Modified KP and Discrete KP

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The aim of this note is to show that the so-called discrete KP, or 1-Toda lattice hierarchy (see Adler and van Moerbeke \[AvM 98\], Ueno and Takasaki (UT 84) ) is the same as a properly defined modified KP hierarchy. Virtually, the relationship between them is of the same sort as between the GD (or the \(n\)th KdV) hierarchy based on \(n\)th order differential operators and the hierarchy based on matrix first-order operators. This relationship is given by the Drinfeld-Sokolov reduction \[DS 81\]. In \[D 81\] (see also \[D 91\] and \[D 97\]) the following version of this reduction was exploited. Linear transformations in the space of differential and pseudodifferential operators (for example, the Adler mapping) can be represented by matrices in a basis of \(\partial^k\). Thus, the matrices in the DS reduction must be treated as linear transformations not in an abstract linear space but in a concrete space of pseudodifferential operators, which gives nice formulas. Similarly, here we deal with the space of series in \(z\) and \(z^{-1}\) where the Baker functions live, or, alternatively, the space of series in \((\partial, \partial^{-1})\) for dressing operators. Linear transformations of this space involved in the modified KP can be represented by matrices of infinite order, and then we get the discrete KP.

There exists several definitions of the modified KP. All of them are trying to imitate the relationship between the KdV (GD) and the modified KdV hierarchy. A complete analogy is impossible since a pseudodifferential operator cannot be factorized in a product of infinitely many first order linear differential operators, a kind of the Miura transform. Palliative solutions of this problem in the existing definitions have, in our opinion, serious shortcomings both, aesthetical and practical. First of all, they are lacking symmetry. The factors are of different character, some of them are first order differential and some pseudodifferential operators, or all pseudodifferential, but in this latter case there are introduced manifestly too many variables. Some of definitions contain an artificial additional integer parameter.

We believe that the discrete KP (1-Toda lattice) hierarchy is the best generalization of the modified KdV and we try to show this in the present article. This is its main point.

The modified KP is obtained by a successive extension of a simple, ordinary KP joining to it new and new unknown functions (“fields”) by Bäcklund (or Darboux) transformations. Independent variables (“times”) remain the same. The old fields, which already existed before an act of extension, just do not feel this act. This is a difference between this type
of extension and that in the case of the Zakharov-Shabat multipole hierarchy (see [D 94])
where an additional pole increased both, the number of fields and the number of times, so
that old fields depended on new times, too.

The paper is self-contained, all definitions are given.

1. Modified KdV.

The modified \( n \)th KdV (or GD) hierarchy is well-known. We will follow our paper [D 93]
in its description.

Let \( v_1, ..., v_n \) be generators of a differential algebra \( A_v \) with a relation \( v_1 + ... + v_n = 0 \).
We shall also use \( v_k \) with any integer subscript \( k \) assuming that \( v_k + n = v_k \). Let
\[
L_i := (\partial + v_{i-1})(\partial + v_1)(\partial + v_n)...(\partial + v_i) = (\partial + v_{i+n-1})(\partial + v_i) \tag{1.1}
\]

Each \( L_i \) can also be represented as
\[
L_i = \partial^n + u_1^{[i]}\partial^{n-1} + ... + u_n^{[i]}.
\]
The coefficients \( \{u_k^{[i]}\}, k = 1, ..., n \) are elements of \( A_v \)
\[
u_k^{[i]} = F_k(v_i, ..., v_{i+n-1}), \ k = 1, ..., n.
\]
These functions define the Miura transformation. They determine an embedding of the
differential algebra \( A_{u^{[i]}} \) of all differential polynomials in \( u_k^{[i]} \) into \( A_v \),
\[
A_{u^{[i]}} \subset A_v.
\]

It is obvious that
\[
(\partial + v_i)L_i = L_{i+1}(\partial + v_i) \tag{1.2}
\]
whence
\[
(\partial + v_i)L_i^{k/n} = L_{i+1}^{k/n}(\partial + v_i)
\]
and
\[
(\partial + v_i)(L_i^{k/n})_{+} - (L_i^{k/n}_{i+1})_{+}(\partial + v_i) = -(\partial + v_i)(L_i^{k/n})_{-} + (L_i^{k/n}_{i+1})_{-}(\partial + v_i).
\]
As usual, the subscript + denotes the positive (differential) part of a pseudodifferential operator, and the subscript - the rest of it. The r.-h.s. is a at most zero-order pseudodifferential operator (ΨDO) while the l.-h.s. is a differential operator. Hence, we proved:

**Lemma 1.1.**
\[
(\partial + v_i)(L_i^{k/n})_{+} - (L_i^{k/n}_{i+1})_{+}(\partial + v_i)
\]
is a zero-order differential operator, i.e., a function (an element of \( A_v \)).

**Corollary 1.2.** The system of equations for \( v_k \):
\[
\partial_k v_i = (L_i^{k/n})_{+}(\partial + v_i) - (\partial + v_i)(L_i^{k/n})_{+}, \ \partial_k = \partial/\partial t_k, \tag{1.3}
\]
where $t_k$ are parameters, and $L_i$ are defined by (1.1), makes sense.

**Definition 1.3.** The system (1.3), (1.1) is called the nth modified KdV, mKdV (or $mGD$).

**Proposition 1.4.** The system (1.3), (1.1) guarantees that all $L_i$ satisfy the n-th KdV equation.

Proof. We have

$$\partial_k L_i = \partial_k (\partial + v_{i+n-1})\cdots(\partial + v_i)$$

$$= \sum_{l=i}^{i+n-1} (\partial + v_{i+n-1})\cdots(\partial + v_l) \left((L_i^{k/n})_+ (\partial + v_l) - (\partial + v_l)(L_i^{k/n})_+ \right) (\partial + v_{l-1})\cdots(\partial + v_i)$$

$$= (L_i^{k/n})_+ L_i - L_i (L_i^{k/n})_+ = [(L_i^{k/n})_+, L_i]. \quad \Box$$

Each operator $L_i$ can be represented in a dressing form $L_i = \hat{w}_i \partial^\omega \hat{w}_i^{-1}$ where $\hat{w}_i = \sum_0^\infty w_{i,l} \partial^{-l}$ with $w_{i,0} = 1$. Operators $\hat{w}_i$ are determined up to multiplication on the right by constant series $c_i(z) = \sum_0^\infty c_{i,l} \partial^{-l}$.

**Lemma 1.5.** Properly choosing $c_i(z)$, one can always achieve the equality:

$$(\partial + v_i) \cdot \hat{w}_i = \hat{w}_{i+1} \cdot \partial. \quad (1.4)$$

Proof. From (1.2) we have $\hat{w}_{i+1} \partial \hat{w}_{i+1}^{-1} (\partial + v_i) = (\partial + v_i) \hat{w}_i \partial \hat{w}_i^{-1}$ or

$$\partial \hat{w}_{i+1}^{-1} (\partial + v_i) \hat{w}_i = \hat{w}_{i+1}^{-1} (\partial + v_i) \hat{w}_i \partial,$$

i.e., $\hat{w}_{i+1}^{-1} (\partial + v_i) \hat{w}_i$ commutes with $\partial$ being a first order $Ψ$DO. It has a form $\partial (1 + \sum_0^\infty c_{i,k} \partial^{-k})$. If we replace $\hat{w}_{i+1}$ by $\hat{w}_i (1 + \sum_0^\infty c_{i,k} \partial^{-k})$, then $\hat{w}_{i+1}^{-1} (\partial + v_i) \hat{w}_i = \partial$, i.e., (1.4). Now we can start with some $i$ and improve in succession $\hat{w}_{i+1}, \hat{w}_{i+2}, \ldots, \hat{w}_{i+n}$. We have

$$\hat{w}_{i+n} = (\partial + v_{i+n-1})\cdots(\partial + v_i) \hat{w}_i \partial^{-n} = L_i L_i^{-1} \hat{w}_i = \hat{w}_i;$$

and $\hat{w}_i$ depends on the index $i$ periodically, like $L_i$. \Box

It is not difficult to show that $\partial_i$ defined by (1.3) commute.

Baker functions $w_i(t, z)$ corresponding to the dressing operators $\hat{w}_i(t, \partial)$ are $w_i(t, z) = \hat{w}_i(t, \partial) \exp \xi(t, z)$ where $\xi(t, z) = \sum_1^\infty t_i z^i$. The Eq. (1.4) is equivalent to

$$(\partial + v_i) w_i(t, z) = zw_{i+1}(t, z). \quad (1.5)$$

**Remark.** It is worth mentioning that in terms of Grassmannians the relations (1.4) or (1.5) mean the following. If $V_i$ are elements of the Grassmannian related to $w_i$ then $zV_{i+1} \subset V_i$ (see [D 93]). Grassmannians considerations help to build examples; for instance, let $H$ be the space $L_2$ on the circle $|z| = 1$, and

$$V_i = \{ f(z) = \sum_{-N}^\infty f_k z^k \mid f(a_l) = e^i a_l f(e a_l); \ l = 1, \ldots, N, \ e^n = 1 \}. \quad (1.6)$$
Functions $f$ are supposed to be prolonged into the circle and $a_l$ are distinct points, $0 < |a_l| < 1$, while $a_l$ are arbitrary non-zero numbers. It is easy to see that all properties are satisfied.

A transition from one solution $L_i$ of KdV to the others, $L_j$, is a Bäcklund (or Darboux) transformation, see Adler [A 81].

2. Modified KP.

There exists several definitions of this hierarchy given by various authors. The first was suggested by Kuperschmidt [K 89], see also Yi Cheng [Y 93] and Gestezy and Unterkofler [GU 95]. All the definitions are trying to transfer the relationship between KdV and mKdV to the KP situation and, first of all, to factorize the KP operator. And there is a big obstacle on this way, even insurmountable one. A pseudodifferential operator cannot be represented as a product of first order differential operators. Suggested palliatives like a product of a finite number of operators with one ΨDO factor and the rest of them being first order differential operators have a disadvantage being not symmetric and not allowing a Bäcklund transformation. We abandon the very idea of the factorization and we will base our definition on a concept of a collection of KP operators connected by the equation (1.2) (see below). There are too many variables here, all of them cannot be independent. The first problem is to find a complete set of independent variables. We suggest the following construction.

Let $L_0$ be a KP operator and $v_i$ where $i \in \mathbb{Z}$ variables. For $i > 0$ let

$$L_i = (\partial + v_{i-1})...(\partial + v_0)L_0(\partial + v_0)^{-1}...(\partial + v_{i-1})^{-1}$$

and

$$L_{-i} = (\partial + v_{-i})^{-1}...(\partial + v_{-1})^{-1}L_0(\partial + v_{-1})...(\partial + v_{-i})$$

Evidently,

$$L_{i+1}(\partial + v_i) = (\partial + v_i)L_i, \quad i \in \mathbb{Z}. \quad (2.1)$$

Thus, we take the collection of all coefficients of $L_0$ along with the set of all $v_i$ as independent variables. Instead, we could take coefficients of some other $L_{i_0}$ and the same set of $v_i$; it would be a different system of variables. We suggest the following construction.

**Definition 2.1.** The modified KP hierarchy is a system of equations:

$$\partial_k L_0 = [(L_0^k)_+, L_0], \quad \partial_k v_i = (L_{i+1}^k)_+(\partial + v_i) - (\partial + v_i)(L_i^k)_+ \quad (2.2)$$

**Proposition 2.2.** For all $i \in \mathbb{Z}$ the equation

$$\partial_k L_i = [(L_i^k)_+, L_i] \quad (2.3)$$

holds.

**Proof.** We use induction. Let $i > 0$ and for all smaller non-negative indices the eq. (2.3) be proven. We have:

$$L_i = (\partial + v_{i-1})L_{i-1}(\partial + v_{i-1})^{-1}$$
and
\[
\partial_k L_i = -(\partial + v_{i-1})L_{i-1}(\partial + v_{i-1})^{-1}\{(L_i^k)_+ + (\partial + v_{i-1}) - (\partial + v_{i-1})(L_i^k)_+\}(\partial + v_{i-1})^{-1}
\]
\[
+ \{(L_i^k)_+ (\partial + v_{i-1}) - (\partial + v_{i-1})(L_i^k)_+\} L_{i-1}(\partial + v_{i-1})^{-1}
\]
\[
+ (\partial + v_{i-1}) \left[(L_i^k)_+, L_{i-1}\right](\partial + v_{i-1})^{-1}
\]
\[
= -L_i(L_i^k)_+ + (\partial + v_{i-1})L_{i-1}(L_i^k)_+(\partial + v_{i-1})^{-1}
\]
\[
+ (L_i^k)_+ L_i - (\partial + v_{i-1})(L_i^k)_+ L_{i-1}(\partial + v_{i-1})^{-1}
\]
\[
+ (\partial + v_{i-1}) \left[(L_i^k)_+, L_{i-1}\right](\partial + v_{i-1})^{-1} = [(L_i^k)_+, L_i].
\]
Similarly,
\[
L_{-i} = (\partial + v_{-i})^{-1}L_{-i+1}(\partial + v_{-i})
\]
and
\[
\partial_k v_{-i} = (L_{-i+1}^k)_+(\partial + v_{-i}) - (\partial + v_{-i})(L_{-i}^k)_+
\]
Then,
\[
\partial_k L_{-i} = -(\partial + v_{-i})^{-1}\{(L_{i-1}^k)_+ (\partial + v_{-i}) - (\partial + v_{-i})(L_{i-1}^k)_+\}(\partial + v_{-i})^{-1}L_{-i+1}(\partial + v_{-i})
\]
\[
+ (\partial + v_{-i})^{-1}L_{-i-1}\{(L_{i-1}^k)_+ (\partial + v_{-i}) - (\partial + v_{-i})(L_{i-1}^k)_+\}
\]
\[
+ (\partial + v_{-i})^{-1}[L_{i-1}^k)_+, L_{-i-1}](\partial + v_{-i})
\]
\[
= -(\partial + v_{-i})^{-1}(L_{-i+1}^k)_+ L_{-i+1}(\partial + v_{-i}) + (L_{-i+1}^k)_+ L_{-i}
\]
\[
+ (\partial + v_{-i})^{-1}L_{-i} L_{-i+1}(\partial + v_{-i}) - L_{-i}(L_{-i}^k)_+
\]
\[
+ (\partial + v_{-i})^{-1}[L_{-i+1}^k)_+, L_{-i+1}](\partial + v_{-i}) = [(L_{-i}^k)_+, L_{-i}]. \square
\]

**Proposition 2.3.** Derivations $\partial_k$ commute.

**Proof.** The fact that $\partial_k \partial L_i = \partial_l \partial_k L_i$ is known; this is a property of the KP hierarchy. Now,
\[
\partial_k \partial v_i - \partial_l \partial_k v_i = \partial_k \left\{(L_{i+1}^k)_+(\partial + v_i) - (\partial + v_i)(L_i^k)_+\right\} - (k \leftrightarrow l)
\]
\[
= [(L_{i+1}^k)_+, L_{i+1}^k_+] (\partial + v_i) - (\partial + v_i)(L_{i+1}^k)_+ (L_{i+1}^k)_+
\]
\[
+ (L_{i+1}^k)_+ \left\{(L_{i+1}^k)_+(\partial + v_i) - (\partial + v_i)(L_{i+1}^k)_+\right\}
\]
\[
- \left\{(L_{i+1}^k)_+(\partial + v_i) - (\partial + v_i)(L_{i+1}^k)_+\right\} (L_{i+1}^k)_+ - (k \leftrightarrow l)
\]
\[
= \left\{(L_{i+1}^k)_+ (L_{i+1}^k)_+ - (L_{i+1}^k)_+ (L_{i+1}^k)_+\right\} (\partial + v_i) - (\partial + v_i) \left\{L_{i+1}^k (L_{i+1}^k)_+ - L_{i+1}^k (L_{i+1}^k)_+ \right\} - (k \leftrightarrow l)
\]
\[
= \left\{L_{i+1}^k (L_{i+1}^k)_+ - L_{i+1}^k (L_{i+1}^k)_+ \right\} (\partial + v_i) - (\partial + v_i) \left\{L_{i+1}^k L_{i+1}^k - L_{i+1}^k L_{i+1}^k \right\} = 0. \square
\]

Dressing operators can be introduced:
\[
\hat{w}_i(t, \partial) = \sum_{\alpha=0}^\infty w_{i\alpha} \partial^\alpha, \quad w_{i0} = 1
\]  
(2.4)
such that
\[ L_i = \hat{w}_i \partial \hat{w}_i^{-1}, \quad (\partial + v_i) \cdot \hat{w}_i = \hat{w}_{i+1} \cdot \partial, \tag{2.5} \]

Baker functions
\[ w_i(t, z) = \hat{w}_i(t, \partial) \exp \xi(t, z), \quad L_i w_i(t, z) = z w_i(t, z), \quad (\partial + v_i) w_i(t, z) = z w_{i+1}(t, z) \quad \tag{2.6} \]

and conjugate Baker functions:
\[ w^*_i(t, z) = (\hat{w}_i(t, \partial)^{-1})^* \exp(-\xi(t, z)), \quad L^*_i w^*_i(t, z) = z w^*_i(t, z), \quad (\partial - v_i) w^*_i(t, z) = -z w^*_i(t, z). \quad \tag{2.7} \]

Here the asterisk * in \( w^*_i \) just belongs to the notation while in \( \hat{w}^*_i \) and \( L^*_i \) it means formal conjugate of operators.

**Proposition 2.4.** There is an automorphism of the mKP:
\[
\begin{align*}
t_k &\mapsto -t_k = (-1)^{k-1} t_k, \quad v_i &\mapsto -v_{i-1}, \quad L_i &\mapsto -L^*_{-i}, \\
\hat{w}_i &\mapsto (\hat{w}^{-1}_i)^*, \quad w_i &\mapsto w^*_i, \quad z &\mapsto -z. \quad \tag{2.8}
\end{align*}
\]

Indeed, it is easy to see that the equations defining the hierarchy tolerate this transformation. □

Notice that any streak of equations (2.2), finite of semi-infinite, \( 0 \leq i < i_1 \) or \( i_1 < i \leq 0 \) (in particular, one equation, \( i = 0 \)) form a closed system. Especially interesting are the semi-infinite cases, \( 0 \leq i < \infty \) or \( -\infty < i \leq 0 \). These are one-sided mKP’s: mKP+ and mKP-. The automorphism (2.8) interchanges them.

Also notice that the mKdV is a restriction of mKP to the case when \( v_{i+n} = v_i \) and \( L_0 = (\partial + v_{n-1})... (\partial + v_0) \).

**The rest of this section is not used in the proof of the main theorem (3.7) and can be skipped.**

It is well-known that the so-called Hirota-Sato bilinear identity gives an equivalent description of the KP hierarchy. It is based on the following fundamental lemma (see [D 91] or [D 97]):

**Lemma 2.5.** Let \( P \) and \( Q \) be two \( \Psi DO \), then
\[
\text{res}_z [(Pe^{xz}) \cdot (Qe^{-xz})] = \text{res}_\partial PQ^*
\]

where \( Q^* \) is the formal adjoint to \( Q \).

**Proof.** The left-hand side is
\[
\text{res}_z [(Pe^{xz}) \cdot (Qe^{-xz})] = \text{res}_z \left( \sum p_i z^i \sum q_j (-z)^j \right) = \sum_{i+j=-1} (-1)^j p_i q_j,
\]

and the right-hand side is
\[
\text{res}_\partial (P \cdot Q^*) = \text{res}_\partial \sum p_i \partial^i (-\partial)^j q_j = \sum_{i+j=-1} p_i q_j. \quad \square
\]
**Proposition 2.6 (bilinear identity).** If \( w \) is a Baker function of the KP hierarchy, then the identity

\[
\text{res}_z \left( \partial^{k_1}_t \ldots \partial^{k_m}_m w \right) \cdot w^* = 0
\]

holds for any \((k_1, \ldots, k_m)\).

**Proof.** Since \( \partial_t w = L^k_t w \), it suffices to consider only the case when \( m = 1 \). Then

\[
\text{res}_z \left( \partial^k w \right) \cdot w^* = \text{res}_z \left( \partial^k \hat{w} e^{\xi(t,z)} \right) \left( \hat{w}^* \right)^{-1} e^{-\xi(t,z)}
\]

\[
= \text{res}_z \left( \partial^k \hat{w}^x \right) \left( \hat{w}^* \right)^{-1} e^{-xz} = \text{res}_\theta \partial^k \hat{w} \cdot \hat{w}^{-1} = \text{res}_\theta \partial^k = 0. \quad \Box
\]

Notice that if the hierarchy equations are not used and it is only assumed that identity will be obtained, \( \text{res}_z \partial^k w \cdot w^* = 0 \).

Return to our mKP hierarchy.

**Proposition 2.7.** If \( w_i \) are Baker functions of the mKP hierarchy then the identity

\[
\text{res}_z \left( z^{i-j} \partial^{k_1}_t \ldots \partial^{k_m}_m w_i \right) \cdot w_j^* = 0, \quad \text{when } i \geq j
\]

(2.9)

holds for any \((k_1, \ldots, k_m)\).

**Proof.** Again, it suffices to take \( m = 0 \). Then

\[
\text{res}_z z^{i-j} \left( \partial^k w_i \right) \cdot w_j^* = \text{res}_z \left( \partial^k \hat{w}_i \partial^{i-j} e^{\xi(t,z)} \right) \left( \hat{w}_j^* \right)^{-1} e^{-\xi(t,z)}
\]

\[
= \text{res}_z \left( \partial^k (\partial + v_{i-1}) \ldots (\partial + v_j) \hat{w}_j e^{xz} \right) \left( \hat{w}_j^* \right)^{-1} e^{-xz} = \text{res}_\theta \partial^k (\partial + v_{i-1}) \ldots (\partial + v_j) = 0. \quad \Box
\]

The converse is also true.

**Proposition 2.8.** Let

\[
w_i = \sum_{\alpha} w_{i\alpha} z^{-\alpha} e^{\xi(t,z)}, \quad w_i^* = \sum_{\alpha} w_{i\alpha}^* z^{-\alpha} e^{-\xi(t,z)}
\]

be formal expansions where \( w_{i\alpha} \) and \( w_{i\alpha}^* \) are functions of variables \( t_k \), and \( w_{i0} = w_{i0}^* = 1 \). Let

\[
\text{res}_z z^{i-j} \left( \partial^{k_1}_t \ldots \partial^{k_m}_m w_i \right) \cdot w_j^* = 0, \quad \text{when } i = j, j + 1
\]

hold for any multiindex. Then \( w_i \) and \( w_i^* \) are the Baker and the adjoint Baker functions of the mKP hierarchy.

**Proof.** When \( i = j \), this is a well-known theorem that states that all \( w_i \)s and \( w_i^* \)s are Baker and conjugate Baker functions of KP (see, e.g., [D 91] or [D 97]). When \( i = j + 1 \) we have

\[
0 = \text{res}_z z^{i-j} \partial^k w_{j+1} w_j^* = \text{res}_z \partial^k \hat{w}_{j+1} \partial e^{\xi(\hat{w}_j^*)^{-1} e^{-\xi}} = \text{res}_\theta \partial^k \hat{w}_{j+1} \partial \hat{w}_j^{-1}
\]

which yields that \( (\hat{w}_{j+1} \partial \hat{w}_j^{-1})_+ = 0 \) and \( \hat{w}_{j+1} \partial \hat{w}_j^{-1} \) is a first-order differential operator: \( \hat{w}_{j+1} \partial \hat{w}_j^{-1} = (\partial + v_j) \hat{w}_j = (\partial + v_j) \hat{w}_j \).

Theorem 2.9
and this proves the proposition. □

3. Discrete KP.

We deal with a linear space $H$ of $\Psi DO$: \( \{ a = \sum_{-\infty} a_i \partial^i \} \); the series are one-way infinite. A dual space $H^*$ is \( \{ b = \sum_{-\infty} \partial^{-i-1} b_i \} \); a coupling is given by \( < a, b > = \text{res}_\theta ab = \sum_{-\infty} a_i b_i \).

Infinite matrices $M = (m_{ij})$ where $i, j \in \mathbb{Z}$ will be considered as matrices of a change of the basis: $M_i = \sum_{-\infty} m_{i\alpha} \partial^\alpha$ are new basis vectors, instead of $\partial^i$. The same matrices also can be treated as matrices of basis change in the dual space, $M^j = \sum_{-\infty} \partial^{-\beta-1} m_{\beta j}$ being new basis vectors instead of $\partial^{-j-1}$. Product $P = MN$ of two matrices can be found as $P_{ij} = \text{res}_\theta M_i N^j$. All the matrices will be triangular, $m_{ij} = 0$ when $i > j + \text{const}$, and all the sums make sense. Thus, the rows are associated with operators $M_i$ and the columns with operators $M^P$.

Let $W$ be a matrix with rows $W_i = w_i \partial^i$ where $\hat{w}_i$ are dressing operators (2.4). Let

\[
\hat{w}_j^{-1} = \sum_{\beta=0}^\infty \partial^{-\beta} \hat{w}_{\beta j}, \quad w_{0j} = 1
\]

and let $\tilde{W}$ be a matrix with columns $\tilde{W}^j = \partial^{-j-1} \hat{w}_j^{-1}$.

**Lemma 3.1.** $\tilde{W} = W^{-1}$.

*Proof.* The matrix elements of $W$ are $W_{ij} = w_{i,j-1}$ when $i \geq j$ and 0 otherwise. The matrix elements of $\tilde{W}$ are $\tilde{W}_{ij} = \tilde{w}_{i,j+1}$ if $i \geq j$ and 0 otherwise. Both, $W$ and $\tilde{W}$, are lower triangular matrices with unities on their diagonals. So is their product $W\tilde{W}$, its subdiagonal elements are

\[
(W\tilde{W})_{ij} = \text{res}_\theta W_i \tilde{W}^j = \text{res}_\theta \hat{w}_i \partial^i \partial^{-j-1} \hat{w}_{j+1}^{-1}, \quad i > j.
\]

Eq. (2.5) implies that if $i > j$ then

\[
\hat{w}_i = (\partial + v_{i-1})\hat{w}_{i-1} \partial^{-1} = \ldots = (\partial + v_{i-1})\ldots(\partial + v_{j+1})\hat{w}_{j+1} \partial^{-i+j+1}
\]

whence

\[
(W\tilde{W})_{ij} = \text{res}_\theta (\partial + v_{i-1})\ldots(\partial + v_{j+1})\hat{w}_{j+1} \partial^{-i+j+1} \partial^{-j-1} \hat{w}_{j+1}^{-1} = \text{res}_\theta (\partial + v_{i-1})\ldots(\partial + v_{j+1}) = 0.
\]

Thus, $W\tilde{W} = I$, the matrix unity, and $\tilde{W} = W^{-1}$. □

**Definition 3.2.** If $M$ is a matrix then $M_+$ is a matrix such that

\[
(M_+)_{ij} = \begin{cases} M_{ij}, & \text{when } i \geq j \\ 0, & \text{when } i < j \end{cases}
\]

\footnotetext{1}{In the same manner we treated the Drinfeld-Sokolov reduction in [D 81], see also [D 91] or [D 97]: differential operators $M_i = \sum_{\alpha} m_{i\alpha} \partial^\alpha$ were associated with the rows and integral operators $M^j = \sum_{\beta} \partial^{-\beta-1} m_{\beta j}$ with the columns of finite matrices.}
and $M_- = M - M_+$. 

It is easy to see that 

$$(M_+)_i = (M_i \partial^{-i})_+ \partial^i, \quad (M_-)_i = (M_i \partial^{-i})_- \partial^i. \quad (3.2)$$

Let $\Lambda$ be the matrix $\Lambda_{ij} = \delta_{i,j-1}$.

**Lemma 3.3.** If $M = (M_{ij})$ is a matrix then an operator associated with a row of a matrix $\Lambda M$ is $M_i \partial$.

**Proof.** 

$$(\Lambda M)_i = \sum_j (\Lambda M)_{ij} \partial^j = \sum_j M_{i,j-1} \partial^j = \sum_j M_{i,j-1} \partial^{j-1} \partial = M_i \partial. \quad \square$$

Let us dress the matrix $\Lambda$ with the help of $W$: $L = W \Lambda W^{-1}$.

**Proposition 3.4.** By virtue of the mKP equations (2.2), the following is true:

$$\partial_k W = -(L^k)_- W. \quad (3.3)$$

**Proof.**

$$\partial_k W_i = \partial_k \hat{w}_i \partial^j = -(L^k_i)_- \hat{w}_i \partial^j,$$

$$(\partial_k W_i \cdot W^{-1})_{ij} = \text{res}_\partial(-(L^k_i)_- \hat{w}_i \partial^j \partial^{-j-1} \hat{w}^{-1}_{j+1}) = -\text{res}_\partial(L^k_i)_-(\partial + v_{i-1})...(\partial + v_{j+1}).$$

On the other hand,

$$(L^k)_{ij} = (W \Lambda^k W^{-1})_{ij} = \text{res}_\partial(W \Lambda^k)_i(W^{-1})^j = \text{res}_\partial \hat{w}_i \partial^j \partial^{k-j-1} \hat{w}^{-1}_{j+1}$$

$$= \text{res}_\partial L^k_i \hat{w}_i \partial^{j-1} \hat{w}^{-1}_{j+1} = \text{res}_\partial L^k_i (\partial + v_{i-1})...(\partial + v_{j+1}) = \text{res}_\partial(L^k_i)_-(\partial + v_{i-1})...(\partial + v_{j+1}). \quad \square$$

**Corollary 3.5.** The mKP equations (2.2) imply

$$\partial_k L = [(L^k)_+, L]. \quad (3.4)$$

**Proof.**

$$\partial_k L = \partial_k (W \Lambda W^{-1}) = -(L^k)_- W \Lambda W^{-1} + W \Lambda W^{-1}(L^k)_- WW^{-1}$$

$$= -[(L^k)_-, L^k] = [(L^k)_+, L]. \quad \square$$

**Definition 3.6.** The equation (3.4) where $L = \Lambda$ (lower triangular) is the discrete KP.

**Theorem 3.7.** The discrete KP (3.4) is equivalent to the modified KP (2.2).

**Proof.** Since this is already proven in one direction, it remains to show that each solution to (3.4) can be obtained from a solution to (2.2) in the above described way.
Thus, given a matrix \( \mathbf{L} = \Lambda + (\text{lower triangular}) \), satisfying (3.4). The matrix \( \mathbf{L} \) can represented in a form \( \mathbf{L} = \mathbf{W} \Lambda \mathbf{W}^{-1} \). Lower triangular dressing matrices \( \mathbf{W} \) with unities on the main diagonal are not uniquely determined; they can be multiplied on the right by a matrix commuting with \( \Lambda \), i.e., constant on all diagonals. Using this freedom, it is always possible to satisfy (3.3) (there remains a little freedom even after this operation: \( \mathbf{W} \) can be multiplied on the right by a constant matrix commuting with \( \mathbf{L} \), this can be fixed with the initial conditions). Thus, one can consider (3.3) as a discrete KP equation. The variable \( t_1 \) we will identify with \( x \) and \( \partial_1 = \partial \).

Let \( \mathbf{W}_i = \hat{w}_i \partial^i \) be operators associated with rows of \( \mathbf{W} \) and \( (\mathbf{W}^{-1})^j \partial^i = \partial^{-j-1} \hat{w}_{j+1} \) operators associated with columns of \( \mathbf{W}^{-1} \). In the next three lemmas there will be assumed that \( \mathbf{W} \) satisfies (3.3).

**Lemma 3.8.** There exist quantities \( v_i \) such that

\[
(\partial + v_i)\hat{w}_i = \hat{w}_{i+1} \partial.
\]

**Proof.** The equation (3.3) for \( k = 1 \) reads

\[
\partial_1 \mathbf{W} = -\mathbf{L} \mathbf{W} + \mathbf{L}_+ \mathbf{W} = -\mathbf{W} \Lambda + (\Lambda - \mathbf{V}) \mathbf{W}
\]

where \( \mathbf{V} \) is a diagonal matrix. Let us take operators associated with the \( i \)th row on the left and on the right of this equality taking into account the Lemma 3.3. \( \partial (\hat{w}_i \partial^i) = -\hat{w}_i \partial \partial^i + \hat{w}_{i+1} \partial^{i+1} - v_i \hat{w}_i \partial^i \) or \( (\partial + v_i)\hat{w}_i = \hat{w}_{i+1} \partial \). □

**Lemma 3.9.**

\[
\hat{w}_{j+1} = w_{j+1}^{-1}.
\]

**Proof.** We have \( (\mathbf{W} \mathbf{W}^{-1})_{ij} = 0 \) when \( i > j \), therefore

\[
0 = \text{res}_{\partial} \hat{w}_i \partial^i \partial^{-j-1} \hat{w}_{j+1} = \text{res}_{\partial} (\partial + v_{i-1})(\partial + v_{j+1}) \hat{w}_{j+1} \hat{w}_{j+1}.
\]

This can be true for all \( i > j \) only if \( (\hat{w}_{j+1} \hat{w}_{j+1})_+ = 0 \). Then \( \hat{w}_{j+1} \hat{w}_{j+1} = 1 \) and \( \hat{w}_{j+1} = \hat{w}_{j+1}^{-1} \). □

**Lemma 3.10.** The operators \( \hat{w}_i \) satisfy KP.

**Proof.** The \( ij \)th element of the matrix equality \( \partial_k \mathbf{W} \cdot \mathbf{W}^{-1} = -(\mathbf{W} \Lambda^k \mathbf{W}^{-1})_+ \) is

\[
\text{res}_{\partial} \partial_k (\hat{w}_i) \partial^i \partial^{-j-1} \hat{w}_{j+1} = -\text{res}_{\partial} \hat{w}_i \partial^k \partial^{i-j-1} \hat{w}_{j+1}, \ i > j
\]

or

\[
\text{res}_{\partial} (\partial_k \hat{w}_i + \hat{w}_i \partial^k \hat{w}_i^{-1})(\partial + v_{i-1})(\partial + v_{j+1}) = 0
\]

for a given \( i \) and all \( j < i \). This implies \( (\partial_k \hat{w}_i + \hat{w}_i \partial^k \hat{w}_i^{-1})_+ = 0 \) and, finally, \( \text{res}_{\partial} \partial_k \hat{w}_i \cdot \hat{w}_i^{-1} = -(\hat{w}_i \partial^k \hat{w}_i^{-1})_+ \) which is KP. □

This also completes the proof of the theorem since \( (\partial + v_i) = \hat{w}_{i+1} \partial \hat{w}_i \) and

\[
\partial_k v_i = -(L_{i+1}^k)_-(\partial + v_i) + (\partial + v_i)(L_i^k)_- = (L_{i+1}^k)_+(\partial + v_i) - (\partial + v_i)(L_i^k)_+. \quad \Box
\]

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