MULTIPLE COMPLEX-VALUED SOLUTIONS FOR THE NONLINEAR SCHRÖDINGER EQUATIONS INVOLVING MAGNETIC POTENTIALS

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(Communicated by Shoji Yotsutani)

Abstract. This paper is concerned with the following nonlinear Schrödinger equations with magnetic potentials

\[
\left( \frac{\nabla}{i} - \alpha A(|x|) \right)^2 u + (1 + \alpha V(|x|)) u = |u|^{p-2} u, \quad u \in H^1(\mathbb{R}^N, \mathbb{C}),
\]

where \(2 < p < \frac{2N}{N-2}\) if \(N \geq 3\) and \(2 < p < +\infty\) if \(N = 2\). \(\alpha\) can be regarded as a parameter. \(A(\cdot) = (A_1(\cdot), A_2(\cdot), \ldots, A_N(\cdot))\) is a magnetic field satisfying that \(A_j(|x|) > 0\) is a real \(C^1\) bounded function on \(\mathbb{R}^N\) and \(V(|x|) > 0\) is a real continuous electric potential. Under some decaying conditions of both electric and magnetic potentials which are given in section 1, we prove that the equation has multiple complex-valued solutions by applying the finite reduction method.

1. Introduction and main result. In this paper, we investigate the existence of a standing wave solution \(\psi(x,t) = e^{-\frac{iEt}{\hbar}}u(x)\), \(E \in \mathbb{R}, u : \mathbb{R}^N \to \mathbb{C}\) to the time-dependent nonlinear Schrödinger equation with an external electromagnetic field

\[
i\hbar \frac{\partial \psi}{\partial t} = \left( \frac{\hbar}{i} \nabla - A(x) \right)^2 \psi + G(x)\psi - |\psi|^{p-2}\psi, \quad x \in \mathbb{R}^N,
\]

which arises in many fields of physics, in particular condensed matter physics and nonlinear optics (see [30]). The function \(\psi(x,t)\) takes on complex values, \(\hbar\) is the Planck constant, \(i\) is the imaginary unit. Here \(A\) denotes a magnetic potential and the Schrödinger operator is defined by

\[
\left( \frac{\hbar}{i} \nabla - A(x) \right)^2 \psi := -\hbar^2 \Delta \psi - \frac{2\hbar}{i} A \cdot \nabla \psi + |A|^2 \psi - \frac{\hbar}{i} \psi \text{div} A.
\]

Actually, in general dimension, the magnetic field \(B\) is a 2-form where \(B_{k,j} = \partial_j A_k - \partial_k A_j\); in the case \(N = 3\), \(B = \text{curl} A\). The function \(G\) represents an electric potential.

2000 Mathematics Subject Classification. Primary: 35J10, 35B99; Secondary: 35J60.

Key words and phrases. Lyapunov-Schmidt reduction, magnetic potentials, nonlinear Schrödinger equations, multiple complex-valued solutions.

The authors would like to thank the referees for their suggestions of this work. The paper is supported by the fund from NSFC (No.11601139).

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Substituting \( \psi(x,t) = e^{-\frac{iEt}{\hbar}} u(x) \) into (1.1), denoting for convenience \( \epsilon = \hbar \) and \( V(x) = (G(x) - E) \), one is led to solve the following complex equation in \( \mathbb{R}^N \)
\[
\left( \frac{i}{\epsilon} \nabla - A(x) \right)^2 u + V(x)u = |u|^{p-2}u, \quad x \in \mathbb{R}^N. \tag{1.2}
\]

In recent years, much interest has been paid to the search of the existence of standing wave solutions for the nonlinear Schrödinger equation (1.2) without magnetic field \( A(x) \). We refer the reader to the papers [5–7, 14, 24–26, 31, 32] and the references therein for some recent work.

When \( A(x) \neq 0 \), (1.2) has been extensively investigated in the literatures based on various assumptions on the potentials \( A(x) \) and \( V(x) \). Compared to the case \( A(x) \equiv 0 \), the case \( A(x) \neq 0 \) becomes more complex since magnetic potential appears. Semiclassical multi-peak solutions for (1.2) with bounded vector potentials was constructed in [11] by Cao and Tang. In [10], using a penalization procedure, Cingolani and Secchi extended the result in [11] to the case of a vector potential \( A \), possibly unbounded. The penalization approach was also used in [2] by Bartsch, Dancer and Peng to obtain multi-bump semiclassical bound states for problem (1.2) with more general nonlinear term \( f(x,u) \). In [13], Ding and Wang mainly considered that the magnetic potential is only continuous and the electric potential may vanish and may change sign somewhere, and the nonlinearity is superlinear with subcritical and supercritical growth as \( |u| \to \infty \). Recently, in [8], Cosmo and Schaftingen considered semiclassical stationary states for (1.2). Their results cover unbounded domains, fast-decaying electric potential and unbounded electromagnetic fields. For more related results, one can refer to [3, 4, 12, 20–22, 27–29, 33] and the references therein.

The aim of this paper is to show the existence of multiple complex-valued solutions to (1.2). We extend the results of Long and Peng in [18] to the nonlinear Schrödinger equation involving magnetic fields. In [18], where \( A(x) \equiv 0 \) and (1.2) was considered as a real-valued problem, multiple non-radial positive solutions were found.

If \( \epsilon = 1 \), \( A(x) := \alpha A(x) \) and \( V(x) := 1 + \alpha V(x) \), then (1.2) is reduced to the following complex problem
\[
\left( \frac{i}{\epsilon} \nabla - \alpha A(x) \right)^2 u + (1 + \alpha V(x))u = |u|^{p-2}u, \quad u \in H^1(\mathbb{R}^N, \mathbb{C}). \tag{1.3}
\]
Throughout this paper, we assume \( A(x) \) and \( V(x) \) are radial. That is to say, \( A(x) = A(|x|) \) and \( V(x) = V(|x|) \). In order to state our main result, we give the conditions imposed on \( A(|x|) \), \( V(|x|)(r = |x|) \):

(V) There are constants \( a \in \mathbb{R} \) and \( b \in (0,1] \), such that
\[ V(r) \sim r^a e^{-br}, \]
as \( r \to +\infty \).

(A) There are constants \( c \in \mathbb{R} \) and \( d \in (0,\frac{1}{2}] \), such that
\[ |A(r)| \sim r^c e^{-dr}, \]
as \( r \to +\infty \).

Our main result is the following:

**Theorem 1.1.** If \( A(|x|) \) satisfies (A) and \( V(|x|) \) satisfies (V), for any fixed integer \( k \geq 1 \), provided \( k \) and \( \alpha \) satisfy one of the following conditions: (1)If \( 4 \sin^2 \frac{\pi}{2k} > \min\{b,2d\} > 2 \sin^2 \frac{\pi}{2k} \), then \( \alpha > \alpha_1 \) for an appropriate large \( \alpha_1 > 0 \); (2)If \( \min\{b,2d\} <
2\sin \frac{x}{k}, then 0 < \alpha < \alpha_2 for an appropriate small \alpha_2 > 0, where \alpha_1 and \alpha_2 dependent on b, d, k and N. Then problem (1.3) possesses a non-radial complex-valued solution.

Denote \( w : \mathbb{R}^N \to \mathbb{R} \) as the unique solution of the problem
\[
\begin{cases}
-\Delta u + u = u^{p-1}, & u > 0 \text{ in } \mathbb{R}^N, \\
u(x) \to 0, & \text{as } |x| \to +\infty.
\end{cases}
\] (1.4)

It is well-known that the unique solution \( w \) of (1.4) satisfies \( w(x) = w(|x|) \) and \( w' < 0 \) (see [16]). One can check that \( e^{i\sigma}w(x) \) (\( \sigma \in [0, 2\pi] \)) are solutions of
\[
-\Delta u + u = |u|^{p-2}u, \quad u \in H^1(\mathbb{R}^N, \mathbb{C}).
\] (1.5)

Denoting the functionals related to (1.5) by \( I_0(u) \),
\[
I_0(u) = \frac{1}{2} \int_{\mathbb{R}^N} |(\nabla u|^2 + |u|^2) - \frac{1}{p} \int_{\mathbb{R}^N} |u|^p, \quad \forall u \in H^1(\mathbb{R}^N, \mathbb{C}).
\]

From [9, 10] we know that \( e^{i\sigma}w(x) \) is non-degenerate, that is
\[
\ker I_0''(e^{i\sigma}w) = \text{span}_\mathbb{R} \left\{ \frac{\partial(e^{i\sigma}w)}{\partial x_1}, \ldots, \frac{\partial(e^{i\sigma}w)}{\partial x_N}, \frac{\partial(e^{i\sigma}w)}{\partial \sigma} \right\}.
\]

Let \( y_j = \left( r \cos \frac{2(j-1)\pi}{k}, r \sin \frac{2(j-1)\pi}{k}, 0 \right), \quad j = 1, 2, \ldots, k, \)
where 0 is the zero vector in \( \mathbb{R}^{N-2} \), \( r \) will be defined in section 2.

Set \( x = (x', x'') \), where \( x' \in \mathbb{R}^2, x'' \in \mathbb{R}^{N-2} \). Define
\[
E = \left\{ u : u \in H^1(\mathbb{R}^N, \mathbb{C}), u \text{ is even in } x_h, h = 2, \ldots, N, \right. \\
u(r \cos \theta, r \sin \theta, x'') = u(r \cos \left( \theta + \frac{2\pi j}{k} \right), r \sin \left( \theta + \frac{2\pi j}{k} \right), x'') \right\}.
\]

Denote
\[
w_{y_j}(x) = w(x - y_j), \quad u_{y_j}(x) = e^{i\sigma}w_{y_j}(x)
\]
and
\[
U_{r, \sigma}(x) = \sum_{j=1}^{k} e^{i\sigma}w_{y_j}(x) := e^{i\sigma}U_y,
\]
where \( \sigma \in [0, 2\pi] \).

In order to prove Theorem 1.1, we will show the following result.

**Theorem 1.2.** If \( A(|x|) \) satisfies (A) and \( V(|x|) \) satisfies (V), \( \min\{b, 2d\} \neq 2\sin \frac{\pi}{k} \), provided \( \alpha \) satisfies one of the following conditions: (1) If \( 4\sin \frac{\pi}{k} > \min\{b, 2d\} > 2\sin \frac{\pi}{k} \), then \( \alpha > \alpha_1 \) for an appropriate large \( \alpha_1 > 0 \); (2) If \( \min\{b, 2d\} < 2\sin \frac{\pi}{k} \), then \( 0 < \alpha < \alpha_2 \) for an appropriate small \( \alpha_2 > 0 \), where \( \alpha_1 \) and \( \alpha_2 \) dependent on \( b, d, k \) and \( N \). Then (1.3) has a complex-valued solution \( Z_k \) of the form
\[
Z_k = e^{i\sigma_k}U_{y_k}(x) + \varphi_k,
\]
where \( \sigma_k \in [0, 2\pi], \varphi_k \in E, \) and as \( \alpha \to +\infty(\text{or } 0^+) \), \( ||\varphi_k|| \to 0. \)

Our paper is organized as follows. In section 2, we will carry out the reduction. Then, we will study the reduced finite dimensional problem in section 3. Some technical estimates are left in the appendix.

**Notations:**

1. The complex conjugate of any number \( u \in \mathbb{C} \) will be denoted by \( \bar{u} \).
2. The real part of a number \( a \in \mathbb{C} \) will be denoted by \( Re \).
3. The ordinary inner product between two vectors \( a, b \in \mathbb{R}^N \) will be denoted by \( a \cdot b \).
4. \( C, \tilde{C}, c_i \) denote generic constants, which may vary inside a chain of inequalities.
5. We use \( O(t), o(t) \) to mean \( |O(t)| \leq C|t|, \frac{o(t)}{t} \rightarrow 0 \) as \( t \rightarrow 0 \); \( o(1) \) denotes quantities that tend to 0 as \( |y| \rightarrow \infty \).

2. The reduction. We assume
\[
r \in \Lambda_k := \begin{cases} (1 - \tau) \ln \alpha + (1 + \tau) \ln \alpha, \quad \text{if } b < 2d, \
(1 - \tau) \ln \alpha + (1 + \tau) \ln \alpha, \quad \text{if } b > 2d, \
b - 2 \sin \frac{\pi}{d}, \quad \text{if } b < 2d, \
\end{cases}
\]
where \( \tau > 0 \) is a small constant. Define
\[
H = \{ \varphi : \varphi \in E, Re \int_{\mathbb{R}^N} e^{i\sigma} \frac{\partial u_1}{\partial r} u_p \bar{w} \bar{\varphi} = 0 \quad \text{and} \quad Re \int_{\mathbb{R}^N} e^{i\sigma} w_p \bar{\varphi} = 0, \quad j = 1, 2, \ldots, k \}. 
\]
By the assumptions of \( A(|x|) \) and \( V(|x|) \), the norm of \( H^1(\mathbb{R}^N, \mathbb{C}) \) can be defined as follows:
\[
\|v\| = \sqrt{\langle v, v \rangle}, \quad \forall v \in H^1(\mathbb{R}^N, \mathbb{C}),
\]
which is induced by the inner product
\[
\langle v_1, v_2 \rangle = Re \int_{\mathbb{R}^N} \nabla v_1 \nabla v_2 + (1 + \alpha V(|x|)) v_1 \bar{v}_2, \quad \forall v_1, v_2 \in H^1(\mathbb{R}^N, \mathbb{C}).
\]
Define
\[
I(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left[ \left( \sum_{i} \frac{\nabla}{i} - \alpha A(|x|) \right) u_1 \right] \left( \sum_{i} \frac{\nabla}{i} - \alpha A(|x|) \right) \bar{v}_2 + Re \int_{\mathbb{R}^N} (1 + \alpha V(|x|)) v_1 \bar{v}_2
\]
\[
- \left[ (p - 2) Re \int_{\mathbb{R}^N} |U_{r,\sigma}|^{p-4} Re(U_{r,\sigma} \bar{v}_2)U_{r,\sigma} \bar{v}_1 + \int_{\mathbb{R}^N} |U_{r,\sigma}|^{p-2} Re(v_1 \bar{v}_2) \right], \quad \forall v_1, v_2 \in H,
\]
is a bounded bi-linear functional in \( H \). Hence, by the Lax-Milgram Theorem there is a bounded linear operator \( \mathcal{L} \) from \( H \) to \( \mathcal{K} \), such that
\[
\langle \mathcal{L} v_1, v_2 \rangle = Re \int_{\mathbb{R}^N} \left[ \left( \sum_{i} \frac{\nabla}{i} - \alpha A(|x|) \right) v_1 \right] \left( \sum_{i} \frac{\nabla}{i} - \alpha A(|x|) \right) \bar{v}_2 + Re \int_{\mathbb{R}^N} (1 + \alpha V(|x|)) v_1 \bar{v}_2
\]
\[
- \left[ (p - 2) Re \int_{\mathbb{R}^N} |U_{r,\sigma}|^{p-4} Re(U_{r,\sigma} \bar{v}_2)U_{r,\sigma} \bar{v}_1 + \int_{\mathbb{R}^N} |U_{r,\sigma}|^{p-2} Re(v_1 \bar{v}_2) \right], \quad \forall v_1, v_2 \in H.
\]
The following result implies that \( \mathcal{L} \) is invertible in \( H \).

**Lemma 2.1.** There exists a positive constant \( C \), for any \( r \in \Lambda_k, \sigma \in [0, 2\pi) \) and provided \( \alpha \) satisfies one of the following conditions: (1) If \( 4 \sin \frac{\pi}{d} > \min\{b, 2d\} > 2 \sin \frac{\pi}{d} \), then \( \alpha > \alpha_1 \) for an appropriate large \( \alpha_1 > 0 \); (2) If \( \min\{b, 2d\} < 2 \sin \frac{\pi}{d} \), then \( 0 < \alpha < \alpha_2 \) for an appropriate small \( \alpha_2 > 0 \), where \( \alpha_1 \) and \( \alpha_2 \) dependent on \( b, d, k \) and \( N \). Then
\[
\| \mathcal{L} \varphi \| \geq C \| \varphi \|, \quad \forall \varphi \in H.
\]
Proof. We only prove the lemma for the case $4\sin \frac{\pi}{b} > 2\sin \frac{\pi}{a}$, since the other case is similar. Here we prove it by a contradiction argument. Suppose to the contrary that there exist $n \to +\infty$, $r_n \in \Lambda_k$ and $\varphi_n \in E$ with
$$
\|L\varphi_n\| = o(1)\|\varphi_n\|.
$$
Then we have
$$
\langle L\varphi_n, \phi \rangle = o(1)\|\varphi_n\|\|\phi\|, \ \forall \phi \in E. \quad (2.1)
$$
We may assume that $\|\varphi_n\|^2 = k$.

Denote
$$
\Omega_j = \left\{ x = (x', x'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2} : \left\langle \frac{x'}{|x'|}, \frac{y_j'}{|y_j'|} \right\rangle \geq \cos \frac{\pi}{k}, j = 1, 2, \ldots, k. \right\}
$$
By (2.1) we have
$$
\begin{align*}
\Re \int_{\Omega_1} \left[ \left( \nabla - \alpha A(|x|) \right) \varphi_n \right] \left( \nabla - \alpha A(|x|) \right) \phi + \Re \int_{\Omega_1} (1 + \alpha V(|x|)) \varphi_n \bar{\phi} \\
- \left[ (p - 2) \Re \int_{\Omega_1} |U_{r, \sigma}|^{p-4} \Re(\varphi_n) U_{r, \sigma} \varphi_n + \int_{\Omega_1} |U_{r, \sigma}|^{p-2} \Re(\varphi_n) \bar{\phi} \right] \\
= \frac{1}{k} \langle L\varphi_n, \phi \rangle = o(1) \frac{1}{\sqrt{k}} \|\phi\|, \ \forall \phi \in H. \quad (2.2)
\end{align*}
$$
Particularly, choosing $\phi = \varphi_n$ we get
$$
\int_{\Omega_1} \left| \left( \nabla - \alpha A(|x|) \right) \varphi_n \right|^2 + (1 + \alpha V(|x|))|\varphi_n|^2 - [(p - 2)|U_{r, \sigma}|^{p-4}(\Re(|U_{r, \sigma}|\varphi_n))^2 \\
+ |U_{r, \sigma}|^{p-2}|\varphi_n|^2] = o(1)
$$
and
$$
\int_{\Omega_1} |\nabla \varphi_n|^2 + (1 + \alpha V(|x|))|\varphi_n|^2 = 1. \quad (2.3)
$$
Let $\tilde{\varphi}_n(x) = \varphi_n(x + y_1)$. It is easy to check that for any $R > 0$, we can choose $k$ large enough such that $B_R(y_1) \subset \Omega_1$. Consequently, (2.3) yields that
$$
\int_{B_R(0)} \left| \left( \nabla - \alpha A(|x|) \right) \tilde{\varphi}_n \right|^2 + (1 + \alpha V(|x|))|\tilde{\varphi}_n|^2 \leq 1.
$$
Thus we may assume that there exists a $\varphi \in H^1(\mathbb{R}^{N}, \mathbb{C})$ such that as $n \to +\infty$,
$$
\tilde{\varphi}_n \rightharpoonup \varphi, \text{ weakly in } H^1_{loc}(\mathbb{R}^{N}, \mathbb{C})
$$
and
$$
\tilde{\varphi}_n \rightharpoonup \varphi, \text{ strongly in } L^2_{loc}(\mathbb{R}^{N}, \mathbb{C}).
$$
Noting that $\tilde{\varphi}_n$ is even in $x_h, h = 2, \ldots, N$, then $\varphi$ is even in $x_h, h = 2, \ldots, N$. On the other hand, from
$$
\Re \int_{\mathbb{R}^N} e^{i\sigma} w_{y_1}^{p-1} \overline{\varphi_n} = 0 \quad \text{and} \quad \Re \int_{\mathbb{R}^N} \frac{\partial(e^{i\sigma} w_{y_1})}{\partial r} w_{y_1}^{p-2} \overline{\varphi_n} = 0,
$$
we obtain
$$
\Re \int_{\mathbb{R}^N} e^{i\sigma} w^{p-1} \overline{\varphi} = 0 \quad \text{and} \quad \Re \int_{\mathbb{R}^N} \frac{\partial(e^{i\sigma} w)}{\partial y_1} w^{p-2} \overline{\varphi} = 0.
$$
So $\varphi$ satisfies
$$
\Re \int_{\mathbb{R}^N} e^{i\sigma} w^{p-1} \overline{\varphi} = 0 \quad \text{and} \quad \Re \int_{\mathbb{R}^N} \frac{\partial(e^{i\sigma} w)}{\partial y_1} w^{p-2} \overline{\varphi} = 0. \quad (2.4)
$$
Define
\[ \hat{H} = \{ \phi : \phi \in H^1(\mathbb{R}^N, \mathbb{C}), Re \int_{\mathbb{R}^N} e^{i\sigma w} \partial(e^{i\sigma w}) = 0 \} \]

For any \( R > 0 \), let \( \phi \in C^\infty_0(B_R(0), \mathbb{C}) \cap \hat{H} \) be any function, satisfying that \( \phi \) is even in \( x_h, h = 2, \cdots, N \). Then \( \phi_n(x) = \phi(x - y_1) \in C^\infty_0(B_R(0), \mathbb{C}) \). Now, we give some estimates which are used later.

\[
\left| Re \int_{B_R(0)} \alpha V(|x|) \varphi_n \overline{\phi} \right| \leq \alpha \int_{B_R(0)} V(|x|)||\varphi_n|| \overline{\phi} \leq C\alpha^\alpha e^{-br} \| \phi \| = C\alpha^{\frac{-2\alpha}{b-2\alpha}} \| \phi \| = o(1) \quad (2.5)
\]

and

\[
\left| Re \int_{B_R(0)} \alpha^2 |A(|x|)|^2 \varphi_n \overline{\phi} \right| \leq \alpha^2 \int_{B_R(0)} |A(|x|)|^2 ||\varphi_n|| \overline{\phi} \leq C\alpha^{2\alpha} e^{-2br} \| \phi \| \leq C\alpha^{2\alpha} e^{-2br} \| \phi \| = C\alpha^{\frac{-4\alpha}{b-4\alpha}} \| \phi \| = o(1).
\]

Similarly,

\[
\left| Re \int_{B_R(0)} \alpha A(|x|) \cdot \nabla \varphi_n \overline{\phi} \right| = o(1) \quad \text{and} \quad \left| Re \int_{B_R(0)} \alpha \varphi_n A(|x|) \cdot \nabla \overline{\phi} \right| = o(1). \quad (2.7)
\]

Substituting \( \phi_n \) into (2.2), by Lemma 4.1, (2.5)-(2.7) and choosing \( \alpha, R \) big enough, then we find

\[
Re \int_{\mathbb{R}^N} \nabla \varphi \nabla \overline{\phi} + Re \int_{\mathbb{R}^N} \varphi \overline{\phi} - \left[ (p-2) Re \int_{\mathbb{R}^N} w^{p-4} Re(e^{i\sigma w}) e^{i\sigma w} + \int_{\mathbb{R}^N} w^{p-2} Re(e^{i\sigma w}) \right] = 0. \quad (2.8)
\]

Furthermore, since \( \varphi \) is even in \( x_h, h = 2, \cdots, N, (2.8) \) is true for any function \( \phi \in C^\infty_0(\mathbb{R}^N, \mathbb{C}) \), which is odd in \( x_h, h = 2, \cdots, N \). Therefore, (2.8) is true for any \( \phi \in C^\infty_0(B_R(0), \mathbb{C}) \cap \hat{H} \). Since \( C^\infty_0(\mathbb{R}^N, \mathbb{C}) \) is dense in \( H^1(\mathbb{R}^N, \mathbb{C}) \), it is easy to prove that

\[
Re \int_{\mathbb{R}^N} \nabla \varphi \nabla \overline{\phi} + Re \int_{\mathbb{R}^N} \varphi \overline{\phi} - \left[ (p-2) Re \int_{\mathbb{R}^N} w^{p-4} Re(e^{i\sigma w}) e^{i\sigma w} + \int_{\mathbb{R}^N} w^{p-2} Re(e^{i\sigma w}) \right] = 0, \quad \forall \phi \in \hat{H}. \quad (2.9)
\]

But (2.9) is true for \( \phi = c_1 e^{i\sigma w} + c_2 e^{i\sigma w} \frac{\partial(e^{i\sigma w})}{\partial x_1} \). Thus (2.9) is true for any \( \phi \in H^1(\mathbb{R}^N, \mathbb{C}) \), which means \( \varphi \in ker I''_0(e^{i\sigma w}) \), and hence \( \varphi = c_1 e^{i\sigma w} + c_2 e^{i\sigma w} \frac{\partial(e^{i\sigma w})}{\partial y_1} \) because \( \varphi \) is even in \( x_h, h = 2, \cdots, N \). By (2.4), we find \( c_1 = c_2 = 0 \), and \( \varphi = 0 \).

Consequently,

\[
\int_{B_R(y_1)} \varphi_n^2 = o(1), \quad \forall R > 0.
\]

Moreover, Lemma 4.1 implies that for any small \( \eta > 0 \), there is a positive constant \( C \) such that

\[
w_{y_1} \leq Ce^{-\eta \sin \frac{\pi}{2} e^{-(1-\eta)|x-y_1|}}, \quad y \in \Omega_1. \quad (2.10)
\]

By Cauchy’s inequality, we have

\[
|2\alpha Re \int_{\Omega_1} \frac{1}{i} \nabla \varphi_n \cdot A(|x|) \overline{\varphi_n}| \leq 2 \int_{\Omega_1} ||\nabla \varphi_n|| |A(|x|| ||\varphi_n||
\]
\[ \leq \frac{1}{2} \int_{\Omega} |\nabla \varphi_n|^2 + 2 \int_{\Omega} |\alpha A(|x|)|^2 |\varphi_n|^2. \]

Meanwhile,
\[
(p - 2)\text{Re} \int_{\Omega} |U_{r, \sigma}|^{p-4} |\text{Re}(U_{r, \sigma})\varphi_n|^2 \leq (p - 2) \int_{\Omega} |U_{r, \sigma}|^{p-4} |(U_{r, \sigma})\varphi_n|^2
\]
\[
\leq (p - 2) \int_{\Omega} U_{r, \sigma}^{p-2} |\varphi_n|^2
\]

Thus, by (2.2), (2.10), (A), (V) and the above inequalities we have
\[
o(1) = \int_{\Omega} \left\{ \left( \frac{\nabla}{i} - \alpha A(|x|) \right) \varphi_n \right\}^2 + (1 + \alpha V(|x|)) |\varphi_n|^2
\]
\[
- [(p - 2)|U_{r, \sigma}|^{p-4}(\text{Re}(U_{r, \sigma})\varphi_n))^2 + |U_{r, \sigma}|^{p-2}|\varphi_n|^2 \right\}
\]
\[
\geq \int_{\Omega} \left[ |\nabla \varphi_n|^2 + \alpha^2 |A(|x|)|^2 |\varphi_n|^2 - 2\alpha\text{Re} \left( \frac{1}{i} \nabla \varphi_n \cdot A(|x|) \varphi_n \right) \right]
\]
\[
+ \int_{\Omega} (1 + \alpha V(|x|)) |\varphi_n|^2 - \int_{\Omega} (p - 2)U_{r, \sigma}^{p-2} |\varphi_n|^2
\]
\[
\geq \frac{1}{2} \int_{\Omega} |\nabla \varphi_n|^2 + (1 + \alpha V(|x|)) |\varphi_n|^2 - \int_{\Omega} |\alpha A(|x|)| |\varphi_n|^2
\]
\[
+ O(e^{-(1-\eta)(p-2)^2}) \int_{\Omega} |\varphi_n|^2 + o(1)
\]
\[
\geq \frac{1}{2} \int_{\Omega} |\nabla \varphi_n|^2 + (1 + \alpha V(|x|)) |\varphi_n|^2 + o(1).
\]

This is a contradiction to (2.3).

Let us recall the following results which are used later.

Lemma 2.2. (See [1], Lemma 3.7) Given \( u, u' : \mathbb{R}^N \to \mathbb{R} \) two positive continuous radial functions such that:
\[ u(x) \sim |x|^a e^{-b|x|}, \quad u'(x) \sim |x|^{a'} e^{-b'|x|} \quad (x \to \infty) \]
where \( a, a' \in \mathbb{R}, \ b > 0, b' > 0 \). Let \( \xi \in \mathbb{R}^N \) tend to infinity. Then, the following asymptotic estimates hold:

(1) If \( b < b' \),
\[
\int_{\mathbb{R}^N} u \xi u' \sim |\xi|^a e^{-b|\xi|}.
\]

(2) If \( b = b' \), suppose, for simplicity, that \( a \geq a' \). Then:
\[
\int_{\mathbb{R}^N} u \xi u' \sim \left\{ \begin{array}{ll}
|\xi|^{a+a'} N+1 e^{-b|\xi|}, & a' > -\frac{N+1}{2}, \\
|\xi|^a e^{-b|\xi|} \log |\xi|, & a' = -\frac{N+1}{2}, \\
|\xi|^a e^{-b|\xi|}, & a' < -\frac{N+1}{2}.
\end{array} \right.
\]

Lemma 2.3. (Lemma 2.3, [19]) For \( q > 1 \), there exists \( C > 0 \) such that for any \( a, b \in \mathbb{C} \) and \(|a| > |b|\),
\[
||a + b|^q(\bar{a} + \bar{b}) - |a|^q \bar{a} - |b|^q \bar{b} - |a|^q |b|^q \text{Re}(\bar{a}b)\bar{a} + |a|^q |b|^q| \leq C|a|^{q-1}|b|^2.
\]
Lemma 2.4. (Lemma 2.4, [19]) There exists $C > 0$ such that for any $a, b \in \mathbb{C}$ and $|a| > |b|$,  
\[
\|a + b^q - |a|^q - q|a|^{q-2}Re(ab) - \frac{q}{2}[(q-2)|a|^{q-4}(Re(ab))^2 + |a|^{q-2}|b|^2]\n\leq \begin{cases} 
C|a|^{q-3}|b|^3, & \text{if } q > 3, \\
C|a|^{3-q}|b|^q, & \text{if } 2 < q \leq 3.
\end{cases}
\]

Proof. Denote $\phi |\phi (Lemma 2.4, \[19\])$, there exists a positive integer $k_0$ such that for each $k \geq k_0$, there exists a $C^1$ map with respect to $(r, \sigma)$ from $\Lambda_k \times [0, 2\pi]$ to $H$: $\varphi = \varphi(r, \sigma)$, satisfying $\varphi \in H$, and  
\[
\left\langle \frac{\partial J(\varphi)}{\partial \varphi}, \varphi \right\rangle = 0, \quad \forall \varphi \in H.
\]

Moreover, there is a small $\tau > 0$, such that  
\[
\|\varphi\| \leq C(e^{-\min\left(p-1-\tau, 2-\tau\right)}r \sin \frac{\pi}{2} + \alpha^2 e^{-(1-\tau)2\tau}r + \alpha e^{-(1-\tau)\tau}r + \alpha e^{-(1-\tau)\tau}r) \quad (2.11)
\]
and $\varphi(r, \sigma) = \varphi(r, \sigma + 2\pi)$.

Proof. Denote  

\[
J(\varphi) = I(U_{r, \sigma} + \varphi), \varphi \in H.
\]

By direct computation, we have  
\[
J(\varphi) = I(U_{r, \sigma} + \varphi) = \frac{1}{2} \int_{\mathbb{R}^N} \left[ \left( \frac{\nabla}{i} - \alpha A(|x|) \right) U_{r, \sigma} \right]^2 + (1 + \alpha V(|x|))|U_{r, \sigma}|^2 \right] \right] - \frac{1}{p} \int_{\mathbb{R}^N} |U_{r, \sigma}|^p 
+ Re \int_{\mathbb{R}^N} \left( \frac{\nabla}{i} - \alpha A(|x|) \right) U_{r, \sigma} \left( \frac{\nabla}{i} - \alpha A(|x|) \right) \varphi + Re \int_{\mathbb{R}^N} (1 + \alpha V(|x|))U_{r, \sigma} \varphi 
- \int_{\mathbb{R}^N} |U_{r, \sigma}|^{p-2} \left( ReU_{r, \sigma} \varphi \right) 
+ \frac{1}{2} \left\{ \int_{\mathbb{R}^N} \left[ \left( \frac{\nabla}{i} - \alpha A(|x|) \right) \varphi \right]^2 + (1 + \alpha V(|x|))|\varphi|^2 \right] 
- (p-2) \int_{\mathbb{R}^N} |U_{r, \sigma}|^{p-4} (ReU_{r, \sigma})^2 \right] - \int_{\mathbb{R}^N} |U_{r, \sigma}|^{p-2} |\varphi|^2 
- \left\{ \frac{1}{p} \int_{\mathbb{R}^N} \left[ |U_{r, \sigma} + \varphi|^p - |U_{r, \sigma}|^p - pRe|U_{r, \sigma}|^{p-2} U_{r, \sigma} \varphi \right] 
- \frac{1}{p} \int_{\mathbb{R}^N} \left[ (p-2)|U_{r, \sigma}|^{p-4} (Re(U_{r, \sigma}) \varphi)^2 + |U_{r, \sigma}|^{p-2} |\varphi|^2 \right] \right\}.
\]

Hence,  
\[
J(\varphi) = J(0) + f(\varphi) + \frac{1}{2} \left\langle \mathcal{L}\varphi, \varphi \right\rangle - R(\varphi),
\]
where  
\[
f(\varphi) = Re \int_{\mathbb{R}^N} \left( \frac{\nabla}{i} - \alpha A(|x|) \right) U_{r, \sigma} \left( \frac{\nabla}{i} - \alpha A(|x|) \right) \varphi + Re \int_{\mathbb{R}^N} (1 + \alpha V(|x|))U_{r, \sigma} \varphi 
- \int_{\mathbb{R}^N} |U_{r, \sigma}|^{p-2} \left( ReU_{r, \sigma} \varphi \right). \quad (2.12)
\]

$\mathcal{L}$ is the bounded linear map from $H$ to $H$ in Lemma 2.1, and  
\[
R(\varphi) = \frac{1}{p} \int_{\mathbb{R}^N} \left[ |U_{r, \sigma} + \varphi|^p - |U_{r, \sigma}|^p - pRe|U_{r, \sigma}|^{p-2} U_{r, \sigma} \varphi \right.
\]
where $\tau < \tau_1$. Hence Lemma 2.6 below implies

$$
-\frac{1}{2}b[(p - 2)|U_{r,\sigma}|^{p-4}(Re(U_{r,\sigma}\varphi))^2 + |U_{r,\sigma}|^{p-2}|\varphi|^2].
$$

It is not difficult to verify that $f(\varphi)$ is a bounded linear functional in $H$, so there exists an $f_k \in H$ such that

$$
f(\varphi) = \langle f_k, \varphi \rangle.
$$

Thus, to find a critical point for $f(\varphi)$, we only need to solve

$$
f_k + \mathcal{L}\varphi - R'(\varphi) = 0.	ag{2.13}
$$

From Lemma 2.1 we know that $\mathcal{L}$ is invertible. Therefore, (2.13) can be rewritten as

$$
\varphi = \mathcal{A}(\varphi) := -\mathcal{L}^{-1}f_k + \mathcal{L}^{-1}R'(\varphi).
$$

Set

$$
\mathcal{N} = \left\{ \varphi : \varphi \in H, \|\varphi\| \leq e^{-\min(p-1-\tau_2-\tau_1)r \sin \frac{\pi}{\tau_1}} + \alpha^2e^{-(1-\tau_1)2dr} + \alpha e^{-(1-\tau_1)dr} + \alpha e^{-(1-\tau_1)br} \right\},
$$

where $\tau < \tau_1 < 1$.

When $2 < p \leq 3$, we can verify that

$$
\|R'(\varphi)\| \leq C\|\varphi\|^{p-1}.
$$

Hence Lemma 2.6 below implies

$$
\|\mathcal{A}(\varphi)\| \leq C\|f_k\| + C\|\varphi\|^{p-1}
\leq C(e^{-\min(p-1-\tau_2-\tau_1)r \sin \frac{\pi}{\tau_1}} + \alpha^2e^{-(1-\tau_1)2dr} + \alpha e^{-(1-\tau_1)dr} + \alpha e^{-(1-\tau_1)br})
+ C(e^{-\min(p-1-\tau_2-\tau_1)r \sin \frac{\pi}{\tau_1}} + \alpha^2e^{-(1-\tau_1)2dr}
+ \alpha e^{-(1-\tau_1)dr} + \alpha e^{-(1-\tau_1)br})^{p-1}
\leq e^{-\min(p-1-\tau_2-\tau_1)r \sin \frac{\pi}{\tau_1}} + \alpha^2e^{-(1-\tau_1)2dr} + \alpha e^{-(1-\tau_1)dr} + \alpha e^{-(1-\tau_1)br}.	ag{2.14}
$$

Thus, $\mathcal{A}$ maps $\mathcal{N}$ into $\mathcal{N}$ when $2 < p \leq 3$.

Meanwhile, when $2 < p \leq 3$, we see

$$
\|R''(\varphi)\| \leq C\|\varphi\|^{p-2}.
$$

Thus,

$$
\|\mathcal{A}(\varphi_1) - \mathcal{A}(\varphi_2)\| = \|\mathcal{L}^{-1}R'(\varphi_1) - \mathcal{L}^{-1}R'(\varphi_2)\|
\leq C\|R'(\varphi_1) - R'(\varphi_2)\| \leq C\|R''(\varepsilon\varphi_1 + (1-\varepsilon)\varphi_2)\||\varphi_1 - \varphi_2|
\leq C(\|\varphi_1\|^{p-2} + \|\varphi_2\|^{p-2})\|v_1 - v_2\| \leq \frac{1}{2}\|\varphi_1 - \varphi_2\|,
$$

where $\varepsilon \in (0, 1)$.

Thus, we have proved that when $2 < p \leq 3$, $\mathcal{A}$ is a contraction map.

When $p > 3$, noting the fact that for any $a \in C$, $|\text{Re}a| \leq |a|$, then by Lemma 2.4, the Hölder inequality and the Sobolev inequality, we get

$$
|\langle R'(\varphi), \xi \rangle| = |Re \int_{\mathbb{R}^N} |U_{r,\sigma} + \varphi|^{p-2}(\overline{U_{r,\sigma} + \varphi})\xi - Re \int_{\mathbb{R}^N} |U_{r,\sigma}|^{p-2}\overline{U_{r,\sigma}}\xi
- Re \int_{\mathbb{R}^N} [(p - 2)|U_{r,\sigma}|^{p-4}Re(\overline{U_{r,\sigma}}\varphi)\overline{U_{r,\sigma}}\xi + |U_{r,\sigma}|^{p-2}\varphi\xi]|
\leq C \int_{\mathbb{R}^N} |U_{r,\sigma} + \varphi|^{p-2}(\overline{U_{r,\sigma} + \varphi}) - |U_{r,\sigma}|^{p-2}\overline{U_{r,\sigma}}
$$
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Moreover, it follows from (2.14) and (2.15) that (2.11) holds. Hence, by Remark 4.2, we get

\[
\|A_{r,\sigma}\| \leq C \left[ \int_{\mathbb{R}^N} |(U_{r,\sigma}|^{p-3}) \phi |^2 \right]^{\frac{p-1}{p}} \|\phi\|.
\]

To finish the proof, we only need to prove that

\[
\|R'(\phi)\| \leq C \left[ \int_{\mathbb{R}^N} |(U_{r,\sigma}|^{p-3}) \phi |^2 \right]^{\frac{p-1}{p}} \|\phi\|.
\]

In order to estimate \(\|R''(\phi)\|\), by the Hölder inequality and the Sobolev inequality, we have

\[
\|R''(\phi)(\xi,\eta)\| = \left| \text{Re} \int_{\mathbb{R}^N} \left[ (p-2) |U_{r,\sigma}|^{p-4} \text{Re}(\bar{U}_{r,\sigma}\phi)U_{r,\sigma}\eta + |U_{r,\sigma}|^{p-2}\phi \eta \right] \right|
\]

which implies

\[
\|R''(\phi)\| \leq C\|\phi\|.
\]

Thus, we have

\[
\|A(\phi)\| \leq C\|f_k\| + C\|\phi\|^2
\]

and

\[
\|A(\phi_1) - A(\phi_2)\| = \|L^{-1}R'(\phi_1) - L^{-1}R'(\phi_2)\|
\]

where \(\varepsilon \in (0, 1)\). Hence, \(A\) is also a contraction map from \(\mathcal{N}\) to \(\mathcal{N}\).

Now applying the contraction mapping theorem, we can find a unique \(\phi\) such that (2.13) holds. Moreover, it follows from (2.14) and (2.15) that (2.11) holds.

To finish the proof, we only need to prove that \(\phi(r, \sigma)\) is \(2\pi\)-periodic with respect to \(\sigma\). Replacing \(\sigma\) by \(\sigma + 2\pi\) in the above reduction process, we get \(\phi(r, \sigma + 2\pi)\). Since \(U_{r,\sigma}\) is \(2\pi\)-periodic, by the uniqueness of \(\phi(r, \sigma)\), we see \(\phi(r, \sigma) = \phi(r, \sigma + 2\pi)\).
Lemma 2.6. There exists a small $\tau > 0$, such that
\[
\|f_k\| \leq C \left( e^{-\min\{p-1,\tau-2\}r \sin \frac{\pi}{2}} + \alpha^2 e^{-(1-\tau)2dr} + \alpha e^{-(1-\tau)dr} + \alpha e^{-(1-\tau)br} \right).
\]

Proof.
\[
f(v) = \text{Re} \int_{\mathbb{R}^N} \left( \frac{\nabla}{i} - \alpha A(|x|) \right) U_{r,\sigma} \left( \frac{\nabla}{i} - \alpha A(|x|) \right) \varphi + \text{Re} \int_{\mathbb{R}^N} (1 + \alpha V(|x|)) U_{r,\sigma} \varphi
\]
\[
- \int_{\mathbb{R}^N} |U_{r,\sigma}|^{p-2} \left( \text{Re} U_{r,\sigma} \right)
\]
\[
= \sum_{j=1}^k \text{Re} \int_{\mathbb{R}^N} |e^{i\sigma w_{ij}}|^p |^{-2} e^{i\sigma w_{ij}} \varphi - \text{Re} \int_{\mathbb{R}^N} |U_{r,\sigma}|^{p-2} U_{r,\sigma} \varphi
\]
\[
+ \text{Re} \int_{\mathbb{R}^N} \alpha V(|x|) U_{r,\sigma} \varphi - \text{Re} \int_{\mathbb{R}^N} \frac{1}{i} \alpha \nabla U_{r,\sigma} \cdot A(|x|) \varphi
\]
\[
+ \text{Re} \int_{\mathbb{R}^N} \frac{\alpha}{i} U_{r,\sigma} A(|x|) \cdot \nabla \varphi + \text{Re} \int_{\mathbb{R}^N} \alpha^2 U_{r,\sigma} |A(|x|)|^2 \varphi \tag{2.16}
\]

Considering the symmetry of the problem and using the same argument in (4.2) and (4.3), we have
\[
\left| \text{Re} \int_{\mathbb{R}^N} \alpha V(|x|) U_{r,\sigma} \varphi \right| \leq \alpha \int_{\mathbb{R}^N} V(|x|) U_{r,\sigma} |\varphi| = k \alpha \int_{\mathbb{R}^N} V(|x|) w_{y_1} |\varphi|
\]
\[
\leq k \alpha \left( \int_{\mathbb{R}^N} V^2 (x + y_1) w_1 \right)^{\frac{1}{2}} \|\varphi\| \leq C \alpha e^{-(1-\tau)br} \|\varphi\|. \tag{2.17}
\]

Also we have
\[
\left| \text{Re} \int_{\mathbb{R}^N} \alpha^2 U_{r,\sigma} |A(|x|)|^2 \varphi \right| \leq k \int_{\mathbb{R}^N} \alpha^2 w_{y_1} |A(|x|)|^2 |\varphi|
\]
\[
\leq k \alpha^2 \left( \int_{\mathbb{R}^N} w_{y_1}^2 |A(|x|)|^4 \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} |\varphi|^2 \right)^{\frac{1}{2}}
\]
\[
\leq k \alpha^2 \left( \int_{\mathbb{R}^N} w_1^2 |A(x + y_1)|^4 \right)^{\frac{1}{2}} \|\varphi\|
\]
\[
\leq C \alpha^2 e^{-(1-\tau)2dr} \|\varphi\|. \tag{2.18}
\]

and
\[
\left| \text{Re} \int_{\mathbb{R}^N} \frac{1}{i} U_{r,\sigma} A(|x|) \cdot \nabla \varphi \right| \leq k \alpha \left( \int_{\mathbb{R}^N} w_{y_1}^2 |A(|x|)|^2 \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} |\nabla \varphi|^2 \right)^{\frac{1}{2}}
\]
\[
\leq C \alpha e^{-(1-\tau)dr} \|\varphi\|. \tag{2.19}
\]

By the same argument, we can get
\[
\left| \text{Re} \int_{\mathbb{R}^N} \frac{1}{i} \alpha \nabla U_{r,\sigma} \cdot A(|x|) \varphi \right| \leq \int_{\mathbb{R}^N} \alpha |\nabla U_{r,\sigma}| |A(|x|)||\varphi|
\]
\[
\leq C \alpha e^{-(1-\tau)dr} \|\varphi\|. \tag{2.20}
\]

Since it follows from (3.18) and (3.19) in [24],
\[
\left| \sum_{j=1}^k \text{Re} \int_{\mathbb{R}^N} |e^{i\sigma w_{ij}}|^p |^{-2} e^{i\sigma w_{ij}} \varphi - \text{Re} \int_{\mathbb{R}^N} |U_{r,\sigma}|^{p-2} U_{r,\sigma} \varphi \right|
\]
The proof of the main result.

3. Where

\[ \tau > \frac{1}{p} \]

Hence, combining all the estimates above we have

\[ \left( \frac{1}{p} \right)^{\frac{p-1}{p}} \int_{\mathbb{R}^N} |w_{y_j}|^{\frac{p-2}{p}} |w_{y_j}|^{\frac{p}{p-1}} |\varphi| \]

When \( 2 < p \leq 3 \), by Lemma 2.2, we have

\[ \left( \frac{1}{p} \right)^{\frac{p-1}{p}} \int_{\mathbb{R}^N} |w_{y_j}|^{\frac{p-2}{p}} |w_{y_j}|^{\frac{p}{p-1}} |\varphi| \]

\[ \leq C \left( \int_{\mathbb{R}^N} \sum_{l \neq j} |w_{y_l}|^{\frac{p}{p-2}} |w_{y_l}|^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \left( \int_{\mathbb{R}^N} |\varphi|^p \right)^{\frac{1}{p}} \]

\[ \leq C \left( \int_{\mathbb{R}^N} \sum_{l \neq j} |w_{y_l}|^{\frac{p}{p-2}} |w_{y_l}|^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \left( \int_{\mathbb{R}^N} |\varphi|^p \right)^{\frac{1}{p}} \]

\[ \leq C \left( \sum_{l \neq j} e^{-\left(\frac{p}{p-2}\right) |y_1 - y_j|} \right)^{\frac{p-1}{p}} \left( \int_{\mathbb{R}^N} |\varphi|^p \right)^{\frac{1}{p}} \]

where \( \tau > 0 \) is any small fixed constant.

When \( p > 3 \), we also have

\[ \left( \frac{1}{p} \right)^{\frac{p-1}{p}} \int_{\mathbb{R}^N} |w_{y_j}|^{\frac{p-2}{p}} |w_{y_j}|^{\frac{p}{p-1}} |\varphi| \]

\[ \leq C \left( \int_{\mathbb{R}^N} \sum_{l \neq j} |w_{y_l}|^{\frac{p}{p-2}} |w_{y_l}|^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \left( \int_{\mathbb{R}^N} |\varphi|^p \right)^{\frac{1}{p}} \]

\[ \leq C \left( \sum_{l \neq j} e^{-\left(\frac{p}{p-2}\right) |y_1 - y_j|} \right)^{\frac{p-1}{p}} \left( \int_{\mathbb{R}^N} |\varphi|^p \right)^{\frac{1}{p}} \]

where \( \tau > 0 \) is any small fixed constant.

Hence, combining all the estimates above we have

\[ \|f_m\| \leq C \left( e^{-\min(p-1, r-\tau) r \sin \frac{\pi}{2}} + \alpha^2 e^{-(1-r)2dr} + \alpha e^{-(1-r)dr} + \alpha e^{-(1-r)br} \right). \]

3. The proof of the main result. In this section we will prove Theorem 1.2.

Proof of Theorem 1.2. Let \( \varphi(r, \sigma) \) be the map obtained in Proposition 2.5. Define

\[ F(r, \sigma) = I(U_{r, \sigma} + \varphi(r, \sigma)), \forall (r, \sigma) \in \Lambda_k \times [0, 2\pi]. \]
It is well-known that if \((r, \sigma)\) is a critical point of \(F(r, \sigma)\), then \(U_{r, \sigma} + \varphi(r, \sigma)\) is a solution of (1.3) (see [11]). As a consequence, in order to complete the proof of the proposition, we only need to prove that \(F(r, \sigma)\) has a critical point in \(\Lambda_k \times [0, 2\pi]\).

By Proposition 2.5 and Lemma 4.3, we have

\[
\mathcal{F}(r) = I(U_{r, \sigma}) + f(\varphi) + \frac{1}{2} (L\varphi, \varphi) - R(\varphi)
= I(U_{r, \sigma}) + O(\|\varphi\|^2 + \|\varphi\|^2)
= A + \frac{\alpha}{2} B_2 r^a e^{-br} + \frac{\alpha^2}{2} B_1 r^2 e^{-2dr} - Br \frac{1-N}{\pi} e^{-2r \sin \frac{\pi}{f}}
+ O(\alpha e^{-(2\pi)br}) + O(\alpha^2 e^{-(2\pi)2dr}) + O(e^{-(2\pi)2r \sin \frac{\pi}{f}}),
\]

where \(A, B, B_1\) and \(B_2\) are defined in Lemma 4.3.

We only prove the theorem for the case \(b < 2d\), since the other case is similar. If \(b < 2d\), then

\[
\mathcal{F}(r) = A + \frac{\alpha}{2} B_2 r^a e^{-br} - Br \frac{1-N}{\pi} e^{-2r \sin \frac{\pi}{f}} + O(\alpha e^{-2r \pi}) + O(e^{-2r \sin \frac{\pi}{f}}).
\]

For the case \(b < 2\sin \frac{\pi}{k}\), we consider the following maximum respect to \(r\) :

\[
\max \{ \mathcal{F}(r) : r \in \Lambda_k \}. \tag{3.1}
\]

Assume that (3.1) is achieved by some \(r_k\) in \(\Lambda_k\), we will prove that \(r_k\) is an interior point of \(\Lambda_k\).

Investigating the following smooth function in \(\Lambda_k\),

\[
g(r) := \frac{\alpha}{2} B_2 r^a e^{-br} - Br \frac{1-N}{\pi} e^{-2r \sin \frac{\pi}{f}},
\]

It is easy to check that \(g(r)\) has a maximum point \(\tilde{r}_k = \frac{(1+o(1))\ln \alpha}{b - 2\sin \frac{\pi}{f}},\) satisfying

\[
e^{-(b-2\sin \frac{\pi}{f})} = \frac{1}{\alpha} \frac{B_2}{b-a} \frac{1}{2 \sin ^2 \frac{\pi}{f} - 1 + \alpha \left( \frac{1}{\tilde{r}_k} \right)} > 0.
\]

With

\[
g(\tilde{r}_k) = \frac{\alpha}{2} B_2 r_k^a e^{-b\tilde{r}_k} - Br \frac{1-N}{\pi} e^{-2\tilde{r}_k \sin \frac{\pi}{f}}
= \frac{\alpha}{2} B_2 r_k^a e^{-2\tilde{r}_k \sin \frac{\pi}{f}} + \frac{1}{2} \frac{2\tilde{r}_k \sin \frac{\pi}{f} - \tilde{r}_k}{b-a} - B \tilde{r}_k \frac{1-N}{\pi} e^{-2\tilde{r}_k \sin \frac{\pi}{f}} \tag{3.2}
\]

By direct computation, we deduce that

\[
\mathcal{F}(r_k) \geq \mathcal{F}(\tilde{r}_k) \geq A + g(\tilde{r}_k) + O\left( \frac{2 \sin \frac{\pi}{f} + \delta}{b - 2 \sin \frac{\pi}{f}} \right) > A. \tag{3.3}
\]

On the other hand, we suppose that \(r_k = \left( \frac{1+\tau}{b-2\sin \frac{\pi}{f}} \right) \ln \alpha\), then

\[
\mathcal{F}(r_k) = A + \frac{\alpha}{2} B_2 r_k^a e^{-br_k} - Br \frac{1-N}{\pi} e^{-2r_k \sin \frac{\pi}{f}} + O(\alpha e^{-(1+\tau)br_k}) + O(e^{-(1+\tau)2r_k \sin \frac{\pi}{f}})
= A + \frac{\alpha}{2} B_2 r_k^a e^{-br_k} - Br \frac{1-N}{\pi} e^{-2r_k \sin \frac{\pi}{f}} + O\left( \frac{2 \sin \frac{\pi}{f} + \delta}{b - 2 \sin \frac{\pi}{f}} \right) < A. \tag{3.4}
\]
This is a contradiction to (3.3). Similarly
\[ F\left(\frac{1 - \tau}{b - 2\sin \frac{\tau}{2}}\right) \ln \alpha < A. \]

Hence we can check that (3.1) is achieved by some \( r_k \), which is in the interior of \( \Lambda_k \). As a result, \( r_k \) is a critical point of \( F(r) \). Therefore
\[ U_{r_k,\sigma_k} + \varphi(r_k, \sigma_k) \]
is a solution of (1.3).

For the case \( 4\sin \frac{\pi}{2} > b > 2\sin \frac{\pi}{2} \), we consider the following minimum respect to \( r \):
\[ \min\{F(r) : r \in \Lambda_k\}. \tag{3.5} \]
Assume that (3.5) is achieved by some \( r_k \) in \( \Lambda_k \), we will prove that \( r_k \) is an interior point of \( \Lambda_k \).

Investigating the following smooth function in \( \Lambda_k \),
\[ g(r) := \frac{\alpha}{2} B_2 r^a e^{-br} - B_2 r^{\frac{1-N}{2}} e^{-2r\sin \frac{\pi}{2}}, \]

It is easy to check that \( g(r) \) has a minimum point \( \tilde{r}_k \) satisfying
\[ e^{-(b-2\sin \frac{\pi}{2})r} = \frac{1}{2} \frac{B \frac{1-N}{2} - a\left(\frac{N-1}{2} + 2r\sin \frac{\pi}{2}\right)}{br - a}, \]
with
\[ g(\tilde{r}_k) = \frac{\alpha}{2} B_2 \tilde{r}_k^a e^{-b\tilde{r}_k} - B_2 \tilde{r}_k^{\frac{1-N}{2}} e^{-2\tilde{r}_k\sin \frac{\pi}{2}} \]
\[ = \frac{\alpha}{2} B_2 \tilde{r}_k^a e^{-2\tilde{r}_k\sin \frac{\pi}{2}} \frac{1}{2} \frac{2B \tilde{r}_k^{\frac{1-N}{2}} - a\left(\frac{N-1}{2} + 2\tilde{r}_k\sin \frac{\pi}{2}\right)}{br - a} - B_2 \tilde{r}_k^{\frac{1-N}{2}} e^{-2\tilde{r}_k\sin \frac{\pi}{2}} \tag{3.6} \]
\[ = B_2 \tilde{r}_k^{\frac{1-N}{2}} e^{-2\tilde{r}_k\sin \frac{\pi}{2}} \left( \frac{2\sin \frac{\pi}{2}}{b} - 1 + o\left(\frac{1}{\tilde{r}_k}\right) \right) < 0. \]

By direct computation, we deduce that
\[ F(r_k) \leq F(\tilde{r}_k) \leq A + g(\tilde{r}_k) + O\left(\frac{2\sin \frac{\pi}{2} + 4}{2\sin \frac{\pi}{2}}\right) < A. \tag{3.7} \]

On the other hand, we suppose that \( r_k = \left(\frac{1+b}{b-2\sin \frac{\pi}{2}}\right) \ln \alpha \), then
\[ F(r_k) \]
\[ = A + \frac{\alpha}{2} B_2 \tilde{r}_k^a e^{-b\tilde{r}_k} - B_2 \tilde{r}_k^{\frac{1-N}{2}} e^{-2\tilde{r}_k\sin \frac{\pi}{2}} + O(\alpha e^{-(1+\tau)b\tilde{r}_k}) + O(e^{-(1+\tau)2\tilde{r}_k\sin \frac{\pi}{2}}) \]
\[ = A + \frac{\alpha}{2} B_2 \tilde{r}_k^a e^{-b\tilde{r}_k} - B_2 \tilde{r}_k^{\frac{1-N}{2}} e^{-2\tilde{r}_k\sin \frac{\pi}{2}} + O\left(\frac{2\sin \frac{\pi}{2} + 4}{2\sin \frac{\pi}{2} - 8}\right) \]
\[ > A. \tag{3.8} \]

This is a contradiction to (3.3). Similarly
\[ F\left(\frac{1 - \tau}{b - 2\sin \frac{\tau}{2}}\right) \ln \alpha > A. \]

Hence we can check that (3.5) is achieved by some \( r_k \), which is in the interior of \( \Lambda_k \). As a result, \( r_k \) is a critical point of \( F(r) \). Therefore
\[ U_{r_k,\sigma_k} + \varphi(r_k, \sigma_k) \]
is a solution of (1.3). □
4. Some technical estimates. In this section, we will give some estimates of the energy expansion for the approximate solutions. Firstly, we recall
\[
y_j = \left( r \cos \frac{2(j-1)\pi}{k}, r \sin \frac{2(j-1)\pi}{k}, 0 \right), \quad j = 1, \ldots, k,
\]
\[
\Omega_j = \left\{ x = (x', x'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2} : \left( \frac{x'}{|x'|}, \frac{y_j'}{|y_j'|} \right) \geq \cos \frac{\pi}{k}, \quad j = 1, 2, \ldots, k. \right\}
\]

At the beginning, we give the following basic estimate.

Lemma 4.1. (See [17, Lemma 2.2]) For any \( x \in \Omega_j \) and \( \eta \in (0, 1) \), there is a positive constant \( C \) such that
\[
w_{y_i} \leq Ce^{-\eta r \sin \frac{\pi}{2} e^{-\lambda(1-\eta)|x-y_i|}}, \quad l \neq j.
\]

Remark 4.2. It follows from Lemma 4.1 that \( |U_{r,\sigma}| \) is bounded.

Lemma 4.3. We have
\[
I(U_{r,\sigma}) = A + \frac{\alpha}{2}B_2^\alpha e^{-\lambda r} + \frac{\alpha^2}{2}B_1^2e^{-2\lambda r} - B_r \frac{1}{2} \frac{\lambda}{2} e^{-2\lambda r} + O(\sigma e^{-1+\lambda r}) + O(\alpha e^{-1+\lambda r}) + O(e^{-1+\lambda r}),
\]
where \( A = \left( \frac{1}{2} - \frac{1}{p} \right) k \int_{\mathbb{R}^N} w^p \), \( B_1, B_2 \) and \( B \) are positive constants.

Proof. By a direct computation, we have
\[
I(U_{r,\sigma}) = \frac{1}{2} \int_{\mathbb{R}^N} \left[ \left( \frac{\nabla}{|x|} - \alpha A(|x|) \right) U_{r,\sigma} \right]^2 + (1 + \alpha V(|x|))|U_{r,\sigma}|^2 - \frac{1}{p} j \int_{\mathbb{R}^N} |U_{r,\sigma}|^p
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^N} |\nabla U_{r,\sigma}|^2 + \frac{1}{2} \int_{\mathbb{R}^N} |U_{r,\sigma}|^2 + \frac{\alpha}{2} \int_{\mathbb{R}^N} V(|x|)|U_{r,\sigma}|^2 - \frac{1}{p} \int_{\mathbb{R}^N} |U_{r,\sigma}|^p
\]
\[
- \text{Re} \int_{\mathbb{R}^N} \frac{1}{2} \overline{U_{r,\sigma}} \nabla U_{r,\sigma} \cdot A(|x|) + \frac{1}{2} \alpha^2 \int_{\mathbb{R}^N} |A(|x|)|^2 |U_{r,\sigma}|^2. \tag{4.1}
\]
Noting the symmetry, we have
\[
\frac{1}{2} \int_{\mathbb{R}^N} |\nabla U_{r,\sigma}|^2 + \frac{1}{2} \int_{\mathbb{R}^N} |U_{r,\sigma}|^2 = \frac{1}{2} \sum_{j=1}^{k} \sum_{i=1}^{k} \int_{\Omega_i} w_{y_j - y_i} w_{y_j} = \frac{k}{2} \int_{\mathbb{R}^N} w^p + \frac{k}{2} \sum_{i=2}^{k} \int_{\mathbb{R}^N} w_{y_i - y_i} w_{y_i}.
\]
By Lemma 4.1, we have
\[
\int_{\mathbb{R}^N} |A(|x|)|^2 w_{y_i} w_{y_j} \leq C k \int_{\Omega_i} |A(|x|)|^2 w_{y_i} e^{-\eta r \sin \frac{\pi}{2} e^{-\lambda(1-\eta)|x-y_i|}}
\]
\[
\leq C k e^{-\eta r \sin \frac{\pi}{2}} \int_{\mathbb{R}^N} |A(x-y_i)|^2 w e^{-\lambda(1-\eta)|x|}
\]
\[
= C k e^{-\eta r \sin \frac{\pi}{2}} \int_{B(1-\delta)(0)} |A(x-y_i)|^2 w e^{-\delta(1-\eta)|x|}
\]
\[
+ C k e^{-\eta r \sin \frac{\pi}{2}} \int_{\mathbb{R}^N \setminus B(1-\delta)(0)} |A(x-y_i)|^2 w e^{-\delta(1-\eta)|x|}
\]
\[
\leq C k e^{-\eta r \sin \frac{\pi}{2}} (r^{2e^{-2\delta r}} + e^{-\lambda(1-\eta)|x|}) \leq C e^{-1+\lambda r}, \tag{4.2}
\]
where we choose \((2-\eta)(1-\delta) > d\).
Using the same argument as [18], we know
\[
\int_{\mathbb{R}^N} |A(|x|)|^2 u_{ij}^2 = \int_{\mathbb{R}^N} |A(x - y_i)|^2 u_{ij}^2
\]
\[
= (1 + o(1)) \int_{B(1 - \delta), (0)} |x - y_i|^{2e^{2d|x - y_i|}} u_{ij}^2 + \int_{\mathbb{R}^N \setminus B(1 - \delta), (0)} |A(x - y_i)|^2 u_{ij}^2
\]
\[
= \tilde{B} e^{-2\tau r} + O(e^{-(1+\tau)2d r}),
\]
(4.3)
where \( \tilde{B} \) is a positive constant, \( \tau > 0 \) small enough.

Hence, we obtain
\[
\int_{\mathbb{R}^N} |A(|x|)|^2 |U_{r,\sigma}|^2 = \int_{\mathbb{R}^N} |A(|x|)|^2 u_{ij}^2
\]
\[
= \sum_{j=1}^{k} \int_{\mathbb{R}^N} |A(|x|)|^2 u_{ij}^2 + \sum_{j \neq j} \int_{\mathbb{R}^N} |A(|x|)|^2 u_{ij} w_{ij}
\]
\[
= B_1 e^{-2\tau r} + O(e^{-(1+\tau)2d r}),
\]
where \( B_1 \) is a positive constant, \( \tau > 0 \) small enough.

Similarly,
\[
\int_{\mathbb{R}^N} V(|x|)|U_{r,\sigma}|^2 = B_2 e^{-\tau r} + O(e^{-(1+\tau)\tau r}),
\]
where \( B_2 \) is a positive constant, \( \tau > 0 \) small enough.

We also have
\[
Re \int_{\mathbb{R}^N} 1 = 2 \alpha U_{r,\sigma} \nabla U_{r,\sigma} \cdot A(|x|) = -Re \int_{\mathbb{R}^N} i\alpha \nabla U_{r,\sigma} \cdot A(|x|) = 0.
\]

For the estimate of \( \int_{\mathbb{R}^N} |U_{r,\sigma}|^p \), we use the following inequalities:

1. If \( a, b \in \mathbb{C} \), for \( 2 < q \leq 3 \) and \( |a| > |b| \)
\[
||a + b|^q - |a|^q - |b|^q - q|a|^{q-2}(Re\bar{a}) - q|b|^{q-2}(Re\bar{b})| \leq C|b|^{q-1}|a|.
\]

2. If \( a, b \in \mathbb{C} \), for \( q > 3 \)
\[
||a + b|^q - |a|^q - |b|^q - q|a|^{q-2}(Re\bar{a}) - q|b|^{q-2}(Re\bar{b})| \leq C(|a|^{q-2}|b|^2 + |b|^{q-2}|a|^2).
\]

And by Lemma 2.2, we have:
\[
\int_{\mathbb{R}^N} |U_{r,\sigma}|^p - \int_{\mathbb{R}^N} \sum_{j=1}^{k} |u_{ij}|^p
\]
\[
= \int_{\mathbb{R}^N} |u_{ij}|^p + \sum_{j=2}^{k} |u_{ij}|^p - \int_{\mathbb{R}^N} \sum_{j=1}^{k} |u_{ij}|^p
\]
\[
\leq \int_{\mathbb{R}^N} |u_{ij}|^p + \sum_{j=2}^{k} |u_{ij}|^p + p \int_{\mathbb{R}^N} |u_{ij}|^{p-2} Re \left( u_{ij} \sum_{j=2}^{k} \bar{u}_{ij} \right)
\]
\[
+ p \int_{\mathbb{R}^N} \sum_{j=2}^{k} |u_{ij}|^{p-2} Re \left( \bar{u}_{ij} \left( \sum_{j=2}^{k} u_{ij} \right) \right)
\]
\[
+ C \int_{\mathbb{R}^N} |u_{ij}|^{p-1} \sum_{j=1}^{k} |u_{ij}| - \int_{\mathbb{R}^N} \sum_{j=1}^{k} |u_{ij}|^p
\]
From Lemma 2.2, we have

\[
\int_{\mathbb{R}^N} |u_{y_j}|^p - \int_{\mathbb{R}^N} \sum_{j=1}^{k} |u_{y_j}|^p + p \int_{\mathbb{R}^N} |u_{y_1}|^{p-2} Re\left( u_{y_1} \sum_{j=2}^{k} \bar{u}_{y_j} \right)
\]

\[+ p \int_{\mathbb{R}^N} \left| \sum_{j=2}^{k} u_{y_j} \right|^{p-2} Re\left( u_{y_1} \left( \sum_{j=2}^{k} \bar{u}_{y_j} \right) \right) + O\left( e^{- (1+\tau) 2r \sin \frac{\pi}{T}} \right) .
\]

By repeated applications of the above inequalities, we obtain

\[
\int_{\mathbb{R}^N} |U_{r,\sigma}|^p = \int_{\mathbb{R}^N} \sum_{j=1}^{k} |u_{y_j}|^p + p \int_{\mathbb{R}^N} \sum_{l \neq j} |u_{y_l}|^{p-2} Re\left( u_{y_l} \bar{u}_{y_j} \right)
\]

\[+ p \int_{\mathbb{R}^N} \sum_{l=1}^{k-1} Re\left( \bar{u}_{y_l} \sum_{j=l+1}^{k} u_{y_j} \right) \left| \sum_{j=l+1}^{k} u_{y_j} \right|^{p-2} + O\left( e^{- (1+\tau) 2r \sin \frac{\pi}{T}} \right) .
\]

(4.5)

From Lemma 2.2, we have

\[
\left| \int_{\mathbb{R}^N} \sum_{l=1}^{k-1} Re\left( \bar{u}_{y_l} \sum_{j=l+1}^{k} u_{y_j} \right) \left| \sum_{j=l+1}^{k} u_{y_j} \right|^{p-2} \right| \leq \int_{\mathbb{R}^N} \sum_{l=1}^{k-1} \left( w_{y_l} \left( \sum_{j=l+1}^{k} w_{y_j} \right)^{p-1} \right)
\]

\[= O\left( \int_{\mathbb{R}^N} \sum_{l=1}^{k-1} w_{y_l} \sum_{j=l+1}^{k} w_{y_j}^{p-1} \right) = O\left( e^{- (1+\tau) 2r \sin \frac{\pi}{T}} \right) .
\]

(4.6)

Using the symmetry, Lemma 2.2, (4.5) and (4.6), we have

\[
\int_{\mathbb{R}^N} |U_{r,\sigma}|^p = \int_{\mathbb{R}^N} \sum_{j=1}^{k} |u_{y_j}|^p + p \int_{\mathbb{R}^N} \sum_{l \neq j} w_{y_l}^{p-1} u_{y_j} + O\left( e^{- (1+\tau) 2r \sin \frac{\pi}{T}} \right)
\]

\[= \int_{\mathbb{R}^N} \sum_{j=1}^{k} |u_{y_j}|^p + kp \int_{\mathbb{R}^N} \sum_{j=2}^{k} w_{y_1}^{p-1} u_{y_j} + O\left( e^{- (1+\tau) 2r \sin \frac{\pi}{T}} \right) .
\]

(4.7)

Hence, we get

\[
I(U_{r,\sigma}) = \frac{1}{2} \int_{\mathbb{R}^N} \left| \nabla U_{r,\sigma} \right|^2 + \frac{1}{2} \int_{\mathbb{R}^N} |U_{r,\sigma}|^2 + \frac{\alpha}{2} \int_{\mathbb{R}^N} V(|x|)|U_{r,\sigma}|^2 - \frac{1}{p} \int_{\mathbb{R}^N} |U_{r,\sigma}|^p
\]

\[- Re \int_{\mathbb{R}^N} \frac{1}{i} \alpha U_{r,\sigma} \nabla U_{r,\sigma} \cdot A(|x|) + \frac{1}{2} \alpha^2 \int_{\mathbb{R}^N} |A(|x|)|^2 |U_{r,\sigma}|^2
\]

\[= \frac{k}{2} \int_{\mathbb{R}^N} w^p + \frac{k}{2} \int_{\mathbb{R}^N} \sum_{i=2}^{k} w_{y_i}^{p-1} w_{y_i} + \frac{\alpha}{2} B_2 r^ae^{-br} + O(\alpha e^{-(1+\tau)br})
\]

\[+ \frac{1}{2} \alpha^2 B_1 r^{2e^{-2dr}} + O(\alpha^2 e^{-(1+\tau)2dr})
\]

\[- \frac{1}{p} \int_{\mathbb{R}^N} \sum_{j=1}^{k} |u_{y_j}|^p - k \int_{\mathbb{R}^N} \sum_{j=2}^{k} w_{y_1}^{p-1} w_{y_j} + O\left( e^{- (1+\tau) 2r \sin \frac{\pi}{T}} \right)
\]

\[= \left( \frac{1}{2} - \frac{1}{p} \right) k \int_{\mathbb{R}^N} w^p + \frac{\alpha}{2} B_2 r^ae^{-br} + \frac{\alpha^2}{2} B_1 r^{2e^{-2dr}} - B r^{\frac{1-N}{2}} e^{-2r} \sin \frac{\pi}{T}
\]

\[+ O(\alpha e^{-(1+\tau)br}) + O(\alpha^2 e^{-(1+\tau)2dr}) + O\left( e^{- (1+\tau) 2r \sin \frac{\pi}{T}} \right) .
\]

(4.8)

\[\Box\]
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Received September 2016; revised April 2017.

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