Von Neumann’s Quantization of General Relativity

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(Dated:)

Von Neumann’s procedure is applied for quantization of General Relativity. We quantize the initial data of dynamical variables at the Planck epoch, where the Hubble parameter coincides with the Planck mass. These initial data are defined via the Fock simplex in the tangent Minkowskian space-time and the Dirac conformal interval. The Einstein cosmological principle is applied for the average of the spatial metric determinant logarithm over the spatial volume of the visible Universe. We derive the splitting of the general coordinate transformations into the diffeomorphisms (as the object of the second Nöther theorem) and the initial data transformations (as objects of the first Nöther theorem). Following von Neumann, we suppose that the vacuum state is a quantum ensemble. The vacuum state is degenerated with respect to quantum numbers of non-vacuum states with the distribution function that yields the Casimir effect in gravidynamics in analogy to the one in electrodynamics. The generation functional of the perturbation theory in gravidynamics is given as a solution of the quantum energy constraint. We discuss the region of applicability of gravidynamics and its possible predictions for explanation of the modern observational and experimental data.

PACS numbers: 11.55.Hx, 13.60.Hb, 25.20.Lj

I. INTRODUCTION

In 1915 Albert Einstein introduced the field equations of gravity \cite{1}. The mathematical basis of these equations is their invariance with respect to the general coordinate transformations in the Riemannian space-time. Solutions of these equations express the Riemannian space metric components through the matter energy-momentum ones. These solutions contain additional constants known as certain initial data. Einstein supposed that the general coordinate transformations have the same physical status as the initial data transformations under the Galileo group or its relativistic generalization known as the Poincaré symmetry group in the Special Relativity (SR).

In the same 1915 David Hilbert derived the Einstein equations by variation of the corresponding action with respect to the metric components \cite{2}. The Hilbert action contained new information in comparison with the equations. Along this line, Emmy Nöther proved two theorems \cite{3}. In accordance with these two Nöther’s theorems, there are two different symmetry groups. The first of them is the global Poincaré group of the initial data transformations as the object of the first Nöther theorem about the conservation laws that follow from a global group. The second symmetry group is the set of general coordinate transformations (i.e. diffeomorphisms) as the object of the second Nöther’s theorem asserting that the time-time and time-space components of the Einstein equations are constraints of the initial data. Therefore one should distinguish between the Einstein diffeomorphisms and the initial data transformations.

The difference between the Einstein diffeomorphisms and the transformations of reference frames was found in the approach to the General Relativity (GR) developed by Vladimir Fock \cite{4}. Fock introduced a diffeo-invariant
orthogonal basis in order to include fermions in the GR. In this way, he replaced the metric components in the GR by the diffeo-invariant simplex components in the tangent Minkowskian space-time. By using the Fock’s modification of the Einstein’s approach, one can introduce the Poincaré symmetry group of the initial data transformations in the tangent space-time.

In the framework of the Hilbert’s “Foundations of physics” [2], the next steps forward in the conception of the initial data transformations were made by Dirac [3] and by Arnowitt, Deser, and Misner [6]. Namely, they formulated the so-called Dirac–ADM Hamiltonian approach to constrained gravitational systems described by the Hilbert action. In this case, the Einstein diffeomorphism group transformations are reduced to its kinemetric subgroup ones [7] that include the Riemannian time reparameterizations. The reparameterization time invariance means that there are two invariant evolution parameters: the geometrical and dynamical ones. The first parameter is the geometrical interval, and the second can be associated with the zeroth mode of one of the dynamical fields in the Wheeler–DeWitt (WDW) field superspace of events [8, 9]. Then the energy of events in the WDW field superspace space is a solution of the energy constraint with respect to the canonical momentum of the dynamical time-like field.

Ch.W. Misner [10] supposed that the dynamical evolution time can be identified with the cosmological scale factor logarithm that can be considered as the zeroth mode of the spatial metric determinant logarithm in the Dirac–ADM Hamiltonian approach to the constrained gravitation system. The scalar field zeroth mode is defined by means of averaging over finite spatial volume in accordance with the cosmological principle suggested by Einstein in 1917 [11]. Misner’s definition of the dynamical evolution time by means of the cosmological principle becomes diffeo-invariant, if we restrict spatial diffeomorphisms by uniform transformations that preserve the spatial metric determinant.

Following Dirac [12], one can call the Einstein spatial metric determinant logarithm as a dilaton field. Dirac and Deser [13] replaced the Einstein intervals by the dilaton-free conformal intervals and identified these intervals with the observable distances. In this case, the dilaton field describes both the redshift in Cosmology and relativistic effects, such as the Mercury anomaly or double Eddington angle of the photon deviation in the Sun gravitational field. If the Hubble parameter and all masses are equal zero, then the Einstein’s theory becomes conformal invariant. Therefore, one can say that the dilaton field is the Goldstone mode accompanying the spontaneous conformal symmetry breaking.

The Goldstone mode approach to the Einstein GR was developed in paper [14]. The authors obtained the Hilbert action as a joint nonlinear realization of the affine symmetry group of all linear transformations [A(4)] and conformal symmetry transformations [C=SO(4,2)] by analogy with the Schwinger–Weinberg phenomenological Chiral Lagrangian as a nonlinear realization of the chiral symmetry in hadron physics [15]. This nonlinear realization supposed that the dilaton and all metric components are Goldstone modes. This approach collects all previous modification discussed before (including Hilbert’s action, Fock’s simplex, and Dirac’s dilaton) and distinguishes the so called normal coordinates in the field of the Goldstone mode space along geodesic lines.

Papers [10 18] aduce theoretical and observational arguments in favor of the separation between the initial data symmetry group and the subgroup of the general uniform coordinate transformations (D-diffeomorphisms). In the present paper we use this separation in order to construct the perturbation theory generating functional in terms of the D-diffeo-invariant independent variables and observable coordinates.

The features of our approach to the GR are:

1) the choice of normal coordinates along geodesic lines in the field space of the Goldstone modes, (i.e. the exponent parameterization of the diagonal metric component [15]);

2) the splitting of the Einstein (E) general coordinate transformations onto the Dirac (D) diffeomorphisms and the initial data transformations;

3) introduction of the D-diffeo-invariant independent variables and coordinates.

These features allow us to construct unambiguously the diffeo-invariant Wheeler-DeWitt equation and solve it by the analogy with the irreducible unitary representations of the Poincaré group [19].

The Dirac definition of observable quantities (as the D-diffeo-invariant ones) distinguishes between physical approximations in terms of invariants and a mathematical ansatz beyond the invariants. There are two approximations: static and dynamical. In the static one we neglect the independent variables in order to obtain the static interactions of external sources. The basic order of the dynamical approximation means that far from external sources we can keep only D-diffeo-invariant independent variables. The next orders of the dynamical approximation represent the gravidynamics perturbation theory.

In the following we consider both these cases and compare the obtained results with the classical homogeneous Cosmology as a Friedmann model of the Universe that takes into account only the dilaton zeroth mode associated with the
cosmological scale factor. The structure of the paper is as follows. Section II defines \textit{gravidynamics} as the Einstein General Relativity in terms of D-diffeo-invariant coordinates and independent variables with corresponding initial data. Section III is devoted to the generating functional of the quantum perturbation theory as quantum solutions of the energy constraint in the GR. Section IV defines the basic approximation of the quantum \textit{gravidynamics} (QGD). Section V estimates the predictable possibilities of the QGD in the context of observational facts and phenomena.

II. DIFFEO-INvariant INITIAL DATA IN GENERAL RELATIVITY

A. Nonlinear realization of the initial data symmetry group

The history of equations of motion is associated with the names of Newton, Maxwell, Einstein, Klein-Gordon, Dirac, and the authors of the Standard Model and the modern string theory. However, physical facts and phenomena are described by solutions of those equations that contain initial data. The initial data history is more simple than the history of equations. Initial data symmetries can be associated with only the 10-parametric Galileo group transformations and the Poincaré ones obtained from Galileo ones via the replacement of velocity translations by the Lorentz antisymmetric rotations

\[
v_1 \to v_1 + v^0 \to x_1 \to x_4 + L_{[0]} t, t \to t + L_{[0]} x_4.\]

In any case all the above mentioned equations for free fields can be classified in the form of the unitary irreducible representations of the Poincaré group obtained by Wigner \cite{19}. In other words, Wigner postulated the priority of the initial data transformation groups and derived all equations as invariant structure relations of the initial data transformation groups.

The Wigner construction can be extended onto the 20-parametric affine group \(A(4)\). The affine group \(A(4)\) generalizes the 10-parametric Poincaré group (that contains 4 translation generators \(P_{(a)}\) and 6 Lorentz ones \(L_{(a)(b)}\) by 10 proper affine generators \(R_{(a)(b)}\). The latter yield 10 symmetric linear transformations of the Minkowskian space-time coordinates

\[
x^\mu \to x^\mu = x^\mu + y^\mu + L_{[\mu\nu]} x^\nu + R_{[\mu\nu]} x^\nu.\]

Ogievetsky proved that the 20-parametric affine group \(A(4)\) and 15-parametric conformal one are a closer of the general coordinate transformation group \cite{20}. Ogievetsky and Borisov identified 4-coordinates \(x^\mu\) and 10 proper affine parameters \(h_{\mu\nu}\) with the Goldstone modes. They considered the motion of the orthogonal simplex in the coset \(K = \frac{A(4)}{L}\), where \((L)\) is the Lorentz subgroup \cite{14}.

The shifts and the rotation of the simplex are given by the Maurer-Cartan linear differential forms \(\omega_{(a)}^P\), \(\omega_{(a)(b)}^R\), and \(\omega_{[a](b)}^L\), respectively. The dependence of these Maurer-Cartan forms on the Goldstone modes \(h_{\mu\nu}\) is determined by the affine group algebra of commutation relations using the transformations

\[
G = e^{iP_{\mu} x^\mu},
\]

\[
G^{-1} dG = i \left[ P_{(a)} \omega_{(a)}^P + R_{(a)(b)} \omega_{(a)(b)}^R + L_{(a)(b)} \omega_{[a](b)}^L \right],
\]

\[
\omega_{[a](b)}^L [h] = \epsilon_{(a)\mu} d x^\mu,
\]

\[
\omega_{(a)(b)}^R [h] = (1/2) \{ \epsilon_{(a)\mu} d e^\mu_{(b)} - \epsilon_{(b)\mu} d e^\mu_{(a)} \},
\]

\[
\omega_{(a)}^P [h] = (1/2) \{ \epsilon_{(a)\mu} d e^\mu_{(b)} + \epsilon_{(b)\mu} d e^\mu_{(a)} \}.
\]

These linear forms and the conformal symmetry principle yield unambiguously the Hilbert action of Einstein’s gravitational theory

\[
W_H = - \int d^4 x \sqrt{-g} \frac{R(4)(g)}{6}\]

with the interval expressed in terms of the Fock simplex components

\[
ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \underbrace{\varepsilon^{2D}}_{Einstein(1915)} \underbrace{\omega_{(a)} \otimes \omega_{(b)} \eta^{(a)(b)}}_{Dirac(1973)} \underbrace{\omega_{(a)} \otimes \omega_{(b)} \eta^{(a)(b)}}_{Fock(1929)}.
\]

Here we use the definition of the conformal Fock simplex components

\[
\omega_{(a)} = e^D \omega_{(a)}^P(h).
\]

This nonlinear realization of affine and conformal symmetry groups not only yields the Hilbert action but justifies the identification of the Dirac conformal intervals

\[
\frac{d^2}{ds^2} = \omega_{(a)} \otimes \omega_{(b)} \eta^{(a)(b)} = e^{2D} \frac{ds^2}{Einstein(1915)}
\]

with the observable ones. This nonlinear realization yields physical variables as their normal coordinates along geodesic lines in the coset \(K = A(4)/L\) in terms of diffeo-invariants in the tangent space-time marked by the round bracket indexes \((a)\) \cite{15}.

\[\text{Here and in the following we use the natural units} \ M_{Pl} \sqrt{3/(8\pi)} \equiv M_{Pl}^* = c = \hbar = 1.\]
B. Diffeo-invariant $3+1$ foliation

The initial data problem supposes a concrete frame of reference known as the Dirac–ADM foliation $\mathcal{F}, \mathcal{G}$, $4 = 3 + 1$. In the frame one can convince that the conformal Fock simplex $\mathcal{F}$ takes the form

$$\omega(0) = e^{-2D}N dx^0,$$
$$\omega(b) = e_{(b)i}[dx^i + N^i dx^0],$$

where $N$ is the lapse function,

$$N^j = N^{j\perp} + N^{j||}; \quad \partial_t N^{j\perp} = 0$$

are three shift vectors, $D$ is the dilaton field, and $e_{(b)i}$ are five symmetric triads with the unit determinant $|e| = 1$.

The Fock simplex separates Einstein’s diffeomorphisms (i.e. general coordinate transformations)

$$x^\mu \rightarrow \tilde{x}^\mu = x^\mu(x^\mu), \quad g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu}(\tilde{x}) = g_{\alpha\beta} \frac{dx^\alpha}{dx^\tilde{\alpha}} \frac{dx^\nu}{dx^\tilde{\nu}}$$

onto the Riemannian time reparameterization and uniform spatial coordinate transformations (i.e. D-diffeomorphisms)

$$x^0 \rightarrow x^0 = x^0(x^0), \quad e_{(b)ij} \rightarrow \tilde{e}_{(b)ij}(\tilde{x}) = e_{(b)jk}(x) \frac{dx^k}{dx^\tilde{j}}$$

and the initial data transformations. The latter include shifts of the dilaton $D(x) \rightarrow D(x) + \text{Constant}$ and the Lorentz rotations of Fock’s simplex components.

The Riemannian time reparameterization separates the time-like dynamical evolution parameter in the field space of event. This evolution parameter can be identified with the zeroth dilaton mode via the volume average in terms of D-diffeo-invariant forms

$$(D) \equiv V_0^{-1} \int V_0 \omega_{(1)} \wedge \omega_{(2)} \wedge \omega_{(3)}D.$$  

according to Einstein’s cosmological principle [11]. Here $V_0 = \int V_0 \omega_{(1)} \wedge \omega_{(2)} \wedge \omega_{(3)}$ is the finite D-diffeo-invariant volume. This zeroth dilaton mode coincides with the cosmological scale factor logarithm

$$\langle D \rangle = -\ln a = \ln(1 + z).$$

This quantity is known in the observational astrophysics as luminosity. The spatial average (7) can be called as the global projection operator, while the orthogonal projection operator

$$\mathcal{P} = D - \langle D \rangle$$

is a local one. The local functions obey the constraint $\langle \mathcal{P} \rangle \equiv 0$. This classification of functions is the consequence of the D-diffeomorphisms.

C. The physical content of the Hilbert action

The direct substitution of the Fock simplex components [3] and [4] into the Hilbert action determines its diffeo-invariant physical content according to the Dirac–ADM $3 + 1$ foliation [16]:

$$W_H = \underbrace{W_{\text{cosmology}}}_{=0 \text{ for } V_0=\infty} + W_{\text{wave}} + W_{\text{gravity}},$$

$$W_{\text{cosmology}} = - \int d^4x \left( \frac{d}{dx^0} \right)^2 \frac{1}{N},$$

$$W_{\text{wave}} = \int d^4x \left[ \frac{N}{6} [v^{(a)(b)} v^{(a)(b)} - R^{(3)}(e)e^{-4D}] \right],$$

$$W_{\text{gravity}} = \int d^4x \left[ -\frac{2}{D^4} \frac{4}{3} N e^{-7D/2} \Delta^{(3)}(e^{-D/2}) \right].$$

Here $R^{(3)}(e)$ is the three dimensional curvature

$$R^{(3)}(e) = -2 \partial_i [\epsilon_{(b)j}] (c||c||c) - \sigma_{(c)}(b)(b)(c)\sigma_{(a)}(b)(a) + \sigma_{(c)}(d)(f)\sigma_{(f)}(d)(c),$$

where

$$\sigma_{(c)}(a)(b) = \omega^R_{(a)(b)}(\partial(c)) + \omega^R_{(a)(c)}(\partial(b)) - \omega^R_{(b)(c)}(\partial(a)),$$

and

$$v^{(a)(b)} = \frac{1}{N} \left[ \frac{\omega_{(a)(b)}(\partial_0 - \partial_t N)}{\partial_t N} + \partial_0 N^{(b)} + \partial_t N^{(a)} \right],$$

are velocities of the gravitational waves and non-zeroth dilaton modes. One can see that the Hilbert action describes three classes of phenomena: Cosmology, waves and potentials of gravity. The Hilbert action [16] shows us also that its kinetic part does not depend on the antisymmetric forms $\omega^R_{(a)(b)}$. These antisymmetric forms are not dynamical variables and they can have zero initial data

$$\omega^R_{(a)(b)}(d) = 0.$$

Following the Dirac’s approach [3] to constrained systems, one can impose the second class constraints

$$\partial_t e^\nu_{(a)} = \omega^R_{(a)(b)}(\partial(b)) = 0,$$

$$v^\nu = 0.$$

They mean the zero initial data for the longitudinal components and the minimal hyper-surface condition [19]. The Hilbert action [16] distinguishes three spaces: the Riemannian one, the tangent one, and the field one.
D. Diffeo-invariant coordinates

The cosmological part of the Hilbert action [10] determines the global lapse function

\[ N_0^{-1} = \langle N^{-1} \rangle \quad (20) \]
as the average over a finite volume. The Hilbert action [10] expressed in terms of the linear Maurer-Cartan forms gives a possibility to introduce the D-diffeo-invariant spatial-time coordinates

\[ d\tau = N_0(x^0)dx^0, \quad X_{(a)} = x^a e_{(a)}. \quad (21) \]

The differential of the diffeo-invariant spatial coordinates [22]

\[ dX_{(a)} = e_{(a)}dx^i + x^i e_{bi}e_{(b)}^j d\epsilon_{j(a)} \quad (23) \]
expresses them via the spatial components of the Fock simplex in Eq. [22] and the diffeo-invariant graviton \( \omega^R_{(b)(a)} \),

\[ \tilde{\omega}_{(a)} \equiv e_{(a)}dx^i = dX_{(a)} - X_{(b)}\omega^R_{(b)(a)}(d) \quad (24) \]

According to the Dirac’s approach to constrained systems the equalities [21] and [22] are weak ones. These equalities can be used only after the variation of the Hilbert action in terms of the Riemannian space-time coordinates. Meanwhile the diffeo-invariant tangent space-time coordinates are needed to compare the solutions of the equations with diffeo-invariant observational data. According to the Dirac concept of observable quantities the observable coordinates should be D-diffeo-invariant. Thus, the “Foundations of the GR” are the Hilbert action [10] and the Dirac conformal interval

\[ \tilde{ds}^2 = e^{-4D}N^2 d\tau^2 - [dX_{(a)} - X_{(b)}\omega^R_{(b)(a)} + N_{(a)}d\tau] \quad (25) \]

Just these “Foundations” lead us to diffeo-invariant variables. The Dirac concept of observability has very important consequences. The diffeo-invariant coordinates [21] and [22] allow us to treat the Maurer-Cartan forms [21] \( \tilde{\omega}_{(a)} \) and \( \omega^R_{(b)(a)}(\partial_{(c)}) \) as D-diffeo-invariant independent degrees of freedom. The classical initial data admit zero values of the diffeo-invariant quantities

\[ \ln N = 0, \quad N_{(a)} = 0, \quad D = 0, \quad \omega^R_{(a)(b)}(\partial_{(c)}) = 0. \quad (26) \]

These values yield the flat metrics

\[ \tilde{ds}^2 = d\eta^2 - [dX_{(a)}]^2 \quad (27) \]
in accord with the correspondence principle.

However, in quantum theory this solution is not stable since

\[ \langle D \rangle \neq 0, \quad \omega^R_{(a)(b)}(\partial_{(c)}) \neq 0. \quad (28) \]

Below we discuss a possibility to construct a stable solution with the help of the uncertainty principle for the strong gravitation waves in the approximation

\[ \tilde{ds}^2 = d\eta^2 - [dX_{(a)} - \omega^R_{(a)(b)}(\partial_{(c)})]^2. \quad (29) \]

In this case, the diffeo-invariant coordinates \( X_{(a)} \) given by Eq. [24]

\[ dX_{(a)} = \tilde{\omega}_{(a)}(d) + X_{(b)}\omega^R_{(b)(a)}(d). \quad (30) \]
are the functionals of the wave solutions expressed via the spin-connection coefficients \( \omega^R_{(a)(b)}(\partial_{(c)}) \) and the Fock simplex components. The latter satisfy the independence constraints

\[ \tilde{\omega}_{(a)}(\partial_{(c)}) = e_{(a)}dx^i = (\partial_{(c)}) \quad (31) \]

of the type of the flat space ones \( dx^i/dx^j = \delta^i_j \). Using conditions [31] one can rewrite Eq. [30] in the form

\[ \frac{dX_{(a)}}{\tilde{\omega}_{(a)}(d)} = \delta_{(a)(b)} + X_{(c)}\omega^R_{(c)(a)}(\partial_{(b)}). \quad (32) \]

In the case of \( (a) = (b) \) this form reduces to

\[ \frac{dX_{(a)}}{\tilde{\omega}_{(a)}(d)} = 1, \quad (33) \]
while in the case of \( (a) \neq (b) \) Eq. [32] yields the equation

\[ \frac{dX_{(a)}}{\tilde{\omega}_{(a)}(d)} = X_{(c)}\omega^R_{(c)(a)}(\partial_{(b)}) \quad (34) \]

that allows us to express diffeo-invariant coordinates via the gravitation waves as we will show below in the Section IV.

E. Diffeo-invariant variables in tangent space-time

Both action [10] and interval [22] are bilinear quantities with respect to the Maurer-Cartan forms. The proposed choice of coordinates strongly simplifies resolution of the Einstein equations in terms of D-diffeo-invariant independent variables. These variables are only two gravitons \( \omega^R_{(a)(b)}(\partial_{(c)}) \) and the dilaton zeroth mode \( D \). The independence means that they can have non-zero initial data after their quantization. The quantization requires canonical momenta of these independent variables that are calculated from the Lagrangian
representation of the action $W = \int d^4 x L = \int dx^0 L$ by the standard way
\[
P_{(D)} = \frac{\partial L}{\partial (\partial_0(D))} = -2V_0 \frac{d(D)}{d\tau}, \tag{35}
\]
\[
p^R_{(a)(b)} = e_{(a)j} \frac{\partial L}{\partial (\partial_0 e_{(b)j})} = \frac{v_{(a)(b)}}{3}. \tag{36}
\]

The Poisson brackets take the form
\[
\{P_{(D)}, \langle D \rangle\} = 1, \tag{37}
\]
\[
\{p^R_{(a)(b)}(x^0, x), \omega_{(c)(d)}(x^0, y)(\partial_k)\} = \Pi_{(a)(b)(c)(d)} \partial_k \delta^3(x - y), \tag{38}
\]

where $\Pi_{(a)(b)(c)(d)}$ is the projection operator. The rest fields $\bar{D}$, and
\[
N = \frac{N}{N_0(x^0)}, \tag{39}
\]
\[
N^{(a)} = \frac{e_{(a)i}N^i}{N_0(x^0)} \tag{40}
\]

are the D-diffeo-invariant potentials. The variation of the Hilbert action with respect to the potentials yields the constraints \[45\]. In particular, the shift vector $N^j$ leads to three momentum constraints, the lapse function logarithm $\ln N$ yields the energy constraint. In perturbation theory these potentials are beyond the basic approximation, they determine only static interactions of external sources.

### F. The momentum constraint

For the explicit solution of the momentum constraint
\[
\frac{\delta W_H}{\delta N^j} = 0 \tag{41}
\]

it is convenient to use the expansion
\[
N_{(b)} = N_{(b)}^I + N_{(b)}^L, \tag{42}
\]
\[
\partial_j N_{(b)}^I = \partial_j N^j, \tag{43}
\]
\[
\partial_j N_{(b)}^L = 0, \tag{44}
\]
\[
p_{(a)(b)} = p_{(a)(b)}^R + \partial_j f_{(a)}^L + \partial_j f_{(a)}^R, \tag{45}
\]
\[
\partial_j f_{(a)}^R = 0, \tag{46}
\]

where $f_{(a)}^L$ satisfies the equation
\[
\left[ \Delta f_{(a)}^L + \partial_0 (x^0) f_{(a)}^L \right] = p_{(b)(c)}^R (x^0) \partial_0 \omega_{(b)(c)}^R \tag{47}
\]

which follows from the momentum constraint \[41\] after the substitution \[45\].

### G. The energy constraint

The invariance of the Hilbert action \[10\] with respect to the time-reparameterizations means that its variation under the lapse function $\frac{\delta W_H}{\delta \ln N} = 0$ is the energy density constraint:
\[
\frac{1}{N} \left[ \frac{d(D)}{d\tau} \right]^2 = N\mathcal{H}, \tag{48}
\]

where
\[
\mathcal{H} = -\delta [W_{\text{wave}} + W_{\text{gravity}}] \frac{\delta N}{\delta N} \tag{49}
\]

\[
= 6 \left[ p_{(a)(b)}^R + \partial_j (x^0) f_{(a)}^L + \partial_j f_{(a)}^R \right]^2 \tag{50}
\]

\[
+ \frac{e^{-4D}}{6} \left[ \omega_{(a)(c)}(x^0) (\partial_j b_j) - \omega_{(b)(c)}(x^0) (\partial_j a_j) \right]^2 \tag{51}
\]

\[
- \frac{4}{3} e^{-7D/2} \nabla e^{-D/2} \tag{52}
\]

is the energy component of the energy-momentum tensor and $f^R_{(a)}$ is given by Eq. \[47\]. This energy density constraint determines both the diffeo-invariant time interval \[21\] and the diffeo-invariant lapse function \[29\]
\[
N = \frac{\langle \sqrt{H} \rangle}{\sqrt{H}} \rightarrow \langle N^{-1} \rangle \equiv 1. \tag{53}
\]

The average of this energy density constraint \[48\] over the space volume $\langle \frac{\delta W_H}{\delta \ln N} \rangle = 0$ leads to the energy constraint in the form of the Friedmann-type equation
\[
\left[ \frac{d(D)}{d\tau} \right]^2 = \langle \sqrt{H} \rangle^2. \tag{54}
\]

The solution of this equation yields the relation between the diffeo-invariant time interval and the dilaton zeroth mode:
\[
\tau - \tau_I = \int \langle D \rangle^{-1} d(D) \langle \sqrt{H} \rangle^{-1}. \tag{55}
\]

This means that the dilaton zeroth mode is a time-like variable.

In the Hamiltonian approach the energy constraint equation takes the form
\[
P_{(D)}^2 - E_U^2 = 0, \tag{56}
\]

where
\[
E_U = 2 \int v_0 \int \omega_1 \wedge \omega_2 \wedge \omega_3 \sqrt{H}. \tag{57}
\]
We can treat quantity \( \mathcal{H} \) as the energy of the Universe in the field space of events.

The dilaton non-zero modes are not independent degrees of freedom, therefore its velocity \( v_\mathcal{H} \) and momentum \( P_\mathcal{H} = 2v_\mathcal{H} \) are equal zero

\[
P_\mathcal{H} = 0. \tag{55}
\]

This equation is in agreement with the Dirac constraint of the minimality of three dimensional hyper-surface at embedding it into the four dimensional Riemannian space \( \mathbb{R}^4 \).

Thus, the constraint-shell Hilbert action takes the form

\[
W_{c-shell} = \int d^4x p_i^a \partial_0 e_{(a)j} + \int dx^0 P_{(D)} \partial_0 (D) \tag{56}
\]

where \( \partial_0 (D) = \frac{d}{d(D)} = (\sqrt{\mathcal{H}}) \frac{d}{d\tau} \). The constraint-shell action requires D-diffeo-invariant initial finite volume and initial data including an initial dilaton value \( \langle D \rangle_I \) as certain input parameters given at the instance of the Universe creation. Here we collect and reproduce the known concepts just since we start from the standard Hilbert action. The new facts are only the definitions of D-diffeo-invariant coordinates and variables with their initial data. It is just the first goal of our approach to the Einstein GR. The next step is to compare the classical choice of unstable dilaton value \( \langle D \rangle_I = 0 \) and the infinite volume value with their quantum values restricted by the uncertainty principle. We call this quantum theory the quantum gravidynamics (QGD) by analogy with electrodynamics and chromodynamics.

We have to define the region of validity of QGD and its predictable possibilities.

\[ T = T - \langle T \rangle = 0. \tag{61} \]

If the dilaton field is split into the zeroth mode and the non-zeroth ones \( D = \langle D \rangle + D \), then equation (58) also is split into the equations for zeroth mode and the non-zeroth modes restricted by constraint (55).

The solution of the zeroth dilaton mode equation

\[
\frac{\delta W_H}{\delta D} = 0 \tag{60}
\]

takes the form

\[
T = T_D - \langle T_D \rangle = 0. \tag{61}
\]

It determines \( \overline{D} \) as one of static potentials.

Here we just adapted the Dirac–ADM Hamiltonian formulation [2] to the Maurer–Cartan forms. In terms of these forms, the Hilbert action becomes a bilinear functional. This means that the action of such a theory describes a physical system of the type of a squeezed oscillator [17]. It gives a hope to construct a quantum theory of such a system at the level of Cartan’s forms and to describe quantum processes. We treat the Einstein’s General Relativity in terms of diffeo-invariant variables and potentials as a candidate for formulation of the Quantum Gravidynamics by the analogy with to Quantum Electrodynamics (QED).

We can treat quantity (\[ \mathcal{H} \]) as the energy of the Universe in the field space of events.

The dilaton non-zero modes are not independent degrees of freedom, therefore its velocity \( v_\mathcal{H} \) and momentum \( P_\mathcal{H} = 2v_\mathcal{H} \) are equal zero

\[
P_\mathcal{H} = 0. \tag{55}
\]

This equation is in agreement with the Dirac constraint of the minimality of three dimensional hyper-surface at embedding it into the four dimensional Riemannian space \( \mathbb{R}^4 \).

Thus, the constraint-shell Hilbert action takes the form

\[
W_{c-shell} = \int d^4x p_i^a \partial_0 e_{(a)j} + \int dx^0 P_{(D)} \partial_0 (D) \tag{56}
\]

where \( \partial_0 (D) = \frac{d}{d(D)} = (\sqrt{\mathcal{H}}) \frac{d}{d\tau} \). The constraint-shell action requires D-diffeo-invariant initial finite volume and initial data including an initial dilaton value \( \langle D \rangle_I \) as certain input parameters given at the instance of the Universe creation. Here we collect and reproduce the known concepts just since we start from the standard Hilbert action. The new facts are only the definitions of D-diffeo-invariant coordinates and variables with their initial data. It is just the first goal of our approach to the Einstein GR. The next step is to compare the classical choice of unstable dilaton value \( \langle D \rangle_I = 0 \) and the infinite volume value with their quantum values restricted by the uncertainty principle. We call this quantum theory the quantum gravidynamics (QGD) by analogy with electrodynamics and chromodynamics.

We have to define the region of validity of QGD and its predictable possibilities.

\[ T = T - \langle T \rangle = 0. \tag{61} \]

If the dilaton field is split into the zeroth mode and the non-zeroth ones \( D = \langle D \rangle + D \), then equation (58) also is split into the equations for zeroth mode and the non-zeroth modes restricted by constraint (55).

The solution of the zeroth dilaton mode equation

\[
\frac{\delta W_H}{\delta D} = 0 \tag{60}
\]

takes the form

\[
T = T_D - \langle T_D \rangle = 0. \tag{61}
\]

It determines \( \overline{D} \) as one of static potentials.

Here we just adapted the Dirac–ADM Hamiltonian formulation [2] to the Maurer–Cartan forms. In terms of these forms, the Hilbert action becomes a bilinear functional. This means that the action of such a theory describes a physical system of the type of a squeezed oscillator [17]. It gives a hope to construct a quantum theory of such a system at the level of Cartan’s forms and to describe quantum processes. We treat the Einstein’s General Relativity in terms of diffeo-invariant variables and potentials as a candidate for formulation of the Quantum Gravidynamics by the analogy with to Quantum Electrodynamics (QED).

### III. QUANTUM GRAVIDYNAMICS

#### A. Formulation

Quantum Electrodynamics is treated as the ideal to formulate any physical theory, including QGD. Einstein used the Maxwell equations of classical electrodynamics as the example of a same type field equations in the gravitation theory. QGD keeps this analogy. If one neglects dynamical variables, in both the theories, there arise the static Colombian type interactions of external sources. In our case, if we neglect dynamical variables (the luminosity \( (D) \) and the graviton \( \omega^R \)) we obtain the black-hole type solution in the isotropic coordinates [10]. In the opposite case, far from external sources, the set of constraints and equations of the QED describe only two transverse photons as independent degrees of freedom. In our case far from heavy bodies the set of constraints and equations of QGD describes only two gravitons \( \omega^R \) and the luminosity \( (D) = -ln a = \ln(1 + z) \), where \( z \) is identified with the redshift in the observational Cosmology.

The Einstein’s cosmological principle [11] determines two classes of functions distinguished by two projection operators. The first class determines the concept of the energy of the Universe, while
the second class determines the physical content of this energy in the form of primordial gravitation waves and their vacuum expectation value. If we neglect gravitons, the set of Einstein equations has only the trivial solution \( \langle D \rangle = 0 \). If we neglect the dilaton, the set of Einstein equations has also the trivial solution \( \omega^R = 0 \). However, both these approximations are only classical ones. In quantum theory both trivial solutions are impossible due to the uncertainty principle. To obtain consequences of the uncertainty principle in quantum gravodynamics we need its perturbation theory generating functional.

### B. Generating functional of the quantum gravodynamics perturbation theory

In the Hamiltonian approach the energy constraint equation \( (63) \) can be quantized. In quantum theory the canonical variables \( \hat{P}_{(D)} \) and \( \langle D \rangle \) become operators with the commutation relation \( \{ \hat{P}_{(D)}, \langle D \rangle \} = \iota, \) and the energy constraint becomes the equation of the Wheeler–DeWitt type \( (64) \):

\[
\left[ \hat{P}_{(D)}^2 - E_0^2 \right] \Psi_{\langle D \rangle} = 0. \tag{62}
\]

By analogy with the unitary irreducible representation of the Poincaré group in quantum field theory, we get a general operator solution of the Wheeler–DeWitt equation for the universe as a sum of two exponent orders with respect to the field evolution parameter \( \langle D \rangle \):

\[
\Psi_{\langle D \rangle} = \hat{A}_{(D)}^+ |0\rangle + \frac{1}{\sqrt{2E_0U}} + \hat{A}_{(D)}^- |0\rangle, \tag{63}
\]

\[
\hat{U}_I = T_{(D)} \exp \left\{ -i \int_{\langle D \rangle} d\langle D \rangle E_0 U \right\}. \tag{63}
\]

It is a unitary operator of the universe evolution in the space of events \( |\langle D \rangle | F \rangle \) relatively to the field evolution parameter \( |\langle D \rangle | \). The WDW functional describes the creation of the universe at the time \( \langle D \rangle \) and its evolution from \( \langle D \rangle \) till the present day moment \( \langle D \rangle_0 \). The two terms correspond to the positive and negative energy, where \( \hat{A}_{(D)}^+ \) can be interpreted as the operator of creation of the universe at the moment \( \langle D \rangle_0 \), and \( \hat{A}_{(D)}^- \) is the correspondent annihilation operator with a commutation relation

\[
\{ \hat{A}_{(D)}^+, \hat{A}_{(D)}^- \} = 1.
\]

Negative energy is removed by the second quantization. After the procedure of normal ordering the field Hamiltonian takes the form

\[
\tilde{H} = \rho_{\text{Cas}} + : H :. \tag{64}
\]

where

\[
\rho_{\text{Cas}} = \sum_{k \neq 0} \frac{\omega_k}{2} f_k \tag{65}
\]

is the density of the Casimir energy of gravitons with the one particle energy \( \omega_k \) that can be determined in finite space by the analogy with the Casimir energy of photons \( 21 \). The divergent integral in the Casimir energy can be regularized by introduction of a simple distribution function

\[
f_k = \left[ 1 - \exp \left( d \sqrt{k^2} \right) \right]^{-1}, \tag{66}
\]

where \( d \) is an input parameter as the volume dimension. This distribution function reflects the degeneration of the vacuum state over momenta.

Recall that the Casimir energy is the result of normal ordering of the product of field operators in the free Hamiltonian. This means that the graviton occupation number cannot be equal zero. It is natural to suppose that at the Beginning the Casimir energy of the Universe

\[
E_U^f = 2 \int_{V_0} d^3 x \sqrt{\rho_{\text{Cas}}} \tag{67}
\]

dominated. Following Kasner \( 23 \), we suppose that the region of validity of the GR is the creation and evolution of the Universe as the whole.

Moreover, the key assumption of our model is the equality of the Casimir dimension \( d(a) \) and the Universe size defined as the horizon:

\[
d(a) = r(a), \quad a = e^{(D)}. \tag{68}
\]

The Casimir energy of the Universe in the volume \( V_0 = \frac{4\pi}{3H_0^2} \) is proportional to the Hubble constant \( H_0 \). The solution of equation \( (68) \) given in refs. \( 17 \), \( 18 \) corresponds to the rigid state \( p = +1 \), \( \rho \). The rigid state dominance is in agreement with the SNe Ia data \( 26 \), \( 27 \), in the framework of conformal GR where the long conformal intervals are identified with observational distances \( 28 \), \( 29 \). This agreement means that the long-distance SNe Ia data testify to an almost empty Universe during all time of its evolution with the dominant energy density contribution from the Casimir energy.

The rigid state dominance means that SNe Ia data correspond to the primordial value of the Hubble parameter and the dilaton initial data \( \langle D \rangle_1 \) at the Planck scale, where the Hubble parameter value coincides with the Planck mass one. In this case the primordial value of the Hubble parameter as the input parameter and the set of Eqs. \( (35)-(61) \) completely defines the perturbation theory.

\[ \text{The set of the vacuum degeneration parameters can be treated as a von Neumann’s “Statistishe Gesamtheinten” or the Blokhintsev’s “Quantum Ensemble” } \]
C. Perturbation theory

An expansion of the evolution generator can be constructed as

\[ \mathbf{E}_U = 2 \int d^3 x \sqrt{\rho \text{Cas}} : \mathcal{H} : = \mathbf{E}_U^L + \mathbf{H}_{\text{QFT}} \sqrt{\rho \text{Cas}} + \cdots \]  

(69)

\[ \mathbf{E}_U^L = 2 \int d^3 x \sqrt{\rho \text{Cas}}, \]

\[ \mathbf{H}_{\text{QFT}} = \int d^3 x : \mathcal{H} :. \]

We can see that this expansion coincides with the quantum field perturbation theory

\[ \mathcal{U} = \mathcal{U}_{\text{Cas}} \cdot T_{i} \exp \left\{ -i \int_{i_{0}}^{i_{f}} d\tilde{t} \mathbf{H}_{\text{QFT}} \right\}, \]

where \( d\tilde{t} = \frac{d(D)}{\sqrt{\rho \text{Cas}}} \) coincides with the conformal time interval as it will be shown later. This expansion is made under the assumption of the dominance of the Casimir energy of graviton.

IV. VACUUM ENERGY OF DIFFEO-ININVARIANT GRAVITONS

A. Quantum graviton

One can see that the Hilbert action contains the global dilaton part (11) (that solves the time-energy problem in the complete theory), the graviton part (12) (that describes diffeo-invariant transverse gravitons), and the potential part (3) (that describes the black hole type solution induced by heavy massive bodies).

If there are no massive bodies at the Beginning of Universe, then the last term (3) can be neglected. In this case the second terms describes an exactly solvable model of diffeo-invariant gravitons. Therefore, in the region far from heavy bodies we can consider the gravitons on an equal footing with the Standard Model quantum fields.

The Hilbert action (10) is a bilinear functional of diffeo-invariant gravitons. Therefore, in the region far from heavy bodies we can consider the gravitons on an equal footing with the Standard Model quantum fields.

The graviton Hamiltonian takes the form

\[ H^{\mathfrak{g}}_{\tau} = \sum_{k^{2} \neq 0} \frac{g_{\mathfrak{g}}^{k k} - g_{\mathfrak{g}}^{k - k} + e^{-4(D)} k^{2} \mathcal{R}_{k} \mathcal{R}_{-k}}{2}. \]

(71)

Solutions of the corresponding graviton equations can be obtained in terms of the conformal time interval (18) \( d\eta = a^{2}(r) d\tau \), where \( a = e^{-(D)} \) and

\[ g_{k}^{\pm} = g_{1k}^{\pm} \exp \{ \pm i |k| (\eta - \eta_{i}) \}. \]

(72)

These solutions describe standard free quantum fields in a finite volume. The Hamiltonian vacuum expectation value yields the Casimir energy with the single input parameter \( H_{0} \) given by the SNe Ia data (26, 27).

B. The single graviton case

We consider the case of the metric

\[ ds^{2} = d\eta^{2} - dX(3) - [\omega^{2}(1) + \omega^{2}(2)] \]

(73)

when the spin-connection coefficients ta\( ke \) the form

\[ \omega_{(1) (1)} (d) = -\omega_{(2) (2)} (d) = dD_{1} (\eta_{-}), \]

\[ \omega_{(1) (2)} (d) = \omega_{(2) (1)} (d) = dD_{2} (\eta_{-}). \]

(74)

(75)

Here the functions \( D_{1}, D_{2} \) depend on only the light cone coordinate

\[ \eta_{(-)} = \eta - dX(3). \]

(76)

It is just the case of a strong gravitation wave, when the spin-connection coefficients are the solution of the exact Einstein equations for the metric (73).

Let us pass to the complex coordinates, linear differential forms and fields

\[ z = x^{1} + i x^{2}, \quad Z(D) = X^{(1)} + i X^{(2)}, \]

\[ D = D_{1} + i D_{2}, \quad \omega (d) = \omega^{(1)} + i \omega^{(2)}. \]

(77)

This means that the integral form \( Z^{*} (D) \) of the tangent coordinates can be obtained via the Eq. (74)

\[ \frac{dZ(D)}{dD} = Z^{*} (D), \quad Z(0) = Z^{0} \]

(78)

that allows us to calculate their time dependence in the field of the strong gravitation wave \( D(i \eta_{-}) \). The solution of Eq. (78)

\[ Z(D) = R(D) e^{i \theta (D)} \]

(79)

describes also the rotations of any vectors in the complex plane (77), where all measurable quantities.
are diffeo-invariants, including the photon spin (polarization). The substitution of (79) into Eq. (78) yields

\[ e^{2\theta(D)}d \left[ \ln \frac{R(D)}{R(0)} + i\theta(D) \right] \\
= \cos 2\theta d\ln \frac{R}{R_0} + \frac{i}{2} d\cos 2\theta \\
+ i\sin 2\theta d\ln \frac{R}{R_0} + \frac{i}{2} d\sin 2\theta \\
= d[D_1 + iD_2]. \quad (80) \]

Using the independence condition \(dX(1)/dX(2) = 0\) one can write the solution of these equations in the form

\[ R(D) = R_0 \exp \left\{ \sqrt{\tilde{D}_1^2 + \tilde{D}_2^2} - \frac{1}{2} \right\}, \quad (81) \]

\[ \cos 2\theta(D) = \frac{\tilde{D}_1}{\sqrt{\tilde{D}_1^2 + \tilde{D}_2^2}}, \quad (82) \]

\[ \sin 2\theta(D) = \frac{\tilde{D}_2}{\sqrt{\tilde{D}_1^2 + \tilde{D}_2^2}}, \quad (83) \]

\[ \tilde{D}_1 = D_1 + \frac{1}{2} \cos 2\theta_0, \]

\[ \tilde{D}_2 = D_1 + \frac{1}{2} \sin 2\theta_0. \]

Here \(R_0, \theta_0\) are initial data compatible with the initial values

\[ R(D = 0) = R_0, \quad (84) \]

\[ \cos 2\theta(D = 0) = \cos 2\theta_0, \quad (85) \]

\[ \sin 2\theta(D = 0) = \sin 2\theta_0. \quad (86) \]

and the perturbation series

\[ Z(0) = Z_0, \quad Z(D) = Z_0 + Z_1D + O(D^2). \]

C. The CMB temperature test of the space-time foam in QGD

The Einstein interval in the Friedmann homogeneous approximation

\[ \left[ \frac{ds}{dt} \right]^2 = 1 - a^2(t) \left( \frac{dX}{dt} \right)^2 = [1 - v^2], \quad (87) \]

defines the cosmic evolution of photon velocities

\[ \frac{dX}{dt} = \frac{dX}{d\eta} = \frac{v_\eta}{a} \]

and, therefore, the cosmic evolution of the CMB photon temperature

\[ T_{\text{CMB}} = \frac{T_{0\text{CMB}}}{a(t)}, \quad T_{0\text{CMB}} = 2.725\text{K}. \quad (88) \]

The QGD basic approximation identifies observational velocities with the conformal interval ones obtained as the squared interval vacuum expectation value

\[ \langle 0 | \left( \frac{d\eta}{d\tau} \right)^2 \rangle_{\text{space-time foam}} = 1 - \left| \sqrt{\tilde{X}} \right|^2 \widetilde{X}^{(1)} \widetilde{X}^{(1)} = \frac{1}{3} \left| \mathcal{X}^2 \right|^2 \omega^2 \Omega_0(1+z)^2 = \Delta T_{0\text{CMB}}, \quad (89) \]

where \(\Omega_{\tilde{g}}\) is of order of unit.

Instead of the Friedmann CMB temperature (88), the QGD basic approximation yields the CMB temperature evolution

\[ T_{\text{CMB}}^{\text{QGD}} = T_{0\text{CMB}} + \Delta T_{0\text{CMB}}(1 + z)^2. \quad (90) \]

If the present day ratio \(\Delta T_{0\text{CMB}}/T_{0\text{CMB}} \sim 10^{-6}\), then the QGD basic approximation predicts that at the time of the recombination \(z_{\text{recomb}} \sim 10^3\), when the CMB radiation decouples from the massive matter, the quantum space-time foam anisotropy was of order of the isotropic part \(T_{0\text{CMB}}\). This means that at the time of the recombination the quantum graviton space-time foam anisotropy can be able to form galaxies and large-scale massive matter structures in the Universe. At the present day time \(z \ll 1\) the QGD basic approximation yields the flat conformal interval that is identified with the measurable distance in the Dirac version of the Einstein theory. In this conformal version the temperature history of the Universe becomes the history of elementary particle masses. The origin of these masses and their cosmic evolution are the objects of the next Sections.

V. OBSERVATIONAL ARGUMENTS

A. SNe Ia data as an evidence of vacuum energy dominance

Observations of the so-called standard candles supernovae [26, 27] demonstrated that they are turned out to be too far away from us. Cosmologists had to introduce one more inflation in the Universe evolution scenario, explaining this inflation by the \(\Lambda\) term or by a modification of the GR. Meanwhile, we have shown [28, 29] that the observed supernova data (SNe Ia) can be explained if one just switches to conformal variables in the standard Friedmann equations. Fitting the data in this way shows that the dominant contribution to the Universe energy density should have the rigid equation of state \(p = +1 \cdot \rho\). In general, the rigid state can be realized in different ways, e.g. with the help of an additional scalar field, see ref. [29]. Here we adapt the possibility to identify the dominant rigid state
with the Casimir vacuum energy as it was suggested in ref. [18]. In fact, the vacuum energy dominance corresponds to the rigid state with the cosmological evolution

$$H(z) = \frac{d(D)}{d\eta} = \frac{H_0}{(1 + z)^2}, \quad (91)$$

$$d\eta = \frac{d\tau}{(1 + z)^2}, \quad (93)$$

where $d\eta$ is the conformal time and $H_0$ is the present day value of the Hubble constant.

**B. The Planck scale cosmic hierarchy**

The Planck epoch corresponds to the redshift value $z_f$ for which the primordial Planck mass value

$$M_{Pl}^* \equiv M_{Pl}^*(z_f) = M_{0Pl}(1 + z_f)^2 \quad (94)$$

coincides with the the primordial Hubble constant

$$H_f \equiv H(z_f) = M_{Pl}^* = M_{0Pl}(1 + z_f)^2. \quad (95)$$

Thus the value of the primordial redshift is of the order

$$z_f \simeq 10^{15}. \quad (96)$$

As discussed in [18], this definition of the Planck epoch follows also from the Planck least action principle applied at the initial moment of the universe evolution.

So, it is quite natural to assume that at the initial moment of a quantum universe creation there is just a single energy scale introduced as the initial data. The present Hubble constant value and the Planck mass are related to each other just by the age of the Universe expressed in terms of the redshift. The evolution of an energy scale of a physical quantity from the initial moment to the present day is obviously defined by the corresponding dimension $d$ and conformal weight $w$. In particular besides the evolution of the Hubble constant and the Planck mass, we can consider evolution of a particle mass which takes today the value $m_0$

$$H_0 \simeq H_f \cdot (1 + z_f)^{-2}, \quad \{d = 1, w = 2\}, \quad (97)$$

$$m_0 \simeq H_f \cdot (1 + z_f)^{1}, \quad \{d = 1, w = -1\}, \quad (98)$$

$$M_{Pl}^* \simeq H_f \cdot (1 + z_f)^{2}, \quad \{d = 1, w = -2\}. \quad (99)$$

Using the known values of $H_0$ and $M_{Pl}^*$ together with the defined above magnitude of $z_f$, we see that $m_0 \simeq 300 \text{ GeV}$, i.e. it is just of the order of the electroweak energy scale. Obviously, we still have to demonstrate how does a massive particle emerge in the conformally symmetric model and how its mass is related to the primary energy scale $H_f$. It is worth to note that in our construction $m_0$ provides the order of the maximal possible mass of an elementary particle in the given universe, in accordance with the hypothesis about the so-called maximon proposed by M.A. Markov [30, 31].

We limit ourselves by the set of particles existing in the Standard Model. Obviously, the electroweak scale is related both to the Higgs boson mass (and its vacuum expectation value) and to the top quark mass. Note that the coincidence of the these energy scales is an unresolved puzzle within the SM. Here we assume that the mechanism of dimensional transmutation [32] is working in the sector of the SM, which contains the most intensive interactions of this model, see the next Section. At the Planck epoch renormalization energy scale is naturally given by $H_f$. The Coleman-Weinberg mechanism then provides non-zero masses and condensates for both scalar and fermion fields. Since the two coupling constants are both of the order of unity, the emerged masses and condensates are of the same order as $H_f$. In particular, using Eq. (97), we can relate the top quark mass with the Hubble constant and the Planck mass:

$$m_t \simeq H_0 [1 + z_f]^3, \quad m_t \simeq M_{Pl}^*[1 + z_f]^{-1}. \quad (100)$$

On the other hand, the top quark condensate should have the same energy scale so

$$\langle \bar{t}t \rangle = -\gamma_t m_t^3 \quad (101)$$

where $\gamma_t$ is a dimensionless constant of the order of unity. To get a concrete number for parameter $\gamma_t$, we have to regularize the divergent integral. It can be done by introduction of a simple distribution function:

$$\gamma_t = -\langle \bar{t}t \rangle / m_t^3 = 4N_c \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\sqrt{p^2 + 1}} f_{1+}(p) \approx 0.39,$$

$$f_{1\pm}(p) = \left[1 \pm \exp \left(\sqrt{p^2 + 1} - 1\right)\right]^{-1}. \quad (102)$$

This distribution function reflects the degeneration of the vacuum state over momenta. The set of the vacuum degeneration parameters can be treated as von Neumann’s “Statistishe Gesamtheiten” [23] or the Blokhintsev’s “Quantum Ensemble” [24].

**C. Conformal symmetry breaking in the Standard Model**

In ref. [33] a simple reduction of the SM to a conformal-invariant theory was suggested. As shown in [34, 35], the infrared instability of the theory leads

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3 Fermion condensates are negative by construction since they are integrals over a closed fermion loop.
to quantum anomalies which break the conformal symmetry spontaneously. This effect is clearly seen in the one-loop approximation for effective potential of the Higgs boson. We briefly repeat the main steps of this construction and extend it by using a direct calculation of the top quark condensate value.

Let us look at the Higgs boson sector of the SM taken without the tachyon mass term. For the first approximation we take the most intensive terms of Higgs boson interactions, i.e. the self-interaction and the Yukawa coupling with the top quark

\[ \mathcal{L}_{\text{int}}(\phi) \approx -\frac{\lambda}{4} \phi^4 - \frac{y_t}{2} \phi \bar{t} t. \]  

(103)

Note that we have here already the single neutral component of the primary complex doublet scalar field. Following the standard Brout–Englert–Higgs mechanism (and the Einstein’s cosmological principle), we split \( \phi \) into its mean field \( \langle \phi \rangle \) and particle-like non-zeroth harmonics \( h \)

\[ \phi = \langle \phi \rangle + h, \quad \int d^4 x h = 0. \]  

(104)

In the same manner, normal ordering of the fermion pair in the Yukawa interaction term in Eq. (103) can be decomposed as \( \bar{t} t = \hat{t} t + (\hat{t} \hat{t}) \). By construction \( m_t = \langle y_t / \sqrt{2} \rangle \langle \phi \rangle \) where \( y_t \approx 0.99 \) is the Yukawa coupling of the top quark, and \( v \approx 246.22 \) GeV is the Higgs boson vacuum expectation value.

One can see that the top quark condensate density supersedes the phenomenological negative square mass term in the Higgs potential. As a consequence, we have a non-trivial minimum in the Higgs field potential

\[ V(\phi) = \frac{\lambda}{4} \phi^4 + \frac{y_t}{\sqrt{2}} \phi \langle \hat{t} t \rangle, \]

\[ \frac{dV(\phi)}{d\phi} \bigg|_{\phi=v} = \lambda v^3 + \frac{y_t}{\sqrt{2}} \langle \hat{t} t \rangle = 0, \]

\[ \frac{d^2 V(\phi)}{d\phi^2} \bigg|_{\phi=v} = m_h^2 = 3\lambda v^2. \]  

(105)

So the Higgs particle mass in the tree-level approximation is defined as

\[ m_{h,0}^2 = \frac{3y_t \langle \hat{t} t \rangle}{\sqrt{2} v} \approx 131 \text{ GeV}, \]  

(106)

where we used Eq. (104) to define the top quark condensate value \( \langle \hat{t} t \rangle \approx -(126 \text{ GeV})^3 \) via the known \( m_t \approx 173 \) GeV. One can see that the standard Brout–Englert–Higgs mechanism is reproduced except that the value of the self-coupling constant \( \lambda \) is 3/2 times less than in the SM. Note that this quantity will be measured experimentally only at high-precision future linear \( e^+ e^- \) colliders. Note also the there is no any experimental data on the values of heavy quark condensates since they do not contribute to hadronic observables.

The above results are obtained in the lowest semi-classical approximation. Certainly, they can be shifted by radiative corrections. But, contrary to the case of the Standard Model, we do not have quadratically divergent corrections to the Higgs boson mass because they are cured by the conformal symmetry in the classical Lagrangian.

VI. CONCLUSIONS

We have shown that the Einstein theory supplemented by the electroweak Standard Model in terms of diffeo-invariant coordinates and variables predicts not only relativistic effects, such as the Mercury anomaly or the double Eddington’s angle of the photon deviation in the Sun gravitational field, but also the recent observational facts in Cosmology and the set of quantum effects. They include:

1. The arrow of time arises due to the vacuum postulate at the Planck scale

\[ (1 + z_f) = \left( \frac{M_{Pl}}{H_0} \right)^{1/4} \approx 10^{15}. \]

2. The SNe Ia data justify the dominance of the Casimir vacuum energy with the rigid-state cosmic evolution and the primordial Hubble parameter value \( H_I = H_0 (1 + z_f)^2 \).

3. The evolution of the vacuum condensates of elementary fields yields the proper order of the Higgs boson mass \( M_{\text{Higgs}} \approx H_0 (1 + z_f)^3 \).

4. The intensive vacuum creation of electroweak bosons [18] and their consequent decays yield the present day number of the CMB photons \( N_\gamma \approx \alpha_{\text{EM}}^6 (1 + z_f)^3 \approx 10^{87} \).

The presented above model for the definition of the Planck epoch is based on the assumption that the conformal symmetry is a fundamental property of Nature. A spontaneous breaking of this symmetry due to a quantum conformal anomaly provides a single energy scale. We have shown that the energy scales of the Hubble constant, the Planck mass, and the SM can be received by the cosmological evolution from the single scale taking into account different conformal weights of the corresponding quantities. As concerning the origin of the QCD energy scale, one can try to look for its relation to the electroweak one, but that goes beyond the scope of this article.

Some other aspects of the discussed above model were considered in our earlier papers [14, 18, 36]. Further work on construction and verification of the model is in order.

It is interesting to perform a super-analog of the Borisov-Ogievetsky nonlinear realization [14] in order to obtain the corresponding number of Goldstone fields for both the general relativity and the minimal Standard Model of the electroweak and strong interactions (10 tensors, 48 vectors, 96...
fermions, and 4 scalars). We have shown in ref. [37] that this super-nonlinear realization as the last unification of the GR and SM can be based on the quaternionic super-extension of the supertwistor construction given in ref. [38]. According to this quaternionic supertwistor construction, the number of fermionic fields can be extended beyond the pure supertwistor one with coherent super-quaternionic states spanning the super-Hilbert spaces.

ACKNOWLEDGMENT

The authors are grateful for useful discussions to participants of the seminars of the Institute for Gravitation and Cosmology, the People's Friendship University of Russia (PFUR), Moscow, and the Russian Gravitational Association (VNIIMS), Moscow. A. Pavlov is grateful to the Joint Institute for Nuclear Research for hospitality. NSH was supported by Nafosted under grant number 103.03-2012.02. A. Arbuzov thanks the Dynasty foundation for a

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