THE UNIVERSAL MINIMAL SPACE FOR GROUPS OF HOMEOMORPHISMS OF H-HOMOGENEOUS SPACES

ELI GLASNER AND YONATAN GUTMAN

To Anatoliy Stepin with great respect.

Abstract. Let $X$ be a $h$-homogeneous zero-dimensional compact Hausdorff space, i.e. $X$ is a Stone dual of a homogeneous Boolean algebra. It is shown that the universal minimal space $M(G)$ of the topological group $G = \text{Homeo}(X)$, is the space of maximal chains on $X$ introduced in [Usp00]. If $X$ is metrizable then clearly $X$ is homeomorphic to the Cantor set and the result was already known (see [GW03]). However many new examples arise for non-metrizable spaces. These include, among others, the generalized Cantor sets $X = \{0,1\}^\kappa$ for non-countable cardinals $\kappa$, and the corona or remainder of $\omega$, $X = \beta\omega \setminus \omega$, where $\beta\omega$ denotes the Stone-Čech compactification of the natural numbers.

1. Introduction

The existence and uniqueness of a universal minimal $G$ dynamical system, corresponding to a topological group $G$, is due to Ellis (see [Ell69], for a new short proof see [GL11]). He also showed that for a discrete infinite $G$ this space is never metrizable, and the latter statement was generalized to the locally compact non-compact case by Kechris, Pestov and Todorcevic in the appendix to their paper [KPT05]. For Polish groups this is no longer the case and we have such groups with $M(G)$ being trivial (groups with the fixed point property or extremely amenable groups) and groups with metrizable, easy to compute $M(G)$, like $M(G) = S^1$ for the group $G = \text{Homeo}_+(S^1)$.
and $M(G) = LO(\omega)$, the space of linear orders on a countable set, for $S_\infty(\omega)$ (GW02). Following Pestov’s work Uspenskij has shown in [Usp00] that the action of a topological group $G$ on its universal minimal system $M(G)$ (with $\text{card } M(G) \geq 3$) is never 3-transitive so that, e.g., for manifolds $X$ of dimension $> 1$ as well as for $X = Q$, the Hilbert cube, and $X = K$, the Cantor set, $M(G)$ can not coincide with $X$. Uspenskij proved his theorem by introducing the space of maximal chains $\Phi(X)$ associated to a compact space $X$. In [GW03] the authors then showed that for $X$ the Cantor set and $G = \text{Homeo}(X)$, in fact, $M(G) = \Phi$. It turns out that this group $G$ is a closed subgroup of $S_\infty(\omega)$ and in [KPT05] Kechris, Pestov and Todorcevic unified and extended these earlier results and carried out a systematic study of the spaces $M(G)$ for many interesting closed subgroups of $S_\infty$.

In the present work we go back to [GW03] and generalize it in another direction. We consider the class of h-homogeneous spaces $X$ and show that for every space in this class the universal minimal space $M(G)$ of the topological group $G = \text{Homeo}(X)$ is again Uspenskij’s space of maximal chains on $X$. If $X$ is metrizable then clearly $X$ is homeomorphic to the Cantor set and the result of [GW03] is retrieved (although even in this case our proof is new, as we make no use of a fixed point theorem). However, many new examples arise when one considers non-metrizable spaces. These include, among others, the generalized Cantor sets $X = \{0, 1\}^\kappa$ for non-countable cardinals $\kappa$, and the widely studied corona or remainder of $\omega$, $X = \beta\omega \setminus \omega$, where $\beta\omega$ denotes the Stone-Čech compactification of the natural numbers. As in [GW03] the main combinatorial tool we apply is the dual Ramsey theorem.

### 1.1. H-homogeneous spaces and homogeneous Boolean algebras.

The following definitions are well known (see e.g. [HNV04] Section H-4):

1. A zero-dimensional compact Hausdorff topological space $X$ is called **h-homogeneous** if every non-empty clopen subset of $X$ is homeomorphic to the entire space $X$.

2. A Boolean algebra $B$ is called **homogeneous** if for any nonzero element $a$ of $B$ the relative algebra $B \upharpoonright a = \{x \in B : x \leq a\}$ is isomorphic to $B$. 
Using Stone’s Duality Theorem (see [BS81] IV §4) a zero-dimensional compact Hausdorff \( h \)-homogeneous space \( X \) is the Stone dual of a homogeneous Boolean Algebra, i.e. any such space is realized as the space of ultrafilters \( B^* \) over a homogeneous Boolean algebra \( B \) equipped with the topology given by the base \( N_a = \{ U \in B^* : a \in U \} \), \( a \in B \). Here are some examples of \( h \)-homogeneous spaces (see [SR89]):

1. The countable atomless Boolean algebra is homogeneous. It corresponds by Stone duality to the Cantor space \( K = \{0, 1\}^\mathbb{N} \).

2. Every infinite free Boolean algebra is homogeneous. These Boolean algebras correspond by Stone duality to the generalized Cantor spaces, \( \{0, 1\}^\kappa \), for infinite cardinals \( \kappa \).

3. Let \( P(\omega) \) be the Boolean algebra of all subsets of \( \omega \) (the first infinite cardinal) and let \( fin \subset P(\omega) \) be the ideal comprising the finite subsets of \( \omega \). Define the equivalence relations \( A \sim_{fin} B \), \( A, B \in P(\omega) \), if and only if \( A \Delta B \) is in \( fin \). The quotient Boolean algebra \( P(\omega)/fin \) is homogeneous. This Boolean algebra corresponds by Stone duality to the corona \( \omega^* = \beta\omega \setminus \omega \), where \( \beta\omega \) denotes the Stone-Čech compactification of \( \omega \).

4. A topological space \( X \) is called a Parovičenko space if:
   
   (a) \( X \) is a zero-dimensional compact space without isolated points and with weight \( c \),
   
   (b) every two disjoint open \( F_\sigma \) subsets in \( X \) have disjoint closures, and
   
   (c) every non-empty \( G_\delta \) subset of \( X \) has non-empty interior.

Under CH Parovičenko proved that every Parovičenko space is homeomorphic to \( \omega^* \) ([Par63]).

In [DM78] van Douwen and van Mill show that under \( \neg \) CH, there are two non-homeomorphic Parovičenko spaces. Their second example of a Parovičenko space is the corona \( X = \beta Y \setminus Y \), where \( Y \) is the \( \sigma \)-compact space \( \omega \times \{0, 1\}^\mathbb{N} \). It is not hard to see that in \( Y \) the clopen sets are of the form \( L = \bigcup_{a \in A} \{a\} \times C_a \) for some \( A \subset \omega \), where for all \( a \in A \), \( C_a \) is non-empty and clopen. If \( |A| = \infty \) then \( L \cong Y \) and if \( |A| < \infty \) then \( Cl_{\beta Y}(L) \subset Y \). These facts imply in a
straightforward manner that $X$ is $h$-homogeneous. In [DM78] it is pointed out that under MA $X$ is not homeomorphic to $\omega^*$. Thus under $\neg\text{CH+MA}$, this example provides another weight $\mathfrak{c}$ $h$-homogeneous space.

(5) Let $\kappa$ be a cardinal. By a well-known theorem of Kripke ([Kri67]) there is a homogeneous countably generated complete Boolean algebra, the so called {f collapsing algebra} $C(\kappa)$ such that if $A$ is a Boolean algebra with a dense subset of power at most $\kappa$, then there is a complete embedding of $A$ in $C(\kappa)$.

(6) It is not hard to check that the product of any number of $h$-homogeneous spaces is again $h$-homogeneous.

1.2. The universal minimal space. A compact Hausdorff $G$-space $X$ is said to be {f minimal} if $X$ and $\emptyset$ are the only $G$-invariant closed subsets of $X$. By Zorn’s lemma each $G$-space contains a minimal $G$-subspace. These minimal objects are in some sense the most basic ones in the category of $G$-spaces. For various topological groups $G$ they have been the object of intensive study. Given a topological group $G$ one is naturally interested in describing all of them up to isomorphism. Such a description is given (albeit in a very weak sense) by the following construction: as was mentioned in the introduction one can show there exists a minimal $G$-space $M(G)$ unique up to isomorphism such that if $X$ is a minimal $G$-space then $X$ is a factor of $M(G)$, i.e., there is a continuous $G$-equivariant mapping from $M(G)$ onto $X$. $M(G)$ is called the {f universal minimal $G$-space}. Usually this minimal universal space is huge and an explicit description of it is hard to come by.

1.3. The space of maximal chains. Let $K$ be a compact Hausdorff space. We denote by $\text{Exp}(K)$ the space of closed subsets of $K$ equipped with the Vietoris topology. A subset $C \subset \text{Exp}(K)$ is a {f chain} in $\text{Exp}(K)$ if for any $E, F \in C$ either $E \subset F$ or $F \subset E$. A chain is {f maximal} if it is maximal with respect to the inclusion relation. One verifies easily that a maximal chain in $\text{Exp}(K)$ is a closed subset of $\text{Exp}(K)$, and that $\Phi(K)$, the space of all maximal chains in $\text{Exp}(K)$, is a closed subset of $\text{Exp}(\text{Exp}(K))$, i.e. $\Phi(K) \subset \text{Exp}(\text{Exp}(K))$ is a compact space. Note that a $G$-action on $K$ naturally induces a $G$-action on $\text{Exp}(K)$ and $\Phi(K)$. This is true in particular for
$K = M(G)$. As the $G$-space $\Phi(M(G))$ contains a minimal subsystem it follows that there exists an injective continuous $G$-equivariant mapping $f : M(G) \to \Phi(M(G))$. By investigating this mapping Uspenskij in $[\text{Usp00}]$ showed that for every topological group $G$, the action of $G$ on the universal minimal space $M(G)$ is not 3-transitive. As a direct consequence of this theorem only rarely the natural action of the group $G = \text{Homeo}(K)$ on the compact space $K$ coincides with the universal minimal $G$-action (as is the case for $X = S^1$). In $[\text{Gut08}]$ it was shown that for $G = \text{Homeo}(X)$, where $X$ belongs to a large family of spaces that contains in particular the Hilbert cube, the action of $G$ on the universal minimal space $M(G)$ is not 1-transitive.

It is easy to see that every $c \in \Phi(K)$ has a first element $F$ which is necessarily of the form $F = \{x\}$. Moreover, calling $x \triangleq r(c)$ the root of the chain $c$, it is clear that the map $\pi : \Phi(K) \to K$, sending a chain to its root, is a homomorphism of dynamical systems.

1.4. **The main result.** In $[\text{GW03}]$ it was shown that the universal minimal space of the group of homeomorphisms of the Cantor set, equipped with the compact-open topology, is the space of maximal chains over the Cantor set. Our goal is to prove the following generalization:

**Theorem.** Let $X$ be a $h$-homogeneous zero-dimensional compact Hausdorff topological space. Let $G = \text{Homeo}(X)$ equipped with the compact-open topology, then $M(G) = \Phi(X)$, the space of maximal chains on $X$.

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2. Preliminaries

2.1. Clopen covers. Let $X$ be a zero-dimensional compact Hausdorff space. Denote by $\mathcal{D}$ ($\hat{\mathcal{D}}$) the directed set (semilattice) consisting of all finite ordered (unordered) clopen partitions of $X$ which are necessarily of the form $\alpha = (A_1, A_2, \ldots, A_m)$ ($\hat{\alpha}$ = \{A_1, A_2, \ldots, A_m\}), where $\cup_{i=1}^n A_i = X$ (disjoint union). The relation is given by refinement: $\alpha \preceq \beta$ ($\hat{\alpha} \preceq \hat{\beta}$) iff for any $B \in \beta$ ($\hat{B} \in \hat{\beta}$), there is $A \in \alpha$ ($A \in \hat{\alpha}$) so that $B \subset A$. The join (least upper bound) of $\alpha$ and $\beta$, $\alpha \lor \beta = \{A \cap B : A \in \alpha, B \in \beta\}$, where the ordering of indices is given by the lexicographical order on the indices of $\alpha$ and $\beta$ ($\hat{\alpha} \lor \hat{\beta} = \{A \cap B : A \in \hat{\alpha}, B \in \hat{\beta}\}$). It is convenient to introduce the notations $\mathcal{D}_k = \{\alpha \in \mathcal{D} : |\alpha| = k\}$ and $\hat{\mathcal{D}}_k = \{\alpha \in \hat{\mathcal{D}} : |\hat{\alpha}| = k\}$. We denote the natural map $(A_1, A_2, \ldots, A_m) \mapsto \{A_1, A_2, \ldots, A_m\}$ by $\tilde{t} : \mathcal{D} \rightarrow \hat{\mathcal{D}}$. There is a natural $G$-action on $\mathcal{D}$ ($\hat{\mathcal{D}}$) given by $g(A_1, A_2, \ldots, A_m) = (g(A_1), g(A_2), \ldots, g(A_m))$ and $g(A_1, A_2, \ldots, A_m) = \{g(A_1), g(A_2), \ldots, g(A_m)\}$. Let $S_k$ denote the group of permutations of $\{1, \ldots, k\}$. $S_k$ acts naturally on $\mathcal{D}_k$ by $\sigma(B_1, B_2, \ldots, B_k) = (B_{\sigma(1)}, B_{\sigma(2)}, \ldots, B_{\sigma(k)})$ for any $\beta = (B_1, B_2, \ldots, B_k) \in \mathcal{D}_k$ and $\sigma \in S_k$. This action commutes with the action of $G$, i.e., $\sigma g \beta = g \sigma \beta$ for any $\sigma \in S_k$ and $g \in G$. Notice that one can identify $\hat{\mathcal{D}}_k = \mathcal{D}_k / S_k$.

2.2. Partition Homogeneity. Let us introduce a new definition:

Definition 2.1. A zero-dimensional compact Hausdorff space $X$ is called partition-homogeneous if for every two finite ordered clopen partitions of the same cardinality, $\alpha, \beta \in \mathcal{D}_m$, $\alpha = (A_1, A_2, \ldots, A_m)$, $\beta = (B_1, B_2, \ldots, B_m)$ there is $h \in \Homeo(X)$ such that $hA_i = B_i$, $i = 1, \ldots, k$.

Proposition 2.2. Let $X$ be an infinite zero-dimensional compact Hausdorff space. $X$ is h-homogeneous if and only if $X$ is partition-homogeneous.

Proof. Assume $X$ is h-homogeneous. Let $\alpha, \beta \in \mathcal{D}_m$, $\alpha = (A_1, A_2, \ldots, A_m)$, $\beta = (B_1, B_2, \ldots, B_m)$. Select homeomorphisms $h_{A,i}, h_{B,i}$, $i = 1, \ldots, m$ with $h_{A,i} : A_i \rightarrow X$, $h_{B,i} : B_i \rightarrow X$. Define $g \in \Homeo(X)$ by $g(x) = h_{B,i}^{-1} \circ h_{A,i}(x)$ for $x \in A_i$. Trivially $gA_i = B_i$. Assume now $X$ is partition-homogeneous. Let $A \neq X$ be a clopen set in $X$. We distinguish between two cases:
(1) \( A \) is a singleton. As \( X \) is partition-homogeneous there exists \( h \in \text{Aut}(X) \) with \( hA = A^c \) and \( hA^c = A \). We conclude \( X \) is a two point space contradicting the assumption that \( X \) is infinite.

(2) \( A \) is not a singleton. Because \( X \) is a compact Hausdorff zero-dimensional space we can find disjoint clopen sets \( A_1, A_2 \) such that \( A = A_1 \cup A_2 \). Let \( h_1 \in G \) so that \( h_1A_1 = A_1 \cup A^c \) and \( h_1A^c_1 = A_2 \). Define the homeomorphism \( h : A \to X \).

\[
h(x) = \begin{cases} h_1(x) & x \in A_1 \\ x & x \in A_2 \end{cases}
\]

\[\square\]

3. Basic properties of h-homogeneous spaces

3.1. Induced orders. Let \( X \) be a compact Hausdorff zero-dimensional h-homogeneous space and denote \( G = \text{Homeo}(X) \). As \( X \) is either trivial or infinite, we will assume from now onward, w.l.o.g. that \( X \) is infinite. Let \( \nu \in \Phi(X) \) and \( D \subset X \) a closed set. Define

\[
D_\nu = \bigcap_{A \in \nu : A \cap D \neq \emptyset} A
\]

By maximality of \( \nu \), one has \( D_\nu \in \nu \). By a standard compactness argument \( D_\nu \cap D \neq \emptyset \) and trivially it is the minimal element of \( \nu \) that intersects \( D \). Similarly for \( D \subset X \) a closed set with \( r(\nu) \in D \), define:

\[
D^\nu = \bigcup_{A \in \nu : A \subset D} A
\]

The maximal element of \( \nu \) that is contained in \( D \).

**Definition 3.1.** Let \( \nu \in \Phi(X) \) and \( \tilde{\alpha} = \{A_1, A_2, \ldots, A_m\} \in \tilde{\mathcal{D}} \). Define \( <_{\nu|\tilde{\alpha}} = <_{\nu} \), the induced order on \( \tilde{\alpha} \) by \( \nu \):

\[
A_i <_{\nu} A_j \iff (A_i)_\nu \subseteq (A_j)_\nu
\]

Similarly for \( \nu \in \Phi(X) \) and \( \alpha \in \mathcal{D} \), define the induced order \( <_{\nu|\alpha} = <_{\nu|\{\alpha\}} \). Denote by \( t^*_\nu : \tilde{\mathcal{D}} \to \mathcal{D} \) the map \( \{A_1, A_2, \ldots, A_m\} \mapsto (A_1, A_2, \ldots, A_m) \) where \( i < j \) if and only if
Let $\beta \in D$, define $t^*_v(\beta) = t^*_v(\tilde{t}(\beta))$. Notice that for all $\sigma \in S_k$, $v \in \Phi(X)$ and $\beta \in D$, 

$$t^*_v(\sigma t^*_v(\beta)) = t^*_v(\beta)$$

**Lemma 3.2.** $gt^*_v(\tilde{\beta}) = t^*_{gv}(g\tilde{\beta})$.

**Proof.** Let $\alpha = (A_1, A_2, \ldots, A_m) = t^*_v(\tilde{\beta})$. By definition $i < j$ if and only if $(A_i)_v \subseteq (A_j)_v$. Notice $(gA_i)_v = \bigcap_{g \in \mathcal{G}} gA_i \neq \emptyset gA = g \bigcap_{A \in \mathcal{V} A \cap A_i \neq \emptyset} A = g(A_i)_v$. Therefore $i < j$ if and only if $(gA_i)_v \subseteq (gA_j)_v$, and we conclude $g\alpha = t^*_v(g\tilde{\beta})$. \hfill $\square$

**Proposition 3.3.** Let $v \in \Phi(X)$ and $\bar{\alpha} = \{A_1, A_2, \ldots, A_m\} \in \tilde{D}$. $\bar{\alpha} <_v \bar{\alpha}$ is a linear order on $\bar{\alpha}$. The ordering $A_{i_1} <_v \bar{\alpha} A_{i_2} <_v \bar{\alpha} \ldots <_v \bar{\alpha} A_{i_m}$ is characterized by $(A_{i_k})_v \setminus (A_{i_1} \cup \ldots \cup A_{i_{k-1}})^v = \{x_k\}$ for $k = 1, 2, \ldots, m$ and suitable $x_k \in A_k$.

**Proof.** Let $D \subset X$ be clopen so that $r(v) \in D$, then it is easy to see that $v_{|D^c} \triangleq \{A \setminus D| D^c \subseteq A \in v\}$ is a maximal chain in $D^c$ and in particular has a root $r(v_{|D^c}) = x_0 \in D^c$. Let $i_1$ be such that $r(v) \in A_{i_1}$. Inductively let $i_{k+1}$ be such that $r(v_{|(A_{i_1} \cup A_{i_2} \cup \ldots \cup A_{i_k})^c}) \in A_{i_{k+1}}$. It is easy to see $A_{i_1} <_v A_{i_2} <_v \ldots <_v A_{i_m}$. This implies both that $<_v$ is a linear order and $(A_{i_k})_v \setminus (A_{i_1} \cup \ldots \cup A_{i_{k-1}})^v = \{x_k\}$ for some $x_k \in A_k$, $k = 1, 2, \ldots, m$. \hfill $\square$

### 3.2. Minimality and proximality of natural actions.

The basis for the Vietoris topology for the compact Hausdorff space $\text{Exp}(X)$ is given by open sets of the form:

$$\mathcal{U} = \langle A_1, \ldots, A_k \rangle = \{F \in \text{Exp}(X) : \forall i F \cap A_i \neq \emptyset \text{ and } F \subset \bigcup A_i\}$$

where $A_i \subset X$ is clopen. It is easy to see that a basis of clopen neighborhood of a maximal chain $v \in \Phi(X)$ is given by

$$\mathcal{U}_\alpha = \langle \mathcal{U}_1, \ldots, \mathcal{U}_n \rangle$$

where $\alpha = (A_1, A_2, \ldots, A_n) \in \mathcal{D}$ and

$$\mathcal{U}_j = \langle A_1, \ldots, A_j \rangle, \quad j = 1, 2, \ldots, n,$$

The following lemma is straightforward:
Lemma 3.4. Let $\alpha = (A_1, A_2, \ldots, A_n) \in D$ and $\upsilon \in \Phi(X)$. Let $\upsilon_{|\alpha}$ be the induced order of $\upsilon$ on $\alpha$, then $\upsilon \in \Upsilon_\alpha$ if and only if $\upsilon_{|\alpha} = \upsilon$, where $\upsilon$ is the usual order on \{1, 2, \ldots, n\}. In particular $\upsilon \in \Upsilon_{\upsilon_{|\alpha}}$.

Theorem 3.5. (1) The system $(X, G)$ is minimal.

(2) The system $(X, G)$ is extremely proximal; i.e. for every closed set $\emptyset \neq F \subsetneq X$ there exists a net $\{g_i\}_{i \in I} \in G$ such that we have $\lim_{i \in I} g_i F = \{x_0\}$ for some point $x_0 \in X$ (see [Gla74]).

(3) The minimal system $(X, G)$ is not isomorphic to the universal minimal system $(M(G), G)$.

(4) $(\Phi(X), G)$ is minimal.

(5) $(\Phi(X), G)$ is proximal.

Proof.

(1) Since $X$ is h-homogeneous, then by Proposition 2.2, $G$ acts transitively on non-trivial (i.e. not $\emptyset, X$) clopen sets. Since $G$ acts transitively on the above mentioned basis, it follows that for every $U \in \mathcal{U}$ we have $\cup\{\alpha(U) : \alpha \in G\} = X$. This property is equivalent to the minimality of the system $(X, G)$.

(2) Fix some $x_0$ in $X$ such that $x_0 \notin F$. For an arbitrary basic clopen neighborhood $U = A$ of $x_0$ which is disjoint from $F$ choose $\alpha_U \in G$ such that $\alpha_U(A^c) = A$. Then $\alpha$ satisfies $\alpha_U(F) \subset U$. Clearly now $\{\alpha_U : U \text{ a neighborhood of } x_0\}$ is the required net.

(3) As the system $(X, G)$ is certainly 3-transitive this claim follows from Uspenskiy’s theorem [Usp00]. For completeness we provide a direct proof. Suppose $(X, G)$ is isomorphic to the universal minimal $G$ system. Let $Y \subset \Phi$ be a minimal subset of $\Phi$. Then, by the coalescence of the universal minimal system (every $G$-endomorphism $\phi : (M(G), G) \to (M(G), G)$ (which is necessarily onto) is an isomorphism, see [GL11] and [Usp00]), the restriction $\pi : Y \to X$, sending a chain to its root, is an isomorphism. Fix $c_0 \in Y$ and let $p_0 \in X$ be its root; i.e. $\pi(c_0) = p_0$. Let $H = \{\alpha \in G : \alpha p_0 = p_0\}$, the stability group of $p_0$. Since $\pi$ is an isomorphism we also have $H = \{\alpha \in G : \alpha c_0 = c_0\}$. Choose
Let \( p_0 \in c_0 \) such that \( \{p_0\} \subsetneq F \subsetneq X \) and let \( p_0 \neq a \in F \) (recall \( X \) is infinite).

Choose a clopen partition of \((P, A, B)\) of \( X \) with \( B \cap F = \emptyset, P \cap F \neq \emptyset \) and \( A \cap F \neq \emptyset \). Using the fact that \( X \) is partition homogeneous, one can find \( g \in G \) so that \( gP = P, gA = B \) and \( gB = A \). One redefines \( g \) so that \( g\mid_P = Id \). As \( g(A \cup P) \cap A = \emptyset \), we have \( F \setminus gF \neq \emptyset \). As \( gA = B \) we have \( gF \setminus F \neq \emptyset \). Conclude that \( F \) and \( gF \) are not comparable. On the other hand \( g(p_0) = p_0 \) means \( g \in H \) whence also \( gc_0 = c_0 \). In particular \( gF \in c_0 \) and as \( c_0 \) is a chain one of the inclusions \( F \subset gF \) or \( gF \subset F \) must hold. This contradiction shows that \((X, G)\) cannot be the universal minimal \( G \)-system.

(4) Let \( \nu', \nu \in \Phi(X) \) and \( \nu' \in \mathcal{U}_\alpha \) for some \( \alpha = (A_1, A_2, \ldots, A_n) \in \mathcal{D} \). Let \( <_\nu \) be the induced order of \( \nu \) on \( \alpha \). Let \( \sigma \in S_\alpha \) be such that for any \( i < j \), \( A_{\sigma(i)} <_\nu A_{\sigma(j)} \). As \( X \) is partition homogeneous we can choose \( g \in G \) so that \( gA_{\sigma(i)} = A_i \). Clearly \( g\nu \in \mathcal{U}_\alpha \).

(5) Let \( \nu_1, \nu_2 \in \Phi(X) \). Fix some \( \nu' \in \mathcal{U}_\alpha \) for some \( \alpha = (A_1, A_2, \ldots, A_n) \in \mathcal{D} \). Let \( < \) be the usual order on \( \{1, 2, \ldots, n\} \). Inductively we will construct \( g \in G \) so that \( <_{g\nu_1|_\alpha} = <_{g\nu_2|_\alpha} = < \). Using Lemma 3.4 this implies \( g\nu_1 \in \mathcal{U}_\alpha \) and \( g\nu_2 \in \mathcal{U}_\alpha \).

As \( \mathcal{U}_\alpha \) is arbitrary, this establishes proximality. Indeed let \( g_1 \in G \) so that \( g_1(r(\nu_1)), g_1(r(\nu_2)) \in A_1 \). Assume we have constructed \( g_k \in G \). Define \( g_{k+1} \in G \) so that \( g_{k+1}|_{A_1 \cup A_2 \cup \ldots \cup A_k} = g_k|_{A_1 \cup A_2 \cup \ldots \cup A_k} \) and \( g_{k+1}(r((g_k\nu_1)((A_1 \cup A_2 \cup \ldots \cup A_k)')')) \), \( g_{k+1}(r(g_k\nu_2)((A_1 \cup A_2 \cup \ldots \cup A_k)')')) \in A_{i+1} \). It is easy to see that \( g = g_n \) has the desired properties.

\[ \square \]

4. Calculation of the universal minimal space

4.1. Overview. The goal of this section is to generalize the main theorem of [GW03]: the universal minimal space of the group of homeomorphisms of the Cantor set, equipped with the compact-open topology, is the space of maximal chains over the Cantor set. We prove the following theorem:
**Theorem 4.1.** Let $X$ be a $h$-homogeneous zero-dimensional compact Hausdorff topological space. Let $G = \text{Homeo}(X)$ equipped with the compact-open topology, then $M(G) = \Phi(X)$.

The proof borrows heavily from the proof in [GW03]. The new ideas (that build on ideas in [GW03]) are presented in subsections 4.2, 4.3, 4.5.

### 4.2. Order topology.

Recall that given a set $Y$ and a linear order $<$ on $Y$ there is a topology generated by the basis of open intervals $(a, b) = \{ y \in Y : a < y < b \}$ where $a, b \in Y$ and equality is allowed on the left (right) if $a$ ($b$) is the smallest (biggest) element of $Y$. This topology is called the order topology on $(Y, <)$. For more details see [Mun75] Section 2.3. One of the most important ingredients in the proof in [GW03] is the fact that the topology on the cantor set $K$ is the order topology associated with the natural order $<$ on $K \subset [0, 1])$. A natural approach to generalizations of the result in the case of $X = \omega^*$ the corona, is to look for an order that will generate the topology on the corona. However, as the following proposition shows this is impossible.

**Proposition 4.2.** The topology on $\omega^*$ is not an order topology.

*Proof.* Assume for a contradiction that the topology on $\omega^*$ is an order topology associated with a linear order $<$. As $\omega^*$ has no isolated points we can find (with no loss of generality) an increasing bounded sequence of points $p_1 < p_2 < p_3 < \cdots < b$. By compactness this sequence admits a least upper bound $p = \text{l.u.b} \{ p_k : k = 1, 2, \ldots \}$. It is easy to check that $p = \lim_{k \to \infty} p_k$, so that the set $\{ p_k : k = 1, 2, \ldots \} \cup \{ p \}$ is a closed subset of $\omega^*$. However, it is well known that the remainder $\omega^*$ has no nontrivial converging sequences; e.g. one can use the fact that the closure of the set $\{ p_k : k = 1, 2, \ldots \}$, like the closure of any infinite discrete countable set in $\beta \omega$, is homeomorphic to $\beta \omega$ (see e.g. [Eng78] Theorem 3.6.14)).

### 4.3. The spaces $\Omega_k$ and $\tilde{\Omega}_k$ and a cocycle equation.

The following subsection is a generalization of Section 3 of [GW03]. Fix $\alpha = (A_1, A_2, \ldots, A_k) \in \mathcal{D}_k$ and define the
clopen subgroup $H_\alpha = \{g \in G : gA_i = A_i, i = 1, \ldots, k\} \subset G$. Consider the discrete homogeneous space of right cosets $H_\alpha \backslash G = \{H_\alpha g : g \in G\}$. There is a natural bijection $\phi : H_\alpha \backslash G \to D_k$ given by $\phi(H_\alpha g) = g^{-1}\alpha$. Let $\Omega_k = \{1, -1\}^{D_k}$ equipped with the product topology. This is a $G$-space under the action $g\omega(\beta) = \omega(g^{-1}\beta)$ for any $\omega \in \Omega_k$, $\beta \in D_k$ and $g \in G$.

Set $\mathcal{T}^k = \{1, -1\}^{S_k}$. We refer to the elements of $\mathcal{T}^k$ as tables. Denote $\tilde{\Omega}_k = (\mathcal{T}^k)^{D_k}$ equipped with the product topology. This is a $G$-space under the action $\cdot : G \times \tilde{\Omega}_k \to \tilde{\Omega}_k$ given by $g \cdot \omega(\tilde{\beta})(\sigma) = \omega(g^{-1}\tilde{\beta})(\sigma)$ for any $\omega \in \Omega_k$, $\tilde{\beta} \in \tilde{\Omega}_k$ and $g \in G$.

There is a natural family of homeomorphisms $\pi_c : \Omega_k \to \tilde{\Omega}_k$, $c \in \Phi(X)$ given by $\omega \mapsto \omega^c$, (also denoted $\tilde{\omega}$ when no confusion arises) where for $\tilde{\beta} = \{B_1, B_2, \ldots, B_k\} \in \tilde{D}_k$ and $\sigma \in S_k$, $\tilde{\omega}(\tilde{\beta})(\sigma) = \omega(\sigma^{-1}t^*_c(\tilde{\beta}))$ ($t^*_c(\cdot)$ is defined after Definition 3.1). In order for $\pi_c$ to be a $G$-homeomorphism we need to equip $\tilde{\Omega}_k$ with a different $G$-action than the natural $G$-action mentioned above. Namely $\bullet_c : G \times \tilde{\Omega}_k \to \tilde{\Omega}_k$, is defined by

$$g \bullet_c \tilde{\omega}(\tilde{\beta})(\sigma) = \omega(g^{-1}\tilde{\beta})(\rho_c(g, \tilde{\beta})\sigma) = \omega(\sigma^{-1}\rho_c(g, \tilde{\beta})^{-1}t^*_c(g^{-1}\tilde{\beta}))$$

where $\rho_c : G \times \tilde{\Omega}_k \to S_k$ is defined uniquely by the equation:

$$\rho_c(g, \tilde{\beta})^{-1}t^*_c(g^{-1}\tilde{\beta}) = g^{-1}t^*_c(\tilde{\beta})$$

As $g \bullet_c \tilde{\omega}(\tilde{\beta})(\sigma) = \omega(\sigma^{-1}g^{-1}t^*_c(\tilde{\beta}))$, we have the equality $g \bullet_c \tilde{\omega}(\tilde{\beta})(\sigma) = \tilde{\omega}(\tilde{\beta})(\sigma)$ which makes $\pi_c : (G, \Omega_k) \to (G, \tilde{\Omega}_k)$ a $G$-homeomorphism (and formally proves $g \bullet_c$ is indeed a $G$-action).

**Lemma 4.3.** $\rho_c : G \times \tilde{\Omega}_k \to S_k$ obeys the following **cocycle** equation:

$$\rho_c(gh, \tilde{\beta}) = \rho_c(g, \tilde{\beta})\rho_c(h, g^{-1}\tilde{\beta})$$

**Proof.** By definition we have $gh \bullet_c \tilde{\omega}(\tilde{\beta})(\sigma) = \tilde{\omega}(\tilde{\beta})(\sigma) = g \bullet_c \tilde{\omega}(\tilde{\beta})(\sigma)$. Notice

$$gh \bullet_c \tilde{\omega}(\tilde{\beta})(\sigma) = \omega(\sigma^{-1}\rho_c(gh, \tilde{\beta})^{-1}t^*_c(h^{-1}g^{-1}\tilde{\beta})),$$

whereas

$$g \bullet_c \tilde{\omega}(\tilde{\beta})(\sigma) = h\omega(\sigma^{-1}\rho_c(g, \tilde{\beta})^{-1}t^*_c(g^{-1}\tilde{\beta})) = \omega(\sigma^{-1}h^{-1}\rho_c(g, \tilde{\beta})^{-1}t^*_c(g^{-1}\tilde{\beta})).$$
This implies
\[ \rho_c(gh, \tilde{\beta})^{-1}t^*_c(h^{-1}g^{-1}\beta) = h^{-1}\rho_c(g, \tilde{\beta})^{-1}t^*_c(g^{-1}\tilde{\beta}). \]
As \( \rho_c(h, g^{-1}\tilde{\beta})^{-1}t^*_c(h^{-1}g^{-1}\beta) = h^{-1}t^*_c(g^{-1}\tilde{\beta}) \), we have
\[ \rho_c(gh, \tilde{\beta})^{-1} = \rho_c(h, g^{-1}\tilde{\beta})^{-1}\rho_c(g, \tilde{\beta})^{-1}. \]
Taking the inverses we get \( \rho_c(gh, \tilde{\beta}) = \rho_c(g, \tilde{\beta})\rho_c(h, g^{-1}\tilde{\beta}) \) \( \square \)

Note that in the end of Section 3 of [GW03] it was mistakenly claimed that \( g \cdot \omega \tilde{\beta}(\sigma) = g \cdot \omega(\tilde{\beta})(\sigma) \), for \( c_0 = \{ [0, t] \cap K \}_{t \in [0, 1]} \) where \( K \), the Cantor set, is embedded naturally in \([0, 1]\).

4.4. The Dual Ramsey Theorem. A partition \( \gamma = (C_1, \ldots, C_k) \) of \( \{1, \ldots, s\} \) into \( k \) nonempty sets is naturally ordered if for any \( 1 \leq i < j \leq k \), \( \min(C_i) < \min(C_j) \).
We denote by \( \Pi(\kappa) \) the collection of naturally ordered partitions of \( \{1, \ldots, s\} \) into \( k \) nonempty sets.

**Definition 4.4.** Let \( \beta = (B_1, \ldots, B_s) \in \Pi(\kappa) \) and \( \gamma = (C_1, \ldots, C_k) \in \Pi(\mu) \) define the amalgamated partition \( \gamma_\beta = (G_1, \ldots, G_s) \in \Pi(\mu) \) by:
\[ G_j = \bigcup_{i \in B_j} C_i \]
Notice \( \gamma_\beta \) is naturally ordered and \( (P_\gamma)_\beta = P_{\gamma_\beta} \). Similarly for \( \alpha = (A_1, A_2, \ldots, A_m) \in \mathcal{D} \) define the amalgamated clopen cover \( \alpha_\gamma = (G_1, G_1, \ldots, G_k) \), where \( G_j = \bigcup_{i \in C_j} A_i \). Notice that \( (\alpha_\gamma)_\beta = \alpha_{\gamma_\beta} \).

We denote by \( \tilde{\Pi}(\kappa) \) the collection of unordered partitions of \( \{1, \ldots, s\} \) into \( k \) nonempty sets. Notice there is a natural bijection \( \tilde{\Pi}(\kappa) \leftrightarrow \Pi(\kappa) \).

**Theorem 4.5.** [The dual Ramsey Theorem] Given positive integers \( k, m, r \) there exists a positive integer \( N = DR(k, m, r) \) with the following property: for any coloring of \( \tilde{\Pi}(N)_k \) by \( r \) colors there exists a partition \( \alpha = \{A_1, A_2, \ldots, A_m\} \in \tilde{\Pi}(N)_m \) of \( N \) into \( m \) non-empty sets such that all the partitions of \( N \) into \( k \) non-empty sets (i.e. elements of \( \tilde{\Pi}(N)_k \)) whose atoms are measurable with respect to \( \alpha \) (i.e. each equivalence class is a union of elements of \( \alpha \)) have the same color.
Definition 4.6. Let $\beta \in \mathcal{D}$ and $c \in \Phi(X)$, define the $\beta$-ratio of $c$, to be the unique element $\theta_\beta(c) \in S_k$ so that:

$$\theta_\beta(c)\beta = t^*_c(\beta)$$

Lemma 4.7. The following holds:

1. $\theta_\beta(c) = \theta_{g\beta}(gc)$ for $c \in \Phi(X)$, $g \in G$ and $\beta \in \mathcal{D}$.
2. $\theta_{\sigma^{-1}t^*_c(\tilde{\beta})}(c) = \sigma$ for $\sigma \in S_k$, $\tilde{\beta} \in \bar{\mathcal{D}}$ and $c \in \Phi(X)$.

Proof.

1. By definition $\theta_{g\beta}(gc)\beta = t^*_c(g\beta)$. By Lemma 3.2, $gt^*_c(\beta) = t^*_c(g\beta)$ and therefore one has $\theta_{g\beta}(gc)\beta = gt^*_c(\beta)$. As the $G$ and $S_k$ actions commute it implies $\theta_{g\beta}(gc)\beta = t^*_c(\beta)$. By definition $\theta_\beta(c)\beta = t^*_c(\beta)$ and we conclude $\theta_\beta(c) = \theta_{g\beta}(gc)$.

2. $\theta_{\sigma^{-1}t^*_c(\tilde{\beta})}(c)\sigma^{-1}t^*_c(\tilde{\beta}) = t^*_c(\sigma^{-1}t^*_c(\tilde{\beta}))$

Let $T \in T^k$. Define $\phi_T : \Phi(X) \to \Omega_k$ by

$$\phi_T(c)(\beta) = T(\theta_\beta(c))$$

Lemma 4.8. $\phi_T : \Phi(X) \to \Omega_k$ is continuous and $G$-equivariant.

Proof. We start by showing that $\phi_T$ is continuous. Let $n \in \mathbb{N}$, $\epsilon_1, \epsilon_2, \ldots \epsilon_n \in \{\pm 1\}$, $\beta_1, \beta_2, \ldots \beta_n \in \mathcal{D}_k$. Let $V$ be an open set of $\Omega_k$ so that $V = \{\omega \in \Omega_k : \omega(\beta_i) = \epsilon_i\}$ and assume $V \neq \emptyset$. Let $c_1 \in \phi_T^{-1}(V)$. Denote $\mathfrak{U} = \bigcap_{i=1}^n \mathfrak{U}_{i,\epsilon_i}(\beta_i)$. By Lemma 3.4, $c_1 \in \mathfrak{U}$ so $\mathfrak{U} \neq \emptyset$. We claim $\phi_T(\mathfrak{U}) \subset V$. Indeed let $c_2 \in \mathfrak{U}$ and fix $i$. By assumption $c_2 \in \mathfrak{U}_{i,\epsilon_i}(\beta_i)$. By Lemma 3.4 $c_2 \in \mathfrak{U}_{i,\epsilon_i}(\beta_i)$. Conclude $t^*_c(\beta) = t^*_c(\beta)$, which implies $\theta_\beta(c_1) = \theta_\beta(c_2)$. This in turn implies $\phi_T(c_1)(\beta_i) = \phi_T(c_2)(\beta_i) = \epsilon_i$. 

Proof. This is Corollary 10 of [GR71].

4.5. Minimal symbolic systems. In the beginning of Section 4 of [GW03] a family of mappings $\phi_T : (G, \Phi(X)) \to (G, \Omega_k), T \in T^k$ are introduced. We will introduce a generalized family but using a different description.

Proof. This is Corollary 10 of [GR71].
To show $G$-equivariance one has to show $g\phi_T(c)(\beta) = \phi_T(c)(g^{-1}\beta) = \phi_T(gc)(\beta)$. By definition $\phi_T(c)(g^{-1}\beta) = T(\theta_{g^{-1}\beta}(c))$ whereas $\phi_T(gc)(\beta) = T(\theta_\beta(gc))$. By Lemma 4.7 $\theta_\beta(gc) = \theta_{g^{-1}\beta}(c)$. □

Let $c_0 \in \Phi(X)$. We will investigate $\pi_{c_0} \circ \phi_T$. By definition $\hat{\omega}^{c_0}(\tilde{\beta})(\sigma) = \omega(\sigma^{-1}t^*_{c_0}(\tilde{\beta}))$ and therefore we have $\hat{\phi}_T(c)(\tilde{\beta})(\sigma) = T(\theta_{\sigma^{-1}t^*_{c_0}(\tilde{\beta})}(c))$. By Lemma 4.7

$$\hat{\phi}_T(c_0)(\tilde{\beta})(\sigma) = \sigma$$

In particular $\hat{\phi}_T(c_0)(\tilde{\beta})(\sigma)$ does not depend on $\tilde{\beta}$ and we denote it by $\hat{\omega}_T$.

The following theorem is based on Theorem 4.1 of [GW03]:

**Theorem 4.9.** Every minimal subsystem of $(G, \Omega_k)$ is a factor of $(G, \Phi(X))$.

**Proof.** Fix a minimal subset $\Sigma \subset \Omega_k$. We shall construct a homomorphism $\phi : (G, \Phi(X)) \rightarrow (G, \Sigma)$. Moreover it will be shown that $\phi = \phi_T$ for some $T \in \mathcal{T}^k$. Fix a point $\omega \in \Sigma$ and $c_0 \in \Phi(X)$. We consider $\hat{\omega}^{c_0}$ as a coloring of elements of $\hat{D}_k$ by $r = |\mathcal{T}^k|$ where the colors are the tables of $\mathcal{T}^k$. For $\tilde{\beta} \in \hat{D}_k$, we thus denote by $\hat{\omega}^{c_0}(\tilde{\beta})$ the element in $\mathcal{T}^k$. Let $m \in \mathbb{N}$ and fix $\alpha \in \mathcal{D}_m$. Let $\beta \in \mathcal{D}$ such that $\alpha \preceq \beta$, $t^*_{c_0}(\beta) = \beta$ and $|\beta| = N = DR(k, m, r)$ as in Theorem 4.5

We define the coloring map to be $f : \Pi(m)_k \rightarrow \mathcal{T}^k$ where $\gamma \mapsto \hat{\omega}^{c_0}(\tilde{\iota}(\beta, \gamma))$. According to Theorem 4.5 there exists $\eta \in \Pi(m)_m$ and $T_\alpha \in \mathcal{T}^k$ so that for any $\tau \in \Pi(m)_k$, $f(\eta_{\tau}) = T_\alpha$. Let $g_\alpha \in G$ be such that $g_\alpha^{-1}t^*_{c_0}(\alpha) = \beta_\eta$. Denote $\hat{\omega}^{c_0}_{g_\alpha} = g_\alpha \bullet c_0 \hat{\omega}^{c_0}$. Notice

$$\hat{\omega}^{c_0}_{g_\alpha}(\tilde{\iota}(t^*_{c_0}(\alpha)_{\tau}))(\sigma) = \omega(\sigma^{-1}g_\alpha^{-1}(t^*_{c_0}(\alpha)_{\tau}))$$

$$= \omega(\sigma^{-1}(g_\alpha^{-1}t^*_{c_0}(\alpha))_{\tau})$$

$$= \omega(\sigma^{-1}(\beta_\eta)_{\tau}) = \omega(\sigma^{-1}\beta_{\eta_{\tau}})$$

for any $\tau \in \Pi(m)_m$. We also have

$$T_\alpha = f(\eta_{\tau}) = \hat{\omega}^{c_0}(\tilde{\iota}(\beta_{\eta_{\tau}}))(\sigma) = \omega(\sigma^{-1}t^*_{c_0}(\tilde{\iota}(\beta_{\eta_{\tau}}))) = \omega(\sigma^{-1}\beta_{\eta_{\tau}})$$

as $t^*_{c_0}(\beta) = \beta$. Conclude $\hat{\omega}^{c_0}_{g_\alpha}(\tilde{\iota}(t^*_{c_0}(\alpha)_{\tau})) = T_\alpha$. Let $\tilde{v} \in \Sigma$ be an accumulation point of the net $\{\hat{\omega}^{c_0}_{g_\alpha}\}_{\alpha \in \mathcal{D}}$. Let $\tilde{\xi}_1, \tilde{\xi}_2 \in \hat{D}_k$. Let $\alpha$ be a common ordered refinement. By the
calculations we have just performed for any $\gamma \geq \alpha$, $\tilde{\xi}_1 = \tilde{t}(t_{c_0}(\gamma), \tau_1)$ and $\tilde{\xi}_2 = \tilde{t}(t_{c_0}(\gamma), \tau_2)$ for some $\tau_1, \tau_2 \in \Pi([\gamma])$, we have $\tilde{\omega}_{g_1}(\tilde{\xi}_1) = \tilde{\omega}_{g_2}(\tilde{\xi}_2)$. This implies there exists $T \in T^k$ such that for any $\tilde{\xi} \in \tilde{D}_k$, $\tilde{\nu}(\tilde{\xi}) = T$, i.e. $\tilde{\nu} = \tilde{\omega}_T$ defined above. We conclude $\Sigma = \phi_T(\Phi(X))$. 

4.6. Calculation of the universal minimal space. We now proceed as in [GW03].

Lemma 4.10. If $Y$ is zero-dimensional compact Hausdorff topological space then the topological group $\text{Homeo}(Y)$ equipped with the compact-open topology has a clopen basis at the identity.

Proof. See the proof of Lemma 3.2 of [MS01]. The clopen basis is given by $\{H_\alpha\}_{\alpha \in D}$ where $H_\alpha$ is defined in Subsection 4.3.

Theorem 4.11. Let $H$ be a topological group. If the topology of $H$ admits a basis for neighborhoods at the identity consisting of clopen subgroups, then $M(H)$ is zero dimensional.

Proof. This follows from Proposition 3.4 of [Pes98] where it is shown that under the same conditions the greatest ambit of $H$ is zero-dimensional.

We now give the proof of Theorem 4.1.

Proof. The proof is a reproduction of the proof appearing in [GW03] that $M(G) = \Phi(K)$, where $K$ is the Cantor set and $G = \text{Homeo}(K)$ is equipped with the compact-open topology. By Theorem 3.5 $(G, \Phi(X))$ is minimal and therefore there is an epimorphism $\pi : (G, M(G)) \to (G, \Phi(X))$. Fix $c_0 \in \Phi(X)$ and let $m_0 \in M(G)$ so that $\pi(m_0) = c_0$. By Lemma 4.10 and Theorem 4.11 $M(G)$ is zero-dimensional. Let $D \subset M(G)$ be a clopen subset and define the continuous function $F_D = 21_D - 1$, where $1_D$ is the indicator function of $D$. If $H = \{g \in G : gD = D\}$ then $H$ is a clopen subgroup of $G$ and hence it contains $H_\alpha$ for some $\alpha \in D_k$ for some $k \in \mathbb{N}$ (see proof of Lemma 4.10). It follows that the map $\psi_D(m) = (F_D(gm))_{g \in G}$, $m \in M(G)$ can be defined as a mapping into $\{1, -1\}^{H_\alpha \setminus G} = \Omega_k$ and thus we have
\( \psi_D : (G, M(G)) \to (G, \Omega_k) \), so that if we set \( Y_D = \psi_D(M(G)) \), the system \((Y_D, G)\) is a minimal symbolic subsystem of \( \Omega_k \). Denote \( y_D = \psi_D(m_0) \).

Apply Theorem 4.9 to define a \( G \)-homomorphism \( \phi_D : \Phi \to \Omega_k \), with and \( y_D' = \phi_D(c_0) \). Given a clopen subset \( D \subset M(G) \) consider the following diagram:

\[
\begin{array}{ccc}
(M(G), m_0) & \xrightarrow{\pi} & (\Phi, c_0) \\
\psi_D \downarrow & & \downarrow \phi_D \\
(Y_D, y_D) & & (Y_D, y_D')
\end{array}
\]

The image \((\psi_D \times (\phi_D \circ \pi))(M(G), m_0) = (W, (y_D, y_D'))\), with \( W \subset Y_D \times Y_D \), is a minimal subset of the product system \((Y_D \times Y_D, G)\). By Theorem 3.5(5) \((Y_D, G)\) is proximal. Therefore the diagonal \( \Delta = \{(y, y) : y \in Y_D\} \) is the unique minimal subset of the product system and we conclude that \( y_D = y_D' \), so that the above diagram is replaced by

\[
\begin{array}{ccc}
(M(G), m_0) & \xrightarrow{\pi} & (\Phi, c_0) \\
\psi_D & & \phi_D \\
(Y_D, y_D) & &
\end{array}
\]

Next form the product space

\[ \Pi = \prod \{Y_D : D \text{ a clopen subset of } M(G)\}, \]

and let \( \psi : M(G) \to \Pi \) be the map whose \( D \)-projection is \( \psi_D \) (i.e. \( (\psi(m))_D = \psi_D(m) \)). We set \( Y = \psi(M(G)) \) and observe that, since clearly the maps \( \psi_D \) separate points on \( M(G) \), the map \( \psi : M(G) \to Y \) is an isomorphism, with \( \psi(m_0) = y_0 \), where \( y_0 \in Y \) is defined by \( (y_0)_D = y_D \). Likewise define \( \phi : \Phi(X) \to Y \) by \( (\phi(m))_D = \phi_D(m) \), so that also \( \phi(c_0) = y_0 \). These equations force the identity \( \psi = \phi \circ \pi \) in the diagram

\[
\begin{array}{ccc}
(M(G), m_0) & \xrightarrow{\pi} & (\Phi, c_0) \\
\psi & & \phi \\
(Y, y_0) & &
\end{array}
\]

Since \( \psi \) is a bijection it follows that so are \( \pi \) and \( \phi \) and the proof is complete. \( \square \)
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Eli Glasner, School of Mathematical Sciences, Tel Aviv University, Ramat Aviv, Tel Aviv 69978, Israel.
E-mail address: glasner@math.tau.ac.il

Yonatan Gutman, Laboratoire d’Analyse et de Mathématiques Appliquées, Université de Marne-la-Vallée, 5 Boulevard Descartes, Cité Descartes - Champs-sur-Marne, 77454 Marne-la-Vallée cedex 2, France.
E-mail address: yonatan.gutman@univ-mlv.fr