Abstract. In this paper, we study the effect of small Brownian noise on a switching dynamical system which models a first-order DC/DC buck converter. The state vector of this system comprises a continuous component whose dynamics switch, based on the ON/OFF configuration of the circuit, between two ordinary differential equations (ODE), and a discrete component which keeps track of the ON/OFF configurations. Assuming that the parameters and initial conditions of the unperturbed system have been tuned to yield a stable periodic orbit, we study the stochastic dynamics of this system when the forcing input in the ON state is subject to small white noise fluctuations of size $\varepsilon$, $0 < \varepsilon \ll 1$. For the ensuing stochastic system whose dynamics switch at random times between a small noise stochastic differential equation (SDE) and an ODE, we prove a functional law of large numbers which states that in the limit of vanishing noise, the stochastic system converges to the underlying deterministic one on time horizons of order $\mathcal{O}(1/\varepsilon^{\nu})$, $0 \leq \nu < 2/3$.

1. Introduction

Ordinary differential equations (ODE) and dynamical systems play a fundamental role in modelling and analysis of various phenomena arising in science and engineering. In many applications, however, the smooth evolution of the ODE dynamics is punctuated by discrete instantaneous events which give rise to switching or non-smooth behaviour. Examples include instantaneous switching between different governing ODE in a power electronic circuit [BKYY, BV01, dBGGV], discontinuous change in velocity for an oscillator impacting a boundary [SH83, Nor91], etc. In such instances, the dynamical system involves functions which are not smooth, but only piecewise-smooth in their arguments. Such piecewise-smooth dynamical systems [dBBCK] display a wealth of phenomena not seen in their smooth counterparts, and have hence been the subject of much current research.

Dynamical systems arising in practice are almost always subject to random disturbances, owing perhaps to fluctuating external forces, or uncertainties in the system, or unmodelled dynamics, etc. A more accurate picture can therefore be obtained by modelling such systems (at least in the continuous-time case) using stochastic differential equations (SDE); intuitively, this corresponds to adding a “noise” term to the ODE. For cases where the perturbing noise is small, it is natural to ask whether the stochastic (perturbed) system converges to the deterministic (unperturbed) one in the limit of vanishing noise, and if yes, how the asymptotic behaviour of the fluctuations may be quantified. Such questions have played a significant role in the development of limit theorems for stochastic processes; see, for instance [DZ98, EK86, FW12, PS08].

Although smooth dynamical systems perturbed by noise have been analysed in great depth over the past few decades, the effect of random noise on non-smooth or switching dynamical systems remains, with some exceptions (see, for instance, CL07, CL11, HBB1, HBB2, SK1, SK2, SK3),
randomly perturbed switching dynamics of a dc/dc converter

relatively unexplored. One of the challenges in such an undertaking is that even in the absence of noise, the dynamics of switching systems can prove rather difficult to analyze. Part of the reason is the frequently encountered intractability of such systems to analytic computation [BC dBGGV], even in cases when the component subsystems are linear.

Our primary interest is the study of stochastic processes which arise due to small random perturbations of (non-smooth) switching dynamical systems. These problems are of immediate relevance in the analysis of DC/DC converters in power electronics—naturally susceptible to noise—in which time-varying circuit topology leads to mathematical models characterised by switching between different governing ODE. In the purely deterministic setting, the dynamics of these systems have been extensively studied, with much of the work focussing on buck converters [BKYY dBGGV HD92 FO96]; these are circuits used to transform an input DC voltage to a lower output DC voltage. Perhaps the simplest of these is the first-order buck converter; this is a system which switches between two linear first-order ODE. While this circuit is pleasantly amenable to some explicit computation, it nevertheless displays rich dynamics in certain parameter regimes. Periodic orbits, bifurcations and chaos for this converter have been studied in [BKYY HD92].

In the present paper, we study small random perturbations of a switching dynamical system which models a first-order buck converter. The state vector of this system comprises a continuous component (the inductor current) governed by one of two different ODE, and a discrete component which takes values 1 or 0 depending on whether the circuit is in the ON versus OFF configurations, respectively. Assuming that the parameters and initial conditions of the unperturbed system have been tuned to yield a stable periodic orbit, we study the stochastic dynamics of this system when the forcing (input DC voltage) is subject to small white noise fluctuations of size \( \varepsilon \), with \( 0 < \varepsilon \ll 1 \).

Our main result is a functional law of large numbers (FLLN) which states that, as \( \varepsilon \to 0 \), the solution of the stochastically perturbed system converges to that of the underlying deterministic system, over time horizons \( T_\varepsilon \) of order \( O(1/\varepsilon^\nu) \) for any \( 0 \leq \nu < 2/3 \). Part of the novelty of this work, in the context of the literature on switching diffusions (see, e.g., [BBG99 LM07 YZ1 YZ2]), is that the switching in our problem is not driven by a discrete-state stochastic process (whose transitions may occur at a rate depending on the continuous component of the state); rather, the switching occurs whenever the continuous component of the state hits a threshold (ON \( \to \) OFF), or upon the arrival of a time-periodic signal (OFF \( \to \) ON). Our switching is thus entirely determined by the continuous component, together with a periodic clock signal. We also note that since the input DC voltage in the buck converter influences the inductor current only in the ON state [BKYY], the perturbed system has alternating stochastic and deterministic evolutions: the dynamics switch at random times between an SDE driven by a small Brownian motion of size \( \varepsilon \) in the ON state, and an ODE in the OFF state. The import of our results is that even in the presence of small stochastic perturbations, one may expect the buck converter to function close to its desired operation for “reasonably long” times.

The rest of the paper is organised as follows. In Section 2, we describe the switching systems (deterministic and stochastic) in some detail, and we pose our problem of interest. Next, in Section 3, we state our main result (Theorem 3.1) and outline the steps to the proof through a sequence of auxiliary lemmas and propositions. A few of these auxiliary results are proved in Section 3 with the remainder (the slightly lengthier ones) being deferred to Section 4.

2. Problem Description

In this section, we formulate our problem of interest. We start with a description, including the governing ODE’s and the switching mechanism, of a dynamical system modelling a first-order buck converter in Section 2.1. Random perturbations of this system, which lead to a switching SDE/ODE
a stable periodic orbit, we pose our questions of interest. In Section 2.2 we obtain explicit formulas for solutions to both the SDE and the ODE’s between switching times, and we piece these together at switching times to obtain expressions describing the overall evolution of both the perturbed (stochastic) and unperturbed (deterministic) switching systems. Finally, after showing in Section 2.4 how problem parameters can be tuned and initial conditions chosen to ensure that the unperturbed system has a stable periodic orbit, we pose our questions of interest.

Before proceeding further, we note that we have a hybrid system. Indeed, the full state of the system is specified by a vector \( z = (x, y) \) taking values in \( \mathcal{Z} = \mathbb{R} \times \{0, 1\} \); here, \( x \in \mathbb{R} \) is the continuous component of the state—corresponding to the inductor current in the buck converter—while the discrete component \( y \) takes values 1 or 0 depending on whether the switch is ON or OFF.

2.1. Deterministic switching system. As noted above, the state of our system at time \( t \in [0, \infty) \) will be specified by a vector \( z(t) = (x(t), y(t)) \) taking values in \( \mathcal{Z} = \mathbb{R} \times \{0, 1\} \). We will assume that the dynamics of \( x(t) \) when \( y(t) = 1 \) (ON configuration) are governed by the ODE

\[
\frac{dx}{dt} = -\alpha_{on} x + \beta,
\]

while the dynamics of \( x(t) \) when \( y(t) = 0 \) (OFF configuration) are described by

\[
\frac{dx}{dt} = -\alpha_{off} x.
\]

Here, \( \beta, \alpha_{on}, \alpha_{off} \) are fixed positive parameters with \( \beta \) representing the (rescaled) input voltage of an external power source, while \( \alpha_{on} \) and \( \alpha_{off} \) denote the (rescaled) resistances in the ON and OFF configurations, respectively.\(^1\)

The switching between the ON and OFF configurations is effected as follows. A reference level \( x_{ref} \in (0, \beta/\alpha_{on}) \) is fixed. Suppose the system starts in the ON configuration, i.e., \( y(0) = 1 \), with \( x(0) \in (0, x_{ref}) \). The current \( x(t) \) increases according to (1), with \( y(t) \) staying at 1, until \( x(t) \) hits the level \( x_{ref} \). At this point, an ON \( \rightarrow \) OFF transition occurs: \( y(t) \) jumps to 0 and \( x(t) \) now evolves according to (2). This continues until the next arrival of a periodic clock signal with period 1 (which arrives at times \( n \in \mathbb{N} \)) triggers an OFF \( \rightarrow \) ON transition: \( y(t) \) jumps back to 1, \( x(t) \) again evolves according to (1), and the cycle continues. Note, that if a clock pulse arrives in the ON configuration, it is ignored. Of course, if one starts in the OFF configuration, \( x(t) \) evolves according to (2) until the next clock pulse, at which point the system goes ON, and the subsequent dynamics are as described above. An important assumption in our analysis is that \( x(t) \) is continuous across switching times.

2.2. Random perturbations. We now suppose that the forcing term \( \beta \) in (1) is subjected to small white noise perturbations of size \( \varepsilon, 0 < \varepsilon \ll 1 \); for the buck converter, this corresponds to small random fluctuations in the input voltage. In this setting, the state of the system at time \( t \in [0, \infty) \) is given by a stochastic process \( Z_t^\varepsilon \triangleq (X_t^\varepsilon, Y_t^\varepsilon) \) taking values in \( \mathcal{Z} \). The dynamics of \( X_t^\varepsilon \) in the ON configuration \( (Y_t^\varepsilon = 1) \) are now governed by the SDE

\[
\frac{dX_t^\varepsilon}{d\tau} = (-\alpha_{on} X_t^\varepsilon + \beta)d\tau + \varepsilon dW_t,
\]

where \( W_t \) is a standard one-dimensional Brownian motion, while evolution of \( X_t^\varepsilon \) in the OFF state \( (Y_t^\varepsilon = 0) \) is governed by the ODE (2), as before. The switching mechanism is similar to that in the unperturbed case, but with the stochastic processes \( X_t^\varepsilon, Y_t^\varepsilon \) playing the roles of \( x(t), y(t) \). Note,

\( ^1 \)More precisely, \( \beta = V_{in}/L, \alpha_{on} = R/L \) and \( \alpha_{off} = (R + r_d)/L \), where \( V_{in} \) is the input voltage, \( R \) denotes the load resistance, and \( r_d \) is the diode resistance [BKYY].
in particular, that the times for on → off transitions are given by passage times of $X_t^\varepsilon$ (governed by (3)) to the level $x_{\text{ref}}$. As before, $X_t^\varepsilon$ is assumed to be continuous across switching times.

### 2.3. Explicit formulas

The foregoing discussion makes clear how $z(t) = (x(t), y(t))$ and $Z_t^\varepsilon = (X_t^\varepsilon, Y_t^\varepsilon)$ are to be obtained, once an initial condition $z_0 = (x_0, y_0) \in \mathcal{Z}$ has been specified: the solutions of $x(t)$ and $X_t^\varepsilon$ are given, respectively, by concatenating solutions to (1) and (2), and solutions to (3) and (2), at the respective switching times, maintaining continuity. The function $y(t)$ and the sample paths of $Y_t^\varepsilon$—which are piecewise constant and take values in $\{0, 1\}$—will be assumed to be right-continuous.

Below, we obtain expressions for $z(t)$ and $Z_t^\varepsilon$ starting from initial condition $z_0 = (x_0, 1)$ where $x_0 \in (0, x_{\text{ref}})$. We note that starting with $y_0 = 1$ entails no real loss of generality; indeed, as will become apparent, the expressions below can be easily modified to accommodate the case when $y_0 = 0$.

In the sequel, we will use $1_A$ to denote the indicator function of the set $A$, and for real numbers $a, b$, we let $a \wedge b$ and $a \vee b$ denote the minimum and maximum of $a$ and $b$, respectively.

#### 2.3.1. Solution of deterministic switching system

As indicated above, we fix an initial condition $z_0 = (x_0, 1)$ with $x_0 \in (0, x_{\text{ref}})$. Let $s_0 \equiv 0$, and set $x^{0,\text{off}}(t) \equiv x_0$. Next, define $x^{1,\text{on}}(t) \equiv 1_{[s_0, \infty)}(t) \cdot \{\beta/\alpha_{\text{on}} + (x_0 - \beta/\alpha_{\text{on}}) e^{-\alpha_{\text{on}} t}\}$ for $t \geq 0$. Let $t_1 \equiv \inf\{t > 0 : x^{1,\text{on}}(t) = x_{\text{ref}}\}$ be the first time that $x^{1,\text{on}}(t)$ reaches level $x_{\text{ref}}$ and define $x^{1,\text{off}}(t) \equiv 1_{[t_1, \infty)}(t) \cdot x_{\text{ref}} e^{-\alpha_{\text{off}}(t-t_1)}$. Let $s_1 \equiv \inf\{t > t_1 : t \in \mathbb{Z}\}$ be the time of arrival of the next clock pulse. The solution of the deterministic switching system on the interval $[s_0, s_1)$ is now given by $x^1(t) \equiv x^{1,\text{on}}(t) \cdot 1_{[s_0, t_1)}(t) + x^{1,\text{off}}(t) \cdot 1_{[t_1, s_1)}(t)$.

In general, given the solution over $[s_0, s_{n-1})$, the solution $x^n(t)$ over $[s_{n-1}, s_n)$ is obtained as follows. We let

$$ t_n \equiv \inf\{t > s_{n-1} : x^{n,\text{on}}(t) = x_{\text{ref}}\}, $$

$$ x^{n,\text{off}}(t) \equiv 1_{[t_n, \infty)}(t) \cdot x_{\text{ref}} e^{-\alpha_{\text{off}}(t-t_n)} \quad \text{for } t \geq 0, $$

$$ s_n \equiv \inf\{t > t_n : t \in \mathbb{Z}\}, $$

$$ x^n(t) \equiv x^{n,\text{on}}(t) \cdot 1_{[s_{n-1}, t_n)}(t) + x^{n,\text{off}}(t) \cdot 1_{[t_n, s_n)}(t). $$

The evolution of the deterministic switching system over $[0, \infty)$ is now given by

$$ z(t) = (x(t), y(t)) \quad \text{where } x(t) \equiv \sum_{n \geq 1} x^n(t), \quad y(t) \equiv \sum_{n \geq 1} 1_{[s_{n-1}, t_n)}(t). $$

We have thus decomposed the evolution into a sequence of on/off cycles with the switching times $t_n$ and $s_n$ corresponding to the $n$-th on/off and off/on transitions, respectively; next, we have solved the ODE (1) and (2) between switching times, and then linked the pieces together while maintaining continuity of $x(t)$ at switching times.

#### 2.3.2. Solution of stochastic switching system

We now provide a similar detailed construction of the stochastic process $Z_t^\varepsilon = (X_t^\varepsilon, Y_t^\varepsilon)$ starting from the same initial condition $z_0 = (x_0, 1)$. Let $W = \{W_t, \mathcal{F}_t : 0 \leq t < \infty\}$ be a standard one-dimensional Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We introduce, for each $n \in \mathbb{N}$, the processes $X_t^{n,\varepsilon,\text{on}}, X_t^{n,\varepsilon,\text{off}}, X_t^{n,\varepsilon}$ and random switching times $\tau_t^\varepsilon, \sigma_t^\varepsilon$, which are defined recursively as follows. Set $\sigma_0^\varepsilon = 0$ and define $X_0^{0,\varepsilon,\text{off}} \equiv x_0$. Now, let

$$ X_t^{1,\varepsilon,\text{on}} \equiv 1_{[\sigma_0^\varepsilon, \infty)}(t) \cdot \{\beta/\alpha_{\text{on}} + (x_0 - \beta/\alpha_{\text{on}}) e^{-\alpha_{\text{on}} t} + \varepsilon \int_0^t e^{-\alpha_{\text{off}}(t-u)} dW_u\} $$

for $t \geq 0$, and let

$$ \tau_1^\varepsilon \equiv \inf\{t > 0 : X_t^{1,\varepsilon,\text{on}} = x_{\text{ref}}\} \quad \text{be the first passage time of } X_t^{1,\varepsilon,\text{on}} \text{ to level } x_{\text{ref}}. $$

We next define
Our stochastic process of interest is now given by 

\[ x_t = \int_{t-1}^t e^{\alpha(t-\tau)} \mathrm{d}W_t \]

for general \( n \in \mathbb{N} \), let \( I = \{ t : 0 \leq t < \infty \} \) be the process defined by 

\[ I_t = \int_0^t e^{\alpha(t-\tau)} \mathrm{d}W_t \]

for \( t \geq 0 \). Note that \( I \) is a continuous, square-integrable Gaussian martingale.

We now define, for each \( n \in \mathbb{N} \),

\[ X_t^{n,\varepsilon,\text{on}} = \int_{\tau_n}^{\tau_{n+1}} e^{-\alpha(x-\tau)} \mathrm{d}W_t \],

\[ \tau_n = \inf \{ t > \tau_{n-1} : X_t^{n,\varepsilon,\text{on}} = x_{\text{ref}} \} \]

(6)

\[ X_t^{n,\varepsilon,\text{off}} = \int_{\tau_n}^{\tau_{n+1}} e^{-\alpha(x-\tau)} \mathrm{d}W_t \],

\[ \sigma_n = \inf \{ t > \tau_n : t \in \mathbb{Z} \} \]

Our stochastic process of interest is now given by

(7)

\[ Z_t = (X_t^{n,\varepsilon}, Y_t^{n,\varepsilon}) \]

where \( X_t^{n,\varepsilon} = \sum_{n \geq 1} X_t^{n,\varepsilon} \), \( Y_t^{n,\varepsilon} = \sum_{n \geq 1} 1_{[\tau_n, \tau_{n+1})}(t) \).

Once again, the evolution comprises a sequence of ON/OFF cycles, with the quantities above admitting a natural interpretation which parallels the unperturbed (deterministic) case.

### 2.4. Stable periodic orbit

We now describe the assumptions on the problem parameters that ensure the existence of a stable periodic solution to (4), (5). The argument proceeds by analysing the stroboscopic map \[ \text{BKYY} \] which takes the system state at one clock instant to the state at the next. Map-based techniques are used extensively in analysing the switching dynamics of power electronic circuits; see also [dBGGV, HDJ92].

**Assumption 2.1.** Fix \( x_{\text{ref}} > 0, 0 < \alpha_{\text{on}} < \log 2 \). Select \( \beta > 0 \) such that

(8)

\[ 2x_{\text{ref}} \alpha_{\text{on}} < \beta < \left( \frac{e^{\alpha_{\text{on}}}}{e^{\alpha_{\text{on}}} - 1} \right) x_{\text{ref}} \alpha_{\text{on}} \]

Let \( \alpha_{\text{off}} > 0 \) such that

(9)

\[ \alpha_{\text{on}} < \alpha_{\text{off}} < \left( \frac{\beta/\alpha_{\text{on}} - x_{\text{ref}}}{x_{\text{ref}}} \right) \alpha_{\text{on}} \]

We now define a map \( f : [0, x_{\text{ref}}] \rightarrow [0, x_{\text{ref}}] \) which maps \( x_0 \in [0, x_{\text{ref}}] \) to the solution \( x(t) \) at time 1, subject to the initial condition \( (x_0, 1) \), i.e., \( f : x_0 \mapsto x(1; x_0, 1) \). We are interested in the case when \( f \) is only piecewise smooth. Put another way, if we let \( x_{\text{border}} = \beta/\alpha_{\text{on}} + (x_{\text{ref}} - \beta/\alpha_{\text{on}}) e^{\alpha_{\text{on}}} \) be the particular value of \( x_0 \) for which the corresponding \( x_{\text{on}}(t) \) satisfies \( x_{\text{on}}(1) = x_{\text{ref}} \), we would like \( x_{\text{border}} \in (0, x_{\text{ref}}) \). It is easily seen that the upper bound on \( \beta \) in (8) ensures that such is indeed the case. The map \( f : x_0 \mapsto x(1; x_0, 1) \) is seen to be given by

(10)

\[ f(x) = \begin{cases} \frac{\beta}{\alpha_{\text{on}}} + (x - \frac{\beta}{\alpha_{\text{on}}}) e^{-\alpha_{\text{on}}} & \text{if } 0 \leq x \leq x_{\text{border}}, \\ x_{\text{ref}} e^{-\alpha_{\text{off}}} \left( \frac{\beta/\alpha_{\text{on}} - x}{\beta/\alpha_{\text{on}} - x_{\text{ref}}} \right) \alpha_{\text{off}}/\alpha_{\text{on}} & \text{if } x_{\text{border}} < x \leq x_{\text{ref}}. \end{cases} \]

**Proposition 2.2.** Suppose Assumption 2.1 holds. Then, the mapping \( f : [0, x_{\text{ref}}] \rightarrow [0, x_{\text{ref}}] \) has a unique fixed point \( x^* \) which lies in the interval \( (x_{\text{border}}, x_{\text{ref}}) \). Further, \( |f'(x^*)| < 1 \), implying that \( x^* \) is a stable fixed point of the discrete-time dynamical system \( x_n \mapsto f(x_n) \).
Proof. Let \( h(x) \triangleq f(x) - x \). Note that \( h(0) = f(0) > 0 \), \( h(x_{\text{ref}}) = f(x_{\text{ref}}) - x_{\text{ref}} = x_{\text{ref}}(e^{-\alpha_{\text{off}}} - 1) < 0 \). Since \( h \) is continuous, there exists \( x^* \in (0, x_{\text{ref}}) \) such that \( h(x^*) = 0 \), i.e., \( f(x^*) = x^* \). Since \( f(x) > x \) for all \( x \in [0, x_{\text{border}}] \), we must have \( x^* \in (x_{\text{border}}, x_{\text{ref}}) \). Further, since \( f(x) \) decreases on \( (x_{\text{border}}, x_{\text{ref}}) \) as \( x \) increases over this same interval, \( f \) can have at most one fixed point. It is easily checked that for \( x \in (x_{\text{border}}, x_{\text{ref}}) \), we have

\[
|f'(x)| = \frac{\alpha_{\text{off}} f(x)}{\beta - \alpha_{\text{on}} x} \leq \frac{\alpha_{\text{off}} x_{\text{ref}}}{\beta - \alpha_{\text{on}} x_{\text{ref}}} < 1,
\]

where the last inequality follows from the upper bound on \( \alpha_{\text{off}} \) in (9). This proves stability of \( x^* \).

We can now pose our principal questions of interest. Suppose \( z(\cdot) \), \( Z_\varepsilon^z \) are obtained from (5), (7), respectively, with initial conditions \( z(0) = Z_0^\varepsilon = (x^*, 1) \), where \( x^* \) is as in Proposition 2.2

- For any fixed \( T \in \mathbb{N} \), do the dynamics of \( Z_\varepsilon^z \) converge to those of \( z(\cdot) \) in a suitable sense as \( \varepsilon \searrow 0 \)?
- If yes, can the results be strengthened to the case when \( T = T_\varepsilon \in \mathbb{N} \) grows to infinity, but “not too fast”, as \( \varepsilon \searrow 0 \)?

In the next section, we will show that both these questions can be answered in the affirmative, provided \( T_\varepsilon = O(1/\varepsilon^\nu) \) with \( 0 \leq \nu < 2/3 \).

3. Main Result

Recall that the state space for the evolution of \( z(t) \) and \( Z_t^z \) is \( \mathcal{X} = \mathbb{R} \times \{0, 1\} \), which inherits the metric \( r(z_1, z_2) \triangleq \{(x_1 - x_2)^2 + (y_1 - y_2)^2\}^{1/2} \) for \( z_1 = (x_1, y_1), \ z_2 = (x_2, y_2) \in \mathcal{X} \), from \( \mathbb{R}^2 \). If \( I \) is a closed subinterval of \( [0, \infty) \), we let \( D(I; \mathcal{X}) \) be the space of functions \( z : I \to \mathcal{X} \) which are right-continuous with left limits. This space can be equipped with the Skorokhod metric \( d_I \) [Bil EK86], which renders it complete and separable. If \( z \in D(I; \mathcal{X}) \) and \( J \) is a closed subinterval of \( I \), then the restriction of \( z \) to \( J \) is an element of \( D(J; \mathcal{X}) \) which, for simplicity of notation, will also be denoted by \( z \). For our switching systems of interest, we note that the function \( z(t) \) in (5), and the sample paths of the process \( Z_t^z \) in (7), belong to \( D([0, t_\varepsilon); \mathcal{X}) \). Our goal here is to study the convergence, as \( \varepsilon \searrow 0 \), of \( Z_\varepsilon^z \) to \( z(\cdot) \) in the space \( D([0, T_\varepsilon); \mathcal{X}) \) for time horizons \( T_\varepsilon = O(1/\varepsilon^\nu) \) where \( 0 \leq \nu < 2/3 \).

We start by defining the Skorokhod metric \( d_I \) on the space \( D(I; \mathcal{X}) \), where \( I = [0, T] \) for some \( T > 0 \). Let \( \Lambda_T \) be the set of all strictly increasing continuous mappings from \([0, T]\) onto itself\(^2\) and let \( \Lambda_T \) be the set of functions \( \lambda \in \Lambda_T \) for which

\[
\gamma_T(\lambda) \triangleq \sup_{0 \leq s < t \leq T} \left| \log \left( \frac{\lambda(t) - \lambda(s)}{t - s} \right) \right| < \infty.
\]

For \( z_1, z_2 \in D([0, T]; \mathcal{X}) \), we now define

\[
d_{[0,T]}(z_1, z_2) \triangleq \inf_{\lambda \in \Lambda_T} \left\{ \gamma_T(\lambda) \vee \sup_{0 \leq s \leq T} r(z_1(t), z_2(\lambda(t))) \right\}.
\]

Note that if \( d_{[0,T]}^0(z_1, z_2) \triangleq \sup_{0 \leq t \leq T} r(z_1(t), z_2(t)) \) is the uniform metric on \( D([0, T]; \mathcal{X}) \), then \( d_{[0,T]}(z_1, z_2) \leq d_{[0,T]}^0(z_1, z_2) \). Indeed, the latter corresponds to the specific choice \( \lambda(t) \equiv t \).

We now state our main result.

\(^2\)See [Bil EK86] for the case \( I = [0, \infty) \).
\(^3\)Thus, we have \( \lambda(0) = 0 \) and \( \lambda(T) = T \) for all \( \lambda \in \Lambda_T \).
**Theorem 3.1** (Main Theorem). Fix $\mathcal{T} \in \mathbb{N}$, $0 \leq \nu < 2/3$. For $\varepsilon \in (0, 1)$, let $T_\varepsilon \in \mathbb{N}$ such that $T_\varepsilon \leq \mathcal{T}/\varepsilon^\nu$. Suppose $z(\cdot), Z^\varepsilon$ are given by (5), (7), respectively, with initial conditions $z(0) = Z^\varepsilon_0 = (x^*, 1)$, where $x^*$ is as in Proposition 2.2. Then, for any $p \in [1, \infty)$, we have that $d_{[0,T_\varepsilon]}(Z^\varepsilon, z) \to 0$ in $L^p$, i.e.,

$$\lim_{\varepsilon \to 0} \mathbb{E}\left[(d_{[0,T_\varepsilon]}(Z^\varepsilon, z))^p\right] = 0.$$  

**Remark 3.2.** Of course, Theorem 3.1 implies that $d_{[0,T_\varepsilon]}(Z^\varepsilon, z)$ converges to 0 in probability, i.e., for any $\vartheta > 0$, we have $\lim_{\varepsilon \to 0} \mathbb{P}\{d_{[0,T_\varepsilon]}(Z^\varepsilon, z) \geq \vartheta\} = 0$.

To explain the intuition behind Theorem 3.1, we note that when $\varepsilon \ll 1$, the likely behaviour of $X^\varepsilon_t$ is to closely track $x(t)$. Therefore, one expects that with high probability, we have $\tau_n^\varepsilon \approx t_n$, $\sigma_n^\varepsilon = s_n = n$ for each $1 \leq n \leq T_\varepsilon$ (at least if $T_\varepsilon$ is not too large). On this “good” event, a random time-deformation $\lambda$, for which $\gamma T_\varepsilon(\lambda)$ is small, can be used to align the jumps of $Y^\varepsilon_t$ and $y(t)$ so that $Y^\varepsilon_{\lambda(t)} \equiv y(t)$. Continuity now ensures that $X^\varepsilon_{\lambda(t)}$ is close to $x(t)$, and we get that $d_{[0,T_\varepsilon]}(Z^\varepsilon, z)$ can be bounded above by a term which goes to zero as $\varepsilon \to 0$. It now remains to show that the probability of the complement of this event, i.e., the event where one or more of the $\tau_n^\varepsilon$ differ from $t_n$ by a significant amount, is small.

Our thoughts are organised as follows. First, we introduce an additional scale $\delta \downarrow 0$ to quantify proximity of $\tau_n^\varepsilon$ to $t_n$; we will later take $\delta = \varepsilon^\varsigma$ for suitable $\varsigma > 0$. Now, for $\varepsilon, \delta \in (0, 1)$, set $G^\varepsilon,\delta_0 \triangleq \Omega$, and for $n \geq 1$, define

$$G^\varepsilon,\delta_n \triangleq \{\omega \in G^\varepsilon,\delta_{n-1} : |\tau_n^\varepsilon(\omega) - t_n| \leq \delta\}$$

$$B^\varepsilon,\delta_n \triangleq \{\omega \in G^\varepsilon,\delta_{n-1} : |\tau_n^\varepsilon(\omega) - t_n| > \delta\} = G^\varepsilon,\delta_{n-1} \setminus G^\varepsilon,\delta_n.$$  

Note that the $G^\varepsilon,\delta_n$s are decreasing, i.e., $G^\varepsilon,\delta_0 \supset G^\varepsilon,\delta_1 \supset \ldots$ and that the $B^\varepsilon,\delta_n$s are pairwise disjoint. Consequently, $\bigcup_{n=1}^{T_\varepsilon} G^\varepsilon,\delta_n \supset G^\varepsilon,\delta_{T_\varepsilon}$ and $\mathbb{P}\left(\bigcup_{n=1}^{T_\varepsilon} B^\varepsilon,\delta_n\right) = \sum_{n=1}^{T_\varepsilon} \mathbb{P}\left(B^\varepsilon,\delta_n\right)$.

We now outline the principal steps in proving Theorem 3.1. First, in Proposition 3.3, we derive a path-wise estimate for $d_{[0,T_\varepsilon]}(Z^\varepsilon, z)$ and its positive powers. This result assures us that our quantity of interest is indeed small on the event $G^\varepsilon,\delta_{T_\varepsilon}$ and of order 1 on its complement. Then, in Proposition 3.6, we obtain an upper bound on $\mathbb{P}\left(B^\varepsilon,\delta_{T_\varepsilon}\right)$ in terms of the tail of the standard normal distribution. These two propositions enable us to complete the proof of Theorem 3.1. Both Propositions 3.3 and 3.6 are proved through a series of Lemmas; the proofs of the latter are deferred to Section 4.

We start by introducing some notation. Let $t_{on} \triangleq t_n - s_{n-1}$ and $t_{off} \triangleq s_{n-1} - t_n$ denote the fractions of time in each interval $[n, n+1]$ for which the deterministic system is in the ON and OFF states respectively, and let $t_{min} \triangleq t_{on} \wedge t_{off}$.

**Proposition 3.3.** For every $p > 0$, there exists a constant $C_p > 0$ such that for all $\varepsilon, \delta \in (0, 1)$, $T_\varepsilon \in \mathbb{N}$ satisfying $0 < \delta \leq t_{min}/(4T_\varepsilon)$, we have

$$d_{[0,T_\varepsilon]}(Z^\varepsilon, z)_t^p \leq C_p \left\{(T_\varepsilon \delta)^p + \delta^p + \frac{1}{\Omega \setminus G^\varepsilon,\delta_{T_\varepsilon}} + \varepsilon^p \sup_{0 \leq t \leq T_\varepsilon} |W_t^p|\right\}.$$
To prove Proposition 3.3, we will employ the (random) time-deformation $\lambda^\varepsilon: \Omega \to \Lambda_{T^\varepsilon}$ defined by

$$
\lambda^\varepsilon(t) \triangleq \sum_{n \geq 1} 1_{[s_{n-1}, t_n)}(t) \cdot \left\{ s_{n-1} + \left( \frac{\tau^\varepsilon_n(\omega) - s_{n-1}}{t_n - s_{n-1}} \right) (t - s_{n-1}) \right\} + \sum_{n \geq 1} 1_{[t_n, s_n)}(t) \cdot \left\{ \tau^\varepsilon_n(\omega) + \left( \frac{s_n - \tau^\varepsilon_n(\omega)}{s_n - t_n} \right) (t - t_n) \right\} \quad \text{if } \omega \in G^\varepsilon_{T^\varepsilon},
$$

and

$$
\lambda^\varepsilon(\omega) \triangleq t \quad \text{if } \omega \in \Omega \setminus G^\varepsilon_{T^\varepsilon}.
$$

Note that in actuality, $\lambda^\varepsilon = \lambda^\varepsilon,\delta$. However, we have suppressed the $\delta$-dependence to reduce clutter and also because we will eventually take $\delta = \delta(\varepsilon)$. The first step is to show that $\gamma_{T^\varepsilon}(\lambda^\varepsilon)$ is small; this is accomplished in Lemma 3.4 below. Next, in Lemma 3.5 we estimate $\sup_{0 \leq t \leq T^\varepsilon} |X^\varepsilon_{\lambda^\varepsilon} - x(t)|^p$ and $\sup_{0 \leq t \leq T^\varepsilon} |Y^\varepsilon_{\lambda^\varepsilon} - y(t)|^p$ for $p > 0$.

**Lemma 3.4.** Let $T^\varepsilon \in \mathbb{N}$. If $0 < \delta \leq t_{\min}/(4T^\varepsilon)$, then for each $\omega \in \Omega$, we have

$$
\gamma_{T^\varepsilon}(\lambda^\varepsilon(\omega)) \leq \frac{4T^\varepsilon\delta}{t_{\min}}.
$$

**Lemma 3.5.** For every $p > 0$, there exists a constant $c_p > 0$ such that for all $\varepsilon, \delta \in (0, 1)$, we have

$$
\sup_{0 \leq t \leq T^\varepsilon} |X^\varepsilon_{\lambda^\varepsilon} - x(t)|^p \leq c_p \left\{ \delta^p 1_{G^\varepsilon_{T^\varepsilon}} + 1_{G^\varepsilon_{T^\varepsilon}} \sup_{0 \leq t \leq T^\varepsilon} |W_t|^p \right\},
$$

$$
\sup_{0 \leq t \leq T^\varepsilon} |Y^\varepsilon_{\lambda^\varepsilon} - y(t)|^p \leq 1_{G^\varepsilon_{T^\varepsilon}}.
$$

Lemmas 3.4 and 3.5 are proved in Section 4. We now have

**Proof of Proposition 3.3.** Let $\lambda^\varepsilon \in \Lambda_{T^\varepsilon}$ be as in (14), (15). It is easily seen from (11) that

$$(d_{[0, T^\varepsilon]}(Z^\varepsilon, z))^p \leq 3^p \left\{ \gamma_{T^\varepsilon}(\lambda^\varepsilon))^p + \sup_{0 \leq t \leq T^\varepsilon} |X^\varepsilon_{\lambda^\varepsilon} - x(t)|^p + \sup_{0 \leq t \leq T^\varepsilon} |Y^\varepsilon_{\lambda^\varepsilon} - y(t)|^p \right\}.$$  

The claim (13) now follows from Lemmas 3.4, 3.5. \qed

We now estimate $\mathbb{P}\left( B_{n}^{\varepsilon,\delta} \right)$. Let $\mathcal{F}$ denote the right tail of the normal distribution; i.e.,

$$
\mathcal{F}(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-t^2/2} dt \quad \text{for } x \geq 0.
$$

A simple integration by parts yields

$$
\mathcal{F}(x) \leq \frac{3}{\sqrt{2\pi}} x^2 e^{-x^2/2}, \quad \text{for } x \geq 1.
$$

**Proposition 3.6.** There exists $\delta_0 \in (0, 1)$ and $K > 0$ such that whenever $0 < \delta < \delta_0$, $\varepsilon \in (0, 1)$, and $n \geq 1$, we have

$$
\mathbb{P}\left( B_{n}^{\varepsilon,\delta} \right) \leq 3\mathcal{F}\left( K_{\delta}^{\varepsilon} \right).
$$
Lemma 3.7. There exists $K_\rightarrow > 0$ such that for any $n \geq 1$, $\varepsilon, \delta \in (0, 1)$, we have
\begin{equation}
\mathbb{P}(B_n^{\varepsilon, \delta}) \leq 2K_\rightarrow \frac{\delta}{\varepsilon}.
\end{equation}

Lemma 3.8. There exists $\delta_+ \in (0, 1)$ and $K_+ > 0$ such that whenever $0 < \delta < \delta_+$, $\varepsilon \in (0, 1)$, $n \geq 1$, we have
\begin{equation}
\mathbb{P}(B_n^{\varepsilon, \delta}) \leq \mathcal{T}(K_+ \delta) / \varepsilon.
\end{equation}

We now provide

Proof of Proposition 3.6. From Lemmas 3.7 and 3.8, we take $K \triangleq K_\rightarrow \land K_+$, $\delta_0 \triangleq \delta_+$. Now, noting that $\mathcal{T}$ is strictly decreasing, we use (20) and (21) to get (19).

Finally, we have

Proof of Theorem 3.7. Fix $p \in [1, \infty)$ and let $\delta \triangleq \varepsilon^\nu$ where $\nu < \varsigma < 1$. By the Burkholder-Davis-Gundy inequalities [KS91] Theorem 3.3.28, there exists a universal positive constant $k_p/2$ such that $\mathbb{E}\sup_{0 \leq t \leq \tau_{\varepsilon}} |W_t|^p \leq k_p/2 \mathbb{E}|\varepsilon |^{3p/2}/2$. Noting that $\mathbb{E}[\mathbb{E}[1_{\Omega_\varepsilon}]^p] = \mathbb{E}(B_n^{\varepsilon, \delta})$, we see from Propositions 3.3 and 3.6 that for $\varepsilon \in (0, 1)$ small enough \footnote{One can check that $0 < \varepsilon < (\frac{4n}{\varepsilon})^{1/(\varepsilon-\nu)} \land \delta_0^{1/\varsigma}$ will suffice.}
\begin{align*}
\mathbb{E}[\mathbb{E}^{\varepsilon, \delta}] &\leq C_p \left\{ 3\mathcal{T}^{\varepsilon, \delta} + \mathcal{T}^{\varepsilon, \delta} K_+ \frac{1}{\varepsilon^{\frac{1}{\varsigma}}} \right\} + k_p/2 \mathbb{E}^{3p/2}/2(1-3\nu/2)p.
\end{align*}
Since $0 \leq \nu < 2/3$ and $\nu < \varsigma < 1$, straightforward calculations using (18) yield (12). \hfill \Box

4. PROOFS OF LEMMAS

Proof of Lemma 3.4. For $\omega \in \Omega \setminus G_{\text{T}_{\varepsilon}}^\varepsilon$, we have $\tau_\varepsilon(\lambda^\varepsilon(\omega)) = 0$, implying (16). So, fix $\omega \in G_{\text{T}_{\varepsilon}}^\varepsilon$. Note that the function $\lambda^\varepsilon(\omega)$ is piecewise-linear with “corners” at $0 = s_0 < t_1 < s_1 < \cdots < t_\varsigma < s_\varsigma = T_{\varepsilon}$. Since $\omega \in G_{\text{T}_{\varepsilon}}^\varepsilon$, we have $\tau_n^{\varepsilon}(\omega) \in [t_n - \delta, t_n + \delta]$ for $1 \leq n \leq T_{\varepsilon}$. Recalling that $\lambda_n^{\varepsilon}(\omega) = \tau_n^{\varepsilon}(\omega)$, it is now easy to see that
\begin{equation}
\max_{1 \leq n \leq T_{\varepsilon}} \left\{ \frac{|\lambda_n^{\varepsilon}(\omega) - \lambda_{n-1}^{\varepsilon}(\omega)|}{t_n - s_{n-1}} - 1 \bigg| + \frac{|\lambda_n^{\varepsilon}(\omega) - \lambda_{n+1}^{\varepsilon}(\omega)|}{s_n - t_n} - 1 \right\} \leq \frac{\delta}{\text{t}_{\text{min}}}.\end{equation}
Now, let $0 \leq s < t \leq T_{\varepsilon}$ and let $\{u_0, u_1, \ldots, u_k\}$ be a sequential enumeration of all corners starting just to the left of $s$ and ending just to the right of $t$, i.e., $u_0 \leq s < u_1 < \cdots < u_{k-1} < t \leq u_k$. By the triangle inequality, we have
\begin{align*}
|\lambda_n^\varepsilon(\omega) - \lambda_n^\varepsilon(\omega) - (t-s)| &\leq |t - u_{k-1}| \frac{|\lambda_n^\varepsilon(\omega) - \lambda_{n-1}^\varepsilon(\omega)|}{t - u_{k-1}} - 1 \\
&+ \sum_{i=2}^{k-1} |u_{i-1} - u_i| \frac{|\lambda_n^\varepsilon(\omega) - \lambda_{n-1}^\varepsilon(\omega)|}{u_i - u_{i-1}} - 1 + |u_1 - s| \frac{|\lambda_n^\varepsilon(\omega) - \lambda_n^\varepsilon(\omega)|}{u_1 - s} - 1.
\end{align*}
Noting that \( |t - u_{k-1}|, |u_{k-1} - u_{k-2}|, \ldots, |u_1 - s| \) are less than \( |t - s| \), recalling the piecewise-linear nature of \( \lambda_t^\xi(\omega) \), and using (22), we get

\[
\left| \frac{\lambda_t^\xi(\omega) - \lambda_t^\delta(\omega)}{t - s} - 1 \right| \leq \sum_{i=1}^k \left| \frac{\lambda_{u_i}(\omega) - \lambda_{u_{i-1}}^\delta(\omega)}{u_i - u_{i-1}} - 1 \right| \leq \frac{2T \varepsilon \delta}{t_{\min}}.
\]

Thus,

\[
\log \left( 1 - \frac{2T \varepsilon \delta}{t_{\min}} \right) \leq \log \left( \frac{\lambda_t^\xi(\omega) - \lambda_t^\delta(\omega)}{t - s} \right) \leq \log \left( 1 + \frac{2T \varepsilon \delta}{t_{\min}} \right).
\]

Using the estimate \(|\log(1 \pm x)| \leq 2|x|\) for \( |x| \leq 1/2 \), we get that for \( 0 < \delta \leq t_{\min}/4T \varepsilon \), we have

\[
-\frac{4T \varepsilon \delta}{t_{\min}} \leq \log \left( \frac{\lambda_t^\xi(\omega) - \lambda_t^\delta(\omega)}{t - s} \right) \leq \frac{4T \varepsilon \delta}{t_{\min}}.
\]

Since \( s, t \) are arbitrary, (16) follows.

**Proof of Lemma 3.2.** We first bound \( \sup_{0 \leq t \leq T \varepsilon} |X_{t,}^\xi - x(t)|^p \). Write \( X_{t,}^\xi - x(t) = \sum_{n=1}^N L_t^{\xi n} \) where

\[
L_t^{\xi n} = \sum_{n \geq 1} 1_{[\sigma_{n-1}^\xi, \sigma_n^\xi)} (\lambda_t^\xi) : \left\{ \begin{array}{l}
\frac{\beta}{\alpha_{on}} + \left( \lambda_{n-1, \xi, off} - \frac{\beta}{\alpha_{on}} \right) e^{-\alpha_{on}(t - s_{n-1})} \\
- \sum_{n \geq 1} 1_{[s_{n-1}, t_n)} (t) : \left\{ \begin{array}{l}
\frac{\beta}{\alpha_{on}} + \left( \lambda_{n-1, off} - \frac{\beta}{\alpha_{on}} \right) e^{-\alpha_{on}(t - s_{n-1})} \\
- \sum_{n \geq 1} 1_{[t_n, s_n)} (t) : x_{ref} e^{-\alpha_{off}(t - t_n)} \end{array} \right. \end{array} \right.
\]

We start by noting that \( L_t^{\xi n} \) are bounded for all \( t, \omega \in [0, T \varepsilon] \times \Omega \). We will now show that for \( \omega \in G_{T \varepsilon}^{\xi, \delta} \), \( L_t^{\xi n} \) are in fact of order \( \delta \). We note that if \( \omega \in G_{T \varepsilon}^{\xi, \delta} \) then \( \lambda_t^\xi(\omega) \in [\sigma_{n-1}^\xi(\omega), \sigma_n^\xi(\omega)] \) if \( t \in [s_{n-1}, t_n) \) for all \( 1 \leq n \leq N \). Thus, we have for each \( t \in [0, T \varepsilon] \),

\[
1_{G_{T \varepsilon}^{\xi, \delta}} \cdot L_t^{\xi n} = 1_{G_{T \varepsilon}^{\xi, \delta}} \sum_{n \geq 1} 1_{[s_{n-1}, t_n)} (t) : \left\{ \begin{array}{l}
\left( \lambda_{n-1, \xi, off} - \frac{\beta}{\alpha_{on}} \right) e^{-\alpha_{on}(t - s_{n-1})} \\
- \left( \lambda_{n-1, off} - \frac{\beta}{\alpha_{on}} \right) e^{-\alpha_{on}(t - s_{n-1})} \end{array} \right.
\]

It is now easy to see that

\[
1_{G_{T \varepsilon}^{\xi, \delta}} \cdot L_t^{\xi n} \leq 1_{G_{T \varepsilon}^{\xi, \delta}} \sum_{n \geq 1} 1_{[s_{n-1}, t_n)} (t) : \left\{ \begin{array}{l}
x_{ref} \alpha_{off} \delta + \left( \frac{\beta}{\alpha_{on}} - x^* \right) \alpha_{on}\delta \end{array} \right.
\]

Hence, there exists \( K_1 > 0 \) such that for all \( t \in [0, T \varepsilon] \),

\[
|L_t^{\xi n}(\omega)| \leq 1_{G_{T \varepsilon}^{\xi, \delta}(\omega)} K_1 \delta + 1_{\Omega \setminus G_{T \varepsilon}^{\xi, \delta}(\omega)} K_1.
\]
Turning to $L^2_{\epsilon,\delta}$, we note that

$$1_{G^{\epsilon,\delta}_{\epsilon}} \cdot L^2_{\epsilon} = 1_{G^{\epsilon,\delta}_{\epsilon}} \cdot \sum_{n \geq 1} 1_{[t_n,s_n)}(t) \cdot x_{\text{ref}} \left\{ e^{-\alpha_{\text{off}}(\lambda^\epsilon_t - \tau^\epsilon_n)} - e^{-\alpha_{\text{off}}(t-t_n)} \right\},$$

which gives

$$\left| 1_{G^{\epsilon,\delta}_{\epsilon}} \cdot L^2_{\epsilon} \right| \leq 1_{G^{\epsilon,\delta}_{\epsilon}} \cdot \sum_{n \geq 1} 1_{[t_n,s_n)}(t) \cdot x_{\text{ref}} \alpha_{\text{off}} |\lambda^\epsilon_t - \tau^\epsilon_n - t + t_n| \leq 2x_{\text{ref}} \alpha_{\text{off}} \delta.$$

Hence, there exists $K_2 > 0$ such that for all $t \in [0,T_{\epsilon}]$,

$$\left| L^2_{\epsilon}(\omega) \right| \leq 1_{G^{\epsilon,\delta}_{\epsilon}}(\omega)K_2 \delta + 1_{\Omega \setminus G^{\epsilon,\delta}_{\epsilon}}(\omega)K_2. \quad (24)$$

Turning now to $L^3_{\epsilon}$, we use integration by parts to get $I_t = e^{\alpha_{\text{on}}s} W_t - \alpha_{\text{on}} \int_0^t W_s e^{\alpha_{\text{on}}s} ds$. It now follows that for any $t \in [0,T_{\epsilon}]$,

$$L^3_{\epsilon} = \epsilon \sum_{n \geq 1} 1_{[\tau^\epsilon_n,s^\epsilon_n)}(\lambda^\epsilon_t) \cdot \left[ W_{\lambda^\epsilon_t} - e^{-\alpha_{\text{on}}[\lambda^\epsilon_t - \lambda^\epsilon_{\tau^\epsilon_n} \cdot \sigma^\epsilon_{n-1}] - 1} - \alpha_{\text{on}}e^{-\alpha_{\text{on}}\lambda^\epsilon_t} \int_{\lambda^\epsilon_t \cdot \sigma^\epsilon_{n-1}} W_s e^{\alpha_{\text{on}}s} ds \right],$$

whence $|L^3_{\epsilon}| \leq \epsilon (2 + \alpha_{\text{on}}) T_{\epsilon} \sup_{0 \leq t \leq T_{\epsilon}} |W_t|$. Recalling $(23)$ and $(24)$, we see that for $p > 0$, there exists $c_p > 0$ such that the first line of $(17)$ holds.

To bound $\sup_{0 \leq t \leq T_{\epsilon}} |Y^\epsilon_{\lambda^\epsilon_t}(\tau - t) - \alpha_{\text{on}}|$, note that for $\omega \in G^{\epsilon,\delta}_{\epsilon}$, we have $Y^\epsilon_{\lambda^\epsilon_t} = \alpha_{\text{on}}(t)$ for all $t \in [0,T_{\epsilon}]$. Consequently, $\sup_{0 \leq t \leq T_{\epsilon}} |Y^\epsilon_{\lambda^\epsilon_t} - \alpha_{\text{on}}| \leq 1_{\Omega \setminus G^{\epsilon,\delta}_{\epsilon}} \cdot \sup_{0 \leq t \leq T_{\epsilon}} |Y^\epsilon_{\lambda^\epsilon_t} - \alpha_{\text{on}}| \leq 1_{\Omega \setminus G^{\epsilon,\delta}_{\epsilon}}. \quad \square$

To state and prove Lemmas 3.7 and 3.8, we will need some notation. For $n \in \mathbb{N}$, $\xi \in (0,x_{\text{ref}})$, we let $a_n(t;\xi) \triangleq 1_{[n-1,\infty)}(t) \cdot \{ \beta/\alpha_{\text{on}} + (\xi - \beta/\alpha_{\text{on}}) e^{-\alpha_{\text{on}}(t-(n-1))} \}$. It is now easily checked that for $t \geq n-1$ and $\alpha \in (0,1)$ small enough,

$$a_n(t;\alpha) - \alpha \leq a_n(t;\alpha) \leq a_n(t;\alpha) \leq a_n(t;\alpha) \leq a_n(t;\alpha) + \alpha.$$

We will also find it helpful to express the continuous square-integrable martingale $I_t = \int_0^t e^{\alpha_{\text{on}}u} dW_u$ as a time-changed Brownian motion. The quadratic variation process of $I_t$ given by $\langle I \rangle_t = \int_0^t e^{2\alpha_{\text{on}}u} du = (e^{2\alpha_{\text{on}}t} - 1)/(2\alpha_{\text{on}})$ for $t \geq 0$ is strictly increasing with $\lim_{t \to \infty} \langle I \rangle_t = \infty$. It therefore follows from Theorem 3.4.6] that the process $V = \{V_s, \mathcal{G}_s : 0 \leq s < \infty \}$ defined by $V_s \triangleq I_{\langle I \rangle_s}$, $\mathcal{G}_s \triangleq \mathcal{F}_{T(s)}$ where $T(s) \triangleq \inf\{ t \geq 0 : \langle I \rangle_t > s \} = [\log (1 + 2\alpha_{\text{on}}s)]/(2\alpha_{\text{on}})$, is a standard one-dimensional Brownian motion, and furthermore, that $I_t = V_{\langle I \rangle_t}$ for all $t \geq 0$.

Below, we will use the fact that if $\omega \in G^{\epsilon,\delta}_{\epsilon}$ (where $n \in \mathbb{N}$), then $\sigma^\epsilon_{n-1} = n-1$, and $|X_{\sigma^\epsilon_{n-1}}(\omega) - x^*| \leq x_{\text{ref}} \alpha_{\text{off}} |\sigma^\epsilon_{n-1}(\omega) - t_{n-1}| \leq x_{\text{ref}} \alpha_{\text{off}} \delta$. Set

$$\mu \triangleq \beta - (\alpha_{\text{on}} + \alpha_{\text{off}}) x_{\text{ref}}, \quad \varpi \triangleq x_{\text{ref}} \alpha_{\text{off}} \delta.$$

Note that, on account of the upper bound on $\alpha_{\text{off}}$ in $(9)$, we have $\mu > 0$.

**Proof of Lemma 3.7.** We start by noting that for $n \geq 1$,

$$X_{\sigma^\epsilon_{n-1}}(\omega) = 1_{G^{\epsilon,\delta}_{\epsilon}}(\omega) \cdot 1_{[n-1,\infty)}(t) \cdot \left\{ a_n(t; X_{\sigma^\epsilon_{n-1}}(\omega)) + \varepsilon e^{-\alpha_{\text{on}}(t-I_{n-1})} \right\}.$$
Using the fact that for $\omega \in G_{n-1}^{e,\delta}$, we have $X^{n-1,e,\text{off}}(\omega) \in [x^* - \varkappa, x^* + \varkappa]$, together with (25), we get

$$B_n^{e,\delta,-} \subset \left\{ \omega \in G_{n-1}^{e,\delta}: \sup_{t \in [n-1, t_{n-\delta}]} \left( a_n \left( t; X^{n-1,e,\text{off}}_{\sigma_{n-1}^{e}} \right) + \varepsilon \varepsilon e^{-\alpha_{\text{on}} t} (I_t - I_{n-1}) \right) \geq x_{\text{ref}} \right\}$$

$$\subset \left\{ \omega \in G_{n-1}^{e,\delta}: \sup_{t \in [n-1, t_{n-\delta}]} e^{-\alpha_{\text{on}} t} (I_t - I_{n-1}) \geq \frac{x_{\text{ref}} - a_n(t_{n-\delta}; x^*) - \varkappa}{\varepsilon} \right\}$$

where the latter set inclusion uses the fact that $x_{\text{ref}} - a_n(t_{n-\delta}; x^*) - \varkappa \geq \delta \mu$. We now easily get that

$$\mathbb{P} \left( B_n^{e,\delta,-} \right) \leq \mathbb{P} \left\{ \sup_{t \in [n-1, t_{n-\delta}]} (\hat{V}(t_{n-1}) - V(t_{n-1})) \geq \frac{\mu \delta e^{\alpha_{\text{on}} n(n-1)}}{\varepsilon} \right\},$$

Letting $u_n \triangleq \langle I \rangle_{n-1}$, $q \triangleq \langle I \rangle_t - u_n$, $v_n \triangleq \langle I \rangle_{t_{n-\delta}}$, and noting that $\hat{V}_q \triangleq V_{u_n+q} - V_{u_n}$ is a Brownian motion, we get

$$\mathbb{P} \left( B_n^{e,\delta,-} \right) \leq \mathbb{P} \left\{ \sup_{q \in [0, v_n - u_n]} \hat{V}_q \geq \frac{\mu \delta e^{\alpha_{\text{on}} (n-1)}}{\varepsilon} \right\} = 2 \mathcal{F} \left( \frac{\mu \delta}{\varepsilon \sqrt{e^{2\alpha_{\text{on}} (t-\delta)} - 1}} \right),$$

where we have explicitly computed $u_n$, $v_n$, and also used [KS91] Remark 2.8.3. Since $e^{2\alpha_{\text{on}} (t-\delta)} - 1 \leq e^{2\alpha_{\text{on}} t^*}$, we easily get (20) with $K_+ \triangleq \sqrt{2\alpha_{\text{on}} e^{-\alpha_{\text{on}} t^*}} \mu$. \hfill \Box

**Proof of Lemma 3.8.** Using the fact that for $\omega \in G_{n-1}^{e,\delta}$, we have $X^{n-1,e,\text{off}}(\omega) \in [x^* - \varkappa, x^* + \varkappa]$, together with (25), we get

$$B_n^{e,\delta,+} \subset \left\{ \omega \in G_{n-1}^{e,\delta}: \sup_{t \in [n-1, t_{n+\delta}]} \left( a_n \left( t; X^{n-1,e,\text{off}}_{\sigma_{n-1}^{e}} \right) + \varepsilon \varepsilon e^{-\alpha_{\text{on}} t} (I_t - I_{n-1}) \right) < x_{\text{ref}} \right\}$$

$$\subset \left\{ \omega \in G_{n-1}^{e,\delta}: a_n(t_{n+\delta}; x^*) - \varkappa + \varepsilon e^{-\alpha_{\text{on}} (t_{n+\delta})} (I_{n+\delta} - I_{n-1}) \geq x_{\text{ref}} \right\}$$

Recalling Assumption 2.1 a bit of computation reveals that if we let

$$\delta_+ \triangleq \frac{1}{\alpha_{\text{on}}} \log \left[ \frac{2\beta - 2\alpha_{\text{on}}x_{\text{ref}}}{\beta - \alpha_{\text{on}}x_{\text{ref}} + \alpha_{\text{off}}x_{\text{ref}}} \right] > 0,$$

then, for $0 < \delta < \delta_+$, we have

$$x_{\text{ref}} + \varkappa - a_n(t_{n+\delta}; x^*) \varepsilon < - \left( \frac{\mu}{2} \right) \frac{\delta}{\varepsilon} < 0$$

Since $e^{-\alpha_{\text{on}} (t_{n+\delta})} (I_{n+\delta} - I_{n-1}) \sim \mathcal{N} \left( 0, \frac{\varepsilon e^{-2\alpha_{\text{on}} (t-\delta)}}{2\alpha_{\text{on}}} \right)$, a straightforward calculation yields that for $0 < \delta < \delta_+$, (21) holds with $K_+ \triangleq \mu \sqrt{\alpha_{\text{on}}/2}$. \hfill \Box
REFERENCES

[BC] Soumitro Banerjee and Krishnendu Chakrabarty. Nonlinear modeling and bifurcations in the boost converter. *Nonlinear modeling and bifurcations in the boost converter*, vol. 13, no. 2, pp. 252–260, March 1998.

[BKYY] Soumitro Banerjee, M. S. Karthik, Guohui Yuan, and James A. Yorke. Bifurcations in one-dimensional piecewise smooth maps—Theory and applications in switching circuits. *IEEE Transactions on Circuits and Systems—I: Fundamental Theory and Applications*, vol. 47, no. 3, pp. 389–394, March 2000.

[BV01] Soumitro Banerjee and George C. Verghese (editors). *Nonlinear Phenomena in Power Electronics*, Wiley, 2001.

[BBG99] Gopal K. Basak, Arnab Bisi, and Mrinal K. Ghosh. Stability of degenerate diffusions with state-dependent switching. *Journal Math. Anal. Appl.*, 240, pp. 219–248, 1999.

[Bil] Patrick Billingsley. *Convergence of probability measures*, second edition, John Wiley & Sons Inc., 1999.

[CL07] Debasis Chatterjee and Daniel Liberzon. On stability of randomly switched nonlinear systems. *IEEE Transactions on Automatic Control*, vol. 52, no. 12, pp. 2390–2394, 2007.

[CL11] Debasis Chatterjee and Daniel Liberzon. Stabilizing randomly switched systems. *SIAM Journal on Control and Optimization*, vol. 49, no. 5, pp. 2008–2031, 2011.

[dBBCK] Mario di Bernardo, Christopher J. Budd, Alan R. Champneys and Piotr Kowalczyk. *Piecewise-smooth dynamical systems. Theory and applications*. Springer 2008.

[dBGGV] Mario di Bernardo, Franco Garofalo, Luigi Glielmo, and Francesco Vasca. Switchings, bifurcations and chaos in DC/DC converters. *IEEE Transactions on Circuits and Systems—I: Fundamental Theory and Applications*, vol. 45, no. 2, pp. 133–141, February 1998.

[DZ98] Amir Dembo and Ofer Zeitouni. *Large Deviations Techniques and Applications*, second edition, Springer 1998.

[EK86] Stewart N. Ethier and Thomas G. Kurtz. *Markov Processes: Characterization and Convergence*. John Wiley & Sons Inc., New York, 1986.

[FO96] Enric Fossas and Gerard Olivar. Study of chaos in the buck converter, *IEEE Transactions on Circuits and Systems—I: Fundamental Theory and Applications*, vol. 43, no. 1, pp. 13–25, January 1996.

[FW12] Mark I. Freidlin and Alexander D. Wentzell. *Random Perturbations of Dynamical Systems*, Third Edition, Springer, 2012.

[HBB1] Martin Hasler, Vladimir Belykh and Igor Belykh. Dynamics of stochastically blinking systems. Part I: Finite time properties. *SIAM Journal on Applied Dynamical Systems*, vol. 12, no. 2, pp. 1007–1030, 2013.

[HBB2] Martin Hasler, Vladimir Belykh and Igor Belykh. Dynamics of stochastically blinking systems. Part II: Asymptotic properties. *SIAM Journal on Applied Dynamical Systems*, vol. 12, no. 2, pp. 1031–1084, 2013.

[HDJ92] D. C. Hamill, J. H. B. Deane, and D. J. Jeffries. Modeling of chaotic DC-DC converters by iterated nonlinear mappings. *IEEE Trans. Power Electron.*, vol. 7, pp. 25–36, January 1992.

[KS91] Ioannis Karatzas and Steven Shreve. *Brownian Motion and Stochastic Calculus*, Second Edition, Springer-Verlag, 1991.

[LM07] Qi Luo and Xuerong Mao. Stochastic population dynamics under regime switching. *Journal of Mathematical Analysis and Applications*, vol. 334, pp. 69–84, 2007.

[Nor91] A. B. Nordmark. Non-periodic motion caused by grazing incidence in an impact oscillator. *J. Sound and Vibration*, vol. 145, no. 2, pp. 279–297, 1991.

[PS08] Grigorios A. Pavliotis and Andrew M. Stuart. *Multiscale Methods. Averaging and Homogenization*. Springer, 2008.

[SH83] S. W. Shaw and P. J. Holmes. A periodically forced piecewise linear oscillator. *J. Sound and Vibration*, 90:129–144, 1983.

[SK1] D. J. W. Simpson and R. Kuske. Stochastically perturbed sliding motion in piecewise-smooth systems. *Discrete Cont. Dyn. Syst. Ser. B*, 19(9), pp. 2889–2913, 2014.

[SK2] D. J. W. Simpson and R. Kuske. The positive occupation time of Brownian motion with two-valued drift and asymptotic dynamics of sliding motion with noise. *Stoch. Dyn.*, 14(4):1450010, 2014.

[SK3] D. J. W. Simpson and R. Kuske. Stochastic perturbations of periodic orbits with sliding. *J. Nonlin. Sci.* (to appear).

[YZ1] G. Yin and C. Zhu. Properties of solutions of stochastic differential equations with continuous-state-dependent switching. *Journal of Differential Equations*, 249, pp. 2409–2439, 2010.

[YZ2] G. Yin and C. Zhu. *Hybrid switching diffusions. Properties and applications*. Springer, 2010.