G/H M–branes
and AdSp+2 Geometries∗†

Leonardo Castellani1, Anna Ceresole2, Riccardo D’Auria 2,
Sergio Ferrara3, Pietro Fré4 and Mario Trigiante5

1 Dipartimento di Scienze e Tecnologie Avanzate, Universitá di Torino, Sede di Alessandria
and Istituto Nazionale di Fisica Nucleare (INFN) - Sezione di Torino, Italy
2 Dipartimento di Fisica Politecnico di Torino, C.so Duca degli Abruzzi, 24, I-10129 Torino
and Istituto Nazionale di Fisica Nucleare (INFN) - Sezione di Torino, Italy
3 CERN, Theoretical Division, CH 1211 Geneva, 23, Switzerland
4 Dipartimento di Fisica Teorica, Universitá di Torino, via P. Giuria 1, I-10125 Torino,
Istituto Nazionale di Fisica Nucleare (INFN) - Sezione di Torino, Italy
5 Department of Physics, University of Wales Swansea, Singleton Park,
Swansea SA2 8PP, United Kingdom

Abstract

We discuss the class of BPS saturated M–branes that are in one–to–one correspondence with the
Freund–Rubin compactifications of M–theory on either AdS4 × G/H or AdS7 × G/H, where G/H is
any of the seven (or four) dimensional Einstein coset manifo lds with Killing spinors classified long ago
in the context of Kaluza–Klein supergravity. These G/H M–branes, whose existence was previously
pointed out in the literature, are solitons that interpolate between flat space at infinity and the
old Kaluza–Klein compactifications at the horizon. They preserve N/2 supersymmetries where N is
the number of Killing spinors of the AdS × G/H vacuum. A crucial ingredient in our discussion
is the identification of a solvable Lie algebra parametrization of the Lorentzian non compact coset
SO(2, p + 1)/SO(1, p + 1) corresponding to anti–de Sitter space AdSp+2. The solvable coordinates
are those naturally emerging from the near horizon limit of the G/H p–brane and correspond to the
Bertotti–Robinson form of the anti–de Sitter metric. The pull-back of anti–de Sitter isometries on the
p–brane world–volume contain, in particular, the recently found broken conformal transformations.

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1 Introduction

Since the second string revolution [1, 2], the five consistent 10–dimensional superstrings have been reinterpreted as different perturbative limits of a single fundamental theory, named M–theory. While the microscopic quantum definition of this latter is still matter of debate, its low energy effective lagrangian is well known and extensively studied since the end of the seventies: indeed it coincides with $D = 11$ supergravity [3, 4]. As a logical consequence of this new deeper understanding, all aspects of $D = 11$ supergravity must have bearings on string theory and admit a string interpretation.

Until a few months ago the aspect of $D = 11$ supergravity whose consequences on string theory has been investigated most is given by its classical $p$–brane solutions, usually called M–branes [5, 6]. In their simplest formulation M–branes are solutions of the classical field equations where the metric takes the following form:

$$ds_{11}^2 = \left(1 + \frac{k}{r^d}\right)^{-\frac{d}{4}} dx^\mu dx^\nu \eta_{\mu\nu} + \left(1 + \frac{k}{r^d}\right)^{\frac{d}{4}} dX^I dX^J \delta_{IJ}.$$  \hspace{1cm} (1)

In eq.(1) $d \equiv p + 1$ ; $\bar{d} \equiv 11 - d - 2$ are the world–volume dimensions of the $p$–brane and of its magnetic dual,\n
$$r \equiv \sqrt{X^I X^J \delta_{IJ}}$$  \hspace{1cm} (3)\n
is the radial distance from the brane in transverse space, $\mu = 0, \ldots, d - 1$ and $I, J = d, \ldots, 10$.

The isometry group of the 11–dimensional metric (1) is:

$$\mathcal{I}_{p–brane} = ISO(1, p) \otimes SO(11 - d)$$  \hspace{1cm} (4)

There are two fundamental branes of this sort: the electric M2–brane ($p = 2$) and the magnetic M5–brane ($p = 5$). In addition one has a variety of more complicated branes that can be interpreted as intersections and superpositions of the fundamental ones at angles. The basic motivation for the pre-eminence of these M–branes in the recent studies on M–theory is their property of being BPS states, that is of admitting a set of Killing spinors whose existence leads to the saturation of the relevant Bogomol’nyi bound in the mass–charge relation. Hence classical M–brane solutions correspond to exact non–perturbative quantum states of the string spectrum and from the string side there exist descriptions of these states in terms of Dirichlet branes [7] using for instance the technology of boundary states [8, 9, 10].

One point that we would like to stress is that M–branes of type (1) are asymptotically flat and have, at spatial infinity ($r \to \infty$), the same topology as the spatial infinity of 11–dimensional Minkowski space, namely $S^9$.

• Freund–Rubin manifolds and M–branes

One different aspect of D=11 supergravity that was actively investigated in the eighties [11, 12, 13, 14, 15] and that has so far eluded being fully incorporated into M–theory is given
by the Kaluza–Klein compactifications. In this context one has the Freund–Rubin vacua [12] where the 11–dimensional space is either:

\[ M_{11} = AdS_4 \times \left( \frac{G}{H} \right)_7 \] (5)

or

\[ M_{11} = AdS_7 \times \left( \frac{G}{H} \right)_4 \] (6)

having denoted by \( AdS_D = SO(2, D - 1)/SO(1, D - 1) \) anti de Sitter space in dimension \( D \) and by \( \left( \frac{G}{H} \right)_n \) an \( n \)–dimensional coset manifold equipped with a \( G \)–invariant Einstein metric.

The first class of Freund–Rubin vacua (5) is somehow reminiscent of the M2–brane since such a metric solves the field equations under the condition that the 4–form field strength take a constant \( SO(1, 3) \)-invariant vev:

\[ F_{\mu_1 \mu_2 \mu_3 \mu_4} = e \epsilon_{\mu_1 \mu_2 \mu_3 \mu_4} \] (7)

on \( AdS_4 \). Alternatively, the second class of Freund–Rubin vacua (6) is somehow reminiscent of the M5–brane. Indeed in this case the chosen metric solves the field equations if the dual magnetic 7–form acquires an \( SO(1, 6) \)-invariant vev:

\[ \ast F_{\mu_1 \ldots \mu_7} = g \epsilon_{\mu_1 \ldots \mu_7} \] (8)

on \( AdS_7 \).

Despite this analogy, little consideration in the context of M–theory was given to the Freund–Rubin solutions until the end of the last year.

Recently, in [17, 18] an important observation was made that lead us to address the question answered by the present paper.

It had been known in the literature already for some years [19] that near the horizon \( (r \rightarrow 0) \) of an \( Mp \)–brane (defined according to (1)) the exact metric becomes approximated by the metric of the following 11–dimensional space:

\[ M^\text{hor}_p = AdS_{p+2} \times S^{9-p} \] (9)

that has

\[ T^\text{hor}_p = SO(2, p+1) \times SO(10-p) \] (10)

as the isometry group.

It was observed in [17] that for \( p = 2 \) and \( p = 5 \) the Lie algebra of \( T^\text{hor}_p \) can be identified with the bosonic sector of a superalgebra \( SC_p \) admitting the interpretation of conformal superalgebra on the \( p \)–brane world–volume. The explicit identifications are

\[ T^\text{hor}_2 = SO(2, 3) \times SO(8) ; \ SC_2 = Osp(8|4) \] (11)

\[ T^\text{hor}_5 = SO(2, 6) \times SO(5) ; \ SC_5 = Osp(2, 6|4) \]

1After submitting this paper, our attention was called to ref. [16], where membrane solutions of \( D = 11 \) supergravity were found, reducing to \( AdS_4 \times (7 \text{-dim. Einstein space}) \) at the horizon.
where $Osp(8|4)$ is the real section of the complex orthosymplectic algebra $Osp^c(8|4)$ having $SO(8) \times Sp(4, \mathbb{R})$ as bosonic subalgebra, while $Osp(2,6|4)$ is the real section of the same complex superalgebra having $SO(2,6) \times (USp(4) \sim SO(5))$ as bosonic subalgebra.

In [17] it was shown how to realize the transformations of $SC_p$ as symmetries of the linearized $p$–brane world–volume action. In [18] it was instead proved that the non–linear Born Infeld effective action of the $p$–brane is invariant under conformal like transformations that realize the group $T_{phor}^p$. Since these transformations are similar but not identical to the standard conformal transformations the author of [15] named them broken conformal transformations. Further very recent developments in this direction appeared in [29, 30].

What we would like to stress in relation with these developments is that the near horizon manifold (9) is just one instance of Freund–Rubin solution, where the internal coset manifolds have been chosen to be:

\[
\begin{align*}
\left(\frac{G}{H}\right)_7 &= \frac{SO(8)}{SO(7)} \equiv S^7; \\
\left(\frac{G}{H}\right)_4 &= \frac{SO(5)}{SO(4)} \equiv S^4
\end{align*}
\] (12)

and the superconformal algebras (11) are nothing else but the full supersymmetry algebras of these Freund–Rubin vacua.

Therefore we see that ordinary M2 and M5–branes are correctly interpreted as $D=11$ supergravity solitons that interpolate between two vacuum solutions of the theory, the space

\[
M_p^\infty = \text{Minkowski}_{11}
\] (13)

at spatial infinity ($r \to \infty$) and the manifold (8) near the horizon ($r \to 0$), which is nothing else but one of the possible Freund–Rubin manifolds.

These solutions are $\frac{1}{2}$–BPS solitons that interpolate between maximally symmetric geometries, i.e. flat SuperPoincaré and maximally anti de Sitter supergravity at the horizon.

In this paper we consider $M$–theory branes which still interpolate between flat and anti de Sitter spaces, but with $N < N_{\text{max}}$ supersymmetries. This allows solutions which, at the horizon, will look as $AdS_{p+2}$ supergravity $\times M_{D-p-2}$ where $M_{D-p-2}$ is a manifold admitting $N$ Killing spinors, appropriate to a $\frac{N}{2}$–BPS soliton. Solutions of this type were originally considered in [16].

Recently $p$-brane theories giving $AdS_{p+2}$ horizon geometries (for the $p = 3$ case) with less supersymmetries have been considered via an explicit construction of the $p + 1$ world volume theory with $N < 4$ [31, 32, 33, 34].

Here we follow the opposite viewpoint, namely starting directly from $M$ theory we consider the soliton solutions of D=11 Supergravity that interpolate between the other Freund Rubin vacua near a horizon and some flat manifold near spatial infinity. For each Freund Rubin manifold there exists a corresponding soliton. We name it a $G/H$ $M$–brane. As we show in the next section the crucial thing is the existence of an Einstein $G$–invariant metric on $G/H$. These metrics were explicitly constructed for all seven and four–dimensional coset manifolds and can be utilized in the explicit derivation of the new 11–dimensional interpolating soliton metrics. The next question is whether such classical solitons are BPS states, namely whether they admit suitable Killing spinors. The answer is once again provided by the old results.
obtained in the context of Kaluza Klein supergravity. What matters is the number $N_{G/H}$ of (commuting) Killing spinors on $G/H$ defined as the solutions of the following equation:

$$\left[D^G_{m/H} + e \Gamma_m\right] \eta = 0 \quad (14)$$

where $D^G_{m/H}$ is the spinorial covariant derivative on $G/H$ calculated with respect to the $G$–invariant spin–connection and $\Gamma_m$ denotes the Dirac matrices in dimension $\dim G/H$. The parameter $e$ is the Freund–Rubin parameter, namely the vev of the 4–form field strength which sets also the scale of anti de Sitter space. For all Freund Rubin manifolds eq. (14) was thoroughly analysed in the eighties, the number $N_{G/H}$ was determined and the solutions $\eta$ were explicitly constructed. In the next section we show that each of such solutions can be used to construct a Killing spinor for the corresponding interpolating soliton. The new Killing spinor is restricted, just as in the case of ordinary $M$–branes, by the action of a projection operator that halves its 32–components. Hence the conclusion is that for $G/H$ M–branes the number of preserved supersymmetries is given by the following calculation:

$$\# \text{supersymmetries in } G/H \text{ 2–brane } = \frac{1}{2} \times \frac{32}{8} \times N_{G/H} = 2N_{G/H} \quad (15)$$

$$\# \text{supersymmetries in } G/H \text{ 5–brane } = \frac{1}{2} \times \frac{32}{4} \times N_{G/H} = 4N_{G/H} \quad (16)$$

In the above equation the factor $\frac{1}{2}$ accounts for the aforementioned projection, the factors $\frac{32}{8}$ or $\frac{32}{4}$ account for the fact that spinor charges are counted in units of either 8–component spinors for $(G/H)_7$ or 4–component spinors for $(G/H)_4$. In any case the total number of spinor charges preserved by the $G/H$ M–brane is 1/2 of the number of spinor charges preserved by the corresponding Freund–Rubin solution. Indeed the Killing spinors of the Freund-Rubin vacua are $N_{G/H}$ tensor products of a 4–component spinor with an 8-component spinor for the 2–brane case and of an 8–component spinor with a 4–component spinor in the 5–brane case. This is the familiar near horizon doubling of supersymmetries.

Summarizing: all $G/H$ M–branes with $N_{G/H} > 0$ are BPS states.

The bosonic isometry group of these classical solutions is given by

$$\mathcal{I}_{G/H-p-brane} = ISO(1,p) \otimes G \quad (17)$$

which replaces eq. (4). Furthermore, in the case $(\frac{G}{H})_7$, recalling results that were obtained in the early eighties [20], we know that, if the Freund Rubin coset manifold admits $N_{G/H}$ Killing spinors, then the structure of the isometry group $G$ is necessarily the following one:

$$G = G' \otimes SO\left(N_{G/H}\right) \quad (18)$$

so that the factor $SO\left(N_{G/H}\right)$ can be combined with the isometry group $SO(2,3)$ of anti de Sitter space to produce the orthosymplectic algebra $Osp\left(N_{G/H}|4\right)$. The same argument leads to the conclusion that in the case $(\frac{G}{H})_4$, the existence of $N_{G/H}$ Killing spinors should imply:

$$G = G' \otimes Usp\left(N_{G/H}\right) \quad (19)$$
In this way the factor $Usp\left(N_{G/H}\right)$ can be combined with the anti de Sitter group $SO(2,6)$ into the orthosymplectic algebra $Osp\left(2,6|N_{G/H}\right)$.

Therefore as the microscopic effective action of ordinary Mp–branes is invariant under transformations of the superconformal algebras (11), in the same way we can conjecture that for $G/H$ Mp–branes the world–volume action should have the following superconformal symmetries:

$$SC_{2}^{G/H} = Osp\left(N_{G/H}|4\right) \times G'$$
$$SC_{5}^{G/H} = Osp\left(2,6|N_{G/H}\right) \times G'$$

- Dimensional transmigration, anti de Sitter space and solvable algebras

The key point in the above outlined developments is a mechanism that we might describe as a dimensional transmigration. In Freund–Rubin solutions the 11–dimensional space is split in either $4 + 7$ or $7 + 4$, the second number denoting the dimensions of the compactified manifold and the first those of the effective space–time. On the other hand, from the viewpoint of the Mp–brane solution the dimensional split is either $11 = 3 + 8$ or $11 = 6 + 5$ the first number denoting the world–volume dimensions, the second number the dimensions of the transverse space. Hence the existing relation between the superisometry group of the near horizon geometry and the superconformal symmetry of the world–volume action involves the following dimensional transmigration: anti de Sitter space $AdS_{p+2}$ looses one of its $p + 2$ dimensions and becomes the $d = p + 1$ dimensional world–volume of the $p$–brane. The lost dimension is swallowed by the compact manifold $G/H$ that, by absorbing it, becomes the transverse manifold to the $p$-brane. The interpolation between two vacua performed by the M–brane soliton is nothing else but such a dimensional transmigration which occurs smoothly while going from the horizon to spatial infinity. This is a nice counting but it poses an obvious question. How can we intrinsically characterize the 1–dimensional submanifold of anti de Sitter space that can transmigrate to the transverse manifold? As we show in section 4 and more extensively in appendix A, the use of solvable Lie algebras answers this question. Anti de Sitter space is a non compact pseudo–riemanian coset manifold:

$$AdS_{p+2} \equiv \frac{SO(2,p+1)}{SO(1,p+1)}$$

yet, in the same way as all the riemanian non–compact coset manifolds, it can be identified with a solvable group manifold, that is:

$$AdS_{p+2} = \exp [Solv]$$

where $Solv$ denotes an appropriate $p + 2$–dimensional solvable Lie algebra. The structure of this solvable algebra is simple: it contains a $p+1$–dimensional abelian ideal $A$ and a single Cartan generator $C$. The solvable group parameters associated with the abelian ideal $A$ span the submanifold of $AdS_{p+2}$ that can be viewed as the $p$–brane world volume. On the other hand the 1–dimensional submanifold generated by the Cartan operator $C$ is the one that performs
the transmigration. Indeed naming $\rho$ this coordinate we find that it is nothing else but the the square of the radial coordinate (3):

$$\rho = r^2$$  \hspace{1cm} (23)

As we show in the appendix and in section 3 the solvable parametrization of anti de Sitter space is that which is naturally provided by the Bertotti Robinson metric [21]. Originally, Bertotti and Robinson introduced a metric in 4–dimensions which describes the tensor product of an anti de Sitter space $AdS_2$ with a 2–sphere $S^2$. Their metric can be easily generalized to dimensions $D$ and describes the tensor product of an anti de Sitter space $AdS_{p+2}$ with a sphere $S_{D-p-2}$. It reads as follows:

$$ds^2 = \rho^2 (-dt^2 + d\vec{z} \cdot d\vec{z}) + \rho^{-2} d\rho^2 + d\Omega^2_{D-p-2}$$  \hspace{1cm} (24)

where the last term $d\Omega^2_{D-p-2}$ is the invariant metric on the $S_{D-p-2}$ sphere, while the previous ones correspond to a particular parametrization of the anti de Sitter metric on $AdS_{p+2}$. This parametrization is precisely the solvable one, the $p+1$ coordinates $t, \vec{z}$ being those associated with the abelian ideal, while $\rho$ is associated with the Cartan generator.

Upon the identification (23), the $D = 11$ Bertotti Robinson metric (24) is the horizon limit ($r \to 0$) of the ordinary Mp-brane metric (1). The reason why the $S_{D-p-2}$ sphere emerges is the standard fact that, using polar coordinates, flat space $\mathbb{R}^{D-p-1}$ can be viewed as a sphere fibration over the positive real line $\mathbb{R}_+$. Indeed we can write the familiar identity:

$$dX^I dX^J \delta_{IJ} = dr^2 + \rho^2 d\Omega^2_{D-p-2}$$  \hspace{1cm} (25)

The crucial observation for the derivation of $G/H$ M–branes is that as $D - p - 1$-dimensional manifold transverse to the $p$–brane, rather than a sphere fibration we can consider a $G/H$–fibration on $\mathbb{R}_+$, $G/H$ being a $D - p - 2$–dimensional coset manifold. This is made possible by the simultaneous fibered structure of anti de Sitter space. In other words the base–manifold $\mathbb{R}_+$ is shared in the bulk of the soliton solution by both fibres, the world–volume fibre and the transverse $G/H$ fibre. When we approach one of the two limits $r \to 0$ or $r \to \infty$ we reconstruct either the anti de Sitter fibration or the transverse space fibration.

The solvable parametrization of anti de Sitter space is also the key to understand the reinterpretation of the isometry superalgebras (20) as superconformal algebras on the brane world–volume. Although our analysis can be extended to the entire supertransformations let us for the moment focus on the bosonic ones. From the world–volume viewpoint the Cartan coordinate $\rho$ becomes a scalar field that enters the generalized Born–Infeld action. Then the broken conformal transformations found by Maldacena in [18] are nothing else but the ordinary action of the isometry group $SO(2, p+1)$ on the $SO(2, p+1)/SO(1, p+1)$ coset representative when the solvable parametrization is adopted. This we explicitly verify in section 4.

Our paper is organized as follows. In section 2 we derive the $G/H$ p–brane soliton solutions of D=11 supergravity. In section 2.3 we discuss their Killing spinors and we show that they are BPS states. In section 3 we analyse the solvable parametrization of anti de Sitter space $AdS_4$ and we show that it leads to the Bertotti Robinson form of the metric. In section 4 we calculate the Killing vectors representing the $SO(2, 3)$ Lie algebra on the solvable coordinates and from them we exactly retrieve the form of Maldacena broken conformal transformations.
Section 5 contains our conclusions. Finally in appendix A we discuss the generalization of our results to a generic AdSp+2 space and we show how the appropriate solvable Lie algebra can be constructed using Iwasawa decomposition.

2 Derivation of the G/H M–brane solutions

In this section we derive the G/H M–brane solutions advocated in the introduction, whose existence was originally proved in [16]. Such configurations are classical solutions of D = 11 supergravity; yet we find it convenient to start by revisiting the derivation of p–brane solutions in a generic space–time dimension D. So doing we can better illustrate the nature of the generalization we propose. Indeed the assumption that transverse space has the topology of IRD−p−1 now appears to be necessary.

In the most general setting p–branes are thought of as solutions of N–extended supergravity theories. Hence the relevant bosonic lagrangian involves a number f of n–form field strengths

\[ F^\Lambda_{M_1...M_n} = n \partial_{[M_1} A^\Lambda_{M_2...M_n]} \] (M_i = 0, ..., D − 1, \Lambda = 1, ..., f)

and a number s of scalar fields \( \phi^i \) (i = 1, ..., s), the relation between the number of space–like dimensions of the brane and the degree of the field–strength being

\[ n = p + 2 \] (26)

In this setting the elaborated geometry of the scalar sector and its relation with the group of duality transformations plays an important role. One example of this is provided by the discussion of the most general 0–brane solutions (black–holes) in N = 8 supergravity where f = 28 and s = 70 [25].

2.1 The general p–brane action in D–dimensions

It is important to take into account all the field strengths and all the scalar fields in order to study the orbits of the U–duality group and the moduli dependence of the solution. However if we are interested in the space–time structure of the p–brane soliton it is sufficient to restrict our attention to a lagrangian of the following type [3]:

\[
I = \int d^Dx \sqrt{-g} \left[ R - \frac{1}{2} \nabla_M \phi \nabla^M \phi - \frac{1}{2n!} e^{a\phi} F_{[n]}^2 \right] \] (27)

involving only the metric \( g_{MN} \), a single scalar \( \phi \) (the dilaton) and a single \( (n − 1) \)-form gauge potential \( A_{[n−1]} \) with field strength \( F_{[n]} \) ( the parameter a is the scalar coupling). From the point of view of the complete supergravity theory, eq. (27) corresponds to U–rotate the field strength vector to a standard one with a single non–vanishing component and truncate the action to such a sector. Similarly \( \phi \) denotes the combination of scalars that couples to the selected field strength.

The field equations derived from (27) have the following form:

\[ R_{MN} = \frac{1}{2} \partial_M \phi \partial_N \phi + S_{MN} \] (28)
\[ \nabla M_1(e^{a \phi} F^{M_1 \ldots M_n}) = 0 \]  
\[ \Box \phi = \frac{a}{2n!} F^2 \]  

where \( S_{MN} \) is the energy-momentum tensor of the \( n \)-form \( F \):

\[ S_{MN} = \frac{1}{2(n-1)!} e^{a \phi} [F_M \ldots F_N \ldots - \frac{n-1}{n(D-2)} F^2 g_{MN}] \]

\subsection*{2.2 The \( G/H \) electric \( p \)-brane ansatz}

Motivated by the arguments discussed in section 1 we search for solutions of eq.s (28), (29), (30) of the form:

\[ ds^2 = e^{2A(r)} dx^\mu dx^\nu \eta_{\mu\nu} + e^{2B(r)}[dr^2 + r^2 \lambda^{-2} ds_{G/H}^2] \]

\[ A_{\mu_1 \ldots \mu_d} = \epsilon_{\mu_1 \ldots \mu_d} e^{C(r)} \]

\[ \phi = \phi(r) \]

where

1. \( \lambda \) is a constant parameter with the dimensions of length.
2. The \( D \) coordinates \( X^M \) are split as follows: \( X^M = (x^\mu, r, y^m) \), \( \eta^{MN} = \text{diag}(-, +, +, \ldots) \)
3. \( \mu = 0, \ldots, d - 1 \) runs on the \( p \)-brane world-volume \( (d = p + 1) \)
4. \( \bullet \) labels the \( r \) coordinate
5. \( m = d + 1, \ldots, D - 1 \) runs on some \( D - d - 1 \)-dimensional compact coset manifold \( G/H \), \( G \) being a compact Lie group and \( H \subset G \) a closed Lie subgroup.
6. \( ds_{G/H}^2 \) denotes a \( G \)-invariant metric on the above mentioned coset manifold.

Eq.s (32), (33), (34) provide the \( G/H \) generalization of the standard electric \( p \)-brane ansatz extensively considered in the literature (see for instance [5]). Indeed, the only difference with the ordinary case is that we have replaced the invariant metric \( ds^2_{S^{D-d-1}} \) on a sphere \( S^{D-d-1} \) by the more general coset manifold metric \( ds_{G/H}^2 \). Applied to the case of \( D = 11 \) supergravity, the electric ansatz will produce both \( G/H \) M2–brane and M5–brane solutions.

On the other hand it is well known that for ordinary branes there exists also a magnetic solitonic ansatz. The \( G/H \) generalization of such a magnetic ansatz is straightforward but we do not dwell on it in this paper, leaving a more in depth analysis for a future publication.

As anticipated in the introduction the isometry group of the field configuration introduced by the electric ansatz (32), (33), (34) is given by the group \( \mathcal{I}_{G/H-p-brane} \) defined in eq.(17).
2.2.1 The Vielbein

In order to prove that the ansatz (32)-(34) is a solution of the field equations it is necessary to calculate the corresponding vielbein, spin–connection and curvature tensors. We use the convention that tangent space indices are underlined. Then the vielbein components relative to the ansatz (32) are:

\[
E^\mu = e^A dx^\mu; \quad E_\bullet = e^B dr; \quad E^m = e^B r \lambda^{-1} E_m,
\]

\[
g_{\mu\nu} = e^{2A} \eta_{\mu\nu}; \quad g_{\bullet\bullet} = e^{2B}, \quad g_{mn} = e^{2B} r^2 \lambda^{-2} g_{mn}
\]

with \( E_x \equiv G/H \) vielbein and \( g_{mn} \equiv G/H \) metric.

2.2.2 The spin connection

The Levi–Civita spin–connection on our \( D \)-dimensional manifold is defined as the solution of the vanishing torsion equation:

\[
dE^M + \omega^M_{\ N} \wedge E^N = 0 \quad (37)
\]

Solving eq.(37) explicitly we obtain the spin–connection components:

\[
\omega^\mu_\nu = 0, \quad \omega^\bullet_\nu = e^{-B} A'E^\nu, \quad \omega^m_\nu = 0, \quad \omega^\bullet^m = e^{-B} (B' + r^{-1}) E^m.
\]

where \( A' \equiv \partial_\bullet A \) etc. and \( \omega^m_\nu \) is the spin connection of the \( G/H \) manifold.

2.2.3 The Ricci tensor

From the definition of the curvature 2-form:

\[
R^{MN} = d\omega^{MN} + \omega^M_{\ S} \wedge \omega^S^N
\]

we find the Ricci tensor components:

\[
R_{\mu\nu} = -\frac{1}{2} \eta_{\mu\nu} e^{2(A-B)} [A'' + d(A')^2 + \tilde{d}A'B' + (\tilde{d} + 1)r^{-1}A']
\]

\[
R_{\bullet\bullet} = -\frac{1}{2} [d(A'') + (A')^2 - A'B'] + (\tilde{d} + 1)(B'' + r^{-1}B')]
\]

\[
R_{mn} = -\frac{1}{2} g_{mn} \frac{r^2}{\lambda^2} [dA'(B' + r^{-1}) + r^{-1}B' + B'' + \tilde{d}(B' + r^{-1})^2] + R_{mn}
\]

where \( R_{mn} \) is the Ricci tensor of \( G/H \) manifold, and \( \tilde{d} \equiv D - d - 2 \).

2.2.4 The field equations

Inserting the electric ansatz into the field eq.s (28) yields:

\[
A'' + d(A')^2 + \tilde{d}A'B' + (\tilde{d} + 1)A'r^{-1} = \frac{\tilde{d}}{2(D - 2)} S^2
\]
\[ d[A'' + (A')^2 - A'B'] + (\tilde{d} + 1)[B'' + r^{-1}B'] = \frac{\tilde{d}}{2(D-2)} S^2 - \frac{(\phi')^2}{2} \]  
\[ g_{mn}[dA'(B' + r^{-1}) + r^{-1}B' + B'' + \tilde{d}(B' + r^{-1})^2] - 2R_{mn} = \]
\[ -\frac{d}{2(D-2)} g_{mn} S^2 \]  
while eq.s (29)-(30) become:
\[ C'' + (\tilde{d} + 1) r^{-1}C' + (\tilde{d}B' - dA' + C' + a\phi')C' = 0 \]  
\[ \phi'' + (\tilde{d} + 1) r^{-1} \phi' + [dA' + \tilde{d}B'] \phi' = -\frac{a}{2} S^2 \]
with
\[ S \equiv e^{\phi + C - dA} \]

### 2.3 Construction of the BPS Killing spinors in the case of \( D = 11 \) supergravity

At this point we specialize our analysis to the case of \( D = 11 \) supergravity, whose action in the bosonic sector reads:
\[ I_{11} = \int d^{11} x \sqrt{-g} \left( R - \frac{1}{48} F_{[4]^2}^2 + \frac{1}{6} \int F_{[4]} \wedge F_{[4]} \wedge A_{[3]} \right) \]  
and we look for the further restrictions imposed on the electric ansatz by the requirement that the solutions should preserve a certain amount of supersymmetry. This is essential for our goal since we are interested in \( G/H \) M–branes that are BPS saturated states and the BPS condition requires the existence of Killing spinors.

As discussed in ref. \[5\], the above action does not fall exactly in the general class of actions of type (27). Nevertheless, the results of sections 2.1 and 2.2 still apply: indeed it is straightforward to verify that the FFA term in the action (49) gives no contribution to the field equations once the electric or magnetic ansatz are implemented. Moreover no scalar fields are present in (49): this we handle by simply setting to zero the scalar coupling parameter \( a \).

Imposing that the ansatz solution admits Killing spinors allows to simplify the field equations drastically.

We recall the supersymmetry transformation for the gravitino:
\[ \delta \psi_M = \bar{D}_M \epsilon \]  
with
\[ \bar{D}_M = \partial_M + \frac{1}{4} \omega_M^{AB} \Gamma_{AB} - \frac{1}{288} [\Gamma^{PQRS} M + \Gamma^{PQR} \delta_M S] F_{PQRS} \]  
Requiring that setting \( \psi_M = 0 \) be consistent with the existence of residual supersymmetry yields:
\[ \delta \psi_M |_{\psi = 0} = \bar{D}_M \epsilon = 0 \]
Solutions $\epsilon(x, r, y)$ of the above equation are Killing spinor fields on the bosonic background described by our ansatz.

In order to discuss the solutions of (52) we adopt the following tensor product realization of the $(32 \times 32)$ $SO(1, 10)$ gamma matrices:

$$\Gamma_A = [\gamma_\mu \otimes \mathbb{1}_8, \gamma_3 \otimes \mathbb{1}_8, \gamma_5 \otimes \Gamma_m]$$

The above basis (53) is well adapted to our $(3+1+7)$ ansatz. The $\gamma_\mu \ (\mu = 0, 1, 2, 3)$ are usual $SO(1, 3)$ gamma matrices, $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$, while $\Gamma_m$ are $8 \times 8$ gamma matrices realizing the Clifford algebra of $SO(7)$. Thus for example $\Gamma_\bullet = \gamma_3 \otimes \mathbb{1}_8$.

Correspondingly, we split the D=11 spinor $\epsilon$ as follows

$$\epsilon = \varepsilon \otimes \eta(r, y)$$

where $\varepsilon$ is an $SO(1, 3)$ constant spinor, while the $SO(7)$ spinor $\eta$, besides the dependence on the internal $G/H$ coordinates $y^m$, is assumed to depend also on the radial coordinate $r$. Note the difference with respect to Kaluza Klein supersymmetric compactifications where $\eta$ depends only on $y^m$. Computing $\tilde{D}$ in the ansatz background yields:

$$\tilde{D}_\mu = \partial_\mu + \frac{1}{2}e^{-B-2A}\gamma_\mu \gamma_3 [e^{3A}A' - \frac{i}{3}e^C C' \gamma_3 \gamma_5] \otimes \mathbb{1}_8$$

$$\tilde{D}_\bullet = \partial_r + \frac{i}{6}e^{-3A}C' e^C \gamma_3 \gamma_5 \otimes \mathbb{1}_8$$

$$\tilde{D}_m = D^{G/H}_m + \frac{r}{2\lambda}[(A' + r^{-1})i\gamma_3 \gamma_5 + \frac{1}{6}e^{C-3A}C'] \otimes \Gamma_m$$

where all $\gamma_\mu, \Gamma_m$ have tangent space indices. The Killing spinor equation $\tilde{D}_\mu \epsilon = 0$ becomes equivalent to:

$$(1_4 - i\gamma_3 \gamma_5)\epsilon = 0; \quad 3e^{3A}A' = e^C C'$$

Thus half of the components of the 4-dim spinor $\varepsilon$ are projected out. Moreover the second equation is solved by $C = 3A$. Considering next $\tilde{D}_\bullet \epsilon = 0$ leads to the equation (where we have used $C = 3A$):

$$\partial_r \eta + \frac{1}{6}C' \eta = 0$$

whose solution is

$$\eta(r, y) = e^{-C(r)/6} \eta_o(y)$$

Finally, $\tilde{D}_m \epsilon = 0$ implies

$$B = -\frac{1}{6}C + \text{const.}$$

$$[D^{G/H}_m + \frac{1}{2\lambda} \Gamma_m] \eta_o = 0$$

Eq.(60) deserves attentive consideration. If we identify the Freund–Rubin parameter as:

$$e \equiv \frac{1}{2\lambda}$$

11
then eq.(60) is nothing else but the Killing spinor equation for a $G/H$ spinor that one encounters while discussing the residual supersymmetry of Freund–Rubin vacua. The solutions of this equation have been exhaustively studied in the old literature on Kaluza–Klein supergravity (see [22] for a comprehensive review) and are all known.

In this way we have explicitly verified what was mentioned in the introduction, namely that the number of $BPS$ Killing spinors admitted by the $G/H$ M–brane solution is $N_{G/H}$, i.e. the number of Killing spinors admitted by the corresponding Freund–Rubin vacuum.

2.4 M-brane solution

To be precise the Killing spinors of the previous section are admitted by a configuration that has still to be shown to be a complete solution of the field equations. To prove this is immediate. Setting $D = 11, d = 3, \tilde{d} = 6$, the scalar coupling parameter $a = 0$, and using the relations $C = 3A, B = -C/6 + \text{const.} = -A/2 + \text{const.}$ we have just deduced, the field equations (43), (44), (45) become:

\begin{align*}
A'' + 7r^{-1}A' & = \frac{1}{3}S^2 \\
(A')^2 & = \frac{1}{6}S^2 \\
R_{mn} & = \frac{3}{\lambda^2}g_{mn}
\end{align*}

Combining the first two equations to eliminate $S^2$ yields:

\begin{equation}
\nabla^2 A - 3(A')^2 \equiv A'' + \frac{7}{r}A' - 3(A')^2 = 0 \tag{65}
\end{equation}

or:

\begin{equation}
\nabla^2 e^{-3A} = 0 \tag{66}
\end{equation}

whose solution is:

\begin{equation}
e^{-3A(r)} = H(r) = 1 + \frac{k}{r^6} \tag{67}
\end{equation}

We have chosen the integration constant such that $A(\infty) = 0$. The functions $B(r)$ and $C(r)$ are then given by $B = -A/2$ (so that $B(\infty) = 0$) and $C = 3A$. Finally, after use of $C = 3A$, the F-field equation (46) becomes equivalent to (53). The equation (66) determining the radial dependence of the function $A(r)$ (and consequently of $B(r)$ and $C(r)$) is the same here as in the case of ordinary branes, while to solve eq.(71) it suffices to choose for the manifold $G/H$ the $G$–invariant Einstein metric. Each of the Freund–Rubin cosets admits such an Einstein metric which was also constructed in the old Kaluza–Klein supergravity literature (see [15, 22]).

Summarizing: for $D = 11$ supergravity the field equations are solved by the ansatz (32), (33) where the $A, B, C$ functions are

\begin{equation}
A(r) = -\frac{\tilde{d}}{18} \ln \left(1 + \frac{k}{r^4}\right) = -\frac{1}{3} \ln \left(1 + \frac{k}{r^6}\right)
\end{equation}
\[ B(r) = \frac{d}{18} \ln \left( 1 + \frac{k}{r^d} \right) = \frac{1}{6} \ln \left( 1 + \frac{k}{r^6} \right) \]

\[ C(r) = 3A(r) \]

(68)

displaying the same \( r \)-dependence as the ordinary M–brane solution (\( \mathbb{1} \)).

In this way we have illustrated the existence of \( G/H \) M–brane solutions (cf. \( [14] \)). Table 1 displays the Freund–Rubin cosets with non vanishing \( N_{G/H} \). Each of them is associated to a BPS saturated M–brane. The notations are as in ref.s \( [15], [22] \).

| G/H   | G                  | H                  | \( N_{G/H} \) |
|-------|--------------------|--------------------|---------------|
| \( S^7 \) | \( SO(8) \)      | \( SO(7) \)        | 8             |
| squashed \( S' \) | \( SO(5) \times SO(3) \) | \( SO(3) \times SO(3) \) | 1             |
| \( M_{ppr} \) | \( SU(3) \times SU(2) \times U(1) \) | \( SU(2) \times U(1)^2 \) | 2             |
| \( N_{100} \) | \( SU(3) \times SU(2) \) | \( SU(2) \times U(1) \) | 3             |
| \( N_{ppr} \) | \( SU(3) \times U(1) \) | \( U(1)^2 \) | 1             |
| \( Q_{ppp} \) | \( SU(2)^3 \) | \( U(1)^3 \) | 2             |
| \( B'_r \) | \( SO(5) \) | \( SO(3)_{\text{max}} \) | 1             |
| \( V_{5,2} \) | \( SO(5) \times U(1) \) | \( SO(3) \times U(1) \) | 2             |

Table 1: Supersymmetric Freund Rubin Cosets with Killing spinors

3 \( AdS_4 \) parametrization and the Bertotti Robinson metric

In this section we consider the explicit example of the 4–dimensional anti de Sitter space:

\[ AdS_4 \equiv \frac{SO(2,3)}{SO(1,3)}. \]

(69)

Relying on the algebraic derivation explained in appendix \( A \) we claim that this coset manifold can be identified with the exponential of a 4–dimensional solvable Lie algebra \( Solv_4 \). The complex form of the \( SO(2,3) \) Lie algebra is \( B_2 \) and the root system is composed by the eight roots:

\[ \pm \epsilon_1 \pm \epsilon_2 ; \; \pm \epsilon_1 , \; \pm \epsilon_2 \]

(70)

where \( \epsilon_i \) denote the unit vectors in a Euclidean two–dimensional space. Adopting the standard notation \( E_\alpha \) for the step operator associated with the root \( \alpha \) and \( H_\alpha \) for the Cartan generator obtained by commuting \( E_\alpha \) with \( E_{-\alpha} \), the results of the appendix yield the following conclusion.
The solvable Lie algebra $Solv(AdS_4)$ generating 4–dimensional anti de Sitter space is spanned by the following three nilpotent operators

$$\mathcal{T}_+ \equiv E_{e^2} ; \quad \mathcal{T}_- \equiv E_{e^2+\epsilon_1} ; \quad \mathcal{T}_+ \equiv E_{e^2-\epsilon_1} \quad (71)$$

plus the following non–compact Cartan generator

$$\mathcal{C} \equiv H_{e^2} \quad (72)$$

The matrix realization of these generators in the 5 of $SO(2, 3)$ is:

$$\mathcal{C} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \mathcal{T}_+ = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

$$\mathcal{T}_- = \begin{pmatrix} 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \mathcal{T}_+ = \begin{pmatrix} 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (73)$$

The reason for eq.(71) is that the parameter associated with $\mathcal{T}_+, \mathcal{T}_-, \mathcal{T}_+$ will be respectively interpreted as the transverse and light–cone coordinates on the 2–brane world volume. This will be manifest at the end of our calculations. For the time being just take these as convenient names given to the solvable Lie algebra generators. Using such a notation we write the coset representative in the following way:

$$L(a, x, t, w) = \tau(x, t, w) S(a)$$

$$S(a) \equiv \exp [-a \mathcal{C}]$$

$$\tau(x, t, w) = \exp [\sqrt{2} x \mathcal{T}_+ + (t - w) \mathcal{T}_- + (t + w) \mathcal{T}_+] \quad (74)$$

By explicit evaluation we find:

$$S(a) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \cosh[a] & 0 & -\sinh[a] & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -\sinh[a] & 0 & \cosh[a] & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (75)$$

and

$$\tau(x, t, w) = \begin{pmatrix} 1 & -t & 0 & t & 0 \\ t & \frac{2-t^2+w^2+x^2}{2} & w & \frac{t^2-w^2-x^2}{2} & x \\ 0 & w & 1 & -w & 0 \\ t & -\frac{t^2+w^2+x^2}{2} & w & \frac{2+t^2-w^2-x^2}{2} & x \\ 0 & x & 0 & -x & 1 \end{pmatrix} \quad (76)$$
Then it is straightforward to calculate the left invariant 1–form and one obtains:

$$\Omega = L^{-1} dL = \begin{pmatrix}
0 & -(dt e^a) & 0 & dt e^a & 0 \\
(dt e^a) & 0 & dw e^a & -da & dx e^a \\
0 & dw e^a & 0 & -(dw e^a) & 0 \\
(dt e^a) & -da & dw e^a & 0 & dx e^a \\
0 & dx e^a & 0 & -(dx e^a) & 0
\end{pmatrix} \quad (77)$$

With these results we are now in a position to calculate the vielbein, the spin connection and the curvature of our anti de Sitter space in the solvable parametrization.

To this effect it suffices to write a standard basis of generators for the $SO(2,3)$ Lie algebra singling out the $\mathbb{K}$ coset orthogonal subspace from the $\mathbb{H} \equiv SO(1,3)$ subalgebra.

First we recall that in our convention the $SO(2,3)$ group is given by the set of $5 \times 5$ matrices that leave invariant the following diagonal metric:

$$\eta = \text{diag} (-,-,+,+,+) \quad (78)$$

Written in standard form the $SO(2,3)$ Lie algebra is as follows:

$$\left[ M^{AB}, M^{CD} \right] = -\eta^{AC} M^{BD} + \eta^{BC} M^{AD} + \eta^{AD} M^{BC} - \eta^{BD} M^{AC}$$

$$M^{AB} = -M^{BA} \quad ; \quad A, B = 1, \ldots, 5 \quad (79)$$

Furthermore the Lorentz subalgebra $SO(1,3)$ we have chosen is given by the subset of $SO(2,3)$ Lie algebra matrices that are of the following form:

$$\begin{pmatrix}
* & 0 & * & * & * \\
0 & 0 & 0 & 0 & 0 \\
* & 0 & * & * & * \\
* & 0 & * & * & * \\
* & 0 & * & * & *
\end{pmatrix} \quad (80)$$

Correspondingly we can write the orthogonal decomposition of the $SO(2,3)$ Lie algebra:

$$SO(2,3) = \mathbb{H}_{SO(1,3)} \oplus \mathbb{K} \quad (81)$$

where the subalgebra $\mathbb{H}_{SO(1,3)}$ is spanned by the three Lorentz boosts $N_1, N_2, N_3$ and the three angular momenta $J_1, J_2, J_3$. We list below the explicit form of these generators and their
correspondence with the generators $M^{AB}$:

\[
M^{1,3} \equiv N_1 = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
\end{pmatrix},
M^{3,4} \equiv J_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[
M^{1,4} \equiv N_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
M^{3,5} \equiv J_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[
M^{1,5} \equiv N_3 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
M^{4,5} \equiv J_3 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

On the other hand the orthogonal complement $\mathcal{K}$ is spanned by the following four generators:

\[
M^{2,1} \equiv \Pi_0 = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
M^{2,3} \equiv \Pi_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[
M^{2,4} \equiv \Pi_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
M^{2,5} \equiv \Pi_3 = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
\end{pmatrix}
\]

Ordering the ten generators in a ten-vector as follows:

\[
T_A \equiv \{\Pi_0, \Pi_1, \Pi_2, \Pi_3, N_1, N_2, N_3, J_1, J_2, J_3\} \quad , \quad A = 1, \ldots, 10 \tag{84}
\]

we find that they are trace-orthogonal according to:

\[
\text{Tr} (T_A, T_B) = k_{AB} = \text{diag}(-2, 2, 2, 2, 2, 2, 2, -2, -2, -2) \tag{85}
\]

so that, by taking traces, we can easily project the left invariant 1–form along the subspace $\mathcal{K}$ or the subalgebra $\mathfrak{H}$ . Such projections yield the vierbein and the spin connection of anti de Sitter space, respectively. Let us begin with the calculation of the vielbein. By definition we have:

\[
V^0 = -\frac{1}{2} \text{Tr} (\Pi_0 \Omega) \\
V^i = \frac{1}{2} \text{Tr} (\Pi_i \Omega) \quad , \quad i = 1, 2, 3 \tag{86}
\]
and we immediately obtain:

\[
\begin{pmatrix}
V_0 \\
V_1 \\
V_2 \\
V_3
\end{pmatrix} =
\begin{pmatrix}
dt e^a \\
dw e^a \\
-da \\
dx \perp e^a
\end{pmatrix}
\]  

(87)

setting:

\[
\rho \equiv e^a
\]  

(88)

and calculating the metric we obtain:

\[
ds^2 \equiv -V_0 \otimes V_0 + V_1 \otimes V_1 + V_2 \otimes V_2 + V_3 \otimes V_3
\]

\[
= \rho^2 \left( -dt^2 + dw^2 + dx^2 \right) + \rho^{-2} d\rho^2
\]  

(89)

which is the anti de Sitter metric in Bertotti Robinson form.

4 The \(SO(2,3)\) transformation rules

Given the coset parametrization in terms of the solvable Lie algebra parameters \((\rho, x, t, w)\) we can work out the explicit form of the Killing vectors representing the infinitesimal action of \(SO(2,3)\) on a general function of \(y^a \equiv \{\rho, x, t, w\}\). We rely on the general formula \([22]\):

\[
T_A L(y) = k_A^a \frac{\partial}{\partial y^a} L(y) - L(y) T_i W^i_A(y)
\]  

(90)

where

\[
\delta y^a \equiv \epsilon^A k_A^a(y)
\]  

(91)

defines the Killing vectors, \(\epsilon^A (A = 1, \ldots, \text{dim} \, G)\) are the Lie algebra parameters, \(T_A\) and \(T_i\) being the generators of the full Lie algebra \(G = SO(2,3)\) and of the subalgebra \(H = SO(1,3)\), respectively and \(W^i_A(y)\) is the infinitesimal \(H\)-compensator. Finally \(L(y)\) is the coset representative. Using eq.(90) and denoting \(\vec{k}_A = \{\vec{\Pi}, \vec{N}, \vec{J}\}\), for the four translations we obtain the result:

\[
\vec{\Pi}_0 = (\rho \, t) \frac{\partial}{\partial \rho} - (t \, x) \frac{\partial}{\partial x} - \frac{1}{2} (1 + \frac{1}{\rho^2} + t^2 + w^2 + x^2) \frac{\partial}{\partial t} - (t \, w) \frac{\partial}{\partial w}
\]

\[
\vec{\Pi}_1 = (\rho \, w) \frac{\partial}{\partial \rho} - (w \, x) \frac{\partial}{\partial x} - (t \, w) \frac{\partial}{\partial t} + \frac{1}{2} (1 + \frac{1}{\rho^2} - t^2 - w^2 + x^2) \frac{\partial}{\partial w}
\]

\[
\vec{\Pi}_2 = -\rho \frac{\partial}{\partial \rho} + x \frac{\partial}{\partial x} + t \frac{\partial}{\partial t} + w \frac{\partial}{\partial w}
\]

\[
\vec{\Pi}_3 = (\rho \, x) \frac{\partial}{\partial \rho} + \frac{1}{2} (1 + \frac{1}{\rho^2} - t^2 + w^2 - x^2) \frac{\partial}{\partial x} - (t \, x) \frac{\partial}{\partial t} - (w \, x) \frac{\partial}{\partial w}
\]  

(92)
while the three Lorentz boosts take the following form:

\[
\vec{N}_1 = -w \frac{\partial}{\partial t} - t \frac{\partial}{\partial w}
\]

\[
\vec{N}_2 = (\rho t) \frac{\partial}{\partial \rho} - (t x) \frac{\partial}{\partial x} + \frac{1}{2} \left(1 - \frac{1}{\rho^2} - t^2 - w^2 - x^2\right) \frac{\partial}{\partial t} - (t w) \frac{\partial}{\partial w}
\]

\[
\vec{N}_3 = -t \frac{\partial}{\partial x} - x \frac{\partial}{\partial t}
\]

Finally for the three rotation generators we get:

\[
\vec{J}_1 = (\rho w) \frac{\partial}{\partial \rho} - (w x) \frac{\partial}{\partial x} - (t w) \frac{\partial}{\partial t} + \frac{1}{2} \left(-1 + \frac{1}{\rho^2} - t^2 - w^2 + x^2\right) \frac{\partial}{\partial w}
\]

\[
\vec{J}_2 = -w \frac{\partial}{\partial x} + x \frac{\partial}{\partial w}
\]

\[
\vec{J}_3 = -x \frac{\partial}{\partial \rho} + \frac{1}{2} \left(1 - \frac{1}{\rho^2} + t^2 - w^2 + x^2\right) \frac{\partial}{\partial x} + t x \frac{\partial}{\partial t} + w x \frac{\partial}{\partial w}
\]

The corresponding compensating \(SO(1,3)\) matrices are listed below.

For the translations we get:

\[
W(\Pi_0) = \begin{pmatrix}
0 & 0 & w & \frac{1}{\rho} & x \\
0 & 0 & 0 & 0 & 0 \\
\frac{1}{\rho} & 0 & 0 & 0 & 0 \\
x & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
W(\Pi_1) = \begin{pmatrix}
0 & 0 & -t & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & 0 \\
0 & 0 & \frac{1}{\rho} & 0 & 0 \\
0 & 0 & x & 0 & 0
\end{pmatrix}
\]

\[
W(\Pi_2) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
W(\Pi_3) = \begin{pmatrix}
0 & 0 & 0 & 0 & -t \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & w \\
0 & 0 & 0 & 0 & \frac{1}{\rho} \\
-1 & 0 & -w & -1 & 0
\end{pmatrix}
\]

For the Lorentz boosts and the rotation generators the compensating \(SO(1,3)\) rotations are
given below:

\[
W(N_1) = \begin{pmatrix}
  0 & 0 & -1 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 \\
 -1 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[
W(J_1) = \begin{pmatrix}
  0 & 0 & -t & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 \\
 -t & 0 & 0 & -\frac{1}{\rho} & -x \\
  0 & 0 & \frac{1}{\rho} & 0 & 0 \\
  0 & 0 & x & 0 & 0 \\
\end{pmatrix}
\]

\[
W(N_2) = -\begin{pmatrix}
  0 & 0 & w & \frac{1}{\rho} & x \\
  w & 0 & 0 & 0 & 0 \\
 \frac{1}{\rho} & 0 & 0 & 0 & 0 \\
 x & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[
W(J_2) = \begin{pmatrix}
  0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & -1 \\
  0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 1 & 0 & 0 \\
\end{pmatrix}
\]

\[
W(N_3) = \begin{pmatrix}
  0 & 0 & 0 & 0 & -1 \\
  0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 \\
 -1 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[
W(J_3) = \begin{pmatrix}
  0 & 0 & 0 & 0 & t \\
  0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & -w \\
 t & 0 & w & \frac{1}{\rho} & 0 \\
\end{pmatrix}
\]

4.1 Retrieving broken conformal transformations

When the four dimensional anti de Sitter group \(SO(2,3)\) is interpreted as the conformal group in three–dimensional space–time we are lead to split its algebra as follows:

\[
SO(2,3) = \{P_\alpha\} \oplus \{j_{\alpha\beta}\} \oplus \{K_\alpha\} \oplus \mathcal{D}
\]

\[
\{P_\alpha\} = \text{translations} ; \quad \alpha = 0, 1, 2
\]

\[
\{K_\alpha\} = \text{conformal boosts} ; \quad \alpha = 0, 1, 3
\]

\[
\{j_{\alpha\beta}\} = \text{Lorentz rotations} ; \quad \alpha, \beta = 0, 1, 2
\]

\[
\mathcal{D} = \text{Dilatation}
\]

and consider its action the three–dimensional coset manifold

\[
M_{1,2}^{\text{Mink}} = \frac{SO(2,3)}{\mathcal{D} \times ISO(1,2)}
\]

(corresponding to \(1 + 2\) Minkowski space). This leads to the standard formulae for special conformal transformations on Minkowski coordinates. What should be noted, however is that the decomposition (97) is an intrinsic algebraic fact and it can be implemented in any case, also when \(SO(2,3)\) is realized as a group of isometries for four dimensional anti de Sitter space. It just suffices to decide which is the group \(SO(1,2)\) that we want to consider as the three–dimensional Lorentz group. Our choice is the following. We identify the matrices of the
\(SO(1, 2) \subset SO(2, 3)\) subalgebra with those of the following form:

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & \star & \star & 0 & \star \\
0 & \star & \star & 0 & \star \\
0 & 0 & 0 & 0 & 0 \\
0 & \star & \star & 0 & \star
\end{pmatrix}
\]

(99)

and correspondingly, using the notations of eq. (83) and (82), we can easily identify the translations and conformal boost generators as follows

\[
P_x = \Pi_3 + J_3 \quad K_x = \frac{\Pi_3 - J_3}{2}
\]

\[
P_t = -\Pi_0 + N_2 \quad K_t = -\frac{\Pi_0 + N_2}{2}
\]

\[
P_w = \Pi_1 - J_1 \quad K_w = \frac{\Pi_1 + J_1}{2}
\]

(100)

They form two separate three-dimensional abelian subalgebras of the anti de Sitter algebra \(SO(2, 3)\):

\[
[P_\alpha, P_\beta] = [K_\alpha, K_\beta] = 0
\]

(101)

Using the explicit form of the Killing vectors (92), (93) and (94) we can write the transformations induced by the operators

\[
P \equiv p_x P_x + p_t P_t + p_w P_w \quad ; \quad K \equiv k_x K_x + k_t K_t + k_w K_w
\]

(102)

on the variables \(\rho, x, t, w\). For the translations we immediately find:

\[
\delta x = p_x \\
\delta t = p_t \\
\delta w = p_w \\
\delta \rho = 0
\]

(103)

which is the result one would also obtain in ordinary Minkowski space. On the other hand for the conformal boosts one also immediately obtains:

\[
\delta x = k^t t x - k^w w x + \frac{k^x \left( \frac{1}{\rho^2} - t^2 + w^2 - x^2 \right)}{2}
\]

\[
\delta t = -(k^w t w) - k^x t x + \frac{k^t \left( \frac{1}{\rho^2} + t^2 + w^2 + x^2 \right)}{2}
\]

\[
\delta w = k^t t w - k^x x w + \frac{k^w \left( \frac{1}{\rho^2} - t^2 - w^2 + x^2 \right)}{2}
\]

\[
\delta \rho = -\rho \left( k^t t - k^w w - k^x x \right)
\]

(104)
Naming the coordinates and the parameters as follows:

\[ y^\alpha = \{x, t, w\} \quad ; \quad k^\alpha = \{k^x, k^t, k^w\} \]

and using the three dimensional lorentzian metric \( \eta^{\alpha\beta} = \text{diag}(+,-,+) \) to raise and lower indices, eqs (104) can be rewritten as follows:

\[
\delta y^\alpha = -y^\alpha y^\beta k^\beta + \frac{1}{2} k^\alpha \left( \frac{1}{\rho^2} + y^\beta y^\beta \right) \\
\delta \rho = \rho y^\beta k^\beta
\]

Eq.(106) exactly coincides with eq.(2.8) of the recent paper [18] by Maldacena (it suffices to identify the coordinate \( \rho \) with the field \( U \). Indeed in [18] the transformations (106) were interpreted as conformal transformations on the \( p \)-brane world volume with respect to which the microscopic Born Infeld action is invariant. Their form is similar to that of canonical conformal transformations but not exactly equal. For this reason they were named \textit{broken conformal transformations}. Our present discussion reveals their true meaning. They are indeed the transformations generated by those generators of the \( SO(2,3) \) group the act as conformal boosts in the situation where \( SO(2,3) \) is interpreted as conformal group in \( D = 3 \) dimensions. Their action however is not calculated on the three coordinates of the coset (98). It is rather calculated on the four coordinates of anti de Sitter space:

\[ \text{AdS}_4 = \frac{SO(2,3)}{SO(1,3)} \]

which yields the result (106). The catch, however, is that in the \textit{solvable Lie algebra parametrization} of \( \text{AdS}_4 \) the Cartan coordinate \( \rho \) is reinterpreted as a world–volume scalar field. So doing the isometry group of anti de Sitter space acts as a group of field dependent conformal transformations for the world–volume theory. We think that this explains the so far mysterious relation between conformal symmetry of the world volume microscopic theory and the anti de Sitter symmetry of the near horizon geometry. We stress that the clarifying item in this explanation is the choice of the \textit{solvable coordinates}.

5 Conclusions and Perspective

In this paper we have retrieved and discussed in a new perspective the class of BPS saturated classical solutions of M–theory that are in one–to–one correspondence with the old supersymmetric Freund–Rubin compactifications of \( D = 11 \) supergravity and reduce to them on the horizon. We have shown the relation between the anti de Sitter symmetry of the Freund–Rubin compactification and the superconformal symmetry of the world–volume theory.

In particular, our discussion suggests that there should exist microscopic world–volume theories where the superisometry group \( \mathcal{SC}^{G/H} \) (see (21)) of the supergravity theory is realized as a global symmetry group. Furthermore, one should be able to reconstruct the massless states of supergravity that belong to specific representations of \( \mathcal{SC}^{G/H} \), as suitable tensor products
of ‘singleton’ representations of $SC^{G/H}$. This mechanism has already been verified [32] for ordinary branes, where the superconformal group $SC$ is a simple supergroup $Osp(N|4)$. In our case the novelty is the existence of the residual symmetry group $G'$ so that the group theoretical construction of the massless states should agree at the level of both factors. This suggests a microscopic world volume theory with suitable matter multiplets. A search for these conjectured world–volume theories is postponed to the future.

In addition, we should stress that the construction of generalized M–branes, interpolating between Kaluza–Klein vacua at the horizon and flat manifolds at spatial infinity, does not exhaust all the possibilities. Specifically, the ansatz (32) can be further generalized by replacing the angular part of the metric $ds^2_{G/H}$ with a general Einstein metric that admits no continuous isometry (as already noted in [16]). Typically this is achieved by orbifoldizing the coset manifold $G/H$ with respect to the action of some discrete subgroup $\Gamma \subset G$. Quite likely such a procedure produces models similar to those already considered from a microscopic world-volume point of view in [31].

It is now a challenging problem to retrieve a description of these string solitons in string language, in particular in terms of $D$–branes.

A \hspace{1cm} AdS_{p+2} as a solvable group manifold

The aim of the present appendix is to show how anti de Sitter space in $d + 1$ dimensions, non–compact, lorentzian coset manifold

$$AdS_{d+1} \equiv \frac{SO(2, d)}{SO(1, d)}$$

(108)

can be described as a solvable group manifold. As anticipated in the main text, our analysis extends to a pseudo–riemanian case the classical treatment of riemanian non compact homogeneous manifolds [28] we have already extensively utilized to discuss the supergravity scalar sectors [26],[23],[24], [25].

Specifically, in this appendix we describe the structure of the solvable Lie algebra defined by the decomposition:

$$SO(2, d) = SO(1, d) \oplus Solv$$

(109)

Our result is that the structure of $Solv$ can be described as follows:

$$Solv = C_K \oplus ( \sum_{\alpha \in \Delta^+} E_{\alpha}) \cap SO(2, d)$$

(110)

where $C_K$ denotes the one–dimensional space consisting of the unique non–compact Cartan generator of $SO(2, d)$ which is not contained in $SO(1, d)$ and which therefore enters the quotient $SO(2, d)/SO(1, d)$. On the other hand the space $\Delta^+$ consists of all the roots of $SO(2, d)$ that have a strictly positive value on the Cartan generator in $C_K$. The intersection symbol in eq. (110) is used because, in general, the shift operators $E_{\alpha} \in \Delta^+$ do not belong to the $SO(2, d)$ real form of the $SO(2 + d)$ complex algebra. However there are suitable linear combinations of
these operators which do belong to such a real form. Hence shift operators $E_\alpha \in \Delta^+$ will enter the structure of $\text{Solv}$ defined by eq. (10) only through the appropriate linear combinations.

In what follows we first give the explicit representation of the $\text{Solv}$ generators in terms of $(d+2) \times (d+2)$ matrices leaving the metric $\eta = \text{diag} \{ -, -, +, \ldots, + \}$ invariant. Then a derivation of eq. (10) will be illustrated in more abstract terms using the Iwasawa decomposition.

The root system of $SO(2, d)$ can be expressed, with respect to an orthonormal basis of $\mathbb{R}^r$, $r = \text{rank} SO(2, d)$, in the following way:

$$\Phi = \{ \pm \epsilon_i \pm \epsilon_j \ 1 \leq i < j \leq r, \pm \epsilon_i \ i = 1, \ldots, r \} \quad \begin{array}{c}
d+2=2r \\
d+2=2r+1 \end{array}$$

The non compact Cartan generators of $SO(2, d)$ are $\{ H_{\epsilon_1}, H_{\epsilon_2} \}$. Choosing the $SO(1, d)$ subalgebra of $SO(2, d)$ that admits $H_{\epsilon_1}$ as the non–compact Cartan generator, the space $\mathcal{C}_K$ in eq. (10) will consist of $H_{\epsilon_2}$ only. Then the roots in $\Delta^+$ will be:

$$\Delta^+ = \{ \epsilon_2 \pm \epsilon_1 , \epsilon_2 \pm \epsilon_i \ i = 3, \ldots, r \} \quad \begin{array}{c}2+d=2r \\
2+d=2r+1 \end{array}$$

In order to construct the $(d+2)$–dimensional matrix representation of the $SO(2, d)$ generators and therefore of the operators in $\text{Solv}$, we start by defining the non–compact Cartan generators $\{ H_{\epsilon_1}, H_{\epsilon_2} \}$ and the compact ones $\{ iH_{\epsilon_i} \ i = 3, \ldots, r \}$ as $(d+2) \times (d+2)$ matrices whose non vanishing entries are given by:

$$(H_{\epsilon_1})_{1,3} = (H_{\epsilon_1})_{3,1} = 1 ; (H_{\epsilon_2})_{2,4} = (H_{\epsilon_2})_{4,2} = 1$$

$$(iH_{\epsilon_{k+2}})_{2(k+1)+1,2(k+1)+2} = - (iH_{\epsilon_{k+2}})_{2(k+1)+2,2(k+1)+1} = 1 \ k = 1, \ldots, r-2$$

The shift operators are represented by eigenmatrices of the adjoint action of the Cartan operators in eqs. (14). Adopting a suitable convention on the normalization of the shift operators, it follows that the matrices representing $E_{\pm \epsilon_2 \pm \epsilon_1}, \ E_{\epsilon_2}, \ E_{\epsilon_1}$ are in the $SO(2, d)$ real form. In particular, the $E_{\epsilon_2 \pm \epsilon_1}$ matrices are characterized by non vanishing entries only in the upper $5 \times 5$ diagonal blocks which coincide respectively with the $SO(2, 3)$ representation ($\mathcal{T}_3$) of the same operators given in eq. (13). Moreover in the $d$–odd case the matrix realization of $E_{\epsilon_2}$ is defined by the following non–zero entries:

$$(E_{\epsilon_2})_{2,d+2} = (E_{\epsilon_2})_{d+2,2} = \frac{1}{\sqrt{2}}$$

$$(E_{\epsilon_2})_{4,d+2} = -(E_{\epsilon_2})_{d+2,4} = \frac{1}{\sqrt{2}}$$

The operators $E_{\epsilon_2 \pm \epsilon_i} \ i = 3, \ldots, r$ are represented by complex matrices whose real and imaginary parts separately belong to $SO(2, d)$. Moreover we can normalize each matrix so that $E_{\epsilon_2-\epsilon_i} = (E_{\epsilon_2+\epsilon_i})^*$. Thus the generators $E_{\epsilon_2 \pm \epsilon_i} \ i = 3, \ldots, r$ will enter the formula (10) only through the following combinations which single out their real and imaginary parts:

$$X_1 = E_{\epsilon_2+\epsilon_3} + E_{\epsilon_2-\epsilon_3} \ , \ X_2 = -i(E_{\epsilon_2+\epsilon_3} - E_{\epsilon_2-\epsilon_3}) \ldots$$

$$X_{2r-5} = E_{\epsilon_2+\epsilon_{r-1}} + E_{\epsilon_2-\epsilon_{r-1}} \ , \ X_{2r-4} = -i(E_{\epsilon_2+\epsilon_r} - E_{\epsilon_2-\epsilon_r})$$

(16)
The corresponding matrix representation in the $d + 2 = 2r$ case is characterized by the following non-zero entries:

$$
(X_k)_{2,4+k} = (X_k)_{4+k,2} = 1
$$

$$
(X_k)_{4,4+k} = - (X_k)_{4+k,4} = 1 \quad k = 1, \ldots, 2r - 4
$$

(117)
The matrices $X_k$ defined above in the $2 + d = 2r$ case are $d - 2$. In the case $2 + d = 2r + 1$, the set of $d - 2$ matrices $X_k$ is completed by defining $X_{d-2}/\sqrt{2} = E_{e_2}$, whose matrix representation is given in eq. (115). The usefulness of this notation will become apparent once we interpret the parameters of the solvable algebra $Solv$ as the coordinates on the world volume of a $(p = d - 1)$–brane: $(t, w, x^1, \ldots, x^{p-1})$. Indeed the generators of $Solv$, according to the structure described in eq.(110), can now be written in the form:

$$
Solv = \{ H_{e_2}, \mathcal{T}_- = E_{e_{2+\epsilon_1}}, \mathcal{T}_+ = E_{e_{2-\epsilon_1}}, X_1, \ldots, X_{p-1} \}
$$

(118)
Let us define the coset representative of the AdS coset space (108) as a solvable group element generated by a combination of the $Solv$ generators as:

$$
\mathbb{I}(a, t, w, x^1, \ldots, x^{p-1}) = \tau(t, w, x^1, \ldots, x^{p-1}) \cdot S(a)
$$

$$
\tau(t, w, x^1, \ldots, x^{p-1}) = \exp \left[ (t - w)\mathcal{T}_- + (t + w)\mathcal{T}_+ + x^1 X_1 + \ldots + x^{p-1} X_{p-1} \right]
$$

$$
S(a) = \exp [-a H_{e_2}]
$$

(119)
This is the generalization to the generic $d + 1$ dimensional case of eq. (74) of the main text corresponding to the $d = 3$ case. Computing the left invariant 1–form one finds:

$$
\Omega = \mathbb{I}^{-1} d \mathbb{I} = \begin{pmatrix}
0 & dt e^a & -dt e^a & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & dw e^a & dt e^a & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & dx^1 e^a & dw e^a & -dw e^a & 0 & 0 & 0 & 0 & 0 \\
0 & dx^2 e^a & dx^1 e^a & dw e^a & -dw e^a & 0 & 0 & 0 & 0 \\
0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & dx^{p-2} e^a & dx^{p-3} e^a & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & dx^{p-1} e^a & dx^{p-2} e^a & dx^{p-3} e^a & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

(120)
which is also the straightforward generalization of the left invariant 1–form computed in eq. (77) for the $SO(2, 3)$ case. The role of the transverse coordinate $x_\perp$ now is played by the $p - 1$ parameters $x^i$. As in the $p = 2$ case, the parameters $t - w$ and $t + w$ are the light–cone coordinates on the world volume of the $p$–brane. Indeed extending the same procedure previously followed for the $p = 2$ case, it is straightforward to compute the vielbein in the general case of $AdS^{p+2}$ we are considering:

$$
V^0 = -\frac{1}{2} \text{Tr}(\Pi_0 \Omega)
$$

$$
V^k = \frac{1}{2} \text{Tr}(\Pi_k \Omega) \quad k = 1, \ldots, p + 1
$$

(121)
where \( \{\Pi_0, \Pi_k\} \) is the basis of matrices defined by the orthogonal decomposition of \( SO(2, p+1) \) with respect to \( SO(1, p+1) \):

\[
\begin{align*}
\Pi_0 &= M^{1,2} \\
\Pi_k &= M^{1,2+k} \quad k = 1, \ldots, p+1
\end{align*}
\]

\( M^{A,B} \) being the orthogonal basis of \( SO(2, d) \) generators. The expression for the vielbein is:

\[
\begin{pmatrix}
V^0 \\
V^1 \\
V^2 \\
V^3 \\
\vdots \\
V^{p+2}
\end{pmatrix} =
\begin{pmatrix}
dt e^a \\
dw e^a \\
d-ag \\
dx^1 e^a \\
\vdots \\
dx^{p-1} e^a
\end{pmatrix}
\]

Setting \( \rho \equiv e^a \), from the vielbein basis we can compute the metric:

\[
ds^2 = -V^0 \otimes V^0 + \sum_{k=1}^{p+1} V^k \otimes V^k
\]

\[
= \rho^2(-dt^2 + dw^2 + d\vec{x} \cdot d\vec{x}) + \rho^{-2}d\rho^2
\]

which is the \( AdS^{p+2} \) metric in Bertotti–Robinson form. It is worthwhile noticing that it is possible to characterize in an intrinsic geometrical way the coordinates on the world sheet of the p–brane as the parameters of the maximal abelian ideal \( A \) of the solvable Lie algebra generating the \( AdS^{p+2} \) space. Indeed it is straightforward to check that:

\[
A = \{T_+, T_-, X_1, \ldots, X_{p-1}\} \subset Solv
\]

is the maximal abelian ideal of \( Solv \).

### A.1 The solvable algebra and Iwasawa decomposition

To conclude we resume our discussion from the start and we give a derivation of eq. (109) which allows the solvable Lie algebra description of \( AdS^{p+2} \). Let us first consider the Iwasawa decomposition of \( SO(2, d) \) and of \( SO(1, d) \) separately. At this point it is useful to recall the main features of the Iwasawa decomposition of a non–compact semisimple Lie algebra. Any non–compact real form \( G_o \) of a complex semisimple Lie algebra \( G \) can be represented, according to the Iwasawa decomposition, as the direct sum of its maximal compact subalgebra \( H \) and a solvable Lie algebra \( Solv \):

\[
G_o = H \oplus Solv
\]

The structure of \( Solv \) is the following:

\[
Solv = C_{ac} \oplus Nil
\]

\[
Nil = (\sum_{\alpha \in A^p} E_\alpha) \cap G_o
\]
where $C_{nc}$ is the subspace of all the non-compact Cartan generators and the remaining nilpotent part $\mathcal{N}il$ of $\text{Solv}$ is generated by all the shift generators of $G$ associated with roots which are positive with respect to $C_{nc}$ and do not vanish identically on it. Moreover these shift generators have to be suitably combined with each other in order to obtain nilpotent operators in the real form $G_\phi$. The Iwasawa decomposition for $SO(2, d)$ and $SO(1, d)$ reads as follows

$$
SO(2, d) = SO(2) \oplus SO(d) \oplus \text{Solv}_{SO(2,d)}
$$

$$
\text{Solv}_{SO(2,d)} = C_{SO(2,d)} \oplus \mathcal{N}il_{SO(2,d)}
$$

$$
SO(1, d) = SO(d) \oplus \text{Solv}_{SO(1,d)}
$$

$$
\text{Solv}_{SO(1,d)} = C_{SO(1,d)} \oplus \mathcal{N}il_{SO(1,d)}
$$

(128)

where $C_{SO(2,d)} = \{H_{\epsilon_1}, H_{\epsilon_2}\}$ is the space spanned by the non-compact Cartan generators of $SO(2, d)$ and $C_{SO(1,d)} = \{H_{\epsilon_1}\}$ consists of the unique non-compact Cartan generator belonging to the chosen $SO(1, d)$ subgroup of $SO(2, d)$. In order to simplify the notation let us define a set of $d - 2$ nilpotent generators $Y_i$ in the same way as for the $X_i$ generators:

$$
2 + d = 2r
$$

$$
Y_1 = E_{\epsilon_1 + \epsilon_3} + E_{\epsilon_1 - \epsilon_3}, \quad Y_2 = -i(E_{\epsilon_1 + \epsilon_3} - E_{\epsilon_1 - \epsilon_3}) \ldots
$$

$$
Y_{2r-5} = E_{\epsilon_1 + \epsilon_{r-1}} + E_{\epsilon_1 - \epsilon_{r-1}}, \quad Y_{2r-4} = -i(E_{\epsilon_1 + \epsilon_{r-1}} - E_{\epsilon_1 - \epsilon_{r-1}})
$$

(129)

For $2 + d = 2r + 1$ let us define $Y_{d-2}/\sqrt{2} = E_{\epsilon_1}$. The structure of $\text{Solv}_{SO(2,d)}$ and of $\text{Solv}_{SO(1,d)}$ can be described as:

$$
\text{Solv}_{SO(2,d)} = \{H_{\epsilon_1}, H_{\epsilon_3}\} \oplus \mathcal{N}il_{SO(2,d)}
$$

$$
\mathcal{N}il_{SO(2,d)} = \{E_{\epsilon_1 \pm \epsilon_2}, X_i, Y_i, i = 1, \ldots, d - 2\}
$$

$$
\text{Solv}_{SO(1,d)} = \{H_{\epsilon_1}\} \oplus \mathcal{N}il_{SO(1,d)}
$$

$$
\mathcal{N}il_{SO(1,d)} = \{(E_{\epsilon_1 + \epsilon_2} + E_{\epsilon_1 - \epsilon_2}), Y_i, i = 1, \ldots, d - 2\}
$$

(130)

The operators $E_{\epsilon_1 + \epsilon_2}$ and $E_{\epsilon_1 - \epsilon_2} = (E_{-\epsilon_1 + \epsilon_2})^t$, in our matrix representation, enter $SO(1, d)$ only through their sum. This can be seen directly from their matrix representation and from the fact that our choice $SO(1, d) \subset SO(2, d)$ corresponds to the group of $SO(2, d)$ matrices having zero entries along the second row and the second column. Indeed the matrix representation of $E_{\epsilon_1 - \epsilon_2} + E_{\epsilon_1 + \epsilon_2}$ has the following non-zero entries:

$$
(E_{\epsilon_1 - \epsilon_2} + E_{\epsilon_1 + \epsilon_2})_{1,4} = (E_{\epsilon_1 - \epsilon_2} + E_{\epsilon_1 + \epsilon_2})_{4,1} = \frac{1}{2}
$$

$$
(E_{\epsilon_1 - \epsilon_2} + E_{\epsilon_1 + \epsilon_2})_{3,4} = -(E_{\epsilon_1 - \epsilon_2} + E_{\epsilon_1 + \epsilon_2})_{4,3} = \frac{1}{2}
$$

(131)

It is immediate to check that also the combination $E_{-(\epsilon_1 + \epsilon_2)} + E_{-\epsilon_1 + \epsilon_2}$, represented by the transpose of the matrix in eq. (131), belongs to $SO(1, d)$. In the basis defined by the Iwasawa decompositions (128), we can write:

$$
SO(2, d) = SO(1, d) \oplus SO(2) \oplus \{Solv_{SO(2,d)}/Solv_{SO(1,d)}\}
$$

$$
SO(2) = \{g\} = \{E_{\epsilon_1 + \epsilon_2} - E_{\epsilon_1 - \epsilon_2} - E_{-(\epsilon_1 + \epsilon_2)} + E_{-\epsilon_1 + \epsilon_2}\} = K_0
$$

$$
Solv_{SO(2,d)}/Solv_{SO(1,d)} = \{H_{\epsilon_2}, E_{\epsilon_1 + \epsilon_2} - E_{\epsilon_1 - \epsilon_2}, X_i, i = 1, \ldots, d - 2\}
$$

(132)
It is straightforward to verify that we can perform a transformation on the basis defined in eqs. ([132]) by means of which the generator $g$ of $SO(2)$ and the generator $E_{\epsilon_1+\epsilon_2} - E_{\epsilon_1-\epsilon_2}$ in $\text{Solv}_{SO(2,d)}/\text{Solv}_{SO(1,d)}$ are mixed with the two operators \{\(E_{\epsilon_1+\epsilon_2} + E_{\epsilon_1-\epsilon_2}, E_{-(\epsilon_1+\epsilon_2)} + E_{-\epsilon_1+\epsilon_2}\)} in $SO(1,d)$ to obtain two independent combinations represented by the operators $E_{\epsilon_1+\epsilon_2}$ and $E_{-\epsilon_1+\epsilon_2}$. The latter, together with the remaining generators in $\text{Solv}_{SO(2,d)}/\text{Solv}_{SO(1,d)}$ define the following solvable Lie algebra:

\[
\text{Solv} = \{H_{\epsilon_2}, E_{\epsilon_1+\epsilon_2}, E_{-\epsilon_1+\epsilon_2}X_i, i = 1, \ldots, d-2\}
\]  
(133)

This result coincides with the one in eq. ([118]) for $d = p + 1$. Therefore this new basis realizes the decomposition ([109]).

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