THE JUCYS-MURPHY BASIS AND SEMISIMPLIcity CRITERIA FOR THE 

q

-BRAUER ALGEBRA

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Abstract. We construct the Jucys-Murphy elements and the Jucys-Murphy basis for the q-Brauer algebra in the sense of [11, Definition 2.4]. We also give a necessary and sufficient condition for the q-Brauer algebra being (split) semisimple over an arbitrary field.

1. Introduction

In his remarkable work [23], Schur classified the polynomial representations of the complex general linear group by decomposing tensor products of its natural representation. The symmetric group appears naturally in Schur’s work and plays an important role. Later on Brauer [2] studied the related problem for the complex orthogonal group and the complex symplectic group and found a new family of complex associative algebras called the Brauer algebras or Brauer centralizer algebras. These algebras play roles similar to those of symmetric groups.

There are many ways to generalize results in [2, 23]. See [6, 25] where the complex classical groups are replaced by quantum enveloping algebras of classic types. Hecke algebras and Birman-Murakami-Wenzl algebras [11,13] come into the picture. In [24], Wenzl considered tensor products of the natural representation of certain coideal algebras [7] and introduced a new associative algebra, called the q-Brauer algebra \( B_n(q,z) \) for any positive integer \( n \geq 2 \). See also [3] for the explicit Schur-Weyl duality between the \( \hat{\mathfrak{g}} \)-quantum groups of types AI or AII and the q-Brauer algebras. The q-Brauer algebra is isomorphic to an endomorphism algebra of the q-Brauer category [20] and is isomorphic to a quotient of the associative algebra introduced by Molev [12].

The q-Brauer algebra has been used to construct new subfactors of type II \(_1\) factors [27]. It has some application in module categories of fusion categories of type A corresponding to certain symmetric spaces [26]. In [24, Theorem 3.8], Wenzl proved that \( B_n(q,z) \) is free over a commutative ring with rank \((2n-1)!!\). When the ground ring is a field, Wenzl also proved that \( B_n(q,z) \) is semisimple if \( z^2 \neq q^{2k} \) for \( k \in \mathbb{Z}, |k| \leq n \) and \( q^2 \) is not an \( \ell \)-th root of unity, \( \ell \leq n \) [24, Theorem 5.3]. Under the assumption \( z^2 \neq 1 \), Dung Tien Nguyen proved that \( B_n(q,z) \) is a cellular algebra in the sense of [8, Definition 1.1] and classified simple \( B_n(q,z) \)-modules over an arbitrary field [22, Theorem 3.2, Theorem 4.1]. He also gave an explicit condition for \( B_3(q,z) \) being semisimple over an arbitrary field [22, Proposition 5.1]. Dung Tien Nguyen’s conditions on parameters [22, Example 5.7] are different from those for the corresponding Birman-Murakami-Wenzl algebra in [17, Theorem 5.9]. Therefore the q-Brauer algebra is another q-deformation of the Brauer algebra.

We expect to generalize our previous results in [17,18] to the q-Brauer algebra. More explicitly, we want to construct Jucys-Murphy basis and orthogonal basis of the q-Brauer algebra so as to compute Gram determinants, determine explicitly the semisimplicity criteria, classify blocks and compute decomposition matrices (when \( q \) is not a root of unity) etc. The first difficult problem that we faced is to find out a family of commutative elements in \( B_n(q,z) \) called the Jucys-Murphy elements or JM elements in the sense of [11, Definition 2.4]. By studying classical branching rule for the q-Brauer algebra in details, we are able to define the Jucys-Murphy basis of the q-Brauer algebra. Since the summation of Jucys-Murphy elements is not a central element, we have to make extra efforts to prove that Jucys-Murphy elements act on Jucys-Murphy basis as upper-triangular matrices. As a by-product one can use arguments in [11] to define an orthogonal basis of \( B_n(q,z) \) in generic case. In this paper, we do not give details since we have not found out an efficient way to compute Gram
Theorem A. For any $n \geq 2$, let $B_n(q, z)$ be defined over a field $F$ which contains invertible elements $z, z-z^{-1}, q$ and $q-q^{-1}$. Let $e$ be the quantum characteristic of $q^2$. Then $B_n(q, z)$ is (split) semisimple if and only if $e > n$ and $z^2 \neq q^{2a}$ where
\[
a \in \{i \in \mathbb{Z} \mid 4 - 2n \leq i \leq n - 2\} \setminus \{i \in \mathbb{Z} \mid 4 - 2n < i \leq 3 - n, 2 \mid i\}. \tag{1.1}
\]

Here is a brief outline of the content of this paper. In the second section, we recall some basic results on the $q$-Brauer algebra. In section 3, we study the classical branching rule. In section 4, we construct the Jucys-Murphy basis and the Jucys-Murphy elements of the $q$-Brauer algebra. Restriction and induction functors are given in section 5. We describe explicitly the radical of certain cell modules in section 6. This enables us to prove Theorem A in section 7.

2. The $q$-Brauer algebra

In this section, we recall the definitions of the Hecke algebra and the $q$-Brauer algebra and give some basic results on them.

2.1. The Hecke algebra associated to the symmetric group. The symmetric group $S_n$ is the Coxeter group with $\{s_i \mid 1 \leq i \leq n - 1\}$ as its distinguished generators subject to the relations:
\[
s_i^2 = 1, \quad s_is_j = s_js_i, \quad s_ks_{k+1}s_k = s_{k+1}s_ks_k \tag{2.1}
\]
where $1 < |i - j|$ and $1 \leq k \leq n - 2$. In this paper we assume $\mathbb{Z} = \mathbb{Z}[q, q^{-1}]$, the ring of Laurent polynomials in indeterminate $q$ with coefficients in $\mathbb{Z}$. The Hecke algebra $H_n$ associated to $S_n$ is the unital associative $\mathbb{Z}$-algebra generated by $\{T_i \mid 1 \leq i \leq n - 1\}$ subject to the relations
\[
(T_i - q)(T_i + q^{-1}) = 0, \quad T_iT_j = T_jT_i, \quad T_kT_{k+1}T_k = T_{k+1}T_kT_k+1 \tag{2.2}
\]
where $1 < |i - j|$ and $1 \leq k \leq n - 2$.

Suppose $w \in S_n$. Then $s_is_{i_2} \cdots s_{i_k}$ is a reduced expression of $w$ if $w = s_is_{i_2} \cdots s_{i_k}$ and $k$ is minimal. In this case, $k$ is the length of $w$ and is denoted by $l(w)$. Let $T_w = T_{i_1}T_{i_2} \cdots T_{i_k}$ if $s_is_{i_2} \cdots s_{i_k}$ is a reduced expression of $w$. It is known that $T_{w}$ is independent of any reduced expression of $w$ and $\{T_w \mid w \in S_n\}$ is a $\mathbb{Z}$-basis of $H_n$. Let $\tau : H_n \rightarrow \mathbb{Z}$ such that $\tau(\sum_w a_wT_w) = a_1$ where 1 is the identity element of $S_n$. Then $\tau$ is a trace function on $H_n$ such that
\[
\tau(T_xt_y) = \delta_{x,y}^{-1} \tag{2.3}
\]
for all $x, y \in S_n$ \cite[Proposition 1.16]{[9]}

2.2. The $q$-Brauer algebra. Let $k = \mathbb{Z}[q, q^{-1}, z, z^{-1}, (q-q^{-1})^{-1}, (z-z^{-1})^{-1}]$, where $z$ is another indeterminate. In this paper we always assume
\[
\delta = \frac{z - z^{-1}}{q - q^{-1}}. \tag{2.4}
\]
Then $\delta$ is invertible in $k$. The original $q$-Brauer algebra is available over $\mathbb{Z}[q, q^{-1}, z, z^{-1}, (q-q^{-1})^{-1}]$. In this case, $\delta$ may not be invertible. In \cite{[21]}, Dung Tien Nguyen proved that $q$-Brauer algebra is a cellular algebra in the sense of \cite[Definition 1.1]{[8]} under the assumption that $\delta$ is invertible. Since our results depend on the cellular structure of the $q$-Brauer algebra, we need this additional assumption.

Definition 2.1. \cite[Definition 3.1]{[21]} \cite[Definition 3.1]{[22]} Suppose $n \geq 2$. The $q$-Brauer algebra $B_n(q, z)$ or just $B_n$ is a unital associative $k$-algebra generated by $T_1, T_2, \cdots, T_{n-1}$ and $E_1$ subject to the relations in (2.2) together with
\[
\begin{align*}
(1) \quad & E_1^2 = \delta E_1, \\
(2) \quad & T_iE_1 = qE_1 = E_1T_i, \\
(3) \quad & E_1T_iE_1 = zE_1, \\
(4) \quad & T_iE_1 = E_1T_i \text{ if } i > 2, \\
(5) \quad & T_2T_1^{-1}T_2^{-1}E_2(2) = E_2(2) = E_2(2)T_2T_3T_1^{-1}T_2^{-1}, \\
& E_2(2) = E_1T_2T_3T_1^{-1}T_2^{-1}E_1.
\end{align*}
\]

Remarks 2.2. Suppose $z = q^a$ for some $a \in \mathbb{Z} \setminus \{0\}$. It follows from \cite[Remark 3.1]{[21]} that the classical limit of $B_n(q^a)$ is the Brauer algebra $B_n(a)$ over $\mathbb{Z}$. In this case, $\lim_{q \rightarrow 1} \delta = a$ and $T_i$ becomes the simple reflection $s_i$ and $E_1$ can be identified with the corresponding element in $B_n(a)$.
We write $B_0 = B_1 = k$. Thanks to [24] Theorem 3.8], $B_n$ is free over $k$ with rank $(2n − 1)!!$ and the subalgebra of $B_n$ generated by $T_1, T_2, \ldots, T_{n−1}$ is isomorphic to the Hecke algebra $\mathcal{H}_n$. For this reason, one can use symbols $T_i$'s in both $\mathcal{H}_n$ and $B_n$. Moreover,

$$\mathcal{H}_n \cong B_n/(E_1)$$

(2.5)

where $\langle E_1 \rangle$ is the two-sided ideal of $B_n$ generated by $E_1$. Let $F$ be a field containing units $z, q, q^{-1}, z − z^{-1}$. Throughout,

$$B_{n,F} := B_n \otimes_k F,$$

where $F$ can be considered as the left $k$-module on which $z, q \in k$ act via the corresponding $z$ and $q$ in $F$. If there is no confusion, we also denote $B_{n,F}$ by $B_n$.

**Lemma 2.3.** [21 Proposition 3.14] There is a $k$-linear anti-involution $\sigma : B_n \to B_n$ such that $\sigma(x) = x$ for any $x \in \{E_1, T_1, T_2, \ldots, T_{n−1}\}$.

**Definition 2.4.** Suppose $0 \leq k \leq \lfloor \frac{n}{2} \rfloor − 1$ and $1 \leq \ell \leq n − 1$. Define

1. $E^0 = 1$ and $E^{k+1} = E_1 T_{2k+2} E_k$, where $T_{i,j} = T_{s_{i,j}}$ and

$$s_{i,j} = \begin{cases} s_is_{i+1,j} & \text{if } i < j, \\ 1 & \text{if } i = j, \\ s_{i,j+1}s_j & \text{if } i > j. \end{cases}$$

2. $E_\ell = T_{\ell,1} T_{2,\ell+1} E_1 T_{\ell+1,\ell} E_{\ell−1}$.

Definition 2.3(1) was given in [24 (3.2)] where $E^k$ was denoted by $E_i$.

**Lemma 2.5.** [24 (3.2), Lemma 3.3], [21] Lemma 3.4, Proposition 3.14] Suppose $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ and $1 \leq \ell \leq \lfloor \frac{n}{2} \rfloor − 1$.

1. $E^{\ell+1} = E_1 T_{2\ell+2} \ell+1 E_{\ell+1} T_{2\ell+2} E_1$,
2. $\sigma(E^k) = E^k$,
3. $E_1 T_{1−1} T_{3} T_2 E_1 = E^2 = E_1 T_1 T_{3−1} T_{2−1} E_1$,
4. $E^k T_{2k−1} = q E^k$.

**Lemma 2.6.** For all admissible $i$ and $j$, we have:

1. $E_{i+1} = T_{i+1} E_1 T_{i+1} T_{i−1}$,
2. $T_i E_1 = E_1 T_i$ if $|i−j| ≥ 2$,
3. $E_i E_j = E_j E_i$ if $|i−j| ≥ 2$,
4. $E^i = E_1 E_3 \cdots E_{2j−1}$.

**Proof.** Obviously (1) follows immediately from Definition 2.3(2). If $i ≥ j + 2$, then (2) follows from (2.2) and Definition 2.3(4). If $i ≤ j − 2$, then $T_i E_j = T_j T_{j−1} T_{i} E_i T_{j−1} E_{j−1} = E_j T_i$ and (2) follows. It is not difficult to verify

$$E_1 E_3 = E_{(2)} = E_3 E_1.$$  

(2.6)

If $j > 3$, then $E_1 E_j = E_1 T_{j−1} T_{j−1} E_{j−1} T_{j−1}$, where the last equality follows from the induction assumption on $j − 1$ and Definition 2.3(4). Suppose $i > 1$. We can assume $j ≥ i + 2$ without losing of any generality when we prove (3). By (2) and induction assumption on $i−1$, $E_i E_j = T_{i−1} T_{i−1} E_{i−1} = T_{i−1} E_i$, in any case, we have verified (3). Thanks to (2.6), $E^2 = E_1 E_3$. In general, using (3), induction assumption on $j−1$, Lemma 2.5(5) and $\sigma$ in Lemma 2.3 yields

$$E^i = E_1 E_3 \cdots E_{2j−1} = T_{2j−1−1} T_{2j−1} E_1 T_{2j−1} T_{2j−1} E_j^{−1} = T_j T_{j−1} E^i,$$

(proving (4). $E_1 T_2 E_1 = z E_1$ is given in Definition 2.1(3). The required formula for $E_1 T_{2−1} E_1$ follows from the quadratic relation in (2.2) and Definition 2.1(3). In general, by induction assumption on $i−1$, we have

$$E_i T_{i+1} E_i = T_{i−1} T_{i−1} E_{i−1} T_{i+1} T_{i+1} E_{i−1} T_{i−1} T_{i−1} = z \pm 1 E_i.$$
proving (5). Obviously, (6) follows from Definition 2.1(1) and (7) was given in Lemma 3.3]. Finally, (8) follows from the following computation:

\[
E_{2i-1}T_{2i-1}^{-1}T_{2i}^{-1}E_i = T_{2i-1}^{-1}T_{2i}^{-1}E_1T_{2i+1}^{-1}T_{2i-1}^{-1}E_i
\]

\[
=q^{-1}T_{2i-1}^{-1}T_{2i}^{-1}E_1T_{2i+1}^{-1}T_{2i-1}^{-1}E_i
\]

\[
=q^{-1}T_{2i-1}^{-1}T_{2i}^{-1}E_1T_{2i+1}T_{2i-1}^{-1}E_i^{-1}T_{2i}^{-1}E_{2i-1}
\]

\[
=q^{-1}T_{2i-1}^{-1}T_{2i}^{-1}E_iT_{2i-1}^{-1}E_{2i-1}
\]

\[
=z^{-1}q^{-1}T_{2i-1}T_{2i}^{-1}E_i
\]

\[
=z^{-1}q^{-1}E_i
\]

where the forth (resp., fifth, resp., sixth) equation follows from (7) (resp., (5), resp., Lemma 2.5(2)(5)).

\[\square\]

**Proposition 2.7.** Suppose \(n \geq 2\) and \(E_1 = qz^{-1}T_1^{-1}T_2E_1\). Then \(E_1^2 = \tilde{E}_1\). Moreover,

1. There is an algebra homomorphism \(\phi : B_n \rightarrow \tilde{E}_1 B_n \tilde{E}_1\) sending 1, \(E_1, T_i\) to \(\tilde{E}_1, \tilde{E}_1 E_3\) and \(\tilde{E}_1 T_{i+2}, 1 \leq i \leq n - 3\), respectively.
2. \(\phi(E^3) = qz^{-1}T_1^{-1}E_2E_3 + 1\) for any \(0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1\).

**Proof.** It is easy to verify that \(E_1^2 = \tilde{E}_1\), \(\tilde{E}_1 E_3 \tilde{E}_1 = \tilde{E}_1 E_3\) and \(\tilde{E}_1 T_i \tilde{E}_1 = \tilde{E}_1 T_i\) if \(3 \leq i \leq n - 1\). So \(\phi(E_1), \phi(T_1), \ldots, \phi(T_{n-3})\) satisfy (2.2) and Definition 2.1(1)-(4). We remark that the quadratic relation in (2.2) becomes \((E_1 T_i - q \tilde{E}_1)(\tilde{E}_1 T_i + q^3 \tilde{E}_1) = 0, 3 \leq i \leq n - 1\). We have

\[
\phi(E^2) = \tilde{E}_1 E_3 T_3^{-1}T_4^{-1}\tilde{E}_1 E_3
\]

\[
= qz^{-1}T_1^{-1}T_2E_1T_2T_3T_1^{-1}T_2^{-1}E_1T_4T_3^{-1}T_4^{-1}\tilde{E}_1 E_3
\]

\[
= qz^{-1}T_1^{-1}T_2E_1T_2T_3T_4T_5T_1^{-1}T_2^{-1}T_3^{-1}T_4^{-1}E_2
\]

\[
= qz^{-1}T_1^{-1}T_2E_3
\]

where the last equality follows from Definition 2.4(1). So,

\[
\phi(E^2T_3T_1^{-1}T_2^{-1}) = qz^{-1}T_1^{-1}T_2E_3 T_4T_5T_3^{-1}T_4^{-1} = qz^{-1}T_1^{-1}T_2E_3 = \phi(T_2T_3T_1^{-1}T_2^{-1}E_2)
\]

This proves that the images of generators also satisfy Definition 2.1(5) and hence \(\phi\) is an algebra homomorphism. When \(i = 0, 1, (2)\) follows from the definition of \(\phi\). In general, by induction assumption on \(i - 1\) we have

\[
\phi(E^3) = \phi(E_1 T_2T_3T_1^{-1}T_2^{-1}E_i^{-1})
\]

\[
= qz^{-1}\phi(E_1 T_2T_3T_1^{-1}T_2^{-1})T_1^{-1}T_2E_i
\]

\[
= qz^{-2}T_1^{-1}T_2E_1T_2T_3T_1^{-1}T_2^{-1}E_1T_4T_5T_2^{-1}T_3^{-1}T_4^{-1}T_2E_i
\]

\[
= qz^{-2}T_1^{-1}T_2E_1T_2T_3T_4T_5T_1^{-1}T_2^{-1}T_3^{-1}T_4^{-1}T_2E_i
\]

\[
= qz^{-1}T_1^{-1}T_2E_i
\]

where the last equality follows from Definition 2.4(1). This proves (2).

\[\square\]

### 3. The Classical Branching Rule

In subsection 3.1, we recall some well-known results on Hecke algebras. In the remaining part of this section, we study the classical branching rule for the \(q\)-Brauer algebra.

**3.1. Cell filtration of cell modules and permutation modules for Hecke algebras.** For any \(0 \leq f \leq [n/2]\), let \(S_{2f+1,n}\) be the symmetric group on letters \([2f + 1, 2f + 2, \ldots, n]\). When \(n\) is even and \(f = n/2\), we set \(S_{2f+1,n} = 1\). Let \(H_{2f+1,n}\) be the Hecke algebra associated to \(S_{2f+1,n}\). Then \(H_{n-2f} \cong H_{2f+1,n}\). The corresponding isomorphism sends \(T_i\) to \(T_{2f+i}, 1 \leq i \leq n - 2f - 1\).

Recall that a composition \(\lambda\) of a non-negative integer \(d\) is a sequence \((\lambda_1, \lambda_2, \cdots)\) of non-negative integers such that \(|\lambda| := \sum_{i \geq 1} \lambda_i = d\). If \(\lambda_i \geq \lambda_{i+1}\) for all possible \(i\), then \(\lambda\) is called a partition. Given a positive integer \(e\), a partition \(\Lambda\) of \(d\) is called \(e\)-restricted if \(\lambda_i - \lambda_{i+1} < e\) for all possible \(i\). When \(e > d\), any partition of \(d\) is \(e\)-restricted. Let \(\Lambda(d)\) (resp., \(\Lambda^+(d)\), resp., \(\Lambda^+\)) be the set of
all compositions (resp., partitions, resp., $e$-restricted partitions) of $d$, where $e$ is always the quantum characteristic of $q^2$. Then $e$ is the minimal positive integer such that

$$1 + q^2 + \cdots + q^{2e-2} = 0.$$ 

If such a positive integer does not exist, then $e = \infty$. In this case, $q^2$ is not a root of unity. It is known that each of $\Lambda(d)$, $\Lambda^+(d)$ and $\Lambda^+(d)$ is a poset with dominance order $\leq$ defined on it such that $\mu \trianglelefteq \lambda$ if $\sum_{j=1}^i \mu_j \leq \sum_{j=1}^i \lambda_j$ for all possible $i$. If $\mu \trianglelefteq \lambda$ and $\mu \neq \lambda$, we write $\mu < \lambda$.

Each composition $\lambda$ of $n - 2f$ corresponds to the Young diagram $[\lambda]$ such that there are $\lambda_i$ boxes in the $i$-th row of $[\lambda]$. A $\lambda$-tableau $t$ is obtained from $[\lambda]$ by inserting $2f + 1, 2f + 2, \ldots, n$ into $[\lambda]$ without repetition. In this case, we write shape$(t) = \lambda$ and call $\lambda$ the shape of $t$. If the entries in $t$ increase from left to right along row and down column, $t$ is called standard. In this case, $\lambda \in \Lambda^+(n - 2f)$. Let $\mathcal{F}^\text{std}(\lambda)$ be the set of all standard $\lambda$-tableaux. Then $t^\lambda \in \mathcal{F}^\text{std}(\lambda)$, where $t^\lambda$ is obtained from $[\lambda]$ by inserting $2f + 1, 2f + 2, \ldots, n$ successively from left to right along the rows of $[\lambda]$. For any $\lambda \in \Lambda^+(n - 2f)$, let $\mathcal{G}_\lambda$ be the Young subgroup with respect to $\lambda$. Then $\mathcal{G}_\lambda$ is the subgroup of $\mathcal{G}_{2f+1,n}$ which stabilizes the entries in each row of $t^\lambda$. For example, $\mathcal{G}_\lambda$ is the subgroup of $\mathcal{G}_{3,9}$ generated by $s_3, s_4, s_6$ and $s_8$ if $n = 9, f = 1$ and $\lambda = (3, 2, 2)$. In this case,

$$t^\lambda = \begin{pmatrix}
3 & 4 & 5 \\
6 & 7 \\
8 & 9
\end{pmatrix}.$$ 

For each $t \in \mathcal{F}^\text{std}(\lambda)$, there is a distinguished right coset representative $d(t)$ of $\mathcal{G}_\lambda$ in $\mathcal{G}_{2f+1,n}$ such that $t = t^\lambda d(t)$. Suppose $s, t \in \mathcal{F}^\text{std}(\lambda)$. Following [9], define

$$x_{st} = T_{d(s)}^* x\lambda T_{d(t)},$$

where $x\lambda = \sum_{w \in \mathcal{G}_\lambda} q^{\ell(w)} T_w$ and $*$ is the anti-involution on $\mathcal{G}_{2f+1,n}$ fixing all generators $T_j$’s.

**Theorem 3.1.** [9 Theorem 3.20] The Hecke algebra $H_{2f+1,n}$ is free over $\mathcal{Z}$ with basis

$$\{x_{st} \mid s, t \in \mathcal{F}^\text{std}(\lambda), \lambda \in \Lambda^+(n - 2f)\}.$$ 

It is a cellular basis in the sense of [8 Definition 1.1]. The required anti-involution is $*$ as above.

Suppose $(\lambda, \mu) \in \Lambda^+(n - 2f) \times \Lambda^+(n - 2f - 1)$. The partition $\mu$ is obtained from $\lambda$ by removing a removable node, say $p = (k, \lambda_k)$ of $\lambda$ if $\mu_j = \lambda_j$, $j \neq k$ and $\mu_k = \lambda_k - 1$. In this case, $\lambda$ is obtained from $\mu$ by adding the addable node $p$ of $\mu$. We write either $\lambda = \Lambda \cup p$ or $\mu = \lambda \setminus p$. Let $\mathcal{R}_\lambda$ be the set of all partitions obtained from $\lambda$ by removing a removable node and $\mathcal{A}_\lambda$ the set of all partitions obtained from $\lambda$ by adding an addable node.

For any $\lambda \in \Lambda^+(n - 2f)$, let $S^\lambda$ be the cell module of $H_{2f+1,n}$ with respect to the Jucys-Murphy basis in Theorem 3.1. By [8 Definition 2.1] $S^\lambda$ can be identified with the free $\mathcal{Z}$-module with basis $\{x_t \mid t \in \mathcal{F}^\text{std}(\lambda)\}$ where

$$x_t := x\lambda t + H^\lambda_{2f+1,n},$$

and $H^\lambda_{2f+1,n}$ is the free $\mathcal{Z}$-submodule of $H_{2f+1,n}$ spanned by $\{x_{us} \mid u, s \in \mathcal{F}^\text{std}(\mu), \mu \triangleright \lambda\}$. It is also a two-sided ideal of $H_{2f+1,n}$. Suppose $n > 2f$. Write $\mathcal{R}_\lambda = \{\mu(i) \mid 1 \leq i \leq a\}$ for some positive integer $a$ such that

$$\mu(1) \triangleright \mu(2) \triangleright \cdots \triangleright \mu(a).$$

For any standard $\lambda$-tableau $t$, let $t_{i-1}^\lambda$ be obtained from $t$ by removing the entry $n$. Then $t_{i-1}^\lambda \in \mathcal{F}^\text{std}(\mu)$ for some $\mu \in \mathcal{R}_\lambda$. Let $S^\lambda_{i-1} = \mathcal{Z}$-span$\{x_{t_{j-1}^\lambda} \mid t_{j-1}^\lambda \in \mathcal{F}^\text{std}(\mu(j)), 1 \leq j \leq i\}$. Then $S^\lambda_{i-1}$ is a right $H_{2f+1,n-1}$-module such that

$$S^\lambda_{i-1} \supset S^\lambda_{i-2} \supset \cdots \supset S^\lambda_0 = 0.$$ 

**Theorem 3.2.** [9 Proposition 6.1] As $H_{2f+1,n-1}$-modules, $S^\lambda_{i-1}/S^\lambda_{i-2} \cong S^{\mu(i)}_{i-1}$, $1 \leq i \leq a$, where $S^{\mu(i)}_{i}$ is the cell module with respect to the Jucys-Murphy basis of $H_{2f+1,n-1}$ in Theorem 3.1.

Suppose $(\lambda, \mu) \in \Lambda(n - 2f) \times \Lambda(n - 2f)$. Recall that a $\lambda$-tableau of type $\mu$ is obtained from $[\lambda]$ by inserting integers $2f + i$ into $[\lambda]$ such that $2f + i$ appears $\mu_i$ times. A $\lambda$-tableau $S$ of type $\mu$ is called row semi-standard if the entries in each row of $S$ are non-decreasing from left to right. A row semi-standard tableau $S$ is called semi-standard if $\lambda$ is a partition and the entries in each column of $S$ are strictly increasing downward. Let $T^ss(\lambda, \mu)$ be the set of all semi-standard $\lambda$-tableaux of type $\mu$. When $\mu = (1, 1, \ldots, 1) \in \Lambda^+(n - 2f)$, $T^ss(\lambda, \mu)$ is $\mathcal{F}^\text{std}(\lambda)$. 


Lemma 3.5. Suppose that $d_k$ of $\mathcal{H}$.

Lemma 3.6. Suppose $d_k$ of $\mathcal{H}$.

Proof. In this subsection, we assume that $\mathcal{B}_n(q, z)$ is defined over $\mathbb{k}$. Let

$$
\Lambda_n = \left\{ (\ell, \mu) \mid \lambda \in \Lambda^+(n-2f), 0 \leq f \leq \left\lfloor \frac{n}{2} \right\rfloor \right\}.
$$

There is a partial order $\leq$ on $\Lambda_n$ such that $(\ell, \mu) \leq (f, \lambda)$ if $\ell < f$ or $\ell = f$ and $\mu \leq \lambda$. Write $(\ell, \mu) < (f, \lambda)$ if $(\ell, \mu) \leq (f, \lambda)$ and $(\ell, \mu) \neq (f, \lambda)$. Later on, we identify each $(\ell, \mu)$ in $\Lambda_n$ with $\mu$. So $\mu \leq \lambda$ if $|\mu| > |\lambda|$ or $|\mu| = |\lambda|$ and $\mu \leq \lambda$.

Definition 3.4. Define $D_{f,n-1}$ and $h \in \{s_{n-1}, s_{n-2}, \ldots, s_2, s_1\}$ such that $h = s_{i_1} > s_{i_2} > \cdots > s_{i_f}$, $1 \leq i_1 < \cdots < i_f < 2k - 1 \leq j_k < i_k \leq n$, for $1 \leq k \leq f$.

Lemma 3.5. Suppose $d^{-1} \in D_{f,n-1}$ and $h \in \{s_{n-1}, s_{n-2}, \ldots, s_2, s_1\}$. We have $dh = s_{i_1}d_1$ for some integer $1 \leq i_1 \leq n$ and $d_1 \in D_{f,n-2}$.

Proof. Suppose $2f - 1 \leq j_f < i_f \leq n - 1$ and $2f < k \leq n - 1$. We have

$$
s_{j_f,2f-1}^{s_{i_f,2f}k,n} = \begin{cases} 
s_{k,n}^{s_{j_f,2f-1}s_{i_f,2f}} & \text{if } i_f < k, \\
_{k-1,n}^{s_{j_f,2f-1}s_{i_f,2f-1,2f}} & \text{if } i_f \geq k, j_f < k - 1, \\
_{k-2,n}^{s_{j_f,2f-1}s_{i_f,2f-1,2f}} & \text{if } i_f \geq k, j_f \geq k - 1.
\end{cases}
$$

Now the result follows if we use $\left[ \frac{n}{2} \right] f$ times to rewrite $dh$ successively.

Lemma 3.6. For any $0 \leq f \leq \left[ \frac{n}{2} \right]$, $D_{f,n}^{-1} \subset B_{f,n}$, where $D_{f,n}^{-1} = \{d \mid d^{-1} \in D_{f,n}\}$.

Proof. The result for $n = 2$ is trivial. In general, write $d = d_{i_f,j_f}d_{i_{f-1},j_{f-1}} \cdots d_{i_1,j_1}$ for any $d \in D_{f,n}$, where

$$
d_{i_f,j_f} = s_{i_f,j_f}^{s_{i_f,j_f}2f-1,2f}.
$$

Suppose $n = 2f$. Thanks to Definition 3.4(1), $(i_f,j_f) = (n, n-1)$ and $d^{-1} \in D_{f-1,n-1}^{-1} \subset B_{f-1,n-1} \subset B_{f,n}$, where the first inclusion follows from induction assumption on $n - 1$. Suppose $n > 2f$. If $i_f < n$ then $d^{-1} \in D_{f,n-1}^{-1}$ and hence $d^{-1} \in B_{f,n}$ by induction assumption on $n - 1$. If $i_f = n$, then there is a $d_1 \in D_{f-1,n-1}^{-1}$ such that

$$
d^{-1} = d_1s_{j_f,2f-1}s_{i_f,2f} = \begin{cases} 
d_{1}^{s_{n-1}s_{j_f,2f-1}s_{i_f,2f}} & \text{if } j_f < n - 1, \\
_{1}^{s_{n-2}s_{n-1}s_{j_f,2f-1}s_{i_f,2f}} & \text{if } j_f = n - 1.
\end{cases}
$$

We use Lemma 3.5 to rewrite both $d_1s_{n-1}$ and $d_1s_{n-2}$. So there is a $d_1 \in D_{f,n-1}^{-1}$ such that $d^{-1} = t_{n-1}d'_1$. Now, the result follows from induction assumption on $n - 1$. □
Recall that a Brauer diagram is a diagram with $2n$ vertices arranged in two rows and $n$ edges such that each vertex belongs to a unique edge. The vertices in the top (resp., bottom) row are labeled as $n+1, n+2, \ldots, 2n$ (resp., $1, 2, \ldots, n$) from left to right. Then each Brauer diagram $d$ corresponds to a unique partition of $1, 2, \ldots, 2n$ into $n$ pairs $\{(i_k, j_k) \mid 1 \leq k \leq n, i_k < j_k\}$. We denote $\{(i_k, j_k) \mid 1 \leq k \leq n\}$ by $\text{conn}(d)$ and call it the connector of $d$. An edge is called a horizontal edge if two vertices of it are at the same row. Otherwise, it is called a vertical edge. Let $d_f$ be the Brauer diagram such that

$$\text{conn}(d_f) = \{(2i - 1, 2i), (n + 2i - 1, n + 2i) \mid 1 \leq i \leq f\} \cup \{(j, n + j) \mid 2f + 1 \leq j \leq n\}.$$ 

Consider a Brauer diagram on which there are exactly $f$ horizontal edges $(n + 2i - 1, n + 2i), 1 \leq i \leq f$, at the top row and there is no crossing between any two vertical edges. Let $D_{f,n}$ be the set of all such Brauer diagrams. The symmetric group $S_n$ acts transitively on the right of $D_{f,n}$. In [22, (2.11)], Dung Tien Nguyen defined $\mathcal{B}_{0,n} = \{1\}$ and

$$\mathcal{B}_{f,n} = \{w^{-1} \in D_{f,n} \mid d = dfw \in D_{f,n}\}$$

(3.4) for $f > 0$. There is a restriction on the length of $d$ in [22, (2.11)]. However, this restriction is redundant since $d$ automatically satisfies this condition if $w^{-1} \in B_{f,n}$ [24, Lemma 2.1].

**Proposition 3.7.** For any $0 \leq f \leq \lfloor \frac{n}{2} \rfloor$, $D_{f,n} = \mathcal{B}_{f,n}$.

**Proof.** There is a well-known result for Brauer diagrams which says that $dfw \in D_{f,n}$ if $w \in D_{f,n}$. Thanks to Lemma 3.6 and (3.4), $D_{f,n} \subseteq \mathcal{B}_{f,n}$. We have $D_{f,n} = \mathcal{B}_{f,n}$ since the cardinalities of $D_{f,n}$ and $\mathcal{B}_{f,n}$ are $\binom{n}{f}$ (see [19, Lemma 4.3] and [21, Remark 3.18]).

Thanks to Proposition 3.7, we can use $D_{f,n}$ to replace $\mathcal{B}_{f,n}$ when we state the cellular basis of $B_n$ in [22, Theorem 3.2]. This has an advantage when we do some explicit computation later on.

**Theorem 3.8.** [22, Theorem 3.2] The $q$-Brauer algebra $B_n$ is free over $\mathbb{K}$ with cellular basis

$$S = \{C^\lambda_{(w,s),(v,t)} \mid (w,s), (v,t) \in I(\lambda), (f, \lambda) \in \Lambda_n\}$$

in the sense of [8, Definition 1.1], where $I(\lambda) := D_{f,n} \times S^\lambda$ and $C^\lambda_{(w,s),(v,t)} = \sigma(T_w)E^f x_{st} T_v$. The required anti-involution is $\sigma$ in Lemma 3.5.

**Lemma 3.9.** [22, Corollary 3.1] Suppose $0 \leq f \leq \lfloor \frac{n}{2} \rfloor$. Then $E^f B_n + B_n^{f+1}$ is a left $H_{2f+1,n}$-module spanned by $\{E^f T_d + B_n^{f+1} \mid d \in D_{f,n}\}$, where $B_n^{f+1}$ is the two-sided ideal of $B_n$ generated by $E^{f+1}$.

**Theorem 3.10.** Let $\phi : B_{n-2} \to \tilde{E}_1 B_n \tilde{E}_1$ be the algebra homomorphism in Proposition 3.7. Then $\phi$ is an algebra isomorphism.

**Proof.** Thanks to Theorem 3.8, $B_{n-2}$ has basis

$$\{\sigma(T_{d_1})E^f T_w T_{d_2} \mid d_1, d_2 \in D_{f,n-2}, w \in H_{2f+1,n-2}, 0 \leq f \leq \lfloor \frac{n-2}{2} \rfloor\}.$$ 

By Proposition 2.7(2), $\phi(\sigma(T_{d_1})E^f T_w T_{d_2}) = qz^{-1} T_1^{-1} T_2 \sigma(T_{c_1})E^{f+1} T_w T_{c_2}$ where $c_1, c_2, w'$ are obtained from $d_1, d_2, w$ by replacing each factor $s_i$ in $d_1, d_2, w$ by $s_{i+2}$. This proves that $\phi(\sigma(T_{d_1})E^f T_w T_{d_2})$ is a basis element of $B_n$, and hence $\phi$ is a monomorphism. On the other hand, there are some scalars $a, b \in \mathbb{K}$ and $w_1, w_2 \in \Theta_3,n$ such that

$$E_{T_{2i}, T_{1j}, E_t} = a E_1 E_3 T_{w_1} + b E_1 T_{w_2}$$

(3.5) and $a = 0$ unless $i_1 \geq 4$ and $j_1 \geq 3$. In general, since $D_{1,n}$ is a right coset representatives of $\Theta_2 \times \Theta_3,n$, by Definition 2.7(3) and (3.3), $E_1 H_n E_1 = E_1 E_3 H_3,n + E_1 H_3,n$. By Theorem 3.8 and (3.5), $E_1 B_n E_1$, is generated by $E_1 E_3, E_1 T_j, 3 \leq j \leq n - 1$. Consequently, $E_1 B_n E_1$ is generated by $E_1 E_3, E_1 T_j, 3 \leq j \leq n - 1$ and hence $\phi$ is an epimorphism.

**Corollary 3.11.** Suppose $n \geq 2$.

(1) There is an algebra isomorphism $\psi : B_{n-2} \to E_1 B_n E_1$ sending $1, E_1$ and $T_1$ to $\delta^{-1} E_1, \delta^{-1} E_1 E_3$ and $\delta^{-1} E_1 T_{i+2}$ for all $1 \leq i \leq n - 3$.

(2) The right $B_n$-module $E^f B_n$ is a left $E^f B_n E^f$-module generated by $E^f T_d, d \in D_{f,n}$. Further, $E^f B_n E^f$ is generated by $E^{f+1}$ and $E^f T_{2f+1}, 1 \leq i \leq n - 2f - 1$.

**Proof.** (1) follows from arguments in the proof of Theorem 3.10 and (2) follows from (3.5) and Lemma 3.9 successively for $B_n, B_{n-2}$ etc.
3.3. The classical branching rule. In this subsection, we go on assuming that $B_n$ is defined over $k$. For any $(f, \lambda) \in \Lambda_n$, let $C(f, \lambda)$ be the (right) cell module of $B_n$ with respect to the cellular basis in Theorem 3.8. Up to an isomorphism, $C(f, \lambda)$ can be considered as the free $k$-module with basis $\{E^f x_\lambda T_{d(u)} T_v + B_n^{f, \lambda} \mid (v, t) \in I(\lambda)\}$, where

$$B_n^{f, \lambda} = k \text{-span}\{C_{(w,s),(v,t)} \mid (w,s), (v,t) \in I(\mu), (\ell, \mu) \in \Lambda_n, \mu \triangleright \lambda\}.$$ 

Definition 3.12. For any $(f, \lambda) \in \Lambda_n$, let $R_{\lambda}(\lambda)$ be the set of all partitions $\mu \in \Lambda_n \cup \Lambda_{n-1}$ such that $(l, \mu) \in \Lambda_{n-1}$ for some $l \in \mathbb{N}$. Let

$$y_\mu = \begin{cases} E^f x_\lambda T_{a_k,n} & \text{if } \mu = \lambda \setminus (k, \lambda_k), \\ E^f x_\lambda T_{b_k,2f-1} E^f -1 x_\mu & \text{if } \mu = \lambda \cup (k, \lambda_k + 1), \end{cases}$$

where $a_k = 2f + \sum_{i=1}^k \lambda_i$ and $b_k = 2f - 1 + \sum_{i=1}^k \lambda_i$.

Thanks to the defining relations for $B_n$, we can rewrite $y_\mu$ as follows:

$$y_\mu = \begin{cases} \sum_{i=a_k-1}^{a_k-1} q^{b_k-i} T_{i,n} E^f x_\mu & \text{if } \mu = \lambda \setminus (k, \lambda_k), \\ E^f x_\lambda T_{b_k,2f-1} \sum_{i=b_k+1}^{b_k} q^{b_k-i} T_{b_k,i} & \text{if } \mu = \lambda \cup (k, \lambda_k + 1). \end{cases} \quad (3.6)$$

For any $(f, \lambda) \in \Lambda_n$, there is a pair $(a, m) \in \mathbb{N} \times \mathbb{N} \setminus \{0\}$ such that

$$R_{\lambda}(\lambda) = \{\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(a)}\} \cup \{\mu^{(a+1)}, \mu^{(a+2)}, \ldots, \mu^{(m)}\}$$

and $(f, \mu^{(i)}) \in \Lambda_{n-1}, 1 \leq i \leq a$ and $(f-1, \mu^{(i)}) \in \Lambda_{n-1}, a + 1 \leq i \leq m$. Obviously $\mu^{(a+1)}$ occurs only if $f > 0$. We can arrange them so that

$$\mu^{(1)} \triangleright \mu^{(2)} \triangleright \ldots \triangleright \mu^{(a)} \triangleright \mu^{(a+1)} \triangleright \ldots \triangleright \mu^{(m)} \quad (3.7)$$

with respect the partial order $\triangleright$ on $\Lambda_{n-1}$. Note that $\mu^{(m)} = \lambda \cup (k + 1,1)$ if $k = \max\{j \mid \lambda_j \neq 0\}$. So

$$y_{\mu^{(m)}} = E^f x_\lambda T_{n-1,2f-1} T_{n-1,2f-1} \quad (3.8)$$

Definition 3.13. For any $1 \leq k \leq m$, let $N_k = \sum_{j=1}^k y_{\mu^{(j)}} B_n^{-1} + B_n^{f, \lambda}$.

It follows from Definition 3.13 that $N_k, 1 \leq k \leq m$ are $B_n^{-1}$-submodules of $C(f, \lambda)$.

Lemma 3.14. Suppose $1 \leq k \leq a$.

1. The $B_{n-1}$-submodule $N_k$ of $C(f, \lambda)$ is spanned by

$$\{y_{\mu^{(i)}} T_{d(u)} T_w + B_n^{f, \lambda} \mid u \in \mathcal{F}_{\text{std}}(\mu^{(j)}), w \in \mathcal{D}_{f,n-1}, 1 \leq j \leq k\}.$$

2. As $B_{n-1}$-modules, $N_k/N_{k-1} \cong C(f, \mu^{(k)})$.

3. The submodule $N_k$ has basis

$$\{E^f x_\lambda T_{d(u)} T_v + B_n^{f, \lambda} \mid s \in \mathcal{F}_{\text{std}}(\lambda), \text{shape}(s_{\downarrow n-1}) \triangleright \mu^{(k)}, v \in \mathcal{D}_{f,n-1}\}.$$  

Proof. Suppose $b \in B_{n-1}$ and $\mu^{(k)} = \lambda \setminus (\ell, \lambda_k)$. Let $a_k = 2f + \sum_{i=1}^k \lambda_i$. Thanks to Definition 3.13 and Theorem 3.2

$$y_{\mu^{(k)}} b = x_\lambda T_{a_k,n} \sum_{v \in \mathcal{D}_{f,n-1}} a_{v,w} T_v E^f T_w \equiv \sum_{w \in \mathcal{D}_{f,n-1}} b_{a,w} y_{\mu^{(k)}} T_{d(u)} T_w \mod N_{k-1}$$

where $a_{v,w}$ and $b_{a,w}$ are some scalars. This proves (1). Suppose $(u, w) \in \mathcal{F}_{\text{std}}(\mu^{(k)}) \times \mathcal{D}_{f,n-1}$. Then

$$y_{\mu^{(k)}} T_{d(u)} T_w + B_n^{f, \lambda} = E^f x_\lambda T_{d(u)} T_{d(u)} T_w + B_n^{f, \lambda}$$

with $\ell_{\downarrow n-1} = u$. Since the RHS of the above equality is a basis element of $C(f, \lambda)$, by (1), $N_k/N_{k-1}$ is free over $k$ with basis $M_k$, where

$$M_k = \{y_{\mu^{(k)}} T_{d(u)} T_w + N_{k-1} \mid u \in \mathcal{F}_{\text{std}}(\mu^{(k)}), w \in \mathcal{D}_{f,n-1}\}.$$  

(3.9)

There is a well-defined $k$-linear isomorphism $\phi : N_k/N_{k-1} \to C(f, \mu^{(k)})$ such that

$$\phi(y_{\mu^{(k)}} T_{d(u)} T_w + N_{k-1}) = E^f x_\lambda T_{d(u)} T_{d(u)} T_w + B_n^{f, \lambda},$$

(3.10)

for all $(u, w) \in \mathcal{F}_{\text{std}}(\mu^{(k)}) \times \mathcal{D}_{f,n-1}$. It follows immediately from Lemma 3.3 and Theorem 3.2 that $\phi$ is a right $B_{n-1}$-homomorphism and hence $\phi$ is a $B_{n-1}$-isomorphism. This completes the proof of (2). Finally, (3) immediately follows from (2), (3.10) and induction assumption on $k - 1$.  

\[\square\]
From here to the end of this section, we assume \((f, \lambda) \in \Lambda_n\) and \(f > 0\). We are going to deal with \(N_k/N_{k-1}\), \(a + 1 \leq k \leq m\).

**Lemma 3.15.** Let \((f - 1, \mu) \in \Lambda_{n-1}\) such that \(\mu \succ \mu^{(m)}\).

1. Suppose \(\mu \not\in \{\mu^{(1)}, \ldots, \mu^{(m)}\}\). If \((\mathfrak{s}, \mu^{(i)}(\mathfrak{s})) \in \mathcal{T}^{\text{std}}(\mu) \times \mathcal{T}(\mu, \mu^{(m)})\), then \(\text{shape}(\mathfrak{s}_{\downarrow n-2}) \succ \lambda\).

2. Suppose \(\mu = \mu^{(j)} = \lambda \cup (k, \lambda k + 1)\) for some \(a + 1 \leq j \leq m\). Let \(s_j = (\mathfrak{s}^{(j)} s_{b_j, n-1}\), where \(b_j = 2f - 1 + \sum_{i=1}^{k} \lambda_i\). Then \(s_j\) is the unique standard \(\mu^{(j)}\)-tableau \(\mathfrak{s}\) satisfying \(\mu^{(m)}(\mathfrak{s}) \in \mathcal{T}^{\text{std}}(\mu^{(j)}, \mu^{(m)})\). In this case, \(\text{shape}(\mathfrak{s}_{\downarrow n-2}) = \lambda\).

**Proof.** The results has already been given in the proof of [5 Corollary 5.4, Lemma 5.5].

**Lemma 3.16.** Suppose \((\mathfrak{s}, \mu^{(m)}(\mathfrak{s})) \in \mathcal{T}^{\text{std}}(\mu) \times \mathcal{T}^{\text{std}}(\mu, \mu^{(m)})\) for some \((f - 1, \mu) \in \Lambda_{n-1}\). If \(\text{shape}(\mathfrak{s}_{\downarrow n-2}) \succ \lambda\), then \(E_f T_{n,2f-1}^{-1} E_f (T_d(s)) x_{nu} \in \mathcal{B}_{f^{\lambda}}\).

**Proof.** The arguments in the proof of [5 Lemma 5.3] depends only on the braid relations for the Hecke algebras. Therefore, one can imitate arguments there to verify our current result. We leave details to the reader.

**Lemma 3.17.** As \(\mathcal{B}_{n-1}\)-modules, \(C(f, \lambda) = N_m\).

**Proof.** Recall that the cell module \(C(f, \lambda)\) has basis \(\{E_f x_{\lambda} T_d(\mathfrak{t}) T_v + \mathcal{B}_{n}^{\lambda \lambda} \mid (v, \mathfrak{t}) \in I(\lambda)\}\). If \(v \in \mathcal{D}_{f,n-1}\), then

\[
E_f x_{\lambda} T_d(\mathfrak{t}) T_v + \mathcal{B}_{n}^{\lambda \lambda} \in E_f x_{\lambda} T_d(\mathfrak{t}) \mathcal{B}_{n-1} + \mathcal{B}_{n}^{\lambda \lambda} \subseteq N_n.
\]

The last inclusion follows from Lemma 3.14(3). Otherwise \(T_v = T_{2f,n} b\) for some \(b \in \mathcal{H}_{n-1}\). So,

\[
E_f x_{\lambda} T_d(\mathfrak{t}) T_v + \mathcal{B}_{n}^{\lambda \lambda} \in E_f x_{\lambda} T_{2f,n} B_{n-1} + \mathcal{B}_{n}^{\lambda \lambda} \subseteq \sum_{j=2f+1}^{n} E_f x_{\lambda} T_{j,n} B_{n-1} + E_f x_{\lambda} T_{n,2f} B_{n-1} + \mathcal{B}_{n}^{\lambda \lambda}.
\]

By Definitions 3.12, 3.13, 3.38 and Lemma 3.14(3),

\[
E_f x_{\lambda} T_{n,2f} B_{n-1} + \mathcal{B}_{n}^{\lambda \lambda} \in N_m \quad \text{and} \quad \sum_{j=2f+1}^{n} E_f x_{\lambda} T_{j,n} B_{n-1} \in N_n \subseteq N_m.
\]

This implies \(C(f, \lambda) \subseteq N_m\) and hence \(C(f, \lambda) = N_m\) as required.

**Lemma 3.18.** If \(b \in E^{-1} \mathcal{B}_{n-1} \cap \mathcal{B}_{n-1}^{f_{1}}\), then there are \(w \in \mathcal{D}_{f,n-1}\) and \(w \in \mathcal{H}_{f,n+1}\) such that

\[
E_{2f-1} T_{n,2f-1}^{-1} T_{n-1,2f-1} b = \sum_{w} h_{w} E_{f} T_{w} \mod B_{n}^{f_{1}}.
\]

**Proof.** By Theorem 3.5, Lemma 3.9 and Corollary 3.11(2), any \(b \in E^{-1} \mathcal{B}_{n-1} \cap \mathcal{B}_{n-1}^{f_{1}}\) can be written as a linear combination of elements in \(S\) up to an element in \(B_{n}^{f_{1}}\), where

\[
S = \{ T_{j,2f-1} T_{i,2f} E_{f} T_{u} T_{w} \mid T_{u} \in \mathcal{H}_{2f+1,n-1}, w \in \mathcal{D}_{f,n-1}, 2f - 1 \leq j < i \leq n - 1\}.
\]

So it is enough to assume \(b = T_{j,2f-1} T_{i,2f} E_{f} T_{u} T_{w}\) when we verify (3.13).

Suppose \(i = n - 1\). Then \(j < n - 1\). Let \(x = E_{2f-1} T_{n,2f}^{-1} T_{n-1,2f-1} T_{j,2f-1} E_{f}\). By Lemma 2.6(2)(5) and Lemma 2.5(6),

\[
x = T_{j+2,2f+1} E_{2f-1} T_{n-1,2f-1}^{-1} T_{n-2,2f-1} E_{f}
\]

and (3.13) follows. Suppose \(i < n - 1\). By (2.2) and Lemma 2.6(2)

\[
E_{2f-1} T_{n,2f}^{-1} T_{n-1,2f-1} T_{j,2f-1} E_{f} = T_{j+2,2f+1} T_{i+2,2f+2} T_{n-1,2f+1},
\]

where \(y = E_{2f-1} T_{n,2f}^{-1} T_{n-1,2f-1} T_{j,2f-1} E_{f}\). Thanks to (2.2), \(T_{i}^{-1} = T_{i} - a\) for any \(1 \leq i \leq n - 1\), where \(a = q - q^{-1}\). We use it to rewrite \(y\) as follows:

\[
y = E_{2f-1} T_{2f} T_{2f+1} E_{f} - (aq^{-1} + a T_{2f+1}) E_{2f-1} T_{2f-1} E_{f}.
\]
Since \(E^{f+1} = E_1 T_{2,2f} + T_{2,2f-1}^{-1} E^f\), the first term in the RHS of (3.16) is equal to
\[
T_{2f-1,1} T_{2,2f}^{-1} E_1 T_{2,2f} + T_{2,2f-1}^{-1} E^f = T_{2f-1,1} T_{2,2f}^{-1} E^{f+1} + B_n^{f+1}.
\]
(3.17)

Using Lemma 2.6(8) to rewrite the second term in the RHS of (3.16), we see that (3.13) follows immediately from (3.15) and (3.17).

**Lemma 3.19.** As \(B_{n-1}\)-modules, \(N_k/N_{k-1} \cong C(f - 1, \mu^{(k)})\) for any \(a + 1 \leq k \leq m\).

**Proof.** Fix \(k, a + 1 \leq k \leq m\). We want to compute \(y^\lambda_{\mu^{(k)}} h\) for any \(h \in B_{n-1}\). Thanks to Theorem 3.8 and Lemma 3.16, \(E^f h + B_n^{f+1}\) can be written as a linear combination of elements \(\pi := x + B_n^{f+1}\), where \(x \in \{h_d E^f T_d, b\}\), and \(b \in E^f B_{n-1} \cap B_n^{f+1}\), \(d \in D_{f-1, n-1}\) and \(h_d \in H_{2f-1, n-1}\). So
\[
y^\lambda_{\mu^{(k)}} h = \sum_{d \in D_{f-1, n-1}} y^\lambda_{\mu^{(k)}} h_d T_d + E_{2f-1} T_{n-2f}^{-1} T_{2f}^{-1} x_{\mu^{(k)}} b \quad \text{(mod} B_n^{f+1})
\]
(18.3)
Further, \(x_{\mu^{(k)}} h \subseteq x_{\mu^{(k)}} h_d T_d \quad \text{(mod} N_a)\),

where \(a_k\) is given in Lemma 3.15(2). The second equivalence follows from (3.13) and Lemma 3.14(3). Further, \(x_{\mu^{(k)}} h \subseteq x_{\mu^{(k)}} h_d T_d \quad \text{(mod} N_a)\). Applying Lemma 3.3 on \(x_{\mu^{(k)}} h\), we have
\[
y^\lambda_{\mu^{(k)}} h = \sum_{d \in D_{f-1, n-1}} E_{T_{n-2f}^{-1} T_{2f}^{-1}} x_{\mu^{(k)}} h_d T_d \quad \text{(mod} N_a)\).
\]
By Lemma 3.15(1) and 3.16, the terms with respect to \(\mu\) in the RHS of the above equality are in \(B_{n-1}\) if \(\mu \notin \{\mu^{(a+1)}, \mu^{(a+2)}, \ldots, \mu^{(k)}\}\). In the remaining case \(S \in T^{ss}(\mu^{(j)}, \mu^{(m)})\) for some \(a + 1 \leq j \leq m\). Thanks to Lemma 3.15(2), \(\mu^{-1}(S)\) contains a unique element, say \(s_j\) in \(T^{std}(\mu^{(j)})\). So \(x_{\mu^{(k)}} h = q^{-\ell(s_j)} x_{\mu^{(k)}} h_d T_d \quad \text{(mod} N_a)\),

where \(N_k/N_{k-1}\) is the \(k\)-module spanned by
\[
M_k = \{y^\lambda_{\mu^{(k)}} T_d(s_d) T_d + N_{k-1} \mid s \in T^{std}(\mu^{(k)}), d \in D_{f-1, n-1}\}.
\]
(3.19)
By Lemma 3.19, \(C(f, \lambda)\) can be spanned by a set whose cardinality is \(\sum_{k=1}^{m} |M_k|\). Thanks to Theorem 2.3, the rank of cell module for Birman-Murakami-Wenzl algebra with respect to \((f, \lambda)\) is equal to \(\sum_{k=1}^{m} |M_k|\). Since rank \(C(f, \lambda)\) is equal to that for Birman-Murakami-Wenzl algebra, we have rank \(C(f, \lambda) = \sum_{k=1}^{m} |M_k|\) and hence \(M_k\) is linear independent for any \(1 \leq k \leq m\). Consequently, \(M_k\) is a basis of \(N_k/N_{k-1}\) for all \(1 \leq k \leq m\). We have a well-defined \(k\)-linear isomorphism \(\phi: N_k/N_{k-1} \to C(f - 1, \mu^{(k)})\) for any \(a + 1 \leq k \leq m\) such that
\[
\phi(y^\lambda_{\mu^{(k)}} T_d(s_d) T_d + N_{k-1}) = E^f x_{\mu^{(k)}} T_{d(s_d)} T_d + B_n^{f+1}.
\]
(3.20)
for any \(s \in T^{std}(\mu^{(k)})\) and \(d \in D_{f-1, n-1}\). By arguments similar to those above (more explicitly, using Theorem 3.8, Lemmas 3.3 and 3.4) we see that \(\phi\) is a right \(B_n^{f+1}\)-homomorphism.

**Theorem 3.20.** Suppose \((f, \lambda) \in \Lambda_n\). Then there is a filtration \(0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_m = C(f, \lambda)\) of \(B_{n-1}\)-modules such that
\[
N_k/N_{k-1} \cong \begin{cases} 
C(f, \mu^{(k)}) & \text{if } 1 \leq k \leq a, \\
C(f - 1, \mu^{(k)}) & \text{if } a + 1 \leq k \leq m.
\end{cases}
\]

**Proof.** The result follows immediately from Lemmas 3.14 and 3.19.

\[\square\]

4. **The Jucys-Murphy elements and Jucys-Murphy basis**

In this section, we construct the Jucys-Murphy elements and the Jucys-Murphy basis of \(B_n\). Unless otherwise stated, we assume that \(B_n\) is defined over \(k\).
4.1. Jucys-Murphy elements. For any $1 \leq i \leq n$, define
\[ L_i = \begin{cases} 0, & \text{if } i = 1, \\ \sum_{j=1}^{i-1} (j,i) - q^2 z^{-1} \sum_{j=1}^{i-1} E_{j,i}, & \text{if } 2 \leq i \leq n, \end{cases} \tag{4.1} \]
where
\[ (j,i) = T_{i,j-1}T_{i-1,j} - T_{j-1,i}T_{i,j-1} \quad \text{and} \quad E_{i,i} = T_{i,j}^{-1}T_{i,j}E_{i}T_{i}^{-1} \quad \text{if } 1 \leq j \leq i - 1. \tag{4.2} \]
for any $1 \leq j \leq i - 1$. Note that $E_{i,i+1}$ is not the $E_{i}$ in Definition 2.4 except the classical limit case.

**Lemma 4.1.** If $n \geq 2$, then $L_n = T_{n-1}L_{n-1} = T_{n-1} - q^2 z^{-1}E_{n-1,n}$.

**Proof.** The result follows immediately from (4.1). \( \square \)

**Lemma 4.2.** Suppose $n > 3$ and $x_m = E_{m-2,m} + E_{m-1,m}$ for $m \geq 3$. Then
(1) $E_1 E_{n-1,n} = E_{n-1,n}E_1$,
(2) $T_{m-2}x_m = x_m T_{m-2}$,
(3) $T_{n-2}E_{i,n} = E_{i,n}T_{n-2}$ if $1 \leq i \leq n - 3$,
(4) $T_{j}E_{n-1,n} = E_{n-1,n}T_{j}$ if $1 \leq j \leq n - 3$.

**Proof.** Thanks to Definition 2.4(5) and Lemma 2.5(3),
\[ E_1 E_{n-1,n} = T_{3,n}^{-1}T_{n,4}E_{1}T_{2}^{-1}T_{1}^{-1}T_{3}T_{2}E_{1}T_{2,n}T_{n-1}^{-1} \]
\[ = T_{3,n}^{-1}T_{n,4}E_{1}T_{2}^{-1}T_{1}^{-1}T_{3}T_{2}E_{1}T_{4,n}T_{n-1}^{-1} \]
\[ = E_{n-1,n}E_1. \]
This proves (1). One can easily verify (2)-(4) by braid relations in (2.2) and Definition 2.4.

**Lemma 4.3.** Suppose $n \geq 2$. For any $x \in B_{n-1}$, $L_{n}x = xL_{n}$.

**Proof.** We prove the result by induction on $n$. The result is trivial when $n = 2$. We have
\[ L_3 = T_2 T_1 T_2 + T_2 - q^2 z^{-1}(E_{1,3} + E_{2,3}). \tag{4.3} \]
It is easy to verify $E_1 L_3 = L_3 E_1 = 0$. It is known that $T_2 T_1 T_2$ is the corresponding Jucys-Murphy element of $H_{3}$ (e.g., [9]) and hence commutes with $T_1$. Then by Lemma 4.2 and (4.3), $L_3 T_1 = T_1 L_3$. This proves the result when $n = 3$. Suppose $n > 3$. It is enough to prove that $L_n x = x L_n$ for any $x \in \{E_1, T_1, \ldots, T_{n-2}\}$. By induction assumption on $n - 1$, Definition 2.4 and Lemma 4.1, $E_1$ commutes with both $T_{n-1}L_{n-1}T_{n-1} + T_{n-1}$ and $E_{n-1,n}$. By Lemma 4.4, $T_{n}E_{n-1,n} = E_{n-1,n}T_{n}$. By induction assumption on $n - 1$ and Lemma 4.4, $T_1 T_2 \ldots T_{n-3}$ commute with both $T_{n-1}L_{n-1}T_{n-1} + T_{n-1}$ and $E_{n-1,n}$. Using Lemma 4.1 again yields $L_{n}x = xL_{n}$ for any $x \in \{T_1, T_2, \ldots, T_{n-3}\}$. It is known that $T_{n-2} \sum_{i=1}^{n-1} i(i,n) = \sum_{i=1}^{n-1} i(i,n) T_{n-2}$ since $\sum_{i=1}^{n-1} i(n)$ is the corresponding Jucys-Murphy element of $H_{n}$. Then by Lemma 4.2(3), $T_{n-2}L_{n} = L_{n}T_{n-2}$. \( \square \)

**Lemma 4.4.** For any $1 \leq i \leq j \leq n$, $L_{i}L_{j} = L_{j}L_{i}$.

**Proof.** Obviously, $L_{i} \in B_{j-1}$ if $i < j$. Now, the result follows from Lemma 4.3. \( \square \)

Thanks to Lemma 4.4, the subalgebra $L$ of $B_{n}$ generated by $\{L_1, L_2, \ldots, L_{n}\}$ is a commutative subalgebra. By Remark 2.2, the Brauer algebra $B_{n}(a)$ can be considered as the classical limit of $B_{n}(q^{a})$. In this case, $\{L_{i} \mid 1 \leq i \leq m\}$ is the set of corresponding Jucys-Murphy elements in $B_{n}$ and $\sum_{i=1}^{n} L_{i}$ is a central element. However, $\sum_{i=1}^{n} L_{i}$ may not be a central element in general. One can easily find a counterexample in $B_{4}$.

### 4.2. A new basis of a cellular module.

Suppose $(f, \lambda) \in \Lambda_{n}$. Write $\mu \rightarrow \lambda$ if $\mu \in R \Lambda_{n}(\lambda)$. An upper-down tableau $t$ of type $\lambda$ is a sequence of partitions $t = (t_0, t_1, \ldots, t_n)$ such that $t_0 = 0$, $t_i \rightarrow t_{i+1}$, $0 \leq i \leq n - 1$ and $t_n = \lambda$. Let $S_{\mu}^{ad}(\lambda)$ be the set of all upper-down tableaux of type $\lambda$. There is a partial order $\preceq$ on $S_{\mu}^{ad}(\lambda)$ such that $t \preceq s$ if $t_k \triangleleft s_k$ for some $k$ and $t_j \geq s_j$ for all $k < j \leq n$. In this case, we also write $t \succeq s$.

**Definition 4.5.** Suppose $(f, \lambda) \in \Lambda_{n}$, $t \in S_{\mu}^{ad}(\lambda)$ and $t_{n-1} = \mu$. Define $m_{t} = E_{f} x_{L} b_{k}$, where $b_{k} = b_{k_{n}}$ which can be defined inductively as follows:
\[ b_{k_{n}} = \begin{cases} T_{n,k}^{-1}b_{k_{n-1}}, & \text{if } \lambda = \mu \cup (k, \mu_{k+1}), \\ T_{n,k}^{-1} \sum_{j=b_{k_{n-1}}+1}^{k} q^{b_{k_{n-1}}-j} T_{n,k}^{-1}b_{k_{n-1}} & \text{if } \lambda = \mu \setminus (k, \mu_{k}), \end{cases} \]
and $a_{k} = 2f + \sum_{j=1}^{k} \lambda_{j}$ and $b_{i} = 2f - 1 + \sum_{j=1}^{i} \lambda_{j}$, for $i \in \{k - 1, k\}$. 
In fact one can easily check that
\[
    m_{t,n} = \begin{cases} 
        \sum_{k=n-k+1}^{k} T_{j,n} T_{k,n} m_{t,n-1} & \text{if } \lambda = \mu \cup (k, \mu_k + 1), \\
        E_{2f-1} T_{n,2f} T_{b,k,2f-1} m_{t,n-1} & \text{if } \lambda = \mu \setminus (k, \mu_k),
    \end{cases}
\] (4.4)
and \(m_{t,n} = \gamma_{\mu} h_{t,n-1}\) in any case.

**Theorem 4.6.** Suppose \((f, \lambda) \in \Lambda_n\).

1. The cell module \(C(f, \lambda)\) has \(k\)-basis \(\{m_t + B_{n-1}^\mu \mid t \in \mathcal{S}_n^{ud}(\lambda)\}\).
2. For any \(1 \leq j \leq m\), the \(k\)-module \(M_j\) spanned by \(\{m_t + B_{n-1}^\mu \mid t \in \mathcal{S}_n^{ud}(\lambda), t_{n-1} \supseteq \mu(j)\}\) is a \(B_{n-1}\)-submodule of \(C(f, \lambda)\), where \(\mu(j)\) is given in (4.7).
3. For any \(1 \leq j \leq m\), \(M_j = N_j\) where the \(N_j\)'s are given in Definition 3.13.
4. As \(B_{n-1}\)-modules, \(M_j/M_{j-1} \cong C(\ell, \mu(j))\), where \(\ell = f\) if \(1 \leq j \leq a\) and \(\ell = f - 1\) if \(a + 1 \leq j \leq m\). The required isomorphism sends \(m_t + M_{j-1}\) to \(m_{t,n-1} + B_{n-1}^{s, \mu(j)}\) where \(t_{n-1} = (t_0, t_1, \ldots, t_{n-1}) \in \mathcal{S}_n^{ud}(\mu(j))\).
5. The \(k\)-algebra \(B_n\) has cellular basis
\[
\{m_{s,t} \mid s, t \in \mathcal{S}_n^{ud}(\lambda), (f, \lambda) \in \Lambda_n\}
\]
where \(m_{s,t} = \sigma(b_s) E^f x_s b_t\) and the required \(\sigma\) is given in Lemma 2.3.

**Proof.** Thanks to Lemmas 3.13 and 19 the \(k\)-linear map \(\phi : C(\ell, \mu(j)) \rightarrow N_j/N_{j-1}\) sending \(E^f x_{\mu(j)} T_{d(t)} T_d + B_{n-1}^{s, \mu(j)}\) to \(y_{\mu(j)}^\lambda T_{d(t)} T_d + N_{j-1}\) is the \(B_{n-1}\)-isomorphism. In particular,
\[
\phi(E^f x_{\mu(j)} + B_{n-1}^{s, \mu(j)}) = y_{\mu(j)}^\lambda + N_{j-1}.
\]
By induction assumption on \(n - 1\), \(C(\ell, \mu(j))\) has basis
\[
\{E^f x_{\mu(j)} b_u + B_{n-1}^{s, \mu(j)} \mid u \in S_n^{ud}(\mu(j))\}.
\]
Note that \(\phi^{-1}(m_{s,t} + B_{n-1}^{s, \mu(j)}) = m_{s,t} + N_{j-1}\) where \(m_{s,t} = y_{\mu(j)}^\lambda b_u\) and \(t_{n-1} = u\). Using the isomorphism \(\phi\) and induction assumption on \(N_{j-1}\), we see that \(N_j\) has basis \(\{m_{s,t} + B_{n-1}^{s, \mu(j)} \mid t \in \mathcal{S}_n^{ud}(\lambda), t_{n-1} \supseteq \mu(j)\}\). In particular, \(M_j = N_j\) where \(M_j\) is defined in (2). This proves (1)–(4). Note that \(B_n\) is a cellular algebra. Thanks to Lemma 2.3, \(E^f\) is fixed by the anti-involution \(\sigma\). It is easy to verify \(\sigma(x_{\lambda}) = x_{\lambda}\). By (1), we have the corresponding results for left cell modules. This implies (5) by using standard arguments on cellular algebras (see [16] Theorem 2.7) for Brauer algebras. \(\square\)

The basis in Theorem 4.6 (1) (resp., (5)) is called the Jucys-Murphy basis of \(C(\ell, \lambda)\) (resp., \(B_n\)).

**Lemma 4.7.** Suppose \(0 \leq f \leq \lfloor \frac{n}{2}\rfloor\).

1. \(E^f E_{i,n} \in B_{n+1}^{i+1}\) if \(2f + 1 \leq i \leq n - 1\).
2. \(E^f ((2j - 1, n) + (2j, n) - q^2 z^{-1} E_{2j-1, n} - q^2 z^{-1} E_{2j, n}) = 0\) if \(1 \leq j \leq f\).

**Proof.** Since \(i \geq 2f + 1\), we have
\[
E^f E_{i,n} = E^{i} T_{i,1}^{-1} T_{2,n} E_{i} T_{2,n} T_{i,1}^{-1} = T_{i,1}^{-1} E^{i} T_{i,1}^{-1} T_{2,n} E_{i} T_{2,n} T_{i,1}^{-1} = 0 \quad (\text{mod } B_{n+1}^{i+1})
\]
where the last equivalence (1) follows from Lemma 2.3.1. Obviously, (2) follows from 4.6 as follows:
\[
E^f (2f - 1, n) = q E^f (2f, n) T_{2f-1}, \quad E^f E_{2f-1,n} = z E^f (2f, n), \quad E^f E_{2f,n} = z q^{-1} E^f (2f, n) T_{2f-1}. \quad (4.5)
\]
By Lemma 2.3.6,
\[
E^f (2f - 1, n) = E^f T_{2f-1}(2f, n) T_{2f-1} = q E^f (2f, n) T_{2f-1}
\]
proving the first equality in (4.5). By Lemma 2.3.5,
\[
E^f E_{2f-1,n} = E_{3} E_{5} \cdots E_{2f-1} T_{n,3} E_{2f-1} T_{n,2f-2} T_{2f-1} = z E^f (2f, n)
\]
where the last equality follows from Definition \ref{def:1}(3), \ref{def:1}(2) and Lemma \ref{lem:3}(4). This proves the second equality in \eqref{eq:4.5}. By \eqref{eq:4.2}, Lemma \ref{lem:3}(4) and the second equality in \eqref{eq:4.5}, we have
\[
E^f \varepsilon_{2f,n} = E^f T_{2f-1}^{-1} E_{2f-1,n} T_{2f-1}^{-1} = zq^{-1} E^f (2f, n) T_{2f-1}^{-1},
\]
proving the last equality in \eqref{eq:4.5}. \hfill \Box

**Proposition 4.8.** Suppose \((f, \lambda) \in \Lambda_n\) and \(\mu^{(k)} = \lambda \backslash p_k\) in \eqref{eq:3.7}, \(1 \leq k \leq a\). Then
\[
y_{\mu^{(k)}}^\lambda L_n \equiv \frac{q^c(p_k) - 1}{q - q^{-1}} y_{\mu^{(k)}}^\lambda \quad \pmod{N_{k-1}},
\]
where \(c(p_k) = j - i\) if \(p_k\) is in \(i\)-th row and \(j\)-th column of the Young diagram \([\lambda]\).

**Proof.** Thanks to Lemma \ref{lem:5} we immediately have
\[
y_{\mu^{(k)}}^\lambda L_n \equiv y_{\mu^{(k)}}^\lambda \sum_{j=2f+1}^{n-1} (j, n) \quad \pmod{B_n^{f+1}}. \tag{4.6}
\]
Now, the result follows from \eqref{eq:1.6} and well-known results on the action of Jucys-Murphy elements for Hecke algebras in \cite{9} Theorem 3.22\footnote{The Hecke algebra \(H_n\) is defined via \((T_i - q)(T_i + 1) = 0\) in \cite{9}. Our \(q^{-1} T_i\) (resp., \(q^2\)) is \(T_i\) (resp., \(q\)) in \cite{9}.} \hfill \Box

**Lemma 4.9.** Suppose \((f, \lambda) \in \Lambda_n\). We have
\[
E^f x_\lambda T_{n,2f-1}^{-1} x \equiv 0 \quad \pmod{N_a}
\]
if \(x \in \{(i, n) - q\delta_{2f-1,i}, E_{j,n} - z^{-1}(2f-1,j)^{-1}, E_{2f-1,n} - \delta\}\) and \(2f-1 \leq i \leq n-1, 2f-1 < j \leq n-1\).

**Proof.** Thanks to Lemma \ref{lem:2.5}
\[
E^f x_\lambda T_{n,2f-1}^{-1}(i, n) = \begin{cases} 
x_\lambda E^f T_{i+1,n} T_{i,2f}^{-1} & \text{if } i \neq 2f-1, \\
q E^f x_\lambda T_{2f,n} & \text{if } i = 2f-1. 
\end{cases}
\]
By Definition \ref{def:1.12} and Lemma \ref{def:1.14}(3),
\[
E^f x_\lambda T_{n,2f-1}^{-1}(i, n) \in N_a \text{ if } i \neq 2f-1. \tag{4.7}
\]
In the remaining case, using the quadratic relation in \ref{def:1.22} to rewrite \(T_{2f,n}\) and using Lemma \ref{def:1.14}(3) again, we have
\[
E^f x_\lambda T_{2f,n} - E^f x_\lambda T_{n,2f}^{-1} \in N_a. \tag{4.8}
\]
When \(x = (i, n) - q\delta_{2f-1,i}\), the result follows from \eqref{eq:1.7} - \eqref{eq:1.8}. Similarly, by Lemma \ref{lem:2.5}(4)(5), Lemma \ref{lem:2.6}(7), and Definition \ref{def:1.11}(1)(2) we have
\[
E^f x_\lambda T_{n,2f-1}^{-1} E_{2f-1,n} = E^f x_\lambda T_{1,2f-1}^{-1} T_{2f,2} E_{1} T_{2,n} T_{2f-1,1} \\
= \delta E^f x_\lambda T_{2f,2} T_{2f-1,1} T_{2f,n} \\
= \delta E^f x_\lambda T_{2f,n}, \tag{4.9}
\]
and
\[
E^f T_{n,2f}^{-1} E_{2f,n} = E^f T_{1,2f+1}^{-1} T_{2f+1,2} E_{1} T_{2,n} T_{2f,1} \\
= E^f T_{2f,1} T_{1,2f+1}^{-1} E_{1} T_{2,n} T_{2f,1} \\
= E^f T_{2f-1,1} T_{2f+1,1}^{-1} E_{1} T_{2,n} T_{2f,1} \\
= E^f T_{2f-1,1} T_{2f+1,1}^{-1} E_{1} T_{2,n} T_{2f,1} \tag{4.10}
\]
\[
= E^f T_{2f}^{-1} E_{2f-1,1} T_{2f+1,1}^{-1} T_{2,2f} T_{2f-1,1} \\
= z^{-1} E^f T_{2f-1,1} T_{2f+1,1}^{-1} T_{2,2f} T_{2f-1,1} \\
= z^{-1} E^f T_{2f-1,1} T_{2f+1,1}^{-1} T_{2,2f} T_{2f-1,1}.
\]
and
\[
E^f T_{n,2f}^{-1} E_{j,n} = E^f T_{j+1,2f}^{-1} T_{i,j+1}^{-1} T_{n,2} E_1 T_{2,n} T_{j,1}^{-1}
\]
\[
= E^f T_{j+1,2f}^{-1} T_{1,j+1,2} E_1 T_{2,n} T_{j,1}^{-1}
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\]
\[
= E^f T_{2f+1,j+1}^{-1} T_{1,2f+1,2} E_1 T_{2,n} T_{j,1}^{-1}
\]
\[
= E^f T_{2f+1,j+1}^{-1} T_{1,2f+1,2} E_1 T_{2,n} T_{j,1}^{-1}
\]
\[
= E^f T_{2f+1,j+1}^{-1} T_{1,2f+1,2} E_1 T_{2,n} T_{j,1}^{-1}
\]
\[
= E^f T_{2f+1,j+1}^{-1} T_{1,2f+1,2} E_1 T_{2,n} T_{j,1}^{-1}
\]
\[
= E^f T_{2f+1,j+1}^{-1} T_{1,2f+1,2} E_1 T_{2,n} T_{j,1}^{-1}
\]
\[
= E^f T_{2f+1,j+1}^{-1} T_{1,2f+1,2} E_1 T_{2,n} T_{j,1}^{-1}
\]
\[
= E^f T_{2f+1,j+1}^{-1} T_{1,2f+1,2} E_1 T_{2,n} T_{j,1}^{-1}
\]
\[
= E^f T_{2f+1,j+1}^{-1} T_{1,2f+1,2} E_1 T_{2,n} T_{j,1}^{-1}
\]
Definition 4.11. Suppose \((f, \lambda) \in \Lambda_n\). For any \(t \in \mathcal{F}_{n}^{ud}(\lambda)\) and \(1 \leq k \leq n\), define
\[
c_t(k) = \begin{cases} 
\frac{q^{2\langle p \rangle - 1}}{q^2 - q^{-2\langle p \rangle - 1}} & \text{if } t_k = t_{k-1} \cup p, \\
\frac{q^2 - q^{-2\langle p \rangle - 1}}{q^2} & \text{if } t_k = t_{k-1} \setminus p.
\end{cases}
\]

Theorem 4.12. Suppose \((f, \lambda) \in \Lambda_n\). For any \(t \in \mathcal{F}_{n}^{ud}(\lambda)\) and \(1 \leq k \leq n\),
\[
m_kL_k = c_t(k)m_k + \sum_{s \triangleright t} a_sm_s
\]
for some scalars \(a_s \in k\).

Proof. We have \(m_kL_1 = 0\). Suppose \(k \geq 2\). By Theorem 4.10(4) and induction assumption for \(n - 1\),
we have the required formulae on \(m_kL_k\), \(1 \leq k \leq n - 1\). When \(k = n\), the required formula follows
from Propositions 4.8, 4.10 and Theorem 4.6(3)(4). \(\square\)

By Theorem 4.12 \(\{L_1, L_2, \ldots, L_n\}\) are JM elements with respect to the Jucys-Murphy basis of any
cell module \(C(f, \lambda)\) in the sense of Definition 2.4.

Now we assume that \(B_n\) is defined over \(F\). It follows from Definition 2.3 that there is a canonical
invariant form, say \(\psi\) on each cell module \(C(f, \lambda)\). Let
\[
D(f, \lambda) = \frac{C(f, \lambda)}{\text{rad}C(f, \lambda)},
\]
where \(\text{rad}C(f, \lambda)\) is the radical of \(C(f, \lambda)\) with respect to \(\psi\). Then all non-zero \(D(f, \lambda)\) consist of all
pair-wise non-isomorphic simple \(B_n\)-modules [5 Theorem 3.4]. It follows from Theorem 4.12 that
\(D(f, \lambda) \neq 0\) if and only if \(\lambda \in \Lambda_n^+(n - 2f)\).

Corollary 4.13. Suppose \((f, \lambda) \in \Lambda_n\), \(\mu \in \Lambda_n^+(n)\) and \([C(f, \lambda) : D(0, \mu)] \neq 0\). Then there are
\((s, t) \in \mathcal{F}_n^{ud}(\mu) \times \mathcal{F}_n^{ud}(\lambda)\) such that \(c_s(k) = c_t(k)\) for all \(1 \leq k \leq n\).

Proof. Let \(L\) be the abelian subalgebra of \(B_n\) generated by \(L_1, L_2, \ldots, L_n\). Restricting both \(C(f, \lambda)\)
and \(D(0, \mu)\) to \(L\) and using Theorem 4.12 yields the result as required. \(\square\)

5. Restriction and Induction functors

In this section, we consider the \(q\)-Brauer algebra over a field \(F\). Recall the algebra homomorphism
\(\phi : B_{n-2} \rightarrow \tilde{E}_1B_n\tilde{E}_1\) in Theorem 3.10. Identifying \(B_{n-2}\) with \(\tilde{E}_1B_n\tilde{E}_1\) via the algebra isomorphism \(\phi\),
we have two functors \(\mathcal{F} : B_n\text{-mod} \rightarrow B_{n-2}\text{-mod}\) and \(\mathcal{G} : B_{n-2}\text{-mod} \rightarrow B_n\text{-mod}\) such that
\[
\mathcal{F}(M) = M\tilde{E}_1, \quad \text{and } \mathcal{G}(N) = N \otimes_{\tilde{E}_1B_n\tilde{E}_1} \tilde{E}_1B_n
\]
for any \(B_n\)-module \(M\) and any \(B_{n-2}\)-module \(N\).

Proposition 5.1. Suppose \((f, \lambda) \in \Lambda_n\) and \((\ell, \mu) \in \Lambda_{n+2}\).

1. \(\mathcal{F}\mathcal{G} = 1\).
2. \(\mathcal{F}(C(f, \lambda)) \cong C(f - 1, \lambda)\) where \(C(-1, \lambda)\) is defined to be 0,
3. \(\mathcal{G}(C(f, \lambda)) \cong C(f + 1, \lambda)\),
4. \(\text{Hom}_{B_{n+2}}(\mathcal{G}(C(f, \lambda)), C(\ell, \mu)) \cong \text{Hom}_{B_n}(C(f, \lambda), \mathcal{F}(C(\ell, \mu)))\).

Proof. Obviously, (1) follows from Theorem 3.10. Recall \(\mathcal{D}_{f,n}\) in Definition 3.2(1). For any \(d \in \mathcal{D}_{f,n-2}\),
let \(\tilde{d}\) be obtained from \(d\) by replacing each factor \(s_i\) of \(d\) by \(s_{i+2}\). Let \(\mathcal{D}_{f,n}\) be the set of all such \(\tilde{d}\). Then \(\mathcal{D}_{f,n} \subset \mathcal{D}_{f,n+1}\). By Theorems 3.8, 3.10 \(\tilde{E}_1B_n\tilde{E}_1\) has cellular basis
\[
\{qz^{-1}T_1^{-1}T_2\sigma(T_d)E^{\ell+1}x_{st}T_c \mid s, t \in \mathcal{F}^{std}(\lambda), (f, \lambda) \in \Lambda_{n-2}\}
\]
where \(x_\lambda\) corresponds to the index representation of \(H_{2f+3,n}\). If we denote by \(\tilde{C}(f - 1, \lambda)\) the corresponding
cell module of \(\tilde{E}_1B_n\tilde{E}_1\) with respect to \((f - 1, \lambda)\), then
\[
C(f - 1, \lambda) \cong \tilde{C}(f - 1, \lambda)
\]
as \(B_{n-2}\)-modules and the required isomorphism sends any basis element \(E^{f-1}x_2T_{\varphi}\) to
\[
qz^{-1}T_1^{-1}T_2E^{\ell}\varphi(x_\lambda)\varphi(T_d) + \tilde{B}^{\varphi\lambda}
\]
where \(\tilde{B}^{\varphi\lambda}\) is the free \(k\)-module spanned by
\[
\bigcup_{(\ell, \mu) \in \Lambda_n} \{qz^{-1}T_1^{-1}T_2\sigma(T_{c_1})E^{\ell}x_{s_1}T_{c_2} \mid s, t \in \mathcal{F}^{std}(\mu), c_1, c_2 \in \mathcal{D}_{f,1-n}, \mu \triangleright \lambda\}.
\]
Proof. Suppose Lemma 6.2.

\[ C(f + 1, \lambda) \] sending \[ E^j x_Td + B^{\circ \lambda} \to g \cdot qz = \lambda \] for any \( h \in B_{n+2} \).

By definition, \( G(C(f, \lambda)) = E^j x_T \) sending \( E^j x_T \) to \( E^j x_T \) for any \( h \in B_{n+2} \). So the number of all such elements is equal to the rank of \( C(f + 1, \lambda) \).

Since we are considering the \( q \)-Brauer algebra over a field \( F \), \( \dim_F G(C(f, \lambda)) \leq \dim_F C(f + 1, \lambda) \), forcing \( \psi \) to be an isomorphism. Finally, (4) follows from the adjoint associativity of tensor, hom functors and the well-known fact that \( F(C(\ell, \mu)) \cong \hom_{B_n}(\tilde{E}_1 B_n, C(\ell, \mu)) \).

6. The Radical of \( C(1, \mu) \)

In this section we consider the \( q \)-Brauer algebra over the field \( F \) containing invertible \( e \) such that \( e > n \) where \( e \) is the quantum characteristic of \( q^2 \). We are going to describe explicitly the radical of cell modules \( C(1, \mu) \) for all \( (1, \mu) \in \Lambda_n \). This result can be considered as the counterpart of [4, Theorem 3.4] for the Brauer algebra. From Lemma 6.2 to the end of Remark 6.8 we assume that the \( q \)-Brauer algebra is defined over \( k \). The following result is motivated by Doran-Wales-Hanlon’s work on Brauer algebra in [4].

**Lemma 6.1.** Let \( V \) be the free \( k \)-module with basis \( \{ E^j T_n d + B_{n}^2 \mid w \in S_{3,n}, d \in D_{1,n} \} \). Then \( V \) is an \( (H_{3,n}, B_{n}) \)-bimodule with the natural actions induced by the multiplication in \( B_{n} \).

**Proof.** The result follows from Theorem 3.8 and Lemma 3.9. □

In the remaining part of this section, we write \( a = q - q^{-1} \).

**Lemma 6.2.** Suppose \( s_{2,j_1}s_{1,i_1}, s_{2,j_2}s_{1,i_2} \in D_{1,n} \). There are \( h \in \sum_{x \in S_{3,n}} q^x z[a] T_x, w \in S_{3,n} \) and \( b \in \{0, 1, 2\} \) such that

\[
E^1 T_{2,i_1} T_{i_1} T_{i_2} T_{j_2} E_1 \equiv \delta_{(i_1,j_1),(i_2,j_2)} \delta E_1 + (1 - \delta_{(i_1,j_1),(i_2,j_2)}) z^b T_w E_1 + ah E_1 \pmod{B_{n}^2}. \tag{6.1}
\]

**Proof.** Suppose \( k < l \). Thanks to the quadratic relation in (2.2) and Definition 2.1.

\[
T_{k,l} = 1 + a \sum_{k+1 \leq i \leq l} T_{k+1,i} T_{i+1,j} T_{k+1,j}, \text{ and } E^1 T_2 T_3 T_2 E_1 \equiv az E_1 + az q E_1 T_3 \pmod{B_{n}^2}. \tag{6.2}
\]

If \( l \leq i_1 \), then

\[
E^1 T_{2,i_2} T_{2,i_1} T_{3,i_2} E_1 \equiv T_{i_1+1,i_2} E_1 T_{2,i_1} T_{3,i_2} E_1 T_{3,i_2+1}. \tag{6.3}
\]

Let \( x = E^1 T_{2,i_1} T_1 T_{2,i_1} E_1 \). By braid relations in (2.2), Definition 2.1(3) and (6.2),

\[
x = E^1 T_2 T_3 T_1 T_{2,i_1} T_{3,j_1} T_{1,j_1} E_1 \equiv E^1 T_2 T_1 (1 + a \sum_{4 \leq m \leq j_1} T_{m,i_1} T_{2,i_1} E_1 \equiv q^2 z E_1 + a \sum_{4 \leq m \leq j_1} T_{m,i_1} T_{2,i_1} T_{3,i_2} T_{4,m} \equiv q^2 z E_1 + a \sum_{4 \leq m \leq j_1} T_{m,i_1} (az E_1 + az q E_1 T_3) \pmod{B_{n}^2}. \tag{6.4}
\]
Now, we are ready to verify (6.1). Suppose \((i_1, j_1) = (i_2, j_2)\). By (6.2) and Definition (2.13)
\[E_1 T_{2,j_2} T_{i_1,i} T_{i_1,1}, T_{j_1,2}, E_1 = E_1 T_{2,j_2} T_{j_1,2}, E_1 + a E_1 T_{2,j_2} \sum_{2 \leq i \leq i_1} T_{1,2} T_{i_1,1}, T_{j_1,2}, E_1
\]
\[= \delta E_1 + a \sum_{3 \leq k \leq j_1} T_{k,3} E_1 T_{2,j_2} T_{3,k}, + a E_1 T_{2,j_2} \sum_{2 \leq i \leq i_1} T_{1,2} T_{i_1,1}, T_{j_1,2}, E_1
\]
\[= \delta E_1 + a \sum_{3 \leq k \leq j_1} T_{k,3} E_1 T_{2,j_2} T_{3,k}, + a E_1 T_{2,j_2} \sum_{2 \leq i \leq i_1} T_{1,2} T_{i_1,1}, T_{j_1,2}, E_1.
\]
By (6.3)–(6.4), \(E_1 T_{2,j_2} T_{i_1,i} T_{i_1,1}, T_{j_1,2}, E_1\) can be written as the required form in (6.1).

Suppose \((i_1, j_1) \neq (i_2, j_2)\). Applying anti-involution \(\sigma\) on (6.1), we see that it is enough to prove (6.1) in two cases: (1) \(i_1 < i_2 < j_2\), (2) \(i_1 = i_2\) and \(j_1 < j_2\). In the first case, we have
\[E_1 T_{2,j_2} T_{i_1,i} T_{i_1,1}, T_{j_1,2}, E_1 = y T_{3,i_1+2}
\]
where \(y = E_1 T_{2,j_2} T_{i_1,i} T_{i_1,1}, T_{j_1,2}, E_1\). We have
\[y = \begin{cases} T_{i_1,3} T_{j_1,2}, T_{2,j_2}, E_1 T_{2,3} T_{2,j_2}, E_1 T_{4,j_1+2} & \text{if } j_1 < i_2, \\ q z E_1 T_{2,3} + a \sum_{3 \leq m \leq j_1} T_{m,3} T_{j_1,2}, E_1 T_{2,3} T_{2,m+1} T_{3,j_1+1} T_{4,j_1+2} & \text{if } j_1 = i_2, \\ T_{i_1,3} (E_1 T_{2,j_2}, T_{i_1,1}) T_{i_1,1}, T_{j_1,2}, E_1 & \text{if } j_1 > i_2. \end{cases}
\]
(6.6)
Applying anti-involution \(\sigma\) on \(E_1 T_{2,j_2} T_{i_1,i} T_{i_1,1}, T_{j_1,2}, E_1\), we see that it is enough to assume \(j_1 \leq j_2\) when we assume \(j_1 < i_2\). If \(j_1 < j_2\), then
\[E_1 T_{2,j_2} T_{i_1,i} T_{i_1,1}, T_{j_1,2}, E_1 = T_{2,j_2}, E_1 T_{2,j_2} T_{i_1,i} T_{i_1,1}, E_1. \]
(6.7)
Under the assumption (1) except \(j_1 = j_2 > i_2\), (6.1) follows immediately from (6.5)–(6.6) if we use (6.2) to rewrite \(E_1 T_{2,j_2} T_{i_1,i} T_{i_1,1}, E_1\). In the remaining case, since \(E_1 T_{2,j_2} T_{i_1,i} T_{i_1,1}, E_1\) has already been computed in (6.4) and (6.1) follows immediately. In the second case,
\[E_1 T_{2,j_2} T_{i_1,i} T_{i_1,1}, T_{j_1,2}, E_1 = E_1 T_{2,j_2} T_{i_1,i} T_{i_1,1}, T_{j_1,2}, E_1 + a \sum_{2 \leq m \leq i_1} E_1 T_{2,j_2} T_{i_1,i} T_{i_1,1}, T_{j_1,2}, E_1
\]
\[= z E_1 T_{2,j_2} T_{3,j_1+1} + a \sum_{2 \leq m \leq i_1} T_{m+1,3} E_1 T_{2,j_2} T_{3,j_1+1} T_{3,j_1+1} T_{4,j_1+1} T_{3,j_i+1} + a \sum_{2 \leq m \leq i_1} T_{m+1,3} E_1 T_{2,j_2} T_{3,j_i+1} T_{3,j_i+1} T_{4,j_i+1} T_{3,j_i+1}. \]
(6.8)
Rewriting \(E_1 T_{2,j_2} T_{i_1,i} T_{i_1,1}, E_1\) via (6.2) again, we have (6.1) immediately.

**Definition 6.3.** Let \(\phi: V \times V \rightarrow \mathbb{k}\) be the bilinear form such that for any \(w_1, w_2 \in \mathcal{S}_{3,n}\) and \(d_1, d_2 \in D_{1,n}\),
\[\phi(E_1 w_1, T_{d_1} + B_n^{2}, E_1 w_2, T_{d_2} + B_n^{2}) = \tau(h)\]
where \(h \in \mathcal{H}_{3,n}\) such that \(E_1 w_1, T_{d_1} \sigma(T_{d_2}) \sigma(T_{w_2}) E_1 \equiv E_1 h (\text{mod } B_n^{2})\) and \(\tau: \mathcal{H}_{3,n} \rightarrow Z\) is the trace function in (2.3).

Thanks to Theorem (3.3) and Corollary (3.11), the element \(h\) in Definition (6.3) is unique and the bilinear form \(\phi\) is well-defined. The following result follows immediately from Definition 6.3.

**Lemma 6.4.** Let \(\phi: V \times V \rightarrow \mathbb{k}\) be the bilinear form in Definition 6.3. Then \(\phi\) is a symmetric invariant form in the sense that \(\phi(xa, y) = \phi(x, y \sigma(a))\) for all \(x, y \in V\) and \(a \in B_n\).

In order to simplify notation, define
\[v_{w,d_{j,i}} = E_1 w_1, T_{d_{j,i}} + B_n^{2}, \]
(6.9)
where \(d_{j,i} = s_{j} s_{i} \in D_{1,n}\) and \(w \in \mathcal{S}_{3,n}\). Then \(\{v_{w,d_{j,i}} \mid 1 \leq i < j \leq n, w \in \mathcal{S}_{3,n}\}\) is a basis of \(V\).

**Lemma 6.5.** For any \(1 \leq k \leq n - 1\), \(T_k\) acts on the right of \(V\) as a symmetric matrix.

**Proof.** There are some \(a(w, d_{j',i}'), (w, d_{j,i}) \in \mathbb{k}\) and \((w', d_{j',i}') \in \mathcal{S}_{3,n} \times D_{1,n}\) such that
\[v_{w,d_{j,i}}, T_k = \sum_{(w', d_{j',i}')} a(w, d_{j',i}'), (w, d_{j,i}) v_{w', d_{j',i}'}.
\]
Let $A_k = (a_{(w,d,j',i'),(w,d,j,i)})$ be the $m \times m$-matrix where $m$ is the rank of $V$. Then $A_k$ is the matrix with respect to $T_k$. The LHS of (6.9) can be computed explicitly by Definition 2.1. There are five cases: (1) $i = k + 1$, (2) $i = k$, (3) $i > k + 1$, (4) $k > j$ (5) $i < k \leq j$.

Thanks to the braid relations in (2.2) and Definition 2.1(2), we have

$$v_{w,d,j}, T_k = \begin{cases} (q - q^{-1})v_{w,d,j,k+1} + v_{w,d,j,k} & \text{if } i = k + 1, \\ v_{w,d,j,k+1} & \text{if } i = k, j > k + 1, \\ qv_{w,d,j,k} & \text{if } i = k, j = k + 1. \end{cases}$$

When $i > k + 1$,

$$v_{w,d,j}, T_k = \begin{cases} v_{w,s_{k+2},d,j}, & \text{if } \ell(w_{s_{k+2}}) > \ell(w), \\ v_{w,s_{k+2},d,j} + (q - q^{-1})v_{w,d,j} & \text{if } \ell(w_{s_{k+2}}) < \ell(w). \end{cases}$$

If $k > j$, then

$$v_{w,d,j}, T_k = \begin{cases} v_{w,s_k,d,j}, & \text{if } \ell(w_s) > \ell(w), \\ v_{w,s_k,d,j} + (q - q^{-1})v_{w,d,j} & \text{if } \ell(w_s) < \ell(w). \end{cases}$$

Finally, assume $i < k \leq j$. If $k \geq j - 1$, then

$$v_{w,d,j}, T_k = \begin{cases} v_{w,d,j+1,i}, & \text{if } k = j, \\ (q - q^{-1})v_{w,d,j} + v_{w,d,j-1,i} & \text{if } k = j - 1. \end{cases}$$

In the remaining case, $k < j - 1$. We have

$$v_{w,d,j}, T_k = \begin{cases} v_{w,s_{k+1},d,j}, & \text{if } \ell(w_{s_{k+1}}) > \ell(w), \\ (q - q^{-1})v_{w,d,j} + v_{w,s_{k+1},d,j} & \text{if } \ell(w_{s_{k+1}}) < \ell(w). \end{cases}$$

In summary, it follows from explicit computation on $v_{w,d,j}, T_k$ that each diagonal entry of $A_k$ is one of scalar in $\{q, q - q^{-1}, 1\}$ and each off diagonal entry of $A_k$ is either 0 or 1. Further, for any $(w', d', i') \neq (w, d, j), a_{(w', d', i'),(w,d,j)} = 1$ if and only if $a_{(w,d,j),(w', d', i')} = 1$. In particular, $A_k$ is a symmetric matrix, proving the result.

Lemma 6.6. Let $G$ be the Gram matrix with respect to the basis $\{v_{w,d,j,i} | 1 \leq i \leq j, w, d, n \in \mathfrak{S}_n, n \in \mathfrak{S}_n\}$ of $V$ and the symmetric invariant form $\phi$ in Definition 6.7. If $\psi : V \rightarrow V$ is the linear map given by $G$ with respect to the basis above, then $\psi$ is a $(\mathfrak{H}_3, \mathfrak{H}_n)$-bimodule homomorphism.

Proof. Thanks to Lemma 6.4, $GA_k$ is a symmetric matrix. By Lemma 6.7, $GA_k = A_k G = A_k G'$, which is equivalent to saying that $\psi$ is a right $\mathfrak{H}_n$-homomorphism. Suppose $3 \leq i \leq n$ and $(w, d), (w', d') \in \mathfrak{S}_n \times D_{1,n}$. Thanks to Definition 6.3, there is an $h \in \mathfrak{H}_n$ such that

$$\phi(T_i E_i T_{w} T_{d}, E_{i} T_{w} T_{d'}) = \tau(T_i h) \quad \text{and} \quad \phi(E_i T_{w} T_{d}, T_i E_{i} T_{w} T_{d'}) = \tau(h T_i).$$

Since $\tau$ is a trace function, we have $\phi(T_i E_i T_{w} T_{d}, E_{i} T_{w} T_{d'}) = \phi(E_i T_{w} T_{d}, T_i E_{i} T_{w} T_{d'})$, proving that $\psi$ is a left $\mathfrak{H}_3$-homomorphism. Further, $\psi$ is a $(\mathfrak{H}_3, \mathfrak{H}_n)$-homomorphism since two actions are induced by the multiplication on $B_n$.

Proposition 6.7. For any two basis elements $v_{w,d}, v_{w',d'}$ of $V$ in (6.3),

$$\phi(v_{w,d}, v_{w',d'}) = \begin{cases} \delta + azf(q) & \text{if } (w, d) = (w', d'), \\ \langle z, q^b \rangle + azg(q) & \text{otherwise,} \end{cases}$$

for some $b \in \mathbb{Z}$, $c \in \{0, 1\}$ and $f(q), g(q) \in q^\mathbb{Z}[a]$.

Proof. The result follows from (2.3) and (6.1).

Remark 6.8. Suppose $z = q^m$ for some $m \in \mathbb{Z}$. The classical limit of $B_n$ is the Brauer algebra $B_n(m)$ over $\mathbb{Z}$ (see Remark 2.2). In this case, the above bilinear form becomes the corresponding bilinear form on $V$ for $B_n(m)$ by setting $q \rightarrow 1$ and the corresponding Gram matrix is $G_1 = \lim_{q \rightarrow 1} G$. More explicitly, the diagonal (resp., off diagonal) entries of the Gram matrix $G_1$ are of form $\delta$ (resp., 0 or 1) and the linear map $\psi : V \rightarrow V$ induced by $G_1$ in Lemma 6.6 becomes an $(\mathfrak{S}_3, \mathfrak{S}_n)$-bimodule homomorphism.

From here to the end of this section, we assume that $B_n$ is defined over the field $F$ which contains a non-zero $q$ such that $e > n$ where $e$ is the quantum characteristic of $q^2$. 
The transition matrix between these two bases of $V$ is identity matrix and $V_{\psi}$. Each $C_i$ is a right irreducible $H_\gamma$-module. With respect to this basis, the corresponding matrix is a right irreducible $H_\gamma$-module. Where $M$ is the $H_2$-module (resp., $H_{3,n}$-module) with basis $\{E_1\}$ (resp., $\{x_\mu T_\nu(t) + H_{3,n}^{c,\mu} \mid t \in \mathcal{F}^{\text{std}}(\mu)\}$). Since $\mathbb{Z}$ is semisimple with multiplicity $\dim S^\alpha$ with $\alpha = (2)$ (see Definition 2.12)) and $N \cong S^\mathbb{Z}$, we have

$$C(1,\mu) \cong \text{Ind}_{H_2 \otimes H_{3,n}}^{H_{3,n}} S^\alpha \otimes S^\mu$$

as right $H_\gamma$-modules. Using Littlewood-Richardson rule for semisimple Hecke algebra $H_\gamma$, $C(1,\mu) \cong \oplus S^\gamma$ (6.10) as $H_\gamma$-modules, where $\gamma \in \Lambda^+(n)$ obtained from $\mu$ by adding two boxes which are not at the same column. Moreover, the multiplicity of $S^\gamma$ is 1. So $[C(1,\mu) : C(0,\lambda)] \neq 0$ if and only if $[C(1,\mu) : C(0,\lambda)] = 1$. Further, by Corollary 4.13, $\dim S^\alpha = 1$. Further, by Definition 6.9.

Therefore $\lambda$ is $(1,\mu)$-admissible.

We are going to prove $[C(1,\mu) : C(0,\lambda)] \neq 0$ if $\lambda$ is $(1,\mu)$-admissible. Consider the $(H_{3,n},H_\gamma)$-bimodule homomorphism induced by $\psi : V \to V$ in Lemma 6.9 via base change. Since we are assuming that $e > n$, both $H_\gamma$ and $H_{3,n}$ are semisimple. This implies that $V$ is direct sum of cell modules $C(1,\mu)$ with multiplicity $\dim S^\mu$. Since $\psi$ is a left $H_{3,n}$-homomorphism, the restriction of $\psi$ to $C(1,\mu)$ fixes $C(1,\mu)$. Let $e_\mu$ be the primitive idempotent with respect to $S^\mu$. Then $S^\mu = e_\mu H_\gamma$ and $\tau(e_\mu) \neq 0$ (see [10] Lemma 1.6). Since $\phi(E_1 e_\mu, E_1 e_\mu) = \delta(\tau(e_\mu)) \neq 0$, the restriction of $\psi$ to the cell module $C(1,\mu)$ is non-trivial and hence

$$\ker \psi \cap C(1,\mu) \neq C(1,\mu)$$

(6.12)

Thanks to [6.10], $C(1,\mu) = \bigoplus C(1,\mu)_\gamma$ where $C(1,\mu)_\gamma$ is the right $H_\gamma$-submodule such that $C(1,\mu)_\gamma \cong S^\gamma$.

Then $C(1,\mu) e_\gamma H_\gamma = C(1,\mu)_\gamma$. Since $\psi$ is a right $H_\gamma$-homomorphism, $\psi$ fixes each $C(1,\mu)_\gamma$. Pick up a basis of $C(1,\mu)_\gamma$ for each $\gamma$. The disjoint union of such bases forms a new basis of $C(1,\mu)$. Note that each $C(1,\mu)_\gamma$ is a right irreducible $H_\gamma$-module. With respect to this basis, the corresponding matrix with respect to the decomposition of $\psi$ to $C(1,\mu)$ is of form $\text{diag}(h_{1}, I_\gamma)$ where $I_\gamma$ is the $S^\gamma \times S^\gamma$ identity matrix and $h_{\gamma} \in F$.

Recall that $\{v_{w,d} \mid w \in S_{3,n}, d \in D_{1,n}\}$ is a basis of $V$ and the disjoint union of bases of $C(1,\mu)_\gamma$ form a new basis of $V$. Moreover, since our computation only involves Hecke algebra, each entry in the transition matrix between these two bases of $V$ is of form $f(q) \otimes 1_F$ for some $f(q) = \frac{a(q)}{b(q)}$ where $g(q), h(q) \in \mathbb{Z}[q]$ such that $h(q) \otimes 1_F \neq 0$. Abusing of notation, we will denote $a \otimes b$ by $a$. Thanks to Proposition 6.7 each entry in the diagonal (resp., off diagonal) of the matrix $G$ with respect to the basis is of form $\delta + zg(q)$ (resp., $zf(q)$) where $f(q), g(q) \in F$. Let $m = \dim V$. So, there exist $f_i(q) \in F, 1 \leq i \leq m$ with $f_j(q) \neq 0$ for some $j$ such that

$$G(f_1(q), \ldots, f_m(q))^t = h_\gamma (f_1(q), \ldots, f_m(q))^t$$

for each $\gamma$ in [6.10]. Then $(\delta + zg(q)) f_j(q) + zf(q) = h_\gamma f_j(q)$ for some $f(q), g(q) \in F$ and hence

$$h_\gamma = \delta + zf_\gamma(q)$$

(6.13)

for some $f_\gamma(q) \in F$. We explain that there are $z$ and $q$ such that $h_\lambda = 0$. This can be seen as follows.

Let $z = q^d$ for some $b \in \mathbb{Z}$ and consider the classical limit of $B_n$. It is the Brauer algebra $B_n(b) \otimes \mathbb{Z}$. The corresponding Brauer algebra $B_n(b) \otimes \mathbb{C}$ is $B_n(b) \otimes \mathbb{C}$. Unless otherwise stated, we work on Brauer algebra $B_n(b) \otimes \mathbb{C}$ at moment. Then

$$\lim_{q \to 1} f_1(q), \ldots, \lim_{q \to 1} f_m(q),$$
is the corresponding eigenvector of $G_1$ in Remark 6.8 (In fact, it is the corresponding $G_1$ over $\mathbb{C}$). Further, $\lim_{q \to 1} \delta = b$ and $\lim_{q \to 1} h = b + h$, where $\lim_{q \to 1} f_q(\delta) = h$. We may choose $b$ such that $b + h = 0$. This implies that $C(1, \mu)_{\lambda}$ is in the kernel of the $\psi$ in classical limit case. Such a kernel is the corresponding radical of the invariant form $\phi$. By (6.12), $C(1, \mu)_{\lambda}$ is a proper submodule of $C(1, \mu)$. If we denote $\text{rad}C(1, \mu)$ by the radical with respect to the canonical invariant form on $C(1, \mu)$, then $C(1, \mu)_{\lambda} \subset \text{rad}C(1, \mu)$. Standard arguments on cellular algebras in [15] shows that $\text{rad}C(1, \mu)$ is killed by $E_1$ and so is $C(1, \mu)_{\lambda}$. This proves that $C(1, \mu)_{\lambda} \cong C(0, \lambda)$. By [4] Theorem 3.2, $b = 1 - c(p_1) - c(p_2)$ and $h = -b$. Since $\delta \neq 0$, $\lim_{q \to 1} f_q(\delta) = -b \neq 0$. So $f_q(\delta) \neq 0$. Since $\lim_{q \to 1} f_q(\delta)(\lambda)$ exist, $f_q^{-1}f_q(\delta)(\lambda) \neq 1$.

We may choose $z$ and $q$ such that $z^{-2} = 1 - (q - q^{-1})f_q(\delta)(\lambda) \notin \{0, 1\}$. In other words, we can find $z$ and $q$ such that $\delta$ is invertible and $h = 0$. By arguments similar to those for the Brauer algebra above, $C(1, \mu)_{\lambda} \cong C(0, \lambda)$ and hence $z^2 = q^{2 - 2(c(p_1) + c(p_2))}$ (see (6.11)).

7. Proof of Theorem A

In this section, we give a proof of Theorem A. For any $(f, \lambda) \in \Lambda_n$, let $G_{f, \lambda}$ be the Gram matrix associated to the canonical invariant form (see [8] Definition 2.3) on $C(f, \lambda)$ with respect to the cellular basis in Theorem 5.8.

**Proposition 7.1.** For $n \geq 2$, let $B_n$ be defined over $F$ containing non-zero $q, z, q^{-1}, q^{-1}$ such that the quantum characteristic $e$ of $q^2$ is strictly bigger than $n$. Then $B_n$ is (split) semisimple if and only if $\det G_{1, \lambda} \neq 0$ for all $\lambda \in \Lambda^+(k - 2)$ and $2 \leq k \leq n$.

**Proof.** Thanks to Theorem 3.8, $B_n$ is a cellular algebra over an arbitrary field. So any field is a split field of $B_n$. By [8] Theorem 3.8, $B_n$ is (split) semisimple if and only if $\prod_{(f, \lambda) \in \Lambda_n} \det G_{f, \lambda} \neq 0$. Since we are assuming $e > n, \mathcal{H}_n$ is semisimple and hence $\prod_{\lambda \in \Lambda^+(n)} \det G_{0, \lambda} \neq 0$. Consequently, $B_n$ is (split) semisimple if and only if $\prod_{(f, \lambda) \in \Lambda_n} \det G_{f, \lambda} \neq 0$ where $\Lambda_n = \{(f, \lambda) \in \Lambda_n \mid f > 0\}$.

We prove the result by induction on $n$. Since we are assuming that $\delta$ is invertible, $\det G_{1, \emptyset} = \delta \neq 0$. So $B_2$ is always semisimple. In the following we assume $n > 2$ and the result is true for $B_r$ with $r < n - 1$.

“⇐” If the result were false, there should be an $(f, \lambda)$ with $f \geq 2$ such that $\det G_{f, \lambda} = 0$. In this case, there is a simple module $D(\ell, \mu) \subset \text{rad}C(f, \lambda)$. In particular, $(\ell, \mu) < (f, \lambda)$ in the sense that the either $\ell < f$ or $\ell = f$ and $\mu < \lambda$. However, since $D(\ell, \mu) \hookrightarrow C(f, \lambda)$, there is a non zero homomorphism from $C(\ell, \mu)$ to $C(f, \lambda)$. Acting the exact functor $\mathcal{F}$ and using Proposition 5.12-(4) yields a non-zero homomorphism $C(0, \mu) \hookrightarrow C(f - \ell, \lambda)$. We claim $\ell < f$. Otherwise, both $C(0, \mu)$ and $C(0, \lambda)$ are simple modules of the semisimple Hecke algebra $\mathcal{H}_{n-2 f}$. So $\lambda = \mu$ and hence $[C(f, \lambda) : D(f, \lambda)] \geq 2$, a contradiction. Hence $f - \ell > 0$.

If $\ell > 0$, then $\det G_{f - \ell, \lambda} = 0$, a contradiction since $B_{n-2 f}$ is semisimple by induction assumption. Suppose $\ell = 0$. Thanks to Theorem 3.20 there are two boxes $p_1$ and $p_2$ such that either $\{C(f, \lambda \setminus p_1) : C(0, \mu \setminus p_2) \neq 0$ or $\{C(f - 1, \lambda \cup p_1) : C(0, \mu \setminus p_2) \neq 0$. Since we are assuming $f \geq 2$, $f > f - 1 \geq 1$. Consequently, either $\det G_{f, \lambda \setminus p_1} \neq 0$ or $\det G_{f - 1, \lambda \cup p_1} \neq 0$. This is a contradiction since $B_{n-1}$ is semisimple by induction assumption.

“⇒” Suppose $\det G_{1, \lambda} = 0$ for some $\lambda \in \Lambda^+(k), 0 \leq k \leq n - 2$. Then $\lambda \neq 0$ and there is a monomorphism $C(0, \mu) \hookrightarrow C(1, \lambda)$ for some partition $|\mu| = k + 2$. Since $B_n$ is semisimple, we have $k < n - 2$. If $k \equiv n \pmod{2}$, then we apply the functor $\mathcal{G}$ repeatedly so as to get a non-zero homomorphism from $C(f - 1, \mu)$ to $C(f, \lambda)$ where $f = \frac{1}{2}(n - k)$. So $\det G_{f, \lambda} = 0$ and $B_n$ is not semisimple, a contradiction. Suppose $k \equiv n + 1 \pmod{2}$. Since $D(0, \mu) \hookrightarrow C(1, \lambda)$, by Theorem 6.10 $\mu$ is $(1, \lambda)$-admissible. By Definition 8.11 $z^2 = q^{2(1 - c(p_1) + c(p_2))}$ where $\mu = \lambda \cup \{p_1, p_2\}$. The arguments in the proof of [14] Theorem 4.1 shows that there is a pair of partitions $\lambda$ and $\hat{\mu}$ such that $(0, \hat{\mu})$ is $(1, \hat{\mu})$-admissible and $|\lambda| = |\hat{\mu}| \pm 1$. By Theorem 6.10 there is a non-zero homomorphism from $C(0, \hat{\mu})$ to $C(1, \hat{\mu})$. Applying the functor $\mathcal{G}$ repeatedly yields $\det G_{f, \hat{\mu}} = 0$ where $f = \frac{1}{2}(n - |\lambda|)$. So $B_n$ is not semisimple, a contradiction. □

Proof of Theorem A Thanks to (2.5), $\mathcal{H}_n \cong B_n / \langle E_1 \rangle$, where $\langle E_1 \rangle$ is the two-sided ideal generated by $E_1$. Therefore, $B_n$ is not semisimple if $\mathcal{H}_n$ is not semisimple. The later is equivalent to the condition $e \leq n$. In summary, we can assume $e > n$ when we discuss semisimplicity of $B_n$ over an arbitrary field. In this case, Proposition 7.1 gives a necessary and sufficient condition for $B_n$ being semisimple.
over an arbitrary field. Thanks to Definition 6.9 and Theorem 6.10, $B_n$ is semisimple if and only if $z^2 \neq q^{2a}$ where

$$a \in \bigcup_{k=2}^{n} \left\{ 1 - \sum_{p \in \Lambda \setminus \mu} c(p) \mid \lambda \in \Lambda^+(k), \mu \subset \lambda, \lambda \setminus \mu = \{p_1, p_2\} \text{ and } p_1, p_2 \text{ are not at the same column} \right\}$$

By [15, Lemma 2.4], this is equivalent to saying that $a$ is given in (1.1).

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