Detecting random walks on graphs with heterogeneous sensors
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Abstract. We consider the problem of detecting a random walk on a graph, based on observations of the graph nodes. When visited by the walk, each node of the graph observes a signal of elevated mean, which we assume can be different across different nodes. Outside of the path of the walk, and also in its absence, nodes measure only noise. Assuming the Neyman-Pearson setting, our goal then is to characterize detection performance by computing the error exponent for the probability of a miss, under a constraint on the probability of false alarm. Since exact computation of the error exponent is known to be difficult, equivalent to computation of the Lyapunov exponent, we approximate its value by finding a tractable lower bound. The bound reveals an interesting detectability condition: the walk is detectable whenever the entropy of the walk is smaller than one half of the expected signal-to-noise ratio. We derive the bound by extending the notion of Markov types to Gauss-Markov types - sequences of state-observation pairs with a given number of node-to-node transition counts and the nodes’ local sample means. The lower bound has an intuitive interpretation: among all Gauss-Markov types that are asymptotically feasible in the absence of the walk, the bound finds the most typical one under the presence of the walk. Finally, we show by a sequence of judicious problem reformulations that computing the bound reduces to solving a convex optimization problem, which is a result in its own right.

Keywords. Random walk, hypothesis testing, error exponent, large deviations principle, threshold effect, Gauss-Markov type, convex analysis, Lyapunov exponent.

I. INTRODUCTION

Suppose we have a network of $N$ nodes, where each node is equipped with a sensor that measures the network environment. The environment can be in two states: 1) either a certain activity is present (e.g., an intruder, a signal), and the nodes have elevated mean; or 2) the environment is static, and the nodes measure only noise. We assume that the activity has the form of a random walk on the nodes of the graph, with a certain transition matrix $P$. We also assume that the measured signals are embedded in additive white Gaussian noise. The goal is to detect the random walk, based on the network observations.
This detection problem has widespread applicability. In [1], the authors consider the problem of detecting the spin of electrons using magnetic resonance force microscopy (MRFM). The observed signal is modeled as a random telegraph signal (i.e., Markov chain with states 0 and 1). Detecting an intruder by a sensor network (e.g., video network) can also be modeled by this model. In [2], a similar, graphical model methodology is applied for detection of highly oscillatory signals (“chirps”). In this paper, we present an application for random medium access in communications systems with extremely low signal-to-noise ratio (SNR) and unknown frequency selective fading. To establish communication with its associated base station, a user sends an access signal that hops from one frequency to another according to a Markov chain with a specified transition matrix; to detect a user, the base station then implements the corresponding likelihood ratio test (see eq. 3 further ahead). By performing frequency hopping, the effects of frequency selective fading are significantly alleviated, and furthermore without the need for synchronization between the sender and the receiver. This kind of scenario might be of interest for the emerging Narrow Band Internet of Things (IoT) standard, which envisions a similar random access setup for an extremely large number of communicating IoT devices and similar effects of unknown environments.

**Related literature.** Detection of Markov chains in noise has been considered in [3], [1], and [4]. For a more general review of hidden Markov processes, we refer the reader to the excellent overview paper [5]. In [1], the authors consider spin detection and are concerned with deriving efficient detection tests and evaluating and comparing their performance. They show that, at any given time \( t \), the optimal, likelihood ratio test can be conveniently expressed as a product \( \Pi_t \) of \( 2t \) matrices, where the transition matrix \( P \) alternates with independent and identically distributed (i.i.d.) diagonal matrices \( D_t \) defined by network observations.

Papers closest to our work are [4], [5]. In [4], [5], the authors consider Neyman-Pearson detection and study asymptotic performance of the likelihood ratio test, as the number of observations per sensor grows. The assumed performance metric is the error exponent of the probability of a miss, under a fixed constraint on the probability of false alarm. Reference [3] evaluates numerically the error exponent for a two-state Markov chain. Reference [4] shows that finding the error exponent of the probability of a miss is equivalent to computing the Lyapunov exponent of the product \( \Pi_t \) above – a problem well-known to be difficult, see [6]. Assuming identical SNR across nodes, the paper then finds a lower bound on the error exponent and shows by simulations that this bound is very close to the true error exponent value.

In our previous work [7], we also studied products of i.i.d. random matrices. However, in contrast with the problem of evaluating the Lyapunov exponent, in [7] we were concerned with evaluating the large deviations rate for the probability of the event that the product stays away from its limiting matrix (i.e.,
away from the Lyapunov exponent limit).

The setup that we study here is also related to random dynamical systems (iterated random functions) [8]. In particular, the log likelihood ratio, see equation (4) in Section II, can be represented in the form of a random linear dynamical system, in which the random linear transformation has a specific form: a deterministic component of the dynamics - the transition matrix \( P \) – is intertwined with a random one - the measurement dependent matrix \( D_t \).

**Contributions.** So far, random walk detection has only been considered under the assumption that the SNR values on the walk’s path are equal. However, although convenient for analytical purposes, this assumption is often not realistic in practice. For example, in a video network, cameras closer to the intruders path will have better SNR than those further away. A communication signal that uses frequency hopping often experiences frequency selective fading – an effect that the receiver must account for when performing signal detection. Motivated by these practical considerations, in this paper we address the scenario when the random walk signal is observed with different SNRs across nodes of the graph. We find a lower bound on the error exponent for Neyman-Pearson detection. Furthermore, we show that computing this bound is equivalent to solving a convex optimization problem, thus showing that evaluation of our bound is computationally efficient. The latter was not known even in the case of equal SNRs across nodes.

To find the bound on the error exponent under the assumption of heterogeneous sensors, we had to follow a different proof path than [4]. In particular, assuming that the SNRs across sensors are different, it is no longer possible to lump all measurements into a single quantity through summation (and similarly with the probability of the state sequence, see eq. (15) and the following text in [4]). Instead, in our proofs, we increase the level of granularity by using the notion of Markov types – sequences of states with the same transition counts, introduced by Davisson, Longo and Sgarro [9]. Following the intrinsic structure of the problem, where sample means are naturally grouped per each node, we extend the notion of Markov types to what we term Gauss-Markov types. The latter define pairs of state-observation sequences where the state sequences have equal Markov types and the observation sequences have equal per node sample means. We then prove the large deviations principle (LDP) for the measure induced by the random Gauss-Markov type, where the probability space is taken to be a uniform probability space over all possible Markov chain sequences up to time \( t \). This result is the core technical result behind the lower bound on the error exponent.

**Paper organization.** In Section II, we pose the problem. In Section III, we introduce Gauss-Markov typea, give preliminaries, and state the LDP result for Gauss-Markov types. In Section IV we state
and prove the main result on the lower bound of the error exponent, while Section VI proves the LDP for Gauss-Markov types. Section VII proves convexity of the error exponent lower bound and, as a by product, derives a solution by a single letter parametrization. Section VII concludes the paper.

Notation. For $N \in \mathbb{N}$, we denote by $1$ the vector of all ones in $\mathbb{R}^N$, and by $e_i$ the $i$-th canonical vector of $\mathbb{R}^N$ (that has value one on the $i$-th entry and the remaining entries are zero); we denote by $\mathbb{R}_+^N$ the set of vectors in $\mathbb{R}^N$ that have all elements non-negative. For a matrix $A$, we let $[A]_{ij}$ and $A_{ij}$ denote its $i,j$ entry and for a vector $a \in \mathbb{R}^d$, we denote its $i$-th entry by $a_i$, $i,j = 1,\ldots,d$. For a function $f : \mathbb{R}^d \mapsto \mathbb{R}$, we denote its domain by $\mathcal{D}_f = \{ x \in \mathbb{R}^d : -\infty < f(x) < +\infty \}$; $\log$ denotes the natural logarithm and $\log_+$ denotes the function $\max\{0, \log\}$. We let $\| \cdot \|$ denote the spectral norm of a square matrix. For $N$ real numbers $d_1, \ldots, d_N$, we let $\text{diag}\{d_1, \ldots, d_N\}$ denote the diagonal matrix whose $i$th diagonal entry is $d_i$, for $i = 1,\ldots,N$. An open Euclidean ball in $\mathbb{R}^d$ of radius $\rho$ and centered at $x$ is denoted by $B_x(\rho)$; the closure, the interior, the boundary, and the complement of an arbitrary set $G \subseteq \mathbb{R}^d$ are respectively denoted by $\overline{G}$, $G^\circ$, $\partial G$, and $G^c$; $\mathcal{B}(\mathbb{R}^d)$ denotes the Borel sigma algebra on $\mathbb{R}^d$; $\Omega$ denotes the probability space and $\omega$ denotes an element of $\Omega$; $\mathbb{P}$ and $\mathbb{E}$ denote the probability and the expectation operator; $\mathcal{N}(m, S)$ denotes Gaussian distribution with mean vector $m$ and covariance matrix $S$; $\mathcal{M}(V, \pi, P)$ denotes a Markov chain on a finite set of states $V$ with initial distribution $\pi$ and transition matrix $P$. For any integer $t$, $S_t$ denotes the state of the Markov chain at time $t$. For any integer $t$, $S^t$ denotes the sequence of the first $t$ states of the Markov chain, i.e., $S^t = (S_1, S_2, \ldots, S_t)$.

II. Problem setup

We consider testing the hypothesis whether or not there is an object (an agent) following a certain random walk on a given graph $G = (V, E)$ of $N$ nodes, where $V = \{1, 2, \ldots, N\}$ denote the set of nodes and $E$ denotes the set of edges of the graph. The transition matrix of the random walk is known and equals $P$. We assume that $P$ is irreducible and aperiodic, and we denote the (unique) stationary distribution of the walk by $\pi$ (note that uniqueness is due to irreducibility and aperiodicity of $P$). We also assume that the walk starts at node $i$ with probability $\pi_i$, for $i = 1,\ldots,N$.

At any time $t$, we denote by $S_t$ the node that the agent visits at time $t$ (the state of the Markov chain at time $t$). Each node in the graph $i$, at each time $t$, produces a noisy measurement of the activity at its location, which we denote by $X_{i,t} \in \mathbb{R}$. We assume that, whenever there is some activity at node $i$, the measurement of node $i$ is normally distributed with mean $\beta_i$ and is otherwise normally distributed with zero mean; the variance of each node’s measurement equals $\sigma^2 = 1$, both in the presence and in the absence of activity at that node. Thus, the SNR resulting from the presence of the agent performing
the random walk ("activity") is different across different nodes. Summarizing, the two hypothesis we consider are:

\[ H_0 : X_{i,k} \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, 1) \]

\[ H_1 : X_{i,k} | S^t \overset{\text{indep.}}{\sim} \begin{cases} \mathcal{N}(\beta_i, 1), & \text{if } S_k = i \\ \mathcal{N}(0, 1), & \text{if } S_k \neq i \end{cases} \]

where \( S^t = (S_1, ..., S_t) \sim \mathcal{M}(V, \pi, P) \), for \( i = 1, 2, ..., N, k = 1, 2, ..., t \), and where \( \mathcal{M}(V, \pi, P) \) denotes a Markov chain on a finite set of states \( V \) with initial distribution \( \pi \) and transition matrix \( P \). For each \( t \), let the random matrix \( X^t \in \mathbb{R}^{N \times t} \) collect measurements of all nodes up to time \( t \), such that the \((i, k)\) entry of \( X^t \) stores the measurement of node \( i \) at time \( k \), i.e., \( X^t_{i,k} = X_{i,k} \), for each \( i \) and \( k \leq t \).

We denote the probability laws corresponding to \( H_0 \) and \( H_1 \) by \( P_0 \) and \( P_1 \), respectively. Similarly, the expectations with respect to \( P_0 \) and \( P_1 \) are denoted by \( E_0 \) and \( E_1 \), respectively. The probability density functions of \( X^t \) under \( H_1 \) and \( H_0 \) are denoted by \( f_{1,t}(\cdot) \) and \( f_{0,t}(\cdot) \). It will also be of interest to introduce the conditional probability density function of \( X^t \) given \( S^t = s^t \) (i.e., the likelihood functions \([5]\)), which we denote by \( f_{1,t,S^t}(\cdot | s^t) \), for any \( s^t \). Finally, the likelihood ratio at time \( t \) denoted by \( L_t \) and at a given realization of \( X^t \) is computed by \( L_t(X^t) = \frac{f_{1,t}(X^t)}{f_{0,t}(X^t)} \).

**Error exponent.** In this paper, we consider Neyman-Pearson hypothesis testing, and we are interested in computing the error exponent for the probability of a miss, given a threshold \( \alpha \) on the probability of false alarm. For each \( t \), let \( P_{\text{miss, } t}^\alpha \) denote the infimum of probability of a miss among all tests such that the resulting probability of false alarm is below \( \alpha \). Then, it is well known \([10], [11]\) that the rate of decay of probability of a miss is given by

\[
\lim_{t \to +\infty} -\frac{1}{t} \log P_{\text{miss, } t}^\alpha = \zeta,
\]

provided that the following limit in (2) exists, where \( \zeta \) is known as the asymptotic Kullback-Leibler rate,

\[
\zeta := \lim_{t \to +\infty} -\frac{1}{t} \log L_t(X^t).
\]

To compute the error exponent \( \zeta \), we express the likelihood ratio in terms of the likelihood functions \( f_{1,t,S^t} \). For short, we denote (with some abuse of notation) \( P(s^t) = \mathbb{P}_1(S^t = s^t) \), i.e., for any \( s^t \), \( P(s^t) = \pi_{s_1} \prod_{k=1}^{t-1} P_{s_k,s_{k+1}} \). Further, for each \( t \), let \( S^t \) denote the set of all feasible sequences \( s^t \) of length \( t \), i.e., \( S^t = \{ s^t = (s_1, ..., s_t) : P(s^t) > 0 \} \), and let \( C_t \) denote its cardinality, \( C_t = |S^t| \). By conditioning

July 24, 2017 DRAFT
on the random walk realizations up to time $t$, it is easy to see that the likelihood ratio at time $t$ can be expressed as:

$$L_t(X^t) = \sum_{s^t \in S^t} P(s^t) \frac{f_{1,t}(X^t|s^t)}{f_{0,t}(X^t)} = \sum_{s^t \in S^t} P(s^t)e^{\sum_{k=1}^{N} \beta_{sk} X_{sk,k} - \frac{\beta_{sk}^2}{2}}. \quad (3)$$

In [1], [12] the authors show that, seemingly combinatorial in nature, the sum in (3) can in fact be conveniently expressed as a matrix product:

$$L_t(X^t) = 1^\top D_t P D_{t-1} P \ldots D_1 P, \quad (4)$$

where, for each $t = 1, 2, \ldots, D_t$ is a diagonal matrix defined by

$$D_t = \text{diag} \left( e^{\beta_1 X_{1,t} - \frac{\beta_1^2}{2}}, \ldots, e^{\beta_N X_{N,t} - \frac{\beta_N^2}{2}} \right).$$

From (2) it follows that $\zeta$ is equal to the (top) Lyapunov exponent [13], and using the fact that measurements (hence matrices $D_t$) are i.i.d. in time, the existence of the limit in (2) follows by the well-known Furstenberg-Kesten theorem [13]. Furthermore, convergence of the expectations of the log-likelihood ratios to the same limit follows. We formally state these results in Lemma 1. Using the fact that $\pi > 0$, the proof of Lemma 1 consists of verifying the condition of the Furstenberg-Kesten’s theorem, i.e., proving that the expectation $E_0 \left[ \log \parallel PD_t \parallel \right]$ is finite.

**Lemma 1.** The limit in (2) exists almost surely and equals

$$\lim_{t \to +\infty} -\frac{1}{t} E_0 \left[ \log L_t(X^t) \right]. \quad (5)$$

Computing the Lyapunov exponent is known to be a very difficult problem [6], even in the case when the sample space of random matrices consists of only two matrices [6]. Thus, our goal is finding tractable upper and lower bounds for $\zeta$. Similarly as in [4], we base our analysis of the error exponent $\zeta$ on the expression (5).

**Upper bound for $\zeta$.** We end this section with a simple and intuitive upper bound for $\zeta$. Suppose that we know in advance the exact path $s^t$ that the random walk will take and suppose that this path is a typical one (i.e., when $t$ is large, the number of times that the random walk is in state $i$ along $s^t$ is approximately $\pi_i t$). The likelihood ratio then equals $\log L_t(X^t) = e^{\sum_{k=1}^{N} \beta_{sk} X_{sk,k} - \frac{\beta_{sk}^2}{2}}$ and the error exponent (5) is computed by

$$\lim_{t \to +\infty} -\frac{1}{t} E_0 \left[ \sum_{k=1}^{t} \beta_{sk} X_{sk,k} - \frac{\beta_{sk}^2}{2} \right] = \lim_{t \to +\infty} \frac{1}{t} \sum_{k=1}^{t} \frac{\beta_{sk}^2}{2} = \sum_{i=1}^{N} \pi_i \frac{\beta_i^2}{2}. \quad (6)$$

\[ \text{July 24, 2017} \]
where in the last equality we used that \( s^t \) is typical. Now, given that it operates with knowledge of the exact path of the random walk, it is intuitive to expect that the error exponent in (6) will be an upper bound for the error exponent (1) for the random walk detection problem. Proposition 2 which we present next formalizes mathematically this intuitive notion; the proof is given in Appendix A.

**Proposition 2.** There holds \( \zeta \leq \overline{\zeta} \), where

\[
\overline{\zeta} = \sum_{i=1}^{N} \pi_i \frac{\beta_i^2}{2},
\]

which equals one half of the limiting value of the total SNR averaged over the random walk’s path,

\[
\lim_{t \to +\infty} \frac{1}{2t} \mathbb{E}\left[ \beta_S^2 \right].
\]

**III. Gauss-Markov types**

In this section, we review concepts and results from the literature that we used in our study of the error exponent \( \zeta \). We also define some novel concepts that will prove instrumental in the analysis of the error exponent \( \zeta \) in (1). Specifically, building on the notion of Markov types from [9], we introduce the notion of Gauss-Markov types which, to each sequence of states \( s^t \), besides Markov type, associates also the vector of local per-state sample means computed along the sequence. We then state the main result behind the lower bound on the error exponent, Theorem 7, which asserts that the sequence of measures induced by the Gauss-Markov type satisfies the LDP. Given the importance of Theorem 7 in the analysis of the error exponent, we dedicate Section V to its proof.

**Transition counts matrix and Markov types.** Fix \( t \geq 1 \). For any given \( t \) and any given sequence \( s^t \in V^t \), for each \( i = 1, \ldots, N \), we denote by \( K_{t,i}(s^t) \) the number of times \( k \) along the sequence \( s^t \) such that \( s_k = i \), i.e., \( K_{t,i}(s^t) = | \{ 1 \leq k \leq t : s_k = i \} | \). Similarly, for every pair \( (i, j) \), we denote by \( [K_{t}]_{ij}(s^t) \) the number of times along the sequence \( s^t \) when the state switches from \( i \) to \( j \), i.e., \( [K_{t}]_{ij}(s^t) = | \{ 1 \leq k \leq t : s_k = i, s_{k+1} = j \} | \). Matrix \( K_t \) is known as the transition counts matrix [14]. For convenience, for each \( s^t \in V^t \), we will assume (without loss of generality) that \( s_{t+1} = s_1 \). Then, it is easy to verify that for each \( i \) there holds \( K_{t,i} = \sum_{j=1}^{N} [K_t]_{ij} \) and also \( K_{t,i} = \sum_{i=1}^{N} [K_t]_{ij} \) (for any \( s^t \in V^t \)).

**Definition 3** (Markov type [9]). Markov type on the set of states \( V \) at time \( t \) is the matrix mapping \( \Theta_t : V^t \mapsto \mathbb{R}^{N^2} \), where, for each \( s^t \in V^t \), \( \Theta_t(s^t) \) is defined by

\[
[\Theta_t]_{ij}(s^t) := \frac{[K_t]_{ij}(s^t)}{t},
\]

for any \( i, j = 1, \ldots, N \).
For any fixed \( t = 1, 2, \ldots \), we define the set \( \Delta_t \) that contains all possible Markov types at time \( t \):
\[
\Delta_t = \left\{ \theta \in \mathbb{R}^{N^2} : \text{for every } (i, j), \text{ there exists } k_{ij} \in \mathbb{Z} \text{ s.t. } \theta_{ij} = \frac{k_{ij}}{t}, \right\},
\]
where \( \sum_{i,j=1}^{N} k_{ij} = t, \sum_{j=1}^{N} k_{ij} = \sum_{i=1}^{N} k_{ij}, k_{ij} \geq 0 \) and \( k_{ij} = 0 \) if \( (i, j) \notin E \). \( \quad (10) \)

It will also be of interest to introduce the set \( \Delta \) that contains the union of all \( \Delta_t \) sets, \( t \geq 1 \):
\[
\Delta = \left\{ \theta \in \mathbb{R}^{N^2} : \sum_{i,j=1}^{N} \theta_{ij} = 1, \sum_{j=1}^{N} \theta_{ij} = \sum_{i=1}^{N} \theta_{ij}, \theta_{ij} \geq 0 \text{ and } \theta_{ij} = 0 \text{ if } (i, j) \notin E \right\}. \quad (11)
\]

For a given \( \theta \in \Delta \), we let \( \overline{\theta} \) denote the vector of row sums of \( \theta \), \( \overline{\theta} = \theta_1 \); note that \( \overline{\theta} \in \mathbb{R}^N_+ \). For each \( \theta \in \Delta \), we define, for each \( t \geq 1 \), the set of sequences of length \( t \) that have the same Markov type \( \theta \):
\[
S^t_{\theta} = \left\{ s^t \in S^t : K_{ij}(s^t) = \theta_{ij}t, i, j = 1, \ldots, N \right\}. \quad (12)
\]

We let \( C_{t,\theta} \) denote its cardinality, \( C_{t,\theta} = |S^t_{\theta}| \). Note that \( S^t_{\theta} \) is non-empty if and only if \( \theta \in \Delta_t \) (if \( \theta \notin \Delta_t \), by the definition of \( \Delta_t \), we have that the number of transition fractions \( \theta \) is not realizable at time \( t \), i.e., there is no sequence of length \( t \) such that, for each \( i, j \), the number of transitions from \( i \) to \( j \) equals \( \theta_{ij}t \)).

**Entropy functions.** We now define relevant entropy functions \( \text{[16]} \) (see also \( \text{[14], [17], [15], [9]} \)), that we will utilize in estimating the size of the sets \( C_{t,\theta} \), \( \theta \in \Delta_t \). To this end, note that each \( \theta \in \Delta \) defines the respective Markov chain with transition matrix \( Q \), defined by \( Q_{ij} := \frac{\theta_{ij}}{\theta_i} \), when \( \theta_i \neq 0 \), and \( Q_{ij} = 0 \), otherwise, for \( i, j = 1, \ldots, N \). (We note in passing that if \( \theta \in \Delta_t \), then \( Q \) is the empirical transition matrix for the transition count matrix \( K \) that validates the fact that \( \theta \) belongs to \( \Delta_t \), see eq. \( \text{(10)} \)). Then, the entropy of the Markov chain \( Q \) is defined through \( \theta \) by:
\[
H(\theta) = - \sum_{i,j=1}^{N} \theta_{ij} \log \frac{\theta_{ij}}{\theta_i}, \quad (13)
\]
see \( \text{[14]} \). Similarly, the relative entropy of \( Q \) with respect to the assumed random walk with the transition matrix \( P \) is defined by
\[
D(\theta||P) = \sum_{i,j=1}^{N} \theta_{ij} \log \frac{\theta_{ij}}{\theta_iP_{ij}}, \quad (14)
\]

\(^1\)The set of all possible Markov types at time \( t \), \( \Delta^*_t \subset \Delta_t \), compared to \( \Delta_t \) has an extra condition in its definition: it requires that the submatrix obtained by deleting all zero rows and columns from the candidate matrix \( \theta \), is irreducible, see \( \text{[15]} \). However, for our purposes this condition can be omitted, due to the fact that both \( \bigcup_{t=1}^{+\infty} \Delta^*_t \) and \( \bigcup_{t=1}^{+\infty} \Delta_t \) are dense in \( \Delta \) (defined in \( \text{(11)} \)), see \( \text{[15]} \).
see, e.g., [15]. The following lemma asserts that $H$ is concave, and $D(\cdot \| P)$ is convex (in $\theta$). We will use these results in Section [VI] when proving convexity of the error exponent lower bound. A (sketch of the) proof based on a certain matrix decomposition can be found in [14]. We provide more direct proofs here, based on inspection of the Hessian matrix; see Appendix A.

**Lemma 4.**

1) Function $H : \mathbb{R}^N_+ \mapsto \mathbb{R}$ is concave.

2) Function $D(\cdot \| P) : \mathbb{R}^N_+ \mapsto \mathbb{R}$ is convex.

In Lemma [5] that we state next we use the entropy function $H$ to approximate the cardinalities $C_{t, \theta}$ of sets $S^t_\theta$, $\theta \in \Delta_t$. In particular, the result in part [2] asserts that $C_{t, \theta}$ increases exponentially fast in the sequence length $t$, with the rate equal to $H(\theta)$; this result is originally proved by Whittle’s formula, see [13], [14], and [9], but it can also be proved using the asymptotic equipartition property (AEP) for Markov sources, see Chapter 3.1 in [16]. It is also of interest to estimate the cardinality $C_t$ of the set $S^t$ of all feasible sequences until time $t$. The result in part [1] of the lemma states that $C_t$ also increases exponentially in $t$, with the rate being the spectral radius $\rho_0$ of $P_0$; this result easily follows from the fact that $\pi > 0$, which implies that $C_t = 1^\top P_0^{t-1} 1$. For completeness, we provide proofs of both results, see Appendix A.

**Lemma 5.**

1) For each $\epsilon > 0$, there exists $t_0 = t_0(\epsilon, P_0)$ such that

$$\rho_0^t \leq C_t \leq \rho_0^t e^{t\epsilon},$$

(15)

for all $t \geq t_0$, where $\rho_0 > 1$.

2) For each $\epsilon > 0$, there exists $t_1 = t_1(\epsilon, P_0)$ such that, for each $t \geq t_0$ and each $\theta \in \Delta_t$, there holds

$$e^{tH(\theta) - t\epsilon} \leq C_{t, \theta} \leq e^{tH(\theta) + t\epsilon}.$$  

(16)

**Gauss-Markov types.** In our problem, we have with each sequence $s^t$ an associated sequence of Gaussian random variables $X_{s_i, k}$, $k = 1, 2, \ldots, t$. From the expression for the likelihood ratio (3), we see that the likelihood ratio depends on the sequence $X_{s_i, k}$ only through the per node sample means, $\sum_{k: s_k = i} X_{i, k}$. Motivated by this observation, we extend the notion of Markov types to, what we call, Gauss-Markov types, by constructing a pair $(\theta, \xi)$, where $\theta$ is a Markov type associated with the sequence of states $s^t$, and $\xi$ is a vector in $\mathbb{R}^N$ whose $i$-th component corresponds to the sample mean of the random walk signal observations at node $i$. More generally, for any pair $(s^t, Z^t)$ where $s^t = (s_1, \ldots, s_t)$ is a sequence of states in $V$ and $Z^t = (Z_1, \ldots, Z_t)$ is a vector in $\mathbb{R}^t$, the Gauss-Markov type is defined as the pair
\[(\theta, \xi), \text{ where } \theta \text{ is given as in eq. (12) above and, for } i = 1, ..., N, \]
\[
\xi_i = \frac{\sum_{k:s_k=i} Z_k}{K_i(s^t)}. \tag{17}
\]

### A. LDP for Gauss-Markov types

In this subsection, we introduce the probability measure induced by the Gauss-Markov type and show that it satisfies the large deviations principle. We first give a formal definition of the large deviations principle [19],[20].

#### Definition 6 (Large deviations principle [19]).

It is said that a sequence of measures \( \mu_t \) satisfies the large deviations principle with rate function \( I \) if for every measurable set \( G \) the following two conditions hold:

1) \[
\limsup_{t \to +\infty} \frac{1}{t} \log \mu_t(G) \leq -\inf_{x \in G} I(x); \tag{18}
\]

2) \[
\liminf_{t \to +\infty} \frac{1}{t} \log \mu_t(G) \geq -\inf_{x \in G} I(x). \tag{19}
\]

For each fixed \( t \), consider the probability space where the sample of outcomes is the set \( S^t \) of all feasible sequences \( s^t \) of length \( t \), and where elements of \( S^t \) are drawn uniformly at random. As previously, we let \( S^t \) a randomly chosen element of \( S^t \). Further, with each \( s^t \), we have an associated Gaussian random vector \( Z_{s^t} \in \mathbb{R}^t \) that is independent of \( S^t \). On this probability space, we define the following random variables:

\[
[\Theta_t]_{ij}(s^t) = \frac{K_{ij}(s^t)}{t}, \text{ for } i, j = 1, ..., N \tag{20}
\]

\[
[\overline{\Theta}_t]_i(s^t) = \frac{K_i(s^t)}{t}, \text{ for } i = 1, ..., N. \tag{21}
\]

That is, \( \Theta_t \) is the (random) Markov type of the sequence \( s^t \), and \( \overline{\Theta}_t \) satisfies \( \overline{\Theta}_t = \Theta_t t \). Note that for each realization \( s^t \), for each \( i \), \( [\Theta_t]_i \) equals \( \sum_{j=1}^N [\Theta_t]_{ij} \). Also, for each \( \omega \in \Omega \), for each \( t \) and \( s^t \), define

\[
Z_{t,i}(s^t) = \begin{cases} \frac{\sum_{k \in [1, t]: s_k=i} Z_{s^t,k}(\omega)}{t}, & \text{if } i \text{ is s.t. } K_i(s^t) > 0; \\ 0, & \text{otherwise}, \end{cases} \tag{22}
\]

We can see that \( (\Theta_t, Z_{t,i}^\omega) \) is the (random) Gauss-Markov type of the pair \( (s^t, Z_{s^t}) \). Note that \( Z_{t,i}^\omega \) has two sources of randomness: the first originating from the randomly chosen sequence \( s^t \), and the second from the random realization of the Gaussian vector \( Z_{s^t} = (Z_{s^t,1}, ..., Z_{s^t,t}) \in \mathbb{R}^t \). Also, for any given \( s^t \), for each \( i \) such that \( K_i(s^t) > 0 \), \( Z_{t,i}^\omega(s^t) \) is a Gaussian random variable with mean 0 and variance equal...
to \( K_i(s^t)/t^2 = \frac{\Theta_{t,i}(s^t)}{t} \). On the other hand, if for some \( i \), \( K_i(s^t) = 0 \), then \( Z_{t,i}^\omega(s^t) \) is deterministic and thus has zero variance \( K_i(s^t)/t^2 = \frac{\Theta_{t,i}(s^t)}{t} \). Thus, for each given \( s^t \), we can write

\[
Z_{t,i}^\omega(s^t) \sim \mathcal{N} \left( 0, \frac{\Theta_{t,i}(s^t)}{t} \right), \quad \text{for } i = 1, ..., N.
\] (23)

For each realization \( \omega \), let \( Q_t^\omega : B(\mathbb{R}^{N^2 + N}) \) denote the probability measure induced by \((\Theta_t, Z_{t}^\omega)\):

\[
Q_t^\omega(B) := \sum_{s^t \in S^t} \mathbf{1}_{\{\Theta_t, Z_{t}^\omega \in B\}}(s^t) C_t,
\] (24)

for arbitrary \( B \in B(\mathbb{R}^{N^2 + N}) \), where, we recall, \( C_t = |S^t| \). (It is easy to verify that \( Q_t^\omega \) is indeed a probability measure.) The next result asserts that \( Q_t^\omega \) satisfies the LDP and computes the corresponding rate function.

**Theorem 7.** For every measurable set \( G \), the sequence of measures \( Q_t^\omega \), \( t = 1, 2, \ldots \), with probability one satisfies the LDP upper bound \((18)\) and the LDP lower bound \((19)\), with the same rate function \( I : \mathbb{R}^{N^2} \mapsto \mathbb{R} \), equal for all sets \( G \), given by

\[
I(\theta, \xi) = \begin{cases} \log \rho_0 - H(\theta) + J_0(\xi), & \text{if } \theta \in \Delta \text{ and } H(\theta) \geq J_0(\xi) \\ +\infty, & \text{otherwise} \end{cases}
\] (25)

where, for any \( \theta \in \mathbb{R}^{N^2} \), function \( J_0 : \mathbb{R}^N \mapsto \mathbb{R} \) is defined as \( J_0(\xi) := \sum_{i: \theta_i > 0} \frac{1}{\pi_i} \frac{\xi_i^2}{2} \), for any \( \xi \in \mathbb{R}^N \) such that \( \xi_i = 0 \) if \( \theta_i = 0 \), and equals \( +\infty \) otherwise.

Theorem 7 is the core result behind Theorem 8 and we dedicate Section 5 to its proof.

**IV. LOWER BOUND ON \( \zeta \)**

We are now ready to state our main result on the lower bound on \( \zeta \).

**Theorem 8.** There holds \( \underline{\zeta} \leq \zeta \), where \( \underline{\zeta} \) is the optimal value of the following optimization problem

\[
\begin{array}{ll}
\text{minimize} & D(\theta||P) + \sum_{i: \theta_i > 0} \frac{(\beta_i - \xi_i)^2}{2} \\
\text{subject to} & H(\theta) \geq J_0(\xi) \\
& \overline{\theta} = \theta_1 \\
& \theta \in \Delta \\
& \xi \in \mathbb{R}^N.
\end{array}
\] (26)

Before proving Theorem 8 in Subsection 5.1 we first give some interpretations of this result and provide an application example.
A. Interpretations and an application example

Interpretation through Gauss-Markov type. If we inspect the optimization problem \((26)\) through the lenses of the Gauss-Markov type \((17)\), a very intuitive interpretation emerges. First, the last three constraints ensure that any candidate \((\theta, \xi) \in \mathbb{R}^{N+2N}\) is a valid Gauss-Markov type. The constraint \(H(\theta) \geq J_0(\xi)\) then filters out all pairs \((\theta, \xi)\) corresponding to sequences that are, in the long run, infeasible under \(H_0\); see also Case 2 in the proof of Theorem 7, large deviations upper bound, in Subsection V-A.

The latter condition is very intuitive (as, in the long run) under the state of nature \(H = H_1\) wrong decisions \(H = H_0\) can only be made on the set of \(H_0\) feasible types \((\theta, \xi)\). Finally, considering the objective function \((26)\), we see that it consists of two terms. The first term has the objective of choosing the sequence \(s^t\) whose Markov type \(\theta\) is as close as possible to the transition matrix of the random walk – i.e., the “true” transition matrix. The second term has the objective of choosing the Gaussian signal (i.e., the random walk signal) \(Z_{s^t}\) whose per node sample means \(\xi_i/\theta_i\) are as close as possible to the expected sample means \(\beta_i, i = 1, ..., N\). Hence, the objective of \((26)\) aims at finding the Gauss-Markov type whose probability is highest under the state of nature \(H_1\). In summary, among all types that are asymptotically feasible under \(H = H_0\), optimization problem \((26)\) finds the one that has the highest probability (i.e., the slowest probability decay) under \(H = H_1\).

Condition for detectability of the random walk. From \((26)\), it is easy to see that, if

\[
H(P) \geq \sum_{i=1}^{N} \pi_i \frac{\beta_i^2}{2},
\]  

(27)

then \((\theta^*, \xi^*)\) is an optimizer, where \(\theta_{ij}^* = \pi_i P_{ij}\), for \(i, j = 1, ..., N\), and \(\xi_i^* = \pi_i \beta_i\), for \(i = 1, ..., N\). The resulting optimal value of \((26)\) equals zero. Hence, the lower bound \(\zeta\) on the error exponent equals zero, indicating that the random walk is not detectable.

Special case \(\beta_i \equiv \beta\). We next consider the special case when all sensors have the same SNR, i.e., when \(\beta_i \equiv \beta\), for some \(\beta \in \mathbb{R}\). In this case, it can be shown that problem \((26)\) reduces to:

\[
\begin{align*}
\text{minimize} & \quad D(\theta \| P) + \frac{(\beta - \xi)^2}{2} \\
\text{subject to} & \quad H(\theta) \geq \frac{\xi^2}{2}, \\
& \quad \theta \in \Delta \\
& \quad \xi \in \mathbb{R}.
\end{align*}
\]  

(28)

In particular, from \((28)\), we can easily recover the condition from [4] for the random walk to be detectable: if the entropy of the random walk \(H(P)\) is greater than the SNR,

\[
H(P) \geq \frac{\beta^2}{2},
\]  

(29)
then the infimum of (28) is zero. Again, since the error exponent lower bound $\zeta$ is then zero, this indicates that under condition (29) the random walk is not detectable. One can in fact show that optimization problem (28) yields the same value as the error exponent lower bound from [4] (see eq. (29) in [4]). We omit the proof here, but we remark that the equivalence of the two optimization problems can be shown by using the method of limiting functions from [17] together with a single-letter parametrization in (73) of the set of candidate solutions $\theta$ (see also the proof of Lemma 16 in Appendix D for how this can be achieved).

**Application example: Frequency hopping random access.** We now explain how one can apply our methodology to develop a novel, frequency hopping based random access scheme. The motivating practical setting is NarrowBand IoT communications [21]. In this emerging standard, the aim is to design communication protocols for future IoT applications, where a large number of devices (e.g., smart electricity, gas or water meters etc.) transmit their data to a neighboring cellular base station. The intrinsic features in such communications are extremely simple communication protocols and extremely low SNR values (many such devices are deployed in basements). Due to low SNR values, in this kind of systems the standard, repetition based mechanisms for random access may exhibit very low user detection rate. The situation is further exacerbated by the fact that fading can vary significantly over time over the used frequency band. Therefore, it is not possible to determine in advance what is the best frequency to send the access requesting signal. To overcome these issues, we propose a random access scheme based on Markov chain frequency hopping. We explain the setup formally. Let $P \in \mathbb{R}^{N^2}$ be a given Markov chain transition matrix, and let $\mathcal{F} = \{f_1, \ldots, f_N\}$ denote the set of frequencies from the allocated frequency band. Let also $\mathcal{G}$ denote the graph on $\mathcal{F}$ defined by the sparsity pattern of $P$. Then, to ask for medium access, a user transmits a sequence of signals, each at a different frequency, where the statistics of the transitions from one frequency to another are defined by the matrix $P$. The receiver then performs the optimal likelihood ratio test to detect the presence of the medium access signal. We remark that, by performing medium access based on the described Markov chain hopping, some detection power is indeed lost (in comparison with the repetition based scheme), but, on the other hand, the scheme is able to successfully combat (unknown) frequency selective fading, and this is furthermore achieved without the need for signal synchronization between sender and receiver.

**B. Proof of Theorem 8**

For notational convenience, for each $s^t \in \mathcal{S}^t$ denote $\beta(s^t) = \sqrt{\beta_{s_1}^2 + \beta_{s_2}^2 + \ldots + \beta_{s_t}^2}$ and

$$\overline{X}(s^t) = \sum_{k=1}^{t} \beta_{s_k} X_{s_k,k}. \quad (30)$$
For each fixed $t$ define also function $\phi_t : \mathbb{R}^G \mapsto \mathbb{R}$,

$$
\phi_t(x) := -\frac{1}{t} \log \left( \sum_{s^t \in \mathcal{S}} P(s^t) e^{-\frac{\beta(s^t)^2}{2} + x_{s^t}} \right),
$$

(31)

where $x_{s^t}$ is an element of a vector $x = \{x_{s^t} : s^t \in \mathcal{S}^t\} \in \mathbb{R}^G$, whose index is $s^t$. Note (see eq. (3)) that $\zeta$ can be expressed as the limit of expectations of functions $\phi_t$ evaluated at $x = \{\overline{X}(s^t) : s^t \in \mathcal{S}^t\}$, $t = 1, 2, \ldots$ i.e.,

$$
\zeta = \lim_{t \to +\infty} E_0 \left[ \phi_t \left( \overline{X}(s^t) \right) \right].
$$

(32)

Similarly as in [4], we approximate the sum in (32) by dropping the correlations between terms $\overline{X}(s^t)$ that correspond to different sequences $s^t$. Specifically, for each sequence $s^t$, we replace the Gaussian vector $(X_{s^t,1}, \ldots, X_{s^t,t})$ by the Gaussian vector of the same dimension $t$, $Z_{s^t} = (Z_{s^t,1}, \ldots, Z_{s^t,t})$, where each component $Z_{s^t,k}, k = 1, \ldots, t$, is standard Gaussian, and where different components are mutually independent. For each $t$ we assume that vectors $Z_{s^t}$ corresponding to different sequences $s^t$ are independent, and we also assume that vectors from different families $Z_{s^t}$ and $Z_{s^t'}$, where $t \neq t'$, are also independent. Thus, for every different $t$, we have $|\mathcal{S}^t|$ independent $t$-dimensional standard Gaussian vectors, each corresponding to a fixed sequence $s^t$. We denote by $(\Omega, \mathcal{F}, P)$ the probability space that generates $\{Z_{s^t} : s^t \in \mathcal{S}^t\}$, $t = 1, 2, \ldots$, by $\omega$ we denote an element of $\Omega$, and by $E$ the corresponding expectation operator.

Let $\overline{Z}(s^t)$ denote the $Z$-variables counterpart of $\overline{X}(s^t)$:

$$
\overline{Z}(s^t) = \sum_{k=1}^{t} \beta_{s^t,k} Z_{s^t,k}.
$$

(33)

Then the following result holds. The proof of Lemma 9 is given in Appendix B.

**Lemma 9.** For every $t \geq 1$, there holds

$$
E_0 \left[ \phi_t \left( \overline{X} \right) \right] \geq E \left[ \phi_t \left( \overline{Z} \right) \right].
$$

(34)

From (34) we immediately obtain

$$
\zeta \geq \limsup_{t \to +\infty} E \left[ \phi_t \left( \overline{Z} \right) \right].
$$

(35)

The next result implies that the upper limit in the preceding relation is in fact a limit, and moreover, it asserts that this limit equals $\zeta$. The proof of Lemma 10 is given in Appendix B.

**Lemma 10.** There holds:

1) The sequence of random variables $\phi_t \left( \overline{Z} \right)$ is uniformly integrable.
2) With probability one,
\[
\lim_{t \to +\infty} \phi_t \left( \mathbf{Z} \right) = \zeta.
\] (36)

Almost sure convergence of \( \phi_t \), together with its uniform integrability, implies convergence in expectation, i.e., we have
\[
\lim_{t \to +\infty} \mathbb{E} \left[ \phi_t \left( \mathbf{Z} \right) \right] = \zeta.
\] (37)

Using the preceding equality in (35), the claim of Theorem 8 follows.

V. PROOF OF LDP FOR GAUSS-MARKOV TYPES

In this section we prove Theorem 7. The LDP upper bound (18) is proven in Subsection V-A, and the LDP lower bound (19) is proven in Subsection V-B.

A. Proof of the LDP upper bound

Upper bound for boxes. We first show the LDP upper bound for all boxes in \( \mathbb{R}^{N^2+N} \). Fix an arbitrary box \( B = C \times D \), where \( C \) is a box in \( \mathbb{R}^{N^2} \) and \( D \) is a box in \( \mathbb{R}^N \). Fix an arbitrary \( t \geq 1 \). Using the fact that the random matrix \( \Theta_t \) is discrete and that it can only take values in \( \Delta_t \), we have
\[
1_{\{ (\Theta_t, \mathbf{Z}_t) \in B \}}(s^t) = 1_{\{ \Theta_t \in C \}}(s^t)1_{\{ \mathbf{Z}_t \in D \}}(s^t) = \sum_{\theta \in C \cap \Delta_t} 1_{\{ \Theta_t = \theta \}}(s^t)1_{\{ \mathbf{Z}_t \in D \}}(s^t).
\]

Thus,
\[
Q^\omega_t(B) = \sum_{\theta \in C \cap \Delta_t} \sum_{s^t \in S^t_\theta} 1_{\{ \Theta_t = \theta, \mathbf{Z}_t \in D \}}(s^t) \frac{C_t}{C_t,\theta}.
\] (38)

Consider now a fixed \( \theta \in C \cap \Delta_t \) and let \( K = k_{ij} \) be the matrix of integers that verifies the fact that \( \theta \) belongs to \( \Delta_t \); for \( i = 1, \ldots, N \), denote also \( \mathbf{7}_t \), where \( \mathbf{7} = \theta \mathbf{1} \). Recall that \( S^t_\theta \) denotes all sequences (of length \( t \)) such that, for each \( (i, j) \), the number of transitions from \( i \) to \( j \) equals \( \theta_{ij} t = k_{ij} \). Then, we have
\[
\sum_{s^t \in S^t_\theta} 1_{\{ \Theta_t \in C \cap \Delta_t \}}(s^t) = \sum_{s^t \in S^t_\theta} 1_{\{ \mathbf{Z}_t \in D \}}(s^t).
\]

Introducing, for each \( \omega \) and \( \theta \in \Delta_t \), a new probability measure \( Q^\omega_t : \mathcal{B}(\mathbb{R}^N) \mapsto \mathbb{R} \), defined by
\[
Q^\omega_t(D') := \frac{\sum_{s^t \in S^t_\theta} 1_{\{ \mathbf{Z}_t \in D' \}}(s^t)}{C_t,\theta},
\] (38)

for any \( D' \in \mathcal{B}(\mathbb{R}^N) \), we obtain
\[
Q^\omega_t(B) = \sum_{\theta \in C \cap \Delta_t} \frac{C_t,\theta}{C_t} Q^\omega_t(D).
\] (39)
We now analyze the empirical distribution \( Q_{t,\theta}^t \). Since random vectors \( Z_{s^t}, s^t \in S^t \), are independent, we have that the indicator functions in the family \( \{1_{\{Z^t_{s^t} \in D\}}(s^t) : s^t \in S^t\} \) are independent – hence they are independent in the subfamily \( S^t_0 \subseteq S^t \) as well. Further, any sequence \( s^t \in S^t_0 \) has the same Markov type \( \theta \). Thus, for each \( s^t \in S^t_0 \), we have that \( K_i(s^t) = \overline{\theta_i} t \), for each \( i \), where, we recall, \( \overline{\theta} = \theta_1 \). Recalling that, for \( i = 1, ..., N \), \( Z_{t,i}^\omega(s^t) \sim N(0, K_i(s^t)/t^2) \), it follows that, for each fixed \( i \), \( \{Z_{t,i}^\omega(s^t) : s^t \in S^t_\theta\} \) is a family of i.i.d. Gaussian random variables, with mean 0 and variance \( \overline{\theta_i} / t \). Thus, \( \{Z_{t,i}^\omega(s^t) : s^t \in S^t_\theta\} \) is also i.i.d.; we denote by \( q_{t,\theta} : B(\mathbb{R}^N) \mapsto \mathbb{R} \) the corresponding probability measure – i.e., \( q_{t,\theta} \) is a probability measure induced by \( Z_{t,i}^\omega(s^t) \), where \( s^t \) is an arbitrary element of \( S^t_\theta \).

Further, since for each fixed \( s^t \in S^t \) individual components \( Z_{s^t,k} \) of vector \( Z_{s^t} \) are independent, by the disjoint block theorem \([22]\), we have that, for any fixed sequence \( s^t \), the individual elements of random vector \( Z_{t,i}^\omega(s^t) \), \( Z_{t,i}^\omega(s^t) \), \( i = 1, ..., N \), are independent. Let \( [q_{t,\theta}]_i : B(\mathbb{R}) \mapsto \mathbb{R} \) denote marginal probability measures induced by \( Z_{t,i}^\omega(s^t) \), for \( i = 1, ..., N \). Recall that \( D \) is a box, and suppose that \( D = D_1 \times ... \times D_N \), for some arbitrary closed intervals \( D_i \) in \( \mathbb{R} \), \( i = 1, ..., N \). Then, we have

\[
q_{t,\theta}(D) := \mathbb{P} \left( Z_{t}^\omega(s^t) \in D \right) = \prod_{i=1}^{N} [q_{t,\theta}]_i(D_i). \quad (40)
\]

From \([23]\), it is easy to see that

\[
[q_{t,\theta}]_i(D_i) = \begin{cases} 
\frac{t}{\sqrt{2\pi}} \int_{\eta_i \in D_i} e^{-\frac{t^2 \eta_i^2}{2}} d\eta_i & \text{if } k_i \geq 1, \\
1, & \text{if } 0 \in D_i \text{ and } k_i = 0 , \\
0, & \text{if } 0 \notin D_i \text{ and } k_i = 0,
\end{cases} \quad (41)
\]

where we remark that the last two equalities are due to the fact that, when \( k_i = 0 \), then \( Z_{t,i}^\omega(s^t) \) is deterministic and equal to zero.

Going back to the family of indicator functions \( \{1_{\{Z^t_{s^t} \in D\}}(s^t) : s^t \in S^t_\theta\} \), we conclude that these are i.i.d. Bernoulli random variables, each with success probability equal to \( q_{t,\theta}(D) \), and thus, the empirical measure \( Q_{t,\theta}^\omega(D) \) has the expected value equal to this quantity,

\[
\mathbb{E} \left[ Q_{t,\theta}^\omega(D) \right] = q_{t,\theta}(D). \quad (42)
\]

The following lemma upper bounds and computes the exponential decay rate for the probability \( q_{t,\theta}(D) \).

**Lemma 11.** For any \( \epsilon > 0 \), there exists \( t_2 = t_2(\epsilon, D) \) such that, for each \( t \geq t_2 \) and \( \theta \in \Delta_t \),

\[
q_{t,\theta}(D) \leq e^{\epsilon t} e^{-t \inf_{\xi \in D} \mathbb{J}_t(\xi)}. \quad (43)
\]

The proof of Lemma 11 is given in Appendix C.
We now introduce
\[ b := \inf_{(\theta, \xi) \in C \times D} J_\theta(\xi) - H(\theta) \]
and proceed with the proof by separately analyzing the cases: 1) \( b \leq 0 \); and 2) \( b > 0 \).

**Case 1: \( b \leq 0 \).** Fix an arbitrary \( \epsilon > 0 \) and for each \( t \geq 1 \) define
\[ A_t = \{ \omega : Q^\omega_{t,\theta}(D) \geq q_{t,\theta}(D) e^t \epsilon, \text{ for some } \theta \in \Delta_t \} . \]

By the union bound, followed by the Markov’s inequality applied to each of the terms in the obtained sum, we have
\[
\mathbb{P}(A_t) \leq \sum_{\theta \in \Delta_t} \mathbb{P}(Q^\omega_{t,\theta}(D) \geq q_{t,\theta}(D) e^t \epsilon) \\
\leq \sum_{\theta \in \Delta_t} \mathbb{E}[Q^\omega_{t,\theta}(D)] e^{t \epsilon} \\
\leq \frac{(t + 1)^N}{e^{t \epsilon}}, \tag{44}
\]
where in the last inequality we used (42) and the fact that each coordinate \( \theta_{ij} \) takes values in the set \{0, 1, ..., t\}, and therefore the number of points in \( \Delta_t \) is upper bounded by \((t + 1)^N\).

The quantity in (44) decays exponentially fast for any \( \epsilon > 0 \) and, by the Borel-Cantelli lemma, we have \( \mathbb{P}(A_t, \inf \text{ often}) = 0 \). Thus, there exists a set \( \Omega^*_0 \subseteq \Omega \), with \( \mathbb{P}(\Omega^*_0) = 1 \), such that for each \( \omega \in \Omega^*_0 \), for any \( \epsilon > 0 \),
\[ Q^\omega_{t,\theta}(D) \leq q_{t,\theta}(D) e^t \epsilon, \tag{45} \]
for all \( \theta \in \Delta_t \), for all \( t \geq t_3 \), where \( t_3 = t_3(\omega, \epsilon, D) \). Combining with the bounds on \( C_t \) and \( C_t,\theta \) from Lemma 5 together with the upper bound on \( q_{t,\theta} \) from Lemma 11, we obtain that, for any fixed \( \epsilon > 0 \), for all \( t \geq t_4 = t_4(\omega, \epsilon, D, P_0) := \max\{t_0, t_1, t_2, t_3\} \),
\[ \frac{C_t,\theta}{C_t} Q^\omega_{t,\theta}(D) \leq \frac{1}{\rho_0} e^{tH(\theta) - t \inf_{\xi \in D} J_\theta(\xi)} e^{4t \epsilon}, \tag{46} \]
for all \( \theta \in \Delta_t \). Going back to equation (39), and applying (46) for each \( \theta \in C \cap \Delta_t \), we get
\[
Q^\omega_t(B) \leq (t + 1)^N \max_{\theta \in C \cap \Delta_t} Q^\omega_{t,\theta}(D) \frac{C_t,\theta}{C_t} \leq (t + 1)^N e^{4t \epsilon} \max_{\theta \in C \cap \Delta_t} \frac{1}{\rho_0} e^{tH(\theta) - t \inf_{\xi \in D} J_\theta(\xi)} \leq (t + 1)^N e^{4t \epsilon} e^{-t \inf_{\xi \in C \cap \Delta_t, \xi \in D} \log \rho_0 + J_\theta(\xi) - H(\theta)}. \tag{47}
\]
Now, note that the following holds

\[
\inf_{(\theta, \xi) \in C \cap \Delta \times D} \log \rho_0 + J_\theta(\xi) - H(\theta)
= \min \left\{ \inf_{(\theta, \xi) \in C \cap \Delta \times D: H(\theta) \geq J_\theta(\xi)} \log \rho_0 + J_\theta(\xi) - H(\theta), \inf_{(\theta, \xi) \in C \cap \Delta \times D: H(\theta) \leq J_\theta(\xi)} \log \rho_0 + J_\theta(\xi) - H(\theta) \right\}
= \inf_{(\theta, \xi) \in C \cap \Delta \times D} \log \rho_0 + J_\theta(\xi) - H(\theta)
\]

(48)

\[
= \inf_{(\theta, \xi) \in C \cap \Delta \times D} I(\theta, \xi)
\]

(49)

where (48) follows from the fact that \( b \leq 0 \) (note that, since function \( J_\theta(\xi) - H(\theta) \) is lower semi-continuous, the set \( \{ (\theta, \xi) \in C \cap \Delta \times D : H(\theta) \geq J_\theta(\xi) \} \) is non-empty), and (49) holds by the definition of the rate function \( I \). Thus, combining (49) with (47) proves the upper bound in Case 1.

**Case 2:** \( b > 0 \). We now prove the upper bound for the case when \( b > 0 \). Define

\[
B_t = \left\{ \omega : \sum_{s^t \in S^t} 1_{\{Z^t_s \in D\}}(s^t) \geq 1, \text{ for some } \theta \in C \cap \Delta_t \right\}.
\]

Again, by the union bound and Markov’s inequality, we have

\[
P(B_t) \leq \sum_{\theta \in C \cap \Delta_t} C_{t, \theta} q_{t, \theta}(D)
\leq \sum_{\theta \in C \cap \Delta_t} e^{2t\epsilon} e^{t(H(\theta) - \inf_{\xi \in D} J_\theta(\xi))}
\leq (t + 1)^N e^{-t(b - 2\epsilon)},
\]

which holds for any fixed \( \epsilon \) and all \( t \geq t_5 = t_5(\epsilon, P_0, D) := \max\{t_1, t_2\} \). For sufficiently small \( \epsilon \), the latter number decays exponentially fast with \( t \). It follows by the Borel-Cantelli lemma, that, for all \( \theta \in C \cap \Delta_t \), with probability one, \( Q^\omega_{t, \theta}(D) = 0 \), for all \( t \geq t_5 \).

Going back to eq. (39), we have

\[
Q^\omega_t(B) \leq (t + 1)^N \max_{\theta \in C \cap \Delta_t} Q^\omega_{t, \theta}(D) \frac{C_{t, \theta}}{C_t}
= 0,
\]

for all \( t \geq t_5 \). On the other hand, since \( b > 0 \), we have that, for any point \( (\theta, \xi) \in C \times D \), \( J_\theta(\xi) - H(\theta) > 0 \). Hence, we obtain

\[
\inf_{(\theta, \xi) \in C \times D} I(\theta, \xi) = +\infty.
\]

Combining the preceding two identities proves the upper bound for any boxes in Case 2. This completes the proof of the LDP upper bound for boxes.
Upper bound for compact sets. We next extend the LDP upper bound from boxes to all compact sets. Fix a compact set $G \subseteq \mathbb{R}^{N^2+N}$. Fix an arbitrary $\alpha > 0$. For each point $(\theta, \xi) \in G$ draw a box $B$ around $(\theta, \xi)$ of size (width) $\delta = \delta(\theta, \xi)$ such that the infimum of $I$ over $B$ is at least $I(\theta, \xi) - \alpha$,

$$\inf_{(\theta', \xi') \in B(\theta, \xi)(\delta(\theta, \xi))} I(\theta', \xi') \geq I(\theta, \xi) - \alpha.$$  

From the family of boxes $\{B(\theta, \xi) : (\theta, \xi) \in G\}$, we extract a finite cover of $G$, $\{B_l : l = 1, \ldots, M\}$ (note that this is feasible due to the fact that $G$ is compact), where we shortly denote $B_l = B_{\theta_l, \xi_l}(\delta_l)$ and $\delta_l$ is the appropriate box size. Then, we have

$$Q^\omega_t(G) \leq \sum_{l=1}^M Q^\omega_t(B_l).$$

Since $M$ is finite, the above inequality implies

$$\limsup_{t \to +\infty} \frac{1}{t} \log Q^\omega_t(G) \leq \max_{1 \leq l \leq M} \limsup_{t \to +\infty} \frac{1}{t} \log Q^\omega_t(B_l) \leq \max_{1 \leq l \leq M} - \inf_{(\theta, \xi) \in B_l} I(\theta, \xi) \leq - \min_{1 \leq l \leq M} I(\theta_l, \xi_l) + \alpha \leq - \inf_{(\theta, \xi) \in G} I(\theta, \xi) + \alpha.$$  

Noting that $\alpha$ can be chosen arbitrarily small proves the LDP upper bound for compact sets.

Upper bound for closed sets. To complete the proof of the LDP upper bound, it remains to show that the upper bound holds for all closed sets. We do this by showing that the sequence of measures $Q^\omega_t$ is exponentially tight with probability one. By Lemma 1.2.18 from [19], this together with the upper bound for all compact sets yields the upper bound for all closed sets.

Lemma 12. With probability 1, the sequence of measures $Q^\omega_t$, $t = 1, 2, \ldots$, is exponentially tight.

Proof. To show that $Q^\omega_t$ is exponentially tight it suffices to show that the rate function has compact support. To this end, note that the variable $\theta$ must belong to the compact set $\Delta$, as otherwise $I = +\infty$. Second, $I$ is finite at a given point $(\theta, \xi)$ only if there holds $H(\theta) \geq \sum_i \xi_i > 0 \frac{1}{\theta_i} \xi_i^2$ and if $\xi_i = 0$ for any $i$ such that $\theta_i = 0$. Note further that the maximal value of the entropy function $H$ on the compact set $\Delta$ equals $\log N$. From the preceding conditions, we thus obtain that in order for $I$ to be finite at some given $(\theta, \xi)$, $\xi$ must satisfy $\xi_i^2 \leq \overline{\theta}_i \log N$ (note that the case $\overline{\theta}_i = 0$ is automatically accounted for). This therefore proves that $I$ has compact support, and hence proves Lemma 12. \qed
B. Proof of the LDP lower bound

Let $U$ be an arbitrary open set in $\mathbb{R}^{N^2+N}$. Denote $a = \inf_{(\theta, \xi) \in U} I(\theta, \xi)$ and note that $a$ can either be a finite number or $+\infty$. If $a = +\infty$, then the lower bound holds trivially. Thus, in the remainder of the proof we assume that $a \in \mathbb{R}$.

We claim that, for any $\alpha > 0$, there exists a point $(\theta^*, \xi^*) = (\theta^*, \xi^*)(\alpha) \in U$ such that

$$I(\theta^*, \xi^*) \leq a + \alpha, \quad (50)$$

and $H(\theta^*) - J_{\theta^*}(\xi^*) > 0$. \quad (51)

To prove this claim, consider an arbitrary fixed $\alpha > 0$. Then, by the definition of $a$, there must exist $(\theta', \xi') \in U$ such that $(50)$ holds. Note that $H(\theta') - J_{\theta'}(\xi')$ can either be greater than 0 or equal to 0 (if $H(\theta') - J_{\theta'}(\xi') < 0$, this would contradict the fact that $a$ is finite). If $H(\theta') - J_{\theta'}(\xi') > 0$, the claim is proven. Hence, suppose that $H(\theta') - J_{\theta'}(\xi') = 0$. Recall that $U$ is an open set. By the definition of rate function $I$, $\theta'$ must be strictly positive in at least one entry, say $\theta'_1 > 0$. By the fact that $U$ is open, there must exist a point $(\theta'', \xi'') \in U$, where $\theta'' = \theta'$, $\xi''_j = \xi'_j$, for all $j \neq i$, and $\xi''_i$ is chosen such that $|\xi''_i| \leq |\xi'_i|$. For $\xi''$ there holds $J_{\theta''}(\xi'') < J_{\theta'}(\xi')$. Thus, choosing $(\theta^*, \xi^*) = (\theta'', \xi'')$ proves (51).

Next, for each $t \geq 1$, pick an arbitrary point $\theta_t$ from the set of closest neighbors of $\theta^*$ in the set $\Delta_t$, $\theta_t \in \text{Argmin}_{\theta \in \Delta_t, \bar{\theta}_t=0} \| \theta - \theta^* \|$. \quad (52)

Note that, since the set $\Delta_t$ gets denser with $t$, we have that $\theta_t \to \theta^*$, as $t$ goes to infinity. We show that there exists a box $D \subseteq \mathbb{R}^N$ (independent of $t$) such that, for all $t$ sufficiently large, $\{\theta_t\} \times D \subseteq U$. Since $(\theta^*, \xi^*) \in U$ and $U$ is open, we can find a sufficiently small box $B = C \times D$ centered at $(\theta^*, \xi^*)$, where $C$ is a box in $\mathbb{R}^{N^2}$ and $D$ is a box in $\mathbb{R}^N$, such that $B$ entirely belongs to $U$. Since $\theta_t \to \theta^*$, the tail of the sequence $\theta_t$ must belong to $C$. Thus, there exists $t_6 = t_6(\theta^*, C, D)$ such that for all $t \geq t_6$, $\{\theta_t\} \times D \subseteq C \times D \subseteq U$.

Similarly as in eq. (39) in the proof of the upper bound, we have

$$Q_{\omega}^{\omega}(U) \geq Q_{\omega}^{\omega}(\{\theta_t\} \times D) = Q_{\omega}^{\omega}(D) \frac{C_{\omega} \theta_t}{C_{\omega}}.$$  \quad (53)

We first show that, with probability one, the empirical measures $Q_{\omega}^{\omega}(D), \theta \in \Delta_t$, approach their respective expectations $q_{\omega}(D)$.

**Lemma 13.** For any $\epsilon > 0$, with probability one, there exists $t_7 = t_7(\omega, \epsilon, D)$ such that

$$Q_{t, \theta_0}(D) \geq q_{t, \theta_0}(D)(1 - \epsilon),$$  \quad (55)

for all $t \geq t_7$.  \quad (56)
Proof. Fix $\epsilon > 0$ and for each $t \geq 1$ define
\[
C_t = \left\{ \omega : \left| \frac{Q^\omega_{t, \theta_t}(D)}{q_{t, \theta_t}(D)} - 1 \right| \geq \epsilon \right\}.
\]
By Chebyshev’s inequality \cite{22}, we have
\[
P(C_t) \leq \frac{\text{Var} \left[ 1\{Z_t^\omega \in D\} \right]}{\epsilon^2 C_{t, \theta_t, q_{t, \theta_t}(D)}^2}
= \frac{q_{t, \theta_t}(D)(1 - q_{t, \theta_t}(D))}{\epsilon^2 C_{t, \theta_t, q_{t, \theta_t}(D)}^2}
\leq \frac{1}{\epsilon^2 C_{t, \theta_t, q_{t, \theta_t}(D)}}.
\]
(54)
We now lower bound $q_{t, \theta_t}(D)$.

Lemma 14. For each $\epsilon > 0$ there exists $t_8 = t_8(\epsilon, \theta^*, D)$ such that, for all $t \geq t_8$,
\[
q_{t, \theta_t}(D) \geq e^{-t \epsilon} e^{-t \inf_{\eta \in D} J_{\theta_t}(\eta)}.
\]
(55)
The proof of Lemma 14 is given in Appendix C.

Having Lemma 14 we are now ready to complete the proof of Lemma 13. Denote $\vartheta = H(\theta^*) - J_{\theta^*}(\xi^*)$, and recall that, by (51), $\vartheta > 0$. Applying (55) and (16) in (54) for an arbitrary fixed $\epsilon$, we have that, for all $t \geq t_9 = t_9(\omega, \epsilon, \theta^*, D, P_0) := \max\{t_1, t_8\}$,
\[
P(C_t) \leq \frac{1}{\epsilon^2} e^{-2t \epsilon + H(\theta_t) - \inf_{\eta \in D} J_{\theta_t}(\eta)}.
\]
(56)
Since $H$ is a continuous function, and $\theta_t \to \theta^*$, we have that $H(\theta_t) \geq H(\theta^*) - \epsilon$, for all $t$ larger than some $t_{10} = t_{10}(\epsilon, \theta^*)$. Second, because $\xi^* \in D$, for any $\theta_t$, there must hold that $\inf_{\eta \in D} J_{\theta_t}(\eta) \leq J_{\theta_t}(\xi^*)$. Further, since $J_{\theta}(\eta) = \sum_{i} \omega_{t,i} > 0 \frac{\xi^2_{t,i}}{2}$ is continuous in $\theta$ restricted to the coordinates $i$ in which $\theta^*_{t,i} > 0$, we have that for all $t$ larger than some $t_{11} = t_{11}(\epsilon, \theta^*)$, $J_{\theta_t}(\xi^*) \leq J_{\theta_t}(\xi^*) + \epsilon$ (note that for any $i$ such that $\theta^*_{t,i} = 0$ it must be that $\xi^*_{t,i} = 0$ – hence $J_{\theta_t}(\xi^*) = \sum_{i} \omega_{t,i} > 0 \frac{\xi^2_{t,i}}{2}$). Applying the preceding findings in (56) for $\epsilon = \vartheta/5 > 0$, we obtain
\[
P(C_t) \leq \frac{1}{\epsilon^2} e^{4t \epsilon - H(\theta^*) + J_{\theta^*}(\xi^*)}
= \frac{1}{\epsilon^2} e^{-t \vartheta/5},
\]
(57)
(58)
which holds for all $t \geq t_{12} = t_{12}(\omega, \epsilon, \theta^*, D, P_0) := \max\{t_9, t_{10}, t_{11}\}$. The claim of the lemma follows by the Borel-Cantelli lemma.

We next combine the result of Lemma 13 with the lower bounds on $q_{t, \theta_t}(D)$, $t = 1, 2, \ldots$, given in Lemma 14. To this end, fix an arbitrary $\epsilon > 0$. Let $\Omega^*_t$ denote the probability one set that verifies
Lemma 13. We have that, for every \( \omega \in \Omega^*_t \), there exists \( t_{13} = t_{13}(\omega, \epsilon, D, P_0) \) such that for all \( t \geq t_{13} = t_{13}(\omega, \epsilon, D) = \max\{t_0, t_1, t_7\} \)

\[
Q_t^\omega(U) \geq q_t, \theta_t(D)(1 - \epsilon) \frac{C_t, \theta_t}{C_t} \\
\geq e^{-3t} e^{-t \inf_{\xi \in D} J_\theta_t(\xi) + H(\theta_t) - \log \rho_0 (1 - \epsilon)}.
\]

Taking the logarithm and dividing by \( t \), we obtain

\[
\frac{1}{t} \log Q_t^\omega(U) \geq -3 \epsilon - \inf_{\xi \in D} J_\theta_t(\xi) + H(\theta_t) - \log \rho_0 + \frac{\log (1 - \epsilon)}{t}.
\]  

As \( t \to +\infty \), \( \theta_t \to \theta^* \) and we have that \( \inf_{\xi \in D} J_\theta_t(\xi) \to \inf_{\xi \in D} J_\theta(\xi) \), and also \( H(\theta_t) \to H(\theta^*) \). Thus, taking the limit in (59) yields

\[
\liminf_{t \to +\infty} \frac{1}{t} \log Q_t^\omega(U) \geq -3 \epsilon - I(\theta^*, \xi^*)
\]

where in the last inequality we used the fact that \( \xi^* \in D \). The latter bound holds for all \( \epsilon > 0 \), and hence taking the supremum over all \( \epsilon > 0 \) yields

\[
\liminf_{t \to +\infty} \frac{1}{t} \log Q_t^\omega(U) \geq -I(\theta^*, \xi^*)
\]

\[
\geq - \inf_{(\theta, \xi) \in U} I(\theta, \xi) - \alpha.
\]

Recalling that \( \alpha \) is arbitrarily chosen, the lower bound is proven.

VI. CONVEXITY OF \( \zeta \)

In this section we prove that problem (26) can be reformulated as a convex optimization problem and hence it is easily solvable.

Lemma 15. If \( H(P) \geq \sum_{i=1}^N \pi_i \beta_i^2 / 2 \), then the optimal value \( \zeta \) of (26) equals zero. Otherwise, (26) is equivalent to the following convex optimization problem and \( \zeta \) is computed as its optimal value:

\[
\begin{align*}
\text{minimize} & \quad -\sum_{i,j=1}^N \theta_{ij} \log P_{ij} + \sum_{i=1}^N \theta_i \beta_i^2 / 2 - 2 \sqrt{\sum_{i=1}^N \theta_i \beta_i^2} \sqrt{H(\theta)} \\
\text{subject to} & \quad \theta = \theta 1 \\
& \quad \theta \in \Delta
\end{align*}
\]

Proof. We start by taking out the dependence on vector \( \xi \), which we do by solving the inner optimization in (26) over \( \xi \), for a given \( \theta \):

\[
\begin{align*}
\text{minimize} & \quad \sum_{i: \theta_i > 0} \theta_i \frac{(\beta_i - \frac{1}{2} \xi^*)^2}{2} \\
\text{subject to} & \quad H(\theta) \geq \sum_{i: \theta_i > 0} \frac{1}{\theta_i} \xi_i^2
\end{align*}
\]

July 24, 2017 DRAFT
Note that, for any given $\theta$, for each $i$, the corresponding optimal solution $\xi_i^*(\theta)$ has the same sign as $\beta_i$. Hence, for each $i = 1, \ldots, N$, we can introduce the change of variables $x_i = \xi_i^2/2$ (optimal $\xi_i^*$ is then obtained from optimal $x^*$ by $\xi_i^* = \text{sign}(\beta_i) \sqrt{2x_i}$, for any $i = 1, \ldots, N$). Optimization problem (61) can now be written as
\begin{align*}
\text{minimize} & \quad \sum_{i: \beta_i > 0} \frac{\beta_i^2}{2} - \sum_{i=1}^N \beta_i |\beta_i| \sqrt{2x_i} + \sum_{i: \beta_i > 0} \frac{1}{\beta_i} x_i \\
\text{subject to} & \quad H(\theta) \geq \sum_{i: \beta_i > 0} \frac{1}{\beta_i} x_i \\
& \quad x \geq 0
\end{align*}
(62)

It is easy to see that (62) is a convex optimization problem (in $x$) with linear constraints. If $\theta$ is such that $H(\theta) = 0$, then $x^* = 0$ is a solution of (62) and thus the second term in (66) reduces to $\sum_{i: \beta_i > 0} \beta_i^2/2$, showing that the objective function in (60) is correctly evaluated (note that the third term in the objective in (60) vanishes for $H(\theta) = 0$). Consider now those matrices $\theta \in \Delta$ such that $H(\theta) > 0$. Then, there exists at least one feasible point $x$ in the interior of the constraint sets of (62), and thus we can solve (62) by solving the corresponding KKT conditions (23). We dualize only the first constraint and let $\lambda$ denote the Lagrange multiplier associated with this constraint. The corresponding Lagrangian function is given by
\begin{align*}
L(x, \lambda) := \sum_{i: \beta_i > 0} \frac{1}{\beta_i} x_i - \sum_{i: \beta_i > 0} \beta_i |\beta_i| \sqrt{2x_i} + \lambda \left( \sum_{i: \beta_i > 0} \frac{1}{\beta_i} x_i - H(\theta) \right),
\end{align*}
(63)
for $x \in \mathbb{R}^N$ and $\lambda \in \mathbb{R}$. Computing the partial derivatives with respect to $x$, we obtain, for each $i$ such that $\beta_i > 0$, $\frac{\partial}{\partial x_i} L(x, \lambda) = \frac{1}{\beta_i} - \beta_i |\beta_i| \sqrt{2x_i} + \lambda \frac{1}{\beta_i}$ (there is no constraint imposed on $x_i$ such that $\beta_i = 0$). Thus the KKT conditions are:
\begin{align*}
\begin{cases}
(1 + \lambda) = \frac{\beta_i^2 |\beta_i|}{\sqrt{2x_i}}, & \text{for } i \text{ s.t. } \beta_i > 0 \\
H(\theta) \geq \sum_{i: \beta_i > 0} \frac{1}{\beta_i} x_i \\
\lambda \geq 0 \\
\lambda \left( H(\theta) - \sum_{i: \beta_i > 0} \frac{1}{\beta_i} x_i \right) = 0
\end{cases}
\end{align*}
(64)

Simple algebraic manipulations reveal that a solution to (64) is given by
\begin{align*}
x_i^* = \begin{cases}
\frac{\beta_i^2 |\beta_i|}{\sqrt{2x_i}} \frac{H(\theta)}{\sum_{i=1}^N \beta_i^2/2}, & \text{if } H(\theta) < \sum_{i=1}^N \beta_i^2/2 \\
\frac{\beta_i^2 |\beta_i|}{\sqrt{2x_i}} \frac{2}{N}, & \text{otherwise}
\end{cases}
\end{align*}
(65)

From (65) it can be seen that, for those $\theta$ such that $H(\theta) \geq \sum_{i=1}^N \beta_i^2/2$, the optimal value of (62) equals 0, and for $\theta$ such that $H(\theta) \leq \sum_{i=1}^N \beta_i^2/2$, the optimal value of (62) equals
\begin{align*}
\sum_{i=1}^N \frac{\beta_i^2}{2} - \sum_{i=1}^N \beta_i |\beta_i| \sqrt{\frac{H(\theta)}{\sum_{i=1}^N \beta_i^2/2}} + \frac{H(\theta)}{\sum_{i=1}^N \beta_i^2/2} = \frac{N}{2} \left( \frac{\beta_i^2}{2} - 2 \sqrt{\frac{H(\theta)}{\sum_{i=1}^N \beta_i^2/2}} + \frac{H(\theta)}{\sum_{i=1}^N \beta_i^2/2} \right).
\end{align*}
(66)
We can therefore solve (26) by partitioning the candidate space $\theta \in \Delta$ into the following two regions: 1) $\mathcal{R}_1 = \left\{ \theta \in \Delta : H(\theta) \geq \sum_{i=1}^N \theta_i \frac{\beta_i^2}{2} \right\}$; and 2) $\mathcal{R}_2 = \left\{ \theta \in \Delta : H(\theta) \leq \sum_{i=1}^N \theta_i \frac{\beta_i^2}{2} \right\}$, and then finding the minimum, and the corresponding optimizer, among the optimal values of the two optimization problems with the above defined constraint sets. For region $\mathcal{R}_1$, we have to solve:

$$\begin{align*}
\text{minimize} & \quad D(\theta||P) \\
\text{subject to} & \quad H(\theta) \geq \sum_{i=1}^N \theta_i \frac{\beta_i^2}{2} \\
& \quad \theta \in \Delta
\end{align*}$$

(67)

For region $\mathcal{R}_2$, we have to solve

$$\begin{align*}
\text{minimize} & \quad -\sum_{i=1}^N \sum_{j=1}^N \theta_{ij} \log P_{ij} + \sum_{i=1}^N \theta_i \frac{\beta_i^2}{2} - 2\sqrt{\sum_{i=1}^N \theta_i \frac{\beta_i^2}{2}} \sqrt{H(\theta)} \\
\text{subject to} & \quad H(\theta) \leq \sum_{i=1}^N \theta_i \frac{\beta_i^2}{2} \\
& \quad \theta \in \Delta
\end{align*}$$

(68)

where we note that the objective is obtained by cancelling out the term $H(\theta)$ in the relative entropy, $D(\theta||P) = -\sum_{i=1}^N \sum_{j=1}^N \theta_{ij} \log P_{ij} - H(\theta)$, with the one in (66).

We show that, when $H(P) \geq \sum_{i=1}^N \pi_i \frac{\beta_i^2}{2}$, the optimal value of (26) equals zero and the corresponding solution is found by solving (67); otherwise, solution of (26) is found by solving (68).

Suppose that $H(P) \geq \sum_{i=1}^N \pi_i \frac{\beta_i^2}{2}$. It is easy to verify that, $\theta^*_{ij} = P_{ij} \pi_i$, $i, j = 1, \ldots, N$ is a solution to (69). Since in this case $D(\theta||P) = 0$, and since the optimal value of (26) is non-negative, it follows that $\theta^*$ is an optimizer of (26) as well.

Suppose now that $H(P) \leq \sum_{i=1}^N \pi_i \frac{\beta_i^2}{2}$. Applying Lemma 4 we have that problem (67) is convex. Dualizing the first constraint only, the resulting KKT conditions are given as follows:

$$\begin{align*}
\left\{ (1 + \lambda) \log \frac{\theta_{ij}}{\beta_i} - \log P_{ij} + \lambda \frac{\beta_i^2}{2} = 0 \\
H(\theta) \geq R(\theta) \\
\lambda \geq 0 \\
\lambda (R(\theta) - H(\theta)) = 0 \\
\theta \in \Delta
\right. \\
\end{align*}$$

(69)

where we defined $R(\theta) := \sum_{i=1}^N \theta_i \frac{\beta_i^2}{2}$, $\theta \in \mathbb{R}^{N^2}$. In order to apply the KKT conditions theorem, we analyze now the existence of Slater’s point [23]. Suppose first that for all $\theta \in \Delta$ there holds $H(\theta) = R(\theta)$. As this region of points is already contained in the constraint set of optimization problem (68), it follows that (26) can be solved by solving (68).

Suppose now that there exists a point $\theta \in \Delta$ such that $H(\theta) > R(\theta)$. If $\theta > 0$, then $\theta$ is a Slater’s point; if $\theta$ is not strictly positive then, by continuity of $H$ and $R$, we can find $\theta' > 0$ in the close neighborhood of $\theta$ that again satisfies $H(\theta') > R(\theta')$. By the KKT theorem, we conclude that, if such
a point $\theta$ exists, then we can find a solution (67) by solving the KKT conditions (69). Since, by the assumption, $H(P) < \sum_{i=1}^{N} \pi_i \frac{\beta^2}{2}$, then, because of the first three KKT conditions it must be that $\lambda > 0$. When combined with the fourth KKT condition, this yields that, at the optimal point $\theta$, there holds $H(\theta) = R(\theta)$. Similarly as in the analysis in the preceding paragraph, this region of points is already contained in the constraint set of optimization problem (68) and hence a solution of (26) can be found by solving (68).

In the following, we show in fact a stronger claim: when $H(P) < \sum_{i=1}^{N} \pi_i \frac{\beta^2}{2}$, solution of (26) is found by solving the convex relaxation (60) of (68). We do this by showing that, under the latter condition, there exists a solution $\theta^*$ of (60) that satisfies

$$H(\theta^*) \leq \sum_{i=1}^{N} \theta_i^* \frac{\beta^2}{2}. \quad (70)$$

Note that, because (60) is convex, with affine, non-empty constraints, the optimizer is found as a solution to the following KKT conditions:

$$\begin{align*}
- \log P_{ij} + \frac{\beta^2}{2} \left( 1 - \sqrt{\frac{H(\theta)}{R(\theta)}} \right) + \frac{\sqrt{R(\theta)}}{\sqrt{H(\theta)}} \log \frac{\theta_{ij}}{\theta_i} + \mu_i - \mu_j + \nu &= 0 \\
\theta \in \Delta
\end{align*} \quad (71)$$

where $\mu_i$ is the Lagrange multiplier corresponding to the constraint $e_i^\top \theta 1 = 1^\top \theta e_i$, for $i = 1, \ldots, N$, and $\nu$ is the Lagrange multiplier corresponding to the constraint $1^\top \theta 1 = 1$ (recall the definition of set $\Delta$ in (11)). Denoting

$$\alpha = \frac{\sqrt{H(\theta)}}{\sqrt{R(\theta)}}. \quad (72)$$

we obtain from (71) that a solution of (60) must be of the form

$$\theta_{ij} = \frac{P_{ij}^{\alpha} e^{-\alpha (1-\alpha) \frac{\beta^2}{2} v_j(\alpha)}}{v_i(\alpha) \lambda(\alpha)}, \quad (73)$$

where $\lambda(\alpha)$ and $v(\alpha)$ are, respectively, the Perron value and the right Perron vector of the matrix $M(\alpha)$ defined by

$$[M(\alpha)]_{ij} := P_{ij}^{\alpha} e^{-\alpha (1-\alpha) \frac{\beta^2}{2}}, \quad (74)$$

for $i, j = 1, \ldots, N$, and $\alpha \in \mathbb{R}$. Hence, we have that the set of solution candidates can be parameterized by a parameter $\alpha \in \mathbb{R}$, and if for some $\alpha$ condition (72) is satisfied, then the corresponding $\theta$ is a solution of (60).

For $\alpha \in \mathbb{R}$, let $H(\alpha)$ denote the value of the entropy function $H$ for matrix $\theta$ parameterized with $\alpha$ as in (73), i.e., $H(\alpha) = H(\theta(\alpha))$, $\alpha \in \mathbb{R}$, and define, similarly, $R(\alpha) = R(\theta(\alpha))$, $\alpha \in \mathbb{R}$. The following lemma is the core of the proof of the existence of $\theta^*$ that satisfies (70). The proof of this result is given in Appendix D; it is based on the limiting sequence of functions technique, e.g., [17].
Lemma 16. There holds

1) $H(1) = H(P)$ and $R(1) = \sum_{i=1}^{N} \pi_i \beta_i^2$;

2) $H(0) = \log \rho_0$;

3) $H(\alpha) - \alpha^2 R(\alpha)$ is decreasing for $\alpha \in [0, 1]$.

Corollary 17 is immediately obtained from parts 1, 2, and 3 of Lemma 16 and the fact that $\rho_0 > 1$ (proven in Lemma 5). In particular, by the assumption that $H(P) < \sum_{i=1}^{N} \pi_i \beta_i^2$ and the fact that $\log \rho_0 > 0$, we have that $\alpha \mapsto H(\alpha) - \alpha^2 R(\alpha)$ is strictly positive at $\alpha = 0$, strictly negative at $\alpha = 1$, and is decreasing on the interval $[0, 1]$. Hence, the claim of Corollary 17 follows.

Corollary 17. If $H(P) \leq \sum_{i=1}^{N} \pi_i \beta_i^2$, then there exists $\alpha^* \in (0, 1)$ such that $H(\alpha^*) = \alpha^* R(\alpha^*)$.

Corollary 17 asserts that, when $H(P) \leq \sum_{i=1}^{N} \pi_i \beta_i^2$, then there must exist $\alpha^* \in (0, 1)$ that satisfies (72).

We now prove that (60) is convex. The first constraint in (60) is linear, hence convex, and set $\Delta$ is also convex. Also, the first two terms of the objective function are linear, hence convex. Thus, if we show that the last term of the objective is convex, we prove the claim. The latter follows as a corollary of the following more general result, which we prove here.

Lemma 18. Let $g, h : \mathbb{R}^d \mapsto \mathbb{R}$ be two non-negative, concave functions. Then, function $f : \mathbb{R}^d \mapsto \mathbb{R}$ defined by

$$f(x) := -\sqrt{g(x)} \sqrt{h(x)}, \quad x \in \mathbb{R}^d,$$

is convex.

The proof of Lemma 18 is given in Appendix D.

In order to apply Lemma 18 we need to verify that functions $\sum_{i=1}^{N} \theta_i \beta_i^2$ and $H(\theta)$ are concave in $(\theta, \vartheta)$. The former is linear and thus concave, and concavity of the latter is proven in Lemma 4.

VII. Conclusion

We addressed the problem of detecting a random walk on a graph, when the observation interval grows large, under the assumption of heterogeneous graph nodes. We illustrate our methodology by devising a random access mechanism, which exploits Markov chain frequency hopping across different access channels to combat possibly unknown, frequency selective fading. This could be of interest for future NB IoT communications standard for which similar random access model and similar unpredictable communication environments are envisioned.
Using the notion of Gauss-Markov type, we provided a lower bound on the Neyman Pearson error exponent, equal to the asymptotic Kullback-Leibler divergence. The bound is in the form of an optimization problem, and it has a very interesting interpretation in terms of the Gauss-Markov type: over all $H_0$-feasible Gauss-Markov types, the solution is the one which has the highest probability under $H_1$ state of nature, i.e., under the presence of the random walk. We prove convexity of the lower bound thus showing that it is efficiently computable.

APPENDIX

A.

Proof of Proposition 2

Proof. By Jensen’s inequality we have

$$\sum_{s^t \in S^t} P(s^t) e^{\sum_{k=1}^t \beta_{s^t} X_{s^t,k} - \frac{\sigma^2_{s^t}}{2}} \geq e^{\sum_{s' \in S^t} P(s') \left\{ \sum_{k=1}^t \beta_{s'} X_{s',k} - \frac{\sigma^2_{s'}}{2} \right\} }.$$

Taking the logarithm and using monotonicity of the expectation, the latter inequality then implies

$$\zeta \leq - \frac{1}{t} \sum_{s' \in S^t} P(s') \left\{ \sum_{k=1}^t \beta_{s'} X_{s',k} - \frac{\beta^2_{s'}}{2} \right\}.$$

$$= - \frac{1}{t} \sum_{s' \in S^t} P(s') \left\{ \sum_{k=1}^t \beta_{s'} \mathbb{E}_1[X_{s',k}] - \frac{\beta^2_{s'}}{2} \right\}$$

$$= \frac{1}{t} \sum_{s' \in S^t} P(s') \sum_{k=1}^t \frac{\beta^2_{s'}}{2}. \quad (76)$$

We now observe that the sum in (76) equals the expected value of the sum of the SNR values at nodes visited until time $t$, i.e.,

$$\mathbb{E}_1 \left[ \sum_{k=1}^t \frac{\beta^2_{S_k}}{2} \right] = \sum_{s'} P(s') \mathbb{E}_1 \left[ \sum_{k=1}^t \frac{\beta^2_{S_k}}{2} \bigg| S^t = s' \right] \quad (77)$$

$$= \sum_{s'} P(s') \sum_{k=1}^t \frac{\beta^2_{s_k}}{2}. \quad (78)$$

On the other hand, the left hand-side in (77) can also be written as

$$\mathbb{E}_1 \left[ \sum_{k=1}^t \frac{\beta^2_{S_k}}{2} \right] = \mathbb{E}_1 \left[ \sum_{k=1}^t \sum_{i=1}^N \mathbb{1}_{\{S_k = i\}} \frac{\beta^2_{i}}{2} \right]$$

$$= \sum_{k=1}^t \sum_{i=1}^N \mathbb{P}_1(S_k = i) \frac{\beta^2_{i}}{2}$$

$$= \sum_{i=1}^N \left\{ \sum_{k=1}^t \mathbb{P}_1(S_k = i) \right\} \frac{\beta^2_{i}}{2}. \quad (79)$$

July 24, 2017
To prove (8), it only remains to show that, for any \(i = 1, \ldots, N\),

\[
\lim_{t \to +\infty} \frac{1}{t} \sum_{k=1}^{t} P_1(S_k = i) = \pi_i. 
\] (79)

It is easy to see that \(P_1(S_k = i) = v^\top P^{k-1} e_i\). Further, since \(P\) is irreducible and aperiodic, with stationary distribution \(\pi\), we have that the Cesàro averages of \(P^k\) converge to \(1\pi^\top\), i.e.,

\[
\lim_{t \to +\infty} \frac{1}{t} \sum_{k=1}^{t} P^{k-1} = 1\pi^\top.
\]

Finally, noting that \(v^\top 1 = 1\) establishes (79) and hence proves (8).

\[\Box\]

**Proof of Lemma 4**

**Proof.** We show that \(H(\theta)\) is concave by proving that its Hessian is negative semi-definite at \(\theta \geq 0\) (and thus for all \(\theta \in \Delta\)). Recall that

\[
H(\theta) = -\sum_{i=1}^{N} \sum_{j=1}^{N} \theta_{ij} \log \frac{\theta_{ij}}{\sum_{k=1}^{N} \theta_{ik}}.
\]

It can be shown that the first and second order partial derivatives of \(H(\theta)\) are given by

\[
\frac{\partial}{\partial \theta_{ij}} H(\theta) = -\log \theta_{ij} + \log \left(\sum_{k=1}^{N} \theta_{ik}\right), \text{ for every } (i, j),
\]

\[
\frac{\partial^2}{\partial \theta_{ij} \theta_{lm}} H(\theta) = \begin{cases} -\frac{1}{\theta_{ij}} + \frac{1}{\sum_{k=1}^{N} \theta_{ik}}, & \text{if } i = l, j = m \\ \frac{1}{\sum_{k=1}^{N} \theta_{ik}}, & \text{if } i = l, j \neq m \\ 0, & \text{otherwise} \end{cases}.
\] (80)

From (80) we see that the Hessian matrix \(\nabla^2 H(\theta)\) takes the following block-diagonal form

\[
\nabla^2 H(\theta) = \text{Diag}(B_1, \ldots, B_N),
\]

where, for each \(i\), \(B_i\) is an \(N\) by \(N\) matrix given by

\[
B_i = -\text{Diag} \left(\frac{1}{\theta_{i1}}, \frac{1}{\theta_{i2}}, \ldots, \frac{1}{\theta_{iN}}\right) + \frac{1}{\sum_{k=1}^{N} \theta_{ik}} 11^\top.
\]

It suffices to show that each \(B_i\) is, at each \(\theta\), negative semi-definite. To this end, fix an arbitrary \(i\), and pick an arbitrary vector \(q \in \mathbb{R}^N\). We have,

\[
q^\top B_i q = -\sum_{j=1}^{N} \frac{q_j^2}{\theta_{ij}} + \frac{(1^\top q)^2}{\sum_{k=1}^{N} \theta_{ik}}.
\] (81)

It suffices to restrict the attention to an arbitrary \(q\) such that \(q \geq 0\) and \(1^\top q = 1\). Then, (81) simplifies to

\[
q^\top B_i q = -\sum_{j=1}^{N} \frac{q_j^2}{\theta_{ij}} + \frac{1}{\sum_{k=1}^{N} \theta_{ik}}.
\] (82)
Exploting now Jensen’s inequality with convex combination \( q_1, \ldots, q_N \), applied to the function \( z \mapsto 1/z \) at points \( z_j = \theta_{ij}/q_j \), yields
\[
\sum_{j=1}^{N} q_j \frac{q_j}{\theta_{ij}} \geq \frac{1}{\sum_{j=1}^{N} q_j \frac{q_j}{q_j}} = \frac{1}{\sum_{j=1}^{N} \theta_{jk}}.
\] (83)

Thus, the quadratic form \( q^\top B_t q \) is at every \( \theta \) smaller or equal than \( 0 \), proving that \( B_t \) is negative semi-definite. Since \( i \) was arbitrary, it follows that \( \nabla^2 H(\theta) \) is negative semi-definite at every \( \theta \), which finally proves that \( H \) is concave.

To prove part 2, we only need to note that \( H(\cdot||P) \) can be represented as a sum of three convex functions, \(-H(\theta), \sum_{ij} \theta_{ij} \log P_{ij}, \) which is linear, hence convex, and the last is a sum of convex functions \( \bar{\theta}_i \log \bar{\theta}_i \).

**Proof of Lemma 5**

**Proof.** We first prove part 1. Due to the fact that the initial distribution of the Markov chain is \( \pi > 0 \) (and hence the chain can start from any state), it is easy to see that \( C_t = 1^\top P_t^{t-1} 1 \). Thus,
\[
\|P_t^{t-1}\|_\infty \leq C_t \leq N \|P_0^{t-1}\|_\infty.
\]

Let \( \rho \) denote the spectral radius of a square matrix and let \( \rho_0 \) denote the spectral radius of \( P_0 \). Then, by the property of spectral radius that asserts that \( \rho(A) \leq \|A\| \) for any square matrix \( A \) and for any matrix norm \( \| \cdot \| \), we have that \( \rho_0 = \rho \left( P_t^t \right) \leq \|P_t^t\|_\infty \), for any \( t \geq 1 \). Also, by Gelfand’s formula (see Theorem 8.5.1 in [24]), for any \( \epsilon > 0 \), for all \( t \) greater than some \( t_0 = t_0(\epsilon) \), there holds \( \|P_t^t\|_\infty \leq \rho_0^t \epsilon^t \).

Summarizing, the result follows.

To prove that \( \rho_0 > 1 \), note that, since \( P \) is irreducible and aperiodic, it must be that \( P_0 \) is also irreducible and aperiodic. Then, there exists a finite positive integer \( M \) such that \( P_0^M \) is strictly positive. Since each entry of \( P_0^M \) is at least one, using the property that the spectral radius of an arbitrary square matrix is greater or equal than its minimal row sum [24], we have that \( \rho(P_0^M) \geq N \). It follows that \( \rho(P_0) = (\rho(P_0^M))^{1/M} > 1 \).

To prove part 2 we use the following result, the proof of which can be found in [20], Chapter II, Section II.2; we note that the proof of this result is based on finding the number of Euler circuits on the graph with the set of vertices \( V = \{1, \ldots, N\} \) and with the set of arcs such that the number of arcs from \( i \) to \( j \) equals \( k_{ij} \), for each pair \( (i, j) \), \( i, j = 1, \ldots, N \).

**Lemma 19.** For each \( t \geq 1 \) and \( \theta \in \Delta_t \) there holds
\[
\prod_{i,k_i>0}(k_i - 1)! \prod_{i,j} k_{ij}! \leq C_{t,\theta} \leq N \prod_{i,k_i>0} k_i! \prod_{i,j} k_{ij}!, \quad (84)
\]
where \( k_{ij} = k_{ij}(\theta) \), \( i, j = 1, \ldots, N \), is the matrix that verifies the fact that \( \theta \) belongs to \( \Delta_t \), and, for each \( i, \) \( k_i = \sum_{j=1}^{N} k_{ij} \).

To complete the proof of Lemma 5, we use Stirling approximation inequalities, which assert that, for all non-negative integers \( k \), there holds
\[
\sqrt{2\pi} \sqrt{k} \left( \frac{k}{e} \right)^k \leq k! \leq e \sqrt{k} \left( \frac{k}{e} \right)^k . \tag{85}
\]

Fixing an arbitrary \( t \geq 1 \) and \( \theta \in \Delta_t \), we apply (85) to (84). Exploiting now the fact that \( \theta_{ij} = 0 \) if and only if \( k_{ij} = 0 \), and also that for \( \theta_{ij} > 0, 1 \leq k_{ij} \leq t \) and \( 1 \leq k_i \leq t \), we obtain
\[
\frac{1}{t^N} e^{\frac{2\pi}{\sqrt{2}} t^2} e^{tH(\theta)} \leq \sum_{s \in S_t} P(s^t) e^{-\frac{\|x\|^2}{2} + x^t} .
\]

Finally, noting that, on both sides of the preceding inequality, the factors that premultiply \( e^{tH(\theta)} \) are polynomial in \( t \), and hence are dominated by any exponential function \( e^{\epsilon t}, \epsilon > 0 \), for sufficiently large \( t \), the claim in part 2 follows.

\[\square\]

**Proof of Lemma 9**

**Proof.** The main technical result behind Lemma 9 is (a variant of) Slepian’s lemma, see [25].

**Lemma 20. (Slepian’s lemma [25])** Let the function \( \phi : \mathbb{R}^L \to \mathbb{R} \) satisfy
\[
\lim_{\|x\| \to +\infty} \phi(x)e^{-\alpha\|x\|^2} = 0, \text{ for all } \alpha > 0. \tag{86}
\]

Suppose that \( \phi \) has nonnegative mixed derivatives,
\[
\frac{\partial^2 \phi}{\partial x_l \partial x_m} \geq 0, \text{ for } l \neq m. \tag{87}
\]

Then, for any two independent zero-mean Gaussian vectors \( X \) and \( Z \) taking values in \( \mathbb{R}^L \) such that \( \mathbb{E}_X[X_l^2] = \mathbb{E}_Z[Z_l^2] \) and \( \mathbb{E}_X[X_l X_m] \geq \mathbb{E}_Z[Z_l Z_m] \) there holds \( \mathbb{E}_X[\phi(X)] \geq \mathbb{E}_Z[\phi(Z)] \), where \( \mathbb{E}_X \) and \( \mathbb{E}_Z \), respectively, denote expectation operators on probability spaces on which \( X \) and \( Z \) are defined.

We apply Lemma 20 to function \( \phi_t \) in (31) and random variables \( \mathbf{X}(s^t) \) and \( \mathbf{Z}(s^t) \), \( s^t \in S_t \), defined, respectively, in (30) and (33). We first verify the conditions of the lemma. Note that vectors \( \mathbf{X} \) and \( \mathbf{Z} \) are of dimension \( L = C_t \).

Since \( \phi_t \) grows linearly in \( \|x\| \), it is easy to see that the first condition of the lemma is satisfied. Considering the second condition,
\[
\frac{\partial \phi_t}{\partial x_{s^t}} = -\frac{P(s^t) e^{-\frac{\|x_{s^t}\|^2}{2} + x_{s^t}}}{\sum_{s^t} P(s^t) e^{-\frac{\|x_{s^t}\|^2}{2} + x_{s^t}}}.
\]
and so
\[
\frac{\partial^2 \phi_t}{\partial x_s \partial x_{s'}} = \frac{P(s^t)e^{-\frac{\|s^t\|^2}{2} + x_s} P(s^t')e^{-\frac{\|s^t\|^2}{2} + x_{s'}}}{\left(\sum_{s^t} P(s^t)e^{-\frac{\|s^t\|^2}{2} + x_s}\right)^2},
\]
which is non-negative for all \( x \in \mathbb{R}^{C_t} \).

Considering further the conditions on \( \mathcal{X} \) and \( \mathcal{Z} \), we have
\[
E_0[\mathcal{X}(s^t)^2] = E[\mathcal{Z}(s^t)^2] = \sum_{i=1}^N K_i(s^t)^2 \frac{\beta^2}{2},
\]
for each \( s^t \in S^t \). Also, it is easy to verify that
\[
E_0[\mathcal{X}(s^t)\mathcal{X}(s^t')] = \sum_{k: s_k = s'_k} \beta^2_{sk} \geq 0 = E[\mathcal{Z}(s^t)\mathcal{Z}(s^t')].
\]
Hence, all the conditions for applying Lemma 20 to \( \phi_t \) and random vectors \( \mathcal{X} \) and \( \mathcal{Z} \) are fulfilled. Thus, by Lemma 20 the claim of Lemma 9 follows.

**Proof of Lemma 10.**

**Proof.** Part 1 can be proven by a simple modification of the proof in Appendix E of [4], hence we omit the proof here.

We prove part 2 by applying Varadhan’s lemma, where we use the LDP for Gauss-Markov types from Theorem 7. We use the following version of Varadhan’s lemma which assumes LDP with probability one, in the sense of Theorem 7, the proof of Lemma 21 is omitted, but we remark that it follows the line of the proof of Varadhan’s lemma [19] (for deterministic sequences of measures) with each application of LDP bound (upper or lower) being used in the probability one sense, and then finally using the fact that countable intersection of probability one sets is also a probability one set.

**Lemma 21** (Varadhan’s lemma [19]). Suppose that for every measurable set \( G \) the sequence of (random) probability measures \( \mu_t^\omega \) with probability one satisfies, respectively, the LDP upper and lower bound with rate function \( I \). Further, let \( F : \mathbb{R}^D \mapsto \mathbb{R} \) be an arbitrary continuous function. If the tail condition below holds with probability one,
\[
\lim_{M \to +\infty} \limsup_{t \to +\infty} \frac{1}{t} \log \int_{x: F(x) \geq M} e^{tF(x)} d\mu_t^\omega(x) = -\infty, \quad \text{(88)}
\]
then, with probability one,
\[
\lim_{t \to +\infty} \frac{1}{t} \log \sum_x e^{tF(x)} d\mu_t^\omega(x) = \sup_{x \in \mathbb{R}^D} F(x) - I(x). \quad \text{(89)}
\]
We apply Varadhan’s lemma \cite{varadhan} to compute the limit of the sequence $\phi_t(\mathcal{Z})$, $t = 1, 2, \ldots$. To this end, observe that $P(s^t)$, $\beta(s^t)$, and $\mathcal{Z}(s^t)$ can be written in terms of $\Theta_t$ and $\mathcal{Z}_t$ as
\begin{equation}
P(s^t) = \frac{\pi_{s_1}}{P_{s_1,s_1}} e^{\sum_{i,j=1}^N K_{ij}(s^t) \log P_{ij}},
\end{equation}
\begin{equation}
\beta(s^t) = \sum_{i=1}^N K_i(s^t) \beta_i^2,
\end{equation}
\begin{equation}
\mathcal{Z}(s^t) = t \sum_{i=1}^N \beta_i \mathcal{Z}_{i,t}(s^t).
\end{equation}
(We remark that, in (90), it might happen that some feasible sequences $s^t$ begin and end with such $s_1 = j$ and $s_t = i$ for which the transition $s_t$ to $s_1$ has zero probability, $P_{s_1,s_1} = P_{ij} = 0$. Then, since $s^t$ is feasible, it must be that $K_{ij}(s^t) = 1$ (counting exactly this last, artificially added transition from $s_t$ to $s_1$), and we have that the two zero terms $- P_{s_1,s_1}$ and $e^{K_{ij}(s^t) \log P_{ij}} = e^{K_{ij}(s^t)}$ will cancel out.)

Hence, we can write
\begin{equation}
\phi_t = - \frac{1}{t} \log \left( \sum_{s^t \in \mathcal{S}^t} \frac{\pi_{s_1}}{P_{s_1,s_1}} e^{t \left( \sum_{i,j=1}^N [\Theta_t]_{ij} \log P_{ij} - \sum_{i=1}^N \mathcal{Z}_{i,t}(s^t) \frac{\beta_i^2}{2} + \sum_{i=1}^N \beta_i \mathcal{Z}_{i,t}(s^t) \right)} \right)
\end{equation}
\begin{equation}
= - \frac{1}{t} \log C_t - \frac{1}{t} \log \int_\Theta \int_\xi e^{t F(\theta, \xi) Q_t^\omega(d\theta, d\xi)},
\end{equation}
where $F : \mathbb{R}^{N^2 + N} \mapsto \mathbb{R}$ is defined as
\begin{equation}
F(\theta, \xi) := \sum_{i,j=1}^N \theta_{ij} \log P_{ij} - \sum_{i=1}^N \theta_i \beta_i^2 \frac{1}{2} + \sum_{i=1}^N \beta_i \xi_i,
\end{equation}
for $(\theta, \xi) \in \mathbb{R}^{N^2 + N}$. By Lemma \cite{varadhan}, we have that the limit of the first term equals $\log \rho_0$. To apply Varadhan’s lemma to compute the limit of the second term in (91) we first need to verify that $F$ satisfies tail condition \cite{varadhan}. From the conditions that define the support of $I$, it is easy to conclude that the rate function has compact support. More particularly, the support $\mathcal{D}_I$ of the rate function will satisfy $\mathcal{D}_I \subseteq B_0 := \Delta \times D_0$, where $D_0 = [- \log N, + \log N]^N$ (see the proof of exponential tightness of the sequence $Q_t^\omega$, Lemma \cite{varadhan}). It can be shown (similarly as in the proof of the upper bound, Case 2: $b > 0$), that $Q_t^\omega(B_0) = 0$, for all $t$ sufficiently large. Choosing $M_0 := \max_{(\theta, \xi) \in B_0} F(\theta, \xi)$ (note that $F$ is continuous and hence by Weierstrass theorem achieves maximum on a compact set), we have that, for each $M \geq M_0$, with probability one, the integral in (88) equals zero for all $t$ sufficiently large. Thus, the sequence of measures $Q_t^\omega$, $t \geq 1$, with probability one, satisfies condition (88).

By Varadhan’s lemma and Lemma \cite{varadhan} part \cite{varadhan}, we therefore obtain that, with probability one,
\begin{equation}
\lim_{t \to +\infty} \phi_t = \log \rho_0 - \sup_{(\theta, \xi) \in \mathbb{R}^{N^2 + N}} F(\theta, \xi) - I(\theta, \xi).
\end{equation}
It is easy to see that the optimization problem in (93) is equivalent to the one given in eq. (26) of Theorem \cite{varadhan} This proves the claim of part 2 and completes the proof of the lemma.
C.

**Proof of Lemma 11.**

*Proof.* Suppose first that there exists $i$ such that $\overline{\theta}_i = 0$ and $0 \notin D_i$. Then, by (41) and (40), we have $q_{t, \theta}(D) = 0$. On the other hand, by the definition of $J_{\theta}$ (see Theorem 7), we have that in this case $J_{\theta}(\xi) = +\infty$, which proves that (43) is true. Thus, in the remainder of the proof we assume that $0 \in D_i$ for each $i$ such that $\overline{\theta}_i = 0$.

Fix an arbitrary $\epsilon > 0$, and let $D_i = [\xi_i, \xi_i + \delta_i]$, for some $\xi_i$ and $\delta_i > 0$, $i = 1, ..., N$. Let also $C_\delta = \prod_{i: \overline{\theta}_i > 0} \delta_i$ and denote with $N_1$ the number of non-zero elements of $\overline{\theta}$. We show that, for any $\theta \in \Delta_t$,

$$q_{t, \theta}(D) \leq \frac{tN_1}{(2\pi)^{\frac{n}{2}}} C_\delta e^{-t \inf_{\xi \in D} J_{\theta}(\xi)}. \tag{94}$$

It is easy to see that, for the given value of $\epsilon > 0$, there exists a finite $t_2$ (that depends on $D$ and $\epsilon$) such that $\frac{tN_1}{(2\pi)^{\frac{n}{2}}} C_\delta \leq e^{t_2}$, for all $t \geq t_2$. Hence, if we show that the inequality (94) holds, the claim of Lemma 11 is proven.

To this end, fix $D_i$'s of the form described in the preceding paragraph and consider an arbitrarily chosen $\theta \in \Delta_t$. For each $i$, denote $k_i = \overline{\theta}_i t$. Note that, for each $i$ such that $\overline{\theta}_i > 0$, $k_i$ is a positive integer by the fact that $\theta \in \Delta_t$. Consider a fixed $i$ that satisfies the preceding condition. Then, $k_i \geq 1$, and also since, for any $\eta_i \in D_i$, $e^{-\frac{t^2}{2k_i^2}} \leq e^{-t \inf_{\xi \in D_i} \frac{t^2}{k_i^2}}$, we obtain from (41) that

$$[q_{t, \theta}]_i(D_i) \leq \frac{t}{\sqrt{2\pi}} e^{-t \inf_{\eta_i \in D_i} \frac{t^2}{k_i^2}} \int_{\eta_i \in D_i} d\eta_i = \frac{t}{\sqrt{2\pi}} \delta_i e^{-t \inf_{\eta_i \in D_i} \frac{t^2}{k_i^2}}. \tag{51}$$

Applying the preceding inequality in (40) for all $i$ such that $k_i \geq 1$, together with (41) for all $i$ such that $\overline{\theta}_i = 0$ proves (43). 

**Proof of Lemma 14.**

*Proof.* Consider first $i$ such that $\overline{\theta}_i^* = 0$. By (52), we then have $\theta_{t,i} = 0$, for all $t \geq 1$. Further, by (51) we have that $\xi_i^* = 0$, and hence, the constructed interval $D_i$ contains zero. Since $\theta_{t,i} = 0$, by (41) we therefore have $[q_{t, \theta}]_i(D_i) = 1$.

Let $N_1$ be the number of non-zero entries in $\overline{\theta}^*$. Since $\theta_i \to \theta^*$, for any $\delta > 0$, there exists $t'_{\delta} = t'_{\delta}(\delta, \theta^*)$ such that for all $t \geq t'_{\delta}$, there holds $\theta_{i,t} \leq (\theta_i^* - \delta, \theta_i^* + \delta)$. Take $\delta$ such that $\theta_i^* - \delta > 0$ for each $i$ such that $\overline{\theta}_i^* > 0$. Then, for any $t \geq t'_{\delta}$ the set of non-zero elements of $\overline{\theta}_i$ is the same as the set of non-zero elements of $\theta^*$, and hence, for any $\eta \in \mathbb{R}^N$, $J_{\theta_i}(\eta) = \sum_{i: \overline{\theta}_i > 0, \theta_{i,t} > 0} \frac{1}{\theta_{i,t}} \beta_i^2$. Using the fact that $D$ is a box and
decomposing \( q_{t,\theta_i}(D) = \prod_{i} p_{j} > 0 [q_{t,\theta_i}]_i(D_i) \), we see that we prove the claim of the lemma if we show that, for any \( i \), for any \( \epsilon > 0 \), there exists \( t_{8,i} = t_{8,i}(\epsilon, i, D_i, \theta^*) \) such that for all \( t \geq t_{8,i} \) there holds

\[
[q_{t,\theta_i}]_i(D_i) \geq e^{-t \epsilon/N_i} e^{-t \inf_{i \in D_i} \frac{1}{\theta_i} \frac{a_i^2}{2}}.
\]

(95)

We prove (95) by considering separately three cases with respect to \( D_i \): 1) \( \xi_i \geq 0 \); 2) \( \xi_i + \delta_i \leq 0 \); and 3) \( \xi_i \leq 0 \leq \xi_i + \delta_i \). In all three cases, we will be using the well-known bounds on the Q-function, which assert that, for an arbitrary \( \alpha > 0 \) and \( a \in \mathbb{R} \), there holds

\[
\frac{a}{\alpha^2 + a^2} e^{-\alpha a^2/2} \leq \int_{x \geq a} e^{-\alpha x^2/2} dx \leq \frac{1}{\alpha a} e^{-\alpha a^2/2}.
\]

(96)

Fix \( t \geq t'_{8} \). To simplify the notation, we drop index \( t \) and denote \( \theta_i \) by \( \theta \), and similarly for \( \bar{\theta}_i \). As before, we denote \( k_i = \bar{\theta}_i \). Note that due to the fact that \( \theta \in \Delta \), we have that \( k_i \) is an integer upper bounded by \( t \), which together with the assumption that \( \bar{\theta}_i > 0 \) implies \( 1 \leq k_i \leq t \).

**Case 1:** \( \xi_i \geq 0 \). We have (see (41)),

\[
[q_{t,\theta_i}]_i(D_i) = \frac{t}{\sqrt{2 \pi k_i}} \int_{\eta_i \leq \xi_i \leq \xi_i + \delta_i} e^{-\eta_i^2/2} \\
\geq \frac{\sqrt{t}}{\sqrt{2 \pi k_i}} \int_{\eta_i \geq \xi_i} e^{-\eta_i^2/2} - \frac{\sqrt{t}}{\sqrt{2 \pi k_i}} \int_{\eta_i \geq \xi_i + \delta_i} e^{-\eta_i^2/2} \\
\geq \frac{\sqrt{t}}{\sqrt{2 \pi k_i}} \frac{\xi_i}{\xi_i^2 + \xi_i} e^{-\xi_i^2/2} - \frac{\sqrt{t}}{\sqrt{2 \pi k_i}} \frac{1}{\xi_i + \delta_i} e^{-\frac{(\xi_i + \delta_i)^2}{2}} \\
\geq \frac{\sqrt{t}}{\sqrt{2 \pi k_i}} \frac{\xi_i}{\xi_i^2 + \xi_i} e^{-\xi_i^2/2} - \frac{\sqrt{t}}{\sqrt{2 \pi t^4}} \frac{1}{\xi_i + \delta_i} e^{-\frac{(\xi_i + \delta_i)^2}{2}} \\
= \frac{\sqrt{t}}{\sqrt{2 \pi k_i}} \frac{\xi_i}{\xi_i^2 + \xi_i} e^{-t \inf_{\eta_i \in D_i} \frac{1}{\theta_i} \frac{a_i^2}{2}} \left( 1 - \frac{1}{t^{3/2} \xi_i (\xi_i + \delta_i)} e^{-\frac{\beta_i}{2}} \right),
\]

(97)

where \( \alpha_i = \frac{(\xi_i + \delta_i)^2}{2} - \xi_i^2 > 0 \), and where in the first inequality we used that \( k_i \leq t \), and in the second we used that \( 1 \leq k_i \leq t \).

Recall that, for all \( t \geq t'_{8}, \bar{\theta}_i \leq \bar{\theta}_i ' + \delta_i \), thus \( \frac{1}{\bar{\theta}_i} \alpha_i \geq \frac{1}{\bar{\theta}_i + \delta} \alpha_i > 0 \). This further implies that the term in the brackets in (97) must converge to 1 as \( t \) increases, and hence, can be lower bounded by a positive constant between 0 and 1, e.g., by 1/2 for sufficiently large \( t \). On the other hand, the first term in (97) decays slower than exponential, and thus the product of the first and the third term can be lower bounded by \( e^{-t \epsilon/N_i} \), for \( t \) sufficiently large. Thus, (95) holds for all \( t \) greater than some \( t_{8,i} = t_{8,i}(\epsilon, D_i, \theta_i ^*) \).

**Case 2:** \( \xi_i + \delta_i \leq 0 \). Similarly as in the previous case, it can be shown here that

\[
[q_{t,\theta_i}]_i(D_i) \geq \frac{\sqrt{t}}{\sqrt{2 \pi t^4 + (\xi_i + \delta_i)^2}} e^{-t \inf_{\eta_i \in D_i} \frac{1}{\theta_i} \frac{a_i^2}{2}} \left( 1 - \frac{1}{t^{3/2} |\xi_i + \delta_i| |\xi_i|} e^{-\frac{\beta_i}{2}} \right),
\]

where \( \beta_i = \frac{\xi_i^2}{2} - \frac{(\xi_i + \delta_i)^2}{2} > 0 \). From here, the proof is analogous as in Case 1.
Case 3: $\xi_i \leq \xi_i + \delta_i$. In this case, we write

$$[q_{t,\theta}]_i(D_i) = 1 - \frac{t}{\sqrt{2\pi k_i}} \int_{\eta_i \geq |\xi_i|} e^{-\frac{\eta_i^2}{2k_i}} d\eta_i - \frac{t}{\sqrt{2\pi k_i}} \int_{\eta_i \geq \xi_i + \delta_i} e^{-\frac{\eta_i^2}{2k_i}} d\eta_i.$$

$$= 1 - \frac{t}{\sqrt{2\pi k_i}} \frac{1}{k_i^2} \frac{e^{-\frac{t}{k_i} \xi_i^2}}{\xi_i^2} - \frac{e^{-\frac{t}{k_i} \delta_i^2}}{\xi_i + \delta_i} - \frac{t}{\sqrt{2\pi k_i}} \frac{1}{k_i} \frac{1}{\xi_i + \delta_i} \frac{e^{-\frac{t}{k_i} (\xi_i + \delta_i)^2}}{\xi_i + \delta_i}.$$

It is easy to see that the second and the third term in the last equation go to zero as $t$ goes to infinity.

Thus, for sufficiently large $t$, we have

$$[q_{t,\theta}]_i(D_i) \geq 1/2.$$

Noting that $e^{-\inf_{n \in D_i} \frac{1}{\pi_i n_i^2}} = 1$ and that, for any $\epsilon > 0$, $1/2 \geq e^{-\epsilon/N}$ for $t$ sufficiently large, proves the claim.

\[\square\]

D.

Proof of Lemma 16

Proof. Part 1 is trivial, and the proof of part 2 can be found in [17]. We next prove part 3. We first show that the function $\log \lambda(\alpha)$ is convex.

Lemma 22. Function $\alpha \mapsto \log \lambda(\alpha)$ is convex on $\mathbb{R}$.

Proof. For any $\alpha \in \mathbb{R}$, let $f_t(\alpha)$ be defined as

$$f_t(\alpha) = \frac{1}{t} \log \sum_{s^t} P^\alpha(s^t) e^{-\alpha(1-\alpha)\beta^2(s^t)/2}. \quad (98)$$

Recalling the definition of the matrix $M(\alpha)$, it is easy to see that the sum in (98) equals $\pi^\top M(\alpha)^{t-1} d(\alpha)$, where $d(\alpha) \in \mathbb{R}^N$ is a vector whose $i$-th component equals $e^{-\alpha(1-\alpha)\beta_i^2/2}$. Using the fact that $\pi, d(\alpha) > 0$, we obtain by Gelfand’s formula (see Theorem 8.5.1 in [24]):

$$\lim_{t \to +\infty} f_t(\alpha) = \log \lambda(\alpha). \quad (99)$$

Computing the first order derivative of $f_t$, we obtain

$$\frac{d}{d\alpha} f_t(\alpha) = \frac{1}{t} \sum_{s^t} P^\alpha(s^t) e^{-\alpha(1-\alpha)\beta^2(s^t)/2} \left( \log P(s^t) + (2\alpha - 1) \frac{\beta^2(s^t)}{2} \right). \quad (100)$$
To ease the derivations, for any $s^t$, denote $h_s^t(\alpha) = P^\alpha(s^t)e^{-\alpha(1-\alpha)\frac{\beta^2(s^t)}{2}}$ and $g_s^t(\alpha) = \log P(s^t) + (2\alpha - 1)\frac{\beta^2(s^t)}{2}$, and note that $\frac{d}{d\alpha} h_s^t(\alpha) = h_s^t(\alpha) g_s^t(\alpha)$. Then, it can be shown that the second order derivative of $f_t$ is given by

$$
\frac{d^2}{d\alpha^2} f_t(\alpha) = \frac{1}{t} \left( \sum_{s^t} h_s^t(\alpha) \right) \left( \sum_{s^t} h_s^t(\alpha) g_s^t(\alpha) \frac{\beta^2(s^t)}{2} + \frac{d}{d\alpha} h_s^t(\alpha) \right) - \frac{1}{t} \left( \sum_{s^t} h_s^t(\alpha) \right) \left( \sum_{s^t} h_s^t(\alpha) g_s^t(\alpha) \frac{\beta^2(s^t)}{2} \right).
$$

The first summand in the preceding equation is positive due to the positivity of $h_s^t(\alpha)$ and of $\frac{\beta^2(s^t)}{2}$. The second summand is non-negative by Cauchy-Schwartz inequality, applied to vectors $\{\sqrt{h_s^t}\}_{s^t \in S^t}$ and $\{\sqrt{h_s^t} \cdot g_s^t\}_{s^t \in S^t}$. Therefore, we have that, for each $t$, $\frac{d^2}{d\alpha^2} f_t(\alpha) \geq 0$, thus each $f_t$ is convex. Since convexity is preserved under passage to the limit, the claim follows.

For any $\alpha \in \mathbb{R}$, let $g_t(\alpha)$ be defined as

$$
g_t(\alpha) = \frac{1}{t} \log \sum_{s^t} P^\alpha(s^t)e^{-\alpha(1-\alpha)\frac{\beta^2(s^t)}{2}} v_{s^t}(\alpha). \quad (101)
$$

It is easy to see that the sum in (101) equals $\pi^\top M(\alpha)^{t-1} v(\alpha) = \lambda(\alpha)^{t-1} \pi^\top v(\alpha)$. Thus

$$
g_t(\alpha) = \frac{1}{t} \log \pi^\top v(\alpha) + \frac{t-1}{t} \log \lambda(\alpha),
$$

which implies the following pointwise limit, on $\alpha \in \mathbb{R}$:

$$
\lim_{t \to +\infty} g_t(\alpha) = \log \lambda(\alpha). \quad (103)
$$

We next compute the first order derivative of $g_t$,

$$
\frac{d}{d\alpha} g_t(\alpha) = \frac{1}{t} \sum_{s^t} P^\alpha(s^t)e^{-\alpha(1-\alpha)\frac{\beta^2(s^t)}{2}} \left( \left( \log P(s^t) + (2\alpha - 1)\frac{\beta^2(s^t)}{2} \right) v_{s^t}(\alpha) + \frac{d}{d\alpha} v_{s^t}(\alpha) \right) - \frac{(2\alpha - 1)}{t} \sum_{s^t} P^\alpha(s^t)e^{-\alpha(1-\alpha)\frac{\beta^2(s^t)}{2}} \frac{\beta^2(s^t)}{2} v_{s^t}(\alpha) + \frac{1}{t} \sum_{s^t} P^\alpha(s^t)e^{-\alpha(1-\alpha)\frac{\beta^2(s^t)}{2}} \frac{d}{d\alpha} v_{s^t}(\alpha).
$$

We show that the sequence $\frac{d}{d\alpha} g_t(\alpha)$ is pointwise convergent on $\mathbb{R}$. We prove this by separately proving pointwise convergence for each of the three summation terms in (104). The last term converges to zero as $t \to +\infty$. To see this, note that:

$$
-\lambda(\alpha)^t v_{s^0}(\alpha) \delta \leq \sum_{s^t} P^\alpha(s^t) e^{-\alpha(1-\alpha)\frac{\beta^2(s^t)}{2}} \frac{d}{d\alpha} v_{s^t}(\alpha) \leq \lambda(\alpha)^t v_{s^0}(\alpha) \delta, \quad (105)
$$
where $\delta = \max_{i=1,\ldots,N} \frac{1}{v_i(\alpha)} \left| \frac{d}{d\alpha} v_i(\alpha) \right|$. 

Recall now equation (73) and let $Q$ denote the respective matrix that defines the solution $\theta$, i.e., for any $\alpha \in \mathbb{R}$ and any $i, j = 1, \ldots, N$, let:

$$
\left[ Q(\alpha) \right]_{ij} = \frac{P_{ij} e^{-\alpha(1-\alpha)\frac{\theta_j}{2}} v_j(\alpha)}{\lambda(\alpha) v_i(\alpha)}.
$$

(106)

We observe that, for any $\alpha \in \mathbb{R}$, $Q(\alpha)$ respects the sparsity pattern of matrix $P$. Thus, for any $\alpha$, $Q(\alpha)$ is irreducible and aperiodic, and therefore has a unique stationary distribution; to be consistent with (73), we denote the stationary distribution of $Q(\alpha)$ by $\overline{\theta} = \overline{\theta}(\alpha)$, for any $\alpha \in \mathbb{R}$.

For any initial state $s_1$, denote $P \left( s^t | s_1 \right) = P_{s_1 s_2} \cdots P_{s_{t-1} s_t}$, and similarly for $Q \left( s^t | s_1 \right)$. It is easy to verify that, for any $s^t$, $t \geq 2$, the following relation holds between $P \left( s^t | s_1 \right)$ and $Q \left( s^t | s_1 \right)$,

$$
P^\alpha (s^t | s_1) e^{-\alpha(1-\alpha)\frac{\theta_1}{2} v_1(\alpha)} = Q \left( s^t | s_1 \right).
$$

(107)

Considering now the second term in (104), we have for any fixed $s^t$,

$$
P^\alpha (s^t) e^{-\alpha(1-\alpha)\frac{\theta_1}{2} v_1(\alpha)} = \frac{\pi s_1 P^\alpha (s^t | s_1) e^{-\alpha(1-\alpha)\frac{\theta_1}{2} v_1(\alpha)}}{\pi^\alpha v(\alpha)} = \frac{\pi s_1 v_1(\alpha)}{\pi^\alpha v(\alpha)} \lambda(\alpha)^{-1} v s_1(\alpha) = \frac{\pi s_1 v_1(\alpha)}{\pi^\alpha v(\alpha)} \lambda(\alpha)^{-1} v s_1(\alpha) = \frac{\lambda(\alpha)^{-1} v s_1(\alpha)}{\pi^\alpha v(\alpha)}.
$$

(108)

We show that the following limit holds, for any initial state $s_1$:

$$
\lim_{t \to +\infty} \frac{1}{t} \sum_{s^t \setminus s_1} Q \left( s^t | s_1 \right) \frac{\beta^2(s^t-1)}{2} = \sum_{i=1}^N \overline{\theta}_i \frac{\beta^2}{2}.
$$

(109)

By simple algebraic manipulations, we obtain

$$
\sum_{s^t \setminus s_1} Q \left( s^t | s_1 \right) \frac{\beta^2(s^t-1)}{2} = \sum_{s^t \setminus s_1} Q \left( s^t | s_1 \right) \sum_{k=2}^t \frac{\beta^2}{2}.
$$

(110)

where in (110) we use that, for any $s_{k-1}$, $\sum_{s^t \setminus s_{k-1}} Q \left( s^t | s_{k-1} \right) = 1$, and in (111) we use that, for any fixed $s_1$ and $s_{k-1}$, $\sum_{s^t \setminus s_{k-1}} Q \left( s^t | s_{k-1} \right) = Q \left( s_{k-1} | s_1 \right)$. Since $Q$ is stochastic, irreducible and aperiodic, with right Perron vector $\overline{\theta}$, we have $Q^t \to 1 \overline{\theta}^\top$, as $t \to +\infty$. Noting that $Q(s_{k-1} | s_1) = e_{s_1}^\top Q^{k-1} e_{s_{k-1}}$, for any $s_1$ and $s_{k-1}$, the limit (109) follows.
Going back to (104), and combining (108) and (109), we have that the limit of the second term in (104), as $t \to +\infty$, equals
\[
\lim_{t \to +\infty} \frac{(2\alpha - 1)}{t} \sum_{s_1} P^\alpha(s^t)e^{-\alpha(1-\alpha)\frac{\beta^2(s^t-1)}{2}} v_{s_1}(\alpha) \frac{\beta^2(s^t-1)}{2} \lambda(\alpha)^{t-1} \pi^+ v(\alpha) = \lim_{t \to +\infty} \frac{(2\alpha - 1)}{t} \sum_{s_1} \pi_{s_1} v_{s_1}(\alpha) \frac{1}{t} \sum_{s'|s_1} Q(s'|s_1) \frac{\beta^2(s^t-1)}{2} \\
= (2\alpha - 1) \sum_{j=1}^N \pi_j v_j(\alpha) \frac{1}{\pi^+ v(\alpha)} \sum_{i=1}^N \bar{q}_i \frac{\beta^2}{2} = (2\alpha - 1) \sum_{i=1}^N \bar{q}_i \frac{\beta^2}{2},
\]
where the last equality follows from the fact that $\{\pi_j v_j(\alpha) : j = 1, \ldots, N\}$ are convex multipliers.

Finally, we consider the first term in (104). Expanding the term under the logarithm to complete $Q(s'|s_1)$ (see eq (107)), we obtain
\[
P^\alpha(s^t)e^{-\alpha(1-\alpha)\frac{\beta^2(s^t-1)}{2}} v_{s_1}(\alpha) \log P(s^t) \lambda(\alpha)^{t-1} \pi^+ v(\alpha) = \frac{1}{\alpha} \pi_{s_1} v_{s_1}(\alpha) Q(s'|s_1) \log \left( \frac{\pi_{s_1} Q(s'|s_1) \lambda(\alpha)^{t-1} v_{s_1}(\alpha)}{e^{-\alpha(1-\alpha)\frac{\beta^2(s^t-1)}{2}} v_{s_1}(\alpha)} \right) \\
= \frac{\pi_{s_1} v_{s_1}(\alpha)}{\pi^+ v(\alpha)} \left( \frac{1}{\alpha} Q(s'|s_1) \log \pi_{s_1} v_{s_1}(\alpha) \frac{v_{s_1}(\alpha)}{v_{s_1}(\alpha)} + \frac{1}{\alpha} Q(s'|s_1) \log Q(s'|s_1) \right) + \frac{t - 1}{\alpha} Q(s'|s_1) \log \lambda(\alpha) + (1 - \alpha)Q(s'|s_1) \frac{\beta^2(s^t-1)}{2}. \tag{113}
\]
Since $\pi_{s_1} v_{s_1}(\alpha)$ is bounded, it is easy to see that
\[
\lim_{t \to +\infty} \frac{1}{t\alpha} \sum_{s_1} Q(s'|s_1) \log \pi_{s_1} v_{s_1}(\alpha) = 0. \tag{114}
\]
Further, using AEP [16], it can be shown that for any $s_1$,\[
\lim_{t \to +\infty} \frac{1}{t} \sum_{s_1} Q(s'|s_1) \log Q(s'|s_1) = -H(\alpha). \tag{115}
\]
As for the limit corresponding to the last term in (113), we use (109). Summarizing (112), (114), (115), (109) yields that the sequence of first order derivatives of $g_t$ is pointwise convergent with the following limit:
\[
\lim_{t \to +\infty} \frac{d}{d\alpha} g_t(\alpha) = (2\alpha - 1) R(\alpha) - \frac{1}{\alpha} H(\alpha) + \frac{1}{\alpha} \log \lambda(\alpha) + (1 - \alpha) R(\alpha) \tag{116}
\]
\[
= \alpha R(\alpha) - \frac{1}{\alpha} H(\alpha) + \frac{1}{\alpha} \log \lambda(\alpha). \tag{117}
\]
We now recall expression (102), and recall that, from Lemma 22 we know that \( \lambda \) is differentiable and that each component of \( v \) is differentiable. From (102) it is easy then to show that the sequence \( \frac{d}{d\alpha} g_t(\alpha) \) is uniformly differentiable on \( \alpha \in [0, 1] \). By Theorem 7.17 from [26], we have that, for each \( \alpha \in [0, 1] \),

\[
\lim_{t \to +\infty} \frac{d}{d\alpha} g_t(\alpha) = \frac{d}{d\alpha} \lim_{t \to +\infty} g_t(\alpha) = \frac{d}{d\alpha} \log(\lambda(\alpha)).
\]

(118)

Multiplying with \( \alpha \) in (116), and rearranging the terms, we get

\[
H(\alpha) - \alpha^2 R(\alpha) = \log(\lambda(\alpha)) - \alpha \frac{d}{d\alpha} \log(\lambda(\alpha)).
\]

Computing now the first order derivative on both sides yields

\[
\frac{d}{d\alpha} \left( H(\alpha) - \alpha^2 R(\alpha) \right) = -\frac{d^2}{d\alpha^2} \log(\lambda(\alpha)).
\]

By Lemma 22 we know that \( \log(\lambda(\alpha)) \) is convex, implying that the right hand side of the preceding equation is negative. This completes the proof of part 3.

\[\blacksquare\]

Proof of Lemma 18

Proof. We prove that \( f \) is convex by showing that its Hessian is a positive semi-definite matrix at every point \( x \in \mathbb{R}^d \). It is easy to see that the gradient of \( f \) at \( x \), \( \nabla f(x) \), is given by

\[
\nabla f(x) = -\frac{1}{2} \frac{\sqrt{h(x)}}{\sqrt{g(x)}} \nabla g(x) - \frac{1}{2} \frac{\sqrt{g(x)}}{\sqrt{h(x)}} \nabla h(x),
\]

(119)

where \( \nabla g(x) \) and \( \nabla h(x) \) denote the gradients of \( g \) and \( h \), respectively, at \( x \). Further, it can be shown that the Hessian of \( f \) at \( x \), \( \nabla^2 f(x) \), equals

\[
\nabla^2 f(x) = -\frac{1}{2} \frac{\sqrt{h(x)}}{\sqrt{g(x)}} \nabla^2 g(x) - \frac{1}{2} \frac{\sqrt{g(x)}}{\sqrt{h(x)}} \nabla^2 h(x)
\]

(120)

\[
- \frac{1}{2} \nabla g(x) \frac{\frac{1}{2} \frac{\sqrt{g(x)}}{\sqrt{h(x)}} \nabla^\top h(x)}{g(x)} \frac{1}{2} \frac{\sqrt{h(x)}}{\sqrt{g(x)}} \nabla^\top g(x)
\]

\[
- \frac{1}{2} \nabla h(x) \frac{\frac{1}{2} \frac{\sqrt{h(x)}}{\sqrt{g(x)}} \nabla^\top g(x)}{h(x)} \frac{1}{2} \frac{\sqrt{g(x)}}{\sqrt{h(x)}} \nabla^\top h(x).
\]

(121)
Since \(g\) and \(h\) are concave, the first two terms in (120) are positive semi-definite matrices. Rearranging the remaining two terms, we obtain that the sum of the last two terms in (120) equals

\[
\frac{1}{4} \frac{1}{g^2(x)h^2(x)} \left( h^2(x) \nabla g(x) \nabla g(x)^\top - g(x) h(x) \nabla g(x) \nabla h(x)^\top \right. \\
+ g^2(x) \nabla h(x) \nabla h(x)^\top - g(x) h(x) \nabla h(x) \nabla h(x)^\top \left. \right) \\
= \frac{1}{4} \frac{1}{g^2(x)h^2(x)} (h(x) \nabla g(x) - g(x) \nabla h(x)) (h(x) \nabla g(x) - g(x) \nabla h(x))^\top ,
\]

which is also a positive semi-definite matrix. Since \(x\) was arbitrary, this proves that \(f\) is convex. 

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