THE TOPOLOGICAL TVERBERG PROBLEM
BEYOND PRIME POWERS

FLORIAN FRICK AND PABLO SOBERÓN

Abstract. Tverberg-type theory aims to establish sufficient conditions for a
simplicial complex $\Sigma$ such that every continuous map $f: \Sigma \to \mathbb{R}^d$ maps $q$ points
from pairwise disjoint faces to the same point in $\mathbb{R}^d$. Such results are plentiful for $q$ a power of a prime. However, for $q$ with at least two distinct prime
divisors, results that guarantee the existence of $q$-fold points of coincidence are
non-existent—aside from immediate corollaries of the prime power case. Here we
present a general method that yields such results beyond the case of prime powers.
In particular, we prove previously conjectured upper bounds for the topological
Tverberg problem for all $q$.

Erratum

We are grateful to an anonymous referee, who has pointed out a mistake in our proof.
The error is in Claim 5.4, where the induction fails in the case $k = q$. The proof of our
main result Theorem 1.1 relies on Claim 5.4. Thus the proof of Theorem 1.1 is incomplete.
The reduction in Section 4 is unaffected. We apologize for this mistake.

1. Introduction

In 1959 Bryan Birch [Bir59] proved that in any straight-line drawing of the complete
graph $K_{3q}$ on $3q$ vertices there are $q$ pairwise vertex-disjoint 3-cycles that surround a
common point. Equivalently, for $3q$ points in the plane, there is a partition into $q$ sets
whose convex hulls all share a common point. In this phrasing of the result, one may
observe that $3q - 2$ points suffice for such a partition to exist. Helge Tverberg [Tve66]
generalized this result to higher dimensions: Any $(q - 1)(d + 1) + 1$ points in $\mathbb{R}^d$ may be
partitioned into $q$ sets whose convex hulls all share a common point.

One may also wonder whether it is necessary that the drawing of $K_{3q}$ is a straight-line
drawing in Birch’s result. Indeed, Imre Bárány conjectured a topological generalization
of Tverberg’s theorem: Any continuous map $f: \Delta_{(q - 1)(d + 1)} \to \mathbb{R}^d$ from the $(q - 1)(d + 1)$-
dimensional simplex $\Delta_{(q - 1)(d + 1)}$ to $\mathbb{R}^d$ identifies points from $q$ pairwise disjoint faces.
For an affine map $f$ this is Tverberg’s theorem since the affine image of a face is the
convex hull of (the images of) its vertices. Bárány’s topological Tverberg conjecture was
proven for $q$ a prime by Bárány, Shlosman, and Szücs [BSS81] and for $q$ a prime power by
Özaydin [Oza87] and Volovikov [Vol96]. Recently, counterexamples were exhibited for any
$q$ that is not a power of a prime; see [BFZ19, Fri13] and Mabillard and Wagner [MW14,
MW15]. Tverberg-type problems have been of significant interest; see [BBZ16, BS18,
BZ17, DLGMM19] for recent surveys.

These developments leave open the problem of topological generalizations of Birch’s
original result and its higher-dimensional versions. There are no non-trivial upper bounds

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for the topological Tverberg problem beyond the case of prime powers. The purpose of the present manuscript is to prove a continuous generalization of Birch’s result in any dimension and for all integers \( q \geq 2 \).

**Theorem 1.1.** Let \( q \geq 2 \) and \( d \geq 1 \) be integers. Let \( f: \Delta_{q(d+1)-1} \to \mathbb{R}^d \) be a continuous map. Then there are \( q \) pairwise disjoint faces \( \sigma_1, \ldots, \sigma_q \) of \( \Delta_{q(d+1)-1} \) such that \( f(\sigma_1) \cap \cdots \cap f(\sigma_q) \neq \emptyset \).

Avvakumov, Karasev, and Skopenkov [AKS19] show that if \( q \) is not a prime power then there exists a continuous map \( f: \Delta_n \to \mathbb{R}^d \) with \( f(\sigma_1) \cap \cdots \cap f(\sigma_q) = \emptyset \) for any \( q \) pairwise disjoint faces \( \sigma_1, \ldots, \sigma_q \) for \( n = q \left( d + 1 - \left\lceil \frac{d+2}{q+1} \right\rceil \right) - 2 \), which is currently the best lower bound for the topological Tverberg problem for large \( d \). Theorem 1.1 shows that the multiplicative factor for the dimension of the simplex is asymptotically at most \( q \) for \( q \) not a power of a prime.

For \( q \) a prime power, Theorem 1.1 is weaker than the topological generalization of Tverberg’s theorem. If \( q + 1 \) is a prime power, it is a simple consequence thereof; see Section 3. Blagojević, the first author, and Ziegler [BFZ19, Conj. 5.5] conjectured Theorem 1.1. They also conjectured that the bound is optimal, which remains open. The best bounds on \( n \) such that any continuous map \( f: \Delta_n \to \mathbb{R}^q \) exhibits a \( q \)-fold point of coincidence among pairwise vertex-disjoint faces are simply derived by choosing \( n \) sufficiently large to guarantee such an intersection for \( p \) faces, where \( p \geq q \) is a prime power. Indeed, the topological tools used to prove these results—the non-existence of associated equivariant maps—fail beyond the case of prime powers. Theorem 1.1 still reduces to the non-existence of an associated \( \mathbb{Z}/p \)-equivariant map for a prime \( p \), but generally now \( p \) will be much larger than \( q \). The proof method presented here seems to be the first that yields non-trivial upper bounds beyond prime powers, and might turn out to be useful in related contexts. We present the key new idea in Section 4.

The continuous generalization of Birch’s result is a simple consequence of Theorem 1.1:

**Corollary 1.2.** For any drawing of \( K_{3q} \) in the plane, where each 3-cycle is embedded, there are \( q \) vertex-disjoint 3-cycles that surround a common point.

We require that every 3-cycle needs to be embedded to make sense of the notion of a 3-cycle surrounding a point: Every 3-cycle separates \( \mathbb{R}^2 \) into two regions by the Jordan curve theorem, and the 3-cycle surrounds every point within the bounded region.

We present some standard results and terminology in Section 2. Surprisingly, there is a much simpler proof of Theorem 1.1 for \( q \leq 33 \). We present it in Section 3. Section 4 contains the key idea and proofs of Theorem 1.1 and Corollary 1.2. The technical verification that an associated configuration space is highly connected is postponed to Section 5.

In Section 6 we present colorful variations of the topological Tverberg theorem beyond prime powers and open problems.

## 2. Preliminaries

Here we collect some of the standard language, notation, and results used throughout the manuscript. We refer the reader to Matoušek’s book [Mat03] for an introduction.

We denote the \( n \)-dimensional simplex \( \{ x \in \mathbb{R}^{n+1} : x_i \geq 0 \text{ and } \sum x_i = 1 \} \) by \( \Delta_n \). A simplicial complex \( \Sigma \) is a non-empty collection of sets such that for \( \sigma \in \Sigma \) and \( \tau \subset \sigma \) we have \( \tau \in \Sigma \). All simplicial complexes considered in this manuscript will be finite. By considering every set \( \sigma \in \Sigma \) to be a simplex of dimension \( |\sigma| - 1 \) with the natural identifications, \( \Sigma \) is a topological space glued from simplices in a natural way. Our notation
does not distinguish between an (abstract) simplicial complex—a collection of finite sets closed under taking subsets—and this geometric realization. For example, as should be obvious from context, a continuous map \( f : \Sigma \to \mathbb{R}^d \) is defined on the geometric realization of \( \Sigma \). Conversely, we also write \( \Delta_n \) for the simplicial complex of all subsets of \([n + 1] = \{1, 2, \ldots, n + 1\}\). We refer to \( \sigma \in \Sigma \) as a face; its dimension is \(|\sigma| - 1\). A 0-dimensional face is called a vertex, a 1-dimensional face is an edge. A face \( \sigma \in \Sigma \) is a maximal face if no proper superset of \( \sigma \) is a face of \( \Sigma \).

The join of simplicial complexes \( K \) and \( L \) is the simplicial complex
\[
K \ast L = \{ \sigma \times \{1\} \cup \tau \times \{2\} : \sigma \in K, \ \tau \in L \}.
\]
That is, the faces of \( K \ast L \) are unions of faces of \( K \) with faces of \( L \), where we force their vertex sets to be disjoint. The geometric realization of \( K \ast L \) is the join (as topological spaces) of their geometric realizations. The \( n \)-fold join of \( K \) is defined recursively by \( K^{*n} = K \ast K^{*(n-1)} \), where \( K^{*1} = K \). A point \( x \) in the geometric realization of \( K^{*n} \) can thus be represented as an (abstract) convex combination \( x = \lambda_1 x_1 + \cdots + \lambda_n x_n \) for points \( x_i \) in \( K \) and \( \lambda_i \geq 0 \) with \( \sum \lambda_i = 1 \).

The homotopical connectivity of a path-connected topological space \( X \) will be denoted by \( \text{conn} X \), that is, \( \text{conn} X = n \) means that \( \pi_j(X, x_0) \) is trivial for \( j \leq n \) and \( \pi_{n+1}(X, x_0) \) is non-trivial for some (and thus any) choice of basepoint \( x_0 \). We define \( \text{conn} X = -1 \) if \( X \) is non-empty and not path-connected. We say that \( X \) is \( n \)-connected if \( \text{conn} X \geq n \). In particular, a space that is \((n+1)\)-connected is also \( n \)-connected.

**Lemma 2.1** (see [Mat03] Prop. 4.4.3). Let \( K \) and \( L \) be non-empty simplicial complexes. Then
\[
\text{conn} K \ast L = \text{conn} K + \text{conn} L + 2.
\]

We will make repeated use of a simple consequence of the Mayer–Vietoris sequence (and Van Kampen’s theorem); see [Bj695], Lemma 10.3:

**Lemma 2.2.** Let \( K \) and \( L \) be non-empty simplicial complexes. If \( K \) and \( L \) are \( n \)-connected and \( K \cap L \) is \((n-1)\)-connected then \( K \cup L \) is \( n \)-connected.

A group action of the group \( G \) on the space \( X \) is called free if \( g \cdot x \neq x \) for all non-trivial \( g \in G \) and all \( x \in X \). If \( G \) acts on the spaces \( X \) and \( Y \) then a continuous map \( f : X \to Y \) is called \( G \)-equivariant, or \( G \)-map, if \( f(g \cdot x) = g \cdot f(x) \) for all \( g \in G \) and all \( x \in X \). If \( G \) acts on the simplicial complex \( \Sigma \) we will always assume that the action is simplicial, that is, that it maps faces to faces.

**Theorem 2.3** (Dold, 1983 [Dol83]). Let \( G \) be a finite, non-trivial group that acts on the simplicial complex \( \Sigma \) and that acts on \( \mathbb{R}^{n+1} \) by linear maps. Suppose \( \Sigma \) is \( n \)-connected and the action of \( G \) restricts to a free action on \( \mathbb{R}^{n+1} \setminus \{0\} \). Then every \( G \)-map \( f : \Sigma \to \mathbb{R}^{n+1} \) has a zero, \( f(x) = 0 \) for some point \( x \) in \( \Sigma \).

We will need the following often used corollary:

**Corollary 2.4.** Let \( p \) be a prime, and let \( \mathbb{Z}/p \) act on the simplicial complex \( \Sigma \) and on \((\mathbb{R}^n)^p\) by shifting copies of \( \mathbb{R}^n \). Suppose \( \Sigma \) is \([(p-1)n-1]\)-connected. Then any \( \mathbb{Z}/p \)-map \( f : \Sigma \to (\mathbb{R}^n)^p \) maps some point \( x \) in \( \Sigma \) to the diagonal \( D = \{(y_1, \ldots, y_p) \in (\mathbb{R}^n)^p : y_1 = y_2 = \cdots = y_p\} \).

**Proof.** Since \( p \) is a prime, the action of \( \mathbb{Z}/p \) on the orthogonal complement \( D^\perp \) of the diagonal \( D \) is free away from 0. Composing \( f \) with the orthogonal projection along \( D \) onto \( D^\perp \) yields an equivariant map \( \Sigma \to D^\perp \). This map has a zero since the dimension of \( D^\perp \) is \((p-1)n\). \( \square \)
3. Simple proofs for special cases

Theorem 1.1 is an immediate corollary of the topological Tverberg theorem if \( q \) or \( q + 1 \) is a power of a prime. In the latter case we are given a continuous map \( f : \Delta_{d(q+1)-1} \to \mathbb{R}^d \), which we extend continuously to a map \( f^+ : \Delta_{d(q+1)} \to \mathbb{R}^d \) in an arbitrary way by adding a dummy vertex \( v^+ \). Since \( q + 1 \) is a prime power, by the topological Tverberg theorem there are \( q + 1 \) pairwise disjoint faces \( \sigma_1, \ldots, \sigma_{q+1} \) of \( \Delta_{d(q+1)} \) with

\[
f^+(\sigma_1) \cap \cdots \cap f^+(\sigma_{q+1}) \neq \emptyset.
\]

Now simply discard the face that (possibly) contains the dummy vertex \( v^+ \), say face \( \sigma_{q+1} \). This leaves \( q \) pairwise disjoint faces of \( \Delta_{d(q+1)-1} \) with \( f(\sigma_1) \cap \cdots \cap f(\sigma_q) \neq \emptyset \). For \( q + 1 \) a prime, the optimal colored Tverberg theorem of Blagojević, Matschke, and Ziegler [BMZ15] gives additional constraints on the faces \( \sigma_1, \ldots, \sigma_q \).

In this section, we present a particular case of Theorem 1.1 that is easier to prove than the general case—the case that \( 2q + 1 \) is a prime. This already proves Theorem 1.1 for a surprisingly large range of small \( q \). The first positive integer \( q \) that is not a prime power and where \( q + 1 \) is not a prime power and \( 2q + 1 \) is not a prime is 34. Thus the reduction in this section proves Theorem 1.1 in particular for all \( q \leq 33 \).

The main idea of the proof of Theorem 1.1 is to consider many copies of the same map \( f \). This induces a map from a high-dimensional simplex, and with appropriate constraints on the faces we can guarantee that it descends to the desired \( q \)-fold point of coincidence. For the special case of two copies the required constrained topological Tverberg theorem is known. The conditions on this Tverberg theorem translate to \( 2q + 1 \) being prime. The result we need is implicit in a paper of Vučič and Živaljević [VZ93]. Denote the vertices of the simplex \( \Delta_n \) by \( v_1, v_2, \ldots, v_{n+1} \).

**Theorem 3.1** (Vučič and Živaljević, 1993 [VZ93]). Let \( p \) be an odd prime, and let \( d \geq 1 \) be an integer. For any continuous map \( f : \Delta_{(p-1)(d+1)} \to \mathbb{R}^d \) there are \( p \) pairwise disjoint faces \( \sigma_1, \ldots, \sigma_p \) such that \( f(\sigma_1) \cap \cdots \cap f(\sigma_p) \neq \emptyset \) and if \( v_{2j-1} \) is in \( \sigma_j \) then \( v_{2j} \) is in \( \sigma_i \) or \( \sigma_{i+1} \) for all \( j \in \{1, 2, \ldots, \frac{1}{2}(p-1)(d+1)\} \). Here we consider \( \sigma_{p+1} \) to be \( \sigma_1 \).

Let \( f : \Delta_{d(q+1)-1} \to \mathbb{R}^d \) be a continuous map, and suppose that \( p = 2q + 1 \) is a prime. Double all vertices of \( \Delta_{d(q+1)-1} \) to obtain a map \( F \) from the simplex \( \Delta_{2q(d+1)-1} \) with twice as many vertices. That is, \( F \) linearly interpolates between the map \( f \) defined on all vertices \( v_2, v_4, \ldots, v_{2q(d+1)} \) of \( \Delta_{2q(d+1)-1} \) with an even index and the same map \( f \) on the face of \( \Delta_{2q(d+1)-1} \) of all odd index vertices \( v_1, v_3, \ldots, v_{2q(d+1)-1} \). Here we assume that the vertices are ordered in the same way in both copies, that is, if \( q : \Delta_{2q(d+1)-1} \to \Delta_{d(q+1)-1} \) denotes the linear map defined on vertices by \( q(v_{2j-1}) = q(v_{2j}) = v_j \), then \( f \circ q = F \).

As before consider the map \( F^+ : \Delta_{2q(d+1)} \to \mathbb{R}^d \) obtained by extending \( F \) continuously to another dummy vertex \( v^+ = v_{2q(d+1)+1} \). Now use Theorem 3.1 for the map \( F^+ \) to obtain \( 2q + 1 \) pairwise disjoint faces \( \sigma_1, \ldots, \sigma_{2q+1} \) with \( F^+(\sigma_1) \cap \cdots \cap F^+(\sigma_{2q+1}) \neq \emptyset \), and with the additional constraint of Theorem 3.1.

By symmetry we may assume that \( v^+ \) is in \( \sigma_{2q+1} \). We thus get that

\[
F(\sigma_1) \cap \cdots \cap F(\sigma_{2q}) \neq \emptyset.
\]

By definition of \( F \), we get that \( f(q(\sigma_1)) \cap \cdots \cap f(q(\sigma_{2q})) \neq \emptyset \). Since \( q \) identifies pairs of vertices, the faces \( q(\sigma_i) \) will generally not be pairwise disjoint. However, by the additional constraint of Theorem 3.1 retaining every other face \( q(\sigma_1), q(\sigma_3), \ldots, q(\sigma_{2q-1}) \) yields \( q \) pairwise disjoint faces, whose images under \( f \) all share a common point.
4. Key Idea: Sparse, symmetric, highly connected complexes

Given a continuous map \( f: \Delta_n \to \mathbb{R}^d \), the condition \( f(\sigma_1) \cap \cdots \cap f(\sigma_q) \neq \emptyset \) for \( q \) pairwise disjoint faces translates to the following equivariant problem: assign a label from the set \( \{q\} = \{1, 2, \ldots, q\} \) to each of the \( n+1 \) vertices of \( \Delta_n \) and check if, for the induced partition of the vertices of \( \Delta_n \) into \( q \) faces, the images of the faces overlap. This leads us to consider the “configuration space” \([q]^{s(n+1)}\), the \((n+1)\)-fold join of \([q]\), where \([q]\) is given the discrete topology. Now the \( q\)-fold join of \( f \) induces a map \([q]^{s(n+1)} \to (\mathbb{R}^d)^q \subset (\mathbb{R}^{d+1})^q\) that is equivariant with respect to diagonally permuting \([q]\) in the domain and the copies of \( \mathbb{R}^{d+1} \) in the codomain. For a prime power and \( n \geq (q-1)(d+1) \) such a map must hit the diagonal \( \{(y, \ldots, y) \in (\mathbb{R}^{d+1})^q : y \in \mathbb{R}^{d+1}\} \), which finishes the proof.

In order to prove Theorem 1.1, we generalize this setup. Let \( p \geq q \) be a prime, and fix integers \( d \geq 1 \) and \( n \geq 1 \). Assign to every vertex of \( \Delta_n \) a set of labels from \([p]\). Each vertex can only receive sets that form faces of a certain simplicial complex \( \Sigma \) over \([p]\), that satisfies the following properties:

(i) \( \Sigma \) is \( \mathbb{Z}/p\)-invariant, where the generator \( \lambda \) maps \( j \mapsto j + 1 \mod p \) for all vertices \( j \).
(ii) \( \Sigma \) has an independent set of size \( q \), that is, \( q \) vertices such that no two of them form an edge.
(iii) The image of every \( \mathbb{Z}/p\)-equivariant map \( \Sigma^{s(n+1)} \to (\mathbb{R}^{d+1})^p \) contains a point on the diagonal \( \{(y, \ldots, y) \in (\mathbb{R}^{d+1})^p : y \in \mathbb{R}^{d+1}\} \).

For example, if \( p = q \) then property [ii] forces that \( \Sigma = [p] \). Property [iii] is satisfied whenever \( n \geq (p-1)(d+1) \). The larger the prime \( p \) compared to \( q \), the denser the complex \( \Sigma \) may be. Our main observation is that for growing \( p \), the denseness of \( \Sigma \) required by property [iii] may outpace the sparseness imposed by property [ii]. In fact, for every prime \( p \geq q \) with \( p \equiv 1 \mod q \) we will construct a simplicial complex \( \Sigma \) that satisfies properties [ii] and [iii], while also being homotopically \((\frac{p}{q} - O(1))\)-connected. This implies property [iii] for \( n \geq q(d+1) \) and sufficiently large \( p \) by Corollary 2.4 and the following:

Lemma 4.1. Let \( c \geq 0, d \geq 0, q \geq 1, n \geq q(d+1) \), and \( p \geq cq + (c-2)q^2(d+1) \) be integers. Then if \( \Sigma \) is a \((\frac{p}{q} - c)\)-connected simplicial complex, its \((n+1)\)-fold join \( \Sigma^{s(n+1)} \) is \( p(d+1)\)-connected.

Proof. By Lemma 2.1 \( \text{conn} \Sigma^{s(n+1)} \geq (n+1)(\frac{p}{q} - c) + 2n \). Now verify that

\[
(n+1)\left(\frac{p}{q} - c\right) + 2n \geq (q(d+1) + 1)\left(\frac{p}{q} - c\right) + 2q(d+1) = p(d+1) + \frac{p}{q} - c - (c-2)q(d+1) \geq p(d+1),
\]

which completes the proof. \( \square \)

Remark 4.2. The proof of Lemma 4.1 shows more generally that if the complex \( \Sigma \) is only \((\frac{p}{q} - o(p))\)-connected, then \( \Sigma^{s(n+1)} \) is still \( p(d+1)\)-connected for sufficiently large \( p \).

Theorem 4.3. Let \( p \) be a prime, and let \( d \geq 1 \) and \( n \geq 1 \) be integers. Suppose that for these given parameters there is a simplicial complex \( \Sigma \) on \([p]\) that satisfies properties [ii] and [iii] above. Then for any continuous map \( f: \Delta_n \to \mathbb{R}^d \) there are \( p \) faces \( \sigma_1, \ldots, \sigma_p \) of \( \Delta_n \) with \( f(\sigma_1) \cap \cdots \cap f(\sigma_p) \neq \emptyset \) and such that for every vertex \( v \) of \( \Delta_n \) the set \( \{j \in [p] : v \in \sigma_j\} \) is a face of \( \Sigma \).

Proof. Consider the map \( \Phi: (\Delta_n)^p \to (\mathbb{R}^{d+1})^p \) defined by

\[
\Phi(\lambda_1 x_1 + \cdots + \lambda_p x_p) = (\lambda_1, \lambda_1 f(x_1), \ldots, \lambda_p, \lambda_p f(x_p)).
\]
The complex $\left(\Delta_n\right)^p$ is isomorphic to $\left(\Delta_{p-1}\right)^{\ast(n+1)}$. Denote the natural isomorphism by $\iota : \left(\Delta_n\right)^p \to \left(\Delta_{p-1}\right)^{\ast(n+1)}$. Thus $\iota^{-1}(\Sigma^{\ast(n+1)})$ is a $\mathbb{Z}/p$-invariant subcomplex of the domain of $\Phi$. By property (iii), there is a point $x = \lambda_1 x_1 + \cdots + \lambda_p x_p \in \left(\Delta_n\right)^p$ with $\iota(x) \in \Sigma^{\ast(n+1)}$ and $\Phi(x) = (y, \ldots, y)$, or equivalently,

$$\lambda_1 = \lambda_2 = \cdots = \lambda_p \quad \text{and} \quad \lambda_1 f(x_1) = \lambda_2 f(x_2) = \cdots = \lambda_p f(x_p).$$

This implies $\lambda_j = \frac{1}{p}$ for all $j$, and thus $f(x_1) = f(x_2) = \cdots = f(x_p)$.

\[ \begin{array}{cccccc}
\Delta_n & \Delta_n & \Delta_n & \Delta_n & \Delta_n & \Delta_n \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \downarrow \Sigma \subset \Delta_{p-1} \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \downarrow \Sigma \subset \Delta_{p-1} \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \downarrow \Sigma \subset \Delta_{p-1} \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \downarrow \Sigma \subset \Delta_{p-1} \\
\end{array} \]

**Figure 1.** The simplex on vertex set $[n+1] \times [p]$ is both the join of columns $(\Delta_n)^p$ as well as the join of rows $(\Delta_{p-1})^{\ast(n+1)}$. Here $\Sigma$ is a subcomplex of each row. The isomorphism $\iota$ reverses the roles of rows and columns.

Now let $\sigma_j$ be the inclusion-minimal face of $\Delta_n$ that contains $x_j$. Then $f(\sigma_1) \cap \cdots \cap f(\sigma_p) \neq \emptyset$. Moreover $\iota(\sigma_1 \ast \cdots \ast \sigma_p)$ is a face of $\Sigma^{\ast(n+1)}$, but this means that each vertex $v$ may only appear in faces $\sigma_j$ whose indices form a face of $\Sigma$.

**Corollary 4.4.** Let $p$ be a prime, and let $d \geq 1$, $q \geq 2$, and $n \geq 1$ be integers. Suppose that for these given parameters there is a simplicial complex $\Sigma$ on $[p]$ that satisfies properties (i), (ii), and (iii) above. Then for any continuous map $f : \Delta_{n-1} \to \mathbb{R}^d$ there are $q$ pairwise disjoint faces $\sigma_1, \ldots, \sigma_q$ of $\Delta_{n-1}$ with $f(\sigma_1) \cap \cdots \cap f(\sigma_q) \neq \emptyset$.

**Proof.** Let us extend $\Delta_{n-1}$ to $\Delta_n$ by adding a dummy vertex $v^+$. We can extend $f$ as well, turning it into a map $f : \Delta_n \to \mathbb{R}^d$. By Theorem 1.3 there are $p$ faces $\sigma_1, \ldots, \sigma_p$ of $\Delta_n$ with $f(\sigma_1) \cap \cdots \cap f(\sigma_p) \neq \emptyset$ and such that for every vertex $v$ of $\Delta_n$ the set $\{j \in [p] : v \in \sigma_j\}$ is a face of $\Sigma$. Let $\sigma^+ = \{j \in [p] : v^+ \in \sigma_j\}$. Now let $I \subset [p]$ be an independent set of size $q$. We claim there exists an $m \in [p]$ such that $\lambda^m I \cap \sigma^+ = \emptyset$, where $\lambda$ is a generator of $\mathbb{Z}/p$. If $\sigma^+ = \emptyset$ the claim is true, so we may assume $|\sigma^+| \geq 1$. The set $\lambda^m I$ is independent for each $m$. If our claim fails to be true, consider the set

$$P = \{(i, m) : i \in I, m \in [p], \lambda^m i \in \sigma^+\}.$$

For each $m$, we know $|\lambda^m I \cap \sigma^+| \geq 1$ since it is not empty. We also know $|\lambda^m I \cap \sigma^+| \leq 1$ since it is the intersection of a complete set with an independent set in $\Sigma$. Therefore, $|P| = p$. However, for each $i \in I$, we know there are exactly $|\sigma^+|$ values of $m$ such that $(i, m) \in P$. Since $p > q$ is prime, it cannot be equal to $|\sigma^+|q$. Therefore, there exists some $m$ such that $\lambda^m I \cap \sigma^+ = \emptyset$.

The faces $\sigma_j$ with $j \in \lambda^m I$ are pairwise disjoint, and none contains the dummy vertex $v^+$. After possibly renumbering the $\sigma_j$ we thus have that $f(\sigma_1) \cap \cdots \cap f(\sigma_q) \neq \emptyset$ for pairwise disjoint faces $\sigma_1, \ldots, \sigma_q$ of $\Delta_{n-1}$.

To prove Theorem 1.1, for given $q \geq 2$ and $d \geq 1$, we have to find a prime $p$ and a $\mathbb{Z}/p$-invariant complex $\Sigma$ on $[p]$ with an independent set of size $q$ that is $(\frac{2}{q} - O(1))$-connected. We will construct such a complex $\Sigma$ in the next section, thus proving our main result:
Proof of Theorem 4.5. Let the integers $q \geq 2$ and $d \geq 1$ be given. There are infinitely many primes $p$ of the form $p = (a + 1)q + 1$ by Dirichlet’s theorem [Dir37]. By Theorem 5.1 for each such prime $p$ there is a simplicial complex $C^a_p$ that satisfies properties (i) and (iii) and is $(\frac{q}{q} - 4)$-connected. For $n = q(d + 1)$ the $(n + 1)$-fold join $(C^a_p)^{\ast(n+1)}$ is $p(d + 1)$-connected for sufficiently large $p$ by Lemma 4.1. Thus $C^a_p$ satisfies property (iii) as well as Corollary 2.4. Thus for any continuous $f : \Delta_{n-1} \rightarrow \mathbb{R}^d$ there are $q$ pairwise disjoint faces whose images all share a common point by Corollary 4.4.

Proof of Corollary 4.2. Let $f : K_{3q} \rightarrow \mathbb{R}^2$ be a continuous map such that every 3-cycle is embedded. The complete graph $K_{3q}$ is the 1-skeleton of $\Delta_{3q-1}$. Let $\sigma$ be a triangle of $\Delta_{3q-1}$. By the Jordan curve theorem $f(\partial \sigma)$ bounds a disk $D$ in $\mathbb{R}^2$. Continuously extend $f$ to $\sigma$ such that $f(\sigma) \subseteq D$. After extending $f$ onto every triangle of $\Delta_{3q-1}$, continuously extend it to all of $\Delta_{3q-1}$. By Theorem 1.1 there are $q$ pairwise disjoint faces $\sigma_1, \ldots, \sigma_q$ with $f(\sigma_1) \cap \cdots \cap f(\sigma_q) \neq \emptyset$. By the methods presented by Schönenborn and Ziegler [SZ05, Sec. 2] the $\sigma_i$ can be chosen such that $\dim \sigma_i \leq 2$, which completes the proof.

We remark that our construction of sparse, $\mathbb{Z}/p$-symmetric, highly connected complexes in Section 5 is optimal, as otherwise by the same reasoning as above one could prove Tverberg-type results that are too strong to be true:

Theorem 4.5. Let $q \geq 2$ be an integer. Then for any sufficiently large prime $p$ the maximal size of an independent set in any $\mathbb{Z}/p$-invariant complex on $[p]$ that is $(\frac{q}{q} - o(p))$-connected is $q$.

Proof. Suppose $\Sigma$ is a simplicial complex on $[p]$ that is $\mathbb{Z}/p$-invariant, $(\frac{q}{q} - o(p))$-connected, and has an independent set of size $q + 1$. By Lemma 4.1 and Remark 4.2 for $p$ sufficiently large $\Sigma$ satisfies property (iii) above for $n = q(d + 1)$ for any given $d \geq 1$. By the reasoning in the proof of Corollary 4.4 for any continuous map $f : \Delta_{n-1} \rightarrow \mathbb{R}^d$ there are $q + 1$ pairwise disjoint faces $\sigma_1, \ldots, \sigma_{q+1}$ of $\Delta_{n-1}$ with $f(\sigma_1) \cap \cdots \cap f(\sigma_{q+1}) \neq \emptyset$. However, the bound of Tverberg’s theorem is known to be optimal. That is, an affine map $f$ that maps vertices into sufficiently generic position yields a contradiction.

5. Construction of simplicial complexes

In this section we construct the required $\mathbb{Z}/p$-invariant complex $\Sigma$ that has an independent set of size $q$ and is almost $(p/q)$-connected. Throughout this section the integer $q \geq 2$ will be fixed. Suppose $p = (a + 1)q + 1$ is a prime for some integer $a \geq 1$. Observe that there are infinitely many such integers $a$ by Dirichlet’s theorem [Dir37]. A set $\sigma \subseteq \mathbb{Z}/p$ is $q$-stable if for any two $i, j \in \sigma$ the difference $i - j$ is not in $\{1, 2, \ldots, q - 1\}$, that is, the cyclic gap between any two elements of $\sigma$ is at least $q$. Let $C_p$ be the simplicial complex of all $q$-stable subsets of $\mathbb{Z}/p$, and let $C^a_p$ be the subcomplex of $q$-stable sets that can be extended to a $q$-stable set of size at least $a$, that is,

$$C^a_p = \{ \sigma \subseteq \mathbb{Z}/p : \sigma \text{ is } q\text{-stable and there is a } \tau \in C_p \text{ with } \sigma \subseteq \tau \text{ and } |\tau| \geq a \}.$$ 

Clearly $C_p$ is $\mathbb{Z}/p$-invariant, and any $q$ consecutive vertices form an independent set. The rest of this section is devoted to checking that $C^a_p$ is highly connected:

Theorem 5.1. For $p = (a + 1)q + 1$ the complex $C^a_p$ is $(a - 2)$-connected.

We remark that $a - 2 = \frac{p - 1}{q} - 3 \geq \frac{q}{q} - 4$, so the theorem above finishes the proof of the main result. To prove Theorem 5.1 we will first break the cyclic symmetry and study subcomplexes of $C^a_p$ obtained by restricting to $p - q + 1$ successive vertices.
The extendability condition seems necessary to guarantee high connectedness. For instance, if \( q = 2 \), the complex \( C_r \) is the independence complex of a cycle of length \( r \). Its connectedness is \( \lfloor (r - 1)/3 \rfloor - 1 \), as its homotopy type was determined by Kozlov [Koz99] to be a wedge of spheres of dimension \( \lfloor (r - 1)/3 \rfloor \). For general \( q \), the complex \( C_r \) has maximal faces which are roughly \( (r/(2q - 1)) \)-dimensional, which may cause the connectedness to drop.

The complexes \( C_p^a \) have also appeared in the study of minimal manifold triangulations: Kühnel and Lassmann [KL96] study \( C_p^{a+1} \) for \( p = (a + 1)q + 1 \) and certain generalizations since they are triangulations of disk bundles over the circle. In particular, \( C_p^{a+1} \) is not highly connected, while \( C_p^a \) is.

5.1. Linear complexes and their connectedness. Fix an integer \( r \geq q \). We define the simplicial complex \( L_r \) as the family of sets \( \sigma \subset [r] \) such that if \( j \in \sigma \), then none of \( j + 1, \ldots , j + q - 1 \) are in \( \sigma \). (Here addition is not modulo \( r \).) In other words, any two elements of \( \sigma \) have a difference greater than or equal to \( q \). In analogy to their cyclic counterparts, define

\[
L^a_r = \{ \sigma \in L_r : \text{there exists } \tau \in L_n \text{ such that } |\tau| \geq a, \sigma \subset \tau \}.
\]

In other words, the maximal faces of \( L^a_r \) have at least \( a \) vertices. This makes the complexes \( L^a_r \) and \( C_p^a \) highly connected, whereas \( L_r \) and \( C_p \) are not as highly connected.

If \( r \geq aq \), then the vertex set of \( L^a_n \) is \([r]\). However, if \((a - 1)q + 1 \leq n < aq\), not all elements of \([r]\) appear as vertices. For example, \( L^a_{(a-1)q+1} \) has a single face \( \sigma = \{1, q + 1, \ldots , (a-1)q + 1\} \) and no other vertices.

For a simplicial complex \( T \) with vertex set in \([r]\) and an integer \( s \), we define \( T + s \) as the complex obtained by translating each face of \( T \) by \( s \) units, with vertex set in \( \{1 + s, 2 + s, \ldots , r + s\} \).

**Claim 5.2.** The complex \( L^a_r \) is either empty or \((a - 2)\)-connected.

**Proof.** We proceed by induction on \( a \). If \( a = 1 \), the result is clear. Now suppose \( a \geq 2 \) and \( r \geq (a - 1)q + 1 \), so \( L^a_r \) is not empty. Every maximal face of \( L^a_r \) has exactly one of the vertices \( \{1, 2, \ldots , q\} \). Let \( A_i \) be the subcomplex of faces in \( L^a_r \) that contain or can be extended to contain vertex \( i \). Notice that \( A_i \) is contractible or empty (if there are no faces containing vertex \( i \)). Then,

\[
L^a_r = A_1 \cup \cdots \cup A_q.
\]

For \( k = 1, \ldots , q \), let \( B_k = A_1 \cup \cdots \cup A_k \). We show by induction on \( k \) that \( B_k \) is \((a - 2)\)-connected. Since \( L^a_r \) is not empty, then \( A_1 \) is not empty, so \( B_1 = A_1 \), which is contractible. If we know that \( B_k \) is \((a - 2)\)-connected for some \( k < q \), then notice \( B_{k+1} = B_k \cup A_{k+1} \). The first complex is \((a - 2)\)-connected and the second is contractible. We only have to show that \( B_k \cap A_{k+1} \) is \((a - 3)\)-connected. This intersection is the set of faces that can be extended to a face of at least \( a - 1 \) vertices among \( \{q + k + 1, \ldots , r\} \), so

\[
B_k \cap A_{k+1} = L^{a-1}_{r-k-q} + (k + q).
\]

By induction, we know this complex is either empty or \((a - 3)\)-connected. If this intersection is \((a - 3)\)-connected, then \( B_{k+1} \) is \((a - 2)\)-connected. However, if this intersection is empty that means that \( A_{k+1} \) must be empty, so \( B_{k+1} = B_k \). In either case \( B_{k+1} \) is \((a - 2)\)-connected.

A simple corollary of this claim is a lower bound on the connectedness of \( L_r \) for any \( r \).

**Corollary 5.3.** The complex \( L_r \) is \( \left( \left\lfloor \frac{r}{2q-1} \right\rfloor - 2 \right) \)-connected.
Proof. The corollary follows immediately from the fact that, for $a = \left\lfloor \frac{r}{2q-1} \right\rfloor$, $L_r = L^a_r$. The containment $L^a_r \subset L_r$ is true by definition of $L^a_r$. A maximal face of $L_r$ cannot leave a gap of length $2q-1$ between two consecutive elements, or it could be extended. Therefore maximal faces of $L_r$ have at least $a$ elements, implying $L_r \subset L^a_r$. □

Next, we need to characterize all faces of the complex $L^a_{r-1}$ for small values $r$. If $r \leq (a - 2)q$, then $L^a_{r-1}$ is empty. If $(a - 2)q + 1 \leq r \leq (a - 1)q$, then the faces of $L^a_{r-1}$ can be fully characterized. A maximal face has exactly $a - 1$ vertices in $\{1, \ldots, r\}$. These vertices can be partitioned into $a - 1$ blocks $I_1, \ldots, I_{a-1}$, where for $j = 1, \ldots, a - 2$

$I_j = \{(j - 1)q + 1, \ldots, jq\}$

and $I_{a-1} = \{(a - 2)q + 1, \ldots, r\}$.

![Blocks in $L^a_{r-1}$](image)

**Figure 2.** Blocks in $L^a_{r-1}$

A maximal face of $L^a_{r-1}$ must have exactly one element of each block. If the element of the $j$-th block is $(j - 1)q + k_j$, we know that

$$1 \leq k_1 \leq \cdots \leq k_{a-1} \leq r - (a - 2)q \leq q.$$  

We need a particular complex $T^a_m \subset L^a_{m-1}$ for $m = (a - 2)q + k + 2$, where $1 \leq k \leq q - 1$. We define $T^a_m$ by its maximal faces. A maximal face $\tau$ of $T^a_m$ is a maximal face of $T^a_{m-1}$ if $\tau$

- does not contain vertex 1, or
- does not contain any of the last $k$ vertices.

That is, $T^a_m$ is the subcomplex of $L^a_{m-1}$ such that its maximal faces have sequences $(k_1, \ldots, k_{a-1})$ that satisfy either $k_1 \geq 2$ or $2 \geq k_{a-1}$.

**Claim 5.4.** For $m = (a - 2)q + k + 2$ the complex $T^a_m$ is $(a - 3)$-connected.

**Proof.** The complex $T^a_m$ is the union of two sets of maximal faces in $L^a_{m-1}$: Those maximal faces whose sequences satisfy $k_1 \geq 2$ and those whose sequences satisfy $k_{a-1} \leq 2$. The former set of faces form the complex $L^a_{m-1} + 1$, while the latter produce $L^a_{m-k}$. Since $m - k = (a - 2)q + 2$ both of these complexes are non-empty, and thus $(a - 3)$-connected by Claim 5.2. Moreover, since $m - 1 \leq (a - 1)q$ for the range of $k$ we are considering, the characterization of their faces above still holds. Their intersection consists of all those faces of $L^a_{m-1}$ contained in a face of size $a - 1$ with $k_1 \geq 2$ and $k_{a-1} \leq 2$. Since the sequence of $k_i$ is increasing, we have $k_1 = k_2 = \cdots = k_{a-1} = 2$, which implies that the intersection is a simplex, and thus contractible. Thus $T^a_m$ is $(a - 3)$-connected. □

**Claim 5.5.** For $1 \leq k \leq q - 1$ and $r = aq + 2$ the complex $L^a_{r-1} \cup \left(L^a_{r+k-2q} + (q - 1)\right)$ is $(a - 3)$-connected.

**Proof.** Since each of $L^a_{r-1}$ and $\left(L^a_{r+k-2q} + (q - 1)\right)$ is $(a - 3)$-connected, we only have to check that their intersection $K = L^a_{r-1} \cap \left(L^a_{r+k-2q} + (q - 1)\right)$ is $(a - 4)$-connected.

The vertices of the complex $K$ are contained in $\{q, q + 1, \ldots, r + k - q - 1\}$. We first treat the case that $k \leq q - 2$. Then $r + k - 2q \leq (a - 1)q$ and so no face of $L^a_{r+k-2q} + (q - 1)$ can
have more than \( a - 1 \) elements. Thus the maximal faces of \( K \) are exactly those maximal faces of \( L_{r+k-2q}^{a-1} + (q-1) \) that can be extended by one of the vertices \( \{1, \ldots, q-1\} \) or \( \{r+k-q, \ldots, r-1\} \) for it to be in \( L_{r-1}^a \). We claim that \( K \) is \( T_{r+k-2q}^{a-1} + (q-1) \), which is \( (a-3) \)-connected by Claim 5.4.

We first show that \( K \subset T_{r+k-2q}^{a-1} + (q-1) \). Let \( \sigma \) be a maximal face of \( K \). If it can be extended to a face of size \( a \) in \( L_{r-1}^a \) by a vertex in \( \{1, \ldots, q-1\} \), then \( \sigma \) cannot contain the vertex \( q \). Thus \( \sigma \) is in \( T_{r+k-2q}^{a-1} + (q-1) \). If on the other hand \( \sigma \) may be extended by a vertex in \( \{r+k-q, \ldots, r-1\} \), then it cannot contain vertices \( r-q, r-q+1, \ldots, r-q+k-1 \). These are precisely the last \( k \) vertices of the vertex set of \( L_{r+k-2q}^{a-1} + (q-1) \) and thus \( \sigma \) is in \( T_{r+k-2q}^{a-1} + (q-1) \).

Now if \( \sigma \) is a maximal face of \( T_{r+k-2q}^{a-1} + (q-1) \), then it has size \( a - 1 \). If \( \sigma \) does not contain vertex \( q \), then it is contained in the face \( \sigma \cup \{1\} \) of \( L_{r-1}^a \) and thus in \( K \). If \( \sigma \) does not contain any of the last \( k \) vertices of the vertex set of \( T_{r+k-2q}^{a-1} + (q-1) \) then it is contained in the face \( \sigma \cup \{r-1\} \) of \( L_{r-1}^a \) and thus in \( K \). This shows that \( K \) is indeed \( T_{r+k-2q}^{a-1} + (q-1) \) and thus \( (a-3) \)-connected.

Lastly, we deal with the case \( k = q - 1 \). The analysis is the same as above with the only difference that now a maximal face of \( K \) may have \( a \) vertices. Since \( r+k-2q = (a-1)q+1 \), there is a unique such face \( \sigma^* = \{q, 2q, \ldots, aq\} \), and \( K \) is the complex

\[
\sigma^* \cup \left( T_{r+k-2q}^{a-1} + (q-1) \right).
\]

Here we treat \( \sigma^* \) as a simplicial complex and tacitly include all of its subsets as faces. This exhibits \( K \) as the union of a simplex, which is contractible, and an \( (a-3) \)-connected complex. The intersection of \( \sigma^* \) and \( T_{r+k-2q}^{a-1} + (q-1) \) consists of all those faces of \( \sigma^* \) that do not use both \( q \) and \( aq \). Thus their intersection is a bipyramid (suspension of an \( (a-2) \)-simplex), which is contractible. This completes the proof.

5.2. Circular arcs complexes and their connectedness. We now study the complexes \( C_p^a \). There is a striking difference when studying the connectedness of \( L_p^a \) and that of \( C_p^a \). The complex \( L_p^a \) is either empty or highly connected. With \( C_p^a \) this fails to hold. For example, for \( p = aq \), the complex \( C_p^a \) consists of \( q \) pairwise disjoint simplexes, so it is non-empty and disconnected. We need more vertices to be able to guarantee high connectedness.

First, let us relate the linear complexes and the circular complexes:

**Claim 5.6.** For every \( a \) and \( r \) we have

\[
C_{r+q-1}^a = \bigcup_{j=0}^{q-1} (L_r^a + j)
\]

**Proof.** Let \( \sigma \) be a non-empty maximal face of \( C_{r+q-1}^a \). Let \( s \in \{1, 2, \ldots, r+q-1\} \) be the largest element of \( \sigma \). If \( s \leq r \), then \( \sigma \) is in \( L_r^a \). Otherwise, let \( j = s-r \). Note that \( 1 \leq j \leq q-1 \). We claim that \( \sigma \) is in \( L_r^a + j \). For this we only need to verify that \( \sigma \subset \{j+1, \ldots, r+j\} \). By the choice of \( s \), \( \sigma \) does not contain any larger elements. By \( q \)-stability, \( \sigma \) cannot contain any of \( s+1, \ldots, s+q-1 \), all modulo \( r+q-1 \). Notice that \( s+q-1 \) modulo \( r+q-1 \) is \( j \), and so the smallest element vertex of \( \sigma \) is larger than \( j \).

Now let \( \sigma \) be a maximal face of \( L_r^a + j \). The cyclic gap between the smallest possible element \( j+1 \) and the largest possible element \( r+j \) is \( q \), and thus \( \sigma \) is a face of \( C_{r+q-1}^a \).

The following claim shows Theorem 5.1.

**Claim 5.7.** For \( r = aq + 2 \), the complex \( C_{r+q-1}^a \) is \( (a-2) \)-connected.
Proof. By Claim 5.6 we know that
\[ C_{r+q-1}^a = \bigcup_{j=0}^{q-1} (L_r^a + j) \]

For \( k = 0, \ldots, q - 1 \), consider the complex
\[ M_k = \bigcup_{j=0}^{k} (L_r^a + j) \]

We prove by induction on \( k \) that \( M_k \) is \((a - 2)\)-connected.

Base of induction. \( M_0 = L_r^a \), which is \((a - 2)\)-connected.

Inductive step. Assume that \( M_{k-1} \) is \((a - 2)\)-connected for some \( 1 \leq k \leq q - 1 \) and we want to show that \( M_k \) is also \((a - 2)\)-connected. We can write \( M_k \) as
\[ M_k = M_{k-1} \cup (L_r^a + k) \]

The first complex is \((a - 2)\)-connected by induction, and the second is \((a - 2)\)-connected by Claim 5.2 so we only need to look at their intersection. The only vertices we can take in the intersection are contained in \( \{ k + 1, k + 2, \ldots, r + k - 1 \} \). We claim that \( M_{k-1} \cap (L_r^a + k) = (L_r^a - 1 + k) \cup (L_{r+k-2q}^a + q) \).

Let \( \sigma \) be a face of \( M_{k-1} \cap (L_r^a + k) \). Thus \( \sigma \) is contained in a maximal face \( \tau_1 \) of \( (L_r^a + k) \) and in a maximal face \( \tau_2 \) of \( (L_r^a + j) \) for some \( j \in \{ 0, 1, \ldots, k - 1 \} \). If \( r + k \notin \tau_1 \) then \( \tau_1 \) and thus \( \sigma \) is a face of \( (L_r^a - 1 + k) \). If \( r + k \in \tau_1 \), then \( \tau_1 \cap \tau_2 \) has at most \( a - 1 \) vertices and \( r + k - q + 1, r + k - q + 2, \ldots, r + k - 1 \) are not in \( \tau_1 \cap \tau_2 \). If \( \tau_2 \) does not contain one of the first \( k \) vertices, we could choose \( \tau_1 = \tau_2 \), which implies that \( \sigma \) is a face of \( L_{r+k-1}^a + k \); a case that we have already treated above. But if \( \tau_2 \) does contain one of the first \( k \) vertices, then \( \tau_1 \cap \tau_2 \) does not contain any vertex among \( \{ 1, 2, \ldots, q \} \). Thus \( \tau_1 \cap \tau_2 \) is a face of \( (L_{r+k-2q}^a + q) \), and so \( \sigma \) is too.

If \( \sigma \) is a face of \( (L_r^a - 1 + k) \), then it is a face of \( (L_r^a + k) \) and of \( (L_r^a + (k - 1)) \) and thus in \( M_{k-1} \cap (L_r^a + k) \). If \( \sigma \) is a maximal face of \( (L_{r+k-2q}^a + q) \), then \( \sigma \cup \{ r + k \} \) is a face of \( L_r^a + k \), and \( \sigma \cup \{ 1 \} \) is a face of \( L_r^a \). Thus \( \sigma \) is a face of \( M_{k-1} \cap (L_r^a + k) \).

We now show that \( (L_{r-1}^a + k) \cup (L_{r+k-2q}^a + q) \) is \((a - 3)\)-connected. Notice that among the vertices \( \{ 1, 2, \ldots, r + k - 1 \} \) the map \( x \mapsto r + k - x \) induces an isomorphism of the complexes
\[
\begin{align*}
(L_r^a - 1 + k) &\to L_r^a \\
(L_{r+k-2q}^a + q) &\to (L_{r+k-2q}^a + (q - 1))
\end{align*}
\]

Therefore, \( M_{k-1} \cap (L_r^a + k) \cong L_{r-1}^a \cup (L_{r+k-2q}^a + (q - 1)) \), which by Claim 5.5 is \((a - 3)\)-connected. \[\square\]

6. Final remarks

We hope that this manuscript is only the starting point for many variants of Theorem 1.1 beyond prime powers. Combining our main result with the work of Blagojević, the first author, and Ziegler [BFZ14] yields some such variants “for free,” for example the following colorful version follows immediately from combining Theorem 1.1 with [BFZ14]. It extends the colorful Tverberg theorem of Živaljević and Vrećica [ZV92] beyond prime powers at the expense of allowing for a higher-dimensional simplex.
Theorem 6.1. Let \( q \geq 2, c \geq 0, \) and \( d \geq 1 \) be integers. Let \( C_1, \ldots, C_c \) be sets of vertices of \( \Delta_{q(d+c+1)-1} \) each of size at most \( 2q-1 \). Let \( f : \Delta_{q(d+c+1)-1} \to \mathbb{R}^d \) be a continuous map. Then there are \( q \) pairwise disjoint faces \( \sigma_1, \ldots, \sigma_q \) of \( \Delta_{q(d+c+1)-1} \) such that \( f(\sigma_1) \cap \cdots \cap f(\sigma_q) \neq \emptyset \) and each \( \sigma_i \) has at most one vertex in each \( C_j \).

If we wish to reduce the number of points in each color class \( C_j \), the reader may verify that it is sufficient to have \( |C_j| \geq q + \frac{q(d+1)}{c} \), and the domain to be a simplex with \( c \) times as many vertices. Therefore, as \( c \to q(d+1) \), the number of elements required in each color class approaches \( q+1 \). This gives us a continuous Tverberg type-theorem similar in parameters to the “equal coefficients” colorful Tverberg [BFZ14, Sob15] that works beyond prime powers.

We provide a precise statement below. The proof is the same as [BFZ14, Thm. 8.1], now using the new cases beyond prime powers of Theorem 6.1. Let \( C \) be a set of vertices of the simplex \( \Delta_n \). Let \( x \) and \( y \) be two points in \( \Delta_n \), and thus they can be written as convex combination of the vertices \( v_1, \ldots, v_{n+1} \), say \( x = \sum \lambda_i v_i \) and \( y = \sum \mu_i v_i \). We say that \( x \) and \( y \) have equal barycentric coordinates with respect to \( C \) if \( \sum v_i \in C \lambda_i = \sum v_i \in C \mu_i \).

Theorem 6.2. Let \( q \geq 2 \) and \( d \geq 1 \) be integers. Let \( C_1 \sqcup \cdots \sqcup C_{q(d+1)} \) be a partition of the vertex set of \( \Delta_{q(d+1)-1} \) each of size \( q+1 \). Let \( f : \Delta_{q(d+1)-1} \to \mathbb{R}^d \) be a continuous map. Then there are \( x_1, \ldots, x_q \) in \( q \) pairwise disjoint faces of \( \Delta_{q(d+1)-1} \) such that \( f(x_1) = \cdots = f(x_q) \) and all \( x_i \) have the same barycentric coordinates with respect to each \( C_j \).

It remains open if a colorful version of Theorem 6.2 is true. This is a known conjecture by Börényi and Larman [BL92], that has not been settled even for affine maps. It has been verified for \( q+1 \) a prime by Blagojević, Matschke, and Ziegler [BMZ11, BMZ15].

Conjecture 6.3. Let \( q \geq 2, d \geq 1 \) be integers. Let \( C_1, \ldots, C_{d+1} \) be pairwise disjoint sets of vertices of \( \Delta_{q(d+1)-1} \) each of size \( q \). Let \( f : \Delta_{q(d+1)-1} \to \mathbb{R}^d \) be a continuous map. Then there are \( q \) pairwise disjoint faces \( \sigma_1, \ldots, \sigma_q \) of \( \Delta_{q(d+1)-1} \) such that \( f(\sigma_1) \cap \cdots \cap f(\sigma_q) \neq \emptyset \) and each \( \sigma_i \) has at most one vertex in each \( C_j \).

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References

[AKS19] Sergey Avvakumov, Roman Karasev, and Arkadiy Skopenkov, Stronger counterexamples to the topological Tverberg conjecture, arXiv preprint arXiv:1908.08731 (2019).

[BBZ16] Imre Börényi, Pavle V. M. Blagojević, and Günter M. Ziegler, Tverberg’s theorem at 50: extensions and counterexamples, Notices Amer. Math. Soc. 63 (2016), no. 7, 732–739.

[BFZ14] Pavle V. M. Blagojević, Florian Frick, and Günter M. Ziegler, Tverberg plus constraints, Bull. Lond. Math. Soc. 46 (2014), no. 5, 953–967.

[BFZ19] Pavle V. M. Blagojević, Florian Frick, and Günter M. Ziegler, Barycenters of polytope skeleta and counterexamples to the topological Tverberg conjecture, via constraints, J. Europ. Math. Soc. (JEMS) 21 (2019), no. 7, 2107–2116.

[Bir59] Bryan J. Birch, On 3N points in a plane, Math. Proc. Camb. Phil. Soc. 55 (1959), no. 4, 289–293.

[Bjo05] Anders Björner, Topological methods, Handbook of combinatorics, vol. 2, Elsevier Sci. B. V., Amsterdam, 1995, pp. 1819–1872.

[BL92] Imre Börényi and David G. Larman, A Colored Version of Tverberg’s Theorem, J. Lond. Math. Soc. 22-45 (1992), no. 2, 314–320.

[BMZ11] Pavle V. M. Blagojević, Benjamin Matschke, and Günter M. Ziegler, Optimal bounds for a colorful Tverberg-Vrecica type problem, Adv. Math. 226 (2011), no. 6, 5198–5215.
[BMZ15] Pavle V. M. Blagojević, Benjamin Matschke, and Günter M. Ziegler, *Optimal bounds for the colored Tverberg problem*, J. Europ. Math. Soc. (JEMS) **17** (2015), no. 4, 739–754.

[BS18] Imre Bárány and Pablo Soberón, *Tverberg’s theorem is 50 years old: a survey*, Bull. Amer. Math. Soc. **55** (2018), no. 4, 459–492.

[BSS81] Imre Bárány, Senya Shlosman, and András Szücs, *On a topological generalization of a theorem of Tverberg*, J. Lond. Math. Soc. **2** (1981), no. 1, 158–164.

[BZ17] Pavle V. M. Blagojević and Günter M. Ziegler, *Beyond the Borsuk–Ulam Theorem: The Topological Tverberg Story*, A Journey Through Discrete Mathematics: A Tribute to Jiří Matoušek (Martin Loebl, Jaroslav Nešetřil, and Robin Thomas, eds.), Springer International Publishing, Cham, 2017, pp. 273–341.

[Dir37] Peter Gustav Lejeune Dirichlet, *Beweis des Satzes, dass jede unbegrenzte arithmetische Progression, deren erstes Glied und Differenz ganze Zahlen ohne gemeinschaftlichen Factor sind, unendlich viele Primzahlen enthält*, Abh. K. Preuss. Akad. Wiss. **45** (1837), 81.

[DLGMM19] Jesús De Loera, Xavier Goaoc, Frédéric Meunier, and Nabil Mustafa, *The discrete yet ubiquitous theorems of Carathéodory, Helly, Sperner, Tucker, and Tverberg*, Bull. Amer. Math. Soc. **56** (2019), no. 3, 415–511.

[Dol83] Albrecht Dold, *Simple proofs of some Borsuk–Ulam results*, Contemp. Math. **19** (1983), 65–69.

[Fri15] Florian Frick, *Countereamples to the topological Tverberg conjecture*, Oberwolfach Rep. **12** (2015), no. 1, 318–321.

[KL96] Wolfgang Kühnel and Günter Lassmann, *Permutated difference cycles and triangulated sphere bundles*, Discrete Math. **162** (1996), no. 1-3, 215–227.

[Koz99] Dmitry N. Kozlov, *Complexes of Directed Trees*, J. Combin. Theory, Ser. A **88** (1999), no. 1, 112–122.

[Mat03] Jiří Matoušek, *Using the Borsuk–Ulam theorem: Lectures on Topological Methods in Combinatorics and Geometry*, Springer Science & Business Media, 2003.

[MW14] Isaac Mabillard and Uli Wagner, *Eliminating Tverberg points, I. An analogue of the Whitney trick*, Proc. 30th Annual Symp. Comput. Geom. (SOCG) (Kyoto), ACM, 2014, pp. 171–180.

[MW15] Isaac Mabillard and Uli Wagner, *Eliminating higher-multiplicity intersections, I. A Whitney trick for Tverberg-type problems*, arXiv preprint [arXiv:1508.02349] (2015).

[Oza87] Murad Özaydin, *Equivariant maps for the symmetric group*, available at https://minds.wisconsin.edu/bitstream/handle/1793/63829/Ozaydin.pdf.

[Sob15] Pablo Soberón, *Equal coefficients and tolerance in coloured Tverberg partitions*, Combinatorica **35** (2015), no. 2, 235–252.

[SZ05] Torsten Schöneborn and Günter M. Ziegler, *The topological Tverberg theorem and winding numbers*, J. Combin. Theory, Ser. A **112** (2005), no. 1, 82–104.

[Tve66] Helge Tverberg, *A generalization of Radon’s theorem*, J. Lond. Math. Soc. **1** (1966), no. 1, 123–128.

[Vol96] Aleksei Yu. Volovikov, *On a topological generalization of the Tverberg theorem*, Math. Notes **59** (1996), no. 3, 324–326.

[VŽ93] Aleksandar Vučić and Rade T. Živaljević, *Note on a conjecture of Sierksma*, Discrete Comput. Geom. **9** (1993), no. 4, 339–349.

[ŽV92] Rade T. Živaljević and Siniša T. Vrečica, *The colored Tverberg’s problem and complexes of injective functions*, J. Combin. Theory, Ser. A **61** (1992), no. 2, 309–318.

Dept. Math. Sciences, Carnegie Mellon University, Pittsburgh, PA 15213, USA

Email address: frick@cmu.edu

Baruch College, City University of New York, One Bernard Baruch Way, New York, NY 10010, USA

Email address: pablo.soberon-bravo@baruch.cuny.edu