Quantum Quench dynamics in Non-local Luttinger Model: Rigorous Results

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We investigate, in the Luttinger model with fixed box potential, the time evolution of an inhomogeneous state prepared as a localized fermion added to the noninteracting ground state. We proved that, if the state is evolved with the interacting Hamiltonian, the averaged density has two peaks moving in opposite directions, with a constant but renormalized velocity. We also proved that a dynamical ‘Landau quasi-particle weight’ appears in the oscillating part of the averaged density, asymptotically vanishing with large time. The results are proved with the Mattis-Lieb diagonalization method. A simpler proof with the exact Bosonization formulas is also provided.

1. INTRODUCTION

Recent experiments on cold atoms have motivated increasing interest in the dynamical properties of many body quantum systems which are closed and isolated from any reservoir or environment. Nonequilibrium properties can be investigated by quantum quenches, in which the system is prepared in an eigenstate of the non-interacting Hamiltonian and its subsequent time evolution driven by an interacting many-body Hamiltonian is observed. As the resulting dynamical behavior is the cumulative effect of the interactions between an infinite or very large number of particles, the computation of local observables averaged over time-evolved states poses typically great analytical difficulties; therefore, apart for some analysis in two dimensions (see, for instance [3, 4]), the problem is mainly studied in one dimension [5–30]. A major difference with respect to the equilib-
rium case relies on the fact that in such a case a form of universality holds, ensuring that a number of properties are essentially insensitive to the model details. At non-equilibrium the behavior depends instead on model details; for instance integrability in spin chains dramatically affects the non equilibrium behavior \cite{13,40,41} while it does not alter the $T = 0$ equilibrium properties \cite{43}. This extreme sensitivity to the details or approximations asks for a certain number of analytical exact results at non-equilibrium, to provide a benchmark for experiments or approximate computations.

One of the interacting Fermionic system where non-equilibrium properties can be investigated is the Luttinger model \cite{32,33} (see also \cite{34–36}), which provides a great number of information in the equilibrium case. In the Luttinger model model the quadratic dispersion relation of the non relativistic fermions is replaced with a linear dispersion relation, leading to the ”anomaly” in the distribution of the ground states density. This anomaly is proved to be universal for a large class of one dimensional Fermionic system, called the Luttinger liquid \cite{31}. Luttinger model became of great interest in mathematical physics ever since the exact solutions founded by Mattis-Lieb \cite{34} and is a key to investigate the mathematical properties of condensed matter physics.

It is important to stress that there exist two versions of this model, the local Luttinger model (LLN) and the non local Luttinger model (NLLM); in the former a local delta-like interaction is present while in the latter the interaction is short ranged but non local. The finite range of the interaction plays as an ultraviolet cut-off. At equilibrium such two models are often confused as they have similar behavior, due to the above mentioned insensitivity to model details; there is however no reason to expect that this is true also at non equilibrium. It should be also stressed that the LLM is plagued by ultraviolet divergences typical of a QFT and an ad-hoc regularization is necessary to get physical predictions; the short time or distance behavior depends on the chosen regularization.

In this paper we study the evolution of inhomogeneous states in the non-local Luttinger Model with a fixed box potential, with the Mattis-Lieb diagonalization method, which was proved to be mathematically rigorous \cite{35,36}. Then we perform rigorous analysis of the asymptotic behavior in the infinite volume limit. The main result shows that (see Theorem \ref{maintheorem}), when the interaction is turned on, the dynamics is ballistic with a constant but renormalized velocity, and the interaction produces a dynamical ‘Landau quasi-particle weight’ in the oscillating part, asymptotically vanishing with time. The expressions we
get do not require any ultraviolet regularization, and correctly capture also the short time dynamics. We also invite the physically oriented reader to read this article along with a short letter [18], in which we studied the quench dynamics of non-local Luttinger model but without giving full details of the proof. In the current article we put full details of the proof and specialize to the box potential, for which the change of velocity due to the many-body interaction is more transparent; we provide also a simpler proof of the main theorem with the exact Bosonization formulas.

The quantum quench of homogeneous states in the NLLM was derived in [20], [21], in which steady states were found. However mathematical rigor is lacking in these work. The quenched evolution of the NLLM prepared in domain wall initial state was studied in [42] and the universality of the quantum Landauer conductance for the final states was proved, in a mathematically rigorous way.

The plan of the paper is the following. We introduce the NLLM with box potential in §II. In §III we prove Theorem 2.2 with the Mattis-Lieb diagonalization method. Some details of the proof are presented in the Appendix. The proof of Theorem 2.2 based on the Bosonization method is given in §IV.

2. THE LUTTINGER MODEL AND MAIN RESULTS

A. The Luttinger model with box potential

The non-local Luttinger model (NLLM) is defined by the Hamiltonian:

\[ H_\lambda = \int_{-L/2}^{L/2} dx \frac{i}{v_F} \left( \psi_{x,1}^+ \partial_x \psi_{x,1}^- : - : \psi_{x,2}^+ \partial_x \psi_{x,2}^- : \right) + \lambda \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} dy v(x-y) \psi_{x,1}^+ \psi_{x,1}^- : \psi_{y,2}^+ \psi_{y,2}^- : \]  

(2.1)

where \( \psi_{x,\omega}^\pm = \frac{1}{\sqrt{2}} \sum_k a_{k,\omega} e^{\pm ikx} \), \( \omega = 1, 2 \), \( k = \frac{2\pi n}{L}, n \in \mathbb{N} \) are fermionic creation or annihilation operators, :: denotes Wick ordering and \( v_F \) is the Fermi velocity. We are choosing units so that \( v_F = 1 \). The two-body interaction potential \( v(x-y) \) is given by:

\[ v(x-y) = \frac{\sin(x-y)}{x-y}, \]  

(2.2)
whose fourier transform reads:

\[ v(p) = \begin{cases} v_0 & \text{for } p \leq 1, \\ 0 & \text{for } p > 1. \end{cases} \]  

(2.3)

The potential \( v(x) \) or \( v(p) \) is also called the box potential and \( v_0 \) is called the strength of \( v(p) \). Equilibrium Luttinger model with box potential was first considered in [44].

In the Fourier space the Luttinger Hamiltonian can be written as

\[
H = H_0 + V = \sum_{k>0} \frac{\lambda}{L} \sum_{p>0} v(p)(\rho_1(p)\rho_2(-p) + \rho_1(-p)\rho_2(p)) + \frac{\lambda}{L} v(0) \ N_1 N_2
\]

(2.4)

where, for \( p > 0 \),

\[
\rho_\omega(p) = \sum_k a_{k+p,\omega}^+ a_{k,\omega}^-, \quad N_\omega = \sum_{k>0} (a_{k,\omega}^+ a_{k,\omega}^- - a_{-k,\omega}^- a_{-k,\omega}^+).
\]

(2.5)

It is well known that Fock space canonical commutation relations don’t have a unique representation in a system with infinite degree of freedom. So one has to introduce a cutoff function \( \chi_\Lambda(k) \) with \( \Lambda \) a large positive number such that \( \chi_\Lambda(k) = 1 \) for \( |k| \leq \Lambda \) and equals 0 otherwise and the regularized operators \( \rho_\omega(p) \) must be thought as \( \lim_{\Lambda \to \infty} \sum_k \chi_\Lambda(k) \chi_\Lambda(k+p) a_{k+p,\omega}^+ a_{k,\omega}^- \).

The Hamiltonian \( H \) as well as \( \rho_\omega(p) \) can be regarded as operators acting on the Hilbert space \( \mathcal{H} \) constructed as follows. Let \( \mathcal{H}_0 \) be the linear span of vectors obtained by applying finitely many times creation or annihilation operators on \( \chi^0 = \prod_{k \leq 0} a_{k,1}^+ a_{-k,2}^- |\text{vac}> \).

(2.6)

In this way we get an abstract linear space to which we introduced scalar products between any pair of vectors. \( \mathcal{H} \) is defined as the completion of \( \mathcal{H}_0 \) in the scalar product just introduced. Moreover the operators \( H \) and \( \rho_\omega(p) \), regarded as operators on \( \mathcal{H} \) with domain \( \mathcal{H}_0 \), are self adjoint.

The basic property of the Luttinger model is the validity of the following anomalous commutation relations, first proved in [34], for \( p, p' > 0 \)

\[
[\rho_1(-p), \rho_1(p')] = [\rho_2(p), \rho_2(-p')] = \frac{pL}{2\pi} \delta_{p,p'}.
\]

(2.7)
Remark that this commutator acting on the Fock space is not precise due to the infinitely many degrees of freedom of the system. So one should introduce a cutoff $\Lambda$ so that the commutator:

$$\sum_{k=\Lambda+1}^{\Lambda} a_{k,\omega}^+ a_{k,-\omega}^+ - \sum_{k=-\Lambda}^{-\Lambda+1} a_{k,\omega}^+ a_{k,-\omega}^- = \sum_{k=-\Lambda}^{\Lambda-1} a_{k,\omega}^+ a_{k,-\omega}^- - \sum_{k=\Lambda+1}^{\Lambda+p} a_{k,\omega}^+ a_{k,-\omega}^- . \quad (2.8)$$

on any state of $\mathcal{H}$ is equal, in the limit $\Lambda \to \infty$, to $\frac{pL}{2\pi}$.

Moreover one can verify that $\rho_2(p)|0\rangle = 0$ and $\rho_1(-p)|0\rangle = 0$.

Other important commutation relations (see \cite{34,45} for proofs) are as follows:

$$[H_0, \rho_\omega(\pm p)] = \pm \varepsilon_\omega p \rho_\omega(\pm p), \quad [\rho_\omega, \psi^\pm_{\omega,x}] = e^{ipx} \psi^\pm_{\omega,x} \quad (2.10)$$

where $\omega = 1, 2$; $\varepsilon_\omega = 1$ for $\omega = 1$ and $\varepsilon_\omega = -1$ for $\omega = 2$.

**B. The Mattis-Lieb diagonalization**

The Hamiltonian (2.4) can be diagonalized with the method of Lieb-Mattis \cite{34}, as follows. First we introduce an operator

$$T = \frac{1}{L} \sum_{p>0} [\rho_1(p)\rho_1(-p) + \rho_2(-p)\rho_2(p)] \quad (2.11)$$

and write $H = (H_0 - T) + (V + T) = H_1 + H_2$. Note that $H_1$ is already diagonalized in that it commutes with $\rho_\omega$. The key for the diagonalization of $H_2$ is the introduction of a bounded operator $S$ acting on the Hilbert space $\mathcal{H}$:

$$S = \frac{2\pi}{L} \sum_{p \neq 0} \phi(p)p^{-1} \rho_1(p)\rho_2(-p), \quad \tanh \phi(p) = -\frac{\lambda v(p)}{2\pi}. \quad (2.12)$$

Using the following Bogolyubov transformations for the operators $\rho_\omega(\pm p)$:

$$e^{iS}\rho_{1,2}(\pm p)e^{-iS} = \rho_{1,2}(\pm p) \cosh \phi(p) + \rho_{2,1}(\pm p) \sinh \phi, \quad (2.13)$$

we can easily prove that $H_2$ can be written in diagonal form:

$$e^{iS}H_2e^{-iS} = \hat{H}_2$$

$$:= \frac{2\pi}{L} \sum_{p} \text{sech}^2\phi(p)[\rho_1(p)\rho_1(-p) + \rho_2(-p)\rho_2(p)] + E_0. \quad (2.14)$$
By Formula (2.12) we can easily find that the operator $S$ hence the transformation in (2.14) is well defined only for $|\lambda v(p)| < 2\pi$; The model is instable for $|\lambda v(p)| > 2\pi$.

Define

$$D = \tilde{H}_2 - T = \frac{2\pi}{L} \sum_p \sigma(p) [\rho_1(p)\rho_1(-p) + \rho_2(-p)\rho_2(p)] + E_0,$$

we have $[H_0, D] = 0$. The diagonalization formula for the Hamiltonian reads:

$$e^{iS} e^{iH_0 t} e^{-iS} = e^{i(H_0+D)t}.$$ 

(2.16)

C. The time evolution of the one particle state and the main theorem

Define

$$\psi_{\omega, \varepsilon, x, \delta} = \psi_{\omega, x} e^{-iH_0 t},$$

$$= \frac{1}{\sqrt{L}} \sum_k a_{\omega, k} e^{\pm i(kx - \varepsilon \omega kt) - \delta |k|},$$

(2.17)

where $\delta \to 0^+$, $\varepsilon_1 = +$, $\varepsilon_2 = -$. By direct calculation we find that:

$$< 0 | \psi_{\omega, x, \delta} \psi_{\omega, y, \delta} | 0 > = \frac{(2\pi)^{-1}}{i\varepsilon \omega (x-y) - i(t-s) + \delta}.$$ 

(2.18)

The relation between the creation or annihilation Fermionic operators and the quasiparticle operators is

$$\psi_x = e^{ip_{\text{F}}x} \psi_{x, 1} + e^{-ip_{\text{F}}x} \psi_{x, 2},$$

(2.19)

where $p_{\text{F}}$ is the Fermi momentum and we call $e^{ip_{\text{F}}x} \psi_{x, 1} = \tilde{\psi}_{x, 1}$ and $e^{-ip_{\text{F}}x} \psi_{x, -1} = \tilde{\psi}_{x, 2}$. In momentum space this simply means that the momentum $k$ is measured from the Fermi points, that is $c_{k, \omega} = \tilde{c}_{k + \varepsilon_{\text{F}}p_{\text{F}}, \omega}$. The ground state of $H$ is $|GS > = e^{iS}|0 >$, where $|0 >$ is the ground state of $H_0$ and the inhomogeneous one particle initial state is given by:

$$|I_t > = e^{iH_0 t}(\psi_{1, x}^{+} + \tilde{\psi}_{2, x}^{+})|0 >.$$ 

(2.20)

Let $n(z)$ be the density operator, which is defined as the limit $\delta \to 0$, $\varepsilon \to 0$ of the following expression:

$$\frac{1}{2} \sum_{\rho = \pm} (\tilde{\psi}_{1, z + \rho \epsilon}^{+} \tilde{\psi}_{1, z}^{-} + \psi_{2, z + \rho \epsilon}^{+} \psi_{1, z}^{-} + \psi_{2, z + \rho \epsilon}^{+} \psi_{2, z}^{-} + \tilde{\psi}_{1, z + \rho \epsilon}^{+} \tilde{\psi}_{2, z}^{-} + \tilde{\psi}_{1, z + \rho \epsilon}^{+} \tilde{\psi}_{1, z}^{-} + \psi_{2, z + \rho \epsilon}^{+} \psi_{1, z}^{-} + \psi_{2, z + \rho \epsilon}^{+} \psi_{2, z}^{-}).$$

(2.21)
Note that summing over $\rho = \pm$ is the point spitting regularization, which plays the same role as the Wick ordering for avoiding divergences. We are interested in the average value of the density operator w. r. t. the 1-particle initial state $\langle \rho \rangle$, formally defined by:

$$G(x, z, t, \delta) := \langle I_t | n(z) | I_t \rangle := \sum_{\omega, \omega'} [\langle 0 | \tilde{\psi}_{\omega, x} \exp[iHt_{\omega, x+\delta} \tilde{\psi}_{\omega', z} \exp[-iHt_{\omega'} \tilde{\psi}_{\omega', x}] | 0 \rangle + \langle 0 | \tilde{\psi}_{\omega, x} \exp[iHt_{\omega, x+\delta} \tilde{\psi}_{\omega', z} \exp[-iHt_{\omega'} \tilde{\psi}_{\omega', x}] | 0 \rangle ] \tag{2.22}$$

As a first step we consider the non-interacting case. Let $|I_{0,t}\rangle := e^{iH_0t} (\tilde{\psi}^+_{1,x} + \tilde{\psi}^+_{2,x})|0\rangle$, we have:

**Theorem 2.1** When $\lambda = 0$, $H = H_0$, we have

$$\lim_{L \to \infty} \langle I_{0,t}|n(z)|I_{0,t}\rangle = \frac{1}{2\pi^2} \frac{\cos 2p_F(x-y)}{(x-z)^2 - t^2} + \frac{1}{4\pi^2} \left[ \frac{1}{((x-z) - t)^2} + \frac{1}{((x-z) + t)^2} \right]. \tag{2.23}$$

**Proof 2.1** We consider first the term with $\omega = 1$, $\omega' = 2$. Using the explicit expressions of the Fermionic operators and taking the limit $\varepsilon \to 0$, we can easily find that this term is equal to $e^{2ip_F(x-y)}(4\pi^2)^{-1}[(x-z)^2 - t^2]^{-1}$; a similar result is found for the second term. The third and fourth terms are vanishing as $\sum_{\rho \varepsilon} \varepsilon \rho = 0$; similarly the last two term give $(4\pi^2)^{-1}[(x-z) \pm t]^{-2}$. Combine all these terms we can derive Formula (2.23), hence proved this theorem.

**Remark 2.1** The physical meaning of Theorem 2.1 is quite clear: when the interaction is turned off, the average of the density is sum of two terms, an oscillating and a non oscillating part (when the particle is added to the vacuum there are no oscillations $p_F = 0$). At $t = 0$ the density is peaked at $z = x$, where the average is singular. With the time increasing the particle peaks move in the left and right directions with constant velocity $v_F = 1$ (ballistic motion); that is, the average of the density is singular at $z = x \pm t$ and a "light cone dynamics" is found.

When we turn on the interaction and let the system driven by the full interacting Hamiltonian, the ground states and the dynamics will be significantly changed. The explicit
expression of (2.22) can be derived with the Mattis-Lieb diagonalization method followed by a rigorous analysis of the asymptotic behavior for $L \to \infty$ and large $t$. We have

**Theorem 2.2** Let the interacting box potential (see (2.3)) be turned on in the Hamiltonian, let $\gamma_0 = \frac{v_0}{2}$ and $\omega_0 = \sqrt{1 - \left(\frac{v_0}{2\pi}\right)^2}$. The average of the density operator with respect to the one particle initial state $|I_{\lambda,t}\rangle$ in the limit $L \to \infty$ reads:

$$\lim_{L \to \infty} <I_{\lambda,t}|n(z)|I_{\lambda,t}> = \frac{1}{4\pi^2} \left[\frac{1}{((x - z) - t)^2} + \frac{1}{((x - z) + t)^2}\right] + \frac{1}{2\pi^2} \frac{\cos 2p_F(x - z) e^{Z(t)}}{(x - z)^2 - (\omega_0 t)^2}.$$  

(2.24)

where

$$Z(t) = \gamma_0 \int_0^1 \frac{dp}{p} (\cos 2\omega_0 pt - 1)$$  

(2.25)

is the Landau quasi particle factor, such that $Z(0) = 1$ and

$$\exp Z(t) \sim \text{cst}(\frac{1}{2\omega_0 t})^{\gamma_0},$$  

(2.26)

for $t \geq 1$.

### 3. PROOF OF THEOREM 2.2

We consider first the term:

$$\langle 0| \psi_{1,x}^+ e^{iH_0} \psi_{1,x}^+ \psi_{1,z}^+ e^{-iH_0} \psi_{1,z} |0\rangle,$$  

(3.27)

and forget the phase factor $e^{\mp ipF x}$ for the moment for simplicity; these factors are very easy to restore. The rest of this subsection is devoted to the calculation of (3.27).

Let $I$ be an identity operator in $\mathcal{H}$. Using the fact that $e^{-i\varepsilon \sigma} e^{i\varepsilon \sigma} = I$ and $e^{-iH_0 e^{iH_0} |I = s} = I$, we can write (3.27) as

$$\langle 0| \psi_{1,x}^+ e^{-i\varepsilon \sigma} (e^{i\varepsilon \sigma} e^{iH_0} e^{-i\varepsilon \sigma}) (e^{i\varepsilon \sigma} \psi_{1,x}^+ e^{-i\varepsilon \sigma}) \cdot (e^{i\varepsilon \sigma} \psi_{2,z}^+ e^{-i\varepsilon \sigma}) (e^{i\varepsilon \sigma} e^{-iH_0} e^{-i\varepsilon \sigma}) e^{i\varepsilon \sigma} \psi_{2,z}^+ |0\rangle |_{\varepsilon = 1, s = t}$$  

(3.28)

**Lemma 3.1** Let $\hat{I}_1$ be an operator valued function of $p_1(\pm p)$ and $\psi_1^\pm$ and $\hat{I}_2$ be an operator valued function of $p_2(\pm p)$ and $\psi_2^\pm$, then we have the following factorization Formula for (3.27):

$$G_1 = I_1 I_2,$$  

(3.29)

where $I_1 = \langle 0| \hat{I}_1 |0\rangle$ and $I_2 = \langle 0| \hat{I}_2 |0\rangle$.  

Proof 3.1 We shall prove this lemma by deriving the explicit expressions of $\hat{I}_1$ and $\hat{I}_2$.

Using the diagonalization formula (2.16), formula (3.28) can be written as:

$$
\langle 0 | \psi_{1,z}^- e^{-i\xi S} e^{i(H_0+D)t} e^{iS} \psi_{1,z}^+ e^{-i(H_0+D)t} e^{-i\xi S} | 0 \rangle |_{\xi=1, s=t}.
$$

(3.30)

Now we consider the term of $e^{i\xi S} \psi_{1,z}^+ e^{-i\xi S}$. It is a well known result [34] that:

$$
e^{i\xi S} \psi_{1,z}^+ e^{-i\xi S} = \psi_{1,z}^+ W_{1,z}^\pm R_{1,z}^\pm.
$$

(3.31)

where

$$
W_{1,z}^\pm = \exp\left\{ \pm \frac{2\pi}{L} \sum_{p>0} \frac{1}{p} [\rho_1(p) e^{-ipz} - \rho_1(-p) e^{ipz}] (\cosh \varepsilon \phi - 1) \right\}
$$

(3.32)

$$
R_{1,z}^\pm = \exp\left\{ \pm \frac{2\pi}{L} \sum_{p>0} \frac{1}{p} [\rho_2(p) e^{-ipz} - \rho_2(-p) e^{ipz}] \sinh \varepsilon \phi \right\}.
$$

Similarly one has

$$
e^{i\xi S} \psi_{2,z}^\pm e^{-i\xi S} = \psi_{2,z}^\pm W_{2,z}^\pm R_{2,z}^\pm.
$$

(3.33)

where

$$
W_{2,z}^\pm = \exp\left\{ \pm \frac{2\pi}{L} \sum_{p>0} \frac{1}{p} [\rho_1(p) e^{-ipz} - \rho_1(-p) e^{ipz}] \sinh \varepsilon \phi \right\}
$$

(3.34)

$$
R_{2,z}^\pm = \exp\left\{ \pm \frac{2\pi}{L} \sum_{p>0} \frac{1}{p} [\rho_2(p) e^{-ipz} - \rho_2(-p) e^{ipz}] (\cosh \varepsilon \phi - 1) \right\}.
$$

Then we consider the term

$$
e^{-i\xi S} e^{i(H_0+D)t} W_{1,z}^- R_{1,z}^- e^{-i(H_0+D)t} e^{i\xi S},
$$

(3.35)

which, after inserting the identity operator $I = e^{i\xi S} e^{-i\xi S}$ and $I = e^{-i(H_0+D)t} e^{i(H_0+D)t}$, is equal to

$$
[e^{-i\xi S} e^{i(H_0+D)t} W_{1,z}^- e^{-i(H_0+D)t} e^{i\xi S}] \cdot [e^{-i\xi S} e^{i(H_0+D)t} R_{1,z}^- e^{-i(H_0+D)t} e^{i\xi S}].
$$

(3.36)

Let $f(p, t)$ be an arbitrary regular function, define $\sigma(p) = \text{sech}2\phi - 1$ and $\omega(p) = \sigma(p) + 1 = \text{sech}2\phi$, we have the following commutation relation

$$
[H_0 + D, \rho_\omega(\pm p)] = \pm \varepsilon \omega(p(\sigma(p) + 1)\rho_\omega(\pm p), \omega = 1, 2, \varepsilon_1 = +, \varepsilon_2 = -,
$$

(3.37)
which implies that
\[ e^{i(H_0+D)t}e^{f(p,t)\rho_\omega(\pm p)}e^{-i(H_0+D)t} = e^{\pm i\omega t(\sigma+1)p_t}f(p,t)\rho_\omega(\pm p). \] (3.38)

Combining the above formula with (2.13) and (2.17) we find that (3.36) can be written as a product of
\[ e^{-iS}e^{i(H_0+D)t}W_{1,z}^\pm e^{-i(H_0+D)t}e^{iS} \]
\[ = \exp \pm \frac{2\pi}{L} \sum_p \left( \frac{\cosh \phi - 1}{p} \right) \left[ (\rho_1(-p) \cosh \varepsilon \phi - \rho_2(-p) \sinh \varepsilon \phi)e^{ipx+ipt(\sigma+1)} - (\rho_1(p) \cosh \varepsilon \phi - \rho_2(p) \sinh \varepsilon \phi)e^{-ipx+ipt(\sigma+1)} \right] := W_{1,z}^\pm. \] (3.39)

and
\[ e^{-iS}e^{i(H_0+D)t}R_{1,z}^- e^{-i(H_0+D)t}e^{iS} \]
\[ = \exp \pm \frac{2\pi}{L} \sum_p \frac{\sinh \phi}{p} \left[ (\rho_2(-p) \cosh \varepsilon \phi - \rho_1(-p) \sinh \varepsilon \phi)e^{ipy+ipt(\sigma+1)} - (\rho_2(p) \cosh \varepsilon \phi - \rho_1(p) \sinh \varepsilon \phi)e^{-ipy+ipt(\sigma+1)} \right] := R_{1,z}^-. \] (3.40)

Using again (3.31), (3.33) and (3.46), we have:
\[ e^{-iS}e^{i(H_0+D)t}e^{iS}\psi_{1,z}^+ e^{-iS} e^{-i(H_0+D)t}e^{iS} \]
\[ = z_a A_{1+} A_{1-} A_{2+} A_{2-} \psi_{1,z,0}^+ \tilde{W}_{1t}^{-1} \tilde{R}_{1t}^{-1} \tilde{W}_{1t}^{-1} \tilde{R}_{1t}^{-1} B_{1^{-1}} B_{1+} B_{2^{-1}} B_{2+}, \] (3.41)

and
\[ e^{-iS}e^{iH}\psi_{2,z}^- e^{-iS} e^{iS}\psi_{2,z}^- e^{-iS} e^{-iH}\psi_{2,z}^- e^{-iS} e^{iS} \]
\[ = z_b \tilde{W}_{2s}^{-1} \tilde{R}_{2s}^{-1} \tilde{W}_{2s}^{-1} \tilde{R}_{2s}^{-1} \tilde{R}_{2s}^{-1} \tilde{W}_{2s}^{-1} \tilde{R}_{2s}^{-1} B_{1^{-1}} B_{1+} B_{2^{-1}} B_{2+}, \] (3.42)

where \( \tilde{W}_{1t}^{-1}, \tilde{R}_{1t}^{-1} \) and \( \tilde{W}_{1t}^{-1}, \tilde{R}_{1t}^{-1} \) are operators depending on \( \rho_{1,2}(\pm p) \), respectively and \( z_a, z_b \) are functions of \( p \). The explicit expressions of the above factors are given in the Appendix.

Then we can easily find that the terms depending on \( \rho_1(\pm p) \) and \( \psi_{1}^\pm \) are factorized with respect to the terms depending on \( \rho_2(\pm p) \) and \( \psi_{2}^\pm \). Let
\[ I_1 := \langle 0|\tilde{I}_1|0 \rangle \]
\[ := \langle 0|\psi_{1z} A_{1+} A_{1-} \psi_{1,z}^+ \tilde{W}_{1t}^{-1} \tilde{W}_{1t}^{-1} \tilde{W}_{1t}^{-1} \tilde{W}_{2t} \tilde{W}_{2t} \tilde{B}_{1+} B_{1-} |0 \rangle, \] (3.43)
and

\[ I_2 := \langle 0| \hat{I}_2|0 \rangle \]

\[ := \langle 0| A_{2+A}^{-1} \hat{R}_1^{-1} \hat{R}_1^{-1} \hat{R}_2 \hat{R}_2 \psi_{2,zt} B_2 \cdot \hat{B}_2 \cdot \hat{\psi}_{2x}^\dagger |0 \rangle, \quad (3.44) \]

and using the fact that \( z_{a} = z_{b}^{-1} \) we have

\[ G_1 = I_1 I_2, \quad (3.45) \]

So we proved Lemma 3.1.

A. Calculation of \( I_1 \) and \( I_2 \)

In this part we derive the explicit expressions for \( I_1 \) and \( I_2 \). It is also useful to introduce the following proposition, which can be easily proved using (2.10):

**Proposition 3.1** Let \( f(p,t) \) is an arbitrary regular function. Then we have:

\[
e^{iH_0t} e^{f(p,t)} e^{-iH_0t} = e^{\pm i\omega_L \sigma t} e^{\pm z \omega L \sigma t} e^{\mp \omega L \sigma t}, \quad \omega = 1, 2; \quad \varepsilon_1 = +, \varepsilon_2 = -, \quad (3.46)\]

The basic idea to calculate \( I_1 \) and \( I_2 \) is to use repeatedly the Hausdorff to move the operators \( \rho_1(-p) \) and \( \rho_2(p) \) to the right most of the expressions in (3.43) and (3.44), and move \( \rho_1(p), \rho_2(-p) \) to the left most of the above expressions. By formula (2.9) and its adjoint form we know that these operator annihilate \( |0 \rangle \) and \( \langle 0 | \) respectively; the survived terms are those independent of \( \rho_{1,2}(\pm p) \). Setting \( \varepsilon = 1 \), we have:

\[ I_1 = \exp\left\{ \frac{2\pi}{L} \sum_p \frac{1}{p} \left[ (e^{-ip(\sigma+1)(t+s)} - 1)(2 \cosh^2 \phi \sinh^2 \phi + \cosh^2 \phi \sinh \phi) \right. \right. \]

\[ + (e^{ip(\sigma+1)(t+s)} - 1) \cosh \phi \sinh^3 \phi + e^{-ip\sigma t}(- \cosh^2 \phi - \sinh^2 \phi) \]

\[ + e^{ip(x-z)+ip(\sigma+1)s} (\cosh \phi \sinh \phi + \cosh^2 \phi) - e^{ip(x-z)+ips} \]

\[ + e^{ip(x-z)-ip(\sigma+1)t} \left( - \sinh \phi - \sinh^2 \phi \right) \}

\[ \} \langle 0| \psi_{1z} \psi_{1,z,t,\delta}^\dagger |0 \rangle, \quad (3.47) \]

and

\[ I_2 = \langle 0| \psi_{2z}^\dagger \psi_{2x}^\dagger |0 \rangle \exp\left\{ \frac{2\pi}{L} \sum_p \frac{1}{p} \left[ (e^{-ip(\sigma+1)(t+s)} - 1) \cosh \phi \sinh^3 \phi \right. \right. \]

\[ + (e^{ip(\sigma+1)(t+s)} - 1)(\cosh^3 \phi \sinh \phi + 2 \cosh^2 \phi \sinh^2 \phi) \]

\[ + e^{-ip\sigma t}(\cosh^2 \phi + \sinh^2 \phi) - e^{ip(x-z)-ip\sigma t} \]

\[ + e^{ip(x-z)+ip(\sigma+1)s}(- \cosh \phi \sinh \phi - \sinh^2 \phi) \]

\[ + e^{ip(x-z)-ip(\sigma+1)t} (\sinh \phi + \cosh^2 \phi) \}

\[ \}. \quad (3.48) \]
Combining (3.47) with (3.48) and setting \( s = t \), we get:

\[
\langle 0 | \psi_{1,x}^- e^{iHt} \psi_{1,x}^+ e^{-iHt} \psi_{2,x}^- | 0 \rangle \]

\[
= \langle 0 | \psi_{1,x}^+ \psi_{1,x,t,\delta}^- | 0 \rangle \langle 0 | \psi_{2,x,t,\delta}^+ \psi_{2,x}^- | 0 \rangle 
\times \exp \left\{ \frac{1}{p} \left[ (e^{ip(x-z)} + ip(x-z) + ipt) \right. 
+ (e^{ip(x-z)} - ip(x-z) + ipt) 
+ 2 \sinh \phi \cosh \phi (\sinh \phi + \cosh \phi)^2 (\cos 2p \sigma + 1) t - 1 \right] \right\}. 
\]

It is useful to derive the asymptotic behavior for the second line in (3.49) and we have:

\[
\lim_{\delta \to 0} \lim_{L \to \infty} \langle 0 | \psi_{1,x}^+ \psi_{1,x,t,\delta}^- | 0 \rangle \langle 0 | \psi_{2,x,t,\delta}^+ \psi_{2,x}^- | 0 \rangle = \frac{1}{4\pi^2} \frac{1}{(x-z)^2 - t^2} \quad (3.50)
\]

With the same method we can derive the explicit expression for the other terms in (2.22). Restoring the phase factor \( e^{\pm ipF(x-z)} \) and combine all the terms of (2.22), we obtain the following desired result:

\[
< I_{n,t} | n(z) | I_{n,t} > = \frac{1}{4\pi^2} \left[ \frac{1}{(x-z) - t}^2 + \frac{1}{(x-z) + t}^2 \right] + \frac{1}{4\pi^2} \frac{e^{Z(t)}}{(x-z)^2 - t^2} \left[ e^{2ipF(x-z)} e^{Q_a(x,z,t)} + e^{-2ipF(x-z)} e^{Q_b(x,z,t)} \right],
\]

where

\[
Z(t) = \sum_{p} \frac{2}{p} \sinh \phi \cosh \phi (\sinh \phi + \cosh \phi)^2 (\cos 2p \sigma + 1) t - 1, \quad (3.52)
\]

\[
Q_a = \sum_{p} \frac{1}{p} \left[ (e^{ip(x-z)} + ip(x-z) + ipt) \right. 
+ (e^{ip(x-z)} - ip(x-z) + ipt) \right],
\]

\[
Q_b = \sum_{p} \frac{1}{p} \left[ (e^{-ip(x-z)} + ip(x-z) + ipt) \right. 
+ (e^{-ip(x-z)} - ip(x-z) + ipt) \right]. \quad (3.53)
\]

**B. The asymptotic behavior for \( L \to \infty \)**

In this section we shall derive the asymptotic behavior of Formula (3.51) in the limit \( L \to \infty \). Using definitions of the hyper-geometric functions we find that

\[
\text{sech} \phi(p) = \frac{1}{2} \left( \frac{1 + \frac{v(p)}{2} \frac{v(p)}{2\pi}}{\sqrt{1 + \frac{v(p)}{2\pi}}} - 1 \right), \quad \cosh \phi = \frac{1}{2} \left( \frac{1 + \frac{v(p)}{2} \frac{v(p)}{2\pi}}{\sqrt{1 + \frac{v(p)}{2\pi}}} + 1 \right), \quad (3.54)
\]
where $v(p)$ is the box potential with strength $v_0$ (see Formula \((2.3)\)), we have the following expression for the critical exponent:

$$\gamma(p) = 2 \sinh \phi(p) \cosh \phi(p)(\sinh \phi(p) + \cosh \phi(p))^2 = \frac{v(p)}{4\pi}. \quad (3.55)$$

Taking the limit $L \to \infty$ means that we should consider the discrete sum over $p$ as integral over continuous variables. We have:

$$Z(t) = \int_0^\infty \frac{\gamma(p) dp}{p}(\cos 2\omega_0 pt - 1) = \gamma_0 \int_0^1 \frac{dp}{p}(\cos 2\omega_0 pt - 1) = \gamma_0 \int_0^{2\omega_0 t} \frac{d(2\omega_0 p)}{2\omega_0 p}(\cos 2\omega_0 pt - 1), \quad (3.56)$$

where $\gamma_0 := \frac{v_0}{4\pi}$ and $\omega_0 := \sqrt{1 - \left(\frac{v_0}{2\pi}\right)^2}$. The second line is true is due to the fact that $\gamma(p) = 0$ for $p \in (1, \infty]$. Let $y = 2\omega_0 pt$ and $w = 2\omega_0 t$, $Z(t)$ can be written as:

$$\gamma_0 \int_0^1 \frac{dp}{p}(\cos 2\omega_0 pt - 1) = \gamma_0 \int_0^w \frac{dy}{y}(\cos y - 1). \quad (3.57)$$

There are three cases to be considered, depending on the range of $t$:

- when $t \ll 1$, which corresponds to the short time behavior and implies that $y \ll 1$ and $w \ll 1$ (to remember that the $v(p)$ is vanishing for $p > 1$); In this case we have

$$Z(t) = \gamma_0 \int_0^w \frac{dy}{y}(\cos y - 1) \sim \gamma_0 \int_0^w \frac{dy}{y}(\cos y - 1) \ll 1. \quad (3.58)$$

So that $Z(t)$ is well defined for $y \ll 1$. Furthermore, it is vanishing as $y \to 0^+$ and we have $e^{Z(t)}|_{t \to 0^+} \to 1$.

- when $t \in (0, 1]$; In this case we can repeat the analysis as above and easily prove that $Z(t)$ is a bounded function.

- when $t \in [1, \infty]$; let $p_0 > 0$ be the minimal value of $p$ and $u = 2\omega_0 p_0 t$, we have

$$Z(t) = \gamma_0 \int_u^{2\omega_0 t} \frac{dy}{y}(\cos y - 1) = -C - \ln u - \int_0^u \frac{\cos y - 1}{y} dy - [\ln 2\omega_0 t - \ln u] = \gamma_0(- \ln 2\omega_0 t - C - \int_0^u \frac{\cos y - 1}{y} dy), \quad (3.59)$$
where we have used the integral formula
\[
\int_u^\infty \frac{\cos y}{y} dy = -C - \ln u - \int_0^u \frac{\cos y - 1}{y} dy,
\] (3.60)

where \( C = 0.577215 \cdots \) is the Euler constant and \( \int_0^u \frac{\cos y - 1}{y} dy \) is a bounded function.

Remark that (3.59) is well defined for \( u \to 0 \), due to the cancellation of \( \ln u \).

So we have
\[
e^{Z(t)} \sim \text{cst} \cdot \left[ \frac{1}{2\omega_0 t} \right]^{\gamma_0}, \quad \text{for } t \geq 1.
\] (3.61)

Now we derive the asymptotic formula for \( Q_a \) and \( Q_b \). Replacing the discrete sum over \( p \) in (3.53) by integrals and performing the integrations, we can easily find that:
\[
Q_a = Q_b = \ln \left( \frac{(x - z)^2 - t^2}{(x - z)^2 - \omega_0^2 t^2} \right).
\] (3.62)

Collecting all the above terms we have:
\[
\lim_{L \to \infty} \frac{\langle I_{\lambda,t} | n(z) | I_{\lambda,t} \rangle}{L} = \frac{1}{4\pi^2} \left[ \frac{1}{(x - z - t)^2} + \frac{1}{(x - z + t)^2} \right] + \frac{1}{2\pi^2} \frac{\cos 2p_F(x - z) \ e^{Z(t)}}{(x - z)^2 - (\omega_0 t)^2}.
\] (3.63)

So we proved theorem 2.2.

4. THE BOSONIZATION METHOD

While the Lieb-Mattis method for solving Luttinger model is mathematically rigorous, technically it is very complicated. There exist another very popular method for studying the one dimensional interacting Fermions models, called the Bosonizations, which states that certain two dimensional models of fermions are equivalent to the corresponding Bosonic models: the corresponding Fermionic Hilbert space and the Bosonic one are isomorphic and the the Fermionic operator can be expressed in terms of the Bosonic operators. While the Bosonization method can reduce significantly the difficulty for the calculation, it has the reputation of not mathematically rigorous. A Rigorous proof of Bosonization formulas was given very recently in a paper by Langmann and Moosavi [45]. In this section we shall prove Theorem 2.2 with the exact Bosonization formulas in
Using Formula (2.13) we have:

\[ e^{\psi_-} = e^{\psi_-} e^{-i \pi x Q_{x, \omega} L} R_{\omega} e^{i \pi x Q_{x, \omega} L} \times \exp \left\{ \varepsilon_\omega \sum_{p > 0} \frac{2\pi}{Lp} \rho_\omega (p) e^{-ipx - \delta |p|} - \rho_\omega (-p) e^{ipx - \delta |p|} \right\}, \]

where \( \omega, \omega' = 1, 2 \), \( \varepsilon_1 = +, \varepsilon_2 = - \), \( Q_\omega = \rho_\omega (0) \) and \( N_\delta = \left( \frac{1}{L(1 - e^{-\pi \delta / L})} \right) \) is the normalization factor. \( R_{\omega}^+ \) is the Klein factor such that \( R_{\omega}^- = (R_{\omega}^+)^\dagger \). They obey the following commutation relation (see [45] for the detailed derivation):

\[
[\rho_\omega (p), R_{\omega}] = \varepsilon_\omega \delta_{\omega, \omega'} \delta_{p, 0} R_{\omega}, \quad [H_0, R_{\omega}] = \varepsilon_\omega \frac{\pi}{L} \{ \rho_\omega (0), R_{\omega} \},
\]

\[
\langle 0 | R_{\omega}^p R_{\omega}^q | 0 \rangle = \delta_{\omega, \omega'} \delta_{p, 0} \delta_{q, 0}, \quad R_{1}^q R_{2}^p = (-1)^{q_1 q_2} R_{2}^p R_{1}^q,
\]

\[
[Q_\omega, R_{1}^q R_{2}^p] = q_\omega R_{1}^q R_{2}^p, \quad q_\omega \in \mathbb{Z}.
\]

We shall not repeat the proof here and the interested reader is invited to look at [45] for details.

Let \( \hat{Z}_\omega^- = e^{i \pi x Q_{x, \omega} L} R_{\omega} e^{i \pi x Q_{x, \omega} L} \) and \( \hat{Z}_\omega^+ \) be its adjoint, we can write the Fermionic operators as:

\[
\psi_\omega^\pm (x, \delta) = N_\delta \hat{Z}_\omega^\pm e^{i \omega \sum_{p > 0} \frac{2\pi}{Lp} \rho_\omega (p) e^{-ipx - \delta |p|} - \rho_\omega (-p) e^{ipx - \delta |p|}}.
\]

We calculate first the term \( \langle 0 | \psi_{1, x}^- e^{i H t} \psi_{1, x}^+ \psi_{2, x}^- e^{-i H t} \psi_{2, x}^+ | 0 \rangle \) in [2.22]忘记 the phase factor \( e^{ipf(x-z)} \) for the moment. Inserting the identity operators \( I = e^{i H t} e^{-i H t} \) and \( I = e^{i S} e^{-i S} \) we derived Formula (3.28), which is the starting point of our analysis.

First of all, it is easy to find that

\[
e^{i S} \hat{Z}_\omega^\pm e^{-i S} = \hat{Z}_\omega^\pm.
\]

Using Formula (2.13) we have:

\[
e^{i S} \psi_{1, x}^\pm e^{-i S} = N_\delta \hat{Z}_\omega^+ \exp \left( \frac{2\pi}{L} \sum_{p > 0} \frac{1}{p} \left\{ - e^{-\delta p} e^{-ipx} \left[ \cosh \phi_{\rho_1} (p) + \sinh \phi_{\rho_2} (p) \right] + e^{-\delta p} e^{ipx} \left[ \cosh \phi_{\rho_1} (-p) + \sinh \phi_{\rho_2} (-p) \right] \right\},
\]
and
\[ e^{iS \psi_{2,z}^{-1}} e^{-iS} = N_\delta \hat{Z}_2 e^{\frac{2\pi}{L} \sum_{p>0} \frac{1}{p} \left\{ -e^{-i\phi} e^{-ip\phi} [\cosh \phi_2(p) + \sinh \phi_1(p)] \\
+ e^{-i\phi} e^{ip\phi} [\cosh \phi_2(-p) + \sinh \phi_1(-p)] \right\} }. \tag{4.68} \]

Using the fact that:
\[ [H_0 + D, \ R_\omega^\pm] = \pm \frac{2\pi(\sigma(0) + 1)}{L} R_\omega^\pm(2\varepsilon_\omega \rho_\omega(0) + 1), \tag{4.69} \]
and
\[ e^{i(H_0+D)t} R_\omega^\pm e^{-i(H_0+D)t} = R_\omega^\pm \exp \left[ \pm \frac{2\pi(\sigma(0) + 1)}{L} (2\varepsilon_\omega \rho_\omega(0) + 1)t \right], \tag{4.70} \]
we have:
\[ e^{-iS} e^{i(H_0+D)t} e^{iS} \sum_{u_0} e^{-iS} e^{i(H_0+D)t} e^{iS} \]
\[ = N_\delta \hat{Z}_1(t) \exp \frac{2\pi}{L} \sum_{p>0} \frac{1}{p} \left\{ e^{-i\phi} [A_1 \rho_1(p) + A_{-1} \rho_1(-p) + A_2 \rho_2(p) + A_{-2} \rho_2(-p)] \right\}, \tag{4.71} \]
and
\[ e^{-iS} e^{i(H_0+D)t} e^{iS} \sum_{u_0} e^{-iS} e^{i(H_0+D)t} e^{iS} \]
\[ = N_\delta \hat{Z}_2(t) \exp \frac{2\pi}{L} \sum_{p>0} \frac{1}{p} \left\{ e^{-i\phi} [B_1 \rho_1(p) + B_{-1} \rho_1(-p) + B_2 \rho_2(p) + B_{-2} \rho_2(-p)] \right\}, \tag{4.72} \]
where
\[
A_{\pm 1} = \pm e^{i\pi [\pm (\sigma+1)t]} \sinh^2 \phi \mp e^{-i\pi [\pm (\sigma+1)t]} \cosh^2 \phi, \\
A_{\pm 2} = \pm e^{i\pi [\pm (\sigma+1)t]} \sinh \phi \cosh \phi \mp e^{i\pi [\pm (\sigma+1)t]} \cosh \phi \sinh \phi, \\
B_{\pm 1} = \pm e^{i\pi [\pm (\sigma+1)t]} \sinh \phi \cosh \phi \mp e^{i\pi [\pm (\sigma+1)t]} \cosh \phi \sinh \phi, \\
B_{\pm 2} = \pm e^{i\pi [\pm (\sigma+1)t]} \sinh^2 \phi \mp e^{i\pi [\pm (\sigma+1)t]} \cosh^2 \phi, \\
\hat{Z}_1(t) = e^{i\pi x_0(0)/L} \exp \left[ -\frac{2\pi(\sigma(0) + 1)}{L} (2\rho_1(0) + 1) t \right] R_1^{-1} e^{i\pi x_0(0)/L}, \\
\hat{Z}_2(t) = e^{i\pi x_2(0)/L} \exp \left[ -\frac{2\pi(\sigma(0) + 1)}{L} (-2\rho_2(0) + 1) t \right] R_2 e^{i\pi x_2(0)/L}. \tag{4.73} \]

When \( p = 0 \), by using the fact that \( \rho_\omega(0)|0\rangle = 0 \) and \( \langle 0|R_\omega^{q_1} R_\omega^{q_2}|0\rangle = \delta_{\omega,\omega'} \delta_{q_1,0} \delta_{q_2,0} \), we have
\[ \langle 0| \hat{Z}_1 \hat{Z}_1^\dagger(t) \hat{Z}_2(t) \hat{Z}_2^\dagger |0\rangle = 1. \tag{4.74} \]
So the nontrivial contributions come from the $p > 0$ part. Using repeatedly the Hausdorff formula we can factorize the terms depending on $\rho_1(\pm p)$ and $\rho_2(\pm p)$:

\[
\langle 0 | N_\delta \exp \left( \frac{2\pi}{L} \sum_{p>0} \frac{1}{p} \right) [e^{-\delta p e^{-ipx} \rho_1(p)} - e^{-\delta p e^{ipx} \rho_1(-p)}] \times N_\delta \exp \left( \frac{2\pi}{L} \sum_{p>0} \frac{1}{p} \right) \left\{ e^{-\delta p [A_+ \rho_1(p) + A_- \rho_1(-p)]} \right\} \times N_\delta \exp \left( \frac{2\pi}{L} \sum_{p>0} \frac{1}{p} \right) \left\{ e^{-\delta p [B_+ \rho_1(p) + B_- \rho_1(-p)]} \right\} \times N_\delta e^{2\pi \sum_{p>0} \frac{1}{p} [e^{-\delta p e^{ipx} \rho_2(p)} - e^{-\delta p e^{-ipx} \rho_2(-p)}]} \langle 0 \rangle
\]

\[
=: N_\delta^4 I_1 I_2, \quad (4.75)
\]

where

\[
I_1 = \langle 0 | e^{2\pi \sum_{p>0} \frac{1}{p} e^{-\delta p [e^{-ipx} \rho_1(p) - e^{ipx} \rho_1(-p)]} \left[ e^{2\pi \sum_{p>0} \frac{1}{p} e^{-\delta p [A_+ \rho_1(p) + A_- \rho_1(-p)]} \times e^{2\pi \sum_{p>0} \frac{1}{p} e^{-\delta p [B_+ \rho_1(p) + B_- \rho_1(-p)]} \langle 0 \rangle, \quad (4.76)\right]}
\]

\[
I_2 = \langle 0 | e^{2\pi \sum_{p>0} \frac{1}{p} e^{-\delta p [A_+ \rho_2(p) + A_- \rho_2(-p)]} \left[ e^{2\pi \sum_{p>0} \frac{1}{p} e^{-\delta p [B_+ \rho_2(p) + B_- \rho_2(-p)]} \times e^{2\pi \sum_{p>0} \frac{1}{p} e^{-\delta p [e^{-ipx} \rho_2(p) - e^{ipx} \rho_2(-p)]} \langle 0 \rangle. \quad (4.77)\right]
\]

Following exactly the same procedure as section 3A, namely using repeatedly the Hausdorff formula and the annihilation formulas we have:

\[
I_1 I_2 = \exp \left( \frac{2\pi}{L} \sum_{p>0} \frac{1}{p} \right) e^{-2\delta p \left( [e^{ip(x-z)+ip(\sigma+1)t} - 1] + [e^{ip(x-z)-ip(\sigma+1)t} - 1] \right)}
+ 2 \sinh \phi \cosh \phi (\sinh \phi + \cosh \phi)^2 (\cos 2p(\sigma + 1)t - 1). \quad (4.78)
\]

In order to reproduce the expressions in (3.51) we need to extract from the above formula the noninteracting 2-point correlation function (see [45]), as follows. We write the terms $e^{\pm ip(x-z)\pm ip(\sigma+1)t} - 1$ in the above formula as

\[
(e^{ip(x-z)\pm ip(\sigma+1)t} - e^{ip(x-z)\pm ip(\sigma+1)t}) + (e^{ip(x-z)\pm ip(\sigma+1)t} - 1),
\]

while the first term gives the factors $Q$, the second term contributes to the non-interacting correlation function:

\[
N_\delta^4 \exp \left( \frac{2\pi}{L} \sum_{p>0} \frac{1}{p} \right) e^{-2\delta p \left( [e^{ip(x-z)+ipt} - 1] + [e^{-ip(x-z)+ipt} - 1] \right), \quad (4.79)\right]
\]

\[
\]
Now we derive the asymptotic formula for (4.79). Using the Poisson summation formula:

\[
\exp \left( \sum_{p>0} \frac{2\pi}{Lp} e^{-2\delta p} \right) = \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} e^{-4n\pi\delta/L} \right) = LN_\delta^2,
\]

where

\[
N_\delta^2 = \frac{1}{L(1 - e^{-4\pi\delta/L})} \sim \frac{1}{4\pi\delta} \text{ for } L \to \infty,
\]

Formulas (4.79) can be written as:

\[
\lim_{\delta \to 0^+} \lim_{L \to \infty} N_\delta^4 \exp \frac{2\pi}{L} \sum_{p>0} \frac{1}{p} e^{-2\delta p} [\left( e^{ip(x-z)+ipt} - 1 \right) + \left( e^{-ip(x-z)+ipt} - 1 \right)]
\]

\[
= \lim_{\delta \to 0^+} \lim_{L \to \infty} N_\delta^4 \frac{1}{L^2 N_\delta^2} \cdot \frac{1}{1 - e^{-4\pi [2\delta + i(x-z) + it]}} \frac{1}{1 - e^{-4\pi [-2\delta - i(x-z) + it]}}
\]

\[
= \frac{1}{4\pi^2} \frac{1}{(x-z)^2 - t^2}.
\]

Following the same procedure we can calculate all the other terms in (2.22) and derive Formula (3.51). The asymptotic expressions for the terms in the exponential can be derived with the same procedure as in the last section and we shall not repeat it here. So we proved Theorem (2.2) with the exact Bosonization formulas.

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5. APPENDIX

A. Explicit expressions of the factors in Formulas (3.43) and (3.44)

With some very long but elementary calculation we find that the expressions of the terms in formula (3.43) and (3.44) read:

\[ z_a = \exp \frac{2\pi}{L} \sum_p \frac{1}{p} \left\{ (e^{-ip_\sigma t} - 1) \right\}, \]  

(5.83)

\[ A_{1\pm} = \exp \frac{2\pi}{L} \sum_p \frac{1}{p} \rho_1(\pm p) \cosh \phi (\mp e^{\mp ipz \pm ip\tau} \pm e^{\mp ipz \mp ip\tau(\sigma + 1)}) \] 

\[ A_{2\pm} = \exp \pm \frac{2\pi}{L} \sum_p \frac{1}{p} \rho_2(\pm p) \sinh \phi (e^{\mp ipz \pm ip\tau} - e^{\mp ipz \pm ip\tau(\sigma + 1)}). \] 

(5.84)

\[ z_b = \exp \frac{2\pi}{L} \sum_p \frac{1}{p} (1 - e^{-ip_\sigma t}) = z_a^{-1}, \]  

(5.85)

\[ B_{1\pm} = \exp \pm \frac{2\pi}{L} \sum_p \frac{1}{p} \rho_1(\pm p) \sinh \phi (\mp e^{\mp ipz \mp ip\tau} \mp e^{\mp ipz \mp ip\tau(\sigma + 1)}), \] 

\[ B_{2\pm} = \exp \frac{2\pi}{L} \sum_p \frac{1}{p} \rho_2(\pm p) \cosh \phi (\pm e^{\pm ipz \mp ip\tau(\sigma + 1)} \mp e^{\mp ipz \mp ip\tau}). \]

\[ \tilde{W}_{1te}^{-1} = \exp \frac{2\pi}{L} \sum_p \frac{1}{p} \left\{ (\cosh \phi - 1)e^{-ipz + ip\tau} \rho_1(p) - (\cosh \phi - 1)e^{ipz - ip\tau} \rho_1(-p) \right\}, \] 

\[ \tilde{R}_{1te}^{-1} = \exp -\frac{2\pi}{L} \sum_p \frac{1}{p} \left\{ \sinh \phi e^{-ipz + ip\tau} \rho_2(p) - \sinh \phi e^{ipz - ip\tau} \rho_2(-p) \right\}. \] 

(5.86)

\[ \tilde{W}_{1te}^{-1} = \exp \frac{2\pi}{L} \sum_p \frac{1}{p} \left\{ (\cosh \phi - 1) \cosh \phi e^{-ipz + ip(\sigma + 1)t} \rho_1(p) \right\} \] 

\[ - (\cosh \phi - 1) \cosh \phi e^{ipz - ip(\sigma + 1)t} \rho_1(-p) \right\}, \] 

\[ \tilde{R}_{1te}^{-1} = \exp -\frac{2\pi}{L} \sum_p \frac{1}{p} \left\{ (\cosh \phi - 1) \sinh \phi e^{-ipz + ip(\sigma + 1)t} \rho_2(p) \right\} \] 

\[ - (\cosh \phi - 1) \sinh \phi e^{ipz - ip(\sigma + 1)t} \rho_2(-p) \right\}, \] 

\[ \tilde{W}_{1te}^{-1} = \exp -\frac{2\pi}{L} \sum_p \frac{\sinh \phi}{p} \left\{ \rho_1(p) \sinh \phi e^{-ipz - ip(\sigma + 1)t} \right\} \] 

\[ - \rho_1(-p) \sinh \phi e^{ipz + ip(\sigma + 1)t}, \] 

\[ \tilde{R}_{1te}^{-1} = \exp -\frac{2\pi}{L} \sum_p \frac{\sinh \phi}{p} \left\{ \rho_2(p) \cosh \phi e^{-ipz - ip(\sigma + 1)t} \right\} \] 

\[ - \rho_2(-p) \cosh \phi e^{ipz + ip(\sigma + 1)t}. \]
\[ \hat{W}_{2t} = \exp \left\{ -\frac{2\pi}{L} \sum_p \frac{1}{p} \sinh \varepsilon \phi (e^{-ipz-ipt} \rho_1(p) - e^{ipz+ipt} \rho_1(-p)) \right\}, \tag{5.87} \]

\[ \hat{R}_{2t} = \exp \left\{ \frac{2\pi}{L} \sum_p \frac{1}{p} (\cosh \varepsilon \phi - 1)(e^{-ipz-ipt} \rho_2(p) - e^{ipz+ipt} \rho_2(-p)) \right\}. \]

\[ \tilde{W}_{2t} = \exp \left\{ \frac{2\pi}{L} \sum_p \frac{1}{p} \sinh \phi \cosh \varepsilon \phi \left[ e^{-ipz+ip(\sigma+1)t} \rho_1(p) \right. \right. \]
\[ \left. \left. \quad - e^{ipz-ip(\sigma+1)t} \rho_1(-p) \right] \right\}, \tag{5.88} \]

\[ \tilde{W}_{2t} = \exp \left\{ -\frac{2\pi}{L} \sum_p \frac{1}{p} (\cosh \phi - 1) \sinh \varepsilon \phi [\rho_1(p)e^{-ipz-ipt(\sigma+1)} \right. \]
\[ \left. \quad - \rho_1(-p)e^{ipz+ipt(\sigma+1)}] \right\}, \]

\[ \tilde{R}_{2t} = \exp \left\{ -\frac{2\pi}{L} \sum_p \frac{\sinh \phi \sinh \varepsilon \phi}{p} \left[ \rho_2(p)e^{-ipz+ipt(\sigma+1)} \right. \right. \]
\[ \left. \left. \quad - \rho_2(-p)e^{ipz-ip(\sigma+1)} \right] \right\}, \]

\[ \hat{R}_{2t} = \exp \left\{ \frac{2\pi}{L} \sum_p \frac{1}{p} (\cosh \phi - 1) \cosh \varepsilon \phi \left[ \rho_2(p)e^{-ipz-ipt(\sigma+1)} \right. \right. \]
\[ \left. \left. \quad - \rho_2(-p)e^{ipz+ipt(\sigma+1)} \right] \right\}. \]

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