A k-hop Collaborate Game Model: Adaptive Strategy to Maximize Total Revenue

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Abstract—In Online Social Networks (OSNs), interpersonal communication and information sharing are happening all the time, and it is real-time. When a user initiates an activity in OSNs, immediately, he/she will have a certain influence in his/her friendship circle automatically. Then, some users in the initiator’s friendship circle will be attracted to participate in this activity. Based on such a fact, we design a k-hop Collaborate Game Model, which means that an activity initiated by a user can only influence those users whose distance are within k-hop from the initiator in OSNs. Besides, we introduce the problem of Revenue Maximization under k-hop Collaborate Game (RMKCG), which identifies a limited number of initiators in order to obtain benefits as much as possible. Collaborate Game Model describes in detail how to quantify benefit and the logic behind it. We do not know how many followers would be generated for an activity in advance, thus, we need to adopt an adaptive strategy, where the decision who is the next potential initiator depends on the results of past decisions. Adaptive RMKCG problem can be considered as a new stochastic optimization problem, and we prove it is NP-hard, adaptive monotone, but not adaptive submodular. But in some special cases, we prove it is adaptive submodular and an adaptive greedy strategy can obtain a $(1 - 1/e)$-approximation by adaptive submodularity theory. Due to the complexity of our model, it is hard to compute the marginal gain for each candidate user, then, we propose a convenient and efficient computational method. The effectiveness and correctness of our algorithms are validated by heavy simulation on real-world social networks eventually.

Index Terms—Collaborate Game Model, Online Social Networks (OSNs), Adaptive Strategy, Stochastic Optimization, Submodularity, Approximation Algorithm

1 INTRODUCTION

THE online social platforms were developing quickly in the last decades and derived a series of famous technological companies, such as Facebook, Twitter, LinkedIn and Tencent. There are billion of people sharing their emotions and discussing current affairs in these platforms. There are more than 1.52 billion users active daily on Facebook and 321 million users active monthly on Twitter. The logic of social platforms can be represented as an online social network (OSNs), which is an undirected graph, including individuals and their relationship. Domingos and Richardson [1][2] was the first one to propose the concept of “Viral Marketing”, which aims to attract follow-ups as many as possible by giving free or coupon samples to the most influential users in OSNs. This groundbreaking researches has had a profound impact on later generations. Inspired by that, Influence Maximization was proposed to model the spread of trust, advertisements or innovations abstractly. Kempe et al. [3] regarded it as a combinatorial optimization, which aims to select a subset of user such that the expected number of follow-up adoptions can be maximized under the size constraint. Besides, they proposed two classical diffusion models called Independent Cascade model and Linear Threshold model, and prove it is NP-hard, monotone and submodualr under these two models. Since this seminal work, a series of variant problems and models under different scenarios and constraints [4][5][6][7] appear constantly, such as profit maximization, rumor blocking and adaptive submodular problem. And some researchers focus on the extension [8][9][10][11][12] from a practical viewpoint.

However, most existing researches about maximization problem, regardless of influence or profit, all base on counting the number of single user that follows our “influence”. This model is indeed effective and valid in most cases, but it does not cover all scenarios. Considering some user in a social platform, such as Facebook, is invited by some organizations to launch an activity, after lauching, his/her friends, or friends of friends, may be interested in this activity and choose to participate in it. At this time, our influence or profit depends on the benefits from activity, which is related to the number of people involved in this event. Based on that, we propose Collaborate Game Model. Considering a game company, they plan to promote their multiplayer game in some social platform by inviting some users to play this game. Once these users accept this invitation, they initiate this game, called initiator, and they will attract friends from a certain range to participate in it. In our model, we have a k-hop assumption that the maximum influence range is k-hop from the initiator, shown as Fig. 1 as an example. Obviously, the revenue should be calculated on a game-by-game manner. For a single game, the more people involved, the greater the income. But the relation between revenue and the number of participants is not linear, we need to construct a valid quantitative model to compute the benefit. Even that our model is called as Collaborate game model, but here, “game” is abstract concept, which is not just limited to the game. It can be extended to other multiplayer activity circumstances. For example, president election, in order to win a high level of support, each camp will promote and advocate their own presidential candidate by inviting some people to support.
this paper. Our contributions are summarized as follows:

1) Collaborate Game Model is a totally new model, which is a generalization to a class of real problems. Based on that, we propose RMKCG problem and its adaptive version firstly.

2) We prove RMKCG problem is NP-hard, adaptive monotone, but not adaptive submodular.

3) We propose a new method (Algorithm 2) to compute the marginal gain, which overcomes the difficulty of computation, improve efficiency and reduce running time greatly. Then, we prove RMKCG problem is adaptive submodular and obtain $(1 - 1/e)$-approximation by adaptive greedy strategy under some special cases.

4) Our proposed algorithms are evaluated on real world social networks, which verify the effectiveness and correctness of them.

**Organization**: Sec. 2 reviews the related survey. Sec. 3 describes model, background knowledges and problem formulation. The solution for RMKCG are presented in Sec. 4. Sec. 5 is the theoretical analysis for BPMCA. Sec. 6 discusses experiment and Sec. 7 is conclusion.

**2 RELATED WORK**

Domingos and Richardson [1] [2] was the first to study viral marketing and the value of customers in social networks. Kempe et al. [3] generalized viral marketing to influence maximization. It can be considered as a combinatorial optimization problem, and they proposed a greedy algorithm implemented by Monte Carlo simulation. Afterward, there have been many variants of influence maximization, among which, profit maximization is related to us. [14] [15] proposed the problem how to select the most influential seed nodes that can maximize the profit. Lu et al. [14] combined prices and valuation by extending the Linear Threshold model, then they used a heuristic unbudgeted greedy framework to solve this problem. Tang et al. [15] provided a strong approximation guarantee with the help of the methods in unconstrained submodular maximization. Other researches on pricing strategies can refer to [16] [17] [18] [19].

As we know, the approximation ratio of greedy algorithm is $(1 - 1/e)$, proved by Nemhauser et al. [20], when the objective function is monotone non-decreasing and submodular. Golovin et al. [13] extended this work to adaptive version and obtained the same approximation ratio when the objective function is adaptive monotone and adaptive submodular. Applied it to social networks, Tong et al. [21] provided a systematic studies on the adaptive influence maximization problem, where the objective function is not adaptive submodular, and they introduced the concept of regret ratio in designing the seeding strategy. Smith et al. [22] introduced the concept of adaptive primal curvature to obtain an approximation ratio for non-submodular cases. When the objective function is not adaptive monotone, but adaptive submodular, Gotovos et al. [23] extended random greedy algorithm to adaptive version and obtained a $(1/e)$-approximation. Other researches on the application of adaptive strategy can refer to [24] [25] [26] [27].

![Fig. 1. An example that shows a single activity: Here, the red node is initiator, and the yellow node is those users that participate the activity launched by red node.](image-url)
3.1 Collaborate Game

In order to understand our game model, we need to know its background how it works firstly. To generalize our problem, we imagine a social relationship network $G$, from Facebook or WeChat, that a game company want to promote their new collaborator game based on this targeted network $G$. The targeted network $G$ can be defined arbitrarily by game company, for example, sub-network of children, sub-network of students or all users in a network according to the targeted users.

Let us describe how a collaborate game works. First, the game company needs to acquire the targeted network $G$ through some social platforms, such as WeChat, the most popular social software in China. It contains some necessary users’ information we can exploit, for example, age, gender and career. More importantly, most social media software has a real-name certification system, which helps us extract the targeted network according to the properties of game. For Tencent, it is more convenient, because Tencent is both a social software company and a game company. Thus for the game promotion department, such scenes are staged every day. When the company invites a user $u$ to play this game, if he/she accepts it, he/she will do as follows:

1) Initiate this game as initiator.
2) Invite his/her friends to participate in this game.
3) If his/her friends are willing to join the game, they will invite their friends to participate in this game again as before.

Even though this game is a multiplayer collaborate game, it does not mean they can invite their friends to participate without limit. Thus, we have a $k$-Hop assumption, where the invitations can only last for $k$ rounds. Explaining in detail, we call initiator as 0-hop participant. The friends of 1-hop participant that accept 0-hop participant’s invitation are called 1-hop participants. The friends of 1-hop participants that accept 1-hop participants’ invitation are called 2-hop participants, repeated until $k$-hop participants. In other words, the distance of all participants for this game from initiator cannot be larger than $k$-hop in targeted network $G$, which limit the scope of a single game.

However, for the game company, it is unrealistic to send too many invitations to get more initiators because they may not be able to get complete network information and determine who will be a potential initiator. Then, for the promotion, giving too many invitations is sometimes counterproductive, because this will make their game very cheap and lack competitiveness. Relied on the above analysis, our problem is who should be the next initiator that are the most beneficial to maximize the total revenue.

3.2 Network Descriptions

A targeted network can be given by a undirected graph $G=(V,E)$ where $V(G) = \{v_1, v_2, ..., v_n\}$ is the set of $n$ users. $E = \{e_1, e_2, ..., e_m\}$ is the set of $m$ undirected edges, where $e \in E(G)$ indicates that there exists friendship between user $u$ and user $v$. The node set and edge set for graph $G$ can be referred as $V(G)$ and $E(G)$ respectively. For an edge $e = \{u, v\}$, $u$ is a friend of $v$, naturally, $v$ is a friend of $u$ as well. We use $N(v)$ to denote the set of friends of node $v$. Here, we define a probability $p_e \in [0, 1]$ for each edge $e \in E(G)$, which represents the degree of intimacy between user $u$ and user $v$ in our model. In other words, when user $u$ participates in this game, whether user $v$ is willing to participate in this game with user $u$ if $u$ invites $v$. Obvious, the more intimate the two friends are, the more likely they are willing to play games together.

For the company, they cannot know exactly how much the degree of intimacy between two users is, thus, the network information is incomplete. From the perspective of data mining, they can predict the probability between two users by learning their communication log to judge the degree of closeness between them. This is beyond the scope of this paper and we will not discuss here. Once a user $u$ accepts the invitation from game company as an initiator, the statuses of those edges whose distance is $k$-hop from user $u$ are partially known. According to the previous observation, the company can make a decision about which potential initiator it the best next, which is the reason why it is called an adaptive strategy.

Then, there is a natural question whether the user would accept this invitation as an initiator when he/she receives it from the game company. We can define an acceptance probability $\theta_u \in [0, 1]$ for each user $u \in V(G)$, which describes the extent to which users are interested in launching this game when he/she receives the invitation. From here, an acceptance vector $\theta = (\theta_1, \theta_2, ..., \theta_n)$ is formulated to give the acceptance probability for each user. It is complicated because different users have a divergent tendency to this game. For example, a game enthusiast may be more inclined to accept the invitation, or a user with many friends may be happy to accept the invitation due to the fact that this game requires multiple people to participate in. Thus, it is flexible about how to define acceptance vector. In this paper, we assume $\theta_u$ is uniformly distributed in interval $[0, 1]$.

3.3 Problem Definition

The company that promotes a game is naturally hoping to get revenue from it, and how can we characterize the revenue from a game. Obviously, the more people involved, the greater the revenue they can get. However, it is not enough to measure the income just by the number of people playing the game. This is a multiplayer game, the actual number of initiated games is much less than the number of all participants. Thus, we need to consider comprehensively both the number of initiated games and the number of participants in each game. Usually, for each specific game, we believe the marginal benefit is diminishing gradually, in other words, the increasing rate in earnings is gradually slowing down as the number of participants increases. Let us see following concrete example:

Example 1. A user $u$ accepts the invitation from the game company to be an initiator, we assume that this company can obtain 5 units revenue from him. User $u$ invites his/her friends, soon afterward, the company can obtain 4 units revenue from each that accepts the invitation from user $u$, namely, 4 units/1-hop participant. Then, we have 3 units/2-hop participant, 2 units/3-hop participant, and so on until k-hop participant.

This assumption is valid because we believe the promotion effect brought by the people who first join this game is
larger than the people who join the game afterward. Those who join this game first need more people to get involved since they need collaboration, however, people who join this game later do not have this requirement.

In order to quantify the revenue the company obtains, we can define a revenue vector $\mathbf{R} = (R_0, R_1, R_2, \ldots, R_k)$, where $R_i \in \mathbb{Z}^+$ and $i \in [k] = \{0, 1, 2, \ldots, k\}$. Here, $R_i$ denotes the revenue the company can gain from an $i$-hop participant, thus, we have $R_0 \geq R_1 \geq R_2 \geq \ldots \geq R_k$ according to our previous assumption. In the targeted network, the number of initiators (0-hop participants) is clear, but other participants are likely to be ambiguous. For example, a user $u$ is a $i$-hop participant to one initiator and $j$-hop participant to another initiator, where $i < j$. At this moment, we consider user $u$ will choose to be a $i$-hop participant to the company. We call it as “Tendency Assumption”. Based on the above model, for a game company, they aim to give a certain number of invitations to initial users such that maximizing the expected revenue. Then, the problem of Revenue Maximization under $k$-hop Collaborate Game (RMKCG) is formulated as follows:

**Problem 1 (RMKCG).**

Given a targeted network $G = (V, E)$, an acceptance vector $\mathbf{a}$, a budget $b$ and a revenue vector $\mathbf{R}$, we aim to find a subset $D \subseteq V(G)$ and $|D| \leq b$ such that the expected total revenue the company gains can be maximized when inviting those users in $D$ to be initiators.

For adaptive strategy, the RMKCG problem can be transformed to find a policy $\pi$, where the company will send an invitation to user $u \in D$ step by step. When the user $u$ accepts to be an initiator, the partial statuses of edges $k$-hop from $u$ and new participants are known, thus, the network information can be updated. A valid policy $\pi$ can at most send $b$ invitations.

The adaptive strategy can be considered as a stochastic process, whose stochasticity is mainly from the following two aspects: For user $u$, the probability ($\theta_u$) that he/she accepts the invitation from the company as an initiator; For user $v$, the probability ($\phi_{uv}$) that he/she joins the game when his/her friend $u$ invites him/her to play together. Now, we can define the state of targeted network. Given targeted network $G = (V, E)$, for each user $u \in V(G)$, the state of $u$ can be denoted as $X_u \in \{0, 1\} \cup \{?\}$, where $X_u = 1$ means user $u$ accepts to be an initiator because of the company’s invitation and $X_u = 0$ means user $u$ rejects to be an initiator. $X_u = ?$ means user $u$ is unknown, who did not receive an invitation from the company. The states of all users are $\emptyset$ at the beginning. Similarly, for each edge $e = \{u, v\} \in E(G)$, the state of $e$ can be denoted as $Y_e \in \{0, 1\} \cup \{?\}$, where $Y_e = 0$ indicates user $u$ (resp. $v$) did not accept the invitation from user $v$ (resp. $u$) and $Y_e = 1$ indicates user $u$ (resp. $v$) is willing to play the game with user $v$ (resp. $u$). Once determined, it cannot be changed. $Y_e = ?$ indicates there is no invitation that happens between user $u$ and user $v$. The states of all edges are $\emptyset$ at the beginning.

After defining the states of users and edges, we have a function $\phi = \{X_u\}_{u \in V(G)} \cup \{Y_e\}_{e \in E(G)} \rightarrow U$ be all possible state, call a realization (the states of all items in $V(G)$ and $E(G)$). Thus, we say that $\phi(u)$ is the state of user $u \in V(G)$ and $\phi(e)$ is the state of edge $e \in E(G)$ under realization $\phi$. We use $\Phi$ to denote a random realization and $\Pr(\phi) = \Pr[\Phi = \phi]$ as the probability distribution over all realizations. Besides, each realization should be consistent. Here, each user can only be one of state in $X_u \in \{0, 1\} \cup \{?\}$, and each edge be one of state in $\{0, 1\} \cup \{?\}$ identically. Considering the adaptive strategy of RMKCG problem (AS-RMKCG Process), the game company does as follows:

1) Send an invitation to user $u$, see the state $\Phi(u)$.
2) Update $D = D \cup \{u\}$
3) If $\Phi(u) = 0$, go to step (5).
4) Update the states of edges whose distance are less than $k$-hop from user $u$, and record these update as a reference for next decision.
5) If $|D| < k$, go back to step (1); Otherwise, stop.

Let us see Fig. 2 as a concrete example. We define $H(\pi, \phi)$ as the set of all users who are invited by the game company according to strategy $\pi$ under realization $\phi$. After each invitation, shown as above process, the states of partial edges can be updated, our observation so far can be represented as a partial realization $\psi$, which is a function of observed objects to their states. Then, $dom(\psi)$ is referred to as the domain of $\psi$, namely, observed users and edges in $\psi$. A partial realization $\psi$ is consistent with a realization $\phi$ if they are equal everywhere in the domain of $\psi$, denoted as $\phi \sim \psi$. If $\psi$ and $\psi'$ are both consistent with some $\phi$, and $dom(\psi) \subseteq dom(\psi')$, we say $\psi$ is a subrealization of $\psi'$, denoted as $\psi \subseteq \psi'$. Besides, we denote $dx(\psi)$ as the observed users in the domain of $\psi$, and $dy(\psi)$ as the observed edges in the domain of $\psi$.

Let $\pi$ be an adaptive invitation strategy of the company. The total revenue gained according to strategy $\pi$ under realization $\phi$ can be defined as follows:

$$f(H(\pi, \phi), \phi) = \sum_{i \in [k]} \sum_{u \in D_i(\pi, \phi)} R_i$$  \hspace{1cm} (1)
Algorithm 1 Adaptive-Invitation \( (G, f, \theta, R, b, k) \)

1: Initialize: \( H \leftarrow \emptyset, \psi \leftarrow \emptyset \)
2: for \( i = 1 \) to \( b \) do
3: \hspace{1em} for user \( u \in V(G) \) \( \setminus H \) do
4: \hspace{2em} \( \Delta(u|\psi) \leftarrow \text{Compute}(G, f, \theta, R, k, \psi, u) \)
5: end for
6: Select \( u_i \in \arg \max_u \Delta(u|\psi) \)
7: \( H \leftarrow H \cup \{ u_i \} \)
8: if \( u^* \) accepts the invitation then
9: \hspace{1em} Update \( \psi \), the states of the edges whose distance are less than \( k \)-hop from \( u \)
10: end if
11: end for
12: return \( H, \psi \)

where \( [k] = \{0, 1, \ldots, k\} \) and \( D_i(\pi, \phi) \) is the set that contains all \( i \)-hop participants to the company according to strategy \( \pi \) under realization \( \phi \), thus, we have
\[
D_0(\pi, \phi) = \{ u | u \in H(\pi, \phi), \phi(u) = 1 \} \\
D_i(\pi, \phi) = \{ u | \exists u \in D_{i-1}(\pi, \phi), \phi(\{u, v\}) = 1 \} \setminus \bigcup_{j=0}^{i-1} D_j(\pi, \phi)
\]

Finally, we can evaluate the performance of a policy \( \pi \) by its expected revenue, and we have
\[
f_{\text{avg}}(\pi) = E_{\Phi}[f(H(\pi, \Phi, \Phi)]
\]

where the expectation is taken with respect to \( \text{Pr}(\Phi = \phi) \). The goal of RMKCG problem is to find a policy \( \pi^* \) such that \( \pi^* \in \arg \max_\pi f_{\text{avg}}(\pi) \) subject to \( |H(\pi, \phi)| \leq b \) for all realization \( \phi \).

4 ALGORITHM

In this section, we propose our algorithm to solve adaptive RMKCG problem, Adaptive-Invitation Algorithm, and introduce how to compute marginal gain efficiently.

4.1 Adaptive-Invitation Algorithm

According to the description of AS-RMKCG Process, inspired by adaptive greedy policy proposed by [13], Adaptive-Invitation Algorithm is proposed, which can be divided into two steps generally: In the first step, we send an invitation to user \( u \) that can obtain the most increment of expected revenue according to partial realization \( \psi \). This step can be generalized to Conditional Expected Marginal Benefit in [13], we have

Definition 1 (Conditional Expected Marginal Benefit). Given a partial realization \( \psi \) and an user \( u \), the conditional expected marginal benefit of \( u \) conditioned on \( \psi \) is
\[
\Delta(u|\psi) = E[f(dx(\psi) \cup \{ u \}, \Phi) - f(dx(\psi), \Phi)|\Phi \sim \psi]
\]

In our RMKCG problem, \( \Delta(u|\psi) \) is the expected total revenue based on previous invited users and observed edges, and the expectation is taken over all realization that are consistent with current partial realization \( \psi \). After inviting user \( u \) as an initiator, in the second step, we need to observe the state of \( u \). If \( u \) accepts this invitation, it updates current observation, namely, updates the states of edges whose distance within \( k \)-hop from \( u \); If \( u \) rejects this invitation, back to the first step to invite next user. The Adaptive-Invitation Algorithm is shown in Algorithm 1. They will be executed iteratively until the number of invitation by company is larger than \( b \).

4.2 Gain Computation

In line 4 of Algorithm 1 we need to compute the value of \( \Delta(u|\psi) \) given current realization \( \psi \), which is deterministic. In this subsection, we talk about how to compute \( \Delta(u|\psi) \) efficiently, shown as Algorithm 2.

For each user \( u \), we assume there is a map containing a user \( \{ u : a \} \), where \( a \in [0, 1] \) is a probability. First, we initialize a list \( L \), whose elements are maps, whose elements are users and their probabilities. For example, we set \( k = 2 \), and we have \( L = \{ \{u_1 : a_1\}, \{u_2 : a_2, u_3 : a_3\}, \{u_4 : a_4, u_5 : \ldots \} \).
absolutely accurate because of the complexity of RMKCG gain. The algorithm is completed.

In line 17 to 18 of Algorithm 2, we set the acceptance probability to user \( u \) as an initiator, and then, update \( \Delta(u|\psi) \). Beginning from line 19, for all possible i-hop participants, we set their participation probability from line 20 to 32. To understand the idea of this part, let us see following example:

**Example 2.** We assume user \( \{ u : a \} \in L[2], a = 0 \), is a possible 2-hop participator to initiator \( u \), and there are 2 users \( \{ x_1 : a_1 \}, \{ x_2 : a_2 \}, \{ x_3 : a_3 \} \in L[1] \) that existing edges \( \{ x_1, v \}, \{ x_2, v \}, \{ x_3, v \} \in E(G) \). If \( \{ x_1, v \} \in dom(\psi), \psi(\{ x_1, v \}) = 0 \); \( \{ x_2, v \} \in dom(\psi), \psi(\{ x_2, v \}) = 1 \); and \( \{ x_3, v \} \notin dom(\psi) \), we can update a as \( a = 1 - (1 - a_1)(1 - 0)(1 - p_{x_2} \cdot a_3) \).

Then, from line 33 to 37, it aims to compute the revenue gain for all possible i-hop participators. For user \( \{ v, a \} \in L[2], i < h(v|\psi) \), we can obtain the gain \( a \cdot (R[i] - R[h(v|\psi)]) \) according to the assumption in Section 3.3, if \( i \geq h(v|\psi) \), no gain. The algorithm is completed.

It is worth noting that this computational method is not absolutely accurate because of the complexity of RMKCG model. Consider targeted network \( G = \{ e_1 = \{ v_1, v_2 \}, e_2 = \{ v_2, v_3 \}, e_3 = \{ v_1, v_3 \} \}, \theta_{v_1} = 1, k = 2 \), the expected revenue of user \( v_2 \) when sending an invitation to \( v_1 \):

\[
(Pr[\Phi(\{ e_1 \}) = 1] R[1]) + (Pr[\Phi(\{ e_1 \}) = 0] Pr[\Phi(\{ e_2 \}) = 1] Pr[\Phi(\{ e_3 \}) = 1]) R[2]
\]

We neglect the second term in Algorithm 2. In the most cases, the value of the second term is much less than that of the first term. If necessary to calculate \( \Delta(u|\psi) \) accurately, we can run this propagation process many times and then take the average of them, called Monte Carlo simulation. Obviously, the more times the simulation is performed, the more accurate the target value and the longer the running time is. Algorithm 2 improves operational efficiency greatly under the premise of ensuring accuracy.

Actually, in line 3 to 4 of Algorithm 1, we do not need to compute \( \Delta(u|\psi) \) for each user \( u \in V(G) \) again at each iteration, because the value of \( \Delta(u|\psi) \) depends only on the states of edges within k-hop from \( u \). If they do not change, the value of \( \Delta(u|\psi) \) is consistent with last iteration naturally.

In iteration \( i \), provided that the distance between user \( u_{i-1} \) and \( u \) is larger than \( 2k \), the states of edges within k-hop from \( u \) maintain consistency with iteration \( i - 1 \), thus, we do not need to compute \( \Delta(u|\psi) \) again. Therefore, it is necessary to maintain a record for the value of each \( \Delta(\cdot|\psi) \). In iteration \( i \), we check whether the distance between \( u \) and \( u_{i-1} \) is less or equal to \( 2k \), if yes, update the value of \( \Delta(u|\psi) \).

The efficiency of Algorithm 1 greatly.

### 5 Theoretical Analysis

In this section, we talk about the hardness and adaptive submodularity of RMKCG problem, and get an approximation ratio of Algorithm 1.

#### 5.1 Hardness

In order to show the hardness of RMKCG problem, we can start from a classical NP-hard problem, Maximum Coverage (MC) problem, reduce MC to RMKCG problem in polynomial time. The decision version of MC can be defined as follows:

**Definition 2 (MC).** Given an integer \( b \), a collection of sets \( S = \{ S_1, S_2, ..., S_m \} \) and an integer \( Q \), we ask whether it exists a subcollection \( S' \subseteq S \) such that \( |S'| \leq b \) and the number of covered elements \( | \bigcup_{S_i \in S'} S_i | \geq Q \).

**Theorem 1.** RMKCG problem is NP-hard, because its special case can be reduced to MC problem in polynomial time.

**Proof.** The decision version of RMKCG problem: Given an integer \( b \), a targeted network \( G = (V,E) \) and an integer \( Q' \), we ask whether it exists a policy \( \pi \) and \( |H(\pi, \phi)| \leq b \) for all \( \phi \) such that \( f_{avg}(\pi) \geq Q' \). To get a special case, we make some assumptions about the original problem: We set \( k = 1 \), initiator only send invitation to his/her directed friends; the revenue from each participant is \( 1 \), revenue vector \( R = (1,1) \); the acceptence probability for each user is \( 1 \), acceptance vector \( \theta = (1,1,1,1,...1) \); and the probability \( p_s \) for each edge is 1, the realization is unique.

Let us construct the equivalent relation between MC and RMKCG. Given an instance of MC, we can define an instance of RMKCG as: \( W = \bigcup_{S_i \in S} S_i \) for each node \( w_j \in W \), we create a node \( v_j \) in the instance of RMKCG. For each set \( S_i \in S \), we create a node \( v_j \) in the instance of RMKCG. Thus, we have \( V(G) = \{ v_1, v_2, ..., v_{|W|} \} \cup \{ v'_1, v'_2, ..., v'_{|S|} \} \).

For each node \( v_j \in S_i \), we create an undirected edge between \( v_j \) and \( v'_1 \) thus, \( E(G) = \bigcup_{i=1}^{|S|} \left( \bigcup_{j=1}^{S_i}(v'_j, v_j) \right) \). The construction can be done in polynomial time and shown
in Fig. 3. Based on the above assumptions, all users in contracted graph $G$ would accepted the invitations if they receive it. Let $Q' = Q + b$, we have:

MC $\Rightarrow$ RMKCG: Given an instance of MC, $S' \subseteq S$, $|S'| \leq b$ and $|\bigcup_{S_i \subseteq S'} S_i| \geq Q$, as an instance of RMKCG, we make the policy $\pi$ invite the users in $D = \{v'_i | S_i \in S'\}$ in any order. We can know that $|H(\pi, \phi)| \leq b$ for all $\phi$, $\phi$ is unique, and $f_{\text{avg}}(\pi) \geq Q + b = Q'$.

RMKCG $\Rightarrow$ MC: Given an instance of RMKCG, a policy $\pi$ such that $|H(\pi, \phi)| \leq b$ and $f_{\text{avg}}(\pi) \geq Q + b$, we have following observation. First, in the constructed network $G$, if $\pi$ selects a node $v_j$, $v_j \in \{v_1, v_2, ..., v_{|W_i|}\}$, and it does not select a $v'_i \in \{v'_1, v'_2, ..., v'_{|S|}\}$, that connects to $v_j$, then we assume there is another policy $\pi$ that selects one of $v'_i$ that connects to $v_j$ instead of $v_j$. Obviously, we have $f_{\text{avg}}(\pi) \geq f_{\text{avg}}(\pi)$ because the number of neighbors of $v'_i$ is greater or at least equal to that of $v_j$, thus, we can make more users participate in this game when inviting $v'_i$ instead of $v_j$. As an instance of MC, we select the subcollection $S' = \{S_i | v'_i \in H(\kappa, \phi)\}$. We can know that $|S'| \leq b$ and $|\bigcup_{S_i \subseteq S'} S_i| \geq f_{\text{avg}}(\sigma) - b \geq f_{\text{avg}}(\pi) - b \geq Q$.

Therefore, the simplest case of RMKCG can be reduced to MC problem, which means that it has at least the same hardness as MC, RMKCG problem is NP-hard. The proof is completed. $\square$

5.2 Approximation Performance

In [13], Golovin et al. proposed two important concepts: Adaptive Monotonicity and Adaptive Submodularity, and prove we can obtain a $(1 - 1/e)$-approximate solution by adaptive greedy strategy if the adaptive optimization problem satisfies these two properties. Our RMKCG problem is an instance of adaptive optimization problem, thus, supposing our objective function, Equation (4), is adaptive monotone and adaptive submodular, we can obtain a similar result. Based on [13], these two concepts are as follows:

Definition 3 (Adaptive Monotonicity). A function $f$ is adaptive monotone with respect to distribution $Pr(\phi)$ if the conditional expected marginal benefit of any user $u$ is nonnegative, i.e., for all $\psi$ with $Pr(\Phi \sim \psi) > 0$ and all $u \in V(G) \setminus dx(\psi)$, we have

$$\Delta(u|\psi) \geq 0$$

(5)

Definition 4 (Adaptive Submodularity). A function $f$ is adaptive submodular with respect to distribution $Pr(\phi)$ if the conditional expected marginal benefit of any user $u$ does not increase as more states of users and edges are observed, i.e., for all $\psi$ and $\psi'$ such that $\psi \subseteq \psi'$ and a user $u \in V(G) \setminus dx(\psi')$, we have

$$\Delta(u|\psi) \geq \Delta(u|\psi')$$

(6)

Then, we can obtain our theoretical results, which is described as follows:

Lemma 1. The objective function $f$ of the RMKCG problem is adaptive monotone.

Proof. Considering a fixed observation $\psi$, for a user $u \in V(G) \setminus dx(\psi)$, if sending an invitation to $u$, there are two situations that can happen: accept or reject. If user $u$ accept to be an initator, the value of marginal gain is at least equal to $R[0] - R[R(h(u|\psi))]$; otherwise, there is no marginal gain. The marginal gain is nonnegative in both cases. Thus, under any realization $\phi$, we have $f(dx(\psi) \cup \{u\}, \phi) \geq f(dx(\psi), \phi) \cdot$. The expected marginal gain $\Delta(\psi|\psi')$ is the linear combination of each realization, then $\Delta(\psi|\psi') \geq 0$ as well. $\square$

Lemma 2. The objective function $f$ of the RMKCG problem is not adaptive submodular.

Proof. Considering an example that targeted graph is $G = \{\{v_0, v_1\}, \{v_1, v_2\}, \{v_1, v_3\}\}$, $k = 2$, $p_{\{v_0, v_1\}} = 0.1$, $p_{\{v_1, v_2\}} = 0.1$, $p_{\{v_1, v_3\}} = 0.1$, $p_{\{v_2, v_3\}} = 0.1$. We have two partial realization $\psi_1 = \emptyset$ and $\psi_2 = \{v_0, v_0, v_1, v_1, v_2, v_1, v_3\}$. We know they do not receive any invitation from company, and $\psi_2$ visits $v_0$ accepts to be an initiator and observes that $\{v_0, v_1, v_1, v_2\}$ and $\{v_1, v_3\}$ exist. Clearly, $\psi_1 \subsetneq \psi_2$. Relied on Definition 1, $\Delta(\psi_1|\psi_2) = 0.5 \cdot (R_0 + 3 \times 0.1 \times R_1)$ because of $E(\theta_{\psi_1}) = 0.5$. However, $\Delta(\psi_1|\psi_2) = 0.5 \cdot (R_0 - R_1 + 2 \times R_2)$. Here, assume $R = (5, 3, 1)$, we have $\Delta(\psi_1|\psi_2) = 2.95$ and $\Delta(\psi_1|\psi_2) = 3$. Thus, $\Delta(\psi_1|\psi_2) < \Delta(\psi_1, \psi_2)$, $f$ is not adaptive submodular. $\square$

Even though the RMKCG problem is not adaptive submodular, in some special cases, it is adaptive submodular unexpectedly. Let us see

Remark 1. Given two partial realization that $\psi \subseteq \psi'$, we have $h(\psi|\psi') \geq h(\psi'|\psi')$ for each user $\psi$ due to the fact that tendency assumption shown as before.

Lemma 3. If our RMKCG problem conforms one of following two special cases:

1) $k \leq 1$
2) For all $u, v \in E(G)$, $p_{uv} = 1$

The objective function $f$ is adaptive submodular.

Proof. In order to prove adaptive submodularity, we need to show, for any partial realization $\psi, \psi'$ such that $\psi \subseteq \psi'$, we have $\Delta(u|\psi) \geq \Delta(u|\psi')$. Similar to the proof of adaptive monotonicity, we consider two fix observation $\psi, \psi'$ such that $\psi \subseteq \psi'$ and a user $u \in V(G) \setminus dx(\psi')$. Given an observation $\psi$, we define the marginal gain under realization $\phi \sim \psi$ as $\Delta(u|\psi)|\phi|:

$$\Delta(u|\psi, \phi \sim \psi) = f(dx(\psi) \cup \{u\}, \phi) - f(dx(\psi), \phi)$$

(7)

Assuming that there are two realizations such that $\psi \sim \phi$ and $\psi' \sim \phi'$, we have $\phi(\psi) = \phi'(\psi)$ for all $v \notin dx(\psi')$ and $\phi(\{u, v\}) = \phi'(\{u, v\})$ for all $\{u, v\} \notin dom(\phi')$, thus they have the same part $\alpha = \psi \cup (\phi' \setminus \phi)$. Now we can prove

$$\Delta(u|\psi, \phi \sim \psi) \geq \Delta(u|\psi', \phi' \sim \psi')$$

(8)

for all $u \in dx(\psi')$. We will discuss the above two cases seperately as follows:

1) For the first case, when $k = 0$, Equation (8) is established obvious because of Remark 1. When $k = 1$, if $u$ accepts the invitation, we have $R[0] - R[h(u|\psi')] \geq R[0] - R[h(u|\psi')]$, the gain of $u$ under $\psi$ is equal or larger than that under $\psi'$. For each user $v \in N(u)$, if $\{u, v\} \in dom(\alpha)$, the gain of $v$ under $\psi$ is equal or larger than that under $\psi'$ regardless $\alpha(\{u, v\}) = 1$ or 0. If $\{u, v\} \notin dom(\psi')(\psi')$, when $\psi'(\{u, v\}) = 0$, there is no gain of $v$ under $\psi'$; when $\psi'(\{u, v\}) = 1$, $\psi'(\{u, v\}) = 0$;
ψ′(v) = 0 definitely, there is no gain of v under ψ′. Thus, whatever ϕ(u, v) is, the gain of v under ψ is equal or larger than that under ψ′. If u rejects the invitation, there is no gain both ψ and ψ′. Integrating all situations, Equation (8) is established.

2) For the second case, for all \{u, v\} ∈ E(G), \(p_{uv} = 1\), the state of edges under realization \(\phi\) and \(\phi'\) are identical. \(\psi ⊆ \psi'\) means that \(dx(\psi) ⊆ dx(\psi')\), based on Remark 1, it is easy to get Equation (8).

According to Equation (7), Equation (8) and Definition 1, we have as follows: \[\Delta(u|\phi) = \sum_{\phi \sim \psi} \text{Pr}[\phi|\phi \sim \psi] \Delta(u|\psi, \phi \sim \psi) = \sum_{\phi' \sim \psi'} \text{Pr}[\phi'|\phi' \sim \psi'] \sum_{\phi \sim \alpha} \text{Pr}[\phi|\phi \sim \alpha] \Delta(u|\psi, \phi \sim \psi)\]

Since Equation (8) and \(\sum_{\phi \sim \alpha} \text{Pr}[\phi|\phi \sim \alpha] = 1\),

\[\geq \sum_{\phi' \sim \psi'} \text{Pr}[\phi'|\phi' \sim \psi'] \sum_{\phi \sim \alpha} \text{Pr}[\phi|\phi \sim \alpha] \Delta(u|\psi', \phi' \sim \psi') \geq \sum_{\phi' \sim \psi'} \text{Pr}[\phi'|\phi' \sim \psi'] \Delta(u|\psi', \phi' \sim \psi') \sum_{\phi \sim \alpha} \text{Pr}[\phi|\phi \sim \alpha] = \sum_{\phi' \sim \psi'} \text{Pr}[\phi'|\phi' \sim \psi'] \Delta(u|\psi', \phi' \sim \psi') = \Delta(u|\phi')\]

Therefore, \(\Delta(u|\psi) \geq \Delta(u|\phi')\), the proof of adaptive submodularity is completed.

Remark 2. Even though the RMKCG problem is not adaptive submodular, we can know that the objective function \(f\) is getting close to be adaptive submodular, when \(k\) becomes smaller or the probabilities of edges are approaching to 1. Shown as the proof of Lemma 2, it is a typical non-submodular case, the likelihood such case happens is reduced gradually when \(k\) becomes smaller or \(p_{uv} \rightarrow 1\) for each \{u, v\} ∈ E(G).

Theorem 2. The adaptive strategy \(\pi_a\) given by Adaptive-Invitation algorithm, for our RMKCG problem is a \((1-1/e)\)-approximate solution when \(k \leq 1\) or \(p_{uv} = 1\) for each \{u, v\} ∈ E(G). Hence, we have \[f_{avg}(\pi_a) \geq (1-1/e) \cdot f_{avg}(\pi^*)\] (9)

Proof. Based on Lemma 1 and Lemma 3, the objective function \(f\) is adaptive monotone and submodular, adaptive greedy policy is a \((1-1/e)\)-approximation according to the conclusion of [13].

6 Experiment

In this section, we need to validate the effectiveness and correctness of our proposed adaptive polices on several real social networks. The datasets in our experiments are from networkrepository.com [28], which is an website of network repository. Three datasets with different size are used in our experiments. The dataset-1 is a co-authorship network, where each edge is a co-authorship among scientists in network theory and experiments. The dataset-2 is a Wiki network, which is a who-votes-on-whom network collected from Wikipedia. The dataset-3 is a social friendship network extracted from Facebook consisting of people with edges representing friendship ties. The statistics information of the three datasets is represented in table 1.

| Dataset | n   | m   | Type       | Avg. Degree |
|---------|-----|-----|------------|-------------|
| Dataset-1 | 0.4K | 1.01K | undirected | 4.00         |
| Dataset-2 | 1.0K | 3.15K | undirected | 6.00         |
| Dataset-3 | 3.0K | 65.2K | undirected | 48.0         |

6.1 Experimental Settings

As mentioned earlier, our proposed algorithms are based on the following parameters: Hop number \(k\) (at most \(k\)-hop participants follow an initiator), Acceptance vector \(\theta\), Revenue vector \(R\), budget \(b\) and edge probability. For each \(e \in E(G)\), we set \(p_e = 0.5\) and for each user \(u\), \(\theta_u\) is uniformly distributed in [0, 1]. There are two experiments we perform in this part: Algorithm 2 performance and Algorithm 1 performance. For the performance of Algorithm 2, we have shown that the value of \(\Delta(u|\psi)\) computed by Algorithm 2 is not precise completely when \(k \geq 2\), and accurate value can be obtained by Monte Carlo simulation. In this experiment, we aim to evaluate the effectiveness and efficiency of our Algorithm 2 compared with Monte Carlo simulation. We run the Monte Carlo simulation 100 times and take the average of them to compute the value of \(\Delta(u|\psi)\).

For the performance of Algorithm 1 in this experiment, we test our Algorithm 1 with some common heuristic algorithms and compare the results of performance. It aims to evaluate the effectiveness of our adaptive invitation strategy. This experiment can be divided into three parts: \(k = 1\), \(k = 2\) and \(k = 3\), and their revenue vector is set as \((8, 6)\), \((8, 6, 4)\) and \((8, 6, 4, 2)\), which means that the revenue of 0-hop participant is 8 units, 1-hop is 6 units, 2-hop is 4 units and 3-hop is 2 units. Our proposed algorithms are compared with some baseline algorithms:

1) MaxDegree: Invite the user with maximum degree at each step within budget \(b\).
2) Random: Invite a user randomly from \(V(G)\) at each step within budget \(b\).
3) MaxProb: Invite the user with maximum acceptance probability at each step within budget $b$.

We use python programming to test each algorithms. The simulation is run on a Windows machine with a 3.40GHz, 4 core Intel CPU and 16GB RAM.

6.2 Experimental Results

In our experiments, the whole graphs are considered as targeted networks. We run these adaptive algorithms on three datasets, and for each algorithm, we simulate 50 times and take the average of them, besides, we record the standard deviation for Adaptive-Invitation algorithm.

Fig. 4 draws the performance achieved by Adaptive-Invitation algorithm under dataset-1, which aims to compare the result of computing $\Delta(u|\psi)$ by Algorithm 2 and Monte Carlo simulation. In other words, in line 4 of Algorithm 1, one uses Algorithm 2, another one uses Monte Carlo simulation. Shown as Fig. 4, the total revenue achieved by use of Algorithm 2 and Monte Carlo simulation is very close, and it is more apparent when $k = 3$. Even though performance of Monte Carlo simulation is slightly better than that of Algorithm 2, this difference is acceptable and it is related to topological structure of targeted networks. However, the running time for Algorithm 2 is much less than Monte Carlo simulation, and obviously, the larger the graph is, the more significant this gap will be. Therefore, Algorithm 2 makes adaptive strategy for RMKCG problem be scalable to large real social networks.

Fig. 5 and Fig. 6 draw the performance comparison achieved by Adaptive-Invitation and other heuristic algorithms under budget $b$ under dataset-2 and dataset-3. From the left column of Fig. 5 and Fig. 6, we can observe two features: (1) The standard deviation drops with the increase of hop $k$ under the same dataset; (2) The standard deviation drops with the increase of budget $b$ under the same dataset and hop. We can give a valid explanation. The uncertainty mainly comes from the nodes’ acceptance probability and the edges’ probability. When $k$ is smaller, for example, $k = 1$, if a user $u$ does not accept the invitation, the revenue of all his/her neighbors is 0 unless there exists other initiators. Conversely, when $k$ is larger, for example, $k = 3$, his/her neighbors are possible to be 1-hop or 2-hop participants, which lead to the gap of benefits reduced. Thus, larger $k$ leads to smaller standard deviation.

Fig. 5. The performance comparison changes between Adaptive-Invitation and other heuristic algorithms over budget $b$ under dataset-2. Left column is the standard deviation of Adaptive-Invitation; Right column is the performance comparison.

Fig. 6. The performance comparison changes between Adaptive-Invitation and other heuristic algorithms over budget $b$ under dataset-3. Left column is the standard deviation of Adaptive-Invitation; Right column is the performance comparison.
deviation. Then, with budget \( b \) increasing, marginal gain is decremented generally, and the gap brought by the difference of the number of initiators is reduced. Thus, larger \( b \) leads to smaller standard deviation. From the right column of Fig. 5 and Fig. 6, the total revenue returned by Adaptive-Invitation is better than other policies under any dataset and hop number, so its performance is best. Among these heuristic policies, MaxDegree has the best performance. It proves our theoretical analysis in the last section. We have said that the objective function is not adaptive submodular when \( k \geq 2 \), but in the figure, it shows the characteristics of submodularity as well, which means that the degree of submodularity is related to the structure of networks and probability setting.

7 Conclusion

In this paper, we build a new model, Collaborate Game, to model some real scenarios that cannot be covered by existing model. Then, we propose an adaptive RMKCG problem, and prove it is NP-hard, adaptive monotone but not adaptive submodular. Even that, we show that under some special case, \( k \leq 1 \) and \( p_e = 1 \) for all \( e \in E(G) \), the objective function is adaptive submodular, which can be solved within \((1 - 1/e)\)-approximation. Besides, we propose an effective method to overcome the difficulty in computing marginal gain. The good performance of our algorithms is verified by our experiments on three real network datasets. The future work can be divided into two parts: (1) Trying to find a more generalized model, and combine with the technique of data mining to optimize the parameter setting. (2) Trying to solve the general case, in other words, get the theoretical bound for non-submodular cases.

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