PERIOD TRIPLING AND QUINTUPLING RENORMALIZATIONS BELOW $C^2$ SPACE

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Abstract. In this paper, we explore the period tripling and period quintupling renormalizations below $C^2$ class of unimodal maps. We show that for a given proper scaling data there exists a renormalization fixed point on the space of piece-wise affine maps which are infinitely renormalizable. Furthermore, we show that this renormalization fixed point is extended to a $C^{1+Lip}$ unimodal map, considering the period tripling and period quintupling combinatorics. Moreover, we show that there exists a continuum of fixed points of renormalizations by considering a small variation on the scaling data. Finally, this leads to the fact that the tripling and quintupling renormalizations acting on the space of $C^{1+Lip}$ unimodal maps have unbounded topological entropy.

1. Introduction. The concept of renormalization arises in many forms though Mathematics and Physics. Renormalization is a technique to describe the dynamics of a given system at a small spatial scale by an induced dynamical system in the same class. Period doubling renormalization operator was introduced by M. Feigenbaum [6], [7] and by P. Coullet and C. Tresser [3], to study asymptotic small scale geometry of the attractor of one dimensional systems which are at the transition from simple to chaotic dynamics.

The hyperbolicity of unique renormalization fixed point has been proven by O. Lanford [8] for period doubling operator through a computer assisted proof and later M. Lyubich [9] gave the generalization to the other sort of renormalizations, in the holomorphic context. Then it was shown by A. Davie [4], the renormalization fixed point is also hyperbolic in the space of $C^{2+\alpha}$ unimodal maps with $\alpha > 0$. These results further extended by E. de Faria, W. de Melo and A. Pinto [5] to a more general type of renormalization, using the results of M. Lyubich [9].

Later, it was extended by Chandramouli, Martens, de Melo, Tresser [2], to a new class denoted by $C^{2+|\cdot|}$ which is bigger than $C^{2+\alpha}$ (for any positive $\alpha \leq 1$), in which the period doubling renormalization converges to the analytic generic fixed point proving it to be globally unique. Furthermore it was shown that below $C^2$, uniqueness is lost and period doubling renormalization operator has infinite topological entropy.

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One dimensional systems which arise from the real time applications are usually not smooth. In dissipative systems the states are groups in so-called stable manifolds, different states in such a stable manifold have the same future. The packing of the stable manifold usually does not occur in a smooth way. For example, the Lorenz flow is a flow on three dimensional space approximating a convection problem in fluid dynamics. The stable manifolds are two dimensional surfaces packed in a non smooth foliation. This flow can be understood by a map on the interval whose smoothness is usually below \( C^2 \).

In this work, we describe the construction of renormalization fixed point below \( C^2 \) space by considering period tripling and quintupling combinatorics. In the case of period tripling renormalization, for a given proper scaling data, we construct a nested sequence of affine pieces whose end-points lie on the unimodal map and shrinking down to the critical point. Also, we prove that the period tripling renormalization operator defined on the space of piece-wise affine infinitely renormalizable maps, has a fixed point, denoted by \( f_{s^*} \), corresponding to a given proper scaling data. In the next subsection 2.2, we describe the extension of this renormalization fixed point \( f_{s^*} \) to a \( C^{1+Lip} \) unimodal map. Furthermore, in subsection 2.3, we show that the topological entropy of renormalization defined on the space of \( C^{1+Lip} \) unimodal maps is unbounded. In subsection 2.4, we consider an \( \epsilon \)-variation on the scaling data and describe the existence of continuum of renormalization fixed points. In section 3, we describe the construction of period quintupling renormalization fixed point \( g_{s^*} \) and its extension to \( C^{1+Lip} \) space. Finally, we show that the geometry of the invariant Cantor set of the map \( g_{s^*} \) is more complex than the geometry of the invariant Cantor set of \( f_{s^*} \).

We recall some basic definitions. Let \( I = [a, b] \) be a closed interval.

A point \( c \in I \) is called critical point of a map \( f : I \to I \) if either \( f'(c) = 0 \) or \( f'(c) \) does not exist. The critical point \( c \) is said to be non-flat critical point of order \( l \) if \( f \) is \( C^{l+1} \) in a neighborhood of \( c \) and \( f'(c) = f''(c) = \cdots = f^{(l-1)}(c) = 0 \) and \( f^{(l)}(c) \neq 0 \).

A map \( f : I \to I \), is a \( C^1 \) map with a unique critical point \( c \in I \), is called unimodal map.

Let \( c \) be a critical point of a unimodal map \( f \) has a quadratic tip if there exists a sequence \( \{y_n\} \) approaches to \( c \) and a constant \( l > 0 \) such that

\[
\lim_{n \to \infty} \frac{f(y_n) - f(c)}{(y_n - c)^2} = -l.
\]

A interval \( J \subset I \) is called a periodic interval of period \( n \) if \( f^n(J) = J \).

A unimodal map \( f \) is called period \( n \)-renormalizable if there exists a proper subinterval \( J \) of \( I \) and a positive integer \( n \geq 2 \) such that

1. \( f^i(J), \ i = 0, 1, \ldots, n-1 \) have no pairwise interior intersection,
2. \( f^n(J) \subset J \).

Then \( f^n : J \to J \) is called a pre-renormalization of \( f \).

Note that, for \( n = 2 \) the map \( f \) is called period doubling renormalizable map and for \( n = 3 \) the map \( f \) is called period tripling renormalizable map.

A map \( f : I \to I \) is infinitely renormalizable map if there exists an infinite sequence \( \{I_n\}_{n=0}^{\infty} \) of nested intervals and an infinite sequence \( \{k(n)\}_{n=0}^{\infty} \) of positive integers such that \( f^{k(n)}|_{I_n} : I_n \to I_n \) are pre-renormalizations of \( f \) and the length of \( I_n \) tends to zero as \( n \to \infty \).
Let $U$ be the set of unimodal maps and $U_0 \subset U$ contains the set of period doubling renormalizable unimodal maps. Let $f \in U_0$. Then, the period doubling renormalization operator

$$R : U_0 \to U$$

is defined by

$$Rf(x) = h^{-1} \circ f^2 \circ h(x),$$

where $h : [0, 1] \to J$ is the orientation reversing affine homeomorphism. The map $Rf$ is again a unimodal map. The set of period doubling infinitely renormalizable maps is denoted by

$$W = \bigcap_{n \geq 1} R^{-n}(U_0).$$

In the next section, we construct a fixed point of period tripling renormalization.

2. Period tripling renormalization.

2.1. Piece-wise affine renormalizable maps. Let us consider a family of maps $\mathcal{U}_c : [0, 1] \to [0, 1]$ defined by

$$\mathcal{U}_c(x) = 1 - \left| \frac{x - c}{1 - c} \right|^\alpha.$$

Where $\alpha > 1$ is the critical exponent and $c \in [0, \frac{1}{2}]$ is the critical point.

In particular, for $\alpha = 2$, $c$ is a non-flat critical point of order 2 (i.e., folding type critical point) and the subclass of unimodal maps $\mathcal{U}_c(x)$ is denoted by $u_c(x)$.

Define an open set

$$T_k = \left\{ (s_1, s_2, s_3, \ldots, s_k) \in \mathbb{R}^k : s_1, s_2, s_3, \ldots, s_k > 0, \sum_{i=1}^k s_i < 1 \right\}, \ k \in \mathbb{N}.$$

For $k = 3$, each element $(s_1, s_2, s_3)$ of $T_3$ is called a scaling tri-factor.

Define affine maps $\tilde{s}_1$, $\tilde{s}_2$ and $\tilde{s}_3$ induced by a scaling tri-factor,

$$\tilde{s}_1 : [0, 1] \to [0, 1]$$

$$\tilde{s}_2 : [0, 1] \to [0, 1]$$

$$\tilde{s}_3 : [0, 1] \to [0, 1]$$

defined by

$$\tilde{s}_1(t) = s_1(1 - t)$$

$$\tilde{s}_2(t) = u_c(0) + s_2(1 - t)$$

$$\tilde{s}_3(t) = 1 - s_3(1 - t).$$

A function $s : \mathbb{N} \to T_k$ is called a scaling data. For $k = 3$, for all $n \in \mathbb{N}$, the scaling tri-factor $s(n) = (s_1(n), s_2(n), s_3(n)) \in T_3$ induces a triplet of affine maps $(\tilde{s}_1(n), \tilde{s}_2(n), \tilde{s}_3(n))$. Now, for each $n \in \mathbb{N}$, we define the following intervals:

$$I^1_n = \tilde{s}_2(1) \circ \tilde{s}_2(2) \circ \tilde{s}_2(3) \circ \ldots \circ \tilde{s}_2(n - 1) \circ \tilde{s}_1(n)([0, 1]),$$

$$I^2_n = \tilde{s}_3(1) \circ \tilde{s}_2(2) \circ \tilde{s}_2(3) \circ \ldots \circ \tilde{s}_2(n - 1) \circ \tilde{s}_3(n)([0, 1]),$$

$$I^3_n = \tilde{s}_2(1) \circ \tilde{s}_2(2) \circ \tilde{s}_2(3) \circ \ldots \circ \tilde{s}_2(n - 1) \circ \tilde{s}_3(n)([0, 1]).$$
**Definition 2.1.** A scaling data is said to be proper if, for all \( n \in \mathbb{N} \),
\[
d(s(n), \partial T_k) \geq \epsilon,
\]
for some \( \epsilon > 0 \).

Also, we have
\[
|I_j^n| \leq (1 - \epsilon)^n, \quad \text{for } n \geq 1 \text{ and } j = 1, 2, 3.
\]

Given a proper scaling data, define
\[
\{ c \} = \bigcap_{n \geq 1} I_{2}^{n}.
\]

A proper scaling data \( s : \mathbb{N} \to T_3 \) induces the set \( D_s = \bigcup_{n \geq 1} (I_1^n \cup I_3^n) \). Consider a map
\[
f_s : D_s \to [0, 1]
\]
such that \( f_s|_{I_1^n} \) and \( f_s|_{I_3^n} \) are the affine extensions of \( u_c|_{\partial I_1^n} \) and \( u_c|_{\partial I_3^n} \) respectively.

![Figure 1. Intervals of next generations](image)

From Figure 1, the end points of the intervals at each level are labeled by
\[
z_0 = 0, \quad y_0 = 1, \quad I_2^0 = [0, 1]
\]
and for \( n \geq 1 \)
\[
x_n = \partial I_1^n \setminus \partial I_2^{n-1}
\]
\[
y_{2n-1} = \min\{\partial I_2^{2n-1}\}
\]
\[
y_{2n} = \max\{\partial I_2^{2n}\}
\]
\[
z_{2n-1} = \max\{\partial I_2^{2n-1}\}
\]
\[
z_{2n} = \min\{\partial I_2^{2n}\}
\]
\[
w_n = \partial I_3^n \setminus \partial I_2^{n-1}.
\]

**Definition 2.2.** For a given proper scaling data \( s : \mathbb{N} \to T_3 \), a map \( f_s \) is said to be

*period tripling infinitely renormalizable* if for \( n \geq 1 \),

(i) \( [f_s(y_n), 1] \) is the maximal domain containing 1 on which \( f_s^{3^n-1} \) is defined affinely and \( [0, f_s^2(y_n)] \) is the maximal domain containing 0 on which \( f_s^{3^n-2} \) is defined affinely,

(ii) \( f_s^{3^n-1}([f_s(y_n), 1]) = I_2^k \), \( f_s^{3^n-2}([0, f_s^2(y_n)]) = I_2^k \).
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Figure 2. Period triple interval combinatorics ($I^n_2 \to I^n_3 \to I^n_1 \to I^n_2$).

Define $U_\infty = \{ f_s : f_s \text{ is period tripling infinitely renormalizable map} \}$.

The combinatorics for period tripling renormalization is shown in Figure 2.

Let $f_s \in U_\infty$ be given by the proper scaling data $s : \mathbb{N} \to T_3$ and define

\[ \hat{I}^n_2 = [0, u^2_c(y_n)] = [0, f^2_s(y_n)], \]

and

\[ \hat{I}^n_2 = [u_c(y_n), 1] = [f_s(y_n), 1]. \]

Let

\[ h_{s,n} : [0, 1] \to \hat{I}^n_2 \]

be defined by

\[ h_{s,n} = \hat{s}_2(1) \circ \hat{s}_2(2) \circ \hat{s}_2(3) \circ \ldots \circ \hat{s}_2(n). \]

Furthermore, let

\[ \hat{h}_{s,n} : [0, 1] \to \hat{I}^n_2 \text{ and } \hat{h}_{s,n} : [0, 1] \to \hat{I}^n_2 \]

be the affine orientation preserving homeomorphisms.

Then define

\[ R_n f_s : h_{s,n}^{-1}(D_s \cap I^n_2) \to [0, 1] \]

by

\[ R_n f_s(x) = \begin{cases} R^- f_s(x), & \text{if } x \in h_{s,n}^{-1}(\bigcup_{m \geq n+1} I^m_3) \\ R^+ f_s(x), & \text{if } x \in h_{s,n}^{-1}(\bigcup_{m \geq n+1} I^m_3) \end{cases} \]

where

\[ R^- f_s : h_{s,n}^{-1}(\bigcup_{m \geq n+1} I^m_3) \to [0, 1] \]

and

\[ R^+ f_s : h_{s,n}^{-1}(\bigcup_{m \geq n+1} I^m_3) \to [0, 1] \]

are defined by

\[ R^- f_s(x) = h_{s,n}^{-1} \circ f^2_s \circ h_{s,n}(x) \]

\[ R^+ f_s(x) = h_{s,n}^{-1} \circ f_s \circ h_{s,n}(x), \]

which are illustrated in Figure 3.

Let $\sigma : T^n_k \to T^n_k$ be the shift map $\sigma(s)(n) = s(n + 1)$. The above construction gives us the following result,

Lemma 2.3. Let $s : \mathbb{N} \to T_3$ be proper scaling data such that $f_s$ is infinitely renormalizable. Then

\[ R_n f_s = f_{\sigma^n(s)}. \]
Let $f_s$ be infinitely renormalizable, then for $n \geq 0$, we have

$$f_s^{3^n} : D_s \cap I_2^n \to I_2^n$$

is well defined.

Now we define the renormalization $R : U_\infty \to U_\infty$ as

$$Rf_s = h_s^{-1} \circ f_s^3 \circ h_s.$$

The maps $f_s^{3^n-2} : \tilde{I}_2^n \to \tilde{I}_2^n$ and $f_s^{3^n-1} : \tilde{I}_2^n \to \tilde{I}_2^n$ are the affine homeomorphisms whenever $f_s \in U_\infty$. Then we have

**Lemma 2.4.** $R^n f_s : D_{\sigma^n(s)} \to [0, 1]$ and $R^n f_s = R_n f_s$.

The lemma 2.3 and lemma 2.4 give the following result.

**Proposition 1.** There exists a map $f_{s^*} \in U_\infty$, where $s^*$ is characterized by

$$R f_{s^*} = f_{s^*}.$$

In particular, $U_\infty = \{f_{s^*}\}$.

**Proof.** Consider $s : \mathbb{N} \to T_3$ be proper scaling data such that $f_s$ is an infinitely renormalizable. Let $c_n$ be the critical point of $f_{\sigma^n(s)}$. Then from Figures 4 and 5, we have the following

1. $u_{c_n}^2(0) = 1 - s_3(n)$
2. $u_{c_n}^3(0) = u_{c_n}(1 - s_3(n))$
   $$= s_1(n)$$
3. $u_{c_n}^4(0) = s_2(n) + u_{c_n}(0)$
4. $c_{n+1} = \frac{u_{c_n}^4(0) - c_n}{s_2(n)}$.

Since $(s_1(n), s_2(n), s_3(n)) \in T_3$, we have the following conditions

$$s_i(n) > 0, \text{ for } i = 1, 2, 3$$

$$s_1(n) + s_2(n) + s_3(n) < 1$$

As the intervals $I_i^n$, for $i = 1, 2, 3$, are pairwise disjoint, let $g_i^n$ and $g_i^n$ are the lengths of the gaps which are in between $I_i^n$ & $I_2^n$ and $I_2^n$ & $I_3^n$ respectively. The
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Figure 4. Period three cobweb diagram

\[
\begin{array}{c|c|c|c}
I_1^n & I_2^n & I_3^n \\
0 & u_{c_n}^2(0) & u_{c_3}^4(0) & u_{c_n}^2(0) & 1 \\
\hline
s_1(n) & g_l^n & s_2(n) & g_r^n & s_3(n) \\

\end{array}
\]

Figure 5. Length of the intervals gaps

gaps are also shown in Figure 5. Then we have, for $n \in \mathbb{N}$,

\[
g_l^n = u_{c_n}(0) - u_{c_n}^2(0) \equiv G_l(c_n) > 0 \tag{7}
\]

\[
g_r^n = u_{c_n}^2(0) - u_{c_n}^4(0) \equiv G_r(c_n) > 0 \tag{8}
\]

\[
0 < c_n < \frac{1}{2} \tag{9}
\]

Solving Eqns. (1), (2), (3) and (4) by using Mathematica, we get

\[
s_1(n) = 1 - \frac{c_n - 8c_n^2 + 21c_n^3 - 25c_n^4 + 17c_n^5 - 6c_n^6 + c_n^7}{1 - c_n}^2 \equiv S_1(c_n)
\]

\[
s_2(n) = \frac{c_n^2(1 - c_n)^2}{(1 - c_n)^{30}} - \left( (-1 + c_n)^{15} + (c_n - 8c_n^2 + 21c_n^3 - 25c_n^4 + 17c_n^5 - 6c_n^6 + c_n^7)^2 \right)^2
\]

\[
\equiv S_2(c_n)
\]

\[
s_3(n) = \frac{(-1 + 3c_n - 2c_n^2 + c_n^3)^2}{(1 - c_n)^{20}} \equiv S_3(c_n)
\]

\[
c_{n+1} = \frac{(1 - c_n)^{11} - ((1 - c_n)^{15} - (c_n - 8c_n^2 + 21c_n^3 - 25c_n^4 + 17c_n^5 - 6c_n^6 + c_n^7)^2)^2}{c_n^2(1 - c_n)^{28} - ((1 - c_n)^{15} - (c_n - 8c_n^2 + 21c_n^3 - 25c_n^4 + 17c_n^5 - 6c_n^6 + c_n^7)^2)^2}
\]

\[
\equiv R(c_n) \tag{10}
\]

The graphs of $S_1$, $S_2$, $S_3$ and $S_1 + S_2 + S_3$ are shown in Figure 6. Note that the conditions (5), (7) and (8) give the condition (6)

\[
0 < \sum_{i=1}^{3} s_i(n) < 1.
\]
Figure 6. (a), (b), (c) and (d) show the graphs of $S_1(c)$, $S_2(c)$, $S_3(c)$ and $(S_1 + S_2 + S_3)(c)$ respectively.

Therefore, the condition (5) together with (7), (8) and (9) define the feasible domain $F_d$ to be:

$$F_d = \left\{ c \in \left(0, \frac{1}{2}\right) : S_i(c) > 0 \text{ for } i = 1, 2, 3, G_l(c) > 0, G_r(c) > 0 \right\}$$  \hspace{1cm} (11)

To compute the feasible domain $F_d$, we need to find subinterval(s) of $(0, 0.5)$ which satisfies the conditions of (11). By using Mathematica software, we employ the following command to obtain the feasible domain

$$\text{N[Reduce[}\{S_1(c) > 0, S_2(c) > 0, S_3(c) > 0, G_l(c) > 0, G_r(c) > 0, 0 < c < 0.5\},c]\}.$$

This yields:

$$F_d = (0.398039..., 0.430159...) \cup (0.430159..., 0.456310...) \equiv F_{d_1} \cup F_{d_2}.$$

From the Eqn.(10), the graphs of $R(c)$ are plotted in the sub-domains $F_{d_1}$ and $F_{d_2}$ of $F_d$ which are shown in Figure 7.

Figure 7. The graph of $R : F_d \rightarrow \mathbb{R}$ and the diagonal $R(c) = c$. 

(a) $R$ has no fixed point in $F_{d_1}$.  \hspace{1cm} (b) $R$ has only one fixed point in $F_{d_2}$. 

Clearly, the map \( R : F_d \to \mathbb{R} \) is expanding in the neighborhood of the fixed point \( c^* \) which is shown in Figure 7b. By using Mathematica, we compute the only fixed point \( c^* = 0.440262... \) in \( F_d \) such that

\[
R(c^*) = c^*
\]

corresponds to the infinitely renormalizable map \( f_{s^*} \).

The graphs of piece-wise infinitely renormalizable map \( f_{s^*} \) and zoomed part of \( f_{s^*} \) are shown in the Figure 8.

**Figure 8.** The graph of map \( f_{s^*} \).

In other words, consider the scaling data \( s^* : \mathbb{N} \to T_3 \), with

\[
s^*(n) = (s^*_1(n), s^*_2(n), s^*_3(n)) = (u^3_{c^*}(0), u^4_{c^*}(0) - u^2_{c^*}(0), 1 - u^2_{c^*}(0)).
\]

Then \( \sigma(s^*) = s^* \) and using Lemma 2.3 we have

\[
Rf_{s^*} = f_{s^*}.
\]

**Remark 1.** Let \( I^n_2 = [y_n, z_n] \) be the interval containing \( c^* \) corresponding to the scaling data \( s^* = (s^*_1, s^*_2, s^*_3) \) then

\[
f_{s^*}(y_n) = u_{c^*}(y_n).
\]

Hence, \( f_{s^*} \) has a quadratic tip.

**Lemma 2.5.** If \( f_{s^*} \) is the map with a proper scaling data \( s^* = (s^*_1, s^*_2, s^*_3) \), then we have

\[
(s^*_2)^2 = s^*_3.
\]

**Proof.** Let \( \hat{I}^n_2 = f_{s^*}(I^n_2) = [f_{s^*}(y_n), 1] \) and \( \hat{I}^{n+1}_3 = f_{s^*}(I^{n+1}_3) \). Then \( f^{n+1}_{s^*} : \hat{I}^n_2 \to I^n_2 \) is affine, monotone and onto. Further, by construction

\[
f^{n+1}_{s^*} : \hat{I}^n_2 \to I^{n+1}_3.
\]

Hence,

\[
\frac{|\hat{I}^{n+1}_2|}{|I^n_2|} = s^*_3.
\]

Therefore, \( |I^n_2| = (s^*_2)^n \) and \( |\hat{I}^n_2| = (s^*_3)^n \). Since,

\[
f_{s^*}(y_n) = u_{c^*}(y_n).
\]
This implies,
\[ s_3^* = \frac{I_2^{n+1}}{|I_2^n|} = \left( \frac{y_{n+1} - c^*}{y_n - c^*} \right)^2 = \left( \frac{I_2^{n+1}}{|I_2^n|} \right)^2 = (s_2^*)^2. \]

\[ \Box \]

**Remark 2.** A proper scaling data \((s_1^*, s_2^*, s_3^*)\) is satisfying the following properties, for all \(n \in \mathbb{N},\)

1. \( \frac{I_2^{n+1}}{|I_2^n|} = s_2^* \) and \( \frac{I_2^{n+1}}{|I_2^n|} = \frac{I_2^{n+1}}{|I_2^n|} = s_2^* \)
2. \( \frac{I_2^{n+1}}{|I_2^n|} = s_1^* \)
3. \( \frac{I_2^{n+1}}{|I_2^n|} = s_3^* \)

**Remark 3.** The invariant Cantor set of the map \(f_{s^*}\) is more complex than the invariant period doubling Cantor set of piece-wise affine infinitely renormalizable map \([2]\). The complexity in the sense that:

1. The geometry of Cantor set of \(f_{s^*}\) is similar to the geometry of Cantor set of \([2]\), on each scale and everywhere the same scaling ratios are used,
2. But unlike the Cantor set of \([2]\), there are now three ratios at each scale.

Furthermore, the scaling data corresponding to \(f_{s^*}\) is very different from the geometry of Cantor attractor of analytic renormalization fixed point, in which there are no two places where the same scaling ratios are used at all scales and where the closure of the set of ratios itself a Cantor set \([1]\).

2.2. \(C^{1+Lip}\) extension of \(f_{s^*}\). In subsection 2.1, we constructed a piece-wise affine infinitely renormalizable map \(f_{s^*}\) corresponding to the proper scaling data \(s^*\).

Our goal is to construct an extension of \(f_{s^*}\) in the class \(C^{1+Lip}\), consisting of infinitely renormalizable maps.

Let \(F : [0, 1]^2 \to [0, 1]^2\) be the scaling function defined as
\[ F \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} u_{c^*}(0) + s_2^*(1 - x) \\ 1 - s_3^*(1 - y) \end{array} \right) = \left( \begin{array}{c} F_1(x) \\ F_2(y) \end{array} \right) \]

Let \(K\) be the graph of \(F_{s^*}\), which is an extension of \(f_{s^*}\), where \(f_{s^*} : D_{s^*} \to [0, 1].\)

Let \(K^1\) and \(K^2\) be the graph of \(F_{s^*}|_{[1.90]}\) and \(F_{s^*}|_{[0.91]}\), respectively, as shown in Figure 9. Then
\[ K = \cup_{m \geq 0} F^m(K^1 \cup K^2) \]

is the graph of \(F_{s^*}\). We claim that \(F_{s^*}\) is a \(C^{1+Lip}\) unimodal map with quadratic tip.

Let \(B_0 = [0, 1] \times [0, 1]\) and for each \(n \in \mathbb{N}\), define \(B_n = F^n(B_0)\) as: \(B_n = \begin{cases} [y_n, z_n] \times [\tilde{y}_n, 1], & \text{if } n \text{ is odd} \\ [z_n, y_n] \times [\tilde{z}_n, 1], & \text{if } n \text{ is even} \end{cases} \)

Also, let \(p_n\) be the point on the graph of the unimodal map \(u_{c^*}(x)\). For all \(n \in \mathbb{N},\)
\[ p_n = \begin{cases} \left( \frac{y_{n+1}}{y_n}, \frac{y_{n+1}}{y_n} \right), & \text{if } n \text{ is odd} \\ \left( \frac{z_n}{\tilde{z}_n}, \frac{z_n}{\tilde{z}_n} \right), & \text{if } n \text{ is even} \end{cases} \]

where \(\tilde{y}_n = u_{c^*}(y_n)\) and \(\tilde{z}_n = u_{c^*}(z_n)\).

The sequence of points \((p_1, p_2, \ldots, p_n, \ldots)\) are shown in Figure 9.

Then the above construction of boxes \(B_n\) together with the points \(p_n\) will lead to following proposition,
Proposition 2. \( K \) is the graph of \( f_s^* \) which is a \( C^1 \) extension of \( f_s^* \).

Proof. Since \( K^1 \) and \( K^2 \) are the graph of \( f_s^*|_{[z_1,y_0]} \) and \( f_s^*|_{[y_0,z_1]} \), respectively, we obtain \( K^{2n+1} = F^n(K^1) \) and \( K^{2n+2} = F^n(K^2) \) for each \( n \in \mathbb{N} \).

Note that \( K^n \) is the graph of a \( C^1 \) function defined on \( [y_{n-1}, y_n+1] \) if \( n \in 4N-1 \), on \( [y_2, y_0-1] \) if \( n \in 4N \), on \( [z_n+1, y_{n-1}] \) if \( n \in 4N+1 \), and on \( [z_{n-1}, y_2] \) if \( n \in 4N+2 \).

To prove the proposition, we have to check continuous differentiability at the points \( p_n \). Consider a neighborhoods \((y_1 - \epsilon, y_1 + \epsilon)\) around \( y_1 \) and \((z_1 - \epsilon, z_1 + \epsilon)\) around \( z_1 \), the slopes are given by an affine pieces of \( f_s^* \) on the subintervals \((y_1, y_1 + \epsilon)\) and \((z_1, z_1 + \epsilon)\) and the slopes are given by the chosen \( C^1 \) extension on \((y_1 - \epsilon, y_1)\) and \((z_1, z_1 + \epsilon)\). This implies, \( K^1 \) and \( K^2 \) are \( C^1 \) at \( p_2 \) and \( p_1 \), respectively.

Let \( \Upsilon_1 \subset K \) be the graph over the interval \((z_1 - \epsilon, z_1 + \epsilon)\) and \( \Upsilon_2 \subset K \) be the graph over the interval \((y_1 - \epsilon, y_1 + \epsilon)\),

then the graph \( K \) locally around \( p_n \) is equal to \[
\begin{cases} 
F_{\frac{n-1}{2}}(\Upsilon_2) & \text{if } n \text{ is odd} \\
F_{\frac{n-2}{2}}(\Upsilon_1) & \text{if } n \text{ is even}
\end{cases}
\]

This implies, for \( n \in \mathbb{N} \), \( K^{2n-1} \) is \( C^1 \) at \( p_{2n} \) and \( K^{2n} \) is \( C^1 \) at \( p_{2n-1} \).

Hence \( K \) is a graph of a \( C^1 \) function on \([0,1] \setminus \{c^*\}\).

From Lemma 2.5, we observe that the horizontal contraction of \( F \) is smaller than the vertical contraction. This implies that the slope of \( K^n \) tends to zero when \( n \) is large.

Therefore, \( K \) is the graph of a \( C^1 \) function on \([0,1]\). \qed
Proposition 3. Let \( F_{s^*} \) be the function whose graph is \( K \) then \( F_{s^*} \) is a \( C^{1+\text{Lip}} \) unimodal map with a quadratic tip.

Proof. As the function \( F_{s^*} \) is a \( C^1 \) extension of \( f_{s^*} \) and \( f_{s^*}|_{\partial s^*} \) has a quadratic tip, therefore \( F_{s^*} \) has a quadratic tip. We have to show that \( K^n \) is the graph of a \( C^{1+\text{Lip}} \) function

\[
F_{s^*}^n : \text{Dom}(K^n) \to [0, 1]
\]

with an uniform Lipschitz bound.

That is, for \( n \geq 1 \),

\[
\text{Lip}(F_{s^*}^{n+1})' \leq \text{Lip}(F_{s^*}^n)'
\]

Let us assume that \( F_{s^*}^n \) is \( C^{1+\text{Lip}} \) with Lipschitz constant \( \lambda_n \) for its derivatives. We show that \( \lambda_{n+1} \leq \lambda_n \).

For given a point \( \left( \begin{array}{c} a \\ b \end{array} \right) \) on the graph of \( F_{s^*}^n \), there is a point \( \left( \begin{array}{c} \bar{a} \\ \bar{b} \end{array} \right) = F_{s^*}^{-1}(a) \) on the graph of \( F_{s^*}^{n+1} \), this implies

\[
F_{s^*}^{n+1}(\bar{a}) = 1 - s_3^* + s_3^* \cdot F_{s^*}^n(a)
\]

Since \( a = 1 - \frac{\bar{a} - u_{s^*}(0)}{s_2^*} \), we have

\[
F_{s^*}^{n+1}(\bar{a}) = 1 - s_3^* + s_3^* \cdot F_{s^*}^n \left( 1 - \frac{\bar{a} - u_{s^*}(0)}{s_2^*} \right)
\]

Differentiate both sides with respect to \( \bar{a} \), we get

\[
(F_{s^*}^{n+1})'(\bar{a}) = \frac{-s_3^*}{s_2^*} (F_{s^*}^n)' \left( 1 - \frac{\bar{a} - u_{s^*}(0)}{s_2^*} \right)
\]

Therefore,

\[
| (F_{s^*}^{n+1})'(\bar{a}_1) - (F_{s^*}^{n+1})'(\bar{a}_2) | \\
= \left| \frac{-s_3^*}{s_2^*} \right| \cdot \left| (F_{s^*}^n)' \left( 1 - \frac{\bar{a}_1 - u_{s^*}(0)}{s_2^*} \right) - (F_{s^*}^n)' \left( 1 - \frac{\bar{a}_2 - u_{s^*}(0)}{s_2^*} \right) \right| \\
\leq \frac{s_3^*}{(s_2^*)^2} \cdot \lambda(F_{s^*}^n)' |\bar{a}_1 - \bar{a}_2| \tag{12}
\]

From Lemma 2.5, we have \((s_2^*)^2 = s_3^*\). Therefore,

\[
\lambda(F_{s^*}^{n+1})' \leq \lambda(F_{s^*}^n)' \leq \lambda(F_{s^*}^n)'
\]

This completes the proof. \( \square \)

Note that \( f_{s^*} \) is infinitely renormalizable piece-wise affine map and \( F_{s^*} \) is the \( C^{1+\text{Lip}} \) extension of \( f_{s^*} \) which is not a \( C^2 \) map. Then it implies that \( F_{s^*} \) is renormalizable. We observe that \( R F_{s^*} \) is an extension of \( R f_{s^*} \). Therefore \( R F_{s^*} \) is renormalizable. Hence, \( F_{s^*} \) is infinitely renormalizable map. Then we have the following theorem,

Theorem 2.6. There exists a period tripling infinitely renormalizable \( C^{1+\text{Lip}} \) unimodal map \( F_{s^*} \) with a quadratic tip such that

\[
R F_{s^*} = F_{s^*}.
\]
2.3. Topological entropy of renormalization. In this section, we calculate the topological entropy of period tripling renormalization operator.

Let us consider three $C^{1+\text{Lip}}$ maps $\psi_0 : [0, y_1] \cup [z_1, 1] \rightarrow [0, 1]$, $\psi_1 : [0, y_1] \cup [z_1, 1] \rightarrow [0, 1]$, and $\psi_2 : [0, y_1] \cup [z_1, 1] \rightarrow [0, 1]$ which extend $f_{s^*}$. For a sequence $\beta = \{\beta_n\}_{n \geq 1} \in \Sigma_3$, where $\Sigma_3 = \{(x_n)_{n \geq 1} : x_n \in \{0, 1, 2\}\}$ is called full 3-Shift. Now define

$$K^n(\beta) = E^n(\text{graph } \psi_{\beta_n}),$$

we have

$$K(\beta) = \bigcup_{n \geq 1} K^n(\beta).$$

Therefore, we conclude that $K(\beta)$ is the graph of a $C^{1+\text{Lip}}$ map $f_\beta$ having quadratic tip by using the same facts of subsection 2.2.

The shift map $\sigma : \Sigma_3 \rightarrow \Sigma_3$ is defined as

$$\sigma(\beta_1\beta_2\beta_3\ldots) = (\beta_2\beta_3\beta_4\ldots).$$

**Proposition 4.** The map $f_\beta^3 : [y_1, z_1] \rightarrow [y_1, z_1]$ is a unimodal map for all $\beta \in \Sigma_3$. Furthermore, $f_\beta$ is period tripling renormalizable and $Rf_\beta = f_{\sigma(\beta)}$.

**Proof.** We know that $f_\beta : [y_1, z_1] \rightarrow I_3^1$ is a unimodal and onto, $f_\beta : I_3^1 \rightarrow I_1^1$ is onto and affine and also $f_\beta : I_1^1 \rightarrow [y_1, z_1]$ is onto and affine. Therefore $f_\beta$ is renormalizable. The above construction implies

$$Rf_\beta = f_{\sigma(\beta)}.$$ 

This gives us the following theorem.

**Theorem 2.7.** The period tripling renormalization operator $R$ defined on the space of $C^{1+\text{Lip}}$ unimodal maps has positive topological entropy.

**Proof.** From the above construction, we conclude that $\beta \mapsto f_\beta \in C^{1+\text{Lip}}$ is injective. The domain of $R$ contains a subset $\Lambda$ on which $R$ is topological conjugate to the full 3-shift. As topological entropy $h_{\text{top}}$ is an invariant of topological conjugacy. Hence $h_{\text{top}}(R|_\Lambda) \geq \ln 3$. 

In fact, the topological entropy of period tripling operator $R$ on $C^{1+\text{Lip}}$ unimodal maps is unbounded because if we choose $n$ different $C^{1+\text{Lip}}$ maps, say, $\psi_0, \psi_1, \psi_2, \ldots \psi_{n-1}$, which extends $f_{s^*}$, then it will be embedded a full $n$-shift in the domain of $R$.

2.4. An $\epsilon$-variation of the scaling data. In this section, we use an $\epsilon$ variation on the construction of scaling data as presented in subsection 2.1 to obtain the following theorem.

**Theorem 2.8.** There exists a continuum of fixed points of period tripling renormalization operator acting on $C^{1+\text{Lip}}$ unimodal maps.

**Proof.** Consider an $\epsilon$ variation on scaling data and we modify the construction which is described in subsection 2.1. This modification is illustrated in Figure 10.

Define a neighborhood $N_\epsilon$ about the point $(u^2_c(0), u^3_c(0))$ as

$$N_\epsilon(u^2_c(0), u^3_c(0)) = \{(u^2_c(0), \epsilon \cdot u^3_c(0)) : \epsilon > 0 \text{ and } \epsilon \text{ close to } 1\}.$$
From above Figure 10, the scaling data is obtained as

\[ s_3(c, \epsilon) = 1 - u_c^2(0) \]
\[ s_1(c, \epsilon) = \epsilon \cdot u_c^3(0) \]
\[ s_2(c, \epsilon) = u_c(\epsilon \cdot u_c^3(0)) - u_c(0) \]

where \( c \in (0, \frac{1}{2}) \). Also, we define

\[ R(c, \epsilon) = \frac{u_c(\epsilon \cdot u_c^3(0)) - c}{s_2(c, \epsilon)}. \]

From subsection 2.1, we know that the period tripling renormalization operator \( R \) has unique fixed point \( c^* \). Consequently, for a given \( \epsilon \) close to 1, \( R(c, \epsilon) \) has only one unstable fixed point, namely \( c^*_\epsilon \). Therefore, we consider the perturbed scaling data \( s^*_\epsilon : \mathbb{N} \to T_3 \) with

\[ s^*_\epsilon(n) = (\epsilon \cdot u_c^3(0), u_c(\epsilon \cdot u_c^3(0)) - u_c(0), 1 - u_c^2(0)). \]

Then \( \sigma(s^*_\epsilon) = s^*_\epsilon \) and using Lemma 2.3, we have

\[ Rf^*_\epsilon = f^*_\epsilon. \]

Moreover, \( f^*_\epsilon \) is a piece-wise affine map which is infinitely renormalizable. Now we use similar extension described in subsection 2.2, then we get \( F_{s^*_\epsilon} \) is the \( C^{1+Lip} \) extension of \( f^*_\epsilon \). This implies that \( F_{s^*_\epsilon} \) is a renormalizable map. As \( RF_{s^*_\epsilon} \) is an extension of \( Rf^*_\epsilon \). Therefore \( RF_{s^*_\epsilon} \) is renormalizable. Hence, for each \( \epsilon \) close to 1, \( F_{s^*_\epsilon} \) is a fixed point of period tripling renormalization. This proves the existence of a continuum of fixed points of renormalization. \( \square \)

Remark 4. (a) When \( \epsilon = 1 \), the fixed point of \( R(c, \epsilon) \) coincides with the fixed point of \( R(c) \) which is described in subsection 2.1.
(b) On the other hand, we have the following relations
(i) if \( \epsilon < 1 \), then
\[ c^* < c^*_\epsilon, \]
PERIOD TRIPLING AND QUINTUPLING RENORMALIZATIONS BELOW $C^2$ SPACE

(ii) if $\epsilon > 1$, then $c^*_\epsilon < c^*$,

(iii) if $\epsilon_0 < \epsilon_1$, then $c^*_{\epsilon_1} < c^*_{\epsilon_0}$.

The above relations (i) and (ii) are illustrated in Figure 11 by plotting the graphs of $R(c, \epsilon)$ as a thick line and $R(c)$ as a dotted line.

![Figure 11](image)

**Figure 11.** (a): $R(c, \epsilon)$ and $R(c)$ for $\epsilon < 1$ and (b): $R(c, \epsilon)$ and $R(c)$ for $\epsilon > 1$.

**Theorem 2.9.** There exists an infinitely renormalizable $C^{1+\text{Lip}}$ unimodal map $k$ with quadratic tip such that \{c_n\}_{n \geq 0}, where $c_n$ is the critical point of $R^n k$, is dense in a Cantor set.

**Proof.** From subsection 2.1, we conclude that the map $R$ has one fixed point $c^*$ which is expanding. We choose $\epsilon_0 > \epsilon_1 > \epsilon_2$ close to 1. Then $R(c, \epsilon_0)$, $R(c, \epsilon_1)$ and $R(c, \epsilon_2)$ have the expanding fixed points $c^*_0$, $c^*_1$ and $c^*_2$, respectively. In fact,

$$\frac{\partial R}{\partial c}(c^*_i, \epsilon_i) > 2, \text{ for each } i = 0, 1, 2.$$

From Remark 4(iii), we have $c^*_0 < c^*_1 < c^*_2$. Therefore, there exists three intervals $A_0 = [c^*_0, a_0]$, $A_1 = [a_1, b_1] \supset c^*_1$ and $A_2 = [a_2, c^*_2]$, such that the maps

$$R_i : A_i \rightarrow [c^*_0, c^*_2] \supset A_i, \text{ for each } i = 0, 1, 2,$$

are expanding diffeomorphisms, where $R_i(c) = R(c, \epsilon_i)$. Therefore, we get a horseshoe map. We use the following coding for the invariant Cantor set of the horseshoe map

$$c : \Sigma_3 \rightarrow [c^*_0, c^*_2]$$

with

$$c(\sigma(\alpha)) = R(c(\alpha), \epsilon_{\alpha_n}).$$

Given a sequence $\alpha \in \Sigma_3$, the proper scaling data $s : \mathbb{N} \rightarrow T_3$ is defined as

$$s(n) = (s_1(c(\sigma^n \alpha), \epsilon_{\alpha_n}), s_2(c(\sigma^n \alpha), \epsilon_{\alpha_n}), s_3(c(\sigma^n \alpha), \epsilon_{\alpha_n})).$$

Consequently, we define a piece-wise affine map

$$f_\alpha : D_s \rightarrow [0, 1].$$

Before we continue the proof, let us discuss the following lemma which is very significant to the proof,
Lemma 2.10. There exists $\rho > 0$ such that
\[ \frac{1}{\rho} \leq \frac{\hat{I}_2^n}{|I_2^n|^2} \leq \rho. \]

Proof. We have
\[ I_2^n = \begin{cases} [y_n, z_n], & \text{for } n = 1, 3, 5, \ldots \\ [z_n, y_n], & \text{for } n = 2, 4, 6, \ldots \end{cases} \]
and
\[ I_3^n = \begin{cases} [w_n, y_{n-1}], & \text{for } n = 1, 3, 5, \ldots \\ [y_{n-1}, w_n], & \text{for } n = 2, 4, 6, \ldots \end{cases} \]
where $y_n, z_n$ and $w_n$ are defined in subsection 2.1.

Let
\[ \hat{I}_2^n = u_c(I_2^n) = [u_c(y_n), 1] = [\hat{y}_n, 1] \]
and
\[ \hat{I}_3^n = [\hat{y}_n, \hat{w}_{n+1}] \subseteq \hat{I}_2^n \]
such that
\[ |\hat{I}_3^{n+1}| = s_2(n)|\hat{I}_2^n|. \]
Define an affine homeomorphism $f_\alpha|_{I_3^{n+1}} : I_3^{n+1} \to \hat{I}_3^{n+1}$ such that
\[ f_\alpha(y_n) = u_c(y_n) = \hat{y}_n. \]

Also, we have $c(n) = c(\sigma^n \alpha) \in [c_0^n, c_2^n]$ which is a small neighborhood of $c^*$. This implies, there exists $\rho_0 \equiv \rho_0(n) > 0$ such that
\[ \frac{1}{\rho_0} \leq \frac{|c - y_n|}{|I_2^n|^2} \leq \rho_0. \]

Then
\[ \frac{|\hat{I}_2^n|}{|I_2^n|^2} = \frac{|u_c(y_n), 1|}{|I_2^n|^2} = \frac{1}{|I_2^n|^2} \frac{(y_n - c)^2}{(1 - c)^2} \]
from Eqs. (13) and (14), we have
\[ \frac{1}{\rho_0^2(1 - c)^2} \leq \frac{|\hat{I}_2^n|}{|I_2^n|^2} \leq \frac{\rho_0^2}{(1 - c)^2}. \]

Since $(1 - c)^2 \leq \frac{1}{(1-c)^2}$, choose $\rho = \frac{\rho_0^2}{(1-c)^2}$. Then we have
\[ \frac{1}{\rho} \leq \frac{|\hat{I}_2^n|}{|I_2^n|^2} \leq \rho. \]

□
To continue the proof of the theorem, we define $F^n : [0, 1]^2 \to [0, 1]^2$ by

$$F^n \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} F^n_1(x) \\ F^n_2(y) \end{array} \right)$$

where $F^n_1$ be the affine orientation preserving homeomorphism and $F^n_2$ be the affine homeomorphism with $F^n_1(1) = y_n$. Therefore, the image of $F^n$ is $B_n$.

Let $K^n = (F^n)^{-1}(\text{graph } f_\alpha) = \gamma^1_n \cup \gamma^3_n$. This is the graph of $f_\alpha$. We extend the function $f_\alpha$ and its graph $F_n$ on the gaps $[u^4_c (z_{n-1}), u^4_c (z_{n-1})]$ and $[u^4_c (z_{n-1}), u^4_c (z_{n-1})]$. Observe that $u^4_c (z_{n-1})$ and $Df_\alpha (u^4_c (z_{n-1}))$ vary within a compact family. This allows us to choose from a compact family of $C^{1+\text{Lip}}$ diffeomorphisms the extensions

$$k^1_n : [z_{n-1}, u^4_c (z_{n-1})] \to [f_\alpha (z_{n-1}), f_\alpha (u^4_c (z_{n-1}))]$$

and

$$k^3_n : [u^4_c (z_{n-1}), 1] \to [0, f_\alpha (u^4_c (z_{n-1}))]$$

of the maps $f_\alpha_n$ on $I^1_1$ and $I^3_3$, respectively. Let $K^1_n$ and $K^3_n$ are the graphs of $k^1_n$ and $k^3_n$ respectively, as shown in Figure 13. Therefore,

$$K = \bigcup_{n \geq 0} F^n(K^1_n \cup K^3_n).$$

Then $K$ is the graph of a unimodal map $k_\alpha : [0, 1] \to [0, 1]$ which extends $f_\alpha$. Notice that $k_\alpha$ is $C^1$. Since $f_\alpha$ has a quadratic tip, $k_\alpha$ also has a quadratic tip. Also, $F^n(K^n)$ is the graph of a $C^{1+\text{Lip}}$ diffeomorphism. For a similar reason as of Eqn. (12) in subsection 2.2, the Lipschitz bound $\lambda_n$ satisfies

$$\lambda_n \leq \frac{|I^4_2|}{|I^2_2|} P_0$$

for some $P_0 > 0$. Using Lemma 2.10, we get

$$\lambda_n \leq P_0 \cdot \rho.$$ 

Thus $k_\alpha$ is a $C^{1+\text{Lip}}$ unimodal map with quadratic tip. The construction implies that $k_\alpha$ is infinitely renormalization and

$$\text{graph}(R^n k_\alpha) \supset K^n.$$
By choosing $\alpha \in \Sigma_3$ such that the orbit under the shift $\sigma$ is dense in the invariant Cantor set of the horseshoe map.

3. Period quintupling renormalization. In this section, we describe the construction of period quintupling renormalizable fixed point on the space of piece-wise affine infinitely renormalizable maps. Further, we describe the extension of piece-wise affine renormalizable map to a $C^{1+\text{Lip}}$ unimodal map. Then, we discuss the entropy of period quintupling renormalization operator acting on the same space.

Consider $T_k$ as defined in subsection 2.1, for $k = 5$, each element $(s_1, s_2, s_3, s_4, s_5)$ is called a scaling quint-factor. Define affine maps $\tilde{s}_1, \tilde{s}_2, \tilde{s}_3, \tilde{s}_4$ and $\tilde{s}_5$ induced by a scaling quint-factor,

$$\tilde{s}_i : [0,1] \to [0,1] \text{ for } i = 1, 2, \ldots, 5,$$

defined by

$$\tilde{s}_1(t) = s_1(1-t)$$
$$\tilde{s}_2(t) = u_c^2(0) + s_2(1-t)$$
$$\tilde{s}_3(t) = u_c(0) + s_3(1-t)$$
$$\tilde{s}_4(t) = u_c^2(0) - s_4(1-t)$$
$$\tilde{s}_5(t) = 1 - s_5(1-t)$$

The scaling quint-factor $s(n) = (s_1(n), s_2(n), s_3(n), s_4(n), s_5(n)) \in T_5$ induces a quintuplet of affine maps $(\tilde{s}_1(n), \tilde{s}_2(n), \tilde{s}_3(n), \tilde{s}_4(n), \tilde{s}_5(n))$.

For each $n \in \mathbb{N}$ and $i \in \{1, 2, \ldots, 5\}$, we define the following intervals:

$$I_n^i = \tilde{s}_2(1) \circ \tilde{s}_2(2) \circ \tilde{s}_2(3) \circ \ldots \circ \tilde{s}_2(n-1) \circ \tilde{s}_i(n)([0,1]).$$

Given a proper scaling data, define

$$\{c\} = \bigcap_{n \geq 1} I_n^2.$$

A proper scaling data $s : \mathbb{N} \to T_5$ induces the set $D_s = \bigcup_{n \geq 1} (I_1 \cup I_3 \cup I_4 \cup I_5^6)$. Consider a map

$$g_s : D_s \to [0,1]$$

such that $g_s|I_1^i$ is the affine extensions of $u_c|\partial I_1^i$, for each $i \in \{1, 3, 4, 5\}$.

Figure 14. The intervals of next generations.
From Figure 14, the end points of the intervals at each level are denoted by

\[ z_0 = 0, \ y_0 = 1, \ I_2^0 = [0, 1] \]

and for \( n \geq 1 \),

\[ a_n = \partial I_1^n \setminus \partial I_2^{n-1} \]

\[ y_{2n-1} = \min\{\partial I_2^{2n-1}\}, \quad y_{2n} = \max\{\partial I_2^{2n}\} \]

\[ z_{2n-1} = \max\{\partial I_2^{2n-1}\}, \quad z_{2n} = \min\{\partial I_2^{2n}\} \]

\[ d_{2n-1} = \min\{\partial I_4^{2n-1}\}, \quad d_{2n} = \max\{\partial I_4^{2n}\} \]

\[ e_{2n-1} = \max\{\partial I_4^{2n-1}\}, \quad e_{2n} = \min\{\partial I_4^{2n}\} \]

\[ f_{2n-1} = \min\{\partial I_4^{2n-1}\}, \quad f_{2n} = \max\{\partial I_4^{2n}\} \]

\[ g_{2n-1} = \max\{\partial I_4^{2n-1}\}, \quad g_{2n} = \min\{\partial I_4^{2n}\} \]

\[ h_n = \partial I_1^n \setminus \partial I_2^{n-1} \]

**Definition 3.1.** For a given proper scaling data \( s : \mathbb{N} \rightarrow T_5 \), a map \( g_s \) is said to be period quintupling infinitely renormalizable if for \( n \geq 1 \),

1. \( g_s(0), g_s(y_n) \) is the maximal domain containing 1 on which \( g_s^{5^n-1} \) is defined affinely and \( [0, g_s^4(y_n)] \) is the maximal domain containing 0 on which \( g_s^{5^n-2} \) is defined affinely,
2. \( g_s(0), g_s^2(y_n) \) is the maximal domain on which \( g_s^{5^n-3} \) is defined affinely and \( [g_s^3(y_n), g_s^2(0)] \) is the maximal domain on which \( g_s^{5^n-4} \) is defined affinely,
3. \( g_s^{5^n-1}([g_s(y_n), 1]) = I_2^n, \)

\[ g_s^{5^n-2}([0, g_s^4(y_n)]) = I_2^n, \]

\[ g_s^{5^n-3}([g_s(0), g_s^3(y_n)]) = I_2^n, \]

\[ g_s^{5^n-4}([g_s^3(y_n), g_s^2(0)]) = I_2^n. \]

Define \( U_\infty = \{g_s : g_s \text{ is a period quintupling infinitely renormalizable}\} \).

Let \( g_s \in U_\infty \) be given by the proper scaling data \( s : \mathbb{N} \rightarrow T_5 \) and define

\[ \hat{I}_2^n = [u_c(y_n), 1] = [g_s(y_n), 1], \]

\[ \hat{I}_2^{n,+} = [0, u_c^2(y_n)] = [0, g_s^2(y_n)], \]

\[ \hat{I}_2^{n,-} = [u_c(0), u_c^2(y_n)] = [g_s(0), g_s^3(y_n)], \]

\[ \hat{I}_2^{n,-} = [u_c^4(y_n), u_c^2(0)] = [g_s^3(y_n), g_s^4(0)]. \]

Let

\[ h_{s,n} : [0, 1] \rightarrow I_2^n \]

be defined by

\[ h_{s,n} = \hat{s}_2(1) \circ \hat{s}_2(2) \circ \hat{s}_2(3) \circ \cdots \circ \hat{s}_2(n). \]

Furthermore, let

\[ \hat{h}_{s,n}^+ : [0, 1] \rightarrow \hat{I}_2^{n,+}, \quad \hat{h}_{s,n}^{++} : [0, 1] \rightarrow \hat{I}_2^{n,++}, \]

\[ \hat{h}_{s,n}^- : [0, 1] \rightarrow \hat{I}_2^{n,-}, \quad \hat{h}_{s,n}^{--} : [0, 1] \rightarrow \hat{I}_2^{n,--} \]

be the affine orientation preserving homeomorphisms. Then define

\[ R_n g_s : h_{s,n}^{-1}(D_s \cap I_2^n) \rightarrow [0, 1] \]
by

\[
R_{n}g_s(x) = \begin{cases} 
R_{n}^{+}g_s(x) = (\hat{h}_{s,n}^{++})^{-1} \circ g_s^2 \circ h_{s,n}(x), & \text{if } x \in h_{s,n}^{-1} \left( \bigcup_{m \geq n+1} I_{m}^{m} \right) \\
R_{n}^{-}g_s(x) = (\hat{h}_{s,n}^{-})^{-1} \circ g_s^4 \circ h_{s,n}(x), & \text{if } x \in h_{s,n}^{-1} \left( \bigcup_{m \geq n+1} I_{m}^{m} \right) \\
R_{n}^{-}g_s(x) = (\hat{h}_{s,n}^{-})^{-1} \circ g_s^3 \circ h_{s,n}(x), & \text{if } x \in h_{s,n}^{-1} \left( \bigcup_{m \geq n+1} I_{m}^{m} \right) \\
R_{n}^{+}g_s(x) = (\hat{h}_{s,n}^{++})^{-1} \circ g_s^2 \circ h_{s,n}(x), & \text{if } x \in h_{s,n}^{-1} \left( \bigcup_{m \geq n+1} I_{m}^{m} \right) 
\end{cases}
\]

where,

\[
R_{n}^{+}g_s : h_{s,n}^{-1} \left( \bigcup_{m \geq n+1} I_{m}^{m} \right) \rightarrow [0, 1]
\]

\[
R_{n}^{-}g_s : h_{s,n}^{-1} \left( \bigcup_{m \geq n+1} I_{m}^{m} \right) \rightarrow [0, 1]
\]

and

\[
R_{n}^{-}g_s : h_{s,n}^{-1} \left( \bigcup_{m \geq n+1} I_{m}^{m} \right) \rightarrow [0, 1],
\]

which are illustrated in Figure 15.

**Figure 15**

**Lemma 3.2.** Let \( s : \mathbb{N} \rightarrow T_{5} \) be proper scaling data such that \( g_s \) is infinitely renormalizable. Then

\[
R_{n}g_s = g_{s}^{n}(s).
\]

Let \( g_s \) be infinitely renormalization, then for \( n \geq 0 \), we have

\[
g_{s}^{5n} : D_{s} \cap I_{2}^{n} \rightarrow I_{2}^{n}
\]

is well defined. Define the renormalization \( R : U_{\infty} \rightarrow U \) as

\[
Rg_s = h_{s,1}^{-1} \circ g_s^5 \circ h_{s,1}.
\]

The maps \( g_{s}^{5n-1} : \hat{I}_{2,n}^{n} \rightarrow I_{2}^{n} \), \( g_{s}^{5n-2} : \hat{I}_{2,++}^{n} \rightarrow I_{2}^{n} \), \( g_{s}^{5n-3} : \hat{I}_{2,-}^{n} \rightarrow I_{2}^{n} \), and \( g_{s}^{5n-4} : \hat{I}_{2,-}^{n} \rightarrow I_{2}^{n} \) are the affine homeomorphisms, whenever \( g_s \in U_{\infty} \). Then
Lemma 3.3. We have $R^n g_s : D_{\sigma^n(s)} \to [0, 1]$ and $R^n g_s = R_n g_s$.

The lemma 3.2 and lemma 3.3 give us the following result,

**Proposition 5.** There exists a map $g_s^* \in U_\infty$, where $s^*$ is characterized by $R g_s^* = g_s^*$.

**Proof.** Consider $s : \mathbb{N} \to T_5$ be proper scaling data such that $g_s$ is infinitely renormalizable. Let $c_n$ be the critical point of $g_{\sigma^n(s)}$.

The combinatorics for period quintupling renormalization is shown in Figure 16.

![Figure 16. Period quintuple combinatorics ($I_2^n \to I_5^n \to I_1^n \to I_3^n \to I_4^n \to I_2^n$).](image)

![Figure 17. Length of the intervals and gaps](image)

From Figures 16 and 17, we have the following

\begin{align}
 u_{c_0}^1(0) &= 1 - s_5(n) \\
 u_{c_0}^5(0) &= u_{c_0}(1 - s_5(n)) \\
 &= s_1(n) \\
 u_{c_0}^6(0) &= u_{c_0}(s_1(n)) \\
 &= s_3(n) + u_{c_0}(0) \\
 u_{c_0}^7(0) &= u_{c_0}(s_3(n) + u_{c_0}(0)) \\
 &= u_{c_0}^2(0) - s_4(n) \\
 u_{c_0}^8(0) &= u_{c_0}(u_{c_0}^2(0) - s_4(n)) \\
 &= s_2(n) + u_{c_0}^3(0) \\
 c_{n+1} &= \frac{u_{c_0}^8(0) - c_n}{s_2(n)} \equiv \mathcal{R}(c_n).
\end{align}

We use Mathematica for solving the Eqns. (15), (16), (17), (18) and (19), and we obtain the expressions for $s_1(n)$, $s_2(n)$, $s_3(n)$, $s_4(n)$ and $s_5(n)$. 
Let $s_i(n) \equiv S_i(c_n)$ for $i = 1, \ldots, 5$. Since these expressions are too lengthy, we just plotted the graph of each $S_i(c)$. The graphs of $S_i(c_n)$ are shown in Figures 18.

Since $(s_1(n), s_2(n), s_3(n), s_4(n), s_5(n)) \in T_5$, then we have the following conditions.

\begin{align}
    s_i(n) &> 0, \quad \text{for each } i \in \{1, 2, \ldots, 5\} \tag{21} \\
    \sum_{i=1}^{5} s_i(n) &< 1 \tag{22} \\
    0 &< c_n < \frac{1}{2} \tag{23}
\end{align}

As intervals $I_i^n$ for $i = 1, 2, \ldots, 5$ are pairwise disjoint, let $g_l^n$ be the length of the gap which is in between $I_i^n$ & $I_{i+1}^n$ and let $g_r^n$ be the length of the gap which is in between $I_{i+1}^n$ and $I_{i+2}^n$ for $i = 1, 2, 3$. The gaps are also illustrated in Figure 17.
Then we have

\[ g^n_l = u^3_{c_n}(0) - u^5_{c_n}(0) \equiv G_l(c_n) > 0 \]  
\[ g^n_{r_1} = u^8_{c_n}(0) - u^9_{c_n}(0) \equiv G_{r_1}(c_n) > 0 \]  
\[ g^n_{r_2} = u^7_{c_n}(0) - u^6_{c_n}(0) \equiv G_{r_2}(c_n) > 0 \]  
\[ g^n_{r_3} = u^4_{c_n}(0) - u^2_{c_n}(0) \equiv G_{r_3}(c_n) > 0 \]  

Note that the conditions (eq. (21)), (24), (25) to (27) implies the condition (22). Therefore, the conditions (21) and (23) together with the gaps conditions (24) to (27) define the feasible domain \( f_d \) to be:

\[
f_d = \{ c \in (0, 0.5) : S_1(c) > 0 \forall i = 1, \ldots, 5, \ G_l(c) > 0, \ G_{r_j}(c) > 0 \forall j = 1, 2, 3 \}.
\]

To compute the feasible domain \( f_d \), we need to find subinterval(s) of \((0, 0.5)\) which satisfies the conditions of (28). By using Mathematica, we employ the following command to obtain the feasible domain

\[
\text{N[Reduce}\{S_1(c) > 0, S_2(c) > 0, S_3(c) > 0, S_4(c) > 0, S_5(c) > 0, G_l(c) > 0, G_{r_1}(c) > 0, G_{r_2}(c) > 0, G_{r_3}(c) > 0, 0 < c < 0.5\}, c]]
\]

This yields:

\[
f_d = (0.379765..., 0.384772...) \cup (0.384772..., 0.390436...) \equiv f_{d_1} \cup f_{d_2}.
\]

From the Eqn.(20), the graphs of \( \mathcal{R}(c) \) in the sub-domains \( f_{d_1} \) and \( f_{d_2} \) of \( f_d \) are shown in Figure 19.

![Graphs](image)

(a) \( \mathcal{R} \) has no fixed point in \( f_{d_1} \).

(b) \( \mathcal{R} \) has only one fixed point in \( f_{d_2} \).

**Figure 19.** The graph of \( \mathcal{R} : f_d \rightarrow \mathbb{R} \) and the diagonal \( \mathcal{R}(c) = c \).

The map \( \mathcal{R} : f_d \rightarrow \mathbb{R} \) is expanding in the neighborhood of the fixed point \( c^* \) which is illustrated in Figure 19b. By Mathematica computations, we observe that \( \mathcal{R} \) has a unique fixed point \( c^* = 0.387226... \) in \( f_{d_2} \). Therefore, we conclude that \( \mathcal{R} \) has only one fixed point in \( f_d \) such that

\[ \mathcal{R}(c^*) = c^* \]

corresponds to an infinitely renormalizable map \( g_{s^*} \).
In other words, consider the scaling data \( s^* : \mathbb{N} \to T_5 \) with 
\[
s^*(n) = (s_1^*(n), s_2^*(n), s_3^*(n), s_4^*(n), s_5^*(n))
\]
\[
= (u_{c^*}^5(0), u_{c^*}^3(0) - u_{c^*}^5(0), u_{c^*}^6(0) - u_{c^*}^3(0), u_{c^*}^2(0) - u_{c^*}^6(0), 1 - u_{c^*}^2(0)).
\]
Then \( \sigma(s^*) = s^* \) and using Lemma 3.2 we have 
\[ Rg_{s^*} = g_{s^*}. \]

**Remark 5.** Let \( I^n_2 = [y_n, z_n] \) be the interval containing \( c^* \) corresponding to the scaling data \( s^* = (s_1^*, s_2^*, s_3^*, s_4^*, s_5^*) \) then 
\[ g_{s^*}(y_n) = u_{c^*}(y_n). \]

Hence, \( g_{s^*} \) has a quadratic tip.

**Lemma 3.4.** If \( g_{s^*} \) is the map with a proper scaling data \( s^* = (s_1^*, s_2^*, s_3^*, s_4^*, s_5^*) \) corresponding to \( c^* \), then we have 
\[ (s_5^*)^2 = s_5^*. \]

**Proof.** Let \( \hat{I}_{2,+}^n = g_{s^*}(I^n_2) = [g_{s^*}(y_n), 1] \) and \( \hat{I}_{2,+}^{n+1} = g_{s^*}(I_{5}^{n+1}) \). Then \( g_{s^*} : \hat{I}_{2,+}^n \to \hat{I}_{2,+}^n \) is affine, monotone and onto. Further, by construction 
\[ g_{s^*}^{-1} : \hat{I}_{2,+}^{n+1} \to \hat{I}_{2,+}^n. \]

Hence, 
\[ \frac{\hat{I}_{2,+}^{n+1}}{\hat{I}_{2,+}^n} = s_5^*. \]
Therefore, \( |I^n_2| = (s_2^*)^n \) and \( |\hat{I}_{2,+}^n| = (s_5^*)^n \). Since, 
\[ g_{s^*}(y_n) = u_{c^*}(y_n). \]
This implies, 
\[ s_5^* = \frac{|\hat{I}_{2,+}^{n+1}|}{|\hat{I}_{2,+}^n|} = \left( \frac{y_{n+1} - c^*}{y_n - c^*} \right)^2 = \left( \frac{|I_{2,+}^{n+1}|}{|I^n_2|} \right)^2 = (s_2^*)^2. \]

**Remark 6.** A proper scaling data \((s_1^*, s_2^*, s_3^*, s_4^*, s_5^*)\) is satisfying the following properties, for all \( n \in \mathbb{N} \),

(i) \( \frac{|I_{2,+}^{n+1}|}{|I^n_2|} = s_2^* \) and \( \frac{|I_{3,+}^{n+1}|}{|I^n_2|} = \frac{|I_{4,+}^{n+1}|}{|I^n_2|} = \frac{|I_5^{n+1}|}{|I_5^n|} = s_2^* \)

(ii) \( \frac{|I_{2,+}^{n+1}|}{|I^n_2|} = s_1^* \)

(iii) \( \frac{|I_{4,+}^{n+1}|}{|I^n_2|} = s_3^* \)

(iv) \( \frac{|I_{4,+}^{n+1}|}{|I^n_2|} = s_4^* \)

(v) \( \frac{|I_{4,+}^{n+1}|}{|I^n_2|} = s_5^* \)

**Remark 7.** The invariant Cantor set of the map \( g_{s^*} \) is next in the complexity to the invariant period tripling Cantor set of piece-wise affine map \( f_{s^*} \) which is described in subsection 2.1. But unlike the Cantor set of \( f_{s^*} \), there are now five ratios at each scale.
In subsection 2.2, we constructed $C^{1+\text{Lip}}$ extension of piece-wise affine map $f_*$ to the $C^{1+\text{Lip}}$ unimodal map $F_*$. A similar construction leads the following result.

**Theorem 3.5.** Let $G_*$ be a $C^{1+\text{Lip}}$ extension of $g_*$. Then $G_*$ is a period quintupling infinitely renormalizable $C^{1+\text{Lip}}$ unimodal map with a quadratic tip such that $R^k G_* = G_*$. 

As we discussed the topological entropy in subsection 2.3 and the $\epsilon-$variation on scaling data in subsection 2.4, in the similar way, the following results hold for period quintupling renormalization.

**Theorem 3.6.** The period quintupling renormalization defined on $C^{1+\text{Lip}}$ unimodal maps has infinite entropy and it has a continuum of fixed points.

**Theorem 3.7.** There exists an infinitely renormalizable $C^{1+\text{Lip}}$ unimodal map $k_\alpha$ with quadratic tip such that $\{c_n\}_{n \geq 0}$ is dense in a Cantor set, where $c_n$ is the critical point of $R^n k_\alpha$ and $R$ is the period quintupling renormalization operator.

**Remark 8.** The similar construction can be done for other two possible period 5 combinatorics, which are $(I_n^3 \to I_n^2 \to I_n^1 \to I_n^2 \to I_n^3 \to I_n^4)$ and $(I_n^4 \to I_n^5 \to I_n^1 \to I_n^2 \to I_n^3 \to I_n^4)$. Consequently, we obtain the above theorems.

4. **Conclusions.** In this work, we showed that the period tripling and period quintupling renormalization operators both have a fixed point corresponding to the given proper scaling data. Also, we notice that the geometry of the invariant Cantor set of the map $g_*$ is more complex than the geometry of the invariant Cantor set of $f_*$. Furthermore, the piece-wise affine period tripling and quintupling renormalizations fixed points are extended to the space of $C^{1+\text{Lip}}$ unimodal maps. Finally, we showed that the period tripling and period quintupling renormalizations defined on this space have positive entropy. In fact, the topological entropy of renormalization operator $R$ is unbounded. We proved the existence of an infinitely renormalizable $C^{1+\text{Lip}}$ unimodal map $k_\alpha$ with quadratic tip such that $\{c_n\}_{n \geq 0}$ is dense in a Cantor set. This gives us the existence of continuum of fixed points of period tripling and period quintupling renormalizations. This shows non-rigidity of period tripling and quintupling renormalizations.

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