Special 2-cocycles and 3–3 Pachner move relations in Grassmann algebra

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Abstract

Grassmann-algebraic relations, corresponding naturally to Pachner move 3–3 in four-dimensional topology, are presented. They involve 2-cocycles of two specific forms, and some more homological objects.

1 Introduction

This short note presents a construction of a 3–3 Pachner move relation in Grassmann algebra — the four-dimensional analogue of pentagon relation known in three-dimensional algebraic topology. Below:

• in Section 2, the Grassmann–Berezin calculus of anticommuting variables is briefly recalled,
• in Section 3, Pachner move 3–3 is explained, together with a possible form of algebraic relation corresponding to it,
• in Section 4, actual four-simplex Grassmann weights are presented satisfying this relation, and a cocyclic property of their coefficients is stated,
• in Section 5, a generalization of our Grassmann weights is proposed involving new coefficients, also, apparently, of homological nature,
• and in Appendix on page 6 different Grassmann weights are presented, whose parameterization uses Jacobi elliptic functions. According to numerical evidence, these Grassmann weights also satisfy the same 3–3 relation.

2 Grassmann–Berezin calculus

A Grassmann algebra over a field $\mathbb{F}$ of characteristic $> 2$ is an associative $\mathbb{F}$-algebra with unity, generators $x_i$ and relations

$$x_i x_j = -x_j x_i.$$
In particular, $x_i^2 = 0$, so an element of a Grassmann algebra is a polynomial of degree $\leq 1$ in each $x_i$. If it consists only of monomials of even (odd) total degree, it is called an even (odd) element.

The exponent is defined by its Taylor series. For instance,

$$\exp(x_1x_2 + x_3x_4) = 1 + x_1x_2 + x_3x_4 + x_1x_2x_3x_4.$$ 

The Berezin integral $[1]$ in a variable ($= \text{generator}$) $x_i$ is, by definition, the $\mathbb{F}$-linear operator

$$f \mapsto \int f \, dx_i$$

in Grassmann algebra satisfying

$$\int dx_i = 0, \quad \int x_i \, dx_i = 1, \quad \int gh \, dx_i = g \int h \, dx_i,$$

if $g$ does not contain $x_i$; multiple integral is understood as iterated one, according to the following model:

$$\int\int xy \, dy \, dx = \int x \left( \int y \, dy \right) \, dx = 1.$$

### 3 Pachner move 3–3 and a proposed form of algebraic relation

Pachner moves $[8]$ are elementary local rebuildings of a manifold triangulation. A triangulation of a piecewise-linear manifold can be transformed into another triangulation using a finite sequence of Pachner moves, see $[6]$ for a pedagogical introduction.

There are five (types of) Pachner moves in four dimensions, of which move 3–3 is, in some informal sense, central. It transforms a cluster of three four-simplices situated around a two-face into a cluster of three other four-simplices situated around another two-face, and occupying the same place in the manifold. We say that these clusters form the left- and right-hand sides of Pachner move, respectively. There are six vertices in each cluster, we denote them $1 \ldots 6$, and the four-simplices will be 12345, 12346 and 12356 in the l.h.s., and 12456, 13456 and 23456 in the r.h.s. Thus, the common inner two-face is 123 in the l.h.s., and 456 in the r.h.s.

An algebraic relation whose l.h.s. and r.h.s. can be said to correspond naturally to the l.h.s. and r.h.s. of a Pachner move gives hope of constructing an invariant of piecewise-linear manifolds.

**Remark.** Four other Pachner moves in four-dimensional topology are $2 \leftrightarrow 4$ and $1 \leftrightarrow 5$. Experience shows that if an interesting formula related to move 3–3 has been discovered, then there are also formulas corresponding to other moves.
The Grassmann-algebraic Pachner move relations proposed in this paper have
the following form:

\[ f_{123} \int \mathcal{W}_{12345} \mathcal{W}_{12346} \mathcal{W}_{12356} \, dx_{1234} \, dx_{1235} \, dx_{1236} \]

\[ = \pm f_{456} \int \int \int \mathcal{W}_{12456} \mathcal{W}_{13456} \mathcal{W}_{23456} \, dx_{1456} \, dx_{2456} \, dx_{3456}. \] (1)

Here Grassmann variables \( x_{ijkl} \) are attached to all three-faces \( ijk \); the Grassmann
weight \( \mathcal{W}_{ijklm} \) of a four-simplex \( ijk \) depends on (i.e., contains) the variables
on its three-faces, e.g., \( \mathcal{W}_{12345} \) depends on \( x_{1234}, x_{1235}, x_{1245}, x_{1345} \) and \( x_{2345} \).

The integration goes in variables on inner three-faces in the corresponding side
of Pachner move, while the result depends on the variables on boundary faces.
Also, there are numeric factors \( f_{ijk} \) before the integrals, thought of as attached
to the respective inner two-faces \( ijk = 123 \) or \( 456 \).

4 Grassmann weights satisfying the 3–3 relation, and a cocyclic property of coefficients

We now present Grassmann four-simplex weights \( \mathcal{W}_{ijklm} \) and factors \( f_{ijk} \), sat-
sifying the 3–3 algebraic relation [1]. These will depend on the coordinates of
vertices: we attach to each vertex \( i \) two numbers \( \xi_i, \eta_i \in \mathbb{F} \) that must be generic
enough so that the expressions (3) below never vanish.

Remark. Or we can take indeterminates over \( \mathbb{F} \) — algebraically independent elements — for \( \xi_i \) and \( \eta_i \).

We define \( \mathcal{W}_{ijklm} \) as the following Grassmann–Gaussian exponent:

\[ \mathcal{W}_{ijklm} = \exp \Phi_{ijklm}, \] (2)

where

\[ \Phi_{ijklm} = p_{ijklm} \sum_{\text{over 2-faces } abc \text{ of } ijk \text{lm}} \epsilon_{ijklm}^{d_1abcd_2} \varphi_{abc} x_{\{abcd_1\}} x_{\{abcd_2\}}, \] (3)

and below we explain the notations in (3).

First, both \( p_{ijklm} \) and \( \epsilon_{ijklm}^{d_1abcd_2} \) are signs. The first of them reflects the consistent
orientation of four-simplices, namely, for the left-hand side

\[ p_{12345} = 1, \quad p_{12346} = -1, \quad p_{23456} = 1, \]

and for the right-hand side

\[ p_{12456} = 1, \quad p_{13456} = -1, \quad p_{23456} = 1. \]
As for the epsilon, it is the sign of permutation between the sequences of its subscripts and superscripts.

Second, the value $\varphi_{abc}$ is defined as follows:

$$\varphi_{abc} = \begin{vmatrix} 1 & 1 & 1 \\ \xi_a & \xi_b & \xi_c \\ \eta_a & \eta_b & \eta_c \end{vmatrix}.$$ \hspace{1cm} (4)

Thus, $\varphi_{abc}$ belongs to an oriented two-face: for instance, $\varphi_{abc} = -\varphi_{bac}$.

And third, the curly brackets in (3) serve to emphasize that the Grassmann variable $x_{\{abcd\}}$ does not depend on the order of indices $a, b, c, d$.

**Theorem 1.** The weights $W_{ijklm}$ defined in this Section satisfy the relation (1), with

$$f_{ijk} = \frac{1}{\varphi_{ijk}},$$

and the sign before the right-hand side is minus.

**Proof.** Direct calculation. I used our package PL \cite{3} for manipulations in Grassmann algebra. \hfill $\Box$

**Statement.** Values $\varphi_{abc}$ form a 2-cocycle: for a tetrahedron $abcd$,

$$\varphi_{bcd} - \varphi_{acd} + \varphi_{abd} - \varphi_{abc} = 0.$$

**Proof.** Simple calculation using (4). \hfill $\Box$

## 5 A generalization involving still more homological objects

We are now going to generalize our Grassmann four-simplex weights using still more objects of, apparently, homological nature.

Calculations show that the exponent (2) has actually no terms of degree $> 2$, that is,

$$\exp \Phi_{ijklm} = 1 + \Phi_{ijklm}.$$ \hspace{1cm} (5)

We now change the definition (2) to the following:

$$W_{ijklm} = h_{ijklm} + \Phi_{ijklm},$$ \hspace{1cm} (6)

where $h_{ijklm}$ is some numeric coefficient (or, more generally, an even element of Grassmann algebra).
Theorem 2. The relation \(1\) holds also for weights defined according to \(6\), provided the coefficients \(h_{ijklm}\) in its r.h.s. are expressed through those in its l.h.s. as follows:

\[
\begin{align*}
\varphi_{456} h_{12456} &= \varphi_{345} h_{12345} - \varphi_{346} h_{12346} + \varphi_{356} h_{12356}, \\
\varphi_{456} h_{13456} &= \varphi_{245} h_{12345} - \varphi_{246} h_{12346} + \varphi_{256} h_{12356}, \\
\varphi_{456} h_{23456} &= \varphi_{145} h_{12345} - \varphi_{146} h_{12346} + \varphi_{156} h_{12356}.
\end{align*}
\]

Proof. Direct calculation.

There appears to be analogy between the constructions in this paper and those in [4]. Recall that second homologies, in their exotic form, do enter in Grassmanian 3–3 relations in the mentioned paper. Our present relations look substantially simpler, which may be important for constructing a TQFT. Their homological nature is still to be clarified.

The 3–3 relations proposed here have been found while trying to generalize the pentagon relation in [5] to four-dimensional case.

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Appendix: A relation with a quadratic form of rank 4

The explanation of relation (5) for the Grassmanian quadratic form $\Phi_{ijklm}$ introduced in Section 4 lies in the fact that $\Phi_{ijklm}$ has rank 2. A generic Grassmanian quadratic form of five variables has, however, rank 4 — the maximal rank of an antisymmetric $5 \times 5$ matrix. It is natural to expect (for instance, from a comparison with the pentagon relation in [5]) that relations with quadratic forms of rank 4 will expose richer mathematical structure than those of rank 2.

In this Appendix, I report new Grassmann weights $W_{ijklm}$, with $\Phi_{ijklm}$ of rank 4. They are constructed almost exactly as in Section 4, except that we specify the field $\mathbb{F}$ to be the field of complex numbers, $\mathbb{F} = \mathbb{C}$, and $\varphi_{abc}$ is defined not by (4), but as follows:

$$\varphi_{abc} = \text{sn}(\alpha_a - \alpha_b) \text{sn}(\alpha_b - \alpha_c) \text{sn}(\alpha_c - \alpha_a).$$

(7)

Here $\text{sn}(\cdot) = \text{sn}(\cdot, k)$ is the Jacobi elliptic sine of some fixed modulus $k$, and there is just one complex number $\alpha_i$ at each vertex $i$.

**Experimental result.** The weights $W_{ijklm}$ with $\varphi_{abc}$ defined according to (7) satisfy the same relation (1), refined according to the same Theorem 1.

Also, it is not hard to show that the values (7) have the same cocyclic property as described in the Statement after Theorem 1.

These new Grassmann weights were found by guess-and-try method combined with some theoretical ideas that are to be disclosed later. By now, I was only able to check numerically the validity of relation (1) for these weights.