Quantum cosmic models and thermodynamics

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Received 22 April 2008, in final form 10 July 2008
Published 19 August 2008
Online at stacks.iop.org/CQG/25/175023

Abstract
The current accelerating phase of the evolution of the universe is considered by
constructing the most economical cosmic models that use just general relativity
and some dominating quantum effects associated with the probabilistic
description of quantum physics. Two such models are explicitly analyzed.
They are based on the existence of a sub-quantum potential and correspond to
a generalization of the spatially flat exponential model of de Sitter space. The
thermodynamics of these two cosmic solutions is discussed, using the second
principle as a guide to choose which among the two is more feasible. The
paper also discusses the relativistic physics on which the models are based,
their holographic description, some implications from the classical energy
conditions and an interpretation of dark energy in terms of the entangled energy
of the universe.

PACS numbers: 95.36.+x, 98.80.−k

1. Introduction
The last few years have seen an influx of ideas and developments conceived to try to find
a cosmic model that is able to consistently predict the observational data that each time
more clearly implies that the current universe is accelerating (see [1] for a recent review).
Nevertheless, none of these models can be shown to simultaneously satisfy the following
two requirements: (1) exactly predicting what observational data point out and (2) an
economic principle according to which one should not include unnecessary ingredients such
as mysterious cosmic fluids or fields nor modifications of the very-well-tested background
theories such as general relativity. The use of scalar fields in quintessence or k-essence
scenarios is notwithstandingly quite similar to including an inflaton in inflationary theories
for the early universe [2]. Even though, owing to the success of the inflationary paradigm
which actually shares its main characteristics with those of the present universal acceleration,
many could take this similarity to be a reason enough to justify the presence of a scalar field
also pervading the current universe; it could well be that a cosmic Occam’s razor principle would turn out to be over and above the nice coincidence between predictions of usual models for inflation and what has been found in cosmic observations such as the measurement of background anisotropies. After all, the medieval opinion that the simplest explanation must be the correct explanation has proved to be extremely fruitful so far, and on the other hand, the paradigm of inflation by itself still raises some deep criticisms. Occam’s razor is also against the idea of modifying gravity by adding to the relativistic Lagrangian some convenient extra terms.

Besides general relativity, quantum theory is the other building block which can never be ignored while constructing a predicting model for any physical system. Although it is true that a quantum behavior must in general be expected to manifest for small-size systems, cosmology is providing us with situations where the opposite really holds. In fact, fashionable phantom models for the current universe are all characterized by an energy density which increases with time, making, in this way, the curvature larger as the size of the universe becomes greater. In such models, quantum effects should be expected to more clearly manifest at the latest times where the universe becomes largest. Thus, it appears that quantum theory should necessarily be another ingredient in our task to build up an economical theory of current cosmology without contravening the Occam’s razor philosophy.

A cosmological model satisfying all the above requirements has been recently advanced [3]. It was in fact constructed using just a gravitational Hilbert–Einstein action without any extra terms and taking into account the probabilistic quantum effects on the trajectories of the particles but not the dynamical properties of any cosmic field such as quintessence or k-essence. The resulting most interesting cosmic model describes an accelerating universe with an expansion rate that goes beyond that of the de Sitter universe into the phantom regime where the tracked parameter of the universal state equation becomes slightly less than \(-1\), and the future is free from any singularity. Such a model, although still a toy one, will thus describe what can be dubbed a *benigner* phantom universe because, besides being regular along its entire evolution, it does not show the violent instabilities driven by a non-canonical scalar-field kinetic term as by construction the model does not have a negative kinetic term nor does it classically violate the dominant energy condition which guarantees the stability of the theory, contrary to what the customary phantom models do. Another cosmic model was also obtained which describes an initially accelerating universe with equation of state parameter always greater than \(-1\), that eventually becomes decelerating for a while, to finally contract down to a vanishing size asymptotically at infinity. The latter model seems to be less adjustable to current observational data although we are not completely sure as this is a toy model.

We know very little about the theoretical nature and origin of dark energy. Therefore, it is worth exploring its thermodynamic properties seeking a deeper understanding, in the hope that this consideration will shed some light on the properties of dark energy and help us understand its rather elusive nature. Actually, some attention has been paid to the subject of thermodynamics of dark energy when this is interpreted as a radiation field [4] and a phantom field [5]. Other authors have also studied a variety of dark energy properties related to thermodynamics [6–10]. Besides reviewing the essentials of it, in this paper, we are going to deal with two fundamental aspects of the benigner phantom scenario. On the one hand, we shall investigate in some detail the basic physics on which it is grounded, and on the other hand, we shall consider some thermodynamical aspects of the benigner phantom scenario, putting special emphasis on general functions, such as entropy, enthalpy as well as temperature, and study the implied holographic description, some consequences from the quantum violation of
the classical conditions on energy and finally the interpretation of the models in terms of the entanglement energy of the accelerating universe.

The paper can be outlined as follows. In section 2, we briefly review the cosmic quantum models, and in section 3, we discuss the thermodynamics that can be associated with such models and its implications in the violation of the classical energy conditions, the cosmic holography and their connection to the notion of entanglement entropy for an accelerating universe. We conclude and add some further comments in section 4. An appendix is added where new material is presented on the consistency of the cosmic quantum models and the quantum aspects that we must include in the theory of special relativity on which such models are based.

2. The quantum cosmic models

In this section, we briefly review the basic ideas and formulae of the cosmic quantum models which were considered in [3]. (For a previous work from which the ideas provided in [3] were derived, see [11].) These models are a quantum extension from the known tachyon dark energy model [12, 13]. The latter scenario is physically grounded on the relativistic Lagrangian for a particle of mass \( m_0 \), i.e.,

\[
L = -m_0 \sqrt{1 - v^2} \quad \text{(with } v = \dot{q} \text{ the particle velocity),}
\]

up-grading the coordinate \( q \) to a scalar field \( \phi \), the squared velocity to \( \partial_i \phi \partial_i \phi = \dot{\phi}^2 \) and the rest mass to the scalar field potential \( V(\phi) \). In order to introduce the cosmic quantum models, we first derive the Lagrangian that corresponds to a particle which is subject to the usual quantum effects. Thus, we apply the Klein–Gordon equation to a general quasi-classical wave function \( \Psi = R(q, t) \exp(iS(q, t)/\hbar) \), and obtain from the resulting real part the expression for the momentum

\[
p = \sqrt{E^2 + \tilde{V}_{SQ}^2 - m_0^2},
\]

where \( E \) is the classical energy and \( \tilde{V}_{SQ} = \hbar \sqrt{(\nabla^2 R - \ddot{R})/R} \) is the sub-quantum potential, so that the Lagrangian becomes

\[
\tilde{L} = \int dq \dot{q} p = -m_0 E \left( \arcsin \sqrt{1 - \dot{\phi}^2}, \sqrt{1 - \frac{\tilde{V}_{SQ}^2}{m_0^2}} \right),
\]

in which \( E(x, k) \) is the elliptic integral of the second kind. Following Bagla et al [12], we then upgrade the quantities entering Lagrangian (2.2) to scalar field quantities in such a way that \( \dot{q}^2 = \dot{\phi}^2 \to \dot{\phi} \dot{\phi} \equiv \dot{\phi}^2 \), and \( m_0 \to \tilde{V}(\phi) \), with \( \tilde{V}(\phi) \) the scalar field potential, and hence we obtain \( \tilde{L} = -\tilde{V}(\phi) E(x(\phi), k(\phi)) \), with \( x(\phi) = \arcsin \sqrt{1 - \dot{\phi}^2} \) and \( k(\phi) = \sqrt{1 - \tilde{V}_{SQ}^2/\tilde{V}(\phi)^2} \). Now, it was shown in [3] that for the model to imply an accelerating universe characterized by an energy density and pressure which depend both on the sub-quantum potential only and vanish (when no cosmological constant is present) in the limit \( \hbar \to 0 \), the above Lagrangian must be expressed as a Lagrangian density to read [3]

\[
L = -\tilde{V}(\phi) [E(x, k) - \sqrt{1 - \dot{\phi}^2}],
\]

where we have subtracted the tachyonic Lagrangian density derived from classical special relativity and \( k \) can be written as \( k = \sqrt{1 - \tilde{V}_{SQ}^2/\tilde{V}(\phi)^2} \), with \( \tilde{V}_{SQ} = \tilde{V}_{SQ}/a^3 \) and \( \tilde{V}(\phi) = \tilde{V}(\phi)/a^3 \) the respective sub-quantum and scalar field potential energy densities, \( a \) being the scale factor of the universe. Lagrangian density (2.3) in fact vanishes in the limit \( \hbar \to 0 \) and from it one can derive the pressure \( p \) and energy density \( \rho \) as

\[
p = L, \quad \rho = \tilde{V}(\phi),
\]

where \( L \) is the classical energy and finally the interpretation of the models in terms of the entanglement energy of the accelerating universe.

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\[ \rho = V(\phi) \left( \sqrt{\phi^2 + \frac{V_{SQ}}{V(\phi)}(1 - \phi^2)} \phi + E(x, k) - \frac{1}{\sqrt{1 - \phi^2}} \right). \] (2.5)

Letting the equation of state parameter \( w = \rho / \rho \) to be time-dependent and using the general expression [12, 13] \( \rho / \rho = -3H(1 + w) = 2\dot{H} / H \) with \( H = \dot{a} / a \), one can obtain [3]

\[ \rho = 6\pi G (\dot{H}^{-1} H V_{SQ})^2, \] (2.6)

\[ p = w(t) \rho = -\left(1 + \frac{2\dot{H}}{3H^2}\right) \rho, \] (2.7)

with

\[ \dot{H} = \pm 4\pi G V_{SQ}, \quad H = H_0 \pm 4\pi G \dot{V}_{SQ} t. \] (2.8)

Regularity requirements for \( \dot{\phi} \) on the equation of motion derived from the Lagrangian density (2.3) leads, by manipulating [3], the Friedmann equations and the above equations, to the condition \( \dot{\phi}^2 = 1 \) and to the simpler expressions

\[ \rho = 6\pi G (\dot{H}^{-1} H V_{SQ})^2, \] (2.9)

\[ p = w(t) \rho = -\left(1 + \frac{2\dot{H}}{3H^2}\right) \rho, \] (2.10)

where

\[ \dot{H} = \pm 4\pi G V_{SQ}, \quad H = H_0 \pm 4\pi G \dot{V}_{SQ} t, \] (2.11)

denote erasing all traces of the scalar field \( \phi \). What remains instead are some constants and a time-dependence which vanishes when \( \hbar \to 0 \), i.e., if we disregarded the integration constant \( H_0 \) (which plays the role of a cosmological constant), only purely quantum effects are left. It is worth remarking that we do not expect the sub-quantum potential \( \dot{V}_{SQ} \) appearing in (2.2) to remain constant along the universal expansion but to increase like the volume \( V = a^3 \) of the universe does, with \( a \) the scale factor. It is the sub-quantum potential density \( V_{SQ} \) appearing in (2.3) that should be expected to remain constant at all cosmic times.

Finally integrating (2.11), we obtain for the scale factor of the universe

\[ a(t) = a_0 e^{H_0 t \pm 2\pi G V_{SQ} t^2}, \] (2.12)

with \( a_0 \) the initial value \( a_0 = a(0) \). From the set of solutions implied by (2.12), we shall disregard from the onset the one corresponding to \( H_0 = 0 \) and \( t = \sqrt{\frac{\ln(a_0/a)}{2\pi G V_{SQ}}} \) (which corresponds to the sign—for the term containing the sub-quantum potential) as it would predict the unphysical case of a universe which necessarily is currently contracting. The chosen solutions are depicted in figure 1 as compared to the usual de Sitter solution. Both of these solutions become flat de Sitter in the classical limit \( \hbar \to 0 \). Besides, we should mention that the de Sitter limit is not exactly identical to the observable universe. By putting matter-energy momentum in the theory, we would then expect different features from the ones found in the de Sitter universe, such as instability, or that the exclusion limit could vary with matter inside, at least during the current epoch of mixture of matter and dark energy. Some \( w > -1 \) branches could survive or some \( w < -1 \) branches could be excluded. One can thus draw the conclusion that pure quantum probability effects on the particles filling the universe make, by themselves, the universe accelerate quicker or slower than what is predicted by a
cosmological constant, but do not induce a future big rip singularity in any case. In the next section we shall see that it is the phantom regime \((w < -1)\) predicted by the solution with the + sign that agrees with the thermodynamic second law and therefore gives rise to what we can name a benigner phantom regime that is free from singularities or unphysical negative kinetic terms in the Lagrangian. We should expect that the inclusion of a very little proportion of matter would not change the above conclusion. In section 4, it will also be shown that these quantum effects can be interpreted as a cosmic entanglement energy.

The reader who may be interested in a discussion on further aspects that re-enforce the consistency of the models considered above and on the quantum modifications that such a description entails in the background relativistic theory is addressed to the appendix.

3. Benigner phantom thermodynamics

3.1. Thermodynamics

The thermodynamical description of dark energy has offered an alternative route to investigate the evolution of the current universe [5–10]. However, whereas well-defined expressions can be obtained for dark energy models with equations of state \(p = w \rho\), where \(w > -1\); in the phantom regime characterized by \(w < -1\), either the temperature or the entropy must be definite negative. In what follows, we shall discuss the thermodynamical properties of the benigner cosmic models in which it will be seen that these problems are alleviated. By using the above equations, we now proceed to derive expressions for the thermodynamical functions.
according to the distinct models implied by the sign ambiguity in (2.12) and the possibility that the cosmological term be zero or not, only for the solution branches that correspond to a positive time \( t > 0 \). On the one hand, the translational energy that can be associated with the scalar field would be proportional to \( \phi^2 \), and therefore because \( \phi^2 = 1 \) [3], the essentially quantum temperature associated with the sub-quantum models must be generally given by

\[
T_{SQ} = \kappa a^3, \tag{3.1}
\]

with \( \kappa \) as a given positive constant whose value will be determined later. It is worth noting that, unlike for phantom energy models [14], in this case the temperature is definite positive even though the value of the state equation parameter \( w \) is less than \(-1\). Moreover, this temperature is an increasing function of the scale factor, and hence it will generally increase with time. It must also be stressed that \( T_{SQ} \) must be a quantum temperature as it comes solely from the existence of a sub-quantum potential.

On the other hand, one can define the entropy and the enthalpy. If, since the universe evolves along an irreversible way, following the general thermodynamic description for dark energy [4, 5], one defines the total entropy of the sub-quantum medium as

\[
S_{SQ}(a) = \frac{\rho V}{T_{SQ}},
\]

with \( V = a^3 \), the volume of the universe, then in the case that we choose for the scale factor the simplest expanding solution (without a cosmological constant)

\[
a + a_0 \exp(2\pi G V_{SQ} t),
\]

with \( V_{SQ} \) the sub-quantum potential density, we obtain the increasing, positive quantity

\[
S_{SQ}(a + a_0) = V_{SQ} \kappa \ln \left( \left( \frac{a + a_0}{a_0} \right)^3 \right). \tag{3.2}
\]

This definition of entropy satisfies the second law of thermodynamics.

For the kind of systems we are dealing with, one may always define a quantity which can be interpreted as the total enthalpy of the universe by using the same expression as for entropy, but referred to the internal energy which, in the present case, is given by \( \rho + p \), instead of just \( \rho \). Thus, we can write for the enthalpy

\[
H_{SQ} = \frac{(\rho + p) V}{T_{SQ}}
\]

which leads to the same cosmic solution to the constant, negative definite quantity

\[
H_{SQ}(a) = -\frac{V_{SQ}}{\kappa}, \tag{3.3}
\]

whose negative sign actually implies a quantum violation of the dominant energy condition and indicates that we are in the phantom regime.

The consistency of the above definitions of entropy and enthalpy will be guaranteed in what follows because the expressions that we obtain from them in the limit \( V_{SQ} \to 0 \) are the same as for de Sitter space.

Since the third power of the ratio \( a_+ / a_0 \) must be proportional to the number of states in the whole universe, the mathematical expression of the entropy given by (3.2) could still be interpreted to be just the statistical classical Boltzmann’s formula, provided we take the constant \( V_{SQ}/\kappa \) to play the role of the Boltzmann’s constant \( k_B \), or in other words, \( k_B \) is taken to be given by \( k_B = V_{SQ}/\kappa \), in such a way that the temperature becomes \( T_{SQ}(a_+) = V_{SQ} a_+^3 / k_B \), which consistently vanishes at the classical limit \( h \to 0 \). If we let \( h \to 0 \), then it would be \( T_{SQ}(a_+) \) but not \( S_{SQ}(a_+) \) which vanishes. In this way, (3.3) becomes

\[
H_{SQ}(a_+) = -k_B. \tag{3.4}
\]

The negative value of this enthalpy can be at first sight taken as a proof of an unphysical character. However, one could also interpret \( H_{SQ}(a_+) \) in the same way as Schrödinger did [15] with the so-called ‘negentropy’ as a measure of the information available in the given system, which in the present case is the universe itself.
The above results correspond to the case in which the universe is endowed with a vanishing cosmological constant. If we allow a nonzero cosmological term $H_0$ to exist, i.e., if we first choose the solution $a_\ast = a_0 \exp(H_0 t - 2\pi G V_{SQ}/t^2)$, then we have for the expressions of the entropy and enthalpy that correspond to a universe which, if $H_0 > \sqrt{4\pi G V_{SQ}}$, first expands in an accelerated way with $w > -1$, then expands in a decelerating way to finally progressively contract all the way down until it fades out at an infinite time,

$$S_{SQ}(a_\ast, H_0) = \frac{3H_0^2}{8\pi G \kappa} - \frac{V_{SQ}}{\kappa} \ln \left[ \left( \frac{a_\ast}{a_0} \right)^3 \right], \quad (3.5)$$

and again for this case

$$H_{SQ}(a_\ast, H_0) = \frac{V_{SQ}}{\kappa} = k_B, \quad (3.6)$$

which is now positive definite.

Equation (3.5) contains two different terms. The first term, $S_{ds} = 3H_0^2 k_B / (8\pi G V_{SQ})$, corresponds to a de Sitter quantum entropy which diverges in the classical limit $h \to 0$. The second one is the same as the statistical-mechanic entropy in (3.2), but with the sign reversed. It would be worth comparing the first entropy term with the Hawking formula for de Sitter spacetime which is given by the horizon area in Planck units, $S_H \propto H_0^{-2} k_B / (\ell^2_P)$ [16]. At first sight, the entropy term $S_{ds}$ appears to be proportional to just the inverse of the Hawking’s formula. However, one can re-write $S_{ds}$ as $S_{ds} = k_B / (2GH_0 \bar{V}_{SQ})$, where $\bar{V}_{SQ} = V_{SQ} V_{ds}$, with $V_{ds}$ the equivalent volume occupied by de Sitter spacetime with horizon at $r = H_0^{-1}$. Now, $\bar{V}_{SQ}$ is the amount of sub-quantum energy contained in that equivalent de Sitter volume, so that we must have $\bar{V}_{SQ} = hH_0$. It follows that $S_{ds}$ actually becomes as given by the horizon area in Planck units too. It is worth noticing that the temperature $T_{SQ}(a_\ast, H_0)$ can similarly be decomposed into two parts: one of which is given by the Gibbons–Hawking expression [16] $hH_0/k_B$, and the other corresponds to the negative volume deficit that the factor $\exp(-2\pi G V_{SQ}/t^2)$ introduces in the de Sitter spacetime volume.

We also note that for this kind of solution a universe with $T_{SQ}(a_\ast, H_0) = V_{SQ}a_0^3 / k_B$ and $S_{SQ}(a_\ast, H_0) = S_{ds}$ is left when we set $t = 0$. If we let $h \to 0$, then $T_{SQ}(a_\ast, H_0) \to 0$ and $S_{SQ}(a_\ast, H_0) \to \infty$. On the other hand, it follows from (3.5) that, as the universe evolves from the initial size $a_0$, the initially positive entropy $S_{SQ}(a_\ast, H_0)$ progressively decreases until it vanishes at a time $t = t_\ast = H_0 / (4\pi G V_{SQ})$, after which the entropy becomes negative. This would mean a violation of the second law of thermodynamics even on the current evolution of the universe which is induced by quantum effects. Therefore, the model that corresponds to (3.5) and (3.6) appears to be prevented by the second law.

Finally, we consider the remaining solution $a_\ast = a_0 \exp(H_0 t + 2\pi G V_{SQ}/t^2)$ which predicts a universe expanding in a super-accelerated fashion all the time up to infinity with $w < -1$. In this case, we obtain

$$S_{SQ}(a_\ast, H_0) = \frac{3H_0^2}{8\pi G \kappa} + \frac{V_{SQ}}{\kappa} \ln \left[ \left( \frac{a_\ast}{a_0} \right)^3 \right], \quad (3.7)$$

with $3H_0^2 / (8\pi G \kappa) = 3H_0^2 k_B / (8\pi G V_{SQ}) \propto S_H$, and

$$H_{SQ}(a_\ast, H_0) = -\frac{V_{SQ}}{\kappa} = -k_B. \quad (3.8)$$

All the above discussion on the relation of the sub-quantum thermodynamical functions with the Hawking temperature and entropy holds also in this case, with the sole difference that now $S_{SQ}(a_\ast, H_0)$ and $T_{SQ}(a_\ast, H_0)$ are larger than their corresponding Hawking counterparts. Again, for this solution, a universe with $T_{SQ}(a_\ast, H_0) = \kappa a_0^3$ and $S_{SQ}(a_\ast, H_0) = S_{ds}$ is left
when we set \( t = 0 \), whereas \( T_{\text{SQ}}(a_+, H_0) \rightarrow 0 \) and \( S_{\text{SQ}}(a_+, H_0) \rightarrow \infty \) in the classical limit \( \hbar \rightarrow 0 \). Moreover, such as it happens when \( H_0 = 0 \), here there is no violation of the second law for \( S_{\text{SQ}}(a_+, H_0) \), but \( H_{\text{SQ}}(a_+, H_0) \) is again a negative constant interpretable like a negative entropy that would mark the onset of existing structures in the universe which are capable to store and process information [15].

In any case, we have shown that the thermodynamical laws derived in this study appear to preclude any model with \( w > -1 \) and so leave only a kind of phantom universe with \( w < -1 \) as the only possible cosmological alternative compatible with such laws. That kind of model does not show however the sort of shortcomings, including instabilities, negative kinetic field terms or the future singularities named big rips, that the usual phantom models have [17]. Since we have dealt with an essentially quantum system, the violation of the dominant energy condition that leads to the negative values of the enthalpy \( H_{\text{SQ}} \) in the thermodynamically allowed models appears to be a rather benign problem from which one could even get some interpretational advantages. In fact, from (2.9)–(2.11), we notice that the violation of the dominant energy condition (DEC)

\[
\rho + p = -V_{\text{SQ}}
\]

has an essentially quantum nature so that such a violation vanishes in the classical limit where \( \hbar \rightarrow 0 \). In fact, it is currently believed that, even though classical general relativity cannot be accommodated for a violation of the dominant energy condition [18], such a violation can be admitted quantum mechanically, at least temporarily. Moreover, since the violating term \(-V_{\text{SQ}}\) is directly related to the negentropy \( H_{\text{SQ}} = -k_B \), it is really tempting to establish a link between that violation and the emergence of life in the universe. After all, one cannot forget that if living beings are fed on with negative entropy [15], then we ought to initially have some amount of negentropy to make the very emergence of life a more natural process which by itself satisfies the second law.

3.2. Violation of classical DEC

Thus, the quantum violation of the dominant energy condition has no classical counterpart, and therefore is physically allowable. We shall investigate in what follows the sense in which that violation would permit the formation of Lorentzian wormholes. Choosing the simplest mixed energy–momentum tensor components and the ansatz that correspond to a static, spherically symmetric wormhole spacetime with vanishing shift function, \( ds^2 = -dt^2 + e^{2\lambda} dr^2 + r^2 d\Omega_2^2 \) (where \( d\Omega_2^2 \) is the metric on the unit two-sphere), we can obtain a wormhole spacetime solution from the corresponding Einstein equations containing the extra sub-quantum energy density and pressure, that is,

\[
-\frac{\lambda'}{r} e^{-\lambda} - \frac{1}{r^2} (e^{-\lambda} - 1) = -\frac{8\pi G}{3} \left( \frac{9r_0^2}{8\pi G r^4} + \rho \right)
\]

\[
-\frac{1}{r^2} (e^{-\lambda} - 1) = \frac{8\pi G}{3} \left( \frac{3r_0^2}{8\pi G r^4} + p \right)
\]

\[
-\frac{1}{2} e^{-\lambda} \frac{\lambda'}{r} = \frac{8\pi G}{3} \left( \frac{3r_0^2}{8\pi G r^4} + p \right),
\]

supplemented by the condition \( \rho + p = -V_{\text{SQ}} \) to obtain

\[
ds^2 = -dt^2 + \frac{dr^2}{1 - \frac{r_0^2}{r^2} + \frac{\ell^2}{r^2} V_{\text{SQ}} r^2} + r^2 d\Omega_2^2,
\]

\[
(3.10)
\]
with $r_0$ the radius of the spherical wormhole throat and $\ell_P$ the Planck length. Note that if $\rho + p$ was positive, then no cosmic wormhole could be obtained, such as it happens for the de Sitter space. Metric (3.10) is by itself nevertheless an actual cosmic wormhole because if that metric is written as

$$\text{d}s^2 = -\text{d}t^2 + \ell_P^2 + r^2 \text{d}\Omega_2^2,$$

then the new parameter $[19]$

$$\ell = \pm \int_{r_0}^r \frac{r'}{r^2 - r_0^2 + \ell_P^2 V_{SQ} r^4} \text{d}r'$$

$$= \pm \frac{1}{2\ell_P \sqrt{V_{SQ}}} \ln \left( \frac{2\ell_P \sqrt{V_{SQ}} \sqrt{r^2 - r_0^2 + \ell_P^2 V_{SQ} r^4} + 2\ell_P^2 V_{SQ} r^2 + 1}{1 + 4\ell_P^2 V_{SQ} r_0^2} \right) \quad (3.12)$$

goes from $-\infty$ (when $r = +\infty$) to zero (at $r = r_0$) and finally to $+\infty$ (when $r = \infty$ again), such as it is expected for a wormhole with a throat at $r = r_0$ which is traversable and can be converted into a time machine. It can be readily checked that for $\rho + p > 0$, there is no metric like (3.12) which can show these properties.

### 3.3. Holographic models

Holographic models which are related with the entropy of a dark energy universe have been extensively considered [20, 21]. We shall discuss now the main equation that would govern the holographic model for the quantum cosmic scenario. If we try to adjust that model to the Li’s holographic description for dark energy [20], then we have to define the holographic sub-quantum model by the relation

$$H^2 = \frac{8\pi G \rho}{3} = 4\pi G V_{SQ} \mu(t)^2 \ln (8G V_{SQ} R_h^2),$$

(3.13)

where the future event horizon $R_h = a(t) \int_{t_0}^{\infty} \text{d}t'/a(t')$ is given by

$$R_h = \frac{e^{\Phi(x)}}{\sqrt{8G V_{SQ}}} \left[ 1 - \Phi(x) \right],$$

(3.14)

with $\Phi(x)$ the probability integral [22],

$$x = \frac{H_0}{\sqrt{8\pi G V_{SQ}}} + \sqrt{2\pi G V_{SQ} t},$$

(3.15)

and

$$\mu(t)^2 = \frac{1}{1 + 3(1 + w(t)) \ln \left[ 1 - \Phi \left( -\frac{1}{1 + w(t)} \right) \right]}.$$

(3.16)

Note that: (1) $R_h \to \infty$ as $t \to \infty$ or $V_{SQ} \to 0$, (2) in the latter limit, $H^2 \to 0$, (3) $\mu(t)^2$ is no longer a constant because we are dealing with a tracking model where the parameter $w$ depends on time, and (4) the holographic model has no problems posed by the usual holographic phantom energy models. However, this formulation does not satisfy the general holographic equation originally introduced by Li which reads [20] $\rho \propto H^2 \propto c^2/R^2$ (where $R$ is the proper radius of the holographic surface and $c$ is a parameter of order unity that depends on $w$ according to the relation $w = -(1+2/c)/3$), and therefore seems not satisfactory enough.

A better and quite simpler holographic description which comes from saturating the original bound on entropy [23] and conforms to the general holographic equation stems directly from...
the very definitions of the energy density (2.9) and the entropy (3.7). Such a definition would read
\[ \rho = \kappa S_{5Q}(a_+, H_0) = \frac{3H^2}{8\pi G} = \frac{3}{8\pi G R_H^2}. \] (3.17)
It appears that if the last equality in (3.18) holds, then the holographic screen is related to the Hubble horizon rather than the future event horizon or particle horizon. In order to confirm that identification, we derive now the vacuum metric that can be associated to our ever-accelerating cosmic quantum model with the ansatz \( ds^2 = -e^{\nu} dt^2 + e^{\lambda} dr^2 + r^2 d\Omega_2^2 \). For an equation of state \( p = w\rho \), the Einstein equations then are
\[ e^{-\lambda} \left( \frac{\lambda'}{r} - \frac{1}{r^2} \right) + \frac{1}{2r^2} = 8\pi G \rho, \] (3.18)
\[ e^{-\lambda} \left( \frac{\nu'}{r} + \frac{1}{r^2} \right) - \frac{1}{2r^2} = 8\pi G w\rho. \] (3.19)
We finally get the non-static metric
\[ ds^2 = -(1 - H^2 r^2)^{(1+3w)/2} dt^2 + \frac{dr^2}{1 - H^2 r^2} + r^2 d\Omega_2^2, \] (3.20)
which consistently reduces to the de Sitter static metric for \( w = -1 \). It follows that there exists a time-dependent apparent horizon at \( r = H^{-1} \) playing in fact the role of a Hubble horizon, like in the de Sitter case.

This holographic model has several advantages over the previous Li model [20] and other models [21], including its: naturalness (it has been many times stressed that choosing the Hubble horizon is more natural than using, for the sake of mathematical consistency, particle or future event horizons), simplicity (no \textit{ad hoc} assumption has been made), implication of an IR cutoff depending on time, formal equivalence with Barrow’s hyper inflationary model [24] (but here respecting the thermodynamical second law as, in this case, \( S_{5Q}(a_+, H_0) \) increases with time) and allowance of a unification between the present model and that for dark energy from vacuum entanglement [25].

3.4. Quantum cosmic models and entanglement entropy

The latter property deserves some further comments. In fact, if we interpret \( a^3 V_{5Q} \) as the total entanglement energy of the universe, due to the additiveness of the entanglement entropy, one can then add up [25] the contributions from all existing individual fields in the observable universe so that the entropy of entanglement \( S_{\text{Ent}} = \beta R_H^2 \) (see comment after (3.8)), with \( \beta \) a constant including the spin degrees of freedom of quantum fields in the observable volume of radius \( R_H \) and a numerical constant of order unity. On the other hand, the presence of a boundary at the horizon leads us to infer that the entanglement energy ought to be proportional to the radius of the associated spherical volume, i.e., \( E_{\text{Ent}} = a R_H \) [25], with \( a \) a given constant. We have then,
\[ E_{\text{Ent}} = a^3 V_{5Q} = a R_H, \] (3.21)
\[ S_{\text{Ent}} = \beta R_H^2. \] (3.22)
It is worth noticing that one can then interpret the used temperature as the entanglement temperature so that \( E_{\text{Ent}} = k_B T(a_+) \). Now, integrating over \( R_H \) the expression for \( dE_{\text{Ent}} \) derived by Lee et al [25] from the saturated black hole energy bound [26],
\[ dE_{\text{Ent}} = T_{\text{Ent}} dS_{\text{Ent}}, \] (3.23)
where $T_{\text{Ent}} = (2\pi R_H)^{-1}$ is the Gibbons–Hawking temperature, we consistently recover expression (3.22) for $\alpha = \beta/\pi$. This result is also consistent with the holographic expression introduced before. It follows therefore that the quantum cosmic holographic model considered in the present paper can be consistently interpreted as an entangled dark energy holographic model similar to the one discussed in [25], with the sub-quantum potential $V_{SQ}$ playing the role of the entanglement energy density.

Before closing up this section, it would be worth mentioning that the recent data [27] seem to point to a value $w < -1$, with $\dot{w}$ small and positive, just the result predicted in the present letter. We in fact note that from (2.7) we obtain that $\dot{w} = 4H^2/(3H^3) \propto t^{-3}$ at sufficiently large time.

4. Conclusions and comments

This paper deals with two new four-dimensional cosmological models describing an accelerating universe in the spatially flat case. The ingredients used for constructing these solutions are minimal as they only specify a cosmic relativistic field described by just Hilbert–Einstein gravity and the probabilistic quantum effects associated with particles in the universe. While one of the models is ruled out on general thermodynamical grounds as being unphysical, the other model corresponds to an equation of state $p = w\rho$ with parameter $w < -1$ for its entire evolution, that is to say, this solution is associated with the so-called phantom sector, showing however a future evolution of the universe which is free from most of the problems confronted by usual phantom scenarios, namely, violent instabilities, future singularities and classical violations of energy conditions. We have shown furthermore that the considered phantom model implies a more consistent cosmic holographic description and the equivalence between the discussed models and the entangled dark energy model of the universe. Therefore, we name our phantom model a benign phantom model.

Indeed, if the ultimate cause for the current speeding up of the universe is quantum entanglement associated with its matter and radiation contents, then one would expect that the very existence of the current universe implied violation of the Bell’s inequalities, and hence the quantum probabilistic description related to the sub-quantum potential considered in this work, or the collapse of the superposed cosmic quantum state into the universe we are able to observe, or its associated complementarity between cosmological and microscopic laws, any other aspects that may characterize a quantum system. The current dominance of quantum repulsion over attractive gravity started at a given coincidence time would then mark the onset of a new quantum region along the cosmic evolution, other than that prevailing at the big bang and early primeval universe, this time referring to the quite macroscopic, apparently classical, large universe which we live in. Thus, quite the contrary to what is usually believed, quantum physics does not just govern the microscopic aspects of nature but also the most macroscopic domain of it in such a way that we can say that current life is forming part and is a consequence of a true quantum system.

Observational data are being accumulated, which each time more accurately point to an equation of state for the current universe which corresponds to a parameter whose value is very close to that of the case of a cosmological constant, but still being less than $-1$ [27]. It appears that one of the models considered in this paper would adjust perfectly to such a requirement, while it does not show any of the shortcomings that the customary phantom or modified-gravity scenarios now at hand actually have. Therefore, one is tempted to call for more developments to be made on such benign cosmicosmological model, aiming at trying to construct a final scenario which would consistently describe the current universe and could
presumably shed some light on what really happened during the primordial inflationary period as well.

Acknowledgments

This work was supported by MEC under Research Project No. FIS2005-01181. ARF acknowledges support from MEC FPU grant No. AP2004-6979. The authors benefited from discussions with C Sigüenza and G Readman.

Appendix A

In this appendix, we shall consider new fundamental aspects that strengthen the consistency and provide further physical motivation to the general model reviewed in section 2. These new aspects concern both the use of a sub-quantum potential model derived from the application of the Klein–Gordon equation and the background relativistic theory associated with the cosmic quantum models.

A.1. The Klein–Gordon sub-quantum model

We note here that, although for some time in the past it was generally believed that the Klein–Gordon equation was unobtainable from the Bohm formalism [28], in recent years the Klein–Gordon equation has found satisfactory causal formulations. The solution presented in [29] by Horton et al has to introduce the causal description of time-like flows in an Einstein–Riemann space (otherwise the probability current can assume negative values of its zeroth component and is not generally time-like). However, there exists a causal Klein-Gordon theory in Minkowski space [30] where this is achieved by introducing a cosmological constant as an additional assumption which is justified in view of recent observations. Therefore, it makes perfect sense to use a Klein–Gordon equation in our model [3]. Moreover, the nonclassical character of the current whose continuity equation is derived from the purely imaginary part of the expression resulting from the application of the Klein–Gordon equation to the wavefunction is guaranteed by the fact that one can never obtain the classical limit by making $\hbar \to 0$. Thus, no classical verdict concerning the current of the kind pointed out by Holland [28] can be established. On the other hand, having a material object whose trajectory escapes out the light cone [28] cannot be used as an argument in favour of the physical unacceptability of the model. Quite the contrary, it expresses its actual essentially quantum content, much as the quite fashionable entangled states of sharp quantum theory seemed at first sight to violate special relativity, and then turned out to be universally accepted. In both cases, physics is preserved because we are not dealing with real signaling. Actually, in section 3, we have shown that our cosmic models can also be interpreted as having originated from the entanglement energy of the whole universe, without invoking any other cause.

Appendix B. Quantum theory of special relativity

Consistent tachyonic theories for dark energy are grounded on special theory of relativity in such a way that all the physics involved at them stems from Einstein relativity. Our cosmic quantum models actually come from a generalization from tachyonic theories for which the corresponding background relativistic description ought to contain the quantum probabilistic footprint. Thus, in order to check their consistency, viability and properly motivate the models reviewed in section 2, one should investigate the characteristics of the quantum relativistic
Actually, there are two ways for defining the action of a free system endowed with a rest mass $m_0$ [31]. The first one is by using the integral expression for the Lagrangian $L = \int p \, dv$, with the momentum $p$ derived from the Hamilton–Jacobi equation, and inserting it in the expression $S = \int_{t_1}^{t_2} L \, dt$. The second procedure stems from the definition $S = \beta \int_{a}^{b} ds$, where $ds$ is the line element and the proportionality constant $\beta = m_0 c$ is obtained by going to the non-relativistic limit. The strategy that we have followed here is to apply the first procedure to derive an integral expression for $S$ in the case of a Hamilton-Jacobi equation containing an extra quantum term and then obtain the expression for $ds$ by comparing the resulting expression for $S$ with that given by the second procedure.

As mentioned above, a Hamilton–Jacobi equation with the quantum extra term can be obtained by applying the Klein–Gordon equation to a quasiclassical wave function $\Psi_1 = R(r, t) \exp(iS(r, t)/\hbar)$ [32], where $R(r, t)$ is the quantum probability amplitude and $S(r, t)$ is the classical action. By the second of the above procedures and $L_Q = -m_0 c^2 E(\phi,k)$, we immediately get for the general spacetime metric

$$ds = E(\phi,k) \, dt,$$

which consistently reduces to the metric of special relativity in the limit $\hbar \to 0$. If we take the above line element as invariant, then we obtain for time dilation

$$dt = \frac{E(k) \, dt_0}{E(\phi,k)},$$

in which $E(k)$ is the complete elliptic integral of the second kind [22].

A key question that arises now is: do the quantum relativistic description and hence our cosmic quantum models satisfy Lorentz invariance? What should be invariant in the present case is the quantity

$$I = ct E \left( \arcsin \sqrt{\frac{c^2 t'^2 - x'^2}{c^2 t^2}}, k \right).$$

If we would choose a given transformation group in terms of hyperbolic or elliptic functions which leaves invariant (such as it happens for Lorentz transformations) the usual relativistic combination $c^2 t^2 - x^2 = c^2 t'^2 - x'^2$, then we obtain

$$I = c Q(t', x') E \left( \arcsin \frac{\sqrt{c^2 t'^2 - x'^2}}{c Q(t', x')}, k \right),$$

where $Q(t', x') \equiv Q(t', x', \Psi)$ is the expression for the transformation of time $t$ in terms of hyperbolic or elliptic functions. It would follow

$$\left( \frac{I}{c Q(t', x')} \right)^{-1} = \frac{\sqrt{c^2 t'^2 - x'^2}}{c Q(t', x')},$$

with $()^{-1}$ denoting the inverted function associated to the elliptic integral of the second kind, generally one of the Jacobian elliptic functions or a given combination of them [22]. Thus, the quantity $I$ can only be invariant under the chosen kind of transformations in the classical limit where $k = 1$. Therefore, a quantum relativity built up in this way would clearly violate Lorentz invariance, at least if we take usual classical values for the coordinates.

In order to obtain the wanted transformation equations, we first notice that if we take the coordinate transformation formulae in terms of the usual hyperbolic or some elliptic functions
of the rotation angle $\Phi$, one can always re-express the invariant quantity $I$ of Einstein special relativity in the form

$$I = cQ(t', x')E \left( \arcsin \left( \frac{\sqrt{c^2t'^2 - x'^2}}{cQ(t', x')} \right), k \right). \quad (B.6)$$

From (B.6), one can write

$$\left( \frac{I}{cQ(t', x')} \right)^{-1} = \left( \frac{\sqrt{c^2t'^2 - x'^2}}{cQ(t', x')} \right)^{-1},$$

and hence,

$$I = \sqrt{c^2t'^2 - x'^2} = ct'E \left( \arcsin \left( \frac{\sqrt{c^2t'^2 - x'^2}}{ct'} \right), k \right), \quad (B.7)$$
i.e., $I$ would in fact have the form of the Einstein relativistic invariant. If we interpret the coordinates entering (B.7) as quantum-mechanical coordinates, then our quantum expression for the invariant $I$ given by (B.3) can be directly obtained from the last equality by making the replacement

$$\sqrt{1 - \frac{x^2}{c^2t^2}} = E \left( \arcsin \sqrt{1 - \frac{x^2}{c^2t^2}}, k \right) \quad (B.8)$$

or

$$\left( \sqrt{1 - \frac{x^2}{c^2t^2}} \right)^{-1} = \sqrt{1 - \frac{x^2}{c^2t^2}} \quad (B.9)$$

where the notation $()^{-1}$ again means inverted function of the elliptic integral of the second kind, and if the coordinates entering the right-hand side are taken to be classical coordinates, then those on the left-hand side must still be considered to be quantum-mechanical coordinates. Classical coordinates are those coordinates used in Einstein special relativity and set the occurrence of a classical physical event in that theory. By quantum coordinates, we mean those coordinates which are subject to quantum probabilistic uncertainties and would define what one may call a quantum physical event, i.e., that event which is quantum mechanically spread throughout the whole existing spacetime with a given probability distribution fixed by the boundaries specifying the extent and physical content of the system.

In what follows, we will always express all equations in terms of classical coordinates, and therefore, for the sake of simplicity, we shall omit the subscript ‘clas’ from them. The equivalence relation given by expressions (B.8) and (B.9) is equally valid for primed and non-primed coordinates and should be ultimately related with the feature that for a given unique time, $t$ or $t'$, the position coordinate, $x$ or $x'$, must be quantum mechanically uncertain. From the equalities (B.8) and (B.9) for primed coordinates, we then get an expression for $I'$ in terms of classical coordinates

$$I' = ct'E \left( \arcsin \frac{\sqrt{c^2t'^2 - x'^2}}{ct'}, k \right), \quad (B.10)$$

which shows the required invariance and in fact becomes the known relativistic result $I' = \sqrt{c^2t'^2 - x'^2}$ in the classical limit $\hbar \to 0$.

From expressions (B.8) and (B.9), we also have

$$1 - \frac{V^2}{c^2} = E(\varphi, k)^2 \to \frac{V}{c} = \sqrt{1 - E(\varphi, k)^2} = \tanh \Phi. \quad (B.11)$$
where $V$ is velocity, $\varphi = \arcsin \sqrt{1 - \frac{v^2}{c^2}}$ and we have specialized to using the usual hyperbolic functions. Whence $\cosh \Phi = 1/E(\varphi, k)$, $\sinh \Phi = \sqrt{1 - E(\varphi, k)^2}/E(\varphi, k)$, and from the customary hyperbolic transformation formulae for coordinates

$$x = x' \cosh \Phi + c t' \sinh \Phi, \quad ct = ct' \cosh \Phi + x' \sinh \Phi,$$

we derive the new quantum relativistic transformation equations

$$x = \frac{x' + ct' \sqrt{1 - E(\varphi, k)^2} E(\varphi, k)}{E(\varphi, k)}, \quad ct = \frac{ct' + x' \sqrt{1 - E(\varphi, k)^2}}{E(\varphi, k)}.$$

Had we started with formulae expressed in terms of the Jacobian elliptic functions [22], such that:

$$\frac{V}{c} = \text{sn}(\Phi, k) = \sqrt{1 - E(\varphi, k)^2},$$

$$x = x' \text{nc}(\Phi, k) + ct' \text{sc}(\Phi, k), \quad ct = ct' \text{nc}(\Phi, k) + x' \text{sc}(\Phi, k),$$

then we would have obtained equations (B.13), so confirming the quantum-mechanical character of the coordinates entering the left-hand side of (B.8) and (B.9). The above derived expressions are not yet the wanted expressions as they still contain an unnecessary element of classicality due to the feature that when using quantum-mechanical coordinates for the derivation of the velocity $V$ setting $x = 0$, the unity on the left-hand side of (B.8) would correspond to the complete elliptic integral of the second kind $E(k)$ [22]. Thus, we finally get for the transformation equations

$$x = \frac{(x' + ct' \sqrt{1 - E(\varphi, k)^2}) E(k)}{E(\varphi, k)}, \quad ct = \frac{(ct' + x' \sqrt{1 - E(\varphi, k)^2}) E(k)}{E(\varphi, k)},$$

which are the wanted final expressions in terms of classical coordinates which in fact reduce to the known Lorentz transformations in the classical limit $\hbar \to 0$. From the formula for time transformation, we in fact get time dilation to be the same as that (B.2) directly obtained from the metric when referring to two events occurring at one and the same point $x'$, i.e.,

$$\Delta t = \frac{E(k) \Delta t_0}{E(\varphi, k)},$$

and from that for space transformation, the formula for length contraction referred to one and the same time $t'$

$$\Delta \ell = \frac{E(\varphi, k) \Delta \ell_0}{E(k)}.$$

In any case, the quantum effects would be expected to be very small, i.e., usually $k$ is generally very close to unity for sufficiently large rest masses of the particles.

For the sake of completeness, we shall derive in what follows the transformation of velocity components. One can also derive from the coordinate transformations (B.16) that if space and time themselves are subject to the quantum-mechanical uncertainties, they should now be given as

$$v_x = \frac{v'_x + c \sqrt{1 - E(\varphi, k)^2}}{1 + \frac{c_t}{c} \sqrt{1 - E(\varphi, k)^2}},$$

$$v_y = \frac{v'_y E(\varphi, k)}{E(k) \left(1 + \frac{c_t}{c} \sqrt{1 - E(\varphi, k)^2}\right)},$$

$$v_z = \frac{v'_z E(\varphi, k)}{E(k) \left(1 + \frac{c_t}{c} \sqrt{1 - E(\varphi, k)^2}\right)}.$$

(B.19)
which reduce once again to the well-known velocity transformation law of Einstein special relativity. Even though they are quantitatively distinct of the latter transformation law, equations (B.19) behave qualitatively in a similar fashion and produce the analogous general velocity addition law as in Einstein special relativity.

We finally turn to the essentials of the relativistic mechanics and find the formulae for momentum and energy that must be satisfied by the cosmic quantum models to be given by

\[ p = \frac{\partial L}{\partial v} = \frac{m_0 c \sqrt{1 - k^2 (1 - \frac{v^2}{c^2})}}{\sqrt{1 - \frac{v^2}{c^2}}}, \]  

\[ E = p v - L = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} \left[ \frac{v}{c} \sqrt{1 - k^2 \left(1 - \frac{v^2}{c^2}\right)} + \sqrt{1 - \frac{v^2}{c^2}} E(\varphi, k) \right]. \]  

(B.20)  

(B.21)

Obviously, these expressions reduce to \( p = \frac{m_0 v}{\sqrt{1 - v^2/c^2}} \) and \( E = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}} \), respectively, in the limit \( \hbar \to 0 \). Moreover, if we set \( v = 0 \), then \( p = V_0/c \) and \( E = m_0 c^2 E(k) \) which become, respectively, 0 and \( m_0 c^2 \) when \( \hbar \to 0 \). It follows then that our quantum special relativistic model has the expected good limiting behavior.

Unless for rather extreme cases, the value of parameter \( k \) is very close to unity and therefore the corrections to the customary expressions induced by the present model should be expected to be very small locally. However, they could be perhaps detectable in specially designed experiments using extremely light particles.

The main conclusion that can be drawn from the above discussion is that whereas Lorentz invariance appears to be violated in our quantum description if classical coordinates are considered, such an invariance is preserved when one uses quantum coordinates in that description.

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17