SHEN’S CONJECTURE ON GROUPS WITH GIVEN SAME ORDER TYPE

L. JAFARI TAGHVASANI AND M. ZARRIN

Abstract. For any group $G$, we define an equivalence relation $\sim$ as below:

$$\forall g, h \in G \quad g \sim h \iff |g| = |h|$$

the set of sizes of equivalence classes with respect to this relation is called the same-order type of $G$ and denote by $\alpha(G)$. In this paper, we give a partial answer to a conjecture raised by Shen. In fact, we show that if $G$ is a nilpotent group, then $|\pi(G)| \leq |\alpha(G)|$, where $\pi(G)$ is the set of prime divisors of order of $G$. Also we investigate the groups all of whose proper subgroups, say $H$ have $|\alpha(H)| \leq 2$.

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1. Introduction and results

Let $G$ be a group, define an equivalence relation $\sim$ as below:

$$\forall g, h \in G \quad g \sim h \iff |g| = |h|$$

the set of sizes of equivalence classes with respect to this relation is called the same-order type of $G$. For instance, the same-order type of the quaternion group $Q_8$ is \{1, 6\}. The only groups of type \{1\} are 1, $\mathbb{Z}_2$. In [3], Shen showed that a group of same-order type \{1, n\}\{(1, m, n)\} is nilpotent (solvable, respectively). Furthermore he gave the structure of these groups. In this paper, we give a partial answer to a conjecture raised by Shen in [3] and we prove that if $G$ is a nilpotent group, then $|\pi(G)| \leq |\alpha(G)|$.

Given a class of groups $\mathcal{X}$, we say that a group $G$ is a minimal non-$\mathcal{X}$-group, or an $\mathcal{X}$-critical group, if $G \not\in \mathcal{X}$, but all proper subgroups of $G$ belong to $\mathcal{X}$. It is clear that detailed knowledge of the structure of minimal non-$\mathcal{X}$-groups can provide insight into what makes a group belong to $\mathcal{X}$. For instance, minimal non-nilpotent groups were analysed by Schmidt [2] and proved that such groups are solvable (see also [3]). Suppose that $t$ be a positive integer and $\mathcal{Y}_t$ be the class of all groups in which $|\alpha(G)| \leq t$. Here, we determine the structure of minimal non-$\mathcal{Y}_2$-group.

Denote by $\phi$ and $S_n$ the Euler’s function and the number of elements of order $n$ in a group $G$ respectively. $X_n$ is the set of all elements of order $n$ in a group $G$. We use symbols $\pi_e(G)$ for the set of element orders.

2. Shen’s conjecture

In [3], Shen posed a conjecture as follows:
Let $G$ be a group with same-order type \{1, n_2, \cdots, n_r\}. Then $|\pi(G)| \leq r$.
Here we give a partial answer to a this conjecture. Note that by Lemma 3 of [4],

$$|\pi(G)| \leq r$$

where $r$ is the number of elements of order $n_i$ in $G$.

In the next section, we give a partial answer to a conjecture raised by Shen.

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the set of sizes of equivalence classes with respect to this relation is called the same-order type of $G$. For instance, the same-order type of the quaternion group $Q_8$ is \{1, 6\}. The only groups of type \{1\} are 1, $\mathbb{Z}_2$. In [3], Shen showed that a group of same-order type \{1, n\}\{(1, m, n)\} is nilpotent (solvable, respectively). Furthermore he gave the structure of these groups. In this paper, we give a partial answer to a conjecture raised by Shen in [3] and we prove that if $G$ is a nilpotent group, then $|\pi(G)| \leq |\alpha(G)|$.

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Here we give a partial answer to a this conjecture. Note that by Lemma 3 of [4],
we can assume that $G$ is finite. To prove Shen’s conjecture we need the following interesting lemmas.

**Lemma 2.1.** Suppose that $G$ is a nilpotent group, $m, n \in \pi_e(G)$ and $(m, n) = 1$. Then

$$S_{mn}^G = S_m^G S_n^G.$$ 

**Proof.** Let $g \in X_{mn}$. As $(m, n) = 1$, so there exist $y, z \in G$, such that $o(y) = m$, $o(z) = n$ and $g = yz$. So $g \in X_m X_n$ and $X_{mn} \subseteq X_m X_n$. On the other hand, if $y \in X_m$ and $z \in X_n$, then, as $G$ is nilpotent, we can obtain that $yz = zy$ and so $o(yz) = o(zy) = o(z) o(y) = mn$. It follows that $X_m X_n \subseteq X_{mn}$ and so $X_{mn} = X_m X_n$. 

**Corollary 2.2.** Let $G$ be a nilpotent group, $m \in \pi_e(G)$ and $m = p_1^{b_1} p_2^{b_2} \cdots p_t^{b_t}$. Then

$$S_m^G = S_{p_1}^{G_{p_1}} S_{p_2}^{G_{p_2}} \cdots S_{p_t}^{G_{p_t}}.$$ 

**Theorem 2.3.** Let $G$ be a nilpotent group. Then

1. If $|\pi(G)| \leq 2$, then $|\pi(G)| \leq |\alpha(G)|$.
2. If $|\pi(G)| \geq 3$, then $\pi(G) \leq |\alpha(G)| - 1$.

**Proof.** (1). If $|\pi(G)| = 1$, then $G$ is a $p$-group and obviously $|\pi(G)| \leq |\alpha(G)|$. Let $\pi(G) = \{p, q\}$. Since $G$ is nilpotent, $G = P \times Q$, where $|P| = p^n$ and $|Q| = q^m$ are $p$-sylow and $q$-sylow subgroups of $G$, respectively. If $p = 2$ and $q = 1$, then $G \cong Z_2 \times Q$. Clearly $\alpha(G) = \alpha(Q)$. Now if $exp(Q) = q$, then $s_q^Q = q^{m - 1}$. So $|\alpha(G)| = |\alpha(Q)| = |\pi(G)| = 2$. Otherwise if $exp(Q) \neq q$, then there exists $x \in Q$ such that $o(x) = q^2$ and since $S_q \neq S_p$, so $|\alpha(G)| \geq 3$ and $|\alpha(G)| \geq 3 > |\pi(G)|$. In other values of $p$ and $n$, in view of Lemma 2.1 the conclusion is trivial.

(2). By hypothesis since $G$ is nilpotent, so $G = P_1 \times \cdots \times P_n$, where $P_i$’s are $p_i$-sylow subgroups of $G$ and $p_1 < p_2 < \cdots < p_n$. We prove by induction on $n$. If $n = 3$, the $\alpha(P_1) \cup \alpha(P_2) \cup \alpha(P_3) \supseteq \{r, t\}$, for distinct numbers $r$ and $t$, so $\alpha(G) \supseteq \{1, r, t, t\}$, as desired.

Now assume the conclusion is true for $G_{n-1} = P_1 \times \cdots \times P_{n-1}$. Let for any $1 \leq i \leq n - 1$, $\alpha(P_i) = \{n_i, 1, \cdots, n_i\}$ and $S_{p_i}$, for $1 \leq i \leq n - 1$ be the maximum number of the set $\alpha(P_i)$. Now for any $l \in \pi_e(G_{n-1})$, assume that $l = p_i^{\beta_1} \cdots p_r^{\beta_r}$, where $1 \leq r \leq n - 1$. By the maximality of $S_{p_i}$’s, we have

$$S_l = S_{p_1^{\beta_1} \cdots p_r^{\beta_r}} = S_{p_1^{\beta_1}} \cdots S_{p_r^{\beta_r}} \leq S_{p_1^{b_1}} \cdots S_{p_n^{b_n}}$$

Besides, $S_{p_n}^{G_{n-1}} \neq 0$ and since $\phi(p_n) = p_n - 1 \mid S_{p_n}$, so $S_{p_n} \neq 1$. Hence we have

$$S_l \leq S_{p_1^{b_1}} \cdots S_{p_n^{b_n}} \leq S_{p_1^{b_1}} \cdots S_{p_n^{b_n}} S_{p_n} = S_{p_1^{b_1} \cdots p_n^{b_n}}$$

It follows that $S_{p_1^{b_1} \cdots p_n^{b_n}} \in \alpha(G_n) \setminus \alpha(G_{n-1})$. Therefore

$$|\alpha(G_n)| = |\alpha(G)| \geq |\alpha(G_{n-1})| + 1$$
and so by induction hypothesis:
\[ |\pi(G)| = n = n - 1 + 1 < |\alpha(G_{n-1})| + 1 \leq |\alpha(G_n)| = |\alpha(G)|. \]
and the conclusion is proved. \(\square\)

3. On same-order type of subgroups of a group

In this section, we determine the structure of minimal non-\(\mathcal{Y}_2\)-group, as follows.

**Theorem 3.1.** Let \(G\) be minimal non-\(\mathcal{Y}_2\)-group. Then \(G\) is a Frobenius or 2-Frobenius group.

**Proof.** Let \(H\) be a non-trivial proper subgroup of \(G\) and \(p \in \pi(H)\). Suppose, on the contrary, that \(q \in \pi(G)\) and \(q \neq p\). Since \(p \mid 1 + s^H_p\) and \(q \mid 1 + s^H_q\), so \(s^H_p, s^H_q \neq \{0, 1\}\), hence \(s^H_p = s^H_q = n_H\). Now as \(H\) is nilpotent, according to Lemma 2.1, we have \(s^H_{pq} = s^H_p s^H_q = n^2_H\), a contradiction. Thus \(H\) is a \(p\)-group. On the other hand, since
\[ p \mid 1 + s^H_p + s^H_p, \]
so \(s^H_{p^2} \neq \{1, n_H\}\), since otherwise \(p \mid 1\), a contradiction. Hence \(s^H_{p^2} = 0\). It follows that every proper subgroup of \(G\) is \(p\)-group of exponent \(p\). If \(p, q \in \pi(G)\), then \(G\) has no element of order \(pq\). If \(G\) is nilpotent, then \(G\) is a \(p\)-group of exponent \(p\) and it is easy to see that such groups are in \(\mathcal{Y}_2\), a contrary. If \(G\) is non-nilpotent, then, as proper subgroup of \(G\) has the same-order type \(\{1, n\}\), Theorem 2.1 of Shen follows that \(G\) is a Schmidt group and so \(|\pi(G)| = 2\). Now, as \(G\) has no element of order \(pq\), Theorem A of [1], completes the proof. \(\square\)

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