Influence factors analysis on traffic jam for a new lattice hydrodynamic model on gyroidal road

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Abstract. In the paper, an extended one-dimensional lattice hydrodynamic model is proposed to investigate the related factors on a gyroidal road (i.e. a curved road with a slope). We investigate the related influences on uniform traffic jam analytically and numerically. Based on control theory, condition for no traffic jam is obtained analytically. Finally, simulations are carried out to verify the new lattice hydrodynamic model and numerical simulations are consistent with the theoretical results. Results show that the related factors including the friction coefficient, radius of curvature and slope of a gyroidal road have major effect on the stability of traffic flow respectively.

1. Introduction
Due to the importance of traffic, traffic flow has attracted considerable attention. In order to make a better understanding for traffic flow, various traffic flow models were proposed including continuum models [1-3], car following models [4-6], cellular automaton models [7-8] and lattice hydrodynamic models [9-10] by many scholars with different backgrounds.

Among these models, lattice hydrodynamic model, which was deduced based on the car following models and the continuum models, was firstly put forward by Nagatani. See Ref. [10] for details. Based on the model proposed by Nagatani, many achievements had been reached with the consideration of different factors like anticipation effect [11], driver's characteristics [12], forward and backward looking effect [13] and so on.

In researches of lattice models, people mainly focused on studying traffic characteristics on the normal road which is thought as even (no slope) or straight (no curvature). However, to our knowledge, in real traffic environment, not every traffic road is straight, it may be a curved road or a gradient road. And when turning a corner, the track of vehicles is curved line. Considering the curved road situation, in 2008, Li et al. [14] firstly studied the phase transition on speed limited traffic flow with a slope based on car following model. The next year, Komada et al. [15] discussed the gravitational influence on the traffic flow by considering different slopes on a curved road. Besides, Liang and Xue [16] firstly investigated the motion of a vehicle on a circular road based on a cellular automaton model numerically. In 2012, Zhu and Yu [17] studied the characteristics of traffic flow by taking into account the slope on an uneven road in the car following model. The stability of the traffic flow was analyzed and the KdV equation and modified KdV equation were obtained in order to describe characteristics of traffic flow wave.

Furthermore, the effects of friction coefficient and radius of curvature on traffic flow were analyzed by Zhu et al. [18]. Then, Cao and Shi [19] developed a novel lattice model for curved road by taking the effect of friction coefficient and curvature radius of the curved road based on the lattice...
model proposed by Nagatani. Recently, Zhu and Zhang [20] proposed a novel lattice model with a slope effect. Based on the novel lattice model proposed by Zhu and Zhang, Arvind Kumar Gupta et al. [21] improved the lattice hydrodynamic model on a gradient highway with different slopes by considering the optimal current difference effect. The effect of slope on uphill or downhill highway was studied using linear stability analysis and results showed that the slope significantly affects the stability region on the phase diagram. In 2016, Zhou and Shi [22] investigated the effects of radian and angle going into curved road on traffic flow by using an extended lattice hydrodynamic model; the results showed that these two factors had an important influence on the stability of traffic flow. Then, Zhou et al. [23] extended the lattice model for curved road to two lane-traffic system by incorporating the effect of lane changing.

From the above studies, it can be easily found the effect of slope, radius of curvature play an important role in traffic flow on a curved road. Unfortunately, researchers only considered the curved road situation or gradient condition in the above researches. However, in the real traffic, the road structure not only has curves but also slopes and the effects of these factors on traffic stability are more complex. So it is useful to study the behavior of traffic flow on a curved road with a slope.

2. New lattice hydrodynamic model

Now, we consider the situation in which vehicles run on a single-lane gyroidal road.

Figure 1 shows the illustration of vehicles running on a gyroidal road. A gravitational force and a centripetal force act on the moving cars at the same time on an uphill curved road, where $\theta$ is the slope of the curved road, $g$ represents the gravitational acceleration and $m$ refers to the weight of a vehicle. Moreover, a horizontal force $mg \sin \theta$ and a vertical force $mg \cos \theta$ are working on the running vehicle. Meanwhile a central force $f = \mu mg \cos \theta$ acts upon the vehicle, where $\mu$ is the lateral friction coefficient of the gyroidal road section.

According to the relationship between the arc and the central angle, that is $\Delta x_j(t) = R \Delta \phi_j(t) = 1/\rho_j(t)$. Based on the lattice model proposed by Nagatani [10], the new lattice hydrodynamic model on gyroidal road is given as follows:

$$\rho_j(t+\tau) - \rho_j(t) + \tau \rho_0 (\rho_j R \omega_j - \rho_{j-1} R \omega_{j-1}) = 0$$

(1)

$$\rho_j(t+\tau) R \omega_j(t+\tau) = \rho_0 V(\rho_{j+1})$$

(2)

where $\rho_0$ is the total average density and it is the reciprocal of average headway $\delta$, that is $\rho_0 = 1/\delta$. $\tau$ is the delay time and $\alpha = 1/\tau > 0$ is the sensitivity of a driver; $\rho_{j+1}$ denotes the local
density \(j+1\) at time \(t\). where \(\omega_j(t) = d\varphi_j(t)/dt\) is the angular velocity on site \(j\) at time \(t\). \(V(\rho_j(t))\) is the optimal velocity function at site \(j\) at time \(t\). \(V(\rho_j(t))\) is adopted as follows:

\[
V(\rho_j(t)) = \frac{\theta_{\text{max}}}{2} + \frac{v_{\text{g, max}}}{2} \left[ \tanh \left( \frac{2}{\rho_0} \rho_j(t) - \frac{1}{\rho_0 \rho_j(\theta)} \right) + \tanh \left( \frac{1}{\rho_0 \rho_j(\theta)} \right) \right]
\]

(3)

where \(\theta_{\text{max}}\) is the maximal angular velocity. \(\rho_0(\theta)\) is the inverse of the safety distance on a road on the gyroidal road. ‘-’ represents an uphill and a downhill situation respectively. \(y_s(\theta) = 1/\rho_0(\theta)\) is the safety arc distance of the vehicle as that used in paper [22].

\[ y_s(\theta) = y_s(1 + \alpha \sin \theta) \]

where \(\alpha\) (we take, \(\alpha = 1\)) represents a constant and for simplicity. \(y_s\) is the safety distance on a road without any slope. \(v_{\text{g, max}}\) is the maximal reduced or enhanced velocity as follows

\[ v_{\text{g, max}} = \frac{mg \sin \theta}{\gamma} \]

where \(\gamma\) defines a longitudinal friction coefficient (for simplicity, \(mg/\gamma = 1\)). As we know the angular velocity is the derivative of the radian for the moving vehicle. Based on the centripetal force formula, we have

\[ mR\frac{\omega_{\text{max}}^2}{\rho_0} = \mu mg \cos \theta \]

(4)

From equation (4), we have \(\omega_{\text{max}} = \sqrt{\mu g \cos \theta / R}\). In the real traffic situation, the maximal angular velocity is less than the theoretic values. So we introduce a modified coefficient \(k\) into the maximal velocity formula on a gyroidal road, where \(k(0 < k \leq 1)\) is a constant coefficient. Therefore, the optimal velocity function is adopted as

\[
V(\rho_j(t)) = k_\text{max} \left[ \tanh \left( \frac{2}{\rho_0} \rho_j(t) - \frac{1}{\rho_0 \rho_j(\theta)} \right) + \tanh \left( \frac{1}{\rho_0 \rho_j(\theta)} \right) \right]
\]

(5)

For simplicity, we set \(A(\theta) = k_\text{max} \sqrt{\mu g R \cos \theta + \sin \theta / 2}\),

\[ V_0(\rho_j(t)) = \left[ \tanh \left( \frac{2}{\rho_0} \rho_j(t) - \frac{1}{\rho_0 \rho_j(\theta)} \right) + \tanh \left( \frac{1}{\rho_0 \rho_j(\theta)} \right) \right], \]

and other parameters are the same as the aforementioned and for the sake of simplicity, \(\rho_j(t)\) is simply denoted as \(\rho\). Then, the optimal velocity function is rewritten as \(V(\rho_j(t)) = A(\theta) V_0(\rho_j(t))\) and this function has a turning point at \(\rho = \rho_0(\theta)\) when \(\rho_0 = \rho_0(\theta)\). From equation (5) one can easily find that the maximum velocity of the vehicle varies with the lateral friction coefficients, the radii of the curvature and the slope of the curved road. When the slope of the road is taken as \(\theta = 0^\circ\) the model has the same form as that in paper [19].

3. Stability analysis based on control theory

In this section, based on control theory, the stability condition is carried out for the new lattice model. Assume that the desired density and the flux \(q_j(t) = R \rho_j(t) \omega_j(t)\) of the traffic flow system have steady-state uniform flow solution

\[ [\rho_j(t), q_j(t)] = [\rho^*, q^*] \]

(6)

The linearized system of equations (1-2) around the steady state (6), can be rewritten as

\[ \rho_j^0(t + \tau) - \rho_j^0(t) + \tau \rho_0(q_j^0 - q_{j-1}^0) = 0 \]

(7)
\[ q_j^0(t + \tau) = \rho_0 A(\theta) A_\rho \rho_j^0, \quad (8) \]

where \( A_\rho = \frac{\partial V_0(\rho_{j+1})}{\partial \rho_{j+1}} \mid_{\rho_{j+1} = \rho^*} \), \( \rho_{j+1}^0 = \rho_{j+1} - \rho^* \), \( q_j^0 = q_j - q^* \), \( q_{j+1}^0 = q_{j+1} - q^* \). By performing Laplace transform, we have

\[ e^{\alpha} P_j(s) - P_j(s) + \tau \rho_0 (Q_j(s) - Q_{j-1}(s)) = 0 \quad (9) \]

\[ e^{\alpha} Q_j(s) = \rho_0 A(\theta) A_\rho P_j(s) \quad (10) \]

where \( L(\rho_j) = P_j(s), L(\rho_{j+1}) = P_{j+1}(s), L(q_{j+1}) = Q_{j+1}(s), L(q_j) = Q_j(s) \).\( L(.) \) denotes the Laplace transform. By eliminating density in equations (9) and (10), we can derive the relation between the site \( j+1 \) flux disturbance and the \( j \) flux disturbance as follows:

\[ Q_j(s) = \frac{\tau \rho_0^2 A(\theta) A_\rho}{e^{\alpha}(e^{\alpha} - 1) - \tau \rho_0^2 A(\theta) A_\rho} Q_{j+1}(s) \quad (11) \]

Taking \( e^{\alpha} = 1 + \frac{3}{2}(\pi)^2 + \frac{3}{6}, \) and inserting into equation (11), the transfer function is obtained as follows:

\[ G(s) = \frac{\tau \rho_0^2 A(\theta) A_\rho}{3/2(\pi)^2 + \pi - \tau \rho_0^2 A(\theta) A_\rho}. \]

The characteristic polynomial \( d(s) \) is equal to \( 3/2(\pi)^2 + \pi - \tau \rho_0^2 A(\theta) A_\rho \). The traffic jam never occurs in the lattice model if the characteristic polynomial \( d(s) \) is stable and \( \|G(s)\|_\infty \leq 1 \). According to the Hurwitz stability criterion, the polynomial is stable if all the coefficients of polynomial have the same sign. Therefore, the polynomial \( d(s) \) will be stable if \( A_\rho < 0 \). Considering the following relationship \( \|G(s)\|_\infty = \sup_{\omega \in [0, \infty)} |G(j \omega)| \),

\[ |G(j \omega)| = \sqrt{G(j \omega) \cdot G(-j \omega)} = \sqrt{\frac{(-\tau \rho_0^2 A(\theta) A_\rho)^2}{(-3/2(\pi \omega)^2 - \tau \rho_0^2 A(\theta) A_\rho)^2 + (\pi \omega)^2}}, \omega \in [0, \infty) \quad (12) \]

We set \( g(\omega) = \frac{(-\tau \rho_0^2 A(\theta) A_\rho)^2}{(-3/2(\pi \omega)^2 - \tau \rho_0^2 A(\theta) A_\rho)^2 + (\pi \omega)^2} \). Obviously, \( g(0) = 1 \), then we should let

\[ g(\omega) \leq 1, \omega \in (0, \infty) \] and one obtains,

\[ 9/4(\pi \omega)^2 + 3\tau \rho_0^2 A(\theta) A_\rho + 1 \geq 0. \]

It can be easily found \( A_\rho < 0 \) and the satisfied condition can be represented as follows:

\[ \tau < \frac{1}{-3\rho_0^2 A(\theta) A_\rho} = \frac{2}{-3\rho_0^2 A_{\rho}(k \sqrt{\mu g R \cos \theta \mp \sin \theta})} \quad (13) \]

or

\[ a > \frac{1}{-3\rho_0^2 A(\theta) A_\rho} = \frac{2}{-3\rho_0^2 A_{\rho}(k \sqrt{\mu g R \cos \theta \mp \sin \theta})} \quad (14) \]

Thus, the neutral stability curve is given by

\[ \tau_c = \frac{1}{a_c} = \frac{2}{-3\rho_0^2 A_{\rho}(k \sqrt{\mu g R \cos \theta \mp \sin \theta})} \quad (15) \]

Equation (13) clearly shows that the slope \( \theta \) on a gyroidal road plays an important role on the stability of traffic flow on a single lane. When \( \theta = 0^\circ \), the result of stability condition is the same as that of Ref. [23].
Figure 2. The neutral stability curves with different slopes in two situations: pattern (a)-uphill situation and pattern (b)-downhill situation.

Curves in Figure 2(a) and 2(b) are the neutral stability lines with different value of $\theta$. Patterns (a) and (b) are corresponding to uphill and downhill situations, respectively. The dimensionless parameters are set $a=1$, $\rho_c=0.4$, $k=0.14$, $R=120\cos\theta^o$, $\mu=0.3$ and $g=10$. The apex of each curve indicates the critical point $(\rho_c, a_c)$. The phase plane is divided into two regions: stable and unstable. The areas above the neutral stability lines are stable regions, and below these curves are unstable regions.

In Figure 2(a) and 2(b), curves coincide perfectly with each other, for instance $\rho=90/\cos4^o$, $\mu=0.8$ and $\rho=120/\cos4^o$, $\mu=0.6$. Put another way, two curves with the same parameters $\mu, \rho$ are the same. It is depicted in Figure 2(a) and 2(b) that in an uphill situation the stable region increases with an increase of the slopes which means that larger value of the slope leads to enlargement of stability region. But in the downhill situation, stable region decreases with the increase of slopes.

Figure 3 shows the neutral stability lines for different friction coefficients and radii of curved roads in two situations (patterns (a) and (b) are corresponding to an uphill and a downhill situation respectively) when $\rho_c=0.4$, $k=0.14$, $R=120/\cos4^o$. The varying tendencies of the stable area with these two variable parameters are identical. Figure 3 shows four curves with different frictions and radii of curvature at the same time. From Figure 3, it can be concluded that with an increase of friction or radius, the stability of the traffic flow decreases both in an uphill situation and in a downhill situation. Traffic flow is more stable for a shorter radius of curved road or smaller friction coefficients. Due to the increase of the radii of curved road and friction coefficient, the maximum velocity becomes higher. The increase of maximum velocity leads to the decrease in stability for traffic flow.
Figure 3. The neutral stability curves for different friction coefficients and radii of curvature in two situations: pattern (a)-uphill situation and pattern (b)-downhill situation.

4. Summary
In the paper, we have proposed a new lattice hydrodynamic model of traffic flow on a gyroidal road by considering the current difference effect. Based on the control theory, the stability of the new model was obtained. According to the analytical results, we found that stability condition for traffic flow on a gyroidal road was influenced by the friction coefficient, radius of curvature, slope of curved road, respectively. With an increase of the slopes, the stability of traffic flow increases in an uphill situation and decreases in a downhill situation when other three factors are invariant. While with an increase of friction coefficient or radius of curvature the stability of the traffic flow decreases both in an uphill situation and in a downhill situation.

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