UNIFORM RESOLVENT ESTIMATES FOR SCHRÖDINGER OPERATOR WITH AN INVERSE-SQUARE POTENTIAL

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Abstract. We study the uniform resolvent estimates for Schrödinger operator with a Hardy-type singular potential. Let \( L_V = -\Delta + V(x) \) where \( \Delta \) is the usual Laplacian on \( \mathbb{R}^n \) and \( V(x) = V_0(\theta)r^{-2} \) where \( r = |x|, \theta = x/|x| \) and \( V_0(\theta) \in C^1(S^{n-1}) \) is a real function such that the operator \( -\Delta_\theta + V_0(\theta) + (n-2)^2/4 \) is a strictly positive operator on \( L^2(S^{n-1}) \). We prove some new uniform weighted resolvent estimates and also obtain some uniform Sobolev estimates associated with the operator \( L_V \).

Key Words: Uniform resolvent estimate, inhomogeneous Strichartz estimate, Sobolev inequality, inverse-square potential

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1. Introduction and main results

In this paper, we study the uniform resolvent estimates and their applications to the Sobolev inequalities and to the global-in-time inhomogeneous Strichartz estimates with non-admissible pairs. Consider the Schrödinger operator

\[
L_V = -\Delta + V(x)
\]
on \( L^2(\mathbb{R}^n) \) with \( n \geq 3 \) where the operator \( \Delta \) is the usual Laplacian on \( \mathbb{R}^n \) and the potential \( V(x) = V_0(\theta)r^{-2} \) with \( r = |x|, \theta = x/|x| \) and \( V_0(\theta) \in C^1(S^{n-1}) \) is a real function. The inverse-square potential is a typical example of critical decaying potentials, which is on a borderline for the validity of the resolvent and Strichartz estimates; we refer to [11, 17].

This paper is motivated by recent work of Bouclet and the first author [4] and the first author [32] in which the effect of decaying potentials in uniform resolvent estimates and global-in-time Strichartz estimates were investigated. In [4], the weighted resolvent estimates \( \|w(L_V - z)^{-1}w^*\|_{L^2 \to L^2} \) uniformly in \( z \) were proved to hold with \( w \) being a large class of weight functions in Morrey-Campanato spaces. The full set of global-in-time Strichartz estimates including the endpoint case was also obtained in [4], but non-admissible inhomogeneous cases were not considered there. The class of potentials we consider here includes the inverse-square type potentials. In [32], the uniform Sobolev estimates for the resolvent were proved under the assumption that zero energy is neither an eigenvalue nor a resonance in a suitable sense for the operator \( L_V \). The first author also proved global-in-time inhomogeneous Strichartz estimates hold for some non-admissible pairs. But one needs the requirement that \( V \in L^{\frac{n}{2}}(\mathbb{R}^n) \) with \( n \geq 3 \) which is not satisfied by the inverse-square potential. In light of this observation, the purpose of this paper is to study the uniform resolvent estimates, the
Sobolev inequalities and the non-admissible inhomogeneous Strichartz estimates which are associated with Schrödinger operator with an inverse-square decaying potential.

The uniform resolvent estimates play a fundamental role in the establishment of time-decay estimates or Strichartz estimates, see [25, 26, 38]. When the potential \( V \) is smooth enough and decays sufficiently fast at infinity, for example \( V \) belongs to Kato class (see [36]), there is a number of literature on the resolvent estimates of the Schrödinger operator with potentials and their applications to global-in-time dispersive estimates, such as time-decay estimates, or Strichartz estimates, in the past decades; see e.g. [16, 24, 37] for the resolvent estimates; [1, 2, 10, 15, 12] for the dispersive and Strichartz estimates and the references therein.

In this paper, as mentioned above, we focus on the Schrödinger operator \( \mathcal{L}_V \) given in \((1.1)\) which appears frequently in mathematics and physics. The study of the operator is connected with the combustion theory to the Dirac equation with Coulomb potential, and the study of perturbations of classic space-time metrics such as Schwarzschild and Reissner–Nordström; see [6, 7, 31, 35, 27, 42] and the references therein.

The Strichartz estimates and time-decay estimates for the dispersive equations with an inverse-square potential were studied in [6, 7, 31, 35]. In particular, Burq et al. [7] established the weighted uniform resolvent estimate

\[
\| |x|^{-1}(\mathcal{L}_V - \sigma)^{-1}|x|^{-1}\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \leq C, \tag{1.2}
\]

and then used it to prove the full set of the Strichartz estimates excluding the double-endpoint inhomogeneous estimates which were proved in [4] later. To prove the inhomogeneous Strichartz estimates for non-admissible pairs and to obtain more Sobolev inequality, the above uniform resolvent estimate \((1.2)\) is not enough. For our purpose, we have to generalize \((1.2)\) to \((1.4)\) stated below in our first result.

Before stating our first result, we introduce some notation. Let \( \nu_0 > 0 \) be the positive square root \([\nu_0]\) of the smallest eigenvalue of the operator \(-\Delta_\theta + V_0(\theta) + (n - 2)^2/4\) where \( \Delta_\theta \) is the usual Laplacian on the sphere \( \mathbb{S}^{n-1} \). We define the interval \( R_{\nu_0} \subset \mathbb{R} \) depending on \( \nu_0 \) by

\[
R_{\nu_0} = \begin{cases} \left( \frac{1}{2}, \frac{3}{2} \right), & \nu_0 > 1/2; \\ \left( 1 - \frac{\nu_0^2}{2\nu_0^2}, 1 + \frac{\nu_0^2}{2\nu_0^2} \right), & 0 < \nu_0 \leq 1/2. \end{cases} \tag{1.3}
\]

**Theorem 1.1** (Weighted resolvent estimates). Let \( n \geq 3 \) and let \( \mathcal{L}_V \) be the operator on \( L^2(\mathbb{R}^n) \) in \((1.1)\). Suppose the real function \( V_0(\theta) := \rho^2 V(x) \in C^1(\mathbb{S}^{n-1}) \) and the smallest eigenvalue of the operator \(-\Delta_\theta + V_0(\theta) + (n - 2)^2/4\) on \( L^2(\mathbb{S}^{n-1}) \) is \( \nu_0^2 > 0 \). Let \( \alpha \in R_{\nu_0} \) be defined in \((1.3)\). Then there exists a constant \( C \) such that the uniform weighted resolvent estimates hold

\[
\sup_{\sigma \in \mathbb{R}^+} \| r^{-\alpha}(\mathcal{L}_V - \sigma)^{-1}r^{-2+\alpha}f \|_{L^2(\mathbb{R}^n)} \leq C \| f \|_{L^2(\mathbb{R}^n)}, \quad f \in C_0^\infty(\mathbb{R}^n). \tag{1.4}
\]

**Remark 1.1.** This is a generalization of [4, Theorem 2.1] in which they proved \((1.4)\) with \( \alpha = 1 \). The smallest eigenvalue \( \nu_0^2 \) plays an important role in \((1.4)\).

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1 To ensure \( \nu_0 > 0 \), it is enough to choose \( V_0(\theta) \) such that \(-\Delta_\theta + V_0(\theta) + (n - 2)^2/4\) is a strictly positive operator on \( L^2(\mathbb{S}^{n-1}) \). For example, one can take \( V_0(\theta) \geq a \) where \( a > -(n - 2)^2/4 \) to guarantee \( \nu_0 > 0 \).
Remark 1.2. Let \( \mathcal{L}_V = -\Delta_g + V \) be defined on a manifold and \( \langle x \rangle = (1 + |x|^2)^{1/2} \).

On the asymptotically Euclidean space, Bony-Hähner [3] proved the resolvent estimates at low frequency

\[
\| \langle x \rangle^{-\alpha} (\mathcal{L}_V - \sigma)^{-1} \langle x \rangle^{-\beta} \|_{L^2 \to L^2} \leq C, \quad |\sigma| \leq 1
\]

provided \( \alpha, \beta > 1/2 \) and \( \alpha + \beta > 2 \) when \( V = 0 \). On the asymptotically conic manifold, Bouclet-Royer [5] showed the similar result of Theorem 1.3 below for Schrödinger operator

\[
(1.7) \quad \mathcal{L}_V\frac{1}{\langle x \rangle^2}.
\]

for Schrödinger operator

\[
(1.8) \quad \mathcal{L}_V \langle x \rangle^2.
\]

The result here is on Euclidean space but with flexible weights such as \( |x|^{-\alpha} \) and also includes the high frequency estimates.

Remark 1.3. One can use the same argument to derive the similar resolvent estimates (1.4) on a metric cone as the last two authors did in [44]. It would be interesting to show a similar result of Theorem 1.3 below for Schrödinger operator \( \mathcal{L}_V \) on the metric cone, for which the last two authors proved the Strichartz estimates in [44][46].

But there is an obstacle to obtain (1.5) below on the metric cone since the metric of section cross is so general that the conjugated points could appear. The difficulties arise from the conjugated points.

When \( V \equiv 0 \), the following uniform Sobolev inequality was proved by Kenig-Ruiz-Sogge [29] and Gutiérrez [19]:

\[
(1.5) \quad \|(-\Delta - \sigma)^{-1} f\|_{L^{p,2}(\mathbb{R}^n)} \leq C|\sigma|^{\frac{2}{n} - 1}\|f\|_{L^{p,2}(\mathbb{R}^n)}, \quad \sigma \notin \mathbb{R}^+, \quad f \in C_0^\infty(\mathbb{R}^n),
\]

where \( n \geq 3 \) and \((p,q)\) satisfies

\[
(1.6) \quad \frac{2}{n + 1} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{2}{n}, \quad \frac{2n}{n + 3} < p < \frac{2n}{n + 1}, \quad \frac{2n}{n - 1} < q < \frac{2n}{n - 3},
\]

and \( L^{p,q}(\mathbb{R}^n) \) is the usual Lorentz space. Precisely speaking, they proved (1.5) with \( L^{p,2}, L^{q,2} \) replaced by \( L^p, L^q \), respectively. However, (1.5) is an immediate consequence of their results and real interpolation theory. Note that the condition (1.6) is known to be sharp (see [19]). It is also worth noting that the uniform Sobolev inequality is a powerful tool in spectral and scattering theory for Schrödinger equations (see [23][29]), as well as nonlinear elliptic equations such as the Ginzburg-Landau equation (see [19]).

As a second result, we extend (1.5) to the operator \( \mathcal{L}_V \). Let us set

\[
\mu_0 = \begin{cases} 1/2, & \nu_0 \geq 1/2; \\ \frac{\nu_0^2}{1 - 2\nu_0}, & 0 < \nu_0 < 1/2. \end{cases}
\]

Theorem 1.2 (Uniform Sobolev inequality). Let \( \mathcal{L}_V \) be given as above and suppose

\[
(1.7) \quad \frac{2}{n + 1} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{2}{n}, \quad \frac{2n}{n + 2(1 + \mu_0)} < p < \frac{2n}{n + 1}, \quad \frac{2n}{n - 1} < q < \frac{2n}{n - 2(1 + \mu_0)}.
\]

Then there exists a positive constant \( C \) such that

\[
(1.8) \quad \| (\mathcal{L}_V - \sigma)^{-1} f \|_{L^{p,2}(\mathbb{R}^n)} \leq C|\sigma|^{\frac{2}{n} - 1}\|f\|_{L^{p,2}(\mathbb{R}^n)}, \quad \sigma \notin \mathbb{R}^+, \quad f \in C_0^\infty(\mathbb{R}^n).
\]
Remark 1.4. When $\nu_0 \geq 1/2$, (1.7) coincides with (1.6) (see Figure 1 below) in which case Theorem 1.2 gives the full range of uniform Sobolev inequalities for $L^p$. Uniform Sobolev inequalities for Schrödinger operators have been recently studied in several papers. Bouclet and the first author [4] and the first author [31] showed (1.8) for $L^p$ under (1.7) and $1/p + 1/q = 1$. For the special case $(p, q) = (2n/3, 2n/2)$, Guillarmou and Hassell [18] showed such estimates to the Laplace operator on nontrapping asymptotically conic manifolds, and Hassell and the second author [22] extended it to potential perturbations with smooth potentials decaying at infinity like $|x|^{-3}$ and without 0 resonance or eigenvalue. Compared with these results, we here prove more results ($p, q$ may not be dual each other) on $\mathbb{R}^n$ for potentials with weaker decay at infinity and critical singularity at the origin.

Finally we state the result about inhomogeneous Strichartz estimates for non-admissible pairs. Before stating the result, we recall the background of the Strichartz estimates without potential. Consider the Cauchy problem for the inhomogeneous Schrödinger equation

\begin{align}
\left\{
\begin{array}{l}
\imath \partial_t u + \Delta u = F(t, x), \quad t \in \mathbb{R}, \, x \in \mathbb{R}^n; \\
u(0) = u_0(x).
\end{array}
\right.
\end{align}

**Figure 1.** The condition (1.6) corresponds to the trapezium $ABB'A'$ with two closed line segments $AB$, $B'A'$ removed, while the condition (1.7) with $\mu_0 < 1/2$ corresponds to the shaded region surrounded by the polygon $CDB'B'C'$ with 4 closed line segments $CD$, $DB$, $B'D'$ and $D'C'$ removed. Here $A = (n+1/2n, n^{-3/2n}), \, B = (n+1/2n, n^{2+3n+1}/2n(n+1)), \, C = (n+2/2n, n^{-2(1+\mu_0)/2n}), \, D = (n+1/2n, n^{-2(1+\mu_0)/2n})$ and $A', B', C', D'$ are dual points of $A, B, C, D$, respectively.
By Duhamel’s formula, the solution $u$ is given by
\begin{equation}
(1.10) \quad u(t) = e^{it\Delta}u_0 - i \int_0^t e^{i(t-s)\Delta}F(s)ds.
\end{equation}

R. Strichartz [39] in 1977 proved that there exists a constant $C$ such that
\begin{equation}
(1.11) \quad \|u(t)\|_{L^q(\mathbb{R};L^r(\mathbb{R}^n))} \leq C \left( \|u_0\|_{L^2} + \|F\|_{L^{q'}(\mathbb{R};L^{r'}(\mathbb{R}^n))} \right)
\end{equation}
with $q = r = \tilde{q} = \tilde{r} = 2(n+2)/n$ when $u_0 \in L^2(\mathbb{R}^n), F \in L^{q'}(\mathbb{R};L^{r'}(\mathbb{R}^n))$. From then, there are many works devoted to this type of a priori estimates, so called the Strichartz estimate, for solutions to the Schrödinger equation in which $q$ is possibly not equal to the exponent $r$; we refer the readers to [14, 28] and the references therein. The Strichartz estimates have been used to prove rich results on the well-posed theory and nonlinear scattering theory for the semi-linear Schrödinger equations on Euclidean space, for example, see [14, 41] and the references therein.

In particular, if $F = 0$, the Strichartz estimate becomes
\begin{equation}
(1.12) \quad \|e^{it\Delta}u_0\|_{L^q(\mathbb{R};L^r(\mathbb{R}^n))} \leq C\|u_0\|_{L^2}
\end{equation}
and if $u_0 = 0$, then
\begin{equation}
(1.13) \quad \left\| \int_0^t e^{i(t-s)\Delta}F(s)ds \right\|_{L^q(\mathbb{R};L^r(\mathbb{R}^n))} \leq C\|F\|_{L^{q'}(\mathbb{R};L^{r'}(\mathbb{R}^n))}.
\end{equation}

The first one is known as a homogeneous Strichartz estimate and the second one is called inhomogeneous Strichartz estimate. If $(q, r)$ satisfies
\begin{equation}
(1.14) \quad q, r \in [2, \infty], \quad 2/q = n(1/2 - 1/r), \quad (q, r, n) \neq (2, \infty, 2),
\end{equation}
we say $(q, r)$ is a Schrödinger admissible pair, denoted by $(q, r) \in \Lambda_0$. From [28], the homogeneous estimate (1.12) holds if and only if $(q, r) \in \Lambda_0$. But there are some differences for the inhomogeneous estimates. It has been known that if both $(q, r)$ and $(\tilde{q}, \tilde{r})$ are admissible pairs, the inhomogeneous estimate (1.13) holds. Furthermore, it is known that there exist the exponent pairs $(q, r)$ and $(\tilde{q}, \tilde{r})$ which do not satisfy the admissible condition, but the inhomogeneous estimate can still be valid; we refer the reader to T. Cazenave and F. Weissler [8] and T. Kato [26] for Schrödinger and to Harmse [20] and Oberlin [33] for wave with $q = r$. After that, D. Foschi [13] and M. Vilela [43] independently and greatly extended the range of the exponent pairs $(q, r)$ and $(\tilde{q}, \tilde{r})$ for which the inhomogeneous Strichartz estimate holds. R. J. Taggart [40] generalized the inhomogeneous Strichartz estimate in an abstract mechanism. For more results on the inhomogeneous Strichartz estimate, we refer to Y. Koh [30] and R. Schippa [38]. However, the problem of finding all possible exponents pairs $(q, r)$ such that the inhomogeneous estimate (1.13) is available remains open.

It is worth remarking here that the argument is based on the method introduced in Keel-Tao [28] and most of the inhomogeneous Strichartz estimates are established there under the assumption that the propagator satisfies the energy estimate
\begin{equation}
(1.15) \quad \|U(t)\|_{L^2 \to L^2} \leq C
\end{equation}
and the dispersive estimate
\begin{equation}
\|U(t)U^*(s)\|_{L^1 \to L^\infty} \leq C|t-s|^{-\sigma}, \quad t \neq s.
\end{equation}

In particular, for the Schrödinger operator without potential, $U(t) = e^{it\Delta}$ and $\sigma = n/2$. It is known that the Strichartz estimate still holds when the pairs $(q,r)$ and $(\tilde{q},\tilde{r})$ are admissible pairs even though the dispersive estimate (1.16) fails. For example, Burq et al. [6] proved the Strichartz estimates for the operator $-\Delta + a|x|^{-2}$ on $\mathbb{R}^n$ with $a > -(n-2)^2/4$ and $n \geq 3$, but the dispersive estimate fails due to the negative inverse-square potential, e.g. see [12, 35]; and the Strichartz estimates including endpoints still hold on non-trapping asymptotically conic manifold or in a conic space (see [22, 44, 46]) but the dispersive estimate fails due to the conjugated points (e.g. see [21]). In the light of those Strichartz estimates were proved for admissible pairs even without the dispersive estimate, it is natural to ask whether the inhomogeneous Strichartz estimates hold for some non-admissible pairs. Due to the inverse-square potential, the usual dispersive estimate (1.16) fails, however we also want to prove inhomogeneous Strichartz estimates for some non-admissible pairs. More precisely, we obtain the following result on the inhomogeneous Strichartz estimate.

**Theorem 1.3** (Inhomogeneous Strichartz estimate). Let $L_V = -\Delta + V(x)$ be given as above. Then the inhomogeneous Strichartz estimate holds for a constant $C$ and $s \in A_{\nu_0}$

\begin{equation}
\left\| \int_0^t e^{i(t-\sigma)L_V} F(\sigma) d\sigma \right\|_{L^2(\mathbb{R};L^{2n/(n+2s)}_{\nu_0})} \leq C\|F\|_{L^2(\mathbb{R};L^{2n/(n+2s)}_{\nu_0})},
\end{equation}

where

\begin{equation}
A_{\nu_0} = \left[ \frac{n}{2(n-1)}, \frac{3n-4}{2(n-1)} \right] \cap R_{\nu_0}.
\end{equation}

**Remark 1.5.** The set $A_{\nu_0}$ is an intersection of two sets, the first set is related to the known result of the inhomogeneous Strichartz estimates in [13, 30, 33, 38] when $V = 0$ and the second set $R_{\nu_0}$ is from Theorem 1.1. The picture of inhomogeneous Strichartz estimate is far to be completed even in the case without potential.

Finally we introduce some notations. We use $A \lesssim B$ to denote $A \leq CB$ for some large constant $C$ which may vary from line to line and depend on various parameters, and similarly we use $A \ll B$ to denote $A \leq C^{-1}B$. We employ $A \sim B$ when $A \lesssim B \lesssim A$. If the constant $C$ depends on a special parameter other than the above, we shall denote it explicitly by subscripts. For instance, $C_\epsilon$ should be understood as a positive constant not only depending on $p,q,n$, and $M$, but also on $\epsilon$. Throughout this paper, pairs of conjugate indices are written as $p,p'$, where $\frac{1}{p} + \frac{1}{p'} = 1$ with $1 \leq p \leq \infty$.

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2. The proof of the weighted resolvent estimate

In this section, we prove the uniform weighted resolvent estimates which are the key point to prove the other two theorems.

The proof of Theorem 1.1. To prove Theorem 1.1 although we follow the idea in [7], some modifications and improvements are required due to the reason that we have to replace the multiplier \( re^{-2\tau \phi(r)} \partial_r \bar{v} \) by \( r^\beta e^{-2\tau \phi(r)} \partial_r \bar{v} \) which brings much harder treating terms in the weighted Hardy’s inequality. By the duality, we only need to prove (1.4) with \( R_{\nu_0} \geq \alpha \geq 1 \). Indeed, if we could prove

\[
\| r^{-\alpha} (L_V - \sigma)^{-1} r^{-2+\alpha'} \|_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)} \leq C, \quad 1 \leq \alpha < \alpha_0
\]

(2.1)

where \( \alpha_0 = 3/2 \) or \( 1 + \frac{\nu_0^2}{1-2\nu_0} \), by taking the adjoint of this estimate and replacing \( \sigma \) by \( \bar{\sigma} \), we also have

\[
\| r^{-\alpha} (L_V - \sigma)^{-1} r^{-2+\alpha'} \|_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)} \leq C, \quad 1 \leq \alpha < \alpha_0
\]

(2.2)

which shows

\[
\| r^{-\alpha'} (L_V - \sigma)^{-1} r^{-2+\alpha'} \|_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)} \leq C, \quad 2 - \alpha_0 \leq \alpha' \leq 1,
\]

(2.3)

where \( \alpha' = 2 - \alpha \). So we only need to prove (1.4) with \( 1 \leq \alpha < \alpha_0 \).

Let \( z = \sqrt{-\sigma} \) with the branch such that \( \text{Re} z = \tau > 0 \). Then given \( f \in L^2(\mathbb{R}^n) \) and \( \sigma \in \mathbb{C} \setminus \mathbb{R}^+ \), consider the Helmholtz equation

\[
L_V u + z^2 u = f.
\]

By density argument, we can take \( f \in C_0^\infty(\mathbb{R}^n) \). Then \( u \) is a classical solution of (2.4)

and define \( v(r, \theta) : (0, \infty) \times S^{n-1} \rightarrow \mathbb{C} \) by

\[
v(r, \theta) = r^{\frac{n+1}{2}} e^{rz} u(r, \theta).
\]

Then we see that

\[
\partial_r v = r^{\frac{n-1}{2}} e^{rz} \left( \frac{n-1}{2r} u + zu + \partial_r u \right),
\]

\[
-\partial_r^2 v = r^{\frac{n-1}{2}} e^{rz} \left( -\partial_r^2 u - 2\left( \frac{n-1}{2r} + z \right) \partial_r u - \left( \frac{(n-1)(n-3)}{4r^2} + \frac{(n-1)z}{r} + z^2 \right) u \right),
\]

\[
z \partial_r v = r^{\frac{n-1}{2}} e^{rz} \left( z \partial_r u + \left( \frac{(n-1)z}{2r} + z^2 \right) u \right).
\]

Therefore, \( v \) satisfies

\[
-\partial_r^2 v + 2z \partial_r v + \left( \frac{(n-1)(n-3)}{4} - \Delta \theta + V_0(\theta) \right) u \left( \frac{n-1}{r} \partial_r u + \left( -\Delta \theta + V_0(\theta) + z^2 \right) u \right)
\]

(2.5)

\[
=r^{\frac{n-1}{2}} e^{rz} f.
\]

For fixed \( M > m > 0 \), let \( \phi = \phi_{m,M}(r) \) be a smooth cut-off function such that \( 0 \leq \phi \leq 1 \) with being zero outside \([0, M+1]\) and equaling to 1 on \([m, M]\). By multiplying (2.3)
by $r^\beta e^{-2r\tau} \phi(r) \partial_r \bar{v}$ with $\beta$ being chosen later and taking the real part, we show that

\[
- \frac{1}{2} r^\beta e^{-2r\tau} \phi(r) \partial_r \partial_r v^2 + 2\tau r^\beta e^{-2r\tau} \phi(r) |\partial_r v|^2 \\
+ \frac{1}{2} r^{2-\beta} e^{-2r\tau} \phi'(r) (\frac{(n-1)(n-3)}{4} + V_0(\theta)) \partial_r |v|^2 + \frac{1}{2} r^{2-\beta} e^{-2r\tau} \phi(r) \Re(-\Delta v \partial_r \bar{v}) \\
= r^{\frac{n-1}{2} + \beta} \phi(r) \Re(e^{r(z-2r)} \partial_r \bar{v} f). 
\]

Integrating the above formula on $(0, \infty) \times \mathbb{S}^{n-1}$ but with volume $drd\theta$ and performing the integration by parts, we have

\[
\frac{1}{2} \int_0^\infty \int_{\mathbb{S}^{n-1}} \partial_r \left( r^\beta e^{-2r\tau} \phi(r) \right) |\partial_r v|^2 drd\theta \\
+ 2\tau \int_0^\infty \int_{\mathbb{S}^{n-1}} r^\beta e^{-2r\tau} \phi(r) |\partial_r v|^2 drd\theta \\
- \frac{1}{2} \int_0^\infty \int_{\mathbb{S}^{n-1}} \partial_r \left( r^{2+\beta} e^{-2r\tau} \phi(r) \right) \left( \frac{(n-1)(n-3)}{4} + V_0(\theta) \right) |v|^2 drd\theta \\
- \frac{1}{2} \int_0^\infty \int_{\mathbb{S}^{n-1}} \partial_r \left( r^{2+\beta} e^{-2r\tau} \phi(r) \right) |\nabla v|^2 drd\theta \\
= \int_0^\infty \int_{\mathbb{S}^{n-1}} r^{\frac{n-1}{2} + \beta} \phi(r) \Re(e^{r(z-2r)} \partial_r \bar{v} f) drd\theta. 
\]

Furthermore we compute that

\[
\frac{1}{2} \int_0^\infty \int_{\mathbb{S}^{n-1}} e^{-2r\tau} \phi(r)^{\beta - 1} (\beta - 2r\tau) |\partial_r v|^2 drd\theta + \frac{1}{2} \int_0^\infty \int_{\mathbb{S}^{n-1}} r^\beta e^{-2r\tau} \phi'(r) |\partial_r v|^2 drd\theta \\
+ 2\tau \int_0^\infty \int_{\mathbb{S}^{n-1}} r^\beta e^{-2r\tau} \phi(r) |\partial_r v|^2 drd\theta \\
+ \frac{1}{2} \int_0^\infty \int_{\mathbb{S}^{n-1}} r^{-3+\beta} (2-\beta + 2r\tau) \left( \frac{(n-2)^2}{4} \right) |v|^2 drd\theta \\
- \frac{1}{2} \int_0^\infty \int_{\mathbb{S}^{n-1}} \partial_r \left( r^{-2+\beta} e^{-2r\tau} \phi'(r) \right) \left( \frac{(n-2)^2}{4} + V_0(\theta) \right) |v|^2 drd\theta \\
- \frac{1}{2} \int_0^\infty \int_{\mathbb{S}^{n-1}} \partial_r \left( r^{-2+\beta} e^{-2r\tau} \phi'(r) \right) |\nabla v|^2 drd\theta \\
= \int_0^\infty \int_{\mathbb{S}^{n-1}} r^{\frac{n-1}{2} + \beta} \phi(r) \Re(e^{r(z-2r)} \partial_r \bar{v} f) drd\theta. 
\]
Therefore we show

\[ \frac{1}{2} \int_0^\infty \int_{S^{n-1}} e^{-2r\tau} \phi(r) \cdot \beta - 1 \left( (\beta + 2r\tau) |\partial_r v|^2 - ((2 - \beta) + 2r\tau) \frac{|v|^2}{4r^2} \right) \, dr\,d\theta \]

\[ + \frac{1}{2} \int_0^\infty \int_{S^{n-1}} e^{-2r\tau} \phi(r) r^{-3+\beta} ((2 - \beta) + 2r\tau) \left( \frac{(n-2)^2}{4} + V_0(\theta) \right) |v|^2 + |\nabla \theta v|^2 \, dr\,d\theta \]

\[ + \frac{1}{2} \int_0^\infty \int_{S^{n-1}} r^\beta e^{-2r\tau} \phi'(r) \left( |\partial_r v|^2 + \frac{1}{4r^2} |v|^2 - \frac{1}{r^2} (|\nabla \theta v|^2 + (V_0(\theta) + \frac{(n-2)^2}{4})|v|^2) \right) \, dr\,d\theta \]

\[ = \int_0^\infty \int_{S^{n-1}} r^{n+\beta} \phi(r) \Re \left( e^{r(z-2\tau)} \partial_r f \right) \, dr\,d\theta. \]

On the other hand, \(-\Delta \theta + V_0(\theta) + (n-2)^2/4\) is positive on \(S^{n-1}\) with the smallest eigenvalue \(\nu_0^2 > 0\), that is,

\[ \int_{S^{n-1}} \left( \frac{(n-2)^2}{4} + V_0(\theta) \right) |v|^2 + |\nabla \theta v|^2 \, d\theta \geq \nu_0^2 \int_{S^{n-1}} |v(r, y)|^2 \, d\theta \geq 0. \]

Hence we show for \(\forall \epsilon > 0\)

\[ \frac{1}{2} \int_0^\infty \int_{S^{n-1}} e^{-2r\tau} \phi(r) \cdot \beta - 1 \left( (\beta + 2r\tau) |\partial_r v|^2 - ((2 - \beta) + 2r\tau) \nu_0^2 - \frac{1}{4} \frac{|v|^2}{r^2} \right) \, dr\,d\theta \]

\[ + \frac{1}{2} \int_0^\infty \int_{S^{n-1}} r^\beta e^{-2r\tau} \phi'(r) \left( |\partial_r v|^2 + \frac{1}{4r^2} |v|^2 - \frac{1}{r^2} (|\nabla \theta v|^2 + (V_0(\theta) + \frac{(n-2)^2}{4})|v|^2) \right) \, dr\,d\theta \]

\[ \leq \int_0^\infty \int_{S^{n-1}} r^{n+\beta} \phi(r) \Re \left( e^{r(z-2\tau)} \partial_r f \right) \, dr\,d\theta \]

\[ \leq \frac{1}{4\epsilon^2} \|r^{\frac{n+\beta}{2}} f\|_{L_2}^2 + \epsilon^2 \int_0^\infty \int_{S^{n-1}} \phi(r) e^{-2r\tau} r^{\beta - 1} |\partial_r v|^2 \, dr\,d\theta. \]

For our purpose, we first need the following lemma.

**Lemma 2.1.** Let \(0 \leq \beta \leq 1\), we have following estimate for \(m \to 0, M \to \infty\)

\[ \int_0^\infty \int_{S^{n-1}} r^\beta e^{-2r\tau} \phi'(r) \left( |\partial_r v|^2 + \frac{1}{4r^2} |v|^2 - \frac{1}{r^2} (|\nabla \theta v|^2 + (V_0(\theta) + \frac{(n-2)^2}{4})|v|^2) \right) \, dr\,d\theta \geq 0. \]

We postpone the proof in the next subsection.

By taking the limits \(m \to 0\) and \(M \to \infty\) and using Lemma 2.1 and (2.8), we have

\[ \frac{1}{2} \int_0^\infty \int_{S^{n-1}} e^{-2r\tau} r^{\beta - 1} \left( (\beta + 2r\tau - 2\epsilon^2) |\partial_r v|^2 \right. \]

\[ + ((2 - \beta) + 2r\tau) \left( \nu_0^2 - \frac{1}{4} \frac{|v|^2}{r^2} \right) \, dr\,d\theta \leq \frac{1}{4\epsilon^2} \|r^{\frac{n+\beta}{2}} f\|_{L_2}^2. \]
Furthermore we obtain for $0 < \beta \leq 1$

\[
(1 - \frac{2\epsilon}{\beta}) \int_0^\infty \int_{S^{n-1}} e^{-2r^\beta r\beta^{-1} (\beta + 2r^\tau) |\partial_r v|^2} drd\theta \\
+ (\nu_0^2 - \frac{1}{4}) \int_0^\infty \int_{S^{n-1}} e^{-2r^\tau r\beta^{-1} ((2 - \beta) + 2r^\tau) \frac{|v|^2}{r^2}} drd\theta 
\leq \frac{1}{2\epsilon^2} \|r^{\frac{\beta}{2}} f\|^2_{L^2}.
\]

**Case 1:** $\nu_0 > 1/2$. Since $0 < \beta \leq 1$ and $r^\tau > 0$, we have

\[
\int_0^\infty \int_{S^{n-1}} e^{-2r^\tau r\beta^{-1} (\beta + 2r^\tau) \frac{|v|^2}{r^2}} drd\theta 
\leq C \nu_0 \|r^{\frac{\beta}{2}} f\|^2_{L^2},
\]

which implies

\[
\|r^{-\alpha} u\|_{L^2(\mathbb{R}^n)} \leq C \nu_0 \|r^{2-\alpha} f\|_{L^2(\mathbb{R}^n)}, \quad \beta = 3 - 2\alpha.
\]

Since $0 < \beta \leq 1$, we have showed that if $\nu_0 > 1/2$ and $1 \leq \alpha < 3/2$.

**Case 2:** $0 < \nu_0 \leq 1/2$. In this case, we need a weighted Hardy’s inequality

**Lemma 2.2** (Weighted Hardy’s inequality). Let $w \in C^2(\mathbb{R}^+ \setminus \{0\}; \mathbb{R})$ satisfy

\[
w(r) \geq 0, \quad w'(r) \leq 0, \quad r(w'(r)^2 + 2w(r)w''(r)) \geq 2w(r)w'(r), \quad \forall r \geq 0.
\]

Let $g : \mathbb{R}^+ \rightarrow \mathbb{C}$ be such that

\[
\int_0^\infty (w^2(r)|g'|^2 + (w'(r))^2|g|^2) \, dr < +\infty
\]

and

\[
\liminf_{r \to 0} w(r)w'(r)|g(r)|^2 = 0.
\]

Then

\[
\int_0^\infty w^2 \frac{|g(r)|^2}{r^2} \, dr \leq 4 \int_0^\infty w^2 |g'(r)|^2 \, dr.
\]

Next we use the modified weighted Hardy’s inequality to show

**Lemma 2.3.** Let $\max \{0, 1 - 2\nu_0\} < \beta \leq 1$, then we have

\[
\int_0^\infty \int_{S^{n-1}} e^{-2r^\tau r\beta^{-1} (\beta + 2r^\tau) |\partial_r v|^2} drd\theta \\
\geq \frac{1}{4} \int_0^\infty \int_{S^{n-1}} e^{-2r^\tau r\beta^{-1} (\beta + 2r^\tau) \frac{|v|^2}{r^2}} drd\theta.
\]

We postpone the proof of the two lemmas at the end of this section.
Using Lemma 2.3 and (2.10), we obtain
\[
C \frac{1}{2\epsilon^2} \|r^{\frac{1+\beta}{2}} f\|_{L^2}^2 \\
\geq \frac{1}{4}(1 - \frac{2\epsilon^2}{\beta}) \int_0^\infty \int_{S^{n-1}} e^{-2r\tau} r^{\beta-1} (\beta + 2r\tau) \left| \frac{|v|^2}{r^2} \right| drd\theta \\
+ (\nu_0^2 - \frac{1}{4}) \int_0^\infty \int_{S^{n-1}} e^{-2r\tau} r^{\beta-1} ((2 - \beta) + 2r\tau) \left| \frac{|v|^2}{r^2} \right| drd\theta \\
\geq \frac{\beta}{4}(1 - \frac{2\epsilon^2}{\beta}) \int_0^\infty \int_{S^{n-1}} e^{-2r\tau} r^{\beta-1} \left| \frac{|v|^2}{r^2} \right| d\theta dr \\
+ (2 - \beta)(\nu_0^2 - \frac{1}{4}) \int_0^\infty \int_{S^{n-1}} e^{-2r\tau} r^{\beta-1} \left| \frac{|v|^2}{r^2} \right| drd\theta,
\]
which implies
\[
\|r^{-\alpha} u\|_{L^2(\mathbb{R}^n)} \leq C_{\nu_0} \|r^{2-\alpha} f\|_{L^2(\mathbb{R}^n)}, \quad \beta = 3 - 2\alpha
\]
provided that
\[
\frac{\beta}{4} > (2 - \beta)(\frac{1}{4} - \nu_0^2) \iff \beta > 1 - \frac{2\nu_0^2}{1 - 2\nu_0^2}.
\]
Thus we have shown
\[
(2.19) \quad \|r^{-\alpha}(\mathcal{L}_V - z^2)^{-1} r^{\alpha-2}\|_{L^2 \to L^2} \leq C, \quad 1 \leq \alpha < 1 + \frac{\nu_0^2}{1 - 2\nu_0^2}.
\]
Therefore we conclude the proof of Theorem 1.1 if we could prove Lemma 2.1, Lemma 2.2 and Lemma 2.3.

2.1. Proof of Lemma 2.1

Before proving Lemma 2.1, we show the following lemmas.

Lemma 2.4.

Let \( k \neq \frac{n}{2} \). There holds
\[
(2.22) \quad \int_{\mathbb{R}^n} \frac{|\partial_r u|^2}{|x|^{1+\beta}} dx + \int_{\mathbb{R}^n} \frac{|u|^2}{|x|^{3-\beta}} dx < +\infty.
\]
Here \( \beta > \max\{0, 1 - 2\nu_0\} \) where \( \nu_0 \) is the positive square root of the smallest eigenvalue of the operator \(-\Delta_\theta + V_0(\theta) + (n - 2)^2/4\).

To prove Lemma 2.4, we first show the modified Hardy inequality.

Lemma 2.5. Let \( k \neq \frac{n}{2} \). There holds
\[
(2.23) \quad \int_0^\infty \int_{S^{n-1}} \frac{|f|^2}{r^{2k}} r^{n-1} drd\theta \leq \frac{4}{(n - 2)^2} \int_0^\infty \int_{S^{n-1}} \frac{|\partial_r f|^2}{r^{2k-2}} r^{n-1} drd\theta.
\]
Proof. First, by the sharp Hardy’s inequality [27], we have
\begin{equation}
\int_0^\infty \frac{|f(r,\theta)|^2}{r^{2k}} r^{n-1} dr \leq \frac{4}{(n-2)^2} \int_0^\infty \left| \partial_r \left( \frac{f(r,\theta)}{r^{k-1}} \right) \right|^2 r^{n-1} dr \\
= \frac{4}{(n-2)^2} \int_0^\infty \frac{1}{r^{2k-2}} |\partial_r f - (k-1)f(r,\theta)r^{-1}|^2 r^{n-1} dr.
\end{equation}
Noting that
\begin{equation}
|\partial_r f - (k-1)r^{-1}f|^2 = |\partial_r f|^2 - 2(k-1)r^{-1} \cdot \text{Re}(f\partial_r f) + (k-1)^2 \frac{|f|^2}{r^2},
\end{equation}
we get
\begin{align*}
\int_0^\infty \frac{1}{r^{2k-2}} |\partial_r f - (k-1)f(r,\theta)r^{-1}|^2 r^{n-1} dr \\
= \int_0^\infty \left( |\partial_r f|^2 - 2(k-1)r^{-1} \cdot \text{Re}(f\partial_r f) + (k-1)^2 \frac{|f|^2}{r^2} \right) r^{n-1} dr \\
= \int_0^\infty \frac{|\partial_r f|^2}{r^{2k-2}} r^{n-1} dr + (k-1)(n-k-1) \int_0^\infty \frac{|f|^2}{r^{2k}} r^{n-1} dr.
\end{align*}
Plugging this into (2.23) yields
\begin{equation}
\int_0^\infty \frac{|f(r,\theta)|^2}{r^{2k}} r^{n-1} dr \\
\leq \frac{1}{(n-2)^2/4 - (k-1)(n-k-1)} \int_0^\infty \frac{|\partial_r f|^2}{r^{2k-2}} r^{n-1} dr \\
= \frac{4}{(n-2k)^2} \int_0^\infty \frac{|\partial_r f|^2}{r^{2k-2}} r^{n-1} dr,
\end{equation}
and so (2.22) follows. Therefore, we integrate on \( S^{n-1} \) to conclude the proof of Lemma 2.5.

Proof of Lemma 2.4. We first consider the case \( \sigma = 0 \), that is, \( u \) solves
\begin{equation}
-\Delta u + r^{-2} V_0(\theta) u = f.
\end{equation}
Multiplying the above equality by \( \bar{u} \) and integrating on \( \mathbb{R}^n \), we obtain
\[ \int_{\mathbb{R}^n} \left( |\nabla u|^2 + V_0(\theta) r^{-2} |u|^2 \right) dx = \text{Re} \int_{\mathbb{R}^n} f \bar{u} \ dx. \]
By Young’s inequality and [3] Proposition 1, we have
\[ \|u\|^2_{H^1} \leq \| \sqrt{\mathcal{E}} u \|^2_{L^2} = \int_{\mathbb{R}^n} \left( |\nabla u|^2 + V_0(\theta) r^{-2} |u|^2 \right) dx \leq C(\epsilon) \int_{\mathbb{R}^n} r^2 |f|^2 dx + \epsilon \int_{\mathbb{R}^n} |\nabla u|^2 dx. \]
Hence,
\[ \int_{\mathbb{R}^n} |\nabla u|^2 dx \leq C \int_{\mathbb{R}^n} r^2 |f|^2 dx, \]
and so \( u \in H^1 \).
Next, multiplying (2.25) by \( \bar{u} \), integrating in \( \mathbb{R}^n \) and taking real part, we obtain
\[
\int_0^\infty \int_{S^{n-1}} |\nabla u|^2 r^{n-1} drd\theta = \frac{1}{2} \int_0^\infty \int_{S^{n-1}} |u|^2 \Delta (r^{\beta-1}) r^{n-1} drd\theta \\
+ \int_0^\infty \int_{S^{n-1}} \frac{V_0(\theta)}{r^{3-\beta}} |u|^2 r^{n-1} drd\theta = \text{Re} \left( \int_0^\infty \int_{S^{n-1}} \frac{\bar{u}}{r^{1-\beta}} |u|^2 r^{n-1} drd\theta \right),
\]
which implies
\[
\int_0^\infty \int_{S^{n-1}} |\nabla u|^2 r^{n-1} drd\theta = \int_0^\infty \int_{S^{n-1}} (V_0(\theta) + \frac{(1-\beta)(n-3+\beta)}{2}) |u|^2 r^{n-1} drd\theta.
\]
Noting that \( \nabla = (\partial_r, r^{-1} \nabla_\theta) \), (2.7) with \( u = v \) implies
\[
\int_0^\infty \int_{S^{n-1}} |\partial_r u|^2 r^{n-1} drd\theta + \int_0^\infty \int_{S^{n-1}} (\nu_0^2 - \frac{(n-2)^2}{4} + \frac{(1-\beta)(n-3+\beta)}{2}) |u|^2 r^{n-1} drd\theta \\ \leq C(\epsilon) \int_0^\infty \int_{S^{n-1}} |f|^2 r^{1+\beta} r^{n-1} drd\theta + \epsilon \int_0^\infty \int_{S^{n-1}} |u|^2 r^{n-1} drd\theta.
\]
Using Lemma 2.5 with \( 2k = 3 - \beta \), one has
\[
\int_0^\infty \int_{S^{n-1}} |u|^2 r^{n-1} drd\theta \leq \frac{4}{(n-3+\beta)^2} \int_0^\infty \int_{S^{n-1}} |\partial_r u|^2 r^{n-1} drd\theta.
\]
Hence, for
\[
\nu_0^2 - \frac{(n-2)^2}{4} + \frac{(1-\beta)(n-3+\beta)}{2} > -\frac{(n-3+\beta)^2}{4}, \quad \Leftrightarrow \quad \nu_0^2 > \frac{(1-\beta)^2}{4},
\]
there holds
\[
\int_0^\infty \int_{S^{n-1}} |\partial_r u|^2 r^{n-1} drd\theta + \int_0^\infty \int_{S^{n-1}} |u|^2 r^{n-1} drd\theta \\ \leq C \int_0^\infty \int_{S^{n-1}} |f|^2 r^{1+\beta} r^{n-1} drd\theta.
\]
Next we consider the case \( \sigma \neq 0 \). Multiplying (2.20) by \( \bar{u} \) and integrating in \( \mathbb{R}^n \), let \( \sigma = \sigma_1 + i\sigma_2 \), we get
\[
\int_{\mathbb{R}^n} \left( |\nabla u|^2 + |V_0(\theta)| r^{-2} |u|^2 \right) dx - \sigma_1 \int_{\mathbb{R}^n} |u|^2 dx = \text{Re} \int_{\mathbb{R}^n} f \bar{u} dx,
\]
\[
-\sigma_2 \int_{\mathbb{R}^n} |u|^2 dx = \text{Im} \int_{\mathbb{R}^n} f \bar{u} dx.
\]
**Case 1:** \( \sigma_2 \neq 0 \). It follows from (2.29) that
\[
|\sigma_2| \int_{\mathbb{R}^n} |u|^2 dx \leq \frac{2}{|\sigma_2|} \int_{\mathbb{R}^n} |f|^2 dx + \frac{|\sigma_2|}{4} \int_{\mathbb{R}^n} |u|^2 dx \Rightarrow u \in L^2(\mathbb{R}^n).
Combining this with (2.28), we obtain
\begin{equation}
\|\nabla u\|^2_{L^2(\mathbb{R}^n)} \sim \|\sqrt{\mathcal{L}} u\|^2_{L^2} = \int_{\mathbb{R}^n} \left( |\nabla u|^2 + V_0(\theta)r^{-2}|u|^2 \right) dx
\end{equation}
\begin{equation}
= \sigma_1 \int_{\mathbb{R}^n} |u|^2 dx - \text{Re} \int_{\mathbb{R}^n} f \bar{u} dx < \infty
\end{equation}
Hence \( u \in \dot{H}^1 \). Multiplying (2.21) by \( \frac{\bar{u}}{r^{1-\beta}} \) and integrating in \( \mathbb{R}^n \), we get
\begin{equation}
\int_0^\infty \int_{\mathbb{S}^{n-1}} \frac{|u|^2}{r^{1-\beta}} r^{n-1} dr d\theta + \int_0^\infty \int_{\mathbb{S}^{n-1}} (V_0(\theta) + \frac{(1-\beta)(n-3+\beta)}{2}) \frac{|u|^2}{r^{3-\beta}} r^{n-1} dr d\theta
\end{equation}
\begin{equation}
- \sigma_1 \int_0^\infty \int_{\mathbb{S}^{n-1}} \frac{|u|^2}{r^{1-\beta}} r^{n-1} dr d\theta = \text{Re} \int_0^\infty \int_{\mathbb{S}^{n-1}} \frac{f \bar{u}}{r^{1-\beta}} r^{n-1} dr d\theta.
\end{equation}
and
\begin{equation}
- \sigma_2 \int_0^\infty \int_{\mathbb{S}^{n-1}} \frac{|u|^2}{r^{1-\beta}} r^{n-1} dr d\theta = \text{Im} \int_0^\infty \int_{\mathbb{S}^{n-1}} \frac{f \bar{u}}{r^{1-\beta}} r^{n-1} dr d\theta.
\end{equation}
This together with Young’s inequality yields \( \int_{\mathbb{R}^n} \frac{|u|^2}{r^{1-\beta}} dx < +\infty \). Applying this fact to (2.32), and by the same argument as (2.27), we obtain that if \( \nu_0^2 > \frac{(1-\beta)^2}{4} \), there holds
\begin{equation}
\int_0^\infty \int_{\mathbb{S}^{n-1}} \frac{\partial_r |u|^2}{r^{1-\beta}} r^{n-1} dr d\theta + \int_0^\infty \int_{\mathbb{S}^{n-1}} \frac{|u|^2}{r^{3-\beta}} r^{n-1} dr d\theta
\end{equation}
\begin{equation}
\leq C \int_0^\infty \int_{\mathbb{S}^{n-1}} |f|^2 r^{1+\beta} r^{n-1} dr d\theta.
\end{equation}
Case 2: \( \sigma_2 = 0 \). In this case, we have \( \sigma_1 < 0 \) due to \( \sigma \notin \mathbb{R}^+ \). Using (2.28), we obtain \( u \in H^1 \). So (2.21) follows from (2.33).

With Lemma 2.4 in hand, we now prove Lemma 2.1. Note the fact that the support of \( \phi^\prime(r) \) is compact and belongs to \([0, m] \cup [M, M + 1] \). One has \( 0 \leq \phi^\prime \leq C/m \) on \([0, m] \) and \( -C \leq \phi^\prime \leq 0 \) on \([M, M + 1] \). Thus by (2.7) it suffices to show the negative terms
\begin{equation}
\int_M^{M+1} \int_{\mathbb{S}^{n-1}} r^\beta e^{-2r\tau} \left( |\partial_r v|^2 + \frac{1}{4r^2} |v|^2 \right) dr d\theta \rightarrow 0, \quad \text{as } M \rightarrow \infty;
\end{equation}
and
\begin{equation}
\int_0^m \int_{\mathbb{S}^{n-1}} r^\beta e^{-2r\tau} \frac{1}{r^2} \left( |\nabla \theta v|^2 + (V_0(\theta) + \frac{(n-2)^2}{4}) |v|^2 \right) dr d\theta \rightarrow 0, \quad \text{as } m \rightarrow 0.
\end{equation}
It is enough to show that there exists a sequence \( M_n \rightarrow \infty \) along which (2.34) holds. We note that
\[ \partial_r v = r^{-\frac{n-1}{2}} e^{r\tau} (\partial_r u + z u + \frac{n-1}{2r} u). \]
By using the modified Hardy inequality \((2.22)\), we have
\[
\int_M^{M+1} \int_{S_{n-1}} r^\beta e^{-2r} \left( |\partial_r v|^2 + \frac{1}{4r^2} |v|^2 \right) dr d\theta \\
\leq C \int_M^{M+1} \int_{S_{n-1}} r^\beta (|\partial_r u|^2 + |u|^2) d\theta r^{n-1} dr \\
\leq C(n, |z|) \int_M^{M+1} r^\beta g(r) dr,
\]
where
\[
g(r) = r^{n-1} \int_{S_{n-1}} (|\partial_r u|^2 + |u|^2) d\theta.
\]
By Lemma 2.4 we get
\[
\int_0^\infty g(r) dr < +\infty.
\]
It thus follows that, given \(\mu_j > 0\), there exists a sequence \(M_n^{(j)} \to \infty\) such that
\[
\int_{M_n^{(j)}} g(r) dr < \frac{\mu_j}{M_n^{(j)}},
\]
because otherwise the integral \(\int_0^\infty g(r) dr\) would diverge. Using a diagonal argument it thus follows that there exists a sequence \(M_n \to \infty\) such that for \(\beta \leq 1\) (i.e. \(\alpha \geq 1\))
\[
\int_{M_n} r^\beta g(r) dr \to 0 \quad \text{as} \quad n \to \infty,
\]
which implies \((2.34)\) along a sequence.

On the other hand, using Lemma 2.3 we have for \(\beta \geq 0\)
\[
\int_0^m \int_{S_{n-1}} r^{\beta-2} e^{-2r} \left( |\nabla \theta v|^2 + |v|^2 \right) dr d\theta \\
\leq \int_0^m \int_{S_{n-1}} r^{-2} (|\nabla \theta u|^2 + |u|^2) d\theta r^{n-1} dr \\
\lesssim \int_{r \leq m} \left( |\nabla u|^2 + \frac{|u|^2}{r^2} \right) dx \\
\to 0 \quad \text{as} \quad m \to 0.
\]

2.2. Proof of Lemmas 2.2 and 2.3.

The proof of Lemma 2.2. This is a modification of the weighted Hardy inequality in [7] Lemma 2.2]. We just modify the argument in [7] to prove it. Let the operator \(G\) be defined as
\[
G := \frac{1}{r} (w \partial_r + \frac{1}{2} w')
\]
It follows from \((2.15)\) that there exists a sequence \(\{r_j\} : r_j \to 0\) such that
\[
\lim_{j \to \infty} w(r_j) w'(r_j) |g(r_j)|^2 = 0.
\]
This together with (2.14) and (2.13) with \(ww' \leq 0\) yields that
\[
\|Gg\|_{L^2(\mathbb{R}^+)}^2 = \int_0^\infty \left( w^2 |g'|^2 + \frac{1}{4}(w')^2|g|^2 + \frac{1}{2}ww'\partial_r|g|^2 \right) dr
\]
\[
= \lim_{j \to \infty} \left\{ \int_{r_j}^\infty \left( w^2 |g'|^2 + \frac{1}{4}(w')^2|g|^2 - \frac{1}{2}\partial_r(ww')|g|^2 \right) dr + \frac{1}{2}w(r)w'(r)|g(r)|^2 \right\}_{r = r_j}^{r = +\infty}
\]
\[
\leq \int_0^\infty \left( w^2 |g'|^2 - \left( \frac{1}{4}(w')^2 + \frac{1}{2}ww'' \right)|g|^2 \right) dr.
\]
On the other hand, for the function \(m(r) = -\frac{w(r)}{2r}\), a simple computation shows that
\[
0 \leq \| (G - \imath m)g \|_{L^2(\mathbb{R}^+)}^2 = \|Gg\|_{L^2(\mathbb{R}^+)}^2 - \langle (wm' - m^2)g, g \rangle
\]
\[
\leq \int_0^\infty \left( w^2 |g'|^2 - \left( \frac{1}{4}(w')^2|g|^2 + \frac{1}{2}ww'' \right)|g|^2 \right) dr - \langle (wm' - m^2)f, f \rangle
\]
(2.36) \[
= \int_0^\infty \left( w^2 |g'|^2 - \left( \frac{1}{4}(w')^2 + \frac{1}{2}ww'' + wm' - m^2 \right)|g|^2 \right) dr.
\]
Noting that \(m = -\frac{w}{2r}\), we have by (2.13)
\[
\frac{1}{4}(w')^2 + \frac{1}{2}ww'' + wm' - m^2 = \frac{1}{4}(w')^2 + \frac{1}{2}ww'' - \frac{w''}{2r} + \frac{w^2}{4r^2} \geq \frac{w^2}{4r^2}.
\]
Plugging this into (2.36), we obtain
\[
\int_0^\infty w^2 \frac{|g|^2}{r^2} dr \leq 4 \int_0^\infty w^2 |g'(r)|^2 dr.
\]

\[\square\]

**The proof of Lemma 2.3**: Let
(2.37) \[
w(r) = e^{-\tau r}r^{\frac{\beta - 1}{2}}(\beta + 2\tau r)^{1/2}.
\]
We first verify the assumption (2.13) on \(w(r)\) when \(0 < \beta \leq 1\). A simple computation shows
(2.38) \[
w'(r) = e^{-\tau r}(\beta - 1/2)(\beta + 2\tau r)^{-1/2}[-2\tau^2 r + \frac{1}{2}(\beta - 1)\beta r^{-1}] \leq 0
\]
for \(0 < \beta \leq 1\). Secondly we have
\[
w''(r) = e^{-\tau r}(\beta - 1/2)(\beta + 2\tau r)^{-3/2}[-2\tau^2 r + \frac{1}{2}(\beta - 1)\beta r^{-1}]^2
\]
\[
+ e^{-\tau r}(\beta - 1/2)(\beta + 2\tau r)^{-3/2}[2\tau(\beta - 1)\beta r^{-1} - 2\tau^2 \beta - \frac{1}{2}(\beta - 1)\beta^2 r^{-2}]
\]
which implies
\[
\begin{align*}
  w'(r)^2 + 2w(r)w''(r) &= e^{-2\tau r}g^{-1}(\beta + 2\tau r)^{-1/2}[-2\tau^2 r + \frac{1}{2}(\beta - 1)\beta r^{-1}]^2 \\
  &+ e^{-2\tau r}g^{-1}(\beta + 2\tau r)^{-3/2}[-2\tau^2 r + \frac{1}{2}(\beta - 1)\beta r^{-1}]^2 \\
  &+ e^{-2\tau r}g^{-1}(\beta + 2\tau r)^{-3/2}[-2\tau(\beta - 1)\beta r^{-1} - 2\tau^2 \beta - \frac{1}{2}(\beta - 1)\beta^2 r^{-2}] \\
  &= e^{-2\tau r}g^{-1}(\beta + 2\tau r)^{-1/2}[-2\tau^2 r + \frac{1}{2}(\beta - 1)\beta r^{-1}]^2 \\
  &+ e^{-2\tau r}g^{-1}(\beta + 2\tau r)^{-3/2}[-2\tau^2 r + \frac{1}{2}(\beta - 1)\beta r^{-1}]^2 \\
  &+ e^{-2\tau r}g^{-1}(\beta + 2\tau r)^{-3/2}[-2\tau(\beta - 1)\beta r^{-1} - 2\tau^2 \beta - \frac{1}{2}(\beta - 1)\beta^2 r^{-2}] \\
  &\geq -2\tau^2 \beta e^{-2\tau r}g^{-1}(\beta + 2\tau r)^{-3/2}.
\end{align*}
\]
In addition, we have
\[
-2w(r)w'(r) = e^{-2\tau r}g^{-1}(\beta + 2\tau r)^{-1/2}[4\tau^2 r - (\beta - 1)\beta r^{-1}] \\
  = e^{-2\tau r}g^{-1}(\beta + 2\tau r)^{-3/2}[4\tau^2 r(\beta + 2\tau r) - (\beta - 1)\beta r^{-1} \cdot 2\tau r] \\
  = e^{-2\tau r}g^{-1}(\beta + 2\tau r)^{-3/2}[4\tau^2 \beta r + 8\tau^3 r^2 - (\beta - 1)\beta r^{-1} \cdot 2\tau r] \\
  \geq 4\tau^2 \beta r e^{-2\tau r}g^{-1}(\beta + 2\tau r)^{-3/2}.
\]
Hence, for \(0 < \beta \leq 1\), we obtain
\[
 r \left( w'(r)^2 + 2w(r)w''(r) \right) - 2w(r)w'(r) \geq 2\tau^2 \beta r e^{-2\tau r}g^{-1}(\beta + 2\tau r)^{-3/2} \geq 0.
\]
Let
\[
(2.39) \quad g(r) = \left( \int_{S^{n-1}} |v(r, \theta)|^2 \, d\theta \right)^{1/2}.
\]
Then, we have
\[
\begin{align*}
g'(r) &= \left( \int_{S^{n-1}} |v(r, \theta)|^2 \, d\theta \right)^{-\frac{1}{2}} \int_{S^{n-1}} \text{Re}(v \partial_r v) \, d\theta \\
|g'(r)|^2 &\leq \int_{S^{n-1}} |\partial_r v|^2 \, d\theta.
\end{align*}
\]
Next we need to verify the assumption (2.14) and (2.15) on \(g\).
For the choice of \(w\) as in (2.37) and \(g\) as in (2.39), we have
\[
\begin{align*}
&\int_0^\infty (|w'|^2|g|^2 + w^2|g'|^2) \, dr \\
&\leq \int_0^\infty e^{-2\tau r}g^{-1}(\beta + 2\tau r)^{-1/2}[4\tau^2 r + \frac{1}{2}(\beta - 1)\beta r^{-1}]^2 \int_{S^{n-1}} |v(r, \theta)|^2 \, d\theta \, dr \\
&+ \int_0^\infty e^{-2\tau r}g^{-1}(\beta + 2\tau r) \int_{S^{n-1}} |\partial_r v|^2 \, d\theta \, dr.
\end{align*}
\]
Recall that
\[ v(r, y) = r^{n-1} e^{rx} u(r, y), \]
and
\[ \partial_r v = r^{n-1} e^{rx} \left( \partial_r u + zu + \frac{n-1}{2r} u \right), \]

hence
\[
\int_0^\infty \left( (w')^2 |g|^2 + u^2 |g'|^2 \right) \, dr \\
\leq C(\tau) \int_0^\infty \int_{S^{n-1}} \left( \frac{|u|^2}{r^{3-\beta}} + \frac{|\partial_r u|^2}{r^{1-\beta}} \right) \, d\theta r^{n-1} \, dr \\
+ C(\tau) \int_0^\infty \int_{S^{n-1}} \left( \frac{|u|^2}{r^{3-\beta}} + \frac{|\partial_r u|^2}{r^{1-\beta}} \right) \, d\theta r^{n-1} \, dr.
\]

The boundedness of (2.40) and (2.41) follow from (2.21).

Now we verify (2.15). For the choice of \( w \) as in (2.37) and \( g \) as in (2.39), we have
\[
w(r) w'(r) |g(r)|^2 = e^{-2r^2}\,r^{\beta-1}[-2r^2 + \frac{1}{2}(\beta - 1)\beta r^{-1}] |g(r)|^2.
\]

We are reduced to show that
\[
limit_{r \to 0} \tilde{g}(0) = 0
\]
with
\[
\tilde{g}(r) := r^{\frac{\beta-2}{2}} f(r) = r^{\frac{\beta-2}{2}} \left( \int_{S^{n-1}} |v(r, y)|^2 \, d\theta \right)^{\frac{1}{2}}.
\]

Recall that
\[ v(r, y) = r^{n-1} e^{rx} u(r, y). \]

For the above \( \beta \in (\max\{0, 1 - 2\nu_0\}, 1] \), we can choose \( \beta_0 \in (\max\{0, 1 - 2\nu_0\}, 1) \) such that \( \beta_0 < \beta \), and let \( \epsilon = \beta - \beta_0 \).

Moreover, by using (2.21) with \( \beta_0 \in (\max\{0, 1 - 2\nu_0\}, 1) \), we have
\[
\int_0^1 \frac{\tilde{g}(r)^2}{r^{1+\epsilon}} \, dr = \int_0^1 r^{\beta-3-\epsilon} \int_{S^{n-1}} |v(r, y)|^2 \, d\theta \, dr \\
= \int_0^1 \int_{S^{n-1}} e^{2r^2} \frac{|u(r, y)|^2}{r^{3-\beta_0}} \, d\theta \, r^{n-1} \, dr \\
\leq C(\tau) \int_{r \leq 1} \frac{|u|^2}{r^{3-\beta_0}} \, dx < +\infty
\]
which shows that
\[
limit_{r \to 0} \tilde{g}(r) = 0,
\]
otherwise the integral \( \int_0^1 \frac{\tilde{g}(r)^2}{r^{1+\epsilon}} \, dr \) diverges.
Therefore we have verified the condition of Lemma 2.2. By Lemma 2.2 we obtain
\[ \int_0^\infty \int_{S^{n-1}} e^{-2\tau r^\beta -1} (\beta + 2\tau r) \left| \frac{v}{r} \right|^2 dr d\theta = \int_0^\infty \frac{w^2 |g|^2}{r^2} dr \]
\[ \leq 4 \int_0^\infty w^2 |g'(r)|^2 dr \]
\[ \leq 4 \int_0^\infty e^{-2\tau r^\beta -1} (\beta + 2\tau r) \int_{S^{n-1}} |\partial_r v|^2 d\theta dr \]
\[ = 4 \int_0^\infty \int_{S^{n-1}} e^{-2\tau r^\beta -1} (\beta + 2\tau r) |\partial_r v|^2 d\theta dr, \]
which implies (2.14), and so we conclude the proof of Lemma 2.3. \(\square\)

3. The Sobolev inequality and inhomogeneous Strichartz estimate

In this section, we prove Theorem 1.2 and Theorem 1.3.

3.1. The proof of Theorem 1.2. We set \( R_0(\sigma) = (-\Delta - \sigma)^{-1} \) and \( R(\sigma) = (L_V - \sigma)^{-1} \). The proof follows a similar line as in [31] based on the iterated resolvent identity (3.1)
\[ R(\sigma) = R_0(\sigma) - R_0(\sigma)VR_0(\sigma) + R_0(\sigma)VR(\sigma)VR_0(\sigma) \]
which follows from the standard resolvent formulas
\[ R(\sigma) = R_0(\sigma) - R_0(\sigma)VR(\sigma) = R_0(\sigma) - R(\sigma)VR_0(\sigma). \]

Let \( f \in C_0^\infty(\mathbb{R}^n) \), \( \sigma \in \mathbb{R}^+ \) and \((p, q)\) satisfy (1.7). Thanks to (1.5), it suffices to deal with the second and third terms of the right hand side of (3.1).

For the second term, we choose \( \tilde{p} \geq 2 \) such that \( 1/\tilde{p} - 1/q = 2/n \). Since \((\tilde{p}, q)\) satisfies (1.6) and \( V \in L^{n/2, \infty} \), we can use (1.5) to obtain
\[ \|R_0(\sigma)VR_0(\sigma)f\|_{L^{n/2, 2}} \lesssim \|VR_0(\sigma)f\|_{L^{2, 2}} \lesssim \|V\|_{L^{n/2, \infty}} \|R_0(\sigma)f\|_{L^{n/2, 2}} \]
\[ \lesssim |\sigma|^{\frac{\tilde{p}}{2} - \frac{1}{q} - 1} \|f\|_{L^{p, 2}}. \]

For the third part, we divide the proof into two cases: \( 1/p - 1/q = 2/n \) and otherwise. Let us first suppose \( 1/p - 1/q = 2/n \). It is easy to see that
\[ \{(p, q) \mid (p, q) \text{satisfies (1.7)} \text{ and } 1 - 1/q < 1 + \mu_0 \} = \{(p_s, q_s) \mid 1 - \mu_0 < s < 1 + \mu_0 \}, \]
where \( p_s = \frac{2n}{n + 2(1 - s)} \), \( q_s = \frac{2n}{n - 2s} \). By (1.5) and Hölder’s inequality in Lorentz spaces,
\[ \|R_0(\sigma)w_1 f\|_{L^{p_s, 2}} \lesssim \|w_1 f\|_{L^{p_s, 2}} \lesssim \|w_1\|_{L^{2, \infty}} \|f\|_{L^2}, \]
\[ \|w_2 R_0(\sigma)f\|_{L^{q_s, 2}} \lesssim \|w_2\|_{L^{2, \infty}} \|R_0(\sigma)f\|_{L^{q_s, 2}} \lesssim \|w_2\|_{L^{2, \infty}} \|f\|_{L^{p_s, 2}} \]
for all \( w_1 \in L^{\frac{n}{n + 2}, \infty} \), \( w_2 \in L^{\frac{n}{n - 2s}, \infty} \) and \( 1/2 < s < 3/2 \). These two estimates, together with (1.4) and the fact \( r^{-\alpha} \in L^{n/\alpha, \infty} \) and \( r^2 V \in L^\infty \), imply for \( 1 - \mu_0 < s < 1 + \mu_0 \),
\[ \|R_0(\sigma)VR(\sigma)VR_0(\sigma)f\|_{L^{n/2, 2}} \lesssim \|R_0(\sigma)r^{-2+s}\|_{L^2 \rightarrow L^{n/2, 2}} \|r^{-s}R(\sigma)VR_0(\sigma)f\|_{L^2} \]
\[ \lesssim \|r^{-s}R(\sigma)r^{-2+s}\|_{L^2} \|r^{-s}R_0(\sigma)f\|_{L^2} \]
\[ \lesssim \|f\|_{L^{p_s, 2}}, \]
which completes the proof of (1.8) for the case when \( 1/p - 1/q = 2/n \).
Consider next the case when \(2/(n+1) \leq 1/p - 1/q < 2/n\). One can find a point \((p_0, q_0)\) satisfying \(p_0 < p, q < q_0, (1.7)\) and \(1/p_0 - 1/q_0 = 2/n\). Since \((p_0, q_0)\) and \((p, q)\) satisfy (1.6), (1.5) and Hölder’s inequality then show the iterated Duhamel identity argument in [4]. Recall \(L\) imply the above two estimates (3.2), (3.3) combined with (1.8) for (3.5) 

\[
\|R_0(\sigma)V f\|_{L^{q/2}} \lesssim |\sigma|^{\frac{n}{2} - \frac{1}{q} - \frac{1}{q}} \|V f\|_{L^{p_0,2}} \lesssim |\sigma|^{\frac{n}{2} - \frac{1}{q} - \frac{1}{q}} \|V f\|_{L^{\infty}},
\]

\[
\|V R_0(\sigma)f\|_{L^{p_0,2}} \lesssim \|R_0(\sigma)f\|_{L^{q_0,2}} \lesssim |\sigma|^{\frac{n}{2} - \frac{1}{q} - \frac{1}{q}} \|V f\|_{L^{p,2}}.
\]

Since

\[
\left(\frac{n}{2} \left(\frac{2}{p_0} - \frac{1}{q}\right) - 1 + \frac{n}{2} \left(\frac{2}{p} - \frac{1}{q}\right)\right) - 1 = \frac{n}{2} \left(\frac{2}{p_0} - \frac{1}{q}\right) + \frac{n}{2} \left(\frac{2}{p} - \frac{1}{q}\right) - 2 = \frac{n}{2} \left(\frac{1}{p} - \frac{1}{q}\right) - 1,
\]

the above two estimates (3.2), (3.3) combined with (1.8) for \((p_0, q_0)\) proved just above, imply

\[
\|R_0(\sigma)VR_0(\sigma)f\|_{L^{q/2}} \lesssim \|R_0(\sigma)V\|_{L^{q_0,2} \to L^{q/2}} \|R_0(\sigma)f\|_{L^{q_0,2}} \lesssim |\sigma|^{\frac{n}{2} - \frac{1}{q} - \frac{1}{q}} \|f\|_{L^{p,2}}.
\]

This completes the proof of Theorem 1.2.

3.2. The proof of Theorem 1.3 We prove Theorem 1.3 by using Theorem 1.1 and the iterated Duhamel identity argument in [4]. Recall \(L_V = -\Delta + V\) and \(L_0 = -\Delta\), define the operators

\[
N_0 F(t) = \int_0^t e^{i(t-s)L_0} F(s) ds, \quad \mathcal{N} F(t) = \int_0^t e^{i(t-s)L} V F(s) ds.
\]

Setting \(u(t) = e^{i(t-s)L} V F(s)\), we can write

\[
u(t) = e^{i(t-s)L} F(s) - i \int_s^t e^{i(t-\tau)L} \left(V e^{-i(t-\tau)s} L V F(s)\right) d\tau.
\]

Integrating in \(s \in [0,t]\), we have by Fubini’s formula

\[
\mathcal{N} F(t) = \int_0^t e^{i(t-s)L} V F(s) ds = \int_0^t e^{i(t-s)L} F(s) ds - \int_0^t \int_s^t e^{i(t-\tau)L} \left(V e^{-i(t-\tau)s} L V F(s)\right) d\tau ds = N_0 F(t) - i \int_0^t \int_0^\tau e^{i(t-\tau)L} \left(V e^{-i(t-\tau)s} L V F(s)\right) ds d\tau = N_0 F(t) - i \int_0^t e^{i(t-\tau)L} \left(V \int_0^\tau e^{-i(t-\tau)s} L V F(s) ds\right) d\tau = N_0 F(t) - i N_0 \left(V (\mathcal{N} F)\right)(t).
\]

Therefore

\[
\mathcal{N} F(t) = N_0 F(t) - i N_0 \left(V (\mathcal{N} F)\right)(t).
\]

On the other hand, by similar argument, we have

\[
N_0 F(t) = \mathcal{N} F(t) + i \mathcal{N} \left(V (N_0 F)\right)(t),
\]
hence
\[ \mathcal{N}F(t) = \mathcal{N}_0 F - i\mathcal{N} \left( V(\mathcal{N}_0 F) \right)(t). \]
Plugging it into (3.5), we obtain
\[ \mathcal{N}F = \mathcal{N}_0 F - i\mathcal{N}_0 \left( V(\mathcal{N}_0 F) \right) - \mathcal{N}_0 \left( V(\mathcal{N}(\mathcal{N}_0 F)) \right), \]
that is
\[ (3.6) \quad \mathcal{N}F = \mathcal{N}_0 F - i(\mathcal{N}_0 V\mathcal{N}_0) F - \mathcal{N}_0 (VNV)\mathcal{N}_0 F. \]
To prove (1.17), we need to estimate
\[ \|\mathcal{N}F\|_{L^2(\mathbb{R}; L^{\frac{2n}{n-2}})} \]
\[ \leq \|\mathcal{N}_0 F\|_{L^2(\mathbb{R}; L^{\frac{2n}{n-2}})} + \| (\mathcal{N}_0 V\mathcal{N}_0) F\|_{L^2(\mathbb{R}; L^{\frac{2n}{n-2}})} + \|\mathcal{N}_0 (VNV)\mathcal{N}_0 F\|_{L^2(\mathbb{R}; L^{\frac{2n}{n-2}})}. \]
By the inhomogeneous Strichartz estimate in [13, 30, 43, 38] for the free Schrödinger equation, we have
\[ (3.7) \quad \|\mathcal{N}_0 F\|_{L^2(\mathbb{R}; L^{\frac{2n}{n-2}})} \lesssim \|F\|_{L^2(\mathbb{R}; L^{\frac{2n}{n-2}})} \quad \text{for} \quad \frac{n}{2} \leq s \leq \frac{3n-4}{2(n-1)}. \]
Since \( V = r^{-2} V_0(\theta) \) with \( V_0 \in C^\infty(\mathbb{S}^{n-1}) \), one has \( V \in L^2_\infty \). From the Strichartz estimate (3.7), we obtain
\[ (3.8) \quad \| (\mathcal{N}_0 V\mathcal{N}_0) F\|_{L^2(\mathbb{R}; L^{\frac{2n}{n-2}})} \lesssim \|V\mathcal{N}_0 F\|_{L^2(\mathbb{R}; L^{\frac{2n}{n-2}})} \lesssim \|F\|_{L^2(\mathbb{R}; L^{\frac{2n}{n-2}})}. \]
To estimate the final term \( \|\mathcal{N}_0 (VNV)\mathcal{N}_0 F\|_{L^2(\mathbb{R}; L^{\frac{2n}{n-2}})} \), we need two lemmas.

**Lemma 3.1.** Let \( s \in [n/2(n-1), (3n-4)/2(n-1)] \). Then
\[ (3.9) \quad \|\mathcal{N}_0 r^{-2+\delta} F\|_{L^2(\mathbb{R}; L^{\frac{2n}{n-2}})} \lesssim \|F\|_{L^2(\mathbb{R}; L^2)}, \]
and
\[ (3.10) \quad \|r^{-\delta}\mathcal{N}_0 F\|_{L^2(\mathbb{R}; L^2)} \lesssim \|F\|_{L^2(\mathbb{R}; L^{\frac{2n}{n-2}})}. \]

**Proof.** It follows from (3.7) and the Hölder inequality that
\[ (3.11) \quad \|\mathcal{N}_0 r^{-2+\delta} F\|_{L^2(\mathbb{R}; L^{\frac{2n}{n-2}})} \lesssim \|r^{-2+\delta} F\|_{L^2(\mathbb{R}; L^{\frac{2n}{n-2}})} \]
\[ \lesssim \|r^{-2+\delta}\|_{L^\infty} \|F\|_{L^2(\mathbb{R}; L^2)} \lesssim \|F\|_{L^2(\mathbb{R}; L^2)} \]
and
\[ (3.12) \quad \|r^{-\delta}\mathcal{N}_0 F\|_{L^2(\mathbb{R}; L^2)} \lesssim \|r^{-\delta}\|_{L^\infty} \|\mathcal{N}_0 F\|_{L^2(\mathbb{R}; L^{\frac{2n}{n-2}})} \lesssim \|F\|_{L^2(\mathbb{R}; L^{\frac{2n}{n-2}})}. \]

**Lemma 3.2.** Let \( \alpha \in \mathbb{R}_{V_0} \) be defined in (1.3), then we have
\[ (3.13) \quad \|r^{-\alpha} \int_0^t e^{i(t-s)\mathcal{L}_V} r^{-2+\alpha} F ds\|_{L^2(\mathbb{R}; L^2)} \leq C\|F\|_{L^2(\mathbb{R}; L^2)}. \]
Proof. This is a consequence of D’Ancona’s proof [9] and the weighted resolvent estimate [1,2]. □

Note that \( r^2 V = V_0(\theta) \in L^\infty(S^{n-1}) \). By using Lemma 3.1 and Lemma 3.2, since \( s \in A_{\nu_0} \), we prove

\[
\|N_0(VN^2VN)F\|_{L^2(\mathbb{R};L^{\frac{n+2}{n+2-s}})} \\
= \|(N_0 r^{-2+s})(r^2 V)(r^{-s}N^r^{-2+s})(r^2 V)(r^{-s}N_0)F\|_{L^2(\mathbb{R};L^{\frac{2n}{n+2-s}})} \\
\lesssim \|(r^{-s}N^r^{-2+s})(r^2 V)(r^{-s}N_0)F\|_{L^2(\mathbb{R};L^2)} \lesssim \|F\|_{L^2(\mathbb{R};L^{\frac{n+2}{n+2-s}})^2}.
\]

Finally we collect (3.7), (3.8) and (3.14) to obtain (1.17). Thus we prove Theorem 1.3.

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