Natural Density and The Quantifier $Most$

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Abstract. This paper considers the quantified simple sentences by $Most$, sometimes referred to as proportional, sometimes the majority. The sentence form: $Most A are B$ where $A$ and $B$ are plural nouns. $A$ and $B$ range over elements of $P(\mathbb{N})$. Moreover, $A$ and $B$ may appear complemented (i.e., as $Non - A$ and $Non - B$). Two different but equivalent semantics are for $Most A are B$ as (i) $|A \cap B| > |A \setminus B|$ and (ii) $C(A \cap B) > \frac{C(A)}{2}$ where $C(X)$ is the cardinality of the set $X$. Both semantics work well on finite sets but exhibit problematic behaviors on infinite sets since division is undefined on cardinal arithmetic. Although semantics (i) is more descriptive than semantics (ii), it also produces insensitivity for certain sets. “Most” has a solid cardinal structure under the interpretation of the majority, and has the more statistical structure with proportional interpretation, and this statistical interpretation provides more flexible range of motion. For all these reasons, we introduce a new semantics with natural density for the sentences ranging over $\mathbb{N}$. We also give an axiomatization of this logic.

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1. Introduction

The fact that the cardinalities of the set of natural numbers and the set of positive even numbers are the same is surprising for most of beginner mathematicians. They can not keep themselves from thinking “how this equality can be when a set is a proper subset of the another”. It is a kind of adventure
in which intuition is difficult at first but later ends up with the mathematical satisfaction with coming from Cantor’s definition \[1\] which constructs bijective functions among infinite sets. Naturally, the issue of counting was that the viewpoint of the infinite sets with Cantor’s interpretation should be different from the finite sets. With this line, different philosophies and mathematical structures were needed to compare the sets according to their various properties. On the other hand, in order to be able to interpret the natural language mathematically and to make its fragments calculable, it is first necessary to emphasize what these fragments are meant to describe. While semantics of natural languages serve this understanding, it makes sense and representations on the finite parts of mathematics and models. When it comes to interacting with infinite parts of models of natural languages, we must revise our point of view as Cantor did, so that it is not too separated. In this sense we are introducing a blended semantics that flows from the finite to infinity, taking into account the features of the quantifier “most”. As Moss declared \[2\], we must not hesitate to apply applied logic and applied mathematics so that of the blessings of mathematics, natural language, and logic to solve each other’s problems.

2. “Most” and Its Historical Surroundings

Aristotle’s logical study involves the oldest formal work on logic. This respectful work is a frequent place where today’s logic studies are circulating. Although his objects belong to the real world, the effort of creating a rule base allows abstraction to have a significant place among the firsts, and forms the basis for this respect. His quantifiers (All, Some, and No) became the defining footprint of the language of the first order logic beyond propositional logic. These quantifiers were regarded as operators from Aristotle to Frege, but Frege treated them as a reasoning between relations. Thus, Frege became the first person to have quantifiers in a modern sense \[3\]. This relational treatment led not only to the old quantifiers but also to other quantifiers such as Not All, Many, More, At least \(n\), and At Most \(n\). Please see Sher’s publications \[4, 5\] for more detail about historical developments and notes of semantics and quantifiers. Other researchers on Aristotelian logic, are Łukasiewicz, Corcoran and Smiley, enlarged the traditional research topics into soundness and completeness of the ancient system \[6, 7, 8\]. Thompson extended the ancient system with “Many, More, and Most” faithful to the original and gave five-quantifier square of opposition \[9\]. Briefly, syllogistic theories have been taking place in wide applications of different areas such as in natural language theory and generalized quantifiers \[10, 12, 13, 14, 15, 17, 18\], in algebraic structures \[19, 20, 21, 22\], in formal logic \[23, 24, 25, 26, 27\], in unconventional systems \[28, 29, 30\], and on infinite sets \[31\].

Returning to the quantifier most, Endrullis and Moss introduced a syllogistic logic with the quantifier on finite sets \[32\]. They gave semantics of the sentence Most \(A\) are \(B\) in a given model as the following:
(1) *Most A are B* is true if and only if $C(A \cap B) > \frac{C(A)}{2}$

where $A$ and $B$ are plural common nouns, and also the function $C$ refers to be size of set. They stated that sentences by quantified *most* have strict majority graph property in proof search in this logic. On the other hand, Westerståhl’s interpretation of *most* and branching quantifiers [33, 34, 35] as follow:

(2) *Most A are B* is true if and only if $C(A \cap B) > C(A \setminus B)$

Notice that semantics (1) and (2) agree in every finite model. The author omits the generalization of the agreement because it is out of the scope of this paper and it is a philosophical and linguistic problem.

Zadeh in fuzzy logic and Hackl in formal semantics took the quantifier *most* as a proportional operator (determiner) [36, 37, 38, 39]. Hackl pointed out that:

The proportional quantifier most, in particular, supplied the initial motivation for adopting Generalized Quantifier Theory (GQT) because its meaning is definable as a relation between sets of individuals, which are taken to be semantic primitives in GQT.

and Zadeh pointed out that:

This representation, then, provides a basis for inference from premises which contain fuzzy quantifiers. For example, from the propositions *Most U’s are A’s and Most A’s are B’s*, it follows that *Most² U’s are B’s*, where *Most²* is the fuzzy product of the fuzzy proportion most with itself.

Consequently, Zadeh and Hackl showed a similar approach even though evaluating "*most*" under different interpretations.

All components discussed in this section were built on the finite models. We will concentrate on the behavior of "*most*" on infinite models and the meaning of "*most*" in the next sections.

3. Finiteness, Infinity and Quantifiers

Throughout history, the notion of infinity has been one of the focal points of philosophers, mathematicians, and even physicists. The existence of infinity, belonging to infinity, being a part of infinity, larger or smaller infinities, types of infinity are still controversial issues today. Every science has tried to find strident or flexible answers to these questions from time to time in their own discipline. Infinity must exist according to Aristotle but potential not actual. So, infinity is divided into two in his account of infinity as potential and actual. The actual infinite contains unending sets or “things” in space that is a has start and end but a collection of infinitely many members. The potential infinite is a group of numbers or “things” that repeat itself without a recognizable ending point. In this sense, although Aristotle did not know $\aleph_0$ and natural numbers, he showed such an approach with his potential infinity.
(for more detail [40] [41] [42]). When we return to today’s science, there are Cantor based definitions [1] for the distinction between finite and infinite sets and for the comparison operations described on them. Of course, it is easy to arrive at precise, understandable and computable judgments while making comparisons of finite sets. It is quite easy to decide on the relationships between them, such as intersection, union, difference, cardinality comparison, etc., when the sets have the finite number of elements. There are confusing difficulties in the comparison of sets as they are infinite. It is necessary to obtain functions such as those described by Cantor in order to count elements of these sets and to compare element numbers. Formally, two sets $A$ and $B$ have the same cardinality, written $|A| = |B|$, if there exists a bijective function $f : A \rightarrow B$. If no such bijective function exists, then the sets have unequal cardinalities, that is, $|A| \neq |B|$. In this sense, there are bijections between the set of odd $(\mathbb{O})$ or even positive numbers $(\mathbb{E})$ and set of natural numbers whereas these sets appear to be half the size of the set of natural numbers $\mathbb{N}$. Having the same number of elements of $\mathbb{E}$ (or $\mathbb{O}$) and $\mathbb{N}$ is the reflection of Cantor’s definition of bijection even though there is a odd (even) number between both pairs (or $\mathbb{O}$) $\mathbb{E}$ in $\mathbb{N}$. Another example for cardinal equality is set $\mathbb{K} = \{3k + 1 : k \in \mathbb{N}\}$, that is $\text{Card}(\mathbb{K}) = \aleph_0$ again.

The classical quantifiers called polyadic in modern sense [16] All, Some, and No examine inclusion, intersection, and disjointness of sets, respectively. These examinations do not discriminate sets to have finite or infinite cardinality and give precise ideas. Other non-classical polyadic quantifiers such as More Than and At least that considers cardinality of sets use aleph definitions on infinite sets by Cantor and known classical set size comparison on the finite sets. As to “most”, it compares both intersection and cardinality together.

A flexible way of saying “strictly more than half” to avoid giving exact numbers on a daily basis is used the quantifier “most”. Although it is desirable to establish flexible clauses, it is basically an intuitive to compare a cardinality. It would not be necessary to test for cardinality if only the intersection test was observed. There would be no need to test for the intersection if only the cardinality comparisons would have been observed. Even the comparisons we talked about would not make any decision on the finite sets. But if we are talking about infinite sets, we should be more careful when using the quantifier most. The size of $\mathbb{K}$ is the same as the size of $\mathbb{N}$, although only one of the three natural numbers belongs to $\mathbb{K}$ as we have already mentioned. If we consider the set of natural numbers as $\{0, 1, 2\}, \{3, 4, 5\}, \{6, 7, 8\},...$as subsets in turn, we find only one element of $\mathbb{K}$ in each subset of $\mathbb{N}$. We could say intuitively that in this case $\mathbb{K}$ takes up space in $\mathbb{N}$ with the ratio of one to three. On the other hand, if we consider the complement set of $\mathbb{K}$, $\mathbb{K}^c = \{0, 2, 3, 5, 6, 8,...\}$, takes up space in $\mathbb{N}$ with ratio of two to three. In other words, the difference between the two consecutive elements of $\mathbb{K}$, as number theorists have said “gap”, is always 3. That is, there are 3 elements
belonging to $\mathbb{N}$ sets but not $\mathbb{K}$’s. We would like to say that most of the elements of $\mathbb{N}$ are elements of $\mathbb{K}^c$ as well due to the ratio of two to three. But we can not say it because $\mathbb{K}^c \setminus \mathbb{N} = \emptyset$ and half of cardinality of $\mathbb{N}$ is undefined. Furthermore, most of $\mathbb{N}$ is not $\mathbb{K}^c$ even though $\mathbb{K}^c \subseteq \mathbb{N}$. Let’s give some more examples and deepen the issue. We will use $\text{Most}(A, B)$ instead of $\text{Most} A$ are $B$ for the abbreviation and saving space.

Example 1. Let $\mathbb{N}^-$ denote $\mathbb{N} \setminus \{1, 2, 3\}$.

(i) $\text{Most}(\mathbb{N}, \mathbb{N})$ is false with (1) because $\frac{C(\mathbb{N})}{2} = \aleph_0$.

(ii) $\text{Most}(\mathbb{N}, \mathbb{N}^-)$ is false with (1) because $\frac{C(\mathbb{N})}{2} = \aleph_0$.

(iii) $\text{Most}(\mathbb{N}^-, \mathbb{N})$ is false with (1) because $\frac{C(\mathbb{N}^-)}{2} = \aleph_0$.

(iv) $\text{Most}(\mathbb{N}, \mathbb{K})$ is false with semantics (1) because $\frac{C(\mathbb{N})}{2} = \aleph_0$.

(v) $\text{Most}(\mathbb{K}, \mathbb{N})$ is false with semantics (1) because $\frac{C(\mathbb{K})}{2} = \aleph_0$.

Notice that additions/subtractions finite elements to/from $\mathbb{N}$, even a “most”-based comparison of $\mathbb{N}$ with itself by semantics (1) leads to wrong evaluations since division of $\aleph_0$ by a finite number is equal to $\aleph_0$ as can be seen from (i) to (v) in Example 1.

Example 2. Let $\mathbb{N}^-$ denote $\mathbb{N} \setminus \{1, 2, 3\}$ and $\mathbb{K}$ denote $\{3k + 1 : k \in \mathbb{N}\}$ again.

(i) $\text{Most}(\mathbb{N}, \mathbb{N})$ is true with semantics (2) because $C(\mathbb{N} \cap \mathbb{N}) = \aleph_0 > C(\mathbb{N} \setminus \mathbb{N}) = 0$.

(ii) $\text{Most}(\mathbb{N}, \mathbb{N}^-)$ is true with semantics (2) because $C(\mathbb{N} \cap \mathbb{N}^-) = \aleph_0 > C(\mathbb{N} \setminus \mathbb{N}^-) = 3$.

(iii) $\text{Most}(\mathbb{N}^-, \mathbb{N})$ is true with semantics (2) because $C(\mathbb{N}^- \cap \mathbb{N}) = \aleph_0 > C(\mathbb{N}^\setminus \mathbb{N}) = 0$.

(iv) $\text{Most}(\mathbb{N}, \mathbb{K}^c)$ is false with semantics (2) because $C(\mathbb{N} \cap \mathbb{K}^c) = \aleph_0 = C(\mathbb{N} \setminus \mathbb{K}^c) = \aleph_0$.

The statement works well from (i) to (iv) in Example 2 with semantics (2). On the other hand, semantics (2) does not help for a dependable comparison unless there is a finite difference between the compared sets. We can see that semantics (2) is more advantageous than semantics (1) although (iv) in Example 2 is supposed to be true.

Proposition 3.1. If $A$ and $B$ are non-empty finite sets and $A \subseteq B$, then $\text{Most}(A, B)$ is true with semantics (1) and (2).

Proof. Trivial. □

Proposition 3.2. If $A$ and $B$ are infinite sets and $A \subseteq B$, then $\text{Most}(A, B)$ is not true always with semantics (1) and (2).
Proof. It is sufficient to give a counter-example. Suppose that \( B = \mathbb{N} \) and \( A = \mathbb{K}^c \). Then the equality \( C(\mathbb{K}^c) = C(\mathbb{K}) \) falsifies the statement with semantics (2), and also we have \( \frac{C(\mathbb{K}^c)}{2} \) for semantics (1). Unfortunately, division operation is not defined in cardinal arithmetic if we assume the Axiom of Choice is true.

Proposition 3.3. Most\((A, A)\) is true with semantics (1) and (2) if \( A \) is a non-empty finite set.

Proof. Trivial.

Proposition 3.4. Most\((A, A)\) is false with semantics (1) and, true with semantics (2) if \( A \) is an infinite set.

Proof. Trivial.

Proposition 3.5. Most\((A, B)\) is false with semantics (1) if \( A \) is an infinite set.

Proof. Trivial.

Proposition 3.6. Let \( A \) and \( B \) be non-empty infinite subsets of \( \mathbb{N} \) and \( C(A \cap B) = \aleph_0 \). Most\((A, B)\) is true with semantics (1) and (2) if \( A \setminus B \) is finite.

Proof. It is easy to see that if \( A \setminus B \) is finite, the \( \aleph_0 > n \) for any finite number.

As can be seen the propositions from 2.1 to 2.6 and the examples, semantics (1) gives much more meaningful results than semantics (2) in many respects, but the resulting misrepresentations and inexplicable results force to introduce a new semantics for our statement. On the other hand, it is often the case that we claim that sentences are false because the semantics are mathematically undefined under defined operations. In such a case, it is a matter of losing from the beginning. How Cantor reflects on his works that the operations defined on finite and infinite sets are separated from each other and must have different aspects of view, we would not be able to exhibit this kind of behavior for the “most” that does not have a cardinality account with uncompleted meaning, unlike other quantifiers here.

4. A Semantics: Natural Density

With Cantor interpretation, injective or bijective functions must be obtained to find the cardinality of the subsets of \( \mathbb{N} \). The fact that functions of this type are dependent on a sequence, that is, has a production rule, is very important for the determination of the existence of the functions to be found. In other words, finding bijective functions from sets of arithmetic progressions to \( \mathbb{N} \) is pretty easy because the progressions consider the sequence of numbers such that the difference of any two successive members is constant. Thus, one can make a decision what cardinality of these sets are. \( \mathbb{K} \) is an example of arithmetic progression and its cardinality is \( \aleph_0 \). As we saw that Most\((\mathbb{N}, \mathbb{K}^c)\)
is false by both semantics (2). However, ratio of the space occupied by \( K, \frac{1}{3} \), in \( N \) is less than ratio of the space occupied by \( K^c, \frac{2}{3} \), in \( N \). Most of the elements of \( N \) must belong to \( K^c \) with these ratios. That is, more than half of \( N \) must be \( K^c \). The ratio \( \frac{2}{3} \) is a meaningful as it is more than \( \frac{1}{2} \). Furthermore, it is much more meaningful if we evaluate the space occupied by \( N \) in \( N \) to be 1. However, we have \( \aleph_0 = c(K^c) > c(K) = \aleph_0 \) by the Cantorian approach, and this does not correspond to the spirit of “most”. As a result, a new semantics so-called natural density (asymptotic density) is proposed in the context of complete inadequacy of semantics (1), and advantages and disadvantages of semantics (2), and the richness and the compatibility of meaning provided by the concept of gap and ratio.

Natural density \[\text{[46]}\] is a common method to measure the size of a subset of the natural numbers, unlike Cantor’s approach. In other words, the natural density is one of the possibilities to measure how large a subset of the set of natural numbers is. There are other densities such as logarithmic, weighted, uniform, exponential and generalizations for different purposes. We assume that the number 0 does not belong to \( N \) in order to adhere our position as number and model (set)-theoretic approach, i.e., we will take \( N \) as a set of positive integers. Here, we give some definitions and properties of natural density and then we discuss the new semantics.

**Definition 4.1.** Set \( A \) is to set \( B \), written \( A \sim B \), means that symmetric difference \( A \triangle B \) is finite.

**Definition 4.2.** \( A \subseteq N \) such that

\[
 d(A) = \lim_{n \to \infty} \frac{| A \cap \{1, 2, ..., n\} |}{n}
\]

if the limit exists, \( d(A) \) is called natural density of \( A \).

It is a kind of “measure” attributing the exact value to an (infinite) arithmetic sequence of natural numbers like \( d(\{k, 2k, 3k, 4k, ...\}) = \frac{1}{k} \) for each finite natural number \( k \). Definition \[\text{[4.2]}\] emphasizes that the natural density is a limit that may do not exist. For this reason, we are doing research on the sets and models that have the density and we will continue assuming that each set has a density. This may seem like a weakness of the semantics we will exhibit, but the difficulties of generalizability of sets whose density cannot be calculated (or not exist) are also obvious. That is, the Cantorian approach has difficulties in terms of cardinality calculations and other problems of these sets. Therefore, the current situation forces the sets to be arithmetic in general. There are, of course, sets that are not arithmetic but have a density, and we will not ignore these sets for sampling. There are also more valid and detailed definitions, such as lower and upper, in this type of density in number theoretical and statistical (probabilistic) sense. Natural density has some axioms as follows:

Let \( d : P(\mathbb{N}) \to [0, 1] \) and \( A, B \) in \( \mathbb{N} \). \[\text{[33]}\]
(1) $0 \leq d(A) \leq 1$, for all $A$.
(2) $d(\mathbb{N}) = 1$ and $d(\emptyset) = 0$.
(3) If $A \sim B$, then $d(A) = d(B)$.
(4) If $A \cap B = \emptyset$, then $d(A) + d(B) \leq d(A \cup B)$.
(5) $d(A) + d(B) \leq 1 + d(A \cap B)$ for all $A, B$.

The notion of asymptotic in Definition 4.1 is one of the most important points of our study. This notion and also (iii) and (iv) in Example 2 are compatible because both are true, we will see that the finite differences of both $\mathbb{N} \setminus \mathbb{N}^-$ and $\mathbb{N}^- \setminus \mathbb{N}$ are true for semantics (2) and the axiom (3).

Useful properties. Some of the following properties, which are a result of the works of Grekos [44], Bucks [45] and Niven [47], will support our study.

(i) $d(A) = 1 - d(A^c)$.
(ii) $d(A) = 0$, if $A$ is a finite subset of $\mathbb{N}$.
(iii) $d(A \cap B) = \min(d(A), d(B))$.
(iv) if $A \subseteq B$, then $d(A) \leq d(B)$.

It is not the main purpose of this paper to establish a logical system but we will mention the completeness and soundness results. We will build a model to introduce the language and the semantics.

Syntax. We will use the following types of expressions to keep our language and the expressions close to syllogistic form. We start with a set of variables $A, B, ..., \text{and also their complements} (A^c \text{ as } non - A, B^c \text{ as } non - B, ...)$ representing plural common nouns. We also have name $U$. Then we consider sentences the following very restricted forms:

Most $A$ are $U$, Most $U$ are $B$, Most $U$ are $U$.

We call this language $\mathcal{L}$(Most, $d$).

Semantics. We are ready to introduce the new semantics. Let $U$ be an universe that is $\mathbb{N}$ and for all variable $A$ in the language, $[[A]] \subseteq U$ infinite sets which have natural density. We assume the sets in the universe that are both themselves and their complements are infinite. An interpretation function satisfies axioms and theorems of natural density $d : P(\mathbb{N}) \to [0, 1]$ for all subsets of $U$. Note that we refer to all subsets of $U$ that have natural density. The semantics allows to be the intersection $[[A]] \cap [[B]]$ and the set difference $[[A \setminus B]] = [[A]] \setminus [[B]]$ for each noun $A$ and $B$. This gives a model $\mathcal{M}(U, d)$ (in short, $\mathcal{M}$). Then we define truth in model so that at least one of $A$ or $B$ will be $U$ as the follow:

$\mathcal{M} \models Most(A,B)$ if and only if $d([[A]] \cap [[B]]) > d([[A]] \setminus [[B]])$.

To remember, $Most(\mathbb{N}, K^c)$ is false with semantics (2) because $C(\mathbb{N} \cap K^c) = \aleph_0 = C(\mathbb{N} \setminus K^c) = \aleph_0$ in (iv) of Example 2. We are looking for the answer to the question with this new semantics in which semantics (2) can not answer. The question returns whether the inequality $d([[\mathbb{N}] \cap [[K^c]]) > d([[\mathbb{N}] \setminus [[K^c]])$ is true with the new semantics, the sentence “Most of $\mathbb{N}$ are $\mathbb{K}^c$”, which we already think is intuitively correct. The intersection $[[\mathbb{N}]] \cap [[[K^c]]]$ is equal to
If the asymptotic density of \([K^c]\) is not known, that is \(d([K^c])\), we have already a useful propriety, (i), \(d(A) = 1 - d(A^c)\). Then, \(d([K^c]) = 1 - d([K^c]) = 1 - d([K])\). It is easy to compute the asymptotic destiny of \([K] = \frac{1}{3}\), and hence \(d([K^c]) = \frac{2}{3}\). We have \(d([K^c]) > d([K])\) by simplifying the inequality. Finally, we obtain \(\frac{2}{3} > \frac{1}{3}\). We saw that \(Most(A,B)\) is true by both semantics (1) and (2) if \(A \setminus B\) is finite in Proposition 3.6. On the other hand, we have removed the necessity of this situation with this semantics because \([N] \setminus [K^c]\) is not finite. So we have showed the correctness of this sentence with the new semantics. Doubtless, this semantics does not work on finite sets. Taking a set \(A = \{1, 2, 3\}\), neither \(Most(U, [A])\) nor \(Most(A, U)\) is possible to be true since \(d([U] \cap [A]) = d([U] \cap [A]) = 0\).

Another important point is that the new semantics also does work with sets whose complements are finite as mentioned is in (2) and (3) of Example 2. Truths of \(Most(N, N^-)\) and \(Most(N^-, N)\) are satisfied with the new semantics. Indeed, \(d([N] \cap [N^-]) = 1\) and \(d([N^-] \cap [N]) = 1\), and also \(d([N] \setminus [N^-]) = 0\) and \(d([N^-] \setminus [N]) = 0\).

An extra example could be the set of prime numbers. It is well-known that the occupation proportion of primes (say \(P\)) in all natural numbers is 0. The semantics shows that \(Most(N, P)\) and \(Most(P, N)\) are false as it is supposed to be. Furthermore, the truths of “most natural numbers are non-prime numbers” and “most non-prime numbers are natural numbers” are supported by the semantics. The primes could possibly be an exit gate for larger studies of the sets have not any arithmetic progression because the primes contain arbitrarily long arithmetic progression \([15]\).

**Remark 4.3.** Indeed, we forced the sets (nouns in the language) in the universe that are both themselves and their complements to be infinite by saying for all variable \(A\) in the language, \([A] \subseteq U\) infinite sets ” because as we mentioned that the truth of the sentences \(Most(N, A)\) and \(Most(A, N)\) are always false where \(A\) is finite. If we allow the sets or their complements to be finite, then we contradict our assumption “for all variable \(A\) in the language, \([A] \subseteq U\) infinite sets ” because we possibly would use a finite set in a sentence of the language. This situation would cause a sentence belonging to a given set of sentences to be false. Therefore, the system would not be sound. We have another chance to construct the language. We would allow only infinite sets to be in the set of premises and the finite sets to be in the derived sentences. Tough \(Most(N^-, N)\) and \(Most(N, N^-)\) are the favorable sentences, we did not prefer to include the sentences in the language for these reasons. If we allow the finite set interpretations to be in the derived sentences, the logical system will not change but the model construction and the completeness proof since the finite sets will be taken account.

It is worth remembering some definitions. We say that \(M \models \Gamma \iff M \models \varphi\) for every \(\varphi \in \Gamma\). We say that \(\Gamma \models \varphi\) iff for all \(M\): if \(M \models \Gamma\), then also \(M \models \varphi\). We read this as \(\Gamma\) logically implies \(\varphi\), or \(\Gamma\) semantically implies \(\varphi\), or that \(\varphi\)
is a semantic consequence of $\Gamma$. The proof system is sound for the semantics if whenever $\Gamma \vdash \varphi$, we also have $\Gamma \models \varphi$. The proof system is complete for the semantics if whenever $\Gamma \models \varphi$, we also have $\Gamma \vdash \varphi$.

The main semantic definition is the *consequence relation*. We take it that $\Gamma \vdash \varphi$ should mean that for all $M$, if all sentences in $\Gamma$ are true in $M$, then so is $\varphi$. For the natural density function $d$, if all sentences in $\Gamma$ are true in $M$, then so is $\varphi$.

\[
\begin{align*}
\text{Most}(A, U) & \quad \text{(Axiom)} \quad \text{Most}(A^c, U) \quad \text{Most}(A, U) \\
& \quad \varphi \quad \quad (X_1) \\
\text{Most}(U, A^c) \quad \text{Most}(U, A) & \quad \varphi \quad \quad (X_2)
\end{align*}
\]

**Figure 1.** The Logic of $\mathcal{L}(\text{Most}, d)$

Figure 1 says that if $\Gamma$ is consistent and $\Gamma \vdash \text{Most}(N, A)$, then $\Gamma \not\vdash \text{Most}(N, A^c)$ (for $(X_1)$), and if $\Gamma$ is consistent and $\Gamma \vdash \text{Most}(A, N)$, then $\Gamma \not\vdash \text{Most}(A^c, N)$ (for $(X_2)$).

Figure 1 has three rules, the one is an axiom and the others are *Ex falso quadlibet* rules since the natural density are not defined for comparisons in which $N$ are not included. Thus, for every sentence form $\varphi$ except $\text{Most}(A, U)$ (Axiom) and a consistent $\Gamma$, $\Gamma \vdash \varphi$ if and only if $\varphi \in \Gamma$, and equivalently, $\Gamma \not\vdash \varphi$ if and only if $\varphi \not\in \Gamma$, $\Gamma \models \varphi$ for all $\varphi$ in $\Gamma$. After that point, it is obvious to see that “$\Gamma \vdash \varphi$ if and only if $M \models \varphi$” because $\Gamma \models \varphi$ for all $\varphi$ in $\Gamma$, which also means the completeness.

5. Conclusion

The semantics in this paper is in the scope of natural density. We have applied the proportional construction of the quantifier “Most” to the natural asymptote which is defined on the subsets of countable infinite sets. We have shown that this semantics works much better than the semantics based on the cardinality of the set difference and only based on the half of the set cardinality. Even though the semantics that considers the cardinality of the set difference work much better than the semantics based solely on the half of cardinality of the set, we have seen that this semantics does not work on the sets which both itself and its complement are infinite. Moreover, the new semantics work well on this type of sets. We also have given an axiomatization of $\mathcal{L}(\text{Most}, d)$ at end of the paper.

The author could not encounter any analogical logical or linguistic studies in the literature to make comparison with this paper on a controversial
ground until completed the paper. If we do not have any logic but we have a set of sentences, what do we think the semantics is mathematically or logically compatible with its sentence? As Corcoran said in the 1973 of the paper [49], these “logics” are models.

The results here should extend many with syllogistic systems, the classical boolean operations and cardinality comparisons.

It is considerable that other density definitions such as “Dirichlet, logarithmic, weighted, uniform, exponential and generalizations” depending on the different meanings (purposes) of the different quantifiers. This study is limited on sets with natural density. There is still no different technique to study sets that do not have this density. But this study can be extended with computably enumerable sets.

Some may find these results to be surprising. Hopefully, mathematicians, linguists, computer scientists, philosophers, and logicians might be interested in the results which may help with other results in several areas.

[51] can be had a look at for a discussion that does not involve natural density but general.

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