Dispersionless Pfaff-Toda hierarchy and elliptic Löwner equation

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May 2016

Abstract

We show that one-variable reductions of the Pfaff-Toda integrable hierarchy in the dispersionless limit are described by a system of coupled elliptic Löwner (Komatu-Goluzin) equations.

1 Introduction

In this paper we consider the dispersionless limit of the Pfaff-Toda hierarchy [1, 2] in the elliptic parametrization [3, 4]. The aim of the paper is to describe one-variable reductions of the hierarchy in the spirit of the Gibbons-Tsarev approach [5, 6], see also [7]-[12], where it was shown that consistent reductions of the KP, modified KP and Toda hierarchies are obtained from solutions to the chordal and radial versions of the Löwner equation known in the classical complex analysis [13].

A similar description of reductions of the Pfaff-KP hierarchy (also known as the Pfaff lattice) is given in [3], where it has been shown that in the Pfaff-KP case the Löwner equation should be substituted by its elliptic analogue (the Komatu-Goluzin equation [14, 15], see also [16]-[21]). Here we show that in the Pfaff-Toda case the reductions are described by solutions to a system of coupled equations looking like the elliptic Löwner ones. The proof is based on some non-trivial identities for theta functions.

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2 The dispersionless Pfaff-Toda hierarchy

The set of hierarchical times in the dispersionless Pfaff-Toda (dPfaff-Toda) hierarchy is \( t = \{ \ldots, t_2, t_1, t_0, t_0, t_1, t_2, \ldots \} \). Below we deal with the real form of the hierarchy, where \( \bar{t}_k \) is the complex conjugate of \( t_k \) for all \( k = 0, 1, 2, \ldots \). Let us introduce the differential operators

\[
D(z) = \sum_{k \geq 1} z^{-k} \partial_{t_k}, \quad \bar{D}(z) = \sum_{k \geq 1} z^{-k} \partial_{\bar{t}_k}
\]

and

\[
\nabla(z) = \partial_{t_0} + D(z), \quad \bar{\nabla}(z) = \partial_{\bar{t}_0} + \bar{D}(z).
\]

Clearly, \( \bar{D}(z) = \bar{D}(\bar{z}), \nabla(z) = \nabla(\bar{z}) \). In the dispersionless Hirota formulation, the dependent variable of the dPfaff-Toda hierarchy is a real function \( F = F(t) \). Introducing the auxiliary functions

\[
P(z) = z e^{-(\partial_{t_0} + \partial_{\bar{t}_0}) \nabla(z) F}, \quad W(z) = z e^{-(\partial_{t_0} - \partial_{\bar{t}_0}) \nabla(z) F},
\]

\[
\bar{P}(z) = z e^{-(\partial_{t_0} + \partial_{\bar{t}_0}) \nabla(z) F}, \quad \bar{W}(z) = z e^{(\partial_{t_0} - \partial_{\bar{t}_0}) \nabla(z) F},
\]

one can present equations of the hierarchy in the form \([2]\)

\[
e^{D(z)D(\zeta) F} \left( 1 - \frac{1}{P(z) P(\zeta)} \right) = \frac{W(z) - W(\zeta)}{z - \zeta} e^{(\partial_{t_0} - \partial_{\bar{t}_0}) \partial_{t_0} F}
\]

\[
e^{D(z)\bar{D}(\zeta) F} \left( 1 - \frac{1}{P(z) P(\zeta)} \right) = \frac{\bar{P}(z) - \bar{P}(\zeta)}{z - \zeta} e^{(\partial_{t_0} + \partial_{\bar{t}_0}) \partial_{t_0} F}
\]

\[
e^{D(z)\bar{D}(\zeta) F} \left( 1 - \frac{1}{P(z) P(\zeta)} \right) = 1 - \frac{1}{W(z) W(\zeta)}
\]

\[
e^{\bar{D}(z)\bar{D}(\zeta) F} \left( W(z) - \bar{W}(\zeta) \right) = \left( \bar{P}(z) - \bar{P}(\zeta) \right) e^{2\partial_{t_0} \partial_{t_0} F}.
\]

Here \( P(\zeta) := \bar{P}(z) \), \( W(\zeta) := W(\bar{z}) \). There are also two equations which are complex conjugate to the first and the second equation in the list. (The third and the fourth ones are self-conjugate.) The differential equations of the Pfaff-Toda hierarchy are obtained by expanding \([4]\) in powers of \( z \) and \( \zeta \).

Dividing the second equation in \([4]\) by the first one, we get the relation

\[
W(z) + W^{-1}(z) - e^{2\partial_{t_0} \partial_{t_0} F} \left( P(z) + P^{-1}(z) \right) = W(\zeta) + W^{-1}(\zeta) - e^{2\partial_{t_0} \partial_{t_0} F} \left( P(\zeta) + P^{-1}(\zeta) \right).
\]

It means that \( C := W(z) + W^{-1}(z) - e^{2\partial_{t_0} \partial_{t_0} F} \left( P(z) + P^{-1}(z) \right) \) does not depend on \( z \). The constant \( C \) can be found by performing the limit \( z \to \infty \): \( C = 2 e^{-(\partial_{t_0} - \partial_{\bar{t}_0}) \partial_{t_0} F} \partial_{t_0} \partial_{t_1} F \).

Dividing the fourth equation in \([4]\) by the third one, we get the relation

\[
W(z) + W^{-1}(z) - e^{2\partial_{t_0} \partial_{t_0} F} \left( P(z) + P^{-1}(z) \right) = \bar{W}(\zeta) + \bar{W}^{-1}(\zeta) - e^{2\partial_{t_0} \partial_{t_0} F} \left( P(\zeta) + P^{-1}(\zeta) \right)
\]

which states that \( C \) is real, i.e., \( e^{\partial_{t_0} F} \partial_{t_0} \partial_{t_1} F = e^{\partial_{t_0} F} \partial_{t_0} \partial_{t_1} F \) (this is in fact one of equations of the hierarchy). As a result, the auxiliary functions \( P(z), W(z) \) satisfy the algebraic equation \([2]\)

\[
W(z) + W^{-1}(z) - R^2 \left( P(z) + P^{-1}(z) \right) = C
\]

(5)
with the real coefficients
\[ R^2 = e^{2\partial_0 \partial_0 F}, \quad C = 2e^{\partial_0 (\partial_0 - \partial_0) \partial_0 \partial_1 F}. \]  
(6)

The functions \( \bar{P}, \bar{W} \) satisfy the same equation. This equation defines an elliptic curve, with \( P \) and \( W \) being functions on it. The local parameter around \( \infty \) is \( z^{-1} \). As is readily seen from (3), both \( P \) and \( W \) have a simple pole at infinity.

Along with \( P, W \) consider the functions
\[ f(z) = \sqrt{P(z)W(z)} = ze^{-\partial_0 \nabla(z) F}, \quad g(z) = \frac{P(z)}{W(z)} = e^{-\partial_0 \nabla(z) F}. \]  
(7)

The function \( f \) has a simple pole at \( \infty \) while \( g \) is regular there. Their complex conjugate functions are \( \bar{f}(\bar{z}) = \bar{f}(\bar{z}) = ze^{-\partial_0 \nabla(\bar{z}) F}, \) \( \bar{g}(\bar{z}) = \bar{g}(\bar{z}) = e^{-\partial_0 \nabla(\bar{z}) F}. \) In terms of \( f, g \) the equation of the elliptic curve (5) reads
\[ R^2(f^2g^2 + 1) + Cfg = f^2 + g^2. \]  
(8)

The functions \( \bar{f}(\bar{z}), \bar{g}(\bar{z}) \) obey the same equation. Note the symmetry \( f \leftrightarrow g \).

We uniformize the elliptic curve (5) with the help of the theta functions \( \theta_a(u) = \theta_a(u|\tau), a = 1, 2, 3, 4 \) (see the appendix for their definition):

\[ f(z) = \frac{\theta_4(u(z))}{\theta_1(u(z))}, \quad g(z) = \frac{\theta_4(u(z) + \eta)}{\theta_1(u(z) + \eta)}. \]  
(9)

The equation (5) of the curve is then satisfied identically if
\[ R = \frac{\theta_1(\eta)}{\theta_4(\eta)}, \quad C = 2\frac{\theta_2(0) \theta_3(\eta) \theta_3(\eta)}{\theta_2(\eta) \theta_2(0) \theta_3(0)}. \]  
(10)

The modular parameter \( \tau \) of the curve (with \( \text{Im} \tau > 0 \)) and the parameter \( \eta \) (a point on the elliptic curve) are dynamical variables which depend on all the times: \( \eta = \eta(t), \) \( \tau = \tau(t). \) We assume that \( \eta \) is real and \( \tau \) is purely imaginary. This assumption is consistent with reality of \( R \) and \( C. \) We fix the expansions of the functions \( u(z) = u(z, t), \) \( \bar{u}(z) = \bar{u}(z, t) \) around \( \infty \) as follows:

\[ u(z, t) = \frac{c_1(t)}{z} + \frac{c_2(t)}{z^2} + \ldots, \quad \bar{u}(z, t) = \frac{\bar{c}_1(t)}{z} + \frac{\bar{c}_2(t)}{z^2} + \ldots. \]  
(11)

In contrast to the dispersionless Pfaff-KP hierarchy the coefficients \( c_i \) are complex.

After the uniformization only two of the four equations in (4) remain independent. Following [4], we represent them in the elliptic form. Let us rewrite the first and the third equations as
\[ (z_1^{-1} - z_2^{-1}) e^{\nabla_1 \nabla_2 F} = R^{-1} g_1g_2 \frac{W_1 - W_2}{1 - P_1 P_2}, \]
\[ e^{\nabla_1 \nabla_2 F} = R^{-1} g_1g_2 \frac{1 - W_1 \bar{W}_2}{1 - P_1 P_2}, \]
where \( \nabla_i = \nabla(z_i), \nabla_i = \nabla(\bar{z}_i), g_i = g(z_i), \) etc. The identities
\[ \frac{W_1 - W_2}{1 - P_1 P_2} = \frac{\theta_1(\eta) \theta_1(u_1 + \eta) \theta_1(u_2 + \eta)}{\theta_4(\eta) \theta_4(u_1 + \eta) \theta_4(u_2 + \eta)} \cdot \frac{\theta_1(u_1 - u_2)}{\theta_4(u_1 - u_2)}, \]
\[ \frac{1 - W_1 \bar{W}_2}{1 - P_1 P_2} = \frac{\theta_2(\eta) \theta_3(\eta) \theta_2(0) \theta_3(0)}{\theta_2(0) \theta_3(0)}. \]
allow us to represent the equations in the form \[4\]

\[(z_1^{-1} - z_2^{-1}) e^{\nabla (z_1) \nabla (z_2) F} = \frac{\theta_1(u(z_1))}{\theta_4(u(z_1))} (u(z_1) - u(z_2)),\]

\[e^{\nabla (z_1) \nabla (z_2) F} = \frac{\theta_1(u(z_1) + \bar{u}(z_2) + \eta)}{\theta_1(u(z_1) + \bar{u}(z_2) + \eta)},\]

\[(z_1^{-1} - z_2^{-1}) e^{\nabla (z_1) \nabla (z_2) F} = \frac{\theta_1(\bar{u}(z_1) - \bar{u}(z_2))}{\theta_1(\bar{u}(z_1) - \bar{u}(z_2))}.\]

Note that the first equation (a “half” of the dPfaff-Toda hierarchy with fixed times \(\bar{t}_k\)) looks like the equation for the dispersionless Pfaff-KP hierarchy in the elliptic parametrization \[3, 4\] (however, with complex times \(t_k\)). The third equation is conjugate of the first one. It represents another copy of the dispersionless Pfaff-KP hierarchy, with respect to the times \(\bar{t}_k\) with fixed \(t_k\)'s. The second equation contains mixed \(t_k\)- and \(\bar{t}_k\)-derivatives. It just couples the two hierarchies into the bigger one.

The limit \(z_2 \to \infty\) of equations \[12\] yields:

\[e^{\theta_0 \nabla (z) F} = z \frac{\theta_1(u(z))}{\theta_4(u(z))}, \quad e^{\theta_0 \nabla (z) F} = \frac{\theta_1(u(z) + \eta)}{\theta_1(u(z) + \eta)}.\]

(and complex conjugate equations). These are expressions for the functions \(f\) and \(g\) \[9\] combined with their definition \[7\]. The further expansion of the first relation as \(z \to \infty\) gives

\[\rho := e^{\theta_0^2 F} = \pi c_1 \theta_2(0) \theta_3(0),\]

where we have used the identity \[341\] from the appendix. This quantity is used for calculations in the next section.

It is convenient to introduce the function

\[S(u) = S(u|\tau) := \log \frac{\theta_1(u|\tau)}{\theta_4(u|\tau)} \]

(up to a constant, it is logarithm of the elliptic sinus function \(sn\)). It has the quasi-periodicity properties \(S(u + 1) = S(u) + i\pi\), \(S(u + \tau) = S(u)\). Its \(u\)-derivative is given by

\[S'(u) = \partial_u S(u|\tau) = \pi \theta_1^2(0) \frac{\theta_2(u) \theta_3(u)}{\theta_1(u) \theta_2(u)}.\]

From \[10\] it follows that

\[R = e^{S(\eta)}, \quad \frac{C}{R} = \frac{2S'(\eta)}{\pi \theta_2(0) \theta_3(0)}.\]

Taking logarithms of equations \[12\] and applying \(\partial_{\theta_0}\) and \(\partial_{\bar{\theta}_0}\), we represent the dPfaff-Toda hierarchy in the following form:

\[\nabla (z_1) S(u(z_2)) = \partial_{\theta_0} S(u(z_1) - u(z_2)), \quad \nabla (z_1) S(u(z_2) + \eta) = \partial_{\bar{\theta}_0} S(u(z_1) - u(z_2)),\]

\[\bar{\nabla} (\bar{z}_1) S(u(z_2)) = \partial_{\theta_0} S(\bar{u}(\bar{z}_1) + u(z_2) + \eta), \quad \bar{\nabla} (\bar{z}_1) S(u(z_2) + \eta) = \partial_{\bar{\theta}_0} S(\bar{u}(\bar{z}_1) + u(z_2) + \eta).\]
together with complex conjugate equations. In particular, in the limit 
\( z_2 \to \infty \) we have
\[
\nabla(z) \log R = \partial_{\bar{z}} S(u(z)) = \partial_{\bar{z}} S(u(z) + \eta).
\]

Equations (16) is the starting point for investigating one-variable reductions.

3 One-variable reductions

One may look for solutions of the dPfaff-Toda hierarchy such that

\[ u(z, t), \eta(t) \text{ and } \tau(t) \]

depend on the times through a single variable \( \lambda = \lambda(t) \):

\[ u(z, t) = u(z, \lambda(t)), \quad \eta(t) = \eta(\lambda(t)), \quad \tau(t) = \tau(\lambda(t)). \]

Such solutions are called one-variable reductions. Our goal is to characterize the class of functions \( u(z, \lambda), \eta(\lambda), \tau(\lambda) \) that are consistent with the hierarchy. For simplicity, in what follows we put \( \lambda = \tau \).

In this section we use the notation
\[
E^{(a)}(u) = E^{(a)}(u|\tau) = \partial_{\bar{u}} \log \theta(u|\tau)
\]

for logarithmic derivatives of the theta functions. For brevity we also set
\[
E(u) := E^{(1)}(u|\tau) + E^{(4)}(u|\tau) = E^{(1)}(u|\frac{\tau}{2}).
\]

For partial \( \tau \)-derivatives of the \( S \)-function we write \( \dot{S}(u) = \partial_\tau S(u|\tau) \). In the case when the argument of the \( S \)-function depends on \( \tau \) the full \( \tau \)-derivative is given by
\[
\frac{dS(u)}{d\tau} = S'(u) \partial_\tau u + \dot{S}(u).
\]

In what follows we need the identities
\[
4\pi i \dot{S}(u) = 2S'(u) E^{(2)}(u) + \pi^2 \theta_4^4(0)
\]

and
\[
S'(x_1 - x_2) \left( -E(x_1) + E(x_2) + 2E^{(2)}(x_1 - x_2) \right) + \pi^2 \theta_4^4(0) = S'(x_1) S'(x_2).
\]

These identities are proved in [3]. Plugging (18) into (17), we have
\[
4\pi i \frac{dS(u)}{d\tau} = S'(u) \left( 4\pi i \partial_\tau u + 2E^{(2)}(u) \right) + \pi^2 \theta_4^4(0).
\]

Assuming the one-variable reduction, one can see that after the substitutions
\[
\begin{align*}
4\pi i \partial_\tau \eta &= E(\xi - \eta) - E(\xi) \\
4\pi i \partial_\tau u &= -E(u + \xi) + E(\xi) \\
4\pi i \partial_{\bar{\tau}} \bar{u} &= -E(\bar{u} + \bar{\xi}) + E(\bar{\xi})
\end{align*}
\]

with the condition
\[
\xi + \bar{\xi} = \eta
\]
appendix. As a result, we obtain

\[ \xi(\tau) = \frac{\eta(\tau)}{2} + i\kappa(\tau), \quad \bar{\xi}(\tau) = \frac{\eta(\tau)}{2} - i\kappa(\tau), \]

(23)

where \( \kappa(\tau) \) is an arbitrary real-valued function which plays the role of the “driving function”. Then the equations (21) acquire the form

\[
\begin{align*}
4\pi i \partial_t \eta(\tau) &= -E\left(\frac{\eta}{2} + i\kappa\right) - E\left(\frac{\eta}{2} - i\kappa\right) \\
4\pi i \partial_t u(z, \tau) &= -E(u + \frac{\eta}{2} + i\kappa) + E\left(\frac{\eta}{2} + i\kappa\right) \\
4\pi i \partial_t \bar{u}(z, \tau) &= -E(u + \frac{\eta}{2} - i\kappa) + E\left(\frac{\eta}{2} - i\kappa\right)
\end{align*}
\]

(24)

These equations are sufficient conditions for the functions \( u(z, \tau), \bar{u}(z, \tau) \) and \( \eta(\tau) \) to be compatible with the infinite dPfaff-Toda hierarchy. Given \( \kappa(\tau) \), one should solve the first equation for \( \eta(\tau) \) and substitute it to the other two equations.

Equations of the reduced hierarchy are written for the dependent variable \( \tau \). In order to obtain them, we need the relations

\[
\nabla(z)\tau = \frac{d\tau}{d\log \rho} \nabla(z) \log \rho = \frac{dS(u(z))}{d\log \rho} \partial_u \tau,
\]

(25)

\[
\nabla(\bar{z})\tau = \frac{d\tau}{d\log R} \nabla(\bar{z}) \log R = \frac{dS(\bar{u}(\bar{z}))}{d\log \rho} \partial_u \tau
\]

(26)

which are easily obtained using the chain rule of differentiating. Their right hand sides can be further transformed with the help of formulas (36), (37), (39), (41) from the appendix. As a result, we obtain

\[
\nabla(z)\tau = \frac{S'(u(z) + \xi)}{S'(\xi)} \partial_u \tau, \quad \nabla(\bar{z})\tau = -\frac{S'(\bar{u}(\bar{z}) + \bar{\xi})}{S'(\xi)} \partial_u \tau.
\]

(27)

These are the generating equations for the infinite reduced hierarchy of equations of hydrodynamic type. In order to write them explicitly, we employ the expansion

\[ S(u(z) + v) = S'(v) + \sum_{k \geq 1} \frac{z^{-k}}{k} B_k'(v), \quad k \geq 1. \]

(28)

(The functions \( B_k = B_k(v|\tau) \) are elliptic analogues of the Faber polynomials, \( B_k'(v) = \partial_v B_k(v) \).) Then the equations are as follows:

\[
\frac{\partial \tau}{\partial t_k} = \phi_k(\xi(\tau)|\tau) \frac{\partial \tau}{\partial t_0}, \quad \frac{\partial \tau}{\partial t_k} = \psi_k(\xi(\tau)|\tau) \frac{\partial \tau}{\partial t_0},
\]

(29)

where

\[
\phi_k(\xi(\tau)|\tau) = \frac{B_k'(\xi(\tau)|\tau)}{S'(\xi(\tau)|\tau)}, \quad \psi_k(\xi(\tau)|\tau) = -\frac{\bar{B}_k'(\bar{\xi}(\tau)|\tau)}{S'(\bar{\xi}(\tau)|\tau)}.
\]

(30)
Formally we can extend this system to the value \( k = 0 \) by setting \( B'_0(u) = S'(u) \). At \( k = 0 \) we get the equation
\[
S'(_{\xi}\partial_{\tau}) + S'(_{\xi}\partial_{\tau}) = 0.
\] (31)

The common solution to these equations can be represented in the hodograph form:
\[
\sum_{k \geq 1} t_k\phi_k(_{\xi}(\tau)) + \sum_{k \geq 0} \bar{t}_k\psi_k(_{\xi}(\tau)) = \Phi(\tau).
\] (32)

Here \( \Phi \) is an arbitrary function of \( \tau \).

4 Appendix

Theta functions

The Jacobi’s theta functions \( \theta_a(u) = \theta_a(u|\tau), a = 1, 2, 3, 4 \), are defined by the formulas
\[
\begin{align*}
\theta_1(u) &= -\sum_{k \in \mathbb{Z}} \exp \left( \pi i \tau (k + \frac{1}{2})^2 + 2\pi i (u + \frac{1}{2})(k + \frac{1}{2}) \right), \\
\theta_2(u) &= \sum_{k \in \mathbb{Z}} \exp \left( \pi i \tau (k + \frac{1}{2})^2 + 2\pi i u(k + \frac{1}{2}) \right), \\
\theta_3(u) &= \sum_{k \in \mathbb{Z}} \exp \left( \pi i \tau k^2 + 2\pi i u k \right), \\
\theta_4(u) &= \sum_{k \in \mathbb{Z}} \exp \left( \pi i \tau k^2 + 2\pi i (u + \frac{1}{2}) k \right)
\end{align*}
\] (33)

with \( \text{Im} \tau > 0 \). The function \( \theta_1(u) \) is odd, the other three functions are even. Shifts by the half-periods relate the different theta functions to each other. We also mention the identity
\[
\theta_1'(0) = \pi \theta_2(0) \theta_3(0) \theta_4(0).
\] (34)

Many useful identities for the theta functions can be found in [22].

Some details of the calculations

Let us start with the first equation in (16),
\[
\nabla(z_1)S\left(u(z_2)\right) = \partial_{\tau}S\left(u(z_1) - u(z_2)\right).
\]
Applying the chain rule of differentiating we can write its left hand side as
\[
\nabla(z_1)S(u_2) = \nabla(z_1)\tau \frac{dS(u_2)}{d\tau} = \nabla(z_1)\log\rho \left( \frac{d\log\rho}{d\tau} \right)^{-1} \frac{dS(u_2)}{d\tau}
\]
(where \( \rho \) is defined in (14) and we have put \( u_i \equiv u(z_i) \) for brevity). In its turn, \( \nabla(z_1)\log\rho \) can be found from the \( z_2 \to \infty \) limit of the first equation in (16):
\[
\nabla(z_1)\log\rho = \partial_{\tau}S(u_1) = \partial_{\tau}\tau \frac{dS(u_1)}{d\tau}.
\]
Therefore, the left hand side is
\[ \nabla(z_1)S(u_2) = \partial_\tau \frac{dS(u_1)}{d\tau} \frac{dS(u_2)}{d\tau} \left( \frac{d\log \rho}{d\tau} \right)^{-1}. \]

The right hand side of the first equation in (16) reads
\[ \partial_\tau S(u_1 - u_2) = \partial_\tau \frac{dS(u_1 - u_2)}{d\tau}. \]

Assuming that \( \partial_\tau \neq 0 \) we see that the first equation in (16) becomes
\[ \frac{dS(u_1)}{d\tau} \frac{dS(u_2)}{d\tau} = \frac{d\log \rho}{d\tau} \frac{dS(u_1 - u_2)}{d\tau}. \] (35)

Each full \( \tau \)-derivative here can be further transformed with the help of (20). Substituting \( \partial_\tau u \) from (21) and using identity (19), we can prove that
\[ 4\pi i \frac{dS(u(z))}{d\tau} = S'(u(z) + \xi)S'(\xi) \] (36)
and, in particular (at \( z \to \infty \)),
\[ 4\pi i \frac{d\log \rho}{d\tau} = (S'(\xi))^2. \] (37)

Therefore, we see that equation (35) is identically satisfied:
\[ S'(u_1 + \xi)S'(\xi) \cdot S'(u_2 + \xi)S'(\xi) = (S'(\xi))^2 \cdot S'(u_1 + \xi)S'(u_2 + \xi). \]

The other equations in (16) can be processed in a similar way. Consider the second equation,
\[ \nabla(z_1)S\left(u(z_2) + \eta\right) = \partial_\tau S\left(u(z_1) - u(z_2)\right). \]

In the left hand side we have:
\[ \nabla(z_1)S(u_2 + \eta) = \partial_\tau \frac{dS(u_2 + \eta)}{d\tau} \]
\[ = \nabla(z_1) \log R \frac{dS(u_2 + \eta)}{d\tau} \left( \frac{d\log R}{d\tau} \right)^{-1} \]
\[ = \partial_\tau S(u_1) \frac{dS(u_2 + \eta)}{d\tau} \left( \frac{d\log R}{d\tau} \right)^{-1} \]
\[ = \partial_\tau \frac{dS(u_1)}{d\tau} \frac{dS(u_2 + \eta)}{d\tau} \left( \frac{d\log R}{d\tau} \right)^{-1}. \]

The right hand side is
\[ \partial_\tau S(u_1 - u_2) = \partial_\tau \frac{dS(u_1 - u_2)}{d\tau}. \]
Recall also that \( \log R = S(\eta) \). Therefore, the second equation in (16) becomes

\[
\frac{dS(u_1)}{d\tau} \frac{dS(u_2 + \eta)}{d\tau} = \frac{dS(\eta)}{d\tau} \frac{dS(u_1 - u_2)}{d\tau}.
\]

Again, substituting \( \partial_\tau u \) from (21) and using identity (19), we can prove that

\[
4\pi i \frac{dS(u_2 + \eta)}{d\tau} = S'(u_2 + \xi)S'(\xi - \eta),
\]

\[
4\pi i \frac{dS(u_1 - u_2)}{d\tau} = S'(u_1 + \xi)S'(u_2 + \xi),
\]

and, in particular (at \( z \to \infty \) in (39)),

\[
4\pi i \frac{dS(\eta)}{d\tau} = S'(\xi)S'(\xi - \eta) = S'(\bar{\xi})S'(\bar{\xi} - \eta)
\]

(the last equality holds because \( \xi + \bar{\xi} = \eta \)). Therefore, we see that equation (38) is identically satisfied:

\[
S'(u_1 + \xi)S'(\xi) \cdot S'(u_2 + \xi)S'(\xi - \eta) = S'(\xi)S'(\xi - \eta) \cdot S'(u_1 + \xi)S'(u_2 + \xi).
\]

Now consider the third equation in (16),

\[
\bar{\nabla}(\bar{z}_1)S(u(z_2)) = \partial_{\bar{t}_0}S(\bar{u}(\bar{z}_1) + u(z_2) + \eta).
\]

Its left hand side is

\[
\bar{\nabla}(\bar{z}_1)S(u_2) = \bar{\nabla}(\bar{z}_1) \tau \frac{dS(u_2)}{d\tau}
\]

\[
= \bar{\nabla}(\bar{z}_1) \log R \frac{dS(u_2)}{d\tau} \left( \frac{d \log R}{d\tau} \right)^{-1}
\]

\[
= \partial_{\bar{t}_0}S(\bar{u}_1) \frac{dS(u_2)}{d\tau} \left( \frac{d \log R}{d\tau} \right)^{-1}
\]

\[
= \partial_{\bar{t}_0} \tau \frac{dS(\bar{u}_1)}{d\tau} \frac{dS(u_2)}{d\tau} \left( \frac{d \log R}{d\tau} \right)^{-1}.
\]

When passing from the second line to the third one we have used the complex conjugate of the second equation in (16) in the limit \( z_2 \to \infty \). The right hand side is

\[
\partial_{\bar{t}_0}S(\bar{u}_1 + u_2 + \eta) = \partial_{\bar{t}_0} \tau \frac{dS(\bar{u}_1 + u_2 + \eta)}{d\tau}.
\]

The third equation in (16) becomes

\[
\frac{dS(\bar{u}_1)}{d\tau} \frac{dS(u_2)}{d\tau} = \frac{dS(\eta)}{d\tau} \frac{dS(\bar{u}_1 + u_2 + \eta)}{d\tau}.
\]

(42)
Substituting here \( \partial_\tau u \) from (21) and using identity (19), we can prove that
\[
4\pi i \frac{dS(\bar{u}(\bar{z}))}{d\tau} = S'(\bar{u}(\bar{z}) + \bar{\xi})S'(\bar{\xi})
\] (43)
and
\[
4\pi i \frac{dS(\bar{u}_1 + u_2 + \eta)}{d\tau} = S'(\bar{u}_1 + \bar{\xi})S'(-u_2 - \xi).
\] (44)
Therefore, we see that equation (42) is identically satisfied:
\[
S'(\bar{u}_1 + \bar{\xi})S'(\bar{\xi}) \cdot S'(u_2 + \xi)S'(\xi) = S'(\xi)S'(\xi - \eta) \cdot S'(\bar{u}_1 + \bar{\xi})S'(-u_2 - \xi)
\]
(we use the fact that \( S'(u) \) is odd function and the constraint \( \xi + \bar{\xi} = \eta \)).

The calculations for the remaining case of the fourth equation in (16) are similar.

Acknowledgements

We thank T. Takebe for discussions. This work has been funded by the Russian Academic Excellence Project “5-100”. Results of section 3 has been obtained under support of the RSF grant 16-11-10160. Research of both authors has also been supported by RFBR grant 14-02-00627. The work of A.Z. has also been partially supported by joint RFBR grant 15-52-50041-YaF.

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