Generalized versus selected descriptions of quantum LC-circuits

E. Papp†
Department of Theoretical Physics, West University of Timisoara, 300223, Romania

C. Micu†
Physics Department, North University of Baia Mare, 430122, Romania

O. Borchin
Department of Theoretical Physics, West University of Timisoara, 300223, Romania.

L. Aur
Department of Theoretical Physics, West University of Timisoara, 300223, Romania.

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Proofs are given that the quantum-mechanical description of the LC-circuit with a time dependent external source can be readily established by starting from a more general discretization rule of the electric charge. For this purpose one resorts to an arbitrary but integer-dependent real function $F(n)$ instead of $n$. This results in a nontrivial generalization of the discrete time dependent Schrödinger-equation established before via $F(n) = n$, as well as to modified charge conservation laws. However, selected descriptions can also be done by looking for a unique derivation of the effective inductance. This leads to site independent inductances, but site dependent ones get implied by accounting for periodic solutions to $F(n)$ in terms of Jacobian elliptic functions. Many-charge generalizations of quantum circuits, including the modified continuity equation for total charge and current densities, have also been discussed.

Keywords: Quantum LC-circuits; Charge discretization; Discrete Schrödinger equations; Many-charge generalizations

I. INTRODUCTION

A fundamental concept which is responsible for sensible effects in electronic devices is the discreteness of the electric charge [1]. Quantizations of the conductance in units of $e^2/hc$ [2,3], or of the magnetic flux in units of $hc/e$ [4], can also be mentioned. Note that the charge of the electron is $-e$, as usual. Handling the discretized charge also means that the application of the discrete calculus, such as done by left ($\nabla$) - and right-hand ($\Delta$) discrete derivatives, is rather suitable. Looking for explanations, we have to realize that looking for low dimensional nanoscale systems on semiconductor quantum wells and nanoelectronic devices [1,6]. In latter cases the coherence length gets larger than sample dimensions, which leads to wealthy manifestations of quantum interference phenomena. Much progress has also been done in the field of miniaturization of circuits. In this context it has been found that the quantum mechanical description of LC [7-10]-, $L$ [10,11]- and RLC [12,13]-circuits can be done by resorting once more again to the charge discretization. Studies in such fields are promising, as they produce ideas for further technological developments.

We shall then use this opportunity to discuss in some more detail the quantum-mechanical description of the mesoscopic LC-circuit with a time dependent voltage source $V_s(t)$. So far the discrete Schrödinger-equation characterizing the LC-circuit has been established by starting from the concrete charge eigenvalue equation [7,8,10,11]

$$Q_q |n| = nq_e |n| \quad ,$$  

where $Q_q$ denotes the Hermitian charge operator and $n$ is an integer playing the role of the discrete coordinate. Equation (1) shows, of course, that the electric charge is quantized in units of the elementary electric charge $q_e = e$. One could also say that $q_e = 2e$ when dealing with Cooper-pairs [14,15]. However, more general discrete Schrödinger-equations concerning LC-circuits can also be derived. For this purpose we shall begin by performing a charge mapping like $Q_q \rightarrow Q_{\tilde{q}} = F(Q_q)$, where $F(n)$ is a real function of $n$. Applying discrete derivatives to the eigenvalue equation of $Q_{\tilde{q}}$, i.e. to

$$\tilde{Q}_{\tilde{q}} |n| = F(Q_q) |n| = F(n) q_e |n| \quad ,$$  

results, surprisingly enough, in a generalized counterpart of the discrete Schrödinger-equation relying on (1), as well as in non-trivial modifications of the charge conservation law.
The first task concerns the derivation of the canonically conjugated observable, i.e. of suitable magnetic flux operators. We shall then obtain a pair of non-Hermitian but conjugated magnetic flux operators. The product of such operators is then responsible for the kinetic energy-term, i.e. for the Hermitian operator of the square magnetic flux. Within the next stage of our approach we shall look, however, for selected realizations of $F(n)$ yielding a reliable description from the physical point of view. This proceeds by establishing in a well defined manner the effective impedance for the quantum circuit one deals with. Many-charge generalizations of quantum-circuit equations can also be readily established.

II. PRELIMINARIES AND NOTATIONS

We have to recall that the classical $RLC$-circuit is described by the balance equation

$$L\frac{dI}{dt} + IR + \frac{Q}{C} = V_s(t)$$ \hspace{1cm} (3)

in accord with Kirchhoff’s law, where the current, the inductance, the capacitance and the resistance are denoted by $I = dQ/dt$, $L$, $C$ and $R$, respectively. Periodic modulations of the voltage like $V_s(t) = V_0 \cos(\omega t)$ are frequently used, in which case the circuit is characterized by the impedance

$$Z = R + i\left(\omega L - \frac{1}{\omega C}\right)$$ \hspace{1cm} (4)

Inserting, for convenience, $R = 0$, leads to the Hamiltonian

$$H_c\left(Q, \frac{\Phi}{c}\right) = \frac{\Phi^2}{2Lc^2} + \frac{Q^2}{2C} - QV_s(t)$$ \hspace{1cm} (5)

where $\Phi = ILc$ stands for the magnetic flux. Accordingly, (3) is produced by the Hamiltonian equations of motion

$$I = \frac{dQ}{dt} = \frac{\partial H}{\partial (\Phi/c)} = \frac{\Phi}{Lc}$$ \hspace{1cm} (6)

and

$$\frac{d}{dt}\left(\frac{\Phi}{c}\right) = -\frac{\partial H}{\partial Q} = -\frac{Q}{C} + V_s(t)$$ \hspace{1cm} (7)

as usual. This also means that the electric charge and the magnetic flux divided by $c$, i.e. $Q$ and $\Phi/c$, are canonically conjugated variables. This result suggest that the quantization of the $LC$-circuit should be done in terms of the canonical commutation relation

$$[Q, \Phi] = i\hbar c$$ \hspace{1cm} (8)

in which case one gets faced with the flux-operator [16]

$$\Phi = -i\hbar c\frac{\partial}{\partial Q}$$ \hspace{1cm} (9)

However, a such realization is questionable because the electric charge, such as defined by (1) is not a continuous observable. This means that the introduction of a discretized version of (9) like

$$\Phi_q = -\frac{i\hbar c}{q_c} \Delta$$ \hspace{1cm} (10)

for which $\Phi_q^+ = -i\hbar c \nabla / q_c$ is in order. The Hermitian time-dependent Hamiltonian of the quantum $LC$-circuit can then be established as

$$H_q = \frac{\Phi_q^+ \Phi_q}{2Lc^2} + \frac{\tilde{Q}^2}{2C} - \tilde{Q}V_s(t)$$ \hspace{1cm} (11)

in which $H_q^{(0)} = \Phi_q^+ \Phi_q/2Lc^2$ has the meaning of the kinetic energy. This amounts to solve the discrete Schrödinger-equation

$$\mathcal{H}_q < n | \Psi(t) > = i\hbar \frac{\partial}{\partial t} < n | \Psi(t) >$$ \hspace{1cm} (12)

working, of course, within the charge-number representation. The Hermitian momentum operator an also be readily introduced as $P_q = (\Phi_q + \Phi_q^+)/2$.

Note that right- and left-hand discrete derivatives referred to above proceed as [17]

$$\Delta f(n) = f(n+1) - f(n) = (\exp(\partial/\partial n) - 1) f(n)$$ \hspace{1cm} (13)

and

$$\nabla f(n) = f(n) - f(n-1) = (1 - \exp(-\partial/\partial n)) f(n)$$ \hspace{1cm} (14)

so that $\Delta^+ = -\nabla$ and $\nabla \Delta = \Delta - \nabla$. In addition, one has the product rule

$$\nabla (f(n)g(n)) = g(n) \nabla f(n) + f(n-1) \nabla g(n)$$ \hspace{1cm} (15)

and similarly for $\Delta$.

III. GENERALIZED VERSIONS OF THE ELECTRIC CHARGE QUANTIZATION

Next let us apply discrete derivatives presented above both to (2) and $|n\rangle$. One finds
\[ \tilde{Q}_q \Delta = q_e F(n + 1) \Delta + q_e \Delta F(n) \] 

and

\[ \nabla \tilde{Q}_q = q_e F(n - 1) \nabla + q_e \nabla F(n) \] 

Performing the Hermitian conjugation gives \( \nabla \tilde{Q}_q = q_e F(n) \nabla \) and \( \Delta \tilde{Q}_q = q_e F(n) \Delta \), where \( \tilde{Q}_q^+ = \bar{\tilde{Q}}_q \). Accordingly

\[ [\tilde{Q}_q, \Delta] = q_e \Delta F(n) (1 + \Delta) \] 

and

\[ [\tilde{Q}_q, \nabla] = q_e \nabla F(n) (1 - \nabla) \] 

Now we are ready to introduce rescaled magnetic flux operators like

\[ \tilde{\Phi}_q = \frac{-i\hbar c}{q_e} \left( \frac{1}{\Delta F(n)} \right) \] 

which can be viewed as the generalized counterparts of (10) and

\[ \tilde{\Phi}_q^+ = \frac{-i\hbar c}{q_e} \left( \frac{1}{\nabla F(n)} \right) \] 

Accordingly, the interaction-free Hamiltonian is given by

\[ \tilde{\mathcal{H}}^{(0)}_q \to \tilde{\mathcal{H}}^{(0)} = \frac{\tilde{\Phi}_q^+ \tilde{\Phi}_q}{2Lc^2} \] 

which can be rewritten equivalently as

\[ \tilde{\mathcal{H}}^{(0)} = -\frac{\hbar^2}{2L(n) q_e^2} \left( \tilde{\Delta} - \nabla \right) \] 

This time the inductance gets rescaled as

\[ L \to \tilde{L}(n) = L \left( \nabla F(n) \right)^2 \] 

whereas the discrete right hand derivative \( \Delta \) is replaced anisotropically by

\[ \tilde{\Delta} = (1 - G(n)) \Delta \] 

One has

\[ G(n) = 1 - \left( \frac{\nabla F(n)}{\Delta F(n)} \right)^2 \] 

which leads to sensible effects. Under such conditions the anisotropic discrete Schrödinger-equation for the single-charge amplitude \( C_n(t) = <n | \Psi(t)> \) is given by

\[ -\frac{\hbar^2(1 - G(n))}{2L(n) q_e^2} C_{n+1}(t) - \frac{\hbar^2}{2L(n) q_e^2} C_{n-1}(t) + \frac{\hbar^2}{L(n) q_e^2} \left( 1 - \frac{G(n)}{2} \right) + \frac{q_e^2}{2C} F^2(n) - q_e F(n) V_s(t) \] 

\[ C_n(t) = \frac{i\hbar}{\partial t} C_n(t) \]

which works in accord with (2) and (12). It is clear that (27) reproduces the usual result [7]

\[ -\frac{\hbar^2}{2Lq_e^2} \nabla \Delta C_n(t) + \left( \frac{q_e^2}{2C} n^2 - q_e n V_s(t) \right) C_n(t) = \frac{i\hbar}{\partial t} C_n(t) \]

via \( F(n) \to n \).

**IV. MODIFIED CHARGE CONSERVATION LAWS**

One sees that (27), which differs in a sensible manner from (28), has a rather complex structure such as involved by the \( n \)-dependence of coefficient functions. Such structures exhibit a certain similarity to Schrödinger equations with a position dependent effective mass [18]. Furthermore, we have to realize that (27) as it stands provides useful insights for more general descriptions. Indeed, (27) produces a modified continuity equation like

\[ \frac{\partial}{\partial t} \rho_n(t) + \Delta J_n(t) = g_n(t) \] 

where

\[ \rho_n(t) = q_e | C_n(t) |^2 \] 

denotes the usual charge density, whereas

\[ J_n(t) = \frac{\hbar}{L(n) q_e} \text{Im} \left( C_n(t) C_{n-1}^*(t) \right) \]

stands for the related current density. The additional term in the continuity equation is

\[ g_n(t) = q_e G(n) \frac{\tilde{L}(n + 1)}{L(n)} J_{n+1}(t) \]
which shows that there are additional effects which are able to affect the time dependence of the charge density. This results in the onset of an extra charge density like
\[ \rho_n^{(diff)}(t) = -G(n) \frac{\tilde{L}(n+1)}{L(n)} \int_{-\infty}^{t} J_{n+1}(t')dt' \],
which relying typically on the nonlinear attributes of the generalized charge discretization function. The total charge density is then given by
\[ \rho_{n}^{(tot)}(t) = \rho_{n}(t) + \rho_{n}^{(diff)}(t) \],
in which it has been assumed that \( \rho_{n}^{(diff)}(t) \to 0 \) when \( t \to -\infty \).

V. INTRODUCING THE EFFECTIVE IMPEDANCE

Equation (27) can also be interpreted in terms of an effective anisotropic inductance by assuming three different realizations, namely \( L_2(n) = \tilde{L}(n)/(1 - G(n)) \), \( L_3(n) = \tilde{L}(n)/(1 - G(n)/2) \). However, the isotropy can be restored via
\[ L_1(n) = L_3(n) = \tilde{L}(n) \],
in which case
\[ G(n) = 0 \].

Accordingly, one should have
\[ \nabla F(n) = \pm \Delta F(n) \],
by virtue of (26), so that
\[ \nabla \Delta F(n) = F(n + 1) - 2F(n) + F(n - 1) = 0 \],

or
\[ F(n + 1) = F(n - 1) \],

respectively. Under such conditions the modifications to the continuity equation are ruled out, as one might expect.

Equation (38) has two kinds of solutions. First there is the linear realization
\[ F^{(1)}(n) = \alpha_1 n + \beta_1 \],
where \( \alpha_1 \) and \( \beta_1 \) are parameters, for which the effective inductance is \( \tilde{L}_1 = \alpha_2^2 L \). The rational charge quantization is performed in terms of the fixings \( \alpha_1 = 1/P \) and \( \beta_1 = 0 \), where \( P \) is a non-zero integer. Periodic functions with unit period could eventually be considered. However, in such cases one has \( \nabla F(n) = \Delta F(n) = 0 \), which means in turn that such solutions can not be accepted. Note that (40) produces sensible modifications going beyond (28). Indeed, the wave function acquires an additional phase via
\[ C_n(t) \to C_n(t) \exp \left( \frac{i}{\hbar} \beta_1 q_c \int_0^t V_s(t')dt' \right) \],
whereas the voltage is supplemented by an additional dc-component, as indicated by the superposition
\[ V_s^{(1)}(t) = \alpha_1 q_c n \left( \frac{\beta_1 q_c}{C} - V_s(t) \right) \].

This time the shifted harmonic oscillator term is given by
\[ V_{HO}(n) = \frac{q_c^2 \alpha_2^2}{2C} n^2 + \frac{q_c^2 \beta_1^2}{2C} \],
so that the total potential energy reads \( V^{(tot)} = V_s^{(1)} + V_{HO}(n) \). Moreover, we are in a position to introduce the effective \( n \)-independent impedance
\[ \tilde{Z}_1 = i \left( \omega \tilde{L}_1 - \frac{1}{\omega C} \right) \],
which proceeds in accord with (4), (24) and (40), where now \( R = 0 \).

Equation (39) has to be solved in terms of periodic functions of double period 2, i.e. in terms of trigonometric and/or Jacobian elliptic functions. In the former case we can propose the solution
\[ F^{(2)}(n) = \alpha_2 \sin(\pi n + \delta_2) + \beta_2 \],
producing an oscillatory charge, for which
\[ \nabla F^{(2)}(n) = 2\alpha_2 (-1)^n \sin \delta_2 \].
The corresponding \( n \)-independent effective inductance is given by \( \tilde{L}_2 = 4\alpha_2^2 \sin^2 \delta_2 \), so that the impedance, say \( \tilde{Z}_2 \), can be readily established in a close analogy with (44). A further solution working in terms of Jacobian elliptic functions such as given by
\[ F^{(3)}(n) = \alpha_3 \sin(2nK + \delta_3) + \beta_3 \],

(47)
can also be proposed. Here \( sn(u) \) denotes the sine amplitude, \( u \) stands for the argument, whereas \( K = K(k) \) is the well known complete elliptic integral of modulus \( k \) \[19\]. Just note that \( sn( -u ) = -sn(u), sn(u + 4K) = sn(u) \) and \( sn(u + 2K) = -sn(u) \). Now one has

\[
\nabla F^{(3)}(n) = -2\alpha_3 sn(2nK + \delta_3),
\]

which shows that this time one deals with the effective sumed. The present charge eigenfunctions are expressed as

\[
\tilde{Z}_3(n) = 4\alpha_3^2 sn^2(2nK + \delta_3)L,
\]

so that the same concerns the related impedance

\[
\bar{Z}_3(n) = i \left( \frac{\omega \tilde{L}_3(n)}{1 - \frac{1}{\omega C}} \right).
\]

For convenience, we have restricted ourselves to periodic \( F_i(n) \)-functions \((i = 2, 3)\) defined in terms of odd functions, as shown by (45) and (47). This corresponds to the linear \( \alpha_i n \)-term in (40), but the flux dependence of persistent currents in Aharonov-Bohm rings could also be invoked \[20\]. Further clarifications concerning this point remain, however, desirable. Two-point impedances can also be established \[21\], which proceeds in terms of eigenvalues of related Laplacian matrices.

VI. MANY CHARGE GENERALIZATIONS OF QUANTUM LC-CIRCUITS

Many-charge generalizations of (11) like

\[
\mathcal{H}^{(MC)}_q = \sum_{j=1}^{N} \mathcal{H}^{(j)}_q
\]

where

\[
\mathcal{H}^{(j)}_q = \frac{\Phi^{(j)}_q + \Phi^{(j)*}_q}{2L_jC_j} + \frac{\tilde{Q}_q^{(j)2}}{2C_j} - \tilde{Q}_q^{(j)}V_s(t)
\]

can also be proposed. The charge operators, the inductances and the capacitances are denoted by \( \tilde{Q}_q^{(j)} \), \( L_j \) and \( C_j \), respectively, where now \( j = 1, 2, \ldots, N \). Such Hamiltonians are synonymous to many-body counterparts of (11). Accordingly, (2) gets generalized as

\[
\tilde{Q}_q^{(j)} | n; N > = q_j F_j(\tilde{Q}_q^{(j)}) | n; n_j > = q_j F_j(n_j) | n_j >
\]

where the \( n_j \)'s are integers which are responsible for the charge eigenvalues. For the sake of generality, several charge scales, say \( q_j \) instead of \( q_e \), have also been assumed. The present charge eigenfunctions are expressed by products like

\[
\mathcal{H}_q^{(MC)} C_{n:N}(t) = i\hbar \frac{\partial}{\partial t} C_{n:N}(t)
\]

by accounting for the factorization ansatz

\[
C_{n:N}(t) \equiv n_1, n_2, \ldots, n_N | \Psi(t) > = \prod_{j=1}^{N} C_{n_j}^{(j)}(t)
\]

Having obtained single charge amplitudes via

\[
\mathcal{H}^{(j)}_q C_{n_j}^{(j)}(t) = i\hbar \frac{\partial}{\partial t} C_{n_j}^{(j)}(t)
\]

opens the way to establish the \( N \)-charge amplitude in terms of (59). This separation produces a unique solution if \( N = 2 \). This means that (59) has to be understood as a reasonable extrapolation of the well defined \( N = 2 \)-result towards \( N \gg 3 \). The many charge version of (27) is then given by

\[
\mathcal{H}_q^{(MC)} C_{n:N}(t) = \frac{h^2(1 - G_j(n_j))}{2L_j(n_j)q_j^2} C_{n_j+1}^{(j)}(t) - \frac{h^2}{2L_j(n_j)q_j^2} C_{n_j-1}^{(j)}(t) +
\]

\[
\frac{h^2}{L_j(n_j)q_j^2} \left[ 1 - \frac{G_j(n_j)}{2} \right] + \frac{q_j^2}{2C_j} F_j^2(n_j) - q_j F_j(n_j) V_s(t)
\]

where

\[
\cdot C_{n_j}^{(j)}(t) = i\hbar \frac{\partial}{\partial t} C_{n_j}^{(j)}(t)
\]
for $j = 1, 2, \ldots, N$, where

$$\bar{L}_j(n_j) = L_j (\nabla_j F_j(n_j))^2$$  \hspace{1cm} (62)

and

$$G_j(n_j) = 1 - \left( \frac{\nabla_j F_j(n_j)}{\Delta_j F_j(n_j)} \right)^2.$$  \hspace{1cm} (63)

Repeating the same steps as before leads to the generalized continuity equation

$$\frac{\partial}{\partial t} \rho_{n_j}^{(j)}(t) + \Delta_j J_{n_j}^{(j)}(t) = g_{n_j}^{(j)}(t)$$  \hspace{1cm} (64)

where

$$\rho_{n_j}^{(j)}(t) = q_j |C_{n_j}^{(j)}|^2$$  \hspace{1cm} (65)

and

$$J_{n_j}^{(j)}(t) = \frac{\hbar}{L_j(n_j)q_j} \text{Im} \left( C_{n_j}^{(j)}(t) C_{n_j+1}^{(j)*}(t) \right)$$  \hspace{1cm} (66)

and

$$g_{n_j}^{(j)}(t) = q_j G_j(n_j) \frac{\bar{L}_j(n_j+1)}{L_j(n_j)} J_{n_j+1}^{(j)}(t).$$  \hspace{1cm} (67)

Our next step is to perform the $j$-summation in (64). This leads to the derivation of total charge and current densities as

$$\rho_{n}^{(tot)}(t) = \sum_{j=1}^{N} \rho_{n_j}^{(j)}(t)$$  \hspace{1cm} (68)

and

$$\bar{J}_{n}^{(tot)}(t) = \left\{ J_{n_1}^{(1)}(t), J_{n_2}^{(2)}(t), \ldots, J_{n_N}^{(N)}(t) \right\}.$$  \hspace{1cm} (69)

respectively. Correspondingly, the continuity equation reads

$$\frac{\partial}{\partial t} \rho_{n}^{(tot)}(t) + \Delta \cdot \bar{J}_{n}^{(tot)}(t) = G_{n}^{(tot)}(t)$$  \hspace{1cm} (70)

where $\Delta = \{ \Delta_1, \Delta_2, \ldots, \Delta_N \}$ and

$$G_{n}^{(tot)}(t) = \sum_{j=1}^{N} g_{n_j}^{(j)}(t).$$  \hspace{1cm} (71)

The conservation of the total charge would then occur when $G_{n}^{(tot)}(t) = 0$ irrespective of $t$. This happens if $F_j(n_j) \rightarrow \alpha_j n_j + \beta_j$, but the same concerns realizations complying with (45) or (47), respectively. Alternatively, there are mutual cancellation effects leading to $G_{n}^{(tot)}(t) = 0$, which are worthy of being considered in some more detail.

**VII. CONCLUSIONS**

In this paper we succeeded to establish a more general quantum-mechanical description of $LC$-circuits by starting from a generalized discretization rule for the electric charge. To this aim one resorts to a real, but integer dependent function $F(n)$ instead of $n$. This leads to the generalized discrete Schrödinger-equation (27), which reproduces the usual result as soon as $F(n) = n$. A such generalized equation is able to incorporate additional effects going beyond the charge conservation proceeding usually in terms of ingoing and outgoing electron flows. Selected realizations of such generalized descriptions are able to be done by resorting to additional physical requirements. For this purpose we found it suitable to look for a unique derivation of the effective inductance, as shown by (35). This leads to a linear realization of $F(n)$ such as indicated by (40), but additional periodic solutions with double period 2 can also be proposed. Accordingly, one gets faced both with trigonometric and elliptic solutions. Such solutions are illustrated by (45) and (47), but other selections can also be done specifically. One sees that the effective inductance remains independent of $n$ both in terms of (40) and (45), but (49) exhibits clearly a non-trivial $n$-dependence. Many charge generalizations of quantum $LC$-circuits can also be done, as indicated in section 6. The modified continuity equation concerning total charge and current densities has also been discussed.

It is worthy of being mentioned that the capacitance is sensitive to the discreteness of the electronic charge, too [22]. This means that the conduction is suppressed at low temperatures and small applied voltages, but this phenomenon of “Coulomb blockade”can be removed by periodically modulated capacitive charging. Such effects have also been discussed by resorting to pure capacity-design circuits [8].

The charge discretization is also able to serve to the quantum description of other systems which are relevant in nanoelectronics, namely miniaturized $LC$ ladder-circuits. First steps along this direction have already been done, but further investigations are in order [13, 23]. Such circuits contain cells coupled capacitively, so that currents ($I_n$), voltages ($V_n$), charges ($Q_n$) and magnetic fluxes ($\Phi_n$) are site dependent. Within the linear regime the current obeys the equation

$$\nabla \Delta I_n = L C \frac{d^2 I_n}{dt^2},$$  \hspace{1cm} (72)

and similarly for $V_n$. We have to remark that these equations are equivalent to a linear Toda lattice [24]. However, nonlinearities may occur, in which case $\Phi_n = L i_0 f_{NL}(I_n/i_0)$ instead of $\Phi_n = L I_n$, where $i_0$ denotes a current scale. This yields the modified equation

$$\nabla \Delta I_n = i_0 L C f_{NL}(I_n/i_0) \frac{d^2 I_n}{dt^2},$$  \hspace{1cm} (73)
which shows that a site dependent inductance such as given by $L_{NL}(n) = L f_{NL}(I_n/v_0)$ has to be accounted for effectively. The same remains valid for a nonlinear capacitance given by $C_{NL}(n) = C g_{NL}(V_n/v_0)$, in which case

$$\nabla \Delta V_n = v_0 LC \frac{d^2 g_{NL}(V_n/v_0)}{dt^2}, \quad (74)$$

where $v_0$ is the voltage scale. We have to realize that the present $n$-dependent inductance $L(n)$ may be related or even identified to $L_{NL}(n)$. So we found a possibility to handle charge and/or field dependent parameters of the quantum $LC$-circuit in terms of corresponding parameters of the nonlinear Toda-lattice. It is understood that time dependent charges for which $I_n = dQ_n/dt \equiv d\tilde{Q}_n/dt$ should be approached within the Heisenberg representation.

Preserving, however, both anisotropy and generality of (27), means that the effective inductance is given by (24), so that the effective impedance is $Z(n) = i(\omega L(n) - 1/\omega C)$. Under such conditions general nonlinear realizations of the charge discretization function $F(n)$, although interesting from the mathematical point of view, are not easily tractable. Indeed, they lead to position dependent hopping amplitudes, to anharmonic effects as well as to complex valued energy dispersion laws. Moreover, in such cases the equivalence between the $L$-ring circuit and the electron on the 1D lattice under the influence of the induced time dependent electric field is lost and the same concerns dynamic localization conditions [10, 25]. However, there are reasons to emphasize that (27) is a promising starting point towards applications concerning the complex motion of electrons or of other carriers under modified charge conservation laws.

Unusual commutation relations like

$$[\tilde{Q}_q, \tilde{\Phi}_q] = -i\hbar c \left(1 + i \frac{q\epsilon}{\hbar c} \Delta F(n) \tilde{\Phi}_q \right), \quad (75)$$

and

$$[\tilde{Q}_q, \tilde{\Phi}_q^+] = i\hbar c \left(\frac{q\epsilon}{\hbar c} \nabla F(n) \tilde{\Phi}_q - \frac{\nabla F(n)}{\Delta F(n)} \right), \quad (76)$$

have also to be mentioned. Such relationships can be viewed as non-Hermitian versions of generalized canonical commutation relations acting on non-commutative spaces [26, 27], which looks rather challenging. Going back to (40) yields, however, a closed algebra encompassing the kinetic energy, the momentum and the charge, as indicated before [10].

Finally let us address the question of whether mesoscopic systems like quantum circuits should lie definitely under the incidence of usual quantum electrodynamics (QED) and of usual condensed matter theory or not. Strictly speaking the answer seems to be negative. Indeed, being mesoscopic or respectively nanosized is rather different from being a macroscopic condensed matter system. First, the number of constituents is by now rather small. However, the main point is that the miniaturization makes the coherence length to be larger than the sample dimensions. This opens the way to the occurrence of unexpected quantum interference phenomena, such as Aharonov-Bohm oscillations of the conductance with respect to the external fields, persistent currents, or Coulomb-blockade effects. It should be stressed that such effects work in conjunction with the discreteness of the charge. Moreover, there are parity dependent period doubling effects in the oscillations of persistent currents in Aharonov-Bohm rings, but when such rings are discretized only [5,28]. In addition, one deals with nontrivial dynamic localization effects characterizing electrons under the influence of a time dependent electric field [25] when the 1D line is replaced again by the 1D lattice. These latter effects show that the discreteness of the space has to be accounted for, too. In other words, one deals specifically with new physics relying on a new quantum phase [29], for which neither the thermodynamic limit nor the ensemble averaging remain valid. Nevertheless, signatures of the many-body Kondo effect are still able to be identified, such as found before in the case of junctions between Aharonov-Bohm rings and leads [30,31]. In other words we have to account for interplays between new and former effects, as one might expect. Furthermore, the application of usual relativistic QED to miniaturized composites looks rather unsuitable. This time there are novel effects relying on non-local currents and quantum non-locality [32] or on the advent of nonlinear relationships to the detriment of Ohm’s law [33], which prevent usual QED from being relevant to mesoscopic systems.

So we are in a position to realize that being discrete opens the way to a deeper understanding of mesoscopic phenomena. The canonical quantization should then be done by applying discrete derivatives instead of usual ones. This results in a promising perspective towards a suitable description of mesoscopic systems, now in terms developments provided by the application of quantum mechanics to low dimensional systems on discrete spaces. The same concerns, of course, the successful description of new phenomena characterizing mesoscopic structures in terms of appropriate tight binding hopping models.

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