Uniformly spread measures and vector fields

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Abstract

We show that two different ideas of uniform spreading of locally finite measures in the $d$-dimensional Euclidean space are equivalent. The first idea is formulated in terms of finite distance transportations to the Lebesgue measure, while the second idea is formulated in terms of vector fields connecting a given measure with the Lebesgue measure.

1 Introduction

This text aims to disentangle and make explicit some ideas implicit in our work [9]. It can be read independently of [9].

Given a locally finite non-negative measure $\nu$ on the Euclidean space $\mathbb{R}^d$, we are interested to know how evenly is the measure $\nu$ spread over $\mathbb{R}^d$? First, we consider counting measures for discrete subsets $X \subset \mathbb{R}^d$: $\nu_X = \sum_{x \in X} \delta_x$ where $\delta_x$ is a unit measure sitting at $x$. Following Laczkovich [7, 8], we say that the set $X$ (and the measure $\nu_X$) are uniformly spread in $\mathbb{R}^d$ if there exists a bijection $S: \mathbb{Z}^d \to X$ such that $\sup\{|S(z) - z|: z \in \mathbb{Z}^d\} < \infty$. Equivalently, there exists a measurable map $T: \mathbb{R}^d \to X$ called the marriage between the $d$-dimensional Lebesgue measure $m_d$ and $\nu_X$ (a.k.a. “matching”, “allocation”) that pushes forward the Lebesgue measure $m_d$ to $\nu_X$ and such that $\sup\{|T(x) - x|: x \in \mathbb{R}^d\} < \infty$.

To extend the notion of uniform spreading to arbitrary measures on $\mathbb{R}^d$, we use the idea of the mass transfer that goes back to G. Monge and L. V. Kantorovich [5, Chapter VIII, §4]. Let $\nu_1$ and $\nu_2$ be locally finite positive measures on $\mathbb{R}^d$. We call a positive locally finite measure $\gamma$ on $\mathbb{R}^d \times \mathbb{R}^d$ a transportation from $\nu_1$ to $\nu_2$, if $\gamma$ has marginals $\nu_1$ and $\nu_2$, that is

$$\int\int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi(x, y) \, d\gamma(x, y) = \int_{\mathbb{R}^d} \varphi(x) \, d\nu_1(x),$$

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and
\[ \int \int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi(y) \, d\gamma(x, y) = \int_{\mathbb{R}^d} \varphi(y) \, d\nu_2(y) \]
for all continuous functions \( \varphi: \mathbb{R}^d \to \mathbb{R}^1 \) with a compact support. Note that if there exists a map \( \tau: \mathbb{R}^d \to \mathbb{R}^d \) that pushes forward \( \nu_1 \) to \( \nu_2 \), then the corresponding transportation \( \gamma_{\tau} \) is defined as follows:
\[ \int \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x, y) \, d\gamma_{\tau}(x, y) = \int_{\mathbb{R}^d} \psi(x, \tau(x)) \, d\nu_1(x) \]
for an arbitrary continuous function \( \psi: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^1 \) with a compact support.

The better \( \gamma \) is concentrated near the diagonal of \( \mathbb{R}^d \times \mathbb{R}^d \), the closer the measures \( \nu_1 \) and \( \nu_2 \) must be to each other. We shall measure such a concentration in the \( L^\infty \)-norm and set
\[ \text{Tra}(\nu_1, \nu_2) = \inf_{\gamma} ||x - y||_{L^\infty(\gamma)} = \inf_{\gamma} \sup \{|x - y|: x, y \in \text{spt}(\gamma)\} \in [0, \infty], \]
where the infimum is taken over all transportations \( \gamma \), and ‘spt’ denotes the closed support. Clearly, \( \text{Tra}(\nu_1, \nu_2) + \text{Tra}(\nu_2, \nu_3) \geq \text{Tra}(\nu_1, \nu_3) \). By \( \text{Tra}(\nu) = \text{Tra}(\nu, m_d) \) we denote the transportation distance between the measure \( \nu \) and the Lebesgue measure \( m_d \). If \( \nu = \nu_X \) with a discrete set \( X \in \mathbb{R}^d \), then
\[ \text{const} \cdot \text{Tra}(\nu) \leq \inf_{S} \sup_{x \in \mathbb{Z}^d} |S(x) - x| \leq \text{Const} \cdot \text{Tra}(\nu) \]
where the infimum is taken over all bijections \( S: \mathbb{Z}^d \to X \). This follows, for instance, from the locally finite marriage lemma discussed two paragraphs below. Throughout the paper, ‘Const’ and ‘const’ mean positive constants that depend only on the dimension \( d \). The values of these constants can be changed at each occurrence.

There exists a dual definition of the transportation distance \( \text{Tra}(\nu_1, \nu_2) \). The distance \( \text{Di}(\nu_1, \nu_2) \) is defined as the infimum of \( r \in (0, \infty) \) such that
\[ \nu_1(B) \leq \nu_2(B_{+r}), \quad \text{and} \quad \nu_2(B) \leq \nu_1(B_{+r}), \quad (1.1) \]
for each bounded Borel set \( B \subset \mathbb{R}^d \). Here, \( B_{+r} \) is the closed \( r \)-neighbourhood of \( B \) (actually, for our purposes, we could take open neighbourhoods as well). The distance \( \text{Di} \) ranges from 0 to \( +\infty \), the both ends are included. We define the discrepancy of the measure \( \nu \) as \( D(\nu) = \text{Di}(\nu, m_d) \). The following duality is classical.

**Theorem 1.2.** \( \text{Tra}(\nu_1, \nu_2) = \text{Di}(\nu_1, \nu_2) \). In particular, \( \text{Tra}(\nu) = D(\nu) \).
For finite measures $\nu_1$ and $\nu_2$, it follows from a result of Strassen [10, Theorem 11] and Sudakov [11]. If the measures $\nu_1$ and $\nu_2$ are counting measures of discrete sets $X_1$ and $X_2$, then it follows from a locally finite version of the marriage lemma due to M. Hall and R. Rado, see Laczkovich [8]. Note that the locally finite marriage lemma asserts existence of a bijection between the sets $X_1$ and $X_2$ which is more than a transportation from $\nu_1$ to $\nu_2$. Theorem 1.2 is also mentioned in Gromov [3, Section 3.1], though the exposition there is quite sketchy. For the reader’s convenience, we recall the proof in Appendix.

A different idea of connecting the measures $\nu$ and $m_d$ comes from the potential theory. We say that a locally integrable vector field $v$ connects the measures $\nu$ and $m_d$ if $\text{div} \, v = \nu - m_d$ (in the weak sense), that is
\[
\int_{\mathbb{R}^d} \langle v(x), \nabla \varphi(x) \rangle \, dm_d(x) = -\int_{\mathbb{R}^d} \varphi(x) \, d(\nu - m_d)(x)
\]
for all smooth compactly supported functions $\varphi : \mathbb{R}^d \to \mathbb{R}^1$. It is easy to see that such a field always exists. For instance, we can take $v = \nabla h$, where $h$ is a solution to the Poisson equation $\Delta h = \nu - m_d$ in $\mathbb{R}^d$ which exists due to a subharmonic version of Weierstrass’ representation theorem [4, Theorem 4.1].

Let $B(x; r)$ be a ball of radius $r$ centered at $x$, and $rB = B(0; r)$. Set $\chi_r = \frac{1}{m_d(rB)} I_{rB}$ where $I_{rB}$ is the indicator function of the ball $rB$. We measure the size of the field $v$ as follows.

**Definition 1.3.** For a locally integrable vector field $v$ on $\mathbb{R}^d$, we set
\[
\text{Ra}(v) = \inf_{r > 0} \left\{ r + \| v \ast \chi_r \|_{\infty} \right\} \quad \text{and} \quad \tilde{\text{Ra}}(v) = \inf_{r > 0} \left\{ r + \| |v| \ast \chi_r \|_{\infty} \right\},
\]
where $\ast$ denotes the convolution.

Evidently, $\text{Ra}(v) \leq \tilde{\text{Ra}}(v) \leq \|v\|_{\infty}$. Note that the multiplicative group $\mathbb{R}_+$ acts by scaling on the measures and vector fields: $\nu_t(B) = \nu(tB)$, $\nu_t(x) = t^{-1} v(tx)$. These actions are ‘coordinated’: if $\text{div} \, v = \nu - m_d$, then $\text{div} \, v_t = \nu_t - m_d$, and they are respected by our definitions of $\text{Tra}$, $\text{Ra}$ and $\tilde{\text{Ra}}$: $\text{Tra}(\nu_t) = t^{-1} \text{Tra}(\nu)$, $\text{Ra}(\nu_t) = t^{-1} \text{Ra}(v)$ and $\tilde{\text{Ra}}(\nu_t) = t^{-1} \tilde{\text{Ra}}(v)$.

**Theorem 1.4.** Let $\nu$ be a non-negative locally finite measure on $\mathbb{R}^d$. Then
\[
\text{const} \cdot \inf_v \tilde{\text{Ra}}(v) \leq \text{Tra}(\nu) \leq \text{Const} \cdot \inf_v \text{Ra}(v),
\]
where the infimum is taken over all vector fields $v$ connecting the measures $\nu$ and $m_d$.  

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This is the main result of this note. In the proof of the upper bound we use duality and actually prove that $D(\nu) \leq \text{Const} \cdot \text{Ra}(\nu)$. For this reason, our technique gives no idea how transportations $\gamma$ may look like in the case when $\text{Tra}(\nu)$ is finite.

**Corollary 1.5.** Let $u$ be a $C^2$-function on $\mathbb{R}^d$ such that $\Delta u = \nu - m_d$. Then

$$\text{Tra}(\nu) \leq \text{Const} \sqrt{\|u\|_{\infty}}.$$ 

One can juxtapose this corollary with classical discrepancy estimates due to Erdős and Turán and Ganelius. In [1] Ganelius proved that if $\nu$ is a probability measure on the unit circumference $\mathbb{T} \subset \mathbb{C}$, and $m$ is the normalized Lebesgue measure on $\mathbb{T}$, then

$$\sup_I |\nu(I) - m(I)| \leq \text{Const} \sqrt{\sup_\mathbb{T} U^\nu},$$

where the supremum is taken over all arcs $I \subset \mathbb{T}$, and

$$U^\nu(z) = \int \log |z - \zeta| d\nu(\zeta)$$

is the logarithmic potential of the measure $\nu$. Since $U^m$ vanishes on $\mathbb{T}$, we can rewrite this as

$$\sup_I |\nu(I) - m(I)| \leq \text{Const} \sqrt{\sup_\mathbb{T} U^\nu - m}.$$ 

Note the supremum on the right-hand side, not the supremum of the absolute value as in our result.

**Proof of Corollary 1.5.** Consider the convolution $u_r = u * \chi_r$. We have

$$\nabla u_r = u * \nabla \chi_r, \quad \text{and} \quad \Delta u_r = \text{div} \nabla u_r = \nu * \chi_r - m_d.$$ 

Noting that $\nabla \chi_r$ is a finite vector measure of total variation

$$\|\nabla \chi_r\|_1 = \|\nabla \chi_1\|_1 \cdot r^{-1} = \text{Const} \cdot r^{-1},$$

we have

$$\text{Ra}(\nabla u_r) \leq \|\nabla u_r\|_\infty \leq \|u\|_\infty \cdot \|\nabla \chi_r\|_1 = \frac{\text{Const}}{r} \cdot \|u\|_\infty,$$

and

$$\text{Tra}(\nu) \leq \text{Tra}(\nu * \chi_r) + \text{Tra}(\nu * \chi_r) \leq r + \text{Const} \cdot \text{Ra}(\nabla u_r) \leq r + \frac{\text{Const}}{r} \cdot \|u\|_\infty.$$ 

Choosing $r = \sqrt{\|u\|_\infty}$, we get the result. \hfill $\Box$
This corollary immediately yields a seemingly more general result (cf. [9, Theorem 4.3]).

**Corollary 1.6.** Let \( u \) be a locally integrable function in \( \mathbb{R}^d \) such that \( \Delta u = \nu - m_d \) weakly. Then

\[
\text{Tra}(\nu) \leq \text{Const} \cdot \inf_{r>0} \{ r + \sqrt{\|u \ast \chi_r\|_{\infty}} \}.
\]

(1.7)

**Proof of Corollary 1.6.** Denote by \( \tilde{\chi}_r \) the 3-rd convolution power of \( \chi_r \) and put \( u_r = u \ast \tilde{\chi}_r \). Then \( u_r \) is a \( C^2 \)-function and \( \Delta u_r = \nu \ast \tilde{\chi}_r - m_d \). Since the function \( \tilde{\chi}_r \) is supported by the ball \( 3r \mathbb{B} \), we have \( \text{Tra}(\nu) \leq 3r + \text{Tra}(\nu \ast \tilde{\chi}_r) \).

Corollary 1.5 applied to the smoothed potential \( u_r \) yields \( \text{Tra}(\nu \ast \tilde{\chi}_r) \leq \text{Const} \sqrt{\|u \ast \chi_r\|_{\infty}} \). At last, note that \( \|u_r\|_{\infty} \leq \|u \ast \chi_r\|_{\infty} \cdot \|\chi_r \ast \chi_r\|_1 = \|u \ast \chi_r\|_{\infty} \) completing the argument. \( \square \)

## 2 Proof of Theorem 1.4

### 2.1 The lower bound

Here, we construct a vector field \( v \) that connects the measure \( \nu \) with the Lebesgue measure \( m_d \) and such that \( \text{Ra}(v) \leq \text{Const} \cdot \text{Tra}(\nu) \).

Let \( r > \text{Tra}(\nu) \). For any \( x, y \in \mathbb{R}^d \) such that \( |x - y| \leq r \), there exists a vector field \( v_{x,y} \) concentrated on the ball \( \mathbb{B}(\frac{x+y}{2}; r) \) such that \( \text{div} v_{x,y} = \delta_x - \delta_y \) (as usual, \( \delta_x \) is a point measure at \( x \) of the unit mass), and

\[
\int_{\mathbb{R}^d} |v_{x,y}(\xi)| \, dm_d(\xi) \leq \text{Const} \cdot r.
\]

(In order to see that such a field \( v \) exists, first, consider a special case \( r = 1 \); then the general case follows by rescaling.)

Now, we take

\[
v = \int_{\mathbb{R}^d \times \mathbb{R}^d} v_{x,y} \, d\gamma(x, y)
\]

where the transportation \( \gamma \) connects the measures \( \nu \) and \( m_d \), and is concentrated on the set \( \{(x, y) : |x - y| \leq r\} \). Then

\[
\text{div} v = \int_{\mathbb{R}^d \times \mathbb{R}^d} (\delta_x - \delta_y) \, d\gamma(x, y) = \nu - m_d,
\]

and for every \( z \in \mathbb{R}^d \)

\[
\int_{\mathbb{B}(z;r)} |v(\xi)| \, dm_d(\xi) \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} d\gamma(x, y) \int_{\mathbb{B}(z;r)} |v_{x,y}(\xi)| \, dm_d(\xi) \leq \text{Const} \cdot r \cdot \int \, d\gamma(x, y),
\]

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where the latter integral is taken over such \((x, y)\) that \(B\left(\frac{x+y}{2}; r\right) \cap B(z; r) \neq \emptyset\), which implies \(|y - z| \leq \frac{5}{2} r\). Thus,

\[
\int_{B(z; r)} |v(\xi)| \, dm_d(\xi) \leq \text{Const} \cdot r \cdot \int_{\mathbb{R}^d} \mathbb{1}_{B(z; 5r/2)}(y) \, d\gamma(x, y)
\]

\[
= \text{Const} \cdot r \cdot \int_{B(z; 5r/2)} dm_d(y) \leq \text{Const} \cdot r^{d+1},
\]

that is, \(\widetilde{R}(v) \leq \text{Const} \cdot r\), q.e.d.

Note that in the argument given above, the Lebesgue measure \(m_d\) can be replaced with any measure \(\mu\) satisfying \(\mu \leq \text{Const} \cdot m_d\). The other inequality \(\text{Tra}(\nu) \leq \text{Const} \cdot \text{Ra}(v)\) does not permit such a replacement. Indeed, if \(\eta_x\) is a normalized volume within the unit ball centered at \(x\), then for \(|x - y| \geq 2\) we have \(\text{Tra}(\eta_x, \eta_y) \geq \text{const} \cdot |x - y|\), whereas it is easy to construct a vector field \(v\) connecting the measures \(\eta_x\) and \(\eta_y\) with \(||v||_\infty \leq \text{Const}\). Just take \(v = (\nabla E) \ast (\eta_x - \eta_y)\), \(E\) being a fundamental solution for the Laplacian in \(\mathbb{R}^d\).

### 2.2 The upper bound

In what follows, by a unit cube we mean \(Q = \prod_{i=1}^d [a_i, a_i + 1], \ a_i \in \mathbb{Z}, \ 1 \leq i \leq d\). The proof of the upper bound relies on the following.

**Lemma 2.1 (Laczkovich).** Suppose that for any set \(U \subset \mathbb{R}^d\) which is a finite union of the unit cubes, we have

\[
|\nu(U) - m_d(U)| \leq \rho m_{d-1}(\partial U) \tag{2.2}
\]

with \(\rho \geq 1\). Then \(D(\nu) \leq \text{Const} \cdot \rho\).

In [8], Laczkovich proved this lemma for the counting measure \(\nu_X\) of a discrete set \(X \subset \mathbb{R}^d\). For the reader’s convenience, will recall the proof of this lemma in [A-2]

Now, the upper bound in Theorem 1.4 will readily follow from the divergence theorem. We need to show that \(D(\nu) \leq \text{Const} \cdot \text{Ra}(v)\). A simple scaling argument shows that it suffices to consider only the case \(\text{Ra}(v) = 1\). Then there exists \(r \leq 2\) such that \(\|v \ast \chi_r\|_\infty \leq 2\). Note that \(\text{div}(v \ast \chi_r) = v \ast \chi_r + m_d\).

let \(U \subset \mathbb{R}^d\) be a finite union of the unit cubes. Then denoting by \(n\) the
outward unit normal to $U$, we have
\[
\| (\nu \ast \chi_r)(U) - m_d(U) \| = \left| \int_U \text{div}(v \ast \chi_r) \, dm_d \right|
\]
\[
= \left| \int_{\partial U} \langle v \ast \chi_r, n \rangle \, dm_{d-1} \right| \leq \| v \ast \chi_r \|_{\infty} m_{d-1}(\partial U) \leq 2m_{d-1}(\partial U),
\]
whence, by Laczkovich’s lemma, $D(\nu \ast \chi_r) \leq \text{Const}$, and finally, $D(\nu) \leq r + D(\nu \ast \chi_r) \leq \text{Const}$.

Appendix

A-1 Transportation supported by a given set

Here, we shall prove a somewhat more general result than Theorem 1.2. Let $F \subset \mathbb{R}^d \times \mathbb{R}^d$ be a closed symmetric set such that $F \cap (\mathbb{R}^d \times B)$ is bounded whenever $B$ is bounded. (A-1.1)

For $U \subset \mathbb{R}^d$, set $U + F = \{ x \in \mathbb{R}^d : \exists y \in U \ (x, y) \in F \}$. If $C \subset \mathbb{R}^d$ is a compact set, then the set $C + F$ is compact as well.

**Definition A-1.2.**

(i) $\text{Tra}(F)$ is a set of all pairs $(\nu_1, \nu_2)$ of locally finite positive measures $\nu_1, \nu_2$ on $\mathbb{R}^d$ such that there exists a transportation $\gamma$ with $\text{spt}(\gamma) \subset F$.

(ii) $\text{Di}(F)$ is a set of all pairs $(\nu_1, \nu_2)$ of locally finite positive measures $\nu_1, \nu_2$ on $\mathbb{R}^d$ such that

$$\nu_1(C) \leq \nu_2(C + F) \quad \text{and} \quad \nu_2(C) \leq \nu_1(C + F)$$

for any compact subset $C \subset \mathbb{R}^d$.

**Theorem A-1.3.** For any closed symmetric set $F \subset \mathbb{R}^d \times \mathbb{R}^d$ satisfying (A-1.1), $\text{Tra}(F) = \text{Di}(F)$.

See also Kellerer [6, Corollary 2.18 and Proposition 3.3] for a wide class of non-closed sets $F$.

Theorem 1.2 follows immediately from Theorem A-1.3: just take a closed symmetric set $F_r = \{ (x, y) \in \mathbb{R}^d \times \mathbb{R}^d : |x - y| \leq r \}$. Then

$$(\nu_1, \nu_2) \in \text{Tra}(F_r) \iff \text{Tra}(\nu_1, \nu_2) \leq r$$

and

$$(\nu_1, \nu_2) \in \text{Di}(F_r) \iff \text{Di}(\nu_1, \nu_2) \leq r.$$
Proof of Theorem [A-1.3] The inclusion $\text{Tra}(F) \subset \text{Di}(F)$ is rather obvious:

$$\nu_1(C) = \gamma(C \times \mathbb{R}^d) = \gamma(C \times C_{+F}) \leq \gamma(\mathbb{R}^d \times C_{+F}) = \nu_2(C_{+F}),$$

and the same for the other inequality.

The proof of the opposite inclusion $\text{Di}(F) \subset \text{Tra}(F)$ is based on duality. Consider a linear space $C_0(\mathbb{R}^d)$ of continuous functions with compact support in $\mathbb{R}^d$ endowed with standard convergence: $f_n \to f$ in $C_0(\mathbb{R}^d)$ if there is a ball $B$ such that $\text{spt}(f_n) \subset B$ for all $n$, and the sequence $f_n$ converges uniformly to $f$. The dual space of continuous linear functionals $M(\mathbb{R}^d)$ consists of signed measures of locally finite variation on $\mathbb{R}^d$ with a usual pairing $\nu(f) = \int f \, d\nu$. If a linear functional $\nu$ on $C_0(\mathbb{R}^d)$ is positive (e.g. $\nu(f) \geq 0$ whenever the function $f$ is non-negative), then it is continuous and is represented by a non-negative locally finite measure. The same facts are true for the linear space $C_0(F)$ of continuous functions with a compact support in $F$, and its dual space $M(F)$.

Consider a mapping $\pi: M(F) \to M(\mathbb{R}^d) \oplus M(\mathbb{R}^d)$ acting as $\pi \gamma = (\nu_1, \nu_2)$, where $\nu_1$ and $\nu_2$ are the marginals of the measure $\gamma$. The mapping $\pi$ is well-defined due to our assumption (A-1.1). The conjugate mapping $\pi': C_0(\mathbb{R}^d) \oplus C_0(\mathbb{R}^d) \to C(F)$ is $\pi'(f,g)(x, y) = f(x) + g(y)$ for $(x, y) \in F$. Assume, that $(\nu_1, \nu_2) \in \text{Di}(F)$. We need to show that the pair $(\nu_1, \nu_2)$ belongs to the image of the cone of positive measures $M_+(F)$ under $\pi$; in other words, that there exists $\gamma \in M_+(F)$ such that

$$\gamma(\pi'(f, g)) = (\nu_1, \nu_2)(f, g) = \int f \, d\nu_1 + \int g \, d\nu_2. \quad (A-1.4)$$

We shall check below that condition $(\nu_1, \nu_2) \in \text{Di}(F)$ ensures that the RHS of $(A-1.4)$ defines a positive linear functional on a linear subspace $L = \pi'(C_0(\mathbb{R}^d) \times C_0(\mathbb{R}^d))$ of $C_0(F)$. The linear space $C_0(F)$ is subordinated to its linear subspace $L$; i.e. for any $\varphi \in C_0(F)$ there are functions $f, g$ in $C_0(\mathbb{R}^d)$ such that

$$|\varphi(x, y)| \leq f(x) + g(y), \quad (x, y) \in F.$$

Then by the classical M. Riesz’ extension theorem (see e.g. [2, Chapter II, §6, Theorem 3]) we can extend this linear functional to a positive linear functional on the whole space $C_0(F)$.

It remains to check that the linear functional is well-defined and positive. Assume that it does not hold; i.e. there is a pair of functions $f, g \in C_0(\mathbb{R}^d)$ such that

$$f(x) + g(y) \geq 0, \quad (x, y) \in F,$$
however,
\[ \int f\,d\nu_1 + \int g\,d\nu_2 < 0. \]
Replacing \( g \) by \( -g \), we get a pair of functions such that
\[ f(x) \geq g(y), \quad (x, y) \in F, \] (A-1.5)
and
\[ \int f\,d\nu_1 < \int g\,d\nu_2. \] (A-1.6)
Then, by virtue of (A-1.5),
\[ \{ y : g(y) \geq t \}_+ \subset \{ x : f(x) \geq t \}, \]
\[ \{ x : f(x) \leq t \}_+ \subset \{ y : g(y) \leq t \}. \]
Using, at last, condition \((\nu_1, \nu_2) \in \text{Di}(F)\), we get
\[ \nu_2(\{ y : g(y) \geq t \}) \leq \nu_1(\{ x : f(x) \geq t \}), \quad t > 0, \]
\[ \nu_2(\{ y : g(y) \leq t \}) \geq \nu_1(\{ x : f(x) \leq t \}), \quad t < 0. \]
Then
\[ \int g\,d\nu_2 \leq \int f\,d\nu_1 \]
which contradicts (A-1.6) and completes the proof of the theorem.

### A-2 Proof of lemma of Laczkovich

We check that, for any bounded Borel set \( V \subset \mathbb{R}^d \),
\[ \nu(V) \leq m_d(V+C_\rho), \] (A-2.1)
\[ m_d(V) \leq \nu(V+C_\rho). \] (A-2.2)

Take \( M = [2\rho d] + 1 \) and denote by \( Q_M \) the collection of all cubes of edge length \( M \),
\[ Q = \prod_{i=1}^d [a_iM, (a_i + 1)M] \]
with \( a_i \in \mathbb{Z}, 1 \leq i \leq d \). Given a bounded Borel set \( V \), consider the cubes \( Q_1, \ldots, Q_n \) from \( Q_M \) that intersect the set \( V \), and denote by \( Q'_i = 3Q_i \) the cube concentric with \( Q_i \) of thrice bigger size, \( 1 \leq i \leq n \). Set
\[ A = \bigcup_{i=1}^n Q_i, \quad B = \bigcup_{i=1}^n Q'_i. \]
We’ll need a simple geometric claim.
Claim A-2.3.

\[ m_{d-1}(\partial A) \leq \frac{2d}{M} m_d(B \setminus A), \quad m_{d-1}(\partial B) \leq \frac{2d}{M} m_d(B \setminus A). \]

Proof of Claim A-2.3. First, we consider the boundary of the set \( A \): \( \partial A = \bigcup_{j=1}^{r} F_j \) where \( F_j \) is a face of some cube \( Q_{ij} \). By \( P_j \) we denote the cube obtained by reflection of \( Q_{ij} \) in \( F_j \); clearly, for all \( j \), \( P_j \subset B \setminus A \). Each cube can be listed at most 2d times in the list of cubes \( P_1, \ldots, P_r \) (since every \( P_j \) cannot have more than 2d neighbours among the cubes \( Q_1, \ldots, Q_n \)). Thus,

\[ 2dm_d(B \setminus A) \geq \sum_{j=1}^{r} m_d(P_j) = rM^d = M \cdot rM^{d-1} = Mm_{d-1}(\partial A). \]

This gives us the first inequality. To estimate \( m_{d-1}(\partial B) \), we note that \( B \setminus A = \bigcup_{j=1}^{s} R_j \) where \( R_1, \ldots, R_s \) are different cubes from the collection \( Q_M \), and that \( \partial B \subset \bigcup_{j=1}^{s} \partial R_j \). Whence,

\[ m_{d-1}(\partial B) \leq \sum_{j=1}^{s} m_{d-1}(\partial R_j) \leq s \cdot 2dM^{d-1} = \frac{2d}{M} sM^d = \frac{2d}{M} m_d(B \setminus A) \]

proving the claim.

Now, we readily finish the proof of the lemma. We choose a constant \( C \) (depending on the dimension \( d \)) so big that \( B \subset V_{+C\rho} \). Then

\[ \nu(V) \leq \nu(A) \leq m_d(A) + \rho m_{d-1}(\partial A) \leq m_d(A) + \frac{2d\rho}{M} m_d(B \setminus A) \leq m_d(B) \leq m_d(V_{+C\rho}), \]

whence (A-2.1); and

\[ \nu(V_{+C\rho}) \geq \nu(B) \geq m_d(B) - \rho m_{d-1}(\partial B) \geq m_d(B) - \frac{2d\rho}{M} m_d(B \setminus A) \geq m_d(B) - m_d(B \setminus A) = m_d(A) \geq m_d(V), \]

whence (A-2.2).

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