A new smoothness result for Caputo-type fractional ordinary differential equations

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Abstract

We present a new smoothness result for Caputo-type fractional ordinary differential equations, which reveals that, subtracting a non-smooth function that can be obtained by the information available, a non-smooth solution belongs to $C^m$ for some positive integer $m$.

Keywords: Caputo, fractional differential equation, smoothness.

1 Introduction

Let us consider the following model problem: seek $0 < h \leq a$ and

$$y \in \left\{ v \in C[0, h] : \|v - c_0\|_{C[0, h]} \leq b \right\}$$

such that

$$\begin{aligned}
D^\alpha y &= f(x, y), \quad 0 \leq x \leq h, \\
y(0) &= c_0,
\end{aligned}$$

(1.1)

where $a > 0$, $b > 0$, $0 < \alpha < 1$, $c_0 \in \mathbb{R}$, and

$$f \in C([0, a] \times [c_0 - b, c_0 + b]).$$

Above, the Caputo-type fractional differential operator $D^\alpha : C[0, h] \to C_0^\infty(0, h)'$ is given by

$$D^\alpha z := DJ^{1-\alpha}(z - z(0))$$

(1.2)

for all $z \in C[0, h]$, where $D$ denotes the well-known first order generalized differential operator, and the Riemann-Liouville fractional integral operator $J^{1-\alpha} : C[0, h] \to C[0, h]$ is defined by

$$J^{1-\alpha}z(x) := \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-t)^{-\alpha}z(t) \, dt, \quad 0 \leq x \leq h,$$

for all $z \in C[0, h]$.

By [2, Lemma 2.1], the above problem is equivalent to seeking solutions of the following Volterra integration equation:

$$y(x) = c_0 + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}f(t, y(t)) \, dt.$$  

(1.3)

Diethelm and Ford [2] proved that, if $f$ is continuous, then (1.3) has a solution $y \in C[0, h]$ for some $0 < h \leq a$, and this solution is unique if $f$ is Lipschitz continuous. A natural question arises whether $y$ can be smoother than being continuous. This is not only of theoretical value, but also of great importance in developing numerical methods for (1.3).

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To this question, Miller and Feldstein [5] gave the first answer: if $f$ is analytic, then $y$ is analytic in $(0, h)$ for some $0 < h \leq a$. Then Lubich [4] considered the behavior of the solution near 0. He showed that, if $f$ is analytic at the origin, then there exists a function $Y$ of two variables that is analytic at the origin such that

$$y(x) = Y(x, x^a), \quad 0 \leq x \leq h,$$

for some $0 < h \leq a$. The above work suggests that non-smoothness of the solution to (1.1) is generally unavoidable. However, Diethelm [1] established a sufficient and necessary condition under which $y$ is analytic on $[0, h]$ for some $0 < h \leq a$. But, since we have already seen that non-smoothness of $y$ is generally unavoidable, it is not surprising that this condition is unrealistic. Recently, Deng [3] proposed two conditions: under the first condition the solution belongs to $C^m$ for some positive integer $m$; under the second one the solution is a polynomial. It should be noted that, the second condition is just the one proposed in [1], and the first condition is also unrealistic.

The main result of this paper is that, although the solution $y$ of (1.1) does not generally belong to $C^m$ for some positive integer $m$, we can still construct a non-smooth function of the form

$$S(x) := c_0 + \sum_{j=1}^{n} c_j x^\gamma_j,$$

such that

$$y - S \in C^m,$$

provided $f$ is sufficiently smooth. Most importantly, given $c_0$ and $f$, we can obtain $S$ by a simple computation. This is significant in the development of numerical methods for (1.1). In addition, we obtain a sufficient and necessary condition under which $y \in C^m$. We note that this condition is essentially the same as the first condition mentioned already in [3, Theorem 2.8], but the necessity was not considered therein.

The rest of this paper is organized as follows. In Section 2 we introduce some basic notation and preliminaries. In Section 3 we state the main results of this paper, and present their proofs in Section 4.

## 2 Notation and Preliminaries

Let $0 < h < \infty$. We use $C[0, h]$ to denote the space of all continuous real functions defined on $[0, h]$. For any $k \in \mathbb{N}_{>0}$ and $0 \leq \gamma \leq 1$, define

$$C^k[0, h] := \left\{ v \in C[0, h] : v^{(j)} \in C[0, h] \text{ for } j = 1, 2, \ldots, k \right\},$$

$$C^{k, \gamma}[0, h] := \left\{ v \in C^k[0, h] : \max_{0 \leq x < y \leq h} |v^{(j)}(x)| < \infty \right\},$$

and endow the above two spaces with two norms respectively by

$$\|v\|_{C^k[0, h]} := \max_{0 \leq j \leq k} \max_{0 \leq x \leq h} |v^{(j)}(x)|$$

for all $v \in C^k[0, h]$, and

$$\|v\|_{C^{k, \gamma}[0, h]} := \max \left\{ \|v\|_{C^k[0, h]}, |v|_{C^{k, \gamma}[0, h]} \right\}$$

for all $v \in C^{k, \gamma}[0, h]$.

Here the semi-norm $|.|_{C^{k, \gamma}[0, h]}$ is given by

$$|v|_{C^{k, \gamma}[0, h]} := \sup_{0 \leq x < y \leq h} \frac{|v^{(k)}(x) - v^{(k)}(y)|}{(y - x)^\gamma}$$

for all $v \in C^{k, \gamma}[0, h]$, and it is obvious that $C^k[0, h]$ coincides with $C^{k, 0}[0, h]$.

For any $s \in \mathbb{N}_{>0}$, define

$$\Lambda_s := \left\{ \beta = (\beta_1, \beta_2, \ldots, \beta_s) \in \{1, 2\}^s \right\},$$

and, for any $\beta \in \Lambda_s$, we use the following notation:

$$\partial_{(\beta)}g := \frac{\partial}{\partial x_{\beta_s}} \frac{\partial}{\partial x_{\beta_{s-1}}} \cdots \frac{\partial}{\partial x_{\beta_1}} g(x_1, x_2),$$

for all $g : \mathbb{R}^2 \to \mathbb{R}$.
where $g$ is a real function of two variables. In addition, we define

$$\Lambda_0 := \{0\},$$

and denote by $\partial_0$ the identity mapping.

3 Main Results

Let us first make the following assumption on $f$.

**Assumption 1.** There exist a positive integer $n$, and a positive constant $M$ such that

$$f \in C^n ([0, a] \times [c_0 - b, c_0 + b]),$$

$$\max_{(x, y) \in [0, a] \times [c_0 - b, c_0 + b]} \max_{0 \leq j \leq n} \left| \frac{\partial^j f}{\partial x^j} (x, y) \right| \leq M.$$

Throughout this paper, we assume that the above assumption is fulfilled.

Define $J \in \mathbb{N}$ and a strictly increasing sequence $\{\gamma_j\}_{j=1}^J$ by

$$\{\gamma_j : 1 \leq j \leq J\} = \{i + j\alpha : i, j \in \mathbb{N}, \ 0 < i + j\alpha < m\}, \quad (3.1)$$

where

$$m := \max \{j \in \mathbb{N} : j < n\alpha\}. \quad (3.2)$$

Define $c_1, c_2, \ldots, c_J \in \mathbb{R}$ by

$$Q(x) - S(x) + c_0 \in \text{span} \{x^{i+j\alpha} : i, j \in \mathbb{N}, \ i + j\alpha \geq m\}, \quad (3.3)$$

where

$$Q(x) := \sum_{s=0}^{n-1} \sum_{\beta \in \Lambda_s} \frac{\partial^s f(0, c_0)}{\Gamma(\alpha)} \int_0^x (x - t_0)^{s-1} dt_0 \prod_{k=1}^s \int_0^{t_k-1} \left[ 1 + (-1)^{s_k+1} \frac{\beta_k}{2} \right] + \frac{1}{2} \sum_{j=1}^J \gamma_j c_j \gamma_j^{s-1} dt_k, \quad (3.4)$$

and

$$S(x) := c_0 + \sum_{j=1}^J c_j x^{\gamma_j}. \quad (3.5)$$

Above and throughout, a product of a sequence of integrals should be understood in expanded form. For example, (3.4) is understood by

$$Q(x) := \sum_{s=0}^{n-1} \sum_{\beta \in \Lambda_s} \frac{\partial^s f(0, c_0)}{\Gamma(\alpha)} \int_0^x (x - t_0)^{s-1} dt_0 \int_0^{t_0} \left[ 1 + (-1)^{s_1+1} \frac{\beta_1}{2} \right] + \frac{1}{2} \sum_{j=1}^J \gamma_j c_j \gamma_j^{s_1-1} dt_1$$

$$\int_0^{t_1} \left[ 1 + (-1)^{s_2+1} \frac{\beta_2}{2} \right] + \frac{1}{2} \sum_{j=1}^J \gamma_j c_j \gamma_j^{s_2-1} dt_2$$

$$\ldots$$

$$\int_0^{t_{s-1}} \left[ 1 + (-1)^{s_s+1} \frac{\beta_s}{2} \right] + \frac{1}{2} \sum_{j=1}^J \gamma_j c_j \gamma_j^{s_s-1} dt_s.$$

**Remark 3.1.** It is easy to see that we can express $Q$ in the form

$$Q(x) = \sum_{j=1}^L d_j x^{\gamma_j},$$

where $\{\gamma_j\}_{j=1}^L$ is a strictly increasing sequence such that $\gamma_j < \gamma_{j+1}$ and

$$\{\gamma_j : 1 \leq j \leq L\} = \{i + j\alpha : i, j \in \mathbb{N}, \ i \leq n - 1, \ 1 \leq j \leq 1 + (n - 1)\gamma_j\}.$$
Moreover, for $1 \leq j \leq J$, the value of $d_j$ only depends on $c_0$, $c_1$, $c_2$, $c_3$, and $f$ (more precisely, $\partial_{\beta} f(0, c_0)$, $\beta \in \Lambda_s$, $1 \leq s \leq n - 1$). Obviously, there exist(s) uniquely $c_1$, $c_2$, $c_3$, and $c_4$, such that (3.3) holds, and hence $c_1$, $c_2$, $c_3$, $c_4$ are/is well-defined. Furthermore, if $\gamma J + \alpha - m > 0$, then

$$Q - S \in C^{m, \gamma_J + \alpha - m}[0, a];$$

and if $\gamma J + \alpha - m = 0$, then

$$Q - S \in C^{m, \alpha}[0, a].$$

**Remark 3.2.** Note that, $S$ only depends on $c_0$ and

$$\{\partial_{\beta} f(0, c_0) : \beta \in \Lambda_s, 0 \leq s < n \}.$$

Since $c_0$ and $f$ are already available, we can obtain $S$ by a simple calculation.

Define

$$h^* := \min \left\{ a, \left( \frac{b \Gamma(1 + \alpha)}{M} \right)^{\frac{1}{\alpha}} \right\}.$$

By [2, Theorem 2.2] we know that there exists a unique solution $y^* \in C[0, h^*]$ to (1.1). Now we state the most important result of this paper in the following theorem.

**Theorem 3.1.** There exist two positive constant $C_0$ and $C_1$ that only depends on $a$, $\alpha$ and $M$, such that, for any $0 < h \leq h^*$ and $K > 0$ such that

$$\|(Q - S)'\|_{C^{m-1}[0, h]} + C_1 h^\alpha + C_0 h^\alpha \sum_{j=1}^{m} K_j \leq K,$$

we have $y^* - S \in C^m[0, h]$ and

$$\|(y^* - S)'\|_{C^{m-1}[0, h]} \leq K.$$  \hfill (3.6)

**Corollary 3.1.** There exists $0 < h \leq h^*$ such that $y^* \in C^m[0, h]$ if, and only if,

$$\frac{\partial^i}{\partial x^i} f(0, c_0) = 0 \quad \text{for all } 0 \leq i < m.$$  \hfill (3.7)

**Remark 3.3.** Corollary 3.1 states that $y^* \in C^1[0, h]$ for some $0 < h \leq h^*$ if and only if $f(0, c_0) = 0$. So we only have $y^* \in C[0, h] \setminus C^1[0, h]$, if $f(0, c_0) \neq 0$. This yields great difficulty in developing high order numerical methods for (1.1), although $y^* \in C^m[0, h]$. Many numerical methods for (1.1) may not even converge theoretically, since they require that $y^* \in C^m[0, h]$ for some positive integer $m$. However, we can obtain the numerical values of $y^*$ at some left-most nodes by solving the following problem ($y^* = y + S$): seek $y \in C^m[0, \tilde{h}]$ such that

$$y(x) = c_0 - S(x) + \frac{1}{\Gamma(\alpha)} \int_0^x (x - t)^{\alpha - 1} f(t, y(t) + S(t)) \, dt, \quad 0 \leq x \leq \tilde{h},$$

where $\tilde{h} \ll h$. Then we start the numerical methods for (1.1).

**Remark 3.4.** Assuming that $f$ satisfies $f(x, c_0) = 0$ for all $0 \leq x \leq a$, it is easy to see that

$$c_i = 0 \quad \text{for all } 1 \leq i \leq J,$$

and hence $S = c_0$. Then Theorem 3.1 implies $y^* \in C^m[0, h]$. Actually, in this case, it is easy to see that $y^* = c_0$.

**Remark 3.5.** Put

$$\Theta := \{1 \leq j \leq J : \gamma_j \notin \mathbb{N}\}.$$

Obviously,

$$\sum_{j \in \Theta} c_j x^{\gamma_j}$$

is the singular part (compared to the $C^m$ regularity) in $S$, and thus the singular part in $y^*$. Corollary 3.1 essentially claims that (3.7) holds if and only if $c_j = 0$ for all $j \in \Theta$. Since (3.7) is rare, we can consider singularity as an intrinsic property of solutions to fractional differential equations. In addition, we have the following result: that $c_j = 0$ for all $1 \leq j \leq J$ is equivalent to that $c_j = 0$ for all $j \in \Theta$. This is contained in the proof of Corollary 3.1 in Section 4.3.
4 Proofs

Let $0 < h < \infty$. For any $k \in \mathbb{N}$ and $\gamma \in [0, 1]$, define
\begin{equation}
C^k\gamma[0, h] := \left\{ v \in C^k\gamma[0, h] : v^{(j)}(0) = 0, \quad j = 0, 1, 2, \ldots, k \right\},
\end{equation}
\begin{equation}
\tilde{C}^k\gamma[0, h] := \left\{ v \in C^k\gamma[0, h] : \|v + S - c_0\|_{C[0, h]} \leq b \right\}.
\end{equation}
In particular, we use $C^k[0, h]$ and $\tilde{C}^k[0, h]$ to abbreviate $C^k[0, h]$ and $\tilde{C}^k[0, h]$ respectively for $k \in \mathbb{N}_{>0}$, and use $C[0, h]$ and $\tilde{C}[0, h]$ to abbreviate $C^0[0, h]$ and $\tilde{C}^0[0, h]$ respectively. In addition, for a function $v$ defined on $(0, h]$ with $h > 0$, by $v \in C^k\gamma[0, h]$ we mean that, setting $v(0) := 0$, the function $v$ belongs to $C^k\gamma[0, h]$.

In the remainder of this paper, unless otherwise specified, we use $C$ to denote a positive constant that only depends on $\alpha$, $a$, and $M$, and its value may differ at each occurrence. By the definitions of $c_1$, $c_2$, $\ldots$, $c_J$, it is easy to see that $|c_j| \leq C$ for all $1 \leq j \leq J$, and we use this implicitly in the forthcoming analysis.

4.1 Some Auxiliary Results

We start by introducing some operators. For $0 < h \leq a$, define $P_{1,h} : \tilde{C}^m[0, h] \to C[0, h]$, $P_{2,h} : \tilde{C}^m[0, h] \to C[0, h]$, and $P_{3,h} : \tilde{C}^m[0, h] \to C[0, h]$, respectively, by
\begin{equation}
P_{1,h}z(x) := \frac{1}{\Gamma(\alpha)} \int_0^x (x - t)^{\alpha - 1} \tilde{G}_{1,h}z(t) \, dt,
\end{equation}
\begin{equation}
P_{2,h}z(x) := \frac{1}{\Gamma(\alpha)} \int_0^x (x - t)^{\alpha - 1} \tilde{G}_{2,h}z(t) \, dt,
\end{equation}
\begin{equation}
P_{3,h}z(x) := \frac{1}{\Gamma(\alpha)} \int_0^x (x - t)^{\alpha - 1} \tilde{G}_{3,h}z(t) \, dt,
\end{equation}
for all $z \in \tilde{C}^m[0, h]$, where $\tilde{G}_{1,h}z$, $\tilde{G}_{2,h}z$, $\tilde{G}_{3,h}z \in C[0, h]$ are given respectively by
\begin{align}
\tilde{G}_{1,h}z(t_0) &:= \sum_{s=1}^n \sum_{\beta_1, \ldots, \beta_n \geq 2} \prod_{k=1}^{s-1} \int_0^{t_{k-1}} \frac{1 + (-1)^{\beta_k + 1}}{2} + \frac{1 + (-1)^{\beta_k}}{2} \sum_{j=1}^J \gamma_j c_j t_j^{\gamma_j - 1} \, dt_k \\
&\quad \int_0^{t_{k-1}} z'(t_s) \partial_\beta f(t_s, z(t_s) + S(t_s)) \, dt_s,
\end{align}
\begin{align}
\tilde{G}_{2,h}z(t_0) &:= \sum_{\beta_1, \ldots, \beta_n \geq 2} \prod_{k=1}^{n-1} \int_0^{t_{k-1}} \frac{1 + (-1)^{\beta_k + 1}}{2} + \frac{1 + (-1)^{\beta_k}}{2} \sum_{j=1}^J \gamma_j c_j t_j^{\gamma_j - 1} \, dt_k \\
&\quad \int_0^{t_{n-1}} \partial_\beta f(t_n, z(t_n) + S(t_n)) \sum_{j=1}^J \gamma_j c_j t_j^{\gamma_j - 1} \, dt_n,
\end{align}
\begin{align}
\tilde{G}_{3,h}z(t_0) &:= \sum_{\beta_1, \ldots, \beta_n \geq 2} \prod_{k=1}^{n-1} \int_0^{t_{k-1}} \frac{1 + (-1)^{\beta_k + 1}}{2} + \frac{1 + (-1)^{\beta_k}}{2} \sum_{j=1}^J \gamma_j c_j t_j^{\gamma_j - 1} \, dt_k \\
&\quad \int_0^{t_{n-1}} \partial_\beta f(t_n, z(t_n) + S(t_n)) \, dt_n,
\end{align}
for all $0 \leq t_0 \leq h$.

Then let us present the following important results for the above operators.

Lemma 4.1. Let $0 < h \leq a$. For any $z \in \tilde{C}^m[0, h]$, we have
\begin{equation}
\frac{1}{\Gamma(\alpha)} \int_0^x (x - t)^{\alpha - 1} f(t, z(t) + S(t)) \, dt = Q(x) + P_{1,h}z(x) + P_{2,h}z(x) + P_{3,h}z(x)
\end{equation}
for all $0 \leq x \leq h$. 
Proof. Let $\beta \in \Lambda_s$ with $1 \leq s < n$. For any $0 < t_s \leq h$, applying the fundamental theorem of calculus yields

$$
\partial_\beta f(t_s, z(t_s) + S(t_s)) = \partial_\beta f(\epsilon, z(\epsilon) + S(\epsilon)) + \int_\epsilon^{t_s} \partial_\beta f(t_{s+1}, z(t_{s+1}) + S(t_{s+1})) dt_{s+1} + 
$$

$$
\int_\epsilon^{t_s} \left( z'(t_{s+1}) + \sum_{j=1}^J \gamma_j c_j t_j^{(j-1)/2} \right) \partial_\beta f(t_{s+1}, z(t_{s+1}) + S(t_{s+1})) dt_{s+1}
$$

for all $0 < \epsilon \leq t_s$, where $\tilde{\beta} := (\beta_1, \beta_2, \ldots, \beta_s, 1)$ and $\tilde{\pi} := (\beta_1, \beta_2, \ldots, \beta_s, 2)$. Taking limits on both sides of the above equation as $\epsilon$ approaches $0^+$, we obtain

$$
\partial_\beta f(t_s, z(t_s) + S(t_s)) = \partial_\beta f(0, c_0) + \int_0^{t_s} \partial_\beta f(t_{s+1}, z(t_{s+1}) + S(t_{s+1})) dt_{s+1} + 
$$

$$
\int_0^{t_s} \left( z'(t_{s+1}) + \sum_{j=1}^J \gamma_j c_j t_j^{(j-1)/2} \right) \partial_\beta f(t_{s+1}, z(t_{s+1}) + S(t_{s+1})) dt_{s+1}.
$$

Using this equality repeatedly, we easily obtain (4.9). This completes the proof.

\[ \square \]

**Lemma 4.2.** Let $0 < h \leq a$. For any $z \in \hat{C}^m[0, h]$, we have $P_{1,h}z \in C^{m,a}[0, h]$ and

$$
\| (P_{1,h}z)' \|_{C^{m-1}[0, h]} \leq C h^\alpha \sum_{j=1}^m \| z_j'' \|_{C^{m-1}[0, h]},
$$

(4.10)

$$
\left| (P_{1,h}z)^{(m)} \right|_{C^{m,a}[0, h]} \leq C \sum_{j=1}^m \| z_j'' \|_{C^{m-1}[0, h]}.
$$

(4.11)

**Lemma 4.3.** Let $0 < h \leq a$. For any $z \in \hat{C}^m[0, h]$, we have $P_{2,h}z, P_{3,h}z \in C^{m,a}[0, h]$ and

$$
\| (P_{2,h}z)' \|_{C^{m-1}[0, h]} + \| (P_{3,h}z)' \|_{C^{m-1}[0, h]} \leq C h^\alpha,
$$

(4.12)

$$
\left| (P_{2,h}z)^{(m)} \right|_{C^{m,a}[0, h]} + \left| (P_{3,h}z)^{(m)} \right|_{C^{m,a}[0, h]} \leq C.
$$

(4.13)

To prove the above two lemmas, we need several lemmas below.

**Lemma 4.4.** Let $0 < h \leq a$ and $g \in C^m[0, h]$. We have $w \in C^{m,a}[0, h]$ and

$$
\| w' \|_{C^{m-1}[0, h]} \leq C h^\alpha \| g' \|_{C^{m-1}[0, h]},
$$

(4.14)

$$
\left| w^{(m)} \right|_{C^{m,a}[0, h]} \leq C \left| g^{(m)} \right|_{C[0, h]},
$$

(4.15)

where

$$
w(x) := \int_0^x (x-t)^{\alpha-1} g(t) dt, \quad 0 \leq x \leq h.
$$

**Proof.** Since $g \in C^m[0, h]$ we have

$$
w^{(i)}(x) = \int_0^x (x-t)^{\alpha-1} g^{(i)}(t) dt, \quad 1 \leq i \leq m.
$$

Then $w \in C^m[0, h]$ and (4.14) follow, and (4.15) follows from [6, Theorem 3.1]. This completes the proof.

\[ \square \]

**Lemma 4.5.** Let $0 < h \leq a$, and $k, l \in \mathbb{N}$ such that $k \leq m$ and $l \alpha \leq 1$. For any $g \in C^{k,l\alpha}[0, h]$, define

$$
w(x) := \int_0^x \sum_{j=1}^J \gamma_j c_j t_j^{\alpha-1} g(t) dt, \quad 0 < x \leq h.
$$

Then we have the following results:
Then, by the simple estimate 
\[ \|w\|_{C^{k,(l+1)\alpha}} \leq C \|g\|_{C^{k,\alpha}}. \]

- If \((l+1)\alpha \leq 1\), then we have \(w \in C^{k,(l+1)\alpha}[0,a]\) and 
\[ \|w\|_{C^{k,(l+1)\alpha}} \leq C \|g\|_{C^{k,\alpha}}. \]

- If \((l+1)\alpha > 1\), then we have \(w \in C^{k+1,(l+1)\alpha-1}[0,a]\) and 
\[ \|w\|_{C^{k+1,(l+1)\alpha-1}} \leq C \|g\|_{C^{k,\alpha}}. \]

For any \(0 < h \leq a\), \(w \in C[0,h]\), and \(\beta \in \Lambda_s\) with \(1 \leq s \leq n\), define \(T_{w,\beta,h} : \hat{C}^m[0,h] \to C[0,h]\) by 
\[ T_{w,\beta,h}z(x) := w(x)\partial_\beta f(x, z(x) + S(x)), \]
for all \(z \in \hat{C}^m[0,h]\).

**Lemma 4.6.** For \(0 \leq k \leq m\), we have \(T_{w,\beta,h}z \in C^{\min\{k,n-s\}}[0,h]\) and 
\[ \|T_{w,\beta,h}z\|_{C^{\min\{k,n-s\}}[0,h]} \leq C \|w\|_{C^{k}[0,h]} \sum_{j=0}^{\min\{k,n-s\}} \|z\|_{C^{m-1}[0,h]}^j \tag{4.16} \]
for all \(0 < h \leq a\), \(w \in C^k[0,h]\), \(\beta \in \Lambda_s\) with \(1 \leq s \leq n\), and \(z \in \hat{C}^m[0,h]\).

The proofs of Lemmas 4.5 and 4.6 are presented in Appendix A. In the rest of this subsection, we give the proofs of Lemmas 4.2 and 4.3.

**Proof of Lemma 4.2.** By (4.3), (4.6), and Lemma 4.4, it suffices to show that, for each \(\beta \in \Lambda_s\) with \(\beta_s = 2\), we have \(g_0 \in C^m[0,h]\) and 
\[ \|g_0\|_{C^m[0,h]} \leq C \sum_{j=1}^{\min\{m,n-s+1\}} \|z\|_{C^{m-1}[0,h]}^j, \tag{4.17} \]
where, if \(s = 1\), then 
\[ g_0(x) := \int_0^x z(t)\partial_2 f\left(t, z(t) + S(t)\right) dt; \]
if \(2 \leq s \leq n\), then 
\[
\begin{align*}
g_0(x) &:= \int_0^x \left(\frac{1}{2} + \frac{(-1)^{\beta_1+1}}{2} + \frac{1}{2} \sum_{j=1}^J \gamma_j c_j t^{\gamma_j-1}\right) g_1(t) \, dt, \\
g_1(x) &:= \int_0^x \left(\frac{1}{2} + \frac{(-1)^{\beta_2+1}}{2} + \frac{1}{2} \sum_{j=1}^J \gamma_j c_j t^{\gamma_j-1}\right) g_2(t) \, dt, \\
&\vdots \\
g_{s-2}(x) &:= \int_0^x \left(\frac{1}{2} + \frac{(-1)^{\beta_{s-2}+1}}{2} + \frac{1}{2} \sum_{j=1}^J \gamma_j c_j t^{\gamma_j-1}\right) g_{s-1}(t) \, dt, \\
g_{s-1}(x) &:= \int_0^x z(t)\partial_\beta f\left(t, z(t) + S(t)\right) dt.
\end{align*}
\]

To do so, we proceed as follows. If \(s = 1\), then by Lemma 4.6 we obtain \(g_0 \in C^m[0,h]\) and (4.17). Let us suppose that \(2 \leq s \leq n\). By Lemma 4.6 it follows \(g_{s-1} \in C^{\min\{m,n-s+1\}}[0,h]\) and 
\[ \|g_{s-1}\|_{C^{\min\{m,n-s+1\}}[0,h]} \leq C \sum_{j=1}^{\min\{m,n-s+1\}} \|z\|_{C^{m-1}[0,h]}^j, \]
Then, by the simple estimate 
\[ (n - s + 1) + (s - 1)\alpha > m, \]
applying Lemma 4.5 to \( g_{s-2}, g_{s-3}, \ldots, g_0 \) successively yields \( g_0 \in C^m[0,h] \) and (4.17). This completes the proof of Lemma 4.2. \( \blacksquare \)

**Proof of Lemma 4.3.** Let us first show that \( G_{2,h} \in C^m[0,h] \) and

\[
\|G_{2,h}\|_{C^m[0,h]} \leq C. \tag{4.18}
\]

By (4.7) it suffices to show that, for any \( \beta \in \Lambda_0 \) with \( \beta_n = 2 \), we have \( g_0 \in C^m[0,h] \) and

\[
\|g_0\|_{C^m[0,h]} \leq C, \tag{4.19}
\]

where

\[
g_0(x) := \int_0^x \left( \frac{1 + (-1)^{\beta_1+1}}{2} + \frac{1 + (-1)^{\beta_1}}{2} \sum_{j=1}^J \gamma_j c_j t^{\gamma_j-1} \right) g_1(t) \, dt,
\]

\[
g_1(x) := \int_0^x \left( \frac{1 + (-1)^{\beta_2+1}}{2} + \frac{1 + (-1)^{\beta_2}}{2} \sum_{j=1}^J \gamma_j c_j t^{\gamma_j-1} \right) g_2(t) \, dt,
\]

\[
\vdots
\]

\[
g_{n-2}(x) := \int_0^x \left( \frac{1 + (-1)^{\beta_{n-1}+1}}{2} + \frac{1 + (-1)^{\beta_{n-1}}}{2} \sum_{j=1}^J \gamma_j c_j t^{\gamma_j-1} \right) g_{s-1}(t) \, dt,
\]

\[
g_{n-1}(x) := \int_0^x \partial_\beta f(t, z(t) + S(t)) \sum_{j=1}^J \gamma_j c_j t^{\gamma_j-1} \, dt,
\]

for all \( 0 \leq x \leq h \). Noting the fact that

\[
\partial_\beta f(\cdot, z(\cdot) + S(\cdot)) \in C[0,h],
\]

and \( \gamma_j \geq \alpha \) for all \( 1 \leq j \leq J \), we easily obtain \( g_{n-1} \in C^{0,\alpha}[0,h] \) and

\[
\|g_{n-1}\|_{C^{0,\alpha}[0,h]} \leq C.
\]

Then, applying Lemma 4.5 to \( g_{n-2}, g_{n-3}, \ldots, g_0 \) successively, and using the fact \( n\alpha > m \), we obtain \( g_0 \in C^m[0,h] \) and (4.19). Thus we have showed \( G_{2,h} \in C^m[0,h] \) and (4.18).

Similarly, we can show that \( G_{3,h} \in C^m[0,h] \) and \( \|G_{3,h}\|_{C^m[0,h]} \leq C \). Consequently, by (4.4), (4.5), and Lemma 4.4, we infer that \( P_{2,h} \), \( P_{3,h} \in C^{m,\alpha}[0,h] \), and (4.12) and (4.13) hold. This completes the proof. \( \blacksquare \)

### 4.2 Proof of Theorem 3.1

By Lemmas 4.2 and 4.3 there exist two positive constants \( C_0 \) and \( C_1 \) that only depend on \( a, \alpha \) and \( M \), such that

\[
\|(P_{1,h}z)'\|_{C^m[0,h]} \leq C_0 h^\alpha \sum_{j=1}^m \|z_j\|_{C^{m-1}[0,h]}, \tag{4.20}
\]

\[
\|(P_{2,h}z)'\|_{C^{m-1}[0,h]} + \|(P_{3,h}z)'\|_{C^{m-1}[0,h]} \leq C_1 h^\alpha, \tag{4.21}
\]

for all \( 0 < h \leq a \) and \( z \in \hat{C}^m[0,h] \). Let \( 0 < h \leq h^* \) and \( K > 0 \) such that

\[
\|(Q - S)'\|_{C^{m-1}[0,h]} + C_1 h^\alpha + C_0 h^\alpha \sum_{j=1}^m K^j \leq K. \tag{4.22}
\]

Define \( \mathcal{J} : V \to C[0,h] \) by

\[
\mathcal{J}z(x) := c_0 - S(x) + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t, z(t) + S(t)) \, dt,
\]

for all \( z \in V \) and \( x \in [0,h] \), where

\[
V := \left\{ v \in \hat{C}^m[0,h] : \|v'\|_{C^{m-1}[0,h]} \leq K \right\}. \tag{4.23}
\]
Remark 4.1. It is clear that $V$ is a bounded, closed, convex subset of $C^m[0, h]$.

Remark 4.2. Let $\delta > 0$. If we put

$$K := \|(Q - S)'\|_{C^{m-1}[0, h]} + C_1 a^\alpha + \delta,$$

$$h := \min \left\{ h^*, \left( \delta^{-1}C_0 \sum_{j=1}^{m} K^j \right)^{-\frac{1}{\alpha}} \right\},$$

then (4.22) holds.

For the operator $J$, we have the following key result.

Lemma 4.7. For each $z \in V$, we have $J z \in V$ and

$$\left| (J z)^{(m)} \right|_{C^{0, \gamma}[0, h]} \leq \left| (Q - S)^{(m)} \right|_{C^{0, \gamma}[0, h]} + C \sum_{j=0}^{m} K^j,$$

(4.25)

where $\gamma := \alpha$ if $\gamma_j + \alpha = m$, and $\gamma := \gamma_j + \alpha - m$ if $\gamma_j + \alpha > m$.

Proof. Let us first show $J z \in V$. Using (4.23) and the fact $h \leq \left( \frac{M(1 + m)}{M} \right)^{\frac{1}{\alpha}}$, we have

$$|J z(x) + S(x) - c_0| = \frac{1}{\Gamma(\alpha)} \left| \int_{0}^{x} (x - t)^{\alpha-1} f(t, z(t) + S(t)) \, dt \right| \leq \frac{M h^\alpha}{\Gamma(1 + \alpha)} \leq b$$

for all $x \in [0, h]$, and so

$$\|J z + S - c_0\|_{C[0, h]} \leq b.$$

By Lemma 4.1 we have

$$J z(x) = c_0 - S(x) + Q(x) + P_{1,h} z(x) + P_{2,h} z(x) + P_{3,h} z(x),$$

(4.26)

and then, by Lemmas 4.2 and 4.3, and the fact $c_0 - S + Q \in C^m[0, h]$, we obtain $J z \in C^m[0, h]$. It remains, therefore, to show that

$$\| (J z)' \|_{C^{m-1}[0, h]} \leq K.$$

(4.27)

To this end, note that, by (4.26), (4.20) and (4.21) we obtain

$$\| (J z)' \|_{C^{m-1}[0, h]} \leq \| (Q - S)' \|_{C^{m-1}[0, h]} + C_1 h^\alpha + C_0 h^\alpha \sum_{j=0}^{m} K^j,$$

and then (4.27) follows from (4.22). We have thus showed $J z \in V$.

Finally, let us show (4.25). By Lemmas 4.2 and 4.3 we obtain

$$\left| (P_{1,h} z)^{(m)} \right|_{C^{\alpha-\gamma}[0, h]} + \left| (P_{2,h} z)^{(m)} \right|_{C^{\alpha-\gamma}[0, h]} + \left| (P_{3,h} z)^{(m)} \right|_{C^{\alpha-\gamma}[0, h]} \leq C \sum_{j=0}^{m} \|z\|_{C^{m-1}[0, h]} \leq C \sum_{j=0}^{m} K^j.$$

From the fact $\gamma \leq \alpha$ it follows

$$\left| (P_{1,h} z)^{(m)} \right|_{C^{\alpha-\gamma}[0, h]} + \left| (P_{2,h} z)^{(m)} \right|_{C^{\alpha-\gamma}[0, h]} + \left| (P_{3,h} z)^{(m)} \right|_{C^{\alpha-\gamma}[0, h]} \leq C \sum_{j=0}^{m} K^j.$$

Using this estimate and the fact that $(Q - S)^{(m)} \in C^{\alpha, \gamma}$ by the definitions of $Q$ and $S$, the desired estimate (4.25) follows from (4.26). This completes the proof.

By the famous Arzelà-Ascoli Theorem and Lemma 4.7, it is evident that $J : V \to V$ is a compact operator, where $V$ is endowed with norm $\|\cdot\|_{C^m[0, h]}$. Therefore, since $V$ is a bounded, closed, convex subset of $C^m[0, h]$, using the Schauder Fixed-Point Theorem gives that there exists $z \in V$ such that

$$J z = z.$$
Putting
\[ y(x) := z(x) + S(x), \quad 0 \leq x \leq h, \]
we obtain
\[ y(x) = c_0 + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t, y(t)) \, dt, \quad 0 \leq x \leq h. \]
By [2, Lemma 2.1], the above \( y \) is a solution of (1.1), and then, since \( y^* \) is the unique solution of (1.1) on \([0, h^*]\), we have \( y^* = y \) on \([0, h]\). Therefore, it is obvious that \( y^* - S \in C^m[0, h] \) and (3.6) hold. This completes the proof of Theorem 3.1.

4.3 Proof of Corollary 3.1
Let us first state the following fact. For each \( 1 \leq j \leq J \), by the definition of \( c_j \), a straightforward computing yields
\[ c_j = \sum_{t \in \Upsilon_{j,1} \cup \Upsilon_{j,2}} t, \quad (4.28) \]
where
\[ \Upsilon_{j,1} := \bigcup_{1 \leq s < n \atop s + 1 = t} \left\{ \frac{B(\alpha, 1 + s) \partial^s f(0, c_0)}{\Gamma(\alpha)} \right\}, \quad (4.29) \]
\[ \Upsilon_{j,2} := \bigcup_{s=1}^{n-1} \bigcup_{k=1}^s \bigcup_{\beta \in \Lambda_s \atop \#\beta = k} \left\{ \frac{B(\alpha, 1 + s - k + \sum_{i=1}^k \gamma_i) \partial^k f(0, c_0)}{\Gamma(\alpha) \prod_{i=1}^k \gamma_i} \prod_{i=1}^{\#\beta} c_i^{\gamma_i} : (i_1, i_2, \ldots, i_k) \in \Xi_{\beta, j} \right\}. \quad (4.30) \]
Above, \( B(\cdot, \cdot) \) denotes the standard beta function, and
\[ \#\beta := \sum_{1 \leq i \leq s \atop \beta_i = 2} 1, \]
\[ \Xi_{\beta, j} := \left\{ (i_1, i_2, \ldots, i_{\#\beta}) : \alpha + s - \#\beta + \sum_{j=1}^{\#\beta} \gamma_i = \gamma_j \right\}, \]
for all \( 1 \leq s < n \) and \( \beta \in \Lambda_s \).

To prove Corollary 3.1, by Theorem 3.1 it suffices to show that (3.7) is equivalent to
\[ c_j = 0 \quad \text{for all } j \in \Theta, \quad (4.31) \]
where
\[ \Theta := \{ 1 \leq j \leq J : \gamma_j \not\in \mathbb{N} \}. \]
But, by (4.28), (4.29) and (4.30), an obvious induction gives
\[ c_j = 0 \quad \text{for all } 1 \leq j \leq J, \quad (4.32) \]
if (3.7) holds. Therefore, it remains to show that (4.31) implies (3.7).

To this end, let us assume that (4.31) holds. Note that we have (4.32). If this statement was false, then let
\[ j_0 := \min \{ 1 \leq j \leq J : c_j \neq 0 \}. \]
Obviously, we have \( j_0 > 1 \) and \( \gamma_{j_0} \in \mathbb{N} \), and in this case, \( \Upsilon_{j_0,1} \) is empty. Thus, by (4.28) we have
\[ c_{j_0} = \sum_{t \in \Upsilon_{j_0,2}} t. \]
But, by the definition of \( \Upsilon_{j_0,2} \) and the fact that \( c_j = 0 \) for all \( 1 \leq j < j_0 \), it is straightforward that \( c_{j_0} = 0 \), which is contrary to the definition of \( j_0 \). Therefore (4.32) holds indeed. Using this result, from (4.28) and (4.30) it follows
\[ c_j = \sum_{t \in \Upsilon_{j,1}} t \quad \text{for all } 1 \leq j \leq J, \]
and then, using (4.32) again, we obtain (3.7). This completes the proof of Corollary 3.1.
Appendix A  Proofs of Lemmas 4.5 and 4.6

To prove Lemma 4.5, we need the following two lemmas.

**Lemma A.1.** Let $h > 0$, $\gamma > 0$ and $g \in C^1[0, h]$. We have $w \in C^1[0, h]$ and

$$w'(x) = \int_0^x t^\gamma g(t) \, dt,$$

where

$$w(x) := \int_0^x t^\gamma g(t) \, dt, \quad 0 < x < h.$$

Since the proof of this lemma is straightforward, it is omitted.

**Lemma A.2.** Let $0 < h < a$, and $l \in \mathbb{N}_{> 0}$ such that $l \alpha \leq 1 < (l + 1) \alpha$. For any $g \in C^{0,l\alpha}[0, h]$, we have $w \in C^{0,(l+1)\alpha-1}[0, h]$ and

$$\|w\|_{C^{0,(l+1)\alpha-1}[0, h]} \leq C \|g\|_{C^{0,l\alpha}[0, h]},$$

where

$$w(x) := \sum_{j=1}^J \gamma_j c_j x^\gamma j^j g(x), \quad 0 < x < h.$$  

**Proof.** It suffices to prove that, for any $1 \leq j \leq J$, we have $v \in C[0, h]$ and

$$\|v\|_{C^{0,(l+1)\alpha-1}[0, h]} \leq C \|g\|_{C^{0,l\alpha}[0, h]},$$

where $v(x) := x^\gamma j^j g(x)$, $0 < x < h$. Noting the fact that $l \alpha + \gamma_j > 1$ and $g \in C^{0,l\alpha}[0, h]$, we easily obtain $v \in C[0, h]$ and

$$\|v\|_{C[0, h]} \leq C \|g\|_{C^{0,l\alpha}[0, h]}.$$

It remains, therefore, to prove that

$$|v(y) - v(x)| \leq C |y - x|^{(l+1)\alpha - 1} \|g\|_{C^{0,l\alpha}[0, h]}$$

for all $0 < x < y < h$. Moreover, since it holds

$$|v(y) - v(x)| = |y^\gamma j^j g(y) - x^\gamma j^j g(x)| = |y^\gamma j^j (g(y) - g(x)) + (y^\gamma j^j - x^\gamma j^j) g(x)| \leq (y^\gamma j^j - x^\gamma j^j) \|g\|_{C^{0,l\alpha}[0, h]},$$

by the fact $g \in C^{0,l\alpha}[0, h]$, we only need to prove that

$$y^\gamma j^j - x^\gamma j^j \leq C |y - x|^{(l+1)\alpha - 1} \tag{A.2}$$

for all $0 < x < y < h$.

Let us first consider the case of $\gamma_j < 1$. A simple algebraic calculation gives

$$(x^\gamma j^j - y^\gamma j^j)x^\alpha = (y - x)^{\alpha + \gamma_j - 1} (A^\gamma j^j - (1 + A)^{\gamma_j - 1}) A^\alpha,$$

where $A := \frac{x}{y - x}$. If $0 \leq A \leq 1$, then by the fact $l \alpha + \gamma_j - 1 > 0$ we have

$$(A^\gamma j^j - (1 + A)^{\gamma_j - 1}) A^\alpha < A^{\alpha + \gamma_j - 1} \leq 1.$$  

If $A > 1$, then using the Mean Value Theorem and the fact $l \alpha + \gamma_j - 2 < 0$ gives

$$(A^\gamma j^j - (1 + A)^{\gamma_j - 1}) A^\alpha < (1 - \gamma_j) A^{\alpha + \gamma_j - 2} < (1 - \gamma_j) < 1.$$  

Consequently, we obtain

$$(x^\gamma j^j - y^\gamma j^j)x^\alpha < (y - x)^{\alpha + \gamma_j - 1}.$$  

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which, together with the trivial estimate
\[ y^{\gamma_j-1} (y-x)^l \alpha < (y-x)^{\gamma_j-1} (y-x)^l \alpha = (y-x)^{l+\gamma_j-1}, \]
yields (A.2).

Then, since (A.2) is evident in the case of \( \gamma_j = 1 \), let us consider the case of \( 1 < \gamma_j < 2 \). Since \( 0 < \gamma_j - 1 < 1 \), we have
\[ y^{\gamma_j-1} - x^{\gamma_j-1} < (y-x)^{\gamma_j-1}. \]
By the definition of \( \gamma_j \) it is clear that
\[ \gamma_j - 1 \geq (l+1)\alpha - 1. \]
Using the above two estimates, we obtain
\[ |y^{\gamma_j-1} - x^{\gamma_j-1}| x^{l\alpha} \leq C(y^{\gamma_j-1} - x^{\gamma_j-1}) \leq C(y-x)^{(l+1)\alpha-1}, \]
which, together with the estimate
\[ y^{\gamma_j-1} (y-x)^l \alpha \leq C(y-x)^l \alpha \leq C(y-x)^{(l+1)\alpha-1}, \]
indicates (A.2).

Finally, let us consider the case of \( \gamma_j \geq 2 \). Using the Mean Value Theorem gives
\[ |y^{\gamma_j-1} - x^{\gamma_j-1}| x^{l\alpha} \leq C(y-x)^{(l+1)\alpha-1}. \]
and then, by the obvious estimate
\[ y^{\gamma_j-1} (y-x)^l \alpha \leq C(y-x)^{(l+1)\alpha-1}, \]
we obtain (A.2). This completes the proof. ■

**Proof of Lemma 4.5** Since \( g \in C^{k,l\alpha}[0, h] \), by Lemma A.1 we have \( w \in C^k[0, h] \) and
\[ w(i)(x) = \int_0^x \sum_{j=1}^J \gamma_j c_j x^{\gamma_j-1} g(i)(t) \, dt, \quad i = 0, 1, 2, \ldots, k. \tag{A.3} \]
It follows
\[ \|w\|_{C^k[0, h]} \leq C \|g\|_{C^{k,l\alpha}[0, h]} . \]
Therefore, it remains to prove that
\[ \left\|w^{(k)}\right\|_{C^{0,(l+1)\alpha-1}[0, h]} \leq C \|g\|_{C^{k,l\alpha}[0, h]} \tag{A.4} \]
if \( (l+1)\alpha \leq 1 \); and that \( w^{(k+1)} \in C^{0,(l+1)\alpha-1}[0, h] \) and
\[ \left\|w^{(k+1)}\right\|_{C^{0,(l+1)\alpha-1}[0, h]} \leq C \|g\|_{C^{k,l\alpha}[0, h]} \tag{A.5} \]
if \( (l+1)\alpha > 1 \).

Let us first consider (A.4). Noting the fact that \( g^{(k)} \in C^{0,l\alpha}[0, h] \) and \( \gamma_j \geq \alpha \) for all \( 1 \leq j \leq J \), by (A.3) a simple computing gives that
\[ \left|w^{(k)}(y) - w^{(k)}(x)\right| \leq C \left\|g^{(k)}\right\|_{C^{0,l\alpha}[0, h]} \left| (y-x)^{(l+1)\alpha} \right| \]
for all \( 0 \leq x < y \leq h \), which implies (A.4).

Then let us consider (A.5). Since \( g^{(k)} \in C^{0,l\alpha} \), by Lemma A.2 we have \( v \in C^{0,(l+1)\alpha-1}[0, h] \) and
\[ \|v\|_{C^{0,(l+1)\alpha-1}[0, h]} \leq C \left\|g^{(k)}\right\|_{C^{0,l\alpha}[0, h]}, \]
where
\[ v(x) := \sum_{j=1}^J \gamma_j c_j x^{\gamma_j-1} g^{(k)}(x), \quad 0 < x \leq h. \]
Then, by (A.3) we readily obtain \( w^{(k+1)} \in C^{0,(l+1)\alpha-1} \) and (A.5), and thus complete the proof of this lemma. ■

Before proving Lemma 4.6, let us introduce the following lemma.
Lemma A.3. Let \( 0 < h \leq a \) and \( \gamma > 0 \). For any \( g \in C^k[0, h] \) with \( 1 \leq k \leq m \), we have \( w \in C^{k-1}[0, h] \) and

\[
\|w\|_{C^{k-1}[0, h]} \leq C \left\| g^{(k)} \right\|_{C[0, h]},
\]

where

\[
w(x) := g(x)x^{\gamma-1}, \quad 0 < x \leq h,
\]

and \( C \) is a positive constant that only depends on \( a, k \) and \( \gamma \).

Proof. If \( k = 1 \), then, by the Mean Value Theorem and the fact \( g(0) = 0 \), this lemma is evident. Thus, below we assume that \( 2 \leq k \leq m \). In the rest of this proof, for ease of notation, the symbol \( C \) denotes a positive constant that only depends on \( a, k \) and \( \gamma \), and its value may differ at each occurrence.

Let us first show that, for \( 0 \leq i < k \), we have \( w_i \in C[0, h] \) and

\[
\|w_i\|_{C[0, h]} \leq C \left\| g^{(k)} \right\|_{C[0, h]}, \tag{A.6}
\]

where

\[
w_i(x) := w^{(i)}(x), \quad 0 < x \leq h.
\]

To this end, let \( 0 \leq i < k \), and note that an elementary computing gives

\[
w_i(x) = \sum_{j=0}^{i} c_{ij} g^{(j)}(x)x^{\gamma-1-i+j}, \quad 0 < x \leq h, \tag{A.7}
\]

where \( c_{ij} \) is a constant that only depends on \( \gamma, i \) and \( j \), for all \( 0 \leq j \leq i \). Since \( g \in C^k[0, h] \), we have \( g^{(j)} \in C^{k-j}[0, h] \), and then, applying Taylor’s formula with integral remainder yields

\[
g^{(j)}(x) = \frac{1}{(k-j-1)!} \int_0^x (x-t)^{k-j-1} g^{(k)}(t) \, dt, \quad 0 \leq x \leq h.
\]

It follows that

\[
\left\| g^{(j)}(x)x^{\gamma-1-i+j} \right\| \leq \frac{\|g^{(k)}\|_{C[0, h]} x^{\gamma+k-(i+1)}}{(k-j)!}, \quad 0 < x \leq h. \tag{A.8}
\]

Since \( \gamma + k - (i + 1) \geq \gamma > 0 \), this implies \( g^{(j)}(x)x^{\gamma-1+i+j} \in C[0, h] \) and

\[
\left\| g^{(j)}(x)x^{\gamma-1+i+j} \right\|_{C[0, h]} \leq C \left\| g^{(k)} \right\|_{C[0, h]}.
\]

Therefore, by (A.7) it follows \( w_i \in C[0, h] \) and (A.6).

Then let us proceed to prove this lemma. Let \( i < k - 1 \). Note that by (A.7) we have

\[
w_i'(x) = w_{i+1}(x), \quad 0 < x \leq h.
\]

Since we have already proved that \( w_i, w_{i+1} \in C[0, h] \), by the Mean Value Theorem it is evident that \( w_i \in C^1[0, h] \) and

\[
w_i'(x) = w_{i+1}(x), \quad 0 \leq x \leq h.
\]

It follows \( w_0 \in C^{k-1}[0, h] \) and

\[
w_0^{(i)} = w_i, \quad 0 \leq i < k,
\]

and hence, by (A.6) we have

\[
\|w_0\|_{C^{k-1}[0, h]} \leq C \left\| g^{(k)} \right\|_{C[0, h]}.
\]

Noting the fact \( w = w_0 \), this completes the proof. \( \blacksquare \)

Proof of Lemma 4.6 Below we employ the well-known principle of mathematical induction to prove this lemma. Firstly, it is clear that (4.16) holds in the case \( k = 0 \). Secondly, assuming that (4.16) holds for \( k = l \) where \( 0 \leq l < m - 1 \), let us prove that (4.16) holds for \( k = l + 1 \). To this end, a straightforward computing gives

\[
(\mathcal{T}_{w, \beta, h}z)'(x) = \mathcal{T}_{w', \beta, h}z(x) + \mathcal{T}_{w, \beta, h}z(x) + \mathcal{T}_{\mathcal{w}, \beta, h}z(x) \tag{A.9}
\]
for all $0 < x \leq h$, where $\tilde{\beta} := (\beta_1, \beta_2, \ldots, \beta_s, 1)$, $\tilde{\beta} := (\beta_1, \beta_2, \ldots, \beta_s, 2)$, and

$$\tilde{w}(x) := w(x) + \left( z'(x) + \sum_{j=1}^{s} \gamma_j c_j x^{\gamma_j} \right).$$

Since $w \in C^k[0, h]$, we have $w' \in C^{k-1}[0, h]$, and by Lemma A.3 we have $\tilde{w} \in C^{k-1}[0, h]$; consequently, $T_{w, \beta, h}z$ and $T_{\tilde{w}, \tilde{\beta}, h}z$ are well-defined, and they both belong to $C[0, h]$. Therefore, by the Mean Value Theorem, and the fact $T_{w, \beta, h}z \in C[0, h]$, it follows that $T_{w, \beta, h}z \in C^1[0, h]$, and (A.9) holds for all $0 \leq x \leq h$.

By our assumption, we have the following results: $T_{w, \beta, h}z \in C^{\min\{k-1, n-s\}}[0, h]$ and

$$\|T_{w, \beta, h}z\|_{C^{\min\{k-1, n-s\}}[0, h]} \leq C \|w\|_{C^k[0, h]} \sum_{j=0}^{\min\{k-1, n-s\}} \|z'\|_{C^{n-1}[0, h]};$$

$T_{\tilde{w}, \tilde{\beta}, h}z \in C^{\min\{k-1, n-s-1\}}[0, h]$ and

$$\|T_{\tilde{w}, \tilde{\beta}, h}z\|_{C^{\min\{k-1, n-s-1\}}[0, h]} \leq C \|\tilde{w}\|_{C^k[0, h]} \sum_{j=0}^{\min\{k-1, n-s-1\}} \|z'\|_{C^{n-1}[0, h]};$$

$T_{\tilde{w}, \tilde{\beta}, h}z \in C^{\min\{k-1, n-s\}}[0, h]$ and

$$\|T_{\tilde{w}, \tilde{\beta}, h}z\|_{C^{\min\{k-1, n-s\}}[0, h]} \leq C \|\tilde{w}\|_{C^k[0, h]} \sum_{j=0}^{\min\{k-1, n-s\}} \|z'\|_{C^{n-1}[0, h]}.$$

In addition, by Lemma A.3 we easily obtain

$$\|\tilde{w}\|_{C^{k-1}[0, h]} \leq C \|w\|_{C^k[0, h]} \left( 1 + \|z'\|_{C^{k-1}[0, h]} \right).$$

As a consequence, we obtain $T_{w, \beta, h}z \in C^{\min\{k, n-s\}}$ and

$$\|(T_{w, \beta, h}z)'\|_{C^{\min\{k-1, n-s\}}[0, h]} \leq C \|w\|_{C^k[0, h]} \sum_{j=0}^{\min\{k, n-s\}} \|z'\|_{C^{n-1}[0, h]}.$$

Then (4.16) follows from the obvious estimate

$$\|T_{w, \beta, h}z\|_{C[0, h]} \leq C \|w\|_{C[0, h]}.$$

This completes the proof of Lemma 4.6.

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