Quantum limits of photothermal and radiation pressure cooling of a movable mirror

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\textbf{Abstract.} We present a general quantum-mechanical theory for the cooling of a movable mirror in an optical cavity when both radiation pressure self-cooling and photothermal cooling effects are present, and show that these two mechanisms may bring the oscillator close to its quantum ground state, although in quite different regimes. Self-cooling caused by coherent exchange of excitations between the cavity mode and the mirror vibrational mode is shown to dominate in the good-cavity regime—when the mechanical resonance frequency is larger than the cavity decay rate, whereas photothermal-induced cooling can be made predominant in the bad-cavity limit. Both situations are compared, and the relevant physical quantities to be optimized in order to reach the lowest final excitation number states are extracted.

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1. Introduction

Cooling a mechanical oscillator to its quantum ground state is a compelling goal to get new insights into the behavior of macroscopic quantum objects, but is also essential for many applications for high-sensitivity measurements [1]–[3], gravitational wave detection interferometers [4], quantum information protocols involving optomechanical coupling [5]–[7], etc. Various schemes can be used to bring a mechanical oscillator down to its quantum ground state using both active [3, 8] and passive [9]–[11] optomechanical coupling with a cavity field, and, recently, many groups have made substantial progress in reducing the thermal noise of various mechanical resonators [12]–[21]. While all these schemes rely on the optomechanical coupling between the mirror and light to achieve cooling of one vibrational mode, there are, however, important differences in the physical mechanisms involved. In [9]–[19] one makes use of the so-called back-action or self-cooling mechanism [22]: the oscillator vibrational mode is coupled preferentially to the cavity field mode, and in a manner quite similar to the cavity cooling of atoms [23], thermal excitations are scattered into the cavity field mode, thus reducing the thermal motion of the mirror when the light leaks out of the cavity. In [13, 20, 21] the cooling results from absorption of the cavity photons by the mechanical oscillator. In this case thermal noise is extracted from the mirror vibrational mode via the coupling to absorption channels. In all the experiments so far, however, the resulting equilibrium state of the oscillator has been a classical one, in the sense that the final mean excitation number is still much larger than one. It is therefore essential to investigate whether or not the quantum ground state may be reached using either of these mechanisms. Moreover, since both effects are usually simultaneously present in experiments, it is important to characterize their range of predominance, as well as to quantify their respective effect on the cooling process. These are the issues that we address in the present paper. We present a full quantum-mechanical treatment for the cooling of a mechanical oscillator, which extends and generalizes the results of [11] by including both self-cooling and photothermal effects. We derive exact expressions for the mirror fluctuations in steady state which allow to investigate the effect of each mechanism and choose the parameters in order to optimize the cooling. The outline of the paper is the following: in section 2 we introduce the ‘cavity + absorbing mirror’ system considered and derive the coupled evolution equations for
the fluctuations of the cavity light and the mirror modes. In section 3, we compute the variances of the mirror position and momentum in steady state, and discuss the good- and bad-cavity situations. We show that optimal cooling in the good-cavity limit is essentially due to purely radiative radiation pressure effects and may bring the oscillator to its quantum ground state, provided that the cavity is appropriately detuned and the coherent coupling with the cavity field is strong enough. In the bad-cavity limit, on the other hand, photothermal effects may dominate and can also cool the oscillator down close to its quantum ground state. In both situations we give simple analytical expressions for the relevant quantities, such as the cooling rates for instance, which can be measured and optimized in experiments.

2. Evolution equations for the quantum fluctuations

We consider a linear cavity constituted by a fixed mirror—which plays the role of a partially transmitting output coupler with intensity transmission $T$—and by a movable mirror which absorbs a fraction $A$ of the intracavity photons. This configuration corresponds to the most typical experimental situation, but the theory also applies to other systems such as the silica toroidal resonators microcavity of \cite{18, 24} for instance. We limit ourselves to a frequency range in the vicinity of a mechanical resonance $\Omega_M$ of the movable mirror and treat the latter as a harmonic oscillator with effective mass $M$, damping rate $\Gamma$ and quality factor $Q = \Omega_M / \Gamma$ \cite{25}. We only consider Gaussian states; the mirror quantum state is standardly described by the position and momentum operators of the vibrational mode considered, $q$ and $p$, satisfying $[q, p] = i\hbar$. The position operator of the movable mirror obeys the following equation of motion

$$M\ddot{q}(t) + \Gamma \dot{q}(t) + M\Omega_M^2 q(t) = F_T(t) + F_{\text{rad}}(t),$$

where $F_{\text{rad}}$ designates the force exerted on the mirror by the intracavity light, which we will decompose into two components: a purely radiative optomechanical force proportional to the intracavity intensity, and a photothermal-induced force due to the absorption of cavity photons by the mirror

$$F_{\text{rad}}(t) = \frac{\sqrt{2}\hbar k}{\tau} I(t) + \beta \frac{\hbar k}{\sqrt{2}} \int_{-\infty}^{t} \frac{dt'}{\tau_c} e^{-\frac{(t-t')}{\tau_c}} I_{\text{abs}}(t').$$

$\hbar k$ is the momentum of the cavity photons, $\tau$ their round trip time, $\beta$ the fraction of absorbed photons contributing to the photothermal effect and $I_{\text{abs}}$ the intensity absorbed by the mirror \cite{26}. To simplify the discussion we assume that there is only one absorption channel, modeled by mirror losses, and that the whole absorption process takes place with a time constant $\tau_c$. Multiple channels could easily be included, but the conclusions are expected to remain qualitatively the same. We will also assume $\beta = 1$ in the rest of the paper. In this case there is a factor 2 difference in the coefficients for the radiative force and the photothermal one, in accordance with the fact that a cavity photon bouncing on and off the mirror exchanges an overall impulsion $2\hbar k$ with the mirror versus only $\hbar k$ for an absorption process.

In equation (1) $F_T$ is the thermal force accounting for the thermal Brownian noise of the mirror, with zero-mean value and a correlation function given by

$$(F_T(t)F_T(t')) = 2\hbar \Gamma M \int d\Omega e^{-i\Omega(t-t')} \Omega \coth \left( \frac{\hbar \Omega}{2k_B T_M} \right),$$
with \( k_B \) the Boltzmann constant and \( T_M \) the temperature of the mirror. As seen from the intracavity field the absorption process is equivalent to having a partially transmitting mirror with transmission \( A \), in addition to the input coupler with transmission \( T \). The cavity field decay rate is thus equal to

\[
\kappa = \kappa_1 + \kappa_2 = \frac{T + A}{2\tau}.
\]  

(4)

We introduce the cavity field mode \( a \) in the rotating frame associated with the laser frequency \( \omega_L \). Denoting by \( \Delta_c = \omega_c - \omega_L \) the cavity detuning, the equation of motion for the cavity field reads

\[
\dot{a}(t) = -(\kappa + i\Delta_c)a(t) + \frac{k}{\sqrt{2}} q(t)a(t) + \sqrt{2\kappa_1 a_{\text{in}}(t)} + \sqrt{2\kappa_2 b_{\text{in}}(t)}.
\]  

(5)

\( a_{\text{in}}(t) \) is the input field annihilation operator associated with the field injected into the cavity by the (fixed) input coupler. The (movable) absorbing mirror being modeled as a lossy beamsplitter one also has an annihilation operator \( b_{\text{in}}(t) \) associated with the vacuum fluctuations entering the cavity from the absorbing mirror port [26]. With this notation the intracavity intensity is

\[
I(t) = a^\dagger(t)a(t) + \sqrt{2\kappa_1 a_{\text{in}}(t)} + \sqrt{2\kappa_2 b_{\text{in}}(t)}.
\]  

2.1. Steady state

In steady state one has \( \langle b_{\text{in}} \rangle = 0 \), and the mean displacement and the mean intracavity field amplitude \( \alpha = \langle a \rangle \) are given by the following relations

\[
\langle q \rangle = \frac{\sqrt{2\hbar k}}{M\Omega_M^2\tau} \left( 1 + \frac{A}{2} \right) \alpha^2, \quad \alpha = \frac{\sqrt{2\kappa_1 \langle a_{\text{in}} \rangle}}{\kappa + i\Delta},
\]  

(7)

where the detuning, \( \Delta = \Delta_c - \Delta_{\text{nl}} \), is the sum of the empty cavity detuning and a nonlinear dephasing proportional to the intracavity intensity

\[
\Delta_{\text{nl}} = \frac{\hbar k^2}{M\Omega_M^2\tau^2} \left( 1 + \frac{A}{2} \right) \alpha^2.
\]  

(8)

Without loss of generality we also choose the phase of the input field such that \( \alpha \) is real. \( \alpha \) is thus a solution of a standard third-order algebraic equation, which may give rise to multistability [27]. A necessary (static) condition for the working point is then that

\[
\kappa^2 + \Delta^2 - 2\Delta \Delta_{\text{nl}} > 0,
\]  

(9)

which is always satisfied when the nonlinear dephasing is small compared with the cavity decay rate (\( \Delta_{\text{nl}} < \max(\kappa, \Delta) \)). There exists another (dynamic) stability condition which is related to radiation pressure-induced heating of the oscillator [27, 28]. Let us point out that the present theory can also be used to investigate such instabilities, but in this paper we will only focus on the stable regime which is favorable to cooling.
2.2. Linearization

Linearizing equations (1) and (5) in the Fourier domain yields

\[ (\kappa + i\Delta + i\Omega)\delta a[\Omega] = \frac{ik\omega}{\tau \sqrt{2}} \delta q[\Omega] + \sqrt{2\kappa_1} \delta a_{in}[\Omega] + \sqrt{2\kappa_2} \delta b_{in}[\Omega]. \]  
(10)

\[ \chi[\Omega]^{-1} \delta q[\Omega] = \delta F_T[\Omega] + \delta F_{\text{rad}}[\Omega], \]  
(11)

with the standard susceptibility defined as

\[ \chi[\Omega]^{-1} = M\Omega_M^2 [1 - \Omega^2 / \Omega_M^2 + iQ^{-1}\Omega / \Omega_M]. \]  
(12)

Using the fact that \( \delta I[\Omega] = \alpha (\delta a[\Omega] + \delta a^\dagger[\Omega]) \) and \( \delta I_{\text{abs}}[\Omega] = \sqrt{2\kappa_2} \alpha (\delta a_{\text{abs}}[\Omega] + \delta a_{\text{abs}}^\dagger[\Omega]) \), one derives the evolution of the mirror position fluctuations

\[ \tilde{\chi}[\Omega]^{-1} \delta q[\Omega] = \delta F_T[\Omega] + \delta F_R[\Omega], \]  
(13)

with an effective susceptibility

\[ \tilde{\chi}^{-1} = \chi[\Omega]^{-1} - M\Omega_M^2 \frac{2\Delta\theta}{D[\Omega]D[-\Omega]} \left( 1 + \frac{A/2}{1 + i\Omega\tau_c} \right), \]  
(14)

with \( D[\Omega] = \kappa + i\Delta + i\Omega, \ \theta = \Delta_{\text{al}} / (1 + A/2) \) and an effective fluctuating force containing both the radiation pressure- and the photothermal-induced fluctuations

\[ \delta F_R[\Omega] = \left( \frac{\sqrt{2\hbar\omega}}{\tau} \right) \left[ \left( 1 + \frac{A/2}{1 + i\Omega\tau_c} \right) \left( \sqrt{2\kappa_1} \frac{D[-\Omega]}{D[\Omega]} \delta a_{in}[\Omega] + \sqrt{2\kappa_2} \frac{D[-\Omega]}{D[\Omega]} \delta b_{in}[\Omega] \right) \right. \]
\[ + \frac{\sqrt{2\kappa_2}}{D[\Omega]} \left( 1 + \frac{A/4}{1 + i\Omega\tau_c} \frac{\kappa_2 - \kappa_1 - i\Delta - i\Omega}{\kappa_2} \right) \delta b_{in}[\Omega] \]
\[ + \frac{\sqrt{2\kappa_2}}{D[-\Omega]} \left( 1 + \frac{A/4}{1 + i\Omega\tau_c} \frac{\kappa_2 - \kappa_1 + i\Delta - i\Omega}{\kappa_2} \right) \delta b_{in}^\dagger[\Omega] \right]. \]  
(15)

3. Steady state noise spectrum and variances

3.1. Exact result

We standardly define the mirror noise spectrum by the relation

\[ \langle \delta q[\Omega] \delta q'[\Omega'] \rangle = 2\pi \delta(\Omega + \Omega') S_q[\Omega]. \]  
(16)

Using equation (13), one gets

\[ S_q[\Omega] = |\tilde{\chi}[\Omega]|^2 \left( S_{F_T}[\Omega] + S_{F_R}[\Omega] \right). \]  
(17)

where \( S_{F_T} \) and \( S_{F_R} \) are the thermal noise and radiation pressure noise spectra, respectively. The steady state variance can then be calculated by integration of the total noise spectrum

\[ \Delta q^2 = \int \frac{d\Omega}{2\pi} S_q[\Omega]. \]  
(18)

Since one has \( p = \dot{q} / \Omega_M \) one can compute the momentum variance by

\[ \Delta p^2 = \int \frac{d\Omega}{2\pi} \left( \frac{\Omega}{\Omega_M} \right)^2 S_q[\Omega]. \]  
(19)
In order to compute these integrals it is convenient to make the parameters dimensionless; we normalize the frequencies of interest to the cavity field decay rate $\kappa$ and introduce:

$$ b = \Omega_M/\kappa, \quad \varphi = \Delta/\kappa, \quad \varphi_{\text{nl}} = \theta/\kappa, \quad c = \Omega_M t_c. $$

We express the intracavity field losses as relative weights between output coupler transmission and photothermal absorption

$$ \frac{\kappa_1}{\kappa} = \frac{T}{T+A}, \quad \frac{\kappa_2}{\kappa} = \frac{A}{T+A} $$

and define normalized position and momentum operators $\tilde{q} = q / \sqrt{\hbar/M\Omega_M}$ and $\tilde{p} = 2p / \sqrt{\hbar M\Omega_M}$, so that their commutator is similar to that of the field operators: $[\tilde{q}, \tilde{p}] = 2i$. Reaching the ground state then implies $\Delta \tilde{q}^2, \Delta \tilde{p}^2 \to 1$. With this notation an exact expression for the mirror position normalized variance is

$$ \Delta \tilde{q}^2 = \int \frac{d\omega}{\pi} \frac{1}{(1 - \omega^2 + \delta\omega^2)^2 + ((\omega/\kappa) + \delta\Gamma)^2} \left[ 1 + \frac{2n_{\text{nl}}(\omega)}{Q} + \varphi_{\text{nl}} \tilde{S}_{F_R}[\omega] \right], $$

where $\omega = \Omega/\Omega_M$ and $n_{\text{nl}}(\omega) = [\exp(\hbar\Omega_M\omega/k_B T_M) - 1]^{-1}$. The quantities $\delta\omega$ and $\delta\Gamma$ are proportional to the change in the real and imaginary parts of the susceptibility, respectively,

$$ \delta\omega = \frac{-2\varphi_{\text{nl}}}{(1 - b^2\omega^2 + \varphi^2)^2 + 4b^2\omega^2} \left( 1 - b^2\omega^2 + \varphi^2 \right) \left( 1 + \frac{A/2}{1 + c^2\omega^2} \right) - \frac{A b c \omega^2}{1 + c^2\omega^2}, $$

$$ \delta\Gamma = \frac{2\varphi_{\text{nl}}}{(1 - b^2\omega^2 + \varphi^2)^2 + 4b^2\omega^2} \left( 1 - b^2\omega^2 + \varphi^2 \right) \frac{A c \omega/2}{1 + c^2\omega^2} + 2b\omega \left( 1 + \frac{A/2}{1 + c^2\omega^2} \right). $$

$\delta\omega$ is related to the frequency shift in the mechanical resonance frequency and $\delta\Gamma$ is related to the modification of the mirror damping rate, as we will see in the following. The normalized radiation pressure noise spectrum is given by

$$ \tilde{S}_{F_R} = \frac{1}{(1 - b^2\omega^2 + \varphi^2)^2 + 4b^2\omega^2} \left[ \frac{2T}{T + A} \left| 1 + \frac{A/2}{1 + i\omega} \right|^2 (1 + b^2\omega^2 + \varphi^2) + \frac{2A}{T + A} \left| 1 + \frac{A/2}{1 + i\omega} \right|^2 \varphi^2 \right. $$

$$ \left. + \frac{2A}{T + A} \left| 1 + ib\omega + \frac{(T + A)}{4(1 + i\omega)} ((A - T)(1 + ib\omega) + ib\omega + b^2\omega^2 - \varphi^2) \right|^2 \right]. $$

This exact result extends that of [11] and includes photothermal absorption on the quantum noise of the mirror. The effect of the various parameters can be investigated directly by numerical computation of the integral of equation (22), but in order to gain some insight into the problem and distinguish the different regimes, we derive some analytical results in the next sections.

### 3.2. Approximate analytical expression

Depending on the regime considered cooling—or heating—may occur owing to the two different processes considered: resonant coupling by radiation pressure with the cavity mode (self-cooling) or photothermal cooling. It is, therefore, important to establish which effect dominates in which regime, and if it is then possible to reach the quantum ground state. For clarity of the discussion, we will consider that the frequency response of the mechanical
oscillator stays peaked around the mechanical frequency and that the effective quality factor stays large, even when the damping is increased. As stated in section 2.1, this is valid in a low intensity regime in which nonlinear effects are negligible. In this case one can interpret the real and imaginary parts of the susceptibility as a resonance frequency shift and an effective damping, respectively. With the dimensionless units introduced previously, the frequency shift and the effective damping rate are given by

\[ \delta \omega = \frac{-2 \varphi \varphi_{nl}}{(1 - b^2 + \varphi^2)^2 + 4b^2} \left( \frac{1}{1 - b^2 + \varphi^2} + \frac{A/2}{1 + c^2} \right) - \frac{A b c}{1 + c^2}, \]

\[ \tilde{\Gamma} = \Gamma \left[ 1 + \frac{2 \varphi \varphi_{nl} Q}{(1 - b^2 + \varphi^2)^2 + 4b^2} \left( \frac{1}{1 - b^2 + \varphi^2} + \frac{A/2}{1 + c^2} \right) + 2b \left( 1 + \frac{A/2}{1 + c^2} \right) \right]. \]

For \( A = 0 \), one recovers the purely radiative situation examined in [11]. The terms arising from the photothermal effect are proportional to \( A \). Their respective contributions have a different dependence on the mechanical resonance frequency/cavity bandwidth ratio, as we discuss later. However, regardless of the dominant effect considered, one must have a positive detuning \( \varphi > 0 \) in order to have cooling. Note that, if \( \varphi < 0 \), one can easily have \( \tilde{\Gamma} < 0 \) and obtain heating. It can be shown rigorously that \( \tilde{\Gamma} > 0 \) corresponds to the second condition for the steady state to be stable, in addition to the one mentioned in section 2.1.

Necessary conditions to obtain a strong cooling are also that the frequency shift stays small, \( \delta \omega \ll 1 \), the effective damping significantly broadens the natural resonance width, \( \tilde{\Gamma} \gg \Gamma \) and that the vibrational mode still has a high effective quality factor \( \tilde{Q} \gg 1 \). Indeed, by neglecting the frequency shift and taking the expression of the noise spectra around \( \omega = 1 \), the evaluation of the integral in equation (22) yields an approximate expression for the mirror variance

\[ \Delta \tilde{q}^2 \simeq \frac{\Gamma^2}{\Gamma} \left[ 1 + 2n_i(\omega = 1) + \varphi_{nl} Q \tilde{S}_{\tilde{f}_h}(\omega = 1) \right]. \] (23)

The damping in the thermal noise is then given by the ratio \( \Gamma / \tilde{\Gamma} \), which can in principle be made quite large. On the other hand, for perfect ground state cooling, the ‘quantum’ radiation pressure noise represented by the second term in equation (23) should also be kept small so that the total variance may tend to 1.

In the next sections, we examine the good/bad-cavity limits, in which the mechanical resonance frequency is large/small with respect to the cavity field decay rate. Figure 1 shows the effective damping rate, as well as the frequency shift, in these two situations, when the coupling is purely radiative (\( A = 0 \)) or in the presence of substantial photothermal absorption. One can see that, in the good-cavity limit, photothermal effects play almost no role, whereas they actually dominate in the bad-cavity limit. Note that for the parameters chosen the frequency shifts are indeed small, in accordance with our harmonic oscillator assumption.

### 3.3. Good-cavity limit

In the good-cavity limit \( b > 1 \), the absorption plays little role in the effective damping rate, the frequency shift and the radiation pressure force spectrum which can be approximated by their purely radiative expressions

\[ \tilde{\Gamma} \simeq \Gamma \left[ 1 + \varphi_{nl} Q \frac{4 \varphi b}{(1 - b^2 + \varphi^2)^2 + 4b^2} \right], \] (24)
Figure 1. Variation of $\bar{\Gamma}/\Gamma$ and $\delta \omega$ with the normalized cavity detuning $\varphi$, in the good-cavity limit (left-hand side, $b = 5$) and in the bad-cavity limit (right-hand side, $b = 0.01$). The dashed and plain curves are for $A = 0$ and 0.2, respectively. The other parameters are $Q = 10^4$, $n_i = 10^2$, $\varphi_{nl} = 0.1$, $c = 1$.

\begin{align}
\delta \omega & \simeq \varphi_{nl} \frac{-2\varphi(1-b^2+\varphi^2)}{(1-b^2+\varphi^2)^2+4b^2}, \\
\bar{\delta} S_F & \simeq \frac{2(1+b^2+\varphi^2)}{(1-b^2+\varphi^2)^2+4b^2\omega^2}.
\end{align}

The effective damping is maximized for $\varphi \simeq b \gg 1$, such that $\bar{\Gamma} \simeq \Gamma(1+\varphi_{nl}Q)$ and $|\delta \omega| \simeq \varphi_{nl}/2b \ll 1$. The overall variance can then be written as

$$\Delta \bar{q}^2 \simeq (1-\eta) \Delta \bar{q}^2_T + \eta \Delta \bar{q}^2_R.$$  \hfill (27)

In equation (27) the quantity

$$\eta = \frac{f \varphi_{nl} Q}{1+f \varphi_{nl} Q}, \quad \text{with} \quad f = \frac{4b\varphi}{(1-b^2+\varphi^2)^2+4b^2},$$

represents the weight between thermal noise- and radiation pressure noise-induced fluctuations and quantifies the amount of thermal excitations taken from the mirror vibrational mode and scattered into the cavity mode. The thermal noise contribution $\Delta \bar{q}^2_T = 1+2n_i(\omega = 1)$ is thus damped by a factor $\sim \varphi_{nl} Q$. The radiation pressure-induced noise is equal to

$$\Delta \bar{q}^2_R = \frac{1+b^2+\varphi^2}{2\varphi b}.$$  \hfill (29)
It is also minimized when \( \phi \simeq b \gg 1 \), such that the overall variance tends to unity when \( \varphi_{nl}Q \) is sufficiently large. As pointed out in [11] \( \eta \) can be interpreted as a cooling efficiency, since under optimal self-cooling conditions, the mirror final temperature can be evaluated via \( \Delta \tilde{q}^2 \equiv 1 + 2n_t \simeq (1 - \eta)(1 + 2n_i) + \eta \), which gives

\[
\eta = 1 - \frac{n_t}{n_i} = \frac{\varphi_{nl}Q}{1 + \varphi_{nl}Q}.
\]

The interpretation of purely radiative self-cooling of a movable mirror is quite analogous to that for cavity cooling of atoms. An appropriately detuned cavity allows to freeze the mirror motion by scattering vibrational excitations preferentially into the cavity mode. In order to take energy from the mirror in a resonant manner the cavity detuning has to match the mechanical resonance frequency (\( \phi \sim b \)), so that both harmonic oscillators efficiently exchange their fluctuations. In this picture the cavity field acts as a thermal noise eater for the mirror and the parameter \( \eta \) is a quantum state transfer efficiency, quite similarly to atom–field quantum state transfer processes [29]. The quantity \( \varphi_{nl}Q \) is an analog of the cooperativity parameter—commonly used within the context of cavity QED—that quantifies the efficiency of the quantum state transfer.

### 3.4. Bad-cavity limit

The bad-cavity limit \( b < 1 \) is not favorable to purely radiative self-cooling \textit{a priori}, since the purely radiative radiation pressure effect on the damping is weaker than for a good cavity. However, the photothermal absorption may considerably change the imaginary part of the susceptibility, even though the frequency shift stays negligible. Indeed, for \( b \ll 1 \) and when the absorption dominates the cavity losses, the effective damping rate, the frequency shift and the radiation pressure noise spectrum can be approximated by

\[
\tilde{\Gamma} \simeq \Gamma \left[ 1 + \varphi_{nl}Q A - \varphi \frac{c}{1 + \varphi^2} + 4\varphi_{nl}Qb \frac{\varphi}{(1 + \varphi^2)^2} \right],
\]

\[
\delta \omega \simeq \frac{-2\varphi_{nl}\varphi}{(1 + \varphi^2)^2},
\]

\[
\bar{S}_F \simeq \frac{2}{1 + \varphi^2} + \frac{2}{(1 + \varphi^2)^2} \left[ 1 - \left( \frac{\varphi^2 A}{4(1 + i\epsilon)} \right)^2 \right].
\]

Again, if \( \varphi_{nl} \ll 1 \), one can neglect the frequency shift. The effective damping rate, however, has two contributions and it can easily be seen that the photothermal effect (second term in (31)) dominates over self-cooling effects (third term in (31)) if \( Ac/(1 + c^2) \gg 4b/(1 + \varphi^2) \). The previous equations also tell what conditions should be fulfilled in order to observe strong photothermal effects: the photothermal-induced damping is maximized when the cavity is detuned by half a cavity bandwidth, \( \varphi \sim 1 \), and when the thermal absorption rate matches the mechanical resonance frequency of the oscillator, \( c \sim 1 \). Satisfying the previous conditions ensures an optimal reduction of the thermal noise, by a factor \( \tilde{\Gamma}/\Gamma \sim \varphi_{nl}Q A/4 \). However, the evaluation of the resulting radiation pressure-induced noise needs to be done carefully, since it includes contributions from both the coherent coupling and the absorption processes. The overall variance is shown to obey a similar equation to equation (29)

\[
\eta = \frac{g\varphi_{nl}Q}{1 + g\varphi_{nl}Q}, \quad \text{with} \quad g = A \frac{\varphi}{1 + \varphi^2} \frac{c}{1 + c^2}.
\]
The resulting radiation pressure noise is given this time by

\[ \Delta q_R^2 \simeq \frac{1}{g(1+\varphi^2)} + \frac{1}{g(1+\varphi^2)^2} \left[ \left( 1 - \frac{\varphi^2 A}{4(1+ic)} \right)^2 - 1 \right]. \tag{35} \]

As can be seen from the dependence with \( A \) and \( \varphi \) there is a trade-off between optimizing the thermal damping, i.e. maximizing \( \eta \), and increasing the resulting radiation pressure noise, which is inversely proportional to \( A \) (assumed small with respect to 1 in our cavity treatment). For instance, \( \varphi \sim 1 \) yields the best thermal damping, but the radiation pressure noise is then \( \sim 4/A \), which always gives a variance substantially above the ground state value. However, increasing the detuning helps (up to a point) in decreasing the radiation pressure noise. Indeed, for large \( \varphi \), the resulting radiation pressure noise is given by

\[ \Delta q_R^2 \sim \frac{4}{A\varphi} + \frac{A\varphi}{8}. \tag{36} \]

If the cooperativity parameter \( \varphi_{nl}Q \) is large enough to ensure good thermal noise damping, the optimum then consists in choosing \( \varphi \sim 4\sqrt{2}/A \), which gives a final limit of \( \Delta q^2 \simeq \sqrt{2} \) for the normalized variance. Small final excitation numbers can thus also be reached in this regime, by optimizing the cavity detuning.

3.5. Numerical results

In figure 2, we plot the normalized variance as a function of the cavity detuning, in the two situations considered in figure 1. The results are obtained from an exact numerical calculation of the integral (22) and it can be seen that they agree well with the analytical conclusions drawn in the harmonic oscillator limit of the previous sections. In the good-cavity limit, the self-cooling is seen to be almost entirely of radiative origin and one can get very close to the ground state by ensuring \( \varphi \sim b \). In the bad-cavity limit, photothermal absorption plays an essential role in damping the thermal noise and allows much stronger cooling than a purely radiative coupling in the same situation. However, the final temperature is comparatively higher than in the purely radiative situation, on account of a fundamental increase in the radiation pressure noise. Figure 3 shows the variation of the variance with the parameter \( c \), i.e. the photothermal absorption characteristic time (normalized to the mechanical resonance frequency). When photothermal effects dominate the cooling is seen to improve when the absorption characteristic time is of the same order as the mirror response time, \( c \sim 1 \).

4. Conclusion

We have presented a full quantum-mechanical derivation of the cooling of a mechanical resonator via radiation pressure, including both absorption and coherent coupling with the cavity light. Thermal noise from the mirror vibrational mode can be extracted in two ways: in the good-cavity limit, when the coherent coupling dominates over the absorption, energy is taken from the mirror when the light is resonantly tuned to the cavity blue-sideband. As shown in [11], in this situation, the amount of thermal noise extracted from the mirror—and therefore the resulting temperature—can be inferred from a homodyne detection of the field reflected by the cavity. In the bad-cavity limit, when photothermal absorption is dominant, the thermal noise...
Figure 2. Normalized variance versus normalized cavity detuning $\varphi$, (a) in the good-cavity limit $b = 5$ and (b) in the bad-cavity limit $b = 0.01$. ♦ and ⋆ are for $A = 0$ and $A = 0.2$, respectively. The other parameters are $Q = 10^4$, $n_i = 10^2$, $\varphi_{nl} = 0.1$, $c = 1$, $T = 0.01$.

Figure 3. Normalized variance versus $c$. ♦ and ⋆ are for $A = 0$ and 0.2, respectively. The other parameters are $Q = 10^4$, $n_i = 10^2$, $\varphi_{nl} = 0.1$, $b = 0.01$, $T = 0.01$, $\varphi = 3$. 
can be transferred to other vacuum modes via the coupling with the light, which also results in efficient cooling. In both cases, appreciable cooling is predicted and lower than unity occupation numbers can in principle be reached provided that suitable frequency matching conditions are satisfied.

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