INTERNAL FEEDBACK STABILIZATION FOR PARABOLIC SYSTEMS COUPLED IN ZERO OR FIRST ORDER TERMS

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Abstract. We consider systems of $n$ parabolic equations coupled in zero or first order terms with $m$ scalar controls acting through a control matrix $B$. We are interested in stabilization with a control in feedback form. Our approach relies on the approximate controllability of the linearized system, which in turn is related to unique continuation property for the adjoint system. For the unique continuation we establish algebraic Kalman type conditions.

1. Introduction. In this paper we study the local feedback stabilization for systems of parabolic equations in one dimension, i.e. on a bounded interval $\Omega \subset \mathbb{R}$. The equations are coupled in either first or zero order terms and we consider general boundary conditions, arbitrarily mixing Dirichlet, Neumann and Robin conditions. Under algebraic conditions of Kalman type concerning the coupling matrices of coefficients, we establish finite dimensional feedback stabilization with internal controls distributed in a subdomain $\omega \subset \Omega$ and acting in part of the equations through a control matrix.

Since the case of general coupled systems, and especially those which are coupled only in first order terms, seems more delicate to treat, we consider systems in space dimension one. In our paper we are interested in fact in stabilization through a finite number of finite dimensional feedback controls and the strategy is as follows.

First we linearize the nonlinear system around the stationary state and we prove approximate controllability for it. For systems of two equations we treat the case of couplings in both zero and first order terms and homogeneous Dirichlet boundary conditions. For systems of $n \geq 3$ equations, we will treat separately the cases of first or zero order couplings. The approximate controllability is obtained by proving the unique continuation property for the adjoint system under corresponding Kalman type conditions satisfied by the coupling matrix and the control matrix.

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We consider an abstract formulation for the given problem as an evolution problem in a Hilbert space. With the result of approximate controllability for the linearized system at hand, we use a spectral decomposition of this Hilbert space with respect to the elliptic operator in a direct sum of closed and invariant subspaces for the semigroup. Moreover, one of these subspaces is finite dimensional, corresponding to the eigenvalues with positive real part (that is the unstable subspace) and the other one is infinite dimensional but stable. With this decomposition of the space we consider the controlled system projected onto these subspaces and we study the controllability of the finite dimensional system. The approximate controllability gives the exact controllability for the system in any time in the finite dimensional subspace and, consequently, complete stabilization for it. We stabilize by a feedback control the finite dimensional projection of the system and we prove, using the norm given by the solution to an appropriate Lyapunov equation, that this finite dimensional feedback control stabilizes the whole nonlinear system.

Concerning previous results on the stabilization of parabolic and parabolic like equations (Navier-Stokes for example) one may consult the monograph of V.Barbu [4]. We mention also for example the papers of V.Barbu and G.Wang [3, 10] concerning stabilization of parabolic equations and the papers of V.Barbu, I.Lasiecka, R.Triggiani [9, 5, 6, 7] for the stabilization of Navier-Stokes equations (with either internal or boundary controls). In these references the authors use a Riccati based approach. In order to obtain finite dimensional stabilization they use a Kalman type condition for a finite dimensional projection; this in turn is obtained through a unique continuation property for systems of eigenfunctions of the elliptic part. There are also studies of stabilization of time dependent controls, e.g. the papers [8, 16], but the approach there is different when dealing with the open loop stabilization for the linearized system.

In our approach we use as an essential argument unique continuation for linear parabolic equations and this is in fact related to global Carleman estimates and the controllability problem. We mention here the controllability of linear parabolic equations with internally distributed controls, which was established by O.Yu.Imanuvilov, A.V.Fursikov (see [12] and references therein), through such observability inequalities for the adjoint equation.

The study of controlled systems of parabolic equations with fewer controls than equations need appropriate Carleman estimates, with partial observations for the adjoint to the linearized system and, in some cases, one may provide algebraic conditions concerning the couplings and control operators in order to obtain appropriate observability inequalities. Kalman type conditions for obtaining Carleman estimates and controllability for systems of parabolic equations coupled in zero order terms, with constant or time depending coupling coefficients, were established in [2, 1]. The case of cascade like systems with one control and space depending couplings was studied in [13]. A more recent paper considering coupled linear systems (not only parabolic) and corresponding observation estimates is the paper of E.Zuazua and P.Lissy [17] where the equations in the system are linearly coupled with constant coefficients in the dominant part and/or in the zero order terms. Observability of the system is obtained in terms of a Kalman condition satisfied by the pair of coupling matrix and respectively the control matrix.

2. Preliminaries and main result. Let $l > 0$, $\Omega = (0, l)$ an open interval and let $\omega \subset\subset \Omega$ an open nonempty subset of $\Omega$. We consider the controlled parabolic
system
\[
\begin{aligned}
\begin{cases}
\partial_t y - Ly + F(D_x y, y) = g + B\chi\omega u, & t > 0, x \in \Omega, \\
(\text{BC}) : \begin{aligned}
\Gamma_1 D_x y(t, 0) + \Gamma_2 y(t, 0) &= 0, \\
\Gamma_3 D_x y(t, l) + \Gamma_4 y(t, l) &= 0,
\end{aligned} & t > 0, \\
y(0, x) = y^0(x), & x \in \Omega,
\end{cases}
\end{aligned}
\tag{2.1}
\]
where \( y = (y_1, \cdots, y_n)^\top \), \( Ly = (Ly_1, \cdots, Ly_n)^\top \) with \( L \) an uniformly elliptic operator of second order, \( Ly = D_x^2 y + \eta_1(x) D_x y + \eta_0(x) y, \quad \eta_0 \in L^\infty(\Omega), \eta_1 \in W^{1,\infty}(\Omega). \)

We used the same symbol for the differential operator \( L \) applied to either vector or scalar functions. The coupling is given through a \( C^\infty \) function, \( F(\zeta, y) = (f_1(\zeta, y), \cdots, f_n(\zeta, y))^\top, \quad F : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \) and in the right-hand side the free term \( g = (g_1, \cdots, g_n)^\top \) is \( L^\infty(\Omega) \). The control is given by \( B\chi\omega u, \quad B \in \mathcal{M}_{n \times n}(\mathbb{R}), \quad u \in L^2((0, T); [L^2(\omega)]^m) \), where \( \chi\omega u \) is the extension of \( u \) by 0 to the whole \( \Omega \).

The boundary conditions are in general form, homogeneous, mixing Dirichlet, Neumann, Robin boundary conditions for each equation of the system. For a general formulation of these boundary conditions we chose diagonal matrices \( \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 \in \mathcal{M}_{n \times n}(\mathbb{R}) \) with the properties
\[
\begin{aligned}
\Gamma_i &= \text{diag}(\gamma_i), \quad \forall i = 1, 4, j = 1, n, \\
\text{rank}[\Gamma_1, \Gamma_2] &= \text{rank}[\Gamma_3, \Gamma_4] = n.
\end{aligned}
\tag{2.2}
\]

We consider a stationary solution of the uncontrolled system, denoted by \( \tilde{y} \). The aim of the paper is to find a control in feedback form \( u = K(y - \tilde{y}) \), such that it stabilizes the controlled system (2.1) around the stationary state \( \tilde{y} \) with respect to a topology to be precised later. This will lead to the study of approximate controllability for a linear system following the unique continuation property for a linear adjoint system. Practically, the unique continuation property for the adjoint linear system will be the main point in proving the stability result.

For this purpose we consider the following linear system obtained through linearization of the nonlinear system around the stationary state:
\[
\begin{aligned}
\begin{cases}
\partial_t w - Lw + A_1(x) D_x w + A_0(x) w = B\chi\omega u, & t > 0, x \in \Omega, \\
\Gamma_1 D_x w(t, 0) + \Gamma_2 w(t, 0) = \Gamma_3 D_x w(t, l) + \Gamma_4 w(t, l) &= 0 & t > 0,
\end{cases}
\end{aligned}
\tag{2.3}
\]
where \( A_0(x), A_1(x) \in \mathcal{M}_{n \times n}(\mathbb{R}) \),
\[
A_1(x) = \left( \frac{\partial f_i}{\partial \zeta_j}(D_x \tilde{y}, \tilde{y}) \right)_{i, j = 1, n}, A_0(x) = \left( \frac{\partial f_i}{\partial y_j}(D_x \tilde{y}, \tilde{y}) \right)_{i, j = 1, n}.
\]

In order to give the problem an abstract formulation, let us consider the Hilbert space \( H = [L^2(\Omega)]^n \) and the operators
\[
\mathcal{A} : D(\mathcal{A}) \subset H \rightarrow H, D(\mathcal{A}) = \{ y \in [H^2(\Omega)]^n \mid y \text{ satisfies } (\text{BC}) \text{ of (2.1)} \}, \mathcal{A} y = Ly, \quad \mathcal{A} = D(\mathcal{A}), \mathcal{A} y = Ly - A_1(x) D_x y - A_0(x) y,
\]
and the control operator
\[
\mathcal{B} : L^2(\omega)^m \rightarrow H, \quad \mathcal{B} u = B\chi\omega u.
\]

By standard computations for adjoints to realizations of linear differential operators, one finds that
\[
D(\mathcal{A}^*) = \{ v \in [H^2(\Omega)]^n : \tilde{\Gamma}_1 v'(0) + \tilde{\Gamma}_2 v(0) = \tilde{\Gamma}_3 v'(l) + \tilde{\Gamma}_4 v(l) = 0 \},
\tag{2.4}
\]
$$\tilde{\Gamma}_1 = \Gamma_1, \tilde{\Gamma}_2 = \Gamma_2 + \Gamma_1 [A_1^T(0) - \eta_1(0)I_\Omega],$$
$$\tilde{\Gamma}_3 = \Gamma_3, \tilde{\Gamma}_4 = \Gamma_4 + \Gamma_3 [A_1^T(l) - \eta_1(l)I_\Omega],$$
\hspace{1cm} (2.5)
and
$$A^*w = L^*w + D_x(A_1(x)w) - A_0(x)w, \ w \in D(A^*).$$
\hspace{1cm} (2.6)

Remark 1. The operator \(A\) is generator of a strongly continuous semigroup but this may not be a contraction semigroup. Also, the operator \(A\) generates an analytic semigroup in \(H\). Indeed, \(A\) is sum of a selfadjoint operator \((y \to D^2_y y)\) with domain \(D(A)\) and a lower order perturbation (see [19]). Compactness of the resolvent of \(A\) follows in a standard way by using Rellich embedding theorem.

The abstract formulation of the nonlinear system reads:
$$\begin{cases}
D_t y - ky + F(D_x y, y) = g + Bu \\
y(0) = y^0
\end{cases}$$
\hspace{1cm} (2.7)
and the linearized system (2.3) becomes
$$\begin{cases}
D_tw = Aw + Bu \\
w(0) = w^0.
\end{cases}$$
\hspace{1cm} (2.8)

The strategy to stabilize the nonlinear system (also found, for example, in [15]) is to construct a finite dimensional feedback control that stabilizes the linear system and then prove, by using an argument based on the Lyapunov equation, that the same control stabilizes the nonlinear system too.

A first step towards proving stabilization of the linearized system is obtaining approximate controllability in arbitrary time \(T > 0\) for the linearized system; this in turn is equivalent to the unique continuation property for the adjoint system. The adjoint to the linearized problem is the following:
$$\begin{cases}
-D_tp - A^*p = 0, \ t \in (0, T), \\
p(T) = p^T.
\end{cases}$$
\hspace{1cm} (2.9)

The unique continuation property in time \(T\) for the above system is expressed in the following form:
$$\text{(UCP)}: B^*p(t) = 0 \text{ for } t \in (0, T) \implies p \equiv 0 \text{ in } (0, T),$$
\hspace{1cm} (2.10)
where \(B^*p = B^T p|_\omega\), with \(B^T\) denoting the transposed of matrix \(B\).

Our paper is divided into two parts:

The first part is dedicated to approximate controllability in given arbitrary time \(T > 0\) and thus to unique continuation property for the adjoint system. This will be discussed in several settings:
- systems of two equations and coupling matrices \(A_0(x), A_1(x)\), possibly nonconstant, and homogeneous Dirichlet boundary conditions \((\Gamma_1 = \Gamma_3 = O_\Omega, \Gamma_2 = \Gamma_4 = I_\Omega)\);
- systems of \(n\) equations with couplings in zero order terms and constant coupling matrix \((A_1 = 0, A_0 = \text{cst})\);
- systems of \(n\) equations with couplings in first order terms and constant coupling matrix \((A_0 = 0, A_1 = \text{cst})\).
Kalman type conditions for unique continuation are established.

The second part of the paper is devoted to feedback stabilization and the result is common to all situations when unique continuation property (UCP) is true.

We describe first the results concerning approximate controllability of coupled linear equations.

**Systems of two linear equations with first and zero order couplings.** We consider systems of two coupled equations for $w = (w_1, w_2)$, with differential operator $L$ with constant coefficients $\eta_0, \eta_1 \in \mathbb{R}$ and possibly nonconstant couplings, under Dirichlet homogeneous boundary conditions:

$$
\begin{align*}
D_t w_1(t, x) - L w_1(t, x) + a(x) D_x w_1 + b(x) D_x w_2 & = \chi \omega u, \quad t > 0, x \in \Omega, \\
D_t w_2(t, x) - L w_2(t, x) + c(x) D_x w_1 + d(x) D_x w_2 & = 0, \quad t > 0, x \in \Omega,
\end{align*}
$$

(2.11)

The adjoint system in $(0, T) \times \Omega$ for the adjoint variable $p = (p_1, p_2)$ is

$$
\begin{align*}
-D_{t} p_1 - L^* p_1 - a D_x p_1 - c D_x p_2 + (\alpha - D_x a)p_1 + (\gamma - D_x c)p_2 & = 0, \\
-D_{t} p_2 - L^* p_2 - b D_x p_1 - d D_x p_2 + (\beta - D_x b)p_1 + (\delta - D_x d)p_2 & = 0,
\end{align*}
$$

(2.12)

If we consider the notations,

$$
\begin{align*}
h(x) & := \frac{\gamma(x) - c'(x)}{c(x)}, \\
k(x) & := -h^2(x) - h'(x) + [\eta_1 - d(x)]h(x) - \eta_0 + \delta(x) - d'(x),
\end{align*}
$$

(2.13)

for $c(x) \neq 0$ in $\omega$, then we have the following results concerning the approximate controllability for the above linear systems:

**Theorem 2.1.** For the linear system (2.11), with $\eta_0, \eta_1 \in \mathbb{R}$, if $c(x) \neq 0$ for $x \in \omega$ and one of the following hypotheses holds true

- (H1) the coefficients of the system are constants in the whole domain, 
  \[ a, b, c, d, \alpha, \beta, \gamma, \delta \in \mathbb{R}; \]
  
- (H2) the coupling coefficients are continuous in $\Omega$, maybe nonconstant, and the function $k = k(x)$ is not constant in $\omega$;  
  
then the linear system (2.11) is approximately controllable in time $T$.

We mention here the paper of M.Duprez and P.Lissy [11] where similar systems of two parabolic equations are considered in higher space dimension with Dirichlet homogeneous boundary conditions and null controllability is proved. When null controllability occurs the approximate controllability is also obtained, but the conditions we ask for approximate controllability are easier to verify than those in [11] and our approach works also in the case of general boundary conditions as mentioned in Remark 5 below.
Systems of \( n \) equations with constant couplings. We consider now systems of \( n \) coupled parabolic equations in either zero order or first order terms, in space dimension one with constant coupling matrices \( A_0 \) or \( A_1 \).

**Remark 2.** The linearized system (2.3) has constant coupling matrices \( A_0, A_1 \) when linearization is around a constant stationary solution to system (2.1). Of course, constant solutions to (2.1) are the solutions to algebraic equation \( F(z, 0) - \eta_0 z = 0 \) satisfying the boundary conditions. Nontrivial constant solutions to (2.1) may exist only if Neumann boundary conditions at both endpoints of the interval are imposed on at least one equation of the system; this is only a necessary condition.

The approximate controllability will be obtained under algebraic conditions of Kalman type, involving the pairs \((A_0, B)\) or \((A_1, B)\). We recall here, for two matrices \( M \in \mathcal{M}_{n \times n}(\mathbb{R}), B \in \mathcal{M}_{n \times m}(\mathbb{R}) \), the definition of the Kalman matrix \( [M|B] \in \mathcal{M}_{n \times nm}(\mathbb{R}) \):

\[
[M|B] := [B, MB, M^2 B, \ldots, M^{n-1} B].
\] (2.14)

**Parabolic systems with zero order couplings.** We consider for the beginning the case where \( A_1 = 0, A_0 \in \mathcal{M}_{n \times n}(\mathbb{R}) \). The result of approximate controllability in this case is:

**Theorem 2.2.** Consider the linear system (2.3) with constant coefficients \( \eta_0, \eta_1 \) and with constant couplings of order zero, \( A_1 \equiv 0 \). If the following Kalman condition holds,

\[
\text{rank}[A_0|B] = n,
\] (2.15)

then the linear system (2.3) is approximately controllable in time \( T \).

**Remark 3.** Controllability results and Carleman estimates for such controlled systems were established in [1] under homogeneous Dirichlet boundary conditions and in this case of zero order couplings our result is a consequence of the cited paper. Here we use general homogeneous boundary conditions and we give a direct proof by reducing the question to the unique continuation result from [20].

**Parabolic systems with first order couplings.** Consider now the case where \( A_0 = 0, A_1 \in \mathcal{M}_{n \times n}(\mathbb{R}) \). Approximate controllability result in this case is:

**Theorem 2.3.** Consider the linear system (2.3) with constant coefficients \( \eta_0, \eta_1 \) and with constant couplings of order one, \( A_0 \equiv 0, A_1 \in \mathcal{M}_{n \times n}(\mathbb{R}) \). Suppose also that the following algebraic conditions concerning coupling matrix and matrices entering the boundary conditions are satisfied:

\[
\text{rank}[A_1|B] = n,
\] (2.16)
\[
\ker B^\top \cap \ker(\Gamma_2 + \Gamma_4(A_1^\top + \eta_1 I)) \cap \ker(\Gamma_4 + \Gamma_3(A_1^\top + \eta_1 I)) = \{0\}
\] (2.17)

then the linear system (2.3) is approximately controllable in time \( T \).

**Remark 4.** Hypothesis (2.17) is automatically satisfied under homogeneous Dirichlet boundary conditions in one boundary point, *e.g.* in \( x = 0 \), as in this case one has \( \Gamma_1 = 0, \Gamma_2 = I_n \).
Finite dimensional feedback stabilization of coupled parabolic systems.

**Stabilization of the linearized system.** In either of the cases when approximate controllability is verified, we prove the following feedback stabilization result for the linearized system:

**Theorem 2.4.** For the linear system (2.3), in the framework of either of Theorems 2.1, 2.2 or 2.3, for any \( \delta > 0 \) there exists \( C = C(\delta) > 0 \), a finite dimensional subspace \( U \subset L^2(\omega) \) and a bounded linear operator \( K \in L(H,U) \) such that the operator \( \mathcal{A} + \mathcal{B}K \) generates an analytic semigroup of negative type that satisfies

\[
\| e^{(\mathcal{A}+\mathcal{B}K)t} \|_H \leq C e^{-\delta t}, \quad t > 0.
\]  

(2.18)

**Stabilization of the nonlinear parabolic system.** Based on the stabilization results in the linear case, we prove by using an argument related to Lyapunov equation, a local feedback stabilization result:

**Theorem 2.5.** Consider the nonlinear system (2.1) and suppose that we are in the framework of either of Theorems 2.1, 2.2 or 2.3 for the linearized system.

Let \( \nu \in (\frac{3}{4}, 1) \). Then there exist \( \varepsilon > 0, \delta > 0, C > 0 \), such that if \( y^0 \in H^{2\nu} \) verifies the boundary conditions BC in (2.1) and \( \| y^0 - \overline{y} \|_{H^{2\nu}} \leq \varepsilon \) then, taking in (2.1) the feedback constructed in Theorem 2.4

\[
u = K(y - \overline{y})
\]  

(2.19)

one has exponential stabilization:

\[
\| y(t) - \overline{y} \|_{H^{2\nu}} + \| y(t) - \overline{y} \|_{L^\infty} \leq C e^{-\delta t} \| y^0 - \overline{y} \|_{H^1}, \quad t > 0.
\]  

(2.20)

3. Approximate controllability for linear systems.

3.1. Systems of two equations. Proof of theorem 2.1. We consider the homogeneous backward adjoint system (2.12), (2.9), and we study the unique continuation property for this system since this property is equivalent to the approximate controllability of the linear system (2.11).

Now, suppose that \( p_1 \equiv 0 \) in \((0, T) \times \omega\). We will treat each case corresponding to hypothesis (H1) and (H2) of Theorem 2.1.

Case 1. We are in the conditions of the hypothesis (H1) of Theorem 2.1, meaning that all coupling coefficients are constants. Since \( c \not\equiv 0 \) in \( \omega \), \( c \) will not be zero in the entire \( \Omega \). Then, the system (2.12) becomes

\[
\begin{aligned}
-D_t p_1 - L^* p_1 - aD_x p_1 - cD_x p_2 + \alpha p_1 + \gamma p_2 & = 0, \quad \text{in } (0, T) \times \Omega, \\
-D_t p_2 - L^* p_2 - bD_x p_1 - dD_x p_2 + \beta p_1 + \delta p_2 & = 0, \quad \text{in } (0, T) \times \Omega, \\
p_i(t, 0) & = p_i(t, l) = 0, \quad i = 1, 2, \quad \text{in } (0, T),
\end{aligned}
\]  

(3.1)

From the first equation of system (3.1) considered on \((0, T) \times \omega\) we get

\[
p_2(t, x) = p_2(t, x_0) e^{h(x-x_0)}, \quad \text{on } (0, T) \times \Omega, \quad \text{with } x_0 \in \omega,
\]  

(3.2)

where \( h = \frac{c}{\alpha} \) since the coefficients of the system are constants.

Looking at the second equation of (3.1) on \((0, T) \times \omega\), we obtain that

\[
-D_t p_2(t, x_0) + [-h^2 + (\eta_1 - \delta) h + (\eta_2 - \gamma)] p_2(t, x_0) = -D_t p_2(t, x_0) + k p_2(t, x_0) = 0
\]

giving that on the subdomain \( \omega \) we have a separation of variables

\[
p_2(t, x) = p_2(t_0, x_0) e^{k(t-t_0)} e^{h(x-x_0)} \quad \text{on } (0, T) \times \omega, \quad t_0 \in (0, T), x_0 \in \omega.
\]
Now, if we denote by
\[
\tilde{p}_1 := p_1 e^{-kt} e^{-hx}, \quad \tilde{p}_2 := p_2 e^{-kt} e^{-hx}
\]
then \( \tilde{p}_1, \tilde{p}_2 \) satisfy
\[
\begin{aligned}
-\partial_t \tilde{p}_1 - L^* \tilde{p}_1 + (-2h - a)D_x \tilde{p}_1 - cD_x \tilde{p}_2 \\
+([d - a]h + \alpha - \delta + \eta_0)[\tilde{p}_1] = 0, \\
-\partial_t \tilde{p}_2 - L^* \tilde{p}_2 - bD_x \tilde{p}_1 + (-2h - d)D_x \tilde{p}_2 \\
+(-bh + \beta)\tilde{p}_1 + \eta_0 \tilde{p}_2 = 0,
\end{aligned}
\]
\[
\begin{aligned}
\tilde{p}_1(t,0) = \tilde{p}_1(t,1) = 0, & \quad i = 1, 2, \\
\tilde{p}_1(t,0) = 0 & \quad \text{in } (0,T) \times \Omega,
\end{aligned}
\]
(3.3)
with
\[
\tilde{p}_1 \equiv 0, \quad \tilde{p}_2 = \text{cst} \text{ in } (0,T) \times \omega, \text{cst } \in \mathbb{R}.
\]
Since the coefficients are constant, the functions \( D_t \tilde{p}_1 \) and \( D_t \tilde{p}_2 \) verify both the system (3.4) and the additional equalities
\[
D_t \tilde{p}_1 = 0, \quad D_t \tilde{p}_2 = 0 \text{ in } (0,T) \times \omega.
\]
Then the unique continuation property for homogeneous parabolic system verified by \( D_t \tilde{p}_1 \) and \( D_t \tilde{p}_2 \) gives that \( D_t \tilde{p}_1 \equiv 0 \) in \( (0,T) \times \Omega \) and \( D_t \tilde{p}_2 \equiv 0 \) in \( (0,T) \times \Omega \), meaning that \( \tilde{p}_1 \) and \( \tilde{p}_2 \) depend only on \( x \) in \( (0,T) \times \Omega \).

Also, \( D_x \tilde{p}_1 \) and \( D_x \tilde{p}_2 \) verify the elliptic system
\[
\begin{aligned}
-\partial_t \tilde{p}_1 - L^* \tilde{p}_1 + (-2h - a)D_x \tilde{p}_1 - cD_x \tilde{p}_2 \\
+([d - a]h + \alpha - \delta + \eta_0)(D_x \tilde{p}_1) = 0, \\
-\partial_t \tilde{p}_2 - L^* \tilde{p}_2 - bD_x \tilde{p}_1 + (-2h - d)D_x \tilde{p}_2 \\
+(-bh + \beta)(D_x \tilde{p}_1) + \eta_0 (D_x \tilde{p}_2) = 0,
\end{aligned}
\]
\[
\begin{aligned}
\tilde{p}_1 \equiv 0, & \quad \text{in } (0,T) \times \omega, \\
\tilde{p}_2 \equiv 0 & \quad \text{in } (0,T) \times \omega,
\end{aligned}
\]
(3.5)
with
\[
D_x \tilde{p}_1 \equiv 0, \quad D_x \tilde{p}_2 \equiv 0 \text{ in } (0,T) \times \omega,
\]
giving the only possibility, by unique continuation result in [20], that \( D_x p_1 \equiv 0, D_x p_2 \equiv 0 \text{ in } (0,T) \times \Omega \) and thus
\[
\tilde{p}_1 \equiv 0 \text{ in } (0,T) \times \Omega, \quad \tilde{p}_2 \equiv \text{cst} \text{ in } (0,T) \times \Omega.
\]
We see now that the boundary conditions forces \( \tilde{p}_2 \) to be zero which implies \( \tilde{p}_1, \tilde{p}_2 \equiv 0 \) in \( (0,T) \times \Omega \) and thus \( p_1, p_2 \equiv 0 \) which concludes the proof of approximate controllability.

**Case 2.** We argue now under hypothesis (H2) of Theorem 2.4, meaning that the function \( k \) is not constant in the subdomain \( \omega \), then, for \( p_1 \equiv 0 \) in \( (0,T) \times \omega \), the first equation of the system (2.12) gives
\[
cD_x p_2 = (\gamma - c')p_2 \text{ in } (0,T) \times \omega
\]
and so
\[
p_2(t,x) = p_2(t,x_0)e^{H(x)}, \quad (0,T) \times \omega
\]
where \( H(x) = \int_{x_0}^x \frac{\gamma(\xi) - c'(\xi)}{\alpha(\xi)}d\xi = \int_{x_0}^x h(\xi)d\xi. \)

Then \( p_2 \) verifies the second equation of (2.12)
\[
\frac{d}{dt}p_2(t,x_0) = p_2(t,x_0)k(x) \text{ in } (0,T) \times \omega
\]
and since \( k \) is not constant there, then \( p_2(t,x_0) \) must be zero in \( (0,T) \times \omega \). Then, invoking again the unique continuation property for the parabolic system (2.12) we can conclude that \( p_1 \equiv 0, p_2 \equiv 0 \text{ in } (0,T) \times \Omega \), thus obtaining the unique continuation property in this case.
Remark 5. Observe that the conclusion of Theorem 2.1 under hypothesis (H2) remains valid under general boundary conditions described in (2.1): the proof makes no use of the imposed Dirichlet homogeneous conditions.

We may also consider Theorem 2.1 under hypothesis (H1) with general boundary conditions in (2.1), (2.2). However, in this case, in order to have the conclusion on approximate controllability and correspondingly on unique continuation for the adjoint problem, one has to insure that the problem (3.4) has as unique solution with first component \( \tilde{p}_1 = 0 \) and second component constant \( \tilde{p}_2 = \text{cst} \) the null solution. Considering that \( p \) satisfies boundary conditions described in (2.4), one sees that such \( \tilde{p} \) satisfies

\[
\tilde{\Gamma}_1 D_t \tilde{p}(t,0) + (h\tilde{\Gamma}_1 + \tilde{\Gamma}_2) \tilde{p}(t,0) = 0, \quad t > 0,
\]

\[
\tilde{\Gamma}_3 D_x \tilde{p}(t,l) + (h\tilde{\Gamma}_3 + \tilde{\Gamma}_4) \tilde{p}(t,l) = 0, \quad t > 0.
\]

Consequently, unique continuation is verified for the adjoint system if \((0,1)^T\) does not belong to at least one of the kernels of matrices \((h\tilde{\Gamma}_1 + \tilde{\Gamma}_2)\) or \((h\tilde{\Gamma}_3 + \tilde{\Gamma}_4)\).

3.2. Systems of \( n \) equations with constant coupling coefficients.

3.2.1. Zero order couplings. Proof of theorem 2.2. We consider system (2.3) with \( A_1 = 0 \). Approximate controllability in arbitrary time \( T > 0 \) is equivalent to unique continuation property for the the adjoint system (2.9):

\[
D_t p + L^* p - A_0^\top p = 0, \quad \text{in } (0,T) \times \Omega \quad \text{and} \quad B^\top p = 0 \quad \text{in } (0,T) \times \omega \Rightarrow p \equiv 0 \quad \text{in } \Omega \times (0,T).
\]

(3.6)

From the adjoint equation above one obtains, after multiplying the adjoint system with \( B^\top \), that

\[
B^\top A_0^\top p \equiv 0 \quad \text{in } (0,T) \times \omega.
\]

By induction, multiplying adjoint equation with matrix \( B(A_0^\top)^k, \; k = 1, \cdots, n-2 \) we obtain that

\[
B^\top (A_0^\top)^k p \equiv 0 \quad \text{in } (0,T) \times \omega, \; k = 0, \cdots, n-1.
\]

(3.7)

The Kalman rank condition (2.15): rank \([A_0|B]\) = \( n \) applied to (3.7) gives that

\[
p \equiv 0 \quad \text{in } (0,T) \times \omega.
\]

which, by unique continuation for systems of parabolic equation (see, for example, the paper of J.-C. Saut and B. Scheurer [20]) gives that

\[
p \equiv 0 \quad \text{in } (0,T) \times \Omega.
\]

3.2.2. First order couplings. Proof of theorem 2.3. For simplifying notations we write the proof with \( \eta_l = 0 \). This is not a loss of generality as we may replace \( A_1 \) by \( A_1 + \eta I \). We consider the controlled system (2.3) with \( A_0 = 0 \):

\[
\begin{align*}
D_t w - L w + A_1 D_x w &= B \chi_\omega u, \\
\Gamma_1 D_x w(t,0) + \Gamma_2 w(t,0) &= \Gamma_3 D_x w(t,l) + \Gamma_4 w(t,l) = 0
\end{align*}
\]

(3.8)

The adjoint problem reads

\[
\begin{align*}
D_t p + L^* p + A_0^\top D_x p &= 0, \quad (t,x) \in (0,T) \times \Omega \\
\Gamma_1 D_x p(t,0) + (\Gamma_2 + \Gamma_1 A_0^\top)p(t,0) &= 0, \quad t \in (0,T) \\
\Gamma_3 D_x p(t,l) + (\Gamma_4 + \Gamma_3 A_0^\top)p(t,l) &= 0, \quad t \in (0,T).
\end{align*}
\]

(3.9)

Approximate controllability in arbitrary time \( T > 0 \) for (3.8) is equivalent to unique continuation property for the the adjoint system:
\[ B^T p = 0 \text{ in } (0, T) \times \omega \implies p = 0 \text{ in } (0, T) \times \Omega. \]

We proceed as in the case of zero order couplings and we multiply the adjoint system (3.9) with \( B^T \) and we obtain
\[ B^T A_1^T (D_x p) = 0 \text{ in } (0, T) \times \omega. \]

Now, for \( k = 2, \ldots, n-2 \), we multiply \( k \) times in \( x \) the adjoint equation:
\[ D_t (D_x^k p) + L^*(D_x^k p) + A_1^T (D_x^{k+1} p) = 0 \text{ in } (0, T) \times \Omega, \tag{3.10} \]
and we multiply by \( B^T (A_1^T)^k \) to obtain
\[ B^T (A_1^T)^k D_x^k p = 0 \text{ in } (0, T) \times \omega, k = 0, \ldots, n-1. \tag{3.11} \]

In order to use the Kalman condition (2.16) we compute derivatives with respect to \( x \) in (3.11) and obtain
\[ B^T (A_1^T)^k D_x^{n-k} p = 0 \text{ in } (0, T) \times \omega, k = 0, \ldots, n-1, \tag{3.12} \]
and find that
\[ D_x^{n-k} p = 0 \text{ in } (0, T) \times \omega, \]
which by unique continuation for systems of parabolic equations applied to (3.10) with \( k = n-1 \) gives
\[ D_x^{n-1} p = 0 \text{ in } (0, T) \times \Omega. \]

This means that \( p \) is a polynomial of degree at most \( n-2 \) of the form
\[ p(t, x) = d_0(t) + \sum_{k=1}^{n-2} d_k(t) x^k. \tag{3.13} \]

Plugging the expression of \( p \) in the adjoint equation in (3.9) we obtain
\[ d'_k + \eta_0 d_k + (k + 1) A_1^T d_{k+1} + (k + 2)(k + 1) d_{k+2} = 0, \tag{3.14} \]
\[ k = 0, n-4, \]
\[ d'_{n-3} + \eta_0 d_{n-3} + (n - 2) A_1^T d_{n-2} = 0 \]
\[ d'_{n-2} + \eta_0 d_{n-2} = 0. \]

Now, we denote by \( c_k := d_k e^{\eta_0 t} \), \( \tilde{c}_k := c_k(0) \). Observe that \( \{c_k\}_k \) satisfy
\[ c'_k = a_k A_1^T c_{k+1} + b_k c_{k+2}, k = 0, \ldots, n-4, \tag{3.15} \]
\[ c'_{n-3} = a_{n-3} A_1^T c_{n-2} \]
\[ c'_{n-2} = 0 \]
where we denoted by \( a_k := -(k + 1), k = 0, \ldots, n-3, \]
\[ b_k = -(k + 2)(k + 1), k = 0, \ldots, n-4. \]

Now, one may easily see that \( c_k \) must be polynomials in \( t \), with vector coefficients in \( \mathbb{R}^n \) and degree at most \( n-2-k \). Denote by \( \text{Dom}(c_k) \) the coefficient of \( t^{n-2-k} \) of the polynomial \( c_k \) (this may be zero if the degree of \( c_k \) is strictly smaller that \( n-2-k \)).

As \( B^T p(t, x) = 0, x \in \omega \) one finds that
\[ B^T c_k(t) = 0, t \in (0, T), k = 0, \ldots, n-2, \tag{3.16} \]
and thus, letting \( t = 0 \) we obtain
\[ B^T \tilde{c}_k \equiv 0, k = 0, \ldots, n-2. \tag{3.17} \]
Consequently, with $P_n(t)$ a polynomial of degree at most $l$. Indeed, take for example Remark 6. Hypothesis (2.17) is essential for approximate controllability of system (3.9) in $x = l$ for the corresponding boundary conditions and find that:

$$
\sum_{k=0}^{n-3} ([k+1)(\Gamma_3c_{k+1}(t) + (\Gamma_4 + \Gamma_3A_1^{\top})c_k(t)] ) t^k + (\Gamma_4 + \Gamma_3A_1^{\top})c_{n-2} = 0 \quad (3.21)
$$

Observe that this is again a polynomial in $t$ of degree at most $n - 2$ and thus all coefficients are null and in particular the coefficient of $t^{n-2}$ is zero; this latter coefficient appears only in $c_0$, and thus

$$
(j_2 + \Gamma_1A_1^{\top})c_{n-2} = 0. \quad (3.20)
$$

Now we look at the other end, $x = l$ for the corresponding boundary conditions and find that:

$$
\text{Dom}(c_n) = \frac{1}{(n-2-k)!} a_k \cdot \ldots \cdot a_{n-3}(A_1^{\top})^{n-2-k} \tilde{c}_{n-2} \quad (3.19)
$$

and thus

$$
c_k(t) = \frac{1}{(n-2-k)!} a_k \cdot \ldots \cdot a_{n-3}(A_1^{\top})^{n-2-k} \tilde{c}_{n-2} = \tilde{c}_{n-2}.
$$

Now we use the boundary condition from (3.9) in $x = 0$, which is satisfied also by $p(t, x)e^{\eta_0 t}$, and we find:

$$
\Gamma_1c_1(t) + (\Gamma_2 + \Gamma_1A_1^{\top})c_0(t) = 0.
$$

This is a polynomial of degree at most $n - 2$ and thus all coefficients are zero, in particular the coefficient of $t^{n-2}$ which is

$$
(j_2 + \Gamma_1A_1^{\top})c_{n-2} = 0.
$$

Remark 6. Hypothesis (2.17) is essential for approximate controllability of system (3.8). Indeed, take for example $n = 3$, $Lu = D_2^2w + w$ ($\eta_1 = 0, \eta_0 = 1$) and

- $A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ such that (2.16) is clearly satisfied.

- $\Gamma_1 = \Gamma_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\Gamma_2 = \Gamma_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

One proves by induction using exactly the argument above that in fact $c_0 = c_1 = \ldots = c_{n-2} = 0$ and thus $p = 0$. □
Take now
\[ c = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad p(t, x) = ce^{-t}\]
and observe that \( p \) is solution to (3.9) satisfying the boundary conditions because \( Dx_p = 0 \) and \( (\Gamma_2 + \Gamma_3 A_1^T)c = (\Gamma_4 + \Gamma_3 A_1^T)c = 0 \). Moreover, \( B^T p = B^T A_1^T p = 0 \).

In this case problem (3.8) is not stabilizable. Indeed, multiplying scalarly in \( L^2(0, t; \mathbb{R}^n) \) problem (3.8) by \( p \) one obtains
\[ \langle D_t w, p \rangle_{L^2} - \langle Aw, p \rangle_{L^2} = \langle B(\chi_{\omega}u, p) \rangle_{L^2}, \]
which is equivalent to
\[ D_t \langle w, p \rangle_{L^2} - \langle w, D_t p + AA^* p \rangle_{L^2} = \int_{\omega} \langle u, B^T p \rangle_{\mathbb{R}^m}, \]
and this implies that
\[ \langle w, p \rangle_{L^2} = w_1(t)e^{-t} = \text{cst}. \]

4. Feedback stabilization for the linear system. For the stabilization, since \( A \) generates an analytic semigroup in \( H \), we have by Rellich embedding theorem that resolvent \( \rho(A) \) of \( A \) is compact and the spectrum \( \sigma(A) \) is discrete and lies in a cone \( V_{\delta, \phi} := \{ z \in \mathbb{C}, |\arg (z - \lambda)| \in [\phi - \delta, \phi + \delta] \} \) for some \( \lambda \in \mathbb{R}, \phi \in (0, \pi) \), with no finite accumulation points. This structure allows a separation of the spectrum. Let \( \delta > 0 \) and \( \bar{\delta} > \delta \) such that \( \sigma(A) \cap \{ \text{Re} \lambda = -\bar{\delta} \} = \emptyset \).
\[ \sigma_1 = \sigma(A) \cap \{ \lambda \in \mathbb{C}, \text{Re} \lambda > -\bar{\delta} \} \]
\[ \sigma_2 = \sigma(A) \cap \{ \lambda \in \mathbb{C}, \text{Re} \lambda < -\bar{\delta} \}. \tag{4.1} \]
Observe that \( \sigma_1(A) \) contains a finite number of elements and, among them, those \( \lambda \in \mathbb{C} \) with \( \text{Re} \lambda > 0 \), corresponding to the unstable states. One can find some \( \rho' \in (0, \pi) \) such that \( \sigma_2(A) \) lies in the cone \( V_{-\delta, \rho'} \).

We attach to \( \sigma_1(A), \sigma_2(A) \) corresponding subspaces \( H_1, H_2 \) in the complexified space \( H^c \) such that \( H_1 \oplus H_2 = H^c \) and such \( H_1, H_2 \) are invariant under the complexified operator denoted also \( A \). Moreover, \( \sigma(A|_{H_1}) = \sigma_1(A), \sigma(A|_{H_2}) = \sigma_2(A) \).

Observe that \( H_1 \) is finite dimensional and is generated by eigenfunctions and generalized eigenfunctions of the elliptic operator corresponding to eigenvalues from \( \sigma_1(A) \).

We consider \( P \) to be the projection on \( H_1 \) corresponding to the direct sum \( H_1 \oplus H_2, Q := I - P \) and \( A_1 = PA, A_2 = QA \). Project now the equation (2.3) on \( H_1, H_2 \) and, for some \( w \) solution of (2.3), we denote by \( W_1 = Pw, W_2 = Qw \). Then we have the two problems:
\[ W_1' = A_1 W_1 + PW_2 on H_1 \]
\[ W_2' = A_2 W_2 + (I - P)W_2 on H_2. \tag{4.2} \]
Since we have approximate controllability in time \( T \) for the linear system in \( H \), we have that the reachable set
\[ \{ W^u(T, \cdot), u \in L^2(0, T; L^2(\omega)) \}, \]
with \( W^u \) solution to (2.3), is dense in \( H \). Now, the subspace \( H_1 \) is finite dimensional, so the projection of the reachable set on \( H_1 \) is the whole space \( H_1 \). This implies
exact controllability in any time $T$ for the equation in $H_1$, giving that the first equation is completely stabilizable:

$$\forall \delta > 0, \ \exists K_1 : H_1 \to L^2(\omega, \mathbb{C}), \ \exists c = c(\delta)$$

such that

$$\|e^{(A_1 + PBK_1)t}\|_{L(H_1)} \leq ce^{-\delta t}. \quad (4.3)$$

Now we define the operator $\tilde{K} := K_1 \circ P$ and we denote by $W_1^{\tilde{K}}$ a solution for the equation in $H_1$ stabilized by $\tilde{K}_1$. Then we have the estimate for the solution,

$$\|W_1^{\tilde{K}}(t)\|_H \leq Ce^{-\delta t}\|W_0\|_H. \quad (4.4)$$

We construct the feedback control

$$u := \tilde{K}W_1 = \tilde{K}PW$$

and we use it in the second equation. Then the solution of the equation in $H_2$ is given by the variation of constants formula,

$$W_2^{\tilde{K}}(t) = e^{tA_2}W_2^0 + \int_0^t e^{(t-s)A_2}(I - P)\mathbb{B}KW_1^{\tilde{K}}(s)ds. \quad (4.5)$$

We pass to norms in the above formula, using that $A_2$ is the generator of a stable semigroup on $H_2$,

$$\|e^{tA_2}W_2^0\|_H \leq Ce^{-\delta t}\|W_2^0\|_H, \quad (4.6)$$

giving

$$\|W_2^{\tilde{K}}(t)\|_H \leq Ce^{-\delta t}\|W_2^0\|_H + \int_0^t Ce^{-\delta(t-s)}e^{-\tilde{\delta}s}\|W_1^{\tilde{K}}\|_H, \quad (4.7)$$

where we have used the estimate obtained on $W_1$ and $e^{tA_2}$. Now, if we choose $\tilde{\delta} > \delta$ when stabilizing $W_1$, then there exists $C = C(\delta, \tilde{\delta})$ such that

$$\|W_2^{\tilde{K}}(t)\|_H \leq Ce^{-\tilde{\delta}t}\|W_2^0\|_H. \quad (4.8)$$

We consider the real part of the system and we take

$$K := \text{Re} \tilde{K}$$

to find that $K$ stabilizes the linear system (2.3):

$$\|e^{t(A + BK)}\|_{L(H)} \leq Ce^{-\tilde{\delta}t}, \quad t > 0. \quad (4.9)$$

5. Local stabilization of the nonlinear system. In order to prove stabilization of the nonlinear system (2.1) around the stationary state $\overline{y}$ with the feedback control $u = K(y - \overline{y})$, we will study an equivalent property, the stability in zero for the system satisfied by $z := y - \overline{y}$:

$$\begin{cases}
z' - Az + F(D\overline{y} + Dy, \overline{y} + z) - F(\overline{y}, D\overline{y}) = \mathbb{B}Kz, \ t > 0 \\
z(0) = z_0 = y_0 - \overline{y}.
\end{cases} \quad (5.1)$$

Since we are interested in local stabilization and in order to avoid blow-up phenomena, we will study first a system obtained through truncation and then show that for initial data small enough in appropriate norm the solution to the truncated feedback controlled system is solution to the initial nonlinear controlled system.

For this purpose we consider the cutoff function

$$\rho_R \in C^\infty(\mathbb{R}^n); \quad \rho_R(w) = 1, \text{ if } |w|_2 \leq R; \quad \rho_R(w) = 0, \text{ if } |w|_2 \geq 2R$$
Regarding the operator $R$ following estimates

\[
|z_2|_2 \text{ is the Euclidean norm in } \mathbb{R}^{2n}, \quad R > 2\|\overline{g}\|_{L^\infty} \quad \text{and we will consider } F_R = \rho_R F \text{ instead of } F \text{ in the above system (5.1). Then, the system we will work with takes the form}
\]

\[
\begin{align*}
    z' &= Az + BKz + Rz = \hat{A}z + Rz \\
    z(0) &= z_0
\end{align*}
\]

where for $w \in H^1(\Omega)$,

\[ |Rw|(x) = R_{D\overline{\rho},\overline{\pi}}(Dw(x), w(x)), \quad x \in \Omega, \]

and $R_{D\overline{\rho},\overline{\pi}}(\zeta, y)$ is the remainder of a first order Taylor development of $F_R(\zeta, y)$:

\[ R_{D\overline{\rho},\overline{\pi}}(\zeta, y) = F_R(\overline{\rho} + \zeta, y + \overline{\rho}) - F_R(\overline{\rho}, y) - \frac{\partial}{\partial \zeta} F_R(\overline{\rho}, y)\zeta - \frac{\partial}{\partial y} F_R(\overline{\rho}, y)y. \]

If we also consider the Taylor expansion of order two for $F_R(\zeta, y)$ and taking into account that $F_R$ is compactly supported, we find further the following for the remainder $R_{D\overline{\rho},\overline{\pi}}(\zeta, y)$:

\[ R_{D\overline{\rho},\overline{\pi}}(\zeta, y) = \frac{1}{2} \left[ \frac{\partial^2}{\partial \zeta^2} F_R(D\overline{\rho}, y)\zeta^2 + 2 \frac{\partial^2}{\partial \zeta \partial y} F_R(D\overline{\rho}, y)\zeta y + \frac{\partial^2}{\partial y^2} F_R(D\overline{\rho}, y)y^2 \right] + O(|\zeta|^2 + |y|^2). \]

We have thus that

\[ R_{D\overline{\rho},\overline{\pi}}(\zeta, y) \leq \min\{ |\zeta|_2 + |y|_2, |\zeta|^2_2 + |y|^2_2 \}. \]

Regarding the operator $Rz = R_{D\overline{\rho},\overline{\pi}}(Dz, z)$, using the linear growth we obtain the following estimates

\[ \|Rz\|_{L^2(\Omega)} \leq C\|z\|_{H^1(\Omega)}, \quad \|Rz\|_{L^\infty(\Omega)} \leq C\|z\|_{W^{1,\infty}(\Omega)} \]

and using the quadratic growth we get

\[ \|Rz\|_{L^2(\Omega)} \leq C\|z\|^2_{W^{1,4}(\Omega)} \]

with $C$ a constant depending only on $R$. In order to estimate the solution to nonlinear problem (5.2) we use norms given by a selfadjoint operator $P$ which is solution to a Lyapunov equation and in this sense we need the following Lemma whose proof was communicated to us by C. Lefter. This result appears also in [14] (see also [15]) but under the more restrictive assumption $D(\tilde{\mathcal{A}}) = D(\mathcal{A}^*)$ which is no more needed in this proof.

**Lemma 5.1.** For $\beta \geq 0$ there exists $P$ an unbounded selfadjoint operator in $H$ such that

\[ (Pz, z)_H = \int_0^\infty \|(-\tilde{\mathcal{A}})^{\beta + \frac{1}{2}}e^{t\mathcal{A}}z\|^2_H dt, \quad z \in D(P) \]

and $(Pz, z)_H^{\frac{1}{2}}$ defines an equivalent norm in $D((-\tilde{\mathcal{A}})^{\beta})$.

Moreover, if $\beta \in [0, 1)$, then $D((-\tilde{\mathcal{A}})^{2\beta}) \subset D(P)$ with continuous embedding, that is:

\[ \exists C > 0 \text{ such that for } z \in D((-\tilde{\mathcal{A}})^{2\beta}), \|Pz\|_H \leq C\|(-\tilde{\mathcal{A}})^{2\beta}z\|_H. \]

In addition, the following Lyapunov algebraic equation is satisfied by $P$

\[ (Pz, \tilde{\mathcal{A}}z) = -\frac{1}{2}\|(-\tilde{\mathcal{A}})^{\beta + \frac{1}{2}}z\|^2_H, \quad \text{for } z \in D((-\tilde{\mathcal{A}})^{\max(2\beta, 1)}). \]
Proof. Let \( Q(z) = \int_0^\infty \|(-\hat{A})^{\beta+\frac{1}{2}} e^{t\hat{A}} z\|^2_H dt \) with \( D(Q) = \{ z \mid Q(z) < +\infty \} \). Observe that \( Q \) is densely defined in \( H \) as it is well defined on \( \bigcap_{n \geq 1} D(-\hat{A})^n \), which is a dense subspace of \( H \). Moreover, \( Q \) is a quadratic form as it is easy to see that
\[
Q(z+w) + Q(z-w) = 2(Q(z) + Q(w)), \quad z, w \in D(Q).
\]
Denote also by \( Q \) the bilinear form
\[
Q(z, w) = \frac{1}{4} (Q(z+w) - Q(z-w)), \quad z, w \in D(Q).
\]
In fact one has
\[
Q(z, w) = \int_0^\infty \langle (-\hat{A})^{\beta+\frac{1}{2}} e^{t\hat{A}} z, (-\hat{A})^{\beta+\frac{1}{2}} e^{t\hat{A}} w \rangle_H dt.
\]
There exists a selfadjoint operator \( \mathbf{P} \) in \( H \) with domain
\[
D(\mathbf{P}) = \{ z \in H \mid \exists C > 0, |Q(z, w)| \leq C \|w\|_H \}
\]
and
\[
Q(z) = \langle \mathbf{P} z, z \rangle, \quad \forall z \in D(\mathbf{P}). \quad (5.8)
\]
Observe that \( D(\hat{A}) = D(A) = D(\hat{A}) \). For \( \theta \in (0, 1) \), consider the real interpolation space \((H, D(\hat{A}))_{\theta, 2}\) with norm
\[
\|z\|^2_{(H, D(\hat{A}))_{\theta, 2}} = \int_0^\infty t^{1-2\theta} \|\hat{A} e^{t\hat{A}} z\|^2_H dt. \quad (5.9)
\]
One also knows that, as \( \hat{A} \) is a negative operator generator of an analytic semigroup in a Hilbert space one has equalities between spaces of real and complex interpolation and domains of powers of the operator \( -\hat{A} \) (see [18]):
\[
(H, D(\hat{A}))_{\theta, 2} = [H, D(\hat{A})]_{\theta} = D((-\hat{A})^{\theta}). \quad (5.10)
\]
Thus, for \( \theta = \frac{1}{2} \) we find
\[
\|z\|^2_{(H, D(\hat{A}))_{\frac{1}{2}}} = \int_0^\infty \|\hat{A} e^{t\hat{A}} z\|^2_H dt \sim \|(-\hat{A})^{\frac{1}{2}} z\|^2_H. \quad (5.11)
\]
This is the result for \( \beta = \frac{1}{2} \) and in this case \( D(Q) = [H, D(\hat{A})]_{\frac{1}{2}} \).

Suppose now that \( \beta \neq \frac{1}{2} \) and denote by
\[
Q_{\frac{1}{2}} z = \int_0^\infty \|\hat{A} e^{t\hat{A}} z\|^2_H dt, \quad z \in [H, D(\hat{A})]_{\frac{1}{2}}.
\]
Observe that for \( z \in D((-\hat{A})^\beta) \)
\[
Q(z) = Q_{\frac{1}{2}} ((-\hat{A})^{\beta-\frac{1}{2}} z) = \|(-\hat{A})^{\beta-\frac{1}{2}} z\|_{[H, D(\hat{A})]_{\frac{1}{2}}} \sim \|(-\hat{A})^\beta z\|_H.
\]
and the first statement of the lemma is proved, which is the fact that \( Q \) defines an equivalent norm in \( D((-\hat{A})^\beta) \) which coincides with \( D((-A)^\beta) \) for \( \beta \in [0, 1] \).

We want now to prove that \( D((-\hat{A})^{2\beta}) \subset D(P) \) with continuous embedding and for this we need to prove that there exists \( C > 0 \) such that
\[
|Q(z, w)| \leq \|(-\hat{A})^{2\beta} z\|_H \|w\|_H \quad \text{for } z \in D((-\hat{A})^{2\beta}). \quad (5.12)
\]
Indeed, for \( z, w \in \bigcap_{n \geq 0} D((-\hat{A})^n) \)
\[
Q(z, w) = \int_0^\infty \langle (-\hat{A})^{\frac{1}{2}+\beta} e^{t\hat{A}} z, (-\hat{A})^{\beta} e^{t\hat{A}} (-\hat{A})^{\frac{1}{2}} e^{t\hat{A}} w \rangle_H dt =
\]
\[= \int_0^\infty \langle (-\tilde{A}^+)^{1-\beta} e^{\tilde{A}t} (\tilde{A})^{1-\beta} z, (-\tilde{A})^{1-\beta} e^{\tilde{A}t} w \rangle \, dt.\]

Using Hölder’s inequality,
\[
\left| Q(z, w) \right| \leq \left[ \int_0^\infty \| (-\tilde{A}^+)^{1-\beta} e^{\tilde{A}t} (\tilde{A})^{1-\beta} z \|_{L^2} \, dt \right]^2 \left[ \int_0^\infty \| (-\tilde{A})^{1-\beta} e^{\tilde{A}t} w \|_{L^2} \, dt \right]^2 \leq C \left[ \int_0^\infty \left( \| (-\tilde{A}^+)^{1-\beta} e^{\tilde{A}t} (\tilde{A})^{1-\beta} z \|_{L^2} \right)^2 \, dt \right]^2 \| w \|_H.
\]

Here we used the equivalence of norms when \( \beta = 0 \). In order to estimate the last integral above we write first \( \beta = \beta_1 + \beta_2 \) with both \( \beta_1, \beta_2 < \frac{1}{2} \) to obtain
\[
I := \int_0^\infty \| (-\tilde{A}^+)^{\beta_1} e^{\tilde{A}t} (\tilde{A})^{\beta_1} z \|_{L^2} \, dt
\]
\[
= \int_0^\infty \| (-\tilde{A}^+)^{\beta_1} e^{\tilde{A}t} (\tilde{A})^{\beta_1} z \|_{L^2} \, dt
\]
\[
\leq C \int_0^\infty \frac{1}{t^{2\beta_1}} \| (-\tilde{A}^+)^{\beta_1} e^{\tilde{A}t} (\tilde{A})^{\beta_1} z \|_{L^2} \, dt.
\]

For the last estimate we used Th. 6.13 from [19].

Since \( \beta_2 < \frac{1}{2} \) we have that \( D((-\tilde{A}^+)^{\beta_2}) = D((-\tilde{A})^{\beta_2}) \) and \( (-\tilde{A}^+)^{\beta_2} (-\tilde{A})^{-\beta_2} \) defines a linear continuous operator in \( H \). We obtain thus
\[
I \leq C \int_0^\infty \frac{1}{t^{2\beta_1}} \| e^{\tilde{A}t} (-\tilde{A})^{\beta_1 + \beta_2} z \|_{L^2} \, dt = C \int_0^\infty \frac{1}{t^{2\beta_1}} \| (-\tilde{A})^{\beta_1 + \beta_2} z \|_{L^2} \, dt.
\]

For \( \theta = \frac{1}{2} + \beta_1 \), we estimate the last term using the definition of norms in interpolation spaces and obtain
\[
I \leq C \| (-\tilde{A})^{-\frac{1}{2} + \beta_1 + \beta_2} z \|_{H, D((\tilde{A}))^\theta} \leq C \| (-\tilde{A})^{\frac{1}{2} + \beta_1 - \frac{1}{2} + \beta_1 + \beta_2} z \|_{H, D((\tilde{A}))^\theta} = \| (\tilde{A})^{2\beta} z \|_{H}^2,
\]
which inserted in (5.13) gives the final conclusion of the Lemma.

We return to the equation (5.2) and multiply it scalarly in \( H \) by \( Pz \) and obtain:
\[
\frac{1}{2} \frac{d}{dt} \langle Pz, z \rangle_H = \langle \tilde{A}z, Pz \rangle_H + \langle Rz, Pz \rangle_H.
\]

Considering \( \beta = \frac{1}{2} \) and using the estimate (5.6) and the properties of the Lyapunov operator, we have that there exists \( C > 0 \) such that
\[
\frac{d}{dt} \| z \|_{H^1}^2 + \| z \|_{H^2}^2 \leq C \| z \|_{W^{1,4}}^2 \| z \|_{H^2}.
\]

Since
\[
H^2(\Omega) \subset H^\frac{1}{2}(\Omega) \subset W^{1,4}(\Omega) \subset H^1(\Omega)
\]
and we have the interpolation inequality,
\[
\| z \|_{W^{1,4}} \leq C \| z \|_{H^1}^\frac{1}{2} \| z \|_{H^2}^\frac{1}{2}
\]
we get
\[
\frac{d}{dt} \| z \|_{H^1}^2 + \| z \|_{H^2}^2 \leq C \| z \|_{H^1} \| z \|_{H^2}^2.
\]
Now, for some $\tilde{\delta} > 0$ small, if we place the system in the neighborhood of zero with $\|z\|_{H^1} < \delta$, the solution remains there and satisfies, for some $\delta > 0$

$$\frac{d}{dt}\|z\|_{H^1}^2 \leq -\delta\|z\|_{H^2}^2,$$

giving exponential decay for the norm $H^1$:

$$\|z\|_{H^1} \sim \langle Pz, z \rangle_H^2 \leq Ce^{-\delta t}\|z_0\|_{H^1}.$$  \hspace{1cm} (5.16)

Now since the dimension of the space we work in is one, we have $H^1(\Omega) \subset L^\infty(\Omega)$ and the decay in the $L^\infty$-norm follows. The truncated system stabilizes in $L^\infty$ norm.

We now prove that for initial data small enough in $D((-\tilde{A})^{\nu})$ for given $\nu \in (\frac{1}{4}, 1)$, the solution to the truncated problem is solution to the initial problem. To do this we need estimates of $z$ in $W^{1,\infty}(\Omega)$; for this purpose we estimate $z$ in $D((-\tilde{A})^{\nu})$ and we take into account the continuous embeddings $D((-\tilde{A})^{\nu}) \subset W^{2\nu,2}(\Omega) \subset W^{1,\infty}(\Omega)$. One has for fixed $\epsilon > 0$ and $\tau \in \mathbb{R}_+$

$$z(\tau + \epsilon) = e^{\epsilon \tilde{A}}z(\tau) + \int_\tau^{\tau + \epsilon} e^{(\tau + s - \epsilon)\tilde{A}}Rz(s)ds$$

and thus

$$(-\tilde{A})^{\nu} z(\tau + \epsilon) = (-\tilde{A})^{\nu} e^{\epsilon \tilde{A}}z(\tau) + \int_0^\epsilon (-\tilde{A})^{\nu} e^{(\epsilon - s)\tilde{A}}Rz(s + \tau)ds.$$  \hspace{1cm} (5.17)

We pass to the norm in $H$ in the equality above and obtain for some constant $C$ depending on the operator $\tilde{A}$

$$\|z(\tau + \epsilon)\|_{D((-\tilde{A})^{\nu})} \leq \frac{C}{\epsilon}\|z(\tau)\|_{H} + \int_0^\epsilon \frac{C}{(\epsilon - s)^\nu}\|Rz(\cdot + \tau)\|_{L^\infty(0,\epsilon;H)}ds \leq$$

$$\leq \frac{C}{\epsilon}\|z(\tau)\|_{H^1} + \int_0^\epsilon \frac{C}{(\epsilon - s)^\nu}\|z(\cdot + \tau)\|_{L^\infty(0,\epsilon;H)}ds.$$  \hspace{1cm} (5.18)

Using now the stability estimate in $H^1$ and integrability in $0$ of $s^{-\nu}$, we find that

$$\|z(\tau + \epsilon)\|_{D((-\tilde{A})^{\nu})} \leq C(\tilde{A}, \epsilon)\|z(0)\|_{H^1}e^{-\delta \epsilon t}.$$  \hspace{1cm} (5.19)

It is clear now that for $\|z(0)\|_{D((-\tilde{A})^{\nu})}$ small enough $z$ is solution to the nonlinear system (5.1) for time $t \geq \epsilon$. Regarding the estimates for $t \in [0, \epsilon]$, using classical arguments concerning continuous dependence of the solution on initial data, we may say that if one chooses $\|z(0)\|_{D((-\tilde{A})^{\nu})}$ small enough then $\|z\|_{L^\infty(0,\epsilon;D((-\tilde{A})^{\nu}))}$ is small enough. Correspondingly, one has $\|z\|_{L^\infty([0,\epsilon];W^{1,\infty}(\Omega))} < R$ and thus $z$ is solution to (5.1) on $[0, \infty)$; stabilization for truncated equation is in fact stabilization for initial nonlinear equation when initial data is small.

Stabilization in $H^2$ or in $D(\tilde{A})$ also follows as above using again the regularizing effect of the analytic semigroup.

We mention here that since the initial data $y^0$ satisfies the boundary conditions and belongs to $H^{2\nu}$, by a classic result of R. Seeley (see [21]), $y_0 \in D((-\tilde{A})^{\nu})$ and the stability estimates obtained in terms of norms in fractional spaces reduce to the estimates using Sobolev norms.

\begin{remark}
Observe that if we are in the framework of Theorem 2.5, the feedback stabilizing (2.1) (or equivalently (2.7)) stabilizes also the system with $\tilde{A}$ perturbed by a small lower order operator: $\tilde{A} + T$ where $T$ is for example a first order linear\end{remark}
differential operator with coefficients having small enough $L^\infty$ norm. As a consequence, if we have to stabilize nonconstant solutions then the linearized system has nonconstant coupling matrices $A_0(x), A_1(x)$. We may then consider a modified linear controlled system with constant coupling matrices $\overline{A}_0 = \frac{1}{l} \int_0^l A_0(x)dx$ and $\overline{A}_1 = \frac{1}{l} \int_0^l A_1(x)dx$. Following the procedure in this paper one may prove that if the modified controlled system is feedback stabilizable and $\|A_0(\cdot) - \overline{A}_0\|_{L^\infty}$ and $\|A_1(\cdot) - \overline{A}_1\|_{L^\infty}$ are small enough, then the same feedback stabilizes (2.1).

REFERENCES

[1] F. Ammar Khodja, A. Benabdallah, C. Dupaix and M. González-Burgos, A generalization of the Kalman rank condition for time-dependent coupled linear parabolic systems, Differ. Equ. Appl., 1 (2009), 427–457.

[2] F. Ammar-Khodja, A. Benabdallah, C. Dupaix and M. González-Burgos, A Kalman rank condition for the localized distributed controllability of a class of linear parabolic systems, J. Evol. Equ., 9 (2009), 267–291.

[3] V. Barbu and G. Wang, Internal stabilization of semilinear parabolic systems, J. Math. Anal. Appl., 285 (2003), 387–407.

[4] V. Barbu, Controllability and Stabilization of Parabolic Equations, Progress in Nonlinear Differential Equations and their Applications Vol. 90, Birkhäuser/Springer, Cham, 2018.

[5] V. Barbu, I. Lasiecka and R. Triggiani, Abstract settings for tangential boundary stabilization of Navier-Stokes equations by high- and low-gain feedback controllers, Nonlinear Anal., 64 (2006) 2704–2746.

[6] V. Barbu, I. Lasiecka and R. Triggiani, Tangential boundary stabilization of Navier-Stokes equations, Mem. Amer. Math. Soc., 181 (2006), 128 pp.

[7] V. Barbu, I. Lasiecka and R. Triggiani, Local exponential stabilization strategies of the Navier-Stokes equations, $d = 2, 3$, via feedback stabilization of its linearization, in Control of Coupled Partial Differential Equations, Internat. Ser. Numer. Math., Vol. 155, Birkhäuser, Basel, 2007.

[8] V. Barbu, S. S. Rodrigues and A. Shirikyan, Internal exponential stabilization to a nonstationary solution for 3D Navier-Stokes equations, SIAM J. Control Optim., 49 (2011), 1454–1478.

[9] V. Barbu and R. Triggiani, Internal stabilization of Navier-Stokes equations with finite-dimensional controllers, Indiana Univ. Math. J., 53 (2004), 1443–1494.

[10] V. Barbu and G. Wang, Feedback stabilization of periodic solutions to nonlinear parabolic-like evolution systems, Indiana Univ. Math. J., 54 (2005), 1521–1546.

[11] M. Duprez and P. Lissy, Positive and negative results on the internal controllability of parabolic equations coupled by zero- and first-order terms, J. Evol. Equ., 18 (2018), 659–680.

[12] A. V. Fursikov and O. Yu. Imanuvilov, Controllability of evolution equations, Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1996.

[13] M. González-Burgos and L. de Teresa. Controllability results for cascade systems of $m$ coupled parabolic PDEs by one control force, Port. Math., 67 (2010), 91–113.

[14] C. Lefter, Feedback stabilization of 2D Navier-Stokes equations with Navier slip boundary conditions, Nonlinear Anal., 70 (2009), 553–562.

[15] C. Lefter, Feedback stabilization of magnetohydrodynamic equations, SIAM J. Control Optim., 49 (2011), 963–983.

[16] C. Lefter, Internal feedback stabilization of nonstationary solutions to semilinear parabolic systems, J. Optim. Theory Appl., 170 (2016), 960–976.

[17] P. Lissy and E. Zuazua, Internal observability for coupled systems of linear partial differential equations, SIAM J. Control Optim., 57 (2019), 832–853.

[18] A. Lunardi, Interpolation theory, third edition Appunti. Scuola Normale Superiore di Pisa (Nuova Serie), Vol. 16, Edizioni della Normale, Pisa, 2018.
[19] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences, Vol. 44, Springer-Verlag, New York, 1983.

[20] J.-C. Saut and B. Scheurer, *Unique continuation for some evolution equations*, *J. Differential Equations*, 66 (1987), 118–139.

[21] R. Seeley, *Norms and domains of the complex powers $A^Bz$*, *Amer. J. Math.*, 93 (1971), 299–309.

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