Universal Painlevé VI Probability Distribution in Pfaffian Persistence and Gaussian First-Passage Problems with a sech-Kernel

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We recast the persistence probability for the spin located at the origin of a half-space arbitrarily $m$-magnetized Glauber-Ising chain as a Fredholm Pfaffian gap probability generating function with a sech-kernel. This is then spelled out as a tau-function for a certain Painlevé VI transcendent, the persistence exponent $\theta(m)/2$ emerging as an asymptotic decay rate. Using a known yet remarkable correspondence that relates Painlevé equations to Bonnet surfaces, the persistence probability also acquires a geometric meaning in terms of the mean curvature of the latter, and even a topological one at the magnetization-symmetric point in terms of Gauss intrinsic curvature. Since the same sech-kernel with an underlying Pfaffian structure shows up in a variety of Gaussian first-passage problems, our Painlevé VI provides their universal first-passage probability distribution, in a manner exactly analogous to the famous Painlevé II Tracy-Widom laws. The tail behavior in the magnetization-symmetric case of our full scaling function allows to recover the exact persistence exponent $\theta(0)/2 = 3/16$ for the 2d-diffusing random field or for random real Kac’s polynomials, a particular result found very recently by Poplavskyi and Schehr [Phys. Rev. Lett. 121, 150601 (2018)]. Our Painlevé VI tau-function persistence distribution also bears a correspondence with a $c=1$ conformal field theory, the monodromy parameters giving the dimensions of the associated primary fields. This yields $\theta(0) = 3\beta$, with $\beta = 1/8$ the Onsager-Yang magnetization exponent for the critical 2d Ising model, a plausible conjecture relating a nonequilibrium exponent to ordinary static critical behavior in one more space dimension, that suggests more generally that methods of boundary conformal field theory should be helpful in determining the critical properties of other unsolved nonequilibrium 1d processes.

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I. INTRODUCTION AND GENERAL MOTIVATIONS

What is the chance for a fluctuating quantity to have always remained above its long-term tendency or, conversely, the likelihood to first cross its average value at a given time? The study of the first-passage properties of a random process revolves around such questions. It is a basic problem in probability [2, 3], with innumerable applications in the natural sciences [4, 5]. The usual playground is to consider a centered Gaussian and stationary process $\{Y(t)\}$, thus uniquely determined by its two-time translationally invariant correlation function $A(\tau) = \langle Y(t_1)Y(t_2)\rangle$. If this correlator vanishes sufficiently fast, physical intuition dictates that $P_0(\tau) = \exp(-\theta(T))$, with an asymptotic
decay rate $\theta$. Leaving aside the Markovian memoryless case, where one can show that $P_0(T) = \frac{1}{\pi} \arcsin |A(T)|$, with
necessarily then $A(T) = e^{-\theta |T|}$ at all times, and despite a huge number of works accumulated over decades, notably in
the signal theory or applied probability literature \cite{10, 11}, there exists hardly a handful of stationary Gaussian processes for
which the decay rate $\theta$ has been determined without approximations, and still less so for the full first-passage probability $-dP_0(T)/dT$. Such quantities, that embody infinite-order correlation functions, are sensitive to the whole
history of the process, as encoded into finer analytical details of the correlator. Without an additional structure present or some approximations made, there is therefore little hope to be able to calculate them.

Starting from the nineties, studies of simple models of phase-ordering kinetics have triggered a renewed interest on
first-passage properties under the wording of persistence \cite{11, 13}. In the context of many-body interacting nonequi-
librium systems, such as those displaying coarsening \cite{14}, the issue is to understand how a local degree of freedom
can maintain its initial orientation as domains of globally aligned spins form and grow forever. Taking for simplicity ± Ising spins quenched from a disordered initial state to zero temperature, the persistence probability $p_0(t)$ is then
defined as the probability that a given spin has always remained in its initial state up to time $t$. This also has a geo-
metric meaning, being the fraction of spins which have never flipped up to time $t$, while $-dP_0(t)/dt$ is the probability
that a domain wall first sweeps over a particular location in space. Part of the initial activity on this subject, and
the fascination for it, was triggered by the discovery that even for the simplest possible models \cite{15, 16}, the algebraic
decay $p_0(t) \propto t^{-\theta}$ takes place with an exponent $\theta$ which does not seem to be related to other known static or dynamic
critical exponents, yet subsuming in a simple number the everlasting evolution of the interwoven mosaic of coarsening
domains.

Early studies culminated in two apparently unrelated climaxes. On one hand it was the finding \cite{17, 18} that for
the simple diffusion equation evolving from random initial conditions — a popular model of phase-ordering where
one considers the sign of the (Gaussian) diffusing field as the local spin variable \cite{20} — the persistence exponent
is a non-trivial function $\theta(d)$ in any space dimension $d$, unrelated to the generic diffusive lengthscale $L(t) \propto t^{1/2}$.
With hindsight, this is less surprising \cite{21}, since the algebraic decay $p_0^{\text{diff}}(t) \propto t^{-\theta(d)}$ of the persistence probability
merely corresponds, when viewed on a logarithmic timescale $T = \ln t$ where the statistical self-similarity of coarsening
domains in the scaling regime appears stationary, to the decay rate for the no-crossing probability $P_0^{\text{diff}}(T) = p_0^{\text{diff}}(e^T) \propto e^{-\theta(d)T}$ of a stationary Gaussian process with a never Markovian correlator \cite{22}. Later, it has also
been realized that the particular case of the $d = 2$ diffusion equation plays a distinguished role. Indeed, if one
considers the Gaussian instantaneous value $X(t) = \varphi(t, 0)$ of the local diffusing field $\partial_t \varphi = \nabla_2^2 \varphi$ (at $r = 0$ without
restriction by translational invariance of the Gaussian random initial condition), and the associated normalized process
$Y(T) = X(e^T)/[(X^2(e^T))]^{1/2}$ (still Gaussian), then the corresponding correlator, $A(T) = \langle Y(0)Y(T) \rangle = 1/\cosh(T/2)$, a smooth,
even function of $T$, appears in one guise or in another into questions as diverse as the determination of the
probability that a random Kac polynomial has no real roots, or that of the gap probability for eigenvalues of truncated
random orthogonal matrices \cite{4, 24, 29}. For all these Gaussian problems that share the same correlator and are thus
actually identical (at least within some appropriate scaling regime), the persistence exponent $\hat{\theta}(2) = 0.1875(10)$
taking the "best" numerically-determined value from \cite{30} therefore represents the universal decay rate of their
common first-passage probability.

On the other hand, and in a genuine tour-de-force, the authors of \cite{19} managed to provide an exact expression
(valid at all times) for the more general persistence probability $p_0^{\text{Potts}}(t_1, t_2; q)$ that a “colored” $q$-state Potts spin
on a 1d chain evolving under zero-temperature Glauber dynamics has never flipped between times $t_1$ and $t_2$, and to
extract (for large $t_2/t_1$) the corresponding algebraic decay $p_0^{\text{Potts}}(t_1, t_2; q) \propto (t_2/t_1)^{-\hat{\theta}(q)}$. Pivotal in the derivation of their results is the consideration of the persistence probability $p_0^{\text{SemiP}}(t_1, t_2; q)$ in a specific geometry, the semi-infinite
chain, and for the particular spin located at the origin here. This suffices to reconstruct the full persistence probability
on an infinite chain, because if any spin there does not flip, the motion of the domain walls to the left and to its
right occurs independently. Conversely, $p_0^{\text{SemiP}}(t_1, t_2; q) = [p_0^{\text{Potts}}(t_1, t_2; q)]^{1/2}$, thus decaying with a halved exponent.

The crucial observation of \cite{19} is that $p_0^{\text{SemiP}}(t_1, t_2; q)$ can be obtained by uncovering a certain algebraic structure,
a Pfaffian. This allows — and this only for the particular spin located at the origin of the semi-infinite chain — to
express the probability that it has the same value at an arbitrary number of fixed times just in terms of combinations
of a basic building block, constituted by the two-body, no-meeting probability between a pair of random walkers.
Once the dust has settled, after many more technical hurdles to be overcome, the eventual upshot is a persistence
exponent $\hat{\theta}(q)$ with a complicated but explicit expression \cite{16, 19}, never rational for any finite $1 < q < \infty$, except for
$q = 2$ Ising spins, for which one finds the strikingly simple number $\hat{\theta}(2) = 3/8$. Yet one cannot help but muse over the
puzzling numerical proximity between the thereby determined persistence exponent for the semi-infinite Ising chain,
$\hat{\theta}(2)/2 = 0.1875$, and the value $\hat{\theta}(2) = 0.1875(10)$ found for the 2d-diffusing random field. But how these two model
systems, apparently so dissimilar, and that do not even live in an ambient space with the same physical dimension,
could possibly be related at the level of a quantity so sensitive to details as the persistence exponent?
II. SUMMARY OF RESULTS

In this note we demonstrate, among other things, how one can answer such a question. Our central result is that, when viewed in the scaling regime on the logarithmic timescale $T$ where the coarsening is stationary, the persistence probability $P_{0,\text{half}}^T(T;m)$ for the particular ± Ising spin located at the origin of an arbitrarily $m$-magnetized half-space chain—a quantity simply related to $p_0^{\text{Semip}}(t_1, t_2; q)$ for $q$-state Potts spins and thus sufficient to rebuild up the full persistence probability—can be recast at all times $T$ as a universal probability distribution involving a tau-function for a member of the highest hierarchy of Painlevé transcendents, a Painlevé VI (PVI). The particular PVI which shows up is the function $y = y(x)$ defined by the solution of

$$
\frac{d^2 y}{dx^2} = \frac{1}{2} \left( \frac{1}{y + 1} + \frac{1}{y - 1} + \frac{1}{y - x} \right) \left( \frac{dy}{dx} \right)^2 - \left( \frac{1}{y + 1} + \frac{1}{y - 1} \right) \left( \frac{dy}{dx} \right) + \frac{1}{x^2 (x - 1)^2} \left( \frac{y - 1}{8 y (y - 1)^2} + \frac{3 x (x - 1)}{8 (y - x)^2} \right),
$$

(1)

with striking numerical values for its parameters, visible in and determined from the non-differential part. The independent variable $x$ of Eq. (1) in rational coordinates corresponds to $x = (t_2/t_1)^2 = e^{2T}$, with $T = \ln(t_2/t_1)$ the logarithmic timescale for the persistence problem. Therefore just the branch $y(x)$ of the solution to Eq. (1) with $x \in (1, +\infty)$ is relevant. The monodromy parameters of our PVI, the four $\{\vartheta_\nu\}_{\nu}$, (a customary notation for PVI, yet an unfortunate one in the context of persistence), that encode (up to a sign) the eigenvalues $\pm \vartheta_\nu/2$ around each of the four fixed singularities $\nu = 0, 1, x, \infty$ (and up to any homographic transformation permuting their locations), are therefore:

$$
\{\vartheta_{\infty}, \vartheta_0^2, \vartheta_1^2, \vartheta_x^2\} = \left\{ 0, 0, 1, \frac{1}{4}, \frac{1}{4} \right\},
$$

(2)

Notice also that even though there exists a choice of signs that makes the sum $\sum_{\nu} \vartheta_\nu$ an integer, corresponding to the classical hypergeometric solutions for the PVI equation [31], one can check that the required physical solution to define a properly normalized persistence probability distribution is a genuinely transcendental PVI.

We now briefly sketch the main steps of our derivation, and their consequences. It uses extensively methods developed in the context of Random Matrix Theory (RMT), although it does not take advantage of the structure imposed a priori by some random matrix ensembles. It turns out however that lurking in the background is the Ginibre Orthogonal Ensemble (GinOE) for real random matrices, usually considered as the most complicated of all standard RMT ensembles [32, 71], and that has already appeared in the exact determination of the distribution of domain sizes for the 1d-Potts model [33, 34]. The precise relationship between the complex-valued Pfaffian point process formed by the eigenvalue distribution of the GinOE and the computation of temporal properties such as the persistence is another (long) story, that we figured out only when this work was already under completion. This will be discussed elsewhere [35]. Notice also that the sech-kernel, Eq. (3), at the heart of this work, can be viewed as a very particular case of a much more general hypergeometric kernel, that had appeared in the representation theory of some big, infinite-dimensional groups [36], figured out there through Riemann-Hilbert techniques. We do not use any such high-level ideas or tools here, and our approach will be a much more bottom-up, pedestrian one. At the end of our journey, beyond sizable technical and conceptual difficulties, the main take-home message is simple. It is that on the exemplary value of the persistence probability, Painlevé transcendents, that reside at the crossroad of so many branches of mathematics, are capable of fulfilling the Holy Grail of statistical physics: the exact integration through a non-Markovian temporal process of the spatial degrees of freedom in a many-body interacting system.

III. PERSISTENCE AS A FREDHOLM PFAFFIAN GAP PROBABILITY GENERATION FUNCTION

The first part of our analysis is to identify that, in the most convenient variables ($T = \ln(t_2/t_1), m = -1 + 2/q$), the expression obtained in [19] for the persistence probability $p_0^{\text{Semip}}(t_1, t_2; q)$, can be recast as a Fredholm Pfaffian gap probability generating function (with parameter $\xi = 1 - m^2$) for the restriction $K_T$ to the symmetric interval $[-T/2, T/2]$ of a certain continuous integral operator $K(x, y)$ dubbed, for obvious reasons, the sech-kernel:

$$
K(x, y) = K(x - y) = \frac{1}{2\pi} \text{sech}(|x - y)/2|,
$$

(3)

(recall that sech = 1/cosh). This kernel corresponds precisely to the stationary Gaussian correlator $A(T) = \rho_{0}^{-1}K(T)$ for the 2d-diffusing random field, $\rho_{0} = 1/(2\pi)$ being the average density of zero-crossings for the latter, i.e. the inverse of the mean spacing distance (on the logarithmic timescale) between changes of sign of $\{Y_{\nu}\}$ [17, 18]. The crucial quantities to consider to show this are $P_0^T(T; m)$, the probabilities for the Ising spin located at the origin of a
half-space $m$-magnetized chain to have always been in the $+$ state (resp. $-$) during a length of logarithmic time $T$. These two persistence probabilities obey simple symmetry and sum-rule relations, both for all $T \geq 0$ and $m \in (-1, 1)$:

$$P_0^+(T; m) = P_0^-(T; -m),$$

$$P_0^+(T; m) + P_0^-(T; m) = P_0^{\text{Half}}(T; m).$$

Eq. (4) is obtained by reversing globally the initial condition, and that does not affect the dual dynamics of the coalescing random walkers determining the probability that the spin at the origin has not flipped, while Eq. (5) is the full persistence probability for the Ising spin located at the origin of the half-space $m$-magnetized chain, irrespective of the state in which it stayed for a length of time $T$. Thus one of these two probabilities, say $P_0^+(T; m)$ determines everything, and it is then related [37] to the central result obtained in [19] (Eq. (29) there) for the persistence probability $P_0^{\text{semiP}}(t_1, t_2; q)$ of the $q$-state Potts spin at the origin of a semi-infinite chain by:

$$P_0^+(T = T_2 - T_1; m) = \frac{1}{q} P_0^{\text{semiP}}(e^{T_1}, e^{T_2}; q)|_{1/q = (1 + m)/2}.$$

The factor $1/q$ on the right-hand-side of Eq. (6), with $1/q = (1 + m)/2$, comes from relabelling a particular Potts color, occurring with probability $1/q$ in the random initial condition, as a $+$ Ising spin, while the $q - 1$ other colors are lumped together as $-$ Ising spins, induced therefore an overall magnetization $m = (1) - (1 - 1/q)$ for this semi-infinite or half-space $m$-magnetized Ising chain. The equivalence of these two viewpoints with a random initial condition in both cases is well-known, see e.g. [38]. The way the persistence exponent $\theta(q)$ was then extracted in [19] from the right-hand-side of Eq. (6) is from Szeg"o-Kac-Akhiezer formula giving the asymptotic behavior of truncated Wiener-Hopf operators (the continuous version of Toeplitz determinants). The continuous kernel which shows up originates from taking the limit of a dense set of times for the spin not to flip between times $t_1$ and $t_2$, and can be derived from the basic Pfaffian building block formed by the probability $c(s, t)$ that pair of $1d$ random walkers starting from the origin at times $s < t$ and going backwards in time while wandering on the positive half-space have not met down to the initial condition. Using the scaling form $c(s, t) \approx (2/\pi)(\arctan \sqrt[2]{s} - \arctan \sqrt[2]{t})$ (Eq. (15) of [19], rewritten), we observe now that the kernel appearing in [19] is simply related to the sech-kernel in the logarithmic variables $s = e^x, t = e^y$, with $1/q = (1 + m)/2$:

$$\left[ \delta(s-t) + 2 \frac{q - 1}{q^2} \frac{d}{ds} c(s, t) \right] dt = \left[ \delta(x-y) - (1-m^2) K(x-y) \right] dy.$$

Equipped with Eq. (6) and Eq. (7), our original contribution then amounts to rewrite in the $(T, m)$-language, and following the framework of [68] valid for any even difference kernel on a symmetric interval, the (complicated) expression found in [19] for the amplitude of the right-hand-side of Eq. (6). The outcome is that the two persistence probabilities $P_0^+(T; m)$ can be expressed in a remarkably compact manner in terms of the even and odd parts, $D_{e, o}(T; \xi)$, of the Fredholm determinant $D(T; \xi) = \prod_{k=0}^{\infty} (1 - \xi \lambda_k(T))$ for $\xi K_T$, with $\xi = 1 - m^2$, viz.

$$P_0^+(T; m) = \frac{D_{e}(T; 1 - m^2) \pm m D_{o}(T; 1 - m^2)}{2},$$

an expression holding true down to $T = \ln (t_2/t_1) \rightarrow 0^+$, where it identifies with $\frac{1 - m}{1 + m}$, the probability for a spin to be either $\pm$ on a $m$-magnetized chain (any arbitrary initial magnetization being preserved by zero-temperature Glauber dynamics at all subsequent times [32]). An immediate consequence of Eq. (8), due to the sum-rule Eq. (5) for the two persistence probabilities, is that

$$P_0^{\text{Half}}(T; m) = D_{e}(T; 1 - m^2) = \exp \left\{ - \int_0^{T/2} dx [R(x, x) + R(x, -x)] \right\},$$

where we have used for the last equality a well-known formula (see e.g. [69]), giving a Fredholm determinant in terms of the matrix elements of the resolvent operator $R$ for $\xi K_T$ (with still $\xi = 1 - m^2$). We recall that the resolvent as an operator is defined by $1 + R = (1 - \xi K_T)^{-1}$, and we denote its matrix elements for $x, y \in [-T/2, T/2]$ as $R(x, y) = R(x, y; \xi) = \langle x||R||y \rangle$ (using Dirac’s bra-ket notations, with in particular $\delta(x-y) = \langle x||\delta||y \rangle$). Eq. (9) is therefore the Fredholm determinant generating function for the even part $K_e$ of the sech-kernel: $K_e(x, y) = (K(x-y) + K(x+y))/2$. The whole derivation can actually be rendered independent of the precise form Eq. (3) of the kernel, a fact already hinted at in [19], when leaving the expression of $\theta(q)$ in terms of the scaling form of $c(s, t)$ in the original time variables. It just depends on the intrinsic Pfaffian structure of the persistence probability, and an equation such as Eq. (9) holds true when expressed in the logarithmic variables for any even-difference kernel on a symmetric interval.
In fact, Eq. (9) is the pristine analog for the sech-kernel of a celebrated result obtained in the context RMT by Gaudin for the sine-kernel [70], who computed in terms of the (even part) of the corresponding Fredholm determinant the exact spacing distribution in the "bulk" scaling limit of the Gaussian Orthogonal Ensemble (GOE), showing thereby that Wigner’s surmise was indeed a very good approximation for real random matrices. Twenty years after, it was recognized by the Kyoto School, although in a somewhat terse manner [71, 72] at the detour of a long and profound article on monodromy preserving deformations [63], that this very same gap spacing Fredholm determinant can be expressed as a tau-function, involving the non-autonomous Hamiltonian evaluated on the equations of motion associated to a certain Painlevé V (P V) function.

IV. COMPUTATION OF FREDHOLM DETERMINANTS FOR THE INTEGRABLE SECH-KERNEL

The second main step of our analysis is to perform the genuine computation of the Fredholm determinants for the sech-kernel in terms of which the persistence probability is expressed. A difference with the standard kernels of RMT is that there does not seem to exist here a bilinearization formula, that permits to find a differential operator $\mathcal{L}$ commuting with the sech-kernel, and thus to compute the Fredholm determinant for $K_T$ with the help of the eigenfunctions of $L$, as was done with prolate spheroidal functions for the bulk GOE in [70], or for the Airy and Bessel kernels [65, 66]. Yet if one rewrites the sech-kernel with the help of the innocent-looking, algebraic identities:

$$\frac{1}{\cosh[(x-y)/2]} = \frac{2\sinh[(x-y)/2]}{\sinh[(x-y)]} = \frac{e^{3x/2}e^{y/2} - e^{3y/2}e^{y/2}}{e^{2x} - e^{2y}}; \quad (10)$$

one recognizes that it gives rise to a so-called integral integrable operator [73], to which one can still apply the formidable functional-analytic machinery developed by Tracy and Widom in the nineties to compute the corresponding Fredholm determinant for $K_T$ as a function of the endpoints of the interval where it is defined [63, 66].

The situation here even corresponds to a case briefly considered in [67] for the finite-matrix size $N$ sine-kernel of the Circular Unitary Ensemble (CUE) of RMT, $K_N(x,y) = (1/N)\sin[N(x-y)/2]/\sin[(x-y)/2]$, up to some analytic continuation $x, y \to 2x, 2y$, and yet with $N = 1/2$! Incidentally, this very observation provided the impetus some years ago for this work, since it is known from the existing literature [18, 50] that the finite-$N$ CUE gives rise to a P V transcendental where the matrix size just appears as a parameter in its monodromy exponents, and which coalescences when $N \to \infty$ to the Jimbo-Miwa-Mori-Sato P V associated to the sine-kernel. We thereby had a clue for the possible occurrence of a non-trivial Painlevé transcendental associated to the sech-kernel, hidden somewhere between the above two in the hierarchy, up to some analytic continuation that ought to be relatively innocuous, since these functions are mathematical objects living intrinsically in the complex (projective) plane. At any rate, using the exponential Christoffel-Darboux decomposition provided by Eq. (10), one can follow the nearly algorithmic and by now well-trodden footsteps of [67]. After differential elimination of intermediate quantities, one arrives at a closed second-order second-degree nonlinear ODE for $H(T) = R(T/2, T/2)$, the resolvent function at coincident points, that we express in terms of the physical variable $T$, i.e. the length of the interval where our kernel acts:

$$\left(\frac{H''}{H'} + 2\coth T\right)^2 - 4\left(\frac{H^2}{H'\sinh^2 T} + 2HH'\coth T + H'\right) = 1. \quad (11)$$

[Compare Eq. (11) with Eq. (5.70) of [67]: they do indeed match with $t \to iT/2$ and $N^2 = 1/4$, as anticipated.] As in studies of standard RMT kernels, the only place where appears the $\xi$-dependence of the resolvent for the diluted kernel $\xi K_T$ is in the initial condition: by the Neumann expansion near $T = 0$ of the resolvent, one finds that $H(0) = \rho_0(1-m^2)$ (that can already be seen at the level of Eq. (7)). Thereby, from Eq. (11), order by order all the coefficients of the Taylor-expansion for $H(T)$ near the origin can be expressed in terms of $H(0)$, with in particular $H'(0) = H^2(0)$ (cancelling the spurious pole at the origin due to our rewriting of Eq. (11)). Therefore there should exist a unique regular solution to Eq. (11) on the whole positive real axis with local Cauchy initial datum $\{H(0) = \rho_0(1-m^2), H'(0) = H^2(0)\}$, joining one having a finite limiting value as $T \gg 1$, this last condition being required through Eq. (11) by the very existence of a persistence exponent. In other words, the formula for the persistence exponent should be rigidly determined by the analytic structure. By rewriting in terms of $m$ and in a somewhat more symmetric form the expression found in [18, 19] for $\tilde{\theta}(q)$, we recast the corresponding expression for $\theta(m)$ as:

$$\theta(m) = \tilde{\theta}(q = 2/(1+m)) = \frac{1}{2}\left\{\arccos\left(\frac{m}{\sqrt{2}}\right) - \arccos\left(\frac{1}{\sqrt{2}}\right)\right\}^2. \quad (12)$$
Notice that \( \theta(m) \) is not an even function of \( m \). This regular solution \( H(T) = H(T; H(0), H^2(0)) \) also permits to define (for a fixed value of \( m \)) two well-normalized probability density functions \( p_1, p_2 \) on \((0, +\infty)\), with (tail) probability distribution functions \( F_1, F_2 \). The first of these two is simply the Fredholm determinant for \((1 - m^2)K_T\):

\[
F_2(T) = \det [1 - (1 - m^2)K_T] = \int_T^{\infty} d\ell p_2(\ell) = \exp \left[ - \int_0^T d\ell H(\ell) \right],
\]

while the second is a rewriting of Eq. (9) for the persistence probability, using the so-called Gaudin’s relation \( dR(x, x)/dx = R^2(x, -x) \), again valid for any even difference kernel\

\[
F_1(T) = P_{\theta}^{\text{Halm}}(T; m) = \int_T^{\infty} d\ell p_1(\ell) = [F_2(T)]^{1/2} \exp \left[ - \frac{1}{2} \int_0^T d\ell \sqrt{H'(\ell)} \right]
\]

Thus the knowledge of \( F_2 \), i.e. that of \( H \) (and of its derivative) is sufficient to determine \( F_1 \). Notice that, obviously, \( F_2(0) = 1 \), and \( F_2(T) \propto e^{-\theta(m)T} \to 0 \) as \( T \to \infty \), since \( H(T) \to \theta(m) \) there. Plots of the distribution \( F_2 \) and of its density should probably more accessible from the Fredholm determinant representation Eq. (13), rather than solving directly the ODE Eq. (11). As for the distribution \( F_1 \), namely the persistence probability for a fixed value of \( m \), its behavior for large \( T \) is more subtle, since

\[
F_1(T) \propto \exp \left\{ -\min[\theta(m), \theta(-m)]T/2 \right\},
\]

because either \( P_{\theta}^{+}(T; m) \propto e^{-\theta(m)T/2} \) or \( P_{\theta}^{-}(T; m) \propto e^{-\theta(-m)T/2} \) dominates the sum Eq. (9) according to the sign of \( m \) (the relation for the respective decay rates coming from Eq. (1)), with balanced contributions only for symmetric initial conditions. By varying \( m \) one therefore observes a cusp at the origin for the overall decay rate of \( F_1(T) \). Yet one can explore the full range of values for \( \theta(m) \) given by Eq. (12) by following continuously one of the two branches \( \theta(\pm m) \), and that was precisely what has been done for the corresponding \( \theta(q) \) in [19], or simply by a further conditioning of \( P_{\theta}^{\pm}(T; m) \) with respect to the value the spin had at \( T = 0 \). This being said, for a fixed value of \( m \) Eqs. (13)-(14) are the analogs for our sech-kernel to the famous Painlevé II (\( P_{II} \)) Tracy-Widom distributions [65], defined in terms of Fredholm determinants for the Airy kernel, and that give respectively the (inter-related) distributions of the maximum eigenvalue in the \( \beta = 2 \) GUE and the \( \beta = 1 \) GOE when properly scaled at the soft edge at the spectrum. Eq. (11) is the equivalent (for each value of \( m \)) to the \( P_{II} \) Hastings-McLeod solution that appears in the Tracy-Widom distributions.

V. THE PAINLEVÉ VI BONNET SURFACE

Αγεμωμετητους μηδενες εισπατω. (Let no one who cannot think geometrically enter here.)

[Legendary inscription written at the entrance of Aristotle’s classroom in Plato’s Academy.]

The third stage of our work consists in the precise identification of the underlying \( P_{VI} \) function \( y(x) \) for which the above probability distribution \( F_2 \) is a tau-function, and especially the determination of its four associated monodromy exponents. Given the 2nd-order 2nd-degree nonlinear ODE satisfied by \( H(T) \), there are many ways to do this correspondence [31, 74], and to express the rational transformation relating a \( P_{VI} \) function \( y(x) \) and its derivative \( y'(x) \) to one of its Hamiltonian \( H(T) \propto H_{VI}(q(s), p(s), s) \) evaluated on the equations of motion, with appropriate relations between the respective independent variables \( x, s, T \) in each case. These are straightforward methods, although a bit heavy [74]. Here we shall use a marvelous geometric shortcut — a remark that we owe to Robert Conte [52] — that permits to recognize in our equation established for \( H(T) \), Eq. (11), the so-called Hazzidakis equation, that appeared as early in 1897 [62], in a question of differential geometry of surfaces stated (and answered) by Pierre-Ossian Bonnet in 1867 [51]. Given the two fundamental Gauss’ quadratic forms of a surface defined in conformal coordinates \((z, \overline{z})\), how to determine all the surfaces that are applicable on each other? It turns out that, as shown by Bonnet, their mean curvature function \( \mathcal{H}_M \) has to satisfy a certain third-order nonlinear ODE depending just on \( \mathbb{R}z \), for which Hazzidakis found an integral of motion, of which the right-hand-side of Eq. (11) is a particular case. It took then one more century until Bobenko and Eitner [53] identified Hazzidakis’ equation for \( \mathcal{H}_M(\mathbb{R}z) \) as a Hamiltonian for a certain one-parameter \( P_{VI} \). The latter remarkable correspondence uses the fact that the Gauss-Codazzi equations for the moving frame of a Bonnet surface can be retranscribed as a Lax pair for \( P_{VI} \) [66, 57]. As shown in 2017 by Robert Conte [63, 64], the corresponding codimension-three Bonnet-P\( P_{VI} \) can be extrapolated to the “full” \( P_{VI} \) with four arbitrary monodromy parameters to obtain, due to its geometric origin, probably the ”best” (more symmetric) Lax pair.
For our particular case, this rather unexpected incarnation of the coarsening motto, according to which the motion of the domain walls proceeds by mean curvature, yields the explicit relation \( H(T) = -2\mathcal{H}_M(\mathbb{R}_z) \) (compare our Eq. (11) with Eq. (B.20) from \[61\]), with \( \mathbb{R}_z = T \), and the specific Bonnet-P\_V\_1, Eq. (1) for \( y = y(x) \), with \( x = e^{2T} \), and its accompanying monodromy exponents, Eq. (2) through:

\[
H(T) = -\frac{(x - 1)x^2}{y(y - 1)(y - x)} \left( \frac{1}{2} \left( \frac{dy}{dx} \right)^2 - \frac{1}{8} \frac{y^2}{x^2(y - x)} \right) .
\]

Hence \( F_2(T) \) is indeed a P\_V\_1-tau function. We mention that this implies that a formula equivalent to Eq. (12) is actually buried as early in 1982 in the so-called Jimbo-Fricke cubic of monodromy invariants exhibited in \[61\], since there the full connexion for the tau-function of P\_V\_1 was solved \[53\]. This also gives a precise meaning to both intuitive viewpoints advocated in the introduction of the persistence exponent as a decay rate endowed with a geometric interpretation, the former aspect for an admittedly complicated time-dependent P\_V\_1 Hamiltonian evaluated on the equations of motion, yielding nevertheless a sort of (exact) Kramers’ formula. A tau-function characterization holds also true for \( F_1(T) \), with additional significant complications \[52, 53\].

This geometric correspondence also gives a flurry of results, among which a precise relationship between the two asymptotic curvatures \( \kappa_1, \kappa_2 \) of the underlying Painlevé-Bonnet (saddle-like) surface and the persistence exponent as \( T = \mathbb{R}_z \to \infty \):

\[
\frac{\kappa_1 + \kappa_2}{2} = -\frac{\theta(m)}{2} .
\]

Eq. (17) thus provides an explanation for the appearance of angles in Eq. (12), as a sort of non-linear quadratic Buffon’s needle formula (see \[44\] for a beautiful exposition of this problem in the spirit of modern random geometry, as well as, incidentally, for a relation with the zeros of Kac’s polynomials, as testified by the very title of that work). Particularly noticeable are the consequences of this Painlevé-Bonnet correspondence for the distinguished value \( m = 0 \) of the persistence exponent. For Bonnet surfaces, it turns out that the mean curvature function also determines its first fundamental quadratic form, a.k.a. the metric, through:

\[
(ds)^2 = \frac{d\varpi}{H'(T) \sinh^2 T} ,
\]

with still \( T = \mathbb{R}_z \), while \( \Im(z) \) is related to the spectral parameter in the Lax pair representation of the moving frame \[55, 61\], the piece \( 1/\sinh^2 T \) in Eq. (18) being the so-called Hopf factor of the surface. From Eq. (18), the metric therefore becomes singular when \( H'(T) \) vanishes. This corresponds to a so-called umbilic point, where the two local curvatures coincide, and where therefore the mean curvature \( \mathcal{H}_M = (\kappa_1 + \kappa_2)/2 \) is also identical to the square-root of Gauss’ curvature, \( \mathcal{X}_G = \kappa_1 \kappa_2 \). The latter is an intrinsic quantity for a surface, independent from any local conformal reparametrization, a result that Gauss called his Theorema Egregium ("Remarkable Theorem"). A vanishing of the derivative of the mean curvature function is precisely what happens for \( m \to 0 \), because for large \( T \):

\[
\exp \left[-\int_0^{T/2} dx R(x, -x) \right] \approx \exp \left[-\frac{1}{2} \int_0^T d\ell \sqrt{H'(\ell)} \right] \to \sqrt{|m|} ,
\]

this after carefully transcribing in the \( (T, m) \)-language the singular behavior \( \propto \sqrt{q(2-q)} \) for \( q \to 2^- \) found in \[19\] for the asymptotic amplitude of \( \tilde{p}_{\text{semiP}}(t_1, t_2; q) \). This was one of the significant technical hurdle that had to be overcome there, in particular to show by a careful analytic continuation that \( \theta(q) \) maintained the same expression for \( q > 2 \), as already hinted at. The above loss of differentiability around \( m = 0 \) can actually be traced back to the fact that the largest eigenvalue of the Fredholm integral equation for the sech-kernel \( \lambda_0(T) \to 1 \) as \( T \gg 1 \), a value corresponding to the maximum of its (self-dual) Fourier transform \( \int_0^\infty dv e^{-ikt} K(v) = \operatorname{sech}(\pi k) \), itself attained at \( k = 0 \). The last phenomenon is also responsible for the "cuspy" behavior evidenced by Eq. (15). The reward for these two facets of singular behavior is a gem constituted by an intrinsic, topological meaning à la Gauss-Bonnet for the persistence exponent of Ising spins with symmetric initial conditions:

\[
\mathcal{H}_M = \sqrt{\mathcal{X}_G} = \kappa = -\frac{\theta(0)}{2} = -\frac{3}{16} .
\]

VI. UNIVERSALITY ASPECTS AND CONCLUSIONS

In the final stage of this note, time is ripe now to glean the consequences for universality of our Pfaffian P\_V\_1 characterization of the persistence probability. We shall also spice this up with a conjecture, that we view as extremely
plausible, corresponding to Eq. (25), and that has probably already been proved in the conformal field theory or integrable literature by more knowledgeable people, although we have not managed to track it down, or simply to understand the corresponding result had we read it.

The first part of our considerations on universality is an immediate (and exact) consequence of our Fredholm Pfaffian generating function expression Eq. (2), when rewritten in terms of the square-root of the determinant of a $2 \times 2$ matrix block operator $K_2$ as in [68]. Copying Eq. (7) from [68], replacing simply there the GOE sine-kernel by our sech-kernel, one has in coordinate-free, short-hand notations:

$$P^{\text{Hilf}}_0(T; m) = \left\{ \det \left[ s_2 - (1 - m^2) \chi_T \left( \begin{array}{cc} K & DK \\ \epsilon K - \epsilon & K \end{array} \right) \chi_T \right] \right\}^{1/2},$$

where $\epsilon(x, y) = (1/2) \text{sgn}(x - y)$, $D = d/dx$, $K$ is the kernel Eq. (3), and $\chi_T$ the characteristic function of the interval $[-T/2, T/2]$. The right-hand-side is nothing but the Fredholm Pfaffian of $j_2 + (1 - m^2)(Jk_2)$, with $J = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)$. Expanding the gap probability generating function, the resulting $n$-point correlation functions $\rho_n(x_1, x_2, \ldots, x_n)$ are found to coincide with the same Pfaffian expressions as those discovered in [29] for the intensities of the Pfaffian Gaussian process $\{f(u)\}_{-1 < u < 1}$ formed by the limiting random real Kac polynomial conditioned by $f(u) = 0$. This holds up to the by-now standard changes of variable [25] $u \approx 1 - 1/t \approx 1 - e^{-T}$, that permits to identify in the vicinity of $u = 1$ the zero-crossings properties of this limiting random Kac’s polynomial, to those of the $2d$-randomly diffusing field $\{X(t)\}$ for $t \gg 1$, or equivalently those for large $T$ of the stationary process $\{Y(T)\}$ with correlator $A(T) = \text{sech}(T/2)$. Resumming back the gap probability generating function specialized to $m = 0$, we obtain from Eq. (19) and then from Eq. (15):

$$P^{\text{Hilf}}_0(T; m = 0) = P^{\text{Diff}_0}(T|Y(0) = 0) \implies \frac{\theta(0)}{2} = \frac{\tilde{\theta}(2)}{16} = \frac{3}{16}.$$  

A subtle but important point concerns the conditioning to an initial value $Y(0) = 0$ appearing in Eq. (22). It can be read off from the equality of Eq. (13) relating $F_1$ to its density $p_1$, and by a comparison with Gaudin’s result for the bulk GOE gap spacing distribution, $p_1^{\text{GOE, bulk}}(\ell) = \frac{d^2}{d\tau^2} \det (1 - K_{\text{GOE}}^\text{bulk} \rho_{\ell})$, which involves instead the second-derivative, twice-conditioned spacing distribution between eigenvalues [62] [70]. In fact, this one-conditioning of the first-passage probability $P_0(T)$ to the initial value $Y(0) = 0$ was implicit in our approach: it corresponds, for an otherwise stationary process, invariant under $T \rightarrow -T$, in making a choice for the origin of the logarithmic timescale $T$ where the last-crossing of zero occurred, and to measure a (positive) length of time from it. Notice that in this respect, Eq. (11) is left unchanged under $H(T) \rightarrow -H(-T)$. In perhaps more intuitive physical terms, the choice for the origin of $T$ and its sign (“the arrow of time”) is unambiguously set by the microscopic timescale $t_{\text{mic}} \sim 1$ beyond which the underlying nonequilibrium coarsening process establishes, and that is thereafter rendered stationary by viewing it on the logarithmic timescale.

Eq. (22) therefore yields the proof of a long-standing conjecture [75] for the value of the persistence exponent of the $2d$-diffusing random field and of allied problems, a result that had actually been obtained before and independently a few months ago while this work was in progress in the remarkable paper [45], using notably the connection with truncated random orthogonal matrices. The techniques used by Poplavskiy and Schelhr also allow them to derive the equality as temporal processes between the sign of the (conditioned) $2d$ diffusing field, and the instantaneous value of the Ising spin located at the origin of a semi-infinite chain with symmetric initial conditions, which entails in particular the first equality of Eq. (22). While we were aware since March 2018 that these authors had made this breakthrough, our work had followed a completely different path and, as far as we can tell, none of the results presented here relies on any reported there. Notice in particular that in our approach the exact $\theta(2) = 3/16$ is an ancillary consequence of the tail behavior in the magnetization-symmetric case of our full Pfaffian $P_{\text{VI}}$ scaling distribution, whose nature was not identified in [43], let alone the geometric interpretation for $\theta(0)/2$, Eq. (20), unveiled. Our expressions for the full scaling function valid at all times and for all $m$ should also grant access to other quantities than those studied in [43]. In this respect, note that our approach allows to recover immediately the algebraic decay for $s < 0$ of $\langle e^{sN_u} \rangle \propto 1/n^{\tilde{\theta}(e^s)}$, the exponential generating function for the number $N_u$ of real zeroes of Kac’s polynomials, that was computed explicitly with rather involved Pfaffian techniques in [43], through the simple correspondence $n \mapsto \ln (T/2)$ [25] [26]; $e^s \mapsto m$ in Eq. (15).

At any rate, for a symmetric initial condition $m = 0$, our distribution $F_1(T)$ represents the exact universal scaling first-passage probability distribution $P_0(T|Y(0) = 0)$ for all Gaussian Pfaffian processes with a stationary sech-kernel correlator, explicitly given by Eq. (14), with $H(T)$ related to the $P_{\text{VI}}$ function $y(x)$, Eq. (11), through Eq. (15).

The second part of our conclusions deal with the tentative exploitation of another remarkable correspondence established recently in an impressive series of works [77] [80] (see also [78], Eq. (5.17), and [81]), and that relates a generic $P_{\text{VI}}$ to conformal field theories with central charge $c = 1$. Briefly, a tau-function for $P_{\text{VI}}$ provides the generating
function of conformal blocks of four Virasoro primary fields, with conformal dimensions $\Delta_\mu = \vartheta_2^2/4$ in terms of the monodromy parameters. Using our Eq. (2) gives here, up to any permutation of the four fixed singularities:

$$\{\Delta_1, \Delta_2, \Delta_3, \Delta_4\} = \left\{0, 0, \frac{1}{16}, \frac{1}{16}\right\},$$

(23)
in which one is told to recognize the so-called sine-Gordon quantum field theory at the free-fermion point, itself equivalent to the square of the 2d (static) Ising model at the critical temperature, admittedly another famous Pfaffian process. More prosaically, we have noticed that the very same sech-kernel, Eq. (3), occurs systematically in quantum field-theoretical approaches for the 2d Ising model: it represents the so-called "heart of the kernel" in form-factor expansions, see e.g. [82]. Contemplating some of the expressions in the latter work or in others [83, 84], as well as the numerical values present in Eq. (23) and Eq. (1), and trying to figure out (literally) all this, we speculate that our PV$_V$ tau-function $F_2(T)$ corresponds, for the critical 2d Ising model with appropriate boundary conditions (a certainly crucial point that we deliberately overlook), to an average between two spins operators, $\sigma$, and a disorder operator $\mu$ [83], i.e. a domain wall, each having at criticality conformal dimensions 1/16. Recalling that the rational variable $x = e^{2T}$ for our PV$_V$ then also corresponds to the cross-ratio of the location of the four operators with dimensions $\Delta_\mu$, the above guess gives, for large separation $x$,

$$F_2(T) \propto \langle 1 \sigma \sigma \mu \rangle_{\text{Ising} T_x} \propto \frac{1}{|x|^{1/16}} \propto e^{-3T/8} \propto (t_1/t_2)^{3/8},$$

(24)
a back-of-the-envelope argument that at least has the virtue of giving the correct answer for $\theta(0)$. Believing in Eq. (24), and since in terms of the Onsager-Yang spontaneous magnetization critical exponent $\beta = 1/8$, the conformal dimension at criticality of both spin- and disorder-operator is $\beta/2$, this would relate in a noticeable fashion the nonequilibrium persistence exponent for the 1d-Ising model (with symmetric initial conditions) to ordinary static critical behavior for the 2d-Ising model:

$$\theta(0) = 3\beta = 3/8.$$ 

(25)

Given the way the $m$-dependence appears in Eq. (12), it could even be possible to recover the general expression for $\theta(m)$ by twisting properly each of the two underlying 2d-Ising models in the conformal field theory description, in order to accomodate the $m$-dependence through the corresponding continuously-varying magnetization exponent in some wedge-shaped geometry [86]. More generally, we believe that methods of boundary conformal field theory should be helpful in determining the critical properties of other unsolved nonequilibrium 1d-reaction-diffusion processes, for instance the one studied in [87], by unfolding in one more space dimension a temporal quantity corresponding to a (non-local) observable here, and connecting it to a suitable and simpler observable up there.

As an overall general conclusion, the specific PV$_V$ probability distribution functions exhibited here in the context of persistenee provide a particularly vivid example in the emerging field of stochastic integrability [88] or integrable probability [89]. There are also other indications [59] that this is just the tip of the iceberg of many other non-trivial universal limit distributions for correlated random variables involving Painlevé transcendents that remain to be discovered. It is also hoped that their universal behavior will one day incarnates into genuine nonequilibrium physical systems, with experiments as striking as those conducted for KPZ-growing interfaces associated to the P$_{1\text{II}}$ Tracy-Widom distributions [41, 42]. The correspondence between a generic PV$_V$ and Bonnet surfaces established in [60, 61] should also explain [90], by the classical confinement along the Painlevé hierarchy all the way down to P$_{1\text{II}}$, why geometry-dependent universality classes are observed in KPZ growth, where these nonequilibrium interfaces forever remembers the initial curvature they had.

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assumption, just valid when $1 \ll t_1 \ll t_2$, when one uses the scaling for the no-meeting probability between a pair of random walks. Note however that Eq. (8) yields the correct answer down to $T \to 0^+$. The eventual expression obtained for the persistence probability $P_{\text{Half}}^{1, r}(T;m)$ does also correspond to a well-normalized distribution function in the $T$ variable, with $F_1(0) = 1$ and $F_1(T) \to 0$ as $T \to \infty$. This probably means that the space-continuous limit of the voter model \[46\], where one replaces coalescing random walks on a lattice by Brownian paths starting at every point of the real line, is also operative to compute the time-continuous limit implicit in the determination of the persistence probabilities, at least when one starts from an entrance law for the spin initial configuration that is either completely random or with sufficient short-ranged correlations with overall magnetization $m$. We leave the justification of this fact as an open mathematical problem.

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