Field Theoretic Realizations for Cubic Supersymmetry

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Abstract

We consider a four dimensional space-time symmetry which is a non trivial extension of the Poincaré algebra, different from supersymmetry and not contradicting a priori the well-known no-go theorems. We investigate some field theoretical aspects of this new symmetry and construct invariant actions for non-interacting fermion and non-interacting boson multiplets. In the case of the bosonic multiplet, where two-form fields appear naturally, we find that this symmetry is compatible with a local $U(1)$ gauge symmetry, only when the latter is gauge fixed by a 't Hooft-Feynman term.

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1 Introduction

Over several decades, supersymmetry (SUSY) has gradually gained the status of a cornerstone in the search for a unified description of elementary particle physics, and more generally of the four fundamental interactions. Despite the present absence of any direct SUSY signatures, some indirect experimental evidence together with a wealth of theoretically appealing features can be embedded in the so-called minimal supersymmetric standard model (MSSM) and its constrained as well as extended versions (see e.g. [4]), thus sharing the tremendous experimental success of the Standard Model of particle physics and predicting new physics at the $O(1) TeV$ energy scale. Moreover, from a purely algebraic point of view, the consideration of supersymmetric theories found its mathematical insight in some extensions of Lie algebras called Lie superalgebras, in particular, evading the conditions of validity of the Coleman-Mandula theorem [5] and leading to a non-trivial extension of the Poincaré algebra [6]. In addition, the introduction of Lie superalgebras has lead to new powerful mathematical tools.

In supersymmetric theories, the extensions of the Poincaré algebra are obtained from a “square root” of the translations, “$QQ \sim P$”. It is tempting to consider other alternatives where the new algebra is obtained from yet higher order roots. The simplest alternative which we will consider in this paper is “$QQQ \sim P$”. It is important to stress that such structures are not Lie (super)algebras (even though they contain a Lie sub-algebra), and as such escape a priori the Coleman-Mandula [5] as well as the Haag-Lopuszanski-Sohnius no-go theorems [6]. Furthermore, as far as we know, no no-go theorem associated with such types of extensions has been considered in the literature. This can open interesting possibilities to search for a field theoretic realization of a non trivial extension of the Poincaré algebra which is not the supersymmetric one. If successful, this might throw a new light on how to construct physical models.

Regarding the algebra per se, several possibilities have already been considered in the literature. Here we are focusing on one of the possible extensions called fractional supersymmetry (FSUSY), [7] – [21]. Basically, in such extensions, the generators of the Poincaré algebra are obtained as $F$-fold symmetric products of more fundamental generators, leading to the “$F$th-root” of translation: “$Q^F \sim P$” with $F$ a positive integer. The $F$-Lie algebras, the structures which underlie FSUSY, are defined in [20, 21] in full analogy with SUSY and its underlying Lie superalgebra structure.

The aim of this paper is to provide the first field theoretic construction in (1 + 3) dimensions of an FSUSY with $F = 3$, which we will refer to as cubic supersymmetry 3SUSY. For this purpose we will first determine finite dimensional fermion and boson multiplet representations of the 3SUSY. Then, we will seek explicit 3SUSY invariant actions and discuss their specific features. We find that the fermion multiplets are made of three definite chirality fermions which are degenerate in mass, while the boson multiplets contain Lorentz scalars, vectors and two-forms. It turns out that cubic SUSY forbids canonical kinetic terms for fermion fields and leads either to higher order derivative ones or requires interaction with some constant background fields. A striking feature for the boson multiplets is the compatibility of 3SUSY with gauge symmetry only when the latter is gauge fixed in the usual
way. We provide also a short discussion of the Noether currents. The rest of the paper is organized as follows: in section 2 we give the cubic supersymmetry algebra and its finite dimensional representations. In section 3 we construct the fermion multiplets and discuss the corresponding allowed actions. Section 4 is devoted to the boson multiplets and to the construction of gauge actions compatible with 3SUSY. Noether currents are briefly discussed in section 5 where some extra comments are made. The conclusions and outlook are summarized in section 6. More extended material for the algebraic construction of 3SUSY can be found in appendix A. Appendix B contains some notations and conventions.

2 Non trivial extensions of the Poincaré algebra

The FSUSY algebra is generated by the usual generators of the Poincaré algebra together with additional “supercharges”. These new generators have to be in some representation of the Lorentz algebra. Historically, it was firstly believed that FSUSY could only apply in low dimensional systems \((D \leq 1 + 2)\), where representations which are neither bosonic nor fermionic exist \([10, 13, 14, 19]\). A next step in the understanding of FSUSY was achieved by the discovery that FSUSY could be extended to any space time-dimension \([20]\), considering infinite dimensional representations of the Lorentz algebra (fractional spin). However, these representations being not exponentiable (they are representations of the Lie algebra but not representations of the Lie group), all the results are valid at the level of the Lie algebra, and consequently the principle of equivalence of special relativity is lost. These results were looking like a breakthrough for the construction of FSUSY in dimension higher than three. In the meantime, it was understood that finite dimensional \(F\)-Lie algebras \((i.e.\) involving representations which are not infinite dimensional and which are exponentiable) could be obtained by an inductive process starting from any simple Lie (super)algebra \([21]\). Among these families of examples, it was observed that, under an Inönü-Wigner contraction of some of the \(F\)-Lie algebras, non-trivial extensions of the Poincaré algebra (which are not the usual supersymmetric ones) can be obtained.

In this paper, we will be interested in one of these extensions, the 3SUSY. It is constructed from the Poincaré generators (in any space-time dimension) \(L_{mn}\) and \(P_m\) together with some additional supercharges \(Q_m\) in the vector representation, satisfying the trilinear relations

\[
Q_m Q_n Q_r + Q_m Q_r Q_n + Q_n Q_m Q_r + Q_r Q_m Q_n + Q_r Q_n Q_m = \eta_{mn} P_r + \eta_{nr} P_m + \eta_{mr} P_n,
\]

with \(\eta_{mn}\) the Minkowski metric. This algebra can be compared in some sense with the algebraic extension studied in \([22, 23]\).

The algebraic features of this new structure is summarized in appendix A and leads to the following algebraic extension of the Poincaré algebra

\[
\begin{align*}
[L_{mn}, L_{pq}] &= \eta_{np} L_{pm} - \eta_{mq} L_{pn} + \eta_{mp} L_{nq} - \eta_{mn} L_{pq}, \\
[L_{mn}, P_p] &= \eta_{np} P_m - \eta_{mp} P_n, \\
[L_{mn}, Q_p] &= \eta_{np} Q_m - \eta_{mp} Q_n, \\
[P_m, Q_n] &= 0, \\
\{Q_m, Q_n, Q_r\} &= \eta_{mn} P_r + \eta_{mr} P_n + \eta_{rn} P_m.
\end{align*}
\]

(2.2)
where \{Q, Q, Q\} stands for the symmetric product of order 3 (as defined in (2.1)). It has to be emphasized that such an extension is not the parasupersymmetric one considered in [24], even if it involves also a trilinear bracket.

The aim of the present paper is to try to implement this new structure in a field theoretical setting. As noted in appendix A, we have found a twelve-dimensional matrix representation of the algebra (2.2)

\[
Q_m = \begin{pmatrix}
0 & \Lambda^{1/3} \gamma_m & 0 \\
0 & 0 & \Lambda^{1/3} \gamma_m \\
\Lambda^{-2/3} P_m & 0 & 0
\end{pmatrix},
\]

(2.3)

with \(\Lambda\) a parameter with mass dimension and \(\gamma_m\) the 4D Dirac matrices. The \(Q\)’s being in the vector representation of the Lorentz algebra, we also have (see appendix A)

\[
J_{mn} = \frac{1}{4} (\gamma_m \gamma_n - \gamma_n \gamma_m) + i (x_m P_n - x_n P_m), \quad [J_{mn}, Q_r] = \eta_{nr} Q_n - \eta_{mr} Q_m,
\]

(2.4)

with \(P_m = -i\partial_m\). However, this representation is reducible, leading to the two inequivalent 6–dimensional representations:

\[
Q_m = \begin{pmatrix}
0 & \Lambda^{1/3} \sigma_m & 0 \\
0 & 0 & \Lambda^{1/3} \bar{\sigma}_m \\
\Lambda^{-2/3} P_m & 0 & 0
\end{pmatrix},
\]

(2.5)

and

\[
Q_m = \begin{pmatrix}
0 & \Lambda^{1/3} \bar{\sigma}_m & 0 \\
0 & 0 & \Lambda^{1/3} \sigma_m \\
\Lambda^{-2/3} P_m & 0 & 0
\end{pmatrix}.
\]

(2.6)

See appendix B for the conventions for the \(\sigma\)–matrices and some useful relations. These two representations are \(CPT\) conjugate of each other. As already mentioned, the two representations (2.5) and (2.6) are certainly not the only possible ones. It might as well be that other matrix representations could also lead to some interesting physical results.

We would like to end up this section by some comparison with the ordinary supersymmetric extension of the Poincaré algebra. Recall that if one tries to construct representations of the SUSY algebra by considering Clifford algebra of polynomial (with 8 variables), one ends up with \(16 \times 16\) matrices. Introducing the Grassmann algebra (i.e. the \(\theta_\alpha, \bar{\theta}_{\dot{\alpha}}\) and their derivative \(\partial_{\theta_\alpha}, \partial_{\bar{\theta}_{\dot{\alpha}}}\)) in matrix representation, these matrices reduce then to the supercharges in the superspace language. Moreover, the number of variables being twice as many as the number of independent variables, leads automatically to reducible representations in the superspace approach. However, within this approach, the matrix representation can be forgotten and only the algebra between the \(\theta\)’s and the \(\partial_\theta\) is useful. It is indeed also possible, for FSUSY,
to introduce some additional variable, \( \eta^m = \begin{pmatrix} 0 & \sigma^m & 0 \\ 0 & 0 & \bar{\sigma}^m \\ 0 & 0 & 0 \end{pmatrix} \) and \( \partial \eta_m = \begin{pmatrix} \bar{\sigma}_m & 0 & 0 \\ 0 & \sigma_m & 0 \end{pmatrix} \) such that we have \( Q_m = \partial \eta_m + f_{mpq} \left( \eta^n \eta^p + \eta^p \eta^n \right) P_q \) with \( f_{mpq} = \eta_{mn} \delta_r^s + \eta_{mr} \delta_n^s + \eta_{rn} \delta_m^s \).

However, we were not able within these variables to introduce some adapted version for a superspace. We will then continue with the matrices (2.5) and (2.6).

### 3 Fermion multiplets

In this section we construct an invariant action under the 3SUSY algebra (2.2) and the representations (2.5) and (2.6). As usual, the content of the representation is not only specified by the form of the matrix representation, but also by the behavior of the vacuum under Lorentz transformations. If we denote \( \Omega \) the vacuum, which is in some specified representation of the Lorentz algebra, with \( \Sigma_{mn} \) the corresponding Lorentz generators, then \( J_{mn} \) given in (2.4) is replaced by \( J_{mn} + \Sigma_{mn} \). In the case, where \( \Omega \) is a Lorentz scalar, one sees that the multiplet of representation (2.5) contains two left-handed and one right-handed fermions, while the multiplet of the representation (2.6) contains one left-handed and two right-handed fermions. These two multiplets are CPT conjugate. At first sight, it might seem surprising that a multiplet contains fields of the same statistics (here only fermions). Indeed, this comes from the fact that we are considering a supercharge \( Q \) in the vector representation, in order to extract the “cubic root of the translation”. In supersymmetric theories the square root is extracted using spinors, and consequently representations of SUSY contain both fermions and bosons. If we would expect something similar in 3SUSY, the \( Q \) have to be in the spin 1/3 representation of the Lorentz algebra (see [20]). But this representation is firstly infinite dimensional, and secondly cannot be exponentiated (see e.g. [25]). Therefore, it does not define a representation of the Lie group \( SO(1,3) \) (in \((1+2)\) dimensions, however, a similar extension is possible [19] and such representations describe relativistic anyons [26].)

Consider the multiplet associated with the matrices (2.5). If we denote \( \Psi = \begin{pmatrix} \psi_1 \alpha \\ \bar{\psi}_2 \\ \psi_3 \alpha \end{pmatrix} \), then under a 3SUSY transformation we have \( \delta_\varepsilon \Psi = \varepsilon^m Q_m \Psi \) and we obtain (see appendix B for spinor notations)

\[
\begin{align*}
\delta_\varepsilon \psi_1 &= \varepsilon^n \Lambda^{1/3} \sigma_n \bar{\psi}_2 \\
\delta_\varepsilon \bar{\psi}_2 &= \varepsilon^n \Lambda^{1/3} \bar{\sigma}_n \psi_3 \\
\delta_\varepsilon \psi_3 &= \varepsilon^n \Lambda^{-2/3} P_n \psi_1
\end{align*}
\]  

(3.1)

Using \( \left( \sigma_{m\alpha\beta} \right)^* = \sigma_{m\beta\alpha} \) and the relations in appendix B, we find the following transformations for the conjugate fields
\[
\begin{align*}
\delta_\varepsilon \bar{\psi}_1 &= \varepsilon^n \Lambda^{1/3} \bar{\sigma}_n \psi_2 \\
\delta_\varepsilon \psi_2 &= \varepsilon^n \Lambda^{1/3} \sigma_n \bar{\psi}_3 \\
\delta_\varepsilon \bar{\psi}_3 &= \varepsilon^n \Lambda^{-2/3} P_n \bar{\psi}_1
\end{align*}
\] (3.2)

with \(\varepsilon^n\), the parameter of the 3SUSY transformation, a purely imaginary number with mass dimension \(-1/3\). At this point it can be observed that if one considers 4D Majorana spinors \(\psi_i = (\psi_{i\alpha} \bar{\psi}_{\dot{\alpha}})\), instead of 2D Weyl spinors, the matrices (2.3) ensure that \(\delta_\varepsilon \psi_i\) are also Majorana (as can be explicitly seen in the appendix B). This means that an invariant 3SUSY action can also be constructed with 4D Majorana spinors. In the sequel we will consider both representations.

Finally, notice that one can associate a grade (or degree) to each of the fermions in the following manner: \(\psi_1\) is of grade \(-1\), \(\psi_2\) of grade 0 and \(\psi_3\) of grade 1. Then \(Q\) turns out to be of grade \(-1\) and the transformations properties (3.1) and (3.2) are compatible with this grading: \(\psi_1 \xrightarrow{Q} \psi_3 \xrightarrow{Q} \psi_2 \xrightarrow{Q} \psi_1\). Moreover, if one wants that both sides of (3.1) and (3.2) have the same grade, then one has to assign a grade \(-1\) to the parameter \(\epsilon\).

We will try now to construct 3SUSY invariant Lagrangians up to a surface term. That is, we will seek a Lagrangian \(\mathcal{L}\) such that

\[
\delta_\varepsilon \mathcal{L} = \varepsilon_m \partial^m (\cdots).
\]

Since, under 3SUSY only \(\psi_3\) transforms as a total derivative (3.1), (3.2), and given the grading structure, the only possible candidates bilinear in the fermion fields involve couplings of either \(\psi_1\) to \(\psi_3\) or \(\psi_2\) to itself. However, this is not sufficient. For instance, one can easily show that the simplest kinetic term

\[
\mathcal{L} = (\psi_1 i \sigma^m \partial_m \bar{\psi}_3 + \psi_3 i \sigma^m \partial_m \bar{\psi}_1) + (\bar{\psi}_1 i \sigma^m \partial_m \psi_3 + \bar{\psi}_3 i \sigma^m \partial_m \psi_1) + \psi_2 i \sigma^m \partial_m \bar{\psi}_2 + \bar{\psi}_2 i \sigma^m \partial_m \psi_2
\] (3.3)

does not transform as a surface term. The reason for this being that the Pauli matrices do not commute. This means that unconventional kinetic terms have to be considered. However, as we will see, conventional mass terms are still allowed. More generally, let us consider the following Lagrangian (in the 4D formalism for notational convenience),

\[
\mathcal{L} = \bar{\psi}_1 \mathcal{O} \psi_3 + \bar{\psi}_3 \mathcal{O} \psi_1 + \bar{\psi}_2 \mathcal{O} \psi_2
\] (3.4)

where \(\mathcal{O}\) is a \(4 \times 4\) hermitian matrix operator and \(\psi_i\) are three Majorana spinors transforming as in (B.15). The variation of this Lagrangian under 3SUSY gives

\[
\delta_\varepsilon \mathcal{L} = \varepsilon^n \{-\bar{\psi}_2 \gamma_n \mathcal{O} \psi_3 + \bar{\psi}_1 \mathcal{O} P_n \psi_1\} + \varepsilon^n \{P_n \bar{\psi}_1 \mathcal{O} \psi_1 + \bar{\psi}_3 \mathcal{O} \gamma_n \psi_2\} + \varepsilon^n \{-\bar{\psi}_3 \gamma_n \mathcal{O} \psi_2 + \bar{\psi}_2 \mathcal{O} \gamma_n \psi_3\}
\] (3.5)
For $\delta \mathcal{L}$ to be a surface term, the operator $\mathcal{O}$ should fulfill

$$[\mathcal{O}, \partial_n] = [\mathcal{O}, \gamma_n] = 0$$

(3.6)

The most general form compatible with (3.6) is found to be

$$\mathcal{O} = \left( \begin{array}{cc}
[m + c^n \partial_n] \times I & 0 \\
0 & [m + c^n \partial_n] \times I
\end{array} \right)$$

(3.7)

where $I$ is the two by two identity matrix, $m$ is a (constant) mass and $c^m$ a Lorentz vector operator commuting with $\partial_n$.

If the $\psi$’s are Majorana spinor fields then, $c_m$ should contain a derivative ($c^m = \frac{\partial m}{\Lambda} f(\frac{\rho}{\Lambda})$) otherwise the derivative part of $\mathcal{L}$ corresponding to $\psi_1, \psi_2$ reduces to a surface term. The simplest 3SUSY Lagrangian reads (when $f$ is the identity function)

$$\mathcal{L}_1 = \bar{\psi}_1 \nabla \psi_3 + \bar{\psi}_3 \nabla \psi_1 + \bar{\psi}_2 \nabla \psi_2 + m(\bar{\psi}_1 \psi_3 + \bar{\psi}_3 \psi_1 + \bar{\psi}_2 \psi_2)$$

(3.8)

Note that for the kinetic term we used the natural mass scale appearing in the representation of the algebra (2.3) while $m$ is an additional mass parameter. Some remarks are in order here. It is straightforward to redefine the fields $\psi_1, \psi_3$ in terms of positive squared mass $(m\Lambda)$ eigenstates. However, the classical equation of motion from $\mathcal{L}_1$ leads only to a Klein-Gordon type equation, thus determining the mass eigenvalue but not the spin content (the equation corresponding to the Pauli-Lubanski Casimir is missing). This means that the non-interacting $\psi$ fields behave like ghosts (anti-commuting spin zero fields). The spinorial character could then be restored by the inclusion of interaction terms with additional fields.

There is another possibility to construct invariant actions involving fermions. This is achieved by introducing an extra fermionic multiplet

$$\Lambda^n = \Psi \otimes \Omega^n = \begin{pmatrix}
\lambda^n_1 \\
\lambda^n_2 \\
\lambda^n_3
\end{pmatrix}$$

(3.9)

where $\Psi$ is a triplet of Majorana fermions transforming under 3SUSY as in (3.1 3.2) and $\Omega^n$ (the vacuum) is a Lorentz vector and a 3SUSY singlet. Consequently, $\Lambda^n$ describes a triplet of Rarita-Schwinger fields transforming under 3SUSY as spin 1/2 fields (3.1 3.2), with an extra vector index attached (e.g. $\delta_n \lambda^n_1 = \epsilon^m \gamma_m \lambda^n_2$). It is then straightforward to show that the Lagrangian

$$\mathcal{L}_3 = \bar{\psi}_1 \partial_n \lambda^n_3 + \bar{\psi}_3 \partial_n \lambda^n_1 + \bar{\psi}_2 \partial_n \lambda^n_2 + \partial_n \bar{\lambda}^n_1 \psi_1 + \partial_n \bar{\lambda}^n_3 \psi_3 + \partial_n \bar{\lambda}^n_2 \psi_2$$

$$+ m(\bar{\psi}_1 \psi_3 + \bar{\psi}_3 \psi_1 + \bar{\psi}_2 \psi_2) + M(\bar{\lambda}^n_1 \lambda_3^n + \bar{\lambda}^n_3 \lambda_1^n + \bar{\lambda}^n_2 \lambda_2^n)$$

(3.10)
transforms as a surface term. Again, as in the previous Lagrangian, the kinetic part of the
Rarita-Schwinger is non-conventional. Here, Rarita-Schwinger just means we are considering
a spin 3/2 representation of the Lorentz group. Let us now project out the spin-half content
of the $\lambda^a_k$’s which will be the dynamical degrees of freedom. We choose the following form
for reasons which will be justified a posteriori.

\[ \lambda^a_k = a^0_0 \lambda_0^{(k)} + a^1_1 \chi_1^{(k)} + a^2_2 \chi_2^{(k)} + i\gamma^n \chi_4^{(k)} \tag{3.11} \]

where $a^0_0, a^1_1, a^2_2$ are real constant vectors which we assume, without loss of generality, to be
orthogonal to each other and not light-like, and the $\chi^{(k)}$’s Majorana fields, so that the $\lambda^a_k$’s
are real. Note first that if we keep only the term $i\gamma^n \chi_4^{(k)}$ in (3.11), then the Rarita-Schwinger
field would be constrained by $\lambda^a = \frac{1}{4} \gamma^n \gamma_q \lambda^q$. One can easily show that the latter constraint
is not preserved by 3SUSY. It is thus important to keep enough terms in (3.11) so that $\lambda^a_k$ are
not constrained a priori. One way of doing this is to choose a sufficient number of non-zero
vector (and/or two-form) components so that there is a one-to-one correspondence between
the $\chi$’s and the $\lambda$’s. In eq. (3.11) for each $k$, the four Majorana $\chi$’s have as many degrees
of freedom as $\lambda^a$, that is 16 real components. To get a one-to-one correspondence, it is then
sufficient to require that $a^0_0, a^1_1$ and $a^2_2$ be acolinear (we actually choose them orthogonal).
Then, the system expressing the components of $\lambda$ in terms of those of the four $\chi$’s is of rank 16.
In this specific case, using $\gamma^n = \frac{a^0_{0}}{a^0_{0}, a_0} \tilde{\alpha}_0 + \frac{a^1_{1}}{a^1_{1} a_1} \tilde{\alpha}_1 + \frac{a^2_{2}}{a^2_{2} a_2} \tilde{\alpha}_2 + \frac{a^3_{3}}{a^3_{3} a_3} \tilde{\alpha}_3$, with $a_3$ a fourth vector orthogonal to $a_0, a_1$ and $a_2$ and $\tilde{\alpha} = a^n \gamma_n$ we get

\[ \chi_{l}^{(k)} = \frac{1}{a_l} (a_l \lambda_{l} - i \tilde{\alpha}_l \chi_4^{(k)}), \quad l = 0, 1, 2 \]

\[ \chi_4^{(k)} = -i \frac{\tilde{\alpha}_3}{a_3} a_3 \lambda_{l} \tag{3.12} \]

This ansatz allows to determine unambiguously the 3SUSY transformation laws of the $\chi$’s and
thus to obtain a 3SUSY invariant action involving only spin one-half fermions, with canonical
kinetic terms and additional couplings to background constant fields. Furthermore, one has
to identify the various physical fields as well as possible auxiliary ones in $\mathcal{L}_3$. In general, the
induced 3SUSY transformations for the $\chi_j$’s will depend on the constant vectors $a^0_0, a^1_1, a^2_2$,
thus making a physical interpretation rather difficult.

It is not difficult to obtain the 3SUSY transformation corresponding to (3.12)

\[ k = 1, 2 \]

\[ \begin{cases} \\
\delta \varepsilon \chi_l^{(k)} = \Lambda^{1/3} \left( \frac{\varepsilon \chi_l^{(k+1)}}{a_l} + \left( 2 i \frac{\varepsilon a_l}{a_0 a_l} - 2 i \frac{\varepsilon a_3}{a_3 a_l} a_l a_l \right) \chi_4^{(k+1)} \right) \\
\delta \varepsilon \chi_4^{(k)} = \Lambda^{1/3} \left( 2 a_3 \varepsilon a_3 a_3 \chi_4^{(k+1)} - \hat{\varepsilon} \right) \chi_4^{(k+1)} \\
\delta \varepsilon \chi_l^{(3)} = \varepsilon^n \Lambda^{-2/3} \chi_l^{(1)} \end{cases} \tag{3.13} \]

However, it is interesting to note that if we choose the 3SUSY parameter $\varepsilon^n$ along the
direction orthogonal to the hyperplane formed by $(a_0, a_1, a_2)$ in the Minkowski 4D space ($\varepsilon = a_3$), then the 3SUSY transformations of the $\chi$’s read

\[ \text{This means that terms like } b^m \varepsilon \gamma_m \chi^{(k)}, b_n \gamma^n \chi^{(k)}, \text{ with } \chi \text{ a Majorana spinors etc. could also be introduced.} \]
\begin{equation}
\begin{aligned}
\delta_\varepsilon \chi_l^{(k)} &= \Lambda^{1/3} \left( \varepsilon_1 \chi_l^{(k+1)} + 2i \varepsilon_2 \chi_4^{(k+1)} \right) \\
\delta_\varepsilon \chi_4^{(k)} &= \Lambda^{1/3} \chi_4^{(k+1)} \\
\delta_\varepsilon \chi_1^{(3)} &= \varepsilon_n \Lambda^{-2/3} P_m \chi_1^{(1)}.
\end{aligned}
\end{equation}

The usual 3SUSY transformations are thus retrieved in the limit of large \( a_i \)'s. This shows that at the level of the generating functional for the correlation functions, one can treat the \( a_i \)'s as background fields. Formally, in the limit discussed above, their net effects would be Dirac delta terms in the action leading to 3SUSY consistent constraints among the \( \psi_l \) and \( \chi_l^{(k)} \) fields. 

[Of course, the above ansatz, taken here for illustration, is not necessarily unique.]

## 4 Boson multiplets

In the previous section, we were considering the fundamental representation associated with the matrices (2.3) and (2.6), say \( \Psi = \begin{pmatrix} \psi_1^\alpha \\ \bar{\psi}_2^{\dot{\alpha}} \\ \psi_3^\alpha \end{pmatrix} \) and \( \Psi' = \begin{pmatrix} \psi'_1^{\dot{\alpha}} \\ \bar{\psi}'_2^\alpha \\ \psi'_3^{\dot{\alpha}} \end{pmatrix} \), \( i.e. \) with the vacuum \( \Omega \) in the trivial representation of the Lorentz algebra. In this section, boson multiplets, will be introduced, corresponding to a vacuum in the spinor representations of the Lorentz algebra. This means that four types of boson multiplets can be introduced: \( \Psi \otimes \Omega^\alpha, \Psi' \otimes \Omega^\dot{\alpha}, \) with \( \Omega^\alpha \) a left-handed spinor and \( \Psi \otimes \bar{\Omega}_\dot{\alpha}, \Psi' \otimes \bar{\Omega}_\alpha \) with \( \bar{\Omega}_\dot{\alpha} \) a right-handed spinor.

For the multiplet associated with \( \Psi^\beta = \Psi \otimes \Omega^\beta \), we have \( \Psi^\beta = \begin{pmatrix} \rho_1^{\alpha \beta} \\ \bar{\rho}_2^{\dot{\alpha} \beta} \\ \rho_3^{\alpha \beta} \end{pmatrix} \) and the transformation under 3SUSY is \( \delta_\varepsilon \Psi^\beta = \varepsilon^m Q_m \Psi^\beta \) with \( Q_m \) given in (2.5). This leads to

\begin{equation}
\begin{aligned}
\delta_\varepsilon \rho_1^{\alpha \beta} &= \Lambda^{1/3} \varepsilon^m \sigma_{m \alpha \alpha} \bar{\rho}_2^{\dot{\alpha} \beta} \\
\delta_\varepsilon \bar{\rho}_2^{\dot{\alpha} \beta} &= \Lambda^{1/3} \varepsilon^m \bar{\sigma}_m^{\dot{\alpha} \alpha} \rho_3^{\alpha \beta} \\
\delta_\varepsilon \rho_3^{\alpha \beta} &= \Lambda^{-2/3} \varepsilon^m P_m \rho_1^{\alpha \beta}.
\end{aligned}
\end{equation}

Notice that \( \rho_1, \bar{\rho}_2 \) and \( \rho_3 \) are not an irreducible representation of \( \mathfrak{sl}(2, \mathbb{C}) \cong \mathfrak{so}(1, 3) \). We therefore define

\begin{equation}
\begin{aligned}
\rho_1 &= \varphi I_2 + \frac{1}{2} B_{mn} \sigma^{nm} \\
\bar{\rho}_2 &= A^m \bar{\sigma}_m \\
\rho_3 &= \tilde{\varphi} I_2 + \frac{1}{2} \tilde{B}_{mn} \sigma^{nm}
\end{aligned}
\end{equation}

with \( I_2 \) the two by two identity matrix, \( A^m \) a vector, \( \varphi, \tilde{\varphi} \) two scalars and \( B_{mn}, \tilde{B}_{mn} \) two self-dual two-forms.
Using the following dictionary to convert the spinor indices to vector indices and vice versa, together with the trace properties of the $\sigma$ matrices, we have

$$\varphi = \frac{1}{2} \text{Tr} [\rho_1], \quad \tilde{\varphi} = \frac{1}{2} \text{Tr} [\rho_3],$$

$$B_{mn} = \text{Tr} [\sigma_{mn} \rho_1], \quad \tilde{B}_{mn} = \text{Tr} [\sigma_{mn} \rho_3],$$

$$A_m = \frac{1}{2} \text{Tr} [\sigma_m \rho_2],$$

with $\text{Tr}$ the trace over the two by two matrices and spinors indices contraction as in appendix B. The tensor field $B_{mn}$ is self-dual ($^* B_{mn} \equiv \frac{1}{2} \varepsilon_{mpq} B^{pq} = iB_{mn}$) because of the property of self-duality of $\sigma_{mn}$, see the appendix B .

Using the correspondence (4.3), the transformations (4.1) become

$$\delta_\varepsilon \varphi = \frac{1}{3} \varepsilon_m A_m$$

$$\delta_\varepsilon B_{mn} = -\frac{1}{3} (\varepsilon_m A_n - \varepsilon_n A_m) + \frac{1}{3} i \varepsilon_{mpq} \varepsilon^p A^q$$

$$\delta_\varepsilon A_m = \frac{1}{3} \left( \varepsilon_m \tilde{\varphi} + \varepsilon_n \tilde{B}_{mn} \right)$$

$$\delta_\varepsilon \tilde{\varphi} = -\frac{2}{3} \varepsilon_m P_m \varphi$$

$$\delta_\varepsilon \tilde{B}_{mn} = -\frac{2}{3} \varepsilon^p P_p B_{mn}$$

In a similar way, we consider a multiplet (CPT conjugate of the previous) constructed from

$$\Psi_\beta = \Psi' \otimes \Omega_\beta = \begin{pmatrix} \tilde{\rho}_1 \tilde{\alpha}_\beta \\ \tilde{\rho}_2 \alpha_\beta \\ \tilde{\rho}_3 \tilde{\alpha}_\beta \end{pmatrix}.$$ 

As before, we introduce the following fields

$$\tilde{\rho}_1 = \varphi' \tilde{I}_2 + \frac{1}{2} \tilde{B}'_{mn} \tilde{\sigma}^{nm}$$

$$\rho_2 = A^m \sigma_m$$

$$\tilde{\rho}_3 = \tilde{\varphi} \tilde{I}_2 + \frac{1}{2} \tilde{B}_{mn} \tilde{\sigma}^{nm}$$

$$\varphi' = \frac{1}{2} \text{Tr} [\tilde{\rho}_1],$$

$$\tilde{\varphi}' = \frac{1}{2} \text{Tr} [\tilde{\rho}_3],$$

$$B'_{mn} = \text{Tr} [\tilde{\sigma}_{mn} \tilde{\rho}_1], \quad \tilde{B}'_{mn} = \text{Tr} [\tilde{\sigma}_{mn} \tilde{\rho}_3],$$

$$A'_m = \frac{1}{2} \text{Tr} [\tilde{\sigma}_m \tilde{\rho}_2].$$

Their transformations are found to be

$$\delta_\varepsilon \varphi' = \frac{1}{3} \varepsilon_m A'_m$$

$$\delta_\varepsilon B'_{mn} = -\frac{1}{3} (\varepsilon_m A'_m - \varepsilon_n A'_m) - \frac{1}{3} i \varepsilon_{mpq} \varepsilon^p A'^q$$

$$\delta_\varepsilon A'_m = \frac{1}{3} \left( \varepsilon_m \tilde{\varphi}' + \varepsilon_n \tilde{B}'_{mn} \right)$$

$$\delta_\varepsilon \tilde{\varphi}' = -\frac{2}{3} \varepsilon_m P_m \varphi'$$

$$\delta_\varepsilon \tilde{B}'_{mn} = -\frac{2}{3} \varepsilon^p P_p B'_{mn}.$$
\( \rho^{\alpha_\beta}, (\rho_{2\alpha\beta})^* = \bar{\rho}_{2\alpha\beta} \text{ and } (\bar{\rho}_{\dot{\alpha}\dot{\beta}})^* = \rho^{\dot{\alpha}\dot{\beta}}. \) [\( B^* \), the complex conjugate of \( B \), is not to be confused with \( ^*B \), the dual of \( B \).] This means, paying attention to the position of the indices and the conventions given in appendix B, that we have

\[
\begin{align*}
\varphi'^* & = -\varphi, & \tilde{\varphi}'^* & = -\tilde{\varphi}, \\
B_{mn}'^* & = -B_{mn}, & \tilde{B}_{mn}'^* & = -\tilde{B}_{mn}, \\
A_{m}'^* & = A_{m}.
\end{align*}
\] (4.7)

These relations are compatible with the transformations laws given in (4.4) and (4.6) since \( \varepsilon_{n}^* = -\varepsilon_{n} \) and \( P_{n} = -i\frac{\partial}{\partial x_{n}}. \)

The two-forms which appear in the bosonic multiplet can be treated either as matter fields [27] or as gauge fields in the context of extended objects [28]. Here we will consider only the latter interpretation since the two-forms sit in the same multiplet as the \( A \)'s which we attempt to identify with the usual gauge fields. However, we will not consider in this paper the coupling to any matter fields, whether point-like or not. Notice that the gauge transformation \( B_{mn} \rightarrow B_{mn} + \partial_{m}\chi_{n} - \partial_{n}\chi_{m} \) which preserves the associated field strength \( H \) (see below) does not preserve the self-duality of \( B \). Conversely, a self-duality preserving transformation \( (B_{mn} \rightarrow B_{mn} + (\partial_{m}\chi_{n} - \partial_{n}\chi_{m}) - i\varepsilon_{mnpq}\partial^{p}\chi^{q}) \) does not preserve the field strength. (Similar remarks hold for \( B' \).)

To identify the \( A \)'s with some gauge fields we introduce the real-valued fields

\[
\begin{align*}
A_{-} & = i\frac{A - A'}{\sqrt{2}}, & A_{+} & = \frac{A + A'}{\sqrt{2}}, \\
B_{-} & = \frac{B - B'}{\sqrt{2}}, & B_{+} & = i\frac{B + B'}{\sqrt{2}}, \\
\tilde{B}_{-} & = \frac{\tilde{B} - \tilde{B}'}{\sqrt{2}}, & \tilde{B}_{+} & = i\frac{\tilde{B} + \tilde{B}'}{\sqrt{2}}, \\
\varphi_{-} & = \frac{\varphi - \varphi'}{\sqrt{2}}, & \varphi_{+} & = i\frac{\varphi + \varphi'}{\sqrt{2}}, \\
\tilde{\varphi}_{-} & = \frac{\tilde{\varphi} - \tilde{\varphi}'}{\sqrt{2}}, & \tilde{\varphi}_{+} & = i\frac{\tilde{\varphi} + \tilde{\varphi}'}{\sqrt{2}}.
\end{align*}
\] (4.8)

These new fields form now one (reducible) multiplet of 3SUSY, with \( ^*B_{-} = B_{+} \). The corresponding two- and three-form field strengths read

\[
\begin{align*}
F_{\pm mn} & = \partial_{m}A_{\pm n} - \partial_{n}A_{\pm m}, \\
H_{\pm mnp} & = \partial_{m}B_{\pm np} + \partial_{n}B_{\pm pm} + \partial_{p}B_{\pm mn}.
\end{align*}
\] (4.9)

They are invariant under the gauge transformations

\[
\begin{align*}
\varphi_{\pm} & \rightarrow \varphi_{\pm} \\
A_{\pm m} & \rightarrow A_{\pm m} + \partial_{m}\chi_{\pm} \\
B_{\pm mn} & \rightarrow B_{\pm mn} + (\partial_{m}\chi_{\pm n} - \partial_{n}\chi_{\pm m})
\end{align*}
\] (4.10)

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where $\chi_{\pm}$ ($\chi_{\pm}^m$) are arbitrary scalar (vector) functions ($\chi_{-}^m$ and $\chi_{+}^m$ can still be related in order to preserve the duality relations between $B_{-}$ and $B_{+}$). 

In a similar way we introduce the field strength $\tilde{H}_{-mnp}, \tilde{H}_{+mnp}$, as well as the dual fields $*H_{-m}, *H_{-m}, *H_{+m}$ (where $*H_m \equiv i \epsilon_{mnpq} H^{npq}$. For instance $*H_{-m} = i \partial^n B_{mn}$). We consider now the following local gauge invariant and zero graded Lagrangian,

$$L = \partial_m \varphi_- \partial^m \tilde{\varphi}_- - \partial_m \varphi_+ \partial^m \tilde{\varphi}_+ + \frac{1}{4} F_{-mn} F_{mn}^m - \frac{1}{4} F_{+mn} F_{mn}^m$$

(4.11)

$$+ \frac{1}{12} H_{-mnp} H_{-mnp} - \frac{1}{12} H_{+mnp} H_{+mnp} - \frac{1}{2} *H_{-m} *H_{-m} + \frac{1}{2} *H_{+m} *H_{+m}$$

After some algebra, one obtains the 3SUSY variation of the Lagrangian, up to a surface term,

$$\delta_{\varepsilon} L = (\delta_{\varepsilon} \partial_n A_{-n}) \partial_n A_{-n} - (\delta_{\varepsilon} \partial_n A_{+n}) \partial_n A_{+n}$$

(4.12)

meaning that the gauge-invariant Lagrangian (4.11) is not 3SUSY invariant. We checked that this result remains unavoidable even if we added the two remaining boson multiplets $\Psi' \otimes \Omega$ and $\Psi \otimes \Omega_{\beta}$.

It is now interesting to note, that up to surface terms,

$$\delta_{\varepsilon} L = \frac{1}{2} \delta_{\varepsilon} (\partial_n A_{-n})^2 - \frac{1}{2} \delta_{\varepsilon} (\partial_n A_{+n})^2$$

(4.13)

This seems to indicate that 3SUSY invariance requires the usual 't Hooft-Feynman gauge fixing term:

$$L_{fsusy} = L + L_{g.f}(\xi = 1),$$

(4.14)

where $L$ is defined in (4.11) and

$$L_{g.f}(\xi) = -\frac{1}{2\xi} (\partial_n A_{-n})^2 + \frac{1}{2\xi} (\partial_n A_{+n})^2$$

(4.15)

leading to

$$\delta_{\varepsilon} L_{fsusy} = 0 \text{ (up to a surface term.)}$$

We do not dwell further here on this intriguing result, that is one symmetry (3SUSY) leads naturally to a physically necessary gauge fixing of another (namely $U(1)$) associated with the

\footnote{Note that the gauge transformations (4.10) correspond naturally to the zero-, one- and two-form character of the components of the 3SUSY gauge multiplet.}
vector field $A$) and thus eliminating the intrinsic unphysical redundancies endemic to gauge theories\(^3\).

One should note, though, the relative minus signs in front of the kinetic terms of the vector fields in (4.11) which endanger a priori the boundedness from below of the energy density of the “electromagnetic” fields. This difficulty does not have a clear physical interpretation as long as interaction terms have not been included, and necessitates a more careful study of the field manifold associated with the density energy. However, it may suggest that some field combinations are dynamically driven to very large values and should be reinterpreted as decoupling from the physical system.

Before concluding this section, we point out that one can partially couple in a 3SUSY invariant way the two-forms associated with the two CPT conjugate multiplets. Namely, defining

$$
\mathcal{L}’_B = H_{mnp}\tilde{H}^{mnp} + H'_{mnp}\tilde{H}^{mnp},
\star\mathcal{L}’_B = \star H_{mnp}\star\tilde{H}^{mnp} + \star H'_{mnp}\star\tilde{H}^{mnp},
$$

one finds that the Lagrangian

$$
\mathcal{L}' = \frac{1}{12}\mathcal{L}'_B - \frac{1}{2}\star\mathcal{L}'_B
$$

is by itself gauge and 3SUSY invariant.

Including similar couplings in the $\varphi$ and $A$ sectors, it is possible to find combinations with the Lagrangian in (4.11) where the vector field kinetic terms have the correct signs. However, the so obtained Lagrangian is no more 3SUSY invariant even including gauge fixing terms.

5 A comment on Noether currents

The 3SUSY algebra we have studied has one main difference with the usual Lie (super)algebra: it does not close through quadratic, but rather cubic, relations. Moreover, it might be possible that some usual results of Lie (super)algebra do not apply straightforwardly. One example is the Noether currents and their associated algebra. Indeed, according to Noether theorem, to all the symmetries correspond conserved currents. The symmetries are then generated by charges which are expressed in terms of the fields. By the spin-statistics theorem, fields having integer spin close with commutators whilst fields with half-integer spin under anticommutators. So it seems that the current algebra automatically leads to Lie (super)algebras. In our case, however, things might look different since we are not dealing with

\(^3\)For comments on interesting proposals to tackle the question of gauge redundancies, in the context of standard SUSY, see for instance [29], section 4.4. It is also worth noting that, even in the absence of (super)symmetries, such gauge fixing conditions appeared naturally in the action-at-a-distance formulation of coupling of a point-like particle (resp. a string) to a vector (resp. a two-form). In particular, in the case of the vector field this leads to the Fermi Lagrangian (which is precisely $-\frac{1}{4}F_{mn}F^{mn} + \mathcal{L}_{\varphi,\xi}(\xi = 1)$, up to a surface-term), and in the case of the two-form to a Lagrangian which is analogous to $\frac{1}{12}H_{++mnp}\tilde{H}^{++mnp} - \frac{1}{2}H_{--m}\star\tilde{H}_{--m}$, see Kalb and Ramond in [28].
a Lie algebra and yet we have only bosonic operators. It is thus interesting to understand how this unusual feature translates.

We will actually start from the classical field theory case, then go to the quantum case through the usual canonical quantization procedure. Let us consider a general Lagrangian $L$, at the classical level, and construct through the standard procedure the Noether charges $\hat{Q}_A$

$$\varepsilon^A \hat{Q}_A = \int d^3x \frac{\delta L}{\delta \partial_0 \Psi_i} \delta \varepsilon \Psi_i$$  \hspace{1cm} (5.1)

associated with the symmetry transformations

$$\delta \varepsilon \Psi_i = \varepsilon^A Q_A \Psi_i$$ \hspace{1cm} (5.2)

Here $Q_A$ generate the symmetry algebra in some appropriate (matrix) representation and $\Psi_i$ is a generic field. In particular, $Q$ could be associated with the 3SUSY symmetry transformations. Upon use of eq.(5.2) in eq.(5.1), one gets

$$\hat{Q}_A = \int d^3x \Pi_i(x) Q_A \Psi_i(x)$$ \hspace{1cm} (5.3)

where $\Pi_i(x) \equiv \frac{\delta L}{\delta \partial_0 \Psi_i}$ is the conjugate momentum. Equation (5.3) is the general relation between $\hat{Q}_A$ and $Q_A$. Now from eqs.(5.2) [5.3] one readily gets

$$\{ \varepsilon^B \hat{Q}_B, \Psi_i(x) \}_{\text{p.b}} = \delta \varepsilon \Psi_i = \varepsilon^A Q_A \Psi_i$$ \hspace{1cm} (5.4)

meaning that at the classical level the Noether charges $\hat{Q}_B$ generate the considered algebra through Poisson brackets. The quantum case is then obtained in the usual canonical way by replacing the Poisson bracket by the commutator $[,]$ in eqs.(5.4) where now $\hat{Q}$, $\hat{P}$ and $\Psi_i$ operate on some Fock space, leading to

$$[\varepsilon^A \hat{Q}_A, \Psi_i(x)] = \delta \varepsilon \Psi_i$$ \hspace{1cm} (5.5)

Before going further in the 3SUSY case, let us make a quick digression and recall some features related to the (more conventional) Lie (super)algebra case. If $Q_a$ and $\hat{Q}_a$ were associated with Lie (super)algebra with structure constants $f^{abc}$, the algebra would have been realized as follows

$$[\delta_a, \delta_b] \Psi_i \equiv \delta_a (\delta_b \Psi_i) - \delta_b (\delta_a \Psi_i) = f^{abc} \delta_c \Psi_i$$ \hspace{1cm} (5.6)

where $\Psi_i$ is now a field operator and $\delta$ is given by (5.5). Equivalently, equation (5.6) reads then

$$[\hat{Q}_a, [\hat{Q}_b, \Psi_i]] - [\hat{Q}_b, [\hat{Q}_a, \Psi_i]] = f^{abc} [\hat{Q}_c, \Psi_i]$$ \hspace{1cm} (5.7)

Now due to Jacobi identity ($[A, [B, C]] + \text{cyclic} = 0$), one can recast the left-hand side of Eq.(5.7) in the form $[[\hat{Q}_a, \hat{Q}_b], \Psi_i]$ to get

$$[[\hat{Q}_a, \hat{Q}_b], \Psi_i] = f^{abc} [\hat{Q}_c, \Psi_i] \quad (\text{for any } \Psi_i)$$ \hspace{1cm} (5.8)
This means that
\[
[\hat{Q}_a, \hat{Q}_b] = f_{ab}^c \hat{Q}_c
\]  
(5.9)
at least on some (sub-)space of field operators $\Psi_i$.

For the 3SUSY algebra we are considering, the analogy with the steps described above stops at Eq. (5.7). Although we do have (generalized) Jacobi identities (see Eq. (A.1)), these do not help in going from equations
\[
(\delta_m.\delta_n.\delta_r + \text{perm}) \, \Psi = [\hat{Q}_m, [\hat{Q}_n, [\hat{Q}_r, \Psi]]] + \text{perm} = \eta_{mn} \hat{P}_r, \Psi + \eta_{mr} \hat{P}_n, \Psi + \eta_{rn} \hat{P}_m, \Psi
\]  
(5.10)
which is the analog of Eq. (5.7) (with $\hat{P}_m$ the generators of the Poincaré translations), to an equation of the form
\[
[\{\hat{Q}_m, \hat{Q}_n, \hat{Q}_r\}, \Phi] = \{\eta_{mn} \hat{P}_r + \eta_{mr} \hat{P}_n + \eta_{rn} \hat{P}_m, \Phi\}
\]  
(5.11)
which would have been the analog of Eq. (5.8). In other words, the quantized version of the Noether charges algebra is just (5.10) and cannot be cast simply in a $\Psi$ independent form\footnote{Note that this aspect is not due to the quantization procedure. It is already present at the classical level.}. We should stress at this level that the difference with the conventional algebras we are pointing out, does not mean the absence of a realization of the 3SUSY algebra in terms of Noether charges. Indeed, starting with the abstract algebra (2.2), we can represent it by some matrices as in section 2 (see e.g. (2.3)). In this case the product of two transformations will be given by $\delta_n.\delta_m \Psi = Q_n Q_m \Psi$ and the algebra will be realized as in (2.1). In the case of Noether charges, the product of two transformations is given by $\delta_n.\delta_m \Psi = [\hat{Q}_n, [\hat{Q}_m, \Psi]]$ (see (5.7)), leading to the realization (5.10) of the algebra (2.1). [We explicitly checked (5.10) on particular Lagrangians such as (4.11) using the usual canonical commutators.]

The point which is potentially tricky is the fact that (5.10) cannot be made $\Psi$ independent in general. Thus the construction of 3SUSY representation states in the Fock space requires some care. Actually (5.10) becomes equivalent to (2.1) (with $Q \rightarrow \hat{Q}, P \rightarrow \hat{P}$), when acting on one particle Fock states. Thus one can construct one particle state representations using a $\Psi$ independent form of the algebra. Now the difference with the conventional algebras is that N-particle state representations will not be trivially obtained from the usual tensor product of one-particle states (a difference which holds even at the classical level). However, one can

\[
\left\{\hat{Q}_m, \left\{\hat{Q}_n, \left\{\hat{Q}_r, \Psi_i\right\}_{p.b}\right\}_{p.b}\right\}_{p.b} + \text{perm.} = \left\{\eta_{mn} \hat{P}_r + \eta_{mr} \hat{P}_n + \eta_{rn} \hat{P}_m, \Psi_i\right\}_{p.b}
\]
show that when acting on N-particle states, the deviation from (2.1) (with \( Q \to \hat{Q}, P \to \hat{P} \)) is expressed in terms of the action of the algebra on (N-1)-particle states. The construction is thus obtained iteratively. We will not detail further here these issues which are outside of the scope of the present paper.

\section{Conclusion}

In the present paper we have constructed the first four dimensional examples of field theories which are invariant under a non trivial extension of the Poincaré algebra that is \textit{not} the supersymmetric one. In particular, we constructed representations in the form of fermion and boson multiplets and their associated invariant actions. In the fermionic case we identified potential difficulties: for the simplest invariant action, the fermionic fields behave like anticommuting spinless fields \textit{i.e.} ghosts. This could be resolved through couplings to background fields. For the bosonic multiplet, we were able to construct a \( U(1) \) gauge invariant action compatible with our symmetry provided that a ’t Hooft-Feynman gauge fixing term is added. The natural appearance of 2-form fields could be a hint that both point particle and one-dimensional extended matter objects should be included in the theory. Strictly speaking, the existence proof of such theories is not complete until a model with interaction terms has been constructed.

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\section{A Algebraic foundation of FSUSY}

We summarize here for completeness the salient algebraic features of FSUSY and its underlying \( F \)-Lie algebra structures. More details can be found in \[20, 21\].

\subsection*{A.1 The algebra}

The general definition of \( F \)-Lie algebras, the abstract algebraic structure underlying FSUSY, was given in \[20, 21\] together with an inductive way to construct \( F \)-Lie algebras associated with \textit{any} Lie algebra or Lie superalgebra. We do not want to go into the detailed definition of this structure here and will only recall the basic points, useful for the sequel. More details can be found in \[20, 21\]. We consider \( F \) a positive integer and define \( q = e^{\frac{2\pi i}{F}} \). The algebra \( g = g_0 \oplus g_1 \) is called an \( F \)-Lie algebra if the following four properties hold:

1. \( g_0 \) is a Lie algebra;
2. \( g_1 \) is a representation of \( g_0 \);
3. There exists a multilinear, \( g_0 \)-equivariant \( F \)-fold \( (i.e. \) which respect the action of \( g_0 \)) map \( \{ \cdots \} : S^F(g_1) \to g_0 \) from \( S^F(g_1) \) into \( g_0 \). In other words, we assume that some of the elements of the Lie algebra \( g_0 \) can be expressed as \( F \)-th order symmetric products
of “more fundamental generators”. Here $S^F(g_1)$ denotes the $F-$fold symmetric product of $g_1$. It can be easily seen that this bracket simply corresponds to the anticommutator, so to Lie superalgebras, when $F = 2$.

4. For $b_i \in g_0$ and $a_j \in g_1$ the following “Jacobi identities” hold:

\[
\begin{align*}
[[b_1, b_2], b_3] + [[b_2, b_3], b_1] + [[b_3, b_1], b_2] &= 0 \\
[[b_1, b_2], a_3] + [[b_2, a_3], b_1] + [[a_3, b_1], b_2] &= 0 \\
[b, \{a_1, \ldots, a_{F}\}] &= \{[b, a_1], \ldots, a_F\} + \cdots + \{a_1, \ldots, [b, a_F]\} \\
\sum_{i=1}^{F+1} [a_i, \{a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{F+1}\}] &= 0.
\end{align*}
\]

It can be seen that $F-$Lie algebras admit a $\mathbb{Z}_F$ grading: $g_0$, being of grade 0 and $g_1$ of grade 1. This means that there exists a grading map $g$ such that $gag^{-1} = qa, a \in g_1$ and $gbg^{-1} = b, b \in g_0$, with $q$ a primitive root of unity ($q^F = 1$). This notion can also be introduced in a more formal way, see [20, 21] for more details. From now on, $g_0$ will be called the bosonic sector of $g$ and $g_1$ the graded sector $^5$.

Having defined the structure of $F-$Lie algebras, we could construct in a systematic way, explicit examples of $F-$Lie algebras associated with Lie (super)algebras [21]. Then, among these families of finite-dimensional examples one can identify $F-$Lie algebras that could generate a non-trivial extension of the Poincaré algebra. Two such examples were given in [21]. Here, we just consider one of these examples, where $g_0 = sp(4, \mathbb{R})$ and $g_1 = \text{ad}(sp(4, \mathbb{R}))$, with $\text{ad}$, the adjoint representation of $sp(4, \mathbb{R})$. If we denote $J_a, a = 1, \ldots, 10$ a basis of $sp(4, \mathbb{R})$, $A_a$ the corresponding basis for $g_1$, and $g_{ab} = \text{Tr}(A_a A_b)$ the Killing form of $sp(4, \mathbb{R})$, then the $F-$Lie algebra of order 3 $g = g_0 \oplus g_1$ reads

\[
[J_a, J_b] = f_{ab} \ c J_c, \quad [J_a, A_b] = f_{ab} \ c A_c, \quad \{A_a, A_b, A_c\} = g_{ab} J_c + g_{ac} J_b + g_{bc} J_a \quad (A.2)
\]

where $f_{ab} \ c$ are the structure constant of $g_0$.

Observing that $so(1, 3) \subset so(2, 3) \cong sp(4)$, and that the $(1 + 3)D$ Poincaré algebra is related to $sp(4)$ through an Inönü-Wigner contraction, one can expect to obtain, from the $F-$Lie algebra $^6$ an extension of the Poincaré algebra. Using vector indices of $so(1, 3)$ coming from the inclusion $so(1, 3) \subset so(2, 3) \cong sp(4)$, the bosonic part of $g$ is generated by $J_{mn}, J_{m4}$, with $m, n = 0, 1, 2, 3$ and the graded part by $A_{mn}, A_{4m}$ ($J_{mn} = - J_{nm}$ and $A_{mn} = - A_{nm}$). Letting $\lambda \to 0$ after the Inönü-Wigner contraction,

\[
\begin{align*}
J_{mn} &\to L_{mn}, \quad J_{m4} \to \frac{1}{\lambda} P_m \\
A_{mn} &\to \frac{1}{\sqrt{\lambda}} Q_{mn}, \quad A_{4m} \to \frac{1}{\sqrt{\lambda}} Q_m.
\end{align*}
\]

$^5$In general, an $F-$Lie algebra admits the decomposition $g = g_0 \oplus g_1 \oplus \cdots \oplus g_{F-1}$ see [20, 21].

$^6$In the same way $N-$extended supersymmetric algebra can be obtained through an Inönü-Wigner contraction of the superalgebra $sp(N|4)$. 

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The trilinear symmetric brackets have the simple form:

\[ [L_{mn}, L_{pq}] = \eta_{mq}L_{pm} - \eta_{mq}L_{pm} + \eta_{mp}L_{mq} - \eta_{mp}L_{mq}, \quad [L_{mn}, P_p] = \eta_{np}P_m - \eta_{mp}P_n, \quad [L_{mn}, Q_p] = \eta_{mp}Q_m - \eta_{mp}Q_n, \quad [P_m, Q_n] = 0, \quad \{Q_m, Q_n, Q_r\} = \eta_{mn}P_r + \eta_{mr}P_n + \eta_{rn}P_m, \]

(A.4)

where \( \eta_{mn} \) is the Minkowski metric. We should mention that this algebra can also be considered in any space-time dimensions. For the purpose of this paper, we consider only the \((1 + 3)\) dimensional case.

### A.2 Representations

The next step in the construction of an invariant action under 3SUSY transformations, is to construct the representations of the 3SUSY algebra \((A.4)\). A representation of an \(F\)–Lie algebra \(g\) is a linear map \(\rho : g \to \text{End}(H)\) \((\text{End}(H)\) being the space of linear applications of \(H\) into \(H)\), and an automorphism \(\hat{g}\) such that \(\hat{g}^F = 1\). It satisfies

\[
\begin{align*}
\text{a)} & \quad \rho([x, y]) = \rho(x)\rho(y) - \rho(y)\rho(x) \\
\text{b)} & \quad \rho\{a_1, \ldots, a_F\} = \sum_{\sigma \in S_F} \rho(a_{\sigma(1)}) \cdots \rho(a_{\sigma(F)}) \\
\text{d)} & \quad \hat{g}\rho(s)\hat{g}^{-1} = \rho\left(g(s)\right)
\end{align*}
\]

\((S_F\) being the group of permutations of \(F\) elements). As a consequence, since the eigenvalues of \(\hat{g}\) are \(F\)th– roots of unity, we have the following decomposition

\[ H = \bigoplus_{k=0}^{F-1} H_k, \]

where \(H_k = \{ |h\rangle \in H : \hat{g}|h\rangle = q^k|h\rangle \}\). The operator \(N \in \text{End}(H)\) defined by \(N|h\rangle = k|h\rangle\) if \(|h\rangle \in H_k\) is the “number operator” (obviously \(q^N = \hat{g}\)). Since \(\hat{g}\rho(b) = \rho(b)\hat{g}, \forall b \in g_0\) each \(H_k\) provides a representation of the Lie algebra \(g_0\). Furthermore, for \(a \in g_1\), \(\hat{g}\rho(a) = q\rho(a)\hat{g}\) and so we have \(\rho(a)H_k \subseteq H_{k+1(\text{mod } F)}\)

Before constructing representations of the algebra \((A.4)\), some general comments are in order. Firstly, observing that the operator \(P^2\) is a Casimir operator, all states in an irreducible representation have the same mass. Secondly, writing \(g = e^{i\frac{2\pi}{N}}\), with \(N\) the number operator, and using the cyclicity of the trace it is easy to prove that \(\text{Tr}(g) = 0\) \([19]\). Thus all the \(H_i\) have the same dimension. Here we assume that we have a finite dimensional representation in order not to have problems with the trace of infinite dimensional matrices (see below).
To obtain representations of the algebra \([A.4]\), we rewrite the RHS of the trilinear bracket as \(\{Q_m, Q_n, Q_r\} = f_{mnr} = f_{mnr}^s P_s\), with \(f_{mnr}^s = \eta_{mn} \delta_r{}^s + \eta_{mr} \delta_n{}^s + \eta_{rn} \delta_m{}^s\). This substitution shows that to the symmetric tensor \(f_{mnr}\) is associated the cubic polynomial \(f(v^0, v^1, v^2, v^3) = f_{mnr}v^m v^n v^r = 3(v.P)(v.v)\). Moreover, the algebra \([A.4]\) simply means that \(f(v) = (v^m Q_m)^3\), as can be verified by developing the cube and identifying all terms, using the trilinear bracket. The generators \(Q_m, m = 0, \cdots, 3\) which are associated with the variables \(v^m, m = 0, \cdots, 3\), then generate an extension of the Clifford algebra called Clifford algebra of the polynomial \(f\) (denoted \(C_f\)). This means that the \(Q\)'s allow to “linearize” \(f\). This algebra is known to the mathematicians and was introduced in 1969 by N. Roby (this structure can be generalized to any polynomial) [30]. However, this algebra is very different from the usual Clifford algebra. Indeed, \(C_f\) is defined through 3-rd order \((n-\text{th order, in general})\) constraints, and consequently the number of independent monomials increases with the polynomial’s degree (for instance, \((Q^1)^2 Q^2\) and \(Q^1 Q^2 Q^1\) are independent). This means that we do not have enough constraints among the generators to order them in some fixed way and, as a consequence, \(C_f\) turns out to be an infinite dimensional algebra. However, it has been proved that for any polynomial a finite dimensional (non-faithful) representation can be obtained [31]. But, for polynomial of degree higher than two, we do not have a unique representation, and inequivalent representations of \(C_f\) (even of the same dimension) can be constructed (see, for instance, [32] and below for the special cubic polynomials). Furthermore, the problem of classification of the representations of \(C_f\) is still open, though it has been proved that the dimension of the representation is a multiple of the degree of the polynomial [33]. For more details one can see [34] and references therein. Thus, the study of the representation of the algebra \([22]\) reduces to a study of the representation of the Clifford algebra of the polynomial \(f(v) = 3P.v(v.v)\).

As mentioned above, representations of the Clifford algebras of polynomials are not classified and only some special matrix representation are known. For \(C_f\) we have found two types of representations. The first one is constructed with the usual Dirac matrices (and it can be extended in any space-time dimension),

\[
Q_m = \begin{pmatrix}
0 & \Lambda^{1/3} \gamma_m & 0 \\
0 & 0 & \Lambda^{1/3} \gamma_m \\
\Lambda^{-2/3} P_m & 0 & 0
\end{pmatrix}.
\]  

(A.6)

The second representation is obtained by linearizing firstly the polynomial \(P.v((v^0)^2 - (v^3)^2) - P.v((v^1)^2 + (v^2)^2)\),

\[
v.P((v^0)^2 - (v^3)^2) - v.P((v^1)^2 + (v^2)^2) = \begin{pmatrix}
0 & \Lambda^{-2/3} v.P & 0 \\
0 & 0 & \Lambda^{1/3}(v^0 + v^3) \\
\Lambda^{1/3}(v^0 - v^3) & 0 & 0
\end{pmatrix}^3 + \begin{pmatrix}
0 & \Lambda^{-2/3} v.P & 0 \\
0 & 0 & \Lambda^{1/3}(-v^1 + iv^2) \\
\Lambda^{1/3}(v^1 + iv^2) & 0 & 0
\end{pmatrix}^3
\]  

(A.7)

with subsequent linearization of this sum of perfect cubes by means of the twisted tensorial product [32]. Similar matrices also appear in a different context in [35] where the cubic root of the Klein-Gordon equation is studied. The first representation is 12-dimensional and
the second representation is 9—dimensional. It is interesting to notice that in the above two matrix representations, in order to have matrix elements of the same dimension, a parameter \( \Lambda \) with mass dimension naturally appears. This means that the 3SUSY extension we are studying automatically contains a mass parameter.

By definition of the algebra (A.4), the \( Q \)'s are in the vector representation of \( \mathfrak{so}(1, 3) \). This means that \( SO(1, 3) \) is an outer automorphism of the 3SUSY algebra. A natural question we should address is whether this automorphism is an inner automorphism. When we have an inner automorphism, this enables us to write down the Lorentz transformations (specified by the matrix \( \Lambda \)) as \( \Lambda^m_n Q^n = S(\Lambda) Q^m S^{-1}(\Lambda) \). At the infinitesimal level this reduces to the possibility of finding the generators \( J_{mn} \) such that

\[
[J_{mn}, Q_r] = \eta_{mr} Q_n - \eta_{mr} Q_m.
\]

For the second representation (A.7), it can be directly checked that there does not exist any \( 9 \times 9 \) matrix \( J_{mn} \) satisfying (A.8). Therefore, for this representation \( SO(1, 3) \) is an outer automorphism. Thus this representation breaks down Lorentz invariance. This problem was also encountered in [35]. In contrast, for the first representation (A.7), it is easy to see that

\[
J_{mn} = \frac{1}{4} (\gamma_m \gamma_n - \gamma_n \gamma_m) + i(x_m P_n - x_n P_m)
\]

(A.9)

with \( P_m = -i \frac{\partial}{\partial x_m} \) are the appropriate Lorentz generators acting on the \( Q \)'s.

\section*{B Conventions and useful relations}

In this appendix, we collect useful relations and conventions used in this paper. The metric is taken to be

\[
\eta_{mn} = \text{diag}(1, -1, -1, -1)
\]

(B.1)

The Levi-Civita tensors \( \varepsilon_{mnpq} \) and \( \varepsilon^{mnpq} = \varepsilon_{rstu} \eta^{mr} \eta^{ns} \eta^{pt} \eta^{qu} \) are normalized as follow

\[
\varepsilon_{0123} = 1, \quad \varepsilon^{0123} = -1
\]

(B.2)

In the \( \mathfrak{sl}(2, \mathbb{C}) \) notations of dotted and undotted indices for two-dimensional spinors, the spinor conventions to raise/lower indices are as follow (we have minor differences as compared to the notations of Wess and Bagger [36]): \( \psi_\alpha = \varepsilon_{\alpha \beta} \psi^\beta, \quad \psi^\alpha = \varepsilon^{\alpha \beta} \psi_\beta, \quad \bar{\psi}_\dot{\alpha} = \varepsilon_{\dot{\alpha} \dot{\beta}} \bar{\psi}^\dot{\beta}, \quad \bar{\psi}^\dot{\alpha} = \varepsilon^{\dot{\alpha} \dot{\beta}} \bar{\psi}_\dot{\beta} \) with \( (\psi_\alpha)^* = \bar{\psi}_{\dot{\alpha}}, \quad \varepsilon_{12} = \varepsilon_{\dot{1}\dot{2}} = -1, \quad \varepsilon^{12} = \varepsilon^{\dot{1}\dot{2}} = 1. \)

The 4D Dirac matrices, in the Weyl representation, are

\[
\gamma_m = \begin{pmatrix} 0 & \sigma_m \\ -\sigma_m & 0 \end{pmatrix},
\]

(B.3)

with

\[
\sigma_m \alpha \dot{\alpha} = \left(1, \sigma_i \right), \quad \bar{\sigma}_m \dot{\alpha} \alpha = \left(1, -\sigma_i \right),
\]

(B.4)
where the $\sigma_i$'s, $i = 1, 2, 3$, are the Pauli matrices. The following relation holds,

$$\bar{\sigma}_m \dot{\alpha}_\alpha = \sigma_{m\beta\dot{\beta}} \varepsilon^{\alpha\beta} \varepsilon^{\dot{\alpha}\dot{\beta}}.$$  \hspace{1cm} (B.5)

Furthermore, the Lorentz generators for the spinors representation are given by

$$\sigma_{mn\alpha\beta} = \frac{1}{4} \left( \sigma_{ma\dot{\alpha}} \sigma_n \dot{\beta} - \sigma_{na\dot{\alpha}} \sigma_m \dot{\beta} \right)$$

$$\bar{\sigma}_{mn} \dot{\alpha}_{\dot{\alpha}} = \frac{1}{4} \left( \bar{\sigma}_{m\dot{\alpha}} \sigma_n \dot{\beta} - \bar{\sigma}_n \sigma_{m\dot{\alpha}} \dot{\beta} \right)$$  \hspace{1cm} (B.6)

We adopt the usual spinor summation convention

$$\psi\lambda = \psi^\alpha \lambda_\alpha, \quad \bar{\psi}\bar{\lambda} = \bar{\psi}^\dot{\alpha} \bar{\lambda}^{\dot{\alpha}}$$  \hspace{1cm} (B.7)

leading to the following Fierz rearrangement:

$$\psi\lambda = \lambda \psi$$

$$\bar{\psi}\bar{\lambda} = \bar{\lambda} \bar{\psi}$$

$$\bar{\psi}\bar{\sigma}_m \lambda = -\lambda \sigma_m \bar{\psi}$$

$$\psi \sigma_{mn} \lambda = -\lambda \sigma_{mn} \psi$$

$$\bar{\psi} \bar{\sigma}_{mn} \bar{\lambda} = -\bar{\lambda} \bar{\sigma}_{mn} \bar{\psi}$$  \hspace{1cm} (B.8)

With our convention for the Levi-Civita tensor, we have

$$\frac{1}{2} \varepsilon_{mnpq} \sigma^{pq} = i \sigma_{mn}$$

$$\frac{1}{2} \varepsilon_{mnpq} \bar{\sigma}^{pq} = -i \bar{\sigma}_{mn}$$  \hspace{1cm} (B.9)

that is, $\sigma_{pq}$ is a self-dual two-form and $\bar{\sigma}_{pq}$ is an antiself-dual two-form.

It can also be observed that

$$\sigma_{mn}^{\alpha\beta} = \sigma_{mn\gamma}^{\beta} \varepsilon^{\alpha\gamma}$$

$$\bar{\sigma}_{mn}^{\dot{\alpha}\dot{\beta}} = \bar{\sigma}_{mn}^{\dot{\alpha}} \varepsilon^{\dot{\beta}\dot{\gamma}}$$  \hspace{1cm} (B.10)

are both symmetric in their spinorial indices.

Moreover, We have the following identities

$$\text{Tr} (\sigma_{mn} \sigma_{pq}) = -\frac{1}{2} (\eta_{mp} \eta_{nq} - \eta_{mq} \eta_{np}) + \frac{i}{2} \varepsilon_{mnpq}$$  \hspace{1cm} (B.11)

$$\text{Tr} (\bar{\sigma}_{mn} \bar{\sigma}_{pq}) = -\frac{1}{2} (\eta_{mp} \eta_{nq} - \eta_{mq} \eta_{np}) - \frac{i}{2} \varepsilon_{mnpq}$$
Starting from Weyl spinors, a Majorana bispinor is given by

$$\psi_M = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^\dot{\alpha} \end{pmatrix}$$

which satisfies the Majorana condition

$$\psi_M^* = \bar{C}\bar{\psi}_M^t,$$  \hspace{1cm} \text{(B.13)}

where the charge conjugation matrix is defined by

$$C = \begin{pmatrix} C_{\alpha\beta} & 0 \\ 0 & C^{\dot{\alpha}\dot{\beta}} \end{pmatrix} = \begin{pmatrix} \varepsilon_{\alpha\beta} & 0 \\ 0 & \varepsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix}$$

Finally the 3SUSY transformations for Majorana fermions read

$$\delta_\varepsilon \psi_1 = \Lambda^{1/3} \varepsilon^n \gamma_n \psi_2, \quad \delta_\varepsilon \bar{\psi}_1 = -\Lambda^{1/3} \bar{\psi}_2 \varepsilon^n \gamma_n,$$

$$\delta_\varepsilon \psi_2 = \Lambda^{1/3} \varepsilon^n \gamma_n \psi_3, \quad \delta_\varepsilon \bar{\psi}_2 = -\Lambda^{1/3} \bar{\psi}_3 \varepsilon^n \gamma_n,$$

$$\delta_\varepsilon \psi_3 = \Lambda^{-2/3} \varepsilon^n P_n \psi_1, \quad \delta_\varepsilon \bar{\psi}_3 = \Lambda^{-2/3} \bar{\psi}_1 \varepsilon^n P_n.$$

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