A new effective-one-body description of coalescing nonprecessing spinning black-hole binaries

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We present a new, tunable effective-one-body (EOB) model of the motion and radiation of coalescing black hole binaries with arbitrary mass ratio and aligned spins. The most novel feature of our formalism is the introduction, and systematic use, of the (gauge-invariant) concept of centrifugal radius $r_c$. In the spinning small mass-ratio limit, the main radial potential expressed in terms of $r_c$ differs very little (and only multiplicatively so) from the usual Schwarzschild potential $1 - 2M/r_c$. This motivates a new, multiplicative way of blending finite-mass-ratio deformations with spin-deformations. In the present exploratory work we consider a minimal version of our spinning EOB model (containing essentially only two adjustable parameters: one in the Hamiltonian and one in the waveform) and calibrate its (dominant mode) waveform against a sample of fifteen equal-mass, equal-spin waveforms produced by the SXS collaboration, and covering the dimensionless spin range $-0.95 \leq \chi \leq +0.98$. The numerical relativity / EOB phasing disagreement remains remarkably small ($\lesssim \pm 0.15$ rad) over the entire spin range.

I. INTRODUCTION

Coalescing black hole binaries are the most promising sources for the network of ground-based, kilometer-size interferometric gravitational wave (GW) detectors that is about to come online at improved sensitivities [1, 2]. It has been pointed out long ago [3] that there is a bias favoring the detection of black hole systems with large, aligned spins. In view of the large parameter space of spinning binaries ($m_1, m_2, S_1, S_2$) and the need to have in hands tens of thousands of waveform templates for detection purposes, it is useful to develop semi-analytic techniques to produce accurate templates. One promising method towards this goal is the analytical effective-one-body (EOB) formalism [4–7]. Several different ways of incorporating spin effects in the EOB formalism have been explored [3, 8–14].

Thanks to recent advances in Numerical Relativity (NR) (see e.g. Refs. [15, 16]), it has been possible, over the last years, to incorporate crucial nonperturbative information in the EOB formalism so as to produce improved “EOBNR” waveforms. This has been done both for nonspinning EOB models [17–30] and for spinning (precessing) configurations [31–34]. Progress in gravitational self-force theory has also allowed one to acquire useful strong-field information [35–43].

Here, we shall propose a new way of defining an EOB Hamiltonian containing spin-orbit and spin-spin interactions. In addition, we will calibrate our new spinning EOB (SEOB) model by using a sample of Caltech-Cornell Simulating eXtreme Spacetime (SXS) [15, 44–50] NR waveforms covering the dimensionless spin range $-0.95 \leq \chi \leq +0.98$. As a proof of principle we shall only consider here the simplest case of equal mass $m_1 = m_2$, equal, parallel (or anti parallel) spin ($\chi_1 = \chi_2 = \chi$) where $\chi_A \equiv S_A / m_A^2$ ($G = c = 1$). We leave to future work a more extensive comparison/calibration of our new SEOB Hamiltonian.

II. REVISITING THE KERR HAMILTONIAN

Here, we reexamine the structure of the Hamiltonian of a test particle on a Kerr background of mass $M$ and spin parameter $a = S/M$, to motivate our definition of a new SEOB model. The Hamiltonian of a spinning test particle of mass $\mu$ and spin $S_\perp$ can be decomposed in “orbital” and “spin-orbit” parts:

$$H_{\text{Kerr}} = H_{\text{orb}} + H_{\text{so}}^S(S) + H_{\text{so}}^\perp(S_\perp).$$

(1)

Here the orbital Hamiltonian (which also contains all interactions that are even in spins such as spin-square effects $\propto S^2$) is

$$H_{\text{orb}}^{\text{Kerr}} = \sqrt{A(r, \theta)} \left( \mu^2 + \frac{\Delta(r)p_r^2}{r^2 + a^2 \cos^2 \theta} + \frac{p_\theta^2}{r^2 + a^2 \cos^2 \theta} + \frac{r^2 + a^2 \cos^2 \theta}{R^4(r) + a^2 \Delta(r) \cos^2 \theta \sin^2 \theta} \frac{p_\phi^2}{r^2 + a^2 \cos^2 \theta} \right).$$

(2)

where

$$\Delta(r) \equiv r^2 - 2Mr + a^2$$

$$R^4(r) \equiv (r^2 + a^2)^2 - a^2 \Delta(r)$$

$$= r^4 + r^2 a^2 + 2Mr a^2$$

$$A(r, \theta) \equiv \frac{\Delta(r)(r^2 + a^2 \cos^2 \theta)}{R^4 + a^2 \Delta(r) \cos^2 \theta},$$

(3)

(4)

(5)
and where the part of the spin-orbit Hamiltonian linked to the background spin $S$ reads

$$H_{so}^S(S) \equiv G_S(r, \theta) \mathbf{L} \cdot \mathbf{S}. \quad (6)$$

Here $\mathbf{L}$ denotes the orbital angular momentum of the particle, while the gyro-gravitomagnetic function entering $H_{so}^S(S)$ reads

$$G_S(r, \theta) = \frac{2r}{R^2(r) + a^2 \Delta(r) \cos^2 \theta}. \quad (7)$$

Methods for computing the spin-orbit Hamiltonian $H_{so}^S(S_*)$ linked to the spin of the particle have been discussed in [9, 11, 51, 52].

The structure of this Hamiltonian can be clarified by introducing the (gauge-invariant) concept of “centrifugal radius” $r_c(r)$, defined so that the orbital part of the Hamiltonian ruling equatorial orbits ($\theta = \pi/2$) can be written as

$$H_{orb,eq}^\text{Kerr}(r, p_r, p_\phi) = \sqrt{A^\text{eq}(r) \left( \mu^2 + \frac{p^2_r}{r^2_c} + \frac{p^2_\phi}{B^\text{eq}(r)} \right)}, \quad (8)$$

with the usual, relativistic centrifugal energy term $\mu^2 + \frac{p^2_r}{r^2_c}$. By comparing with Eq. (2) one obtains

$$r^2_c \equiv \frac{R^2(r)}{r^2} = r^2 + a^2 + \frac{2M a^2}{r}. \quad (9)$$

In addition, we have

$$A_{eq}(r) = \frac{\Delta(r)}{r^2_c} = \left( 1 - \frac{2M}{r_c} \right) \left( 1 + \frac{2M}{r_c} \right), \quad (10)$$

and $B^\text{eq}(r) = r^2/\Delta$ so that

$$A^\text{eq}(r)B^\text{eq}(r) = \frac{r^2}{r^2_c}. \quad (11)$$

We also note that

$$G^S_{so}(r_c) = \frac{2}{r^2_c}. \quad (12)$$

Equation (10) displays a remarkable fact, which is rather hidden in usual formulations of the Kerr Hamiltonian: while the usual (gauge-dependent) Boyer-Lindquist radius of the outer horizon ($\Delta(r^+_{\text{h}}) = 0$), which is also the location where the main radial $A$ potential enters the equatorial dynamics has a zero, strongly depends on the spin parameter $a$, $r^+_{\text{h}} = M + \sqrt{M^2 - a^2}$, the corresponding (gauge-invariant) value of the centrifugal radius $r_c$ does not depend on $a$ and is simply equal to the usual Schwarzschild-coordinate value, $r_c = 2M$. In other words, when using $r_c$ as radial coordinate, the scalar $A$ potential factorizes in the product

$$A_{eq}(r_c) = A_{Schw}(r_c) \hat{A}(r_c) \quad (13)$$

of the usual Schwarzschild $A_{Schw}(r_c) = 1 - 2M/r_c$ and of a correcting factor $\hat{A}$

$$\hat{A}(r_c) = \frac{1 + \frac{2M}{r_c}}{1 + \frac{2M}{r}} \approx \frac{1}{1 + \frac{Ma^2}{r^3_c} + \frac{3Ma^4}{4r^5_c} + \frac{5M^2a^4}{2r^7_c} + \mathcal{O} \left( \frac{1}{r^9_c} \right)} \quad (14)$$

This natural factorization of the Kerr $A$ potential will be the crucial motivation for the definition of our new SEOB formalism below.

The correcting factor $\hat{A}(r_c)$ embodies the multipolar structure of the Kerr hole, e.g. at lowest order in a PN expansion one finds

$$A_{PN}(r_c) = \left( 1 - \frac{2M}{r_c} \right) \left( 1 - \frac{Ma^2}{r^3_c} + \ldots \right) \approx 1 - \frac{2M}{r_c} - \frac{Ma^2}{r^3_c}. \quad (15)$$

where $-Ma^2/r^3_c$ is the quadrupole gravitational potential term (seen in the equatorial plane).

The (multipolar) correcting factor $\hat{A}(r_c)$ is everywhere smaller than one and larger than $2/3$, a value reached only at the horizon $r_c = 2M$ and for maximum spin $a = M$. The derivative of the function $r_c(r)$ is equal to

$$\frac{dr_c}{dr} = \frac{r_c}{r_c \left( 1 - \frac{Ma^2}{r^3_c} \right)} \quad (16)$$

so that $r_c(r)$ reaches a minimum at $r = r_{\text{min}} = (Ma^2)^{1/3}$. One sees that, when $a < M$, $r_{\text{min}} < M$ so that $r_{\text{min}}^6 < 2M$. However, when $a \to M$, $r_{\text{min}} \to M$ and $r_{\text{min}}^6 \to 2M$. The inverse function expressing the original Boyer-Lindquist radial coordinate $r$ in terms of the centrifugal radius $r_c$ is well defined on the interval $r_c \geq 2M$, i.e. outside the horizon, and reads

$$r = 2 \sqrt{\frac{r_c^2 - a^2}{3}} \cos \left( \frac{\pi}{6} + \frac{1}{3} \arcsin \frac{Ma^2}{\left( \frac{r^2_c - a^2}{3} \right)^{3/2}} \right). \quad (17)$$

The multiplicative modification of $A_{eq}(r)$ by a factor, depending on $a^2$, that does not change the location of the horizon, but introduces extra attractive forces linked to
the quadrupole and higher moments of Kerr is illustrated in Fig. 1. It is striking how small is the modification of the Kerr $A$ potential due to the $a^2$ terms when plotted as a function of the inverse centrifugal radius $^2 u_c = M/r_c$. Numerically, the difference $A(u_c, a) - A(u_c, 0)$ (which vanishes both at $u_c = 0$ and $u_c = \frac{1}{2}$) reaches, around $u_c \approx 0.4$, an extremum approximately equal to $-0.004$ for $|a| = 0.5M$, to $-0.01$ for $|a| = 0.8M$ and to $-0.016$ for $a = 1M$. This indicates that the most important physical effects determining the energetics of circular orbits comes from the interplay between $A_{\text{Schw}}(r_c) = 1 - 2M/r_c$ and the spin-orbit coupling $(G_S(r_c)LS)$, with only a relatively small contribution of (quadrupolar) spin-squared effects.

Similar multiplicative modifications occur in the (equatorial) gyro-gravitomagnetic function which reads

$$G_S^{\text{eq}}(r_c) = \frac{2}{r_c^2} = \frac{2}{r_c^2} \tilde{G}_S(r_c),$$

leading to a spin-orbit coupling

$$H_{\text{so,eq}}^S = \frac{2}{r_c^3} \tilde{G}_S M a L,$$

where

$$\tilde{G}_S(r_c) = \frac{r_c}{r} \approx 1 + \frac{a^2}{2r_c^2} + \frac{M a^2}{r_c^3} + \mathcal{O}\left(\frac{1}{r_c^4}\right).$$

Finally, let us note that, when considering general nonequatorial orbits all the functions appearing in the Hamiltonian naturally factorize as the product of their equatorial (radial-dependent) value with a $\cos \theta$-dressing factor of the type

$$1 + f_1(r) \cos^2 \theta \over 1 + f_2(r) \cos^2 \theta.$$ (21)

For instance, the $A(r, \theta)$ function reads

$$A(r, \theta) = A_{\text{eq}}(r) \frac{1 + \frac{a^2 \cos^2 \theta}{r^2}}{1 + \frac{a^2 \Delta \cos^2 \theta}{r^2 r_c^2}}.$$ (22)

III. DYNAMICS: A NEW SEOB HAMILTONIAN

Some of the main challenges in previous definitions of SEOB models were: (i) blending Kerr spin effects in $\Delta(r) = r^2 - 2Mr + a^2 = r^2 (1 - 2M/r + a^2/r^2)$ with finite-mass-ratio deformations (depending on $\nu \equiv m_1 m_2/(m_1 + m_2)^2$ in $A = 1 - 2M/r + 2\nu(M/r)^3 + \ldots$) and (ii) resuming spin-orbit and spin-spin effects. A first suggestion [3] to address issue (i) was to deform the Kerr $\Delta(r)$ function (considered in Boyer-Lindquist-type coordinates) into the Padé-resummed function $\Delta(r) = r^2 P_{21}^\nu[r_\text{EOB}^2(1 - 2\nu(M/r)^3 + \ldots)]$ where $u = M/r$. More recently, it was suggested [10] to enforce the existence of two zeros in $\Delta(r)$ of the form $r_{\text{EOB}}^\nu = (M \pm \sqrt{M^2 - a^2})/(1 - \nu r)$ (where $K$ is an adjustable parameter) by factoring $\Delta(r)$ as

$$\Delta(r) = \frac{r^2 a^2}{M^2} \left(u - \frac{M}{r_{\text{EOB}}^+}\right) \left(u - \frac{M}{r_{\text{EOB}}^-}\right) \times \left[1 + \nu \Delta_0 + \nu \ln(1 + \Delta_1 u + \Delta_2 u^2 + \Delta_3 u^3 + \Delta_4 u^4)\right].$$

Here, we introduce a novel (and more natural) way of blending the $a^2$-deformation effects with the $\nu$-deformation ones based on the new way of considering the Kerr $A$-potential explained in the previous section. Namely, we shall extend the definition of a centrifugal radius $r_c$ to the spinning, comparable-mass case and decompose its (equatorial) radial $A$-potential as the product of the usual $\text{EOB}, \nu$-deformed, nonspinning (Padé-resummed) $A$-potential $A_{\text{orb}}(u_c, \nu)$ expressed as a function of $u_c \equiv M/r_c$, by a $\nu$-deformed version of the multipolar correcting factor $A(\nu_c)$, Eq. (14).

Concerning the challenge of resuming spin-orbit effects, it was previously addressed in Refs. [9, 10, 12, 13]. Here, we shall first follow the Damour-Jaranowski-Schäfer (DJS) [6, 9] philosophy of working in a gauge that eliminates the $p^2$ dependence to keep only a dependence on $1/r$ and $p^2_{\text{orb}}$. In addition, as previous work [9, 10, 12–14, 33] has shown the importance of controlling the natural taming of the gyrogravitomagnetic couplings by higher-order PN corrections, we shall re-
sum them by means of \( P_m^n \) (i.e., inverse Taylor) Padé approximants.

In the present work, we limit ourselves, for simplicity and orientation, to considering nonprecessing (parallel) spins. This means that we shall define a comparable-mass deformation of the equatorial-Hamiltonian-dynamics of Sec. II, leaving to future work the task of \( \nu \)-deforming the various \( \cos \theta \) dressing factors of Eq. (21). We recall that the full EOB Hamiltonian \( H \) is expressed in terms of the effective Hamiltonian \( H_{\text{eff}} \) (which generalizes the Hamiltonian of a test-particle) as

\[
H = M \left[ 1 + 2\nu \left( \frac{H_{\text{eff}}}{\mu} - 1 \right) \right].
\]  

(23)

Here and henceforth, \( M \equiv m_1 + m_2, \mu = \frac{m_1 m_2}{M} \) and \( \nu = \mu/M \), by contrast with the previous section where \( M \) denoted the large mass and \( \mu \) the small one. We decompose the effective Hamiltonian \( H_{\text{eff}} \) in an orbital part \( H_{\text{eff}}^{\text{orb}} \) (which gathers all the terms that are even in spins) and in two spin-orbit pieces (that are odd in spins):

\[
H_{\text{eff}} = H_{\text{eff}}^{\text{orb}} + G_S \mathbf{L} \cdot \mathbf{S} + G_S, \mathbf{L} \cdot \mathbf{S},
\]  

(24)

where

\[
H_{\text{eff}}^{\text{orb}} = \left[ p_r^2 + A(r, \nu, S_1, S_2) \left( \mu^2 + \frac{L^2}{c^2} + Q_4 \right) \right].
\]  

(25)

Here, \( \mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{L} \mathbf{e}_\phi \) denotes the orbital angular momentum (with modulus \( L = p_\phi \)) and the comparable-mass analogs of the background and (rescaled) test spins \((\mathbf{S}, \mathbf{S})\) are, following Ref. [9], defined as

\[
\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2,
\]  

(26)

\[
\mathbf{S}_\ast = \frac{m_2}{m_1} \mathbf{S}_1 + \frac{m_1}{m_2} \mathbf{S}_2.
\]  

(27)

The building blocks, \( A, r_c^2, Q_4, G_S, G_{S} \) entering the effective Hamiltonian (24) are defined next.

### A. Effective orbital Hamiltonian

Hereafter we will often (but not always \(^3\)) work with rescaled variables: \( r \equiv r^{\text{phys}}/M, \mathbf{p} \equiv \mathbf{p}^{\text{phys}}/\mu, p_\phi \equiv L \equiv L^{\text{phys}}/(\mu M) \) as well as \( H \equiv H^{\text{phys}}/M \) and \( \hat{H}_{\text{eff}} \equiv H^{\text{phys}}/\mu \). We also introduce \( u = M/r^{\text{phys}} = 1/r \) and \( u_c = M/r^{\text{phys}} = 1/r_c \). The 5PN-log-accurate PN expansion of the orbital (nonspinning) part of the \( A \) potential, considered as a function of \( r_c \), reads

\[
A_{\text{orb}}^{\text{PN}}(u_c) = 1 - 2u_c + 2\nu u_c^3 + \nu a_4 u_c^4 + \nu (a_5^c + a_5^d \ln u_c) u_c^5 + \nu (a_6^c + a_6^d \ln u_c) u_c^6,
\]  

(28)

where the comparable-mass version of the spin-dependent function \( u_c(u) \) relating \( u_c \) to the Boyer-Lindquist type variable \( u = 1/r \) will be introduced below. As indicated here, though we conceptually view \( A \) as a function of \( r_c \), it will be convenient to express the Hamiltonian dynamics in terms of a Boyer-Lindquist type radius \( r \), and, as usual, of a related tortoise radial momentum \( p_{r_c} \). The other building elements of \( H_{\text{eff}} \) are defined through

\[
D \equiv AD = \frac{\nu^2}{r_c^2} D_{\text{orb}}(u_c),
\]  

(32)

with

\[
D_{\text{orb}}(u_c) = \frac{1}{1 + 6\nu u_c^3 + 2(26 - 3\nu)\nu u_c^3},
\]  

(33)

(note that the coefficient of \( \nu u_c^3 \) here was misprinted as \( 2(23 - 3\nu) \) in Ref. [56]) and

\[
p_{r_c}^2 = \frac{A}{B} p_{r_c}^2,
\]  

(34)

\[
Q_4 = 2\nu(4 - 3\nu)p_{r_c}^2 u_c^2.
\]  

(35)

\(^3\) The context will make it clear whether rescaled variables are used or not.
B. Spin-orbit interaction terms

On the model of the multiplicative structure (18) of the gyrogravitomagnetic coupling $G_S$ entering the equatorial Kerr Hamiltonian, we write

$$G_S = G^0_S \hat{G}_S,$$

$$G_{S_\nu} = G^0_{S_\nu} \hat{G}_{S_\nu},$$

where

$$G^0_S = 2uu^2,$$

$$G^0_{S_\nu} = \frac{3}{2} u^4_N.$$ (36-40)

Here, $(\hat{G}_S, \hat{G}_{S_\nu})$ are PN correcting factors $\hat{G}_S = 1 + O(1/c^2)$, $\hat{G}_{S_\nu} = 1 + O(1/c^2)$. We take them in DJS gauge (i.e., as function of $1/r_c$ and $p_r^2$ only), resumming their NNLO PN expansion [12, 13] by means of inverse Taylor, $P^0_m$, Padé approximants. In addition, we include two more $\nu = 0$ terms in $\hat{G}_{S_\nu}$, coming from the circular limit of a spinning test-particle in a Schwarzschild background [11, 43]

$$\hat{G}^{\text{circ}}_{S_\nu}(\nu = 0) = \frac{2}{1 + \frac{1}{\sqrt{1 - 3u_c}}},$$

$$\approx \left(1 + \frac{3}{4}u_c + \frac{27}{16}u^2_c + \frac{135}{32}u^3_c + \frac{2835}{256}u^4_c + \ldots\right)^{-1}.\quad (41)$$

Their explicit expressions read

$$\hat{G}_S = \left(1 + c_{10}u_c + c_{20}u^2_c + c_{30}u^3_c\right.$$

$$+ c_{20}p^2_r + c_{12}u_c p^2_r + c_{40}p^4_r\right)^{-1},$$

$$\hat{G}_{S_\nu} = \left(1 + c_{10}^*u_c + c_{20}^*u^2_c + c_{30}^*u^3_c + c_{40}^*u^4_c\right.$$

$$+ c_{20}^*p^2_r + c_{12}^*u_c p^2_r + c_{40}^*p^4_r\right)^{-1}.\quad (42-43)$$

Starting from Eqs. (55) and (56) of Ref. [12, 58] and taking their $P^0_m$ approximants one explicitly obtains

$$c_{10} = \frac{5}{16},$$

$$c_{20} = \frac{51}{8} + \frac{41}{256} \nu^2,\quad (44-45)$$

$$c_{30} = \nu c_3,$$ (46)

$$c_{20} = \frac{27}{16},$$ (47)

$$c_{12} = \frac{12}{16} \nu - \frac{49}{128} \nu^2,\quad (48-49)$$

$$c_{02} = \frac{5}{16},$$ (49)

$$c_{12} = \frac{5}{16} + \frac{169}{256} \nu^2,\quad (49)$$

$$c^*_{10} = \frac{3}{4} + \frac{\nu}{2},\quad (50)$$

$$c^*_{20} = \frac{27}{16} + \frac{29}{4} \nu + \frac{3}{8} \nu^2,\quad (51)$$

$$c^*_{02} = \frac{5}{4} + \frac{3 \nu}{2},\quad (52)$$

$$c^*_{30} = \frac{135}{32} + \nu c_3,\quad (53)$$

$$c^*_{40} = \frac{2835}{256},\quad (54)$$

$$c^*_{12} = \frac{4}{12} + 11 \nu - \frac{7}{8} \nu^2,\quad (55)$$

$$c^*_{04} = \frac{5}{48} + \frac{25}{12} \nu + \frac{3}{8} \nu^2.\quad (56)$$

We included $\nu$-dependent tunable next-to-next-to-leading-order (NNNLO) contributions both in $(\hat{G}_S, \hat{G}_{S_\nu})$. They are taken here as being simply proportional to $u^3_c$. In this exploratory investigation we shall conflate these two (apriori independent) terms in a common NNNLO contribution parametrized by the single parameter $c_3$ entering both $c_{30}$ and $c_{30}^*$. Eqs. (46), (53). Note that in $c^*_{30}$ (and $c^*_{40}$) we included also the $\nu$-independent (spinning test-particle) contribution coming from Eq. (41). Recently, combined progress in numerical and analytical gravitational self force calculations of spin-orbit effects [42, 43] has brought an improved knowledge of $G_{S_\nu}$. We leave to future work the incorporation of this knowledge in the SEOB model.

C. Spin-spin interaction

We have seen above that in the Kerr Hamiltonian expressed in $r_c$ coordinates the quadrupole moment of the Kerr black hole was modifying the main radial potential by $\delta A^{\text{quadrupole}} = -Ma^2_{Kerr}/r_c^3$. It was shown in [3] that, to leading PN order, the combination of quadrupolar $S^2_1$, $S^2_2$ effects with spin-spin $S_1 S_2$ effects, could be incorporated simply by changing the Kerr spin parameter $a$ entering the Kerr Hamiltonian by the leading-order
effective Kerr parameter\textsuperscript{4}
\[
a \equiv a_1 + a_2 = \frac{S_1}{m_1} + \frac{S_2}{m_2} = \frac{S + S_c}{m_1 + m_2}.
\] (57)

In other words, we can include the leading order effect of spin-spin interactions by defining the functional link between the SEOB centrifugal radius \(r_c\) and its Boyer-Lindquist type analog \(r\) by the relation
\[
[r_c^{\text{LO}}(r, S_1, S_2)]^2 = r^2 + a^2 + 2m\frac{a^2}{r},
\] (58)
where, now, \(M = m_1 + m_2\) and \(a\) is given by Eq. (57).

At the next-to-leading (NLO) order in spin-spin effects we need to correct the quadrupolar and spin-spin interaction terms \(\sim MS^2/r_c^3\) (viewed in the equatorial plane) by adding PN corrections \(\sim \nu S^2/r_c^3\left(p^2 + p_r^2 + \frac{1}{r}\right)\). Starting from the EOB reformulation \cite{ref14} of the PN-expanded NLO spin-spin results of Refs. \cite{ref59, ref60}, one finds that, when including for simplicity only the circular, equatorial (spin-aligned) contributions, we can take into account NLO spin-spin effects by simply modifying the definition of the function \(r_c^{\text{NLO}}(r)\) as
\[
[r_c^{\text{NLO}}(r, S_1, S_2)]^2 = r^2 + a^2 + 2m\frac{a^2}{r} + \delta a^2(r),
\] (59)
where \(\delta a^2(r)\) is
\[
\delta a^2(r) = \frac{M^3}{r}\left\{(a_{11} + c_{11})\chi_1^2 + (a_{22} + c_{22})\chi_2^2 + (a_{12} + c_{12})\chi_1\chi_2\right\}
\] (60)
with \(\chi_A = a_A/m_A = S_A/m_A^2\) and \cite{ref14}
\[
a_{11} = \frac{\nu}{16} \left[-32\nu - 22\nu^2 + X_{1/2}(21 - 44\nu - 44\nu^2) + (X_{1/2})^2(-21\nu - 22\nu^2)\right],
\] (61)
\[
a_{12} = \frac{\nu}{8} [24 - 53\nu - 44\nu^2 + (X_{1/2} + X_{2/1})(-32\nu - 22\nu^2)],
\] (62)
\[
a_{22} = \frac{\nu}{16} \left[-32\nu - 22\nu^2 + X_{2/1}(21 - 44\nu - 44\nu^2) + (X_{2/1})^2(-21\nu - 22\nu^2)\right],
\] (63)
\[
c_{11} = \frac{\nu}{16} \left[88\nu + 14\nu^2 + X_{1/2}(-117 + 196\nu + 28\nu^2) + (X_{1/2})^2(117\nu + 14\nu^2)\right],
\] (64)
\[
c_{12} = \frac{\nu}{8} \left[-120 + 229\nu + 28\nu^2 + (X_{1/2} + X_{2/1})(112\nu + 14\nu^2)\right],
\] (65)
\[
c_{22} = \frac{\nu}{16} \left[88\nu + 14\nu^2 + X_{2/1}(-117 + 196\nu + 28\nu^2) + (X_{2/1})^2(117\nu + 14\nu^2)\right].
\] (66)

Here
\[
X_{1/2} \equiv \frac{X_1}{X_2}, \quad X_{2/1} \equiv \frac{X_2}{X_1},
\] (67)
with
\[
X_1 \equiv \frac{m_1}{M} = \frac{1}{2}(1 + \sqrt{1 - 4\nu}), \quad X_2 \equiv \frac{m_2}{M} = 1 - X_1,
\] (68)
where we use the convention \(m_1 \geq m_2\).

In the equal-mass, equal-spin case, \(m_1 = m_2\), \(\chi_1 = \chi_2 = \chi\) (so that \(a \equiv a_1 + a_2 = M\chi\)), Eq. (60) gives
\[
\delta a^2 = -\frac{9Ma^2}{8r}\quad \text{when replacing the LO term}
\]
\[
2\frac{Ma^2}{r}\quad \text{in Eq. (58) by}\quad \frac{7Ma^2}{8r}.
\]

In the present work we shall use everywhere the \(H_{\text{eff}}^{\text{orb}}\)-related NLO functional link \(r_c(r)\) given by Eq. (59).

Note, however, that this use means that we are introducing specific corresponding \(\nu\)-dependent, spin-quadratic, NLO corrections to other parts of the Hamiltonian (such as \(G_S, G_{S_c}\)).

\section{IV. Radiative sector: Radiation reaction and waveforms}

\subsection{A. Flux and waveform at future null infinity}

In absence of a robust strategy for resumming the analytically predicted \cite{ref61} radial contribution to the radiation reaction, we set it to zero, \(F_r = 0\) (as we had done in most of our previous EOB work). As a preparation for defining the azimuthal contribution \(F_\varphi\) to radiation reaction, as well as the waveform, especially during the post-Keplerian plunge \cite{ref62}, we define a non-Keplerian “azimuthal” velocity \(v_\varphi\) as
\[
v_\varphi \equiv r\Omega_\varphi.
\] (69)

Here
\[
r\Omega_\varphi \equiv \left\{(r_c^2\psi_c)^{-1/2} + \frac{G}{H}\right\}_{p_r=0}^{-2/3},
\] (70)
with
\[
\psi_c = -\frac{2}{A} \left( u_c' + \frac{G'}{u_c A} \sqrt{\frac{A}{p_c^2}} + u_c^2 A \right),
\]
where
\[
G = G_S S + G_{S*, S*},
\]
and the prime indicates derivatives with respect to \( r \). The variables \((u, u_c, p_c)\) that appear in the definition of \( r_\Omega \) (after having set \( p_c \to 0 \) as indicated in Eq. (70)) are evaluated along the EOB dynamics. The definition (70) is such that, during the adiabatic circular inspiral, one has a usual looking Kepler’s law:
\[
1 = \Omega^2 r^3.
\]
In the nonspinning limit, Eq. (70) reduces to the definition of \( r_\infty \) given around Eq. (19) in Ref. [56].

The azimuthal component of radiation reaction is identified with the total mechanical angular momentum loss given by
\[
F_\varphi = F_\varphi^\infty + F_\varphi^H = -J^\infty - J_1^H - J_2^H,
\]
where the angular momentum flux at infinity \( J^\infty \) is re-summed according to the multipolar waveform resummation introduced in [7] for nonspinning binaries and extended in [29] to the spinning case. The horizon flux contributions \( J_1^H, J_2^H \) are obtained by combining the results of Refs. [63, 64].

More precisely, \( J^\infty \) is given by
\[
J^\infty = \frac{\dot{F}^\infty}{2\Omega} = \frac{1}{8\pi} \sum_{\ell=2}^{\infty} \sum_{m=1}^{\ell} m^2 \Omega |R h_{\ell m}(x)|^2,
\]
where \( R \) denotes the distance from the source. The multipoles \( h_{\ell m}(x) \) are written in factorized form as
\[
h_{\ell m}(x) = h_{\ell m}^{(N, r)}(u_c) h_{\ell m}^{\text{tail}}(y) S_{\ell m}^{(c)}(v_c, S_1, S_2) h_{\ell m}^{\text{NQC}},
\]
where we indicated the (main) arguments used in several factors of the waveform [in particular \( y \equiv (\Omega \ell)^{2/3} \)]. Note that in our EOB/NR comparisons below we shall work with a Zerilli-normalized multipolar waveform \( \Psi_{\ell m} = (R/M) h_{\ell m}/\sqrt{(\ell + 2)(\ell + 1)\ell (\ell - 1)} \). Here \( \epsilon = 0, 1 \) is the parity of the considered multipole (i.e. the parity of \( \ell + m \)). The first factor, \( h_{\ell m}^{(N, r)} \), is the Newtonian wavefunction,
\[
h_{\ell m}^{(N, r)}(x) = \frac{\nu}{R} c_{\ell m}^{(c)} Y_{\ell m}(\hat{r}, \theta, \varphi) r^{\ell - 1} e^{-m \varphi} \left( \frac{\pi}{2} \right)^{\frac{\ell}{2}},
\]
where \( Y_{\ell m}(\theta, \varphi) \) are scalar spherical harmonics, \( \varphi \) is the orbital phase, and where the functions \( n_{\ell m}^{(c)} \) and \( c_{\ell m}^{(c)} \) are given in Eqs. (5), (6) and (7) of Ref. [7]. Note in particular that
\[
c_{\ell m}^{(c)} = X_{2}^{\ell + 1 - \epsilon} + (-)^m X_{1}^{\ell + 1 + \epsilon}.
\]

The second factor is the effect of tails [7, 20, 65]:
\[
h_{\ell m}^{\text{tail}}(y) \equiv T_{\ell m}(y) e^{i \delta_{\ell m}(y)},
\]
where the residual phase corrections \( \delta_{\ell m}(y) \) are given as in Ref. [56], without adding possible spin-dependent corrections in this exploratory study. The third factor, \( S_{\ell m}^{(c)} \), is a parity-dependent source term defined as \( S_{\ell m}^{(0)} = \hat{H}_{\ell m} \) and \( S_{\ell m}^{(1)} = p_c/(r_\Omega v_c) \). The fourth factor \( f_{\ell m}(v_c, S_1, S_2) \) is taken, when \( m \) is even, as \( f_{\ell m} = \rho_{\ell m} \) with
\[
\rho_{\ell m} = \rho_{\ell m}^{\text{orb}} + \rho_{\ell m}^S.
\]
Here, we use the orbital contributions \( \rho_{\ell m}^{\text{orb}} \) at the \( 3^{+}2\)PN accuracy as given in Ref. [56]. The spin-dependent contributions \( \rho_{\ell m}^S \) are mostly taken from Ref. [66] (modulo some new, additional contributions mentioned below):
\[
\rho_{\ell m}^S = c_{\ell m}^{30} v^3 + c_{\ell m}^{34} v^3 + c_{\ell m}^{34} v^5,
\]
where
\[
\rho_{\ell m}^S = -\frac{4\nu}{3(3\nu - 1)} \chi S v^2,
\]
\[
\rho_{\ell m}^S = -\frac{1}{15(1 - 3\nu)} \times \left[ (42\nu^2 - 41\nu + 10)\chi S + (10 - 39\nu)\delta m \chi A \right] v^3,
\]
\[
\rho_{\ell m}^S = -\frac{1}{15(1 - 3\nu)} \times \left[ (78\nu^2 - 59\nu + 10)\chi S + (10 - 21\nu)\delta m \chi A \right] v^3.
\]

The \( \rho_{\ell m}^S \) term has two spin-orbit contributions (at LO and NLO) and a LO spin-spin one \(^5\) involving the square of the effective Kerr parameter \( a = a_1 + a_2 = (S + S_*)/M \). Explicitly we have
\[
c_{\ell m}^{30} = \frac{2}{3} \left[ \chi S (1 - \nu) + \chi A \delta m \right],
\]
\[
c_{\ell m}^{34} = \left( \frac{34}{21} + \frac{49 \nu}{18} + \frac{209 \nu^2}{126} \right) \chi S
\]
\[
+ \left( \frac{34}{21} - \frac{19 \nu}{42} \right) \delta m \chi A,
\]
\[
c_{\ell m}^{34} = \frac{1}{2} a^2,
\]
where we used the notation
\[
\chi A = \frac{1}{2} (\chi_1 - \chi_2), \quad \chi S = \frac{1}{2} (\chi_1 + \chi_2),
\]
\[
\delta m = X_1 - X_2.
\]

\(^5\) There are no explicit spin-quadratic contributions to multipole moments. This spin-spin term comes from the indirect effect of the spin-spin Hamiltonian contribution [67].
The NLO, $\mathcal{O}(v^5)$ spin-orbit contribution to $\rho_{2n}^S$ is new. It was deduced by us (together with the $\mathcal{O}(v^3)$ contribution to $f_S^2$, discussed below) from the decomposition in partial multipoles (which was kindly provided to us by Guillaume Faye) of the total energy flux given in Eq. (5) of the 2010 Erratum [68] of Ref. [69]. On the other hand, when $m$ is odd, we use (essentially as in Ref. [33]) an additive expression for $f_{m}\nu(v_\nu, S_1, S_2)$, which defactorizes the singular factor $\delta m$ (which vanishes in the equal-mass limit) present in the Newtonian prefactor $[\epsilon_{\ell+m}]_{\epsilon=1} \propto \delta m$, Eq. (78):

$$\delta m f_m(v_\nu, S_1, S_2) = \delta m \left(\rho_{\ell+m}^{\text{orb}}\right)^{1/2} + \tilde{f}_m^S,$$  

(90)

where the $\delta m$-rescaled spin contributions $\tilde{f}_m^S$ are\footnote{Beware, however, of a misprint in Eq. (A15c) of [33], giving $f_{31}$, with respect to the (corrected) results of [66].}

\begin{align*}
\tilde{f}_{21}^S &= -\frac{3}{2} \delta m \chi S + \chi A) v \\
&+ \left[ \left( \frac{61}{12} + \frac{79}{84} \nu \right) \delta m \chi S + \left( \frac{61}{12} + \frac{131}{84} \nu \right) \chi A \right] v^3, \\
\tilde{f}_{33}^S &= - \left[ \delta m \chi S \left( 2 - \frac{5}{2} \nu \right) + \chi A \left( 2 - \frac{19}{2} \nu \right) \right] v^3, \\
\tilde{f}_{31}^S &= - \left[ \delta m \chi S \left( 2 - \frac{13}{2} \nu \right) + \chi A \left( 2 - \frac{11}{2} \nu \right) \right] v^3, \\
\tilde{f}_{43}^S &= - \frac{5}{2(2\nu - 1)} (\delta m \chi S - \chi A) v, \\
\tilde{f}_{41}^S &= \tilde{f}_{43}^S.
\end{align*}

(91-95)

Here the $\mathcal{O}(v^3)$ term in $\tilde{f}_{21}^S$ is new (see discussion above of the NLO contribution to $\rho_{2n}^S$). Finally the NQC multipolar factor $\tilde{h}_{\ell m}^\text{NQC}$ in Eqs. (76) depends on 4 real parameters, 2 for the amplitude, $a_{i}^{\ell m}, i = 1, 2$, and 2 for the phase $\hat{h}_{i}^{\ell m}, i = 1, 2$ and reads

$$\tilde{h}_{\ell m}^\text{NQC} = \left( 1 + \sum_{j=1}^{2} a_{i j}^{\ell m} n_{j} \right) \exp \left( i \sum_{j=1}^{2} h_{j i}^{\ell m} n_{j} \right),$$

(96)

where the $n_i$'s factors are chosen here to be

$$n_1 = \left( \frac{P_{r_0}}{r_0 \Omega} \right)^2,$$

(97a)

$$n_2 = \left( \frac{(\hat{r})^{(0)}}{r \Omega} \right)^2,$$

(97b)

$$n_1' = \frac{P_{r'}}{r \Omega},$$

(97c)

$$n_2' = \frac{P_{r'}}{r_0 \Omega} = n_1' (r \Omega)^2.$$  

(97d)

Here, the superscript $(0)$ on the right-hand side of the definition of $n_2'$ means that the second time derivative of $r$ is evaluated along the conservative dynamics (i.e.

neglecting the contributions proportional to $\mathcal{F}$, see Appendix of [56] for a discussion). For simplicity in the present work we include NQC factors only for $\ell = m = 2$ (because of the complete mass and spin symmetry the next multipoles $(2, 1), (3, 3)$ exactly vanish).

B. Horizon flux contributions

The horizon flux contribution $\mathcal{F}_\nu^H$ to the azimuthal radiation reaction force $\mathcal{F}_\varphi$, Eq. (74), is written as

$$\mathcal{F}_\nu^H = -\frac{32}{5} \nu^2 \Omega^5 \tilde{r}_1^4 \left( \hat{j}_1^H + \hat{j}_2^H \right),$$

(98)

where

$$\hat{j}_1^H = \frac{\nu_5^3}{4} X_1^4 \left( 1 + 3 \chi_1^2 \right) - \chi_1^2 + 2 \hat{F}_{22}^{(H, 0)} \left( 1 + \sqrt{1 - \chi_1^2} \right) X_1 \nu_5^3$$

(99)

is the result derived in Ref. [63] modified by including the (resummed, hybridized) PN amplification factor $\hat{F}_{22}^{(H, 0)}$ given by Eq. (35) of Ref. [64]. We leave to future work the inclusion of the PN corrections to the spin-odd leading order term, $\propto \nu_5^5$ (see Refs. [70, 71]).

C. Ringdown modelization

Contrary to previous EOB work, the ringdown is here modeled using the new analytical representation recently introduced in Ref. [72]. Let us recall that the EOB waveform is made of the juxtaposition of two distinct waveforms: the inspiral-plus-plunge waveform on the EOB time interval $-\infty < t_{\text{EOB}} < t_{\text{EOB}}^{\text{match}}$ and the ringdown waveform for $t_{\text{EOB}}^{\text{match}} < t_{\text{EOB}}$. In the notation used below the matching time will be

$$t_{\text{EOB}}^{\text{match}} = t_{\text{EOB}}^{\text{NQC}}$$

(100)

which corresponds, on the NR time axis, to the instant $t_{\text{match}}^{\text{NR}} = t_{\text{EOB}}^{\text{match}} + 2M$, where $A_{22} = |h_{22}^{\text{NR}}|$ is the amplitude of the NR quadrupolar waveform $h_{22}^{\text{NR}}$.

V. NR COMPLETION OF THE NEW SPINNING EOB MODEL

A. Alignment of the EOB and NR time axes and determination of NQC corrections

In early (nonspinning) EOB work the correspondence between the EOB and NR time axes was defined by identifying the peak of the orbital EOB frequency $\nu_{\text{EOB}}^\text{peak}$ with the peak of the $\ell = m = 2$ NR amplitude $\nu_{\text{EOB}}^{\text{peak}}$. However, later work [29, 33, 56] introduced as a free parameter a time-lag between the peak of the orbital $\nu_{\text{EOB}}^{\text{peak}}$ and
Here we shall make use of this flexibility but we shall parametrize it in a different way by using as main “anchor point” on the EOB time axis the peak \(t_{\text{peak}}\) of the pure orbital frequency

\[
\Omega_{\text{orb}} = \frac{M}{\mathcal{H}} \frac{\partial \mathcal{H}_{\text{eff}}}{\partial L} = \frac{p_z h_z^2 A}{\mathcal{H} H_{\text{eff}}/m} \tag{101}
\]

The motivation for this choice is the following: while, in the test mass limit, \(t_{\text{peak}}\) differed from \(t_{\text{NQC}}\) by an amount which was getting as large as several tens of \(M\) for large and positive spin [73], it was found that (in the test-mass limit) \(t_{\text{peak}}\) differed from \(t_{\text{NQC}}\) only by a few \(M\)'s for most spin values [74]. In the present work we parametrize this flexibility by introducing the quantity \(\Delta t_{\text{NQC}}\) defined so that

\[
t_{\text{NQC}} \equiv t_{\text{peak}} - \Delta t_{\text{NQC}}. \tag{102}
\]

In addition we define the correspondence between the EOB and NR time axes by requiring that

\[
t_{\text{NQC}}^{\text{EOB}} \leftrightarrow t_{\text{NQC}}^{\text{NR}}, \tag{103}
\]

where we choose the NR extraction point \(t_{\text{extr}}^{\text{NR}}\) to be \(2M\) on the right of the peak of the NR \(h_{22}\) waveform, i.e.

\[
t_{\text{extr}}^{\text{NR}} = t_{\text{extr}}^{\text{NR}} + 2M. \tag{104}
\]

This choice means that the peak on \(h_{22}\) on the NR time axis corresponds to

\[
t_{\text{extr}}^{\text{EOB}} \leftrightarrow t_{\text{extr}}^{\text{NQC}} - 2M - \Delta t_{\text{NQC}}. \tag{105}
\]

Analogously to Ref. [56] (mutatis mutandis), the degree of osculation between the EOB and NR waveforms at \(t_{\text{EOB}}^{\text{EOB}} \leftrightarrow t_{\text{EOB}}^{\text{NR}}\) is defined by imposing the following four conditions

\[
\begin{align}
A_{\ell m}^{\text{EOB}} (t_{\text{EOB}}^{\text{EOB}}) &= A_{\ell m}^{\text{NR}} (t_{\text{EOB}}^{\text{NR}}), \tag{106a} \\
A_{\ell m}^{\text{EOB}} (t_{\text{EOB}}^{\text{EOB}}) &= A_{\ell m}^{\text{NR}} (t_{\text{EOB}}^{\text{NR}}), \tag{106b} \\
\omega_{\ell m}^{\text{EOB}} (t_{\text{EOB}}^{\text{EOB}}) &= \omega_{\ell m}^{\text{NR}} (t_{\text{EOB}}^{\text{NR}}), \tag{106c} \\
\omega_{\ell m}^{\text{EOB}} (t_{\text{EOB}}^{\text{EOB}}) &= \omega_{\ell m}^{\text{NR}} (t_{\text{EOB}}^{\text{NR}}), \tag{106d}
\end{align}
\]

which yield both a \(2 \times 2\) linear system to be solved to obtain the \(a_{\ell m}^c\)'s, and, separately, a \(2 \times 2\) linear system to be solved for the \(b_{\ell m}^c\)'s. Note that the values of the \(a_{\ell m}^c\)'s affect the modulus of the inspiral-plus-plunge waveform, which then affects the computation of the radiation reaction force (through the angular momentum flux). In turn, this modifies the EOB dynamics itself, and, consequently, the determination of the \((a_{\ell m}^c, b_{\ell m}^c)'s\). This means that one must bootstrap, by iteration, the determination of the \((a_{\ell m}^c, b_{\ell m}^c)'s\) until convergence (say at the third decimal digit) is reached. This typically requires three iterations.

\[
\begin{align}
a_{0}^{c} &= -129, \tag{107} \\
\Delta t_{\text{NQC}}(\chi = 0) &= 1M. \tag{108}
\end{align}
\]

\[
\begin{align*}
\Delta t_{\text{NQC}}(\chi = 0) &= 1M. 
\end{align*}
\]

B. Analytical freedom of the model and its NR calibration

In summary, the adjustable parameters of the present spinning EOB model are: \((a_{0}^{c}, c_{3}, \Delta t_{\text{NQC}})\). As said above the radiation reaction used in the present EOB model differs from the one of Ref. [56] by setting to zero the radial component \(F_r = 0\). This change in radiation reaction obliged us to (marginally) recalibrate the nonspinning model of Ref. [56], which means, in the presently considered equal-mass case, the values of \(a_{0}^{c}\) and \(\Delta t_{\text{NQC}}\). In doing so, we could also simplify the NQC factor of Ref. [56], reducing the NQC parameters from the six used there (3 for the amplitude and 3 for the phase) to the four we use here. We found that the values

\[
\begin{align}
a_{0}^{c} &= -129, \tag{107} \\
\Delta t_{\text{NQC}}(\chi = 0) &= 1M. \tag{108}
\end{align}
\]

give a EOBNR phase agreement comparable to the one of [56], as illustrated in Fig. 2. The NR/EOB alignment time window is indicated by the two dash-dotted vertical lines during inspiral indicate the alignment time window is indicated by the two dashed vertical lines during inspiral.
merger. The values of \((c_3, \Delta t_{\text{NQC}})\) we used are listed in Table I, together with the corresponding \(a^{(22)}_1, h^{(22)}_1\)'s.

The values of \(c_3\) leading to a good phase agreement were found to depend (in a roughly piecewise linear manner) on \(\chi\). Such a \(\chi\) dependence of the effective NNNLO spin-orbit parameter \(c_3\) can be intuitively understood if one remembers that we have parametrized NNNLO spin-orbit effects by a purely \(r\)-dependent term \(\nu c_3^2/r^3\) while we could have also included, at the same PN order, terms of order \(\nu c_3^2 p_2^2/r^2\). In other words the single, effective, value of \(c_3\) we use can be roughly considered as being of order \(c_3 \approx c_0^2 + (r p_2^2 c_3^2)\) where \((r p_2^2)\) is some average value of \(r p_2^2\) during the plunge. When \(\chi > 0\) the plunge phase becomes shorter and shorter as \(\chi\) increases, while when \(\chi < 0\) the plunge phase becomes more prominent as \(\chi \to -1\). Correspondingly we expect that the average value \((r p_2^2)\) will depend on \(\chi\) and take significantly larger values for \(\chi \to -1\). These considerations might explain why we had to use a \(\chi\)-dependent value (increasing when \(\chi \to -1\)) for the effective NNNLO parameter \(c_3\).

We leave to future work a study including more general ways of parametrizing NNNLO effects, possibly including \(p^2\)-dependent terms in \((G_S, G_{S_2})\).

These time-domain comparisons are done by suitably determining a relative time and phase shift between the two phases \(\phi_{22}^{\text{NR}}(t^{\text{NR}})\) and \(\phi_{22}^{\text{EOB}}(t^{\text{EOB}})\). These shifts are estimated by minimizing the time integral of the square of the phase difference on a time interval corresponding to a given frequency interval \([M \omega_L, M \omega_R]\). Following Refs. [29, 56, 75], usually we perform this waveform alignment on the long inspiral phase. More precisely, we take \(M \omega_R = 0.041\) for all waveforms, while \(M \omega_L\) ranges between 0.031 and 0.033 for all spin values except for \(\chi = +0.44\), where we take \(M \omega_L = 0.0386\). Figure 3 illustrates the EOB/NR agreement for the \(\ell = m = 2\) waveform (both in phase and amplitude) for the representative values of \(\chi = (-0.95, -0.44, +0.44, +0.98)\). The dashed vertical line on the plots indicates the NR NQC extrac-
TABLE I. Parameters of the model: effective NNNLO spin-orbit coupling, $c_3$, NR/EOB time-shift $\Delta t_{NQC}$ and next-to-quasi-circular ($a_{i}^{(m)}$, $b_{i}^{(m)}$) parameters for the sixteen values of the spin considered. The NQC parameters are obtained by imposing occlusion conditions between EOB and NR waveform amplitudes and frequencies around merger.

| $\chi$ | $c_3$ | $\Delta t_{NQC}$ | $a_1^{22}$ | $a_2^{22}$ | $b_1^{22}$ | $b_2^{22}$ |
|-------|-------|------------------|------------|------------|------------|------------|
| -0.94905 | 92.5 | 1 | 0.143 | 0.337 | 0.163 | 3.489 |
| -0.8996 | 89 | 1 | 0.139 | 0.364 | 0.165 | 3.362 |
| -0.7998 | 84 | 1 | 0.135 | 0.442 | 0.161 | 2.895 |
| -0.5999 | 73 | 1 | 0.129 | 0.614 | 0.156 | 2.090 |
| -0.43756 | 67 | 1 | 0.128 | 0.786 | 0.150 | 1.583 |
| -0.2000 | 59 | 1 | 0.126 | 1.032 | 0.147 | 1.018 |
| 0 | ... | 1 | -0.075 | 1.496 | 0.148 | 0.914 |
| +0.2000 | 26 | 1 | 0.106 | 1.574 | 0.134 | 0.564 |
| +0.436554 | 17 | 1 | -0.023 | 1.862 | 0.086 | 0.263 |
| +0.6000 | 16.5 | 1 | -0.129 | 1.598 | 0.083 | 1.016 |
| +0.7999 | 8.5 | 1 | -0.233 | 0.632 | 0.572 | 7.947 |
| +0.8498 | 5.5 | 1 | -0.313 | -0.244 | 0.809 | 16.061 |
| +0.8997 | 5.5 | -1 | 0.079 | 0.214 | 0.768 | 12.607 |
| +0.9496 | 4.5 | -3 | 0.432 | 0.338 | 0.678 | 10.900 |
| +0.9695 | 4.5 | -4 | 0.612 | 0.562 | 0.525 | 8.328 |
| +0.9794 | 3.5 | -4 | 0.608 | 0.214 | 0.638 | 12.058 |

The alignment time window corresponding to $[M\omega_L, M\omega_R]$ is indicated by the two dash-dotted vertical lines in the inspiral phase.

The amplitude agreement (see dashed, orange online, lines) is excellent all over, including during the ringdown. The phasing disagreement is always remarkably small during the inspiral phase.

In the present exploratory study, we considered a minimal version of our new SEOB model having only three adjustable parameters: $a_0^t$, $\Delta t_{NQC}$ and $c_3$. The nonspinning limit of this model differs (from the analytical point of view) from our previous nonspinning model [56] essentially in having set the radial component of the radiation reaction $F_r$ to zero. This led us to recalibrate the value of $a_0^t$ against nonspinning SXS NR waveforms, which yielded $a_0^t = -129$ (instead of our previous preferred value $a_0^t = -101.876$). Such a calibration (together with a modified choice of $\Delta t_{NQC}$) yields a $l = m = 2$ waveform whose phasing agrees within $\pm 0.08$ rad with the $q = 1$, nonspinning, SXS NR waveform.

Then, by calibrating (for each value of the spin $\chi$) the single, effective, next-to-next-to-leading-order spin-orbit parameter $c_3$ we were able to obtain a good EOB/NR phasing agreement over the full time span of each of the fifteen, presently catalogued spinning SXS waveforms. The range of spin values spanned by these waveforms is $-0.95 \leq \chi \leq 0.98$. The longest waveforms have about 50 GW cycles up to merger. In the case of $\chi = 0.98$ the accumulated EOB/NR phasing disagreement over the 50 GW cycles before merger (corresponding to $\Delta t \sim 6420M$) is of the order of 0.1 rad.

The present study leaves room for many possible improvements and further investigations such as: (i) including the recently determined 4PN Hamiltonian [57, 76] as well as the gravitational-self-force knowledge of the spin-orbit coupling [42, 43]; (ii) the role of the various ways of gauge-fixing the spin-orbit couplings so as to eliminate the $\chi$-dependence of the effective spin-orbit parameter $c_3$ or at least to introduce some additional analytical flexibility (e.g., of the type $c_3'$ mentioned above) leading to a smooth, and therefore fittable, $\chi$-dependence.

VI. CONCLUSIONS

We have presented a new (nonprecessing) spinning EOB model. The most novel feature of our scheme is the use (for equatorial dynamics) of the concept of centrifugal radius $r_c$. We showed the interest of this concept for the test-mass Kerr Hamiltonian and used its comparable-mass-case generalization to define a new way of blending the spin-deformation with the mass-ratio-deformation. In addition, we have used a recently proposed new way to attach the ringdown waveform, based on a new (NR-fitted) way of parametrizing the ringdown signal.

In the present exploratory study, we considered a minimal version of our new SEOB model having only three adjustable parameters: $a_0^t$, $\Delta t_{NQC}$ and $c_3$. The nonspinning limit of this model differs (from the analytical point of view) from our previous nonspinning model [56] essentially in having set the radial component of the radiation reaction $F_r$ to zero. This led us to recalibrate the value of $a_0^t$ against nonspinning SXS NR waveforms, which yielded $a_0^t = -129$ (instead of our previous preferred value $a_0^t = -101.876$). Such a calibration (together with a modified choice of $\Delta t_{NQC}$) yields a $l = m = 2$ waveform whose phasing agrees within $\pm 0.08$ rad with the $q = 1$, nonspinning, SXS NR waveform.

Then, by calibrating (for each value of the spin $\chi$) the single, effective, next-to-next-to-leading-order spin-orbit parameter $c_3$ we were able to obtain a good EOB/NR phasing agreement over the full time span of each of the fifteen, presently catalogued spinning SXS waveforms. The range of spin values spanned by these waveforms is $-0.95 \leq \chi \leq 0.98$. The longest waveforms have about 50 GW cycles up to merger. In the case of $\chi = 0.98$ the accumulated EOB/NR phasing disagreement over the 50 GW cycles before merger (corresponding to $\Delta t \sim 6420M$) is of the order of 0.1 rad.

The present study leaves room for many possible improvements and further investigations such as: (i) including the recently determined 4PN Hamiltonian [57, 76] as well as the gravitational-self-force knowledge of the spin-orbit coupling [42, 43]; (ii) the role of the various ways of gauge-fixing the spin-orbit couplings so as to eliminate the $\chi$-dependence of the effective spin-orbit parameter $c_3$ or at least to introduce some additional analytical flexibility (e.g., of the type $c_3'$ mentioned above) leading to a smooth, and therefore fittable, $\chi$-dependence.

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