A PAC RL Algorithm for Episodic POMDPs

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Abstract

Many interesting real world domains involve reinforcement learning (RL) in partially observable environments. Efficient learning in such domains is important, but existing sample complexity bounds for partially observable RL are at least exponential in the episode length. We give, to our knowledge, the first partially observable RL algorithm with a polynomial bound on the number of episodes on which the algorithm may not achieve near-optimal performance. Our algorithm is suitable for an important class of episodic POMDPs. Our approach builds on recent advances in method of moments for latent variable model estimation.

1 INTRODUCTION

A key challenge in artificial intelligence is how to effectively learn to make a sequence of good decisions in stochastic, unknown environments. Reinforcement learning (RL) is a subfield specifically focused on how agents can learn to make good decisions given feedback in the form of a reward signal. In many important applications such as robotics, education, and healthcare, the agent cannot directly observe the state of the environment responsible for generating the reward signal, and instead only receives incomplete or noisy observations.

One important measure of an RL algorithm is its sample efficiency: how much data/experience is needed to compute a good policy and act well. One way to measure sample complexity is given by the Probably Approximately Correct framework: an RL algorithm is said to be PAC if with high probability, it selects a near-optimal action on all but a number of steps (the sample complexity) which is a polynomial function of the problem parameters. There has been substantial progress on PAC RL for the fully observable setting (Brafman and Tennenholtz, 2003; Strehl and Littman, 2005; Kakade, 2003; Strehl et al., 2012; Latimore and Hutter, 2012), but to our knowledge there exists no published work on PAC RL algorithms for partially observable settings.

This lack of work on PAC partially observable RL is perhaps because of the additional challenge introduced by the partial observability of the environment. In fully observable settings, the world is often assumed to behave as a Markov decision process (MDP). An elegant approach for proving that a RL algorithm for MDPs is PAC is to compute finite sample error bounds on the MDP parameters. However, because the states of a partially observable MDP (POMDP) are hidden, the naive approach of directly treating the POMDP as a history-based MDP yields a state space that grows exponentially with the horizon, rather than polynomial in all POMDP parameters (Even-Dar et al., 2005).

On the other hand, there has been substantial recent interest and progress on method of moments and spectral approaches for modeling partially observable systems (Anandkumar et al., 2012, 2014; Hsu et al., 2008; Littman et al., 2001; Boots et al., 2011). The majority of this work has focused on inference and prediction, with little work tackling the control setting. Method of moments approaches to latent variable estimation are of particular interest because for a number of models they obtain global optima and provide finite sample guarantees on the accuracy of the learned model parameters.

Inspired by this work, we propose a POMDP RL algorithm that is, to our knowledge, the first PAC POMDP RL algorithm for episodic domains (with no restriction on the policy class). Our algorithm is applicable to a restricted but important class of POMDP
settings, which include but are not limited to information gathering (Boutilier, 2002), dialogue management (Ko et al., 2010), and medical diagnosis before decision making (Amato and Brunskill, 2012). Our work builds on method of moments (MoM) work for estimating HMMs (Anandkumar et al., 2012) in order to efficiently instantiating our bounds would yield an impractical number of samples for a real application. Nevertheless, we believe understanding the sample complexity may help to guide the amount of data required for a task, and also similar to PAC MDP RL work, may motivate new practical algorithms that build on these ideas.

2 BACKGROUND AND RELATED WORK

The inspiration for pursuing PAC bounds for POMDPs came about from the success of PAC bounds for MDPs (Brafman and Tennenholtz, 2003; Strehl and Littman, 2005; Kakade, 2003; Strehl et al., 2012; Lattimore and Hutter, 2012). While algorithms have been developed for POMDPs with finite sample bounds (Peshkin and Mukherjee, 2001; Even-Dar et al., 2005), unfortunately these bounds are not PAC as they have an exponential dependence on the horizon length.

Alternatively, Bayesian methods (Ross et al., 2011; Doshi-Velez, 2012) are very popular for solving POMDPs. For MDPs, there exist Bayesian methods that have PAC bounds (Kolter and Ng, 2009; Abe and Warmuth, 2001). However, there have been no PAC bounds for Bayesian methods for POMDPs. That said, Bayesian methods are optimal in the Bayesian sense of making the best decision given the posterior over all possible future observations, which does not translate to a frequentist finite sample bound.

We build on method of moments (MoM) work for estimating HMMs (Anandkumar et al., 2012) in order to provide a finite sample bound for POMDPs. MoM is able to obtain a global optimum, and has finite sample bounds on the accuracy of their estimates, unlike the popular Expectation-Maximization (EM) that is only guaranteed to find a local optima, and offers no finite sample guarantees. MLE approaches for estimating HMMs (Abe and Warmuth, 1992) also unfortunately do not provide accuracy guarantees on the estimated HMM parameters. As POMDP planning methods typically require us to have estimates of the underlying POMDP parameters, it would be difficult to use such MLE methods for computing a POMDP policy and providing a finite sample guarantee.

Aside from the MoM method in Anandkumar et al. (2012), another popular spectral method involves using Predictive State Representations (PSRs) (Littman et al., 2004; Boots et al., 2011), to directly tackle the control setting; however it only has asymptotic convergence guarantees and no finite sample analysis. There is also another method of moments approach to transfer across a set of bandits tasks, but the latent variable estimation problem is substantially simplified because the state of the system is unchanged by the selected actions (Azar et al., 2013).

Fortunately, due to the polynomial finite sample bounds from MoM, we can achieve a PAC (polynomial) sample complexity bound for POMDPs.

3 PROBLEM SETTING

We consider a partially observable Markov decision process (POMDP) which is described as the tuple (S, A, R, T, Z, b, H) where we have a set of discrete states S, discrete actions A, discrete observations Z, discrete rewards R, initial belief b (more details below), and episode length H. The transition model is represented by a set of |A| matrices T_a(i, j) : |S| × |S| where the (i, j)-th entry is the probability of transitioning from s_i to s_j under action a. With a slight abuse of notation, we use Z to denote both the finite set of observations and the observation model captured by the set of |A| observation matrices, Z_a where the (i, j)-th entry represents the probability of observing z_i given the agent took action a and transitioned to state s_j. We similarly do a slight abuse of notation

\[ H \]

\[ \text{Abe and Warmuth (1992)'s MLE approach guarantees that the estimated probability over } H \text{-length observation sequences has a bounded KL-divergence from the true probability of the sequence under the true parameters, which is expressed as a function of the number of underlying data samples used to estimate the HMM parameters. We think it may be possible to use such estimates in the control setting when modeling hidden state control systems as PSRs, and employing a forward search approach to planning; however, there remain a number of subtle issues to address to ensure such an approach is viable and we leave this as an interesting direction for future work.} \]
and let $R$ denote both the finite set of rewards, and the reward matrices $R_a$ where the $(i, j)$-th entry in a matrix denotes the probability of obtaining reward $r_i$ when taking action $a$ in state $s_j$. Note that in our setting we also treat the reward as an additional observation.\cite{inprogress}

The objective in POMDP planning is to compute a policy $\pi$ that achieves a large expected sum of future rewards, where $\pi$ is a mapping from histories of prior sequences of actions, observations, and rewards, to actions. In many cases we capture prior histories using a sufficient statistic called the belief $b$ where $b(s)$ represents the probability of being in a particular state $s$ given the prior history of actions, observations and rewards. One popular method for POMDP planning involves representing the value function by a finite set of $\alpha$-vectors, where $\alpha(s)$ represents the expected sum of future rewards of following the policy associated with the $\alpha$-vector from initial state $s$. POMDP planning then proceeds by taking the first action associated with the policy of the $\alpha$-vector which yields the maximum expected value for the current belief state, which can be computed for a particular $\alpha$-vector using the dot product $\langle b, \alpha \rangle$.

In the reinforcement learning setting, the transition, observation, and/or reward model parameters are initially unknown. The goal is to learn a policy that achieves large sum of rewards in the environment without advance knowledge of how the world works.

We make the following assumptions about the domain and problem setting:

1. We consider episodic, finite horizon partially observable RL (PORL) settings.
2. It is possible to achieve a non-zero probability of being in any state in two steps from the initial belief.
3. For each action $a$, the transition matrix $T_a$ is full rank, and the observation matrix $Z_a$ and reward matrix $R_a$ are full column rank.

The first assumption on the setting is satisfied by many real world situations involving an agent repeatedly doing a task: for example, an agent may sequentially interact with many different customers each for a finite amount of time. The key restrictions on the setting are captured in assumptions 2 and 3. Assumption 2 is similar to a mixing assumption and is necessary in order for MoM to estimate dynamics for all states. Assumption 3 is necessary for MoM to uniquely determine the transition, observation, and reward dynamics. The second assumption may sound quite strong, as in some POMDP settings states are only reachable by a complex sequence of carefully chosen actions, such as in robotic navigation or video games. However, assumption 2 is commonly satisfied in many important POMDP settings that primarily involve information gathering. For example, in preference elicitation or user modeling, POMDPs are commonly used to identify the, typically static, hidden intent or preference or state of the user, before taking some action based on the resulting information (Boutilier 2002). Examples of this include dialog systems (Ko et al., 2010), medical diagnosis and decision support (Amato and Brunskill 2012), and even human-robot collaboration preference modeling (Nikolaidis et al., 2015). In such settings, the belief commonly starts out non-zero over all possible user states, and slowly gets narrowed down over time. The third assumption is also significant, but is still satisfied by an important class of problems that overlap with the settings captured by assumption 2. Information gathering POMDPs where the state is hidden but static automatically satisfy the full rank assumption on the transition model, since it is an identity matrix. Assumption 3 on the observation and reward matrices imply that the cardinality of the set of observations (and rewards) is at least as large as the size of the state space. A similar assumption has been made in many latent variable estimation settings (e.g. (Anandkumar et al., 2012, 2014; Song et al., 2010)) including in the control setting (Boots et al., 2011). Indeed, when the observations consist of videos, images or audio signals, this assumption is typically satisfied (Boots et al., 2011), and such signals are very common in dialog systems and the user intent and modeling situations covered by assumption 2. Satisfying that the reward matrix has full rank is typically trivial as the reward signal is often obtained by discretizing a real-valued reward. Therefore, while we readily acknowledge that our setting does not cover all generic POMDP reinforcement learning settings, we believe it does cover an important class of problems that are relevant to real applications.

\section{Algorithm}

Our goal is to create an algorithm that can achieve near optimal performance from the initial belief on each episode. Prior work has shown that the error in the POMDP value function is bounded when us-
However, this work takes a sensitivity analysis perspective, and does not address how such model estimation errors themselves could be computed or bounded.

In contrast, many PAC RL algorithms for MDPs have shown that exploration is critical in order to get enough data to estimate the model parameters. However in MDPs, algorithms can directly observe how many times every action has been tried in every state, and can use this information to steer exploration towards less explored areas. In partially observable settings it is more challenging, as the state itself is hidden, and so it is not possible to directly observe the number of times an action has been tried in a latent state. Fortunately, recent advances in method of moments (MoM) estimation procedures for latent variable estimation (see e.g. (Anandkumar et al., 2012, 2014)) have demonstrated that in certain uncontrolled settings, including many types of hidden Markov models (HMMs), it is still possible to achieve accuracy estimates of the underlying latent variable model parameters as a function of the amount of data samples used to perform the estimation. For some intuition about this, consider starting in a belief state $b$ which has non-zero probability over all possible states. If one can repeatedly take the same action $a$ from same belief $b$, given a sufficient number of samples, we will have actually taken action $a$ in each state many times (even if we don’t know the specific instances on which action $a$ was taken in a state $s$).

The control setting is more subtle than the uncontrolled setting which has been the focus of the majority of recent MoM spectral learning research, because we wish to estimate not just the transition and observation models of a HMM, but to estimate the POMDP model parameters. Our ultimate interest is in being able to select good actions. A naive approach is to independently learn the transition, observation, and reward parameters for each separate action, by restricting the POMDP to only execute a single action, thereby turning the POMDP into an HMM. However, this simple approach fails because the returned parameters can correspond to a different labeling of the hidden states are provided, which removes the need for latent variable estimation.

Algorithm 1: EEPORL

| input: $S, A, Z, R, H, N, \pi, \pi_{rest}$ |
| 1 Let $\pi_{explore}$ be the policy where $a_1, a_2$ are uniformly random, and $p(a_{t+2}|a_t) = \frac{1}{|A|^2}$; |
| 2 $X \leftarrow \emptyset$; |
| // Phase 1: |
| 3 for episode $i \leftarrow 1$ to $N$ do |
| 4 | Follow $\pi_{explore}$ for 4 steps ; |
| 5 | Let $x_t = (a_t, r_t, z_t, a_{t+1})$; |
| 6 | $X \leftarrow X \cup \{(x_1, x_2, x_3)\}$; |
| 7 | Execute $\pi_{rest}$ for the rest of the steps ; |
| // Phase 2: |
| 8 | Get $\hat{T}, \hat{O}, \hat{w}$ for the induced HMM from $X$ through our extended MoM method ; |
| 9 | Using the labeling from Algorithm 2 with $\hat{O}$, compute estimated POMDP parameters.; |
| 10 | Call Algorithm 3 with estimated POMDP parameters to estimate a near optimal policy $\hat{\pi}$; |
| 11 | Execute $\hat{\pi}$ for the rest of the episodes ; |

Algorithm 2: LabelActions

| input: $\hat{O}$ |
| 1 foreach column $i$ of $\hat{O}$ do |
| 2 | Find a row $j$ such that $\hat{O}(i,j) = \frac{2}{|R| |Z|}$; |
| 3 | Let the observation associated with row $j$ be $(a, r', z', a')$ label column $i$ with $(a, a')$; |

Algorithm 3: FindPolicy

| input: $\hat{\beta}(s_{(a_0,a_1)}), \hat{\beta}(z|a, s_{(a,a')}), \hat{r}|s_{(a,a')}, a')$, $\hat{p}(r|s_{(a,a')}, a')$, $\hat{p}(s_{(a,a')}|s_{(a,a')}, a')$ |
| 1 $∀a, a' A, \Gamma_{t-1}^{a,a'} = \{\beta_{t}^{a,a'}(s_{(a-a)})\}$; |
| 2 for $t \leftarrow 2$ to $H$ do |
| 3 $∀a, a' A, \Gamma_{t-1}^{a,a'} = \emptyset$; |
| 4 for $a, a' A$ do |
| 5 | for $f_t(r, z) \in (|R| \times |Z| \rightarrow \Gamma_{t-1}^{a,a'})$ do |
| 6 | // all mappings from an observation pair to a previous $\beta$-vector |
| 7 | $∀a, a' A, \Gamma_t^{a,a'} = \Gamma_{t-1}^{a,a'} \cup \{\beta_t^{a,a'}(s_{(a-a)})\}$; |
| 8 | Return arg max $\Gamma_{t-1}^{a,a'}(\hat{b} \cdot \beta_H)$; |

Figure 1: POMDP (left) analogous to induced HMM (right). Gray nodes show fully observed variables, whereas white nodes show latent states.
states. For example, the first column of the transition matrix for action $a_1$ may actually correspond to the state $s_2$, while the first column of the transition matrix for action $a_2$ may truly correspond to $s_3$. We require that the labeling must be consistent for all actions since we wish to compute what happens when different actions are executed consecutively. An unsatisfactory way to match up the labels for different actions is by requiring that the initial belief state have probabilities that are unique and well separated per state. Then we can use the estimated initial belief from each action to match up the labels. However, this is a very strong assumption on the starting belief state which is unlikely to be realized.

To address this challenge of mismatched labels, we transform our POMDP into an induced HMM (see Figure 1) by fixing the policy to $\pi_{\text{explore}}$ (for a few steps, during a certain number of episodes), and create an alternate hidden state representation that directly solves the problem of alignment of hidden states across actions. Specifically, we make the hidden state at time $t$ of the induced HMM, denoted by $h_t$, equal to the tuple of the action at time step $t$, the next state, and the subsequent action, $h_t = (a_t, s_{t+1}, a_{t+1})$. We denote the observations of the induced HMM by $x$, and the observation associated with a hidden state $h_t$ is the tuple $x_t = (a_t, r_t, z_t, a_{t+1})$. Figure 1 shows how the graphical model of our original POMDP is related to the graphical model of the induced HMM. In making this transformation, our resulting HMM still satisfies the Markov assumption: the next state is only a function of the prior state, and the observation is only a function of the current state. But, this transformation also has the desired property that it is now possible to directly align the identity of states across selected actions. This is because HMM parameters now depend on both state and action, so there is a built-in correlation between different actions. We will discuss this more in the theoretical analysis.

We are now ready to describe our algorithm for episodic finite horizon reinforcement learning in POMDPs, EEPORL (Explore then Exploit Partially Observable RL), which is shown in Algorithm 1. Our algorithm is model-based and proceeds in two phases. In the first phase, it performs exploration to collect samples of trying different actions in different (latent) states. After the first phase completes, we extend a MoM approach ([Anandkumar et al., 2012]) to compute estimates of the induced HMM parameters. We use these estimates to obtain a near-optimal policy.

### 4.1 Phase 1

The first phase consists of the first $N$ episodes. Let $\pi_{\text{explore}}$ be a fixed open-loop policy for the first four actions of an episode. In $\pi_{\text{explore}}$ actions $a_1, a_2$ are selected uniformly at random, and $p(a_{t+2}|a_1) = \frac{1}{1+c(|A|)}$ where $c$ can be any positive real number. For our proof, we pick $c = O(1/|A|)$. Note that $\pi_{\text{explore}}$ only depends on previous actions and not on any observations. The definition of $p(a_{t+2}|a_1)$ for what will work for the proof only requires it to be full-rank and having some minimum probability over all actions. We chose a perturbed identity matrix for simplicity. Since $\pi_{\text{explore}}$ is a fixed policy, the POMDP process reduces to a HMM for these first four steps. During these steps we store the observed experience as $(x_1, x_2, x_3)$, where $x_t = (a_t, r_t, z_t, a_{t+1})$ is an observation of our previously defined induced HMM. The algorithm then follows policy $\pi_{\text{rest}}$ for the remaining steps of the episode. All of these episodes will be considered as potentially non-optimal, and so the choice of $\pi_{\text{rest}}$ does not impact the theoretical analysis. However, empirically $\pi_{\text{rest}}$ could be constructed to encourage near optimal behavior given the observed data collected up to the current episode.

### 4.2 Parameter Estimation

After Phase 1 completes, we have $N$ samples of the tuple $(x_1, x_2, x_3)$. We then apply our extension to the MoM algorithm for HMM parameter estimation by [Anandkumar et al., 2012]. Our extension computes estimates and bounds on the transition model $\hat{T}$, observation matrix $\hat{O}$, and belief vector $\hat{w}$ for the induced HMM. The belief $\hat{w}$ is over the second hidden state, $h_2$.

As mentioned before as one major challenge, labeling of the states $h$ of the induced HMM is arbitrary; however it is consistent between $\hat{T}, \hat{O}, \hat{w}$ since this is a single HMM inference problem. Recall that a hidden state in our induced HMM is defined as $h_t = (a_t, s_{t+1}, a_{t+1})$. Since the actions are fully observable, it is possible to label each state $h = (a, s', a')$ (i.e. the columns of $\hat{O}$, the rows and columns of $\hat{T}$, and the rows of $\hat{w}$) with two actions $(a, a')$ that are associated with that state. This is possible because the true observation matrix entries for the actions of a hidden state must be non-zero, and the true value of all other entries (for other actions) must be zero; therefore, as long as we have sufficiently accurate estimates of the observation matrix, we can use the observation matrix parameters to augment the states $h$ with their associated action pair. This procedure is performed by Algorithm 2. This labeling provides a connection between the HMM state $h$ and the original POMDP state. For a particular pair of actions $a, a'$, there are exactly $|S|$ HMM states that correspond to them. Thus looking at
the columns of $\tilde{O}$ from left-to-right, and only picking out the columns that are labeled with $a,a'$ results in a specific ordering of the states $(a,i,a')$, which is a permutation of the POMDP states, which we denote as $\{s_{(a,a'),1},s_{(a,a'),2},\ldots,s_{(a,a'),|S|}\}$. We will also use the notation $s_{(a,a')}$ to implicitly refer to a vector of states in the order of the permutation.

The algorithm proceeds to estimate the original POMDP parameters in order to perform planning and compute a policy. Note that the estimated parameters use the computed $s_{(a,a')}$ permutations of the state. Let $\hat{O}^{a,a'}$ be the submatrix where the rows and columns correspond to the actions $(a,a')$ and $\hat{T}^{a,a',a''}$ be the submatrix where the rows correspond to the actions $(a',a'')$ and columns correspond to the actions $(a,a')$. Then the estimated POMDP parameters can be computed as follows:

\[
\hat{b}(s_{(a_0,a_1)}) = \text{normalize}((\hat{T}^{-1}I\hat{T}^{-1}\tilde{w})(a_0,\cdot,a_1)),
\]

\[
\hat{p}(z|a,s_{(a,a')}) = \text{normalize}(\sum_r \hat{O}^{a,a'})
\]

\[
\hat{p}(r|s_{(a,a')},a') = \text{normalize}(\sum_z \hat{T}^{a,a',a''})
\]

\[
\hat{p}(s_{(a',a'')}|s_{(a,a')} , a') = \text{normalize}(\hat{T}^{a,a',a''})
\]

Note that we require an additional normalize() procedure since the MoM approach we leverage is not guaranteed to return well formed probability distributions. The normalization procedure just divides by the sum to make them into valid probability distributions (if there are negative values we can either set them to zero or even just use the absolute value).

Algorithm 3 then uses these estimated POMDP parameters to compute a policy. The algorithm constructs $\beta$-vectors (see Definition 1) that represent the expected sum of rewards of following a particular policy starting with action $a'$ given an input permuted state $s_{(a,a')}$. Aside from this slight modification, $\beta$-vectors are analogous to $\alpha$-vectors in standard POMDP planning. The $\beta$-vectors form an approximate value function for the underlying POMDP and can be used in a similar way to standard $\alpha$-vectors.

### 4.3 Phase 2

In phase 2, after estimating the POMDP parameters and $\beta$-vectors, we use the estimated POMDP value function to extract a policy for acting, and we will shortly prove sufficient conditions for this policy to be near-optimal for all remaining episodes.

The policy followed depends on the computed value function. If computationally tractable, one can compute $\beta$-vectors incrementally for all possible $H$-step policies. In this case, control proceeds by finding the best $\beta$-vector for the estimated initial belief $\hat{b}(s_{(a_0,a_1)})$ (largest dot product of the $\beta$-vector with the initial belief) and then following the associated policy $\hat{\pi}$. $\pi$ is then followed for the entire episode with no additional belief updating required as the policy itself encodes the conditional branching.

However, in practical circumstances, it will not be possible to enumerate all possible $H$-step policies. In this case, one can use point-based approaches or other methods that use $\alpha$-vectors to enumerate only a subset of possible policies. In this case there will be an additional error $\epsilon_{\text{planning}}$ in the final error bound due to finite set of policies considered. In our analysis we omit $\epsilon_{\text{planning}}$ for simplicity and assume that we enumerate all $H$-step policies.

**Definition 1.** A $\beta$-vector taking as input $s_{(a,a')}$ with root action $a'$ and $t$-step conditional policies $f_t(r,z)$ for each observation pair $(r,z)$ is defined as

\[
\beta^{\alpha}(s_{(a,a')}) = \sum_r p(r|s_{(a,a')},a) \cdot r
\]

\[
\beta^{\alpha}_{t+1}(s_{(a,a')}) = \sum_{r,z,s_{(a',f_t(r,z))}} (r + \gamma \beta^{\alpha}_{t}(s_{(a',f_t(r,z)))}) \cdot p(r|s_{(a,a')},a)p(z|s_{(a',f_t(r,z))},a)p(s_{(a,a')}|s_{(a',f_t(r,z)})
\]

where $f_t(r,z)$ can also denote the root action of the policy $f_t(r,z)$ used in terms like $s_{(a,f_t(r,z))}$.

### 5 THEORY

#### 5.1 PAC Theorem Setup

We now state our primary result. For full details, please refer to our tech report\[1\]. Before doing so, we define some additional notation. Let $V^*(b) = \sum_{t=1}^{H} r_t$ starting from belief $b$ be the total undiscounted reward following policy $\pi$ for an episode. Let $\sigma_1(B) = \max_a \sigma_1(B)$ and similarly for $\sigma_1(R_a)$ and $\sigma_1(Z_a)$. Let $\sigma_\alpha(T_a) = \min_a \sigma_\alpha(T_a)$ and similarly for $\sigma_\alpha(R_a)$ and $\sigma_\alpha(Z_a)$. Assume $\sigma_\alpha(T_a)$, $\sigma_\alpha(R_a)$, and $\sigma_\alpha(Z_a)$ are all at most 1 (otherwise each term can be replaced by 1 in the final sample complexity bound below).

#### 5.2 PAC Theorem

**Theorem 1.** For POMDPs that satisfy the stated assumptions defined in the problem setting, executing EEPOPRL will achieve an expected episodic reward of $V(b_0) \geq V^*(b_0) - \epsilon$ on all but a number of episodes that is bounded by

\[
O\left(H^4V_2^2|A|^2|R|^4|Z|^4|S|^2 \left(1 + \sqrt{\log(\frac{1}{\delta})} \right)^2 \log \left( \frac{1}{\delta} \right) \right)
\]

\[
\frac{C_{d,d,l}}{2} \frac{\delta_{\alpha}}{3} \frac{\delta_{\beta}}{3} \frac{\delta_{\gamma}}{3} \frac{\delta_{\sigma}}{3} \frac{\delta_{\tau}}{3} \frac{\delta_{\nu}}{3} \frac{\delta_{\phi}}{3} \frac{\delta_{\epsilon}}{3}
\]

[1] http://www.cs.cmu.edu/~zguo/#publications
with probability at least $1 - \delta$, where

$$C_{d,d,d}(\delta) = \min(C_{1,2,3}(\delta), C_{1,3,2}(\delta))$$

$$C_{1,2,3}(\delta) = \min\left(\frac{\min_{i \neq j} \|M_3(\hat{e}_i - \hat{e}_j)\|_2 \cdot \sigma_k(P_{1,2})^2}{\|P_{1,2,3}\|_2 \cdot k^3 \cdot \kappa(M_1)^4}, \frac{\delta}{\log(k/\delta)}\right)^{1/4}$$

$$C_{1,3,2}(\delta) = \min\left(\frac{\min_{i \neq j} \|M_2(\hat{e}_i - \hat{e}_j)\|_2 \cdot \sigma_k(P_{1,3})^2}{\|P_{1,2,3}\|_2 \cdot k^3 \cdot \kappa(M_1)^4}, \frac{\delta}{\log(k/\delta)}\right)^{1/4}$$

The quantities $C_{1,2,3}, C_{1,3,2}$ directly arise from using the previously referenced MoM method for HMM parameter estimation (Anandkumar et al. 2012) and involve singular values of the moments of the induced HMM and the induced HMM parameters (see Anandkumar et al. 2012 for details).

We now briefly overview the proof. Detailed proofs are available in the supplemental material. We first show that by executing EEPORL we obtain parameter estimates of the induced HMM, and bounds on these estimates, as a function of the number of data points (Lemma 2). We then prove that we can use the induced HMM to obtain estimated parameters of the underlying POMDP (Lemma 3). Then we show that we can compute policies that are equivalent (in structure and value) to those from the original POMDP (Lemma 4). Then we bound the error in the resulting value function estimates of the resulting policies due to the use of approximate (instead of exact) model parameters (Lemma 5). This allows us to compute a bound on the number of required samples (episodes) necessary to achieve near-optimal policies, with high probability, for use in phase 2.

We commence the proof by bounding the error in estimates of the induced HMM parameters. In order to do that, we introduce Lemma 1 which proves that samples taken in phase 1 belong to an induced HMM where the transition and observation matrices are full rank. This is a requirement for being able to apply the MoM HMM parameter estimation procedure of Anandkumar et al. (2012).

**Lemma 1.** The induced HMM has the observation and transition matrices defined as

$$O(a_{i,t}^q, h_{t+1}^q) = \delta(a_{i,t}^q, a_{i+1,t}^q) \delta(s_{i+1,t}^q | s_{i,t}^q, h_{i,t}^q) p(x_{i+1,t} | s_{i+1,t}^q, a_{i+1,t}^q)$$

$$T(h_{i,t}^q, h_{i+1,t}^q) = \delta(a_{i+1,t}^q, a_{i+1+1,t}^q) p(s_{i+1,t}^q | s_{i,t}^q, h_{i,t}^q) p(a_{i+1,t+1} | a_{i,t}^q)$$

where $i$ is the index over the rows and $j$ is the index over the columns, and $x_{i,t}^q = (a_{i,t}^q, s_{i,t+1}^q, r_{i,t+1}^q, a_{i+1,t}^q)$, $h_{i+1,t}^q = (a_{i+1,t}^q, s_{i+1,t+2}, a_{i+1+1,t}^q)$. $T$ and $O$ are both full rank and $w = p(h_2)$ has positive probability everywhere. Furthermore the following terms are bounded: $|T|_2 \leq \sqrt{|\mathcal{S}|}$, $\|T^{-1}\|_2 \leq \frac{2(1 + \epsilon|\mathcal{A}|)}{\epsilon^2}$, $\sigma_{\min}(O) \geq \sigma_{\min}(R_{a}) \mathcal{g}_{a}(Z_{a})$, and $\|O\|_2 = \sigma_1(O) \leq |\mathcal{S}|$.

Next, we use Lemma 2 which is an extension of the method of moments by Anandkumar et al. (2012) that provides a bound on the accuracies of the estimated induced HMM parameters in terms of $N$, the number of samples collected. Our extension involves computing $\hat{T}$ (the original method only had $\bar{T}$ and $\bar{O}$) and bounding its accuracy.

**Lemma 2.** Given an HMM such that $p(h_2)$ has positive probability everywhere, the transition matrix is full rank, and the observation matrix is full column rank, then by gathering $N$ samples of $(x_1, x_2, x_3)$, the estimates $\hat{T}, \hat{O}, \hat{w}$ can be computed such that

$$||\hat{T} - T||_2 \leq 18|\mathcal{A}| |\mathcal{S}|^4 (\sigma_{\min}(R_{a}) \mathcal{g}_{a}(Z_{a}))^{-4} \epsilon_1$$

$$||\hat{O} - O||_2 \leq |\mathcal{A}| |\mathcal{S}|^{0.5} \epsilon_1$$

$$||\hat{w} - w||_2 \leq 14|\mathcal{A}|^2 |\mathcal{S}|^{2.5} (\sigma_{\min}(R_{a}) \mathcal{g}_{a}(Z_{a}))^{-4} \epsilon_1$$

where $|| \cdot ||_2$ is the spectral norm for matrices, and the euclidean norm for vectors, and $w$ is the marginal probability of $h_2$, with probability $1 - \delta$, as long as

$$N \geq O \left(\frac{|\mathcal{A}|^2 |\mathcal{S}| |\mathcal{R}| (1 + \sqrt{\log(1/\delta)})^2 \cdot \epsilon_1^2}{C_{d,d,d}(\delta)^2 \cdot \log\left(\frac{1}{\delta}\right)}\right)$$

Next we proceed by showing how to bound the error in the estimates of the POMDP parameters. The following Lemma 3 is a prerequisite for computing the submatrices of $\hat{O}$ and $\hat{T}$ needed for the estimates of the POMDP parameters.

**Lemma 3.** Given $\hat{O}$ with max-norm error $\epsilon_0 \leq \frac{1}{32 |\mathcal{S}|^2 |\mathcal{R}|}$, then the columns which correspond to HMM states of the form $h = (a, s', a')$ can be labeled with their corresponding $a, a'$ using Algorithm 2.

With the correct labels, the submatrices of $\hat{O}$ and $\hat{T}$ allow us to compute estimates of the original POMDP parameters in terms of these permutations $s_{(a,a')}$. Lemma 4 bounds the error in these resulting estimates.

**Lemma 4.** Given $\hat{T}, \hat{O}, \hat{w}$ with max-norm errors $\epsilon_T, \epsilon_O, \epsilon_w$ respectively, then the following bounds hold on the estimated POMDP model parameters with prob-
ability at least $1 - \delta$

$$\left| \hat{p}(s', a')|s(a, a'), a'\right| - p(s', a')|s(a, a'), a'| \leq 4|S||T|\epsilon_a$$

$$\left| \hat{p}(z|a, s(a, a')) - p(z|a, s(a, a')) \right| \leq 4\epsilon_z$$

$$\left| \hat{p}(r|s(a, a'), a') - p(r|s(a, a'), a') \right| \leq 4\epsilon_R$$

$$\left| \hat{b}(s_{a(a, a)}) - b(s_{a(a, a)}) \right| \leq 4|A|^4|S|(|T|^{-1})^2\epsilon_w + 6|T|^{-1}|S|^2\epsilon_T$$

where $\epsilon_a = \Theta(1/|A|)$

We proceed by bounding the error in computing the estimated $\beta$-vectors. Lemma 5 states that $\beta$-vectors are equivalent under permutation to $\alpha$-vectors.

**Lemma 5.** Given the permutation of the states $s(a, a'), j = s(\phi(s(a, a'), j))$, $\beta$-vectors and $\alpha$-vectors over the same policy $\pi_t$ are equivalent i.e. $\beta_t^{\pi}(s(a, a'), j) = \alpha_t^{\pi}(s(\phi(s(a, a'), j))$

The following lemma bounds the error in the resulting $\alpha$-vectors obtained by performing POMDP planning, and follows from prior work [Fard et al., 2008; Ross et al., 2009].

**Lemma 6.** Suppose we have approximate POMDP parameters with errors $|\hat{p}(s'|s, a) - p(s'|s, a)| \leq \epsilon_T$, $|\hat{p}(z|s, a') - p(z|s, a')| \leq \epsilon_z$, and $|\hat{p}(r|s, a) - p(r|s, a')| \leq \epsilon_R$. Then for any t-step conditional policy $\pi_t$

$$\left| \alpha_t^{\pi}(s) - \hat{\alpha}_t^{\pi}(s) \right| \leq t^2R_{\max}( |R| \epsilon_R + |S| \epsilon_T + |Z| \epsilon_Z)$$

We next prove that our EEPORL algorithm computes a policy that is optimal for the input parameters.

**Lemma 7.** Algorithm 3 finds the policy $\tilde{\pi}$ which maximizes $V_{\tilde{\pi}}(b(s_1))$ for a POMDP with parameters $\hat{b}(s_1), \hat{p}(z|s, a), \hat{p}(r|s, a)$, and $\hat{p}(s'|s, a)$.

We now have all the key pieces to prove our result.

**Proof.** (Proof sketch of Theorem 1). Lemma 4 shows that the error in the estimates of the POMDP parameters can be bounded in terms of the error in the induced HMM parameters, which is itself bounded in terms of the number of samples (Lemma 1). Lemma 5 and Lemma 6 together bound the error in computing the estimated value function (as represented by $\beta$-vectors) using estimated POMDP parameters.

We then need to bound the error from executing $\tilde{\pi}$ that Algorithm 3 returns compared to the optimal policy $\pi^*$. We know from Lemma 7 that Algorithm 3 correctly identifies the best policy for the estimated POMDP. Then let the initial beliefs $b, \hat{b}$ have error $||b - \hat{b}||_{\infty} \leq \epsilon_b$, and the bound over $\alpha$-vectors of any policy $\pi$, $||\pi - \hat{\pi}||_{\infty} \leq \epsilon_\alpha$ be given. Then

$$V_{\hat{\pi}}(\hat{b}) = \hat{b} \cdot \alpha_{\hat{\pi}} \geq \hat{b} \cdot \alpha_{\pi} - \epsilon_\alpha$$

The following bound is due to the triangle inequality, the fourth inequality is by the triangle inequality, the fifth inequality is since $\alpha$ is at most $V_{\max}$. Next

$$V_{\tilde{\pi}}(b) \geq b \cdot \alpha_{\tilde{\pi}} - \epsilon_b V_{\max}$$

where the first inequality is by triangle inequality, the second inequality is because $\alpha$ is at most $V_{\max}$, the third inequality is triangle inequality, and the fourth inequality is due to $||\hat{b}||_1 = 1$. Putting those two together results in

$$V_{\tilde{\pi}}(b) \geq V^*(b) - 2\epsilon_b V_{\max} - 2\epsilon_\alpha$$

Letting $\epsilon = 2\epsilon_b V_{\max} + 2\epsilon_\alpha$, and setting the number of episodes $N$ to the value specified in the theorem will ensure that the resulting errors $\epsilon_b$ and $\epsilon_\alpha$ are small enough to obtain an $\epsilon$-optimal policy as desired.

**6 CONCLUSION**

We have provided a PAC RL algorithm for an important class of episodic POMDPs, which includes many information gathering domains. To our knowledge this is the first RL algorithm for partially observable set-tant class of episodic POMDPs, which includes many information gathering domains. To our knowledge this is the first RL algorithm for partially observable settings that has a sample complexity that is a polynomial function of the POMDP parameters.

There are many areas for future work. We are interested in reducing the set of currently required assumptions, thereby creating PAC PORL algorithms that are suitable to more generic settings. Such a direction may also require exploring alternatives to method of moments approaches for performing latent variable estimation. We also hope that our theoretical results will lead to further insights on practical algorithms for partially observable RL.

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A Appendix Overview

This appendix is organized as follows. Section B contains a few generic helper lemmas. Section C contains pseudo-code of the algorithm. Section D contains some small lemmas to be used later. Section E states the main lemmas, which are also stated in the main paper. Finally, section F contains the main theorem and proof of the paper.

B Helper Lemmas

B.1 Matrix Norms Lemma

Taken from the matrix norms section in the matrix cookbook Petersen et al. (2015).

1. Induced norms (e.g. \( ||\cdot||_2 \)) are sub-multiplicative: \( ||AB||_2 \leq ||A||_2||B||_2 \)
2. \( ||A||_2 \leq ||A||_F \)
3. \( ||A||_{\text{max}} \leq ||A||_2 \) where \( ||A||_{\text{max}} = \max_{ij} |A_{ij}| \)
4. \( ||A||_{\text{max}} \leq ||A||_F \)
5. \( ||A||_F \leq \sqrt{d}||A||_2 \) where \( A \) has rank \( d \)
6. \( ||A||_F \leq \sqrt{mn}||A||_{\text{max}} \) where \( A \) is \( m \times n \)
7. \( ||A||_2 \leq \sqrt{mn}||A||_{\text{max}} \)
8. \( ||AB||_{\text{max}} \leq \sqrt{mn^2k}||A||_{\text{max}}||B||_{\text{max}} \) derived from \( ||\cdot||_2 \) where \( B \) is \( n \times k \)

B.2 Perturbed Inverse Lemma

Taken from MoM paper Anandkumar et al. (2012), which was taken from Theorem 2.5, p. 118 in Stewart et al. (1990).

If

1. \( A, E \in \mathbb{R}^{k \times k} \), \( A \) is invertible
2. \( ||A^{-1}E||_2 < 1 \) (\( ||A^{-1}||_2||E||_2 < 1 \) is sufficient)
3. \( \hat{A} = A + E \)

Then

1. \( \hat{A} \) is invertible
2. \[ ||\hat{A}^{-1} - A^{-1}||_2 \leq \frac{||E||_2||A^{-1}||_2^2}{1 - ||A^{-1}E||_2} \leq \frac{||E||_2||A^{-1}||_2^2}{1 - ||A^{-1}||_2||E||_2} \] (1)

B.3 Submatrix Eigenvalue Extreme Lemma

Let \( M \) be a \( n \times n \) symmetric matrix, which can be viewed as a block matrix

\[ M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \] (2)

where \( A \) is \( k \times k \) and \( B, C, D \) have appropriate dimensions. Then \( \lambda_n(M) \leq \lambda_k(A) \leq \lambda_1(A) \leq \lambda_1(M) \) i.e. the min and max eigenvalues of \( A \) are bounded in-between the min and max eigenvalues of \( M \).
Proof. The Rayleigh quotient with $M$ is

$$R(M, x) = \frac{x^T M x}{x^T x}$$

(3)

such that $\min_x R(M, x)$ is equal to the smallest eigenvalue $\lambda_n(M)$ and $\max_x R(M, x)$ is equal to the largest eigenvalue $\lambda_1(M)$. Let $x = [y^T z^T]$ where $y$ is column vector of size $k$. Note that $A$ is symmetric as well. Then

$$R(M, x) = \frac{[y^T z^T]}{[y^T z^T]} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix}$$

(4)

$$= y^T A y + z^T C y + y^T B z + z^T D z$$

(5)

$$\Rightarrow R(M, [y^T 0]) = \frac{y^T A y}{y^T y} = R(A, y)$$

(6)

(7)

This means that $\min_x R(M, x) \leq \min_y R(M, [y^T 0]) = \min_y R(A, y) \leq \max_y R(A, y) = \max_y R(M, [y^T 0]) \leq \max_x R(M, x)$. Thus $\lambda_n(M) \leq \lambda_k(A) \leq \lambda_1(A) \leq \lambda_1(M)$. □

B.4 Normalization Lemma

Given a nonnegative, nonzero vector $x \in \mathbb{R}^n$ and an estimated vector $\hat{x} \in \mathbb{R}^n$ such that $\|x - \hat{x}\|_\infty \leq \epsilon \leq \frac{\|x\|_1}{2n}$, let the normalization function be $\hat{x} = \text{normalize}(x)$ where $\hat{x}_i = \frac{x_i}{\|x\|_1}$. Then

$$\|\text{normalize}(x) - \text{normalize}(\hat{x})\|_\infty \leq \frac{2(x_1 n + \|x\|_1)\epsilon}{\|x\|_1^2}$$

(8)

Proof. Let $\|\text{abs}(\hat{x})\|_i = |\hat{x}_i|$. Then $|\text{abs}(\hat{x})_i - x_i| = |\hat{x}_i - x_i| \leq |\hat{x}_i - x_i|$ since $x_i \geq 0$. This implies that $\|\text{abs}(\hat{x}) - x\|_\infty \leq \|x - \hat{x}\|_\infty \leq \epsilon$. Next note that $\sum_i |x_i - \text{abs}(\hat{x})_i| \leq \sum_i |x_i - \text{abs}(\hat{x})_i|$ which means $\|x_1 - \|\text{abs}(\hat{x})\|_1 \| \leq \|x - \text{abs}(\hat{x})\|_1 \leq n \epsilon$. Note that $\|\text{abs}(\hat{x})\|_1 = \|\hat{x}\|_1$. Then

$$|\text{normalize}(x)_i - \text{normalize}(\hat{x})_i| = \left| \frac{x_i}{\|x\|_1} - \frac{\hat{x}_i}{\|\hat{x}\|_1} \right|$$

(9)

$$\leq \frac{|x_i| \hat{x}_i \|x\|_1 - |x_i - \|x\|_1|}{\|x\|_1 \|\hat{x}\|_1}$$

(10)

$$\leq \frac{|x_i| \hat{x}_i \|x\|_1 - |x_i - \|x\|_1|}{\|\hat{x}\|_1(\|x\|_1 - \epsilon)}$$

(11)

$$\leq \frac{x_i \epsilon + \|x\|_1 |x\|_1^2}{\|x\|_1^2}$$

(12)

$$\leq \frac{2(x_1 n + \|x\|_1)\epsilon}{\|x\|_1^2}$$

(13)

where the numerator of the third inequality follows from $\|x - \text{abs}(\hat{x})\|_1 \leq n \epsilon$ that was derived above. □
Algorithm 1: EEPORL

\textbf{input:} $S, A, Z, R, H, N, c, \pi_{rest}$

1. Let $\pi_{explore}$ be the policy where $a_1, a_2$ are uniformly random, and $p(a_{t+2}|a_t) = \frac{1}{1+|A|}(I+c1_{|A|\times|A|})$;
2. $X \leftarrow \emptyset$;
   // Phase 1:
   // Collect samples for an induced HMM
3. for episode $i \leftarrow 1$ to $N$
   4. Follow $\pi_{explore}$ for 4 steps;
   5. Let $x_t = (a_t, r_t, z_t, a_{t+1})$;
      // collect observation sample tuple
   6. $X \leftarrow X \cup \{x_1, x_2, x_3\}$;
   7. Execute $\pi_{rest}$ for the rest of the steps;
   // Phase 2:
   // Compute estimated POMDP parameters
8. Get $\hat{T}, \hat{O}, \hat{w}$ for the induced HMM from $X$ through an adapted MoM method;
9. Call Algorithm 2 with $\hat{O}$ to label the columns with their corresponding actions;
10. Let $s(a,a',i)$ be a permutation of states by following the ordering of the columns of $\hat{O}$ corresponding to $h = (a, \cdot, a')$;
11. Let $\hat{O}^{a,a'}$ be the submatrix where the rows and columns correspond to the actions $(a, a')$;
12. Let $\hat{T}^{a,a',a''}$ be the submatrix where the rows correspond to the actions $(a', a'')$ and columns correspond to the actions $(a, a')$;
13. Compute the following estimates of the POMDP parameters:
   \begin{align*}
   \hat{b}(s(a_0,a_1)) &= \text{normalize}((\hat{T}^{-1}\hat{T}^{-1}\hat{w})(a_0, \cdot, a_1)) \\
   \hat{p}(z|a, s(a,a')) &= \text{normalize}(\sum_r \hat{O}^{a,a'}) \\
   \hat{p}(r|s(a,a'), a') &= \text{normalize}(\sum_z \hat{O}^{a,a'}) \\
   \hat{p}(s(a',a'')|s(a,a'), a') &= \text{normalize}(\hat{T}^{a,a',a''})
   \end{align*}
   Call Algorithm 3 with the above quantities to estimate a policy $\hat{\pi}$;
14. Execute $\hat{\pi}$ for the rest of the episodes;

Algorithm 2: LabelActions

\textbf{input:} $\hat{O}$

1. foreach column $i$ of $\hat{O}$ do
2. Find a row $j$ such that $\hat{O}(i,j) \geq \frac{2}{3|A||Z|}$;
3. Let the observation associated with row $j$ be $(a, r', z', a')$, then label column $i$ with $(a, a')$;

D Small Lemmas

D.1 Spectral Norm Lemma

Lemma. For any $a \in A$
\begin{align*}
||T_a||_2 &\leq \sqrt{|S|} \quad (14) \\
||Z_a||_2 &\leq \sqrt{|S|} \quad (15) \\
||R_a||_2 &\leq \sqrt{|S|} \quad (16)
\end{align*}
**Algorithm 3: FindPolicy**

**input:**

\[
\begin{align*}
&\hat{b}(s_{(a_0,a_1)}) \\
&\hat{p}(z|a,s_{(a,a')}) \\
&\hat{p}(r|s_{(a,a')},a') \\
&\hat{p}(s_{(a',a'')}|s_{(a,a')},a')
\end{align*}
\]

// First, incrementally compute \(\beta\)-vectors for all \(t\)-step policies

\[
\begin{align*}
&\text{// } \Gamma_t^{a_{-a}} \text{ contains all } \beta\text{-vectors for } \Gamma_t^{a_{-a}} \text{ that take as input the permutation } \quad s_{(a_{-a})} \\
1 &\forall a_{-}, a \in A, \quad \Gamma_1^{a_{-a}} = \{\hat{\beta}_1^a(s_{(a_{-a})})\} ; \\
2 &\text{for } t \leftarrow 2 \text{ to } H \text{ do} \\
3 &\quad \forall a,a' \in A, \quad \Gamma_t^{a_{-a'}} = \emptyset ; \\
4 &\quad \text{for } a,a' \in A \text{ do} \\
5 &\quad \quad \text{for } f_t(r,z) \in (|R| \times |Z| \rightarrow \Gamma_{t-1}^{a_{-a'}}) \text{ do} \\
6 &\quad \quad \quad \forall a_{-} \in A, \quad \Gamma_t^{a_{-a'}} = \Gamma_t^{a_{-a'}} \cup \{\hat{\beta}_t^{a_{-a'}}(s_{(a_{-a})})\} ; \\
7 &\quad \text{Return arg max}_{a_0,a_1,\beta_H(s_{(a_0,a_1)})} \sum_{s_{(a_0,a_1)}} \hat{b}(s_{(a_0,a_1)}) \cdot \beta_H(s_{(a_0,a_1)}) ;
\end{align*}
\]

Proof. Using \(||A||_2 \leq ||A||_F = ||vec(A)||_2 \leq \sqrt{||vec(A)||_1}\) where \(vec(A)\) is the vector of the entries of the matrix \(A\), and the last inequality holds because all entries of \(A\) are at most 1

\[
\begin{align*}
||T_a||_2 &\leq \sqrt{|S|} \quad (17) \\
||Z_a||_2 &\leq \sqrt{|S|} \quad (18) \\
||R_a||_2 &\leq \sqrt{|S|} \quad (19)
\end{align*}
\]

\[\square\]

**D.2 Exploration Policy Lemma**

**Lemma.** Let \(\Pi\) be the \(|A| \times |A|\) matrix that \(\pi_{\text{explore}}\) is based on where

\[
p(a_{t+2}|a_t) = \Pi(a_{t+2},a_t)
\]

\[
= \frac{1}{1 + c|A|} (I + c1_{|A| \times |A|})
\]

and \(c > 0\) is an open parameter, and \(1_{|A| \times |A|}\) is a \(|A| \times |A|\) matrix of all ones. Then it follows that

\[
||\Pi||_2 \leq 1 \quad (22)
\]

\[
||\Pi^{-1}||_2 \leq 2(1 + c|A|) \quad (23)
\]

\[
e_a = \min_{a''} \Pi(a'',a) = \frac{c}{1 + c|A|} \quad (24)
\]

**Proof.** \(\Pi\) is a perturbed identity matrix

\[
\Pi = \frac{1}{1 + c|A|} (I + c1_{|A| \times |A|})
\]

where \(c > 0\) is a real number we can choose, and \(1_{|A| \times |A|}\) is a \(|A| \times |A|\) matrix of all ones. This yields \(\Pi\) such that each column is a probability distribution, where the off diagonal entries are all equal, and the diagonal entries
are equal and greater than the off diagonal entries. Then

$$\epsilon_a = \min_{a'' \neq a} \Pi(a'' \neq a) = \frac{c}{1 + c|A|}$$  \hspace{1cm} (26)$$

Since $1_{|A| \times |A|}$ has only a rank of 1, it can have at most one nonzero eigenvalue. Note $1_{|A| \times |A|}1_{|A|} = |A|1_{|A|}$ where $1_{|A|}$ is a column vector of ones. Then $1_{|A| \times |A|}$ has only one nonzero eigenvalue of $|A|$. Thus $\|1_{|A| \times |A|}\|_2 = |A|$. Then

$$\|\Pi\|_2 = \left\| \frac{1}{1 + c|A|}(I + c1_{|A| \times |A|}) \right\|_2 \leq \frac{1}{1 + c|A|}(\|1\|_2 + c\|1_{|A| \times |A|}\|_2) \leq \frac{1}{1 + c|A|}(1 + c|A|) \leq 1$$  \hspace{1cm} (27)$$

Next, by the Sherman-Morrison formula

$$(I + c1_{|A| \times |A|})^{-1} = (I + c1_{|A|}1_{|A|}^T)^{-1} = \frac{1}{(1 + c1_{|A|}1_{|A|}^T)}(I - c1_{|A|}1_{|A|}1_{|A|}^T)$$  \hspace{1cm} (28)$$

Then

$$\|\Pi^{-1}\|_2 = \left\| \frac{1}{1 + c|A|}(I + c1_{|A| \times |A|})^{-1} \right\|_2 \leq (1 + c|A|)\left\| (I + c1_{|A| \times |A|})^{-1} \right\|_2 \leq (1 + c|A|)\left\| (I - c1_{|A|}1_{|A|}1_{|A|}^T) \right\|_2 \leq (1 + c|A|)\left(\|I\|_2 + \frac{c}{1 + c|A|}\|1_{|A| \times |A|}\|_2 \right) \leq (1 + c|A|)(1 + 1) \leq 2(1 + c|A|)$$  \hspace{1cm} (29)$$

E Main Lemmas

E.1 Lemma 1 and Proof

**Lemma 1.** The induced HMM has the observation and transition matrices defined as

$$O(x_t^i, h_t^j) = \delta(a_t^i, a_t^j)\delta(a_{t+1}^i, a_{t+1}^j)p(z_{t+1}^i | a_t^i, s_{t+1}^j)p(r_{t+1}^i | s_{t+1}^j, a_{t+1}^j)$$

$$T(h_{t+1}^i, h_t^j) = \delta(a_{t+1}^i, a_{t+1}^j)p(s_{t+2}^i | s_{t+1}^j, a_{t+1}^j)p(a_{t+2}^i | a_{t+1}^j)$$
where \( i \) is the index over the rows and \( j \) is the index over the columns, and \( x_t^i = (a_t^i, z_{t+1}^i, r_{t+1}^i, a_{t+1}^i), h_t^i = (a_t^i, s_{t+1}^i, a_{t+1}^i) \). \( T \) and \( O \) are both full rank and \( w = p(h_2) \) has positive probability everywhere. Furthermore

\[
\|T\|_2 \leq \sqrt{|S|} \tag{41}
\]

\[
\|T^{-1}\|_2 \leq \frac{2(1+|A|)}{\sigma_2(T)} \tag{42}
\]

\[
\sigma_{\min}(O) \geq \sigma_a(R_a)\sigma_a(Z_a) \tag{43}
\]

\[
\|O\|_2 = \sigma_1(O) \leq |S| \tag{44}
\]

**Proof:**

First we will show that the Markov property holds for the newly defined hidden state. Then we will show that the middle belief has positive probability everywhere. Then we will show that the observation and transition matrices of this induced HMM are as stated and have full column rank and have bounded singular values.

### E.1.1 HMM

The HMM state is defined as \( h_t = (a_t, s_{t+1}, a_{t+1}) \), i.e. a state along with the previous and current actions. The new HMM observation will be \( x_t = (a_t, z_{t+1}, r_{t+1}, a_{t+1}) \). Note that the action is defined to only depend on the action 2 steps ago i.e. \( p(a_{t+2}|a_t) \) and is independent of everything else. First, the Markov property will be shown to hold

\[
p(h_{t+1}|\ldots, h_t) = p((a'_{t+1}, s'_{t+2}, a'_{t+2})|\ldots, (a_t, s_{t+1}, a_{t+1}))
\]

\[
= p(a'_{t+2}|a_t)p(a'_{t+1}|a_{t+1})p(s'_{t+2}|s_t, a_t)
\]

\[
= p(h_{t+1}|h_t) \tag{47}
\]

Note that \( p(a'_{t+1}|a_{t+1}) \) is just a delta function. Next, the HMM observation will be shown to only directly depend on the current HMM state

\[
p(x_t|h_{t}, \ldots, h_t) = p(a_t', z_{t+1}, r_{t+1}, a_{t+1}'), p(a_t|s_{t+1}, a_{t+1}, a_{t+1}, \ldots)
\]

\[
= p(a_t'|a_t)p(a'_{t+1}|a_{t+1})p(z_{t+1}|a_t, s_{t+1})p(r_{t+1}|s_{t+1}, a_{t+1})
\]

\[
= p(x_t|h_t) \tag{50}
\]

where \( p(a_t'|a_t) \) and \( p(a'_{t+1}|a_{t+1}) \) are essentially delta functions. Thus this formulation gives rise to an HMM.

### E.1.2 Middle Belief

By the assumptions on the initial belief of the POMDP, after any \( a_2 \), all states are reachable for \( s_3 \). This means \( p(a_2, s_3) \) has positive probability for all values. Furthermore, \( a_3 \) only depends on \( a_1 \). Since \( a_1 \) is uniformly random, by the choice of \( \Pi \), \( a_3 \) is also uniformly random. Thus \( p(a_2, s_3, a_3) = p(h_2) \) has positive probability for all values.

### E.1.3 Observation Matrix

Next, the HMM observation matrix will be derived

\[
O(x_t^i, h_t^i) = p(a_t^i, z_{t+1}^i, r_{t+1}^i, a_{t+1}^i|a_t^i, s_{t+1}^i, a_{t+1}^i)
\]

\[
= \delta(a_t^i, a_t^i)\delta(a_{t+1}^i, a_{t+1}^i)p(z_{t+1}^i|a_t^i, s_{t+1}^i)p(r_{t+1}^i|s_{t+1}^i, a_{t+1}^i) \tag{52}
\]

where \( i \) is the index over the rows and \( j \) is the index over the columns, and \( x_t^i = (a_t^i, z_{t+1}^i, r_{t+1}^i, a_{t+1}^i), h_t^i = (a_t^i, s_{t+1}^i, a_{t+1}^i) \). Also note that the \( t \) here doesn’t really matter (since the matrix is the same for all \( t \)), and is only used in a relative way to distinguish between current and next time steps.

The next claim is that \( O \) is full column rank. To see this, first permute the columns and rows of \( O \) so that they are grouped by \( (a_t^i, a_{t+1}^i) \) for the rows and \( (a_t^i, a_{t+1}^i) \) for the columns. Then the only nonzero blocks are when \( (a_t^i, a_{t+1}^i) = (a_t^i, a_{t+1}^i) \) i.e. \( O \) becomes a block diagonal matrix with rectangular blocks. Next, fix
Next, the HMM transition matrix will be derived. Then the block corresponding to \((a_t, a_{t+1})\) is made up of the entries
\[ p(s^i_{t+1}|a_t, s^j_t)p(r^i_{t+1}|s^j_t, a_{t+1}). \]
Call this block \(O^i_{a_t, a_{t+1}}\). Consider looking only at the rows where we fix \(r^i_{t+1} = r\), then the columns are just a scaled version of the columns of \(Z_a\). Since \(Z_a\) has full column rank, this implies that this block also has linearly independent columns. Since all of the diagonal blocks of \(O\) have full column rank, this implies \(O\) itself has full column rank.

Finally, consider the singular values of \(O\). Recall \(O\) can be viewed as a block diagonal matrix with rectangular blocks \(O^i_{a_t, a_{t+1}}\). Then \((O^T O)\) is a block diagonal matrix with square blocks of \(((O^i_{a_t, a_{t+1}})^T O^i_{a_t, a_{t+1}})\) on the diagonal. The eigenvalues of \((O^T O)\) are therefore just the eigenvalues of \(((O^i_{a_t, a_{t+1}})^T O^i_{a_t, a_{t+1}})\) i.e. the singular values of \(O^i_{a_t, a_{t+1}}\). Note that \(O^i_{a_t, a_{t+1}}((z, r), (s, s)) = p(z|a_t, s)p(r|s, a_{t+1})\) and can be thought of as the kronecker product \((R_{a_{t+1}} \otimes Z_a)((r, z), (s_1, s_2)) = p(z|a_t, s_1)p(r|s_2, a_{t+1})\) but with all the columns removed except for the columns in which \(s_1 = s_2 = s\). In other words the kronecker product can be thought of as the following block matrix
\[
R_{a_{t+1}} \otimes Z_a = [O^i_{a_t, a_{t+1}} B]
\]  

where \(B\) is a matrix containing the columns of the kronecker product not present in \(O^i_{a_t, a_{t+1}}\). Then
\[
(R_{a_{t+1}} \otimes Z_a)^T (R_{a_{t+1}} \otimes Z_a) = \begin{bmatrix} ((O^i_{a_t, a_{t+1}})^T O^i_{a_t, a_{t+1}}) & \cdots \\ \vdots & \ddots & \vdots \\ \cdots & \cdots & \cdots \end{bmatrix}
\]  

Then by the submatrix eigenvalue extreme lemma, it follows that \(\sigma_{\min}(R_{a_{t+1}} \otimes Z_a) \leq \sigma_{\min}(O^i_{a_t, a_{t+1}}) \leq \sigma_1(O^i_{a_t, a_{t+1}}) \leq \sigma_1(R_{a_{t+1}} \otimes Z_a)\). Then
\[
\sigma_{\min}(O) = \min_{a, a'} \min_{a, a'} \sigma_{\min}(O^i_{a_t, a_{t+1}}) \geq (\min_{a, a'} \min_{a, a'} \sigma_{\min}(O^i_{a_t, a_{t+1}})) \sigma_{\min}(Z_{a'}) \geq \sigma_{a, a'}(R_a)\sigma_{a'}(Z_a) \sigma_{\min}(O) = \sigma_{1}(O^i_{a_t, a_{t+1}}) \leq (\max_{a, a'} \sigma_1(R_a)\sigma_{a'}(Z_a)) \leq |S|
\]

where the last inequality follows from Lemma [D.1]

E.1.4 Transition Matrix

Next, the HMM transition matrix will be derived
\[
T(h^i_{t+1}, h^j_t) = p(a^j_{t+1}, s^j_{t+2}, a^j_{t+2}|a^i_{t+1}, s^i_{t+1}, a^i_{t+1}) = \delta(a^j_{t+1}, a^j_{t+1}) p(s^j_{t+2}|a^j_{t+1}, s^j_{t+1}, a^j_{t+1}) \leq (\max_{a, a'} \sigma_1(R_a)\sigma_{a'}(Z_a)) \leq |S|.
\]  

where \(i\) is the index over the rows and \(j\) is the index over the columns, and \(h^i_{t+1} = (a^i_{t+1}, s^i_{t+2}, a^i_{t+2})\), \(h^j_t = (a^j_i, s^j_{t+1}, a^j_{t+1})\).

To see that \(T\) is full rank, first permute the rows to be grouped by \(a^i_{t+1}\), and permute the columns to be grouped by \(a^j_{t+1}\). Then \(T\) becomes a block diagonal matrix, with each diagonal block composed of \(p(s^j_{t+2}|s^j_{t+1}, a^i_{t+1})p(a^j_{t+2}|a^i_{t+1})\), where \(a^i_{t+1} = a^j_{t+1} = a^i_{t+1}\). This diagonal block is actually the kronecker product of \(T_{a_{t+1}}\) and \(\Pi\). Since \(T_{a_{t+1}}\) and \(\Pi\) are both full rank, their kronecker product is also full rank. Since all the diagonal blocks are full rank, that means \(T\) is full rank.

Finally, consider the singular values of \(T\). Since \(T\) is a block diagonal matrix, the singular values are exactly the singular values of all the diagonal blocks. Each block is kronecker product \(T_a \otimes \Pi\), thus the singular values of each block is made up of the product of singular values of \(T_a\) and \(\Pi\). Therefore \(\sigma_1(T) = \max_a \sigma_1(T_a)\sigma_1(\Pi)\) and
\[ \sigma_{\text{min}}(T) = \min_a \sigma_{\text{min}}(T_a) \sigma_{\text{min}}(\Pi). \]

Thus
\[
\|T\|_2 = \max_a \|T_a\|_2 \|\Pi\|_2 \\
\leq \sqrt{|S|} \quad (64)
\]
\[
\|T^{-1}\|_2 = \frac{1}{\sigma_{\text{min}}(T)} \\
\leq \frac{1}{\min_a \sigma_{\text{min}}(T_a) \sigma_{\text{min}}(\Pi)} \quad (65)
\]
\[
\|T^{-1}\|_2 = \frac{1}{\sigma_a(T_a)} \quad (66)
\]
\[
\|T - I\|_2 \leq 18|A||S| \left( \sigma_a(R_a) \sigma_a(Z_a) \right)^{-4} \epsilon_1 \quad (67)
\]
\[
\|\hat{O} - O\|_2 \leq 4|A| |S|^{0.5} \epsilon_1 \quad (68)
\]
\[
\|\hat{O} - O\|_{\text{max}} \leq \epsilon_1 \quad (69)
\]
\[
\|\hat{w} - w\|_2 \leq 14|A|^2 |S|^{2.5} \left( \sigma_a(R_a) \sigma_a(Z_a) \right)^{-4} \epsilon_1 \quad (70)
\]

where \(\|\cdot\|_2\) is the spectral norm for matrix arguments, and the Euclidean norm for vector arguments, and \(w\) is the marginal probability of \(h_2\), with probability \(1 - \delta\), as long as
\[
\mathcal{O} \left( \frac{|A|^2 |Z| |R|(1 + \sqrt{\log(1/\delta)})^2}{(C_{d, d, d}(\delta))^2 \cdot \epsilon_1} \log \left( \frac{1}{\delta} \right) \right) \leq N \quad (71)
\]

**E.2 Lemma and Proof**

**Lemma 2.** Given an HMM such that the marginal probability of \(h_2\) has positive probability everywhere, the transition matrix is full rank, and the observation matrix is full column rank, then by gathering \(N\) samples of \((x_1, x_2, x_3)\), the estimates \(\hat{T}, \hat{O}, \hat{w}\) can be computed such that
\[
\|\hat{T} - T\|_2 \leq 18|A||S|^{4} \left( \sigma_a(R_a) \sigma_a(Z_a) \right)^{-4} \epsilon_1 \quad (72)
\]
\[
\|\hat{O} - O\|_2 \leq 4|A| |S|^{0.5} \epsilon_1 \quad (73)
\]
\[
\|\hat{w} - w\|_2 \leq 4|A|^2 |S|^{2.5} \left( \sigma_a(R_a) \sigma_a(Z_a) \right)^{-4} \epsilon_1 \quad (74)
\]

where \(\|\cdot\|_2\) is the spectral norm for matrix arguments, and the Euclidean norm for vector arguments, and \(w\) is the marginal probability of \(h_2\), with probability \(1 - \delta\), as long as
\[
\mathcal{O} \left( \frac{|A|^2 |Z| |R|(1 + \sqrt{\log(1/\delta)})^2}{(C_{d, d, d}(\delta))^2 \cdot \epsilon_1} \log \left( \frac{1}{\delta} \right) \right) \leq N \quad (75)
\]

**Proof:**

The proof will first use prior method of moments work to derive bounds on the estimation error on \(O\) and \(OT\) of the induced HMM. Then using those estimates, we will show how to compute an estimate of \(T\) and bound its error. Then, we will show how to estimate \(w\) and its error. Finally we will give a sufficient lower bound on \(N\) in order to achieve an estimation error of \(\epsilon_1\) on the estimated matrices.

**E.2.1 Method of Moments**

Theorem 3.1 from Anandkumar et al. (2012) states that given \(\epsilon_2\), if the following is satisfied (i.e. there are at least this many samples of \((x_1, x_2, x_3)\))
\[
\frac{1 + \sqrt{\log(1/\delta)}}{\sqrt{N}} \leq C \cdot C_{1, 2, 3}(\delta) \cdot \epsilon_3 \quad (76)
\]
\[
\Leftrightarrow \frac{1 + \sqrt{\log(1/\delta)}}{C \cdot C_{1, 2, 3}(\delta) \cdot \epsilon_3} \leq \sqrt{N} \quad (77)
\]
\[
\Leftrightarrow \frac{(1 + \sqrt{\log(1/\delta)})^2}{C^2(C_{1, 2, 3}(\delta))^2 \cdot \epsilon_3^2} \leq N \quad (78)
\]

and also given \(\epsilon_2\), if the following is also satisfied
\[
\frac{(1 + \sqrt{\log(1/\delta)})^2}{C^2(C_{3, 1, 2}(\delta))^2 \cdot \epsilon_2^2} \leq N \quad (79)
\]
where

\[ C_{1,2,3}(\delta) = \min \left( \frac{\min_{i \neq j} \| M_2(\bar{e}_i - \bar{e}_j) \|_2 \cdot \sigma_k(P_{1,2})^2}{\| P_{1,2,3} \|_2 \cdot k^3 \cdot \kappa(M_1)^4}, \frac{\delta}{\log(k/\delta)} \cdot \frac{\sigma_k(P_{1,3})^2}{1} \right) \]  

(79)

\[ C_{1,3,2}(\delta) = \min \left( \frac{\min_{i \neq j} \| M_2(\bar{e}_i - \bar{e}_j) \|_2 \cdot \sigma_k(P_{1,3})^2}{\| P_{1,3,2} \|_2 \cdot k^3 \cdot \kappa(M_1)^4}, \frac{\delta}{\log(k/\delta)} \cdot \frac{\sigma_k(P_{1,2})^2}{1} \right) \]  

(80)

\[ C_{d,d,d}(\delta) = \min(C_{1,2,3}(\delta), C_{1,3,2}(\delta)) \]  

(81)

then

\[ \| col(\hat{O}) - col(O) \|_2 \leq \epsilon_2 \]  

(82)

\[ \| col(\hat{O}T) - col(OT) \|_2 \leq \epsilon_3 \]  

(83)

with probability 1 − δ. In other words, the estimated parameters are close in the Euclidean norm. Note that the columns of \( \hat{O} \) and \( \hat{O}T \) may be permuted from \( O \) and \( OT \), however \( \hat{O} \) and \( \hat{O}T \) are permuted in the same unknown way to match. No knowledge of the permutation is necessary.

Then by matrix norms

\[ \| \hat{O} - O \|_2 \leq \| \hat{O} - O \|_F \]  

(84)

\[ \leq |A| \sqrt{|S|} \epsilon_2 \]  

(85)

\[ \| \hat{O} - O \|_{\infty} \leq \max_i \| \text{col}_i(\hat{O}) - \text{col}_i(O) \|_{\infty} \]  

(86)

\[ \leq \max_i \| \text{col}_i(\hat{O}) - \text{col}_i(O) \|_2 \]  

(87)

\[ \leq \epsilon_2 \]  

(88)

\[ \| \hat{O}T - OT \|_2 \leq |A| \sqrt{|S|} \epsilon_3 \]  

(89)

using the fact that from the Euclidean distances of the columns we can conclude that the Frobenius norm is \( \sqrt{\sum_{\text{ncols}} \epsilon_2^2} = \epsilon_2 \sqrt{|A|^2|S|} \).

### E.2.2 Constructing Transition Matrix Estimate

The pseudoinverse of \( O \) can be used to compute the HMM transition matrix \( T \) since \( O \) is full column rank

\[ O^+ OT = (O^T O)^{-1} O^T OT \]  

(90)

\[ = T \]  

(91)

Then following the same procedure, the HMM transition matrix estimate can be computed

\[ \hat{T} = \hat{O}^+ \hat{O}T \]  

(92)

\[ = (\hat{O}^T \hat{O})^{-1} \hat{O}^T \hat{O}T \]  

(93)

Next, to compute the error

\[ \| \hat{T} - T \|_2 = \| (\hat{O}^T \hat{O})^{-1} \hat{O}^T \hat{O}T - (O^T O)^{-1} O^T OT \|_2 \]  

(94)

first some intermediate quantities will be computed. To begin, because we will be computing inverses, we need to bound the accuracy \( \epsilon_2 \) in order to use the perturbed inverse lemma; therefore we first assume the following condition

**Condition 1.**

\[ \epsilon_2 \leq \frac{1}{6|A||S|^{1.5}(\mathbb{C}(R_a)\mathbb{C}(Z_a))^{-4}} \]  

(95)
Let $\hat{\sigma}$ be the distribution of $|O(T)|^{-1}$ combined with the above condition. Now the error in estimating the HMM transition matrix will be computed

$$||\hat{T} - T||_2$$

where we substitute in values from lemma 1 and the estimation errors calculated above. The next simplification is by letting $\epsilon_2 = \epsilon_3 = \epsilon_1$, then

$$||\hat{T} - T||_2$$

$$\leq 6A||S|^{1.5}||O(T)^{-1}||_2^2 \epsilon_1 + |A||S|^{0.5} \epsilon_1 ||S|^{1.5}||O(T)^{-1}||_2 + |S||A||S|^{0.5} \epsilon_1 ||O(T)^{-1}||_2$$

E.2.3 Estimating $w$

Let $X_2$ be random variable for $x_2$ i.e. the HMM observation of the second step. Let $p$ be the distribution of $x_2$. Let $\hat{p}$ be the empirical distribution made from the counts of the samples from $x_2$ obtained from the exploring
episodes. Then from [Weissman et al. 2003] we can bound the deviation
\[
\delta_p = P(||\hat{\theta} - \theta||_1 \geq \epsilon_1) \leq \left(2|A|^2|R||Z| - 2\right)e^{-N \epsilon_1^2 / 2}
\]  
(122)

\[
\iff N \geq O \left( \log \left( \frac{1}{\delta_p} \right) \frac{|A|^2|R||Z|}{\epsilon_1^2} \right)
\]  
(123)

and get a sufficient condition on how many samples we need. \(w = p(h_2)\) can be computed by
\[
p = Ow
\]  
(124)

\[
\implies w = (O)^+p = ((O)^T(O))^{-1}(O)^Tp
\]  
(125)

and thus estimated using \(\hat{O}\) and \(\hat{p}\). Next, the error of the estimate will be bounded. Let \(||\hat{\theta} - \theta||_2 = \epsilon_1\). Note that \(||(O^T\hat{O})^{-1} - (O^TO)^{-1}||_2\) has been computed in the previous section.

\[
||\hat{w} - w||_2 = ||(O^T\hat{O})^{-1}O^T\hat{p} - (O^TO)^{-1}O^Tp||_2
\]  
(127)

\[
\leq ||(O^T\hat{O})^{-1}O^T\hat{p} - (O^TO)^{-1}O^Tp||_2 + ||(O^TO)^{-1}O^T\hat{p} - (O^TO)^{-1}O^Tp||_2
\]  
(128)

\[
\leq ||O^T||_2||\hat{p}||_2||\hat{O}^{-1} - (O^TO)^{-1}||_2 + ||(O^TO)^{-1}||_2||\hat{O}^{-1} - (O^TO)^{-1}||_2
\]  
(129)

\[
\leq ||O^T||_2||\hat{O}^{-1} - (O^TO)^{-1}||_2 + ||(O^TO)^{-1}||_2||\hat{O}^{-1} - (O^TO)^{-1}||_2
\]  
(130)

\[
\leq ||O||_2 + ||O^T - O||_2 + ||(O^TO)^{-1}||_2 + ||(O^TO)^{-1}||_2 + ||(O^TO)^{-1}||_2 + ||(O^TO)^{-1}||_2
\]  
(131)

\[
\leq 14|A|^2|S|^{2.5}||O^TO||^{-1}||\hat{\theta}||^2 \epsilon_1
\]  
(132)

\[
\leq 14|A|^2|S|^{2.5}(\sigma_{a}(R_a)\sigma_{a}(Z_a))^{-4} \epsilon_1
\]  
(133)

where the first inequality is from using triangle inequality; the third inequality uses the fact that \(||\hat{\theta}||_1 \leq 1\) and also another triangle inequality; the fifth inequality substitutes in values computed from previous sections; the sixth inequality comes from assuming that \(||(O^TO)^{-1}||_2 \leq ||(O^TO)^{-1}||_2^2\), which is implied by assuming that \(\sigma_{a}(R_a) \geq 1\) and \(\sigma_{a}(Z_a) \geq 1\). To be precise, we should instead use the quantities \(\max(\sigma_{a}(R_a)^{-1}, 1)\) and \(\max(\sigma_{a}(Z_a)^{-1}, 1)\). In the final sample complexity bound, we will implicitly use the versions with the max operator wherever the quantities \(\sigma_{a}(R_a)^{-1}\) and \(\sigma_{a}(Z_a)^{-1}\).

### E.2.4 Combining the Conditions

Recall eqn (78)
\[
\frac{(1 + \sqrt{\log(1/\delta)})^2}{C^2(C_{d,1,2}(\delta))^2} \leq N
\]  
(136)

which needs to be satisfied in order to get estimation error bounds on \(\epsilon_2\) (and respectively \(\epsilon_3\)). Combining these requirements with the one from estimating \(w\) (eqn (123)), and letting \(\epsilon_2 = \epsilon_3 = \epsilon_1\), we get a single requirement of
\[
O \left( \frac{|A|^2|R||Z|(1 + \sqrt{\log(3/\delta)})^2}{C^2(C_{d,d,d}(\delta/3))^2} \log \left( \frac{3}{\delta} \right) \right) \leq N
\]  
(137)

where we give each of the three requirements an error probability of \(\delta/3\).

### E.3 Lemma and Proof

**Lemma 3.** Given \(\hat{O}\) with max-norm error \(\epsilon_0 \leq \frac{1}{\sqrt{|Z||R|}}\), then the columns which correspond to HMM states of the form \(h = (a, s', a')\) can be labeled with their corresponding \(a, a'\) using Algorithm 2.
Proof:

Recall $O(i, j) = p(x_t = i | h_t = j) = \delta(a_i, a'_i) \delta(a'_{i+1}, a_{i+1}) p(z'_{i+1} | a'_i, s'_{i+1}) p(r_{i+1} | s'_{i+1}, a'_{i+1})$. Consider a column of $O$. The column corresponds to some HMM state $j = (a'_i, s'_{i+1}, a'_{i+1})$, but the $(a'_i, s'_{i+1}, a'_{i+1})$ are unknown and only the index $j$ is known. However the rows, which correspond to HMM observation $i = (a_i, z'_i, r'_i, s'_{i+1}, a'_{i+1})$, are known since the observations are fully observed. The only entries in this column that can be nonzero are the rows where the actions match i.e. $(a_i, a_{i+1}) = (a'_i, a'_{i+1})$. There are $|Z||R|$ number of these entries. The nonzero entries form a probability distribution and must sum to one. Therefore, the largest nonzero value in this column must be at least $\frac{1}{|Z||R|}$. Since by assumption $\epsilon_O \leq \frac{1}{3(|Z||R|)}$, the largest nonzero value in $O$ must be at least $\frac{2}{3|Z||R|}$. Thus there exist at least one entry in this column of $O$ that is at least $\frac{2}{3|Z||R|}$ and where the actions match, so it is possible to encounter this during Algorithm 2. Also, Algorithm 2 cannot pick entries from $\hat{O}$ that are zero in $O$, since those entries would be at most only $\frac{1}{3|Z||R|}$. Thus Algorithm 2 will always pick an entry with matching actions, and correctly label the HMM state corresponding to each column with the matching actions.

E.4 Lemma 4 and Proof

Lemma 4. Given $\hat{T}, \hat{O}, \hat{w}$ with max-norm errors $\epsilon_T, \epsilon_O, \epsilon_w$ respectively, then the following bounds hold on the estimated POMDP model parameters with probability at least $1 - \delta$:

$$|\tilde{P}(s(a', a''), s(a, a')) - p(s(a', a''), s(a, a'))| \leq \frac{4|S|\epsilon_T}{\epsilon_a}$$ (138)

$$|\tilde{P}(z|a, s(a, a')) - p(z|a, s(a, a'))| \leq 4Z|R|\epsilon_O$$ (139)

$$|\tilde{P}(r|s(a, a'), a') - p(r|s(a, a'), a')| \leq 4Z|R|\epsilon_O$$ (140)

$$|\tilde{b}(s(a_0, a_1)) - b(s(a_0, a_1))| \leq 4A^4|S|(|T^{-1}|2\epsilon_w + 6||T^{-1}||^2\epsilon_T)$$ (142)

where $\epsilon_a = \Theta(1/|A|)$

Proof:

First we will show the errors for the estimated observation, reward, and transition parameters. Then we will show the errors for the estimated initial belief.

E.4.1 Observation, Reward, and Transition

By the normalization lemma for $\sum_r \hat{O}_{HMM,a,a'}((z, r), s(a, a'))$ over $z$

$$|\tilde{P}(z|a, s(a, a')) - p(z|a, s(a, a'))|$$

$$= \|\text{normalize } \left( \sum_r O_{HMM,a,a'}((z, r), s(a, a')) \right) - \text{normalize } \left( \sum_r \hat{O}_{HMM,a,a'}((z, r), s(a, a')) \right) \|_{\infty}$$ (144)

$$\leq \frac{2(|Z| + 1)|R|\epsilon_O}{1}$$ (145)

By the normalization lemma for $\sum_z \hat{O}_{HMM,a,a'}((z, r), s(a, a'))$ over $r$

$$|\tilde{P}(r|a, s(a, a')) - p(r|a, s(a, a'))|$$

$$= \|\text{normalize } \left( \sum_z O_{HMM,a,a'}((z, r), s(a, a')) \right) - \text{normalize } \left( \sum_z \hat{O}_{HMM,a,a'}((z, r), s(a, a')) \right) \|_{\infty}$$ (148)

$$\leq \frac{2(|R| + 1)|Z|\epsilon_O}{1}$$ (149)
For the transition estimates, first note
\[
\sum_{s(a',a'')} T_{HMM,a,a',a''}(s(a',a'')|s(a,a')) = \sum_{s(a,a'')} p(s(a',a'')|s(a',a),a')p(a''|a) = p(a''|a) = \epsilon_a
\] (151)
(152)
(153)

Then by the Normalization Lemma
\[
|p(s(a',a'')|s(a',a),a') - \bar{p}(s(a',a'')|s(a',a),a')| = \|\text{normalize}(T_{HMM,a,a',a''}(s(a',a'')|s(a',a)j)) - \text{normalize}(\tilde{T}_{HMM,a,a',a''}(s(a',a'')|s(a',a')j))\|_\infty \leq \frac{2(|S| + \epsilon_a)\epsilon_T}{\epsilon_a^2}
\] (154)
(155)
(156)
(157)

E.4.2 Initial distribution

First, consider \(||T^{-1} - \tilde{T}^{-1}||_2\). In order to apply the perturbed inverse lemma, we start be assuming the following condition holds

Condition 2.
\[
\epsilon_T \leq \frac{1}{2||T^{-1}||_2}
\] (158)

Then it follows by the perturbed inverse lemma that
\[
||\tilde{T}^{-1} - T^{-1}||_2 \leq \frac{\epsilon_T||T^{-1}||_2^3}{1 - ||T^{-1}||_2^2\epsilon_T} \leq 2\epsilon_T||T^{-1}||_2^2
\] (159)
(160)
(161)

Then (note \(|| \cdot ||_2\) is Euclidean when the inside is a vector, and the operator norm when the inside is a matrix)
\[
||T^{-1}T^{-1}w - \tilde{T}^{-1}\tilde{T}^{-1}\tilde{w}||_2 \leq ||T^{-1}T^{-1}w - T^{-1}T^{-1}\tilde{w}||_2 + ||T^{-1}T^{-1}\tilde{w} - \tilde{T}^{-1}\tilde{T}^{-1}\tilde{w}||_2 \leq ||T^{-1}T^{-1}||_2||w - \tilde{w}||_2 + ||T^{-1}T^{-1} - \tilde{T}^{-1}\tilde{T}^{-1}||_2||\tilde{w}||_2 \leq ||T^{-1}||_2^2\epsilon_w + ||T^{-1}T^{-1} - \tilde{T}^{-1}\tilde{T}^{-1}||_2 \cdot 1
\] (162)
(163)
(164)
(165)

then that inner term can be simplified
\[
||T^{-1}T^{-1} - \tilde{T}^{-1}\tilde{T}^{-1}||_2 \leq ||T^{-1}T^{-1} - T^{-1}\tilde{T}^{-1}||_2 + ||T^{-1}\tilde{T}^{-1} - \tilde{T}^{-1}\tilde{T}^{-1}||_2 \leq ||T^{-1}||_2||T^{-1} - \tilde{T}^{-1}||_2 + ||\tilde{T}^{-1}||_2||T^{-1} - \tilde{T}^{-1}||_2 \leq (||T^{-1}||_2 + ||\tilde{T}^{-1}||_2)||T^{-1} - \tilde{T}^{-1}||_2 \leq (2||T^{-1}||_2 + ||T^{-1} - \tilde{T}^{-1}||_2)||T^{-1} - \tilde{T}^{-1}||_2 \leq (2||T^{-1}||_2 + 2\epsilon_T||T^{-1}||_2^2)2\epsilon_T||T^{-1}||_2^2 \leq (2||T^{-1}||_2 + ||T^{-1}||_22\epsilon_T)||T^{-1}||_2^2 \leq 6||T^{-1}||_2^2\epsilon_T \leq 6||T^{-1}||_2^2
\] (166)
(167)
(168)
(169)
(170)
(171)
(172)
(173)
(174)
Thus plugging that back that inner term results in
\[
\|T^{-1}T^{-1}w - \hat{T}^{-1}\hat{T}^{-1}\hat{w}\|_2 \leq \|T^{-1}\|_2^2\epsilon_w + 6\|T^{-1}\|_2^3\epsilon_T \tag{175}
\]
\[
\implies \|T^{-1}T^{-1}w - \hat{T}^{-1}\hat{T}^{-1}\hat{w}\|_\infty \leq \|T^{-1}\|_2^2\epsilon_w + 6\|T^{-1}\|_2^3\epsilon_T \tag{176}
\]
\[
\implies |p(a_0, s(a_0, a_1), a_1) - \hat{p}(a_0, s(a_0, a_1), a_1)| \leq \|T^{-1}\|_2^2\epsilon_w + 6\|T^{-1}\|_2^3\epsilon_T \tag{177}
\]
so now the final step is to extract out only the entries corresponding to \((a_0, a_1)\) and normalize (the normalization will be over \(s(a_0, a_1)\)). Assuming that the following condition holds

**Condition 3.**
\[
(\|T^{-1}\|_2^2\epsilon_w + 6\|T^{-1}\|_2^3\epsilon_T) \leq \frac{1}{2|A|^2|S|} \tag{178}
\]
then by the Normalization Lemma
\[
|\hat{p}(s(a_0, a_1)) - p(s(a_0, a_1))| = |\hat{p}(a_0, s(a_0, a_1), a_1) - p(a_0, s(a_0, a_1), a_1)| \tag{179}
\]
\[
\leq 2|A|^4(|S| + \frac{1}{|A|^2})(\|T^{-1}\|_2^2\epsilon_w + 6\|T^{-1}\|_2^3\epsilon_T) \tag{180}
\]
noting that \(\sum_{s(a_0, a_1)} p(a_0, s(a_0, a_1), a_1) = p(a_0, a_1) = p(a_0)p(a_1) = \frac{1}{|A|^2}\) (since the first two actions are uniformly random).

**E.5 Lemma 5 and Proof.**

**Lemma 5.** Given the permutation of the states \(s(a,a',j) = s_{\phi(a,a'),j}\), \(\beta\)-vectors and \(\alpha\)-vectors over the same policy \(\pi_t\) are equivalent i.e.
\[
\beta_t^{\pi_t}(s(a,a'),j) = \alpha_t^{\pi_t}(s_{\phi(a,a'),j}) \tag{182}
\]

**Proof:**

Note that we are using the notation \(a_-\) as just another variable for actions, just like the notation \(a\). First the base case
\[
\beta_0^a(s(a_-,a),j) = \sum_r p(r|s(a_-,a),j) \cdot r \tag{183}
\]
\[
= \sum_r p(r|s_{\phi(a_-,a),j}) \cdot r \tag{184}
\]
\[
= \alpha_0^a(s_{\phi(a_-,a)}) \tag{185}
\]
Next is the induction step. The induction hypothesis is that this equivalence holds for all \(\beta\)-vectors with \(t\)-step policies i.e. \(\beta_t^{\pi_t}(s(a_-,a)) = \alpha_t^{\pi_t}(s_{\phi(a_-,a)})\) where \(a\) is the root action of \(\pi_t\). Also note that \(f_t(r,z)\) is the rest of conditional policy \(\pi_t\) after executing \(a\) and seeing \((r,z)\). Then
\[
\beta_{t+1}^{a,f_t}(s(a_-,a),j) \tag{186}
\]
\[
= \sum_{r,z} p(r|s(a_-,a),a)p(s(a,f_t(r,z))|s(a_-,a),a)p(z|s(a,f_t(r,z))a)(r + \gamma\beta_{t+1}^{a,f_t}(s(a,f_t(r,z),k))) \tag{187}
\]
\[
= \sum_{r,z} p(r|s_{\phi(a_-,a),j},a)p(s_{\phi(a,f_t(r,z),k)}|s_{\phi(a_-,a),j},a)p(z|s_{\phi(a,f_t(r,z),k)},a)(r + \gamma\alpha_{t+1}^{a,f_t}(s_{\phi(a,f_t(r,z),k)})) \tag{188}
\]
where we use the induction hypothesis. In order to simplify further, consider the partial term

$$g(r, z) = \sum_{s(a, f_t(r, z))} p(r|s(a_{-a}, a_j), a) p(s_{a_{a}, f_t(r, z), k}) | s_{a_{a}, a_j}, a) p(z|s_{a_{a}, a_j}, a) (r + \gamma f_t(r, z) (s_{a_{a}, f_t(r, z), k}))$$

Since this is a sum, the order of summation does not matter. In particular, the summation can be done in the order of the original state \(s_i\) where \(i = \phi(a, f_t(r, z), k)\).

$$g(r, z) = \sum_{s_i} p(r|s(a_{-a}, a_j), a) p(s_i|s(a_{-a}, a_j), a) p(z|s_i, a) (r + \gamma f_t(r, z) (s_i))$$

Then

$$\beta_{t+1}^{n, f_t}(s(a_{-a}, a_j)) = \sum_{r, z} g(r, z)$$

$$= \sum_{r, z, s_i} p(r|s(a_{-a}, a_j), a) p(s_i|s(a_{-a}, a_j), a) p(z|s_i, a) (r + \gamma f_t(r, z) (s_i))$$

$$= \alpha_{t+1}^{n, f_t}(s(a_{-a}, a_j))$$

### E.6 Lemma and Proof

**Lemma 6.** Suppose we have approximate POMDP parameters with errors

$$|\hat{p}(s'|s, a) - p(s'|s, a)| \leq \epsilon_T$$

$$|\hat{p}(z|a, s') - p(z|a, s')| \leq \epsilon_Z$$

$$|\hat{p}(r|s, a) - p(r|s, a')| \leq \epsilon_R$$

then for any \(t\)-step conditional policy \(\pi_t\)

$$|\alpha_{t}^{\pi_t}(s) - \hat{\alpha}_{t}^{\pi_t}(s)| \leq t^2 R_{max}(|R|\epsilon_R + |S|\epsilon_T + |Z|\epsilon_Z).$$

**Proof:**

First the base case for 1-step policies

$$|\alpha_{1}^{\pi_1}(s) - \hat{\alpha}_{1}^{\pi_1}(s)| = \left| \sum_r p(r|s, a) \cdot r - \sum_r \hat{p}(r|s, a) \cdot r \right|$$

$$\leq \sum_r |p(r|s, a) - \hat{p}(r|s, a)| \cdot r$$

$$\leq \sum_r |p(r|s, a) - \hat{p}(r|s, a)| \cdot r$$

$$\leq \sum_r |p(r|s, a) - \hat{p}(r|s, a)|$$

$$\leq |R| R_{max} \epsilon_R$$

$$\leq R_{max}(|R|\epsilon_R + |S|\epsilon_T + |Z|\epsilon_Z)$$

Next the induction step, where \(V_{max}(t)\) is the upper bound on the value for \(t\) steps, and where the induction
Algorithm 3 finds the policy \( \hat{\alpha}_{f_t(r,z)}(s) \), which maximizes \( V^\hat{\pi}(\hat{b}(s_1)) \) for a POMDP with parameters \( \hat{b}(s_1), \hat{p}(s|a,s'), \hat{p}(r|s,a), \) and \( \hat{p}(s'|s,a) \).

**Proof:**

The outer loop builds up the set \( \Gamma_{t-1}^{a_{t-1},a} \) of beta vectors that take as input \( s(a_{t-1},a) \) with \( a \) as the root action. This is true in the base case for \( \Gamma_{t-1}^{a_{t-1},a} \). For the induction step, fix \( (a_{t-1},a) \). Then \( f_t(r,z) \) is taken from all possible mappings from an observation pair to \( \beta_{t-1}(s(a_{t-1},a')) \in \Gamma_{t-1}^{a_{t-1},a'} \) and all possible next actions \( a' \). Thus all possible \( \beta_t(s(s(a_{t-1},a)),a) \) are computed.

The final step is an argmax. Since \( \hat{p}(s_1(a_0,a_1)) \) is just a permutation of \( \hat{p}(s_1) \), using it will not change the dot product \( \hat{p}(s_1(a_0,a_1)) \cdot \beta_H(s(a_0,a_1)) \). Thus the argmax is correctly finding the policy associated with \( V^*(\hat{p}(s_1)) \).

**E.7 Lemma 7 and Proof**

**Lemma 7.** Algorithm 3 finds the policy \( \hat{\pi} \) which maximizes \( V^\hat{\pi}(\hat{b}(s_1)) \) for a POMDP with parameters \( \hat{b}(s_1), \hat{p}(s|a,s'), \hat{p}(r|s,a), \) and \( \hat{p}(s'|s,a) \).

**Proof:**

The outer loop builds up the set \( \Gamma_{t-1}^{a_{t-1},a} \) of beta vectors that take as input \( s(a_{t-1},a) \) with \( a \) as the root action. This is true in the base case for \( \Gamma_{t-1}^{a_{t-1},a} \). For the induction step, fix \( (a_{t-1},a) \). Then \( f_t(r,z) \) is taken from all possible mappings from an observation pair to \( \beta_{t-1}(s(a_{t-1},a')) \in \Gamma_{t-1}^{a_{t-1},a'} \) and all possible next actions \( a' \). Thus all possible \( \beta_t(s(s(a_{t-1},a)),a) \) are computed.

The final step is an argmax. Since \( \hat{p}(s_1(a_0,a_1)) \) is just a permutation of \( \hat{p}(s_1) \), using it will not change the dot product \( \hat{p}(s_1(a_0,a_1)) \cdot \beta_H(s(a_0,a_1)) \). Thus the argmax is correctly finding the policy associated with \( V^*(\hat{p}(s_1)) \).

**F Main Theorem**

**Theorem 1.** For POMDPs that satisfy the stated assumptions defined in the problem setting, executing EEPORL will achieve an expected episodic reward of \( V(b_0) \geq V^*(b_0) - \epsilon \) on all but a number of episodes that is bounded by

\[
O \left( \frac{H^4 V_{max}^2 |A|^2 |R|^4 |Z|^4 |S|^2 (1 + \sqrt{\frac{3}{\delta}}) \left( \frac{3}{\delta} \right) \log \left( \frac{3}{\delta} \right)}{C_{d,d,d} \left( \frac{T_T}{\delta} \right) \sigma_r(T_a)^6 \sigma_r(R_a)^8 \sigma_r(Z_a)^8 \epsilon^2} \right)
\]
Lemma 4

Note that with probability at least $1 - \delta$, where

$$C_{d,d}(\delta) = \min(C_{1,2,3}(\delta), C_{1,3,2}(\delta))$$

where

$$C_{1,2,3}(\delta) = \min \left( \min_{\pi \neq \hat{\pi}} \left( \frac{\|M_b(\hat{\pi}, \hat{\pi})\|_2 \cdot \sigma_k(P_{1,2})}{\|P_{1,2,3}\|_2 \cdot k^5 \cdot \kappa(M_1)^4} \cdot \frac{\delta}{\log(k/\delta)} \cdot \frac{\sigma_k(P_{1,3})}{1} \right) \right)$$

$$C_{1,3,2}(\delta) = \min \left( \min_{\pi \neq \hat{\pi}} \left( \frac{\|M_b(\hat{\pi}, \hat{\pi})\|_2 \cdot \sigma_k(P_{1,2})}{\|P_{1,3,2}\|_2 \cdot k^5 \cdot \kappa(M_1)^4} \cdot \frac{\delta}{\log(k/\delta)} \cdot \frac{\sigma_k(P_{1,2})}{1} \right) \right)$$

Proof:

Let the initial beliefs $b, \hat{b}$ and error $\|b - \hat{b}\|_\infty \leq \epsilon_b$, and the bound over $\alpha$-vectors of any policy $\pi, \|\alpha^\pi - \hat{\alpha}^\pi\|_\infty \leq \epsilon_\alpha$ be given. Let $\hat{\pi}$ be the policy returned by Algorithm 3 i.e the optimal policy for $\hat{b}$ and $\hat{\alpha}$. Let $\pi^*$ be the optimal policy for $b$ and $\alpha$. Then

$$\hat{V}^\pi(\hat{b}) = \hat{b} \cdot \hat{\alpha}^\pi$$

(217)

$$\geq \hat{b} \cdot \hat{\alpha}^\pi - [\hat{b} \cdot \alpha^\pi - \hat{b} \cdot \hat{\alpha}^\pi]$$

(218)

$$\geq \hat{b} \cdot \alpha^\pi - \epsilon_\alpha$$

(219)

$$\geq b \cdot \alpha^\pi - [b \cdot \alpha^\pi - \hat{b} \cdot \hat{\alpha}^\pi] - \epsilon_\alpha$$

(220)

$$\geq b \cdot \alpha^\pi - \epsilon_\alpha V_{\max} - \epsilon_\alpha$$

(221)

$$\geq V^*(b) - \epsilon_\alpha V_{\max} - \epsilon_\alpha$$

(222)

Putting those two together results in

$$\hat{V}^\pi(\hat{b}) \geq V^*(b) - 2\epsilon_\alpha V_{\max} - 2\epsilon_\alpha$$

(223)

Plugging in $\epsilon_b$ and $\epsilon_\alpha$ from Lemma 4 and Lemma 3 gets us

$$\hat{V}^\pi(\hat{b}) \geq V^*(b) - 2(4|A|^4|S|(|T^{-1}|^2 \epsilon_w + 6|T^{-1}|^3 \epsilon_T)) V_{\max} - 2(H^2 R_{\max}(|R| \epsilon_R + |S| \epsilon_T + |Z| \epsilon_Z))$$

(224)

We know $|T^{-1}|^2 = 2(1 + c|A|)(\sigma_a(T_a))^{-1}$ and $\epsilon_a = \frac{c}{1 + c|A|}$ from Lemma 4 and the exploration policy lemma. Note that $c = O(1/|A|)$ and so $\epsilon_a = O(1/|A|)$. Now let’s carefully substitute in quantities from Lemma 2 and Lemma 4

$$\hat{V}^\pi(\hat{b}) \geq V^*(b) - 2(4|A|^4|S|(|T^{-1}|^2 \epsilon_w + 6|T^{-1}|^3 \epsilon_T)) (\sigma_a(T_a))^{-4} \epsilon_1$$

(225)

$$\leq |T^{-1}|^2 (14|A|^2|S|^2 \sigma_a(R_a) \sigma_a(Z_a))^{-4} \epsilon_1$$

(226)

$$+ 6|T^{-1}|^3 (18|A||S|^2 \sigma_a(R_a) \sigma_a(Z_a))^{-4} \epsilon_1$$

(227)

$$\leq 122 |T^{-1}|^2 |A|^2|S|^4 (\sigma_a(R_a) \sigma_a(Z_a))^{-4} \epsilon_1$$

(228)

$$\leq 122(8(1 + c|A|^2)^3 |a|^3 |Z| |S|^4 (\sigma_a(R_a) \sigma_a(Z_a))^{-4} \epsilon_1$$

(229)

$$\leq O(|A|^2|S|^4 (\sigma_a(T_a)^{-3}|a|^3 |Z| |S|^4 (\sigma_a(R_a) \sigma_a(Z_a))^{-4} \epsilon_1$$

(230)
Lemma 2 to get the following requirement on $N$

\[
\begin{align*}
&\leq (|R| |Z||R|\epsilon_O) + |S| \frac{4|S|\epsilon_T}{\epsilon_0^2} + |Z| (|Z||R|\epsilon_O)) \\
&\leq |R|(4|Z||R|\epsilon_1) + |S| \frac{18 \cdot 4}{\epsilon_0^2} |S| |A| |S|^4 (\sigma_a(R_a)\sigma_a(Z_a))^{-4}\epsilon_1 + |Z| |Z||R|\epsilon_1 \\
&\leq (18 \cdot 4 + 8) \frac{1}{\epsilon_0^2} |A| |S|^6 |R|^2 |Z|^2 (\sigma_a(R_a)\sigma_a(Z_a))^{-4}\epsilon_1 \\
&= O (|A|^3 |S|^6 |R|^2 |Z|^2 (\sigma_a(R_a)\sigma_a(Z_a))^{-4}\epsilon_1)
\end{align*}
\]

now putting that back into the bound

\[
V^\pi(b) \geq V^*(b) - O (|S|^2 |R|^2 |Z|^2 (\sigma_a(R_a)\sigma_a(Z_a))^{-4}\epsilon_1)
\]

now if we let that error be equal to $\epsilon$ i.e. $V^\pi(b) \geq V^*(b) - \epsilon$, then we can substitute $\epsilon$ for $\epsilon_1$ into eqn 74 from Lemma 2 to get the following requirement on $N$

\[
O \left( \frac{H^4V^2_{\max}|A|^4 |R|^2 |Z|^2 |S|^4 (\sigma_a(T_a))^{-6} (\sigma_a(R_a)\sigma_a(Z_a))^{-8}(1 + \sqrt{\log(3/\delta)})^2 \log \left( \frac{3}{\delta} \right)}{(C_{d,d,d}(\delta/3))^2} \right) \leq N
\]

where

\[
\begin{align*}
C_{d,d,d}(\delta) &= \min(C_{1,2,3}(\delta), C_{1,3,2}(\delta)) \\
C_{1,2,3}(\delta) &= \min \left( \frac{|M_2(\tilde{e}_i - \tilde{e}_j)|}{|P_{1,2,3}|} \right) \cdot \frac{\delta}{\log(k/\delta)} \\
C_{1,3,2}(\delta) &= \min \left( \frac{|M_2(\tilde{e}_i - \tilde{e}_j)|}{|P_{1,3,2}|} \right) \cdot \frac{\delta}{\log(k/\delta)}
\end{align*}
\]

are the quantities from MoM from Anandkumar et al. (2012).

And that is the final sample complexity bound (eqn 246). Note that we assume $\sigma_a(R_a)^{-1} \geq 1$, $\sigma_a(Z_a)^{-1} \geq 1$ and $\sigma_a(T_a)^{-1} \geq 1$ or else we can just replace those quantities by 1 in the bound.

One final thing to do is to give sufficient conditions on $N$ so that the conditions made on all of the $\epsilon_1, \epsilon_2, \ldots$ in the lemmas hold. We then note that the final sample complexity bound is a sufficient condition on $N$ (we only consider $\epsilon \leq 1$ to be interesting and don’t consider $\epsilon > 1$).

From Lemma 3 the condition is that $\epsilon_O \leq \frac{1}{3|R||Z|}$. This translates into

\[
\epsilon_1 \leq \frac{1}{3|R||Z|}
\]

where the first inequality is due to $\epsilon_1 = \epsilon_O$ from Lemma 2; the second inequality is from substituting the first inequality into eqn 74 from Lemma 2. The second inequality condition is already satisfied by the final bound (eqn 246). Next up is the condition (eqn 95) made during the proof of Lemma 2 where

\[
\epsilon_1 \leq \frac{1}{6|A||S|^1.5 (\sigma_a(R_a)\sigma_a(Z_a))^{-4}}
\]

\[
\epsilon_1 \leq O \left( \frac{|A|^4 |S|^3 |Z||R|(\sigma_a(R_a)\sigma_a(Z_a))^{-8}(1 + \sqrt{\log(3/\delta)})^2 \log \left( \frac{3}{\delta} \right)}{(C_{d,d,d}(\delta/3))^2} \right) \leq N
\]

\[
\epsilon_1 \leq \frac{1}{6|A||S|^1.5 (\sigma_a(R_a)\sigma_a(Z_a))^{-4}}
\]

\[
\epsilon_1 \leq O \left( \frac{|A|^4 |S|^3 |Z||R|(\sigma_a(R_a)\sigma_a(Z_a))^{-8}(1 + \sqrt{\log(3/\delta)})^2 \log \left( \frac{3}{\delta} \right)}{(C_{d,d,d}(\delta/3))^2} \right) \leq N
\]
where the second inequality is obtained from substituting the first inequality into eqn 254. The second inequality is also already satisfied by the final bound. Then the condition made during the proof of Lemma 4 (eqn 158) is

$$\epsilon_T \leq \frac{1}{2\|T^{-1}\|_2}$$

(254)

$$\iff \epsilon_1 \leq \frac{1}{36|A||S|^4(\sigma_a(R_a)\sigma_a(Z_a))^{-4(2|A|+|A|+|A|)}\sigma_a(T_a)}$$

(255)

$$\iff \epsilon_1 \leq O\left(\frac{1}{|A||S|^4(\sigma_a(R_a)\sigma_a(Z_a))^{-4(\sigma_a(T_a))^{-1}}}\right)$$

(256)

$$\iff O\left(\frac{|A|^4|S|^8|Z||R||(\sigma_a(R_a)\sigma_a(Z_a))^{-8(\sigma_a(T_a))^{-2}}(1 + \sqrt{\log(3/\delta)})^2 \log \left(\frac{3}{\delta}\right)}{(C_{d,d,d}(\delta/3))^2 \log(3/\delta)}\right) \leq N$$

(257)

where the second inequality comes from using eqn 236. The third inequality is from eqn 74 and is also satisfied by the final bound. Finally the second condition from the proof of Lemma 4 (eqn 178) is

$$\epsilon_T \leq \frac{1}{36|A||S|^4(\sigma_a(R_a)\sigma_a(Z_a))^{-4(2|A|+|A|+|A|)}\sigma_a(T_a)}$$

(258)

$$\iff \epsilon_1 \leq O\left(\frac{1}{|A||S|^5\sigma_a(T_a)^{-3(\sigma_a(R_a)\sigma_a(Z_a))^{-4}}}\right)$$

(259)

$$\iff O\left(\frac{|A|^{10}|S|^{10}|Z||R||\sigma_a(T_a)^{-6(\sigma_a(R_a)\sigma_a(Z_a))^{-8(\sigma_a(T_a))^{-2}}}(1 + \sqrt{\log(3/\delta)})^2 \log \left(\frac{3}{\delta}\right)}{(C_{d,d,d}(\delta/3))^2 \log(3/\delta)}\right) \leq N$$

(260)

where the second inequality comes from using eqn 236. The third inequality is from eqn 74 and is also satisfied by the final bound. Thus, the given final sample complexity bound is sufficient.