Free Fock parafermions in the tight-binding model with dissipation

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Parafermions that generalize (Majorana or usual) fermions appear as interacting quasi-particles because of their nature. Although attempts to develop models with free (non-interacting) parafermions have been undertaken, existing proposals require unphysical conditions such as realizing purely non-Hermitian systems. Here we present a way for the realization of free Fock parafermions in the tight-binding model with controlled dissipation of a very simple form. Introducing dissipation transforms an originally non-integrable quantum model to an exactly solvable classical one.

I. INTRODUCTION

The combination of fundamental importance and potential applications puts analyzing quantum systems hosting exotic quasiparticle excitations to the forefront of ongoing research. Specifically, anyons that exhibit richer exchange statistics have received a significant attention [1–26]. Potential applications of non-Abelian anyons are mainly related to the usage of their braiding statistics for topologically protected quantum information processing [27]. Finding an experimentally relevant setup to host such exotic quasiparticles is a challenge. One of the options is to use systems, in which non-Abelian anyons arise from Majorana zero modes bound to vortices [28–32], such as a fractional quantum Hall state at 5/2 filling [33], unconventional $p + ip$ superconductors [34–38], and semiconductor wires [39–43]. However, it is impossible to realize the universal set of quantum gates in such cases since they cannot be approximated by braiding operations [43–45]. Several proposals to overcome this difficulty have been suggested [46, 47], but realistic ones require non-topological operations killing the main feature of such systems — their immunity from decoherence [27].

An important step is to go beyond existing models to the ones having more complex braiding statistics and allowing to manipulate quantum states more efficiently. We note that in addition to solid-state systems, programmable quantum simulators can be used for studying exotic quasiparticles. In particular, Rydberg quantum simulator allows realizing Ising-type models [48, 49] and more complex $Z_n$ lattice models related to the $n$-state Potts or $Z_n$ chiral clock model. Recently, Rydberg ensembles have been studied in the context of mesonic and baryonic quasiparticle excitations in the $Z_3$ case [50], and more intriguing features could appear in the $Z_n$ case. We note that lattice models based on a discrete group $Z_n$ were discovered in Ref. [51]. Critical properties of these models are described by a family of minimal models of 2D conformal field theories which possesses current fields with non-integer spins, called parafermions in Ref. [52].

Over the past decade parafermions have been a subject of intensive research [53–56] in particular from the perspective of quantum computing [57]. Specifically, there is a significant interest to Fock parafermions (FPF), which are anyonic quasi-particles that generalize usual identical fermions in a simple particle-like picture [58] (standard parafermions generalize Majorana fermions). Various properties of standard and Fock parafermions [60–63] and schemes for their realization [44, 55, 59, 64–78] have been considered. The simplest setup for Fock parafermions is a one-dimensional (1D) tight-binding model, which is however shown to be interacting and non-integrable [79]. Numerical simulations based on exact diagonalization and on the density-matrix renormalization group have predicted bound states in the spectrum due to its peculiar anyonic properties [79]. Although considerable efforts have been made in searching for free (i.e. non-interacting) parafermionic models [54], existing proposal require implementing purely non-Hermitian systems. For instance, this is the case with the Baxter non-Hermitian Hamiltonian [80], which possesses a free-particle complex spectrum. Many properties of the Baxter Hamiltonian have been discussed in Ref. [54]. It should be noted that in the case of standard parafermions, free particle behaviour may take place in the the low-energy sector of some models [52, 78]. However, the question of the very existence of physically relevant models hosting free FPF remains open.

Here we positively answer to this question by demonstrating a 1D model with free FPF with dissipation. Including dissipation of a very simple form transform a non-integrable model of Fock parafermion to an exactly-solvable one by natural canceling the Hermitian conjugate part of the Baxter Hamiltonian. We expect that experimental realization of the suggested model should be possible using solid-state realization and programmable quantum simulators.

II. FOCK PARAFERMIONS

Let us make a brief overview of Fock parafermions and some of their properties before discussing the model. FPFs are anyonic quasi-particles that generalize usual
identical fermions. First of all, the difference is in the dimensionality of the Fock space. Whereas for fermions the local Hilbert space (say, on a lattice site) is two dimensional, for \( \mathbb{Z}_n \) FPFs it is \( n \)-dimensional. Thus, introducing the FPF creation and annihilation operators, \( F_j^\dagger \) and \( F_j \) correspondingly, one has

\[
(F_j^\dagger)^m |0\rangle = |m_j\rangle, \quad 0 \leq m \leq n - 1, \tag{1}
\]

where \( |0\rangle \) is a vacuum, \( |m_j\rangle \) is a Fock state with \( m \) FPFs on the \( j \)-th site and nothing on the rest of the chain. In other words, one can have up to \( n \) FPFs on each site. For a general Fock state, the action of \( F_j^\dagger \) and \( F_j \) is a bit more involved [58]:

\[
F_j^\dagger |m_j\rangle = \omega^{-\sum_{k=1}^j m_k} |m_j + 1\rangle, \quad F_j |m_j\rangle = \omega\sum_{k=1}^j m_k |m_j - 1\rangle, \tag{2}
\]

where \( \omega = e^{\pi i/n} \), FPF creation and annihilation operators satisfy the following relations:

\[
(F_j^\dagger)^n F_j = 0, \quad (F_j^\dagger)^m F_j^m + F_j F_j^m (F_j^\dagger)^{n-m} = 1, \tag{3}
\]

with \( 1 \leq m \leq n - 1 \). As an immediate consequence of Eq. (3), one has \( F_j^m (F_j^\dagger)^m F_j = F_j^m \). For operators acting on different sites, one has

\[
F_j^\dagger F_k = \omega^{\text{sgn}(k-j)} F_k F_j^\dagger, \quad F_j^\dagger F_k = \omega^{-\text{sgn}(k-j)} F_k F_j^\dagger, \tag{4}
\]

where \( \text{sgn}(x) \) is the sign function. Thus, Eqs. (3) and (4) allow one to bring any monomial in FPFs to the normal order with all creation operators being to the left from all annihilation operators. Further, one introduces the FPF number operator

\[
N_j = \sum_{m=1}^{n-1} (F_j^\dagger)^m F_j^m, \tag{5}
\]

which acts on the Fock states as expected:

\[
N_j |m_j\rangle = m_j |m_j\rangle, \tag{6}
\]

and satisfies the usual commutation relations

\[
[N_j, F_j] = -F_j, \quad [N_j, F_j^\dagger] = F_j^\dagger. \tag{7}
\]

One can easily check that Eqs. (3)–(5) reduce to the standard fermionic relations for \( n = 2 \). On the other hand, for \( n > 2 \), from Eq. (4) one clearly sees that FPFs are anyons with the statistical parameter \( 2/n \). In this case, the factor of \( \omega^{\text{sgn}(k-j)} \) in Eq. (4) comes into play, which makes a crucial difference between FPFs and fermions and has far-reaching consequences. For instance, it makes FPFs intrinsically interacting and the Fourier components of \( F_j \) defined as

\[
F_k = \frac{1}{\sqrt{L}} \sum_{j=1}^L e^{ijk} F_j, \tag{8}
\]

are not FPFs themselves, as they do not satisfy the relations (4). As a result, even if some quadratic FPF Hamiltonian can be written in the momentum space as \( H = \sum_k \varepsilon_k F_k^\dagger F_k \), it is not diagonal since different momentum modes are coupled. Indeed, it is easy to check that the commutator \([F_k^\dagger F_k, F_q^\dagger F_q]\) is not zero [79].

Just like the usual spinless fermions are related to the spin-1/2 Pauli operators via the Jordan-Wigner transformation, FPFs can be obtained from the generalized \( \mathbb{Z}_n \) Pauli operators \( X_j \) and \( Z_j \). The latter are unitary and satisfy

\[
X_j^n = Z_j^n = 1, \quad X_j^\dagger = X_j^{-1}, \quad Z_j^\dagger = Z_j^{-1}, \tag{9}
\]

along with the following relations:

\[
X_j^k Z_j^l = \omega^{kl} Z_j^l X_j^k, \quad X_j Z_k = Z_k X_j \quad (j \neq k). \tag{10}
\]

The operators \( X_j \) and \( Z_j \) act non-trivially on \( j \)-th site and their matrix representations are \( X_j = 1 \otimes \ldots \otimes X \otimes 1 \ldots \) and \( Z_j = 1 \otimes \ldots \otimes Z \otimes \ldots \), where \( X \) and \( Z \) are \( n \times n \) matrices with the following elements: \( Z_{p,q} = \delta_{p,q} \omega^{p-1} \) and \( X_{p,q} = \delta_{p,q} \omega^{p-1} \). Then, using a generalization of the Jordan-Wigner transformation, the so-called Fradkin-Kadanoff transformation [81], for the FPF annihilation operator one obtains

\[
F_j = \left( \prod_{k=1}^{j-1} Z_k \right) \Sigma_j^-, \tag{11}
\]

where \( \Sigma_j^- \) is the lowering operator. In terms of \( X_j \) and \( Z_j \) the latter is given by

\[
\Sigma_j^- = \frac{n-1}{n} X_j - \frac{1}{n} X_j \sum_{m=1}^{n-1} Z_m^m. \tag{12}
\]

Its matrix representation is \( \Sigma_j^- = 1 \otimes \ldots 1 \otimes \Sigma_j^- \otimes 1 \ldots \), with the elements of \( \Sigma_j^- \) being \( \delta_{p,q} \omega^{p-1} \).

For completeness, we briefly comment on the notion of standard parafermions. We stress once more that Fock parafermions generalize usual complex fermions, which are related to real Majorana fermions by a linear transformation. In turn, Majorana fermions are generalized by parafermions \( \gamma_j \). The latter satisfy the generalized Clifford algebra relations

\[
\gamma_j^n = 1, \quad \gamma_j^\dagger = \gamma_j^{-1}, \quad \gamma_j \gamma_k = \omega^{\text{sgn}(k-j)} \gamma_k \gamma_j, \tag{13}
\]

and can be written in terms of \( \mathbb{Z}_n \) matrices as

\[
\gamma_{2j-1} = \prod_{k=1}^{j-1} Z_k X_j, \quad \gamma_{2j} = \omega^{(n-1)/2} \gamma_{2j-1} Z_j. \tag{14}
\]

The relation between Fock parafermions \( F_j \) and parafermions \( \gamma_j \) can be easily obtained from Eqs. (11)–(14), and it reads as

\[
F_j = \frac{n-1}{n} \gamma_{2j-1} - \frac{1}{n} \sum_{m=1}^{n-1} \omega^{m/2} (\gamma_{2j-1})^\dagger \gamma_{2j}. \tag{15}
\]
were we took into account that $Z^m_j = \omega^{n^2/2} \gamma_{2j+1}^j \gamma_{2j}^j$, as follows from Eq. (14). Let us also mention that one can construct FPF coherent states (eigenstates of the annihilation operator $F_j$) using an appropriate generalization of Grassmann variables, see e.g. Ref. [82].

III. THE MODEL

We study the FPF tight-binding model on a chain of $L$ sites with open boundary conditions. Since one can have up to $n - 1$ FPFs on each lattice site, we allow hopping processes involving $1 \leq m \leq n - 1$ particles, generalizing a usual single-particle hopping. The Hamiltonian reads

$$H = -\sum_{m=1}^{n-1} \sum_{j=1}^{L-1} \left( \alpha_m (F_j^\dagger)^m F_{j+1}^m + \text{H.c.} \right) - \sum_{m=1}^{n-1} \mu_m \sum_{j=1}^{L} (F_j^m)^m F_j^m,$$

with the Hermitian part $H$ given by Eq. (16). We then take the jump operators of the following form:

$$L_j^{(m)} = F_j^m + \Delta_m F_{j+1}^m, \quad 1 \leq m \leq n - 1,$$

where $\Delta_m \neq 0$ is a complex parameter. The jump operator (19) is (quasi)local and acts on a single link between two adjacent sites [84]. We choose the values of $\Delta_m$ that satisfy

$$\alpha_m = i \gamma_m \Delta_m / 2.$$

Then, Eqs. (16) and (18) yield

$$H_{\text{eff}} = -\sum_{m=1}^{n-1} \left\{ i \gamma_m \Delta_m \sum_{j=1}^{L-1} (F_j^m)^m F_{j+1}^m + \frac{\gamma_m}{2} \sum_{j=1}^{L-1} \left( (F_j^m)^m F_j^m + |\Delta_m|^2 (F_{j+1}^m)^m F_{j+1}^m \right) \right\} + \mu_m \sum_{j=1}^{L} (F_j^m)^m F_j^m.$$

Note that the effective Hamiltonian (21) has a very simple structure. Indeed, taking into account Eq. (11) we have $(F_j^m)^m F_{j+1}^m = (\Sigma_j^+)^m Z_j^m (\Sigma_j^-)^m$ and $(F_j^m)^m F_{j+1}^m = (\Sigma_j^+)^m (\Sigma_j^-)^m$, with $\Sigma_j^+ = (\Sigma_j^-)^\dagger$ being the raising operator. Then, using the expressions for the matrix elements of $Z_j$ and $\Sigma_j^-$ written after Eqs. (10) and (12) correspondingly, we see that $H_{\text{eff}}$ has a lower-triangular structure in the Fock basis. Moreover, the term on the first line of Eq. (21) does not have diagonal matrix elements, whereas the remaining terms [the second and the third lines in Eq. (21)] are diagonal and contribute to the eigenvalues. Therefore, the spectrum of $H_{\text{eff}}$ can be readily obtained and it reads

$$H_{\text{eff}} |m_1, \ldots, m_L\rangle = E_{m_1, \ldots, m_L} |m_1, \ldots, m_L\rangle,$$

where $|m_1, \ldots, m_L\rangle$ with $0 \leq m_j \leq n - 1$ form the basis in the $n^L$-dimensional Fock space and the eigenvalues are given by

$$E_{m_1, \ldots, m_L} = -\sum_{j=1}^{L} \sum_{p=1}^{m_j} \left( \mu_p + \frac{i \Gamma_p(p)}{2} \right),$$

where $\Gamma_p(p) = \gamma_p$, $\Gamma_p(0) = \gamma_p |\Delta_p|^2$, and $\Gamma_p(\Gamma_p(\Gamma_p + 1))$ for $2 \leq j \leq L - 1$. As expected, the spectrum (23) is complex, except for the ground state, since $H_{\text{eff}}$ is neither Hermitian nor PT-symmetric [85]. Remarkably, we see that Eq. (23) is nothing else than the spectrum of free Fock parafermions, since it is a linear combination of single-particle energy levels. Moreover, as we show below, the spectrum and the eigenstates of the full Liouvillian possess the same free-particle structure.
IV. LIOUVILLIAN SPECTRUM AND FREE FOCK PARAFERMIIONS

The non-Hermitian Hamiltonian $H_{\text{eff}}$ from Eq. (18) is used routinely in the so-called quantum trajectories technique, which numerically solves a stochastic differential equation for a wavefunction, rather than for a density matrix [86–89]. The quantum trajectories technique is quite successful in obtaining steady state averages of local observables. However, this method provides a rather poor approximation of an open system described by the full Liouvillian $\mathcal{L}$, if one is interested in e.g. the spectral properties. Interestingly, there are some exceptional cases in which $H_{\text{eff}}$ does provide a complete description of a dissipative quantum system. For instance, this situation occurs if the following conditions are met: (i) there is an observable $Q$ that commutes with the Hamiltonian, $[H, Q]=0$; (ii) all jump operators $L_j$ correspond to either pure loss or pure gain, and (iii) jump operators satisfy $[L_i, Q] \propto L_j$. In this case the Liouvillian spectrum is given by $\lambda_{m,n}=-i(E_n-E_m)$, where $E_m$ belong to the energy spectrum of the non-Hermitian Hamiltonian (18). In the same way the Liouvillian eigenstates are constructed from the eigenstates of $H_{\text{eff}}$ [90]. Recently, an exact solution for the dissipative one-dimensional Hubbard model was obtained with the help of the outlined approach [91]. Importantly, it is also applicable in our case. Indeed, the Hamiltonian (16) possesses the $U(1)$ symmetry and conserves the number of particles since $[H, \mathcal{N}]=0$, where $\mathcal{N} = \sum_{j=1}^{L} N_j$ is the total number operator. Further, the jump operator $L_j$ from Eq. (19) clearly describes pure losses. Finally, taking into account Eq. (7), we see that $[\mathcal{N}, L_j]=-L_j$. Therefore, for the Liouvillian $\mathcal{L}$ from Eq. (17), with the Hamiltonian (16) and the jump operator (19), we immediately find the spectrum that reads

$$\lambda_{m,p}=-i\,(E_m-E_p),$$

(24)

where $E_m = E_{m_1}, \ldots, m_L$ is given by Eq. (23). Similarly, one can obtain the Liouvillian eigenstates from the eigenstates of $H_{\text{eff}}$ [90].

Let us discuss the spectrum (24) in more detail. First of all, taking into account Eq. (23) we see that there is always a unique zero eigenvalue $\lambda_{0,0}=0$. For $\gamma_1 \neq 0$, the remaining Liouvillian eigenvalues have a finite negative real part, which means that the steady state of Eq. (17) is a vacuum. However, the situation drastically changes if in Eq. (16) we take $\alpha_1 = 0$ and in Eq. (21) put $\gamma_1 = 0$. In other words, we forbid the single-particle hopping and losses. In this case it follows from Eqs. (23) and (24) that the states with no more than one particle per lattice site form a decoherence-free subspace consisting of $2^L - 1$ “dark states”.

V. CONCLUSIONS

To summarize, we have found a simple open quantum system that hosts free (non-interacting) Fock parafermions. Starting from the non-integrable FPF tight-binding model (16) and including a Markovian dissipation described by the jump operator of a very simple form (19), we have shown that for a specific values of the system parameters the Liouvillian spectrum has a free-particle structure and is given by Eqs. (23) and (24). In contrast to all previously known free $\mathbb{Z}_n$ models, we had accurately and consistently treated the dissipation by working with the Lindblad equation (17). Thus, our results provide a realistic physical system that hosts exotic and long-sought free Fock parafermions. We expect that our predictions can be probed on the basis of currently available experimental facilities using solid-state systems or programmable quantum simulators.

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