FINITE SETS WITH FAKE OBSERVABLE CARDINALITY

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Abstract. Let $X$ be a compact metric space and let $|A|$ denote the cardinality of a set $A$. We prove that if $f : X \to X$ is a homeomorphism and $|X| = \infty$, then for all $\delta > 0$ there is $A \subset X$ such that $|A| = 4$ and for all $k \in \mathbb{Z}$ there are $x, y \in f^k(A)$, $x \neq y$, such that $\text{dist}(x, y) < \delta$. An observer that can only distinguish two points if their distance is greater than $\delta$, for sure will say that $A$ has at most 3 points even knowing every iterate of $A$ and that $f$ is a homeomorphism. We show that for hyper-expansive homeomorphisms the same $\delta$-observer will not fail about the cardinality of $A$ if we start with $|A| = 3$ instead of 4. Generalizations of this problem are considered via what we call $(m, n)$-expansiveness.

Introduction

Since 1950, when Utz [16] initiated the study of expansive homeomorphism, several variations of the definition appeared in the literature. Let us recall that a homeomorphism $f : X \to X$ of a compact metric space $(X, \text{dist})$ is expansive if there is an expansive constant $\delta > 0$ such that if $x \neq y$, then $\text{dist}(f^k(x), f^k(y)) > \delta$ for some $k \in \mathbb{Z}$. Some variations of this definition are weaker, as for example continuum-wise expansiveness [6] and $N$-expansiveness [9] (see also [3,8,13]). A branch of research in topological dynamics investigates the possibility of extending known results for expansive homeomorphisms to these versions. See for example [2,5,10,12,14].

Other related definitions are stronger than expansiveness as for example positive expansiveness [15] and hyper-expansiveness [1]. Both definitions are so strong that their examples are almost trivial. It is known [15] that if a compact metric space admits a positive expansive homeomorphism, then the space has only a finite number of points. Recall that $f : X \to X$ is positive expansive if there is $\delta > 0$ such that if $x \neq y$, then $\text{dist}(f^k(x), f^k(y)) > \delta$ for some $k \geq 0$. Therefore, we have that if the compact metric space $X$ is not a finite set, then for every homeomorphism $f : X \to X$ and for all $\delta > 0$ there are $x \neq y$ such that $\text{dist}(f^k(x), f^k(y)) < \delta$ for all $k \geq 0$. This is a very general result about the dynamics of homeomorphisms of compact metric spaces.
Another example of this phenomenon is given in [1], where it is proved that no uncountable compact metric space admits a hyper-expansive homeomorphism (see Definition 3). Therefore, if $X$ is an uncountable compact metric space, as for example a compact manifold, then for every homeomorphism $f: X \to X$ and for all $\delta > 0$ there are two compact subsets $A, B \subset X$, $A \neq B$, such that $\text{dist}_H(f^k(A), f^k(B)) < \delta$ for all $k \in \mathbb{Z}$. The distance $\text{dist}_H$ is called Hausdorff metric and its definition is recalled in equation (3) below.

According to Lewowicz [7] we can explain the meaning of expansiveness as follows. Let us say that a $\delta$-observer is someone that cannot distinguish two points if their distance is smaller than $\delta$. If $\text{dist}(x, y) < \delta$ a $\delta$-observer will not be able to say that the set $A = \{x, y\}$ has two points. But if the homeomorphism is expansive, with expansive constant greater than $\delta$, and if the $\delta$-observer knows all of the iterates $f^k(A)$ with $k \in \mathbb{Z}$, then he will find that $A$ contains two different points, because if $\text{dist}(f^k(x), f^k(y)) > \delta$, then he will see two points in $f^k(A)$. Let us be more precise.

**Definition 1.** For $\delta \geq 0$, a set $A \subset X$ is $\delta$-separated if for all $x \neq y, x, y \in A$, it holds that $\text{dist}(x, y) > \delta$. The $\delta$-cardinality of a set $A$ is

$$|A|_\delta = \sup\{|B| : B \subset A \text{ and } B \text{ is } \delta\text{-separated} \},$$

where $|B|$ denotes the cardinality of the set $B$.

Notice that the $\delta$-cardinality is always finite because $X$ is compact. The $\delta$-cardinality of a set represents the maximum number of different points that a $\delta$-observer can identify in the set.

In this paper we introduce a series of definitions, some weaker and other stronger than expansiveness, extending the notion of $N$-expansiveness of [9]. Let us recall that given $N \geq 1$, a homeomorphism is $N$-expansive if there is $\delta > 0$ such that if $\text{diam}(f^k(A)) < \delta$ for all $k \in \mathbb{Z}$, then $|A| \leq N$. In terms of our $\delta$-observer we can say that $f$ is $N$-expansive if there is $\delta > 0$ such that if $|A| = N + 1$, a $\delta$-observer will be able to say that $A$ has at least two points given that he knows all of the iterates $f^k(A)$ for $k \in \mathbb{Z}$, i.e., $|f^k(A)|_\delta > 1$ for some $k \in \mathbb{Z}$. Let us introduce our main definition.

**Definition 2.** Given integer numbers $m > n \geq 1$ we say that $f: X \to X$ is $(m, n)$-expansive if there is $\delta > 0$ such that if $|A| = m$, then there is $k \in \mathbb{Z}$ such that $|f^k(A)|_\delta > n$.

The first problem under study is the classification of these definitions. We prove that $(m, n)$-expansiveness implies $N$-expansiveness if $m \leq (N + 1)n$. In particular, if $m \leq 2n$, then $(m, n)$-expansiveness implies expansiveness. These results are stated in Corollary 1.7. It is known that even on surfaces, $N$-expansiveness does not imply expansiveness for $N \geq 2$, see [2]. Here we show that $(m, n)$-expansiveness does not imply expansiveness if $n \geq 2$. For example, Anosov diffeomorphisms are known to be expansive and a consequence of Theorem 5.1 is that Anosov diffeomorphisms are not $(m, n)$-expansive for all $n \geq 2$. 

It is a fundamental problem in dynamical systems to determine which spaces admit expansive homeomorphisms (or Anosov diffeomorphisms). In this paper we prove that no Peano continuum admits a \((m, n)\)-expansive homeomorphism if \(2m \geq 3n\), see Theorem 3.2. We also show that if \(X\) admits a \((n + 1, n)\)-expansive homeomorphism with \(n \geq 3\), then \(X\) is a finite set. Examples of \((3, 2)\)-expansive homeomorphisms are given on countable spaces (hyper-expansive homeomorphisms), see Theorem 4.1.

The article is organized as follows. In Section 1 we prove basic properties of \((m, n)\)-expansive homeomorphisms. In Section 2 we prove the first statement of the abstract, i.e., no infinite compact metric space admits a \((4, 3)\)-expansive homeomorphism. In Section 3 we show that no Peano continuum admits a \((m, n)\)-expansive homeomorphism if \(2m \geq 3n\). In Section 4 we show that hyper-expansive homeomorphisms are \((3, 2)\)-expansive. Such homeomorphisms are defined on compact metric spaces with a countable number of points. In Section 5 we prove that a homeomorphism with the shadowing property and with two points \(x, y\) satisfying

\[
0 = \liminf_{k \to \infty} \text{dist}(f^k(x), f^k(y)) < \limsup_{k \to \infty} \text{dist}(f^k(x), f^k(y))
\]

cannot be \((m, 2)\)-expansive if \(m > 2\).

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1. Separating finite sets

Let \((X, \text{dist})\) be a compact metric space and consider a homeomorphism \(f : X \to X\). Let us recall that for integer numbers \(m > n \geq 1\) a homeomorphism \(f\) is \((m, n)\)-expansive if there is \(\delta > 0\) such that if \(|A| = m\), then there is \(k \in \mathbb{Z}\) such that \(|f^k(A)|_\delta > n\). In this case we say that \(\delta\) is a \((m, n)\)-expansive constant. The idea of \((m, n)\)-expansiveness is that our \(\delta\)-observer will find more than \(n\) points in every set of \(m\) points if he knows all of its iterates.

Remark 1.1. From the definitions it follows that a homeomorphisms is \((N + 1, 1)\)-expansive if and only if it is \(N\)-expansive in the sense of [9]. In particular, \((2, 1)\)-expansiveness is equivalent with expansiveness.

Remark 1.2. Notice that if \(X\) is a finite set, then every homeomorphism of \(X\) is \((m, n)\)-expansive.

Proposition 1.3. If \(n' \leq n\) and \(m - n \leq m' - n'\), then \((m, n)\)-expansive implies \((m', n')\)-expansive with the same expansive constant.

Proof. The case \(|X| < \infty\) is trivial, so, let us assume that \(|X| = \infty\). Consider \(\delta > 0\) as a \((m, n)\)-expansive constant. Given a set \(A\) with \(|A| = m'\) we will show that there is \(k \in \mathbb{Z}\) such that \(|f^k(A)|_\delta > n'\), i.e., the same expansive constant works. We divide the proof in two cases.
First assume that $m' \geq m$. Let $B \subset A$ with $|B| = m$. Since $f$ is $(m, n)$-expansive, there is $k \in \mathbb{Z}$ such that $|f^k(B)|_\delta > n$. Therefore $|f^k(A)|_\delta > n \geq n'$, proving that $f$ is $(m', n')$-expansive.

Now suppose that $m' < m$. Given that $|A| = m'$ and $|X| = \infty$ there is $C \subset X$ such that $A \cap C = \emptyset$ and $|A \cup C| = m$. By $(m, n)$-expansiveness, there is $k \in \mathbb{Z}$ such that $|f^k(A \cup C)|_\delta > n$. Then, there is a $\delta$-separated set $D \subset f^k(A \cup C)$ with $|D| > n$. Notice that $|f^k(A) \cap D| = |A| \delta + \epsilon \leq |A| \delta + |B| \delta - |A \cap B|$. 

As a consequence of Proposition 1.3 we have that

1. $(m, n)$-expansive implies $(m + 1, n)$-expansive and
2. $(m, n)$-expansive implies $(m - 1, n - 1)$-expansive.

In Table 1 below we can easily see all these implications. The following proposition allows us to draw more arrows in this table, for example: $(4, 2) \Rightarrow (2, 1)$.

| Table 1. Basic hierarchy of $(m, n)$-expansiveness. Each pair $(m, n)$ in the table stands for “$(m, n)$-expansive”. In the first position, $(2, 1)$, we have expansiveness. The first line, of the form $(N + 1, 1)$, we have $N$-expansive homeomorphisms. |
| --- |
| $(2, 1)$ $\Rightarrow$ $(3, 1)$ $\Rightarrow$ $(4, 1)$ $\Rightarrow$ ... |
| $\uparrow$ $\uparrow$ $\uparrow$ |
| $(3, 2)$ $\Rightarrow$ $(4, 2)$ $\Rightarrow$ $(5, 2)$ $\Rightarrow$ ... |
| $\uparrow$ $\uparrow$ $\uparrow$ |
| $(4, 3)$ $\Rightarrow$ $(5, 3)$ $\Rightarrow$ $(6, 3)$ $\Rightarrow$ ... |
| $\uparrow$ $\uparrow$ $\uparrow$ |
| ... ... ... |

Proposition 1.4. If $a, n \geq 2$ and $f : X \rightarrow X$ is an $(an, n)$-expansive homeomorphism, then $f$ is $(a, 1)$-expansive.

In order to prove it, let us introduce two previous results.

Lemma 1.5. If $A, B \subset X$ are finite sets and $\delta > 0$ satisfies $|A| = |A|_\delta$ and $|B|_\delta = 1$, then for all $\epsilon > 0$ it holds that

$$|A \cup B|_{\delta + \epsilon} \leq |A|_{\epsilon} + |B|_{\delta} - |A \cap B|.$$ 

Proof. If $A \cap B = \emptyset$, then the proof is easy because

$$|A \cup B|_{\delta + \epsilon} \leq |A|_{\delta + \epsilon} + |B|_{\delta + \epsilon} \leq |A|_{\epsilon} + |B|_{\delta}.$$
Proposition 1.4 we have that $f$ is expansive. Since $|A| = |A|_\delta$, we have that $A$ is $\delta$-separated. Therefore $|A \cap B| = 1$ because $|B|_\delta = 1$. Assume that $A \cap B = \{y\}$. Let us prove that $|A \cup B|_{\delta + \varepsilon} \leq |A|_\varepsilon$ and notice that it is sufficient to conclude the proof of the lemma.

Let $C \subset A \cup B$ be a $(\delta + \varepsilon)$-separated set such that $|C| = |A \cup B|_{\delta + \varepsilon}$. If $C \subset A$, then

$$|A \cup B|_{\delta + \varepsilon} = |A|_{\delta + \varepsilon} \leq |A|_\varepsilon.$$ 

Therefore, let us assume that there is $x \in C \setminus A$. Define the set

$$D = (C \cup \{y\}) \setminus \{x\}.$$

Notice that $|C| = |D|$ and $D \subset A$.

We will show that $D$ is $\varepsilon$-separated. Take $p, q \in D$ and arguing by contradiction assume that $p \neq q$ and $\text{dist}(p, q) \leq \varepsilon$. If $p, q \in C$ there is nothing to prove because $C$ is $(\delta + \varepsilon)$-separated. Assume now that $p = y$. We have that $\text{dist}(x, p) \leq \delta$ because $x, p \in B$ and $|B|_\delta = 1$. Thus

$$\text{dist}(x, q) \leq \text{dist}(x, p) + \text{dist}(p, q) \leq \varepsilon + \delta.$$ 

But this is a contradiction because $x, q \in C$ and $C$ is $(\varepsilon + \delta)$-separated. □

**Lemma 1.6.** If $f$ is $(m + l, n + 1)$-expansive, then $f$ is $(m, n)$-expansive or $(l, 1)$-expansive.

**Proof.** Let us argue by contradiction and take an $(m + l, n + 1)$-expansive constant $\alpha > 0$. Since $f$ is not $(m, n)$-expansive for $\varepsilon \in (0, \alpha)$ there is a set $A \subset X$ such that $|A| = m$ and $|f^k(A)|_\varepsilon \leq n$ for all $k \in \mathbb{Z}$. Take $\delta > 0$ such that $|A| = |A|_\delta$ and $\delta + \varepsilon < \alpha$.

Since $f$ is not $(l, 1)$-expansive there is $B$ such that $|B| = l$ and $|f^k(B)|_\delta = 1$ for all $k \in \mathbb{Z}$. By Lemma 1.5 we have that

$$|f^k(A \cup B)|_{\delta + \varepsilon} \leq |f^k(A)|_\varepsilon + |f^k(B)|_\delta - |A \cap B| \leq n + 1 - |A \cap B|$$

for all $k \in \mathbb{Z}$. Also, we know that $|A \cup B| = m + l - |A \cap B|$. If we denote $r = |A \cap B|$, then $f$ is not $(m + l - r, n + 1 - r)$-expansive. And by Proposition 1.3 we conclude that $f$ is not $(m + l, n + 1)$-expansive. This contradiction proves the lemma. □

**Proof of Proposition 1.4.** Assume by contradiction that $f$ is not $(a, 1)$-expansive. Since $f$ is $(an, n)$-expansive, by Lemma 1.6 we have that $f$ has to be $(a(n - 1), n - 1)$-expansive. Arguing inductively we can prove that $f$ is $(a(n - j), n - j)$-expansive for $j = 1, 2, \ldots, n - 1$. In particular, $f$ is $(a, 1)$-expansive, which is a contradiction that proves the proposition. □

**Corollary 1.7.** If $m \leq an$ and $f$ is $(m, n)$-expansive, then $f$ is $(a, 1)$-expansive (i.e., $(a - 1)$-expansive in the sense of [9]). In particular, if $m \leq 2n$ and $f$ is $(m, n)$-expansive, then $f$ is expansive.

**Proof.** By Proposition 1.3 we have that $f$ is $(an, n)$-expansive. Therefore, by Proposition 1.4 we have that $f$ is $(a, 1)$-expansive. □
2. Separating 4 points

In this section we prove that \((n+1, n)\)-expansiveness with \(n \geq 3\) implies that \(X\) is finite.

**Theorem 2.1.** If \(X\) is a compact metric space admitting a \((4, 3)\)-expansive homeomorphism, then \(X\) is a finite set.

**Proof.** By contradiction assume that \(f\) is a \((4, 3)\)-expansive homeomorphism of \(X\) with \(|X| = \infty\) and take an expansive constant \(\delta > 0\). We know that \(f\) cannot be positive expansive (see \([4, 7]\) for a proof). Therefore there are \(x_1, x_2\) such that

\[
\text{dist}(f^k(x_1), f^k(x_2)) < \delta
\]

for all \(k \geq 0\). Analogously, \(f^{-1}\) is not positive expansive, and we can take \(y_1, y_2\) such that

\[
\text{dist}(f^k(y_1), f^k(y_2)) < \delta
\]

for all \(k \leq 0\). Consider the set

\[A = \{x_1, x_2, y_1, y_2\}.
\]

We have that \(2 \leq |A| \leq 4\) (we do not know if the 4 points are different). By inequalities (1) and (2) we have that \(|f^k(A)|_\delta < |A|\) for all \(k \in \mathbb{Z}\). If \(n = |A|\), then we have that \(f\) is not \((n, n-1)\)-expansive. In any case, \(n = 2, 3\) or \(4\), by Proposition 1.3 (see Table 1) we conclude that \(f\) is not \((4, 3)\)-expansive. This contradiction finishes the proof. \(\square\)

**Remark 2.2.** If \(X\) is a compact metric space admitting a \((n+1, n)\)-expansive homeomorphism with \(n \geq 3\), then \(X\) is a finite set. It follows by Theorem 2.1 and Proposition 1.3.

**Corollary 2.3.** If \(f : X \to X\) is a homeomorphism of a compact metric space and \(|X| = \infty\), then for all \(\delta > 0\) and \(m \geq 4\) there is \(A \subset X\) with \(|A| = m\) such that \(|f^k(A)|_\delta < |A|\) for all \(k \in \mathbb{Z}\).

**Proof.** It is just a restatement of Remark 2.2. \(\square\)

3. On Peano continua

In this section we study \((m, n)\)-expansiveness on Peano continua. Let us start recalling that a **continuum** is a compact connected metric space and a **Peano continuum** is a locally connected continuum. A singleton space \(|X| = 1\) is a **trivial** Peano continuum. For \(x \in X\) and \(\delta > 0\) define the **stable** and **unstable** set of \(x\) as

\[
W^s_\delta(x) = \{y \in X : \text{dist}(f^k(x), f^k(y)) \leq \delta \ \forall \ k \geq 0\},
\]

\[
W^u_\delta(x) = \{y \in X : \text{dist}(f^k(x), f^k(y)) \leq \delta \ \forall \ k \leq 0\}.
\]

**Remark 3.1.** Notice that \((m, n)\)-expansiveness implies continuum-wise expansiveness for all \(m > n \geq 1\). Recall that \(f\) is **continuum-wise expansive** if there is \(\delta > 0\) such that if \(\text{diam}(f^k(A)) < \delta\) for all \(k \in \mathbb{Z}\) and some continuum \(A \subset X\), then \(|A| = 1\).
Theorem 3.2. If $X$ is a non-trivial Peano continuum, then no homeomorphism of $X$ is $(m,n)$-expansive if $2m \geq 3n$.

Proof. Let $\delta$ be a positive real number and assume that $f$ is $(m,n)$-expansive. As we remarked above, $f$ is a continuum-wise expansive homeomorphism. It is known (see [5, 6]) that for such homeomorphisms on a Peano continuum, every point has non-trivial stable and unstable sets. Take $n$ different points $x_1, \ldots, x_n \in X$ and let $\delta' \in (0,\delta)$ be such that $\text{dist}(x_i, x_j) > 2\delta'$ if $i \neq j$. For each $i = 1, \ldots, n$, we can take $y_i \in W^s_\delta(x_i)$ and $z_i \in W^u_\delta(x_i)$ with $x_i \neq y_i$ and $x_i \neq z_i$. Consider the set $A = \{x_1, y_1, z_1, \ldots, x_n, y_n, z_n\}$. Since $\text{dist}(x_i, x_j) > 2\delta'$ if $i \neq j$, and $y_i, z_i \in B_{\delta'}(x_i)$ we have that $|A| = 3n$. If $A_i$ denotes the set $\{x_i, y_i, z_i\}$ we have that $|f^k(A_i)|_{\delta'} \leq 2$ for all $k \in \mathbb{Z}$. This is because if $k \geq 0$, then $\text{dist}(f^k(x_i), f^k(y_i)) \leq \delta'$ and if $k \leq 0$, then $\text{dist}(f^k(x_i), f^k(z_i)) \leq \delta'$. Therefore $|f^k(A)|_{\delta'} \leq 2n$, and then $|f^k(A)|_{\delta} \leq 2n$. Since $\delta > 0$ and $n \geq 1$ are arbitrary, we have that $f$ is not $(3n, 2n)$-expansive for all $n \geq 1$. Finally, by Proposition 1.3, we have that $f$ is not $(m,n)$-expansive if $2m \geq 3n$. □

Corollary 3.3. If $f : X \to X$ is a homeomorphism and $X$ is a non-trivial Peano continuum, then for all $\delta > 0$ there is $A \subset X$ such that $|A| = 3$ and $|f^k(A)|_{\delta} \leq 2$ for all $k \in \mathbb{Z}$.

Proof. By Theorem 3.2 we know that $f$ is not $(3,2)$-expansive. Therefore, the proof follows by definition. □

4. Hyper-expansive homeomorphisms

Denote by $\mathcal{K}(X)$ the set of compact subsets of $X$. This space is usually called as the hyper-space of $X$. We recommend the reader to see [11] for more on the subject of hyper-spaces and the proofs of the results that we will cite below. In the set $\mathcal{K}(X)$ we consider the Hausdorff distance $\text{dist}_H$ making $(\mathcal{K}(X), \text{dist}_H)$ a compact metric space. Recall that

\begin{equation}
\text{dist}_H(A, B) = \inf \{ \varepsilon > 0 : A \subset B_{\varepsilon}(B) \text{ and } B \subset B_{\varepsilon}(A) \},
\end{equation}

where $B_{\varepsilon}(C) = \bigcup_{x \in C} B_{\varepsilon}(x)$ and $B_{\varepsilon}(x)$ is the usual ball of radius $\varepsilon$ centered at $x$. As usual, we let $f$ to act on $\mathcal{K}(X)$ as $f(A) = \{f(a) : a \in A\}$.

Definition 3. We say that $f$ is hyper-expansive if $f : \mathcal{K}(X) \to \mathcal{K}(X)$ is expansive, i.e., there is $\delta > 0$ such that given two compact sets $A, B \subset X$, $A \neq B$, there is $k \in \mathbb{Z}$ such that $\text{dist}_H(f^k(A), f^k(B)) > \delta$ where $\text{dist}_H$ is the Hausdorff distance.

In [1] it is shown that $f : X \to X$ is hyper-expansive if and only if $f$ has a finite number of orbits (i.e., there is a finite set $A \subset X$ such that $X = \bigcup_{k \in \mathbb{Z}} f^k(A)$) and the non-wandering set is a finite union of periodic points which are attractors or repellers. Recall that a point $x$ is in the non-wandering set if for every neighborhood $U$ of $x$ there is $k > 0$ such that $f^k(U) \cap U \neq \emptyset$. A point $x$ is periodic if for some $k \geq 0$ it holds that $f^k(x) = x$. The orbit
\( \gamma = \{x, f(x), \ldots, f^{k-1}(x)\} \) is a periodic orbit if \( x \) is a periodic point. A periodic orbit \( \gamma \) is an attractor (repeller) if there is a compact neighborhood \( U \) of \( \gamma \) such that \( f^k(U) \to \gamma \) in the Hausdorff distance as \( k \to \infty \) (resp. \( k \to -\infty \)).

**Theorem 4.1.** If \( f : X \to X \) is a hyper-expansive homeomorphism and \( |X| = \infty \), then \( f \) is \((m, n)\)-expansive for some \( m > n \geq 1 \) if and only if \( m \leq 3 \).

**Proof.** Let us start with the direct part of the theorem. Let \( P_a \) be the set of periodic attractors, \( P_r \) the set of periodic repellers and take \( x_1, \ldots, x_j \) one point in each wandering orbit. (Recall that, as we said above, hyper-expansiveness implies that \( f \) has just a finite number of orbits.) Define \( Q = \{x_1, \ldots, x_j\} \).

Take \( \delta > 0 \) such that

1. if \( p, q \in P_a \cup P_r \) and \( p \neq q \), then \( \text{dist}(p, q) > \delta \).
2. if \( x_i \in Q \), then \( B_\delta(x_i) = \{x_i\} \) (recall that wandering points are isolated by [1]).
3. if \( p \in P_a \), \( x_i \in Q \) and \( k \leq 0 \), then \( \text{dist}(p, f^k(x_i)) > \delta \).
4. if \( q \in P_r \), \( x_i \in Q \) and \( k \geq 0 \), then \( \text{dist}(p, f^k(x_i)) > \delta \) and
5. if \( x, y \in Q \) and \( k > 0 > l \), then \( \text{dist}(f^k(x), f^l(y)) > \delta \).

Let us prove that such \( \delta \) is a \((3, 2)\)-expansive constant. Take \( a, b, c \in X \) with \( |\{a, b, c\}| = 3 \). The proof is divided by cases:

- If \( a, b, c \in P = P_a \cup P_r \), then item 1 above concludes the proof.
- If \( a, b \in P \) and \( c \notin P \), then there is \( k \in \mathbb{Z} \) such that \( f^k(c) \in Q \). In this case items 1 and 2 conclude the proof.
- Assume now that \( a \in P \) and \( b, c \notin P \). Without loss of generality let us suppose that \( a \) is a repeller. Let \( k_a, k_c \in \mathbb{Z} \) be such that \( f^{k_a}(b), f^{k_c}(c) \in Q \). Define \( k = \min\{k_a, k_c\} \). In this way: \( \text{dist}(f^k(a), f^{k_a}(b)), \text{dist}(f^k(a), f^{k_c}(c)) \geq \delta \) by item 4 and \( \text{dist}(f^{k}(b), f^{k_a}(c)) \geq \delta \) by item 2.
- If \( a, b, c \notin P \), then consider \( k_a, k_b, k_c \in \mathbb{Z} \) such that \( f^{k_a}(a), f^{k_b}(b), f^{k_c}(c) \in Q \). Assume, without loss, that \( k_a \leq k_b \leq k_c \). Take \( k = k_b \). In this way, items 2 and 5 finishes the direct part of the proof.

To prove the converse, we will show that \( f \) is not \((m, 3)\)-expansive for all \( m > 3 \). Take \( \delta > 0 \). Notice that since \( X = \infty \) there is at least one wandering point \( x \). Without loss of generality assume that \( \lim_{k \to \infty} f^k(x) = p_a \) an attractor fixed point and \( \lim_{k \to -\infty} f^k(x) = p_r \) a repeller fixed point. Take \( k_1, k_2 \in \mathbb{Z} \) such that \( \text{dist}(f^{k_1}(x), p_a) < \delta \) for all \( k \leq k_1 \) and \( \text{dist}(f^{k_2}(x), p_a) < \delta \) for all \( k \geq k_2 \).

Let \( l = k_2 - k_1 \) and define \( x_1 = f^{-k_1}(x) \), and \( x_{i+1} = f(x_i) \) for all \( i \geq 1 \). Consider the set \( A = \{x_1, \ldots, x_m\} \). By construction we have that \( |A| = m \) and \( |f^k(A)| \leq 3 \) for all \( k \in \mathbb{Z} \). Thus, proving that \( f \) is not \((m, 3)\)-expansive if \( m > 3 \). \( \Box \)

**Remark 4.2.** In light of the previous proof one may wonder if a smart \( \delta \)-observer will not be able to say that \( A \) has more than \( 3 \) points. We mean, we are assuming that a \( \delta \)-observer will say that \( A \) has \( n' \) points with

\[ n' = \max_{k \in \mathbb{Z}} |f^k(A)|_\delta. \]
According to the dynamic of the set $A$ in the previous proof, we guess that with more reasoning a smarter $\delta$-observer will find that $A$ has more than 3 points.

Theorem 4.1 gives us examples of (3,2)-expansive homeomorphisms on infinite countable compact metric spaces. A natural question is: does (3,2)-expansiveness implies hyper-expansiveness? I do not know the answer, but let us remark some facts that may be of interest. If $f$ is (3,2)-expansive, then:

- For all $x \in X$ either the stable or the unstable set must be trivial. It follows by the arguments of the proof of Theorem 3.2.
- If $x, y$ are doubly asymptotic, i.e., $\text{dist}(f^k(x), f^k(y)) \to 0$ as $k \to \pm\infty$, then they are isolated points of the space. Suppose that $x$ were an accumulation point. Given $\delta > 0$ take $k_0$ such that if $|k| > k_0$, then $\text{dist}(f^k(x), f^k(y)) < \delta$. Take a point $z$ close to $x$ such that $\text{dist}(f^k(x), f^k(z)) < \delta$ if $|k| \leq k_0$ (we are just using the continuity of $f$). Then $x, y, z$ contradicts (3,2)-expansiveness.

**Proposition 4.3.** There are (4,2)-expansive homeomorphisms that are not (3,2)-expansive.

*Proof.* Let us prove it giving an example. Consider a countable compact metric space $X$ and a homeomorphism $f: X \to X$ with the following properties:

1. $f$ has 5 orbits,
2. $a, b, c \in X$ are fixed points of $f$,
3. there is $x \in X$ such that $\lim_{k \to -\infty} f^k(x) = a$ and $\lim_{k \to +\infty} f^k(x) = b$,
4. there is $y \in X$ such that $\lim_{k \to -\infty} f^k(y) = b$ and $\lim_{k \to +\infty} f^k(y) = c$.

In order to see that $f$ is not (3,2)-expansive consider $\varepsilon > 0$. Take $k_0 \in \mathbb{Z}$ such that for all $k \geq k_0$ it holds that $\text{dist}(f^k(x), b) < \varepsilon$ and $\text{dist}(f^{-k}(y), b) < \varepsilon$. Define $u = f^{k_0}(x)$ and $v = f^{-k_0}(y)$. In this way $\|\{f^k(u), b, f^k(v)\}\| \leq 2$ for all $k \in \mathbb{Z}$. This proves that $f$ is not (3,2)-expansive.

Let us now indicate how to prove that $f$ is (4,2)-expansive. Consider $\varepsilon > 0$ such that if $i \geq 0$ and $j \in \mathbb{Z}$, then $\text{dist}(f^{-i}(x), f^j(y)) > \varepsilon$ and $\text{dist}(f^j(x), f^j(y)) > \varepsilon$. Now, a similar argument as in the proof of Theorem 4.1, shows that $f$ is (4,2)-expansive. \hfill $\square$

5. With the shadowing property

In this section we prove that an important class of homeomorphisms are not $(m,n)$-expansive for all $m > n \geq 2$. In order to state this result let us recall that a $\delta$-pseudo orbit is a sequence $\{x_k\}_{k \in \mathbb{Z}}$ such that $\text{dist}(f(x_k), x_{k+1}) \leq \delta$ for all $k \in \mathbb{Z}$. We say that a homeomorphism has the shadowing property if for all $\varepsilon > 0$ there is $\delta > 0$ such that if $\{x_k\}_{k \in \mathbb{Z}}$ is a $\delta$-pseudo orbit, then there is $x$ such that $\text{dist}(f^k(x), x_k) < \varepsilon$ for all $k \in \mathbb{Z}$. In this case we say that $x$ $\varepsilon$-shadows the $\delta$-pseudo orbit.
Theorem 5.1. Let $f : X \to X$ be a homeomorphism of a compact metric space $X$. If $f$ has the shadowing property and there are $x, y \in X$ such that

$$0 = \liminf_{k \to \infty} \text{dist}(f^k(x), f^k(y)) < \limsup_{k \to \infty} \text{dist}(f^k(x), f^k(y)),$$

then $f$ is not $(m, n)$-expansive if $m > n \geq 2$.

Proof. By Proposition 1.3 it is enough to prove that $f$ cannot be $(m, 2)$-expansive if $m > 2$. Consider $\varepsilon > 0$. We will define a set $A$ with $|A| = \infty$ such that for all $k \in \mathbb{Z}$, $f^k(A) \subset B_\varepsilon(f^k(x)) \cup B_\varepsilon(f^k(y))$, proving that $f$ is not $(m, 2)$-expansive for all $m > 2$.

Consider two increasing sequences $a_l, b_l \in \mathbb{Z}$, $\rho \in (0, \varepsilon)$ and $\delta > 0$ such that

$$a_1 < b_1 < a_2 < b_2 < a_3 < b_3 < \cdots,$$

$$\text{dist}(f^{a_l}(x), f^{a_l}(y)) < \delta,$$

$$\text{dist}(f^{b_l}(x), f^{b_l}(y)) > \rho$$

for all $l \geq 1$ and assume that every $\delta$-pseudo orbit can be $(\rho/2)$-shadowed. For each $l \geq 1$ define the $\delta$-pseudo orbit $z^l_k$ as

$$z^l_k = \begin{cases} f^k(x) & \text{if } k < a_l, \\ f^k(y) & \text{if } k \geq a_l. \end{cases}$$

Then, for every $l \geq 1$ there is a point $w^l$ whose orbit $(\rho/2)$-shadows the $\delta$-pseudo orbit $\{z^l_k\}_{k \in \mathbb{Z}}$. Let us now prove that if $1 \leq l < s$, then $w^l \neq w^s$. We have that $a_l < b_l < a_s$. Therefore $z^l_{b_l} = f^{b_l}(y)$ and $z^s_{b_s} = f^{b_s}(x)$. Since $w^l$ and $w^s$ $(\rho/2)$-shadow the pseudo orbits $z^l$ and $z^s$, respectively, we have that

$$\text{dist}(f^{b_l}(w^l), f^{b_l}(y)), \text{dist}(f^{b_s}(w^s), f^{b_s}(x)) < \rho/2.$$

We conclude that $w^l \neq w^s$ because $\text{dist}(f^{b_l}(x), f^{b_l}(y)) > \rho$. Therefore, if we define $A = \{w^l : l \geq 1\}$ we have that $|A| = \infty$. Finally, since $\rho < \varepsilon$, we have that $f^k(A) \subset B_\varepsilon(f^k(x)) \cup B_\varepsilon(f^k(y))$ for all $k \in \mathbb{Z}$. Therefore, $|f^k(A)|_\varepsilon \leq 2$ for all $k \in \mathbb{Z}$. \hfill \Box

Remark 5.2. Examples where Theorem 5.1 can be applied are Anosov diffeomorphisms and symbolic shift maps. Also, if $f : X \to X$ is a homeomorphism with an invariant set $K \subset X$ such that $f : K \to K$ is conjugated to a symbolic shift map, then Theorem 5.1 holds because the $(m, n)$-expansiveness of $f$ in $X$ implies the $(m, n)$-expansiveness of $f$ restricted to any compact invariant set $K \subset X$ as can be easily checked.

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