The generalized reciprocal distance matrix of graphs

Gui-Xian Tian\textsuperscript{a}, Mei-Jiao Cheng\textsuperscript{a}, Shu-Yu Cui\textsuperscript{b}

\textsuperscript{a}Department of Mathematics, Zhejiang Normal University, Jinhua 321004, China
\textsuperscript{b}Xingzhi College, Zhejiang Normal University, Jinhua 321004, China

Abstract

Let $G$ be a simple undirected connected graph with the Harary matrix $RD(G)$, which is also called the reciprocal distance matrix of $G$. The reciprocal distance signless Laplacian matrix of $G$ is $RQ(G) = RT(G) + RD(G)$, where $RT(G)$ denotes the diagonal matrix of the vertex reciprocal transmissions of graph $G$. This paper intends to introduce a new matrix $RD_\alpha(G) = \alpha RT(G) + (1-\alpha)RD(G)$, $\alpha \in [0, 1]$, to track the gradual change from $RD(G)$ to $RQ(G)$. First, we describe completely the eigenvalues of $RD_\alpha(G)$ for some special graphs. Then we obtain several basic properties of $RD_\alpha(G)$ including inequalities that involve the spectral radii of the reciprocal distance matrix, reciprocal distance signless Laplacian matrix and $RD_\alpha$-matrix of $G$. We also provide some lower and upper bounds of the spectral radius of $RD_\alpha$-matrix. Finally, we depict the extremal graphs with maximal spectral radius of the $RD_\alpha$-matrix among all connected graphs of fixed order and precise vertex connectivity, edge connectivity, chromatic number and independence number, respectively.

AMS classification: 05C50 15A18

Keywords: reciprocal distance matrix; reciprocal distance signless Laplacian matrix; $RD_\alpha$-matrix; spectral radius; extremal graph

1 Introduction

All the graphs studied in this paper are simple undirected connected graphs. Let $G = (V(G), E(G))$ be a connected graph of order $n$, where its vertex set is $V(G) = \{ v_1, v_2, \ldots, v_n \}$ and its edge set is $E(G)$. We use $v_iv_j$ to denote that two vertices $v_i$ and $v_j$ are neighbors in the graph $G$. The adjacency matrix of $G$ is the matrix $A(G) = (a_{ij})_{n \times n}$, where $a_{ij} = 1$ if $v_iv_j \in E(G)$ and 0, otherwise. The degree diagonal matrix of $G$ is denoted by $\overrightarrow{D}(G)$. Then the signless Laplacian matrix and Laplacian matrix are $Q(G) = \overrightarrow{D}(G) + A(G)$ and $L(G) = \overrightarrow{D}(G) - A(G)$, respectively. The sum of distances of $v$ from all other vertices of $G$ are denoted by $Tr_G(v)$, which is called as the vertex transmission of $v$ in $G$. The distance matrix of $G$ is denoted by $D(G) = (d_{ij})$, where $d_{ij}$ represents the distance between $v_i$ and $v_j$. Let the diagonal matrix $\text{Tr}(G)$ be the vertex transmissions matrix with the $(i,i)$th entry of $\text{Tr}(G)$ being $\text{Tr}_G(v_i)$. Similarly, the distance signless Laplacian matrix and distance Laplacian matrix of graph $G$ is defined by $Q^D(G) = \text{Tr}(G) + D(G)$ and $L^D(G) = \text{Tr}(G) - D(G)$, respectively.

In [17], Plavšić et al. introduced the conception of Harary matrix $RD(G) = (RD_{ij})$ of a graph $G$, which is also the so-called reciprocal distance matrix and defined by

$$RD_{ij} = \begin{cases} \frac{1}{d_{ij}}, & \text{if } i \neq j, \\ 0, & \text{otherwise}. \end{cases}$$

The reciprocal transmission $\text{RT}_{G}(v_i)$ of a vertex $v_i$ in $G$ is given by the sum of the reciprocal distance of $v_i$ from all other vertices of $G$, equivalently, $\sum_{v_j \in V(G) \setminus \{v_i\}} \frac{1}{d_{ij}}$. We say that a graph $G$ is $r$-reciprocal transmission regular whenever its reciprocal transmission $\text{RT}_{G}(v_i) = k$ for any $v_i \in V(G)$. Let $n \times n$
diagonal matrix $RT(G)$ be the vertex reciprocal transmissions of graph $G$. The reciprocal distance Laplacian matrix, denoted by $RQ(G) = RT(G) - RD(G)$, was defined by Bapat and Panda in [2]. It was proved [2] that, given a connected graph $G$ of order $n$, the spectral radius of its reciprocal distance Laplacian matrix $\rho(RQ(G)) \leq n$ if and only if its complement graph, denoted by $\overline{G}$, is disconnected. The reciprocal distance signless Laplacian matrix, denoted by $RQ(G) = RT(G) + RD(G)$, was introduced in [11]. Recently, the lower and upper bounds of the spectral radii of the reciprocal distance matrices and reciprocal distance signless Laplacian matrices of graphs were given in [7][22] and [11][12], respectively. Su et al. [19] determined the extremal graphs with maximal spectral radius of the $RD$-matrix among all graphs of order $n$ and precise vertex connectivity, edge connectivity, chromatic number and independence number, respectively. In [10], Huang et al. determined the extremal graphs with maximal spectral radius of the $RD$-matrix of the graphs with given matching number, bipartite graphs with given matching number, graphs with given cut edge number and so on.

In [16], the matrix $A_\alpha(G) = \alpha D(G) + (1 - \alpha)A(G)$ for $\alpha \in [0, 1]$, which is a convex linear combination of the degree diagonal matrix $D(G)$ and adjacency matrix $A(G)$, was proposed by Nikiforov. Clearly, $A_0(G) = A(G)$, $A_{1/2}(G) = \frac{1}{2}Q(G)$ and $A_1(G) = D(G)$. In 2019, to study the gradual change from $D(G)$ to $Q^\alpha(G)$, Cui et al. [6] proposed the matrix $D_\alpha(G) = \alpha Tr(G) + (1 - \alpha)D(G)$ with $\alpha \in [0, 1]$. It is easy to see that $D_0(G) = D(G)$, $D_{1/2}(G) = \frac{1}{2}Q(G)$ and $D_1(G) = Tr(G)$. Thus, the matrix $A_\alpha(G)$ is the unified way of $A(G)$ and $Q(G)$. In the same way, the matrix $D_\alpha(G)$ connects $D(G)$ to $Tr(G)$ with $\frac{1}{2}Q(G)$ situating in the middle of the interval. Recently, some spectral properties of $D_\alpha$-matrix and $A_\alpha$-matrix have been studied extensively and several results have been published. For more details on $A_\alpha$-matrix and $D_\alpha$-matrix, see [6][8][11][16] and the cited references therein.

In this paper, using a strategy similar to the one above, the convex linear combinations of the matrices $RT(G)$ and $RD(G)$ are studied, which is defined by

$$RD_\alpha(G) = \alpha RT(G) + (1 - \alpha)RD(G), \quad 0 \leq \alpha \leq 1.$$ 

Since $RD_0(G) = RD(G)$, $RD_{1/2}(G) = \frac{1}{2}RQ(G)$ and $RD_1(G) = RT(G)$, then $RD_{1/2}(G)$ and $RQ(G)$ have same spectral properties. Thus $RD_\alpha(G)$ may form a unified theory of $RD(G)$ and $RQ(G)$. To this extent these matrices $RD(G)$, $RT(G)$ and $RQ(G)$ may be understood from a completely new perspective, and some interesting topics arise. Especially in spectral extremal graph theory, characterize the extremal graphs with maximal (or minimal) spectral radius among a given class of graphs is of great interest and importance. For the reciprocal distance matrix $RD(G)$, some spectral extremal graphs with fixed structure parameters have been characterized in [10][19]. It is natural to ask whether these result can be generalized to $RD_\alpha(G)$. If it is available, then we shall go straight to the extremal graphs with maximal (or minimal) spectral radius of $RQ(G)$ for a graph $G$. Otherwise we can see some interesting differences.

Given a connected graph $G$, all eigenvalues of $RD_\alpha(G)$ forms the $RD_\alpha$-spectrum of $G$, use the notation $\sigma(RD_\alpha(G)) = \{\lambda_1(RD_\alpha(G)), \lambda_2(RD_\alpha(G)), \ldots, \lambda_n(RD_\alpha(G))\}$, where all eigenvalues are arranged in descending order. The maximum eigenvalue $\lambda_1(RD_\alpha(G))$ is called the spectral radius of the matrix $RD_\alpha(G)$, denoted by $\rho(RD_\alpha(G))$. Similarly, the spectrum of any matrix $M$ of order $n$ is denoted by $\sigma(M) = \{\lambda_1(M), \lambda_2(M), \ldots, \lambda_n(M)\}$ and its spectral radius is denoted by $\rho(M)$. In general, the notations $C_n$, $P_n$, $K_n$ and $K_{1,n-1}$ represent the cycle, path, complete graph and star of order $n$, respectively.

This paper is organized as follows. Section 2 presents some basics and spectra of $RD_\alpha(G)$ for some special graphs, such as join graphs, complete split graphs, complete multipartite graphs and so on. Section 3 presents some spectral properties of the matrix $RD_\alpha(G)$ and obtains some lower and upper bounds on spectral radius of $RD_\alpha(G)$ for a connected graph $G$. Section 4 is dedicated to spectral extremal problems, we determine the extremal graphs with maximal spectral radius of $RD_\alpha(G)$ in several types of simple connected graphs of order $n$ and precise vertex connectivity, edge connectivity, chromatic number and independence number, respectively. Finally, we sum up our previous work and put forward some problems for further research in Section 5.
2 Spectra of $RD_{\alpha}(G)$ for some special graphs

In this section, we describe completely the $RD_{\alpha}$-spectra of some special graphs. At the first, we consider some easy cases. Clearly, $RD_{\alpha}(K_n) = D_{\alpha}(K_n)$. This implies that $\sigma(RD_{\alpha}(K_n)) = \{n-1, (an-1)^{n-1}\}$, where $a[b]$ means that the multiplicity of $a$ is $b$.

Given an $r$-regular graph $G$ of order $n$ with diameter $2$, based on the relationships between elements of $RD(G)$ and $A(G)$, we have $RD(G) = \frac{1}{2}(J_n - I_n + A(G))$, where $J_n$ and $I_n$ are the all ones matrix and identity matrix of size $n$, respectively. It is easy to get that

$$RD_{\alpha}(G) = \frac{1}{2}((an + \alpha r - 1)I_n + (1 - \alpha)J_n + (1 - \alpha)A(G)).$$

Assume that $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is the spectrum of $A(G)$ and $1_n = (1, 1, \ldots, 1)^T$, $x_2, \ldots, x_n$ are the corresponding eigenvectors. Since

$$RD_{\alpha}(G)1_n = \frac{1}{2}(n + r - 1)1_n.$$

Then $1_n = (1, 1, \ldots, 1)^T$ is the eigenvector of $RD_{\alpha}(G)$ with respect to the eigenvalue $\frac{1}{2}(n + r - 1)$. Noting that $x_i \perp 1_n$, then one has

$$RD_{\alpha}(G)x_i = \frac{1}{2}((an + \alpha r - 1) + (1 - \alpha)\lambda_i)x_i.$$

Thus, $\frac{1}{2}((an + \alpha r - 1) + (1 - \alpha)\lambda_i)$ is exactly an eigenvalue of $RD_{\alpha}(G)$ for arbitrary $i = 2, 3, \ldots, n$. So we can find the relationship between the $RD_{\alpha}$-eigenvalues of a regular graph with diameter $2$ and its adjacency eigenvalues.

In the following some known conclusions about a square matrix are given. We use $\Phi_M(\lambda) = \det(\lambda I_n - M)$ to represent the characteristic polynomial of $n \times n$ matrix $M$, which is also called as the $\lambda$-characteristic polynomial of $M$. The coronal, denoted by $\Gamma_M(\lambda)$, of a matrix $M$ is the sum of all the elements of the matrix $(\lambda I_n - M)^{-1}$. It means that

$$\Gamma_M(\lambda) = \frac{\Phi_M(\lambda)}{\lambda^t}.$$

It was proved [5] that, if all the row sum of matrix $M$ are equal to $t$, then $\Gamma_M(\lambda) = \frac{\Phi_M(\lambda)}{\lambda^t}$. For two given column vectors $x$ and $y$, if a matrix $M$ is invertible, then $\det(M + xy^T) = (1 + y^TM^{-1}x)\det(M)$, which is also called as the known Matrix Determinant Lemma in [15].

Given two graphs $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$, the graph $G_1 \cup G_2$ has vertex set $V(G_1 \cup G_2) = V(G_1)^T \cup V(G_2)$ and edge set $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. The join graph $G_1 \cup G_2$ is obtained from $G_1 \cup G_2$ by connecting every vertex in $G_1$ to every vertex in $G_2$. In what follows, we give the $RD_{\alpha}$-characteristic polynomial of $G_1 \cup G_2$.

**Lemma 2.1.** For $i = 1, 2$, assume that $G_i$ is a connected graph of order $n_i$ and

$$M_i = \frac{1}{2}[(n_i\alpha - 1)I_{n_i} + (1 - \alpha)J_{n_i} + A_{\alpha}(G_i)],$$

where $A_{\alpha}(G_i)$ is the $A_{\alpha}$-matrix of $G_i$. Then the $RD_{\alpha}$-characteristic polynomial of $G_1 \cup G_2$ is

$$\Phi_{RD_{\alpha}}(\lambda) = \Phi_{M_1}(\lambda - \alpha n_2)\Phi_{M_2}(\lambda - \alpha n_1)[1 - (1 - \alpha)^2\Gamma_{M_1}(\lambda - \alpha n_2)\Gamma_{M_2}(\lambda - \alpha n_1)].$$

**Proof.** Since the diameter of $G_1 \cup G_2$ is $2$, then

$$RD_{ij}(G_1 \cup G_2) = \begin{cases} 1 & \text{if } v_i v_j \in E(G_1 \cup G_2), \\ 0 & \text{if } i = j, \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

With a suitable label of the vertices of $G_1 \cup G_2$, one gets

$$RD_{\alpha}(G_1 \cup G_2) = \left( \begin{array}{ccc} \alpha n_2 I_{n_1} & M_1 & (1 - \alpha)J_{n_1 \times n_2} \\ (1 - \alpha)J_{n_2 \times n_1} & \alpha n_1 I_{n_2} & M_2 \end{array} \right).$$

The remaining proof is exactly similar to that of Lemma 2.1 in [11], we omit the detail. \(\square\)
Theorem 2.2. For \( i = 1, 2 \), suppose that \( G_i \) is an \( r_i \)-regular graph with \( n_i \) vertices. Then the spectrum of \( RD_\alpha(G_1 \lor G_2) \) consists precisely of:

i) The eigenvalue \( \frac{1}{2}(\alpha(n + n_2 + r_i) + (1 - \alpha)\lambda_j(A(G_i)) - 1) \) for any \( 2 \leq j \leq n_1; \)

ii) The eigenvalue \( \frac{1}{2}(\alpha(n + n_1 + r_2) + (1 - \alpha)\lambda_k(A(G_2)) - 1) \) for any \( 2 \leq k \leq n_2; \)

iii) The rest eigenvalues are the roots of \( (\lambda - \alpha n_2 - \frac{1}{2}n_1 - \frac{1}{2}r_1 + \frac{1}{2})(\lambda - \alpha n_1 - \frac{1}{2}n_2 - \frac{1}{2}r_2 + \frac{1}{2}) - (1 - \alpha)^2n_1n_2 = 0 \).

Proof. For \( i = 1, 2 \), we take \( M_i = \frac{1}{2}[(n_i, \alpha - 1)I_{n_i} + (1 - \alpha)J_{n_i} + A_\alpha(G_i)] \), where \( A_\alpha(G_i) \) is the \( \alpha \)-matrix of \( G_i \). First we calculate \( \Phi_{M_i}(\lambda - \alpha n_2) = \det((\lambda - \alpha n_2)I_{n_i} - \frac{1}{2}(n_i, \alpha - 1)I_{n_i} + (1 - \alpha)J_{n_i} + A_\alpha(G_i)) \). For the sake of convenience, let \( N_1 = \frac{1}{2}(n_1 + 2an_2 - 1)I_{n_1} + \frac{1}{2}A_\alpha(G_1) \). Now, from the Matrix Determinant Lemma, we can obtain

\[
\Phi_{M_i}(\lambda - \alpha n_2) = (1 - \frac{1}{2}(1 - \alpha)1^T_{n_1}(\lambda I_{n_1} - N_1)^{-1}1_{n_1}) \det(\lambda I_{n_1} - N_1) = (1 - \frac{1}{2}(1 - \alpha)\Gamma_{N_1}(\lambda))(\frac{1}{2})^{n_1} \Phi_{A_\alpha(G_1)}(2\lambda - \alpha n_1 - 2an_2 + 1). \tag{2}
\]

Since \( G_i \) is an \( r_i \)-regular graph, then the matrix \( N_1 \) has the same row sum \( \frac{1}{2}(an_1 + 2an_2 + r_i - 1) \). This implies that

\[
\Gamma_{N_1}(\lambda) = \frac{1}{2}^{n_1}(\lambda I_{n_1} - N_1)^{-1}1_{n_1} = \frac{\frac{n_1}{\lambda - \frac{1}{2}(an_1 + 2an_2 + r_i - 1)}}{\lambda - \frac{1}{2}(an_1 + 2an_2 + r_i - 1)}. \tag{3}
\]

Substituting (3) into (2), one has

\[
\Phi_{M_i}(\lambda - \alpha n_2) = \frac{1}{2}^{n_1} \frac{\lambda - \alpha n_2 - \frac{1}{2}n_1 - \frac{1}{2}r_1 + \frac{1}{2}}{\lambda - \frac{1}{2}(an_1 + 2an_2 + r_i - 1)} \Phi_{A_\alpha(G_1)}(2\lambda - \alpha n_1 - 2an_2 + 1). \tag{4}
\]

Similarly,

\[
\Phi_{M_2}(\lambda - \alpha n_1) = \frac{1}{2}^{n_2} \frac{\lambda - \alpha n_1 - \frac{1}{2}n_2 - \frac{1}{2}r_2 + \frac{1}{2}}{\lambda - \frac{1}{2}(an_2 + 2an_1 + r_2 - 1)} \Phi_{A_\alpha(G_2)}(2\lambda - \alpha n_2 - 2an_1 + 1). \tag{5}
\]

Since \( G_i \) is an \( r_i \)-regular graph for \( i = 1, 2 \). Then the matrix \( M_i \) has the same row sum \( \frac{1}{2}(n_i + r_i - 1) \). Thus,

\[
\Gamma_{M_i}(\lambda - \alpha n_1) = \frac{n_1}{\lambda - \alpha n_1 - \frac{1}{2}(n_1 + r_1 - 1)}, \quad \Gamma_{M_2}(\lambda - \alpha n_1) = \frac{n_2}{\lambda - \alpha n_1 - \frac{1}{2}(n_2 + r_2 - 1)}. \tag{6}
\]

Substituting (4), (5), (6) back into (1) in Lemma 2.1, we obtain the \( RD_\alpha \)-characteristic polynomial of \( G_1 \lor G_2 \)

\[
\Phi_{RD_\alpha}(\lambda) = \frac{1}{2}^{n_1+n_2} \Phi_{A_\alpha(G_1)}(2\lambda - 2an_2 - an_1 + 1)\Phi_{A_\alpha(G_2)}(2\lambda - 2an_1 - an_2 + 1) \frac{f(\lambda)}{(\lambda - \alpha an_2 - \frac{1}{2}n_1 - \frac{1}{2}r_1 + \frac{1}{2})(\lambda - \alpha an_1 - \frac{1}{2}n_2 - \frac{1}{2}r_2 + \frac{1}{2})} \]

where \( f(\lambda) = (\lambda - \alpha n_2 - \frac{1}{2}n_1 - \frac{1}{2}r_1 + \frac{1}{2})(\lambda - \alpha n_1 - \frac{1}{2}n_2 - \frac{1}{2}r_2 + \frac{1}{2}) - (1 - \alpha)^2n_1n_2. \) Notice that \( A_\alpha(G_i) = \alpha r_i J_{n_i} + (1 - \alpha)A(G_i) \) for the \( r_i \)-regular graph \( G_i \). Thus \( \alpha r_i + (1 - \alpha)\lambda \) is an eigenvalue of \( A_\alpha(G_i) \) whenever \( \lambda \) is an eigenvalue of \( G_i \) for \( i = 1, 2 \). Therefore, the required result follows. \( \square \)

Obviously, Theorem 2.2 implies immediately the \( RD_\alpha \)-spectra of the following graphs.

Corollary 2.3. For \( a, b \geq 1 \), let \( K_{a,b} = K_a \lor \overline{K_b} \) be a complete bipartite graph with \( a + b = n \) vertices. Then the spectrum of \( RD_\alpha(K_{a,b}) \) consists precisely of:

\[
\frac{\alpha(n + b) - 1}{2} \begin{cases} a - 1 \end{cases}, \quad \frac{\alpha(n + a) - 1}{2} \begin{cases} b - 1 \end{cases},
\]

and the rest eigenvalues are

\[
(\alpha + \frac{1}{2})n - 1 \pm \frac{\sqrt{(\alpha - \frac{1}{2})^2(a - b)^2 + 4(1 - \alpha)^2ab}}{2}.
\]
Corollary 2.4. For $a \geq 1, b \geq 2$, let $CS_{a,b} = K_a \lor K_b$ be a complete split graph with $a + b = n$ vertices. Then the spectrum of $RD_\alpha(CS_{a,b})$ is given by $(\alpha n - 1)^{a-1}, (\alpha(a+2) - 1)^{b-1}$ and
\[
(\alpha + 1)n - \frac{b}{2} - \frac{3}{2} \pm \sqrt{((\alpha - 1)(a - b) - \frac{b}{2} b + \frac{1}{2})^2 + 4(1 - \alpha)^2 ab}
\]

A wheel graph $W(n)$ is the graph constructed by the join operation between a isolated vertex $K_1$ and a cycle $C_{n-1}$, that is, $W(n) = K_1 \lor C_{n-1}$.

Corollary 2.5. The $RD_\alpha$-eigenvalues of the wheel graph $W(n) = K_1 \lor C_{n-1}$ are
\[
(\alpha n + 3) - 1 + 2(1 - \alpha) \cos\left(\frac{2\pi j}{n-1}\right), \text{ for } 1 \leq j \leq n - 2,
\]
and
\[
(\alpha + \frac{1}{2})n \pm \sqrt{(\alpha(n - 2) - \frac{n}{2})^2 + 4(1 - \alpha)^2 (n - 1)}
\]

The following proposition establishes a relationship between the $RD_\alpha$-spectrum and some particular vertex subset of $G$, which will be used to give the spectrum of a graph with given clusters. The proof of this proposition is similar to that of Proposition 11 in [6], the detail is omitted.

Proposition 2.6. Suppose that $G$ be a connected graph and $C \subseteq V(G)$ such that any two vertices in $C$ having same neighborhood in $V(G) \setminus C$, where $|C| = c$. Then

(i) If $C$ is an independent set, then $\alpha t + \frac{\alpha - 1}{2}$ is an eigenvalue of $RD_\alpha(G)$ with repeated at least $c - 1$ times, where $t$ is the reciprocal transmission in $G$ of the vertices in $C$.

(ii) If $C$ is a clique, then $\alpha(t + \frac{1}{2}c + \frac{1}{2}) - 1$ is an eigenvalue of $RD_\alpha(G)$ with repeated at least $c - 1$ times.

The following definition comes from [3][14]. Let $G$ be a connected graph of order $n$. A *cluster* in graph $G$, denoted by $(C,S)$, is a pair of vertex subsets $C$ and $S$, where $|C| = c$, $|S| = s$, and $C$ is a set of cardinality $|C| = c \geq 2$ of pairwise co-neighbor vertices sharing the same set $S$ of $s$ neighbors. Clearly, $C$ is an independent set of $G$ and the reciprocal transmissions $K_{trG}(v)$ are the same for any $v \in C$. Graph $G(K_c)$ is the graph constructed by replacing the independent set $C$ in graph $G$ with the complete graph $K_c$.

For simplicity, let $C = \{v_1, \ldots, v_c\}$, $S = \{v_{c+1}, \ldots, v_{c+s}\}$ and $\{v_{c+s+1}, \ldots, v_n\}$ be the set of the remaining vertices in $G$. Also, let $t$ be the reciprocal transmission of the vertices in $C$ and $\beta = 1 - \alpha$. Then the matrices $RD_\alpha(G)$ and $RD_\alpha(G(K_c))$ can be given by:
\[
RD_\alpha(G) = \begin{pmatrix} U & W \\ W^T & Z \end{pmatrix} \quad \text{and} \quad RD_\alpha(G(K_c)) = \begin{pmatrix} V & W \\ W^T & Z \end{pmatrix},
\]
where
\[
U = U_{exe} = \begin{pmatrix} \alpha t + \frac{1}{2} \beta & \ldots & \frac{1}{2} \beta \\ \frac{1}{2} \beta & \alpha t & \ldots \\ \vdots & \ddots & \ddots & \ddots \\ \frac{1}{2} \beta & \ldots & \frac{1}{2} \beta & \alpha t \end{pmatrix},
\]
\[
V = V_{exe} = \begin{pmatrix} \alpha(t + \frac{1}{2}c - \frac{1}{2}) & \beta & \ldots & \beta \\ \beta & \alpha(t + \frac{1}{2}c - \frac{1}{2}) & \ldots \\ \vdots & \ddots & \ddots & \ddots \\ \beta & \ldots & \beta & \alpha(t + \frac{1}{2}c - \frac{1}{2}) \end{pmatrix},
\]
\[ W = \beta \begin{pmatrix} \frac{1}{d_{1,c+1}} & \frac{1}{d_{1,c+2}} & \cdots & \frac{1}{d_{1,n}} \\ \frac{1}{d_{1,c+1}} & \frac{1}{d_{1,c+2}} & \cdots & \frac{1}{d_{1,n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{d_{1,c+1}} & \frac{1}{d_{1,c+2}} & \cdots & \frac{1}{d_{1,n}} \end{pmatrix} \]

and

\[ Z = Z_{(n-c) \times (n-c)} = \begin{pmatrix} \alpha RT_G(v_{c+1}) & \frac{\beta}{d_{c+1,c+2}} & \cdots & \frac{\beta}{d_{c+1,n}} \\ \frac{\beta}{d_{c+2,c+1}} & \alpha RT_G(v_{c+2}) & \cdots & \frac{\beta}{d_{c+2,n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\beta}{d_{n,c+1}} & \frac{\beta}{d_{n,c+2}} & \cdots & \alpha RT_G(v_n) \end{pmatrix}. \]

**Theorem 2.7.** Let \( G \) be a connected graph with \( n \) vertices and having a cluster \((C, S)\). Assume that the reciprocal transmissions \( RT_T G(v) = t \) for any \( v \in C \). Then we have the following statement.

(i) \( at + \frac{\alpha - 1}{2} \) is an eigenvalue of \( RD_\alpha(G) \) with multiplicity \( c - 1 \), and the rest \( n - c + 1 \) eigenvalues of \( RD_\alpha(G) \) are the eigenvalues of the following matrix

\[ Q_1 = \begin{pmatrix} \alpha(t + \frac{1}{2}) - \frac{1}{c} + \frac{1}{2}c(1 - \alpha) & x^T \\ \frac{1}{d_{1,c+1}} & \frac{1}{d_{1,c+2}} & \cdots & \frac{1}{d_{1,n}} \end{pmatrix}. \]

where

\[ x = (1 - \alpha) \begin{pmatrix} \frac{1}{d_{1,c+1}} & \frac{1}{d_{1,c+2}} & \cdots & \frac{1}{d_{1,n}} \end{pmatrix}^T. \]

(ii) \( \alpha(t + \frac{1}{2}c + \frac{1}{2}) - 1 \) is an eigenvalue of \( RD_\alpha(G(K_c)) \) with repeated \( c - 1 \) times, and the rest \( n - c + 1 \) eigenvalues of \( RD_\alpha(G(K_c)) \) are the eigenvalues of the following matrix

\[ Q_2 = \begin{pmatrix} \alpha(t + \frac{1}{2}) - 1 + \frac{1}{2}c(1 - \alpha) & x^T \\ \frac{1}{d_{1,c+1}} & \frac{1}{d_{1,c+2}} & \cdots & \frac{1}{d_{1,n}} \end{pmatrix}. \]

**Proof.** Proposition 2.6 implies that \( at + \frac{\alpha - 1}{2} \) is an eigenvalue of \( RD_\alpha(G) \) with repeated \( c - 1 \) times. Similarly, \( \alpha(t + \frac{1}{2}c + \frac{1}{2}) - 1 \) is an eigenvalue of \( RD_\alpha(G(K_c)) \) with repeated \( c - 1 \) times.

In what follows, suppose that \( \pi : V(G) = V_1 \cup \{v_{c+1}\} \cup \{v_{c+2}\} \cup \cdots \cup \{v_n\} \) is a partition of the vertex set \( V(G) \), where the vertex subset \( V_1 = C = \{v_1, \ldots, v_c\} \). By observation, this partition \( \pi \) is an equitable partition for the matrix \( RD_\alpha(G) \) and \( RD_\alpha(G(K_c)) \). It is easy to see that \( Q_1 \) and \( Q_2 \) are the quotient matrices corresponding to \( RD_\alpha(G) \) and \( RD_\alpha(G(K_c)) \) with respect to the partition \( \pi \). Therefore, the eigenvalues of \( Q_1 \) and \( Q_2 \) are the \( (n - c + 1) \) remaining eigenvalues of \( RD_\alpha(G) \) and \( RD_\alpha(G(K_c)) \), respectively. \( \square \)

A vertex with only one neighbor is called a **pendent vertex** in a graph \( G \). The unique neighbor of a pendent vertex is called a **quasi-pendent vertex** in \( G \). Clearly, a pair of each maximal set \( C \) of pendent vertices and the corresponding quasi-pendent vertex \( v \) forms a cluster \((C, \{v\})\).

**Remark 1.** Taking \( \alpha = 0 \) and \( \alpha = \frac{1}{2} \) in Theorem 2.7, we can get some spectral information about the reciprocal distance matrix \( RD(G) \) and reciprocal distance signless Laplacian of a graph with clusters, respectively. For example, let graph \( G \) with a cluster \((C, S)\) is a connected graph and cardinality \(|C| = c\). Theorem 2.7 implies that \(-\frac{d}{2}\) is an eigenvalue of \( RD(G) \) repeated at least \( c - 1 \) times. Now assume that \( G \) has \( p \) pendent vertices and \( q \) quasi-pendent vertices. Apply Theorem 2.7 repeatedly, it turns out that \(-\frac{d}{2}\) is an eigenvalue of \( RD(G) \) with repeated at least \( p - q \) times.

Next we give the spectrum of \( RD_\alpha(G) \) for a complete \( r \)-partite graph \( G = K_{n_1, n_2, \ldots, n_r} \) with \( n = n_1 + n_2 + \cdots + n_r \) vertices. A complete \( r \)-partite graph \( G = K_{n_1, n_2, \ldots, n_r} \) is the graph formed by partitioning its vertex set into \( r \) subsets \( V_1, V_2, \ldots, V_r \) with cardinality \(|V_i| = n_i| \), and joining two vertices by an edge if and only if they are located in different subsets.
Theorem 2.8. For a complete r-partite graph $G = K_{n_1, n_2, \ldots, n_r}$ of order $n = n_1 + n_2 + \cdots + n_r$ with $n \geq 4$, we have

(i) $\alpha(n) - \frac{n}{2} - \frac{1}{2}$ is an eigenvalue of $RD_{\alpha}(G)$ with repeated $n_i - 1$ times, for any $i \in \{1, 2, \ldots, r\}$.

(ii) the eigenvalues of the following matrix $T$ are the rest eigenvalues of $RD_{\alpha}(G)$, where

$$T = \begin{pmatrix}
\alpha(n - n_1) + \frac{1}{2}(n_1 - 1) & (1 - \alpha)n_2 & \cdots & (1 - \alpha)n_r \\
(1 - \alpha)n_1 & \alpha(n - n_2) + \frac{1}{2}(n_2 - 1) & \cdots & (1 - \alpha)n_r \\
\vdots & \vdots & \ddots & \vdots \\
(1 - \alpha)n_1 & (1 - \alpha)n_2 & \cdots & \alpha(n - n_r) + \frac{1}{2}(n_r - 1)
\end{pmatrix}.$$ 

Proof. Let $\pi : V_1 \cup V_2 \cup \ldots \cup V_r$ be the partition of vertex set $V(G)$ with cardinality $|V_i| = n_i$ for $i \in \{1, 2, \ldots, r\}$. Notice that $RT_{\alpha}(v) = n - \frac{n_i}{2} - \frac{1}{2}$ for any vertex $v \in V_i$. It follows from Proposition 2.6 that $\alpha(n) - \frac{n}{2} - \frac{1}{2}$ is an eigenvalue of $RD_{\alpha}(G)$ with repeated $n_i - 1$ times. On the other hand, the partition $\pi$ is an equitable partition of $RD_{\alpha}(G)$ for the complete r-partite graph $G$. With a suitable labeling, the quotient matrix of $RD_{\alpha}(G)$ can be described as the matrix $T$ in the statement of this theorem. Hence, the $r$ remaining eigenvalues of $RD_{\alpha}(G)$ can be derived from the eigenvalues of $T$. \[\square]\n
3 Basic properties of $RD_{\alpha}(G)$

Lemma 3.1. For two Hermitian matrices $A_{n \times n}$ and $B_{n \times n}$, let $C = A + B$ and $1 \leq i, j \leq n$. Then

$$\lambda_i(C) \leq \lambda_j(A) + \lambda_{i-j+1}(B) \quad \text{for} \quad j \leq i,$$

and

$$\lambda_i(C) \geq \lambda_j(A) + \lambda_{i-j+n}(B) \quad \text{for} \quad i \leq j.$$ 

Moreover, each one of the equality holds iff there is a nonzero vector that is an eigenvector to each of the three eigenvalues involved.

Lemma 3.1 implies that

$$\lambda_i(A) + \lambda_{\min}(B) \leq \lambda_i(C) \leq \lambda_i(A) + \lambda_{\max}(B).$$

The following proposition implies that, for $\frac{1}{2} \leq \alpha \leq 1$, each one of the eigenvalues of $RD_{\alpha}(G)$ does not decrease when an edge is added between two nonadjacent vertices in a graph $G$.

Proposition 3.2. Suppose that $G$ is a graph with $n$ vertices and $\frac{1}{2} \leq \alpha \leq 1$. For an edge $e \notin E(G)$, let the graph $\tilde{G} = G + e$. Then we have

$$\lambda_i(RD_{\alpha}(\tilde{G})) \geq \lambda_i(RD_{\alpha}(G)) \quad \text{for} \quad 1 \leq i \leq n. \quad (7)$$

Proof. For any two vertices $u$ and $v$ of graph $G$, we find that $d_{\tilde{G}}(u, v) \leq d_{\tilde{G}}(u, v)$, equivalently, $\frac{1}{d_{\tilde{G}}(u, v)} \geq \frac{1}{d_G(u, v)}$. Then $RD_{\alpha}(\tilde{G}) = RD_{\alpha}(G) + M$, where

$$M = \begin{pmatrix}
am_1 & (1 - \alpha)m_{1,2} & \cdots & (1 - \alpha)m_{1,n} \\
(1 - \alpha)m_{2,1} & am_2 & \cdots & (1 - \alpha)m_{2,n} \\
\vdots & \vdots & \ddots & \vdots \\
(1 - \alpha)m_{n,1} & (1 - \alpha)m_{n,2} & \cdots & am_n
\end{pmatrix}$$

and $m_i = \sum_{j=1,j\neq i}^n m_{i,j}$ for $i = 1, 2, \ldots, n$. It is clear that, for $\alpha \in [\frac{1}{2}, 1]$, the matrix $M$ is diagonally dominant. At that time, $M$ is also a positive semidefinite matrix with nonnegative diagonal entries. Thus the minimum eigenvalue of $M$ is greater than or equal to zero. Now, Lemma 3.1 implies that the required result follows. \[\square\]
In the light of Proposition 3.2, an upper bound for the $k$-th largest eigenvalue of $RD_\alpha(G)$ for a connected graph $G$ of order $n$ is given by

$$\lambda_i(RD_\alpha(G)) \leq \lambda_i(RD_\alpha(K_n)) = an - 1.$$ 

where $2 \leq i \leq n$ and $\frac{1}{2} \leq \alpha \leq 1$.

What is noteworthy is that the inequality (7) is strict for $i = 1$ and $\alpha \in [0, 1)$. Indeed, given a connected graph $G$, $RD_\alpha(G)$ is a nonnegative irreducible matrix for $\alpha \in [0, 1)$.

Proposition 3.3. Suppose that $G$ is a connected graph. For an edge $e \notin E(G)$, let $\tilde{G} = G + e$. Then, for $\alpha \in [0, 1)$,

$$\rho(RD_\alpha(\tilde{G})) > \rho(RD_\alpha(G)).$$

(8)

Corollary 3.4. Let $G$ be a connected bipartite graph with $n$ vertices, where $n \geq 2$. Suppose that the two disjoint parts have $a$ and $n - a$ vertices with $1 \leq a \leq \lfloor \frac{n}{2} \rfloor$, then

$$\rho(RD_\alpha(G)) \leq \frac{(\alpha + \frac{1}{2})n - 1 + \sqrt{(\alpha - \frac{1}{2})^2(2a - n)^2 + 4(1 - \alpha)^2a(n - a)}}{2}.$$

Then the equality holds if and only if $G = K_{a,n-a}$.

Proof. It follows from Proposition 3.3 that $\rho(RD_\alpha(K_{a,n-a})) \geq \rho(RD_\alpha(G))$. Moreover, the equality holds if and only if $G = K_{a,n-a}$. On the basis of Corollary 2.3, we have

$$\rho(RD_\alpha(K_{a,n-a})) = \frac{(\alpha + \frac{1}{2})n - 1 + \sqrt{(\alpha - \frac{1}{2})^2(2a - n)^2 + 4(1 - \alpha)^2a(n - a)}}{2},$$

as required.

The following proposition 3.5 implies that, for any connected graph $G$, the eigenvalue $\lambda_i(RD_\alpha(G))$ is increasing in $\alpha$, where $0 \leq \alpha \leq 1$ and $2 \leq i \leq n$.

Proposition 3.5. Let $G$ be a connected graph with $n$ vertices. If $1 \geq \alpha > \beta \geq 0$ and $1 \leq i \leq n$, then

$$\lambda_i(RD_\alpha(G)) \geq \lambda_i(RD_\beta(G)).$$

(9)

The equality holds in (8) if and only if $i = 1$ and graph $G$ is reciprocal transmission regular.

Proof. As we all known, $RD_\alpha(G) = RD_\alpha(G) + (\alpha - \beta)(RT(G) - RD(G)) = RD_\beta(G) + (\alpha - \beta)RL(G)$. Applying Lemma 3.1 with $C = RD_\alpha(G)$, $A = RD_\beta(G)$ and $B = (\alpha - \beta)RL(G)$, we have

$$\lambda_i(RD_\alpha(G)) \geq \lambda_i(RD_\beta(G)) + (\alpha - \beta)\lambda_n(RL(G)).$$

Since the matrix $RL(G)$ is positive semidefinite, then $\lambda_n(RL(G)) \geq 0$ (in fact, $\lambda_n(RL(G)) = 0$ (2)). Hence,

$$\lambda_i(RD_\alpha(G)) \geq \lambda_i(RD_\beta(G)).$$

Now suppose that the equality holds in (2). According to Lemma 3.1, $\lambda_i(RD_\alpha(G))$, $\lambda_i(RD_\beta(G))$ and $\lambda_n(RL(G))$ have the same eigenvector $1_n$ because $1_n$ is the unique eigenvector of $\lambda_n(RL(G))$ (see Remark 2.1 in [1]). This means that $i = 1$ and the graph $G$ is reciprocal transmission regular. To the contrary, let $i = 1$ and $G$ be reciprocal transmission regular. Then, it can be verified that the equality in (2) holds.

Proposition 3.5 implies that the following corollary is immediate.
Corollary 3.6. Let $G$ be a connected graph with $n$ vertices and $0 \leq \alpha \leq 1$. Then
\[
\lambda_k(RD_{\alpha}(G)) \leq \lambda_k(RD_{\alpha}(G)) \leq \lambda_k(RT(G)) = RT_{rk} \quad \text{for} \quad 1 \leq k \leq n,
\]
where $RT_{rk}$ is the $k$-th largest reciprocal transmission of $G$.

Remark that Corollary 3.6 gives an upper bound on $\lambda_n(RD_{\alpha}(G))$, but this bounds can be improved in the following proposition.

Proposition 3.7. Suppose that $G$ is a connected graph with $n$ vertices. Then
\[
\lambda_n(RD_{\alpha}(G)) \leq \alpha RT_r \quad \text{and} \quad \rho(RD_{\alpha}(G)) \geq \alpha RT_r.
\]
where $RT_r$ and $RT_r$ are the maximum reciprocal transmission and minimum reciprocal transmission of $G$, respectively.

Proof. Assume that $u$ is a vertex with the minimum reciprocal transmission $RT_r$ of graph $G$. If $e_u$ is the characteristic vector of the vertex $u$, then
\[
\lambda_n(RD_{\alpha}(G)) = \min_{\|x\|_2=1} x^T RD_{\alpha}(G)x \leq e_u^T RD_{\alpha}(G)e_u = \alpha RT_r.
\]
Similarly, we easily obtain that $\rho(RD_{\alpha}(G)) \geq \alpha RT_r$.

Further, Using Lemma 3.1 to the definition of $RD_{\alpha}(G)$, we can get the following bounds on the $k$-th largest eigenvalue $\lambda_k(RD_{\alpha}(G))$ in terms of the $k$-th largest eigenvalue $\lambda_k(RD(G))$, maximum reciprocal transmission $RT_r$ and minimum reciprocal transmission $RT_r$ of a connected graph $G$.

That is to say,
\[
\alpha RT_r + (1 - \alpha)\lambda_k(RD(G)) \leq \lambda_k(RD_{\alpha}(G)) \leq \alpha RT_r + (1 - \alpha)\lambda_k(RD(G)).
\]

Observe that $RD_{\alpha}(G) + RD_{1-\alpha}(G) = RQ(G)$. From Lemma 3.1 again, we can obtain a relation for the spectral radii of three matrices below:
\[
\rho(RD_{\alpha}(G)) + \rho(RD_{1-\alpha}(G)) \geq \rho(RQ(G)).
\]

Remark that this relation can been performed the transformation between the upper and lower bounds of the spectral radius $\rho(RD_{\alpha}(G))$. For example, applying this relation and Lemma 3.1, we can get the following inequalities on $\rho(RD_{\alpha}(G))$ in terms of $\rho(RD(G))$, $\rho(RQ(G))$ and the maximum reciprocal transmission $RT_r$ of $G$. This statement and its proof is analogous to an existing result related to $\rho(D_{\alpha}(G))$ (see Propositions 7 and 8 in [8]), these details of the proof are omitted.

Proposition 3.8. Let $G$ be a connected graph of order $n$ with maximum reciprocal transmission $RT_r$. Then

(i) For $0 \leq \alpha \leq \frac{1}{2}$, we have
\[
(1 - \alpha)\rho(RQ(G)) + (2\alpha - 1)RT_r \leq \rho(RD_{\alpha}(G)) \leq \alpha \rho(RQ(G)) + (1 - 2\alpha)\rho(RD(G)).
\]

Moreover, if $G$ is not reciprocal transmission regular, then the first equality holds if and only if $\alpha = \frac{1}{2}$, and the second equality holds if and only if $\alpha = 0$ or $\alpha = \frac{1}{2}$.

(ii) For $\frac{1}{2} \leq \alpha \leq 1$, we have
\[
\alpha \rho(RQ(G)) + (1 - 2\alpha)\rho(RD(G)) \leq \rho(RD_{\alpha}(G)) \leq (1 - \alpha)\rho(RQ(G)) + (2\alpha - 1)RT_r.
\]

Moreover, if $G$ is not reciprocal transmission regular, then the first equality holds if and only if $\alpha = \frac{1}{2}$, and the second equality holds if and only if $\alpha = \frac{1}{2}$ or $\alpha = 1$.

In what follows, we turn our attention to the positive semidefiniteness of the matrix $RD_{\alpha}(G)$. First notice that the matrix $RD_0(G)$ is not positive semidefinite, but the matrix $RD_{\frac{1}{2}}(G)$ is positive semidefinite. According to the Proposition 3.5, the function $f(\alpha) = \lambda_{\min}(RD_{\alpha}(G))$ is continuous and increasing in $\alpha$. Then there exists the minimum $\alpha_0 \in (0, \frac{1}{2}]$ such that $f(\alpha_0) = 0$. Thus the matrix $RD_{\alpha}(G)$ is a positive semidefinite matrix if and only if $\alpha \in [\alpha_0, 1]$. This observation raises the following problem:
Problem 3.9. For a connected graph, determine the minimum \( \alpha_0 \in (0, \frac{1}{2}] \) such that the matrix \( RD_\alpha(G) \) is a positive semidefinite matrix for \( \alpha \in [\alpha_0, 1] \).

Proposition 3.10. Let \( G \) be a \( k \)-reciprocal transmission regular graph with \( n \) vertices. Then the matrix \( RD_\alpha(G) \) is positive semidefinite if and only if \( \alpha \geq \alpha_0 = \frac{-\lambda_{\min}(RD(G))}{k-\lambda_{\min}(RD(G))} \).

Proof. Clearly, \( RD_\alpha(G) = akI_n + (1-\alpha)RD(G) \) because \( G \) is \( k \)-reciprocal transmission regular. This implies that \( ak + (1-\alpha)\lambda_{\min}(RD(G)) = \lambda_{\min}(RD_\alpha(G)) \). So, \( \lambda_{\min}(RD_\alpha(G)) \geq 0 \) if and only if \( \alpha \geq \alpha_0 = \frac{-\lambda_{\min}(RD(G))}{k-\lambda_{\min}(RD(G))} \), as required.

Theorem 3.11. For a complete bipartite graph \( K_{a,n-a} \) of order \( n \geq 4 \) with \( 1 \leq a \leq \lfloor \frac{n}{2} \rfloor \), the matrix \( RD_\alpha(K_{a,n-a}) \) is a positive semidefinite matrix if and only if \( \alpha \in [\alpha_0, 1] \), where the smallest \( \alpha_0 = \frac{n-1+3a(n-a)}{2(n-1)+4a(n-a)} \).

Proof. In accordance with Corollary 2.3, the eigenvalues of \( RD_\alpha(K_{a,n-a}) \) are

- \( \frac{(2a-\alpha-1)}{2} \), repeated \( a-1 \) times,
- \( \frac{(n+a-\alpha-1)}{2} \), repeated \( n-a-1 \) times,
- the remaining two eigenvalues

\[
\frac{(\alpha + \frac{1}{2})n - 1 \pm \sqrt{(\alpha - \frac{1}{2})^2(2a - n)^2 + 4(1 - \alpha)^2a(n-a)}}{2}.
\]

Observe that \( \frac{(n+a-\alpha-1)}{2} \leq \frac{(2a-\alpha-1)}{2} \) as \( a \leq \lfloor \frac{n}{2} \rfloor \). Thus, the minimum eigenvalue of \( RD_\alpha(K_{a,n-a}) \) is \( \frac{(n+a-\alpha-1)}{2} \) or \( \frac{(n+a-\alpha-1)}{2} \) and the remaining two eigenvalues. Consider the following functions \( \phi(x) = \varphi(x) = \frac{(\alpha + \frac{1}{2})n - 1 \pm \sqrt{(\alpha - \frac{1}{2})^2(2a - n)^2 + 4(1 - \alpha)^2a(n-a)}}{2} \) for \( 0 \leq x \leq 1 \). By a simple calculation, the zero of the function \( \varphi(x) \) is \( x_0 = \frac{n-1+3a(n-a)}{2(n-1)+4a(n-a)} \), but \( \phi(x_0) > 0 \). On the other hand, Proposition 3.5 implies that \( \phi(x) \) and \( \varphi(x) \) are strictly increasing in \( x \). Therefore, \( x_0 \) is the smallest value \( \alpha_0 \) for \( \alpha \in [0, 1] \) such that \( RD_\alpha(K_{a,n-a}) \) is a positive semidefinite matrix.

Theorem 3.12 Let \( W(n) \) be a wheel graph, where \( n \geq 4 \). Then the matrix \( RD_\alpha(W(n)) \) is a positive semidefinite matrix if and only if \( \alpha \in [\alpha_0, 1] \), where the smallest \( \alpha_0 = \frac{2}{n+3} \) for \( n = 2k+1 \), or

\[
\alpha_0 = \frac{1 - 2 \cos \left( \frac{2k\pi}{2k+1} \right)}{n+3 - 2 \cos \left( \frac{2k\pi}{2k+1} \right)}, \quad \text{for } n = 2k+2.
\]

Proof. From Corollary 2.5, the eigenvalues of the matrix \( RD_\alpha(W(n)) \) are

\[
\alpha(n+3) - 1 + 2(1 - \alpha) \cos \left( \frac{2j\pi}{n-1} \right), \quad \text{for } 1 \leq j \leq n-2,
\]

and

\[
\frac{(\alpha + \frac{1}{2})n \pm \sqrt{(\alpha(n-2) - \frac{3}{2})^2 + 4(1 - \alpha)^2(n-1)}}{2}.
\]
Now we need to consider the minimum eigenvalue of the matrix $RD_\alpha(W(n))$. First we suppose that $n = 2k + 1$. Then the candidates to be the minimum eigenvalue of $RD_\alpha(W(n))$ are $\frac{\alpha(n+5)-3}{2}$ and $(\alpha+\frac{3}{4})n+\sqrt{(\alpha(n-2)-\frac{3}{4})^2+4(1-\alpha)^2(n-1)}$. Set $\phi(x) = (n+5)x-3$ and
\[
\varphi(y) = \frac{(y+\frac{3}{4})n-\sqrt{((n-2)y-\frac{3}{4})^2+4(1-y)^2(n-1)}}{2}.
\]
After a simple calculation, the zero of the function $\phi(x)$ is $x_0 = \frac{3}{n+5}$, but $\varphi(x_0) > 0$. Proposition 3.5 implies that $\phi(x)$ and $\varphi(x)$ are strictly increasing in $x$. Therefore, $x_0$ is the smallest value of $\alpha$ for $\alpha \in [0,1]$, such that $RD_\alpha(W(n))$ is a positive semidefinite matrix.

Next assume that $n = 2k + 2$. We can find that the eigenvalue $\frac{\alpha(n+3)-1+2(1-\alpha)\cos(\frac{2\pi}{2k+1})}{2}$ or $(\alpha+\frac{1}{4})n-\sqrt{(\alpha(n-2)-\frac{1}{4})^2+4(1-\alpha)^2(n-1)}$ will be the minimum eigenvalue of $RD_\alpha(W(n))$. Set
\[
\phi(x) = \frac{(n+3)x-1+2(1-x)\cos(\frac{2\pi}{2k+1})}{2}
\]
and
\[
\varphi(y) = \frac{(y+\frac{1}{4})n-\sqrt{((n-2)y-\frac{1}{4})^2+4(1-y)^2(n-1)}}{2}.
\]
After a simple calculation, the zero of the function $\phi(x)$ is $x_0 = \frac{1-2\cos(\frac{2\pi}{2k+1})}{n+3-2\cos(\frac{2\pi}{2k+1})}$, and the zero of the function $\varphi(y)$ is $y_0 = \frac{-1}{n+4}$. It is clear that $x_0 \geq y_0$ whenever $n = 2k + 2$. From Proposition 3.5, $\phi(x)$ and $\varphi(x)$ are strictly increasing in $x$. Therefore, $x_0$ is the smallest value of $\alpha$ for $\alpha \in [0,1]$, such that $RD_\alpha(W(n))$ is a positive semidefinite matrix.

Next, we shall give some lower and upper bounds on the spectral radius of $RD_\alpha(G)$ in terms of the reciprocal transmission sequence $\{RTr_G(v_1), RTr_G(v_2), \ldots, RTr_G(v_n)\}$ of a graph $G$.

**Lemma 3.13.** [21] Let $M = (m_{i,j})_{n \times n}$ be a complex irreducible matrix and $l_k = |\{m_{k,j} : m_{k,j} \neq 0, j \in \{1, 2, \ldots, n\} \setminus \{k\}\}|$. Then
\[
\rho(M) \leq \max_{1 \leq i \leq n} \left\{ |m_{i,i}| + \sqrt{\sum_{l=1, l \neq i}^{n} l_k |m_{l,i}|^2} \right\}. \tag{11}
\]
Moreover, if the equality holds in (11), then for $\forall i, j \in \{1, 2, \ldots, n\}$,
\[
|m_{i,i}| + \sqrt{\sum_{l=1, l \neq i}^{n} l_k |m_{l,i}|^2} = |m_{j,j}| + \sqrt{\sum_{l=1, l \neq j}^{n} l_k |m_{l,j}|^2}.
\]

Now, applying Lemma 3.13 to $RD_\alpha(G)$, we may conclude the following.

**Theorem 3.14.** Let $G$ be a connected graph with $n$ vertices. Then
\[
\rho(RD_\alpha(G)) \leq \max_{1 \leq i \leq n} \left\{ \alpha RTr_G(v_i) + (1-\alpha) \sqrt{(n-1) \sum_{l=1, l \neq i}^{n} \left( \frac{1}{d_G(v_l,v_i)} \right)^2} \right\}. \tag{12}
\]
If the equality holds in (12), then $\alpha RTr_G(v_i) + (1-\alpha) \sqrt{(n-1) \sum_{l=1, l \neq i}^{n} \left( \frac{1}{d_G(v_l,v_i)} \right)^2}$ are all equal for any $i \in \{1, 2, \ldots, n\}$. 


Lemma 3.15. Let $M = (m_{i,j})_{n \times n}$ be a complex irreducible matrix. Then
\[
\min_{1 \leq i \leq n} \left\{ \sum_{j=1}^{n} m_{i,j} \right\} \leq \rho(M) \leq \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^{n} m_{i,j} \right\}. \tag{13}
\]
Moreover, each one of the equalities holds in (13) if and only if all row sums of $M$ are equal.

Theorem 3.16. Let $G$ be a connected graph of order $n$. Then
\[
\min_{1 \leq i \leq n} \left\{ \alpha RT_r G(v_i) + \frac{(1 - \alpha)RT_i}{RT_r G(v_i)} \right\} \leq \rho(RD_\alpha(G)) \leq \max_{1 \leq i \leq n} \left\{ \alpha RT_r G(v_i) + \frac{(1 - \alpha)RT_i}{RT_r G(v_i)} \right\}, \tag{14}
\]
where $RT_i = \sum_{j=1, j \neq i}^{n} RD_{i,j} RT_r G(v_j)$. Moreover, for $\frac{1}{2} \leq \alpha \leq 1$, the equalities hold in (14) if and only if graph $G$ is reciprocal transmission regular.

Proof. Let $(RT(G))^{-1}$ be the inverse matrix of $RT(G)$. By a simple calculation, the $i$-th row sum of $(RT(G))^{-1}RD_\alpha(G)RT(G)$ is equal to
\[
\alpha RT_r G(v_i) + (1 - \alpha) \frac{RT_i}{RT_r G(v_i)}.
\]
Use Lemma 3.15 by taking $M = (RT(G))^{-1}RD_\alpha(G)RT(G)$, the inequality (14) is immediate.

Assume that graph $G$ is $k$-reciprocal transmission regular. It is easy to verify that $\rho(RD_\alpha(G)) = k$ and $\alpha RT_r G(v) + (1 - \alpha) \frac{RT_i}{RT_r G(v)} = k$ for any $v \in V(G)$. Hence, the equality holds in (14).

Conversely, assume either of the equalities holds in (14). At that time, Lemma 3.15 implies that all row sums of $(RT(G))^{-1}RD_\alpha(G)RT(G)$ are equal. Now, for $\forall u, v \in V(G)$, one obtains that
\[
\alpha RT_r G(u) + (1 - \alpha) \frac{RT_u}{RT_r G(u)} = \alpha RT_r G(v) + (1 - \alpha) \frac{RT_v}{RT_r G(v)}. \tag{15}
\]
Without loss of generality, let $RT_r G(u) = RT_r \max$ and $RT_r G(v) = RT_r \min$, then we have $RT(u) \geq RT_r \max RT_r \min$, $RT(v) \leq RT_r \max RT_r \min$. According to (15), it is easy to see that
\[
\alpha RT_r \max + (1 - \alpha)RT_r \min \leq \alpha RT_r \min + (1 - \alpha)RT_r \max.
\]
This implies that $RT_r \min = RT_r \max$ for $\frac{1}{2} \leq \alpha < 1$. Hence, the graph $G$ is reciprocal transmission regular.

Lemma 3.17. Let $M = (m_{i,j})_{n \times n}$ be a nonnegative irreducible symmetric matrix with the $i$-th row sum $M_i$. Then
\[
\sqrt{\frac{1}{n} \sum_{j=1}^{n} M_i^2} \leq \rho(M) \leq \max_{1 \leq i \leq n} \sqrt{\frac{1}{n} \sum_{j=1}^{n} m_{i,j} M_i}. \tag{16}
\]
Each one of the equalities in (16) holds if and only if either all row sums of $M$ are equal, or there exists a permutation matrix $Q$ such that
\[
Q^T MQ = \begin{pmatrix} 0 & C \\ C^T & 0 \end{pmatrix},
\]
where all row sum of $C$ are equal.
Theorem 3.18. Let \( G \) be a connected graph of order \( n \), where \( n \geq 2 \). Then
\[
\sqrt{\frac{\sum_{i=1}^{n} RT_{\alpha}(v_i)^2}{n}} \leq \rho(RD_{\alpha}(G)) \leq \max_{1 \leq i \leq n} (\alpha RT_{\alpha}(v_i) + (1 - \alpha) \sum_{j=1, j \neq i}^{n} \frac{1}{d_G(v_i, v_j)} \sqrt{RT_{\alpha}(v_j)}).
\] (17)
Either of the equalities holds in (17) if and only if graph \( G \) is reciprocal transmission regular.

Proof. Since \( RD_{\alpha}(G) \) is a nonnegative irreducible symmetric matrix. Then, it follows from Lemma 3.17 that the inequality (17) is immediate. Clearly, there is not a permutation matrix \( Q \) such that
\[Q^T RD_{\alpha}(G)Q = \begin{pmatrix} 0 & C^T \\ C & 0 \end{pmatrix}.
\]
So, the required result follows. \( \Box \)

Applying Theorem 3.18, we easily obtain the following result.

Corollary 3.19. Let \( G \) be a connected graph with \( n \geq 2 \) vertices and Harary index \( H(G) \). Then
\[
\rho(RD_{\alpha}(G)) \geq \frac{2H(G)}{n}.
\] (18)
The equality holds in (18) if and only if graph \( G \) is reciprocal transmission regular.

Proof. Using the Cauchy-Schwarz inequality to the left-side of the inequality (17), we have
\[
\rho(RD_{\alpha}(G)) \geq \sqrt{\frac{\sum_{i=1}^{n} RT_{\alpha}(v_i)^2}{n}} \geq \sqrt{\frac{\sum_{i=1}^{n} RT_{\alpha}(v_i)}{n}} = \frac{2H(G)}{n}.
\]
Clearly, the equality holds if and only if \( G \) is reciprocal transmission regular. \( \Box \)

Remark that, if \( \alpha = 0 \), then this result reduces to \( \rho(RD(G)) \geq \frac{2H(G)}{n} \). Similarly, let \( \alpha = \frac{1}{2} \), then this result reduces to \( \rho(RQ(G)) \geq \frac{4H(G)}{n} \). So, Corollary 3.19 generalizes the results in [1] and [22].

4 Graphs with fixed structure parameters

Given a connected graph \( G = (V(G), E(G)) \), its vertex connectivity \( \kappa(G) \) is the minimum number of vertices whose removal yields a disconnected graph. An edge cut \( [S, \overline{S}] \) of \( G \) is a subset of \( E(G) \) between \( S \) and \( \overline{S} \), where \( S \) is a nonempty proper subset of \( V(G) \) and \( \overline{S} = V(G) \setminus S \). A k-edge cut is an edge cut of \( k \) edges in \( G \). The edge connectivity \( \kappa'(G) \) of \( G \) is the minimum cardinal of the edge cuts in \( G \). Obviously, for the minimum vertex degree \( \delta(G) \), vertex connectivity \( \kappa(G) \) and edge connectivity \( \kappa'(G) \) in the graph \( G \), we know that \( \kappa(G) \leq \kappa'(G) \leq \delta(G) \).

Let \( G_{\alpha} \) and \( \overline{G}_{\alpha} \) denote the set of all graphs of order \( n \) with vertex connectivity \( r \) and edge connectivity \( r \), respectively. Obviously, \( G_{\alpha}^{n-1} = \overline{G}_{\alpha}^{n-1} = K_n \).

Let \( RD_{\alpha}(G) \) be the \( RD_{\alpha} \)-matrix of graph \( G \) and \( x = (x_1, x_2, \ldots, x_n)^T \) be a column vector of order \( n \). As we all know,
\[x^T RD_{\alpha}(G)x = \alpha \sum_{i=1}^{n} RT_{\alpha}(v_i)x_i^2 + 2(1 - \alpha) \sum_{1 \leq i < j \leq n} \frac{1}{d_G(v_i, v_j)}x_ix_j.
\]

Then
\[
x^T RD_{\alpha}(G')x - x^T RD_{\alpha}(G)x = \alpha \sum_{i=1}^{n} (RT_{\alpha}(v_i)^2 - RT_{\alpha}(v_i))x_i^2
\]
\[+ 2(1 - \alpha) \sum_{1 \leq i < j \leq n} \left( \frac{1}{d_G(v_i, v_j)} - \frac{1}{d_G(v_i, v_j)} \right)x_ix_j.
\] (19)
Theorem 4.1. Let $n$ and $r$ be given integers with $1 \leq r \leq n - 2$. Then, for $0 \leq \alpha < 1$, the graph $K_r \lor (K_1 \cup K_{n-r-1})$ is the unique graph with maximal spectral radius of the matrix $RD_\alpha(G)$ in $G \in G^r_n$.

Proof. Assume that $G$ is a graph with maximal spectral radius $\rho(RD_\alpha(G))$ of $RD_\alpha(G)$ in $G \in G^r_n$. Proposition 3.2 implies that the graph $G$ must be isomorphic to $K_r \lor (K_{n_1} \cup K_{n_2})$, where $n_1 + n_2 = n - r$. Without loss of generality, suppose that $1 \leq n_1 \leq n_2$.

Suppose towards contradiction that $n_1 > 1$. Let $x$ be the perron eigenvector with respect to the spectral radius $\rho(RD_\alpha(G))$ of $RD_\alpha(G)$ and $\|x\| = 1$. According to the equivalence of vertices in $G$, the perron eigenvector $x$ can be written as

$$x^T = (x_1, x_2, x_3, \ldots, x_r) .$$

Let $v_1 \in V(K_{n_1})$. Construct a new graph $G'$ from graph $G$ by deleting the edges $\{v_1 v_i : v_i \in V(K_{n_1'})\} \setminus \{v_1\}$ and adding the edges $\{v_1 v_j : v_j \in V(K_{n_2})\}$, that is,

$$G' = G - \bigcup_{v_i \in V(K_{n_1}) \setminus \{v_1\}} v_1 v_i + \bigcup_{v_j \in V(K_{n_2})} v_1 v_j .$$

Obviously, $G' = K_r \lor (K_{n_1 - 1} \cup K_{n_2 + 1})$. Since

$$\rho(RD_\alpha(G')) - \rho(RD_\alpha(G)) \geq x^T RD_\alpha(G')x - x^T RD_\alpha(G)x,$$

According to [14], we only need to compute $\frac{1}{d_{G'}(v_1, v_j)} - \frac{1}{d_G(v_1, v_j)}$ and $RT r_{G'}(v_i) - RT r_{G}(v_i)$ for any $v_i, v_j \in V(G)$. After some careful observations and calculations, one gets that

(i) $\frac{1}{d_{G'}(v_1, v_j)} - \frac{1}{d_G(v_1, v_j)} = -\frac{1}{2}$ for any $v_j \in V(K_{n_1}) \setminus \{v_1\}$;

(ii) $\frac{1}{d_{G'}(v_1, v_j)} - \frac{1}{d_G(v_1, v_j)} = \frac{1}{2}$ for any $v_j \in V(K_{n_2})$;

(iii) $RT r_{G'}(v_i) - RT r_{G}(v_i) = \frac{n_2 - (n_1 - 2)}{2}$;

(iv) $RT r_{G'}(v_i) - RT r_{G}(v_i) = -\frac{1}{2}$ for any $v_i \in V(K_{n_1}) \setminus \{v_1\}$;

(v) $RT r_{G'}(v_j) - RT r_{G}(v_j) = \frac{1}{2}$ for any $v_j \in V(K_{n_2})$.

Plugging the above into (19), we have

$$x^T RD_\alpha(G')x - x^T RD_\alpha(G)x = \alpha \left[ \frac{n_2 - (n_1 - 1)}{2} x_1^2 - \frac{(n_1 - 1)}{2} x_2^2 + \frac{n_2}{2} x_2^2 \right] + (1 - \alpha) \left[ n_2 x_1 x_2 - (n_1 - 1) x_1^2 \right]$$

$$= \frac{\alpha n_2}{2} (x_1^2 + x_2^2) - (n_1 - 1) x_1^2 + (1 - \alpha) n_2 x_1 x_2 .$$

As a matter of convenience, let $\rho = \rho(RD_\alpha(G))$. It follows from $RD_\alpha(G)x = \rho x$ that

$$\rho x_1 = \alpha (n - 1) - \frac{n_2}{2} x_1 + (1 - \alpha) [(n_1 - 1) x_1 + \frac{n_2}{2} x_2 + r x_3]$$

and

$$\rho x_2 = \alpha (n - 1) - \frac{n_1}{2} x_1 + (1 - \alpha) \left[ \frac{n_1}{2} x_1 + (n_2 - 1) x_2 + r x_3 \right],$$

or equivalently,

$$\rho x_1 - \alpha (n - 1) - \frac{n_2}{2} x_1 - (1 - \alpha) [(n_1 - 1) x_1 - (1 - \alpha) \frac{n_2}{2} x_2 = (1 - \alpha) r x_3$$

14
\[ \rho x_2 - \alpha (n-1 - \frac{n_1}{2})x_2 - (1-\alpha)(n_2-1)x_2 - (1-\alpha)\frac{n_1}{2}x_1 = (1-\alpha)\rho x_3. \]

By combining the above two equations, we easily obtain that
\[ (\rho - \alpha n + \frac{n_1 + n_2}{2}\alpha + 1 - \frac{n_1}{2})x_1 = (\rho - \alpha n + \frac{n_1 + n_2}{2}\alpha + 1 - \frac{n_2}{2})x_2. \] (20)

Since \((1-\alpha)\frac{n_1}{2}x_1 + (1-\alpha)\frac{n_2}{2}x_2 > 0\), then \((\rho - \alpha n + \frac{n_1 + n_2}{2}\alpha + 1 - \frac{n_1}{2})x_1 > 0\). It follows from (20) that \(\frac{x_1}{x_2} \geq 1\) as \(n_2 \geq n_1\). Thus,
\[ \frac{\alpha n_2}{2}(x_1^2 + x_2^2) - (n_1-1)x_1^2 + (1-\alpha)n_2x_1 x_2 = x_1^2\frac{\alpha n_2}{2}(\frac{x_2}{x_1} - 1)^2 + (n_2\frac{x_2}{x_1} - n_1 + 1) > 0, \]
which leads to \(\rho(RD_{\alpha}(G')) > \rho(RD_{\alpha}(G))\). This contradicts our previous assumption. Hence, \(G = K_r \cup (K_1 \cup K_{n-r-1})\).

In what follows, we intends to give the extremal graphs with maximal spectral radius of the \(RD_{\alpha}\)-matrix among all connected graphs of fixed edge connectivity. To do this, we first recall the following lemma.

**Lemma 4.2.** \([20]\) Let \(S\) be a nonempty vertex subset of a graph \(G\) with minimum vertex degree \(\delta(G)\). If \(|S| < \delta(G)\), then \(|S| > \delta(G)\).

**Theorem 4.3.** For \(1 \leq r \leq n-2\) and \(0 \leq \alpha < 1\), the graph \(K_r \cup (K_1 \cup K_{n-r-1})\) is the unique graph with maximal spectral radius of the matrix \(RD_{\alpha}(G)\) in \(G \in \mathcal{G}_n\).

**Proof.** Let \(G\) be the graph with maximal spectral radius of the matrix \(RD_{\alpha}(G)\) in \(G \in \mathcal{G}_n\). It is known that \(\delta(G) \geq r\). If there exists a vertex \(v\) in graph \(G\) with degree \(r\), then \(\{(v) \cup V(G) \cup \{v\}\}\) is an \(r\)-edge cut of \(G\). Proposition 3.2 implies that the induced subgraph of \(V(G) \cup \{v\}\) must be a complete graph. So, the required result follows.

Now assume that \(\delta(G) > r\) and \(|S, S'|\) is an \(r\)-edge cut of \(G\) with \(|S| = n_1\) and \(|S'| = n_2\). Let \(G_1 = G[S]\) and \(G_2 = G[S']\) be two induced subgraphs of \(G\) with respect to \(S\) and \(S'\), respectively. It follows from Proposition 3.2 that \(G_1\) and \(G_2\) are complete graphs. Noting that \(\delta(G) > r\), then \(n_1 > 1\) and \(n_2 > 1\). Let \(V(G_1) = \{v_1, v_2, \ldots, v_n\}\) and \(V(G_2) = \{v_{n_1+1}, v_{n_1+2}, \ldots, v_n\}\). Also let \(x = (x_1, x_2, \ldots, x_n, x_{n+1}, x_{n+2}, \ldots, x_n)\) be the Perron eigenvector with respect to \(\rho(RD_{\alpha}(G))\), where \(x_i\) be the component of \(x\) corresponding to the vertex \(v_i\) in \(V(G)\). Without loss of generality, let \(x_1 = \min_{1 \leq i \leq n} x_i\) and \(x_1 \leq x_2 \leq \cdots \leq x_n\). We may assume that \(v_1\) is adjacent to \(t\) vertices of \(G_2\). Obviously, \(t \leq \min\{r, n_2\}\).

Now consider the following two cases: \(t = r\) and \(t < r\).

**Case 1:** \(t = r\). In this case, the edges between \(v_1\) and \(G_2\) are exactly the cut edges between \(G_1\) and \(G_2\). At that time, we have \(n_2 \geq r + 2\). Indeed, if \(n_2 = r\), or \(n_2 = r + 1\), then there must exists a vertex with degree \(r\) in \(G_2\). This contradicts our hypothesis \(\delta(G) > r\). Now construct a new graph \(G'\) from \(G\) as follows:

\[ G' = G - \bigcup_{v_i \in V(G_1) \setminus v_1} v_1v_i + \bigcup_{v_i \in V(G_1) \setminus v_1, v_j \in V(G_2)} v_1v_j. \]

It is clear that \(G' = K_r \cup (K_1 \cup K_{n-r-1})\). Let \(A = V(G_1) \setminus \{v_1\}\), \(B = \{v_j \in V(G_2) : v_1v_j \in E(G)\}\) and \(C = V(G_2) \setminus B\). According to [19], we only need to calculate \(\frac{1}{d_{G'}(v_i, v_j)} - \frac{1}{d_{G}(v_i, v_j)}\) and \(RT_{G'}(v_1) - RT_{G}(v_1)\) for any \(v_i, v_j \in V(G)\). By some observations and calculations, one has

(i) \(\frac{1}{d_{G'}(v_i, v_j)} - \frac{1}{d_{G}(v_i, v_j)} = -\frac{1}{2}\) for any \(v_i \in A\);

(ii) \(\frac{1}{d_{G'}(v_i, v_j)} - \frac{1}{d_{G}(v_i, v_j)} = \frac{1}{2}\) for any \(v_i \in A, v_j \in B\);
(iii) $\frac{1}{d_G(v_i, v_k)} - \frac{1}{d_G(v_i, v_j)} = \frac{2}{5}$ for any $v_i \in A, v_k \in C$;
(iv) $RT_{r}G_i'(v_i) - RT_{r}G_i(v_i) = -\frac{|A|}{2}$;
(v) $RT_{r}G_i'(v_i) - RT_{r}G_i(v_i) = -\frac{1}{2} + \frac{1}{3}|B| + \frac{2}{3}|C|$ for any $v_i \in A$;
(vi) $RT_{r}G_i'(v_j) - RT_{r}G_i(v_j) = \frac{1}{3}|A|$ for any $v_j \in B$;
(vii) $RT_{r}G_i'(v_k) - RT_{r}G_i(v_k) = \frac{2}{5}|A|$ for any $v_k \in C$.

Now let us substitute the above into (19),

$$
\rho(RD_n(G') - \rho(RD_n(G)) \geq x^T RD_n(G')x - x^T RD_n(G)x
$$

$$
= \alpha \left[ -\frac{1}{2} |A| x_1^2 + (\frac{1}{2} + \frac{1}{2} |B| + \frac{2}{3} |C|) \sum_{v_i \in A} x_i^2 + \frac{2}{3} |A| \sum_{v_j \in B} x_j^2 + \frac{2}{3} |A| \sum_{v_k \in C} x_k^2 \right]
$$

$$
+ (1 - \alpha) \left[ - \sum_{v_i \in A} x_i x_i + \sum_{v_i \in A, v_j \in B} x_i x_j + \sum_{v_i \in A, v_k \in C} \frac{4}{3} x_i x_k \right]
$$

$$
> 0,
$$

where the last inequality holds because $x_1 = \min_{1 \leq i \leq n} \{x_i\}$. So, $\rho(RD_n(G')) > \rho(RD_n(G))$, which yields a contradiction.

Case 2: $t < r$. In this case, Lemma 4.2 implies that $n_2 > r + 1$ as $\delta(G) > r$. Let $W = \{v \in V(G_2) : vw \notin E(G), w \in V(G_1)\} \{v_i\}$. Since $|[[V(G_1) \{v_i\}, V(G_2)]| = r - t$ and $n_2 > r + 1$, then $|W| > 0$. Let $V_F = \{v_2, \ldots, v_{n_1 - r + 1}\} \subseteq V(G_1)\} \{v_i\}$. For all $v_j \in V_F, \forall v_j \in G_2$, vertex $v_j$ is not adjacent to $v_i$. Let $V_F' = V(G_1) \{\{v_1\} \cup V_F\}$. It is clear that $|[[V_F', V(G_2)]| = r - t$. Now construct a new graph $G''$ from $G$ as below:

$$
G'' = G - \bigcup_{v_j \in V(F)} v_1 v_j + \bigcup_{v_j \in V(G_1)\{v_1\}, v_j \in V(G_2), v_j \notin E(G)} v_1 v_j.
$$

Obviously, $G'' = K_2 \cup (K_1 \cup K_{n_1 - r - 1})$. In the following, we can write $\frac{1}{d_G(v_i, v_j)} - \frac{1}{d_G(v_i, v_j)}$ and $RT_{r}G''(v_i) - RT_{r}G(v_i)$ for any $v_i, v_j \in V(G)$.

(i) $\frac{1}{d_G(v_i, v_j)} - \frac{1}{d_G(v_i, v_j)} = -\frac{4}{3}$ for any $v_j \in V_F$;

(ii) $\frac{1}{d_G(v_i, v_j)} - \frac{1}{d_G(v_i, v_j)} = 1 - \frac{1}{d_G(v_i, v_j)}$ for any $v_i \in V(G_1) \{v_1\}, v_j \in V(G_2)$;

(iii) $RT_{r}G''(v_i) - RT_{r}G(v_i) = -\frac{|F|}{3}$;

(iv) $a_1 = RT_{r}G''(v_j) - RT_{r}G(v_j) = -\frac{1}{2} + \sum_{v_j \in V(G_2)} (1 - \frac{1}{d_G(v, v_j)})$ for any $v_j \in V_F$;

(v) $a_2 = RT_{r}G''(v_j) - RT_{r}G(v_j) = \sum_{v_j \in V(G_2)} (1 - \frac{1}{d_G(v, v_j)})$ for any $v_j \in V_F$;

(vi) $a_3 = RT_{r}G''(v_j) - RT_{r}G(v_j) = \sum_{v_j \in V(G_1) \{v_1\}} (1 - \frac{1}{d_G(v, v_j)})$ for any $v_j \in V(G_2)$.

Again, substitute the above into (19),

$$
\rho(RD_n(G'') - \rho(RD_n(G)) \geq x^T RD_n(G'')x - x^T RD_n(G)x
$$

$$
= \alpha \left[ -\frac{|F|}{2} x_1^2 + \sum_{v_j \in V_F} a_1 x_j^2 + \sum_{v_j \in V_F} a_2 x_j^2 + \sum_{v_j \in V(G_2)} a_3 x_j^2 \right]
$$

$$
+ (1 - \alpha) \left[ - \sum_{v_j \in V_F} x_i x_j + \sum_{v_j \in V(G_1) \{v_1\}, v_j \in V(G_2)} 2(1 - \frac{1}{d_G(v, v_j)}) x_i x_j \right].
$$

(21)
Since every vertex in \( V_F \) is not adjacent to any vertex of \( G_2 \), then \( d_G(v_f, v_j) \geq 2 \). This implies that \( a_1 > \frac{1}{4}(n_2 - 1). \)

Recall that \( n_2 > r + 1 \) and \( d_G(v_f, v_j) \geq 2 \) for any \( v_f \in V_F, v_j \in V(G_2) \). Thus,

\[
-\frac{|F|}{2}x_1^2 + \sum_{v_f \in V_F} a_1x_f^2 > \frac{1}{2}|F|(n_2 - 2)x_1^2 > 0,
\]

and

\[
\sum_{v_f \in V_F} 2(1 - \frac{1}{d_G(v_f, v_j)})x_f x_j = \sum_{v_f \in V_F} x_1 x_f
\]

\[
= \sum_{v_f \in V_F, v_j \in V(G_2)} 2(1 - \frac{1}{d_G(v_f, v_j)})x_f x_j + \sum_{v_f \in V_F, v_j \in V(G_2)} 2(1 - \frac{1}{d_G(v_f, v_j)})x_f x_j - \sum_{v_f \in V_F} x_1 x_f
\]

\[
\geq \sum_{v_f \in V_F, v_j \in V(G_2)} x_f x_j + \sum_{v_f \in V_F, v_j \in V(G_2)} 2(1 - \frac{1}{d_G(v_f, v_j)})x_f x_j - \sum_{v_f \in V_F} x_1 x_f
\]

\[
> 0.
\]

It follows from (21) that \( \rho(RD_n(G'')) - \rho(RD_n(G)) > 0 \), that is, \( \rho(RD_n(G'')) > \rho(RD_n(G)) \), a contradiction. To sum up, \( K_r \lor (K_1 \lor K_{n-r-1}) \) is the unique graph with maximal spectral radius of the matrix \( RD_n(G) \) in \( G \in G_n \).

For a connected graph \( G \), the chromatic number \( \chi(G) \) is the smallest number of colors of \( V(G) \) such that any two adjacent vertices with different colors. The Turán graph, denoted by \( T_{n, r} \), is the complete \( r \)-partite graph with \( n \) vertices and each part has \( \lceil \frac{n}{r} \rceil \) or \( \lceil \frac{n}{2} \rceil \) vertices. Denote the set of all connected graphs of order \( n \) with chromatic number \( \chi \) by \( \tilde{G}_n^\chi \).

**Theorem 4.4.** For \( n \geq 2 \) and \( 0 \leq \alpha \leq \frac{1}{16} \), the Turán graph \( T_{n, \chi} \) is the unique graph with maximal spectral radius of the matrix \( RD_n(G) \) in \( G \in \tilde{G}_n^\chi \).

**Proof.** Let \( G \) be a graph with the maximal spectral radius of the matrix \( RD_n(G) \) in \( G \in \tilde{G}_n^\chi \). The conclusion is trivial when \( \chi = n \), that is, \( G \) is a complete graph \( K_n \). Also noting that the unique connected non-complete graph is the path \( P_3 \) whenever \( n = 2 \) or 3. In the following, assume that \( 2 \leq \chi \leq n - 1, n \geq 4 \) and \( G \) has a partition \( \pi : V_1 \cup V_2 \cup \cdots \cup V_{\chi} \) of \( V(G) \), where \( |V_i| = n_i \) and \( \sum_{i=1}^{\chi} n_i = n \). Proposition 3.2 implies that \( G = K_{n_1, n_2, \ldots, n_\chi} \). We may assume that \( n_1 = \max_{1 \leq i \leq \chi} \{n_i\} \) and \( n_2 = \min_{1 \leq i \leq \chi} \{n_i\} \). Next, we only need to prove that \( n_1 - n_2 < 1 \).

Suppose towards contradiction that \( n_1 - n_2 \geq 1 \). Let \( x \) be the perron eigenvector corresponding to the spectral radius \( \rho(RD_n(G)) \) of \( RD_n(G) \). According to the equivalence of the vertices in \( G \), the vector \( x \) can be written as

\[
x^T = (x_1, \ldots, x_1, x_2, \ldots, x_2, \ldots, x_{\chi}, \ldots, x_{\chi}).
\]

Now, select any vertex \( v_1 \in V_1 \), construct graph \( G' \) by deleting the edges \( v_1 v_j \) for any \( v_j \in V_2 \), and adding the edges \( v_1 v_i \) for any \( v_i \in V_1 \setminus \{v_1\} \) in graph \( G \), that is,

\[
G' = G - \bigcup_{v_j \in V_2} v_1 v_j + \bigcup_{v_i \in V_1 \setminus v_1} v_1 v_i.
\]

Obviously, \( G' = K_{n_1-1, n_2+1, n_3, \ldots, n_\chi} \).

In light of (10), we first calculate \( \frac{1}{d_G(v_i, v_j)} - \frac{1}{d_G(v_i, v_j)} \) and \( RT_{G'}(v_i) - RT_{G'}(v_i) \) for \( v_i, v_j \in V(G) \). These values are listed below:

(i) \( \frac{1}{d_G(v_i, v_j)} - \frac{1}{d_G(v_i, v_j)} = \frac{1}{2} \) for every \( v_i \in V_1 \setminus \{v_1\} \);
(ii) \( \frac{1}{d_G(v_i, v_j)} - \frac{1}{d_G(v_1, v_j)} = -\frac{1}{2} \) for every \( v_j \in V_2 \);

(iii) \( RT_G(v_1) - RT_G(v_i) = \frac{\alpha_1 - n_2 - 1}{2} \);

(iv) \( RT_G(v_i) - RT_G(v_1) = \frac{1}{2} \) for every \( v_i \in V_1 \setminus \{v_1\} \);

(v) \( RT_G(v_j) - RT_G(v_j) = -\frac{1}{2} \) for every \( v_j \in V_2 \).

Substitute the above into (19) to obtain

\[
\rho(RD_\alpha(G')) - \rho(RD_\alpha(G)) \geq x^T RD_\alpha(G')x - x^T RD_\alpha(G)x = \alpha \left[ \frac{n_1 - n_2 - 1}{2} x_1^2 + \frac{(n_1 - 1) - n_2}{2} x_2^2 \right]
+ (1 - \alpha) [(n_1 - 1)x_1^2 - n_2 x_2^2]
= (n_1 - 1)x_1^2 - n_2 x_2^2 - \frac{\alpha n_2}{2} (x_1 - x_2)^2.
\]

(22)

For the sake of convenience, let \( \rho = \rho(RD_\alpha(G)) \). From \( RD_\alpha(G)x = \rho x \), we easily find that, for \( 1 \leq i \leq \chi \),

\[
\rho x_i = \alpha (n - \frac{n_i + 1}{2}) x_i + (1 - \alpha) \left( \sum_{k=1}^{\chi} n_k x_k - \frac{n_i + 1}{2} x_i \right).
\]

It means that

\[
(\rho - \alpha n + \frac{n_i + 1}{2}) x_i = (1 - \alpha) \sum_{k=1}^{\chi} n_k x_k.
\]

(23)

Therefore, it follows from (23), along with \( n_1 \geq n_k \), that \( x_1 \leq x_k \) for \( k = 2, 3, \ldots, \chi \) and

\[
(\rho - \alpha n + \frac{n_i + 1}{2}) x_1 = (\rho - \alpha n + \frac{n_2 + 1}{2}) x_2.
\]

(24)

Now, plugging (24) into (22), one obtains that

\[
\rho(RD_\alpha(G')) - \rho(RD_\alpha(G)) \geq (n_1 - 1)x_1^2 - n_2 x_1 x_2 - \frac{\alpha n_2}{2} (x_1 - x_2)^2
= x_1^2 \left[ (n_1 - 1) - n_2 \frac{x_2}{x_1} - \frac{\alpha n_2}{2} \left( 1 - \frac{x_2}{x_1} \right)^2 \right]
= \frac{x_1^2}{\rho - \alpha n + \frac{n_2 + 1}{2}} \left[ (n_1 - 1) - n_2 \left( \rho - \alpha n + \frac{n_1 + 1}{2} \right) \right]
- \frac{(n_1 - 1)(n_1 - n_2)}{2} - \frac{\alpha n_2 (n_2 - n_1)^2}{8 (\rho - \alpha n + \frac{n_2 + 1}{2})}.
\]

(25)

As we all know,

\[
\rho(RD_\alpha(G)) \geq \min_{1 \leq i \leq n} \left( \sum_{j=1}^{n} RD_\alpha(G)_{i,j} \right) = n - \frac{n_1 + 1}{2},
\]

(26)

where \( RD_\alpha(G)_{i,j} \) is the \((i,j)\)th entry of \( RD_\alpha(G) \). According to (26) and \( n_1 > n_2 + 1 \), we have
\( \rho - \alpha n + \frac{\alpha n + \frac{1}{2}}{n} > 2(1 - \alpha)n_2 > 2\alpha n_2 \) for \( 0 \leq \alpha \leq \frac{7}{16} \), which implies that
\[
(n_1 - 1 - n_2)(\rho - \alpha n + \frac{n_1 + 1}{2}) - \frac{\alpha n_2}{2} - \frac{\alpha n - 2}{16} (n_2 - n_1)^2
\]
\[
> (n_1 - 1 - n_2)(\rho - \alpha n + \frac{n_1 + 1}{2}) - \frac{\alpha n_2}{2} - \frac{\alpha n - 2}{16} (n_2 - n_1)^2
\]
\[
= (n_1 - 1 - n_2)(\rho - \alpha n + \frac{1}{2} - \frac{n_1 + 1 - n_2}{16}) - \frac{n_2 - 1}{16}
\]
\[
= (n_1 - 1 - n_2) \left( \left( 1 - \alpha \right) \sum_{k=1}^{\chi} n_k - \frac{n_1}{2} - \frac{n_1 + 1 - n_2}{16} \right) - \frac{n_2 - 1}{16} \quad \text{(by (28))}
\]
where the last inequality follows because \( n_1 - 1 - n_2 \geq 1 \) and \( x_1 \leq x_k \) for \( k = 2, 3, \ldots, \chi \). Finally, it follows from this, along with (28), that \( \rho(RD_\alpha(G')) - \rho(RD_\alpha(G)) > 0 \) for \( 0 \leq \alpha \leq \frac{7}{16} \). This contradicts our previous assumption. Therefore, \( G = T_{n,\chi} \).

A vertex subset \( S \in V(G) \) is called an independent set of a graph \( G \) if any two vertices of \( S \) is not adjacent in \( G \). The independent number of \( G \) is the number of elements in the maximum independent set of graph \( G \).

**Theorem 4.5.** Let \( G \) be a simple connected undirected graph of \( n \) vertices with independence number \( k \) and \( 0 \leq \alpha < 1 \). Then

\[
\rho(RD_\alpha(G)) \leq \frac{(1 + \alpha)n - \frac{1}{4} - \frac{n}{2} + \sqrt{((1 - \alpha)n + 2\alpha k - \frac{4\alpha}{3} - \frac{1}{2})^2 + 4(1 - \alpha)^2k(n - k)}}{2}
\]

Furthermore, the equality holds if and only if \( G = K_k \vee K_{n-k} \).

**Proof.** Since \( G \) is a connected graph with independence number \( k \). Then, Proposition 3.2 implies that \( \rho(RD_\alpha(G)) \leq \rho(RD_\alpha(K_k \vee K_{n-k})) \). Now, according to Corollary 2.4, the equation

\[
\rho(RD_\alpha(K_k \vee K_{n-k})) = \frac{(1 + \alpha)n - \frac{1}{4} - \frac{n}{2} + \sqrt{((1 - \alpha)n + 2\alpha k - \frac{4\alpha}{3} - \frac{1}{2})^2 + 4(1 - \alpha)^2k(n - k)}}{2}
\]

is true.

**5 Concluding remarks**

This work is mainly concerned with spectral properties of the generalized reciprocal distance matrix \( RD_\alpha(G) \), which is the unified way of the reciprocal distance matrix \( RD(G) \) and reciprocal distance signless Laplacian matrix \( RQ(G) \). Several basic spectral properties of \( RD_\alpha(G) \) and some bounds on generalized reciprocal distance spectral radius are established. We completely determine the generalized reciprocal distance spectra of some special graphs, which are used to discuss the positive semidefinite properties of \( RD_\alpha(G) \). We also characterize the extremal graphs with maximal generalized reciprocal distance spectral radius in several kinds of simple connected graphs with precise graph invariants, vertex connectivity, edge connectivity, chromatic number, independence number and so on.

It is worth mentioning that there is a small amount of research for the new introduced matrix \( RQ(G) \) (see [11, 12]). Studying spectral properties of \( RD_\alpha(G) \) not only generalizes the related results of \( RD(G) \), but also promotes the spectral research of \( RQ(G) \). For example, if we take \( \alpha = 0 \) in Theorem 4.1, then \( K_r \vee (K_1 \cup K_{n-r-1}) \) has the maximal reciprocal distance spectral radius among connected graphs with precise vertex connectivity, which is exactly a result in [19]. If we take \( \alpha = \frac{1}{2} \) in Theorem 4.1, then we arrive at:
Theorem 5.1. The graph $K_r \vee (K_1 \cup K_{n-r-1})$ is the unique graph with maximal reciprocal distance signless Laplacian spectral radius in $G \in G'_n$ with $1 \leq r \leq n - 2$.

However, that doesn’t always seem to happen. For example, it is proved in our Theorem 4.4 that the Turán graph $T_{n, \chi}$ has maximal spectral radius of $RD_\alpha(G)$ in $G \in \tilde{G}_\chi^n$ with $0 \leq \alpha \leq \frac{7}{16}$. We don’t know if the result holds for $\alpha = \frac{1}{2}$. This observation raises the following problem:

1. Whether there exists $\alpha_0 \in \left[\frac{1}{2}, 1\right)$ such that $T_{n, \chi}$ has maximal spectral radius of $RD_\alpha(G)$ in $G \in \tilde{G}_\chi^n$ for $0 \leq \alpha \leq \alpha_0$, or determine the graphs with maximal spectral radius of $RD_\alpha(G)$ in $G \in \tilde{G}_\chi^n$ for $\alpha \in (\frac{7}{16}, 1)$.

There are, of course, many other problems to be considered in the future work, for instance:

2. Characterize the graphs with minimal or maximal spectral radius of $RD_\alpha(G)$ in several kinds of graphs, such as trees, unicyclic graphs, the graphs with given matching number and so on.

3. Given some special graph classes, determine the smallest $\alpha_0 \in (0, \frac{1}{2}]$ such that $RD_\alpha(G)$ is positive semidefinite whenever $\alpha_0 \leq \alpha \leq 1$.

Finally, we point out that, the generalized reciprocal distance energy for a connected graph $G$ of order $n$ can be defined as

$$E_{RD_\alpha}(G) = \sum_{i=1}^{n} \left| \lambda_i(RD_\alpha(G)) - \frac{2\alpha H(G)}{n} \right|.$$

It is clear that $E_{RD_0}(G)$ is the reciprocal distance energy of $G$ [9], while $E_{RD_{1/2}}(G)$ is exactly the reciprocal distance signless Laplacian energy of $G$ [1, 13]. Therefore, it is entirely possible for us to focus on research these energies in a unified way.

References

[1] A. Alhevaz, M. Baghipur, H.S. Ramane, Computing the reciprocal distance signless Laplacian eigenvalues and energy of graphs, Matematiche LXXIV (I) (2019) 49-73.

[2] R. Bapat, S.K. Panda, The spectral radius of the reciprocal distance Laplacian matrix of a graph, Bull. Iran. Math. Soc. 44(5) (2018) 1211-1216.

[3] D.M. Cardoso, M.A.A. de Freitas, E. Martins, M. Robbiano, Spectra of graphs obtained by a generalization of the join graph operation, Discrete Math. 313 (2013) 733-741.

[4] D.M. Cardoso, O. Rojo, Edge perturbation on graphs with clusters: adjacency, Laplacian and signless Laplacian eigenvalues, Linear Algebra Appl. 512 (2017) 113-128.

[5] S-Y. Cui, G-X. Tian, The spectrum and the signless Laplacian spectrum of coronae, Linear Algebra Appl. 437 (2012) 1692-1703.

[6] S-Y. Cui, J-X. He, G-X. Tian, The generalized distance matrix, Linear Algebra Appl. 563 (2019) 1-23.

[7] K.Ch. Das, Maximum eigenvalue of the reciprocal distance matrix, J. Math. Chem. 47 (2010) 21-28.

[8] R.C. Díaz, G. Pastén, O. Rojo, New Results on the $D_\alpha$-matrix of connected graphs, Linear Algebra Appl. 577 (2019) 168-185.

[9] A.D. Güngör, A.S. Çevik, On the Harary energy and Harary estrada index of a graph, MATCH Commun Math Comput Chem. 64 (2010) 281-296.
[10] F. Huang, X. Li, S. Wang, On graphs with maximum Harary spectral radius, Appl. Math. Comput. 266 (2015) 937-945.

[11] H. Lin, J. Xue, J. Shu, On the $D_\alpha$-spectra of graphs, Linear Multilinear Algebra 69 (2021) 997-1019.

[12] L. Medina, M. Trigo, Upper bounds and lower bounds for the spectral radius of reciprocal distance, reciprocal distance Laplacian and reciprocal distance signless Laplacian matrices, Linear Algebra Appl. 609 (2021) 386-412.

[13] L. Medina, M. Trigo, Bounds on the Reciprocal distance energy and Reciprocal distance Laplacian energies of a graph, Linear Multilinear Algebra (2020) DOI: 10.1080/03081087.2020.1825607.

[14] R. Merris, Laplacian matrices of graphs: a survey, Linear Algebra Appl. 197-198 (1994) 143-176.

[15] H. Minc, Nonnegative Matrices, John Wiley & Sons, New York, 1988.

[16] V. Nikiforov, Merging the $A$– and $Q$-spectral theories, Appl. Anal. Discrete Math. 11 (2017) 81-107.

[17] D. Plavšić, S. Nikolić, N. Trinajstić, Z. Mihalić, On the Harary index for the characterization of chemical graphs, J. Math. Chem. 12 (1993) 235-250.

[18] W. So, Commutativity and spectra of Hermitian matrices, Linear Algebra Appl. 212/213 (1994) 121-129.

[19] L. Su, H. Li, M. Shi, J. Zhang, On the spectral radius of the reciprocal distance matrix, Adv. Math. (China) 43(4) (2014) 551-558.

[20] D.B. West, Introduction to Graph Theory, Second Edition, Upper Saddle River, Prentice Hall, 2001.

[21] L. You, Y. Shu, X-D. Zhang, A sharp upper bound for the spectral radius of a nonnegative matrix and applications, Czechoslov. Math. J. 66 (2016) 701-715.

[22] B. Zhou, N. Trinajstić, Maximum eigenvalues of the reciprocal distance matrix and the reverse Wiener matrix, Int. J. Quant. Chem. 108 (2008) 858-864.