Reduced Hamiltonians

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ABSTRACT: We resurrect a standard construction of analytical mechanics dating from the last century. The technique allows one to pass from any dynamical system whose first order evolution equations are known, and whose bracket algebra is not degenerate, to a system of canonical variables and a non-zero Hamiltonian that generates their evolution. We advocate using this method to infer a canonical formalism, as a prelude to quantization, for systems in which the naive Hamiltonian is constrained to vanish. The construction agrees with the usual results for gauge theories and can be applied as well to gravity, even when the spatial manifold is closed. As an example, we construct such a reduced Hamiltonian in perturbation theory around a flat background on the manifold $T^3 \times R$. The resulting Hamiltonian is positive semidefinite and agrees with the A.D.M. energy in the limit that deviations from flat space remain localized as the toroidal radii become infinite. We also obtain closed form expressions for the reduced Hamiltonians of two minisuperspace truncations. Although our results are classical they can be formally quantized to give the naive functional formalism. This is not only an effective starting point for calculations, it also seems to provide a formulation of the quantum theory which is non-perturbative, at least in principle. The marriage we advocate between the old technique and canonical quantization seems to have profound implications for quantum gravity, especially as regards the conservation of energy, statistical mechanics, and the problem of time.

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1. Introduction

It is traditional to develop the canonical formalism from a Lagrangian. One first identifies variables whose Poisson bracket algebra is canonical, then the Hamiltonian is constructed and used to generate first order evolution equations. However, the imposition of constraints can result in a system of dynamical variables whose bracket algebra is not canonical and whose time evolution, while completely determined, is not generated by a known Hamiltonian. This possibility is especially relevant to systems in which gravity is dynamical and the spatial manifold is closed because then the naive Hamiltonian is constrained to vanish [1]. The ubiquity of canonical quantization and its evident failure in this case has led to much puzzlement over how these perfectly acceptable classical systems should be quantized [2]. We wish here to advocate a straightforward solution: impose the constraints to obtain directly a set of first order evolution equations and a bracket algebra for the reduced variables, then construct a canonical formalism which reproduces this system. The resulting reduced canonical formalism can then be quantized as usual.

Our motive in suggesting this step is the highly unsatisfactory situation which currently prevails in the formulation of canonical quantum gravity. Those who approach this field from the perspective of other disciplines have long been frustrated by its failure to obey certain obvious correspondence limits. There are at least four problems of this nature:

(1) The Paradox of Second Coordinatization — In classical gravitation we know that fixing the lapse and shift determines the coordinate system up to deformations of the initial value surface. We also think we understand pretty well how to infer physics from the behavior of the metric field in these coordinates. In quantum gravity on a closed spatial manifold we are told that even after fixing the lapse and the shift we must determine what time is all over again. We are also told that physical information can only be gleaned from observables which are manifestly coordinate invariant, that we don’t possess any such observables, and that even if we did it would not be possible to
compute their expectation values because an inner product cannot be given until after the meaning of time has been clarified. How can changing \( h \) from zero so thoroughly confuse the way we infer physics from the field variables?

(2) The Paradox of Dynamics — A similarly striking issue concerns the limit in which gravity becomes non-dynamical. Since the Hamiltonians of pure matter theories are not typically zero, even on closed spatial manifolds, these theories can be quantized canonically. But when gravity is made dynamical on a spatially closed manifold it is asserted that we no longer know even what the inner product is, no matter how weak the gravitational interaction might be relative to other forces. Note that the alleged difficulty is unrelated to the non-renormalizability of Einstein’s theory; it would occur even in an ultraviolet finite theory of quantum gravity. How can changing Newton’s constant from zero affect the basic structure of quantum mechanics?

(3) The Paradox of Topology — Although topology is not a continuous parameter the comparison between manifolds with differing topologies is disquieting. For spatially open manifolds there is no obstacle to canonical quantization because the naive Hamiltonian can be non-zero \([3,4]\). Of course there remains the highly non-trivial problem of finding a theory whose dynamics are consistent, but there is no confusion about basic issues such as the meaning of time or how to define inner products. When the spatial manifold is closed it is asserted that we no longer understand these issues. Yet there is no local experiment which can distinguish a closed space from an open one, as witness the fact that either possibility might describe our own universe. How can removing a point from the other end of the universe affect our ability to provide a quantum mechanical description of local observations which have a vanishingly small probability for even being in causal contact with the boundary?

(4) The Paradox of Stability — Finally, there is the peculiarity encountered in passing from pure quantum mechanics to statistical mechanics on a spatially closed manifold
when gravity is dynamical. Since the energy is constrained to be zero it follows that all states are degenerate. For example, the total energy needed to add a particle-anti particle pair to any state is zero because the negative gravitational interaction energy cancels the positive energy of the pair. Simple considerations of entropy seem then to require that the microcanonical ensemble should be concentrated around what would otherwise be thought of as very highly excited states. To see this note that for every “empty” state there are a countably infinite number of degenerate states which contain a single particle-anti particle pair moving apart with various momenta. There are even more states which contain two pairs, and more yet which contain three, etc. So what prevents such a universe from evaporating into a maelstrom of pairs?

We resolve the paradox of second coordinatization by denying that there is any fundamental distinction between classical and quantum measurement beyond that usually imposed by the uncertainty principle. What time means in quantum gravity is determined by fixing the lapse and the shift, just as in the classical theory. Any question that can be answered classically by studying a functional of the metric with fixed initial value data can be studied as well in the quantum theory using the expectation value of this same functional in the presence of a state whose probability is concentrated around the classical initial value data. We can base observations on a non-invariant such as the metric because the gauge has been fixed: any quantity becomes invariant when it is defined in a unique coordinate system.

We resolve the paradoxes of dynamics and of topology by showing that for any constrained system which can be reduced — that is, for which the residual gauge freedom can be fixed and the constraints imposed — there is a canonical choice of reduced dynamical variables and a non-zero Hamiltonian which generates their time evolution. In fact there are many such Hamiltonians, each corresponding to a different identification of canonical variables in the reduced dynamical system. For gravity these Hamiltonians seem to have
the property that the closed space versions go over to the known open space ones when
the coordinate volume is taken to infinity in such a way that the initial value data are only
locally disturbed from a background which obeys the appropriate asymptotic conditions.*
When gravity is coupled to matter the non-zero reduced Hamiltonians seem to agree with
the corresponding pure matter Hamiltonians in the limit that Newton’s constant vanishes.

Although the Hamiltonians we are discussing generate time evolution they are not
generally conserved. Outside of a few special cases the generally conserved energy really
is zero, so the universe can potentially suffer from the instability described in (4). We see
a mostly empty universe today because the balance implicit in the $H = 0$ constraint is
maintained by means of global, negative energy modes, and the rate at which these modes
can be excited is highly suppressed by causality and by the weakness of the gravitational
interaction. There has simply not been enough time for the universe to evaporate into
pairs. Depending on the topology and the causal structure of the initial state there may
never be enough time for this to happen.

Section 2 presents the construction by means of which any known system of first order
evolution equations and (not necessarily canonical) brackets can be used to identify a set
of canonical variables and a Hamiltonian that generates their time evolution. This is not
original work; the construction was given about a century ago [5,6]. What does seem to be
original is the idea of applying it generally to gravity. In taking this step we are following
the work in 2 + 1 dimensions of Moncrief [7], Hosoya and Nakao [8], and Carlip [9]. We
believe that their constructions are special cases of the general technique; we believe the
same is true of the lovely formalism for open, asymptotically flat spaces which was devised
by Arnowitt, Deser and Misner (A.D.M.) [3, and references therein].

* For [3] this background would be flat space, for [4] it would be the de Sitter geometry.
Note that topological obstructions — for example, the inability to impose a flat metric on
$S^3$ — need have no physical relevance because we might be able to use half of the infinite
coordinate volume to approach the asymptotic geometry of the open manifold and the
other half to deviate so as to reconcile the topology of the closed manifold.
In section 3 we apply the method to a standard gauge in scalar electrodynamics. This is difficult to do in gravity because one can seldom obtain explicit solutions for the constraints. However, the method is simple enough to carry out in perturbation theory or when the dynamics is suitably truncated. In section 4 we give a perturbative construction for a flat space background on the manifold $T^3 \times R$; in section 5 we apply the method to several models of minisuperspace. We show in section 6 that if the unreduced theory exists then quantization along the lines we advocate results in the usual functional formalism. In this form the technique is not only simple to exploit for the purposes of perturbative calculations, it also provides a definition of the theory that is valid beyond perturbation theory, at least in principle. Section 7 summarizes our resolutions to the aforementioned paradoxes and discusses the implications our view has for statistical mechanics and the conservation of energy.

2. The Construction

Consider a set of $2N$ dynamical variables, $\{v^i(t) : R \rightarrow R^{2N}\}$. Suppose that the time evolution of these variables is determined by first order equations of the form:

$$\dot{v}^i(t) = f^i(v, t) \quad (2.1)$$

where the $f^i$'s are known functions of the $v^i$'s and possibly also of time. We do not assume that the $v^i$'s are necessarily canonical but rather that they obey the following bracket algebra:

$$\{v^i(t), v^j(t)\} = J^{ij}(v, t) \quad (2.2)$$

where the bracket matrix $J^{ij}$ is antisymmetric, invertible and obeys the Jacobi identity. We shall denote the inverse bracket matrix by the symbol, “$J_{ij}$”:

$$J_{ij} J^{jk} = \delta_i^k \quad (2.3)$$

* To emphasize that the dynamical variables are functions of both the canonical positions and their conjugate momenta we have adopted the neutral symbol, “$v^i$, ” for “variable.”
Taking the time derivative of (2.2) and substituting (2.1) reveals the following relation:

\[ J^{ik} f^{j}_{,k} - J^{jk} f^{i}_{,k} = J^{ij}_{,k} f^{k} + \frac{\partial J^{ij}}{\partial t} \]  

(2.4a)

where a comma denotes differentiation. The analogous relation for the inverse is:

\[ J^{ik} f^{j}_{,j} - J^{jk} f^{i}_{,i} = -J^{ij}_{,i} f^{k} - \frac{\partial J^{ij}}{\partial t} \]  

(2.4b)

Since brackets obey the Jacobi identity we have the following differential relation for the elements of the bracket matrix:

\[ J^{i\ell} J^{j\ell}_{,k} + J^{j\ell} J^{k\ell}_{,i} + J^{k\ell} J^{ij}_{,\ell} = 0 \]  

(2.5a)

The analogous result for the inverse is:

\[ J^{ij}_{,k} + J^{jk}_{,i} + J^{ki}_{,j} = 0 \]  

(2.5b)

This last relation is reminiscent of the Bianchi identities in electromagnetism and allows us to write the inverse bracket matrix as the curl of a “vector potential,” \( \phi_i \):

\[ J^{ij} = \phi_{j,i} - \phi_{i,j} \]  

(2.6)

Just as in electrodynamics \( \phi_i \) is undetermined up to the gradient of a scalar function; similarly, one may need several coordinate patches if the dynamical variables map into some space other than \( \mathbb{R}^{2N} \). A convenient representation for the vector potential is:

\[ \phi_i(v, t) = -\int_{v_0}^{v} d\tau \tau J_{ij}(v_0 + \tau \Delta v, t) \Delta v^j \]  

(2.7)

where \( v_0^i \) is any fixed point and \( \Delta v^i \equiv v^i - v_0^i \). Note that under a change in the dynamical variables \( J^{ij} \) transforms contravariantly while \( J^{ij} \) transforms covariantly; \( \phi_i \) transforms covariantly up to the addition of a gradient.

The simplest possibility for reconstructing a Hamiltonian is that the evolution equations are integrable. The condition for this is that the bracket matrix should be free of explicit time dependence:

\[ \exists H(v, t) \ni \dot{v}(t) = \{ v^i, H \} \text{ iff } \frac{\partial J^{ij}}{\partial t} = 0 \]  

(2.8)
For the proof note that, if it exists, the Hamiltonian is determined up to a function of time by the equations:

$$\frac{\partial H}{\partial v^i} = J_{ij} f^j$$  \hspace{1cm} (2.9)

Since we must have $H_{ij} = H_{ji}$ the integrability condition is:

$$0 = \left( J_{ik} f^k \right)_j - \left( J_{jk} f^k \right)_i$$  \hspace{1cm} (2.10a)

$$= - \left( J_{ki,j} + J_{jk,i} \right) f^k + \left( J_{ik} f^k_j - f^k_i J_{jk} \right)$$  \hspace{1cm} (2.10b)

$$= - \frac{\partial J_{ij}}{\partial t}$$  \hspace{1cm} (2.10c)

where we have used (2.4b) and (2.5b) to reach the final line. The theorem is proved upon noting that the bracket matrix is invertible. When the integrability condition is met it is simple to check that:

$$H(v, t) - H(v_0, t) = \int_0^1 d\tau \Delta v^i J_{ij} \left( v_0 + \tau \Delta v, t \right) f^j \left( v_0 + \tau \Delta v, t \right)$$  \hspace{1cm} (2.11)

gives an explicit representation for the Hamiltonian.

To identify canonical variables in the local neighborhood of any point $v_i^0$ we construct $2N$ functions $Q^a(v, t)$ and $P_b(v, t)$ ($a, b = 1, 2, \ldots, N$) which are instantaneously invertible at any time and which obey:

$$\phi_i(v, t) = P_a(v, t) \frac{\partial Q^a}{\partial v^i}(v, t)$$  \hspace{1cm} (2.12)

Note that invertibility implies:

$$\begin{pmatrix} \delta^a_b & 0 \\ 0 & \delta^b_a \end{pmatrix} = \begin{pmatrix} \frac{\partial Q^a}{\partial v^i} & \frac{\partial Q^a}{\partial v^j} \\ \frac{\partial P_a}{\partial v^i} & \frac{\partial P_a}{\partial v^j} \end{pmatrix}$$  \hspace{1cm} (2.13a)

$$\delta^i_j = \frac{\partial v^i}{\partial Q^a} \frac{\partial Q^a}{\partial v^j} + \frac{\partial v^i}{\partial P_a} \frac{\partial P_a}{\partial v^j}$$  \hspace{1cm} (2.13b)

The problem of showing that such functions exist is known as Pfaff’s problem in honor of the German mathematician J. F. Pfaff [10]. This problem was solved long ago by G. Frobenius and J. G. Darboux [11,12,13]. That the transformation:

$$q^a(t) = Q^a(v(t), t), \quad p_a(t) = P_a(v(t), t)$$  \hspace{1cm} (2.14)
gives canonical variables is straightforward to see. First, substitute (2.12) into (2.6) to obtain:

\[
J_{ij} = \frac{\partial P_a}{\partial v^i} \frac{\partial Q^a}{\partial v^j} - \frac{\partial Q^a}{\partial v^i} \frac{\partial P_a}{\partial v^j}
\]  

(2.15)

Now use this relation and (2.13a) to show that the bracket matrix “raises” indices on the transformation matrices:

\[
\frac{\partial v^i}{\partial Q^a} = J_{ij} J_{jk} \frac{\partial v^k}{\partial Q^a}
\]  

(2.16a)

\[
= J_{ij} \left( \frac{\partial P_b}{\partial v^j} \frac{\partial Q^b}{\partial v^k} - \frac{\partial Q^b}{\partial v^j} \frac{\partial P_b}{\partial v^k} \right) \frac{\partial v^k}{\partial Q^a}
\]  

(2.16b)

\[
= J_{ij} \frac{\partial P_a}{\partial v^j}
\]  

(2.16c)

\[
\frac{\partial v^i}{\partial P_a} = J_{ij} J_{jk} \frac{\partial v^k}{\partial P_a}
\]  

(2.17a)

\[
= J_{ij} \left( \frac{\partial P_b}{\partial v^j} \frac{\partial Q^b}{\partial v^k} - \frac{\partial Q^b}{\partial v^j} \frac{\partial P_b}{\partial v^k} \right) \frac{\partial v^k}{\partial P_a}
\]  

(2.17b)

\[
= -J_{ij} \frac{\partial Q^a}{\partial v^j}
\]  

(2.17c)

Finally, take the brackets using (2.2), (2.13a), (2.14), (2.16) and (2.17):

\[
\{q^a, q^b\} = \frac{\partial Q^a}{\partial v^i} J_{ij} \frac{\partial Q^b}{\partial v^j} = -\frac{\partial Q^a}{\partial v^i} \frac{\partial v^i}{\partial P_b} = 0
\]  

(2.18a)

\[
\{q^a, p_b\} = \frac{\partial Q^a}{\partial v^i} J_{ij} \frac{\partial P_b}{\partial v^j} = \frac{\partial Q^a}{\partial v^i} \frac{\partial v^i}{\partial Q^b} = \delta^a_b
\]  

(2.18b)

\[
\{p_a, p_b\} = \frac{\partial P_a}{\partial v^i} J_{ij} \frac{\partial P_b}{\partial v^j} = \frac{\partial P_a}{\partial v^i} \frac{\partial v^i}{\partial Q^b} = 0
\]  

(2.18c)

A constructive — and slightly different — procedure for finding canonical coordinates is credited to the French mathematician Darboux and can be found in the text by Arnold [14]. The older derivation is discussed in the text by Whittaker [15].

It is possible to give explicit formulae for \( N = 1 \). For this case we can write down the inverse bracket matrix:

\[
\begin{pmatrix}
J_{11} & J_{12} \\
J_{21} & J_{22}
\end{pmatrix}
= \begin{pmatrix}
0 & -\frac{1}{J} \\
\frac{1}{J} & 0
\end{pmatrix}
\]  

(2.19)
and we further know that the function $J^{12}(v, t)$ is non-zero. The most convenient “gauge” for the vector potential is an axial one:

$$
\phi_1(v^1, v^2, t) = \int_0^{v^2} \frac{ds}{J^{12}(v^1, s, t)}
$$

(2.20a)

$$
\phi_2(v^1, v^2, t) = 0
$$

(2.20b)

Comparison with (2.12) reveals that $Q$ cannot depend upon $v^2$ and that the choice of $Q$ uniquely determines $P$. The simplest choice is:

$$
Q(v^1, v^2, t) = v^1
$$

(2.21a)

$$
P(v^1, v^2, t) = \int_0^{v^2} \frac{ds}{J^{12}(v^1, s, t)}
$$

(2.21b)

Note that although the general construction is only local — and so must be built up in patches — the formulae for $N = 1$ are valid globally if the domain of $(v^1, v^2)$ is $\mathbb{R}^2$.

Of course the mapping to canonical variables can only be unique up to canonical transformations — even in the Lagrangian formalism. This shows up slightly in the integration constants which come from solving (2.12); for the case of $N = 1$ above we had the freedom to choose $Q$ to be any function of $v^2$ and $t$ which was instantaneously invertible on $v^2$. By far the larger source of variation is the freedom to add the gradient of a scalar to $\phi_i$. From the instantaneous relation:

$$
\phi_i \, dv^i = p_a \, dq^a
$$

(2.22)

we see that the “gauge transformation,” $\phi_i \rightarrow \phi_i - \partial_i \theta$, induces a canonical transformation whose generating function is $-\theta$.

Once the brackets are canonical they are also independent of time, so we can get a Hamiltonian in the new coordinate system by using (2.11). The evolution equations in these coordinates are:

$$
\dot{q}^a(t) = f^a(q(t), p(t), t)
$$

(2.23a)
\[ \dot{p}_a(t) = f_a(q(t), p(t), t) \]  

(2.23b)

where we define:

\[ f^a(Q(v, t), P(v, t), t) \equiv \frac{\partial Q^a}{\partial v^i}(v, t) f^i(v, t) + \frac{\partial Q^a}{\partial t}(v, t) \]  

(2.24a)

\[ f_a(Q(v, t), P(v, t), t) \equiv \frac{\partial P_a}{\partial v^i}(v, t) f^i(v, t) + \frac{\partial P_a}{\partial t}(v, t) \]  

(2.24b)

Substituting into (2.11) gives:

\[
H(q, p, t) - H(q_0, p_0, t) = \int_0^1 d\tau \left\{ \Delta p_a f^a(q_0 + \tau \Delta q, p_0 + \tau \Delta p, t) - \Delta q^a f_a(q_0 + \tau \Delta q, p_0 + \tau \Delta p, t) \right\} \]  

(2.25)

where \((q_0, p_0)\) is any point in phase space and we define \(\Delta q^a \equiv q^a - q_0^a\) and \(\Delta p_a \equiv p_a - p_0a\).

Note that the Hamiltonian (2.25) only generates the evolution (2.23) of the canonical coordinates. Its bracket with \(v^i(t)\) does not generally give \(\dot{v}^i(t)\). If we call the inverse transformation \(V^i(q, p, t)\) then the relation:

\[ v^i(t) = V^i(q(t), p(t), t) \]  

(2.26)

implies that the original dynamical variables acquire only a portion of their time dependence from that of the canonical coordinates:

\[
\dot{v}^i(t) = \frac{\partial V^i}{\partial q^a} \dot{q}^a + \frac{\partial V^i}{\partial p_a} \dot{p}_a + \frac{\partial V^i}{\partial t} \]  

(2.27a)

\[
= \left\{ v^i(t), H \right\} + \frac{\partial V^i}{\partial t} \]  

(2.27b)

We have already seen that the final term in (2.27b) must be non-zero if the \(J^{ij}\)'s contain explicit time dependence.

Far from being a problem, this extra source of time dependence is a blessing. It is what keeps the evolution of the original dynamical variables fixed if we alter what is meant by the canonical variables. As long as there is a clear procedure for inferring physics from the \(v^i(t)\)'s it does not matter that we can choose the canonical variables differently. This
is a subtle point and a crucially important one. It is also counterintuitive because any choice of canonical coordinates does offer a complete description of the degrees of freedom available to the system. The point is that what these degrees of freedom mean physically changes as we change our representation of them.

The possibility for this sort of subtlety is always present in physics because time dependent canonical transformations can be made in any classical theory and, with proper attention to operator ordering, on the quantum level as well. For example, in perturbative quantum field theory there is a well known transformation, called as the “interaction representation,” which takes one to a set of fields whose time evolution and commutation relations are free. Consider the theory of a scalar field, \( \phi(t, \vec{x}) \), whose action has the following form:

\[
S[\phi] = \int d^4x \left\{ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \right\} + S_I[\phi] \tag{2.28}
\]

where \( S_I[\phi] \) is an ultralocal interaction. Since \( S_I[\phi] \) is free of derivatives the momentum canonically conjugate to \( \phi(t, \vec{x}) \) is just its time derivative. The only non-vanishing equal time bracket is:

\[
\left\{ \phi(t, \vec{x}), \dot{\phi}(t, \vec{y}) \right\} = \delta^3(\vec{x} - \vec{y}) \tag{2.29}
\]

The field equation:

\[
(\Box - m^2) \phi + \frac{\delta S_I[\phi]}{\delta \phi} = 0 \tag{2.30}
\]

can be integrated to give a new field which obeys the Klein-Gordon equation:

\[
\Phi[\phi] \equiv \phi + \frac{1}{\Box - m^2} \frac{\delta S_I[\phi]}{\delta \phi} \tag{2.31}
\]

If we define the inverse of \( \Box - m^2 \) so that it and its first derivative vanish at \( t = 0 \) then we have also:

\[
\Phi(0, \vec{x}) = \phi(0, \vec{x}) \tag{2.32a}
\]

\[
\dot{\Phi}(0, \vec{x}) = \dot{\phi}(0, \vec{x}) \tag{2.32b}
\]
It follows that $\Phi(t,\vec{x})$ not only *evolves* like a free field, it also obeys the same bracket algebra. We do not conclude that all such theories are free, even perturbatively, because we insist upon inferring physics from the original variables. The only difference between this standard situation and the previous discussion of this section is that even the original scalar is canonical whereas we do not make this assumption about the $v^i$'s.

Why the imposition of constraints tends to result in a non-canonical bracket matrix is obvious to anyone who has ever constructed a Dirac bracket [16]. Even though the gauge and constrained variables are irrelevant to physics, the unreduced canonical formalism is an artificial construct in which the unphysical variables play an essential role in keeping the brackets canonical under time evolution. After reduction these variables are no longer independent and this changes the bracket algebra. Unless what we call the reduced variables are chosen very carefully they will not have a canonical bracket algebra. This does not mean that other choices are “wrong.” In the sense of providing a complete description of physics a non-canonical set of variables can be as “right” as a canonical set. Non-canonical reduced variables can even be highly preferred on account of bearing a simpler relation than any canonical variables to the original degrees of freedom from which physics is inferred.

In the next section we use scalar electrodynamics to provide an explicit example of how reduction can yield a non-canonical bracket matrix. For now we can understand the issues in a simple but somewhat contrived fashion through a model of two coupled oscillators whose Lagrangian is:

$$L = \frac{1}{2}m\left(\dot{q}_1^2 + \dot{q}_2^2\right) - \frac{1}{2}m\omega^2\left(\frac{5}{4}q_1^2 + q_1q_2 + \frac{5}{4}q_2^2\right)$$  \hspace{1cm} (2.33)

The resulting canonical formalism will serve as our model for an unreduced canonical formalism, in spite of the fact that the Lagrangian possesses no continuous symmetries and gives rise to no constraint equations. We shall make up for the absence of constraints by imposing them *ad hoc.*
It is straightforward to show that the canonical variables of the coupled oscillator system have the following time evolution:

\[ q_1(t) = \frac{1}{2}(\hat{q}_1 + \hat{q}_2) \cos\left(\frac{3}{2}\omega t\right) + \frac{1}{2}(\hat{q}_1 - \hat{q}_2) \cos\left(\frac{1}{2}\omega t\right) \]
\[ + \frac{1}{3m\omega}(\hat{p}_1 + \hat{p}_2) \sin\left(\frac{3}{2}\omega t\right) + \frac{1}{m\omega}(\hat{p}_1 - \hat{p}_2) \sin\left(\frac{1}{2}\omega t\right) \]  
(2.34a)

\[ p_1(t) = -\frac{3}{4}m\omega(\hat{q}_1 + \hat{q}_2) \sin\left(\frac{3}{2}\omega t\right) - \frac{1}{4}m\omega(\hat{q}_1 - \hat{q}_2) \sin\left(\frac{1}{2}\omega t\right) \]
\[ + \frac{1}{2}(\hat{p}_1 + \hat{p}_2) \cos\left(\frac{3}{2}\omega t\right) + \frac{1}{2}(\hat{p}_1 - \hat{p}_2) \cos\left(\frac{1}{2}\omega t\right) \]  
(2.34b)

\[ q_2(t) = \frac{1}{2}(\hat{q}_1 + \hat{q}_2) \cos\left(\frac{3}{2}\omega t\right) + \frac{1}{2}(-\hat{q}_1 + \hat{q}_2) \cos\left(\frac{1}{2}\omega t\right) \]
\[ + \frac{1}{3m\omega}(\hat{p}_1 + \hat{p}_2) \sin\left(\frac{3}{2}\omega t\right) + \frac{1}{m\omega}(-\hat{p}_1 + \hat{p}_2) \sin\left(\frac{1}{2}\omega t\right) \]  
(2.34c)

\[ p_2(t) = -\frac{3}{4}m\omega(\hat{q}_1 + \hat{q}_2) \sin\left(\frac{3}{2}\omega t\right) - \frac{1}{4}m\omega(-\hat{q}_1 + \hat{q}_2) \sin\left(\frac{1}{2}\omega t\right) \]
\[ + \frac{1}{2}(\hat{p}_1 + \hat{p}_2) \cos\left(\frac{3}{2}\omega t\right) + \frac{1}{2}(-\hat{p}_1 + \hat{p}_2) \cos\left(\frac{1}{2}\omega t\right) \]  
(2.34d)

In these formulae we have denoted the initial values by a hat. It is these initial values that are the independent degrees of freedom of this system; in particular, only they have independent Poisson brackets. It is straightforward to verify that if the initial variables have Poisson brackets then the time evolution given in (2.34) preserves this feature. That is, the only non-zero equal time brackets are:

\[ \{ q_1(t), p_1(t) \} = 1 = \{ q_2(t), p_2(t) \} \]
(2.35)

We emphasize that the bracket algebras for different times are not independently specifiable. Since the variables at time \( t \) are uniquely determined in terms of their initial values, the Poisson bracket algebra at time \( t \) is determined by time evolution and by the initial bracket algebra.

The model at this stage should be thought of as analogous to temporal gauge QED or Yang-Mills, or to synchronous gauge gravity. A constraint in these models is a relation between the initial value variables, and such constraints are reduced by imposing a gauge condition on the initial value surface. It is only in exceptional cases that we can find
gauge conditions which are preserved under time evolution. Initial value gauge conditions always imply some relation between the later canonical variables, but very seldom the same relation. Unless we choose what we call the variables of the reduced theory to compensate for this change, the reduced brackets will become non-canonical because at any instant they are the Dirac brackets associated with different gauge conditions.

Let us suppose that the reduction of our oscillator model is accomplished by setting \( \hat{q}_2 = 0 = \hat{p}_2 \). Let us further suppose that we take as our reduced variables, \( v^1(t) \equiv q_1(t) \) and \( v^2(t) \equiv p_1(t) \). Since the constraint and its conjugate gauge condition affect only the initial values we see from (2.34) that the reduced variables have the following evolution:

\[
\begin{align*}
v^1(t) &= \frac{1}{2}\hat{q}_1 \cos \left( \frac{3}{2} \omega t \right) + \frac{1}{2}\hat{q}_1 \cos \left( \frac{1}{2} \omega t \right) + \frac{1}{3m\omega} \hat{p}_1 \sin \left( \frac{3}{2} \omega t \right) + \frac{1}{m\omega} \hat{p}_1 \sin \left( \frac{1}{2} \omega t \right) \\
v^2(t) &= -\frac{3}{4} m\omega \hat{q}_1 \sin \left( \frac{3}{2} \omega t \right) - \frac{1}{4} m\omega \hat{q}_1 \sin \left( \frac{1}{2} \omega t \right) + \frac{1}{2} \hat{p}_1 \cos \left( \frac{3}{2} \omega t \right) + \frac{1}{2} \hat{p}_1 \cos \left( \frac{1}{2} \omega t \right)
\end{align*}
\]

(2.36a) (2.36b)

It follows that the only non-zero equal time Poisson bracket is:

\[
\{ v^1(t), v^2(t) \} = \frac{4}{3} \cos^2 \left( \frac{1}{2} \omega t \right) - \frac{1}{3} \cos^2 \left( \omega t \right)
\]

(2.37)

Although it is canonical at \( t = 0 \), it is not so later on, and it can even pass through zero!

If we ignore the fact that the bracket matrix becomes non-invertible at \( \omega t \approx (2n + 1)\pi \pm .12\pi \) then the construction gives the following canonical variables:

\[
\begin{align*}
q(t) &\equiv v^1(t) \\
p(t) &\equiv \frac{3v^2(t)}{4 \cos^2 \left( \frac{1}{2} \omega t \right) - \cos^2 \left( \omega t \right)}
\end{align*}
\]

(2.38a) (2.38b)

The associated Hamiltonian is:

\[
H(t, q, p) = \left[ \frac{4}{3} \cos^2 \left( \frac{1}{2} \omega t \right) - \frac{1}{3} \cos^2 \left( \omega t \right) \right] \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2 \left[ \frac{4}{3} \cos^2 \left( \frac{1}{2} \omega t \right) - \frac{1}{3} \cos^2 \left( \omega t \right) \right]^{-2} \]

(2.39)
Note that although this Hamiltonian generates time evolution for $q(t)$ and $p(t)$, it does not do so for either the reduced variables or for the original ones. The variable $v^2(t)$ acquires additional time dependence from relation (2.38b), and the original variables have the following expressions in terms of $q(t)$ and $p(t)$:

$$q_1(t) = q(t) \quad (2.40a)$$

$$p_1(t) = \left[ \frac{4}{3} \cos^2 \left( \frac{1}{2} \omega t \right) - \frac{1}{3} \cos^2 \left( \omega t \right) \right] p(t) \quad (2.40b)$$

$$q_2(t) = \frac{-2 \sin \left( \frac{1}{2} \omega t \right) \sin \left( \frac{3}{2} \omega t \right)}{4 \cos^2 \left( \frac{1}{2} \omega t \right) - \cos^2 \left( \omega t \right)} q(t) + \frac{4}{3} \sin^2 \left( \frac{1}{2} \omega t \right) \sin \left( \omega t \right) \frac{p(t)}{m \omega} \quad (2.40c)$$

$$p_2(t) = \frac{-3 \cos^2 \left( \frac{1}{2} \omega t \right) \sin \left( \omega t \right)}{4 \cos^2 \left( \frac{1}{2} \omega t \right) - \cos^2 \left( \omega t \right)} m \omega q(t) + \frac{2}{3} \sin \left( \frac{1}{2} \omega t \right) \sin \left( \frac{3}{2} \omega t \right) p(t) \quad (2.40d)$$

Although the preceding canonical formalism is valid, it is not simple because the reduced variables were badly chosen. A much more convenient choice is:

$$v^1(t) \equiv q_1(t) + q_2(t) = \hat{q}_1 \cos \left( \frac{3}{2} \omega t \right) + \frac{2}{3m \omega} \hat{p}_1 \sin \left( \frac{3}{2} \omega t \right) \quad (2.41a)$$

$$v^2(t) \equiv p_1(t) + p_2(t) = -\frac{3}{2} m \omega q_1 \sin \left( \frac{3}{2} \omega t \right) + \hat{p}_1 \cos \left( \frac{3}{2} \omega t \right) \quad (2.41b)$$

The resulting reduced bracket algebra is canonical:

$$\{ v^1(t), v^2(t) \} = 1 \quad (2.42a)$$

and the Hamiltonian is time independent:

$$H = \frac{1}{2m} \left( v^2 \right)^2 + \frac{9}{8} m \omega^2 \left( v^1 \right)^2 \quad (2.42b)$$

This Hamiltonian generates the evolution of $v^1(t)$ and $v^2(t)$ but not that of the original variables. They acquire additional time dependence through the relations:

$$q_1(t) = \left[ 2 \cos^2 \left( \frac{1}{2} \omega t \right) - \cos^2 \left( \omega t \right) \right] v^1(t) - \frac{4}{3} \sin^2 \left( \frac{1}{2} \omega t \right) \sin \left( \omega t \right) \frac{v^2(t)}{m \omega} \quad (2.43a)$$
\[
p_1(t) = \cos^2\left(\frac{1}{2}\omega t\right) \sin(\omega t) m\omega v^1(t) + \left[\frac{2}{3} \cos^2\left(\frac{1}{2}\omega t\right) + \frac{1}{3} \cos^2(\omega t)\right] v^2(t) \tag{2.43b}
\]
\[
q_2(t) = \left[2 \sin^2\left(\frac{1}{2}\omega t\right) - \sin^2(\omega t)\right] v^1(t) + \frac{4}{3} \sin^2\left(\frac{1}{2}\omega t\right) \sin(\omega t) \frac{v^2(t)}{m\omega} \tag{2.43c}
\]
\[
p_2(t) = -\cos^2\left(\frac{1}{2}\omega t\right) \sin(\omega t) m\omega v^2(t) + \left[\frac{2}{3} \sin^2\left(\frac{1}{2}\omega t\right) + \frac{1}{3} \sin^2(\omega t)\right] v^1(t) \tag{2.43d}
\]

Another convenient choice is:

\[
v^1(t) \equiv q_1(t) - q_2(t) = \tilde{q}_1 \cos\left(\frac{1}{2}\omega t\right) + \frac{2}{m\omega} \tilde{p}_1 \sin\left(\frac{1}{2}\omega t\right) \tag{2.44a}
\]
\[
v^2(t) \equiv p_1(t) - p_2(t) = -\frac{1}{2} m\omega \tilde{q}_1 \sin\left(\frac{1}{2}\omega t\right) + \tilde{p}_1 \cos\left(\frac{1}{2}\omega t\right) \tag{2.44b}
\]

As before, the resulting reduced bracket algebra is canonical:

\[
\{v^1(t), v^2(t)\} = 1 \tag{2.45a}
\]

and the Hamiltonian is time independent:

\[
H = \frac{1}{2m} \left(v^2\right)^2 + \frac{1}{8} m\omega^2 \left(v^1\right)^2 \tag{2.45b}
\]

This Hamiltonian generates the evolution of \(v^1(t)\) and \(v^2(t)\). The original variables acquire additional time dependence through the relations:

\[
q_1(t) = \left[\frac{2}{3} \cos^2\left(\frac{1}{2}\omega t\right) + \frac{1}{3} \cos^2(\omega t)\right] v^1(t) + \frac{4}{3} \sin^2\left(\frac{1}{2}\omega t\right) \sin(\omega t) \frac{v^2(t)}{m\omega} \tag{2.46a}
\]
\[
p_1(t) = -\cos^2\left(\frac{1}{2}\omega t\right) \sin(\omega t) m\omega v^1(t) + \left[2 \cos^2\left(\frac{1}{2}\omega t\right) - \cos^2(\omega t)\right] v^2(t) \tag{2.46b}
\]
\[
q_2(t) = \left[-\frac{2}{3} \sin^2\left(\frac{1}{2}\omega t\right) - \frac{1}{3} \sin^2(\omega t)\right] v^1(t) + \frac{4}{3} \sin^2\left(\frac{1}{2}\omega t\right) \sin(\omega t) \frac{v^2(t)}{m\omega} \tag{2.46c}
\]
\[
p_2(t) = -\cos^2\left(\frac{1}{2}\omega t\right) \sin(\omega t) m\omega v^1(t) + \left[-2 \sin^2\left(\frac{1}{2}\omega t\right) + \sin^2(\omega t)\right] v^2(t) \tag{2.46d}
\]

Three points deserve mention. First, although the latter two formulations are significantly simpler than the first, all three give precisely the same evolution for \(q_i(t)\) and \(p_i(t)\). As long as we infer physics from these variables the seemingly different theories are in fact identical. Physicists are so conditioned to regard the Hamiltonian as an observable that
this point can not be overemphasized. A glance at the first Hamiltonian (2.39) suggests a harmonic oscillator with time dependent mass and frequency:

\[ m(t) = \frac{m}{\frac{4}{3} \cos^2\left(\frac{1}{2}\omega t\right) - \frac{1}{3} \cos^2(\omega t)} \quad (2.47a) \]

\[ \omega^2(t) = \left[ \cos^2\left(\frac{1}{2}\omega t\right) + \frac{1}{4} \cos^2(\omega t) \right] \omega^2 \quad (2.47b) \]

The second Hamiltonian (2.42b) is that of an oscillator of mass \( m \) and frequency \( \frac{3}{2} \omega \), while the third Hamiltonian (2.45b) is that of an oscillator with mass \( m \) and frequency \( \frac{1}{2} \omega \). In fact the system is none of these things, it is rather a set of coupled oscillators which has been subjected to a constraint. This only becomes apparent by ignoring the Hamiltonian and concentrating instead upon the \( q_i(t) \)'s and the \( p_i(t) \)'s.

Note that although the reduced canonical variables provide a complete and minimal representation of the system’s dynamical degrees of freedom, there is nothing wrong with using as observables the overcomplete representation provided by the original variables. These will include some pure gauge degrees of freedom and some constrained ones. Of course the former are unphysical but the latter encode perfectly valid information, even if this information can be recovered from the reduced variables. The longitudinal electric field offers a familiar example. We really do not need this quantity since it can be recovered from a knowledge of the positions of all the charges, but there is nothing wrong with regarding it as an observable since it can be measured using the Lorentz force law.

The second point is that the reduced canonical variables do have a role to play. In the quantum theory they tell us how to describe states, and how the original variables act as operators upon these states. For example, suppose we quantize the first representation of the coupled oscillator system. A state in the Schrödinger picture position representation is described by a square integrable wavefunction, \( \psi(q,t) \). The time evolution of this wavefunction is generated by the Hamiltonian (2.39). The operators \( q_i(t) \) and \( p_i(t) \) act on such
a state through relations (2.40), where \( q(t) \psi(q, t) = q \psi(q, t) \) and \( p(t) \psi(q, t) = -i \frac{\partial \psi(q, t)}{\partial q} \).

Note that even in the Schrödinger picture the observables \( q_i(t) \) and \( p_i(t) \) have time dependence. In the Heisenberg picture the state is time independent and the evolution of the reduced canonical variables, \( q(t) \) and \( p(t) \), is generated by (2.39).

The final point concerns energy. We have stated that none of the Hamiltonians — (2.39), (2.42b) and (2.45b) — plays the usual role of the energy as an observable, so what does? The answer is the Hamiltonian of the original theory, after the imposition of our “gauge condition” and “constraint,” \( \hat{q}_2 = 0 = \hat{p}_2 \):

\[
E = \frac{\hat{p}_1^2}{2m} + \frac{\hat{p}_2^2}{2m} + \frac{1}{2} m \omega^2 \left( \frac{5}{4} \hat{q}_1^2 + \hat{q}_1 \hat{q}_2 + \frac{5}{4} \hat{q}_2^2 \right) \tag{2.48a}
\]

\[
\rightarrow \quad \frac{\hat{p}_2^2}{2m} + \frac{5}{8} m \omega^2 \hat{q}_1^2 \tag{2.48b}
\]

This energy does not generate time evolution for the reduced canonical variables in any of the three representations given above. For example, in the last representation we can use relations (2.34) and (2.46) to express the energy as:

\[
E(v_1, v_2, t) = \left[ \frac{9}{16} + \frac{7}{16} \cos(\omega t) \right] \frac{1}{2} m \omega^2 \left( v_1^1 \right)^2 \]

\[
- \frac{15}{8} \sin(\omega t) \omega v_1^1 v_2^2 + \left[ \frac{17}{8} - \frac{15}{8} \cos(\omega t) \right] \left( \frac{v_2^2}{2m} \right)^2 \tag{2.49}
\]

From (2.40a) it follows that:

\[
\{ E, v^1(t) \} = -\frac{15}{8} \sin(\omega t) \omega v^1(t) + \left[ \frac{17}{8} - \frac{15}{8} \cos(\omega t) \right] \frac{v^2(t)}{m} \neq \frac{v^2(t)}{m} = \dot{v}^1(t) \tag{2.50a}
\]

\[
\{ E, v^2(t) \} = -\left[ \frac{9}{16} + \frac{7}{16} \cos(\omega t) \right] m \omega^2 v^1(t) + \frac{15}{8} \sin(\omega t) \omega v^2(t) \neq -\frac{1}{4} m \omega^2 v^1(t) = \dot{v}^2(t) \tag{2.50b}
\]

In spite of the fact that the physical energy \( E \) does not generate time evolution, it is conserved:

\[
\frac{dE}{dt} = \frac{\partial E}{\partial v^1} \dot{v}^1 + \frac{\partial E}{\partial v^2} \dot{v}^2 + \frac{\partial E}{\partial t} = 0 \tag{2.51}
\]
This potential for a disagreement between the physical energy and the Hamiltonian which generates time evolution is a general feature of reduced canonical systems.

3. Correspondence With Gauge Theories

The purpose of this section is to study a simple model in which the various features of the construction and the context in which it takes place can be clearly understood. The model is scalar QED in flat space, the Lagrangian for which is:

\[ L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \left( \partial_\mu - ie A_\mu \right) \phi^* \left( \partial^\mu + ie A^\mu \right) \phi \]  

(3.1)

Here \( F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \) is the field strength tensor, \( \phi \) is a complex scalar field, \( e \) is the electromagnetic coupling constant, and we are using a spacelike metric. The Lagrangian is invariant under a gauge transformation:

\[ A_\mu(x) \mapsto A_\mu(x) - \partial_\mu \theta(x) \]  

(3.2a)

\[ \phi(x) \mapsto \exp \left[ ie \theta(x) \right] \phi(x) \]  

(3.2b)

which is parametrized by an arbitrary real scalar function, \( \theta(x) \). By imposing temporal gauge:

\[ A_0(x) = 0 \]  

(3.3)

one obtains a constrained canonical system which still possesses invariance under gauge transformations parametrized by time independent functions \( \theta(x) \). We will show how this system can be reduced to a completely gauge fixed system containing only dynamical degrees of freedom, and we will explain why the resulting Poisson brackets can fail to be canonical. We will then change variables to obtain canonical Poisson brackets and infer the Hamiltonian which generates time evolution for the reduced system.

It is a simple matter to show that in temporal gauge the momenta canonically conjugate to \( A_i \), \( \phi \) and \( \phi^* \) are, respectively:

\[ E_i(t, \vec{x}) = \dot{A}_i(t, \vec{x}) \]  

(3.4a)
\[ \pi(t, \vec{x}) = \dot{\phi}^*(t, \vec{x}) \] (3.4b)

\[ \pi^*(t, \vec{x}) = \dot{\phi}(t, \vec{x}) \] (3.4c)

What it means to be “canonical” is that the non-zero equal time Poisson brackets are:

\[ \{ A_i(t, \vec{x}), E_j(t, \vec{y}) \} = \delta_{ij} \delta^3(\vec{x} - \vec{y}) \] (3.5a)

\[ \{ \phi(t, \vec{x}), \pi(t, \vec{y}) \} = \delta^3(\vec{x} - \vec{y}) \] (3.5b)

\[ \{ \phi^*(t, \vec{x}), \pi^*(t, \vec{y}) \} = \delta^3(\vec{x} - \vec{y}) \] (3.5c)

The Hamiltonian is:

\[ H = \int d^3x \left\{ \frac{1}{2} E_i E_i + \frac{1}{4} F_{ij} F_{ij} + \pi^* \pi + \left( \partial_i - ie A_i \right) \phi^* \left( \partial_i + ie A_i \right) \phi \right\} \] (3.6)

It is straightforward to check that this functional generates time evolution. That is, the Poisson brackets of the Hamiltonian with the canonical coordinates give relations (3.4a) through (3.4c), and the Poisson brackets of the conjugate momenta give the canonical formulation of the dynamical Euler-Lagrange equations:

\[ \dot{E}_i = \{ E_i, H \} = \partial_j F_{ji} + ie \phi^* \left( \partial_i + ie A_i \right) \phi - ie \phi \left( \partial_i - ie A_i \right) \phi^* \] (3.7a)

\[ \dot{\pi} = \{ \pi, H \} = \left( \partial_i - ie A_i \right) \left( \partial_i - ie A_i \right) \phi^* \] (3.7b)

\[ \dot{\pi}^* = \{ \pi, H \} = \left( \partial_i + ie A_i \right) \left( \partial_i + ie A_i \right) \phi \] (3.7c)

The non-dynamical Euler-Lagrange equation — the one obtained by variation with respect to \( A_0 \) — is not realized through the definition of time evolution. It must be imposed as a constraint:

\[ \partial_i E_i + ie \left( \pi \phi - \pi^* \phi^* \right) = 0 \] (3.8)

Of course the left hand side of this constraint is also the generator of infinitesimal, time-independent gauge transformations.
Although this system possesses a time independent gauge symmetry, it is still complete in the sense that the equations of evolution uniquely determine the fields at any time in terms of those on an initial value surface. To economize the notation let us refer to a general field as \( \psi_a(t, \vec{x}) \) and an initial value configuration as \( \hat{\psi}_a(\vec{x}) \). The evolution equations (3.4) and (3.7) uniquely determine the former in terms of the latter. Even though we cannot exhibit this relation as we could for the coupled oscillator system of the previous section — cf. relations (2.34) — we can still represent it functionally:

\[
\psi_a(t, \vec{x}) = \Psi[\hat{\psi}]_a(t, \vec{x})
\]

(3.9)

Just as with the coupled oscillator system discussed in the previous section, it is the \( \hat{\psi}_a(\vec{x}) \)'s that represent the dynamical degrees of freedom of the unconstrained system. Only their Poisson brackets are independent, for example:

\[
\{\psi_a(t, \vec{x}), \psi_b(t, \vec{y})\} = \int d^3u \int d^3v \frac{\delta \Psi[\hat{\psi}]_a(t, \vec{x})}{\delta \hat{\psi}_c(\vec{u})} \{\hat{\psi}_c(\vec{u}), \hat{\psi}_d(\vec{v})\} \frac{\delta \Psi[\hat{\psi}]_b(t, \vec{y})}{\delta \hat{\psi}_d(\vec{v})}
\]

(3.10)

The equal time bracket algebra remains canonical because of the way time evolution acts; in our notation, because of the special way the functionals \( \Psi[\hat{\psi}]_a(t, \vec{x}) \) depend upon the initial value configurations.

The constraint (3.8) represents a relation between the \( \hat{\psi}_a(\vec{x}) \)'s. The theory is reduced by identifying a gauge condition on the \( \hat{\psi}_a(\vec{x}) \)'s that can be imposed by a unique, field dependent transformation of the residual symmetry group. Together the constraint and this gauge condition serve to eliminate a conjugate pair of the \( \hat{\psi}_a(\vec{x}) \)'s. This does not change the way time evolution makes the fields depend upon their initial configurations, it only fixes the values of a conjugate pair of these initial configurations. The result is to change the equal time bracket algebra (3.10) since the Poisson brackets of the initial configurations are replaced with Dirac brackets.

After reduction the full set of \( \psi_a(t, \vec{x}) \)'s provides an overcomplete description of the system at any time. Although it is always possible to identify a minimal set of reduced
variables whose equal time bracket algebra is canonical, making such an identification is not necessarily easy and we often settle for a set of reduced variables whose bracket algebra is not degenerate but also not canonical. (If the bracket algebra becomes degenerate it means that the reduced variables do not provide a complete description of physics. That is, we cannot use the reduced set of $\psi_a$’s to recover the reduced $\hat{\psi}_a$’s which are the true dynamical degrees of freedom.) This set is typically the evolution of whatever initial configurations remain after reduction. For example, in reducing the coupled oscillator system of the last section we eliminated $\hat{q}_2$ and $\hat{p}_2$, leaving $\hat{q}_1$ and $\hat{p}_1$. The natural first choice for reduced variables was accordingly $q_1(t)$ and $p_1(t)$. We then discovered that it would have been better to choose either $q_1(t) + q_2(t)$ and $p_1(t) + p_2(t)$ or $q_1(t) - q_2(t)$ and $p_1(t) - p_2(t)$ because these variables have a canonical equal time bracket algebra while the original choice does not. We shall now witness a similar phenomenon in reducing scalar electrodynamics.

A natural gauge condition compatible with (3.8) is:

$$\partial_i A_i(0, \vec{x}) = 0 \quad (3.11)$$

From this and relation (3.4a) we infer the following evolution for the divergence of the vector potential:

$$\partial_i A_i(t, \vec{x}) = -ie \int_0^t ds \left[ \pi(s, \vec{x}) \phi(s, \vec{x}) - \pi^*(s, \vec{x}) \phi^*(s, \vec{x}) \right] \quad (3.12)$$

Of course (3.11) does not completely fix the gauge; it is still possible to make time independent, harmonic transformations. This freedom can be eliminated with a surface condition. A typical choice for the latter would be setting the normal component of $A_i(0, \vec{x})$ to zero on the “surface at infinity,” which could realized as the limit of successively larger spheres. Rather than burdening the formalism with a cumbersome limiting procedure we will instead subsume the surface condition into the asymptotic fall-off usually assumed for the
gauge invariant part of the vector potential in order to make the magnetic field energy finite.

There is a very close relation between fixing the harmonic gauge freedom and dividing the vector potential — and all other vector fields — into “transverse” and “longitudinal” components. With the just-stated convention the transverse and longitudinal parts of a vector field $f_i(t, \vec{x})$ which obeys the asymptotic fall-off condition are is defined as follows:

$$f^T_i(t, \vec{x}) \equiv f_i(t, \vec{x}) - \frac{\partial}{\partial x^i} \int d^3y \, G(\vec{x}; \vec{y}) \partial_j f_j(t, \vec{y}) \quad (3.13a)$$

$$f^L_i(t, \vec{x}) \equiv \frac{\partial}{\partial x^i} \int d^3y \, G(\vec{x}; \vec{y}) \partial_j f_j(t, \vec{y}) \quad (3.13b)$$

where the Green’s function is, $G(\vec{x}; \vec{y}) \equiv -\left(4\pi \|\vec{x} - \vec{y}\|\right)^{-1}$.

Since the longitudinal components of the electric field and the vector potential are entirely constrained it is natural to take the transverse components to be among the reduced dynamical variables, along with the scalar field and its conjugate momentum. Although the Dirac brackets of these variables are initially canonical:

$$\{A^T_i(0, \vec{x}), E^T_j(0, \vec{y})\} = \delta_{ij} \delta^3(\vec{x} - \vec{y}) + \frac{\partial}{\partial x^i} \frac{\partial}{\partial y^j} G(\vec{x}; \vec{y}) \quad (3.14a)$$

$$\{\phi(0, \vec{x}), \pi(0, \vec{y})\} = \delta^3(\vec{x} - \vec{y}) \quad (3.14b)$$

$$\{\phi^*(0, \vec{x}), \pi^*(0, \vec{y})\} = \delta^3(\vec{x} - \vec{y}) \quad (3.14c)$$

time evolution makes the equal time bracket algebra non-canonical. We cannot exhibit the relation in closed form the way we did for the coupled oscillator system of the previous section but we can evaluate the second time derivative at $t = 0$ easily enough. This is because reduction affects only the initial value fields; the time evolution equations are unchanged from (3.4) and (3.7). We can therefore use the evolution equations to reduce any number of time derivatives of an initial bracket to brackets of the initial value fields,
which can then be evaluated using (3.14). For example, we compute:

\[
\left( \frac{\partial}{\partial t} \right)^2 \{ \phi(t, \vec{x}), \pi(t, \vec{y}) \} \bigg|_{t=0} = \frac{\partial}{\partial t} \left( \left\{ \pi^*(t, \vec{x}), \pi(t, \vec{y}) \right\} + \left\{ \phi(t, \vec{x}), (\partial_i - ieA_i)^2 \phi^*(t, \vec{y}) \right\} \right) \bigg|_{t=0} \\
= \left\{ (\partial_i + ieA_i)^2 \phi(0, \vec{x}), \pi(0, \vec{y}) \right\} + \left\{ \pi^*(0, \vec{x}), (\partial_i - ieA_i)^2 \phi^*(0, \vec{y}) \right\} \\
+ \left\{ \pi^*(0, \vec{x}), (\partial_i - ieA_i)^2 \phi^*(0, \vec{y}) \right\} + \left\{ \phi(0, \vec{x}), (\partial_i - ieA_i)^2 \pi(0, \vec{y}) \right\} \\
+ \left\{ \phi(0, \vec{x}), -ieE_i (\partial_i - ieA_i) \phi^*(0, \vec{y}) \right\} + \left\{ \phi(0, \vec{x}), -ie (\partial_i - ieA_i) E_i \phi^*(0, \vec{y}) \right\} \\
= -e^2 \frac{\partial G(\vec{y}; \vec{x})}{\partial y^i} \hat{\phi}(\vec{x}) \left[ \frac{\partial}{\partial y^i} - ie \hat{A}_i^T(\vec{y}) \right] \hat{\phi}^*(\vec{y}) \\
- e^2 \left[ \frac{\partial}{\partial y^i} - ie \hat{A}_i^T(\vec{y}) \right] \left\{ \hat{\phi}^*(\vec{y}) \frac{\partial G(\vec{y}; \vec{x})}{\partial y^i} \right\} \hat{\phi}(\vec{x})
\]

Of course the failure of the \( \phi - \pi \) bracket to remain canonical is due to the fact that reduction makes the longitudinal components of the vector potential and the electric field depend upon the charge fields.

One consequence of (3.15) is that the bracket matrix contains explicit time dependence in addition to being non-canonical. To see this note that the bracket matrix must be time dependent because the second derivative of one of its components fails to vanish. If this time dependence were exclusively implicit — that is, if it derived only from the bracket matrix’s dependence upon the time dependent dynamical variables — then the initial matrix elements would depend upon the reduced dynamical variables. Because they do not we infer that the bracket matrix must harbor explicit as well as implicit time dependence.

We saw in the discussion associated with (2.8) that explicit time dependence in the bracket matrix precludes the existence of a Hamiltonian which generates time evolution for the reduced dynamical variables. This is a little strange because (3.6) is the conserved, gauge invariant energy functional for scalar electrodynamics. However, it is easy to verify that the change reduction effects in the longitudinal electric field prevents even the initial
brackets from agreeing with (3.4) and (3.7):

\[
\left\{ \phi(0, \vec{x}), H \right\} = \pi^*(0, \vec{x}) \\
+ e^2 \int d^3y \left[ \pi(0, \vec{y}) \phi(0, \vec{y}) - \pi^*(0, \vec{y}) \phi^*(0, \vec{y}) \right] G(\vec{y}; \vec{x}) \phi(0, \vec{x})
\]  

(3.16a)

\[
\left\{ \pi(0, \vec{x}), H \right\} = \left[ \frac{\partial}{\partial x^i} - ie A^T_i(0, \vec{x}) \right] \left[ \frac{\partial}{\partial x^i} - ie A^T_i(0, \vec{x}) \right] \phi^*(0, \vec{x}) \\
- e^2 \int d^3y \left[ \pi(0, \vec{y}) \phi(0, \vec{y}) - \pi^*(0, \vec{y}) \phi^*(0, \vec{y}) \right] G(\vec{y}; \vec{x}) \pi(0, \vec{x})
\]  

(3.16b)

Note again that our result is stronger than just that (3.6) fails to generate time evolution: there does not exist a Hamiltonian which generates time evolution for these reduced variables.

As was explained in the last section, the problem with finding a Hamiltonian that generates time evolution derives from an unfortunate choice of the reduced variables. We emphasize that there is nothing dynamically wrong with the choice we made. It is the natural one, and it does provide a complete and minimal description of the physics of scalar electrodynamics. Our choice is not even particularly inconvenient for computation; we will see in section 6 that it can be written in the functional formalism almost as simply as any other choice. The only problem comes if we insist on an explicit operator formalism in which time evolution is generated by a Hamiltonian. The cure for this problem is a necessarily time dependent field redefinition to canonical variables.

There are many sets of canonical variables but the simplest is surely that which is related to our set of reduced variables through a time dependent gauge transformation. The gauge parameter is:

\[
\theta(t, \vec{x}) = \int_0^t ds \alpha_0(s, \vec{x})
\]

(3.17a)

\[
\alpha_0(t, \vec{x}) \equiv ie \int d^3y G(\vec{x}; \vec{y}) \left\{ \pi(t, \vec{y}) \phi(t, \vec{y}) - \pi^*(t, \vec{y}) \phi^*(t, \vec{y}) \right\}
\]

(3.17b)

We will adopt the convention that the Greek or Latin letter which denotes one of the old variables goes over in the new variables to the corresponding Latin or Greek letter
respectively. The new variables are:

\[ \alpha_i^T(t, \bar{x}) \equiv A_i^T(t, \bar{x}) \]  
(3.18a)

\[ \epsilon_i^T(t, \bar{x}) \equiv E_i^T(t, \bar{x}) \]  
(3.18b)

\[ f(t, \bar{x}) \equiv \exp \left[ ie \theta(t, \bar{x}) \right] \phi(t, \bar{x}) \]  
(3.18c)

\[ p(t, \bar{x}) \equiv \exp \left[ -ie \theta(t, \bar{x}) \right] \left\{ \pi(t, \bar{x}) - ie \alpha_0(t, \bar{x}) \phi^*(t, \bar{x}) \right\} \]  
(3.18d)

\[ f^*(t, \bar{x}) \equiv \exp \left[ -ie \theta(t, \bar{x}) \right] \phi^*(t, \bar{x}) \]  
(3.18e)

\[ p^*(t, \bar{x}) \equiv \exp \left[ ie \theta(t, \bar{x}) \right] \left\{ \pi^*(t, \bar{x}) + ie \alpha_0(t, \bar{x}) \phi(t, \bar{x}) \right\} \]  
(3.18f)

Since the quantity \( \alpha_0(t, \bar{x}) \) is a gauge invariant it has the same form in terms of \( f \) and \( p \) as \( \phi \) and \( \pi \):

\[ \alpha_0(t, \bar{x}) = ie \int d^3y G(\bar{x}; \bar{y}) \left\{ p(t, \bar{y}) f(t, \bar{y}) - p^*(t, \bar{y}) f^*(t, \bar{y}) \right\} \]  
(3.19)

By differentiating these relations and then using (3.4) and (3.7) we infer the following evolution equations:

\[ \dot{\alpha}_i^T(t, \bar{x}) = \epsilon_i^T(t, \bar{x}) \]  
(3.20a)

\[ \dot{\epsilon}_i^T(t, \bar{x}) = \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^j} \alpha_i^T(t, \bar{x}) + ie f^*(t, \bar{x}) \left[ \partial_i + ie \alpha_i^T(t, \bar{x}) \right] f(t, \bar{x}) \]  
(3.20b)

\[ -ie f(t, \bar{x}) \left[ \partial_i - ie \alpha_i^T(t, \bar{x}) \right] f^*(t, \bar{x}) \]

\[ \dot{f}(t, \bar{x}) = \pi^*(t, \bar{x}) - ie \alpha_0(t, \bar{x}) f(t, \bar{x}) \]  
(3.20c)

\[ \dot{p}(t, \bar{x}) = ie \alpha_0(t, \bar{x}) p(t, \bar{x}) + \left[ \frac{\partial}{\partial x^i} - ie \alpha_i^T(t, \bar{x}) \right] \left[ \frac{\partial}{\partial x^i} - ie \alpha_i^T(t, \bar{x}) \right] f^*(t, \bar{x}) \]  
(3.20d)

\[ \dot{f}^*(t, \bar{x}) = \pi(t, \bar{x}) + ie \alpha_0(t, \bar{x}) f^*(t, \bar{x}) \]  
(3.20e)

\[ \dot{p}^*(t, \bar{x}) = -ie \alpha_0(t, \bar{x}) p^*(t, \bar{x}) + \left[ \frac{\partial}{\partial x^i} + ie \alpha_i^T(t, \bar{x}) \right] \left[ \frac{\partial}{\partial x^i} + ie \alpha_i^T(t, \bar{x}) \right] f(t, \bar{x}) \]  
(3.20f)

Of course these relations are generated by the Hamiltonian (3.6), which in the new variables has the form:

\[ H = \int d^3x \left\{ \frac{1}{2} \epsilon_i^T \epsilon_i^T + \frac{1}{2} \partial_i \alpha_j^T \partial_j \alpha^T + \frac{1}{2} \partial_i \alpha_0 \partial_i \alpha_0 + p^* p + \left( \partial_i - ie \alpha_i^T \right) f^* \left( \partial_i + ie \alpha_i^T \right) f \right\} \]  
(3.21)
The reason the old Hamiltonian can generate evolution for these variables and not for our first choice of reduced dynamical variables is that the field redefinition contains explicit time dependence through the time integration in (3.17a).

It should now be apparent — and it can be, with some difficulty, checked using relations (3.18) — that the bracket algebra is canonical. That is, the non-zero equal time brackets are:

\[
\{ \alpha^T_i (t, \vec{x}), \epsilon^T_j (t, \vec{y}) \} = \delta_{ij} \delta^3 (\vec{x} - \vec{y}) + \frac{\partial}{\partial x^i} \frac{\partial}{\partial y^j} G(\vec{x}; \vec{y}) \quad (3.22a)
\]

\[
\{ f(t, \vec{x}), p(t, \vec{y}) \} = \delta^3 (\vec{x} - \vec{y}) \quad (3.22b)
\]

\[
\{ f^*(t, \vec{x}), p^*(t, \vec{y}) \} = \delta^3 (\vec{x} - \vec{y}) \quad (3.22c)
\]

It should also be apparent that the canonical formalism we have constructed is just that which follows from the invariant action (3.1) by the imposition of Coulomb gauge:

\[
\partial_i A_i (t, \vec{x}) = 0 \quad (3.23)
\]

Relation (3.19) comes from solving the constraint equation:

\[
\partial_i \partial_i A_0 (t, \vec{x}) = ie \left[ \dot{\phi}^* (t, \vec{x}) \phi (t, \vec{x}) - \dot{\phi} (t, \vec{x}) \phi^* (t, \vec{x}) \right] \quad (3.24)
\]

subject to a surface condition at spatial infinity which fixes the freedom to perform time dependent, harmonic gauge transformations.

4. Perturbative Gravity Around Flat Space On \( T^3 \times R \)

This section is divided into four parts. In the first we describe the canonical formalism for gravity in a general closed spatial manifold. The second part introduces the mode and tensor decompositions we shall use for \( T^3 \times R \). In the third part we apply this mode decomposition to perturbation theory around flat space. It is here that we impose the constraints and fix the gauge to obtain the reduced theory. In the final part we obtain
a reduced canonical formalism and we show that our result agrees in the limit of infinite toroidal radius and localized initial value data with that obtained by A.D.M. [3] for open, asymptotically flat space.

(1) — Description of Canonical Formalism.

We define the lapse $N^0$ and the shift $N^i$ via the invariant interval:

$$ds^2 = - \left(N^0\right)^2 dt^2 + \gamma_{ij} \left(dx^i + N^i dt\right) \left(dx^j + N^j dt\right). \quad (4.1)$$

This implies the 4–metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ are:

$$g_{\mu\nu} = \begin{pmatrix} - (N^0)^2 + N^k N^l \gamma_{kl} & N^k \gamma_{kj} \\ \gamma_{ik} N^k & \gamma_{ij} \end{pmatrix} \quad (4.2)$$

$$g^{\mu\nu} = \frac{1}{(N^0)^2} \begin{pmatrix} -1 & N^j \\ N^i & (N^0)^2 \gamma_{ij} - N^i N^j \end{pmatrix} \quad (4.3)$$

The usual Hilbert action for gravity

$$S = \int d^4x \left[ \frac{1}{\kappa^2} R \sqrt{-g} \right] \quad (4.4a)$$

can be written in canonical form as:

$$S = \int d^4x \left[ -\dot{\pi}^{ij} \gamma_{ij} - N^\mu \mathcal{H}_\mu \right] \quad (4.4b)$$

An integration by parts was used to arrive at (4.4b) from (4.4a) and the following definitions were used:

$$\pi^{ij} \equiv \frac{\sqrt{\gamma}}{2N^0 \kappa^2} \left( \gamma^{ik} \gamma^{jl} - \gamma^{ij} \gamma^{kl} \right) \left( \dot{\gamma}_{kl} - N_{k;l} - N_{l;k} \right) \quad (4.5a)$$

$$\mathcal{H}_0 \equiv \kappa^2 \sqrt{\gamma} \left( \gamma_{ik} \gamma_{jl} - \frac{1}{2} \gamma_{ij} \gamma_{kl} \right) \pi^{ij} \pi^{kl} - \frac{\sqrt{\gamma}}{\kappa^2} R \quad (4.5b)$$

$$\mathcal{H}_i \equiv -2 \gamma_{ij} \pi^{jl}_{;l} \quad (4.5c)$$
In the previous expressions a semicolon indicates covariant differentiation on the spatial sections using the connection compatible with the 3–metric, \( \gamma_{ij} \); \( \mathcal{R} \) is the Ricci scalar formed from \( \gamma_{ij} \).

In these variables the Hamiltonian is,

\[
H = \int d^3x \, N^\mu \mathcal{H}_\mu \tag{4.6}
\]

Variations of it with respect to \( \pi^{ij} \) and \( \gamma_{ij} \) give us the evolution equations,

\[
\dot{\gamma}_{ij} = \frac{2\kappa^2}{\sqrt{\gamma}} N^0 \left( \pi_{ij} - \frac{1}{2} \gamma_{ij} \pi \right) + N_{i; j} + N_{j; i} \tag{4.7a}
\]

\[
\dot{\pi}^{ij} = -\frac{\sqrt{\gamma}}{\kappa^2} N^0 \left( \mathcal{R}^{ij} - \frac{1}{2} \gamma^{ij} \mathcal{R} \right) + \frac{\kappa^2}{2 \sqrt{\gamma}} N^0 \gamma^{ij} \left( \pi^{lm} \pi_{lm} - \frac{1}{2} \pi^2 \right)
\]

\[
- \frac{2\kappa^2}{\sqrt{\gamma}} N^0 \left( \pi^{il} \pi^{lj} - \frac{1}{2} \pi \pi^{ij} \right) + \frac{\sqrt{\gamma}}{\kappa^2} \left( N^{0;ij} - \gamma^{ij} N^{0;l}_{\ ;l} \right)
\]

\[
+ \left( \pi^{ij} N^l \right)_{;l} - N^{i}_{\ ;l} \pi^{lj} - N^{j}_{\ ;l} \pi^{li} \tag{4.7b}
\]

while variation with respect to \( N^\mu \) gives the constraint equation,

\[
\mathcal{H}_\mu = 0 \tag{4.7c}
\]

Of course (4.7a) is just a restatement of the definition (4.5a) of the conjugate momentum. Relations (4.7b) are canonical versions of the six \( g_{ij} \) Euler-Lagrange equations; the constraints (4.7c) are linear combinations of the four \( g_{\mu 0} \) equations.

We imagine the volume gauge to have been fixed by specifying the lapse and shift, possibly as functionals of the 3-metric and its conjugate momentum. Such a gauge condition eliminates the ability to perform diffeomorphisms which are locally time dependent, as witness the fact that the Cauchy problem has a unique solution for fixed (and non-degenerate) lapse and shift. Just as with temporal gauge in scalar electrodynamics, our
gravitational gauge leaves a residual symmetry of transformations which are completely
caracterized by their action on the initial value surface, and by the condition that they
do not affect the lapse and shift.

Suppose we represent a general infinitesimal diffeomorphism, \( x^\mu \mapsto x^\mu + \theta^\mu(x) \), using
the parameter \( \theta^\mu(x) \). It is a simple exercise to show that the 4-metric is changed by the
following amount:

\[
\delta_\theta(g^{\mu\nu}) = \theta^{\mu\rho} g^{\rho\mu} + \theta^\nu_{\ ,\rho} g^{\mu\rho} - g^{\mu\rho} \theta^{\rho\cdot}.
\]

By requiring that \( \delta_\Theta g^{\mu0} = 0 \) we see that the residual transformations, \( \Theta[\hat{\theta}](t, \vec{x}) \), are
characterized by their initial values, \( \hat{\theta}(\vec{x}) \), and by the following evolution equations:

\[
\dot{\Theta}^0 = \Theta^0 \cdot N^j - \frac{N^0}{N^0} \Theta^\rho
\]

As with temporal gauge scalar electromagnetism, the constraints generate residual symmetry transformations. That is, if we define,

\[
\mathcal{H}[\hat{\theta}] = \int d^3x \left\{ \hat{\theta}^0(\vec{x}) N^\mu(0, \vec{x}) \mathcal{H}_\mu(0, \vec{x}) + \hat{\theta}^i(\vec{x}) \mathcal{H}_i(0, \vec{x}) \right\},
\]

then explicit calculation shows that

\[
\{ \hat{\gamma}_{ij}, \mathcal{H}[\hat{\theta}] \} = \hat{\gamma}_{ij}^k \hat{\gamma}_{kj} + \hat{\gamma}_{kj} \hat{\gamma}_{ik} + \hat{\theta}^0 \cdot \hat{\gamma} \cdot \hat{\gamma}^0 + \hat{\theta}^0 \cdot i \hat{\gamma}_{jk} \hat{N}^k + \hat{\theta}^0 \cdot j \hat{\gamma}_{ik} \hat{N}^k
\]

\[
= -\delta_\Theta[\hat{\theta}](\hat{\gamma}_{ij})
\]

\[
\{ \hat{\pi}_{ij}, \mathcal{H}[\hat{\theta}] \} = \hat{\pi}^{ij} \hat{\theta}^0 + \sqrt{\hat{\gamma}} \left( \hat{\theta}^0 \cdot i \hat{N}^0 + \hat{\theta}^0 \cdot j \hat{N}^0 \cdot j + \hat{\theta}^0 \cdot j \hat{N}^0 \cdot j \right)
\]

\[
- \frac{\sqrt{\hat{\gamma}}}{\kappa^2} \hat{\gamma}^{ij} \left( \hat{\theta}^0 \cdot k \hat{N}^0 + 2\hat{\theta}^0 \cdot k \hat{N}^0 \cdot k + \hat{\theta}^0 \cdot k \hat{N}^0 \cdot k \right) - \left( \hat{N}^i \hat{\pi}^k \hat{N}^j + \hat{N}^j \hat{\pi}^i \hat{N}^k \right) \hat{\theta}^0 \cdot k
\]

\[
+ \hat{\pi}^i \hat{N}^k \hat{\theta}^0 \cdot k - \hat{\theta}^0 \cdot i \hat{\pi}^k \hat{N}^j - \hat{\theta}^0 \cdot j \hat{\pi}^i \hat{N}^k + (\hat{\pi}^i \hat{\pi}^k) \hat{\theta}^0 \cdot k
\]

\[
= -\delta_\Theta[\hat{\theta}](\hat{\pi}^{ij}).
\]
Note that this is not a definition. The left hand sides of (4.11) and (4.12) are defined by (4.10), (4.5b) and (4.5c), while the identifications on the right hand side are made by applying (4.8) to the canonical coordinates and taking any time derivatives from (4.9).

(2) — Mode Decomposition on $T^3 \times R$.

Now that the canonical formalism for a general space with closed spatial sections has been described it will be specialized to the treatment of $T^3 \times R$. The coordinate ranges are $t \in R$ and $0 \leq x^i < L$. The points $x^i = 0$ and $x^i = L$ are identified. Any function $f(t, \vec{x})$ can be decomposed in modes in the following way:

$$f(t, \vec{x}) = L^{-3/2} \sum_{n_1 = -\infty}^{\infty} \sum_{n_2 = -\infty}^{\infty} \sum_{n_3 = -\infty}^{\infty} \exp \left[ \frac{2\pi}{L} \vec{n} \cdot \vec{x} \right] \tilde{f}(t, \vec{n})$$

(4.13a)

$$\tilde{f}(t, \vec{n}) \equiv (2\pi)^{-3} L^{-3/2} \int_0^L dx_1 \int_0^L dx_2 \int_0^L dx_3 \exp \left[ -\frac{2\pi}{L} \vec{n} \cdot \vec{x} \right] f(t, \vec{x})$$

(4.13b)

Note that when $f(t, \vec{x})$ is real we have $\tilde{f}^*(t, \vec{n}) = \tilde{f}(t, -\vec{n})$.

In representing tensors such as $\gamma_{ij}$ and $\pi^{ij}$ it is convenient to decompose the index structure in a way that depends on the mode number. Let us define the 3-momentum, the transverse projection operator and the longitudinal inversion operator as follows:

$$\vec{k} \equiv \frac{2\pi}{L} \vec{n}$$

(4.14a)

$$T_{ij} \equiv \delta_{ij} - k_i k_j \frac{k^2}{k^2}$$

(4.14b)

$$L_{ij} \equiv \delta_{ij} - \frac{k_i k_j}{2k^2}$$

(4.14c)

For $\vec{k} \neq 0$ one can decompose any symmetric 2–tensor into three component pieces:

$$\tilde{f}_{jk} = \tilde{f}_{jk}^{tt} + \tilde{f}_{jk}^t + i \left( \tilde{f}_{j} k_k + \tilde{f}_{k} k_j \right)$$

(4.15a)

$$\tilde{f}_{ij}^{tt} \equiv \left( T_{ik} T_{jl} - \frac{1}{2} T_{ij} T_{kl} \right) \tilde{f}_{kl}$$

(4.15b)

$$\tilde{f}_{ij}^t \equiv \frac{1}{2} T_{ij} T_{kl} \tilde{f}_{kl} \equiv \frac{1}{2} T_{ij} \tilde{f}^t$$

(4.15c)
\[
\tilde{f}_j \equiv -\frac{i}{k^2} L_{jk} \tilde{f}_{jk}
\]

(4.15d)

Note that for each \( \vec{k} \neq 0 \) there are two independent transverse traceless components \( \tilde{f}_{ij}^{tt} \), three longitudinal components \( \tilde{f}_i \), and one independent transverse component \( \tilde{f}^t \).

Of course for \( \vec{k} = 0 \) all components satisfy the transversality condition. We therefore decompose the zero mode tensor into five transverse traceless components and one trace:

\[
\tilde{f}_{ij}(t, 0) = \tilde{f}_{ij}^{tt}(t, 0) + \frac{1}{3} \delta_{ij} \tilde{f}^{tr}(t)
\]

(4.16)

We can carry the decomposition over into position space through the inverse transform as follows:

\[
f_{ij} = \frac{1}{3} f^{tr} \delta_{ij} + f_{ij}^{tt} + f_{ij}^t + (f_{i,j} + f_{j,i})
\]

(4.17a)

\[
f^{tr}(t) \equiv L^{-3/2} \tilde{f}^{tr}(t)
\]

(4.17b)

\[
f_{ij}^{tt}(t, \vec{x}) \equiv L^{-3/2} \sum_{\vec{n}} \exp \left[ \frac{2\pi}{L} \vec{n} \cdot \vec{x} \right] \tilde{f}_{ij}^{tt}(t, \vec{n})
\]

(4.17c)

\[
f_{ij}^t(t, \vec{x}) \equiv L^{-3/2} \sum_{\vec{n} \neq 0} \exp \left[ \frac{2\pi}{L} \vec{n} \cdot \vec{x} \right] \tilde{f}_{ij}^t(t, \vec{n})
\]

(4.17d)

\[
f_i(t, \vec{x}) \equiv L^{-3/2} \sum_{\vec{n} \neq 0} \exp \left[ \frac{2\pi}{L} \vec{n} \cdot \vec{x} \right] \tilde{f}_i(t, \vec{n})
\]

(4.17e)

Note that the longitudinal and transverse components contain no spatial zero modes while \( f^{tr} \) is all zero mode. The transverse traceless components alone contain both zero and non-zero modes.

(3) — Perturbing Around Flat Space

Since \( \eta_{\mu\nu} \) is a solution of Einstein’s equations in \( T^3 \times R \) we can perturb around flat space, \( g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \). (We define the constant \( \kappa^2 \equiv 16\pi G \).) The corresponding expansions for the various canonical variables are:

\[
\gamma_{ij} = \delta_{ij} + \kappa h_{ij}
\]

(4.18a)

\[
\pi^{ij} = \frac{1}{\kappa} p^{ij}
\]

(4.18b)
\[ N^0 = 1 + \kappa n^0 \]  
\[ N^i = 0 + \kappa n^i \]  

We refer to \( h_{ij}, p^{ij}, n^0 \) and \( n^i \) collectively as the weak fields. By convention the background metric is used to raise and lower indices on the weak fields. Since the background metric in this case is \( \eta_{\mu\nu} \) it is irrelevant whether the spatial indices of weak fields are up or down, and raising a temporal index merely flips the sign. Note that the placement of \( \kappa \)'s in (4.18a) and (4.18b) implies that \( h_{ij} \) and \( p^{ij} \) have the same bracket or commutation relations as \( \gamma_{ij} \) and \( \pi^{ij} \).

If we expand the equations of time evolution, (4.7a) and (4.7b), and then segregate according to tensor components, the following equations result:

\[ \dot{h}^{tt}_{ij} = 2p^{tt}_{ij} + \mathcal{O}(\kappa) \]  
\[ \dot{p}^{tt}_{ij} = \frac{1}{2} \nabla^2 h^{tt}_{ij} + \mathcal{O}(\kappa) \]  
\[ \dot{h}^t = 0 + \mathcal{O}(\kappa) \]  
\[ \dot{p}^t = -2\nabla^2 n^0 + \mathcal{O}(\kappa) \]  
\[ \nabla^2 \dot{h}_i + \dot{h}_{j,ji} = -2p^t_{,i} + \nabla^2 n_i + n_{j,ji} + \mathcal{O}(\kappa) \]  
\[ \nabla^2 \dot{p}_i + \dot{p}_{j,ji} = 0 + \mathcal{O}(\kappa) \]  
\[ \dot{h}^{tr} = -p^{tr} + \mathcal{O}(\kappa) \]  
\[ \dot{p}^{tr} = 0 + \mathcal{O}(\kappa) \]

In these relations we have implicitly regarded the various weak fields as being of order one. This is not really correct because not all the fields are independent. Even in a theory without local symmetries we could use the equations of time evolution to express the weak fields at any time as functionals of the initial weak fields. It is traditional in this case to develop perturbative solutions as though the initial value configurations are of order one.
in the coupling constant. The scheme is more complicated in a theory which possesses local symmetries because then one must impose a volume gauge condition in order to define a canonical formalism. Further, the canonical formalism so obtained possesses a set of constraints upon the initial value configurations and also, typically, a local but time independent residual symmetry. This residual symmetry is fixed by imposing gauge conditions on the initial weak field configurations. In our case we shall find it convenient to imagine that the surface gauge conditions are of order one, but we shall allow for the possibility of higher order terms in the volume gauge conditions. The constraints are solved perturbatively on the initial value surface to express the initial values of the constrained fields as power series expansions in functionals of the initial values of the unconstrained fields, regarding the latter as of order one. One then solves the perturbative equations of time evolution as for a theory without constraints but remembering that not all the initial configurations are of order one, and that the volume gauge conditions may also supply higher order terms.

The four constraints can be expanded as follows in powers of the weak fields:

\[ \mathcal{H}_0 = \frac{1}{\kappa} \left( h_{,ii} - h_{ij,ij} \right) + \left( \frac{1}{2} h_{,i} h_{,i} - \frac{1}{2} h h_{,j} h_{,j} - h_{,j} h_{,i} - h_{ij} h_{ij,} + h_{i} h_{i,j} + h_{ij} h_{,i,j} \right),_k \]
\[ + \left( p_{ij} p_{ij} - \frac{1}{2} p^2 \right) + \left( -\frac{1}{4} h_{,i} h_{,i} + \frac{1}{2} h_{,i} h_{,i,j} + \frac{1}{4} h_{ij,k} h_{ij,k} - \frac{1}{2} h_{ij,k} h_{,k,j,i} \right) + \mathcal{O}(\kappa) (4.23a) \]
\[ \mathcal{H}_i = -\frac{2}{\kappa} p_{ij,j} - 2 \left( h_{ij,j} p_{jk} \right),_k + h_{jk,i} p_{jk} + \mathcal{O}(\kappa) \quad (4.23b) \]

Substitution of the tensor decomposition (4.17) reveals that the \( \mathcal{H}_0 \) constraint determines the weak field \( h^t \):

\[ \nabla^2 h^t = \kappa \mathcal{Q}_0 \left[ h^{tt}, p^{tt}; h^t, p^t; h, p; h^{tr}, p^{tr} \right] \quad (4.24a) \]

\[ \mathcal{Q}_0 \equiv \left( -\frac{1}{2} h_{,i} h_{,i} + \frac{1}{2} h h_{,j} h_{,j} + h_{,j} h_{,i} + h_{ij} h_{ij,} - h_{i} h_{i,j} + h_{ij} h_{,i,j} \right),_k + \left( -p_{ij} p_{ij} + \frac{1}{2} p^2 \right) \]
\[ + \left( \frac{1}{4} h_{,i} h_{,i} - \frac{1}{2} h_{,i} h_{ij,j} - \frac{1}{4} h_{ij,k} h_{ij,k} + \frac{1}{2} h_{ij,k} h_{,k,j,i} \right) + \mathcal{O}(\kappa) \quad (4.24b) \]
Similarly, the $\mathcal{H}_i$ constraint gives an equation for the weak field $p_i$:

$$\nabla^2 p_i + p_{j,ji} = \kappa \mathcal{Q}_i \left[ h^{tt}, p^{tt}; h^t, p^t; h, p; h^{tr}, p^{tr} \right]$$

(4.25a)

$$\mathcal{Q}_i \equiv -\left(h_{ij} p_{jk}\right)_{,k} + \frac{1}{2} h_{jk,i} p_{jk} + \mathcal{O}(\kappa)$$

(4.25b)

We can solve perturbatively for $h^t$ and $p_i$ because these weak field components contain no zero modes and the Laplacian is therefore a negative definite operator. However, we must first subtract off the zero mode parts of the sources $\mathcal{Q}_\mu$. For any function $f(t, \vec{x})$ we define its non-zero mode part as:

$$f_{NZ}(t, \vec{x}) = f(t, \vec{x}) - \mathcal{L}^{-3} \int_0^L dy_1 \int_0^L dy_2 \int_0^L dy_3 f(t, \vec{y})$$

(4.26)

To solve for $h^t$ and $p_i$ one simply inverts the Laplacian on the non-zero mode sectors of (4.24a) and (4.25a):

$$h^t = \frac{\kappa}{\sqrt{2}} \mathcal{Q}^{NZ}_0 \left[ h^{tt}, p^{tt}; h^t, p^t; h, p; h^{tr}, p^{tr} \right]$$

(4.27a)

$$p_i = \frac{\kappa}{\sqrt{2}} L_{ij} \mathcal{Q}^{NZ}_j \left[ h^{tt}, p^{tt}; h^t, p^t; h, p; h^{tr}, p^{tr} \right]$$

(4.27b)

and then substitutes the resulting equations to re-express any $h^t$’s or $p_i$’s which appear in the sources. For example the first iteration gives:

$$h^t = \frac{\kappa}{\sqrt{2}} \mathcal{Q}^{NZ}_0 \left[ h^{tt}, p^{tt}; \frac{\kappa}{\sqrt{2}} \mathcal{Q}^{NZ}_0 p^t; h, \frac{\kappa}{\sqrt{2}} L_{ij} \mathcal{Q}^{NZ}_j h^{tr}, p^{tr} \right]$$

(4.28a)

$$p_i = \frac{\kappa}{\sqrt{2}} L_{ij} \mathcal{Q}^{NZ}_j \left[ h^{tt}, p^{tt}; \frac{\kappa}{\sqrt{2}} \mathcal{Q}^{NZ}_0 p^t; h, \frac{\kappa}{\sqrt{2}} L_{kl} \mathcal{Q}^{NZ}_l h^{tr}, p^{tr} \right]$$

(4.28b)

Of course there are still $h^t$’s and $p_i$’s inside the new sources — though space presents us from displaying it explicitly — but whereas these fields might appear at order $\kappa$ on the right hand side of (4.27) they cannot appear before order $\kappa^2$ on the right hand side of (4.28). Because each iteration moves them to a higher order in $\kappa$ we can obtain in this way an asymptotic series solution as a functional of $h^{tt}, p^{tt}, p^t, h, h^{tr}$ and $p^{tr}$.
Although we have just seen that the constraints completely determine $h^t$ and $p_i$ it is not quite true that constraining $h^t$ and $p_i$ completely enforces the constraints. There remain the zero modes. One can see by direct integration that although the zero mode constraints are free of terms linear in the weak fields they are not trivial at the next order, even when $h^t$ and $p_i$ are set to their constrained values:

$$\int d^3x H_0 = \int d^3x Q_0$$

$$= \int d^3x \left\{ -\frac{1}{6} (p^{tr})^2 + p_{ij}^{tt} p_{ij}^{tt} + 2p_{i,j} p_{i,j} - 2p_{i,i} p^t + \frac{1}{4} h_{ij,k}^{tt} h_{ij,k}^{tt} + \frac{1}{8} h^t_i h^t_i \right\} + O(\kappa)$$  \hspace{1cm} (4.29a)

$$\rightarrow -\frac{1}{6} L^3 (p^{tr})^2 + \int d^3x \left\{ p_{ij}^{tt} p_{ij}^{tt} + \frac{1}{4} h_{ij,k}^{tt} h_{ij,k}^{tt} \right\} + \kappa C_0 \left[ h^{tt}, p^{tt}; h, p^t; h^{tr}, p^{tr} \right]$$  \hspace{1cm} (4.29b)

$$\int d^3x H_i = \int d^3x Q_i$$

$$= \int d^3x \left\{ p_{jk}^{tt} h_{jk,i}^{tt} + \frac{1}{2} p^t h_{i,j}^t + p_{j,j} h_{k,ki} h_{k,ki} - 2p_{j,k} h_{j,ki} \right\} + O(\kappa)$$  \hspace{1cm} (4.30a)

$$\rightarrow \int d^3x p_{jk}^{tt} h_{jk,i}^{tt} + \kappa C_i \left[ h^{tt}, p^{tt}; h, p^t; h^{tr}, p^{tr} \right]$$  \hspace{1cm} (4.30b)

(The functionals $C_\mu \left[ h^{tt}, p^{tt}; h, p^t; h^{tr}, p^{tr} \right]$ are of cubic order and higher in the remaining weak fields.) A consequence is that there are solutions to the linearized field equations which can not be perturbatively corrected to give asymptotic solutions to the full field equations. This phenomenon is known as linearization instability, and it afflicts gravitational perturbation theory whenever the background possesses Killing vectors* on a spatially closed manifold [17,18,19].

The linearization instability is sometimes regarded as a non-trivial obstacle to the development of perturbation theory. This is not correct. We need merely to restrict to those linearized solutions which satisfy the first non-trivial parts of the four zero mode constraints

* The number of constraints which lack linear terms is equal to the number of Killing vectors. We have four because only the four translations give global Killing vectors for flat space on $T^3 \times R$. Lorentz transformations — which also give Killing vectors for flat space on $R^3 \times R$ — do not respect the identification of $T^3 \times R$. 

37
and then develop systematic corrections as usual. Because our strategy is different for the
global Hamiltonian constraint (4.29) than for the global momentum constraint (4.30) we
shall discuss them separately.

At quadratic order in the remaining independent weak fields the global Hamiltonian
constraint is the difference of two manifestly positive quantities. This means we can solve
it explicitly as follows:

\[ p_{tr} = \pm \sqrt{6}L^{-3/2} \left[ \int d^3x \left( p_{tj}^t p_{tj}^t + \frac{1}{4} h_{tj,k}^{tt} h_{tj,k}^{tt} \right) + \kappa C_0 \left[ h_{tj}^{tt}, p_{tj}^t; h, p^t; h_{tr}^{tt}, p_{tr}^t \right] \right]^{1/2} \]  

(4.31)

The issue of choosing the sign in (4.31) commands considerably more attention than it
deserves. The constraint equation does not fix it, and either choice is allowed classically —
the positive sign corresponds to a contracting universe while the negative sign gives an
expanding universe. If we are to avoid imposing extraneous conditions, and especially if
quantum mechanics is to recover classical results in the correspondence limit then we must
include both signs. This is achieved by using a multi-component wavefunction:

\[ \Psi \longrightarrow \begin{pmatrix} \Psi^+ \\ \Psi^- \end{pmatrix} \]  

(4.32)

The action of the the wholly constrained operator \( p_{tr}^t \) on \( \Psi^\pm \) is defined by (4.31) with the
plus or minus sign respectively. A purely contracting universe would be represented by
\( \Psi^+ = 0 \) whereas a purely expanding universe would have \( \Psi^- = 0 \). We shall see in section 6
that the straightforward application of the Faddeev-Popov technique for gauge fixing re-
results in the absolute value of an operator which causes the inner product to segregate into
a manifestly positive contribution from each component. *

There is also the issue of perturbatively iterating (4.31) to achieve an asymptotic series
solution which is free of dependence upon \( p_{tr}^t \). We first define the zeroth order energy:

\[ E_0 \equiv \int d^3x \left\{ p_{tj}^t p_{tj}^t + \frac{1}{4} h_{tj,k}^{tt} h_{tj,k}^{tt} \right\} \]  

(4.33)

* We wish to suggest that the same procedure be applied whenever discrete choices must be
made in solving constraints.

38
and then expand the square root:

\[ p_{tr} = \pm \sqrt{6L^{-3/2}} \sqrt{E_0} \left\{ 1 + \frac{1}{2} \kappa \frac{C_0}{E_0} - \sum_{n=2}^{\infty} \frac{(2n-3)!!}{n!} \left( -\kappa \frac{C_0}{2E_0} \right)^n \right\} \]

Assuming that it is fair to regard the ratio \( C_0/E_0 \) as of first plus higher orders in the weak fields, we then obtain an asymptotic series solution by iteration. For example, the first iteration gives:

\[ p_{tr} = \pm \sqrt{6L^{-3/2}} \left\{ \sqrt{E_0} + \kappa S_0 \left[ h^{tt}, p^{tt}; h, p^t; h^{tr}, p^{tr} \right] \right\} \]

As with \( h^t \) and \( p_i \), successive iterations push to ever higher orders any dependence of the right hand side upon \( p_{tr} \).

Of course the iterative solution for \( p_{tr} \) will result in nonsense, even perturbatively, if \( E_0 \) can be made to vanish without \( C_0 \) vanishing at least as rapidly. It turns out that this cannot happen for three reasons. First, we will shortly see that \( h_i \) and \( p^t \) can be gauged to zero. Second, \( E_0 \) is a sum of squares of all the remaining variables except for \( h^{tr} \) and the zero modes of \( h^{tt} \). Finally, the dependence \( C_0 \) inherits from \( \mathcal{H}_0 \) implies that each of its terms must vanish at least quadratically with \( p_{ij}^{tt} \) and/or the non-zero modes of \( h_i^{tt} \). To see this last point note from substituting the weak field expansions (4.18) into (4.5b) that \( \mathcal{H}_0 \) consists of terms quadratic in \( p_{ij} \) with any number of \( h_{ij} \)'s and other terms which are free of \( p_{ij} \) but contain at least one differentiated \( h_{ij} \). Upon integration over \( T^3 \) each of the pure \( h_{ij} \) terms must contain at least two non-zero modes of \( h_{ij} \). Constraining \( h^t \) and \( p_i \) to zero can result in terms which have any even power of the remaining components of \( p_{ij} \) but it cannot result in odd powers of the momentum nor can it introduce pure \( h_{ij} \) terms which fail to possess at least two non-zero modes. Upon gauging \( p^t \) to zero we see that every term in \( C_0 \) must either contain a positive even power of \( p_{ij}^{tt} \) or \( p^{tr} \), or it must contain at least two non-zero modes of \( h_{ij}^{tt} \). It follows that whenever \( E_0 \) vanishes \( C_0 \) must
vanish at least as rapidly, so the ratio $c_0/E_0$ can be legitimately regarded as of order one and higher in the weak fields.

The three global momentum constraints cannot be imposed this way because we see from (4.30c) that their quadratic parts are not differences of squares. Our strategy is therefore to leave them as constraints upon the classical initial value data or, in the quantum theory, upon the space of states. We can get away with this for three reasons. First, their imposition is not necessary in order to construct a reduced canonical formalism with a non-zero Hamiltonian. This was obviously not the case for the global Hamiltonian constraint. Second, the global momentum constraints remove no negative energy modes, unlike the global Hamiltonian constraint. Finally, the symmetry generated by the global momentum constraints consists of constant spatial translations on $T^3$. Since these form a compact group there is no need to gauge fix them in the functional formalism. Additional support for the viability of not attempting to reduce the global momentum constraints can be found in the close parallel with the $N = \tilde{N}$ constraint of closed bosonic string theory in lightcone gauge [20].

We turn now to the issue of gauge fixing. Since we wish in the end to compare our results with those of A.D.M. [3] we shall of course need to follow them in the choice of gauge. Their perspective was slightly different from ours: whereas we impose the volume gauge by choosing the lapse and shift, and then fix (most of) the residual symmetry with gauge conditions on the initial value surface, A.D.M. impose a volume gauge condition on the weak fields $h_{ij}$ and $p^{ij}$ and then use the evolution equations for the frozen variables to determine the lapse and the shift. We can obtain the same result by merely choosing our lapse and shift, and our auxiliary surface conditions, so as to agree with A.D.M. The distinction between the two methods is important only to Faddeev-Popov gauge fixing in the quantum theory which was developed years after A.D.M. wrote. In our notation the
conditions favored by A.D.M. [3] are:

\[ h_i(t, \vec{x}) = 0 \] (4.36a)

\[ p^t(t, \vec{x}) = 0 \] (4.36b)

We shall accordingly begin by showing that the residual symmetry allows the perturbative imposition \( \hat{h}_i = 0 \) and \( \hat{p}^t = 0 \). We then argue that \( n^0(0, \vec{x}) \) and \( n^i(0, \vec{x}) \) can be chosen so as to perturbatively enforce the A.D.M. condition (4.36).

To properly organize the notion of a perturbative transformation we insert a factor of \( \kappa \) into the infinitesimal transformation parameter, \( \theta^\mu \equiv \kappa \tau^\mu \). By substituting this and the perturbative expansions (4.18), and by iterating (4.11) we obtain the following expressions for the non-infinitesimal but perturbatively small transformations of \( \hat{h}_{ij} \) and \( \hat{p}_{ij} \):

\[
\hat{h}'_{ij} - \hat{h}_{ij} = - (\hat{\tau}_{i,j} + \hat{\tau}_{j,i}) \\
+ \kappa \left( \frac{1}{2} (\hat{\tau}_{k,i} \hat{\tau}_{k,j} + \hat{\tau}_{k,j} \hat{\tau}_{k,i} + \frac{1}{2}(\hat{\tau}_{k,j} \hat{\tau}_{k,i})_{,i} - \hat{\tau}_{k,i} \hat{\tau}_{k,j} - \hat{\tau}_{k,j} \hat{\tau}_{k,i} - \hat{h}_{ik} - \hat{h}_{jk} + \hat{\tau}_{k} \right) \\
- \hat{\tau}_i \hat{\tau}_j - \hat{\tau}_j \hat{\tau}_i + \hat{\tau}_0 \hat{n}_i + \hat{\tau}_0 \hat{n}_j - 2 \hat{\tau}_{ij} \hat{\tau}_0 - 2 \hat{\tau}_{ij} \hat{\tau}_0 + \delta_{ij} \hat{p}^0 \hat{\tau}_0 + O(\kappa^2) \quad (4.37a)
\]

\[
\hat{p}_{ij} - \hat{p}_{ij} = -(\hat{\tau}_0 \hat{n}_{ij} - \delta_{ij} \hat{\tau}_0) + \kappa \mathcal{H}_{ij} \left[ \hat{h}^{tt}, \hat{\tilde{p}}^{tt}; \hat{\tilde{h}}^{tt}, \hat{\tilde{p}}^{tt}; \hat{h}; \hat{\tilde{h}}; \hat{p}; \hat{\tilde{p}}; \hat{h}^{tr}, \hat{\tilde{h}}^{tr}, \hat{p}^{tr}, \hat{\tilde{p}}^{tr}; \hat{\tau}_0, \hat{\tau} \right] \quad (4.37b)
\]

\[
\hat{\tau}_i = \hat{h}_i + \frac{\kappa}{\sqrt{2}} L_{ij} \partial_k \mathcal{H}_{jk}^{NZ} \left[ \hat{h}^{tt}, \hat{\tilde{p}}^{tt}; \hat{\tilde{h}}^{tt}, \hat{\tilde{p}}^{tt}; \hat{h}; \hat{\tilde{h}}; \hat{p}; \hat{\tilde{p}}; \hat{h}^{tr}, \hat{\tilde{h}}^{tr}, \hat{p}^{tr}, \hat{\tilde{p}}^{tr}; \hat{\tau}_0, \hat{\tau} \right] \quad (4.39)
\]

The zero modes of \( \hat{\tau}_i \) are not fixed because they are conjugate to the global momentum constraints which are not being reduced. We use the non-zero modes of \( \hat{\tau}_0 \) to perturbatively enforce \( \hat{p}' = 0 \) by iterating the equation:

\[
\hat{\tau}_0 = - \frac{1}{2} \hat{p}^t - \frac{\kappa}{2 \sqrt{2}} T_{ij} \mathcal{P}^{NZ}_{ij} \left[ \hat{h}^{tt}, \hat{\tilde{p}}^{tt}; \hat{\tilde{h}}^{tt}, \hat{\tilde{p}}^{tt}; \hat{h}; \hat{\tilde{h}}; \hat{p}; \hat{\tilde{p}}; \hat{h}^{tr}, \hat{\tilde{h}}^{tr}, \hat{p}^{tr}, \hat{\tilde{p}}^{tr}; \hat{\tau}_0, \hat{\tau} \right] \quad (4.40)
\]

* The component fields \( h_i \) and \( p^t \) used by A.D.M. actually contain zero modes, unlike ours, and these zero modes were given non-zero values determined by the asymptotically flat boundary conditions [3].
We will henceforth drop the prime and assume that \( \hat{h}_i = 0 = \hat{p}^t \).

There remains the zero mode of \( \hat{\tau}^0 \). We shall use this to enforce \( \hat{h}^{tr} = 0 \) although the argument for being able to impose this condition is more subtle. First, note that since the Wheeler-DeWitt symmetry must be gauge fixed [21], and since the subgroup of constant time translations is not compact, we do not have the option of declining to enforce some zero mode gauge condition. Second, note from (4.37a) that \( h^{tr} \) is the quantity affected to lowest order by a constant time translation. Let us label such a transformation by the single parameter \( \hat{z} \):

\[
\hat{z} \equiv L^{-3} \int d^3x \hat{\theta}^0(\vec{x})
\]  

(4.41)

Whereas the parameter \( \hat{\theta}^\mu \) which enforced \( \hat{h}_i = 0 = \hat{p}^t \) was of order \( \kappa \) we need \( \hat{z} \) to be of order one. Even so, the fact that \( p^{tr} \) is constant to lowest order allows us to obtain a perturbative expression for the result of a non-infinitesimal shift:

\[
\hat{h}^{tr'} - \hat{h}^{tr} = \frac{1}{6} \hat{p}^{tr} + \kappa Z[\hat{h}^{tt}, \hat{p}^{tt}; \hat{h}^{tr}, \hat{p}^{tr}; \hat{z}]
\]  

(4.42)

We see that the desired condition can be imposed formally by iterating the equation:

\[
\hat{z} = -6 \frac{\hat{h}^{tr}}{\hat{p}^{tr}} - \frac{6\kappa}{\hat{p}^{tr}} Z[\hat{h}^{tt}, \hat{p}^{tt}; \hat{h}^{tr}, \hat{p}^{tr}; \hat{z}]
\]  

(4.43)

We include the qualifier “formally” because the transformation is obviously singular when \( \hat{p}^{tr} \) vanishes. Of course \( \hat{p}^{tr} \) is not an independent degree of freedom; and we see from (4.31) that it is about as protected from vanishing as it is invariantly possible to get. However, if all the modes of \( \hat{p}^{tt}_{ij} \) and all the non-zero modes of \( \hat{h}^{tt}_{ij} \) vanish then \( \hat{p}^{tr} \) vanishes, but we can have a non-zero \( \hat{h}^{tr} \). In this case both \( h_{ij}(t, \vec{x}) \) and \( p_{ij}(t, \vec{x}) \) are constant in space and time, and no temporal translation exists which will enforce \( \hat{h}^{tr'} = 0 \).

Our procedure is to go ahead and impose \( \hat{h}^{tr} = 0 \) anyway. Considerable justification for this course derives from the close analogy to imposing the gauge \( q^0(\tau) = t \) for a massless free particle whose position and momenta are \( q^\mu(\tau) \) and \( p_\mu(\tau) \) respectively. In this system
the constrained variable, $p_0(\tau)$ has an ambiguous sign which necessitates a 2-component wavefunction. Just as with gravity, the gauge condition conjugate to this constrained variable is singular for constant field configurations. This resolves itself in the quantum theory by the gauge fixed inner product acquiring a Faddeev-Popov determinant which endows the troublesome sector of configuration space with zero measure [21].

\[
\langle \psi | \delta(q^0 - t) \text{ abs}(p_0) | \phi \rangle = -\frac{i}{2} \int d^3 q \left\{ \psi^{\ast*}(t, \vec{x}) \phi^+(t, \vec{x}) - \psi^{\ast*}(t, \vec{x}) \phi^+(t, \vec{x}) \right\} + \frac{i}{2} \int d^3 q \left\{ \psi^{-\ast}(t, \vec{x}) \phi^-(t, \vec{x}) - \psi^{-\ast}(t, \vec{x}) \phi^-(t, \vec{x}) \right\}
\]

(4.44)

We will see at the end of section 6 that Faddeev-Popov gauge fixing endows gravity with the same sort of inner product. The result obtained for the free particle has such universal acceptance that we shall henceforth ignore the completely analogous problem with imposing $\hat{h}^{tr} = 0$ on constant field configurations.

It remains to show that we can choose the lapse and shift so as to enforce the A.D.M. gauge conditions (4.36) for all time. To see this it suffices to apply the constraints to evolution equations (4.20b), (4.21a) and (4.22a):

\[
p^t = -2\nabla^2 n^0 + \kappa N^0 \left[ h^{tt}, p^{tt}; n^0, n; p^t, h, h^{tr} \right]
\]

(4.45a)

\[
\nabla^2 \dot{h}_i + \dot{h}_{j,j} = \nabla^2 n_i + n_{j,j} + \kappa N_i \left[ h^{tt}, p^{tt}; n^0, n; p^t, h, h^{tr} \right]
\]

(4.45b)

\[
\dot{h}^{tr} = \mp L^{-3/2} \sqrt{6E_0} + \kappa \left[ h^{tt}, p^{tt}; n^0, n; p^t, h, h^{tr} \right]
\]

(4.45c)

Since we already have $p^t(0, \vec{x}) = 0 = h_i(0, \vec{x})$ we will have the A.D.M. condition (4.36) if the weak field lapse and shift are obtained by iterating the equations:

\[
n^0 = \frac{\kappa}{2\nabla^2} N^0_{NZ} \left[ h^{tt}, p^{tt}; n^0, n; 0, 0, h^{tr} \right]
\]

(4.46a)

\[
n_i = -\frac{\kappa}{\nabla^2} L_{ij} N^0_{NZ} \left[ h^{tt}, p^{tt}; n^0, n; 0, 0, h^{tr} \right]
\]

(4.46b)

* We have taken the liberty to correct an error that appeared in formula (23c) of [21].
Note that these equations only determine the non-zero modes of the lapse and shift. We propose that the zero modes be left one and zero, respectively, to all orders. Of course while (4.45c) — and the initial condition, \( h^{tr}(0, \vec{x}) = 0 \) — determines \( h^{tr}(t, \vec{x}) \), this component field does not vanish after \( t = 0 \).

(4) — The Reduced Canonical Formalism and Its and Correspondence Limit.

In the previous section we succeeded in reducing the theory to the point where only the transverse-traceless fields survive. It is convenient, however, to view the system that results when \( p^{tr} \) and \( \hat{\theta}^0 \) are not yet reduced. Because special care must be given to the zero modes we will do this in \( k \)-space:

\[
\overset{\cdot}{\tilde{p}}_{ij}^{tt} = -\frac{1}{2} k^2 \tilde{h}_{ij}^{tt} + \kappa \left[ \tilde{U}_{ij}(h^{tt}, p^{tt}) + \frac{1}{12} k^2 h^{tr} \tilde{h}_{ij}^{tt} - \frac{1}{3} p^{tr} \tilde{p}_{ij}^{tt} \right] + \mathcal{O}(\kappa^2) \quad (4.47a)
\]

\[
\overset{\cdot}{\tilde{h}}_{ij}^{tt} = 2 \tilde{p}_{ij}^{tt} + \kappa \left[ 2 \tilde{W}_{ij}(h^{tt}, p^{tt}) + \frac{1}{3} h^{tr} \tilde{p}_{ij}^{tt} + \frac{1}{3} p^{tr} \tilde{h}_{ij}^{tt} \right] + \mathcal{O}(\kappa^2) \quad (4.47b)
\]

The explicit forms of \( \tilde{U}_{ij} \) and \( \tilde{W}_{ij} \) are:

\[
\tilde{U}_{ij} = \frac{1}{(2\pi)^3 L^3/2} \left( T_{in} T_{jr} - \frac{1}{2} T_{ij} T_{nr} \right) \int d^3 x \left\{ e^{-i\vec{k} \cdot \vec{x}} \left[ \frac{1}{4} h_{lm,n}^{tt} h_{lm,r}^{tt} - \frac{1}{2} h_{lm,n}^{tt} h_{lr,m}^{tt} - \frac{1}{2} h_{nl,m}^{tt} h_{rl,m}^{tt} + \frac{1}{2} h_{nl,m}^{tt} h_{rm,l}^{tt} \right] \right\} (4.48a)
\]

\[
\tilde{W}_{ij} = \frac{1}{(2\pi)^3 L^3/2} \left( T_{in} T_{jr} - \frac{1}{2} T_{ij} T_{nr} \right) \int d^3 x \left[ e^{-i\vec{k} \cdot \vec{x}} \left( p_{nm}^{tt} h_{mr}^{tt} + p_{rm}^{tt} h_{mn}^{tt} \right) \right] (4.48b)
\]

and the evolution equations for the zero modes are obtained by setting \( k = 0 \) after changing \( T_{ij} \to \delta_{ij} \) in the expressions for \( \tilde{U}_{ij} \) and \( \tilde{W}_{ij} \).

This system is canonical for the same reason that the A.D.M. system is: the surface and volume gauges have been chosen so that the variables conjugate to each of the constrained variables — that is, \( p^{t} \) for \( h^{t} \) and \( h_{i} \) for \( p_{i} \) — remain zero for all time. The act of
reducing \( p^{tr} \) and \( h^{tr} \) spoils canonicity because \( h^{tr} \) does not remain zero after the initial time. The mechanism is the same as we found in section 3 where the evolution of a non-zero longitudinal vector potential in temporal gauge broke the canonicity of scalar electrodynamics. Because it is only the trace components which break canonicity we know that it will be restored if we can transform to variables \((\tilde{X}, \tilde{P})\) with the same evolution equations except for lack of dependence on \( p^{tr} \) and \( h^{tr} \). This transformation will necessarily be time dependent and non-local. The time dependence arises because the transformation must give \((\tilde{h}^{tt}, \tilde{p}^{tt})\) at \( t = 0 \) so that at this time \((\tilde{X}, \tilde{P})\) obey the same commutation relations obeyed by \((\tilde{h}^{tt}, \tilde{p}^{tt})\); but it must deviate from this at later times since we wish to eliminate \((h^{tr}, p^{tr})\) from equations (4.47). The non-locality enters merely because the transformations must depend on \((h^{tr}, p^{tr})\) and these are non-local, as witness equation (4.31).

It is trivial to check that to the order we are working the following transformations possess the properties mentioned above:

For \( \omega \equiv |\vec{k}| \neq 0 \):

\[
\tilde{P}_{ij} = \tilde{p}_{ij}^{tt} + \kappa \left\{ \frac{1}{4} p^{tr} \left[ \frac{\sin(\omega t)}{\omega} \right] \left[ \tilde{p}_{ij}^{tt} \cos(\omega t) + \frac{\omega}{2} \tilde{h}_{ij}^{tt} \sin(\omega t) \right] - \frac{1}{12} h^{tr} \tilde{p}_{ij}^{tt} \right\} \quad (4.49a)
\]

\[
\tilde{X}_{ij} = \tilde{h}_{ij}^{tt} - \kappa \left\{ \frac{1}{4} p^{tr} \left[ \frac{\sin(\omega t)}{\omega} \right] \left[ \tilde{h}_{ij}^{tt} \cos(\omega t) - \frac{2}{\omega} \tilde{p}_{ij}^{tt} \sin(\omega t) \right] - \frac{1}{12} h^{tr} \tilde{h}_{ij}^{tt} \right\} \quad (4.49b)
\]

while for \( \omega = 0 \):

\[
\tilde{P}_{ij}(0) = \tilde{p}_{ij}^{tt}(0) + \kappa \left\{ \frac{1}{4} t p^{tr} \tilde{p}_{ij}^{tt}(0) - \frac{1}{12} h^{tr} \tilde{p}_{ij}^{tt}(0) \right\} \quad (4.50a)
\]

\[
\tilde{X}_{ij}(0) = \tilde{h}_{ij}^{tt}(0) - \kappa \left\{ \frac{1}{4} t p^{tr} \left[ \tilde{h}_{ij}^{tt}(0) - t \tilde{p}_{ij}^{tt}(0) \right] - \frac{1}{12} h^{tr} \tilde{h}_{ij}^{tt}(0) \right\} \quad (4.50b)
\]

Applying the transformations (4.49) and (4.50) to the evolution equations (4.47) results, as required, in equations independent of both \( h^{tr} \) and \( p^{tr} \):

\[
\dot{\tilde{P}}_{ij} = -\frac{1}{2} k^2 \tilde{X}_{ij} + \kappa \tilde{U}_{ij}(X, P) + \mathcal{O}(\kappa^2) \quad (4.51a)
\]
\[ \dot{X}_{ij} = 2 \bar{P}_{ij} + 2 \kappa \bar{W}_{ij}(X, P) + \mathcal{O}(\kappa^2) \quad (4.51b) \]

and again, the equations for the zero modes are obtained by setting \( k = 0 \) in the above after changing \( T_{ij} \rightarrow \delta_{ij} \) in equations (4.48).

As previously mentioned, the variables \( \tilde{X} \) and \( \tilde{P} \) are canonical at \( t = 0 \) since at this time they are simply equal to \( \tilde{h}^{tt} \) and \( \tilde{p}^{tt} \). Furthermore, they will remain canonical at later times since the evolution equations (4.51) are just those of the A.D.M. variables.

The Hamiltonian that generates equations (4.51) in terms of the \( x \)-space variables is:

\[
H = \int d^3x \left\{ P_{ij} \dot{P}_{ij} + \frac{1}{4} X_{ij,l} X_{ij,l} + \kappa \left[ -X_{lm,i} X_{lj,m} X_{ij} - \frac{1}{4} X_{lm} X_{ij, lm} X_{ij} + \frac{1}{2} X_{il,m} X_{jl,m} X_{ij} - \frac{1}{2} X_{im,l} X_{jl,m} X_{ij} + 2 P_{il} P_{lj} X_{ij} \right] \right\} \quad (4.52)
\]

there are two reasons for writing the Hamiltonian in terms of \((X, P)\) as opposed of \((\tilde{X}, \tilde{P})\). The first one is merely that it is in terms of these variables that the form of \( H \) is the simplest. The second reason is that in this form it is obvious that the Hamiltonian density implied by (4.52) is local. However, in order to derive the evolution equations (4.51) while still treating the zero modes properly, one must work in momentum space. Here the non-zero bracket relations are:

For \( \vec{k}, \vec{q} \neq 0 \),

\[
\{ \tilde{X}_{ij}(\vec{k}), \bar{P}_{lm}(-\vec{q}) \} = \frac{1}{2} \delta_{k,q} \left[ \left( T_{il} T_{jm} - \frac{1}{2} T_{ij} T_{lm} \right) + \left( T_{im} T_{jl} - \frac{1}{2} T_{ij} T_{lm} \right) \right] \quad (4.53a)
\]

For \( \vec{k} = \vec{q} = 0 \)

\[
\{ \tilde{X}_{ij}(0), \bar{P}_{lm}(0) \} = \frac{1}{2} \left[ \left( \delta_{il} \delta_{jm} - \frac{1}{2} \delta_{ij} \delta_{lm} \right) + \left( \delta_{im} \delta_{jl} - \frac{1}{2} \delta_{ij} \delta_{lm} \right) \right] \quad (4.53b)
\]

So far we have succeeded in reducing the theory and extracting the proper Hamiltonian. We will next prove that in the appropriate limit the Hamiltonian of equation (4.52) goes to
that obtained by following the A.D.M. procedure. The appropriate limit is that in which a configuration in $T^3 \times R$ goes to the same configuration in an open space with flat boundary conditions. Explicitly, the limit in which the two treatments agree is that in which we take $L \to \infty$ with localized initial perturbations. We refer to this limit as the open space limit. It should be obvious that the proof reduces to showing that in the open space limit both $p^{tr}$ and $h^{tr}$ vanish; since if this is the case $(X, P)$ become just $(h^{tt}, p^{tt})$ respectively and the Hamiltonian (4.52) already has the correct form.

Let us begin then by examining $E_0$ as defined by equation (4.33):

$$E_0 \equiv \int d^3x \left\{ p^{tt}_{ij} p^{tt}_{ij} + \frac{1}{4} h^{tt}_{ij,k} h^{tt}_{ij,k} \right\}$$  \hspace{1cm} (4.33)

Note that $E_0$ remains finite in the open space limit, even though the range of integration increases from $[0, L)$ to $(-\infty, \infty)$. The reason for this is that localized initial perturbations guarantees that the integrand above has finite support.

Now let us inspect the evolution equation for $h^{tr}$ (equation (4.22)) together with the constraint equation for $p^{tr}$ (equation (4.31)):

$$\dot{h}^{tr} = -p^{tr} + \kappa L^{-3} \int d^3x \left\{ A_1[h^{tt}, p^{tt}; h^{tr}, p^{tr}] \right\}$$  \hspace{1cm} (4.54a)

$$p^{tr2} = 6L^{-3} \left\{ E_0 + \kappa \int d^3x A_0[h^{tt}, p^{tt}; h^{tr}, p^{tr}] \right\}$$  \hspace{1cm} (4.54b)

Equations (4.54) are iterative relations for $h^{tr}$ and $p^{tr}$ in terms of the $tt$ fields after the gauge has been fixed and the constraints for $h^t$ and $p_i$ have been enforced. Equation (4.54a) can be integrated (again iteratively) to give:

$$h^{tr} = \int_0^t d\tau \left[ -p^{tr} + \kappa L^{-3} \int d^3x \left\{ A_1[h^{tt}, p^{tt}; h^{tr}, p^{tr}] \right\} \right]$$  \hspace{1cm} (4.55)

to see that both $h^{tr}$ and $p^{tr}$ vanish in the open space limit we must examine $L^{-3} \int d^3x A_1$ and $L^{-3} \int d^3x A_0$ closer. Let us then explore the $L$ dependance of each of these two terms separately.
\( A_1 \) is of second order and higher in the fields; therefore the highest power of \( L \) in \( \int d^3 x A_1 \) occurs when the integral acts on constants (since both \( h^{tt} \) and \( p^{tt} \) have finite support). In the open space limit we can then replace \( \int d^3 x A_1 \) with \( L^3 M_1 \) where \( M_1 \) is at least of second order in \( h^{tr} \) and/or \( p^{tr} \).

The form of \( A_0 \) is at least of third order in the fields (remembering that \( E_0 \) is independent of \( L \) in the limit). Similar considerations as those mentioned for the case of \( A_1 \) reveal that in the open space limit \( \int d^3 x A_0 \) can be replaced by \( L^3 M_0 \) where \( M_0 \) is of degree three and higher in \( h^{tr} \) and/or \( p^{tr} \).

We can then, in the open space limit, write equations (4.55) and (4.54b) as:

\[
\begin{align*}
    h^{tr} &= \int_0^t d\tau \left\{ -p^{tr} + \kappa M_1 \left[ h^{tr}, p^{tr} \right] \right\} \quad (4.56a) \\
    p^{tr2} &= L^{-3} E_0' + \kappa M_0' \left[ h^{tr}, p^{tr} \right] \quad (4.56b)
\end{align*}
\]

where the primes have been put in to absorb an irrelevant factor of 6 that would otherwise appear multiplying the right hand side of equation (4.56b). At this point it should be obvious to the reader that the perturbative solutions to equations (4.56) are \( h^{tr} = 0 \) and \( p^{tr} = 0 \). For those that still have some doubts let us take the \( L \to \infty \) limit and re-write equations (4.56) as:

\[
\begin{align*}
    h^{tr} &= \int_0^t d\tau \left\{ -p^{tr} + \kappa \alpha_{nm} \left( h^{tr} \right)^n \left( p^{tr} \right)^m \right\} \quad (4.57a) \\
    p^{tr2} &= \kappa \alpha'_{n'm'} \left( h^{tr} \right)^{n'} \left( p^{tr} \right)^{m'} \quad (4.57b)
\end{align*}
\]

with \( n + m \geq 2 \) and \( n' + m' \geq 3 \). Each successive iteration of equations (4.57) brings with it positive powers of \( \kappa \). Therefore, to any order in perturbation in powers of \( \kappa \) both \( h^{tr} \) and \( p^{tr} \) vanish in the open space limit. Thereby proving the correspondence between our method and that of A.D.M.

48
We conclude this section by re-stating the result: Our method of reduction gives a precise meaning to time and, perhaps more importantly, this time evolution coincides in the appropriate limit with that obtained by A.D.M. for open space.

5. Exact Results For Minisuperspace

In this section we examine two simple models. The first one is that of pure gravity with a cosmological constant. The second example deals with the case a gravity coupled to a scalar field. In both of these models we will truncate the form of the metric to allow only certain zero modes to be present so as to render the models exactly solvable.

— Zero modes of gravity with a cosmological constant.

For the first example we choose the form of the 3-metric to be such as to have two homogeneous degrees of freedom:

\[
 ds^2 = -N^2(\tau)d\tau^2 + b^{2/3}(\tau)\left[e^{a(\tau)}dx^2 + e^{-a(\tau)}dy^2 + dz^2\right]
\]  
(5.1)

the action in canonical form is:

\[
 S = \int d\tau \left[p_b \dot{b} + p_a \dot{a} - N \mathcal{R}\right]
\]  
(5.2)

with \( \mathcal{R} \) defined as:

\[
 \mathcal{R} \equiv \frac{\kappa^2}{2b^2} p_a^2 + \frac{2\Lambda}{\kappa^2} b - \frac{3\kappa^2}{8} b p_b^2
\]  
(5.3)

The equations of motion before the reduction is implemented are obtained by varying the action of (5.2) with respect to \( p_a, a, p_b \) and \( b \) respectively:

\[
 \dot{a} = N \frac{\kappa^2}{b^2} p_a \]  
(5.4a)

\[
 \dot{p}_a = 0 \]  
(5.4b)

\[
 \dot{b} = -N \frac{3\kappa^2}{4} b p_b \]  
(5.4c)

\[
 \dot{p}_b = N \left(-\frac{\kappa^2}{b^2} p_a^2 + \frac{2\Lambda}{\kappa^2} b - \frac{3\kappa^2}{8} b p_b^2\right) \]  
(5.4d)
while varying $N$ results in the constraint $\mathcal{R} = 0$:

$$\frac{\kappa^2}{2b^2} p_a^2 + \frac{2 \Lambda}{\kappa^2} b - \frac{3\kappa^2}{8} b p_b^2 = 0 \quad (5.4e)$$

Since the constraint $(5.4e)$ will be enforced by singling out $p_b$ we wish to choose the volume gauge by simplifying the equation for $\dot{b}$ as much as possible. The obvious choice is:

$$N = \frac{1}{b |p_b|} \quad (5.5)$$

Now that the volume gauge has been fixed we fix the constraint and use equation $(5.4c)$ to fix $b$ in the following manner$*$:

$$p_b = -\sqrt{\frac{16 \Lambda}{3 \kappa^4} + \frac{4}{3 b^3} p_a^2} \quad (5.6a)$$

$$b = 1 + \frac{3 \kappa^2}{4} \tau \quad (5.6b)$$

The reduction is now complete. The equations of motion for the physical fields become:

$$\dot{a} = \frac{\kappa^2 p_a}{b(t)^3} \left[ \sqrt{\frac{16 \Lambda}{3 \kappa^4} + \frac{4}{3 b(t)^3} p_a^2} \right]^{-1} \quad (5.7a)$$

$$\dot{p}_a = 0 \quad (5.7b)$$

the above equations are integrable and the Hamiltonian can be obtained from them:

$$H = \frac{3 \kappa^2}{4} \sqrt{\frac{16 \Lambda}{3 \kappa^4} + \frac{4}{3 b(t)^3} p_a^2} \quad (5.8)$$

Before going to the next example we wish to clarify one point: The choice of $N \neq 1$ signifies that the time evolution implied by equation $(5.8)$ is not that corresponding to

$*$ For simplicity of exposition we have made a definite choice for the sign of $p_b$, that corresponding to an expanding space. As in section 4 the wave function really consists of two components, one for each of the two signs. Note also that we fix the surface gauge condition by choosing $b(0) = 1$. 

50
time evolution in flat space (we will see this point more clearly in the next example when we take the limit $\kappa \to 0$). $N$ was chosen so as to make equation (5.4c) exactly solvable. It was by no means a unique choice; for example had we chosen:

$$N = \frac{1}{|p_b|}$$

(5.9)
equation (5.4c) would still be easy to solve but equations (5.4a $- b$) would have a different form:

$$a' = \kappa^2 e^{3/2\kappa^2 t} \frac{p_a}{p_b}$$

(5.10)

$$p'_a = 0$$

where prime denotes differentiation with respect to the new time parameter $t$. The Hamiltonian then would be:

$$H' = \frac{3\kappa^2}{4} e^{-3/4\kappa^2 t} \sqrt{\frac{16 \Lambda}{3 \kappa^4} + \frac{4}{3 b(t)^3} p^2_a}$$

(5.11)

This gauge dependance of the Hamiltonian should come as no surprise since changing how we gauge fix $N$ changes what we mean by time, thereby changing what we mean by time evolution. In our next example we will show how despite this freedom we can make contact with the results obtained in a theory for which gravity is not dynamical ($i.e.$., in the limit $\kappa \to 0$) and $N = 1$ always.

— Zero mode of gravity coupled to a scalar field

We start by looking at the zero modes of a massive scalar field coupled to gravity and we allow the 3–metric to have only one degree of freedom:

$$ds^2 = -N^2(\tau) d\tau^2 + e^{\frac{2}{3} a(\tau)} d\vec{x}^2$$

(5.12)

The action in canonical form is:

$$S = \int d\tau \left[ \pi \dot{\phi} + \dot{p}_\phi - N \mathcal{R} \right]$$

(5.13)
where \((\pi, p)\) are the variables conjugate to \((\phi, a)\) respectively and:

\[
\mathcal{R} \equiv \left[ \frac{1}{2} \pi^2 - \frac{1}{2} \alpha^2 p^2 \right] e^{-a} + \left[ \frac{1}{2} m^2 \phi^2 \right] e^{a} \quad (5.14)
\]

with \(\alpha^2 \equiv 12\pi G\).

By varying this action we obtain the unreduced equations of motion:

\[
\dot{a} = -\alpha^2 Ne^{-a} p \quad (5.15a)
\]

\[
\dot{p} = N \left[ \frac{1}{2} \pi^2 - \frac{1}{2} \alpha^2 p^2 \right] e^{-a} - N \left[ \frac{1}{2} m^2 \phi^2 \right] e^{a} \quad (5.15b)
\]

\[
\dot{\phi} = N e^{-a} \pi \quad (5.15c)
\]

and

\[
\dot{\pi} = -N m^2 \phi e^{a} \quad (5.15d)
\]

while variation with respect to \(N\) gives the constraint equation

\[
\left[ \frac{1}{2} \pi^2 - \frac{1}{2} \alpha^2 p^2 \right] e^{-a} + \left[ \frac{1}{2} m^2 \phi^2 \right] e^{a} = 0 \quad (5.15e)
\]

We must now select a volume gauge condition to impose. Setting \(N = 1\) will not do because one would be left with the task of solving equation (5.15a) explicitly for \(a\). A gauge choice that simplifies this task is:

\[
N = \frac{e^{a}}{|p|} \quad (5.16)
\]

It is obvious that the above choice makes the job of solving equation (5.15a) a trivial one. Note; however, that \(N\) does not approach 1 in the limit \(\kappa \to 0\). We will have to account for this when comparing our results to those obtained for flat space.

Having fixed the volume gauge we now proceed to reduce the theory by enforcing the constraint (5.15e) and fixing the value of \(a\) at \(\tau = 0\) *:

\[
p = -\frac{1}{\alpha} \sqrt{\pi^2 + m^2 \phi^2 e^{2a}} \quad (5.17a)
\]

\[
a = \alpha^2 \tau \quad (5.17b)
\]

* We again chose the sign of the constrained variable to give increasing \(a\) for increasing \(\tau\)
where we have chosen $a(0) = 0$.

The equations of motion for the remaining variables are, after implementing reduction:

\[
\dot{\phi} = \frac{\alpha \pi}{\sqrt{\pi^2 + m^2 \phi^2 e^{2a}}} \tag{5.18a}
\]

\[
\dot{\pi} = -\frac{\alpha m^2 \phi e^{2a}}{\sqrt{\pi^2 + m^2 \phi^2 e^{2a}}} \tag{5.18b}
\]

The gauge choice (5.16) leaves the variables canonical. The Hamiltonian obtained by integrating equations (5.18) is:

\[
H_\tau = \alpha \sqrt{\pi^2 + m^2 \phi^2 e^{2a \tau}} \tag{5.19}
\]

the subscript $\tau$ is there to remind us that this Hamiltonian describes evolution with respect to $\tau$ which in the limit $\alpha \to 0$ does not go to the flat space $t$ simply because $N$ does not go to 1 in that limit.

To see how to recover the flat space result in the limit $\alpha \to 0$. Let us examine the $\tau$ evolution of $\phi$:

\[
\frac{\partial \phi}{\partial \tau} = \frac{\partial H_\tau}{\partial \pi} \tag{5.20a}
\]

\[
N(\tau(t)) \frac{\partial \phi}{\partial t} = \frac{\partial H_{\tau(t)}}{\partial \pi} \tag{5.20b}
\]

\[
\frac{\alpha e^{\alpha^2 \tau(t)}}{\sqrt{\pi^2 + m^2 \phi^2 e^{2\alpha^2 \tau(t)}}} \frac{\partial \phi}{\partial t} = \frac{\partial}{\partial \pi} \left( \alpha \sqrt{\pi^2 + m^2 \phi^2 e^{2\alpha^2 \tau(t)}} \right) \tag{5.20c}
\]

which in the limit $\alpha \to 0$ can be written as:

\[
\frac{\partial \phi}{\partial t} \to \sqrt{\pi^2 + m^2 \phi^2} \frac{\partial}{\partial \pi} \sqrt{\pi^2 + m^2 \phi^2} \tag{5.21a}
\]

\[
= \frac{\partial}{\partial \pi} \left( \frac{1}{2} \pi^2 + \frac{1}{2} m^2 \phi^2 \right) \tag{5.21b}
\]
which of course is the correct limit.

We end this section by pointing out what we hoped to accomplish with these two examples. The first example was meant as a simple illustration of the method for an admittedly simple model. In it we made evident the fact that the form of the reduced Hamiltonian rests on the choice of the lapse function (i.e., the Hamiltonian is gauge dependent). The second example was used to show how the paradox of dynamics is resolved by reduction. It was used to show that the limit of $\kappa \to 0$ gives the same results obtained for flat space.

6. Correspondence With The Functional Formalism

The purpose of this section is to show that expectation values and matrix elements in the reduced canonical theory can be expressed very simply in terms of the naive functional formalism of the unconstrained theory. The key to obtaining this result is that reduction affects only the allowed initial values of Heisenberg operators, not their subsequent time evolutions. We can therefore enforce reduction by gauge fixing on the initial value surface and use the unconstrained Hamiltonian to implement time evolution. Since evolution is carried out with the known Hamiltonian of the unconstrained theory we get the usual functional formalism and we need never go to the trouble of actually constructing a reduced Hamiltonian or inferring how the original dynamical variables depend upon the reduced canonical variables. We first derive the result for a general constrained canonical formalism, then we explain how it applies to the coupled oscillator model of section 2, to scalar QED in temporal gauge, and to quantum general relativity with fixed lapse and shift.

We require first a notation for the unconstrained canonical formalism whose reduction leads to the $v^i(t)$’s of section 2. In a gauge theory this would be the canonical formalism which results from the imposition of a volume gauge condition, that is, one set of conditions for each spacetime point. Fixing the volume gauge is enough to make time evolution unique but it typically leaves a residual symmetry of gauge transformations of the initial value
data. These residual symmetry transformations are generated by constraints which restrict
the allowed initial value data. It is the act of fixing the residual symmetry and imposing
the constraints which leads to the reduced dynamical system characterized by the \( v^i(t) \)'s.

Let us consider the unconstrained system to be described by coordinates \( x^\alpha(t) \) and
momenta \( \pi_\alpha(t) \) where \( \alpha = 1, 2, \ldots, L \), and \( L = N + K \). The variables are canonical, that
is, the only non-zero equal time commutators are:

\[
\left[ x^\alpha(t), \pi_\beta(t) \right] = i \delta^\alpha_\beta \quad (6.1)
\]

Time evolution is generated by a possibly time dependent Hamiltonian, \( H(x, \pi, t) \):

\[
\dot{x}^\alpha(t) = -i \left[ x^\alpha(t), H(x(t), \pi(t), t) \right] \quad (6.2a)
\]
\[
\dot{\pi}_\alpha(t) = -i \left[ \pi_\alpha(t), H(x(t), \pi(t), t) \right] \quad (6.2b)
\]

Equations of this form serve to determine the variables at any time in terms of the initial
values, \( \hat{x}^\alpha \) and \( \hat{\pi}_\alpha \):

\[
x^\alpha(t) = X^\alpha \left( \hat{x}, \hat{\pi}, t \right) \quad (6.3a)
\]
\[
\pi_\alpha(t) = \Pi_\alpha \left( \hat{x}, \hat{\pi}, t \right) \quad (6.3b)
\]

Finally, we assume the system is subject to a set of \( K \) constraints:

\[
C_k \left( x(t), \pi(t), t \right) = 0 \quad , \quad k = 1, 2, \ldots, K \quad (6.4)
\]

To accommodate the coupled oscillator model of section 2 we have broken the usual con-
tvention of forbidding explicit time dependence in the constraints. We do require, however,
that if \( C_k \left( \hat{x}, \hat{\pi}, 0 \right) = 0 \) for all \( k \) then \( C_k \left( x(t), \pi(t), t \right) = 0 \) as well.

States in the unconstrained formalism can be represented by their wavefunctions in
the basis of position eigenkets at some fixed time:

\[
\left| \psi; t \right\rangle = \int d^L \xi \, \psi(\xi) \left| \xi; t \right\rangle \quad (6.5a)
\]
\[ x^\alpha(t) |\xi; t\rangle = \xi^\alpha |\xi; t\rangle \]  

(6.5b)

Note that these are Heisenberg states so they do not evolve in time, despite the fact that the wavefunction might be given in terms of the eigenkets of the position operator at any time. The inner product between two such states is the usual one:

\[ \langle \psi_2; t | \psi_1; t \rangle = \int d^L x \, \psi_2^*(x) \psi_1(x) \]  

(6.6)

The Heisenberg evolution operator:

\[ U(t_2, t_1) \equiv T\left\{ \exp \left[ i \int_{t_1}^{t_2} dt \, H(x(t), \pi(t), t) \right] \right\} \]  

(6.7a)

The symbol \( T \) denotes the ordering convention in which canonical operators at later times appear to the left of those at earlier times; coordinates stand to the right of momenta at equal times) enters when we wish to study operators at different times:

\[ x^\alpha(t_2) = U(t_2, t_1) x^\alpha(t_1) U^\dagger(t_2, t_1) \]  

(6.7b)

or when we wish to specify the wavefunctions in terms of the eigenkets of operators at different times:

\[ |\xi; t_2\rangle = U(t_2, t_1) |\xi; t_1\rangle \]  

(6.7c)

In these cases it is convenient to subsume the evolution operator into a functional integral. Suppose we wish to study some functional \( \mathcal{O}[x, \pi] \) of the canonical operators defined between times \( t_1 \) and \( t_2 > t_1 \). We can take the matrix element of its time-order product between states specified at \( t_2 \) and \( t_1 \) using the following formula:* 

\[ \langle \psi_2; t_2 | T(\mathcal{O}[x, \pi]) | \psi_1; t_1 \rangle = \left[ \int_{t_2 \geq t_1} dt \right] \left[ \int_{t_1 \geq t_2} dt' \right] \psi_2^*(x(t_2)) \mathcal{O}[x, \pi] \psi_1(x(t_1)) \]  

\[ \times \exp \left[ i \int_{t_1}^{t_2} dt \left\{ \pi_\alpha(t) \dot{x}^\alpha(t) - H(x(t), \pi(t), t) \right\} \right] \]  

(6.8)

* Functional integrals are sufficiently familiar in particle physics that we shall omit specification of the detailed skeletonization which would be necessary to make our formulae well-defined [22]. This could only be done for quantum mechanics and free field theory in any case. Complete rigor has not yet been obtained for interacting quantum field theories in four dimensions, although it is completely obvious how to proceed in regulated perturbation theory.
Note that the symbols \(x^\alpha(t)\) and \(\pi^\alpha(t)\) stand for operators on the left hand side of this equation while they are \(\mathbb{C}\)-number functions in the functional integral to the right. If the Hamiltonian is quadratic in the momenta then we can perform the Gaussian integrations over \(\pi^\alpha(t)\) and pass from the phase space functional formalism to the more familiar configuration space version.

If we wish to specify both states at the same time — say \(t_1\) — then it is necessary to first evolve forward to some arbitrary point \(t_2\), beyond the final observation in \(\mathcal{O}[x, \pi]\), and then evolve backwards to \(t_1\). The necessary formalism was worked out three decades ago by Schwinger [23] and has been studied more recently by Jordan [24].** If we denote the fields which implement forward evolution with a superscript or subscript \(+\), and those which generate backward evolution with a superscript or subscript \(−\), then the relevant functional integral is:

\[
\langle \psi_2; t_1 | T(\mathcal{O}[x, \pi]) | \psi_1; t_1 \rangle = \int \left[ \frac{dx^-(t)}{t_2 \geq t_1} \frac{d\pi^-(t')}{t_2 > t' \geq t_1} \frac{dx^+(t)}{t_2 \geq t_1} \frac{d\pi^+(t')}{t_2 > t' \geq t_1} \right] \delta(x^-(t_2) - x^+(t_2)) \times \psi^*_2(x^-(t_1)) \exp \left[ -i \int_{t_1}^{t_2} dt \left\{ \pi^-\alpha(t) \dot{x}^\alpha(t) - \mathcal{H}(x^-(t), \pi^-(t), t) \right\} \right] \times \mathcal{O}[x^+, \pi^+] \exp \left[ i \int_{t_1}^{t_2} dt \left\{ \pi^+\alpha(t) \dot{x}^\alpha(t) - \mathcal{H}(x^+(t), \pi^+(t), t) \right\} \right] \psi_1(x(t_1))
\]

Note that we must use \(\text{“}+\text{”}\) variables in the functional \(\mathcal{O}\) if the functional integral is to give the expectation value of the time-ordered product; using the \(\text{“}−\text{”}\) variables would give the anti-time-ordered product.

The unconstrained matrix elements and expectation values we have described are inadequate in two ways: they are typically divergent for the only interesting states, and they include information we don’t want to know about unphysical operators. The first problem arises because one enforces the constraints (6.4) by requiring the states to be annihilated by

---

** Although Schwinger and Jordan assumed that the initial and final states were free vacuum and that \(t_1\) was in the asymptotic past, the generalization to arbitrary states and times is trivial.
the $C_k$’s. The inner product (6.6) between two such states can be divergent on account of the integration over residual gauge transformations. *This is even true when, as in the case for gravity, the residual gauge transformation of a coordinate $x^\alpha$ involves the momentum $\pi_\beta$ [21].* The second problem arises because it is really operators in the reduced canonical formalism whose matrix elements and expectation values we wish to study. These reduced operators have the same time evolution as the unconstrained ones, but they depend upon $2K$ fewer initial value operators.

It turns out that by implementing reduction we can also solve the problem with the inner product. The unconstrained theory is reduced by identifying $K$ residual gauge conditions on an initial value surface which we take to be $t = 0$ for definiteness:

$$G^k(\hat{x}, \hat{\pi}) = 0$$

(6.10)

These conditions are arbitrary except for the requirement that the Faddeev-Popov matrix formed from commutation with the constraints:

$$M_{k}^{\ell}(\hat{x}, \hat{\pi}) \equiv -i\left[C_k(\hat{x}, \hat{\pi}, 0), G^\ell(\hat{x}, \hat{\pi})\right]$$

(6.11)

should have a non-zero determinant.

We can use the conditions $G^k = 0$ and $C^\ell = 0$ to decompose the $2L$ operators of the unconstrained formalism into two commuting sets of canonical variables:

$$\left\{\hat{x}, \hat{\pi}\right\} \longrightarrow \left\{\hat{q}, \hat{p}; \hat{c}, \hat{g}\right\}$$

(6.12)

The $N \hat{q}^a$’s and $N \hat{p}_b$’s commute canonically and are of course the quantum versions of the reduced canonical variables whose construction was described in section 2. The $K \hat{g}^k$’s form similar conjugate pairs with the $K \hat{c}_\ell$’s. The two sets of variables commute with one another so that the only non-zero commutators are:

$$\left[\hat{q}^a, \hat{p}_b\right] = i\delta^a_b$$

(6.13a)
The \(\hat{g}^k\)'s are pure gauge variables which have the property of vanishing with the gauge conditions:

\[
\hat{g}^k \delta^K \left[ G(\hat{x}, \hat{\pi}) \right] = 0
\]  

The \(\hat{c}_\ell\)'s are constrained variables; the constraint equations determine them as functions \(\kappa_\ell(\hat{q}, \hat{p})\) of the reduced canonical variables. The Coulomb potential of electrodynamics and the Newtonian potential of gravitation are examples of constrained variables. When acting on invariant states — that is, upon states which are annihilated by the constraint operators — there is no difference between the \(\hat{c}_\ell\)'s and these functions:

\[
\left| \psi_{\text{inv}}; t \right\rangle \equiv \kappa_\ell(\hat{q}, \hat{p}) \left| \psi_{\text{inv}}; t \right\rangle
\]  

Except for the fact that we have distinguished between the constrained variables and the constraints themselves — that is, we have allowed for the possibility that \(\kappa_\ell \neq 0\), and that the reduced canonical variables might not commute with the constraints — the decomposition (6.12) is a standard one in the theory of constrained quantization [25]. The reduced operators whose matrix elements and expectation values we really wish to study are:

\[
x_\alpha^\ell(t) \equiv X_\alpha^{(\hat{x}, \hat{\pi}, t)} \bigg|_{\hat{g}=0, \hat{c}=\kappa}
\]

\[
\pi_\alpha^\ell(t) \equiv \Pi_\alpha^{(\hat{x}, \hat{\pi}, t)} \bigg|_{\hat{g}=0, \hat{c}=\kappa}
\]

Note that they have the same time evolution as the unconstrained operators (6.3).

Now consider the inner product of two states which are annihilated by the constraints. Since the potential for divergences in the naive inner product comes from integrating over residual gauge transformations we can avoid the problem by surface gauge fixing. In the
case where the gauge conditions and the Faddeev-Popov determinant depend only upon the coordinates this is accomplished by formally inserting unity in the form:

$$1 = \int d^K \theta \exp \left[ i \theta_k C_k \right] \delta^K \left[ G(\hat{x}, \hat{\pi}) \right] \abs{ \det M_{k\ell}(\hat{x}, \hat{\pi}) } \exp \left[ -i \theta_k C_k \right]$$  (6.17)

One then changes variables and drops the typically divergent integration over gauge parameters. Although the question of operator ordering arises forcefully when either the gauge conditions or the Faddeev-Popov determinant are allowed to depend upon non-commuting operators, it is natural to expect that the same procedure works generally. This expectation is fulfilled in several interesting cases, including that of perturbative quantum gravity [21]. We therefore propose the following reduced inner product:

$$\langle \psi_2; t_2 \mid \psi_1; t_1 \rangle_G = \langle \psi_2; t_2 \mid \delta^K \left[ G(\hat{x}, \hat{\pi}) \right] \abs{ \det M_{k\ell}(\hat{x}, \hat{\pi}) } \mid \psi_1; t_1 \rangle \quad (6.18)$$

where it is understood that the issue, if there is one, of operator ordering is to be resolved on a case-by-case basis. Note that in view of (6.17) this inner product is independent of the choice of gauge for states which are annihilated by the constraints. In fact it amounts to a realization in the unconstrained formalism of the standard inner product of the reduced theory [25]. Enforcing this correspondence is an important determinant of the ordering convention for the gauge fixing paraphernalia; another important condition is the persistence of Hermiticity when we drop the requirement that the wave functions are annihilated by the constraints.

The relation we seek between invariant operators in the unconstrained theory and the corresponding reduced operators holds in the gauge fixed inner product (6.18). Although invariant operators generally depend upon the $\hat{g}^k$’s and the $\hat{c}_\ell$’s in addition to the reduced canonical variables, when such an invariant operator acts upon an invariant state we can use the constraints $\hat{c}_\ell = \kappa_\ell \left[ \hat{q}, \hat{p} \right]$:

$$O_{\text{inv}} \left[ x, \pi \right] \left| \psi_{\text{inv}}; t \right\rangle = O_{\text{inv}} \left[ x, \pi \right] \left| \psi_{\text{inv}}; t \right\rangle_{\hat{c} = \kappa}$$  (6.19a)
\[ \left\langle \psi_{\text{inv}}; t \right| \mathcal{O}_{\text{inv}} \left[ x, \pi \right] = \left\langle \psi_{\text{inv}}; t \right| \mathcal{O}_{\text{inv}} \left[ x, \pi \right] \bigg|_{c=\kappa} \quad (6.19b) \]

Note that this is true even when the \( \tilde{c}_\ell \)'s are not themselves invariant. Once the \( \tilde{c}_\ell \)'s have been eliminated there is no obstacle to commuting the \( \tilde{g}^k \)'s through the \( \tilde{q}^a \)'s and \( \tilde{p}_b \)'s to act on the gauge fixing delta function. We therefore obtain the result that the expectation value or matrix element of an invariant operator in the presence of invariant states is the same as the expectation value or matrix element of the corresponding reduced operator:

\[ \left\langle \psi_{\text{inv}}; t \right| \mathcal{O}_{\text{inv}} \left[ x, \pi \right] \left| \psi_{\text{inv}}'; t' \right\rangle_G = \left\langle \psi_{\text{inv}}; t \right| \mathcal{O}_{\text{inv}} \left[ x_r, \pi_r \right] \left| \psi_{\text{inv}}'; t' \right\rangle_G \quad (6.20) \]

To reach the functional formalism we consider the action of the gauge fixing paraphernalia on the ket to be a state in the unconstrained theory:

\[ \delta^K \left[ G \left( \bar{x}, \bar{\pi} \right) \right] \text{ abs}\left\{ \det \left[ M_{k\ell} \left( \bar{x}, \bar{\pi} \right) \right]\right\} \left| \psi_{\text{inv}}'; t' \right\rangle \equiv \left| \psi'; t' \right\rangle \quad (6.21) \]

and apply the functional formulae (6.8) and (6.9). It is clearly convenient for this purpose to either assume that \( t' = 0 \) or else to fix the residual gauge on the surface \( t = t' \). The relevant formulae for matrix elements and expectation values are, respectively:

\[ \left\langle \psi_2; t_2 \left| T \left( \mathcal{O} \left[ x, \pi \right] \right) \right| \psi_1; t_1 \right\rangle_G = \left[ \int_{t_2 \geq t \geq t_1} \left[ dx(t) \right] \left[ d\pi(t) \right] \psi_2^\dagger \left( x(t_2) \right) \mathcal{O} \left[ x, \pi \right] \right. \left. \times \exp \left( i \int_{t_1}^{t_2} dt \left\{ \pi_\alpha(t) \dot{x}_\alpha(t) - H \left( x(t), \pi(t), t \right) \right\} \right) \delta^K \left[ G \left( x(t_1), \pi(t_1) \right) \right] \text{ abs}\left\{ \det \left[ M_{k\ell} \left( x(t_1), \pi(t_1) \right) \right]\right\} \psi_1 \left( x(t_1) \right) \right) \quad (6.22a) \]

\[ \left\langle \psi_2; t_2 \left| T \left( \mathcal{O} \left[ x, \pi \right] \right) \right| \psi_1; t_1 \right\rangle_G = \left[ \int_{t_2 \geq t \geq t_1} \left[ dx_- (t) \right] \left[ d\pi_- (t) \right] \left[ dx_+ (t) \right] \left[ d\pi_+ (t) \right] \delta \left( x_- (t_2) - x_+ (t_2) \right) \right. \left. \times \psi_2^\dagger \left( x_- (t_1) \right) \exp \left( -i \int_{t_1}^{t_2} dt \left\{ \pi_\alpha (t) \dot{x}_- (t) - H \left( x_- (t), \pi_- (t), t \right) \right\} \right) \right. \left. \times \mathcal{O} \left[ x_+, \pi^+ \right] \exp \left( i \int_{t_1}^{t_2} dt \left\{ \pi_\alpha^+ (t) \dot{x}_+ (t) - H \left( x_+ (t), \pi_+ (t), t \right) \right\} \right) \delta^K \left[ G \left( x(t_1), \pi(t_1) \right) \right] \text{ abs}\left\{ \det \left[ M_{k\ell} \left( x(t_1), \pi(t_1) \right) \right]\right\} \psi_1 \left( x(t_1) \right) \right) \quad (6.22b) \]

61
The reader should be aware that these expressions may require modification on the initial value surface to account for whatever operator ordering prescription is imposed upon the paraphernalia of surface gauge fixing. If both the time-ordered product of \( \mathcal{O}[x, \pi] \) and the two states are invariant then relation (6.20) shows that (6.22a) and (6.22b) give the matrix element and expectation value, respectively, of the reduced operator \( T\left(\mathcal{O}[x_r, \pi^r]\right) \).

If \( \mathcal{O}[x, \pi] \) is invariant before time-ordering then the necessary ordering corrections are the simple ones of the unconstrained theory, not the potentially complicated ones of the reduced theory. That is, we first convert the reduced operator \( \mathcal{O}[x_r, \pi^r] \) into the analogous unconstrained operator, \( \mathcal{O}[x, \pi] \), inside the matrix element or expectation value. Then we write it as the time-ordered product, \( T\left(\mathcal{O}[x, \pi]\right) \), plus ordering corrections, and we use (6.22) to evaluate the matrix element or expectation value of the time-ordered product. Since the ordering corrections are also operators in the unconstrained theory we time order them and apply (6.22) again, repeating the procedure as often as is necessary to reduce everything to functional integrals.

If the states and the operator in (6.22) are invariant then these expressions are independent of the residual gauge condition \( G^k = 0 \). However, even when the wavefunctions or the functional \( \mathcal{O}[x, \pi] \) are not manifestly invariant the right hand sides of (6.22a) and (6.22b) still represent the matrix elements or expectation values of some invariant operator in the presence of some invariant states. That this must be so follows from the fact that the gauge has been completely fixed and any quantity becomes invariant when it is defined in a unique gauge. To find which invariants one employs the residual gauge condition to “invariantize” the wavefunctions and the operator [21,26].

It is to some extent pointless to expend effort in order to discover a manifestly invariant state or operator when the only way physical information can be extracted from these objects is by taking gauge fixed inner products. The aesthetic advantage to using manifestly invariant wavefunctions and operators is that then matrix elements and expectation values
are independent of the choice of gauge. If we understand physics in a particular gauge — as we often do — then this is not much of an advantage. The practical advantage is that manifest invariance allows us, through relation (6.20), to compute the matrix element or expectation value of a reduced operator $O[x_r, \pi^r]$ using the same matrix element or expectation value of the unconstrained operator $O[x, \pi]$. We can therefore avoid the need to ever construct the reduced Hamiltonian which, as we have seen, can be a formidable task. Of course we can always invariantize non-invariant states and operators, but then the non-locality and complicated field dependence this typically entails makes it difficult to compute the time ordering corrections. However, we emphasize that the process is simple enough to carry out perturbatively — witness section 4 — and the very fact that only operator ordering corrections are needed to relate $O[x, \pi]$ to $O[x_r, \pi^r]$ inside gauge fixed inner products shows that we are well and truly free of the formal paralysis described as the paradox of second coordinatization.

We turn now to a survey of how the general formalism manifests itself in the three dynamical systems we have studied. For the coupled oscillator model of section 2 the $x^\alpha(t)$’s are $q_1(t)$ and $q_2(t)$ while the $\pi_\alpha(t)$’s are $p_1(t)$ and $p_2(t)$. The evolution functions $X^\alpha(\hat{x}, \hat{\pi}, t)$ and $\Pi_\alpha(\hat{x}, \hat{\pi}, t)$ can be read off from (2.34). The unconstrained Hamiltonian is:

$$H = \frac{1}{2m}p_1^2 + \frac{1}{2m}p_2^2 + \frac{1}{2}m\omega^2 \left(\frac{5}{4}q_1^2 + q_1 q_2 + \frac{5}{4}q_2^2\right)$$ (6.23)

Because it is quadratic in the momenta we can convert the canonical functional integrals of (6.22) into the more familiar, configuration space functional integrals.

Recall that since the coupled oscillator model is not really a gauge theory we imposed $\hat{p}_2 = 0$ as an ersatz “constraint.” We might therefore identify the single constraint by inverting (2.34) to give $\hat{p}_2$ in terms of the time evolved variables:

$$C = \frac{1}{2} \left[p_1(t) + p_2(t)\right] \cos\left(\frac{3}{2}\omega t\right) + \frac{1}{2} \left[-p_1(t) + p_2(t)\right] \cos\left(\frac{1}{2}\omega t\right)$$

$$+ \frac{3}{4}m\omega \left[q_1(t) + q_2(t)\right] \sin\left(\frac{3}{2}\omega t\right) + \frac{1}{4}m\omega \left[-q_1(t) + q_2(t)\right] \sin\left(\frac{1}{2}\omega t\right)$$ (6.24)
The most general invariant operator is a function of $\hat{q}_1$, $\hat{p}_1$ and $\hat{p}_2$. We can of course express these initial value operators in terms of the evolved operators. The result for $\hat{p}_2$ is just (6.24); the results for $\hat{q}_1$ and $\hat{p}_1$ are:

$$\hat{q}_1 = \frac{1}{2} \left[ q_1(t) + q_2(t) \right] \cos \left( \frac{3}{2} \omega t \right) + \frac{1}{2} \left[ q_1(t) - q_2(t) \right] \cos \left( \frac{1}{2} \omega t \right) + \frac{1}{3m\omega} \left[ p_1(t) + p_2(t) \right] \sin \left( \frac{3}{2} \omega t \right) + \frac{1}{m\omega} \left[ -p_1(t) + p_2(t) \right] \sin \left( \frac{1}{2} \omega t \right)$$  (6.25a)

$$\hat{p}_1 = \frac{1}{2} \left[ p_1(t) + p_2(t) \right] \cos \left( \frac{3}{2} \omega t \right) + \frac{1}{2} \left[ p_1(t) - p_2(t) \right] \cos \left( \frac{1}{2} \omega t \right) + \frac{3}{4} m\omega \left[ q_1(t) + q_2(t) \right] \sin \left( \frac{3}{2} \omega t \right) + \frac{1}{4} m\omega \left[ q_1(t) - q_2(t) \right] \sin \left( \frac{1}{2} \omega t \right)$$  (6.25b)

“Invariant” states are those whose wavefunctions are independent of $q_2$ in the $t = 0$ basis of position eigenkets. Since invariant operators are independent of $\hat{q}_2$ it is clear why any $\hat{p}_2$’s can also be neglected when acting on an invariant state.

Since the residual gauge condition is $G = \hat{q}_2$ the associated Faddeev-Popov matrix is:

$$M = -1$$  (6.26)

The single canonical pair $(\hat{q}, \hat{p})$ is of course just $(\hat{q}_1, \hat{p}_1)$; the pure gauge variable is $\hat{g}^k \rightarrow \hat{q}_2$, and the constrained variable is $\hat{c}_\ell \rightarrow \hat{p}_2$. Note that the unconstrained operators $q_i(t)$ and $p_i(t)$ are not invariant because they depend upon $\hat{q}_2$. This means that products of them do not give products of the analogous reduced operators in matrix elements and expectation values. For example, it is straightforward to show that in this gauge:

$$\langle \psi_{\text{inv}}; t | q_1(s) q_1(s') | \psi'_{\text{inv}}; t' \rangle_G = \langle \psi_{\text{inv}}; t | q_1^r(s) q_1^r(s') | \psi'_{\text{inv}}; t' \rangle_G$$

$$- \frac{i}{m\omega} \left[ \sin \left( \frac{3}{2} \omega s \right) - \sin \left( \frac{1}{2} \omega s \right) \right] \left[ \frac{1}{2} \cos \left( \frac{3}{2} \omega s' \right) - \frac{1}{2} \cos \left( \frac{1}{2} \omega s' \right) \right] \langle \psi_{\text{inv}}; t | \psi'_{\text{inv}}; t' \rangle_G$$  (6.27)

The terms on the extreme right represent unphysical and gauge dependent information that is included in the expectation value or matrix element of the unconstrained, non-invariant operator.

For scalar QED in temporal gauge the $x^\alpha(t)$’s are the fields $\phi(t, \vec{x})$, $\phi^*(t, \vec{x})$ and $A_i(t, \vec{x})$; the $\pi^\alpha(t)$’s are $\pi(t, \vec{x})$, $\pi^*(t, \vec{x})$ and $E_i(t, \vec{x})$. Owing to the interaction, there is no closed
form expression for the evolution functions $X^\alpha(\vec{x}, \vec{\pi}, t)$ and $\Pi_\alpha(\vec{x}, \vec{\pi}, t)$. The unconstrained Hamiltonian is some ordering of (3.6). Since it is quadratic we can convert the canonical functional formalism into the configuration space version.

The constraint — of which there is one for each space point $\vec{x}$ — is some ordering of (3.8). A typical invariant operator is:

$$O_{\text{inv}} = \phi(t, \vec{x}) \exp \left[ ie \int d^3x' G(\vec{x}; \vec{x}') \partial_i A_i(t, \vec{x}) \right]$$

$$\times \left[ E_j(u, \vec{y}) \pi(v, \vec{z}) \exp \left[ -ie \int d^3z' G(\vec{z}; \vec{z}') \partial_k A_k(v, \vec{z}) \right] \right]$$

(6.28)

where $G(\vec{x}; \vec{x}')$ is the Green’s function first used in (3.13). The exponential term associated with each of the charged fields creates the associated longitudinal Coulomb field. A typical invariant wavefunctional is that of the free vacuum which, up to a normalization factor, is:

$$\begin{align*}
\Psi_{\text{inv}}[A, \phi, \phi^*] &\propto \exp \left[ -\int d^3x \Phi^*(\vec{x}) \sqrt{-\nabla^2} \Phi(\vec{x}) \right] \\
&\times \exp \left[ -\frac{1}{4} \int d^3xF_{ij}(\vec{x}) \frac{1}{\sqrt{-\nabla^2}} F_{ij}(\vec{x}) \right] \\
\Phi(\vec{x}) &\equiv \phi(\vec{x}) \exp \left[ ie \int d^3x' G(\vec{x}; \vec{x}') \partial_i A_i(\vec{x}) \right]
\end{align*}$$

(6.29a)

(6.29b)

The apparent coupling between the vector potential and the charged fields is a gauge fiction; it disappears when the instantaneous Coulomb gauge condition (3.11) is used. This is an example of why it is pointless to struggle to achieve manifest invariance in an expression which must ultimately rest against a gauge fixing delta functional. Note as well that (6.28) and (6.29) are only invariant with respect to the residual, time independent gauge symmetry.

For instantaneous Coulomb gauge, (3.11), the Faddeev-Popov matrix is:

$$M(\vec{x}, \vec{y}) = -\frac{\partial}{\partial x^i} \frac{\partial}{\partial y^i} \delta^3(\vec{x} - \vec{y})$$

(6.30)
Note that it has a non-singular determinant on the appropriate function space. The most natural identification of canonical, gauge and constrained operators on the initial value surface is:

\[
\hat{q}^a \longrightarrow \{ \hat{A}^T_i(\vec{x}), \hat{\phi}(\vec{y}), \hat{\phi}^*(\vec{z}) \} \tag{6.31a}
\]

\[
\hat{p}_a \longrightarrow \{ \hat{E}^T_i(\vec{x}), \hat{\pi}(\vec{y}), \hat{\pi}^*(\vec{z}) \} \tag{6.31b}
\]

\[
\hat{g}^k \longrightarrow A^L_i(\vec{x}) \tag{6.31c}
\]

\[
\hat{c}^k \longrightarrow E^L_i(\vec{x}) \tag{6.31d}
\]

Note that although these initial operators do commute canonically — on the appropriate function space — we saw in section 3 that their time evolved versions do not. If we assume that \( t > u > v > 0 \) then the invariant operator (6.28) is also time ordered and we can write:

\[
\langle \Psi_{\text{inv}}; t \mid T(\mathcal{O}_{\text{inv}}) \mid \Psi_{\text{inv}}; 0 \rangle_G = \left[ \int dA_\mu(s, \vec{w}) \left[ \phi(s, \vec{w}) \right] \left[ d\phi^*(s, \vec{w}) \right] \delta[A_0(s, \vec{w})] \right]_{t \geq s \geq 0, \vec{w} \in \mathbb{R}^3}
\times \Psi_{\text{inv}}^*[A(t, \vec{w}), \phi(t, \vec{w}), \phi^*(t, \vec{w})] \phi(t, \vec{x}) \exp \left[ ie \int d^3x' G(\vec{x}; \vec{x}') \partial_i A_i(t, \vec{x}) \right]
\times F_{0ij}(u, \vec{y}) \phi^*(v, \vec{z}) \exp \left[ -ie \int d^3z' G(\vec{z}; \vec{z}') \partial_k A_k(v, \vec{z}) \right]
\times \exp \left[ i \int_0^t ds \int d^3w \mathcal{L} \delta[\partial_i A_i(0, \vec{w})] \Psi_{\text{inv}}[A(0, \vec{w}), \phi(0, \vec{w}), \phi^*(0, \vec{w})] \right] \tag{6.32}
\]

where expression (3.1) defines the Lagrangian \( \mathcal{L} \). Note that we have integrated out the momenta and that we have introduced \( A_0 \) as an integral over the volume gauge fixing delta functional. Since the operator (6.28) is invariant, as are the states, it follows that this functional integral represents the matrix element of the corresponding reduced operator. The interested reader can worked out examples from standard QED in ref. [26].

For quantum general relativity at fixed lapse and shift the \( x^\alpha(t) \)'s are the 3-metrics, \( \gamma_{ij}(t, \vec{x}) \); the \( \pi_\alpha(t) \)'s are their conjugate momenta, \( \pi^{ij}(t, \vec{x}) \). It is no more possible than for
scalar QED to give explicit forms for the evolution functions, \( X^\alpha(\vec{x}, \vec{\pi}, t) \) and \( \Pi_\alpha(\vec{x}, \vec{\pi}, t) \).

The Hamiltonian is some operator ordering of the classical result:

\[
H[\gamma, \pi](t) = \int d^3x \; N^\mu[\gamma, \pi](t, \vec{x}) \; \mathcal{H}_\mu[\gamma, \pi](t, \vec{x})
\]

(6.33)

Note that the lapse and shift may depend upon time and also upon the dynamical variables; in fact dependence upon the dynamical variables is necessary classically in order to have a chance of avoiding the evolution of coordinate singularities. Although no one has ever exhibited a gauge which is classically free of coordinate singularities its existence seems obvious if a sufficient amount of field dependence and non-locality is permitted in the lapse and shift. In any case we shall assume that such a gauge exists.

Note that if the lapse and shift do depend upon the \( \pi^{ij} \)'s then the canonical action may not be quadratic in the momenta and we would not be able to reach the usual configuration space functional formalism. Although inconvenient, there is no inconsistency in this. It could happen as well in scalar QED if we had allowed \( A_0 \) to depend upon \( E_i, \pi \) or \( \pi^* \). Note also that even if the lapse and shift depend only upon \( \gamma_{ij} \), integrating out the momenta will give a field dependent measure factor. If the lapse depends non-locally upon \( \gamma_{ij} \) then this measure factor can be non-local, and it may therefore give non-zero contributions even in dimensional regularization. Again, this is not specific to gravity; we could have made it happen as well for scalar QED by permitting \( A_0 \) to depend non-locally upon the fields.

There are four constraints for each space point \( \vec{x} \). Without regard to operator ordering they are:* 

\[
\mathcal{H}_0 = \frac{\kappa^2}{\sqrt{\gamma}} \left( \gamma_{ik} \gamma_{j\ell} - \frac{1}{2} \gamma_{ij} \gamma_{k\ell} \right) \pi^{ij} \pi^{k\ell} - \frac{1}{\kappa^2} \left( \mathcal{R} - 2\Lambda \right) \sqrt{\gamma}
\]

(6.34a)

\[
\mathcal{H}_i = -2 \gamma_{ij} \pi^{j\ell} \pi^{k\ell}
\]

(6.34b)

* It is futile to pay much attention to ordering so long as the problem with renormalization precludes being able to take the unregulated limit of inner products [27].
It is reasonably straightforward to construct a basis of operators which transform as scalar densities under an arbitrary diffeomorphism. The method is to evaluate Riemann tensors at the ends of geodesic segments emanating from a central origin, with the indices of each Riemann tensor defined in the local inertial frame field of the origin as obtained by parallel transport along the connecting geodesic [28]. If the coordinate time generated by (6.33) can be extended infinitely then full diffeomorphism invariance results when the origin is integrated over $R \times M^3$. Since such operators are invariant under all diffeomorphisms they are necessarily invariant under those of the residual symmetry group which preserve the fixed lapse and shift.

No one has obtained closed form expressions for normalizable invariant states, but one can of course construct them order by order in perturbation theory. It is the fact that the asymptotic functional formalism does this automatically which explains the observation in [26] and [28] that all of Einstein’s equations are obeyed in regulated perturbation theory. Note also that there is a considerable phenomenological advantage to building invariant states perturbatively: one knows what they mean this way. If given a solution of the constraints it is difficult to tell what level of excitation it represents since on a closed spatial manifold all states are degenerate with zero energy. However, if this solution is perturbatively related to the invariantized free vacuum then one can assume — at least as a first guess — that it represents empty space. Similar comments apply to the wavefunctionals built up from other Fock space states.

In section 4 we reduced the theory in two steps. In the first step we enforced the constraints that fixed $h^t$ and $p_i$ and surface gauge fixed their conjugate variables. Explicitly:

The $C_k$’s are:

\[ h^t = 0 + \mathcal{O}(\kappa^1) \]  \hspace{1cm} (6.35a)

\[ p_i = 0 + \mathcal{O}(\kappa^1) \]  \hspace{1cm} (6.35b)
while the $G_k$’s are:

$$
\hat{p}^t = 0 \quad (6.36a)
$$

$$
\hat{h}_i = 0 \quad (6.36b)
$$

The Faddeev-Popov determinant arising from this first step of reduction is just a $\mathbb{C}$-number to lowest order. This is not the case for the second step. In the second step we enforced the constraint on $p^{tr}$ and surface gauge fixed $h^{tr}$ using:

$$
(p^{tr})^2 = 6 E_0 + \mathcal{O}(\kappa^1) \quad (6.37a)
$$

$$
\hat{h}^{tr} = 0 \quad (6.37b)
$$

We see now that the Faddeev-Popov matrix for this, the second step is $2\hat{p}^{tr}$. The choice of variables on the initial value surface goes as follows:

$$
\hat{q}^a \rightarrow \hat{h}^{tt} \quad (6.38a)
$$

$$
\hat{p}_a \rightarrow \hat{p}^{tt} \quad (6.38b)
$$

$$
\hat{g}^k \rightarrow \left\{ \hat{p}^{t}, \hat{h}^{i}, \hat{h}^{tr} \right\} \quad (6.38c)
$$

$$
\hat{c}_k \rightarrow \left\{ \hat{h}^{t}, \hat{p}^{i}, \hat{p}^{tr} \right\} \quad (6.38d)
$$

We will see next how the inner product breaks up in two parts. One part for negative $p^{tr}$ and one for positive, representing and expanding and a contracting space respectively.

Let us look at the inner product defined in (6.18):

$$
\langle \psi_2; t_2 \| \psi_1; t_1 \rangle_G = \langle \psi_2; t_2 \| \delta^K \left[ G(\hat{x}, \hat{\pi}) \right] \text{abs} \left\{ \text{det} \left[ M_{k\ell}(\hat{x}, \hat{\pi}) \right] \right\} \psi_1; t_1 \rangle \quad (6.18)
$$

$$
\simeq \langle \psi_2; t_2 \| \delta \left[ \hat{h}^{tr} \right] \text{abs} \left\{ 2p^{tr} \right\} \psi_1; t_1 \rangle \quad (6.39)
$$

Where we used $\simeq$ because we are disregarding overall multiplicative factors and we are working to lowest order in $\kappa$. As mentioned in section 4 each wavefunction is divided in two parts depending on the action of $p^{tr}$ upon them:

$$
|\psi; t\rangle = |\psi^+; t\rangle + |\psi^-; t\rangle \quad (6.40)
$$
The inner product of (6.39) can then be written as:

\[
\langle \psi_2; t_2 | \psi_1; t_1 \rangle_G = \langle \psi_2; t_2 \left| \Theta \left( p^{tr} \right) \delta \left( h^{tr} \right) p^{tr} \Theta \left( p^{tr} \right) + \Theta \left( p^{tr} \right) p^{tr} \delta \left( h^{tr} \right) \Theta \left( p^{tr} \right) - \Theta \left( -p^{tr} \right) \delta \left( h^{tr} \right) p^{tr} \Theta \left( -p^{tr} \right) - \Theta \left( -p^{tr} \right) p^{tr} \delta \left( h^{tr} \right) \Theta \left( -p^{tr} \right) \right| \psi_1; t_1 \rangle
\]  

(6.41)

Where we have chosen a Hermitian ordering. Using (6.40) equation (6.41) becomes:

\[
\langle \psi_2; t_2 | \psi_1; t_1 \rangle_G = \langle \psi_2^+; t_2 \left| \delta \left( h^{tr} \right) |p^{tr}| + |p^{tr}| \delta \left( h^{tr} \right) \right| \psi_1^+; t_1 \rangle \\
+ \langle \psi_2^-; t_2 \left| \delta \left( h^{tr} \right) |p^{tr}| + |p^{tr}| \delta \left( h^{tr} \right) \right| \psi_1^-; t_1 \rangle
\]  

(6.42)

We see that, as previously promised, the inner product breaks up into two parts. Both of these parts are present quantum mechanically; however, classically either \( \psi^+ = 0 \) or \( \psi^- = 0 \).

7. Conclusions

The context of our analysis is that of a gauge theory in which the ability to perform local, time dependent transformations has been fixed but there is still a residual gauge symmetry characterized by how it acts on the initial value surface. Examples of such theories include scalar electrodynamics in temporal gauge, which was studied in section 3, and general relativity with fixed lapse and shift, which was studied in section 4. The initial value problem has a unique solution in this setting — which it does not in the fully invariant theory — because residual symmetry transformations cannot change the time evolved dynamical variables without also changing their initial values. We assume that a canonical formalism describes this initial value problem; we assume as well that the residual gauge symmetry is generated in the usual way by functions of the dynamical variables which are constrained to vanish as a consequence of the Euler-Lagrange equations.
of the (volume) gauge fixed variables. This is what we mean by the “unconstrained theory,” and its generic dynamical variables are the $x^\alpha(t)$'s and $\pi_\beta(t)$'s of section 6.

The generic reduced theory is obtained by fixing the residual gauge freedom, imposing the constraints, and identifying a subset, $\{v^i(t)\}$, of the original dynamical variables which gives a complete and minimal representation of physics. What this means is that the constraints, the residual gauge conditions and the values of the $v^i(t)$'s at any time $t$ uniquely determine the $x^\alpha(t)$'s and the $\pi_\beta(t)$'s. The reduced dynamical variables inherit their time evolution and their equal time bracket (or commutation) algebra from the unconstrained theory. This is true even when the unconstrained Hamiltonian vanishes after reduction; it is even true if *no* Hamiltonian exists which generates the time evolution of the $v^i(t)$'s.

Section 2 described a standard construction of the last century for identifying canonical variables $q^a(t)$ and $p_b(t)$ in the reduced theory, and for finding the non-zero Hamiltonian which generates their time evolution. The identification of a reduced canonical formalism is not unique. On the classical level we can vary it by performing canonical transformations; with suitable attention to operator ordering there is a similar class of transformations on the quantum level. What keeps *physics* unchanged is the fact that measurement theory is based upon the $x^\alpha(t)$'s and the $\pi_\beta(t)$'s, considered as functions of the reduced canonical variables. As our identification of the reduced canonical variables changes, the functional dependence of the unreduced canonical variables changes so as to keep the $x^\alpha(t)$'s and $\pi_\beta(t)$'s the same.

An immediate consequence of the multiplicity of reduced canonical formalisms is that the associated Hamiltonians have no physical significance independent of whatever meaning attaches to the reduced canonical variables whose evolution they generate. Although this may seem strange, quantum field theorists are familiar with the phenomenon through the “interaction representation.” This is a field redefinition in which any perturbatively well defined interacting quantum field theory is transformed into the corresponding free
quantum field theory. We do not conclude that all perturbatively well defined quantum field theories are free because we insist that physics be inferred from the original fields.

The construction of a reduced canonical formalism is irrelevant to most issues in classical physics. This is because we infer physics from the $x^\alpha(t)$’s and $\pi_\beta(t)$’s, and we may as well solve for these variables directly in the unconstrained theory, starting from initial value data which obey the constraints. The principle motivation for erecting a reduced canonical formalism is as a prelude to applying canonical quantization. This is especially relevant to systems, such as gravity on a spatially closed manifold, for which reduction causes the unconstrained Hamiltonian to vanish. As has been noted, we did not invent the construction discussed in section 2; this was done by the classical physicists of the previous century [5,6,10-15]. Our contribution is rather to propose that quantum gravity should be defined by canonically quantizing a reduced canonical formulation of whatever is the correct theory of gravity.

We constructed explicit reduced canonical formalisms for the coupled oscillator model of section 2, and for scalar electrodynamics in section 3. Although no complete construction seems possible for general relativity we showed in section 4 how to do it perturbatively around a flat background on $T^3 \times R$. We also gave explicit constructions for a handful of minisuperspace truncations in section 5. Our inability to give an explicit, non-perturbative construction for the complete theory of general relativity does not pose a practical barrier — even for the study of non-perturbative phenomena — because of the relation we were able to obtain in section 6 between the relatively simple quantum mechanics of the unconstrained theory and that of the reduced theory. Equation (6.20) asserts that the matrix elements or expectation values of invariant functionals of the reduced operators are equal to the matrix elements or expectation values of the same functionals of the unconstrained operators in the presence of invariant states. Explicit functional integral representations exist for unconstrained matrix elements and expectation values. These representations are
as simple to evaluate as it seems possible for anything to be in an interacting quantum field theory, and they have at least the potential for extension beyond perturbation theory. Thus it is not really necessary to explicitly construct a reduced canonical formalism; it suffices to know that such a construction exists and that we can study it using the far simpler formalism of the unconstrained theory.

It is worth reviewing how the formulation we propose for canonical quantum gravity avoids the four correspondence paradoxes mentioned in section 1. Our resolution to the paradox of second coordinatization is that the lapse and the shift determine what is meant by time evolution in quantum gravity the same way they do for classical gravity.* It is neither necessary, nor even particularly desirable, to try doing the job twice by identifying some other variable as “time” and then attempting to interpret the Wheeler-DeWitt constraint as a Schrödinger evolution equation. Our method works because the Heisenberg field operators depend upon the time fixed by the lapse and the shift, whether or not we restrict the initial value data by fixing the residual gauge and imposing the constraints. The conventional method will only work if one of the Heisenberg operators is an invertible function of time, and the method will only produce tractable results if this time depen-

* A tangential point concerns the order of gauge fixing. Our volume gauge condition determines the lapse and shift as functionals of $\gamma_{ij}(t, \vec{x})$, $\pi^{ij}(t, \vec{x})$, and possibly also of time and space; we then impose a surface gauge condition upon the $\gamma_{ij}$’s and $\pi^{ij}$’s, and use the constraints to determine the evolution of some components of the $\gamma_{ij}$’s and $\pi^{ij}$’s. Many researchers [3,7,8,9] prefer to impose a volume gauge condition upon $\gamma_{ij}(t, \vec{x})$ and $\pi^{ij}(t, \vec{x})$; they then use the constraint equations, with some surface gauge condition, to solve for the lapse and shift. An example of our method in scalar electrodynamics is fixing $A_0(t, \vec{x}) = 0$ as the volume gauge condition, and then using the constraints and the surface condition, $\partial_i A_i(0, \vec{x}) = 0$, to determine the longitudinal field components. An example of the other procedure would be fixing $\partial_i A_i(t, \vec{x}) = 0$ as the volume gauge, and then using the constraint equation plus the freedom to perform time dependent, harmonic gauge transformations, to determine $A_0(t, \vec{x})$. Our method gives the best chance of defining a successful evolution in gravity because it allows one to adjust the rate of evolution in response to what the fields are doing. In this way we can avoid coordinate singularities, and also — by merely slowing down before they form — the evolution of physical singularities. Of course if the other method gives a successful evolution it can always be transcribed into our form by merely regarding the derived solutions for the lapse and shift as volume gauge conditions.
dence is relatively simple. Gravity is a formidably complex dynamical system and it may somewhere harbor such an operator; it is certain though that no one has found it yet, despite years of search.

A key point of our proposal is that one should infer physics the same way in the quantum theory as in the classical theory: by studying the reduced variables, \( x_\alpha^r(t) \) and \( \pi^r_{\beta}(t) \). The fact that these variables are not manifestly invariant before gauge fixing is irrelevant. Any quantity can be made gauge invariant by defining it in a particular gauge and this is just what fixing the residual gauge freedom does. No classical relativist would demur from modeling planetary orbits using the geodesics of the Schwarzschild metric just because the latter was reported in Schwarzschild coordinates. Our formalism permits the same indifference on the quantum level. Correspondence in the limit that \( \hbar \) vanishes is manifest through relation (6.20) and the functional expressions (6.22) for unconstrained matrix elements and expectation values.

Our resolution to the paradox of dynamics is that proper correspondence should pertain in the limit that \( G \) vanishes if that the reduced canonical variables are taken to include the canonical variables of the pure matter theory. In this case the Hamiltonian of the reduced canonical formalism for gravity plus matter will go over into that of free gravitation around some background, plus the pure matter Hamiltonian for that background. The same correspondence will obviously apply for inner products. We have no general proof that a suitable choice exists for the reduced canonical variables but, if it does, then the result follows automatically from the correspondence limit of the evolution equations.* We feel it is plausible that such a choice exists, and we have given an explicit example in the minisuperspace truncations of section 5.

Our resolution to the paradox of topology is that a correspondence limit should exist between the non-zero Hamiltonians of infinite, spatially open manifolds [3,4] and the re-

* That the field equations obey the correct correspondence limit is not in doubt.
duced Hamiltonians of spatially closed manifolds. This correspondence will exist when the reduced canonical variables are chosen to include the canonical variables of the open space theory, and when the coordinate volume of the closed space is taken to infinity in such a way that the initial value data obey the asymptotic condition of the open space. The idea is that in such a situation causality prevents the closed topology from reaching around the universe to influence the localized initial disturbances. As with the paradox of dynamics, the desired correspondence follows immediately from the field equations provided that a suitable choice can be made for the reduced canonical variables. We have no general proof that such a choice exists but we did show in section 4 that it does for gravity on the manifold $T^3 \times R$.

We comment that if the aforementioned topological correspondence limit holds generally then the $2 + 1$ dimensional constructions of Moncrief [7], Hosoya and Nakao [8] and of Carlip [9] would be special cases of the general formalism described in section 2. The same would be true of the $3 + 1$ dimensional constructions of A.D.M.[3], and of Deser and Abbott [4]. Indeed, the method of section 2 seems to provide the long sought unifying principle needed to define energy on a space of arbitrary topology.

We do not avoid the paradox of stability by appealing to the fact that reduced Hamiltonians have non-trivial spectra. As has been noted, what the reduced Hamiltonian is depends upon how we identify the reduced canonical variables, so its spectrum for an arbitrary identification has no intrinsic significance. Nor are reduced Hamiltonians typically conserved. They do not therefore fill the usual role of constraining the way in which a theory can excite its various degrees of freedom. In fact for gravity on a spatially closed manifold the only generally conserved energy functional is the unconstrained Hamiltonian which vanishes upon reduction. This means that all states really are degenerate and hence that the universe is liable to evaporate into pairs in the manner of section 1.

That we are here is a consequence — assuming that our spatial manifold is closed —
of causality and of the weakness of the gravitational interaction. The \( H = 0 \) constraint is not met, as is sometimes supposed, by a vast reduction in the number of possible states compared with gravity on an open space. The perturbative analysis of section 4 shows that there are at least as many positive energy graviton modes on \( T^3 \times R \) as on \( R^3 \times R \); in fact the \( H = 0 \) constraint is enforced by the global negative energy mode, \( p^{tr} \). We conjecture that this is the case generally; that is, positive energy modes can only be excited by corresponding excitations of a global negative energy mode. But a global mode by definition pervades the spatial manifold, so exciting it requires a similarly extensive process. On large manifolds causality imposes a formidable barrier to such excitation. There is an additional barrier in the fact that the global mode can only be excited gravitationally. Since gravitational interactions are typically very weak in our current universe they must proceed slowly. Note that neither barrier would apply to a strongly gravitating system of small physical volume. One consequence of our work is the prediction that such systems ought to be unstable.

The barrier causality imposes against instability becomes absolute in the limit that the coordinate volume goes to infinity while the initial value data are only locally disturbed from a vacuum solution. In this limit the global mode decouples, and both its (negative) energy and the (positive) energy of the local modes become separately conserved. This is how we can approach the conserved Hamiltonians \([3,4]\) of spatially open manifolds. The gravitational barrier becomes similarly infinite in the limit that \( G \) vanishes. In this limit the negative mode again decouples — along with all the other gravitational modes — and the energies of matter and of each gravitational mode become separately conserved. This is how we can approach the conserved Hamiltonians of pure matter theories.

We should comment as well on a venerable argument sometimes used to deny the possibility of non-zero Hamiltonians for gravity on closed spatial manifolds. The argument begins with the observation that any such Hamiltonian would have to be the integral of
a local function of the metric and its first derivative. If it is to have physical significance such a Hamiltonian must also be an invariant, but there are no invariant functions of the metric and its first derivative. Therefore the Hamiltonian of any theory with dynamical gravity must be zero when the spatial manifold is closed. We evade this argument by the simple device of permitting the Hamiltonian to be non-invariant. It generates time evolution in a particular coordinate system — the one provided by the fixed lapse and shift — so it can and should be a gauge dependent object. The physical energy, on the other hand, must really be an invariant, and we have seen that it is zero. There is no contradiction between these two facts because the non-zero Hamiltonian that generates time evolution for a reduced dynamical system need not be the physical energy, neither must it be necessarily conserved. We cobbled together a simple example of this with the coupled oscillator system of section 2, and the phenomenon is generic in gravity.

We close with an admonition to those who would prefer to continue searching for a conventional solution to the “problem of time” — i.e., the identification of some component of the metric as physical time and the interpretation of the Wheeler-DeWitt constraint as a Schrödinger evolution equation. Though our frank admission that time is a gauge choice and our frequent appeal to perturbation theory in implementing this gauge choice may seem ugly, the construction of section 2 is not intrinsically tied to perturbation theory — witness the functional representations of section 6 — and it has the great advantage that one can tailor the coordinate system to the state and the operator under study. There is no reason why a “magic bullet” should even exist for identifying physical time, much less that it should be a simple functional of the metric. Nor is it in accord with our experience in other branches of physics to assume that a single calibration should govern all conceivable measurements. We do not insist, for example, that condensed matter experimentalists use the same thermometers in the microkelvin regime as in the heart of an exploding, high field magnet. Nor do we deny the validity or utility of measuring temperature just because
different techniques are employed in different regimes. What we insist upon is rather that any two methods should agree in those environments for which they both apply. There is no reason to impose a more stringent standard in the vastly more varied and extreme environments imaginable to a theorist in quantum gravity.

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