Fourier Cosine and Sine Transform on fractal space

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Abstract: In this paper, we establish local fractional Fourier Cosine and Sine Transforms on fractal space, considered some properties of local fractional Cosine and Sine Transforms, show applications of local fractional Fourier Cosine and Sine transform to local fractional equations with local fractional derivative.

Keywords: fractal space; local fractional Fourier Sine Transforms; local fractional Fourier Cosine Transforms; local fractional equation; local fractional derivative

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1 Introduction

The fractional Fourier transform has been investigated in a number of papers and has been proved to be very useful in solving engineering problems [1-7]. It is important to deal with the continuous fractal functions, which are irregular in the real world. Recently, Yang-Fourier transform based on the local fractional calculus was introduced [8] and Yang continued to study this subject [9-10]. The importance of Yang-Fourier transform for fractal functions derives from the fact that this is the only mathematic model which focuses on local fractional continuous functions derived from local fractional calculus. More recently, some model for engineering derived from local fractional derivative was proposed [11-12].

The purpose of this paper is to establish local fractional Cosine and Sine Transforms based on the Yang-Fourier transforms and consider its application to local fractional equations with local fractional derivative. This paper is organized as follows. In section 2, local fractional Cosine and Sine Transforms is derived; Section 3 presents Properties of local fractional Fourier Cosine and Sine Transforms; Applications of local fractional Fourier Cosine and Sine Transforms are discussed in section 4.

2 local fractional Cosine and Sine Transforms

In this section, we start with the following result [9,10]:
\[ f(x) = \frac{1}{(2\pi)^\alpha} \int_{-\infty}^{\infty} C_k E_\alpha(i^\alpha x^\alpha \omega^\alpha)(d\omega)^\alpha, \tag{2.1} \]

where

\[ C_k = \frac{1}{\Gamma(1 + \alpha)} \int_{-\infty}^{\infty} f(x) E_\alpha(-i^\alpha x^\alpha \omega^\alpha)(dx)^\alpha. \tag{2.2} \]

From (2.2), the Yang-Fourier transform of \( f(x) \) is given by [9,10]

\[ F_\alpha\{f(x)\} = f_\omega^\alpha(\omega) := \frac{1}{(2\pi)^\alpha} \int_{-\infty}^{\infty} E_\alpha(-i^\alpha \omega^\alpha x^\alpha) f(x)(dx)^\alpha. \tag{2.3} \]

And its inverse formula of Yang-Fourier’s transforms as follows

\[ f(x) = F_\alpha^{-1}(f_\omega^\alpha(\omega)) := \frac{1}{(2\pi)^\alpha} \int_{-\infty}^{\infty} E_\alpha(i^\alpha \omega^\alpha x^\alpha) f_\omega^\alpha(\omega)(d\omega)^\alpha. \tag{2.4} \]

Now, by (2.1) and (2.2), we have

\[ f(x) = \frac{1}{(2\pi)^\alpha} \int_{-\infty}^{\infty} \left[ \frac{1}{\Gamma(1 + \alpha)} \int_{-\infty}^{\infty} f(\xi) E_\alpha(-i^\alpha \xi^\alpha \omega^\alpha)(d\xi)^\alpha \right] E_\alpha(i^\alpha \omega^\alpha x^\alpha)(d\omega)^\alpha. \tag{2.5} \]

Here, we named (2.5) the Yang-Fourier integral formula.

We express the exponential factor \( E_\alpha(i^\alpha \omega^\alpha(x^\alpha - \xi^\alpha)) \) in (2.5) in terms of trigonometric functions on fractal set of fractal dimension and use the even and odd nature of the cosine and the sine functions respectively as functions of \( \omega \), so that (2.5) can be written as

\[ f(x) = \frac{1}{(2\pi)^\alpha} \int_{-\infty}^{\infty} \left[ \frac{1}{\Gamma(1 + \alpha)} \int_{-\infty}^{\infty} f(\xi) E_\alpha(-i^\alpha \xi^\alpha \omega^\alpha)(d\xi)^\alpha \right] E_\alpha(i^\alpha \omega^\alpha x^\alpha)(d\omega)^\alpha \]

\[ = \frac{1}{(2\pi)^\alpha} \int_{-\infty}^{\infty} \left[ \frac{1}{\Gamma(1 + \alpha)} \int_{-\infty}^{\infty} f(\xi) E_\alpha(i^\alpha \omega^\alpha(x^\alpha - \xi^\alpha))(d\xi)^\alpha \right] (d\omega)^\alpha \]

\[ = \frac{2}{(2\pi)^\alpha} \int_{0}^{\infty} \left[ \frac{1}{\Gamma(1 + \alpha)} \int_{-\infty}^{\infty} f(\xi) \cos_\alpha(\omega^\alpha(x^\alpha - \xi^\alpha))(d\xi)^\alpha \right] (d\omega)^\alpha. \]

Hence, we have

\[ f(x) = \frac{2}{(2\pi)^\alpha} \int_{0}^{\infty} \left[ \frac{1}{\Gamma(1 + \alpha)} \int_{-\infty}^{\infty} f(\xi) \cos_\alpha(\omega^\alpha(x^\alpha - \xi^\alpha))(d\xi)^\alpha \right] (d\omega)^\alpha. \tag{2.6} \]

This is another version of the Yang-Fourier integral formula.

We now assume that \( f(x) \) is an even function and expand the cosine function in (2.6) to obtain

\[ f(x) = f(-x) = \frac{4}{(2\pi)^\alpha} \frac{1}{\Gamma(1 + \alpha)} \int_{0}^{\infty} \cos_\alpha(\omega^\alpha x^\alpha)(d\omega)^\alpha \int_{0}^{\infty} f(\xi) \cos_\alpha(\omega^\alpha \xi^\alpha)(d\xi)^\alpha. \tag{2.7} \]

This is called the local fractional Fourier cosine integral formula.

Similarly, for an odd function \( f(x) \), we obtain the local fractional Fourier sine integral formula

\[ f(x) = -f(-x) = \frac{4}{(2\pi)^\alpha} \frac{1}{\Gamma(1 + \alpha)} \int_{0}^{\infty} \sin_\alpha(\omega^\alpha x^\alpha)(d\omega)^\alpha \int_{0}^{\infty} f(\xi) \sin_\alpha(\omega^\alpha \xi^\alpha)(d\xi)^\alpha. \tag{2.8} \]
The local fractional Fourier cosine integral formula (2.7) leads to the local fractional Fourier cosine transform and its inverse defined by

\[
F_{\alpha,c}\{f(x)\} = f_{\omega,c}^{F,\alpha}(\omega) := \frac{2}{\Gamma(1 + \alpha)} \int_0^\infty f(x) \cos(\omega^\alpha x^\alpha) \, dx^\alpha, \tag{2.9}
\]

\[f(x) = F_{\alpha,c}^{-1}\{f_{\omega,c}^{F,\alpha}(\omega)\} := \frac{2}{(2\pi)^\alpha} \int_0^\infty f_{\omega,c}^{F,\alpha}(\omega) \cos(\omega^\alpha x^\alpha) \, d\omega^\alpha, \tag{2.10}\]

where \(F_{\alpha,c}\) is the local fractional Fourier cosine transform operator and \(F_{\alpha,c}^{-1}\) is its inverse operator.

Similarly, the local fractional Fourier sine integral formula (2.8) leads to the local fractional Fourier sine transform and its inverse defined by

\[
F_{\alpha,s}\{f(x)\} = f_{\omega,s}^{F,\alpha}(\omega) := \frac{2}{\Gamma(1 + \alpha)} \int_0^\infty f(x) \sin(\omega^\alpha x^\alpha) \, dx^\alpha, \tag{2.11}
\]

\[f(x) = F_{\alpha,s}^{-1}\{f_{\omega,s}^{F,\alpha}(\omega)\} := \frac{2}{(2\pi)^\alpha} \int_0^\infty f_{\omega,s}^{F,\alpha}(\omega) \sin(\omega^\alpha x^\alpha) \, d\omega^\alpha, \tag{2.12}\]

where \(F_{\alpha,s}\) is the local fractional Fourier cosine transform operator and \(F_{\alpha,s}^{-1}\) is its inverse operator.

**Example 1.** Show that \(a > 0\)

\[
F_{\alpha,s}\{E_\alpha(-at^\alpha)\} = \frac{2\omega^\alpha}{a^2 + \omega^2}, \quad a > 0, \tag{2.13}
\]

\[
F_{\alpha,c}\{E_\alpha(-at^\alpha)\} = \frac{2a}{a^2 + \omega^{2\alpha}}, \quad a > 0. \tag{2.14}
\]

By local fractional Fourier cosine transform, we have

\[
F_{\alpha,s}\{E_\alpha(-at^\alpha)\} = \frac{2}{\Gamma(1 + \alpha)} \int_0^\infty E_\alpha(-at^\alpha) \sin(\omega^\alpha x^\alpha) \, dx^\alpha.
\]

\[
= \frac{2\omega^\alpha}{a^2} \int_0^\infty E_\alpha(-at^\alpha) \sin(\omega^\alpha x^\alpha) \, dx^\alpha
\]

\[
= \frac{2\omega^\alpha}{a^2} \int_0^\infty E_\alpha(-at^\alpha) \sin(\omega^\alpha x^\alpha) \, dx^\alpha = F_{\alpha,s}\{E_\alpha(-at^\alpha)\}. \tag{2.15}
\]

By (2.15), we obtain

\[
F_{\alpha,s}\{E_\alpha(-at^\alpha)\} = \frac{2\omega^\alpha}{a^2 + \omega^{2\alpha}}.
\]

Similarly, we obtain (2.14)

### 3 Properties of local fractional Fourier Cosine and Sine Transforms

**Theorem 3.1** If \(F_{\alpha,c}\{f(x)\} = f_{\omega,c}^{F,\alpha}(\omega)\) and \(F_{\alpha,s}\{f(x)\} = f_{\omega,s}^{F,\alpha}(\omega)\), then

\[
F_{\alpha,c}\{f(ax)\} = \frac{1}{a^\alpha} f_{\omega,c}^{F,\alpha}\left(\frac{\omega}{a}\right), \tag{3.1}
\]

\[
F_{\alpha,s}\{f(ax)\} = \frac{1}{a^\alpha} f_{\omega,s}^{F,\alpha}\left(\frac{\omega}{a}\right). \tag{3.2}
\]
Proof. As a direct application of the local fractional Fourier Cosine transform, we derive the following identity

\[
F_{\alpha,c} \{ f(ax) \} = \frac{2}{\Gamma(1 + \alpha)} \int_0^\infty f(ax) \cos(\omega^\alpha x^\alpha)(dx)^\alpha
\]

(3.3)

taking \( y = ax \) in (3.3) implies that

\[
\frac{2}{a^\alpha \Gamma(1 + \alpha)} \int_0^\infty f(ax) \cos(\omega^\alpha x^\alpha)(dx)^\alpha
\]

\[
= \frac{2}{a^\alpha \Gamma(1 + \alpha)} \int_0^\infty f(y) \cos(\omega^\alpha \frac{y}{a}\alpha)(dy)^\alpha = \frac{1}{a^\alpha} f_{\omega,c}^{F,\alpha} \left( \frac{\omega}{a} \right).
\]

Similarly, we obtain (3.2)

Under appropriate conditions, the following properties also hold:

\[
F_{\alpha,c} \{ f^{(\alpha)}(x) \} = \omega^\alpha F_{\omega,c}^{F,\alpha}(\omega) - 2f(0),
\]

(3.4)

\[
F_{\alpha,c} \{ f^{(2\alpha)}(x) \} = -\omega^{2\alpha} F_{\omega,c}^{F,\alpha}(\omega) - 2f^{(\alpha)}(0),
\]

(3.5)

\[
F_{\alpha,s} \{ f^{(\alpha)}(x) \} = -\omega^\alpha F_{\omega,s}^{F,\alpha}(\omega),
\]

(3.6)

\[
F_{\alpha,s} \{ f^{(2\alpha)}(x) \} = -\omega^{2\alpha} F_{\omega,s}^{F,\alpha}(\omega) + 2\omega^\alpha f(0),
\]

(3.7)

These results can be generalized for the cosine and sine transforms of higher order derivatives of a function.

Theorem 3.2 (Convolution Theorem for the local fractional Fourier Cosine Transform) If \( F_{\alpha,c} \{ f(x) \} = f_{\omega,c}^{F,\alpha}(\omega) \) and \( F_{\alpha,c} \{ g(x) \} = g_{\omega,c}^{F,\alpha}(\omega) \), then

\[
\frac{2}{(2\pi)^\alpha} \int_0^\infty F_{\omega,c}^{F,\alpha}(\omega) g_{\omega,c}^{F,\alpha}(\omega) \cos(\omega^\alpha x^\alpha)(d\omega)^\alpha
\]

\[
= \frac{1}{\Gamma(1 + \alpha)} \int_0^\infty f(\xi)g(x + \xi) + g(|x - \xi|)(d\xi)^\alpha.
\]

(3.8)

Or, equivalently,

\[
\int_0^\infty f_{\omega,c}^{F,\alpha}(\omega) g_{\omega,c}^{F,\alpha}(\omega) \cos(\omega^\alpha x^\alpha)(d\omega)^\alpha
\]

\[
= \frac{(2\pi)^\alpha}{2\Gamma(1 + \alpha)} \int_0^\infty f(\xi)[g(x + \xi) + g(|x - \xi|)](d\xi)^\alpha.
\]

(3.9)
Proof. Using the definition of the inverse local fractional Fourier cosine transform, we have

\[
F_{\alpha,c}^{-1}(f_{\omega,c}^{\alpha}(\omega)g_{\omega,c}^{\alpha}(\omega)) = \frac{2}{(2\pi)^\alpha} \int_0^\infty f_{\omega,c}^{\alpha}(\omega)g_{\omega,c}^{\alpha}(\omega) \cos_\alpha(\omega^\alpha x^\alpha)(dw)^\alpha
\]

\[
= \frac{2}{(2\pi)^\alpha} \int_0^\infty g_{\omega,c}^{\alpha}(\omega) \cos_\alpha(\omega^\alpha x^\alpha)(dw)^\alpha \frac{2}{\Gamma(1+\alpha)} \int_0^\infty f(x) \cos_\alpha(\omega^\alpha x^\alpha)(dx)^\alpha
\]

\[
= \frac{4}{(2\pi)^\alpha \Gamma(1+\alpha)} \int_0^\infty g_{\omega,c}^{\alpha}(\omega) \cos_\alpha(\omega^\alpha x^\alpha)(dw)^\alpha \int_0^\infty f(x) \cos_\alpha(\omega^\alpha x^\alpha)(dx)^\alpha.
\]

Hence, we obtain

\[
F_{\alpha,c}^{-1}(f_{\omega,c}^{\alpha}(\omega)g_{\omega,c}^{\alpha}(\omega)) = \frac{4}{(2\pi)^\alpha \Gamma(1+\alpha)} \int_0^\infty g_{\omega,c}^{\alpha}(\omega) \cos_\alpha(\omega^\alpha x^\alpha)(dw)^\alpha \int_0^\infty f(x) \cos_\alpha(\omega^\alpha x^\alpha)(dx)^\alpha.
\]

in which the definition of the inverse local fractional Fourier cosine transform is used. This proves (3.8). It also follows from the proof of Theorem 2 that

\[
\int_0^\infty f_{\omega,c}^{\alpha}(\omega)g_{\omega,c}^{\alpha}(\omega)(dw)^\alpha = \frac{(2\pi)^\alpha}{2\Gamma(1+\alpha)} \int_0^\infty f(\xi)[g(x+\xi) + g(|x-\xi|)](d\xi)^\alpha.
\]

This proves result (3.9).

Putting \(x = 0\) in (3.9), we obtain

\[
\int_0^\infty f_{\omega,c}^{\alpha}(\omega)g_{\omega,c}^{\alpha}(\omega)(dw)^\alpha = \frac{(2\pi)^\alpha}{\Gamma(1+\alpha)} \int_0^\infty f(\xi)g(\xi)(d\xi)^\alpha = \frac{(2\pi)^\alpha}{\Gamma(1+\alpha)} \int_0^\infty f(x)g(x)(dx)^\alpha.
\]

Substituting \(g(x) = \overline{f(x)}\) gives, since \(f_{\omega,c}^{\alpha}(\omega) = \overline{g_{\omega,c}^{\alpha}(\omega)}\),

\[
\int_0^\infty |f_{\omega,c}^{\alpha}(\omega)|^2(dw)^\alpha = \frac{(2\pi)^\alpha}{\Gamma(1+\alpha)} \int_0^\infty |f(x)|^2(dx)^\alpha.
\]

This is the Parseval relation for the local fractional Fourier cosine transform.

Similarly, we obtain

\[
\int_0^\infty f_{\omega,s}^{\alpha}(\omega)g_{\omega,s}^{\alpha}(\omega) \cos_\alpha(\omega^\alpha x^\alpha)(dw)^\alpha
\]

\[
= \int_0^\infty f_{\omega,s}^{\alpha}(\omega)g_{\omega,s}^{\alpha}(\omega) \cos_\alpha(\omega^\alpha x^\alpha)(dw)^\alpha
\]

\[
= \frac{2}{\Gamma(1+\alpha)} \int_0^\infty g_{\omega,s}^{\alpha}(\omega) \cos_\alpha(\omega^\alpha x^\alpha)(dw)^\alpha \int_0^\infty f(\xi) \sin_\alpha(\omega^\alpha \xi^\alpha)(d\xi)^\alpha.
\]
which is, by interchanging the order of integration,
\[
\int_0^\infty \int_0^\infty \frac{f_{\omega,s}(\omega)g_{\omega,s}(\omega)}{\Gamma(1+\alpha)} \cos(\omega^\alpha x^\alpha)(d\omega)^\alpha d\xi^\alpha = \frac{2}{\Gamma(1+\alpha)} \int_0^\infty f(\xi) \sin(\omega^\alpha \xi^\alpha)(d\xi)^\alpha \int_0^\infty g(\omega, s)(\omega^\alpha x^\alpha)(d\omega)^\alpha.
\]

in which the inverse local fractional Fourier sine transform is used. Thus, we find
\[
\int_0^\infty \int_0^\infty \frac{f_{\omega,s}(\omega)g_{\omega,s}(\omega)}{\Gamma(1+\alpha)} \cos(\omega^\alpha x^\alpha)(d\omega)^\alpha d\xi^\alpha = \frac{2}{\Gamma(1+\alpha)} \int_0^\infty f(\xi)(d\xi)^\alpha \int_0^\infty g(\omega, s)(\omega^\alpha x^\alpha)(d\omega)^\alpha.
\]

4 Applications

Use the local fractional Fourier sine transform to solve the following differential equation:
\[y^{(2\alpha)} - 9y(t) = 50E_\alpha(-2t^\alpha), \quad y(0) = y_0.\]

Since we are interested in positive + region, we can take \(y(t)\) to be an odd function and take local fractional Fourier sine transforms. It is clear from its definition that local fractional Fourier sine transform is linear
\[
F_{\alpha,s}(a f_1(x) + b f_2(x)) = a F_{\alpha,s}(f_1(x)) + b F_{\alpha,s}(f_2(x)).
\]
Using this property and taking local fractional Fourier sine transform of both sides of the differential equation, we have

\[ F_{\alpha,s}\{y^{(2\alpha)}\} - 9F_{\alpha,s}\{y(t)\} = 50F_{\alpha,s}\{E_\alpha(-2t^\alpha)\}. \]

Since

\[ F_{\alpha,s}\{y^{(2\alpha)}(t)\} = -\omega^{2\alpha}F_{\alpha,s}\{y(t)\} + 2\omega^{\alpha}y(0). \]

So

\[ -\omega^{2\alpha}F_{\alpha,s}\{y(t)\} + 2\omega^{\alpha}y(0) - 9F_{\alpha,s}\{y(t)\} = 50F_{\alpha,s}\{E_\alpha(-2t^\alpha)\}. \]

which, after collecting terms, becomes

\[ (\omega^{2\alpha} + 9)F_{\alpha,s}\{y(t)\} = -50\frac{2\omega^{\alpha}}{4 + \omega^{2\alpha}} + 2\omega^{\alpha}y_0. \]

Thus

\[ F_{\alpha,s}\{y(t)\} = -50\frac{2\omega^{\alpha}}{4 + \omega^{2\alpha}} - \frac{1}{9} + y_0\frac{2\omega^{\alpha}}{\omega^{2\alpha} + 9}. \]

With partial fraction of

\[ \frac{1}{(4 + \omega^{2\alpha})(\omega^{2\alpha} + 9)} = \frac{1}{5(4 + \omega^{2\alpha})} - \frac{1}{5(\omega^{2\alpha} + 9)}, \]

we have

\[ F_{\alpha,s}\{y(t)\} = -10\frac{2\omega^{\alpha}}{(4 + \omega^{2\alpha})} + 10\frac{2\omega^{\alpha}}{5(\omega^{2\alpha} + 9)} + y_0\frac{2\omega^{\alpha}}{\omega^{2\alpha} + 9} \]

\[ = (y_0 + 10)\frac{2\omega^{\alpha}}{\omega^{2\alpha} + 9} - 10\frac{2\omega^{\alpha}}{(4 + \omega^{2\alpha})} \]

\[ = (y_0 + 10)F_{\alpha,s}\{E_\alpha(-3t^\alpha)\} - 10F_{\alpha,s}\{E_\alpha(-2t^\alpha)\}. \]

Taking the inverse local fractional Fourier sine transform, we get the solution

\[ y(t) = (y_0 + 10)E_\alpha(-3t^\alpha) - 10E_\alpha(-2t^\alpha). \]

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