Random Forest Estimation of Conditional Distribution Functions and Conditional Quantiles

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Abstract. We propose a theoretical study of two realistic estimators of conditional distribution functions and conditional quantiles using random forests. The estimation process uses the bootstrap samples generated from the original dataset when constructing the forest. Bootstrap samples are reused to define the first estimator, while the second requires only the original sample, once the forest has been built. We prove that both proposed estimators of the conditional distribution functions are consistent uniformly a.s. To the best of our knowledge, it is the first proof of a.s. consistency and including the bootstrap part. The consistency result holds for a large class of functions, including additive models and products.

1. Introduction

Conditional distribution functions and conditional quantiles estimation is an important task in several domains including environment, insurance or industry. It is also an important tool for Quantile Oriented Sensitivity Analysis (QOSA), see e.g., [15, 21, 9]. In order to estimate conditional quantiles, various methods exist such as kernel based estimation or quantile regression [19] but they present some limitations. Indeed, the performance of kernel methods strongly depends on the bandwidth parameter selection and quickly breaks down as the number of covariates increases. On the other hand, quantile regression as presented in [19] may not be adapted in a non-gaussian setting since the true conditional quantile is not necessarily a linear combination of the input variables [22]. To overcome these issues, we propose to explore the random forest estimation of conditional quantiles [23].

Random forest algorithms allow a flexible modeling of interactions in high dimension by building a large number of regression trees and averaging their predictions. The most famous random forest algorithm is that of [6] whose construction is based on the seminal work of [1, 18, 11]. Breiman’s random forest estimate is a combination of two essential components: Bagging and Classification And Regression Trees (CART)-split criterion [8]. Bagging for bootstrap-aggregating was proposed by [5] in order to improve the performance of weak or unstable learners. Two randomness are included: a bootstrap sample is used to construct each tree and at each split step, some input variables are chosen randomly in \{X_1, \ldots, X_d\}.

Random forests are also related to some local averaging algorithms such as nearest neighbors methods [20, 3] or kernel estimators [30]. More precisely, thanks to [20], the random forest method can be seen as an adaptive neighborhood regression procedure and therefore the prediction (estimation of the conditional mean) can be formulated as a weighted average of the observed response variables.

Based on that approach, we develop a Weighted Conditional Empirical Cumulative Distribution Function (W_{C,ECDF}) approximating the Conditional Cumulative Distribution Function (C_{CDF}). Then, \(\alpha\)-quantile estimates are obtained by

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using $W_{ECDF}$ instead of $C_{CDF}$. [23] defined a $W_{ECDF}$ with weights using the original dataset whereas we allow to construct the weights using the bootstrap samples, as it is done practically in regression random forests. We prove the almost sure consistency of these estimators. The main hypothesis are:

1. the convergence to 0 of the variation of the conditional distribution function on leaves;
2. a control of the size of the leaves, implying that the trees are not fully developed.

In Section 5, we prove that the first hypothesis is satisfied for a large class of models, including additive models and product functions. Moreover, the first hypothesis is also satisfied for some modified version of the CART algorithm, such as in [23]. The second hypothesis may be seen as a stopping rule in the algorithm, so that it should be controlled by practitioners. An implementation of both algorithms is made available within a Julia package called ConditionalDistributionForest [14] as well as a python package named qosa-indices. [13].

The $C_{CDF}$ can be seen as a regression function. Several authors such as [7, 4, 35, 32, 25, 34, 16] have established asymptotic properties of particular variants and simplifications of the original Breiman’s random forest algorithm. Facing some theoretical issues with the bootstrap, most studies replace it by subsampling, assuming that each tree is grown with $s_n < n$ observations randomly chosen without replacement from the original dataset. Most of the time, in order to ensure the convergence of the simplified model, the subsampling rate $s_n/n$ is assumed to tend to zero at some prescribed rate, assumption that excludes the bootstrap mode. Besides, the consistency is generally showed by assuming that the number of trees goes to infinity which is not fully relevant in practice. Under some conditions, [23] showed that if the infinite random forest regression estimator is $L^2$ consistent then so does the finite random forest regression estimator when the number of trees goes to infinity in a controlled way.

Recent attempts to bridge the gap between theory and practice, provide some results on random forest algorithms at the price of fairly strong conditions. For example, [32] showed the $L^2$ consistency of random forests in an additive regression framework by replacing the bootstrap step by subsampling. Their result rests on a fundamental lemma developed in [31] which reviews theoretical random forest, the additive assumption is required here. Furthermore, consistency and asymptotic normality of the whole algorithm were recently proved under strong conditions by [34] replacing bootstrap by subsampling and simplifying the splitting step. One of the strong conditions used in the Theorem 3.1. of [34] is that the individual trees satisfy a condition called honesty. An example of an honest tree given by the authors is one where the tree is grown using one subsample, while the predictions at the leaves of the tree are estimated using a different subsample. Due to this assumption, the authors admit that their theorems are not valid for the practical applications most of the time because almost all implementations of random forests use the training sample twice.

Thus, despite an active investigation during the last decade, further consistency results are still welcome. Our major contribution is the proof of the almost sure uniform convergence of the estimator $W_{ECDF}$ both using the bootstrap samples (Theorem 4.1) or the original one (Theorem 4.2). Remark that [23] gave a proof of the consistency in probability of the $W_{ECDF}$ for a simplified model where the weights are considered as constant while they are indeed random variables heavily data-dependent. We provide an a.s. consistency proof under realistic assumptions for a method based on bootstrap samples. This consistency result holds for a large class of functions called ♠-class which contains additive functions, product
functions e.g. So that our result is much broader than previous ones. Also, a
sub-product of the consistency proof is an asymptotic proximity result between
empirical and theoretical trees (see Proposition 5.4), which is interesting in itself
for further theoretical studies on random forests.

The paper is organized as follows. Breiman’s random forest algorithm is detailed
in Section 2 and notations are stated. The random forest estimations of CDF based
both on bootstrap samples and the original dataset are introduced in Section 3
as a natural generalization of regression random forests. The main consistency
results are presented in Section 4 and a deep discussion on the variation of the
conditional distribution function on leaves is proposed in Section 5. The main
proofs are gathered in Section 6 and some of them are postponed in Appendix A. A
short simulation study is given in Appendix B and a conclusion is given in Section 7.

2. Breiman’s random forest

The aim of this section is to present Breiman’s random forest algorithm as well
as notations used throughout this paper.

Random forest is a generic term to name an aggregation scheme of decision trees
allowing to deal with both supervised classification and regression tasks. We are
only concerned with the regression task.

The general framework is the nonparametric regression estimation where an input
dimensional random vector \( \mathbf{X} \in \mathcal{X} = \prod_{i=1}^{d} [u_i, v_i] \subset \mathbb{R}^d \) is observed and a response \( Y \in \mathbb{R} \) is
predicted by estimating the regression function \( m(\mathbf{x}) = \mathbb{E}[Y|X = \mathbf{x}] \). We assume
that we are given a training sample \( \mathcal{D}_n = (\mathbf{X}_j, Y_j)_{j=1,...,n} \) of independent random
variables distributed as the prototype pair \( (\mathbf{X}, Y) \) which is a \( (d+1) \)-dimensional
random vector. The purpose is to use the dataset \( \mathcal{D}_n \) to construct an estimator
\( m_n : \mathcal{X} \rightarrow \mathbb{R} \) of the function \( m \).

Random forests proposed by [6] build a predictor consisting of a collection of \( k \)
randomized regression trees grown based on the CART algorithm.

The CART-split criterion of [8] is used in the construction of the individual trees
to recursively partition the input space \( \mathcal{X} \) in a dyadic manner. More precisely,
at each step of the partitioning, a part of the space is divided into two sub-parts
according to the best cut perpendicular to the axes. This best cut is selected in
each node of the tree by optimizing the CART-split criterion over the \( d \) variables,
i.e. minimizing the prediction squared error in the two child nodes. The trees
are then grown recursively until reaching a stopping rule. There are several rules,
but one generally proposed is that the tree construction continues while leaves
contain at least \( \text{min}_\text{samples}_\text{leaf} \) elements. This criterion is implemented in the
\texttt{RandomForestRegressor} class of the python package \texttt{Scikit-Learn} [26] or in the
\texttt{build_forest} function of the Julia [2] package \texttt{DecisionTree}.

Building several different trees from a single dataset requires to randomize the
tree building process. [6] proposed to inject some randomness both in the dataset
and in the tree construction. First of all, prior to the construction of each tree,
a resampling step is done by bootstrapping [12] from the original dataset, that is,
by choosing uniformly at random \( n \) times from \( n \) observations with replacement.
Only these bootstrap observations are taken into account in the tree building.
Accordingly, the \texttt{min}_\text{samples}_\text{leaf} hyperparameter introduced previously refers, in
the random forest method, to the minimum number of bootstrap observations con-
tained in each leaf of a tree. Secondly, at each step of the tree construction, instead
of optimizing the CART-split criterion over the \( d \) variables, a number of variables
called \textit{max\_features} is selected uniformly at random among the \(d\) variables, the set of selected variables is denoted \(\mathcal{M}_{\text{try}}\). Then, the best split is chosen as the one optimizing the CART-split criterion as follows: for a given node \(A = \prod_{i=1}^{d} [a_i, b_i]\), the CART-split is given by maximizing over \(j \in \mathcal{M}_{\text{try}}\) and \(z \in A^j = [a_j, b_j]\)

\[
(2.1) L^A_\ell(j, z) = \frac{1}{N^\ell_n(A)} \sum_{i=1}^{n} \left( Y_i - \bar{Y}_A \right)^2 \mathbb{I}(x_i \in A) - \frac{1}{N^\ell_n(A)} \sum_{i=1}^{n} \left( Y_i - \bar{Y}_{A_L} \mathbb{I}\{x_i^j \leq z\} - \bar{Y}_{A_R} \mathbb{I}\{x_i^j > z\} \right)^2 \mathbb{I}(x_i \in A),
\]

where \(A_L = \{x \in A, x^j \leq z\}\) (resp. \(A_R\)) is the left (resp. right) child of \(A\), \(\bar{Y}_B\) is the mean of the \(Y_i^j\)'s in the bootstrap sample with \(X^j \in B\) and \(N^b_n(B)\) denotes the number of elements in the bootstrap sample belonging to \(B\).

Now, for any query point \(x \in X\), the \(\ell\)-th tree estimates \(m(x)\) as follows

\[
m^\ell_n(x; \Theta_\ell, D_n) = \sum_{j \in D_n^*(\Theta_\ell)} \frac{\mathbb{I}\{X^j \in A_n(x; \Theta_\ell, D_n)\} Y^j}{N^b_n(x; \Theta_\ell, D_n)},
\]

where:
- \(\Theta_\ell, \ell = 1, \ldots, k\) are independent random vectors, distributed as a generic random vector \(\Theta = (\Theta^1, \Theta^2)\) and independent of \(D_n\). \(\Theta^1\) contains indexes of observations that are used to build each tree, i.e. the bootstrap sample and \(\Theta^2\) indexes of splitting candidate variables in each node,
- \(D^*_n(\Theta_\ell)\) is the bootstrap sample selected prior to the tree construction,
- \(A_n(x; \Theta_\ell, D_n)\) is the tree cell (subspace of \(X\)) containing \(x\),
- \(N^b_n(x; \Theta_\ell, D_n)\) is the number of elements of \(D^*_n(\Theta_\ell)\) that fall into \(A_n(x; \Theta_\ell, D_n)\).

The trees are then combined to form the finite forest estimator

\[
m_{k,n}^b(x; \Theta_1, \ldots, \Theta_k, D_n) = \frac{1}{k} \sum_{\ell=1}^{k} m^\ell_n(x; \Theta_\ell, D_n).
\]

We may now present the conditional distribution function estimators.

3. Conditional Distribution Forests

We aim to estimate \(F(y|x) = \mathbb{P}(Y \leq y | X = x)\). Two estimators may be defined. One uses the bootstrap samples both in the forest construction and in the estimation. The other uses the original sample in the estimation part. Once the distribution function has been estimated, the conditional quantiles may be estimated straightforwardly.

3.1. Bootstrap samples based estimator. First of all, let us define the random variable \(B_j(\Theta_\ell^1, D_n)\) as the number of times that the observation \((X^j, Y^j)\) has been drawn from the original dataset for the \(\ell\)-th tree construction. Thanks to it, the conditional mean estimator in Equation (2.3) may be rewritten as

\[
m^b_{k,n}(x; \Theta_1, \ldots, \Theta_k, D_n) = \sum_{j=1}^{n} \left( \frac{1}{k} \sum_{\ell=1}^{k} B_j(\Theta_\ell^1, D_n) \mathbb{I}\{X^j \in A_n(x; \Theta_\ell, D_n)\} \right) Y^j
\]

\[
= \sum_{j=1}^{n} w^b_{n,j}(x; \Theta_1, \ldots, \Theta_k, D_n) Y^j,
\]
where the weights are defined by

\[ w_{n,j}^b(\mathbf{x}; \Theta_1, \ldots, \Theta_k, \mathcal{D}_n) = \frac{1}{k} \sum_{\ell=1}^k B_j(\Theta_1, \mathcal{D}_n) \mathbb{I}_{\{Y_j \in A_n(\mathbf{x}; \Theta_\ell, \mathcal{D}_n)\}} \frac{N_{\ell}^b(\mathbf{x}; \Theta_\ell, \mathcal{D}_n)}{N_{\ell}^b(\mathbf{x}; \Theta_\ell, \mathcal{D}_n)} . \]

Note that the weights \( w_{n,j}^b(\mathbf{x}; \Theta_1, \ldots, \Theta_k, \mathcal{D}_n) \) are nonnegative random variables as functions of \( \Theta_1, \ldots, \Theta_k, \mathcal{D}_n \) and their sum for \( j = 1, \ldots, n \) equals 1.

The random forest estimator (3.1) can be seen as a local averaging estimate. Indeed, as mentioned by [29], the regression trees make an average of the observations located in a neighborhood of \( \mathbf{x} \), this neighborhood being defined as the leaf of the tree containing \( \mathbf{x} \). The forest, which aggregates several trees, also operates by calculating a weighted average of the observations in a neighborhood of \( \mathbf{x} \). However, in the case of forests, this neighborhood results from the superposition of the neighborhoods of each tree, and therefore has a more complex shape. Several works have tried to study the random forest algorithm from this point of view (local averaging estimate) such as [20] who was the first to point out the connection between the random forest and the adaptive nearest-neighbors methods, further developed by [3]. Some works such as [31] have also studied random forests through their link with the kernel methods.

We are interested in the Conditional Cumulative Distribution Function (C_CDF) of \( Y \) given \( X = \mathbf{x} \) in order to obtain the conditional quantiles. Pairing the following equality

\[ F(y|X = \mathbf{x}) = \mathbb{P}(Y \leq y|X = \mathbf{x}) = \mathbb{E}[\mathbb{I}_{\{Y \leq y\}}|X = \mathbf{x}] , \]

with the weighted approach described above, we propose to estimate the C_CDF as follows

\[ F_{k,n}^b(y|X = \mathbf{x}; \Theta_1, \ldots, \Theta_k, \mathcal{D}_n) = \sum_{j=1}^n w_{n,j}^b(\mathbf{x}; \Theta_1, \ldots, \Theta_k, \mathcal{D}_n) \mathbb{I}_{\{Y_j \leq y\}} . \]

Hence, given a level \( \alpha \in ]0,1[ \), the conditional quantile estimator \( \hat{q}^\alpha(Y|X = \mathbf{x}) \) is defined as follows

\[ \hat{q}^\alpha(Y|X = \mathbf{x}) = \inf \{ Y_p, p = 1, \ldots, n : F_{k,n}^b(Y|X = \mathbf{x}; \Theta_1, \ldots, \Theta_k, \mathcal{D}_n) \geq \alpha \} . \]

Let us turn now to the estimator using the original sample.

### 3.2. Original sample based estimator.

Trees are still grown with their respective bootstrap sample \( \mathcal{D}_n^* \)(\( \Theta_\ell \)), \( \ell = 1, \ldots, k \). But instead of considering them in the estimation, we may use the original sample \( \mathcal{D}_n \). Consider the weights

\[ w_{n,j}^o(\mathbf{x}; \Theta_1, \ldots, \Theta_k, \mathcal{D}_n) = \frac{1}{k} \sum_{\ell=1}^k \mathbb{I}_{\{Y_j \in A_n(\mathbf{x}; \Theta_\ell, \mathcal{D}_n)\}} \frac{N_{\ell}^o(\mathbf{x}; \Theta_\ell, \mathcal{D}_n)}{N_{\ell}^o(\mathbf{x}; \Theta_\ell, \mathcal{D}_n)} , \]

where \( N_{\ell}^o(\mathbf{x}; \Theta_\ell, \mathcal{D}_n) \) is the number of points of \( \mathcal{D}_n \) that fall into \( A_n(\mathbf{x}; \Theta_\ell, \mathcal{D}_n) \). As previously, the weights \( w_{n,j}^o(\mathbf{x}; \Theta_1, \ldots, \Theta_k, \mathcal{D}_n) \) are nonnegative random variables as functions of \( \Theta_1, \ldots, \Theta_k, \mathcal{D}_n \) and their sum over \( j = 1, \ldots, n \) equals 1.

It was proposed in [23] to estimate the C_CDF with

\[ F_{k,n}^o(y|X = \mathbf{x}; \Theta_1, \ldots, \Theta_k, \mathcal{D}_n) = \sum_{j=1}^n w_{n,j}^o(\mathbf{x}; \Theta_1, \ldots, \Theta_k, \mathcal{D}_n) \mathbb{I}_{\{Y_j \leq y\}} . \]

The conditional quantiles are then estimated by plugging \( F_{k,n}^o(y|X = \mathbf{x}; \Theta_1, \ldots, \Theta_k, \mathcal{D}_n) \) instead of \( F(y|X = \mathbf{x}) \) as before.

A python library named qosa-indices has also been developed to perform the numerical estimations of conditional distributions and quantiles for both methods. It is available at [13] and uses Scikit-Learn, Numpy, Numba. Both approaches...
are also implemented in a Julia package based on the library DecisionTree and that is available at [14].

It has to be noted that a package called quantregForest has been made available in R [27] and can be found at [24]. The estimation method currently implemented in quantregForest is different from the method described in [23]. As a matter of fact, for a new observation \( x \) and the \( \ell \)-th tree, one element of \( D_n = (X_j, Y_j)_{j=1,\ldots,n} \) falling into the leaf node \( A_{n}(x; \Theta_{\ell}, D_n) \) is chosen at random. This gives, \( k \) values of \( Y \) and allows to estimate the conditional distribution function with the classical Empirical Cumulative Distribution Function associated with the empirical measure.

4. Consistency results

In this section, we state our main results, which are the uniform a.s. consistency of both estimators \( F_{b,k,n} \) and \( F_{o,k,n} \) of the conditional distribution function. It constitutes the most interesting result of this paper because it handles the bootstrap component and gives the almost sure uniform convergence. Indeed, most of the studies \[32, 34, 16\] replaces the bootstrap by subsampling without replacement in order to avoid the mathematical difficulties induced by this one and therefore differ slightly from the procedure used in practice.

[23] showed the uniform convergence in probability of a simplified version of the estimator \( F_{o,k,n} \). In [23], the weights \( w_{o,n,j}^\ell (x; \Theta_1, \ldots, \Theta_k, D_n) \) are in fact considered to be non-random while they are indeed random variables depending on \( (\Theta_{\ell})_{\ell=1,\ldots,k} \) and \( D_n \).

Overall, proving the consistency of the forest methods whose construction depends both on the \( X_j \)'s and on the \( Y_j \)'s is a difficult task. This feature makes the resulting estimate highly data-dependent, and therefore difficult to analyze. A simplification widely used by most authors from a theoretical point of view is to work with random forest estimates whose form of the tree depends only on \( X_j \)'s which \[10\] called the \( X \)-property but the \( Y_j \)'s are still used to compute the prediction, either the conditional mean or the conditional distribution function for example. One of the first results dealing with data-dependent random forest estimator of the regression function is \[32\] who showed the \( L^2 \) consistency in an additive regression framework by replacing the bootstrap by subsampling. Thanks to the following assumptions, we go further by showing the a.s. consistency of our estimators in a general framework and not only in the additive regression scheme.

**Assumption 4.1.** For all \( \ell \in [1, k] \), we assume that the variation of the conditional cumulative distribution function within any cell goes to 0:

\[
\forall x \in X, \forall y \in \mathbb{R}, \sup_{z \in A_{n}(x; \Theta_{\ell}, D_n)} |F(y|z) - F(y|x)| \xrightarrow{n \to \infty} 0.
\]

We shall discuss further on Assumption 4.1. In particular in Section 5, we show that this assumption is verified for a large class of functions. Also, let us remark that Assumption 4.1 is satisfied, for example, provided that the diameter of each tree cell goes to zero and for all \( y \), \( F(y|\cdot) \) is continuous. This is satisfied, for example if the splitting rules in the construction of the trees implicate that each direction \( j = 1, \ldots, d \) is chosen with positive probability at each split and a positive proportion of the subsample goes in each child node (see [23] Assumption 3 and Lemma 2). Imposing that each direction is chosen with a positive probability at each split seems unrealistic since even non informative variables will be chosen. This is why we propose an approach where no additional assumptions on the splitting rules are needed, see Section 5.
Assumption 4.2. We shall make the following assumptions on \( k \) (number of trees) and \( N_n^b(x; \Theta, D_n) \) (number of bootstrap observations in a leaf node):

1. \( k = \mathcal{O}(n^\alpha) \), with \( \alpha > 0 \).
2. \( \forall x \in \mathcal{X}, \quad N_n^b(x; \Theta, D_n) = \Omega\left(\sqrt{n} (\ln(n))^{\beta}\right), \quad \text{with} \quad \beta > 1, \ a.s. \) or
3. \( \forall x \in \mathcal{X}, \quad \mathbb{E}[N_n^b(x; \Theta, D_n)] = \Omega\left(\sqrt{n} (\ln(n))^{\beta}\right), \quad \text{with} \quad \beta > 1, \) and

\[
\forall x \in \mathcal{X}, \quad \text{CV}^2(N_n^b(x; \Theta, D_n)) = \mathcal{O}\left(\frac{1}{n^{\alpha/2} (\ln(n))^{\gamma/2}}\right), \quad \text{with} \quad \gamma > 1.
\]

Remark 4.1. In order to prove our main consistency result, either Assumption 4.2, item 2. or item 3. is needed. Item 2. may seem much stronger than item 3. but it has to be noted that the number of bootstrap observations in a tree leaf is a construction parameter of the forest, so that it can be controlled. Using item 2. simplifies the proof but item 3. is sufficient.

Assumption 4.3. For every \( x \in \mathcal{X} \), the conditional cumulative distribution function \( F(y | X = x) \) is continuous and strictly increasing in \( y \).

The two theorems below give the uniform a.s. consistency of our two estimators.

**Theorem 4.1.** Consider a random forest which satisfies Assumptions 4.1 to 4.3. Then,

\[
\forall x \in \mathcal{X}, \quad \sup_{y \in \mathbb{R}} |F_{k,n}^b(y | X = x) - F(y | X = x)| \xrightarrow{a.s.} 0.
\]

**Theorem 4.2.** Consider a random forest which satisfies Assumptions 4.1 to 4.3. Then,

\[
\forall x \in \mathcal{X}, \quad \sup_{y \in \mathbb{R}} |\text{CV} \left( F_{k,n}^b(y | X = x) - F(y | X = x) \right)| \xrightarrow{a.s.} 0.
\]

Remark 4.2. Using standard arguments, the consistency of quantile estimates stems from Assumption 4.3 as well as the uniform convergence of the conditional distribution function estimators obtained above.

Let us make some comments on the assumptions above.

Assumption 4.1 ensures a control on the approximation error of the estimators. It is drawn from Proposition 2 of [32] which shows the consistency of Breiman’s random forest estimate in an additive regression framework. Their Proposition 2 allows to manage the approximation error of the estimator by showing that the variation of the regression function \( m \) within a cell of a random empirical tree is small provided \( n \) is large enough. This result is based on Lemma 1 of [32] which states that the variation of the regression function \( m \) within a cell of a random theoretical tree goes to zero for an additive regression model. A random theoretical tree is grown following the same rules as a random empirical tree, except that the theoretical equivalent of the empirical CART-split criterion (2.1) on a node \( A \) below is used to choose the best split

\[
L^*_A(i, z) = \text{Var}(Y | X \in A) - \mathbb{P}(X_i < z | X \in A) \text{Var}(Y | X_i < z, X \in A) - \mathbb{P}(X_i \geq z | X \in A) \text{Var}(Y | X_i \geq z, X \in A).
\]

Hence, a theoretical tree is obtained thanks to the best consecutive cuts \((i^*, z^*)\) optimizing the previous criterion \( L^*(\cdot, \cdot) \).

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1. \( f(n) = \Omega(g(n)) \iff \exists k > 0, \exists n_0 > 0 \quad \forall n \geq n_0 \quad |f(n)| \geq k \cdot |g(n)| \)
2. \( \text{CV}(X) = \sigma_X / \mathbb{E}[X] \)
General results on standard partitioning estimators whose construction is independent of the label in the training set (see Chapter 4 in [17] or Chapter 6 in [10]) state that a sufficient condition to prove the consistency is that the diameter of the cells tends to zero as \( n \to \infty \). Instead of such a geometrical assumption, Proposition 2 in [32] ensures that the variation of \( m \) inside a node is small thanks to their Lemma 1. But the cornerstone of the Lemma 1 is the Technical Lemma 1 of [31] recalled below for completeness.

**Lemma** (Technical). Assume that:

- \( Y = m(X) + \varepsilon \) with \( m(X) = \sum_{i=1}^{d} m_i(X_i) \), \( X \sim \mathcal{U}([0,1]^d) \) and \( \varepsilon \sim \mathcal{N}(0,\sigma^2) \),
- \( L_A^*(i,z) = 0 \; \forall i = 1, \ldots, d, \forall z \in [a_i,b_i] \) \( (0 \leq a_i < b_i \leq 1) \),

then the regression function \( m \) is constant on \( A \).

This lemma states that if the theoretical split criterion is zero for all cuts in a node, then the regression function \( m \) is constant on this node, i.e. the variation of \( m \) on the cell is zero. But, examples for which \( L_A^*(i,z) = 0 \; \forall i, \forall z \in [a_i,b_i] \) and the regression function is not constant can be easily constructed. Let us consider a two-dimensional example, let \( A = A_1 \times A_2 = [a_1,b_1] \times [a_2,b_2] \) and suppose that the response \( Y \) is

\[
Y = X_1X_2 + c_1X_1 + c_2X_2 + \varepsilon =: m(X_1, X_2) + \varepsilon,
\]

with

- \( X = (X_1, X_2) \) independent random inputs,
- \( c_1 = -\frac{\mathbb{E}[X_2 \mathbb{1}_{(X_2 \in A_2)}]}{\mathbb{P}(X_2 \in A_2)} \) and \( c_2 = -\frac{\mathbb{E}[X_1 \mathbb{1}_{(X_1 \in A_1)}]}{\mathbb{P}(X_1 \in A_1)} \),
- and \( \varepsilon \) a centered noise independent of \( X \).

It can be shown for this model that within the node \( A \), \( L^* = 0 \) for all \( i \in \{1,2\} \), for all \( z \in [a_i,b_i] \) and yet the regression function \( m \) is not constant.

Accordingly, the technical lemma above is well-designed for an additive regression framework. But this context is far from reality for many concrete examples. Section 5 uses a deep analysis of the theoretical trees to enlarge the class of functions for which Assumption 4.1 holds without additional assumptions on the splits during the tree construction.

On the other hand, Assumption 4.2 allows us to control the estimation error of our estimators and expresses that cells should contain a sufficiently large number of points so that averaging among the observations is effective. Finally, Assumption 4.3 is used to get uniform convergence of the estimators.

5. On variation in Random Forest leaves

Our purpose is to propose conditions under which Assumption 4.1 on the variation of \( F(y|.) \) on leaves is satisfied. An interesting sub-product of this study is Proposition 5.4 which shows the asymptotic proximity of empirical and theoretical trees.

In what follows, \( C, C', C_1, C_2, \ldots \) denote any positive constants so that we may write \( C + C' = C, uC_1 = C, \) with \( u > 0 \), \( \ldots \). If \( A = \prod_{j=1}^{d} [a_j, b_j] \subset \mathcal{X} \) is a rectangle, we denote by \( A^{-j} = \prod_{k \neq j} [a_k, b_k] \) and \( A^J = \prod_{k \in J} [a_k, b_k] \) for any \( J \subset \{1, \ldots, d\} \). Also, given \( x \in \mathbb{R}^d \), \( x^{-j} \) is the vector of \( \mathbb{R}^{d-1} \) where the \( j \)-th coordinate has been removed.
and $X^j$ is the vector of $\mathbb{R}^J$ whose coordinates are $x_j$, $j \in J$. We shall denote by $\mathcal{A}$ the set of rectangles.

We consider the following model.

**Assumption 5.1.**

- $Y = m(X) + \varepsilon$;
- $X = (X_1, \ldots, X_d)$ is a continuous random vector with independent coordinates;
- $\varepsilon$ and $X$ are independent, $\varepsilon$ is a continuous, centered random variable with increasing distribution function, $\varepsilon$ has light tails i.e. there exists $0 < \theta < 1$ such that for any $D > 0$, $\mathbb{P}(|\varepsilon| > D) \leq C \theta^D$.
- $X$ takes its values in $X$ which is assumed to be a compact hyper-rectangle of $\mathbb{R}^d$: $X = \prod_{i=1}^d [u_i, v_i]$, $-\infty < u_i \leq v_i < \infty$;
- for any $y, x \mapsto F(y|x)$ is continuous.

Our result will hold for functions $m$ which belong to a certain class denoted $\blacklozenge$-class, which is defined by its complementary below.

**Definition 5.1.** Let $f : X \rightarrow \mathbb{R}$, it does NOT belong to the $\blacklozenge$-class if there exists a rectangle $A = \prod_{j=1}^d [a_j, b_j] \subset X$, with $a_j \leq b_j$ such that for all $j = 1, \ldots, d$, $z \mapsto \mathbb{E} \left[ f(z, X^{-j}) 1_{\{X^{-j} \in A^{-j} \}} \right]$ is constant on $[a_j, b_j]$ and $f$ is not constant on $A$.

For example the function $m$ defined in (4.2) does not belong to the $\blacklozenge$-class.

**Remark 5.1.** From Definition 5.1, if $f$ belongs to the $\blacklozenge$-class then either for any rectangle $A$, there exists $j = 1, \ldots, d$ such that $z \mapsto \mathbb{E} \left[ f(z, X^{-j}) 1_{\{X^{-j} \in A^{-j} \}} \right]$ is not constant on $[a_j, b_j]$ or if for some rectangle $A$, $z \mapsto \mathbb{E} \left[ f(z, X^{-j}) 1_{\{X^{-j} \in A^{-j} \}} \right]$ is constant for all $j = 1, \ldots, d$ then $f$ is constant on $A$.

Let us remark that the $\blacklozenge$-class contains:

- additive functions $f(x) = \sum_{j=1}^d f_j(x_j)$,
- product functions $f(x) = \prod_{j=1}^d f_j(x_j)$ provided that for all $j = 1, \ldots, d$, $\mathbb{E} \left[ f_j(X_j) 1_{\{X_j \in [a_j, b_j] \}} \right] \neq 0$,
- sums of product functions $f(x) = \sum_{i \in I} \prod_{j \in I} f_j(x_j)$ with $I$ a partition of $\{1, \ldots, d\}$ provided that for all $j = 1, \ldots, d$, $\mathbb{E} \left[ f_j(X_j) 1_{\{X_j \in [a_j, b_j] \}} \right] \neq 0$,
- sums of product functions $f(x) = \sum_{i \in I} \prod_{j \in I} f^i_j(x_j)$ with $I$ a family of subsets of $\{1, \ldots, d\}$ provided that the $f^i_j$ are either all increasing or all decreasing and they are either all positive or all negative,
- other functions, such as - for example - in dimension 2 with $(X_1, X_2) \rightsquigarrow \mathcal{U}([0, 1]^{2})$: $f(x, y) = \ln(x + y)$.

**Remark 5.2.** Constructing a function not belonging to the $\blacklozenge$-class, such as (4.2), e.g., requires some combinatorics. This leads us to claim that the $\blacklozenge$-class is dense (in $L^\infty$ sense).

We are interested in $F(y|\cdot)$, remark that $F(y|\cdot)$ is constant on $A$ if and only if $m(\cdot)$ is constant on $A$. Indeed,

$$F(y|x) = \mathbb{P}(Y \leq y|X = x) = \mathbb{P}(\varepsilon \leq y - m(x)) = F_\varepsilon(y - m(x)).$$

We shall use theoretical trees construction as in [32], the theoretical CART-split is done by maximizing $L^\lambda(t, z)$ defined in (4.1). If $L^\lambda$ admits several maxima, then
Theorem 5.1. Let $\Theta \in \mathcal{E}$ inspired by [31] and [32]. It is done with the following steps.

1. Let $A_h(\Theta)$ be a leaf of a theoretical tree stopped at level $h$, let $\Delta(F(y|\cdot), A_h(\Theta))$ be the variation of $F(y|\cdot)$ on $A_h(\Theta)$, then $\Delta(F(y|\cdot), A_h(\Theta))$ goes to 0 as $h$ goes to infinity.

2. Consider $S = \{s_1, \ldots, s_h\}$ with $s_i \in \{L, R\}$, it describes a path of length $h$ in a tree. $L$ means left child and $R$ means right child. Let $A_h^h(S, \Theta)$ be the corresponding node in an empirical tree and $A_h(S, \Theta)$ the corresponding node in a theoretical tree. Then for some theoretical tree, $\text{diam}(A_h^h(S, \Theta) \setminus A_h(S, \Theta))$ goes to zero as $n$ goes to infinity.

3. Finally, we prove that the variation of $F(y|\cdot)$ on $A_h^h(S, \Theta)$ goes to zero as $n$ goes to infinity.

The following lemma proves item 1.

**Lemma 5.2.** Assume that the regression function $m$ belongs to the $\mathcal{A}$-class, let $S^\infty = (s_j, j = 1, \ldots)$ with $s_j \in \{L, R\}$, it describes an infinite path in a theoretical tree, let $S^h = (s_j, j = 1, \ldots, h)$, it describes a path in a theoretical tree of height $h$, $A_h(S^h, \Theta)$ be the corresponding leaf. Then the variation of $F(y|\cdot)$ on $A_h(S^h, \Theta)$ goes to 0 as $h$ goes to infinity.

**Proof.** Since $x \mapsto F(y|x)$ is assumed to be continuous, then the result holds if the diameter of $A_h(S^h, \Theta)$ goes to zero. $A_{\infty}(S^\infty, \Theta) = \bigcap_{h \geq 1} A_h(S^h, \Theta)$, it is a decreasing intersection of rectangles, if its diameter is non zero then it writes: for a non empty subset $J \subset \{1, \ldots, d\}$, $(x_j, j \in J^c)$, $(x_j, j \in J)$, $(x_j, j \in J^c, j \in J) \in \mathcal{X}$

\[ A_{\infty}(S^\infty, \Theta) = \{x_j, j \in J^c\} \times \prod_{j \in J} [a_j, b_j] = \{x_j, j \in J^c\} \times A^J, \text{ with } a_j < b_j. \]

Remark that for any rectangle $A = \prod_{k=1}^d [a_k, b_k]$

\[ L_A^*(i, z) = \mathbb{P}(X_i < z|X \in A)(\mathbb{E}[Y|X_i < z, X \in A] - \mathbb{E}[Y|X \in A])^2 + \mathbb{P}(X_i \geq z|X \in A)(\mathbb{E}[Y|X_i \geq z, X \in A] - \mathbb{E}[Y|X \in A])^2 \]

can be seen as a continuous function of $\alpha_k, \beta_k, k = 1, \ldots, d, j, z$.

For $j \in J$, $L_{A_{\infty}(S^\infty)}^*(j, z)$ rewrites

\[ L_{A_{\infty}(S^\infty)}^*(j, z) = \mathbb{P}(X_j < z|X^J \in A^J)(\mathbb{E}[Y|X_j < z, X^J \in A^J] - \mathbb{E}[Y|X \in A])^2 + \mathbb{P}(X_j \geq z|X^J \in A^J)(\mathbb{E}[Y|X_j \geq z, X^J \in A^J] - \mathbb{E}[Y|X \in A])^2. \]

Also, for $j \in J^c$, $L_{A_{\infty}(S^\infty)}^*(j, z) = 0$.

Write $A_h(S^h, \Theta) = \prod_{j=1}^d [a^h_j, b^h_j]$, then for $j \in J$, $a^h_j \rightarrow a_j$ and $b^h_j \rightarrow b_j$ as $h \rightarrow \infty$, also $L_{A_{\infty}(S^\infty)}^*(j, z) \rightarrow L_{A_{\infty}(S^\infty)}^*(j, z)$.

Let $(j^*, z^*) \in \text{argmax} L_{A_{\infty}(S^\infty)}^*(j, z)$, if $j^* \in J^c$ then $L_{A_{\infty}(S^\infty)}^*(j^*, z^*) = 0$ and
thus \( L_{A_n(S)}(j, z) = 0 \) for any \( j, z \). Assume \( j^* \in J \) and consider a subsequence \( (h_p)_{p \in \mathbb{N}} \) such that \( j^* \in \mathcal{M}_{\text{try}} \) at each level \( h_p \), it exists a.s. because \( \Theta \) gives a positive probability to each \( j \in \{1, \ldots, d\} \) of belonging to \( \mathcal{M}_{\text{try}} \) at each level. Then, consider \( (j^p, z^p) \in \arg\max L_{A_n(S)}^{j^p}(j, z) \) and \( (j^\infty, z^\infty) \) any limit point of the sequence \( (j^p, z^p) \), which means that for a subsequence \( p_0 \to \infty, (j^{p_0}, z^{p_0}) \to (j^\infty, z^\infty) \), thus for \( q \) large enough, \( j^{p_0} = j^\infty \). It easily follows that \( L_{A_n(S)}^{j^\infty}(j^\infty, z^\infty) \leq \max L_{A_n(S)}(j^p, z^p) \). Now, if \( j^\infty \in J \), each \( z^k \) is either \( a^h \) or \( b^h \), \( z^\infty \) is either \( a_j \) or \( b_j \). So that either \( \{z < l, x^j \in A_j\} = \{x^j \in A_j\} \) and \( \{z \geq l, x^j \in A_j\} = \emptyset \) or \( \{z < l, x^j \in A_j\} = \emptyset \) and \( \{z \geq l, x^j \in A_j\} = \{x^j \in A_j\} \), thus
\[
L_{A_n(S)}^{j^\infty}(j^\infty, z^\infty) = \mathbb{P}(X_j < z | x^j \in A_j) \left( \mathbb{E}[Y | x^j < z, x^j \in A_j] - \mathbb{E}[Y | x^j \in A_j] \right)^2 \\
+ \mathbb{P}(X_j \geq z | x^j \in A_j) \left( \mathbb{E}[Y | x^j \geq z, x^j \in A_j] - \mathbb{E}[Y | x^j \in A_j] \right)^2 \\
= 0.
\]
If \( j^\infty \in J \), then \( L_{A_n(S)}^{j^\infty}(j^\infty, z^\infty) = 0 \). Finally, since \( L_{A_n(S)}^{j^\infty}(j, z) \leq L_{A_n(S)}^{j^\infty}(j^\infty, z^\infty) = 0 \), we conclude that \( L_{A_n(S)}^{j^\infty}(j, z) = 0 \) for all \( (j, z) \). This is equivalent to: for all \( i = 1, \ldots, d, z \in [a_i, b_i] \),
\[
\mathbb{E}[Y | x^i = z, x \in A_{\infty}] - \mathbb{E}[Y | x \in A_{\infty}] = 0 \Leftrightarrow \mathbb{P}(x^i = z, x \in A_{\infty}) \mathbb{E}[Y | x \in A_{\infty}] = \mathbb{P}(x^i = z, x \in A_{\infty}) \mathbb{E}[Y | x \in A_{\infty}].
\]
By deriving with respect to \( z \), we may see that it is equivalent to \( z \mapsto \mathbb{E} \left[ m \left( z^\infty, x^i, x \in A_{\infty} \right) \right] \) is constant for all \( i = 1, \ldots, d \). Since we assumed that \( m \) belongs to the \( \mathcal{M}_{\text{class}, A} \), either \( m \) is constant on \( A_{\infty}(S^\infty) \) or the diameter of \( A_{\infty}(S^\infty) \) is zero. In both cases, the variation of \( F(y) \) on \( A_{\infty}(S^\infty, \Theta) \) goes to 0 as \( k \) goes to infinity.

Let us turn to item 2. The main tool for this step is the following result which has an intrinsic interest since it proves that the theoretical and empirical cost functions used for the tree constructions are uniformly close, provided that the leave sizes are not too small. Recall that \( L_n^A(j, z) \) is the empirical cost on the rectangle \( A \) on direction \( j \) and at level \( z \).

**Proposition 5.3.** Let Assumption [5.3] be satisfied. Let \( \beta > \frac{1}{2} \), let \( A \) be a rectangle in \( \mathcal{X} \), we shall say that \( (A, j, z) \in \mathcal{A}^\beta \) if the numbers \( N_{A_L}, N_{A_R} \) of elements of \( \mathcal{D}_n \) belonging to \( A_L := A \cap \{x_j \leq l\} \) and \( A_R := A \cap \{x_j > l\} \) are greater than \( C\sqrt{n}(\ln n)^{\beta} \). We have
\[
sup_{(A, j, z) \in \mathcal{A}^\beta} \left| L_n^A(j, z) - L_n^A(j, z) \right| \overset{a.s.}{\to} 0.
\]

**Proof.** Let \( (A, j, z) \in \mathcal{A}^\beta \) be fixed. Rewrite the difference \( |L_n^A(j, z) - L_n^A(j, z)| = |T_L + T_R| \) with
\[
T_L = T_{1, 1} + T_{1, 2} = \frac{N_{A_L}}{N_A} \left( (Y_A - Y_{A_L})^2 - (\mathbb{E}[Y | x \in A] - \mathbb{E}[Y | x \in A_L])^2 \right) \\
+ (\mathbb{E}[Y | x \in A] - \mathbb{E}[Y | x \in A_L])^2 \frac{N_{A_R}}{N_A} \left( P(X \in A_L | x \in A) - P(X \in A) \right).
\]
The term \( T_R \) is defined in the same way by using \( A_R \) instead of \( A_L \). In order to prove the proposition, we shall prove that \( \sup_{A, j, z} T_{1, 1} \) and \( \sup_{A, j, z} T_{1, 2} \) go to 0 a.s.
The same holds in the same way for $T_{R,1}$ and $T_{R,2}$. Using Vapnik-Chervonenkis theory on rectangles in $A$ (see Lemma A.2 in Appendix A) we have:

$$(5.1) \quad P\left( \sup_{B \in A} \frac{N^b_B}{n} - \mathbb{P}(X \in B) > \kappa \right) \leq 16(n + 1)^{2d}e^{-n\kappa^2/128}.$$ 

$T_{L,2}$ decomposes into:

$$|T_{L,2}| \leq (\mathbb{E}(Y|X \in A) - \mathbb{E}(Y|X \in A_L))^2 \times \left[ \frac{n}{N^b_A} \mathbb{P}(X \in A_L) + \frac{1}{N^b_A} \mathbb{P}(X \in A_L) \right].$$

Remark that for $B \in A$, if

$$(5.2) \quad \left| \frac{N^b_B}{n} - \mathbb{P}(X \in B) \right| \leq \frac{C (\ln n)^\beta}{\sqrt{n}}$$

and $N^b_B \geq C\sqrt{n}(\ln n)^\beta$, then $P(X \in B) \geq \frac{C (\ln n)^\beta}{n}$. So that, for $(A,j,z) \in A^n$, we have, provided that $(5.2)$ holds for $A$,

$$\mathbb{E}(Y|X \in A) \leq D + \mathbb{E}(Y^n) \frac{\mathbb{P}(Y > D)^{\frac{1}{2}}}{\mathbb{P}(X \in A)^{1/2}} \leq (\ln n)^\gamma + C e^{(\ln n)^\gamma} \frac{n^{\frac{1}{2}(1 - \frac{1}{2})}}{(\ln n)^{\gamma(1/2)}} \leq C(\ln n)^\gamma,$$ 

by taking $D = (\ln n)^\gamma$, $\gamma > 1$, $p, q, r > 0$ with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. Now, Equation $(5.1)$ gives that $(5.2)$ is satisfied for $A$ and $A_L$ with probability greater than $1 - 16(n + 1)^{2d}e^{-C(\ln n)^{2\beta}}$ and the condition on $N^b_{A_L}$ for $(A,j,z) \in A^n$ gives that

$$P(T_{L,2} > \kappa) \leq 16(n + 1)^{2d}e^{-C(\ln n)^{2\beta}} + C(n + 1)^{2d}e^{-Cn^2(\ln n)^{2\beta}}.$$ 

Then, Borel-Cantelli Lemma gives that $\sup_{(A,j,z) \in A^n} T_{L,2}$ goes to zero a.s. provided that $2\beta - 4\gamma > 1$.

The term $T_{L,1}$ is treated in the same way, instead of using $(5.1)$, we use Lemma A.2 in Appendix A to get that for any $D > 0$, for any $\kappa > 0$, $A \in A$ and $\frac{1}{p} + \frac{1}{q} = 1$

$$P\left( \left| \frac{1}{n} \sum_{i=1}^{n} Y_i^* \mathbb{1}_{X_i \in A} - \mathbb{E}(Y^*|X \in A) \right| > \kappa \right) \leq 6 \left( \frac{24eD}{\kappa} \ln \left( \frac{48eD}{\kappa} \right) \right)^{2d} e^{-n\kappa^2/(128D^2)} + \frac{C \mathbb{E}(Y^n)^{\frac{1}{2}} \mathbb{P}(Y > D)^{\frac{1}{2}}}{\kappa},$$

where the $(X_i^*, Y_i^*)$’s is the bootstrap sample from $(X_1, Y_1), \ldots, (X_n, Y_n)$ and we take $D = (\ln n)^\gamma$ as before.

The last stone for the proof of Theorem 5.1 is to prove that at each level $h$, each node of the empirical tree $A^n(S)$ is close to a level $h$ node of a theoretical tree.

**Proposition 5.4.** Let Assumption 5.1 be satisfied. Assume that for $\beta > \frac{5}{2}$, $N^b_n(\Theta_\ell, D_n) \geq C\sqrt{n}(\ln n)^\beta$. For $h \in \mathbb{N}$, let $S \in \{L,R\}^h$ describe a path of length $h$ in a binary tree, let $A^n(S)$ and $A(S)$ be corresponding nodes in empirical and theoretical trees. Denote

$$A(S) = \prod_{j=1}^{d} [a_j, b_j] \text{ and } A^n(S) = \prod_{j=1}^{d} [a^n_j, b^n_j].$$
Then, for any $\varepsilon > 0$ there exists $n_h$ such that for $n > n_h$, and a theoretical tree
\begin{equation}
\max_{j=1,\ldots,d} \max \{|a_j - a_j^n|, |b_j - b_j^n|\} \leq \varepsilon \sum_{p=0}^{h} \frac{1}{2^p}.
\end{equation}

**Proof.** We proceed by induction on $h$. If $h = 0$ then $A(S) = A^n(S) = \mathcal{X}$ and the assertion holds. Let us assume that the result holds for $h \in \mathbb{N}$, let $S' \in \{L, R\}^{h+1}$ describe a path of length $h + 1$ in a binary tree, $S' = (S, u)$ with $S \in \{L, R\}^h$ and $u \in \{L, R\}$. Let $(j^n, z^n) \in \argmax L^n_{A^n(S)}(j, z)$, then
\[A^n(S') = A^n(S) \cap \{x_n \leq z^n\} \text{ or } A^n(S') = A^n(S) \cap \{x_n > z^n\}.
\]

Our hypothesis on the construction of the empirical tree implies that for all $n \in \mathbb{N}$, $(A^n(S), j^n, z^n) \in A^n$ so that by Proposition 5.3
\[|L^n_{A^n(S)}(j^n, z^n) - L^n_{A^n(S)}(j^n, z^n)| \rightarrow 0 \text{ a.s. as } n \rightarrow \infty.\]

The induction hypothesis implies that $a_j^n, b_j^n$’s go to $a_j, b_j$ with $A(S) = \prod_{j=1}^{d} [a_j, b_j]$ a node in a theoretical tree of height $h$. Let $(j^n, z^n)$ be any limit point of $(j^n, z^n)$, the continuity of $L^n$ as a function of $(a_j, b_j, j, z)$, leads to $L^n_{A^n(S)}(j^n, z^n) \rightarrow L^n_{A^n(S)}(j^n, z^n)$, as $n$ goes to infinity, taking if needed a subsequence. Remark that $(L^n_{A^n(S)}, j^n, z^n)$, the continuity of $L^n$ as a function of $(a_j, b_j, j, z)$, leads to $L^n_{A^n(S)}(j^n, z^n) \rightarrow L^n_{A^n(S)}(j^n, z^n)$, as $n$ goes to infinity, taking if needed a subsequence.

Remark that $A(S)$ writes, for $J \subset \{1, \ldots, d\}$, $x_j = a_j = b_j$ for $j \notin J$ and $a_j = b_j$ for $j \in J$,
\[A(S) = \prod_{j \notin J} \{x_j\} \times \prod_{j \in J} [a_j, b_j] =: \prod_{j \in J} \{x_j\} \times A^J \text{ and for } j \in J,
\]
\[L^n_{A(S)}(j, z) = \mathbb{P}(X_j < z | X^j \in A^J) \left( \mathbb{E}[Y | X_j < z, X^j \in A^J] - \mathbb{E}[Y | X^j \in A^J] \right)^2
\]
\[+ \mathbb{P}(X_j \geq z | X^j \in A^J) \left( \mathbb{E}[Y | X_j \geq z, X^j \in A^J] - \mathbb{E}[Y | X^j \in A^J] \right)^2,
\]
for $j \notin J$, $L^n_{A(S)}(j, z) = 0$. Now, consider any $(j, z)$, $j \in J$, if $(A^n(S), j, z) \not\in A^n$ for $n$ large enough, then $L^n_{A^n(S)}(j, z) = 0$ because, in that case, either $\mathbb{P}(X_j < z, X^j \in A^J) = 0$ and $\mathbb{P}(X_j \geq z, X^j \in A^J) = \mathbb{P}(X^j \in A^J)$ (using (5.1)). Otherwise, $(A^n, j, z) \in A^n$ for infinitely many $n$ and using again Proposition 5.3, we have $L^n_{A^n(S)}(j^n, z^n) \geq L^n_{A^n(S)}(j, z)$. We conclude that $(j^n, z^n)$ is argmax $L^n_{A^n(S)}(j, z)$ and thus $A(S) \cap \{x^n \leq z^n\}$ and $A(S) \cap \{x^n \leq z^n\}$ are level $h + 1$ nodes of a theoretical tree. The induction is satisfied by choosing $n_{h+1}$ so that for $n > n_{h+1}$, $|z^n - z| \leq \frac{\varepsilon}{\sqrt{n}}$ for one $z^n$.

**End of proof of Theorem 5.1** The end of the proof of Theorem 5.1 is done by noting that if all leaves of the empirical trees verify:
\[C_1 \sqrt{n} \ln(n)^{\beta} \leq A_n^b(\Theta_c, D_n) \leq C_2 \sqrt{n} \ln(n)^{\beta}\]
then the height $h$ of the trees satisfies: $\ln n \leq h \ln 2 + \ln(C_2 \sqrt{n} \ln(n)^{\beta})$, so that $h \rightarrow \infty$ as $n$ goes to infinity. Fix $\varepsilon > 0$ and using Lemma 5.2 choose $h$ so that for any $|S| = h$, the variation of $F(|y|)$ on $A(S)$ is less than $\varepsilon$. Choose $\eta$ so that if $|z - \eta| < \eta$ then $|F(y|z) - F(y|\eta)| < \varepsilon$. Using Proposition 5.4, choose $n$ such that (5.4) is satisfied for $A_n(S)$. Then, the variation of $F(|y|)$ on $A_n(S)$ is less than $3\varepsilon$.

The next section is devoted to the proof of Theorem 4.1. The proof of Theorem 4.2 is similar and left to the reader.
6. Proofs of the main theorems

The proofs of Theorems 4.1 and 4.2 are close. We only provide that of Theorem 4.1 below.

6.1. Proof of Theorem 4.1. The main ingredient of the proof is to use a second sample \( \mathcal{D}_n^\circ \) in order to deal with the data-dependent aspect. Thus, we first define a dummy estimator based on two samples \( \mathcal{D}_n \) and \( \mathcal{D}_n^\circ \) which will be used below. The trees are grown using \( \mathcal{D}_n \), but we consider another sample \( \mathcal{D}_n^\circ \) (independent of \( \mathcal{D}_n \) and \( \Theta \)) which is used to define a dummy estimator

\[
F_{k,n}^\circ (y \mid x; \Theta_1, \ldots, \Theta_k, \mathcal{D}_n^\circ, \mathcal{D}_n) = \sum_{j=1}^n w_{n,j}^\circ (x; \Theta_1, \ldots, \Theta_k, x^{\circ 1}, \ldots, x^{\circ n}, \mathcal{D}_n) \mathbb{I}_{\{y^{\circ} \leq y\}},
\]

where the weights are

\[
w_{n,j}^\circ (x; \Theta_1, \ldots, \Theta_k, x^{\circ 1}, \ldots, x^{\circ n}, \mathcal{D}_n) = \frac{1}{n} \sum_{f=1}^k \frac{\mathbb{I}_{\{x^{\circ f} \in A_n(x; \Theta_f, \mathcal{D}_n)\}}}{N_n^\circ (x; \Theta_f, x^{\circ 1}, \ldots, x^{\circ n}, \mathcal{D}_n), \ j = 1, \ldots, n},
\]

with \( N_n^\circ (x; \Theta_f, x^{\circ 1}, \ldots, x^{\circ n}, \mathcal{D}_n) \), the number of elements of \( \mathcal{D}_n^\circ \) that fall into \( A_n(x; \Theta_f, \mathcal{D}_n) \). Throughout this section, we shall use the convention \( \frac{0}{0} = 0 \) in case \( N_n^\circ (x; \Theta_f, x^{\circ 1}, \ldots, x^{\circ n}, \mathcal{D}_n) = 0 \) and thus \( \mathbb{I}_{\{x^{\circ f} \in A_n(x; \Theta_f, \mathcal{D}_n)\}} = 0 \) for \( j = 1, \ldots, n \). The weights \( w_{n,j}^\circ (x; \Theta_1, \ldots, \Theta_k, x^{\circ 1}, \ldots, x^{\circ n}, \mathcal{D}_n) \) are nonnegative random variables, as function of \( \Theta_1, \ldots, \Theta_k, x^{\circ 1}, \ldots, x^{\circ n}, \mathcal{D}_n \). To lighten the notation in the sequel, we will simply write \( F_{k,n}^\circ (y \mid x) = \sum_{j=1}^n w_{n,j}^\circ (x) \mathbb{I}_{\{y^{\circ} \leq y\}} \) instead of (6.1).

Let \( x \in \mathcal{X} \) and \( y \in \mathbb{R} \), we have

\[
\left| F_{k,n}^b (y \mid x) - F (y \mid x) \right| \leq \left| F_{k,n}^\circ (y \mid x) - F (y \mid x) \right| + \left| F_{k,n}^\circ (y \mid x) - F_{k,n}^b (y \mid x) \right|.
\]

The convergence of the two right-hand terms is handled separately into the following Proposition 6.1 and Lemma 6.2.

**Proposition 6.1.** Consider a random forest which satisfies Assumptions 4.1 and 4.2. Then,

\[
\forall x \in \mathcal{X}, \forall y \in \mathbb{R}, \quad F_{k,n}^\circ (y \mid x) \xrightarrow{n \to \infty} F (y \mid x).
\]

Hence, Proposition 6.1 establishes the consistency for a random forest estimator based on a second sample \( \mathcal{D}_n^\circ \) independent of \( \mathcal{D}_n \) and \( \Theta \). [34] proved that estimators built from honest forests are asymptotically Gaussian. Remark that in [34], it is also required to control the proportion of chosen observations at each split and in each direction. In our case, going through a kind of honest trees is just a theoretical tool. We go one step further with the following lemma by showing that the estimators built with honest and non-honest trees are close.

**Lemma 6.2.** Consider a random forest which satisfies Assumption 4.2. Then,

\[
\forall x \in \mathcal{X}, \forall y \in \mathbb{R}, \quad \left| F_{k,n}^\circ (y \mid x) - F_{k,n}^b (y \mid x) \right| \xrightarrow{n \to \infty} 0.
\]

Hence, according to Proposition 6.1 and Lemma 6.2 we get

\[
\forall x \in \mathcal{X}, \forall y \in \mathbb{R}, \quad F_{k,n}^b (y \mid x) \xrightarrow{n \to \infty} F (y \mid x).
\]

Now, thanks to Dini’s second theorem, let us sketch how to obtain the almost sure uniform convergence relative to \( y \) of the estimator.
Note that \( \{ Y' \leq y \} = \{ U_j \leq F_{Y|X=x}(y) \} \) under Assumption 4.3 with \( U_j = F_{Y|X=x}(Y'j), j = 1, \ldots, n \) which are i.i.d random variables. Then, (6.2) is equivalent to
\[
\forall x \in \mathcal{X}, \forall s \in [0, 1], \sum_{j=1}^{n} w_j^b(x) I_{\{U_j \leq s\}} \xrightarrow{a.s.} \mathcal{D}_{\infty} s.
\]
As in the proof of Glivenko–Cantelli’s Theorem, using that \( s \mapsto \sum_{j=1}^{n} w_j^b(x) I_{\{U_j(\omega) \leq s\}} \) is increasing and Dini’s second theorem, we get the uniform convergence almost everywhere, which concludes the proof of the theorem.

We now turn to the proofs of Proposition 6.1 and Lemma 6.2. To that aim, the following lemma, based on Vapnik-Chervonenkis classes [33], is a key tool.

**Lemma 6.3.** Consider \( \mathcal{D}_n \) and \( \mathcal{D}_n^0 \), two independent datasets of independent \( n \) samples of \((X,Y)\). Build a tree using \( \mathcal{D}_n \) with bootstrap and bagging procedure driven by \( \Theta \). As before, \( N^b(\mathcal{A}_n(\Theta)) = N^b(\mathcal{X}; \mathcal{D}_n) \) is the number of bootstrap observations of \( \mathcal{D}_n \) that fall into \( \mathcal{A}_n(\Theta) \) and \( N^o(\mathcal{A}_n(\Theta)) = \sum_{j=1}^{n} I_{\{X \in A_n(\Theta)\}} \), the number of observations of \( \mathcal{D}_n^0 \) that fall into in \( \mathcal{A}_n(\Theta) \).

\[
\forall \varepsilon > 0, \quad \mathbb{P}\left( |N^b(\mathcal{A}_n(\Theta)) - N^o(\mathcal{A}_n(\Theta))| > \varepsilon \right) \leq 24(n + 1)^{2d}e^{-\varepsilon^2/288n}.
\]

**Proof of Lemma 6.3**

Let \( \varepsilon > 0 \) and \( x \in \mathcal{X} \), we have
\[
\begin{align*}
& \mathbb{P}\left( \left| \frac{N^b(\mathcal{A}_n(\Theta))}{n} - \frac{1}{n} \sum_{j=1}^{n} I_{\{X \in A_n(\Theta)\}} \right| > \frac{\varepsilon}{3n} \right) + \mathbb{P}\left( \left| \frac{1}{n} \sum_{j=1}^{n} I_{\{X \in A_n(\Theta)\}} - \mathbb{P}(X \in A_n(\Theta)) \right| > \frac{\varepsilon}{3n} \right) \\
& + \mathbb{P}\left( \sup_{A \in \mathcal{B}} \left| \frac{1}{n} \sum_{j=1}^{n} B_j(\Theta, \mathcal{D}_n) I_{\{X \in A\}} - \frac{1}{n} \sum_{j=1}^{n} I_{\{X \in A\}} \right| > \frac{\varepsilon}{3n} \right)
\end{align*}
\]

with \( \mathcal{B} = \left\{ \prod_{i=1}^{d} [a_i, b_i] : a_i, b_i \in \mathbb{R} \right\} \). The last two right-hand terms are handled thanks to a direct application of the Theorem of [33] over the class \( \mathcal{B} \) whose Vapnik-Chervonenkis dimension is \( 2d \). This class is nothing more than an extension of the class \( \mathcal{R} \) of rectangles in \( \mathbb{R}^d \). Following the lines of the proof of Theorem 13.8 in [10], one sees that the classes \( \mathcal{R} \) and \( \mathcal{B} \) have the same Vapnik-Chervonenkis dimension.

The first right hand term is treated by applying Vapnik-Chervonenkis’ Theorem under the conditional distribution given \( \mathcal{D}_n \) as in the proof of Lemma A.1.

Finally, we get the overall upper bound
\[
\mathbb{P}\left( |N^b(\mathcal{A}_n(\Theta)) - N^o(\mathcal{A}_n(\Theta))| > \varepsilon \right) \leq 24(n + 1)^{2d}e^{-\varepsilon^2/288n}.
\]

\( \square \)

**Lemma 6.3** is the main ingredient of the proof of Proposition 6.1.
Proof of Proposition 6.3

We aim to prove

\[ \forall x \in \mathcal{X}, \forall y \in \mathbb{R}, \quad P \left( F_{k,n}^\circ (y | X = x) \rightarrow F(y | X = x) \right) = 1. \]

Let \( x \in \mathcal{X} \) and \( y \in \mathbb{R} \), we have

\[ |F_{k,n}^\circ (y | x) - F(y | x)| \leq \left| \sum_{j=1}^n w_j^\circ(x) \left( I_{\{Y_j \leq y\}} - F(y | X^\circ) \right) \right| + \frac{1}{n} \sum_{j=1}^n w_j^\circ(x) \left( F(y | X^\circ) - F(y | x) \right). \]

Define

\[ W_n = n \sum_{j=1}^n w_j^\circ(x) \left( I_{\{Y_j \leq y\}} - F(y | X^\circ) \right) = n \sum_{j=1}^n w_j^\circ(x) Z_j^\circ \]

with \( Z_j^\circ = I_{\{Y_j \leq y\}} - F(y | X^\circ) \), \( n \) i.i.d random variables and

\[ V_n = \sum_{j=1}^n w_j^\circ(x) \left( F(y | X^\circ) - F(y | x) \right). \]

Remark that \( E \left[ Z_j^\circ | X^\circ \right] = 0 \).

We first show that \((W_n)_n \geq 1\) goes to 0 a.s. in the case of Assumption 4.2 item 2. This is achieved by adapting Hoeffding inequality’s proof to our random weighted sum context. For any \( \varepsilon > 0 \), \( t \in \mathbb{R}_+^* \), we have

\[ P(W_n > \varepsilon) \leq E \left[ e^{tW_n} \right] \cdot e^{-t\varepsilon}. \]

We shall make use of the folklore lemma below.

**Lemma.** Let \( X \) be a centred random variable, a.s. bounded by 1. Then, for any \( t \in \mathbb{R}, E \left[ e^{tX} \right] \leq e^{t^2/2} \).

Let \( t > 0 \), we have

\[ E \left[ e^{tW_n} \right] = E \left[ \prod_{j=1}^n e^{t w_j^\circ(x) Z_j^\circ} \right] = E \left[ \prod_{j=1}^n e^{t w_j^\circ(x) Z_j^\circ} \right] \left| D_n, \Theta_1, \ldots, \Theta_k, x^\circ, \ldots, x^n \right|. \]

conditionally to \( D_n, \Theta_1, \ldots, \Theta_k, x^\circ, \ldots, x^n \), the \( w_j^\circ \) are constant and the \( Z_j^\circ \) are centred, independent and bounded by 1. Thus, using the folklore lemma,

\[ E \left[ e^{tW_n} \right] \leq E \left[ \prod_{j=1}^n e^{tw_j^\circ(x)^2/2} \right]. \]

Let \( K > 0 \) be such that for all \( \ell = 1, \ldots, k, N_n^\circ(A_n(\ell)) = N_n^\circ(x; \Theta_\ell, D_n) \geq K \sqrt{n} \left( \ln(n) \right) ^{\beta} \) a.s. by using Assumption 4.2 item 2. Denote \( \Gamma(\ell) \) the event \( \{ N_n^\circ(A_n(\ell)) < K \sqrt{n} \left( \ln(n) \right) ^{\beta} \} \). Remark that \( \Gamma(\ell) \subset \{ |N_n^\circ(A_n(\ell)) - N_n^\circ(A_\ell) | \geq K \sqrt{n} \left( \ln(n) \right) ^{\beta} \} \). Thus, using Lemma 6.3, we have that \( P(\Gamma(\ell)) \leq 24(n+1)^{2\delta} \exp \left[ -K^2(\ln(n))^{2\beta} \right] \).

We have

\[ \sum_{j=1}^n w_j^\circ(x)^2 \leq \sum_{j=1}^n \frac{w_j^\circ(x)}{k} \left( \sum_{\ell=1}^k I_{\{x^\circ \in A_n(\ell)\}} \left( I_{\{\Gamma(\ell)\}} + I_{\{\Gamma(\ell)\}} \right) \right) \]

\[ \leq \sum_{j=1}^n w_j^\circ(x) \left( \frac{2}{K \sqrt{n \ln(n)}} + \frac{1}{k} \sum_{\ell=1}^k I_{\{x^\circ \in A_n(\ell)\}} I_{\{\Gamma(\ell)\}} \right). \]
So that,
\[
\mathbb{E} \left[ \prod_{j=1}^{n} e^{t^2 w_j^2(x)/2} \right] \leq \exp \left[ \frac{t^2}{2} \left( K \sqrt{n} (\ln(n))^\beta \right) \right] \times \mathbb{E} \left[ \exp \left( \frac{t^2}{2} \cdot 1_{\{\cup_{k=1}^{\infty} \Gamma(\ell)\}} \right) \right] \\
\leq \exp \left[ \frac{t^2}{2} \left( K \sqrt{n} (\ln(n))^\beta \right) \right] \times \left( 1 + \epsilon^{t^2/2} \sum_{\ell=1}^{k} \mathbb{P} (\Gamma(\ell)) \right) \\
\leq \exp \left[ \frac{t^2}{2} \left( K \sqrt{n} (\ln(n))^\beta \right) \right] \times \left( 1 + 24k (n + 1)^{2d} \exp \left[ \frac{t^2}{2} - \frac{K^2 (\ln(n))^{2\beta}}{1152} \right] \right).
\]

Taking \( t^2 = \frac{K^2 (\ln(n))^{2\beta}}{576} \) leads to
\[
\mathbb{P}(W_n > \epsilon) \leq \left( 1 + 24k (n + 1)^{2d} \right) \exp \left[ \frac{K (\ln(n))^\beta}{576} - \frac{\epsilon K (\ln(n))^\beta}{24} \right].
\]

The same upper bound is obtained for \( \mathbb{P}(W_n < -\epsilon) \) by using that \( \mathbb{P}(W_n < -\epsilon) = \mathbb{P}(-W_n > \epsilon) \).

Thus, by using Assumption [4.2] item 1., \( k = O(n^\alpha) \) so that the right hand side is summable, we conclude that \( W_n \) goes to 0 almost surely.

In the case where Assumption [4.2] item 3. is satisfied, the proof that \( \langle W_n \rangle_{n \geq 1} \) goes to 0 a.s. is done in a similar spirit following the steps below.

1. First show that \( \langle W_n \rangle_{n \geq 1} \) goes to 0 a.s. This is achieved by decomposing
\[
\mathbb{E} \left[ (W_n)^2 \right] = \mathbb{E} \left[ \left( \sum_{j=1}^{n} w_j^2(x) Z_j^2 \right)^2 \right] \\
= \sum_{j=1}^{n} \sum_{m=1}^{n} \mathbb{E} \left[ w_j^2(x) w_m^2(x) Z_j^2 Z_m^2 \right] \\
= \sum_{j=1}^{n} \mathbb{E} \left[ w_j^2(x) Z_j^2 \right]^2 + \sum_{1 \leq j, m \leq n, j \neq m} \mathbb{E} \left[ w_j^2(x) w_m^2(x) Z_j^2 Z_m^2 \right] \\
\overset{\text{def}}{=} I_n + J_n
\]

Bienaimé–Tchebychev’s inequality, Lemma 6.3 and Assumption 4.2 items 1. and 3., give that there exist \( C, K \) and \( M \) positive constants such that
\[
I_n \leq k \mathbb{P} (|N^b(A_n(\Theta)) - N^o(A_n(\Theta))| > \lambda) + \frac{4}{\mathbb{E}[N^b(A_n(\Theta))]} + 4k \left( \text{CV} (N^b(A_n(\Theta))) \right)^2 \\
\leq 24Cn^\alpha(n + 1)^{2d} \exp \left[ -\frac{K^2 (\ln(n))^{2\beta}}{4608} \right] + \frac{4}{K \sqrt{n} (\ln(n))^\beta} + \frac{4CM^2}{n (\ln(n))^7}.
\]

Then, the trick of using a second sample \( D_n^o \) independent of the first-one and the random variable \( \Theta \) is really important to handle the \( J_n \) term. Indeed, we have \( J_n = 0 \) because \( \mathbb{E}[Z_n^m \mid X^{\infty n}] = 0 \) while the equivalent term encountered in the proof of the Theorem 2 developed by [32] is handled using a conjecture regarding the correlation behavior of the CART algorithm that is difficult to verify (cf. assumption (H2) of [32]). Finally,
\[
\forall \epsilon > 0, \quad \mathbb{P}(|W_n| > \epsilon) \leq \frac{\mathbb{E}[W_n^2]}{\epsilon^2} = \frac{I_n}{\epsilon^2}.
\]
We have,
\[ \forall \varepsilon > 0, \quad \mathbb{P} \left( \limsup_{n \to \infty} \{|W_{n,z}| \geq \varepsilon\} \right) = 0, \]
which implies that \( W_{n,z} \xrightarrow{a.s.} 0 \).

(2) Show that \( (W_n)_{n \geq 1} \) converges almost surely to 0. Bienaimé-Tchebychev inequality, Lemma 6.3 and Assumption 4.2 items 1. and 3. as well as Borel-Cantelli Lemma give that
\[ \forall \varepsilon > 0, \quad \mathbb{P} \left( \limsup_{n \to \infty} \{|W_n - W_{p,z}| \geq \varepsilon\} \right) = 0, \]
for \( p = p(n) = \lfloor \sqrt{n} \rfloor \). From this, we deduce that \( (W_n)_{n \geq 1} \) goes to 0 a.s.

Finally, we show that \( (V_n)_{n \geq 1} \) goes to 0 a.s. which easily follows from Assumption 4.1. This allows us to conclude that
\[ \forall x \in \mathcal{X}, \forall y \in \mathbb{R}, \quad F_{k,n}^o (y \mid X = x) \xrightarrow{a.s.} F (y \mid X = x). \]

Let us now turn to the proof of Lemma 6.2 which shows that the dummy estimator \( F_{k,n}^o \) is close to the interesting one \( F_{k,n}^b \).

**Proof of Lemma 6.2**

Let \( x \in \mathcal{X} \) and \( y \in \mathbb{R} \), we have that
\[
|F_{k,n}^o (y \mid X = x) - F_{k,n}^b (y \mid X = x)| = \left| \frac{1}{k} \sum_{\ell=1}^{k} \left( \sum_{j=1}^{n} \frac{1_{\{x_j \in A_n(x;\Theta_\ell,D_n)\}} 1_{\{Y_j \leq y\}} - \sum_{j=1}^{n} B_j (\Theta_\ell,D_n) 1_{\{x_j \in A_n(x;\Theta_\ell,D_n)\}} 1_{\{Y_j \leq y\}}}{N_n^o (x;\Theta_\ell,D_n)} \right) \right|
\]
\[
= \left| \frac{1}{k} \sum_{\ell=1}^{k} \left( \# \left\{ j \leq J^o / X^{(j)} \in A_n (\Theta_\ell) \right\} \right) - \sum_{j \in S} \frac{B_j (\Theta_\ell,D_n)}{N_n^b (A_n (\Theta_\ell))} \right|
\]
where we denote \( A_n (\Theta_\ell) = A_n (x;\Theta_\ell,D_n) \), \( N^o (A_n (\Theta_\ell)) = N_n^o (x;\Theta_\ell,X^{o1},...,X^{on},D_n) \) and \( N^b (A_n (\Theta_\ell)) = N_n^b (x;\Theta_\ell,D_n) \). \( J, J^o \) are such that \( Y^{(j)} \leq y < Y^{(j+1)} \) and \( Y^{(J)} \leq y < Y^{(J+1)} \), with \( Y^{(j)} \) (resp. \( Y^{(j)} \)) the order statistics of \( (Y^{1},...,Y^{on}) \) (resp. \( (Y^{1},...,Y^{n}) \)) and the \( X^{(j)} \) (resp. \( X^{(j)} \)) the corresponding \( X^{op} \)'s (resp. \( X^p \)'s).

Let us consider for some \( \ell \in [1,k] \),
\[
G = \# \left\{ j \leq J^o / X^{(j)} \in A_n (\Theta_\ell) \right\} \frac{\sum_{j \in S} B_j (\Theta_\ell,D_n)}{N_n^b (A_n (\Theta_\ell))} \defeq N_J^o (A_n (\Theta_\ell)) / N_n^b (A_n (\Theta_\ell)).
\]

We have,
\[
|G| \leq \frac{\left| N^o (A_n (\Theta_\ell)) - N^b (A_n (\Theta_\ell)) \right|}{N_n^b (A_n (\Theta_\ell))} + \frac{\left| N^o_J (A_n (\Theta_\ell)) - N_J (A_n (\Theta_\ell)) \right|}{N_n^b (A_n (\Theta_\ell))}
\]
\[
\defeq |G_1| + |G_2|.
\]

We continue the proof below in the case where Assumption 4.2 item 3. is satisfied. The case where item 2. is verified is done easier following the same lines. Let \( \varepsilon > 0 \).
We are now going to show the almost everywhere convergence to 0 for each term \( G_1 \) and \( G_2 \). Let us start with \( G_1 \).

\[
\Pr (|G_1| > \varepsilon) = \Pr \left( \frac{|N^b (A_n (\Theta)) - N^b (A_n (\Theta))|}{N^b (A_n (\Theta))} > \varepsilon \right)
\]
\[
= \Pr (|N^\circ (A_n (\Theta)) - N^b (A_n (\Theta))| > \varepsilon N^b (A_n (\Theta)), N^b (A_n (\Theta))^1) + \Pr (|N^\circ (A_n (\Theta)) - N^b (A_n (\Theta))| > \varepsilon N^b (A_n (\Theta)), N^b (A_n (\Theta))^2) \leq \lambda)
\]

where \( \lambda = \frac{\mathbb{E} [N^b (A_n (\Theta))]^2}{2} \)

\[
\leq \Pr (|N^\circ (A_n (\Theta)) - N^b (A_n (\Theta))| > \varepsilon \lambda) + \Pr (N^b (A_n (\Theta)) \leq \lambda)
\]

Thanks to Borel–Cantelli Lemma,

\[
\Pr (N^b (A_n (\Theta)) \leq \lambda) \leq 4 \frac{\text{Var} (N^b (A_n (\Theta)))}{\mathbb{E} [N^b (A_n (\Theta))]^2} \leq 4 \left( \text{CV} (N^b (A_n (\Theta))) \right)^2
\]

(6.3)

Now, using Lemma 6.3 and Assumption 4.2, we get

\[
\Pr (|G_1| > \varepsilon) \leq \Pr (|N^\circ (A_n (\Theta)) - N^b (A_n (\Theta))| > \varepsilon \lambda) + 4 \left( \text{CV} (N^b (A_n (\Theta))) \right)^2 
\]

\[
\leq 24(n + 1)^{2d} \exp \left[ -\frac{\varepsilon^2 K^2 (\ln \( n \))^2 \beta}{1152} \right] + \frac{4M^2}{n^{n+1} (\ln \( n \))^2} .
\]

Then, thanks to Borel–Cantelli Lemma

\[
\forall \varepsilon > 0, \quad \Pr \left( \limsup_{n \to \infty} \{|G_1| > \varepsilon\} = 0 \right),
\]

which implies \( G_1 \xrightarrow{a.s.} n^\infty \to 0 \).

The \( G_2 \) term is treated by using again the Vapnik-Chervonenkis theory. By considering the class \( \mathcal{B} = \left\{ \prod_{i=1}^d [a_i, b_i] \times [-\infty, y] : a_i, b_i \in \mathbb{R} \right\} \), it gives (following the lines of proof of Lemma A.1)

\[
\Pr (|G_2| > \varepsilon) \leq \Pr (|N^\circ_j (A_n (\Theta)) - N^b (A_n (\Theta))| > \varepsilon \lambda) + 4 \left( \text{CV} (N (A_n (\Theta))) \right)^2 
\]

\[
\leq 24(n + 1)^{2d} \exp \left[ -\frac{\varepsilon^2 K^2 (\ln \( n \))^2 \beta}{1152} \right] + \frac{4M^2}{n^{n+1} (\ln \( n \))^2} .
\]

Thanks to Borel–Cantelli Lemma, we get

\[
\forall \varepsilon > 0, \quad \Pr \left( \limsup_{n \to \infty} \{|G_2| > \varepsilon\} = 0 \right),
\]

which implies that \( G_2 \xrightarrow{a.s.} n^\infty \to 0 \).

We conclude that \( G \) goes to 0 for all \( \ell \), thus,

\[
\forall x \in \mathcal{X}, \forall y \in \mathbb{R}, \quad \left| F_{k,n}^\circ (y | X = x) - F_{k,n}^b (y | X = x) \right| \xrightarrow{a.s.} n^\infty \to 0 .
\]

In the case where Assumption 4.2 item 2. is verified, it exists \( K > 0 \) such that

\[
N^b (A_n (\Theta)) \geq K \sqrt{n} (\ln \( n \))^2 a.s.
\]

So that \( \Pr (|G_1| > \varepsilon) \) and \( \Pr (|G_2| > \varepsilon) \) are bounded above respectively by

- \( \Pr \left( \left| N^b (A_n (\Theta)) - N^\circ (A_n (\Theta)) \right| > \varepsilon K \sqrt{n} (\ln \( n \))^2 \right) \),

- and \( \Pr \left( \left| N^\circ_j (A_n (\Theta)) - N^b (A_n (\Theta)) \right| > \varepsilon K \sqrt{n} (\ln \( n \))^2 \right) \).
A simple application of Lemma 6.3 and an adaptation of it to $N_1(A_n(\Theta_\ell))$ show that $G_1$ and $G_2$ go to 0 a.s.

This conclude the proof of Theorem 4.1. The proof of Theorem 4.2 which is a bit simpler is left to the reader.

7. Conclusion

This article proposes two conditional distribution functions and conditional quantiles approximations based on random forests. The former is a natural generalisation of the random forest estimator of the regression function making use of the bootstrap samples, while the latter is based on a variant using only the original dataset.

The consistency of the bootstrap samples based estimator is shown under realistic assumptions and constitutes the major contribution of this paper. Indeed, this is the first consistency result handling the bootstrap component in a random forest method whereas it is usually replaced by subsampling. As for the second estimator, the consistency proof established in [23] for a simplified random forest model is extended to a realistic one by taking into account all the randomness used in the procedure. The two estimators have close performances on our toy example. A specific interest of the bootstrap estimation is that the Out-Of-Bag samples could be used for cross-validation and / or back-testing procedures.

The estimators developed in this paper rest on trees grown with the CART-split criterion. But the assumptions providing the consistency results are detached from the split procedure used. Thus, the theoretical tools developed here could be useful for a large class of methods by just changing the splitting scheme. An ambitious additional work would be to develop a theoretical analysis for obtaining convergence rates and also to construct confidence intervals.

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We provide some technical Lemmas used in Section 5 and Section 6.

**Lemma A.1.** Let $A$ be the class of rectangles in $\mathbb{R}^d$, let $\kappa > 0$, for any $n$,$$
abla \left( \sup_{B \in A} \left| N_B n - \mathbb{P}(X \in B) \right| > \kappa \right) \leq 16(n + 1)^{2d} e^{-n\kappa^2/128} .$$
Proof. Denote by \( N_B \) the number of elements of the original sample \( D_n \) that are in \( B \). We have:

\[
\mathbb{P} \left( \sup_{B \in A} \left| \frac{N_B}{n} - \mathbb{P}(X \in B) \right| > \kappa \right) \\
\leq \mathbb{P} \left( \sup_{B \in A} \left| \frac{N_B}{n} - \mathbb{P}(X \in B) \right| > \kappa/2 \right) + \mathbb{P} \left( \sup_{B \in A} \left| \frac{N_B}{n} - \frac{N_B}{n} \right| > \kappa/2 \right) \\
= \mathbb{P} \left( \sup_{B \in A} \left| \frac{N_B}{n} - \mathbb{P}(X \in B) \right| > \kappa/2 \right) + \mathbb{E} \left( \mathbb{P} \left( \sup_{B \in A} \left| \frac{\sum_{i=1}^{n} I_{\{X_i \in B\}}}{n} - \mathbb{P}(X^* \in B|D_n) \right| > \kappa/2 \right) \right).
\]

where \( X^* \) is the bootstrap sample, whose distribution conditionally to \( D_n \) is uniform on \( \{X^1, \ldots, X^n\} \). We apply Vapnik-Chervonenkis ([33]) inequality conditionally to \( D_n \) to get that the second term is bounded above by \( 8(n + 1)^2 e^{-\kappa^2/128} \) and Vapnik-Chervonenkis inequality to get that the first term admits the same above bound.

\( \square \)

Lemma A.2. Let \((X^*_i, Y^*_i)\) be the bootstrap sample from \((X_1, Y_1), \ldots, (X_n, Y_n)\). For any \( D > 0 \), for any \( \kappa > 0 \), \( A \in A \) and \( \frac{1}{p} + \frac{1}{q} = 1 \). Assume that \( \mathbb{E}|Y|^p < \infty \) then

\[
\mathbb{P} \left( \left| \frac{1}{n} \sum_{i=1}^{n} Y^*_i I_{\{X^*_i \in A\}} - \mathbb{E}(Y I_{\{X \in A\}}) \right| > \kappa \right) \\
\leq \mathbb{E} \left( (Y - Z) I_{\{X \in A\}} \right) + \frac{1}{n} \sum_{i=1}^{n} \left( Y^*_i - Z^*_i \right) I_{\{X^*_i \in A\}} + \mathbb{E} \left( Z I_{\{X \in A\}} \right).
\]

The probability that the sum of the first two terms is greater than \( \kappa \) is bounded above by \( C \mathbb{E}|Y|^p \mathbb{P}(Y > D)^{\frac{1}{2}} \). The probability that the last term is greater than \( \kappa \) is bounded by following the proof of Theorem 9.6 in [17] page 155, once conditionally to \( D_n \) to take into account the bootstrap sample and then unconditionally. \( \square \)

In order to illustrate the theoretical results, we provide a numerical example.

Appendix B. Numerical example

The convergence of the estimators, introduced in Section [3] is illustrated on the following toy example

\[(B.1) \quad Y = X_1 + X_2 + X_3 + \varepsilon,\]

where \( X = (X_1, X_2, X_3) \) are three independent random variables with \( X_1 \sim GPD(1.5, 0.25) \) (a Generalised Pareto Distribution), \( X_2 \sim LN(1.1, 0.6) \) (a Log Normal Distribution), \( X_3 \sim \Gamma(2, 0.6) \) (a Gamma Distribution) and \( \varepsilon \) is an independent centered Gaussian noise with variance \( \sigma^2 = 4 \).

The accuracy of the conditional distribution function estimators will be evaluated first, then that of the conditional quantile estimators.
B.1. Conditional distribution function. Let us start by assessing the performance of the C_CDF estimators using the Kolmogorov-Smirnov distance recalled below

\[ KS(x) = \max_y \left| \hat{F}(y|x) - F(y|x) \right| , \]

with \( \hat{F}(y|x) \) being either \( F^b_{k,n}(y|x) \) or \( F^o_{k,n}(y|x) \) here. \( F(y|x) \) has the following expression in our example,

\[ F(y|X = x) = \Phi \left( \frac{y - (x_1 + x_2 + x_3)}{\sigma} \right) , \]

with \( \Phi \), the distribution function of the standard normal distribution \( \mathcal{N}(0, 1) \).

For two randomly chosen points \( x \), the C_CDF estimates are built with a sample of size \( n = 10^4 \), a forest grown with \( n_{trees} = 500 \) and the minimum number of samples required to be at a leaf node set to \( min_{samples\_leaf} = \lfloor \sqrt{n} \cdot \log(n) / 250 \rfloor / 250 \). These experiments are replicated \( s = 500 \) times. Figure 1 below shows the C_CDF approximations computed with the estimator using the original dataset on the left-hand side and those of the estimator calculated with the bootstrap samples on the right-hand side. On each graph, the orange plain line is the true C_CDF, while the blue ones represent the 95% quantiles of the replications. Figure 1 therefore allows to display and compare approximated curves visually.

From a quantitative perspective, the quality of the estimators is measured at each point \( x \) using the following average Kolmogorov-Smirnov distance

\[ K\bar{S}(x) = \frac{1}{s} \sum_{j=1}^{s} KS_j(x) . \]

According to the numerical results displayed in Figure 1, the estimators perform well for points that are well represented in the training sample but, the performance decreases for extreme points. In order to reflect the overall performance of the estimators, we define in the sequel an averaged version of the previous metric and compute it with \( p = 5 \times 10^4 \) randomly chosen points \( x \)

\[ M.K\bar{S} := \frac{1}{p} \sum_{j=1}^{p} K\bar{S}(x^j) . \]

We get \( M.K\bar{S} = 0.1344 \) for the estimator \( F^b_{k,n}(y|x) \) and \( M.K\bar{S} = 0.1295 \) for \( F^o_{k,n}(y|x) \). Thus, it seems that both estimators have a good accuracy for estimating the C_CDF of most points \( x \).

Let us now assess the performance of the conditional quantile estimators.

B.2. Conditional quantiles. The analytic value of the \( \alpha \)-quantile conditionally to \( x = (x_1, x_2, x_3) \) is easy to calculate,

\[ q^\alpha(Y|x) = x_1 + x_2 + x_3 + \sigma \times z_\alpha , \]

with \( z_\alpha \), \( \alpha \)-quantile of the standard normal distribution \( \mathcal{N}(0, 1) \).

Figure 2 shows for two specific points \( x \) and for several levels \( \alpha \) ranging from 0.1 to 0.9, the distribution of the estimators of the conditional quantiles computed with the original dataset on the left-hand side and with the bootstrap samples on the right-hand side. The estimates have been calculated with the following setting: a sample of size \( n = 10^4 \), a forest grown with \( n_{trees} = 500 \) and the minimum number of samples required to be at a leaf node set to \( min_{samples\_leaf} = \lfloor \sqrt{n} \cdot \log(n) / 250 \rfloor / 250 \). In order to assess the quality of the estimators at these points, the
Figure 1. Estimation of the conditional distribution function for two different values $x$ by using the original sample (on the left side) and the bootstrap samples (on the right side). On each graph, the orange line is the true value along with the 95% confidence bands in blue.

The following indicators are computed by repeating the experiment $s = 500$ times.

\[
RMSE(x) = \sqrt{\frac{1}{s} \sum_{j=1}^{s} (\hat{q}_j^a(Y|x) - q^a(Y|x))^2},
\]

\[
Bias(x) = \frac{1}{s} \sum_{j=1}^{s} \hat{q}_j^a(Y|x) - q^a(Y|x),
\]

\[
Variance(x) = \frac{1}{s} \sum_{j=1}^{s} \left( \hat{q}_j^a(Y|x) - \frac{1}{s} \sum_{j=1}^{s} \hat{q}_j^a(Y|x) \right)^2,
\]

where $\hat{q}_j^a(x)$ is the estimator on the $j$'s sample, $j = 1, \ldots, s$.

Based on the graphs obtained in Figure 2, it seems difficult to say if one estimator is better than the other. It appears that the performance of the two estimators differs a bit depending on the observation $x$.

In order to get global measures of both estimators, we define an averaged version of the previous indicators computed with $p = 5 \times 10^4$ randomly chosen points $x$. 
Figure 2. Distribution of the conditional quantile approximations computed for three different values $x$ by using the original sample (on the left side) and the bootstrap samples (on the right side). On each graph, the values above the boxplots are $R$ for $RMSE(x)$, $B$ for $Bias(x)$ and $V$ for $Variance(x)$.

By using the same setting as previously for the estimators, the numerical results for these three measures are listed in Table 1 for several levels $\alpha$. First of all, both estimators have an equivalent RMSE, whereas the original dataset based estimator is the one with the smallest variance, which may seem surprising. Indeed, random forest method is an ensemble learning method that begins with bagging (the bootstrapped aggregation of regression tree predictions) in order to reduce the variance of the prediction function. Thus, for the particular case of the conditional quantile approximation, it is observed the opposite phenomenon on our example. In contrast, the bootstrap samples based estimator has the smallest bias for all $\alpha$. The bias-variance tradeoff of both methods could be handled by tuning the hyperparameters of the random forest such as $min_{samples,leaf}$, which would also allow to improve their accuracy. Finally, it has to be noted that the performance of the two estimators depends on the level $\alpha$. 

According to the following formulas

\[
M_{\cdot RMSE} := \frac{1}{p} \sum_{j=1}^{p} RMSE(x^j),
\]

\[
M_{\cdot Bias} := \frac{1}{p} \sum_{j=1}^{p} Bias(x^j),
\]

\[
M_{\cdot Variance} := \frac{1}{p} \sum_{j=1}^{p} Variance(x^j).
\]
| \( \alpha = 0.1 \) | 0.6382 | 0.2382 | 0.2826 | \( \alpha = 0.2 \) | 0.5868 | 0.2011 | 0.2565 |
| \( \alpha = 0.3 \) | 0.5640 | 0.1791 | 0.2519 | \( \alpha = 0.4 \) | 0.5521 | 0.1638 | 0.2544 |
| \( \alpha = 0.5 \) | 0.5470 | 0.1530 | 0.2615 | \( \alpha = 0.6 \) | 0.5489 | 0.1482 | 0.2758 |
| \( \alpha = 0.7 \) | 0.5602 | 0.1530 | 0.3053 | \( \alpha = 0.8 \) | 0.5901 | 0.1666 | 0.3659 |
| \( \alpha = 0.9 \) | 0.6786 | 0.2443 | 0.6526 | | | | |

| Original sample | Bootstrap samples |
|-----------------|-------------------|
| \( M_{RMSE} \)  | \( M_{RMSE} \)  |
| \( M_{Bias} \)   | \( M_{Bias} \)   |
| \( M_{Variance} \) | \( M_{Variance} \) |

Table 1. Results of the averaged \( RMSE(x) \), \( Bias(x) \) and \( Variance(x) \) computed over \( p = 5 \times 10^4 \) observations of \( x \).

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