Failures of model-dependent generalization bounds for least-norm interpolation

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Abstract

We consider bounds on the generalization performance of the least-norm linear regressor, in the over-parameterized regime where it can interpolate the data. We describe a sense in which any generalization bound of a type that is commonly proved in statistical learning theory must sometimes be very loose when applied to analyze the least-norm interpolant. In particular, for a variety of natural joint distributions on training examples, any valid generalization bound that depends only on the output of the learning algorithm, the number of training examples, and the confidence parameter, and that satisfies a mild condition (substantially weaker than monotonicity in sample size), must sometimes be very loose—it can be bounded below by a constant when the true excess risk goes to zero.

1 Introduction

Deep learning methodology has revealed some striking deficiencies of classical statistical learning theory: large neural networks, trained to zero empirical risk on noisy training data, have good predictive accuracy on independent test data. These methods are overfitting (that is, fitting to the training data better than the noise should allow), but the overfitting is benign (that is, prediction performance is good). It is an important open problem to understand why this is possible.

The presence of noise is key to why the success of interpolating algorithms is mysterious. Generalization of algorithms that produce a perfect fit in the absence of noise has been studied for decades (see [Haussler, 1992] and its references). A number of recent papers have provided generalization bounds for interpolating algorithms in the absence of noise, either for deep networks or in abstract frameworks motivated by deep networks [Li and Liang, 2018, Arora et al., 2019, Cao and Gu, 2019, Feldman, 2020]. The generalization bounds in these papers either do not hold or become vacuous in the presence of noise: Assumption A1 in [Li and Liang, 2018] rules out noisy data; the data-dependent bound in Arora et al. [2019, Theorem 5.1] becomes vacuous when independent noise is added to the \( y_i \); adding a constant level of independent noise to the \( y_i \) in [Cao and Gu, 2019, Theorem 3.3] gives an upper bound on excess risk that is at least a constant; and the analysis in [Feldman, 2020] concerns the noise-free case.

There has also been progress on bounding the gap between average loss on the training set and expected loss on independent test data, based on uniform convergence arguments that bound the complexity of classes of real-valued functions computed by deep networks. For instance, the results in [Bartlett, 1998] for sigmoid nonlinearities rely on \( \ell_1 \)-norm bounds on parameters throughout the network, and those in [Bartlett et al., 2017] for ReLUs rely on spectral norm bounds of the weight matrices throughout the network (see also the refinements in [Bartlett and Mendelson, 2002, Neyshabur et al., 2015, Bartlett et al., 2017, Golowich et al.].
These bounds involve distribution-dependent function classes, since they depend on some measure of the complexity of the output model that may be expected to be small for natural training data. For instance, if some training method gives weight matrices that all have small spectral norm, the bound in [Bartlett et al., 2017] will imply that the gap between empirical risk and predictive accuracy will be small. But while it is possible for these bounds to be small for networks that trade off fit to the data with complexity in some way, it is not clear that a network that interpolates noisy data could ever have small values of these complexity measures. This raises the question: are there any good data-dependent bounds for interpolating networks?

Zhang et al. [2017] claimed, based on empirical evidence, that conventional learning theoretic tools are useless for deep networks, but they considered the case of a fixed class of functions defined, for example, as the set of functions that can be computed by a neural network, or that can be reached by stochastic gradient descent with some training data, no matter how unlikely. These observations illustrate the need to consider distribution-dependent notions of complexity in understanding the generalization performance of deep networks. The study of such distribution-dependent complexity notions has a long history in nonparametric statistics, where it is central to the problem of model selection [see, for example, Bartlett et al., 2002, and its references]; uniform convergence analysis over a level in a complexity hierarchy is part of a standard outline for analyzing model selection methods.

Nagarajan and Kolter [2019] provided an example of a scenario where, with high probability, an algorithm generalizes well, but two-sided uniform convergence fails for any hypothesis space that is likely to contain the algorithm’s output. Their analysis takes an important step in allowing distribution-dependent notions of complexity, but only rules out the application of a specific set of tools: uniform convergence over a model class of the absolute differences between expectations and sample averages. Indeed, in their proof, the failure is an under-estimation of the accuracy of a model—a model has good predictive accuracy, but performs poorly on a sample (one obtained as a transformed but equally likely version of the sample that was used to train the model). However, in applying uniform convergence tools to show good performance of an algorithm, uniform bounds are only needed to show that bad models are unlikely to perform well on the training data. So if one wishes to prove bounds that guarantee that an algorithm has poor predictive accuracy, Nagarajan and Kolter [2019] provided an example where uniform convergence tools will not suffice. In contrast, we are concerned with understanding what tools can provide guarantees of good predictive accuracy of interpolating algorithms.

In this paper, motivated by the phenomenon of benign overfitting in deep networks, we consider a simpler setting where the phenomenon occurs, that of linear regression. (Lemma 5.2 of [Negrea et al., 2020] adapts the construction of [Nagarajan and Kolter, 2019] to show a similar failure of uniform convergence in this context, and similarly cannot shed light on tools that can or cannot guarantee good predictive accuracy.) We study the minimum norm linear interpolant. Earlier work [Bartlett et al., 2020] provides tight upper and lower bounds on the excess risk of this interpolating prediction rule under suitable conditions on the probability distribution generating the data, showing that benign overfitting depends on the pattern of eigenvalues of the population covariance (and there is already a rich literature on related questions [Liang and Rakhlin, 2020, Belkin et al., 2019b, a, Hastie et al., 2019a, b, Negrea et al., 2020, Dereziński et al., 2019, Li et al., 2020, Tsigler and Bartlett, 2020]). These risk bounds involve fine-grained properties of the distribution. Is this knowledge necessary? Is it instead possible to obtain data-dependent bounds for interpolating prediction rules? Already the proof in [Bartlett et al., 2020] provides some clues that this might be difficult: when benign overfitting occurs, the eigenvalues of the empirical covariance are a very poor estimate of the true covariance eigenvalues—all but a small fraction (the largest eigenvalues) are within a constant factor of each other.
In this paper, we show that in these settings there cannot be good risk bounds based on data-dependent function classes in a strong sense: For linear regression with the minimum norm prediction rule, any “bounded-antimonotonic” model-dependent error bound that is valid for a sufficiently broad set of probability distributions must be loose—too large by an additive constant—for some (rather innocuous) probability distribution. The bounded-antimonotonic condition formalizes the mild requirement that the bound does not degrade very rapidly with additional data. Aside from this constraint, our result applies for any bound that is determined as a function of the output of the learning algorithm, the number of training examples, and the confidence parameter. This function could depend on the level in a hierarchy of models where the output of the algorithm lies. Our result applies whether the bound is obtained by uniform convergence over a level in the hierarchy, or in some other way.

The intuition behind our result is that benign overfitting can only occur when the test distribution has a vanishing overlap with the training data. Indeed, interpolating the data in the training sample guarantees that the conditional expectation of the prediction rule’s loss on the training points that occur once must be at least the noise level. Using a Poissonization method, we show that a situation where the training sample forms a significant fraction of the support of the distribution is essentially indistinguishable from a benign overfitting situation where the training sample has measure zero. Since we want a data-dependent bound to be valid in both cases, it must be loose in the second case.

2 Preliminaries and main results

We consider prediction problems with patterns \(x \in \ell_2\) and labels \(y \in \mathbb{R}\), where \(\ell_2\) is the space of square summable sequences of real numbers. In fact, all probability distributions that we consider have support restricted to a finite dimensional subspace of \(\ell_2\), which we identify with \(\mathbb{R}^d\) for an appropriate \(d\). For a joint distribution \(P\) over \(\mathbb{R}^d \times \mathbb{R}\) and a hypothesis \(h : \mathbb{R}^d \to \mathbb{R}\), define the risk of \(h\) to be

\[
R_P(h) = \mathbb{E}_{(x,y) \sim P}[(y - h(x))^2].
\]

Let \(R_P^*\) be the minimum of \(R_P\) over measurable functions.

For any positive integer \(k\), a distribution \(D\) over \(\mathbb{R}^k\) is sub-Gaussian with parameter \(\sigma\) if, for any \(u \in \mathbb{R}^k\),

\[
\mathbb{E}_{x \sim D}[\exp(u \cdot (x - \mathbb{E}x))] \leq \exp\left(\frac{\|u\|^2 \sigma^2}{2}\right).
\]

A joint distribution \(P\) over \(\mathbb{R}^d \times \mathbb{R}\) has unit scale if \((X_1, ..., X_d, Y) \sim P\) is sub-Gaussian with parameter 1. It is innocuous if

- it is unit scale,
- the marginal on \((X_1, ..., X_d)\) is Gaussian, and
- the conditional of \(Y\) given \((X_1, ..., X_d)\) is continuous.

A sample is a finite multiset of elements of \(\mathbb{R}^d \times \mathbb{R}\). A least-norm interpolation algorithm takes as input a sample, and outputs \(\theta \in \mathbb{R}^d\) that minimizes \(\|\theta\|\) subject to

\[
\sum_i (\theta \cdot x_i - y_i)^2 = \min_{\hat{y}_1, ..., \hat{y}_n} \sum_i (\hat{y}_i - y_i)^2.
\]

We will refer both to the parameter vector \(\theta\) output by the least-norm interpolation algorithm and the function \(x \to \theta \cdot x\) parameterized by \(\theta\) as the least-norm interpolant.
A function $\epsilon(h, n, \delta)$ mapping a hypothesis $h$, a sample size $n$ and a confidence $\delta$ to a positive real number is a uniform model-dependent bound for unit-scale distributions if, for all unit-scale joint distributions $P$ and all sample sizes $n$, with probability at least $1 - \delta$ over the random choice of $S \sim P^n$, we have

$$R_P(h) - R_P^* \leq \epsilon(h, n, \delta).$$

The bound $\epsilon$ is $c$-bounded antimonotonic for $c \geq 1$ if for all $h$, $\delta$, $n_1$ and $n_2$, if $n_2/2 \leq n_1 \leq n_2$ then $\epsilon(h, n_2, \delta) \leq c\epsilon(h, n_1, \delta)$. This requires that the bound cannot get too much worse too quickly with more data. If $\epsilon(h, \cdot, \delta)$ is monotone-decreasing for all $h$ and $\delta$, then it is 1-bounded antimonotonic.

A set $B \subseteq \mathbb{N}$ is $\beta$-dense if $\liminf_{N \to \infty} \frac{|B \cap \{1, \ldots, N\}|}{N} \geq \beta$.

Say that $B \subseteq \mathbb{N}$ is strongly $\beta$-dense beyond $n_0$ if, for all $s \in \mathbb{N}$ such that $s^2 \geq n_0$,

$$\frac{|B \cap \{s^2, \ldots, (s + 1)^2 - 1\}|}{2s + 1} \geq \beta.$$ 

(Notice that if a set is strongly $\beta$-dense beyond $n_0$, then it is $\beta$-dense.)

The following is our main result.

**Theorem 1** If $\epsilon$ is a bounded-antimonotonic, uniform model-dependent bound for unit-scale distributions, then there are constants $c_0, c_1, c_2, c_3, c_4 > 0$ and innocuous distributions $P_1, P_2, \ldots$, such that for all $0 < \delta < c_1$, the least-norm interpolant $h$ satisfies, for all large enough $n$,

$$\Pr_{S \sim P^n} \left[ R_P(h) - R_P^* \leq c_0/\sqrt{n} \right] \geq 1 - \delta$$

but nonetheless, the set of $n$ such that

$$\Pr_{S \sim P^n} \left[ \epsilon(h, n, \delta) > c_2 \right] \geq \frac{1}{2}$$

is strongly $(1 - \frac{c_3}{\log(1/\delta)})$-dense beyond $c_4 \log(1/\delta)$.

### 3 Proof of Theorem 1

Our proof uses the following lemma [Birch, 1963] (see also [Feller, 1968, p. 216] and [Batu et al., 2000, 2013]), which has become known as the “Poissonization lemma”. We use $\text{Poi}(\lambda)$ to denote the Poisson distribution with mean $\lambda$: For $t \sim \text{Poi}(\lambda)$ and $k \geq 0$,

$$\Pr[t = k] = \frac{\lambda^k e^{-\lambda}}{k!}.$$ 

**Lemma 2** If, for $t \sim \text{Poi}(n)$, you throw $t$ balls independently uniformly at random into $m$ bins,

- the numbers of balls falling into the bins are mutually independent, and
- the number of balls falling in each bin is distributed as $\text{Poi}(n/m)$.
For each \( n \), our proof uses three distributions: \( D_n, Q_n \) and \( P_n \). The first, \( D_n \), is used to define \( Q_n \) and \( P_n \); it is chosen so that the least-norm interpolant performs well on \( D_n \). The distribution \( Q_n \) is defined so that the least-norm interpolant performs poorly on \( Q_n \). The distribution \( P_n \) is defined so that, when the least-norm interpolant performs well on \( D_n \), it also performs well on \( P_n \). Crucially, the least norm interpolants that arise from \( Q_n \) and \( P_n \) are closely related.

For each \( n \), the joint distribution \( D_n \) on \((x, y)\)-pairs is defined as follows. Let \( s = \lceil \sqrt{n} \rceil \), \( N = s^2 \), \( d = N^2 \). Let \( \theta^* \) be an arbitrary unit-length vector. Let \( \Sigma_n \) be an arbitrary covariance matrix with eigenvalues \( \lambda_1 = 1/81, \lambda_2 = \cdots = \lambda_d = 1/d^2 \). The marginal of \( D_n \) on \( x \) is then \( \mathcal{N}(0, \Sigma_n) \). For each \( x \in \mathbb{R}^d \), the distribution of \( y \) given \( x \) is \( \mathcal{N}(\theta^* \cdot x, 1/81) \). For \( d \geq 9 \), since \((x, y)\) is Gaussian, \( \|\Sigma_n\| \leq 1/81 \), and the variance of \( y \) is \( 1/81 \), each \( D_n \) is innocuous.

For an absolute constant positive integer \( b \), we get \( Q_n \) from \( D_n \) through the following steps.

1. Sample \((x_1, y_1), \ldots, (x_{bn}, y_{bn}) \sim D_n^{bn}\).
2. Define \( Q_n \) on \( \mathbb{R}^d \times \mathbb{R} \) so that its marginal on \( \mathbb{R}^d \) is uniform on \( U = \{x_1, \ldots, x_{bn}\} \) and its conditional distribution of \( Y | X \) is the same as \( D_n \).

**Definition 3** For a sample \( S \), the compression of \( S \), denoted by \( C(S) \), is defined to be

\[
C(S) = ((u_1, v_1), \ldots, (u_k, v_k)),
\]

where \( u_1, \ldots, u_k \) are the unique elements of \( \{x_1, \ldots, x_n\} \), and, for each \( i \), \( v_i \) is the average of \( \{y_j : 1 \leq j \leq n, x_j = u_i\} \).

For the least-norm interpolation algorithm \( A \), for any pair \( S \) and \( S' \) of samples such that \( C(S) = C(S') \), we have \( A(S) = A(S') \). (This is true because the least-norm interpolant \( A(S) \) is uniquely defined by the equality constraints specified by the compression \( C(S) \).)

So that a generalization bound often must apply to \( Q_n \), we need to show that it is likely to be unit scale. The proof of this lemma is in Appendix A.

**Lemma 4** There is a positive constant \( c_5 \) such that, for all large enough \( n \), with probability \( 1 - \frac{c_5}{n} \), \( Q_n \) has unit scale.

We can show that the least-norm interpolant is bad for \( Q_n \) by only considering the points in the support of \( Q_n \) that the algorithm sees exactly once.

**Lemma 5** For any constant \( c > 0 \), there are constants \( c_6, c_7 > 0 \) such that, for all sufficiently large \( n \), almost surely for \( Q_n \) chosen randomly as described above, if \( t \) is chosen randomly according to \( \text{Poi}(cn) \) and \( S \) consists of \( t \) random draws from \( Q_n \), then with probability at least \( 1 - e^{-c_6n} \) over \( t \) and \( S \),

\[
E_{(x, y) \sim Q_n} [(A(C(S))(x) - y)^2] - E_{(x, y) \sim Q_n} [(f^*(X) - Y)^2] \geq c_7
\]

where \( f^* \) is the regression function for \( D_n \) (and hence also for \( Q_n \)).

**Proof:** Recall that \( U = \{x_1, \ldots, x_{bn}\} \) is the support of the marginal of \( Q_n \) on the independent variables. With probability 1, \( U \) has cardinality \( bn \). Define \( h = A(C(S)) \). If some \( x \in U \) appears exactly once in \( S \), then \( h(x) \) is a sample from the distribution of \( y \) given \( x \) under \( D_n \). Thus, for such an \( x \), the expected quadratic loss of \( h(x) \) on a test point is the squared difference between two independent samples from this distribution, which is twice its variance, i.e. twice the expected loss of \( f^* \), which is \( 2 \times 1/2 = 1 \). On any
\(x \in U\), whether or not it was seen exactly once in \(S\), by definition, \(f^*(x)\) minimizes the expected loss given \(x\).

Lemma 2 shows that, conditioned on the random choice of \(Q_n\), the numbers of times the various \(x\) in \(S\) are mutually independent and, the probability that \(x \in U\) is seen exactly once in \(S\) is \(\frac{c}{b} \exp\left(-\frac{c}{b}\right) \geq \frac{c - e^{-c}}{2}\). Applying a Chernoff bound (see, for example, Theorem 4.5 in Mitzenmacher and Upfal [2005]), the probability that fewer than \(ce - c_n/2\) members of \(U\) are seen exactly once in \(S\) is at most \(e^{-c_n}n\) for an absolute constant \(c_0\). Thus if \(U_1\) is the (random) subset of points in \(U\) that were seen exactly once, we have

\[
Q_n \left[ (h(X) - Y)^2 \right] - Q_n \left[ (f^*(X) - Y)^2 \right] \\
= \sum_{x \in U} Q_n \left[ ((h(X) - Y)^2 - (f^*(X) - Y)^2)1_{X=x} \right] \\
\geq \sum_{x \in U_1} \mathbb{E}[(f^*(X) - Y)^2]1_{X=x} \\
= \frac{|U_1|}{bn}.
\]

Since, with probability \(1 - e^{-c_n}\), \(|U_1| \geq c e^{-c}/2\), this completes the proof.

**Definition 6** Define \(P_n\) as follows.

1. Set the marginal distribution of \(P_n\) on \(\mathbb{R}^d\) the same as that of \(D_n\).

2. To generate \(Y\) given \(X = x\) for \((X,Y) \sim P_n\), first sample \(a\) random variable \(Z\) whose distribution is obtained by conditioning a draw from a Poisson with mean \(\frac{c}{b}\) on the event that it is at least 1, then sample \(Z\) values \(V_1, \ldots, V_Z\) from the conditional distribution \(D_n(Y|X = x)\), and set \(Y = \frac{1}{Z} \sum_{i=1}^{Z} V_j\).

Note that, since \(D_n\) has a density, \(x_1, \ldots, x_r\) are almost surely distinct and hence \(S\) drawn from \(P_n\) has \(C(S) = S\) a.s.

The following lemma implies that the bounds for \(P_n\) tend to be as big as those for \(Q_n\).

**Lemma 7** Define \(Q_n\) as above let \(Q_n\) be the resulting distribution over the random choice of \(Q_n\). Suppose \(P_n\) is defined as in Definition 6. Let \(c > 0\) be an arbitrary constant. Choose \(S\) randomly by choosing \(t \sim \text{Poi}(cn), Q_n \sim Q_n\), and \(S \sim Q_n^n\). Choose \(T\) by choosing \(r \sim B\left(bn, 1 - \exp\left(-\frac{c}{b}\right)\right)\) and \(T \sim P_n\). Then \(C(S)\) and \(T\) have the same distribution. In particular, for all \(\delta > 0\), for any function \(\psi\) of the least norm interpolant \(h\), a sample size \(r\), and a confidence parameter \(\delta\), we have

\[
\mathbb{E}_{\mathbf{r} \sim \text{Poi}(cn), Q_n \sim Q_n}[\mathbb{E}_{S \sim Q_n} [\psi(h(S), |C(S)|, \delta)]] = \mathbb{E}_{\mathbf{r} \sim B\left(bn, 1 - \exp\left(-\frac{c}{b}\right)\right)}[\mathbb{E}_{T \sim P_n} [\psi(h(T), r, \delta)]].
\]

**Proof:** Let \(C\) be the probability distribution over training sets obtained by picking \(Q_n\) from \(Q_n\), picking \(t\) from \(\text{Poi}(cn)\), picking \(S\) from \(Q_n^n\) and compressing it. Let \(C = C(S)\) be a random draw from \(C\). Let \(n_C\) be the number of examples in \(C\).

We claim that \(n_C\) is distributed as \(B\left(bn, 1 - \exp\left(-\frac{c}{b}\right)\right)\). Conditioned on \(Q_n\), and recalling that \(U\) is the support of \(Q_n\), for any \(x \in U\), Lemma 2 implies that for each \(x \in U\), the probability that \(x\) is not seen is the probability, under a Poisson with mean \(\frac{c}{b}\), of drawing a 0. Thus, the probability that \(x\) is seen is \(1 - \exp\left(-\frac{c}{b}\right)\). Since the numbers of times different \(x\) are seen in \(S\) are independent, the number seen is distributed as \(B\left(bn, 1 - \exp\left(-\frac{c}{b}\right)\right)\).
Now, for each \( x \in U \), the event that it is in \( C(S) \) is the same as the event that \( x \) appears at least once in \( S \). Thus, conditioned on the event that \( x \) appears in \( S \), the number of \( y \) values that are used to compute the \( y \) value in \( C(S) \) is distributed as a Poisson with mean \( \frac{\alpha}{7} \), conditioned on having a value at least 1.

Let \( D_{n,X} \) be the marginal distribution of \( D_n \) on the \( x \)'s. If we make \( n \) independent draws from \( D_{n,X} \), and then independently reject some of these examples, to get \( n_C \) draws, the resulting \( n_C \) examples are independent. (We could first randomly decide the number \( n_C \) of examples to keep, and then draw those independently from \( D_n \), and we would have the same distribution."

The last two paragraphs together, along with the definition of \( Q_n \), imply that the distribution over \( T \) obtained by sampling \( r \) from \( B(\beta n, 1 - e^{-c/b}) \) and \( T \) from \( P^c_n \) is the same as the distribution over \( C \) obtained by sampling \( Q_n \) from \( Q_n \), \( t \) from \( \text{Poi}(cn) \), then sampling \( S \) from \( Q^t_n \) and compressing it. Thus, the distributions of \( T \) and \( (C(S)) \) are the same, and hence the distributions of \( (h(T), |T|) \) and \( (h(S), |C(S)|) \) are the same, because \( h(S) = h(C(S)) \).

We will use the following bound on the Poisson distribution.

**Lemma 8** ([Canonne, 2017]) For any \( \lambda, \alpha > 0 \), \( \Pr_{r \sim \text{Poi}(\lambda)}(r \geq (1 + \alpha)\lambda) \leq \exp \left( -\frac{\alpha^2}{2(1+\alpha)} \right) \).

Armed with these tools, we now show that \( \epsilon \) must often have a large value.

**Lemma 9** Then there are positive constants \( c_1, c_2, c_3, c_4 \) such that, for all \( 0 < \delta < c_1 \), the set of \( n \) such that

\[
\Pr_{S \sim P_n}[\epsilon(h, n, \delta) > c_2] \geq \frac{1}{2}
\]

is strongly \( (1 - \frac{c_3}{\log(1/\delta)}) \)-dense beyond \( c_4 \log(1/\delta) \).

**Proof:** We will think of the natural numbers as being divided into bins \([1, 2), [2, 4), [4, 7), ... \) Let us focus our attention on one bin: \( \{s^2, ..., (s + 1)^2 - 1\} \). Let \( n \) denote the center of the bin, \( n = s^2 + s \), so that \( s \sim \sqrt{n} \).

For a constant \( c_8 > 0 \) and any \( \delta > 0 \), Lemma [7] implies

\[
\mathbb{E}_{r \sim B(\beta n, 1 - e^{-c/b})}[\Pr_{T \sim P^c_n}[\epsilon(h(T), r, \delta) \leq c_8]] = \mathbb{E}_{r \sim \text{Poi}(\beta cn), Q_n \sim Q_n}[\Pr_{S \sim Q^t_n}[\epsilon(h(S), |C(S)|, \delta) \leq c_8]]
\]

(1)

Suppose that \( \epsilon \) is \( B' \)-bounded-antimonotonic. Fix \( B > 0 \) such that \( B > B' \). Then

\[
\mathbb{E}_{r \sim \text{Poi}(\beta cn), Q_n \sim Q_n}[\Pr_{S \sim Q^t_n}[\epsilon(h, |C(S)|, \delta) \leq c_8]] = \mathbb{E}_{r \sim \text{Poi}(\beta cn), Q_n \sim Q_n}[\Pr_{S \sim Q^t_n}[B\epsilon(h, |C(S)|, \delta) \leq c_8 B]]
\]

\[
\leq \mathbb{E}_{r \sim \text{Poi}(\beta cn), Q_n \sim Q_n}[\Pr_{S \sim Q^t_n}[R_{Q_n}(h) - R^*_{Q_n} > B\epsilon(h, |C(S)|, \delta)]] + \mathbb{E}_{r \sim \text{Poi}(\beta cn), Q_n \sim Q_n}[\Pr_{S \sim Q^t_n}[R_{Q_n}(h) - R^*_{Q_n} \leq c_8 B]].
\]

(2)

For each sample size \( t \) and any \( Q_n \) that has unit scale

\[
\Pr_{S \sim Q^t_n}[R_{Q_n}(h) - R^*_{Q_n} > B\epsilon(h, |C(S)|, \delta)]
\]

\[
\leq \Pr_{S \sim Q^t_n}[R_{Q_n}(h) - R^*_{Q_n} > \epsilon(h, t, \delta)] + \Pr_{S \sim Q^t_n}[B\epsilon(h, |C(S)|, \delta) \leq \epsilon(h, t, \delta)]
\]

\[
\leq \delta + \Pr_{S \sim Q^t_n}[|C(S)| < t/2]
\]

where the second inequality follows from the fact that \( \epsilon \) is a valid \( B' \)-bounded-antimonotonic uniform model-dependent bound for unit-scale distributions and \( B > B' \). Combining this with Lemma [4] we have

\[
\mathbb{E}_{r \sim \text{Poi}(\beta cn), Q_n \sim Q_n}[\Pr_{S \sim Q^t_n}[R_{Q_n}(h) - R^*_{Q_n} > B\epsilon(h, |C(S)|, \delta)]] \leq \delta + \frac{c_5}{n} + \Pr_{S \sim Q^t_n}[|C(S)| < t/2].
\]
Now by a union bound, for some constant $c_9 > 0$,
\[
\mathbb{E}_{t \sim \text{Poi}(cn), Q_n \sim Q_n} [\Pr_{S \sim Q_n^c} |C(S)| < t/2] \\
\leq \mathbb{E}_{t \sim \text{Poi}(cn), Q_n \sim Q_n} [\Pr_{S \sim Q_n^c} |C(S)| < c_9 n] + \Pr_{t \sim \text{Poi}(cn)} [t/2 \geq c_9 n] \\
= \Pr_{Z \sim B(bn, 1 - e^{-c/b})} [Z < c_9 n] + \Pr_{t \sim \text{Poi}(cn)} [t/2 \geq c_9 n] \\
\leq \delta,
\]
where the last inequality follows from a Chernoff bound and from Lemma 8 with $n = \Omega(\log(1/\delta))$ and provided we can choose $c_9$ to satisfy $c/2 < c_9 < b(1 - e^{-c/b})$. Our choice of $b$ and $c$, specified below, will ensure this. In that case, we have that
\[
\mathbb{E}_{t \sim \text{Poi}(cn), Q_n \sim Q_n} [\Pr_{S \sim Q_n^c} \epsilon(h, |C(S)|, \delta) \leq c_8] \\
\leq 2\delta + \frac{c_5}{n} + \mathbb{E}_{t \sim \text{Poi}(cn), Q_n \sim Q_n} [\Pr_{S \sim Q_n^c} |R_{Q_n}(h) - R_{Q_n^c}(h)| \leq c_8 B].
\]
Applying Lemma 5 to bound the RHS, if $n$ is large enough and $c_8 B < c_7$, then
\[
\mathbb{E}_{t \sim \text{Poi}(cn), Q_n \sim Q_n} [\Pr_{S \sim Q_n^c} \epsilon(h, |C(S)|, \delta) \leq c_8] \leq 3\delta + \frac{c_5}{n}.
\]
Returning to (1), we get
\[
\mathbb{E}_{t \sim B(bn, 1 - e^{-c/b})} [\Pr_{T \sim P_n} \epsilon(h, r, \delta) \leq c_8] \leq 3\delta + \frac{c_5}{n}.
\]
Let us now focus on the case that $b = 2$ and $c = 2 \ln 2$, so that
\[
\mathbb{E}_{t \sim B(bn, 1 - e^{-c/b})} [r] = (1 - e^{-c/b})bn = n.
\]
(And note that $c/2 = \ln 2 < 1 = b(1 - e^{-c/b})$, as required for (3).) Chebyshev’s inequality implies
\[
\Pr_{t \sim B(bn, 1 - e^{-c/b})} [r \notin [n - s, n + s]] \leq c_{10}
\]
for an absolute positive constant $c_{10}$. Returning now to (4), Markov’s inequality implies
\[
\Pr_{r \sim B(bn, 1 - e^{-c/b})} [\Pr_{T \sim P_n} \epsilon(h, r, \delta) \leq c_8] > 1/2] \leq c_{11} \left( \delta + \frac{1}{n} \right).
\]
Further, it is known [Stud [1977], Box et al. [1978]] that there is an absolute constant $c_{12}$ such that, for all large enough $n$ and all $r_0 \in [n - s, n + s]$,
\[
\Pr_{r \sim B(bn, 1 - e^{-c/b})} [r = r_0] \geq \frac{c_{12}}{\sqrt{n}}.
\]
Combining this with (5) and recalling that $s$ and $s'$ are $\Theta(\sqrt{n})$, we get
\[
\frac{|\{r \in [n - s, n + s] : \Pr_{T \sim P_n} \epsilon(h, r, \delta) \leq c_8] > 1/2 |}{2s + 1} \leq c_{13} \left( \delta + \frac{1}{n} \right) \leq \frac{c_{14}}{\log(1/\delta)}.
\]
for $n \geq c_4 \log(1/\delta)$ and small enough $\delta$. Since, for all $r \in [n - s, n + s]$, we have $P_r = P_n$, this completes the proof.

The following bound can be obtained through direction application of the results in [Bartlett et al. [2020]]. The details are given in Appendix [3].

**Lemma 10** There is a constant $c$ such that, for all large enough $n$, with probability at least $1 - \delta$, for $S \sim P_n$, the least-norm interpolant $h$ satisfies $R_{P_n}(h) - R_{P_n}^* \leq c\sqrt{\frac{\log(1/\delta)}{n}}$.

Combining this with Lemma 9 proves Theorem 1.
4 Acknowledgements

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A Proof of Lemma 4

To prove Lemma 4, we will need some lemmas. The first is from [Buldygin and Kozachenko, 1980] (see [Rivasplata, 2012]).

**Lemma 11** If $X_1$ is a sub-Gaussian random variable with parameter $\sigma_1$, and $X_2$ is a (not necessarily independent) sub-Gaussian random variable with parameter $\sigma_2$, then $X_1 + X_2$ is sub-Gaussian with parameter $\sigma_1 + \sigma_2$.

This immediately implies the following.

**Lemma 12** If $X_1$ is a sub-Gaussian random vector with parameter $\sigma_1$, and $X_2$ is a (not necessarily independent) sub-Gaussian random vector with parameter $\sigma_2$, then $X_1 + X_2$ is sub-Gaussian with parameter $\sigma_1 + \sigma_2$.

**Proof:** Any projection of $X_1 + X_2$ is the sum of the projections of $X_1$ and $X_2$, so this follows from Lemma 11.

**Lemma 13** For a random vector $X = (X_1, \ldots, X_k)$, if $X_1$ is sub-Gaussian with parameter $1/3$, $X_2$ is sub-Gaussian with parameter $1/3$, and $(X_3, \ldots, X_k)$ is sub-Gaussian with parameter $1/3$, then $X$ is sub-Gaussian with parameter 1.

**Proof:** Embedding a random vector into a higher-dimensional space by adding components that always evaluate to zero does not affect whether it is sub-Gaussian, or its sub-Gaussian parameter. Since

$$X = (X_1, 0, \ldots, 0) + (0, X_2, 0, \ldots, 0) + (0, 0, X_3, \ldots, X_k),$$

applying Lemma 12 above completes the proof.

Now, for $U \sim D_n^m$, the uniform distribution $Q$ over $U$, and $(X_1, \ldots, X_d, Y) \sim Q$, we now would like to show that $X_1$ is sub-Gaussian with parameter $1/3$. We will use the following known sufficient condition, which can be recovered by tracing through the constants in the proof of Proposition 2.5.2 of [Vershynin, 2018].
Lemma 14  If a random variable $X$ satisfies $E \left[ \exp \left( \frac{18X^2}{e} \right) \right] \leq 2$, then $X$ is sub-Gaussian with parameter $1/3$.

Now we are ready to analyze the marginal distribution of the first component.

Lemma 15  For $U$ obtained from $m$ independent samples from $\mathcal{N}(0, \sigma^2)$ for $\sigma \leq 1/9$ if $Q$ is the uniform distribution over $U$, then, with probability at least $1 - \frac{3}{m}$, $Q$ is sub-Gaussian with parameter $1/3$.

Proof: Define $a = 18/e$ and let $Z = E_{x \sim Q[\exp(a x^2)]}$.

We have

$$E_{S \sim \mathcal{N}(0, \sigma)^m}[Z] = E_{S \sim \mathcal{N}(0, \sigma)^m}[E_{x \sim Q[\exp(ax^2)]}] = \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} e^{ax^2} \exp \left( -\frac{x^2}{2\sigma^2} \right) dx = \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} \exp \left( - \frac{1}{2\sigma^2} - a \right) x^2 \right) dx = \frac{1}{\sqrt{2\pi\sigma}} \times \sqrt{\frac{1}{2\sigma^2} - a} = \frac{1}{\sqrt{1 - 2a\sigma^2}}.$$ 

Similarly

$$\text{Var}_S[Z] = \text{Var}_S[E_{x \sim Q[\exp(ax^2)]}] = \frac{1}{m} \text{Var}_{x \sim \mathcal{N}(0, \sigma)[\exp(ax^2)]} \leq \frac{1}{m} E_{x \sim \mathcal{N}(0, \sigma)[\exp(2ax^2)]} = \frac{1}{m \sqrt{1 - 4a\sigma^2}}.$$ 

By Chebyshev’s inequality,

$$\Pr \left[ \frac{Z}{\sqrt{1 - 2a\sigma^2} + \frac{1}{\sqrt{3(1 - 4a\sigma^2)^{1/4}}}} \leq \frac{3}{m} \right].$$

For $\sigma \leq 1/9$, recalling that $a = 18/e$ shows that $\Pr[Z \geq 2] \leq \frac{3}{m}$ and applying Lemma 14 completes the proof.

Armed with these lemmas, we are now ready to prove Lemma 4. For $S \sim D_n^m$, let $Q$ be the uniform over $S$. For $(X_1, \ldots, X_d, Y) \sim Q$, Lemma 15 implies that, with probability $1 - 6/m$, $X_1$ and $Y$ are both sub-Gaussian with parameter $1/3$. It remains to analyze $(X_2, \ldots, X_d)$. Let $S'$ be the projections of the elements of $S$ onto these coordinates. With probability at least $1 - 3/m$, for all $s' \in S'$, $||s'|| \leq \log(em^2/3)/\sqrt{d};$
see [Lovász and Vempala, 2007, Lemma 5.17]. Recalling that $d = \Theta(n^2)$, if $m = bn$, then, for all large enough $n$, with probability $1 - 3/m$, $\max_{s' \in S^+} ||s|| \leq 1/6$, which implies that $(X_2, ..., X_d)$ is sub-Gaussian with parameter $1/3$. Putting this together with the analysis of $X_1$ and $Y$, and applying Lemma 13, completes the proof.

### B Proof of Lemma 10

The lemma follows from Theorem 1 of [Bartlett et al., 2020]; before showing how to apply it, let us first restate a special case of the theorem for easy reference.

#### B.1 A useful upper bound

The special case concerns the least-norm interpolant applied to training data $(x_1, y_1), ..., (x_n, y_n)$ drawn from a joint distribution $P$ over $(x, y)$ pairs. The marginal distribution of $x$ is Gaussian with covariance $\Sigma$. There is a unit length $\theta^*$ such that, for all $x$, the conditional distribution of $y$ given $x$ has mean $\theta^* \cdot x$ is sub-gaussian with parameter $1$ and variance at most $1$.

We will apply an upper bound in terms of the eigenvalues $\lambda_1 \geq \lambda_2 \geq ...$ of $\Sigma$. The bound is in terms of two notions of the effective rank of the tail of this spectrum:

$$r_k(\Sigma) = \frac{\sum_{i > k} \lambda_i}{\lambda_{k+1}}, \quad R_k(\Sigma) = \frac{(\sum_{i > k} \lambda_i)^2}{\sum_{i > k} \lambda_i^2}.$$  

The rank of $\Sigma$ is assumed to be greater than $n$.

**Lemma 16** There are $b, c, c_1 > 1$ for which the following holds. For all $n$, $P$ and $\Sigma$ defined as above, write $k^* = \min\{k \geq 0 : r_k(\Sigma) \geq bn\}$. Suppose that $\delta < 1$ with $\log(1/\delta) < n/c$. If $k^* < n/c_1$, then, with probability at least $1 - \delta$, the least-norm interpolant $h$ satisfies

$$R_P(h) - R^*_P \leq c \left( \max \left\{ \sqrt{\frac{r_0(\Sigma)}{n}}, \frac{r_0(\Sigma)}{n}, \sqrt{\frac{\log(1/\delta)}{n}} \right\} + \log(1/\delta) \left( \frac{k^*}{n} + \frac{n}{R_{k^*}(\Sigma)} \right) \right).$$

#### B.2 The proof

To prove Lemma 10 we need to show that $P_\alpha$ satisfies the requirements on $P$ in Lemma 16 and evaluate the effective ranks $r_k$ and $R_k$ of $P_\alpha$'s covariance $\Sigma_\alpha$. Define $\alpha = 1/d^2$. We have

$$r_0 = \frac{1/81 + (d - 1)\alpha}{1/81} = 1 + 81(d - 1)\alpha$$

(which is bounded by a constant) and

$$R_0 = \frac{(1/81 + (d - 1)\alpha)^2}{1/81^2 + (d - 1)\alpha^2}.$$  

For $k > 0$,

$$r_k = R_k = d - k.$$
Since $d$ grows faster than $n$, for large enough $n$, $k^* := \min \{ k : r_k \geq bn \} = 1$. So

$$R_{k^*} = d - 1 = \Omega(n^2).$$

Each sample from the distribution of $Y$ given $X = x$ has a mean of $\theta^* \cdot x$, and is sub-Gaussian with parameter at most $\frac{1}{9}$, and with variance at most $1/81$ (because increasing $Z$ only decreases the variance of $Y$).

Evaluating Lemma 16 on $P_n$ then gives Lemma 10.