Predictive resource allocation for flexible loads with local QoS

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Abstract—Loads that can vary their power consumption without violating their Quality of service (QoS), that is flexible loads, are an invaluable resource for grid operators. Utilizing flexible loads as a resource requires the grid operator to incorporate them into a resource allocation problem. Since flexible loads are often consumers, for concerns of privacy it is desirable for this problem to have a distributed implementation. Technically, this distributed implementation manifests itself as a time varying convex optimization problem constrained by the QoS of each load. In the literature, a time invariant form of this problem without all of the necessary QoS metrics for the flexible loads is often considered. Moving to a more realistic setup introduces additional technical challenges, due to the problems’ time-varying nature. In this work, we develop an algorithm to account for the challenges introduced when considering a time varying setup with appropriate QoS metrics.

I. INTRODUCTION

Relying more and more on renewable generation is the envisioned future for the power grid. However, this goal is not without its challenges; renewable sources, such as solar and wind, are highly volatile. Moreover, supply and demand of power must always be in equilibrium, and when renewable generation cannot ensure this, controllable generation sources must ramp to ensure equilibrium. Economically, for a Balancing Authority (BA) (the institution responsible for ensuring supply and demand are balanced in a given geographical area), ramping generators or utilizing batteries for this is not feasible. This has motivated the recent investigation of a new resource to help where conventional generators and batteries fall short: flexible loads.

Flexible loads can deviate from a baseline level of consumption without violating the Quality of Service (QoS) of the load. From the perspective of the BA, flexible loads deviating from baseline are identical to a battery discharging and charging. Due to this, flexible loads are often said to provide “Virtual Energy Storage” (VES) [1]. More importantly, grid support from flexible loads is more cost effective than batteries [2]. Some examples of flexible loads include residential air conditioners [3], water heaters [4], refrigerators [5], commercial HVAC systems [6], and pumps for irrigation [7] and pool cleaning [8].

To utilize flexible loads, the BA in some way must incorporate them into a resource allocation problem. In a centralized framework, the resource allocation problem involves a central authority accounting for all of its resources and their constraints, and then allocating its needs to each resource based on the constraints. The problem is typically solved for a specific future duration. For instance, the BA allocates its resources for the next day [9].

In contrast to the BA solving a centralized resource allocation problem, it is possible to decentralize and have each flexible load solve a portion of the centralized problem. Furthermore, this distributed algorithm can run in real time. The advantage of a distributed solution is that (i) privacy is protected, as each load only needs to know its own QoS and (ii) the solution is more robust to modeling error as no one entity is making decisions for the ensemble based on models of the ensemble; each member of the ensemble makes decisions for itself based on a combination of its local and global information.

Solving the resource allocation problem in a distributed fashion and real time falls under the framework of time-varying optimization. There are two main challenges in this framework: (C1) shifting to a real time solution is problematic for constraints with “memory”, e.g. dynamic systems or rate constraints that require past state values to evaluate, and (C2) at each instant in time typically only one iteration of the optimization algorithm can be applied. While the effects of point (C2) are indirectly/directly analyzed in virtually all works on real time optimization, point (C1) is often not considered. That is, most works on time varying optimization focus only on static constraint maps [10] or unconstrained problems [11]. Unfortunately, the QoS of flexible loads is specified by constraints with memory.

In addition to the literature on time-varying optimization, there is a subfield of literature focused on the distributed resource allocation for flexible loads [12]–[19] in the smart grid. While there is a library of work [20]–[22] on how to model the QoS of flexible loads for the purpose of resource allocation, only a few works on distributed resource allocation take this into account [12], [16].

To summarize, much of the past work on time-varying optimization is focused on problems of different structure than the resource allocation problem for flexible loads. Thus the algorithms developed are not directly applicable. Whereas, many of the past works focused on distributed resource allocation for flexible loads do not account for the entirety of the loads’ QoS.

In this paper, we develop an algorithm for distributed resource allocation that allows loads to account for a wide
variety of QoS metrics. In doing so, our algorithm incorporates principled techniques to overcome the challenges (C1) and (C2) listed above. To overcome (C1), we employ a state augmentation technique that augments past fictitious state values that act as surrogates for the previous states. To overcome (C2), we utilize predictions of the time varying quantities to facilitate a benefit similar to warm start technique in centralized optimization. With all features of the algorithm accounted for, we prove an Input to State stability (ISS) result for when the time varying aspect is arbitrary (but in some sense bounded). This stability result is guaranteed under gain conditions that are specified in terms of the readily available problem data.

In numerical experiments we validate our theoretical results and compare our proposed method to a past method in the literature. In the time-varying setting our proposed method is able to successfully have flexible loads solve the resource allocation problem in a distributed/hierarchical fashion. Additionally, it is shown that the past method, based on dual ascent, can lead to integrator windup in the same time-varying setting.

The paper proceeds as follows: in Section II the problem setup and requirements are described. In Section III the resource allocation problem is introduced, as well as past ways it has been posed as an optimization problem. In Section IV our proposed method is introduced and it is analyzed in Section V. We give numerical examples in Section VI and conclude in Section VII.

II. NEEDS OF THE LOADS AND THE POWER GRID

A. Notation

We let \( \mathbb{N} \) and \( \mathbb{R} \) denote the natural and real numbers, respectively. We let the index \( i \in \{1, \ldots, N\} \) denote the \( i^{th} \) load, where \( N \) is the total number of loads. The index \( t \in \mathbb{N} \) is the discrete time index. The index \( t \) will only appear as a subscript, while \( i \) will only appear as a superscript. In the sequel, unless specified \( \| \cdot \| \) will refer to the 2-norm of a vector on the appropriate dimension vector space. We reserve lowercase letters for vectors/scalars and uppercase letters for matrices. The notation \( x[j] \), when \( x \) is a vector, will refer to the \( j^{th} \) element of the vector \( x \).

The power consumed by load \( i \) at time \( t \) is denoted \( d_{i|t} \). Furthermore, the quantity \( d_{i+j|t} \) is the power consumption that at time \( t \) load \( i \) predicts it will consume at time \( t + j \), where \( j \leq N_p \) and \( N_p \) is the prediction horizon. For convenience, we define \( N_p^{-} := N_p - 1 \). The required total power from all loads, i.e., the reference signal, at time \( t \) is denoted \( s_t \).

We consider two “stacked” vectorized versions of the scalar quantities \( d_{i+j|t} \). The first is the load perspective stacking where we stack the scalars \( d_{i+j|t} \) into a vector and denote it as \( x_{i|t} \triangleq [d_{i|t}, \ldots, d_{i+N_p^{-}|t}]^T \). The second is the grid perspective stacking where we stack over all loads, forming \( x_{[i|t]} \triangleq [d_{i+j|t}, \ldots, d_{N_i|t}]^T \). In any case, for a fixed \( N_p \) we refer to the following \( x_t \triangleq [(x_1)^T, \ldots, (x_N)^T]^T \), which contains all the elements of \( x_{i|t} \) and \( x_{j|t} \). The purpose for introducing both stacked forms is for ease of exposition.

B. BA’s Needs: Reference tracking (global goal)

The BA employs support from flexible loads to help mitigate supply and demand mismatch. Using the previously defined variables, this goal is captured by requiring the following to be small:

\[
e_{\tau|t} = \sum_{i=1}^{N} d_{i|t}^e - s_{\tau}, \quad J_G(x_{\tau|t}) = e_{\tau|t}^2, \quad \tau \leq t + N_p^{-}.
\]

C. Individual Needs: The QoS set (local constraints)

We describe the requirements of the loads through a QoS set. These constraints are taken from the vast literature on “capacity characterization” of flexible loads [20]–[22]. The constraints on the power for the \( i^{th} \) load, \( x_{i|t} \), are:

\[
D^i(d_{i-1|t-1}^i) \triangleq \left\{ x_i^i : \forall j \in \{t, \ldots, t + N_p^{-}\}, \right. \\
\left. \begin{array}{l}
\text{Power :} \quad d_L^i \leq d_{i|t}^i \leq d_H^i, \\
\text{Rate :} \quad r_L^i \leq d_{j|t}^i - d_{j-1|t}^i \leq r_H^i, \quad j > t \right. \\
\left. \text{Rate-IC :} \quad r_L^i \leq d_{j|t}^i - d_{i+1|t-1}^i \leq r_H^i, \quad j > t, \right. \\
\left. \text{Energy :} \quad e_{i|t}^i \leq \sum_{j=t}^{t+N_p^{-}} d_{j|t}^i \leq e_H^i \right\}.
\]

Each constraint (2)-(6) has a specific meaning as illustrated by the labels given. An additional Rate-IC constraint is included to emphasize that previous data is required to evaluate this constraint. Furthermore, it is necessary to define the QoS set \( \{3\} \) over a time horizon, otherwise enforcing the constraint \( \{6\} \) would not be possible. The constraints (2)-(6) model various classes of flexible loads, e.g., batteries, HVAC systems in commercial buildings, thermostatically controlled loads (TCLs) [21], and pool pumps [9].

However, while the QoS set specifies maximum limits it does not mean that it is desirable to operate at these limits.
Thus, the loads are also interested in making the following quantity small,

\[ J_L(x_{t|t}) = \sum_{i=1}^{N} (d^i_{t|t})^2 \xi^i, \quad \tau \leq t + N_p, \tag{7} \]

where \( \xi^i > 0 \) for all \( i \in \{1, \ldots, N\} \). The quantity (7) can be thought of as a regularization term.

**Proposition 1.** For each \( i \) the set \( \mathcal{D}^i(d^i_{t\!\!-\!1|t\!\!-\!1}) \) is compact, convex, and non empty.

There are two important points about the set \( \mathcal{D}^i(d^i_{t\!\!-\!1|t\!\!-\!1}) \): (i) the constraints (11)-(12) require more than one instant of time to appropriately evaluate, and (ii) the constraint (3) has memory, at time \( t \) the set \( \mathcal{D}^i(d^i_{t\!\!-\!1|t\!\!-\!1}) \) is a function of \( d^i_{t\!\!-\!1|t\!\!-\!1} \).

**Comment 1.** The constraint set (2) in its abstract form captures the heterogeneity of the load. In fact, other than it being convex, compact, and each load having an independent constraint set, we require no more assumptions for this set. For example, load \( i = 10 \) could be a Walmart and load \( i = 5 \) could be a classroom on a university campus, both shifting their load to help the grid. Put explicitly, our proposed optimization problem and solution method tolerate arbitrary high degrees of heterogeneity.

**D. Information structure**

The information structure considered is depicted in Figure 1 which is a hierarchical communication structure with distributed computation. For each time \( t \), the loads are allowed to communicate exactly once to the BA in order to receive global information, the signal \( e_{i|t} \) (11). The loads can then use this global information to apply one iteration of an optimization algorithm to achieve the global goal, tracking the reference \( s_t \). However, at the next time, \( t+1 \), the reference \( s_t \) will change and hence the optimization problem the loads are attempting to solve is operating in “real time.”

**III. RESOURCE ALLOCATION AND OPTIMIZATION BASICS**

The goal of the resource allocation problem is to set up one problem that combines both the grid’s and individual needs, as specified in Section II. Additionally, we seek a distributed and real time solution to the resource allocation problem. As stated in the introduction, the combination of the requirements in Section II with a real time and distributed implementation is often not considered.

To better understand our contribution we review resource allocation problems considered in past literature, and comment on how these methods lead to challenges when faced with the more realistic problem specifications here. However, before any of this we review how to solve a constrained optimization problem, of special structure, in a distributed fashion using projected gradient descent.

### A. Solving a constrained optimization problem

A distributed algorithm for solving the following time varying structured convex problem,

\[ \min_{x \in \mathcal{Z}} f(z; t), \quad z \in \mathbb{R}^q, \quad \mathcal{Z} = \mathcal{Z}^1 \times \cdots \times \mathcal{Z}^q, \tag{8} \]

with \( z^i \in \mathcal{Z}^i \) only, is the so-called projected gradient descent method,

\[ z^i_{t+1} = \Pi_{\mathcal{Z}^i}(z^i_t - \alpha \nabla f^i(z^i_t)), \quad \forall i \in \{1, \ldots, q\}, \tag{9} \]

\[ \nabla f^i(z^i_t) \triangleq \frac{\partial f(z; t)}{\partial z^i_t} \bigg|_{z^i_t = z^i}, \quad \Pi_{\mathcal{X}}(x) \triangleq \arg \min_{y \in \mathcal{X}} \|y - x\|, \]

with \( \alpha > 0 \) a step size. The projected gradient method applied to time invariant problems has its origins in [23]. For an introduction to time varying convex optimization the paper [24] is a good reference. As we will see, the resource allocation problem naturally has a similar structure to (8).

### B. Example resource allocation 1: Dual ascent

A commonly encountered resource allocation problem is,

\[ \beta^*_t = \min_{x_t} \beta(x_t) = \frac{1}{2} \left( J_L(x_{t|t}) + J_G(x_{t|t}) \right) \tag{10} \]

\[ \text{s.t.} \quad d^i_{t|t} \in [d^i_L, d^i_H], \quad \forall i \in \{1, \ldots, N\}, \tag{11} \]

\[ e_{i|t} = 0 \leftrightarrow \lambda_t, \tag{12} \]

where \( \leftrightarrow \) refers to the association of the dual variable \( \lambda_t \) (i.e., the Lagrange multiplier). In this setting the prediction horizon, \( N_p \), is zero making the decision variable for load \( i \) only \( d^i_{t|t} \). Given feasibility and strong convexity, one can solve this problem in a distributed/hierarchical fashion with the so called “dual ascent” method,

\[ \lambda_t = \lambda_{t-1} + \gamma e_{t\!\!-\!1|t\!\!-\!1}, \tag{13} \]

\[ d^i_{t|t} = \Pi_{[d^i_L, d^i_H]} \left( \frac{\lambda_t}{\xi^i} \right), \tag{14} \]

with \( \gamma \) a stepsize. The general derivation of these equations can be found in most introductory optimization textbooks [26].

Immediately, we see that this method will have some problems. Firstly, the resource allocation (10)-(12) do not account for all of the constraints in the QoS set. Secondly, if \( e_{i|t} \) cannot be made small (ideally zero) then the solution method (13) will suffer from the so called “integrator windup” phenomenon. For a time-invariant optimization problem, under the appropriate assumptions, it is straightforward to ensure zero steady state error, i.e., \( e_{i|t} \to 0 \). However, when the optimization problem is non-stationary it may be possible that for some time the problem is feasible and for other periods of time the problem is not feasible. When the problem is non-feasible the dual update equation (13) will continue to integrate non-zero error. When the problem becomes feasible again, the Lagrange multiplier will be far away from the optimal Lagrange multiplier for the newly feasible problem. Thirdly, knowing that the resource
allocation \(^{(10)}\) will be feasible for all time is centralized knowledge, or requires the solution to a centralized optimization problem.

C. Example resource allocation 2: predictive resource allocation

Another resource allocation formulation is the predictive resource allocation problem, which is described by the following optimization problem at time \(t \in \mathbb{N}\):

\[
\kappa^*_t = \min_{x_t} \kappa(x_t) = \frac{1}{2} \left( \sum_{\tau=t}^{t+N_p^*} J_L(x_{\tau|t}) + J_G(x_{\tau|t}) \right),
\]

s.t. \(x_t^i \in D_t^i(d_{t-1}^i|t-1), \quad \forall \ i \in \{1, \ldots, N\}\). \((15)\)

with \(N_p^* > 0\). This formulation allows for the incorporation of an appropriate QoS set \((3)\), however the constraint set is time varying and state dependent. So, while this problem may appear to be in the form amendable for the algorithm \((9)\), this is not the case. As the algorithm \((9)\) requires a fixed constraint set, and the constraint set \((16)\) is not fixed. So as it stands, there is no clear way to specify a distributed algorithm to solve \((15)\).

This problem is considered in \([14]\), however the focus there is not a real time implementation. As a result, the challenges we face here were not present in \([14]\).

The formulation of a resource allocation problem with all the appropriate QoS constraints, such as \((15)\) - \((16)\) - and an algorithm for its solution are the focus of the rest of the paper.

IV. PROPOSED METHOD

Largely, the limitation of the past resource allocation problem is that they do not consider appropriate load QoS metrics (dual ascent resource allocation, Section III-B). Further, we see that when including the appropriate metrics the constraint set becomes time varying and state dependent (predictive resource allocation, Section III-C). So that if we wish to use the appropriate QoS set, modifications to the resource allocation must be done to make the set fixed. We handle this limitation with a state augmentation technique, which we describe next.

A. Predictive Resource Allocation with memory

We define the memory objective at time \(t \in \mathbb{N}\) as follows:

\[
J_m(x_{t-1}|t) = \sum_{i=1}^{N} (d_i^{t-1}|t)-d_i^{t-1}|t-1\right)^2 + \left( \sum_{i=1}^{N} d_i^{t-1}|t-s_{t-1} \right)^2.
\]

We have introduced the variable \(d_i^{t-1}|t\), which is a fictitious variable at time \(t\) that we desire to be close to \(d_i^{t-1}|t-1\) (treated as a constant at time \(t\)), where close is defined by \((d_i^{t-1}|t)-d_i^{t-1}|t-1\))\(^2\). The augmented decision variable, \(z_t\), containing \(d_i^{t-1}|t\) is then:

\[
\begin{align*}
\hat{z}_t^i & \triangleq (d_i^{t-1}|t, (x_t^i)^T)^T, \\
z_t & \triangleq ([\hat{z}_t^1]^T, \ldots, ([\hat{z}_t^N]^T)^T,
\end{align*}
\]

where, by construction, \(z_t\) contains all the elements in \(x_{j|t}\), so where convenient we refer to \(x_{j|t}\) however, within the scope of an optimization problem, the relevant decision variable is \(z_t\). With \(z_t^i\) it is now possible to redefine the QoS set \((2)\) as independent of the previous state value. We denote this new set as:

\[
D_t^i \triangleq \left\{ z_t^i : \text{s.t. \((3), (4)\), and \((6)\)} \right\}.
\]

Comment 2. In \((20)\) the constraint \((5)\) is evaluated with the decision variable \(d_t^{i,t-1|t}\) and not an externally specified variable/parameter. Hence, there is no need to distinguish between the rate and rate-IC constraint.

With this, the predictive resource allocation problem with memory is the following:

\[
\min_{z_t} \eta(z_t) = \frac{1}{2} \left( \sum_{\tau=t}^{t+N_p^*} J_L(x_{\tau|t}) + J_G(x_{\tau|t}) + J_m(x_{t-1|t}) \right)
\]

s.t. \(z_t^i \in D_t^i, \quad \forall \ i \in \{1, \ldots, N\}\). \((21)\)

We see that \((21)\) is in a form applicable to the example algorithm \((9)\). The solution to \((21)\) is denoted \(z_t^*\) with optimal value \(\eta^* = \eta(z_t^*)\).

B. Proposed algorithm

To solve the problem \((21)\), we propose the following algorithm. The \(i^{th}\) load updates its state with:

\[
z_{i,t+1} = \Pi_{D_t^i} \left( \hat{P} (z_t^i - \alpha \nabla \eta^i(z_t^i)) \right) = \Pi_{D_t^i} \left( \hat{P} \psi_t^i \right),
\]

\[
\psi_t^i \triangleq z_t^i - \alpha \nabla \eta^i(z_t^i),
\]

where \(\alpha > 0\) is a step size common to all loads, and \(\hat{P}\) is the following matrix,

\[
\hat{P} = \begin{bmatrix}
0_{N_p \times 1} & I_{N_p} \\
0_{1 \times N_p}
\end{bmatrix}.
\]

Including the matrix \(\hat{P}\) will be elaborated on in section \(\ref{sec:IV-C}\). However, its primary purpose is to “shift” the data to facilitate a benefit similar to warm start techniques in optimization. In fact, a flavor of this idea was included in \((27)\), among others, to speed up the solution time for real time Model Predictive Control.

Recall, for each load \(i\), the quantity \(z_t^i\) is a vector in \(\mathbb{R}^{N_p+1}\) where \(N_p\) is the prediction horizon. The algorithm \((22)\) is an update rule for the entire vector \(z_t^i\), the value that the load \(i\) actually consumes at time \(t\) is then \(z_{i|2}^t = d_t^{i|t}\).

The ensemble dynamics, i.e., the vectorized form of the algorithm \((22)\) are,

\[
z_{t+1} = \Pi_{D_t^i} \left( \hat{P} \psi_t^i \right), \quad \Pi_{D_t^i} \left( \hat{P} \psi_t^i \right)^T = \Pi_D \left( P \psi_t^i \right),
\]

\[
D = D_1 \times \cdots \times D_N, \quad \psi_t = ([\psi_1^i]^T, \ldots, ([\psi_N^i]^T)^T,
\]

\[
P = I_N \otimes \hat{P},
\]
where $\times$ denotes Cartesian product, $\otimes$ denotes matrix Kronecker product [28], and $z_t$ is a vector in $\mathbb{R}^{(N_p+1)N}$. Since the Cartesian product operation preserves convexity and for each $i$ we have $D^i$ is convex, the set $D$ is also convex. The vectorized form (24) is useful for analysis, however during implementation each load has the ability to update its own local variable $z_t^i$ by solely using (22).

**Proposition 2.** Let $z^*_t$ be the optimal solution to problem (21) at time $t \in \mathbb{N}$, then we have that

$$z^*_t = \Pi_D\left(z_t - \alpha \nabla \kappa(z_t)\right) = \Pi_D(\psi^*_t).$$

In the above, the set $D$ is the same set that the algorithm (24) uses during implementation. We will see that this facet of Proposition 2 is important for the stability analysis of the proposed algorithm (24).

**C. Contribution**

Our proposed resource allocation method and algorithm have three key contributions over the past literature: (i) we accurately account for all the QoS of the loads, (ii) the inclusion of predictions and consequently the “horizon shifting” matrix $\hat{P}$, and (iii) the inclusion of the term:

$$J_m(x_{t-1}|t)$$

in the objective of (21). These improvements have been stated prior, but now with the developed math and notation they can be better exposed. The advantages of point (i) are explicit, so we focus on points (ii) and (iii).

Elaborating on point (ii), multiplying the content of the projection operator by the matrix $\hat{P}$ in (22) is consistent with “shifting the horizon” of data. For instance, at time $t$ the $i^{th}$ load produces a trajectory of demand consumptions from time $t$ to time $t + N_p - 1$. The initial condition at time $t + 1$ is then the value predicted for time $t + 1$ at time $t$. However, this value has already gone through at least one iteration thus speeding up the convergence in a way similar to “warm start” techniques in centralized optimization.

Elaborating on point (iii), we now examine what would happen when (27) is not included in the objective. We are concerned with the predictive resource allocation problem, as described in Section III-C. In this scenario the appropriate fixed point definition for the optimal trajectory in terms of the algorithm is now

$$x_t^* = \Pi_{D_t^i}\left(x_t - \alpha \nabla \kappa(x_t)\right) = \Pi_{D_t^i}(\Psi^*_t),$$

$$D_t^i \triangleq D^1(d_{t-1|t-1}^i) \times \cdots \times D^N(d_{t-1|t-1}^N),$$

thus $D_t^i$ is the constraint set computed with the previous optimal values, and $\Psi^*_t$ is the content of the projection operator in (28). Suppose now we attempt to use the prototype algorithm (9) to solve (15), i.e.,

$$x_{t+1} = \Pi_{D_t}\left(x_t - \alpha \nabla \kappa(x_t)\right) = \Pi_{D_t}(\Psi_t),$$

$$D_t \triangleq D^1(d_{t-1|t-1}^i) \times \cdots \times D^N(d_{t-1|t-1}^N).$$

Typically, one is interested in bounding $\|x_{t} - x^*_t\|$, which is usually performed using the non-expansive property of the projection operator. However since we have that $D_t \neq D_t^i$, we see that,

$$\|x_t - x^*_t\| = \|\Pi_{D_t}(\Psi_t) - \Pi_{D_t^i}(\Psi^*_t)\| \not\leq \|x_t - x^*_t\|,$$

since the non-expansive property requires that $D_t = D_t^i$. Thus showing convergence or boundedness will be greatly complicated, and likely lead to a lackluster bound.

**V. STABILITY**

**A. Preliminaries**

We list a string of results that will be useful for the analysis of the proposed algorithm (24).

**Proposition 3.** The Hessian $\nabla^2 \eta$ and gradient $\nabla \eta(z_t)$ can be expressed in the following form, letting $H^i \triangleq \text{diag}([\zeta^i, \zeta^i, \ldots, \zeta^i]) \in \mathbb{R}^{N_p+1}$, for all $z_t \in \mathbb{R}^{(N_p+1)N}$

(i) : $\nabla^2 \eta = 1_N \otimes (1_N \otimes I_{N_p+1}) + \bigoplus_{i=1}^N H^i,$

(ii) : $\nabla \eta(z_t) = (\nabla^2 \eta) z_t - u_t,$

where $\bigoplus$ denotes the Kronecker sum of matrices [28], $\text{diag}(a)$ denotes the diagonal matrix of the vector $a$. $1_N \in \mathbb{R}^N$ is the column vector of all ones, and the vector $u_t \in \mathbb{R}^{(N_p+1)N}$ is,

$$u_t = [(u_1^T, \ldots, u_N^T)^T]$$

with,

$$u_t^i = [d_{t-1|t-1}^i + s_{i-1}, s_{i}, \ldots, s_{i+N_p}]^T.$$  

We have dropped the dependence of $z_t$ on the Hessian, as the Hessian is a constant matrix, where additionally, based on the form given in Proposition 3, it is symmetric, i.e., $\nabla^2 \eta = (\nabla^2 \eta)^T$ and positive definite.

**Proposition 4.** Let $\hat{\zeta}^i = \zeta^i$ for all $i \in \{1, \ldots, N\}$, then

$$\|P \nabla^2 \eta - \nabla^2 \eta P\| = 0.$$  

**Lemma 1** (Theorem 2.1, [29]). For any $s, \tau \in \mathbb{N}$, the following bound holds,

$$\frac{1}{N} \|z^*_s - z^*_\tau\| \leq \frac{\tilde{g}^*_s}{\lambda_{\min}(\nabla^2 \eta)}.$$

where $\tilde{u}^*_{s,\tau} = \|u^*_s - u^*_\tau\|.$

**Proof.** See [29].

**Lemma 2.** For all $t \in \mathbb{N}$ the following holds,

$$\frac{1}{N} \|Pz^*_{t-1} - z^*_t\| \leq \frac{\tilde{g}^*_t}{\lambda_{\min}(\nabla^2 \eta)},$$

where $\tilde{g}^*_t = \tilde{u}^*_{t-1} + 2\tilde{u}^*_t$, $\tilde{u}^*_t = \|u^*_{t-1} - u^*_t\|$ and $u^*_{t-1}$ is the value that produces an optimal solution of all zeros.

**Proof.** See appendix.

This result will render itself useful for the stability analysis. Also necessary in our stability results is the class of $\mathcal{K}$ and $\mathcal{KL}$ functions, that hold their usual definitions as seen, e.g. in [30].
TABLE I: Simulation Parameters

| Par. | Unit value | Par. | Unit value |
|------|------------|------|------------|
| N    | hundred    | α    | 0.1, 4     |
| r    | kW         | e_H  | 0, 10      |
| z    | kW         | z    | -0.50, 0.50|

B. Stability: Main result

Our main theoretical results for our proposed algorithm (22) is summarized in Theorem 1. If we treat the value \( \| z_t - z^*_t \| \) as the “state” and an upper bound on the time varying aspects to the optimization problem as the “input”, then Theorem 1 is a global input to state stability (ISS) result.

Practically, we want the magnitude \( \| z_t - z^*_t \| \) to be small, as the optimal solution \( z^*_t \) represents the value that optimally satisfies all of the specified criteria.

The theorem below requires the following boundedness assumptions:

\begin{itemize}
  \item[A1:] for all \( t \in \mathbb{N} \), \( \tilde{g}^*_t < \tilde{g} < \infty \),
  \item[A2:] for all \( t \in \mathbb{N} \), \( \ell < t \), \( \| P u_{t-\ell} - u^*_t \| < \Delta < \infty \).
\end{itemize}

Then we denote \( \bar{u} \triangleq \left( \frac{N \tilde{g}}{\alpha \max (\text{sign}(\tilde{g}))} + \Delta \right) \).

Theorem 1 (Global-ISS). If assumptions A1 and A2 are satisfied, the step size \( \alpha \) satisfies,

\[ \alpha \in \left( 0, \frac{1}{\zeta_{\max} + N} \right) \],

where \( \zeta_{\max} = \max_{1 \leq i \leq N} \zeta_i \),

and \( \tilde{\zeta}_i = \zeta_i \) for all \( i \in \{1, \ldots, N\} \), then there exists a \( \Gamma \in \mathbb{K} \) and an \( \Omega \in \mathbb{K} \) such that

\[ \| z_t - z^*_t \| \leq \Omega(\| z_0 - z^*_0 \|, t) + \Gamma(\bar{u}) \]

where \( z_0 \) is the initial iterate of (24).

Proof. See appendix.

In Theorem 1 we have developed conditions on the step-size in terms of the readily available problem data that will give a stability result for time varying reference signals.

VI. NUMERICAL EXAMPLES

Here we offer numerical examples to validate the result from Theorem 1. This involves simulating the algorithm (22) on various types of data. We provide two scenarios for this: Scenario 1 (S1) a step reference that makes problem (10) not feasible so to illustrate the integrator windup of the dual ascent method and Scenario 2 (S2) our proposed method tracking Bonneville Power Administration (BPA) balancing reserves deployed (BRD) signal to illustrate the effectiveness of our algorithm tracking a time varying signal.

In both scenarios: (i) each load is given a set of parameter values obtained by a linear spacing between the maximum and minimum values found (along with the other relevant simulation parameters) in Table I and (ii) the sampling time is \( T_s = 5 \) minutes.

A. Scenario 1: Integrator Windup of dual ascent

The first example we illustrate is the “integrator windup” behavior that the dual algorithm suffers when problem (10) is not feasible, as described in Section III-B. The result of this is shown in Figure 2. When the resource allocation problem (10) is not feasible, the dual variable update (13) will continue to integrate non-zero area. It then takes dual ascent time to reach zero steady state error once feasibility is regained. It is worth noting that the two regions of integrated area in Figure 2 are equivalent.

For comparison we also utilize our proposed algorithm with solely the magnitude constraints (3) and \( N_p = 0 \), which does not suffer from integrator windup.

![Fig. 2: Integrator Windup of dual ascent with step response reference.](image)

B. Scenario 2: Tracking BPA’s BRD

With our proposed method, we track a time-varying reference with a prediction horizon of \( N_p = 5 \); see Figure 3. Since the data obtained from BPA is on the order of GW, we scale the reference down to satisfy the magnitude constraint. However, this is not required for the success of the algorithm, only to aid in exposition of the results.

The 1-norm tracking error of the signal in Figure 3 is 16.3%, and can be attributed to 2 factors: (i) the reference is only guaranteed to satisfy the magnitude constraint (3) so it may not be feasible for the other constraints and (ii) the algorithm only guarantees ISS and not asymptotic tracking. However, from experience we believe (i) to be the contributing factor. Other numerical experiments conducted suggest that it is possible to make the error quite small by increasing \( N_p \) if the constraints are all feasible.

VII. CONCLUSION

We propose a real time optimization algorithm with distributed computation and hierarchical communication structure for the resource allocation of flexible loads in the smart grid. Our algorithm has two key innovations: (i) the utilization of predictions and (ii) a state augmentation technique to handle dynamic constraints.

Future work includes: (i) analyzing further the effects of the state augmentation technique, similar to the penalty method technique applied in [16] and (ii) the development of asymptotic results for constrained time varying optimization.
Fig. 3: Tracking the time varying reference with the proposed method.

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APPENDIX

A. Proof of Lemma 2

Proceeding directly, by the triangle inequality we have

$$\frac{1}{N} \| Pz_{t-1}^* - z_t^* \| \leq \frac{1}{N} \| z_{t-1}^* - z_t^* \| + \frac{1}{N} \| Pz_{t-1}^* - z_{t-1}^* \|,$$

$$\leq \frac{\bar{u}_{t-1}^*}{\lambda_{\min}(\nabla^2 \eta)} + \frac{2}{\lambda_{\min}(\nabla^2 \eta)} \| z_{t-1}^* \|. \quad (33)$$

We can bound \( \| z_{t-1}^* \| \) by using Lemma 1 where \( z_{t-1}^* \) will be zero when \( u_t^* = u_t^{*,0} \), yielding

$$\frac{1}{N} \| Pz_{t-1}^* - z_t^* \| \leq \frac{\bar{u}_{t-1}^*}{\lambda_{\min}(\nabla^2 \eta)} + \frac{2\bar{u}_t^*}{\lambda_{\min}(\nabla^2 \eta)} = \frac{\bar{g}_t^*}{\lambda_{\min}(\nabla^2 \eta)},$$

which is the desired result.

B. Proof of Theorem 7

We start with the developed vectorized notation,

$$\| z_t - z_t^* \| = \| P(\bar{y}_{t-1}) - P(\bar{y}_t^*) \| \leq \| P(\bar{y}_{t-1}) - \bar{y}_t^* \| + \| \bar{y}_t^* - z_t^* \|$$

$$= \| P(\bar{y}_{t-1}) - \bar{y}_t^* \| + \alpha \left( P(\nabla \eta(z_{t-1}) - \nabla \eta(z_t^*)) \right),$$
where the inequality is by the non-expansive property of the projection operator. Working with the gradient terms we have from Proposition 3 that,

\[ P(\nabla \eta(z_{t-1})) = P((\nabla^2 \eta)z_{t-1} - u_{t-1}) \]

so that \( P\nabla \eta(z_{t-1}) - \nabla \eta(z^*_t) = \nabla^2 \eta (Pz_{t-1} - z^*_t) + (u^*_t - Pu_{t-1}) \]

+ \( P((\nabla^2 \eta) - (\nabla^2 \eta)P) z_{t-1} \)

Substituting this result into the original quantity of interest and applying the triangle inequality, we have:

\[
\|z_t - z^*_t\| \leq M(\alpha)\|Pz_{t-1} - z^*_t\| + \alpha\|(Pu_{t-1} - u^*_t)\|
\]

+ \( \alpha\left\| (P(\nabla^2 \eta) - (\nabla^2 \eta)P)\right\|\|z_{t-1}\|, \)

\[
\leq M(\alpha)\|Pz_{t-1} - z^*_t\| + \alpha\|(Pu_{t-1} - u^*_t)\|,
\]

where \( M(\alpha) = \|I - \alpha \nabla^2 \eta\|. \) The third term above is eliminated from our choice of \( \zeta^t = \zeta^t. \) As the Hessian is positive definite, it is possible to pick an \( \alpha \) so that \( M(\alpha) < 1. \)

We take this fact for granted now, and later in the proof provide the bound found in the theorem. Now utilizing the triangle inequality we have that,

\[
\|z_t - z^*_t\| \leq M(\alpha)\left(\|z_{t-1} - z^*_{t-1}\| + \|Pz^*_{t-1} - z^*_t\|\right)
\]

+ \( \alpha\|(Pu_{t-1} - u^*_t)\|. \)

Now applying the results of Lemma 2 we have,

\[
\|z_t - z^*_t\| \leq M(\alpha)\left(\|z_{t-1} - z^*_{t-1}\| + \frac{N\tilde{g}_t^*}{\lambda_{\min}(\nabla^2 \eta)}\right)
\]

+ \( \alpha\|(Pu_{t-1} - u^*_t)\|. \)

We iterate this backwards a total of \( t \) times to reach \( t = 0, \) yielding:

\[
\|z_t - z^*_t\| \leq M^t(\alpha)\|z_0 - z^*_0\|
\]

+ \( \alpha \sum_{\ell=1}^{t} M^{t-\ell}(\alpha)\left(\|Pu_{t-\ell} - u^*_{t-\ell}\| + \frac{N\tilde{g}^*_{t-\ell}}{\alpha\lambda_{\min}(\nabla^2 \eta)}\right). \)

Now, from our assumptions we can bound the quantity in parentheses in the summation by \( \bar{u} \) yielding,

\[
\|z_t - z^*_t\| \leq M^t(\alpha)\|z_0 - z^*_0\| + \frac{\alpha\bar{u}}{1 - M(\alpha)},
\]

\[
\leq \Omega(\|z_0 - z^*_0\|, t) + \Gamma(\bar{u}),
\]

where, as desired, it can be easily verified that \( \Omega \in KL \) and \( \Gamma \in KC \) as long as \( M(\alpha) < 1, \) which we ensure next.

Now that the ISS result has been obtained, we show how the range on \( \alpha \) is obtained to guarantee \( M(\alpha) = \|I - \alpha \nabla^2 \eta\| < 1. \) By definition we have, \( \|I - \alpha \nabla^2 \eta\| = \max\{ |\lambda_{\min}(I - \alpha \nabla^2 \eta)|, |\lambda_{\max}(I - \alpha \nabla^2 \eta)| \}, \)

since \( I - \alpha \nabla^2 \eta \) is symmetric and where \( \lambda_{\max}(A) \) and \( \lambda_{\min}(A) \) are the maximum and minimum eigenvalue of the matrix \( A, \) respectively. If we denote \( \lambda_i(\nabla^2 \eta) \) the \( i^{th} \) eigenvalue of \( \nabla^2 \eta, \) then \( \lambda_i(I - \alpha \nabla^2 \eta), \) the \( i^{th} \) eigenvalue of \( I - \alpha \nabla^2 \eta, \) is

\[ \lambda_i(I - \alpha \nabla^2 \eta) = 1 - \alpha \lambda_i(\nabla^2 \eta), \]

which is obtained by considering the eigendecomposition of \( \nabla^2 \eta. \) Thus, we seek to guarantee \( M(\alpha) < 1, \) and it is sufficient to require

\[ 0 < \alpha \lambda_{\max}(\nabla^2 \eta) < 1, \]

which immediately leads to the lower bound \( \alpha > 0, \) since the Hessian is positive definite. To obtain the upper bound we apply the Gershgorin circle theorem. This is readily applicable based on the structure of the Hessian found in Proposition 3. This yields the following sufficient lower and upper bound on \( \alpha \) for \( M(\alpha) < 1, \)

\[ \alpha \in \left(0, \frac{1}{\xi_{\max} + N}\right). \]