COMPUTING MOTIVIC ZETA FUNCTIONS ON LOG SMOOTH MODELS

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Abstract. We give an explicit formula for the motivic zeta function in terms of a log smooth model. It generalizes the classical formulas for snc-models, but it gives rise to much fewer candidate poles, in general. This formula plays an essential role in recent work on motivic zeta functions of degenerating Calabi-Yau varieties by the second-named author and his collaborators. As a further illustration, we explain how the formula for Newton non-degenerate polynomials can be viewed as a special case of our results.

1. Introduction

Denef and Loeser’s motivic zeta function is a subtle invariant associated with hypersurface singularities over a field $k$ of characteristic zero. It can be viewed as a motivic upgrade of Igusa’s local zeta function for polynomials over $p$-adic fields. The motivic zeta function is a power series over a suitable Grothendieck ring of varieties, and it can be specialized to more classical invariants of the singularity, such as the Hodge spectrum. The main open problem in this context is the so-called monodromy conjecture, which predicts that each pole of the motivic zeta function is a root of the Bernstein polynomial of the hypersurface.

One of the principal tools in the study of the motivic zeta function is its explicit computation on a log resolution [DL01, 3.3.1]. While this formula gives a complete list of candidate poles of the zeta function, in practice most of these candidates tend to cancel out for reasons that are not well understood. Understanding this cancellation phenomenon is the key to the monodromy conjecture. The aim of this paper is to establish a formula for the motivic zeta function in terms of log smooth models instead of log resolutions (Theorem 4.3.1). These log smooth models can be viewed as partial resolutions with toroidal singularities. Our formula generalizes the computation on log resolutions, but typically gives substantially fewer candidate poles (Proposition 4.4.2). A nice bonus is that, even for log resolutions, the language of log geometry allows for a cleaner and more conceptual proof of the formula for the motivic zeta function in [NS07], and to extend the results to arbitrary characteristic (Corollary 4.3.2). We will also indicate in Section 7.2 how our formula gives a conceptual explanation for the determination of the set of poles of the motivic zeta function of a curve singularity; this is the only dimension in which the monodromy conjecture has been proven completely. A special case of our formula has appeared in the literature under a different guise, namely, the calculation of motivic zeta functions of hypersurfaces that are non-degenerate with respect to their Newton

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polyhedron \([\text{Gu02}]\). We will explain the precise connection (and correct some small errors in \([\text{Gu02}]\)) in Section 7.3.

Our results apply not only to Denef and Loeser’s motivic zeta function, but also to the motivic zeta functions of degenerations of Calabi-Yau varieties that were introduced in \([\text{HN11}]\). Here the formula in terms of log smooth models is particularly relevant in the context of the Gross-Siebert program on toric degenerations and Mirror Symmetry, where log smooth models appear naturally in the constructions. We have already applied our formula to compute the motivic zeta function of the degeneration of the quartic surface in \([\text{NOR16}]\) and we are considering the higher-dimensional case in an ongoing project. Our formula is also used in an essential way in \([\text{HN16}]\) to prove an analog of the monodromy conjecture for a large and interesting class of degenerations of Calabi-Yau varieties (namely, the degenerations with monodromy-equivariant Kulikov models).

The main results in this paper form a part of the first author’s PhD thesis \([\text{Bu15a}]\). They were announced in \([\text{Bu15b}]\).

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Notations and conventions. Unless explicitly stated otherwise, all logarithmic structures in this paper are defined with respect to the Zariski topology. We will discuss a generalisation of our results to the Nisnevich topology in Section 5.2. All the log schemes are assumed to be Noetherian and fs (fine and saturated). Log schemes will be denoted by symbols of the form \(\mathcal{X}^i\), and the underlying scheme will be denoted by \(\mathcal{X}\). We write \(M_{\mathcal{X}^i}\) for the sheaf of monoids on \(\mathcal{X}^i\). We will follow the convention in \([\text{GR15}]\) and speak of regular log schemes and smooth morphisms between log schemes instead of log regular log schemes and log smooth morphisms. When we refer to geometric properties of the underlying schemes instead, this will always be clearly indicated.

2. MONOIDS AND FANS

2.1. Generalities. For general background on the algebraic theory of monoids, we refer to \([\text{GR15}]\) §4]. All monoids are assumed to be commutative, and we will usually use the additive notation \((M, +)\), with neutral element 0. For every monoid \(M\), we denote by \(M^\times\) the submonoid of invertible elements of \(M\). The monoid \(M\) is called sharp when \(M^\times = \{0\}\). We denote by \(M^\sharp\) the sharpification of \(M\), that is, the quotient \(M/M^\times\). We set \(M^+ = M\setminus M^\times\), the unique maximal ideal of \(M\). We write \(M^{\text{gp}}\) for the groupification of \(M\), and \(M^{\text{int}}\) for the integral monoid associated with \(M\) (that is, the image of the canonical morphism \(M \to M^{\text{gp}}\)). We denote by \(M^{\text{sat}}\) the saturation of \(M^{\text{int}}\) in \(M^{\text{gp}}\). For every monoid \(M\), we denote by \(M^\vee\) its dual monoid: \(M^\vee = \text{Hom}(M, \mathbb{N})\). We will also consider the submonoid \(M^\vee,\text{loc}\) of \(M^\vee\) consisting of local homomorphisms \(M \to \mathbb{N}\), that is, morphisms \(\varphi : M \to \mathbb{N}\) such that \(\varphi(m) \neq 0\) for every \(m \in M^+\).

When working with monoids, it is useful to keep in mind the following more concrete description: if \(M\) is a fine, saturated and torsion free monoid, then we can
identify $M$ with the monoid of integral points of the convex rational polyhedral cone $\sigma$ generated by $M$ in the vector space $M^{\mathbb{R}} \otimes_{\mathbb{Q}} \mathbb{R}$. The monoid $M$ is sharp if and only if $\sigma$ is strictly convex.

A monoidal space is a topological space $T$ endowed with a sheaf $M$ of monoids. If $(T, M)$ is a monoidal space, we will denote by $T\# = T$ equipped with the sheaf $M\# = M/M \times$ for every open subspace $U$ in $T$. This construction applies, in particular, to logarithmic schemes $X^\dagger$ when viewed as monoidal spaces $(X, M_{X^\dagger})$.

### 2.2. The root index

**Definition 2.2.1.** Let $M$ be a fine monoid endowed with a morphism of monoids $\varphi : \mathbb{N} \to M$. The root index of $\varphi$ is defined to be 0 if $\varphi(1)$ is invertible. Otherwise, it is the largest positive integer $\rho$ such that the residue class of $\varphi(1)$ in $M\#$ is divisible by $\rho$.

Note that such a largest $\rho$ exists because $M\#$ is a submonoid of the free abelian group of finite rank $(M^{\mathbb{Z}})^{\mathbb{Z}}$. The importance of the root index lies in the following properties.

**Proposition 2.2.2.** Let $M$ be a fine and saturated monoid, and let $\varphi : \mathbb{N} \to M$ be a morphism of monoids. Denote by $\rho$ the root index of $\varphi$. For every positive integer $d$, we consider the monoid $M(d) = \left( M \oplus_{\mathbb{N}} \frac{1}{d} \mathbb{N} \right)$.

1. The monoid $M(d)$ is fine for every $d > 0$.
2. If $d$ divides $\rho$, then $M^\times \to (M(d)^{\text{sat}})^\times$ is an isomorphism, and the morphism $\varphi_d : \frac{1}{d}\mathbb{N} \to M(d)^{\text{sat}}$

has root index $\rho/d$.
3. If $d$ is prime to $\rho$, then the morphism

$$M^\times \to (M(d)^{\text{sat}})^\times$$

is an isomorphism. In particular, if $M$ is sharp, then so is $M(d)^{\text{sat}}$.

**Proof.**

1. It is obvious that $M(d)$ is finitely generated. It is also integral because we can apply the criteria of [Ka89 4.1] to the morphism $\mathbb{N} \to (1/d) \mathbb{N}$.

2. Assume that $d$ divides $\rho$. We may suppose that $M$ is sharp, because the morphism

$$(M(d)^{\text{sat}})^\times \to (M(d)^{\text{sat}})^\times$$

is an isomorphism by the universal properties of sharpification, coproduct and saturation. Let $m$ be an element of $M$ such that $\rho m = \varphi(1)$. Then

$$d((\rho/d)m - \varphi_d(1/d)) = 0,$$

so that $(\rho/d)m - \varphi_d(1/d)$ is a unit in $M(d)$ and $(\rho/d)m = \varphi_d(1/d)$ in $(M(d)^{\text{sat}})^\times$. Using the universal properties of the coproduct, saturation and sharpification, together with the fact that $M$ is sharp and saturated, we obtain a morphism of
monoids $(M(d)^{\text{sat}})^{\sharp}_{\pm} \to M$ that sends the residue class of $\varphi_d(1/d)$ to $(\rho/d)m$ and that is inverse to $M \to (M(d)^{\text{sat}})^{\sharp}$. It follows at once that

$$\varphi_d : \frac{1}{d}\mathbb{N} \to M(d)^{\text{sat}}$$

has root index $\rho/d$.

Assume that $d$ is prime to $\rho$ and let $x$ be an invertible element in $M(d)^{\text{sat}}$. We must show that $x$ lies in $M$. Since $M \to M(d)^{\text{sat}}$ is exact by [GR15, 4.4.42(vi)], it is enough to prove that $x \in M_{\text{sp}}$. We can write $x$ as $(m,i/d)$ with $m \in M_{\text{sp}}$ and $i \in \{0, \ldots, d-1\}$. Since $M$ is saturated, the element $dx$ lies in $M$, and hence in $M^\times$ because we can apply the same argument to the inverse of $x$. This means that $dm = -i\varphi(1)$ in $M^\times$. But $d$ is prime to $\rho$, so that $d$ divides $i$. Hence, $i = 0$ and $x \in M_{\text{sp}}$.

3. Smooth log schemes

3.1. Fans and log stratifications. Let $\mathcal{X}^\dagger$ be a regular log scheme, and consider its associated fan $F(\mathcal{X}^\dagger)$ in the sense of [Ka94, §10]. This is a sharp monoidal space whose underlying topological space is the subspace of $\mathcal{X}$ consisting of the points $x$ such that $M_{\mathcal{X}^\dagger,x}^+$ generates the maximal ideal of $\mathcal{O}_{\mathcal{X},x}$. The sheaf of monoids on $F(\mathcal{X}^\dagger)$ is the pullback of the sheaf

$$M_{\mathcal{X}^\dagger} = M_{\mathcal{X}^\dagger}/\mathcal{O}_{\mathcal{X}}^\times$$

on $\mathcal{X}$. The natural morphism of monoidal spaces $F(\mathcal{X}^\dagger) \to (\mathcal{X}^\dagger)^{\sharp}$ has a canonical retraction $\pi : (\mathcal{X}^\dagger)^{\sharp} \to F(\mathcal{X}^\dagger)$ that sends a point $x$ of $\mathcal{X}$ to the point of $F(\mathcal{X}^\dagger)$ that corresponds to the prime ideal of $\mathcal{O}_{\mathcal{X},x}$ generated by $M_{\mathcal{X}^\dagger,x}^+$. With the help of this retraction, we can enhance the construction of the Kato fan to a functor from the category of regular log schemes to the category of fans, by sending a morphism of regular log schemes $\varphi : \mathcal{Y}^\dagger \to \mathcal{X}^\dagger$ to the morphism of fans

$$F(\mathcal{Y}^\dagger) \longrightarrow (\mathcal{Y}^\dagger)^{\sharp} \overset{h^\dagger}{\longrightarrow} (\mathcal{X}^\dagger)^{\sharp} \overset{\pi}{\longrightarrow} F(\mathcal{X}^\dagger).$$

See [GR15 §10.6] for additional background.

For every point $\tau$ of $F(\mathcal{X}^\dagger)$, we denote by $r(\tau)$ the dimension of the monoid $M_{F(\mathcal{X}^\dagger),\tau}$, and we call this number the rank of $\tau$. The fiber $\pi^{-1}(\tau)$ is an irreducible locally closed subset of $\mathcal{X}$ of pure codimension $r(\tau)$, and it is a regular subscheme of $\mathcal{X}$ if we endow it with its reduced induced structure [GR15, 10.6.9(iii)]. We denote this subscheme by $E(\tau)^{\circ}$, and we write $E(\tau)$ for its schematic closure in $\mathcal{X}$. The collection of subschemes

$$\{E(\tau)^{\circ} \mid \tau \in F(\mathcal{X}^\dagger)\}$$

is a stratification of $\mathcal{X}$, which is called the log stratification of $\mathcal{X}^\dagger$. For every point $x$ of $E(\tau)^{\circ}$, the characteristic monoid $M_{\mathcal{X}^\dagger,x}^+$ is canonically isomorphic to $M_{F(\mathcal{X}^\dagger),\tau}$. We say that $E(\tau)^{\circ}$ is a stratum of rank $r(\tau)$.

The boundary of $\mathcal{X}^\dagger$ is the locus of points where the log structure is non-trivial; this is a closed subset of $\mathcal{X}$ of pure codimension one, and it is equal to the union of strata $E(\tau)^{\circ}$ such that $M_{F(\mathcal{X}^\dagger),\tau} \neq 0$. We denote by $D$ the reduced Weil divisor supported on the boundary of $\mathcal{X}^\dagger$. Then the log structure on $\mathcal{X}^\dagger$ coincides with the divisorial log structure induced by $D$ [Ka94, 11.6]. If we denote by $E_i$, $i \in I$ the prime components of $D$, then a point $\tau$ of $\mathcal{X}$ belongs to $F(\mathcal{X}^\dagger)$ if and only if it is a...
Example 3.1.1. Let \( X \) be a quasi-compact regular scheme and let \( D \) be a strict normal crossings divisor on \( X \). Let \( X^\dagger \) be the log scheme that we obtain by endowing \( X \) with the divisorial log structure induced by \( D \). Then \( X^\dagger \) is log regular and its boundary coincides with \( D \). Conversely, if \( X^\dagger \) is a regular log scheme, then the underlying scheme \( X \) is regular if and only if \( M_{F(X^\dagger),\tau} \) is isomorphic to \( \mathbb{N}^{\tau} \) for every \( \tau \) in \( F(X^\dagger) \) \cite[10.5.35]{GR15}; in that case, the boundary divisor \( D \) of \( X^\dagger \) has strict normal crossings.

Let \( F' \) be a fine and saturated proper subdivision of the fan \( F(X^\dagger) \) in the sense of \cite[9.7]{Ka94}. Such a subdivision gives rise to a proper étale morphism of log schemes \( h : Y^\dagger \to X^\dagger \) such that \( F(Y^\dagger) \) is isomorphic to \( F' \) over \( F(X^\dagger) \). Moreover, \( h \) is an isomorphism over the log trivial locus of \( X^\dagger \). More precisely, the discriminant locus of \( h : Y \to X \) is the union of strata \( E(\tau) \) in \( X \) such that the morphism of monoidal spaces \( F' \to F(X^\dagger) \) is not an isomorphism over any open neighbourhood of \( \tau \) in \( F(X^\dagger) \). Let \( \tau' \) be a point of \( F(Y^\dagger) \) and denote by \( \tau \) its image in \( F(X^\dagger) \). Denote by \( N \) the kernel of the surjective morphism of monoids \( M_{F(X^\dagger),\tau}^\text{red} \to M_{F(Y^\dagger),\tau'}^\text{red} \). It follows immediately from the construction of the morphism \( h : Y^\dagger \to X^\dagger \) in the proof of \cite[9.9]{Ka94} that \( E(\tau')^o \) is a torsor over \( E(\tau)^o \) with translation group

\[
\text{Spec } \mathbb{Z}[N] \cong G_{\tau(\tau)-\tau(\tau')}^m,\mathbb{Z}.
\]

3.2. Log modifications and ramified base change. Let \( R \) be a complete discrete valuation ring with residue field \( k \) and quotient field \( K \). We write \( S^\dagger \) for the scheme \( S = \text{Spec } R \) endowed with its standard log structure (the divisorial log structure induced by the closed point \( \text{Spec } k \)). We fix a uniformizer \( t \) in \( R \). For every positive integer \( n \) we denote by \( R[n] \) the extension \( R[u]/(u^n - t) \) of \( R \). We write \( S(n)^\dagger \) for the scheme \( S(n) = \text{Spec } R(n) \) with its standard log structure.

Let \( X^\dagger \) be a smooth log scheme over \( S^\dagger \). Such a log scheme is also regular, by \cite[8.2]{Ka94}, so that we can apply the constructions from Section 3.1 to \( X^\dagger \). Let \( \tau \) be a point in \( F(X^\dagger) \). We say that \( \tau \) is a vertical point if it is contained in the special fiber \( X_\kappa \) of \( X \); otherwise, we call \( \tau \) horizontal. Assume that \( \tau \) is vertical. Then the structural morphism \( X^\dagger \to S^\dagger \) induces a local morphism of monoids

\[
\varphi : \mathbb{N} \to M_{F(X^\dagger),\tau}.
\]

We define the root index \( \rho(\tau) \) to be root index of this morphism \( \varphi \).

We set \( \rho = \rho(\tau) \) and we denote by \( Y^\dagger \) be the fibered product

\[
X^\dagger \times_{S^\dagger} S(\rho)^\dagger
\]

in the category of fine and saturated log schemes. Then \( Y^\dagger \) is smooth over \( S(\rho)^\dagger \) because smoothness is preserved by fs base change. The underlying scheme \( Y \) is the normalization of \( X \times_{S} S(\rho) \), because it is normal (by regularity of \( Y^\dagger \)) and the morphism \( Y \to X \times_{S} S(\rho) \) is finite and birational (it is an isomorphism over the locus where the log structure is trivial). We set

\[
\bar{E}(\tau)^o = \left( Y \times_{X} E(\tau)^o \right)_{\text{red}}.
\]
This is a union of logarithmic strata of $\mathcal{Y}^\dagger$, each of which has characteristic monoid

$$(M_{F(\mathcal{X}^\dagger)},\tau \oplus \frac{1}{\rho} \mathbb{N})^{\text{sat,}^\dagger}$$

(the proof of [Na97, 2.1.1] remains valid for log structures on the Zariski site). By Proposition 2.2.2(2), we know that the natural morphism

$$M_{F(\mathcal{X}^\dagger),\tau} \to (M_{F(\mathcal{X}^\dagger),\tau \oplus \frac{1}{\rho} \mathbb{N})^{\text{sat,}^\dagger}$$

is an isomorphism.

**Example 3.2.1.** Let $\mathcal{X}^\dagger$ be a smooth log scheme over $S^\dagger$. Assume that the underlying scheme $\mathcal{X}$ is regular. We write

$$\mathcal{X}_k = \sum_{i \in I} N_i E_i$$

where $E_i$, $i \in I$ are the prime components of $\mathcal{X}_k$ and the coefficients $N_i$ are their multiplicities. Let $\tau$ be a vertical point of $F(\mathcal{X}^\dagger)$ and let $J$ be the set of indices $j \in I$ such that $\tau$ lies on $E_j$. Then there exists an isomorphism of monoids $M_{F(\mathcal{X}^\dagger),\tau} \cong \mathbb{N}^h \times \mathbb{N}^h$ for some integer $h \geq 0$ such that $\varphi(1) = (\langle N_j \rangle_{j \in J}, 0)$. Here $h$ is the number of irreducible components of the boundary of $\mathcal{X}^\dagger$ that pass through $\tau$ and that are horizontal, i.e., not contained in the special fiber $\mathcal{X}_k$. The root index $\rho(\tau)$ is the greatest common divisor of the multiplicities $N_j$, $j \in J$. If $\rho = \rho(\tau)$ is not divisible by the characteristic of $k$, then $\tilde{E}(\tau)^o \to E(\tau)^o$ has a canonical structure of a $\mu_\rho$-torsor, which is described explicitly in [Ni13, §2.3].

The following results constitute a key step in the calculation of the motivic zeta function.

**Lemma 3.2.2.** Let $\mathcal{X}^\dagger$ be a smooth log scheme over $S^\dagger$ and let $\tau$ be a vertical point of $F(\mathcal{X}^\dagger)$ of root index $\rho = \rho(\tau)$. Let $m$ be any positive multiple of $\rho$ and denote by $\mathcal{Z}^\dagger$ be the fibered product

$$\mathcal{X}^\dagger \times_{S^\dagger} S(m)^\dagger$$

in the category of fine and saturated log schemes. Then the natural morphism

$$(\mathcal{Z} \times_{\mathcal{X}} E(\tau)^o)_{\text{red}} \to \tilde{E}(\tau)^o$$

is an isomorphism.

**Proof.** We set

$$\mathcal{Y}^\dagger = \mathcal{X}^\dagger \times_{S^\dagger} S(\rho)^\dagger,$$

as before. We have already recalled that the log scheme $\mathcal{Y}^\dagger$ is regular and that $\tilde{E}(\tau)^o$ is a union of logarithmic strata with characteristic monoid

$$M = (M_{F(\mathcal{X}^\dagger),\tau} \oplus \frac{1}{\rho} \mathbb{N})^{\text{sat,}^\dagger} \cong M_{F(\mathcal{X}^\dagger),\tau}.$$

The monoid $M$ is endowed with a local morphism $\varphi : (1/\rho)\mathbb{N} \to M$ induced by $\mathcal{Y}^\dagger \to S(\rho)^\dagger$, and the morphism $\varphi$ has root index 1 by Proposition 2.2.2(2).

Locally at every point of $\tilde{E}(\tau)^o$, we can find a chart for $\mathcal{Y}^\dagger$ of the form $\mathcal{Y}^\dagger \to \text{Spec} \mathbb{Z}[M]$, by the proof of [GR15, 10.1.36(i)]. Hence, we can also find a chart for
the morphism $\mathcal{Y}^\dagger \to S(\rho)^\dagger$ of the form

$$
\begin{array}{ccc}
\mathcal{Y}^\dagger & \longrightarrow & \text{Spec } \mathbb{Z}[M \times \mathbb{Z}] \\
\downarrow & & \downarrow \\
S(\rho)^\dagger & \longrightarrow & \text{Spec } \mathbb{Z}[\frac{1}{\rho}N]
\end{array}
$$

where $(1/\rho)N \to M \times \mathbb{Z}$ is a morphism of monoids such that the composition with the projection $M \times \mathbb{Z} \to M$ coincides with $\varphi$. If we set

$$
N = (M \times \mathbb{Z}) \oplus \frac{1}{\rho}N,
$$

then, locally around every point of $\bar{E}(\tau)^\circ$, the underlying scheme of the fs fibered product $Z^\dagger$ is given by

$$
Z = \left(\mathcal{Y} \times_{R(\rho)} R(m)\right) \times_{\mathbb{Z}[N]} \mathbb{Z}[N^\text{sat}],
$$

and $(Z \times_{X} E(\tau)^\circ)_{\text{red}}$ is the zero locus of the maximal ideal $(N^\text{sat})^+ = N^\text{sat} \setminus (N^\text{sat})^\times$. Hence, the $E(\tau)^\circ$-scheme $(Z \times_{X} E(\tau)^\circ)_{\text{red}}$ is isomorphic to

$$
\bar{E}(\tau)^\circ \times_{\mathbb{Z}[N]} \mathbb{Z}[(N^\text{sat})^\times] \cong \bar{E}(\tau)^\circ,
$$

where the last isomorphism follows from Proposition 3.2.2.

**Proposition 3.2.3.** Let $\mathcal{X}^\dagger$ be a smooth log scheme over $S^\dagger$. Let $\psi : F' \to F$ be a fine and saturated proper subdivision of $F = F(\mathcal{X}^\dagger)$, and denote by $h : (\mathcal{X}'^\dagger) \to \mathcal{X}^\dagger$ the corresponding morphism of log schemes. Let $\tau'$ be a point of $F'$ and set $\tau = \varphi(\tau')$. Then there exists a natural morphism of $E(\tau)^\circ$-schemes

$$
\bar{E}(\tau')^\circ \to \bar{E}(\tau)^\circ
$$

such that $\bar{E}(\tau')^\circ$ is a $\mathbb{G}_{m,k}^{r(\tau') - r(\tau)}$-torsor over $\bar{E}(\tau)^\circ$.

**Proof.** Let $m$ be a positive integer that is divisible by both $\rho(\tau)$ and $\rho(\tau')$ and set

$$
Z^\dagger = \mathcal{X}^\dagger \times_{S^\dagger} S(m)^\dagger, \quad (Z')^\dagger = (\mathcal{X}')^\dagger \times_{S^\dagger} S(m)^\dagger.
$$

By Lemma 3.2.2, the morphism $h_m : (Z')^\dagger \to Z^\dagger$ obtained from $h$ by fs base change induces a morphism of $E(\tau)^\circ$-schemes $E(\tau')^\circ \to E(\tau)^\circ$.

We will prove that the morphism $h$ induced by the subdivision $\psi$ is compatible with fs base change, in the following sense. The refinement $\psi : F' \to F$ induces a refinement

$$
\psi_m : F(Z^\dagger) \times_{F'} F' \to F(Z^\dagger)
$$

where the fibered product is taken in the category of fine and saturated fans. We claim that the morphism of log schemes induced by this refinement is precisely the morphism $h_m : (Z')^\dagger \to Z^\dagger$. Since $E(\tau)^\circ$ is a union of logarithmic strata of rank $r(\tau)$ in $Z^\dagger$ and $E(\tau')^\circ$ is a union of logarithmic strata of rank $r(\tau')$ in $(Z')^\dagger$, this implies that $E(\tau')^\circ$ is a $\mathbb{G}_{m,k}^{r(\tau') - r(\tau)}$-torsor over $E(\tau)^\circ$.

It remains to prove that $h_m$ is indeed the morphism induced by the refinement $\psi_m$. The morphism induced by $\psi_m$ is characterized by the following universal property [Ka94, 9.9]: it is a final object in the category of logarithmic schemes
\( W^\dagger \) endowed with a morphism \( W^\dagger \to Z^\dagger \) and a morphism of monoidal spaces \( \pi' : W^\dagger \to F(Z^\dagger) \times_{F'} F' \) such that the diagram

\[
\begin{array}{ccc}
W^\dagger & \longrightarrow & Z^\dagger \\
\pi' \downarrow & & \downarrow \pi \\
F(Z^\dagger) \times_{F'} F' & \longrightarrow & F(Z^\dagger)
\end{array}
\]

commutes. If \( W^\dagger \) is such a final object, then we have a canonical morphism \( (Z')^\dagger \to W^\dagger \) of log schemes over \( Z^\dagger \). Conversely, applying the universal properties for the morphism \( (X')^\dagger \to X^\dagger \) and the fs base change to \( S(m)^\dagger \), we obtain a morphism \( W^\dagger \to (Z')^\dagger \) of log schemes over \( Z^\dagger \). These two morphisms are mutually inverse, so that \( W^\dagger \) is isomorphic to \( (Z')^\dagger \) over \( Z^\dagger \).

For the applications in Section 7 we will also need the following result, which relates log regularity and log smoothness.

**Proposition 3.2.4.** Assume that \( k \) has characteristic zero. Let \( X^\dagger \) be a log scheme of finite type over \( S^\dagger \) (with respect to the étale topology) such that \( X \) is flat over \( S \). Then \( X^\dagger \) is smooth over \( S^\dagger \) if and only if \( X^\dagger \) is regular.

**Proof.** As we have already recalled, every smooth log scheme over \( S^\dagger \) is regular by [Ka94, 8.2]. So we only need to prove the converse implication. Assume that \( X^\dagger \) is regular. Since \( k \) has characteristic zero, we can take arbitrary roots of invertible functions on \( X \) locally in the étale topology on \( X \). This implies that, étale locally around every geometric point \( x \) on \( X_k \), we can find a chart for \( X^\dagger \to S^\dagger \) if of the form

\[
\begin{array}{ccc}
X^\dagger & \longrightarrow & \text{Spec } Z[M] \\
\downarrow & & \downarrow \\
S^\dagger & \longrightarrow & \text{Spec } Z[N]
\end{array}
\]

where \( M = M^\dagger_{X^\dagger} \times_{X^\dagger} x \) and \( N \to M \) is the morphism of monoids induced by \( X^\dagger \to S^\dagger \). The morphism \( N \to M \) is injective by flatness of \( X \) over \( S \), and the morphism of schemes

\[
X \to \text{Spec } Z[M] \times_{\text{Spec } Z[N]} S
\]

is smooth over a neighbourhood of \( \tau \) by the local description of regular log schemes in [Ka94, 3.2(1)]. Now it follows from Kato’s logarithmic criterion for smoothness [Ka89, 3.5] that \( X^\dagger \to S^\dagger \) is smooth.

Beware that Proposition 3.2.4 does not extend to the case where \( k \) has characteristic \( p > 0 \). The problem is that we cannot take \( p \)-th roots of all invertible functions locally in the étale topology. A sufficient condition for log smoothness is given by the following statement. Let \( X^\dagger \) be a regular log scheme of finite type over \( S^\dagger \) (with respect to the étale topology) such that \( X \) is flat over \( S \). Suppose moreover that \( k \) is perfect, the log structure on \( X^\dagger \) is vertical, \( X_k \) is smooth over \( K \) and the multiplicities of the components in the special fiber are prime to \( p \). Then \( X^\dagger \) is smooth over \( S^\dagger \). This follows from the same argument as in the proof of Proposition 3.2.4.
4. Motivic zeta functions

We denote by $R$ a complete discrete valuation ring with residue field $k$ and quotient field $K$. We assume that $k$ is perfect and we fix a uniformizer $t$ in $R$. For every positive integer $n$, we write $R(n) = R[u]/(u^n - t)$ and $K(n) = K[u]/(u^n - t)$. We write $S^1$ and $S(n)^1$ for the schemes $S = \text{Spec } R$ and $S = \text{Spec } R(n)$ endowed with their standard log structures.

4.1. Grothendieck rings and geometric series. If $R$ has equal characteristic, then for every noetherian $k$-scheme $X$, we denote by $\mathcal{M}_X$ the Grothendieck ring of varieties over $X$, modified ring of varieties by its universally homeomorphic $X$-schemes of finite type – see [NS11, §3.8]. In the calculation of the motivic zeta function, we will need to consider some specific geometric series in $L^{-1}$. The standard technique is to pass to the completion $\hat{\mathcal{M}}_X$ of $\mathcal{M}_X$ with respect to the dimensional filtration. However, since it is not known whether the completion morphism $\mathcal{M}_X \to \hat{\mathcal{M}}_X$ is injective, we will use a different method to avoid any loss of information. We start with an elementary lemma.

Lemma 4.1.1. Let $M$ be a sharp, fine and saturated monoid of dimension $d$. For every morphism of monoids $u : M \to \mathbb{N}$, we denote by $u^\text{gp} : M^\text{gp} \to \mathbb{Z}$ the induced morphism of groups. Let $m$ be an element of $M^\text{gp}$ and let $n$ be an element of $M \setminus \{0\}$. Let $u_1, \ldots, u_r$ be the generators of the one-dimensional faces of $M^\lor$, and denote by $I$ the set of indices $i$ in $\{1, \ldots, r\}$ such that $u_i(n) > 0$. We assume that $u_j^\text{gp}(m) = 1$ for every $j \not\in I$. Then the series

$$\sum_{u \in M^\lor, \text{loc}} L^{-u^\text{gp}(m)} T^{u(n)}$$

in the variables $L$ and $T$ lies in the subring

$$\mathbb{Z}[L, L^{-1}, T] \left[ \frac{T}{1 - L^{-u^\text{gp}(m)} T^{u(n)}} \right]_{i \in I}$$

of $\mathbb{Z}[L^{-1}, T][L]$.

Proof. A sharp, fine and saturated monoid is called simplicial if its number of one-dimensional faces is equal to its dimension. We can subdivide $M^\lor$ into a fan of simplicial monoids without inserting new one-dimensional faces, and such a subdivision gives rise to a partition of $M^\lor, \text{loc}$. Thus we may assume from the start that $M^\lor$ is simplicial, so that $d = r$ and $u_1, \ldots, u_r$ form a basis for the $\mathbb{Q}$-vector space $(M^\lor)^\text{gp} \otimes_\mathbb{Z} \mathbb{Q}$. Denote by $P$ the fundamental parallelepiped

$$P = \{ \lambda_1 u_1 + \ldots + \lambda_r u_r \in M^\lor, \text{loc} \mid \lambda_i \in \mathbb{Q} \cap (0, 1) \}$$

in $M^\lor, \text{loc}$. Then $P$ is a finite set, and we have

$$\prod_{u \in P} = (L - 1)^{r-1} \left( \sum_{u \in P} L^{-u^\text{gp}(m)} T^{u(n)} \right)^{r-1} \prod_{i=1}^r \frac{1}{1 - L^{-u_i^\text{gp}(m)} T^{u_i(n)}}.$$

Now the result follows from the assumption that for every $i$, either $u_i(n) > 0$ or $u_i^\text{gp}(m) = 1$; note that at most $r - 1$ of the values $u_i(n)$ vanish, because $n \neq 0$. ■
Keeping the notations and assumptions of Lemma 4.1.1 we define

$$(L - 1)^{d-1} \sum_{u \in M^\vee, loc} L^{-u^{gp}(m)} T^u(n)$$

as the value of

$$(L - 1)^{d-1} \sum_{u \in M^\vee, loc} L^{-u^{gp}(m)} T^u(n)$$

at $L = \mathbb{L}$. Lemma 4.1.1 guarantees that this is a well-defined element of $\mathcal{M}_k[T]$.

### 4.2. Definition of the motivic zeta function

Let $\mathcal{X}$ be an $R$-scheme of finite type with smooth generic fiber $\mathcal{X}_K$, and let $\omega$ be a volume form on $\mathcal{X}_K$ (that is, a nowhere vanishing differential form of maximal degree on each connected component of $\mathcal{X}_K$). A Néron smoothing of $\mathcal{X}$ is a morphism of finite type $h : \mathcal{Y} \to \mathcal{X}$ such that $\mathcal{Y}$ is smooth over $R$, $h_K : \mathcal{Y}_K \to \mathcal{X}_K$ is an isomorphism, and the natural map $\mathcal{Y}(R') \to \mathcal{X}(R')$ is a bijection for every finite unramified extension $R'$ of $R$. Such a Néron smoothing always exists, by [BLR90 3.1.3]. For every connected component $C$ of $\mathcal{Y}_k$, we denote by $\text{ord}_C \omega$ the unique integer $a$ such that $t^{-a} \omega$ extends to a relative volume form on $\mathcal{Y}$ locally around the generic point of $C$.

**Definition 4.2.1** (Loeser-Sebag). The motivic integral of $\omega$ on $\mathcal{X}$ is defined by

$$\int_{\mathcal{X}} |\omega| = \sum_{C \in \pi_0(\mathcal{Y}_k)} [C] L^{- \text{ord}_C \omega} \in \mathcal{M}_{\mathcal{X}_k}$$

where $\mathcal{Y} \to \mathcal{X}$ is any Néron smoothing and $\pi_0(\mathcal{Y}_k)$ is the set of connected components of $\mathcal{Y}_k$.

It is a deep fact that this definition does not depend on the choice of a Néron smoothing; the proof relies on the theory of motivic integration [LS03]. Definition 4.2.1 can be interpreted as a motivic upgrade of the integral of a volume form on a compact $p$-adic manifold [LS03 §4.6].

The motivic zeta function of the pair $(\mathcal{X}, \omega)$ is a generating series that measures how the motivic integral in Definition 4.2.1 changes under ramified extensions of $R$. For every positive integer $n$, we set $\mathcal{X}(n) = \mathcal{X} \times_R R(n)$, and we denote by $\omega(n)$ the pullback of $\omega$ to the generic fiber of $\mathcal{X}(n)$.

**Definition 4.2.2.** The motivic zeta function of the pair $(\mathcal{X}, \omega)$ is the generating series

$$Z_{\mathcal{X}, \omega}(T) = \sum_{n>0} \left( \int_{\mathcal{X}(n)} |\omega(n)| \right) T^n \in \mathcal{M}_{\mathcal{X}_k}[T].$$

Beware that this definition depends on the choice of the uniformizer $t$, except when $k$ has characteristic zero and contains all the roots of unity: in that case, $K(n)$ is the unique degree $n$ extension of $K$, up to $K$-isomorphism.

If $h : \mathcal{X}' \to \mathcal{X}$ is a proper morphism of $R$-schemes such that $h_K : \mathcal{X}'_K \to \mathcal{X}_K$ is an isomorphism, then it follows immediately from the definition that we can recover $Z_{\mathcal{X}, \omega}(T)$ from $Z_{\mathcal{X}', \omega}(T)$ by specializing the coefficients with respect to the forgetful group homomorphism $\mathcal{M}_{\mathcal{X}'_k} \to \mathcal{M}_{\mathcal{X}_k}$. Thus we can compute $Z_{\mathcal{X}, \omega}(T)$ after a suitable proper modification of $\mathcal{X}$. The principal aim of this paper is to establish an explicit formula for $Z_{\mathcal{X}, \omega}(T)$ in the case where $\mathcal{X}$ is smooth over $S^1$ with respect to a suitable choice of log structure on $\mathcal{X}$. 

4.3. Explicit formula on a log smooth model. Let $\mathcal{X}^\dagger$ be a smooth log scheme of finite type over $S^\dagger$, and denote by $D$ its reduced boundary divisor. We write $F = F(\mathcal{X}^\dagger)$ for the fan associated with $\mathcal{X}^\dagger$, and we denote by $e_\ell$ the image of the uniformizer $t$ in the monoid of global sections on $F$. We write $F^\text{vert}$ for the set of vertical points in $F$.

Let $\omega$ be a differential form of maximal degree on $\mathcal{X}_K$ that is nowhere vanishing on $\mathcal{X}_K \setminus D$. Then we can view $\omega$ as a rational section of the relative canonical bundle $\omega_{\mathcal{X}^\dagger/S^\dagger}$. As such, it defines a Cartier divisor on $\mathcal{X}$, which we denote by $\text{div}_{\mathcal{X}^\dagger}(\omega)$. This divisor is supported on $D$. Let $\tau$ be a point of $F$. For every element $u$ of $M_{F,\tau}^{\rho,\text{loc}}$, we set $u(\omega) = u_{\text{gp}}(\mathcal{F}) \in \mathbb{Z}$, where $\mathcal{F}$ is the residue class in $M_{F,\tau}^{\rho}$ of any element $f \in M_{\mathcal{X}^\dagger,\tau}^{\rho}$ such that $\text{div}(f) = \text{div}_{\mathcal{X}^\dagger}(\omega)$ locally at $\tau$. This definition does not depend on the choice of $f$. Note that $u(\omega) > 0$ if $\tau$ is not a vertical point of $F$, because $\text{div}_{\mathcal{X}^\dagger}(\omega) \geq 0$ on $\mathcal{X}_K$.

**Theorem 4.3.1.** Let $\mathcal{X}^\dagger$ be a smooth log scheme of finite type over $S^\dagger$. We assume that the generic fiber $\mathcal{X}_K$ is smooth over $K$ (but we allow the log structure on $\mathcal{X}^\dagger$ to be non-vertical). Let $\omega$ be a volume form on $\mathcal{X}_K$. Then for every $\tau$ in $F^{\text{vert}}$, the expression

$$ (L - 1)^{r(\tau) - 1} \sum_{u \in M_{F,\tau}^{\rho,\text{loc}}} L^{-u(\omega)} T^u(e_\ell) $$

(2)

is well-defined in $\mathcal{M}_{\mathcal{X}_K}(T)$, and the motivic zeta function of $(\mathcal{X}, \omega)$ is given by

$$ Z_{\mathcal{X}, \omega}(T) = \sum_{\tau \in F^{\text{vert}}} [\mathcal{E}(\tau)\omega](L - 1)^{r(\tau) - 1} \sum_{u \in M_{F,\tau}^{\rho,\text{loc}}} L^{-u(\omega)} T^u(e_\ell) $$

(3)

in $\mathcal{M}_{\mathcal{X}_K}(T)$.

**Proof.** We break up the proof into four steps.

**Step 1:** the expression (2) is well-defined. Since $\omega$ is a volume form $\mathcal{X}_K$, the horizontal part of the divisor $\text{div}_{\mathcal{X}^\dagger}(\omega)$ coincides with the horizontal part of the reduced boundary divisor $D$ of $\mathcal{X}^\dagger$. This means that $u(\omega) = 1$ for every $\tau \in F^{\text{vert}}$ and every generator $u$ of a one-dimensional face of $M_{F,\tau}$ such that $u(e_\ell) = 0$. Hence, Lemma 4.1.1 guarantees that (2) is a well-defined element of $\mathcal{M}_{\mathcal{X}_K}(T)$.

**Step 2:** invariance under log modifications. We will show that the right hand side of (3) is invariant under the log modification $h : (\mathcal{X}')^\dagger \to \mathcal{X}^\dagger$ induced by any fs proper subdivision $\psi : F' \to F$ that is an isomorphism over $F \cap \mathcal{X}_K$, or equivalently, such that $h_K : \mathcal{X}'_K \to \mathcal{X}_K$ is an isomorphism.

Let $\tau$ be a vertical point in $F$. Then, by the definition of a proper subdivision, the morphism $\psi$ induces a bijection between $M_{F',\tau}^{\rho,\text{loc}}$ and the disjoint union of the sets $M_{F,\tau'}^{\rho,\text{loc}}$ where $\tau'$ runs through the set of points in $\psi^{-1}(\tau)$. Since $h$ is an étale morphism of log schemes, the pullback of $\text{div}_{\mathcal{X}^\dagger}(\omega)$ to $\mathcal{X}'$ coincides with $\text{div}_{(\mathcal{X}')^\dagger}(\omega)$. Thus if $u$ is an element of $M_{F',\tau'}^{\rho,\text{loc}}$ for some $\tau'$ in $\psi^{-1}(\tau)$, then the value $u(h_K^*(\omega))$ computed on $(\mathcal{X}')^\dagger$ coincides with the value $u(\omega)$ computed on $\mathcal{X}^\dagger$. The same is obviously true for $u(e_\ell)$. Moreover, by Proposition 3.2.3 we have

$$ [\mathcal{E}(\tau')\omega] = (L - 1)^{r(\tau') - r(\tau)} [\mathcal{E}(\tau)\omega] $$

(4)

every point $\tau'$ in $\psi^{-1}(\tau)$. Thus the right hand side of (3) does not change if we replace $\mathcal{X}^\dagger$ by $(\mathcal{X}')^\dagger$. 

As a side remark, we observe that our assumption that \( h_K \) is an isomorphism has only been used to ensure that \( h_K^*\omega \) is a volume form on \( X_K' \), so that the right hand side of (3) is still well-defined in \( \mathcal{M}_{X_k}[T] \) if we replace \( X' \) by \( (X')^\dagger \). Our proof actually shows that the right hand side of (3), viewed as an element of 

\[
\mathcal{M}_{X_k}[T] \left[ \frac{1}{i! - 1} \right]_{i > 0},
\]

is invariant under any proper subdivision \( \psi : F' \to F \).

Step 3: compatibility with fs base change. We will prove that the formula (3) is compatible with fs base change, in the following sense. Let \( n \) be a positive integer and denote by \( F(n) \) the fan of the smooth log scheme \( X^\dagger \times_{S^\dagger} S(n)^\dagger \) over \( S(n)^\dagger \). Let \( t(n) \) be a uniformizer in \( R(n) \). Then for every positive integer \( i \), the coefficient of \( T^i \) in the expression

\[
\sum_{\tau' \in F(n)^\text{vert}} (L - 1)^{i(i - 1)} [\bar{E}(\tau')^\circ] \sum_{u' \in M_{F(n), \tau'}} L^{-u'(\omega(n))} T^{u'(e_i(n))}
\]

is equal to the coefficient of \( T^{iu} \) in the right hand side of (3).

To see this, we first observe that Lemma 3.2.2 implies that, for every point \( \tau \) of \( F^\text{vert} \), the \( k \)-scheme \( \bar{E}(\tau)^\circ \) is isomorphic to the disjoint union of the \( k \)-schemes \( \bar{E}(\tau)^\circ \) where \( \tau' \) runs over the points of \( F(n)^\text{vert} \) that are mapped to \( \tau \) under the morphism of fans \( F(n) \to F \). Moreover, \( M_{F(n), \tau'} \) is canonically isomorphic to

\[
(M_{F, \tau} \oplus [1/n] \mathbb{N})^\text{sat, k},
\]

which yields a bijective correspondence between the local morphisms \( u' : M_{F(n), \tau'} \to \mathbb{N} \) such that \( u'(e_i(n)) = i \) and the local morphisms \( u : M_{F, \tau} \to \mathbb{N} \) such that \( u(e_i) = in \). Now it only remains to notice that \( u'(\omega(n)) = u(\omega) \) because

\[
\omega_{X^\dagger \times_{S^\dagger} S(n)^\dagger / S(n)^\dagger}
\]

is canonically isomorphic to the pullback of \( \omega_{X^\dagger / S^\dagger} \), by the compatibility of relative log differentials with fs base change.

Hence, in order to prove Theorem 3, it suffices to show that the coefficient of \( T \) in the right hand side of (3) equals the motivic integral \( \int_X \omega \) in \( \mathcal{M}_{X_k} \).

Step 4: proof of the formula. By [CH15, 4.6.31], we can find a proper subdivision \( F' \to F \) such that, if we denote by \( (X')^\dagger \to X^\dagger \) the associated morphism of log schemes, the scheme \( X' \) is regular and the morphism \( X'_K \to X_K \) is an isomorphism. Thus, by Step 2, we may assume right away that \( X \) itself is regular. We write \( \text{Sm}(X) \) for the \( R \)-smooth locus of \( X \). Then the open immersion \( \text{Sm}(X) \to X \) is a Néron smoothening, by [BLR90, 3.1.2] and the subsequent remark. By Step 3 and the definition of the motivic integral, we only need to prove that the coefficient of \( T \) in the right hand side of (3) is equal to

\[
\sum_{C \in \pi_0(\text{Sm}(\mathcal{X})_k)} [C] L^{-\text{ord}_C \omega}.
\]

Let \( \tau \) be a point in \( F^\text{vert} \). By Example 3.2.1, the point \( \tau \) lies in \( \text{Sm}(\mathcal{X})_k \) if and only if there exists a local morphism \( u : M_{F, \tau} \to \mathbb{N} \) with \( u(e_i) = 1 \). In that case, the root index of \( \tau \) is equal to 1, so that \( \bar{E}(\tau)^\circ = E(\tau)^\circ \). Since the strata \( E(\tau)^\circ \) with \( \tau \in \text{Sm}(\mathcal{X})_k \) form a partition of \( \text{Sm}(\mathcal{X})_k \), it suffices to prove the following
property: let \( \tau \) be a point in \( F \cap \text{Sm}(X)_k \) and denote by \( C(\tau) \) the unique connected component of \( \text{Sm}(X)_k \) containing \( \tau \). Then we have

\[
(L - 1)^{r(\tau) - 1} \sum_{u \in M_{F,\tau}^{\text{loc}}, u(\epsilon_i) = 1} L^{-u(\omega)} = L^{-\text{ord}_{C(\tau)} \omega}
\]

in \( \mathcal{M}_k \) (here we again use Lemma 4.1.1 to view the left hand side as an element of \( \mathcal{M}_k \)).

First, we consider the case where the log structure at \( \tau \) is vertical. Then \( \tau \) is the generic point of \( C(\tau) \) and \( M_{F,\tau} \) is isomorphic to \( \mathbb{N} \) as an \( \mathbb{N} \)-monoid. Thus the only morphism \( u \) contributing to the sum in the left hand side of (5) is the identity morphism \( u : \mathbb{N} \to \mathbb{N} \). Now the equality follows from the fact that locally around \( \tau \), we have a canonical isomorphism \( \omega_{X/S} \cong \omega_{X'/S'} \) because the morphism \( X' \to S' \) is strict at \( \tau \).

Now we generalize the result to the case where the log structure at \( \tau \) is not vertical. By Example 3.2.1 there exists an isomorphism \( M_{F,\tau} \to \mathbb{N} \times \mathbb{N}^h \) for some integer \( h \geq 0 \) such that the morphism \( \mathbb{N} \to M_{F,\tau} \) is given by \( 1 \mapsto (1, 0) \). In this case, restriction to \( \mathbb{N}^h \) defines a bijection between the set of local morphisms \( u : M_{F,\tau} \to \mathbb{N} \) mapping \( e_t \) to 1 and the set of local morphisms \( u' : \mathbb{N}^h \to \mathbb{N} \). Since \( \omega \) is a volume form on \( X_K \), we have \( u(\omega) = \text{ord}_{C(\tau)} \omega + u'(1, \ldots, 1) \). Hence,

\[
(L - 1)^{r(\tau) - 1} \sum_{u \in M_{F,\tau}^{\text{loc}}, u(\epsilon_i) = 1} L^{-u(\omega)} = L^{-\text{ord}_{C(\tau)} \omega} (L - 1)^h \sum_{u' \in (\mathbb{N}^h)^{\text{loc}}} L^{-u'(1, \ldots, 1)}
\]

\[
= L^{-\text{ord}_{C(\tau)} \omega} (L - 1)^h \sum_{i_1, \ldots, i_h > 0} L^{-i_1 + \ldots + i_h}
\]

\[
= L^{-\text{ord}_{C(\tau)} \omega}
\]

in \( \mathcal{M}_k \).

As a special case of Theorem 4.3.1 we recover a generalization to arbitrary characteristic of the formula for strict normal crossings models from [NS07, 7.7]. Beware that in [NS07], the motivic integrals were renormalized by multiplying them with \( \mathbb{L}^{-d} \), where \( d \) is the relative dimension of \( X \) over \( R \).

**Corollary 4.3.2.** Let \( X \) be a regular flat \( R \)-scheme of finite type such that \( X_k \) is a strict normal crossings divisor, and write

\[
X_k = \sum_{i \in I} N_i E_i.
\]

Denote by \( X' \) the log scheme obtained by endowing \( X \) with the divisorial log structure induced by \( X_k \), and assume that \( X' \) is smooth over \( S' \) (this is automatic when \( k \) has characteristic zero, by Proposition 3.2.4). Let \( \omega \) be a volume form on \( X_K \).

For every non-empty subset \( J \) of \( I \), we set

\[
E_J = (\bigcap_{j \in J} E_j) \setminus \bigcup_{i \notin J} E_i
\]

and \( N_J = \gcd\{N_j \mid j \in J\} \). We denote by \( \tilde{E}_J \) the inverse image of \( E_J \) in the normalization of \( X \times_R R(N_J) \). Let \( \nu_i \) be the multiplicity of \( E_i \) in the divisor \( \text{div}_X(\omega) \),
for every $i$ in $I$. Then the motivic zeta function of $(X, \omega)$ is given by

$$Z_{X,\omega}(T) = \sum_{\emptyset \neq J \subseteq I} \left[ \tilde{E}_J^a \right] (L - 1)^{|J| - 1} \prod_{j \in J} \frac{L^{-\nu_j}T^{N_j}}{1 - L^{-\nu_j}T^{N_j}}$$

in $M_{X_k}[T]$.

Proof. In view of Example 3.2.1, this is a particular case of the formula (3) in Theorem 4.3.1. ■

4.4. Poles of the motivic zeta function. Theorem 4.3.1 yields interesting information on the poles of the motivic zeta function. Since the localized Grothendieck ring of varieties is not a domain, the notion of a pole requires some care; see [RV03]. To circumvent this issue, we introduce the following definition.

**Definition 4.4.1.** Let $X$ be a Noetherian $k$-scheme and let $Z(T)$ be an element of $M_{X}[T]$. Let $P$ be a set of rational numbers. We say that $P$ is a set of candidate poles for $Z(T)$ if $Z(T)$ belongs to the subring $M_{X} \left[ T, \frac{1}{1 - L^aT^b} \right]_{(a,b) \in \mathbb{Z} \times \mathbb{Z}^{\geq 0}, a/b \in P}$ of $M_{X}[T]$.

For any reasonable definition of a pole (in particular, the one in [RV03]), the set of rational poles is included in every set of candidate poles.

**Proposition 4.4.2.** Let $X^\dagger$ be a smooth log scheme of finite type over $S^\dagger$ such that $X_K$ is smooth over $K$. Let $\omega$ be a volume form on $X_K$. Write $X_k = \sum_{i \in I} N_i E_i$ and denote by $\nu_i$ the multiplicity of $E_i$ in $\text{div}_{X^\dagger}(\omega)$, for every $i \in I$. Then $P(X) = \{-\frac{\nu_i}{N_i} | i \in I\}$ is a set of candidate poles for $Z_{X,\omega}(T)$.

Proof. Let $\tau$ be a vertical point of $F = F(X^\dagger)$. In view of Theorem 4.3.1, it suffices to show that $P(X)$ is a set of candidate poles for

$$\sum_{u \in M^{\text{loc}}_{F,\tau}} L^{-u(\omega)T^{u(e_i)}}.$$

The one-dimensional faces of $M^{\text{loc}}_{F,\tau}$ correspond canonically to the irreducible components of the boundary divisor $D$ of $X^\dagger$ passing through $\tau$. If $u$ is a generator of a one-dimensional face and $E$ is the corresponding component of $D$, then $u(\omega)$ and $u(e_i)$ are the multiplicities of $E$ in $\text{div}_{X^\dagger}$ and $X_k$, respectively. In particular, if $D$ is not included in $X_k$, then $u(e_i) = 0$ and $u(\omega) = 1$, because $\omega$ is a volume form on $X_k$. Thus the result follows from Lemma 4.1.1. ■

Proposition 4.4.2 tells us that, in order to find a set of candidate poles of $Z_{X,\omega}(T)$, it is not necessary to take a log resolution of the pair $(X, X_k)$, which would introduce many redundant candidate poles. This observation is particularly useful in the context of the monodromy conjecture for motivic zeta functions; see Section 7.
5. Generalizations

5.1. Formal schemes. The definition of the motivic zeta function (Definition 4.2.2) can be generalized to the case where \( \mathcal{X} \) is a formal scheme satisfying a suitable finiteness condition (a so-called special formal scheme in the sense of \([\text{Be}09]\), which is also called a formal scheme formally of finite type in the literature). This generalization is carried out in \([\text{Ni}09]\), and it is not difficult to extend our formula from Theorem 4.3.1 to this setting. The main reason why we have chosen to work in the category of schemes in this article is the lack of suitable references for the basic properties of logarithmic formal schemes on which the proof of our formula relies. However, the proofs for log schemes carry over easily to the formal case, so that the reader who would want to apply Theorem 4.3.1 to formal schemes should have no difficulties in making the necessary verifications.

5.2. Nisnevich log structures. We will now show how Theorem 4.3.1 can be adapted to log schemes in the Nisnevich topology. This allows us to compute motivic zeta functions on a larger class of models with components with “mild” self-intersections in the special fiber. This generality is needed, for instance, for the applications to motivic zeta functions of Calabi-Yau varieties in \([\text{HN}16]\). We will explain in Example 5.2.3 what is the advantage of the Nisnevich topology over the étale topology when computing motivic zeta functions.

Let \( \mathcal{X} \) be a smooth log scheme of finite type over \( S \) with respect to the Nisnevich topology. Then the sheaf of monoids \( M_{\mathcal{X}}^1 \) is constructible on the Nisnevich site of \( \mathcal{X} \), by the same proof as in \([\text{GR}15]\) 10.2.21. We choose a partition \( \mathcal{P} \) of \( \mathcal{X}_k \) into irreducible locally closed subsets \( S \) such that the restriction of \( M_{\mathcal{X}}^1 \) to the Nisnevich site on \( S \) is constant. We denote by \( P \) the set consisting of the generic points of all the strata \( S \) in \( \mathcal{P} \). For every point \( \tau \) in \( P \) we will write \( E(\tau) \) for the unique stratum in \( \mathcal{P} \) containing \( \tau \), and we denote by \( r(\tau) \) the dimension of the monoid \( M_{\mathcal{X},\tau}^1 \). We define the root index \( \rho(\tau) \) and the scheme \( \tilde{E}(\tau) \) in exactly the same way as before, and we write \( e_t \) for the image of \( t \) in the monoid of global sections of \( M_{\mathcal{X},\tau}^1 \). If \( \mathcal{X}_K \) is smooth over \( K \) and \( \omega \) is a volume form on \( \mathcal{X}_K \), then we can simply copy the definition of the value \( u(\omega) \) for every local morphism \( u : M_{\mathcal{X},\tau}^1 \rightarrow \mathbb{N} \).

**Theorem 5.2.1.** Let \( \mathcal{X} \) be a smooth log scheme of finite type over \( S \) (with respect to the Nisnevich topology). We assume that the generic fiber \( \mathcal{X}_K \) is smooth over \( K \). Let \( \omega \) be a volume form on \( \mathcal{X}_K \). Then the motivic zeta function of \( (\mathcal{X}, \omega) \) is given by

\[
Z_{\mathcal{X},\omega}(T) = \sum_{\tau \in \mathcal{P}} (\mathbb{L} - 1)^{r(\tau)} \tilde{E}(\tau) \sum_{u \in \{M_{\mathcal{X},\tau}^1\}^{\nu,\text{loc}}} \mathbb{L}^{-u(\omega)} T^{u(e_t)}
\]

in \( \mathcal{M}_{\mathcal{X}_K}[T] \).

**Proof.** One can reduce to the Zariski case by observing that the motivic zeta function \( Z_{\mathcal{X},\omega}(T) \) is local with respect to the Nisnevich topology, in the following sense: If \( h : \mathcal{U} \rightarrow \mathcal{X} \) is an étale morphism of finite type and \( Y \) is a subscheme of \( \mathcal{X}_k \) such that \( Y' = \mathcal{U} \times_{\mathcal{X}} Y \rightarrow Y \) is an isomorphism, then \( Z_{\mathcal{X},\omega}(T) \) and \( Z_{\mathcal{U},h^*_\omega}(T) \) have the same image under the base change morphisms \( \mathcal{M}_{\mathcal{U}}[T] \rightarrow \mathcal{M}_Y[T] \) and \( \mathcal{M}_{\mathcal{X}_K}[T] \rightarrow \mathcal{M}_Y[T] \), respectively. This is an immediate consequence of the definition of the motivic integral. Moreover, if \( \{Y_1, \ldots, Y_r\} \) is a finite partition of \( \mathcal{X}_k \) into
subschemas and we denote by $Z_i(T)$ the image of $Z_{X,\omega}(T)$ under the composition

$$\mathcal{M}_{X_k}[T] \to \mathcal{M}_Y[T] \to \mathcal{M}_{X_k}[T]$$

(base change followed by the forgetful morphism), then

$$Z_{X,\omega}(T) = Z_1(T) + \ldots + Z_r(T).$$

Thus we can compute $Z_{X,\omega}(T)$ on a Nisnevich cover of $X$ where the log structure becomes Zariski. Since the right hand side of (11) satisfies the analogous localization property with respect to the Zariski topology, the result now follows from the Zariski case that was proven in Theorem 4.3.1.

Corollary 5.2.2. Let $X^\dagger$ be a smooth log scheme of finite type over $S^\dagger$ (with respect to the Nisnevich topology) such that $X_K$ is smooth over $K$. Let $\omega$ be a volume form on $X^\dagger_K$. Write $X_k = \sum_{i \in I} N_i E_i$ and denote by $\nu_i$ the multiplicity of $E_i$ in $\text{div}_{X^\dagger}(\omega)$, for every $i \in I$. Then

$$\mathcal{P}(X) = \{ -\frac{\nu_i}{N_i} | i \in I \}$$

is a set of candidate poles for $Z_{X,\omega}(T)$.

Proof. The proof is almost identical to the proof of the Zariski case (Proposition 4.4.2), using the formula in Theorem 5.2.1 instead of Theorem 4.3.1. We no longer have a bijective correspondence between the generators $u$ of one-dimensional faces of $(M^\triangledown_{X^\dagger,\tau})^\vee$ and the irreducible components of the boundary $D$ containing $\tau$, in general, because one component may have multiple formal branches at $\tau$ and each of these will give rise to a one-dimensional face of $(M^\triangledown_{X^\dagger,\tau})^\vee$. However, it remains true that for every generator $u$ of a one-dimensional face of $(M^\triangledown_{X^\dagger,\tau})^\vee$, there exists an irreducible component $E$ of $D$ such that $u(e_i)$ equals the multiplicity of $D$ in $X_k$ and $u(\omega)$ equals the multiplicity of $E$ in $\text{div}_{X^\dagger}(\omega)$. This is sufficient to prove the result.

The following example shows that the formula in Theorem 5.2.1 may fail if we replace the Nisnevich topology by the étale topology.

Example 5.2.3. Let $R = \mathbb{R}[t]$ and set

$$X = \text{Spec } R[x, y]/(x^2 + y^2 - t).$$

We denote by $X^\dagger$ the étale log scheme we get by endowing $X$ with the divisorial log structure induced by $X_\mathbb{R}$. Then $X^\dagger$ is log smooth over $R$, since the base change to $R' = \mathbb{C}[t]$ is isomorphic to $\text{Spec } R'[u, v]/(uv - t)$. However, the log structure is not Nisnevich (that is, the étale sheaf $M_{X^\dagger}$ is not the pullback of a sheaf in the Nisnevich topology).

The étale sheaf $M^\triangledown_{X^\dagger}$ is locally constant on the complement of the origin $O$ of $X_\mathbb{R}$, with geometric stalk $\mathbb{N}$. The geometric stalk of $M^\triangledown_{X^\dagger}$ at $O$ is isomorphic to $\mathbb{N}^2$. The line bundle $\omega_{X^\dagger/S^\dagger}$ is trivial on $X$ with generator

$$\omega = \frac{1}{t}(ydx - xdy).$$

Blowing up $X$ at $O$, we obtain a regular $R$-scheme whose special fiber has strict normal crossings. Using Corollary 4.3.2 one computes that the image of $Z_{X,\omega}(T)$
under the forgetful morphism $\mathcal{M}_{X_R}[T] \to \mathcal{M}_R[T]$ is equal to

\[
(C - \text{Spec } \mathbb{C}) \frac{T^2}{1 - T^2} + (\text{Spec } \mathbb{C})(L - 1) \frac{T}{1 - T} + (\text{Spec } \mathbb{C})(L - 1) \frac{T^3}{(1 - T)(1 - T^2)},
\]

where $C$ is a geometrically connected smooth projective rational curve over $\mathbb{R}$ without rational point.

The right hand side of (6) equals

\[
[\text{Spec } \mathbb{C}] \frac{T}{1 - T} + (L - 1) \frac{T^2}{(1 - T)^2},
\]

which does not agree with our expression for $Z_{X,\omega}(T)$. Indeed,

\[
[C - \text{Spec } \mathbb{C}] \neq L - 1
\]
in $\mathcal{M}_R$, as can be seen by applying the étale realization morphism

\[
\mathcal{M}_R \to K_0(\mathbb{Q}_\ell[\mathbb{Z}/2\mathbb{Z}]) : [X] \mapsto \sum_{i \geq 0} (-1)^i [H_{\text{et},c}^i(X \times_X \mathbb{C}, \mathbb{Q}_\ell)]
\]

for any prime $\ell$.

Similar examples can be constructed when $k$ is algebraically closed, for instance by considering

\[
\mathcal{X} = \text{Spec } R[x, y, z, z^{-1}]/(x^2 + zy^2 - t)
\]

with $R = k[t]$.

6. The monodromy action

6.1. Equivariant Grothendieck rings. Let $X$ be a Noetherian scheme and let $G$ be a finite group scheme over $\mathbb{Z}$ that acts on $X$; unless explicitly stated otherwise, we will always assume that group schemes act on schemes from the left. Suppose that the action of $G$ on $X$ is good, which means that $X$ can be covered by $G$-stable affine open subschemes. Then the Grothendieck group $K^G_0(\text{Var} X)$ of $X$-schemes with $G$-action is the abelian group defined by the following presentation:

- Generators: Isomorphism classes $[Y]$ of $X$-schemes $Y$ of finite type endowed with a good action of $G$ such that the morphism $Y \to X$ is $G$-equivariant; here the isomorphism class is taken with respect to $G$-equivariant isomorphisms.
- Relations:
  1. If $Y$ is an $X$-scheme of finite type with good $G$-action and $Z$ is a closed subscheme of $Y$ that is stable under the $G$-action, then
     \[
     [Y] = [Z] + [Y \setminus Z].
     \]
  2. If $Y$ is an $X$-scheme of finite type with good $G$-action and $A \to Y$ is an affine bundle of rank $r$ endowed with an affine lift of the $G$-action on $Y$, then
     \[
     [A] = [A^r_{\mathbb{Z}} \times_X Y]
     \]
     where $G$ acts trivially on $A^r_{\mathbb{Z}}$.

We define a ring structure on $K^G_0(\text{Var} X)$ by means of the multiplication rule $[Y] \cdot [Y'] = [Y \times_X Y']$, where $G$ acts diagonally on $Y \times_X Y'$. We write $\mathbb{L}$ for the class $[A^1_{\mathbb{Z}} \times_X X]$ and we set $M^G_0 = K^G_0(\text{Var} X)[L^{-1}]$.

We will use this definition in the case where $G = \mu_n$, the group scheme of $n$-th roots of unity, for some positive integer $n$. If $m$ is a positive multiple of $n$, then
the \((m/n)\)-th power map \(\mu_n \to \mu_n\) induces a ring morphism \(\mathcal{M}^{\mu_n}_{X} \to \mathcal{M}^{\mu_n}_{X}\). We denote by \(\hat{\mu}\) the profinite group scheme of roots of unity and we set
\[
\mathcal{M}^{\hat{\mu}}_X = \lim_{n \to 0} \mathcal{M}^{\mu_n}_X
\]
where the positive integers \(n\) are ordered by the divisibility relation. An action of \(\hat{\mu}\) on a Noetherian scheme is called good if it factors through a good action of \(\mu_n\) for some \(n > 0\). We will need the following elementary result.

**Proposition 6.1.1.** Let \(Y \to X\) be an equivariant morphism of Noetherian schemes with a good \(\mu_n\)-action, for some \(n > 0\). Assume that \(Y\) is a \(\mathbb{G}_{m,Z}^r\)-torsor over \(X\) and that the action
\[
\mathbb{G}_{m,Z}^r \times Z Y \to Y
\]
is \(\mu_n\)-equivariant, where \(\mu_n\) acts trivially on \(\mathbb{G}_{m,Z}^r\). Then we have
\[
[Y] = [X](\mathbb{L} - 1)^r
\]
in \(K_0^{\mu_n}(\text{Var}_X)\).

**Proof.** The torsor \(Y\) can be decomposed as a product
\[
\mathcal{L}_1^* \times X \cdots \times X \mathcal{L}_r^*
\]
where \(\mathcal{L}_1, \ldots, \mathcal{L}_r\) are \(\mu_n\)-equivariant line bundles on \(X\) and \(\mathcal{L}_i^*\) is obtained from \(\mathcal{L}_i\) by removing the zero section. Now the relations in the equivariant Grothendieck ring immediately imply that
\[
[Y] = [X](\mathbb{L} - 1)^r
\]
in \(K_0^{\mu_n}(\text{Var}_X)\). \(\blacksquare\)

### 6.2. Monodromy action on the motivic zeta function

Let \(K\) be a field of characteristic zero and set \(R = k[[t]]\) and \(K = k((t))\). Let \(X\) be an \(R\)-scheme of finite type with smooth generic fiber \(X_K\), and let \(\omega\) be a volume form on \(X_K\). Then the definitions of the motivic zeta function \(Z_{X,\omega}(T)\) (Definition 4.2.2) can be refined in the following way. For every positive integer \(m\), the finite group scheme \(\mu_m\) of \(m\)-th roots of unity acts on \(S(m) = \text{Spec} R[u]/(u^n - t)\) from the right via multiplication on \(u\):
\[
R[u]/(u^n - t) \to R[u]/(u^n - t) \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta]/(\zeta^n - 1) : u \mapsto \zeta u.
\]
We invert this action to obtain a left action on \(S(m)\). This induces a left action of \(\mu_m\) on \(X(m)\). One can use this action to upgrade the motivic integral
\[
\int_{X(m)} |\omega(n)|
\]
to an element in the equivariant Grothendieck ring \(\mathcal{M}_{X_m}^{\mu_m}\) of \(X_m\)-varieties with \(\mu_m\)-action — see [Ha15], where one can remove the assumption that \(k\) contains all the \(m\)-th roots of unity, since it is not needed in the arguments. This equivariant motivic integral can be computed by taking a quasi-projective \(\mu_m\)-equivariant Néron smoothening \(\mathcal{Y} \to X(m)\) over \(R(m)\): then
\[
\int_{X(m)} |\omega(n)| = \sum_{i \in \mathbb{Z}} [C(i)] \mathbb{L}^{-i} \in \mathcal{M}_{X_m}^{\mu_m}
\]
where \(C(i)\) is the union of the connected components \(C\) of \(Y_k\) such that \(\text{ord}_C(\omega(n)) = i\); note that \(C(i)\) is stable under the action of \(\mu_m\), because \(\omega(n)\) is defined over \(K\).
A quasi-projective $\mu_n$-equivariant smoothening $Y \rightarrow X(n)$ can always be produced by means of the smoothening algorithm described in the proof of [BLR90] 3.4.2; quasi-projectivity implies that the $\mu_n$-action on $Y$ is good.

Now we can view the motivic zeta function

$$Z_{X,\omega}(T) = \sum_{n>0} \left( \int_{X(n)} |\omega(n)| \right) T^n$$

as an object in $\mathcal{M}_{X,K}^\mu[T]$. We will use the new notation $Z_{X,\omega}^\mu(T)$ to indicate that we take the $\mu$-action into account. On the other hand, the schemes $\tilde{E}(\tau)^n$ appearing in Theorems 4.3.1 and 6.2.1 also carry an obvious action of the group scheme $\tilde{\mu}$, because $\mu_n$ acts on the fs base change

$$X^\dagger \times_\mathcal{S} S(n)^\dagger$$

for every $n > 0$ via the left action on $S(n)$.

**Theorem 6.2.1.** If $k$ has characteristic zero, then Theorems 4.3.1 and 5.2.1 are valid already for the equivariant motivic zeta function $Z_{X,\omega}^\mu(T)$ in $\mathcal{M}_{X,K}^\mu[T]$.

**Proof.** We can follow a similar strategy as in the proof of Theorem 4.3.1. Let $k^n$ be an algebraic closure of $k$ and set $(S^n)^\dagger = \text{Spec } k^n[t]$ with its standard log structure. To compute the degree $n$ coefficient of $Z_{X,\omega}(T)$ we can choose a regular subdivision of the fan $F^n(n)$ of $X(n)$ and $S^n$ that is equivariant with respect to the actions of $\mu_n(k^n)$ and the Galois group $\text{Gal}(k^n/k)$. This can be achieved by canonical equivariant resolution of singularities for toroidal embeddings (see for instance the remark on p. 33 of [Wan97]). The induced log modification of $X(n)$ is a regular scheme and its smooth locus is a $\mu_n$-equivariant Néron smoothening of $X(n)$. Then a similar computation as in the last step of the proof of Theorem 4.3.1 yields the desired result.

The only step that requires further clarification is the equality (4): we need to show that it remains valid in the equivariant Grothendieck ring $\mathcal{M}_{X,K}^\mu$. For every fixed point $\sigma$ of the $\mu_n(k^n)$-action on $F^n(n)$, the group $\mu_n(k^n)$ also acts trivially on the stalk of $\mathcal{M}_{F^n(n)}$ at $\sigma$ by [Na97] 2.1.1. In the notation of (4), this means that the natural morphism $\tilde{E}(\tau)^n \rightarrow \tilde{E}(\tau)$ is a $\mu_n$-equivariant torsor with translation group $\mathbb{G}_{m,k}^{r(\tau) - r(\tau')}$, where $\mu_n$ acts trivially on $\mathbb{G}_{m,k}^{r(\tau) - r(\tau')}$. Now it follows from Proposition 6.1.1 that

$$[\tilde{E}(\tau)^n] = (L - 1)^{r(\tau) - r(\tau')} [\tilde{E}(\tau)^0]$$

in $\mathcal{M}_{X,K}^\mu$.

The definition of a set of candidate poles (Definition 4.4.1) can be generalized to elements of $\mathcal{M}_{X,K}^\mu[T]$ in the obvious way. Then we can deduce the following result from Theorem 6.2.1

**Corollary 6.2.2.** Assume that $k$ has characteristic zero. Let $X^\dagger$ be a smooth log scheme of finite type over $S^\dagger$ (with respect to the Zariski or Nisnevich topology) such that $X_K$ is smooth over $K$. Let $\omega$ be a volume form on $X_K$. Write $X_K = \sum_{i \in I} N_i E_i$ and denote by $\nu_i$ the multiplicity of $E_i$ in $\text{div}_{X^\dagger}(\omega)$, for every $i \in I$. Then

$$P(X) = \left\{ -\frac{\nu_i}{N_i} \mid i \in I \right\}$$

is a set of candidate poles for $Z_{X,\omega}^\mu(T)$. 
The motivic zeta function of Denef-Loeser.

Let $k$ be a field of characteristic zero, let $X$ be an irreducible smooth $k$-variety and let

$$f : X \to \mathbb{A}^1_k = \text{Spec } k[t]$$

be a dominant morphism of $k$-schemes. We set $X_0 = f^{-1}(0)$. In [DL01], Denef and Loeser have defined the **motivic zeta function** $Z_f(T)$ of $f$, which is a power series in $\mathcal{M}_{X_0}^k[T]$ that can be viewed as a motivic upgrade of Igusa’s local zeta function for polynomials over $p$-adic fields. The famous **monodromy conjecture** predicts that the set of roots of the Bernstein polynomial of $f$ is a set of candidate poles for $Z_f(T)$ (in the sense of Definition 4.4.1). This has been proven when $X$ has dimension 2 and for some specific classes of singularities, but the conjecture is wide open in general. In fact, to our best knowledge, the proofs of the dimension 2 case in the literature consider a slightly weaker conjecture, dealing only with the so-called “naive” motivic zeta function, which can be viewed as the quotient of $Z_f(T)$ by the action of $\hat{\mu}$ (up to multiplication by a factor $\mathbb{L} - 1$). We will explain below how the argument can be refined to prove the conjecture for $Z_f(T)$ (Corollary 7.2.2).

Let $R = k[[t]]$, $K = k((t))$ and $X = X \times_{k[t]} R$. Let us recall how one can rewrite $Z_f(T)$ as the motivic zeta function of $(X, \omega)$ for a suitable volume form $\omega$ on $X_K$. Since the definition of the motivic zeta function is local on $X$, we can assume that $X$ carries a volume form $\phi$ over $k$. To this volume form, one can attach a so-called **Gelfand-Leray form** $\omega = \phi/df$, which is a volume form on $X_K$. Theorem 9.10 in [NS07, 9.5]. Theorem 9.10 in [NS07] states that

$$Z_{X, \omega}(T) = Z_f(\mathbb{L}T)$$

in $\mathcal{M}_{X_0}^k[T]$ (where we forget the $\hat{\mu}$-action on the right hand side). This can also be deduced from Corollary 4.3.2 and Denef and Loeser’s formula for the motivic zeta function in terms of a log resolution for $f$ [DL01, 3.3.1]. Using Theorem 6.2.4, one can moreover show that this equality holds already for the equivariant motivic zeta function $Z^\alpha_{X, \omega}(T)$ in $\mathcal{M}_{X_0}^k[T]$. More precisely, let $h : Y \to X$ be a log resolution for the pair $(X, X_0)$, and write $h^*X_0 = \sum_{i \in \mathbb{I}} N_i E_i$ and $K_{Y/X} = \sum_{i \in \mathbb{I}} (v_i - 1) E_i$. Set $Y = Y \times_{k[[t]]} R$ and endow it with the divisorial log structure induced by $Y_k$. Then the multiplicity of $E_i$ in $\text{div}_Y(\omega)$ equals $v_i - N_i$, so that the expression in Corollary 4.3.2 is precisely Denef and Loeser’s formula for $Z_f(\mathbb{L}T)$. Hence, we obtain that

$$Z^\alpha_{X, \omega}(T) = Z_f(\mathbb{L}T)$$

in $\mathcal{M}_{X_0}^k[T]$.

Thus Theorem 6.2.4 and Corollary 6.2.2 also apply to the motivic zeta function of Denef and Loeser. As an illustration, we will apply these results to two particular situations: the case where $X$ has dimension 2, and the case where $f$ is a polynomial that is non-degenerate with respect to its Newton polyhedron. These cases have been studied extensively in the literature; we will explain how some of the main results can be viewed as special cases of Theorem 4.3.1.
7.2. The surface case. Assume that \( X \) has dimension 2, and let \( h: Y \to X \) be a log resolution for the pair \((X, X_0)\). We write \( h^*X_0 = \sum_{i \in I} N_i E_i \) and \( K_{Y/X} = \sum_{i \in I} (\nu_i - 1)E_i \). The numbers \( N_i \) and \( \nu_i \) are called the numerical data associated with the component \( E_i \). It follows from Denef and Loeser’s formula [DL01, 3.3.1] that

\[
P = \left\{ -\frac{\nu_i}{N_i} \mid i \in I \right\}
\]

is a set of candidate poles for \( Z_f(T) \). However, it is known that many of these candidate poles are not actual poles of \( Z_f(T) \). In [Ve97], Veys has provided a conceptual explanation for this phenomenon by providing a formula for the topological zeta function (a coarser predecessor of the motivic zeta function) in terms of the relative log canonical model of \((X, X_0)\) over \( X \). We can now upgrade this result to the motivic zeta function and understand it as a special case of Theorem 4.3.1.

**Theorem 7.2.1.** For every \( i \in I \) such that \( E_i \) is exceptional with respect to \( h \), we denote by \( k_i \) the field \( H^0(E_i, \mathcal{O}_{E_i}) \). We define \( I_0 \) to be the subset of \( I \) consisting of the indices \( i \) such that \( E_i \) is an exceptional component of \( h^*X_0 \) satisfying \( (E_i)^2 \geq -2[k_i : k] \). Then

\[
P' = \left\{ -\frac{\nu_i}{N_i} \mid i \in I \setminus I_0 \right\}
\]

is still a set of candidate poles of \( Z_f(T) \in \mathcal{M}_{\kappa_0}^2[T] \).

**Proof.** Set \( Y = Y \times_{k[t]} R \). Contracting all the components \( E_i \) with \( i \in I_0 \) yields a new model \( Z \) of \( X_\kappa \) that is proper over \( X \), namely, the log canonical model of \((X, X_\kappa)\) over \( X \). We endow \( Z \) with the divisorial log structure induced by \( Z_\kappa \). It follows from [LS15, §3] that the resulting log scheme \( Z^1 \) is regular with respect to the étale topology, and since \( k \) has characteristic zero, this implies that \( Z^1 \) is smooth over \( S^1 \) with respect to the étale topology (Proposition 6.2.4).

The log structure on \( Z^1 \) fails to be Zariski precisely at the self-intersection points of components in the strict transform of \( X_0 \). If \( k \) is algebraically closed, then the log structure is Nisnevich at these points (because they have algebraically closed residue field) and our result follows immediately from Corollary 6.2.2. For general \( k \), we can make the log structure Zariski by blowing up \( Z^1 \) at each of the self-intersection points (see the proof of [N06, 5.4]). These blow-ups are log blow-ups, so that the resulting morphism of log schemes \( W^1 \to Z^1 \) is étale. Therefore, blowing up at a self-intersection point of a component \( E_i \) yields an exceptional divisor with numerical data \( N = 2N_i \) and \( \nu = 2\nu_i \). This implies that \( \mathcal{P}(W^1) = \mathcal{P}' \), so that the result follows from Corollary 6.2.2 (applied to the smooth Zariski log scheme \( W^1 \)).

**Corollary 7.2.2.** There exists a set of candidate poles for \( Z_f(T) \) that consists entirely of roots of the Bernstein polynomial of \( f \). Thus the monodromy conjecture for \( Z_f(T) \in \mathcal{M}_\kappa^2[T] \) holds in dimension 2.

**Proof.** Loeser has proven in [Lo88, III.3.1] that every element of \( \mathcal{P}' \) is a root of the Bernstein polynomial of \( f \).

Analogous results have previously appeared in the literature for the \( p \)-adic zeta function [Lo88, SL83], the topological zeta function [Ve97] and the so-called “naïve” motivic zeta function [Ro04].
7.3. Non-degenerate polynomials. As a second illustration, we will use our results to recover the formula for the motivic zeta function of a polynomial that is non-degenerate with respect to its Newton polyhedron [Gu02, §2.1] (see also [BV10 §10] for a calculation of the local “na"ive” motivic zeta function at the origin). In fact, our computations show that the formula in [Gu02, 2.1.3] has some flaws; we will explain in Remark 7.3.3 what needs to be corrected. Let

\[ f = \sum_{m \in \mathbb{N}^n} a_m x^m \]

be a non-constant polynomial in \( k[x_1, \ldots, x_n] \), where we use the multi-index notation for \( x = (x_1, \ldots, x_n) \). We assume that \( f(0, \ldots, 0) = 0 \). The support \( S(f) \) of \( f \) is the set of \( m \in \mathbb{N}^n \) such that \( a_m \) is non-zero, and the Newton polyhedron \( \Gamma(f) \) of \( f \) is the convex hull of

\[ \bigcup_{m \in S(f)} (m + \mathbb{R}_{\geq 0}). \]

For every face \( \gamma \) of \( \Gamma(f) \), we set

\[ f_\gamma = \sum_{m \in \gamma \cap \mathbb{N}^n} a_m x^m. \]

Then \( f \) is called non-degenerate with respect to its Newton polyhedron if, for every face \( \gamma \) of \( \Gamma(f) \), the polynomial \( f_\gamma \) has no critical points in the torus \( G_{m,k}^n \) (this includes the case \( \gamma = \Gamma(f) \)). This condition was introduced by Kushnirenko in [Ko76]. It guarantees that many interesting invariants of the singularities of \( f \) can be computed from the Newton polyhedron in a combinatorial way. In particular, every regular subdivision of the dual fan of \( \Gamma(f) \) defines a toric modification of \( k^n_k \) that is a log resolution for the pair \( (k^n_k, \text{div}(f)) \). Moreover, if we fix the support \( S(f) \), then \( f \) is Newton non-degenerate for a generic choice of coefficients \( a_m \).

We denote by \( \Sigma \) the dual fan of \( \Gamma(f) \) and by \( h : Y \to \mathbb{A}^n_k \) the toric modification associated with the subdivision \( \Sigma \) of \((\mathbb{R}_{>0})^n \). We view \( Y \) as a \( k[t] \)-scheme via the morphism \( f \circ h : Y \to \text{Spec}k[t] \) and we set \( \mathcal{Y} = Y \times_{k[t]} R \). We denote by \( H \) the pullback to \( \mathcal{Y} \) of the union of the coordinate hyperplanes in \( k^n_k \). We endow \( \mathcal{Y} \) with the divisorial Zariski log structure induced by the divisor \( \mathcal{Y}_k + H \). The result is a Zariski log scheme \( \mathcal{Y}^\dagger \) over \( S^\dagger \).

Proposition 7.3.1. The log scheme \( \mathcal{Y}^\dagger \) is smooth over \( S^\dagger \).

Proof. By Proposition 7.2.4, we only need to show that \( \mathcal{Y}^\dagger \) is regular. Since \( f \) has no critical points on \( G_{m,k}^n \), by the non-degeneracy assumption, we only need to check regularity at the points \( y \) on \( H \cap \mathcal{Y}_k \). Let \( \sigma \) be the cone of \( \Sigma \) such that \( y \) lies on the associated torus orbit \( O(\sigma) \) in \( Y \), and denote by \( \gamma \) the face of \( \Gamma(f) \) corresponding to \( \sigma \). We denote by \( M \) the characteristic monoid \( M = M_{\mathcal{Y}^\dagger,y}^\gamma \) of \( \mathcal{Y}^\dagger \) at \( y \).

If \( y \) does not lie on the strict transform of \( \text{div}(f) \), then \( M = (\sigma^\vee \cap \mathbb{Z}^n)^{\gamma} \) so that \( M^\text{gp} \) has rank \( r = \dim(\sigma) \). Locally at \( y \), the maximal ideal of \( M_{\mathcal{Y}^\dagger,y} \) defines the toric orbit \( O(\sigma) \), which is regular of codimension \( r \) in \( \mathcal{Y} \). Hence, \( \mathcal{Y}^\dagger \) is regular at \( y \).

Now suppose that \( y \) lies on the strict transform of \( \text{div}(f) \). If \( v \) is any lattice point on \( \gamma + (\sigma^\vee \cap \mathbb{Z}^n)^{\gamma} \), then the strict transform of \( \text{div}(f) \) at \( y \) is defined by \( f' = (f_v/x^v) + g \) where \( g \) is an element of \( O_{\mathcal{Y},y} \) that vanishes along \( O(\sigma) \). The monoid \( M \) is the submonoid of \( O_{\mathcal{Y},y}/O_{\mathcal{Y},y}^{\mathcal{Y}^\dagger,y} \) generated by \( (\sigma^\vee \cap \mathbb{Z}^n)^{\gamma} \) and the residue class of \( f' \). In particular, \( M^\text{gp} \) has rank \( r = \dim(\sigma) + 1 \). Moreover, locally at \( y \),
In that case, we set

\[ \text{Theorem 7.3.2.} \]

Let \( \rho \) be a non-constant polynomial in \( k[x_1, \ldots, x_n] \) such that \( f \) vanishes at the origin and \( f \) is non-degenerate with respect to its Newton polyhedron \( \Gamma(f) \). Set \( X_0 = f^{-1}(0) \). Then the motivic zeta function \( \zeta_f(T) \) can be written as

\[
\sum_{\gamma} \left( \frac{[X_0(0)]_{\text{L}}^{L^{-1}T}}{1 - L^{-1}T} + [X_0(1)] \right) \sum_{u \in 0, \ldots, n} L^{-\nu(u)}T^{N(u)}
\]

in \( M^n(X_0)[T] \), where the sum is taken over all the faces \( \gamma \) of \( \Gamma(f) \).
Proof. We will explain how this can be interpreted as a special case of Theorem 4.3.1. We set
\[ \omega = \frac{dx_1 \wedge \ldots \wedge dx_n}{df}, \]
viewed as a volume form on \( Y_k \). For notational convenience, we denote by \( D \) the strict transform of \( \text{div}(f) \) on \( Y \).

Let \( \gamma \) be a face of \( \Gamma(f) \). The points in \( F(Y)^{\text{vert}} \) that lie on \( O(\sigma_{\gamma}) \) are the generic points of \( O(\sigma_{\gamma}) \cap D \) and, provided that \( v_\gamma \neq 0 \), also the generic point of \( O(\sigma_{\gamma}) \).

Let \( \tau \) be a point of \( F(Y)^{\text{vert}} \cap O(\sigma_{\gamma}) \) and set \( M = M_{\gamma_{E}, \tau}^{1} \). As usual, we write \( e_t \) for the class of \( t = f \circ h \) in \( M \).

If \( \tau \) lies on \( D \) then we have \( M \cong M_{\gamma} \otimes \mathbb{N} \) and this yields an isomorphism
\[ M^{\gamma, \text{loc}} \cong (\tilde{\sigma}_{\gamma} \cap \mathbb{N}^n) \otimes (\mathbb{N} \setminus \{0\}). \]

The element \( e_t \) of \( M \) corresponds to \((v_{\gamma}, 1)\). It follows that the root index of \( \mathbb{N} \to M : 1 \mapsto e_t \) is one, so that \( E(\tau)^0 = E(\tau)^0 \) is the connected component of \( D \) that contains \( \tau \). We have seen in the proof of Proposition 7.3.1 that \( X_{\gamma}(0) \) is isomorphic to \((D \cap O(\sigma_{\gamma})) \times_k \mathbb{A}_{m,k}^{\dim(\sigma)} \). Moreover, for every element \( u' = (u, n) \) in \( M^{\gamma, \text{loc}} \) we have \( u'(e_t) = u(v_{\gamma}) + n = N(u) + n \) and \( u'(\omega) = \nu(u) - N(u) \). Thus the contribution of \( F(Y)^{\text{vert}} \cap O(\sigma_{\gamma}) \cap D \) to the formula for \( Z_f(T) = Z_{Y, \omega}(L^{-1}T) \) in Theorem 4.3.1 is equal to
\[ [X_{\gamma}(0)] \frac{L^{-1}T}{1 - L^{-1}T} \sum_{u \in \sigma_{\gamma} \cap \mathbb{N}^n} L^{-\nu(u)} T^{N(u)}. \]

If \( \tau \) does not lie on \( D \), then \( M \) is canonically isomorphic to \( M_{\gamma}^0 \) so that we can identify \( M^{\gamma, \text{loc}} \) with \( \tilde{\sigma}_{\gamma} \cap \mathbb{N}^n \). The element \( e_t \) of \( M \) is equal to \( v_{\gamma} \), so that \( u(e_t) = u(v_{\gamma}) = N(u) \) for every \( u \in \sigma_{\gamma} \cap \mathbb{N}^n \). We also have \( u(\omega) = \nu(u) - N(u) \). Thus, in order to match the formula in the statement of the theorem with the one in Theorem 4.3.1 it suffices to show that
\[ [X_{\gamma}(1)] = [E(\tau)^0](L - 1)^{n - \dim(\gamma)} - 1 \]
in \( M_{\gamma}^0 \). We write \( v_\gamma = \rho_{\gamma}^{\text{prim}} \) for a positive integer \( \rho \) and a primitive vector \( e_\gamma^{\text{prim}} \) in \( M_{\gamma}^0 \). Then \( \rho \) is the root index of the morphism of monoids \( \mathbb{N} \to M_{\gamma}^0 \) that maps 1 to \( e_t \). In the proof of Proposition 7.3.1 we have explained that locally at every point of \( E(\tau)^0 = O(\sigma_{\gamma}) \setminus D \), we can write \( f \circ h = x^v((f_\gamma/x^v) + g) \) where \( v \) is any lattice point on \( \gamma + M_{\gamma}^0 \) and \( g \) is a regular function on \( Y \) that vanishes along \( O(\sigma_{\gamma}) \).

We can choose \( v \) in such a way that it is divisible by \( \rho \), because \( v_\gamma \) is divisible by \( \rho \).

Now it easily follows from the definition that \( E(\tau)^0 \) is the cover of \( E(\tau)^0 \) defined by taking a \( \rho \)-th root of the unit \( f_\gamma/x^v \):
\[ E(\tau)^0 \cong \text{Spec } k[M_{\gamma}^0, T, T^{-1}]/((f_\gamma/x^v) - T^\rho). \]
The group scheme \( \mu_\rho \) acts on \( E(\tau)^0 \) from the left by the inverse of multiplication on \( T \), that is, \( T * \zeta = \zeta^{-1}T \).

We define a \( \mu_\rho \)-equivariant morphism of \( X_0 \)-schemes \( X_\gamma(1) \to E(\tau)^0 \) by means of the morphism of \( k \)-algebras
\[ k[M_{\gamma}^0, T, T^{-1}]/((f_\gamma/x^v) - T^\rho) \to k[\mathbb{Z}^n]/(f_\gamma - 1) \]
that maps $T$ to $x^{-v/\rho}$ and that maps $x^m$ to itself, for every $m \in M_k^n$. The morphism $X_\gamma(1) \to E(\tau)^\circ$ is an equivariant torsor with translation group $\text{Spec} \, \mathbb{Z}[\mathbb{Z}/L]$, where $L$ is the saturated sublattice of $\mathbb{Z}^n$ generated by $\gamma \cap \mathbb{Z}^n$. The equality \eqref{eq:main} now follows from Proposition \ref{prop:main}.

**Remark 7.3.3.** The calculation of $Z_f(T)$ in [Gar02, §2.1] contains the following flaws: the $\mu$-action on the schemes $X_\gamma(1)$ is ill-defined; the term involving $[X_\gamma(1)]$ should be omitted if $v_\gamma = 0$; the factor $(L-1)$ after $[X_\gamma(0)]$ should be omitted; the $X_0$-scheme structure on $X_\gamma(0)$ and $X_\gamma(1)$ is not specified.

**Corollary 7.3.4.** Let $f$ be a non-constant polynomial in $k[x_1, \ldots, x_n]$ such that $f$ vanishes at the origin and $f$ is non-degenerate with respect to its Newton polyhedron $\Gamma(f)$. Denote by $R(f)$ the set of primitive generators of the rays of the dual fan of $\Gamma(f)$ (that is, the inward pointing primitive normal vector on the facets of $\Gamma(f)$). Then the set

$$
\{-1\} \cup \{ -\frac{\nu(u)}{N(u)} \mid u \in R(f), N(u) \neq 0 \}
$$

is a set of candidate poles of $Z_f(T)$.

**Proof.** This follows from Theorem 7.3.2 in the same way as in the proof of Proposition 4.4.2.

Note that this set of candidate poles is substantially smaller than the set of candidates we would get from a toric log resolution of $(\mathbb{A}^n_k, X_0)$: the latter set would include the candidate poles associated with all the rays in a regular subdivision of the dual fan of $\Gamma(f)$. An analogous result for Igusa's $p$-adic zeta function was proven in [DH01].

The same method of proof yields similar results for the local motivic zeta function $Z_{f,O}(T)$ of $f$ at the origin $O$ of $\mathbb{A}^n_k$. This zeta function is defined as the image of $Z_f(T)$ under the base change morphism $\mathcal{M}_{\mathbb{A}^n_k}^{\hat{\mu}} \to \mathcal{M}_{O}^{\hat{\mu}} = \mathcal{M}_k^{\hat{\mu}}$. In fact, we only need to assume that $f$ is non-degenerate with respect to the compact faces of its Newton polyhedron. This means that for every compact face $\gamma$ of $\Gamma(f)$, the polynomial $f_\gamma$ has no critical points in $\mathbb{G}_m^n$.

**Theorem 7.3.5.** We keep the notations of Theorem 7.3.2 but we replace the non-degeneracy assumption on $f$ by the weaker condition that $f$ is non-degenerate with respect to the compact faces of $\Gamma(f)$. Let $O$ be the origin of $\mathbb{A}^n_k$. Then the motivic zeta function $Z_{f,O}(T)$ of $f$ at $O$ can be written as

$$
\sum_{\gamma} \left( \left[ X_\gamma(0) \right] \frac{L^{-1}T}{1-L^{-1}T} + [X_\gamma(1)] \right) \sum_{u \in \sigma_{\gamma} \cap \mathbb{N}^n} L^{-\nu(u)}T^{N(u)}
$$

in $\mathcal{M}_k^{\hat{\mu}}$, where the sum is taken over all the compact faces $\gamma$ of $\Gamma(f)$.

**Proof.** The non-degeneracy condition on $f$ guarantees that $\mathcal{Y}^1$ is smooth over $S^1$ at every point of $h^{-1}(O)$, by the same arguments as in the proof of Proposition 7.3.1. The remainder of the argument is identical to the proof of Theorem 7.3.2; we only need to take into account that $O(\sigma_{\gamma})$ lies in $h^{-1}(O)$ if $\gamma$ is compact, and has empty intersection with $h^{-1}(O)$ otherwise.

**Remark 7.3.6.** The monodromy conjecture for non-degenerate polynomials in at most 3 variables has been proven for the topological zeta function [LVP11] and the
$p$-adic and naïve motivic zeta functions \cite{BV16} (in a weaker form, replacing roots of the Bernstein polynomial by local monodromy eigenvalues). See also \cite{Lo90} for partial results in arbitrary dimension in the $p$-adic setting.

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