THE EVOLUTION AND LUMINOSITY FUNCTION OF QUASARS 
FROM COMPLETE OPTICAL SURVEYS

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ABSTRACT
We use several quasar samples (Large Bright QSO Survey, Homogeneous Bright QSO Survey, Durham/AAT, and Edinburgh QSO Survey) to determine the density functions and the luminosity evolution of quasars. Combining these different samples and accounting for varying selection criteria require tests of correlation and the determination of density functions for multiply truncated data. We describe new nonparametric techniques for accomplishing these tasks, which have been developed recently by Efron & Petrosian. With these methods, the luminosity evolution can be found through an investigation of the correlation of the bivariate distribution of luminosities and redshifts. Here, one assumes a cosmological model to convert the observed fluxes into luminosities. We use matter-dominated models with either zero cosmological constant or zero spatial curvature. Of the two most commonly used models for luminosity evolution, \( L \propto a^{k(z)} \) and \( L \propto (1 + z)^k \), we find that the second functional form is a better description of the data at all luminosities; we find \( k = 2.58 \) ([2.14, 2.91] 1 \( \sigma \) region) for the Einstein–de Sitter cosmological model. Using this form of luminosity evolution, we determine a global luminosity function and the evolution of the comoving density for the two types of cosmological models. For the Einstein–de Sitter cosmological model we find a relatively strong increase in comoving density up to \( z \lesssim 2 \), at which point the density peaks and begins to decrease rapidly. This is in agreement with results from high-redshift surveys. However, we find that pure luminosity evolution, i.e., constant comoving density, is possible for some cosmological models for \( 3 \lesssim z \lesssim 2 \). We find that the local luminosity function \( \Phi(L_o) \) exhibits the usual double power-law behavior. The luminosity density \( \mathcal{L}(z) = \int L \Psi(L, z) dL \), where \( \Psi(L, z) \) is the luminosity function, is found to increase rapidly at low redshift and to reach a peak at around \( z \approx 2 \). Our results for \( \mathcal{L}(z) \) are compared to results from high-redshift surveys and to the variation of the star formation rate with redshift.

Subject headings: cosmology: theory — galaxies: evolution — galaxies: luminosity function, mass function — quasars: general

1. INTRODUCTION
Investigations of the evolution of the quasar population have played a major role in the development of our ideas about the nature of these sources and their connection to other extragalactic objects. Ever since the first complete survey of 3C radio quasars by Schmidt (1968) and the subsequent survey of 4C quasars by Lynds & Wills (1972), it has been evident that the population of quasars as a whole has undergone rapid evolution. Using the so-called \( \langle V/V_{\text{max}} \rangle \) method, these authors interpreted the evolution with redshift \( z \) as caused by an increase in the comoving density of quasars with redshift. However, both the source counts (Giacconi et al. 1979; Tananbaum et al. 1979) and the redshift distribution of optically selected samples of quasars (see, e.g., Marshall 1985) clearly showed that such pure density evolution models, for which the luminosity function is separable as

\[
\Psi(L, z) = \psi(L) \rho(z), \tag{1}
\]

cannot be correct. As more data was accumulated, the pure luminosity evolution model, with

\[
\Psi(L, z) = \psi[L/g(z)]/g(z), \tag{2}
\]

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gained more popularity. The function \( g(z) \) describes the luminosity evolution of the population, and \( L_o = L/g(z) \) is the luminosity adjusted to its present epoch (\( z = 0 \)) value. This model, while providing a better fit to the data than that of pure density evolution, also appears to be inadequate in many cases (see, e.g., Petrosian 1973; Schmidt & Green 1978; Koo & Kron 1988; Caditz & Petrosian 1990). Without loss of generality, we can write the luminosity function as

\[
\Psi(L, z) = \rho(z) \psi[L(g(z), \alpha_i)]/g(z). \tag{3}
\]

With \( \psi \) normalized such that \( \int_0^\infty \psi(L, z) dL = 1 \), \( \rho(z) \) gives the density and its evolution and \( \psi(L_o, \alpha_i) \) describes the local luminosity function [with \( g(0) = 1 \)]. Here we explicitly include the \( \alpha_i \) parameters such as spectral index and break luminosity which describe the shape of the luminosity function. In general, these parameters may vary with redshift. A surprising, and a priori unexpected, result has been the absence of evidence for strong shape variation.

Such results imply that the density and luminosity functions \( \rho(z) \) and \( g(z) \) describe the physical evolution of sources; e.g., the rate of birth and death of sources and the changes in source luminosity with time. Cavaliere and colleagues (see Cavaliere & Padovani 1988, and references therein) were the first to emphasize this fact. A more complete description of the relation between the physical evolution and the functions describing the generalized luminosity function (or statistical evolution) can be found in Caditz & Petrosian (1990). Unfortunately, this relation is not unique;
thus any test of quasar models via their expected evolution with cosmic epoch must usually involve additional assumptions. For example, quasars could be long-lived (compared to the Hubble time) sources created during a relatively short period at high-redshift undergoing continuous luminosity evolution. This is the model used in Cadiz, Petrosian, & Wandel (1991). Alternatively, quasars could be short-lived phenomena with birth rate, death rate, and luminosity that vary systematically with redshift (see Siemiginowska & Elvis 1997). Most of the work along these lines assumes what is now the standard model for the source of energy of quasars and other active galactic nuclei (AGNs); namely, accretion onto a massive black hole.

Another interesting aspect of quasar evolution is the relation between the evolution of the luminosity density of quasars, $L = \int L\Psi(L, z) dL$, and similar functions describing the evolution of galaxies, such as the star formation rate (SFR). As shown by the high-redshift surveys of Schmidt, Schneider, & Gunn (1995) and others (see, e.g., Warren, Hewett, & Osmer 1994), the rapid rise with redshift of the luminosity of quasars stops around redshift 2 or 3 and is expected to drop, perhaps mimicking the SFR (see Shaver et al. 1998; Cavaliere & Vittorini 1998; Hawkins & Vérón 1996).

In this paper we determine the luminosity function and its evolution for combined samples of quasars from various surveys described in § 2. For a complete review of the various ways of accomplishing this task, see Petrosian (1992). We use here nonparametric methods based on Lynden-Bell’s (1971) idea that was generalized by Efron & Petrosian (1992) to allow not only a determination of the functional forms but also a test of correlation between redshifts and luminosities. This is essential for any determination of the functional form of the luminosity evolution $g(z)$. The above papers deal with magnitude-limited samples with only an upper magnitude (lower flux) limit. However, because of observational constraints, some of the samples are truncated in redshift space or have a lower magnitude (upper flux) limit. The methods described in the above papers cannot be used for such multiply truncated data. New methods for treating this type of data were developed by Efron & Petrosian (1999). In § 3 we describe these new techniques and their relation to the older methods. The choice of cosmological model also plays a crucial role in such determinations. It is well known that one cannot determine both the evolution of the luminosity function and the parameters of the cosmological model from magnitude-limited samples alone. Only by assuming values for the cosmological parameters such as matter density, curvature, and cosmological constant can one calculate the form of $\Psi(L, z)$. Alternatively, with some assumptions about $\Psi(L, z)$ one can test various cosmological models. In § 4 we describe the cosmological model parameters, and in § 5 we present the results of applications of the new statistical methods to the data described in § 2. Finally, in § 6 we summarize these results and present our conclusions.

2. THE DATA

There have been many surveys of quasars, ranging from the original radio-selected samples of the 3C (Schmidt 1963) and 4C (Lynds & Wills 1972) surveys to a variety of other optically and X-ray–selected samples. In this paper, we will focus only on optically selected data. We use data from four samples with relatively similar selection criteria that provide a somewhat homogeneous data set spanning a large area of the $L$-$z$ plane. In order to combine the samples, all magnitudes were transformed to the $B$ band. Some of the samples were artificially truncated in order to insure completeness within the truncation limits. The combined sample consisted of 1552 objects in the range $15.5 < B < 21.2$ and $0.3 < z < 2.2$, with well-defined upper and lower magnitude limits for each object.

Figure 1 shows the distribution of the complete surveys in the $B$-$z$ plane, which at first sight shows little evidence for a Hubble relation. However, as we shall discuss below, there is evidence for cosmological dimming with redshift of the sources.

2.1. The Large Bright QSO Survey

The Large Bright QSO survey (LBQS) contains 1055 QSOs in the magnitude range $16.0 < B < 18.85$ and redshift range $0.2 < z < 3.4$ (Hewett, Foltz, & Chaffee 1995). The faint magnitude limits for the 18 fields range from 18.41 to 18.85, and the bright magnitude limit is 16.0 for the entire sample. In order to transform from $B_j$ to $B$ magnitudes, the color equation of Blair & Gilmore (1982),

$$B = B_j + 0.28(B - V),$$

was used assuming an average $B - V$ of 0.3 for the entire sample. When we combine this sample with the Anglo-Australian Telescope (AAT) sample and others, we introduce artificial cutoffs at $z = 0.3$ and $z = 2.2$, which are the completeness limits for the AAT data. In addition, we removed all objects brighter than $B = 16.5$ in order to insure completeness near the bright magnitude limit, leaving a total of 871 objects.

2.2. The Homogeneous Bright QSO Survey

Data from the six deepest fields of the Homogeneous Bright QSO Survey (HBQS) were published by Cristiani et al. (1995). The six deepest fields of the HBQS contain either $B_j$ or $B'$ magnitudes for 285 QSOs in the range $15.5 <
with redshift, if any. This last aspect is a higher order effect on the data and fits to some parametric forms of such a determination. The most common method is to bin limited sample. Several different techniques exist that give a determination of the functional forms of the local luminosity function in a different regime than the previous ones. The faint magnitude limit ranges from 20.25 to 21.27, and the bright magnitude limit ranges from 16.4 to 18.0. The data are given in the "b" magnitude system, which according to Boyle et al. (1990) may be converted to the Durham/AAT data. As described below, we combine all of these data and determine both the luminosity evolution and the density evolution.

2.3. The Durham/AAT Survey

The Durham/AAT survey contains 419 QSOs in the magnitude range 17 < B < 21.27 (Boyle et al. 1990) and redshift range 0.3 < z < 2.2, giving information about the QSO luminosity function in a different regime than the previous two samples. The faint magnitude limit ranges from 20.25 to 21.27, and the bright magnitude limit ranges from 16.4 to 18.0. The data are given in the "b" magnitude system, which according to Boyle et al. (1990) may be converted into the B system by the relation

\[ B = B' + 0.11(B - V) \]  

of Blair & Gilmore (1982) was used assuming \( B - V = 0.3 \). Again, when combining this sample only objects in the redshift range 0.3 < z < 2.2 were used, leaving a total of 254 objects.

2.4. The Edinburgh QSO Survey

A subsample of the Edinburgh QSO Survey (EQS) consisting of 12 QSOs brighter than \( B = 16.5 \) was published by Goldschmidt et al. (1992). Of these, the eight that fall in the redshift range 0.3 < z < 2.2 were added to the combined samples to give information about the luminosity function at the bright end.

There have been several previous analyses of the above quasar surveys. Boyle et al. (1990) used binning techniques to fit the Durham/AAT data to the pure luminosity evolution model. More recently, La Franca & Cristiani (1997) fitted the LBQS and HBQS data to a more complex luminosity function (involving three or more parameters) with no density evolution. Hatziminaoglou, Van Waerbeke, & Mathiez (1998) examined the pure luminosity evolution and pure density evolution cases separately using both the Durham/AAT and LBQS data. As described below, we combine all of these data and determine both the luminosity evolution \( g(z) \) and the density evolution \( \rho(z) \).

3. STATISTICAL METHODS

The statistical problem at hand is the determination of the true distribution of luminosities and redshifts of the sources from a biased or truncated data set, such as a flux-limited sample. Several different techniques exist that give such a determination. The most common method is to bin the data and fit to some parametric forms of \( \psi(L) \) or \( \rho(z) \). However, it is preferable to avoid binning and to use non-parametric methods whenever possible. For a review of the various methods, see Petrosian (1992). Assuming the general form of equation (3) for the luminosity function, we must determine the functional forms of the local luminosity function \( \psi(L_0, \alpha_i) \), density evolution \( \rho(z) \), and the luminosity evolution \( g(z) \), as well as the changes in the parameters \( \alpha_i \) with redshift, if any. This last aspect is a higher order effect and will not be within the scope of this paper. We will discuss the certainty with which this can be ignored in the final analysis.

All nonparametric techniques for determining the distribution in a bivariate setting require that the data be expressed in terms of two uncorrelated variables, i.e., that we use variables \( x \) and \( y \) for which the density function is separable: \( \Psi(x, y) = \rho(x)\psi(y) \). Thus before applying non-parametric methods one must first determine the degree of correlation of the data in the \( x \) \( y \) plane. This determination and the process of removing the correlation is equivalent to a determination of the functional form of the luminosity evolution \( g(z) \) and the subsequent transformation \( L \rightarrow L_0 = L/g(z) \). This cannot be accomplished easily by nonparametric methods. Therefore, we chose two parametric forms for the luminosity evolution, \( g_k(z) \) and \( g_k(z) \), and find the values of the parameters \( k \) and \( k' \) for which \( L_0 \) and \( z \) are uncorrelated. Once this is done, the nonparametric methods described in Petrosian (1992) may be used to determine \( \rho(z) \) and \( \psi(L_0) \).

The methods normally used to test correlation and determine the distributions are suited for simple truncations, such as \( y < f(x) \) [which by defining \( x' = f(x) \) can be reduced to the generic case of \( y < x' \)]. This is sufficient for simple flux-limited data. However, the majority of astronomical data, and quasar data in particular, suffer from more than one truncation. The data may have an upper as well as a lower truncation, \( f^{-}(x) < y < f^{+}(x) \), or there may be similar truncations in the value of the other variable. In addition, the functions \( f^{-}(x) \) and \( f^{+}(x) \) may not be continuous or even single valued. In general, multiply truncated redshift-magnitude data may be written as \( \{ z_i, m_i, [z_i^-, z_i^+] \} \}_{i=1}^M \), where \( [z_i^-, z_i^+] \) are the observational limits on \( z \) and \( m_i \) for the \( i \)th object, respectively. Given a cosmological model \( \Omega \), this gives data of the form \( \{ z_i, L_i, [z_i^-, z_i^+] \} \}_{i=1}^M \), which is the case in the majority of our analyses, then the problem is to test the correlation and distribution of \( x \) and \( y \) from a data set \( \{ x_i, y_i \}_{i=1}^N \) given truncation limits \( [y_i^-, y_i^+] \) for each point. The previous methods developed for this test (see Petrosian 1992 and Efron & Petrosian 1992) are suited for one-sided truncations. In a more recent work (Efron & Petrosian 1999), we have developed methods, which are a generalization of the earlier methods, for dealing with doubly truncated data. We will briefly review these new methods of testing correlation and nonparametrically determining the density evolution and luminosity function.

3.1. Tests of Correlation and Determination of Luminosity Evolution

3.1.1. Untruncated Data

If \( x \) and \( y \) are independent, then the rank \( R_x \) of \( x \) in an untruncated sample (i.e., a sample truncated parallel to the \( x \) and \( y \) axes, so that \( y_x^\pm \) are independent of \( z_x \)) will be distributed uniformly between 1 and \( N \) with an expected mean \( E = (1/2)(N + 1) \) and variance \( V = (1/12)(N^2 - 1) \). One may then normalize \( R_x \) to have a mean of 0 and a variance of 1 by defining the statistic \( T_x = (R_x - E)/V \). One then rejects or accepts the hypothesis of independence based on the distribution of the \( T_x \).

One simple way of doing so is by defining a single statistic \( t_{data} \) based on the \( T_x \) with a mean of 0 and a variance of 1.
One then rejects the hypothesis of independence if \( |t_{\text{data}}| \) is too large (e.g., \( t_{\text{data}} \geq 1 \)) for rejection of independence at the 1 \( \sigma \) level. The quantity

\[
\tau = \frac{\sum_i (R_i - E)}{\sqrt{\sum_i V}}
\]

is one choice of such a test statistic. This \( \tau \) is equivalent to Kendall’s \( \tau \) statistic, which is defined in the following manner (see, e.g., Press et al. 1990). Consider all possible pairings \( \mathcal{P} = \{(i, j)\} \) between data points and call a pairing \((i, j)\) positive if \((x_i - x_j)(y_i - y_j) > 0\) and negative if \((x_i - x_j)(y_i - y_j) < 0\). If there are no ties, then the \( \tau \) of equation (7) is equivalent to Kendall’s \( \tau \) statistic

\[
\tau = \frac{\text{number positive (i, j) } \in \mathcal{P} - \text{number negative (i, j) } \in \mathcal{P}}{\text{number (i, j) } \in \mathcal{P}}.
\]

### 3.2. Nonparametric Determination of Distribution Functions

Once a function \( g \) is found that removes the correlation between \( x \) and \( y \) (or \( z \) and \( L \)), the task is to find the underlying distributions \( p(x) \) (the density evolution) and \( \psi(y) \) (the local luminosity function) given uncorrelated data \( \{x_i, y_i\} \) and truncation limits \([y_i^-, y_i^+]\). If the truncation is one-sided (e.g., \( y_i^+ = \infty \) for all \( i \)), then a variety of nonparametric methods can be used to determine the univariate distribution functions. As shown by Petrosian (1992), all nonparametric methods reduce to Lynden-Bell’s (1971) method. As with the tests of correlation described above, the gist of this method is to find for each point the comparable set defined in equation (9) and the number \( N_i \) of points in this set. For example, the cumulative distribution in \( y' \), \( \Phi(y') = \int_{y'}^{\infty} \psi(t)\,dt \), is given by

\[
\ln \Phi(y') = \sum_{j < i} \ln \left( 1 + \frac{1}{N_j} \right).
\]
For doubly truncated data, the comparable set is not completely observed; thus a simple analytic method such as the one described above is not possible. However, it turns out that a simple iterative procedure can lead to a maximum likelihood estimate of the distributions. Here we give a brief description of this method; for details see Efron & Petrosian (1999).

Assume that the underlying density function \( \psi(y) \) is discretely distributed over the \( N \) observed values of \( y \). If we let \( \psi_i = \psi(y_i) \) be the probability density at \( y_i \), then \( \Phi_i = \sum_j \psi_j \), where the summation includes all data points for which \( y_j \in [y_i^-, y_i^+] \) is the total probability density for the truncation region \([y_i^-, y_i^+]\) . If we define the matrix \( J \) by

\[
J_{ij} = \begin{cases} 
1, & \text{if } y_j \in [y_i^-, y_i^+] \\
0, & \text{if } y_j \notin [y_i^-, y_i^+] 
\end{cases},
\]

then the definition of \( \Phi_i \) is equivalent to \( \Phi = J \cdot \Psi \), where \( \Psi = (\psi_1, \ldots, \psi_N) \) and \( \Phi = (\Phi_1, \ldots, \Phi_N) \). This matrix \( J \) contains all of the information about the data and the truncation limits needed to find the vector of probability densities \( \Psi \). We also have the normalization condition on \( \Psi \),

\[
\sum_{i=1}^{N} \psi_i = 1.
\]

From the definition of \( \Phi_i \), it follows that the conditional probability \( \psi(y_i|[y_i^-, y_i^+]) \) of observing a value \( y_i \) within the truncation region \([y_i^-, y_i^+]\) is

\[
\psi(y_i|[y_i^-, y_i^+]) = \begin{cases} 
\psi_i/\Phi_i, & \text{if } y_i \in [y_i^-, y_i^+] \\
0, & \text{if } y_i \not\in [y_i^-, y_i^+] 
\end{cases}.
\]

The final condition on \( \Psi \) is determined by maximizing the likelihood of observing the actual data,

\[
P_{\text{data}} = \prod_{i=1}^{N} \psi(y_i|[y_i^-, y_i^+]) = \prod_{i=1}^{N} \sum_j J_{ij} \psi_j.
\]

By setting \( \partial P_{\text{data}}/\partial \Psi = 0 \), it follows that \( \psi_k^{-1} = \sum_j J_{jk} \Phi_j^{-1} \), \( k = 1, \ldots, N \). This may be written compactly as

\[
\frac{1}{\Psi} = J^*- \frac{1}{J \cdot \Psi},
\]

with the notation \( 1/a = (a_1^{-1}, \ldots, a_N^{-1}) \). Thus we have reduced the problem of finding the density function for data with arbitrary truncation to the “moral equivalent” of an eigenvalue problem for the matrix \( J \).

In practice, this condition may be used as a recursive formula to determine \( \Psi \). One starts with an initial guess for the density vector \( \Psi^0 \). Equation (19) then gives the recursion relation

\[
\frac{1}{\Psi^{(j+1)}} = J^* \frac{1}{\Psi^{(j)}} + c^{(j)},
\]

where the constant \( c^{(j)} \) is determined by the normalization condition \( \sum \psi_i^{(j+1)} = 1 \). One may use as an initial guess the untruncated solution \( \psi_0 = 1/N \). In most problems, however, one of the two truncations will have a more pronounced effect than the other. In this case one may ignore the weaker truncation and use the result based on the one-sided method as an initial guess. We found that with this initial guess and data confined to one region of the \( L-z \) plane, the sequence of \( \Psi^{(j)} \) defined by equation (20) usually converged quickly. However, for combined samples spanning different regions of the \( L-z \) plane it was helpful to use an algorithm to accelerate the convergence of the series of \( \Psi^{(j)} \). For this purpose we used Aitken’s \( \delta^2 \) method, which gives an improved estimate for the terms of series by assuming approximately geometric convergence (see, e.g., chap. 5.1 of Press et al. 1990).

4. COSMOLOGY AND MODELS OF LUMINOSITY EVOLUTION

In order to determine the intrinsic parameters of an object from the observed data, one must assume a certain cosmological model. Many cosmological models may be described in terms of a few fundamental parameters, which (see, e.g., Peebles 1993) for a matter-dominated (nonrelativistic) universe are the matter density \( \rho_0 \), the cosmological constant \( \Lambda \), and the curvature of space \( k \) (which is \(+1\), 0, or \(-1\) for closed, flat, or open universes, respectively). Using Hubble’s constant \( H_0 \), these parameters may be written in dimensionless form as

\[
\Omega_M = \frac{8\pi G \rho_0}{3H_0^2}, \quad \Omega_k = -\frac{k c^2}{(H_0 R_0)^2}, \quad \text{and } \Omega_\Lambda = \frac{\Lambda}{3H_0^2},
\]

where \( R_0 \) is the value of the expansion parameter of the universe at the present epoch. These three parameters are related via the Friedman-Lemaître equation \( \Omega_\Lambda + \Omega_k + \Omega_M = 1 \) (see, e.g., Peebles 1993), allowing us to eliminate one of them in favor of Hubble’s constant. For example, the curvature term may be written

\[
\Omega_k = 1 - (\Omega_M + \Omega_\Lambda).
\]

We will consider the two classes of cosmological models given by \( \Omega_\Lambda = 0 \) (no cosmological constant) and \( \Omega_\Lambda = 0 \) (flat universe with cosmological constant). For calculations of the luminosity function, we will pay particular attention to the two cases \( \Omega_k = \Omega_\Lambda = 0 \) and \( \Omega_k = 0.85, \Omega_\Lambda = 0.15 \). The first of these is the standard Einstein–de Sitter model, and the second is an inflationary model with parameters in accordance with current observations. For definiteness, Hubble’s constant was assumed to be \( H_0 = 70 \) km s\(^{-1}\) Mpc, although most results are independent of this assumption.

Once the values of the cosmological parameters are fixed, calculations of intrinsic parameters are relatively straightforward (Peebles 1993). The absolute luminosity, for example, takes the form

\[
L = 4\pi d_L^2 K(z),
\]

where \( f \) is the observed flux, the luminosity distance \( d_L \) is

\[
d_L = \frac{c}{H_0} \left(1 + \frac{1}{z}ight) \frac{\sin \sqrt{k} u(z)}{\sqrt{k}},
\]

and \( K(z) \) is the \( K \)-correction term. The comoving coordinate distance is

\[
u(z) = \int_0^z \frac{dz}{\sqrt{\Omega_M(1+z)^3 + \Omega_k(1+z)^2 + \Omega_\Lambda}},
\]

and the comoving volume contained within a sphere of radius corresponding to redshift \( z \) is

\[
V(z) = 4\pi \left( \frac{c}{H_0} \right)^2 \int_0^z \frac{du}{u} \frac{d\nu}{dz} \frac{dz}{(1+z)^3}.
\]
In order to determine the \( K \)-correction term \( K(z) \), one must make an assumption about the quasar spectrum in the optical region. The general practice here is to assume a power-law spectrum \( I_{\text{optical}} \propto \nu^\alpha \) with spectral index \( \alpha \approx -0.5 \), which gives \( K(z) = (1 + z)^{1 + \alpha} \approx \sqrt{(1 + z)}. \)

Two models for luminosity evolution were used: \( g(z) \propto e^{K(z)} \) and \( g(z) \propto (1 + z)^{b} \). The first of these assumes an exponential dependence on the fractional look-back time \( t(z) \), which is defined as \( t(z) = 1 - T(z)/T(0) \), where \( T(z) \) is the age of the universe at redshift \( z \),

\[
T(z) = H_0^{-1} \int_z^{\infty} \frac{du}{dz} (1 + z).
\] (27)

The second model assumes a power-law dependence on the scale factor of the universe (or the expansion parameter \( R \)), which is independent of the cosmological parameters. Analyses of earlier data (see, e.g., Caditz & Petrosian 1990) have traditionally given estimates of quasar evolution for these two parametric forms of \( g(z) \approx e^{K(z)} \) and \( g(z) \approx (1 + z)^{b} \).

5. THE EVOLUTION OF THE LUMINOSITY FUNCTION

In what follows we apply the procedures of § 3 to determine the correlation between the luminosities and redshifts and test parametric forms for the evolution of the luminosity, the function \( g(z) \). Then we transform all luminosities to their present epoch values \( L_0 = L/g(z) \) and determine the comoving density evolution \( \rho(z) \) and the present epoch luminosity function \( \rho(L_0) \) for the cosmological models described in § 4. We apply these tests to the surveys described in § 2 individually and in various combinations. Before presenting these results, we discuss briefly the redshift-magnitude data, i.e., the Hubble diagram shown in Figure 1.

5.1. The Quasar Hubble Diagram

As is evident from Figure 1, at first glance there seems to be very little correlation between the redshifts and magnitudes (or fluxes) of quasars; i.e., there is no obvious evidence for a Hubble-type relation. This result is well known. For example, a preliminary test of correlation between \( m \) and \( z \) in a small subsample by Efron & Petrosian (1992, Fig. 6) showed no correlation. Earlier, the absence of a clear Hubble relation was used as an argument against the cosmological origin of quasar redshift (see, e.g., Burbidge & O’Dell 1973). This is not the only possible interpretation, however. One would not expect a simple Hubble diagram for sources with a broad luminosity function (nonstandard candle sources; see, e.g., Petrosian 1974). The absence of an obvious Hubble relation can also arise from approximate cancellation between cosmological dimming and luminosity evolution. Exact cancellation of these two effects is highly implausible and could bring into question the basic assumptions about the distribution of sources. To clarify this situation we have applied the correlation tests described in § 3 to the surveys of § 2. First ignoring the high flux (low-magnitude) limit, i.e., with \( m_{\text{min}} = \infty \), we use the one-sided tests and find the results labeled \( \tau \) shown in Table 1. When using the double-sided tests, we find the results \( \tau_1 \), which indicate slightly less correlation, as is expected because the one-sided methods ignore the slight truncation induced by the high flux limits. These tests, when applied to the combined data sets, give a correlation of \( \tau = 3.63 \). This result rejects the hypothesis of independence between \( B \) and \( z \) at the 99.97% confidence level and is independent of any cosmological parameters. In addition, we may test the parametric fit \( B(z) = B - \beta \log [d_L^2(z)K(z)] + \text{constant} \) for the data using the methods for multiple truncations to determine the best value of \( \beta \), i.e., the value for which \( B(z) \) and \( z \) are uncorrelated (\( \tau = 0 \)). A value of \( \beta \approx 2.5 \) is what one would expect for standard candle sources with a very narrow luminosity function, while a value of \( \beta = 0 \) would mean the complete absence of a Hubble relation and the exact cancellation described above. The results shown in Table 1 indicate that \( \beta \), while clearly less than 2.5, differs significantly from 0 for the cosmological models discussed in § 4. The best parametric fit for the Einstein-de Sitter cosmological model (\( \beta = 0.84 \)) is shown in Figure 1 along with the expected relation for standard candle sources (\( \beta = 2.5 \)).

We now turn to a determination of the evolution of the luminosity.

5.2. The Luminosity Evolution \( g(z) \)

We examine the two commonly used parametric forms for the luminosity evolution, the exponential, and power-law forms. These two different forms emphasize the evolution in different regions of the luminosity-redshift (\( L-z \)) plane. A correct parameterization will have the same value for its parameters when applied to samples with different limits (i.e., different coverage of the \( L-z \) plane). This fact can be used to test a given parametric form. Table 2 summarizes the results described in the subsequent sections. Figure 2 gives an example of the variation of the test statistic \( \tau \) as a

| Sample          | \( N \) | \( \tau_1 \) | \( P_1 \) | \( \tau_2 \) | \( P_2 \) | \( \beta_1 \) | \( \beta_2 \) |
|-----------------|--------|------------|---------|------------|---------|------------|------------|
| Durham/AAT      | 419    | -0.65      | 48.71   | -0.45      | 35.02   | -0.22      | -0.18      |
| LBQS            | 871    | 4.65       | 99.99   | 3.75       | 99.98   | 1.11       | 0.92       |
| HBQS            | 254    | 1.37       | 82.92   | 1.27       | 79.42   | 0.71       | 0.59       |
| LBQS and HBQS   | 1125   | 4.80       | 99.99   | 3.98       | 99.99   | 1.01       | 0.83       |
| Combined Data   | 1552   | 3.98       | 99.99   | 3.63       | 99.97   | 0.84       | 0.70       |

* \( N \) is the number of data points in each of the data sets of § 2. The correlation \( \tau \) and the probability value \( P \) for rejection of independence between \( B \) magnitude and redshift \( z \) are given. The first set of values \((\tau_1, P_1)\) are found using the one-sided method of Efron & Petrosian 1992, and the second set of values \((\tau_2, P_2)\) are found using the general method for doubly truncated data presented in § 3. The parameters \( \beta_1 \) and \( \beta_2 \) are found with this general method by fitting to the equation \( B(z) = B - \beta \log [d_L^2(z)K(z)] + \text{constant} \) for the two cosmological models \( \Omega_m = 1 \), \( \Omega_m = 0.15 \), \( \Omega_m = 0.85 \), respectively.
function of the evolution parameter $k'$. The optimal value of $k$, i.e., the value which indicates that $L_0$ and $z$ are not correlated, is given by the condition $\tau = 0$, and the $1\sigma$ range of this parameter is given by the condition $|\tau| < 1$. These values are shown in Figure 2.

5.2.1. Evolution of the Form $g_1(z) = e^{\Omega(z)}$

Figures 3a and 3b display the best values and the $1\sigma$ ranges for the evolution parameter $k$ assuming both $\Omega_\Lambda = 0$ and $\Omega = 0$ cosmological models for the different samples. The Durham/AAT data exhibit much stronger evolution than the other data, indicating that this form of evolution does not adequately describe the data in the entire $L-z$ plane. The values of $k$ found for most of the data sets are considerably less than the previously determined value $k \approx 7.5$ by Caditz & Petrosian (1990), which was dominated by the Durham/AAT data.

5.2.2. Evolution of the Form $g_2(z) = (1 + z)^{k'}$

Figures 4a and 4b give results for the evolution parameter $k'$ for the same cosmological models as above. The allowed ranges obtained from the different samples are much closer; thus this form of evolution is shown to be a closer approximation to the actual evolution than the exponential form. For the Einstein–de Sitter model, the best value of evolution parameter is $k' = 2.58$ with a $1\sigma$ range of $k' \in [2.14, 2.91]$. As with the previous case, the values of $k'$ are somewhat less than the previously found value of $k' \approx 3$ of Caditz & Petrosian (1990).

It is clear that a better fit can be achieved with a different functional form for $g(z)$ with two or more parameters. However, in what follows we assume the simpler form of evolution $g(z) = (1 + z)^k$, with $k(\Omega)$ the optimal value of $k$.

### TABLE 2

| Sample           | $N$  | $k$  | $k_{\text{min}}$ | $k_{\text{max}}$ | $k'$  | $k'_{\text{min}}$ | $k'_{\text{max}}$ |
|------------------|------|------|------------------|------------------|------|------------------|------------------|
| Durham/AAT ...... | 419  | 8.72 | 6.66             | 10.07            | 3.53 | 2.57             | 5.05             |
| HBQS ............. | 254  | 5.39 | $-\infty$        | 6.49             | 3.20 | $-\infty$        | 3.94             |
| LBQS ............. | 871  | 4.28 | 2.66             | 5.17             | 2.02 | 1.24             | 2.53             |
| Combined Data .....| 1552 | 5.15 | 4.36             | 5.70             | 2.58 | 2.14             | 2.91             |

* These parameters refer to luminosity evolutions $g_1(z) \propto e^{\Omega(z)}$ and $g_2(z) \propto (1 + z)^{k'}$, respectively. The numbers $k$ and $k'$ are the optimal values for a given data set (i.e., those for which $\tau = 0$). The numbers $k_{\text{min}}, k_{\text{max}}$ and $k'_{\text{min}}, k'_{\text{max}}$ are the minimum and maximum values allowed at the $1\sigma$ level (i.e., those for which $|\tau| = 1$). Note the variation in $k$ and the near constancy of $k'$.
for a given cosmological model $\Omega$. We therefore transform the data to $\{[L_0(z), z]\}_{i=1}^{N}$ with $L_0(z, m, \Omega) = L(z, m, \Omega)/(1 + z)^k$ and apply the method of § 3 to find nonparametric estimates for the density functions $\rho(z)$ and $\psi(L_0)$. This method now gives directly the cumulative functions $\sigma(z) = \int_0^z \rho(z)(dV/dz)dz$ and $\Phi(L_0) = \int_0^{L_0} \psi(L)dL$.

5.3. The Density Evolution $\rho(z)$

The cumulative density function $\sigma(z)$ is the total number of objects within the angular area of the survey up to redshift $z$. If there is no density evolution, i.e., $\rho(z) = \rho_0$ is a constant, then $\sigma \propto V$, where $V(z)$ is the comoving volume up to redshift $z$. We determine $\sigma(z)$ and $\rho(z)$ using the new method for doubly truncated data. In order to determine if density evolution exists, we fit $\sigma$ to $V$ by a simple power law $\sigma(z) \propto V^\lambda$, where $\lambda \neq 1$ indicates the presence of density evolution. If the density increases with redshift we expect $\lambda > 1$, and if the density decreases with redshift we expect $\lambda < 1$. Even if $\lambda = 1$, however, density evolution may be present: the density may increase and decrease in such a way as to cancel and give a fit of $\lambda = 1$. Figures 5a and 5b show the variation of $\sigma$ and $\rho$, with $V$ for the combined sample for three different cosmological models. The dotted lines show the best fits to the form $\sigma \propto [V(z, \Omega)]^{\lambda}$. For the Einstein–de Sitter model (top lines in Fig. 5), we have $\lambda = 1.19$, indicating that the comoving density increases with redshift roughly as $\sigma \propto V^{1.19}$. This would indicate a simple power-law density evolution $\rho \propto V^{0.19}$. The density evolution shown in Figure 5b exhibits this average behavior, but in detail is more complex: the density increases more rapidly at low $z$, reaches a plateau at $z \approx 2$, and possibly decreases at higher $z$. This behavior will be discussed below in more detail and for higher redshift data.

As mentioned in § 1, one cannot determine the evolution of sources [the function $\rho(z)$] and the evolution of the universe (the parameters $\Omega$) simultaneously. Given one of these (e.g., the cosmological model), the other (the density evolution) may be determined from the data. The variation of $\lambda = d \ln \sigma/d \ln V$ with $\Omega_m$ for the two different classes of cosmological models is shown in Figures 6a and 6b. It is
to an increase in density with redshift. Pure luminosity evolution, \( j \) corresponds to a decrease in density with redshift, and of space contained within the sphere of radius \( R \) is defined by the equation

\[
V(R) = \frac{4}{3} \pi R^3 \Omega_0.
\]

Results for cosmological models with \( \Omega_0 = 0.3 \) and de Sitter cosmological model. Two sets of results are given. The first of these \( (\Omega_0 = 0.3 \Omega_L \approx 0, \Omega_\Lambda \approx 1) \) and \( \Omega_M \approx 0.15, \Omega_\Lambda \approx 0.85 \). This second set of parameters is quite close to those currently favored by many observations. As above, this does not imply the complete absence of density evolution. The lower two curves in Figure 5b show the variation of \( \rho \) for these two models. Clearly, there is less variation than for the Einstein–de Sitter model, but the general behavior is similar.

To further analyze the variation of \( \rho(z) \), in Figure 7 we show our nonparametric determination of \( \rho(z) \) for the Einstein–de Sitter cosmological model. Two sets of results are given. The first of these \( (\text{squares}) \) shows \( \rho(z) \) in the region \( 0.3 < z < 2.2 \) from the combined data. The second \( (\text{triangles}) \) shows \( \rho(z) \) in the region \( 0.3 < z < 3.3 \) from the LBQS data (which do not have a high-redshift cutoff at \( z = 2.2 \)) alone. As evident in Figure 7, the density increases relatively slowly at low redshift \( [\rho \sim (1 + z)^{\lambda}] \) before reaching a peak at \( z \approx 2 \) and decreasing rapidly \( [\rho \sim (1 + z)^{-5}] \) at higher redshift. As discussed further in § 5.5, the decrease in density present in this data at redshift of about 2 is in agreement with high-redshift survey results (Schmidt et al. 1995; Warren et al. 1994).

5.4. The Luminosity Function \( \psi(L_0) \)

In a similar fashion we may obtain the cumulative luminosity function \( \Phi(L_0) \) from the uncorrelated data set \{\( L_0, z \)\}. Figures 8a and 8b show \( \Phi(L_0) \) for the combined data set with \( k' = 2.58 \), along with the best fits to a double power law form

\[
\Phi(L_0) = \frac{\Phi_0}{(L_0/L_\ast)^{k_1} + (L_0/L_\ast)^{k_2}}.
\]

We used the cosmological models \( \Omega_M = 1, \Omega_\Lambda = 0 \) (Einstein–de Sitter model) and \( \Omega_M = 0.15, \Omega_\Lambda = 0.85 \) (pure luminosity evolution with cosmological constant). In both cases, the results for the combined samples exhibit roughly double power-law dependence on \( L_0 \) with similar values of the fitting parameters. The primary differences between the two models are that the Einstein–de Sitter model gives a slightly gentler slope above the break luminosity, \( k_2 = 3.17 \) as opposed to \( k_2 = 3.59 \), and a lower break luminosity, \( L_\ast = 6.19 \times 10^{37} \text{ ergs s}^{-1} \text{ Hz} \) as opposed to \( L_\ast = 9.49 \times 10^{39} \text{ ergs s}^{-1} \text{ Hz} \).

We check for possible variation in the shape of \( \psi(L_0) \) by dividing the data into three redshift bins: \( 0.3 < z < 0.86, 0.86 < z < 1.48, \) and \( 1.48 < z < 2.2 \). We then find the differential luminosity function \( \psi(L_0) \) for these three redshift bins, first assuming no luminosity evolution \( [g(z) = \text{constant}] \) and then assuming luminosity evolution \( g(z) = (1 + z)^{\lambda} \).

These luminosity functions (with arbitrary vertical normalization) are shown in Figures 9a and 9b, respectively. The presence of a strong shift to higher luminosities is clearly evident for \( g(z) = \text{constant} \). However, when the evolution \( g(z) = (1 + z)^{\lambda} \) is taken out, the luminosity function seems to exhibit little variation; the slopes appear roughly the same at low \( L_0 \) and high \( L_0 \), and the break luminosity does not vary as much with redshift. Although imprecise,
combined data set is shown as a function of for two di†er-

tions. We Ðrst note the good agreement in the data, which is claimed to be complete up to this redshift.

We extend this further to a redshift of 3.3 using the LBQS above results, we can evaluate 

$\rho(z)$, both of which are related to the formation of galaxies and the variation of the accretion rate, both of which are related to the formation of galaxies and their evolution through mergers or collisions. Using the above results, we can evaluate $\mathcal{L}(z)$ up to a redshift of 2.2. We extend this further to a redshift of 3.3 using the LBQS data, which is claimed to be complete up to this redshift. These results, with arbitrary vertical normalization, are shown in Figure 10. We first note the good agreement in the $z < 2$ region, indicating that perhaps the LBQS result at higher redshift is a representative behavior. We may also use high-redshift surveys of quasars (Schmidt et al. 1995; Warren et al. 1994) to study $\mathcal{L}(z)$ in this range. Unfortunately, the selection of high-$z$ quasars in these samples is more complicated, and the subsequent analyses involve more assumptions. For example, Schmidt et al. (1995) use the V/ $\nu_{\text{max}}$ method to determine $\rho(z)$, tacitly assuming that $g(z) = \text{constant}$ (as well as $\alpha_i = \text{constant}$) so that $\mathcal{L}(z) \propto \rho(z)$. We show these results (again with arbitrary vertical normalization) in Figure 10. These results agree with the general trend of decline in $\mathcal{L}(z)$ at high redshifts.

It has been claimed (Cavaliere & Vittorini 1998; Shaver et al. 1998) that this rise and fall of $\mathcal{L}(z)$ with redshift is similar to the behavior of the SFR, which has recently been extended to high redshifts (see, e.g., Madau 1997; Hughes et al. 1998). We have shown these rates in Figure 10 as well. Although the general trend of rise and fall of the SFR and $\mathcal{L}(z)$ is the same, there is considerable difference in the

5.5. The Luminosity Density $\mathcal{L}(z)$

Finally, we determine the luminosity density function $\mathcal{L}(z)$. This quantity is defined as the total rate of energy production by quasars in the optical range as a function of redshift; $\mathcal{L}(z) = \int_0^\infty L \Phi(L) dL$. If the shape of the luminosity function is invariant, then $\mathcal{L}(z) \propto \rho(z) g(z)$. This rate depends in a complicated way on the distribution of masses of the central black holes and the variation of the accretion rate, both of which are related to the formation of galaxies and their evolution through mergers or collisions. Using the above results, we can evaluate $\mathcal{L}(z)$ up to a redshift of 2.2. We extend this further to a redshift of 3.3 using the LBQS data, which is claimed to be complete up to this redshift. These results, with arbitrary vertical normalization, are shown in Figure 10.
vertical normalizations are arbitrary. All of the above results indicate that Warren et al. (1994) are given by the letters s and w, respectively. The Squares and triangles give our results for the combined data and the LBQS.

Consider the many differences between star formation and the generation of energy by quasars, the observed difference between the SFR and $\mathcal{L}(z)$ in Figure 10 is not surprising.

6. SUMMARY AND CONCLUSIONS

There have been several analyses of quasar evolution in the past. Our results differ from these previous results in two important respects:

1. We have used nonparametric statistical methods for multiply truncated data that allow us to combine samples with different selection criteria.
2. We have used the data to investigate the simplest evolutionary models of the luminosity function described by only a luminosity evolution $g(z)$ and a density evolution $\rho(z)$. We do not rely on more complex predetermined forms such as that found in $\chi^2$-limited surveys and can account for selection biases in generally truncated data where each data point has different truncation limits. In particular, these methods can treat samples with both upper and lower flux limits and redshift limits.
3. This versatility allows one to combine data from different surveys with different selection criteria.

4. The first of our techniques, a generalized nonparametric test of independence, allows one to determine the degree of correlation between luminosity and redshift, giving an indication of the luminosity evolution in the luminosity function. The evolution form may then be determined parametrically, and from the goodness of the fit one can distinguish between different forms.

5. The second of our techniques provides a nonparametric estimate for the univariate distributions in redshift and luminosity, i.e., the comoving density evolution and the global luminosity function, respectively.

We have applied these methods to the combined data from several large surveys and determined the luminosity evolution $g(z)$, the density evolution $\rho(z)$, and the luminosity function $\psi[L_0 = L/g(z)]$ of the generalized luminosity function of equation (3) for flat ($\Omega_k = 0$ and $\Omega_M = 1 - \Omega_k$) and zero cosmological constant ($\Omega_k = 0$ and $\Omega_M = 1 - \Omega_k$) cosmological models. We assume a shape invariant luminosity function, $\alpha_i = \text{constant}$. More complex luminosity functions, $\alpha_i \neq \text{constant}$, or those with luminosity-dependent density evolution, etc., can be tested if the simpler prescription used here is not consistent with all of the data. We found that the scenario of equation (3) provides an adequate description of the existing data.

Our results may be summarized as follows:

1. We found a strong correlation between luminosity and redshift, indicating the presence of rapid luminosity evolution.
2. The parametric model of luminosity evolution $(1 + z)^k$ provides a better description of the data than the model $e^{\alpha z}$, although neither parameterization perfectly removes the correlation in all areas of the $L$-$z$ plane. In order to better model this evolution, future analyses of quasar evolution could consider parametric forms with more than one free parameter.

More complex forms of luminosity evolution have sometimes been expressed in terms of luminosity-dependent luminosity (or density) evolution. These more complicated forms can be turned into a simple luminosity-independent luminosity evolution form with more than one parameter. For example, the form with the exponent $k' = k_1 - k_2(z)$ and $L > L_*$ used by La Franca & Cristiani (1997) is the same as the simpler luminosity-independent luminosity evolution with $k' = k_1/[1 + k_2 \ln [(1 + z)/(1 + z_*)]]$ for $z$ greater than some $z_*$. The value of $k' = 3.53$ we find for the Durham/AAT sample is similar to the values found by others (3.45, Boyle 1992; greater than 3.2, Caditz & Petrosian 1990). However, the value 2.58 for the combined data is smaller than the previous estimates (e.g., 3.26, La Franca & Cristiani 1997), most of which assume a true luminosity evolution, i.e., a constant $\rho(z)$. The exponent $k'$ of the luminosity evolution is somewhat coupled to the strength of the density evolution (see also below). An incorrect assumption about the latter will result in an incorrect value for $k'$. In our method we derive the two evolutions independently; we determine the degree and form of the correlation between $L$ and $z$ (i.e., the luminosity evolution) without making any assumptions about the existence, form, or strength of the density evolution.

Given the form of the luminosity evolution, we make the simple transformation of all luminosities to their hypothe-
of these previous works and partly due to the incompleteness at high redshift of the data used in our analysis.

The cumulative local luminosity function $\Phi(L_\alpha) = \int_{L_\alpha}^{\infty} \psi(x)dx$ has the double power-law form found previously. In the Einstein–de Sitter model, the break luminosity is $L_\alpha = 6 \times 10^{29}$ ergs s$^{-1}$ Hz, and the low- and high-luminosity power-law indices are $k_1 = 1.05$ and $k_2 = 3.17$. These are consistent with values of 1.35–1.50 and 3.6–3.9 obtained for the same exponent from the determinations of the differential luminosity function (Caditz & Petrosian 1990; La Franca & Cristiani 1996). There appears to be little variation with redshift of the shape of the cumulative and differential luminosity functions; thus the $z_i = \text{constant}$ prescription seems adequate. With more data one could determine precisely the variation with redshift of the shape of $\Phi(L_\alpha)$.

The above description of the luminosity function allows us to determine the rate of energy generation per unit comoving volume of quasars as a function of redshift. We show that this function $\mathcal{L}(z) \propto \rho(z)g(z)$ increases rapidly with $z$ at low redshift, peaks around $z \approx 2$, and then decreases. This is also in rough agreement with the high-$z$ survey results mentioned above. This variation of $\mathcal{L}(z)$ is similar to but significantly different from recent determinations of the SFR, whose behavior at high redshifts is still controversial (Hughes et al. 1998).

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