The Complete Black Brane Solutions in $D$-Dimensional Coupled Gravity System

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March 28, 2022

Abstract

In this paper, we use only the equation of motion for an interacting system of gravity, dilaton and antisymmetric tensor to study the black brane solutions. By making use of the property of Schwarzian derivative, we obtain the complete solution of this system of equations. For some special values we obtain the well-known BPS brane and black brane solutions.
1 Introduction

Superstring theory \[1, 2\] is the leading candidate for a theory unifying all matter and forces (including gravity, in particular). Unfortunately, there are five such consistent theories, namely, \(SO(32)\) type I, type IIA, type IIB, \(E_8 \otimes E_8\) heterotic and \(Spin(32)/Z_2\) heterotic theories. This richness is an embarrassment for pure theorists. One hope is that all these five superstring theories are just different solutions of an underlying theory. In the last few years this hope turns out to be true. The underlying theory is the so called M theory \[3, 4, 5\]. One formulation of M theory is given in terms of Matrix theory \[6\]. Nevertheless this formulation is not background independent. It is difficult to use it to discuss non-perturbative problems and still we should rely on the study of BPS states. A thorough study of BPS states in any theory is a must for an understanding of non-perturbative phenomena.

In superstring theory and M theory, there is a plethora of BPS states. These BPS states have the special property of preserving some supersymmetry. In a low energy limit they are special solutions of the low energy supergravity theory with Poincaré invariance. However there exists also some other \(p\)-branes which don’t preserve any supersymmetry and they are also no longer Poincaré invariant. In finding these (soliton) solutions one either resort to supersymmetry or make some simple plausible assumptions. There is no definite reasoning that the so obtained solution is unique. The purpose of this paper is to fill this gap.

It is also interesting to study soliton solutions for its own sake without using any supersymmetric argument. Quite recently there are some interests \[7, 8, 9, 11\] in studying branes in type 0 string theories which have no fermions and no supersymmetry in 10 dimensions \[2\]. In these string theories, we can’t use supersymmetric arguments. So it is important to push other symmetric arguments to their limits.

In this paper we will study the coupled system of gravity \((g_{MN})\), dilaton \((\phi)\) and anti-symmetric tensor \((A_{M_1M_2\ldots M_{n-1}})\) in any dimensions. In a previous letter \[12\] we announced the complete solution of this system. Here we will give more details of the derivation of the complete solution and also extend the solution to include black branes. After making a quasi-Poincaré invariant ansatz for the metric and either electric or magnetic ansatz for the anti-symmetric tensor, we derive all the equations of motion in some details. A system of six ordinary differential equations are obtained for five unknown functions (of one variable). After making some changes of the unknown
functions we solved four equations explicitly. As we shown in [4], these remaining two equations (for one unknown functions) are mutually compatible and a unique solution can always be obtained with appropriate boundary conditions. The last equation can also be solved in the most generic case by exploiting some properties of Schwarzian derivative. The complete solutions are given explicitly. By requiring that the solution approaches the flat space-time for $r \to \infty$, we found that there remains four free parameters. For special values of these four parameters we got the well-known BPS brane solution and black brane solution [14, 15, 16]. We will not discuss the physical property of these solutions [17]. For previous studies of soliton and brane solutions in supergravity and string theories, see for example the reviews [18, 19, 20, 21, 22]. We note that some solutions are also generated via sigma-models in [23]. These solutions are only special cases of our complete solutions. In fact there is a large number of papers dealing with the brane solution in one way or another. We apologize in advance that this paper is not a review and we can’t cite all of them because the purpose of this paper is to obtain the most complete solution which wasn’t achieved before.

2 The Equations of Motion and the Ansatz

Our starting point is the following action for the coupled system of gravity $g_{MN}$, dilaton $\phi$ and anti-symmetric tensor $A_{M_1 \cdots M_{n-1}}$:

$$I = \int d^{D}x \sqrt{-g} \left( R - \frac{1}{2} g^{MN} \partial_M \phi \partial_N \phi - \frac{1}{2} \cdot n! e^{a \phi} F^2 \right),$$  

(1)

where $a$ is a constant and $F$ is the field strength: $F = dA$.

The equations of motion can be easily derived from the above action (1):

$$R_{MN} = \frac{1}{2} \partial_M \phi \partial_N \phi + S_{MN},$$  

(2)

$$\frac{1}{\sqrt{-g}} \partial_M (\sqrt{-g} e^{a \phi} F_{M_1 M_2 \cdots M_n}) = 0,$$  

(3)

$$\frac{1}{\sqrt{-g}} \partial_M (\sqrt{-g} g^{MN} \partial_N \phi) = \frac{a}{2n!} e^{a \phi} F^2,$$  

(4)

where

$$S_{MN} = \frac{1}{2(n-1)!} e^{a \phi} \left( F_{M_1 M_2 \cdots M_n} F_{N M_2 \cdots M_n} - \frac{n-1}{n(D-2)} F^2 g^{MN} \right).$$  

(5)
Of course it is impossible to solve the above system of equations in their full generality. To get some meaningful solution we will make some assumptions by using symmetric arguments.

Our ansatz for a $p$-dimensional black brane is as follows:

$$ds^2 = -e^{2A_0(r)}dt^2 + \sum_{\alpha=1}^p e^{2A_\alpha(r)}(dx^\alpha)^2 + e^{2B(r)}dr^2 + e^{2C(r)}d\Omega_{\tilde{d}+1}^2,$$  

(6)

where $d\Omega_{\tilde{d}+1}^2$ is the square of the line element on the unit $\tilde{d}+1$ sphere which can be written as follows:

$$d\Omega_{\tilde{d}+1}^2 = d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + \cdots + (\sin \theta_1 \cdots \sin \theta_{\tilde{d}})^2 d\theta_{\tilde{d}+1}^2.$$  

(7)

The brane is extended in the directions $(t, x^\alpha)$. When all $A_\alpha$ (including $A_0$) are equal, the brane is Poincaré invariant in the space-time spanned by $(t, x^\alpha)$. We will call the ansatz (6) quasi-Poincare invariant when all $A_\alpha$ are equal except $A_0$.

For the anti-symmetric tensor $A$, we have 2 different choices. The first choice is the electric case and we take the following form for $A$:

$$A = \pm e^{\Lambda(r)} dx^0 \wedge dx^1 \wedge \cdots \wedge dx^p.$$  

(8)

The second choice is the magnetic case and we take the following form for the dual potential $\tilde{A}$:

$$\tilde{A} = \pm e^{\Lambda(r)} dx^0 \wedge dx^1 \wedge \cdots \wedge dx^p.$$  

(9)

We note that the relation between the antisymmetric tensor field strength $F$ and its dual field strength $\tilde{F}$ ($\equiv d\tilde{A}$) is:

$$F^{M_1 \cdots M_n} = \frac{1}{\sqrt{-g}} \frac{1}{(D-n)!} e^{M_1 \cdots M_n N_1 \cdots N_{D-n}} \tilde{F}_{N_1 \cdots N_{D-n}}.$$  

(10)

By using this relation the ansatz (9) transform to an ansatz for $F$:

$$F = \pm \Lambda' \exp \left[ \Lambda - a\phi - \sum_{\alpha=0}^p A_\alpha - B + (\tilde{d} + 1) C \right] \omega_{\tilde{d}+1},$$  

(11)

where $\omega_{\tilde{d}+1}$ is the volume form of the sphere $S^{\tilde{d}+1}$ with unit radius. As it is well-known from duality, the equation of motion (3) becomes the Bianchi
identity for $\tilde{F}$ which is satisfied automatically. Nevertheless the Bianchiih identity for $F$ becomes the equation of motion for $\tilde{F}$ which is given as follows:

$$\frac{1}{\sqrt{-g}} \partial_{N_1} (\sqrt{-g} e^{-a\phi} \tilde{F}^{N_1 N_2 \cdots N_D}) = 0.$$ (12)

From now on we will discuss the electric case only.

Denoting the coordinate index as $(M) = (t, \alpha, r, i)$ and tangent index as $(A) = (\bar{t}, \bar{\alpha}, \bar{r}, \bar{i})$, i.e. a bar over coordinate index, we introduce the following moving frame:

$$e^{\bar{t}} = e^{A(r)} dt = -e_{\bar{t}},$$ (13)

$$e^{\bar{\alpha}} = e^{A_{\alpha}(r)} dx^{\bar{\alpha}} = e_{\bar{\alpha}},$$ (14)

$$e^{\bar{r}} = e^{B(r)} dr = e_{\bar{r}},$$ (15)

$$e^{\bar{i}} = e^{C(r)} \sin \theta_1 \cdots \sin \theta_{i-1} d\theta_i = e_{\bar{i}}.$$ (16)

From [13] we have the following results for the Riemann curvature:

$$R^{AB} = f^{AB} e^A \wedge e^B,$$ (17)

where there is no summation ove $A$ and $B$. $f^{AB}$ can be chosen to be symmetric and is 0 when $A = B$. By definition

$$R^{AB} = \frac{1}{2} R^{AB}_{MN} dx^M \wedge dx^N,$$ (18)

we have

$$R^{AB}_{MN} = f^{AB}(e^A_M e^B_N - e^A_N e^B_M),$$ (19)

and

$$R^A_M = R^{AB}_{MN} e^N_B = \sum_B f^{AB} e^A_M \equiv f^A e^A_M,$$ (20)

where we have defined $f^A$ as

$$f^A = \sum_B f^{AB},$$ (21)

which can be computed to give the following results:

$$f^{\bar{t}} = -e^{-2B} \left( A''_0 + A'_0 \left( \sum_{\alpha=0}^p A'_\alpha - B' + (\bar{d} + 1)C'' \right) \right).$$ (22)
\[ f^\alpha = -e^{-2B} \left( A''_\alpha + A'_\alpha \left( \sum_{\beta=0}^{p} A'_{\beta} - B' + (\tilde{d} + 1)C' \right) \right), \]  
(23)

\[ f^{\bar{r}} = -e^{-2B} \left( \sum_{\alpha=0}^{p} A''_\alpha + (\tilde{d} + 1)C'' + \sum_{\alpha=0}^{p} (A'_{\alpha})^2 \right. \]
\[ + (\tilde{d} + 1)(C'')^2 - B' \left( \sum_{\alpha=0}^{p} A'_{\alpha} + (\tilde{d} + 1)C' \right), \]  
(24)

\[ f^{\bar{\imath}} = -e^{-2B} \left( C'' + C' \left( \sum_{\alpha=0}^{p} A'_{\alpha} - B' + (\tilde{d} + 1)C' \right) \right. \]
\[ - \tilde{d} e^{-2C+2B}. \]  
(25)

Notice that the metric in (23) is a diagonal metric we have then

\[ R_{MN} = R^A_M e_{AN} = f^A \delta^A_M g_{MN}, \]  
(26)

\[ R_{MN} dx^M \otimes dx^N = -f^\bar{r} e^{\bar{r}} \otimes e^{\bar{r}} + \sum_{\alpha=1}^{p} f^{\bar{\imath}} e^{\bar{\imath}} \otimes e^{\bar{\imath}} \]
\[ + f^{\bar{\imath}} e^{\bar{\imath}} \otimes e^{\bar{\imath}} + f^{\bar{\imath}} d\Omega_{\tilde{d}+1}, \]  
(27)

To obtain the equation of motion from eqs. (2) to (4) we also need to know some expressions involving \( F \). We have (for elementary solution)

\[ A = \pm e^{A(r)} dx^0 \wedge dx^1 \ldots \wedge dx^{n-2}, \]  
(28)

\[ F = dA = \pm A' e^{A(r)} dr \wedge dx^0 \wedge dx^1 \ldots \wedge dx^{n-2}, \]  
(29)

\[ F_{MN} \equiv F_M^{MN} F_N^{M_2 \ldots M_n}, \]  
(30)

\[ F_{rr} = -(n - 1)! (A' e^{A(r)})^2 e^{-2 \sum_{\alpha=0}^{n} A_{\alpha}}, \]  
(31)

\[ F_{\mu\nu} = -g_{\mu\nu}(n - 1)! (A' e^{A(r)})^2 e^{-2 \sum_{\alpha=0}^{n} A_{\alpha} - 2B}, \]  
(32)

\[ F_{MN} = 0, \]  
(33)

and

\[ F^2 = g^{MN} F_{MN} = -n! (A' e^{A(r)})^2 e^{-2 \sum_{\alpha=0}^{n} A_{\alpha} - 2B}, \]  
(34)

The equation of motion for \( g^{MN} \) gives the following equations:

\[ f^{\bar{r}} g_{rr} = \frac{1}{2} (\partial_r \phi)^2 + \frac{1}{2} \frac{1}{(n - 1)!} e^a \phi \left( F_{rr} - \frac{(n - 1)}{n(D - 2)} F^2 e^{2B} \right), \]  
(35)
\[ f^\mu e^{2A} \eta_{\mu \nu} = \frac{1}{2 \cdot (n-1)!} e^{\alpha \phi} \left( F_{\mu \nu} - \frac{(n-1)}{n(D-2)} F^2 e^{2A} \eta_{\mu \nu} \right), \quad (36) \]
\[ f^i g_{ij} = \frac{1}{2 \cdot (n-1)!} e^{\alpha \phi} \left( F_{ij} - \frac{(n-1)}{n(D-2)} F^2 g_{ij} \right), \quad (37) \]

for the \((rr)\), \((\mu \nu)\) and \((ij)\) components respectively. Substituting all \(f^A\) and \(F_{MN}\) into the above equations and setting \(C = B + \ln r\), \(A_0 = A + \frac{1}{2} \ln f\) and \(A_\alpha = A (\alpha = 1, \cdots, p)\), we obtain the following four equations:

\[ A'' + d(A')^2 + \tilde{d} A'B' + \frac{\tilde{d} + 1}{r} A' + \frac{1}{2} (\ln f)' A' = \frac{\tilde{d}}{2(D-2)} S^2, \quad (38) \]

\[ A'' + d(A')^2 + \tilde{d} A'B' + \frac{\tilde{d} + 1}{r} A' + \frac{1}{2} (\ln f)'' + \frac{1}{2} (\ln f)' \times \left((d+1)A' + \frac{1}{2} (\ln f)' + \tilde{d} B' + \frac{\tilde{d} + 1}{r}\right) = \frac{\tilde{d}}{2(D-2)} S^2, \quad (39) \]

\[ B'' + dA'B' + \frac{d}{r} A' + \tilde{d}(B')^2 + \frac{1}{2} (\ln f)' \left(B' + \frac{1}{r}\right) + \frac{2\tilde{d} + 1}{r} B' = -\frac{1}{2} \frac{d}{D-2} S^2, \quad (40) \]

\[ dA'' + (\tilde{d} + 1)B'' + d(A')^2 + \frac{\tilde{d} + 1}{r} B' - dA'B' + \frac{1}{2} (\ln f)'' + \frac{1}{4} (\ln f)'^2 + \frac{1}{2} (\phi')^2 = \frac{1}{2} \frac{\tilde{d}}{D-2} S^2, \quad (41) \]

where \(d = p + 1 = n - 1\) and

\[ S = \frac{\Lambda'}{f^{\frac{D}{2}}} e^{\frac{1}{2} \alpha \phi + A - dA}. \quad (42) \]

The equation of motion for \(\phi\) is

\[ \phi'' + \left(dA' + \tilde{d} B' + \frac{\tilde{d} + 1}{r} + \frac{1}{2} (\ln f)' \right) \phi' = -\frac{a}{2} S^2, \quad (43) \]

while the equation of motion for \(F\) is

\[ \left( \frac{\Lambda'}{f^{\frac{D}{2}}} e^{A + a \phi - dA + \tilde{d} B - \frac{1}{2}} \right)' = 0. \quad (44) \]
These six equations, eqs. (38)-(41), (43) and (44), consist of the complete system of equations of motion for five unknown functions: \( f(r), A(r), B(r), \phi(r) \) and \( \Lambda(r) \). We will solve these equations completely in the next three sections.

3 A First Try at the Solution

In this section we will try our best to solve the above system of equations by elementary means. First it is easy to integrate eq. (44) to get

\[
\frac{\Lambda'}{f^2(r)} e^{\Lambda(r) + a\phi(r) - dA(r) + dB(r)} r^{d+1} = C_0, \quad (45)
\]

where \( C_0 \) is a constant of integration. If we know the other four functions \( f(r), A(r), B(r) \) and \( \phi(r) \), this equation can be easily integrated to give \( \Lambda(r) \):

\[
e^{\Lambda(r)} = C_0 \int r \frac{f^2(r)}{r^{d+1}} e^{-a\phi(r) + dA(r) - dB(r)} dr. \quad (46)
\]

By using eq. (45), \( S \) can be written as follows:

\[
S(r) = C_0 \frac{e^{-a\phi(r) - dB(r)}}{r^{d+1}}. \quad (47)
\]

In order to solve the other equations we make a change of functions from \( f(r), A(r), B(r) \) and \( \phi(r) \) to \( \xi(r), \beta(r), \eta(r) \) and \( Y(r) \):

\[
\xi(r) = dA(r) + dB(r) + \frac{1}{2} \ln f(r), \quad (48)
\]
\[
\beta(r) = dA(r) + dB(r) - \frac{1}{2} \ln f(r), \quad (49)
\]
\[
\eta(r) = \phi(r) + a (A(r) - B(r)), \quad (50)
\]
\[
Y(r) = A(r) - B(r). \quad (51)
\]

The equations are then changed to

\[
\xi'' + (\xi')^2 + \frac{2d + 1}{r} \xi' = 0, \quad (52)
\]
\[
\beta'' + \left( \xi' + \frac{d + 1}{r} \right) \beta' + \frac{d}{r} \xi' = 0, \quad (53)
\]
\[ \eta'' + \left( \xi' + \frac{\tilde{d} + 1}{r} \right) \eta' - \frac{a}{r} \xi' = 0, \quad (54) \]

\[ Y'' - \frac{\Delta}{2} (Y')^2 + \left[ \frac{\tilde{d} - d}{2(D - 2)} (\xi + \beta)' + \frac{\tilde{d} + 1}{r} + \frac{a}{r} \eta' - \frac{1}{2} (\xi - \beta)' \right] Y' \]

\[ -\frac{1}{2} (\eta')^2 - \xi'' + \frac{(\xi' + \beta')^2}{4(D - 2)} - \frac{1}{4} (\xi' - \beta')^2 = 0, \quad (55) \]

\[ Y'' + \left( \xi' + \frac{\tilde{d} + 1}{r} \right) Y' - \frac{\xi'}{r} = \frac{1}{2} S^2, \quad (56) \]

where

\[ \Delta = \frac{2 \tilde{d} \tilde{d}}{D - 2} + a^2. \quad (57) \]

The general solutions for \( \xi, \beta \) and \( \eta \) can be obtained easily from eqs. (52), (53) and (54) and we have

\[ \xi = \ln \left| C_1 + C_2 r^{-2\tilde{d}} \right|, \quad (58) \]

\[ \eta' = \frac{2C_2a + C_3r^{\tilde{d}}}{r(C_2 + C_1 r^{2\tilde{d}})}, \quad (59) \]

\[ \beta' = \frac{-2C_2 \tilde{d} + C_4r^{\tilde{d}}}{r(C_2 + C_1 r^{2\tilde{d}})} \quad (60) \]

where \( C_i \)'s are constants of integration. To make sense of the above expressions, \( C_1 \) and \( C_2 \) can’t be zero simultaneously.

Substituting the above expressions into eqs. (53) and (54), we get

\[ Y'' - \frac{\Delta}{2} (Y')^2 + Q(r) Y' = R(r) \quad (61) \]

and

\[ S^2 = \Delta \left( Y' - \frac{2C_2 + \frac{1}{\Delta}(aC_3 + \frac{\tilde{d}C_4}{D-2})r^{\tilde{d}}}{r(C_2 + C_1 r^{2\tilde{d}})} \right)^2 + \frac{K}{(C_2 + C_1 r^{2\tilde{d}})^2} r^{2\tilde{d}-2}, \quad (62) \]

where

\[ Q(r) = \frac{\tilde{d} + 1}{r} + \frac{2C_2(\Delta - \tilde{d}) + (a C_3 + \frac{\tilde{d}C_4}{D-2})r^{\tilde{d}}}{r(C_2 + C_1 r^{2\tilde{d}})}, \quad (63) \]
\[ R(r) = \frac{1}{r^2(C_2 + C_1 r^{2\tilde{d}})^2} \times \left[ 2C_2^2(\Delta - \tilde{d}) + 2C_2 \left( a C_3 + \frac{\tilde{d} C_4}{D - 2} \right) r^{\tilde{d}} \right. \\
+ \left. \left( 2C_1 C_2 \tilde{d}(2\tilde{d} + 1) + \frac{1}{2} C_3^2 + \frac{(D - 3)}{4(D - 2)} C_4^2 \right) r^{2\tilde{d}} \right], \tag{64} \]

\[ K = C_3^2 - \frac{1}{\Delta} \left( a C_3 + \frac{\tilde{d} C_4}{D - 2} \right)^2 \]

\[ + 8\tilde{d}(\tilde{d} + 1) C_1 C_2 + \frac{(D - 3)}{2(D - 2)} C_4^2. \tag{65} \]

Notice that our system of equations of motion is an over determined system: five unknown functions satisfying six equations. We have solved four equations and there are two equations, eqs. (61) and (62), remaining with one unknown function \( Y(r) \). These two equations have the same form as the equations derived in [13]. So the same proof given in [13] can be used here to show that these two equations actually give no constraints on \( Y(r) \) and effectively there is only one equation, i.e., we can solve either one of them and the other one will be satisfied automatically. In next section we will solve eq. (61) completely.

### 4 A Special Solution

Now we start to solve eq. (61). Let

\[ g = \int d\,r \, e^{\Delta Y - \int Q(r)dr}, \tag{66} \]

or,

\[ Y = \frac{1}{\Delta} \left( \ln(g') + \int Q(r)dr \right), \tag{67} \]

eq. (61) becomes

\[ \left( \frac{g''}{g'} \right)' - \frac{1}{2} \left( \frac{g''}{g'} \right)^2 = \tilde{R}(r). \tag{68} \]

Here

\[ \tilde{R}(r) = \Delta R(r) - Q'(r) - \frac{1}{2} Q^2(r) = -\frac{d^2 - 1}{2r^2} + \frac{\tilde{\Lambda} r^{2\tilde{d} - 2}}{2(C_2 + C_1 r^{2\tilde{d}})^2}. \tag{69} \]
with
\[ \tilde{\Lambda} = \Delta K - 4 \tilde{d}^2 C_1 C_2 \] (70)

The left-hand side of eq. (68) is the well-known Schwarzian derivative of
the function \( g \) defined as:
\[ S(g) \equiv \left( \frac{g''}{g'} \right)' - \frac{1}{2} \left( \frac{g''}{g'} \right)^2. \] (71)

Thus, in order to solve eq. (68), one must find a function \( g \) such that
\[ S(g) = -\frac{\tilde{d}^2 - 1}{2r^2} + \frac{\tilde{\Lambda}}{2} \frac{r^{2\tilde{d}-2}}{(C_2 + C_1 r^{2\tilde{d}})^2}. \] (72)

### 4.1 Notations and Conventions

Let \( f \) be an arbitrary function of \( r \) and \( f' \) be its derivative. The notation \( f(r) \) always represents the value of the function \( f \) at \( r \). Similarly, \( f'(r) \) is the value of the function \( f' \) at \( r \).

For two functions \( f \) and \( g \), \( fg \) is their product and \( \frac{f}{g} \) is their division, while the composition of two functions \( f \) and \( g \) is denoted by \( f \circ g \). These functions are defined by
\[ (fg)(r) = f(r)g(r), \quad (f\frac{g}{g})(r) = \frac{f(r)}{g(r)}, \quad (f \circ g)(r) = f(g(r)). \] (73, 74, 75)

For product of three or more functions we assume that the composition of two functions has a higher rank of precedence of associativity than the products of functions. Nevertheless the symbol \( \frac{f}{g} \) is considered to be a pure entity and can’t be breaked. To understand these conventions we have the following examples:
\[ fg \circ h = g \circ h \circ f = f \circ (g \circ h), \quad (\frac{f}{g} \circ h) = (\frac{f}{g}) \circ h. \] (76, 77)

\(^1\text{We denote } f f \text{ by } f^2, f f f \text{ by } f^3, \text{ and so on.}\)
If \( k \in \mathbb{C} \) or \( \mathbb{R} \), we define a function \( l_k \) as the multiplication of its variable with the number \( k \), i.e.,

\[
l_k(r) = kr.
\]

For convenience we also consider \( k \) itself as a constant function taking the value \( k \):

\[
k(r) = k.
\]

Other functions such as the power function \( r^s \) with real number \( s \) and the exponential function \( e^r \) are denoted by \( p_s \) and \( \exp \) and other elementary functions are denoted by their standard mathematical symbols.

With these conventions, we can derive the following derivative rules:

\[
\begin{align*}
    l'_k &= k, & k' &= 0, \\
    p'_k &= s p_{s-1}, & \arctan' &= \frac{1}{1+p^2}, \\
    \exp' &= \exp, & \ln' &= p_{-1},
\end{align*}
\]

and the derivative rule for composition of function is

\[
(f \circ g)' = g' f' \circ g
\]

### 4.2 Some Properties of the Schwarzian Derivative

Now we list some elementary properties of the Schwarzian derivative.

1. If \( f \) and \( g \) are two functions, we have

\[
S(f \circ g) = (g')^2 S(f) \circ g + S(g).
\]

2. If \( f = \frac{l_a + b}{l_c + d} \) for some numbers \( a, b, c \) and \( d \), namely, \( f(r) = \frac{ar + b}{cr + d} \), then

\[
S\left( \frac{l_a + b}{l_c + d} \right) = 0.
\]

This is the well-known fact that the fractional linear transformation is a global conformal transformation of the complex sphere. By using this result we have

\[
S\left( \frac{l_a + b}{l_c + d} \circ g \right) = S(g).
\]

That is to say, if \( g \) is a special solution of the equation \( S(f) = R \), the general solution will be

\[
f = \frac{l_a + b}{l_c + d} \circ g = \frac{ag + b}{cg + d}.
\]
with constants $a, b, c, d$ such that $ad - bc = 1$.

(3) For $s \in \mathbb{R}$, we have

$$S(l_k) = 0,$$  \hspace{1cm} (84)

$$S(p_s) = -\frac{s^2 - 1}{2p_2},$$  \hspace{1cm} (85)

$$S(\exp) = -\frac{1}{2},$$  \hspace{1cm} (86)

$$S(\ln) = \frac{1}{2p_2},$$  \hspace{1cm} (87)

$$S(\tan) = 2,$$  \hspace{1cm} (88)

$$S(\arctan) = -\frac{2}{(1 + p_2)^2}.$$  \hspace{1cm} (89)

### 4.3 A Special Solution of Eq. (72)

The function $\tilde{R}$ in Eq. (72) is

$$\tilde{R} = \tilde{d}^2 - 1 + \frac{\tilde{d}^2 - 2}{2p_2} \frac{1}{(C_2 + C_1 p_2)^2} \circ p_{\tilde{d}}.$$  \hspace{1cm} (90)

For $C_1 C_2 > 0$, we have

$$\tilde{R} = S(p_{\tilde{d}}) + (p_{\tilde{d}}')^2 S(h_1) \circ p_{\tilde{d}} = S(h_1 \circ p_{\tilde{d}})$$  \hspace{1cm} (91)

where $h_1$ is a yet-unknown function such that

$$S(h_1) = \frac{\tilde{\Lambda}}{2d^2 C_2^2} \frac{1}{(1 + \frac{C_1}{C_2} p_2)^2} = S(l \sqrt{C_1/C_2}) + (l' \sqrt{C_1/C_2})^2 \frac{\tilde{\Lambda}}{2d^2 C_1 C_2} \frac{1}{(1 + p_2)^2} \circ l \sqrt{C_1/C_2} = S(h_2 \circ l \sqrt{C_1/C_2}).$$  \hspace{1cm} (92)

Now we want to find a function $h_2$ such that

$$S(h_2) = \frac{\tilde{\Lambda}}{2d^2 C_1 C_2} \frac{1}{(1 + p_2)^2}$$
\[ S(\arctan) + \frac{\Delta K}{2d^2C_1C_2} \frac{1}{(1 + p_2)^2} \]
\[ = S(\arctan) + (\arctan')^2 \frac{\Delta K}{2d^2C_1C_2} \circ \arctan \]
\[ = S(h_3 \circ \arctan) \quad (93) \]

where \( K \) is the constant in eq. (65). Note that in the third line we have considered \( \frac{\Delta K}{2d^2C_1C_2} \) to be a constant function. The function \( h_3 \) in the above is easy to find to be \( \tan \circ l_k \) with
\[ k = \frac{1}{2d} \sqrt{\frac{\Delta K}{C_1C_2}}, \quad (94) \]

because
\[ S(h_3) = \frac{\Delta K}{2d^2C_1C_2} \]
\[ = S(l_k) + (l_k')^2 S(\tan) \circ l_k \]
\[ = S(\tan \circ l_k). \quad (95) \]

Combining all the above steps, one finds that
\[ \tilde{R} = S(\tan \circ l_k \circ \arctan \circ \arctan \circ p_2 \circ \arctan). \quad (96) \]

and a special solution of eq. (72) is found to be:
\[ g_0 = \tan \circ l_k \circ \arctan \circ \sqrt{\frac{C_1}{C_2}} \circ p_2 \circ \arctan. \quad (97) \]

namely,
\[ g_0(r) = \tan \left( k \arctan \sqrt{\frac{C_1}{C_2}} \right). \quad (98) \]

The special case considered in [13] corresponds to \( k = 1 \). By using Mathematica, we have checked that the above function is indeed a solution of eq. (72).

The general solution \( f \) of eq. (72) can be obtained according to eq. (83). Here we have three independent constants. This is the right number for a third order ordinary differential equation. So we obtain the complete solution to eq. (72). Substituting these into Eq. (77), one can write down the complete solution to eq. (61) which we turn in the next section.
5 The Complete Solution

The general solution of eq. (72) is obtained from the above special solution, eq. (98), by an arbitrary $SL(2, R)$ transformation:

$$ g(r) = \frac{a_0 g_0(r) + b_0}{c_0 g_0(r) + d_0}, \quad (99) $$

where $a_0$, $b_0$, $c_0$ and $d_0$ consist of an $SL(2, R)$ matrix:

$$ a_0 d_0 - b_0 c_0 = 1. \quad (100) $$

With this general solution in hand one can check that the other equation (62) is also satisfied. Setting

$$ h(r) = \arctan \sqrt{\frac{C_1}{C_2}} r^{\hat{d}}, \quad (101) $$

the complete solutions are:

$$ \begin{align*}
\xi(r) &= \ln \left| C_1 + C_2 r^{-2\hat{d}} \right|, \\
\eta(r) &= C_5 - \frac{a}{d} \xi(r) + \frac{C_3}{d\sqrt{C_1 C_2}} h(r), \\
\beta(r) &= C_6 + \xi(r) + \frac{C_4}{d\sqrt{C_1 C_2}} h(r), \\
Y(r) &= -\frac{2}{\Delta} \ln \left| C_7 \cos(k h(r)) + C_8 \sin(k h(r)) \right| - \frac{1}{\hat{d}} \xi(r) + \frac{a C_3 + \frac{\Delta C_4}{D-2}}{d\Delta \sqrt{C_1 C_2}} h(r),
\end{align*} \quad (102) \quad (103) \quad (104) \quad (105) $$

and

$$ \begin{align*}
f(r) &= \exp \left[ -C_6 - \frac{C_4}{d\sqrt{C_1 C_2}} h(r) \right], \\
A(r) &= \frac{C_6}{2(D-2)} + \frac{a C_3 + \left(1 + \frac{a^2}{2\hat{d}}\right) C_4}{(D-2)\Delta \sqrt{C_1 C_2}} h(r) \\
&\quad - \frac{2\hat{d}}{\Delta(D-2)} \ln \left| C_7 \cos(k h(r)) + C_8 \sin(k h(r)) \right|, \quad (106) \quad (107)
\end{align*} $$
\[ B(r) = \frac{C_6}{2(D - 2)} + \frac{1}{d} \xi(r) + \frac{a (-2d C_3 + a C_4)}{2d \Delta (D - 2) \sqrt{C_1 C_2}} h(r) \]
\[ + \frac{2d}{\Delta (D - 2)} \ln |C_7 \cos(k h(r)) + C_8 \sin(k h(r))|, \quad (108) \]
\[ \phi(r) = C_5 + \frac{2d C_3 - a C_4}{(D - 2) \Delta \sqrt{C_1 C_2}} h(r) \]
\[ + \frac{2a}{\Delta} \ln |C_7 \cos(k h(r)) + C_8 \sin(k h(r))|, \quad (109) \]
\[ e^{\Lambda(r)} = C_0 e^{\left(- \frac{a C_5 + \frac{d C_6}{D-2}}{D-2}\right)} \left[ C_9 + \frac{2}{\sqrt{\Delta K}} \right. \]
\[ \times \left. \frac{\sin(k h(r))}{C_7 (C_7 \cos(k h(r)) + C_8 \sin(k h(r)))} \right], \quad (110) \]

with
\[ C_0^2 e^{-a C_5 - \frac{d C_6}{D-2}} = K (C_7^2 + C_8^2). \quad (111) \]

From the above explicit result we see that there are 9 constants of integration. Most of them can be removed by choosing appropriate space-time transformations. In the next section we will consider the case where \( C_1 C_2 < 0 \) and requiring that the solution goes to flat space-time for \( r \to \infty \).

### 6 The case for \( C_1 C_2 < 0 \).

In this case the function \( h(r) \) is changed to
\[ h(r) = \ln \left[ \frac{1 - (r_0/r)^d}{1 + (r_0/r)^d} \right], \quad (112) \]
by omitting an overall constant and setting \( C_2 = -r_0^2 \hat{d}, \quad (C_1 = 1) \). The complete solution is
\[ \xi(r) = \ln \left[ 1 - \left( \frac{r_0}{r} \right)^{2 \hat{d}} \right], \quad (113) \]
\[ \eta(r) = -\frac{a}{d} \xi(r) + c_1 h(r) \quad (114) \]
\[ \beta(r) = \xi(r) + c_2 h(r), \quad (115) \]
\[ Y(r) = -\frac{1}{d} \xi(r) + \frac{1}{\Delta} \left( a c_1 + \frac{\tilde{d} c_2}{D-2} \right) h(r) \]
\[ -\frac{2}{\Delta} \ln \left[ \cosh(\tilde{k} h(r)) + c_3 \sinh(\tilde{k} h(r)) \right], \quad (116) \]

where \( c_i \)'s are free constants and
\[ \tilde{k}^2 = -\frac{\Delta}{4} \tilde{K}, \quad (117) \]
\[ \tilde{K} = c_1^2 - \frac{1}{\Delta} \left( a c_1 + \frac{\tilde{d} c_2}{D-2} \right)^2 - \frac{2(\tilde{d} + 1)}{d} + \frac{(D-3) c_2^2}{2(D-2)}, \quad (118) \]

where \( \tilde{K} \) is assumed to be negative and

\[ f(r) = \left[ \frac{1 - \left( \frac{r_0}{r} \right)^d}{1 + \left( \frac{r_0}{r} \right)^d} \right]^{-c_2} \quad (119) \]
\[ A(r) = \frac{\tilde{d} \left( a c_1 + \left( 1 + \frac{a^2}{2d} \right) c_2 \right)}{\Delta(D-2)} h(r) \]
\[ -\frac{2 \tilde{d}}{\Delta(D-2)} \ln \left[ \cosh(\tilde{k} h(r)) + c_3 \sinh(\tilde{k} h(r)) \right], \quad (120) \]
\[ B(r) = \frac{1}{d} \xi(r) - \frac{a (2 \tilde{d} c_1 - a c_2)}{2 \Delta(D-2)} h(r) \]
\[ + \frac{2 \tilde{d}}{\Delta(D-2)} \ln \left[ \cosh(\tilde{k} h(r)) + c_3 \sinh(\tilde{k} h(r)) \right], \quad (121) \]
\[ \phi(r) = \frac{\tilde{d} (2 \tilde{d} c_1 - a c_2)}{\Delta(D-2)} h(r) \]
\[ + \frac{2 a}{\Delta} \ln \left[ \cosh(\tilde{k} h(r)) + c_3 \sinh(\tilde{k} h(r)) \right], \quad (122) \]
\[ e^{\Lambda(r)} = c_0 \frac{\sinh(\tilde{k} h(r))}{\cosh(\tilde{k} h(r)) + c_3 \sinh(\tilde{k} h(r))} + \text{const.}, \quad (123) \]

and
\[ c_0^2 = \frac{4}{\Delta} (c_3^2 - 1). \quad (124) \]

For \( \tilde{K} > 0 \), we set \( \tilde{k}^2 = \frac{\Delta}{4} \tilde{K} \), and the solution is obtained by the simple substitution: \( \cosh \to \cos \) and \( \sinh \to \sin \). The last equation (124) is changed
to

$$c_0^2 = \frac{4}{\Delta} (c_3^2 + 1).$$

The black brane solution found in [14, 15, 16] is a special case of the above solution with \( c_1 = \frac{a}{d} \) and \( c_2 = -2 \). The \( \tilde{k} \) can be computed from (117) and we get \( k = 1 \). To make contact with the previous result, we should change the isotropic coordinates used in this paper to the Schwarzschild-type coordinates. In Schwarzschild-type coordinate, the black brane solution obtained in [14, 15, 16] is

$$d\tilde{s}^2 = f_{-}^\frac{4}{\Delta} \left[ -\frac{f_+}{f_-} dt^2 + dx^i dx^i \right]$$

$$+ f_{-}^\frac{4}{\Delta} \left[ \frac{1}{f_+ f_-} d\tilde{r}^2 + \tilde{r}^2 d\Omega_{d+1}^2 \right],$$

$$\phi = -\frac{2a}{\Delta} \ln f_-, \quad f_\pm = 1 - (\frac{r_\pm}{\tilde{r}})^d.$$}

The transformation from the isotropic coordinate \( r \) to the Schwarzschild-type coordinate \( \tilde{r} \) is

$$r = \tilde{r} \left( \frac{\sqrt{f_+} + \sqrt{f_-}}{2} \right)^\frac{d}{2},$$

$$\frac{dr^2}{r^2} = \frac{d\tilde{r}^2}{\tilde{r}^2} \frac{1}{f_+ f_-},$$

and we have

$$\ln \left[ 1 - \left( \frac{r_0}{r} \right)^d \right] = \frac{1}{2} \ln f_+ + \ln \left[ \frac{2}{\sqrt{f_+} + \sqrt{f_-}} \right],$$

$$\ln \left[ 1 + \left( \frac{r_0}{r} \right)^d \right] = \frac{1}{2} \ln f_- + \ln \left[ \frac{2}{\sqrt{f_+} + \sqrt{f_-}} \right],$$

$$\ln(\cosh h(r) + c_3 \sinh h(r)) = -\frac{1}{2} (\ln f_+ + \ln f_-),$$

with

$$r_0^d = \frac{1}{4} \left( r_+^d - r_-^d \right),$$

$$c_3 = -\frac{r_+^d + r_-^d}{r_+^d - r_-^d}.$$
We have then

\[ f(r) = \frac{f_+}{f_-}, \quad (135) \]

\[ A(r) = \frac{2 \tilde{d}}{\Delta (D-2)} (h(r) + \ln(\cosh h(r) + c_3 \sinh h(r))) \]

\[ = \frac{2 \tilde{d}}{\Delta (D-2)} \ln f_-, \quad (136) \]

\[ B(r) = \frac{1}{d} \xi(r) - \frac{a^2}{\Delta d} h(r) + \frac{2 d}{\Delta (D-2)} \ln(\cosh h(r) + c_3 \sinh h(r)) \]

\[ = \frac{2}{d} \ln \frac{2}{\sqrt{f_+} + \sqrt{f_-}} + \frac{a^2}{\Delta d} \ln f_-, \quad (137) \]

\[ \phi(r) = \frac{2 a}{\Delta} (h(r) + \ln(\cosh h(r) + c_3 \sinh h(r))) = -\frac{2 a}{\Delta} \ln f_- . \quad (138) \]

The above results gave exactly the black brane solution (126).

From the above results we see that the black brane solution is a two-parameter solution. Our complete solution is a four-parameter solution. A detail study of the physical property of this more general solution is left for a future publication.

**Acknowledgments**

We would like to thank Han-Ying Guo, Yi-hong Gao, Ke Wu, Ming Yu, Zhu-jun Zheng and Zhong-Yuan Zhu for discussions. This work is supported in part by funds from Chinese National Science Fundation and Pandeng Project.

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