GOOD AND BAD BEHAVIOUR OF THE LINEARITY DEFECT

HOP D. NGUYEN

ABSTRACT. The linearity defect introduced by Herzog and Iyengar is a numerical measure for the complexity of minimal free resolutions over local rings and graded algebras. We study the behaviour of linearity defect along short exact sequences. Typically, bad behaviour may occur, in sharp contrast to the Castelnuovo-Mumford regularity. However, along fairly large classes of exact sequences involving Koszul modules, the behaviour of linearity defect is under good control. We also generalize the notion of Koszul filtrations due to Conca, Trung and Valla to local rings. Among the applications, we prove that specializations of absolutely Koszul algebras are again absolutely Koszul, answering a question due to Conca, Iyengar, Nguyen and Römer. Several Koszulness criteria for local rings and modules over them are provided.

1. Introduction

The linearity defect, introduced by Herzog and Iyengar [18], measures how far a module is from having a linear free resolution. The notion was inspired by work of Eisenbud, Floystad and Schreyer [13] on free resolutions over the exterior algebra. Let us recall this notion. Throughout, we denote by \((R, \mathfrak{m}, k)\) either a noetherian local ring with maximal ideal \(\mathfrak{m}\) and residue field \(k = R/\mathfrak{m}\), or a standard graded \(k\)-algebra with graded maximal ideal \(\mathfrak{m}\). (Sometimes, we omit \(k\) and write simply \((R, \mathfrak{m})\).) Let \(M\) denote a finitely generated (graded) \(R\)-module. Let the minimal (graded) free resolution of \(M\) over \(R\) be

\[
F : \cdots \to F_i \xrightarrow{\partial} F_{i-1} \xrightarrow{\partial} \cdots \to F_1 \xrightarrow{\partial} F_0 \to 0.
\]

By definition, the differential maps \(F_i\) into \(\mathfrak{m}F_{i-1}\). Then \(F\) has a filtration \(\mathcal{G}^i F\) given by \((\mathcal{G}^n F)_i = \mathfrak{m}^{n-i}F_i\) for all \(n, i\) (where \(\mathfrak{m}^j = R\) if \(j \leq 0\)), and the map

\[
(\mathcal{G}^n F)_i = \mathfrak{m}^{n-i}F_i \to (\mathcal{G}^n F)_{i-1} = \mathfrak{m}^{n-i+1}F_{i-1}
\]

is induced by the differential \(\partial\). The associated graded complex induced by the filtration \(\mathcal{G}^i F\), denoted by \(\text{lin}^R F\), is called the linear part of \(F\). We define the linearity defect of \(M\) as the number

\[
\text{ld}_R M = \sup\{i : H_i(\text{lin}^R F) \neq 0\}.
\]

We set the linearity defect of the zero module to be \(-\infty\). We say that \(M\) is a Koszul module if \(\text{ld}_R M = 0\). Furthermore, \(R\) is called a Koszul ring if \(\text{ld}_R k = 0\). In the graded case, \(R\) is a Koszul algebra (i.e. \(k\) has a linear free resolution as an \(R\)-module) if and only if \(R\) is a Koszul ring, or equivalently, if and only if \(\text{ld}_R k < \infty\) [18]. This is reminiscent of the result due to Avramov, Eisenbud and Peeva [4], [7] saying that \(R\) is a Koszul algebra if and only if \(k\) has Castelnuovo-Mumford regularity \(\text{reg}_R k < \infty\). It is not clear whether the analogous statement for local rings, that \(\text{ld}_R k < \infty\) implies \(R\) is Koszul, holds true; see [2], [31] for the current

The author was supported by the CARIGE foundation.
progress on this question, and [13], [1], [25], [28], [33], [34] for some other directions of study. For recent surveys related to free resolutions and Koszul algebras, we refer to [9] and [27].

It always holds that $\ld_R M \leq \pd_R M$, the projective dimension of $M$. Moreover, in the graded case, if $\ld_R M$ is finite, then so is the Castelnuovo-Mumford regularity $\reg_R M$ ([18, Proposition 1.12]). From this, one might expect that linearity defect would behave along exact sequences as well as the regularity and the projective dimension. However, complicated behaviour do occur. Let $0 \to M \to P \to N \to 0$ be an exact sequence of finitely generated $R$-modules. Then it may happen that $\ld_R P = \infty$ while $\ld_R M = \ld_R N = 0$; see 3.1 for more pathological examples.

One of the purposes of this paper is to analyse the behaviour of linearity defect along short exact sequences. Not surprisingly, we need a characterization of the linearity defect that is resolution-free, e.g. one that involves only Tor or Ext modules. Such a characterization, which turns out to be a useful one, was discovered by Segà [31, Theorem 2.2]:

$$\ld_R M = \inf \left\{ t : \Tor^R_i(R/m^{s+1}, M) \to \Tor^R_i(R/m^s, M) \text{ is the zero map} \right\}.$$

In Section 2, we start with general bounds on linearity defects of modules in a short exact sequence (Proposition 2.5). Unfortunately, these bounds do not prohibit pathological behaviour of linearity defect along short exact sequences, as shown in Section 3. In Section 4, two instances of good behaviour are described for short exact sequence involving Koszul modules. The main results of this section, Theorems 4.1 and 4.5, form the technical heart of this paper. We quote Theorem 4.1 and its proof.

**Theorem A.** Let $0 \to M' \to P' \to N' \to 0$ be an exact sequence of non-trivial finitely generated $R$-modules where

(i) $M'$ is a Koszul module,
(ii) $M' \cap m^j P' = m^i M'$ (e.g., $M'$ is a pure submodule of $P'$).

Then the natural map $\Tor^R_i(k, M') \to \Tor^R_i(k, P')$ is injective for all $i \geq 0$. And we have either $\ld_R N' = \ld_R P'$ if $\ld_R P' \geq 1$ or $\ld_R N' \leq 1$ if $\ld_R P' = 0$.

Furthermore (see Green and Martínez-Villa [15]), the necessary and sufficient condition for $\ld_R N' = 0$ is that $\ld_R P' = 0$ and $M' \cap m^s P' = m^s M'$ for all $s \geq 1$.

The other main result of Section 4 and its proof are summarized in the following theorem. One can view it as a dual statement of Theorem A.

**Theorem B.** Let $0 \to M \to P \to N \to 0$ be an exact sequence of non-trivial finitely generated $R$-modules where

(i) $P$ is a Koszul module,
(ii) $M \subseteq m^s P$.

Then the natural map $\Tor^R_i(k, M) \to \Tor^R_i(k, P)$ is zero for all $i \geq 0$. And we have either $\ld_R N = \ld_R M + 1$ if $\ld_R M \geq 1$ or $\ld_R N \leq 1$ if $\ld_R M = 0$.

Furthermore, the necessary and sufficient condition for $\ld_R N = 0$ is that $\ld_R M = 0$ and $M \cap m^{s+1} P = m^s M$ for all $s \geq 1$.

Using Theorems A and B, we study in Section 5 the behaviour of linearity defect under change of rings. The most precise statement that we can prove is contained in Theorem 5.2.
Theorem C. Let \((R, \mathfrak{m}) \to (S, \mathfrak{n})\) be a surjection of local rings such that \(\text{ld}_R S = 0\). Then for any finitely generated \(S\)-module \(N\), there is an equality \(\text{ld}_R N = \text{ld}_S N\).

Following Iyengar and Römer [20], \(R\) is said to be absolutely Koszul if every finitely generated \(R\)-module has finite linearity defect. For the sake of examples, if \(Q\) is a complete intersection of quadrics and \(Q \to R\) is a Golod map of graded \(k\)-algebras (i.e. \(\text{reg}_Q R = 1\)), then \(R\) is absolutely Koszul (see [18, Proposition 5.8, Theorem 5.9]). Other examples of absolutely Koszul algebras include, when \(\text{char} k = 0\), Veronese subrings of a polynomial ring over \(k\) with dimension at most 3 (see [10, Corollary 5.4]). It is not clear whether one can remove the restriction on the dimension in the last example. The reader may consult [10], [20] for more examples and questions concerning absolutely Koszul rings. As a corollary of Theorem C, we show that absolutely Koszul algebras are stable under taking quotient ring modulo a regular linear form (Corollary 5.6). This answers in the positive a question raised in [10, Remark 3.10]. On the negative side, many examples are provided to contrast between the linearity defect and the regularity, the behaviour of the first one under change of rings often seems to be much wilder than that of the second.

An efficient method to establish Koszulness of graded algebras is constructing Koszul filtrations [12]; see also, e.g., [8], [11], [17]. In Section 6, we generalize the Koszul filtration method from the graded case to the local setting. The notion of ideals with linear quotients is very useful to prove that certain ideals have linear resolutions. In the same section, we introduce the local version of this notion, and prove that it enjoys the same property as in the graded case. Finally, the end of Section 6 is devoted to several other applications of linearity defect to the Koszul property of local rings and modules over such rings. Among the applications, we have:

(i) results of Lu-Zhou [21, Theorem 3.7], Sharifan-Varbaro [32, Corollary 2.4], Murai [23, Proposition 3.7], saying that modules with linear quotients are componentwise linear in the sense of Herzog and Hibi [16] (see Proposition 6.7 for additional details);
(ii) parts of results of Avramov-Iyengar-Şega [5], [6] on Koszul property of rings with Conca generators and rings with short Koszul modules (see Propositions 6.8, 6.14, 6.15);
(iii) large parts of Ahangari’s work [1] on the Koszul property of rings with minimal multiplicity or certain artinian rings with \(\mathfrak{m}^3 = 0\) (see Corollary 6.10, Proposition 6.11, Theorem 6.12).

Moreover, our results hold in the generality of local rings, usually surpassing the scope of the previous results; see Theorem 6.3, Propositions 6.7 and 6.15.

For unexplained notations and terminology, we refer to Avramov’s monograph [3] and the book of Peeva [26]. Note that we only state our results for local rings but the analog for graded algebras are straightforward and will be freely used.

2. General bounds

Background. Let \((S, \mathfrak{n})\) be a standard graded algebra over a field \(k\). Let \(N\) be a finitely generated graded \(S\)-module. The Castelnuovo-Mumford regularity of \(N\) over \(S\) is

\[
\text{reg}_S N = \max\{j - i : \text{Tor}_i^S(k, N)_j \neq 0\}.
\]

We say that \(S\) is a Koszul algebra, if \(k = S/\mathfrak{n}\) has linear free resolution over \(S\), equivalently \(\text{reg}_S k = 0\). The standard graded polynomial ring \(k[x_1, \ldots, x_n]\) (where \(n \geq 1\)) is a Koszul algebra: \(k\) is resolved by the Koszul complex, which is a linear resolution.
Let $M$ be a finitely generated (graded) $R$-module, where $(R, \mathfrak{m})$ is our local ring (or standard graded $k$-algebra). The associated graded module of $M$ with respect to the $\mathfrak{m}$-adic filtration is

$$\text{gr}_\mathfrak{m} M = \bigoplus_{i=0}^\infty \frac{\mathfrak{m}^i M}{\mathfrak{m}^{i+1} M}.$$ 

It is a graded module with generators in degree 0 over the associated graded ring $\text{gr}_\mathfrak{m} R$. Recall that Koszul modules are related to linear free resolutions by the following result; we refer the reader to [22, Theorem 2.5] and [18, Proposition 1.5].

**Proposition 2.1.** Let $M \neq 0$ be a finitely generated $R$-module. The following are equivalent:

(i) $M$ is a Koszul $R$-module, i.e., $\text{ld}_R M = 0$;

(ii) The graded $\text{gr}_\mathfrak{m} R$-module $\text{gr}_\mathfrak{m} M$ has 0-linear free resolution.

**Definition 2.2.** We say that $R$ is a Koszul ring if the residue field $k = R/\mathfrak{m}$ is a Koszul module. For example, any regular local ring is Koszul, since $\text{gr}_\mathfrak{m} R$ is isomorphic to a standard graded polynomial ring over $k$.

For convenience of our arguments, sometimes we work with the invariant

$$\text{gl ld} R = \sup\{\text{ld}_R M : M \text{ is a finitely generated (graded) } R\text{-module}\},$$

which is called the global linearity defect of $R$. The following lemma is taken from [10, Corollary 6.4]; we include an argument here for the sake of completeness.

**Lemma 2.3.** Let $f \neq 0$ be a quadratic form in the polynomial ring $k[x_1, \ldots, x_n]$ (where $n \geq 1$). Then $\text{gl ld}(k[x_1, \ldots, x_n]/(f)) = n - 1$.

**Proof.** Denote $R = k[x_1, \ldots, x_n]/(f)$. Passing to a field extension of $k$ does not affect $\text{gl ld} R$, hence we can assume that $k$ is infinite. Choose a regular sequence of linear forms $l_1, \ldots, l_{n-1}$. Denoting $S = R/(l_1, \ldots, l_{n-1})$, then applying [20, Theorem 2.11] for the map $R \to S$, we get

$$\text{gl ld} R \leq \text{gl ld} S + n - 1.$$

Since $R$ has minimal multiplicity, so does $S$. Therefore $S \cong k[y]/(y^2)$, where $k[y]$ is a polynomial ring. This implies $\text{gl ld} S = 0$ and the last inequality yields $\text{gl ld} R \leq n - 1$.

For the reverse inequality, note that $l_1^2, \ldots, l_{n-1}^2$ is also an $R$-regular sequence. Direct computations show that

$$\text{ld}_R R/(l_1^2, \ldots, l_{n-1}^2) = n - 1.$$

In particular, $\text{gl ld} R \geq n - 1$, as desired. \hfill $\square$

**Bounding the linearity defect.** The starting point for our investigation is the following result due to Şega. It was stated for the local case but the proof works equally well in the graded case, taking advantage of the grading.

**Theorem 2.4** (Şega, [31, Theorem 2.2]). For any non-trivial finitely generated $R$-module $M$, the following are equivalent:

(i) $\text{ld}_R M \leq t$;

(ii) The natural morphism $\text{Tor}^R_i(R/\mathfrak{m}^{s+1}, M) \to \text{Tor}^R_i(R/\mathfrak{m}^{s}, M)$ is zero for all $i > t, s \geq 0$.

The main result of this section is
Proposition 2.5. Let $0 \to M \to P \to N \to 0$ be a short exact sequence of non-trivial finitely generated $R$-modules. Define the (possibly infinite) numbers:

\[ d_M = \inf \{ m \geq 0 : \text{the connecting map } \text{Tor}^R_i(k, N) \to \text{Tor}^R_i(k, M) \text{ is zero for all } i \geq m \}, \]
\[ d_P = \inf \{ m \geq 0 : \text{the natural map } \text{Tor}^R_i(k, M) \to \text{Tor}^R_i(k, P) \text{ is zero for all } i \geq m \}, \]
\[ d_N = \inf \{ m \geq 0 : \text{the natural map } \text{Tor}^R_i(k, P) \to \text{Tor}^R_i(k, N) \text{ is zero for all } i \geq m \}. \]

Then there are inequalities

(i) \[ \text{ld}_R N \leq \max \{ \min \{ d_P, d_M + 1 \}, \text{ld}_R P, \text{ld}_R M + 1 \}, \]
(ii) \[ \text{ld}_R P \leq \max \{ \min \{ d_M, d_N \}, \text{ld}_R M, \text{ld}_R N \}, \]
(iii) \[ \text{ld}_R M \leq \max \{ \min \{ d_N - 1, d_P \}, \text{ld}_R N - 1, \text{ld}_R P \}. \]

Several comments are in order.

Remark 2.6. (i) If $P$ is a free module, then $\text{Tor}^R_i(k, P) = 0$ for all $i \geq 1$, therefore $d_P, d_N \leq 1$ in this case. Similar things happen if $M$ or $N$ is a free module. In general, we have $d_P \leq \min \{ \text{pd}_R M + 1, \text{pd}_R P + 1 \}$ and two analogous inequalities.

(ii) Since $\text{Tor}^R_i(k, M) \to \text{Tor}^R_i(k, P) \to \text{Tor}^R_i(k, N) \to \text{Tor}^R_{i-1}(k, M)$ is an exact sequence for all $i$, we also have other interpretations for the numbers $d_M, d_N, d_P$. For example,

\[ d_M = \inf \{ m \geq 0 : \text{the natural map } \text{Tor}^R_i(k, M) \to \text{Tor}^R_i(k, P) \text{ is injective for all } i \geq m \}. \]

Therefore, the two numbers $d_P$ and $d_M$ are not simultaneously finite unless $\text{pd}_R M < \infty$. Similar statements hold for the pairs $d_P$ and $d_N, d_M$ and $d_N$.

(iii) The above interpretation of $d_M$ indicates that the first inequality of (2.5) relates $\text{ld}_R N$ with asymptotic properties of the map $\text{Tor}^R_i(k, M) \to \text{Tor}^R_i(k, P)$. Similar comments apply to the inequalities for linearity defects of $M$ and $P$.

Example 2.7. In general, none of the numbers $d_M, d_N, d_P$ is finite, even if $R$ is Koszul and $M, N, P$ are Koszul modules. For example, take $R = k[x, y]/(xy)$. Consider the exact sequence

\[ 0 \to (x^3, y^2) \to (x^2, y^2) \to \frac{(x^2, y^2)}{(x^3, y^2)} \to 0. \]

The (2-periodic) minimal free resolution of $k$ over $R$ is given by

\[ F : \cdots \to R^2 \xrightarrow{\begin{pmatrix} y & 0 \\ x & 0 \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} R \to 0. \]

Let $P = (x^2, y^2)$, we want to compute $\text{Tor}^R_i(k, P)$. Note that $P \otimes_R F_i = P \oplus P$ for $i \geq 1$. Fix $i \geq 2$, the map $P \otimes_R F_{2i} \to P \otimes_R F_{2i-1}$ is given by

\[ (a, b) \mapsto (ya, xb) \]

and the map $P \otimes_R F_{2i+1} \to P \otimes_R F_{2i}$ is given by

\[ (u, v) \mapsto (xu, yv). \]
Let \( \partial \) be the differential of \( P \otimes_R F \), then
\[
\text{Ker } \partial_{2i} = (x^2) \oplus (y^2),
\]
\[
\text{Im } \partial_{2i+1} = (x^3) \oplus (y^3).
\]
Therefore \( \text{Tor}_2^R(k, P) \cong (x^2)/(x^3) \oplus (y^2)/(y^3) \). Similarly, setting \( M = (x^3, y^2) \), then it holds that \( \text{Tor}_2^R(k, M) \cong (x^3)/(x^4) \oplus (y^2)/(y^3) \). In particular,
\[
\text{Ker}(\text{Tor}_2^R(k, M) \to \text{Tor}_2^R(k, P)) = ((x^3)/(x^4))
\]
\[
\text{Im}(\text{Tor}_2^R(k, M) \to \text{Tor}_2^R(k, P)) = ((y^2)/(y^3)).
\]
This implies that \( d = \infty \). Denote \( N = P/M \), then from the exact sequence of \( \text{Tor} \), we also infer that \( d_M = d_N = \infty \). Note that \( N \cong R/(x, y) = k \), so \( \text{ld}_R N = 0 \). One can check that \( M, P \) are Koszul modules: By Lemma 2.3, \( \text{ld}_R R/U \leq 1 \) for any ideal \( U \subseteq m \). Hence \( \text{ld}_R U = 0 \).

Now we are going to prove Proposition 2.5. First we have several simple but very useful observations. For simplicity, the obvious superscripts in \( \text{Tor} \) modules are omitted.

**Lemma 2.8.** Let \( M \to P \) be an \( R \)-linear map between finitely generated \( R \)-modules.

(i) If for some \( \ell \geq \text{ld}_R M + 1 \), the map \( \text{Tor}_{\ell-1}(k, M) \to \text{Tor}_{\ell-1}(k, P) \) is injective, then the map
\[
\text{Tor}_i(R/m^s, M) \to \text{Tor}_i(R/m^s, P)
\]
is injective for all \( i \geq \ell \) and all \( s \geq 0 \).

(ii) If for some \( \ell \geq \text{ld}_R P + 1 \), the map \( \text{Tor}_{\ell-1}(k, M) \to \text{Tor}_{\ell-1}(k, P) \) is zero, then the map
\[
\text{Tor}_i(R/m^s, M) \to \text{Tor}_i(R/m^s, P)
\]
is zero for all \( i \geq \ell \) and all \( s \geq 0 \).

**Proof.** Consider the following commutative diagram
\[
\begin{array}{ccc}
\text{Tor}_i(R/m^s, M) & \xrightarrow{\alpha_i^M} & \text{Tor}_{i-1}(m^s/m^{s+1}, M) \\
\text{Tor}_i(R/m^s, P) & \xrightarrow{\alpha_i^P} & \text{Tor}_{i-1}(m^s/m^{s+1}, P).
\end{array}
\]

(i) By induction on \( i \) and using the above diagram for \( s = 1 \), we see that \( \text{Tor}_i(k, M) \to \text{Tor}_i(k, P) \) is injective for all \( i \geq \ell - 1 \). Note that as \( i \geq \text{ld}_R M + 1 \), by Theorem 2.4, the map \( \alpha_i^M \) is injective. Next let \( s \geq 0 \) be arbitrary, again using the diagram and the fact that \( m^s/m^{s+1} \) is either 0 (equivalently, \( m^s = 0 \)) or isomorphic to a direct sum of copies of \( k \), we deduce that \( \rho \) is also injective.

(ii) Similarly, by induction on \( i \) and using the diagram for \( s = 1 \), \( \text{Tor}_i(k, M) \to \text{Tor}_i(k, P) \) is the zero map for all \( i \geq \ell - 1 \). Note that since \( i \geq \text{ld}_R P + 1 \), \( \alpha_i^P \) is injective. Then for arbitrary \( s \geq 0 \), using the diagram, we see that \( \rho \) is the zero map as well. \( \square \)

**Proof of Proposition 2.5.** (i) For the proof of the inequality \( \text{ld}_R N \leq \max\{d_P, \text{ld}_R P, \text{ld}_R M + 1\} \), we may assume that \( \ell = \max\{d_P, \text{ld}_R P, \text{ld}_R M + 1\} < \infty \). For each \( i > \ell, s \geq 0 \), from the exact sequence
\[
0 \to m^s/m^{s+1} \to R/m^{s+1} \to R/m^s \to 0,
\]
we get the commutative diagram with exact rows and columns

\[
\begin{array}{cccc}
\text{Tor}_i(R/m^s, M) & \rightarrow & \text{Tor}_{i-1}(m^s/m^{s+1}, M) & \rightarrow \text{Tor}_{i-1}(R/m^{s+1}, M) \\
0 & \rightarrow & \text{Tor}_i(R/m^s, P) & \rightarrow \text{Tor}_{i-1}(m^s/m^{s+1}, P) & \rightarrow \text{Tor}_{i-1}(R/m^{s+1}, P) \\
\text{Tor}_i(R/m^s, N) & \rightarrow & \text{Tor}_{i-1}(m^s/m^{s+1}, N) & \rightarrow \gamma \\
0 & \rightarrow & \text{Tor}_{i-1}(R/m^s, M) & \rightarrow & \text{Tor}_{i-2}(m^s/m^{s+1}, M)
\end{array}
\]

By Šega’s theorem 2.4 and the fact that \( i \geq \max\{\text{ld}_R P + 1, \text{ld}_R M + 2\} \), we have \( \alpha^i_P, \alpha^{i-1}_M \) are injective. Note that \( m^s/m^{s+1} \) is either zero if \( m^s = 0 \) or otherwise a direct sum of copies of \( k \), therefore by hypothesis, we have \( \psi_{i-1} = 0 \). Now we need to show that \( \alpha^i_N \) is also injective. This is a simple diagram chasing. Hence \( \text{ld}_R N \leq \ell \).

Next we want to show that \( \text{ld}_R N \leq \max\{d_M + 1, \text{ld}_R P, \text{ld}_R M + 1\} \). We lose nothing by assuming that the later is finite. Take \( i \geq \max\{d_M + 1, \text{ld}_R P, \text{ld}_R M + 1\} + 1 \). Look at the exact sequence \( \text{Tor}_{i-1}(k, N) \rightarrow \text{Tor}_{i-2}(k, M) \rightarrow \text{Tor}_{i-2}(k, P) \). Since \( i - 2 \geq d_M \), the first map is zero. Hence the second map is injective. Now \( i - 1 \geq \text{ld}_R M + 1 \), hence by Lemma 2.8(i), \( \kappa \) is injective. Therefore by diagram chasing, again \( \text{ld}_R N < i \).

(ii), (iii): The proofs are similar to part (i).

We record a few consequences of Proposition 2.5. Interestingly, we can extract information about the linearity defect from any free resolution, minimal or not, of a module: If \( P \) is a free resolution of \( N \), then \( \text{ld}_R N = r \geq 1 \) if and only if \( r \) is the minimal number \( i \) such that \( N_i \) is Koszul, where \( N_i = \text{Im}(P_i \rightarrow P_{i-1}) \) for \( i \geq 1 \). If \( N \) is a Koszul module then so is \( N_i \) for every \( i \geq 1 \).

**Corollary 2.9.** Let \( 0 \rightarrow M \rightarrow P \rightarrow N \rightarrow 0 \) be an exact sequence of non-trivial finitely generated \( R \)-modules. Then

(i) \( \text{ld}_R N \leq \min\{\max\{\text{pd}_R P + 1, \text{ld}_R M + 1\}, \max\{\text{ld}_R P, \text{pd}_R M + 1\}\} \),

(ii) \( \text{ld}_R P \leq \min\{\max\{\text{pd}_R M + 1, \text{ld}_R N\}, \max\{\text{ld}_R M, \text{pd}_R N + 1\}\} \),

(iii) \( \text{ld}_R M \leq \min\{\max\{\text{ld}_R N - 1, \text{pd}_R P\}, \max\{\text{pd}_R N, \text{ld}_R P\}\} \).

In particular, we have:

(a) If \( P \) is free, then \( \text{ld}_R M = \text{ld}_R N - 1 \) if \( \text{ld}_R N \geq 1 \) and \( \text{ld}_R M = 0 \) otherwise.

(b) If one of the modules has finite projective dimension, then the other two have both finite or both infinite linearity defects.

**Proof.** For (i): using Proposition 2.5, we get

\[ \text{ld}_R N \leq \max\{d_P, \text{ld}_R P, \text{ld}_R M + 1\}. \]

Recalling that \( d_P \leq \max\{\text{pd}_R P + 1, \text{pd}_R M + 1\} \) and \( \text{ld}_R M \leq \text{pd}_R M \), the desired inequalities follow.
Similarly, for (ii), we combine Proposition 2.5 with the fact that \( d_N \leq \text{pd}_R N + 1 \) and \( d_M \leq \text{pd}_R M + 1 \). Finally, for (iii), the fact we need is \( d_N \leq \max\{\text{pd}_R N + 1, \text{pd}_R P + 1\} \).

For (a): since \( \text{pd}_R P = 0 \), from (i) and (iii), we get the inequalities
\[
\text{ld}_R N \leq \text{ld}_R M + 1,
\]
\[
\text{ld}_R M \leq \max\{\text{ld}_R N - 1, 0\}.
\]
This yields the conclusion of (a). The remaining assertion is a consequence of (i)–(iii). \(\square\)

3. Examples

We give instances where bad behaviour of linearity defect along short exact sequences of modules occur even for Koszul algebras.

Example 3.1. Let \( R = k[x, y, z, t]/((x, y)^2 + (z, t)^2) \), \( \mathfrak{m} \) its graded maximal ideal. The Hilbert series of \( R \) is \( H_R(\nu) = (1 + 2\nu)^2 \). By result of Roos [29, Theorem 2.4], there exists a graded \( R \)-module with infinite linearity defect. Explicitly, by [29, Formula (5.2)] and [18, Proposition 1.8], the cokernel of the map \( R(-1)^3 \to R^2 \) given by the matrix
\[
\begin{pmatrix}
y & x + 3t & t \\
z & -t & x + t
\end{pmatrix}
\]
is such a module. Let \( F = R(-1)^3, G = R^2, M = \text{Ker}(F \to G) \) and \( N = \text{Im}(F \to G) \). Note that \( F \) is the projective cover of \( N \). Since \( N \subseteq \mathfrak{m}G \), we have \( \mathfrak{m}^2 N = 0 \) (note that \( \mathfrak{m}^3 = 0 \)). Clearly \( \text{ld}_R N = \text{ld}_R M = \infty \).

(i) The \( R \)-module \( N \) is an extension of Koszul \( R \)-modules. Indeed, we have an exact sequence
\[
0 \to \mathfrak{m}N \to N \to N/\mathfrak{m}N \to 0.
\]
Now \( \mathfrak{m}N \) and \( N/\mathfrak{m}N \) are both annihilated by \( \mathfrak{m} \), so they are Koszul modules. So there is an extension of Koszul \( R \)-modules which has infinite linearity defect.

(ii) Since \( M \subseteq \mathfrak{m}F \), we also have an exact sequence
\[
0 \to M \to \mathfrak{m}F \to \mathfrak{m}N \to 0.
\]
Now \( \mathfrak{m}F \) is a Koszul module and \( \mathfrak{m}N \) is also Koszul as noted above. So the kernel of a surjection of Koszul modules may have infinite linearity defect.

(iii) Now \( N \) is an \((R/\mathfrak{m}^2)\)-module so we can take the beginning of the minimal graded \((R/\mathfrak{m}^2)\)-free resolution of \( N \), say (without grading notation)
\[
0 \to D \to (R/\mathfrak{m}^2)^r \to N \to 0.
\]
So \( D \) is annihilated by \( \mathfrak{m} \), hence \( D \) is a Koszul \( R \)-module. Also \( \text{ld}_R(R/\mathfrak{m}^2) = 1 \) but \( \text{ld}_R N = \infty \).

We do not know if there exists a short exact sequence in which the first two modules are Koszul but the cokernel has infinite linearity defect.
4. Sequences involving Koszul modules

We describe quite concretely the behaviour of linearity defects for some sequences involving Koszul modules without assuming anything about the ground ring. Firstly, using results in Section 2, we can control the linearity defect for certain “pure” extensions of a Koszul module. The first main result of this section is as follow.

**Theorem 4.1.** Let \( 0 \to M' \to P' \to N' \to 0 \) be a short exact sequence of non-zero finitely generated \( R \)-modules where

(i) \( M' \) is a Koszul module;
(ii) \( M' \cap \mathfrak{m}P' = \mathfrak{m}M' \).

Then there are inequalities \( \text{ld}_R P' \leq \text{ld}_R N' \leq \max\{\text{ld}_R P', 1\} \). In particular, \( \text{ld}_R N' = \text{ld}_R P' \) if \( \text{ld}_R P' \geq 1 \) and \( \text{ld}_R N' \leq 1 \) if \( \text{ld}_R P' = 0 \).

Moreover (see Green and Martínez-Villa \[15, Propositions 5.2 and 5.3\]), \( \text{ld}_R N' = 0 \) if and only if \( P' \) is a Koszul module and \( M' \cap \mathfrak{m}^sP' = \mathfrak{m}^sM' \) for all \( s \geq 1 \).

**Proof.** (a) We will show that \( d_{M'} = 0 \), or equivalently, \( \text{Tor}_i^R(k, M') \to \text{Tor}_i^R(k, P') \) is injective for each \( i \geq 0 \).

This is clear for \( i = 0 \) thanks to the equality \( M' \cap \mathfrak{m}P' = \mathfrak{m}M' \). Now using Lemma 2.8(i) where \( \text{ld}_R M' = 0, \ell = 1 \), we get the desired claim.

Alternatively, let \( F, G \) be the minimal free resolution of \( M', P' \) respectively. There is a lifting \( \varphi \colon F \to G \) of the inclusion \( M' \to P' \). We prove that \( \varphi \) maps \( F \) injectively to a direct summand of \( G \). Equivalently, we show by induction on \( i \) that \( \overline{\varphi_i} = \varphi_i \otimes k \) is an injection. The case \( i = 0 \) is obvious since \( M'/\mathfrak{m}M' \to P'/\mathfrak{m}P' \) is injective.

Now assume that \( i \geq 1 \) and we have \( \overline{\varphi_{i-1}} \) is injective. Look at the diagram

\[
\begin{array}{ccc}
F_i/\mathfrak{m}F_i & \xrightarrow{\overline{\varphi_i}} & G_i/\mathfrak{m}G_i \\
\downarrow \overline{\varphi} & & \downarrow \\
\mathfrak{m}F_{i-1}/\mathfrak{m}^2F_{i-1} & \xrightarrow{\overline{\varphi_{i-1}}} & \mathfrak{m}G_{i-1}/\mathfrak{m}^2G_{i-1}.
\end{array}
\]

Since \( M' \) is a Koszul module, \( \overline{\varphi} \) is an injection. Using the induction hypothesis that \( F_{i-1} \) is a free direct summand of \( G_{i-1} \), the below horizontal map, also denoted by \( \overline{\varphi_{i-1}} \), is injective. Hence the “diagonal” map of the diagram is injective and so is \( \overline{\varphi_i} \).

Finally, using Proposition 2.5 where \( d_{M'} = 0 \) and \( \text{ld}_R M' = 0 \), we obtain that

\[ \text{ld}_R N' \leq \max\{1, \text{ld}_R P'\}, \]

and that

\[ \text{ld}_R P' \leq \text{ld}_R N'. \]

The first part of the result is already proved. Next we give a new proof for the result of Green and Martínez-Villa.
(b) Now assume that $P'$ is a Koszul module and $M' \cap m^sP' = m^sM'$ for all $s \geq 1$. We show that $\text{ld}_R N' = 0$. Consider the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \text{Tor}_1(R/m^s, M') & \xrightarrow{\alpha M'} & \text{Tor}_0(m^s/m^{s+1}, M') & \longrightarrow & \text{Tor}_0(R/m^{s+1}, M') \\
| & & | & & | & & | \\
0 & \longrightarrow & \text{Tor}_1(R/m^s, P') & \xrightarrow{\alpha P'} & \text{Tor}_0(m^s/m^{s+1}, P') & \longrightarrow & \text{Tor}_0(R/m^{s+1}, P') \\
| & & | & & | & & | \\
\text{Tor}_1(R/m^s, N') & \xrightarrow{\alpha N'} & \text{Tor}_0(m^s/m^{s+1}, N') & \longrightarrow & 0 \\
| & & | & & | & & | \\
\text{Tor}_0(R/m^s, M') & \longrightarrow & 0 \\
\end{array}
$$

We know that $\text{ld}_R N' \leq 1$ by the preceding part, so by Theorem 2.4, it is enough to show that $\text{Tor}_1^R(R/m^s, N') \rightarrow \text{Tor}_0^R(m^s/m^{s+1}, N')$ is injective for all $s \geq 1$. Clearly $\text{Im} \gamma = (M' \cap m^sP')/m^sM' = 0$, so $\pi$ is surjective. According to the hypothesis, $\psi$ is injective. By the snake lemma, $\text{Ker} \alpha_{P'} \rightarrow \text{Ker} \alpha_{N'} \rightarrow \text{Coker} \alpha_{M'} \rightarrow \text{Coker} \alpha_{P'}$ is exact. But $\text{Ker} \alpha_{P'} = 0 = \text{Ker} \kappa$, hence $\text{Ker} \alpha_{N'} = 0$.

(c) Finally, assume that $\text{ld}_R N' = 0$, then by the first part, $\text{ld}_R P' \leq \text{ld}_R N' = 0$. Assume that on the contrary, $M'/m^{s+1}M' \rightarrow P'/m^{s+1}P'$ is not injective for some $s \geq 1$, we will show that $\text{ld}_R N' = 1$. Choose $s$ minimal with this property. Again in the above diagram, $\text{Im} \gamma = 0$ by the choice of $s$. Using the snake lemma, we get $\text{Ker} \alpha_{N'} \cong \text{Ker} \kappa \neq 0$. Therefore $\text{ld}_R N' \geq 1$, a contradiction. The proof of the theorem is completed.

\[\square\]

\textbf{Remark 4.2.} (i) The theorem is not true if $M'$ is not a Koszul module or $M' \cap mP' \neq m^sM'$. Firstly, consider the sequence

$$
0 \rightarrow (x^2, y^2) \rightarrow (x^2, y^2, xz) \rightarrow \frac{(x^2, y^2, xz)}{(x^2, y^2)} \rightarrow 0
$$

over $R = k[x, y, z]$. Set $M' = (x^2, y^2)$, $P' = (x^2, y^2, xz)$ and $N' = (x^2, y^2, xz)/(x^2, y^2)$. Then $N' \cong R/(x)$, so $\text{ld}_R N' = 0$. It is clear that $M' \cap mP' = m^sM'$, $M'$ is not Koszul, and $\text{ld}_R P' = 1 > \text{ld}_R N'$.

Secondly, consider the sequence

$$
0 \rightarrow D \rightarrow (R/m^2)^r \rightarrow N \rightarrow 0
$$

in Example 3.1(iii). Note that $D$ is Koszul, and $D \subseteq m(R/m^2)^r$, hence the condition (ii) of Theorem 4.1 is not satisfied. In this case, we also have $\text{ld}_R N' = \infty > \max\{1, \text{ld}_R(R/m^2)^r\} = 1$.

(ii) In the situation of Theorem 4.1, it may happen that $\text{ld}_R P' = 0$ but $\text{ld}_R N' = 1$. Consider the exact sequence of $(R = k[x, y]$-modules

$$
0 \rightarrow (x^2) \rightarrow (x^2, y) \rightarrow (x^2, y)/(x^2) \rightarrow 0.
$$

Clearly $\text{ld}_R(x^2) = \text{ld}_R(x^2, y) = 0$, while $N' = (x^2, y)/(x^2) \cong R/(x^2)$, so $\text{ld}_R N' = 1$. 

Remark 4.3. The fact that Tor$_i^R(k, M') \to$ Tor$_i^R(k, P')$ is always injective for all $i \geq 0$ was shown by Martínez-Villa and Zacharia [22, Proposition 3.2] by different means. Note that therein, it is not necessary to assume that $R$ is a Koszul ring. Similar remark applies when comparing Corollary 4.4 below with [22, Corollary 3.3].

We also obtain interesting information about behaviour of projective dimension and regularity for sequences of type in Theorem 4.1.

Corollary 4.4 (See [22, Corollary 3.3]). With the hypothesis of Theorem 4.1, there is an equality

$$\text{pd}_R P' = \max\{\text{pd}_R M', \text{pd}_R N'\}.$$ 

If $R$ is a standard graded algebra and $M', P', N'$ are finitely generated graded $R$-modules, then

$$\text{reg}_R P' = \max\{\text{reg}_R M', \text{reg}_R N'\}.$$ 

Proof. For each $i \geq 0$, we have a short exact sequence

$$0 \to \text{Tor}_i^R(k, M') \to \text{Tor}_i^R(k, P') \to \text{Tor}_i^R(k, N') \to 0.$$ 

This clearly implies our desired equalities. \hfill \Box

We also have the control over linearity defect for “small inclusion” in a Koszul module. The next result demonstrates that if $N$ is any finitely generated $R$-module and for any Koszul $R$-module $P$ which surjects onto $N$ in a way that $M = \operatorname{Ker}(P \to N) \subseteq mP$, the module $M$ behaves as if it was the first syzygy module of $N$. See Corollary 4.9 for another result of this type.

Theorem 4.5. Let $0 \to M \to P \to N \to 0$ be a short exact sequence of non-zero finitely generated $R$-modules where

(i) $P$ is a Koszul module;
(ii) $M \subseteq mP$.

Then there are inequalities $\text{ld}_R N - 1 \leq \text{ld}_R M \leq \max\{0, \text{ld}_R N - 1\}$. In particular, $\text{ld}_R N = \text{ld}_R M + 1$ if $\text{ld}_R M \geq 1$ and $\text{ld}_R N \leq 1$ if $\text{ld}_R M = 0$.

Furthermore, $\text{ld}_R N = 0$ if and only if $M$ is a Koszul module and $M \cap m^{s+1}P = m^sM$ for all $s \geq 0$.

Remark 4.6. The result is false in general if $P$ is not Koszul or $M \not\subseteq mP$.

(i) Firstly, look at the sequence

$$0 \to D \to (R/m^2)^r \to N \to 0$$

in Example 3.1(iii). It is easy to verify that $D \subseteq m(R/m^2)^r$ but $\text{ld}_R(R/m^2)^r = 1$, and $\text{ld}_R D = 0$ while $\text{ld}_R N = \infty > \text{ld}_R D + 1 = 1$.

(ii) Secondly, look at the sequence

$$0 \to M \to mF \to mN \to 0$$

in Example 3.1(ii). We know that $mF$ is Koszul, but $M \not\subseteq m^2F$, otherwise $mM = 0$ and thus $M$ would be Koszul, while in fact $\text{ld}_R M = \infty$. We also know that $\max\{0, \text{ld}_R(mN) - 1\} = 0 < \text{ld}_R M = \infty$.

Before going to the proof, we begin with two lemmas.
Lemma 4.7. Let $M \neq 0$ be a Koszul module, and
\[
\cdots \to F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} M \to 0
\]
be its minimal free resolution. For each $i \geq 1$, denote by $\Omega_i(M) = \ker \partial_{i-1}$ the $i$-th syzygy module of $M$. Then
\[
\Omega_i(M) \cap m^s F_{i-1} = m^{s-1} \Omega_i(M)
\]
for all $i \geq 1$ and all $s \geq 1$.

Proof. Since each syzygy modules of $M$ is Koszul, it is enough to consider the case $i = 1$.

Since passing to completion is a faithfully exact functor, we can assume that $R$ is complete. Denote $K = F_1, F = F_0, N = \ker(F_0 \to M) = \Omega_1(M)$. Then $N \subseteq mF$ so $m^{s-1} N \subseteq N \cap m^s F$ and it remains to establish the reverse inclusion. We can also assume that $s > 1$.

Take $x \in N \cap m^s F$. Since $M$ is Koszul, the following sequence is exact
\[
\frac{m^{s-1} K}{m^s K} \to \frac{m^s F}{m^{s+1} F} \to \frac{m^s M}{m^{s+1} M}.
\]
Since $\partial(x) = 0$, there exist $x_1 \in m^{s+1} F$ and $y_1 \in m^{s-1} K$ such that $x = \partial(y_1) + x_1$. Now $x_1 \in N \cap m^{s+1} F$, so similarly, there exist $x_2 \in m^{s+2} F$ and $y_2 \in m^s K$ such that $x_1 = \partial(y_2) + x_2$.

In particular, $x = \partial(y_1) + \partial(y_2) + x_2$.

Continuing this process, for each $p \geq 0$, there exist $x_p \in N \cap m^{s+p} F$ and $y_p \in m^{s+p-2} K$ such that $x = \partial(y_1 + \cdots + y_p) + x_p$. Since $R$ is complete, we can let $y = y_1 + y_2 + \cdots \in m^{s-1} K$, and then $x = \partial(y) \in m^{s-1} \partial(K) = m^{s-1} N$. The proof of the lemma is finished. 

An argument in the proof of Theorem 4.5 uses

Lemma 4.8 (Lifting Lemma). Let $P$ be a non-zero Koszul $R$-module and $M \subseteq m^s P$ a submodule of $P$ (where $s \geq 1$). Let $F,G$ be the minimal free resolution of $M,P$, respectively. Then there exists a lifting $\varphi : F \to G$ of the inclusion $M \to P$ such that
\[
\varphi(F) \subseteq m^s G.
\]

Proof. We use induction on $i$ to show that there exists a lifting $\varphi_i : F_i \to G_i$ compatible with the maps $\varphi_{i-1}, \ldots, \varphi_0$ such that
\[
\varphi_i(F_i) \subseteq m^s G_i.
\]

Let $\varphi_{-1} = 0$. Since $M \subseteq m^s P$, $\varphi_0$ can be immediately chosen. Now assume that we have constructed $\varphi_i$, where $i \geq 0$. Take $x \in \Omega_{i+1}(M) \subseteq mF_i$. Then
\[
\varphi_i(x) \subseteq m\varphi_i(F_i) \cap \Omega_{i+1}(P) \subseteq m^{s+1} G_i \cap \Omega_{i+1}(P) = m^s \Omega_{i+1}(P).
\]
The last equality is due to Lemma 4.7. Hence we can construct $\varphi_{i+1}$ compatible with the previous morphisms and such that $\varphi_i(F_i) \subseteq m^s G_i$. The proof of the lemma is completed.

Now we are ready for

Proof of Theorem 4.5. (a) We claim that $d_P = 0$, i.e., $\text{Tor}^R_i(k,M) \to \text{Tor}^R_i(k,P)$ is the zero map for each $i \geq 0$. Directly, we can use Lemma 2.8(ii) since in this case $\text{ld}_R P = 0$ and $\text{Tor}^R_0(k,M) \to \text{Tor}^R_0(k,P)$ is the zero map.

For a second proof using lifting property of Koszul modules, let $F,G$ be the minimal free resolutions of $M,P$ respectively. Since $M \subseteq mP$, from Lifting Lemma 4.8, there exist a morphism $\varphi : F \to G$ which lifts the inclusion $M \to P$ and such that $\varphi(F) \subseteq mG$. 
Now the morphisms $\text{Tor}_i^R(k, M) \to \text{Tor}_i^R(k, P)$ are induced by the morphism of complex

$$k \otimes_R \varphi : k \otimes F \to k \otimes G.$$ 

But $\varphi(F) \subseteq mG$ thus $k \otimes_R \varphi = 0$ and we get that $d_P = 0$.

Now using Proposition 2.5 where $d_P = 0$ and the fact that $P$ is Koszul, we see that $\text{ld}_R N \leq \max\{0, 0, \text{ld}_R M + 1\} = \text{ld}_R M + 1$,

and $\text{ld}_R M \leq \max\{0, \text{ld}_R N - 1, 0\} = \max\{0, \text{ld}_R N - 1\}$.

This gives the first part of the result.

(b) Now assume that $M$ is a Koszul module and $M \cap m^{s+1}P = m^sM$ for all $s \geq 0$. Since $M \subseteq mP$, there is an exact sequence

$$0 \to M \to mP \to mN \to 0.$$ 

We show that the induced sequence of graded $gr_m R$-modules

$$0 \to (\text{gr}_m M)(-1) \to \text{gr}_m P \to \text{gr}_m N \to 0$$

is exact. Indeed, since $M \subseteq mP$, we have $0 \to P/mP \to N/mN \to 0$ is exact. For each $s \geq 1$, we prove that the sequence below is exact

$$0 \to \frac{m^{s-1}M}{m^sM} \to \frac{m^sP}{m^{s+1}P} \to \frac{m^sN}{m^{s+1}N} \to 0.$$ 

In fact, denote by $g$ the surjection map $P \to N$. Let $\bar{x} \in \text{Ker } \pi$ where $x \in m^sP$. Then $g(x) \in m^{s+1}N$, and so $g(x - y) = 0$ for some $y \in m^{s+1}P$, as $g$ is surjective. This implies that $x - y \in M \cap m^sP = m^{s-1}M$; the last equality holds by the hypothesis. Now $y \in m^{s+1}P$, therefore $\bar{x} \in \frac{m^{s-1}M}{m^sM}$, as desired. The exactness on the left follows from the equality $M \cap m^{s+1}P = m^sM$. So the sequence (4.1) is exact.

Denote $A = \text{gr}_m R$. Now the first two modules in (4.1) have linear $A$-free resolutions, moreover $\text{reg}_A(\text{gr}_m M)(-1) = 1$ and $\text{reg}_A \text{gr}_m P = 0$. Therefore $\text{gr}_m N$ also has 0-linear $A$-free resolution. So $N$ is a Koszul $R$-module by Proposition 2.1.

(c) Finally, assume that $\text{ld}_R N = 0$. Since $M \subseteq mP$, we have an induced isomorphism $P/mP \cong N/mN$. Denote $\lambda_{i,s}(M) = \ell(\text{Tor}_i^R(R/m^s, M))$, where $\ell(\cdot)$ signifies the length. Now as $M \subseteq mP$, $\text{Tor}_0^R(k, M) \to \text{Tor}_0^R(k, P)$ is the zero map, hence by Lemma 2.8(ii), the map $\text{Tor}_i^R(R/m^s, M) \to \text{Tor}_i^R(R/m^s, P)$ is zero for all $i \geq 1, s \geq 0$. Therefore we have an exact sequence

$$0 \to \text{Tor}_1^R(R/m^s, P) \to \text{Tor}_1^R(R/m^s, N) \to \text{Tor}_0^R(R/m^s, M) \to \text{Tor}_0^R(R/m^s, P) \to \text{Tor}_0^R(R/m^s, N) \to 0.$$ 

Evaluating the length, we get

$$\lambda_{0,s}(M) = -\lambda_{1,s}(P) + \lambda_{0,s}(P) + \lambda_{1,s}(N) - \lambda_{0,s}(N).$$

Similarly, as $P$ is Koszul, for the sequence $0 \to mP \to P \to P/mP \to 0$, we have an equality

$$\lambda_{0,s}(mP) = -\lambda_{1,s}(P) + \lambda_{0,s}(P) + \lambda_{1,s}(P/mP) - \lambda_{0,s}(P/mP).$$
We also have such an equality for $N$. In particular, as $P/mP \cong N/mN$, we have
\[ \lambda_{0,s}(mP) - \lambda_{0,s}(mN) = \lambda_{0,s}(M). \]

Now from the exactness of
\[ 0 \to M \to mP \to mN \to 0, \tag{4.2} \]
we conclude that the map $\text{Tor}_0(R/m^s, M) \to \text{Tor}_0(R/m^s, mP)$ is injective. In other words, $M \cap m^{s+1}P = m^sM$ for all $s \geq 0$. By the first part of the result, $\text{ld}_R M \leq \max\{0, \text{ld}_R N - 1\} = 0$. The proof of the theorem is now finished. $\square$

**Corollary 4.9.** With the hypothesis of Theorem 4.5, there is an equality
\[ \text{pd}_R N = \max\{\text{pd}_R M + 1, \text{pd}_R P\}. \]

If $R$ is a standard graded algebra and $M, P, N$ are finitely generated graded $R$-modules then
\[ \text{reg}_R N = \max\{\text{reg}_R M - 1, \text{reg}_R P\}. \]

**Proof.** For each $i \geq 0$, we have a short exact sequence
\[ 0 \to \text{Tor}_i^R(k, P) \to \text{Tor}_i^R(k, N) \to \text{Tor}_{i-1}^R(k, M) \to 0. \]
This desired conclusion follows. $\square$

The following result generalizes a well-known fact about modules with linear free resolution.

**Corollary 4.10** (Green, Martínez-Villa [15, Proposition 5.5]). Let $R$ be a Koszul local ring. Let $M \neq 0$ be a Koszul $R$-module. Then $m^iM$ is also a Koszul module for all $i \geq 1$.

**Proof.** First proof: It is enough to consider the case $i = 1$. Look at the exact sequence
\[ 0 \to \text{Tor}_1^R(k, P) \to \text{Tor}_1^R(k, N) \to \text{Tor}_{1-1}^R(k, M) \to 0. \]
Note that $M/mM$ is an $R/m$-module, so $\text{ld}_R M/mM = 0$ since $R$ is a Koszul ring. Using the first part of Theorem 4.5, we get $\text{ld}_R(M/mM) = 0$ as well.

Second proof: let $0 \to L \to F \to M \to 0$ be the beginning of the minimal free resolution of $M$. We have an induced exact sequence
\[ 0 \to L \to mF \to mM \to 0. \]
Now $\text{ld}_R(mF) = \text{ld}_R m = 0$ since $R$ is Koszul ring. Since $M$ is Koszul module, by Lemma 4.7, $L \cap m^sF = m^{s-1}L$ for all $s \geq 1$. Therefore applying Theorem 4.1 to the above sequence, $mM$ is a Koszul module. $\square$

5. **Base change**

Recall the following well-known change of rings statement concerning regularity (see, for example, [9, Proposition 3.3]).

**Proposition 5.1.** If $R \to S$ is a surjection of standard graded $k$-algebras. Let $N$ be a finitely generated graded $S$-module. Then:

(i) It always holds that $\text{reg}_R N \leq \text{reg}_R S + \text{reg}_S N$.
(ii) If $\text{reg}_R S \leq 1$ then $\text{reg}_S N \leq \text{reg}_R N$.
(iii) In particular, if $\text{reg}_R S = 0$ then $\text{reg}_R N = \text{reg}_S N$.

Now we deduce from Theorem 4.5 the following analog of Proposition 5.1(iii).
Theorem 5.2. Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a surjection of local rings such that $\text{ld}_R S = 0$. Then for any finitely generated $S$-module $N$, there is an equality
\[ \text{ld}_R N = \text{ld}_S N. \]

In particular, $\text{glld } S \leq \text{glld } R$.

Proof. We claim that $\text{ld}_R N = 0$ if and only if $\text{ld}_S N = 0$. Denote $A = \text{gr}_R S, B = \text{gr}_R S, U = \text{gr}_R N$ we get $\text{reg}_A B = 0$ by hypothesis. Hence applying Proposition 5.1, we get that $\text{reg}_B U = \text{reg}_A U$. The claim then follows from the last equality.

To prove that $\text{ld}_R N = \text{ld}_S N$, firstly consider the case $\text{ld}_R N = \ell < \infty$. We prove by induction on $\ell$. The case $\ell = 0$ was treated above.

Assume that $\ell \geq 1$, then by the claim, it follows that $\text{ld}_S N \geq 1$. Let $0 \to M \to P \to N \to 0$ be the beginning of the minimal $S$-free resolution of $N$. Since $M \subseteq \mathfrak{m}P$ and $\text{ld}_R P = \text{ld}_R S = 0$, we get from Theorem 4.5 that $\text{ld}_R M = \ell - 1$. Since $\text{ld}_S N \geq 1$, we also have $\text{ld}_S M = \text{ld}_S N - 1$. By induction hypothesis, $\text{ld}_R M = \text{ld}_S M$, thus $\text{ld}_R N = \text{ld}_S N$.

Now consider the case $\text{ld}_R N = \infty$ and by way of contradiction, assume that $\text{ld}_S N < \infty$. Again looking at the syzygy modules of $N$ as an $S$-module and using Theorem 4.5, we reduce the general situation to the case $\text{ld}_R N = \infty$ and $\text{ld}_S N = 0$. This contradicts the claim above. So in any case $\text{ld}_R N = \text{ld}_S N$.

The remaining assertion is obvious. \hfill \Box

Example 5.3. The following example shows that in Theorem 5.2, one cannot weaken the hypothesis that $R \to S$ is surjective to “$R \to S$ is a finite morphism”. Take $R = k$ and $S = k[x, y]/(x^2, y^2)$. Then $S$ is a finite, free $R$-module so $\text{ld}_R S = 0$. On the other hand, by [18, Theorem 6.7], $\text{glld } S = \infty$ and $\text{glld } R = 0$. Hence the conclusion of Theorem 5.2 does not hold for $R \to S$.

Remark 5.4. The analog of Proposition 5.1(i) for linearity defect is completely false: even if $R \to S$ is a Golod map of Koszul algebras (hence $\text{ld}_R S = 1$), it is possible for some Koszul $S$-module $N$ to have infinite linearity defect over $R$. For example, take $R = k[x, y, z, t]/(x^2 + (z, t)^2)$ as in Example 3.1. Consider the map $R \to R/\mathfrak{m}^2$. Since $R$ is Koszul, $R \to S$ is a Golod map. Consider the $R$-module $N$ in Example 3.1. Recall that $N$ is also an $S$-module, and of course $\text{ld}_S N = 0$. On the other hand, we know that $\text{ld}_R N = \infty$.

This example also shows that Theorem 5.2 does not apply if $\text{ld}_R S \geq 1$.

Remark 5.5. In view of Proposition 5.1(ii), we can ask:

Let $R \to S$ be a surjection of local rings such that $\text{ld}_R S \leq 1$. Is it true that $\text{ld}_S N \leq \text{ld}_R N$ for any finitely generated $S$-module $N$?

But the answer is no, even if $R$ and $S$ are Koszul. Indeed, take $R = k[x, y]/(x^2)$ and $S = R/(y^2)$, then $\text{ld}_R S = 1$ and from Lemma 2.3, $\text{glld } R = 1$. However as noted above, $\text{glld } S = \infty$. Hence the question is not true. If we do not insist that $S$ is Koszul, we can take $R = k[x, y]$ and $S = k[x, y]/(x^3)$. Then $\text{ld}_R S = 1, \text{ld}_S S = \infty$ while $\text{ld}_R k = 0$.

As a corollary to Theorem 5.2, we prove that specializations of absolutely Koszul algebras are again absolutely Koszul; see further questions in [10, Remark 3.10]. Recall from [20] that $R$ is called absolutely Koszul if for every finitely generated $R$-module $M$, we have $\text{ld}_R M < \infty$. By [20, Theorem 2.11], if $R$ is a graded absolutely Koszul algebra and $x \in R_1$ an $R$-regular linear form such that $R/(x)$ is absolutely Koszul, then so is $R$. The converse is given by
Corollary 5.6. Let \((R, \mathfrak{m})\) be an absolutely Koszul local ring and \(x \in \mathfrak{m} \setminus \mathfrak{m}^2\) be such that \(\overline{x} \in \mathfrak{m}/\mathfrak{m}^2\) is \(\mathrm{gr}_x R\)-regular. Then \(R/(x)\) is also absolutely Koszul.

Proof. Since \(\overline{x}\) is \(\mathrm{gr}_x R\)-regular, we get that \(\ld_R R/(x) = 0\). The result follows from Theorem 5.2. \(\square\)

It was asked in [10, Remark 3.10] whether if \(R\) is an absolutely Koszul algebra and \(x\) a regular element of degree 2, then \(R/(x)\) is also absolutely Koszul. We provide a partial answer to this question.

Proposition 5.7. Let \((R, \mathfrak{m})\) be an absolutely Koszul algebra, and \(x\) an \(R\)-regular element of degree 2. Assume that for any finitely generated graded \(R/(x)\)-module \(N\) with the property that \(\ld_R N = 1\), we have \(\ld_{R/(x)} N < \infty\). Then \(R/(x)\) is absolutely Koszul.

Proof. Denote \(S = R/(x)\). Let \(N\) be any finitely generated graded \(S\)-module, we use induction on \(\ld_S N\) to show that \(\ld_S N < \infty\).

If \(\ld_S N = 0\), denoting \(U = \mathrm{gr}_x N\), then \(U\) has linear \(R\)-free resolution. Since \(\reg_S S = 1\), we also have \(\reg_S U \leq \reg_R U = 0\) by Proposition 5.1(ii). In particular \(\ld_S N = 0 < \infty\). If \(\ld_S N = 1\) then by the hypothesis, it follows that \(\ld_S N < \infty\). Now assume that \(\ld_S N \geq 2\). Let \(0 \to M \to P \to N \to 0\) be the beginning of the minimal graded \(S\)-free resolution of \(N\). Since \(\pd_R P = 1\), applying Corollary 2.9, we have

\[
\begin{align*}
\ld_S N &\leq \max\{2, \ld_R M + 1\}, \\
\ld_R M &\leq \max\{1, \ld_R N - 1\}.
\end{align*}
\]

Since \(\ld_R N \geq 2\), we get from the second inequality that \(\ld_R M \leq \ld_R N - 1\). Together with the first one, we deduce that there are two cases: either \(\ld_R N = 2\) and \(\ld_R M = 0\), or \(\ld_R N = \ld_R M + 1\). In the first case, we know from above that \(\ld_S M = 0\), hence \(\ld_S N \leq 1\). In the second case, using induction hypothesis for \(M\), we obtain \(\ld_S M < \infty\). This yields \(\ld_S N \leq \ld_S M + 1 < \infty\). The proof is completed. \(\square\)

Remark 5.8. Unfortunately, there is no easy way to bound \(\ld_{R/(x)} N\) when \(\ld_R N = 1\) in the previous proposition. For this, we only need to consider \(R = k[x, y]/(x^2)\) and \(S = R/(y^2)\). From the proof and the equalities \(\mathrm{gl} \ld R = 1\), \(\mathrm{gl} \ld S = \infty\) (Remark 5.5), we see that for any \(n \geq 1\), there exists a finitely generated graded \(S\)-module \(N\) such that \(\ld_R N = 1\) and \(\ld_S N \geq n\).

It is natural to ask

Question 5.9. Let \(R\) be an absolutely Koszul algebra and \(x\) an \(R\)-regular element of degree 2. Is it true that for any finitely generated graded \(R/(x)\)-module \(N\) such that \(\ld_R N = 1\), we have \(\ld_{R/(x)} N < \infty\)?

Example 5.10. Let \((R, \mathfrak{m}) \to (S, \mathfrak{n})\) be a finite, flat morphism of local rings. One may ask whether for any finitely generated \(S\)-module \(N\) such that \(\ld_S N = 0\), we also have \(\ld_R N = 0\)? This is true if \(\pd_S N = 0\): in that case \(\pd_R N = 0\). But in general, this is far from the true. For any \(n \geq 1\), take \(R = k[x_1, \ldots, x_n]\) and \(S = k[x_1, \ldots, x_n, y_1, \ldots, y_n]/(y_1^2, \ldots, y_n^2)\). We have a surjection \(S \to k[x_1, \ldots, x_n]/(x_1^2, \ldots, x_n^2)\) given by

\[
\begin{align*}
x_i &\mapsto x_i, \\
y_i &\mapsto x_i,
\end{align*}
\]
for $1 \leq i \leq n$. The kernel is $(x_1 - y_1, \ldots, x_n - y_n)$. Since $x_1 - y_1, \ldots, x_n - y_n$ is an $S$-regular sequence, we see that $\text{ld}_S k[x_1, \ldots, x_n]/(x_1^2, \ldots, x_n^2) = 0$. On the other hand, direct computations with the Koszul complex show that $\text{ld}_R k[x_1, \ldots, x_n]/(x_1^2, \ldots, x_n^2) = n$.

6. Filtrations

**Koszul filtrations.** In the graded setting, the notion of Koszul filtrations in [12] has proved to be useful to detect Koszul property of algebras. We extend this notion to the local setting in the present section.

**Definition 6.1.** Let $(R, m, k)$ be a local ring. Let $\mathcal{F}$ be a collection of ideals of $R$ satisfying the following conditions:

(i) $0, m \in \mathcal{F}$,

(ii) for every ideal $I \in \mathcal{F}$ and all $s \geq 1$, we have $I \cap m^{s+1} = m^s I$,

(iii) for every ideal $I \neq 0$ of $\mathcal{F}$, there exist a finite filtration $0 = I_0 \subset I_1 \subset \cdots \subset I_n = I$ and elements $x_j \in I_j \setminus I_{j-1}$, such that for each $j = 1, \ldots, n$, $I_j \in \mathcal{F}$, $I_j = I_{j-1} + (x_j)$ and $I_{j-1} : x_j \in \mathcal{F}$.

Then we call $\mathcal{F}$ a Koszul filtration of $R$.

**Remark 6.2.** (i) It is straightforward to check that the usual notion of Koszul filtration for standard graded algebras satisfies the conditions of Definition 6.1.

(ii) Condition (iii) in our definition of Koszul filtration is more involved than the corresponding condition in [12, Definition 1.1]; the reason behind is to make the induction process in the proof of Theorem 6.3 below to work. In the case of graded Koszul filtrations, the condition is automatically satisfied.

The following theorem extends a well-known result about algebras with Koszul filtration [12].

**Theorem 6.3.** Let $(R, m, k)$ be a local ring with a Koszul filtration $\mathcal{F}$. Then

(i) for any ideal $I \in \mathcal{F}$, $R/I$ is a Koszul $R$-module;

(ii) $R$ is a Koszul ring;

(iii) moreover, $R/I$ is a Koszul ring for any $I \in \mathcal{F}$.

**Proof.** (i) We may assume that $m \neq 0$, otherwise $R$ is a field and $\mathcal{F} = \{(0)\}$. We prove by induction on $i \geq 1$ and on inclusion relation in $\mathcal{F}$ that the connecting map

$$\text{Tor}_i^R(R/m^s, R/I) \to \text{Tor}_{i-1}^R(m^s/m^{s+1}, R/I)$$

is injective for all $s \geq 0$.

Firstly, assume that either $i = 1$ or $I = 0$. For $i = 1$, since $\text{Tor}_1^R(R/m^s, R/I) = (I \cap m^s)/m^s I$, the natural map

$$\text{Tor}_1^R(R/m^{s+1}, R/I) \to \text{Tor}_1^R(R/m^s, R/I)$$

is zero by condition (ii) for Koszul filtrations. Hence the connecting map is injective. For $I = 0$, the statement is trivial.

Now consider $i \geq 2$ and $I \neq 0$. By definition of Koszul filtration, one can choose $J$ in $\mathcal{F}$ and $x \in I \setminus J$ such that $I = J + (x)$ and $J : x \in \mathcal{F}$. We have an exact sequence

$$0 \to R/(J : x) \xrightarrow{x} R/J \to R/I \to 0.$$
We have $m/m^2 \cong k^t$ for some $t \geq 1$. Consider the commutative diagram

$$
\begin{array}{c}
\text{Tor}^R_\ell(R/m, R/J : x) \xrightarrow{\rho_\ell} \text{Tor}^R_\ell(R/m, R/J) \\
\downarrow \quad \quad \quad \quad \quad \quad \downarrow \\
\text{Tor}^R_{\ell-1}(m/m^2, R/J : x) \xrightarrow{\rho^{\ell-1}_\ell} \text{Tor}^R_{\ell-1}(m/m^2, R/J).
\end{array}
$$

By induction on $\ell$, we see that $\rho_\ell$ is the zero map for all $\ell \leq i$. The case $\ell = 0$ follows since $R/(J : x) \subseteq x(R/J)$. The induction step follows by using the above diagram and the fact that $\tau_\ell$ is injective (the hypothesis of the induction on $i$ and inclusion).

Now consider the diagram

$$
\begin{array}{c}
0 \longrightarrow \text{Tor}_i(R/m^s, R) \xrightarrow{\alpha^2_i} \text{Tor}_{i-1}(m^s/m^{s+1}, R/J) \\
\downarrow \quad \quad \quad \quad \quad \quad \downarrow \\
\text{Tor}_i(R/m^s, R/I) \xrightarrow{\alpha^3_i} \text{Tor}_{i-1}(m^s/m^{s+1}, R/I) \\
\downarrow \quad \quad \quad \quad \quad \quad \downarrow \\
0 \longrightarrow \text{Tor}_{i-1}(R/m^s, R/J : x) \xrightarrow{\alpha^{i-1}_i} \text{Tor}_{i-2}(m^s/m^{s+1}, R/J : x)
\end{array}
$$

Since $J \subseteq I$, by induction hypothesis, $\alpha^2_i, \alpha^{i-1}_i$ are injective. We know from the previous paragraph that $\rho_{i-1}$ is the zero map. Hence by a snake lemma argument, $\alpha^3_i$ is also injective, as desired.

(ii) From (i), taking $I = m$, we get that $\text{ld}_R k = 0$. This shows that $R$ is Koszul.

(iii) Since $R/I$ is a Koszul $R$-module, the ring $\text{gr}_m(R/I)$ has linear resolution over $\text{gr}_m R$. This shows that

$$
\text{reg}_{\text{gr}_m(R/I)} k = \text{reg}_{\text{gr}_m R} k = 0,
$$

where the first inequality follows from Proposition 5.1(iii), and the second from part (ii). Therefore $\text{gr}_m(R/I)$ is a Koszul algebra, equivalently, $R/I$ is a Koszul ring.

**Modules with Koszul quotients.** Recall the following notion due to Herzog and Hibi [16].

**Definition 6.4 (Componentwise linear modules).** Let $R$ be a standard graded $k$-algebra. Let $M$ be a finitely generated graded $R$-module. Then $M$ is said to be componentwise linear if for every $d \in \mathbb{Z}$, the submodule $M_{(d)} = (m \in M : \deg m = d) \subseteq M$ has $d$-linear resolution as an $R$-module.

Römer proved in his thesis [28] the following characterization of componentwise linear ideals over Koszul algebras; see [20, Theorem 5.6] for a proof.

**Theorem 6.5 (Römer).** Assume that $R$ is a Koszul algebra. Then for any finitely generated graded $R$-module $M$, the following are equivalent:

(i) $M$ is componentwise linear;
(ii) $M$ is a Koszul module over $R$. 
We will give a criterion for Koszul modules over a local ring $R$. First we introduce the following notion, generalizing the notion of ideals with linear quotients [19, Section 1]. The later are an ideal-theoretic analog of rings with Koszul filtrations.

**Definition 6.6 (Modules with Koszul quotients).** Let $M \neq 0$ be a finitely generated $R$-module with minimal generators $m_1, \ldots, m_t$. Let $I_i = (m_1, \ldots, m_{i-1}) :_R m_i$. We say that $M$ has Koszul quotients if for each $i = 1, \ldots, t$, the cyclic module $R/I_i$ is a Koszul module.

In view of Römer’s theorem 6.5, the following result is a generalization of [21, Theorem 3.7], [32, Corollaries 2.4, 2.7], [23, Propostion 3.7]. A notable feature is that no assumption on the ring is needed, while in the three precedent results just cited, $R$ has to be at least a Koszul algebra.

**Proposition 6.7.** Let $M \neq 0$ be a module with Koszul quotients and $m_1, \ldots, m_t$ are the minimal generators of $M$ satisfying the condition of Koszul quotients. Then each of the submodule $(m_1, \ldots, m_i)$ of $M$ is a Koszul module for $1 \leq i \leq t$. In particular, $M$ is a Koszul module.

Moreover, we have

$$
\beta_s(M) = \sum_{i=1}^{t} \beta_s(R/I_i) \text{ for all } s \geq 0,
$$

$$
pd_R M = \max_{1 \leq i \leq t} \{pd_R(R/I_i)\}.
$$

If $R$ is a graded algebra, $M$ a graded module, $\deg m_i = d_i$ for $1 \leq i \leq t$, then we also have

$$
\beta_{s,j}(M) = \sum_{i=1}^{t} \beta_{s,j-d_i}(R/I_i) \text{ for all } s,j \geq 0,
$$

$$
reg_R M = \max_{1 \leq i \leq t} \{reg_R(R/I_i) + d_i\}.
$$

**Proof.** Denote $M_i = (m_1, \ldots, m_i)$. Observe that $(m_i)/(m_i) \cap M_{i-1} = (m_i)/I_i m_i \cong R/I_i$ for each $1 \leq i \leq t$. In fact, this follows since if $xm_i \in I_i m_i \subseteq (m_1, \ldots, m_{i-1})$ then $x \in (m_1, \ldots, m_{i-1}) :_R m_i = I_i$. Since $m_1, \ldots, m_t$ are minimal generators, we have $M_{i-1} \cap mM_i = mM_{i-1}$. Therefore using induction on $i$, the short exact sequence

$$
0 \to M_{i-1} \to M_i \to R/I_i \to 0,
$$

and Theorem 4.1, we conclude that $M_i$ is a Koszul module for every $1 \leq i \leq t$.

For the remaining statements, we note that from the proof of Corollary 4.4, the sequence

$$
0 \to \text{Tor}_s^R(k, M_{i-1}) \to \text{Tor}_s^R(k, M_i) \to \text{Tor}_s^R(k, R/I_i) \to 0
$$

is exact for every $i$ and every $s$. In the graded case, we use the corresponding facts for the exact sequence

$$
0 \to M_{i-1} \to M_i \to (R/I_i)(-d_i) \to 0.
$$

The proof is finished. \qed
Some applications. Firstly, we present a slight improvement of a result due to Avramov, Iyengar and Şega (which corresponds to the case \( q \) is a principal ideal \((x)\) in the notation of the next result).

Proposition 6.8 (See [5, Theorem 3.2]). Let \((R, m, k)\) be a local ring. Let \( q \subseteq m \) be an ideal such that \( m^2 = qm \) and \( q^2 = 0 \). Let \( y_1, \ldots, y_e \) be minimal generators of \( q \) where \( y_i \in m \). Then the collection of ideals

\[
F = \{0, (y_1), (y_1, y_2), \ldots, (y_1, \ldots, y_{e-1})\} \cup \{I \subseteq m: I \text{ contains } q\}
\]

is a Koszul filtration for \( R \). Moreover, any non-trivial finitely generated \( R \)-module \( M \) that satisfies the condition \( qM = 0 \) is a Koszul module.

Proof. The case \( q = 0 \) is trivial as the reader may check, so we assume that \( q \neq 0 \). Clearly \( F \) contains 0 and \( m \). We begin by checking the condition (ii) for Koszul filtrations. Firstly consider the case \( I \neq 0 \) is an ideal containing \( q \). As \( m^3 = 0 \), the condition is trivial for \( s \geq 2 \). For \( s = 1 \), \( m^2 \subseteq q \subseteq I \), hence \( m^2 \cap I = m^2 = mq \subseteq mI \subseteq m^2 \cap I \). In particular, all containments in the last string are in fact equalities.

Next consider the case \( I = (y_1, \ldots, y_i) \) where \( 1 \leq i \leq e - 1 \). Take \( x \in I \cap m^2 = I \cap qm \), then

\[
x = r_1y_1 + \cdots + r_iy_i = s_1y_1 + \cdots + s_ey_e \text{ where } r_i \in R, s_i \in m.
\]

Then we have \((r_1 - s_1)y_1 + \cdots + (r_i - s_i)y_i - s_{i+1}y_{i+1} - \cdots - s_ey_e = 0\). But \( y_1, \ldots, y_e \) are linearly independent modulo \( mq \), therefore \( r_j - s_j \in m \) for all \( 1 \leq j \leq i \). Hence \( r_j \in m \) for all \( 1 \leq j \leq i \), and so \( x \in mI \), as desired.

Now we verify condition (iii). Let \( I \subseteq m \) be an ideal of \( R \) containing \( q \). Let \( z_1, \ldots, z_n \) be an irredundant set of elements of \( I \) such that \( I = q + (z_1, \ldots, z_n) \). Define \( I_0 = 0, I_1 = (y_1), I_2 = (y_1, y_2), \ldots, I_e = (y_1, \ldots, y_e) = q, I_{e+1} = q + (z_1), \ldots, I_{e+n} = q + (z_1, \ldots, z_n) = I \). Observe that \( I_j : y_{j+1} \) are proper ideals containing \( q \) for \( 0 \leq j \leq e \) and \( I_{e+t-1} : z_t = m \) for \( 1 \leq t \leq n \). This argument also implies that the condition (iii) holds if \( I \) is among ideals of the type \((y_1, \ldots, y_i)\) where \( 1 \leq i \leq e - 1 \). Hence \( F \) is a Koszul filtration of \( R \). In particular, \( R/I \) is a Koszul \( R \)-module if \( I \subseteq m \) is an ideal containing \( q \).

Let \( M \) be a finitely generated \( R \)-module with \( qM = 0 \). Let \( m_1, \ldots, m_t \) be a system of minimal generators of \( M \). Immediately, we get \((m_1, \ldots, m_{i-1}) : m_i \) is a proper ideal containing \( q \) for each \( i = 1, \ldots, t \). Therefore by the first part of the result, \( M \) has Koszul quotients. In particular, \( M \) is a Koszul module by Proposition 6.7.

Remark 6.9. Note that in the previous result, if \( q \) is a principal ideal, using the machinery in [5] one obtains more information about modules over the local ring \( R \): every finitely generated \( R \)-module has a Koszul syzygy module.

The following result is suggested by Ahangari’s Theorem 2.13 in [1].

Corollary 6.10. Let \((R, m, k)\) and \( q \) be as in Proposition 6.8. Then

(i) for any finitely generated \( R \)-module \( M \) and any ideal \( I \subseteq m \) containing \( q \), \( IM \) is a Koszul module,

(ii) for any ideal \( I \subseteq m \), \( R/I \) is a Koszul ring.

Proof. (i) Since \( qM \) is annihilated by \( q \), Proposition 6.8 yields that \( qM \) is a Koszul module. For the same reason \( IM/qM \) is Koszul. Consider the exact sequence

\[
0 \to qM \to IM \to IM/qM \to 0.
\]
We wish to show that $qM \cap m/I = mqM$; Theorem 4.1 then yields $\text{ld}_R IM \leq \text{ld}_R (IM/qM) = 0$. Note that $mq \subseteq mI \subseteq m^2 = mq$, so all inclusions are in fact equality. Hence the desired equality is clear.

(ii) Denote by $\bar{\cdot}$ the residue class in $R/I$. Then from $m^2 = \bar{mq}$ and $\bar{m}^2 = 0$ in $\bar{R}$, we conclude using Proposition 6.8 that $\bar{R}$ is Koszul. □

For local rings with minimal multiplicity, we have

**Proposition 6.11** (See Ahangari [1, Proposition 2.14]). Let $(R, m, k)$ be a Cohen-Macaulay local ring with positive dimension and minimal multiplicity. Assume that $k$ is infinite. Let $J$ be a minimal reduction of $m$. Then $R$ has a Koszul filtration and for any finitely generated $R$-module $M$ such that $JM = 0$, we have $\text{ld}_R M = 0$.

**Proof.** There exist a set of minimal generators $\{a_1, \ldots, a_d\}$ of $J$ such that $\bar{a_1}, \ldots, \bar{a_d} \in m/m^2$ is a regular $gr_m R$-sequence (see [30, Corollary 2.6]). By a theorem of Valabrega and Valla (loc. cit., Corollary 1.1), we see that

- (i) $a_1, \ldots, a_d$ is an $R$-regular sequence,
- (ii) $(a_1, \ldots, a_i) \cap m^{s+1} = m^s(a_1, \ldots, a_i)$ for every $1 \leq i \leq d$ and every $s \geq 0$.

We will show that the following family is a Koszul filtration for $R$:

$F = \{0, (a_1), (a_1, a_2), \ldots, (a_1, \ldots, a_{d-1}), \} \cup \{I \subseteq m : J \subseteq I\}$.  

Clearly $0, m \in F$. Thanks to the second equality above, any ideal $(a_1, \ldots, a_i)$ satisfies condition (ii) of Koszul filtrations. For every proper ideal $I \supseteq J$ and every $s \geq 0$, we have

$I \cap m^{s+1} = I \cap m^s J \subseteq m^s I \subseteq I \cap m^{s+1}$,

where the first equality follows from [30, Corollary 2.6]. Hence condition (ii) is also fulfilled by $J$. The verification of property (iii) follows from the fact that $a_1, \ldots, a_d$ is an $R$-regular sequence. Hence $F$ is a Koszul filtration.

For the remaining assertion, we only have to choose any set of minimal generators $\{m_1, \ldots, m_t\}$ of $M$. Then $M$ has Koszul quotients simply because $(m_1, \ldots, m_{i-1}) : m_i$ is a proper ideal containing $J$, for all $1 \leq i \leq t$. Proposition 6.7 then implies that $M$ is Koszul. □

The last main result of [1], which rests on Fitzgerald’s paper [14], can also be recovered using our machinery.

**Theorem 6.12** (See [1, Theorem 2.15]). Let $(R, m)$ be a local ring and $M$ a finitely generated $R$-module. Assume that for any $x \in m \setminus m^2$, there is an equality $m^2 = \text{ann}(x)m$. Then

- (i) The family of ideals $\{I \subseteq m : I$ is generated by elements in $m \setminus m^2\}$ forms a Koszul filtration for $R$,
- (ii) $M$ is Koszul if $\text{ann}(x)M = 0$ for some $x \in m \setminus m^2$,
- (iii) for any sequence $x_1, \ldots, x_r$ of elements belonging to $m \setminus m^2$, the module $(x_1, \ldots, x_r)M$ is Koszul.
- (iv) for any ideal $I \subseteq m$, $R/I$ is a Koszul ring.

**Proof.** We have the following

**Observation:** Consider a proper ideal $I$ such that there exists an element $x \in m \setminus m^2$ with $\text{ann}(x) \subseteq I$. Then for any minimal generator $y$ of $I$, it holds that $y \in m \setminus m^2$.  

Indeed, assume that \( y \in m^2 \). Since \( x \in m \setminus m^2 \), the hypothesis tells us \( m^2 = \text{ann}(x) m \subseteq mI \). In particular, \( y \in mI \), so it cannot be a minimal generator.

(i) Let \( y_1, \ldots, y_n \) be a minimal system of generators of \( m \). Since \( y_i \notin m^2 \) for all \( i \), we have \( m^2 \subseteq \cap_{i=1}^{n} \text{ann}(y_i) \), and thus \( m^3 = 0 \). The first condition for Koszul filtrations is satisfied, since \( 0, m \in \mathcal{F} \). For the second condition, it suffices to prove that \( I \cap m^2 = mI \) for any ideal \( I \in \mathcal{F} \). But since each \( I \in \mathcal{F} \) is generated by elements in \( m \setminus m^2 \), the last equality is clear.

To verify condition (iii), let \( I \neq 0 \) be an ideal of \( \mathcal{F} \) and \( a_1, \ldots, a_d \) are elements in \( m \setminus m^2 \) which minimally generated \( I \). Set \( I_j = (a_1, \ldots, a_j) \) for \( 0 \leq j \leq d \). Then \( I_{j-1} : a_j \supseteq \text{ann}(a_j) \), hence by the observation, \( I_j : a_j \in \mathcal{F} \). Therefore \( I \) satisfies condition (iii) for Koszul filtrations. Putting everything together, we get that \( \mathcal{F} \) is a Koszul filtration.

(ii) We show that \( M \) has Koszul quotients. Let \( m_1, \ldots, m_t \) be minimal system of generators of \( M \). Then for every \( 1 \leq i \leq t \), \( (m_1, \ldots, m_{i-1}) : m_i \supseteq \text{ann}(x) \). The observation then implies that \( (m_1, \ldots, m_{i-1}) : m_i \in \mathcal{F} \) for all such \( i \), namely \( M \) has Koszul quotients.

(iii) We show that \( (x_1, \ldots, x_r)M \) has Koszul quotients. We can choose a minimal system of generators \( y_1, \ldots, y_e \) of \( (x_1, \ldots, x_s)M \) such that for each \( 1 \leq i \leq e \), \( y_i = x_r m_i \) for certain \( 1 \leq r_i \leq r \) and \( m_i \in M \). Now for every \( 1 \leq i \leq e \), observe that \( (y_1, \ldots, y_{i-1}) : y_i \supseteq \text{ann}(x_{r_i}) \). From the observation above, one sees that \( (y_1, \ldots, y_{i-1}) : y_i \in \mathcal{F} \) for every \( i \). This means that \( M \) has Koszul quotients.

(iv) See the proof of [1, Theorem 2.15].

Remark 6.13. It is worth mentioning that in Theorem 2.13 and Theorem 2.15 in Ahangari’s paper mentioned above, one also has information on the regularity of the \( \text{gr}_m R \)-module \( \text{gr}_m M \).

We can also recover partly another result of Avramov, Iyengar and Šega.

**Proposition 6.14** (See [5, Theorem 4.1]). Let \((R, m, k)\) be a local ring with \( m^3 = 0 \) and \( \dim_k m^2 = 1 \). If \( \dim_k (0 : m) \leq \dim_k (m/m^2) - 1 \) then \( R \) has a Koszul filtration. In particular, \( R \) is a Koszul ring.

**Proof.** Let \( x_1, \ldots, x_n \) be a minimal system of generators of \( m \). Since \( m^2 \) is generated by \( x_i x_j, 1 \leq i \leq j \leq n \), one can assume that one of the elements \( x_i^2, x_i x_1, \ldots, x_i x_n \) is non-zero. In particular, \( m^2 = x_1 m \). If \( x_1^2 = 0 \) then applying Proposition 6.8, we see that \( R \) has a Koszul filtration.

Consider the case \( x_1^2 \neq 0 \). As \( \dim_k m^2 = 1 \), for each \( i \geq 2 \), we have \( x_1 x_i = t_i x_1^2 \) for some \( t_i \in R \). Replacing \( x_i \) by \( x_i - t_i x_1 \), we can assume that \( x_1 x_2 = x_1 x_3 = \cdots = x_1 x_n = 0 \). We claim that there exist \( i, j \geq 2 \) such that \( x_i x_j \neq 0 \); the argument is given at the end of the current proof. Assume for the moment that this is true, then \( x_i x_j \) is a \( k \)-basis for \( m^2 \). Hence \( x_1^2 \in (x_i x_j) \). We claim that

\[ \mathcal{F} = \{ 0, (x_2), (x_2, x_3), \ldots, (x_2, \ldots, x_n) \} \bigcup \{ I \subseteq m : x_1 \in I \} \]

is a Koszul filtration for \( R \). Of course \( 0, m \in \mathcal{F} \).

Firstly, we check condition (ii) for Koszul filtrations. Since \( m^3 = 0 \) this reduces to proving that \( I \cap m^2 = mI \) for each \( I \in \mathcal{F} \). Since \( mI \subseteq I \cap m^2 \subseteq m^2 \) and \( \dim_k m^2 = 1 \), it is enough to show that if \( I \cap m^2 \neq 0 \) then \( mI \neq 0 \). This is trivial if \( x_1 \in I \), namely \( 0 \neq x_1^2 \in mI \). Now if \( I \) is one of the ideals \( (x_2), \ldots, (x_2, \ldots, x_n) \), as \( I \cap m^2 \neq 0 \), clearly \( x_1^2 \in I \). Denote \( I = (x_2, \ldots, x_m) \) then \( x_1^2 = r_2 x_2 + \cdots + r_m x_m \) where \( r_j \in R \). But \( x_1, x_2, \ldots, x_n \) are linearly independent modulo \( m^2 \), so \( r_2, \ldots, r_m \in m \) and \( x_1^2 \in mI \), as desired.
Secondly, we verify condition (iii). Observe that we have the following equality

\[ 0 : x_1 = (x_2, \ldots, x_n), \]
Indeed, clearly \(0 : x_i = (x_1, x_2, \ldots, x_n)\). But \(x_1^2 \in (x_1, x_2)\) hence the desired equality holds true. Note that the ideals \(0 : (x_2, (x_2) : x_3, \ldots, (x_2, \ldots, x_{n-1}) : x_n\) all contain \(x_1\). So the condition (iii) is valid for the ideals \(0, (x_2), (x_2, x_3), \ldots, (x_2, \ldots, x_n).\) Similarly, condition (iii) is valid for each of the ideals containing \(x_1\). Hence \(F\) is a Koszul filtration of \(R\).

Now assume that there exist no \(i\) and \(j\), both at least 2, such that \(x_i x_j \neq 0\); equivalently \((x_2, \ldots, x_n)^2 = 0\). We will derive a contradiction. Observe that \(x_1^2, x_2, \ldots, x_n\) is a generating set of the \(k\)-vector space \((0 : m)\). But \(\dim_k (0 : m) \leq n - 1\), hence we have a relation \(s_1 x_1^2 + s_2 x_2 + \cdots + s_n x_n = 0\) where each \(s_i\) is in \(R\) and at least one of them is a unit. Since \(x_1, x_2, \ldots, x_n\) are linearly independent modulo \(m^2\), we obtain that \(s_j \in m\) for all \(j \geq 2\). Therefore \(s_1 \notin m\). But then \(x_1^2 \in m(x_2, \ldots, x_n) = 0\), a contradiction.

The proof of the result is now completed. \(\square\)

Now we generalize a Koszul criterion via short Koszul modules discovered by Avramov, Iyengar and Şega [6]. Note that we can do away with the hypothesis that \(R\) is a graded algebra in loc. cit.

**Proposition 6.15** (See [6, Theorem 1.6]). Let \((R, m)\) be a local ring. Assume that there exists a non-zero finitely generated \(R\)-module \(M\) such that \(M\) is Koszul and \(m^2 M = 0\). Then \(R\) is a Koszul ring.

We present two proofs of this proposition; the first one utilizes arguments from Section 4.

**First proof.** If \(mM = 0\) then \(M\) is a direct sum of copies of \(k\); so \(\text{ld}_R k = \text{ld}_R M = 0\). Therefore \(R\) is a Koszul ring. Assume that \(mM \neq 0\). We prove by induction on \(i \geq 1\) that the natural map \(\text{Tor}^R_i (R/m^i, k) \rightarrow \text{Tor}^R_{i-1} (m^i/m^{i+1}, k)\) is injective for all \(s \geq 0\). By Theorem 2.4, this is equivalent to \(R\) being Koszul. The claim is clear for \(i = 1\): the image of the second map in the following display is zero

\[
\substack{m^{i+1}/m^{i+2} \cong \text{Tor}^R_i (R/m^i, k) \rightarrow \text{Tor}^R_i (R/m^i, k) \cong m^i/m^{i+1}}
\]
Assume that \(i \geq 2\) and the statement is true for \(i - 1\).

Consider the short exact sequence

\[ 0 \rightarrow mM \rightarrow M \rightarrow M/mM \rightarrow 0. \quad (6.1) \]
Since \(m^2 M = 0\), we have \(mM \cong k^m\) where \(m \geq 1\). *A priori*, \(M/mM \cong k^n\) for some \(n \geq 1\).

Consider the diagram induced by the sequence (6.1)

\[
\begin{array}{ccc}
0 & \longrightarrow & \text{Tor}^1 (R/m^1, M) \\
\downarrow & & \downarrow \\
\text{Tor}^1 (R/m^1, k^n) & \longrightarrow & \text{Tor}^1 (R/m^1, k^n)
\end{array}
\]

\[
\begin{array}{ccc}
0 & \longrightarrow & \text{Tor}^2 (R/m^2, k^m) \\
\downarrow & & \downarrow \\
\text{Tor}^2 (R/m^2, k^m) & \longrightarrow & \text{Tor}^2 (R/m^2, k^m)
\end{array}
\]

\[
\begin{array}{ccc}
0 & \longrightarrow & \text{Tor}^1 (R/m^1, k^m) \\
\downarrow & & \downarrow \\
\text{Tor}^1 (R/m^1, k^m) & \longrightarrow & \text{Tor}^1 (R/m^1, k^m)
\end{array}
\]

\[
\begin{array}{ccc}
0 & \longrightarrow & \text{Tor}^1 (R/m^1, k^m) \\
\downarrow & & \downarrow \\
\text{Tor}^1 (R/m^1, k^m) & \longrightarrow & \text{Tor}^1 (R/m^1, k^m)
\end{array}
\]
By induction hypothesis, $\alpha_i^{i-1}$ is injective. Since $M$ is Koszul the map $\alpha_1^1$ is injective as well. Now $\rho_{i-1}$ is induced by the maps $k^m \cong \mathfrak{m}M \subseteq M$ and $M$ is Koszul, so using the proof of Theorem 4.5, we obtain that $\rho_{i-1}$ is the zero map. Therefore by a simple diagram chasing, $\alpha_{i-1}'$ is also injective, as desired. \hfill \Box

Here is the second proof: It is easy to see that $R/\mathfrak{m}^2$ is always a Koszul ring. Hence the desired result follows from applying the below proposition for the morphism $R \to R/\mathfrak{m}^2$.

**Proposition 6.16.** Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a surjection of local rings, where $S$ is a Koszul ring. Assume that there exists a non-zero finitely generated $S$-module $M$ such that $M$ is a Koszul $R$-module. Then $R$ is also a Koszul ring.

**Proof.** Denote $A = \text{gr}_R R, B = \text{gr}_S S \cong \text{gr}_R R$. We have an induced surjection $A \to B$ and $B = \text{gr}_S S$ is a Koszul algebra. Since $M$ is a Koszul $R$-module, by [18, Proposition 1.5], $\text{gr}_R M$ is a non-zero $B$-module which has linear $A$-free resolution. That is to say $\text{reg}_A \text{gr}_R M = 0$. By [24, Theorem 1.4], this implies that $A$ is also a Koszul algebra. Hence $R$ is a Koszul ring. \hfill \Box

**Acknowledgements**

We would like to thank Aldo Conca, Srikanth Iyengar and Tim Römer for some inspiring discussions related to the content of this paper.

**References**

[1] Rasoul Ahangari Maleki, *On the regularity and Koszulness of modules over local rings*. Comm. Algebra **42** (2014), 3438–3452.

[2] Rasoul Ahangari Maleki and Maria Evelina Rossi, *Regularity and linearity defect of modules over local rings*. to appear in J. Commut. Algebra.

[3] ______, *Infinite free resolution*. in *Six lectures on Commutative Algebra* (Bellaterra, 1996), 1–118, Progr. Math., **166**, Birkhäuser (1998).

[4] Luchezar L. Avramov and David Eisenbud, *Regularity of modules over a Koszul algebra*. J. Algebra. **153** (1992), 85–90.

[5] Luchezar L. Avramov, Srikanth B. Iyengar and Liana M. Šega, *Free resolutions over short local rings*. J. Lond. Math. Soc. **78** (2008), no. 2, 459–476.

[6] Luchezar L. Avramov, Srikanth B. Iyengar and Liana M. Šega, *Short Koszul modules*. J. Commut. Algebra **2** (2010), no. 3, 249–279.

[7] Luchezar L. Avramov and Irena Peeva, *Finite regularity and Koszul algebras*. Amer. J. Math. **123** (2001), 275–281.

[8] Stephan Blum, *Initially Koszul algebras*. Beiträge Algebra Geom. **41** (2000), 455–467.

[9] Aldo Conca, Emanuela de Negri and Maria Evelina Rossi, *Koszul algebra and regularity*. in *Commutative Algebra: expository papers dedicated to David Eisenbud on the occasion of his 65th birthday*, I. Peeva (ed.), Springer (2013), 285–315.

[10] Aldo Conca, Srikanth B. Iyengar, Hop D. Nguyen and Tim Römer, *Absolutely Koszul algebras and the Backelin-Roos property*. submitted preprint (2014).

[11] Aldo Conca, Maria Evelina Rossi and Giuseppe Valla, *Gröbner flags and Gorenstein algebras*. Compositio Math. **129** (2001), 95–121.

[12] Aldo Conca, Ngo Viet Trung and Giuseppe Valla, *Koszul property for points in projective space*. Math. Scand. **89** (2001), 201–216.

[13] David Eisenbud, Gunnar Fløystad and Frank-Olaf Schreyer, *Sheaf cohomology and free resolutions over exterior algebras*. Trans. Amer. Math. Soc. **355** (2003), 4397–4426.

[14] Robert M. Fitzgerald, *Local Artinian rings and the Fröberg relation*, Rocky Mountain J. Math. **26** (1996), 1351–1369.
[15] Edward L. Green and Roberto Martínez-Villa, Koszul and Yoneda algebras. in: Representation Theory of Algebras (Cocoyoc, 1994), in: CMS Conf. Proc. Vol. 18, American Mathematical Society, Providence (1996), 247–297.

[16] Jürgen Herzog and Takayuki Hibi, Componentwise linear ideals. Nagoya Math. J. 153 (1999), 141–153.

[17] Jürgen Herzog, Takayuki Hibi and Gaetana Restuccia, Strongly Koszul algebras. Math. Scand. 86 (2000), 161–178.

[18] Jürgen Herzog and Srikanth B. Iyengar, Koszul modules. J. Pure Appl. Algebra 201 (2005), 154–188.

[19] Jürgen Herzog and Yukihide Takayama, Resolution by mapping cones. Homology Homotopy Appl. 4 (2002), no. 2, part 2, 277–294.

[20] Srikanth B. Iyengar and Tim Römer, Linearity defects of modules over commutative rings. J. Algebra 322 (2009), 3212–3237.

[21] Dancheng Lu and Dexu Zhou, Componentwise linear modules over a Koszul algebra. Taiwanese J. Math. 17 (2013), no. 6, 2135–2147.

[22] Roberto Martínez-Villa and Dan Zacharia, Approximations with modules having linear free resolutions. J. Algebra 266 (2003), 671–697.

[23] Satoshi Murai, Free resolutions of lex-ideals over a Koszul toric ring. Trans. Amer. Math. Soc. 363 (2011), no. 2, 857–885.

[24] Hop D. Nguyen and Thanh Vu, Regularity over homomorphisms and a Frobenius characterization of Koszul algebras. submitted (2014), http://arxiv.org/abs/1303.5160.

[25] Ryota Okazaki and Kohji Yanagawa, Linearity defects of face rings. J. Algebra 314 (2007), no. 1, 362–382.

[26] Irena Peeva, Graded syzygies. Algebra and Applications. Volume 14, Springer, London (2011).

[27] Irena Peeva and Mike Stillman, Open problems on syzygies and Hilbert functions. J. Commut. Algebra 1 (2009), 159–195.

[28] Tim Römer, On minimal graded free resolutions. Ph.D. dissertation, University of Essen (2001).

[29] Jan-Erik Roos, Good and bad Koszul algebras and their Hochschild homology. J. Pure Appl. Algebra 201 (2005), no. 1–3, 295–327.

[30] Maria Evelina Rossi and Giuseppe Valla, Hilbert functions of filtered modules. Lecture Notes of the Unione Matematica Italiana 9, Springer, UMI, Bologna (2010).

[31] Liana M. Şega, On the linearity defect of the residue field. J. Algebra 384 (2013), 276–290.

[32] Leila Sharifan and Matteo Varbaro, Graded Betti numbers of ideals with linear quotients, Le Matematiche 63 (2008), no. 2, 257–265.

[33] Kohji Yanagawa, Castelnuovo-Mumford regularity for complexes and weakly Koszul modules. J. Pure Appl. Algebra 207 (2006), no. 1, 77–97.

[34] _____, Linearity defect and regularity over a Koszul algebra. Math. Scand. 104 (2009), no. 2, 205–220.

Institut für Mathematik, Friedrich-Schiller-Universität Jena, Ernst-Abbe-Platz 2, 07743 Jena
E-mail address: ngdhop@gmail.com