Comment on the Renormalization Group Improvement in Exclusive $b \to c$ Transitions

N.G. Uraltsev

St. Petersburg Nuclear Physics Institute, Gatchina, St. Petersburg 188350, Russia

Abstract

Using rather general consideration I argue that the numerical impact of the existing next-to-leading logarithmic summations of terms $\log m_b/m_c$ for perturbative factors $\eta_A$, $\eta_V$ in the exclusive zero recoil semileptonic transitions is irrelevant, and adopting corresponding corrections beyond the exact one loop result for both estimating the values of these formfactors and their theoretical uncertainty, is misleading. The central theoretical value if taken literally is then $\eta_A \simeq 0.97$. 
Weak decays of beauty particles allow one the accurate theoretical description based on QCD, which utilizes the existence of the large mass scale $m_b$; this description in many instances is model-independent to a large extent. The first step in a practical incarnation of this idea is to account for the interaction physics associated with the high scale, which can be done in a standard way expanding in $\alpha_s(m_Q)$. Typically the first radiative corrections $\sim \alpha_s(m_b)$ are just those effects that produce the largest impact in decays of $b$ hadrons, at least in the absence of accidental cancellations. Among the exclusive decays, most attention has been paid to $b \to c$ transitions, and in particular in the zero recoil kinematics, where the most informative conclusions can be done. For example, rather accurate experimental determination of the KM mixing parameter $|V_{cb}|$ is possible using the decay rate of $B \to D^* \ell \nu$ (and, possibly, $B \to D \ell \nu$) extrapolated to the zero recoil point. The corresponding theoretical accuracy of such determination can be as high as 10% or even somewhat better, following only the way to extract $|V_{cb}|$ from inclusive semileptonic decay widths [1, 2].

Perturbative radiative corrections to the $b \to c$ semileptonic amplitude at zero recoil have been calculated in one loop at least seven years ago [3] and appeared to be rather small, at the level of 3%:

$$\eta_A = 1 + \frac{\alpha_s}{\pi} \left( \frac{m_b + m_c}{m_b - m_c} \log \frac{m_b}{m_c} \frac{8}{3} \right)$$

$$\eta_V = 1 + \frac{\alpha_s}{\pi} \left( \frac{m_b + m_c}{m_b - m_c} \log \frac{m_b}{m_c} - 2 \right)$$

where subscripts $V$ and $A$ mark vector and axial currents. In particular, for $\eta_A$ this was a result of a partial cancellation that existed for the real masses of $b$ and $c$ quarks. In $b \to c$ transitions there exists the region of loop momenta between $m_c$ and $m_b$ that contributes to the renormalization of the matrix elements of weak currents, which happens to yield the correction of the sign opposite to the correction to $\eta_A$ at equal masses. It is obvious, though, that the numerical cancellation in the first order correction does not mean that the whole perturbative one is also suppressed; for there are next order corrections governed by the factor $\alpha_s^2(m_c)$ that generically are as large as the literal one loop correction. Later some formal renormalization group improvement of the V-S calculation has been done where next-to-leading corrections have been summed up in the logarithmic approximation, i.e. accounting for the terms $\alpha_s^{n+1} \log^n \frac{m_b}{m_c}$.

In this note I discuss in more detail arguments pointed out recently in Refs. [1, 2], which show the numerical irrelevance of this renormgroup improvement for obtaining the actual values of $\eta_A$ and $\eta_V$, and examplify the statement made there that using NLA-“improved” expressions for estimating the theoretical uncertainty in the perturbative renormalization factors is misleading. More consistent treatment outlined here allows one to obtain the numerical result of the NLA (next-to-leading approximation) accuracy without explicit calculations. This more complete discussion seems to be worthwhile in view of some confusion appearing in literature (see,
e.g. Refs. [4]). I confirm that further theoretical improvement of the one loop result for $\eta_A$ can come only from a true two loop calculation, or at least if the two loop correction is exactly calculated at $m_c = m_b$.

2. The main object of our discussion will be two renormalization factors $\eta_A, \eta_V$ defined perturbatively through the “forward” matrix elements as follows:

$$\langle c(p')|\bar{c}\gamma_\mu\gamma_5 b(p) \rangle = \eta_A \bar{c}(p')\gamma_\mu\gamma_5 b(p)$$

$$\langle c(p')|\bar{c}\gamma_\mu b(p) \rangle = \eta_V \bar{c}(p')\gamma_\mu b(p)$$

where $c(p'), b(p)$ are the perturbative on shell heavy quark states (in the lhs) and the corresponding spinors (in the rhs). It is important for what follows that $\eta_A, V$ are infrared finite: this is a consequence of the fact that inelastic decay probability vanishes at the maximal recoil $q^2_{\text{lept}} = (m_b - m_c)^2$.

Straightforward calculation of the one loop quark diagrams leads to the first order result [3]

$$\eta_A = 1 + \frac{\alpha_s}{\pi} C_A^{(1)} \simeq 0.965 \quad \eta_A^{(1)} = \frac{m_b + m_c}{m_b - m_c} \log \frac{m_b}{m_c} - \frac{8}{3}$$

$$\eta_V = 1 + \frac{\alpha_s}{\pi} C_V^{(1)} \simeq 1.02 \quad \eta_V^{(1)} = \frac{m_b + m_c}{m_b - m_c} \log \frac{m_b}{m_c} - 2$$

The value of $\alpha_s$ in the one loop calculation must be evaluated at a scale of the order of $m_c$ or $m_b$. The common point of view is that to eliminate this scale ambiguity and improve the accuracy of the estimate one has to make the summation of the terms $(\log m_b/m_c)^n$ in higher orders of perturbation theory. In real world the value of $\log m_b/m_c \simeq 1.2$ is not a large parameter. It is true, however, that the actual relevance of such an expansion parameter is difficult to judge a priori. Before turning to concrete calculations let us analyse the perturbative series appearing in the logarithmic summation, from a more general perspective.

The most simple observation which, as is shown below, underlies all numerics in the problem, is that $\eta_A, V$ are to be symmetric as functions of $m_b$ and $m_c$. Therefore it is advantageous to introduce the geometric average mass

$$\tilde{m} = (m_c m_b)^{1/2} ,$$

$$m_b = x \tilde{m} , \quad m_c = \frac{1}{x} \tilde{m} , \quad x = \sqrt{m_b/m_c}$$

and use $\bar{\alpha}_s \equiv \alpha_s(\tilde{m})$ as an expansion parameter for perturbative series. For the sake of definiteness I will discuss the case of axial current and omit subscript “$A$” in what follows, where it is unimportant. The same consideration obviously applies to both cases.
Starting from the point $\bar{m}$ one needs to sum up powers of $\log m_b/\bar{m}$ and $\log \bar{m}/m_c$ which both equal to $\log x$. Using the above notations one then can write the generic expansion for $\eta$ as

$$\eta = 1 + \frac{\alpha_s(\bar{m})}{\pi} c^{(1)}(x) + f_0(\frac{\alpha_s(\bar{m})}{\pi} \log x) + \frac{\alpha_s(\bar{m})}{\pi} f_1(\frac{\alpha_s(\bar{m})}{\pi} \log x) +$$

$$+ \left( \frac{\alpha_s(\bar{m})}{\pi} \right)^2 f_2(\frac{\alpha_s(\bar{m})}{\pi} \log x) + \ldots.$$  \hspace{1cm} (5)

$f_0$ is the matter of the leading log approximation (LLA), $f_1$ is obtained in the next-to-leading summation, etc. It is assumed in eq.(5) that $f_0(r) \sim O(r^2)$ and $f_1(r) \sim O(r)$ at small $r$, for the corresponding terms are explicitly accounted for by the exact zero and first order coefficients.

The function $c^{(1)}(x)$ is, of course, symmetric under the transformation $x \rightarrow 1/x$. The fact that

$$\eta(\bar{m}, x) = \eta(\bar{m}, 1/x)$$  \hspace{1cm} (6)

therefore implies that all $f_i(x)$ are even functions of their arguments:

$$f_i(r) = f_i(-r).$$  \hspace{1cm} (7)

If $f_i(r)$ were analytical at $r = 0$, eq.(7) would have meant that all “NLA” effects due to $f_1$, $f_2$ \ldots produce practically vanishing impact on $\eta$ at $\log \sqrt{m_b/m_c} \simeq 0.6$ and $r = \frac{\alpha_s}{\pi} \log \sqrt{m_b/m_c} \simeq 0.05$. For then the expansion of $f_1(r)$ would have started with $r^2$ with the term

$$\sim \frac{\bar{\alpha}_s}{\pi} \left( \frac{\bar{\alpha}_s}{\pi} \log x \right)^2 \simeq 2 \cdot 10^{-4},$$  \hspace{1cm} (8)

which is parametrically smaller than non-log $(\alpha_s/\pi)^2$ terms omitted in the NLA, even irrespective of the exact value. However, the functions $f_i(r)$ obtained in a log summation via an expansion in $1/\log (m_b/m_c)$ are not analytical in general at $r = 0$. It is examplified, in particular, by the one loop expression: in terms of the variable $r$ its mass dependent part is written as

$$\coth \left( \frac{\pi}{\bar{\alpha}_s} r \right) \cdot 2r$$  \hspace{1cm} (9)

which is nothing but $2|r|$ in the LLA approach.

In fact, the LLA function $f_0$ is well known. Exponentiating the one loop expression eq.(3) one readily obtains the standard LLA asymptotics

$$\eta^{LLA} = \left( \frac{\alpha_s(m_c)}{\alpha_s(m_b)} \right)^\frac{2}{\hat{\gamma}} = \left( 1 + \frac{b_{LLA}}{2\hat{\gamma}} |\log x| \right)^\frac{2}{\hat{\gamma}},$$
\[
b = \frac{11}{3} N_c - \frac{2}{3} n_f = \frac{25}{3} . \tag{10}
\]

Therefore,
\[
f_0(r) = \left(1 + \frac{\beta}{2|r|} \right)^{\frac{\mu}{2}} - 1 - 2|r| \simeq 2r^2 + \frac{697}{54} |r|^3 + \frac{643}{27} r^4 + \ldots \tag{11}
\]

where it is explicitly shown that the term linear in \(\alpha_s \log\) is absent (this is a feature of the improved version of LLA used in eq. (4), which includes the first loop correction exactly). Numerically the leading term in the \(\text{rhs}\) constitutes approximately \(4.6 \cdot 10^{-3}\) whereas the second one is \(1.4 \cdot 10^{-3}\); though, obviously, both can be neglected for practical purposes.

Now let us consider the NLA terms in eq. (5) represented by \(f_1\). The second, analytical term in the expansion of \(f_1\) at small \(r\) is already of the order of \((\bar{\alpha}_s/\pi)^3 \log^2 \sqrt{m_b/m_c} \approx 2 \cdot 10^{-4}\) and definitely is to be neglected. Therefore the only real impact could be identified with the first term linear in \(|\log m_b/m_c|\) in the expansion of \(f_1\). However this term is explicitly non-analytical as a function of masses at \(m_c = m_b\) and, therefore, cannot give any resemblance to the true correction.

Let us dwell on this particular point. Neglecting terms \((\alpha_s/\pi)^3\) and higher in eq. (5), \(\eta\) can be written in the form
\[
\eta - 1 - \frac{\bar{\alpha}_s}{\pi} c^{(1)}(x) - 2 \left( \frac{\bar{\alpha}_s}{\pi} \log x \right)^2 = \left( \frac{\bar{\alpha}_s}{\pi} \right)^2 c_2 + f^{(1)}_1 \left( \frac{\bar{\alpha}_s}{\pi} \right)^2 |\log x| + \mathcal{O}(\alpha_s^3) \tag{12}
\]

where \(c_2\) denotes the value of the second order correction to \(\eta\) at \(m_b = m_c\) and \(f^{(1)}_1\) is the coefficient in the first term of the expansion of the NLA function \(f_1\) at small argument. NLA pretends to evaluate the difference in the left hand side calculating the leading correction to it.

As was shown above, the only numerical impact of the “NLA improvement” for \(b \to c\) decays at zero recoil is the second term in the \(\text{rhs}\) of eq. (12). However, the difference in the \(\text{lhs}\) is an analytical function at \(m_b = m_c\). Therefore, either \(f^{(1)}_1\) must vanish or, if it does not, it cannot be trusted as being overwhelmed by neglected terms. For example, the derivative of \(\eta\) with respect to \(x\) is a continuous function in any particular order in \(\alpha_s(\bar{m})\), as well as the whole formfactor. If a term in the series is not analytical in some region of positive \(m_c/m_b\), it must be subleading.

In the complete expansion in \(1/\log x\) eq. (12) such non-analytical terms can easily emerge, for example, in the form \(\sqrt{1 + \log^2 x}\), or in a way similar to eq. (9), therefore non-vanishing \(f^{(1)}_1\) does not mean that an arithmetic mistake has been made. However, log expansions like eq. (8) are asymptotic only. If it appears that the non-analytical terms like \(|\log x|\) are dominant, it clearly signals that one has gone too far beyond the asymptotic region so that the difference between the true value of the function and its truncated series is already larger than the last accounted term. It is not surprising at all that this happens for \(b \to c\) transitions: the relevant “large” parameter of the expansion, \(\log \sqrt{m_b/m_c}\), appears to be only \(0.6!\)
Let us return for the moment to the LLA. Its naive application would produce a large numerical correction to \( \eta_A \), \( \delta_\eta \simeq \frac{2\alpha_s}{\pi} \log \frac{m_b}{m_c} \simeq +0.10 \) practically irrespective of whether one uses the whole LLA function \( (\alpha_s(m_c)/\alpha_s(m_b))^{6/25} - 1 \), or only its first term \( \frac{2\alpha_s}{\pi} |\log x| \) — for its other terms are safely below 1%. According to the general reasoning above in reality this first term is to be practically absent, and one gets much better approximation \( \eta_A \approx 1 \) merely neglecting it at all! The exact calculation of the one loop correction clearly supports our general conclusion. This failure of the naive LLA has been erroneously interpreted in papers [4] as a necessity and relevance of the NLA log corrections obtained in a standard renormgroup treatment. In fact the true correction at a few percent accuracy is given by the exact one loop expression eq. (3) and, of course, is quadratic in \( \log \sqrt{m_b/m_c} \) being thus rather small. One can formally “improve” the naive LLA result using the more consistent LLA expansion according to eq. (5) which makes use of the exact one loop coefficient. Then at least the leading term of the remaining LLA correction, eq. (11), has a proper analytical form, and problems start with only the next term which is practically negligible. Though it goes without saying that the size of both these terms lies below the “noise level” and they cannot be taken seriously under any circumstances.

The very same consideration applies to the “NLA improvement” of Refs. [4]. The result obtained there has the form

\[
\eta_A = \alpha^2 \left\{ 1 + \frac{\alpha_s(m_b) - \alpha_s(m_c)}{\alpha_s(m_c)} Z_4 - \frac{8\alpha_s(m_c)}{3\pi} + \frac{1}{x^2} \left( \frac{25}{54} - \frac{14}{27} \alpha^3 - \frac{1}{18} \alpha^4 + \frac{8}{3} \log \alpha \right) + \frac{4\alpha_s(\mu)}{\pi} \frac{\log x}{x^2(x^2 - 1)} \right\},
\]

\[
\alpha = \left( \frac{\alpha_s(m_c)}{\alpha_s(m_b)} \right)^{3/25}, \quad Z_4 \simeq -1.5608
\]

and \( \mu \) is some scale between \( m_c \) and \( m_b \). This expression, formally valid, as usually, at \( \log m_b/m_c \gg 1 \), \( \alpha_s \cdot \log m_b/m_c \sim 1 \), allows one to determine the NLA function \( f_1 \) in eq. (3). In particular at small argument it would be

\[
f_1(r) = \left( -\frac{25}{3} Z_4 + \frac{50}{3} s - \frac{2596}{225} \right) \cdot |r| + \mathcal{O}(r^2), \quad \mu \equiv m_c m_b^{1-s}
\]

and the “wrong” non-analytical term is the leading one here for any intermediate scale \( \mu \) at \( 0 \leq s \leq 1 \). For this reason it is clear that such numerical predictions for the perturbative corrections to \( \eta_A \) beyond the exact first loop result have no any relevance to the real value of \( \eta_A \). The true result that can be seen from real two loop calculations (when they are made) is to be quadratic in \( \log \sqrt{m_b/m_c} \) and, therefore, absolutely different numerically from the one following from eq. (3) — or both must be smaller than neglected terms.

Similar to the case of the LLA, the problem with the leading term in the NLA can be cured by an explicit calculation of the two loop correction, which would eliminate...
the spurious part proportional to $|\log x|$ from the NLA function $f_1$. However, it is clear that the exact second loop calculation would provide one with enough accuracy \textit{per se} and any further summation is not needed then.

It is worth to emphasize that this is not that has been done in Refs. [4] where the NLA result is adjusted to fit exactly the one loop correction when expanded in $\alpha_s$. In fact, only the first three terms in the curly brackets in eq. (13) can (and have been) directly obtained by summing the leading logs; other terms are a rather arbitrary extrapolation made to get the correct one loop result when $\alpha_s \log$ is small.

The theoretical analysis above elucidate the underlying problem faced in the first two Refs. [4], that makes it impossible a consistent derivation of eq. (13) in its literal form. For to have any practical relevance the calculation must include terms like $m_c/m_b$ that vanish in any order of the expansion in $1/\log (m_b/m_c)$ being exponentially suppressed in this parameter. On the other hand, it is the presence of these terms that allows the analytical formfactor to get non-analytical leading term. Therefore their inclusion is to be mandatory if one considers the case when $\log \sqrt{m_b/m_c}$ is not large, – which seems to be impossible directly withing the orthodox renormalgroup expansion used in Refs. [4].

It is interesting to note that, as it follows from the symmetry arguments, the main impact is expected from the value of the second order correction at the symmetric point $m_c = m_b$; the dependence on the ratio takes the quadratic form $\sim \log^2 \sqrt{m_b/m_c} \approx 0.35$ and may well appear to be weak numerically. However, all corrections to the vector formfactor vanish at this point, and therefore the proper one loop result for $\eta_V$ is expected to be more reliable.

3. Let us now discuss the anatomy of numbers in eq. (13) that lead to the precise numerical prediction quoted in Refs. [4]. It is easily checked that the literal expression in eq. (13) is approximated by its perturbative expansion in $\alpha_s/\pi$ through the second order with a better than 1% accuracy; for simplicity we then neglect terms $\sim (\alpha_s/\pi)^3$. The first order term exactly coincides with $c_A^{(1)}$, and keeping only this one would give the numerical one loop result

$$\eta_A \approx 0.965$$

for the parameters adopted in Refs. [4]: $m_c/m_b = 0.3$ and $\Lambda_{\overline{MS}}(4) = 0.25$ GeV. We then consider the second order coefficient that follows from eq. (13). It takes the form

$$c^{(2)}(x) = \left(2 + \frac{55x^2 + 300s - 133}{9x^2(x^2 - 1)}\right) \log^2 x + \frac{1386x^{-2} - 1875Z_4 - 3007}{225} \log x$$

$\text{Note that introducing the variable } \log (m_b/m_c) \text{ instead of } (\frac{m_b}{m_c} - 1) \text{ increases the radius of convergence of the Taylor expansion of the first loop coefficient from } 0 < \frac{m_b}{m_c} \leq 2 \text{ to } e^{-2\pi} \leq \frac{m_b}{m_c} \leq e^{2\pi} \approx 500. \text{ Then one can, for example, use only the first non-trivial term } \frac{1}{3} \log^2 \frac{m_b}{m_c} \text{ in the expansion of } c^{(1)} \text{ in eqs. (1)} \text{ to get its value at } m_c \neq m_b \text{ with more than enough accuracy. The second order coefficient as it would come from eq. (13) is much more singular as will be seen shortly, that does not seem to reflect reality.}$
at \( x \geq 1 \); if \( m_c > m_b \) \( c^{(2)}(x) \) is to be calculated using eq. (13) with the substitution \( x \to 1/x \) (the first term in the brackets is the “improved LLA” one). The scale parameter \( s \) is defined in eq. (14). Again it is advantageous to use the scale variable \( y = -\log x \); the plot of \( c^{(2)}(y) \) as given by eq. (15) is shown in Fig. 1 for \( s = 1 \) (\( \mu = m_c \)) and \( s = 0 \) (\( \mu = m_b \)). The central value of \( c^{(2)} \) at \( m_c/m_b \simeq 0.3 \) is approximately 2.7 which translates into the correction +0.016 to \( \eta_A \). As it is seen from Fig. 1 this value of \( c^{(2)} \) is largely determined by the linear term \( \propto |y| \). However, just this linear term cannot be present in the formfactor! Moreover, if one varies \( s \) from 0 to 1 \( c^{(2)} \) varies by \( \pm 0.8 \) which produces a \( \pm 0.005 \) variation in \( \eta_A \), i.e. practically the whole of 0.006 estimated in Refs. \[4\] as the theoretical uncertainty in \( \eta_A \). Therefore, the major part of the theoretical uncertainty was deduced from the term that would be absent at all in a consistent calculation.

The central value for \( \eta_A \) given by eq. (13) at \( s = 1/2 \) is

\[
\eta_A = 0.976
\]

and the difference with the one loop value is mainly given by the shift 0.016 due to the second order term (the remaining difference is given by the \( (\alpha_s/\pi)^3 \) terms in eq. (13), which are also \textit{ad hoc}; in particular, they vary extremely sharp). The larger difference, 0.02 occurs when \( \mu = m_c \) is assumed (the scale preferred in Refs. \[4\]), and is almost completely due to the “wrong” term in the second order coefficient.

It is curious to note that if one decides to adjust the scale \( \mu \) in eq. (13) in such a way as to get rid of the unphysical fracture in \( c^{(2)}(y) \) at \( m_b = m_c \), viz. take \( \mu \) somewhat above \( m_b \) \((s \simeq -0.09)\) then the difference between the one loop and the NLA expression eq. (13) appears to be only about 0.003 \[4\], i.e. essentially smaller than the accuracy pretended on even in Refs. \[4\]. This is not surprizing, of course, for the value of \( c^{(2)} \) at \( m_c = m_b \), which only plays a role for a smooth function, is set zero. Thus one can see that any noticeable deviation of the estimates inferred from the NLA expression (13), from the “educated one loop” result does not have any justification.

What is wrong, conceptually, in the estimate based on eq. (13)? The main flaw is that it assumes \( c^{(2)} \) to vanish\[4\] at \( m_b = m_c \) (it is worth reminding that \( c^{(2)} \) depends on the renormalization scheme used to define the strong coupling). It is clear that in reality the main part of the second order correction to \( \eta_A \) is to be given just by \( c^{(2)}(m_b = m_c) \). For this point is a regular one for the formfactor and it must be a smooth even function of \( y \); therefore its variation up to the real value of the mass ratio is not expected to be large for \( y^2 \simeq 0.4 \). No apparent reason is seen why this

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2It is quoted in Refs. \[4\] as 0.986 although such a value is obtained only at extreme values of all parameters, viz. \( m_c/m_b = 0.25 \) and the “worst” choice of \( \mu = m_c \), whereas the literal dependence on the strong coupling is not pronounced.

3Actually even this smallish difference is almost completely saturated by the “improved LLA” term.

4This happens in fact for any order perturbative coefficient stemming from eq. (13), except for the first loop one.
value cannot be as large as ±4 – if eq. (13) gives, though rather arbitrarily, just values of a similar size; this would shift the estimate for $\eta_A$ by ±0.025. From this perspective the claim made in the last of Refs. [4] that the corrections in $c^{(2)}$ that are not accounted for by eq. (13), are not expected to exceed unity, seems to be not self-consistent.

The second failure of eq. (13) is that at small $y$ it has a large linear in $|y|$ term that dominates the value of the function – but cannot be there in reality. This fact comes from the ad hoc method of calculating the “exponential” terms $\log x$ within renormgroup approach, suggested in the first two Refs. [4]. At least in the problem under consideration it appears to be inconsistent with general symmetry properties of the formfactor; in particular it leads to a rather sharp variation in the perturbative coefficients at $|y| \approx 0.6$, which can hardly be expected from true functions.

The arguments given above make it clear that not only an attempt to determine a more precise value of $\eta_{A,V}$ by a next-to-leading summation of $\log (m_b/m_c)$, but even an estimate of a possible theoretical uncertainty varying parameters entering this calculation (the way adopted in the original papers [4]), are completely misleading. Instead, one could have used, in principle, for the latter purpose the one loop result eq. (3) which has been shown to give the same accuracy from the very beginning. However, the particular cancellation in the first loop coefficient in $\eta_A$ at real masses $m_b$ and $m_c$ makes it unsafe the standard way of varying the scale at which $\alpha_s$ is evaluated. From this point of view it is more reliable to consider the case when $m_b = m_c$ and to vary the scale of the strong coupling near $\mu = m_c$ for this quantity. This way suggests the uncertainty of about $\pm 2 \div 3\%$ in $\eta_A$ – the number that seems to be more reasonable as an account for the physics at the scale $m_c$, and is supported by the numerical discussion above. The real clarification of the value of $\eta_A$ as well as a more confident estimate of its theoretical uncertainty at the level of a few percent is possible only by the exact calculation of the two loop correction to $\eta_A$.

In the absence of real two loop calculations the best one has at present for the perturbative correction factors is the one loop expression of Ref. [3] where the scale of the strong coupling is taken to be $\sqrt{m_c m_b}$:

$$\eta_A = 1 + \frac{\alpha_s(\sqrt{m_b m_c})}{\pi} \left( \frac{m_b + m_c}{m_b - m_c} \log \frac{m_b}{m_c} - \frac{8}{3} \right) \simeq 0.97$$

$$\eta_V = 1 + \frac{\alpha_s(\sqrt{m_b m_c})}{\pi} \left( \frac{m_b + m_c}{m_b - m_c} \log \frac{m_b}{m_c} - \frac{2}{3} \right) \simeq 1.02 \quad (16)$$

where it is put $m_c/m_b = 0.27$ and $\bar{\alpha}_s = 0.25$, with the uncertainty of at least about 3% for the axial current. This uncertainty must be added to the possible variation in the impact of non-perturbative corrections in the total zero recoil formfactor of $B \to D^*$ decay[9].

4. To summarize, in the present note I discussed in more detail the arguments

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5 The method of adding the various theoretical uncertainties in quadrature adopted in Refs. [4] seems to be unjustified leading to an essential overestimate of the existing theoretical accuracy.
that lay behind the short comment on the existed perturbative calculations formu-
lated previously in Refs. [1, 2]. They suggest that a consistent LLA or NLA in the
exclusive semileptonic $b \to c$ decays, if possible at all, cannot produce any trust-
worthy impact on $\eta_{A,V}$ above a permille level. This is noticeably below a typical
size of neglected ordinary second order corrections. Neither in such a way existing
theoretical uncertainty can be estimated. The most general arguments demonstrate
that using the strong coupling normalized at the average scale $\mu^2 = m_cm_b$ in the
exact one loop expression, eq. (16), is as good as (or even better than) the existing
next-to-leading formulae [4], and the phenomenological application of the latter is
misleading beyond the information contained in eqs. (14). The only reliable way to
go beyond the one loop relations is to calculate directly two loop corrections to the
above radiative factors.

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to M. Neubert as early as in 1991, when discussing the proper form of the axial
formfactor of the $b \to c$ transitions at zero recoil.

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Preprint CERN-TH.7395/94, August 1994.

Figure Caption

Fig. 1 The second order coefficient in $\eta_A$ as a function of the mass ratio $m_c/m_b$ as
it would follow from the “NLA” formulae of Refs. [4], eq. (13). The upper curve corresponds to $\mu = m_c$ and the lower one is for $\mu = m_b$. 
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-ph/9409382v1
Fig. 1