conjugate’ in misère play, is obtained by turning the board around, that is interchanging the options (and followers) of the two players. For the Cheating-Robot model, this is not completely possible. If Right is the Cheating Robot in $G$ and Left the Cheating Robot in $H$ then in $G + H$ play never starts as each waits for the other.

As Lemma 1 showed, there are no negatives in $\mathbb{CR}$ but we use the negative sign for the conjugate for convenience.

Definition 7. [Conjugate of $G$] Let $G$ be an $\mathbb{CR}$ game. Then $-G = \{-G^R \mid -G^S \mid -G^L\}$, but Right is still the Cheating Robot.

Lemma 2. Let $G \in \mathbb{CR}$ and $G \neq 0$ then $0 \succ G - G$.

Proof. By Corollary 1 $G - G \neq 0$. For any $X$, if Right can win (draw) $X$ then by playing the corresponding move in the $G$ and $-G$ pair then Right wins (draws) $G - G + X$ as well. Therefore $o_{cr}(X) \geq o_{cr}(G - G + X)$, which proves $0 \succ G - G$. $\square$

Consequently from Lemma 2 we have the following results immediately.

Corollary 2. Let $G$ be a $\mathbb{CR}$ game.

- If $G \prec 0$ then $o_{cr}(G) \leq D$. 

• If $G > 0$ then $o_{cr}(G) \geq D$.

A game $G$ is a dicot\(^3\) if either both players have a move or the game is over \([18]\). Dicot games are examples of cases where $0 > G$ does not mean that $o_{cr}(G) = \mathcal{R}$.

**Lemma 3.** Let $G \in \mathbb{CR}$ be a simultaneous dicot game and $G \neq 0$, then $o_{cr}(G) = D$ and $0 > G$.

**Proof.** Let $G \in \mathbb{CR}$ be a simultaneous play dicot. For any $X \in \mathbb{CR}$, if Right wins or draws $X$ then by replying locally in $G + X$, Right will always have a response to Left’s move. In the case of Right winning $X$, eventually Right will have a move in a follower of $X$ but Left will not. Thus $0 > G$. \(\square\)

We can now define and name some other special games.

**Definition 8.** [Integers] For $n > 0$, $n = \{n-1 \mid \cdot \}$; for $n < 0$, $n = \{\cdot \mid n+1\}$, where $0 = 0$.

For $n > 0$, $n$ is the game in which Left has $n$ moves and Right has none, but Left can only use the $n$ moves in a disjunctive sum.

In addition to dicots, $G = \{-1 \mid 0 \mid 0\}$ is an example of a game where $o_{cr}(G) = D$ (and we know $G \neq 0$) where $o_{cr}(n \cdot G) = \mathcal{R}$, $n > 1$.

**Theorem 9.** Let $i, j, k$ be positive integers.

1. $j + k = j + k$ and the same is true if $j$ and $k$ are negative integers.
2. $k > k - 1$. Also $-k + 1 > -k$;
3. $\{k-1 \mid k \mid k+i\} \equiv k$ and $\{k-i \mid -k \mid -k+1\} \equiv -k$.

**Proof.** (1) Note that this is an equality not an equivalence. In the next equations, the first equality is from the definition of disjunctive sum and the second by induction.

\[
\{k-1 \mid \cdot \} + \{j-1 \mid \cdot \} = \{k+j-1, k-1+j \mid \cdot \} = \{k+j-1 \mid \cdot \} = k+j
\]

A similar argument holds if $j$ and $k$ are negative.

(2) For any $X \in \mathbb{CR}$, suppose Right can win $1 + X$ we claim Right can win $X$. In $X$, Left can only move to $X^L$ then in $1 + X^L$ Right has a winning move to $1 + X^S$. Now by

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\(^3\)In normal play these are called all-small but the term refers to the fact that the value of the game is an infinitesimal. This is not true under other winning conventions and the term ‘dicotic’ was first coined for this class of games but in the literature authors have shortened it to dicot.
induction Right can win $X^S$. Moreover, if Right can win $X$ then, clearly, Right can win $-1 + X$ showing $0 \succ -1$.

Suppose Right can win $k + X$. In response to Left’s move to $k - 1 + X$, Right has a winning response $k - 1 + X^R$. Now in $k - 1 + X$, if Left moves to $k - 2 + X$ then Right moves to $k - 2 + X^R$. By induction, $k - 1 > k - 2$ and so Right wins $k - 2 + X^R$. If Left moves to $k - 1 + X^L$ then in $k + X^L$ Right has a winning move to $k + X^S$. By induction, Right wins $k - 1 + X^S$ if he wins $k + X^S$. Since Left has no moves in a negative integer then if Right can win $-k + 1 + X$ then he can win $-k + X$.

Note $k \neq k - 1$ since $k - k$ is a draw but $k - 1 - k$ is a Right win.

(3) We want to show \( \{k - 1 \mid \frac{\_\_}{\_\_} \mid k + i\} \equiv \frac{\_\_}{\_\_} \). Fix $k, i > 0$. Let $G = \{k - 1 \mid \frac{\_\_}{\_\_} \mid k + i\}$. We first show that $G \succ k$. Suppose Right can win $G + X$. Now in $k + X$ Left has two options:

i. $k - 1 + X$: Right considers Left moving to $G^L + X$ and Right has winning responses. If one is to $G^L + X = k - 1 + X^R$ then in $k - 1 + X$ Right can play to $k - 1 + X^R$ and win. If there are no such winning moves then, in $G^L + X$ Right wins by moving to $k + X$, i.e., Right wins $k + X$.

Now we show that $G \succeq k$. Suppose Right can win $k + X$. In $G + X$ Left has two moves.

i. $G^L + X$: in $k - 1 + X$, Right has a winning move to $k - 1 + X^R$. In $G^L + X$, Right can also move to the same position, $G^L + X^R$, and therefore Right wins.

ii. $G + X^L$: In $k + X^L$, Right has a winning move to $k + X^S$. Now, by induction, Right wins $G + X^S$.

Now let $G = \{-k - i \mid -k \mid -k + 1\}$. Suppose Right can win $-k + X$. In $G + X$, Left has two moves:

i. $G^L + X$: Here Right moves to $-k + X$ and wins by induction.

ii. $G + X^L$: If Right wins $-k + X^L$ by moving to $-k + X^S$ then Right wins $G + X$, by induction, by playing to $G + X^S$. If Right wins $-k + X^L$ by moving to $-k + 1 + X^L$ he wins $G + X^L$ by moving in $G$ to $-k + 1 + X^L$.

Suppose Right can win $G + X$. In $-k + X$, Left has one move to $-k + X^L$. If Right wins $G + X^L$ by moving to $G + X^S$ then Right wins $-k + X^S$ by induction. If Right wins $G + X^L$ by moving to $-k + 1 + X^L$ then he wins $-k + X^L$ by also moving to $-k + 1 + X^L$. \( \square \)
Observation 1. The CR games born by day 1 are: \( 0 = \{ \cdot | \cdot | \cdot \} \), \( \frac{1}{2} = \{0 | \cdot | \cdot \} \), \( -1 = \{ \cdot | \cdot | 0 \} \), and \( \overline{3} = \{0 | 0 | 0 \} \). This list is exhaustive as day 1 games can only have options born by day 0 and so options are either empty or 0. The partial order on these games is \( 1 \succ 0 \succ \frac{1}{2} \succ -1 \). Theorem 9 shows that \( 1 \succ 0 \succ -1 \). Lemma 3 shows that \( 0 \succ \frac{1}{2} \). Suppose Right can win \( \frac{1}{2} + X \) then Right can win \( -1 + X \) by playing the corresponding moves. The exception is if the winning response in \( \overline{3} + X \) is to \( 0 + X \) but then in \( -1 + X \) Right also moves to \( 0 + X \) and wins.

3 Simple Hot Games

In this section we build a sub-class of games in which both Left and Right want to move or the game is an integer. This sub-class is closed under disjunctive sum and options. It is a generalization of the positions found in Question 1. The theory presented here is applied to BW SIMULTANEOUS TOPPLING DOMINOES in Section 3.1.

Definition 9. Let \( a, b, c \) be integers then \( G = \{a | b | c\} \) is a simple hot game if \( a \geq 0 \geq c \) and \( a \geq b \geq c \). The sub-class \( \mathbb{CR}_{SH} \) consists of all disjunctive sums of simple hot games and integers.

For example,
\[
\{10 | 8 | -3\} + \{2 | -2 | -4\} + \{2 | \cdot | \cdot\} + \{\cdot | \cdot | -1\}.
\]
This sub-class has nicer properties than \( \mathbb{CR} \). Importantly, ‘integer’ games add the same as ‘addition’ integers and there are games equivalent to 0.

Lemma 4. Let \( a \) and \( c \) be integers. In \( \mathbb{CR}_{SH} \), \( a + c \equiv a + c \).

Proof. If \( a \) and \( c \) are both positive or both negative, the result follows from Theorem 9. We now suppose that \( a \geq 0 \geq c \) and we must prove that for all \( X \), \( o_{cr}(a + c + X) = o_{cr}(a + c + X) \). We will proceed on the birthday of the disjunctive sum of the simple hot games in \( X \).

First, suppose \( X \) is a disjunctive sum of integers—this is the base case. By Theorem 9 \( X \equiv p + q \) for some \( p \geq 0 \geq q \). In both \( a + c + X \) and \( a + c + X \), each player can only move in ‘their’ integers and Left has exactly \( a + p \) moves and Right \( c + q \). Thus \( o_{cr}(a + c + X) = o_{cr}(a + c + X) \).

Suppose that \( X \) is the disjunctive sum of integers and simple hot games. By Theorem 9 \( X \equiv p + q + X \) where \( X \) is the disjunctive sum of simple hot games. Now \( o_{cr}(a + c + X) = o_{cr}(a + p + c + q + X') \) and \( o_{cr}(a + p + c + q + X') = o_{cr}(a + p + c + q + X') \).

Thus, for the rest of the proof, we may now assume that \( X \) is not a disjunctive sum of integers.

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Next, we prove $1 + -1 \equiv 0$. Let $X \in \mathbb{CR}_{SH}$. First, we prove $o_{cr}(X) \geq o_{cr}(1 + -1 + X)$. If $o_{cr}(X) = L$ then the result is true. We may now assume that $o_{cr}(X) < L$. If Left plays to $-1 + X$ Right responds to $X$ and trivially $o_{cr}(X) \geq o_{cr}(X)$. If Left plays to $1 + -1 + X^L$ then in $X^L$ Right has a winning/drawing play to $X^S$. Right then plays $1 + -1 + X^L$ to $1 + -1 + X^S$ and the result follows by induction.

Suppose Right’s winning/drawing move is to $1 + X^L$. If $\{a | b | c\}$ is the summand in which Left moves then $1 + X^L \equiv 1 + a + X'$. Whereas if Right responded in $\{a | b | c\}$ then the resulting position is $1 + -1 + b + X' \equiv b + X'$ where the equivalence follows by induction. Since $a \geq b$ then $1 + a + X' \geq b + X'$ and Right wins/draws $1 + a + X'$ by assumption thus Right can win/draw $b + X'$. However, from $X^L$ Right can then respond to $b + X'$ thus $o_{cr}(1 + -1 + X) \geq o_{cr}(X)$.

Now suppose that $a \geq 1$ and $-1 \geq c$, then by Theorem 9 again,

$$a + c \equiv \sum_{i=1}^{a} 1 + \sum_{j=1}^{c} -1 \equiv 1 + -1 + \sum_{i=1}^{a-1} 1 + \sum_{j=1}^{c-1} -1$$

By the first part of the proof $1 + -1 \equiv 0$ thus

$$1 + -1 + \sum_{i=1}^{a-1} 1 + \sum_{j=1}^{c-1} -1 \equiv \sum_{i=1}^{a-1} 1 + \sum_{j=1}^{c-1} -1.$$ 

Pairing off the $1$ and $-1$ summands gives $a + c \equiv a + c$. 

Note: This result proves that integer games add as integer numbers so we drop the special notation.

**Definition 10.** Let the integer stop of $G$ be

$$IS(G) = \begin{cases} k, & \text{if } G \text{ is the integer } k; \\ \max_i \min_j IS(G^{S_{i,j}}) & \text{otherwise} \end{cases}$$

**Observation 2.** The integer stop is the sum of the integers when all of the summands have been reduced to integers. The integer stop is a refinement of the outcome since if $IS(G) > 0$, respectively $= 0$, $< 0$ then Left wins, the game is a draw, Right wins.

It is intuitively obvious that, in a disjunctive sum, nobody wants to play in an integer if there is a non-integer summand. We prove the result next, and it gives an insight into how Left needs to play.
Lemma 5. Let $d, a_i, b_i, c_i$ be integers with $a_i \geq 0 \geq c_i$, and $a_i > b_i > c_i$, $i = 1 \ldots n$. For all $i$, let $G_i = \{a_i \mid b_i \mid c_i\}$ be a simple hot game and let $G = d + \sum_{i=1}^{n} G_i$. Moreover, label the summands so that $a_1 - c_1 \geq a_i - c_i$ for all $i$. Neither Left nor Right can force $IS(G)$ if he/she plays in $d$.

Proof. By Lemma 4 we can assume that there is at most one integer in the disjunctive sum. We first consider $d \geq 0$.

Left has three different moves in $G = d + \{a_1 \mid b_1 \mid c_1\} + \sum_{i=2}^{n} G_i$. Let $I = IS(\sum_{i=2}^{n} G_i)$ and $I_j = IS(\sum_{i=2,i\neq j}^{n} G_i)$.

Left can move to $d - 1 + \{a_1 \mid b_1 \mid c_1\} + \sum_{i=2}^{n} G_i$ and Right moves to $H_1 = d - 1 + c_1 + \sum_{i=2}^{n} G_i$. Thus, by induction, $IS(H_1) = d - 1 + c_1 + I$.

Left can also move to $d + \{a_1 \mid b_1 \mid c_1\} + \sum_{i=2}^{n} G_i$. Right has two moves. He can move $H_2 = d + b_1 + \sum_{i=2}^{n} G_i$ and $IS(H_2) = d + b_1 + I$. Since $b_1 > c_1$ then $IS(H_2) \geq IS(H_1)$. Right can also move to $H_3 = d + a_1 + c_j + \sum_{i=2,i\neq j}^{n} G_i$. In $H_1$ if we assume that Right only responds in $G_j$ after a Left move in $G_j$ then

$$IS(H_1) \leq d - 1 + c_1 + b_j + IS \left( \sum_{i=2,i\neq j}^{n} G_i \right).$$

By assumption, $a_1 - c_1 \geq a_j - c_j$ which gives $a_j + c_1 \leq a_1 - c_j$, and, since $b_j < a_j$ then $c_1 + b_j < c_1 + a_j$. Together they give $c_1 + b_j < c_1 + a_j$ and

$$IS(H_1) \leq d - 1 + c_1 + b_j + IS \left( \sum_{i=2,i\neq j}^{n} G_i \right)$$

$$< d + a_1 + c_j + IS \left( \sum_{i=2,i\neq j}^{n} G_i \right) = IS(H_3).$$

Thus in all cases, Left does better if she does not play in the integer.

Now assume that $d < 0$. Let $I_j = IS(\sum_{i=1,i\neq j}^{n} G_i)$. Left can move to $d + \{a_j \mid b_j \mid c_j\} + \sum_{i=1,i\neq j}^{n} G_i$ and suppose Right responds to $K_1 = d + 1 + a_j + \sum_{i=1,i\neq j}^{n} G_i$. Now $IS(K_1) = d + 1 + a_j + I_j$. However, if Right responds in $G_j$ which gives $K_2 = d + b_j + \sum_{i=1,i\neq j}^{n} G_i$ then $IS(K_2) = d + b_j + I_j$. Since $a_j > b_j$, $K_2$ is a better option than $K_1$, i.e., Right does better by not playing in the integer.

\[ \square \]

Lemmas 4 and 5 show that in the analysis of a disjunctive sum $G = \sum_{i=1}^{n} G_i$ we can assume that each summand is not an integer. As hinted at in the proof of Lemma 5 there is an ordering of the summands, i.e., $i < j$ then $a_i - c_i > a_j - c_j$, that is, all the Human (Left)
needs to know for her strategy. When Left makes a move, Right has to find a corresponding summand in which to respond.

We show that there are optimal strategies, for both players, Left’s is the aforementioned ordering, Right’s corresponds to the solutions of an optimization problem.

Let \(K_G\) be a complete graph, with loops and weighted edges, where \(V(K_G) = \{1, 2, \ldots, n\}\) and for \(i < j\), weight \(w(ij) = a_i - c_j\) and \(w(ii) = b_i\). The weight corresponds to the result of Left playing in \(G_i\) and Right responding \(G_j\), including responding in the same summand.

Consider the strategies where Left plays in the summand corresponding to the least index of the summands not yet having been played. In these, when Left plays in \(i\), Right’s response in \(m(i)\) has \(i \leq m(i)\). Conversely, a matching \(M\) (loops are allowed) on \(K_G\) corresponds to a strategy for Right, called the matching strategy, as follows:

- Suppose \(ii \in M\) then if Left plays in \(i\) then Right responds in \(i\);
- Suppose \(ij \in M\) then if Left plays on \(i\) then Right plays in \(j\).

**Theorem 10.** Let \(a_i > b_i > c_i\) and \(a_i > 0 > c_i\), \(i = 1 \ldots n\), be integers and let \(G_i = \{a_i \mid b_i \mid c_i\}\) and \(G = \sum_{i=1}^{n} G_i\). The optimal strategies for Right in \(G\) correspond to matching strategies from the solutions to the minimum weighted matching on \(K_G\).

The general minimum weight matching (dual to the maximum weight matching problem) problem can be solved in \(O(mn)\) with some improvements for integer weight to either \(mn^{3/4} \ln N\) and \(Nm\sqrt{n}\). If \(N\) is small then this can be reduced to \(mn^{1+\epsilon}\). See [11] for a survey.

We already have one feasible strategy/matching, that is, the one in which Right responds in the same summand. This is the \(\Delta\)-strategy and let \(Sc(\Delta)\) be the score of Right playing \(\Delta\). Instead of proving Theorem 10 we show that a graph with fewer edges is all that Right needs to consider in order to improve on the \(\Delta\)-strategy.

Let \(D_G\) be a graph, with weighted edges, where \(V(D_G) = \{1, 2, \ldots, n\}\) and for \(i < j\), weight \(w(ij) = a_i + c_j - b_i - b_j\). If Right responds in the same summand then when playing \(i\) and \(j\) the score is \(b_i + b_j\). If Right responds in \(j\) to Left playing \(i\) the score is \(a_i + c_j\) and the difference (positive being good for Right) is \(w(ij)\). Part of the proof needs to show that when Left plays \(i\) Right can only play in other summands with indices greater than \(i\).

**Theorem 11.** Let \(a_i > b_i > c_i\) and \(a_i > 0 > c_i\), \(i = 1 \ldots n\), be integers and let \(G_i = \{a_i \mid b_i \mid c_i\}\) and order the \(G_i\) so that if \(i < j\) then \(a_i - c_i > a_j - c_j\). Let \(G = \sum_{i=1}^{n} G_i\). The optimal strategies of \(G\) correspond to the order strategy for Left and for Right, a matching strategy from the solutions to the maximum weighted matching problem on \(D_G\).

**Proof.** Let \(W\) be the weight of the maximum weighted matching problem on \(D_G\).

A matching strategy \(\alpha\) for \(G\) gives a matching in \(D_G\), which is a candidate for the solution to the maximum weighted matching problem in \(D_G\). Thus \(Sc(\alpha) \geq Sc(\Delta) - W\).
Now let $M$ be a maximum weight matching on $D_G$. If Left and Right play the matching strategy from $M$ the score is $Sc(\Delta) - W$.

Suppose that Left does not play an order strategy but chooses the least index, say $i$, in a matching edge. At worst Right can play $m(i)$ and obtain the score $Sc(\Delta) - W$. He might be able to do even better. If Left choose $m(i)$ instead of $i$ then Right responds in $i$. The improvement in Right’s saving is $a_{m(i)} + c_i - b_i - b_{m(i)}$. If Left plays in $i$ then Right responds in $m(i)$ and Right’s saving is $a_i + c_{m(i)} - b_i - b_{m(i)}$. Since $a_i - c_i \geq a_{m(i)} - c_{m(i)}$ then $a_{m(i)} + c_i - b_i - b_{m(i)} < a_i + c_{m(i)} - b_i - b_{m(i)}$ thus Right does better if Left plays $m(i)$.

Thus Right can guarantee at least $Sc(\Delta) - W$ by using a solution to the maximum weighted matching problem and Left can guarantee no worse then $Sc(\Delta) - W$ by playing a matching strategy. Left needs to play the components in order, as described using the above algorithm.

3.1 Toppling Dominoes—A Case Study

We consider positions that consist of $p$ black dominoes followed by $q$ white dominoes. Recall that the Left dominoes are on the left and we write $(p, q)$ and, for brevity, refer to this game as TD$_2$. It seems clear that both players should move to topple only one of their dominoes and topple it toward the opponent’s dominoes. This will prove that any TD$_2$ position is a simple hot game, which is the next result.

Lemma 6. The TD$_2$ position $(p, q)$ is equivalent to $\{p - 1 | p - q | 1 - q\}$ in CR$_{SH}$.

Proof. Consider the Left options of $G = (p, q)$. If Left topples a domino to the left, then toppling the leftmost domino leaves more Left options. If Left topples a domino to the right, similarly, toppling the rightmost domino is best. The same holds for Right’s options.

From the integer stops, toppling your domino which is closest to your opponent’s, in the direction of your opponent, is best. Now we can use Theorem\[11\] We already know that $G = (p_1, q_1) + (p_2, q_2) + \ldots + (p_n, q_n)$ is indexed so that if $i < j$ then $p_i + q_i \geq p_j + q_j$ Left should play in $(p_1, q_1)$. The problem is to determine Right’s best response.

Right has the matching strategy from Theorem\[11\] however, we can further reduce the number of edges in the graph so that Right might be able to handle or at least get close to an optimal strategy with back-of-the-envelope or no visible calculations at all. This is for the case where Right is not a robot but a human with inside information.

The next observation allows us to reduce the number of edges in $D_G$.

Lemma 7. Let $G = (p_1, q_1) + (p_2, q_2) + \ldots + (p_n, q_n)$. Suppose there are indices $i < j < k < \ell$ with $i \ell, jk \in D_G$ then there is an optimal solution that does not include $i \ell$.

Proof. Assume the components of $G$ are ordered by largest sum and assume that $i < j < k < \ell$ with $i \ell, jk \in D_G$. Suppose that $i \ell$ and $jk$ are in an assignment solution for $D_G$. The
saving for Right from matching $i$ with $\ell$ and $j$ with $k$ is $p_\ell - q_\ell + p_k - q_j = p_\ell - q_j + p_k - q_i$, which is the same as the contribution of the edges $ik$ and $j\ell$.

Assume now that whenever $i\ell$ is in an assignment solution for $D_G$ that $jk$ is not.

First, suppose neither of $j$ and $k$ are in a matching edge. Let $r$ be the savings from playing the assignment solution for $D_G$ except for those involving $i, \ell$. The matching strategy gives a result of $Sc(\Delta) - r - (p_\ell - q_\ell)$. Now, responding in $k$ to Left’s move in $j$ gives the result $Sc(\Delta) - r - (p_\ell - q_\ell) - (p_k - q_k)$. Since $p_j - q_k \geq 0$ then $p_j - q_k + p_k - q_j \geq p_k - q_j$ and thus matching $j$ and $k$ gives smaller solution contradicting our assumption.

Second, suppose that $h < i$ and $hj$ is a matching edge. The improvement over $Sc(\Delta)$ that $i\ell, hj$ contribute is $p_j - q_h + p_\ell - q_i$ which is the same as $ij$ and $h\ell$. The argument is similar if $h > \ell$ or if $k$ is in a matching edge.

Thus in all cases, if $i < j < k < \ell$ with $i\ell, jk \in D_G$ there is an optimal solution which does not include $i\ell$.

For a disjunctive sum $G = (p_1, q_1) + (p_2, q_2) + \ldots + (p_n, q_n)$ it is possible for $|E(D_G)|$ to be quadratic in $n$. For example, if $p_1 > k > q_i$, for some $k$ and for all $i$, then $D_G$ is a complete graph. However, the reductions from Lemma 7 result in graph where the number of edges is linear in the number of summands.

**Lemma 8.** Let $G$ be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ with edges satisfying the condition: there is no set of four vertices $\{v_i, v_j, v_k, v_\ell\}$, $i < j < k < \ell$ with both $v_i v_\ell$ and $v_j v_k$ being edges. Then $|E(G)| \leq 2n - 3$.

**Proof.** For $n \leq 3$ the result is true since any graph on that number of vertices has, respectively, at most 0, 1, 3 edges.

Let $v_p v_r$ be an edge in $G$. The technique is to transform $G$ into 3 subgraphs, keeping the number of edges constant at each step. One subgraph will be on vertices $v_1$ to $v_p$, one on $v_p$ to $v_r$, and one on $v_r$ to $v_n$. We will bound the total number of edges on the first and third subgraphs by induction and count the edges in the second.

By assumption, if $p < q < r$ and if $v_q v_p$ is an edge then either $t \leq p$ or $q \leq t$.

Let $q$ be the least index $q > p$ and consider the set $N = \{v_t : v_tv_q \in E(G) \text{ and } t < q\}$. Note that having $x, y \in N$ and both adjacent to $v_p$ violates the assumption on the edges.

First, suppose $p < r - 1$. Let $G'$ have the same vertices and edges as $G$ except for $t \in N$, $v_tv_q$ is replaced by $v_tv_p$. The edge $v_tv_p$ may be duplicated. If $G'$ did not satisfy the assumption then neither would $G$. Therefore we repeat the process. We apply the corresponding process to vertices $v_q$ adjacent to vertices with indices greater than $r$. Let $G''$ be the final result. It satisfies the assumption. The induced graph on $\{v_1, \ldots, v_p\}$ has at most $2p - 3 + (r - p - 1)$ edges, the first part by induction and the second counts the duplicated edges. Similarly, the induced graph on $\{v_r, \ldots, v_n\}$ has at most $2(n + 1 - r) - 3 + (r - p - 1)$ edges. The vertices in the induced graph on $\{v_p, \ldots, v_r\}$ only have edges to
\( v_p \) and \( v_r \), i.e., the total number of edges is \( 2(r-p-1) \) from the vertices not equal to \( v_p \) or \( v_r \) and an extra edge between \( v_p \) and \( v_r \). The total number of edges is bounded above by
\[
[2p-3+(r-p-1)]+[2(n+1-r)-3]+[(r-p-1)+2(r-p-1)+1] = 2(n+p-r)-3 < 2n-3.
\]

If \( p = r - 1 \) then the process of the previous paragraph gives
\[
|E(G')| \leq [2p-3+1] + [2(n+1-p-1)-3+1] + [1] = 2n - 3.
\]

Thus \( G \) has at most \( 2n - 3 \) edges. \( \square \)

Let \( G = (p_1, q_1) + (p_2, q_2) + \ldots + (p_n, q_n) \) and let the graph \( D'_G \) be \( D_G \) reduced by Lemma[7] until no further reductions are possible.

**Corollary 3.** Let \( G = (p_1, q_1) + (p_2, q_2) + \ldots + (p_n, q_n) \) then \( D'_G \) has at most \( 2n - 3 \) edges.

Returning to our example, Left will want to order the components in decreasing order of sums,

\[(56, 7) + (37, 11) + (16, 29) + (8, 33) + (24, 15) + (20, 17) + (18, 18) + (16, 14) + (12, 18) + (8, 19)\].

Right on the other hand will want to determine an optimal solution to the optimization problem. The \( \Delta \)-strategy yields an overall score of 34; meaning the sum is in favour of Left, when Left plays in components based on largest sum and Right responds in the same component. Exploring alternatives, Right examines what happens when he plays in matched components instead of always following the \( \Delta \)-strategy. The graph with all edges is shown in Figure [5]. In order to simplify the number of edges within the graph before looking for an optimal solution, we use Lemma[7] the graph then reduces as shown in Figure [6]. From here, we have two triangles, several components which are optimized by using the \( \Delta \)-strategy, and a few edges which are not simplified. We must check all possibilities to determine which is optimal for Right.

Right wants to minimize the overall score, and so we examine the matchings and choose the lowest overall score (optimal for Right). Recall \( Sc(\Delta) = 34 \).

- If \( (37, 11) \) and \( (24, 15) \) are paired off then \( (56, 7) \) will be matched to \( (16, 29) \) and \( (20, 17) \) to \( (18, 18) \). This yields a matching score of \( Sc(\Delta) - 13 - 9 - 1 = 11 \).
- If \( (37, 11) \) and \( (20, 17) \) are paired off then \( (56, 7) \) will be matched to \( (16, 29) \) and \( (24, 15) \) to \( (18, 18) \). This yields a matching score of \( Sc(\Delta) - 9 - 9 - 3 = 13 \).
- The remaining possibility is to optimize a matching in each of the triangles, i.e., \( (37, 11) \) with \( (56, 7) \) and \( (24, 15) \) with \( (20, 17) \), which gives \( Sc(\Delta) - 30 - 5 = -1 \).
Figure 5: Options for Right via matchings which are better than the $\Delta$-strategy.

Figure 6: The reduced graph after applying Lemma 7.
Both players can quickly realize that even if Left playing her best strategy, Right will choose to match over the triangles to obtain an overall score of \(-1\). Thus, even though \(Sc(\Delta)\) is a relatively large (in this context) positive number, Right can win, therefore the answer to Question 1 is that Left should gracefully decline to play.

**Question 2.** Is there a constant time algorithm to determine Right’s best response to Left’s move in \(TD_2\)?

4 Discussion

The class \(\mathbb{CR}\) is reminiscent of the class of short misère games in that (i) the equivalence class of 0 has one element, and (ii) there are no inverses. However, we showed that there is a sub-class with more algebraic structure. As in the other models of combinatorial games, the important questions are (i) to find a test for \(G \succ H\) that involves only the followers of \(G\) and \(H\); (ii) find sub-classes in which some/all games have inverses and in these find reductions that result in a canonical form; (iii) extend the class of combinatorial games to those where the winner has the greater score. In alternating combinatorial game theory, games that have positive or non-negative incentive have the richest algebraic structure, see Milnor [16] and [1, 4, 18] respectively. Is there a rich theory in the corresponding simultaneous hot games, that is where \(G^L \succeq G^S \succeq G^R\) for every \(G^L\), \(G^S\) and \(G^R\)?

The Cheating-Robot model is based on zero-sum matrix games. Is there an interesting theory if the Cheating-Robot model is applied to non-zero-sum matrix games?

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