ON ONE CLASS OF FRACTAL SETS

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Abstract. In the present article a new class Υ of all sets represented in the following form is introduced:

\[ S(s,u) \equiv \left\{ x : x = \Delta_{\alpha_1-1}^{u_1} \alpha_1 u_2 \cdots \alpha_n \in A_0, \alpha_n \neq u, \alpha_n \neq 0 \right\}, \]

where \( 2 < s \in \mathbb{N} \) and \( u \in A \) are fixed parameters, \( A = \{0, 1, \ldots, s - 1\} \), and \( A_0 = A \backslash \{0\} \). Topological, metric, and fractal properties of these sets are studied. The problem of belonging to these sets of normal numbers is investigated. The theorem on the calculation of the Hausdorff-Besicovitch dimension of an arbitrary set whose elements have restrictions on using of digits or combinations of digits in own \( s \)-adic representations is formulated and proved.

1. Introduction

A fractal in the wide sense is a set whose the topological dimension and the Hausdorff-Besicovitch dimension are different, and in the narrow sense is a set that has the fractional Hausdorff-Besicovitch dimension. In 1977, the notion “fractal” was introduced by B. Mandelbrot in [3] but such sets one can to see in earlier mathematics researches. For example, the Cantor set

\[ C[3, \{0, 2\}] = \{ x : x = \Delta_{\alpha_1}^{3} \alpha_2 \cdots \alpha_n, \alpha_n \in \{0, 2\} \} \]

is a fractal. Fractals are applied in modeling of biological and physical processes, and the notion of “fractals” unites various mathematical objects such as continuous nowhere monotonic functions, nowhere differentiable functions, singular distributions, etc. [1, 2].

In the present article a new class of certain fractal sets whose elements have a functional restriction on using of symbols in own \( s \)-adic representations is introduced, and topological, metric, and fractal properties of such sets are studied. In 2012, the results of this article were presented by the author in the abstracts [4, 5, 6].

An expansion of a real number \( x_0 \in [0; 1] \) in the form

\[ x_0 = \frac{\alpha_1}{s} + \frac{\alpha_2}{s^2} + \cdots + \frac{\alpha_n}{s^n} + \ldots, \quad (1) \]

2010 Mathematics Subject Classification. 28A80, 11K55, 26A09.

Key words and phrases. Fractals, Hausdorff-Besicovitch dimension, self-similar sets, \( s \)-adic representation, normal numbers.
is called the \textit{s-adic expansion of }$x_0$. Here $1 < s$ is a fixed positive integer, $\alpha_n \in A = \{0, 1, \ldots, s - 1\}$. By $x_0 = \Delta_s^{\alpha_1 \alpha_2 \ldots \alpha_n}$ denote the s-adic expansion of $x_0$. The notation $\Delta_s^{\alpha_1 \alpha_2 \ldots \alpha_n}$ is called the \textit{s-adic representation of }$x_0$.

A number whose the s-adic representation is periodic with the period (0) is called an \textit{s-adic-rational number}. Any s-adic-rational number has two different s-adic representations, i.e.,

$$\Delta_s^{\alpha_1 \alpha_2 \ldots \alpha_n - 1 \alpha_n (0)} = \Delta_s^{\alpha_1 \alpha_2 \ldots \alpha_n - 1 \alpha_n - 1 (s - 1)}.$$

The other numbers in $[0; 1]$ are called \textit{s-adic-irrational numbers}. These numbers have the unique s-adic representation.

### 2. The Object of Research

Let $2 < s$ be a fixed positive integer, $A = \{0, 1, \ldots, s - 1\}$ be an alphabet of the s-adic number system, $A_0 = A \setminus \{0\} = \{1, 2, \ldots, s - 1\}$, and

$$L \equiv (A_0)^\infty = (A_0) \times (A_0) \times \ldots$$

be the space of one-sided sequences of elements of $A_0$.

Consider a new class $\Upsilon_s$ of sets $S_{(s, u)}$ represented in the form

$$S_{(s, u)} \equiv \left\{ x : x = \Delta_s^{u \ldots \alpha_n \alpha_{n-1} \ldots \alpha_1}, (\alpha_1, \ldots, \alpha_n) \in L, \alpha_n \neq u, \alpha_n \neq 0 \right\}, \quad (2)$$

where $u = 0, s - 1$, $u$ and $s$ are fixed for the set $S_{(s, u)}$. That is the class $\Upsilon_s$ contains the sets $S_{(s, 0)}, S_{(s, 1)}, \ldots, S_{(s, s - 1)}$. We say that $\Upsilon$ is a class of sets such that contains the classes $\Upsilon_3, \Upsilon_4, \ldots, \Upsilon_n, \ldots$.

It is easy to see that the set $S_{(s, u)}$ can be defined by the s-adic expansion in the following form

$$S_{(s, u)} \equiv \left\{ x : x = \frac{u}{s - 1} + \sum_{n=1}^{\infty} \frac{\alpha_n - u}{s^{\alpha_1 + \ldots + \alpha_n}}, (\alpha_1, \ldots, \alpha_n) \in L, \alpha_n \neq u, \alpha_n \neq 0 \right\}. \quad (3)$$

### 3. Topological and Metric Properties of $S_{(s, 0)}$

By $x_0 = \Delta_s^{c_1 c_2 \ldots}$ denote the following equality

$$S_{(s, 0)} \ni x_0 = \frac{c_1}{s^{c_1}} + \frac{c_2}{s^{c_1 + c_2}} + \frac{c_3}{s^{c_1 + c_2 + c_3}} + \ldots + \frac{c_n}{s^{c_1 + c_2 + \ldots + c_n}} + \ldots.$$

It follows from the definition of the set $S_{(s, 0)}$ that s-adic-rational numbers do not belong to $S_{(s, 0)}$. Hence each element of $S_{(s, 0)}$ has the unique s-adic representation.

**Lemma 1.** The set $S_{(s, 0)}$ is an uncountable set.

**Proof.** Let the mapping $f : S_{(s, 0)} \to C[s, A_0]$, where $C[s, A_0] = \{ x : x = \Delta_s^{\alpha_1 \alpha_2 \ldots \alpha_n}, \alpha_n \in A_0 \}$, be given by

$$\forall (\alpha_n) \in L : x = \sum_{n=1}^{\infty} \frac{\alpha_n}{s^{\alpha_1 + \ldots + \alpha_n}} \to \sum_{n=1}^{\infty} \frac{\alpha_n}{s^n} = y = f(x), \quad (4)$$
i.e.,
\[ x = \Delta_{c_1 \cdots c_n}^s(0_{e_1-1}0_{e_2-1} \cdots 0_{e_n-1}) \rightarrow \Delta_{c_1 \cdots c_n}^s \cdot y = f(x). \] (5)

From (5) it follows that (4) is a bijection. Since \( C[s, A_0] \) is a uncountable set, we see that \( S_{(s, 0)} \) is a uncountable set. \( \square \)

To investigate topological and metric properties of \( S_{(s, 0)} \), we shall study properties of cylinders.

**Definition 1.** A cylinder \( \Delta_{c_1 \cdots c_n}' \) of rank \( n \) with base \( c_1 \cdots c_n \) is a set formed by all numbers of \( S_{(s, 0)} \) with \( s \)-adic representations in which the first \( n \) non-zero digits are fixed and coincide with \( c_1, c_2, \ldots, c_n \), respectively.

**Lemma 2.** Cylinders have the following properties:

1. \( \inf \Delta_{c_1 \cdots c_n}' = \Delta_{c_1 \cdots c_n}'(s-1) = g_n + \frac{s-1}{s^{c_1 + c_2 + \cdots + c_n}(s^s - 1)} \),
2. \( \sup \Delta_{c_1 \cdots c_n}' = \Delta_{c_1 \cdots c_n}'(1) = g_n + \frac{1}{s^{c_1 + c_2 + \cdots + c_n}(s-1)} \),
   where \( g_n = \sum_{k=1}^{n} \frac{c_k}{s^{c_1 + c_2 + \cdots + c_n}} \).
3. \( \Delta_{c_1 \cdots c_n}' \subseteq I_{c_1 \cdots c_n} \),
   where endpoints of the segment \( I_{c_1 \cdots c_n} \) coincide with endpoints of \( \Delta_{c_1 \cdots c_n}' \),
   i.e.,
   \[ I_{c_1 \cdots c_n} = \left[ g_n + \frac{s-1}{(s^s - 1)s^{c_1 + c_2 + \cdots + c_n}}, g_n + \frac{1}{(s-1)s^{c_1 + c_2 + \cdots + c_n}} \right]. \] (6)
4. \( \Delta_{c_1 \cdots c_n}' \subseteq \bigcup_{i=1}^{s-1} \Delta_{c_1 \cdots c_n}' \), \( \forall c_n \in A_0, n \in \mathbb{N} \).
5. Suppose that \( d(\cdot) \) is the diameter of a set. Then
   \[ d(\Delta_{c_1 \cdots c_n}') = \frac{s^{s-1} - 1 - (s-1)^2}{(s-1)(s^{s-1} - 1)s^{c_1 + c_2 + \cdots + c_n}}. \]
6. \[ \frac{d(\Delta_{c_1 \cdots c_n}')}{d(\Delta_{c_1 \cdots c_n}')} = \frac{1}{s^{c_n+1}}. \]
(7) The following inequality
\[
\inf \Delta'_{c_1c_2\ldots c_{n-1}p} > \sup \Delta'_{c_1c_2\ldots c_{n-1}[p+1]}
\]
holds for all \( p \in N_{s-2} = \{1, 2, \ldots, s-2\} \).

(8) An arbitrary interval of the form
\[
T_{c_1\ldots c_{n-1}p} = (\sup \Delta'_{c_1\ldots c_{n-1}[p+1]}; \inf \Delta'_{c_1\ldots c_{n-1}p}) = (b_{p+1}; a_p),
\]
where
\[
b_{p+1} = \frac{p + 1}{s^{c_1+c_2+\ldots+c_{n-1}+p+1}} + \frac{1}{(s-1)s^{c_1+c_2+\ldots+c_{n-1}+p+1}} + \sum_{k=1}^{n-1} \frac{c_k}{s^{c_1+c_2+\ldots+c_k}}
\]
and
\[
a_p = \frac{p}{s^{c_1+c_2+\ldots+c_{n-1}+p}} + \frac{s - 1}{(s-1)s^{c_1+c_2+\ldots+c_{n-1}+p}} + \sum_{k=1}^{n-1} \frac{c_k}{s^{c_1+c_2+\ldots+c_k}},
\]
satisfies the condition \( T_{c_1\ldots c_{n-1}p} \cap S_{(s,0)} = \emptyset \).

(9) \( \Delta'_{c_1c_2\ldots c_{n-1}p} \cap \Delta'_{c_1c_2\ldots c_{n-1}[p+1]} = \emptyset \) \( \forall p \in N_{s-2} \).

(10) If \( x_0 \in S_{(s,0)} \), then
\[
x_0 = \bigcap_{n=1}^{\infty} \Delta'_{c_1c_2\ldots c_n}.
\]

Proof. Prove that the first property is true. Since \( \frac{a}{s^k} > \frac{b}{s^k} \) holds for \( a < b \), we have
\[
\inf \Delta'_{c_1c_2\ldots c_n} = \frac{1}{s^{c_1+c_2+\ldots+c_n}} \left( \frac{s-1}{s^{c_1+c_2+\ldots+c_n}} + \frac{s-1}{s^{2(s-1)+1}} + \ldots \right) + \sum_{k=1}^{n} \frac{c_k}{s^{c_1+c_2+\ldots+c_k}} =
\]
\[
= \left( \sum_{k=1}^{n} \frac{c_k}{s^{c_1+c_2+\ldots+c_k}} \right) + \frac{s-1}{s^{c_1+c_2+\ldots+c_n}(s-1)} = \Delta'_{c_1c_2\ldots c_n(s-1)},
\]
\[
\sup \Delta'_{c_1c_2\ldots c_n} = \frac{1}{s^{c_1+c_2+\ldots+c_n}} \left( \frac{1}{s} + \frac{1}{s^2} + \ldots \right) + \sum_{k=1}^{n} \frac{c_k}{s^{c_1+c_2+\ldots+c_k}} =
\]
\[
= \left( \sum_{k=1}^{n} \frac{c_k}{s^{c_1+c_2+\ldots+c_k}} \right) + \frac{1}{s^{c_1+c_2+\ldots+c_n}} \frac{1}{s-1} = \Delta'_{c_1c_2\ldots c_n(1)}.
\]
The second property follows from Property 1. Property 3 follows from the first and the second properties, and
\[
\Delta'_{0\ldots 0 c_1 0\ldots 0 c_2\ldots 0\ldots 0 c_n} = \left[ \sum_{k=1}^{n} \frac{c_k}{s^{c_1+c_2+\ldots+c_k}} \cdot \frac{1}{s^{c_1+c_2+\ldots+c_n}} + \sum_{k=1}^{n} \frac{c_k}{s^{c_1+c_2+\ldots+c_k}} \right].
\]

Properties 4-6 follow from the last properties.
Let us prove that Property 7 is true. Consider the difference
\[
\inf \Delta'_{c_1c_2\ldots c_{n-1}p} - \sup \Delta'_{c_1c_2\ldots c_{n-1}[p+1]} = \left(\sum_{k=1}^{n-1} \frac{c_k}{s^1+c_1+c_2+\ldots+c_k}\right) + \frac{p}{s^{1+c_1+c_2+\ldots+c_{n-1}+p}} + \frac{1}{s^{1+c_1+c_2+\ldots+c_{n-1}+p}} \cdot \frac{s-1}{s} - \left(\sum_{k=1}^{n-1} \frac{c_k}{s^1+c_1+c_2+\ldots+c_k}\right) - \frac{p+1}{s^{1+c_1+c_2+\ldots+c_{n-1}+p+1}} - \frac{1}{s} = \frac{(s^{s-1} - 1)(p(s-1)^2 - s) + s(s-1)^2}{s^{s-1} - 1} > 0.
\]
For proving Property 8, let us show that the following inequalities hold.
1) \(\inf \Delta'_{c_1\ldots c_{n-1}p} - \inf \Delta'_{c_1\ldots c_{n-1}p\cap c_n} \leq 0, \ p \in N_{s-2}^1\).
2) \(\sup \Delta'_{c_1\ldots c_{n-1}[p+1]} - \sup \Delta'_{c_1\ldots c_{n-1}[p+1]c_n} \geq 0, \ p \in N_{s-2}^1\).

Similarly,
2) \(\left(\sum_{k=1}^{n-1} \frac{c_k}{s^1+c_1+c_2+\ldots+c_k}\right) + \frac{p+1}{s^{1+c_1+c_2+\ldots+c_{n-1}+p+1}} - \frac{1}{s} = \frac{s^{c_{n+1}-1} - c_{n+1}(s-1)}{(s-1)s^{c_{n+1}+1}+1} \geq 0.
\]
Here the last inequality is an equality whenever \(c_{n+1} = 1\).

Property 9 follows from Property 8.

Prove that Property 10 is true. If \(x_0 = \Delta'_{c_1c_2\ldots c_n} \in S(s,0)\), then
\[x_0 = \Delta'_{c_1c_2\ldots c_n} \in \Delta'_{c_1} \cap \Delta'_{c_1c_2} \cap \ldots \cap \Delta'_{c_1c_2\ldots c_n} \cap \ldots .\]

Hence,
\[x_0 \in I_{c_1} \cap I_{c_1c_2} \cap \ldots \cap I_{c_1c_2\ldots c_n} \cap \ldots .\]
From properties of cylinders it follows that
\[I_{c_1} \supset I_{c_1c_2} \supset \ldots \supset I_{c_1c_2\ldots c_n} \supset \ldots .\]
Therefore,

\[ x_0 = \bigcap_{n=1}^{\infty} \Delta^\prime_{c_1c_2...c_n}. \]

\[ \square \]

**Corollary 1.** The condition

\[ S_{(s,0)} \subset I_0 = \left[ \frac{s-1}{s^{s-1}-1}, \frac{1}{s^{s-1}-1} \right]. \]

is satisfied for any positive integer \( s > 2 \).

**Theorem 1.** The set \( S_{(s,0)} \) is a perfect and nowhere dense set of zero Lebesgue measure.

**Proof.** Let us prove that the set \( S_{(s,0)} \) is a nowhere dense set. From the definition it follows that there exist cylinders \( \Delta^\prime_{c_1c_2...c_n} \) of rank \( n \) in an arbitrary subinterval of the segment \( I_0 \). Since Property 7 and Property 8 hold for these cylinders, we have that for any subinterval of \( I_0 \) there exists a subinterval such that does not contain points from \( S_{(s,0)} \). So \( S_{(s,0)} \) is a nowhere dense set.

Show that \( S_{(s,0)} \) is a set of zero Lebesgue measure. Suppose that

\[ S_{(s,0)} = \bigcap_{k=1}^{\infty} E_k, \]

where

\[ E_1 = I_1 \cup I_2 \cup ... \cup I_{s-1}, \]
\[ E_2 = I_{11} \cup ... \cup I_{[s-1][s-1]}, \]
\[ E_3 = I_{111} \cup ... \cup I_{[s-1][s-1][s-1]}, \]

\[ \vdots \]
\[ E_k = I_{1...1} \cup ... \cup I_{[s-1][...[s-1][s-1]} \]

and \( I_{c_1c_2...c_n} \) is a closed interval whose endpoints coincide with endpoints of the cylinder \( \Delta^\prime_{c_1c_2...c_n} \). In addition, since \( E_{k+1} \subset E_k \), we have

\[ E_k = E_{k+1} \cup E_{k+1}. \]

Let \( I_0 \) be an initial closed interval such that \( \lambda(I_0) = d_0 \) and \( S_{(s,0)} \subset I_0 \). Then

\[ \lambda(E_1) = \left( \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3} + \cdots + \frac{1}{s^{s-1}} \right) d_0. \]

If

\[ \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3} + \cdots + \frac{1}{s^{s-1}} = \sigma, \]

then

\[ \lambda(E_1) = d_0 - \sigma d_0 = d_0(1 - \sigma). \]
Similarly, 
\[
\lambda(\bar{E}_2) = d_i^{(1)} - \sigma d_i^{(1)} = d_i^{(1)}(1 - \sigma) = \left( \frac{1}{s} + \frac{1}{s^2} + \cdots + \frac{1}{s^{i-1}} \right) d_0(1 - \sigma) = \\
= \sigma d_0(1 - \sigma), \text{ where } i = 1, s - 1,
\]
and
\[
\lambda(\bar{E}_3) = d_i^{(2)} - \sigma d_i^{(2)} = d_i^{(2)}(1 - \sigma) = \sigma d_i^{(1)}(1 - \sigma) = \sigma^2 d_0(1 - \sigma).
\]
So we obtain that the sequence
\[
d_0(1 - \sigma), \sigma d_0(1 - \sigma), \sigma^2 d_0(1 - \sigma), \ldots, \sigma^{n-1} d_0(1 - \sigma), \ldots,
\]
is an infinitely decreasing geometric progression. Hence,
\[
\lambda(S(s,0)) = d_0 - \sum_{k=1}^{\infty} \lambda(\bar{E}_k) = d_0 - \sum_{j=1}^{\infty} \sigma^{j-1} d_0(1 - \sigma) = d_0 - \frac{d_0(1 - \sigma)}{1 - \sigma} = 0.
\]
So \(S(s,0)\) is a set of zero Lebesgue measure.

Prove that \(S(s,0)\) is a perfect set. Since \(E_k = I_{1} \cup \cdots \cup I_{s-1} \cup I_{[s - 1][s - 1]} \) is a closed sets \((E_k\) is a union of segments), we see that
\[
S(s,0) = \bigcap_{k=1}^{\infty} E_k
\]
is a closed set.

Let \(x \in S(s,0), P\) be any interval that contains \(x\), and \(J_n\) be a segment of \(E_n\) that contains \(x\). Choose a number \(n\) such that \(J_n \subset P\). Suppose that \(x_n\) is the endpoint of \(J_n\) such that the condition \(x_n \neq x\) holds. Hence \(x_n \in S(s,0)\) and \(x\) is a limit point of the set.

Since \(S(s,0)\) is a closed set and does not contain isolated points, we obtain that \(S(s,0)\) is a perfect set.


4. Topological and metric properties of \(S(s,u)\), where \(u \in A_0\)

By \(x_0 = \Delta_{c_1 \ldots c_n \ldots}\) denote the equality
\[
x_0 = \frac{u}{s - 1} + \sum_{k=1}^{\infty} \frac{c_k - u}{s^{1+\ldots+c_k}}.
\]
That is
\[
x_0 = \Delta_{c_1 \ldots c_n \ldots} = \Delta_{c_1 \ldots u \ldots c_2 \ldots u \ldots c_n \ldots}^{c_1^{-1} \ldots c_2^{-1} \ldots c_n^{-1}}.
\]

**Lemma 3.** The set \(S(s,u)\) is an uncountable set for an arbitrary \(u \in A_0\).
Proof. Define the map \( f : \mathcal{S}_{(s,u)} \to C[s, A_0 \setminus \{u\}] \) by the rule

\[
\forall (\alpha_n) \in L : x = \left( \frac{u}{s-1} + \sum_{n=1}^{\infty} \frac{\alpha_n - u}{s^{\alpha_1 + \alpha_2 + \ldots + \alpha_n}} \right) \sum_{n=1}^{\infty} \frac{\alpha_n}{s^n} = y = f(x),
\]

i.e.,

\[
x = \Delta_s^{\alpha_1} \Delta_s^{\alpha_2} \cdots \Delta_s^{\alpha_n} \cdot \sum_{n=1}^{\infty} \frac{\alpha_n}{s^n} = f(x).
\]

Assume that there exist \( x_1 \neq x_2 \) from the set \( \mathcal{S}_{(s,u)} \) such that \( f(x_1) = f(x_2) \). Since \( \inf \Delta \neq 0 \) and \( \sup \Delta \neq 0 \) for any positive integer \( n \), we obtain that \( f \) is a bijection. Hence \( \mathcal{S}_{(s,u)} \) is uncountable, since \( C[s, A_0 \setminus \{u\}] \) is uncountable. \( \square \)

Definition 2. A cylinder \( \Delta_{c_1 \ldots c_n}^{(s,u)} \) of rank \( n \) with base \( c_1c_2\ldots c_n \) is a set of the following form

\[
\Delta_{c_1 \ldots c_n}^{(s,u)} = \left\{ x : x = \left( \sum_{k=1}^{n} \frac{c_k - u}{s^{c_1 + \cdots + c_k}} + \sum_{i=n+1}^{\infty} \frac{\alpha_i - u}{s^{\alpha_1 + \cdots + \alpha_i}} \right) \sum_{n=1}^{\infty} \frac{\alpha_n}{s^n} = y \right\},
\]

where \( c_1, c_2, \ldots, c_n \) are fixed s-adic digits, \( \alpha_n \neq u, \alpha_n \neq 0 \), and \( 2 < s \in \mathbb{N}, n \in \mathbb{N} \).

Lemma 4. Cylinders \( \Delta_{c_1 \ldots c_n}^{(s,u)} \) have the following properties:

\[
\begin{align*}
\inf \Delta_{c_1 \ldots c_n}^{(s,u)} &= \begin{cases} 
\tau + \frac{1}{s^{c_1 + \cdots + c_n}} \left( \frac{s-1-u}{s^{c_1 + \cdots + c_n}} \right), & \text{if } u \in \{0, 1\} \\
\tau + \frac{1}{s^{c_1 + \cdots + c_n}} \frac{1}{s^{c_1 + \cdots + c_n}}, & \text{if } u \in \{2, 3, \ldots, s-1\}, \\
\tau + \frac{1}{s^{c_1 + \cdots + c_n}} \frac{1}{s^{c_1 + \cdots + c_n}}, & \text{if } u = 0
\end{cases}, \\
\sup \Delta_{c_1 \ldots c_n}^{(s,u)} &= \begin{cases} 
\tau + \frac{1}{s^{c_1 + \cdots + c_n}} \left( \frac{1}{s^{c_1 + \cdots + c_n}} + \frac{u}{s^{c_1 + \cdots + c_n}} \right), & \text{if } u \in \{1, 2, \ldots, s-2\} \\
\tau + \frac{1}{s^{c_1 + \cdots + c_n}} \left( 1 - \frac{1}{s^{c_1 + \cdots + c_n}} \right), & \text{if } u = s-1
\end{cases}
\end{align*}
\]

where

\[
\tau = \sum_{k=1}^{n} \frac{c_k - u}{s^{c_1 + \cdots + c_k}} + \sum_{k=1}^{n} \frac{u}{s^k}.
\]
(2) If \( d(\cdot) \) is the diameter of a set, then
\[
d(\Delta_{c_1\ldots c_n}) = \frac{1}{s^{c_1 + \cdots + c_n}} d(S_{(s,u)});
\]
(3)
\[
\frac{d(\Delta_{c_1\ldots c_{n+1}}^{(s,u)})}{d(\Delta_{c_1\ldots c_n}^{(s,u)})} = \frac{1}{s^{c_{n+1}}};
\]
(4)
\[
\Delta_{c_1c_2\ldots c_n}^{(s,u)} = \bigcup_{i=1}^{s-1} \Delta_{c_1c_2\ldots c_n}^{(s,u)} \forall c_n \in A_0, \ n \in \mathbb{N}, \ i \neq u.
\]
(5) The following relationships are satisfied:
(a) if \( u \in \{0,1\} \), then
\[
\inf \Delta_{c_1\ldots c_n}^{(s,u)} > \sup \Delta_{c_1\ldots c_n}^{(s,u)[p+1]};
\]
(b) if \( u \in \{2,3,\ldots,s-3\} \), then
\[
\begin{cases}
\sup \Delta_{c_1\ldots c_n}^{(s,u)} < \inf \Delta_{c_1\ldots c_n}^{(s,u)[p+1]} & \text{for all } p+1 \leq u \\
\inf \Delta_{c_1\ldots c_n}^{(s,u)} > \sup \Delta_{c_1\ldots c_n}^{(s,u)[p+1]} & \text{for all } u < p;
\end{cases}
\]
(c) if \( u \in \{s-2,s-1\} \), then
\[
\sup \Delta_{c_1\ldots c_n}^{(s,u)} < \inf \Delta_{c_1\ldots c_n}^{(s,u)[p+1]};
\]

Proof. The first property follows from the equalities:
\[
\inf S_{(s,u)} = \begin{cases}
\sum_{k=1}^{\infty} \frac{\max \{\alpha_k\} - u}{z_{\max \{\alpha_k\}}} + \frac{u}{z_{\max \{\alpha_k\}}} & \text{if } u \in \{0,1\} \\
\sum_{k=1}^{\infty} \frac{\min \{\alpha_k\} - u}{z_{\min \{\alpha_k\}}} + \frac{u}{z_{\min \{\alpha_k\}}} & \text{if } u \in \{2,3,\ldots,s-1\}, \text{ or } u = 0 \\
\sum_{k=1}^{\infty} \frac{\max \{\alpha_k\} - s + 1}{z_{\max \{\alpha_k\}}} + 1 & \text{if } u = s - 1.
\end{cases}
\]
\[
\sup S_{(s,u)} = \begin{cases}
\sum_{k=1}^{\infty} \frac{u}{z_{\max \{\alpha_k\}}} + \frac{u}{z_{\min \{\alpha_k\}}} & \text{if } u \in \{1,2,\ldots,s-2\} \\
\sum_{k=1}^{\infty} \frac{\max \{\alpha_k\} - s + 1}{z_{\max \{\alpha_k\}}} + 1 & \text{if } u = s - 1.
\end{cases}
\]
These equalities follow from the definition of \( S_{(s,u)} \).

It is easy to see that the second property follows from the first property, the third property is a corollary of the second property, and Property 4 follows from the definition.

Let us show that Property 5 is true. We now prove that the first inequality holds for \( u = 1 \). In fact,
\[
\inf \Delta_{c_1\ldots c_n}^{(s,1)} - \sup \Delta_{c_1\ldots c_n}^{(s,1)[p+1]} = \left( \sum_{k=1}^{\infty} \frac{c_k - 1}{s^{c_1 + \cdots + c_k}} \right) + \left( \sum_{k=1}^{\infty} \frac{1}{s^k} \right).
\]
Prove that the second inequality is true. Here \( 1 < p < s - 1 \) and \( s > 2 \).

Consider the system of inequalities. Let us prove that the first inequality of the system is true. Here \( p + 1 \leq u \). In fact,

\[
\sup_{c_1, \ldots, c_n} \Delta^{(s,u)} - \inf_{c_1, \ldots, c_n} \Delta^{(s,u)} = \frac{p}{s^{c_1+\cdots+c_n+p+1}} + \frac{1}{s^{c_1+\cdots+c_n+p+1}} \left( \frac{1}{s^{u+1}-1} + \frac{u}{s-1} \right) - \frac{p + 1}{s^{c_1+\cdots+c_n+p+1}} - \frac{1}{s^{c_1+\cdots+c_n+p+1}} \left( \frac{1}{s^{u+1}-1} + \frac{u}{s-1} \right) = \]

\[
= \frac{1}{s^{c_1+\cdots+c_n+p+1}} \left( \frac{1}{s^{u+1}-1} + \frac{u}{s-1} - u - \frac{p + 1}{s} - \frac{1}{s(s-1) + p} \right) < 0.
\]

Prove that the second inequality is true. Here \( p > u \), i.e., \( p - u \geq 1 \). Similarly,

\[
\sup_{c_1, \ldots, c_n[p+1]} \Delta^{(s,u)} - \inf_{c_1, \ldots, c_n[p+1]} \Delta^{(s,u)} = \frac{p}{s^{c_1+\cdots+c_n+p+1}} + \frac{1}{s^{c_1+\cdots+c_n+p+1}} \left( \frac{1}{s^{u+1}-1} + \frac{u}{s-1} \right) - \frac{p + 1}{s^{c_1+\cdots+c_n+p+1}} - \frac{1}{s^{c_1+\cdots+c_n+p+1}} \left( \frac{1}{s^{u+1}-1} + \frac{u}{s-1} \right) = \]

\[
= \frac{1}{s^{c_1+\cdots+c_n+p+1}} \left( \frac{1}{s^{u+1}-1} + \frac{u}{s-1} - u - \frac{p + 1}{s} - \frac{1}{s(s-1) - u} \right) > 0.
\]

The last inequality of Property 5 is a corollary of the system of inequalities and is proved by analogy.
Corollary 2. The following equalities are hold:

\[
\inf S_{s,u} = \begin{cases} 
\sum_{k=1}^{\infty} \frac{1-u}{s^{k-1}} + \frac{u}{s^{k+1}} = \frac{1}{s-1}, & \text{if } u \in \{0,1\} \\
\sum_{k=1}^{\infty} \frac{1-u}{s^{k-1}} + \frac{u}{s^{k+1}} = \frac{1}{s-1}, & \text{if } u \in \{2,3,\ldots,s-1\},
\end{cases}
\]

\[
\sup S_{s,u} = \begin{cases} 
\frac{1}{s-1}, & \text{if } u = 0 \\
\sum_{k=1}^{\infty} \frac{1-u}{s^{k+1}} + \frac{u}{s^{k+1}} = \frac{1}{s-1}, & \text{if } u \in \{1,2,\ldots,s-2\}
\end{cases}
\]

\[
1 - \frac{1}{s-1}, & \text{if } u = s-1.
\]

Theorem 2. The set $S_{s,u}$ is a perfect and nowhere dense set of zero Lebesgue measure.

Proof. Since $S_{s,u} \subset C[0,1]$ for all $u = \frac{1}{s} - \frac{1}{s-1}$, where $A_0 = \{1,2,\ldots,s-1\}$, we see that $S_{s,u}$ is a nowhere dense set of zero Lebesgue measure. The fact that $S_{s,u}$ is a perfect set can be proved by analogy with the proof of this fact for $S_{s,0}$.

5. Fractal properties of $S_{s,u}$

Theorem 3. The set $S_{s,u}$ is a self-similar fractal and the Hausdorff-Besicovitch dimension $\alpha_0(S_{s,u})$ of this set satisfies the following equation

\[
\sum_{p_i \neq u, p_i \in A_0} \left(\frac{1}{s}\right)^{p_i \alpha_0} = 1.
\]

Proof. Consider the set $S_{s,0}$. Since $S_{s,0} \subset I_0$ and $S_{s,0}$ is a perfect set, we obtain that $S_{s,0}$ is a compact set. In addition,

\[
S_{s,0} = [I_1 \cap S_{s,0}] \cup [I_2 \cap S_{s,0}] \cup \ldots \cup [I_{s-1} \cap S_{s,0}]
\]

and

\[
[I_1 \cap S_{s,0}] \overset{s^{-1}}{\sim} S_{s,0}, [I_2 \cap S_{s,0}] \overset{s^{-2}}{\sim} S_{s,0}, \ldots, [I_{s-1} \cap S_{s,0}] \overset{s^{-(s-1)}}{\sim} S_{s,0}.
\]

Since the set $S_{s,0}$ is a compact self-similar set of space $\mathbb{R}^1$, we have that the self-similar dimension of this set is equal to the Hausdorff-Besicovitch dimension of $S_{s,0}$. So the set $S_{s,0}$ is a self-similar fractal, and its Hausdorff-Besicovitch dimension $\alpha_0(S_{s,0})$ satisfies the equation

\[
\left(\frac{1}{s}\right)^{\alpha_0} + \left(\frac{1}{s}\right)^{2\alpha_0} + \left(\frac{1}{s}\right)^{3\alpha_0} + \cdots + \left(\frac{1}{s}\right)^{(s-1)\alpha_0} = 1.
\]

Fractal properties of the set $S_{s,u}$ ($u \in A_0$) can be formulated and proved by analogy with the case of $S_{s,0}$. However in the case of $S_{s,u}$, the condition $\alpha_n \neq u$ for all $n \in \mathbb{N}$ should be taken into account.
Consider the set of all numbers whose $s$-adic representations contain only combinations of $s$-adic digits that is using in the $s$-adic representations of elements of $S_{(s,u)}$.

A set $\tilde{S}$ is the set of all numbers whose $s$-adic representations contain only combinations of $s$-adic digits from the set
\[ \{1, 02, 003, \ldots, u\ldots uc, \ldots, (s - 1)(s - 1)(s - 2)\}, \]
where $c \in A_0, u \in A, c \neq u$.

**Theorem 4.** The set $\tilde{S}$ is an uncountable, perfect, and nowhere dense set of zero Lebesgue measure.

Properties of the set $\tilde{S}$ follow from the following fractal properties of this set.

**Theorem 5.** The set $\tilde{S}$ is a self-similar fractal, and its Hausdorff-Besicovitch dimension $\alpha_0$ satisfies the following equation
\[
\left( \frac{1}{s} \right)^{\alpha_0} + (s - 1) \left( \frac{1}{s} \right)^{2\alpha_0} + (s - 1) \left( \frac{1}{s} \right)^{3\alpha_0} + \cdots + (s - 1) \left( \frac{1}{s} \right)^{(s-1)\alpha_0} = 1.
\]

**Proof.** The $s$-adic representation of an arbitrary element from $\tilde{S}$ contains combinations of digits from the following tuple:

\[
\begin{align*}
02, 003, \ldots, & 0\ldots00(s - 1); \\
1, 12, 113, \ldots, & 1\ldots11(s - 1); \\
223, 2224, \ldots, & 2\ldots22(s - 1); \\
\ldots & \\
u2, u3, \ldots, & uu(u - 1), u\ldots uu(u + 1), \ldots, uu(s - 1); \\
\ldots & \\
(s - 1)2, (s - 1)(s - 1)3, \ldots, & (s - 1)(s - 1)(s - 1)(s - 2).
\end{align*}
\]

Here $s^2 - 3s + 3$ combinations of $s$-adic digits, i.e., the unique 1-digit combination and $s - 1$ k-digit combinations for all $k = \frac{2}{1}, s - 1$. Our statement follows from Theorem 5.

**Theorem 6.** Let $E$ be a set whose elements represented by finite number of fixed combinations $\sigma_1, \sigma_2, \ldots, \sigma_m$ of $s$-adic digits in the $s$-adic number system. Then the Hausdorff-Besicovitch dimension $\alpha_0$ of $E$ satisfies the following equation:
\[
N(\sigma_m^1) \left( \frac{1}{s} \right)^{\alpha_0} + N(\sigma_m^2) \left( \frac{1}{s} \right)^{2\alpha_0} + \cdots + N(\sigma_m^k) \left( \frac{1}{s} \right)^{k\alpha_0} = 1,
\]
where $N(\sigma_m^k)$ is a number of $k$-digit combinations $\sigma_m^k$ from the set $\{\sigma_1, \sigma_2, \ldots, \sigma_m\}, k \in \mathbb{N}$, and $N(\sigma_m^1) + N(\sigma_m^2) + \cdots + N(\sigma_m^k) = m$. 
Proof. Let \( \{\sigma_1, \sigma_2, \ldots, \sigma_m\} \) be a set of fixed combinations of s-adic digits, and the s-adic representation of any number from \( E \) contains only such combinations of digits.

It is easy to see that there exist combinations \( \sigma', \sigma'' \) from the set \( \{\sigma_1, \sigma_2, \ldots, \sigma_m\} \) such that \( \Delta_{\sigma'} = \inf E, \Delta_{\sigma''} = \sup E \), and
\[
d(E) = \sup E - \inf E = \Delta_{\sigma''} - \Delta_{\sigma'}.
\]

A cylinder \( \Delta_{\sigma_1, \sigma_2, \ldots, \sigma_n} \) of rank \( n \) with base \( \sigma_1, \sigma_2, \ldots, \sigma_n \) is a set formed by all numbers of \( E \) with s-adic representations in which the first \( n \) combinations of digits are fixed and coincide with \( \{\sigma_1, \sigma_2, \ldots, \sigma_n\} \) respectively (\( \sigma_j \in \{\sigma_1, \sigma_2, \ldots, \sigma_m\} \) for all \( j = 1, n \)).

It is easy to see that
\[
d(\Delta_{\sigma_1, \sigma_2, \ldots, \sigma_n}) = \frac{1}{s^{N(\sigma_1+\sigma_2+\ldots+\sigma_n)}}d(E),
\]
where \( N(\sigma_1+\sigma_2+\ldots+\sigma_n) \) is a number of s-adic digits in the combination \( \sigma_1 \sigma_2 \ldots \sigma_n \).

Since \( E \) is a Cantor type set, \( E \subset \inf E; \sup E \), and
\[
d(\Delta_{\sigma_1, \sigma_2, \ldots, \sigma_n}) = \frac{1}{s^{N(\sigma_1+\sigma_2+\ldots+\sigma_n+\sigma_{n+1})}}d(E),
\]where \( I_{\sigma_1} = [I_{\sigma_1} \cap E] \cup [I_{\sigma_2} \cap E] \cup \ldots \cup [I_{\sigma_m} \cap E] \),
where \( I_{\sigma_i} = \inf \Delta_{\sigma_i}; \sup \Delta_{\sigma_i} \) and \( i = 1, 2, \ldots, m \), we have
\[
[I_{\sigma_1} \cap E]^{s-1} E, [I_{\sigma_2} \cap S]^{s-2} E, [I_{\sigma_3} \cap E]^{s-3} E, \ldots, [I_{\sigma_m} \cap E]^{s-k} E.
\]
This completes the proof. \( \square \)

Remark 1. The following statements are true:
- if there exists \( k \in \mathbb{N} \) such that \( N(\sigma_m^k) = s^k \), then \( \alpha_0(E) = 1 \);
- if \( m = 1 \), then \( \alpha_0(E) = 0 \).

6. Normal numbers on \( \bigcup_{u=0}^{s-1} S_{(s,u)} \)

Suppose that
\[
S_0 = \bigcup_{u=0}^{s-1} S_{(s,u)}.
\]
The frequency of a digit \( i \) in the s-adic expansion of a number \( x \) is a limit
\[
v_i = \lim_{k \to \infty} \frac{N_i(x,k)}{k} \quad \text{(if there exists a limit from the right),}
\]
where \( N_i(x,k) \) is a number of the digit \( i \) \( (i = 0, s - 1) \) up to the \( k \)-th place inclusively in the s-adic expansion, and \( k^{-1}N_i(x,k) \) is the relative frequency of using of the digit \( i \) in the s-adic expansion of \( x \).

A number \( x \in [0; 1] \) is called a normal number on the base \( s \) whenever for each \( i \in \{0, 1, \ldots, s - 1\} \) there exists the frequency and \( v_i(x) = s^{-1} \).
Theorem 7. Let $s > 2$ be a fixed positive integer; then the following statements are true:

- if $s = 3$, then the set of normal numbers from $S_0$ is a continuum set (a subset of $S_{(3,0)}$), and its Hausdorff-Besicovitch dimension $\alpha_0$ satisfies the following inequalities
  \[ \frac{1}{3} \log_3 2 \leq \alpha_0 \leq \log_3 \frac{2}{\sqrt{5} - 1}; \]
- if $s \neq 3$, then the set of normal numbers from $S_0$ is an empty set.

Proof. If a number $x$ does not belong to the set of normal numbers from $S_0$, then there exists sequence (at least one) of digits of $x$ such that the frequency of one of the digits does not exist or is not equal to $s - 1$. Find such sequence $(k_n)$.

Suppose that $k_n = ns$, i.e., choose $ns$ digits. Here $n \in \mathbb{N}$, $s \geq 3$, and

\[ \triangle k = k_{n+1} - k_n = ts, \]

where $t$ is a fixed positive integer.

Let $x$ be a normal number. Then

\[ \lim_{k_n \to \infty} N_1(x, k_{n+1}) = \lim_{k_n \to \infty} N_2(x, k_{n+1}) = \cdots = \lim_{k_n \to \infty} N_{s-1}(x, k_{n+1}) = 1, \]

where $k_{n+1} = (n + t)s$. Hence,

\[ N_1(x, k_{n+1}) \to (n + t), \]
\[ N_2(x, k_{n+1}) \to (n + t), \]
\[ \cdots \cdots \cdots \]
\[ N_{s-1}(x, k_{n+1}) \to (n + t). \]

Find $N_0(x, k_{n+1})$ by the definition of $S_{(s,0)}$. Since $N_0(x, k_{n+1}) \neq 0$ holds only for $S_{(s,0)}$ (i.e., the set of normal numbers of $S_0$ is a subset of $S_{(s,0)}$), we obtain

\[ N_0(x, k_{n+1}) \to A = N_2(x, k_{n+1}) + 2N_3(x, k_{n+1}) + \cdots + (s - 2)N_{s-1}(x, k_{n+1}), \]

\[ A = \sum_{j=1}^{s-2} [j(n + t)] = \frac{(s - 2 + 1)(s - 2)(n + t)}{2} = \frac{(s - 2)(s - 1)(n + t)}{2}. \]

Therefore,

\[ \lim_{k_n \to \infty} \frac{N_0(x, k_{n+1})}{k_{n+1}} = \lim_{(n+t) \to \infty} \frac{(s - 2)(s - 1)(n + t)}{2s(n + t)} = \frac{(s - 2)(s - 1)}{2s} \neq \frac{1}{s} \]

for any $s > 3$. If $s = 3$, then we have the equality.

By $S_N$ denote the set of normal numbers of $S_0$. It is easy to see that

\[ \alpha_0(S_3) \leq \alpha_0(S_N) \leq \alpha_0(S_{(3,0)}), \]
where \( S'_3 = \{ x : x = \Delta^3_{\sigma_1\sigma_2...\sigma_n...} \in \{021,102\} \} \). It follows from Theorem 6 that 
\[
\alpha_0(S_{(3,0)}) = \log_3 \frac{2}{\sqrt{5} - 1} \text{ and the Hausdorff-Besicovitch dimension of } S'_3 \text{ satisfies the equation }
\[
2 \left( \frac{1}{3} \right)^{3\alpha_0} = 1.
\]
So \( \alpha_0(S'_3) = \frac{1}{3} \log_3 2 \). □

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