SHARP BOUNDS OF FIFTH COEFFICIENT AND HERMITIAN-TOEPLITZ DETERMINANTS FOR SAKAGUCHI CLASSES

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Abstract. For the classes of analytic functions \( f \) defined on the unit disk satisfying
\[
\frac{2zf'(z)}{f(z) - f(-z)} \prec \varphi(z) \quad \text{and} \quad \frac{(2zf'(z))'}{(f(z) - f(-z))'} \prec \varphi(z),
\]
denoted by \( S^*_s(\varphi) \) and \( C_s(\varphi) \), respectively, the sharp bound of the \( n^{th} \) Taylor coefficients are known for \( n = 2, 3 \) and \( 4 \). In this paper, we obtain the sharp bound of the fifth coefficient. Additionally, the sharp lower and upper estimates of the third order Hermitian Toeplitz determinant for the functions belonging to these classes are determined. The applications of our results lead to the establishment of certain new and previously known results.

1. Introduction

Let \( \mathcal{H} \) be the class of holomorphic functions in the unit disk \( \mathbb{D} \) and \( A \subset \mathcal{H} \) represent the class of functions \( f \) satisfying \( f(0) = f'(0) - 1 = 0 \). Let \( \mathcal{S} \subset A \) be the class of univalent functions. A function \( f \in \mathcal{H} \) is said to be starlike with respect to symmetric point if for \( r \) less than and sufficiently close to 1 and every \( z_0 \) on \( |z| = r \), the angular velocity of \( f(z) \) about the point \( f(-z_0) \) is positive at \( z = z_0 \) as \( z \) traverses the circle \( |z| = r \) in the positive direction. Sakaguchi [20] showed that a function \( f \in A \) is starlike with respect to symmetrical point if and only if
\[
\Re \frac{zf'(z)}{f(z) - f(-z)} > 0.
\]
The class of all such functions is denoted by \( \mathcal{S}^*_s \). It is noted that the class of functions univalent and starlike with respect to symmetric points includes the classes of convex functions and odd functions starlike with respect to the origin.
[20]. Afterwards, Das and Singh [5] introduced the class $K_n$ of $f \in A$, known as convex functions with respect to symmetric points, which satisfy

$$\text{Re} \left( \frac{(2zf''(z))'}{(f(z) - f(-z))'} \right) > 0.$$  

The functions in the class are convex and Das and Singh proved that the $n^{th}$ coefficient of functions in $K_n$ is bounded by $1/n$, $n \geq 2$.

Incorporating the notion of subordination, Ravichandran [19] generalized these classes as

$$S_n^*(\varphi) = \left\{ f \in A : \frac{2zf'(z)}{f(z) - f(-z)} \prec \varphi(z) \right\},$$

$$C_n(\varphi) = \left\{ f \in A : \frac{(2zf'(z))'}{(f(z) - f(-z))'} \prec \varphi(z) \right\},$$

where $\varphi(z)$ is an analytic univalent function in $D$ satisfying (i) $\varphi(D)$ is symmetric about the real axis, (ii) $\varphi(D)$ is starlike with respect to $\varphi(0) = 1$ (iii) $\varphi''(0) > 0$ and (iv) $\text{Re} \varphi(z) > 0$ for all $z \in D$. Let us take

$$\varphi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \cdots, \quad B_1 > 0.$$  

They obtained certain convolution conditions and growth and distortion estimates for functions belonging to these classes. Later, Shanmugam et al. [21] found the sharp bound of Feketo-Szegő functional, $|a_3 - \mu a_2^2|$ for the classes $S_n^*(\varphi)$ and $C_n(\varphi)$, which easily provides the bound for initial coefficients $|a_3|$ and $|a_4|$. Further, the sharp bound of $|a_4|$ was determined by Khatter et al. [9] and for certain important choices of $\varphi$ such as

$$S_{n,e}^* := S_n^*(-e^z), \quad S_{n,L}^* := S_n^*(\sqrt{1+z}), \quad S_{n,RL}^* := S_n^*(\sqrt{2} - (\sqrt{2} - 1)\sqrt{(1 - z)/(1 + 2(\sqrt{2} - 1)z)}),$$

the sharp bound of $|a_5|$ was also established. The sharp bound of $|a_6|$ for functions belonging to the classes $S_n^*(\varphi)$ and $C_n(\varphi)$ was still unknown. We get this bound in Section 2. Recently, Gangania and Kumar [6] studied generalized Bohr Rogosinski type inequalities for the classes $S_n^*(\varphi)$ and $C_n(\varphi)$. Kumar and Kumar [11] obtained the sharp bound of second and third order Hermitian-Toeplitz determinant for Sakaguchi functions and the classes defined in (1.2).

For $f \in A$ and $m, n \in \mathbb{N}$, the Hermitian-Toeplitz determinant of order $m$ is given by

$$T_m(n)(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+m-1} \\ a_{n+1} & a_n & \cdots & a_{n+m-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+m-1} & a_{n+m-2} & \cdots & a_n \end{vmatrix}.$$  

It can be easily seen that the determinant of $T_{m,1}(f)$ is rotationally invariant that is determinant of $T_{m,1}(f)$ and $T_{m,1}(f_0)$ are same, where $f_0 = e^{-\theta}f(e^{i\theta}z)$. 

$$\text{Re} \left( \frac{(2zf''(z))'}{(f(z) - f(-z))'} \right) > 0.$$
and $\theta \in \mathbb{R}$. Since for $n = 1$ and $f \in \mathcal{A}$, $a_1 = 1$. Thus, the third order Hermitian-Toeplitz determinant is
\begin{equation}
T_{3,1}(f) = 1 - 2|a_2|^2 + 2 \Re(a_2^*a_3) - |a_3|^2.
\end{equation}
Ye and Lim [23] proved that any $n \times n$ matrix over $\mathbb{C}$ generically can be written as the product of some Toeplitz matrices or Hankel matrices. The applications of Toeplitz matrices and Toeplitz determinants can be seen in the field of pure as well as applied mathematics. They arise in algebraic geometry, numerical integration, numerical integral equations and queueing networks. For more applications, we refer to [23] and the references cited therein.

Numerous papers have recently focused on finding the sharp upper and lower bounds of the Hermitian Toeplitz determinants for functions in $\mathcal{A}$. Cudna et al. [4] initiated this work by determining the sharp lower and upper estimates for $T_{2,1}(f)$ and $T_{3,1}(f)$ for the class of starlike and convex functions of order $\alpha$, $0 \leq \alpha < 1$. The bounds of $T_{2,1}(f)$ and $T_{3,1}(f)$ for the class $\mathcal{S}$ and its certain subclasses were derived by Obradović and Tuneski [18]. For more recent work on this topic, we refer to [1, 10, 12–14] and the references cited therein.

The aim of this paper is to derive the bound of $|a_5|$ and third order Hermitian Toeplitz determinant for $f$ belonging to the classes $S^*_\phi$ and $C_\phi(\phi)$.

2. Fifth coefficient bound

Let $\mathcal{P}$ be the class of Carathéodory functions $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ satisfying $\Re p(z) > 0$ ($z \in \mathbb{D}$). The subsequent lemmas are used in order to prove the bound of $|a_5|$.

**Lemma 2.1** ([17]). If the functions $1 + \sum_{n=1}^{\infty} p_n z^n$ and $1 + \sum_{n=1}^{\infty} q_n z^n$ are members of $\mathcal{P}$, then the same is true of the function
\[
1 + \sum_{n=1}^{\infty} \frac{p_n q_n}{2} z^n.
\]

**Lemma 2.2** ([17]). Let $h(z) = 1 + \beta_1 z + \beta_2 z^2 + \cdots$ and $1 + H(z) = 1 + b_1 z + b_2 z^2 + \cdots$ be functions in $\mathcal{P}$, and set
\[
\gamma_n = \frac{1}{2^n} \left[ 1 + \frac{1}{2} \sum_{\nu=1}^{n} \binom{n}{\nu} \beta_\nu \right], \quad \gamma_0 = 1.
\]
If $A_n$ is defined by
\[
\sum_{n=1}^{\infty} (-1)^{n+1} \gamma_{n-1} H^n(z) = \sum_{n=1}^{\infty} A_n z^n,
\]
then $|A_n| \leq 2$.

It is worth recalling the Möbius function $\Psi_\xi$, which maps the unit disk $\mathbb{D}$ onto itself and given by
\begin{equation}
\Psi_\xi(z) = \frac{z - \xi}{1 - \xi z}, \quad \xi \in \mathbb{D}.
\end{equation}
Lemma 2.3 ([3]). If \( p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathcal{P} \), then for some \( \xi_1, \xi_2, \xi_3 \in \overline{\mathbb{D}} \),

\[
\begin{aligned}
p_1 &= 2\xi_1, \\
p_2 &= 2\xi_1^2 + 2(1 - |\xi_1|^2)\xi_2, \\
p_3 &= 2\xi_1^3 + 4(1 - |\xi_1|^2)\xi_1\xi_2 - 2(1 - |\xi_1|^2)\xi_2^2 + 2(1 - |\xi_1|^2)(1 - |\xi_2|^2)\xi_3.
\end{aligned}
\]

(2.2)

Further, for \( \xi_1, \xi_2 \in \mathbb{D} \) and \( \xi_3 \in \mathbb{T} := \partial \mathbb{D} = \{ z \in \mathbb{C} : |z| = 1 \} \), there is a unique function \( p(z) = (1 + \omega(z))/(1 - \omega(z)) \in \mathcal{P} \) with \( p_1, p_2 \) and \( p_3 \) as in (2.2), where

\[
\omega(z) = z\Psi_{-\xi_1}(z\Psi_{-\xi_2}(\xi_3 z)),
\]

that is

\[
p(z) = \frac{1 + (\xi_2\xi_3 + \xi_1\xi_2 + \xi_1)z + (\xi_1\xi_3 + \xi_1\xi_2^2 + \xi_2)z^2 + \xi_3 z^3}{1 + (\xi_2\xi_3 + \xi_1\xi_2 - \xi_1)z + (\xi_1\xi_3 - \xi_1\xi_2^2 - \xi_2)z^2 - \xi_3 z^3}.
\]

Conversely, for given \( \xi_1, \xi_2 \in \mathbb{D} \) and \( \xi_3 \in \overline{\mathbb{D}} \), we can construct a (unique) function \( p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathcal{P} \) such that \( p_1, p_2 \) and \( p_3 \) satisfy the identities in (2.2). For this, we define

\[
\omega(z) = \omega_{\xi_1, \xi_2, \xi_3}(z) = z\Psi_{-\xi_1}(z\Psi_{-\xi_2}(\xi_3 z)).
\]

Moreover, if we define \( p(z) = (1 + \omega(z))/(1 - \omega(z)) \), then \( p_1, p_2 \) and \( p_3 \) satisfy the identities in (2.2) (see the proof of [3, Lemma 2.4]).

Assumption 2.4. Let \( \varphi(z) \) be given by (1.1). The following conditions on coefficients of \( \varphi \) helps us to prove the result.

\(\textbf{C1} \) : \( |B_1^3 - 2B_1B_2 + 2B_2^2| < |2B_1^3 - B_1^2 - 2B_1B_2| \),

\(\textbf{C2} \) : \( |B_1^3 - B_1^2B_2 + 3B_2^2 - 3B_1B_3| < 3|B_1^3 - B_1^2 + B_2^2| \),

\(\textbf{C3} \) : \( |B_1^3 - B_1^2(8B_2 + 3) - 6B_1^2(B_2(3B_2 + 2B_3 + 2) - 6B_3 + 9B_4) + B_1^3(7B_2(2B_2 + 4) - 24B_3 + 18B_4) + 6B_1^2(B_2^2 - 2B_2^2 + 8B_2B_3 - 3B_3^2 + 6(B_2 + 1)B_4 - 6B_1B_2(3B_2^2 - 6B_3 + B_2^2(4B_3 - 6) + 6B_2(B_2 + 2B_3))) + 18B_1^2(-2B_2^2 + B_2((B_2 - 2)B_2 + 2B_4)) + B_1^2B_2(B_2(5B_2 + 6) - 24B_3 + 18B_4) - 36(2B_3 + B_4)) < 2((B_1 - 2)B_1 + 2B_2) \)

\[
(B_1(2B_1 + B_2 - 3) + 3B_3)(4B_1^2 + 6B_2^2 - B_1^2(2B_2 + 3) - 3B_1B_3),
\]

\(\textbf{C4} \) : \( 0 < (2B_1 - B_2^2 - 2B_2)/(2(B_1 - B_2)) < 1 \).

Theorem 2.5. If \( f(z) = z + a_2z^2 + a_3z^3 + \cdots \in \mathcal{S}_S^\prime(\varphi) \) and coefficients of \( \varphi(z) \) satisfy the conditions \( \textbf{C1}, \textbf{C2}, \textbf{C3} \) and \( \textbf{C4} \), then

\[
|a_n| \leq \frac{B_1}{4}.
\]

The bound is sharp.
Proof. Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S^*_\varphi \). Then there exists a Schwarz function \( \omega(z) \) such that

\[
\frac{2zf'(z)}{f(z) - f(-z)} = \varphi(\omega(z)).
\]

By the one-to-one correspondence between the class of Schwarz functions and the class \( \mathcal{P} \), we obtain

\[
(2.5) \quad 2zf'(z) = \varphi \left( \frac{p(z) - 1}{p(z) + 1} \right)
\]

for some \( p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathcal{P} \). On the comparison of the same powers of \( z \) with the series expansions of functions \( f(z) \), \( \varphi(z) \) and \( p(z) \), the above equation yields

\[
(2.6) \quad a_5 = \frac{B_4}{8} (\Upsilon_1 p_4^4 + \Upsilon_2 p_2^2 p_4 + \Upsilon_3 p_1 p_3 + \Upsilon_4 p_2^4 + p_4),
\]

where

\[
\begin{align*}
\Upsilon_1 &= \frac{B_4^2 - 2B_1 + 6B_2 - 2B_1 B_2 + B_2^2 - 6B_3 + 2B_4}{16B_4}, \\
\Upsilon_2 &= \frac{3B_1 - B_4^2 - 6B_2 + B_1 B_2 + 3B_3}{4B_4}, \\
\Upsilon_3 &= \frac{B_2 - B_1}{B_4}, \\
\Upsilon_4 &= \frac{B_4^2 - 2B_1 + 2B_2}{4B_4}.
\end{align*}
\]

Let us consider that \( q(z) = 1 + \sum_{n=1}^{\infty} \kappa_n z^n \) and \( h(z) = 1 + \sum_{n=2}^{\infty} \nu_n z^n \) are the members of \( \mathcal{P} \), then by Lemma 2.1 for \( p \in \mathcal{P} \), we have

\[
(2.8) \quad 1 + H(z) := 1 + \sum_{n=1}^{\infty} \frac{p_n \kappa_n}{2} z^n \in \mathcal{P}.
\]

For \( h \in \mathcal{P} \) and the function \( 1 + H(z) \) given in (2.8), Lemma 2.2 gives

\[
(2.9) \quad A_4 = \frac{1}{2} \gamma_0 p_4^4 - \frac{1}{4} \gamma_1 \kappa_1 p_2^2 + \frac{1}{2} \gamma_1 \kappa_1 \kappa_3 p_1 p_3 + \frac{3}{8} \gamma_2 \kappa_2 \kappa_2 p_2^2 p_2 - \frac{1}{16} \gamma_3 \kappa_1^2 p_4^4,
\]

where \( \gamma_0 = 1 \),

\[
(2.10) \quad \gamma_1 = \frac{1}{2} \left( 1 + \frac{1}{2} \nu_1 \right), \quad \gamma_2 = \frac{1}{4} \left( 1 + \nu_1 + \frac{1}{2} \nu_2 \right), \quad \gamma_3 = \frac{1}{8} \left( 1 + \frac{3}{2} \nu_1 + \frac{3}{2} \nu_2 + \frac{1}{2} \nu_3 \right)
\]

and

\[
(2.11) \quad |A_4| \leq 2.
\]

Now, in order to establish the required bound, we construct functions \( h(z) \) and \( q(z) \) such that

\[
(2.12) \quad A_4 = \Upsilon_1 p_4^4 + \Upsilon_2 p_1^2 p_2 + \Upsilon_3 p_1 p_3 + \Upsilon_4 p_2^2 + p_4,
\]
where \( \Upsilon \)'s and \( A_4 \) are given in (2.7) and (2.9), respectively. For \( 0 < \tau < 1 \), define
\[
q(z) = \frac{1 + 2\tau z + 2\tau^2 z^2 + 2\tau^3 z^3 + z^4}{1 - z^4},
\]
which yields
\[
(2.13) \quad \kappa_1 = \kappa_3 = 2\tau, \quad \kappa_2 = 2\tau^2 \quad \text{and} \quad \kappa_4 = 2.
\]
From [2, Theorem 1], we have \( q \in \mathcal{P} \). To construct function \( h(z) \), using Lemma 2.3, let
\[
h(z) = \frac{1 + \omega_1(z)}{1 - \omega_1(z)}
\]
such that
\[
(2.14) \quad \omega_1(z) = z\Psi_{-\varepsilon_1}(z\Psi_{-\varepsilon_2}(\varepsilon_3 z)),
\]
where \( \varepsilon_1, \varepsilon_2 \in \mathbb{D} \) and \( \varepsilon_3 \in \mathbb{D} \). Thus, we have
\[
\nu_1 = 2\varepsilon_1, \quad \nu_2 = 2\varepsilon_1^2 + 2(1 - |\varepsilon_1|^2)\varepsilon_2, \quad \nu_3 = 2\varepsilon_1^3 + 4(1 - |\varepsilon_1|^2)\varepsilon_1\varepsilon_2 - 2(1 - |\varepsilon_1|^2) \varepsilon_1^2 \varepsilon_2 + 2(1 - |\varepsilon_1|^2)(1 - |\varepsilon_2|^2)\varepsilon_3.
\]
The above set of equations may be satisfied by many \( \varepsilon \)'s. For our purpose, we impose some restriction on \( \varepsilon \)'s and take all \( \varepsilon \)'s as real numbers. Therefore,
\[
(2.15) \quad \begin{cases} 
\nu_1 = 2\varepsilon_1, \\
\nu_2 = 2\varepsilon_1^2 + 2(1 - \varepsilon_1^2)\varepsilon_2, \\
\nu_3 = 2\varepsilon_1^3 + 4(1 - \varepsilon_1^2)\varepsilon_1\varepsilon_2 - 2(1 - \varepsilon_1^2)\varepsilon_1^2 \varepsilon_2 + 2(1 - \varepsilon_1^2)(1 - \varepsilon_2^2)\varepsilon_3.
\end{cases}
\]
In addition, if we define
\[
\varepsilon_1 = \frac{B_1^3 - 4B_1 B_2 + 2B_2^2}{2B_1^2 - B_1 - 2B_2}, \quad \varepsilon_2 = \frac{B_1^3 - B_1^2 B_2 + 3B_2^2 - 3B_4 B_3}{3(-B_1^2 + B_1 + B_2^2)}, \quad \varepsilon_3 = \frac{B_1^3 - B_1^6(8B_2 + 3) - 6B_1^4(2B_2 + 2B_3 + 2) - 6B_3 + 9B_4}{B_1^3(7B_2(2B_2 + 4) - 24B_3 + 18B_4) + 6B_1^2(B_2^3 - 2B_2^2 + 8B_3 - 3B_4^2 + 6B_2(2B_3 - 4)) + 18B_2^2(-2B_1^2 + B_2((2B_2 - 2)B_2 + 2B_4)) + B_1^6(2B_1^2 - 2B_1 B_2) + B_2(6 + 5B_2 - 24B_3 + 18B_4))}
\]
and
\[
\tau = \sqrt{\frac{2B_1 - B_1^2 - 2B_2}{2(B_1 - B_2)}},
\]
then by Assumption 2.4, we have $|\varepsilon_1| < 1$, $|\varepsilon_2| < 1$, $|\varepsilon_3| < 1$ and $0 < \tau < 1$. Putting these defined $\varepsilon$’s in (2.15), we obtain $\nu_i$’s, which in turn together with (2.10) yields

\begin{equation}
\begin{aligned}
\gamma_1 &= -\frac{(B_1 - B_2)^2}{B_1(B_1^2 - 2B_1 + 2B_2)}, \\
\gamma_2 &= -\frac{(B_1 - B_2)^2(B_1^2 + 6B_2 - B_1(3 + B_2) - 3B_3)}{3B_1(B_1^2 - 2B_1 + 2B_2)^2}, \\
\gamma_3 &= -\frac{(B_1 - B_2)^2(B_1^2 + 6B_2 + B_2^2 - 2B_1(1 + B_2) - 6B_3 + 2B_4)}{4B_1(B_1^2 - 2B_1 + 2B_2)^2}.
\end{aligned}
\end{equation}

(2.16)

On putting the values of $\nu_i$’s and $\gamma_i$’s from (2.13) and (2.16), respectively, in (2.9), we get (2.12). Using the bound $|A_4| \leq 2$ in (2.12), we get

$$|\Upsilon_1 p_1^2 + \Upsilon_2 p_1^2 p_2 + \Upsilon_3 p_1 p_3 + \Upsilon_4 p_2^2 + p_4| \leq 2,$$

which together with (2.6) gives the desired bound of $|a_5|$.

Consider the function $\tilde{f}_5(z) = z + \sum_{n=2}^{\infty} \tilde{a}_n z^n$ in the unit disk satisfying

$$\frac{2z \tilde{f}_5'(z)}{\tilde{f}_5(z) - \tilde{f}_5(-z)} = \varphi(z^4),$$

where $\varphi(z)$ is given by (1.1). Clearly, $\tilde{f}_5 \in S_\varphi^*$. Equating the coefficients in the above equation, we obtain $\tilde{a}_2 = \tilde{a}_3 = \tilde{a}_4 = 0$ and $\tilde{a}_5 = B_1/4$, that demonstrates the sharpness of the bound. \(\square\)

For $-1 \leq B < A \leq 1$, consider the classes $S_\varphi^*[A, B] := S_\varphi^*([1 + A z]/(1 + B z))$ and $S_\varphi^*_{SG} := S_\varphi^*(2/(1 + e^{-z}) - 3B - 2)(A(B(5B - 31) - 27) - 3)) < 2|A - B|^4(A - 3B - 2)(A(B - 2) - 4B^2 + 2B + 3)(A(B + 4) + 2B(B - 2) - 3)| + 15) + A(B(B(5B - 31) - 27) - 3)) < 2|A - B|^4(A

then

\begin{equation}
\begin{aligned}
C_1 : & (A - B)^2(A + B + 2B^2) < |(A - 3B - 2)(A - B)^2|, \\
C_2 : & |(A - B)^3(B + 1)| < 3|(A - B)^2(A - 1 + (B - 1)B)|, \\
C_3 : & |(A - B)^5(B + 1)| < 3|(A - B)^2(A - 1 + (B - 1)B)|, \\
C_4 : & 0 < (3B - A + 2)/(2B + 2) < 1,
\end{aligned}
\end{equation}

(2.17)

The bound is sharp.
Example 2.7. For $A = 0$ and $B = -1/2$, all conditions in Corollary 2.6 are satisfied. Thus, if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S^*_{s}[0, -1/2]$, then $|a_5| \leq 1/8$.

Corollary 2.8. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S^*_{s,s,L}$, then $|a_5| \leq 1/8$ and the bound is sharp.

In case of the classes $S^*_{s,L}$ and $S^*_{s,RL}$, the coefficients of corresponding $\varphi$ satisfy the conditions $C_1$, $C_2$, $C_3$ and $C_4$. Theorem 2.5 yields the following result for these classes:

Remark 2.9. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S^*_{s,L}$, then $|a_5| \leq 1/8$ [9, Theorem 5(a)].

Remark 2.10. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S^*_{s,RL}$, then $|a_5| \leq (5 - 3\sqrt{2})/8$ [9, Theorem 5(b)].

Theorem 2.11. If $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in C_5(\varphi)$ and coefficients of $\varphi(z)$ satisfy the conditions $C_1$, $C_2$, $C_3$ and $C_4$, then

$$|a_5| \leq \frac{B_1}{20}.$$  

The bound is sharp.

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in C_5(\varphi)$. Then there exists a Schwarz function $\omega(z)$ such that

$$\frac{(2zf'(z))'}{(f(z) - f(-z))'} = \varphi(\omega(z)).$$

Comparing the coefficients of the same powers of $z$ after applying the series expansion of $f(z)$, $\varphi(z)$ and $p(z)$ leads to

$$a_5 = \frac{B_1}{20}(\Upsilon_1 p_1 + \Upsilon_2 p_1^2 + \Upsilon_3 p_1 p_3 + \Upsilon_4 p_2 + p_4),$$

where $\Upsilon_i$’s are given in (2.7). Since, $\Upsilon_i$’s are the same as in the case of $S^*_{s}(\varphi)$, therefore following the same methodology as in Theorem 2.5, we get the bound of $|a_5|$.

To see the sharpness, consider the function $\tilde{g}_5(z) = z + \sum_{n=2}^{\infty} \tilde{a}_n z^n$ in $D$ such that

$$\frac{(2z\tilde{g}_5'(z))'}{(\tilde{g}_5(z) - \tilde{g}_5(-z))'} = \varphi(z^4).$$

Comparison of coefficients of same powers yields $\tilde{a}_2 = \tilde{a}_3 = \tilde{a}_4 = 0$ and $\tilde{a}_5 = B_1/20$, which proves the sharpness of the bound. \qed
We can define the classes $C_{[A,B]}, C_{s.e}, C_{s,SG}, C_{s,L}$ and $C_{s,RL}$ in a similar manner as $S_{k}^{*}[A, B], S_{k}^{*}S_{k}^{*}, S_{k}^{*}S_{k}^{*}, S_{k}^{*}S_{k}^{*}$ and $S_{k}^{*}S_{k}^{*}$, respectively. For these classes, Theorem 2.11 yields the following:

**Corollary 2.12.** (1) If $f(z) = z + \sum_{n=2}^{\infty} a_{n}z^{n} \in C_{k}[A, B]$ such that $A$ and $B$ satisfy the conditions given in Corollary 2.6, then $|a_{5}| \leq (A - B)/20$.

(2) If $f(z) = z + \sum_{n=2}^{\infty} a_{n}z^{n} \in C_{s,e}$, then $|a_{5}| \leq 1/20$.

(3) If $f(z) = z + \sum_{n=2}^{\infty} a_{n}z^{n} \in C_{s,L}$, then $|a_{5}| \leq 1/40$.

(4) If $f(z) = z + \sum_{n=2}^{\infty} a_{n}z^{n} \in C_{s,RL}$, then $|a_{5}| \leq (5 - 3\sqrt{2})/40$.

(5) If $f(z) = z + \sum_{n=2}^{\infty} a_{n}z^{n} \in C_{s,SG}$, then $|a_{5}| \leq 1/40$.

All these bounds are sharp.

### 3. Hermitian-Toeplitz determinant

Shanmugam et al. [21] obtained the following bounds of $|a_{3} - \mu a_{2}^{2}|$ for $f(z) = z + \sum_{n=2}^{\infty} a_{n}z^{n}$ belonging to the classes $S_{k}^{*}(\varphi)$ and $C_{k}(\varphi)$.

**Lemma 3.1** ([21, Theorem 2.1]). If $f(z) = z + \sum_{n=2}^{\infty} a_{n}z^{n} \in S_{k}^{*}(\varphi)$, then

$$|a_{3} - \mu a_{2}^{2}| \leq \begin{cases} \frac{1}{2} \left( B_{2} - \frac{\mu}{2} B_{1}^{2} \right) & \text{if } \mu \leq \nu_{1}, \\ \frac{B_{1}}{2} & \text{if } \nu_{1} \leq \mu \leq \nu_{2}, \\ -\frac{1}{2} \left( B_{2} - \frac{\mu}{2} B_{1}^{2} \right) & \text{if } \mu \geq \nu_{2}, \end{cases}$$

where $\nu_{1} = (2(B_{2} - B_{1}))/B_{1}^{2}$ and $\nu_{2} = (2(B_{2} + B_{1}))/B_{1}^{2}$. The bound is sharp.

**Lemma 3.2** ([21, Corollary 2.4]). If $f(z) = z + \sum_{n=2}^{\infty} a_{n}z^{n} \in C_{k}(\varphi)$, then

$$|a_{3} - \mu a_{2}^{2}| \leq \begin{cases} \frac{1}{6} \left( B_{2} - \frac{3}{8} \mu B_{1}^{2} \right) & \text{if } \mu \leq \nu_{1}, \\ \frac{B_{1}}{6} & \text{if } \nu_{1} \leq \mu \leq \nu_{2}, \\ -\frac{1}{6} \left( B_{2} - \frac{3}{8} \mu B_{1}^{2} \right) & \text{if } \mu \geq \nu_{2}, \end{cases}$$

where $\nu_{1} = (8(B_{2} - B_{1}))/B_{1}^{2}$ and $\nu_{2} = (8(B_{2} + B_{1}))/B_{1}^{2}$. The bound is sharp.

For $\mu = 0$, the following bounds for $|a_{3}|$ directly follow, which help us to prove the results:

**Lemma 3.3.** If $f(z) = z + a_{2}z^{2} + a_{3}z^{3} + \cdots \in S_{k}^{*}(\varphi)$ and $B_{1} \leq |B_{2}|$, then

$$|a_{3}| \leq \frac{|B_{2}|}{2}.$$  

**Lemma 3.4.** If $f(z) = z + a_{2}z^{2} + a_{3}z^{3} + \cdots \in C_{k}(\varphi)$ and $B_{1} \leq |B_{2}|$, then

$$|a_{3}| \leq \frac{|B_{2}|}{6}.$$
Theorem 3.5. If \( f \in S^*_x(\varphi) \) and \( B_1 \leq |B_2| \), then
\[
T_{3,1}(f) \leq 1.
\]
The bound is sharp.

Proof. Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S^*_x(\varphi) \). Then
\[
T_{3,1}(f) = 1 - 2|a_2|^2 - |a_3|^2 + 2 \Re(a_2^3 a_3).
\]
Applying the inequality \( 2 \Re(a_2^3 a_3) \leq 2|a_2^2||a_3| \) in the last equation, we obtain
\[
T_{3,1}(f) \leq 1 - 2|a_2|^2 - |a_3|^2 + 2|a_2|^2|a_3| =: g(x),
\]
where \( g(x) = 1 - 2|a_2|^2 - x^2 + 2|a_2|^2 x \) with \( x = |a_3| \). For \( f \in S^*_x(\varphi) \), we have \( |a_2| \leq B_1/2 \) and from Lemma 3.3, \( |a_3| \leq |B_2|/2 \). Thus \( |a_2| \in [0, 1] \) and \( x = |a_3| \in [0, 1] \). As \( g'(x) = 0 \) at \( x = |a_2|^2 \) and \( g''(x) < 0 \) for all \( x \in [0, 1] \). Consequently, we have
\[
T_{3,1}(f) \leq \max g(x) = g(|a_2|^2) = (|a_2|^2 - 1)^2 \leq 1.
\]
Since the identity function \( f(z) = z \) is a member of the class \( S^*_x(\varphi) \) and for this function, we have \( a_2 = 0, a_3 = 0 \) and \( T_{3,1}(f) = 1 \), which shows that the bound is sharp. \( \square \)

Theorem 3.6. If \( f \in C_1(\varphi) \) and \( B_1 \leq |B_2| \), then
\[
T_{3,1}(f) \leq 1.
\]
The result is sharp.

Proof. Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in C_1(\varphi) \). Then using the inequality \( \Re(a_2^3 a_3) \leq |a_2|^2|a_3| \) in (1.4) for \( f \in C_1(\varphi) \), we obtain
\[
T_{3,1}(f) \leq 1 - 2|a_2|^2 - |a_3|^2 + 2|a_2|^2|a_3| =: g(x),
\]
where \( g(x) = 1 - 2|a_2|^2 - x^2 + 2|a_2|^2 x \). Since \( |a_2| \leq B_1/4 \) and from Lemma 3.4, we have \( |a_3| \leq |B_2|/6 \), therefore \( |a_2| \in [0, 1/2] \) and \( |a_3| \in [0, 1/3] \). Also, note that \( g(x) \) attains its maximum value at \( x = |a_2|^2 \). Hence
\[
T_{3,1}(f) \leq \max g(x) = g(|a_2|^2) = (|a_2|^2 - 1)^2 \leq 1.
\]
The equality case holds for \( f(z) = z \). \( \square \)

Theorem 3.7. If \( f \in S^*_x(\varphi) \) such that \( B_1^2 > 2B_2 \), then the following estimates hold:
\[
T_{3,1}(f) \geq \begin{cases} 
\min \left\{ 1 - \frac{B_1^2}{4}, 1 - \frac{B_1^2}{2} + \frac{B_1^2 B_2}{4} - \frac{B_2^2}{4} \right\}, & \sigma_1 \notin [0, 4], \\
1 - \frac{B_1^2}{2} + \frac{B_1^2 B_2}{4} - \frac{B_2^2}{4}, & \sigma_1 = 4, \\
1 - \frac{B_1^2}{16(B_1^2 + B_1^2(2B_2 - 1) - 2B_1 B_2 - B_2^2)}, & \sigma_1 \in (0, 4),
\end{cases}
\]
where
\[ \sigma_1 = \frac{2B_1(B_1^2 - 2B_2)}{(B_1^2 - B_1 - B_2)(B_1 + B_2)}. \]

First two inequalities are sharp.

Proof. Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S_*^\prime(\varphi). \) Then from (2.5), we obtain
\[ a_2 = \frac{B_1p_1}{4} \quad \text{and} \quad a_3 = \frac{1}{8}(-B_1p_1^2 + B_2p_1^2 + 2B_1p_2). \]

Since the class \( S_*^\prime(\varphi) \) and the class \( \mathcal{P} \) is rotationally invariant, therefore we can take \( p_1 = p \in [0, 2] \). Moreover, Libera et al. [15] showed that \( 2p_2 = p_1^2 + (4 - p_1^2)\zeta, \zeta \in \mathbb{D} \) for \( p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathcal{P} \). Thus, we have
\[ -|a_3|^2 = -\frac{1}{64} \left( B_2^2p_1^4 + B_1^2(4 - p_1^2)^2|\zeta|^2 + 2B_1B_2p_1^2(4 - p_1^2)\text{Re} \bar{\zeta} \right), \]
\[ 2\text{Re}(a_2^2a_3) = \frac{1}{64} B_2^2p_1^2 \left( (B_2 - B_1)p_1^2 + B_1(p_1^2 + (4 - p_1^2)\text{Re} \bar{\zeta}) \right). \]

Taking these into account in (1.4), we get
\[ T_{3,1}(f) = \frac{1}{64} \left( (B_2^2 - B_2)B_2p_1^4 - B_1^2(4 - p_1^2)^2|\zeta|^2 + B_1(B_2^2 - 2B_2)p_1^2(4 - p_1^2)\text{Re} \bar{\zeta} \right) - \frac{B_1^2p_1^2}{8} + 1 \]
\[ =: F(p_1, |\zeta|, \text{Re} \bar{\zeta}). \]

It can be seen that \( F(p_1, |\zeta|, \text{Re} \bar{\zeta}) \geq F(p_1, |\zeta|, -|\zeta|) =: G(x, y) \) by considering \( p_1^2 = x \) and \( |\zeta| = y \), where
\[ G(x, y) = \frac{1}{64} \left( (B_1^2 - B_2)B_2x^2 - B_1^2(4 - x)^2y^2 - B_1(B_1^2 - 2B_2)x(4 - x)y \right) \]
\[ - \frac{B_2^2x}{8} + 1. \]

Whenever \( B_1^2 > 2B_2 \), we have
\[ \frac{\partial G}{\partial y} = \frac{1}{64} (-2B_2^2(4 - x)^2y - B_1(B_1^2 - 2B_2)x(4 - x)) \leq 0 \]
for \( x \in [0, 4] \) and \( y \in [0, 1] \), which means that \( G(x, y) \) is a decreasing function of \( y \) and \( G(x, y) \geq G(x, 1) =: I(x) \) with
\[ I(x) = \frac{1}{64} (B_1^3 + B_1^2(B_2 - 1) - 2B_1B_2 - B_2^2)x^2 + \frac{B_1}{16} (2B_2 - B_1^2)x - \frac{B_1^2}{4} + 1. \]

An easy computation yields that \( I'(x) = 0 \) at
\[ x_0 = \frac{2B_1(B_1^2 - 2B_2)}{(B_1^2 - B_1 - B_2)(B_1 + B_2)}. \]
and

\[ I''(x_0) = \frac{1}{32}(B_1^3 - B_1 - B_2)(B_1 + B_2). \]

Since \( B_1^2 > 2B_2 \), therefore numerator of \( x_0 \) is always positive. Moreover, denominator of \( x_0 \) and numerator of \( I''(x_0) \) are same, therefore \( x_0 < 0 \) (or \( x_0 > 0 \)) if and only if \( I''(x_0) < 0 \) (or \( I''(x_0) > 0 \)). Here we discuss the following cases:

**Case I:** Whenever \( x_0 \in (0, 4) \), then \( I''(x_0) > 0 \). Thus \( I(x) \) attains its minimum value at \( x_0 \), which gives

\[ T_{3,1}(f) \geq I(x_0) = 1 - \frac{B_1^3(B_1^3 + 4B_1^2 - 4B_1 - 8B_2)}{16(B_1^3 + B_1^2(B_2 - 1) - 2B_1B_2 - B_2^2)}. \]

**Case II:** When \( x_0 < 0 \) or \( x_0 > 4 \), which indicates that \( I(x) \) does not have any critical point, therefore

\[ T_{3,1}(f) \geq \min\{I(0), I(4)\} = \min\left\{ 1 - \frac{B_1^2}{4}, 1 - \frac{B_1^2}{2} + \frac{B_1^2B_2}{4} - \frac{B_2^2}{4} \right\}. \]

For \( x_0 = 4 \), \( T_{3,1}(f) \geq I(4) \).

Functions \( \tilde{f}_2 \in S^*_s(\varphi) \) and \( \tilde{f}_3 \in S^*_s(\varphi) \) given by

\[ \frac{2\varphi \tilde{f}_2(z)}{\tilde{f}_2(z) - \tilde{f}_2(-z)} = \varphi(z), \quad \frac{2\varphi \tilde{f}_3(z)}{\tilde{f}_3(z) - \tilde{f}_3(-z)} = \varphi(z^2) \]

show that these bounds are sharp as

\[ T_{3,1}(\tilde{f}_2) = 1 - \frac{B_1^2}{2} + \frac{B_1^2B_2}{4} - \frac{B_2^2}{4} \quad \text{and} \quad T_{3,1}(\tilde{f}_3) = 1 - \frac{B_1^2}{4}, \]

which completes the proof. \( \square \)

**Theorem 3.8.** If \( f \in C_s(\varphi) \) and \( 3B_1^2 \geq 8B_2 \), then the following estimates hold:

\[ T_{3,1}(f) \geq \begin{cases} \min\left\{ 1 - \frac{B_1^2}{36}, 1 - \frac{B_1^2}{8} + \frac{B_1^2B_2}{48} - \frac{B_2^2}{36} \right\}, & \sigma_2 \notin [0, 4], \\ 1 - \frac{B_1^2}{8} + \frac{B_1^2B_2}{48} - \frac{B_2^2}{36}, & \sigma_2 = 4, \\ 1 - \frac{B_1^2(12B_1^2 + 12B_2^2 + 4B_1 - 32B_2)}{64(3B_1^3 + B_1^2(3B_2 - 4) - 8B_1B_2 - 4B_2^2)}, & \sigma_2 \in (0, 4), \end{cases} \]

where

\[ \sigma_2 = \begin{cases} 2B_1(3B_1^2 + 10B_1 - 8B_2), & \sigma_2 \notin [0, 4], \\ \frac{2B_1(3B_1^2 + 10B_1 - 8B_2)}{3B_1^3 + 3B_1^2B_2 - 4B_1^2 - 8B_1B_2 - 4B_2^2}, & \sigma_2 \in (0, 4). \end{cases} \]

First two inequalities are sharp.
Proof. Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in C_{\ast}(\varphi) \). Then from (2.18), we obtain

\[
(3.2) \quad a_3 = \frac{B_1 p_1}{576} - \frac{1}{576} (B_1 p_1^2 + B_2 (4 - p_1^2)^2 |\zeta|^2 + 2B_1 B_2 p_1^2 (4 - p_1^2) \text{Re} \bar{\zeta}),
\]

\[
2 \text{Re}(a_3^2) = \frac{B_1^2 p_1^2}{768} (4 - B_1^2 + B_2 p_1^2 + B_1 (p_1^2 + (4 - p_1^2) \text{Re} \bar{\zeta})).
\]

These above values together with (1.4) leads to

\[
T_{3,1}(f) = \left( \frac{B_1^2 B_2}{768} - \frac{B_2^2}{576} \right) p_1^4 - \frac{1}{576} B_1^2 (4 - p_1^2)^2 |\zeta|^2 + \left( 3 \frac{B_1^2 - 8 B_1 B_2}{2304} \right) p_1^2 (4 - p_1^2) \text{Re} \bar{\zeta} - \frac{B_1^2 p_1^4}{32} + 1
\]

\[
=: F(p_1, |\zeta|, \text{Re} \bar{\zeta}).
\]

As \( \text{Re} \bar{\zeta} \geq -|\zeta| \), hence \( F(p_1, |\zeta|, \text{Re} \bar{\zeta}) \geq F(p_1, |\zeta|, -|\zeta|) := G(x, y) \), where

\[
G(x, y) = \left( \frac{B_1^2 B_2}{768} - \frac{B_2^2}{576} \right) x^2 - \frac{1}{576} B_1^2 (4 - x^2) y^2 - \left( 3 \frac{B_1^2 - 8 B_1 B_2}{2304} \right) x (4 - x) y
\]

\[
- \frac{B_1^2}{32} x + 1
\]

for \( x = p_1^2 \in [0, 4] \) and \( y = |\zeta| \in [0, 1] \). Whenever \( 3B_1^2 \geq 8B_1 B_2 \), we have

\[
\frac{\partial G(x, y)}{\partial y} = -\frac{2}{88} B_1^2 (4 - x^2) y - \left( 3 \frac{B_1^2 - 8 B_1 B_2}{2304} \right) x (4 - x) \leq 0.
\]

Therefore, \( G(x, y) \) is a decreasing function of \( y \) and \( G(x, y) \geq G(x, 1) =: I(x) \), where

\[
I(x) = 1 - \frac{B_1 (3B_1^2 + 10B_1 - 8B_2)}{576} x - \frac{B_1^2}{36}
\]

\[
+ \frac{x^2 (3B_1^2 + 3B_1^2 B_2 - 4B_1^2 - 8B_1 B_2 - 4B_2^2)}{2304}.
\]

An elementary calculation reveals that \( I'(x) = 0 \) at

\[
x_0 = \frac{2B_1 (3B_1^2 + 10B_1 - 8B_2)}{3B_1^2 + 3B_1^2 B_2 - 4B_1^2 - 8B_1 B_2 - 4B_2^2}
\]

and

\[
I''(x) = \frac{3B_1^2 + 3B_1^2 B_2 - 4B_1^2 - 8B_1 B_2 - 4B_2^2}{1152}.
\]

Since \( 3B_1^2 \geq 8B_2 \) and \( B_1 > 0 \), therefore numerator of \( x_0 \) is always positive. Also, note that, denominator \( x_0 \) and numerator of \( I''(x) \) is same, therefore sign of \( x_0 \) and \( I''(x) \) changes simultaneously. Here, two cases arise:
Case I: $0 < x_0 < 4$. In this case $I''(x) > 0$, so the minimum of $I(x)$ attains at $x_0$, which gives

$$T_{3.1}(f) \geq I(x_0) = 1 - \frac{B_1^2(B_1^3 + 12B_1^2 + 4B_1 - 32B_2)}{64(3B_1^3 + B_1^2(3B_2 - 4) - 8B_1B_2 - 4B_2^2)}.$$

Case II: $x_0 < 0$ or $x_0 > 4$. In this case $I(x)$ has no critical point. Thus

$$T_{3.1}(f) \geq \min\{I(0), I(4)\} = \min \left\{ 1 - \frac{B_1^2}{36}, 1 - \frac{B_1^2}{16} - \frac{B_1^2B_2}{48} - \frac{B_2^2}{36} \right\}.$$

For the case $x_0 = 4$, we have $T_{3.1}(f) \geq I(4)$. The sharpness of these bounds follows from the functions $\tilde{g}_2(z)$ and $\tilde{g}_3(z)$ defined by

$$\frac{(2z\tilde{g}_3(z))'}{(\tilde{g}_2(z) - \tilde{g}_2(-z))'} = \varphi(z) \quad \text{and} \quad \frac{(2z\tilde{g}_3(z))'}{(\tilde{g}_3(z) - \tilde{g}_3(-z))'} = \varphi(z^2),$$

respectively. Since

$$T_{3.1}(\tilde{g}_2) = 1 - \frac{B_1^2}{8} + \frac{B_1^2B_2}{48} - \frac{B_2^2}{36} \quad \text{and} \quad T_{3.1}(\tilde{g}_3) = 1 - \frac{B_1^2}{36},$$

which completes the proof.  

4. Some special cases

If $\varphi(z) = (1 + Az)/(1 + Bz)$, the classes $S^*_\varphi$ and $C_\varphi$ reduce to the classes $S^*_\varphi[A, B]$ and $C_\varphi[A, B]$, respectively. Theorems 3.5 and 3.6 immediately give the following sharp bound for the class $S^*_\varphi[A, B]$ and $C_\varphi[A, B]$.

Corollary 4.1. (1) If $f \in S^*_\varphi[A, B]$ and $A - B \leq |B^2 - AB|$, then $T_{3.1}(f) \leq 1$.
(2) If $f \in C_\varphi[A, B]$ and $A - B \leq |B^2 - AB|$, then $T_{3.1}(f) \leq 1$.

Theorems 3.7 and 3.8 yield the following lower bound of $T_{3.1}(f)$ for these classes.

Corollary 4.2. If $f \in S^*_\varphi[A, B]$ such that $A^2 - B^2 > 0$, then the following estimates hold:

$$T_{3.1}(f) \geq \begin{cases} 
\min \left\{ 1 - \frac{1}{4}(A - B)^2, 1 - \frac{1}{4}(A - B)^2(AB + 2) \right\}, & \sigma_1 \notin [0, 4], \\
1 - \frac{1}{4}(A - B)^2(AB + 2), & \sigma_1 = 4, \\
\frac{1}{16(1 - A)(1 - B)} \left( A^2 - 2A(B - 2) + B^2 + 4B - 4(A - B)^2 \right), & \sigma_1 \in (0, 4), 
\end{cases}$$

where

$$\sigma_1 = -\frac{2(A + B)}{(1 - A)(1 - B)}.$$ 

First two inequalities are sharp.
Corollary 4.3. If $f \in C_s[A, B]$ and $3A^2 + 2AB - 5B^2 \geq 0$, then the following estimates hold:

$$T_{3,1}(f) \geq \begin{cases} \frac{\min \left( 1 - \left( A - B \right)^2, 1 - \frac{(A - B)^2(B^2 + 3AB + 18)}{144} \right)}{1 - \frac{(A - B)^2(B^2 + 3AB + 18)}{144}}, & \sigma_2 \notin [0, 4], \\ 1 - \frac{(A^2 - 2A(B - 6) + B^2 + 20B + 4)(A - B)^2}{64(1 - B)(3A + B - 4)}, & \sigma_2 = 4, \\ 1 - \frac{(A^2 - 2A(B - 6) + B^2 + 20B + 4)(A - B)^2}{64(1 - B)(3A + B - 4)}, & \sigma_2 \in (0, 4), \end{cases}$$

where

$$\sigma_2 = \frac{2(3A + 5(B + 2))}{(1 - B)(3A + B - 4)}.$$ 

First two inequalities are sharp.

For $\varphi(z) = (1 + (1 - 2\alpha)z)/(1 - z)$ and $(1 + z)/(1 - z)$ in $S'_s(\varphi)$, we obtain the class $S'_s(\alpha)$ and Sakaguchi’s class, $S'_s$, respectively, where $\alpha \in [0, 1]$. For more detail of these classes, we refer to [16,22]. Theorems 3.5 and 3.7 yield the following sharp lower and upper bounds of $T_{3,1}(f)$ for these classes, proved by Kumar and Kumar [11].

Remark 4.4. (1) If $f \in S'_s(\alpha)$, then $(3 - 2\alpha)\alpha^2 \leq T_{3,1}(f) \leq 1$ [11, Theorem 2.2].

(2) If $f \in S'_s$, then $0 \leq T_{3,1}(f) \leq 1$ [11, Corollary 2.3].

For other subclasses of $S'_s$, the following sharp bounds follow from Theorem 3.7.

Corollary 4.5. If $f \in S'_{s,SG}$, then $T_{3,1}(f) \geq 2009/2304$.

Remark 4.6. (1) If $f \in S'_{s,L}$, then $T_{3,1}(f) \geq 221/256$ [11, Theorem 3.1].

(2) If $f \in S'_{s,RL}$, then $T_{3,1}(f) \geq (863 - 444\sqrt{2})/256$ [11, Theorem 3.3].

Theorems 3.6 and 3.8 give the following corollaries for different subclasses of $C_c$.

Corollary 4.7. (1) If $f \in C_s[A, B]$ and $A - B \leq |B^2 - AB|$, then $T_{3,1}(f) \leq 1$.

(2) If $f \in C_s(\alpha)$, then $T_{3,1}(f) \leq 1$.

(3) If $f \in C_s$, then $T_{3,1}(f) \leq 1$.

All these bounds are sharp.

Corollary 4.8. (1) If $f \in C_{s,SG}$, then $T_{3,1}(f) \geq 31/32$.

(2) If $f \in C_{s,L}$, then $T_{3,1}(f) \geq 4459/4608$.

(3) If $f \in C_{s,RL}$, then $T_{3,1}(f) \geq (-3731 + 5835\sqrt{2})/4608$.

All these bounds are sharp.

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