C\(_{\lambda}\)-extended oscillator algebra and parasupersymmetric quantum mechanics *

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Abstract

The C\(_{\lambda}\)-extended oscillator algebra is generated by \(\{1, a, a^\dagger, N, T\}\), where \(T\) is the generator of the cyclic group \(C_\lambda\) of order \(\lambda\). It can be realized as a generalized deformed oscillator algebra (GDOA). Its unirreps can thus be easily exhibited using the representation theory of GDOAs and their carrier space show a \(Z_\lambda\)-grading structure. Within its infinite-dimensional Fock space representation, this algebra provides a bosonization of parasupersymmetric quantum mechanics of order \(p = \lambda - 1\).

1 Introduction

Deformations of the oscillator algebra have found a lot of applications to physical problems. Some of them are the description of systems with non-standard statistics, algebraic realizations of some one-dimensional exactly solvable potentials (e.g. the Calogero model), the use of their coherent and squeezed states to describe experimental properties of light (e.g. in a Kerr medium).

Supersymmetric quantum mechanics (SSQM) and the concept of shape invariance have proved very useful for generating exactly solvable potentials. SSQM establishes also a nice symmetry between bosons and fermions, which has been enlarged to symmetry between bosons and parafermions to give rise to parasupersymmetric quantum mechanics (PSSQM).

The purpose of this communication is to introduce a new oscillator algebra, the C\(_{\lambda}\)-extended oscillator algebra, where \(C_\lambda\) is a cyclic group of order \(\lambda\), and to show a possible bosonization of PSSQM using its elements. This is one of the prospected results announced in Ref. [1].

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2 Definitions, Fock-space representation and bosonic Hamiltonian

The Calogero-Vasiliev algebra $[2]$ is generated by the operators $\{1, a, a^\dagger, N, K\}$ satisfying

$$[a, a^\dagger] = 1 + \nu K, \quad \nu \in \mathbb{R}, \quad K^2 = 1,$$

$$[N, a] = -a, \quad [N, a^\dagger] = a^\dagger, \quad [N, K] = 0, \quad aK = -Ka, \quad a^\dagger K = -Ka^\dagger,$$

$$N^\dagger = N, \quad (a^\dagger)^\dagger = a, \quad K^\dagger = K^{-1}. \quad \tag{4}$$

Klein operator $K$ can be interpreted as the generator of the symmetric group $S_2$. This led to a generalization of the Calogero-Vasiliev algebra known as the $S_N$-extended oscillator algebra, used in the study of the $N$-particle Calogero problem.

On the other hand, one can see $K$ as the generator of the cyclic group $C_2$ of order 2. As proposed in Ref. $[1]$, this leads to another generalization, the $C_{\lambda}$-extended oscillator algebra ($\lambda \in \{2, 3, 4, \ldots\}$) generated by $\{1, a, a^\dagger, N, T\}$, satisfying the relations

$$[a, a^\dagger] = 1 + \frac{\lambda - 1}{\lambda} \sum_{\mu=1}^{\lambda-1} \kappa_\mu T^\mu, \quad \kappa_\mu \in \mathbb{C}, \quad T^\lambda = 1, \quad \tag{5}$$

$$[N, a] = -a, \quad [N, a^\dagger] = a^\dagger, \quad [N, T] = 0, \quad aT = e^{i2\pi/\lambda} Ta, \quad a^\dagger T = e^{-i2\pi/\lambda} Ta^\dagger, \quad \tag{6}$$

$$N^\dagger = N, \quad (a^\dagger)^\dagger = a, \quad T^\dagger = T^{-1}. \quad \tag{7}$$

Here $T$ is the generator of a unitary representation of the cyclic group of order $\lambda$, $C_\lambda = \{1, T, T^2, \ldots, T^{\lambda-1}\}$, and the $\lambda - 1$ complex parameters $\kappa_\mu$ are restricted by the condition $\kappa_\mu^* = \kappa_{\lambda-\mu}$ to preserve the Hermiticity properties of (5), so that there are only $\lambda - 1$ independent real parameters in the definition of the algebra. It is worth noting that the annihilation and creation operators (resp. $a$ and $a^\dagger$) commute with $T$, with $q$ being a $\lambda$-th root of unity. For $\lambda = 2$, this reduces to the anticommutation (3) of $a$ and $a^\dagger$ with $K$.

$C_\lambda$ possesses $\lambda$ inequivalent one-dimensionnal unirreps labelled by $\mu$, and such that $\Gamma^\mu (T^\nu) = \exp(i2\pi \mu \nu / \lambda)$, $\mu, \nu = 0, 1, 2, \ldots, \lambda - 1$. The projector $P_\mu$ on the carrier space of the $\mu$-indexed unirrep takes the form

$$P_\mu = \frac{1}{\lambda} \sum_{\nu=0}^{\lambda-1} (\Gamma^\mu (T^\nu))^* T^\nu = \frac{1}{\lambda} \sum_{\nu=0}^{\lambda-1} e^{-i2\pi \mu \nu / \lambda} T^\nu. \quad \tag{9}$$

It is straightforward to show that these operators satisfy the projection operator defining relations, i.e., $P_\mu P_\nu = \delta_{\mu,\nu} P_\mu$, and $\sum_{\mu=0}^{\lambda-1} P_\mu = 1$. Conversely, $T^\nu$ can be expressed in terms of $P_\mu$,

$$T^\nu = \sum_{\mu=0}^{\lambda-1} e^{i2\pi \mu \nu / \lambda} P_\mu, \quad \tag{10}$$
so that the $C_\lambda$-extended oscillator algebra can be redefined in terms of $P_\mu$ instead of $T^\mu$,

$$[a, a^\dagger] = 1 + \sum_{\mu=0}^{\lambda-1} \alpha_\mu P_\mu, \quad \alpha_\mu \in \mathbb{R}, \quad \sum_{\mu=0}^{\lambda-1} \alpha_\mu = 0, \quad (11)$$

$$[N, a] = -a, \quad [N, a^\dagger] = a^\dagger, \quad [N, P_\mu] = 0, \quad (12)$$

$$aP_\mu = P_{\mu-1}a, \quad a^\dagger P_\mu = P_{\mu+1}a^\dagger, \quad P_\mu P_\nu = \delta_{\mu,\nu} P_\mu, \quad (13)$$

$$N^\dagger = N, \quad (a^\dagger)^\dagger = a, \quad (P_\mu)^\dagger = P_\mu, \quad \sum_{\mu=0}^{\lambda-1} P_\mu = 1, \quad (14)$$

where $\mu, \nu = 0, 1, 2, \ldots, \lambda - 1$, and we use the convention that for any parameter or operator $X$ involved in a $C_\lambda$-extended oscillator algebra, $X_y \equiv X_{y \mod \lambda}$ (e.g. $P_{-1} = P_{\lambda-1}$). The reality and sum conditions on $\alpha_\mu$ in (11) follow from their dependence on $\kappa_\mu$, i.e., $\alpha_\mu = \sum_{\nu=1}^{\lambda-1} \exp(i2\pi \mu\nu/\lambda)\kappa_\nu$, $\mu = 0, 1, \ldots, \lambda - 1$.

$T$ can be realized in terms of the number operator $N$. There exist other realizations (using a function of spin matrices, for instance), but here, only that involving $N$ is considered. Explicitly

$$T = e^{i2\pi N/\lambda}, \quad P_\mu = \frac{1}{\lambda} \sum_{\nu=0}^{\lambda-1} e^{i2\pi \nu(N-\mu)/\lambda}, \quad \mu = 0, 1, \ldots, \lambda - 1, \quad (15)$$

assuming the spectrum of $N$ is made of integers only. Within this realization, the $C_\lambda$-extended oscillator algebra can be seen as a generalized deformed oscillator algebra (GDOA), and its unirreps can be developed using the results of Ref. [3].

A GDOA is characterized by an analytic deformation function $G(N)$ such that $[a, a^\dagger]^q = G(N)$, and a structure function $F(N)$ vanishing for $N = 0$, and solution of the functional equation $F(N+1) - qF(N) = G(N)$. A GDOA Casimir operator is $C = q^{-N}(F(N) - a^\dagger a)$. Here, (11) together with (15) give $q = 1$, $G(N) = 1 + \sum_{\mu=0}^{\lambda-1} \alpha_\mu P_\mu$, and

$$F(N) = N + \sum_{\mu=0}^{\lambda-1} \beta_\mu P_\mu, \quad \beta_\mu = \sum_{\nu=0}^{\mu-1} \alpha_\nu, \quad \beta_0 = 0, \quad (16)$$

with $F(N+1) = N + 1 + \sum_{\mu=0}^{\lambda-1} \beta_{\mu+1} P_\mu$.

A Fock-space representation is characterized by the existence of a normalized simultaneous eigenvector $|0\rangle$ of $N$ and $C$ with both eigenvalues equal to 0, and which is destroyed by $a$. In such a unirrep, $a^\dagger a = F(N)$, $aa^\dagger = F(N+1)$, and $P_\mu|0\rangle = \delta_{\mu,0}|0\rangle$. Its carrier space is spanned by the vectors

$$|n\rangle = \mathcal{N}_n^{-1/2} \left(a^\dagger\right)^n |0\rangle, \quad n = 0, 1, 2, \ldots, d - 1, \quad (17)$$

where $d$ may be finite or infinite, and the normalization coefficients are given by $\mathcal{N}_n = \prod_{\mu=1}^{n} F(\mu) = \prod_{\mu=1}^{n} (\mu + \beta_\mu)$. If $|n\rangle \equiv |k\lambda + \mu\rangle$, where $k \in \mathbb{Z}^+$ and $\mu = n \mod \lambda$, does exist, then the generators of the algebra act on it as

$$N |k\lambda + \mu\rangle = (k\lambda + \mu) |k\lambda + \mu\rangle, \quad P_\nu |k\lambda + \mu\rangle = \delta_{\nu,\mu} |k\lambda + \mu\rangle, \quad (18)$$

$$a |k\lambda + \mu\rangle = \sqrt{k\lambda + \mu + \beta_\mu} |k\lambda + \mu - 1\rangle, \quad (19)$$

$$a^\dagger |k\lambda + \mu\rangle = \sqrt{k\lambda + \mu + 1 + \beta_{\mu+1}} |k\lambda + \mu + 1\rangle. \quad (20)$$
The existence condition for $|n\rangle$ is the unitarity condition of the representation, which is fulfilled if and only if $F(\mu) \geq 0$, $\forall \mu = 1, 2, \ldots, d$. There are thus two types of Fock space representations:

1. finite-dimensional unirreps of dimension $d < \lambda$
   \[ \iff F(\mu) > 0, \forall \mu = 1, 2, \ldots, d - 1, \text{ and } F(d) = 0, \]

2. infinite-dimensional bounded from below (BFB) unirrep
   \[ \iff F(\mu) > 0, \text{ or } \beta_\mu = \sum_{\nu=0}^{\mu-1} \alpha_\nu > -\mu, \forall \mu = 1, 2, \ldots, \lambda - 1. \]

From now on, only the bosonic Fock space representation, corresponding to the second type, will be considered.

It is worth noting the Fock-space $\mathbb{Z}_\lambda$-grading: for each $\mu = 0, 1, \ldots, \lambda - 1$, let $\mathcal{F}_\mu = \{|k\lambda + \mu\rangle, k \in \mathbb{Z}^+\}$; then the entire carrier Fock space is $\mathcal{F} = \sum_{\mu=0}^{\lambda-1} \oplus \mathcal{F}_\mu$, and the action of the algebra generators on a vector of $\mathcal{F}$ depends on which $\mathcal{F}_\mu$ subspace it belongs.

The bosonic oscillator Hamiltonian is defined by $H_0 = \frac{1}{2}\{a^\dagger, a\} = \frac{1}{2}(F(N) + F(N + 1))$. By using (16), $H_0$ can be rewritten as

\[ H_0 = N + \frac{1}{2} + \sum_{\mu=0}^{\lambda-1} \gamma_\mu P_\mu, \quad \gamma_\mu = \sum_{\nu=0}^{\mu-1} \alpha_\nu + \alpha_\mu/2 \quad (\gamma_0 = \alpha_0/2), \]

and it is obvious that

\[ H_0 |n\rangle = E_n |n\rangle, \quad \text{where} \quad E_n = n + \frac{1}{2} + \gamma_{n \text{ mod } \lambda}. \]

The spectrum of the bosonic Hamiltonian possesses $\lambda$ families of equally spaced eigenvalues (within each $\mathcal{F}_\mu$, $H_0$ is harmonic). Because of this very interesting property, the $C_\lambda$-extended oscillator algebra provides an algebraic realization for the recently introduced cyclic shape invariant potentials \[4\]. This will not be developed here, for more details see Refs. \[1, 5\].

### 3 Bosonization of (para)supersymmetric quantum mechanics

Supersymmetric quantum mechanics (SSQM) is characterized by supercharge operators $Q$, $Q^\dagger$ such that

\[ Q^2 = 0, \quad \{Q^\dagger, Q\} = \mathcal{H}, \quad (Q^\dagger)^\dagger = Q, \]

so that the supersymmetric Hamiltonian $\mathcal{H}$ commutes with the supercharges ($[\mathcal{H}, Q] = 0$). There exists a realization of this algebra wherein the supercharges are the product of mutually commuting fermionic (Pauli matrices) and bosonic operators. An alternative realization of SSQM, without fermionic matrices, has been provided using the Calogero-Vasiliev (or $C_2$-extended oscillator) algebra generators \[6\]. The choice $Q = a^\dagger P_1$ (so that $\mathcal{H} = a^\dagger a P_0 + aa^\dagger P_1$) corresponds to an unbroken SSQM (except for the ground
state, all the states are two-fold degenerate), while $Q = a^\dagger P_0$ (and $H = aa^\dagger P_0 + a^\dagger a P_1$) describes a broken SSQM (all the states are two-fold degenerate).

SSQM has been generalized to parasupersymmetric quantum mechanics (PSSQM) of order $p$ by Khare [7]:

$$Q^{p+1} = 0, \quad Q^n \neq 0, \quad n = 1, 2, \ldots, p, \quad (24)$$

and

$$[H, Q] = 0, \quad (25)$$

so that SSQM is PSSQM of order 1. PSSQM can be realized in terms of mutually commuting parafermionic (matrices) and bosonic operators. A property of PSSQM of order $p$ is that the energy levels above the $(p - 1)$th one are $(p + 1)$-fold degenerate. This fact, and the bosonization of SSQM by the $C_{p+1}$-extended oscillator algebra hint at a possibility of describing PSSQM in terms of the generators of the $C_{p+1}$-extended oscillator algebra, whose bosonic Hamiltonian possesses $p + 1$ families of equally spaced eigenvalues, which may be shifted to coincide from some excited state onwards.

Taking $Q_\sigma = \sum_{\nu=0}^p \sigma_\nu a^\dagger P_\nu$, $\sigma_\nu \in \mathbb{C}$, where all the operators belong to the $C_{p+1}$-extended oscillator algebra, Eq. (24) is obtained iff for only one $\nu$, $\sigma_\nu = 0$. Hence good candidates for parasupercures are

$$Q_{\eta,\mu} = \sum_{\nu=1}^p \eta_{\mu+\nu} a^\dagger P_{\mu+\nu}, \quad (27)$$

with $\mu = 0, 1, \ldots, p$, and $\eta_{\mu+\nu} \in \mathbb{C}_0$.

For a given $\mu$, the ansatz

$$H_\mu = \frac{1}{2} \{a^\dagger, a\} + \frac{1}{2} \sum_{\nu=0}^p r_{\nu,\mu} P_\nu, \quad (28)$$

where $r_{\nu,\mu} \in \mathbb{R}$, commutes with $Q_{\eta,\mu}$ iff $[H_\mu, a^\dagger P_{\nu+\mu}] = 0$, $\forall \nu = 1, 2, \ldots, p$, which is equivalent to

$$r_{\nu,\mu} \in \mathbb{R}, \quad r_{\nu+\mu,\mu} = 2 + \alpha_{\mu+\nu} + \alpha_{\mu+\nu+1} + r_{\mu+\nu+1,\mu}, \quad \nu = 1, 2, \ldots, p. \quad (29)$$

It now remains to impose the nonlinear relation (26) between $Q_{\eta,\mu}$ and $H_\mu$, defined in (27) and (28), respectively. Such a relation is valid on every state of the BFB Fock-space representation of the associated $C_{p+1}$-extended oscillator algebra iff

$$\sum_{\nu=0}^{p-1} |\eta_{\mu+\nu+1}|^2 = 2p, \quad (30)$$

$$\sum_{\nu=1}^{p-1} |\eta_{\mu+\nu+1}|^2 \left( \nu + \sum_{\rho=0}^{\nu-1} \alpha_{\mu+\rho+2} \right) = p \left( 1 + \alpha_{\mu+2} + r_{\mu+2,\mu} \right). \quad (31)$$

It can be shown that Khare PSSQM of order $p$ can be bosonized with any $C_{p+1}$-extended oscillator algebra admitting a BFB Fock representation, or in other words that Eqs. (29), (30), and (31) admit solutions for $\eta_{\mu+\nu}$ and $r_{\nu,\mu}$ for any allowed values of the algebra.
parameters. For instance, for the choice $|\eta_{\mu+\nu}|^2 = 2$, $\nu = 1, 2, \ldots, p$, satisfying (30), one finds

$$r_{\mu+2,\mu} = \frac{1}{p} \left\{ (p - 2) \alpha_{\mu+2} + 2 \sum_{\nu=\mu+3}^{\mu+p} (p + \mu - \nu + 1) \alpha_{\nu} + p(p - 2) \right\},$$

(32)

with the remaining $r_{\nu,\mu}$’s taking definite values in terms of $r_{\mu+2,\mu}$ and $\alpha_p$. For $\mu = 0$, PSSQM is unbroken, otherwise it is broken with a $(\mu + 1)$-fold degenerate ground state. All the excited states are $(p + 1)$-fold degenerate. For $\mu = 0, 1, \ldots, p - 2$, the ground state energy may be positive, null, or negative depending on the parameters, whereas for $\mu = p - 1$ or $p$, it is always positive.

In Ref. [1], it was shown that for $p = 2$, Beckers-Debergh PSSQM [8] can only be bosonized with those $C_3$-extended oscillator algebras for which $\alpha_{\mu+2} = -1$. In such a case, Khare (or Rubakov-Spiridonov) PSSQM is simultaneously realized.

## 4 Conclusion

A new deformation of the oscillator algebra, called $C_\lambda$-extended oscillator algebra, has been introduced, and seen to be a generalization of the Calogero-Vasiliev oscillator algebra.

It provides an algebraic realization of supersymmetric cyclic shape invariant potentials. By using its generators, PSSQM of order $\lambda - 1$ can be realized without parafermionic matrices.

Some open and very interesting questions concern the possibilities of representing the algebra in terms of differential operators and providing it with a Hopf structure. Other investigations related to realizations of pseudosupersymmetric and orthosupersymmetric QM are investigated.

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