SYNCHRONIZED ABANDONMENT IN DISCRETE-TIME RENEWAL INPUT QUEUES WITH VACATIONS

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Abstract. We consider a discrete-time infinite buffer renewal input queue with multiple vacations and synchronized abandonment. Waiting customers get impatient during the server’s vacation and decide whether to take service or abandon simultaneously at the vacation completion instants. Using the supplementary variable technique and difference operator method, we obtain an explicit expression to find the steady-state system-length distributions at pre-arrival, random, and outside observer’s observation epochs. We provide the stochastic decomposition structure for the number of customers and discuss the various performance measures. With the help of numerical experiments, we show that the method formulated in this work is analytically elegant and computationally tractable. The results are appropriate for light-tailed inter-arrival distributions and can also be leveraged to find heavy-tailed inter-arrival distributions.

1. Introduction. Discrete-time queues with vacation have been experiencing increased attention over the last few decades due to their applications in the performance analysis of digital telecommunication systems [6, 28]. With the enormous growth of telecommunication systems, provisioning enough quality of service (QoS) during the network infrastructure has turned into a necessity. Their significance has increased further due to the development of the digital communication network, which can work based on a slotted system such as data, voice, video, etc. A comprehensive study of discrete-time queues with server vacations and their applications has been focused in the works of Bruneel and Kim [6], Hunter [15], Takagi [24], Woodward [29] and Alfa [2]. Further, several authors extended the simplest discrete-time vacation queues with geometric arrivals and services to include uncorrelated arbitrary arrivals as a well-known generalization. A study on discrete-time GI/Geo/1 queue with exhaustive service and multiple vacations was initiated by Tian and Zhang [26]. This model has been studied in several directions to include multiple working vacations and vacation interruption [18], single working vacation [7], Bernoulli-schedule-controlled vacation and vacation interruption [25], priority customers [22], bulk arrivals [13, 5], N policy [19], finite buffer capacity

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Server vacations may have mixed effects on the system performance. On the one hand, the vacation period can be effectively used to perform certain secondary tasks instead of wasting the resources by keeping the server idle. On the other hand, customers joining the system during vacation have to wait in the queue till the server return to resume operation. Sometimes waiting longer during vacation causes frustration to customers, and they become impatient, which further leads to customers deciding to abandon the system. It is important to study customers’ impatience in the discrete-time vacation queues as these systems are useful in rendering basic structure for the study and effective design of diverse practical situations such as inventory problems, manufacturing system, communication network, data/voice transmission, etc.

Vacation queueing systems with impatient customers, where the cause of impatience is vacation, have been studied earlier in a Markovian queueing setting by Altman and Yechiali. In their works, customers get impatient when the server is on vacation and perform independent abandonments. Motivated by the remote system applications, several researchers extended Altman and Yechiali’s model to include geometric and synchronized abandonments. Synchronized abandonment became an attractive area to study in queueing literature because of its various industrial applications. However, very few works are available in this direction due to the analytic complexities that arise as a result of customers’ synchronized abandonments. The synchronized abandonment in continuous-time queues are studied in Adan et al. discussed M/M/1 and M/G/1 queue with multiple vacations and synchronized abandonments. Economou and Kapodistria analyzed an unreliable queue with a single server in which customers get impatient during the server’s failure and execute synchronized abandonments. Kapodistria examined a single server Markovian queue with impatient customers doing synchronized abandonment of two types. In the first type, all present customers get impatient and carry out synchronized abandonments, and in the second type, all customers excluding the customer in service perform synchronized abandonment. Kapodistria et al. discussed both the transient and stationary analysis of birth/immigration-death processes with binomial catastrophes. Yajima and Phung-Duc analyzed the Markov-modulated M^X/M/∞ queue with binomial catastrophes. The works mentioned above are limited to the continuous-time queues.

As the arrivals and departures occur near slot boundaries, discrete-time queues are an ideal model for telecommunication systems contrary to continuous-time queues. Further, one can obtain the continuous-time result from a discrete-time queue in the limiting case, but the converse is not true. Analysis and modeling of discrete-time renewal input queue with vacations and synchronized abandonment are more engrossed than the equivalent continuous-time counterpart. In the real world, an example of this model is the customer care services rendered by most major corporations globally. While the exact performance may vary based on the corporation, the basic assumption is that service agents are allocated to customers based on their availability. If all representatives are busy, then customers are given the option to stay in a queue until a representative is free. At any moment, multiple customers can independently decide to leave the queue. Discrete-time queues with heavy-tailed distributions have been noticed in numerous studies of network traffic and are of great significance for traffic characteristics analysis and network capacity planning (Harris et al. [14]). Despite the several advantages of discrete-time queues and their increasing use in digital communication, computer networking,
synchronized abandonment in discrete-time queues

satellite, and mobile communications, there is a scarcity of works related to customer abandonment. To the best of our knowledge, there is no work reported on the discrete-time renewal input queueing system with vacations and synchronized abandonment. It motivates us to study the renewal input discrete-time queues with multiple vacations and synchronized abandonment.

Our model may be helpful in the operation and management of service systems like agricultural firms, call centers, blood bank systems, and inventory management systems. In these systems, the vital performance measure is the customer’s abandonment. It has a significant economic impact on the service provider and provides an insight into customer satisfaction. There is an apparent similarity between perishable inventory systems and queuing systems with customer’s abandonment (Nahmias [21]). For perishable items such as fresh vegetables and fruits, wholesomeness is essential, and that may involve the demand and competitiveness as time passes. Consider an inventory store that deals with several goods of perishable products. The arrival of the fresh vegetables and fruit items in the inventory store can be regarded as a renewal input process and has unlimited storage capacity. The available inventory can be considered as a queue, and the order processed as completion of service. When demand arrives, the need is met instantly on an oldest first basis (first-come, first-served discipline); backorders are not allowed. When all the demands consume, the server takes a vacation and continues to remain on until at least one request arrives at the service desk. The fresh vegetables and fruit items in the inventory are the products that lose their value over time, up until the point they become worthless. The food inspector will execute an inspection to assure that he will take things out from the inventory whose lifetime has perished to preserve the freshness of the vegetables and fruit items in stock. He checks the perishing items individually to decide whether they should remain or get rid of from the inventory.

In this paper, we study the discrete-time infinite buffer renewal input queue with multiple vacations and synchronized abandonment. We find the steady-state system-length distributions at pre-arrival, random, and outside observer’s observation epochs using the supplementary variable method and difference equation techniques. The customers get impatient during the absence of the server and abandon the system simultaneously instead of one by one. We obtain the stochastic decomposition structure for the number of customers and find various performance measures. We derive many non-existing/existing queueing models as special cases of our model. Numerical results show the influence of different parameters on the system performance characteristics. Our analysis is suitable and computationally efficient for both light-tailed and heavy-tailed inter-arrival distributions. The main contributions of our work are:

- steady-state analysis of a discrete-time renewal input queue with vacations and synchronized abandonment. This work is the first of its kind in discrete-time queues with customer abandonment.
- characterization of the stochastic decomposition structure for the number of customers.
- computation of performance measures and show their dependence on system parameters through various numerical experiments.

This paper is organized as follows. Section 2 describes the model characteristics and specifies the parameters in detail. Section 3 deals with the steady-state analysis of the model and the system length distributions at pre-arrival, random, and outside
observer’s observation epochs. We provide the stochastic decomposition results of the average system sizes in Section 4. Section 5 presents important performance measures, while Section 6 provides some special cases of our model. Section 7 illustrate the impact of system parameters through numerical results and Section 8 concludes the paper.

2. Model formulation. We study the discrete-time GI/Geo/1 queue with multiple vacations and synchronized abandonment under a late arrival system with delayed access (LAS-DA). In LAS-DA models, an arrival takes precedence over a departure, whereas the opposite happens in an early arrival system (EAS). For more details on the definitions and differences between EAS and LAS-DA, one may refer to Hunter [15]. These policies play a significant role to determine the steady-state probabilities of the number in the system at various epochs. We assume that the time axis is slotted into intervals of equal length with the length of a slot being unity and the boundaries are marked by 0, 1, 2, …, t, … . According to LAS-DA policy, potential arrivals takes place in (t−, t) and potential departures occur in (t, t+). The vacation can begin or finish at t+, that is, just after a departure. The various time epochs at which events take place are shown in Fig. 1.

Now, we consider a single server infinite buffer discrete-time queue wherein inter-arrival times of successive arrivals are independent, identically distributed (i.i.d.) random variables with probability mass function (p.m.f.) \( a_n = P(A = n), \) \( n \geq 1, \) probability generating function (p.g.f.) \( A(z) = \sum_{n=1}^{\infty} a_n z^n \) and mean inter-arrival time \( a = \frac{dA(z)}{dz} |_{z=1}. \) The server has two operating modes: busy (active) or vacation (inactive). However, customers are served individually during the active mode, and no service is provided during vacation. The service times are assumed to be independent and geometrically distributed random variables with p.m.f. \( s_i = \bar{\mu}_i^{i-1} \mu, i \geq 1, \) \( 0 < \mu < 1 \) and mean service time \( 1/\mu. \) We denote \( \bar{x} = 1 - x \) for any real number \( x \in [0, 1]. \) Upon arrival, customer joins a single queue of infinite buffer and are served in a first-come-first-served discipline. The server takes a vacation when the system becomes empty and continues taking vacations until it finds customers waiting at a vacation completion epoch. The vacation times \( V \) are independent and geometrically distributed with common p.m.f. \( P(V = i) = \bar{\phi}^{i-1} \phi, i \geq 1, \) \( 0 < \phi < 1 \) and mean vacation time \( 1/\phi. \) We assume the inter-arrival times, vacation times, and service times are mutually independent. The service (vacation) can start only

![Figure 1. Various time epochs in a late arrival system with delayed access (LAS-DA)](image)
at slot boundaries, and their duration are integral multiples of a slot. The traffic intensity is given by \( \rho = \lambda/\mu \), and \( \rho < 1 \) assures the stability of the system under steady-state.

Waiting customers become impatient during vacation and abandon the system at vacation completion epochs which coincide with the arrival instants of a secondary transport facility. Based on the arrival frequency of the transport facility, two different analysis models, like unique abandonment epoch (UAE) and multiple abandonment epochs (MAE), are found. The details about the UAE and MAE models may be found in Adan et al. [1]. We consider the UAE model, in which the abandonment epochs (MAE), are found. The details about the UAE and MAE models may be found in Adan et al. [1]. We consider the UAE model, in which the transport facility arrives at the end of a vacation, and the impatient customers if any, decide whether to avail of the facility or not. Thus, at the abandonment epoch, the present customers remain in the system with probability \( q \) or leave the system with \( p = \bar{q} \), independent of the others. Thus, the number in the system decreases according to a binomial distribution at each vacation completion epoch.

3. Analysis of the model. Let us consider a GI/Geo/1 queue with multiple vacations and synchronized abandonment under the LAS-DA policy. The state of the system prior to a potential arrival (at \( t^- \)), that is, before the beginning of the slot, is depicted by the following random variables:

\[
N_t = \text{number of customers in the system};
\]

\[
U_t = \text{remaining interarrival time for the next arrival};
\]

\[
J_t = \text{state of the server, i.e., } = \begin{cases} 1, & \text{if the server is busy;} \\ 0, & \text{if the server is on vacation.} \end{cases}
\]

where for the sake of notational simplicity, we have used \( t \) in place of \( t^- \). Let us define their joint probabilities as

\[
P_{i,j}(u,t) = P[N_t = i, U_t = u, J_t = j], \quad u \geq 0, \quad i \geq 0, \quad j = 0, 1.
\] (1)

In steady-state, let

\[
P_{i,j}(u) = \lim_{t \to \infty} P_{i,j}(u,t), \quad i \geq 0, \quad j = 0, 1.
\] (2)

We formulate the forward difference equations using the remaining inter-arrival time as the supplementary variable to find system length distributions at pre-arrival and arbitrary epochs. Associating the states of the system at two consecutive epochs \( t^- \) and \( (t + 1)^- \), we have in steady-state for \( u \geq 1 \)

\[
P_{0,0}(u - 1) = P_{0,0}(u) + \mu P_{1,1}(u) + \phi \sum_{j=1}^{\infty} \binom{j}{j} p^j P_{j,0}(u),
\] (3)

\[
P_{n,0}(u - 1) = \bar{\phi} P_{n,0}(u) + \bar{\phi} a(u) P_{n-1,0}(0), \quad n \geq 1,
\] (4)

\[
P_{1,1}(u - 1) = \bar{\mu} P_{1,1}(u) + \mu P_{2,1}(u) + \mu a(u) P_{1,1}(0) + \phi \sum_{j=1}^{\infty} \binom{j}{j-1} q^j p^{j-1} P_{j,0}(u)
\]

\[
+ \phi a(u) \sum_{j=1}^{\infty} \binom{j}{j-1} q^j p^{j-1} P_{j,0}(u),
\] (5)

\[
P_{n,1}(u - 1) = \bar{\mu} P_{n,1}(u) + \mu P_{n+1,1}(u) + \bar{\mu} a(u) P_{n-1,1}(0) + \phi \sum_{j=n}^{\infty} \binom{j}{j-n} q^n p^{j-n} P_{j,0}(u)
\]

\[
+ \mu a(u) P_{n,1}(0) + \phi a(u) \sum_{j=n-1}^{\infty} \binom{j}{j-n+1} q^{n-1} p^{j-n+1} P_{j,0}(u), \quad n \geq 2,
\] (6)
Define \( P^*_{n,j}(z) = \sum_{n=0}^{\infty} P_{n,j}(u)z^n \) and \( P^*_{n,j}(1) = P_{n,j}, n \geq 0, j = 0, 1, \) where \( P_{n,j} \) refers the probability of \( n \) customers in the system when the server is in state \( j \) at an arbitrary epoch. Multiplying (3) to (4) by \( z^n \) and summing over \( u \) from 1 to \( \infty \), we obtain

\[
(z - 1)P^*_{0,0}(z) = \mu P^*_{1,1}(z) + \phi \sum_{j=1}^{\infty} \left( \begin{array}{c} j \\ j \end{array} \right) p^j P^*_{j,0}(z) - P_{0,0}(0) - \mu P_{1,1}(0)
\]

Putting \( P^*_{0,0}(z) = \phi \sum_{j=1}^{\infty} \left( \begin{array}{c} j \\ j \end{array} \right) p^j P^*_{j,0}(z) \), we obtain

\[
(z - \tilde{\phi})P^*_{0,0}(z) = \tilde{\phi} A(z)P_{n-1,0}(0) - \tilde{\phi} P_{n,0}(0),
\]

(8)

\[
(z - \bar{\mu})P^*_{1,1}(z) = \mu P^*_{2,1}(z) + \mu A(z)P_{1,1}(0) + \phi \sum_{j=1}^{\infty} \left( \begin{array}{c} j \\ j \end{array} \right) q p^{j-1} P^*_{j,0}(z) - \bar{\mu} P_{1,1}(0)
\]

(9)

\[
(z - \bar{\mu})P^*_{n,1}(z) = \mu P^*_{n+1,1}(z) + \mu A(z)P_{n,1}(0) + \bar{\mu} A(z)P_{n-1,1}(0) - \bar{\mu} P_{n,1}(0) - \mu P_{n+1,1}(0)
\]

(10)

Adding (7) to (10), we get

\[
\sum_{n=0}^{\infty} P^*_{n,0}(z) + \sum_{n=1}^{\infty} P^*_{n,1}(z) = \frac{A(z) - 1}{z - 1} \left\{ \sum_{n=0}^{\infty} P_{n,0}(0) + \sum_{n=1}^{\infty} P_{n,1}(0) \right\}. \tag{11}
\]

It can be noted that the left-hand side of (12) presents the probability that an arrival is about to happen or, it is the arrival rate of customers.

3.1. **Steady-state system size distribution.** Now to find the explicit expressions for the steady-state probabilities \( P_{n,i} = P^*_{n,i}(0) \) we formulate a recursive method. To find these, we will first compute \( \{P_{n,0}(0)\}_{n=0}^{\infty} \) and \( \{P_{n,1}(0)\}_{n=1}^{\infty} \).

Putting \( z = \tilde{\phi} \) in (8), we have

\[
P_{n,0}(0) = \bar{\phi} P_{n-1,0}(0), n \geq 1. \tag{13}
\]

On recursive use of (13), we have

\[
P_{n,0}(0) = \alpha^n P_{0,0}(0), n \geq 1, \tag{14}
\]

where \( \alpha = A(\tilde{\phi}) \). Using (14) in (8), we obtain

\[
P^*_{n,0}(z) = \frac{\bar{\phi}\alpha^{n-1}(A(z) - \alpha)}{z - \tilde{\phi}} P_{0,0}(0), n \geq 1. \tag{15}
\]

By using the shift operator \( E \) defined by \( E x_n = x_{n+1} \) for all \( n \), (10) can be written as

\[
[z - (\bar{\mu} + \mu E)] P^*_{n,1}(z) = \bar{\mu} A(z) P_{n-1,1}(0) + \mu A(z) P_{n,1}(0) - \bar{\mu} P_{n,1}(0) - \mu P_{n+1,1}(0)
\]
The difference equation (17) is given by

\[ C \]

where

\[ \phi \sum_{j=n}^{\infty} \left( \frac{j}{j-n} \right) q^n p^{j-n} P_{j,0}(z) - \phi \sum_{j=n}^{\infty} \left( \frac{j}{j-n} \right) q^n p^{j-n} P_{j,0}(0) \]

\[ + \phi A(z) \sum_{j=n+1}^{\infty} \left( \frac{j}{j-n+1} \right) q^{n-1} p^{j-n+1} P_{j,0}(0), \quad n \geq 2. \]

Putting \( z = \bar{\mu} + \mu E \) in (16) and applying equations (14) and (15), we have

\[
[E - A(\bar{\mu} + \mu)](\bar{\mu} + \muE)P_{n,1}(0) / E
\]

\[ = \phi \sum_{j=n}^{\infty} \left( \frac{j}{j-n} \right) q^n p^{j-n} \tilde{\phi}(\alpha)^{j-1} (A(\bar{\mu} + \muE) - \alpha) P_{0,0}(0) - \phi \sum_{j=n}^{\infty} \left( \frac{j}{j-n} \right) q^n p^{j-n} \]

\[ \alpha^j P_{0,0}(0) + \phi A(\bar{\mu} + \muE) \sum_{j=n-1}^{\infty} \left( \frac{j}{j-n+1} \right) q^{n-1} p^{j-n+1} \alpha^j P_{0,0}(0) \]

\[ = \frac{\phi \tilde{\phi}(A(\bar{\mu} + \muE) - \alpha)}{\alpha - \bar{\mu} + \muE - \phi} \frac{\sigma^n}{(1-p\alpha)} P_{0,0}(0) + \frac{\phi(A(\bar{\mu} + \muE) - \sigma)}{1-p\alpha} P_{0,0}(0) \]

\[ \alpha^j P_{0,0}(0) \]

\[ \alpha \sum_{j=n-1}^{\infty} \left( \frac{j}{j-n+1} \right) q^{n-1} p^{j-n+1} \alpha^j P_{0,0}(0) \]

\[ (E - A(\bar{\mu} + \muE))(\bar{\mu} + \muE)P_{n,1}(0) = \frac{\phi \tilde{\phi}(A(\bar{\mu} + \muE) - \alpha)}{\alpha - \bar{\mu} + \muE - \phi} \frac{\sigma^n}{(1-p\alpha)} P_{0,0}(0) \]

\[ + \frac{\phi(A(\bar{\mu} + \muE) - \sigma)}{1-p\alpha} P_{0,0}(0) \]

\[ \alpha^j P_{0,0}(0) \]

where \( \sigma = \frac{p\alpha}{1-p\alpha} \). The complementary solution of homogeneous difference equation \((E - A(\bar{\mu} + \muE))(\bar{\mu} + \muE)P_{n,1}(0) = 0\) of equation (17) is given by

\[ P_{i,b}^{(c)}(0) = C\gamma^i, \]

where \( \gamma \) is a real root inside the unit circle of the equation \( z - A(\bar{\mu} + \muz) = 0 \) and \( C \) is an arbitrary constant. Applying Rouché's theorem it is seen that equation has one real root inside the unit circle for \( \rho < 1 \). The particular solution of the difference equation (17) is given by

\[ P_{i,b}^{(p)}(0) = \phi \frac{\sigma^{n+1}}{\alpha - \bar{\mu} + \muE - \phi} \left[ \frac{\phi(A(\bar{\mu} + \muE) - \alpha)}{\alpha(\bar{\mu} + \muE - \phi)} \right] \]

\[ \frac{\sigma^n}{\alpha(\bar{\mu} + \muE - \phi)} - \frac{1}{\sigma} \]

Thus, the general solution of (17) is given by

\[ P_{n,1}(0) = C\gamma^n - K\sigma^{n+1} P_{0,0}(0) \quad n \geq 1, \]

where \( K = \frac{\phi}{(1-p\alpha)(\bar{\mu} + \muE)} \left[ \frac{1}{\sigma} - \frac{\tilde{\phi}(A(\bar{\mu} + \muE) - \alpha)}{\alpha(\bar{\mu} - \muE)(A(\bar{\mu} + \muE) - \sigma)} \right] \). Substituting (20) into (16) leads to

\[
[z - (\bar{\mu} + \muE)] P_{n,1}(z) = (\bar{\mu} + \muE)(A(z) - \gamma)C\gamma^n - (\bar{\mu} + \muE)(A(z) - \sigma)K\sigma^n P_{0,0}(0)
\]

\[ + \frac{\phi P_{0,0}(0)}{1-p\alpha} \frac{\phi(A(z) - \alpha)}{\alpha(z - \phi)} \sigma^n + \frac{\phi P_{0,0}(0)}{1-p\alpha} (A(z) - \sigma)\sigma^{n-1}, \quad n \geq 2 \]

The general solution of (21) is given by

\[ P_{n,1}(z) = \frac{C(\bar{\mu} + \muE)(A(z) - \gamma)\gamma^{n-1}}{z - (\bar{\mu} + \muE)} - \frac{K(\bar{\mu} + \muE)(A(z) - \sigma)\sigma^n P_{0,0}(0)}{z - (\bar{\mu} + \muE)} \]

\[ + \frac{\phi P_{0,0}(0)}{1-p\alpha}(z - (\bar{\mu} + \muE)) \left( \frac{\sigma\phi(A(z) - \alpha)}{\alpha(z - \phi)} + (A(z) - \sigma) \right), \quad n \geq 2. \]
Again setting \( z = \bar{\mu} \) in (5), and using (14), (15), (20), (22), we have \[ C = K\sigma P_{0,0}(0). \]

Setting \( z = 1 \) in (7) and using (14) and (15), after simplification, we have
\[
P_{1,1} = \left[ \frac{q + p\phi}{\mu(1-p\alpha)} + K\sigma(\gamma - \sigma) \right] P_{0,0}(0). \tag{23}
\]

Now, all the steady-state probabilities, \( P_{n,j}(0), n \geq j \) are derived in terms of the only unknown \( P_{0,0}(0) \), which can be computed from the normalization equation as
\[
P_{0,0}(0) = \frac{\lambda(1-\alpha)(1-\gamma)}{(1-\gamma) + Kq\alpha(\gamma - \sigma)}.	ag{24}
\]

The steady-state probabilities are summed up in the following Lemma.

**Lemma 3.1.** If \( \sigma \neq \gamma \), the steady-state probabilities in the GI/Geo/1 queue with multiple vacations and synchronized abandonment under the equilibrium condition \( \rho < 1 \), are
\[
P_{n,j}(0) = \begin{cases} \alpha^n P_{0,0}(0), & j = 0; \ n \geq 1, \\ K\sigma(\gamma^n - \sigma^n)P_{0,0}(0), & j = 1; \ n \geq 1, \end{cases}
\]
with \( P_{0,0}(0) \) given in (24).

**Remark 1.** In the above section, we present the detailed calculation for the case \( \sigma \neq \gamma \). Whenever \( \sigma = \gamma \), the particular solution of the non-homogeneous linear difference equation (17) is given by
\[
P_{n,1}^{\text{part}}(0) = -K A^*(\mu \bar{\sigma}) - \sigma \frac{A^*(1) (\mu \bar{\sigma}) + 1}{\mu A^*(1) (\mu \bar{\sigma}) + 1} (n+1)\sigma^n P_{0,0}(0), \ n \geq 1,
\]
and all the corresponding derivations can be similarly obtained.

**3.2. Pre-arrival and arbitrary epoch probabilities.** Let \( \{P_{n,j}^-(0)\}, n \geq j, j = 0, 1 \) denote the pre-arrival epoch probability, that is, an arrival sees \( n \) customers in the system and the server is in state \( j \) at arrival epoch. Applying Bayes’ theorem, we have
\[
P_{n,j}^-(0) = \lim_{t \to \infty} \frac{P[N_t = n, J_t = j, U_t = 0]}{P[U_t = 0]}, \ j = 0, 1.
\]

Further, using (12) in the above expression, we obtain
\[
P_{n,j}^- = \frac{1}{\lambda} P_{n,j}(0), \ n \geq j, j = 0, 1. \tag{25}
\]

We obtain pre-arrival epoch probability from the equations (24), (14) and (20) as
\[
P_{0,0}^- = \frac{(1 - \alpha)(1 - \gamma)}{(1 - \gamma) + Kq\alpha(\gamma - \sigma)}, \quad P_{n,0}^- = \alpha^n P_{0,0}^-, \ n \geq 1,
\]
\[
P_{n,1}^- = K\sigma(\gamma^n - \sigma^n)P_{0,0}^-, \ n \geq 1.
\]

The arbitrary epoch probabilities, \( P_{n,j}, n \geq j, j = 0, 1 \) are computed from the corresponding expressions of \( P_{n,j}^+(0) \). Setting \( z = 1 \) in equations (15) and (22), we obtain the arbitrary epoch probabilities as
\[
P_{n,0} = \frac{\lambda \phi}{\phi} \frac{\alpha^{n-1}(1-\alpha)}{P_{0,0}^-}, \ n \geq 1. \tag{26}
\]
Lastly, we find

\[ n_{\text{ABANDONMENT \ ARE \ GIVEN \ BY}} \]

\[ (27) \]

Outside observer's distribution. \[ \text{Theorem 3.2. If } \rho < 1, \text{ the steady-state system length distributions at pre-arrival and arbitrary instants in a GI/Geo/1 queue with multiple vacations and synchronized abandonment are given by} \]

\[ P_{n,j} = \begin{cases} 
\alpha^n P_{0,0}, & n \geq 1, j = 0 \\
K\sigma(\gamma - \sigma^n)P_{0,0}, & n \geq 1, j = 1. 
\end{cases} \]

\[ K\sigma(\gamma - \sigma)(1 - \gamma)(1 - \sigma) P_{0,0}, \quad n = 0, j = 0 \]

\[ \frac{\lambda\alpha}{\phi} \alpha^{n-1}(1 - \alpha) P_{0,0}^-, \quad n \geq 1, j = 0 \]

\[ \rho P_{0,0}^- K\sigma \{(\mu + \mu\gamma)\gamma^{n-1} - (\mu + \mu\sigma)\sigma^{n-1}\} + \frac{q + p\phi}{1 - p\alpha} \sigma^{n-1}, \quad n \geq 1, j = 1. \]

where \[ P_{0,0}^- = \frac{\lambda\alpha}{\phi} \frac{1 - \alpha(1 - \gamma)}{1 - \gamma + Kp\alpha(\gamma - \sigma)}, \quad \sigma \neq \gamma. \]

3.3. Outside observer's distribution. In LAS-DA, the outside observer's observation epoch is more likely to fall in the interval \((t, (t + 1) -)\). Since nothing occurs (no arrival and no departure) in this interval, the outside observer's distribution \[ P_{n,j}^o \]

will be the same as the distribution of number in system observed at \((t + 1) -\) and hence \[ P_{n,j}^o = P_{n,j}. \]

4. Stochastic decomposition results. We notice that the queues with vacation have the stochastic decomposition structures for queue length \([11, 20]\). In numerous cases, it is of interest to decompose the measures into several factors to compare with the existing models. Phung-Duc \([23]\) studied the conditional stochastic decomposition structure for queue length in a Markovian multi-server system with setup times and presented a physical interpretation for the generating functions. Here, the stationary queue length can be decomposed into the sum of two independent random variables, which are the representing queue length in the classical GI/Geo/1 queue without vacation and additional queue length owed to the multiple vacations and synchronized abandonment effect. We define the stationary queue length of the renewal input queue with vacations and synchronized abandonment, the queue length distribution of the classical GI/Geo/1 queue, and the additional queue length due to the multiple vacations and synchronized abandonment as the non-negative random variables \(N_s, N\) and \(N_d\), respectively. Let the probability generating functions of \(N_s, N\) and \(N_d\) be \(N_s(z), N(z)\) and \(N_d(z)\), respectively. From \((26) - (28)\), we have the distribution of \(N_s\) as

\[ P\{N_s = 0\} = P_{0,0} = T P_{0,0}^-. \]
Given, respectively, by
\[ P\{N_s = j\} = P_{j,0} + P_{j,1} = \left[ \frac{\lambda \phi(1-\alpha)\alpha^{j-1}}{\phi} + K \rho \sigma ((\bar{\mu} + \mu \gamma)\gamma^{j-1} - (\bar{\mu} + \mu \sigma)\sigma^{j-1}) \right. \]
\[ + \left. \frac{\rho(q + p\phi)\sigma^{j-1}}{1 - p\alpha} \right] P_{0,0}, \quad j \geq 1. \]

Taking the p.g.f. of the distribution of \( N_s \), we get
\[ N_s(z) = \sum_{j=0}^{\infty} P\{N_s = j\} z^j \]
\[ = \left[ T + \frac{\lambda \phi(1-\alpha)}{\phi} \sum_{j=1}^{\infty} z^j \alpha^{j-1} + K \rho \sigma \sum_{j=1}^{\infty} z^j ((\bar{\mu} + \mu \gamma)\gamma^{j-1} - (\bar{\mu} + \mu \sigma)\sigma^{j-1}) \right. \]
\[ + \left. \frac{\rho(q + p\phi)}{1 - p\alpha} \sum_{j=1}^{\infty} z^j \sigma^{j-1} \right] P_{0,0} \]
\[ = \psi^{-1} \left[ T + \frac{\lambda \phi(1-\alpha)}{\phi(1 - \alpha z)} + K \rho \sigma \left( \frac{(\bar{\mu} + \mu \gamma)}{1 - \gamma z} - \frac{(\bar{\mu} + \mu \sigma)}{1 - \sigma z} \right) + \frac{\rho(q + p\phi)z}{(1 - p\alpha)(1 - \sigma z)} \right] \]
\[ = \left( \frac{1 - \gamma}{1 - \gamma z} \right) \times N_d(z) = N(z)N_d(z), \]
where \( \psi = \frac{(1-\alpha)}{(1-\gamma) + Kq\alpha(\gamma - \sigma)}. \) Here, \( N(z) \) is the p.g.f. of the stationary distribution of the queue length at an arbitrary instant of the classical GI/Geo/1 queue, and \( N_d(z) \) is the p.g.f. of the additional queue length due to the multiple vacations and synchronized abandonment and given by
\[ N_d(z) = \psi \left[ T + \frac{\lambda \phi\alpha}{\phi(1-\alpha z)} + K \rho \sigma z(\mu + \bar{\mu}z)(\gamma - \sigma) + \frac{\rho(q + p\phi)z}{(1 - p\alpha)(1 - \sigma z)} \right]. \]

By rearranging the terms of \( N_d(z) \),
\[ N_d(z) = \psi T + \psi z \left( \frac{\lambda \phi\alpha}{\phi} + \frac{\rho(q + p\phi + Kq\alpha(\gamma - \sigma))}{1 - p\alpha} - T\gamma \right) \]
\[ + \psi \sum_{j=2}^{\infty} z^j \left( \frac{\lambda \phi\alpha(\alpha - \gamma)}{\phi} \alpha^{j-2} + \frac{\rho(\sigma - \gamma)(q + p\phi - Kq\alpha(\bar{\mu} + \mu \sigma))}{1 - p\alpha} \sigma^{j-2} \right) \]
Comparing the coefficients of \( z^j \), we get the stationary distribution of \( N_d \) as
\[ P\{N_d = 0\} = \psi T \]
\[ P\{N_d = 1\} = \psi \left( \frac{\lambda \phi\alpha}{\phi} + \frac{\rho(q + p\phi + Kq\alpha(\gamma - \sigma))}{1 - p\alpha} - T\gamma \right) \]
\[ P\{N_d = j\} = \psi \left( \frac{\lambda \phi\alpha(\alpha - \gamma)}{\phi} \alpha^{j-2} + \frac{\rho(\sigma - \gamma)(q + p\phi - Kq\alpha(\bar{\mu} + \mu \sigma))}{1 - p\alpha} \sigma^{j-2} \right), \quad j \geq 2. \]
The expected values of \( N_d \) and \( N_s \) from the stochastic decomposition property are given, respectively, by
\[ E(N_d) = \psi \left[ \frac{\lambda \phi\alpha - T\gamma + \rho(q + p\phi + Kq\alpha(\gamma - \sigma))}{1 - p\alpha} + \frac{\lambda \phi\alpha(\alpha - \gamma)(2 - \alpha)}{\phi(1-\alpha)^2} \right. \]
\[ + \left. \frac{\rho(\sigma - \gamma)(q + p\phi - Kq\alpha(\bar{\mu} + \mu \sigma))(2 - \sigma)}{(1 - p\alpha)(1 - \sigma)^2} \right], \]
\[ E(N_s) = \frac{\gamma}{1 - \gamma} + E(N_d). \]
5. **Performance measures.** The probability that the system is on a vacation \( P\{J = 0\} \) and the probability that the system is in a busy period \( P\{J = 1\} \) are, respectively, given by

\[
P\{J = 0\} = \sum_{n=0}^{\infty} P_{n,0} = P_{0,0}^- \left[ \frac{\mu - \lambda q - \lambda p\phi}{\mu(1-\alpha)} + \frac{K\sigma(\gamma - \sigma)(1-\rho)}{(1-\gamma)(1-\sigma)} \right],
\]

\[
P\{J = 1\} = \sum_{n=1}^{\infty} P_{n,1} = \rho P_{0,1}^- \left[ \frac{K\sigma(\gamma - \sigma)}{(1-\gamma)(1-\sigma)} + \frac{q + p\phi}{1 - \rho\alpha} \right].
\]

The average system length \( (L_s) \) is given by

\[
L_s = \sum_{n=0}^{\infty} n P_{n,0} + \sum_{n=1}^{\infty} n P_{n,1}
= \left[ \frac{\bar{\phi}}{\phi(1-\alpha)} + \frac{(q + p\phi)(1-\rho)}{\mu(1-\alpha)^2} + \frac{\rho K\sigma(\gamma - \sigma)}{\mu(1-\gamma)(1-\sigma)} \left( \bar{\mu} + \frac{1 - \gamma\sigma}{(1-\gamma)(1-\sigma)} \right) \right] \lambda P_{0.0}^-.
\]

Using Little’s formula, the average sojourn time in the system \( (W_s) \) is

\[
W_s = \frac{L_s}{\lambda} = \left[ \frac{\bar{\phi}}{\phi(1-\alpha)} + \frac{(q + p\phi)(1-\rho)}{\mu(1-\alpha)^2} + \frac{\rho K\sigma(\gamma - \sigma)}{\mu(1-\gamma)(1-\sigma)} \left( \bar{\mu} + \frac{1 - \gamma\sigma}{(1-\gamma)(1-\sigma)} \right) \right] P_{0.0}^-.
\]

The average abandonment rate \( (AR) \) is given by

\[
AR = \sum_{n=1}^{\infty} \sum_{j=n}^{\infty} \left( j - n \right) q^n p^{j-n} P_{j,0} = \frac{\lambda \bar{\phi} q}{\phi(1-\alpha)} P_{0,0}^-.
\]

6. **Some special models.** In this section, by taking particular values of the model parameters, we derive the results of some special models.

**Geo/Geo/1 queue with synchronized reneging and vacations.** Considering the the inter-arrival times as geometric distribution, our model reduces to the Geo/Geo/1 queue with multiple vacations and synchronous abandonment. The inter-arrival p.g.f. reduces to \( A(z) = \frac{\lambda z}{1 - \lambda z} \) and the characteristic root \( \gamma = \frac{\lambda \bar{\mu}}{\rho\phi} \).

Further, \( \alpha = A(\phi) = \frac{\lambda \bar{\phi}}{\phi + \lambda \bar{\phi}} \) and \( \sigma = \frac{\lambda \phi}{\phi + \lambda \phi} \). The steady-state system-length distribution at pre-arrival and arbitrary epochs match due to the BASTA property and is given by

\[
P_{n,0} = \left( \frac{\lambda \bar{\phi}}{\phi + \lambda \bar{\phi}} \right)^n P_{0,0}^-, \quad n \geq 1,
\]

\[
P_{n,1} = K \left( \frac{q\lambda \bar{\phi}}{\phi + q\lambda \phi} \right)^n \left( \left( \frac{\lambda \bar{\mu}}{\phi + q\lambda \phi} \right)^n - \left( \frac{q\lambda \bar{\phi}}{\phi + q\lambda \phi} \right)^n \right) P_{0,0}^-, \quad n \geq 1,
\]

where \( K = \frac{(\phi + q\lambda \bar{\phi})(\phi + \lambda \bar{\phi})(\phi q(\mu - \lambda - 1) - \phi^2 \bar{\mu} - \lambda \phi^2 q^2)}{q\lambda \phi(\mu + q\lambda \phi)((\mu - \lambda)q\phi - \phi \bar{\mu})} \), and

\[
P_{0,0}^- = \frac{(\mu - \lambda)(\phi + q\lambda \phi)(\phi + \lambda \bar{\phi}) + Kq\lambda \phi(\phi \lambda \bar{\mu} + q\lambda \phi(\lambda - \mu))}{(\mu - \lambda)(\phi + q\lambda \phi)(\phi + \lambda \bar{\phi}) + Kq\lambda \phi(\phi \lambda \bar{\mu} + q\lambda \phi(\lambda - \mu))}, \quad \rho \neq \sigma.
**GI/Geo/1 queue with multiple vacations.** Taking \( p = 0 \), the model reduces to GI/Geo/1 queue with multiple vacations. In this case, system-length distribution at pre-arrival epoch is

\[
P_{0,0}^- = \frac{(1 - \gamma)(\phi - \mu + \mu \alpha)}{\phi - \mu + \mu \gamma},
\]

\[
P_{n,0}^- = \frac{(1 - \gamma)(\phi - \mu + \mu \alpha)\alpha^n}{\phi - \mu + \mu \gamma}, \quad n \geq 1,
\]

\[
P_{n,1}^- = \frac{\phi(1 - \gamma)}{\phi - \mu + \mu \gamma} (\gamma^n - \alpha^n), \quad n \geq 1.
\]

The system-length distribution at an arbitrary epoch is

\[
P_{n,0} = \lambda \tilde{\phi}(1 - \alpha)(1 - \gamma)(\phi - \mu + \mu \alpha)\alpha^{n-1}, \quad n \geq 1,
\]

\[
P_{n,1} = \frac{\lambda(1 - \gamma)}{\mu(\phi - \mu + \mu \gamma)} (\phi (\tilde{\mu} + \mu \gamma)\gamma^{n-1} - \tilde{\phi}\mu (1 - \alpha)\alpha^{n-1}), \quad n \geq 1.
\]

The average system length (\( L_s \)) is

\[
L_s = \frac{\lambda(1 - \gamma)}{\phi + \mu - \mu \gamma} \left( \frac{\phi(\tilde{\mu} + \mu \gamma)}{\mu(1 - \gamma)^2} - \frac{\tilde{\phi}\mu}{\phi} \right).
\]

Our results match analytically with the results studied in Tian et al. [26].

**The classical GI/Geo/1 queue.** Taking \( p = 0 \) and \( \phi \to 1 \), the model reduces to the classical GI/Geo/1 queue. In this case, \( K = 1 \). The system-length distribution at a pre-arrival epoch is given as

\[
P_{0,0}^- = (1 - \gamma),
\]

\[
P_{n,1}^- = (1 - \gamma)\gamma^n, \quad n \geq 1.
\]

The system-length distribution at arbitrary epoch is

\[
P_{n,1} = \frac{\lambda(1 - \gamma)\gamma^{n-1}}{\mu}, \quad n \geq 1,
\]

and \( P_{0,0} = 1 - \sum_{n=1}^{\infty} P_{n,1} = 1 - \rho \). It is to be noted that \( P_{n,0}, n \geq 1 \) will not occur in this case. The average system length is \( L_s = \frac{\rho}{1 - \rho} \). The results match with the results of GI/Geo/1 queue presented in the literature [27].

**GI/M/1 queue with vacations and synchronized abandonment.** We want to derive the results for the continuous-time counterpart from the GI/Geo/1 queueing system with multiple vacations and synchronized abandonment under certain limiting conditions. For the continuous-time model, we assume the random variable \( \tilde{A} \) as the inter-arrival times between two consecutive arrivals which are i.i.d. with distribution function \( \tilde{A}(y) \), Laplace-Stieltjes transform (L.S.T) \( \tilde{A}^*(s) \) and mean arrival rate \( \tilde{\lambda} \). The service (vacation) times are assumed to be independent and exponentially distributed with mean service (vacation) time \( 1/\tilde{\mu} \) (1/\( \tilde{\phi} \)). Let us assume that the time axis is divided into intervals of equal length \( \Delta \), such that \( \Delta > 0 \) and is sufficiently small. Thus, we have, \( a_n = Pr((n - 1)\Delta < \tilde{A} < n\Delta), \quad n \geq 1 \) and
$E(\hat{A}) = E(A)|\Delta, \lambda = \hat{\lambda}\Delta, \mu = \hat{\mu}\Delta$ and $\phi = \hat{\phi}\Delta$. Taking $\mu = \hat{\mu}\Delta$ and $\Delta \to 0$, we have

$$A(1 - \mu + \mu z) = \lim_{\Delta \to 0} (1 - \hat{\mu}\Delta + \hat{\mu}\Delta z)$$

$$= \lim_{\Delta \to 0} \sum_{n=1}^{\infty} Pr\left((n-1)\Delta < \hat{A} \leq n\Delta\right) (1 - \hat{\mu}\Delta + \hat{\mu}\Delta z)^n$$

$$= \int_0^\infty e^{-\hat{\mu}(1-z)y} d\hat{A}(y) = A^*(\hat{\mu}(1-z)).$$

Thus, $A^*(\hat{\mu}(1-z)) - z = 0$ depicts the characteristic equation corresponding to the continuous-time queue, which has a single root $\hat{\gamma}$ inside the circle $|s| = 1$. Similarly, putting $\phi = \hat{\phi}\Delta$ and taking $\Delta \to 0$, we have in continuous-time $\alpha = \lim_{\Delta \to 0} A(1 - \phi\Delta) = A^*(\hat{\phi})$ and $\sigma = \lim_{\Delta \to 0} \frac{qA(1-\phi\Delta)}{1-pA(1-\phi\Delta)} = \frac{qA^*(\hat{\phi})}{1-pA^*(\hat{\phi})}$.

$$P_{0,0}^- = \frac{(1-\alpha)(1-\hat{\gamma})(1-\sigma)}{(1-\hat{\gamma})(1-\sigma) + \hat{K}\sigma(\hat{\gamma} - \sigma)(1-\alpha)},$$

$$P_{n,0}^- = \alpha^n P_{0,0}^-, n \geq 1,$$

$$P_{n,1}^- = \hat{K}\sigma(\hat{\gamma}^n - \sigma^n)P_{0,0}^-, n \geq 1,$$

where $\hat{K} = \frac{\phi}{\alpha(1-p\alpha)} \frac{A^*(\mu\sigma)^{1-\alpha}}{A^*(\mu\sigma)^{1-\sigma}}$, $\sigma \neq \hat{\gamma}$.  

7. Numerical results. In this section, several numerical experiments are discussed to show the effect of system parameters on the performance measures. We compare the steady-state distribution of the number of customers in the system for both light-tailed and heavy-tailed inter-arrival distributions with the following set of parameters $\rho = 0.5, \phi = 0.1, q = 0.5, p = 0.5$ and $\rho = 0.537416, \phi = 0.1, q = 0.5, p = 0.5$, respectively. In Table 1, we consider geometric (Geo) and deterministic (D) inter-arrival distributions with p.g.f.s $A(z) = \frac{\lambda z}{1-(1-\lambda)z}$ and $A(z) = z^n$, respectively. In the Geo/Geo/1 queue, the steady-state probabilities at pre-arrival epochs and arbitrary epochs exactly match in both the server vacation state and the busy state. It is because of the BASTA property associated with Bernoulli arrivals. The last two rows in the table show the sum of the corresponding probabilities ($\Sigma$), the average number of the customers in the system ($L_s$), and the mean sojourn time ($W_s$).

The steady-state probabilities and their related average values for the negative binomial (NB) and discrete phase-type (D-PH) inter-arrival distributions are presented in Table 2. We consider an NB distribution with p.g.f. $A(z) = \left(\frac{\lambda z}{1-(1-\lambda)z}\right)^5$ and a discrete phase-type distribution with representation $(\beta, S)$, where $\beta = (0.1, 0.6, 0.3)$.

$S = \begin{bmatrix} 0.32065 & 0.45198 & 0.18769 \\ 0.15019 & 0.23952 & 0.5599 \\ 0.20742 & 0.11843 & 0.25 \end{bmatrix}$, $A(z) = \frac{0.926585z + 1.4934}{z^3 - 1.591511z^2 + 29.490691z - 36.400621}$ is the p.g.f. It is seen that with various light-tailed inter-arrival distributions having the same arrival rate, customers following geometric arrivals have a shorter delay and discrete phase-type arrivals have a longer delay. Further, we observed that the steady-state probabilities in the deterministic system are equal to that of the negative binomial system.
Table 1. System-length distributions at pre-arrival and arbitrary epochs for geometric and deterministic arrival distributions

| n  | P_{n,0}         | P_{n,1}         | P_{n,0}         | P_{n,1}         |
|----|-----------------|-----------------|-----------------|-----------------|
| 0  | 0.313492        | 0.313492        | 0.288979        | 0.138613        |
| 1  | 0.201531        | 0.064165        | 0.201531        | 0.064165        |
| 2  | 0.129555        | 0.030556        | 0.129555        | 0.030556        |
| 3  | 0.083286        | 0.014474        | 0.083286        | 0.014474        |
| 4  | 0.053541        | 0.006856        | 0.053541        | 0.006856        |
| 5  | 0.034419        | 0.003248        | 0.034419        | 0.003248        |
| 6  | 0.022127        | 0.001538        | 0.022127        | 0.001538        |
| 7  | 0.014224        | 0.000729        | 0.014224        | 0.000729        |
| 8  | 0.009144        | 0.000345        | 0.009144        | 0.000345        |
| 9  | 0.005878        | 0.000164        | 0.005878        | 0.000164        |
| 10 | 0.003779        | 0.000077        | 0.003779        | 0.000077        |
| 15 | 0.000406        | 0.000046        | 0.000406        | 0.000046        |
| Σ  | 0.877778        | 0.122222        | 0.877778        | 0.122222        |

In the above experiments (in Tables 1 and 2), we discuss the effectiveness of our method by considering several light-tailed discrete distributions as inter-arrival distributions. Now, we aim to show that our method is also suitable for heavy-tailed distributions, a class of probability distributions characterized by slower decay rate than the geometric distribution. For this, we consider the standard log normal (SLN) and Weibull (Wb) distribution as two candidates for our numerical experiments with discrete heavy-tailed distributions in Table 3. We consider the
SLN distribution with p.m.f. \( a_i = c e^{-\ln(i)^2/2}, i \geq 1 \), p.g.f. \( A(z) = \sum_{n=1}^{\infty} a_n z^n \) where \( c \) is the normalization constant given by \( c^{-1} = \sum_{i=1}^{\infty} e^{-\ln(i)^2/2} \) and the Weibull distribution with p.m.f. \( a_n = \frac{M^\sqrt{n}}{c - 1}, n \geq 1 \), p.g.f. \( A(z) = \sum_{n=1}^{\infty} a_n z^n \) where \( c = \sum_{n=0}^{\infty} M^{-\sqrt{n}} \). In the GI/Geo/1 queue experiments with SLN and Weibull inter-arrival distributions, the system parameters are set to \( \rho = 0.537416, \phi = 0.1, q = 0.5, p = 0.5 \). The steady-state system-length distributions at pre-arrival epochs and arbitrary epochs, their sums (\( \Sigma \)), the average number of the customers in the system (\( L_s \)) and the mean sojourn time (\( W_s \)) are presented in Table 3.

Table 3. System-length distributions at pre-arrival and arbitrary epochs for discrete standard log normal (SLN) and discrete Weibull (Wb) arrival distributions

| \( n \) | \( P_{n,0} \) | \( P_{n,1} \) | \( P_{n,0} \) | \( P_{n,1} \) |
|---|---|---|---|---|
| 0 | 0.181276 | 0.248765 | 0.103221 | 0.132529 |
| 1 | 0.123225 | 0.108283 | 0.112313 | 0.104447 | 0.085123 | 0.113309 | 0.082833 | 0.109539 |
| 2 | 0.083763 | 0.076346 | 0.097393 | 0.084544 | 0.070198 | 0.088184 | 0.068309 | 0.084544 |
| 3 | 0.056939 | 0.051897 | 0.073931 | 0.059909 | 0.057890 | 0.062539 | 0.056332 | 0.059909 |
| 4 | 0.038705 | 0.035277 | 0.048965 | 0.042080 | 0.047740 | 0.043931 | 0.046455 | 0.042080 |
| 5 | 0.026310 | 0.023980 | 0.030504 | 0.029529 | 0.039369 | 0.036283 | 0.038310 | 0.029529 |
| 6 | 0.017885 | 0.016301 | 0.018276 | 0.016130 | 0.032467 | 0.021630 | 0.031593 | 0.016130 |
| 7 | 0.012157 | 0.011081 | 0.010658 | 0.014537 | 0.026774 | 0.015177 | 0.026054 | 0.014537 |
| 8 | 0.008826 | 0.007532 | 0.006904 | 0.010200 | 0.022080 | 0.010649 | 0.021486 | 0.010200 |
| 9 | 0.005618 | 0.004385 | 0.003342 | 0.007157 | 0.018208 | 0.007472 | 0.017718 | 0.007157 |
| 10 | 0.003819 | 0.002146 | 0.001910 | 0.005021 | 0.015016 | 0.005242 | 0.014612 | 0.005021 |
| 15 | 0.000554 | 0.000505 | 0.000691 | 0.008584 | 0.005727 | 0.000891 | 0.005573 | 0.008584 |
| 20 | 0.000080 | 0.000004 | 0.000004 | 0.001415 | 0.002184 | 0.000152 | 0.002126 | 0.001415 |
| 30 | 0.000002 | 0.000002 | 0.000002 | 0.000004 | 0.000318 | 0.000004 | 0.000309 | 0.000004 |
| 40 | 0 | 0 | 0 | 0 | 0.000046 | 0 | 0.000045 | 0 | 0.000046 | 0 | 0.000045 | 0 |
| 60 | 0 | 0 | 0 | 0 | 0.000001 | 0 | 0.000001 | 0 | 0.000001 | 0 | 0.000001 | 0 | 0.000001 | 0 |
| \( \Sigma \) | 0.566667 | 0.493993 | 0.599480 | 0.400520 | 0.588712 | 0.411288 | 0.604956 | 0.395044 |

Next, we discuss several numerical experiments to illustrate the effect of system parameters such as service rate, vacation rate, and synchronized abandonment on the performance measures of discrete-time queues with the following set of parameters \( \lambda = 0.2, \mu = 0.6, \phi = 0.3, q = 0.6 \). In Figs. 2, 4 and 6, we compare the behavior of the abandonment rate under varying service rates, vacation rates, and abandonment probabilities in the above four discrete-time queues with a mean arrival rate of 0.2. Customer abandonment rates are directly proportional to the service rate and the probability of remaining in the system, but inversely proportional to the vacation rate. Customer abandonment rate increases linearly with abandonment probabilities, and the values are lower for geometric arrivals and higher for deterministic or negative binomial arrival systems. In Fig. 4, abandonment rate decreases with increasing vacation rate, which is intuitive as longer vacations encourage customer impatience to decide in favor of abandonment. From Figs. 2 and 4, the service rate
Figure 2. Effect of $\mu$ on customer abandonment in different queueing systems with $\lambda = 0.2, \phi = 0.3, q = 0.6$

Figure 3. Effect of $\mu$ on the mean sojourn time in different queueing systems with $\lambda = 0.2, \phi = 0.3, q = 0.6$

has less impact on abandonment than the customer’s non-abandonment probability. As the source of customer impatience is the server vacation, extended vacation periods encourage the impatient customer in favor of abandonment.

In Figs. 3, 5 and 7, we observe that mean sojourn times are decreasing functions of service rate and vacation rate, which is intuitive as the expected delay is higher when the server is slow as well as server vacation period is longer. Based on the arrival process, one may adjust the service speed for system efficiency. Queues with constant arrivals have the lowest waiting time, whereas geometric arrivals wait longer for service. There is a very minimal difference among the mean values for lower vacation rates, and the gap increases with an increase in vacation rate. In Fig. 7, the mean system delay increases when more customers decide to remain in the system at vacation completion epochs. The more the possibility to stay in the system, the less the chance of abandonment, which leads to an increase in waiting times of queueing customers. Lastly, in Figs. 8-9, the effect of arrival rate on the average number in the system and on the abandonment rate is illustrated for deterministic and geometric arrivals with the parameters $\mu = 0.6, \phi = 0.3, q = 0.5$. The average system content increases with more arrivals, but the rate of increase is higher with the geometric arrivals than the constant ones in a congested system. The abandonment rate is a concave function of the arrival rate.

8. Conclusion. We have analyzed an infinite buffer GI/Geo/1 queueing system with multiple vacations and synchronized abandonment. The customers get impatient when the server is unavailable and decide randomly whether to take service or abandon simultaneously at the vacation completion instants. Using the supplementary variable technique and difference operator method, we obtained an explicit expression to find the steady-state system-length distributions at pre-arrival, random, and outside observer’s observation epochs. The remarkable aspect of our approach is that it does not need the structure of any complex transition probability matrix that typically appears in schematic methods. We have discussed the stochastic
decomposition structure for the number of customers and provided the various performance measures. Then, we depicted numerical results to illustrate the impact of system parameters on performance measures. With some numerical experiments, we express that the analytical results presented in this paper can be carried out readily and give precise results for both light-tailed and heavy-tailed inter-arrival distributions. This analysis is useful for heavy-tailed distributions which have applications in network traffic models. We have shown that the discrete-time result gets converted into the corresponding continuous-time results in the limiting case. For
the practitioners, the presented model has applications in telecommunication systems and call centers, wherein the absence of the server is the cause of customers’ abandonment. One may use the discussed model as an alternative approach for many special cases available/non-available in the literature. Further, this work may be extended to include bulk arrival and bulk service systems with renewal or non-renewal inputs, different vacation policies, heterogeneous customers, and synchronized abandonment which are left for future investigations. Analysis of the model from an economic perspective using the concepts from game theory will be a very complex and challenging piece of work.

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