The Rank of Elliptic Surfaces in Unramified Abelian Towers

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Abstract. Let $\mathcal{E} \to C$ be an elliptic surface defined over a number field $K$. For a finite covering $C' \to C$ defined over $K$, let $\mathcal{E}' = \mathcal{E} \times_C C'$ be the corresponding elliptic surface over $C'$. In this paper we give a strong upper bound for the rank of $\mathcal{E}'(C'/K)$ in the case of unramified abelian coverings $C' \to C$ and under the assumption that the Tate conjecture is true for $\mathcal{E}'/K$. In the case that $C$ is an elliptic curve and the map $C' = C \to C$ is the multiplication-by-$n$ map, the bound for $\text{rank}(\mathcal{E}'(C'/K))$ takes the form $O\left(\frac{n^\kappa}{\log \log n}\right)$, which may be compared with the elementary bound of $O(n^2)$.

1. Introduction

It is a longstanding problem to describe the variation of the rank of the Mordell-Weil group in families of elliptic curves. There are many variations on this theme. One can study elliptic curves over number fields or over function fields and one can study the rank for a fixed base field and varying elliptic curve or for a fixed elliptic curve and varying base field.

In this note we begin with a number field $K$, a curve $C/K$ defined over $K$, and a (nonconstant) elliptic curve $\mathcal{E}$ defined over the function field $K(C)$ of $C$. Equivalently, we consider an elliptic surface $\mathcal{E} \to C$ defined $K$. Our main result is the following (conditional) upper bound for the rank of $\mathcal{E}(K(C'))$ for finite abelian unramified extensions $K(C')/K(C)$.

Theorem 1. With notation as above, let $A = \text{Gal}(K(C')/K(C))$. Notice that $G_{K/K}$ acts on $A$. Assume that the Tate conjecture is true for the elliptic surface associated to $\mathcal{E}/C'$, and let $\mathfrak{M}(\mathcal{E}')$ denote its
conductor. Then
\[
\text{rank } \mathcal{E}(K(C')) \leq \frac{(\text{Number of } G_{K/K} \text{ orbits of } A)}{|A|} \cdot (|\mathfrak{N}(\mathcal{E}')| + 4g' - 4),
\]
where $g'$ is the genus of the curve $C'$.

We remark that there is an elementary geometric upper bound for the rank, coming from cohomological considerations, which says that
\[
(1) \quad \text{rank } \mathcal{E}(K(C')) \leq |\mathfrak{N}(\mathcal{E}')| + 4g' - 4.
\]
Thus the gain in Theorem 1 comes from nontrivial action of $G_{K/K}$ on the Galois group of the field extension $K(C'/K(C))$, or equivalently from nontrivial action of $G_{K/K}$ on the group of deck transformations of the finite unramified covering $C' \to C$.

As an interesting special case of the theorem, we take $C = C'$ to be an elliptic curve and consider the coverings $[n] : C \to C$ given by the multiplication-by-$n$ maps. Then $A = C[n]$ is the group of $n$ torsion points on $C$ and Serre’s theorem tells us that the action of $G_{K/K}$ on $A$ is highly nontrivial as $n$ increases. Combining this with the theorem gives our second main result.

**Theorem 2.** Let $C/K$ be an elliptic curve defined over a number field, let $\mathcal{E}/K(C)$ be an elliptic curve, and for each $n \geq 1$, let $K_n$ be the extension field of $K(C)$ corresponding to the multiplication-by-$n$ map $[n] : C \to C$. Assume that Tate’s conjecture is true for the elliptic surface $\mathcal{E}_n$ associated to $\mathcal{E}/K_n$. Then there is a $\kappa > 0$ and an $n_0 = n_0(K, C, \delta)$ such that
\[
\text{rank } \mathcal{E}(K_n) \leq |\mathfrak{N}(\mathcal{E}_n)|^{\kappa/\log \log n} \quad \text{for all } n > n_0.
\]

We also show in this situation that the average rank of $\mathcal{E}(K_n)$ is smaller than a multiple of the logarithm of its conductor. (See Theorem 16.) Thus in an unramified abelian tower over an elliptic base, the rank grows much more slowly than the elementary geometric bound.

There are many interesting questions one might ask, for example:

- Can the rank go to infinity in an unramified abelian tower?
- What is the best upper bound for the rank in terms of the conductor?

In the case that the number field $K$ is replaced by a finite field $\mathbb{F}_q$, results of Shioda [17], Brumer [2] and Ulmer [21] provide a definitive answer.

**Theorem 3.** Let $\mathcal{E}$ be a nonconstant elliptic curve defined over a function field $\mathbb{F}_q(C)$ over a finite field.
The geometric rank of $E$ is bounded by
\[ \text{rank } E(\overline{\mathbb{F}_q}(C)) \leq |\mathcal{N}(E)| + 4g - 4. \]
Further, there exist examples with conductor of arbitrarily high degree for which this bound is sharp. (See [17, 21].)

(b) The arithmetic rank of $E$ is bounded by
\[ \text{rank } E(\mathbb{F}_q(C)) \leq \frac{|\mathcal{N}(E)| + 4g - 4}{2 \log_q |\mathcal{N}(E)|} + O \left( \frac{|\mathcal{N}(E)|}{(\log_q |\mathcal{N}(E)|)^2} \right). \]
(See [2].) Further, there exist examples with conductor of arbitrarily high degree for which the main term in this bound is sharp. (See [21].)

See [22, Section 4.3] for a detailed discussion of these results. We also observe that Brumer’s proof of the upper bound (Theorem 3) is modeled after a result of Mestre [9] that deals with elliptic curves over $\mathbb{Q}$.

Returning now to the case of an elliptic curve $E$ over $K(C)$ when $K$ is a number field, we note that the geometric bound given in (1) holds more generally when the number field $K$ is replaced by its algebraic closure, that is,
\[ \text{rank } E(\overline{K}(C')) \leq |\mathcal{N}(E')| + 4g' - 4. \]
This is analogous to the bound in Theorem 3(a). Ulmer [22, Section 8] has asked if this bound can be improved for the group $E(K(C'))$. For example, one might be tempted to make the following conjecture as the analog of the bound in Theorem 3(b) and of Mestre’s result [9].

**Conjecture 4.** Let $K(C)$ be the function field of a curve over a number field, and let $E/K(C)$ be a non-constant elliptic curve (i.e., $j(E) \notin K$). Then
\[ \text{rank } E(K(C')) \ll \frac{|\mathcal{N}(E)|}{\log |\mathcal{N}(E)|}, \]
where the implied constant depends on $K$ and $C$.

More precisely, there is an absolute constant $\alpha > 0$ so that
\[ \text{rank } E(K(C')) \leq \alpha \frac{|\mathcal{N}(E)| + 4g - 4}{\log |\mathcal{N}(E)|} \cdot \log |\text{Disc}(K/\mathbb{Q})|. \]

It is unclear to what extent Theorems 1 and 2 should be considered as providing evidence for this conjecture and to what extent they simply suggest that the rank over unramified towers is unusually small.

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2. Setup for a Single Elliptic Surface

2.1. Notation. We set the following notation.

\begin{align*}
K/\mathbb{Q} &\quad \text{a number field.} \\
q_p &\quad \text{the norm of an ideal } p \text{ of } K. \\
C/K &\quad \text{a smooth projective curve of genus } g. \\
\mathcal{E}/K &\quad \text{a nonconstant elliptic surface } \mathcal{E} \to C \text{ defined over } K. \text{ In particular, the assumption that } j(\mathcal{E}) \notin K \text{ implies that } \mathcal{E} \text{ does not split as a product even after base extension of } C. \\
\mathcal{E}(C/K) &\quad \text{the group of sections of } \mathcal{E} \to C \text{ defined over } K. \\
N(\mathcal{E}/C) &\quad \text{the conductor of the elliptic surface } \mathcal{E} \to C. \text{ The conductor is a divisor on } C; \text{ we denote its degree by } |N(\mathcal{E}/C)|. \\
\mathcal{E}(C/K) &\quad \text{the group of sections of } \mathcal{E} \to C \text{ defined over } K.
\end{align*}

Remark 1. We fix a finite set of primes \(S\) such that for all \(p \notin S\), the elliptic surface \(\mathcal{E} \to C\) has good reduction at \(p\). That is, \(\mathcal{E} \to C\) has a model over the ring of \(S\)-integers of \(K\). We enlarge \(S\) further so that for each prime \(p \notin S\), the conductor of \(\mathcal{E}/C/\mathbb{F}_p\) is the reduction modulo \(p\) of the conductor of \(\mathcal{E}/C/K\). We will write \(\sum_p\) to mean the sum over all primes of \(K\) that are not in \(S\). From time to time, we may enlarge the set \(S\).

2.2. A Rank Estimate and a Rank Formula. An elementary upper bound for the rank of \(\mathcal{E}\) can be obtained from the cohomology of \(\mathcal{E}(\mathbb{C})\). We call this the geometric bound, and observe that it automatically provides an upper bound for the rank of \(\mathcal{E}(C/K)\) over the number field \(K\).

Theorem 5 (Geometric Rank Bound).

\[
\text{rank } \mathcal{E}(C/K) \leq \text{rank } \mathcal{E}(C/\bar{K}) \leq |N(\mathcal{E}/C)| + 4g - 4.
\]

Proof. See [18, Corollary 2].

In order to improve the geometric bound, we use a local-global formula originally proposed by Nagao [12]. The following notation is needed in order to state the required result.

For each prime ideal \(p\) of \(K\) and each point \(R \in C(\mathbb{F}_p)\), let \(\mathcal{E}_R\) denote the fiber of \(\mathcal{E}\) over \(R\) and let

\[
a_p(\mathcal{E}_R) = \begin{cases} 
q_p + 1 - |\mathcal{E}_R(\mathbb{F}_p)| & \text{if } \mathcal{E}_R/\mathbb{F}_p \text{ is smooth,} \\
0 & \text{if } \mathcal{E}_R/\mathbb{F}_p \text{ is singular.}
\end{cases}
\]
If $\mathcal{E}_R/\mathbb{F}_p$ is smooth, then $a_p(\mathcal{E}_R)$ is the trace of Frobenius and satisfies the usual Weil bound $|a_p(\mathcal{E}_R)| \leq 2\sqrt{q_p}$. The “average” of these values over the fibers will be denoted

$$A_p(\mathcal{E}/C) = \frac{1}{q_p} \sum_{R \in C(\mathbb{F}_p)} a_p(\mathcal{E}_R).$$

The Weil bound gives $|A_p(\mathcal{E}/C)| \leq 2\sqrt{q_p}(1 + o(1))$ but a theorem of Deligne (with a further improvement by Michel) gives a much better estimate.

**Theorem 6.**

$$|A_p(\mathcal{E}/C)| \leq |\mathfrak{M}(\mathcal{E}/C)| + 4g - 4 + O(1/\sqrt{q_p}),$$

where the implied constant depends on $\mathcal{E}/C/K$, but is independent of $p$.

**Proof.** The weaker upper bound

$$|A_p(\mathcal{E}/C)| \leq 2(\# \text{ of singular fibers}) + 4g - 4$$

follows in a straightforward manner from an equidistribution result of Deligne [3], see also [8, (3.6.3)]. Each singular fiber contributes at least one to the conductor, so this yields

$$|A_p(\mathcal{E}/C)| \leq 2|\mathfrak{M}(\mathcal{E}/C)| + 4g - 4$$

However, for an elliptic surface, since one knows exactly the monodromy action around the points of semistable reduction, Michel [10, Section 4] explains how to save one factor of the semistable part of the conductor, which yields the stated result. See also Michel [11] and Fisher [6] for similar results and extensions. \(\Box\)

The following analytic version of a conjecture of Nagao [12] gives a local-global formula for the rank of $\mathcal{E}(C/K)$.

**Theorem 7.** Assume that the Tate conjecture is true for the surface $\mathcal{E}/K$. Then

$$\text{rank } \mathcal{E}(C/K) = \text{res}_{s=1} \sum_p -A_p(\mathcal{E}/C) \frac{\log q_p}{q_p^s},$$

where the sum is over all prime ideals of $K$ not in $S$.

**Proof.** This is proven in [13, Theorem 1.3]. \(\Box\)
3. Elliptic Surfaces in Unramified Abelian Towers

We continue with the notation from the previous section. In particular, \( C \) is a curve and \( \mathcal{E} \rightarrow C \) is an elliptic surface, both defined over a number field \( K \). For any finite cover \( C' \rightarrow C \), we obtain a new elliptic surface via pullback,

\[
\mathcal{E}' = \mathcal{E} \times_C C' \rightarrow C'.
\]

We consider covers \( C' \rightarrow C \) satisfying the following conditions:

- The curve \( C' \) and the map \( C' \rightarrow C \) are defined over \( K \), so \( \mathcal{E}'/C' \) is likewise defined over \( K \).
- The map \( C' \rightarrow C \) is a Galois cover with abelian Galois group \( A \). In other words, \( A \) is an abelian subgroup of \( \text{Aut}(C'/\bar{K}) \) and the map \( C' \rightarrow C \) induces an isomorphism \( C'/A \cong C \).
- The map \( C' \rightarrow C \) is unramified.

The conductor of an elliptic surface behaves nicely under an unramified pullback of the base curve.

**Proposition 8.** Let \( \mathcal{E} \rightarrow C \) be an elliptic surface and let \( f : C' \rightarrow C \) be an unramified map.

(a) The conductor of the pullback \( \mathcal{E}' = \mathcal{E} \times_C C' \) is given by

\[
\mathfrak{N}(\mathcal{E}'/C') = f^*(\mathfrak{N}(\mathcal{E}/C)).
\]

(b) The canonical divisors on \( C \) and \( C' \) are related by

\[
\mathcal{K}_{C'} = f^*(\mathcal{K}_C).
\]

(c) In particular,

\[
|\mathfrak{N}(\mathcal{E}'/C')| + 4g' - 4 = \deg(f)(|\mathfrak{N}(\mathcal{E}/C)| + 4g - 4),
\]

where \( g \) and \( g' \) are the genera of \( C \) and \( C' \), respectively.

**Proof.** (a) Immediate from the fact that a minimal equation for \( \mathcal{E} \) over the local ring \( \mathcal{O}_R \) of a point \( R \in C \) remains a minimal Weierstrass equation over the local ring \( \mathcal{O}_P \) for any point \( P \in f^{-1}(R) \), since \( \mathcal{O}_P \) is an unramified extension of \( \mathcal{O}_R \).

(b) This is [7, Proposition IV.2.3] with trivial ramification divisor.

(c) Immediate from (a), (b), and \( \deg(\mathcal{K}_C) = 2g - 2 \) [7, IV.1.3.3].

Each element of \( A \) is an automorphism \( a : C' \rightarrow C' \). These automorphisms need not be defined over \( K \), but the fact that \( C' \) is defined over \( K \) implies that \( G_{K/K} \) acts on \( A \). Since \( A \) is finite, we can choose a finite extension \( L/K \) so that \( G_{L/K} \) acts on \( A \). Note that \( G_{L/K} \) need not be abelian.
Example 1. If $C$ is an elliptic curve, we can take $C' = C$ and use the multiplication by $n$ map $[n] : C \to C$. In this case the group $A$ is the group of $n$-torsion point $C[n]$ on $C$, and the action of $G_{K/K}$ on $A$ is the usual Galois action on the $n$-torsion points of an elliptic curve.

Example 2. More generally, we can embed $C$ into its Jacobian variety $C \to J$ and let $C'$ be the pullback of $C$ via the multiplication-by $n$ map $[n] : J \to J$. Then $C' \to C$ is an abelian unramified cover with group $A = J[n]$ having the natural $G_{K/K}$ action.

Example 3. There is a partial converse to the previous example. If $C' \to C$ is any abelian unramified cover, then there is an isogeny $F : J' \to J$ of their Jacobians so that $C' = F^{-1}(C)$ and $A = \ker(F)$. See [14, Chapter VI, Section 12, Corollary to Proposition 11].

Example 4. If we drop the requirement that $C' \to C$ be unramified, then an interesting case is $C' = C = \mathbb{P}^1$ with the map $T \to T^n$. This is the situation studied in [19], where it was shown that the rank of $\mathcal{E}'(C'/K)$ is bounded by a generalized divisors-of-$n$ function. If one further puts various sorts of technical restrictions on the discriminant of $\mathcal{E}$, then Shioda, Stiller [20], and Fastenberg [4, 5] have shown that the rank of $\mathcal{E}'(C'/K)$ is bounded independently of $n$. The techniques in the present paper can be adapted to handle coverings $C' \to C$ with limited ramification, but the resulting estimates are somewhat complicated, so we have opted to restrict attention to unramified coverings.

4. Elementary Results about Groups Acting on Sets

In this section we recall and prove some elementary estimates that will be required later.

Lemma 9. Let $G$ be a finite group that acts on a finite set $X$. Then

$$\frac{1}{|G|} \sum_{\sigma \in G} |\{x \in X : \sigma(x) = x\}| = (\text{Number of } G\text{-orbits of } X).$$

Proof. Let $\rho$ be the permutation representation of $G$ acting on $X$ and let $\chi$ be its character [15, Section 1.2]. Then as in [15, Section 2.3],

$$(\text{Number of } G\text{-orbits of } X)$$

$$= (\text{Number of times } \rho \text{ contains the unit representation})$$

$$= \frac{1}{|G|} \sum_{\sigma \in G} \chi(\sigma).$$

This is the desired formula, since from the definition of the permutation representation, it is clear that $\chi(\sigma)$ is equal to the number of elements of $X$ that are fixed by $\sigma$. \qed
Lemma 10. Let $G$ be a finite group that acts on a finite set $X$, and let $H$ be a subgroup of $G$. Then

\[(\text{Number of } H\text{-orbits of } X) \leq (G : H) \cdot (\text{Number of } G\text{-orbits of } X),\]

with equality if and only if $H_x = G_x$ for every $x \in X$. (Here $G_x = \{\sigma \in G : \sigma(x) = x\}$ is the stabilizer of $x$, and similarly for $H_x$.)

Proof.

\[
\begin{align*}
(\text{Number of } H\text{-orbits of } X) &= \sum_{x \in X} \frac{1}{|H_x|} \\
&= \frac{1}{|H|} \sum_{x \in X} |H_x| \\
&= (G : H) \sum_{x \in X} \frac{|H_x|}{|G|} \\
&= (G : H) \sum_{x \in X} \frac{1}{|G_x|} \cdot \frac{|H_x|}{|G_x|} \\
&\leq (G : H) \sum_{x \in X} \frac{1}{|G_x|} \quad \text{since } H_x \subseteq G_x, \\
&= (G : H) \cdot (\text{Number of } G\text{-orbits of } X).
\end{align*}
\]

This proves the desired inequality. Further, we have equality if and only if $|H_x| = |G_x|$ for every $x \in X$. □

5. Subgroups of $A = \text{Aut}(C'/C)$ and Intermediate Curves

Each subgroup $B \subseteq A$ corresponds to a curve $C_B$ satisfying

\[C' \to C_B \to C \quad \text{and} \quad \text{Aut}(C'/C_B) = B.\]

Equivalently, $C_B$ is the quotient curve $C'/B$. Since $A$ is abelian, every subgroup is normal, so the covering $C_B \to C$ is also Galois with automorphism group naturally isomorphic to $A/B$.

Recall that we have fixed a finite extension $L/K$ so that $G_{L/K}$ acts on $A$. All of the curves $C_B$ are defined over $L$. If $G_{L/K}$ normalizes $B$, that is, if

\[b \in B \text{ and } \sigma \in G_{L/K} \implies \sigma(b) \in B,\]

then the curve $C_B$ is defined over $K$.

In subsequent sections, we reduce the groups $B$ and the curves $C_B$ modulo primes $p$ of $K$. In order to do this, we observe that $A$ and its subgroups have a natural structure as group schemes over $\overline{K}$ (cf. Example 3), and in fact as group schemes over the field $L$. They thus extend to group schemes over the ring of $S'$-integers of $L$ for
some finite set of primes $S'$. We adjoin to the set $S$ the primes of $K$ lying below the primes of $S'$. It then makes sense to reduce not only the curves $C_B$, but also the automorphism groups $B = \text{Aut}(C'/C_B)$ and $A/B = \text{Aut}(C_B/C)$, modulo primes not lying above $S$. If $G_{L/K}$ normalizes $B$, then $B$ is a group scheme over $K$, and we may reduce modulo primes of $K$ not in $S$.

6. Abelian Unramified Covers over Finite Fields

We next reduce the unramified abelian covering $C' \to C$ and the elliptic surfaces $E$ and $E'$ modulo a prime ideal $p \not\in S$ of $K$. Adjoining finitely many primes to our set $S$ of excluded primes, we may assume the following conditions:

- The reduced curves $C'/\mathbb{F}_p$ and $C/\mathbb{F}_p$ are nonsingular.
- The map $C' \to C$ over $\mathbb{F}_p$ is unramified with abelian Galois group equal to $A$. In particular, the map is separable and $A(\overline{\mathbb{F}}_p) = A(\mathbb{C})$.
- The elliptic surfaces $E/\mathbb{F}_p$ and $E'/\mathbb{F}_p$ are nonsingular.
- The conductor $\mathfrak{N}(E/C/\mathbb{F}_p)$ of the reduction of $E$ modulo $p$ is equal to the reduction modulo $p$ of the global conductor $\mathfrak{N}(E/C/K)$. More generally, we assume that this is true for each pullback $E \times_C C_B$ for each subgroup $B \subset A$. In particular, it is true for $E'$, which corresponds to taking $B = A$.

**Proposition 11.** Let $\sigma = \sigma_p$ be the Frobenius map, so $\sigma$ generates the Galois group of $\overline{\mathbb{F}}_p/\mathbb{F}_p$. Let $B$ be the subgroup of $A$ defined by

$$B = \{ \sigma(a) \circ a^{-1} : a \in A \}.$$

(a) The group $B$ is defined over $\mathbb{F}_p$, and hence the curve $C_B = C'/B$ is also defined over $\mathbb{F}_p$.

(b) The image of $C'(\mathbb{F}_p)$ in $C(\mathbb{F}_p)$ is the same as the image of $C_B(\mathbb{F}_p)$ in $C(\mathbb{F}_p)$.

(c) Let $R \in C(\mathbb{F}_p)$ be a point in the image of $C'(\mathbb{F}_p)$. Then there are exactly $|A(\mathbb{F}_p)|$ points of $C'(\mathbb{F}_p)$ that map to $R$.

(d) Let $R \in C(\mathbb{F}_p)$ be a point in the image of $C_B(\mathbb{F}_p)$. Then there are exactly $(A : B)$ points of $C_B(\mathbb{F}_p)$ that map to $R$.

**Proof.** (a) It is clear that $\sigma(B) = B$, so the fact that $\sigma$ generates $\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ implies that $B$ is defined over $\mathbb{F}_p$. Since $C'$ is defined over $\mathbb{F}_p$ by assumption, this in turn implies that the quotient $C'/B$ is defined over $\mathbb{F}_p$. 
(b). The composition of maps $C'(\mathbb{F}_p) \to C_B(\mathbb{F}_p) \to C(\mathbb{F}_p)$ shows that the image of $C'(\mathbb{F}_p)$ is contained in the image of $C_B(\mathbb{F}_p)$. Next let $Q \in C_B(\mathbb{F}_p)$ and choose any point $P \in C'(\mathbb{F}_p)$ that maps to $Q$. The fact that $Q$ is defined over $\mathbb{F}_p$ and that $C_B = C'/B$ means that there is an automorphism $b \in B$ such that $\sigma(P) = b(P)$. By definition of $B$, there is an automorphism $a \in A$ such that $b = \sigma(a) \circ a^{-1}$, and hence

$$a^{-1}(P) = \sigma(a)^{-1}(\sigma(P)) = \sigma(a^{-1}(P)).$$

Thus $a^{-1}(P)$ is fixed by $\sigma$, so it is in $C'(\mathbb{F}_p)$. Further, $a^{-1}(P)$ and $P$ both have the same image in $C(\mathbb{F}_p)$, which in turn is the same as the image of $Q$ (since $P \in C'$ maps to $Q \in C_B$). This proves that given any point $Q \in C_B(\mathbb{F}_p)$, its image in $C(\mathbb{F}_p)$ is also the image of a point in $C'(\mathbb{F}_p)$, which gives the other inclusion and completes the proof of (b).

(c). By assumption, there is at least one point $P \in C'(\mathbb{F}_p)$ that maps to $R$. The full inverse image of $R$ is the orbit $AR = \{a(R) : a \in A\}$. The assumption that $C' \to C$ is unramified implies that $AR$ consists of $|A|$ distinct points, or equivalently, that only the identity element of $A$ fixes $R$. Hence

$$a(R) \in C'(\mathbb{F}_p) \iff \sigma(a(R)) = a(R)$$

$$\iff (a^{-1} \circ \sigma(a))(R) = R$$

$$\iff \sigma(a) = a^{-1}$$

$$\iff a \in A(\mathbb{F}_p).$$

This completes the proof that there are exactly $|A(\mathbb{F}_p)|$ points of $C'(\mathbb{F}_p)$ that map to $R$.

(d). Applying (c) to the map $C_B \to C$, we see that $|(A/B)(\mathbb{F}_p)|$ points of $C_B(\mathbb{F}_p)$ are mapped to $\bar{R}$. However, the definition of $B$ implies that $\sigma$ fixes every point in $(A/B)(\mathbb{F}_p)$. To see why this is true, let $aB \in (A/B)(\mathbb{F}_p)$. Then

$$\sigma(aB) = \sigma(a)B = a\sigma(a)a^{-1}B = aB,$$

since $\sigma(a)a^{-1} \in B$. Hence

$$|(A/B)(\mathbb{F}_p)| = |(A/B)(\mathbb{F}_p)| = |A|/|B| = (A : B),$$

which completes the proof of (d).

\hspace{1cm} \Box

**Proposition 12.**

$$|A_p(\mathcal{E}')| \leq \frac{|A(\mathbb{F}_p)|}{|A|} \cdot (|\mathfrak{N}(\mathcal{E}'/C')| + 4q' - 4) + O(1/\sqrt{q}).$$
Remark 2. Notice that if $\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ acts trivially on $A$, then $A(\mathbb{F}_p) = A(\overline{\mathbb{F}}_p)$, so the upper bound in Proposition 12 reduces to the generic upper bound provided by Theorem 6. However, when the action is nontrivial, then Proposition 12 may provide a significant strengthening of the generic bound.

Proof. Let $B = \{\sigma(a) \circ a^{-1} : a \in A\}$ be the subgroup of $A$ described in Proposition 11. To ease notation, we let

$$C'' = C_B = C'/B \quad \text{and} \quad \mathcal{E}'' = \mathcal{E}_B = \mathcal{E} \times_C C_B.$$ 

Thus we have a commutative diagram

$$
\begin{array}{ccc}
\mathcal{E}'(\mathbb{F}_p) & \longrightarrow & \mathcal{E}''(\mathbb{F}_p) \\
\downarrow & & \downarrow \\
C'(\mathbb{F}_p) & \longrightarrow & C''(\mathbb{F}_p)
\end{array}
\longrightarrow
\begin{array}{ccc}
\mathcal{E}(\mathbb{F}_p) & \longrightarrow & \\
\downarrow & & \downarrow \\
C(\mathbb{F}_p) & \longrightarrow &
\end{array}
$$

Proposition 11 tells us that the image of $C'(\mathbb{F}_p)$ in $C(\mathbb{F}_p)$ is the same as the image of $C''(\mathbb{F}_p)$ in $C(\mathbb{F}_p)$ and gives us the multiplicity of each map.

We use the fact that if $P \in C'(\mathbb{F}_p)$ maps to $R \in C(\mathbb{F}_p)$, then the fiber $\mathcal{E}'_P$ is isomorphic to $\mathcal{E}_R$, and similarly if $Q \in C''(\mathbb{F}_p)$ maps to $R \in C(\mathbb{F}_p)$, then $\mathcal{E}_Q'' \cong \mathcal{E}_R$. We compute

$$q_p A_p(\mathcal{E}') = \sum_{P \in C'(\mathbb{F}_p)} a_p(\mathcal{E}'_P) \quad \text{by definition of } A_p,$$

$$= |A(\mathbb{F}_p)| \sum_{R \in \text{Image}[C'(\mathbb{F}_p) \to C(\mathbb{F}_p)]} a_p(\mathcal{E}_R) \quad \text{from Proposition 11(c),}$$

$$= |A(\mathbb{F}_p)| \sum_{R \in \text{Image}[C''(\mathbb{F}_p) \to C(\mathbb{F}_p)]} a_p(\mathcal{E}_R) \quad \text{from Proposition 11(b),}$$

$$= \frac{|A(\mathbb{F}_p)|}{(A : B)} \sum_{Q \in C''(\mathbb{F}_p)} a_p(\mathcal{E}_Q'') \quad \text{from Proposition 11(d),}$$

$$= q_p \cdot \frac{|A(\mathbb{F}_p)|}{(A : B)} A_p(\mathcal{E}'').$$

Applying the conductor estimate given in Theorem 6 to $\mathcal{E}''$ and using the elementary relation (Proposition 8c) between the conductors of $\mathcal{E}''$
and \(E', \) we obtain the upper bound

\[
|A_p(E')| = \frac{|A(F_p)|}{|A : B|} \cdot |A_p(E'')| \leq \frac{|A(F_p)|}{|A : B|} \cdot \frac{1}{|B|} \cdot (|\mathfrak{N}(E'/C')| + 4g' - 4) + O(1/\sqrt{q_p})
\]

\[
= \frac{|A(F_p)|}{|A|} \cdot (|\mathfrak{N}(E'/C')| + 4g' - 4) + O(1/\sqrt{q_p}).
\]

\[
\square
\]

7. An Upper Bound for the Rank

**Theorem 13.** Let \(E \to C\) be an elliptic surface defined over a number field \(K\), let \(C' \to C\) be an unramified abelian covering defined over \(K\) and with automorphism group \(A\), and let \(E' = E \times_C C'\) be the pullback of \(E\) via this covering. Assume that the Tate conjecture is true for \(E'/K\). Then

\[
\text{rank } E'(C'/K) \leq \frac{(\text{Number of } G_{\bar{K}/K} \text{ orbits of } A)}{|A|} \cdot (|\mathfrak{N}(E'/C')| + 4g' - 4).
\]

**Remark 3.** Notice that if \(G_{\bar{K}/K}\) acts trivially on \(A\), then we obtain nothing better than the geometric upper bound given in Theorem 5. However, if the Galois action is nontrivial, as tends to the case in a tower \(C \leftarrow C_1 \leftarrow C_2 \leftarrow \cdots\), then Theorem 13 often gives an upper bound that is considerably better than the geometric bound. We will see an example of this below (Theorem 16) in which for every \(\epsilon > 0\), the upper bound for rank \(E_n(C_n/K)\) is smaller than \(|\mathfrak{N}(E_n/C_n)|^{\epsilon}\) as \(n \to \infty\).

**Proof.** We apply the analytic rank formula in Theorem 7 to the elliptic surface \(E' \to C'\),

\[
\text{rank } E'(C'/K) = \text{res}_{s=1} \sum_p -A_p(E') \frac{\log q_p}{q_p^s}.
\]
Taking absolute values and using the estimate for $|A_p|$ provided by Theorem 12 yields

$$\text{rank } \mathcal{E}'(C'/K) \leq \left( | \mathcal{N}(\mathcal{E}'/C') | + 4g' - 4 \right) \sum_{s=1}^{\text{res}} \frac{|A(\overline{F}_p)|}{|A|} \cdot \frac{\log q_p}{q_p^s} \cdot \frac{1}{q_p^{s+1/2}}.$$

The series in the big-O term converges, so its residue is zero and it may be discarded. Next we observe that the size of $A(\overline{F}_p)$ depends only on the action of $p$-Frobenius on $A$. In other words, if we choose an element $\sigma \in G_{L/K}$, then $|A(\overline{F}_p)|$ is the same for every prime $p$ such that $\sigma$ is in the $p$-Frobenius conjugacy class $(p, L/K) \subset G_{L/K}$. (As always, we have discarded the finitely many primes for which $A$ has bad reduction.) More precisely, if $\sigma \in (p, L/K)$, then

$$|A(\overline{F}_p)| = |\{a \in A : \sigma(a) = a\}|$$

is simply the number of elements of $A$ fixed by $\sigma$. We denote this last quantity by $h^0(\sigma, A)$.

We can thus rewrite the above sum as

$$\text{rank } \mathcal{E}'(C'/K) \leq \left( | \mathcal{N}(\mathcal{E}'/C') | + 4g' - 4 \right) \sum_{\sigma \in G_{L/K}} \frac{h^0(\sigma, A)}{|A|} \sum_{s=1}^{\text{res}} \frac{\log q_p}{q_p^s}.$$

The residue is the degree of the extension $L/K$ (cf. \cite{19}), so

$$\text{rank } \mathcal{E}'(C'/K) \leq \left( | \mathcal{N}(\mathcal{E}'/C') | + 4g' - 4 \right) \frac{1}{|G_{L/K}|} \sum_{\sigma \in G_{L/K}} \frac{h^0(\sigma, A)}{|A|}.$$

Finally, we apply Lemma 9 to obtain the desired bound, which completes the proof of Theorem 13. 

\[ \square \]

8. The Rank in an Elliptic Tower

We now consider the case that the base curve $C/K$ is an elliptic curve and we take the unramified abelian covers $[n] : C \to C$ given by the multiplication-by-$n$ maps. In the notation of Theorem 13, we have $C' = C$, but $\mathcal{E}'$ is most definitely not equal to $\mathcal{E}$. We let $\mathcal{E}_n$ denote the pullback of $\mathcal{E}$ via the map $[n] : C \to C$. The automorphism group $A$ in this case is the group $A = C[n]$ of $n$ torsion points of $C$, with the natural action of $G_{K/K}$.

As is clear from Theorem 13, the nontriviality of our bound for the rank of $\mathcal{E}_n(C/K)$ depends on the degree of nontriviality of the action
of $G_{\bar{K}/K}$ on $C[n]$. A famous theorem of Serre gives us control of that action.

**Theorem 14** (Serre [16]). *Let $C/K$ be an elliptic curve defined over a number field $K$. There is an integer $I(C/K)$ so that for every integer $n \geq 1$, the image of the representation $\rho_{C,n}: G_{\bar{K}/K} \rightarrow \text{Aut}(C[n]) \cong \text{GL}_2(\mathbb{Z}/n\mathbb{Z})$ has index at most $I(C/K)$.*

**Remark 4.** Conjecturally, the index bound in Theorem 14 can be chosen to depend only on the field $K$, and possibly only on the degree $[K : \mathbb{Q}]$.

We also need the following elementary fact concerning the natural action of the general linear group.

**Proposition 15.** *Let $n \geq 1$ and $r \geq 1$ be an integers. Then the natural action of $\text{GL}_r(\mathbb{Z}/n\mathbb{Z})$ on $(\mathbb{Z}/n\mathbb{Z})^r$ has $d(n)$ distinct orbits, where $d(n)$ is the number of divisors of $n$.*

**Proof.** We begin with the case that $n = p^e$ is a prime power. For any vector $v = (v_1, \ldots, v_r) \in (\mathbb{Z}/p^e\mathbb{Z})^r$, let

$$\text{ord}_p(v) = \min \{\text{ord}_p(v_1), \ldots, \text{ord}_p(v_r), e\}.$$  

We claim that $v$ and $w$ have the same $\text{GL}_r$-orbit if and only if $\text{ord}_p(v) = \text{ord}_p(w)$.

First, if $v = Aw$ for any integer matrix $A$, then it is clear from the definition that $\text{ord}_p(v) \geq \text{ord}_p(w)$. If $v$ and $w$ are in the same orbit, then $A$ is invertible, so we get an equality $\text{ord}_p(v) = \text{ord}_p(w)$.

Next suppose that $\text{ord}_p(v) = \text{ord}_p(w)$. If this common value is $e$, then $v = w = 0$ and there is nothing further to be said. So suppose that the common value is $k$ with $k < e$. Then we can write $v = p^k v'$ and $w = p^k w'$, where $\text{ord}_p(v') = \text{ord}_p(w') = 0$. It thus suffices to prove that any two vectors with $\text{ord}_p = 0$ are in the same $\text{GL}_r(\mathbb{Z}/p^e\mathbb{Z})$ orbit, and for that it suffices to show that if $\text{ord}_p(v) = 0$, then $v$ is in the orbit of the unit vector $e = (1, 0, 0, \ldots, 0)$. We are thus reduced to showing that every vector with $\text{ord}_p(v) = 0$ can be placed as the first column of a matrix in $\text{GL}_r(\mathbb{Z}/p^e\mathbb{Z})$. A matrix modulo $p^e$ is invertible if and only if its determinant is prime to $p$, so we are reduced to the case that $e = 1$. Then the condition $\text{ord}_p(v) = 0$ says simply the $v \not\equiv 0 \pmod{p}$, and the desired conclusion follows from the fact that a nonzero vector in a vector space can always be extended to a basis for the vector space. The vectors in this basis, lifted from $(\mathbb{Z}/p\mathbb{Z})^r$ to $(\mathbb{Z}/p^e\mathbb{Z})^r$, form the desired matrix in $\text{GL}_r(\mathbb{Z}/p^e\mathbb{Z})$. 

This proves that $\text{ord}_p(v)$ completely characterizes the orbit of $v$ under the action of $\text{GL}_r(\mathbb{Z}/p^e\mathbb{Z})$. The quantity $\text{ord}_p(v)$ is an integer between $0$ and $e$, which proves that there are $e+1$ distinct orbits. Since $d(p^e) = e + 1$, this proves the proposition when $n = p^e$ is a prime power. Finally, the Chinese Remainder Theorem and the multiplicativity of $d(n)$ gives the result for all integers $n \geq 1$.

Remark 5. Applying Lemma 9 with $G = \text{GL}_r(\mathbb{Z}/n\mathbb{Z})$ and $X = (\mathbb{Z}/n\mathbb{Z})^r$ and using Proposition 15 yields some amusing formulas. For example, take $r = 1$, so $G = (\mathbb{Z}/n\mathbb{Z})^*$, $X = \mathbb{Z}/n\mathbb{Z}$, and the action is multiplication. Then for $a \in (\mathbb{Z}/n\mathbb{Z})^*$,

$$\left| \{ x \in \mathbb{Z}/n\mathbb{Z} : ax \equiv x \pmod{n} \} \right| = \left| \{ x \in \mathbb{Z}/n\mathbb{Z} : n \mid (a-1)x \} \right| = \gcd(a-1, n)$$

This yields

$$\sum_{0 \leq a < n, \gcd(a,n) = 1} \gcd(a-1, n) = d(n)\phi(n).$$

It does not seem obvious, a priori, that there should be any natural relationship that involves taking numbers that are one less than numbers relatively prime to $n$ and looking at their common factors with $n$.

Theorem 16. Let $C/K$ be an elliptic curve defined over a number field $K$, let $E \to C$ be an elliptic surface defined over $K$, and for each integer $n \geq 1$, let $E_n \to C$ be the elliptic surface obtained by pullback via the multiplication-by-$n$ map $[n] : C \to C$. Assume that the Tate conjecture is true for the surfaces $E_n/K$.

(a) Let $I(C/K)$ be the supremum over $n$ of the index of $\rho_n(G_{K/K})$ in $\text{Aut}(C[n])$. (Theorem 14 ensures that $I(C/K)$ is finite.) Then for all $n \geq 1$,

$$\text{rank} E_n(C/K) \leq I(C/K) \cdot \frac{d(n)}{n^2} \cdot |\mathcal{R}(E_n/C)|.$$

(b) The sum

$$\frac{1}{x} \sum_{n \leq x} \frac{\text{rank} E_n(C/K)}{\log |\mathcal{R}(E_n/C)|}$$

is bounded as $x \to \infty$. Thus the average rank of $E_n(C/K)$ is smaller than a fixed multiple of the logarithm of its conductor.

(c) There is a constant $\kappa = \kappa(K,C,E)$ so that for all sufficiently large $n$,

$$\text{rank} E_n(C/K) \leq |\mathcal{R}(E_n/C)|^{\kappa / \log \log |\mathcal{R}(E_n/C)|}.$$
In particular, for any \( \epsilon > 0 \) we have
\[
\text{rank } \mathcal{E}_n(C/K) \ll |\mathfrak{N}(\mathcal{E}_n/C)|^\epsilon,
\]
where the implied constant depends on \( K, C, \mathcal{E}, \) and \( \epsilon, \) but is independent of \( n. \)

Proof. Let 
\[
\rho_n : G_{K/K} \to \text{Aut}(C[n]) \cong \text{GL}_2(\mathbb{Z}/n\mathbb{Z})
\]
be the representation of \( G_{K/K} \) on the \( n \)-torsion of \( C. \) Serre’s theorem (Theorem 14 tells us that the index of \( \rho_n(G_{K/K}) \) in \( \text{GL}_2(\mathbb{Z}/n\mathbb{Z}) \) is at most \( I(C/K), \) where \( I(C/K) \) is independent of \( n. \) It follows from Proposition 10 and Proposition 15 that
\[
\text{(Number of } G_{K/K} \text{ orbits in } C[n])
\]
\[
\leq I(C/K) \cdot \text{(Number of Aut}(C[n]) \text{ orbits in } C[n])
\]
\[
= I(C/K) \cdot \text{(Number of } \text{GL}_2(\mathbb{Z}/n\mathbb{Z}) \text{ orbits in } (\mathbb{Z}/n\mathbb{Z})^2)
\]
\[
= I(C/K)d(n).
\]
Applying our main result (Theorem 13) with \( A = C[n] \) and using the above bound for the number of \( G_{K/K} \) orbits in \( C[n] \) yields
\[
\text{rank } \mathcal{E}_n(C/K) \leq I(C/K) \cdot \frac{d(n)}{n^2} \cdot |\mathfrak{N}(\mathcal{E}_n/C)|.
\]
This completes the proof of (a).

Proposition 8 says that \( \mathfrak{N}(\mathcal{E}_n/C) = n^2 \mathfrak{N}(\mathcal{E}/C), \) so we can rewrite the upper bound as
\[
\text{rank } \mathcal{E}_n(C/K) \leq I(C/K) \cdot d(n) \cdot |\mathfrak{N}(\mathcal{E}/C)|.
\]
The divisor function \( d(n) \) has the properties (see [1, Theorem 3.3 and Theorem 13.12]
\[
\sum_{n \leq x} d(n) \sim x \log x \quad \text{and} \quad \limsup_{n \to \infty} \frac{\log d(n)}{\log n / \log \log n} = \log 2.
\]
The first formula implies that
\[
\frac{1}{x} \sum_{2 \leq n \leq x} \frac{d(n)}{\log n}
\]
is bounded for all \( x \geq 2. \) Hence
\[
\frac{1}{x} \sum_{n \leq x} \text{rank } \mathcal{E}_n(C/K) \log |\mathfrak{N}(\mathcal{E}_n/C)| \leq \frac{1}{x} \sum_{n \leq x} I(C/K) \cdot d(n) \cdot |\mathfrak{N}(\mathcal{E}/C)|
\]
is also bounded. This completes the proof of (b). Finally, let $c_1, c_2, \ldots$ denote absolute constants. Then using the second formula gives

$$\text{rank} \mathcal{E}_n(C/K) \leq c_1 I(C/K)|\mathfrak{N}(\mathcal{E}/C)|n^{c_2/\log \log n}$$

$$\leq c_3 I(C/K)|\mathfrak{N}(\mathcal{E}/C)||\mathfrak{N}(\mathcal{E}_n/C)|^{c_4/\log \log |\mathfrak{N}(\mathcal{E}_n/C)|},$$

which completes the proof of (c).

□

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