Abstract. Consider \( A(x, D) : C^\infty(\Omega, E) \to C^\infty(\Omega, F) \) an elliptic and canceling linear differential operator of order \( \nu \) with smooth complex coefficients in \( \Omega \subset \mathbb{R}^N \) from a finite dimension complex vector space \( E \) to a finite dimension complex vector space \( F \) and \( A^*(x, D) \) its adjoint. In this work we characterize the (local) continuous solvability of the partial differential equation \( A^*(x, D)u = f \) in the distribution sense for a given distribution \( f \); more precisely we show that any \( x_0 \in \Omega \) is contained in a neighborhood \( U \subset \Omega \) in which its continuous solvability is characterized by the following condition on \( f \): for every \( \varepsilon > 0 \) and any compact set \( K \subset U \), there exists \( \theta = \theta(K, \varepsilon) > 0 \) such that the following holds for all smooth function \( \varphi \) supported in \( K \):
\[
\| f(\varphi) \| \leq \theta \| \varphi \|_{W^{\nu-1,1}} + \varepsilon \| A(x, D) \varphi \|_{L^1},
\]
where \( W^{\nu-1,1} \) stands for the homogenous Sobolev space of all \( L^1 \) functions whose derivatives of order \( \nu - 1 \) belongs to \( L^1(U) \).

This characterization implies and extends results obtained before for operators associated to elliptic complex of vector fields (see [12], [13]); we also provide local analogues, for a large range of differential operators, to global results obtained for the classical divergence operator in [11] and [9].

1. Introduction

Consider \( \Omega \subset \mathbb{R}^N \) an open set and \( A(\cdot, D) \) a linear differential operator of order \( \nu \) with smooth complex coefficients in \( \Omega \) denoted by:
\[
A(x, D) = \sum_{|\alpha| \leq \nu} a_\alpha(x) \partial^\alpha : C^\infty(\Omega; E) \to C^\infty(\Omega; F),
\]
where \( E \) is a complex vector space of dimension \( n \) and \( F \) is a complex vector space of dimension \( n' \geq n \).

A series of results concerning on local \( L^1 \) estimates for linear differential operators has been studied by J. Hounie and T. Picon in the setting of elliptic systems of complex vector fields, complexes and pseudocomplexes ([12], [13]). The following characterization of local \( L^1 \) estimates for operators \( A(x, D) \) was proved in [11], namely:

**Theorem 1.1.** Assume, as before, that \( A(\cdot, D) \) is a linear differential operator of order \( \nu \) between the spaces \( E \) and \( F \). The following properties are equivalent:

1. \( A(x, D) \) is elliptic and canceling (see below for a definition of those properties);
   2. every point \( x_0 \in \Omega \) is contained in a ball \( B = B(x_0, r) \subset \Omega \) such that the a priori estimate
      \[
      \| u \|_{W^{\nu-1, N(\nu-1)}} \leq C \| A(x, D) u \|_{L^1},
      \]
      holds for some \( C > 0 \) and all smooth functions \( u \in C^\infty(B; E) \) having compact support in \( B \).

Here, given \( k \in \mathbb{N} \) and \( 1 \leq p \leq \infty \), \( W^{k,p}(\Omega) \) denotes the homogeneous Sobolev space of complex functions in \( L^p(\Omega) \) whose weak derivatives of order \( k \) belong to \( L^p(\Omega) \), endowed with the (semi-)norm
\[
\| u \|_{W^{k,p}} := \sum_{|\alpha| \leq k} \| \partial^\alpha u \|_{L^p}.
\]

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It turns out that elliptic linear differential operators that satisfy an \textit{a priori} estimate like \textup{[13]} can be characterized in terms of properties of their principal symbol $a_\nu(x, \xi) = \sum_{|\alpha| = \nu} a_\alpha(x) \xi^\alpha$. Recall that the \textit{ellipticity} of $A(x, D)$ at $x_0 \in \Omega$ means that for every $\xi \in \mathbb{R}^N \setminus \{0\}$ the map $a_\nu(x_0, \xi) : E \to F$ is injective.

**Definition 1.2.** Let $x_0 \in \Omega$. A linear partial differential operator $A(x, D)$ of order $\nu$ from $E$ to $F$ with principal symbol $a_\nu(x, \xi)$ that satisfies:

\begin{equation}
\bigcap_{\xi \in \mathbb{R}^N \setminus \{0\}} a_\nu(x_0, \xi)[E] = \{0\}
\end{equation}  

is said to be \textit{canceling at} $x_0$. If \textup{(\star)} holds for every $x_0 \in \Omega$ we say that $A(x, D)$ is \textit{canceling}.

Examples of canceling operators satisfying \textup{(\star)} can be founded in \textup{[13]}; this is the case in particular for operators associated to elliptic system of complex vector fields (see \textup{[12], [13]}). The canceling property for linear differential operators was originally defined by Van Schaftingen \textup{[14]} in the setup of homogeneous operators with constant coefficients $A(D)$ and stands out by several applications (and characterizations) in the theory of \textit{a priori} estimates in $L^1$ norm (see for instance \textup{[20]} for a brief description).

In this work, we are interested to study the (local) continuous solvability in the weak sense of the equation:

\begin{equation}
A^*(x, D) v = f,
\end{equation}

where $A(x, D)$ is an elliptic and canceling linear differential operator. We use the notation $A^* := \overline{A^t}$ where $\overline{A^t}$ denotes the operator obtained from $A$ by conjugating its coefficients and $A^t$ is its formal transpose — namely this means that, for all smooth functions $\varphi$ and $\psi$ having compact support in $\Omega$ and taking values in $E$ and $F$ respectively, we have:

\[
\int_\Omega A(x, D) \varphi \cdot \psi = \int_\Omega \varphi \cdot \overline{A^t(x, D)} \psi.
\]

Our main result is the following.

**Theorem 1.3.** Assume $A(x, D)$ is as before. Then every point $x_0 \in \Omega$ admits an open neighborhood $U \subset \Omega$ such that for any $f \in \mathcal{D}'(U)$, the equation \textup{(2)} is continuously solvable in $U$ if and only if $f$ is an $A$-charge in $U$, meaning that for every $\varepsilon > 0$ and every compact set $K \subset \subset U$, there exists $\theta = \theta(K, \varepsilon) > 0$ such that one has:

\begin{equation}
\|f(\varphi)\| \leq \theta \|\varphi\|_{W^{0, 1}} + \varepsilon \|A(x, D) \varphi\|_{L^1},
\end{equation}

for any smooth function $\varphi$ in $U$ vanishing outside $K$.

One simple argument (see Section \textup{3}) shows that the above continuity property on $f$ is a necessary condition for the continuous solvability of equation \textup{(4)} in $U$. Theorem \textup{1.3} asserts that the continuity property \textup{(3)} is also sufficient, under the canceling and ellipticity assumptions on the operator. The proof, which will be presented in Section \textup{3}, is based on a functional analytic argument inspired from \textup{11} and already improved in \textup{[15]} for divergence-type equations associated to complexes of vector fields (observe in particular that one recovers \textup{[11], Theorem 1.2]} when applying Theorem \textup{1.3} to the latter context). However, it should be mentioned here that by allowing in \textup{(2)} a much larger class of (higher order) differential operators, that method of proof had to be very substantially refined, leading to the use of new tools. Applications of Theorem \textup{1.3} are presented in the Section \textup{4}.

Assume for a moment that $A(x, D)$ be elliptic. We point out the canceling assumption - characterized by inequality \textup{[11]} - plays a fundamental role in our argument in the proof of Theorem \textup{1.3}. However, we should emphasize that this property might not be necessary to obtain a characterization of continuous solutions to the equation \textup{(4)} formulated along the previous lines. In the context of the Poisson equation with measure data $\Delta u = \mu$ (where the Laplacean operator is \textit{not} a canceling operator), it follows indeed from a result by Aizenman and Simon \textup{[2], Theorem 4.14] (see also Ponce \textup{[14], Proposition 18.1]}, where this question is studied in a luminous fashion) that, given a smooth, bounded open set $\Omega$ in $\mathbb{R}^n$ and a
measure $\mu$ in $\Omega$, the Dirichlet problem associated with $\Delta u = \mu$ in $\Omega$ has a continuous solution in $\Omega$ if and only if, for every $\varepsilon > 0$, there exists $\theta > 0$ such that for any $\varphi \in C^0_{0}(\Omega)$, one has:

$$\int_{\Omega} \varphi \, d\mu \leq \theta \|\varphi\|_{L^1} + \varepsilon \|\Delta \varphi\|_{L^1}.$$ 

This result, however, is proved using very different techniques than the ones developed here.

2. Preliminaries and notations

We always denote by $\Omega$ an open set of $\mathbb{R}^N$, $N \geq 2$. Unless otherwise specified, all functions are complex valued and the notation $\int_{A} f$ stands for the Lebesgue integral $\int_{\Omega} f(x) \, dx$. As usual, $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$ are the spaces of complex test functions and distributions, respectively. When $K \subset \subset \Omega$ is a compact subset of $\Omega$, we let $\mathcal{D}(\Omega) := \mathcal{D}(\Omega) \cap \mathcal{E}'(K)$, where $\mathcal{E}'(K)$ is the space of all distributions with compact support in $K$. Since the ambient field is $\mathbb{C}$, we identify (formally) each $f \in L^1_{loc}(\Omega)$ with the distribution $T_f \in \mathcal{D}'(\Omega)$ given by $T_f(\varphi) = \int_{\Omega} f \varphi$. We also consider $C(\Omega)$ the space of all continuous functions in $\Omega$. When working with objects in a function space taking values in a finite-dimensional (normed) vector space $E$, we shall indicate it as a second argument (e.g. $C(\Omega, E)$ will denote the space of all $E$-valued continuous vector fields $v : \Omega \to E$). Finally we use the notation $f \preceq g$ to indicate the existence of an universal constant $C > 0$, independent of all variables and unmentioned parameters, such that one has $f \leq Cg$.

Some Sobolev spaces. Given a finite-dimensional (complex, normed) vector space $E$, we denote as before by $W^{k,p}(\Omega; E)$ for $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, the homogeneous Sobolev space of functions in $L^p(\Omega; E)$ whose weak derivatives of order $k$ belong to $L^p(\Omega; E)$, endowed with the (semi-)norm $\|u\|_{W^{k,p}} := \sum_{|\alpha| \leq k} \|\partial^\alpha u\|_{L^p}$; we also denote by $W^{k,p}_c(\Omega; E)$ the space of its elements having compact support in $\Omega$. Given $1 < p \leq \infty$ and $k \in \mathbb{N}^*$ we also define the space $W^{k,p}_0(\Omega; E^*) := \{f \in W^{k,p}(\Omega; E) : \partial F = f\}$ as the space of distributions $f \in \mathcal{D}'(\Omega; E^*)$ enjoying the following property: for all $K \subset \subset \Omega$, there exists $C_K > 0$ such that for all $\varphi \in \mathcal{D}_K(\Omega; E)$, one has:

$$|\langle f, \varphi \rangle| \leq C_K \|\varphi\|_{W^{k,p}_0(\Omega)} = C_K \sum_{|\alpha| = k} \|\partial^\alpha \varphi\|_{p'},$$

where $1 \leq p' < \infty$ is defined by $\frac{1}{p} + \frac{1}{p'} = 1$.

Given $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, one also defines the (classical, inhomogeneous) Sobolev space $W^{k,p}(\Omega; E)$ as the space of complex functions in $L^p(\Omega; E)$ whose weak derivatives up to order $k$ belong to $L^p(\Omega; E)$, endowed with the norm $\|u\|_{W^{k,p}} := \sum_{|\alpha| \leq k} \|\partial^\alpha u\|_{L^p}$. We denote finally by $W^{k,p}_0(\Omega; E)$ the completion of the space $\mathcal{D}(\Omega; E)$ in $W^{k,p}(\Omega; E)$. The space $W^{k,p}_0(\Omega; E)$ is classically reflexive and separable for $1 < p < \infty$ (see e.g. [11] p. 64).

Lemma 2.1. Fix $k \in \mathbb{N}^*$ and $1 < p \leq \infty$. Given $f \in W^{-k,p}_0(\Omega; E^*)$ and $K \subset \subset \Omega$ there exists $(g_\alpha)_{|\alpha| = k} \in L^p(\Omega; E^*)$ for which one has, for all $\varphi \in \mathcal{D}_K(\Omega; E)$:

$$\langle f, \varphi \rangle = \sum_{|\alpha| = k} \int_\Omega g_\alpha \partial^\alpha \varphi.$$ 

Remark 2.2. In the latter expression, and throughout this paper, one uses, given $e \in E$ and $e^* \in E^*$ (and in any finite-dimensional duality setting), the notation $e \ast e^*$ instead of $(e^*, e)$.

Proof. Fix $f \in W^{-k,p}_0(\Omega; E^*)$ and $K \subset \subset \Omega$. Denote by $M$ the number of multi-indices $\alpha \in \mathbb{N}^n$ with $|\alpha| = k$ and let:

$$X := \{ (u_\alpha)_{|\alpha| = k} : \text{there exists } \varphi \in \mathcal{D}_K(\Omega; E) \text{ with } u_\alpha = \partial^\alpha \varphi \text{ for all } |\alpha| = k \},$$

be endowed with the norm $\|(u_\alpha)_{|\alpha| = k}\| := \sum_{|\alpha| = k} \|u_\alpha\|_{p'}$ — we hence see it as a subspace of $L^{p'}(\Omega, E^M)$.

Now define a linear functional $F$ on $X$ in the following way: if $(u_\alpha)_{|\alpha| = k} \in X$ is given, there exists a unique $\varphi \in \mathcal{D}_K(\Omega; E)$ with $u_\alpha = \partial^\alpha \varphi$ for all $|\alpha| = k$; we then let $\langle F, (u_\alpha) \rangle := \langle f, \varphi \rangle$. There holds:

$$|\langle F, (u_\alpha) \rangle| \leq C_K \sum_{|\alpha| = k} \|u_\alpha\|_{p'},$$
Assume $\Omega$ to be an open set and $x$. By the Hahn-Banach theorem, $F$ extends to a continuous linear functional on $L^p(\Omega, E')$ satisfying

$$|\langle F, (u_\alpha) \rangle| \leq C_K \sum_{|\alpha|=k} \|u_\alpha\|_{p'},$$

for all $(u_\alpha) \in L^p(\Omega, E')$; there hence exists $(g_\alpha) \in L^p(\Omega, (E^*)^M) = (L^p(\Omega; E^M))^*$ such that one has:

$$\langle F, (u_\alpha) \rangle = \sum_{|\alpha|=k} \int_{\Omega} g_\alpha \bar{u}_\alpha,$$

for all $(u_\alpha) \in L^p(\Omega, E^M)$. We hence get in particular, for $\varphi \in \mathcal{D}_K(\Omega; E)$:

$$\langle f, \varphi \rangle = \langle f, (\partial^\alpha \varphi)_{|\alpha|=k} \rangle = \sum_{|\alpha|=k} \int_{\Omega} g_\alpha \bar{\partial^\alpha \varphi}.$$ 

The proof is complete.

Example 2.3. Assume $\Omega$ to be an open set and fix $k \in \mathbb{N}$. Then one has $L^N_{\text{loc}}(\Omega; E^*) \subseteq W^{-k, N}_{\text{loc}}(\Omega; E^*)$.

To see this, observe the statement is obvious in case $k = 0$. Hence assume $k \in \mathbb{N}^*$ and fix $f \in L^N_{\text{loc}}(\Omega; E^*)$. Let $K \subset \subset \Omega$ be compact and compute now for $\varphi \in \mathcal{D}_K(\Omega)$:

$$\left| \int_{\Omega} f \bar{\varphi} \right| \leq \|f\|_{L^N(K; E^*)} \|\varphi\|_{L^{N(N-1)}(K; E)} = \|f\|_{L^N(K; E^*)} \|\varphi\|_{L^{N(N-1)}(\mathbb{R}^N; E)}.$$

Yet we get, according to the Sobolev-Gagliardo-Nirenberg inequality (which we shall refer to as the “SGN” inequality in the sequel):

$$\|\varphi\|_{L^{N(N-1)}(\mathbb{R}^N; E)} \leq \kappa(N) \sum_{|\alpha|=1} \|\partial^\alpha \varphi\|_{L^1(\mathbb{R}^N; E)} = \kappa(N) \sum_{|\alpha|=1} \|\partial^\alpha \varphi\|_{L^1(\Omega; E)},$$

and hence we find, for all $\varphi \in \mathcal{D}_K(\Omega)$:

$$\left| \int_{\Omega} f \bar{\varphi} \right| \leq \kappa(N) \|f\|_{L^N(K; E^*)} \|\varphi\|_{W^{1,1}(\Omega; E)}.$$

If $k = 2$, then for all $|\alpha| = 1$ we have, using Hölder’s inequality and SGN inequality again:

$$\|\partial^\alpha \varphi\|_{L^1(\Omega; E)} = \|\partial^\alpha \varphi\|_{L^1(K; E)} \leq |K|^\frac{1}{N} \|\partial^\alpha \varphi\|_{L^{N(N-1)}(\mathbb{R}^N; E)} \leq |K|^\frac{1}{N} \sum_{|\beta|=1} \|\partial^{\alpha+\beta} \varphi\|_{L^1(\mathbb{R}^N; E)} = |K|^\frac{1}{N} \sum_{|\beta|=1} \|\partial^{\alpha+\beta} \varphi\|_{L^1(\Omega; E)},$$

which implies, for all $\varphi \in \mathcal{D}_K(\Omega)$:

$$\left| \int_{\Omega} f \bar{\varphi} \right| \leq \kappa(N) \|f\|_{L^N(K; E^*)} \|\varphi\|_{W^{1,1}(\Omega; E)} = \kappa(N) \|f\|_{L^N(K; E^*)} |K|^\frac{1}{N} \|\varphi\|_{W^{k,1}(\Omega; E)}.$$

One proves inductively for a general $k \in \mathbb{N}^*$ one has:

$$\left| \int_{\Omega} f \bar{\varphi} \right| \leq \kappa(N) \|f\|_{L^N(K; E^*)} |K|^\frac{k-1}{N} \|\varphi\|_{W^{k,1}(\Omega; E)} \leq \kappa(N) \|f\|_{L^N(K; E^*)} |K|^\frac{k-1}{N} \|\varphi\|_{W^{k,N/(N-1)}(\Omega; E)},$$

where Hölder’s inequality is used again; this means finally that one has $f \in W^{-k, N}_{\text{loc}}(\Omega; E^*)$.

Remark 2.4. Using approximation by smooth functions and a recursive use of the SGN inequality as done in the above example, one shows that, given $K \subset \subset \Omega$ a compact set and an integer $k \in \mathbb{N}$, there exists a constant $C(K, k) > 0$ such that for any $g \in W^{k,N/(N-1)}(\Omega; E)$ satisfying $\text{supp} g \subseteq K$, one has $g \in W^{k,N/(N-1)}_{0}(\Omega; E)$ and:

$$\|g\|_{W^{k,N/(N-1)}(\Omega; E)} \leq C(K, k) \|g\|_{W^{k,N/(N-1)}(\Omega; E)}.$$
ON LOCAL CONTINUOUS SOLVABILITY OF EQUATIONS ASSOCIATED TO ELLIPTIC AND CANCELING LINEAR DIFFERENTIAL OPERATORS

3. CANCELING AND ELLIPTIC DIFFERENTIAL OPERATORS

Given \( A(x, D) \) as before, the 2r-order differential operator \( \Delta_A := A^*(\cdot, D) \circ A(\cdot, D) \) may be regarded as an elliptic pseudodifferential operator with symbol in the Hörmander class \( S^{2r}(\Omega) \), so that there exist properly supported pseudodifferential operators \( q, \tilde{q} \in OpS^{-2\nu}(\Omega) \) (parametrixes) and \( r, \tilde{r} \in OpS^{-\infty}(\Omega) \) for which one has, for any \( f \in C^\infty(\Omega, F) \):

\[
\Delta_A q(x, D)f + r(x, D)f = \tilde{q}(x, D)\Delta_A f + \tilde{r}(x, D)f = f.
\]

Writing \( \Delta_A q(x, D)f = A^*(x, D)u \) for \( u = A(x, D)q(x, D)f \) we then get:

\[
A^*(x, D)u - f = r(x, D)f
\]

for every \( f \in C^\infty(\Omega, F) \).

Proposition 3.1. Assume that \( A(x, D) \) is as before. Then for every point \( x_0 \in \Omega \) and any \( 0 < \beta < 1 \), there exist an open ball \( B = B(x_0, \ell) \subset \Omega \) and a constant \( C = C(B) > 0 \) such that, for all \( \varphi \in \mathcal{D}(B, E) \), one has:

\[
\sum_{|\alpha| = \nu - 1} \| \partial^\alpha \varphi \|_{1-\beta,1} \leq C \| A(x, D)\varphi \|_{L^1}.
\]

The previous inequality states the embedding into \( L^1 \) of some version of a fractional Sobolev space \( W^{1-\beta,1}_{c,0}(B) \) that can be defined according to the following procedure. Given \( B = B(x_0, \ell) \) a ball consider \( \tilde{B} = B(x_0, 2\ell) \) the ball with the same center as \( B \) but twice its radius. Let \( \psi \in \mathcal{D}(\tilde{B}) \) satisfy \( \psi(x) \equiv 1 \) on \( B \) and define \( \Lambda_\gamma := \Lambda_\gamma(x, D) \) the pseudodifferential operator with symbol \( \lambda_\gamma(x, \xi) = \psi(x)(1 + 4\pi^2|\xi|^2)^\gamma \in S^\gamma(\mathbb{R}^N) \). Denote then by \( W^{\gamma,p}_{c,0}(B) \) the set of distributions with compact support \( f \in \mathcal{E}'(B) \) such that one has \( \Lambda_\gamma f \in L^p(\mathbb{R}^N) \); one endows it with the semi-norm \( \| f \|_{\gamma,p} := \| \Lambda_\gamma u \|_p \). Note that the space \( W^{\gamma,p}_{c,0}(B) \) is independent of the choice of \( \psi \), i.e. that if \( \psi_2(x), \psi_1(x) \in \mathcal{D}(\tilde{B}) \) satisfy \( \psi_1(x) = \psi_2(x) \equiv 1 \) on \( \tilde{B} \), then \( \| \Lambda_{\gamma,\psi_1} f \|_{L^p} \equiv \| \Lambda_{\gamma,\psi_2} f \|_{L^p} \).

Proof. Fix \( \alpha \) a multi-index with \( |\alpha| = \nu - 1 \). Let \( h = A(x, D)\varphi \). Thanks to identity (11) and to the calculus of pseudodifferential operators we have

\[
\Lambda_{1-\beta} \partial^\alpha \varphi = p(x, D)h + r'(x, D)\varphi,
\]

where \( p(x, D) := \Lambda_{1-\beta} \partial^\alpha q(x, D)A^*(x, D) \in OpS^{-\beta} \) and \( r'(x, D) := \Lambda_{1-\beta} \partial^\alpha r(x, D) \in OpS^{-\infty} \). As a consequence of [11, Theorem 6.1] we have \( \| p(x, D)h \|_{L^1} \leq \| A(x, D)\varphi \|_{L^1} \), which implies:

\[
\| \Lambda_{\beta-1} \partial^\alpha \varphi \|_{L^1} \leq C \| A(x, D)\varphi \|_{L^1} + \| r'(x, D)\varphi \|_{L^1}.
\]

As the second term on the right side may be absorbed (see [12, p. 798]), shrinking the neighborhood if necessary, we obtain the estimate (2) after recalling estimate (11) in Theorem [11] above.

4. FUNCTIONS OF BOUNDED VARIATION ASSOCIATED TO \( A(x, D) \)

4.1. Basic definitions; approximation and compactness. Let \( W^{k,p}(\Omega; E) \) be the linear space of all complex functions in \( W^{k,p}(\Omega; E) \) whose support is a compact subset of \( \Omega \).

The following definition of variation associated to \( A(x, D) \) of \( g \in W^{
u-1,1}_c(\Omega; E) \) recalls the classical definition of variation when \( \nu = 1 \) and \( A(x, D) = \nabla \). It has been formulated for real vector fields by N. Garofalo and D. Nhieu [13] and adapted for complex vector fields in [11].

Definition 4.1. Given \( g \in W^{
u-1,1}_c(\Omega; E) \) and \( U \subseteq \Omega \) an open set, one calls the extended real number:

\[
|D_A g|(U) := \sup \left\{ \int_\Omega g A^*(\cdot, D)v : v \in C^\infty_c(\Omega; F^*), \supp v \subseteq U, \| v \|_\infty \leq 1 \right\},
\]

the (total) \( A \)-variation associated to \( A(x, D) \) of \( g \) in \( U \) and we let \( |D_A g| := |D_A g|(\Omega) \) in case there is no ambiguity on the open set \( \Omega \). We denote by \( BV_{A,c}(\Omega) \) the set of all \( g \in W^{
u-1,1}_c(\Omega; E) \) with \( |D_A g| < +\infty \).

\[1\] Since we will only work with symbols of type \((1,0)\), the type will be omitted in the notation; concerning pseudodifferential operators we refer, for instance, to [11, Chapter 3] and [13].
Given \( g \in BV_{A,c}(\Omega) \), we denote by \( D_A g \) the unique \( F \)-valued Radon measure satisfying:

\[
\int_{\Omega} g A^*(\cdot, D) v = \int_{\Omega} \bar{v} \cdot d[D_A g],
\]

for all \( v \in C_c^\infty(\Omega, F^*) \). It is clear by definition that \( \|D_A g\| \) is also the total variation in \( \Omega \) of \( D_A g \).

**Remark 4.2.** Given \( g \in BV_{A,c}(\Omega) \), one has \( \supp D_A g \subseteq \supp g \). Indeed, given \( x \in \Omega \setminus \supp g \), find a radius \( r > 0 \) for which one has \( B(x, r) \subseteq \Omega \setminus \supp g \). We then have \( D_A g(v) = 0 \). Hence we also get \( D_A g(v) = 0 \) for all \( v \in C_c(B(x, r), F^*) \), which ensures that one has \( x \notin \supp D_A g \) and finishes to show the inclusion \( \supp D_A g \subseteq \supp g \).

**Remark 4.3.** It follows readily from the previous definition that if \( (g_i) \subseteq BV_{A,c}(\Omega) \) converges in \( L^1 \) to \( g \in W^{\nu-1,1}(\Omega) \), one then has \( g \in BV_{A,c}(\Omega) \) and:

\[
\|D_A g\| = \lim_{i \to \infty} \|D_A g_i\|.
\]

We shall refer to this in the sequel as the lower semi-continuity of the \( A \)-variation.

We say that a sequence \((f_i)_i\) of functions with complex values defined on open set \( \Omega \subset \mathbb{R}^N \) is compactly supported in \( \Omega \) if there is a compact set \( K \subset \Omega \) such that one has \( \supp f_i \subseteq K \) for every \( i \).

We shall make an extensive use of the following concept of convergence.

**Definition 4.4.** Given \( g \in W^{\nu-1,1}(\Omega; E) \) and a sequence \((\varphi_i)_i \subseteq \mathcal{D}(\Omega; E) \) we shall write \( \varphi_i \rightharpoonup g \) in case the following conditions hold:

\begin{enumerate}[(i)]
  \item \((\varphi_i)\) converges to \( g \) in \( W^{\nu-1,1} \) norm;
  \item \((\varphi_i)\) is compactly supported in \( \Omega \);
  \item \( \sup |A\varphi_i|_1 < +\infty \).
\end{enumerate}

The following lemma is a Friedrich’s type lemma; in the case where \( A \) is a system of real vector fields, it reminds a result by N. Garofalo and D. Nhieu [A, Lemma A.3]. In order to state it, fix \( \eta \in \mathcal{D}(\mathbb{R}^n) \) a radial function with nonnegative values, satisfying \( \supp \eta \subseteq B(0,1) \) and \( \int_{\mathbb{R}^n} \eta = 1 \), and, for each \( \varepsilon > 0 \), define \( \eta_\varepsilon \in \mathcal{D}(\mathbb{R}^n) \) by \( \eta_\varepsilon(x) := \varepsilon^{-N} \eta(x/\varepsilon) \).

**Lemma 4.5.** For any \( g \in BV_{A,c}(\Omega) \), one has:

\[
\lim_{\varepsilon \to 0} \|A(\eta_\varepsilon * g) - \eta_\varepsilon * (D_A g)\|_{L^1(\Omega)} = 0.
\]

**Proof.** We can assume for simplicity \( E = \mathbb{R}^n \) and \( F = \mathbb{R}^n \). Let us first assume that \( \nu = 1 \) and write \( A = c + \sum_{j=1}^m a_j \partial_j \), where \( c \) and \( a_j \), \( 1 \leq j \leq m \) are locally Lipschitz functions. Write now, for \( x \in \Omega \) and \( \varepsilon > 0 \) small enough so that one has \( |c(y) - c(z)| \leq \delta \) for all \( y, z \in \Omega \) with \( |y - z| \leq \varepsilon \):

\[
|c(x)(\eta_\varepsilon * g)(x) - [\eta_\varepsilon * (cg)](x)| \leq \int_{B(x, \varepsilon)} |c(x) - c(y)||g(y)||\eta_\varepsilon(x-y)| dy \lesssim \delta \int_{B(x, \varepsilon)} |g(y)||\eta_\varepsilon(x-y)| dy.
\]

We hence have, for \( \varepsilon > 0 \) small enough, using Fubini’s theorem:

\[
\|c(\eta_\varepsilon * g) - \eta_\varepsilon * (cg)\|_{L^1(\Omega)} \lesssim \delta \|g\|_{L^1(\Omega)}.
\]

Writing now \( L = \sum_{j=1}^N a_j \partial_j \), it now follows from [A, Lemma A.3] that one also has:

\[
\lim_{\varepsilon \to 0} \|L(\eta_\varepsilon * g) - \eta_\varepsilon * L(g)\|_{L^1(\Omega)} = 0.
\]

This finishes the proof in case \( \nu = 1 \).

In case \( \nu > 1 \), it suffices to consider the case \( A = a \partial^\alpha \) for some multi-index \( \alpha \in \mathbb{N}^n \) with \( |\alpha| \leq \nu \), and some locally Lipschitz function \( a \). The case \( |\alpha| \leq 1 \) being already dealt with using the preceding part of the proof, we can assume \( A = A' \partial_j \) where \( A' = a' \partial^\alpha' \) for \( |\alpha'| \geq 1 \). Since we can now write (using the notation \([\cdot, \cdot]\) for the commutator):

\[
[A, \eta_\varepsilon * (\cdot)] g = [A', \eta_\varepsilon * (\cdot)] \partial_i g,
\]
it follows inductively from the fact that \( \partial_t g \in W^{-2,1}_e(\Omega) \subseteq L^1(\Omega) \) and from the preceding part of the proof, that one has:

\[
\| [A, \eta_* (\cdot)] g \|_{L^1(\Omega)} \to 0, \quad \varepsilon \to 0.
\]

The proof is complete. 

We now obtain an analogous result, in \( BV_{A,c} \), to the standard approximation theorem for \( BV_e \) functions.

**Lemma 4.6.** For any \( g \in BV_{A,c}(U) \), there exists a sequence \( \{\varphi_i\}_i \subseteq \mathcal{D}(U) \) such that one has \( \varphi_i \to g \) and, moreover:

\[
\| D_A g \| = \lim_i \| A(\cdot, D) \varphi_i \|_1.
\]

**Proof.** Fix now \( g \in BV_{A,c}(\Omega) \) and define for \( 0 < \varepsilon < \text{dist}(\text{supp} g, \Omega) \) a function \( g_\varepsilon \in \mathcal{D}(\Omega) \) by the formula:

\[
g_\varepsilon := \eta_\varepsilon \ast g.
\]

It is easy to see that one has \( g_\varepsilon \to g \) in \( L^1(\Omega) \) and that there exists a compact set \( K \subseteq \Omega \) such that one has \( \text{supp} g_\varepsilon \subseteq K \) for all \( \varepsilon > 0 \) small enough.

On the other hand, observe that according to the previous lemma, one can write for \( \varepsilon > 0 \) small enough:

\[
\langle \eta_\varepsilon \ast (D_A g), \varphi \rangle = \langle D_A g, \eta_\varepsilon \ast \varphi \rangle = \int \eta_\varepsilon \ast \varphi \cdot d[DAg] = \int g A^*(\cdot, D)(\eta_\varepsilon \ast \varphi).
\]

so that one also has:

\[
\left| \int \Omega A(\eta_\varepsilon \ast g) \cdot \varphi \right| \leq \left[ \int \Omega g A^*(\eta_\varepsilon \ast \varphi) \right] + \| H_\varepsilon (g) \|_1 \leq \| D_A g \| + \| H_\varepsilon (g) \|_1.
\]

We hence get, by duality:

\[
\| A(\eta_\varepsilon \ast g) \|_1 \leq \| D_A g \| + \| H_\varepsilon (g) \|_1,
\]

and the result follows from the aforementioned property of \( H_\varepsilon (g) \) when \( \varepsilon \) approaches 0, and from the lower semicontinuity property already mentioned.

The following proposition is a compactness result in \( BV_A \).

**Proposition 4.7.** Assume that the open set \( U \subseteq \Omega \) supports a SGN inequality of the type appearing in [Theorem 1.1, (1)] as well as an inequality of type (2) for some \( 0 < \beta < 1 \). If \( (g_i) \subseteq BV_{A,c}(U) \) is compactly supported in \( U \) and if moreover one has:

\[
\sup_i \| D_A g_i \| < +\infty,
\]

then there exists \( g \in BV_{A,c}(U) \) and a subsequence \( (g_i) \subseteq (g_i) \) converging to \( g \) in \( W^{\nu - 1,1}_e \).

**Proof.** Choose a compact set \( K \subseteq U \) for which one has \( \text{supp} g_i \subseteq K \) for all \( i \). Choose also, according to Lemma 4.4, a sequence \( \{\varphi_i\}_i \subseteq \mathcal{D}(U) \) and a compact set \( K' \subseteq U \) satisfying the following conditions for all \( i \):

\[
\text{supp} \varphi_i \subseteq K', \quad \| g_i - \varphi_i \|_{W^{\nu - 1,1}_e} \leq 2^{-i} \quad \text{and} \quad \| A \varphi_i \|_1 \leq \| D_A g_i \| + 1.
\]

We hence have \( \sup_i \| A \varphi_i \|_1 < +\infty \) while it is clear that \( (\varphi_i) \) is compactly supported and satisfies \( \| g_i - \varphi_i \|_{W^{\nu - 1,1}_e} \to 0, \quad i \to \infty \).

Now fix \( 0 < \beta < 1 \), a multi-index \( \alpha \in \mathbb{N} \mathbb{N} \) satisfying \( |\alpha| = \nu - 1 \) and observe that the sequence \( (\psi_i)_i \) also satisfies, according to (2):

\[
\sup_i \| \partial^\alpha \psi_i \|_{1-\beta,1} = \sup_i \| A_{1-\beta} \partial^\alpha \psi_i \|_1 \leq C \sup_i \| A \varphi_i \|_1 < +\infty.
\]

It hence follows from the compactness of the inclusion of \( W^{1-\beta,1}_e(U) \subseteq L^1(U) \) (see [11, Theorem 6.2]) that there exists \( h^\alpha \in L^1_c(U) \) and a subsequence \( (\varphi^\alpha_{i_k}) \subseteq (\varphi_i) \) such that \( \partial^\alpha \varphi^\alpha_{i_k} \) converges to \( h^\alpha \) in \( L^1(U) \). This yields a subsequence \( (\varphi_{i_k}) \subseteq (\varphi_i) \) such that, for all \( \alpha \in \mathbb{N}^N \) with \( |\alpha| = \nu - 1 \), one has \( \partial^\alpha \varphi_{i_k} \to h^\alpha \),
\(k \to \infty\). Using the Rellich-Kondrachov theorem (see e.g. Ziemer [22, Theorem 2.5.1]), we can moreover assume that \(\varphi_{i_k} \to g\) in \(L^1_c(U)\). According to the closing lemma (see Willem [21, Lemma 6.1.5]), we then have \(h^\alpha = \partial^\alpha g\) for all \(\alpha \in \mathbb{N}^N\) with \(|\alpha| = \nu - 1\). This ensures \(g \in W^{\nu-1,1}(U)\) and the convergence of \((g_{i_k})\) to \(g\) in \(W^{\nu-1,1}(U)\). Moreover, the semicontinuity property of the \(A\)-variation yields \(g \in BV_{A,c}(U)\), which terminates the proof.

**Remark 4.8.** According to Theorem [21] and Proposition [20], we see that if one assumes \(A\) to be elliptic and canceling, each point \(x_0 \in \Omega\) is contained a neighborhood \(U \subseteq \Omega\) satisfying the hypotheses of the previous proposition.

### 4.2. A Sobolev-Gagliardo-Nirenberg inequality in \(BV_A\)

As announced we get the following result:

**Proposition 4.9.** Let \(A(x,D)\) be as before. Then every point \(x_0 \in \Omega\) is contained in an open neighborhood \(U \subseteq \Omega\) such that the inequality:

\[
\|g\|_{W^{\nu-1,N/N-1}} \leq C \|D_Ag\|,
\]

holds for all \(g \in BV_{A,c}(U)\), where \(C = C(U) > 0\) is a constant depending only on \(U\).

**Proof.** Fix \(x_0 \in \Omega\). It follows from Theorem [21] that there exists a neighborhood \(U \subseteq \Omega\) of \(x_0\) and \(C = C(U) > 0\) such that, for all \(\varphi \in \mathcal{D}(U;E)\), one has:

\[
\|\varphi\|_{W^{\nu-1,N/N-1}} \leq C \|A(\cdot, D)\varphi\|_1.
\]

Then given \(g \in BV_{A,c}(U)\) consider a sequence \(\{\varphi_i\} \subseteq \mathcal{D}(U;E)\) satisfying (i)-(iii) by Lemma [21]. As a consequence of Fatou’s Lemma and the previous estimate we conclude (extracting if necessary a subsequence) that:

\[
\|g\|_{W^{\nu-1,N/N-1}} \leq \lim_{i \to \infty} \|\varphi_i\|_{W^{\nu-1,N/N-1}} \leq C \lim_{i \to \infty} \|A(\cdot,D)\varphi_i\|_1 = C \|D_Ag\|.
\]

The proof is complete.

### 5. \(A\)-charges and their extensions to \(BV_{A,c}\)

We now get back to the original problem of finding, locally, a continuous solution to (2).

#### 5.1. \(A\)-fluxes and \(A\)-charges

Distributions which allow, in an open set \(\Omega\), to solve continuously (2), will be called \(A\)-fluxes.

**Definition 5.1.** A distribution \(\mathcal{F} \in \mathcal{D}'(\Omega)\) is called an \(A\)-flux in \(\Omega\) if the equation (2) has a continuous solution in \(\Omega\), i.e. if there exists \(v \in C(\Omega,F^*)\) such that one has, for all \(\varphi \in \mathcal{D}(\Omega;E)\):

\[
\mathcal{F}(\varphi) = \int_\Omega \bar{v} \cdot A(\cdot, D)\varphi, \quad \forall \varphi \in \mathcal{D}(\Omega;E).
\]

\(A\)-fluxes satisfy the following continuity condition.

**Lemma 5.2.** If \(\mathcal{F}\) is an \(A\)-flux then \(\lim_i \mathcal{F}(\varphi_i) = 0\) for every sequence \((\varphi_i) \subseteq \mathcal{D}(\Omega;E)\) verifying \(\varphi_i \rightharpoonup 0\).

**Proof.** Let \(\mathcal{F}\) be an \(A\)-flux and let \(v \in C(\Omega,F^*)\) be such that (1) holds. Fix a sequence \((\varphi_i) \subseteq \mathcal{D}(\Omega;E)\) verifying \(\varphi_i \rightharpoonup 0\), let \(c := \sup_i \|A\varphi_i\|_1 < +\infty\) and choose a compact set \(K \subset \subset \Omega\) for which one has \(\operatorname{supp} \varphi_i \subseteq K\) for all \(i\).

Fix now \(\varepsilon > 0\). According to Weierstrass’ approximation theorem, choose a vector field \(w \in \mathcal{D}(\Omega,F^*)\) for which one has \(\sup_K |v - w| \leq \varepsilon\) and compute, for all \(i\):

\[
|\mathcal{F}(\varphi_i)| \leq \left| \int_\Omega (\bar{v} - \bar{w}) \cdot A(\cdot, D)\varphi_i \right| + \left| \int_\Omega \bar{w} \cdot A(\cdot, D)\varphi_i \right| \leq \varepsilon \|A\varphi_i\|_1 + \left| \int_\Omega \varphi_i A^*w \right| \leq c\varepsilon + \|A^*w\|_\infty \|\varphi_i\|_1.
\]

We hence get \(\lim_i |\mathcal{F}(\varphi_i)| \leq c\varepsilon\), and the result follows for \(\varepsilon > 0\) is arbitrary.

The above property suggests the following definition of linear functionals enjoying some continuity property involving the operator \(A\).
Definition 5.3. A linear functional $\mathcal{F} : \mathcal{D}(\Omega; E) \to \mathbb{C}$ is called an $A$-charge in $\Omega$ if $\lim_i \mathcal{F}(\varphi_i) = 0$ for every sequence $(\varphi_i)_i \in \mathcal{D}(\Omega; E)$ satisfying $\varphi_i \to 0$. The linear space of all $A$-charges in $\Omega$ is denoted by $CH_A(\Omega)$.

The following characterization of $A$-charges will be useful in the sequel.

Proposition 5.4. If $\mathcal{F} : \mathcal{D}(\Omega; E) \to \mathbb{C}$ is a linear functional, then the following properties are equivalent

(i) $\mathcal{F}$ is an $A$-charge,
(ii) for every $\varepsilon > 0$ and each compact set $K \Subset \Omega$ there exists $\theta > 0$ such that, for any $\varphi \in \mathcal{D}_K(\Omega; E)$, one has:

\[
|\mathcal{F}(\varphi)| \leq \theta |\varphi|_{W^{1,1}} + \varepsilon \|A(\cdot, D)\varphi\|_1.
\]

Proof. We proceed as in [3, Proposition 2.6].

Since (ii) implies trivially (i), it suffices to show that the converse implication holds. To that purpose, assume (i) holds, i.e. suppose that $\mathcal{F}$ is an $A$-charge. Fix $\varepsilon > 0$ and a compact set $K \Subset \Omega$. By hypothesis, there exists $\eta > 0$ such that for every $\varphi \in \mathcal{D}_K(\Omega; E)$ satisfying $|\varphi|_{W^{1,1}} \leq \eta$ and $\|A\varphi\|_1 \leq 1$, we have $|\mathcal{F}(\varphi)| \leq \varepsilon$. We now define $\theta := \varepsilon/\eta$.

Fix $\varphi \in \mathcal{D}_K(\Omega; E)$ and assume by homogeneity that one has $\|A(\cdot, D)\varphi\|_1 = 1$. If moreover one has $|\varphi|_{W^{1,1}} \leq \eta$, then one computes $|\mathcal{F}(\varphi)| \leq \varepsilon = \varepsilon\|A(\cdot, D)\varphi\|_1$. If on the contrary we have $\|g\|_{W^{1,1}} > \eta$, we define $\hat{\varphi} := \varphi/\|g\|_{W^{1,1}}$. Then we have $|\hat{\varphi}|_{W^{1,1}} = \eta$ as well as $\|A(\cdot, D)\hat{\varphi}\|_1 < 1$, and hence also $|\mathcal{F}(\hat{\varphi})| \leq \varepsilon$; this yields finally $|\mathcal{F}(\varphi)| = |\varphi|_{W^{1,1}}|\mathcal{F}(\hat{\varphi})|/\eta \leq \varepsilon|\varphi|_{W^{1,1}}/\eta = \theta|\varphi|_{W^{1,1}}$.

As we shall see now, $A$-charges can be extended in a unique way to linear forms on $BV_{A, c}$.

Proposition 5.5. An $A$-charge $\mathcal{F}$ in $\Omega$ extends in a unique way to a linear functional $\tilde{\mathcal{F}} : BV_{A, c}(\Omega) \to \mathbb{C}$ satisfying the following property: for any $\varepsilon > 0$ and each compact set $K \Subset \Omega$, there exists $\theta > 0$ such that for any $g \in BV_{A, K}(\Omega)$ one has:

\[
|\tilde{\mathcal{F}}(g)| \leq \theta |g|_{W^{1,1}} + \varepsilon \|D_A g\|.
\]

Proof. Given $g \in BV_{A, c}(\Omega)$, fix $(\varphi_i)_i \in \mathcal{D}(\Omega; E)$ satisfying $\varphi_i \to g$ and observe that it follows from (10) that $(\mathcal{F}(\varphi_i))_i$ is a Cauchy sequence of complex numbers whose limit does not depend on the choice of sequence $(\varphi_i)_i \in \mathcal{D}(\Omega; E)$ satisfying $\varphi_i \to g$. We hence define $\tilde{\mathcal{F}}(g) := \lim_i \mathcal{F}(\varphi_i)$. It now follows readily from (10) and Remark 5.6 that $\tilde{\mathcal{F}}$ satisfies the desired property.

Remark 5.6. If $\tilde{\mathcal{F}} : BV_{A, c}(\Omega) \to \mathbb{C}$ extends the $A$-charge $\mathcal{F}$, it is easy to see from the previous proposition that for any compactly supported sequence $(g_i)_i \in BV_{A, c}(\Omega)$ satisfying $g_i \to 0$, $i \to \infty$ in $W^{1,1}(\Omega; E)$ and $\sup_i |D_A g_i| < +\infty$, one has $\tilde{\mathcal{F}}(g_i) \to 0$, $i \to \infty$.

From now on, we shall identify any $A$-charge with its extension to $BV_{A, c}$ and use the same notation for the two linear forms.

5.2. Examples of $A$-charges. Let us define two important classes of $A$-charges.

Example 5.7. In case $\mathcal{F}$ is the $A$-flux associated to $v \in C(\Omega, F^*)$ according to (10), its unique extension to $BV_{A, c}(\Omega)$ is the $A$-charge:

\[
\Gamma(v) : BV_{A, c}(\Omega) \to \mathbb{C}, g \mapsto \int_{\Omega} \bar{v} \cdot d[D_A g].
\]

To see this, fix $g \in BV_{A, c}(\Omega)$ together with a sequence $(\varphi_i)_i \in \mathcal{D}(\Omega; E)$ satisfying $\varphi_i \to g$ and choose a compact set $K$ satisfying $\text{supp} g \subseteq K \Subset \Omega$ as well as $\text{supp} \varphi_i \subseteq K$ for all $i$. Given $\varepsilon > 0$, choose $w \in \mathcal{D}(\Omega, F^*)$ a smooth vector field satisfying $\text{supp} K |v - w| \leq \varepsilon$ and compute:

\[
|\Gamma(v)(g) - \int_{\Omega} \bar{v} \cdot d[D_A g]| = \lim_i \int_{\Omega} \bar{v} \cdot A(\cdot, D)\varphi_i - \int_{\Omega} \bar{v} \cdot d[D_A g].
\]
On the other hand we have for all $i$:

\[
\begin{align*}
\int_{\Omega} \bar{v} \cdot A(\cdot, D) \varphi_i - \int_{\Omega} \bar{v} \cdot d [D_A g] &\leq \left| \int_{\Omega} (\bar{v} - \bar{w}) \cdot A(\cdot, D) \varphi_i \right| + \left| \int_{\Omega} (\bar{v} - \bar{w}) \cdot d [D_A g] \right| \\
&+ \left| \int_{\Omega} \bar{w} \cdot A(\cdot, D) \varphi_i - \int_{\Omega} \bar{w} \cdot d [D_A g] \right| \leq \varepsilon \| A \varphi_i \|_1 + \varepsilon \| D_A g \| + \left| \int_{\Omega} \varphi_i A^* w - \int_{\Omega} \bar{w} \cdot d [D_L g] \right|.
\end{align*}
\]

Using the properties of $(\varphi_i)_i$ and Lebesgue’s dominated convergence theorem, we thus get:

\[
\lim_i \left| \int_{\Omega} \bar{v} \cdot A(\cdot, D) \varphi_i - \int_{\Omega} \bar{v} \cdot d [D_A g] \right| \leq 2 \varepsilon \| D_A g \| + \left| \int_{\Omega} g A^* (\cdot, D) \bar{w} - \int_{\Omega} \bar{w} \cdot d [D_L g] \right| = 2 \varepsilon \| D_A g \|,
\]

according to (8). The result follows, for $\varepsilon > 0$ is arbitrary.

**Example 5.8.** Assume that $U$ supports a SGN inequality of type (8) for $BV_A$ functions in $U$. Then given $f \in W_{-1}^{(v-1),N}(U;E^*)$, $f$ extends uniquely to an $A$-charge in $U$.

To see this, fix $K \subset U$ and infer from Lemma 5.21 that there exist $(g_\alpha)_{|\alpha|=\nu-1} \subset L^N(U;E^*)$ such that

\[
\langle f, \varphi \rangle = \sum_{|\alpha|=\nu-1} \int_U g_\alpha \partial^\alpha \varphi, \quad \forall \varphi \in \mathcal{D}(U;E).
\]

Now fix $\varepsilon > 0$ and choose $\theta > 0$ large enough for $\left( \int_{K \cap \{ |g_\alpha| \theta \} } |g_\alpha|^N \right)^{1/N} \leq \varepsilon/C$ to hold for any $|\alpha| = \nu - 1$, where $C$ is a positive constant satisfying (8). We then compute, for $\varphi \in \mathcal{D}(U;E)$:

\[
\begin{align*}
\langle f, \varphi \rangle &\leq \theta \left( \sum_{|\alpha|=\nu-1} \int_{K \cap \{ |g_\alpha| \theta \} } |\partial^\alpha \varphi| \right) + \sum_{|\alpha|=\nu-1} \int_{K \cap \{ |g_\alpha| \theta \} } |g_\alpha \partial^\alpha \varphi|, \\
&\leq \theta \| \varphi \|_{W^{\nu-1,1}} + \sum_{|\alpha|=\nu-1} \left( \int_{K \cap \{ |g_\alpha| \theta \} } |g_\alpha|^N \right)^{1/N} \| \partial^\alpha \varphi \|_{N/N-1}, \\
&\leq \theta \| \varphi \|_{W^{\nu-1,1}} + \varepsilon \| A(\cdot, D) \varphi \|_{L^1}.
\end{align*}
\]

The conclusion that $f$ extends to an $A$-charge follows by approximation.

**Example 5.9.** Assume that $U$ supports a SGN inequality of type (8) for $BV_A$ functions in $U$. Given $f \in L_{-1}^{N}(U;E^*)$ we know from Example 5.24 that it defines a distribution in $W_{-1}^{(v-1),N}(U;E^*)$. We then define, for $\varphi \in \mathcal{D}(U;E)$:

\[
\langle \Lambda(f), \varphi \rangle := \langle f, \varphi \rangle = \int_U \hat{f} \hat{\varphi}.
\]

It follows from the previous example and from approximation that $\Lambda(f)$ extends uniquely to an $A$-charge in $U$ verifying, for all $g \in BV_{A,c}(U)$:

\[
\langle \Lambda(f), g \rangle = \int_U \hat{f} g.
\]

**Remark 5.10.** It is easy to see that for any $x_0 \in \Omega$, there exists an open neighborhood $U \subset \Omega$ of $x_0$ such that one has $\Lambda[\mathcal{D}(U;E)] \subseteq \Gamma[C^\infty(U, F^*)]$. Given $\varphi \in \mathcal{D}(U;E)$, thanks to the local solvability of the elliptic operator $\Delta_A = A^* (\cdot, D) \circ A(\cdot, D)$ (see [5.6, Corollary 4.8]), there exists $u \in C^\infty(U)$ a smooth solution to $\Delta_A u = \varphi$ in $U$. Let $v := A(\cdot, D) u$. This yields, for any $g \in BV_{A,c}(U)$:

\[
\Lambda(\varphi)(g) = \int_U \hat{\varphi} g = \int_U g A^* (\cdot, D) v = \int_U \bar{v} \cdot d [D_A g] = \Gamma(v)(g),
\]

for we could, in the computation above, replace $v$ by $v \chi$ where $\chi \in \mathcal{D}(U)$ satisfies $\chi = 1$ in a neighborhood of $\text{supp} \ g$.

It turns out that a linear functional on $BV_{A,c}$ is an $A$-charge if and only if it is continuous with respect to some locally convex topology on $BV_{A,c}$.
5.3. Another characterization of $A$-charges. In the sequel, a locally convex space means a Hausdorff locally convex topological vector space. For any family $\mathcal{O}$ of sets and any set $X$ we denote $\mathcal{O} \subseteq X := \{O \cap X : O \in \mathcal{O}\}$. Following [S, Theorem 3.3] we define the following topology on $BV_{A,c}(\Omega)$, called the localized topology associated to the family of subspaces $BV_{A,K,\lambda}(\Omega)$.

**Definition 5.11.** Let $\mathcal{T}_A$ be the unique locally convex topology on $BV_{A,c}(\Omega)$ such that

(a) $\mathcal{T}_A \subseteq BV_{A,K,\lambda} \subseteq T_{W^{-1,1}} \subseteq BV_{A,K,\lambda}$ for all $K \subset \subset \Omega$ and $\lambda > 0$ where we let:

$$BV_{A,K,\lambda} = \{g \in BV_{A,c}(\Omega) : \text{supp } g \subseteq K, \|D_A g\| \leq \lambda\},$$

and where $T_{W^{-1,1}}$ is the $W^{\nu-1,1}$-topology;

(b) for every locally convex space $Y$, a linear map $f : (BV_{A,c}; \mathcal{T}_A) \rightarrow Y$ is continuous if only if $f \upharpoonright BV_{A,K,\lambda}$ is $W^{\nu-1,1}$ continuous for all $K \subset \subset \Omega$ and $\lambda > 0$.

**Remark 5.12.** Uniqueness of the above topology is easily seen according to property (b). Concerning the existence, one can define the topology $\mathcal{T}_A$ by constructing a basis of neighborhoods $\mathcal{B}_A$ of the origin in the following way: denote by $\mathcal{B}_A$ the collection of all absorbing, balanced and convex subsets $U \subseteq BV_A(\Omega)$ satisfying $U \subseteq BV_{A,K,\lambda} \in T_{W^{-1,1}} \subseteq BV_{A,K,\lambda}$. Calling $\mathcal{T}_A$ the vector topology on $BV_A(\Omega)$ admitting $\mathcal{B}_A$ as a neighborhood basis at the origin, one can see that it satisfies properties (a) and (b) above.

Choosing $(K_k)_{k \in \mathbb{N}}$ an increasing sequence of compact sets exhausting $\Omega$ and defining $X_k := BV_{A,K_k}(\Omega)$ for all $k \in \mathbb{N}$, we have:

$$BV_{A,K_k,k} = \{g \in X_k : \|D_A g\| \leq k\}.$$

Since it is straightforward to see that all the vector spaces $BV_{A,K_k}$ are closed in the $W^{\nu-1,1}$ topology, and that $\|D_A\|$ is a lower semicontinuous norm on $BV_{A,c}(\Omega)$, it now follows readily from [S, Proposition 3.11] that $\mathcal{T}_A$-continuous linear functionals on $BV_A(\Omega)$ are actually the $A$-charges in $\Omega$.

**Proposition 5.13.** A linear functional $\mathcal{F} : BV_{A,c}(\Omega) \rightarrow \mathbb{C}$ is an $A$-charge if and only if it is $\mathcal{T}_A$-continuous.

The following result will be useful in the sequel.

**Corollary 5.14.** Assume that $K \subset \subset \mathbb{R}^n$ is a compact set and that $\lambda > 0$ is a real number. If $(g_i)_{i \in I} \subseteq BV_{A,K,\lambda}(U)$ converges to $0$ as distributions, i.e. if one has $\int_U g_i \varphi \to 0$ for all $\varphi \in \mathcal{D}(U; E^*)$, then the net $(\|g_i\|_{W^{\nu-1,1}})_{i \in I}$ also converges to $0$.

**Proof.** Proceed towards a contradiction, assume that $(g_i)_{i \in I}$ is as in the statement, and that $(\|g_i\|_{W^{\nu-1,1}})_{i \in I}$ fails to converge to $0$, meaning that there is $\varepsilon > 0$ such that for all $i \in I$, one can find $j \in J$ satisfying $j \geq i$ and $\|g_j\|_{W^{\nu-1,1}} \geq \varepsilon$. Define then $J := \{i \in I : \|g_j\|_{W^{\nu-1,1}} \geq \varepsilon\}$. Now, $J$ is a directed set and consider the net $(g_j)_{j \in J}$. Since $BV_{A,K,\lambda}(U)$ is compact in the $W^{\nu-1,1}$ topology according to Proposition 5.11, we know that there exists a cluster point $g \in BV_{A,K,\lambda}(U)$ of $(g_j)_{j \in J}$ in the $W^{\nu-1,1}$ topology. It’s easy to see from the definition of $J$ that one has $g \neq 0$. On the other hand, fix $\varphi \in \mathcal{D}(U; E^*)$. Since $(g_j)_{j \in J}$ converges to $0$ as distributions, we get for $j \in J$:

$$\int_U g_j \varphi \to 0.$$

Yet we should also get for $j \in J$:

$$\left| \int_U g_j \varphi - \int_U g \varphi \right| \leq C \|g_j - g\|_{W^{\nu-1,1}(U)} \to 0,$$

which implies that $\int_U g \varphi = 0$. Since $\varphi \in \mathcal{D}(U; E^*)$ is arbitrary, this means that $g = 0$, which is a contradiction; the proof is complete.

We now turn to proving the key result for obtaining Theorem 5.3.
6. Towards Theorem \[\text{[13]}\]

Throughout this section, we assume that the open set $U \subseteq \Omega$ supports inequalities of type \[\text{[11]}\] and \[\text{[13]}\]; we also assume that one has $\Lambda[\mathcal{G}(U; E)] \subseteq \Gamma(C(U, F^*))$.

Remark 6.1. It follows from Theorem \[\text{[1]}\], Proposition \[\text{[5]}\] and Remark \[\text{[11]}\] that for any $x_0 \in \Omega$, one can find an open neighborhood $U$ of $x_0$ in $\Omega$ satisfying all the above assumptions.

Our intention is to prove the following result.

Theorem 6.2. If $\mathcal{F} : BV_{A,e}(U) \to F$ is an $A$-charge in $U$, then there exists $v \in C(U, F^*)$ for which one has $\mathcal{F} = \Gamma(v)$, i.e. such that one has, for any $g \in BV_{A,e}(U)$:

$$\mathcal{F}(g) = \int_U \bar{v} \cdot d[D_Ag].$$

To prove this theorem, we have to show that the map

$$\Gamma : C(U, F^*) \to CH_A(U), v \mapsto \Gamma(v),$$

is surjective. In order to do this, we endow $C(U, F^*)$ with the usual Fréchet topology of uniform convergence on compact sets, and $CH_A(U)$ with the Fréchet topology associated to the family of seminorms $(\| \cdot \|_K)_K$ defined by:

$$\| \mathcal{F} \|_K := \sup \{|\mathcal{F}(g)| : g \in BV_{A,K}(U), \|D_Ag\| \leq 1\},$$

where $K$ ranges over all compact sets $K \subseteq U$. The surjectivity of $\Gamma$ will be proven in case we show that $\Gamma$ is continuous and verifies the following two facts:

(a) $\Gamma[C(U, F^*)]$ is dense in $CH_A(U)$.

(b) $\Gamma^*[CH_A(U)^*]$ is sequentially closed in the strong topology of $C(U, F^*)^*$.

Indeed, it will then follow from the Closed Range Theorem \[\text{[1]}\], Theorem 8.6.13 together with \[\text{[8]}\], Proposition 6.8 and (b) that $\Gamma[C(U, F^*)]$ is closed in $CH_A(U)$. Using (a) we shall then conclude that one has:

$$\Gamma[C(U, F^*)] = CH_A(U),$$

i.e. that $\Gamma$ is surjective.

The strategy of the proof of (b) follows the lines of De Pauw and Pfeffer’s proof in \[\text{[2]}\]. For the proof of (a), however, the proof presented below is slightly different from their approach; we namely manage to avoid an explicit smoothing process and choose instead to use an abstract approach similar to the one used in \[\text{[13]}\] in order to solve the equation $d\omega = F$.

Let us start by showing that $\Gamma : C(U, F^*) \to CH_A(U)$ is linear and continuous. Indeed given a compact set $K \subseteq U$ and $g \in BV_{A,K}(U)$ we have:

$$|\Gamma(v)(g)| = \left| \int_U \bar{v} \cdot d[D_Ag] \right| \leq \|D_Ag\| \|v\|_{\infty,K},$$

which implies $\|\Gamma(v)\|_K \leq \|v\|_{\infty,K}$. Next we identify the dual space $CH_A(U)^*$.

6.1. Identifying $CH_A(U)^*$.

Proposition 6.3. The map $\Phi : BV_{A,e}(U) \to CH_A(U)^*$ given by $\Phi(g)(F) = F(g)$ is a linear bijection.

First let us check that $\Phi$ is well defined. In fact, given $K \subseteq U$ and $g \in BV_{A,K}(U)$ we have

$$|\Phi(g)(F)| = |F(g)| \leq \|D_Ag\| \|F\|_K,$$

according to the definition of $\| \cdot \|_K$. Hence $\Phi(g)$ is continuous and $\Phi(g) \in CH_A(U)^*$.

To show that $\Phi$ is injective, let $g \in BV_{A,e}(U)$ be such that $\Phi(g) = 0$. Then for any $B \subseteq U$ measurable and bounded and for any $e^* \in E^*$ we have:

$$\int_B e^* g = \int_U \chi_B e^* g = \Lambda(\chi_B e^*) (g) = \Phi(g)(\Lambda(\chi_B e^*)) = 0.$$
Thus \( e^* g = 0 \) a.e. in \( U \), which implies that \( g = 0 \) (and hence that \( \Phi \) injective) since \( e^* \in E^* \) is arbitrary.

The next step is to prove that \( \Phi \) is surjective. To show this property we shall define a right inverse for \( \Phi \), called \( \Psi \).

Let \( \Psi : CH_A(U)^* \to \mathcal{D}'(U; E^*) \) be defined by:
\[
(12) \quad \Psi(\alpha)(\varphi) := \alpha[\Lambda(\varphi)].
\]
We claim that \( \Psi \) is well defined, i.e. that for \( \alpha \in CH_A(U)^* \), we have \( \Psi(\alpha) \in BV_{A,e}(U) \). Indeed, given \( \alpha \in CH_L(U)^* \) there exist \( C > 0 \) and \( K \subset \subset U \) such that for all \( F \in CH_L(U) \) we have \( |\alpha(F)| \leq C |F|_K \). In particular, for every \( \varphi \in \mathcal{D}(U; E^*) \) we have:
\[
|\Psi(\alpha)(\varphi)| \leq C \|\Lambda(\varphi)\|_K
\leq C \sup \{|\Lambda(\varphi)(g)| : g \in BV_{A,K}(U), \|D_A g\|_1 \leq 1\},
\]
from which it already follows that one has \( \sup \Phi(\alpha) \subset \subset K \), since the above inequalities yield \( \Phi(\alpha)(\varphi) = 0 \) if \( \sup \varphi \cap K = \emptyset \). According to Remark 22, we see moreover that for any \( g \in BV_{A,K}(U) \) satisfying \( \|D_A g\|_1 \leq 1 \), one has:
\[
\int_U \varphi g \leq \|\varphi\|_{W^{1,N(N-1)}(U; E)} \|g\|_{W^{1,N(N-1)}(U; E)}
\leq C(K, \nu) \|\varphi\|_{W^{1,N(N-1)}(U; E)} \|g\|_{W^{1,N(N-1)}(U; E)}
\leq \tilde{C}(K, \nu) \|\varphi\|_{W^{1,N(N-1)}(U; E)}^*,
\]
where the latter inequality comes from the SGN inequality in \( BV_{A} \) (Proposition 11). It follows then from the reflexivity of \( W^{1,N(N-1)}(U; E) \) that one has \( g \in W^{1,N(N-1)}(U; E) \) and \( \|g\|_{W^{1,N(N-1)}(U; E)} \leq \|g\|_{W^{1,N(N-1)}(U; E)} \).

Moreover, for any \( \psi \in C^\infty_c(U, F^*) \) we have:
\[
|\Psi(\alpha)(A^*(\cdot,D)\psi)| = |\alpha[A^*(\cdot,D)\psi]|,
\leq C \|A^*(\cdot,D)\psi\|_K
\leq C \sup \left\{ \int_U g A^*(\cdot,D)\psi \ : g \in BV_{A,K}(U), \|D_A g\|_1 \leq 1 \right\}
\leq C \sup \left\{ \|D_A g\|_1 \|\psi\|_\infty \ : g \in BV_{A,K}(U), \|D_A g\|_1 \leq 1 \right\}
\leq C \sup \|\psi\|_\infty,
\]
so that one has \( \Psi(\alpha) \in BV_{A,e}(U) \).

**Lemma 6.4.** The maps \( \Phi \) and \( \Psi \) defined above are inverses, i.e. we have:
(i) \( \Psi \circ \Phi = Id_{BV_{A,e}(U)} \);
(ii) \( \Phi \circ \Psi = Id_{CH_A(U)^*} \) (in particular, \( \Phi \) is surjective).

In order to prove the previous lemma, we shall need some observations concerning the polar sets of some neighborhoods of the origin in \( CH_A(U) \). First, observe that the family of all sets \( V(K, \varepsilon) \) (where \( K \) ranges over all compact subsets of \( U \), and \( \varepsilon \) over all positive real numbers) defined by:
\[
V(K, \varepsilon) := \{ F \in CH_A(U) : \|F\|_K \leq \varepsilon \},
\]
is a basis of neighborhoods of the origin in \( CH_A(U) \).

**Claim 5.** Fix \( K \subset \subset U \) a compact set and a real number \( \varepsilon > 0 \). For any \( \alpha \in V(K, \varepsilon)^0 \), one has:
To prove part (i), assume that \( \varphi \in \mathcal{D}(U) \) satisfies \( K \cap \text{supp} \varphi = \emptyset \). Then, we get for \( \lambda > 0 \):

\[
\| \lambda \Lambda(\varphi) \|_K = \sup \left\{ \lambda \left| \int_U \varphi g \right| : g \in BV_{A,K}, \| D_A g \| \leq 1 \right\} = 0.
\]

In particular this yields \( \lambda \Lambda(\varphi) \in V(K, \varepsilon) \). We hence obtain:

\[
\lambda \alpha[\Lambda(\varphi)] = |\alpha[\lambda \Lambda(\varphi)]| \leq 1,
\]

for any \( \lambda > 0 \). Since \( \lambda > 0 \) is arbitrary, this implies that one has \( \alpha[\Lambda(\varphi)] = 0 \), i.e. that \( \Psi(\alpha)(\varphi) = 0 \). We may now conclude that \( \text{supp} \Psi(\alpha) \subseteq K \). In order to obtain statement (ii), fix \( v \in C^\infty_0(U, F^*) \) satisfying \( \| v \|_\infty \leq 1 \) and compute:

\[
\| \varepsilon \Lambda(A^*(\cdot, D)v) \|_K = \varepsilon \| \Lambda(A^*(\cdot, D)v) \|_K,
\]

\[
= \varepsilon \sup \left\{ \left| \int_U g A^*(\cdot, D)v \right| : g \in BV_{A,K}, \| D_A g \| \leq 1 \right\},
\]

\[
= \varepsilon \sup \left\{ \left| \int_U \varphi g \right| : g \in BV_{A,K}, \| D_A g \| \leq 1 \right\},
\]

\[
\leq \varepsilon \sup \{ \| D_A g \| \cdot \| v \|_\infty : g \in BV_{A,K}, \| D_A g \| \leq 1 \},
\]

\[
\leq \varepsilon,
\]

so that one has \( \varepsilon \Lambda(A^*(\cdot, D)v) \in V(K, \varepsilon) \). It hence follows that:

\[
\varepsilon \| \Psi(\alpha)(A^*(\cdot, D)v) \| = | \alpha[\varepsilon \Lambda(A^*(\cdot, D)v)] | \leq 1,
\]

and we thus get:

\[
\| \Psi(\alpha)(A^*(\cdot, D)v) \| \leq \frac{1}{\varepsilon}.
\]

Since \( v \in \mathcal{D}(U, F^*) \) is an arbitrary vector field satisfying \( \| v \|_\infty \leq 1 \), this yields \( \| D_A \Psi(\alpha) \| \leq \frac{1}{\varepsilon} \), and concludes the proof of the claim. \( \square \)

We now turn to proving Lemma 6.2.

Proof of Lemma 6.2. To prove part (i), fix \( g \in BV_{A,\varepsilon}(U) \) and compute, for \( \varphi \in \mathcal{D}(U; E^*) \):

\[
\Psi[\Phi(g)](\varphi) := \Phi(g)[\Lambda(\varphi)] = \Lambda(\varphi)(g) = \int_U \varphi g,
\]

that is, \( \Psi[\Phi(g)] = g \) in the sense of distributions.

In order to prove part (ii), fix \( \alpha \in CH_A(U) \). We have to show that, for any \( F \in CH_A(U) \), we have:

\[
\Phi[\Psi(\alpha)](F) = \alpha(F),
\]

i.e. that for any \( F \in CH_A(U) \), one has:

\[
F[\Psi(\alpha)] = \alpha(F).
\]

To this purpose, define for any \( F \in CH_A(U) \) a map:

\[
\Delta_F : CH_A(U)^* \rightarrow \mathbb{C}, \alpha \mapsto \Delta_F(\alpha) := F[\Psi(\alpha)].
\]

Claim 6.6. Given \( F \in CH_A(U) \), the map \( \Delta_F \) is weakly* continuous on \( V(K, \varepsilon)^* \) for all \( K \subset U \) and \( \varepsilon > 0 \).

To prove this claim, fix \( K \subset U \), \( \varepsilon > 0 \) and assume that \( (\alpha_i)_{i \in I} \subseteq V(K, \varepsilon)^* \) is a net weak* converging to 0. In particular one gets:

(a) for any \( \varphi \in \mathcal{D}(U; E^*) \), we have \( \Lambda(\varphi) \in CH_A(U) \) and hence the net \( (\Psi(\alpha_i)(\varphi))_{i \in I} = (\alpha_i[\Lambda(\varphi)])_{i \in I} \)

converges to 0; in other terms, the net \( (\Psi(\alpha_i))_{i \in I} \) converges to 0 in the sense of distributions.

According to Claim 6.3, we moreover have:

(b) \( \text{supp} \Psi(\alpha_i) \subseteq K \) for each \( i \in I \);
(c) \( c := \sup_{i \in I} \| D_A \Psi(\alpha_i) \| \leq \frac{1}{\varepsilon} \).
We thus have \((\Psi(\alpha_i))_{i \in I} \subseteq BV_{A;K,F}^\alpha\). It hence follow from Corollary [4, 3] that the net \((\|\Psi(\alpha_i)\|_{W^{p-1,1}})_{i \in I}\) converges to 0. From the fact that \(F\) is an \(A\)-charge we see that the net \((F[\Psi(\alpha_i)])_{i \in I}\) converges to 0 as well. This means, in turn, that \((\Delta_F(\alpha_i))_{i \in I}\) converges to 0, which shows that \(\Delta_F\) is weak*-continuous on \(V(K,\varepsilon)\).

Claim 6.7. For any \(\alpha \in CH_A(U)^*\), we have \(\Delta_F(\alpha) = \alpha(F)\).

To prove the latter claim, observe that according to Claim [4, 3] and to the Banach-Grothendieck theorem [3, Theorem 8.5.1], there exists \(\tilde{F} \in CH_A(U)\) such that for any \(\alpha \in CH_A(U)^*\), we have:

\[
\Delta_F(\alpha) = \alpha(\tilde{F}).
\]

Yet given \(g \in BV_{A;C}(U)\), we then have, according to [Lemma [4, 3], (i)]:

\[
F(g) = F(\Psi(\Phi(g))) = \Delta_F[\Phi(g)] = \Phi(g)(\tilde{F}) = \tilde{F}(g),
\]

i.e. \(F = \tilde{F}\), which proves the claim.

It now suffices to observe that Lemma [4, 3] is proven for we have established the equality \(F[\Psi(\alpha)] = \alpha(F)\) for any \(F \in CH_A(U)\) and \(\alpha \in CH_A(U)^*\).

As a corollary, we get a proof of the density of \(\Lambda[\mathcal{D}(U)]\) and \(\Gamma[C(U,F^*)]\) in \(CH_A(U)\).

**Corollary 6.8.** The space \(\Lambda[\mathcal{D}(U,E^*)]\) is dense in \(CH_A(U)\).

**Proof.** Assuming that \(\alpha \in CH_A(U)^*\) satisfies \(\alpha \uparrow \Lambda[\mathcal{D}(U,E^*)] = 0\), we compute for any \(\varphi \in \mathcal{D}(U,E^*)\):

\[
\Psi(\alpha)(\varphi) = \alpha[\Lambda(\varphi)] = 0.
\]

This means that \(\Psi(\alpha) = 0\), and implies that \(\alpha = \Phi \circ \Psi(\alpha) = \Phi(0) = 0\). The result then follows from the Hahn-Banach theorem.

**Corollary 6.9.** The space \(\Gamma[C(U,F^*)]\) is dense in \(CH_A(U)\).

**Proof.** It follows from the previous corollary that \(\Lambda[\mathcal{D}(U,E^*)]\) is dense in \(CH_A(U)\). Since by hypothesis we also have \(\Lambda[\mathcal{D}(U,F^*)] \subseteq \Gamma[C(U,F^*)] \subseteq CH_A(U)\), it is clear that \(\Gamma(U,F^*)\) is dense in \(CH_A(U)\).

In order to study the range of \(\Gamma^*\), we introduce the following linear operator:

\[
\Xi : BV_{A;C}(U) \to C(U,F^*)^*, g \mapsto \Xi(g),
\]

defined by \(\Xi(g)(v) := \Gamma(v)(g)\) for any \(v \in C(U,F^*)\).

**Claim 6.10.** We have \(\text{Im} \Gamma^* = \Xi \Xi\).

**Proof.** To prove this claim, fix \(\mu \in C(U,F^*)\). If one has \(\mu = \Gamma^*(\alpha)\) for some \(\alpha \in CH_A(U)^*\), then we compute for \(v \in C(U,F^*)\):

\[
\Xi[\Psi(\alpha)](v) = \Gamma(v)(\Psi(\alpha)) = \Phi[\Psi(\alpha)][\Gamma(v)] = \alpha(\Gamma(v)) = \Gamma^*(\alpha)(v) = \mu(v),
\]

so that one has \(\mu = \Xi[\Psi(\alpha)] \in \text{Im} \Xi\). Conversely, if one has \(\mu = \Xi(g)\) for some \(g \in BV_{A;C}(U)\), then we compute for \(v \in C(U,F^*)\):

\[
\Gamma^*[\Phi(g)](v) = \Phi(g)[\Gamma(v)] = \Gamma(v)(g) = \Xi(g)(v) = \mu(v),
\]

so that one has \(\mu = \Gamma^*[\Phi(g)] \in \text{Im} \Gamma^*\).

Consider the set

\[
B := \{v \in C(U,F^*) : \|v\|_{C(U,F^*)} \leq 1\}.
\]

It is clear that \(B\) is bounded in \(C(U,F^*)\). Hence the seminorm:

\[
p : C(U,F^*)^* \to \mathbb{R}_+, \mu \mapsto p(\mu) := \sup_{v \in B} |\mu(v)|,
\]
is strongly continuous (i.e. continuous with respect to the strong topology) on $C(U, F^*)^*$. Observe now that one has, for $g \in BV_A, c(U)$:

$$p[\Xi(g)] = \sup_{v \in B}|\Xi(g)(v)|,$$

$$= \sup_{v \in B}[|\Gamma(v)(g)| : v \in B],$$

$$= \|D_A g\|.$$

**Lemma 6.11.** The set $\text{im} \Xi$ is strongly sequentially closed in $C(U, F^*)^*$.

**Proof.** Fix a sequence $(\Xi(g_k))_{k \in \mathbb{N}} \subseteq \text{im} \Xi$ and assume that, in the strong topology, one has:

$$\Xi(g_k) \to \mu \in C(U, F^*)^*, \quad k \to \infty.$$ 

The strong continuity of $p$ then yields:

$$c := \sup_{k \in \mathbb{N}}\|D_A g_k\| = \sup_{k \in \mathbb{N}}p[\Xi(g_k)] < +\infty.$$

**Claim 6.12.** There exists a compact set $K \subset U$ such that one has $\text{supp} g_k \subseteq K$ for each $k \in \mathbb{N}$.

To prove this claim, let us first prove that the sequence $(\text{supp} D_A g_k)_{k \in \mathbb{N}}$ is compactly supported in $U$ (i.e. that there is a compact subset of $U$ containing $\text{supp} D_A g_k$ for all $k$). To this purpose, we proceed towards a contradiction and assume that it is not the case. Let then $U = \bigcup_{j \in \mathbb{N}} U_j$ be an exhaustion of $U$ by open sets satisfying, for each $j \in \mathbb{N}$, $U_j \subseteq U_{j+1}$ and such that $U_j$ is a compact subset of $U$ for each $j \in \mathbb{N}$.

Since $(\text{supp} D_A g_k)_{k \in \mathbb{N}}$ is not compactly supported, there exist increasing sequences of integers $(j_l)_{l \in \mathbb{N}}$ and $(k_l)_{l \in \mathbb{N}}$ satisfying, for any $l \in \mathbb{N}$:

$$\text{supp}(D_A g_{k_l}) \cap (U_{j_l+1} \setminus \tilde{U}_{j_l}) \neq \emptyset.$$

In particular, there exists for each $l \in \mathbb{N}$ a vector field $v_l \in C_c(U_{j_l+1} \setminus \tilde{U}_{j_l}, F^*)$ with $\|v_l\|_{\infty} < 1$ and:

$$a_l := \left|\int_U \bar{v}_l \cdot d[D_A g_{k_l}]\right| > 0.$$

Let now, for $l \in \mathbb{N}$, $b_l := \max_{0 \leq k \leq l} \frac{1}{a_k}$ and define a bounded set $B' \subseteq C(U, F^*)$ by:

$$B' := \left\{v \in C(U, F^*) : \|v\|_{\infty, U_{j_l+1}} \leq b_l \text{ for each } l \in \mathbb{N}\right\}.$$

It follows from the construction of $B$ that one has $w_l := b_l v_l \in B$ for any $l \in \mathbb{N}$. Moreover the seminorm

$$p' := C(U, F^*)^* \to \mathbb{R}^+, \mu \mapsto \sup_{v \in B'}|\mu(v)|,$$

is strongly continuous. Yet we get for $l \in \mathbb{N}$:

$$p'[\Xi(g_{k_l})] \geq \Xi(g_{k_l})(w_l) = |\Gamma(w_l)(g_{k_l})| = b_l \left|\int_U \bar{v}_l \cdot d[D_A g_{k_l}]\right| = b_l a_l \geq l.$$

Since this yields $p'[\Xi(g_{k_l})] \to \infty$, $l \to \infty$, we get a contradiction with the fact that $p'$ is strongly continuous (recall that $(\Xi(g_{k_l}))_{k \in \mathbb{N}}$ converges in the strong topology).

**Claim 6.13.** Let $V$ be an open set and let $r(x, D) \in S^{-\infty}$ be a regularizing operator. Assume that $g \in L^{N/N-1}(V)$ satisfies $\int_V g[\psi - r(x, D)\psi] = 0$ for all $\psi \in \mathcal{P}(V, F^*)$. Under those assumptions, one has $g = 0$ in $V$.

**Proof.** Fix $x_0 \in V$ and let $\ell > 0$ be such that $B(x_0, \ell) \subseteq V$. Given $\psi \in \mathcal{P}(B(x_0, \ell), F^*)$, begin by observing that, for all $\phi \in \mathcal{P}(B(x_0, \ell), F)$, one has:

$$\left|\int_{B(x_0, \ell)} \phi \cdot r(x, D)\psi\right| \leq C\|\phi\|_{L^1} \|\psi\|_{L^N} \leq C'\ell \|\phi\|_{L^{N/N-1}} \|\psi\|_{L^N},$$

where the first inequality follows from [12, inequality (3.3)]. It hence follows by duality that one has:

$$\|r(x, D)\psi\|_{L^N(B(x_0, \ell))} = \sup_{|\phi|_{L^{N/N-1}} \leq 1} \int_{B(x_0, \ell)} \phi \cdot r(x, D)\psi \leq C\ell \|\psi\|_{L^N}.$$
Assuming hence that $\ell$ is small enough (say $\ell = \ell_0$), this yields $\|r(x, D)\|_{L^N(B(x_0, \ell_0))} < 1$.

Now writing $\psi - r(x, D)\psi = (I - r(x, D))\psi$, and using [3] Exercise 6.14 to infer that $I - r(x, D)$ is a linear isomorphism of $L^N(B(x_0, \ell_0))$, we see that $[I - r(x, D)](\mathcal{D}(B(x_0, \ell_0)))$ is dense in $L^N(B(x_0, \ell_0))$, which is sufficient to conclude that $g = 0$ in $B(x_0, \ell_0)$, and hence that $g = 0$ on $V$ since $x_0$ is arbitrary.

Now choose $K \subset U$ a compact set for which one has $\text{supp}(D_Ag_k) \subseteq K$ for all $k \in \mathbb{N}$ and fix $k \in \mathbb{N}$. Fix also $x_0 \in U \setminus K$, choose $\ell > 0$ so that $V = B(x_0, \ell) \subseteq U$ and fix $f \in \mathcal{D}(V, F^*)$. Define $u = A(\cdot, D)q(\cdot, D)f \in \mathcal{D}(V, F^*)$ satisfying (1). One then has:

$$
\int_V g_k j + r(x, D)f = \int_V g_k A^*(x, D)u = \int_V \tilde{u} \cdot [D_Ag_k] = 0,
$$

since one knows that $V \cap \text{supp}[D_Ag_k] = \emptyset$. Applying the previous claim to $g_k$, $V$ and $-r(\cdot, D)$ we hence get $g_k = 0$ on $V$. It then follows that one has $\text{supp}g_k \subseteq K$, for $x_0$ is an arbitrary point in $U \setminus K$.

Getting back to the proof of Lemma 6.11 observe that, according to Proposition 6.14 there exists a subsequence $(g_{k_l}) \subseteq (g_k)$, $W^{n-1,1}$-converging to $g \in BV_{A,c}(U)$. Using the fact that $\Gamma(v)$ is an $A$-charge, we compute:

$$
\mu(v) = \lim_{l \to \infty} \Xi(g_{k_l})(v) = \lim_{l \to \infty} \Gamma(v)(g_{k_l}) = \Gamma(v)(g) = \Xi(g)(v),
$$

and hence we get $\mu = \Xi(g) \in \text{im} \Xi$.

We hence proved the following theorem.

**Theorem 6.14.** We have $CH_A(U) = \Gamma[C(U, F^*)]$.

## 7. Application: Elliptic Complexes of Vector Fields

Consider $n$ complex vector fields $L_1, \ldots, L_n$, $n \geq 1$, with smooth coefficients defined on an open set $\Omega \subset \mathbb{R}^N$, $N \geq 2$. Naturally, we assume that the vector fields $L_j$, $1 \leq j \leq n$ do not vanish in $\Omega$; in particular, they may be viewed as non-vanishing sections of the vector bundle $\mathbb{C}T(\Omega)$ as well as first order differential operators of principal type.

We impose two fundamental properties on those vector fields in our context; namely, we require that:

(a) $L_1, \ldots, L_n$ are everywhere linearly independent;

(b) the system $\mathcal{L} := \{L_1, \ldots, L_n\}$ is elliptic.

The latter means for any 1-form $\omega$, i.e. any section of $T^*(\Omega)$, the equality $\langle \omega, L_j \rangle = 0$ for $1 \leq j \leq n$ implies that one has $\omega = 0$ — which is equivalent to require that the second order operator: $\Delta_{\mathcal{L}} := L_1^*L_1 + \cdots + L_n^*L_n$ is elliptic. We use the notation $L_j^* := \overline{L_j}$ where $\overline{L_j}$ denotes the vector field obtained from $L_j$ by conjugating its coefficients and let $L_j^*$ denote the formal transpose of $L_j$ for $j = 1, \ldots, n$ — namely this means that, for all (complex valued) $\varphi, \psi \in \mathcal{D}(\Omega)$, we have:

$$
\int_{\Omega} (L_j^*\varphi)\overline{\psi} = \int_{\Omega} \varphi(L_j^*\overline{\psi}).
$$

Consider the gradient $\nabla_{\mathcal{L}} : C^\infty(\Omega) \to C^\infty(\Omega)^n$ associated to the system $\mathcal{L}$ defined by $\nabla_{\mathcal{L}} u := (L_1u, \ldots, L_nu)$ and its formal complex adjoint operator, defined for $v \in C^\infty(\Omega, \mathbb{C}^n)$ by:

$$
\text{div}_{\mathcal{L}, v} := L_1^*v_1 + \cdots + L_n^*v_n.
$$

The following local continuous solvability result is known for divergence-type operators of the previous type; it is borrowed from [11, Theorem 1.2].

**Corollary 7.1.** Assume that the system of vector fields $\mathcal{L}$ satisfies (i) and (ii). Then every point $x_0 \in \Omega$ is contained in an open neighborhood $U \subset \Omega$ such that for any $f \in \mathcal{D}(U)$, the equation:

$$
\text{div}_{\mathcal{L}, v} = f
$$

is continuously solvable in $U$ if and only if $f$ is an $\mathcal{L}$-charge in $U$, meaning that for every $\varepsilon > 0$ and every compact set $K \subset U$, there exists $\theta = \theta(K, \varepsilon) > 0$ such that one has, for every $\varphi \in \mathcal{D}_K(\Omega)$:

$$
|f(\varphi)| \leq \theta \|\varphi\|_1 + \varepsilon \|\nabla_{\mathcal{L}} \varphi\|_1. 
$$
This result can be seen as a direct consequence of Theorem 4.3 applied to the first order operator $A(\cdot, D) := \nabla_L$, which is elliptic and canceling. Indeed, from the fact that $L$ is elliptic we easily see that $\nabla_L$ is elliptic as well. Furthermore, [4 Lemma 4.1] together with the assumption that the system $L$ be linearly independent, shows that $\nabla_L$ is canceling.

Let $C^\infty(\Omega, \Lambda^k \mathbb{R}^n)$ denote the space of $k$-forms on $\mathbb{R}^n$, $0 \leq k \leq n$, with smooth, complex coefficients defined on $\Omega$. Each $f \in C^\infty(\Omega, \Lambda^k \mathbb{R}^n)$ may be written as:

$$f = \sum_{|I|=k} f_I dx_I, \quad dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

where one has $f_I \in C^\infty(\Omega)$ and where $I = \{i_1, \ldots, i_k\}$ is a set of strictly increasing indices with $i_l \in \{1, \ldots, n\}$, $l = 1, \ldots, k$. Consider the differential operators:

$$d_L : C^\infty(\Omega, \Lambda^k \mathbb{R}^n) \to C^\infty(\Omega, \Lambda^{k+1} \mathbb{R}^n)$$

defined by:

$$d_L f := \sum_{|I|=k} (d_L f_I) dx_I$$

for $f \in C^\infty(\Omega)$, and, for $f = \sum_{|I|=k} f_I dx_I \in C^\infty(\Omega, \Lambda^k \mathbb{R}^n)$, $1 \leq k \leq n-1$, by:

$$d_{L,k} f := \sum_{|I|=k} (d_{L,0} f_I) dx_I = \sum_{|I|=k} \sum_{j=1}^n (L_j f_I) dx_I \wedge dx_j,$$

We also define the dual pseudo-complex $d^*_L : C^\infty(\Omega, \Lambda^k \mathbb{R}^n) \to C^\infty(\Omega, \Lambda^k \mathbb{R}^n)$, $0 \leq k \leq n-1$, determined by the following relation for any $u \in C^\infty_c(\Omega, \Lambda^k \mathbb{R}^n)$ and $v \in C^\infty_c(\Omega, \Lambda^{k+1} \mathbb{R}^n)$:

$$\int d_{L,k} u \cdot \overline{v} = \int u \cdot d^*_{L,k} v,$$

where the dot indicates the standard pairing on forms of the same degree. This is to say that given $f = \sum_{|I|=k} f_I dx_I$, one has:

$$d^*_{L,k} f = \sum_{|J|=k} \sum_{j \in J} L^*_j f_I dx_I \wedge dx_J,$$

where, for each $j_I \in J = \{j_1, \ldots, j_k\}$ and $l \in \{1, \ldots, k\}$, $dx_{j_I} \wedge dx_J$ is defined by:

$$dx_{j_I} \wedge dx_J := (-1)^{l+1} dx_1 \wedge \cdots \wedge dx_{j_I-1} \wedge dx_{j_I} \wedge \cdots \wedge dx_{j_k}.$$

Suppose first that $L$ is involutive, i.e. that each commutator $[L_j, L_\ell]$, $1 \leq j, \ell \leq n$ is a linear combination of $L_1, \ldots, L_n$. Then the chain $\{d_{L,k}\}_k$ defines a complex of differential operators associated to the structure $L$, which is precisely the de Rham complex when $n = N$ and $L_j = \partial_{x_j}$ (see [4] for more details). In the non-involutive situation, we do not get a complex in general, and the fundamental complex property $d_{L,k+1} \circ d_{L,k} = 0$ might not hold. On the other hand, this chain still satisfies a “pseudo-complex” property in the sense that $d_{L,k+1} \circ d_{L,k}$ is a differential of order one rather than two, as it is generically expected. We will refer to $(d_{L,k}, C^\infty_c(\Omega, \Lambda^k \mathbb{R}^n))$ as the pseudo-complex $\{d_L\}$ associated with $L$ on $\Omega$.

Consider the operator

$$A(\cdot, D) = (d_{L,k}, d^*_{L,k}) : C^\infty_c(\Omega, \Lambda^k \mathbb{R}^n) \to C^\infty_c(\Omega, \Lambda^{k+1} \mathbb{R}^n) \times C^\infty_c(\Omega, \Lambda^{k-1} \mathbb{R}^n),$$

for $0 \leq k \leq n$. Here the operator $d_{L,-1} = d^*_{L,-1}$ is understood to be zero. The operator $A(\cdot, D)$ is elliptic and canceling for $k \notin \{1, n-1\}$ (see Section 4 [13] for details), so that for each $x_0 \in \Omega$ there exists an neighborhood $U \subset \Omega$ of $x_0$ and $C > 0$ such that the inequality:

$$\|u\|_{L^{N/2} \cap L^1} \leq C(\|d_{L,k} u\|_{L^1} + \|d^*_{L,k-1} u\|_{L^1}),$$

holds for any $u \in \mathcal{D}(U, \Lambda^k \mathbb{R}^n)$ (see [13, Theorem B]).

Now we consider the equation (2) associated to operator $A(\cdot, D)$, i.e. the equation:

$$d^*_{L,k} u + d_{L,k-1} v = f,$$

The following local continuous solvability result for (14) is a consequence of our main theorem.

**Corollary 7.2.** Consider a system of complex vector fields $L = \{L_1, \ldots, L_n\}$, $n \geq 2$ satisfying hypotheses (i)-(ii) above, and the pseudo-complex $\{d_{L,k}\}_k$ associated with $L$ on $\Omega$ with $k \notin \{1, n-1\}$. Then every point $x_0 \in \Omega$ is contained in an open neighborhood $U \subset \Omega$ such that for any $f \in \mathcal{D}'(U, \Lambda^k \mathbb{R}^n)$, the equation (14)
is continuously solvable in $U$ if and only if for every $\varepsilon > 0$ and every compact set $K \subset U$, there exists $\theta = \theta(K, \varepsilon) > 0$ such that one has, for every $\varphi \in \mathcal{D}_K(U, L^k \mathbb{R}^n)$:

$$|f(\varphi)| \leq \theta \|\varphi\|_1 + \varepsilon (\|d_{L,k}\varphi\|_1 + \|d^*_{L,k-1}\phi\|_1).$$

Theorem 4.1 is a direct consequence of the previous result, taking $k = 0$ (recall that one has $d_{L,0} = \nabla_L$ and $d^*_{L,0} = \text{div}_L^*$). We emphasize that the operator is not canceling when $k = 1$ or $k = n - 1$ (see [3, Section 4]).

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