Non-minimally coupled nonlinear spinor field in FRW cosmology

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Within the scope of a FRW cosmological model we have studied the role of spinor field in the evolution of the Universe when it is non-minimally coupled to the gravitational one. We have considered a few types of nonlinearity. It was found that if the spinor field nonlinearity describes an ordinary matter such as radiation, the presence of non-minimality becomes essential and leads to the rapid expansion of the Universe, whereas if the spinor field nonlinearity describes a dark energy, the evolution of the Universe is dominated by it and the difference between the minimal and non-minimally coupled cases become almost indistinguishable.

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I. INTRODUCTION

The discovery and further confirmation of the accelerated expansion of the Universe led to reconsider the existing theories of cosmology. One of the straight forward ways was to introduce some additional component into the right hand side of the Einstein equations with negative pressure which would work as repulsive force thus giving rise to the accelerated mode of expansion. A number of models were proposed by different authors. Model exploiting the spinor field was one of them. For more than two decades spinor field is being widely used in cosmology mainly thanks to its specific behavior in presence of gravitational field. In a number of papers the authors have shown that the nonlinear spinor field can give rise to regular solutions as well as explain the late-time accelerated mode of expansion of the Universe [1–6]. But most of those papers considered the minimal coupling of spinor and gravitational field. It should be noted that along with the dark energy models many authors suggested the modification of the Einstein equations itself. Scalar tensor theory cite Brans-Dicke, theory with non-minimal coupling, $F(R)$ theory [8], $F(R, T)$ theory with $T$ being the trace of energy-momentum tensor (EMT) [9], $F(T)$ theory with $T$ being the torsion [10], $f(G)$ theory are the few to name. The motivation behind this research was to study the influence of spinor field in the evolution of the universe when it is non-minimally coupled to...
the gravitational field. Since spinor field is more sensitive to the gravitational field than the scalar one, in our view it may give rise to some unexpected results. Recently, Carloni et al [12] has considered non-minimally coupled spinor field with the gravitational one. Non-minimally coupled spinor and gravitational fields within the scope of Bianchi type-I metric was studied in [13]. In this report we plan to continue that study for an isotropic and homogeneous space-time given by a FRW metric.

II. BASIC EQUATIONS

We consider the action in the form

$$ S = \int \sqrt{-g} \left[ (\kappa_1 + \lambda_1 S) R + L_{\text{sp}} \right] d\Omega, \quad \kappa_1 = \frac{1}{2\kappa}. \quad (1) $$

where $S = \bar{\psi}\psi$ is a scalar constructed from spinor fields, $\lambda_1$ is the coupling constant. Here $\kappa$ is the Einstein’s constant defined as $\kappa = 8\pi G$, with $G$ being the Newton’s gravitational constant. The spinor field Lagrangian takes the form

$$ L_{\text{sp}} = \frac{i}{2} \left[ \bar{\psi} \gamma^\mu \nabla_\mu \psi - \nabla_\mu \bar{\psi} \gamma^\mu \psi \right] - m\bar{\psi}\psi - \lambda F(S). \quad (2) $$

Note that in general the nonlinear term $F$ may be the arbitrary function of invariant $K$ which takes one of the following expressions: $\{I, J, I + J, I - J\}$. Here $I = \bar{\psi}\psi$ and $J = i\bar{\psi}\gamma^5\psi$. Here $m$ is the spinor mass. $\lambda$ is the self coupling constant that can be positive or negative. Here $\nabla_\mu$ is the covariant derivative of the spinor field

$$ \nabla_\mu \psi = \partial_\mu \psi - \Gamma_\mu \psi, \quad \nabla_\mu \bar{\psi} = \partial_\mu \bar{\psi} + \bar{\psi} \Gamma_\mu. \quad (3) $$

Here $\Gamma_\mu$ is the spinor affine connection.

Variation with respect to metric functions give [13]

$$ R^\nu_\mu - \frac{1}{2} \delta^\nu_\mu R = \frac{1}{(\kappa_1 + \lambda_1 S)} \left[ T^\nu_\mu + \lambda_1 \left( g^{\nu\tau} \nabla_\mu \nabla_\tau - \delta^\nu_\mu \Box \right) S \right]. \quad (4) $$

where $T^\nu_\mu$ is the energy-momentum tensor of the spinor field. The corresponding equations for spinor field we find varying the action with respect to $\psi$ and $\bar{\psi}$ [13]

$$ i\gamma^\mu \nabla_\mu \bar{\psi} - m\bar{\psi} - \lambda F_S \bar{\psi} + \lambda_1 R \bar{\psi} = 0, \quad (5a) $$

$$ i\nabla_\mu \bar{\psi} \gamma^\mu + m\bar{\psi} + \lambda F_S \bar{\psi} - \lambda_1 R \bar{\psi} = 0. \quad (5b) $$

From (5) one finds that $L_{\text{sp}} = SF_S - F$.

We consider the isotropic FRW space-time is given by

$$ ds^2 = dt^2 - a^2 \left( dx_1^2 + dx_2^2 + dx_3^2 \right), \quad (6) $$

with the scale factor $a$ is the functions of time only.

For the metric (6) we choose the tetrad such that they have the following nontrivial components:

$$ e_0^{(0)} = 1, \quad e_i^{(i)} = a, \quad i = 1, 2, 3. \quad (7) $$
From
\[ \Gamma_\mu = \frac{1}{8} [\partial_\mu \gamma_\alpha, \gamma^\alpha] - \frac{1}{8} \Gamma^\beta_{\mu\alpha} [\gamma_\beta, \gamma^\alpha]. \]  

(8)

where \([a,b] = ab - ba\) one finds the following expressions for spinor affine connections:
\[ \Gamma_0 = 0, \quad \Gamma_1 = \frac{\dot{a}}{2} \gamma^1 \gamma^0, \quad \Gamma_2 = \frac{\dot{a}}{2} \gamma^2 \gamma^0, \quad \Gamma_3 = \frac{\dot{a}}{2} \gamma^3 \gamma^0. \]  

(9)

In (8) \(\gamma_\beta = e^{(b)}_\beta \tilde{\gamma}_b\) and \(\gamma^\alpha = e^\alpha_{(a)} \tilde{\gamma}^a\) are the Dirac matrices in curve space-time and \(e^{(a)}_\alpha\) and \(e^{(b)}_\beta\) are the tetrad vectors.

We consider the case when the spinor field depends on \(t\) only. The spinor field equations in this case read
\[ i\gamma^0 \left( \psi + \frac{3}{2} \frac{\dot{a}}{a} \psi \right) - m \psi - \lambda F_S \psi + \lambda_1 R \psi = 0, \]  

(10a)
\[ i \left( \bar{\psi} + \frac{3}{2} \frac{\dot{a}}{a} \bar{\psi} \right) \psi^0 + m \bar{\psi} + \lambda F_S \bar{\psi} - \lambda_1 R \bar{\psi} = 0, \]  

(10b)

where we denote \(F_S = dF / dS\). From (10) one easily finds
\[ S = C_0 / a^3, \quad C_0 = \text{Const}. \]  

(11)

The energy-momentum tensor of the spinor field
\[ T_{\mu}^\rho = \frac{i}{4} g^{\rho\nu} \left( \bar{\psi} \gamma_\mu \nabla_\nu \psi + \bar{\psi} \gamma_\nu \nabla_\mu \psi - \nabla_\mu \bar{\psi} \gamma_\nu \psi - \nabla_\nu \bar{\psi} \gamma_\mu \psi \right) - \delta^\rho_\mu L_{sp}, \]  

(12)

in our case gives the following nontrivial components \([13]\)
\[ T_0^0 = mS + \lambda F(S), \]  

(13a)
\[ T_1^1 = T_2^2 = T_3^3 = \lambda \left( F(S) - SF_S \right). \]  

(13b)

Taking into account that in our case, \(\Box S = \ddot{S} + 3 \frac{\dot{a}}{a} \dot{S}\), in view of
\[ \nabla_\mu \nabla_\nu S = \nabla_\mu \partial_\nu S = \partial_\mu \partial_\nu S - \Gamma^\alpha_{\mu\nu} \partial_\alpha S, \]  

(14a)
\[ \Box S = g^{\alpha\beta} \nabla_\alpha \nabla_\beta S = g^{\alpha\beta} \left( \partial_\alpha \partial_\beta S - \Gamma^\tau_{\alpha\beta} \partial_\tau S \right), \]  

(14b)

for the metric (6) from (4) we find
\[ 2 \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} = \frac{1}{(\kappa_1 + \lambda_1 S)} \left[ \lambda \left( F(S) - SF_S \right) - \lambda_1 \ddot{S} - 2 \lambda_1 \frac{\dot{a}}{a} \dot{S} \right], \]  

(15a)
\[ 3 \frac{\ddot{a}^2}{a^2} = \frac{1}{(\kappa_1 + \lambda_1 S)} \left[ (mS + \lambda F(S)) - 3 \lambda_1 \frac{\dot{a}}{a} \dot{S} \right]. \]  

(15b)

In a recent paper \([13]\) it was shown that if instead of ordinary scalar we deal with \(S = \bar{\psi} \psi\) as component by component, then for the second derivative in (14a) we get an additional term,
namely \( \psi \Gamma_{\nu} \partial_{\mu} \psi - \partial_{\mu} \psi \Gamma_{\nu} \psi \). In our case \( \psi \) is a function of \( t \), moreover \( \Gamma_{0} = 0 \) and \( \Gamma_{i} = (\dot{\psi}/2) \bar{\psi} \psi^{0} \). On account of that we can write \( \psi \Gamma_{\nu} \partial_{\mu} \psi - \partial_{\mu} \psi \Gamma_{\nu} \psi = \frac{1}{2} \left( \psi \bar{\psi} \psi^{0} \psi - \psi \bar{\psi} \psi^{0} \psi \right) \). Further multiplying (10a) by \( \bar{\psi} \bar{\psi} \) from the left and (10b) by \( \bar{\psi} \psi \) from the right and adding them we find \( \left( \psi \bar{\psi} \psi^{0} \psi - \psi \bar{\psi} \psi^{0} \psi \right) \equiv 0 \). Thus in this case we can deal with \( S \) as an ordinary scalar.

In view of (11) from (15b) we find

\[
\dot{a} = \sqrt{\frac{(mS + \lambda F)}{3(\kappa_{1} - 2\lambda_{1}S)}} a = \sqrt{\frac{(mC_{0} + \lambda a^{3}F)}{3(\kappa_{1}a^{3} - 2\lambda_{1}C_{0})}} a. \tag{16}
\]

Further from (11) we find that \( \dot{S} = -3 \frac{a}{a}S + 12 \frac{a^{2}}{a^{2}}S \). Then on account of (16) we rewrite (15a) as

\[
\dot{a} = \left[ \frac{\lambda \left( F - SF_{3} \right)}{(2\kappa_{1} - \lambda_{1}S)} - \frac{(\kappa_{1} + 7\lambda_{1}S) \left( mS + \lambda F \right)}{3(2\kappa_{1} - \lambda_{1}S) \left( \kappa_{1} - 2\lambda_{1}S \right)} \right] a, \tag{17}
\]

in view of (11) which can be written as

\[
\dot{a} = \left[ \frac{\lambda \left( a^{3}F - C_{0}F_{3} \right)}{(2\kappa_{1}a^{3} - \lambda_{1}C_{0})} - \frac{(\kappa_{1}a^{3} + 7\lambda_{1}a^{3}C_{0}) \left( mC_{0} + \lambda a^{3}F \right)}{3(2\kappa_{1}a^{3} - \lambda_{1}C_{0}) \left( \kappa_{1}a^{3} - 2\lambda_{1}C_{0} \right)} \right] a. \tag{18}
\]

### III. NUMERICAL ANALYSIS

In what follows we solve this equation numerically. For simplicity we set \( m = 1 \), \( \kappa_{1} = 1 \) and \( C_{0} = 1 \). We consider two cases, one with non-minimal coupling, another with minimal coupling, so that the role of non-minimal coupling becomes clear.

**Case 1** In this case we consider the non-minimal coupling with nonlinear term (plotted in solid blue line) setting \( \lambda_{1} = 1 \), \( \lambda = 1 \).

**Case 2** As a second case we consider nonlinear spinor field with minimal coupling setting \( \lambda_{1} = 0 \), \( \lambda = 1 \) (plotted in dot red line).

The initial value \( a(0) \) was chosen in such a way that the initial value of \( \dot{a}(0) \) that was determined from (16) remains real. As it was mentioned earlier, the nonlinear spinor field can simulate different types of dark energy. Here we consider different types of nonlinearity and compare the results for there different cases.

**Dust**

Let us begin with linear spinor field. Setting \( \lambda = 0 \) from (13) we find \( T_{0}^{0} = mS \) and \( T_{1}^{1} = T_{2}^{2} = T_{3}^{3} = 0 \). It means the linear spinor field behaves like dust. In Fig. 1 the behavior or scale factor \( a \) is plotted for non-minimal and minimal coupling. As one sees, non-minimal coupling in this case leads to the rapid expansion of the Universe.

**Radiation**

Let us first consider the case when the Universe is filled with radiation. In this case the spinor field nonlinearity is given by [6]

\[
F = S^{1+W}, \quad W = 1/3. \tag{19}
\]

The corresponding solution is given in Fig. 2. Here the blue solid line stand for non-minimal coupling with nonlinear spinor field, and red dot line stands for minimal coupling with nonlinear spinor field. Like in the previous case here too we see that the non-minimal coupling plays significant role in the evolution of the Universe and leads to its rapid expansion.
Let us consider the spinor field nonlinearity which is responsible for quintessence. In this case the spinor field nonlinearity can be given by [6]

$$F = S^{1+W}, \quad W < -1/3.$$  \hspace{1cm} (20)

Let us set $W = -1/2$. The solution to the equation (18) is plotted in the Fig. 3. Here we see that the

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{Fig1.png}
\caption{Plot of scale factor $a$ with the Universe filled with dust. Blue solid line stands for non-minimal coupling, while the red dot line corresponds to minimal coupling.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{Fig2.png}
\caption{Plot of scale factor $a$ with the Universe filled with radiation. Blue solid line stands for non-minimal coupling, while the red dot line corresponds to minimal coupling.}
\end{figure}

\textbf{Quintessence}

Let us consider the spinor field nonlinearity which is responsible for quintessence. In this case the spinor field nonlinearity can be given by [6]

$$F = S^{1+W}, \quad W < -1/3.$$  \hspace{1cm} (20)

Let us set $W = -1/2$. The solution to the equation (18) is plotted in the Fig. 3. Here we see that the
nonlinear term plays the key role in the evolution of the Universe. The presence of non-minimality is hardly distinguishable.

FIG. 3: Plot of scale factor $a$ with the Universe filled with quintessence. Blue solid line stands for non-minimal coupling, while the red dot line corresponds to minimal coupling.

**Chaplygin Gas**

Another choice of spinor field nonlinearity could be the one that describes a Chaplygin gas. As it was shown in [6] spinor field nonlinearity in this case takes the form

$$F = \left(A + S^{1+\alpha}\right)^{1/(1+\alpha)},$$

with $A > 0$ and $0 < \alpha \leq 1$. Inserting it into (18) and setting $A = 1$ and $\alpha = 0.5$ we have solved the equation is question numerically. The result is illustrated in Fig. 4. As in case of quintessence, here too the prime role belongs to the spinor field nonlinearity.

**Modified Quintessence**

The discovery of late time acceleration gives rise a number of problems. One of the problems is the eternal acceleration. To avoid this a modified quintessence was proposed. In this case the spinor field nonlinearity takes the form [6]

$$F = S^{1+W} + \frac{W}{1+W} \varepsilon_{\text{cr}},$$

where $\varepsilon_{\text{cr}}$ is come constant. We set $W = -1/2$ and $\varepsilon_{\text{cr}} = 0.01$ Then the solution to the equation (18) takes the from drawn in Fig. 5. We again see that the evolution of the Universe is dominated by the dark energy given by the spinor field nonlinearity.

**Modified Chaplygin Gas**

We also consider the case when the dark energy is the combination of quintessence and Chaplygin gas. In this case the spinor field nonlinearity takes the form [6]

$$F = \left(\frac{A}{1+W} + S^{(1+W)(1+\alpha)}\right)^{1/(1+\alpha)},$$

with $A > 0$ and $0 < \alpha \leq 1$. Inserting it into (18) and setting $A = 1$ and $\alpha = 0.5$ we have solved the equation is question numerically. The result is illustrated in Fig. 4. As in case of quintessence, here too the prime role belongs to the spinor field nonlinearity.
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FIG. 4: Plot of scale factor $a$ with the Universe filled with Chaplygin gas. Blue solid line stands for non-minimal coupling, while the red dot line corresponds to minimal coupling.

FIG. 5: Plot of scale factor $a$ with the Universe filled with modified quintessence. Blue solid line stands for non-minimal coupling, while the red dot line corresponds to minimal coupling.

with $W < -1/3$ and $A > 0$. We have taken $W = -1/2$ and $\alpha = 2$. The corresponding solution is illustrated in the Fig. 23. Like other previous cases spinor field nonlinearity plays principal role in the evolution of the Universe.
FIG. 6: Plot of scale factor $a$ with the Universe filled with modified Chaplygin gas. Blue solid line stands for non-minimal coupling, while the red dot line corresponds to minimal coupling.

IV. CONCLUSION

Since in a FRW Universe the non-diagonal components of the energy-momentum tensor of the spinor field do not exist the spinor field does not impose any additional restriction on the geometry of the Universe as it takes place for the anisotropic cosmological models. This is true for both cases with minimal and non-minimal coupling. If the spinor field nonlinearity behaves like an ordinary matter, e.g., radiation, the presence of non-minimality becomes significant and in this case the non-minimal coupling leads to the rapid rapid expansion of the Universe, whereas if the spinor field nonlinearity describes a dark energy, the evolution of the Universe is totally dominated by it and the presence of non-minimality remains almost unnoticeable.

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