Harnack inequalities for positive solutions of the heat equation on closed Finsler manifolds

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Abstract

The main goal of this paper is to generalize some Li-Yau type gradient estimates to Finsler geometry in order to derive Harnack type inequalities. Moreover, we obtain, under some curvature assumption, a general gradient estimate for positive solutions of the heat equation when the manifold evolving along the Finsler Ricci flow.

Keywords: Gradient estimates, Harnack inequalities, Finsler Ricci flow, Heat flow.

Mathematics Subject Classification (2010): 53C60 · 35K05 · 53C44

1 Introduction

A very active research topic in geometric analysis is the study of the heat equation on manifolds because of its several applications in physics and natural sciences. In their famous paper [11], P. Li and S-T. Yau have proved the following

**Theorem A.** Let $M$ be an $n$-dimensional complete Riemannian manifold with Ricci curvature bounded from below by $-k$ for some nonnegative constant $k$ and let $u \in C^\infty([0,T] \times M)$ be a positive solution of the heat equation

$$\Delta u - \partial_t u = 0.$$ (1.1)

Then, for any $\alpha > 1$, it holds

$$|\nabla f|^2 - \alpha \partial_t f \leq \frac{n\alpha}{t} + \frac{n\alpha^2 k}{2(\alpha - 1)},$$ (1.2)

where $f = \log u$. Particularly, when $k = 0$, letting $\alpha \to 1$, one obtains

$$|\nabla f|^2 - \partial_t f \leq \frac{n\alpha}{t}.$$ (1.3)

It is worth to mention that the equation (1.2) is not sharp unless $k = 0$. An important application of this estimate is that it provides Harnack type inequalities. There is a rich literature about improvement and generalization of the Li-Yau gradient estimate (1.2), see for example [5, 13, 14, 2] and the references therein. These results do not only concern the heat equation on Riemannian manifolds but also general linear and semi-linear parabolic equations on Kaehler manifolds or Alexandrov spaces. Despite these important efforts done, the question of sharpness of the Li-Yau gradient estimate remain open.

Recently, a general gradient estimate for positive solutions of the heat equation on closed Riemannian manifold which extends many existing estimates was proved [6]. The first purpose of this paper is to generalize this result to Finsler geometry in order to derive Harnack type inequalities. Finsler manifolds are natural generalization of Riemannian manifolds in the sense that each tangent space is endowed with a Minkowskian norm instead of the Euclidean one. Observe that, in this paper, we use the nonlinear Shen’s Finsler Laplacian which is an extension among others of the Laplace-Beltrami operator to Finsler geometry. Because of the lack of linearity, method used in Riemannian setting does not work. To overcome this difficulty, we use the weighted gradient and Laplacian which are linear operators (see Section 2 for the definitions). Our first result is stated as follows.

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Theorem 1.1. Let $(M, F)$ be a closed Finsler manifold of dimension $n$, equipped with a smooth measure $d\mu$, such that the weighted Ricci curvature satisfies $\text{Ric}_N \geq K$ for some $N \in [n, \infty)$ and $K \in \mathbb{R}$ and take a positive global solution $u : [0, T] \times M \to \mathbb{R}$ to the heat equation. Let us consider the functions $\lambda, \beta, \Psi \in C^1((0, T])$ satisfying

\begin{align*}
(B_1) & \quad \beta(t) \in (0, 1), \forall t \in (0, T]; \\
(B_2) & \quad \lim_{t \to 0^+} \lambda(t) = 0 \text{ and } \lambda(t) > 0, \forall t \in (0, T]; \\
(B_3) & \quad \frac{\beta' - 2K^-\beta}{1 - \beta} = (\ln \lambda)' > 0 \text{ on } (0, T]; \\
(B_4) & \quad \limsup_{t \to 0^+} \psi \geq 0; \\
(B_5) & \quad \Psi' + \frac{\beta' - 2K^-\beta}{1 - \beta}\Psi - \frac{N(\beta' - 2K^-\beta)^2}{8\beta(1 - \beta)^2} \geq 0 \text{ for any } t \in (0, T],
\end{align*}

where $K^- := \min\{K, 0\}$. Then, for any $x, y \in M$ and $0 < s < t \leq T$, we have

$$u(s, x) \leq u(t, y) \exp \left\{ \frac{d_F(y, x)^2}{4(t-s)^2} \int_s^t \frac{dt}{\beta} + \int_s^t \Psi \, dr \right\},$$

(1.4)

where $d_F$ is the distance function induced by the Finsler metric $F$.

The related definitions such as weighted Ricci curvature and the Finsler distance function are given in Section 2 below. Recall that Ohta and Sturm [13] have already proved some Harnack type inequalities in Finsler setting. Theorem 1.1 can be viewed as a generalisation of their results.

The Li-Yau type gradient estimates are also investigated on manifolds evolving along geometric flows. The case of the Ricci flow is well known and was initiated by Hamilton in [7] on Riemannian manifolds; see [10] and references therein. In [3], Bao has introduced the notion of Finsler Ricci flow and recently, Lakzian [5] has derived Harnack estimates for positive solutions to the heat equation under this flow. This paper gives also a generalization of Lakzian result.

Theorem 1.2. Let $(M, F)$ be a closed Finsler manifold of dimension $n$ endowed with a smooth measure $d\mu$. Let $(F_t)_{t \in [0, T]}$ be a solution of the Ricci flow on $M$ with $F_0 = F$. Assume that there exists some positive constants $K_i, i = 1, 2, 3, 4$, such that for all $t \in [0, T]$, $F_t$ has isotropic $S$-curvature, $S = \sigma F_t + d\varphi$ for some time dependent functions $\sigma$ and $\varphi$ with $-K_3 F_t \leq d\sigma$ and $-K_4 F_t^2 \leq \text{Hess} \varphi$, and its Ricci curvature satisfies $-K_1 \leq \text{Ric}_{ij} \leq K_2$.

Let $u : [0, T] \times M \to \mathbb{R}$ be a positive solution of the heat equation under the Finsler Ricci flow $(F_t)$. Let $\beta, \lambda \in C^1((0, T])$ satisfying

\begin{align*}
(C_1) & \quad \beta(t) \in (0, 1), \forall t \in (0, T]; \\
(C_2) & \quad \lim_{t \to 0^+} \lambda(t) = 0 \text{ and } \lambda(t) > 0, \forall t \in (0, T]; \\
(C_3) & \quad \frac{\beta' - 2K^-\beta}{1 - \beta} = (\ln \lambda)' < 0 \text{ on } (0, T];
\end{align*}

Then we have, for any $t \in (0, T]$,

$$\beta F(\nabla f)^2 - \partial_t f \leq \frac{n}{2\beta} \left( (\ln \lambda)' - \frac{2\beta'}{1 - \beta} \right) + \frac{n(C_1 + \beta')}{2\beta(1 - \beta)} + \frac{n^{3/2}\sqrt{C_2}}{\beta} + \sqrt{2nC_3},$$

(1.5)

where $C_1 := K_1, C_2 := \max\{K_1^2, K_2^2\}$ and $C_3 = K_3 + K_4$.

We obtain an improvement of [3] Theorem 1.1 with a suitable choice of functions $\beta$ and $\lambda$.

The paper is organised as follows. In section 2, we review some facts about Finsler geometry which we need for the sequel. Thereafter, we prove Li-Yau gradient estimate in section 3. That allows us to show the Theorem 1.1. In the last section, we deal with gradient estimate in time dependent Finsler manifolds where we prove Theorem 1.2.
2 Preliminaries

In this section, we briefly recall some basic concepts of Finsler geometry necessary for further discussions. We refer to [12, 3] and references therein for more details.

Let $M$ be an $n$-dimensional smooth manifold and $\pi : TM \to M$ be the natural projection from the tangent bundle $TM$. A point $(x, y) \in TM$ is such that $y = y'\frac{\partial}{\partial y'}$ in the local coordinates $(x^i, y^i)$ on $TM$. Let $F : TM \to [0, \infty)$ be a Finsler metric on $M$, that is $F$ satisfies

(i) Regularity: $F$ is smooth on $TM \setminus \{0\}$;
(ii) Positive homogeneity: $F(x, \lambda y) = \lambda F(x, y), \forall \lambda > 0$;
(iii) Strong convexity: The fundamental quadratic form

$$g = g_{ij}(x, y) \, dx^i \otimes dx^j; \quad g_{ij}(x, y) := \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}(x, y),$$

is positive definite for any $(x, y) \in TM$ with $y \neq 0$.

For $x, y \in M$, we define the distance from $x$ to $y$ by

$$d_F(x, y) := \inf_{\gamma} \int_0^1 F(\gamma(t), \dot{\gamma}(t)) \, dt,$$

where the infimum is taken over all $C^1$-curves $\gamma : [0, 1] \to M$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Since the Finsler metric is only positively homogeneous, the distance function may be asymmetric. We call geodesic any $C^\infty$-curve $\gamma$ on $M$ which is locally minimising and has constant speed. We also define the exponential map by

$$\exp_x(v) := \gamma(1),$$

where $\gamma : [0, 1] \to M$ is a geodesic with $\dot{\gamma}(0) = v \in T_x M$.

Let $V = v^i \frac{\partial}{\partial x^i}$ be a nonzero vector on an open subset $U \subset M$. Through equation (2.1), one defines a Riemannian metric

$$g_V \left( X^i \frac{\partial}{\partial x^i}, Y^j \frac{\partial}{\partial x^j} \right) := g_{ij}(V)X^iY^j,$$

and a covariant derivative by

$$D^V_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} := \Gamma^k_{ij}(x, V) \frac{\partial}{\partial x^k},$$

where $\Gamma^k_{ij}(x, V)$ are coefficients of the Chern connection.

The flag curvature of the plane spanned by two linearly independent vectors $V, W \in T_x M \setminus \{0\}$ is defined by

$$K(V, W) := \frac{g_V(R^V(W, V)W, V)}{g_V(W, V)g_V(W, W) - g_V(V, V)^2},$$

where $R^V$ is the Chern curvature as follows

$$R^V(X, Y)Z := D_X^V D_Y^V Z - D_Y^V D_X^V Z - D_{[X, Y]}^V Z.$$

Then, the Ricci curvature of $(M, F)$ is given by

$$\text{Ric}(V) := \sum_{i=1}^{n-1} K(V, e_i),$$

where $\{e_1, \ldots, e_{n-1}, e_n := \frac{V}{F(V)}\}$ is an orthonormal basis of $T_x M$ with respect to $g_V$ and the Ricci tensor is defined as follows:

$$\text{Ric}_{ij} := \frac{1}{2} \frac{\partial^2 (F^2 \text{Ric})}{\partial y^i \partial y^j}.$$

Now, let us consider an arbitrary volume form $d\mu = \sigma(x) dx$ on $M$. The distortion of $(M, F, d\mu)$ is defined, for any $y \in T_x M \setminus \{0\}$, by

$$\tau(x, y) := \ln \left( \frac{\sqrt{\det(g_{ij}(x, y))}}{\sigma(x)} \right).$$
Let $y \in T_x M \setminus \{0\}$ and $\gamma : (-\epsilon, \epsilon) \rightarrow \mathbb{R}$ be the geodesic with $\gamma(0) = x$ and $\gamma'(0) = y$. The $S$-curvature defined as

$$S(x, y) := \frac{d}{dt} \left[ r(y(t)) \right]_{t=0},$$

measures the rate of changes of the distortion along geodesics. One also defines

$$\hat{S}(y) := \frac{1}{F(x, y)^2} \frac{d}{dt} \left[ S(y(t), \gamma'(t)) \right]_{t=0}.$$

The weighted Ricci curvature of the Finsler manifold $(M, F, d\mu)$ is defined as follows

$$Ric_n(y) := \begin{cases} \text{Ric}(y) + \hat{S}(y), & \text{for } S(y) = 0, \\ -\infty, & \text{otherwise}, \end{cases}$$

$$Ric_N(y) := \text{Ric}(y) + \hat{S}(y) - \frac{S(y)^2}{(N-n)F(y)^2}, \forall N \in (n, \infty)$$

$$Ric_\infty(y) := \text{Ric}(y) + \hat{S}(y).$$

In analogy with the Ricci flow in Riemannian setting, Bao [3] proposed the following definition of the Finsler Ricci flow:

$$\frac{\partial}{\partial t} g_{ij} = -2Ric_{ij}, \quad g|_{t=0} = g_0,$$  \hspace{1cm} (2.4)

which contracting to $y^i y^j$, via Euler theorem, provides

$$\frac{\partial}{\partial t} \log F = -\text{Ric}, \quad F|_{t=0} = F_0,$$ \hspace{1cm} (2.5)

where $F_0$ is the initial Finsler structure. Short time existence and uniqueness of the Finsler Ricci flow has been studied in some particular cases, see e.g [1, 3].

For their convenient use, let us recall the definitions of the gradient, the Hessian and the Laplacian operators on $(M, F, d\mu)$. Let $u$ be a differentiable function and $V$ a smooth vector field on $M$. We set $M_u := \{ x \in M; du(x) \neq 0 \}$ and $M_V := \{ x \in M; V(x) \neq 0 \}$. The divergence of $V := V^i \frac{\partial}{\partial x^i}$ with respect to the volume form $d\mu := \sigma(x) dx$ is defined by

$$\text{div}(V) := \sum_{i=1}^n \left( \frac{\partial V^i}{\partial x^i} + \frac{V^i}{\sigma} \frac{\partial \sigma}{\partial x^i} \right).$$

Let $\mathcal{L}^* : T^* M \rightarrow TM$ be the Legendre transform which assigns to each $\alpha \in T^*_x M$ the unique element $v \in T_x M$ such that $\alpha(v) = F^*(x)^2$ and $F(v) = F^*(\alpha)$, where $F^*$ stands for the dual norm of $F$. Then the gradient vector and the Laplacian of $u$ are given by

$$\nabla u(x) = \mathcal{L}^*(du(x)), \quad \Delta u(x) := \text{div}(\nabla u).$$

If $V$ does not vanish on $M_u$, one can also define the weighted gradient vector and the weighted Laplacian of $u$ on the Riemannian manifolds $(M, g\nu)$ as

$$\nabla^V u := \begin{cases} g_{ij}(V) \frac{\partial u}{\partial x^i} \frac{\partial}{\partial x^j}, & \text{on } M_V, \\ 0, & \text{on } M \setminus M_V, \end{cases}, \quad \Delta^V u := \text{div}(\nabla^V u).$$

In particular, on $M_u$, we have

$$\nabla u = \nabla^V u, \quad \Delta u = \Delta^V u.$$

In other words, the Laplacian of $u$ can be expressed using the $S$-curvature as

$$\Delta u = \text{tr}_{g\nu^*}(\nabla^2 u) - S(\nabla u),$$ \hspace{1cm} (2.6)

where $\nabla^2 u := D^V u \nabla u$ is the Hessian of $u$. Here the trace is taken with respect to an $g\nu^*$-orthonormal basis. The point-wise Finslerian version of the Bochner-Weitzenbock formulas are given by

$$\Delta^V u \left( \frac{F^2(\nabla u)}{2} \right) - d(\Delta u)(\nabla u) \geq \text{Ric}_N(\nabla u) + \frac{(\Delta u)^2}{N},$$ \hspace{1cm} (2.7)
$$\Delta \nabla u \left( \frac{F^2(\nabla u)}{2} \right) - d(\Delta u)(\nabla u) = Ric(\nabla u) + \|\nabla^2 u\|^2_{HS(\nabla u)}. \quad (2.8)$$

We conclude this section by recalling some notions about the heat equation \( \partial_t u = \Delta u \) associated with the Finsler Laplacian. We consider a Finsler space \((M, F, d\mu)\) where \((M, F)\) is a Finsler space and \(d\mu\) a smooth measure on \(M\).

A function \( u : [0, T] \times M \to \mathbb{R} \) is said to be a global solution of the heat equation \( \partial_t u = \Delta u \) if it satisfies the following:

(i) \( u \in L^2([0, T], H^1_0(M)) \cap H^1([0, T], H^{-1}(M)) \),

(ii) for all \( t \in [0, T] \) and \( \phi \in C_c^\infty(M) \),

$$\int_M \phi \partial_t u_t \ d\mu = -\int_M d\phi(\nabla u_t) \ d\mu,$$

where \( u_t := u(t, \cdot) \).

3 Gradients estimates on static Finsler manifolds

This section deals with the proof of Theorem [13]. Let \((M, F, d\mu)\) be a closed \( n \)-dimensional Finsler manifold equipped with a smooth measure \(d\mu\). Assume that its weighted Ricci curvature \( Ric_N \geq K \) for some \( N \in [n, \infty) \) and \( K \in \mathbb{R} \).

Let us consider a global positive solution \( u : [0, T] \times M \to \mathbb{R} \) to the Finsler heat equation on \((M, F, d\mu)\). We will fix a measurable one-parameter family of non-vanishing vector fields \( \{V_t\}_{t \in [0, T]} \) on \(M\) such that \( V_t = \nabla u_t \). Here, \( u_t(x) := u(t, x), (t, x) \in [0, T] \times M \).

For any \((t, x) \in [0, T] \times M\), we set \( f(t, x) := f_t(x) = \ln u(t, x) \). One can easily check that (see [13] Equation 4.2),

$$\partial_t (F(\nabla f)^2) = 2d(\partial_t f)(\nabla f), \quad (3.1)$$

and for every \( t \in [0, T] \), we have in distributional sense,

$$\Delta f + F(\nabla f)^2 = \partial_t f. \quad (3.2)$$

Suppose further that there exist some functions \( \lambda, \beta, \Psi \in C^1((0, T]) \) such that

(B1) \( \beta(t) \in (0, 1), \forall t \in (0, T]; \)

(B2) \( \lim_{t \to 0^+} \lambda(t) = 0 \) and \( \lambda(t) > 0, \forall t \in (0, T]; \)

(B3) \( \frac{\beta' - 2K^{-}\beta}{1 - \beta} - (\ln \lambda)' > 0 \) on \( (0, T]; \)

(B4) \( \limsup_{t \to 0^+} \psi \geq 0; \)

(B5) \( \Psi' + \frac{\beta' - 2K^{-}\beta}{1 - \beta} \Psi - \frac{N(\beta' - 2K^{-}\beta)^2}{8\beta(1 - \beta)^2} \geq 0 \) for any \( t \in (0, T], \)

where \( K^{-} := \min\{K, 0\} \).

Let us consider the function \( G \) defined on \([0, T] \times M\) by \( G = \beta F(\nabla f)^2 - \partial_t f - \Psi \). We claim the following

Lemma 3.1. \( G \) satisfies

$$\Delta^V G + 2dG(\nabla f) - \partial_t G \geq \frac{\beta' - 2K^{-}\beta}{1 - \beta} G, \quad (3.3)$$

in the sense of distribution on \((0, T).\)
Proof. For all $\phi \in H^1_0((0,T) \times M)$, we have
\[
\int_0^T \int_M \left\{ -d\phi (\nabla V(\partial_t f)) + 2\phi d(\partial_t f)(\nabla f) - \phi \partial_t (\nabla f) \right\} \, d\mu dt
\]
\[
= \int_0^T \int_M \left\{ -d\phi (\nabla V(\partial_t f)) + \phi |2d(\partial_t f)|(\nabla f) - \partial_t (F(\nabla f)^2) - \partial_t (\Delta f) \right\} \, d\mu dt
\]
\[
= \int_0^T \int_M \left\{ -d\phi (\nabla V(\partial_t f)) - \phi \Delta (\partial_t f) \right\} \, d\mu dt
\]
\[
= 0,
\]
where we used equations (3.2) and (3.1).

Furthermore, using again (3.2), (3.1) and the Bochner-Weitzenbock inequality, we have for any nonnegative test function $\phi \in H^1_0((0,T) \times M)$,
\[
\int_0^T \int_M \left\{ -d\phi (\nabla V(F(\nabla f)^2)) + 2\phi d(F(\nabla f)^2)(\nabla f) - \phi \partial_t (F(\nabla f)^2) \right\} \, d\mu dt
\]
\[
= \int_0^T \int_M \left\{ -d\phi (\nabla V(F(\nabla f)^2)) - 2\phi \partial_t (\Delta f)(\nabla f) + \phi (2d(\partial_t f)(\nabla f) - \partial_t (F(\nabla f)^2)) \right\} \, d\mu dt
\]
\[
= \int_0^T \int_M \left\{ -d\phi (\nabla V(F(\nabla f)^2)) - 2\phi \partial_t (\Delta f)(\nabla f) \right\} \, d\mu dt
\]
\[
\geq \int_0^T \int_M 2\phi \{Ric_N(\nabla f) + \frac{(\Delta f)^2}{N} \} \, d\mu dt \geq \int_0^T \int_M 2\phi \{KF(\nabla f)^2 + \frac{(\Delta f)^2}{N} \} \, d\mu dt
\]
\[
\geq \int_0^T \int_M 2\phi \{K^{-1}F(\nabla f)^2 + \frac{(\Delta f)^2}{N} \} \, d\mu dt
\]
Therefore, for every nonnegative $\phi \in H^1_0((0,T) \times M)$,
\[
\int_0^T \int_M \left\{ -d\phi (\nabla V G + 2\phi dG(\nabla f) - \phi \partial_t G) \right\} \, d\mu dt
\]
\[
\geq \int_0^T \int_M \left\{ 2\beta \left( K^{-1}F(\nabla f)^2 + \frac{(\Delta f)^2}{N} \right) - \beta F(\nabla f)^2 + \Psi \right\} \, d\mu dt
\]
\[
= \int_0^T \int_M \phi \left\{ \frac{2\beta}{N}(\Delta f)^2 + \frac{\beta - 2K^{-1}}{1 - \beta} (\Delta f + G + \Psi) + \Psi \right\} \, d\mu dt
\]
\[
\geq \int_0^T \int_M \phi \left\{ \frac{\beta - 2K^{-1}}{1 - \beta} G + \Psi + \frac{\beta - 2K^{-1}}{1 - \beta} \Psi - \frac{N(\beta - 2K^{-1})^2}{8\beta(1 - \beta)^2} \right\} \, d\mu dt
\]
\[
\geq \int_0^T \int_M \frac{\beta - 2K^{-1}}{1 - \beta} \phi G \, d\mu dt
\]
which complete the proof of (3.3).

Now, let us define the function $H(t,x) = \lambda(t)G(t,x)$. From (3.3), we can easily deduce that the following equation
\[
\Delta^V H + 2dH(\nabla f) - \partial_t H \geq \left( \frac{\beta - 2K^{-1}}{1 - \beta} - (\ln \lambda)'' \right) H,
\]
holds in the distributional sense on $(0,T)$.

Consider a point $(t_0,x_0)$ at which $H$ attains its maximum on $[0,T] \times M$.

Lemma 3.2. We have $H(t_0,x_0) \leq 0$.

Proof. Assume by contradiction that $H(t_0,x_0) > 0$. From conditions $(B_1)$, $(B_2)$ and $(B_4)$, we can deduce that $\lim_{t \to 0^+} H \leq 0$ and thus $t_0 > 0$. On other hand, by $(B_3)$,
\[
\left( \frac{\beta'(t_0) - 2K^{-1} \beta(t_0)}{1 - \beta(t_0)} - (\ln \lambda)'(t_0) \right) H(t_0,x_0) > 0
\]
(3.5)
on some neighborhood of \((t_0, x_0)\). Therefore, \(H\) is a strict subsolution of the linear parabolic operator
\[
\text{div}_\mu(\nabla^V H) + 2dH(\nabla f) - \partial_t H,
\]
on such a neighborhood.

This implies that \(H(t_0, x_0)\) is strictly less than the supremum of \(H\) on the boundary of any small parabolic cylinder \([t_0 - \delta, t_0] \times B_\delta(x_0)\), where \(B_\delta(x_0) = \{x \in M : d_F(x_0, x) < \delta\}\). Hence \((t_0, x_0)\) cannot be the maximum point of \(H\).

Hence, we have proved the following gradient estimate:

**Theorem 3.3.** Let \((M, F, d\mu)\) be a closed Finsler space of dimension \(n\) with weighted Ricci curvature satisfying \(\text{Ric}_N \geq K\) for some \(N \in [n, \infty)\) and \(K \in \mathbb{R}\). Let \(u : [0, T] \times M \to \mathbb{R}\) be a positive solution of the associated heat equation. If there are functions \(\lambda, \beta, \Psi \in C^1((0, T])\) satisfying assumptions \((B_1) - (B_3)\), then for any \(t \in (0, T)\), we have
\[
\beta F(\nabla f)^2 - \partial_t f \leq \psi,
\]
where \(f := \ln u\).

Following [6], we obtain this immediate consequence of Theorem 3.3:

**Corollary 3.4.** Let \((M, F, d\mu)\) and \(u\) be as in Theorem 3.3 with \(K < 0\). Let \(b \in C^1((0, T])\) be a positive increasing function on \((0, T]\) such that
\[
\lim_{t \to 0^+} b(t) = 0 \text{ and } \frac{b^2}{b} \in L^1([0, T]).
\]
Then,
\[
\beta F(\nabla f) - \partial_t f \leq \Psi,
\]
where
\[
\beta = 1 + \frac{2K}{b(t)e^{-2Kr}} \int_0^t b(s)e^{-2Ks} \, ds,
\]
and
\[
\Psi = \frac{n}{2b} \int_0^t \frac{b^2(s)}{b\beta}(s) \, ds.
\]

**Remark 3.5.** Taking \(b(t) = (1 - \theta Kt)^{\frac{n}{2} - 1}\) and \(b(t) = \sinh^2(-Kt) + \cosh(-Kt)\sinh(-Kt) + Kt\), one obtains respectively
\[
F(\nabla f)^2 - (1 - \theta Kt)\partial_t f \leq \frac{n(2 - \theta)^2}{16\theta(1 - \theta)t} + \frac{nK^2\theta t}{4} - \frac{nK}{2},
\]
and
\[
F(\nabla f)^2 - \left(1 + \frac{\sinh(-Kt)\cosh(-Kt) - Kt}{\sinh^2(-Kt)}\right) \partial_t f \leq \frac{nK}{2}(\coth(-Kt) + 1).
\]

We can now derive our Harnack type inequality.

**Proposition 3.6.** Under assumptions of Theorem 3.3 we have, for any \(x, y \in M\) and \(0 < s < t \leq T\),
\[
u(s, x) \leq u(t, y) \exp \left\{ \frac{d_F(y, x)^2}{4(t-s)^2} \int_s^t \frac{d\tau}{\beta} + \int_s^t \Psi \, d\tau \right\},
\]
where \(d_F\) is the distance function induced by the Finsler metric \(F\).

**Proof.** Consider the reverse curve \(\eta : [s, t] \ni \tau \mapsto \eta(\tau) := \exp_B((t - \tau)v) \in M\) of the minimal geodesic joining \(y = \eta(t)\) to \(x = \eta(s)\) where \(v \in T_y M\) is a suitable vector. For any \(\tau \in [s, t],\)
we have $F(-\hat{\eta}(\tau)) = d_F(y, x)/(t - s)$. Let $f = \ln u$ and $\sigma(\tau) = -f(\tau, \eta(\tau))$. Then, according to Theorem 3.3 we have

$$\sigma(t) - \sigma(s) = \int_s^t \dot{\sigma}(\tau) \, d\tau$$

$$= \int_s^t \{df(-\hat{\eta}) - \partial_t f\} \, d\tau$$

$$\leq \int_s^t \{F(\nabla f)F(-\hat{\eta}) + \Psi - \beta F(\nabla f)^2\} \, d\tau$$

$$= \int_s^t \{F(\nabla f)\frac{d_F(y, x)}{t - s} + \Psi - \beta F(\nabla f)^2\} \, d\tau$$

$$\leq \int_s^t \left\{\frac{1}{4\beta} \frac{d_F(y, x)^2}{(t - s)^2} + \Psi\right\} \, d\tau$$

where we used in the last line the fact that the polynomial $P(x) = -\beta x^2 + \frac{d_F(y, x)}{t - s} x + \Psi$ attains its maximum at $x_0 = \frac{d_F(y, x)}{2\beta(t - s)}$. □

4 Gradient estimates with time dependant Finsler metrics

In this section, we consider a smooth manifold evolving along the Finsler Ricci flow. Our aim is to prove a general gradient estimate for positive solutions of the heat equation under the Ricci flow, that is functions $u \in L^2([0, T], H^1(M)) \cap H^1([0, T], H^{-1}(M))$ satisfying

$$\int_M \phi \partial_t u_1 \, d\mu = - \int_M d\phi(\nabla u_1) \, d\mu,$$

where the gradient is with respect to the Finsler metric $F_t$. Let us observe that, here we didn’t investigated about existence and regularity of such solutions.

Let $(M, F)$ be a closed $n$-dimensional Finsler manifold endowed with a smooth measure $d\mu$. Let $(F_t)_{t \in [0, T]}$ be a solution of the Ricci flow on $M$ with $F_0 = F$. Let us consider a positive solution $u : [0, T] \times M \to \mathbb{R}$ of the heat equation under the Ricci flow, that is $u \in L^2([0, T], H^1(M)) \cap H^1([0, T], H^{-1}(M))$ and satisfies

$$\int_M \phi \partial_t u_1 \, d\mu = - \int_M d\phi(\nabla u_1) \, d\mu,$$

where the gradient is taken with respect to the Finsler metric $F_t$ and let $f = \ln u$.

We have the following evolution equation

**Lemma 4.1.** [8] Along the Finsler Ricci flow $(F_t)_{t \in [0, T]}$, it holds

$$\partial_t (F(\nabla f)^2) = 2d(\partial_t f)(\nabla f) + 2Ric^{ij}(\nabla f) f_i f_j. \quad (4.1)$$

Let $\beta, \lambda \in C^1((0, T])$ be three functions satisfying the following assumptions:

(C1) $\beta(t) \in (0, 1), \forall t \in (0, T]$;

(C2) $\lim_{t \to 0^+} \lambda(t) = 0$ and $\lambda(t) > 0, \forall t \in (0, T]$;

(C3) $\frac{2\beta'}{1 - \beta} - (\ln \lambda)' < 0$ on $(0, T]$.

On $[0, T] \times M$, define $G := \beta F(\nabla f)^2 - \partial_t f$ and $H := \lambda G$.

**Lemma 4.2.** It holds

$$\Delta^V H + 2dH(\nabla f) - \partial_t H = \left(\frac{\beta'}{1 - \beta} - (\ln \lambda)\right) H + \lambda$$

in the sense of distribution on $(0, T) \times M$, where

$$I(t, x) := \frac{\beta'}{1 - \beta} \Delta f + 2(1 - \beta)Ric^{ij}(\nabla f) f_i f_j + 2\beta(\nabla f) + \frac{\dot{S}(\nabla f) + \|\nabla^2 f\|^2}{2} + 2Ric^{ij}(\nabla f) f_i f_j \quad (4.3)$$
Proof. We have
\[
\partial_t(d\phi(\nabla f)) = d(\partial_t\phi)(\nabla f) + d\phi(\partial_t(L^*(df)))
\]
\[
= d(\partial_t\phi)(\nabla f) + d\phi(L^*(d(\partial_t f))) + d\phi((\partial_t L^*)(\nabla f))
\]
\[
= d(\partial_t\phi)(\nabla f) + d\phi(\nabla(\partial_t f)) + 2g^{jk} Ric^j_k(\nabla f)\phi_i f_j
\]
\[
= d(\partial_t\phi)(\nabla f) + d\phi(\nabla(\partial_t f)) + 2Ric^j(\nabla f)\phi_i f_j
\]
(4.4)

Using this relation, we compute
\[
\int_T^0 \int_M \{ -d\phi(\nabla^V(\partial_t f)) + 2\phi d(\partial_t f)(\nabla f) - \phi \partial_t(\partial_t f) \} \, d\mu dt
\]
\[
= \int_T^0 \int_M \{ -d\phi(\nabla^V(\partial_t f)) + \phi [2d(\partial_t f)(\nabla f) - \partial_t(F(\nabla f)^2)] - \partial_t(\Delta f) \} \, d\mu dt
\]
\[
= \int_T^0 \int_M \{ -d\phi(\nabla^V(\partial_t f)) - \phi [2Ric^j(\nabla f)\phi_i f_j + \partial_t(\Delta f)] \} \, d\mu dt
\]
\[
= \int_T^0 \int_M \{ -\partial_t(d\phi(\nabla f)) + d(\partial_t\phi)(\nabla f) + 2Ric^j(\nabla f)\phi_i f_j
\]
\[
- \phi [2Ric^j(\nabla f)\phi_i f_j + \partial_t(\Delta f)] \} \, d\mu dt
\]
\[
= \int_T^0 \int_M \{ 2Ric^j(\nabla f)\phi_i f_j - 2\phi Ric^j(\nabla f)\phi_i f_j \} \, d\mu dt
\]
\[
= \int_T^0 \int_M -2\phi (Ric^j(\nabla f)\phi_i f_j + Ric^j(\nabla f)\phi_i f_j) \, d\mu dt,
\]
(4.5)

and
\[
\int_T^0 \int_M \{ -d\phi(\nabla^V(F(\nabla f)^2)) + \phi [2d(F(\nabla f)^2)(\nabla f) - \partial_t(F(\nabla f)^2)] \} \, d\mu dt
\]
\[
= \int_T^0 \int_M \{ -d\phi(\nabla^V(F(\nabla f)^2)) + 2\phi [d(F(\nabla f)^2)(\nabla f) - d(\partial_t f)(\nabla f) - Ric^j f_i f_j] \} \, d\mu dt
\]
\[
= \int_T^0 \int_M \{ -d\phi(\nabla^V(F(\nabla f)^2)) - 2\phi [d(\Delta f)(\nabla f) + Ric^j f_i f_j] \} \, d\mu dt.
\]

From the Bochner-Weitzenbock formula, we have
\[
\int_T^0 \int_M \{ -d\phi(\nabla^V(F(\nabla f)^2)) + \phi [2d(F(\nabla f)^2)(\nabla f) - \partial_t(F(\nabla f)^2)] \} \, d\mu dt
\]
\[
= \int_T^0 \int_M 2\phi \{ Ric(\nabla f) + \tilde{S}(\nabla f) + \|\nabla^2 f\|_{L^2}^2 - Ric^j(\nabla f) f_i f_j \} \, d\mu dt.
\]
(4.6)

Hence
\[
\int_T^0 \int_M \{ -d\phi(\nabla^V G) + 2\phi dG(\nabla f) - \phi \partial_t(G) \} \, d\mu dt
\]
\[
= \int_T^0 \int_M \phi \left\{ -\beta' F(\nabla f)^2 + 2\beta \left( Ric(\nabla f) + \tilde{S}(\nabla f) + \|\nabla^2 f\|_{L^2}^2 - Ric^j(\nabla f) f_i f_j \right) \right\}
\]
\[
+ 2 \left( Ric^j(\nabla f) f_i f_j + Ric^j(\nabla f) f_i f_j \right) \, d\mu dt
\]
\[
= \int_T^0 \int_M \phi \left\{ \frac{\beta'}{1 - \beta}(G + \Delta f) + 2(1 - \beta)Ric^j(\nabla f) f_i f_j
\]
\[
+ 2\beta (Ric(\nabla f) + \tilde{S}(\nabla f) + \|\nabla^2 f\|_{L^2}^2) + 2Ric^j(\nabla f) f_i f_j \right\} \, d\mu dt.
\]
(4.7)
Lemma 4.3. The function $t$ satisfies Young’s inequality, we have
\[ t \leq 1 \] Proof. Where we set $C = \{ \} $ Choose a normal coordinate system $(g, T)$ on $\mathbb{R}^n$, then $\Gamma \in \mathbb{R}^n$, $\Gamma \in \mathbb{T}_xM$, for all $x \in T_xM \setminus \{0\}$, Therefore, one can easily check the following
\[ \sum_{i,j=1}^{n} (Ric^{ij})^2 + \beta \sum_{i,j=1}^{n} f_{ij}^2 \right) = 1/n(\Delta f)^2 \] \[ I(t, x) = \frac{\beta}{1-\beta} \Delta f + 2(1-\beta)Ric^{ij}(\nabla f) f_{ij} + 2\beta(Ric(\nabla f) + S(\nabla f) + \|\nabla^2 f\|)^2 + 2Ric^{ij}(\nabla f) f_{ij} \] $\geq \frac{\beta}{1-\beta} \Delta f + 2(1-\beta)Ric^{ij}(\nabla f) f_{ij} + 2\beta(Ric(\nabla f) + S(\nabla f) + \|\nabla^2 f\|)^2 + 2Ric^{ij}(\nabla f) f_{ij} \] \[ \geq \frac{\beta}{1-\beta} \Delta f - 2(1-\beta)Ric^{ij}(\nabla f) f_{ij} + 2\beta(Ric(\nabla f) + S(\nabla f) + \|\nabla^2 f\|)^2 + 2Ric^{ij}(\nabla f) f_{ij} \] $\geq \frac{\beta}{1-\beta} \Delta f - 2(1-\beta)Ric^{ij}(\nabla f) f_{ij} + 2\beta(Ric(\nabla f) + S(\nabla f) + \|\nabla^2 f\|)^2 + 2Ric^{ij}(\nabla f) f_{ij} \] $\geq \frac{\beta}{1-\beta} \Delta f - 2\beta C_1 + 2\beta C_2 - 2\beta C_3 + \frac{1}{\beta} \sum_{i,j=1}^{n} (Ric^{ij})^2 \] where we set $C_1 = K_1, C_2 = \max\{K_1^2, K_2^2\}$ and $C_3 = K_3 + K_4$. \]
Combining (1.2) and (4.11), we obtain
\[
\Delta^V H + 2dH(\nabla f) - \partial_t H \geq \left( \frac{\beta'}{1 - \beta} - (\ln \lambda)' \right) H + \lambda J,
\]
in the distributional sense on \((0, T) \times M\), where
\[
J(t, x) := \frac{\beta}{n} \left( F(\nabla f)^2 - \partial_t f \right)^2 + \frac{\beta'}{1 - \beta} \left( F(\nabla f)^2 - \partial_t f \right) - 2C_1 F(\nabla f)^2 - \frac{n^2 C_2}{\beta} - 2\beta C_3.
\]
Let \((t_0, x_0)\) be the maximizer of \(H\) on \([0, T) \times M\). Without lost of generality, we can assume \(H(t_0, x_0) > 0\), since, otherwise Theorem 1.2 is trivially satisfied. From condition \((C_2)\), we have \(t_0 > 0\). Then, at \((t_0, x_0)\) we have necessary
\[
\left( \frac{\beta'}{1 - \beta} - (\ln \lambda)' \right) H + \lambda J \leq 0.
\]
Indeed, if we assume the contrary, by an argument analogue to the proof of Theorem 3.3 one can show that \(H(t_0, x_0)\) could not be the supremum of \(H\), that is a contradiction.

At \((t_0, x_0)\), we have
\[
H \leq \lambda \left[ \frac{n}{2\beta} \left( (\ln \lambda)' - \frac{2\beta'}{1 - \beta} \right) + \frac{n(C_1 + \beta')}{2\beta(1 - \beta)} + \frac{n^{3/2} \sqrt{C_2}}{\beta} + \sqrt{2nC_3} \right].
\]
Indeed, let \(w := F(\nabla f)^2\) and \(z := \partial_t f\). We have
\[
(w - z)^2 = (\beta w - z)^2 + 2(1 - \beta)w(\beta w - z) + (1 - \beta)^2 w^2
\]
\[
= \frac{H^2}{\lambda^2} + 2(1 - \beta)wH - (1 - \beta)^2 w^2.
\]
Then,
\[
J = \frac{\beta}{n} (w - z)^2 + \frac{\beta'}{1 - \beta} (w - z) - 2C_1 w - \frac{n^2 C_2}{\beta} - 2\beta C_3
\]
\[
= \frac{\beta}{n} \left( \frac{H^2}{\lambda^2} + 2(1 - \beta)wH - (1 - \beta)^2 w^2 \right) + \frac{\beta'}{1 - \beta} \frac{H}{\lambda} - \beta' w - 2C_1 w - \frac{n^2 C_2}{\beta} - 2\beta C_3
\]
\[
= \frac{\beta H^2}{n \lambda^2} + 2 \frac{\beta}{n(1 - \beta)w} \frac{H}{\lambda} + \frac{\beta'}{1 - \beta} \frac{H}{\lambda} + \frac{\beta}{n(1 - \beta)^2 w^2} - 2C_1 w - \frac{n^2 C_2}{\beta} - 2\beta C_3
\]
\[
\geq \frac{\beta H^2}{n \lambda^2} + \frac{\beta'}{1 - \beta} \frac{H}{\lambda} - \frac{n(2C_1 + \beta')}{4\beta(1 - \beta)^2} - \frac{n^2 C_2}{\beta} - 2\beta C_3,
\]
where used have been made of \(ax^2 + bx \geq \frac{b^2}{4a}, \forall x \in \mathbb{R}\) for \(a > 0\), in the last line.

Replacing this estimate in (4.16) yields at \((t_0, x_0)\),
\[
\frac{\beta}{n} H^2 + \lambda \left( \frac{\beta'}{1 - \beta} - (\ln \lambda)' \right) H - \lambda^2 \left( \frac{n^2 C_2}{4\beta(1 - \beta)^2} - \frac{n^2 C_2}{\beta} - 2\beta C_3 \right) \leq 0.
\]
Remark that the left hand side of (4.18) is a quadratic polynomial \(P(x) = ax^2 + bx + c\) in \(H\) with \(a > 0, b < 0\) and \(c < 0\). Hence, using \(x \leq \frac{b + \sqrt{b^2 - 4ac}}{2a} \leq \frac{\beta + \sqrt{\beta^2 - 4\beta^2 C_3}}{4\beta(1 - \beta)^2}\), one obtains (4.17).

This completes the proof of Theorem 1.2 which we state again here

**Theorem 4.4.** Let \((M, F, d\mu)\) be a closed Finsler space and \((F_t)_{t \in [0, T)} \) be a solution to the Finsler Ricci flow with \(F_0 = F\). Assume that there exists some positive constants \(K_i, i = 1, 2, 3, 4\), such that for all \(t \in [0, T)\), \(F_t\) has isotropic \(S\)-curvature, \(S = \sigma F_t + d\varphi\) for some time dependent functions \(\sigma\) and \(\varphi\) with \(-K_3 F_t \leq d\varphi \) and \(-K_4 F_t^2 \leq \text{Hess} \varphi\), and its Ricci curvature satisfies \(-K_1 \leq \text{Ric} \leq K_2\). If \(u : [0, T) \times M \to \mathbb{R}\) is a positive solution of the heat equation under the Finsler Ricci flow \((F_t)\) and \(\beta, \lambda \in C^1([0, T))\) are functions satisfying assumptions \((C_1) - (C_4)\) then for any \(t \in (0, T]\), we have,
\[
\beta F(\nabla f)^2 - \partial_t f \leq \frac{n}{2\beta} \left( (\ln \lambda)' - \frac{2\beta'}{1 - \beta} \right) + \frac{n(C_1 + \beta')}{2\beta(1 - \beta)} + \frac{n^{3/2} \sqrt{C_2}}{\beta} + \sqrt{2nC_3},
\]
where \(C_1 := K_1, C_2 := \max\{K_1^2, K_2^2\}\) and \(C_3 = K_3 + K_4\) and \(f = \ln u\).
Remark 4.5. When all Finsler metrics $F_t$, $t \in [0,T]$, have vanishing $S$-curvature, we have
\[
\beta F(\nabla f)^2 - \partial_t f \leq \frac{n}{2\beta} \left( (\ln \lambda)' - \frac{2\beta'}{1 - \beta} \right) + \frac{n(C_1 + \beta')}{2\beta(1 - \beta)} + \frac{n^{3/2} C_2}{\beta}, \tag{4.20}
\]
Particularly, taking $\beta = \frac{1}{\theta}$ as constant function ($\theta > 1$) and $\lambda(t) = t$ in (4.20), one obtains
\[
F(\nabla f)^2 - \theta \partial_t f \leq \frac{n\theta^2}{2t} + \frac{nC_1\theta^3}{2(\theta - 1)} + n^{3/2}\theta^2 \sqrt{C_2}.
\]
This estimate improves Lakzian result [8, Theorem 1.1].

Corollary 4.6. Let $(F_t)_{[0,t]}$ be a Finsler Ricci flow on a closed $n$-dimensional manifold $M$ such that each Finsler metric $F_t$ has Ricci curvature satisfying $0 \leq \text{Ric}_{ij} \leq C$ for some positive constant $C$ and isotropic $S$-curvature $S_t = \sigma F_t + d\varphi$ where $\sigma$ and $\varphi$ satisfy assumption (4.10).

Let $b \in C^1((0,T])$ be a positive increasing function such that $\lim_{t \to 0^+} b(t) = 0$. Then a positive solution $u$ of the heat equation under the Ricci flow satisfies
\[
F(\nabla f)^2 - (1 + b)\partial_t f \leq n(1 + b)^2 \left( \frac{b'}{b} + C\sqrt{n} + \frac{\sqrt{2nC_3}}{n(1 + b)} \right), \tag{4.21}
\]
where $f = \ln u$.

Particularly, if all Finsler metrics $F_t$ have vanishing $S$-curvature, then we have
\[
F(\nabla f)^2 - (1 + b)\partial_t f \leq n(1 + b)^2 \left( \frac{b'}{b} + C\sqrt{n} \right) \tag{4.22}
\]
Proof. The result follows by choosing $\beta = \frac{1}{\theta t}$ and $\lambda = b$ in Theorem 4.4. Here $C_1 = 0$ and $C_2 = C^2$. \qed

Remark 4.7. Some examples of functions $b$ satisfying assumptions of Corollary 4.6 are $b(t) = \theta t$ with $\theta > 0$ and $b(t) = \sinh(t)$.

Proposition 4.8. Under hypothesis of Theorem 4.4, we have, for any $x, y \in M$ and $0 < s < t \leq T$,
\[
u(s,x) \leq u(t,y) \exp \left\{ A(s,t) + B(s,x,t,y) + (t - s)\sqrt{2nC_3} \right\}, \tag{4.23}
\]
where
\[
A(s,t) = (t - s) \int_0^1 \left\{ \frac{n}{2\beta} \left( (\ln \lambda)' - \frac{2\beta'}{1 - \beta} \right) + \frac{n(C_1 + \beta')}{2\beta(1 - \beta)} + \frac{n^{3/2} \sqrt{C_2}}{\beta} \right\} d\tau,
\]
and
\[
B(s,x,t,y) = \inf \left\{ c \int_0^1 \frac{F_x(c(\tau), \dot{c}(\tau))^2}{\beta(\tau)} \right\},
\]
where the infimum is taken over all the smooth paths $c : [0,1] \to M$ satisfying $c(0) = y$ and $c(1) = x$.

Proof. Let $c : [0,1] \to M$ be a smooth curve with $x = c(1)$ and $y = c(0)$. Let $f = \ln u$ and define $l(\tau) = f(\tau, c(\tau))$ for $\tau \in [0,1]$ with $\tau := (1 - \tau)t + \tau s$. Then, we have
\[
\frac{\partial l(\tau)}{\partial \tau} = (t - s) \left( \frac{dF_x(c(\tau), \dot{c}(\tau))}{t - s} - \partial_t f(\tau, c(\tau)) \right)
\leq (t - s) \left( \frac{F_x(c(\tau), \nabla f) F_x(c(\tau), \dot{c}(\tau))}{t - s} - \partial_t f(\tau, c(\tau)) \right)
\leq (t - s) \left( \frac{1}{2\beta(\tau)} F_x(c(\tau), \dot{c}(\tau))^2 + \frac{\beta(\tau)}{2} \times 2 F_x(c(\tau), \nabla f)^2 - \partial_t f(\tau, c(\tau)) \right)
\leq \frac{F_x(c(\tau), \dot{c}(\tau))^2}{4\beta(\tau)(t - s)}
\ + (t - s) \left\{ \frac{n}{2\beta} \left( (\ln \lambda)' - \frac{2\beta'}{1 - \beta} \right) + \frac{n(C_1 + \beta')}{2\beta(1 - \beta)} + \frac{n^{3/2} \sqrt{C_2}}{\beta} + \sqrt{2nC_3} \right\}
\]
Integration of this inequality yields
\[
\ln \frac{u(s,x)}{u(t,y)} = l(1) - l(0) = \int_0^1 \frac{\partial l(\tau)}{\partial \tau} d\tau
\]
\[
\leq (t-s)\sqrt{2nC_3} + \frac{1}{4(t-s)} \int_0^1 \frac{F_\beta(\varepsilon(\tau),\dot{\varepsilon}(\tau))^2}{\beta(\tau)} d\tau
\]
\[
+ (t-s) \int_0^1 \left\{ \frac{n}{2\beta} \left( \ln \lambda \right)' - \frac{2\beta'}{1 - \beta} + \frac{n(C_1 + \beta')}{2\beta(1 - \beta)} + \frac{n^{3/2} \sqrt{C_2}}{\beta} \right\} d\tau
\]
Exponentiating this inequality gives immediately the required estimate.

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