The aim of this short note is to answer a question by Guoliang Yu of whether the group $\text{EL}_3(\mathbb{Z}\langle x, y \rangle)$, where $\mathbb{Z}\langle x, y \rangle$ is the free (non-commutative) ring, has any faithful finite dimensional linear representations over a field. Recall that for every (associative unitary) ring $R$ the group $\text{EL}_n(R)$ is the subgroup of $\text{GL}_n(R)$ generated by all $n \times n$-elementary matrices $x_{ij}(r) = \text{Id} + re_{ij}$ ($r \in R$, $1 \leq i \neq j \leq n$). Clearly, if $R$ has a faithful finite dimensional linear representation over a field, then the group $\text{EL}_n(R)$ also has a faithful finite dimensional linear representation over the same field. The conclusion is true even if $R$ has an ideal of finite index that has a faithful finite dimensional representation (see Theorem 1).

The converse implication (which would imply the negative answer to G. Yu’s question) should have been known for many years, but we could not find it in the literature. There are many results about isomorphisms between various matrix groups over (mostly commutative) rings from the original results of Mal’cev [Ma] to results of O’Meara [OM] to Mostow rigidity results [Mo].

There are also many results about homomorphisms of one general matrix group into another. Churkin [Ch] proved that the wreath product $\mathbb{Z} \wr \mathbb{Z}^n$ embeds into a matrix group over a field $K$ of characteristic 0 if and only if the transcendence degree of $K$ over its prime subfield is at least $n$ (a similar result is proved in the case of positive characteristic). Hence $\text{SL}_n(K)$ cannot embed into $\text{SL}_{n'}(K')$ if $K$, $K'$ are fields of characteristic 0 and the transcendence degree of $K$ is bigger than the transcendence degree of $K'$. Much stronger non-embeddability results for general linear groups over fields follow from the main result of Borel-Tits [BT] (we are grateful to Yves de Cornulier for pointing out to this reference). See also surveys [JWW] and [HJW].

The main result of the note is the following:

**Theorem 1.** (a) Let $R$ be an associative unitary ring, $k \geq 3$. The group $\text{EL}_k(R)$ has a faithful finite dimensional representation over $\mathbb{C}$ if and only if $R$ has a finite index ideal $I$ that admits a faithful finite dimensional representation over $\mathbb{C}$.

(b) The group $\text{EL}_3(\mathbb{Z}\langle x, y \rangle)$ does not have a faithful finite dimensional representation over any field.
The proof of this theorem is given at the end of the paper (after Remark 10). Part (a) of Theorem 1 does not hold if we replace \( \mathbb{C} \) by a field of positive characteristic (see Remark 11). Note that the “finite index” condition in part (a) of Theorem 1 is necessary because there are finite rings (say, the endomorphism ring of the Abelian group \( \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p^2\mathbb{Z} \) where \( p \) is a prime) that do not have any finite dimensional faithful representations over fields and over any commutative rings. Also parts (a) and (b) of Theorem 1 do not hold for \( \text{EL}_2 \). Indeed, by Exercise 2 of Section 2.7 in [Co], the group \( \text{GE}_2(\mathbb{Q}(x,y)) \) generated by the elementary and diagonal matrices of \( \text{GL}_2(\mathbb{Q}(x,y)) \), is isomorphic to the group \( \text{GE}_2(\mathbb{Q}[x]) \) which is a subgroup of \( \text{GL}_2(\mathbb{C}) \), and so \( \text{EL}_2(\mathbb{Z}(x,y)) < \text{GE}_2(\mathbb{Q}(x,y)) \) is linear.

Let \( \pi : \text{EL}_k(R) \to \text{GL}_n(K) \) be a linear representation of the group \( \text{EL}_k(R) \), where \( K \) is an algebraically closed field, \( k \geq 3 \).

**Definition 2.** Let \( U \) denote the set \( \{ \pi(x_{13}(r)) \mid r \in R \} \) and \( V \) be the Zariski closure of \( U \). By construction \( V \) is an algebraic variety.

**Theorem 3.** There exist two distinguished elements \( 0 \) and \( 1 \) in \( V \) and polynomial maps \( +, \times : V \times V \to V \), \( - : V \to V \), which give \( V \) a structure of an associative ring. Moreover the map \( \rho : R \to U \subset V \) defined by \( \rho(r) = \pi(x_{13}(r)) \) is a ring homomorphism.

**Proof.** Define the “addition” \( + : U \times U \to U \) as follows \( u_1 + u_2 := u_1u_2 \), where the multiplication on the right is the one in the group \( \text{GL}_n(K) \). It is clear that this map is given by some algebraic function therefore it extends to a polynomial map on \( V \times V \). Similarly we can define a map \( - : V \to V \) as the extension of the inversion \( u \to u^{-1} \). Notice that by construction we have the identities

\[
\rho(r_1) + \rho(r_2) = \rho(r_1 + r_2) \quad \text{and} \quad -\rho(r) = \rho(-r),
\]

i.e., the map \( \rho : R \to U \) is a homomorphism between Abelian groups. The identity element of \( \text{GL}_k \) is in \( U \subset V \) and we will denote it as the distinguished element \( 0 \in V \), since it is the identify element of \( U \) with respect to the addition.

These two operations turn \( V \) into an Abelian group: since all the axioms are satisfied on the Zariski dense set \( U \) they are satisfied on the whole variety \( V \).

In order to define the “multiplication” we need to use two special elements \( w_{23} \) and \( w_{12} \) in \( \text{EL}_k(R) \) which have the properties

\[
w_{12}x_{13}(r)w_{12}^{-1} = x_{23}(r) \quad \text{and} \quad w_{23}x_{13}(r)w_{23}^{-1} = x_{12}(r)
\]

The existence of these elements is well known and they can be easily written as product of generators in \( \text{EL}_k(R) \), for example we can take the matrices (embedded in the top left corner if \( \text{EL}_k(R) \)).

\[
w_{12} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad w_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}
\]
Now we can define the algebraic map \( \times : U \times U \to \text{GL}_n(K) \) as follows
\[
u_1 \times u_2 := [w_{23}u_1 w_{23}^{-1}, w_{12}u_2 w_{12}^{-1}]
\]
The commutator relation \([x_{12}(r), x_{23}(s)] = x_{13}(rs)\) implies that
\[
\rho(r_1) \times \rho(r_2) = \rho(r_1r_2),
\]
thus \( \times \) is a map from \( U \times U \) to \( U \) and can be extended to a polynomial map from \( V \times V \) to \( V \). The element \( \rho(1) \) plays the role of the unit with respect to this multiplication and we will call it \( 1 \in V \).

The same argument as before shows that \( 0, 1 \) and the maps +, − and \( \times \) turn \( V \) into an associative ring with a unit.

**Lemma 4.** Let \( V \) be an algebraic variety with two algebraic operations, given by polynomial functions, which turn it into an associative ring with \( 1 \). Then:
(a) any point on \( V \) is non-singular, thus the irreducible components of \( V \) do not intersect;
Let \( V_0 \) denote the irreducible (connected) component of \( 0 \) in \( V \) then:
(b) \( V_0 \) is a two-sided ideal in \( V \);
(c) the quotient \( V/V_0 \) is a finite ring.

**Proof.** (a) The structure of an Abelian group on \( V \) with respect to the addition implies that the automorphism group of the variety \( V \) acts transitively on the points, therefore all points are non-singular;
(b) For any \( v \in V \) the closure of \( v \times V_0 \) is a irreducible sub-variety \( V \) (since it is an image of a irreducible one) which contains \( 0 \), therefore it is a subset of \( V_0 \). This shows that \( V_0 \) is a left ideal in \( V \). Similar argument shows that \( V_0 \) is a right ideal;
(c) It is a classical result that any algebraic variety has only finitely many irreducible components. \( \square \)

**Lemma 5.** Let \( V \) be an algebraic variety over \( \mathbb{C} \), with two algebraic operations which turn it into an associative ring with \( 1 \). Then the irreducible component \( V_0 \) of \( V \) is isomorphic to a finite dimensional algebra over \( \mathbb{C} \), i.e., the ring \( V \) is virtually linear over \( \mathbb{C} \).

**Proof.** Note that the additive group \( V_+ \) of \( V \) is an Abelian Lie group over \( \mathbb{C} \). By [Po] \( V_0 \) is a product of a finite number of copies of \( \mathbb{C} \) and a finite number of 1-dimensional tori. Therefore the fundamental group \( \Gamma \) of \( V_0 \) (based at \( 0 \)) is isomorphic to \( \mathbb{Z}^k \) for some \( k < \infty \), and the product of any two loops in \( \Gamma \) is the same as their point-wise sum in \( V_0 \).

Multiplication by an element in \( V \) induces an endomorphism of \( \Gamma \) and so we have a map \( \phi \) from \( V \) to the endomorphism ring \( \text{End}(\Gamma) \) of \( \Gamma \). This map is continuous and a ring homomorphism because it preserves multiplication

\[\text{We consider the topology on } V \text{ induced by the usual topology on } \mathbb{C}^n, \text{ instead of the Zariski topology. This is one of the reasons why this argument does not work over fields of positive characteristic.}\]
by construction and the distributive law implies that $\phi$ send the sum of
the loops to the point-wise sum of their images. The endomorphism ring is
discrete, therefore the image of $V_0$ is trivial and $\phi$ factors through a map
$\bar{\phi} : V/V_0 \to \text{End}(\Gamma)$. The ring $\text{End}(\Gamma)$ does not have any finite sub-rings
since the characteristic is 0, unless $\Gamma$ is the trivial group, because the order
of the identity in $\text{End}(\Gamma) \simeq \text{Mat}_k(\mathbb{Z})$ is infinite. Thus $\Gamma$ is trivial and $V_0$ is a
simply connected Abelian Lie group over $\mathbb{C}$. Therefore $V_0$ is isomorphic to
a finite dimensional vector space over $\mathbb{C}$. The distributive laws imply that
multiplication on $V_0$ is bilinear, i.e., $V_0$ is a finite dimensional algebra over
$\mathbb{C}$. □

Remark 6. The analog of Lemma 5 is not true in the case of positive
characteristic. It is possible to construct examples where the exponent of
the additive group of $V$ is finite but is not equal to the characteristic of the
field.

Here is one simple example (it is somewhat similar to the example from [Be]
mentioned above): Let $K$ be an infinite field of characteristic 2 and let
$V = K \times K$ with the following operations:

$$(a, b) + (c, d) = (a + c, ac + b + d) \quad (a, b) \times (c, d) = (ac, be^2 + a^2d)$$

One can verify directly that $V$ is a commutative ring. The elements $(0, b)$
form an ideal $I$ with zero multiplication, $V/I$ is isomorphic as a ring to the
field $K$ (identified as a set with $\{(a, 0) \mid a \in K\}$), the action of $V/I$ on $I$
is given by $(a, 0)(0, d) = (0, a^2d)$. Every element of the form $(a, b), a \neq 0,$
is invertible (the inverse is $(a^{-1}, \frac{b}{a^2})$), i.e., $V$ is a local ring with a maximal
ideal $I$. Therefore that ring does not have proper ideals of finite index. This
ring is not linear over any field since all elements of the form $(a, b), a \neq 0,$
have “additive” order 4. Hence $V$ is not virtually linear.

Corollary 7. Let $V$ be an algebraic variety over a field of characteristic 0,
with two algebraic operations which turn it into an associative ring with 1.
Then any ring homomorphism $\phi : \mathbb{Z}(x, y) \to V$ has non-trivial kernel.

Proof. By the previous lemma $V$ is virtually linear therefore it satisfies some
polynomial identity $[R_0]$, but the ring $\mathbb{Z}(x, y)$ does not satisfy any polynomial
identity $[R_0]$. Therefore $\phi$ is not injective. □

Using Lemma 5 we can easily recover a significant part of the result by
Chen [C]. Let $D$ be a (noncommutative) division ring, and consider a group
homomorphism from $\text{SL}_n(D)$ to $G(k)$ with Zariski-dense image, where $G$
is a simple algebraic group defined over a field $k$. Then $D$ must be finite-
dimensional over its center. The following corollary recovers that result in
the case of characteristic 0 and $n \geq 3$.

Corollary 8. Let $D$ be a non-commutative division ring with characteristic
0. Then the group $\text{EL}_n(D), n \geq 3$, has nontrivial finite dimensional repre-
sentations in characteristic 0 if and only if $D$ is finite dimensional over its
center.
Proof. Let $\pi : \text{EL}_n(D) \to \text{GL}_n(k)$ be a nontrivial representation of the group $\text{EL}_n(D)$. Therefore the map $\rho : D \to U$ is a ring homomorphism with a non-trivial image and $\rho$ has to be an isomorphism since $D$ does not have any non-trivial ideals. By Lemma 5 $D$ is linear and thus it satisfies some polynomial identity. Finally by a theorem of Kaplansky [H] every division algebra $D$ satisfying a polynomial identity is finite dimensional over its center. The other direction is obvious. □

**Lemma 9.** Let $V$ be an algebraic variety over a field $K$ (of arbitrary characteristic) with two algebraic operations which turn it into an associative ring with 1. If $V$ is irreducible then the multiplicative group of $V$ is linear over $K$.

**Proof.** Let $A$ denote the ring of germs of rational functions on $V_0$ defined around the point 0. Let $I$ be the maximal ideal in $A$ consisting of germs that are 0 at 0. By Lemma 4 all points of $V$, including the point 0, are non-singular. Therefore $I/I^2$ is a finite dimensional vector space over the field $A/I = K$, and the dimension coincides with the dimension of $V$.

The left multiplication $l_v$ by any $v \in V$ defines an algebraic map $V_0 \to V_0$ which fixes 0 therefore it induces an endomorphism $l_v : A \to A$. It is clear that these maps define a group homomorphism $\psi : V^* \to \text{Aut}(A)$ by $(\psi(v)(f))(x) = f(l_v(x))$, where $V^*$ is the set all invertible elements in $V$. The kernel $S$ of the map $\psi$ consists of all elements $v$ in $V^*$ such that $(v - 1) \times V_0 = 0$, because the triviality of $l_v$ implies that the multiplication by $v$ gives the identity map from $V_0$ to $V_0$. If $V$ is connected then $V_0$ contains 1 thus the only element in the kernel of $\psi$ is the identity.

Consider the maps $\psi_n : V^* \to \text{Aut}(A/I^n)$ induced by $\psi$ and their kernels $S_n = \ker \psi_n$. By construction the sets $S_n$ form a decreasing sequence of sub-varieties of $V$ and that $\cap_n S_n = S$. By the Noetherian property, we have that there exist $M > 0$ such that $S = S_M$, i.e., the map $\psi_M$ is injective.

The group $\text{Aut}(A/I^M)$ is linear over $K$ because it is inside the group of all linear transformations of $A/I^M$ considered as a (finite dimensional) vector space over $K$, i.e., $\psi_M$ is a faithful linear representation of $V$. □

**Remark 10.** Let $V$ be the variety with the ring structure constructed in Remark [6]. The group $\text{EL}_3(V)$ is a subgroup of the multiplicative semigroup of the ring of $3 \times 3$ matrices over $V$, which is an algebraic variety (isomorphic to $K^{18}$, where the addition and the multiplication are given by some polynomial functions of degree 4). Lemma 9 implies that $\text{EL}_3(V)$ is linear group over $K$. Thus, there exists a ring $R$ which is not (virtually) linear over any field, but the group $\text{EL}_3(R)$ is linear. Hence part (a) of Theorem 1 does not hold in the case of positive characteristic.

The result in Corollary 7 also holds in the case of positive characteristic, but the argument is different.
Theorem 11. Let $V$ be an algebraic variety with two algebraic operations which turn it into an associative ring. Then any ring homomorphism $\phi : \mathbb{Z}(x, y) \rightarrow V$ has a non-trivial kernel.

Proof. Let $k$ be the dimension of $V$ and let assume that the map $\phi$ is injective. Let $s_l$ denote the symmetric function on $l$ arguments, i.e.,

$$s_l(x_1, \ldots, x_l) = \sum_{\sigma \in S_n} (-1)^\sigma \prod x_{\sigma(i)}$$

Pick elements $r_1, r_2, \ldots, r_{k+1}$ such that $s_l(r_1, \ldots, r_l)$ is not 0 in the ring $R = \mathbb{Z}(x, y)$ for any $l \leq k+1$ (for example we can take $r_i = xy^{i+1}$). Let $M_l$ denote the $\mathbb{Z}$ span of the elements $r_1, \ldots, r_l$ in the ring $R$ and let $N_l$ be the Zariski closure of $\phi(M_l)$ in $V$.

Lemma 12. The symmetric function $s_{l+1}$ is zero when evaluated on any $l + 1$ elements in $M_l$.

Proof. The polynomial $s_{l+1}$ is linear in every variable and anti-symmetric, and $M_l$ is spanned by less that $l + 1$ elements.

This immediately implies:

Corollary 13. The symmetric function $s_{l+1}$ is zero when evaluated on any $l + 1$ elements in $N_l$.

Lemma 14. For any $l$ we have that $\dim N_l > \dim N_{l-1}$.

Proof. Let $N_{l,i}$ denote the set $i.\phi(r_l) + N_{l-1}$ for a positive integer $i$ (here $i.r$ denotes the sum $r + r + \cdots + r$). Using the fact that the operation $+$ is an algebraic function, we can conclude that this is a sub-variety of $N_l$ and $\dim N_{l,i} = \dim N_{l-1}$ because the algebraic map $v \mapsto i.\phi(r_l) + v$ is a bijection from $V$ to $V$. Let us show that these subvarieties are disjoint: assume that $i_1.\phi(r_l) + v_1 = i_2.\phi(r_l) + v_2$ for some different integers $i_1$ and $i_2$ and some points $v_1, v_2 \in N_{l-1}$. Using the linearity of the symmetric function $s_l$ we have

$$(i_2 - i_1).s_l(\phi(r_1), \ldots, \phi(r_{l-1}), \phi(r_l)) = s_l(\phi(r_1), \ldots, \phi(r_{l-1}), v_1 - v_2) =$$

$$= s_l(\phi(r_1), \ldots, \phi(r_{l-1}), v_1) - s_l(\phi(r_1), \ldots, \phi(r_{l-1}), v_2) = 0$$

because $s_l$ is trivial on $N_{l-1}$. However this contradicts the choice of the elements $r_l$ and the injectivity of $\phi$ because

$$(i_2 - i_1).s_l(\phi(r_1), \ldots, \phi(r_{l-1}), \phi(r_l)) = \phi((i_2 - i_1).s_l(r_1, \ldots, t_l)) \neq 0.$$

Thus $N_l$ contains infinitely many subvarieties of dimension equal to the one of $N_{l+1}$, which is only possible if $\dim N_l > \dim N_{l-1}$.

The above lemma yields:

Corollary 15. The dimension of $N_l$ is greater than or equal to $l$.

This is a contradiction because by construction $N_{k+1} \subset V$ and $\dim V = k < k + 1 \leq \dim N_{k+1}$, which completes the proof of Theorem 11.
Remark 16. It is not clear if it is possible to embed $F_p[x, y]$ into an algebraic variety with a ring structure over a field of positive characteristic. (The above argument only works if the “base ring” contains $\mathbb{Z}$.)

Before completing the proof of Theorem 1, we need to prove a lemma about the Steinberg groups $St_n(R)$ where $R$ is an associative unitary ring. This group has (formal) generators $x_{ij}(r)$ for $1 \leq i \neq j \leq n$ and $r \in R$, which satisfy the following commutator relations:

$$x_{ij}(r)x_{ij}(s) = x_{ij}(r+s)$$
$$[x_{ij}(r), x_{pq}(s)] = 1 \quad \text{if } i \neq q, j \neq p$$
$$[x_{ij}(r), x_{jk}(s)] = x_{ik}(s) \quad \text{if } i \neq k$$

There is a surjection from $St_n(R)$ onto $EL_n(R)$ mapping $x_{ij}(r)$ to $\text{Id} + re_{ij}$. The kernel of this surjection is denoted by $K_{2,n}(R)$.

The following lemma is fairly standard and probably well known. We are grateful to Nikolai Vavilov for correcting the original proof of that lemma.

**Lemma 17.** If $R$ is a finite ring, then the Steinberg group $St_n(R)$ is finite for any $n \geq 3$.

**Proof.** Using results of Vaserstein [Va] and Milnor [Mi] we deduce that in the case of a finite ring $R$, the groups $K_{2,n}(R)$ do not depend on $n \geq 3$ and are finite and central in $St_n(R)$ (because the stable range of a finite ring is 1). Therefore, the Steinberg group $St_n(R)$, $n \geq 3$, is a perfect extension of a finite group $K_{2,n}(R)$ by the finite group $EL_n(R)$ and thus is also finite. □

**Proof of Theorem 1.** (a) Suppose that $G = EL_3(R)$ is linear over a field $\mathbb{C}$. Then by Theorem 3 $R$ embeds into a ring that is a variety over $\mathbb{C}$. By Lemma 5 then $R$ has a finite index ideal that is linear over $\mathbb{C}$.

Suppose now that $R$ has a finite index ideal $I$ that is linear over $\mathbb{C}$. Consider the congruence subgroup $G_I$ of $G$ corresponding to $I$, that is the subgroup generated by all $x_{ij}(r)$, $r \in I$. The subgroup $G_I$ has a finite index in $G$, because the quotient $G/G_I$ is a homomorphic image of the Steinberg group $St_3(R/I)$ which is finite by Lemma 17. Also, $G_I$ is linear over $\mathbb{C}$, therefore $G$ is linear over $\mathbb{C}$ (consider the representation induced by the faithful representation of $G_I$).

(b) Suppose that $G = EL_3(\mathbb{Z}<x, y>)$ is linear over any field $K$. Again by Theorem 3 then $\mathbb{Z}<x, y>$ embeds into a ring that is a finite dimensional algebraic variety over $K$. By Theorem 2 that is impossible, a contradiction. □

The following theorem can be proved in the same manner as Theorem 1.

**Theorem 18.** (a) If the group $St_3(R)$ is linear over $\mathbb{C}$, then $R$ has a finite index ideal that is linear over $\mathbb{C}$.

(b) The group $St_3(\mathbb{Z}<x, y>)$ is not linear over any field $K$. 
## References

| Reference | Description |
|-----------|-------------|
| [Be]      | George M. Bergman, Some examples in PI ring theory. Israel J. Math. 18 (1974), 257–277. |
| [BT]      | Armand Borel, Jacques Tits, Homomorphismes “abstraits” de groupes algébriques simples. Ann. of Math. (2) 97 (1973), 499–571. |
| [C]       | Yu Chen, Homomorphisms from linear groups over division rings to algebraic groups. Group theory, Beijing 1984, 231–265, Lecture Notes in Math., 1185, Springer, Berlin, 1986. |
| [Ch]      | V. A. Churkin, V. A. The representation of groups over rational function fields. Algebra i Logika 7 1968 no. 4, 120–123. |
| [Co]      | P. M. Cohn, Free rings and their relations. Second edition. London Mathematical Society Monographs, 19. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London, 1985. |
| [HJW]     | Alexander J. Hahn, Donald G. James, Boris Weisfeiler, Homomorphisms of algebraic and classical groups: a survey. Quadratic and Hermitian forms (Hamilton, Ont., 1983), 249–296, CMS Conf. Proc., 4, Amer. Math. Soc., Providence, RI, 1984. |
| [H]       | I. N. Herstein, Noncommutative rings. The Carus Mathematical Monographs, No. 15, John Wiley & Sons, Inc., New York 1968. |
| [JWW]     | D. James, W. Waterhouse, and B. Weisfeiler, Abstract homomorphisms of algebraic groups: problems and bibliography, Communications in Algebra, 9:1 (1981), 95–114. |
| [Ma]      | A. I. Mal’cev, The elementary properties of linear groups. 1961 Certain Problems in Mathematics and Mechanics (In Honor of M. A. Lavrent’ev) pp. 110–132 Izdat. Sibirsk. Otdel. Akad. Nauk SSSR, Novosibirsk. |
| [Mi]      | J. Milnor, Introduction to algebraic $K$-theory. Annals of Mathematics Studies, No. 72. Princeton University Press, Princeton, N.J., 1971. |
| [Mo]      | G. D. Mostow, Strong rigidity of locally symmetric spaces. Annals of Mathematics Studies, No. 78. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1973. |
| [OM]      | O. T. O’Meara, Lectures on linear groups. Expository Lectures from the CBMS Regional Conference held at Arizona State University, Tempe, Ariz., March 26–30, 1973. Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics, No. 22. American Mathematical Society, Providence, R.I., 1974. |
| [Po]      | L. S. Pontryagin, Nepryryvnye gruppy. (Russian) [Continuous groups] Fourth edition. “Nauka”, Moscow, 1984. |
| [Ro]      | L. H. Rowen, Polynomial identities in ring theory. Pure and Applied Mathematics, 84. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1980. |
| [Va]      | L. N. Vaserstein, Bass’s first stable range condition. Proceedings of the Luminy conference on algebraic $K$-theory (Luminy, 1983). J. Pure Appl. Algebra 34 (1984), no. 2-3, 319–330. |

Martin Kassabov  
Department of Mathematics  
Cornell University  
Ithaca, NY 14853-4201 USA  
kassabov@math.cornell.edu

Mark V. Sapir  
Department of Mathematics  
Vanderbilt University  
Nashville, TN 37240, USA  
m.sapir@vanderbilt.edu