Anomalous scaling and Lee-Yang zeroes in Self-Organized Criticality.

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We show that the generating functions of probability distributions in SOC models exhibit a
Lee-Yang phenomenon [17]. Namely, their zeroes pinch the real axis at \( z = 1 \), as the system size
goes to infinity. This establish a new link between the classical theory of critical phenomena and
SOC. A scaling theory of the Lee-Yang zeroes is proposed in this setting.

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In 1988, Bak, Tang and Wiesenfeld [1] proposed for the first time a mechanism in which a dynamical system reaches
“spontaneously” a stationary state with some features reminiscent of a critical state. More precisely, by its only internal
reorganization in reaction to (stationary) external perturbations, a system organizes into a state with scale invariance
and power law statistics. This effect, called Self-Organized Criticality (SOC), was quite unexpected, since usually,
the critical state of a thermodynamic system needs a fine tuning of some control parameter (temperature, magnetic
field, etc...) which is at first sight absent from the model introduced by BTW and from the many variants proposed
later [2,3]. Furthermore, the stationary regime corresponds to a non-equilibrium situation where the (stationary)
flux of external perturbation is dissipated in the bulk or at the boundaries, generating a constant flux through the
system. As a consequence, one generally believes that the usual equilibrium statistical mechanics treatments using
the concepts of Gibbs measure, free energy, etc ... cannot be applied for the analysis of SOC systems.

On the other hand, it is also believed that concepts like universality classes, critical exponents, order parameter, etc
…, borrowed from the equilibrium statistical mechanics of phase transitions are still relevant in SOC. Actually, the
identification of universality classes is one of the main goal in the SOC literature. However, since these concepts are
not defined via a thermodynamic analysis, alternative definitions are used. The dynamics of SOC systems occurring in
chain reaction or “avalanche” like events, a set of observables characterizing the avalanches, size (s), duration (t), area
(a), etc … are defined. Fix such an observable, say \( N \), and compute the related probability \( P_L(n) \) at
stationarity for a system of characteristic size \( L \). The numerical simulations show that the graph of \( P_L(n) \) exhibits a
power law behavior over a finite range, with a cut-off corresponding to finite size effects. As \( L \) increases the power law
range increases. This leads to the conjecture that as \( L \to \infty \), \( P_L(n) \) converges to a probability distribution \( P^*(n) \), with
a power law tail having an exponent \( \tau_n \) called the critical exponent of the observable \( n \). It seems commonly admitted
in the SOC community that a classification of the models can be made by the knowledge of their critical exponents
(“universality classes”). Consequently, a large effort has been devoted to the computation of these quantities.

Considerably less efforts have been made to establish a clear foundation of the basic SOC concepts and to clarify their
connection to their classical statistical mechanics counterpart [4,5]. Clearly, this is a hard task since even preliminary
notions like “state” or ”thermodynamic limit”, though intuitively clear, suffer from a lack of precise mathematical
definition in this setting. In discrete automata like the BTW model [6] the state is the unique ergodic probability
measure, \( \mu_L \), of a discrete Markov chain, finite when \( L \) is finite. In continuous dynamical systems like the Zhang

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model there exists typically infinitely many ergodic measures and therefore one has to add additional constraints to define the state \( \mu_L \) in a non ambiguous way. The probability distributions \( P_L \) for the observables \( a, s, t \) ... are directly obtained from \( \mu_L \) but they contain less information. The observable \( a, s, t \) are simply indicators of the dynamics. There is no a priori reason to believe that the knowledge of \( P_L(s), P_L(a), P_L(t) \) or, even, of the joint probability \( P_L(a, s, t) \) gives all the relevant (that is, allowing to classify the models into universality classes) informations about the state \( \mu_L \).

The thermodynamic limit \( L \to \infty \) and the supposed “convergence” of \( \mu_L \) to a “critical state” poses deeper problems since even the proof that there exists indeed a limiting state and that the probability distribution of avalanche observables are still defined in this limit remains to be done, for most models. The usual classical statistical mechanics constructions of the thermodynamic limit like the Dobrushin-Landford-Ruelle’s (DLR) cannot be directly applied because of the a priori absence of a Gibbs formalism. On the other hand, the methods used in interacting particle systems, allowing to define the dynamics in the thermodynamic limit, on the basis of the Hille-Yoshida theorem and the properties of Feller processes, requires locality conditions which are broken in SOC model. One has then to develop new ideas for non-Fellerian Markov processes and this has only been done in a few examples.

But one of the main problem is the treatment of the data obtained from finite size systems simulations itself and the extrapolation to the \( L \to \infty \) limit. Indeed, though it was believed in the earlier SOC papers that this extrapolation can be handled by classical finite-size scaling, further investigations proposed alternative scaling and, at the moment, there is no agreement on which scaling form applies. Consequently, a lot of efforts have yet to be devoted to the understanding and analysis of SOC models.

Though the analogy between self-organized criticality and usual critical phenomena is the core of the SOC paradigm, it is remarkable that, up to now, some well developed techniques of analysis of critical phenomena have not been adapted to the study of SOC models. A phase transition has different manifestation. It is in particular characterized by a singularity of the thermodynamic potential (free energy, pressure). At a phase transition point, and for suitable interactions, the free energy, which is the generating function of the cumulants, exists in the thermodynamic limit but it is not analytic (in a \( k \)th order phase transition it is \( C^{k-1} \) but not \( C^k \)). In many examples, the failure of analyticity is manifested by the Lee-Yang phenomenon. For finite size systems the partition function is a polynomial in a variable \( z \) which typically depends on control parameters like the temperature or the external field. Since all coefficients are positive there is no zero on the positive real axis. However, in the thermodynamic limit, at the critical point, some zeroes pinch the real axis at \( z = 1 \), leading to a singularity in the free energy. The finite-size scaling properties of the leading zeroes and of the density of zeroes near to \( z = 1 \) determine the order of the transition and also the critical exponents in the case of a second order phase transition.

A natural question is whether there exists a similar property in SOC, namely can we exhibit a “free-energy like” function, developing singularities in a similar way in the infinite lattice size limit. Though there exists a huge literature about the Lee-Yang zeroes, there is, to the best of our knowledge, no attempt to study Self-Organized Criticality from this point of view. In this paper, we show that the cumulants generating function of the probability distribution of the observables \( a, s, t, \ldots \) have this property. More precisely, the expected convergence of \( P_L \) to a power law induces
a Lee-Yang phenomenon for the corresponding cumulants generating function (eq. [1]). We show that this effect is related to the observed divergence of the moments. Furthermore, a scaling theory of the Lee-Yang zeroes is proposed.

After some preliminaries (section [I]), we give explicit analytical results (section [II]) in several cases used as guidelines for subsequent analysis of a SOC model (section [III]). We first study the truncated power law case where the cut-off tends to infinity when a parameter $L$, (corresponding to the lattice size in SOC models), tends to infinity (section [II A]). We give in particular an analytic expression for the zeroes. Then, we investigate the effect of a smooth cut off (section [II B]). We first discuss the properties that this cut off must have, extrapolated from numerical simulations, and present some of the ansatz found in the literature (section [II B 1]). We then explicitly compute the Lee-Yang zeroes for a probability distribution obeying the finite-size scaling form proposed in [13] and converging to a power law as $L \to \infty$ (section [II B 2]). We show in particular that, when the power law exponent $\tau$ is larger that 1 there is a violation of the scaling usually observed in classical critical phenomena; namely, there is an anomalous logarithmic dependence on $L$ for the angle that the zeroes do with the real axis in the $t = \log(z)$ plane. We also show that when $\tau > 1$ a bias is artificially induced by the numerical simulations, when the size of the sample used to generate the empirical probability distribution is fixed independently of the lattice size. This effect can be analyzed with the Lee-Yang zeroes (section [II B 3]). We then briefly study some other scaling form proposed in the literature and the effect of a finite size scaling violation on the Lee-Yang zeroes (section [II B 4]). Finally, in the last section, we present numerical simulations for the Lee-Yang zeroes in the Zhang SOC model and compare them to the theoretical results obtained in section [I]. We see no clear cut evidence of finite size scaling violation, but show that this model is quite sensitive to the numerical cut off induced by a lattice size independent sampling. This can raise some doubts on the conclusions about scaling (Finite Size Scaling, multifractal or whatsoever) which can be drawn from some large lattice simulations done on this model in the literature.

I. PROBABILITY DISTRIBUTION AND LEE-YANG ZEROES.

A. The finite size system.

Let $P_L(n) = \text{Prob}(N = n)$ be the probability distribution of the avalanche observable $N \in 1 \ldots \xi_L$, where the index $L$ refers to the characteristic size of the system. $\xi_L$, the maximal value that $N$ takes is finite, whenever $L < \infty$, but diverges as $L \to \infty$. Therefore, the function:

$$Z_L(z) = \sum_{n=1}^{\xi_L} z^n P_L(n) \quad (1)$$

where $z \in \mathbb{C}$, is a polynomial of degree $\xi_L$. In particular, since $Z_L(z)$ is an analytic function of $z$ in the complex plane, all its moments exist. Denote by $E[n^q]_L$ the expectation with respect to $P_L(n)$. Call :

$$m_L(q) = \sum_{n=1}^{\xi_L} P_L(n)n^q \overset{\text{def}}{=} E[n^q]_L \quad (2)$$

where $q$ is a real (positive) number. For integer $q$, the $m_L(q)$’s are the moments of $P_L(n)$. Note that the normalization of $P_L(n)$ imposes $Z_L(1) = m_L(0) = 1$. 

3
For finite \( L \), \( Z_L(z) \) has \( \xi_L \) zeroes in \( \mathbb{C} \), that are either real \( \leq 0 \), or complex conjugate. Denote them by \( z_L(k), k = 1 \ldots \xi_L \) and order them such that \( 0 < |z_L(1) - 1| \leq \ldots \leq |z_L(k) - 1| \leq \ldots \leq |z_L(\xi_L) - 1| \). Note that \( z = 0 \) is a trivial zero, of multiplicity one, since \( P_L(1) > 0 \). Write \( z_L(k) = R_L(k)e^{i\theta_L(k)} = 1 + r_L(k)e^{i\nu_L(k)} \). Since all \( P_L(n) \) are positive, \( Z_L(z) \) has no zero on the positive real axis for finite \( L \). Consequently, the log-generating function \( \log[Z_L(z)] \) is well defined on \( \mathbb{R}^*_+ \). Furthermore:

\[
G_L(t) \overset{\text{def}}{=} \log[Z_L(e^t)]
\]

is an analytic function of \( t \). Call:

\[
\chi_L(q) = \frac{d^q}{dz^q}\log[Z_L(z)] \bigg|_{z=1}
\]

where \( z = e^t \). The quantities \( \chi_L(q) \) are easily expressed in terms of the Lee-Yang zeroes:

\[
\chi_L(q) = (-1)^{q-1}(q-1)! \sum_{k=1}^{\xi_L} \frac{1}{(1-z_L(k))^q}
\]

B. The “thermodynamic” limit \( L \to \infty \).

1. Divergence of the moments and Lee-Yang phenomenon.

As already written in the introduction, a mathematical definition of the thermodynamic limit in SOC is a difficult task, beyond the scope of this paper. However, in [20], we developed a dynamical system approach for the Zhang model. Then, the thermodynamic formalism [21–24] can be used to define the finite size SOC state of as a Gibbs measure in this setting. It is then shown that the joint avalanche size distribution, for example, can be obtained in this formalism via a proper potential. The corresponding generating function for the time correlations, called the topological pressure, is the formal analog to the free energy. In this setting, it is argued that the critical behavior expected in the thermodynamic limit, is manifested by a non analyticity of the topological pressure as \( L \to \infty \), that can be linked to the loss of hyperbolicity characterizing the limit \( L \to \infty \) of the Zhang model [25]. The loss of analyticity can be easily detected by looking at the generating function (1). Indeed, its zeroes exhibit a Lee-Yang phenomenon.

The paper [20] is devoted to dynamical system aspects and to the mathematical foundation of a thermodynamic formalism for the Zhang SOC model, and the link between the scaling theory of Lee-Yang zeroes in classical critical phenomena and general SOC model is not addressed. This is the aim of the present paper. The results developed here are therefore complementary to [20] but are independent.

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1. There is obviously a formal analogy between (1) (resp. (3)) and a partition function (resp. a free energy).
2. The particular structure of the Zhang model allows to symbolically encode the dynamics. In the framework of the thermodynamic formalism a Gibbs measure is a probability measure weighting the symbolic chains encoding the trajectories with an exponential weight called a potential (see [21] for details).
The present paper focuses on the analytic properties of the log-generating functions of the probability of avalanches indicators, when \( L \to \infty \). It intends to analyze the variations in the Lee-Yang zeroes properties if one uses the different scaling forms found in the SOC literature. Consequently, we collected the minimal implicit assumptions used in the SOC literature and we infer the consequences they lead to. This means that the result developed a priori hold for all SOC models.

It is first assumed that \( P_n(L) \) converges to some probability distribution \( P_n^*(n) \), \( n = 1 \ldots \infty \). It is furthermore assumed that \( P_n^*(n) \) has a power law tail\(^3\), namely \( P_n^*(n) = K n^{-\tau} \), for a certain \( n \) range, \( n = n_0 \ldots \infty \) where \( n_0 < \infty, \forall L \). The number \( n_0 \) depends on the model and on the observable and introduces an extra parameter in the characteristics of the probability distribution. In the computations done in this paper there is no loss of generality is assuming that \( n_0 = 1 \). Therefore, in the sequel, \( P_n^*(n) \) will stand for the power law \( K n^{-\tau} \), \( n = 1 \ldots \infty \).

The measured exponent \( \tau \) in SOC belongs to the interval \([1, 2]\). \( K = P_n^*(1) \) is the normalization constant. Consequently, \( K = \frac{1}{\zeta(\tau)} \) where \( \zeta \) is the Riemann \( \zeta \) function\(^4\). Under the above assumptions, the moments \( m_L(q) \) behaves asymptotically like \( \sum_{n=1}^{n=\infty} n^{q-\tau} \). This sum diverges for all \( q > \tau - 1 \). It is numerically observed that \( m_L(q) \) diverges like \( m_L(q) \sim L^{\sigma(q)} \). A central issue is to compute the scaling exponents given by:

\[
\sigma(q) \overset{\text{def}}{=} \lim_{L \to \infty} \frac{\log(m_L(q))}{\log(L)} = \lim_{L \to \infty} \frac{\log(\chi_L(q))}{\log(L)}
\]

\( \sigma(q) \) is a non-decreasing function. Its Legendre transform is found under the name of “multifractal spectrum” in the SOC literature\(^5\) though it has no direct connexion with the fractal geometry of the invariant set.

Since \( P_n^*(n) \) is a probability distribution the limiting generating function:

\[
Z_n^*(z) = \lim_{L \to \infty} Z_L(z) = \sum_{n=1}^{n=\infty} P_n^*(n)z^n
\]

is still an analytic function in the open unit disc in \( \mathbb{C} \). However, the log-generating function of \( P_n^*(n) \) is not analytic near to \( z = 1 \) since the derivative of order \( q > \tau - 1 > 0 \) diverge. The corresponding singularity is related to the behavior of the zeroes in the vicinity of \( z = 1 \). More precisely, fix \( \epsilon > 0 \) arbitrary small, call \( I_L(\epsilon) = \{ i \mid |z_L(i) - 1| < \epsilon \} \) and \( n_L(\epsilon) = \#I_L(\epsilon) \) where \( \# \) denotes the cardinality of a set. Then the divergence of \( \chi_L(q) \) is governed by the zeroes which accumulate in \( I_L(\epsilon) \). Namely, the sum \( \text{(3)} \) contains a singular term:

\[
\gamma_s(L, \epsilon, q) = (-1)^{q-1}2(q-1)! \sum_{k=1}^{n_L(\epsilon)} \cos(q\nu_L(k)) \frac{\cos(q\nu_L(k))}{r^2_L(k)}
\]

which diverges as \( L \to \infty \), while the remaining part in the sum is regular and is bounded by \( \frac{(q-1)!}{\epsilon q} \) as \( L \to \infty \).

\(^3\)Note that the limiting probability distribution is defined only if \( \tau > 1 \).
\(^4\)In general the normalization constant depends on \( n_0 \).
2. Scaling of the zeroes in classical critical phenomena.

In the theory of classical critical phenomena, it is possible to relate the scaling exponents of quantities such as magnetization or latent heat, susceptibility, etc ... to the behavior of the Lee-Yang zeroes near to \( z = 1 \). There exists a scaling theory based on earlier works by Lee and Yang [17], Grossmann and Rosenhauer [26], Abe [27], Suzuki [28], Privman and Fisher [29], Itzykson et al. [30], Glasser et al. [31]. Many analytical and rigorous results are also known (for example [32–35]). A lot of efforts have been devoted to the study of ferromagnetic systems (e.g. Ising or Potts models) though many other examples have also been studied in the literature. In this setting, one distinguishes the zeroes in the complex magnetic field (called Lee-Yang zeroes) from the zeroes in the temperature plane (Fisher zeroes). In the first case, the zeroes lie on the unit circle for a large class of models including the Ising’s one.

The Fisher zeroes usually approach \( t = 0 \) in the \( t = \log(z) \)-complex plane with a constant angle \( \phi \) (this is the case for the Ising model and mean-field ferromagnetic models [31]). This allows to obtain simple scaling expression for the singular part of the free energy \( f^s(t) \) where \( t \) is the reduced temperature. In this setting, an analytic expression for \( f^s(t) \) has been obtained by Grossmann and Rosenhauer [26], and, later on, extended by Itzykson et al. [30] by using the renormalization group theory. This approach has been extended by Glasser et al. [31] to mean-field models. In the thermodynamic limit \( f^s_{\pm} \sim A_{\pm}|t|^{2-\alpha} \) where \( A_{\pm} \) are universal constants (\( \pm \) label the two magnetic phases at low temperature), and \( \alpha \) is the critical exponent for the specific heat. It follows from the renormalization group analysis [29] that the singular part of the free energy obeys a Finite-Size Scaling form:

\[
f^s(t, V) = \frac{1}{V} F[t(AV)^{\frac{1-\alpha}{2}}]
\]

where \( V \) is the finite volume and \( F \) a universal function. Accordingly, the \( n \) first Fisher zeroes are given by:

\[
t_V(n) = \left[ \frac{2\pi}{(A_+^2 + A_-^2 - A_+A_- \cos(\pi\alpha))} \right]^{\frac{1}{2-\alpha}} e^{i(\pi - \phi)}
\]

The angle \( \phi \) is related to \( A_{\pm}, \alpha \) by \( \tan[(2-\alpha)\phi] = \frac{\cos(\pi\alpha) - A_-}{\sin(\pi\alpha)} \). This situation, where the zeroes approach the singularity with a constant angle \( \phi \) and where the modulus scales like the volume to a certain power will be referred to as normal scaling in the sequel.

It seems a general observation [26] that the zeroes lie on a curve or a union of curves dividing the complex plane in different regions of analyticity of \( Z^*(z) \), corresponding to different phases. More precisely, it has been recently proved by Biskup et al. [34] that the zeroes lie on curves with a simple analytic expression and accumulate in the thermodynamic limit on loci where the various branches of the free energy have the same modulus. This last result suggests that a wide extension of the Lee-Yang phenomena can be made toward dynamical systems near to a critical point. We now develop this aspect for the analysis of the log generating function of probability distribution in the SOC framework. In the sequel, we will not distinguish between the Lee-Yang zeroes and the Fisher zeroes and we will use the generic terminology “Lee-Yang” for the zeroes.
II. SCALING THEORY OF SOC AND LEE-YANG ZEROES.

In this section, we establish analytical results for various finite size scaling forms found in the SOC literature. These results are then used in section III for the analysis of the empirical data obtained from a numerical simulation of a SOC model. As a matter of fact, for finite size SOC systems, the power law is truncated by a cut off characterized by a length scale $\Lambda_L$, usually different from $\xi_L$. The starting point is therefore the analysis of a truncated power law with a sharp cut-off at a value $\Lambda_L$. This is a useful example for subsequent analysis since the analytic form of $P_L(n)$ and the cut-off is known.

A. Zeroes of a truncated power law.

Assume that $P_L(n) = \frac{C_L}{n^\tau}, n = 1 \ldots \xi_L$ where $C_L$ is a normalization constant and $\xi_L = \Lambda_L = L^\beta, \tau > 1, \beta > 0$. Furthermore, assume that $\tau < 2$.

1. Scaling of the moments and log generating function.

For $0 \leq q \leq \tau - 1$, $m_L(q) \rightarrow \frac{\xi(\tau - q)}{\xi(\tau)}$ and consequently $\sigma(q) = 0$. The non-zero scaling exponents $\sigma(q)$ can be obtained from the following integral approximation of $m_L(q)$, which becomes exact in the limit $L \rightarrow \infty$, provided $q \geq \tau - 1$:

$$m_L(q) \sim C_L \Lambda_L^{q+1-\tau} \int_{\frac{1}{\xi_L}}^{1} u^{q-\tau} du = \frac{C_L}{q + 1 - \tau} \left(\Lambda_L^{q+1-\tau} - 1\right) \quad (11)$$

Then, $\sigma(q) = \beta(q + 1 - \tau)$ for (real) $q > \tau - 1$. Note however, that for finite size, one has additional $L$ dependent terms which have to be considered when extrapolating from numerical simulations. It is also interesting to note that formula (11) gives useful informations on the rate of convergence of $m_L(q)$ to a constant for $q < \tau - 1$. Indeed, the convergence is not uniform in $q$, namely the closer is $q$ to $\tau - 1$ the slower is the convergence rate. This means that a systematic bias due to finite size is introduced in the numerical simulations when dealing with the $q$’s close to $\tau - 1$. This produces a spurious curvature, near to $\tau - 1$, for the function $\sigma(q)$ extrapolated from numerical data. This effect, which disappears as $L \rightarrow \infty$, can lead to misleading conclusion since it can be interpreted as an evidence of a multifractal scaling.

The scaling exponents can also be obtained from the scaling of the log generating function. Indeed, the generating function writes:

$$Z_L(t) = 1 + \sum_{n=1}^{\xi_L} P_L(n)(e^{tn} - 1) \sim 1 + C_L \Lambda_L^{1-\tau} \int_{\frac{1}{\xi_L}}^{1} u^{-\tau}(e^{t\Lambda_L u} - 1) du \quad (12)$$

Set $t' = \Lambda_L t$ and

$$\psi(t') \overset{\text{def}}{=} \int_{0}^{1} u^{-\tau}(e^{t'u} - 1) du = \sum_{n=1}^{\infty} \frac{t'^n}{n!} \int_{0}^{1} u^{n-\tau} du = \sum_{n=1}^{\infty} \frac{t'^n}{n!(n + 1 - \tau)} \quad (13)$$
Note that since $\tau < 2$, this integral is finite as can easily be checked by integration by part. Therefore the commutation of the integral and the series is allowed.

Consequently,

$$Z_L(t) \sim 1 + C_L \Lambda_L^{1-\tau} \psi(t') = 1 + C_L L^{\beta(1-\tau)} \psi(tL^\beta)$$

(14)

and :

$$G_L(t) = \log(1 + C_L L^{\beta(1-\tau)} \psi(tL^\beta))$$

(15)

which gives the right scaling for the moments by differentiating with respect to $t$ at $t = 0$. One remarks that this scaling form is analogous to the form (9).

2. Lee-Yang zeroes.

From equation (14) the zeroes are approximately given by:

$$\psi(t') = -C_L \Lambda_L^{\tau-1} = -C_L L^{\beta(\tau-1)}$$

(17)

Since $\tau > 1$, $\Lambda_L^{\tau-1}$ diverges. $\psi(t')$ is an increasing function of the real variable $t'$ which vanishes as $t' \to 0$ and tends to $-\infty$ when $t' \to -\infty$. Furthermore, for any $K > 0$, $\psi(t')$ is bounded by $\psi(K)$ in the ball $|t'| < K$ in the complex plane. Consequently, the equation (17) can be fulfilled only if $t'_L(k)$'s have a diverging modulus as $L$ grows. On the other hand, since $t_L(k) = \frac{t'_L(k)}{\Lambda_L}$ converges to zero, $|t'_L(k)|$ must grow slower than $\Lambda_L$. It grows in fact like $\log(\Lambda_L)$ as shown below.

Note that, conversely, when $\tau < 1$, $-C_L L^{\beta(\tau-1)}$ goes to zero in the thermodynamic limit $[\text{5}]$. Then, the zeroes are formally given by :

$$t_L(k) = \frac{1}{\Lambda_L} \psi^{-1}_k(-C_L \Lambda_L^{\tau-1}) \sim \frac{1}{\Lambda_L} \psi^{-1}_k(0)$$

where $\psi^{-1}_k$ is the $k$ th branch of the inverse of $\psi$ in the complex plane. Consequently, the zeroes have a simple scaling in these case similar to (10). In particular, the argument does not depend on $L$.

The observed values of the critical exponents in SOC, $\tau > 1$, induces anomalous scaling that can be observed on the Lee-Yang zeroes. Though the form (14) can be used to compute the Lee-Yang zeroes, it is easier to use:

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5Though the limit probability is not defined it is nevertheless possible to investigate the finite size scaling properties for the zeroes of the finite size generating function.
\[ Z_L(t) = \sum_{n=1}^{N} P_L(n) e^{\imath n} \sim CL \Lambda^{-\tau}_t \int_{\frac{1}{\Lambda_t}}^{1} h(u,t')du \]  

where \( h(u,t') = u^{-\tau} e^{\imath u}. \) There exists several techniques to compute the Lee Yang zeroes in statistical mechanics. A standard way is to argue that the asymptotic free energy admits different analytic continuation in different regions of the complex plane, separated by Stokes lines where the zero accumulate in the thermodynamic limit. Indeed, because of the large number of terms in the polynomial which make up the partition function, the behavior tends to become dominated by some set of the coefficients. Thus we have different analytic functions in different regions of the complex plane. These functions have oscillating phases but smoothly varying amplitude. The zeroes locates then on Stokes boundaries where two types of behavior have comparable magnitude \([30,31,34]\). The Stokes boundaries becomes cuts in the thermodynamic limit. Across the boundaries the free energy has a regular real part and jumps in the imaginary part \([20]\).

Applying this strategy to our formal partition function \([18]\), one identifies easily two regions. For real \( t' \), as \( u \) grows from to 0 to \( \infty \), \( h(u,t') \) first decay like \( u^{-\tau} \) until a minimum \( u_- = \frac{t}{\tau} \). Therefore, \( u_- > \frac{1}{\Lambda_t} \) when \( t < \tau \). For \( u > u_- \), \( h(u,t') \) grows exponentially like \( e^{\imath u} \). Therefore, when \( t' \) is small the integral in \([18]\) is essentially dominated by the algebraic decay \( u^{-\tau} \) and \( \int_{\frac{1}{\Lambda_t}}^{1} h(u,t')du \sim \frac{1}{\tau-1}[\Lambda^{-1}_t - 1] \). On the other hand, for large \( t' \), \( u_- \to 0 \) and the algebraic part is negligible compared to the exponential part. Hence \( \int_{\frac{1}{\Lambda_t}}^{1} h(u,t')du \sim \int_{u_-}^{1} e^{\imath u}du = \frac{1}{\tau} \left[ e^{\imath u} - e^{\imath \tau} \right] \). This argumentation extends to complex \( t' \) and suggests that one can roughly divide the \( t \) complex plane into two regions where \( Z_L(t) \) has a different analytic form: for sufficiently small \( t' \) the algebraic part dominates, while for large \( t' \) the exponential part is dominant. Then the zeroes have to stay at the place where the two forms are of the same order. Therefore an approximate equation for the location of the zeroes is given by:

\[ \frac{1}{t'} \left[ e^{t'} + a_1 t' - a_2 \right] = -\frac{\Lambda^{-1}_t}{\tau - 1} \]  

where \( a_1 = \frac{1}{\tau}, a_2 = e^{\tau}. \)

The solutions of \( \frac{t'}{\tau} = -\frac{\Lambda^{-1}_t}{\tau - 1} \) are given by:

\[ t'_L(k) = \Lambda_L t_L(k) = -W_k(\frac{\tau - 1}{\Lambda^{-1}_L}) \]  

where \( W_k(x) \) is the \( k \) th branch of the Lambert function \([20]\). Note that the Lambert function has infinitely many branch and consequently the equation \( \frac{t'}{\tau} = -\frac{\Lambda^{-1}_t}{\tau - 1} \) has infinitely many solutions. Indeed, in replacing the initial sum by an integral in \([18]\) we have introduced spurious zeroes which have to be removed for finite \( L \). In the sum \([18]\) one has a step of integration \( \frac{1}{\Lambda_t} \) which defines a lower cut-off in the scales one has to consider. Consequently, only the branches \( k = -\frac{1}{\Lambda_t}, \ldots, \frac{1}{\Lambda_t} \), where one takes into account the symmetry of the zeroes with respect to the real axis, are relevant.

The Lambert function \( W_k \) admits the following expansion, for large \( |\log(x)| \) \([20]\):

\[ W_k(x) = \log_k(x) - \log(\log_k(x)) + \sum_{l \geq 0} \sum_{m \geq 1} c_{lm}(\log \log_k(x))^m (\log_k(x))^{l+m} \]  

\[ Z_L(t) \]
where $\log_k$ is the $k$ th branch of the complex logarithm and $c_{lm} = \frac{1}{m}(-1)^l [l+m]_n$ is expressed in terms of the Stirling numbers of the first kind $(-1)^{m+n} [n]_m$.

The double series $\sum_{l \geq 0} \sum_{m \geq 1} c_{lm} \frac{\log \Lambda (x)^m}{\log (\Lambda x)^{l+m}}$ is absolutely convergent for sufficiently large $|\log(x)|$. Since we are only interested in the asymptotic divergence when $L$ grows, one can therefore neglect the series in the asymptotic. Note however that the convergence to the asymptotic regime where the series becomes negligible is faster when the product $\beta \tau - 1$ is larger.

The term $\log_l(\frac{\Lambda - 1}{\Lambda})$ cannot be neglected compared to $\log_l(\frac{\Lambda - 1}{\Lambda})$ since it contains crucial $k$ dependence for the real part of $t_L(k)$ (see below). It is interesting to note that it introduces a $\log\log L$ finite size scaling correction. A similar correction as been found in [37] for the Potts model with $q \geq 4$.

The corrections due to the other terms of eq. (19) become rapidly negligible as can easily be seen by a perturbation expansion.

One finally obtains the following asymptotic form for the Lee Yang zeroes of the truncated power-law:

\[
\Re(t_L(k)) \sim \frac{-\log(\frac{\tau - 1}{\Lambda}) + \frac{1}{2} \log \left( \log^2(\frac{\tau - 1}{\Lambda}) + 4k^2\pi^2 \right)}{\Lambda L} \tag{22}
\]

\[
\Im(t_L(k)) \sim \frac{2k\pi}{\Lambda L} - \frac{1}{\Lambda L} \arctan \left( \frac{2\pi k}{\log \Lambda} \right) \tag{23}
\]

The term $\log_l(\frac{\Lambda - 1}{\Lambda})$ introduces a $k$ dependence which implies in particular that the zeroes (in the $z$ plane) do not lie on circle but on a more complicated curve (see Fig. 3). This dependence remains important, even for the first zeroes, up, to very large $L$, especially if $\tau$ is close to $1$. Indeed, for a fixed $k$ the term $\log^2(\frac{\tau - 1}{\Lambda})$ dominates the term $4k^2\pi^2$ only for $L >> (\tau - 1)e^{2\pi k} \Lambda^{-1}$ (say, for $\tau = 1.25, \beta = 2.67$ this corresponds to a $L >> 1500$).

The arctan term in the imaginary part acts essentially as a phase term $-\frac{1}{\Lambda L} \arctan \left( \frac{2\pi k}{\log \Lambda} \right)$ which is slowly varying (in the $k$ variable) compared to the dominant term $\frac{2k\pi}{\Lambda L}$. Furthermore, since the arctan is bounded above by $\frac{\pi}{2}$ it is rapidly negligible as $k$ grows. Therefore, one can consider with a good approximation that $\Im(t_L(k)) \sim \frac{2k\pi}{\Lambda L}$.

The argument of $t_L(k)$ formally corresponds to the angle that the Fisher zeroes do with the real axis in critical phenomena. For a power law with $\tau < 1$ this angle is independent of $L$ as discussed above. Conversely, for $\tau > 1$ it is given by:

\[
\text{Arg}(t_L(k)) \sim \arctan \left( \frac{2k\pi}{\log(\frac{\tau - 1}{\Lambda}) + \frac{1}{2} \log \left( \log^2(\frac{\tau - 1}{\Lambda}) + 4k^2\pi^2 \right)} \right) \tag{24}
\]

One observes therefore a logarithmic deviation to the normal scaling on the argument of the $t_L(k)$’s. The conclusion is therefore that, though the truncated power law obeys the classical finite size scaling form (3), the Lee-Yang zeroes display nevertheless an anomalous scaling due to the exponent $\tau > 1$ (see Fig. 3).

In the $z$ plane the zeroes $z_L(k) = e^{t_L(k)}$ are approximately given by:

\[
R_L(k) = |z_L(k)| \sim 1 + \Re(t_L(k)) \tag{25}
\]

\[
\theta_L(k) = \Im(t_L(k)) \sim \frac{2k\pi}{\Lambda L} = \frac{2k\pi}{L^\beta} \tag{26}
\]
Therefore the arguments $\theta_L(k)$ of the zeroes in the $z$ complex plane are uniformly distributed in $[-\pi, \pi]$ with a good approximation.

Finally, one can determine the exponents $\tau, \beta$ from the Lee-Yang zeroes. The exponent $\beta$ corresponds to the scaling exponent of the correlation length $\xi_L$. The equation (26) provides a straightforward way to compute it. Furthermore, the exponent $\tau$ can be obtained from eq. (22). The term $4\pi^2k^2$ certainly rapidly dominates the term $\log\left(\frac{\tau - 1}{\Lambda_L}\right)$ in the modulus $R_L(k)$ as $k$ grows. This is a fortiori true for $k \sim \Lambda_L$ which correspond to the zeroes the farthest from $z = 1$. Consequently:

$$R_L(\theta) \sim \left[\frac{\Lambda_L^{-1}}{\tau - 1} \sqrt{\log^2\left(\frac{\tau - 1}{\Lambda_L}\right) + 4\theta^2\Lambda_L^2}\right]^{\frac{1}{\Lambda_L}} \sim \left[\frac{2\theta \Lambda_L}{\tau - 1}\right]^{\frac{1}{\Lambda_L}}$$

(27)

If one takes $k = \frac{\Lambda_L}{2}$ (corresponding to an angle $\pi$), one has:

$$\tau = \lim_{L \to \infty} \frac{\Lambda_L \log(R_L(\pi))}{\log(\Lambda_L)}$$

(28)

which allows a possible determination of $\tau$ from the scaling of the Lee-Yang zeroes. Note however that the convergence is logarithmic in $L$.

3. Numerical checks.

Since it is easy to generate numerically a power law distribution one is a priori free to choose any values for $\tau$ and $\beta$. However, the closer $\tau$ is to 1 the slower is the convergence to the asymptotic regime where the formulas obtained in the previous section hold. More precisely, the rate of convergence is essentially governed by the product $\beta(\tau - 1)$. The closer is $\tau$ to 1 the larger has to be $\beta$. But the larger $\beta$, the faster the degree of the polynomial increases with $L$ and therefore the time needed for the computation of the zeroes increases. On the other hand, since the theory developed here is independent of the actual value of $\beta, \tau$ (provided $\tau > 1$), we mainly studied examples where $\beta = 2$ and $\beta = 2.2$ which gives a reasonable increase in the polynomial degree, and $\tau = 1.9$ such that the product $\beta(\tau - 1) \sim 2$.

We first depicted the pattern of zeroes, for different sizes, in Fig. 1.
One notices the slow convergence to the unit circle, and the shape of the curve near to $z = 1$: this is not a circle.

We plotted Fig. 2 the real and imaginary part of the zeroes in the $t$ plane. For the real part we tried a fit of the form

$$r(k) = \left( -\log(a) + \frac{1}{2} \log(\log^2(a) + 4\pi^2k^2) \right) \Lambda_L$$

where $a$ is a free parameter. The formula (22) gives $a_{th} = \frac{\tau - 1}{\Lambda_L}$ but, in the several approximations we made, we neglected some constants, and one expects $a$ to be different from the theoretical value, with an error that should decrease as $L$ grows. The result of the fitting is represented Fig. 2a for the 200 first zeroes. We found indeed that the experimental value $a_{exp}$ is closer and closer to its theoretical value as $L$ increases and that our approximation is better and better as $L$ increases. For $L = 50$, $a_{th} = 7.87 \times 10^{-4}$, $a_{exp} = 5.6 \times 10^{-4} \pm 4 \times 10^{-5}$; for $L = 100$, $a_{th} = 2.26 \times 10^{-4}$, $a_{exp} = 1.5 \times 10^{-4} \pm 3 \times 10^{-5}$ and for $L = 150$, $a_{th} = 1.09 \times 10^{-4}$, $a_{exp} = 7.1 \times 10^{-5} \pm 810^{-7}$. The imaginary part is plotted Fig. 2 b together with the theoretical prediction $\Im(t_L(k)) = \frac{2\pi k}{\Lambda_L}$ (straight lines). We remarked a slight deviation of the first zeroes to $\frac{2\pi k}{\Lambda_L}$ due to the correction term in (26).

Fig. 3 a, we plotted the argument of $t_L(k)$ as a function of $L$ for $k = 1 \ldots 5$. We notice the logarithmic deviation to the normal scaling as predicted by formula (24). We tried to fit these curves with a fit of the form

$$\arctan\left( \frac{2\pi k}{\alpha(\log(x) + \gamma) + \frac{1}{2} \log(\alpha^2(\log(x) + \gamma)^2 + 4\pi^2k^2)} \right)$$

where $\alpha, \gamma$ are free parameters. In fig. 3 b, we show the different scaling occurring for $\tau < 1$ (normal scaling) or $\tau > 1$ (anomalous scaling).
Finally we argued above that the exponents $\beta$ and $\tau$ can be determined with a good accuracy from the scaling of the zeroes. In fig. a we plotted $\text{Arg}(z_L(5)) = \theta_L(5)$ as function of $L$ for $\beta = 1.5, \tau = 1.6; \beta = 2, \tau = 1.21; \beta = 2.2, \tau = 1.9$ and $L = 20 \ldots 120$. We choose the 5th zero rather than the first ones since for the first zeroes the correction coming from the arctan term in eq. (23) influences slightly the scaling for small $L$. We tried a fit of the form $C x^\beta$ where $C_{th} \sim 31.415 \ldots$. The fits shown on the figure gave respectively: $\beta = 1.507 \pm 0.002; \beta = 2.007 \pm 9 \times 10^{-3}; \beta = 2.205 \pm 0.001$.

For the determination of $\tau$ we used eq. (27) for $\tau = 0.2; 1.21; 1.9$ and $\beta = 2$. We have interpolated the data with a fit form $e^{\beta \tau \log(ax)/x^\beta}$ where $a, \tau$ were free parameters. For $L = 20 \ldots 120$, we found respectively $0.1998(2) \pm 6.10^{-6}; 1.207(5) \pm 7.10^{-5}, 1.89(5) \pm 0.0001$ for the value of $\tau$. This is satisfactory especially when taking into account the smallness of the $L$’s we considered.

B. General case : the structure of the cut off.

The observed probability distribution of avalanches observables in SOC models is in general a power-law truncated by a cut-off function associated to finite-size effects. Except for a few case [38], the analytic form of the cut-off is not known. Consequently, various scaling form have been proposed in the SOC literature. In this section we grasp the most common scaling forms and discuss their effect on the Lee-Yang zeroes. We also investigate the effect of the sampling cut-off inherent to numerical simulations.

1. General assumptions about the cut off function.

Since $P_L(n)$ is expected to converge to a power law $Kn^{-\tau}$ one can write, without loss of generality, the finite size probability under the form:
\[ P_L(n) = \frac{C_L}{n^\tau} f_L(n), \quad 1 \leq n \leq \xi_L \]  

(29)

where \( C_L \) is a normalization constant depending on \( L \). \( f_L(n) \) is the finite size cut-off.

The graph obtained from numerical simulations suggests that \( f_L(n) \) is a regular function which obeys:

\[
(i) \quad \lim_{L \to \infty} f_L(n) = 1, \forall n \leq \xi_L.
\]

\[
(ii) \quad \forall p > 0, \lim_{n \to \infty} n^p f_L(n) = 0, \forall L
\]

The property (i) corresponds to the pointwise convergence to a power law. (ii) characterizes the observed fact that the tail of \( P_L(n) \) decreases faster than any power of \( n \) (e.g. is least exponentially decreasing).

These properties are not sufficient for a scaling theory and further assumptions have to be made. We now discuss various scaling form that one can find in the literature but also the numerically induced cut-off effects.

**Finite-Size Scaling.** In 1990, Kadanoff et al. [13] proposed a Finite-Size Scaling Ansatz where:

\[ f_L(n) = g \left( \frac{n}{\Lambda_L} \right) \]  

(30)

\( g \) being a universal function. \( \Lambda_L = L^\beta \) is the characteristic scale for a lattice of size \( L \) and \( \xi_L = \alpha \Lambda_L \) where \( \alpha \) is a constant. The case of a truncated power law developed in section [11.A] corresponds to the particular case where \( \alpha = 1 \), and \( g(x) \) is equal to 1 for \( x \in [0, 1] \) and zero otherwise. Note that in this case the property (ii) has the consequence that:

\[
(iii) \quad \lim_{x \to 0} g(x) = 1
\]

multifractal scaling. In the same paper, Kadanoff et al. [13] discussed another form of scaling, that is:

\[
\frac{\log(P_L(n))}{\log(L_{\infty})} = f \left( \frac{\log(n)}{\log(L_{\infty})} \right)
\]  

(31)

where \( f \) is universal and does not depend explicitly on \( L \). \( L_0, n_0 \) are some constants that can be omitted by a suitable redefinition of the quantities, then:

\[
P_L(n) = L^f(\log n/\log L)
\]  

(32)

This representation is called by the authors an \( f - \alpha \) representation, \( \alpha \) being the quantity \( \frac{\log(n)}{\log(L_{\infty})} \). In this case one has a whole spectrum of scaling indexes, i.e. all the values taken on by \( \frac{df}{d\alpha} \). In the general case \( f \) is non linear. Then the universality class is given by the function \( f \), rather than by a finite set of critical exponents. In this case, the scaling exponents are non linear functions of \( q \).

**Finite Size Scaling violation and convergence to a power law.** Another scaling form has been introduced by Lise and Paczuski [16] from numerical simulations on the OFC model [8]. Lise and Paczuski analyzed their data with the following form for \( P_L(n) \):

\[^6B.C.\ is\ very\ grateful\ to\ M.\ Paczuski\ for\ illuminating\ discussions\ on\ this\ topic\ in\ Bielefeld.\]
\[ P_L(n) = C_L n^{-\tau} L^{F_L \left( \frac{\log(n)}{\log L} \right)}; \ n = 1 \ldots L^{\beta L} \]  

(33)

where \( \beta_L \) is now \( L \) dependent, \( \beta_L < \beta < \infty \) and \( \beta_L \to \beta \) as \( L \to \infty \). Furthermore, the numerical plot of \( F_L(x) \) in [10] suggests that \( F_L(x) \) converges to a “step” function \( Y(x - \beta) \) as \( L \to \infty \), where \( Y(u) = 0, u \in \mathbb{R} \), \( Y(0) = C \) and \( Y(u) = -\infty, u \in [0, \infty] \). The finite size scaling case corresponds to \( F_L(x) = \frac{\log(g(Lx^{1-\gamma}))}{\log L} \) where \( g \) is defined in [30]. This example is quite interesting since it gives an example of a probability distribution violating (30) but converging nevertheless to a power law (namely the exponents \( \tau \) and \( \beta \) are still meaningful). We remark indeed that the corresponding probability distribution is not multifractal in the sense of [14] since it is easy to check that the scaling exponents are given by \( \sigma(q) = \beta(q + 1 - \tau), \forall q \geq \tau - 1 \) and \( \sigma(q) = 0, \forall q < \tau - 1 \). This case is therefore intermediate between the Finite-Size Scaling [30] and the multifractal case [32].

In the following sections we discuss the behavior of the Lee-Yang zeroes in these different cases.

2. Finite-Size Scaling.

We show in this section that the behavior of the Lee-Yang zeroes in the finite size scaling case is essentially the same as for the truncated power law, provided \( g \) in (30) fulfills the conditions (i),(ii),(iii) in the previous section.

It is well known that the moments obey the same scaling. This can be recovered from the generating function [4]. Provided \( \tau < 2 \) it scales as \( L \to \infty \) like:

\[ Z_L(t) \sim 1 + C_L A_L^{1-\tau} \left[ \Psi_\alpha(t') - \frac{\Psi_\alpha(t')}{\tau} \right] + C_L A_L^{1-\tau} \Psi_\alpha(t') + C_L \psi(t) \sim 1 + C_L A_L^{1-\tau} \Psi_\alpha(t') \]

(34)

for \( t \to 0 \) in the \( t \) complex plane. In this equation \( t' = \Lambda_L t \) and:

\[ \Psi_\gamma(t') \overset{\text{def}}{=} \int_0^{\gamma} u^{-\tau} g(u) (e^{t'u} - 1) du = \sum_{n=1}^{\infty} \frac{t^n}{n!} \int_0^{\gamma} u^{n-\tau} g(u) du \]

(35)

Consequently, near to \( t = 0 \), \( Z_L(t) \sim 1 + C_L L^{\beta(1-\gamma)} \Psi_\alpha(tL^\beta) \) and \( G_L(t) \sim C_L L^{\beta(1-\gamma)} \Psi_\alpha(tL^\beta) \). As expected one obtains the same scaling as for a truncated power law. The only change is that the function \( \Psi \) depends now on the \( g \) function.

The zeroes of \( Z_L(t) \) are therefore well approximated by:

\[ \Psi_\alpha(t') \sim -C_L A_L^{\gamma-1} \]

(36)

which is completely analogous to [17]. Therefore, the same conclusions hold: the equation (36) can be fulfilled only if the solutions, \( t'_L(k) \), have a diverging modulus as \( L \) grows. The computation of the zeroes is essentially the same as in section (IIA) which slight complications due to the presence of the function \( g \).

One can write:

\[ Z_L(t) \sim C_L A_L^{1-\gamma} \int_{\lambda}^{\alpha} h(u, t') du \]

(37)

15
where now \( h(u, t') = u^{-\tau}g(u)e^{t'u} \). Recall that, by hypothesis, \( g(u) \) is an at least exponentially decreasing function. As \( u \) grows from to \( 0 \) to \( \infty \), \( h(u, t') \) first decay like \( u^{-\tau} \) until a minimum \( u_- \) after which \( h(u, t') \) grows exponentially like \( e^{t'u} \). \( u_- \) tends to \( 0 \) as \( t' \to \infty \). More precisely, if \( t' \) is sufficiently large, from (iii) \( g(u) \) is essentially 1 on the interval \([0, u_-]\) and therefore \( u_- \) is approximately given by \( t'u_-' = \tau u^{-\tau-1} \). Consequently \( u_- \sim \frac{\tau}{t'} \). When \( u \) is sufficiently large the decay coming from \( g(u) \) compensates the exponential increase of \( e^{t'u} \). Therefore, there is a maximum \( u_+ \) after which \( h(u, t') \) tends to zero with a rate given by \( g(u) \). Here, \( h(u, t') \) is essentially \( g(u)e^{t'u} \). Therefore \( u_+ \) is given by \( t'g(u) + g'(u) = 0 \) or \( t' = -\frac{d \log(g(u))}{du} \), where \( -\frac{d \log(g(u))}{du} \) is a function which increases faster than \( u \) by (ii). Consequently, \( u_+ \) diverges as \( t' \to \infty \). Since \( \alpha \) is bounded by assumption, \( u_+ > \alpha \) for \( t' \) (resp. \( L \)) sufficiently large. But, from eq. (36) we know that the modulus of the \( t' \) corresponding to the zeroes diverge as \( L \to \infty \). Consequently, provided \( L \) for is sufficiently large, the zeroes lie in a region where \( u_+ > \alpha \). Guided by the wisdom coming from section 1A we can assume that the zeroes accumulate onto a curve \( \gamma_L \) which separates the complex plane into two regions. In the first one, the behavior is dominated by the algebraic part and the integral in (37) is essentially \( \int_{\gamma_L} h(u, t')du \sim \frac{1}{\tau} \left[ \Lambda_L^{-\tau-1} - \alpha^{-\tau-1} \right] \). In the second region, the exponential part dominates. Provided \( t' \) is sufficiently large \( (u_+ \gg \alpha) \), the variations of \( u^{-\tau}g(u) \) are small compared to the increase of \( e^{t'u} \) and this function can be approximated by some constant \( \Gamma_g \). Hence \( \int_{u_-}^\alpha h(u, t')du \sim \Gamma_g \int_{u_-}^\alpha e^{t'u}du \sim \Gamma_g \left[ e^{t'u} - e^u \right] \).

Consequently, for sufficiently large \( L \), the zeroes are well approximated by

\[
\frac{1}{t'} \left[ e^{t'u} + a_1t' - a_2 \right] = -\frac{\Lambda_L^{-1}}{\Gamma_g(t' - 1)}
\]

where \( a_1 = \frac{\alpha^{-\tau-1}}{\tau^{(\tau-1)}}, a_2 = e^\tau \).

This equation is similar to eq. (19). The zeroes are now given by:

\[
t_{L}(k) = -\frac{W_k(\frac{a_1\tau}{\Lambda_L})}{\frac{a_1\tau}{\alpha\Lambda_L}(\tau - 1)}
\]

Consequently, one finds that the pattern of zeroes in the finite-size scaling case is essentially the same as the power law case, up to a correction depending on \( \alpha, \Gamma_g \). More precisely, using the series expansion (21) of the Lambert function one finds:

\[
\Re(t_{L}(k)) \sim \frac{-\log(\frac{\alpha\Gamma_g(\tau - 1)}{\Lambda_L^{-1}}) + \frac{1}{2}\log\left(\log^2(\frac{\alpha\Gamma_g(\tau - 1)}{\Lambda_L^{-1}}) + 4k^2\pi^2\right)}{\alpha\Lambda_L}
\]

(40)

\[
\Im(t_{L}(k)) \sim \frac{2k\pi}{\alpha\Lambda_L}
\]

(41)

where \( k = 1 \ldots \xi_L = \alpha\Lambda_L \).

In the \( z \) plane this essentially results in a trivial re scaling of the argument \( \Im(t_{L}(k)) \) and a slight change in the modulus \( R_{L}(k) \):

\[
R_{L}^{\text{FSS}}(k) \sim \kappa_L R_{L}^{\text{PL}}(k)^{\frac{1}{2}}
\]

(42)

where the superscript FSS (resp. PL) refers to the Finite Size scaling (resp. Power Law) situation. \( \kappa_L = \left( \frac{1}{\alpha\Lambda_L} \right)^{\frac{1}{2\alpha\Lambda_L}} \) is therefore a scaling factor which tends to 1 as expected since the zeroes have to accumulate on the unit circle. In equation (12) we neglected the more complicated dependence coming from the \( \log(\log) \) term. This has a small effect
on the first zeroes but becomes negligible as \( k \) grows. This provides a way to determine \( \kappa_L \). Setting \( R_L(\pi) \) for the farthest zero from \( z = 1 \):

\[
\kappa_L = \frac{R_F^{SS}(\pi)}{R_L(\pi)^2}
\]  

(43)

The argument of \( t_L(k) \)'s is:

\[
\text{Arg}(t_L(k)) \sim \arctan \left( \frac{2k\pi}{-\log(\frac{\alpha \Gamma_{g}(\tau - 1)}{\Lambda_{L}^{-1}}) + \frac{1}{2} \log \left( \log^2(\frac{\alpha \Gamma_{g}(\tau - 1)}{\Lambda_{L}^{-1}}) + 4k^2\pi^2 \right)} \right)
\]  

(44)

The cut off \( g \) modifies therefore the value of the angles but not the scaling.

We numerically checked these result in the following case. We generated a probability distribution given by:

\[
P_L(n) = C_L n^{-\gamma} g(\frac{n}{L^\gamma}); \ n = 1 \ldots \alpha L^\beta
\]

where:

\[
g(x) = e^{-x^\gamma}
\]

(46)

We fixed the parameters to the values: \( \beta = 2, \tau = 1.9, \gamma = 2.5, \alpha = 1 \). We show Fig. 5 the collapse of the curve of zeroes to the corresponding curve for a power law with the same \( \tau, \beta \). \( \kappa \) was computed from the ratio (43). We found \( \kappa = 1.0001 \) for \( L = 100 \).

3. Effect of a sampling cut off.

We would like now to point out a very simple way to violate the scaling form (30), by the only numerical procedure traditionally used in the computation of \( P_L(n) \). This effect is closely related to the anomalous scaling of the Lee-Yang zeroes since it appears for a critical exponent \( \tau > 1 \).
In numerical simulations, one computes an empirical distribution \( P^{\text{exp}}_L(n, \omega) = \frac{\mathcal{N}(n, \omega)}{N} \), where \( \mathcal{N} \) is the total number of avalanche observed during the simulation and \( \mathcal{N}(n, \omega) \) is the number of times an avalanche of size \( n \) was observed. This number is a random variable depending, say, on the initial condition(s), or more generally, on the seed \( \omega \) used in the random generator. However, one expects the system to be ergodic in a strong sense such that \( P^{\text{exp}}_L(n, \omega) \rightarrow P_L(n) \) as \( \mathcal{N} \rightarrow \infty \) for generic choices of \( \omega \) (see [1] for details). However, since \( \mathcal{N} \) is finite, there exists wild fluctuations in the tail of the distribution. Furthermore, the avalanches such that \( P_L(n) < \frac{1}{N} \) have a small (though non zero) probability to be observed in a numerical simulation and this probability decreases as \( n \) increases. Obviously, there are several methods such as smoothing or binning, allowing to reduce the effects of noise in the tail.

There exists however another, more subtle effect. In all the examples of numerical computations that we have found in the SOC literature, the value of \( \mathcal{N} \) is fixed, independently of the system size. This induces a pathological bias affecting the extrapolation to the thermodynamic limit whatever the method used to analyze the empirical distribution. In particular a violation of finite-size scaling can be observed on \( P^{\text{exp}}_L(n, \omega) \) even if the theoretical probability \( P_L(n) \) obeys (30). When \( \mathcal{N} \) is kept fixed while \( L \) increases, the estimation of the maximal value \( \xi_L \) that the random variable can take (defining the exponent \( \beta \)) is more and more biased. Indeed, while the true \( \xi_L \) diverges as \( L \rightarrow \infty \), the empirical value \( \xi_{\text{exp}}^L(\omega) \) converges to a constant. Consequently, the probability distribution extrapolated to the thermodynamic limit from the empirical distribution is biased. The aim of this section is to analyze this effect, not discussed in the literature.

Assume therefore that \( P_L(n) \) is like [2] with a cut off obeying (30). Call \( F_L(n) = \sum_{k=1}^n P_L(k) \) the corresponding repartition function. Assume now that we perform a finite sampling of the probability distribution with \( \mathcal{N} \) trials \( X_1, \ldots, X_\mathcal{N} \) where \( X_i, i = 1 \ldots \mathcal{N} \) are independent identically distributed random variable, with probability \( P_L(n) \). Assume furthermore that \( \mathcal{N} \) is fixed independently of \( L \). Call \( \xi_{\text{exp}}^L = \max \{ X_k, k = 1 \ldots \mathcal{N} \} \) the maximal value observed in the finite sampling. The repartition function of the random variable \( \xi_{\text{exp}}^L \) is \( F_\mathcal{N}^L(x) \). Its average is given by:

\[
E[\xi_{\text{exp}}^L] = \xi_L = \sum_{n=1}^{\xi_L} F_\mathcal{N}^L(n)
\]

(47)

Clearly, were \( L \) fixed while \( \mathcal{N} \rightarrow \infty \), then would \( E[\xi_{\text{exp}}^L] \) converges to \( \xi_L \). In fact the ergodic theorem gives a stronger statement, namely \( \xi_{\text{exp}}^L \rightarrow \xi_L \) almost-surely in this case. This essentially means that for sufficiently small \( L \), \( E[\xi_{\text{exp}}^L] \) gives a good estimate of \( \xi_L \). On the other hand, if \( \mathcal{N} \) is fixed while \( L \) increases, the correction term \( \sum_{n=1}^{\xi_L} F_L^\mathcal{N}(n) \) in (47) becomes more and more important leading to a wrong estimation of \( \xi_L \). To be more precise fix a value \( y \in [0,1] \), such that \( F_L^\mathcal{N} \) is considered as non negligible as soon as \( F_L^\mathcal{N}(n) > y \). Hence \( y \) is somehow arbitrary here (say close to 0). Since \( F_L^\mathcal{N}(n) \) is strictly increasing the equation \( F_L^\mathcal{N}(x) = y \Rightarrow F_L(x) = y^{\frac{1}{\mathcal{N}}} \) has a unique solution

\footnote{However, since the largest non zero value of \( P^{\text{exp}}_L(n, \omega) \) is \( \frac{1}{N} \), the events such that \( P_L(n) << \frac{1}{N} \), even when observed, are given an incorrect probability by the numerical procedure. The discrepancy with the theoretical value increases as \( n \) increases.}
\footnote{We assume here that the trials are independent for simplicity. In SOC models, the avalanches are not independent though, for finite \( L \), the correlation decay can be fast (it is exponential in the finite size Zhang model).}
\( x_L \equiv x_L(y), \forall y \in [0, 1]. \) If \( L \) is small (or if \( \mathcal{N} \) is sufficiently large) \( x_L \sim \xi_L \) and therefore \( F_L^N(n) \) is essentially non zero for \( n \sim \xi_L \). In this case the term \( \sum_{n=1}^{\xi_L} F_L^N(n) \) in (47) is negligible compared to \( \xi_L \).

On the other hand, \( x_L \) is bounded from above by a finite value \( x \) such that \( F^*(x) = K \sum_{n=1}^{x} n^{-\gamma} = y^{\frac{1}{\gamma}}, \) where \( F^* \) is the repartition function of the limiting probability \( P^*(n) \). Consequently, as \( L \) increases, for any \( y \in [0, 1], \frac{x_L(y)}{\xi_L} \to 0. \) Hence, the function \( F_L^N(x) \) is non negligible on a larger and larger interval, whose length scales like \( \xi_L \). Therefore, the sum \( \sum_{n=1}^{\xi_L} F_L^N(n) \) in (47) becomes more and more important, yielding a decrease in the expectation of the empirical maximum.

In the range of \( L \) values where this effect starts to manifest one has \( x_L \sim \xi_L \). Hence, \( \sum_{n=1}^{\xi_L} F_L^N(n) \sim \sum_{n=x_L}^{\xi_L} F_L^N(n) \) and \( F_L^N(n) \) has only to be estimated in the interval \([x_L, \xi_L]\). Furthermore, \( F_L(n) = 1 - \sum_{k=n}^{\xi_L} P_L(n) \sim 1 - C_L \sum_{n=1}^{\xi_L} u^{-\tau} g(\frac{n}{N_L}) du \). When \( x_L \) is close to \( \xi_L \), \( \int_{n_{x_L}}^{\xi_L} u^{-\tau} g(\frac{n}{N_L}) du \sim (\xi_L - n) \xi_L^{-\tau} g(\alpha) \) for \( n \in [x_L, \xi_L] \). Furthermore, in this range, \( 1 - C_L\xi_L^{-\tau} (\xi_L - n) g(\alpha) \) is small compared to 1. Hence, \( F_L^N(n) \sim 1 - NC_L \xi_L^{-\tau} (\xi_L - n) g(\alpha) \). The equation \( F_L^N(x) = y \) has therefore an approximate solution \( x_L = \xi_L - \frac{1 - y}{NC_L g(\alpha)} \xi_L \). Then \( \sum_{n=x_L}^{\xi_L} F_L^N(n) \sim \sum_{n=x_L}^{\xi_L} F_L^N(n) \) can be roughly approximated by a linear interpolation giving \( \sum_{n=x_L}^{\xi_L} F_L^N(n) \sim \frac{1 - y}{NC_L g(\alpha)} \xi_L \).

Consequently, the empirical expectation is, in this approximation:

\[
E[\xi_L^\text{exp} | \xi_L, N] \sim \xi_L \left( 1 - \frac{1 - y}{NC_L g(\alpha)} \xi_L^{-1} \right) \tag{48}
\]

Since \( \tau > 1 \) the correction term increases as \( L \) grows. It eventually becomes of the same order as \( \xi_L \), but when \( L \) increases one has to add higher order corrections to eq. (48). On the other hand, were \( \tau < 1 \), then would the correction term become negligible as \( L \) grows.

The \( L \) value where the effect starts can be estimated by:

\[
L \sim \left( \frac{1 - y}{NC_L g(\alpha)} \right)^{\frac{1}{\tau - 1}} \tag{49}
\]

Consequently, this effect is more prominent when \( \beta(\tau - 1) \) is larger.

The ratio \( \alpha_L = \frac{E[\xi_L^\text{exp}]}{\xi_L} \) is therefore not equal to a constant \( \alpha \) as it must be, but is \( L \) dependent (see Fig. a)). Clearly, for sufficiently large \( L \) the corresponding probability violates the finite size scaling and the data collapse.

Furthermore, the scaling of the moments is also affected by this effect. Indeed the moments are obtained empirically from the formula \( m_L^\text{exp}(q) = \sum_{n=1}^{\xi_L^\text{exp}} n^q P_L^\text{exp}(n) \) and the scaling exponents are extrapolated from the formula:

\[
\sigma^\text{exp}(q) = \lim_{L \to \infty} \frac{\log(m_L^\text{exp}(q))}{\log(L)}
\]

When \( L \) is sufficiently small, \( \xi_L^\text{exp} \sim \alpha L^\beta \) since the correction due to the finite sampling has essentially no effect. Then one obtains the right scaling exponent \( \sigma(q) \) from the data. However, when \( L \) increases, one observes a spurious deviation of the curve \( m_L(q) \) from the theoretical value. This effect is illustrated Fig. in the case \( \tau = 1.9, \beta = 2, \gamma = 2.5, \alpha = 2, g(x) = e^{-x^\gamma} \) where two samples with \( \mathcal{N} = 10^6 \) and \( \mathcal{N} = 10^8 \) were generated. Note that the effect is more prominent when \( q \) increases.
This computation shows therefore that for the values \( \tau > 1 \) numerical problems appears, induced by the finite size sampling. Not only the fluctuations of \( \xi_{L,N}^{\exp} \) increase but also the averaged value \( \mathbb{E}[\xi_{L,N}^{\exp}] \) is biased. To our opinion, the estimation of the correct \( \xi_L \) is the main problem in analyzing the data from SOC simulations.

The analysis of the Lee-Yang zeroes for relatively small sizes can however give a fairly good estimate of the values \( \alpha, \beta \) allowing to extrapolate \( \xi_L \) to larger size. Indeed, eq. (40) suggests that the argument of \( t_L(k) \) is not too sensitive to the fluctuations of \( \xi_{L,N}^{\exp} \) (compared to the fluctuations of the moments which are of order \( (\xi_{L,N}^{\exp})^{q+1-\tau} \)). Hence, it is possible to find the values of \( \alpha, \beta \). We give an example Fig. 7 where the empirical data are the same as those used for the computation of the moments. As for the truncated power law, we found a slight deviation for the first zero. Interpolating the values from \( k = 1 \ldots 10 \) we found \( \alpha = 2.094 \pm 0.006 \pm 0.09, \beta = 2.002 \pm 0.003 \) which gives quite a good estimate.

The real part of Lee-Yang zeroes and consequently, the argument, is more sensitive to non extensive sampling effect. An analytical expression is obtained if one modifies the equations (22, 23, 24) by replacing \( \alpha \) by \( \alpha_L \). The argument of
Arg(t_L(k)) writes now:

$$\text{Arg}(t_L(k)) \sim \arctan\left(\frac{2k\pi}{-\log(\frac{\alpha L \Gamma_g(\tau-1)}{\Lambda_L}) + \frac{1}{2} \log \left(\log^2(\frac{\alpha L \Gamma_g(\tau-1)}{\Lambda_L}) + 4k^2\pi^2\right)}\right)$$

The finite sample effect is illustrated Fig. 8 for the first zero. One observes a deviation from the real curve for \(N = 10^6\). This can be used as an empirical way to define the \(L\) where the empirical distribution is not biased. Note that the curve of the argument of the first empirical zero obtained for \(N = 10^8\) follows the theoretical curve with a good accuracy. This shows that the determination of the zeroes is robust with respect to fluctuations in the coefficients of the polyom (1). This is in fact an easy consequence of the implicit function theorem.

FIG. 8. Effect of a sample cut off on the argument of the zeroes in the \(t\) plane.

Looking at fig. (6b) and fig. (8) one could argue that the moments are less sensitive than the Lee-Yang zeroes to a size independent sampling. However, the sensitivity of the moments increases with the degree \(q\). Therefore, to detect this effect, one has to compute the moments for high \(q\). This can clearly cause numerical problems. On the other hand, the Lee-Yang zeroes integrate informations about all order moments and the sensitivity can be detected easily. We believe that this kind of analysis is prior to any investigation concerning in particular the multifractal nature of the scaling.

Our conclusion is therefore twofold. Firstly, some cautions are required when increasing \(L\) to extrapolate the thermodynamic limit. If \(N\) is kept fixed, too large \(L\) will give wrong estimations. In this case at least, the bigger is not the best. This effect can be compared to the critical slowing down in the literature about critical phenomena. However, to the best of our knowledge we don’t know any example in the SOC literature where this effect has been discussed. Secondly, the Lee-Yang zeroes give nevertheless useful informations. They can be used to determine the range of \(L\) values where the data are not too affected, and, in this range, the simple scaling of the imaginary part allows to determine the scaling of \(\xi_L\). This can used with other methods such as binning, or smoothing to determine the exact distribution from the empirical one.
4. Other scaling.

In this section we investigate briefly the other scaling forms discussed in section II B. Our main conclusion is that the Lee-Yang zeroes are highly sensitive to the changes in the scaling form. This remark opens perspective to develop a general theory allowing to extrapolate the characteristic of probability distributions from the behavior of the Lee-Yang zeroes of the empirical generating function. However, we have not yet been able to provide an equivalent of the analytic forms (20,39) that would be helpful to properly extract the features of the probability distribution from the Lee-Yang zeroes. The development of such a general framework is under investigation and will be published in a separated paper. We give a few numerical examples fig. 9a, b.

We investigated first the case \( F_L(x) = Y(x - \beta_L) \). In this case, the computations done in section II A essentially hold, with a \( \beta \) depending on \( L \). In particular, eq. (26) suggests that the \( L \) dependence of \( \beta \) should be detected on the argument of \( z_L(k), \theta_L(k) \). In fig. 9, we plotted \( \theta_L(5) \) in the case \( \beta_L = \beta(1 - \frac{1}{L}) \), where \( \beta = 2, \tau = 1.9 \), and \( L = 20 \ldots 120 \). The theoretical prediction is \( \theta_L(5) = \frac{10\pi}{L^{\beta_1} - 1} \). We tried a fit of the form \( f(x) = \frac{10\pi}{L^{\beta_1} - 1} \), where \( \alpha, \beta \) are the fit parameters. We found \( \alpha = 1.014(8) \pm 0.0006, \beta = 2.013(9) \pm 0.0001 \), which is quite satisfactory.

We also tried a more general form for \( F_L(x) \) with a non linear \( F_L \) converging to a step function as \( L \to \infty \):

\[
F_L(x) = -(1 + \tanh \left( L^{\alpha_1} \left( x - \beta - \frac{1}{L^{\alpha_2}} \right) \right)); \quad x \leq \beta - \frac{1}{L^{\alpha_2}}
\]

In our simulations, \( \beta = 2, \tau = 1.9, \alpha_1 = 0.1, \alpha_2 = 1 \). \( L^{\alpha_1} \) controls the rate of approach of \( F_L(x) \) to the step function as \( L \) grows. A small \( \alpha_1 \) gives a slow convergence and therefore has an effect up to very large \( L \).

The result for the argument \( \theta_L(5) \) is also represented fig. 9a. We note that the curve is indistinguishable from the previous case and therefore the imaginary part of \( t_L \) is not sensitive to the non linear effect of the tanh, but gives the right \( \alpha_2 \). On the other hand, we noted that the argument of \( t_L \) is sensitive to the non linear effect (fig. 9b).

For the multifractal case we studied the case where the multifractal spectrum has the form \( f(x) = C - \tau x - ax^2 \). This is the lowest degree non linear form of \( f \) compatible with (i),(ii) and with the convexity of \( f \). The values of \( \alpha = 1, \beta = 2, \tau = 1.9 \) were the same as for the previous examples. We observe (Fig. 7a) that \( \theta_L(k) \) is not sensitive to the multifractality and gives therefore the right \( \alpha, \beta \). Consequently, our method to estimate the degree is also valid for a multifractal distribution (32). On the other hand \( Arg(t_L(k)) \) is modified for a multifractal distribution, but we are not yet able to analyze this variation.

All the results are depicted fig. 9a (\( \theta_L(5) = Im(t_L(5)) \)) and fig. 9b (\( Arg(t_L(5)) \)).
III. AN EXAMPLE OF SOC MODEL.

In this section, we study an example of Lee-Yang zeroes computation for a SOC model, the Zhang model, defined as follows.

Let $\Lambda$ be a $d$-dimensional sub-lattice in $\mathbb{Z}^d$, taken as a square of edge length $L$ for simplicity. Call $N = \#\Lambda = L^d$, and let $\partial \Lambda$ be the boundary of $\Lambda$, namely the set of points in $\mathbb{Z}^d \setminus \Lambda$ at distance 1 from $\Lambda$. Each site $i \in \Lambda$ is characterized by its "energy" $X_i$, which is a non-negative real number. Call $X = \{X_i\}_{i \in \Lambda}$ a configuration of energies.

Let $E_c$ be a real, strictly positive number, called the critical energy, and $\mathcal{M} = [0, E_c]^N$. A configuration $X$ is called "stable" if $X \in \mathcal{M}$ and "unstable" otherwise. If $X$ is stable then one chooses a site $i$ at random with probability $\frac{1}{N}$, and add to it energy $\delta$ (excitation). If a site $i$ is over critical or active ($X_i \geq E_c$), it loses a part of its energy in equal parts to its $2d$ neighbors (relaxation). Namely, we fix a parameter $\epsilon \in [0, 1]$ such that the remaining energy of $i$ is $\epsilon X_i$, after relaxation of the site $i$, while the $2d$ neighbors receive the energy $(1-\epsilon)X_i^{2d}$. Note therefore that the energy is locally conserved. If several nodes are simultaneously active, the local distribution rules are additively superposed, i.e. the time evolution of the system is synchronous. The succession of updating leading an unstable configuration to a stable one is called an avalanche. The energy is dissipated at the boundaries of the system, namely the sites of $\partial \Lambda$ have always zero energy. As a result, all avalanches are finite. Consequently, whatever the observable $n, \xi_L \sim \infty$ for finite $L$. The addition of energy is adiabatic. When an avalanche occurs, one waits until it stops before adding a new energy quantum. Further excitations eventually generate a new avalanche, but, because of the adiabatic rule, each new avalanche starts from only one active site. It is conjectured that a critical state is reached in the thermodynamic limit.

Though it has long been believed that the Zhang model obeys finite size scaling (30), a recent paper revised this point of view and claimed that the Zhang model does not even have a multifractal scaling (but no alternative scaling was proposed (15)). We will not solve this debate in this paper. Rather we will come to two conclusions. Firstly, because of high sensitivity of the model to the sample cut off (fig. 12), one has somehow to relativise the conclusions

FIG. 9. Fig. 9a. Argument $\theta_L(5)$ for the truncated power law (POW); a truncated power law with $L$ dependent $\beta$ (V); a probability distribution of the form (33) with $F_L(x)$ given by (51) (NL) and a multifractal distribution (MF). Fig. 9b. Argument of $t_L(5)$.
about the scaling obtained from the numerical simulations. This shows that to draw any reliable conclusion on the scaling one has to increase the sample with the system size (e.g. like $L^{\beta}$). This will clearly be rapidly intractable even for the fastest computers. Secondly, the Lee-Yang zeroes gives rather reliable extrapolations provided the size $L$ is not too large.

We computed the empirical probability distribution of avalanche sizes $P_L^{\text{exp}}(s)$ where the size is the total number of relaxing sites during one avalanche. We did our simulations for lattice sizes from $L = 10$ to $L = 55$ in two dimensions, with $E_c = 2.2, \epsilon = 0.1$ with a statistics over $\mathcal{N} = 10^6$ and $\mathcal{N} = 10^8$ avalanches. Consequently, $\mathcal{N}$ was fixed independently of $L$ as usually done in SOC numerical simulations.

We first present Fig. 10a the experimental zeroes in the $z$ complex plane for $L = 30, \mathcal{N} = 10^6$. In order to see the effect of the noise, we also computed the zeroes of a smoothed version of the empirical probability distribution (fig. (10b)). The smoothing method uses a binning procedure, followed by a spline extrapolation, allowing to fill the “holes” existing in the empirical probability distribution. These holes corresponds to events that didn’t happen during the trial and consequently are given a zero probability. In the numerical computation of the zeroes, these holes corresponds therefore to vanishing coefficients in the polynom (1), that produce problems in the convergence of most of the root finding algorithms. Our numerical procedure seems however to be robust with respect to this effect.

As argued all along in this paper, the argument $\theta_L(k)$ provides a way to determine the exponent $\beta$ characterizing the maximal avalanche size. We plot fig. 11a the $\theta_L(k)$’s for $k = 1 \ldots 5$ versus $L$. We note that $\theta_L(k)$ is quite robust to noise and gives therefore a reliable way to measure $\beta, \alpha$. We used a fit form $\frac{2\pi k}{\alpha L^\beta}$, where $\alpha, \beta$ are fit parameters. We found a slight $k$ dependence for the first zeroes (as expected from eq. (23) if one assumes FSS). We plotted fig. 11b the extrapolated $\alpha, \beta$ versus $k$. For $k > 3$ these value seem to stabilize around $\alpha = 0.62 \pm 0.07$ and $\beta = 2.59 \pm 0.04$. In the finite size scaling ansatz $\beta$ and $\tau$ are related by $\beta(2 - \tau) = 2$ and $\tau = 1.253$ is known from the renormalization group analysis. Therefore the predicted value for $\beta$ is 2.667. Despite the smallness of the $L$ we considered, the computed value is not too far from the predicted one. However, an accurate determination of $\beta, \tau$ and a precise check

![Fig. 10](image-url)
of FSS would demand somewhat larger size systems, that we were unable to generate for this illustration. (Note that the main problem is not the computation of the zeroes since there exist quite fast and precise root finding algorithms, but the generation of $P_L^{\exp}$ itself.)

\[
\log_{10}(k) \quad \log_{10}\{\text{Arg}[z_L(k)]\} \quad k=1 \quad k=4 \quad k=7 \quad k=10
\]

\[
0 \quad 2 \quad 4 \quad 6 \quad 8 \quad 10
\]

\[
0 \quad 1 \quad 2 \quad 3
\]

FIG. 11. a. Argument of the 5 first Lee-Yang zero in the $z$ plane versus $L$, for $Ec = 2.2, \epsilon = 0.1$. The full lines corresponds to a fit. b. Plot of the fit parameters $\alpha, \beta$ versus $k$.

Finally, we investigate the scaling of the argument $t_L(k)$ and a possible size independent sampling effect. The main difficulty here are the wild fluctuations in $\xi_L^{exp}$. Indeed, the real part of $t_L(k)$ is more sensitive than the imaginary part to these fluctuations and consequently $\text{Arg}(t_L(k))$ as wild fluctuations. Only the first zeroes seem to be robust to this effect. As discussed in section II B 3 these fluctuations in $\xi_L^{exp}$ are intrinsic to the empirical computation of $P_L^{\exp(L)}$ and cause problems in the extrapolation to the thermodynamics limit, whatever the method used. We plotted Fig. 12 the argument of the 2 first Lee-Yang zeroes in the $t$ plane, versus $L$. Our simulations suggests that the Zhang model is sensitive to the size independent sampling. Note in particular that the values of $L$ that we used in our computation are quite smaller that the ones found in the literature and the effect is already significant for $N \leq 10^6$.

\[
\log_{10}(N) = 10^8, 10^9
\]

\[
0.4 \quad 0.45 \quad 0.5 \quad 0.55 \quad 0.6 \quad 0.65 \quad 0.7 \quad 0.75 \quad 0.8 \quad 0.85 \quad 0.9
\]

FIG. 12. Argument of Lee-Yang zeroes $t_L(1), t_L(1)$ for $Ec = 2.2, \epsilon = 0.1$. In color full lines are drawn the interpolation of the empirical curves obtained from the sampling with $N = 10^9$.

This shows clearly that a re estimation of the conclusions drawn from the numerics in the Zhang model (and also,
maybe, for some other SOC models) should be done in light of this effect. On the other hand, our approach suggests that there may be no need to go to gigantic sizes provided the finite lattice size effects are carefully handled. From this point of view, analytic formula like (44) might provide a way to analyze these effects.

IV. CONCLUSION.

In this paper, we have shown that the finite size study of the SOC like probability distributions leads to similar Lee Yang or Fisher phenomenon as in statistical physics models near the critical point. This implies that the convergence of the SOC state to a critical state with power law statistics can be analyzed in a similar way as equilibrium statistical mechanics. More precisely, the way the zeros of the partition function accumulate on the real axis, when the size of the system grows up, provides relevant informations on the critical structure of the observed system. In particular, it permits to measure useful critical indexes of the underlying theory.

Moreover, we have shown that the size of the SOC models power exponent, $\tau > 1$, leads to a comprehensive violation of the standard scaling laws. We give a approximate theory of this effect well confirmed by numerical simulations. It is the same characteristic which leads to a specific sensibility of the SOC numerical experiments to size independent sampling effects. We studied carefully this effect on extrapolation to the limit $L \rightarrow \infty$ and show that it could possibly mimics important effects such as multifractality. We notice that the argument of the zeros in the $t = \log(z)$ plane is a good test of this effect.

On one other hand, we show that the arguments of the first zeros in the $z$ plane of the generating function, $G(z)$ is rather insensitive to these effects, statistically robust, and provide a nice way to compute the SOC $\alpha$ and $\beta$ parameters. Using the standard Kadanoff et al. scaling form [13], we verify that the parameter’s values as extracted from numerical simulations where in good agreement with the theoretical input of the model. This last result gives us some confidence to extract the values of these parameters from Zhang’s model numerical data. Notice that these results have been extracted from medium range simulations. This shows up once more the power of the finite size analysis of the critical phenomenon.

This paper is (with [21]) a first step toward a scaling theory of SOC system from the behavior of the Lee-Yang zeroes. The next step would be the definition of the exponents characterizing the approach to criticality, like the exponents $\alpha, \beta, \gamma$ in statistical mechanics and their link to the scaling of the zeroes.

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