Completion of standard-like embeddings

Joel Giedt

Department of Physics, University of California, and Theoretical Physics Group, 50A-5101, Lawrence Berkeley National Laboratory, Berkeley, CA 94720 USA.

Abstract

Inequivalent standard-like observable sector embeddings in $Z_3$ orbifolds with two discrete Wilson lines, as determined by Casas, Mondragon and Muñoz, are completed by examining all possible ways of embedding the hidden sector. The hidden sector embeddings are relevant to twisted matter in nontrivial representations of the Standard Model and to scenarios where supersymmetry breaking is generated in a hidden sector. We find a set of 175 models which have a hidden sector gauge group which is viable for dynamical supersymmetry breaking. Only four different hidden sector gauge groups are possible in these models.

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One of the distasteful aspects of four-dimensional heterotic string phenomenology is the glut of vacua possible in even the most elementary compactification schemes. For instance, the lowly $Z_3$ orbifold \[1\] admits an enormously large number of low energy effective theories, once non-standard embeddings—including discrete Wilson lines (described below)—are allowed. (The embedding dictates how the space group—the transformation group used to construct the orbifold—affects the gauge degrees of freedom in the underlying string theory. For a recent review of heterotic orbifolds, see \[2\].) However, it was pointed out some time ago by Casas, Mondragon and Muñoz (CMM) that most of the embeddings are actually redundant, and only a relatively small set of inequivalent embeddings exist \[3\].

In heterotic $Z_3$ orbifold models with discrete Wilson lines, the embedding is expressed in terms of four sixteen-dimensional vectors: the twist embedding $V$ and three Wilson lines $a_1, a_3$ and $a_5$; each of the four vectors is given by one-third of a vector belonging to the $E_8 \times E_8$ root lattice (denoted here as $\Lambda_{E_8 \times E_8}$):

\[
3V \in \Lambda_{E_8 \times E_8}, \quad 3a_i \in \Lambda_{E_8 \times E_8}, \quad \forall \ i = 1, 3, 5. \tag{1}
\]

(In Appendix A we provide a brief review of the $E_8$ and $E_8 \times E_8$ root systems, including explicit realizations of the respective root lattices $\Lambda_{E_8}$ and $\Lambda_{E_8 \times E_8}$.) It is convenient to denote the vector formed from the first eight entries of $V$ by $V_A$ and the vector formed from the last eight entries of $V$ by $V_B$, so that the twist embedding $V$ may be written as $V = (V_A; V_B)$. Eq. \([1]\) then implies

\[
3V_A \in \Lambda_{E_8}^{(A)}, \quad 3V_B \in \Lambda_{E_8}^{(B)}, \tag{2}
\]

where $\Lambda_{E_8}^{(A)}$ and $\Lambda_{E_8}^{(B)}$ are the two copies of the $E_8$ root lattice used to construct $\Lambda_{E_8 \times E_8}$. Similarly, we write $a_i = (a_{iA}; a_{iB})$ for each $i = 1, 3, 5$. In addition to \([2]\), constraint \([1]\) becomes

\[
3a_{iA} \in \Lambda_{E_8}^{(A)}, \quad 3a_{iB} \in \Lambda_{E_8}^{(B)}, \quad \forall \ i = 1, 3, 5. \tag{3}
\]

The set \(\{V_A, a_{1A}, a_{3A}, a_{5A}\}\) dictates the space group transformation properties of the underlying string degrees of freedom corresponding to the first $E_8$ factor of the gauge group; i.e, the set “embeds the first $E_8$.” Similarly, the set \(\{V_B, a_{1B}, a_{3B}, a_{5B}\}\) embeds the second $E_8$. For discrete Wilson lines constructions, the embedding of the gauge degrees of freedom has the effect of breaking each $E_8$ down to a rank eight subgroup:

\[
E_8(A) \to G_O, \quad E_8(B) \to G_H, \tag{4}
\]

where $G_O$ and $G_H$ are usually coined the “observable” and “hidden” sector gauge groups. Typically, $G_O$ and $G_H$ each contain one or more $U(1)$s, as required to conserve rank. We
note that one must be careful not to take the terms “observable” and “hidden” too literally in these models since twisted fields (twisted and untwisted refer to choices of closed string boundary conditions—properties which also characterize particle states in the field theory limit) charged under the nonabelian factors of $G_O$ are typically also charged under $U(1)$s contained in $G_H$. Thus, gauge interactions between observable and hidden sector fields are generic and are potentially a worrisome feature because of experimental constraints on gauge interactions beyond those of the Standard Model. It is also conceivable that supersymmetry breaking in the hidden sector may be communicated too forcefully to the observable sector via these gauge interactions, even if they are broken at an intermediate scale.

Models with three generations of quarks and leptons can be obtained by choosing the third Wilson line $a_5$ to vanish, as explained in refs. [4]. Consequently, three generation models of this ilk are specified by the set of embedding vectors $\{V, a_1, a_3\}$. For this reason, we will ignore $a_5$ in the remainder of this article. The observable sector gauge group $G_O$ is determined entirely by the set of observable sector embedding vectors $\{V_A, a_{1A}, a_{3A}\}$. Many such sets lead to a standard-like observable sector gauge group $G_O$ of the form

$$G_O = SU(3) \times SU(2) \times U(1)^5.$$  \hspace{1cm} (5)

CMM have determined observable sector embeddings of this type, with the additional requirement of quark doublets—$(3, 2)$ irreducible representations (irreps) under the $SU(3) \times SU(2)$ subgroup of (5)—in the untwisted sector. It is suprising that CMM have found that any observable sector embedding satisfying these two conditions is equivalent to some one of only nine $\{V_A, a_{1A}, a_{3A}\}$; they are displayed in Table I. Although they argue that these nine observable sector embeddings are inequivalent, in Appendix B we show that three more equivalences exist:

$$\text{CMM 3} \simeq \text{CMM 1}, \quad \text{CMM 5} \simeq \text{CMM 4}, \quad \text{CMM 7} \simeq \text{CMM 6}. \hspace{1cm} (6)$$

Thus, the number of inequivalent observable sector embeddings satisfying the CMM conditions is presumably six; we take CMM observable sector embeddings 1, 2, 4, 6, 8 and 9 as representatives of these six. This does not mean that only six models of this type exist. For each choice of the six inequivalent $\{V_A, a_{1A}, a_{3A}\}$ there will be many possible hidden sector embeddings $\{V_B, a_{1B}, a_{3B}\}$, not all of which are equivalent. CMM have left the hidden sector embedding unspecified and the purpose of this paper is to enumerate the allowed ways (up to equivalences) of embedding the hidden sector.

One might wonder whether or not the hidden sector embedding has any phenomenological relevance from the “low energy” ($\lesssim 100$ TeV) point of view. We now point out three ways
in which the hidden sector embedding is crucial to understanding the low energy physics predicted by a given model. Firstly, the mass-shell conditions for twisted sector states in the underlying string theory depend on the full embedding \(\{V, a_1, a_3\}\). It is the solution of the mass-shell conditions which determines the spectrum of particle states below the string scale, roughly \(10^{17}\) GeV for the weakly coupled heterotic string. Thus, the hidden sector embedding is important because the spectrum of twisted sector states, including those charged under the observable sector gauge group \(G_O\), depends on \(\{V_B, a_{1B}, a_{3B}\}\). Secondly, it was mentioned above that twisted sector fields in nontrivial irreps of \(G_O\) are typically charged under \(U(1)\) factors contained in the hidden sector gauge group \(G_H\); the spectrum of hidden \(U(1)\) charges will also depend on the hidden sector embedding. Finally, the hidden sector embedding is relevant to model building because \(G_H\) and the nontrivial matter irreps under nonabelian factors of \(G_H\) play a crucial role in models of dynamical supersymmetry breaking; for example, the authors of ref. \[5\] illustrate how the mass of the gravitino and supersymmetry breaking soft terms are sensitive to the spectrum and dynamics of the hidden sector.

The allowed ways of completing the embeddings of Table I may be determined from the consistency conditions (which ensure world sheet modular invariance—a property which is necessary for the absence of quantum anomalies—of the underlying string theory) presented in ref. \[4\]:

\[
3V_B \in \Lambda_{E_8}, \quad 3a_{iB} \in \Lambda_{E_8},
\]

\[
3V \cdot V \in \mathbb{Z}, \quad 3a_i \cdot a_j \in \mathbb{Z}, \quad 3V \cdot a_i \in \mathbb{Z}.
\]

(The consistency conditions \(\ref{7}\) were already given in \(\ref{2}\) and \(\ref{3}\) above; the last two equations in \(\ref{8}\) must hold for all choices of \(i\) and \(j\).) For example, the first embedding in Table I has \(9V_A \cdot a_{1A} = -2\). Then the hidden sector embeddings which complete CMM 1 must satisfy \(9V_B \cdot a_{1B} = 2 \mod 3\) since

\[
V \cdot a_1 = V_A \cdot a_{1A} + V_B \cdot a_{1B}
\]

and from \(\ref{8}\) we see that \(9V \cdot a_1\) must be a multiple of three.

An infinite number of solutions to \(\ref{7}\) and \(\ref{8}\) exist, even after the CMM conditions of \(\ref{5}\) and untwisted \((3, 2)\) irreps are imposed. This does not imply an infinite number of physically distinct models. For example, trivial permutation redundancies such as

\[
\begin{pmatrix}
V_I^I_B \\
a_{1I}^I_{1B} & a_{3I}^I_{3B}
\end{pmatrix}
\leftrightarrow
\begin{pmatrix}
V_J^I_B \\
a_{1J}^I_{1B} & a_{3J}^I_{3B}
\end{pmatrix},
\forall I, J = 1, \ldots, 8
\]

allow for different embeddings which give identical physics. Redundancies related to the signs of entries also exist (to be addressed later). Moreover, we will see below that an upper bound
may be placed on the magnitude of the entries of the embedding vectors; that is, any embedding with an entry whose magnitude is greater than the bound is equivalent to another embedding which respects the bound. Once these redundancies are eliminated the number of consistent hidden sector embeddings is large ($10^4 \sim 10^5$), though no longer infinite. However, just as with the observable sector embeddings, the equivalence relations pointed out by CMM allow for a dramatic reduction when one determines the physically distinct models.

We have carried out an automated reduction using the equivalence relations of CMM, which they have denoted “(i)” through “(vi)”. Their operations “(ii)” through “(v)” would affect the observable embedding and are thus irrelevant to our analysis. This leaves two equivalence relations, presented here for ease of reference.

(I) The addition of a root lattice vector $\ell \in \Lambda_{E_8}$ to any one of the vectors $V_B, a_{1B}$ or $a_{3B}$; it is important to stress that any one of these embedding vectors may be shifted independently:

$$V_B \rightarrow V_B + \ell \quad \text{or} \quad a_iB \rightarrow a_iB + \ell, \quad i = 1 \text{ or } 3.$$  \hspace{1cm} (11)

(II) A Weyl reflection performed simultaneously on each of the embedding vectors in the set \{\(V_B, a_{1B}, a_{3B}\)\}:

$$V_B \rightarrow V_B - (V_B \cdot e_j)e_j, \quad a_iB \rightarrow a_iB - (a_{iB} \cdot e_j)e_j, \quad i = 1 \text{ and } 3.$$  \hspace{1cm} (12)
In keeping with the notation of Appendix A, $e_j$ is one of the 240 nonzero roots of $E_8$. In what follows we will refer to these as operations (I) and (II).

Operation (I) corresponds to an invariance under translations by elements of the $E_8$ root lattice $\Lambda_{E_8}$. This transformation group is referred to as the lattice group associated with $\Lambda_{E_8}$; we will denote this group as $T$. Since operation (I) allows each vector $V_B, a_{1B}$ and $a_{3B}$ to be shifted by a different $E_8$ root lattice vector, it is actually $T^3 = T \times T \times T$ which is the corresponding invariance group. Operation (II) corresponds to an invariance under the $E_8$ Weyl group, which we denote $W$. To systematically analyze possible equivalences between different hidden sector embeddings under operations (I) and (II), it is therefore vital to have a rudimentary understanding of these two groups and their combined action on the representation space $\mathbb{R}^8$; i.e., real-valued eight-dimensional vectors such as $V_B, a_{1B}$ and $a_{3B}$. It is also helpful to develop a concise notation for certain essential features of $T$ and $W$. For these purposes we now embark on a minor study of these two groups.

It is convenient to notate the elements of $T$ as $T_\ell$, where $\ell$ is the lattice vector by which the translation is performed:

$$T_\ell P = P + \ell, \quad \ell \in \Lambda_{E_8}, \quad \forall \ P \in \mathbb{R}^8. \quad (13)$$

Weyl reflections by any of the 240 nonzero $E_8$ roots belong to $W$; we write these as $W_i$ with the subscript corresponding to the $E_8$ root $e_i$ used in the reflection:

$$W_i : P \rightarrow W_i P = P - (P \cdot e_i)e_i, \quad \forall \ i = 1, \ldots, 240, \quad \forall \ P \in \mathbb{R}^8. \quad (14)$$

It is not difficult to check that for each of these operators $W_i^2 = 1$, so that each is its own inverse; thus, the Weyl group $W$ can be built up by taking all possible products of the 240 $W_i$:

$$W = \{1, W_i, W_iW_j, \ldots\}. \quad (15)$$

The $E_8$ Weyl group is a nonabelian finite group of order (the number of elements) 696 729 600. On the other hand, there are only 240 Weyl reflections $W_i$. Thus, the generic element of $W$ is not a simple reflection (14), but is a product of several such reflections. In what follows, we write generic elements of the Weyl group in calligraphic type: $W_I \in W$, with $I = 1, \ldots, 696 729 600$. Thus, for each element $W_I$ of $W$, Weyl reflections $W_j, W_k, \ldots, W_m$ exist such that

$$W_I = W_jW_k \cdots W_m. \quad (16)$$

We point out one more property of the Weyl group $W$, which we will have occasion to appeal to below: an $E_8$ root lattice vector, when subjected to a Weyl group transformation, yields
back an $E_8$ root lattice vector. Explicitly, if $\ell \in \Lambda_{E_8}$ and $W_I \in W$, then there exists a $k \in \Lambda_{E_8}$ such that

$$W_I \ell = k.$$  \hfill (17)

In mathematical parlance, $W_I$ is an automorphism of $\Lambda_{E_8}$.

With these tools in hand, there is a useful theorem which we can prove.

**Theorem 1** If $W_I \in W$ and $T_\ell \in T$, then there exists a $T_k \in T$ such that $W_I T_\ell = T_k W_I$.

To see this, let $P \in \mathbb{R}^8$ and compute

$$W_I T_\ell P = W_I (P + \ell) = W_I P + W_I \ell.$$ \hfill (18)

The last step follows from the fact that $W_I$ is a linear operator—a property which is evident from (14) and (16). Using (17), the right-handed side of (18) can be rewritten

$$W_I P + W_I \ell = W_I P + k = T_k W_I P.$$ \hfill (19)

I.e., $W_I T_\ell = T_k W_I$, as was to be shown.

A sequence of operations (I) and (II) has the form of a product of various elements of $T$ and $W$. Theorem 1 allows one to rewrite any sequence of operations (I) and (II), whatever the order and number of operations of each type, in the form

$$O = T_\ell W_I, \quad T_\ell \in T, \quad W_I \in W.$$ \hfill (20)

We stress that the element $T_\ell$ may be different for each of the embedding vectors $V_B, a_{1B}$ and $a_{3B}$, but that the Weyl group element $W_I$ acting on these vectors must be the same. Typically, $W_I$ will be a generic element of the Weyl group taking the form (18), corresponding to a string of operations of type (II). Thus, we arrive at the following rather useful conclusion: any sequence of operations (I) and (II), whatever the order and number of operations of each type, is equal in effect to a sequence of operations of type (II), followed by a single operation of type (I), allowing for different shifts for each of the three embedding vectors. Symbolically, we need only consider equivalences of the form

$$O = T_\ell W_j W_k \cdots W_m.$$ \hfill (21)

Suppose two embeddings $\{V_B, a_{1B}, a_{3B}\}$ and $\{V_B', a_{1B}', a_{3B}'\}$. We want to determine whether these two embeddings are equivalent. Based on the results of the last paragraph, we see that it is sufficient to first tabulate all points in the orbit of $\{V_B, a_{1B}, a_{3B}\}$ under $W$, and then to
check whether any of these points are related to \( \{ V'_B, a'_{1B}, a'_{3B} \} \) by operation (I). (The orbit of \( \{ V_B, a_{1B}, a_{3B} \} \) under \( W \) is tabulated by computing the transformations \( \{ W_I V_B, W_I a_{1B}, W_I a_{3B} \} \) for all 696729600 elements \( W_I \) of the \( E_8 \) Weyl group.) If the two embeddings are related in this way, then they are equivalent.

As mentioned above, for a given \( \{ V_A, a_{1A}, a_{3A} \} \), the number of consistent \( \{ V_B, a_{1B}, a_{3B} \} \) is infinite; the following definition exploits operation (I) to immediately and efficiently eliminate enough redundancy to obtain a finite set.

**Definition 1** An embedding \( \{ V_B, a_{1B}, a_{3B} \} \) is in *minimal* form provided:

(a) \( 3V^I_B \in \mathbb{Z}, 3a_{1B}^I \in \mathbb{Z} \) and \( 3a_{3B}^I \in \mathbb{Z} \) for each choice \( I = 1, \ldots, 8 \);

(b) \( |3V^I_B| \leq 2, |3a_{1B}^I| \leq 2 \) and \( |3a_{3B}^I| \leq 2 \) for each choice \( I = 1, \ldots, 8 \);

(c) no more than one entry of each vector \( 3V^I_B, 3a_{1B}^I \) and \( 3a_{3B}^I \) has absolute value two, and any such entry is the left-most nonzero entry.

Any embedding may be reduced to minimal form by means of operation (I). We will demonstrate the veracity of this statement by considering \( V_B \) which are not minimal. It will be understood that similar statements hold for \( a_{1B} \) and \( a_{3B} \) which are not minimal, since operations of type (I) are allowed to act independently on \( V_B, a_{1B} \) and \( a_{3B} \).

From (7) one sees that \( 3V_B \) is an \( E_8 \) root lattice vector. As explained in Appendix A, the entries of an \( E_8 \) root lattice vector are either all integral or all half-integral. In the latter case, part (a) of Definition 1 will not be satisfied. However, operation (I) allows us to shift

\[
3V_B \rightarrow 3V_B + 3\ell, \quad \ell \in \Lambda_{E_8}.
\]

(22)

If we take \( \ell \) to be any lattice vector with half-integral entries, then (22) transforms \( 3V_B \) to a lattice vector with integral entries. Now suppose \( 3V_B \) satisfies part (a) of Definition 1 but \( |3V^I_B| > 2 \) for one or more choices of \( I \). It is in all cases possible to find a lattice vector \( \ell \) such that (22) generates an equivalent \( 3V_B \) which satisfies part (b) of Definition 1. To see this, first note that repeated shifts (22) by vectors

\[
3\ell \in \left\{ \pm(3,3,0,0,0,0,0,0), (3,-3,0,0,0,0,0,0) \right\}
\]

(23)

(underlining indicates that any permutation of entries may be taken) allows \( 3V_B \) to be translated to a form where no entry has absolute value greater than three. If the original \( 3V_B \) satisfied (7), then the translated one will as well, since the sum of two lattice vectors is also a lattice.
vector. As explained in Appendix A, an $E_8$ root lattice vector must have its entries sum to an even number (the final condition in (31)). Then from (7) we know that
\[ \sum_{I=1}^{8} 3V_I^J = 0 \mod 2. \] (24)

If for any $I$ the translated vector has $3V_I^J = \pm 3$, then (24) implies that there must be a $J \neq I$ such that $3V_B^J$ is an odd integer. If $3V_B^J = \pm 3$, then a final shift by one of the vectors in (23) allows us to set $V_I^J \to 0$ and $V_B^J \to 0$. For example:

\[
3V_B = (\ldots, 3, \ldots, 3, \ldots) \quad \text{and} \quad 3\ell = (\ldots, -3, \ldots, -3, \ldots)
\]

gives
\[
3V_B \to 3V_B + 3\ell = (\ldots, 0, \ldots, 0, \ldots).
\] (25)

On the other hand, if $3V_B^J = \pm 1$, then a final shift by one of the vectors in (23) allows us to set $V_I^J \to 0$ and $V_B^J \to \mp 2$. From the above manipulations, it should be clear that a shift (22) by an appropriate vector (23) will eliminate any pair of $\pm 2$s appearing in $3V_B$ in favor of a pair of $\pm 1$s. Similarly, if a $\pm 1$ precedes a $\pm 2$ (reading left to right), the order may be reversed—possibly altering signs—by a shift (22) by an appropriate vector (23). In this way, we are always able to transform any $V_B$ satisfying parts (a) and (b) of Definition 1 into an equivalent form which also satisfies part (c) of Definition 1.

It is a simple exercise to verify that Weyl reflections (14) using $E_8$ roots of the form $e_i = (1, -1, 0, \ldots, 0)$ exchange two entries; it is also easy to check that Weyl reflections using roots of the form $e_i = (1, 1, 0, \ldots, 0)$ exchange two entries and flip both signs. We will refer to these as “integral” Weyl reflections. The second type uses $E_8$ roots of the form $e_i = (\pm 1/2, \ldots, \pm 1/2)$ with an even number of positive entries, and we will refer to these as “half-integral” Weyl reflections. These tend to have more dramatic effects; for example, $3V_B = (1, \ldots, 1)$ can be reflected to $3V_B = (2, 2, 0, \ldots, 0)$ using $e_i = (1/2, 1/2, -1/2, \ldots, -1/2)$. By such manipulations, together with operation (1), it is well-known that only five inequivalent twist embeddings $V = (V_A; V_B)$ exist (including $V = 0$). Consistency with a given CMM $V_A$ restricts $V_B$ to one or two choices. We can eliminate remaining redundancies related to integral Weyl reflections by enforcing ordering and sign conventions on $a_{1B}$ and $a_{3B}$. With this in mind, we make the following definition.

**Definition 2** An embedding \( \{V_B, a_{1B}, a_{3B}\} \) is in canonical form if $3V_B = (2, 1, 1, 0, 0, 0, 0, 0)$ for CMM 1 $\sim$ 7, $3V_B = (1, 1, 0, 0, 0, 0, 0, 0)$ or $3V_B = (2, 1, 1, 1, 0, 0, 0)$ for CMM 8 $\sim$ 9; and, $a_{1B}$ and $a_{3B}$ are first fixed to minimal form, and then subjected to whatever integral Weyl reflections are required such that they satisfy the following conditions:
(a) \( V_B^I = V_B^{I+1} \Rightarrow a_{1B}^I \geq a_{1B}^{I+1}, \ I = 1, \ldots, 7; \)

(b) \( V_B^I = 0 \Rightarrow a_{1B}^I \geq 0, \ I = 3, \ldots, 7; \)

(c) \( a_{7B}^7 = 0 \Rightarrow a_{8B}^8 \geq 0 \) while \( a_{1B}^7 \neq 0 \Rightarrow 3a_{1B}^8 \geq -1; \)

(d) \( V_B^I = V_B^{I+1} \) and \( a_{1B}^I = a_{1B}^{I+1} \Rightarrow a_{3B}^I \geq a_{3B}^{I+1}, \ I = 1, \ldots, 7; \)

(e) \( V_B^I = a_{1B}^I = 0 \Rightarrow a_{3B}^I \geq 0, \ I = 3, \ldots, 6; \)

(f) \( a_{1B}^7 = a_{8B}^8 = 0 \) or \( a_{1B}^6 = a_{7B}^7 = a_{3B}^6 = 0 \Rightarrow a_{3B}^7 \geq 0; \)

(g) \( a_{1B}^6 = a_{7B}^7 = 0 \) and \( a_{6B}^6 \neq 0 \) and \( a_{1B}^8 \neq 0 \Rightarrow 3a_{3B}^7 \geq -1; \)

(h) \( a_{1B}^7 = a_{8B}^8 = a_{3B}^6 = 0 \Rightarrow a_{3B}^8 \geq 0; \)

(i) \( a_{1B}^7 = a_{8B}^8 = 0 \) and \( a_{3B}^7 \neq 0 \Rightarrow 3a_{3B}^8 \geq -1; \)

It is straightforward, though tedious, to verify that any \( a_{1B} \) and \( a_{3B} \) of minimal form can be transformed to satisfy the conditions listed above using the integral Weyl reflections; we do not present a proof here as the manipulations are lengthy and elementary. Transforming all embeddings \( \{V_B, a_{1B}, a_{3B}\} \) to canonical form, we arrive at a set for which no two are related purely by integral Weyl reflections.

With the definition (13), it is not difficult to check

\[
W_i W_j W_i = W_k, \quad e_k = e_j - (e_j \cdot e_i) e_i. \tag{26}
\]

Recall that the entries of \( E_8 \) roots \( e_i \) are either all integral or all half-integral. We denote integral roots with undotted subscripts from the beginning of the alphabet, \( e_a, e_b, \ldots \) and half-integral roots with dotted subscripts from the beginning of the alphabet, \( \hat{e}_a, \hat{e}_b, \ldots \). It should be clear that \( e_a - (e_a \cdot e_a) e_a \) is a half-integral root since \( e_a \cdot e_a \in \mathbb{Z} \). Thus we can specialize (29) to obtain, for example,

\[
W_a W_a W_a = W_{\hat{c}}, \quad e_{\hat{c}} = e_a - (e_a \cdot e_a) e_a. \tag{27}
\]

We can then perform manipulations such as

\[
W_\hat{a} W_a = W_\hat{a} W_a W_\hat{a} W_a W_\hat{a} W_\hat{a} W_a = W_\hat{a} W_{\hat{c}} W_\hat{d}. \tag{28}
\]

\[
W_\hat{a} W_\hat{b} W_a = W_\hat{a} W_a W_\hat{a} W_a W_\hat{a} W_a W_\hat{b} W_a W_\hat{a} W_\hat{c} W_\hat{d}. \tag{29}
\]
where $W_{\dot{c}}$ is defined explicitly in (27) and $W_{\dot{d}} = W_{\dot{a}} W_{\dot{b}} W_{\dot{a}}$ is defined analogously. This illustrates how (27) allows us to write a generic element (16) of the Weyl group $W$ in the form

$$W_I = W_{\dot{a}} \cdots W_{\dot{c}} W_{\dot{a}} \cdots W_{\dot{c}}.$$  \hspace{1cm} (30)

Equivalences related to the string of integral Weyl reflections $W_{\dot{a}} \cdots W_{\dot{c}}$ are eliminated by going to canonical form. From these considerations we find that, given a set of canonical embeddings, equivalences may be identified by the following procedure:

(i) compute the orbit of $\{V_B, a_{1B}, a_{3B}\}$ under strings of half-integral Weyl reflections;

(ii) fix the results of (i) to minimal form by operations of type (I);

(iii) fix the results of (ii) to canonical form by integral Weyl reflections;

(iv) check whether the results of (iii) are related by operation (I) to any other embedding in the original set.

The last step is simply a matter of checking whether the differences $V_B - V'_B$, $a_{1B} - a'_{1B}$ and $a_{3B} - a'_{3B}$ each give lattice vectors, where $\{V_B, a_{1B}, a_{3B}\}$ is a result of step (iii) and $\{V'_B, a'_{1B}, a'_{3B}\}$ is an element of the original set of canonical embeddings.

In our automated analysis, we first generated a list of all possible consistent embeddings of the hidden sector, constraining them to be of canonical form. Since all embeddings can be reduced to canonical form by way of operations (I) and (II), we are assured that this list is complete. The number of “initial” embeddings was at this point already reduced to roughly $10^4$. Using the procedure outlined in the previous paragraph, we removed as many of the redundant embeddings as performing only 1, 2 and 3 half-integral Weyl reflections in step (i) would allow. Because the $E_8$ Weyl group is so large, it proved to be impractical to act on the initial embeddings with each of its elements. It also proved impractical to perform four or more half-integral Weyl reflections. The number of positive half-integral roots is 64 (negative roots generate the same Weyl reflections); four Weyl reflections would have required roughly $10^7$ different operations for each embedding. The initial list was thereby reduced to a mere 192 embeddings. This list is guaranteed to be complete, but entries of the list are not necessarily inequivalent. However, already in going from 2 half-integral Weyl reflections to 3 half-integral Weyl reflections, the list did not shrink by much. It would appear that though there may be some equivalences remaining, there should not be very many. (It is worth pointing out that application of an analogous procedure to the observable sector embeddings turned up equivalences overlooked by CMM, already at the level of one half-integral Weyl reflection.)
We have, in addition, determined the hidden sector gauge group $G_H$ for each of the 192 embeddings. Only five $G_H$ were found to be possible, displayed in Table II. This is remarkable, considering that one might naively expect a large subset of the 112 breakings [6] of $E_8$ to be present. Apparently, the CMM requirements of [7] and untwisted quark doublets significantly affect what is possible in the hidden sector.

In Appendix C, we present lists of the hidden sector embeddings which complete the CMM analysis. We have not displayed Case 5 $G_H$ models, since we do not regard them as affording viable scenarios of hidden sector dynamical supersymmetry breaking. They are, however, available from the author upon request. Eliminating the Case 5 $G_H$ models from the total of 192, we are left with 175 models. Also not included is the enumeration of the spectrum of massless matter for these models, with their $U(1)$ charges. We have performed this analysis and hope to present interesting examples and a summary of general features in a later publication.

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### Appendix A

In this appendix we briefly review some salient aspects of the $E_8$ and $E_8 \times E_8$ root systems.
The material given below can be found in standard textbooks on string theory, such as [7], as well as texts on Lie algebras and groups, such as [8]; it is included here for ease of reference.

A basis in the root space may be chosen such that the $E_8$ root lattice can be written as the (infinite) set of eight-dimensional vectors

$$\Lambda_{E_8} = \left\{ \left( n_1, \ldots, n_8 \right), \left( n_1 + \frac{1}{2}, \ldots, n_8 + \frac{1}{2} \right) \mid n_1, \ldots, n_8 \in \mathbb{Z}, \sum_{i=1}^{8} n_i = 0 \mod 2 \right\}. \quad (31)$$

Note that the components of a given $E_8$ root lattice vector are either all integral or all half-integral. Lattice vectors $\ell \in \Lambda_{E_8}$ which satisfy $\ell \cdot \ell = 2$ (where the ordinary eight-dimensional “dot product” is implied) yield the 240 nonzero $E_8$ roots, which we denote $e_1, \ldots, e_{240}$. By convention, we take as positive roots those $e_i$ whose first nonzero entry (counting left to right) is positive. A simple root is a positive root which cannot be obtained from the sum of two positive roots. Eight simple roots exist for $E_8$, which we denote by $\alpha_1, \ldots, \alpha_8$. These form a basis for the $E_8$ root lattice given in (31), which may alternatively be written as

$$\Lambda_{E_8} = \left\{ \sum_{i=1}^{8} m_i \alpha_i \mid m_1, \ldots, m_8 \in \mathbb{Z} \right\}. \quad (32)$$

The $E_8 \times E_8$ root lattice is constructed by taking the direct sum of two copies of $\Lambda_{E_8}$, which we distinguish by labels $(A)$ and $(B)$:

$$\Lambda_{E_8 \times E_8} = \Lambda_{E_8}^{(A)} \oplus \Lambda_{E_8}^{(B)}. \quad (33)$$

Thus, an $E_8 \times E_8$ root lattice vector $\ell$ is a sixteen-dimensional vector satisfying

$$\ell = (\ell_A; \ell_B), \quad \ell_A \in \Lambda_{E_8}^{(A)}, \quad \ell_B \in \Lambda_{E_8}^{(B)}, \quad (34)$$

where we have denoted the first eight entries of $\ell$ by $\ell_A$ and the last eight entries of $\ell$ by $\ell_B$, as in the main text. The 480 nonzero roots of $E_8 \times E_8$ are given in this notation by $(e_i; 0)$ and $(0; e_i)$, where $e_i$ is one of the 240 nonzero $E_8$ roots. Similarly, the sixteen simple roots of $E_8 \times E_8$ are given by $(\alpha_i; 0)$ and $(0; \alpha_i)$, where $\alpha_i$ is one of the eight $E_8$ simple roots.

**Appendix B**

The equivalences (6) were uncovered using the automated routines developed for the analysis of hidden sector embeddings; any further equivalences between the observable sector embeddings of CMM would require four or more half-integral Weyl reflections, transformations which were not studied for reasons explained above. Because the equivalences (6) are a significant revision
to the results of ref. [3], we have chosen to explicitly demonstrate them in this appendix. In addition to operations (I) and (II) used in the main text, we make use of two redefinitions of the Wilson lines which give equivalent embeddings (cf. ref. [3]):

\[ a_1 \rightarrow a'_1 = -a_1 - a_3, \quad a_3 \rightarrow a'_3 = a_1 - a_3; \]  
\[ a_1 \rightarrow a'_1 = a_1 - a_3, \quad a_3 \rightarrow a'_3 = a_1 + a_3. \]  

In what follows we will ignore the hidden sector embedding vectors, since in the end we complete the observable sector embeddings with all consistent choices.

First consider CMM 3, as given in Table I. We Weyl reflect (operation (II)) by \( e = \frac{1}{2}(1, 1, -1, -1, -1, 1, 1, -1) \) to obtain

\[ 3V_A \rightarrow 3V'_A = (-1, -1, 0, 0, 2, 0, 0), \]
\[ 3a_{1A} \rightarrow 3a'_{1A} = \frac{1}{2}(-1, -1, 1, 1, -3, -1, 3), \]
\[ 3a_{3A} \rightarrow 3a'_{3A} = \frac{1}{2}(-1, -1, 1, 1, 3, 1, 3). \]  

Application of (35) yields

\[ 3a'_{1A} \rightarrow 3a''_{1A} = -3a'_{1A} - 3a'_{3A} = (1, 1, -1, -1, 0, 0, -3), \]
\[ 3a'_{3A} \rightarrow 3a''_{3A} = 3a'_{1A} - 3a'_{3A} = (0, 0, 0, 0, -3, -1, 0). \]  

Finally, we employ operation (I) to shift

\[ 3a''_{1A} \rightarrow 3a''_{1A} = 3a''_{1A} + 3\ell_1, \]
\[ 3a''_{3A} \rightarrow 3a''_{3A} = 3a''_{3A} + 3\ell_3. \]  

where

\[ 3\ell_1 = (0, 0, 0, 0, 3, 0, 0, 3), \]
\[ 3\ell_3 = (0, 0, 0, 0, 0, 3, 3, 0), \]

To obtain

\[ 3a''_{1A} = (1, 1, -1, -1, 2, 0, 0, 0), \]
\[ 3a''_{3A} = (0, 0, 0, 0, 0, 0, 2, 0). \]  

With \( V'_A \) as given in (37), one can see by comparison to Table I that \( \{V'_A, a''_{1A}, a''_{3A}\} \) is precisely the observable sector embedding of CMM 1; thus, we have shown the first equivalence of (3).

Next consider CMM 5. We Weyl reflect by \( e = \frac{1}{2}(1, 1, -1, -1, -1, 1, 1, -1) \) to obtain

\[ 3V_A \rightarrow 3V'_A = (-1, -1, 0, 0, 2, 0, 0), \]
\[ 3a_{1A} \rightarrow 3a'_{1A} = \frac{1}{2}(1, 1, -1, -1, -3, 3, -3), \]
\[ 3a_{3A} \rightarrow 3a'_{3A} = \frac{1}{2}(-1, -1, 1, 1, 3, 3, 1). \]
Application of (36) yields

\[ 3a'_{1A} \rightarrow 3a''_{1A} = 3a'_{1A} - 3a'_{3A} = (1, 1, -1, -1, -1, -3, 0, -2), \]
\[ 3a'_{3A} \rightarrow 3a''_{3A} = 3a'_{1A} + 3a'_{3A} = (0, 0, 0, 0, 0, 3, -1). \]  

(43)

Shifting as in (39), but with

\[ 3\ell_1 = (0, 0, 0, 0, 0, 3, 0, 3), \quad 3\ell_3 = (0, 0, 0, 0, 0, -3, 3), \]  

we obtain

\[ 3a''_{1A} = (1, 1, -1, -1, 0, 0, 1), \quad 3a''_{3A} = (0, 0, 0, 0, 0, 0, 2). \]  

(45)

Performing a Weyl reflection of \( \{V'_{A}, a''_{1A}, a''_{3A}\} \) by the root \( e' = (0, 0, 0, 0, 0, 1, -1) \) interchanges entries seven and eight of each embedding vector:

\[ 3V'_{A} \rightarrow 3V''_{A} = (-1, -1, 0, 0, 0, 2, 0, 0), \]
\[ 3a''_{1A} \rightarrow 3a''_{1A} = (1, 1, -1, -1, -1, 0, 1, 0), \]
\[ 3a''_{3A} \rightarrow 3a''_{3A} = (0, 0, 0, 0, 0, 2, 0). \]  

(46)

Comparing to Table II, we see that \( \{V''_{A}, a''_{1A}, a''_{3A}\} \) is the observable sector embedding of CMM 4; this proves the second equivalence of (8).

Finally consider CMM 7. Weyl reflection by \( e = \frac{1}{2}(1, 1, -1, -1, -1, 1, -1) \) yields

\[ 3V'_{A} \rightarrow 3V'_{A} = (-1, -1, 0, 0, 0, 2, 0, 0), \]
\[ 3a'_{1A} \rightarrow 3a'_{1A} = (-1, -1, 1, 1, 0, -1, 2), \]
\[ 3a'_{3A} \rightarrow 3a'_{3A} = \frac{1}{2}(-1, -1, 1, 1, -1, 3, 1). \]  

(47)

Application of (36) gives

\[ 3a'_{1A} \rightarrow 3a''_{1A} = 3a'_{1A} - 3a'_{3A} = \frac{1}{2}(-1, -1, 1, 1, 1, -5, 3), \]
\[ 3a'_{3A} \rightarrow 3a''_{3A} = 3a'_{1A} + 3a'_{3A} = \frac{1}{2}(-3, -3, 3, 3, -1, 1, 5). \]  

(48)

We shift as in (39), but with

\[ 3\ell_1 = \frac{1}{2}(3, 3, -3, -3, -3, 3, 3, -3), \quad 3\ell_3 = \frac{1}{2}(3, 3, -3, -3, -3, 3, -3, -9), \]  

(49)

to obtain

\[ 3a''_{1A} = (1, 1, -1, -1, 2, -1, 0), \quad 3a''_{3A} = (0, 0, 0, 0, 1, -1, -2). \]  

(50)
Weyl reflection of \( \{ V_A', a_{1A}', a_{3A}' \} \) by \( e' = (0, 0, 0, 0, 0, 1, -1) \) then \( e'' = (0, 0, 0, 0, 0, 1, 1) \) flips the signs of entries seven and eight of each embedding vector, yielding

\[
\begin{align*}
3V_A' &\rightarrow 3V_A'' = (-1, -1, 0, 0, 2, 0, 0), \\
3a_{1A}' &\rightarrow 3a_{1A}'' = (1, 1, -1, -1, 1, 2, 1, 0), \\
3a_{3A}' &\rightarrow 3a_{3A}'' = (0, 0, 0, 0, 1, 1, 2).
\end{align*}
\] (51)

Comparing to Table I, we see that \( \{ V_A'', a_{1A}'', a_{3A}''' \} \) is the observable sector embedding of CMM 6; this demonstrates the third equivalence of (i).

**Appendix C**

To construct the full sixteen-dimensional embedding vectors \( V, a_1, \) and \( a_3 \), simply take the direct sum of a CMM observable sector embedding (labeled by subscript \( A \)) and a hidden sector embedding (labeled by subscript \( B \)) from a corresponding table:

\[
V = (V_A; V_B), \quad a_1 = (a_{1A}; a_{1B}), \quad a_3 = (a_{3A}; a_{3B}).
\] (52)

For instance, the observable sector embedding CMM 1 from Table I may be completed by any of the embeddings in Table III. Any other hidden sector embedding which is consistent with CMM 1 will be equivalent to one of the choices given in Table III. It should be noted that CMM 8 and CMM 9 each allow two inequivalent hidden sector twist embeddings \( V_B \); as a consequence, two hidden sector embedding tables are given for each. We have abbreviated \( G_H \) by the cases defined in Table I.

**Table III: CMM 1, 3V_B = (2,1,0,0,0,0,0,0).**

| # | \( 3a_{1B} \) | \( 3a_{3B} \) | \( G_H \) | # | \( 3a_{1B} \) | \( 3a_{3B} \) | \( G_H \) |
|---|---|---|---|---|---|---|---|
| 1 | (-2,0,0,0,0,0,0,0) | (0,0,0,0,0,0,0,0) | 1 | 2 | (0,2,0,0,0,0,0,0) | (-1,0,0,0,0,0,0,0) | 1 |
| 3 | (0,2,0,0,0,0,0,0) | (1,0,0,0,0,0,0,0) | 1 | 4 | (-2,0,0,0,0,0,0,0) | (0,0,0,2,1,1,1,1) | 2 |
| 5 | (-2,0,0,0,0,0,0,0) | (0,0,0,2,1,1,1,1) | 2 | 6 | (-1,1,0,0,1,0,0,0) | (-1,0,0,0,0,0,0,0) | 2 |
| 7 | (-1,1,0,1,1,0,0,0) | (0,0,1,0,1,0,0,0) | 2 | 8 | (-1,0,1,0,1,0,0,0) | (1,1,0,0,0,0,0,0) | 2 |
| 9 | (0,1,1,1,1,0,0,0) | (0,0,0,1,1,0,0,0) | 2 | 10 | (0,1,1,1,1,0,0,0) | (0,1,1,0,0,0,0,0) | 2 |
| 11 | (0,0,0,0,0,0,0,0) | (0,0,0,2,1,1,1,1) | 2 | 12 | (0,0,0,0,0,0,0,0) | (0,0,0,2,1,1,1,1) | 2 |
| 13 | (-2,0,0,0,0,0,0,0) | (0,0,0,1,1,0,0,0) | 3 | 14 | (-1,1,0,0,1,0,0,0) | (-2,1,1,0,0,0,0,0) | 3 |
| 15 | (-1,1,0,1,1,1,0,0) | (2,1,1,1,1,1,0,0) | 3 | 16 | (0,1,1,1,1,1,0,0) | (-2,1,1,1,1,1,0,0) | 3 |
| 17 | (0,1,1,1,1,1,0,0) | (0,0,0,1,1,1,0,0,0) | 3 | 18 | (0,0,0,0,0,0,0,0) | (0,0,0,1,1,1,0,0,0) | 3 |
Table III: (continued) CMM 1, 3

| #  | $3\alpha_1 B$         | $3\alpha_3 B$         | $G_H$ | #  | $3\alpha_1 B$         | $3\alpha_3 B$         | $G_H$ |
|----|-----------------------|-----------------------|-------|----|-----------------------|-----------------------|-------|
| 19 | (-2,0,0,0,0,0,0,0)    | (0,1,-2,1,1,1,0,0)   | 4     | 20 | (-1,1,0,1,1,0,0,0)    | (-2,1,-1,0,-1,1,0,0)  | 4     |
| 21 | (-1,1,0,1,1,0,0,0)    | (-2,0,1,-1,1,0,0)    | 4     | 22 | (-1,1,0,1,1,0,0,0)    | (-2,1,0,0,0,1,1,-1)   | 4     |
| 23 | (-1,1,0,1,1,0,0,0)    | (-2,1,0,0,1,1,1)     | 4     | 24 | (-1,1,0,1,1,0,0,0)    | (2,1,1,1,0,1,0,0)     | 4     |
| 25 | (-1,1,0,1,1,0,0,0)    | (2,1,0,0,0,1,1,1)    | 4     | 26 | (-1,1,0,1,1,0,0,0)    | (-1,1,1,-1,-1,1,1,-1) | 4     |
| 27 | (0,1,1,1,1,0,0,0)     | (-2,-1,-1,-1,1,0,0)  | 4     | 28 | (0,1,1,1,1,0,0,0)     | (-2,1,0,1,1,1,0,0)    | 4     |
| 29 | (0,1,1,1,1,0,0,0)     | (2,0,-1,-1,1,0,0)    | 4     | 30 | (0,1,1,1,1,0,0,0)     | (2,1,1,1,0,1,0,0)     | 4     |
| 31 | (0,1,1,1,1,0,0,0)     | (-1,1,-1,-1,-1,1,-1) | 4     | 32 | (0,2,0,0,0,0,0,0)     | (-2,0,1,1,1,1,0,0)    | 4     |
| 33 | (0,2,0,0,0,0,0,0)     | (2,-1,1,1,1,0,0)     | 4     |    |                       |                       |       |

Table IV: CMM 2, 3

| #  | $3\alpha_1 B$         | $3\alpha_3 B$         | $G_H$ | #  | $3\alpha_1 B$         | $3\alpha_3 B$         | $G_H$ |
|----|-----------------------|-----------------------|-------|----|-----------------------|-----------------------|-------|
| 1  | (-2,0,-1,1,0,0,0,0)   | (-1,0,-1,0,0,0,0,0)   | 1     | 2  | (-2,0,-1,1,0,0,0,0)   | (1,0,1,0,0,0,0,0)     | 1     |
| 3  | (-2,1,1,0,0,0,0,0)    | (0,1,-1,0,0,0,0,0)    | 1     | 4  | (-2,0,-1,1,0,0,0,0)   | (-2,-1,1,1,0,0,0,0)   | 2     |
| 5  | (-2,0,-1,1,0,0,0,0)   | (2,1,-1,1,1,0,0,0)    | 2     | 6  | (-2,1,1,0,0,0,0,0)    | (0,0,2,1,1,1,-1)      | 2     |
| 7  | (-2,1,1,0,0,0,0,0)    | (0,0,2,1,1,1,1)       | 2     | 8  | (-1,0,1,1,1,1,-1)     | (-1,1,1,1,1,1,-1)     | 2     |
| 9  | (-1,0,1,1,1,1,-1)     | (0,1,-1,0,0,0,0,0)    | 2     | 10 | (-1,1,-1,1,1,0,0)     | (-1,-1,0,0,0,0,0,0)   | 2     |
| 11 | (-1,1,-1,1,1,1,0,0)   | (0,0,0,0,0,0,1,1)     | 2     | 12 | (-1,1,-1,1,1,1,0,0)   | (1,1,0,0,0,0,0,0)     | 2     |
| 13 | (-2,0,-1,1,0,0,0,0)   | (-1,1,-1,1,1,1,1)     | 3     | 14 | (-2,1,1,0,0,0,0,0)    | (0,0,1,1,0,0,0)       | 3     |
| 15 | (-1,0,0,1,1,1,1,-1)   | (0,0,2,1,-1,-1,1)     | 3     | 16 | (-1,0,0,1,1,1,1,-1)   | (0,0,1,1,-1,-2,-1)    | 3     |
| 17 | (-1,1,-1,1,1,1,0,0)   | (-2,-1,1,0,-1,1,0,0)  | 3     | 18 | (-1,1,-1,1,1,1,0,0)   | (2,1,1,1,1,0,0,0)     | 3     |
| 19 | (-2,0,-1,1,0,0,0,0)   | (-2,0,1,1,1,0,0,0)    | 4     | 20 | (-2,0,-1,1,0,0,0,0)   | (-2,1,0,-1,1,1,0,0)   | 4     |
| 21 | (-2,1,1,0,0,0,0,0)    | (0,-1,2,1,1,1,0,0)    | 4     | 22 | (-1,0,1,1,1,1,-1)     | (-2,-1,0,0,-1,-1,0)   | 4     |
| 23 | (-1,0,0,1,1,1,1,-1)   | (2,1,1,0,0,0,-1)      | 4     | 24 | (-1,0,0,1,1,1,1,-1)   | (0,-1,2,0,1,-1,0,0)   | 4     |
| 25 | (-1,0,0,1,1,1,1,-1)   | (0,-1,2,1,1,0,0,-1)   | 4     | 26 | (-1,0,0,1,1,1,1,-1)   | (0,0,0,0,0,1,-1)      | 4     |
| 27 | (-1,1,-1,1,1,1,0,0)   | (-2,-1,1,0,0,1,0,0)   | 4     | 28 | (-1,1,-1,1,1,1,0,0)   | (-2,0,1,0,0,-1,1,-1)  | 4     |
| 29 | (-1,1,-1,1,1,1,0,0)   | (-2,0,1,1,-1,1,0,0)   | 4     | 30 | (-1,1,-1,1,1,1,0,0)   | (-2,1,0,-1,-1,0,0)    | 4     |
| 31 | (-1,1,-1,1,1,1,0,0)   | (-2,1,0,1,1,1,0,0)    | 4     | 32 | (-1,1,-1,1,1,1,0,0)   | (2,1,1,0,0,-1,1,0)    | 4     |
| 33 | (-1,1,-1,1,1,1,0,0)   | (-1,1,1,-1,-1,1,-1)   | 4     |    |                       |                       |       |
Table V: CMM 4, $3V_B = (2,1,1,0,0,0,0,0)$.

| #  | $3a_{1B}$                      | $3a_{3B}$                      | $G_H$ | #  | $3a_{1B}$                      | $3a_{3B}$                      | $G_H$ |
|----|--------------------------------|--------------------------------|-------|----|--------------------------------|--------------------------------|-------|
| 1  | (-2,1,-1,0,0,0,0,0)           | (-1,-1,0,0,0,0,0,0)           | 1     | 2  | (-2,1,-1,0,0,0,0,0)           | (0,-1,1,0,0,0,0,0)            | 1     |
| 3  | (2,-1,1,0,0,0,0,0)            | (1,1,0,0,0,0,0,0)             | 1     | 4  | (-2,0,1,1,0,0,0,0)            | (-2,1,-1,1,0,0,0,0)           | 2     |
| 5  | (-2,0,1,1,0,0,0,0)            | (1,1,0,0,0,0,0,0)             | 2     | 6  | (-2,1,-1,0,0,0,0,0)           | (1,-1,1,1,1,1,1,-1)           | 2     |
| 7  | (-2,1,-1,0,0,0,0,0)           | (1,-1,1,1,1,1,1)              | 2     | 8  | (-1,1,0,1,1,1,1,0)            | (-1,0,-1,0,0,0,0)             | 2     |
| 9  | (-1,1,1,1,1,1,1,1)            | (-1,1,1,-1,-1,1,1)            | 2     | 10 | (-1,1,0,1,1,1,1,0)            | (0,1,0,0,0,0,0,0)             | 2     |
| 11 | (2,-1,1,0,0,0,0,0)            | (1,-1,1,1,1,1,1,-1)           | 2     | 12 | (2,-1,1,0,0,0,0,0)            | (1,-1,1,1,1,1,1)              | 2     |
| 13 | (-2,0,1,1,0,0,0,0)            | (2,1,1,1,1,1,0,0)             | 3     | 14 | (-2,0,1,1,0,0,0,0)            | (-1,1,1,1,1,1,1,-1)           | 3     |
| 15 | (-2,1,1,1,1,1,1,1)            | (-2,1,1,1,1,1,0,0)            | 3     | 16 | (-2,-1,1,1,1,0,0,0)           | (-1,1,1,1,1,1,-1)             | 4     |
| 17 | (-2,1,1,1,1,1,1,0)            | (-2,1,1,0,0,0,0,0)            | 3     | 18 | (-2,1,1,1,1,1,0,0)            | (2,0,0,-1,0,0,0)              | 3     |
| 19 | (-1,0,0,1,0,0,0,0)            | (-1,1,-1,1,1,1,0,0)           | 4     | 20 | (0,1,0,1,0,0,0,0)             | (2,0,0,1,1,0,0)               | 4     |

Table VI: CMM 6, $3V_B = (2,1,1,0,0,0,0,0)$.

| #  | $3a_{1B}$                      | $3a_{3B}$                      | $G_H$ | #  | $3a_{1B}$                      | $3a_{3B}$                      | $G_H$ |
|----|--------------------------------|--------------------------------|-------|----|--------------------------------|--------------------------------|-------|
| 1  | (-1,0,1,0,1,0,0,0)            | (-2,0,-1,1,0,0,0,0)           | 1     | 2  | (0,1,0,1,0,0,0,0)             | (-2,-1,0,1,0,0,0,0)           | 1     |
| 3  | (0,1,0,1,0,0,0,0)             | (0,-2,1,0,0,0,0,0)            | 1     | 4  | (-1,0,0,1,0,0,0,0)            | (-1,0,0,-1,1,1,1,1)           | 2     |
| 5  | (-1,0,0,1,0,0,0,0)            | (2,0,-1,1,0,0,0,0)            | 2     | 6  | (0,1,0,1,0,0,0,0)             | (-2,-1,0,1,0,0,0,0)           | 2     |
| 7  | (0,1,0,1,0,0,0,0)             | (0,-2,1,0,0,0,0,0)            | 2     | 8  | (-2,0,-1,1,1,0,0)            | (-1,-1,1,1,1,1,0)             | 2     |
| 9  | (-2,0,-1,1,1,1,0)            | (0,1,0,1,1,1,1,1)             | 2     | 10 | (-2,0,-1,1,1,1,0)            | (1,1,1,1,1,1,0)               | 2     |
| 11 | (-2,1,1,1,1,0,0,0)            | (0,2,-1,0,1,0,0,0)            | 2     | 12 | (-2,1,1,1,0,0,0,0)            | (0,0,-2,1,1,0,0)              | 2     |
| 13 | (-1,0,0,1,0,0,0,0)            | (-1,0,0,-1,1,1,1,1)           | 3     | 14 | (0,1,0,1,0,0,0,0)             | (-1,-1,1,1,1,1,1,0)           | 3     |
| 15 | (-2,0,-1,1,1,0,0,0)           | (-2,0,-1,1,0,0,0,0)           | 3     | 16 | (-2,0,-1,1,1,0,0,0)           | (-1,-1,1,1,-1,0,0)            | 4     |
| 17 | (-2,1,1,1,1,0,0,0)            | (-2,1,1,0,0,0,0,0)            | 3     | 18 | (-2,1,1,1,1,0,0,0)            | (2,0,0,-1,0,0,0)              | 3     |
| 19 | (-1,0,0,1,0,0,0,0)            | (-1,1,-1,1,1,1,0,0)           | 4     | 20 | (0,1,0,1,0,0,0,0)             | (2,0,0,1,1,0,0)               | 4     |
Table VI: (continued) CMM 6, $3V_B = (2,1,1,0,0,0,0,0)$.

| #  | $3a_{1B}$         | $3a_{3B}$         | $G_H$ | #  | $3a_{1B}$         | $3a_{3B}$         | $G_H$ |
|----|------------------|------------------|-------|----|------------------|------------------|-------|
| 21 | (0,1,1,1,0,0,0,0) | (-1,1,-1,-1,1,0,0) | 4     | 22 | (-2,0,-1,1,1,0,0,0) | (-2,-1,0,0,0,-1,0,0) | 4     |
| 23 | (-2,0,-1,1,1,1,0,0) | (-2,1,0,0,0,0,0,0) | 4     | 24 | (-2,0,-1,1,1,1,0,0) | (-1,1,0,0,-1,1,0,1) | 4     |
| 25 | (-2,0,-1,1,1,1,0,0) | (-1,1,0,0,-1,1,1) | 4     | 26 | (-2,0,-1,1,1,1,0,0,0) | (-1,1,1,0,1,1,0,1) | 4     |
| 27 | (-2,0,-1,1,1,1,0,0) | (2,0,0,0,-1,1,0,0) | 4     | 28 | (-2,0,-1,1,1,0,0,0) | (1,1,0,1,1,1,0,0) | 4     |
| 29 | (-2,1,1,1,1,0,0,0) | (-2,0,-1,0,0,0,0,0) | 4     | 30 | (1,1,0,1,0,0,0,0) | (1,0,0,0,1,0,0,0,0) | 4     |
| 31 | (-2,1,1,1,1,0,0,0) | (2,0,0,1,0,0,0,0) | 4     | 32 | (-2,1,1,1,1,0,0,0) | (1,1,0,1,0,0,0,0) | 4     |

Table VII: CMM 8, $3V_B = (1,1,0,0,0,0,0,0)$.

| #  | $3a_{1B}$         | $3a_{3B}$         | $G_H$ | #  | $3a_{1B}$         | $3a_{3B}$         | $G_H$ |
|----|------------------|------------------|-------|----|------------------|------------------|-------|
| 1  | (0,0,1,1,1,1,0,0) | (0,0,-1,-1,-1,1,0) | 1     | 2  | (0,0,1,1,1,0,0,0) | (-1,-2,0,0,0,-1,0,0) | 2     |
| 3  | (0,0,1,1,1,1,0,0) | (0,0,2,0,0,0,1,-1) | 2     | 4  | (0,0,1,1,1,0,0,0) | (0,0,-1,-1,-1,1,1,-1) | 3     |
| 5  | (0,0,1,1,1,1,0,0) | (0,0,2,0,0,0,1,1)  | 4     |     |                   |                   |       |

Table VIII: CMM 8, $3V_B = (2,1,1,1,1,0,0,0)$.

| #  | $3a_{1B}$         | $3a_{3B}$         | $G_H$ | #  | $3a_{1B}$         | $3a_{3B}$         | $G_H$ |
|----|------------------|------------------|-------|----|------------------|------------------|-------|
| 1  | (-1,0,0,0,-1,1,1,0) | (-1,-1,-1,-1,0,0,1) | 2     | 2  | (-1,0,0,0,-1,1,1,0) | (-1,1,1,-1,0,0,0,0,0,1) | 2     |
| 3  | (0,1,1,0,0,1,1,0)  | (1,-1,-1,1,0,0,0,0) | 2     | 4  | (-1,0,0,0,-1,1,1,0) | (-1,1,0,0,-1,1,1,0) | 3     |
| 5  | (-1,0,0,0,-1,1,1,0) | (0,0,-1,-1,-1,1,0,0) | 3     | 6  | (-1,1,0,0,0,1,0,0,0) | (-2,0,0,-1,0,0,0,0) | 3     |
| 7  | (-1,1,0,0,0,1,0,0)  | (1,1,1,1,1,1,0,0,0) | 3     | 8  | (-1,0,0,0,-1,1,1,0,0) | (-2,0,1,-1,0,0,0,0) | 4     |
| 9  | (-1,0,0,0,-1,1,1,0) | (-2,0,0,0,1,0,0,0,0) | 4     | 10 | (-1,0,0,0,-1,1,1,0,0) | (-2,1,0,0,0,0,0,0,0,1) | 4     |
| 11 | (-1,0,0,0,-1,1,1,0) | (-1,0,-1,1,0,0,0,0) | 4     | 12 | (-1,0,0,0,-1,1,0,0,0) | (-1,1,1,1,1,0,0,0,0) | 4     |
| 13 | (-1,1,1,0,0,0,1,0,0) | (-2,1,0,0,0,0,0,0,0) | 4     | 14 | (-1,1,1,1,0,0,1,0,0) | (-1,-1,1,-1,1,0,0,0) | 4     |
| 15 | (-1,1,1,0,0,0,1,0,0) | (-1,-1,1,1,0,0,0,0,0) | 4     | 16 | (0,1,1,1,0,0,0,0,0)  | (-2,0,0,0,1,-1,0,0,0) | 4     |
| 17 | (0,1,1,1,1,0,0,0,0) | (0,1,0,-2,1,0,0,0,0) | 4     |     |                   |                   |       |
Table IX: CMM 9, $3V_B = (1,1,0,0,0,0,0,0)$.

| #  | $3a_{1B}$         | $3a_{3B}$         | $G_H$ | #  | $3a_{1B}$         | $3a_{3B}$         | $G_H$ |
|----|-------------------|-------------------|-------|----|-------------------|-------------------|-------|
| 1  | (1,0,1,0,0,0,0,0) | (-1,1,-1,1,1,1,1) |       | 1  | (1,0,1,0,0,0,0,0) | (1,2,0,1,1,1,0,0) | 2     |
| 3  | (-1,1,1,1,1,1,-1) | (0,0,2,0,-1,-1,1) | 2     | 4  | (-1,1,1,1,1,1,-1) | (0,0,0,0,-1,-1,0) | 3     |
| 5  | (-1,1,1,1,1,1,-1) | (-1,-2,1,0,0,0,-1) | 4     |

Table X: CMM 9, $3V_B = (2,1,1,1,1,0,0,0)$.

| #  | $3a_{1B}$         | $3a_{3B}$         | $G_H$ | #  | $3a_{1B}$         | $3a_{3B}$         | $G_H$ |
|----|-------------------|-------------------|-------|----|-------------------|-------------------|-------|
| 1  | (-1,0,0,0,1,0,0)  | (-1,0,0,0,-1,0,0) | 2     | 2  | (-1,0,0,0,1,0,0)  | (-1,1,-1,-1,-1,1) | 2     |
| 3  | (-2,0,0,0,-1,1,1) | (-1,-1,-1,-1,1,1) | 2     | 4  | (-1,0,0,0,1,0,0)  | (-2,1,0,0,-1,1,1) | 3     |
| 5  | (-2,0,0,0,-1,1,1) | (-2,0,0,-1,1,1,0) | 3     | 6  | (-1,0,0,0,1,0,0)  | (-2,0,-1,-1,1,1,0) | 4     |
| 7  | (0,1,0,0,1,0,0)   | (-2,1,0,0,-1,1,0) | 4     | 8  | (-2,0,0,0,-1,1,1,-1) | (-2,0,-1,-1,1,0,-1) | 4     |
| 9  | (-2,0,0,-1,1,1,1) | (-2,1,0,0,0,-1,1) | 4     | 10 | (-2,0,0,0,-1,1,1,1) | (-2,1,0,0,-1,1,1) | 4     |
| 11 | (-2,0,0,-1,1,1,1) | (-1,-1,-1,-1,1,1) | 4     | 12 | (-2,0,0,0,-1,1,1,1) | (-1,1,1,-1,1,1,1) | 4     |
| 13 | (-2,0,0,-1,1,1,1) | (-1,1,1,-1,1,1,-1) | 4     | 14 | (-2,0,0,0,-1,1,1,1) | (-1,1,-1,1,1,1,1) | 4     |
| 15 | (-2,0,0,-1,1,1,1) | (1,1,1,1,1,1,-1)  | 4     | 16 | (-2,0,0,0,-1,1,1,1) | (-2,0,-1,1,0,1,0,-1) | 4     |
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