DYNAMICAL QUANTUM GROUPS AT ROOTS OF 1

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1. Introduction

1. The notion of a dynamical quantum group was first suggested by Felder \cite{Fe} in 1994. Namely, Felder considered the quantum dynamical Yang-Baxter equation (also known as the Gervais-Neveu equation), which is a generalization of the usual quantum Yang-Baxter equation, and used the Faddeev-Reshetikhin-Takhtajan method to associate to a solution $R$ of this equation a certain algebra – the dynamical quantum group $F_R$. Felder also considered representations of $F_R$, and showed that although $F_R$ is not a Hopf algebra, its representations form a tensor category.

In 1991, Babelon \cite{Ba} generalized Drinfeld’s twisting method to the dynamical case, introducing the notion of a dynamical twist (see also \cite{BBB}). Given a dynamical twist in a quasitriangular Hopf algebra $U$, one can define a solution of the dynamical Yang-Baxter equation acting on the tensor square of any representation of $U$.

In 1997, it was shown independently in \cite{ABRR}, \cite{JKOS}, \cite{EV1} that one can naturally construct a dynamical twist in the universal enveloping algebra of any simple Lie algebra or its $q$-deformation. Using the method of \cite{BBB}, one can obtain from these twists the solutions of the quantum dynamical Yang-Baxter equation from \cite{Fe}.

At the same time, it was shown in \cite{EV1} that to any dynamical twist $J$, one can associate a Hopf algebroid $F(J)$. In the cases of $F_R$, this Hopf algebroid coincides with $F_R$ as an algebra. In particular (as was shown already in \cite{EV2}), $F_R$ has a structure of a Hopf algebroid. This explains the existence of a tensor product on the category of representations of $F_R$.

In 1999, P. Xu \cite{Xu1} associated to a dynamical twist $J$ on a Hopf algebra $U$, another Hopf algebroid $U(J)$, obtained by twisting $U$ by means of $J$. This Hopf algebroid $U(J)$ is closely related to the quasi-Hopf algebra associated to $(U, J)$ in \cite{BBR}, \cite{JKOS} (see \cite{Xu2}). P.Xu suggested that $U(J)$ should be dual, in an appropriate sense, to $F(J)$ (this is motivated by the duality of their classical limits). However, it is not very convenient to formulate such a statement precisely, because of difficulties with the notion of a dual Hopf algebroid.

Moreover, it was shown in \cite{EV1} that for dynamical twists $J$ constructed in \cite{ABRR}, \cite{JKOS}, \cite{EV1}, the category of representations of $F(J)$ is essentially the same (as a tensor category) as that of $U(g)$ or $U_q(g)$. Thus, it is essentially the same as that of $U(J)$, which suggests that $U(J)$ and $F(J)$ should be not only dual to each other but also isomorphic. In other words, $U(J)$ and $F(J)$ should be selfdual. Note that this would be a fundamentally new property, not satisfied by the usual Drinfeld-Jimbo quantum groups.

Date: March 27, 2000.
2. This paper has two main goals: to make the above picture precise, and to
generalize the theory of dynamical quantum groups to the case when the quantum
parameter \( q \) is a root of unity.

Our first step is that we replace the notion of a Hopf algebroid with the recently
introduced notion of a weak Hopf algebra \([BNSz],[BSz]\). Roughly speaking, a
weak Hopf algebra is an algebra and a coalgebra such that \( \Delta \) is a homomorphism
of algebras but is allowed to map \( 1 \) to some idempotent not equal to \( 1 \otimes 1 \) (see
Section 2 for a precise definition). Every weak Hopf algebra has a natural structure
of a Hopf algebroid, but not vice versa. However, it turns out that dynamical
quantum groups (at roots of unity) are a nice class of Hopf algebroids which do
come from weak Hopf algebras. Moreover, regarding dynamical quantum groups as
weak Hopf algebras rather than Hopf algebroids is convenient for two reasons: first,
the definition of a weak Hopf algebra is much simpler, and second, it is naturally
self-dual.

Our main results can be summarized as follows.
1. We generalize Drinfeld’s twisting theory to weak Hopf algebras (Section 3). In
   particular, we show that twisting of a quasitriangular weak Hopf algebra (defined
   in \([NVT]\)) gives another quasitriangular weak Hopf algebra.
2. For every dynamical twist \( J : T \rightarrow U^\otimes 2 \) of a Hopf algebra \( U \), we define
   two weak Hopf algebras \( H \) and \( D \), the first by analogy with the construction of
   \([Xu1]\), and the second by analogy with the construction of \([EV1]\) (they correspond
to the Hopf algebroids of \([Xu]\), \([EV]\); see Section 4). We show that \( H \) is isomor-
   phic to \( D^* \) with opposite multiplication. We consider the special case when \( U \)
is quasitriangular, and analyze the homomorphism \( H^{\ast \ast} = D \rightarrow H \) defined by the
quasitriangular structure on \( H \). We give a criterion on when this homomorphism
is an isomorphism.
3. We take \( U \) to be quantum group \( U_q(\mathfrak{g}) \) (for a finite dimensional simply laced
   Lie algebra \( \mathfrak{g} \)), when \( q \) is a root of unity (more precisely, the finite dimensional
version considered by Lusztig \([L]\); see Section 5). We show that the known methods
of producing dynamical twists for generic \( q \) (\([ABRR]\), \([ESS]\)) can also be used
to produce dynamical twists when \( q \) is a root of unity. In particular, for every
generalized Belavin-Drinfeld triple \( (\Gamma_1, \Gamma_2, T) \) for \( \mathfrak{g} \), we construct (following \([ESS]\)
) a family of dynamical twists for \( U_q(\mathfrak{g}) \) which depends on \( |\Gamma_1| \) parameters. With
appropriate modifications this construction can be carried out in the non-simply
laced case as well.
4. We show that if \( T \) is an automorphism of the Dynkin diagram of \( \mathfrak{g} \) (in
   particular, if \( T = 1 \)) then the weak Hopf algebras \( H \) and \( D \) associated to the
corresponding twists are isomorphic (via the \( R \)-matrix of \( H \)). In particular, \( D \) is
self-dual (isomorphic to \( D^{\ast \ast} \)), as was expected (for \( T = 1 \)) in the case of generic
\( q \). In particular, this implies that in this case the category of representations of the
algebra \( D = D_T \) corresponding to \( T \) is equivalent (as a tensor category) to \( \text{Rep}(U) \),
and thus to \( \text{Rep}(D_T) \) for any other automorphism \( T' \). In particular, \( \text{Rep}(D_T) \) is
equivalent to \( \text{Rep}(D_1) \).

Note that an analog of the latter result (when \( q \) is generic, \( \mathfrak{g} \) is the affine Lie
algebra \( \widehat{\mathfrak{sl}}_n \), and \( T \) is the rotation of the Dynkin diagram of \( \mathfrak{g} \) by the angle \( 2\pi k/n \),
\((k,n) = 1\)) is proved in \([ES2]\).

Acknowledgements. We thank Ping Xu and Philippe Roche for useful discus-
sions. The first author was supported by the NSF grant DMS-9700477. The second
DYNAMICAL QUANTUM GROUPS AT ROOTS OF 1

author thanks UCLA for providing him a research assistantship and MIT for the warm hospitality during his visit.

2. WEAK HOPF ALGEBRAS AND HOPF ALGEBROIDS

Throughout this paper we work over an algebraically closed field $k$ and use Sweedler’s notation for comultiplication, writing $\Delta(h) = h^{(1)} \otimes h^{(2)}$.

2.1. Weak Hopf algebras.

Definition 2.1.1 ([BNSz], [BSz]). A weak bialgebra is a $k$-vector space $H$ that has structures of an algebra $(H, m, 1)$ and a coalgebra $(H, \Delta, \epsilon)$ such that the following axioms hold:

1. $\Delta$ is a (not necessarily unit-preserving) homomorphism:

$$\Delta(hg) = \Delta(h)\Delta(g);$$

2. The unit and counit satisfy the identities:

$$\epsilon(hgf) = \epsilon(hg^{(1)})\epsilon(g^{(2)}f) = \epsilon(hg^{(2)})\epsilon(g^{(1)}f),$$

$$(\Delta \otimes \text{id})\Delta(h) = (\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1),$$

for all $h, g, f \in H$.

Definition 2.1.2 ([BNSz], [BSz]). A weak Hopf algebra $H$ is a weak bialgebra equipped with a linear map $S: H \to H$, called an antipode, satisfying the following axioms:

$$m(\text{id} \otimes S)\Delta(h) = (\epsilon \otimes \text{id})(\Delta(1)(h \otimes 1)), $$

$$m(S \otimes \text{id})\Delta(h) = (\text{id} \otimes \epsilon)((1 \otimes h)\Delta(1)),$$

$$S(h^{(1)})h^{(2)}S(h^{(3)}) = S(h),$$

for all $h \in H$.

Here axioms (2) and (3) of Definition 2.1.1 are analogous to the bialgebra axioms of $\epsilon$ being an algebra homomorphism and $\Delta$ a unit preserving map, axioms (4) and (5) of Definition 2.1.2 generalize the properties of the antipode with respect to the counit. Also, it is possible to show that given (2) - (4), axiom (5) is equivalent to $S$ being both anti-algebra and anti-coalgebra map.

A morphism of weak Hopf algebras is a map between them which is both an algebra and a coalgebra morphism commuting with the antipode.

The image of a morphism is clearly a weak Hopf algebra. The tensor product of two weak Hopf algebras is defined in an obvious way.

Below we summarize the basic properties of weak Hopf algebras, see [BNSz] for the proofs.

The antipode $S$ of a weak Hopf algebra $H$ is unique; if $H$ is finite-dimensional then it is bijective [BNSz].

The right-hand sides of the formulas (4) and (5) are called the target and source counital maps and denoted $\epsilon_t$, $\epsilon_s$ respectively:

$$\epsilon_t(h) = (\epsilon \otimes \text{id})(\Delta(1)(h \otimes 1)),$$

$$\epsilon_s(h) = (\text{id} \otimes \epsilon)((1 \otimes h)\Delta(1)).$$

The counital maps $\epsilon_t$ and $\epsilon_s$ are idempotents in $\text{End}_k(H)$, and satisfy relations $S \circ \epsilon_t = \epsilon_s \circ S$ and $S \circ \epsilon_s = \epsilon_t \circ S$. 
The main difference between weak and usual Hopf algebras is that the images of the counital maps are not necessarily equal to \( k \). They turn out to be subalgebras of \( H \) called \textit{target} and \textit{source counital subalgebras} or \textit{bases} as they generalize the notion of a base of a groupoid (cf. examples below):

\[
\begin{align}
H_t &= \{ h \in H \mid \varepsilon_t(h) = h \} = \{ (\phi \otimes \text{id})\Delta(1) \mid \phi \in H^* \}, \\
H_s &= \{ h \in H \mid \varepsilon_s(h) = h \} = \{ (\text{id} \otimes \phi)\Delta(1) \mid \phi \in H^* \}.
\end{align}
\]

The counital subalgebras commute and the restriction of the antipode gives an anti-isomorphism between \( H_t \) and \( H_s \).

Any morphism between weak Hopf algebras preserves counital subalgebras, i.e., if \( \Phi : H \rightarrow H' \) is a morphism then its restrictions on the counital subalgebras are isomorphisms: \( \Phi|_{H_t} : H_t \cong H'_t \) and \( \Phi|_{H_s} : H_s \cong H'_s \).

The algebra \( H_t \) (resp. \( H_s \)) is separable (and, therefore, semisimple [^1]) with the separability idempotent \( e_t = (S \otimes \text{id})\Delta(1) \) (resp. \( e_s = (\text{id} \otimes S)\Delta(1) \)), i.e., we have \( m(e_t) = m(e_s) = 1 \) and

\[
\begin{align}
(z_1 \otimes 1)e_t(z_2 \otimes 1) &= (1 \otimes z_2)e_t(1 \otimes z_1), & z_1, z_2 &\in H_t, \\
(y_1 \otimes 1)e_s(y_2 \otimes 1) &= (1 \otimes y_2)e_s(1 \otimes y_1), & y_1, y_2 &\in H_s.
\end{align}
\]

As a consequence of this fact and \( \Delta \) being a homomorphism, we have the following useful identities:

\[
\begin{align}
(1 \otimes z_1)\Delta(h)(1 \otimes z_2) &= (S(z_1) \otimes 1)\Delta(h)(S(z_2) \otimes 1), & z_1, z_2 &\in H_t, \\
(y_1 \otimes 1)\Delta(h)(y_2 \otimes 1) &= (1 \otimes S(y_1))\Delta(h)(1 \otimes S(y_2)), & y_1, y_2 &\in H_s.
\end{align}
\]

Note that \( H \) is an ordinary Hopf algebra if and only if \( \Delta(1) = 1 \otimes 1 \) if and only if \( \varepsilon \) is a homomorphism if and only if \( H_t = H_s = k \).

The dual vector space \( H^* \) has a natural structure of a weak Hopf algebra with the structure operations dual to those of \( H \):

\[
\begin{align}
\langle \phi \psi, h \rangle &= \langle \phi \otimes \psi, \Delta(h) \rangle, \\
\langle \Delta(\phi), h \otimes g \rangle &= \langle \phi, hg \rangle, \\
\langle S(\phi), h \rangle &= \langle \phi, S(h) \rangle,
\end{align}
\]

for all \( \phi, \psi \in H^*, h, g \in H \). The unit of \( H^* \) is \( \varepsilon \) and counit is \( \phi \mapsto \langle \phi, 1 \rangle \).

One can check that if \( S \) is invertible, then the opposite algebra \( H^{\text{op}} \) is a weak Hopf algebra with the same coalgebra structure and the antipode \( S^{-1} \). Similarly, the cooperator coalgebra \( H^{\text{cop}} \) is a weak Hopf algebra with the same algebra structure and the antipode \( S^{-1} \).

It was shown in [^NVT] that modules over any weak Hopf algebra \( H \) form a monoidal category, called the \textit{representation category} and denoted \( \text{Rep}(H) \) with the product of the modules \( V \) and \( W \) being equal to \( \Delta(1)(V \otimes W) \) and the unit object given by \( H_t \) which is an \( H \)-module via \( h \cdot z = \varepsilon_t(hz), h \in H, z \in H_t \).

**Example 2.1.3.** Let \( G \) be a \textit{groupoid} over a finite base (i.e., a category with finitely many objects, such that each morphism is invertible) then the groupoid algebra \( kG \) is generated by morphisms \( g \in G \) with the unit \( 1 = \sum_X \text{id}_X \), where the sum is taken over all objects \( X \) of \( G \), and the product of two morphisms is equal to their composition if the latter is defined and 0 otherwise. It becomes a weak Hopf algebra via:

\[
\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad S(g) = g^{-1}, \quad g \in G.
\]
The counital maps are given by \( \varepsilon_t(g) = gg^{-1} = \text{id}_{\text{target}(g)} \) and \( \varepsilon_s(g) = g^{-1}g = \text{id}_{\text{source}(g)} \).

If \( G \) is finite then the dual weak Hopf algebra \((kG)^*\) is generated by idempotents \( p_g, g \in G \) such that \( p_gp_h = \delta_{g,h}p_g \) and
\[
\Delta(p_g) = \sum_{uv=g} p_u \otimes p_v, \quad \varepsilon(p_g) = \delta_{g,gg^{-1}} = \delta_{g,g^{-1}g}, \quad S(p_g) = p_g^{-1}.
\]

It is known that any group action on a set gives rise to a finite groupoid. Similarly, in the non-commutative situation, one can associate a weak Hopf algebra with every action of a usual Hopf algebra on a separable algebra, see \([NVT]\) for details.

### 2.2. Hopf algebroids

The following notions were introduced in \([Lu]\) (see also \([Xu1]\)).

**Definition 2.2.1.** An algebra \( H \) is called a *total algebra* with a *base algebra* \( R \) if there exist a target map \( \alpha : R \to H \) which is an algebra homomorphism and a source map \( \beta : R \to H \) which is an algebra anti-homomorphism, such that the images \( \alpha(R) \) and \( \beta(R) \) commute in \( H \), i.e.,
\[
\alpha(a)\beta(b) = \beta(b)\alpha(a), \quad \forall a, b \in R.
\]

If \( H \) is a total algebra then above maps define a natural \( R \)-\( R \) bimodule structure on \( H \) via
\[
a \cdot h \cdot b = \alpha(a)\beta(b)h, \quad h \in H, \ a, b \in R.
\]

Note that the bimodule tensor products \( H \otimes_R H, H \otimes_R H \otimes_R H, \ldots \) are \( R \)-\( R \) bimodules in an obvious way.

**Definition 2.2.2.** A *comultiplication* on a total algebra \( H \) is an \( R \)-\( R \) bimodule map \( \Delta : H \to H \otimes_R H \) satisfying \( \Delta(1) = 1 \otimes_R 1 \),
\[
(\Delta \otimes_R \text{id})\Delta = (\text{id} \otimes_R \Delta)\Delta : H \to H \otimes_R H \otimes_R H,
\]
and compatible with the maps \( \alpha, \beta \) and comultiplication in the sense that
\[
\Delta(h)(\beta(a) \otimes 1 - 1 \otimes \alpha(a)) = 0,
\]
\[
\Delta(hg) = \Delta(h)\Delta(g), \quad h, g \in H, \ a \in R.
\]

Note that the right-hand side of equation \((21)\) is well defined in \( H \otimes_R H \) because of condition \((20)\).

**Definition 2.2.3.** A *counit* for \( H \) is an \( R \)-\( R \) bimodule map \( \varepsilon : H \to R \) (where \( R \) is an \( R \)-\( R \) bimodule via multiplication) such that \( \varepsilon(1_H) = 1_R \) and
\[
(\varepsilon \otimes \text{id})\Delta = (\text{id} \otimes \varepsilon)\Delta = \text{id},
\]
where we identify \( R \otimes_R H \cong H \otimes_R R \cong H \).

**Definition 2.2.4.** A *bialgebroid* is a total algebra \( H \) that possesses a comultiplication and counit.

**Definition 2.2.5.** An *antipode* for a bialgebroid \( H \) with a base \( R \) is a map \( \tau : H \to H \) which is an algebra anti-homomorphism such that \( \tau \circ \beta = \alpha \) and satisfies the following properties:

1. \( m(\tau \otimes \text{id})\Delta = \beta \circ \varepsilon \circ \tau; \)
2. There exists a linear map $\gamma : H \otimes_R H \to H \otimes H$ which is a right inverse for the natural projection $H \otimes H \to H \otimes_R H$ such that $m(\text{id} \otimes \gamma) \gamma \Delta = \alpha \circ \varepsilon$.

Note that $m(\tau \otimes \text{id})$ is well defined on $H \otimes R H$ but $m(\text{id} \otimes \tau)$ is not, this is why it is necessary to assume the existence of $\gamma$ in Definition 2.2.5.

**Definition 2.2.6.** A Hopf algebroid $H$ is a bialgebroid with an antipode.

A (base preserving) morphism between Hopf algebroids $H = (H, R, \ldots)$ and $H' = (H', R', \ldots)$ is a pair $(\psi, \Psi)$, where $\psi : R \to R'$ is an algebra isomorphism and $\Psi : H \to H'$ is an algebra homomorphism which is also an $R - R$ bimodule map such that $(\Psi \otimes \Delta) = \Delta' \circ \Psi$ and
\[
\alpha' \circ \psi = \Psi \circ \alpha, \quad \beta' \circ \psi = \Psi \circ \beta, \\
\varepsilon' \circ \Psi = \psi \circ \varepsilon, \quad \tau' \circ \Psi = \Psi \circ \tau.
\]

**2.3. The Hopf algebroid corresponding to a weak Hopf algebra.** It turns out that weak Hopf algebras form a proper subclass of Hopf algebroids.

**Proposition 2.3.1.** Any weak Hopf algebra $H$ (not necessarily finite-dimensional) has a natural structure of a Hopf algebroid (i.e., this assignment defines a functor).

**Proof.** Define the base algebra to be the target subalgebra of $H$, i.e., $R = H_t$, and let $\alpha = \text{id}_{H_t}$, $\beta = S^{-1}|H_t$. Then, clearly, $\alpha(R) = H_t$, $\beta(R) = H_s$, so that images of $\alpha$ and $\beta$ commute.

The comultiplication $\Delta$ regarded as a map to $H \otimes_R H$ is coassociative and compatible with the multiplication. It is a bimodule map since
\[
\Delta(\alpha(a)h) = (\alpha(a) \otimes 1)\Delta(h), \quad \Delta(\beta(a)h) = (1 \otimes \beta(a))\Delta(h),
\]
for all $h \in H$ and $a \in R$. We also have the compatibility condition:
\[
\Delta(h)(\beta(a) \otimes 1 - 1 \otimes \alpha(a)) = h(1)S^{-1}(a) \otimes h(2) - h(1) \otimes h(2)a = 0.
\]
Next, let $\epsilon = \varepsilon_t$, then we have $\epsilon(1) = 1$ and
\[
\epsilon(a \cdot h \cdot b) = \varepsilon_t(aS^{-1}(b)h) = a\epsilon(h)b,
\]
for all $h \in H$, $a, b \in R$, i.e., $\epsilon$ is an $R - R$ module map, also
\[
\begin{align*}
\epsilon(h(1)) \cdot h(2) &= \varepsilon_t(h(1))h(2) = \varepsilon(1(1)h(1))1(2)h(2) = h, \\
h(1) \cdot \epsilon(h(2)) &= S^{-1}(\epsilon(h(2)))h(1) = 1(1)h(1)\varepsilon(S(1(2))h(2)) = h,
\end{align*}
\]
where we used the antipode axioms of a weak Hopf algebra. Thus, $\epsilon$ satisfies the counit axiom.

The antipode $\tau = S$ is an algebra anti-homomorphism satisfying $\tau \circ \beta = \text{id}_{H_s} = \alpha$.

The section $\gamma : H \otimes_R H \to H \otimes H$ is given by
\[
\gamma(h \otimes_R g) = (h \cdot S(1(1))) \otimes (1(2) \cdot g) = \Delta(1)(h \otimes g).
\]
Finally, we verify the antipode properties:
\[
m(\tau \otimes_R \text{id})\Delta = \varepsilon_s = S^{-1} \circ \varepsilon_t \circ S = \beta \circ \varepsilon \circ \tau, \\
m(\text{id} \otimes_R \tau)\gamma \Delta = \varepsilon_t = \alpha \circ \varepsilon.
\]
Thus, $(H, H_t, \text{id}_{H_t}, S^{-1}_{H_t}, \Delta, \varepsilon_t, S)$ is a Hopf algebroid.

If $\Psi : H \to H'$ is a morphism between weak Hopf algebras, then it is clear from our definitions that the pair $(\Psi|_{H_t}, \Psi)$ gives a morphism between the corresponding Hopf algebroids. \qed
Remark 2.3.2. The converse of Proposition 2.3.1 is false even when $H$ is finite dimensional. Indeed, the base algebra $R = H_t$ of $H$ is necessarily separable, on the other hand, for any algebra $A$ the space $A \otimes A^{op}$ has a structure of a Hopf algebroid with the base $A$ (La1, Examples 3.1 and 4.4). In the case when $A$ is not separable, this Hopf algebroid is not a weak Hopf algebra.

3. Weak Hopf algebras coming from twisting

3.1. Twisting. We describe the procedure of constructing new weak Hopf algebras by twisting a comultiplication. Twisting of Hopf algebroids without an antipode was developed in [Xu1] and a special case of twisting of weak Hopf $*$-algebras was considered in [NV].

Definition 3.1.1. A twist for a weak Hopf algebra $H$ is a pair $(\Theta, \bar{\Theta})$, with

\begin{align*}
(\epsilon \otimes \text{id})\Theta &= (\text{id} \otimes \epsilon)\Theta = (\epsilon \otimes \text{id})\bar{\Theta} = (\text{id} \otimes \epsilon)\bar{\Theta} = 1, \\
(\Delta \otimes \text{id})(\Theta)(\text{id} \otimes 1) &= (\text{id} \otimes \Delta)(\Theta)(1 \otimes \Theta), \\
(\bar{\Theta} \otimes 1)(\Delta \otimes \text{id})(\bar{\Theta}) &= (1 \otimes \Theta)(\text{id} \otimes \Delta)(\bar{\Theta}), \\
(\Delta \otimes \text{id})(\bar{\Theta})(\text{id} \otimes \Delta)(\Theta) &= (\Theta \otimes 1)(1 \otimes \bar{\Theta}), \\
(\text{id} \otimes \Delta)(\bar{\Theta})(\Delta \otimes \text{id})(\Theta) &= (1 \otimes \Theta)(\bar{\Theta} \otimes 1).
\end{align*}

For ordinary Hopf algebras this notion coincides with the usual notion of twist and each of the four conditions (25) – (28) implies the other three. But since $\Theta$ and $\bar{\Theta}$ are, in general, not invertible we need to impose all of them.

The next Proposition extends Drinfeld’s twisting construction to the weak case.

Proposition 3.1.2. Let $(\Theta, \bar{\Theta})$ be a twist for a weak Hopf algebra $H$. Then there is a weak Hopf algebra $H_\Theta$ having the same algebra structure and counit as $H$ with a comultiplication and antipode given by

\begin{align*}
\Delta_\Theta(h) &= \bar{\Theta}\Delta(h)\Theta, \\
S_\Theta(h) &= v^{-1}S(h)v,
\end{align*}

for all $h \in H_\Theta$, where $v = m(S \otimes \text{id})\Theta$ is invertible in $H_\Theta$.

Proof. Clearly, $\Delta_\Theta$ is an algebra homomorphism. Its coassociativity follows from axioms (25) and (27).

Observe that the relations between $\Theta$ and $\epsilon$ yield the following identities (recall that $S$ is invertible when restricted on the counital subalgebras, since the latter are finite-dimensional) : \begin{align*}
\epsilon(\Theta(1))\Theta(2) &= 1, \\
\bar{\Theta}(1)\epsilon(\Theta(2)) &= 1.
\end{align*}

Here and in what follows we write $\Theta = \Theta(1) \otimes \Theta(2)$ and $\bar{\Theta} = \Theta(1) \otimes \bar{\Theta}(2)$.

Using these properties and the equation (11) we check that $\epsilon$ is still a counit for $(H, \Delta_\Theta)$:

\begin{align*}
(\epsilon \otimes \text{id})\Delta_\Theta(h) &= \epsilon(\bar{\Theta}(1)h(1)\Theta(1))\bar{\Theta}(2)h(2)\Theta(2) \\
&= \bar{\Theta}(2)S^{-1}\epsilon(\bar{\Theta}(1))h\epsilon(\Theta(1))\Theta(2) = h, \\
(\text{id} \otimes \epsilon)\Delta_\Theta(h) &= \bar{\Theta}(1)\epsilon(\Theta(1))h\bar{\Theta}(2)S^{-1}\epsilon(\Theta(2))\Theta(1) = h.
\end{align*}
We proceed to verify the rest of the axioms of a weak Hopf algebra, writing \( \Delta_\Theta(h) = h_{\Theta(1)} \otimes h_{\Theta(2)} \):

\[
\varepsilon(g h_{\Theta(1)}) \varepsilon(h_{\Theta(2)} f) = \varepsilon(g \Theta^{(1)} h_{(1)} \Theta^{(1)} \Theta^{(2)} h_{(2)} \Theta^{(2)} f) = \varepsilon(g h_{(1)}) \varepsilon(h_{(2)} f) = \varepsilon(g h f),
\]

for all \( g, h, f \in H \).

The axioms involving \( \Delta_\Theta(1) \) are checked using identities (28) and (29) of Definition 3.1.1:

\[
(1 \otimes \Delta_\Theta(1))(\Delta_\Theta(1) \otimes 1) = (1 \otimes \Theta)(\Theta \otimes 1) = (1 \otimes \Theta)(\Theta \otimes 1) \Theta(\otimes 1) = (\Delta_\Theta \otimes \text{id}) \Delta_\Theta(1) = (\Theta \otimes 1)(\Theta \otimes 1) \Theta(\otimes 1) = (\Theta(\otimes 1)(1 \otimes \Theta) = (\Delta_\Theta(1) \otimes 1)(1 \otimes \Delta_\Theta(1)).
\]

We define a new antipode by

\[
S_\Theta(h) = \Theta^{(1)} S(\Theta^{(2)}) S(h) S(\Theta^{(1)}) \Theta^{(2)}
\]

and compute (writing \( \Theta' \), \( \Theta' \) for additional copies of \( \Theta \) and \( \Theta \)):

\[
m(\text{id} \otimes S_\Theta) \Delta_\Theta(h) = \Theta^{(1)} h_{(1)} \Theta^{(1)} \Theta^{(2)} h_{(2)} \Theta^{(2)}
\]

where we used axioms of \( (\Theta, \Theta) \) and properties of the counital maps. Note that for \( h = 1 \) we get the identity

\[
\Theta^{(1)} S(\Theta^{(2)}) : S(\Theta^{(1)}) \Theta^{(2)} = 1,
\]

i.e., \( \Theta^{(1)} S(\Theta^{(2)}) \) is the inverse of \( v = S(\Theta^{(1)}) \Theta^{(2)} \). It can be proven similarly that

\[
m(\text{id} \otimes \Delta_\Theta) \Delta_\Theta(h) = (\text{id} \otimes \varepsilon)((1 \otimes h) \Delta_\Theta(1)), \quad h \in H.
\]

Finally, let us write

\[
(\Delta_\Theta \otimes \text{id}) \Delta_\Theta(h) = \Theta^{(1)} h_{(1)} \Theta^{(1)} \otimes \Theta^{(2)} h_{(2)} \Theta^{(2)} \otimes \Theta^{(3)} h_{(3)} \Theta^{(3)}
\]
and compute

\[ m(m \otimes \text{id})(S_\Theta \otimes \text{id} \otimes S_\Theta)\Delta_\Theta(h) = \]
\[ = v^{-1}S(\Theta^{(1)})S(h_{(1)})\Theta(\Theta^{(1)})\Theta(\Theta^{(2)})v \]
\[ h_{(2)}\Theta(\Theta^{(2)})S(\Theta^{(3)})S(h_{(3)})S(\Theta^{(3)})v = \]
\[ = v^{-1}S(\Theta^{(1)})S(h_{(1)})\varepsilon_x(\Theta^{(1)})h_{(2)}\varepsilon_x(\Theta^{(2)})S(h_{(2)})S(\Theta^{(2)})v \]
\[ = v^{-1}\varepsilon_x(\Theta^{(1)})h_{(1)}S(h_{(2)})S(\Theta^{(2)})v = \]
\[ = v^{-1}S(h_{(1)})\varepsilon_x(\Theta^{(1)})\varepsilon_x(h_{(2)})S(\Theta^{(2)})v = v^{-1}S(h)v = S_\Theta(h), \]

whence \( H_\Theta \) is a weak Hopf algebra.

\[ \square \]

**Remark 3.1.3.** Note that it follows from the proof of Proposition 3.1.2 that the counital maps of the twisted weak Hopf algebra \( H_\Theta \) are given by

\[
(34) \quad (\varepsilon_\lambda)_\Theta(h) = \varepsilon(\Theta^{(1)})h\Theta^{(2)},
\]
\[
(35) \quad (\varepsilon_x)_\Theta(h) = \Theta^{(1)}\varepsilon(h\Theta^{(2)}),
\]
so the counital subalgebras are also getting deformed in general.

**Remark 3.1.4.** If \((\Theta, \bar{\Theta})\) is a twist for \( H \) and \( x \in H \) is an invertible element such that \( \varepsilon_x(x) = \varepsilon_x(x) = 1 \) then \((\Theta^x, \bar{\Theta}^x)\), where

\[
\Theta^x = \Delta(x)^{-1}\Theta(x \otimes x) \quad \text{and} \quad \bar{\Theta}^x = (x^{-1} \otimes x^{-1})\bar{\Theta}\Delta(x),
\]
is also a twist for \( H \). The twists \((\Theta, \bar{\Theta})\) and \((\Theta^x, \bar{\Theta}^x)\) are called *gauge equivalent* and \( x \) is called a *gauge transformation*. Given such an \( x \), the map \( h \mapsto x^{-1}hx \) is an isomorphism between weak Hopf algebras \( H_\Theta \) and \( H_{\Theta^x} \).

**Remark 3.1.5.** If \( \Theta \) is a twist for a weak Hopf algebra \( H \), then it also defines a twist of the corresponding Hopf algebroid constructed as in Proposition 3.1.2, cf. [Xu1].

### 3.2. Quasitriangular weak Hopf algebras

The notion of a *quasitriangular weak Hopf algebra* was introduced and studied in \([NVT]\). It is defined as a triple \((H, R, \bar{R})\) where \( H \) is a weak Hopf algebra with

\[
(36) \quad R \in \Delta^{op}(H \otimes H)\Delta(1), \quad \bar{R} \in \Delta(1)(H \otimes H)\Delta^{op}(1),
\]
\[
(37) \quad R\bar{R} = \Delta^{op}(1), \quad \bar{R}R = \Delta(1), \quad \text{and} \quad \Delta^{op}(h)R = R\Delta(h),
\]
for all \( h \in H \), where \( \Delta^{op} \) denotes the comultiplication opposite to \( \Delta \), and such that \( R \) obeys the following conditions:

\[
(38) \quad (\text{id} \otimes \Delta)R = R_{13}R_{12}, \quad (\Delta \otimes \text{id})R = R_{13}R_{23},
\]
where \( R_{12} = R \otimes 1 \) etc. as usual.

The existence of a quasitriangular structure \( R \) on \( H \) is equivalent to \( \text{Rep}(H) \) being a braided category, and any quasitriangular structure \( R \) is a solution of the quantum Yang-Baxter equation:

\[
(39) \quad R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.
\]

An example of a quasitriangular weak Hopf algebra is given by the Drinfeld double of finite dimensional weak Hopf algebra \([NVT]\).
As in the case of ordinary Hopf algebras, a quasitriangular structure $\mathcal{R}$ on $H$ defines two homomorphisms of weak Hopf algebras:

\begin{align}
(40) & \quad \rho_1 : H^* \ni \phi \mapsto (\text{id} \otimes \phi)(\mathcal{R}) \in H^{\text{op}}, \\
(41) & \quad \rho_2 : H^* \ni \phi \mapsto (\phi \otimes \text{id})(\mathcal{R}) \in H^{\text{cop}},
\end{align}

in particular, when $\mathcal{R}$ has a maximal ($= \dim H$) rank, then it defines isomorphisms $H^* \cong H^{\text{op}} \cong H^{\text{cop}}$.

A twisting of a quasitriangular weak Hopf algebra is again quasitriangular. Namely, if $(\Theta, \bar{\Theta})$ is a twist and $(\mathcal{R}, \bar{\mathcal{R}})$ is a quasitriangular structure for $H$ then the quasitriangular structure for $H_\Theta$ is given by $(\Theta_{21}, \bar{\Theta}_2, \bar{\Theta}_2 \Theta_{21})$. The proof of this fact is exactly the same as for ordinary Hopf algebras.

4. Weak Hopf algebras arising from dynamical twists

We describe two methods of constructing weak Hopf algebras, which are finite-dimensional modifications of constructions proposed in [Xu] and [EV]. It turns out that resulting weak Hopf algebras are dual to each other.

4.1. Dynamical twists on Hopf algebras. Dynamical twists first appeared in the work of Babelon [Ba], see also [BB].

Let $U$ be a Hopf algebra and $\hat{A} = \text{Map}(T, k)$ be a commutative and cocommutative Hopf algebra of functions on a finite Abelian group $T$ which is a Hopf subalgebra of $U$. Let $P_\mu, \mu \in T$ be the minimal idempotents in $A$. We fix this notation for the rest of the paper.

Remark 4.1.1. In [EV] the role of the group $T$ is played by the dual space of a Cartan subalgebra of a simple Lie algebra.

Definition 4.1.2. We say that an element $x$ in $U^{\otimes n}, n \geq 1$ has zero weight if $x$ commutes with $\Delta^a(a)$ for all $a \in A$, where $\Delta^a : A \rightarrow A^{\otimes n}$ is the iterated comultiplication.

Definition 4.1.3. An invertible, zero-weight $U^{\otimes 2}$-valued function $J(\lambda)$ on $T$ is called a dynamical twist for $U$ if it satisfies the following functional equations:

\begin{align}
(42) & \quad (\Delta \otimes \text{id}) J(\lambda)(\lambda + h^{(3)}) \otimes 1 = (\text{id} \otimes \Delta) J(\lambda)(1 \otimes J(\lambda)), \\
(43) & \quad (\varepsilon \otimes \text{id}) J(\lambda) = (\text{id} \otimes \varepsilon) J(\lambda) = 1.
\end{align}

Here an in what follows the notation $\lambda + h^{(i)}$ means that the argument $\lambda$ is shifted by the weight of the $i$-th component, e.g., $J(\lambda + h^{(3)}) = \sum_{\mu} J(\lambda + \mu) \otimes P_\mu \in U^{\otimes 2} \otimes A$.

Remark 4.1.4. If $J(\lambda)$ is a dynamical twist for $U$ and $x(\lambda)$ is an invertible, zero-weight $U$-valued function on $T$ such that $\varepsilon(x(\lambda)) \equiv 1$, then

\[ J^x(\lambda) = \Delta(x(\lambda)^{-1}) J(\lambda) \left( x(\lambda + h^{(2)}) \otimes x(\lambda) \right) \]

is also a dynamical twist for $U$, gauge equivalent to $J(\lambda)$.

Note that for every fixed $\lambda \in T$ the element $J(\lambda) \in U \otimes U$ does not have to be a twist for $U$ in the sense of Drinfeld. It turns out that an appropriate object for which $J$ defines a twisting is a certain weak Hopf algebra that we describe next.
4.2. Twisted weak Hopf algebra \((\text{End}_k(A) \otimes U)_\mathcal{J}\) (cf. [Xu1]). Observe that the simple algebra \(\text{End}_k(A)\) has a natural structure of a weak Hopf algebra given as follows.

Let \(\{E_{\lambda\mu}\}_{\lambda,\mu \in T}\) be a basis of \(\text{End}_k(A)\) such that
\[
(E_{\lambda\mu} f)(\nu) = \delta_{\mu\nu} f(\lambda), \quad f \in A, \lambda, \mu, \nu \in T,
\]
then the comultiplication, counit, and antipode of \(\text{End}_k(A)\) are given by
\[
\Delta(E_{\lambda\mu}) = E_{\lambda\mu} \otimes E_{\lambda\mu}, \quad \varepsilon(E_{\lambda\mu}) = 1, \quad S(E_{\lambda\mu}) = E_{\mu\lambda}.
\]

Define the tensor product weak Hopf algebra \(H = \text{End}_k(A) \otimes U\).

**Proposition 4.2.2.** The elements
\[
\Theta = \sum_{\lambda\mu} E_{\lambda\lambda + \mu} \otimes E_{\lambda\lambda} P_{\mu} \quad \text{and} \quad \bar{\Theta} = \sum_{\lambda\mu} E_{\lambda + \mu\lambda} \otimes E_{\lambda\lambda} P_{\mu}
\]
define a twist for \(H\).

**Proof.** Clearly, we have \(\Theta = \Delta(1)\Theta, \bar{\Theta} = \bar{\Theta}\Delta(1), \text{ and } \Theta\bar{\Theta} = \Delta(1)\). The relations between \(\Theta, \bar{\Theta}\) and counit are straightforward. We also compute
\[
(\Delta \otimes \text{id})(\Theta)(\Theta \otimes 1) = (\text{id} \otimes \Delta)(\Theta)(1 \otimes \Theta)
= \sum_{\lambda\mu\nu} E_{\lambda\lambda + \mu + \nu} \otimes E_{\lambda\lambda + \mu} P_{\nu} \otimes E_{\lambda\lambda} P_{\mu},
\]
\[
(\bar{\Theta} \otimes 1)(\Delta \otimes \text{id})(\bar{\Theta}) = (1 \otimes \bar{\Theta})(\text{id} \otimes \Delta)(\bar{\Theta})
= \sum_{\lambda\mu\nu} E_{\lambda + \mu\lambda + \nu} \otimes E_{\lambda + \mu\lambda} P_{\nu} \otimes E_{\lambda\lambda} P_{\mu}.
\]

One verifies the identities (28) and (29) of Definition 3.1.1 in a similar way. \[\square\]

Thus, according to Proposition 3.1.2, \(H_\Theta = (\text{End}_k(A) \otimes U)_\Theta\) becomes a weak Hopf algebra. It is non-commutative, non-cocommutative, and not a Hopf algebra if \(|T| > 1\). Following [Xu1], we show that it can be further twisted by means of a dynamical twist \(J(\lambda)\) on \(U\).

**Lemma 4.2.3.** Let \(J(\lambda) \in A \otimes U^\otimes 2\) be an invertible, zero weight \(U^\otimes 2\)-valued function on \(T\). Then the following identities hold in \(H^\otimes 3\) :
\[
(\Delta \otimes \text{id})(\Theta)(J(\lambda) \otimes 1) = (J(\lambda + h^{(3)}) \otimes 1)(\Delta \otimes \text{id})(\Theta),
\]
\[
(id \otimes \Delta)(\Theta)(1 \otimes J(\lambda)) = (1 \otimes J(\lambda))(id \otimes \Delta)(\Theta),
\]
\[
(J^{-1}(\lambda) \otimes 1)(\Delta \otimes \text{id})(\Theta) = (\Delta \otimes \text{id})(J^{-1}(\lambda + h^{(3)}) \otimes 1),
\]
\[
(1 \otimes J^{-1}(\lambda))(id \otimes \Delta)(\Theta) = (id \otimes \Delta)(\Theta)(1 \otimes J^{-1}(\lambda)),
\]

where \(J(\lambda) = J^{(1)}(\lambda) \otimes J^{(2)}(\lambda)\) is embedded in \(H \otimes H\) as
\[
J(\lambda) = \sum_{\lambda} E_{\lambda\lambda} J^{(1)}(\lambda) \otimes E_{\lambda\lambda} J^{(2)}(\lambda).
\]
Proof. We check first two identities, leaving the rest as an exercise to the reader. Using the formulas for $\Theta$ and $\bar{\Theta}$ we compute:

$$
(\Delta \otimes \text{id})(\Theta)(J(\lambda) \otimes 1) = \sum_{\lambda\mu} E_{\lambda\lambda+\mu} E_{\mu\nu} J^{(1)}(\nu) \otimes E_{\lambda\lambda+\mu} E_{\mu\nu} J^{(2)}(\nu) \otimes E_{\lambda\lambda} P_{\mu}
$$

$$
= \sum_{\lambda\mu} J^{(1)}(\lambda + \mu) E_{\lambda\lambda+\mu} \otimes J^{(2)}(\lambda + \mu) E_{\lambda\lambda} P_{\mu}
$$

$$
= (J(\lambda + h^{(3)}) \otimes 1)(\Delta \otimes \text{id})(\Theta),
$$

$$
(id \otimes \Delta)(\Theta)(1 \otimes J(\lambda)) = \sum_{\lambda\mu} E_{\lambda\lambda+\mu} \otimes E_{\lambda\lambda} P_{\mu(1)} J^{(1)}(\lambda) \otimes E_{\lambda\lambda} P_{\mu(2)} J^{(2)}(\lambda)
$$

$$
= \sum_{\lambda\mu} E_{\lambda\lambda+\mu} \otimes J^{(1)}(\lambda) E_{\lambda\lambda} P_{\mu(1)} \otimes J^{(2)}(\lambda) E_{\lambda\lambda} P_{\mu(2)}
$$

$$
= (1 \otimes J(\lambda))(\text{id} \otimes \Delta)(\Theta),
$$

where we used the zero weight property of $J(\lambda)$. \qed

### Proposition 4.2.4

If $J(\lambda)$ is a dynamical twist for $U$ then the pair $(F(\lambda), \bar{F}(\lambda))$, where

$$
F(\lambda) = J(\lambda)\Theta \quad \text{and} \quad \bar{F}(\lambda) = \bar{\Theta}J^{-1}(\lambda)
$$

defines a twist for $H = \text{End}_k(A) \otimes U$.

**Proof.** The twist relations involving counit are obvious, for the rest we have, using identities from Lemma 4.2.3:

$$
(\Delta \otimes \text{id})(F(\lambda))(F(\lambda) \otimes 1) = (\Delta \otimes \text{id})(J(\lambda)\Theta)(J(\lambda)\Theta \otimes 1)
$$

$$
= (\Delta \otimes \text{id})(J(\lambda))(J(\lambda + h^{(3)}))(\Delta \otimes \text{id})(\Theta)(\Theta \otimes 1)
$$

$$
= (id \otimes \Delta)(J(\lambda))(1 \otimes J(\lambda))(id \otimes \Delta)(\Theta)(1 \otimes \Theta)
$$

$$
= (id \otimes \Delta)(J(\lambda)\Theta)(1 \otimes J(\lambda)\Theta),
$$

$$
(\bar{F}(\lambda) \otimes 1)(\Delta \otimes \text{id})(\bar{F}(\lambda)) = (\bar{\Theta}J^{-1}(\lambda) \otimes 1)(\Delta \otimes \text{id})(\bar{\Theta}J^{-1}(\lambda))
$$

$$
= (\bar{\Theta} \otimes 1)(\Delta \otimes \text{id})(\bar{\Theta})
$$

$$
= (\bar{\Theta} \otimes 1)(\Delta \otimes \text{id})(\bar{\Theta})(J^{-1}(\lambda + h^{(3)}) \otimes 1)(\Delta \otimes \text{id})(J^{-1}(\lambda))
$$

$$
= (1 \otimes \bar{\Theta})(id \otimes \Delta)(\bar{\Theta})(1 \otimes J^{-1}(\lambda))(id \otimes \Delta)(J^{-1}(\lambda))
$$

$$
= (1 \otimes \bar{\Theta})(id \otimes \Delta)(\bar{\Theta})(\bar{F}(\lambda)),
$$

and one checks other identities similarly. \qed

Thus, every dynamical twist $J(\lambda)$ for a Hopf algebra $U$ gives rise to a weak Hopf algebra $H_J = H_{J(\lambda)\Theta}$.

### Remark 4.2.5

According to Proposition 3.1.4, the antipode $S_J$ of $H_J$ is given by $S_J(h) = v^{-1} S(h) v$ for all $h \in H_J$, where $S$ is the antipode of $H$ and

$$
v = \sum_{\lambda\mu} E_{\lambda+\mu\lambda}(S(J^{(1)})) J^{(2)})(\lambda) P_{\mu}.
$$

### Remark 4.2.6

If $J^x(\lambda)$ is a dynamical twist for $U$ gauge equivalent to $J(\lambda)$ by means of some $x(\lambda)$ as in Remark 4.1.4, then $x = \sum x(\lambda) E_{\lambda\lambda}$ is a gauge transformation of $H$ establishing a gauge equivalence between the twists $(J(\lambda)\Theta, \Theta J^{-1}(\lambda))$ and $(J^x(\lambda)\Theta, \Theta(J^x)^{-1}(\lambda))$.  

4.3. **Dynamical quantum groups of [EV1]**. Suppose that dim \(U < \infty\). We will introduce a weak Hopf algebra \(D_J\) on a vector space \(D = \text{Map}(\mathbb{T} \times T, k) \otimes U^*\) by adapting the construction of [EV1] to the finite-dimensional case and show that this weak Hopf algebra is in fact dual to the twisted weak Hopf algebra \(H_J\).

Let \(\{U_i\}\) and \(\{L_i\}\) be dual bases in \(U\) and \(U^*\), then the element \(L = \sum U_i \otimes L_i\) does not depend on the choice of the bases.

Define the coalgebra structure on \(D_J\) as dual to the algebra structure of \(H\):

\[
\begin{align*}
(id \otimes \Delta)L &= L^{12}L^{13}, \\
(\Delta(f))(\lambda^1, \lambda^2) &= f(\lambda^1 + \mu^1, \lambda^2 + \mu^2), \\
(id \otimes \varepsilon)(L) &= 1, \\
\varepsilon(f) &= \sum_{\lambda} f(\lambda, \lambda),
\end{align*}
\]

for all \(f \in \text{Map}(\mathbb{T} \times T, k)\) and \(\lambda^1, \lambda^2, \mu^1, \mu^2 \in \mathbb{T}\).

The algebra structure of \(D_J\) is given as follows. Observe that the vector space \(U^*\) is bigraded:

\[U^* = \oplus U^*[\alpha^1, \alpha^2], \quad \text{where} \quad U^*[\alpha^1, \alpha^2] = \text{Hom}_k(P_{\alpha^1} U P_{\alpha^2}, k).\]

Let us set

\[
\begin{align*}
f(\lambda^1, \lambda^2)g(\lambda^1, \lambda^2) &= g(\lambda^1, \lambda^2)f(\lambda^1, \lambda^2), \\
f(\lambda^1, \lambda^2)L_{\alpha^1\alpha^2} &= L_{\alpha^1\alpha^2}f(\lambda^1 + \alpha^1, \lambda^2 + \alpha^2), \\
L^{23}L^{13} &= J_{12}^{-1}(\lambda^1)(\Delta \otimes \varepsilon)(L)J_{21}(\lambda^2).\]
\]

where \(f(\lambda^1, \lambda^2), g(\lambda^1, \lambda^2) \in \text{Map}(\mathbb{T} \times T, k)\), and \(L_{\alpha^1\alpha^2} \in U^*[\alpha^1, \alpha^2]\). The equation \(L^{23}L^{13}\) is in \(U \otimes D_J\) and the sign :: ("normal ordering") means that the matrix elements of \(L\) should be put on the right of the elements of \(J^{-1}(\lambda^1), J(\lambda^2)\).

The unit of \(D_J\) is defined in an obvious way:

\[
1 = (\varepsilon \otimes \text{id})L.
\]

Let us consider two \(U\)-valued functions on \(\mathbb{T}\):

\[
\bar{K}(\lambda) = m(\text{id} \otimes S)J^{-1}(\lambda), \quad K(\lambda) = (m(S \otimes \text{id})J)(\lambda - h).
\]

Note that \(K(\lambda)\) and \(\bar{K}(\lambda)\) are inverses of each other and commute with \(A\) since \(J(\lambda)\) is of zero weight. Define the antipode of \(D_J\) by setting

\[
\begin{align*}
(Sf)(\lambda^1, \lambda^2) &= f(\lambda^2, \lambda^1), \\
(id \otimes S)(L) &= : \bar{K}(\lambda^2)(S^{-1} \otimes \text{id})(L)K(\lambda^1) :,
\end{align*}
\]

for all \(f \in \text{Map}(\mathbb{T} \times T, k)\) and \(\lambda^1, \lambda^2 \in \mathbb{T}\) and extending \(S\) to an algebra anti-homomorphism.

Next, we introduce a basis \(\{\mathbb{I}_{\lambda^1\lambda^2}\}\) of delta-functions on \(\mathbb{T} \times \mathbb{T}\), i.e.,

\[
\mathbb{I}_{\lambda^1\lambda^2}(\alpha^1, \alpha^2) = \delta_{\lambda^1 \alpha^1} \delta_{\lambda^2 \alpha^2}.
\]

Define a duality form between \(D_J = \text{Map}(\mathbb{T} \times T, k) \otimes U^* = A \otimes A \otimes U^*\) and \(H = \text{End}_k(A) \otimes U\) by

\[
\langle \mathbb{I}_{\lambda^1\lambda^2}x, E_{\alpha^1\alpha^2}u \rangle = \delta_{\lambda^1 \alpha^1} \delta_{\lambda^2 \alpha^2} \langle x, u \rangle
\]

for all \(x \in U^*\) and \(u \in U\).
Then in terms of the homogeneous elements $L_{\alpha', \alpha''} \in U^*[\alpha, \alpha']$ the relations (68) and (12) defining the multiplication and antipode of $D_J$ can be rewritten as

\[(65) \quad \langle L_{\alpha', \alpha''} L_{\beta', \beta''}, E_{\nu', \nu''} u \rangle = \langle L_{\beta', \beta''} \otimes L_{\alpha', \alpha''}, J^{-1}(\nu') \Delta(u) J(\nu'') \rangle\]

\[(66) \quad \langle S(L_{\alpha', \alpha''}), E_{\nu', \nu''} u \rangle = \langle L_{\alpha', \alpha''}, \bar{K}(\nu'') S^{-1}(u) K(\nu') \rangle,\]

for all $\alpha, \alpha', \beta, \beta', \nu, \nu'' \in \mathbb{T}$ and $u \in U$.

**Theorem 4.3.1.** With the above operations $D_J$ becomes a weak Hopf algebra opposite to $H_J^*$.  

**Proof.** We need to show that the structure operations of $D_J$ are obtained by transposing the corresponding operations of $H_J$. For the unit and counit this is obvious. The product of the elements $I_{\lambda_1 \lambda_2} L_{\alpha', \alpha''}$ and $I_{\mu_1 \mu_2} L_{\beta_1 \beta_2}$ of $D_J$, where $L_{\alpha', \alpha''} \in U^*[\alpha, \alpha']$, can be found by the evaluation against the elements of $H_J$ using formula (43):

\[
\langle I_{\lambda_1 \lambda_2} L_{\alpha', \alpha''}, E_{\nu', \nu''} u \rangle = 
\delta_{\lambda_1 \mu_1} \delta_{\lambda_2 \mu_2} \delta_{\lambda' \nu_1} \delta_{\lambda'' \nu_2} <L_{\beta_1 \beta_2} \otimes L_{\alpha', \alpha''}, J^{-1}(\nu') \Delta(u) J(\nu'')\rangle.
\]

On the other hand, we have

\[
\langle I_{\mu_1 \mu_2} L_{\beta_1 \beta_2} \otimes I_{\lambda_1 \lambda_2} L_{\alpha', \alpha''}, \Delta_J(E_{\nu', \nu''} u) \rangle = 
\sum_{\eta_1, \eta_2} \langle I_{\mu_1 \mu_2} L_{\beta_1 \beta_2}, E_{\nu_1 + \eta_1 \nu_2 + \eta_2} \rangle \langle I_{\lambda_1 \lambda_2} L_{\alpha', \alpha''}, \Delta_J(E_{\nu_1, \nu_2} u) \rangle = 
\delta_{\lambda_1 \mu_1} \delta_{\lambda_2 \mu_2} <L_{\beta_1 \beta_2} \otimes L_{\alpha', \alpha''}, J^{-1}(\nu') \Delta(u) J(\nu'')\rangle,
\]

where $J^{-1} = \bar{J}^1 \otimes \bar{J}^2$. Comparing the last two relations we conclude that the multiplication in $D_J$ is opposite to the one induced by the comultiplication in $H_J$.

Finally, the antipode defined by the formula (43) satisfies

\[
\langle S(I_{\lambda_1 \lambda_2} L_{\alpha', \alpha''}), E_{\nu', \nu''} u \rangle = 
\langle I_{\lambda_1 \lambda_2} L_{\alpha', \alpha''}, \bar{K}(\nu'') S^{-1}(u) K(\nu') \rangle,
\]

for all $u \in U$ and $\lambda, \lambda', \alpha, \alpha', \nu, \nu'' \in \mathbb{T}$, while the transpose of $S_J^{-1}$ gives

\[
\langle I_{\lambda_1 \lambda_2} L_{\alpha', \alpha''}, S^{-1}(E_{\nu', \nu''} u) \rangle = 
\sum_{\mu_1 \mu_2} \langle I_{\lambda_1 \lambda_2}, E_{\nu_2 + \mu_2} \rangle \langle L_{\alpha', \alpha''}, \bar{K}(\nu'') \rangle \langle S^{-1}(u) K(\nu') P_{\mu_2} \rangle,
\]

which completes the proof. \qed
4.4. The case of quasitriangular U. When U is quasitriangular with the universal R-matrix $\mathcal{R}$, the twisted weak Hopf algebra $H_J$ is also quasitriangular by means of the matrices

$\mathcal{R}(\lambda) = \tilde{\Theta}J_{21}^{-1}(\lambda)\mathcal{R}J(\lambda)\Theta$ and $\tilde{\mathcal{R}}(\lambda) = \tilde{\Theta}J_{21}^{-1}(\lambda)\tilde{\mathcal{R}}J_{21}(\lambda)\Theta$.

More explicitly,

$$\mathcal{R}(\lambda) = \sum_{\lambda,\mu,\nu} E_{\lambda+\mu,\nu} P_\mu \mathcal{R}^{(1)}(\lambda) \otimes E_{\lambda+\mu,\nu} \mathcal{R}^{(2)}(\lambda) P_\nu,$$

where $\mathcal{R}^{(i)}(\lambda) = J_{21}^{-1}(\lambda)\mathcal{R}J(\lambda)$. By (67), $\mathcal{R}(\lambda)$ establishes a weak Hopf algebra homomorphism $H^*_J \rightarrow H^*_J$, i.e., a homomorphism $\rho : D_J \rightarrow H_J$.

Remark 4.4.1. Note that $\dim(P_\mu U) = \dim(U|\mathbb{T}|)$ for all $\mu, \nu \in \mathbb{T}$, since any finite dimensional Hopf algebra is free over its Hopf subalgebra $\mathbb{F}$. In particular, $\dim U|\mathbb{T}|$ is an integer.

Proposition 4.4.2. $\rho$ is an isomorphism if and only if the element

$$\mathcal{R}^{(\mu,\nu)}_\mu(\lambda) = P_\mu \mathcal{R}^{(1)}(\lambda) \otimes \mathcal{R}^{(2)}(\lambda) P_\nu \in U \otimes U$$

has the maximal possible rank ($= \dim U|\mathbb{T}|$) for all fixed $\lambda, \mu, \nu \in \mathbb{T}$.

Proof. Since $H$ and $D$ are finite dimensional, $\rho$ is an isomorphism if and only if its image coincides with $H$, i.e.,

$$\text{Image}(\rho) = \text{span}\left\{ \sum_{\nu} E_{\lambda+\nu,\mu} P_\mu \mathcal{R}^{(1)}(\lambda) \phi(\mathcal{R}^{(2)}(\lambda) P_\nu) \mid \lambda, \mu \in \mathbb{T}, \phi \in U^* \right\}$$

$$= \bigoplus_{\lambda,\mu,\nu} E_{\lambda+\nu,\mu} \text{span}\left\{ P_\mu \mathcal{R}^{(1)}(\lambda) \phi(\mathcal{R}^{(2)}(\lambda) P_\nu) \mid \phi \in U^* \right\} = U,$$

which happens precisely when

$$\text{span}\left\{ P_\mu \mathcal{R}^{(1)}(\lambda) \phi(\mathcal{R}^{(2)}(\lambda) P_\nu) \mid \phi \in U^* \right\} = P_\mu U,$$

for all $\lambda, \mu, \nu \in \mathbb{T}$. Since a Hopf algebra $U$ is a free $\text{Map}(\mathbb{T}, k)$-module, we see that the rank of $\mathcal{R}^{(\mu,\nu)}_\mu(\lambda)$ has to be equal to $\dim(P_\mu U)$ and, therefore, to $\dim U|\mathbb{T}|$, by Remark 4.4.1.

5. Dynamical twists for $U_q(\mathfrak{g})$ at roots of 1

5.1. Construction of $J(\lambda)$. Suppose that $\mathfrak{g}$ is a simple Lie algebra of type $A$, $D$ or $E$ and $q$ is a primitive $\ell$th root of unity in $k$, where $\ell \geq 3$ is odd and coprime with the determinant of the Cartan matrix $(a_{ij})_{i,j=1,...,m}$ of $\mathfrak{g}$.

Let $U = U_q(\mathfrak{g})$ be the corresponding quantum group which is a finite dimensional Hopf algebra with generators $E_i, F_i, K_i$, where $i = 1, \ldots, m$ and the following relations $1$:

- $K_i^\ell = 1$, $E_i^\ell = 0$, $F_i^\ell = 0$,
- $K_i K_j = K_j K_i$, $K_i E_j = q^{a_{ji}} E_j K_i$, $K_i F_j = q^{-a_{ij}} F_j K_i$,
- $E_i F_j - F_j E_i = \delta_{ij} \dfrac{K_i - K_i^{-1}}{q - q^{-1}}$,
- $E_i E_j = E_j E_i$, $F_i F_j = F_j F_i$ if $a_{ij} = 0$,
- $E_i^2 E_j - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 = 0$ if $a_{ij} = -1$,
- $F_i^2 F_j - (q + q^{-1}) F_i F_j F_i + F_j F_i^2 = 0$ if $a_{ij} = -1$. 

with the comultiplication, counit, and antipode given by
\[
\Delta(K_i) = K_i \otimes K_i, \quad \Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i,
\]
\[
S(K_i) = K_i^{-1}, \quad S(E_i) = -K_i^{-1}E_i, \quad S(F_i) = -F_iK_i,
\]
\[
\varepsilon(K_i) = 1, \quad \varepsilon(E_i) = \varepsilon(F_i) = 0.
\]
Denote by \((\cdot, \cdot)\) the bilinear form on \(\mathbb{Z}^m\) defined by the Cartan matrix \((a_{ij})\).

Let \(T \cong (\mathbb{Z}/\ell\mathbb{Z})^m\) be the Abelian group generated by \(K_i, i = 1, \ldots, m\). For any \(m\)-tuple of integers \(\gamma = (\gamma_1, \ldots, \gamma_m)\) we will write \(K_\gamma = K_1^{\gamma_1} \cdots K_m^{\gamma_m} \in T\). Denote by \(I\) the set of all integer \(m\)-tuples \(\gamma = (\gamma_1, \ldots, \gamma_m)\) with \(0 \leq \gamma_i < \ell, i = 1, \ldots, m\).

Let \(\Delta^+\) be the set of positive roots of \(\mathfrak{g}\). For each \(\alpha \in \Delta^+\) define \(E_\alpha, F_\alpha \in U_q(\mathfrak{g})\) inductively by setting \(E_\alpha = E_i, F_\alpha = F_i\) for all simple roots \(\alpha_i, i = 1, \ldots, m\) and
\[
E_{\alpha + \alpha'} = q^{-1}E_{\alpha'}E_\alpha - E_\alpha E_{\alpha'} \quad \text{and} \quad F_{\alpha + \alpha'} = qF_\alpha F_{\alpha'} - F_{\alpha'}F_\alpha, \quad \alpha, \alpha' \in \Delta^+.
\]
Let \(\beta_1, \ldots, \beta_N\) be the normal ordering of \(\Delta^+\) and for every \(N\)-tuple of non-negative integers \(a = (a_1, \ldots, a_N)\) introduce the monomials
\[
E_a = E_{a_1}^{a_1} \cdots E_{a_N}^{a_N} \quad \text{and} \quad F_a = F_{a_1}^{a_1} \cdots F_{a_N}^{a_N}.
\]
Then the universal \(R\)-matrix of \(U_q(\mathfrak{g})\) is given by \([\mathbb{R}]_{\mathbb{R}}\) :
\[
(71) \quad R = \frac{1}{\ell^m} \prod_{s=1}^N \left( \sum_{n=0}^{m} q^{-\frac{n(n+1)}{2}} \frac{(1-q^2)^n}{[n]_q!} E_{\beta_\alpha} \otimes F_{\beta\alpha} \right) \left( \sum_{\beta, \gamma \in I} q^{(\beta, \gamma)} K_\beta \otimes K_\gamma \right),
\]
where \([n]_q! = [1]_q[2]_q \cdots [n]_q, \quad [n]_q = \frac{q^n-q^{-n}}{q-q^{-1}}, \quad \text{and}
\]
\[
(72) \quad \Omega = \frac{1}{\ell^m} \sum_{\beta, \gamma \in I} q^{(\beta, \gamma)} K_\beta \otimes K_\gamma,
\]
is the “Cartan part” of \(R\).

Note that the idempotents
\[
P_\beta = \frac{1}{|I|} \sum_{\lambda \in I} q^{(\beta, \lambda)} K_\lambda
\]
generate a commutative and cocommutative Hopf subalgebra \(A = \text{Map}(T, k)\) of \(U\).

Observe that \(U\) is \(\mathbb{Z}^m\)-graded in such a way that a monomial \(X\) in \(E_i, F_i, K_i\) belongs to \(U[\beta - \beta']\) where \(\beta = (\beta_1, \ldots, \beta_m)\) and \(\beta' = (\beta'_1, \ldots, \beta'_m)\) are such that \(E_i\) appears exactly \(\beta_i\) times and \(F_i\) appears exactly \(\beta'_i\) times in \(X\) for each \(i\).

This induces a \(\mathbb{Z}\)-grading of the algebra \(U\) with
\[
(73) \quad \deg(E_i) = 1, \quad \deg(F_i) = -1, \quad \deg(K_i) = 0, \quad i = 1, \ldots, m,
\]
and \(\deg(XY) = \deg(X) + \deg(Y)\) for all \(X\) and \(Y\). Of course, there are only finitely many non-zero components of \(U\) since it is finite dimensional.

Let \(U_+\) be the subalgebra of \(U\) generated by the elements \(E_i, K_i, i = 1, \ldots, m\), \(U_-\) be the subalgebra generated by \(F_i, K_i, i = 1, \ldots, m\), and \(I_{\pm}\) be the kernels of the projections from \(U_{\pm}\) to the elements of zero degree.

For arbitrary non-zero constants \(\Lambda_1, \ldots, \Lambda_m\) define a Hopf algebra automorphism \(\Lambda\) of \(U\) by setting
\[
\Lambda(E_i) = \Lambda_i E_i, \quad \Lambda(F_i) = \Lambda_i^{-1} F_i, \quad \text{and} \quad \Lambda(K_i) = K_i \quad \text{for all} \quad i = 1, \ldots, m.
\]
If for $\beta = (\beta_1, \ldots, \beta_m) \in \mathbb{Z}^m$ we write $\Lambda_\beta = \Lambda_1^{\beta_1} \cdots \Lambda_m^{\beta_m}$ then
\[(74)\]
$\Lambda_{U[\beta]} = \Lambda_\beta \cdot \text{id}_{U[\beta]}$.

**Definition 5.1.1.** We will say that $\Lambda = (\Lambda_1, \ldots, \Lambda_m)$ is *generic* if the spectrum of $\Lambda$ does not contain $\ell$th roots of unity.

For every $K_\lambda \in T$ we introduce the following linear operator on $U \otimes U$:
\[(75)\]
$A^R_L(\lambda)X = (\text{Ad} \, K_\lambda \circ \Lambda \otimes \text{id})(RX \Omega^{-1})$.

**Proposition 5.1.2.** For every generic $\Lambda$ there exists a unique element $J(\lambda) \in 1 + I_+ \otimes I_-$ that satisfies the following ABRR relation [ABRR, ESI, ESS]:
\[(76)\]
$A^R_L(\lambda)J(\lambda) = J(\lambda)$.

*Proof.* Let us write $X = \sum_{j \geq 1} X^j$, where $X^j$ is the sum of all terms having the $\mathbb{Z}$-degree $j$ in the first component. Using the structure of the $R$-matrix of $U$, we can write (70) as a finite system of linear equations labeled by degree $j \geq 1$:
\[(77)\]
$X^j = (\text{Ad} \, K_\lambda \circ \Lambda \otimes \text{id})(\Omega X^j \Omega^{-1}) + \cdots$,
where $\cdots$ stands for the terms involving $X^i$ for $i < j$. Thus, the equation (77) can be solved recursively, starting with $X^0 = 1$, and the solution is unique provided that the operator
\[(78)\]
$id - (\text{Ad} \, K_\lambda \circ \Lambda \otimes \text{id}) \circ \text{Ad} \, \Omega$
is invertible in $\text{End}_U(U \otimes U)$. Let us show that this operator is diagonalizable and compute its eigenvalues.

For all $X_\alpha \in U[\alpha], X_{\alpha'} \in U[\alpha']$, and $K_\beta \in T$ we have
\[
(\text{Ad} \, K_\beta)X_\alpha = q^{(\alpha, \beta)}X_\alpha, \quad (\text{Ad} \, K_\beta)X_{\alpha'} = q^{(\alpha', \beta)}X_{\alpha'},
\]
and therefore
\[
\text{Ad} \, \Omega(X_\alpha \otimes X_{\alpha'}) = q^{(\alpha, \alpha')}(K_{-\alpha} \otimes K_{-\alpha})(X_\alpha \otimes X_{\alpha'}),
\]
whence the eigenvalues of $\text{Ad} \, \Omega$ in $U[\alpha] \otimes U[\alpha']$ are the numbers
\[
d_{\chi \chi'} = q^{(\alpha, \alpha')}(K_{-\alpha})\chi(K_{-\alpha}'),
\]
where $\chi$ and $\chi'$ are characters of $T$. In particular, each $d_{\chi \chi'}$ is an $\ell$th root of unity.

Clearly, the eigenvalue of the operator $(\text{Ad} \, K_\beta \circ \Lambda \otimes \text{id})$ in $U[\alpha] \otimes U[\alpha']$ is $\Lambda_\alpha q^{(\alpha, \lambda)}$. Putting these numbers together, we conclude that (78) is invertible in $U \otimes U$ if and only if
\[
1 - \Lambda_\alpha q^{(\alpha, \lambda+\alpha')}d_{\chi \chi'} \neq 0,
\]
for all $\lambda, \alpha, \alpha', \chi, \chi'$ which is the case for generic $\Lambda$. \qed

**Remark 5.1.3.** If $J(\lambda) \in 1 + I_+ \otimes I_-$ is a solution of (76), then it has zero weight, by the uniqueness result of Proposition 5.1.2, since $(\text{Ad} \, \Delta(K_\beta))J(\lambda)$ is also a solution of (76) for all $K_\beta \in T$.

Our goal is to show that the above element $J(\lambda)$ gives rise to a dynamical twist for $U_q(g)$.

Similarly to (72) define the operator
\[(79)\]
$A^R_L(\lambda)X = (\text{id} \otimes \text{Ad} \, K_{-\lambda} \circ \Lambda^{-1})(RX \Omega^{-1})$
for all $X \in U \otimes U$. 

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DYNAMICAL QUANTUM GROUPS AT ROOTS OF 1

17
Lemma 5.1.4. $A^2_L(\lambda)$ and $A^2_R(\lambda)$ commute.

Proof. For all $X \in U \otimes U$ we have

$$(\text{Ad} K_\lambda \circ \Lambda \otimes \text{id}) R (\text{Ad} K_\lambda \circ \Lambda \otimes \text{Ad} K_{-\lambda} \circ \Lambda^{-1}) (RX \Omega^{-1}) \Omega^{-1} =$$

$$= (\text{id} \otimes \text{Ad} K_{-\lambda} \circ \Lambda^{-1}) R (\text{Ad} K_\lambda \circ \Lambda \otimes \text{Ad} K_{-\lambda} \circ \Lambda^{-1}) (RX \Omega^{-1}) \Omega^{-1},$$

since $(\Lambda \otimes \text{id}) R = (\text{id} \otimes \Lambda^{-1}) R$, whence $A^2_L(\lambda) \circ A^2_R(\lambda) = A^2_R(\lambda) \circ A^2_L(\lambda)$. \qed

Corollary 5.1.5. $J(\lambda)$ is the unique solution of the system

$$(A^2_L(\lambda) X = X \quad \text{and} \quad A^2_R(\lambda) X = X)$$

with $X \in 1 + I_+ \otimes I_-$. 

Proof. We have

$$A^2_L(\lambda) \circ A^2_R(\lambda) J(\lambda) = A^2_R(\lambda) \circ A^2_L(\lambda) J(\lambda) = A^2_R(\lambda) J(\lambda),$$

hence $A^2_R(\lambda) J(\lambda) = J(\lambda)$ by the uniqueness of the solution of (80). \qed

Following [ESS], consider the 3-component operators:

$$(A^2_L(\lambda) \circ \Lambda \otimes \text{id} \otimes \text{id}) R_{13} R_{12} (\text{id} \otimes \text{id} \otimes \text{Ad} K_{-\lambda} \circ \Lambda^{-1}) R_{13} R_{23} =$$

$$= (\text{id} \otimes \text{id} \otimes \text{Ad} K_{-\lambda} \circ \Lambda^{-1}) R_{13} R_{23} (\text{id} \otimes \text{id} \otimes \text{Ad} K_{-\lambda} \circ \Lambda^{-1}) R_{13} R_{12}.$$

If we denote $\hat{R} = (\text{id} \otimes \text{id} \otimes \text{Ad} K_{-\lambda} \circ \Lambda^{-1}) R = (\text{id} \otimes \text{Ad} K_{-\lambda} \circ \Lambda^{-1}) R$ and

$$\hat{R} = (\text{Ad} K_\lambda \circ \Lambda \otimes \text{Ad} K_{-\lambda} \circ \Lambda^{-1}) R,$$

then the above equality translates to

$$\hat{R}_{13} \hat{R}_{12} \hat{R}_{13} \hat{R}_{23} = \hat{R}_{13} \hat{R}_{23} \hat{R}_{13} \hat{R}_{12},$$

which follows from the quantum Yang-Baxter equation after cancelling the first factor. \qed

Lemma 5.1.6. The operators $A^3_L(\lambda)$ and $A^3_R(\lambda)$ commute.

Proof. This statement amounts to showing that

$$(\text{Ad} K_\lambda \circ \Lambda \otimes \text{id} \otimes \text{id}) R_{13} R_{12} (\text{id} \otimes \text{id} \otimes \text{Ad} K_{-\lambda} \circ \Lambda^{-1}) R_{13} R_{23} =$$

$$= (\text{id} \otimes \text{Ad} K_{-\lambda} \circ \Lambda^{-1}) R_{13} R_{23} (\text{Ad} K_\lambda \circ \Lambda \otimes \text{id} \otimes \text{Ad} K_{-\lambda} \circ \Lambda^{-1}) R_{13} R_{12}.$$

If we denote $\hat{R} = (\text{id} \otimes \text{Ad} K_{-\lambda} \circ \Lambda^{-1}) R = (\text{id} \otimes \text{Ad} K_{-\lambda} \circ \Lambda^{-1}) R$ and

$$\hat{R} = (\text{id} \otimes \text{id} \otimes \text{Ad} K_{-\lambda} \circ \Lambda^{-1}) R,$$

then the above equality translates to

$$\hat{R}_{13} \hat{R}_{12} \hat{R}_{13} \hat{R}_{23} = \hat{R}_{13} \hat{R}_{23} \hat{R}_{13} \hat{R}_{12},$$

which follows from the quantum Yang-Baxter equation after cancelling the first factor. \qed

Lemma 5.1.7. If there exists a solution $X$ of the system

$$(A^3_L(\lambda) X = A^3_R(\lambda) X = X)$$

with $X \in I_+ \otimes U \otimes U + U \otimes U \otimes I_-$, then it is unique.

Proof. It is enough to show that such a solution $X$ is unique for the equation $A^3_L A^3_R X = X$. Let us write

$$X = 1 + \sum_{k,l \geq 0, k+l \geq 0} X^{k,l}$$

where $X^{k,l}$ is the sum of all elements having $\mathbb{Z}$-degree $k$ in the first component and $-l$ in the third one. Then the equation $A^3_L(\lambda) A^3_R(\lambda) X = X$ transforms to the system

$$X^{k,l} = (\text{Ad} K_\lambda \circ \Lambda \otimes \text{id} \otimes \text{Ad} K_{-\lambda} \circ \Lambda^{-1}) (WX^{k,l} W^{-1}) + \text{terms depending on } X^{k',l'} \text{ with } k' + l' < k + l,$$
for all $k \geq 0$ and $l \geq 0$ such that $k + l > 0$, where $W = \Omega_{12}\Omega_{23}(\Omega_{13})^2$.

As in Proposition 5.1.2 one can check that the operator

\[
(84) \quad \text{id} - (\text{Ad} K_\lambda \circ \Lambda \otimes \text{id} \otimes \text{Ad} K_{-\lambda} \circ \Lambda^{-1}) \circ \text{Ad} W
\]
is invertible for generic $\Lambda$, therefore the above system can be solved recursively and the solution is unique.

**Theorem 5.1.8.** $J(\lambda)$ satisfies the equations

\[
(85) \quad (\Delta \otimes \text{id})J(\lambda)(J(\lambda + h^{(3)}) \otimes 1) = (\text{id} \otimes \Delta)J(\lambda)(1 \otimes J(\lambda - h^{(1)})),
\]

\[
(86) \quad (\varepsilon \otimes \text{id})J(\lambda) = (\text{id} \otimes \varepsilon)J(\lambda) = 1.
\]

**Proof.** Let us denote the left-hand side of (85) by $Y_L$ and the right-hand side by $Y_R$. We show that both $Y_L$ and $Y_R$ are solutions of the system $A^3_L(\lambda)X = A^3_R(\lambda)X = X$, then the result will follow from Lemma 5.1.7. We have:

\[
A^3_L Y_L = (\text{id} \otimes \text{id} \otimes \text{Ad} K_{-\lambda} \circ \Lambda^{-1})(\mathcal{R}_{13}\mathcal{R}_{23} \Omega_{13}^{-1} \Omega_{23}^{-1})
\]

\[
= (\text{id} \otimes \text{id} \otimes \text{Ad} K_{-\lambda} \circ \Lambda^{-1})(\Delta \otimes \text{id})(\mathcal{R}J(\lambda))(\lambda + h^{(3)})(\Delta \otimes \text{id})\Omega^{-1}
\]

\[
= (\Delta \otimes \text{id})(\text{id} \otimes \text{Ad} K_{-\lambda} \circ \Lambda^{-1})(\mathcal{R}J(\lambda))\Omega^{-1}(\lambda + h^{(3)}) \otimes 1
\]

\[
= (\Delta \otimes \text{id})J(\lambda)(J(\lambda + h^{(3)}) \otimes 1) = Y_L,
\]

\[
A^3_R Y_R = (\text{Ad} K_\lambda \circ \Lambda \otimes \text{id} \otimes \text{Ad})(\mathcal{R}_{13}\mathcal{R}_{12} \Omega_{13}^{-1} \Omega_{12}^{-1})
\]

\[
= (\text{Ad} K_\lambda \circ \Lambda \otimes \text{id} \otimes \text{Ad})(\Delta \otimes \text{id})(\mathcal{R}J(\lambda))(1 \otimes J(\lambda - h^{(1)}))\otimes (\text{id} \otimes \Delta)\Omega^{-1}
\]

\[
= (\text{id} \otimes \Delta)(\text{Ad} K_\lambda \circ \Lambda \otimes \text{id})(\mathcal{R}J(\lambda)\Omega^{-1})(1 \otimes J(\lambda - h^{(1)}))
\]

\[
= (\text{id} \otimes \Delta)J(\lambda)(1 \otimes J(\lambda - h^{(1)})) = Y_R,
\]

where we used that $A^3_L = A^3_R$ and $J$ commutes with $\text{Ad} K_\lambda$.

To establish that $A^3_L Y_L = Y_L$ note that both $A^3_L Y_L$ and $Y_L$ are solutions of the equation $A^3_R Y = X$ and therefore are determined uniquely by their parts of zero degree in the third component. Thus it suffices to compare these parts:

\[
(A^3_L Y_L)^0 = (\text{Ad} K_\lambda \circ \Lambda \otimes \text{id} \otimes \text{id}) \circ \text{Ad} \Omega_{13}(\mathcal{R}_{12}\mathcal{R}_{12} \Omega_{13})^{-1}
\]

\[
= \sum_{\beta} (\text{Ad} K_{\lambda+\beta} \circ \Lambda \otimes \text{id})(\mathcal{R}J(\lambda + h^{(3)})\Omega^{-1}) \otimes P_\beta
\]

\[
= (\text{Ad} K_{\lambda+h^{(3)}} \circ \Lambda \otimes \text{id})(\mathcal{R}J(\lambda + h^{(3)})\Omega^{-1}) = J(\lambda + h^{(3)}) = Y_L^0,
\]

\[
(A^3_R Y_R)^0 = (\text{id} \otimes \text{id} \otimes \text{Ad} K_{-\lambda} \circ \Lambda^{-1}) \circ \text{Ad} \Omega_{13}(\mathcal{R}_{13}\mathcal{R}_{12} \Omega_{13})^{-1}
\]

\[
= \sum_{\beta} P_\beta \otimes (\text{id} \otimes \text{Ad} K_{-\lambda+\beta} \circ \Lambda^{-1})(\mathcal{R}J(\lambda - h^{(1)})\Omega^{-1})
\]

\[
= (\text{id} \otimes \text{Ad} K_{-(\lambda-h^{(1)})} \circ \Lambda^{-1})(\mathcal{R}J(\lambda - h^{(1)})\Omega^{-1}) = J(\lambda - h^{(1)}) = Y_R^0.
\]

The relations between $J(\lambda)$ and $\varepsilon$ are obvious.

**Proposition 5.1.9.** The element

\[
(87) \quad J(\lambda) = J(2\lambda + h^{(1)} + h^{(2)})
\]
is a dynamical twist for $U_q(\mathfrak{g})$ in the sense of Definition 4.1.3.
Proof. We directly compute:
\[
(\Delta \otimes \text{id})J(\lambda)(J(\lambda + h^{(3)}) \otimes 1) = \\
= (\Delta \otimes \text{id})J(2\lambda + h^{(1)} + h^{(2)})(J(2\lambda + h^{(1)} + h^{(2)} + 2h^{(3)}) \otimes 1) \\
= (\text{id} \otimes \Delta)J(2\lambda + h^{(1)} + h^{(2)})(1 \otimes J(2\lambda + h^{(2)} + h^{(3)})) \\
= (\text{id} \otimes \Delta)J(\lambda)(1 \otimes J(\lambda)).
\]
The identities \((\varepsilon \otimes \text{id})J(\lambda) = 1\) and \((\text{id} \otimes \varepsilon)J(\lambda) = 1\) are clear.

Example 5.1.10. Let us give an explicit expression for the twists \(J(\lambda)\) and \(\mathcal{J}(\lambda)\) in the case \(\mathfrak{g} = \mathfrak{sl}(2)\). In this case, \(U_q(\mathfrak{g})\) is generated by \(E, F, K\) with the standard relations. The element analogous to \(J(\lambda)\) for generic \(q\) was computed already in [Ba] (see also [BBB]). If we switch to our conventions, this element will take the form
\[
J(\lambda) = \sum_{n=0}^{\infty} q^{-n(n+1)/2} \frac{(1 - q^2)^n}{[n]_q!} (E^n \otimes F^n) \prod_{\nu=1}^{\infty} \frac{\Lambda q^{2\lambda}}{1 - \Lambda q^{2\lambda + 2\nu}(K \otimes K^{-1})}.
\]
It is obvious that the formula for \(q\) being a primitive \(\ell\)-th root of unity is simply obtained by truncating this formula:
\[
J(\lambda) = \sum_{n=0}^{\ell-1} q^{-n(n+1)/2} \frac{(1 - q^2)^n}{[n]_q!} (E^n \otimes F^n) \prod_{\nu=1}^{\infty} \frac{\Lambda q^{2\lambda}}{1 - \Lambda q^{2\lambda + 2\nu}(K \otimes K^{-1})}.
\]
Therefore,
\[
\mathcal{J}(\lambda) = \sum_{n=0}^{\ell-1} q^{-n(n+1)/2} \frac{(1 - q^2)^n}{[n]_q!} (E^n \otimes F^n) \prod_{\nu=1}^{\infty} \frac{\Lambda q^{4\lambda} K \otimes K}{1 - \Lambda q^{4\lambda + 2\nu}(K^2 \otimes 1)}.
\]
Note that the term of this sum corresponding to \(n = 1\) coincides with the one computed in Section 5.3.

5.2. Dynamical twists arising from generalized Belavin-Drinfeld triples (cf. [ESS]).

Definition 5.2.1. A generalized Belavin-Drinfeld triple for a simple Lie algebra \(\mathfrak{g}\) consists of subsets \(\Gamma_1, \Gamma_2\) of the set \(\Gamma = \{\alpha_1, \ldots, \alpha_m\}\) of simple roots of \(\mathfrak{g}\) together with an inner product preserving bijection \(T : \Gamma_1 \rightarrow \Gamma_2\).

We say that \(T\) is nilpotent if for any \(i = 1, \ldots, m\) there exists a positive integer \(d_i\) such that \(T^{d_i}(\alpha_i) \notin \Gamma_1\). For non-nilpotent \(T\) we define an order of \(T\) (denoted by \(n(T)\)) to be the least common multiple of the lengths of orbits of \(T\).

Let \(Q_1, Q_2, \) and \(Q\) be the free Abelian groups generated by the root sets \(\Gamma_1, \Gamma_2,\) and \(\Gamma\) respectively and
\[
L = \{\lambda \in Q \mid (\lambda, \alpha) = (\lambda, T\alpha) \quad \forall \alpha \in \Gamma_1\}.
\]
Then \(T\) extends to the isomorphism between \(Q_1\) and \(Q_2\) and one can check as in Lemma 3.1 in [ESS] that \(Q_1 \cap L = \{\lambda \in Q_1 \mid T\lambda = \lambda\}\) and \(Q_1 + L\) and \(L^+ + L\) are finite index subgroups of \(Q\). Let
\[
n_1 = [Q : (Q_1 + L)] \quad \text{and} \quad n_2 = [Q : (L^+ + L)]
\]
We assume that \(\ell\) is coprime with both \(n_1\) and \(n_2\) and introduce a homomorphism \(Q_1 + L \rightarrow Q\) also denoted by \(T\), letting
\[
T|_{Q_1} = T, \quad T|_L = \text{id}.
\]
Since $T = \text{id}$ on $Q_1 \cap L$ it follows that $T$ is well-defined.

Factorization by $\ell Q$ yields an automorphism of $\mathbb{T} = Q/\ell Q$:

$$\mathcal{T} : (Q_1 + L)/\ell Q = Q/\ell Q \to Q/\ell Q,$$

which preserves the inner product $(\cdot, \cdot) : \mathbb{T} \times \mathbb{T} \to \mathbb{Z}/\ell \mathbb{Z}$ and extends to the algebra homomorphisms $\mathcal{T}_{\pm} : U_{\pm} \to U_{\pm}$ defined by

$$\mathcal{T}_{\pm}(K_{\lambda}) = K_{T_{\pm}\lambda}, \quad \mathcal{T}_{\pm}(K_{\lambda}) = K_{T_{\mp-1}\lambda},$$

for all $\lambda \in \mathbb{T}$ and $i = 1, \ldots, m$, where $T(i)$ denotes the number such that $T(\alpha_i) = \alpha T(i)$ and $T^{-1}(i)$ the number such that $T^{-1}(\alpha_i) = \alpha T^{-1}(i)$.

Let $\mathbb{T}_L$ and $\mathbb{T}_L^\perp$ be the images of $L$ and $L^\perp$ in $\mathbb{T}$ and

$$\Omega_L = \sum_{K_{\beta}, K_{\gamma} \in \mathbb{T}_L} q^{(\beta, \gamma)} K_{\beta} \otimes K_{\gamma}, \quad \Omega_L^\perp = \sum_{K_{\beta}, K_{\gamma} \in \mathbb{T}_L^\perp} q^{(\beta, \gamma)} K_{\beta} \otimes K_{\gamma}.$$ 

Then $\mathbb{T} = \mathbb{T}_L \oplus \mathbb{T}_L^\perp$ and $\Omega = \Omega_L \Omega_L^\perp$. We define a modification of the operator $A_L^2(\lambda)$ introduced in [4.3]:

$$A_L^2(\lambda)X = (\mathcal{T}_{\pm} \circ \text{Ad} K_{\lambda} \circ \Lambda \otimes \text{id})(R X \Omega_L^{-1}), \quad \lambda \in \mathbb{T}_L,$$

One can show, using the same argument as in Proposition 5.1.3, that if $A_i = \Lambda T(\lambda)$ for all $\lambda \in \mathbb{T}_L$ and the spectrum of $\Lambda$ (in the case $T$ is not nilpotent) does not contain roots of unity of order $n(T)\ell$, then there exists a unique element $J_T(\lambda) \in Z + L_L \otimes L_L^{-1}$, where $Z = ((id - T)^{-1} T \otimes id)\Omega_L$, satisfying the modified ABRR relation (cf. [ESS]):

$$J_T(\lambda)J_T(\lambda) = J_T(\lambda),$$

and commuting with $\text{Ad} \Delta(K_{\lambda})$ for all $\lambda \in \mathbb{T}_L$. Next, if we define

$$A_H^2(\lambda)X = (id \otimes \mathcal{T}_{\pm} \circ \text{Ad} K_{\lambda} \circ \Lambda)(R X \Omega_L^{-1}), \quad \lambda \in \mathbb{T}_L,$$

then Lemma 5.1.4 and Corollary 5.1.5 are still valid because of the identity

$$(\mathcal{T}_{\pm} \circ \text{id})\Omega_L = (id \otimes \mathcal{T}_{\pm})\Omega_L$$

that follows from the inner product preserving property of $T$.

Modifying definitions of $A_L^2(\lambda)$ and $A_H^2(\lambda)$ as in [ESS], it is possible to repeat the proofs of Lemmas 5.1.4, 5.1.7 and Theorem 5.1.8 showing that $J_T(\lambda)$ satisfies (S), and hence that $J_T(\lambda)$ constructed as in Proposition 5.1.9 is a dynamical twist for $U$.

### 5.3. Non-degeneracy of the twisted $R$-matrix and self-duality.

We will show that for the dynamical twist $J(\lambda)$ for a quantum group $U_q(g)$ the corresponding weak Hopf algebras described in sections 4.2 and 4.3 are isomorphic, i.e., that the twisted $R$-matrix

$$R(\lambda) = \sum_{\lambda_{\mu\nu}} E_{\lambda_{\mu\nu}}^* P_{\mu} R^j(1)(\lambda) \otimes E_{\lambda_{\mu\nu}} R^j(2)(\lambda) P_{\nu},$$

where $R^j(\lambda) = J_1^{-1}(\lambda) R J(\lambda)$, establishes an isomorphism between weak Hopf algebras $D_{J(\lambda)} = \text{Map}(\mathbb{T} \times \mathbb{T}, k) \otimes U_q(g)^*$ and $H_{J(\lambda)} = (\text{End}(A) \otimes U_q(g))_{J(\lambda)\Theta} = D_{J(\lambda)}^{op}$.

**Proposition 5.3.1.** Let $\rho : H_{J(\lambda)}^* \to H_{J(\lambda)}^{op}$ defined by $\phi \mapsto (id \otimes \phi)R(\lambda)$ be the homomorphism of weak Hopf algebras given by the $R$-matrix $R(\lambda)$ of $H_{J(\lambda)}$. Then the elements $E_{\lambda_{\mu\nu}}, K_i, E_i, F_i, \lambda, \mu \in \mathbb{T}, i = 1, \ldots, m$ belong to $\text{Image}(\rho)$. 
Proof. It follows from the explicit formula (77) for the universal $R$-matrix $R$ of $\mathcal{U}_q(\mathfrak{g})$, defining equation (76) of $J(\lambda)$, and expression (78) for $\mathcal{R}(\lambda)$ that

$$\mathcal{R}(\lambda) = \sum_{a,b} \mathcal{R}_{a,b}(\lambda),$$

where

$$\mathcal{R}_{a,b}(\lambda) = \sum_{\lambda,\mu,\nu} (E_{\lambda\lambda+\nu} \otimes E_{\lambda+\mu\lambda})(F_a E_b \otimes E_a F_b)(P_\mu \otimes 1)C_{a,b}(\lambda)(1 \otimes P_\nu),$$

the “coefficients” $C_{a,b}(\lambda)$ are $(kT)^{\otimes 2}$-valued functions on $T$, and $a, b$ run over $N$-tuples of non-negative integers.

Note that the terms of $\mathcal{R}(\lambda)$ occurring in $\mathcal{R}_{0,0}(\lambda)$ are linearly independent from the rest and so are those occurring in $\mathcal{R}_{\delta_i,0}(\lambda)$ and $\mathcal{R}_{0,\delta_i}(\lambda)$, where $\delta_i$ is the $N$-tuple with 1 in the position corresponding to the single root $\alpha_i, i = 1, \ldots, m$ and 0’s elsewhere.

Hence, the subspaces

$$V_{a,b} = \{(id \otimes \phi)\mathcal{R}_{a,b}(\lambda) \mid \phi \in H^\vee_T\}, \quad \text{where} \quad (a, b) = (0, 0), (0, \delta_i), \text{or} \ (\delta_i, 0)$$

belong to the image of $\rho$.

In all of the three above cases we will show that $C_{a,b}(\lambda)$ is invertible in $(kT)^{\otimes 2}$ (and, therefore, $(P_\mu \otimes 1)C_{a,b}(\lambda)(1 \otimes P_\nu)$ are non-zero scalars for all $\mu, \nu$) and that the generators of $H_T$ lie in the algebra generated by the above subspaces $V_{a,b}$.

Clearly, $C_{0,0}(\lambda) = \Omega$ is invertible, whence $V_{0,0}$ is spanned by the elements $E_{\lambda\lambda+\nu} P_\mu$, i.e., $E_{\lambda\mu} \in \text{Image}(\rho)$ for all $\lambda, \mu$ and $K_i \in \text{Image}(\rho)$ for $i = 1, \ldots, m$.

Next, $C_{0,\delta_i}(\lambda)$ is the coefficient with $E_i \otimes F_i$ in $\mathcal{R}(\lambda)$. To determine it, note that by (71) and (73) we have

$$J(\lambda) = 1 + \sum_i (E_i \otimes F_i)b_i(\lambda) + \cdots, \quad b_i(\lambda) \in (kT)^{\otimes 2},$$

$$\mathcal{R} = (1 + (q^{-1} - q)\sum_i (E_i \otimes F_i) + \cdots)\Omega,$$

where $\cdots$ stand for the terms of degree $> 1$ in the first component. We use the recursive relation (77) to find $b_i(\lambda), i = 1, \ldots, m$ :

$$(E_i \otimes F_i)b_i(\lambda) = (\text{Ad} K_{\alpha_i} \otimes \text{id}) \circ \text{Ad} \Omega(E_i \otimes F_i)b_i(\lambda) + (q^{-1} - q)(E_i \otimes F_i) = \Lambda_i q^{\langle\lambda,\alpha_i\rangle}(E_i \otimes F_i)(b_i(\lambda)q^2(K_i \otimes K_i^{-1}) + (q^{-1} - q)),$$

from where we obtain

$$b_i(\lambda) = \frac{\Lambda_i q^{\langle\lambda,\alpha_i\rangle}(q^{-1} - q)}{1 - \Lambda_i q^{2\langle\lambda,\alpha_i\rangle + 2}(K_i \otimes K_i^{-1})},$$

and consequently

$$C_{0,\delta_i}(\lambda) = q^{2}(K_i \otimes K_i^{-1})b_i(2\lambda + h^{(1)} + h^{(2)})\Omega + (q^{-1} - q)\Omega$$

$$= (q^{-1} - q)\left(\frac{\Lambda_i q^{2\langle\lambda,\alpha_i\rangle + 2}(K_i^2 \otimes 1)}{1 - \Lambda_i q^{2\langle\lambda,\alpha_i\rangle + 2}(K_i^2 \otimes 1)} + 1\right)\Omega$$

$$= \frac{(q^{-1} - q)\Omega}{1 - \Lambda_i q^{2\langle\lambda,\alpha_i\rangle + 2}(K_i^2 \otimes 1)},$$

which is invertible for all generic $\lambda$.

It follows that $V_{0,\delta_i}$ is spanned by the elements $E_{\lambda\lambda+\nu} P_\mu E_i$, whence $E_i \in \text{Image}(\rho)$. 

Finally,
\[ J_{21}^{-1}(\lambda) = 1 - (F_i \otimes E_i)b_i(\lambda)_{21} + \cdots, \]
therefore, \( C_{\delta,0}(\lambda) = -b_i(2\lambda + h^{(1)} + h^{(2)})_{21} \Omega \) is invertible and \( F_i \in \text{Image}(\rho) \). □

As a corollary, we obtain the following

**Theorem 5.3.2.** The \( R \)-matrix \( R(\lambda) \) defines an isomorphism between weak Hopf algebras \( H_J \) and \( D_J \cong H_J^{op} \).

**Proof.** We saw in Proposition 5.3.1 that the image of \( \rho : H_J^* \rightarrow H_J^{op} \) contains all the generators of the algebra \( H \), therefore \( \text{Image}(\rho) = H \). Since \( H \) is finite dimensional, \( \rho \) is an isomorphism. □

5.4. **Non-degeneracy of the twisted \( R \)-matrix in the case when \( T \) is an automorphism of the Dynkin diagram of \( g \).** We extend the result of Section 5.3 to the case when \( \Gamma_1 = \Gamma_2 = \Gamma \) and \( T \neq \text{id} \).

The dynamical twist \( J_T(\lambda) \) constructed in Section 5.2 then has a form
\[
J_T(\lambda) = Z + \sum_{ij} (E_i \otimes F_j)b_{ij}(\lambda) + \cdots, \quad b_{ij}(\lambda) \in (kT)^{\otimes 2},
\]
where \( b_{ij}(\lambda) = 0 \) if \( \alpha_i \) and \( \alpha_j \) belong to different orbits of \( T \) and \( \cdots \) stands for the terms having the \( Z \)-degree \( > 1 \) in the first component. Similarly, one has
\[
(J_T)^{-1}(\lambda) = Z_{21}^{-1} + \sum_{ij} (F_i \otimes E_j)b_{ij}(\lambda) + \cdots, \quad b_{ij}(\lambda) \in (kT)^{\otimes 2}.
\]
As in Proposition 5.3.1 it is possible to find the coefficients \( b_{ij}(\lambda) \) explicitly:
\[
b_{ij}(\lambda) = \frac{\Lambda_{n-k}(q^{-1} - q)q^{(\lambda,\alpha_i) + (\lambda + \alpha_j, s_{ij})}(K_{s_{ij}} \otimes K_{s_{ij}}^{-1})Z}{1 - \Lambda_{n}q^{\lambda + \alpha_j,s}(K_{s} \otimes K_{s}^{-1})},
\]
where \( n \) is the length of the corresponding orbit, \( k \) is the unique number such that \( T^k(\alpha_i) = \alpha_j \) \((0 \leq k < n)\), \( s_{ij} = T(\alpha_i) + \cdots + T^{n-k-1}(\alpha_j) \), and \( s = \alpha_i + \cdots + T^{n-1}(\alpha_i) \) is the orbit sum. Note that since \( \Lambda_i \) is independent from \( i \) within an orbit and \( \lambda \in \mathbb{T}_L \), the denominator in the right-hand side of the above formula does not depend on \( i \) and \( j \).

The dynamical \( R \)-matrix \( R^{J_T}(\lambda) \) of \( U_q(g) \) has a form
\[
R^{J_T}(\lambda) = \Omega ZZ_{21}^{-1} + \sum_{ij} (F_i \otimes E_j)b_{ij}(\lambda)\Omega Z + \sum_{ij} Z_{21}^{-1}(E_i \otimes F_j)(q^{a_{ij}}(K_j \otimes K_j^{-1})b_{ij}(\lambda) + \delta_{ij}(q^{-1} - q)Z)\Omega + \cdots,
\]
where the listed terms are linearly independent from \( \cdots \) in each component.

Formula (5.3.1) gives an expression for the \( R \)-matrix \( R(\lambda) \) of the weak Hopf algebra \( H_{J_T} \) in terms of \( R^{J_T}(\lambda) \) and it is easy to see that the image of \( R(\lambda) \) contains the matrix units \( E_{\mu \mu'} \), \( \mu, \mu' \in L \) and elements \( K_i \), \( i = 1, \ldots, m \) generating \( \mathbb{T} \). Showing that it also contains generators \( E_i \) (resp. \( F_i \)) amounts to proving that the matrices \( A_{\text{e}q}(\lambda) \) (resp. \( B_{\text{e}q}(\lambda) \)), where
\[
A_{\text{e}q}(\lambda)_{ij} = (P_j \otimes P_j)(q^{a_{ij}}(K_j \otimes K_j^{-1})b_{ij}(\lambda) + \delta_{ij}(q^{-1} - q)Z),
\]
\[
B_{\text{e}q}(\lambda)_{ij} = (P_i \otimes P_i)b_{ij}(\lambda),
\]
and $P_\eta$, $P_\nu$ are minimal idempotents in $kT$, are invertible for all $\eta, \nu, \lambda$. Using the formula for $b_{ij}(\lambda)$ one can show that these matrices are equivalent (up to permuting and multiplying the rows and columns by non-zero constants) to the matrix

$$
\begin{pmatrix}
1 & 1 & \ldots & 1 \\
\Lambda & 1 & \ldots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
\Lambda & \Lambda & \ldots & 1
\end{pmatrix},
$$

(104)

where $\tilde{\Lambda} = \Lambda^{n}q^{(\lambda + \alpha, s)}(P_\eta \otimes P_\nu)(K_s \otimes K_s)Z$ is a multiple of $P_\eta \otimes P_\nu$ independent from $i$ and $j$. The determinant of this matrix is equal to $(\Lambda - 1)^{n-1}$, so it is invertible when $\Lambda^{n(T_\ell)} \neq 1$.

Thus, Theorem 5.3.2 extends to the case when $T$ is an automorphism of the Dynkin diagram of $\mathfrak{g}$:

**Theorem 5.4.1.** For every generalized Belavin-Drinfeld triple $(\Gamma, \Gamma, T)$ the $R$-matrix $R(\lambda)$ defines an isomorphism between weak Hopf algebras $H_{J_T}$ and $D_{J_T} \simeq H^\text{op}_{J_T}$.

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