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Comparative analysis of different variants of the Uzawa algorithm in problems of the theory of elasticity for incompressible materials

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GRAPHICAL ABSTRACT

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DIFFERENT VARIANTS OF THE UZAWA ALGORITHM ARE COMPARED WITH ONE ANOTHER. THE COMPARISON IS PERFORMED FOR THE CASE IN WHICH THIS ALGORITHM IS APPLIED TO LARGE-SCALE SYSTEMS OF LINEAR ALGEBRAIC EQUATIONS. THESE SYSTEMS ARISE IN THE FINITE-ELEMENT SOLUTION OF THE PROBLEMS OF ELASTICITY THEORY FOR INCOMPRESSIBLE MATERIALS. A MODIFICATION OF THE UZAWA ALGORITHM IS PROPOSED.

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Introduction

One widespread method for the solution of elasticity problems is the finite-element method. The application of this method results in a system of linear algebraic equations (SLAE) with a sparse matrix [1–3]. This system includes a large number of equations. This number depends essentially on the dimension of the problem and the fineness of the finite-element mesh. As a rule, the use of a finer mesh results in more accurate solutions. It is important to choose a method that permits one to solve systems of maximum size under limited computational resources. Different types of problems result in matrices of different structures, and different methods are effective for these matrices. A customized approach is necessary for specific problems in order to solve SLAE effectively.

Matrices can be symmetric (for problems of linear elasticity) or nonsymmetric (for nonlinear problems that are linearized using the Newton technique). If elasticity problems are solved in regions with a complicated geometry, the portrait of a matrix can be irregular (a portrait is the set of pairs of indices corresponding to nonzero elements), and the condition number of a matrix can be very large (a larger condition number involves a slower convergence of iterative methods).

Direct methods permit one to determine the exact solution of a system by a finite number of arithmetic operations for the case in which all the arithmetic operations are performed exactly. However, the application of direct methods to large-scale systems involves a very large expense of computer memory for the storage of the matrices that arise at the intermediate stages of the computations, even in the case in which the original matrix is very sparse. If these matrices cannot be stored in the random access memory of a computer, the application of direct methods is practically impossible.

One of the most powerful tools for solving large and sparse systems of linear algebraic equations is a class of iterative methods called Krylov subspace methods [4–8]. These methods are based on the minimization of the norm of the residuals. The conjugate gradient method is effective for systems with symmetric matrices. The biconjugate gradient method and the Generalized Minimal Residual method are used for the nonsymmetric case. The well-known modifications of these methods, the Biconjugate Gradient Stabilized method and the Flexible Generalized Minimal Residual method, permit one to use preconditioners [5].

However, these methods are almost unusable for some classes of problems. For the problems of these classes, these methods usually do not converge or converge very slowly. The potential cause of this effect is that the eigenvalues of the matrix of the system have different signs. Consider now one of these classes.

Consider SLAE arising from the finite-element solution of 3D elasticity problems for bodies made of incompressible materials. In particular, these problems may be formulated on the foundation of the theory of superimposed finite strains [9,10]. These include problems of the stress concentration near holes or inclusions that originate in prestressed bodies [11,12].

These SLAEs have the following form:

\[
\begin{pmatrix}
A & B^T \\
B & 0
\end{pmatrix}
\begin{pmatrix}
u \\
p
\end{pmatrix} =
\begin{pmatrix}
f \\
0
\end{pmatrix}.
\]

Here \(A\) is a symmetric, positive definite matrix, and \(B\) is a rectangular matrix. These systems can be written in the usual form \(Mx = \mathbf{f}\), where

\[
M = \begin{pmatrix}
A & B^T \\
B & 0
\end{pmatrix}, \quad x = \begin{pmatrix}
u \\
p
\end{pmatrix}, \quad R = \begin{pmatrix}
f \\
0
\end{pmatrix}.
\]

The matrices of such systems (systems with saddle points) have eigenvalues of different signs. The direct use of the iterative methods listed above is not effective for such systems. One can solve this problem using modified iterative methods for solving SLAE, in particular, relaxation methods [5,6,13].

Note that systems with saddle points arise from the numerical solution of dynamical problems of incompressible viscous liquids [14–16].

Methodology

The Uzawa method is intended for the solution of SLAE with saddle point matrices [5,13,15,17]. This method is iterative. At each iteration of this method, two SLAEs with the same matrix \(A\) and different right parts are solved. These SLAEs can be solved by direct methods or by the above mentioned iterative methods.

There are some variants of the Uzawa algorithm. [5] These variants are based on different iterative methods of solution of SLAE, such as the simple iteration method (SIter) [18], the minimal residual method (MRes) [19], the steepest descent method (StDes) [19], the conjugate gradient method (CG) [20] (the two- and three-layered schemes are referred to as CG2 and CG3, respectively), and the three-layered conjugate residual method (CRRes) [21]. The formulas for these variants of the Uzawa method are written in analogy with the formulas for the corresponding iterative methods.
A variant of the algorithm that realizes the Uzawa method on the basis of the three-layered scheme of the conjugate gradient method is presented below.

1. Setting the initial approximation \( x^{(0)} = \begin{pmatrix} u^{(0)} \\ p^{(0)} \end{pmatrix} \) and initial values of parameters \( a_0, \tau_0 \).
2. Setting an iteration counter: \( k := 0 \).
3. Computation of the norm of the residual vector \( \|r^{(0)}\| := \|R - Mx^{(0)}\| \).
4. Solution of the system \( Au^{(k+1)} = f - Cp^{(k)} \) with respect to \( u^{(k+1)} \) (the vector \( u^{(k)} \) is chosen as the initial approximation if the system is solved by an iterative method; here and below \( C = B^T \)).
5. Solution of the system \( A_{y^{(k+1)}} = CBu^{(k+1)} \) with respect to \( y^{(k+1)} \) (the zero vector is chosen as the initial approximation if the system is solved by an iterative method).
6. \( \tau_{k+1} := \frac{(Bu^{(k+1)}, Bu^{(k+1)})}{(Bu^{(k+1)}, Bu^{(k+1)})} \).
7. \( p^{(k+1)} := p^{(k)} + \tau_{k+1}Bu^{(k+1)} \).
8. \( z_{k+1} := \left[ 1 - \tau_{k+1}(Bu^{(k+1)}, Bu^{(k+1)})^{-1} \right] \).
9. \( p^{(k+1)} := z_{k+1}p^{(k+1)} + (1 - z_{k+1})p^{(k)} \).
10. Computation of the norm of the residual vector \( \|r^{(k+1)}\| := \|R - Mx^{(k+1)}\| \).
11. If \( \|r^{(k+1)}\| < \varepsilon \|r^{(0)}\| \), then go to item 12, else \( k := k + 1 \) and go to item 4.
12. End.

At the 4th and the 5th steps of this algorithm, the SLAEs are solved. As mentioned above, this solution can be obtained with the use of direct methods or iterative methods.

Note that the expression for the coefficient \( a_{k+1} \) at the 8-th step of the proposed algorithm is widely used for problems of hydrodynamics and gas dynamics. However, computational experiments show that for the problems of solid mechanics this method frequently diverges. It is possible to modify the expression for \( a_{k+1} \) in order to provide the better convergence of this method for the problems of solid mechanics. For the conjugate gradient method, the modified expression for \( a_{k+1} \) is

\[
\begin{align*}
\bar{a}_{k+1} &:= \left[ 1 + \frac{\tau_{k+1}(Bu^{(k+1)}, Bu^{(k+1)})^{-1}}{\tau_k(Bu^{(k)}, Bu^{(k)})z_k} \right].
\end{align*}
\]

Similarly for the Uzawa method based on the conjugate residual method expression for \( a_{k+1} \) is represented as

\[
\begin{align*}
\bar{a}_{k+1} &:= \left[ 1 - \tau_{k+1}(Bu^{(k+1)}, Bu^{(k+1)})^{-1} \right] \frac{1}{\tau_k(Bu^{(k)}, Bu^{(k)})z_k},
\end{align*}
\]

the modified expression for \( a_{k+1} \) is

\[
\begin{align*}
\bar{a}_{k+1} &:= \left[ 1 + \frac{\tau_{k+1}(Bu^{(k+1)}, Bu^{(k+1)})^{-1}}{\tau_k(Bu^{(k)}, Bu^{(k)})z_k} \right].
\end{align*}
\]

A series of computational experiments were performed. These computational experiments show that this modification converges for a range of solid mechanics problems for which the unmodified method diverges. The example is represented in the next section. That’s why this modification could be used in solid mechanics problems.

Results and discussion

The algorithms presented in the previous section were implemented in the finite-element strength analysis system (FIDESYS) [22]. The results of solving these problems by different variants of the Uzawa algorithm were compared. These variants of the Uzawa algorithm are based on StDes, the conjugate gradient method (both CG2 and CG3), and the CRes. The comparison was made for four matrices of different dimensions:

1. 45,442 rows, 39,042 of them accounting for the main block (matrix A);
2. 101,762 rows, 87,362 of them accounting for the main block;
3. 228,242 rows, 195,842 of them accounting for the main block;
4. 439,502 rows, 377,002 of them accounting for the main block.

In the process of the computations, it was assumed that $\varepsilon = 10^{-4}$, i.e., the criterion of termination is that the residue is reduced to 1:10,000 of the initial value. The SLAE at the 4th and the 5th steps of the Uzawa method was solved by direct methods.

The number of iterations required for the solution of a system using different variants of the Uzawa algorithm is shown in Fig. 1 for systems with different matrices (the matrices are ordered with respect to their dimension). The different methods are labeled by different characters.

One can see from Fig. 1 that there is no unique dependence between the dimension of the matrix and the number of iterations that is required for the solution of the system with a given accuracy. In addition, it is clear from Fig. 1 that the most effective and stable variant of the Uzawa method is based on CRRes.

The dependence of the computation time and the number of iterations on the variant of the Uzawa method is presented in Fig. 2 for matrix 2.

One can see from Fig. 2 that the computation time in seconds is one-third of the number of iterations of the Uzawa method. One iteration of the Uzawa method requires about 0.3 s of computation time for matrix 2.

Consider now the model problem of stress distribution around the elliptical hole made of the incompressible neo-Hookean material [23]. The material constant is given for rubber: $C_1 = 0.9$ MPa. The problem is solved for two-dimensional case (plane strain). The body assumes a square shape in the undeformed state, and the body size is $L$ by $L$. The semi-axes of ellipse are $0.1L$ and $0.025L$. The minor and major axes of the ellipse coincide with axes $x$ and $y$, respectively, and the square sides are parallel to these axes. The tensile load 0.05 MPa along the $x$-axis is applied to the sides parallel to the $y$-axis, and it is assumed that the displacement of two other sides in the direction of the $y$-axis is equal to zero.

The problem is solved using FIDESYS CAE-system [22] with geometrical nonlinearity accounted for. The SLAE for this problem is solved using the Uzawa algorithm. The system contains 50,747 rows, 33,978 of them accounting for the main block. Some results of numerical solution of this problem are shown in Fig. 3. The distribution of pressure around the hole is shown in this figure. The stress and strain are also computed.
Comparative analysis of different variants of the Uzawa algorithm

The results of solving this problem by different variants of the Uzawa algorithm are shown in Table 1. CG3Mod and CResMod denote modifications of CG3 and CRes methods, respectively; \( \tau \) is a parameter for the simple iteration method. In the process of the computations, it was assumed that \( \varepsilon = 10^{-5} \), i.e., the criterion of termination is that the residue is reduced to 1:100,000 of the initial value.

One can see from the table that the modified methods CG3Mod and CResMod converge while the unmodified methods CG3 and CRes diverge. The modified methods CG3Mod and CResMod are slower in comparison with the other methods. Nevertheless, the computation time for CG3Mod and CResMod is admissible. The simple iteration method gives the best result for \( \tau = 1 \). However, this method diverges for some other values of the parameter \( \tau \). So, this method requires individual tuning of the parameter \( \tau \) for each specific problem.

For this reason, the simple iteration method may be inconvenient for some users.

The final conclusion is that all the variants of the Uzawa algorithm considered below may be convenient in one case or another.

Conclusions

The comparison of different variants of the Uzawa algorithm is performed for large-scale systems of linear algebraic equations arising from the finite-element solution of elasticity problems for incompressible materials. The modification of the Uzawa algorithm is proposed. The computational experiments show that this modification improves the convergence of the Uzawa algorithm for the problems of solid mechanics. The final conclusion is that each variant of the Uzawa algorithm considered below has its advantages and disadvantages and may be convenient in one case or another.

Conflict of Interest

The authors have declared no conflict of interest.

Compliance with Ethics Requirements

This article does not contain any studies with human or animal subjects.

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References

[1] Levin VA, Zingerman KM, Vershinin AV, Freiman EI, Yangirova AV. Numerical analysis of the stress concentration near holes originating in previously loaded viscoelastic bodies at finite strains. Int J Solids Struct 2013;50:3119–35.
[2] Levin VA, Vershinin AV. Non-stationary plane problem of the successive origination of stress concentrators in a loaded body. Finite deformations and their superposition. Commun Numer Methods Eng 2008;24:2229–39.
[3] Zienkiewicz OC, Taylor RL. The finite element method. The basis, vol. 1. Oxford: Butterworth-Heinemann; 2000.
[4] Balandin M, Chernyshev O, Shurina E. Analysis of methods for solving large-scale non-symmetric linear systems with sparsed matrices. In: Parallel computing technologies. In: Malyskhan V, editor. Lecture notes in computer science, vol. 1277. Berlin: Springer-Verlag; 1997. p. 336–43.
[5] Benzi M, Golub GH, Liesen J. Numerical solution of saddle point problems. Acta Numer 2005;14:1–137.
[6] Bychenkov YV, Chizhonkov EV. Iterative methods for solving saddle-point problems. Moscow: BINOM; 2013 [in Russian].
[7] Liesen J, Tichý P. Convergence analysis of Krylov subspace methods. Mitt Ges Angew Math Mech 2004;27:153–73.
[8] Saad Y. Iterative methods for sparse linear systems. 2nd ed. Philadelphia, USA: SIAM; 2003.
[9] Levin VA. Using the method of successive approximations in problems of superposition of finite deformations. Soviet Appl Mech 1987;23:472–6.
[10] Levin VA. Theory of repeated superposition of large deformations and viscoelastic bodies. Int J Solids Struct 1998;35:2585–600.
[11] Levin VA, Zingerman KM. A class of methods and algorithms for the analysis of successive origination of holes in a prestressed viscoelastic body. Finite strains. Commun Numer Methods Eng 2008;24:2240–51.
[12] Zingerman KM, Levin VA. Redistribution of finite elastic strains after the formation of inclusions. Approximate analytical solution. J Appl Math Mech 2009;73:710–21.
[13] Chizhonkov EV. Relaxation methods for solving saddle-point problems. Moscow: URSS; 2002 [in Russian].
[14] Bychenkov YV, Chizhonkov EV. Optimization of one three-parameter method of solving an algebraic system of the Stokes type. Russian J Numer Anal Math Model 1999;14:429–40.
[15] Kobelkov GM, Olshanskii MA. Effective preconditioning of Uzawa type schemes for a generalized Stokes problem. Numer Math 2000;86:443–70.
[16] Ladyzhenskaya OA. The mathematical theory of viscous incompressible flow. 2nd ed. London: Gordon and Breach; 1969. p. 34–48.
[17] Uzawa H. Iterative methods for concave programming. In: Arrow KJ, Hurwicz L, Uzawa H, editors. Studies in linear and nonlinear programming. Stanford, USA: Stanford University Press. p. 154–65.
[18] Rees T. Preconditioning iterative methods for PDE constrained optimization Ph.D. thesis. University of Oxford; 2010. p. 76–9.
[19] Hu Q, Zou J. Nonlinear inexact Uzawa algorithms for linear abd nonlinear saddle-point problems. SIAM J Optim 2006;16:798–825.
[20] Liesen J, Parlett BN. On nonsymmetric saddle point matrices that allow conjugate gradient iterations. Numer Math 2008;108:605–24.
[21] Klawonn A. An optimal preconditioner for a class of saddle point problems with a penalty term. SIAM J Sci Comput 2000;19:540–52.
[22] FIDESYS Official Site. <http://www.cae-fidesys.com/en> [accessed on 21.03.2016].
[23] Treloar LRG. The physics of rubber elasticity. New York: Oxford University Press; 1975.