ABOUT REMAINDERS IN COMPACTIFICATIONS OF
PARATOPOLOGICAL GROUPS

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Abstract. In this paper, we prove a dichotomy theorem for remainders in compactifications of paratopological groups: every remainder of a paratopological group $G$ is either Lindelöf and meager or Baire. Moreover, we give a negative answer for a question posed by D. Basile and A. Bella in [6], and some questions about remainders of paratopological groups are posed in the paper.

1. Introduction

By a remainder of a space $X$ we understand the subspace $bX \setminus X$ of a Hausdorff compactification $bX$ of $X$. Remainders in compactifications of topological spaces have been studied by some topologists in the last few years. A famous classical result in this study is the following theorem of M. Henriksen and J. Isbell [9]:

(M. Henriksen and J. Isbell) A space $X$ is of countable type if and only if the remainder in any (in some) compactification of $X$ is Lindelöf.

Since topological groups are much more sensitive to the properties of their remainders than topological spaces in general, topologists are mainly interesting in the remainders of topological groups or paratopological groups. For instance, Arhangel’skiĭ has recently proved the following two dichotomy theorems about remainders in compactifications of topological groups:

Theorem 1.1. [1] If $G$ is a topological group, and some remainder of $G$ is not pseudocompact, then every remainder of $G$ is Lindelöf.

Theorem 1.2. [2] Suppose that $G$ is a non-locally compact topological group. Then either every remainder of $G$ has the Baire property, or every remainder of $G$ is $\sigma$-compact.

Moreover, D. Basile and A. Bella have just shown a dichotomy theorem for homogeneous spaces:

Theorem 1.3. [6] The remainder of a homogeneous space is either Baire or meager and realcompact.

D. Basile and A. Bella posed the following question.

Question 1.1. [6] Let $X$ be a homogeneous space and let $bX$ be a compactification of $X$. Is it true that the remainder $bX \setminus X$ is either pseudocompact or realcompact and meager?

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In [6], D. Basile and A. Bella has shown that none of the Arhangel’skii’s dichotomy theorems can be generalized to the case of homogeneous spaces. In [4], Arhangel’skii has given an example to show that the Theorem 1.1 can not be generalized to the case of paratopological groups. Naturally, the following two questions arise:

**Question 1.2.** How about the dichotomy theorem of the remainders in compactifications of paratopological groups?

**Question 1.3.** Can the dichotomy theorem 1.2 be generalized to the case of paratopological groups?

In this paper, we show that, for a paratopological group $G$, every remainder of $G$ is either Lindelöf and meager or Baire, which give an answer for Question 1.2. Also, we give a partial answer for the Question 1.3. Finally, we give a negative answer for the Question 1.1.

2. **Preliminaries**

Recall that a **topological group** $G$ is a group $G$ with a topology such that the product map of $G \times G$ into $G$ associating $xy$ with arbitrary $(x, y) \in G \times G$ is jointly continuous and the inverse map of $G$ onto itself associating $x^{-1}$ with arbitrary $x \in G$ is continuous. A **paratopological group** $G$ is a group $G$ with a topology such that the product map of $G \times G$ into $G$ is jointly continuous. A **semitopological group** $G$ is a group $G$ with a topology such that the product map of $G \times G$ into $G$ is separately continuous. A **quasitopological group** $G$ is a group $G$ with a topology such that it is a semitopological group and the inverse map of $G$ onto itself is continuous.

Recalled that a space is **Baire** if the intersection of a sequence of open and dense subsets is dense. Moreover, a space is called **meager** if it can be represented as the union of a sequence of nowhere dense subsets. Let us call a map $f$ of a space $X$ into a space $Y$ $k$-gentle [4] if for every compact subset of $G$ of countable character in $G$, then $f$ is compact. A semitopological group $G$ will be called $k$-gentle [4] if the inverse map $(x) \mapsto x^{-1}, \forall x \in G$ is $k$-gentle.

A family $A$ of open subsets of a space $X$ is called a **base of $X$ at a set $A$** if $A = \cap A$ and for any neighborhood $U$ of $A$, there is a $V \in A$ such that $A \subset V \subset U$. If $A$ is countable, then we say that $A$ has countable character in $X$. A space $X$ is of **countable type** [8] if every compact subspace $F$ of $X$ is contained in a compact subspace $K \subset X$ with a countable base of open neighborhoods in $X$.

Throughout this paper, all spaces are assumed to be Tychonoff. Denote positively natural number by $N$. We refer the reader to [3, 8] for notations and terminology not explicitly given here.

3. **Remainders of paratopological groups**

Firstly, we give a lemma.

**Lemma 3.1.** [4] Let $G$ be a paratopological group. If there exists a non-empty compact subset of $G$ of countable character in $G$, then $G$ is of countable type.

Now, we give a dichotomy theorem of the remainders in compactifications of paratopological groups.
Theorem 3.1. Let G be a non-locally compact paratopological group. Then either every remainder of G has the Baire property, or every remainder of G is meager and Lindelöf.

Proof. Suppose that $bG$ is a compactification of G such that the remainder $Y = bG \setminus G$ does not have the Baire property. Next, we shall prove that Y is Lindelöf and meager.

Since Y does not have the Baire property, there exists a countable family $\{U_n : n \in \mathbb{N}\}$ of open subsets of Y such that $\bigcap \{U_n : n \in \mathbb{N}\}$ is not dense in Y. Because G is nowhere locally compact, Y is dense in $bG$. For each $n \in \mathbb{N}$, there exists an open subset $V_n$ of $bG$ such that $U_n = V_n \cap Y$. Therefore, we can find a non-empty open subset $U$ of $bG$ such that $(\bigcap_{\gamma} (U \cap G))$ is not dense in $U \cap G$. By Theorem 3.9.6, it is known that every Čech-complete space is of countable type. Since Z is Čech-complete, there exists a non-empty compact subset $F$ of countable character in Z. Because $U \cap G$ is open in G, $F$ is of countable character in G. Obviously, $F$ is compact in G. Therefore, G is of countable type by Lemma 3.1. Therefore, Y is Lindelöf by M. Henriksen and J. Isbell theorem. Moreover, Y is meager by Theorem 1.3. This complete the proof.

Remark Observe that a remainder Y of a non-locally compact paratopological group G cannot have the Baire property, be Lindelöf and meager at the same time. Indeed, it is easy to see that the failure of the Baire property is equivalent to the existence of some non-empty open meager subset. Thus we have the following two corollaries.

Corollary 3.1. Let X be a neither Baire nor meager space. Then X cannot be a remainder in compactifications of any paratopological group.

Corollary 3.2. Let X be a neither Baire nor Lindelöf space. Then X cannot be a remainder in compactifications of any paratopological group.

Remark D. Basile and A. Bella has shown that there exists a homogeneous space such that the remainder of some compactification is neither Baire nor Lindelöf, see [6, Example 3.3]. Hence Theorem 3.1 can not be generalized to the case of homogeneous spaces. However, we have the following question.

Question 3.1. Let X be a non-locally compact semitopological group or quasitopological group, and let $bX$ be a compactification of X. Is it true that the remainder $bX \setminus X$ has the Baire property or is Lindelöf and meager?

Next, we obtain two corollaries from Theorem 3.1. Firstly, we show that the Arhangel’skii’s dichotomy Theorems 1.2 can be generalized to the case of $k$-gentle paratopological groups, which give a partial answer for Question 1.3.

Lemma 3.2. Let G be a $k$-gentle paratopological group such that some remainder of G is Lindelöf. Then G is a topological group.

Corollary 3.3. Let G be a non-locally compact $k$-gentle paratopological group. Then either every remainder of G has the Baire property, or every remainder of G is $\sigma$-compact.
Proof. Suppose that $bG$ is a compactification of $G$, and put $Y = bG \setminus G$. By Theorem 3.1, $Y$ has the Baire property, or is meager and Lindelöf. Suppose that $Y$ does not have the Baire property. Then $Y$ is Lindelöf, and hence $G$ is a topological group by Lemma 3.2. Then $Y$ is $\sigma$-compact by Theorem 1.2. □

It follows from [1] that a remainder in some compactification of a topological group is metacompact iff it is Lindelöf iff it is realcompact. Therefore, we have the following question.

**Question 3.2.** Assume that $G$ is a non-locally compact paratopological group, and put $Y = bG \setminus G$. Are the following conditions equivalent?

1. $Y$ is metacompact;
2. $Y$ is Lindelöf;
3. $Y$ is realcompact.

A space $X$ is called **metacompact** if each open covering of $X$ can be refined by a point-finite open covering. A space $X$ is called **ccc** if every disjoint family of open subsets of $X$ is countable.

**Lemma 3.3.** [7] Every point-finite open collection in a ccc Baire space is countable.

The next corollary gives a partial answer for the Question 3.2.

**Corollary 3.4.** Assume that $G$ is a non-locally compact paratopological group, and put $Y = bG \setminus G$. If $Y$ is metacompact and ccc, then $Y$ is Lindelöf.

**Proof.** By Theorem 3.1, $Y$ has the Baire property, or is meager and Lindelöf. Suppose that $Y$ has the Baire property. Then $Y$ is Lindelöf by Lemma 3.3. Hence $Y$ is Lindelöf. □

Now, we shall give a negative answer for Question 1.1 by Example 3.1.

**Example 3.1.** There exists a paratopological group $X$ such that some compactification $bX$ of $X$ has a remainder which is neither pseudocompact nor meager.

**Proof.** Let $Z = X \cup Y$ be the two-arrows space of P. S. Alexandroff and P. S. Urysohn [8, Exercise 3.10. C], where $X = \{(x, 0) : 0 < x \leq 1\}$ and $Y = \{(x, 1) : 0 \leq x < 1\}$. The space $X$ is the arrow space which is homeomorphic to the Sorgenfrey line, see [8, Example 1.2.2]. $Z$ is a Hausdorff compactification of Sorgenfrey line $X$, and its remainder $Y$ is still a copy of Sorgenfrey line. Moreover, there exists a natural structure of an Abelian group on $Y$ such that the multiplication $(u, v) \mapsto u \cdot v$ is continuous, that is, the space $Y$ admits a structure of a paratopological group. For example, if $u = (x, 1)$ and $v = (y, 1)$ are two points in $Y$, then $u \cdot v = (x + y, 1)$ if $x + y < 1$, and $u \cdot v = (x + y - 1, 1)$ if $x + y \geq 1$. However, Sorgenfrey line is non-pseudocompact; otherwise, Sorgenfrey line is a compact space since it is a Lindelöf space, which is a contradiction. Moreover, since $X$ has the Baire property [8], $X$ is non-meager. Therefore, $Y$ is neither pseudocompact nor meager. □

**Remark** It follows from Example 3.1 that, in Question 1.1, the answer is also negative if we replace the “homogeneous space” by “paratopological group”.

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