One-loop calculation of mass dependent $\mathcal{O}(a)$ improvement coefficients for the relativistic heavy quarks on the lattice

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We carry out the one-loop calculation of mass dependent $\mathcal{O}(a)$ improvement coefficients in the relativistic heavy quark action recently proposed, employing the ordinary perturbation theory with the fictitious gluon mass as an infrared regulator. We also determine renormalization factors and improvement coefficients for the axial-vector current at the one-loop level. It is shown that the improvement coefficients are infrared finite at the one-loop level if and only if the improvement coefficients in the action are properly tuned at the tree level.

1. INTRODUCTION

While, in principle, lattice QCD allows a precise determination of weak matrix elements associated with the $B$ and $D$ mesons, $c$- and $b$-quarks are too heavy to treat directly in current numerical simulations due to large $\mathcal{O}(m_Q a)$ errors.

Recently, a new relativistic approach to control $m_Q a$ errors was proposed from the viewpoint of the on-shell $\mathcal{O}(a)$ improvement program assuming $m_Q \gg \Lambda_{QCD}$. The action is given by

\[ S_q = a^4 \sum_x \bar{q}(x) \left[ \gamma_0 D_0 + \nu \gamma_i D_i + m_0 \right] \]

\[ -ar_t/2D_0^2 - ar_s/2D_i^2 - aigc_B/4\sigma_{ij}F_{ij} \]

\[ -aigc_E/2\sigma_{0i}F_{0i} \] \(q(x)\),

where we are allowed to choose \(r_t = 1\) and other four parameters \(\nu, r_s, c_E\) and \(c_B\) are analytic functions of \(m_Q a\) and the gauge coupling \(g\). Once a nonperturbative adjustment of four parameters \(\nu, r_s, c_E\) and \(c_B\) is achieved, the remaining cutoff effects are reduced to be $\mathcal{O}(a\Lambda_{QCD})^2$.

In this report we present a perturbative determination of $c_E$ and $c_B$ up to the one-loop level (see Ref. 2 for details). In addition we calculate the renormalization factors and the $\mathcal{O}(a)$ improvement coefficients for the axial-vector current at the one-loop level.

2. DETERMINATION OF $c_E$ AND $c_B$ AT THE ONE-LOOP LEVEL

We determine $c_E = c_E^{(0)} + g^2 c_E^{(1)}$ and $c_B = c_B^{(0)} + g^2 c_B^{(1)}$ from the on-shell quark-quark scattering amplitude. In the massless case, $c_{SW}$ is successfully determined up to the one-loop level employing the conventional perturbation theory with the fictitious gluon mass as an infrared regulator.\cite{1}. We extend this calculation to the massive case. Since the tree level determination of $c_E$ and $c_B$ in the massive case is already performed in Ref. \cite{3}, next step is to calculate six types of one-loop diagrams depicted in Fig. 1.

We first consider to calculate $c_E^{(1)}$. Without the space-time symmetry the general form of the off-shell vertex function at the one-loop level is written as

\[ \Lambda_k^{(1)}(p, q, m) \]

\[ = \gamma_k F_{1}^k + \gamma_k \{ \not{q} F_{2}^k + \not{p} F_{3}^k \} + \{ q F_{4}^k + \not{p} F_{5}^k \} \gamma_k \]

\[ + \not{q} \gamma_k \not{p} F_{6}^k + \not{q} \gamma_k \not{p} F_{7}^k + \not{q} \gamma_k \not{p} F_{8} \]

\[ + (p_k + q_k) \left[ H_{L}^k + \not{q} H_{R}^k + \not{q} H_{S}^k \right] \]

\[ + (p_k - q_k) \left[ G_{1}^k + \not{q} G_{2}^k + \not{q} G_{3}^k + \not{q} G_{4}^k + \not{q} G_{5}^k \right] \]

\[ + \mathcal{O}(a^2), \]

where $\Lambda_k = \Lambda_k^{(0)} + g^2 \Lambda_k^{(1)} + \mathcal{O}(g^4)$ and $\not{q} = \sum_{\alpha=0}^{3} p_{\alpha} \gamma_{\alpha}$, $\not{q} = \sum_{\alpha=0}^{3} q_{\alpha} \gamma_{\alpha}$, $\gamma_0 = p_0 \gamma_0$, $\gamma_0 = q_0 \gamma_0$. $p$ and $q$ are incoming and outgoing quark...
momenta, respectively. The coefficients $F_i^k$, $G_i^k$ and $H_i^k$ are functions of $p^2$, $q^2$, $p \cdot q$ and $m$. Sandwicking $\Lambda_{k}^{(1)}(p, q, m)$ by the on-shell quark states $u(p)$ and $\bar{u}(q)$, which satisfy $\bar{q}u(p) = im_p u(p)$ and $\bar{u}(q)q = im_q \bar{u}(q)$, the matrix element is reduced to

$$\bar{u}(q)\Lambda_{k}^{(1)}(p, q, m)u(p)$$

$$= \bar{u}(q)\gamma_{\mu}u(p)\{F_{1}^{k} + im_p 2F_{2}^{k} - m_p^2 F_{3}^{k}\}$$

$$+(p_k + q_k)\bar{u}(q)u(p)\{H_{1}^{k} + im_p 2H_{2}^{k} - m_p^2 H_{4}^{k}\}$$

$$+ O(a^2),$$

where we use the relations among $F_i^k$, $G_i^k$ and $H_i^k$ constrained by the charge conjugation symmetry.

The relevant term for the determination of $c_{B}$ is the last one in eq. (3), whose coefficient can be extracted by setting $p = p^* \equiv (p_0 = im_p, p_i = 0)$ and $q = q^* \equiv (q_0 = im_q, q_i = 0)$ in eq. (2):

$$[H_{1}^{k} + im_p (H_{2}^{k} + H_{3}^{k}) - m_p^2 H_{4}^{k}]^{\text{latt}}_{p = p^*, q = q^*}$$

$$= \frac{1}{8} \text{Tr}\left\{\left[\frac{\partial}{\partial p_k} + \frac{\partial}{\partial q_k}\right] \Lambda_{k}^{(1)}(p^*, q^*, m)(\gamma_4 + 1)\right\}_{i \neq k}$$

(4)

$$- \left[\frac{\partial}{\partial p_i} - \frac{\partial}{\partial q_i}\right] \Lambda_{k}^{(1)}(p^*, q^*, m)(\gamma_4 + 1)\gamma_i \gamma_k.\right\}$$

We should remark that the second term in eq. (3) contains both the lattice artifact of $O(p_k/m, q_k/a)$ and the physical contribution of $O(p_k/m, q_k/m)$. The parameter $c_{B}$ is determined to eliminate the lattice artifact of $O(p_k/a, q_k/a)$:

$$\frac{c_{B}^{(1)} - r_{s}^{(1)}}{2} = [H_{1}^{k} + im_p 2H_{2}^{k} - m_p^2 H_{4}^{k}]^{\text{latt}}_{p = p^*, q = q^*}$$

$$- Z_{q}^{(0)} [H_{1}^{k} + im_p 2H_{2}^{k} - m_p^2 H_{4}^{k}]^{\text{cont}}_{p = p^*, q = q^*}.\right.$$}

The infrared behavior of eq. (4) is investigated by expanding the inner loop momentum in Fig. 1 around zero. The infrared divergence is found to be

$$\left(- \frac{1}{2N_c} + \frac{N_c}{2}\right) (c_{B}^{(0)} - r_{s}^{(0)})L - \left(\frac{N_c}{2}\right) \frac{Z_{q}^{(0)}}{m_p^{(0)}}L$$

where $L = -1/(16\pi^2)\ln|\lambda^2 a^2|$ with $\lambda$ the fictitious gluon mass. If the tree level improvement coefficients are properly tuned as $c_{B}^{(0)} = r_{s}^{(0)}$, we are left with $- (N_c/2)(Z_{q}^{(0)}/m_p^{(0)})L$, which is exactly the same as the infrared divergence in the continuum theory with the correct normalization factor $Z_{q}^{(0)}$.

In the same way, $c_{E}^{(1)}$ is determined free from the infrared divergence. Results for $c_{B}^{(1)}$ and $c_{E}^{(1)}$ with the plaquette action are plotted in Fig. 2. As expected $c_{B}^{(1)}$ and $c_{E}^{(1)}$ approaches to the massless value for $c_{SW}^{(1)}$ as $m_p/a$ vanishes.

3. $O(a)$ IMPROVEMENT OF THE AXIAL-VECTOR CURRENT

We consider the heavy-light axial-vector current $A_{\mu}$. Without space-time symmetry, the renormalized operator $A_{\mu}^{\text{latt}, R}(x)$ with the $O(a)$ improvement is written as

$$A_{\mu}^{\text{latt}, R}(x) = Z_{A}^{\text{latt}} [\bar{q}(x)\gamma_{\mu}\gamma_{5}Q(x)\right]$$
Figure 3. $\Delta_{A_{\mu}}$ in heavy-light system with the plaquette gauge action as a function of heavy quark pole mass $m_Q^{(0)}$.

$$
-g^2 c_{A_{\mu}}^+ \partial_\mu (\bar{q}(x) \gamma_5 Q(x)) - g^2 c_{A_{\mu}}^- \partial_\mu (\bar{q}(x) \gamma_5 Q(x))
$$

$$
-g^2 c_{A_{\mu}} \{ \bar{Q}(x) \gamma_i \gamma_5 \}
$$

$$
-g^2 c_{H_{\mu}} \bar{q}(x) \gamma_i \gamma_5 \{ \bar{Q}(x) \}
$$

$$
-g^2 c_{L_{\mu}} \{ \bar{\chi}(x) \gamma_i \gamma_5 \}
$$

(5)

where $Z_{A_{\mu}}^{\text{latt}}$ and $c_{A_{\mu}}^{(+,-,H,L)}$ depend on the quark masses $m_Q$ and $m_q$. $\partial_\mu^+$ and $\partial_\mu^-$ are defined as $\partial_\mu^+ = \partial_\mu^\rightarrow + \partial_\mu^\leftarrow$ and $\partial_\mu^- = \partial_\mu^\rightarrow - \partial_\mu^\leftarrow$. With the aid of equation of motion we can choose $c_{H_{0}}^+ = c_{A_{0}}^- = 0$.

To extract $Z_{A_{\mu}}^{\text{latt}}$ and $c_{A_{\mu}}^{(+,-,H,L)}$ at the one-loop level from the general form of the vertex function, we have to use both decay and scattering processes. Except for the physical contribution, all parameters are infrared finite, once $\nu^{(0)}$, $r^{(0)}$, $c_{E}^{(0)}$ and $c_{B}^{(0)}$ are properly tuned. In Figs. [3][4] we show numerical results for $\Delta_{A_{\mu}}$ and $c_{A_{\mu}}^{(+,-,H,L)}$ in heavy-light system. The former is defined by $Z_{A_{\mu}}^{\text{latt}} / Z_{A_{\mu}}^{\text{MS}} = \sqrt{Z_{Q}^{(0)} (1 - g^2 \Delta_{A_{\mu}})}$, where we take $m_q = 0$ for the light quark. We observe anticipated features that $c_{A_{\mu}}^{(-,H,L)}$ vanishes as the heavy quark mass decreases, while $\Delta_{A_{\mu}}$ and $c_{A_{\mu}}^+$ become close to their corresponding massless values obtained in Refs. [3][4].

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Figure 4. $c_{A_{0}}^{(+,-),H,L}$ in heavy-light system with the plaquette gauge action.

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