Non-standard solutions of relativistic wave equations and decays of elementary particles

Andrzej Okniński*
Chair of Mathematics and Physics, Politechnika Świętokrzyska,
Al. 1000-lecia PP 7, 25-314 Kielce, Poland

May 23, 2017

Abstract

We carry out a constructive review of non-standard solutions of relativistic wave equations. Such solutions are obtained via splitting of relativistic wave equations written in spinor form. All these solutions are also solutions of the Dirac equation and are non-standard because they involve higher-order spinors. The main finding is that non-standard solutions describe decaying states.

1 Introduction

Recently, we have studied generalized solutions of the Dirac equation which are also solutions of other relativistic wave equations such as spin 0, 1 Duffin-Kemmer-Petiau (DKP) equations or spin 1 Hagen-Hurley equations [1–7]. These solutions have been obtained via splitting of relativistic wave equations written in spinor form. Such generalized solutions of the Dirac equation are non-standard because they involve higher-order spinors. In this work we study solutions and subsolutions of the Dirac equation, DKP equations, Hagen-Hurley equations as well as spin $\frac{3}{2}$ Fierz-Pauli equations. Our approach, based on spinor form of relativistic wave equations, and involving higher-order spinors (multispinors) can be traced back to ideas proposed by Schwinger [8,9].

The aim of the present investigation, based on papers [1–7], is to generalize our study of such non-standard solutions to the case of the Fierz-Pauli equations and obtain a broader and unifying picture.

The paper is organized as follows. In the next Section relativistic wave equations written in standard as well as spinor forms are reviewed. In Section 3 non-standard solutions of relativistic spin equations for $s = 0, \frac{1}{2}, 1, \frac{3}{2}$ are derived and interpreted as decaying states. In Section 4 ideas related to multispinors, Fermi-Bose transformations, and supersymmetry are reviewed. In the

*Email: fizao@tu.kielce.pl
last Section we discuss our results from broader point of view involving theoretical framework discussed in Section 4. In what follows we use notation and conventions described in [1–3].

2 Relativistic wave equations

2.1 Spin $\frac{1}{2}$

The Dirac equation, describing spin $\frac{1}{2}$ elementary particles is:

$$\gamma^\mu p_\mu \Psi = m\Psi,$$

(1)

where $\gamma^\mu$ are $4 \times 4$ matrices fulfilling [10, 11]:

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^\mu\nu I_{4 \times 4},$$

(2)

where $g^\mu\nu = \text{diag} (1, -1, -1, -1)$ and $I_{4 \times 4}$ is a $4 \times 4$ unit matrix. In the spinor representation of the Dirac matrices we have $\Psi = (\xi^A, \eta_{\dot{B}})^T$ [12], where $T$ denotes transposition of a matrix. In spinor formalism the Dirac equation reads:

$$p^A_{\dot{B}} \eta_{\dot{B}} = m \xi^A,$$

$$p_{AB} \xi^A = m \eta_{\dot{B}}$$

(3)

2.2 Spin 0 and 1

Equations considered in this Subsection, describing spin 0 and 1 bosons, are written as:

$$\beta_\mu p^\mu \Psi = m\Psi,$$

(4)

Eq. (4) describes a particle with definite mass if $\beta^\mu$ matrices obey the commutation relations [13–17]:

$$\sum_{\lambda, \mu, \nu} \beta^\lambda \beta^\mu \beta^\nu = \sum_{\lambda, \mu, \nu} g^{\lambda\mu} \beta^\nu,$$

(5)

where we sum over all permutations of $\lambda, \mu, \nu$.

It was noticed in Ref. [18] that $\beta^\mu$ matrices can be realized in form:

$$\beta^\mu = \frac{1}{2} (\gamma^\mu \otimes I_{4 \times 4} + I_{4 \times 4} \otimes \gamma^\mu).$$

(6)

It turns out that such $\beta^\mu$ obey simpler but more restrictive commutation relations [18, 19]:

$$\beta^\lambda \beta^\mu \beta^\nu + \beta^\nu \beta^\mu \beta^\lambda = g^{\lambda\mu} \beta^\nu + g^{\mu\nu} \beta^\lambda,$$

(7)

for which Eq. (4) leads to the Duffin-Kemmer-Petiau (DKP) theory of spin 0 and 1 mesons, see [18–20]. This reducible 16-dimensional representation [4] of $\beta^\mu$ matrices (denoted as 16) can be decomposed as $16 = 10 \oplus 5 \oplus 1$. Explicit formulas for the corresponding $10 \times 10$ (spin 1 case) and $5 \times 5$ (spin 0) matrices
are given in [16–18], while the one-dimensional representation \( \mathbf{1} \) is trivial, i.e. all \( \beta^\mu = 0 \). In the case of more general Eqs. (5) there are also other representations of \( \beta^\mu \) matrices, see [16,17] for a review. For example, there are two representations \( \mathbf{7} \) for which the corresponding \( 7 \times 7 \) matrices \( \beta^\mu \) yield the Hagen-Hurley equations for spin 1 bosons [21–23].

In the DKP theory of spin 0 mesons the representation \( \mathbf{5} \) is used (i.e. with \( 5 \times 5 \) matrices \( \beta^\mu \)) for which Eq. (4) reads:

\[
p^\mu \psi = m \psi^\mu,
p_\nu \psi^\nu = m \psi
\]

(8)

with \( \Psi \) in Eq. (4) defined as:

\[
\Psi = (\psi^\mu, \psi^0)^T = (\psi^0, \psi^1, \psi^2, \psi^3, \psi^1)^T,
\]

(9)

where \( T \) denotes transposition.

Within the spinor formalism Eqs. (8) can be written as:

\[
p^{\hat{A} \hat{B}} \psi = m \psi^{\hat{A} \hat{B}},
p^{\hat{A} \hat{B}} \psi^{\hat{A} \hat{B}} = 2 m \psi
\]

(10)

In the case of spin 1 mesons the representation \( \mathbf{10} \) is used for which Eq. (4) reduces to the Proca equations [24]:

\[
p^\mu \psi_\nu - p^\nu \psi_\mu = m \psi^{\mu \nu},
p_\mu \psi^{\mu \nu} = m \psi^{\nu \mu}
\]

(11)

with \( \Psi \) in Eq. (4) equal:

\[
\Psi = (\psi^{\mu \nu}, \psi^\lambda)^T = (\psi^{01}, \psi^{02}, \psi^{03}, \psi^{23}, \psi^{31}, \psi^{31}, \psi^{01}, \psi^1, \psi^2, \psi^3)^T,
\]

(12)

where \( \psi^\lambda \) are real and \( \psi^{\mu \nu} \) are purely imaginary (alternatively, \( -\partial^\mu \psi^\nu + \partial^\nu \psi^\mu = m \psi^{\mu \nu} \), \( \partial^\mu \psi^{\mu \nu} = m \psi^{\nu \mu} \), where \( \psi^\lambda \), \( \psi^{\mu \nu} \) are real). The spin 1 condition, \( p_\nu \psi^\nu = 0 \), follows from the second of Eqs. (11) due to antisymmetry of tensor \( \psi^{\mu \nu} \). Eqs. (11) can be written in spinor form as [25, 26]:

\[
p^{\hat{A} \hat{B}} \psi = m \psi^{\hat{A} \hat{B}},
p^{\hat{A} \hat{B}} \psi^{\hat{A} \hat{B}} = 2 m \psi
\]

(10)

In the case of spin 1 mesons the representation \( \mathbf{10} \) is used for which Eq. (4) reduces to the Proca equations [24]:

\[
p^\mu \psi_\nu - p^\nu \psi_\mu = m \psi^{\mu \nu},
p_\mu \psi^{\mu \nu} = m \psi^{\nu \mu}
\]

(11)

with \( \Psi \) in Eq. (4) equal:

\[
\Psi = (\psi^{\mu \nu}, \psi^\lambda)^T = (\psi^{01}, \psi^{02}, \psi^{03}, \psi^{23}, \psi^{31}, \psi^{31}, \psi^{01}, \psi^1, \psi^2, \psi^3)^T,
\]

(12)

where \( \psi^\lambda \) are real and \( \psi^{\mu \nu} \) are purely imaginary (alternatively, \( -\partial^\mu \psi^\nu + \partial^\nu \psi^\mu = m \psi^{\mu \nu} \), \( \partial^\mu \psi^{\mu \nu} = m \psi^{\nu \mu} \), where \( \psi^\lambda \), \( \psi^{\mu \nu} \) are real). The spin 1 condition, \( p_\nu \psi^\nu = 0 \), follows from the second of Eqs. (11) due to antisymmetry of tensor \( \psi^{\mu \nu} \). Eqs. (11) can be written in spinor form as [25, 26]:

\[
p^{\hat{A} \hat{B}} \psi = m \psi^{\hat{A} \hat{B}},
p^{\hat{A} \hat{B}} \psi^{\hat{A} \hat{B}} = 2 m \psi
\]

(10)

In the case of spin 1 mesons the representation \( \mathbf{10} \) is used for which Eq. (4) reduces to the Proca equations [24]:

\[
p^\mu \psi_\nu - p^\nu \psi_\mu = m \psi^{\mu \nu},
p_\mu \psi^{\mu \nu} = m \psi^{\nu \mu}
\]

(11)

with \( \Psi \) in Eq. (4) equal:

\[
\Psi = (\psi^{\mu \nu}, \psi^\lambda)^T = (\psi^{01}, \psi^{02}, \psi^{03}, \psi^{23}, \psi^{31}, \psi^{31}, \psi^{01}, \psi^1, \psi^2, \psi^3)^T,
\]

(12)

where \( \psi^\lambda \) are real and \( \psi^{\mu \nu} \) are purely imaginary (alternatively, \( -\partial^\mu \psi^\nu + \partial^\nu \psi^\mu = m \psi^{\mu \nu} \), \( \partial^\mu \psi^{\mu \nu} = m \psi^{\nu \mu} \), where \( \psi^\lambda \), \( \psi^{\mu \nu} \) are real). The spin 1 condition, \( p_\nu \psi^\nu = 0 \), follows from the second of Eqs. (11) due to antisymmetry of tensor \( \psi^{\mu \nu} \). Eqs. (11) can be written in spinor form as [25, 26]:

\[
p^{\hat{A} \hat{B}} \psi = m \psi^{\hat{A} \hat{B}},
p^{\hat{A} \hat{B}} \psi^{\hat{A} \hat{B}} = 2 m \psi
\]

(10)

In the case of spin 1 mesons the representation \( \mathbf{10} \) is used for which Eq. (4) reduces to the Proca equations [24]:

\[
p^\mu \psi_\nu - p^\nu \psi_\mu = m \psi^{\mu \nu},
p_\mu \psi^{\mu \nu} = m \psi^{\nu \mu}
\]

(11)

with \( \Psi \) in Eq. (4) equal:

\[
\Psi = (\psi^{\mu \nu}, \psi^\lambda)^T = (\psi^{01}, \psi^{02}, \psi^{03}, \psi^{23}, \psi^{31}, \psi^{31}, \psi^{01}, \psi^1, \psi^2, \psi^3)^T,
\]

(12)
obtained for $7 \times 7$ matrices $\beta^{\mu}$ corresponding to representation 7 ($\chi_{\mu \nu}$, $\eta_{\mu \nu}$ differ by factor 2 from tensors defined in [25,27]). In equations (14) tensor $\chi_{\mu \nu}$ is selfdual ($\chi_{23} = i \chi_{01}$, $\chi_{31} = i \chi_{02}$, $\chi_{12} = i \chi_{03}$) while $\eta_{\mu \nu}$ in (15) is antiselfdual ($\eta_{23} = -i \eta_{01}$, $\eta_{31} = -i \eta_{02}$, $\eta_{12} = -i \eta_{03}$). Note that since there are two different equations (14), (15) there are also two sets of $7 \times 7$ matrices $\beta^{\mu}$ fulfilling (7).

More exactly, in the case of Eq. (14) we put $\Psi = (\chi_{01}, \chi_{02}, \chi_{03}, \psi_{0}, \psi_{1}, \psi_{2}, \psi_{3})^T$ into Eq. (1) with:

$$
\begin{align*}
\beta_0 &= i (-e_{1,5} - e_{2,6} - e_{3,7} + e_{6,2} + e_{5,1}) \\
\beta_1 &= -i (-e_{1,4} + ie_{2,7} - ie_{3,6} + ie_{7,2} - e_{4,1}) \\
\beta_2 &= -i (ie_{1,7} - e_{2,4} + ie_{3,5} - ie_{5,3} - e_{4,2} + ie_{7,1}) \\
\beta_3 &= -i (ie_{1,6} - ie_{2,5} - e_{3,4} - e_{4,3} + ie_{5,2} - ie_{6,1})
\end{align*}
$$

(16)

where $A_{j,k}$ denotes a non-zero element $A$ of $\beta^{\mu}$ lying on the intersection of $j$-th row and $k$-th column. These matrices are unitarily related to the corresponding $\tilde{\beta}^{\mu}$ matrices of Ref. [16,17] (cf. Eq. (2.24) in part I), i.e. there is such $U$ that $U^\dagger \tilde{\beta}^{\mu} U = -\beta^{\mu}$. And another representation 7 is $\beta^{\mu} = -\beta^{\mu*}$ where $*$ is the complex conjugation.

These equations can be cast into spinor form [28]:

$$
\begin{align*}
p^A_B \psi_{AB} &= m \chi_{BD} , \quad \chi_{BD} = \chi_{DB} \\
p^A_B \chi_{BD} &= -m \psi_{AB}
\end{align*}
$$

(17)

$$
\begin{align*}
p^A_B \varphi_{CB} &= m \eta_{AC} , \quad \eta_{AC} = \eta_{CA} \\
p^A_B \eta_{AC} &= -m \varphi_{AB}
\end{align*}
$$

(18)

with symmetric spinors $\chi_{BD}$, $\eta_{AC}$ corresponding to selfdual and antiselfdual tensors $\chi_{\mu \nu}$, $\varphi_{\mu \nu}$, respectively [25,27]. Equations (17), (18) violate parity [25,27] and hence could be used in the context of weak interactions.

### 2.3 Spin $\frac{3}{2}$

The Rarita-Schwinger equations for spin $\frac{3}{2}$ particles read [12,29]:

$$
\begin{align*}
\gamma^{\mu} p^{\mu} \Phi_{\nu} &= m \Phi_{\nu} , \quad \text{(19a)} \\
\gamma^{\nu} \Phi_{\nu} &= 0 , \quad \text{(19b)}
\end{align*}
$$

where $\Phi_{\nu}$ is a vector-valued spinor and Eq. (19a) is spin 1 constraint. Strictly speaking, there is also another constraint, $\partial^{\mu} \Phi_{\nu} = 0$, which follows from Eqs. (19) [30].

The Fierz-Pauli equations, the spinor version of Eqs. (14), first written by Dirac and studied later by Fierz and Pauli [12,28,31] are:

$$
\begin{align*}
p^{AB} \eta_{ABC} &= 0 \\
p^{AB} \xi_{ABC} &= 0 \\
p^{DC} \eta_{AD} &= m \xi_{AC} \\
p^{DC} \xi_{AD} &= m \eta_{AC}
\end{align*}
$$

(20)

where \( \eta_{\dot{A}D} = \eta_{D\dot{A}}, \zeta_{\dot{A}} = \zeta^{\dot{A}} \) and conditions (20a) exclude presence of spin \( \frac{1}{2} \) particles.

The Rarita-Schwinger or Fierz-Pauli equations can be used to describe spin \( \frac{3}{2} \) particles: a hypothetical gravitino (cf. Section 31.3 of [22]) or baryon resonances (see [33] for a zoo of spin \( \frac{3}{2} \) resonances). Existence of spin \( \frac{3}{2} \) leptons has been also conjectured, see [34] and references therein.

3 Non-standard solutions of relativistic wave equations and decays of elementary particles

3.1 Spin 0

We start with the spin 0 DKP equations written in the spinor formalism as (10). Splitting the last of equations (10), \( p^{AB}\psi_{AB} = p^{11}\psi_{11} + p^{12}\psi_{12} + p^{21}\psi_{21} + p^{22}\psi_{22} = 2m\psi \), we obtain two sets of equations involving components \( \psi_{11}, \psi_{12}, \psi \) and \( \psi_{21}, \psi_{22}, \psi \), respectively:

\[
\begin{align*}
p_{11}\psi &= m\psi_{11} \\
p_{12}\psi &= m\psi_{12} \\
p^{11}\psi_{11} + p^{12}\psi_{12} &= m\psi \\
p_{21}\psi &= m\psi_{21} \\
p_{22}\psi &= m\psi_{22} \\
p^{21}\psi_{21} + p^{22}\psi_{22} &= m\psi
\end{align*}
\]

(21)

(22)

each of which describes particle with mass \( m \) (we check this substituting \( \psi_{11}, \psi_{12} \) or \( \psi_{21}, \psi_{22} \) into the third equations). The splitting preserving \( p_{\mu}p^{\mu}\psi = m^{2}\psi \) is possible due to spinor identities, \( p_{11}p^{11} + p_{21}p^{21} = p_{\mu}p^{\mu}, p_{12}p^{12} + p_{22}p^{22} = p_{\mu}p^{\mu} \), cf. [1]. Thus equations (21), (22) are equivalent to the DKP equations (10). We described similar equations in [2]. From each of equations (21), (22) an identity follows:

\[
\begin{align*}
p_{12}\psi_{11} &= p_{11}\psi_{12} \\
p_{22}\psi_{21} &= p_{21}\psi_{22}
\end{align*}
\]

(23a)

(23b)

Equation (21) and the identity (23a), as well as equation (22) and the identity (23b) can be written in form of the Dirac equations [2]:

\[
\begin{pmatrix}
0 & 0 & p_{11} & p_{21} \\
0 & 0 & p_{12} & p_{22} \\
p^{11} & p^{12} & 0 & 0 \\
p^{21} & p^{22} & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\psi_{11} \\
\psi_{12} \\
\psi \\
0
\end{pmatrix}
= m
\begin{pmatrix}
\psi_{11} \\
\psi_{12} \\
\psi \\
0
\end{pmatrix},
\]

(24)

\[
\begin{pmatrix}
0 & 0 & p_{11} & p_{21} \\
0 & 0 & p_{12} & p_{22} \\
p^{11} & p^{12} & 0 & 0 \\
p^{21} & p^{22} & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\psi_{21} \\
\psi_{22} \\
\psi \\
0
\end{pmatrix}
= m
\begin{pmatrix}
\psi_{21} \\
\psi_{22} \\
\psi \\
0
\end{pmatrix},
\]

(25)
respectively, with one zero component. Since in Eqs. (24), (25) there is the same differential operator we can write these equations as a single Dirac equation. Substituting explicit formulas for the spinors \( p^A, p^B \), we have [6]:

\[
(p^0 \gamma^0 - p^1 \gamma^1 - p^2 \gamma^2 - p^3 \gamma^3) \psi = m \psi,
\]

where \( \psi = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix} \) is a generalized (matrix) wavefunction and \( \gamma^\mu \) matrices read [6]:

\[
\gamma^0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},
\]

\[
\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},
\]

and we use a shorthand \( \hat{\psi} = (\psi^A, \psi^{I_2 \times 2})^T \) where \( I_2 \times 2 \) is the \( 2 \times 2 \) unit matrix. This is the modified spinor representation of matrices \( \gamma^\mu \) with \( \gamma^i \rightarrow -\gamma^i \) \((i = 1, 2, 3)\) and \( \Psi = (\xi^A, \eta^B)^T \rightarrow \Psi = (\eta^B, \xi^A)^T \) with respect to [12].

### 3.2 Spin 1

We shall now describe two-step splitting of the spin 1 DKP equations written in the spinor notation as [13], see Ref. [7].

Equations [13] are split to yield two separate equations for spinors \( \chi_{\hat{A} \hat{B}}, \zeta_{\hat{A} \hat{B}}, \eta_{AC}, \eta_{AC}, \zeta_{\hat{A} \hat{B}} \):

\[
\begin{align*}
p_{\hat{A} \hat{B}} \chi_{C \hat{B}} &= m \eta_{AC}, & \eta_{AC} &= \eta_{CA} \\
p_{\hat{B} \hat{A}} \eta_{AC} &= -m \zeta_{\hat{A} \hat{B}}
\end{align*}
\]

\[
\begin{align*}
p_{\hat{A} \hat{B}} \zeta_{\hat{A} \hat{B}} &= m \chi_{\hat{A} \hat{B}}, & \chi_{\hat{A} \hat{B}} &= \chi_{\hat{B} \hat{A}} \\
p_{\hat{A} \hat{B}} \chi_{\hat{A} \hat{B}} &= -m \zeta_{\hat{A} \hat{B}}
\end{align*}
\]

respectively [11]. This first level of splitting was achieved due to appropriate spinor identities, see Eq. (11) in Ref. [1]. Indeed, solutions of Eqs. (28), (29) obey the DKP equations [13]. We have thus obtained the Hagen-Hurley equations [17], [18] in spinor form with \( \psi_{\hat{A} \hat{B}} = \varphi_{\hat{A} \hat{B}} = \zeta_{\hat{A} \hat{B}} \) (see also Eqs. [14], [15] for the tensor setting).

We shall now split the \( s = 1 \) Hagen-Hurley equations [29]. Substituting expressions for \( p_{\hat{A} \hat{B}} \) and \( p_{\hat{A} \hat{B}} \) into Eqs. (29), cf. [1], we obtain a system of eight
where the condition $\chi_{B\hat{B}} = \chi_{\hat{D}B}$ is not imposed and all equations are arranged into two subsets \((30a), \ (30b)\). Note that each of these equations is the Dirac equation with the same set of $\gamma^\mu$ matrices \([4]\).

We note that solutions of two Dirac equations \((30)\) are non-standard since they involve higher-order spinors rather than spinors $\xi_A, \eta_B$. To interpret Eqs. \((30)\) we put:

$$
\chi_{B\hat{D}} (x) = \eta_B (x) \alpha_{\hat{D}} (x), \quad (31a)
$$
$$
\zeta_{A\hat{B}} (x) = \xi_A (x) \alpha_{\hat{B}} (x), \quad (31b)
$$

where $\alpha_{\hat{A}} (x)$ is the Weyl spinor, describing massless neutrinos, while $\eta_B (x)$, $\xi_A (x)$ are the Dirac spinors. Although neutrinos are massive \([35]\) their masses are very small hence this approximation should not lead to significant errors.

Note that now $\chi_{12} \neq \chi_{21}$ and, accordingly, the spin is not determined — more exactly, the spin is in the $0 \oplus 1$ space. It means that we consider not real but virtual (off-shell) bosons \([36]\). This substitution is in the spirit of the method of fusion of de Broglie \([37, 38]\) (similar ansatz was used in the $s = 0$ case \([5]\). After the substitution of \((31)\) into Eqs. \((30)\) we obtain two equations:

$$
- (p^1 + ip^2) \eta_1 \alpha_{\hat{A}} - (p^0 - p^3) \eta_2 \alpha_{\hat{A}} = -m \xi_A \alpha_{\hat{A}},
$$
$$
(p^0 + p^3) \eta_1 \alpha_{\hat{A}} + (p^1 - ip^2) \eta_2 \alpha_{\hat{A}} = -m \xi_A \alpha_{\hat{A}},
$$
$$
- (p^1 - ip^2) \xi_1 \alpha_{\hat{A}} - (p^0 - p^3) \xi_2 \alpha_{\hat{A}} = m \eta_1 \alpha_{\hat{A}},
$$
$$
(p^0 + p^3) \xi_1 \alpha_{\hat{A}} + (p^1 + ip^2) \xi_2 \alpha_{\hat{A}} = m \eta_2 \alpha_{\hat{A}},
$$

where $\hat{A} = 1, 2$, and, after substituting solution of the Weyl equation

$$
p^{\hat{A}\hat{B}} \alpha_{\hat{B}} = 0, \quad (33)
$$

$\alpha_{\hat{A}} (x) = \hat{\alpha}_{\hat{A}} e^{ik \cdot x}$, $k^\mu k_\mu = 0$, we get a single Dirac — equation for spinors $\xi_A (x)$, $\eta_B (x)$:

$$
- (\hat{p}^1 + i\hat{p}^2) \eta_1 - (\hat{p}^0 - \hat{p}^3) \eta_2 = -m \xi_1,
$$
$$
(\hat{p}^0 + \hat{p}^3) \eta_1 + (\hat{p}^1 - i\hat{p}^2) \eta_2 = -m \xi_2,
$$
$$
- (\hat{p}^1 - i\hat{p}^2) \xi_1 - (\hat{p}^0 - \hat{p}^3) \xi_2 = m \eta_1,
$$
$$
(\hat{p}^0 + \hat{p}^3) \xi_1 + (\hat{p}^1 + i\hat{p}^2) \xi_2 = m \eta_2,
$$

with rescaled momentum operators $\hat{p}^\mu = p^\mu + k^\mu$. 

7
Indeed, let us consider for example the first term in the first of equations (32). It can be written as:

$$-\alpha_1 \dot{A} (p^1 + ip^2) \eta_1 - \eta_1 (p^1 + ip^2) \alpha_1 = -\alpha_1 (p^1 + ip^2) \eta_1$$

and thus Eqs. (32) reduce to a single Dirac equation (34) for spinors $\xi_A(x)$, $\eta_B(x)$ since components $\alpha_1(x)$, $\alpha_2(x)$ cancel out. Equations (33), (34) describe two spin $\frac{1}{2}$ particles, one massless and another massive, with spins coupling to $s = 0$ or $s = 1$.

The above description fits decay of a virtual $W^-$ boson into a lepton and antineutrino, for example [7]:

$$W^- \rightarrow e + \bar{\nu}_e.$$

A good example is provided by the case of a (three-body) mixed beta decay [39]:

$$n (\uparrow) \rightarrow \left\{ \begin{array}{ll}
(p (\downarrow) + [e (\uparrow) \bar{\nu}_e (\uparrow)]) & \text{Gamow-Teller transition} \\
(p (\uparrow) + [e (\uparrow) \bar{\nu}_e (\downarrow)]) & \text{Fermi transition}
\end{array} \right.$$

where products of the $W^-$ boson decay (see (33)) are shown in square brackets and $(\uparrow)$ denotes spin $\frac{1}{2}$ – this seems to correspond well to the proposed transition from Eq. (29) to Eqs. (33), (34). Indeed, we had to assume that the spin was in the $0 \oplus 1$ space to describe the decay, see remark before Eqs. (32). And this agrees well with decay products of the $W^-$ meson with spins coupling to $s = 1$ or $s = 0$ with the spin change absorbed by spin flip of the proton.

### 3.3 Spin $\frac{1}{2}$

The Dirac equation (11) can be written in spinor notation as [12]:

$$\begin{aligned}
 p^{11} \eta_1 + p^{12} \eta_2 &= m \xi^1 \\
 p^{21} \eta_1 + p^{22} \eta_2 &= m \xi^2 \\
 p^{11} \xi^1 + p^{21} \xi^2 &= m \eta^1 \\
 p^{12} \xi^1 + p^{22} \xi^2 &= m \eta^2
\end{aligned}$$

(38)

Obviously, due to relations between components of $p^{AB}$ and $p_{CD}$, $p_{11} = p^{22}$, $p_{12} = -p^{21}$, $p_{21} = -p^{12}$, $p_{22} = p^{11}$, the equation (38) can be rewritten in terms of components of $p^{AB}$ only. Eq. (38) corresponds to (11) in the spinor representation of $\gamma$ matrices and $\Psi = (\xi^1, \xi^2, \eta_1, \eta_2)^T$.

For $m \neq 0$ we can define new higher-order spinors:

$$\begin{aligned}
 p^{11} \xi^1 &= m \psi^{11}_1, & p^{21} \xi^2 &= m \psi^{21}_2, \\
 p^{12} \xi^1 &= m \psi^{12}_1, & p^{22} \xi^2 &= m \psi^{22}_2
\end{aligned}$$

(39)

(40)

where we have:

$$\psi^{11}_1 + \psi^{21}_2 = \eta_1, \quad \psi^{12}_1 + \psi^{22}_2 = \eta_2.$$
The Dirac equations (38) can be now written with help of Eqs. (39), (40) as:

\[
\begin{align*}
p_{11}\xi^1 &= m\psi^{1}_{11} \\
p_{12}\xi^1 &= m\psi^{1}_{12} \\
p_{21}\xi^2 &= m\psi^{2}_{21} \\
p_{22}\xi^2 &= m\psi^{2}_{22}
\end{align*}
\]

where components \( p_{\alpha\beta} \) are used throughout.

It follows from Eqs. (42) that the following identities hold:

\[
\begin{align*}
p_{12}\psi^{1}_{11} &= p_{11}\psi^{1}_{12}, \\
p_{22}\psi^{2}_{21} &= p_{21}\psi^{2}_{22}
\end{align*}
\]

Taking into account the identities (43), (44) we can decouple Eqs. (42) and write them as a system of the following two equations:

\[
\begin{align*}
p_{11}\xi^1 &= m\psi^{1}_{11} \\
p_{12}\xi^1 &= m\psi^{1}_{12} \\
p_{22}\psi^{1}_{11} - p_{21}\psi^{1}_{12} &= m\xi^1 \\
p_{21}\xi^2 &= m\psi^{2}_{21} \\
p_{22}\psi^{2}_{21} + p_{11}\psi^{2}_{22} &= m\xi^2
\end{align*}
\]

System of equations (45), (46) is equivalent to the Dirac equation (38) if the definitions (41) are invoked. Due to the identities (43), (44) equations (45), (46) can be cast into covariant form, cf. Subsection 3.1, note however that some components of spinor \( \psi^{\hat{A}}_{BC} \) are missing.

The problem of missing components of spinor \( \psi^{\hat{A}}_{BC} \) is rather serious because it means that theory is not fully covariant. To solve the problem in the spirit of Ref. [3] we could assume that \( \xi^A (x) = \hat{\alpha}^A \chi (x), \psi^{A}_{BC} = \hat{\alpha}^A \chi_{BC} (x) \) where \( \hat{\alpha}^A \) is a constant spinor. In this work we make more general assumptions:

\[
\begin{align*}
\xi^A (x) &= \alpha^A (x) \chi (x), \\
\psi^{A}_{BC} (x) &= \alpha^A (x) \chi_{BC} (x)
\end{align*}
\]

where \( \alpha^A (x) = \hat{\alpha}^A e^{ik \cdot x}, k^\mu k_\mu = 0, \) is a two-component neutrino spinor, i.e. it fulfills the Weyl equation, \( p_{\alpha\beta} \alpha^A (x) = 0. \) Substituting (47) into Eqs. (45), (46) we get:

\[
\begin{align*}
p_{11}\alpha^1 \chi &= m\alpha^1 \chi_{11} \\
p_{12}\alpha^1 \chi &= m\alpha^1 \chi_{12} \\
p_{22}\alpha^1 \chi_{11} - p_{21}\alpha^1 \chi_{12} &= m\alpha^1 \chi \\
p_{21}\alpha^2 \chi &= m\alpha^2 \chi_{21} \\
p_{22}\alpha^2 \chi &= m\alpha^2 \chi_{22}
\end{align*}
\]

\[
\begin{align*}
p_{12}\alpha^2 \chi_{21} + p_{11}\alpha^2 \chi_{22} &= m\alpha^2 \chi
\end{align*}
\]
and:
\[
\begin{align*}
\tilde{p}_{11}\chi &= m\chi_{11} \\
\tilde{p}_{12}\chi &= m\chi_{12} \\
\tilde{p}_{22}\chi_{11} - \tilde{p}_{11}\chi_{12} &= m\chi \\
\tilde{p}_{21}\chi &= m\chi_{21} \\
\tilde{p}_{22}\chi &= m\chi_{22} \\
\tilde{p}_{12}\chi_{21} + \tilde{p}_{11}\chi_{22} &= m\chi
\end{align*}
\]
(50)
\[
\begin{align*}
\tilde{p}_{21}\chi &= m\chi_{21} \\
\tilde{p}_{22}\chi &= m\chi_{22} \\
\tilde{p}_{12}\chi_{21} + \tilde{p}_{11}\chi_{22} &= m\chi
\end{align*}
\]
(51)

where \(\tilde{p}^{AB} = p^{AB} + k^{AB}\), since components \(\alpha^A\) cancel out. We note that equations (50), (51) are the set of two \(3 \times 3\) equations equivalent to the spin 0 DKP equation with rescaled momentum operators \(\tilde{p}^{AB}\).

We have thus described a transition from the Dirac equation describing a spin \(\frac{1}{2}\) fermion to spin 0 massive boson and spin \(\frac{1}{2}\) Weyl particle. In our previous paper [5] we have suggested that inverse of the described process corresponds to the first stage of the main channel of pion decay:
\[
\pi^- \rightarrow (\mu^- \bar{\nu}_\mu) \rightarrow \mu^- + \bar{\nu}_\mu
\]
(52)

with formation of the intermediate complex \((\mu^- \bar{\nu}_\mu)\). However, the substitutions (47a), (47b) suggest another picture. The transformation from spin \(\frac{1}{2}\) fermion to spin 0 boson and spin \(\frac{1}{2}\) Weyl particle agrees well with two secondary channels of decay of the \(\tau\) lepton [33]:
\[
\begin{align*}
\tau^- &\rightarrow \pi^- + \nu_\tau \ (10.82\ \%) \quad (53a) \\
\tau^- &\rightarrow K^- + \nu_\tau \ (6.96 \times 10^{-3}) \quad (53b)
\end{align*}
\]

3.4 Spin \(\frac{3}{2}\)

We start with Eqs. (20) making the following substitutions:
\[
\begin{align*}
\eta^{\hat{B}}_{AC}(x) &= \psi^{\hat{B}}_A(x) \alpha_C(x) \\
\xi^{\hat{B}\hat{C}}_A(x) &= \psi^{\hat{B}}_A(x) \beta^C(x)
\end{align*}
\]
(54)

where bispinor \((\alpha_A, \beta^C)^T\) is a wavefunction of a spin \(\frac{1}{2}\) fermion and \(\psi^{\hat{B}}_A\) describes a spin 1 boson. We thus give up symmetry of spinors \(\eta^{\hat{B}}_{AC}, \xi^{\hat{B}\hat{C}}_A\) in undotted and dotted indices, respectively.

We have from (20) and (54):
\[
\begin{align*}
p^{AB}\psi^{\hat{A}B}_A\alpha_C &= 0 \\
p^{AB}\psi^{\hat{A}B}_A\beta^C &= 0 \\
p^{DC}\psi^{\hat{B}}_A\alpha_D &= m\psi^{\hat{B}}_A\beta^C \\
p^{DC}\psi^{\hat{B}}_A\beta^C &= m\psi^{\hat{B}}_A\alpha_D
\end{align*}
\]
(55a)
(55b)
Let us assume that $\psi_A^\dot{\beta}(x) = \hat{\psi}_A^\dot{\beta} e^{ik\cdot x}$, $k^\mu k_\mu = \kappa^2$. Moreover, we put $\alpha_A(x) = \hat{\alpha}_A e^{iq\cdot x}$, $q^\mu q_\mu = \lambda^2$. Then we get from (55):

$$\begin{align*}
\bar{\rho}^{AB} \psi_A^{\dot{\beta}} &= 0 \\
\bar{\rho}_{AB} \psi^{\dot{\beta}A} &= 0 \\
\bar{\rho}^{DC} \alpha_D &= m\beta^C \\
\bar{\rho}_{DC} \beta^C &= m\alpha_D
\end{align*}$$

(56a)

(56b)

with rescaled momentum operators $\bar{\rho}^{AB} = \rho^{AB} + q^{AB}$, $\bar{\rho}^{DC} = \rho^{DC} + k^{DC}$. We have thus obtained conditions (56a) and the (modified) Dirac equation for spin $\frac{1}{2}$ fermion (56b). It follows that the theory describes a massive spin $\frac{1}{2}$ fermion (with $m^2 = \kappa^2 + 2q^\mu k_\mu + \lambda^2$) and a massive spin 1 boson ($m^2 = \lambda^2$). Indeed, conditions (20a) reduce via (56a) to:

$$\bar{\rho}^{AB} \psi_{AB} = 0,$$

(57)

i.e. $\bar{\rho}^\mu \psi_\mu = 0$, and this is spin 1 condition.

It turns out that there are no decays of spin $\frac{3}{2}$ particles into massive spin $\frac{1}{2}$ fermion and massive spin 1 boson [33]. However, decay of hypothetical excited spin $\frac{3}{2}$ neutrino, $\nu^* \rightarrow eW$, corresponding to transition from Eqs. (20) to (56), was considered [34]. On the other hand, there are decays with massless spin 1 boson. Therefore we demand that $\lambda = 0$.

Now, transition from Eqs. (20) to (56) is consistent for $\lambda = 0$ with the decay of the spin $\frac{3}{2}$ baryon $\Omega_c (2770)^0$ into the spin $\frac{1}{2}$ baryon $\Omega_0^0$ and a photon [33]:

$$\Omega_c (2770)^0 \rightarrow \Omega_0^0 + \gamma.$$

(58)

Note that the $\Omega_c (2770)^0 - \Omega_0^0$ mass difference is so small that all strong decay channels are excluded [33].

4 Particle decays, Fermi-Bose transformations and supersymmetry

We have demonstrated in the preceding Section that non-standard solutions of the Dirac equation, i.e. solutions involving higher-order spinors, correspond to decaying states of particles, see Eqs. (36), (53), (58). In all these cases there is a Fermi-Bose transition. Of course, all these transitions are consistent with spin-statistics theorem, yet FB transformations occur and deserve a study from a more general, unifying, point of view.

It was noticed by Schwinger that a formalism of multispinors (higher-order spinors) provides a unified description of particles with arbitrary spins and statistics [8]. It was shown in [9] that this approach leads in a natural way to FB transformations and supersymmetry.
There are many ideas connected with fermion-boson (FB) analogies in the literature. For example, there is FB equivalence, FB duality, FB transformations, to name a few. Important step in understanding FB analogy was made by Polyakov who discovered possibility of fermion-boson transmutation of elementary excitations of a scalar field interacting with the topological Chern-Simons term in (2 + 1) dimensions [40]. Recently, the smooth and controlled evolution from a fermionic Bardeen-Cooper-Schrieffer (BCS) superfluid state to a molecular Bose-Einstein condensate (BEC) has been realized in ultracold Fermi gases [41]. More on these ideas can be found in Refs. [42–48].

Supersymmetry provides theoretical framework to describe transformations between particles of different statistics [32]. Recently, several supersymmetric systems, concerned mainly with anyons in 2+1 dimensions [49–56] as well as with the 3+1 dimensional Majorana-Dirac-Staunton theory [57], unifying fermionic and bosonic fields, have been described. Additional information on FB and supersymmetric ideas can be found in [58, 59].

5 Discussion

We have demonstrated that, in the non-interacting case, there are non-standard solutions of spin $0, \frac{1}{2}, 1, \frac{3}{2}$ relativistic equations which are also solutions of the Dirac equation. All these solutions are non-standard since they involve higher-order spinors. The major finding is that such solutions correspond to decaying states, see Eqs. (36), (53), (58). Note that in all these transitions bosons are born from fermions or fermions are born from bosons, all FB transformations consistent with spin-statistics theorem.

There is also a larger picture. The multispinor description of particle fields and sources provides a unification of all spins and statistics [8,9]. More exactly, the formalism involving higher-order spinors allows the treatment of all spins on equal footing and, as was shown by Schwinger, leads to Fermi-Bose transformations and supersymmetry. The present work as well as our earlier papers [1–7] devoted to FB transformations and based on multispinor formalism, belong to this circle of ideas.

Note finally, that there are many other decay channels of baryons which have not been explained by our formalism, see for example decay of $\Delta(1232)$ resonance where the main decay mode is $N\pi$ while the mode $N\gamma$, analogous to (58), is secondary [33].

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.
References

[1] A. Okniński, "Splitting the Kemmer-Duffin-Petiau equations," Proceedings of Institute of Mathematics of NAS of Ukraine, vol. 50, part 2 (2004) 902-908; arXiv: math-ph/0309013 (2003).

[2] A. Okniński, "Supersymmetric content of the Dirac and Duffin-Kemmer-Petiau equations," Int. J. Theor. Phys. 50 (2011) 729-736.

[3] A. Okniński, "Duffin-Kemmer-Petiau and Dirac Equations – A Supersymmetric Connection," Symmetry 4 (2012) 427-440.

[4] A. Okniński, "On the mechanism of fermion-boson transformation," Int J. Theor. Phys. 53 (2014) 2662-2667.

[5] A. Okniński, "Neutrino-assisted fermion-boson transformations," Acta Phys. Polon. B 46 (2015) 221-229.

[6] A. Okniński, "Synthesis of Relativistic Wave Equations: The Noninteracting Case," Adv. Math. Phys. 2015 (2015).

[7] A. Okniński, "Generalized solutions of the Dirac equation, W bosons, and beta decay," Adv. High Energy Phys. 2016 (2016) 2689742.

[8] J. Schwinger, "Particles and Sources," Phys. Rev. 152 (1966) 1219-1226.

[9] J. Schwinger, "Multispinor Basis of Fermi-Bose Transformations," Ann. Phys. 119 (1979) 192-237.

[10] P.A.M. Dirac, "The quantum theory of the electron," Proc. Roy. Soc. (London) A 117 (1928) 610.

[11] P.A.M. Dirac, "The quantum theory of the electron. Part II," Proc. Roy. Soc. (London) A 118 (1928) 351.

[12] V.B. Berestetskii, E.M. Lifshits, L.P. Pitaevskii, Relativistic Quantum Theory, Pergamon, 1974.

[13] K.H. Tzou, Comptes Rendus Acad. Sci. (Paris) 244 (1957) 2137.

[14] K.H. Tzou, "Corpuscular representation of the vector field, and comparison with the field of maximum spin 1 of the fusion theory," Journal de Physique et le Radium (France), changed to J. Phys. (Orsay, Fr.) 18 (1957) 619-624.

[15] A. Okninski, "Effective Quark Equations," Acta Phys. Pol. B 12 (1981) 87-94.

[16] J. Beckers, N. Debergh and A.G. Nikitin, "On Parasupersymmetries and Relativistic Descriptions for Spin One Particles. I. The Free Context," Fortschr. Phys. 43 (1995) 67-80.
[17] J. Beckers, N. Debergh and A.G. Nikitin, "On Parasupersymmetries and Relativistic Descriptions for Spin One Particles. II. The Interacting Context with (Electro)Magnetic Fields," *Fortschr. Phys.* 43 (1995) 81-96.

[18] N. Kemmer, "The particle aspect of meson theory," *Proc. Roy. Soc. (London)* A 173 (1939) 91-116.

[19] R.J. Duffin, "On the Characteristic Matrices of Covariant Systems," *Phys. Rev.* 54 (1938) 1114.

[20] G. Petiau, University of Paris Thesis, published in: *Acad. Royale de Belgique, Classe de Sci., Memoires Coll.*, in 8°, 16 (1936) Fasc. 2.

[21] C.R. Hagen and W.J. Hurley, "Magnetic moment of a particle with arbitrary spin," *Phys. Rev. Lett.* 24 (1970) 1381-1384.

[22] W.J. Hurley, "Relativistic Wave Equations for Particles with Arbitrary Spin," *Phys. Rev. D* 4 (1971) 3605-3616.

[23] W.J. Hurley, "Invariant bilinear forms and the discrete symmetries for relativistic arbitrary-spin fields," *Phys. Rev. D* 10 (1974) 1185-1200.

[24] A. Proca, "Sur la théorie ondulatoire des électrons positifs et négatifs," *Journal de Physique et le Radium (France)*, changed to *J. Phys.(Orsay, Fr.)* 7 (1936) 347-353.

[25] J. Lopuszański, *Rachunek Spinorów (The Calculus of Spinors, in Polish)*, PWN–Polish Scientific Publishers, Warsaw, 1985.

[26] A.H. Taub, "Spinor Equations for the Meson and Their Solution When No Field Is Present," *Phys. Rev.* 56 (1939) 799-810.

[27] J. Lopuszański, "The Representations of the Poincaré Group in the Framework of Free Quantum Fields," *Fortschr. Phys.* 26 (1978) 261-288.

[28] P.A.M. Dirac, "Relativistic Wave Equations," *Proc. Roy. Soc. (London)* A 155 (1936) 447-459.

[29] W. Rarita, J. Schwinger. "On a theory of particles with half-integral spin." *Phys. Rev.* 60 (1941) 61-62.

[30] T. Pilling, Symmetry of Massive Rarita–Schwinger fields, *International Journal of Modern Physics* A 20.13 (2005): 2715-2741.

[31] M. Fierz, W. Pauli, "On relativistic wave equations for particles of arbitrary spin in an electromagnetic field," *Proc. Roy. Soc. (London)* 173 (1939) 211-232.

[32] S. Weinberg, *The Quantum Theory of Fields*, Vol. III, *Supersymmetry*, Cambridge University Press, Cambridge 2000.
[33] C. Patrignani et al. (Particle Data Group), *Chinese Physics C* 40 (2016) 100001.

[34] A. Ozansoy, V. Ari, V. Çetinkaya, Search for Excited Spin-3/2 Neutrinos at LHeC, *Adv. High Energy Phys.* 2016 (2016) 1739027.

[35] J.F. Donoghue, E. Golowich, B.R. Holstein, *Dynamics of the Standard Model*, Cambridge, UK: Cambridge University Press, 2014.

[36] M. Thomson, *Modern particle physics*, Cambridge University Press, 2013.

[37] E.M. Corson, *Introduction to Tensors, Spinors, and Relativistic Wave Equations*, Blackie and Son, London 1953.

[38] L. de Broglie, *Théorie générale des corpuscules à spin*, Gauthier-Villars, Paris, 1943.

[39] K.S. Krane, *Introductory Nuclear Physics*, John Wiley & Sons, New York, 1988.

[40] A.M. Polyakov, ”Fermi-Bose transmutations induced by gauge fields,” *Mod. Phys. Lett. A* 3 (1988) 325-328.

[41] W. Zwerger (ed.), *The BCS–BEC Crossover and the Unitary Fermi Gas*, Lecture Notes in Physics 836, Springer-Verlag 2012.

[42] P. Garbaczewski, *Classical and quantum field theory of exactly soluble nonlinear systems*, World Scientific 1985.

[43] P. Garbaczewski, ”Representations of the CAR Generated by Representations of the CCR. Fock Case,” *Comm.Math. Phys.* 43 (1975) 131-136.

[44] V.M. Simulik, *What is the Electron?*, Apeiron 2005.

[45] V.M. Simulik, I.Yu. Krivsky, Bosonic symmetries of the massless Dirac equation. *Advances in Applied Clifford Algebras* 8 (1998) 69-82.

[46] V.M. Simulik, I.Yu. Krivsky, Bosonic symmetries of the Dirac equation, *Phys. Lett. A* 375 (2011) 2479-2483.

[47] I.Yu. Krivsky, T.M. Zajac, S. Shpyrko, Extension of the Standard CD Algebra in the Axiomatic Approach for Spinor Field and Fermi–Bose Duality, *Adv. Appl. Clifford Algebras*, 27 (2017) 14311458.

[48] W.A. Rodrigues Jr., Bosonization of Fermionic Fields and Fermionization of Bosonic Fields, *Adv. Appl. Clifford Algebras*, 27 (2017) 17691778.

[49] R. Jackiw, V.P. Nair, ”Relativistic wave equation for anyons,” *Phys. Rev. D* 43 (1991) 1933-1942.

[50] M.S. Plyushchay, ”Relativistic particle with torsion, Majorana equation and fractional spin,” *Phys. Lett. B* 262 (1991) 71-78.
[51] M.S. Plyushchay, ”The model of relativistic particle with torsion,” Nucl. Phys. B 362 (1991) 54-72.

[52] M.S. Plyushchay, ”Majorana equation and exotics: Higher derivative models, anyons and noncommutative geometry,” Electron.J.Theor.Phys. 3 (2006) 17-31.

[53] P.A. Horváthy, M.S. Plyushchay, M. Valenzuela, ”Supersymmetry between Jackiw-Nair and Dirac-Majorana anyons,” Phys. Rev. D 81 (2010) 127701.

[54] P.A. Horváthy, M.S. Plyushchay, M. Valenzuela, ”Bosons, fermions and anyons in the plane, and supersymmetry,” Ann. Phys. 325 (2010) 1931-1975.

[55] P.A. Horváthy, M.S. Plyushchay, M. Valenzuela, ”Supersymmetry between Jackiw-Nair and Dirac-Majorana anyons,” Phys. Rev. D 81 (2010) 127701.

[56] P.A. Horváthy, M.S. Plyushchay, M. Valenzuela, ”Supersymmetry of the planar Dirac–Deser–Jackiw–Templeton system and of its nonrelativistic limit,” J. Math.Phys. 51 (2010) 092108.

[57] P.A. Horváthy, M.S. Plyushchay, M. Valenzuela, ”Bosonized supersymmetry from the Majorana-Dirac-Staunton theory and massive higher-spin fields,” Phys. Rev. D 77 (2008) 025017.

[58] F. Wilczek, Fractional statistics and anyon superconductivity, World Scientific 1990.

[59] M. Stone, Bosonization, World Scientific 1994.