Pictures on the second homotopy module of the group from Kronecker product on the representation quaternion group

A M Zakiya\textsuperscript{1}, Y Yanita\textsuperscript{2} and I M Arnawa\textsuperscript{3}

Department of Mathematics Faculty of Mathematics and Natural Science
Andalas University
Kampus Unand Limau Manis, Padang 25163, Indonesia

Corresponding author: yanita@sci.unand.ac.id

Abstract. This paper discusses pictures on the second homotopy module of the group from the Kronecker product on the representation quaternion group. It needs three notions to construct the second homotopy module, which is the presentation of the group, word, and spherical picture. In this paper, it has used the presence of a group that is obtained from the group Kronecker product on the representation of the quaternion group. This presentation has four generators and five relations. Based on this presentation, we got word, and then a picture can be drawn. It uses van Kampen Lemma to construct the picture and operation on picture to obtain a spherical picture. It has presented pictures with one and all of the generators in the presentation of the group. It has used relations in the presentation of a group to make the picture become a spherical picture. We conclude that the spherical picture, which becomes generators built all pictures.

1. Introduction

Second homotopy module is a group with elements are classes equivalence of spherical picture. To construct this group, we need a presentation of the group, which is a group whose elements are represented by generators and relations. This generator and relation is a unity, which then becomes a way of writing each element in the group to be made the presentation. Let $G$ be a group and $P$ is the presentation of $G$, then the second homotopy module is symbolized by $\pi_2(P)$.

In this paper, we use the group obtained from the application of Kronecker product on the representation of the quaternion group. The group has 32 elements [1] and the presentation is $P = \langle a, b, c, d | a^4 = b^4 = c^2 = d^2, a^2 = b^2, ab = ba^{-1}, acd = dca^{-1}, bcd = dcb^{-1} \rangle$ [2]. Based on pictures over $P$, the equivalent spherical picture is determined and then the equivalence classes of the spherical picture can be formed. The classes of the spherical picture are the elements of $\pi_2(P)$.

The purpose of this paper is to describe the picture that is in the second homotopy module. We concluded that the picture is derived from the spherical picture, which is the generator in the second homotopy module. And then, it has shown that the spherical picture generates a picture is related to relations in the presentation of a group and this result in Theorem 3.1 in Section 3.

2. Basic theory

We start the notion with the word, picture, and spherical picture (see [3], [4], [5]).

Definition 2.1 [4] Let $x = \{x_1, x_2, ..., x_n\}$ be a set of distinct elements and $x^{-1} = \{x_1^{-1}, x_2^{-1}, ..., x_n^{-1}\}$ be a set of elements, distinct from each other and the element of $X$. Define $x^{\pm 1} = x \cup x^{-1}$. A word $W$ is a finite string with a form $x_1^{\varepsilon_1} x_2^{\varepsilon_2} ... x_n^{\varepsilon_n}$, $n \geq 0$, $x_i \in x$ and $\varepsilon_i = \pm 1$, $i = 1, 2, ..., n$.  

Content from this work may be used under the terms of the Creative Commons Attribution 3.0 licence. Any further distribution of this work must maintain attribution to the author(s) and the title of the work, journal citation and DOI.

Published under licence by IOP Publishing Ltd
The inverse of $W$ denoted $W^{-1}$ is word $x_n^{\varepsilon_n} x_{n-1}^{\varepsilon_{n-1}} \ldots x_2^{\varepsilon_2} x_1^{\varepsilon_1}$ if $x_i^{\varepsilon_i} \neq x_i^{-\varepsilon_{i+1}}$, $i = 1, \ldots, n - 1$, then we say that it is reduced. Furthermore, it is cyclically reduced if in addition, $x_1^{\varepsilon_1} \neq x_n^{-\varepsilon_n} x_1$.

The set of words will be denoted by $(x^{\pm 1})^*$. We introduce a multiplication $(x^{\pm 1})^* \times (x^{\pm 1})^* \to (x^{\pm 1})^*$ by writing end-to-end the words that are to be multiplied. The length of the product is the sum of the length of the factors. It is clear that this multiplication is associative and that 1 is a two-sided identity. To obtain a group from $(x^{\pm 1})^*$, we shall introduce an equivalence relation in $(x^{\pm 1})^*$.

Two words $V$ and $W$ are equivalent if one of the words can be obtained from the other by a finite succession of insertion and deletion of expressions of the form $x_i^{\varepsilon_i} x_i^{-\varepsilon_i}$, $x_i \in X$, $\varepsilon_i = \pm 1$. If $V$ and $W$ equivalent, then symbolized by $V \sim W$. Relation \(\sim\) is an equivalence relation in $(x^{\pm 1})^*$. Let $[W]$ is the equivalence class of word $W$, then set of equivalence classes $\{[W]; W \in (x^{\pm 1})^*\}$ with binary operation $[V][W] = [VW]$ for every $V, W \in (x^{\pm 1})^*$ thereby becomes a group. This group is called free group on $x$, and symbolized by $F(x)$. The identity element of $F(x)$ is the empty word and the inverse of $[W]$ is $[W^{-1}]$, written $[W]^{-1} = [W^{-1}]$, for every $W \in (x^{\pm 1})^*$. We have the next theorem.

**Theorem 2.2** [6] Let $x$ be a set and $(x^{\pm 1})^*$ is a corresponding set of words built from $x^{\pm 1} = x \cup x^{-1}$. Product operation definition $(x^{\pm 1})^*$ descent in a well-defined fashion to the set $F(x)$ of equivalence classes of a member of $(x^{\pm 1})^*$, and $F(x)$ thereby becomes a group.

Suppose that $G$ is a group. If on $G$ is defined as a set $x$ as in Theorem 2.1 and there is one or some words in $(x^{\pm 1})^*$ which is the same as an empty word called a relator set and symbolized by $R$ ($R$ is a set of words on $(x^{\pm 1})^*$ which is the same as an empty word. Then the system $(x | r)$ is referred to a presentation $G$, and usually written by $P = \langle x | r \rangle$, where $x$ is a set of generators and $r$ is a set of relations. We say that $P$ is finite if $x$ and $r$ are both finite.

If $P = \langle x | r \rangle$ be a presentation group, so there are homotopy group $\pi_1(P)$ and $\pi_2(P)$. It’s known that $\pi_1(P)$ as the first fundamental group and $\pi_2(P)$ as a second fundamental group or second homotopy module. This article discusses $\pi_2(P)$ with

$$P = \langle a, b, c, d | a^4 = b^4 = c^2 = d^2, a^2 = b^2, ab = ba^{-1}, acd = dca^{-1}, bcd = dcb^{-1} \rangle.$$  

Geometric configurations can represent the element of the second homotopy module $\pi_2(P)$ called spherical pictures.

**Definition 2.3** [3] A picture $P$ over $P$ is a geometric configuration consisting of of the following:

a. A disc $D^2$ with basepoint $O$ on $\partial D^2$.

b. Disjoint discs $\Delta_1, \Delta_2, \ldots, \Delta_n$ in the interior of $D^2$. Each $\Delta_i$ has a basepoint $O_i$ on $\partial \Delta_i$.

c. A finite number of disjoint arcs $a_1, a_2, \ldots, a_m$ where each arc lies in the closure of $D^2 - \bigcup_{i=1}^n \Delta_i$ and is either a simple closed curve having trivial intersection with $\partial D^2 \cup \bigcup_{i=1}^n \Delta_i$ or is a simple non-closed curve which joins two points of $\partial D^2 \cup \bigcup_{i=1}^n \Delta_i$, neither point being a basepoint. Each arc has a standard orientation, indicated by a short arrow meeting with the arc transversely and labeled by an element of $x \cup x^{-1}$. (By the discs of $P$ we mean the discs $\Delta_1, \Delta_2, \ldots, \Delta_n$ and not ambient disc $D^2$; $a_1, a_2, \ldots, a_m$ are arcs of $P$. The boundary $\partial D^2$ denoted by $\partial P$.

d. If we travel around $\partial \Delta_i$ once in a clockwise direction starting from $O_i$ and read off the labels on arcs encountered (if we cross an arc, labeled $x$ say, in the direction of its orientation, then we read $x^{-1}$), then we obtain a word which belongs to $r \cup r^{-1}$. We call this word the label of $\Delta_i$. 

2
The picture $P$ over $P$ is spherical if it has at least one disc and no arc of $P$ meets $\partial P$.

A picture $P$ over $P$ becomes a based picture over $P$ when it is equipped with basepoint as follows:
- Each disc $\Delta$ has one basepoint, which is a selected point in the interior of a basic corner $\Delta$.
- Picture $P$ has a global basepoint, which is a selected point in $\partial P$ that does not lie on any arc of $P$.

Two pictures will be said to be equivalent if one can be transformed to the other by a finite number of delete/insert floating circle\(^1\), delete/insert canceling pairs\(^2\), bridge move and replace($X$), where $X$ be a set of based spherical pictures over $P$.

3. The main result

A picture $P$ over $P$ is called a set of the generator of $\pi_2(P)$ if $\{[P]:P \in P\}$ generate $\pi_2(P)$. The author in [3] it is stated that the construction of generator $\pi_2(P)$ is analogous to the Dehn algorithm. The Dehn algorithm is a process of finding a set of $X$ related to a spherical picture on $P$. Therefore, the set generator of $P$ is generator iff each spherical picture over $P$ can be transformed into an empty picture by using operation on the picture.

**Theorem 3.1**

Let $P = \langle a, b, c, d | a^4 = b^4 = c^2 = d^2, ab = ba^{-1}, acd = dca^{-1}, bcd = dcb^{-1} \rangle$ be a presentation of the group. Then generator of $\pi_2(P)$ has generator spherical picture with discs are all of generators and arcs are all of the relations.

We need Theorem 3.2 and Lemma 3.3 to prove this theorem.

**Theorem 3.2** [7] The element $[P] (P \in X)$ generate $\pi_2(P)$ if and only if every spherical picture is equivalent to the empty picture (relative to $X$).

\(^1\) Floating circle in a based picture $\subset$ is an arc of $\subset$ that separates the ambient disc into two components, one of which contains the global basepoint of $\subset$ and all remaining arcs and discs of $\subset$ [3].

\(^2\) Cancelling pair in $\subset$ is a connected spherical subpicture of $\subset$ that contains exactly two discs such that: the two discs are labeled by the same relator and have opposite signs; the based point of the discs lie in the same region; each arc in the subpicture has an endpoint on each disc [3].
We say that $X$ generates $\pi_2(P)$ if the elements $[P]$ ($P \in X$) generate $\pi_2(P)$.

We already to proof the Theorem 1.1. It has known that the $\pi_2(P)$ in this theorem have pictures with disjoint arcs $a$, $b$, $c$, and $d$. These pictures have infinite forms. We need van Kampen Lemma to construct the pictures/spherical pictures.

**Lemma 3.3** (van Kampen Lemma) [8] Let $P(x|r)$ be a presentation defining the group $G$. If $W$ is the word on $x^\pm1$, then $W \approx 1$ if and only if $W \approx 1 U_1r_1^{e_1}U_1^{-1}U_2r_2^{e_2}U_2^{-1}...U_nr_n^{e_n}U_n^{-1}$, where $U_i \in r$ and $r_i^{e_i} \in x$.

**Proof the Theorem 3.1**

Generator of $\pi_2(P)$ with $P = \langle a, b, c, d | a^4 = b^4 = c^2 = d^2, ab = ba^{-1}, acd = dca^{-1}, bcd = dcab^{-1} \rangle$ are

![Diagram](Figure 2.png)

**Figure 2.** The generator of $\pi_2(P)$

Let $P$ be a picture over $P$. If $P$ contains only a picture with arcs is a relation in $P$, so $P$ is a generator of $\pi_2(P)$ based on Theorem 3.1. We use Lemma 3.2 to draw the picture $P$. If $P$ contains pictures with two or more relation in $P$, then use finite operation on picture to get the spherical picture. Thus, this spherical picture that becomes the generator of $\pi_2(P)$.

**References**

[1] Yanita Y, Helmi M R and Zakiya A M 2019 *Asian Journal of Scientific Research* Vol. 12 No. 2 pp. 293-297

[2] Yanita Y 2020 *Journal of Computational and Theoretical Nanoscience* Vol. 17: pp. 1-4

[3] Bogley W A and Pride S J 1993 Calculating generators of $\pi_2$, in *Two-dimensional homotopy and combinatorial group theory* (eds. C. Hog-Angeloni, W. Metzler & A. J. Sieradski) London Math. Soc. Lecture Note Ser. No. 197 (Cambridge University Press) pp. 157-188

[4] Johnson D L 1997 *Presentation of Group* Second Edition London Mathematical Society Student Text 15 Cambridge University Press.

[5] Fine B and Rosenberger G 1999 *Algebraic Generalizations of Discrete Groups: A Path to Combinatorial Group Theory Through One-Relator Product* Marcell Dekker Inc. New York the USA

[6] Knapp A W 2006 *Basic Algebra* Birkh"auser Boston Springer Science+Business Media LLC New York the USA
[7] Pride S J 1991 Identity among relations of a group presentation, in *Group Theory from the geometrical viewpoint – Trieste* World Scientific Publishing Co. Pte. Ltd. Singapore 687-717

[8] Hatcher A 2002 *Algebraic Topology* Cambridge University Press. USA