LIE BRACKET DERIVATION-DERIVATIONS IN BANACH ALGEBRAS

CHOONKIL PARK

ABSTRACT. In this paper, we introduce and solve the following additive-additive \((s, t)\)-functional inequality
\[
\|g(x + y) - g(x) - g(y)\| + \|h(x + y) + h(x - y) - 2h(x)\| 
\leq \left\| s \left( 2g \left( \frac{x + y}{2} \right) - g(x) - g(y) \right) \right\| + \left\| t \left( 2h \left( \frac{x + y}{2} \right) + 2h \left( \frac{x - y}{2} \right) - 2h(x) \right) \right\|,
\]
where \(s\) and \(t\) are fixed nonzero complex numbers with \(|s| < 1\) and \(|t| < 1\). Using the direct method and the fixed point method, we prove the Hyers-Ulam stability of Lie bracket derivation-derivations in complex Banach algebras, associated to the additive-additive \((s, t)\)-functional inequality (0.1) and the following functional inequality
\[
\|g, h\|([x, y]) = \|g, h\|([x, y]) - x[g, h]([y])\| + \|h([x, y]) - h(x)[y] - xh(y)\| \leq \varphi(x, y).
\]

1. INTRODUCTION AND PRELIMINARIES

Let \(A\) be a complex Banach algebra and \(\text{Der}(A)\) be the set of \(\mathbb{C}\)-linear bounded derivations on \(A\). For \(\delta_1, \delta_2 \in \text{Der}(A),\)
\[
\delta_1 \circ \delta_2(ab) = \delta_1 \circ \delta_2(a)b + \delta_2(a)\delta_1(b) + \delta_1(a)\delta_2(b) + a\delta_1 \circ \delta_2(b),
\]
\[
\delta_2 \circ \delta_1(ab) = \delta_2 \circ \delta_1(a)b + \delta_1(a)\delta_2(b) + \delta_2(a)\delta_1(b) + a\delta_2 \circ \delta_1(b)
\]
for all \(a, b \in A\). Let \([\delta_1, \delta_2] = \delta_1 \circ \delta_2 - \delta_2 \circ \delta_1\). Then
\[
[\delta_1, \delta_2](ab) = [\delta_1, \delta_2](a)b + a[\delta_1, \delta_2](b)
\]
for all \(a, b \in A\). Since \([\delta_1, \delta_2] : A \to A\) is \(\mathbb{C}\)-linear, \([\delta_1, \delta_2] \in \text{Der}(A)\) for all \(\delta_1, \delta_2 \in \text{Der}(A)\). Thus \(\text{Der}(A)\) is a Lie algebra with Lie bracket \([\delta_1, \delta_2]\), since \(\delta_1 + \delta_2\) and \(\alpha \delta_1\) are \(\mathbb{C}\)-linear derivations on \(A\) for all \(\delta_1, \delta_2 \in \text{Der}(A)\) and all \(\alpha \in \mathbb{C}\). One can easily show that \(\text{Der}(A)\) is a Banach space, since \(A\) is complete.

In this paper, we introduce and investigate Lie bracket derivation-derivations in complex Banach algebras.

Definition 1.1. Let \(A\) be a complex Banach algebra and \(D, H : A \to A\) be \(\mathbb{C}\)-linear mappings. Let \([D, H](a) = D(H(a)) - H(D(a))\) for all \(a \in A\). A \(\mathbb{C}\)-linear mapping \([D, H] : A \to A\) is called a Lie bracket derivation-derivation in \(A\) if \([D, H]\) and \(H\) (or \(D\)) are derivations in \(A\), i.e.,
\[
[D, H](ab) = [D, H](a)b + a[D, H](b),
\]
\[
H(ab) = H(a)b + aH(b)
\]
for all \(a, b \in A\).
Since \([\delta_1, \delta_2] \in \text{Der}(A)\) for \(\delta_1, \delta_2 \in \text{Der}(A)\), \([\delta_1, \delta_2]\) is a Lie bracket derivation-derivation.

The stability problem of functional equations originated from a question of Ulam [22] concerning the stability of group homomorphisms. Hyers [10] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [20] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Gavruta [9] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. Park [15, 16, 18] defined additive-\(\rho\)-functional inequalities and proved the Hyers-Ulam stability of the additive \(\rho\)-functional inequalities in Banach spaces and non-Archimedean Banach spaces. The stability problems of various functional equations and functional inequalities have been extensively investigated by a number of authors (see [7, 8, 13, 21, 23]).

We recall a fundamental result in fixed point theory.

**Theorem 1.2.** [2, 5] Let \((X, d)\) be a complete generalized metric space and let \(J : X \to X\) be a strictly contractive mapping with Lipschitz constant \(\alpha < 1\). Then for each given element \(x \in X\), either

\[
d(J^n x, J^{n+1} x) = \infty
\]

for all nonnegative integers \(n\) or there exists a positive integer \(n_0\) such that

1. \(d(J^n x, J^{n+1} x) < \infty, \quad \forall n \geq n_0;\)
2. the sequence \(\{J^n x\}\) converges to a fixed point \(y^*\) of \(J;\)
3. \(y^*\) is the unique fixed point of \(J\) in the set \(Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\};\)
4. \(d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy)\) for all \(y \in Y\).

In 1996, Isac and Rassias [11] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [3, 4, 6, 17, 19]).

In this paper, we solve the additive-additive \((s, t)\)-functional inequality (0.1). Furthermore, we investigate Lie bracket derivation-derivations in complex Banach algebras associated to the additive-additive \((s, t)\)-functional inequality (0.1) and the functional inequality (0.2) by using the direct method and by the fixed point method.

Throughout this paper, assume that \(A\) is a complex Banach algebra and that \(s\) and \(t\) are fixed nonzero complex numbers with \(|s| < 1\) and \(|t| < 1\).

## 2. Stability of Additive-Additive \((s, t)\)-Functional Inequality (0.1): A Direct Method

In this section, we solve and investigate the additive-additive \((s, t)\)-functional inequality (0.1) in complex Banach algebras.

**Lemma 2.1.** If mappings \(g, h : A \to A\) satisfy \(g(0) = h(0) = 0\) and

\[
\|g(x + y) - g(x) - g(y)\| + \|h(x + y) + h(x - y) - 2h(x)\| \\
\leq \left\| s \left( 2g \left( \frac{x + y}{2} \right) - g(x) - g(y) \right) \right\| + \left\| t \left( 2h \left( \frac{x + y}{2} \right) + 2h \left( \frac{x - y}{2} \right) - 2h(x) \right) \right\| \tag{2.1}
\]

for all \(x, y \in A\), then the mappings \(g, h : A \to A\) are additive.
Lemma 2.2. [14, Theorem 2.1] Let $\phi : A \to A$ be an additive mapping such that

$$f(\lambda a) = \lambda f(a)$$

for all $\lambda \in T^1 := \{ \xi \in \mathbb{C} : |\xi| = 1 \}$ and all $a \in A$. Then the mapping $f : A \to A$ is $\mathbb{C}$-linear.

Using the direct method, we prove the Hyers-Ulam stability of Lie bracket derivation-derivations in complex Banach algebras associated to the additive-additive $(s,t)$-functional inequality (2.1) and the functional inequality (0.2).

Theorem 2.3. Let $\varphi : A^2 \to [0, \infty)$ be a function such that

$$\sum_{j=1}^{\infty} 4^j \varphi \left( \frac{x}{2^j}, \frac{y}{2^j} \right) < \infty$$

(2.2)

for all $x, y \in A$. Let $g, h : A \to A$ be mappings satisfying $g(0) = h(0) = 0$ and

$$\|g(\lambda(x + y)) - \lambda g(x) - \lambda g(y)\| + \|h(\lambda(x + y)) + h(\lambda(x - y)) - 2\lambda h(x)\|$$

$$\leq \left\| s \left( 2g \left( \frac{\lambda x + y}{2} \right) - \lambda g(x) - \lambda g(y) \right) \right\|$$

$$+ \left\| t \left( 2h \left( \frac{\lambda x + y}{2} \right) + 2h \left( \frac{\lambda x - y}{2} \right) - 2\lambda h(x) \right) \right\| + \varphi(x, y)$$

(2.3)

for all $\lambda \in T^1$ and all $x, y \in A$. If the mappings $g, h : A \to A$ satisfy

$$\|g[h(x) - h(x)] - [g, h](x) - x[g, h](x)\| + \|h(x) - h(x) - xh(y)\| \leq \varphi(x, y)$$

(2.4)

for all $x, y \in A$, then there exist a unique $\mathbb{C}$-linear $D : A \to A$ and a unique derivation $H : A \to A$ such that $[D, H] : A \to A$ is a derivation and

$$\|g(x) - D(x)\| + \|h(x) - H(x)\| \leq \sum_{j=1}^{\infty} 2^{j-1} \varphi \left( \frac{x}{2^j}, \frac{y}{2^j} \right)$$

(2.5)

for all $x \in A$.

Proof. Letting $\lambda = 1$ and $y = x$ in (2.3), we get

$$\|g(2x) - 2g(x)\| + \|h(2x) - 2h(x)\| \leq \varphi(x, x)$$

(2.6)

and so

$$\left\| g(x) - 2g \left( \frac{x}{2} \right) \right\| + \left\| h(x) - 2h \left( \frac{x}{2} \right) \right\| \leq \varphi \left( \frac{x}{2}, \frac{x}{2} \right)$$
for all $x \in A$. Thus
\[
\left\| 2^j g \left( \frac{x}{2^j} \right) - 2^m g \left( \frac{x}{2^m} \right) \right\| + \left\| 2^j h \left( \frac{x}{2^j} \right) - 2^m h \left( \frac{x}{2^m} \right) \right\| 
\leq \sum_{j=l}^{m-1} \left\| 2^j g \left( \frac{x}{2^j} \right) - 2^{j+1} g \left( \frac{x}{2^{j+1}} \right) \right\| + \sum_{j=l}^{m-1} \left\| 2^j h \left( \frac{x}{2^j} \right) - 2^{j+1} h \left( \frac{x}{2^{j+1}} \right) \right\|
\leq \sum_{j=l+1}^{m} 2^{j-1} \phi \left( \frac{x}{2^j} \frac{x}{2^j} \right)
\]
for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in A$. It follows from (2.7) that the sequences \( \{2^k g \left( \frac{x}{2^k} \right) \} \) and \( \{2^k h \left( \frac{x}{2^k} \right) \} \) are Cauchy for all $x \in A$. Since $Y$ is a Banach space, the sequences \( \{2^k g \left( \frac{x}{2^k} \right) \} \) and \( \{2^k h \left( \frac{x}{2^k} \right) \} \) converge. So one can define the mappings $D, H : A \to A$ by
\[
D(x) := \lim_{k \to \infty} 2^k g \left( \frac{x}{2^k} \right), \quad H(x) := \lim_{k \to \infty} 2^k h \left( \frac{x}{2^k} \right)
\]
for all $x \in A$. Moreover, letting $l = 0$ and passing to the limit $m \to \infty$ in (2.7), we get (2.5).

It follows from (2.3) that
\[
\left\| D \left( \lambda(x + y) \right) - \lambda D(x) - \lambda D(y) \right\| + \left\| H \left( \lambda(x + y) \right) + H(\lambda(x - y)) - 2\lambda H(x) \right\|
\leq \lim_{n \to \infty} 2^n \left\| g \left( \frac{\lambda x + y}{2^n} \right) - \lambda g \left( \frac{x}{2^n} \right) \right\| + \lim_{n \to \infty} 2^n \left\| h \left( \frac{\lambda x + y}{2^n} \right) + h \left( \frac{\lambda x - y}{2^n} \right) - 2\lambda h \left( \frac{x}{2^n} \right) \right\|
\leq \lim_{n \to \infty} 2^n \left\| s \left( 2g \left( \frac{\lambda x + y}{2^{n+1}} \right) - \lambda g \left( \frac{x}{2^{n+1}} \right) \right) \right\| + \lim_{n \to \infty} 2^n \varphi \left( \frac{x}{2^n} \frac{y}{2^n} \right)
\leq \left\| t \left( 2h \left( \frac{\lambda x + y}{2^{n+1}} \right) + h \left( \frac{\lambda x - y}{2^{n+1}} \right) - 2\lambda h \left( \frac{x}{2^n} \right) \right) \right\| + \lim_{n \to \infty} 2^n \varphi \left( \frac{x}{2^n} \frac{y}{2^n} \right)
\]
for all $\lambda \in \mathbb{T}^1$ and all $x, y \in A$. So
\[
\left\| D \left( \lambda(x + y) \right) - \lambda D(x) - \lambda D(y) \right\| + \left\| H \left( \lambda(x + y) \right) + H(\lambda(x - y)) - 2\lambda H(x) \right\|
\leq \left\| s \left( 2D \left( \frac{\lambda x + y}{2} \right) - \lambda D(x) - \lambda D(y) \right) \right\| + \left\| t \left( 2H \left( \frac{\lambda x + y}{2} \right) + 2H \left( \frac{\lambda x - y}{2} \right) - 2\lambda H(x) \right) \right\|
\]
for all $\lambda \in \mathbb{T}^1$ and all $x, y \in A$.

Let \( \lambda = 1 \) in (2.8). By Lemma 2.1, the mappings $D, H : A \to A$ are additive.

It follows from (2.8) and the additivity of $D$ and $H$ that
\[
\left\| D \left( \lambda(x + y) \right) - \lambda D(x) - \lambda D(y) \right\| + \left\| H \left( \lambda(x + y) \right) - H(\lambda(x - y)) - 2\lambda H(y) \right\|
\leq \left\| s \left( D \left( \lambda(x + y) \right) - \lambda D(x) - \lambda D(y) \right) \right\| + \left\| t \left( H \left( \lambda(x + y) \right) - H(\lambda(x - y)) - 2\lambda H(x) \right) \right\|
\]
for all $\lambda \in \mathbb{T}^1$ and all $x, y \in A$. Since $|s| < 1$ and $|t| < 1$,
\[
D \left( \lambda(x + y) \right) - \lambda D(x) - \lambda D(y) = 0, \quad H \left( \lambda(x + y) \right) - H(\lambda(x - y)) - 2\lambda H(y) = 0
\]
and so $D(\lambda x) = \lambda D(x)$ and $H(\lambda x) = \lambda H(x)$ for all $\lambda \in \mathbb{T}^1$ and all $x, y \in A$. Thus by Lemma 2.2, the additive mappings $D, H : A \to A$ are $\mathbb{C}$-linear.
It follows from (2.4) and the additivity of $D$ and $H$ that
\[
\| [D, H](xy) - [D, H](x)y - x[D, H](y) \| + \| H(xy) - H(x)y - xH(y) \|
\]
\[
= 4^n \left\| \left[ g, h \right] \left( \frac{xy}{4^n} \right) - \left[ g, h \right] \left( \frac{x}{2^n} \right) \cdot \frac{y}{2^n} - \frac{x}{2^n} \cdot \left[ g, h \right] \left( \frac{y}{2^n} \right) \right\|
\]
\[
+ 4^n \left\| h \left( \frac{xy}{4^n} \right) - h \left( \frac{x}{2^n} \right) \cdot \frac{y}{2^n} - \frac{x}{2^n} \cdot h \left( \frac{y}{2^n} \right) \right\| \leq 4^n \varphi \left( \frac{x}{2^n} : \frac{y}{2^n} \right),
\]
which tends to zero as $n \to \infty$, by (2.2). So
\[
[D, H](xy) - [D, H](x)y - x[D, H](y) = 0, \quad H(xy) - H(x)y - xH(y) = 0
\]
for all $x, y \in A$. Hence the mappings $[D, H] : A \to A$ and $H : A \to A$ are derivations.

**Corollary 2.4.** Let $r > 2$ and $\theta$ be nonnegative real numbers and $g, h : A \to A$ be mappings satisfying $g(0) = h(0) = 0$ and
\[
\| g(λ(x + y)) - λg(x) - λg(y) \| + \| h(λ(x + y)) + h(λ(x - y)) - 2λh(x) \|
\]
\[
\leq \left\| s \left( 2g \left( \frac{λx + y}{2} \right) - λg(x) - λg(y) \right) \right\|
\]
\[
+ \left\| t \left( 2h \left( \frac{λx + y}{2} \right) + 2h \left( \frac{λx - y}{2} \right) - 2λh(x) \right) \right\| + θ(\|x\|^r + \|y\|^r) \tag{2.9}
\]
for all $λ \in \mathbb{T}^1$ and all $x, y \in A$. If the mappings $g, h : A \to A$ satisfy
\[
\| [g, h](xy) - [g, h](x)y - x[g, h](y) \| + \| h(xy) - h(x)y - xh(y) \| \leq θ(\|x\|^r + \|y\|^r) \tag{2.10}
\]
for all $x, y \in A$, then there exist a unique $\mathbb{C}$-linear $D : A \to A$ and a unique derivation $H : A \to A$ such that $[D, H] : A \to A$ is a derivation and
\[
\| g(x) - D(x) \| + \| h(x) - H(x) \| \leq \frac{2θ}{2^r - 2} \|x\|^r
\]
for all $x \in A$.

**Proof.** The proof follows from Theorem 2.3 by $φ(x, y) = θ(\|x\|^r + \|y\|^r)$ for all $x, y \in A$. □

**Theorem 2.5.** Let $φ : A^2 \to [0, \infty)$ be a function and $g, h : A \to A$ be mappings satisfying $g(0) = h(0) = 0$, (2.3), (2.4) and
\[
Φ(x, y) := \sum_{j=0}^{∞} 2^j \varphi(2^j x, 2^j y) < \infty \tag{2.11}
\]
for all $x, y \in A$. Then there exist a unique $\mathbb{C}$-linear $D : A \to A$ and a unique derivation $H : A \to A$ such that $[D, H] : A \to A$ is a derivation and
\[
\| g(x) - D(x) \| + \| h(x) - H(x) \| \leq \frac{1}{2} Φ(x, x) \tag{2.12}
\]
for all $x \in A$.

**Proof.** It follows from (2.6) that
\[
\left\| g(x) - \frac{1}{2} g(2x) \right\| + \left\| h(x) - \frac{1}{2} h(2x) \right\| \leq \frac{1}{2} φ(x, x) \tag{2.13}
\]
for all \( x \in A \). Thus
\[
\left\| \frac{1}{2^l} g \left( \frac{x}{2^l} \right) - \frac{1}{2^m} g \left( \frac{x}{2^m} \right) \right\| + \left\| \frac{1}{2^l} h \left( \frac{2^l x}{2^l} \right) - \frac{1}{2^m} h \left( \frac{2^m x}{2^m} \right) \right\| \leq \sum_{j=1}^{m-1} \left\| \frac{1}{2^j} g \left( \frac{2^j x}{2^j} \right) - \frac{1}{2^{j+1}} g \left( \frac{2^{j+1} x}{2^{j+1}} \right) \right\| + \sum_{j=1}^{m-1} \left\| \frac{1}{2^j} h \left( \frac{2^j x}{2^j} \right) - \frac{1}{2^{j+1}} h \left( \frac{2^{j+1} x}{2^{j+1}} \right) \right\|
\]

(2.14)

for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x \in A \). It follows from (2.14) that the sequences \( \left\{ \frac{1}{2^i} g \left( 2^i x \right) \right\} \) and \( \left\{ \frac{1}{2^i} h \left( 2^i x \right) \right\} \) are Cauchy for all \( x \in A \). Since \( Y \) is a Banach space, the sequences \( \left\{ \frac{1}{2^i} g \left( 2^i x \right) \right\} \) and \( \left\{ \frac{1}{2^i} h \left( 2^i x \right) \right\} \) converge. So one can define the mappings \( D, H : A \to A \) by
\[
D(x) := \lim_{k \to \infty} \frac{1}{2^k} g \left( 2^k x \right), \quad H(x) := \lim_{k \to \infty} \frac{1}{2^k} h \left( 2^k x \right)
\]

for all \( x \in A \). Moreover, letting \( l = 0 \) and passing to the limit \( m \to \infty \) in (2.14), we get (2.12).

By the same reasoning as in the proof of Theorem 2.3, one can show that the mappings \( D, H : A \to A \) are \( \mathbb{C} \)-linear.

It follows from (2.4) and the additivity of \( D, H \) that
\[
\|[D, H](xy) - [D, H](x)y - x[D, H](y)\| + \|H(xy) - H(x)y - xH(y)\| = \frac{1}{4^n} \|[g, h] (4^n xy) - [g, h] (2^n x \cdot 2^n y) - (2^n x)[g, h] (2^n y)\|
\]
\[
+ \frac{1}{4^n} \|h(4^n xy) - h(2^n x \cdot 2^n y - 2^n x \cdot h(2^n y)\| \leq \frac{1}{4^n} \phi(2^n x, 2^n y),
\]

which tends to zero as \( n \to \infty \), by (2.11). So
\[
[D, H](xy) - [D, H](x)y - x[D, H](y) = 0, \quad H(xy) - H(x)y - xH(y) = 0
\]

for all \( x, y \in A \). Hence the mappings \( [D, H] : A \to A \) and \( H : A \to A \) are derivations.

\[\square\]

**Corollary 2.6.** Let \( r < 1 \) and \( \theta \) be nonnegative real numbers and \( g, h : A \to A \) be mappings satisfying \( g(0) = h(0) = 0, \) (2.9) and (2.10). Then there exist a unique \( \mathbb{C} \)-linear \( D : A \to A \) and a unique derivation \( H : A \to A \) such that \( [D, H] : A \to A \) and \( H : A \to A \) are derivations and
\[
\|[g(x) - D(x)]\| + \|h(x) - H(x)\| \leq \frac{2\theta}{2 - 2^r} \|x\|^r
\]

for all \( x \in A \).

**Proof.** The proof follows from Theorem 2.5 by \( \phi(x, y) = \theta(\|x\|^r + \|y\|^r) \) for all \( x, y \in A \). \[\square\]

3. Stability of additive-additive \((s, t)\)-functional inequality (0.1): a fixed point method

Using the fixed point method, we prove the Hyers-Ulam stability of Lie bracket derivation-derivations in complex Banach algebras associated to the additive-additive \((s, t)\)-functional inequality (0.1) and the functional inequality (0.2).
**Theorem 3.1.** Let \( \varphi : A^2 \to [0, \infty) \) be a function such that there exists an \( L < 1 \) with

\[
\varphi \left( \frac{x}{2}, \frac{y}{2} \right) \leq \frac{L}{4} \varphi (x, y) \leq \frac{L}{2} \varphi (x, y) \tag{3.1}
\]

for all \( x, y \in A \). Let \( g, h : A \to A \) be mappings satisfying \( g(0) = h(0) = 0 \), (2.3) and (2.4). Then there exist a unique \( \mathbb{C} \)-linear \( D : A \to A \) and a unique derivation \( H : A \to A \) such that \( [D, H] : A \to A \) is a derivation and

\[
\|g(x) - D(x)\| + \|h(x) - H(x)\| \leq \frac{L}{2(1-L)} \varphi (x, x) \tag{3.2}
\]

for all \( x \in A \).

**Proof.** It follows from (3.1) that

\[
\sum_{j=1}^{\infty} 4^j \varphi \left( \frac{x}{2^j}, \frac{y}{2^j} \right) \leq \sum_{j=1}^{\infty} 4^j \frac{L^j}{2^j} \varphi (x, y) = \frac{L}{1-L} \varphi (x, y) < \infty
\]

for all \( x, y \in A \). By Theorem 2.3, there exist a unique \( \mathbb{C} \)-linear \( D : A \to A \) and a unique derivation \( H : A \to A \) satisfying (2.5) such that \( [D, H] : A \to A \) is a derivation.

Letting \( \lambda = 1 \) and \( y = x \) in (2.3), we get

\[
\|g(2x) - 2g(x)\| + \|h(2x) - 2h(x)\| \leq \varphi (x, x) \tag{3.3}
\]

for all \( x \in A \).

Consider the set

\[
S := \{(g, h) \mid g, h : A \to A, \ g(0) = h(0) = 0\}
\]

and introduce the generalized metric on \( S \):

\[
d((g, h), (g_1, h_1)) = \inf \{ \mu \in \mathbb{R}_+ : \|g(x) - g_1(x)\| + \|h(x) - h_1(x)\| \leq \mu \varphi (x, x), \ \forall x \in A \},
\]

where, as usual, \( \inf \phi = +\infty \). It is easy to show that \((S, d)\) is complete (see [12]).

Now we consider the linear mapping \( J : S \to S \) such that

\[
J(g, h)(x) := \left( 2g \left( \frac{x}{2} \right), 2h \left( \frac{x}{2} \right) \right)
\]

for all \( x \in A \).

Let \((g, h), (g_1, h_1) \in S\) be given such that \( d((g, h), (g_1, h_1)) = \varepsilon \). Then

\[
\|g(x) - g_1(x)\| + \|h(x) - h_1(x)\| \leq \varepsilon \varphi (x, x)
\]

for all \( x \in A \). Since

\[
\left\| 2g \left( \frac{x}{2} \right) - 2g_1 \left( \frac{x}{2} \right) \right\| + \left\| 2h \left( \frac{x}{2} \right) - 2h_1 \left( \frac{x}{2} \right) \right\| \leq 2\varepsilon \varphi \left( \frac{x}{2}, \frac{x}{2} \right) \leq 2\frac{L}{2} \varphi (x, x) = L\varepsilon \varphi (x, x)
\]

for all \( x \in A \), \( d(J(g, h), J(g_1, h_1)) \leq L\varepsilon \). This means that

\[
d(J(g, h), J(g_1, h_1)) \leq Ld((g, h), (g_1, h_1))
\]

for all \((g, h), (g_1, h_1) \in S\).

It follows from (3.3) that

\[
\left\| g(x) - 2g \left( \frac{x}{2} \right) \right\| + \left\| h(x) - 2h \left( \frac{x}{2} \right) \right\| \leq \varphi \left( \frac{x}{2}, \frac{x}{2} \right) \leq \frac{L}{2} \varphi (x, x)
\]

for all \( x \in A \). So \( d((g, h), (Jg, Jh)) \leq \frac{L}{4} \).

By Theorem 1.2, there exist mappings \( D, H : A \to A \) satisfying the following:
(1) \((D, H)\) is a fixed point of \(J\), i.e.,
\[
D(x) = 2D\left(\frac{x}{2}\right), \quad H(x) = 2H\left(\frac{x}{2}\right)
\]
for all \(x \in A\). The mapping \((D, H)\) is a unique fixed point of \(J\) in the set
\[
M = \{g \in S : d((g, h), (g_1, h_1)) < \infty\}.
\]
This implies that \((D, H)\) is a unique mapping satisfying (3.4) such that there exists a \(\mu \in (0, \infty)\) satisfying
\[
\|g(x) - D(x)\| + \|h(x) - H(x)\| \leq \mu \varphi(x, x)
\]
for all \(x \in A\);
(2) \(d(J^l(g, h), (D, H)) \to 0\) as \(l \to \infty\). This implies the equality
\[
\lim_{l \to \infty} 2^l g\left(\frac{x}{2^l}\right) = D(x), \quad \lim_{l \to \infty} 2^l h\left(\frac{x}{2^l}\right) = H(x)
\]
for all \(x \in A\);
(3) \(d((g, h), (D, H)) \leq \frac{1}{1-L} d((g, h), J(g, h))\), which implies
\[
\|g(x) - D(x)\| + \|h(x) - H(x)\| \leq \frac{L}{2(1-L)} \varphi(x, x)
\]
for all \(x \in A\).

The rest of the proof is the same as in the proof of Theorem 2.3.

\[\square\]

**Corollary 3.2.** Let \(r > 2\) and \(\theta\) be nonnegative real numbers and \(g, h : A \to A\) be mappings satisfying \(g(0) = h(0) = 0\), (2.9) and (2.10). Then there exist a unique \(C\)-linear \(D : A \to A\) and a unique derivation \(H : A \to A\) such that \([D, H] : A \to A\) is a derivation and
\[
\|g(x) - D(x)\| + \|h(x) - H(x)\| \leq \frac{2\theta}{2^r - 2} \|x\|^r
\]
for all \(x \in A\).

**Proof.** The proof follows from Theorem 3.1 by taking \(L = 2^{1-r}\) and \(\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)\) for all \(x, y \in A\).

\[\square\]

**Theorem 3.3.** Let \(\varphi : A^2 \to [0, \infty)\) be a function such that there exists an \(L < 1\) with
\[
\varphi(x, y) \leq 4L \varphi\left(\frac{x}{2}, \frac{y}{2}\right)
\]
for all \(x, y \in A\). Let \(g, h : A \to A\) be mappings satisfying \(g(0) = h(0) = 0\), (2.3) and (2.4). Then there exist a unique \(C\)-linear \(D : A \to A\) and a unique derivation \(H : A \to A\) such that \([D, H] : A \to A\) is a derivation and
\[
\|g(x) - D(x)\| + \|h(x) - H(x)\| \leq \frac{1}{2(1-L)} \varphi(x, x)
\]
for all \(x \in A\).
Proof. It follows from (3.5) that
\[\sum_{j=1}^{\infty} \frac{1}{4^j} \varphi(2^j x, 2^j y) \leq \sum_{j=1}^{\infty} \frac{1}{4^j} (4L)^j \varphi(x, y) = \frac{L}{1 - L} \varphi(x, y) < \infty\]
for all \(x, y \in A\). By Theorem 2.5, there exist a unique \(\mathbb{C}\)-linear \(D : A \to A\) and a unique derivation \(H : A \to A\) satisfying (2.12) such that \([D, H] : A \to A\) is a derivation.

Let \((S, d)\) be the generalized metric space defined in the proof of Theorem 3.1.

Now we consider the linear mapping \(J : S \to S\) such that
\[J(g, h)(x) := \left(\frac{1}{2} g(2x), \frac{1}{2} h(2x)\right)\]
for all \(x \in A\).

It follows from (3.3) that
\[\|g(x) - \frac{1}{2} g(2x)\| + \|h(x) - \frac{1}{2} h(2x)\| \leq \frac{1}{2} \varphi(x, x)\]
for all \(x \in A\).

The rest of the proof is similar to the proof of Theorem 3.1. \(\square\)

Corollary 3.4. Let \(r < 1\) and \(\theta\) be nonnegative real numbers and \(g, h : A \to A\) be mappings satisfying \(g(0) = h(0) = 0\), (2.9) and (2.10). Then there exist a unique \(\mathbb{C}\)-linear \(D : A \to A\) and a unique derivation \(H : A \to A\) such that \([D, H] : A \to A\) is a derivation and
\[\|g(x) - D(x)\| + \|h(x) - H(x)\| \leq \frac{2\theta}{2 - 2^r} \|x\|^r\]
for all \(x \in A\).

Proof. The proof follows from Theorem 3.3 by taking \(L = 2^{r-1}\) and \(\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)\) for all \(x, y \in A\). \(\square\)

Acknowledgments
C. Park was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2017R1D1A1B04032937).

Competing interests
The author declares that they have no competing interests.

Authors’ contributions
The author conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.
References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.

[2] L. Cadariu, V. Radu, *Fixed points and the stability of Jensen’s functional equation*, J. Inequal. Pure Appl. Math. **4**, no. 1, Art. ID 4 (2003).

[3] L. Cadariu, V. Radu, *On the stability of the Cauchy functional equation: a fixed point approach*, Grazer Math. Ber. **346** (2004), 43–52.

[4] L. Cadariu, V. Radu, *Fixed point methods for the generalized stability of functional equations in a single variable*, Fixed Point Theory Appl. **2008**, Art. ID 749392 (2008).

[5] J. Diaz, B. Margolis, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Am. Math. Soc. **74** (1968), 305–309.

[6] I. EL-Fassi, *Generalized hyperstability of a Drygas functional equation on a restricted domain using Brzdek’s fixed point theorem*, J. Fixed Point Theory Appl. **19** (2017), 2529–2540.

[7] I. EL-Fassi, *Solution and approximation of radical quintic functional equation related to quintic mapping in quasi-β-Banach spaces*, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. **113** (2019), no. 2, 675–687.

[8] M. Eshaghi Gordji, M.B. Ghaemi and B. Alizadeh, *A fixed point method for perturbation of higher ring derivations in non-Archimedean Banach algebras*, Int. J. Geom. Meth. Mod. Phys. **8** (2011), no. 7, 1611–1625.

[9] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.

[10] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A. **27** (1941), 222–224.

[11] G. Isac, Th. M. Rassias, *Stability of ψ-additive mappings: Applications to nonlinear analysis*, Int. J. Math. Math. Sci. **19** (1996), 219–228.

[12] D. Miheţ, V. Radu, *On the stability of the additive Cauchy functional equation in random normed spaces*, J. Math. Anal. Appl. **343** (2008), 567–572.

[13] I. Nikoufar, *Jordan (θ, φ)-derivations on Hilbert C∗-modules*, Indag. Math. **26** (2015), 421–430.

[14] C. Park, *Homomorphisms between Poisson JC∗-algebras*, Bull. Braz. Math. Soc. **36** (2005), 79–97.

[15] C. Park, *Additive p-functional inequalities and equations*, J. Math. Inequal. **9** (2015), 17–26.

[16] C. Park, *Additive p-functional inequalities in non-Archimedean normed spaces*, J. Math. Inequal. **9** (2015), 397–407.

[17] C. Park, *Fixed point method for set-valued functional equations*, J. Fixed Point Theory Appl. **19** (2017), 2297–2308.

[18] C. Park, *Biderivations and bihomomorphisms in Banach algebras*, Filomat (in press).

[19] V. Radu, *The fixed point alternative and the stability of functional equations*, Fixed Point Theory **4** (2003), 91–96.

[20] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Am. Math. Soc. **72** (1978), 297–300.

[21] L. Székelyhidi, *Superstability of functional equations related to spherical functions*, Open Math. **15** (2017), no. 1, 427–432.

[22] S. M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publ. New York, 1960.

[23] Z. Wang, *Stability of two types of cubic fuzzy set-valued functional equations*, Results Math. **70** (2016), 1–14.

Choonkil Park
Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea
E-mail address: baak@hanyang.ac.kr