**B – L model with D₄ × Z₄ × Z₂ symmetry for fermion mass hierarchies and mixings** *

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**Abstract:** We constructed a gauge B – L model with D₄ × Z₄ × Z₂ symmetry to explain the quark and lepton mass hierarchies and their mixings with realistic CP phases via the type-I seesaw mechanism. Six quark masses, three quark mixing angles, and the CP phase in the quark sector take the central values whereas Yukawa couplings in the quark sector are diluted in a range of difference of three orders of magnitude by the perturbation theory at the first order. Concerning the neutrino sector, a small neutrino mass is achieved by the type-I seesaw mechanism. Both inverted and normal neutrino mass hierarchies are consistent with the experimental data. The predicted sum of neutrino masses for normal and inverted hierarchies, the effective neutrino masses, and the Dirac CP phase are also consistent with recently reported limits.

**Keywords:** flavor symmetries, quark and lepton masses and mixing, extensions of electroweak gauge sector, neutrino mass and mixing, non-standard-model neutrinos, right-handed neutrinos, D₄ discrete symmetry

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**I. INTRODUCTION**

The mass hierarchy problem is one of the most exciting issues in particle physics that requires the extension of the Standard Model (SM). Some experimental data are related to the flavour problem, including the origin of the quark mass hierarchy [1] m_u ≪ m_c ≪ m_t and m_d ≪ m_s ≪ m_b, the hierarchy of the charged lepton mass m_e ≪ m_μ ≪ m_τ, and the origin of the three extremely small quark mixing angles as well as the neutrino mass spectrum and mixings.

Because of the mentioned issues, various SM extensions have been proposed, such as symmetry extensions with scalars and/or fermion fields. The B – L model [2–8] is relevant because it constitutes the simplest approach to add three right-handed neutrinos to generate neutrino masses. Although this model solves many interesting problems, such as dark matter [3], the muon anomalous magnetic moment [4, 8], leptogenesis [5, 6], and gravitational wave radiation [7], it cannot provide a satisfactory explanation for fermion masses and mixing observables. Non-Abelian discrete symmetries are considered the most powerful tool for reproducing the observed mass and mixing patterns of lepton and quarks (see, for example, Ref. [9]). In particular, the D₄ symmetry received much attention because it provides a predictive depiction of the mentioned patterns [10–23]. These previous studies are essentially different from the present study in the following basic points:

1) Ref. [16] is based on symmetries\(^1\) \(G_{BL} \times D_4 \times Z_4\) in which, for the quark sector, up to four \(SU(2)_L\) doublets and three singlets are introduced, and the obtained quark mixing matrix, whose "13", "23", "31", and "32" entries are zero, is not natural because all the elements of the quark mixing matrix must be non-zero [1].

2) Ref. [17] is based on symmetries\(^2\) \(G_{SM} \times D_4 \times Z_2\) in which the realistic quark mixing pattern is not considered and the quark mass hierarchy is not satisfied.

3) Ref. [18] is based on symmetries \(G_{331} \times U(1)_Y \times D_4\) in which five \(SU(3)_L\) triplets are used, and the 1 – 2 mixing of the ordinary quarks is obtained if the \(D_4\) symmetry is violated with 1’ symmetry instead of 1, as usual.

4) Ref. [19] is based on symmetries \(G_{331} \times U(1)_Y \times D_4\) in which a realistic quark mixing matrix is achieved; however, the quark mass hierarchy is not satisfied.

5) In Ref. [20], the obtained quark mixing matrix, whose "13", "23", "31", and "32" entries are zero, is not natural because all the elements of the quark mixing mat-
rix must be non-zero [1], and the quark mass hierarchy is not satisfied.

(6) In Ref. [21], the obtained quark mixing matrix, whose "13", "23", "31" and "32" entries are zero, is not natural because all the elements of the quark mixing matrix must be non-zero [1], and the quark mass hierarchy is not satisfied.

(7) Ref. [22] is based on symmetries $G_{3(1)} \times D_4 \times Z_4 \times Z_a^{(1)} \times Z_b^{(1)} \times Z_6$, in which two $SU(3)_L$ triplets and six $SU(3)_L$ singlets are used.

(8) In Ref. [23], the quark mass hierarchy is slightly unnatural given that the Yukawa couplings spread over the region from $O(10^{-3})$ to $O(1)$ (difference of three orders of magnitude).

Hence, it is desirable to develop another $D_4$ flavor model to overcome the aforementioned limitations of previous studies, especially the quark mass hierarchy, the extremely small quark mixing angles, the neutrino mass spectrum, and the mixing pattern.

In this paper, we propose an alternative $D_4$ model that differs from those of Refs. [13, 16]. We additionally introduce one doublet ($H'$) put in $D_4$ under $D_3$ [13] and use one singlet instead of one doublet in the quark sector [16]. The properties under $D_4$ of the right-handed charged lepton ($l_{1R}$) and right-handed neutrino ($v_{1R}$), as well as the properties under $Z_4$ of right-handed leptons $l_{1R}, l_{2R}, v_{1R}, v_{2R}$ and singlet scalars $\chi, \phi, \phi'$ in the present study are completely different from those of Ref. [13, 16]. As a consequence, the charged leptons, neutrinos, and quark mass hierarchies can be naturally achieved.

The rest of this paper is organized as follows. We describe the model in Section II. Sections III and IV are devoted to the quark and lepton masses and mixings, respectively. Section V presents a numerical analysis. Some conclusions are drawn in Sec. VI.

II. MODEL

The total symmetry of the model is $\Gamma = SU(2)_L \times U(1)_Y \times U(1)_{B-L} \times D_4 \times Z_4 \times Z_6$, where the lepton, quark, and scalar fields under $D_4$ and $Z_4$ are essentially different from those of Refs. [13, 16]. In particular, in this study, the first families of the left-handed quark and right-handed up-and down quarks are assigned in $D_4$; the two other families of quarks are assigned in $Z_4$. To explain the hierarchies of quark masses, one $SU(2)_L$ doublet $H'$ with $B-L = 0$ put in $Z_4$ under $D_4$ together with three flavons $\rho, \phi$ and $\phi'$ with $B-L = 0$ respectively put in $Z_4$ and $D_4$ under $D_4$ are additional introduced, i.e., the considered model contains two $SU(2)_L$ doublets $\phi$. The particle and scalar contents of the model are shown in Table I.

With the given particle content, $\Phi_{l_{1R}}$ transforms as $(2, \bar{2}, 0, 1_{--}, -1)$ and can couple to $(H\phi)_{1_{--}}, \Phi_{l_{1R} u_{aR}} \sim (2, \bar{2}, 2, 0, 1_{--} + 1_{--}, 1_{--} + 1_{--}, 1_{--} + 1_{--})$ can couple to $H, H', (H\phi)_{1_{--}}$ and $(H\phi)_{1_{--}}$, respectively; $\Phi_{l_{1R} u_{aR}} \sim (2, \bar{1}, 0, 2, i)$ can couple to $(H\rho)_{1_{--}}$ and $(H\rho)_{1_{--}}$; and $\Phi_{l_{1R} l_{1R}} \sim (2, \bar{1}, 0, 2, -i)$ can couple to $(H\rho')_{1_{--}}$ and $(H\rho')_{1_{--}}$ to form invariant terms that generate an up-quark mass matrix. The situation is similar for the down quark sector. The Yukawa terms in the quark and lepton sectors are

\[
-L^y = \frac{h_1}{\Lambda} \langle \bar{Q}_{l_{1R}} l_{1R} \rangle_{1_{--}} (H\phi)_{1_{--}} + h_2 \langle \bar{Q}_{l_{1R}} l_{1R} \rangle_{1_{--}} H + h_3 \langle \bar{Q}_{l_{1R}} l_{1R} \rangle_{1_{--}} H' + h_4 \langle \bar{Q}_{l_{1R}} l_{1R} \rangle_{1_{--}} (H\phi)_{1_{--}} + \frac{h_5}{\Lambda} \langle \bar{Q}_{l_{1R}} l_{1R} \rangle_{1_{--}} (H\phi)_{1_{--}} + \frac{h_6}{\Lambda} \langle \bar{Q}_{l_{1R}} l_{1R} \rangle_{1_{--}} (H\phi)_{1_{--}} + \frac{h_7}{\Lambda} \langle \bar{Q}_{l_{1R}} l_{1R} \rangle_{1_{--}} (H\phi)_{1_{--}}
\]

\[
-L^y_{lep} = \frac{h_1}{\Lambda} \langle \bar{Q}_{l_{1R}} l_{1R} \rangle_{1_{--}} (H\phi)_{1_{--}} + h_2 \langle \bar{Q}_{l_{1R}} l_{1R} \rangle_{1_{--}} H + h_3 \langle \bar{Q}_{l_{1R}} l_{1R} \rangle_{1_{--}} H' + h_4 \langle \bar{Q}_{l_{1R}} l_{1R} \rangle_{1_{--}} (H\phi)_{1_{--}} + \frac{h_5}{\Lambda} \langle \bar{Q}_{l_{1R}} l_{1R} \rangle_{1_{--}} (H\phi)_{1_{--}} + \frac{h_6}{\Lambda} \langle \bar{Q}_{l_{1R}} l_{1R} \rangle_{1_{--}} (H\phi)_{1_{--}} + \frac{h_7}{\Lambda} \langle \bar{Q}_{l_{1R}} l_{1R} \rangle_{1_{--}} (H\phi)_{1_{--}}
\]

1) see, for instance [24, 25] for a review of the two-Higgs-doublet model (2HDM).
where $x_{1,2}^{d,d}, y_{1,2}^{d,a}$ and $z_{1,2,3}^{d,a}$ are the Yukawa-like couplings in the quark sector, $h_{1,2,3,4}, x_{1,2,3,4}$ and $y_{1,2,3}$ are the Yukawa-like couplings in the lepton sector, and $\Lambda$ is the cut-off scale of the theory.

Note that the additional discrete symmetries $D_4$, $Z_2$, and $Z_3$ play crucial roles in forbidding undesired terms to obtain the expected quark and lepton mass matrices listed in Table A1. For instance, in the absence of $Z_2$, there will be additional invariant terms, $(\overline{Q}_{1L} l_{aR}) (H \rho_2), (\overline{Q}_{1L} l_{aR}) (H') \rho_2), (\overline{Q}_{2L} l_{bR}) (H \rho_2), and (\overline{Q}_{2L} l_{bR}) (H') \rho_2)$, which contribute to the entries “12”, “13”, “21”, and “31” of the charged lepton matrix. As a result, we cannot obtain the mass of charged leptons as expected because the charged lepton matrix cannot be diagonalized.

The vacuum expectation value (VEV) of the scalar fields takes the form

$$
\langle H \rangle = (0v)^T, \quad \langle H' \rangle = (0v')^T, \quad \langle \phi \rangle = v, \quad \langle \phi \rangle = v, \quad \langle \rho \rangle = (\rho_1, \rho_2) \equiv (v, v').
$$

(3)

In fact, the electroweak symmetry breaking scale is approximately in the order of one hundred GeV, $v^2 + v'^2 = (174 \text{ GeV})^2$. Furthermore, in the 2HDM, the limits of the parameter $t_g = v'/v$ are given by [26] $t_g = v'/v \in [1, 10, 0.0]$ or [27] $t_g = v'/v \in [1.0, 3.0]$. For the purpose of determining the scale of the Yukawa couplings, we consider the case of $t_g = 1.424$, i.e.,

$$
v = 100 \text{ GeV}, \quad v' = 142.40 \text{ GeV}.
$$

(4)

In addition, to satisfy the quark mass hierarchy, the VEV of the singlets and cut-off scale are assumed to be

$$
v = 5 \times 10^{11} \text{ GeV}, \quad v' = 10^{14} \text{ GeV}, \quad \Lambda \approx 10^{13} \text{ GeV}.
$$

(5)

For models with more than one $SU(2)_L$ scalar doublet, as the one proposed in this study, Flavor Changing Neutral Current (FCNC) processes such as $b \rightarrow s \gamma$ exist in the Higgs sector. However, they are suppressed by non-Abelian discrete symmetries [28, 29]. For these processes to be below current experimental limits, some restrictions on the model parameters, such as the Yukawa couplings and large masses for non SM scalars, need to be imposed. The considered model contains many free parameters, which enables assuming that the remaining scalars are sufficiently heavy to fulfill current experimental limits. Furthermore, the first two lines in Eq. (2) imply that the off-diagonal Yukawa couplings in the charged-lepton sector are proportional to $\frac{v}{\Lambda} \sim 10^{-2}$. Therefore, lepton flavor violation (LFV) processes, such as $l_i \rightarrow l_j \gamma$, are suppressed by the extremely small factor $\frac{v}{\Lambda} \ll 1$ associated with the mentioned small Yukawa couplings and large mass scale of the heavy scalars $m_q$ [30–33]. A detailed analysis of FCNC and LFV processes are beyond the scope of this study.

### III. QUARK MASS AND MIXING

Using the Clebsch-Gordan coefficients of the $D_4$ symmetry [34] and Eq. (1), when the scalar fields take the VEVs as in Eq. (3), the up- and down-quark mass matrices take the following forms:

$$
M_q = M_q^{(0)} + \delta M_q (q = u, d),
$$

(6)

where

$$
M_q^{(0)} = \begin{pmatrix}
    a_{1q} & 0 & 0 \\
    0 & a_{2q} + a_{3q} & 0 \\
    0 & 0 & a_{2q} - a_{3q}
\end{pmatrix},
$$

(7)

$$
\delta M_q = \begin{pmatrix}
    0 & c_{1q} & c_{1q} - c_{3q} \\
    c_{2q} + c_{4q} & b_{1q} - b_{2q} & 0 \\
    c_{2q} - c_{4q} & b_{1q} + b_{2q} & 0
\end{pmatrix},
$$

(8)

with

$$
a_{1q} = x_{1q}^2 \frac{v}{\Lambda}, \quad a_{2q} = x_{2q}^2 \frac{v}{\Lambda}, \quad a_{3q} = x_{3q}^2 \frac{v}{\Lambda},
$$

$$
b_{1q} = y_{1q}^2 \frac{v}{\Lambda}, \quad b_{2q} = y_{2q}^2 \frac{v}{\Lambda}, \quad c_{1q} = z_{1q}^2 \frac{v}{\Lambda},
$$

$$
c_{2q} = z_{2q}^2 \frac{v}{\Lambda}, \quad c_{3q} = z_{3q}^2 \frac{v}{\Lambda}, \quad c_{4q} = z_{4q}^2 \frac{v}{\Lambda} (q = u, d).
$$

(9)

Equations (6)–(8) show that, besides the two doublets $H$ and $H'$, one singlet $\varphi$ contributes to $M_q^{(0)}$ while $\delta M_q$ is
due to the contribution of two singlets $\rho$ and $\phi$. Without the contributions of $\rho$ and $\phi$, $\delta M_\ell$ will vanish, and the quark mass matrices $M_\ell$ in Eq. (6) reduce to the diagonal matrices $M_\ell^{(0)}$, i.e., the corresponding quark mixing matrix $V_{CKM} = 3 \times 3$, which was ruled out by recent data. The realistic quark mixing angles are extremely small \cite{1}, which implies that the quark mixing matrix is very close to the identity matrix; thus, the second term $\delta M_\ell$ in Eq. (7) can be considered as a perturbed parameter for generating the quark mixing pattern. As a consequence, a realistic quark mixing pattern can be achieved at the first order of the perturbation theory. Indeed, at this first order, the matrices $\delta M_\ell$ contribute to the eigenvectors but they do not contribute at all to the eigenvalues of the quark mass matrices $M_\ell$. The quark masses are determined as

\[
m_u = a_{1u}, \quad m_c = a_{2u} + a_{3u}, \quad m_t = a_{2u} - a_{3u},
\]
\[
m_d = a_{1d}, \quad m_s = a_{2d} + a_{3d}, \quad m_b = a_{2d} - a_{3d},
\]
and the corresponding perturbed quark mixing matrices are

\[
V_{L}^\prime = U_{L}^\prime = \begin{pmatrix}
1 & c_{1u} + c_{3u} & c_{1u} - c_{3u} \\
c_{4u} + c_{2u} & m_u - m_c & m_u - m_s \\
c_{4u} - c_{2u} & b_{2u} - b_{1u} & m_t - m_c \\
 & m_t - m_s & 1
\end{pmatrix},
\]
\[
V_{R}^\prime = U_{R}^\prime = \begin{pmatrix}
1 & c_{1d} + c_{3d} & c_{1d} - c_{3d} \\
c_{4d} + c_{2d} & m_d - m_b & m_d - m_s \\
c_{4d} - c_{2d} & b_{2d} - b_{1d} & m_b - m_s \\
 & m_b - m_s & 1
\end{pmatrix},
\]
with $b_{1,2q}$ and $c_{1,2,3,4q} (q = u, d)$ given in Eq. (8). For simplicity, we consider the case of $y_{1q} = y_{2q} = y_{3q} (q = u, d)$, $z_{3d} = z_{1d} = z_{d}$, i.e.,

\[
b_{2d} = b_{1d} = b_d, \quad b_{2u} = b_{1u} = b_u, \quad c_{3d} = c_{1d} = c_d.
\]

The quark mixing matrix, $V_{CKM} = V_{T}^\prime V_{L}^\dagger$, includes the following entries:

\[
V_{11}^{CKM} = 1 + 2c_{1u}^2 (c_{1u} + c_{3u}) + c_{4u}^2 + c_{2u}^2 + c_{4d}^2 + c_{2d}^2,
\]
\[
V_{12}^{CKM} = \frac{2b_{1u} c_{1u} + c_{3u}}{(m_u - m_c)(m_u - m_s)} + \frac{c_{4u}^2 + c_{2u}^2 + c_{4d}^2 + c_{2d}^2}{m_u - m_s},
\]
\[
V_{13}^{CKM} = \frac{c_{1u} c_{3u}}{m_t - m_c} + \frac{c_{4u} c_{3u}}{m_t - m_s},
\]
\[
V_{22}^{CKM} = 1 + \frac{4b_{d}^2 c_{4d}}{(m_b - m_s)(m_b - m_s)} + \frac{(c_{2u} + c_{4u})(c_{2d}^2 + c_{4d}^2)}{(m_u - m_u)(m_u - m_u)},
\]
\[
V_{33}^{CKM} = 1 + \frac{(c_{2d} - c_{4d})(c_{2d} + c_{4d})}{(m_b - m_s)(m_b - m_s)}.
\]

Comparing the model results on the quark masses and quark mixing matrix in Eqs. (9) and (12) with their corresponding experimental constraints on $V_{\text{exp}}$ shown in Table 2 (second column), we obtain explicit expressions for $a_{1u,d}, a_{2u,d}, a_{3u,d}, b_{1u,d}, b_{2u,d}, c_{1u,d}, c_{2u,d}, c_{3u,d}$, and $c_{4u,d}$ as functions of quark masses and quark mixing matrix elements as presented in Eqs. (B1) and (B2) of Appendix B.

Equations (8), (11), (B1), and (B2) imply that the model parameters $a_{1u,d}, a_{2u,d}, a_{3u,d}, b_{1u,d}, b_{2u,d}, c_{1u,d}, c_{2u,d}, c_{3u,d}$, and $c_{4u,d}$ depend on the observed parameters in the quark sector, including quark masses $m_u, m_c, m_t, m_s, m_d, m_b$ and quark mixing matrix elements $V_{ij}^{\text{exp}} (i, j = 1, 2, 3)$, which have been determined accurately \cite{1}. At the best-fit points of the mentioned parameters \cite{1} given in Ref.\cite{1}, we obtain a prediction for the quark mixing matrix as well as the parameters of the model in the quark mixing matrix, as shown in Table 2 and Eq. (13), respectively.

\[
a_{1u} = 2.160 \times 10^{-3} \text{ GeV}, \quad a_{2u} = 86.980 \text{ GeV},
\]
\[
a_{3u} = -85.710 \text{ GeV}, \quad b_u = (2.308 + 0.5413i) \text{ GeV},
\]
\[
c_{1u} = 8.414 + 3.028i \text{ GeV}, \quad c_{2u} = (0.564 - 0.511i) \text{ GeV},
\]
\[
c_{3u} = (-8.269 - 3.170i) \text{ GeV}, \quad c_{4u} = (0.754 - 0.353i) \text{ GeV},
\]
\[
a_{1d} = 4.670 \times 10^{-3} \text{ GeV}, \quad a_{2d} = 2.140 \text{ GeV},
\]
\[
a_{3d} = -2.040 \text{ GeV}, \quad b_d = (-8.658 + 0.262i)10^{-2} \text{ GeV},
\]
\[
c_{1d} = (-5.080 + 4.973i)10^{-3} \text{ GeV},
\]
\[
c_{2d} = (0.193 - 0.077i) \text{ GeV},
\]
\[
c_{3d} = (-0.204 + 0.087i) \text{ GeV}.
\]

1) The best-fit points in Table 2 correspond to the Wolfenstein parameters \cite{1}: $\lambda = 0.2250$, $A = 0.826$, $\rho = 0.159$ and $\theta = 0.348$ which correspond to the mixing angles $\sin^2 \theta_{12} = 0.22500$, $\sin^2 \theta_{13} = 0.00369$, $\sin^2 \theta_{23} = 0.04182$ and $\sin^2 \theta_{CP} = 1.444$. 

\[063102-4\]
The Jarlskog invariant in the quark sector, $J_{CP}^q = \text{Im} \{ V_{us} V_{cb} V_{ub}^* V_{cb}^* \}$, is calculated from Eq. (12) with the model results listed in Table 2 (third column), obtaining $J_{CP}^q = 3.08 \times 10^{-5}$, which coincides with the value reported in Ref. [1].

Then, comparing Eqs. (8) and (13) with the aid of Eqs. (4)–(5), we obtain

$$J_{q} \text{CP} = (a_1 + a_3 + a_4 + a_5) = (a_2 - a_3)$$

(17)

which differ by approximately three orders of magnitude.

**IV. LEPTON MASSES AND MIXINGS**

Using the Clebsch-Gordan coefficients of $D_4$ [34] and Eq. (2), when the scalar fields take the VEVs, as in Eq. (3), we obtain the charged leptons ($M_I$) and neutrino (Dirac and right-handed Majorana) mass matrices ($M_D, M_R$) as follows:

$$M_I = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 + a_3 & a_4 + a_5 \\ 0 & a_4 - a_5 & a_2 - a_3 \end{pmatrix},$$

(14)

$$M_D = \begin{pmatrix} 0 & -a_D & a_D + b_D \\ -a_D & 0 & c_D + d_D \\ a_D & c_D & 0 \end{pmatrix},$$

(15)

$$M_R = \begin{pmatrix} a_R & 0 & 0 \\ 0 & b_R & c_R \\ 0 & c_R & b_R \end{pmatrix},$$

(16)

where

$$a_1 = \left( \frac{V_{us}}{\Lambda} \right) v h_1, \quad a_2 = h_2 v, \quad a_3 = h_3 v', \quad a_4 = \left( \frac{V_{us}}{\Lambda} \right) v' h_3,$$

(17)

$$a_D = \left( \frac{V_{us}}{\Lambda} \right) x_1 v, \quad b_D = \left( \frac{V_{us}}{\Lambda} \right) x_2 v', \quad c_D = \left( \frac{V_{us}}{\Lambda} \right) x_3 v,$$

(18)

$$d_D = \left( \frac{V_{us}}{\Lambda} \right) x_4 v', \quad a_R = \left( \frac{V_{us}}{\Lambda} \right) y_1 v_\phi, \quad b_R = y_2 v_\phi,$$

(19)

$$c_R = \frac{y_3}{\Lambda} v_\phi.$$

(20)

- **Charged-lepton sector:** For simplicity, we consider the case of $\arg h_3 = (\arg h_2 + \pi)$ and $\arg h_5 = \arg h_4$, i.e.,
$\arg \alpha_1 = (\arg \alpha_2 + \pi)$ and $\arg \alpha_5 = \arg \alpha_4$. Yukawa couplings $h_i (i = 1 + 5)$ are complex in general; therefore, the matrix $M_t$ and its eigenvalues are complex. Let us first define a Hermitian matrix $m^2_t = M_t^* M_t$ as

$$m^2_t = M_t^* M_t = \begin{pmatrix} A_0 & 0 & 0 \\ 0 & B_0 & D_0 e^{i\theta} \\ 0 & D_0 e^{i\theta} & C_0 \end{pmatrix},$$ (18)

where

$$A_0 = |\alpha_1|^2, \quad B_0 = (|\alpha_2| - |\alpha_1|)^2 + (|\alpha_4| + |\alpha_5|)^2, \quad C_0 = (|\alpha_2| + |\alpha_1|)^2 + (|\alpha_4| - |\alpha_5|)^2,$$

$$D_0 = 2(|\alpha_2||\alpha_4| + |\alpha_5||\alpha_1|) c_a, \quad G_0 = -2(|\alpha_1||\alpha_4| + |\alpha_2||\alpha_5|) s_a, \quad D_0 = \sqrt{D_0^2 + G_0^2}.$$ (19)

$$\theta = \arccos \left( \frac{D_0}{D_0} \right), \quad \alpha = \arg \alpha_2 - \arg \alpha_4.$$ (20)

The matrix $m^2_t$ in Eq. (18) is diagonalized by two mixing matrices $V_{tL,R}$ with $V^*_{tL} m^2_t V_{tR} = \text{diag}(m^2_{\nu_e}, m^2_{\nu_\mu}, m^2_{\nu_\tau})$, where

$$m^2_{\nu_e} = A_0, \quad m^2_{\nu_\mu} = \frac{1}{2} \left( B_0 + C_0 + \sqrt{(B_0 - C_0)^2 + 4D_0^2} \right),$$ (21)

$$V_{tL} = V_{tR} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_\phi & -s_\phi e^{-i\theta} \\ 0 & s_\phi e^{i\theta} & c_\phi \end{pmatrix},$$ (22)

where

$$s_\phi = \frac{1}{\sqrt{2}} \sqrt{1 - \frac{B_0 - C_0}{B_0 - C_0 + \sqrt{(B_0 - C_0)^2 + 4D_0^2}}}.$$ (23)

Equations (19)–(21) and (23) yield the following relations:

$$|\alpha_1| = m_{\nu_e}, \quad |\alpha_2| = |\alpha_2| D_{0, s_a} + |\alpha_4| c_a G_0 \left( |\alpha_4|^2 - |\alpha_5|^2 \right) s_{2a},$$

$$|\alpha_3| = |\alpha_4| c_a G_0 + |\alpha_5| D_{0, s_a} \left( |\alpha_5|^2 - |\alpha_4|^2 \right) s_{2a},$$

$$|\alpha_4| = \frac{a + b}{2}, \quad |\alpha_5| = \frac{a - b}{2},$$ (24)

where

$$a = \sqrt{(B_0 C_0 - x_0 + y_0)^2 - 4B_0 C_0 y_0 + B_0 C_0 - x_0 + y_0 - 4D_0^2},$$

$$b = \sqrt{(B_0 C_0 - x_0 + y_0)^2 - 4B_0 C_0 y_0 + B_0 C_0 + x_0 - y_0 - 4D_0^2},$$ (25)

$$x_0 = \frac{(c_a G_0 + D_0 s_a)^2}{s_{2a}^2}, \quad y_0 = \frac{(c_a G_0 - D_0 s_a)^2}{s_{2a}^2}.$$ (26)

$$B_0 = (m_{\nu_e}^2 - m_{\nu_\mu}^2) s_{\phi}^2 + m_{\nu_\tau}^2, \quad C_0 = (m_{\nu_\mu}^2 - m_{\nu_\tau}^2) s_{\phi}^2 + m_{\nu_e}^2,$$

$$D_0 = (m_{\nu_\mu}^2 - m_{\nu_\tau}^2) c_\phi s_\phi + m_{\nu_e}^2, \quad G_0 = (m_{\nu_\mu}^2 - m_{\nu_\tau}^2) s_\phi c_\phi + m_{\nu_e}^2.$$ (27)

Equations (16) and (24)–(27) imply that $h_1$ depends on $m_{\nu_e}, \alpha, v_\phi$, and $\nu_2$ depends on $v, m_{\nu_\mu}, m_\tau, \psi, \theta$ and $\alpha$; $h_3$ depends on $v', m_{\nu_\mu}, m_\tau, \psi, \theta$ and $\alpha$; and $h_4$ and $h_5$ depend on $v, v', m_{\nu_\mu}, m_\tau, \psi, \theta$ and $\alpha$. We show in Sec. V that, according to the observed charged leptons $m_{\nu_{e,\mu,\tau}}$ [1] and cut-off scale, and the VEV scales of scalar fields defined by Eqs. (4) and (5), there exist possible ranges of the model parameters such that the Yukawa couplings in the charged lepton sector, $h_i (i = 1 + 5)$, differ by approximately two orders of magnitude, i.e., the charged lepton mass hierarchy is satisfied.

- **Neutrino sector**: The effective neutrino mass matrix arises from the type-I seesaw mechanism, that is, according to Eq. (15), $M_\nu = -M_D M_R^{-1} M_D^*$:

$$M_\nu = \begin{pmatrix} A & -B_1 & -B_2 \\ -B_1 & C_1 & C_2 \\ -B_2 & C_2 & C_2 \end{pmatrix},$$ (28)

where

1) In this work, the following notations are used: $s_\phi = \sin \phi, c_\phi = \cos \phi, s_\theta = \sin \theta, c_\theta = \cos \theta, t_\alpha = \tan \alpha, s_\alpha = \tan \theta, s_\beta = \sin \delta_{CP}, s_{ij} = \sin \theta_{ij}, c_{ij} = \cos \theta_{ij}$ and $t_{ij} = \tan \theta_{ij} (j = 12, 13, 23)$. 

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\[
A = \frac{2b_2^2}{b_R + c_R} + \frac{2a_2^2}{b_R - c_R},
\]
\[
B_1 = \frac{(c_D + d_D)(a_D(b_R + c_R) - b_D(b_R - c_R))}{b_R^2 - c_R^2},
\]
\[
B_2 = \frac{(c_D - d_D)(a_D(b_R + c_R) + b_D(b_R - c_R))}{b_R^2 - c_R^2},
\]
\[
C_1 = \frac{b_R(c_D + d_D)^2}{b_R^2 - c_R^2}, \quad C_2 = \frac{b_R(c_D - d_D)^2}{b_R^2 - c_R^2},
\]
\[
C_3 = \frac{c_R(c_D^2 - d_D^2)}{b_R^2 - c_R^2}.
\]  

(29)

The mass matrix \(M_e\) in Eq. (28) presents the following three eigenvalues and corresponding mixing matrix:

\[
\lambda_1 = 0, \lambda_2 = \frac{C_2 - 2B_2n_1 + An_1^2 + n_2(2C_1 - 2B_1n_1 + C_1n_2)}{n_1^2 + n_2^2 + 1},
\]
\[
\lambda_3 = \frac{C_2 - 2B_2t_1 + At_1^2 + t_2(2C_1 - 2B_1t_1 + C_1t_2)}{t_1^2 + t_2^2 + 1}.
\]  

(30)

\[
\begin{pmatrix}
  k_1 & n_1 & t_1 \\
  \sqrt{1 + k_1^2(1 + k_2^2)} & \sqrt{n_1^2 + n_2^2 + 1} & \sqrt{t_1^2 + t_2^2 + 1} \\
  \frac{k_1}{k_2} & \frac{n_1}{n_2} & \frac{t_1}{t_2} \\
  \frac{1}{\sqrt{1 + k_1^2(1 + k_2^2)}} & \frac{1}{\sqrt{n_1^2 + n_2^2 + 1}} & \frac{1}{\sqrt{t_1^2 + t_2^2 + 1}}
\end{pmatrix}
\]

(31)

where the expressions of parameters \(k_{1,2}, n_{1,2}\) and \(t_{1,2}\) are provided in Appendix C, satisfy the following relations:

\[
k_1(n_1 + k_2n_2) + 1 = 0, \quad k_1(t_1 + k_2t_2) + 1 = 0,
\]
\[
n_1t_1 + n_2t_2 + 1 = 0,
\]  

(32)

The neutrino mass matrix \(M_e\) in Eq. (28) is diagonalized as

\[
C_2 = \frac{B_2(k_1 + n_1) + C_3(k_1k_2 + n_2)}{0 + k_1[A_1n_1 + C_1k_2n_2 - B_1(k_2n_1 + n_2)]} = 0,
\]  

(33)

\[
C_2 = \frac{B_2(k_1 + t_1) + C_3(k_1k_2 + t_2)}{0 + k_1[A_1t_1 + C_1k_2t_2 - B_1(k_2t_1 + t_2)]} = 0,
\]  

(34)

\[
C_2 + C_3n_1t_1 - B_1n_2t_1 = 0,
\]  

(35)

\[
C_2 + k_1[2C_3k_2 - 2B_2 + k_1(A - 2B_1k_2 + C_1k_2^2)] = 0.
\]  

(36)

Depending on the sign of \(\Delta m^2_{31}\), the neutrino mass spectrum can exhibit a normal or inverted hierarchy [1]. In the considered model, \(0 = m_1 \geq m_2 \geq m_3 \geq \lambda_3\) for NH and \(0 = m_3 \geq m_1 \geq m_2 \geq \lambda_3\) for IH. Given that the lightest neutrino mass is equal to zero, other neutrino masses and their sum are given by

\[
\left\{ \begin{array}{l}
m_1 = 0, \quad m_2 = \sqrt{\Delta m^2_{21}}, \quad m_3 = \sqrt{\Delta m^2_{31}} \quad \text{for NH,} \\
m_1 = \sqrt{-\Delta m^2_{31}}, \quad m_2 = \sqrt{\Delta m^2_{21} - \Delta m^2_{31}}, \quad m_3 = 0 \quad \text{for IH.}
\end{array} \right.
\]  

(37)

\[
\sum m_n = \left\{ \begin{array}{l}
\sqrt{\Delta m^2_{21} + \Delta m^2_{31}} \quad \text{for NH,} \\
\sqrt{\Delta m^2_{31} - \Delta m^2_{31} + \sqrt{-\Delta m^2_{31}}} \quad \text{for IH.}
\end{array} \right.
\]  

(38)
where $\lambda_2, \lambda_1, k_{1,2}, n_{1,2}$ and $t_{1,2}$ are given in Appendix C.

Equations (30) and (32)–(36) yield

\[
\begin{aligned}
k_1 &= \frac{n_1t_1 + n_2^2 + 1}{t_1 (n_1^2 + n_2^2) + n_1}, &
k_2 &= \frac{n_2(t_1 - n_1)}{n_1t_1 + n_2^2 + 1}, &
t_3 &= \frac{-n_1t_1 + 1}{n_2} \\
2 &= \frac{1 - k_1n_1}{k_1}, &
\lambda_1 &= \frac{k_3(n_1 - k_1k_2)}{k_1 [k_1 + k_2] n_1 - 1}, &
t_2 &= \frac{k_2(k_1 + n_1)}{1 + k_1 [k_1 + k_2] n_1 - 1}, \quad \text{for NH,}
\end{aligned}
\]

(40)

\[
A = -\frac{C_2 - B_2(k_1 + n_1) + C_1 k_1 k_2 n_2 + C_3(k_1 n_2 + n_2) - B_1 k_1(k_2 n_1 + n_2)}{k_1 n_1} \\
\text{(NH and IH),}
\]

(41)

\[
B_1 = \frac{C_3 + C_1 k_1 k_2}{k_1} + \frac{(C_2 - B_2 k_1 + C_1 k_1 k_2)(n_1 - t_1)}{(n_1 t_2 - n_2 t_1) k_1} \\
\text{(NH and IH),}
\]

(42)

\[
B_2 = \frac{C_2}{k_1} + C_3 k_2 \\
\text{(NH and IH),}
\]

(43)

\[
C_1 = \frac{C_3(k_1 - n_1) + (C_2 - B_2 n_1 + C_3 n_2)(k_1 - t_1)}{t_2 - k_1 t_1} \frac{(C_2 - B_2 k_1 + C_1 k_1 k_2)(n_1 - t_1) n_1}{n_2 t_1 - n_1 t_2} \\
\text{(NH and IH),}
\]

(44)

\[
C_2 = \left\{ \begin{array}{ll}
\sqrt{\Delta m_{21}^2} \frac{n_2}{1 + n_1^2 + n_2^2} + \frac{\sqrt{\Delta m_{21}^2}}{n_1^2 (1 + t_1^2) + n_2^2 (1 + t_2^2)} & \text{for NH,} \\
k_1^2 \left( \sqrt{\Delta m_{21}^2} - \sqrt{\Delta m_{31}^2} - \sqrt{\Delta m_{51}^2} \right) \frac{1}{1 + 2k_1 n_1 + k_1^2 [n_1^2 + k_1^2 (1 + n_1^2)]} + \frac{1}{1 + k_2^2 (1 + n_2^2)} & \text{for IH,}
\end{array} \right.
\]

(45)

\[
C_3 = \left\{ \begin{array}{ll}
\sqrt{\Delta m_{21}^2} \frac{n_2}{1 + n_1^2 + n_2^2} - \frac{\sqrt{\Delta m_{21}^2} (1 + n_1 t_1) n_2}{(1 + n_1^2) t_1^2 + n_2^2 (1 + t_2^2)} & \text{for NH,} \\
k_1 k_2 \left( \sqrt{\Delta m_{21}^2} - \sqrt{\Delta m_{31}^2} - \sqrt{-\Delta m_{51}^2} \right) \frac{(k_1 n_1 + 1)}{1 + 2k_1 n_1 + k_1^2 [n_1^2 + k_1^2 (1 + n_1^2)]} - \frac{\sqrt{\Delta m_{31}^2} - \sqrt{\Delta m_{51}^2}}{1 + k_2^2 (1 + n_2^2)} & \text{for IH.}
\end{array} \right.
\]

(46)

The corresponding leptonic mixing matrix is

\[
U = U^\dagger U = \left( \begin{array}{ccc}
k_1 & n_1 & t_1 \\
\sqrt{\Delta m_{21}^2} k_1^2 + 1 & \sqrt{n_1^2 + n_2^2 + 1} & \sqrt{t_1^2 + t_2^2 + 1} \\
\sqrt{k_2^2 + 1} k_1^2 + 1 & e^{-i\theta} (c_\theta e^{i\theta} n_2 + s_\theta) & e^{-i\theta} (s_\theta + c_\theta c_\theta t_2) \\
c_\theta k_2 k_3 + e^{i\theta} s_\theta & \sqrt{n_1^2 + n_2^2 + 1} & \sqrt{t_1^2 + t_2^2 + 1} \\
e^{-i\theta} (c_\theta e^{i\theta} n_2 + s_\theta) & e^{-i\theta} (s_\theta + e^{i\theta} c_\theta t_2) & \sqrt{t_1^2 + t_2^2 + 1} \\
c_\theta k_2 k_3 + e^{-i\theta} s_\theta & \sqrt{n_1^2 + n_2^2 + 1} & \sqrt{t_1^2 + t_2^2 + 1} \\
c_\theta e^{i\theta} n_2 s_\theta & c_\theta e^{i\theta} s_\theta & \sqrt{t_1^2 + t_2^2 + 1} \\
c_\theta e^{i\theta} n_2 s_\theta & \sqrt{n_1^2 + n_2^2 + 1} & \sqrt{t_1^2 + t_2^2 + 1} \\
c_\theta e^{i\theta} s_\theta t_2 & \sqrt{n_1^2 + n_2^2 + 1} & \sqrt{t_1^2 + t_2^2 + 1} \\
c_\theta e^{i\theta} s_\theta t_2 & \sqrt{n_1^2 + n_2^2 + 1} & \sqrt{t_1^2 + t_2^2 + 1} \\
c_\theta e^{i\theta} s_\theta t_2 & \sqrt{n_1^2 + n_2^2 + 1} & \sqrt{t_1^2 + t_2^2 + 1} \\
c_\theta e^{i\theta} s_\theta t_2 & \sqrt{n_1^2 + n_2^2 + 1} & \sqrt{t_1^2 + t_2^2 + 1} \\
c_\theta e^{i\theta} s_\theta t_2 & \sqrt{n_1^2 + n_2^2 + 1} & \sqrt{t_1^2 + t_2^2 + 1} \\
c_\theta e^{i\theta} s_\theta t_2 & \sqrt{n_1^2 + n_2^2 + 1} & \sqrt{t_1^2 + t_2^2 + 1} \\
\end{array} \right)
\]

for NH,

(47)

\[
\text{for IH.}
\]
The lepton mixing matrix $U_{\text{PMNS}}$, according to the standard parametrization, takes the form

$$U_{\text{PMNS}} = \begin{pmatrix}
  c_{13} c_{12} & s_{12} c_{13} & s_{13} e^{i \delta} \\
  -s_{23} s_{12} - e^{i \delta} c_{12} s_{13} s_{23} & c_{12} c_{23} - e^{i \delta} s_{12} s_{13} s_{23} & c_{13} s_{23} \\
  s_{23} s_{12} - e^{i \delta} c_{12} c_{23} s_{13} & -s_{12} s_{23} - e^{i \delta} c_{23} s_{12} s_{13} & c_{13} c_{23}
\end{pmatrix},$$

where $s_{ij} = \sin \theta_{ij}$ and $c_{ij} = \cos \theta_{ij}$ with $\theta_{13}, \theta_{12}$ and $\theta_{23}$ are the reactor, solar, and atmospheric mixing angles, respectively; $\delta_{CP}$ is the Dirac CP violation phase and $\eta_{1,2}$ are the two Majorana CP violating phases. Comparing the entries "12" and "13" of the two mixing matrices in Eqs. (47) and (48), we obtain

$$\eta_1 = 0, \quad \eta_2 = \delta (\text{both NH and IH}).$$

(49)

The lepton mixing angles, obtained from Eqs. (47) and (48), are

$$s_{13}^2 = |U_{e3}|^2 = \begin{cases}
  \frac{t_1^2}{t_1^2 + t_2^2 + 1} & \text{for NH,} \\
  k_1^2 & \text{for IH,}
\end{cases}$$

(50)

$$s_{12}^2 = 1 - |U_{e1}|^2 = \begin{cases}
  \frac{n_1^2 (t_1^2 + t_2^2 + 1)}{(t_1^2 + 1)(n_1^2 + n_2^2 + 1)} & \text{for NH,} \\
  \left[1 + \frac{k_1^2 (1 + k_2^2)}{(1 + k_1^2 k_2^2)(1 + t_1^2 + t_2^2)} \right] t_1^2 & \text{for IH,}
\end{cases}$$

(51)

The Jarlskog invariant in the active sector, determined from Eq. (47), takes the form [1, 35]

$$J_{CP}^{(0)} = \frac{n_1 t_1 (t_2 - n_2) s_{e6} c_{e6}}{(n_1^2 + n_2^2 + 1)(1 + t_1^2 + t_2^2)}$$

(53)

Comparing $J_{CP}^{(0)}$ in Eq. (53) and that of the standard parametrization, $J_{CP}^{(0)} = c_{12} c_{13} c_{23} s_{12} s_{13} s_{23}$, we obtain

$$s_6 = \frac{n_1 t_1 (t_2 - n_2) s_{e6} c_{e6}}{(n_1^2 + n_2^2 + 1)(1 + t_1^2 + t_2^2)}$$

(54)

The effective neutrino masses [36], obtained from Eqs. (37), (39), and (47), exhibit the following forms:

$$\langle m_{ee} \rangle = \sum_{i=1}^{3} |U_{ei}|^2 m_{e} = \begin{cases}
  \frac{\sqrt{\Delta m_{31}^2 n_1^2}}{1 + n_1^2 + n_2^2} + \frac{\sqrt{\Delta m_{31}^2 t_1^2}}{1 + t_1^2 + t_2^2} & \text{for NH,} \\
  \frac{-\sqrt{\Delta m_{31}^2 n_1^2}}{1 + n_1^2 + n_2^2} + \frac{\sqrt{\Delta m_{31}^2 t_1^2}}{1 + t_1^2 + t_2^2} & \text{for IH,}
\end{cases}$$

(55)

$$m_\beta = \sqrt{\sum_{i=1}^{3} |U_{ei}|^2 m_{e}^2} = \begin{cases}
  \frac{\sqrt{\Delta m_{21}^2 n_1^2}}{1 + n_1^2 + n_2^2} + \frac{\Delta m_{21}^2 t_1^2}{1 + t_1^2 + t_2^2} & \text{for NH,} \\
  \frac{\sqrt{\Delta m_{21}^2 n_1^2}}{1 + n_1^2 + n_2^2} - \frac{\Delta m_{21}^2 t_1^2}{1 + t_1^2 + t_2^2} & \text{for IH,}
\end{cases}$$

(56)

According to Eqs. (50)–(52), we can express $n_{12}, t_1$ and $s_6$ in terms of two constrained parameters $\theta_6$ and five observable parameters $\Delta m_{21}^2, \Delta m_{31}^2, s_{12}^2, s_{23}^2, s_{13}^2$ as follows:

- For NH:

$$n_1 = \frac{s_{12}^2 c_{13} t_1^2}{\sqrt{(c_{12}^2 t_1^2 - s_{13}^2) s_{12}^2 c_{12} c_{13} t_1^2 - s_{12}^2 s_{13}^2 t_1^2}}, \quad n_2 = \frac{(1 + n_1 t_1) s_{13}}{\sqrt{c_{12}^2 t_1^2 - s_{13}^2}},$$

(57)
\[ t_1 = t_{13} \sqrt{\frac{s_2^3(s_{23}^2 - c_2^2) + c_2^2(c_{23}^2 + c_2 s_2^2) + 2 \sqrt{c_2^2 s_2^2(s_{23}^2 c_{23}^2 - s_2^2 c_2^2) + 2 c_2^2 c_2 s_2^2}}{(c_2^2 - s_{23}^2)^2}}. \]  

(58)

- For IH:

\[ k_1 = -t_{13} \sqrt{\frac{s_2^3(s_{23}^2 - c_2^2) + c_2^2(c_{23}^2 + c_2 s_2^2) - 2 \sqrt{c_2^2 s_2^2(s_{23}^2 c_{23}^2 - s_2^2 c_2^2)}}{(c_2^2 - s_{23}^2)^2}}. \]

(59)

\[ k_2 = \sqrt{\frac{k_{13}^2 s_{13}^2 - s_{13}^2}{k_{13} s_{13}}}, \quad n_1 = \frac{s_{12} c_{12} c_{13}}{s_{13}^2 + s_{13}^2 c_{13}^2} \sqrt{\frac{k_{13}^2 (k_{13}^2 s_{13}^2 - s_{13}^2) - k_1 c_{13}^2 s_{13} c_{13}}{k_{13}^2 (s_{13}^2 - k_{13}^2)}}. \]

(60)

Equations (40)-(46) and (54)-(60) show that the model parameters \( s_{13}, h_{12}, m_{1,2} \) and \( t_{2,1} \) depend on two constrained parameters \( c_{13}, s_{23} \) and three observable parameters \( s_{12}, s_{23}, s_{13} \) while \( c_{13}, s_{13}, c_{13} \) and \( m_\mu \) depend on two constrained parameters \( c_{13}, s_{23} \) and five observable parameters \( \Delta m_{21}^2, \Delta m_{31}^2, s_{12}, s_{23}, s_{13} \).

V. NUMERICAL ANALYSIS

- Concerning the charged lepton sector, using the values of \( \Lambda \), the observed values of the charged lepton masses \( [1] \), that is, \( m_e = 0.51099 \text{ MeV} \), \( m_\mu = 105.65837 \text{ MeV} \), \( m_\tau = 1776.86 \text{ MeV} \), and the VEV of scalar fields in Eqs. (4) and (5), and with the help of Eqs. (16) and (24)-(27), we obtain \( |h_1| \approx 10^{-2} \), and \( h_{2,3,4,5} \) still depend on three parameters: \( \alpha, \theta, \psi \). In the case of \( s_\theta = -0.95 (\alpha = 288.2^\circ) \), the Yukawa-like couplings \( h_{2,3,4,5} \) depend on two parameters, namely \( \theta \) and \( \psi \), which are plotted in Figs. 1 and 2.

Figures 1 and 2 imply that

\[ |h_2| \approx |h_3| \approx 10^{-2}, \quad |h_4| \approx |h_5| \approx 10^{-1}, \]

(61)

which in turn implies that the Yukawa couplings in the charged lepton sector differ from each other by one order of magnitude for a natural explanation of the charged lepton mass hierarchy.

- Regarding the neutrino sector, Equation (37) shows that the neutrino masses \( m_{2,3} \) for NH and \( m_{1,2} \) for IH) depend on two experimental parameters, \( \Delta m_{21}^2 \) and \( \Delta m_{31}^2 \), which have been measured with high accuracy. In the case of \( \Delta m_{21}^2 \) and \( \Delta m_{31}^2 \), their values lie in the 3\( \sigma \) range [37], i.e., \( \Delta m_{21}^2 \in (69.40, 81.40) \text{ meV}^2 \) and \( \Delta m_{31}^2 \in (1.47, 3.63) \times 10^3 \text{ meV}^2 \). The allowed regions are \( m_{1,2,3} \), \( m_1 = 0 \), \( m_2 \in (8.33, 9.02) \text{ meV} \), \( m_3 = (49.70, 51.30) \text{ meV} \) for NH, and \( m_1 \in (48.70, 50.30) \text{ meV} \), \( m_2 = (49.4, 51.0) \text{ meV} \), \( m_3 = 0 \) for IH. The sum of neutrino masses is predicted to be

\[ \sum m_\nu \text{ (meV)} \in \begin{cases} (58.25, 60.25) \text{ for NH,} \\ (98.50, 101.0) \text{ for IH,} \end{cases} \]

(62)

Fig. 1. (color online) \( 10^3|h_2| \) (left panel) and \( 10^3|h_3| \) (right panel) versus \( c_\theta \) and \( s_\theta \) with \( c_\theta \in (0.29, 0.31) \) and \( s_\theta \in (0.25, 0.65) \).
which is consistent with the limits $[38] \sum m_\nu < 0.15 \text{ eV}$ (NH) and $\sum m_\nu < 0.17 \text{ eV}$ (IH), $\sum m_\nu < 0.14 \text{ eV}$ $[39]$, $\sum m_\nu < 0.152 \text{ eV}$ $[40]$ (minimal $\Lambda CDM + \sum m_\nu$), $\sum m_\nu < 0.118 \text{ eV}$ (high-$l$ polarization), $\sum m_\nu < 0.101 \text{ eV}$ (NP-DDE model), $\sum m_\nu < 0.093 \text{ eV}$ (NP-DDE+r model); the most aggressive bound is $\sum m_\nu < 0.078 \text{ eV}$ (NP-DDE+r with the R16 prior) $[40, 41]$, $\sum m_\nu < 0.183 \text{ eV}$ for IH $[42]$, $\sum m_\nu < 0.13 \text{ eV}$ (the base dataset) and $\sum m_\nu < 0.11 \text{ eV}$ (pol dataset) $[43]$, $\sum m_\nu < 0.19 \text{ eV}$ $[44]$.

To determine the possible ranges of the parameters $k_{1,2}, n_{1,2}, t_{1,2}$ and obtain predictive values for the Dirac $CP$ violating phase $\delta$, we use the observables $\Delta m^2_{21}, \Delta m^2_{31}$, $\sin^2 \theta_{12}$, $\sin^2 \theta_{23}$ and $\sin^2 \theta_{13}$ as input parameters; these observables are given in Table 3.

At the best-fit values of the lepton mixing angles $[37]$, $\sin^2 \theta_{12} = 0.318$ and $\sin^2 \theta_{13} = 2.200 \times 10^{-2}$ for NH while $\sin^2 \theta_{12} = 0.318$ and $\sin^2 \theta_{13} = 2.225 \times 10^{-2}$ for IH; $s_\theta, k_{1,2}, n_{1,2}$ and $t_{1,2}$ depend on two parameters: $c_\theta$ and $s_\theta$. The Dirac $CP$ violating phase $\delta$ (more precisely, $s_\delta$) is a function of two parameters, $c_\theta$ and $s_\theta$, with $c_\theta \in (0.29, 0.31)$ and $s_\theta \in (0.25, 0.65)$ for both IH and NH; it is plotted in Fig. 3. According to this figure, we have that

$$s_\delta \in (-0.95, -0.50), \text{ i.e., } \delta' \in (288.20, 330.00) \text{ (NH and IH).}$$

The dependences of $k_{1,2}, n_{1,2}$ and $t_{1,2}$ on the parameters $c_\theta$ and $s_\theta$, with $c_\theta \in (0.29, 0.31)$ and $s_\theta \in (0.25, 0.65)$ for both IH and NH, are respectively plotted in Figs. 4, 5, 6, 7, 8, and 9.

These figures imply that

$$k_1 \in \begin{cases} (-1.54, -1.42) & \text{for NH,} \\ (-0.215, -0.170) & \text{for IH,} \end{cases}$$

$$k_2 \in \begin{cases} (-0.25, 0.10) & \text{for NH,} \\ (-4.60, -3.20) & \text{for IH,} \end{cases}$$

(63)

Fig. 2. (color online) $|k_4|$ (left panel) and $|k_5|$ (right panel) versus $c_\theta$ and $s_\theta$ with $c_\theta \in (0.29, 0.31)$ and $s_\theta \in (0.25, 0.65)$.

| $n_1$ | $n_2$ |
|------|------|
| $\{ (0.70, 0.875) \}$ for NH, | $\{ (0.20, 0.80) \}$ for NH, |
| $\{ (-4.50, -2.75) \}$ for IH, | $\{ (-3.00, -1.60) \}$ for IH, |

(65)

Table 3. Global analysis of neutrino oscillation data $[37]$. Best-fit point (3$\sigma$ range) (NH) Best-fit point (3$\sigma$ range) (IH)

| $\Delta m^2_{21} \text{ [meV}^2\text{]}$ | $75.0 (69.4 \rightarrow 81.4)$ | $75.0 (69.4 \rightarrow 81.4)$ |
| $|\Delta m^2_{31}|/10^3 \text{ [meV}^2\text{]}$ | $2.55 (2.47 \rightarrow 2.63)$ | $2.45 (2.37 \rightarrow 2.53)$ |
| $\sin^2 \theta_{12}$ | $0.318 (0.271 \rightarrow 0.369)$ | $0.318 (0.271 \rightarrow 0.369)$ |
| $\sin^2 \theta_{23}$ | $0.574 (0.434 \rightarrow 0.610)$ | $0.578 (0.433 \rightarrow 0.608)$ |
| $\sin^2 \theta_{13}/10^{-2}$ | $2.200 (2.00 \rightarrow 2.405)$ | $2.225 (2.018 \rightarrow 2.424)$ |
| $\delta_{CP}/\pi$ | $1.08 (0.71 \rightarrow 1.99)$ | $1.58 (1.11 \rightarrow 1.96)$ |

Fig. 3. (color online) $s_\delta$ versus $c_\theta$ and $s_\theta$ with $c_\theta \in (0.29, 0.31)$ and $s_\theta \in (0.25, 0.65)$ for both NH and IH.
Fig. 4. (color online) $k_1$ versus $c_p$ and $s_p$ with $c_p \in (0.29, 0.31)$ and $s_p \in (0.25, 0.65)$ for NH (left panel) and IH (right panel).

Fig. 5. (color online) $k_2$ versus $c_p$ and $s_p$ with $c_p \in (0.29, 0.31)$ and $s_p \in (0.25, 0.65)$ for NH (left panel) and IH (right panel).

Fig. 6. (color online) $n_1$ versus $c_p$ and $s_p$ with $c_p \in (0.29, 0.31)$ and $s_p \in (0.25, 0.65)$ for NH (left panel) and IH (right panel).
Fig. 7. (color online) $n_2$ versus $c_\theta$ and $s_\theta$ with $c_\theta \in (0.29,0.31)$ and $s_\theta \in (0.25,0.65)$ for NH (left panel) and IH (right panel).

Fig. 8. (color online) $t_1$ versus $c_\theta$ and $s_\theta$ with $c_\theta \in (0.29,0.31)$ and $s_\theta \in (0.25,0.65)$ for NH (left panel) and IH (right panel).

Fig. 9. (color online) $t_2$ versus $c_\theta$ and $s_\theta$ with $c_\theta \in (0.29,0.31)$ and $s_\theta \in (0.25,0.65)$ for NH (left panel) and IH (right panel).
Similarly, to determine the possible ranges of the parameters $A, B_{1,2}, C_{1,2,3}, \langle m_\alpha \rangle$, and $m_\beta$, we fix $\sin^2 \theta_{12}, \sin^2 \theta_{23}$, and $\sin^2 \theta_{13}$ at their best-fit points [37], $c_\varphi = 0.30(\theta = 72.54^\circ)$ and $s_\varphi = 0.40(\psi = 23.58^\circ)$ for both IH and NH, and $\Delta m_{21}^2$ and $\Delta m_{31}^2$ take their values in their corresponding 3 $\sigma$ ranges [37]: $\Delta m_{21}^2 \in (69.4,81.4)\text{meV}^2$ and $\Delta m_{31}^2 \in (2.47,2.63)\times10^3\text{meV}^2$ (NH) while $\Delta m_{31}^2 \in (-2.53,-2.37)\times10^3\text{meV}^2$ (IH). The dependence of $A, B_{1,2}, C_{1,2,3}, \langle m_\alpha \rangle$, and $m_\beta$ on the parameters $\Delta m_{21}^2$ and $\Delta m_{31}^2$ are presented in Figs. 10, 11, 12, 13, 14, 15, 16, and 17, respectively.

Figures 10 and 15 imply that

$$\begin{align*}
A \in & \{(3.70,3.925)\text{meV} \text{ for NH,} \\
& (48.00,49.40)\text{meV} \text{ for IH,} \\
B_1 \in & \{(3.90,4.25)\text{meV} \text{ for NH,} \\
& (-4.775,-4.60)\text{meV} \text{ for IH,} \\
B_2 \in & \{(-7.15,-6.80)\text{meV} \text{ for NH,} \\
& (-5.75,-5.575)\text{meV} \text{ for IH,} \\
C_1 \in & \{(38.40,39.60)\text{meV} \text{ for NH,} \\
& (27.40,28.20)\text{meV} \text{ for IH,} \\
\end{align*}$$

Fig. 10. (color online) $A$ (meV) versus $\Delta m_{21}^2$ and $\Delta m_{31}^2$ with $\Delta m_{21}^2 \in (69.4,81.4)\text{meV}^2$ and $\Delta m_{31}^2 \in (2.47,2.63)\times10^3\text{meV}^2$ for NH (left panel) and $\Delta m_{31}^2 \in (-2.53,-2.37)\times10^3\text{meV}^2$ for IH (right panel).

Fig. 11. (color online) $B_1$ (meV) versus $\Delta m_{21}^2$ and $\Delta m_{31}^2$ with $\Delta m_{21}^2 \in (69.4,81.4)\text{meV}^2$ and $\Delta m_{31}^2 \in (2.47,2.63)\times10^3\text{meV}^2$ for NH (left panel) and $\Delta m_{31}^2 \in (-2.53,-2.37)\times10^3\text{meV}^2$ for IH (right panel).
Fig. 12. (color online) $B_2$ (meV) versus $\Delta m_{21}^2$ and $\Delta m_{31}^2$ with $\Delta m_{21}^2 \in (69.4, 81.4)\text{meV}^2$ and $\Delta m_{31}^2 \in (2.47, 2.63)\times 10^3\text{meV}^2$ for NH (left panel) and $\Delta m_{31}^2 \in (−2.53, −2.37)\times 10^3\text{meV}^2$ for IH (right panel).

Fig. 13. (color online) $C_1$ (meV) versus $\Delta m_{21}^2$ and $\Delta m_{31}^2$ with $\Delta m_{21}^2 \in (69.4, 81.4)\text{meV}^2$ and $\Delta m_{31}^2 \in (2.47, 2.63)\times 10^3\text{meV}^2$ for NH (left panel) and $\Delta m_{31}^2 \in (−2.53, −2.37)\times 10^3\text{meV}^2$ for IH (right panel).

Fig. 14. (color online) $C_2$ (meV) versus $\Delta m_{21}^2$ and $\Delta m_{31}^2$ with $\Delta m_{21}^2 \in (69.4, 81.4)\text{meV}^2$ and $\Delta m_{31}^2 \in (2.47, 2.63)\times 10^3\text{meV}^2$ for NH (left panel) and $\Delta m_{31}^2 \in (−2.53, −2.37)\times 10^3\text{meV}^2$ for IH (right panel).
Fig. 15. (color online) $C_3$ (meV) versus $\Delta m^2_{21}$ and $\Delta m^2_{31}$ with $\Delta m^2_{21}$ ∈ (69.4, 81.4) meV$^2$ and $\Delta m^2_{31}$ ∈ (2.47, 2.63) 10$^3$ meV$^2$ for NH (left panel) and $\Delta m^2_{31}$ ∈ (−2.53, −2.37) 10$^3$ meV$^2$ for IH (right panel).

Fig. 16. (color online) $\langle m_{ee} \rangle$ (meV) versus $\Delta m^2_{21}$ and $\Delta m^2_{31}$ with $\Delta m^2_{21}$ ∈ (69.4, 81.4) meV$^2$ and $\Delta m^2_{31}$ ∈ (2.47, 2.63) 10$^3$ meV$^2$ for NH (left panel) and $\Delta m^2_{31}$ ∈ (−2.53, −2.37) 10$^3$ meV$^2$ for IH (right panel).

Fig. 17. (color online) $m_\beta$ (meV) versus $\Delta m^2_{21}$ and $\Delta m^2_{31}$ with $\Delta m^2_{21}$ ∈ (69.4, 81.4) meV$^2$ and $\Delta m^2_{31}$ ∈ (2.47, 2.63) 10$^3$ meV$^2$ for NH (left panel) and $\Delta m^2_{31}$ ∈ (−2.53, −2.37) 10$^3$ meV$^2$ for IH (right panel).
which are values below the upper limits for \( m_{\nu} \) from KamLAND-Zen [45] \( m_{\nu} < 61 \pm 165 \) meV, GERDA [46] \( m_{\nu} < 104 \pm 228 \) meV, and CUORE [47] \( m_{\nu} < 75 \pm 350 \) meV; the constraints for \( m_p \) are expressed as 8.5 meV < \( m_p \) < 1.1 eV for NH and 48 meV < \( m_p \) < 1.1 eV for IH [1], \( m_p \in (8.90 - 12.60) \) eV [48], and \( m_p \in 0.8 \) eV [49].

\[ C_2 \in \begin{cases} 
(16.00, 16.60) \text{ meV} & \text{for NH,} \\
(23.00, 23.70) \text{ meV} & \text{for IH,} 
\end{cases} 
\]

\[ C_3 \in \begin{cases} 
(-19.00, -18.20) \text{ meV} & \text{for NH,} \\
(-24.70, -24.00) \text{ meV} & \text{for IH.} 
\end{cases} 
\]

\[ \langle m_{\nu} \rangle \in \begin{cases} 
(3.700, 3.925) \text{ meV} & \text{for NH,} \\
(48.00, 49.40) \text{ meV} & \text{for IH,} 
\end{cases} 
\]

\[ m_p \in \begin{cases} 
(8.75, 9.10) \text{ meV} & \text{for NH,} \\
(48.40, 49.80) \text{ meV} & \text{for IH,} 
\end{cases} 
\]

\[ (69) \]

\[ \langle m_{\nu} \rangle < 350 \times \frac{10^{-3}}{\text{eV}} \text{ consistent with recently reported constraints.} \]
APPENDIX B: EXPLICIT EXPRESSIONS OF $a_{1u,d}, a_{2u,d}, a_{3u,d}, b_{u,d}, c_{1u,d}, c_{2u,d}, c_{3u},$ AND $c_{4u,d}$ AS FUNCTIONS OF QUARK MASSES AND QUARK MIXING MATRIX ELEMENTS

The explicit expressions of $a_{1u,d}, a_{2u,d}, a_{3u,d}, b_{u,d}, c_{1u,d}, c_{2u,d}, c_{3u},$ and $c_{4u,d}$ are

$$a_{1u} = m_u + m_t, a_{2u} = \frac{m_c + m_t}{2}, a_{3u} = \frac{m_c - m_t}{2},$$

$$a_{1d} = m_d + m_b, a_{2d} = \frac{m_s + m_b}{2}, a_{3d} = \frac{m_s - m_b}{2},$$

$$c_{1u} = -c_{3u} + \frac{(m_d - m_s)(m_c - m_u)(1 - V_{11}^{\exp})}{2c_{1d}^{*}},$$

$$c_{2u} = \frac{(m_u - m_c)(m_d - m_s)}{2c_{2d}^{*} + c_{1d}^{*}} \left[ \frac{4b_y^{*}b_y}{(m_b - m_c)(m_c - m_u)} + \frac{c_{4u}^{*} + c_{3u}^{*}}{m_c - m_u} + V_{22}^{\exp} - 1 \right],$$

$$c_{3u} = \frac{m_u - m_c}{2} \left[ \frac{V_{11}^{\exp} - 1}{(m_u - m_c)(m_d - m_s)} + \frac{c_{2u}^{*} - c_{3u}^{*}}{m_c - m_u} + V_{13}^{\exp} \right],$$

$$2c_{4u} = \frac{4b_y^{*}b_y}{c_{2d}^{*} + c_{1d}^{*}} \left[ (m_u - m_c)(m_d - m_s) \right],$$

$$b_u = \frac{(m_u - m_c)(m_d - m_s)}{2c_{2d}^{*} + c_{1d}^{*}} \left[ 2c_{1d}^{*} + c_{3u}^{*} \right],$$

$$b_d = \frac{(m_u - m_c)(m_d - m_s)}{2c_{2d}^{*} + c_{1d}^{*}} \left[ 2c_{1d}^{*} + c_{3u}^{*} \right],$$

$$c_{2d} = V_{31}^{\exp} + \frac{(V_{33}^{\exp} - 1)(m_d - m_s)}{c_{3u}^{*}},$$

$$c_{4d} = \left\{ 2c_{1d}^{*} + c_{3u}^{*} \right\},$$

$$(m_u - m_c)(m_d - m_s) \left[ 2c_{1d}^{*} + V_{13}^{\exp} \right],$$

$$V_{21}^{\exp} V_{31}^{\exp} - V_{12}^{\exp} V_{32}^{\exp}.$$  

where

$$F_y = (V_{33}^{\exp} - 1)(V_{13}^{\exp} V_{21}^{\exp} - V_{12}^{\exp}), G_y = V_{33}^{\exp} \left\{ V_{11}^{\exp} + (V_{12}^{\exp} - V_{13}^{\exp}) V_{21}^{\exp} + (V_{21}^{\exp} - V_{23}^{\exp}) \right\},$$

$$H_y = (V_{11}^{\exp} + V_{22}^{\exp})(1 - V_{33}^{\exp}),$$

$$T_y = (1 - V_{11}^{\exp}) V_{21}^{\exp} (1 - V_{33}^{\exp} + V_{23}^{\exp}),$$

$$K_y = \left[ (V_{11}^{\exp} + V_{13}^{\exp} V_{21}^{\exp}) V_{33}^{\exp} - (V_{12}^{\exp} V_{22}^{\exp} V_{33}^{\exp} - V_{13}^{\exp} V_{21}^{\exp} V_{33}^{\exp} - V_{22}^{\exp} V_{31}^{\exp} V_{33}^{\exp}) \right],$$

$$P_y = (V_{22}^{\exp} - V_{12}^{\exp} V_{21}^{\exp} V_{33}^{\exp} + (V_{12}^{\exp} V_{31}^{\exp} V_{33}^{\exp} + V_{13}^{\exp} V_{23}^{\exp} V_{33}^{\exp} + (V_{12}^{\exp} V_{31}^{\exp} V_{33}^{\exp} + V_{23}^{\exp} V_{31}^{\exp} V_{33}^{\exp})).$$

APPENDIX C: EXPLICIT EXPRESSIONS OF $k_{1,2,m_1,2}$ AND $t_{1,2}$ AS FUNCTIONS OF $a_p,b_p,c_p,F_p,G_p,a_R,b_R$ AND $c_R$

The explicit expressions of $k_{1,2,m_1,2}$ and $t_{1,2}$ are

$$a_p,b_p,c_p,F_p,G_p,a_R,b_R$$

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\begin{align}
  k_1 &= \frac{c_D - d_D}{a_D + b_D}, \quad k_2 = \frac{a_D - b_D}{c_D + d_D}, \\
  t_2 &= \frac{(c_D + d_D) \left\{ \left( (a_D + b_D)^2 + (c_D - d_D)^2 \right) c_R^2 + (a_D^2 - b_D^2)c_Rb_R + 2(c_Dd_D - a_Db_D)b_R^2 \\
  &\quad - b_R \sqrt{\Delta} \right\} }{ \left\{ (c_D - d_D) \left[ b_D^2c_R(b_R - c_R) + a_D^2b_R(b_R + c_R) + b_Rc_R(c_D^2 + d_D^2) - c_R \sqrt{\Delta} \right] \right\}}, \\
  t_1 &= \frac{a_D^2(b_D - a_D)(b_R + c_R) + b_D \left[ (b_D^2 + 2c_Dd_D)(b_R - c_R) - (c_D^2 + d_D^2)c_R \right] \\
  &\quad - a_D \left[ b_D^2(b_R - c_R) + c_Rc_R + 2c_Dd_D(b_R + c_R) + c_Rd_D^2 \right] + (b_D - a_D \sqrt{\Delta} \right\} }{ \left\{ (c_D - d_D) \left[ b_D^2c_R(b_R - c_R) + a_D^2b_R(b_R + c_R) + b_Rc_R(c_D^2 + d_D^2) + c_R \sqrt{\Delta} \right] \right\}}, \\
  t_2 &= \frac{(c_D + d_D) \left\{ (a_D + b_D)^2 + (c_D - d_D)^2 \right\} c_R^2 + (a_D^2 - b_D^2)c_Rb_R + 2(c_Dd_D - a_Db_D)b_R^2 \\
  &\quad + b_R \sqrt{\Delta} \right\} }{ \left\{ (c_D - d_D) \left[ b_D^2c_R(b_R - c_R) + a_D^2b_R(b_R + c_R) + b_Rc_R(c_D^2 + d_D^2) + c_R \sqrt{\Delta} \right] \right\}},
\end{align}

where

\begin{align}
  \Delta &= a_D^2(b_R + c_R)^2 + \left( b_D^2b_R - (c_D^2 + d_D^2)c_R \right)^2 + 8a_Db_Dc_Rd_D(c_R^2 - b_D^2) + c_Rd_D^2 \\
  &\quad + 2 \left[ 2b_D^2c_R^2 - b_D^2b_Rc_R + (b_D^2 - c_D^2)c_R^2 \right] d_D^2 + 2a_D^2(b_R + c_R) \left[ b_D^2(b_R - c_R) + c_R(c_D^2 + d_D^2) \right].
\end{align}

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