Conformal Transformations as Observables

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Abstract

$C$ denotes either the conformal group in $3 + 1$ dimensions, $PSO(4, 2)$, or in one chiral dimension, $PSL(2, \mathbb{R})$. Let $U$ be a unitary, strongly continuous representation of $C$ satisfying the spectrum condition and inducing, by its adjoint action, automorphisms of a v.Neumann algebra $A$. We construct the unique inner representation $U^A$ of the universal covering group of $C$ implementing these automorphisms. $U^A$ satisfies the spectrum condition and acts trivially on any $U$-invariant vector.

This means in particular: Conformal transformations of a field theory having positive energy are weak limit points of local observables. Some immediate implications for chiral subnets are given. We propose the name “Borchers-Sugawara construction”.

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1 Introduction

Space-time symmetries are of paramount importance to relativistic quantum field theory. Intuitively we expect such coordinate transformations to be connected to observables. Time translations, for example, should be observable due to their connection with the energy operator. If we have a stress energy tensor in the theory, as it is often the case in models, the energy operator itself is given as an integral of this local quantum field. Yet, the implementation of covariance may be given in abstract terms or may stem from a larger theory into which the theory of interest is embedded (eg Coset models), and it is not always manifest how covariance may be implemented by observables of the subtheory.

More specifically, as a fact of life any observation is of finite extension in space and time and thus we regard the local observables as the constituting objects in quantum field theory. For this reason we shall work with the v.Neumann algebra $A$ which is generated by all local observables, in accordance with the principles given by Haag, Kastler [HK64], and others (cf. [Haa92]). Thereby our setting also includes quantum field theories which are not necessarily described completely by covariant quantum fields, and which
might not possess a stress energy tensor. In fact, the main result is an abstract statement about v. Neumann algebras, without reference to the local structure of a quantum field theory.

We consider representations of such theories which admit a unitary implementation of covariance and the task thus amounts to a search for observable, unitary, implementing operators. Quite obviously these operators can not be local observables, since locality implies that adjoint action of these operators is trivial on algebras which are associated with causally disconnected regions. On the other hand we believe any observation has to be local in nature and we conclude: space-time transformations should be non-local limits of local observables. We take this as definition of global observables.

The problem of identifying space-time symmetry transformations as global observables is of any interest only, if the given representation is reducible. In irreducible representations, such as the vacuum representation, any bounded operator can be represented as a weak limit of local operators. The representations we have in mind are manifestly reducible and the innerness of implementing operators in the global sense promises to be of some use in these circumstances. We return to this point in the latter part of this work.

To our knowledge this problem so far has been dealt with only in the case of abelian groups of translations satisfying the spectrum condition (positivity of energy). Borchers [Bor66] solved this problem relying almost entirely on the spectrum condition and using a deep result on the innerness of norm-continuous connected automorphism groups of v. Neumann algebras [KR67] (Corollary 8). His result is the key building block in our work.

In the abelian case there are many inner implementing representations with different spectral properties. It was a challenging task to ensure existence of an inner implementing representation satisfying the spectrum condition. Arveson [Arv74] gave a proof for a one-parameter group, Borchers and Buchholz [Bor87] (and references therein) succeeded in solving this problem in general.

In this respect the situation for an inner implementing representation of $C$ is different. Because $C$ is identical with its commutator subgroup, the result of our construction is unique and validity of the spectrum condition follows. We show as well that $U$-invariant vectors are left invariant by the action of the inner implementing representation $U^A$. Another result is the proof of complete reducibility of $U^A$ making weak assumptions on the original representation $U$.

In the course of our argument we will construct an inner implementing representation $U^{A'}$ for the commutant of the v. Neumann algebra $A$ as well. We have the following relation: $U(g) = U^A(g) U^{A'}(g), \quad \forall g \in C$. This equation reminds of the Coset construction [GKO86] involving stress energy tensors of chiral current algebras, which are given by the Sugawara construction [Sug68]. It is not difficult to show that our result agrees with the outcome of integrating the respective stress energy tensors.

At this point we stress that, although the relation to Coset constructions as considered by Goddard, Kent and Olive [GKO86] motivated this work, our result is independent of the existence of a stress energy tensor. We make use of this and connect it to a generalised notion of Coset construction, which we discuss briefly.
We make some immediate remarks relating our construction to chiral subtheories, general chiral Coset theories and conformal inclusions in the latter part of this work. We think this gives sufficient evidence for the relevance of an inner implementing representation in studies on chiral theories. We hope it will serve well as a substitute for the Sugawara stress energy tensor in many respects, although there are special features of an inner implementing representation connected to a stress energy tensor. On the other hand we believe our construction is somewhat special to (chiral) conformal field theories as we argue in the discussion concluding this article, and we know that the deeper part of it is due to Borchers. Summing up these thoughts we consider the term “Borchers-Sugawara construction” appropriate.

We treat the cases $C = PSO(4, 2)$ and $C = PSL(2, \mathbb{R})$ explicitly, because detailed results on these groups are readily available. We expect our results to hold true for all conformal groups since we make use of typical features of conformal groups only.

## 2 Preparations and first remarks

We deal with the conformal group in $1 + 3$ dimensions ($PSO(4, 2)$) and in one chiral dimension ($PSL(2, \mathbb{R})$). Since both groups share all the features required here, the symbol $C$ will denote both of them in the following. For geometrical interpretation and some general facts on $C$ we refer to [LM75][Mac77][BGL93][GF93].

We use the symbol $\tilde{C}$ for the universal covering group of $C$ and $p$ for the covering projection from $\tilde{C}$ onto $C$. The following subgroups of $C$ will occur frequently: The group of translations, $T$, of special conformal transformations, $S$, the group of global scaling by a factor $\lambda \in \mathbb{R}_+ \setminus \{0\}$, $D$, and the group of “conformal time” translations, $R$, which is generated by the conformal Hamiltonian. The corresponding subgroups of $\tilde{C}$ will be denoted by $\tilde{T}$, $\tilde{S}$, $\tilde{D}$, $\tilde{R}$. We use parameters on $R$ which make it naturally isomorphic to $\mathbb{R}/2\pi\mathbb{Z}$.

By these conventions we have $R(2\pi) = id$ and the following relation between the generator of rotations, $H$, the generator of “physical time” translations, $P_0$, and the generator of special conformal transformations in direction of “physical time”, $K_0$:

$$2H = P_0 - K_0$$

(1)

In the following $\mathcal{H}$ always stands for a separable HILBERT space, $U$ and $\tilde{U}$ are unitary, strongly continuous representations of $C$, $\tilde{C}$ on $\mathcal{H}$, respectively. We use the physicists’ convention on the abstract LIE algebra of $C$ and will not distinguish between elements of the abstract LIE algebra and corresponding selfadjoint generators of unitary, strongly continuous representations, since this leads to no ambiguities. If not stated otherwise $\mathcal{A}$ stands for a v.NEUMANN algebra of operators on $\mathcal{H}$, $\mathcal{A}'$ for its commutant and $\alpha, \alpha'$ for automorphic actions of $C$ on $\mathcal{A}, \mathcal{A}'$ respectively. We note that any spatial automorphism of $\mathcal{A}$, given by the adjoint action of a unitary operator, induces a spatial automorphism of $\mathcal{A}'$ as well.

We prove a lemma on the spectrum condition first. The result is well known (see e.g [GL96], Lemma B.4) and our proof is not new, presumably, but to our knowledge not
yet accessible in the literature. The argument is short and straightforward; its second part is adapted from [Mac77]. Afterwards we prove uniqueness of the inner implementing representation.

Proposition 1 If any one of the operators \(H, P_0, -K_0\) has positive spectrum, then all three of them. In this case we say that \(\tilde{U}\) satisfies the spectrum condition.

Proof: Assume \(H\) is positive. Take any vector \(\phi\) analytic for the representation \(\tilde{U}\) (cf. eg [BR77]). We have:

\[
0 \leq 2\langle \phi, \tilde{U}(\tilde{D}(\lambda))H\tilde{U}(\tilde{D}(\lambda))^\ast\phi \rangle = \lambda^2 \langle \phi, P_0\phi \rangle + \lambda^{-2} \langle \phi, -K_0\phi \rangle
\]  

(2)

Multiplying by \(\lambda^{\pm 2}\) and taking the appropriate limits \(\lambda \to 0, \infty\) we deduce \(\omega_\phi(P_0) \geq 0\) and \(\omega_\phi(-K_0) \geq 0\). Since the analytic vectors for the representation \(\tilde{U}\) form a core for all generators we may apply criterion 5.6.21 of [KR83].

Now assume \(P_0\) or \(-K_0\) is positive. Special conformal transformations and translations are conjugate in \(\tilde{C}\): \(S(-n) = R(\pi)T(n)R(-\pi)\). Defining \(g_t\) as \(S(n)R(t)T(n)R(-t)\) this identity becomes: \(\lim_{t \to \pi} g_t = id\). Now we see that the corresponding holds true in \(\tilde{C}\), since we know it for \(C\), the relation is continuous in \(n\), and the covering projection is continuous as well. Because conjugation by a unitary operator does not change the spectrum, positivity of \(P_0\) follows from positivity of \(-K_0\) and vice versa. Positivity of \(H\) follows from equation \([\text{1}]\) by criterion 5.6.21 of [KR83] applied as before while discussing equation \([\text{2}]\).

\(\Box\)

Proposition 2 Assume \(\text{Ad}_U\) induces an automorphism group \(\alpha\) on \(A\). If there exists a representation \(U^A\) of \(\tilde{C}\) by unitary operators in \(A\) implementing \(\alpha\) by its adjoint action on \(A\), then this representation is unique.

Proof: Assume there are two such representations, \(U^A_1\) and \(U^A_2\). Then the operators \(U^A_1(g)U^A_2(g)^*\), \(g \in \tilde{C}\), implement the trivial automorphism. For this reason these operators belong to the centre of \(A\). Using this fact it is straightforward to show that the operators \(U^A_1(g)U^A_2(g)^*\) form a representation of \(\tilde{C}\). This representation is abelian and its kernel contains all elements of the form \(g_1g_2g_1^{-1}g_2^{-1}\). Now these elements generate the whole of \(\tilde{C}\) since \(\tilde{C}\) has a simple Lie algebra. Thereby \(U^A_1(g)U^A_2(g)^* = 1\ \forall g \in \tilde{C}\).

\(\Box\)

We call a representation \(U^A\) in the sense of the proposition above an inner implementing representation (corresponding to the pair \((U, A)\)). We immediately have:

Proposition 3 Assume the unique inner implementing representation \(U^A\) to exist. Then \(U^A \equiv U^A\cdot U\cdot \text{op}\) is the unique inner implementing representation corresponding to \((U, A')\). If \(U^A\) is strongly continuous, then so is \(U^A'\).

Proof: First we prove innerness of the operators \(U(g)U^A(g)^*\) by recognising that their adjoint action on \(A\) implements the trivial automorphism. Making use of this it is straightforward to show that these operators do in fact define a representation. The implementation property and unitarity is trivial. Uniqueness follows from proposition \([\text{4}]\) directly.
Continuity is fulfilled, since we are multiplying continuous functions.

3 Realising the construction

This section contains the derivation of our main result. We depend on the following statement:

**Lemma 4** Let $U$ satisfy the spectrum condition and let $\text{Ad}_U$ induce an automorphism group $\alpha$ of $\mathcal{A}$. Then there are strongly continuous, unitary, inner implementing representations $T^A$, $S^A$ for the restrictions of $\alpha$ to the one parameter subgroups of translations and special conformal transformations, respectively.

**Proof:** This is an application of Borchers’ theorem [Bor66] and proposition 1.

At this point we stress that it is not clear at all whether these restricted inner implementing groups form a representation of $\tilde{C}$. We will show that the inner implementing representation may be constructed from any given pair $T^A$, $S^A$. Translations and special conformal transformations together generate the whole of $\tilde{C}$.

**Theorem 5 (main theorem)** Let $U$ be a unitary, strongly continuous representation of $C$ on a separable Hilbert space $\mathcal{H}$ satisfying the spectrum condition, $\mathcal{A}$ a v.Neumann algebra of bounded operators on $\mathcal{H}$. Assume that the adjoint actions of $U$ on $\mathcal{A}$, $\mathcal{A}'$ define groups $\alpha$, $\alpha'$ of automorphisms of $\mathcal{A}$, $\mathcal{A}'$, respectively.

Then there exist unique unitary, strongly continuous, inner implementing representations $U^A$, $U^{A'} \equiv U^{A*} \cdot U \circ p$ of $\tilde{C}$.

**Proof:** We follow arguments given in [BGL95] and look at the unitary group $G$ generated algebraically by the operators $T^A$, $S^A$ of lemma 1. We define for any non trivial relation $\prod_i T^A(x_i)S^A(n_i) = \mathbb{1}$ a corresponding element: $g_\pi := \prod_i T(x_i)S(n_i)$. By the implementation property of lemma 1 we have $\alpha_{g_\pi}(A) = A$ for all $A \in \mathcal{A}$. The elements $g \in C$ having trivial automorphic action $\alpha_g$ on $\mathcal{A}$ form a normal subgroup. But since $C$ has trivial centre ([Mac77], direct calculation on $PSL(2, \mathbb{R})$) and simple Lie algebra it is simple as a group and, therefore, we have $g_\pi = id$.

Thus the mapping $\phi : G \to C$ defined by $T^A(x_i) \mapsto T(x_i)$, $S^A(n_i) \mapsto (n_i)$ extends to a surjective homomorphism as translations and special conformal transformations generate $C$. Now we look at the kernel of $\phi$, $\ker_\phi$, and take arbitrary $V \in \ker_\phi$. Then we have $VAV^* = \alpha_{\phi(V)}(A) = A$ for all $A \in \mathcal{A}$. This implies that $\ker_\phi$ is a central subgroup of $G$.

\footnote{See eg [BGL93] (proposition 1.6), for $PSL(2, \mathbb{R})$ use the Iwasawa decomposition [GF93] and do straight forward calculations.}
Therefore we have the following exact sequence, which defines a central extension of \( C \) by \( \ker \phi \):

\[
\text{id} \rightarrow \ker \phi \rightarrow G \rightarrow C \rightarrow \text{id}
\]

Now we know that \( G \) is a “weak Lie extension” of \( C \) in the sense of [BGL95]. \( C \) has a simple Lie algebra and because of this it is identical with its commutator subgroup and has vanishing second cohomology. By the same argument as for the proof of corollary 1.8 [BGL95] we have: there is a unitary, strongly continuous representation \( U^A \) of \( \tilde{C} \) such that \( \phi \circ U^A = p \). In particular \( U^A \) is inner and implementing.

The remainder follows by proposition 3.

\[\Box\]

Two Remarks: Since we start with a proper representation \( U \) of \( C \), the cocycles of \( U^A \), \( U^{A'} \) as (generalised) ray representations of \( C \) have to be mutually inverse, and common eigenvectors of \( H^A \), \( H^{A'} \) have eigenvalues which sum up to integers.

In [Kös02] an alternative derivation was given for \( PSL(2, \mathbb{R}) \), which applies to representations \( \tilde{U} \) of \( \tilde{C} \) instead of representations \( U \) of \( C \) as well. An explicit continuous mapping from \( \tilde{C} \) into the group of unitaries of \( \mathcal{A} \) is given there, which yields implementers of the automorphic action of \( \tilde{C} \) on \( \mathcal{A} \). These implementers thus form a (generalised) ray representation of \( \tilde{C} \), which can be lifted to a proper representation of \( \tilde{C} \). This approach is complementary to the one used here and agrees with the one used by Buchholz et al. [BDFS00] (appendix) for deriving a representation of the Poincaré group from modular conjugations of wedge algebras.

4 Examining the result

In this section we derive three features of the inner implementing representations which they inherit from the original representation: spectrum condition, invariant vectors, complete reducibility. We consider them in this order.

Corollary 6 Both \( U^A \) and \( U^{A'} \) satisfy the spectrum condition.

Proof: The operators \( U^{A\lor A'}(g, h) := U^A(g)U^{A'}(h) \) define a unitary, strongly continuous representation of \( \tilde{C} \times \tilde{C} \). With respect to \( U^{A\lor A'} \) we have a dense domain of analytic vectors and we take an arbitrary vector \( \psi \) from it. The result follows now as in the proof of proposition [3] from the inequality \( 0 \leq \langle U^A(\tilde{D}(\lambda))^*\psi, P_0U^A(\tilde{D}(\lambda))^*\psi \rangle = \langle \psi, \lambda^2P^A_0\psi \rangle + \langle \psi, P^{A'}_0\psi \rangle \).

\[\Box\]

Corollary 7 Let \( \mathcal{H} \ni \Omega \) be a vector left invariant by \( U \). Then \( U^A \), \( U^{A'} \) both leave \( \Omega \) invariant.

Proof: Since translations and special conformal transformations generate the whole of \( \tilde{C} \) it is sufficient to show invariance of \( \Omega \) for these two subgroups. We consider translations only; the argument for special conformal transformations is the same. We may specialise
further to translations by \(tx, x \in \mathbb{R}^{d+1}, x^2 > 0, t \in \mathbb{R}\), since any vector in spacetime may be represented as difference of two timelike vectors. The generator of translations in direction \(x\) is positive by the spectrum condition. The following argument applies in the case \(C = PSL(2, \mathbb{R})\) directly.

Take arbitrary \(\psi \in \mathcal{H}\). We have \(\langle \psi, U^A(g)\Omega \rangle = \langle \psi, U^A'(g)^*\Omega \rangle\) by assumption. Set \(f_\psi(t) := \langle \psi, U^A(\tilde{T}_x(t))\Omega \rangle\), \(g_\psi(t) := \langle \psi, U^A(\tilde{T}_x(t))^*\Omega \rangle\). Due to the spectrum condition (corollary 6) \(f_\psi\) may be extended to the upper half of the complex plane by means of the Laplace transform (cf. eg [SW64], chapter 2). This continuation is analytic in the interior and of at most polynomial growth for complex arguments. On the real line we have \(|f_\psi| \leq \|\Omega\|\|\psi\|\) and due to the theorem of PHRAGMEN-LINDELÖF [Tit39] (section 5.62) this bound holds true for the continuation of \(f_\psi\) as well.

The same line of argument works for \(g_\psi\) with respect to the lower half of the complex plane. Since \(f_\psi\) and \(g_\psi\) coincide on the real line both are restrictions of an entire function (reflection principle). This entire function is bounded by \(|\Omega||\psi|\), and due to LIOUVILLE’s theorem it is constant. Since the vectors \(U^A(\tilde{T}_x(t))\Omega, U^A(\tilde{T}_x(t))^*\Omega\) are determined by the scalar products \(f_\psi(t)\) and \(g_\psi(t), \psi \in \mathcal{H}\), invariance follows by taking \(t = 0\). □

For the next corollary we prepare ourselves by a lemma and a comment. In the corollary the representation \(U\) is assumed completely reducible with finite multiplicities. Although this is a pretty strong assumption in group theoretical terms, we consider this a rather natural assumption from the quantum field theoretical point of view. In this context it is somewhat weaker than a common nuclearity condition [BGL93]. Nuclearity is desirable for quantum field theories and in our setting it corresponds to demanding the \(H\) eigenspaces to be finite dimensional with degeneracies growing at most exponentially. Typical (integrable) chiral models such as current algebras exhibit this behaviour (cf. eg [GF93]). This implies our assumption as the following lemma clarifies.

\(\tilde{R}(2\pi)\) generates an infinite cyclic group contained in the centre of \(\tilde{C}\). The following lemma shows that complete reducibility of a representation \(\tilde{U}\) of \(\tilde{C}\) satisfying the spectrum condition is equivalent to requiring the representation space to have a decomposition into a direct sum of eigenspaces of \(\tilde{R}(2\pi)\). Due to the infinite order of the central subgroup generated by \(\tilde{R}(2\pi)\) this is not obvious.

**Lemma 8** Assume the spectrum of \(\tilde{U}(\tilde{R}(2\pi))\) to be pure point. Then the spectrum of \(H\) is pure point and \(\tilde{U}\) is completely reducible into a direct sum of irreducible representations.

**Proof:** Let \(\mathcal{H}_i\) denote the eigenspace belonging to eigenvalue \(e^{i2\pi \lambda_i}\). The restriction of \(\tilde{U}(\tilde{R}(t))e^{-i\lambda_i t}\) to \(\mathcal{H}_i\) defines a representation of \(\tilde{U}(1)\). This representation is completely reducible due to the compactness of \(\tilde{U}(1)\) (cf. eg [BR77]). This proves the claim on the spectrum of \(H\).

By the spectrum condition there are vectors of lowest eigenvalue. By the complete analysis of lowest weights in unitary representations of \(\tilde{C}\) [Mac77, Gri93], it is known which
d\footnote{For \(C = P SO(4, 2)\) there is an additional \(\mathbb{Z}_2\) contained in the centre [Mac77], but this poses no problem.}
lowest eigenvalues may occur and that the cyclic representations generated from these
lowest weight vectors are irreducible. Taking such a lowest weight vector, applying to it
the linear span of the $\tilde{U}(g)$, $g \in \tilde{C}$, and taking the completion thus yields an irreducible
representation space. We may reduce by it because of unitarity. We iterate this procedure
and arrive at the second claim since $\mathcal{H}$ is separable.

\[ \square \]

**Corollary 9** Assume $U$ to be completely reducible with finite multiplicities. Then $U^A$
and $U^A'$ are completely reducible.

**Proof:** Denote the lowest weight vectors by $\varphi_{(d,i)}$, $i$ being the multiplicity index and $d$
the eigenvalue of $H$. For any fixed $d$ the $\varphi_{(d,i)}$ span a finite dimensional HILBERT space.
This space is left invariant by the operators $U^A(\tilde{R}(2\pi)), U^A'(\tilde{R}(2\pi))$. Both operators may
be diagonalised on this space simultaneously, the result being a mere relabelling of the
irreducible subrepresentations of $U$. Now $U^A(\tilde{R}(2\pi)), U^A'(\tilde{R}(2\pi))$ both are diagonal on the irreducible subspaces generated from the “new” lowest weight vectors $\varphi'_{(d,i)}$ and thus
on the whole of $\mathcal{H}$. Now the claim follows as in the proof of lemma 8.

\[ \square \]

**Remark:** Nontrivial unitary representations of $\tilde{C}$ are necessarily infinite dimensional and
the multipicity spaces of $U^A$ serve as representation spaces for $U^A'$ and vice versa. The irreducible representations of $U^A$ and $U^A'$ will, therefore, not have finite multiplicities in general.

## 5 Applications to chiral subnets

In this subsection we gather a few immediate implications of the BORCHERS- SUGAWARA construction for chiral subnets. We denote by $B$ a chiral conformal precosheaf in its vacuum representation satisfying common assumptions and properties as given in [GL96]. The symbol $I$ stands for proper intervals contained in $S^1$. Although the mapping $I \rightarrow B(I)$ does not define a net in the proper sense of the term, we will use this term as we want to stress the relation of these models to the concept of local quantum field theories given usually by nets of local algebras.

We consider a chiral subnet $A$ of $B$. The local algebras of $A$ are contained in the ones of $B$ and $A$ satisfies the same assumptions as $B$ except cyclicity of the vacuum. Properties of local algebras $A(I)$ such as weak additivity or factor property can be proved on the basis of modular invariance of $A(I) \subset B(I)$ [Bor00] (lemma VI.1.2.(4.)). The symbol $A$ also denotes the v.NEUMANN algebra generated by all local algebras of the net $A$. Thus all prerequisites for the BORCHERS-SUGAWARA construction are at our disposal. Furthermore we know that for $A \subseteq B$ the projection onto the cyclic subspace associated to $A$ and the vacuum $\Omega$ is not $\mathbb{1}$ (modular covariance of $A$, cf. eg [Bor00]) and it is contained in $A'$ by the REEH-SCHLIEIDER- theorem. This implies that the representation of $A$ is manifestly reducible and the application of the BORCHERS-SUGAWARA construction is not in vain. We collect a few consequences for any chiral subnet first:
Proposition 10 The inner implementing unitaries $U^A(g) \neq 1$ are not elements of any local algebra. $A$ contains non-trivial non-local operators, the vacuum is not faithful for $A$, and the action of $Ad_{U^A}$ on the local operators of the net $A$ is ergodic, if $A \neq \mathbb{C}$. 

Proof: Suppose for some $g \in \widetilde{C}$ the unitary $U^A(g) \neq 1$ is contained in a local algebra. By locality and invariance of the vacuum there is a local algebra $B(I)$ such that all vectors $B\Omega, B \in B(I)$, remain unchanged when acted upon by $U^A(g)$. Thus, by the Reeh-Schlieder property of $B$, $U^A(g)$ has to be trivial and the existence of such operators is denied.

The kernel of $U^A$ has to be different from $\widetilde{C}$, else $A$ is left invariant pointwise by the covariance automorphisms and therefore must be abelian by locality. But local algebras of $A$ have to be factors as elements of a chiral subnet. So $A \neq \mathbb{C}$ requires the existence of operators $U^A(g) \neq 1$. These are not local operators.

Any fixed point of the action of $Ad_{U^A}$ on a local algebra $A(I)$ has to be contained in its centre due to locality. This centre is trivial since $A(I)$ is a factor. $\Omega$ can not be separating, because we have: $(U^A(g) - 1)\Omega = 0$.

Now we discuss the relevance of the Borchers-Sugawara construction in studies on chiral subnets and the relation to the Sugawara construction to some extent. To this end we make some simple considerations on Coset theories.

In a large class of chiral conformal models such as free fermions and chiral current algebras there are explicit constructions for the transformation operators as observables in terms of local quantum fields (cf. eg [FST89]). In both cases the construction yields a representation of the whole Virasoro algebra. For chiral current algebras the construction was given by Sugawara [Sug68] up to a numerical factor. This diffeomorphism invariance is broken in any positive energy representation necessarily; it remains a $\widetilde{C}$ symmetry only.

We have constructed the inner implementation of this remaining symmetry in a completely model independent way. Results of Rehren [Reh00] indicate this inner implementing representation might play a central role in studies on chiral subnets. One situation of particular interest arises if one considers the set of algebras defined by the local relative commutants $C_I := A(I)' \cap B(I)$ of a subnet $A \subset B$. If one can show this set to satisfy isotony, it is in fact a chiral subnet $C \subset B$ itself. We define a Coset theory $C$ associated to a subnet $A \subset B$ to be a chiral subnet $C \subset B$ satisfying $C(I) \subset C_I$. A simple argument leads to the following lemma:

Lemma 11 The maximal Coset theory associated to a subnet $A \subset B$ is defined by $C_{\text{max}}(I) := \{U^A(g), g \in \widetilde{C}\}' \cap B(I)$. It satisfies $C_{\text{max}}(I) = A(I)' \cap B(I)$ as well.

Proof: Obviously this definition yields a subnet $C_{\text{max}} \subset B$. Since the operators of a local algebra of $C_{\text{max}}$ commute with the inner implementation of $A$, we deduce from locality of $B$ that $C_{\text{max}}$ is in fact a Coset theory.

\footnote{For chiral current subalgebras isotony for the local relative commutants (ie the property $C_I \subset C_J$ for $I \subset J$) follows from strong additivity (“v.Neumann density”) in positive energy representations (cf. [Lar97], corollary 1.3.3).}
Let $\mathcal{C}$ be any Coset theory, $I, J$ proper intervals satisfying $I \subset J$ and $I' \cup J = S^1$. By isotony of $\mathcal{C}$, locality and weak additivity for chiral subnets we have: $\mathcal{C}(I) \subset (\mathcal{A}(I') \lor \mathcal{A}(J))' = \mathcal{A}' \subset \{U^A(g), g \in \tilde{C}\}'$.

While the global algebra $\mathcal{A}$ might be a fairly intractable object, the representing operators have a lot of well known features. Therefore the characterisation given above may prove useful. Certainly $U^A$ implements covariance on any Coset theory, but it is not obvious whether $U^A$ itself is contained in the global algebra $\mathcal{C}_{max}$.

It might happen that a subnet $\mathcal{A} \subset \mathcal{B}$ admits no Coset theory at all, i.e. $\mathcal{C}_{max}(I) = \mathbb{C}\mathbb{1}$. In this case we call $\mathcal{A} \subset \mathcal{B}$ a conformal inclusion. This term stems from studies on chiral current algebras. Here we have for both nets $\mathcal{A} \subset \mathcal{B}$ stress energy tensors $\Theta^\mathcal{A}, \Theta^\mathcal{B}$. A simple argument shows that their difference $\Theta^\mathcal{B} - \Theta^\mathcal{A} \equiv \Theta^{coset}$ is a stress energy tensor alike. By the Reeh-Schlieder theorem and the Lüscher-Mack theorem $[FST89]$ $\Theta^{coset}$ vanishes iff its central charge vanishes. Its central charge is completely determined by the finite dimensional Lie algebras from which the current algebras are constructed and by the embedding of the smaller one into the larger one. Its zeros, characterising the notion of conformal embeddings for these models, have been classified $[SW86][BB87]$. The following proposition shows that our definition covers these as special cases.

**Proposition 12** Suppose the inner implementing representation of theorem $[\text{?}]$ for a chiral subnet $\mathcal{A} \subset \mathcal{B}$ satisfies $U = U^A$. Then $\mathcal{A} \subset \mathcal{B}$ is conformal.

**Proof:** By assumption we have $U^A = \mathbb{1}$. Since $U^A$ implements covariance on any Coset theory, the local algebras of $\mathcal{C}_{max}$ have to be trivial by the reasoning given in the proof to proposition $[\text{?}]$.

While given $\mathcal{A}$ and $U$ the inner implementing representation $U^A$ is unique, $U^A$ does not determine the subnet $\mathcal{A} \subset \mathcal{B}$, as examples of conformal embeddings show. In general there will be a lot of subnets transforming covariantly under the action of $U^A$ (transformation property) and a lot of subnets containing the operators of $U^A$ as global observables (generating property). Generically there will be no simple relation such as inclusion or commutativity etc for any pair $\mathcal{M}_1, \mathcal{M}_2$ of chiral subnets having one or both properties. There is, of course, a maximal subnet transforming covariantly and having the generating property. It is given by: $\mathcal{A}_{max}(I) := \{U^A'(g), g \in \tilde{C}\}' \cap \mathcal{B}(I)$. Any subnet $\mathcal{A}$ having both properties defines a conformal inclusion $\mathcal{A} \subset \mathcal{A}_{max}$. Since studies on conformal inclusions form an area of research of their own, $\mathcal{A}_{max}$ should be a generic object to explore.

### 6 Discussion

We have presented a construction applying and generalising the result of Borchers $[\text{Bor66}]$. The result coincides with the corresponding structure in special cases in which there is a stress energy tensor. In particular it generalises, within its natural limits, the Sugawara construction $[\text{Sug68}]$. We have proposed the name “Borchers-Sugawara construction” because of these relations. The construction is completely model
independent and does not require existence of a stress energy tensor. We expect special features of an inner implementing representation connected to a stress energy tensor. This is subject to work in progress.

It is natural to ask if this construction may be applied to other space-time symmetry groups. In our view the key tools in our construction are the following: the original representation satisfies the spectrum condition for some translation subgroups. There are sufficiently many of them to generate the whole group and we have an argument how to derive a representation of the covering group from the unitary group generated by operators constructed by means of Borchers’ theorem.

We have not examined applicability of our strategy to other cases in any detail, but we want to comment on the Poincaré group. Here the translations usually satisfy the spectrum condition. Unfortunately, so to say, they form an invariant subgroup and although one is tempted to generate the group from $PSL(2,\mathbb{R})$ subgroups (as eg in [KW01]) this seems impossible with subgroups satisfying the spectrum condition. Therefore this most important case is still out of reach.

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