On $Z_p$-norms of random vectors

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Abstract

To any $n$-dimensional random vector $X$ we may associate its $L_p$-centroid body $Z_p(X)$ and the corresponding norm. We formulate a conjecture concerning the bound on the $Z_p(X)$-norm of $X$ and show that it holds under some additional symmetry assumptions. We also relate our conjecture with estimates of covering numbers and Sudakov-type minorization bounds.

1 Introduction. Formulation of the Problem.

Let $p \geq 2$ and $X = (X_1, \ldots, X_n)$ be a random vector in $\mathbb{R}^n$ such that $\mathbb{E}|X|^p < \infty$. We define the following two norms on $\mathbb{R}^n$:

$$
\|t\|_{\mathcal{M}_p(X)} := (\mathbb{E}|\langle t, X \rangle|^p)^{1/p} \quad \text{and} \quad \|t\|_{Z_p(X)} := \sup\{|\langle t, s \rangle|: \|s\|_{\mathcal{M}_p(X)} \leq 1\}.
$$

By $\mathcal{M}_p(X)$ and $Z_p(X)$ we will also denote unit balls in these norms, i.e.

$$
\mathcal{M}_p(X) := \{t \in \mathbb{R}^n: \|t\|_{\mathcal{M}_p(X)} \leq 1\} \quad \text{and} \quad Z_p(X) := \{t \in \mathbb{R}^n: \|t\|_{Z_p(X)} \leq 1\}.
$$

The set $Z_p(X)$ is called the $L_p$-centroid body of $X$ (or rather of the distribution of $X$). It was introduced (under a different normalization) for uniform distributions on convex bodies in [9]. Investigation of $L_p$-centroid bodies played a crucial role in the Paouris proof of large deviations bounds for Euclidean norms of log-concave vectors [10]. Such bodies also appears in questions related to the optimal concentration of log-concave vectors [7].

Let us introduce a bit of useful notation. We set $|t| := \|t\|_2 = \sqrt{\langle t, t \rangle}$ and $B_2^n = \{t \in \mathbb{R}^n: |t| \leq 1\}$. By $\|Y\|_p = (\mathbb{E}|Y|^p)^{1/p}$ we denote the $L_p$-norm of a random variable $Y$. Letter $C$ denotes universal constants (that may differ at each occurrence), we write $f \sim g$ if $\frac{1}{C}f \leq g \leq Cf$.

Let us begin with a simple case, when a random vector $X$ is rotationally invariant. Then $X = RU$, where $U$ has a uniform distribution on $S^{n-1}$ and $R = |X|$ is a nonnegative random variable, independent of $U$. We have for any vector $t \in \mathbb{R}^n$ and $p \geq 2$,

$$
\|\langle t, U \rangle\|_p = |t|\|U_1\|_p \sim \sqrt{\frac{p}{n+p}} |t|,
$$

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where $U_1$ is the first coordinate of $U$. Therefore

$$\|t\|_{M_p(X)} = \|U_1\|_p \|R\|_p \|t\| \quad \text{and} \quad \|t\|_{Z_p(X)} = \|U_1\|^{-1}_p \|R\|^{-1}_p \|t\|.$$ 

So

$$\left( \mathbb{E}\|X\|_{Z_p(X)}^p \right)^{1/p} = \|U_1\|^{-1}_p \|R\|^{-1}_p \left( \mathbb{E}|X|^p \right)^{1/p} = \|U_1\|^{-1}_p \sim \sqrt{\frac{n + p}{p}}. \quad (1)$$

This motivates the following problem.

**Problem 1.** Is it true that for (at least a large class of) centered $n$-dimensional random vectors $X$,

$$\left( \mathbb{E}\|X\|_{Z_p(X)}^2 \right)^{1/2} \leq C \sqrt{\frac{n + p}{p}} \quad \text{for } p \geq 2,$$

or maybe even

$$\left( \mathbb{E}\|X\|_{Z_p(X)}^p \right)^{1/p} \leq C \sqrt{\frac{n + p}{p}} \quad \text{for } p \geq 2?$$

Notice that the problem is linearly-invariant, since

$$\|AX\|_{Z_p(AX)} = \|X\|_{Z_p(X)} \quad \text{for any } A \in \text{GL}(n). \quad (2)$$

For any centered random vector $X$ with nondegenerate covariance matrix, random vector $Y = \text{Cov}(X)^{-1/2}X$ is isotropic (i.e. centered with identity covariance matrix). We have $\mathcal{M}_2(Y) = \mathcal{Z}_2(Y) = B^n_2$, hence

$$\mathbb{E}\|X\|_{Z_2(X)}^2 = \mathbb{E}\|Y\|_{Z_2(Y)}^2 = \mathbb{E}|Y|^2 = n.$$ 

Next remark shows that the answer to our problem is positive in the case $p \geq n$.

**Remark 1.** For $p \geq n$ and any $n$-dimensional random vector $X$ we have $(\mathbb{E}\|X\|_{Z_p(X)}^p)^{1/p} \leq 10.$

**Proof.** Let $S$ be a $1/2$-net in the unit ball of $\mathcal{M}_p(X)$ such that $|S| \leq 5^n$ (such net exists by the volume-based argument, cf. [1 Corollary 4.1.15]). Then

$$\left( \mathbb{E}\|X\|_{Z_p(X)}^p \right)^{1/p} \leq 2 \left( \mathbb{E} \sup_{t \in S} |\langle t, X \rangle|^p \right)^{1/p} \leq 2 \left( \mathbb{E} \sum_{t \in S} |\langle t, X \rangle|^p \right)^{1/p} \leq 2 |S|^{1/p} \sup_{t \in S} (\mathbb{E}|t, X\rangle|^p)^{1/p} \leq 2 \cdot 5^{n/p}.$$

$\square$
$L_p$-centroid bodies play an important role in the study of vectors uniformly distributed on convex bodies and a more general class of log-concave vectors. A random vector with a nondegenerate covariance matrix is called log-concave if its density has the form $e^{-h}$, where $h: \mathbb{R}^n \to (-\infty, \infty]$ is convex. If $X$ is centered and log-concave then
\[
\|\langle t, X \rangle\|_p \leq \lambda^p_q \|\langle t, X \rangle\|_q \quad \text{for } p \geq q \geq 2,
\]
where $\lambda = 2$ ($\lambda = 1$ if $X$ is symmetric and log-concave and $\lambda = 3$ for arbitrary log-concave vectors). One of open problems for log-concave vectors [7] states that for such vectors, arbitrary norm $\| \|$ and $q \geq 1$,
\[
(E\|X\|^q_{Z_p(X)})^{1/q} \leq C \left( E\|X\| + \sup_{\|t\|_r \leq 1} \|\langle t, X \rangle\|_q \right).
\]
In particular one may expect that for log-concave vectors
\[
(E\|X\|^q_{Z_p(X)})^{1/q} \leq C \left( E\|X\|_{z_p(X)} + \sup_{t \in \mathcal{M}_p(X)} \|\langle t, X \rangle\|_q \right) \leq C \left( E\|X\|_{z_p(X)} + \frac{\max\{p, q\}}{p} \right).
\]
As a result it is natural to state the following variant of Problem 1.

**Problem 2.** Let $X$ be a centered log-concave $n$-dimensional random vector. Is it true that
\[
(E\|X\|^q_{Z_p(X)})^{1/q} \leq C \left( n/p \right) \quad \text{for } 2 \leq p \leq n, \ 1 \leq q \leq \sqrt{pn}.
\]

In Section 2 we show that Problems 1 and 2 have affirmative solutions in the class of unconditional vectors. In Section 3 we relate our problems to estimates of covering numbers. We also show that the first estimate in Problem 1 holds if the random vector $X$ satisfies the Sudakov-type minorization bound.

## 2 Bounds for unconditional random vectors

In this section we consider the class of unconditional random vectors in $\mathbb{R}^n$, i.e. vectors $X$ having the same distribution as $(\varepsilon_1|X_1|, \varepsilon_2|X_2|, \ldots, \varepsilon_n|X_n|)$, where $(\varepsilon_i)$ is a sequence of independent symmetric $\pm 1$ random variables (Rademacher sequence), independent of $X$.

Our first result shows that formula (1) may be extended to the unconditional case for $p$ even. We use the standard notation – for a multiindex $\alpha = (\alpha_1, \ldots, \alpha_n)$, $x \in \mathbb{R}^n$ and $m = \sum \alpha_i$, $x^\alpha := \prod_i x_i^{\alpha_i}$ and $\binom{m}{\alpha} := m!/(\prod_i \alpha_i!)$. 
Proposition 2. We have for any \( k = 1, 2, \ldots \) and any \( n \)-dimensional unconditional random vector \( X \) such that \( \mathbb{E}|X|^{2k} < \infty \),

\[
\left( \mathbb{E}\|X\|_{\mathbb{Z}_{2k}(X)}^{2k} \right)^{1/(2k)} \leq c_{2k} := \left( \sum_{\|\alpha\|_1=k} \left( \frac{k}{2\alpha} \right)^2 \right)^{1/(2k)} \sim \sqrt{\frac{n+k}{k}},
\]

where the summation runs over all multiindices \( \alpha = (\alpha_1, \ldots, \alpha_n) \) with nonnegative integer coefficients such that \( \|\alpha\|_1 = \sum_{i=1}^n \alpha_i = k \).

Proof. Observe first that

\[
\mathbb{E}|\langle t, X \rangle|^{2k} = \mathbb{E} \left| \sum_{i=1}^n t_i X_i \right|^{2k} = \sum_{\|\alpha\|_1=k} (2k)^{2\alpha} t^{2\alpha} \mathbb{E} X^{2\alpha}.
\]

For any \( t, s \in \mathbb{R}^n \) we have

\[
|\langle t, s \rangle|^k = \sum_{\|\alpha\|_1=k} \left( \frac{k}{\alpha} \right)^2 t^\alpha s^\alpha.
\]

So by the Cauchy-Schwarz inequality,

\[
\|s\|_{\mathbb{Z}_{2k}(X)}^k = \sup \{ |\langle t, s \rangle|^k : \mathbb{E}|\langle t, X \rangle|^{2k} \leq 1 \} \leq \left( \sum_{\|\alpha\|_1=k} \left( \frac{k}{2\alpha} \right)^2 \mathbb{E} X^{2\alpha} \right)^{1/2}.
\]

To see that \( c_{2k} \sim \sqrt{(n+k)/k} \) observe that

\[
\left( \frac{k}{2\alpha} \right)^2 = \left( \frac{2k}{k} \right)^{-1} \prod_{i=1}^n \left( \frac{2\alpha_i}{\alpha_i} \right).
\]

Therefore, since \( 1 \leq \binom{2l}{l} \leq 2^{2l} \), we get

\[
4^{-k} \binom{n+k-1}{k} \leq c_{2k}^{2k} \leq 4^k \binom{n+k-1}{k}.
\]

\[\square \]

Corollary 3. Let \( X \) be an unconditional \( n \)-dimensional random vector. Then

\[
\left( \mathbb{E}\|X\|_{\mathbb{Z}_p(X)}^{2k} \right)^{1/2k} \leq C \sqrt{\frac{n+p}{p}} \quad \text{for any positive integer } k \leq \frac{p}{2}.
\]
Proof. By the monotonicity of $L_{2k}$-norms we may and will assume that $k = \lfloor p/2 \rfloor$. Then by Proposition 2,

$$\left( \mathbb{E} \|X\|_{Z_{p}(X)}^{2k} \right)^{1/2k} \leq \left( \mathbb{E} \|X\|_{Z_{2k}(X)}^{2k} \right)^{1/2k} \leq C \sqrt{\frac{n + k}{k}} \leq C \sqrt{\frac{n + p}{p}}.$$ 

In the unconditional log-concave case we may bound higher moments of $\|X\|_{Z_{p}(X)}$.

**Theorem 4.** Let $X$ be an unconditional log-concave $n$-dimensional random vector. Then for $p, q \geq 2$,

$$\left( \mathbb{E} \|X\|_{Z_{p}(X)}^{q} \right)^{1/q} \leq C \left( \sqrt{\frac{n + p}{p}} + \sup_{t \in M_{p}(X)} \|t, X\|_{q} \right) \leq C \left( \sqrt{\frac{n + p}{p}} + \frac{q}{p} \right).$$

In order to show this result we will need the following lemma.

**Lemma 5.** Let $2 \leq p \leq n$, $X$ be an unconditional random vector in $\mathbb{R}^{n}$ such that $\mathbb{E}|X|^{p} < \infty$ and $\mathbb{E}|X_{i}| = 1$. Then

$$\|s\|_{Z_{p}(X)} \leq \sup_{I \subset [n], |I| \leq p} \|t\|_{M_{p}(X)} \leq \sum_{i \in I} t_{i}s_{i} + C_{1} \sup_{\|t\|_{M_{p}(X)} \leq 1, \|t\|_{2} \leq p^{-1/2}} \sum_{i = 1}^{n} t_{i}s_{i}. \quad (4)$$

**Proof.** We have by the unconditionality of $X$ and Jensen’s inequality,

$$\|t\|_{M_{p}(X)} = \left\| \sum_{i = 1}^{n} t_{i}X_{i} \right\|_{p} \geq \left\| \sum_{i = 1}^{n} t_{i}E|X_{i}| \right\|_{p}.$$ 

By the result of Hitczenko [5], for numbers $a_{1}, \ldots, a_{n},$

$$\left\| \sum_{i = 1}^{n} a_{i} \varepsilon_{i} \right\|_{p} \sim \sum_{i \leq p} a_{i}^{*} + \sqrt{p} \left( \sum_{i > p} |a_{i}^{*}|^{2} \right)^{1/2}, \quad (5)$$

where $(a_{i}^{*})_{i \leq n}$ denotes the nonincreasing rearrangement of $(|a_{i}|)_{i \leq n}$. Thus

$$\sqrt{p} \left( \sum_{i > p} |t_{i}^{*}|^{2} \right)^{1/2} \leq C_{1} \|t\|_{M_{p}(X)}$$

and (4) easily follows. \qed
Proof of Theorem 4. The last bound in the assertion follows by (3). It is easy to see that (increasing $q$ if necessary) it is enough to consider the case $q \geq \sqrt{np}$.

If $q \geq n$ then the similar argument as in the proof of Remark 1 shows that

$$\left( \mathbb{E} \|X\|_{Z_p(X)}^q \right)^{1/q} \leq 2 \cdot 5^{n/q} \sup_{t \in M_p(X)} \|\langle t, X \rangle\|_q \leq 10 \sup_{t \in M_p(X)} \|\langle t, X \rangle\|_q.$$  

Finally, consider the remaining case $\sqrt{pn} \leq q \leq n$. By (2) we may assume that $\mathbb{E}|X_i| = 1$ for all $i$. By the log-concavity $\|\langle t, X \rangle\|_{q_1} \leq C \frac{q_2}{q_1} \|\langle t, X \rangle\|_{q_2}$ for $q_1 \geq q_2 \geq 1$, in particular $\sigma_i := \|X_i\|_2 \leq C$.

Let $\mathcal{E}_1, \ldots, \mathcal{E}_n$ be i.i.d. symmetric exponential random variables with variance 1. By [6, Theorem 3.1] we have

$$\sup_{\|t\|_{M_p(X)} \leq 1, \|t\|_2 \leq p^{-1/2}} \left\| \sum_{i=1}^n t_i X_i \right\|_q$$

$$\leq C \left( \sup_{\|t\|_{M_p(X)} \leq 1, \|t\|_2 \leq p^{-1/2}} \left\| \sum_{i=1}^n t_i \sigma_i \mathcal{E}_i \right\|_1 \right) + \sup_{\|t\|_{M_p(X)} \leq 1, \|t\|_2 \leq p^{-1/2}} \|\langle t, X \rangle\|_q.$$  

We have

$$\sup_{\|t\|_{M_p(X)} \leq 1, \|t\|_2 \leq p^{-1/2}} \|\langle t, X \rangle\|_q \leq \sup_{\|t\|_{M_p(X)} \leq 1} \|\langle t, X \rangle\|_q$$

and

$$\left\| \sum_{i=1}^n t_i \sigma_i \mathcal{E}_i \right\|_1 \leq \frac{1}{\sqrt{p}} \sqrt{\sum_{i=1}^n \sigma_i^2 \mathcal{E}_i^2} \leq \frac{1}{\sqrt{p}} \sqrt{\sum_{i=1}^n \sigma_i^2} \leq C \frac{\sqrt{n}}{\sqrt{p}}.$$  

Thus

$$\sup_{\|t\|_{M_p(X)} \leq 1, \|t\|_2 \leq p^{-1/2}} \left\| \sum_{i=1}^n t_i X_i \right\|_q \leq C \left( \frac{\sqrt{n}}{\sqrt{p}} + \sup_{\|t\|_{M_p(X)} \leq 1} \|\langle t, X \rangle\|_q \right).$$

Let for each $I \subset [n]$, $P_I X = (X_i)_{i \in I}$ and $S_I$ be a 1/2-net in $M_p(P_I X)$ of cardinality at
most $5^{|I|}$. We have

$$\sup_{I \subset [n], |I| \leq p} \left\| \sum_{i \in I} t_i X_i \right\|_q \leq 2 \sup_{I \subset [n], |I| \leq p} \left\| \sum_{i \in I} t_i X_i \right\|_q \leq 2 \left( \sum_{I \subset [n], |I| \leq p} \sum_{t \in S_I} \mathbb{E} \left| \sum_{i \in I} t_i X_i \right|^q \right)^{1/q} \leq 2 \cdot 5^{p/q} \left\{ \sup_{I \subset [n], |I| \leq p} \left| \sum_{t \in S_I} \sum_{i \in I} t_i X_i \right| \right\}^{1/q} \leq 2 \cdot 5^{p/q} \left\{ \sup_{I \subset [n], |I| \leq p} \left| \sum_{t \in S_I} \sum_{i \in I} t_i X_i \right| \right\}^{1/q} \leq 10 \left( \frac{en}{p} \right)^{p/q} \sup_{t \in \mathcal{M}_p(X)} \left\| \sum_{i \in I} t_i X_i \right\|_q \leq C \sup_{t \in \mathcal{M}_p(X)} \left\| \sum_{i \in I} t_i X_i \right\|_q,$$

where the last estimate follows from $q \geq \sqrt{np}$.

Hence the assertion follows from Lemma 5.\hfill \Box

**Corollary 6.** Let $X$ be an unconditional log-concave $n$-dimensional random vector and $2 \leq p \leq n$. Then

$$\frac{1}{C} \sqrt{\frac{n}{p}} \leq \mathbb{E}\|X\|_{Z_p(X)} \leq \left( \mathbb{E}\|X\|_{Z_p(X)}^{\frac{p}{\sqrt{np}}} \right)^{1/\sqrt{np}} \leq C \sqrt{\frac{n}{p}} \quad (6)$$

and

$$\mathbb{P}\left( \|X\|_{Z_p(X)} \geq \frac{1}{C} \sqrt{\frac{n}{p}} \right) \geq \frac{1}{C}, \quad \mathbb{P}\left( \|X\|_{Z_p(X)} \geq C \sqrt{\frac{n}{p}} \right) \leq e^{-t \sqrt{np}} \text{ for } t \geq 1.$$

**Proof.** The upper bound in (6) easily follows by Theorem 4. In fact we have for $t \geq 1$,

$$\left( \mathbb{E}\|X\|_{Z_p(X)}^{t \sqrt{np}} \right)^{1/(t \sqrt{np})} \leq C t \sqrt{\frac{n}{p}},$$

hence the Chebyshev inequality yields the upper tail bound for $\|X\|_{Z_p(X)}$.

To establish lower bounds we may assume that $X$ is additionally isotropic. Then by the result of Bobkov and Nazarov [3] we have $\|\langle t, X \rangle\|_p \leq C(\sqrt{p}\|t\|_2 + p\|t\|_\infty)$. This easily gives

$$\mathbb{E}\|X\|_{Z_p(X)} \geq \frac{1}{C} \sqrt{\frac{n}{p}} \mathbb{E}\|X\|_{[n/2]} \geq \frac{1}{C} \sqrt{\frac{n}{p}},$$

where the last inequality follows by Lemma 7 below.
By the Paley-Zygmund inequality we get

$$P\left(\|X\|_{Z_p(X)} \geq \frac{1}{C} \sqrt{\frac{n}{p}}\right) \geq P\left(\|X\|_{Z_p(X)} \geq \frac{1}{2} E\|X\|_{Z_p(X)}\right) \geq \frac{(E\|X\|_{Z_p(X)})^2}{4E\|X\|_{Z_p(X)}^2} \geq c.$$ 

\[\square\]

**Lemma 7.** Let $X$ be a symmetric isotropic $n$-dimensional log-concave vector. Then $E X^*_{\lfloor n/2 \rfloor} \geq \frac{1}{4}.$

**Proof.** Let $a_i > 0$ be such that $P(X_i \geq a_i) = 3/8.$ Then by the log-concavity of $X_i,$ $P(|X_i| \geq ta_i) = 2P(X_i \geq ta_i) \leq (3/4)^t$ for $t \geq 1$ and integration by parts yields $\|X_i\|_2 \leq Ca_i.$ Thus $a_i \geq c_1$ for a constant $c_1 > 0.$

Let $S = \sum_{i=1}^n I_{\{|X_i| \geq c_1\}}.$ Then $ES = \sum_{i=1}^n P(|X_i| \geq c_1) \geq 3n/4.$ On the other hand $ES \leq \frac{n}{2} + nP(X^*_{\lfloor n/2 \rfloor} \geq c_1),$ so

$$E X^*_{\lfloor n/2 \rfloor} \geq c_1 P(X^*_{\lfloor n/2 \rfloor} \geq c_1) \geq 1/4.$$ 

\[\square\]

The next example shows that the tail and moment bounds in Corollary 6 are optimal.

**Example.** Let $X = (X_1, \ldots, X_n)$ be an isotropic random vector with i.i.d. symmetric exponential coordinates (i.e. $X$ has the density $2^n \exp(-\sqrt{2\|x\|_1})$). Then $(E|X_i|^p)^{1/p} \leq p/2,$ so $\frac{2}{p} e_i \in M_p(X)$ and

$$P\left(\|X\|_{Z_p(X)} \geq t\sqrt{n/p}\right) \geq P(|X_i| \geq t\sqrt{np}/2) \geq e^{-t\sqrt{np}/\sqrt{2}}$$

and for $q = s\sqrt{np},$ $s \geq 1,$

$$\left(\frac{E\|X\|^q_{Z_p(X)}}{p^{1/q}}\right)^{1/q} \geq \frac{2q}{p} \|X_i\|_q \geq cq/p = cs\sqrt{n/p}.$$ 

### 3 General case – approach via entropy numbers

In this section we propose a method of deriving estimates for $Z_p$-norms via entropy estimates for $M_p$-balls and Euclidean distance. We use a standard notation – for sets $T, S \subset \mathbb{R}^n,$ by $N(T, S)$ we denote the minimal number of translates of $S$ that are enough to cover $T.$ If $S$ is the $\varepsilon$-ball with respect to some translation-invariant metric $d$ then $N(T, S)$ is also denoted as $N(T, d, \varepsilon)$ and is called the metric entropy of $T$ with respect to $d.$

We are mainly interested in log-concave vectors or random vectors which satisfy moment estimates

$$\|\langle t, X \rangle\|_p \leq \lambda_p \|\langle t, X \rangle\|_q \quad \text{for} \quad p \geq q \geq 2.$$ 

(7)

Let us start with a simple bound.
Proposition 8. Suppose that \( X \) is isotropic in \( \mathbb{R}^n \) and (7) holds. Then for any \( p \geq 2 \) and \( \varepsilon > 0 \) we have

\[
\left( \mathbb{E} \left\| X \right\|_{Z_p(X)}^2 \right)^{1/2} \leq \varepsilon \sqrt{n} + \frac{e\lambda}{p} \max \{p, \log N(\mathcal{M}_p(X), \varepsilon B_2^n)\}.
\]

Proof. Let \( N = N(\mathcal{M}_p(X), \varepsilon B_2^n) \) and choose \( t_1, \ldots, t_N \in \mathcal{M}_p(X) \) such that \( \mathcal{M}_p(X) \subset \bigcup_{i=1}^N (t_i + \varepsilon B_2^n) \). Then

\[
\|x\|_{Z_p(X)} \leq \varepsilon |x| + \sup_{i \leq N} \langle t_i, x \rangle.
\]

Let \( r = \max\{p, \log N\} \). We have

\[
\left( \mathbb{E} \sup_{i \leq N} |\langle t_i, X \rangle|^2 \right)^{1/2} \leq \left( \mathbb{E} \sup_{i \leq N} \|\langle t_i, X \rangle\| \right)^{1/r} \leq \left( \sum_{i=1}^N \mathbb{E} \|\langle t_i, X \rangle\| \right)^{1/r} \leq N^{1/r} \sup_i \|\langle t_i, X \rangle\| \leq e\lambda \frac{r}{p} \sup_i \|\langle t_i, X \rangle\| p \leq e\lambda \frac{r}{p}.
\]

Remark 9. The Paouris inequality \cite{10} states that for isotropic log-concave vectors and \( q \geq 2 \), \( (\mathbb{E} |X|^q)^{1/q} \leq C(\sqrt{n} + q) \), so for such vectors and \( q \geq 2 \),

\[
\left( \mathbb{E} \|X\|_{Z_p(X)}^q \right)^{1/q} \leq C\varepsilon(\sqrt{n} + q) + \frac{2e}{p} \max\{p, q, \log N(\mathcal{M}_p(X), \varepsilon B_2^n)\}.
\]

Unfortunately, the known estimates for entropy numbers of \( \mathcal{M}_p \)-balls are rather weak.

Theorem 10 (\cite{4} Proposition 9.2.8]). Assume that \( X \) is isotropic log-concave and \( 2 \leq p \leq \sqrt{n} \). Then

\[
\log N \left( \mathcal{M}_p(X), \frac{t}{\sqrt{p}} B_2^n \right) \leq C \frac{n \log^2 p \log t}{t} \quad \text{for } 1 \leq t \leq \min \left\{ \sqrt{p}, \frac{1}{C} \frac{n \log p}{p^2} \right\}.
\]

Corollary 11. Let \( X \) be isotropic log-concave, then

\[
\left( \mathbb{E} \|X\|_{Z_p(X)}^p \right)^{1/p} \leq C \left( \frac{n}{p} \right)^{3/4} \log p \sqrt{\log n} \quad \text{for } 2 \leq p \leq \frac{1}{C} n^{3/7} \log^{-2/7} n.
\]

Proof. We apply Theorem 10 with \( t = (n/p)^{1/4} \log p \log^{1/2} n \) and Proposition 8 with \( \varepsilon = t_p^{-1/2} \).

Remark 12. Suppose that \( X \) is centered and the following stronger bound than (7) (satisfied for example for Gaussian vectors) holds

\[
\|\langle t, X \rangle\|_p \leq \lambda \sqrt{\frac{p}{q}} \|\langle t, X \rangle\|_q \quad \text{for } p \geq q \geq 2.
\]
Then for any $2 \leq p \leq n$,

$$\frac{1}{\lambda} \sqrt{\frac{2n}{p}} \leq \left( \mathbb{E} \|X\|_{Z_p(X)}^2 \right)^{1/2} \leq \left( \mathbb{E} \|X\|_{Z_p(X)}^n \right)^{1/n} \leq 10\sqrt{\frac{n}{p}}.$$  

Proof. Without loss of generality we may assume that $X$ is isotropic. We have

$$\|\langle t, X \rangle\|_p \leq \lambda \sqrt{p/2} \|\langle t, X \rangle\|_2 = \lambda \sqrt{p/2} |t|,$$

so $M_p(X) \supset \lambda^{-1} \sqrt{2/p} B^n_2$ and

$$\left( \mathbb{E} \|X\|_{Z_p(X)}^2 \right)^{1/2} \geq \frac{1}{\lambda} \sqrt{\frac{2}{p}} \left( \mathbb{E} |X|^2 \right)^{1/2} = \frac{1}{\lambda} \sqrt{\frac{2n}{p}}.$$

On the other hand let $S$ be a $1/2$-net in $M_p(X)$ of cardinality at most $5^n$. Then

$$\left( \mathbb{E} \|X\|_{Z_p(X)}^n \right)^{1/n} \leq 2 \left( \mathbb{E} \sup_{t \in S} |\langle t, X \rangle|^n \right)^{1/n} \leq 2 \left( \sum_{t \in S} \mathbb{E} |\langle t, X \rangle|^n \right)^{1/n} \leq 2 |S|^{1/n} \sup_{t \in S} \|\langle t, X \rangle\|_n \leq 10\lambda \sqrt{\frac{n}{p}} \sup_{t \in S} \|\langle t, X \rangle\|_p \leq 10\lambda \sqrt{\frac{n}{p}}.$$

Recall that the Sudakov minoration principle [11] states that if $G$ is an isotropic Gaussian vector in $\mathbb{R}^n$ then for any bounded $T \subset \mathbb{R}^n$ and $\varepsilon > 0$,

$$\mathbb{E} \sup_{t \in T} \langle t, G \rangle \geq \frac{1}{C} \varepsilon \sqrt{\log N(T, \varepsilon B^n_2)}.$$

So we can say that a random vector $X$ in $\mathbb{R}^n$ satisfies the $L_2$-Sudakov minoration with a constant $C_X$ if for any bounded $T \subset \mathbb{R}^n$ and $\varepsilon > 0$,

$$\mathbb{E} \sup_{t \in T} \langle t, X \rangle \geq \frac{1}{C_X} \varepsilon \sqrt{\log N(T, \varepsilon B^n_2)}.$$  

Example. Any unconditional $n$-dimensional random vector satisfies the $L_2$-Sudakov minoration with constant $C \sqrt{\log(n + 1)/(\min_{i \leq n} \mathbb{E}|X_i|)}$.

Indeed, we have by the unconditionality, Jensen’s inequality and the contraction principle,

$$\mathbb{E} \sup_{t \in T} \sum_{i=1}^n t_i X_i = \mathbb{E} \sup_{t \in T} \sum_{i=1}^n t_i |X_i| \geq \mathbb{E} \sup_{t \in T} \sum_{i=1}^n t_i \mathbb{E}|X_i| \geq \min_{i \leq n} \mathbb{E}|X_i| \mathbb{E} \sup_{t \in T} \sum_{i=1}^n t_i \varepsilon_i.$$
On the other hand, the classical Sudakov minoration and the contraction principle yields

\[
\frac{1}{C} \sqrt{\log N(T, \varepsilon B_2^n)} \leq \mathbb{E} \sup_{t \in T} \sum_{i=1}^{n} t_i g_i \leq \mathbb{E} \max_{i \leq n} |g_i| \mathbb{E} \sup_{t \in T} \sum_{i=1}^{n} t_i \varepsilon_i
\]

\[
\leq C \sqrt{\log(n + 1)} \mathbb{E} \sup_{t \in T} \sum_{i=1}^{n} t_i \varepsilon_i.
\]

However the $L_2$-Sudakov minoration constant may be quite large in the isotropic case even for unconditional vectors if we do not assume that $L_1$ and $L_2$ norms of $X_i$ are comparable. Indeed, let $P(X = \pm n^{1/2} e_i) = 1/n$ for $i = 1, \ldots, n$, where $e_1, \ldots, e_n$ is the canonical basis of $\mathbb{R}^n$. Then $X$ is isotropic and unconditional. Let $T = \{t \in \mathbb{R}^n : \|t\|_\infty \leq n^{-1/2}\}$. Then

\[
\mathbb{E} \sup_{t \in T} |\langle t, X \rangle| \leq 1.
\]

However, by the volume-based estimate,

\[
N(T, \varepsilon B_2^n) \geq \frac{\text{vol}(T)}{\text{vol}(\varepsilon B_2^n)} \geq \left( \frac{1}{C \varepsilon} \right)^n,
\]

hence

\[
\sup_{\varepsilon > 0} \varepsilon \sqrt{\log N((T, \varepsilon B_2^n)} \geq \frac{1}{C \sqrt{n}}.
\]

Thus the $L_2$-Sudakov constant $C_X \geq \sqrt{n}/C$ in this case.

Next proposition shows that random vectors with uniformly log-convex density satisfy the $L_2$-Sudakov minoration.

**Proposition 13.** Suppose that a symmetric random vector $X$ in $\mathbb{R}^n$ has the density of the form $e^h$ such that $\text{Hess}(h) \geq -\alpha \text{Id}$ for some $\alpha > 0$. Then $X$ satisfies the $L_2$-Sudakov minoration with constant $C_X \leq C/\sqrt{\alpha}$.

**Proof.** We will follow the method of the proof of the (dual) classical Sudakov inequality (cf. (3.15) and its proof in [8]).

Let $T$ be a bounded symmetric set and

\[
A := \mathbb{E} \sup_{t \in T} |\langle t, X \rangle|.
\]

By the duality of entropy numbers [2] we need to show that $\log^{1/2} N(\varepsilon^{-1} B_2^n, T^o) \leq C \varepsilon^{-1} \alpha^{1/2} A$ for $\varepsilon > 0$ or equivalently that

\[
N(\delta B_2^n, 6AT^o) \leq \exp(C \alpha \delta^2) \quad \text{for } \delta > 0.
\]
To this end let \( N = N(\delta B^n_2, 6AT^o) \). If \( N = 1 \) there is nothing to show, so assume that \( N \geq 2 \). Then we may choose \( t_1, \ldots, t_N \in \delta B^n_2 \) such that the balls \( t_i + 3AT^o \) are disjoint. Let \( \mu = \mu_X \) be the distribution of \( X \). By the Chebyshev inequality,

\[
\mu(3AT^o) = 1 - \mathbb{P} \left( \sup_{t \in T} |\langle t, X \rangle| > 3A \right) \geq \frac{2}{3}.
\]

Observe also that for any symmetric set \( K \) and \( t \in \mathbb{R}^n \),

\[
\mu(t + K) = \int_K e^{h(x-t)} dx = \int_K e^{h(x+t)} dx = \int_K \frac{1}{2} (e^{h(x-t)} + e^{h(x+t)}) dx \geq \int_K e^{(h(x-t)+h(x+t))/2} dx.
\]

By Taylor’s expansion we have for some \( \theta \in [0, 1] \),

\[
\frac{h(x-t) + h(x+t)}{2} = h(x) + \frac{1}{4} (\langle \text{Hess} h(x + \theta t), t \rangle + \langle \text{Hess} h(x - \theta t), t \rangle) \geq h(x) - \frac{1}{2} \alpha |t|^2.
\]

Thus

\[
\mu(t + K) \geq \int_K e^{h(x) - \alpha |t|^2/2} = e^{-\alpha |t|^2/2} \mu(K)
\]

and

\[
1 \geq \sum_{i=1}^{N} \mu(t_i + 3AT^o) \geq \sum_{i=1}^{N} e^{-\alpha |t_i|^2/2} \mu(3AT^o) \geq \frac{2N}{3} e^{-\alpha \delta^2/2} \geq N^{1/3} e^{-\alpha \delta^2/2}
\]

and (9) easily follows.

\[\square\]

**Proposition 14.** Suppose that \( X \) satisfies the \( L_2 \)-Sudakov minoration with constant \( C_X \). Then for any \( p \geq 2 \)

\[
N \left( \mathcal{M}_p(X), \frac{eC_X}{\sqrt{p}} B^n_2 \right) \leq e^p.
\]

In particular if \( X \) is isotropic we have for \( 2 \leq p \leq n \),

\[
\left( \mathbb{E} \|X\|_{Z_p(X)}^2 \right)^{1/2} \leq e \left( C_X \sqrt{\frac{n}{p}} + 1 \right).
\]

**Proof.** Suppose that \( N = N(\mathcal{M}_p(X), eC_Xp^{-1/2}B^n_2) \geq e^p \). We can choose \( t_1, \ldots, t_N \in \mathcal{M}_p(X) \), such that \( \|t_i - t_j\|_2 \geq eC_Xp^{-1/2} \). We have

\[
\mathbb{E} \sup_{i \geq N} (t_i, X) \geq \frac{1}{C_X} eC_Xp^{-1/2} \sqrt{\log N} > e.
\]
However on the other hand,

$$E\sup_{i \geq N} \langle t_i, X \rangle \leq \left( E\sup_{i \geq N} |\langle t_i, X \rangle|^p \right)^{1/p} \leq \left( \sum_{i \geq N} E|\langle t_i, X \rangle|^p \right)^{1/p} \leq N^{1/p} \max_i \|\langle t_i, X \rangle\|_p \leq e.$$  

To show the second estimate we proceed in a similar way as in the proof of Proposition

We choose $T \subset M_p(X)$ such that $|T| \leq e^p$ and $M_p(X) \subset T + eC_Xp^{-1/2}B_2^n$. We have

$$\|X\|_{Z_p(X)} \leq eC_Xp^{-1/2}|X| + \sup_{t \in T} |\langle t, X \rangle|$$

so that

$$\left( E\|X\|_{Z_p(X)}^2 \right)^{1/2} \leq eC_Xp^{-1/2}(E|X|^2)^{1/2} + \left( E\sup_{t \in T} |\langle t, X \rangle|^2 \right)^{1/2}.$$  

Vector $X$ is isotropic, so $E|X|^2 = n$ and since $T \subset M_p(X)$ and $p \geq 2$ we get

$$\left( E\sup_{t \in T} |\langle t, X \rangle|^2 \right)^{1/2} \leq \left( E\sup_{t \in T} |\langle t, X \rangle|^p \right)^{1/p} \leq \left( \sum_{t \in T} E|\langle t, X \rangle|^p \right)^{1/p} \leq |T|^{1/p} \max_{t \in T} \|\langle t, X \rangle\|_p \leq e.$$  

References

[1] S. Artstein-Avidan, A. Giannopoulos and V. D. Milman, *Asymptotic Geometric Analysis. Part I*, Mathematical Surveys and Monographs 202, American Mathematical Society, Providence, RI, 2015.

[2] S. Artstein, V. D. Milman and S. J. Szarek, *Duality of metric entropy*, Ann. of Math. (2) 159 (2004), 1313–1328.

[3] S. Bobkov and F. L. Nazarov, *On convex bodies and log-concave probability measures with unconditional basis*, in: Geometric Aspects of Functional Analysis, 53–69, Lecture Notes in Math. 1807, Springer, Berlin, 2003.

[4] S. Brazitikos, A. Giannopoulos, P. Valettas and B. H. Vritsiou, *Geometry of isotropic convex bodies*, Mathematical Surveys and Monographs 196, American Mathematical Society, Providence, RI, 2014.

[5] P. Hitczenko, *Domination inequality for martingale transforms of a Rademacher sequence*, Israel J. Math. 84 (1993) 161–178.
[6] R. Latała, *Weak and strong moments of random vectors*, Marcinkiewicz Centenary Volume, Banach Center Publ. **95** (2011), 115–121.

[7] R. Latała, *On some problems concerning log-concave random vectors*, Convexity and Concentration, 525–539, IMA Vol. Math. Appl. **161**, Springer, 2017.

[8] M. Ledoux and M. Talagrand, *Probability in Banach Spaces. Isoperimetry and Processes*, Springer, Berlin, 1991.

[9] E. Lutvak and G. Zhang, *Blaschke-Santaló inequalities*, J. Differential Geom. **47** (1997), 1–16.

[10] G. Paouris, *Concentration of mass on convex bodies*, Geom. Funct. Anal. **16** (2006), 1021–1049.

[11] V. N. Sudakov, *Gaussian measures, Cauchy measures and ε-entropy*, Soviet Math. Dokl. **10** (1969), 310–313.

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