Seiberg-Witten Geometry with Various Matter Contents

Seiji Terashima and Sung-Kil Yang

Institute of Physics, University of Tsukuba
Ibaraki 305-8571, Japan

Abstract

We obtain the Seiberg-Witten geometry for four-dimensional $N = 2$ gauge theory with gauge group $SO(2N_c) \ (N_c \leq 5)$ with massive spinor and vector hypermultiplets by considering the gauge symmetry breaking in the $N = 2$ $E_6$ theory with massive fundamental hypermultiplets. In a similar way the Seiberg-Witten geometry is determined for $N = 2$ $SU(N_c) \ (N_c \leq 6)$ gauge theory with massive antisymmetric and fundamental hypermultiplets. Whenever possible we compare our results expressed in the form of ALE fibrations with those obtained by geometric engineering and brane dynamics, and find a remarkable agreement. We also show that these results are reproduced by using $N = 1$ confining phase superpotentials.
1 Introduction

In the seminal paper by Seiberg and Witten (SW), it was discovered that the low-energy behavior of \( N = 2 \) \( SU(2) \) supersymmetric gauge theory in four dimensions is described in terms of the geometry associated with the Riemann surface \([1]\). Extensions of their work to the case of other gauge groups were carried out by several groups \([2]-[10]\). At first sight it was unclear if the Riemann surface in the exact description is an auxiliary object for mathematical setup or a real physical object. It turns out that four-dimensional \( N = 2 \) gauge theory on \( \mathbb{R}^4 \) is realized by an M-theory fivebrane on \( \mathbb{R}^4 \times \Sigma \) embedded in \( \mathbb{R}^{10} \times S^1 \), where \( \Sigma \) is the SW Riemann surface \([11, 12]\).

Gauge theories associated with the configuration \( \mathbb{R}^4 \times \Sigma \) can be analyzed by the brane dynamics \([13]\). So far \( N = 2 \) gauge theories with classical gauge groups containing flavor matters in the fundamental representation have been understood successfully in this framework \([14]\). Toward further developments it is highly desirable to be able to describe matter contents in various representations as well as exceptional gauge symmetry. However, it is still difficult to explain the exceptional gauge symmetry along the lines of \([12]\). Concerning the matter representations other than the fundamentals, our analysis has so far been restricted to the case of \( N = 2 \) \( SU(N_c) \) gauge theory with matters in the symmetric or antisymmetric representations \([15, 16]\). The difficulty lies not only due to the lack of precise knowledge of the brane dynamics, but also due to the lack of the field theory answer.

Our purpose in this paper is to present some field theory answer to the above issue. Staring with the \( N = 2 \) SW geometry with \( E_6 \) gauge group with massive fundamental matters, which we proposed in the previous communication \([17]\), we construct the SW geometry with \( SU(N_c) \) and \( SO(N_c) \) gauge groups with various matter contents. All these geometries we will obtain take the form of a fibration of the ALE spaces over a sphere.

This paper is organized as follows. In sect.2, we explain in detail how to implement the gauge symmetry breaking in the SW geometry by giving appropriate VEV to the adjoint scalar field in the \( N = 2 \) vector multiplet.

In sect.3, breaking the \( E_6 \) symmetry down to \( SO(2N_c) \) \( (N_c \leq 10) \), we derive the SW geometry for \( N = 2 \) \( SO(2N_c) \) theory with massive spinor and vector hypermultiplets. In
the massless limit, our \( SO(10) \) result is in complete agreement with the one obtained by the method of geometric engineering [18]. For \( SO(8) \) it is amusing that the SW geometry with massive spinor and vector matters is symmetric under a part of the \( SO(8) \) triality which exchanges the vector and spinor representations.

In sect.4, the breaking of \( E_6 \) to \( SU(N_c) \) \( (N_c \leq 6) \) is considered. The SW geometry we will find naturally takes the form of the ALE space description. On the other hand, the brane dynamics relevant for \( N = 2 \) \( SU(N_c) \) theory with antisymmetric matters yields the SW geometry which seems apparently distinct from our expression [15, 16]. We will show, however, that the singularity structure exhibited by the complex curve in [15, 16] is also realized in our ALE space description.

In sect.5, it is shown that the results obtained in the previous sections can be rederived by the use of the method of \( N = 1 \) confining phase superpotentials.

Finally in sect.6, we draw our conclusions.

2 Gauge symmetry breaking in Seiberg-Witten geometry

Let us consider four-dimensional \( N = 2 \) supersymmetric gauge theory with gauge group \( G \) and \( N_f \) flavors of \( N = 2 \) hypermultiplets which consist of \( N = 1 \) chiral multiplets \( Q^i, \tilde{Q}_j \) \( (1 \leq i, j \leq N_f) \). The \( N = 2 \) vector multiplet contains an \( N = 1 \) adjoint chiral multiplet \( \Phi \). Let \( Q \) belong to an irreducible representation \( R \) of the gauge group \( G \) with dimension \( d_R \) and \( \tilde{Q} \) to the conjugate representation of \( R \). The tree-level superpotential of this theory is determined by the \( N = 2 \) supersymmetry

\[
W = \sqrt{2} \sum_{i=1}^{N_f} \tilde{Q}_i \Phi_R Q^i + \sqrt{2} \sum_{i=1}^{N_f} m_i \tilde{Q}_i Q^i, \tag{1}
\]

where \( \Phi_R \) is a \( d_R \times d_R \) matrix representation of \( \Phi \) in \( R \) and \( m_i \) is a mass of the \( i \)-th hypermultiplet.

It is convenient for subsequent considerations to fix our notation for the root system. The simple roots of \( G \) are denoted as \( \alpha_i \) where \( 1 \leq i \leq r \) with \( r \) being the rank of \( G \). Any root is decomposed as \( \alpha = \sum_{i=1}^r a^i \alpha_i \). The component indices are lowered by \( a_i = \sum_{j=1}^r A_{ij} a^j \) where \( A_{ij} \) is the Cartan matrix. The inner product of two roots \( \alpha, \beta \) is
then defined by

$$\alpha \cdot \beta = \sum_{i=1}^{r} a_i b_i = \sum_{i,j=1}^{r} a_i A_{ij} b_j,$$

where $\beta = \sum_{i=1}^{r} b_i \alpha_i$.

Classically the VEV of the adjoint Higgs $\Phi$ is chosen to take the values in the Cartan subalgebra. The classical moduli space is then parametrized by a Higgs VEV vector $a = \sum_{i=1}^{r} a_i \alpha_i$. At the generic points in the classical moduli space, the gauge group $G$ is completely broken to $U(1)^r$. However there are singular points where $G$ is broken only partially to $\prod_i G'_i \times U(1)^l$ with $G'_i$ being a simple subgroup of $G$. If we fix the gauge symmetry breaking scale to be large, the theory becomes $N = 2$ supersymmetric gauge theory with the gauge group $\prod_i G'_i \times U(1)^l$ and the initial SW geometry reduces to the one describing the gauge group $G'_i$ after taking an appropriate scaling limit.

We begin with the case of $N = 2$ supersymmetric $SU(r+1)$ gauge theory with fundamental flavors. The SW curve for this theory is given by

$$y^2 = \text{det}_{r+1} (x - \Phi R)^2 - \Lambda_{2(r+1)-Nf} N_f \prod_{i=1}^{N_f} (m_i - x).$$

Choosing the classical value $\langle \Phi R \rangle_{cl}$ as

$$\langle \Phi R \rangle_{cl} = \text{diag} \left( \langle a^1 \rangle, \langle a^2 \rangle - \langle a^1 \rangle, \langle a^3 \rangle - \langle a^2 \rangle, \cdots, \langle a^r \rangle - \langle a^{r-1} \rangle, -\langle a^r \rangle \right) = \text{diag}(M, M, M, \cdots, M, -rM),$$

where $M$ is a constant, we break the gauge group $SU(r+1)$ down to $SU(r) \times U(1)$. Note that this parametrization is equivalent to $\langle a^j \rangle = jM$ which means $\langle a_j \rangle = \delta_{j,r}(r+1)M$. Setting $a_i = \delta_{j,r}(r+1)M + \delta a_i$ and $m_i = M + m'_i$, we take the scaling limit $M \to \infty$ with $\Lambda^{2r-N_f} = \frac{\Lambda^{2(r+1)-N_f}}{(r+1)M^2}$ held fixed. Then we are left with the SW curve corresponding to the gauge group $SU(r)$

$$(y')^2 = \left\{ \left( x' - \delta a^1 \right) \left( x' - (\delta a^2 - \delta a^1) \right) \cdots \left( x' - (-\delta a^{r-1}) \right) \right\}^2 - \Lambda^{2r-N_f} \prod_{i=1}^{N_f} (m'_i - x'),$$

where $y' = \frac{y}{\sqrt{r+1}M}$ and $x' = x - M$. Notice that we must shift the masses $m_i$ to obtain the finite masses of hypermultiplets in the $SU(r)$ theory with $N_f$ flavors.

Now we consider the case of $N = 2$ theory with a simple gauge group $G$. When we assume the nonzero VEV of the adjoint scalar, the largest non-Abelian gauge symmetry
which is left unbroken has rank $r - 1$. As we will see shortly, this largest unbroken gauge symmetry is realized by choosing

$$\langle a_i \rangle = M \delta_{i,i_0}, \quad 1 \leq i \leq r,$$

(6)

where $M$ is an arbitrary constant and $i_0$ is some fixed value. Under this symmetry breaking (6), a gauge boson which corresponds to a generator $E_b$, where the subscript $b = \sum_i b^i \alpha_i$ indicates a corresponding root, has a mass proportional to $\langle a \rangle \cdot b = M b^{i_0}$. This is seen from $[(\langle a \rangle \cdot H, E_b) = (\langle a \rangle \cdot b) E_b$ where $H_i$ are the generators of the Cartan subalgebra. Thus the massless gauge bosons correspond to the roots which satisfy $b^{i_0} = 0$ and the unbroken gauge group becomes $G' \times U(1)$ where the Dynkin diagram of $G'$ is obtained by removing a node corresponding to the $i_0$-th simple root in the Dynkin diagram of $G$. The Cartan subalgebra of $G$ is decomposed into the Cartan subalgebra of $G'$ and the additional $U(1)$ factor. The former is generated by $E_{\alpha_k} \in G$ obeying $[E_{\alpha_k}, E_{\alpha_{-k}}] \simeq \alpha_k \cdot H$ with $k \neq i_0$, while the latter is generated by $\alpha_{i_0} \cdot H$. Therefore, we set

$$a^i = (A^{-1})^{i_0} M + \delta a^i,$$

(7)

where scalars corresponding to $G'$ have been denoted as $\delta a$ with $\delta a^{i_0} = 0$. Note that the $U(1)$ sector decouples completely from the $G'$ sector and the Weyl group of $G'$ naturally acts on $\delta a$ out of which the Casimirs of $G'$ are constructed.

When the gauge symmetry is broken as above, we have to decompose the matter representation $R$ of $G$ in terms of the subgroup $G'$ as well. We have

$$\mathcal{R} = \bigoplus_{s=1}^{n_R} \mathcal{R}_s,$$

(8)

where $\mathcal{R}_s$ stands for an irreducible representation of $G'$. Accordingly $Q^i$ is decomposed into $Q^i_s (1 \leq i \leq N_f, 1 \leq s \leq n_{\mathcal{R}})$ in a $G'$ representation $\mathcal{R}_s$. $\tilde{Q}_i$ is decomposed in a similar manner. After the massive components in $\Phi$ are integrated out, the low-energy theory becomes $N = 2 \ G' \times U(1)$ gauge theory. The $U(1)$ sector decouples from the $G'$ sector and we consider the $G'$ sector only. The semiclassical superpotential for this theory can be read off from (3). We have

$$W = \sum_{i=1}^{N_f} \left( \sqrt{2} \sum_{s=1}^{n_R} \langle a \rangle \cdot \lambda_{\mathcal{R}_s} + m_i \right) \tilde{Q}_{is} Q^i_s + \sqrt{2} \sum_{s=1}^{n_R} \tilde{Q}_{is} \Phi_{\mathcal{R}_s} Q^i_s,$$

(9)
where $\lambda_{R^s}$ is a weight of $R$ which branches to the weights in $R^s$. This implies that we should shift the mass $m_i$ as

$$m_i = -\langle a \rangle \cdot \lambda_{R^s_i} + m_i' = -M \left( \lambda_{R^s_i} \right)^i_0 + m_i'$$

(10)

to obtain the $G'$ theory with appropriate matter hypermultiplets. Note that we can choose $R^s_i$ for each hypermultiplet separately. This enables us to obtain the $N_f$ matters in different representations of $G'$ from the $N_f$ matters in a single representation of $G$. In the limit $M \to \infty$, some hypermultiplets have infinite masses and decouple from the theory. Then the superpotential (9) becomes

$$W = \sqrt{2} \sum_{i=1}^{N_f} m_i' \tilde{Q}_{i, s_i}^i Q_{s_i}^i + \sqrt{2} \sum_{i=1}^{N_f} \tilde{Q}_{i, s_i}^i \Phi_{R^s_i} Q_{s_i}^i,$$

(11)

and the resulting theory becomes $N = 2$ theory with gauge group $G'$ with hypermultiplets belonging to the representation $R^s_i$. Note that $\langle a \rangle \cdot \lambda_{R^s}$ is proportional to its additional $U(1)$ charge.

In the known cases, the low-energy effective theory in the Coulomb phase is described by the Seiberg-Witten geometry which is described by a three-dimensional complex manifold in the form of the ALE space of ADE type fibered over $\mathbb{CP}^1$

$$z + \frac{1}{z} \Lambda^{2h - l(R)} N_f \prod_{i=1}^{N_f} \Delta_{G}^R(x_{1}, x_{2}, x_{3}; a, m_i) - W_{G}(x_{1}, x_{2}, x_{3}; a) = 0,$$

(12)

where $z$ parametrizes $\mathbb{CP}^1$, $h$ is the dual Coxeter number of $G$ and $l(R)$ is the index of the representation $R$ of the matter. Here $W_{G}(x_{1}, x_{2}, x_{3}; a) = 0$ is a simple singularity of type $G$ and $\Delta_{G}^R(x_{1}, x_{2}, x_{3}; a, m_i)$ is some polynomial of the indicated variables. Note that the simple singularity $W_{G}$ depends only on the gauge group $G$, but the $\Delta_{G}^R(x_{1}, x_{2}, x_{3}; a, m_i)$ depends on the matter content of the theory.

Starting with (12) let us consider the symmetry breaking in the SW geometry. In the limit $M \to \infty$, the gauge symmetry $G$ is reduced to the smaller one $G'$. The SW geometry is also reduced to the one with gauge symmetry $G'$ in this limit. We can see this by substituting $a = \langle a \rangle + \delta a$ into (12) and keeping the leading order in $M$. To leave the $j$-th flavor of hypermultiplets in the $G'$ theory, its mass $m_j$ is also shifted as in (10).
After taking the appropriate coordinate \((x'_1, x'_2, x'_3)\) we should have

\[
W_G(x_1, x_2, x_3; a) = M^{h-h'} W_{G'}(x'_1, x'_2, x'_3; \delta a) + o(M^{h-h'}),
\]

\[
X^\mathcal{R}_G(x_1, x_2, x_3; a, m_j) = M^{l(\mathcal{R})-l(\mathcal{R}_{s_j})} X^{\mathcal{R}_{s_j}}_{G'}(x'_1, x'_2, x'_3; \delta a, m'_j) + o(M^{l(\mathcal{R})-l(\mathcal{R}_{s_j})}),
\]

(13)

where \(W_{G'}\) is a simple singularity of type \(G'\), \(X^{\mathcal{R}_{s_j}}_{G'}\) is some polynomial of the indicated variables, \(h'\) is the dual Coxeter number of \(G'\) and \(l(\mathcal{R}_{s_j})\) is the index of the representation \(\mathcal{R}_{s_j}\) of \(G'\). The dependence on \(M\) can be understood from the scale matching relation between theories with gauge group \(G\) and \(G'\)

\[
\Lambda'^{2h'-\sum_{j=1}^{N_f} l(\mathcal{R}_{s_j})} = \frac{\Lambda^{2h-l(\mathcal{R})N_f}}{M^{2(h-h')-(l(\mathcal{R})N_f-\sum_{j} l(\mathcal{R}_{s_j}))}},
\]

(14)

where \(\Lambda'\) is the scale of the \(G'\) theory. Thus, in the limit \(M \to \infty\), the SW geometry becomes

\[
z' + \frac{1}{z'} \Lambda'^{2h'-\sum_{j=1}^{N_f} l(\mathcal{R}_{s_j})} \prod_{j=1}^{N_f} X^{\mathcal{R}_{s_j}}_{G'}(x'_1, x'_2, x'_3; \delta a, m'_j) - W_{G'}(x'_1, x'_2, x'_3; \delta a) = 0,
\]

(15)

where \(z' = z/M^{h-h'}\). In the following two sections we will apply this reduction procedure explicitly to the \(N = 2\) gauge theory with gauge group \(E_6\) with \(N_f\) fundamental hypermultiplets.

### 3 Breaking \(E_6\) gauge group to \(SO(10)\)

There are two ways of removing a node from the Dynkin diagram of \(E_6\) to obtain a simple group \(G'\) (see fig.1). When a node corresponding to \(\alpha_5\) (or \(\alpha_6\)) is removed, we have \(G' = SO(10)\) (or \(SU(6)\)). The former corresponds to the case of \(G' = SO(10)\) and the latter to \(G' = SU(6)\). First we consider the breaking of \(E_6\) gauge group down to \(SO(10)\) by tuning VEV of \(\Phi\) as \(\langle a_i \rangle = M\delta_{i,5}\). Using the inverse of the Cartan matrix we get \(\langle a^i \rangle = (\frac{2}{3}M, \frac{4}{3}M, \frac{5}{3}M, \frac{5}{3}M, \frac{4}{3}M, M)\).

The Seiberg-Witten geometry for \(N = 2\) gauge theory with gauge group \(E_6\) with \(N_f\) fundamental matters is proposed in [17]

\[
z + \frac{1}{z} \Lambda^{24-6N_f} \prod_{i=1}^{N_f} X_6^{27}(x_1, x_2, x_3; w, m_i) - W_{E_6}(x_1, x_2, x_3; w) = 0,
\]

(16)
where

\[ W_{E_6}(x_1, x_2, x_3; w) = x_1^4 + x_2^3 + x_3^2 + w_2 x_1^2 x_2 + w_5 x_1 x_2 + w_6 x_1^2 + w_8 x_2 + w_9 x_1 + w_{12}, \]  

(17)

and

\[ X_{E_6}^{27}(x_1, x_2, x_3; w, m) = 8 \left( m_i^6 + 2w_2 m_i^4 - 8m_i^3 x_1 + \left( w_2^2 - 12x_2 \right) m_i^2 
+ 4w_5 m_i - 4w_2 x_2 - 8(x_1^2 - ix_3 + w_6/2) \right). \]  

(18)

Here \( w_k = w_k(a) \) is the degree \( k \) Casimir of \( E_6 \) made out of \( a_j \) and the degrees of \( x_1, x_2 \) and \( x_3 \) are 3, 4 and 6 respectively. Now, substituting \( a_i = M \delta_i, 5 + \delta a_i \) into \( w_k(a) \) and setting \( \delta a^5 = 0 \), we expand \( W_{E_6} \) and \( X_{E_6}^{27} \) in \( M \). As discussed in the previous section, there should be coordinates \((x'_1, x'_2, x'_3)\) which can eliminate the terms depending upon \( M^l \) (5 ≤ \( l \) ≤ 12) in \( W_{E_6} \). Indeed, we can find such coordinates as,

\[ x_1 = -\frac{2}{27} M^3 - \frac{1}{4} M x'_1 - \frac{1}{6} M w_2, \]

\[ x_2 = \frac{1}{54} M^4 + \frac{1}{12} M^2 x'_1 + \frac{1}{9} M^2 w_2 + \frac{1}{6} x'_2 + \frac{1}{8} w_2, \]

\[ x_3 = -\frac{1}{16} M^2 x'_3. \]  

(19)

Then the \( E_6 \) singularity \( W_{E_6} \) is written as

\[ W_{E_6}(x_1, x_2, x_3; w) = \left( \frac{1}{4} M \right)^4 W_{D_5}(x'_1, x'_2, x'_3; v) + O(M^3), \]  

(20)

where

\[ W_{D_5}(x_1, x_2, x_3; v) = x_1^4 + x_1 x_2^2 - x_3^2 + v_2 x_1^3 + v_4 x_1^2 + v_6 x_1 + v_8 + v_5 x_2, \]  

(21)
and \( v_k = v_k(\delta a) \) is the degree \( k \) Casimir of \( SO(10) \) constructed from \( \delta a_i \). If we represent \( \Phi \) as a \( 10 \times 10 \) matrix of the fundamental representation of \( SO(10) \), we have \( v_{2l} = \frac{1}{2l} \text{Tr} \Phi^{2l} \) and \( v_5 = 2i \text{Pf} \Phi \). Thus we see in the \( M \to \infty \) limit that the SW geometry for \( N = 2 \) pure Yang-Mills theory with gauge group \( E_6 \) becomes that with gauge group \( SO(10) \).

Next we consider the effect of symmetry breaking in the matter sector. The fundamental representation \( 27 \) of \( E_6 \) is decomposed into the representations of \( SO(10) \times U(1) \) as

\[
27 = 16_{-\frac{1}{4}} \oplus 10_{\frac{3}{2}} \oplus 1_{-\frac{3}{4}},
\]

where the subscript denotes the \( U(1) \) charge \( \alpha_5 \cdot \lambda_i \) (\( 1 \leq i \leq 27 \)). The indices of the spinor representation \( 16 \) and the vector representation \( 10 \) are four and two, respectively. Let us first take the scaling limit in such a way that the spinor matters of \( SO(10) \) survive. Then the terms with \( M^l \) \( (l \geq 3) \) in \( X_{E_6}^{27} \) must be absent after a change of variables (19) and the mass shift \( m_i = \frac{1}{3} M + m_{si} \) (see (10)). In fact we find that

\[
X_{E_6}^{27}(x_1, x_2, x_3; w, m_i) = M^2 X_{D_5}^{16}(x'_1, x'_2, x'_3; v, m_{si}) + O(M),
\]

where

\[
X_{D_5}^{16}(x_1, x_2, x_3; v, m) = m^4 + \left(x_1 + \frac{1}{2} v_2\right) m^2 + \frac{1}{2} x_3 - \frac{1}{4} \left(v_4 - \frac{1}{4} v^2_2\right) - \frac{1}{2} v_2 x_1 - \frac{1}{2} x^2_1.
\]

In order to make the vector matter of \( SO(10) \) survive, we shift masses as \( m_i = -\frac{2}{3} M + m_{vi} \). The result reads

\[
X_{E_6}^{27}(x_1, x_2, x_3; v, m_i) = M^2 X_{D_5}^{10}(x'_1, x'_2, x'_3; v, m_{vi}) + O(M^3),
\]

where

\[
X_{D_5}^{10}(x_1, x_2, x_3; v, m) = m^2 - x_1.
\]

Assembling (20), (23), (25) and taking the limit \( M \to \infty \) with

\[
\Lambda^{16-4N_s-2N_v} = 2^{16+3N_s+3N_v} M^{-(8-2N_s-4N_v)} \Lambda^{24-6N_f}
\]

kept fixed, we now obtain the SW geometry for \( N = 2 \) \( SO(10) \) gauge theory with \( N_s \) spinor and \( N_v \) vector hypermultiplets

\[
z + \frac{1}{z} \Lambda^{16-4N_s-2N_v} \prod_{i=1}^{N_s} X_{D_5}^{16}(x_1, x_2, x_3; v, m_{si}) \prod_{j=1}^{N_v} X_{D_5}^{10}(x_1, x_2, x_3; v, m_{vj})
\]

\[-W_{D_5}(x_1, x_2, x_3; v) = 0,
\]

(28)
where $N_f = N_s + N_v$. In the massless case $m_{s_i} = m_{v_j} = 0$, our result agrees with that obtained from the analysis of the compactification of Type IIB string theory on the elliptically fibered Calabi-Yau threefold [18]. This is non-trivial evidence in support of (16). Moreover the SW geometry derived in [18] is only for the massless matters with $N_s - N_v = -2$. Here our expression is valid for massive matters of arbitrary number of flavors.

### 3.1 Breaking $SO(10)$ to $SO(8)$ and $SO(6)$

Next we examine the gauge symmetry breaking in the $N = 2$ $SO(10)$ gauge theory with spinor matters. When $\Phi$ acquires the VEV $\langle a_i \rangle = M \delta_i, 1$, namely $\langle a^i \rangle = (M, M, M, M, M/2, M/2)$, the gauge group $SO(10)$ breaks to $SO(8)$. (we rename $\delta a_i$ to $a_i$ henceforth.) Note that the spinor representation of $SO(10)$ reduces to the spinor $8s$ and its conjugate $8c$ of $SO(8)$.

Upon taking the limit $M \to \infty$ with $a_i = \langle a_i \rangle + \delta a_i$, we make a change of variables in (21)

\[
\begin{align*}
  x_1 &= x'_1, \\
  x_2 &= iM x'_2, \\
  x_3 &= M x'_3.
\end{align*}
\]

In terms of these variables, the $D_5$ singularity is shown to be

\[
W_{D_5}(x_1, x_2, x_3; v) = \left(-M^2\right) W_{D_4}(x'_1, x'_2, x'_3; u) + O(M),
\]

where

\[
W_{D_4}(x_1, x_2, x_3; u) = x_1^3 + x_1 x_2^2 + x_3^2 + u_2 x_1^2 + v_4 x_1 + u_6 + 2i \tilde{v}_4 x_2,
\]

$u_k$ is the degree $k$ Casimir of $SO(8)$ constructed from $\delta a_i$ and $\tilde{v}_4 = $ Pfaffian. The contribution (24) coming from the matters becomes

\[
X^{16}_{D_5}(x_1, x_2, x_3; v, m_{s_i}) = M^2 X^{8s}_{D_4}(x'_1, x'_2, x'_3; u, m'_{s_i}) + O(M^3),
\]

where

\[
X^{8s}_{D_4}(x_1, x_2, x_3; u, m) = m^2 + \frac{1}{2} x_1 - \frac{i}{2} x_2 + \frac{1}{4} u_2.
\]
In the above limit, we have taken $m_{si} = \frac{1}{2}M + m'_{si}$ which corresponds to the spinor representation of $SO(8)$. If we instead take $m_{si} = -\frac{1}{2}M + m'_{si}$, which corresponds to the conjugate spinor representation, then $x_2$ is replaced with $-x_2$ in $X_{D_4}^{8s}$.

If we consider the vector matters of $SO(10)$, we see that a change of variables (29) without the shift of mass does not affect $m_{vi} - x_1$. Therefore, in taking the limit $M \to \infty$ with

$$\Lambda_{SO(8)NsNv}^{12-2Ns-2Nv} = M^{-(4-2Ns)} \Lambda_{SO(10)NsNv}^{16-4Ns-2Nv}$$

being fixed, we conclude that the SW geometry for $N = 2$ $SO(8)$ gauge theory with $N_s$ spinor and $N_v$ vector flavors is

$$z + \frac{1}{z} \Lambda_{SO(8)NsNv}^{12-2Ns-2Nv} \prod_{i=1}^{Ns} X_{D_4}^{8s}(x_1, x_2, x_3; u, m'_{si}) \prod_{j=1}^{Nv} X_{D_4}^{8v}(x_1, x_2, x_3; u, m_{vj})$$

$$- W_{D_4}(x_1, x_2, x_3; u) = 0,$$

where $X_{D_4}^{8v}(x_1, x_2, x_3; u, m) = m^2 - x_1$.

There is a $\mathbb{Z}_2$ action in the triality of $SO(8)$ which exchanges the vector representation and the spinor representation. Accordingly the $SO(8)$ Casimirs are exchanged as

$$v_2 \leftrightarrow v_2,$$

$$v_4 \leftrightarrow -\frac{1}{2}v_4 + 3\text{Pf} + \frac{3}{8}v_2^2,$$

$$\text{Pf} \leftrightarrow \frac{1}{2}\text{Pf} + \frac{1}{4}v_4 - \frac{1}{16}v_2^2,$$

$$v_6 \leftrightarrow v_6 + \frac{1}{16}v_3^2 - \frac{1}{4}v_4v_2 + \frac{1}{2}\text{Pf}v_2.$$

Thus the $\mathbb{Z}_2$ action is expected to exchange $X_{D_4}^{8s}$ and $X_{D_4}^{8v}$ in (33) after an appropriate change of coordinates $x_i$. Actually, using the new coordinates $(x'_1, x'_2)$ introduced by

$$x_1 = -\frac{1}{2}x'_1 + \frac{i}{2}x'_2 - \frac{1}{4}v_2,$$

$$x_2 = -\frac{3}{2}x'_1 + \frac{1}{2}x'_2 - i\frac{1}{4}v_2,$$

we see that the $D_4$ singularity (34) remains intact except for (36) and $X_{D_4}^{8s} \leftrightarrow X_{D_4}^{8v}$.

One may further break the gauge group $SO(8)$ to $SO(6)$ following the breaking pattern $SO(10)$ to $SO(8)$. Suitable coordinates are found to be $x_1 = x'_1$, $x_2 = imx'_2$ and $x_3 = Mx'_3$.  

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The resulting SW geometry for $N = 2$ $SO(6)$ gauge theory with $N_s$ spinor flavors and $N_v$ vector flavors is

$$z + \frac{1}{z} A^{8-N_s-2N_v} \prod_{i=1}^{N_s} (\frac{1}{2} x_2 \pm m_{s_i}) \prod_{j=1}^{N_v} (m_{v_j}^2 - x_1)$$

$$- W_{D_3}(x_1, x_2, x_3; u) = 0,$$

(38)

where $W_{D_3}(x_1, x_2, x_3; u) = x_1^2 + x_1 x_2 + x_3^2 + u_2 x_1 + u_4 + 2i\text{Pf} \Phi x_2$. The sign ambiguity in (38) arises from the two possible choices of the shift of masses in $SO(8)$ theory.

When $N_s = 0$, it is seen that the present $SO(2N_c)$ results yield the well-known curves for $SO(2N_c)$ theory with vector matters [3, 10].

4 Breaking $E_6$ gauge group to $SU(6)$

In this section we wish to break the $E_6$ gauge group down to $SU(6)$ by giving the VEV $\langle a_i \rangle = M\delta_{i,6}$ to $\Phi$, that is, $\langle a_i \rangle = (M, 2M, 3M, 2M, M, 2M)$. As in the previous section, we first substitute $a_i = M\delta_{i,6} + \delta a_i$ into $w_k(a)$ in (19) and set $\delta a^6 = 0$. Then we expand $W_{E_6}$ and $X_{E_6}^{27}$ in $M$, and look for the coordinates $(x'_1, x'_2, x'_3)$ which eliminate the terms depending on $M^l$ ($7 \leq l \leq 12$) in (10). We can find such coordinates as

$$x_1 = -\frac{5}{8} M^2 x'_1 - 3 \frac{1}{4} x'_2 w_2,$$

$$x_2 = \frac{1}{16} M^4 + (\frac{1}{4} x'_2 + \frac{1}{4} x'_1^2 + \frac{1}{12} w_2) M^2,$$

$$x_3 = \frac{1}{160} M^6 + (\frac{1}{8} x'_2 + \frac{3}{160} w_2) M^4$$

$$+ \frac{1}{8} (x'_3 - x'_2^2 - 3x'_1 x'_1^2 - x'_2 w_2 + \frac{2}{15} w_2^2 - 3x'_1^4) M^2 + \frac{1}{2} w_5 x'_1 - \frac{1}{10} w_6,$$

(39)

in terms of which the $E_6$ singularity $W_{E_6}$ is represented as

$$W_{E_6}(x_1, x_2, x_3; w) = \left(\frac{1}{2} M\right)^6 W_{A_5}(x'_1, x'_2, x'_3; v) + O(M^5),$$

(40)

where

$$W_{A_5}(x_1, x_2, x_3; v) = x'_1 + x_2 x_3 + v_2 x_1^{-1} + v_3 x_1^{-2} + \cdots + v_r x_1 + v_{r+1},$$

(41)

and $v_k = v_k(\delta a)$ is the degree $k$ Casimir of $SU(6)$ build out of $\delta a_i$. Hence it is seen in the $M \to \infty$ limit that the SW geometry for $N = 2$ pure Yang-Mills theory with gauge group $E_6$ becomes that with gauge group $SU(6)$. 11
The fundamental representation $27$ of $E_6$ is decomposed into the representations of $SU(6) \times U(1)$ as

$$27 = 15_0 \oplus 6_1 \oplus \bar{6}_{-1}, \quad (42)$$

where the subscript denotes the $U(1)$ charge $\alpha_6 \cdot \lambda_i (1 \leq i \leq 27)$. The indices of the antisymmetric representation $15$ and the fundamental representation $6$ are four and one, respectively. Thus the terms with $M^l$ ($l \geq 3$) in $X_{E_6}^{27}$ must be absent after taking the coordinates $(x_1', x_2', x_3')$ defined in (39). Note that there is no need to shift the mass to make the antisymmetric matter survive. We indeed obtain a desired expression

$$X_{E_6}^{27}(x_1, x_2, x_3; w, m_i) = -M^2 X_{A_5}^{15}(x_1', x_2', x_3'; v, m_i) + O(M), \quad (43)$$

where

$$X_{A_5}^{15}(x_1, x_2, x_3; v, m) = m^4 - 2m^3x_1 + 3 \left( \frac{1}{3}v_2 + x_1^2 + x_2 \right) m^2
+mv_3 - x_3 + x_1^4 + 2v_2x_1^2 + 3x_2x_1^2 + v_3x_1 + x_2^2 + v_2x_2 + v_4. \quad (44)$$

If we shift the mass as $m_i = M + m_{f_i}$ in order to make the vector matter survive, we find that

$$X_{E_6}^{27}(x_1, x_2, x_3; v, m_i) = 2M^5 X_{A_5}^{6}(x_1', x_2', x_3'; v, m_{f_i}) + O(M^4), \quad (45)$$

where $X_{A_5}^{6}(x_1, x_2, x_3; v, m) = m + x_1$. The shift of masses $m_i = -M + m_{f_i}$ also corresponds to making the vector matter survive, but the factor $(-1)$ is needed in the RHS of (15).

From these observations we can obtain the SW geometry for $N = 2$ $SU(6)$ gauge theory with $N_a$ antisymmetric and $N_f'$ fundamental matters by taking the limit $M \to \infty$ while

$$\Lambda_{SU(6)N_aN_f'}^{12-4N_a-N_f'} = (-1)^{N_a}2^{12+2N_f'}M^{-(12-2N_a-5N_f')}\Lambda^{24-6N_f} \quad (46)$$

held fixed. Our result reads

$$z + \frac{1}{z}\Lambda_{SU(6)N_aN_f'}^{12-4N_a-N_f'} \prod_{i=1}^{N_a} X_{A_5}^{15}(x_1, x_2, x_3; v, m_{a_i}) \prod_{j=1}^{N_f'} X_{A_5}^{6}(x_1, x_2, x_3; v, m_{f_j})
-W_{A_5}(x_1, x_2, x_3; v) = 0, \quad (47)$$

where $N_f = N_a + N_f'$. 

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4.1 Breaking $SU(6)$ to $SU(5)$, $SU(4)$ and $SU(3)$

We are now able to break $SU(r+1)$ gauge group to $SU(r)$ successively by putting $\langle a_i \rangle = M\delta_{i,r}$. In sect.2 we have seen that the proper coordinates are chosen to be $x_1 = x'_1 + M/(r+1)$, $x_2 = x'_2$ and $x_3 = Mx'_3$ in terms of which $W_{A_r}(x_1, x_2, x_3; v) = MW_{A_{r-1}}(x'_1, x'_2, x'_3; v') + O(M^0)$. Note that the degrees of $x_1, x_2$ and $x_3$ are 1, 2 and $r - 1$, respectively. The antisymmetric representation of $SU(r+1)$ is decomposed into the antisymmetric and fundamental representations of $SU(r) \times U(1)$ as follows

$$\frac{r(r+1)}{2} = \frac{(r-1)r}{2} \oplus \frac{r-1}{r+1},$$

where the subscript denotes the $U(1)$ charge. After some computations we can see that the SW geometry for $N = 2$ $SU(r+1)$ $(r \leq 5)$ gauge theory with $N_a$ antisymmetric and $N'_f$ fundamental hypermultiplets turns out to be

$$z + \frac{1}{z} A_{SU(r+1)N_aN'_f} 2^{(r+1)-(r-1)N_a-N'_f} \prod_{i=1}^{N_a} X_{A_r}^{\frac{(r+1)}{2}}(x_1, x_2, x_3; v, m_{ai}) \prod_{j=1}^{N'_f} (x_1 - m_{fj})$$

$$-W_{A_r}(x_1, x_2, x_3; v) = 0,$$

where $X_{A_r}^{\frac{(r+1)}{2}}$ is defined as

$$X_{A_r}^{\frac{(r+1)}{2}}(x_j; v, m_{ai}) = \frac{2M}{r+1} + m_{ai}' = M X_{A_{r-1}}^{\frac{(r-1)}{2}}(x_j'; v', m_{ai}') + O(M^0),$$

and $A_{SU(r+1)N_aN'_f} = M^{2-N_a} A_{SU(r)N_aN'_f}$. Explicit calculations yield

$$X_{A_4}^{10}(x_j; v, m_{ai}) = m^3 - m^2 x_1 + (2x_2 + 2x_1^2 + v_2)m + 2x_1^2 - x_3 + x_2 x_1 + v_2 x_1 + v_3,$$

$$X_{A_3}^{6}(x_j; v, m_{ai}) = m^2 + x_2 - x_3 + 2x_1^2 + v_2,$$

$$X_{A_2}^{3}(x_j; v, m_{ai}) = m + x_1 - x_3.$$

We also see that

$$X_{A_5}^{15}(x_j; v, m_{ai}) = \frac{2}{3} M + m_{f_1}' = M^3(x_1' - m_{f_1}') + O(M^2),$$

$$X_{A_4}^{10}(x_j; v, m_{ai}) = \frac{3}{5} M + m_{f_1}' = -M^2(x_1' - m_{f_1}') + O(M^4),$$

$$X_{A_3}^{6}(x_j; v, m_{ai}) = \frac{1}{2} M + m_{f_1}' = M(x_1' - m_{f_1}' - x_3') + O(M^0)$$
by shifting masses in such a way that the fundamental matters remain.

We now check our $SU(N_c)$ results. First of all, for $SU(3)$ gauge group, the antisymmetric representation is identical to the fundamental representation. Thus (49) should be equivalent to the well-known $SU(3)$ curve. In fact, if we integrate out variables $x_2$ and $x_3$, the SW geometry (49) yields the $SU(3)$ curve with $N_a + N_f'$ fundamental flavors.

Let us next turn to the case of $SU(4)$ gauge group. Since the Lie algebra of $SU(4)$ is the same as that of $SO(6)$, the antisymmetric and fundamental representations of $SU(4)$ correspond to the vector and spinor representations of $SO(6)$ respectively. This relation is realized in (49) and (38) as follows. If we set

$$x_1 = \frac{1}{2} x'_2, \quad x_2 = i x'_3 - \frac{1}{2} x'_1 - \frac{1}{4} x'_2^2 - \frac{1}{2} v_2$$

and

$$x_3 = i x'_3 + \frac{1}{2} x'_1 + \frac{1}{4} x'_2^2 + \frac{1}{2} v_2,$$ 

we find

$$W_{A_3}(x_i; v) = -\frac{1}{4} W_{D_3}(x'_i; u),$$

where $u$ is related to $v$ through $u_2 = 2 v_2, u_4 = -4 v_4 + v_2^2$ and $\text{Pf} = i v_3$. Moreover we obtain $X^6_{A_3}(x_j; v, m_{ai}) = m_{ai}^2 - x'_i$ and $x_1 - m_{fj} = \frac{1}{2} x'_2 - m_{fj}$. Thus our $SU(4)$ result is in accordance with what we have anticipated. This observation provides a consistency check of our procedure since both $SO(6)$ and $SU(4)$ results are deduced from the $E_6$ theory via two independent routes associated with different symmetry breaking patterns.

Checking the $SU(5)$ gauge theory result is most intricate. Complex curves describing $N = 2$ $SU(N_c)$ gauge theory with matters in one antisymmetric representation and fundamental representations are obtained in [15, 16] using brane configurations. Let us concentrate on $SU(5)$ theory with one massless antisymmetric matter and no fundamental matters in order to compare with our result (38). The relevant curve is given by (54).

The discriminant of (54) has the form

$$\Delta_{\text{Brane}} = F_0(v) \Lambda^{105} (27 \Lambda^7 v_2^2 + v_3^3) (H_{50}(v, L))^2 (H_{35}(v, L))^6,$$

where $F_0$ is some polynomial in $v$, $H_n$ is a degree $n$ polynomial in $v$ and $L = -\Lambda^7/4$. If we set $v_2 = v_3 = 0$ for simplicity, then

$$H_{50}(v, L) = 65536 v_4^{10} v_5^2 + 1048576 v_4^9 L^2 - 33587200 v_4^7 v_5^3 L + 160000 v_4^5 v_5^6.$$
We have also calculated the discriminant $\Delta_{ALE}$ of our expression (19) with $r = 4$ and found it in the factorized form. Evaluating $\Delta_{Brane}$ and $\Delta_{ALE}$ at sufficiently many points in the moduli space, we observe that $\Delta_{ALE}$ also contains a factor $H_{50}(v, L)$ with $\Lambda^7_{SU(4)_{1,0}} = L$. This fact may be regarded as a non-trivial check for the compatibility of the M-theory/brane dynamics result and our ALE space description. It is thus inferred that only the zeroes of a common factor $H_{50}(v, L)$ in the discriminants represent the physical singularities in the moduli space.\[\]5 N=1 Confining phase superpotentials

In this section we will rederive the SW geometry obtained in the previous sections using the method of $N = 1$ confining phase superpotentials. We will explain the essence of this method in the following. More detailed explanation is presented in \cite{19, 20, 17}. First we add a tree-level superpotential

$$W_{tree} = \sum_{i=1}^{r} g_i s_i(\Phi), \quad (57)$$

to perturb $N = 2$ theory with gauge group $G$ to $N = 1$ theory, where $s_i(\Phi)$ are Casimirs of $G$ built out of $\Phi$ and $g_i$ are coupling parameters. It is then observed that only the singularities of the moduli space where dyons become massless remain as the $N = 1$ vacua. Thus studying this perturbed $N = 1$ theory with a confined photon, which corresponds to unbroken $SU(2) \times U(1)^{r-1}$ vacua classically, we can find the physical singular loci of $N = 2$ moduli space and construct the corresponding SW geometry. Recall that a basis of Casimirs $s_i$ should be chosen judiciously to obtain the correct results. The SW geometry (16) is derived in this manner \cite{17}.

\[\]

\[\]\[\]

* A similar phenomenon is observed in $SU(4)$ gauge theory. We have checked that the discriminant of the curve for $SU(4)$ theory with one massive antisymmetric hypermultiplet proposed in \cite{15} and that of our ALE formula (19) with $r = 3$ carry a common factor.
Following [17] let us now describe the computation in the $N = 1$ confining phase approach. In all the cases considered below, we use $w_i$ to denote the deformation parameters of the standard $ADE$ singularities. (In the previous sections, we have used $u_i$ for $SO(2r)$ and $v_i$ for $SU(r + 1)$ instead of $w_i$.) $P(y_i, s_i)$ is defined as a polynomial which becomes zero if we evaluate $s_i$ and $y_i$ in the classical $SU(2) \times U(1)^{r-1}$ vacua and $X(y_i, s_i, m)$ stands for the “matter factor” which relates the scale of the high-energy theory to that of the low-energy $SU(2)$ Yang-Mills theory taking into account the factor arising from the Higgs effect [20]. The SW geometry is obtained as [17]

\[ z + \frac{1}{z} \Lambda^{2h-l(\mathcal{R})} N_f X^{N_f} + P = 0, \]

where $l(\mathcal{R})$ is the index of the representation $\mathcal{R}$ of the matter and $N_f$ denotes the number of matters in $\mathcal{R}$.

### 5.1 $SU(r+1)$ gauge theory

In this subsection it is convenient to put $y_n = g_{r-n}/g_r$. In $SU(4)$ theory, we should take

\[ s_2 = w_2, \quad s_3 = w_3, \quad s_4 = w_4 - \frac{1}{4} w_2^2. \]  

Then we obtain $P = -y_2^2 + 2 y_2 y_1^2 + s_2 y_2 + s_3 y_1 + s_4$. For hypermultiplets in the antisymmetric representation, we get $X = m^2 + 2y_2$, thereby the ALE expression of the SW geometry is reproduced. On the other hand, an $N = 1$ confining phase superpotential is considered in [21] so as to produce the curve based on an M-fivebrane configuration [15].

In $SU(5)$ theory, we should take

\[ s_2 = w_2, \quad s_3 = w_3, \quad s_4 = w_4 - \frac{1}{4} w_2^2, \quad s_5 = w_5 - \frac{1}{2} w_2 w_3. \]

Then we obtain $P = 2 y_1^2 y_3 - 2 y_2 y_3 + y_2^2 y_1 + s_2 y_2 + s_3 y_1 + s_4 y_1 + s_5$. Considering the antisymmetric flavors, we get $X = m^3 - y_1 m^2 + 2my_2 + 2y_3$.

In $SU(6)$ theory, we should take

\[ s_2 = w_2, \quad s_3 = w_3, \quad s_4 = w_4 - \frac{1}{4} w_2^2, \]

\[ s_5 = w_5 - \frac{1}{2} w_2 w_3, \quad s_6 = w_6 - \frac{1}{2} w_2 w_4 + \frac{1}{8} w_2^3. \]
In this case \( y_3 = y_2 y_1 \) and we find \( P = 2y_1^2 y_4 - 2y_2 y_4 - y_2^2 y_1^2 + s_2 y_4 + s_3 y_2 y_1 + s_4 y_2 + s_5 y_1 + s_6 \). For the antisymmetric flavors, we obtain \( X = m^4 - 2y_1 m^3 + m^2 (y_1^2 + 2y_2) + m s_3 + y_2^2 + 2y_4 \).

In all the cases above, we see that the SW geometry obtained from the confining phase superpotential method is in agreement with (19) after an appropriate change of coordinates.

### 5.2 SO(2r) gauge theory

In \( SO(8) \) theory, we should take
\[
\begin{align*}
  s_2 &= w_2, \quad s_4 = w_4 - \frac{1}{3} w_2^2, \quad s_6 = w_6 - \frac{1}{3} w_4 w_2 + \frac{2}{27} w_2^3, \\
  s'_4 &= \text{Pfaffian}. \tag{62}
\end{align*}
\]

Let \( y_1 = g'_4/g_6, \ y_2 = g_4/g_6 \) and \( y_3 = g_2/g_6 \). In this case \( y_3 = \frac{1}{12} y_1^3 \) and we find \( P = y_2^3 - \frac{1}{4} y_1^2 y_2 + \frac{1}{12} s_2 y_1^2 + s'_4 y_1 + s_4 y_2 + s_6 \). If we consider the antisymmetric flavors, we get
\[
X = m^2 + \frac{1}{4} y_1 + \frac{1}{2} y_2 + \frac{1}{12} s_2, \tag{63}
\]
and for the fundamental representation
\[
X_f = m^2 - y_2 + \frac{1}{3} s_2. \tag{64}
\]

Using the relation \( y_1 = -2ix_2 \) and \( y_2 = x_1 + \frac{1}{3} v_2 \) one can see that this result is equivalent to (35).

In \( SO(10) \) theory, we should take
\[
\begin{align*}
  s_2 &= w_2, \quad s_4 = w_4 - \frac{1}{4} w_2^2, \quad s_5 = \text{Pfaffian}, \quad s_6 = w_6 - \frac{1}{2} w_4 w_2 + \frac{1}{8} w_2^3, \\
  s_8 &= w_8 - \frac{1}{4} w_4^2 + \frac{1}{8} w_4 w_2^2 - \frac{1}{64} w_2^4. \tag{65}
\end{align*}
\]

Let \( y_2 = g_5/g_8, \ y_1 = g_6/g_8, \ y_3 = g_4/g_8 \) and \( y_4 = g_2/g_8 \). There is a relation \( y_4 = y_1 y_3 \) and we find \( P = -y_3^2 + 2y_3 y_1^2 + \frac{1}{4} y_2^2 y_1 + s_2 y_3 y_1 + s_4 y_3 + s_5 y_2 + s_6 y_1 + s_8 \). If we consider the antisymmetric flavors, we get \( X = m^4 + m^2 (\frac{1}{2} s_2 + y_1) - m \frac{1}{2} y_2 - \frac{1}{2} y_3 \). For the fundamental representation, \( X_f = m^2 - y_1 \). Similarly to the case of \( SO(8) \), we can verify the equivalence of this result to (28).
6 Conclusions

Starting with the SW geometry for $N = 2$ supersymmetric gauge theory with gauge group $E_6$ with massive fundamental hypermultiplets, we have obtained the SW geometry for $SO(2N_c)$ ($N_c \leq 5$) theory with massive spinor and vector hypermultiplets by implementing the gauge symmetry breaking in the $E_6$ theory. The other symmetry breaking pattern has been used to derive the SW geometry for $N = 2$ $SU(N_c)$ ($N_c \leq 6$) theory with massive antisymmetric and fundamental hypermultiplets. All the SW geometries we have obtained are of the form of ALE fibrations over a sphere. Whenever possible our results have been compared with those obtained in the approaches based on the geometric engineering and the brane dynamics. It is impressive to find an agreement in spite of the fact that the methods are fairly different. Furthermore the SW geometries derived from the $E_6$ theory have also been obtained from the point of view of $N = 1$ confining phase superpotentials.

Let us mention here that the SW geometry for $N = 2$ $E_7$ gauge theory with massive fundamental matters will be obtained without any essential difficulty by finding an appropriate $N = 1$ confining phase superpotential. The symmetry breaking of $E_7$ will then give rise to $SO(12)$ theory with spinor matters as well as $SU(7)$ theory with antisymmetric matters. For the gauge group $E_8$, however, the situation seems rather subtle in employing the confining phase superpotential technique since there is no distinction between the fundamental and adjoint representations.

Finally, in order to analyze the mass of the BPS states and other interesting properties of the theory, one has to know the Seiberg-Witten three-form and appropriate cycles in the ALE fibration space. For $N = 2$ $SO(10)$ theory with massless spinor and vector hypermultiplets, these objects may be obtained in principle from the Calabi-Yau threefold on which the string theory is compactified [18]. It is important to find the SW three-form and appropriate cycles for the SW geometry when the massive hypermultiplets exist. This issue is left for our future consideration.

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