Energy Transfer, Weak Resonance, and Fermi’s Golden Rule in Hamiltonian Nonlinear Klein-Gordon Equations

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Abstract

This paper focuses on a class of nonlinear Klein-Gordon equations in three dimensions, which are Hamiltonian perturbations of the linear Klein-Gordon equation with potential. The unperturbed dynamical system has a bound state with frequency $\omega$, a spatially localized and time periodic solution. In quantum mechanics, metastable states, which last longer than expected, have been observed. These metastable states are a consequence of the instability of the bound state under the nonlinear Fermi’s Golden Rule.

In this study, we explore the underlying mathematical instability mechanism from the bound state to these metastable states. Besides, we derive the sharp energy transfer rate from discrete to continuum modes, when the discrete spectrum was not close to the continuous spectrum of the Schrödinger operator $H = -\Delta + V + m^2$, i.e. weak resonance regime $\sigma_c(\sqrt{H}) = [m, \infty)$, $0 < 3\omega < m$. This extends the work of Soffer and Weinstein [32] for resonance regime $3\omega > m$ and confirms their conjecture in [32]. Our proof relies on a more refined version of normal form transformation of Bambusi and Cuccagna [3], the generalized Fermi’s Golden Rule, as well as certain weighted dispersive estimates.

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1 Introduction

1.1 Background

The well-known Kolmogorov-Arnold-Moser (KAM) theory is concerned with the persistence of periodic and quasi-periodic motion under the Hamiltonian perturbation of a dynamical system, which has many applications in various research fields, such as the N-body problem in celestial mechanics [19, 22]. For a finite dimensional integrable Hamiltonian system, the aforementioned theory was initiated by Kolmogorov [19] and then extended by Moser [25] and Arnold [1]. Subsequently, many efforts have been focused on generalizing the KAM theory to infinite dimensional Hamiltonian systems (Hamiltonian PDEs), wherein solutions are defined on compact spatial domains, such as [5, 9, 20]. In all of the above results, appropriate non-resonance conditions imply the persistence of periodic and quasi-periodic solutions. See [22, 36] and the references therein for a comprehensive survey.

In this study, we focus on a different phenomenon that resonance conditions lead to the instability of periodic or quasi-periodic solutions. For example, in quantum mechanics, under a small perturbation, an excited state could be unstable with energy shifting to the ground state, free waves and nearby excited states. In this process anomalously long-lived states called metastable states in physics literature were observed. To study this instability mechanism, Dirac, in 1927, considered an unperturbed Hamiltonian $H_0$ with the initial eigenstate $i(x)$, and the perturbing Hamiltonian $H_1$ with the final eigenstate $f(x)$, then calculated the transition probability per unit time from the initial eigenstate to the final eigenstate

$$
\Gamma_{i \to f} = \frac{2\pi}{\hbar} \left| \int_{\mathbb{R}^3} i(x) H_1(x) f(x) dx \right|^2 \cdot \rho_f,
$$

where $\rho_f$ is the density of the final states. This was utilized by Fermi in 1934 to establish his famous theory of beta decay, where he called the formula (1.1) “golden rule No. 2,” thus now known as “Fermi’s Golden Rule.”

The rigorous mathematical analysis of the instability mechanism of bound states and energy transfer rates were acquired quite late and remain largely open. The first significant
result of such problem for a nonlinear PDE was provided by Soffer and Weinstein [32]. Consider the Cauchy problem for a nonlinear Klein-Gordon equation with potential in the whole space,
\[
\begin{aligned}
\begin{cases}
\partial_t^2 u - \Delta u + m^2 u + V(x)u = \lambda u^3, & t > 0, x \in \mathbb{R}^3, \lambda \in \mathbb{R}, \\
u(x, 0) = u_0(x), \quad \partial_t u(x, 0) = u_1(x).
\end{cases}
\end{aligned}
\tag{1.2}
\]
This equation is a Hamiltonian system with energy
\[
E[u, \partial_t u] \equiv \frac{1}{2} \int (\partial_t u)^2 + |\nabla u|^2 + m^2 u^2 + V(x)u^2 dx - \frac{\lambda}{4} \int u^4 dx.
\]
The nonlinearity of \(u^3\) can certainly be replaced by a general Hamiltonian one. See [32] for more details. Assume that the potential function \(V(x)\) satisfies the following properties:

**Assumption 1.1.** Let \(V(x)\) be real-valued function such that

(V1) for \(\delta > 5\) and \(|\alpha| \leq 2, |\partial^{\alpha} V(x)| \leq C_\alpha \langle x \rangle^{-\delta}\).

(V2) zero is not a resonance of the operator \(-\Delta + V\).

(V3) \((-\Delta + V(x))(-\Delta + 1)^{-1}\) is bounded on \(L^2\) for \(|l| \leq N_s\) with \(N_s \geq 10\).

(V4) the operator \(H = -\Delta + V(x) + m^2 \triangleq B^2\) has a continuous spectrum, \(\sigma_c(H) = [m^2, +\infty)\), and a unique strictly positive simple eigenvalue, \(\omega^2 < m^2\) with the associated normalized eigenfunction \(\varphi(x)\):
\[
B^2 \varphi = \omega^2 \varphi.
\]

Under these assumptions, one can observe that the linear Klein-Gordon equation (1.2), with \(\lambda = 0\), has a two-parameter family of spatially localized and time-periodic solutions of the form:
\[
u(t, x) = R \cos(\omega t + \theta)\varphi(x).
\]
For \(\lambda \neq 0\), i.e., Hamiltonian nonlinear perturbation of the linear dispersive equation, instead of the KAM type results, Soffer and Weinstein [32] proved that if \(3\omega > m\) and the following resonance condition (analogous Fermi’s Golden Rule Condition) was met,
\[
\Gamma \equiv \frac{\pi}{3\omega} \left( \mathcal{P}_c \varphi^3(x), \delta(B - 3\omega) \mathcal{P}_c \varphi^3(x) \right) \equiv \frac{\pi}{3\omega} |\mathcal{F}_c \varphi^3(3\omega)|^2 > 0.
\tag{1.3}
\]
where \(\mathcal{P}_c\) denotes the projection onto the continuous spectral part of \(B\) and \(\mathcal{F}_c\) denotes the Fourier transform relative to the continuous spectral part of \(B\), then small global solutions to (1.4) decayed to zero at an anomalously slow rate as time tended to infinity. More precisely, the solution \(u(t, x)\) had the following expansion as \(t \to \pm \infty\):
\[
u(t, x) = R(t) \cos(\omega t + \theta(t))\varphi(x) + \eta(t, x),
\]
where
\[
R(t) = O(|t|^{-\frac{1}{4}}), \quad \theta(t) = O(|t|^{\frac{1}{2}}) \quad \text{and} \quad \|
\eta(t, \cdot)\|_{L^8} = O(|t|^{-\frac{3}{4}}).
\]
This result provides the exact rate of energy transfer from discrete to continuum modes. As a corollary, there are no small global periodic or quasi-periodic solutions to (1.2). Besides, the instability mechanism of the bound state is revealed through the nonlinear analogous Fermi’s
Golden Rule (1.3) as the key resonance coefficient in the dynamical equation of the discrete mode. Thus, this non-negative coefficient reflects the nonlinear resonant interaction between the bound state (eigenfunctions) and the radiation (continuous spectral modes).

The Fermi’s Golden Rule condition (1.3) implies that

\[ 3\omega \in \sigma_c(B) = [m, +\infty), \]

which ensures a strong coupling of the discrete spectrum to the continuous spectrum. For the weak resonance regime, i.e. the discrete spectrum \( \omega \) is not close to the continuous spectrum \( 3\omega < m \), the Fermi’s Golden Rule (1.3) fails, and the method proposed by Soffer and Weinstein [32] cannot be directly applied to derive a similar result. Nevertheless, the authors in [32] conjectured that a similar Fermi’s Golden Rule holds, and the solutions decay to zero at certain rates for the general case. Since then, many efforts have been focused on this conjecture, i.e. the instability mechanism and energy transfer rate of metastable states in a weak resonance regime [2, 3]. We emphasize that the energy transfer rate is crucial to obtain a quantitative description on the lifespan of metastable states.

1.2 Related Works

For the nonlinear Klein-Gordon equations with zero potential, the global well-posedness and scattering theory for small initial data has been established in several works. For instance, see [18] and [29] for the three dimensional case, [26] and [31] for the two dimensional case, and [10], [15] for the one dimensional case. These solutions satisfy the linear dispersive estimates.

When potentials are present \( V(x) \neq 0 \) but the operator \( B \) has no eigenvalues, the same decay results in linear Klein-Gordon equations are true in three space dimensions, see [37]. Several interesting studies focus on quadratic nonlinearities and variable coefficients in one dimensional case, see [14], [23]. The lower bound of the decay rate has also been proved using an alternative approach in a recent work by An–Soffer [2], when \( B \) has one simple eigenvalue lying close to the continuous spectrum, i.e., the case in [32]. And in the recent interesting work [24], the authors extended the results of [32] to quadratic nonlinearities and obtained the sharp decay rates. Finally, in the remarkable work [3], for initial data in \( H^1 \times L^2 \) and a general potential which allows for multiply eigenvalues and arbitrary gaps between the discrete and continuous spectra, Bambusi and Cuccagna proposed a new normal form transformation preserving the Hamiltonian structure and proved that small solutions were asymptotically free under a non-degeneracy hypothesis. Moreover, they discovered the instability mechanism of bound states through a nonlinear Fermi’s Golden Rule, whose coefficient was never negative and generically strictly positive. We mention that for initial data in \( H^1 \times L^2 \), decay estimates do not hold due to the conservation of energy. Nevertheless, the authors in [3] remarked that it was possible to prove appropriate decay rates by restricting initial data to the class in [32]. And in this paper we settle down this problem in our context.

We also mention that the Fermi’s Golden Rule was first introduced by Sigal [30] in mathematical context, wherein the author established the instability mechanism of quasi-periodic solutions to nonlinear Schrödinger and wave equations. This was then further extended to study the asymptotic stability of bound states of nonlinear Schrödinger equations by Tsai–Yau [35], Soffer–Weinstein [33], Gang [11], Gang–Sigal [12], Gang–Weinstein [13]; see also the recent advances by Cuccagna–Maeda [7], their survey [8] and references therein.

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1.3 Main Result

Our main result is the following theorem:

**Theorem 1.2.** Let \((2N-1)\omega < m < (2N+1)\omega\) for some \(N \in \mathbb{N}\) and \(C_0\) be a positive constant. Under Assumption 1.1, if the following resonance condition (\(\gamma\) is defined in Section 4) \(\gamma > 0\) holds, then there exists a sufficiently small constant \(\epsilon > 0\) such that for any initial data \(u_0, u_1\) satisfying

\[
\|u_0\|_{W^{2,2} \cap W^{1,1}} + \|u_1\|_{W^{1,2} \cap W^{1,1}} \leq \epsilon
\]

and

\[
\|P_c u_0\|_{W^{2,2} \cap W^{1,1}} + \|P_c u_1\|_{W^{1,2} \cap W^{1,1}} \leq C_0 \left( \|P_d u_0\|_{L^2} + \|P_d u_1\|_{L^2} \right),
\]

solution \(u(t, x)\) to (1.2) with \(\lambda \neq 0\) has the following expansion as \(t \to \pm \infty:\)

\[
u(t, x) = R(t) \cos(\omega t + \theta(t)) \varphi(x) + \eta(t, x),
\]

where

\[
\frac{1}{2} R(0) \leq R(t) \leq CR(0),
\]

\[
\theta(t) = O(|t|^{1 - \frac{1}{4N}}), \quad \|\eta(t, \cdot)\|_{L^8} = O(|t|^{-\frac{3}{4}}).
\]

for some positive constant \(C > 0\). Here, \(P_d\) and \(P_c\) denote the projections onto the discrete and continuous spectral part of \(B\) respectively.

**Remark 1.3.** Evidently, the case where \(N = 1\) is exactly the work of Soffer-Weinstein [32]. We point out that the case of \(m = (2N + 1)\omega\) is beyond the scope of our work, and it remains unsolved.

**Remark 1.4.** We indicate that \(R(0)\) can be of order one by choosing a sufficiently small \(\lambda\) and a simple scaling argument. The assumption on initial data

\[
\|P_c u_0\|_{W^{2,2} \cap W^{1,1}} + \|P_c u_1\|_{W^{1,2} \cap W^{1,1}} \lesssim \|P_d u_0\|_{L^2} + \|P_d u_1\|_{L^2},
\]

see (1.4), is necessary, which leads to resonance-dominated solutions with the decay rates (1.5) and (1.6). Indeed, this assumption should also be imposed in [32] and other related references. As pointed out by Tsai and Yau in [35], when

\[
\|P_c u_0\|_{W^{2,2} \cap W^{1,1}} + \|P_c u_1\|_{W^{1,2} \cap W^{1,1}} \gg \|P_d u_0\|_{L^2} + \|P_d u_1\|_{L^2},
\]

there exist dispersion-dominated solutions with faster decay rates, even in the case of Soffer and Weinstein [32].

**Remark 1.5.** The decay rate (1.5) is consistent with the numerical prediction in [4].
1.4 Outline of Proof

Now we explain the ideas of our proof. By setting $u_\lambda(t, x) = |\lambda|^{1/2} u(t, x)$, we may assume $\lambda = \pm 1$. We adapt the framework of Hamiltonian method proposed in [3] to derive the dynamical equations of discrete and continuous modes. By changing variables, the nonlinear Klein-Gordon equations (1.2) can be written as the following Hamilton equations (see Section 3 for details)

\begin{align*}
\dot{\xi} &= -i \frac{\partial H}{\partial \bar{\xi}}, \\
\dot{f} &= -i \nabla_{\bar{f}} H.
\end{align*}

with the corresponding Hamiltonian

\begin{align*}
H &= H_L + H_P, \\
H_L &= \omega |\xi|^2 + \langle \bar{f}, B f \rangle, \\
H_P &= \int_{\mathbb{R}^3} \left( \sum \frac{\xi + \bar{\xi}}{\sqrt{2\omega}} \varphi(x) + U(x) \right)^4 dx,
\end{align*}

where $\nabla_{\bar{f}} H$ is the gradient with respect to the $L^2$ metric, and $U = B^{-\frac{1}{2}}(f + \bar{f})/\sqrt{2} \equiv P_c u$. To further simplify the dynamics of discrete mode and decouple the discrete mode from the continuous ones, we proposed a refined Birkhoff normal form transformation compared to the one in [3]. More precisely, we prove that for any $0 \leq r \leq 2N$ there exists an analytic canonical transformation $T_r$ putting the system in normal form up to order $2r + 4$, i.e.

\[ H^{(r)} := H \circ T_r = H_L + Z^{(r)} + R^{(r)} \]

where $Z^{(r)}$ is a polynomial of degree $2r + 2$ which is in normal form (see Definition 3.2), and $R^{(r)}$ consists of error terms of higher order, see Theorem 3.3. The normal form transformation is canonical and preserves the Hamiltonian nature of the system. Our key observation is that the order of normal form is actually increased by two in each step, which is consistent with the theoretic prediction by the non-Hamiltonian method of Soffer and Weinstein [32]. Besides, we explore the explicit forms of these coefficients appeared in error terms, whose structure will be crucial in error estimates.

After applying the normal form transformation, we are able to extract the main terms of the continuous modes $f$ and obtain the dynamics of the discrete mode as follows:

\[ \frac{d}{dt} |\xi|^2 = -2\gamma |\xi|^{4N+2} + 2 Re(\xi \bar{R}_\xi), \]

where

\[ \gamma := (2N + 1) \langle \Phi_{0,2N+1}, \delta(B - (2N + 1)\omega) \bar{\Phi}_{0,2N+1} \rangle \geq 0, \quad \Phi_{0,2N+1} \in \mathcal{S}(\mathbb{R}^3, \mathbb{C}), \]

is the coefficient of nonlinear Fermi’s Golden Rule, and $R_\xi$ is an error term, see (4.6). Thus, under the Fermi’s Golden Rule condition $\gamma > 0$, we could expect that the decay rate of the discrete mode $\xi(t)$ is $O(|t|^{-\frac{4N+2}{\gamma}})$, provided that an appropriate estimate for the error term $R_\xi$ is present.

To close our argument, the estimate of error term $R_\xi$ remains to be performed, which by (4.6) requires deriving the decay estimates of the continuous mode $f$. Once this is achieved, a bootstrap argument would imply Theorem 1.2. Since we are pursuing decay estimates in
time, the space-time dispersive estimates from [3] are not applicable here. Besides, we shall see that the classical \( L^p \) estimates also fail due to the slow decay rate of \( \xi \) and \( f \), especially for the weak resonance regime \( N > 1 \). To illustrate this, consider

\[
\hat{R}_f := -i \int_0^t e^{-iB(t-s)} \partial f \hat{R} ds,
\]

which appears as an error term in the expression of the continuous mode \( f \), see [4,3]. Here \( \hat{R} \) is the error term coming from the normal form transformation \( \hat{R} = \sum_{d=1}^5 \hat{R}_d \). Denote

\[
\hat{R}_f^2 := -i \int_0^t e^{-iB(t-s)} \partial f \hat{R}_2 ds.
\]

When \( N = 1 \) as in [32], using linear dispersive estimates, and by virtue of the structure of \( \hat{R}_d \) in Theorem [33] one has

\[
\|B^{-1/2} \hat{R}_f^2\|_{L^8} \lesssim \int_0^t \min \left\{ |t-s|^{-\frac{3}{2}}, |t-s|^{-\frac{1}{2}} \right\} |s|^{-\frac{1}{2}} [\xi]^2 \left( \|B^{-1/2} f\|_{L^8} + \|B^{1/2} f\|_{L^4} \right) ds,
\]

\[
\|B^{1/2} \hat{R}_f^2\|_{L^4} \lesssim \int_0^t |t-s|^{-\frac{3}{2}} |s|^{-\frac{1}{2}} [\xi]^2 \left( \|B^{-1/2} f\|_{L^8} + \|B^{1/2} f\|_{L^4} \right) ds,
\]

which yield

\[
\|B^{-1/2} \hat{R}_f^2\|_{L^8} \lesssim |t|^{-1+\delta_0} [\xi]^2 \left( [B^{-1/2} f]_{s,\frac{1}{2}} + [B^{1/2} f]_{s,\frac{1}{2}} \right),
\]

\[
\|B^{1/2} \hat{R}_f^2\|_{L^4} \lesssim |t|^{-\frac{1}{2}+\delta_0} [\xi]^2 \left( [B^{-1/2} f]_{s,\frac{1}{2}} + [B^{1/2} f]_{s,\frac{1}{2}} \right),
\]

where \([\xi]_\alpha = \sup_{0 \leq t \leq T} \|\xi(t)\|_{L^\alpha} \), together with decay estimate of \( \xi(t) = O(|t|^{-\frac{1}{4}}) \), these estimates imply the desired error estimate of \( \hat{R}_f \) for small data, which is similar to the approach used in [32]. We emphasize that in the aforementioned estimates, the decay rate \( O(|t|^{-\frac{1}{4}}) \) of \( \xi(t) \) is necessary and cannot be weakened. However, for the weak resonance regime \( N > 1 \), the expected decay rate \( \xi(t) = O(|t|^{-\frac{1}{4N}}) \), which is much slower than required. For instance, the linear dispersive estimates imply

\[
\|B^{1/2} \hat{R}_f^2\|_{L^4} \lesssim \int_0^t |t-s|^{-\frac{3}{2}} |s|^{-\frac{1}{4N}} [\xi]^2 \left( \|B^{-1/2} f\|_{L^8} + \|B^{1/2} f\|_{L^4} \right),
\]

where

\[
\frac{1}{2} + \frac{1}{2N} < 1.
\]

Hence, the decay estimates of \( B^{-1/2} f, B^{1/2} f \) cannot be closed this way.

In this paper, we overcome this difficulty by applying the weighted \( L^p \) estimates, see Lemma [23], which acquires time decay at the cost of the spatial decay. More precisely, instead of estimating \( \|B^{1/2} f\|_{L^4} \), we consider the weighted \( L^4 \) norm \( \langle x \rangle^{-\sigma} \|B^{1/2} f\|_{L^4} \), then

\[
\|\langle x \rangle^{-\sigma} B^{1/2} f\|_{L^4} \lesssim \int_0^t \min \{ |t-s|^{-\frac{3}{2}}, |t-s|^{-\frac{1}{2}} \} |s|^{-\frac{1}{2N}} [\xi]^2 \left( \|\langle x \rangle^{-\sigma} \partial f \hat{R}_2\|_{W^{1,\frac{1}{4}}} \right)
\]

\[
\lesssim (\langle \xi \rangle^{4N} |t|)^{-\frac{2N+3}{4N}} [\xi]^2 \left( \|\langle x \rangle^{-\sigma} B^{1/2} f\|_{s,\frac{1}{2}} + [B^{-1/2} f]_{s,\frac{1}{2}} \right).
\]

Such a technique is based on the spatially localized property of \( \partial f \hat{R}_2 \) to absorb the weight \( \langle x \rangle^\sigma \) on the RHS.
1.5 Structure of the Paper

The remaining part of this paper is organized as follows. In Section 2, we introduce some useful dispersive estimates and weighted inequalities for linear equations with potential, and we present the global existence theory and energy conservation of the nonlinear Klein-Gordon equation. In Section 3, we begin our proof by performing Birkhoff normal form transformation. Then we isolate the key resonant terms in the dynamical equation of the discrete mode in Section 4. In Section 5, we derive the asymptotic behavior of discrete and continuous modes. In Section 6, the error terms are carefully estimated. In Section 7, we prove our main result using apriori estimates and bootstrap arguments.

1.6 Notations

Throughout our paper, we use $C$ to denote an absolute positive constant that may vary from line to line. We write $A \lesssim B$ to mean that $A \leq CB$ for some absolute constant $C > 0$. We will use $A \approx B$ in a similar standard way.

2 Preliminary

In this section, we provide some useful lemmas on the linear analysis for the Klein-Gordon equation with potential and the global well-posedness theory of the nonlinear Klein-Gordon equation (1.2).

2.1 Linear Dispersive Estimates

Consider the Cauchy problem for three dimensional linear Klein-Gordon equation with a potential

\begin{equation}
\begin{aligned}
&\partial_t^2 u - \Delta u + m^2 u + V(x)u = 0, \quad t > 0, x \in \mathbb{R}^3, \\
u(0, x) = u_0, \quad \partial_t u(0, x) = u_1.
\end{aligned}
\end{equation}

Denote $B^2 = -\Delta + m^2 + V(x)$, then equation (2.1) can be solved as

\[ u(t, x) = \cos Bt \ u_0 + \frac{\sin Bt}{B} u_1. \]

For $V(x) = 0$, i.e. free Klein-Gordon case, the standard $L^p$ dispersive estimates follow from an oscillatory integration method and the conservation of the $L^2$ norm. More precisely, the $L^p$ norm of the solution to $u(t, x)$ satisfies the dispersive decay estimate $\|u(t, \cdot)\|_{L^p} \leq C |t|^{-3(\frac{1}{2} - \frac{1}{p})}$.

For $V(x) \neq 0$, if $V(x)$ satisfies some suitable decay and regularity conditions, then the same decay rate of $u$ can be obtained by the $W^{k,p}$-boundedness of the wave operator after being projected on the continuous spectrum of $B$. For instance, see [17], [28], [37].

**Lemma 2.1** ($L^p$ dispersive estimates). Assume that $V(x)$ is a real-valued function and satisfies $(V1),(V2)$. Let $1 < p \leq 2$, $\frac{1}{p} + \frac{1}{p'} = 1$, $0 \leq \theta \leq 1$, $l = 0, 1$, and $s = (4 + \theta)(\frac{1}{2} - \frac{1}{p'})$. Then

\[ \|e^{iBt} B^{-l} P \phi \|_{L^p} \lesssim |t|^{-(2 + \theta)(\frac{1}{2} - \frac{1}{p'})}\|\phi\|_{s,p}, \quad |t| \geq 1, \]

and

\[ \|e^{iBt} B^{-l} P \phi \|_{L^p} \lesssim |t|^{-(2 - \theta)(\frac{1}{2} - \frac{1}{p'})}\|\phi\|_{s,p}, \quad 0 < |t| \leq 1. \]
The following weighted decay estimate of the Klein-Gordon equation was first established by Jensen and Kato \[16\] for the Schrödinger equation and then extended to the Klein-Gordon equation by Komech and Kopylova \[21\]. These estimates are used to reveal the non-resonance structure of nonlinearities in the equation satisfied by the continuous spectrum part.

**Lemma 2.2** (Weighted $L^2$ estimate). Assume $V(x)$ satisfies the hypothesis of Lemma \[2.1\] Then, for $\sigma > \frac{5}{2}$, $l = 0, 1$, we have

$$
\left\| \langle x \rangle^{-\sigma} e^{iBt} P_c \langle x \rangle^{-\sigma} \psi \right\|_{L^2} \lesssim (t)^{-\frac{3}{2}} \| \psi \|_2.
$$

The following weighted $L^p$ estimates, which are consequences of the interpolation between the standard $L^\infty - L^1$ estimates and the weighted $L^2$ estimate, are also necessary:

**Lemma 2.3** (Weighted $L^p$ estimate). Assume that $V(x)$ satisfies the hypothesis of Lemma \[2.1\] Let $1 < p \leq 2$, $\frac{1}{p} + \frac{1}{p'} = 1$, $0 \leq \theta \leq 1$, $l = 0, 1$, and $s = (4 + \theta)(\frac{1}{2} - \frac{1}{p'})$. Then, for $\sigma > \frac{5}{2}$,

$$
\left\| \langle x \rangle^{-\sigma} e^{iBt} P_c \langle x \rangle^{-\sigma} \psi \right\|_{L^p} \lesssim |t|^{-\frac{2}{p'}} \left\| \psi \right\|_{s,p}, \quad |t| \geq 1
$$

and

$$
\left\| \langle x \rangle^{-\sigma} e^{iBt} P_c \langle x \rangle^{-\sigma} \psi \right\|_{L^p} \lesssim |t|^{-\frac{2}{p'}} \left\| \psi \right\|_{s,p}, \quad 0 < |t| \leq 1
$$

hold.

The use of the weighted $L^p$ estimates enables us to obtain a better time decay of the continuous spectrum part of $u$ from its spatial decay, which aids dealing with the asymptotic behavior of solutions to the nonlinear Klein-Gordon equations. In particular, they help us to overcome the loss of the derivative of $\eta_3$. For more details, see Section 7.

### 2.2 Singular Resolvents and Time Decay

The following local decay estimates for singular resolvents $e^{iBt}(B - \Lambda + i0)^{-l}$, which was proved in \[32\], are also significant. Here, $\Lambda$ is a point in the interior of the continuous spectrum of $B(\Lambda > m)$.

**Lemma 2.4** (Decay estimates for singular resolvents). Assume that $V(x)$ is a real-valued function and satisfies (V1)-(V3). Let $\sigma > 16/5$. Then for any point $\Lambda > m$ in the continuous spectrum of $B$, we have for $l = 1, 2$:

$$
\left\| \langle x \rangle^{-\sigma} e^{iBt}(B - \Lambda + i0)^{-l} P_c \langle x \rangle^{-\sigma} \psi \right\|_2 \lesssim (t)^{-\frac{5}{2}} \| \psi \|_{1,2}, \quad t > 0,
$$

and

$$
\left\| \langle x \rangle^{-\sigma} e^{iBt}(B - \Lambda - i0)^{-l} P_c \langle x \rangle^{-\sigma} \psi \right\|_2 \lesssim (t)^{-\frac{5}{2}} \| \psi \|_{1,2}, \quad t < 0.
$$

### 2.3 Global Well-Posedness and Energy Conservation

The global well-posedness of \[1.2\] with small initial data is well-known.
Theorem 2.5. Assume $V \in L^p$ with $p > 3/2$. Then, there exists $\varepsilon_0 > 0$ and $C > 0$, such that for any $\|(u_0, u_1)\|_{H^1 \times L^2} \leq \varepsilon < \varepsilon_0$, equation (1.2) admits exactly one solution $u \in C^0(\mathbb{R}; H^1) \cap C^1(\mathbb{R}; L^2)$ such that $(u(0), \partial_t u(0)) = (u_0, u_1)$. Furthermore, the map $(u_0, u_1) \mapsto (u(t), \partial_t u(t))$ is continuous from the ball $\|(u_0, u_1)\|_{H^1 \times L^2} < \varepsilon_0$ to $C^0(I; H^1) \times C^0(I; L^2)$ for any bounded interval $I$. Moreover, the energy $E[u, \partial_t u] \equiv \frac{1}{2} \int (\partial_t u)^2 + |\nabla u|^2 + m^2 u^2 + V(x)u^2 dx - \frac{\lambda}{4} \int u^4 dx$ is conserved and $\|(u(t), v(t))\|_{H^1 \times L^2} \leq C \|(u_0, v_0)\|_{H^1 \times L^2}$.

We refer to [6] for details.

3 Normal Form Transformation

In this section, we present a new normal form transformation which is a refined version of Theorem 4.9 in [3]. The main differences are as follows: (i) we find that the order of normal form actually increase by two in each step, which is consistent with the result in [32]; (ii) we give explicit forms of these coefficients appeared in error terms, whose structure will be crucial in the subsequent error estimates.

3.1 Hamiltonian Structure

Now we consider 3D nonlinear Klein Gordon equation (NLKG)

$$u_{tt} - \Delta u + Vu + m^2 u = u^3, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3,$$

which is a Hamiltonian perturbation of linear Klein-Gordon equation with potential. More precisely, in $H^1(\mathbb{R}^3, \mathbb{R}) \times L^2(\mathbb{R}^3, \mathbb{R})$ endowed with the standard symplectic form, namely

$$\Omega((u_1, v_1); (u_2, v_2)) := \langle u_1, v_2 \rangle_{L^2} - \langle u_2, v_1 \rangle_{L^2}$$

we consider the Hamiltonian

$$H = H_L + H_P,$$

$$H_L := \int_{\mathbb{R}^3} \frac{1}{2} (v^2 + |\nabla u|^2 + Vu^2 + m^2 u^2) \, dx,$$

$$H_P := \int_{\mathbb{R}^3} \frac{1}{4} u^4 \, dx.$$ 

The corresponding Hamilton equations are $\dot{v} = -\nabla_u H, \dot{u} = \nabla_v H$, where $\nabla_u H$ is the gradient with respect to the $L^2$ metric, explicitly defined by

$$\langle \nabla_u H(u), h \rangle = d_u H(u)h, \quad \forall h \in H^1$$

and $d_u H(u)$ is the Frechét derivative of $H$ with respect to $u$. It is easy to see that the Hamilton equations are explicitly given by

$$\begin{cases} \dot{v} = \Delta u - Vu - m^2 u + u^3, & \dot{u} = V \end{cases} \iff \ddot{u} = \Delta u - Vu - m^2 u + u^3.$$
Write
\[ u = q\varphi + P_cu, \quad v = p\varphi + P_cv, \]
with a slightly abuse of notations, from now on we denote
\[ B := P_c(-\Delta + V + m^2)^{1/2}P_c, \]
and define the complex variables
\[ \xi := \frac{q\sqrt{\omega} + ip}{\sqrt{2}}, \quad f := \frac{B^{1/2}P_cu + iB^{-1/2}P_cv}{\sqrt{2}}. \]

Then, in terms of these variables the symplectic form has the form
\[ \Omega \left( (\xi^{(1)}, f^{(1)}); (\xi^{(2)}, f^{(2)}) \right) = 2 \operatorname{Re} \left[ i \left( \xi^{(1)}\bar{\xi}^{(2)} + \langle f^{(1)}, \bar{f}^{(2)} \rangle \right) \right] \]
and the Hamilton equations take the form
\[ \dot{\xi} = -i\frac{\partial H}{\partial \bar{\xi}}, \quad \dot{f} = -i\nabla_f H. \]

where
\[ H_L = \omega |\xi|^2 + \langle \bar{f}, Bf \rangle, \]
\[ H_P(\xi, f) = \int_{\mathbb{R}^3} \left( \sum \frac{\xi + \bar{\xi}}{\sqrt{2\omega}} \varphi(x) + U(x) \right)^4 dx \]
with \( U = B^{-1/2}(f + \bar{f})/\sqrt{2} \equiv P_cu. \) The Hamiltonian vector field \( X_H \) of a function is given by
\[ X_H(\xi, \bar{\xi}, f, \bar{f}) = \left( -i\frac{\partial H}{\partial \xi}, i\frac{\partial H}{\partial \bar{\xi}}, -i\nabla_f H, i\nabla_{\bar{f}} H \right). \]

The associate Poisson bracket is given by
\[ \{H, K\} := i \left( \frac{\partial H}{\partial \xi} \frac{\partial K}{\partial \bar{\xi}} - \frac{\partial H}{\partial \bar{\xi}} \frac{\partial K}{\partial \xi} \right) + i \langle \nabla_f H, \nabla_{\bar{f}} K \rangle - i \langle \nabla_{\bar{f}} H, \nabla_f K \rangle. \]

Denote \( z = (\xi, f), f = (f, \bar{f}), \) and \( \mathcal{P}^{k,s} = \mathbb{C} \times P_c H^{k,s} (\mathbb{R}^3, \mathbb{C}), \) where
\[ H^{k,s} (\mathbb{R}^3, \mathbb{C}) = \left\{ f : \mathbb{R}^3 \to \mathbb{C} \text{ s.t. } \|f\|_{H^{s,k}} := \|x|^s(-\Delta + 1)^{k/2}f\|_{L^2} < \infty \right\}. \]

3.2 Lie Transform
Consider a function \( \chi \) of the form
\[ \chi(z) \equiv \chi(\xi, f) = \chi_0(\xi, \bar{\xi}) + \sum_{|\mu|+|\nu|=M_0+1} \xi^\mu \bar{\xi}^\nu \int_{\mathbb{R}^3} \Phi_{\mu\nu} \cdot f dx \]
where $\Phi_{\mu \nu} \cdot f := \Phi_{\mu \nu} f + \Psi_{\mu \nu} \tilde{f}$ with $\Phi_{\mu \nu}, \Psi_{\mu \nu} \in S(\mathbb{R}^3, \mathbb{C})$ and where $\chi_0$ is a homogeneous polynomial of degree $M_0 + 2$. The Hamiltonian vector field satisfies $X_\chi \in C^\infty(\mathcal{P}^{-\kappa,-s}, \mathcal{P}^{k,\tau})$ for any $k, \kappa, s, \tau \geq 0$. Moreover we have

$$\|X_\chi(z)\|_{\mathcal{P}^k, \tau} \leq C_{k, s, \kappa, \tau} \|z\|_{\mathcal{P}^{k, \tau}}^{M_0 + 1}. \quad (3.4)$$

Since $X_\chi$ is a smooth polynomial it is also analytic. Denote by $\phi^t$ the flow generated by $X_\chi$. For fixed $\kappa, s$, by (3.4) $\phi^t$ is well defined up to any fixed time $\bar{t}$, in a sufficiently small neighborhood $\mathcal{U}^{-\kappa,-s} \subset \mathcal{P}^{-\kappa,-s}$ of the origin. Set $\phi := \phi^1 \equiv \phi^t|_{t=1}$. The canonical transformation $\phi$ will be called the Lie transform generated by $\chi$.

**Lemma 3.1.** Given a functional $\chi$ of the form (3.3). Assume $\Phi_{\mu \nu}, \Psi_{\mu \nu} \in S(\mathbb{R}^3, \mathbb{C})$ for all $\mu$ and $\nu$. Let $\phi$ be its Lie transform. Denote $z' = \phi(z), z \equiv (\xi, f)$ and $z' \equiv (\xi', f')$. Then, there exists a sufficiently small neighborhood $\mathcal{U}^{-\kappa,-s} \subset \mathcal{P}^{-\kappa,-s}$ of the origin, such that the following expansions hold:

$$\xi' = \xi + \sum_{i=0}^{\infty} \sum_{k=1}^{k} a_{i \mu \nu} \xi^i \xi^\nu \prod_{j=1}^{i} \int \Phi_{\mu \nu}^{ij} \cdot f \: dx, \quad (3.5)$$

$$f' = f + \sum_{i=0}^{\infty} \sum_{k=1}^{k-1} b_{i \mu \nu} \xi^i \xi^\nu \prod_{j=1}^{i} \Lambda_{\mu \nu}^{ij} \cdot f \: dx \Psi_{\mu \nu}^i. \quad (3.6)$$

where $a_{i \mu \nu}$ and $b_{i \mu \nu}$ are constants and $\Phi_{\mu \nu}^{ij}, \Lambda_{\mu \nu}^{ij}, \Psi_{\mu \nu}^i \in S(\mathbb{R}^3, \mathbb{C})$.

**Proof.** Let $z(t) = (\xi(t), f(t)) = \phi^t(z)$, then $z(0) = z, z(1) = z'$. By virtue of the Lie transform and the analyticity of the Hamiltonian vector field $X_\chi$, we have

$$\xi' = \xi + \sum_{k=1}^{\infty} \sum_{i=0}^{\infty} \frac{1}{k!} \{\chi_{r, \ldots, \{\chi_r, \xi\}}\}, \quad (3.7)$$

and it is easy to show by induction that

$$\{\chi_{r, \ldots, \{\chi_r, \xi\}}\} = \sum_{i=0}^{\infty} \sum_{k=1}^{k} a_{i \mu \nu} \xi^i \xi^\nu \prod_{j=1}^{i} \int \Phi_{\mu \nu}^{ij} \cdot f \: dx. \quad (3.8)$$

Similarly, since

$$\frac{df}{dt} = -i \sum_{\mu+\nu=M_0+1} \xi^\mu \xi^\nu \Psi_{\mu \nu}, \quad (3.8)$$

we have

$$f' = f - i \sum_{\mu+\nu=M_0+1} \int_0^1 \xi^\mu(t) \xi^\nu(t) dt \Psi_{\mu \nu}. \quad (3.9)$$

By virtue of the Lie transform, we have

$$\xi^\mu(t) \xi^\nu(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \{\chi_{r, \ldots, \{\chi_r, \xi^\mu \xi^\nu\}}\}, \quad (3.10)$$

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hence

$$f' = f - i \sum_{\mu + \nu = M_0 + 1}^{\infty} \sum_{k=0}^{\infty} \frac{1}{(k + 1)!} \{\chi_r, \ldots, \{\chi_r, \xi^\mu \xi^\nu\}\} \Psi_{\mu \nu}. \quad (3.11)$$

The rest follows similarly. \hfill \Box

### 3.3 Normal Form Transformation

**Definition 3.2.** A polynomial $Z$ is in normal form if we have

$$Z = Z_0 + Z_1$$

where $Z_0$ is a linear combination of monomials $|\xi|^{2\mu}$, and $Z_1$ is a linear combination of monomials of the form

$$\xi^\mu \xi^\nu \int \Phi(x) f(x) dx, \quad \xi^\mu \xi^\nu \int \Phi(x) f(x) dx$$

with indexes satisfying

$$\omega(\mu - \nu) < -m, \quad \omega(\mu' - \nu') > m$$

and $\Phi \in \mathcal{S}\left(\mathbb{R}^3, \mathbb{C}\right)$.  

**Theorem 3.3.** For any $n > 0, s > 0$ and any integer $r$ with $0 \leq r \leq 2N$, there exist open neighborhoods of the origin $U_{r,n,s} \subset \mathcal{P}^{1/2,0}$, $U_{r,-n,-s} \subset \mathcal{P}^{-n,-s}$, and an analytic canonical transformation $T_r : U_{r,n,s} \to \mathcal{P}^{1/2,0}$, such that $T_r$ puts the system in normal form up to order $2r + 4$. More precisely, we have

$$H^{(r)} := H \circ T_r = H_L + Z^{(r)} + R^{(r)}$$

where: (i) $Z^{(r)}$ is a polynomial of degree $2r + 2$ which is in normal form,
(ii) $I - T_r$ extends into an analytic map from $U_{r,-n,-s}$ to $\mathcal{P}^{n,s}$ and

$$\|z - T_r(z)\|_{\mathcal{P}^{n,s}} \lesssim \|z\|^{3}_{\mathcal{P}^{-n,-s}}.$$

(iii) we have $R^{(r)} = \sum_{d=0}^{5} R^{(r)}_d$ with the following properties:

(iii.0) we have

$$R^{(r)}_0 = \sum_{\mu + \nu = 2r + 4} a^{(r)}_{\mu \nu}(\xi) \xi^\mu \xi^\nu$$

where $a^{(r)}_{\mu \nu} \in C^\infty(\mathbb{C}), a^{(r)}_{\mu \nu} = a^{(r)}_{\nu \mu}$ satisfying the following expansion with a sufficiently large integer $M^* > 0$:

$$a^{(r)}_{\mu \nu}(\xi) = \sum_{k=0}^{M^*} \sum_{|\alpha + \beta| = 2k} a^{(r)}_{\mu \nu; \alpha \beta} \xi^\alpha \xi^\beta, \quad (3.12)$$

(iii.1) we have

$$R^{(r)}_{1} = \sum_{\mu + \nu = 2r + 3} \xi^\mu \xi^\nu \int_{\mathbb{R}^3} \Phi^{(r)}_{\mu \nu}(x, \xi) \cdot f(x) dx$$

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where the map

\[ \mathbb{C} \ni \xi \mapsto \Phi_{\mu\nu}(\cdot, \xi) \in (H^{n,s})^2 \] is \( C^\infty \), \( \Phi_{\mu\nu} = (\Phi^{(r)}_{\mu\nu}, \Phi^{(r)}_{\nu\mu}) \)

satisfying the following expansion:

\[ \Phi^{(r)}_{\mu\nu}(\cdot, \xi) = \sum_{k=0}^{M^*} \sum_{[\alpha+\beta]=2k} \Phi^{(r)}_{\mu\nu\alpha\beta}(x) \xi^\alpha \xi^\beta \] (3.13)

with \( \Phi^{(r)}_{\mu\nu\alpha\beta}(x) \in \mathcal{S}(\mathbb{R}^3, \mathbb{C}) \).

(iii.2-4) for \( d = 2, 3, 4 \), we have

\[ R_d^{(r)} = \int_{\mathbb{R}^3} F_d^{(r)}(x, z) [U(x)]^d dx + \sum_{k=1}^{d} \prod_{l=1}^{k} \int_{\mathbb{R}^3} A_{dik}^{(r)}(x, z) \cdot fdx, \] (3.14)

where \( F_d^{(r)} \equiv 1 \); for \( d = 2, 3 \), \( F_d^{(r)}(x, z) \in \mathbb{R} \) is a linear combination of terms of the form

\[ \sum_{k=0}^{M^*} \sum_{i=0}^{\mu+\nu=4-d+2k-i} \xi^\mu \xi^\nu \prod_{j=1}^{i} \int \Phi^{ij}_{\mu\nu}(x) \cdot fdx \Psi^{ij}_{\mu\nu}(x), \] (3.15)

and \( A_{dik}^{(r)}(x, z) = (A_{dik}^{(r)}, A_{dik}^{(r)}) \) \((d = 2, 3, 4)\) is a linear combination of terms of the form

\[ \sum_{k=0}^{M^*} \sum_{i=0}^{\mu+\nu=1+2k-i} \xi^\mu \xi^\nu \prod_{j=1}^{i} \int \Phi^{ij}_{\mu\nu}(x) \cdot fdx \tilde{\Psi}^{ij}_{\mu\nu}(x), \] (3.16)

with \( \Phi^{ij}_{\mu\nu}(x), \tilde{\Phi}^{ij}_{\mu\nu}(x), \Psi^{ij}_{\mu\nu}(x), \tilde{\Psi}^{ij}_{\mu\nu}(x) \in \mathcal{S}(\mathbb{R}^3, \mathbb{C}) \), and the sum over index \( k \) in (3.14) is a finite sum.

(iii.5) for \( d = 5 \), we have

\[ \left\| \nabla_x \xi R_5^{(r)} \right\|_{(P^{n,s})^2} \lesssim |\xi|^{M^*}. \]

Remark 3.4. For \( f \in X = W^{2,2}(\mathbb{R}^3, \mathbb{C}) \cap W^{2,1}(\mathbb{R}^3, \mathbb{C}) \), since \( H^{k,s}(\mathbb{R}^3, \mathbb{C}) \subset X \) for \( k, s \) large, we also have

\[ \| (\xi, f) - T_\mathcal{E}(\xi, f) \|_{C \times X} \lesssim \| (\xi, f) \|_{\mathcal{F}^{n,s}}^{\frac{3}{2}}. \]

Proof. We prove Theorem 3.3 by induction. We note that with some slightly abuses of notations, we denote \( a \) with indexs as some constant, and denote \( \Phi \) or \( \Psi \) with indexs as some Schwartz function, they may change line from line, depending on the context. We also note that the sum with no upper index always denotes finite sum.

(Step 0) First, when \( r = 0 \), Theorem 3.3 holds with \( T_0 = I, Z^{(0)} = 0, R^{(0)} = H_p \). And we have

\[ R_0^{(0)} = \sum_{\mu+\nu=4} \xi^\mu \xi^\nu \int_{\mathbb{R}^3} \frac{\varphi^4}{2 \sqrt{\omega^4}} dx, \]

\[ R_1^{(0)} = \sum_{\mu+\nu=3} \xi^\mu \xi^\nu \int_{\mathbb{R}^3} \frac{3 \varphi^3}{2 \sqrt{\omega^4}} \left( B^{-\frac{1}{2}} f + B^{-\frac{3}{2}} f \right) dx, \]

\[ R_k^{(0)} = \int_{\mathbb{R}^3} F_k^{(0)} U_k^{d} dx, \quad F_k^{(0)} = \sum_{\mu+\nu=4-d} \frac{C_k^d}{2^{1-d} \sqrt{\omega^{4-d}}} \xi^\mu \xi^\nu \varphi^{1-d}(d = 2, 3), F_4^{(0)} = 1. \]
Thus, \( a_{\mu \nu}^{(0)} \triangleq \int_{\mathbb{R}^3} \Phi_1 \chi^4 \) and \( \Phi_{\mu \nu}^{(0)} \triangleq \left( \frac{3}{2 \sqrt{\omega}} B - \frac{1}{2} (\Phi^3), \frac{3}{2 \sqrt{\omega}} B^{-\frac{1}{2}} (\Phi^3) \right) \).

**Step** \( r \to r + 1 \) Now we assume that the theorem holds for some \( 0 \leq r \leq 2N \), we shall prove this for \( r + 1 \). More precisely, define

\[
R_{12}^{(r)} = R_{12}^{(r)} - \sum_{\mu + \nu = 2r + 3} \Phi_{\mu \nu}^{(r)}(x,0) \cdot f(x) dx.
\]

By (3.12) and (3.13), we have

\[
\begin{align*}
R_{12}^{(r)} & = R_{12}^{(r)} - \sum_{\mu + \nu = 2r + 3} \xi^\mu \xi^\nu \int_{\mathbb{R}^3} \Phi_{\mu \nu}^{(r)}(x,0) \cdot f(x) dx.
\end{align*}
\]

where the coefficients \( a_{\mu \nu}^{(r+1)}(\xi), \Phi_{\mu \nu}^{(r+1)}(x,\xi) \) satisfy (3.12)-(3.13) respectively, with \( r \) replaced by \( r + 1 \).

Set

\[
K_{r+1} := \sum_{\mu + \nu = 2r + 4} a_{\mu \nu}^{(r)}(0) \xi^\mu \xi^\nu + \sum_{\mu + \nu = 2r + 3} \xi^\mu \xi^\nu \int_{\mathbb{R}^3} \Phi_{\mu \nu}^{(r)}(x,0) \cdot f(x) dx,
\]

which is real-valued. Then, we solve the following homologic equation

\[
\{H_L, \chi_{r+1}\} + Z_{r+1} = K_{r+1},
\]

with \( Z_{r+1} \) in normal form. Thus,

\[
\begin{align*}
Z_{r+1} & = \sum_{\mu + \nu = 2r + 4} a_{\mu \nu}^{(r)}(0) |\xi|^\mu + \sum_{\mu + \nu = 2r + 3} \xi^\mu \xi^\nu \int_{\mathbb{R}^3} \Phi_{\mu \nu}^{(r)}(x,0) f(x) dx \\
& + \sum_{\mu + \nu = 2r + 3} \omega \cdot (\mu - \nu) > m \xi^\mu \xi^\nu \int_{\mathbb{R}^3} \Phi_{\mu \nu}^{(r)}(x,0) f(x) dx,
\end{align*}
\]

and

\[
\begin{align*}
\chi_{r+1} & = i \sum_{\mu + \nu = 2r + 3} a_{\mu \nu}^{(r)}(0) \omega \cdot (\mu - \nu) \xi^\mu \xi^\nu + i \sum_{\mu + \nu = 2r + 3} \omega \cdot (\mu - \nu) > m \xi^\mu \xi^\nu \int_{\mathbb{R}^3} \Phi_{\mu \nu}^{(r)}(x,0) f(x) dx \\
& - i \sum_{\mu + \nu = 2r + 3} \omega \cdot (\mu - \nu) < m \xi^\mu \xi^\nu \int_{\mathbb{R}^3} \Phi_{\mu \nu}^{(r)}(x,0) f(x) dx,
\end{align*}
\]

where the operator

\[
R_{\mu \nu} := (B - \omega \cdot (\mu - \nu))^{-1}.
\]
Let $\phi_{r+1}$ be the Lie transform generated by $\chi_{r+1}$, i.e. $\phi_{r+1} = \phi_{r+1}|_{t=1}$, where

$$\frac{d\phi^t_{r+1}}{dt} = X_{\chi_{r+1}} = (-i\partial_{\xi}\chi_{r+1}, -i\nabla_f\chi_{r+1}).$$

Then, for $z' = (\xi', f') = \phi_{r+1}(\xi, f)$, Lemma 3.1 holds, i.e.

$$\xi' = \xi + \sum_{k=1}^{\infty} \sum_{i=0}^{k} \sum_{\mu + \nu = (2r+2)k+1-i} a_{i\mu\nu} \xi^\mu \xi^\nu \prod_{j=1}^{i} \Phi^{ij}_{\mu\nu} \cdot f, \quad (3.17)$$

$$f' = f + \sum_{k=1}^{\infty} \sum_{i=0}^{k-1} \sum_{\mu + \nu = (2r+2)k+1-i} \xi^\mu \xi^\nu \prod_{j=1}^{i} \Lambda^{ij}_{\mu\nu} \cdot f dx \Psi^{ij}_{\mu\nu}. \quad (3.18)$$

Recall that

$$R^{(r)} = K_{r+1} + R_{02}^{(r)} + R_{12}^{(r)} + \sum_{d=2}^{5} R_{d}^{(r)},$$

and $K_{r+1} = Z_{r+1} + \{H_L, \chi_{r+1}\}$, we have

$$H^{(r+1)} \triangleq H^{(r)} \circ \phi_{r+1} = H \circ (T_r \circ \phi_{r+1}) \equiv H \circ T_{r+1} = H_L \circ \phi_{r+1} + Z^{(r)} \circ \phi_{r+1} + R^{(r)} \circ \phi_{r+1} + K_{r+1} + Z_{r+1}$$

$$+ [H_L \circ \phi_{r+1} - (H_L + \{\chi_{r+1}, H_L\})] \quad (3.19)$$

$$+ Z^{(r)} \circ \phi_{r+1} - Z^{(r)} \quad (3.20)$$

$$+ (K_{r+1} \circ \phi_{r+1} - K_{r+1}) \quad (3.21)$$

$$+ (R_{02}^{(r)} + R_{12}^{(r)}) \circ \phi_{r+1} \quad (3.22)$$

$$+ \sum_{d=2}^{5} R_{d}^{(r)} \circ \phi_{r+1}. \quad (3.23)$$

We define $Z^{(r+1)} = Z^{(r)} + Z_{r+1}$ in the normal form of order $2r + 4$. For the term $3.19$, similar to the proof of Lemma 3.1, we have

$$H_L \circ \phi_{r+1} - (H_L + \{\chi_{r+1}, H_L\})$$

$$= \sum_{k=2}^{\infty} \frac{1}{k!} \{\chi_{r+1}, \ldots, \chi_{r+1}, H_L\}$$

$$= \sum_{k=2}^{\infty} \sum_{i=0}^{k} \sum_{\mu + \nu = 2(r+1)k+2-i} a_{i\mu\nu} \xi^\mu \xi^\nu \prod_{j=1}^{i} \Phi^{ij}_{\mu\nu} \cdot f dx$$

$$= \sum_{k=2}^{M^*} \left( \sum_{\mu + \nu = 2(r+1)k+2} a_{0\mu\nu} \xi^\mu \xi^\nu + \sum_{\mu + \nu = 2(r+1)k+1} a_{1\mu\nu} \xi^\mu \xi^\nu \int \Phi^{11}_{\mu\nu} \cdot f dx \right)$$

$$+ \sum_{k=2}^{M^*} \sum_{i=2}^{k} \sum_{\mu + \nu = 2(r+1)k+2-i} a_{i\mu\nu} \xi^\mu \xi^\nu \prod_{j=1}^{i} \Phi^{ij}_{\mu\nu} \cdot f dx + O(|\xi|^{M^*}).$$
Thus (3.19) can be absorbed into $R_0^{(r+1)}$, $R_1^{(r+1)}$, $R_2^{(r+1)}$ and $R_3^{(r+1)}$.

The terms (3.20), (3.21) and (3.22) can be handled similarly.

For the term (3.23), denote $f' = f + G_f$, $U' = U + G_U$, then for $d = 2, 3, 4$, we have

$$
R_d^{(r)} \circ \phi_{r+1} = \int_{\mathbb{R}^3} F_d^{(r)}(x, z') (U + G_U)^d \, dx + \sum_{k}^{d} \prod_{l=1}^{k} \int_{\mathbb{R}^3} \Lambda_{d,l,k}^{(r)}(x, z') \cdot (f + G_f) \, dx
$$

$$
= \sum_{j=0}^{d} \left[ \int F_d^{(r)}(x, z') U^j G_U^{d-j} \, dx + \sum_{k,l,i}^{d} \int_{\mathbb{R}^3} \Lambda_{d,l,i,k}^{(r)}(x, z') \cdot f \, dx \prod_{l' \neq l} \int_{\mathbb{R}^3} \Lambda_{d,l,k}^{(r)}(x, z') \cdot G_f \, dx \right]
$$

$$
:= \sum_{j=0}^{d} H_{dj}.
$$

By (3.18), we have

$$
G_f = \sum_{k=1}^{\infty} \sum_{i=0}^{k-1} \sum_{\mu + \nu = (2r + 2)k + 1 - i} \xi^i \bar{\xi}^j \prod_{j=1}^{i} \int \Lambda_{\mu \nu}^{ij} \cdot f \, dx \Psi_{\mu \nu} + O(|\xi|^{M^*}),
$$

and for $2 \leq j \leq d$,

$$
H_{dj} = \int F_d^{(r)}(x, z') U^j G_U^{d-j} \, dx + \sum_{k,l,i}^{d} \int_{\mathbb{R}^3} \Lambda_{d,l,i,k}^{(r)}(x, z') \cdot f \, dx \prod_{l' \neq l} \int_{\mathbb{R}^3} \Lambda_{d,l,k}^{(r)}(x, z') \cdot G_f \, dx
$$

$$
= \sum_{k=0}^{M^*} \sum_{i=0}^{k} \sum_{\mu + \nu = (2r + 2)(d - 1) + 2k - i} a_{i \mu \nu} \xi^i \bar{\xi}^j \sum_{j=1}^{i+1} \prod_{j=1}^{i+1} \Phi_{\mu \nu}^{ij} \cdot f \, dx + O(|\xi|^{M^*}),
$$

where

$$
F_j^{(r+1)} = F_j^{(r)}(x, z') G_U^{d-j} - O(|\xi|^{M^*}) = \sum_{k=0}^{M^*} \sum_{i=0}^{k} \sum_{\mu + \nu = 4 - j + (2r + 2)(d - j) + 2k - i} a_{i \mu \nu} \xi^i \bar{\xi}^j \sum_{l=1}^{i} \prod_{l=1}^{i} \Phi_{\mu \nu}^{il} \cdot f \, dx \psi_{\mu \nu}^{l}(x),
$$

note that $F_4^{(r+1)} \equiv 1$. Thus $H_{dj}$ can be absorbed into $R^{(r+1)}$. Finally, it is direct to see $R_5^{(r)} \circ \phi_{r+1}$ can be absorbed into $R_5^{(r+1)}$. □
4 Isolation of the Key Resonant Terms

Apply Theorem 3.3 for \( r = 2N \), we obtain the new Hamiltonian

\[
H = H_L(\xi, f) + Z_0(\xi) + Z_1(\xi, f) + \mathcal{R},
\]

where

\[
Z_1(\xi, f) := \langle G, f \rangle + \langle \bar{G}, \bar{f} \rangle,
\]

\[
G := \sum_{(\mu, \nu) \in M} \xi^\mu \bar{\xi}^\nu \Phi_{\mu\nu}(x), \Phi_{\mu\nu} \in \mathcal{S}(\mathbb{R}^3, \mathbb{C}),
\]

with

\[
M = \{(\mu, \nu) \mid \mu + \nu = 2r + 1, 0 \leq r \leq 2N, \omega(\mu - \nu) < -m\}.
\]

Then, the corresponding Hamiltonian equations are

\[
\dot{f} = -i(Bf + \bar{G}) - i\partial \bar{f} \mathcal{R},
\]

\[
\dot{\xi} = -i\omega \xi - i\partial \bar{\xi} Z_0 - i \langle \partial \bar{\xi} G, f \rangle - i \langle \partial \bar{\xi} \bar{G}, \bar{f} \rangle - i\partial \bar{\xi} \mathcal{R}.
\]

Define

\[
M_1 = \{(\mu, \nu) \mid \mu + \nu = 2N + 1, \; \omega(\mu - \nu) < -m\},
\]

and

\[
G = \sum_{(\mu, \nu) \in M} \xi^\mu \bar{\xi}^\nu \Phi_{\mu\nu} = \sum_{(\mu, \nu) \in M_1} \xi^\mu \bar{\xi}^\nu \Phi_{\mu\nu} + \sum_{(\mu, \nu) \in M \setminus M_1} \xi^\mu \bar{\xi}^\nu \Phi_{\mu\nu} := \mathcal{M}_G + \mathcal{R}_G.
\]

Since \((2N - 1)\omega < m < (2N + 1)\omega\), we have

\[
\mathcal{M}_G = \bar{\xi}^{2N+1}\Phi_{0,2N+1}.
\]

Let \( \eta = e^{i\omega t} \xi \), by (4.1) and Duhamel’s formula, we have

\[
f(t) = e^{-iBt} f(0) + \int_0^t e^{-iB(t-s)} (-i\bar{G} - i\partial \bar{f} \mathcal{R}) ds
\]

\[
= e^{-iBt} f(0) - i e^{-iBt} \sum_{(\mu, \nu) \in M} \int_0^t e^{i(B - \omega(\nu - \mu))s} \eta'' \bar{\eta}' \Phi_{\mu\nu} ds - i \int_0^t e^{-iB(t-s)} \partial \bar{f} \mathcal{R} ds
\]

\[
:= \mathcal{M}_f + \mathcal{R}_f,
\]

where

\[
\mathcal{M}_f = - \sum_{(\mu, \nu) \in M_1} e^{-i(\nu - \mu)t} \eta'' \bar{\eta}' (B - \omega(\nu - \mu) - i0)^{-1} \Phi_{\mu\nu},
\]

\[
= -\xi^{2N+1}(B - (2N + 1)\omega - i0)^{-1}\Phi_{0,2N+1}
\]
is the main term and
\[ R_f = e^{-iBt}f(0) - i \int_0^t e^{-iB(t-s)} \partial_f R ds \]
\[ + \sum_{(\mu,\nu)\in M_1} \eta(0) \bar{\eta}(0) e^{-iBt} (B - \omega(\nu - \mu) - i0)^{-1} \Phi_{\mu\nu} \]
\[ + e^{-iBt} \sum_{(\mu,\nu)\in M_1} \int_0^t e^{i(B-\omega(\nu-\mu))s} \frac{d}{ds} (\eta^\prime(0) \bar{\eta}^\prime) (B - \omega(\nu - \mu) - i0)^{-1} \Phi_{\mu\nu} ds \]
\[ - i e^{-iBt} \sum_{(\mu,\nu)\in M\setminus M_1} \int_0^t e^{i(B-\omega(\nu-\mu))s} \eta^\prime \bar{\eta}^\prime \Phi_{\mu\nu} ds \]
are error terms, which will be estimated later, see Section 6. Substitute (4.4) into (4.2), we obtain
\[ \dot{\xi} = -i \omega \xi - i \partial_\bar{\xi} Z_0 - i \langle \partial_\bar{\xi} M_G, M_f \rangle - i \langle \partial_\bar{\xi} \bar{M}_G, \bar{M}_f \rangle + R_\xi, \]
where
\[ R_\xi = -i \langle \partial_\bar{\xi} R_G, M_f \rangle - i \langle \partial_\bar{\xi} \bar{R}_G, \bar{M}_f \rangle - i \langle \partial_\bar{\xi} G, R_f \rangle - i \langle \partial_\bar{\xi} \bar{G}, \bar{R}_f \rangle - i \partial_\bar{\xi} R \]
can be treated as error terms. Hence, we compute
\[ \frac{d}{dt} |\xi|^2 = 2 \text{Re}(\bar{\xi} \dot{\xi}) \]
\[ = -2(2N+1)|\xi|^{4N+2} \text{Im}(\Phi_{0,2N+1}, (B - (2N+1)\omega - i0)^{-1} \Phi_{0,2N+1}) + 2 \text{Re}(\bar{\xi} R_\xi). \]
Note that by using Plemelji formula
\[ \frac{1}{x + i0} = \text{P.V.} \frac{1}{x} \pm i\pi \delta(x), \]
we have
\[ \gamma := (2N+1) \text{Im}(\Phi_{0,2N+1}, (B - (2N+1)\omega - i0)^{-1} \Phi_{0,2N+1}) \]
\[ = (2N+1) \langle \Phi_{0,2N+1}, \delta(B - (2N+1)\omega) \Phi_{0,2N+1} \rangle \]
\[ \geq 0, \]
and
\[ \frac{d}{dt} |\xi|^2 = -2\gamma |\xi|^{4N+2} + 2 \text{Re}(\bar{\xi} R_\xi). \]
(4.7)
Throughout this paper, we assume the following non-degenerate assumption.

**Assumption 4.1** (Fermi’s Golden Rule). \( \gamma > 0 \).

**Remark 4.2.** The Fermi’s Golden Rule condition implies that \( |\xi| \approx \frac{|\xi_0|}{(1 + |\xi_0|^{4N})^{\frac{1}{4}}} \) if we could neglect the error term \( 2 \text{Re}(\bar{\xi} R_\xi) \), which will be justified in Section 6.
5 Asymptotic Behavior

In this section, we derive the asymptotic behavior of $\xi$ and $f$. Before proceeding, we introduce some useful notations. Let $T > 0$ be fixed, denote $\langle t \rangle = (1 + t^2)^{1/2}$ and

$$\frac{[\xi]}{4N}(T) = \sup_{0 \leq t \leq T} \langle |\xi_0|^{4N}t \rangle^{1/4N} |\xi(t)|,$$

We define the norm $\| \cdot \|_X$ as

$$\|f\|_X = \|f\|_{W^{2,2}} + \|f\|_{W^{2,1}}.$$

5.1 Dynamics of $\xi$

The following theorem allows to treat $R_\xi$ perturbatively in the dynamics of $\xi$:

**Theorem 5.1.** Suppose that $\gamma > 0$ and the error term $R_\xi$ satisfies

$$|R_\xi(t)| \leq Q_0 (1 + 4N\gamma|\xi_0|^{4N}t)^{-4N\gamma - 1 - \delta}$$

for some small constant $Q_0$ and $\delta > 0$, then

$$|\xi(t)|^{4N} \leq (1 + 4N\gamma|\xi_0|^{4N}t)^{-1} \left( |\xi_0|^{4N} + \frac{C(\gamma^{-1/2}Q_0^{1/2}|\xi_0|^{4N-1/2} + \gamma^{-4N+1}Q_0^{4N})}{(1 + 4N\gamma|\xi_0|^{4N}t)^{\delta/2}} \right)$$

(5.1)

for some absolute constant $C$. In addition, if $Q_0 = O(|\xi_0|^{4N+1+\epsilon})$ and $|\xi_0|$ is sufficiently small, then

$$|\xi(t)|^{4N} \geq (1 + 4N\gamma|\xi_0|^{4N}t)^{-1} \left( |\xi_0|^{4N} - \frac{|\xi_0|^{4N+\epsilon}}{(1 + 4N\gamma|\xi_0|^{4N}t)^{\delta}} \right).$$

(5.2)

**Proof.** We use standard comparison theorem to prove Theorem 5.1. Define $r = |\xi|^{4N}$, by (4.7) we have

$$-4N\gamma r^2 - 4N|R_\xi|r^{\frac{4N-1}{4N}} \leq r' \leq -4N\gamma r^2 + 4N|R_\xi|r^{\frac{4N-1}{4N}},$$

(5.3)

Define

$$h(t) = (1 + 4N\gamma r_0 t)^{-1}(r_0 + C_0 (1 + 4N\gamma r_0 t)^{-\delta/2}),$$

then

$$h'(t) = \frac{-4N\gamma r_0^2}{(1 + 4N\gamma r_0 t)^2} - \frac{4N(1 + \delta/2)\gamma r_0 C_0}{(1 + 4N\gamma r_0 t)^{2+\delta/2}}.$$

Hence,

$$h' + 4N\gamma h^2 = \frac{4N(1 - \delta/2)\gamma r_0 C_0}{(1 + 4N\gamma r_0 t)^{2+\delta/2}} + \frac{4N\gamma C_0^2}{(1 + 4N\gamma r_0 t)^{2+\delta}} \geq \frac{4N\gamma C_0^2}{(1 + 4N\gamma r_0 t)^{2+\delta}}.$$
Note that
\[ h^{\frac{4N-1}{4N}} \leq \left( 1 + 4N\gamma r_0 t \right)^{-\frac{4N-1}{4N}} \left( r_0 + C_0 r_0^2 \right) \left( 1 + 4N\gamma r_0 t \right)^{-\frac{(4N-1)\delta}{8N}}, \]
then
\[ 4N|\mathcal{R}_\xi|h^{\frac{4N-1}{4N}} \leq \frac{4NQ_0 r_0^{\frac{4N-1}{4N}} + 4NQ_0 C_0^{\frac{4N-1}{4N}}}{(1 + 4N\gamma r_0 t)^{2+\delta}}. \]
Hence, if we choose \( C_0 \) such that
\[ 4N\gamma C_0^2 \geq 8NQ_0 r_0^{\frac{4N-1}{4N}} + 8NQ_0 C_0^{\frac{4N-1}{4N}}, \]
then
\[ h' + 4N\gamma h^2 > 4N|\mathcal{R}_\xi|h^{\frac{4N-1}{4N}}. \] (5.4)

By comparing (5.3) and (5.5) and note that \( r_0 \leq h_0 \), we get \( r(t) \leq h(t) \) and the desired estimate is proved. To achieve the condition (5.4), it suffices to choose \( C_0 = C(\gamma^{-1/2}Q_0^{1/2} r_0^{\frac{4N-1}{8N}} + \gamma^{-\frac{4N}{8N}} r_0^{\frac{4N}{8N+1}}) \) for some large constant \( C \).

For the lower bound of \(|\xi|\), note that
\[ r' \geq -4N\gamma r^2 - 4N|\mathcal{R}_\xi| r^{\frac{4N-1}{4N}}. \] (5.6)
Choose
\[ \tilde{h}(t) = (1 + 4Nr_0 \gamma t)^{-1} \left( r_0 - r_0^{(1+\varepsilon)/8N} (1 + 4Nr_0 \gamma t)^{-\delta} \right), \]
similarly we compute
\[ \tilde{h}' + 4N\gamma \tilde{h}^2 = -\frac{4N\gamma(1-\delta)r_0^{2+\frac{\delta}{8N}}}{(1 + 4N\gamma r_0 t)^{2+\delta}} + \frac{4N\gamma r_0^{2+\frac{\delta}{8N}}}{(1 + 4N\gamma r_0 t)^{2+2\delta}} \lesssim -\frac{r_0^{2+\frac{\delta}{8N}}}{(1 + 4N\gamma r_0 t)^{2+\delta}}, \]
and
\[ \tilde{h}^{\frac{4N-1}{4N}} < (1 + 4N\gamma r_0 t)^{-\frac{4N-1}{4N}} r_0^{\frac{4N-1}{4N}}, \]

hence
\[ 4N|\mathcal{R}_\xi|\tilde{h}^{\frac{4N-1}{4N}} \lesssim \frac{r_0^{2+\frac{\delta}{8N}}}{(1 + 4N\gamma r_0 t)^{2+\delta}}. \]
Therefore if we choose \(|\xi_0|\) to be sufficiently small, then
\[ \tilde{h}' < -4N\gamma \tilde{h}^2 - 4N|\mathcal{R}_\xi|\tilde{h}^{\frac{4N-1}{4N}}. \] (5.7)
Using comparison theorem we get \( r \geq \tilde{h} \). □
5.2 Dynamics of $f$

By (4.3), $f$ can be decomposed as

$$f(t) = e^{-iBt} f(0) + \int_0^t e^{-iB(t-s)}(-iG - i\partial_f \mathcal{R})ds.$$ 

Denote

$$\hat{\mathcal{R}}_f := -i \int_0^t e^{-iB(t-s)} \partial_f \mathcal{R}ds.$$

**Proposition 5.2.** Let $\sigma > \frac{5}{2}$, suppose that the error term $\hat{\mathcal{R}}_f$ satisfies

\begin{align*}
\|B^{-1/2} \hat{\mathcal{R}}_f\|_{L^8} &\lesssim \langle t \rangle^{-2N+1} \|f(0)\|_X + \langle \xi_0 \rangle^{4N} t^{-\frac{2N+1}{4N+1}} \mathcal{P}_1 + \langle \xi_0 \rangle^{4N} t^{-\frac{2N+1}{4N+1}} \mathcal{P}_2, \quad (5.8) \\
\|\langle x \rangle^{-\sigma} B^{1/2} \hat{\mathcal{R}}_f\|_{L^4} &\lesssim \langle t \rangle^{-\frac{1}{2}} \|f(0)\|_X + \langle \xi_0 \rangle^{4N} t^{-\frac{1}{4N+1}} \mathcal{P}_3 + \langle \xi_0 \rangle^{4N} t^{-\frac{1}{4N+1}} \mathcal{P}_4. \quad (5.9)
\end{align*}

for some constants $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$, then the following estimates hold

\begin{align*}
\|B^{-1/2} e^{-iBt} f(0)\|_{L^8} &\lesssim \langle t \rangle^{-9/8} \|B^{-1/2} f(0)\|_{W^{2, \frac{4}{9}}} \lesssim \langle t \rangle^{-9/8} \|B^{-1/2} f(0)\|_X. \\
\|B^{1/2} e^{-iBt} f(0)\|_{L^4} &\lesssim \langle t \rangle^{-1/2} \|B^{1/2} f(0)\|_{W^{1, \frac{4}{9}}} \lesssim \langle t \rangle^{-1/2} \|B^{-1/2} f(0)\|_X.
\end{align*}

Thus, it suffices to prove that

\begin{align*}
\left\| B^{-1/2} \int_0^t e^{-iB(t-s)} \tilde{G}ds \right\|_{L^8} &\lesssim \langle \xi_0 \rangle^{4N} t^{-\frac{2N+1}{4N+1}} \| \xi \|^{2N+1}_{\frac{1}{4N}}, \quad (5.10) \\
\left\| \langle x \rangle^{-\sigma} B^{1/2} \int_0^t e^{-iB(t-s)} \tilde{G}ds \right\|_{L^4} &\lesssim \langle \xi_0 \rangle^{4N} t^{-\frac{2N+1}{4N+1}} \| \xi \|^{2N+1}_{\frac{1}{4N}}. \quad (5.11)
\end{align*}

Note that by definition,

$$G = \sum_{(\mu, \nu) \in \mathcal{M}} \xi^{\mu} \overline{\xi}^{\nu} \Phi_{\mu \nu}(x), \Phi_{\mu \nu} \in S \left( \mathbb{R}^3, \mathbb{C} \right),$$

with $\mathcal{M} = \{(\mu, \nu) \mid \mu + \nu = 2r + 1, 0 \leq r \leq 2N, \omega(\mu - \nu) < -m\}$. Since $(2N - 1)\omega < m < (2N + 1)\omega$, we have

$$\mu + \nu \geq 2N + 1,$$

which implies (5.10) and (5.11) by using Lemma A.1 in Appendix.
6 Error Estimates

In this section, we estimate error terms \( \tilde{\mathcal{R}}_f \) and \( \mathcal{R}_\xi \) by using the asymptotic behavior of \( \xi \) and \( f \). Once this was achieved, we can use bootstrap arguments to finish our proof, see Section 7.

To proceed, we assume that there exist positive constants \( A_f(T), B_f(T), C_f(T), D_f(T) \) such that for \( 0 \leq t \leq T \), it holds that

\[
\| B^{-1/2} f \|_{L^8} \lesssim \langle t \rangle^{-\frac{2N+1}{4N}} A_f(T) + \langle |\xi_0|^{4N} t \rangle^{-\frac{2N+1}{4N}} B_f(T), \tag{6.1}
\]

\[
\| \langle x \rangle^{-\sigma} B^{1/2} f \|_{L^4} \lesssim \langle t \rangle^{-\frac{1}{2}} C_f(T) + \langle |\xi_0|^{4N} t \rangle^{-\frac{1}{2}} D_f(T). \tag{6.2}
\]

Recall that

\[
\tilde{\mathcal{R}}_f = -i \int_0^t e^{-iB(t-s)} \partial_f \mathcal{R} ds,
\]

we first prove estimates of \( \partial_f \mathcal{R} = \sum_{d=1}^5 \partial_f \mathcal{R}_d \).

**Lemma 6.1.** The following estimates hold:

\[
\| B^{-1/2} \partial_f \mathcal{R}_1 \|_{W^{2,8/7}} \lesssim \langle \xi_0 \rangle^{\frac{4N+3}{4N}} \| \xi \|^{\frac{4N+3}{4N}},
\]

\[
\| B^{-1/2} \partial_f \mathcal{R}_2 \|_{W^{2,8/7}} \lesssim \langle \xi_0 \rangle^{\frac{4N+3}{4N}} \| \xi \|^{\frac{4N+3}{4N}} \left( \| B^{-1/2} f \|_{L^8} + \| \langle x \rangle^{-\sigma} B^{1/2} f \|_{L^4} \right),
\]

\[
\| B^{-1/2} \partial_f \mathcal{R}_3 \|_{W^{2,8/7}} \lesssim \langle \xi_0 \rangle^{\frac{4N+3}{4N}} \| \xi \|^{\frac{4N+3}{4N}} \left( \| B^{-1/2} f \|_{L^8} + \| \langle x \rangle^{-\sigma} B^{1/2} f \|_{L^4} \right),
\]

\[
\| B^{-1/2} \partial_f \mathcal{R}_4 \|_{W^{2,8/7}} \lesssim \| B^{1/2} f \|_{L^8} \| B^{-1/2} f \|_{L^4} \left( \| B^{-1/2} f \|_{L^8} + \| \langle x \rangle^{-\sigma} B^{1/2} f \|_{L^4} \right),
\]

\[
\| B^{-1/2} \partial_f \mathcal{R}_5 \|_{W^{2,8/7}} \lesssim \langle \xi_0 \rangle^{\frac{4N+3}{4N}} \| \xi \|^{\frac{4N+3}{4N}}.
\]

**Proof.** The estimates of \( \mathcal{R}_1 \) and \( \mathcal{R}_5 \) are direct. For \( \mathcal{R}_d, d = 2, 3 \), by Theorem 5.3, \( f_d^{(2N)}(x, z) \), are linear combinations of terms of the form \( \xi^\mu \xi^\nu \prod_j \Phi_j \cdot f dx \Psi \), where \( \mu + \nu \geq 4 - d + 2k - i, 0 \leq i \leq k, \Phi_j, \Psi \in S(\mathbb{R}^3, \mathbb{C}) \). And \( \Lambda_d^{(2N)}(x, z) \) are linear combinations of terms of the form \( \xi^\mu \xi^\nu \prod_j \Phi_j \cdot f dx \Psi \), where \( \mu + \nu \geq 1 + 2k - i, 0 \leq i \leq k, \Phi_j, \Psi \in S(\mathbb{R}^3, \mathbb{C}) \). Hence, \( \partial_f \mathcal{R}_d \) are linear combinations of terms of forms

\[
\xi^\mu \xi^\nu \prod_j \Phi_j \cdot f dx \int \Psi U^d dx \Psi', \quad \mu + \nu \geq 4 - d + 2(k + 1) - i, 0 \leq i \leq k,
\]

or

\[
\xi^\mu \bar{\xi}^\nu \prod_j \Phi_j \cdot f dx B^{-1/2} \left( \Psi U^{d-1} \right), \quad \mu + \nu \geq 4 - d + 2k - i, 0 \leq i \leq k,
\]

or

\[
\xi^\mu \xi^\nu \prod_j \Phi_j \cdot f dx \Psi, \quad \mu + \nu \geq 4 - d + 2k - i, 0 \leq i \leq k.
\]
In each form, \( \mu + \nu \geq 4 - d \) and \( f \) has at least order \( d - 1 \). Then the rest follows directly by Hölder’s inequality and Leibnitz rule. For \( d = 4 \), we have

\[
\|B^{-1/2} \left( B^{-1/2} f + B^{-1/2} \tilde{f} \right) \|^3_{W^{2,8/7}} \\
\lesssim \|B^{-1/2} f + B^{-1/2} \tilde{f} \|^3_{W^{1,8/7}} \\
\lesssim \|B^{-1/2} f\|_{W^{1,2}}^4 \|B^{-1/2} f\|_{L^5}^{5/3}.
\]

\[\square\]

**Corollary 6.2.** Assuming (6.1) and (6.2), it holds that

\[
\|B^{-1/2} \partial_f R_1\|_{W^{2,8/7}} \lesssim \langle |\xi_0|^{4N} t \rangle - \frac{4N + 3}{4N} \langle |\xi|^{4N + 3} \rangle ;
\]

\[
\|B^{-1/2} \partial_f R_2\|_{W^{2,8/7}} \lesssim \langle |\xi_0|^{4N} t \rangle - \frac{1}{4N} \langle t \rangle - \frac{2N + 1}{4N} \langle |\xi|^{2} \rangle \left( A_f + |\xi_0|^{-1} C_f \right) + \langle |\xi_0|^{4N} t \rangle - \frac{2N + 2}{4N} \langle |\xi|^{2} \rangle (B_f + D_f),
\]

\[
\|B^{-1/2} \partial_f R_3\|_{W^{2,8/7}} \lesssim \langle |\xi_0|^{4N} t \rangle - \frac{1}{4N} \langle t \rangle - \frac{2N + 1}{4N} \langle |\xi|^{2} \rangle \left( A_f(A_f + C_f + D_f) + \langle |\xi_0|^{4N} t \rangle - \frac{2N + 2}{4N} \langle |\xi|^{2} \rangle B_f(B_f + C_f + D_f),
\]

\[
\|B^{-1/2} \partial_f R_4\|_{W^{2,8/7}} \lesssim \langle t \rangle - \frac{N + 4}{4N} \|B^{1/2} f\|_{L^2}^2 A_f^{3/5} + \langle |\xi_0|^{4N} t \rangle - \frac{2N + 2}{4N} \|B^{1/2} f\|_{L^2}^2 B_f^{5/3},
\]

\[
\|B^{-1/2} \partial_f R_5\|_{W^{2,8/7}} \lesssim \langle |\xi_0|^{4N} t \rangle - \frac{4M^*}{4N}. \]

**Proof.** The proof is direct by substituting (6.1) and (6.2) into Lemma 6.1. We have used the fact that \( \langle |\xi_0|^{4N} t \rangle - \frac{1}{4N} \langle t \rangle \leq \langle |\xi_0|^{4N} \rangle \). \[\square\]

Similarly, we have the following lemma. The proof also relies on the explicit forms of \( \partial_f R_d \) and standard Hölder’s inequality. Here we omit the details.

**Lemma 6.3.** Let \( \sigma > \frac{5}{2} \), the following estimates hold:

\[
\|\langle x \rangle^\sigma B^{1/2} \partial_f R_1\|_{W^{1,4/3}} \lesssim \langle |\xi_0|^{4N} t \rangle - \frac{4N + 3}{4N} \langle |\xi|^{4N + 3} \rangle ;
\]

\[
\|\langle x \rangle^\sigma B^{1/2} \partial_f R_2\|_{W^{1,4/3}} \lesssim \langle |\xi_0|^{4N} t \rangle - \frac{N}{4N} \langle |\xi|^{2} \rangle \left( \|B^{-1/2} f\|_{L^2} + \|\langle x \rangle^{-\sigma} B^{1/2} f\|_{L^1} \right),
\]

\[
\|\langle x \rangle^\sigma B^{1/2} \partial_f R_3\|_{W^{1,4/3}} \lesssim \langle |\xi_0|^{4N} t \rangle - \frac{N}{4N} \langle |\xi|^{2} \rangle \|B^{-1/2} f\|_{L^2} \left( \|B^{-1/2} f\|_{L^2} + \|\langle x \rangle^{-\sigma} B^{1/2} f\|_{L^1} \right),
\]

\[
\|B^{1/2} \partial_f R_4\|_{W^{1,4/3}} \lesssim \|B^{1/2} f\|_{L^2}^2 \|B^{-1/2} f\|_{L^2}^2,
\]

\[
\|\langle x \rangle^\sigma B^{1/2} \partial_f R_5\|_{W^{1,4/3}} \lesssim \langle |\xi_0|^{4N} t \rangle - \frac{M^*}{4N} \langle |\xi|^{M^*} \rangle ;
\]

**Corollary 6.4.** Let \( \sigma > \frac{5}{2} \) and assume (6.1) and (6.2) hold, then

\[
\|\langle x \rangle^\sigma B^{1/2} \partial_f R_1\|_{W^{1,4/3}} \lesssim \langle |\xi_0|^{4N} t \rangle - \frac{4N + 3}{4N} \langle |\xi|^{4N + 3} \rangle ,
\]

\[
\|\langle x \rangle^\sigma B^{1/2} \partial_f R_2\|_{W^{1,4/3}} \lesssim \langle |\xi_0|^{4N} t \rangle - \frac{N}{4N} \langle |\xi|^{2} \rangle \left( A_f + C_f \right) + \langle |\xi_0|^{4N} t \rangle - \frac{2N + 2}{4N} \langle |\xi|^{2} \rangle (B_f + D_f),
\]

\[
\|\langle x \rangle^\sigma B^{1/2} \partial_f R_3\|_{W^{1,4/3}} \lesssim \langle |\xi_0|^{4N} t \rangle - \frac{N}{4N} \langle |\xi|^{2} \rangle A_f(A_f + C_f + D_f) + \langle |\xi_0|^{4N} t \rangle - \frac{2N + 2}{4N} \langle |\xi|^{2} \rangle B_f(B_f + C_f + D_f),
\]

\[
\|B^{1/2} \partial_f R_4\|_{W^{1,4/3}} \lesssim \langle t \rangle - \frac{N + 4}{4N} \|B^{1/2} f\|_{L^2} A_f^2 + \langle |\xi_0|^{4N} t \rangle - \frac{2N + 2}{4N} \|B^{1/2} f\|_{L^2} B_f^2,
\]

\[
\|\langle x \rangle^\sigma B^{1/2} \partial_f R_5\|_{W^{1,4/3}} \lesssim \langle |\xi_0|^{4N} t \rangle - \frac{M^*}{4N} \langle |\xi|^{M^*} \rangle .
\]
Proposition 6.5. For $\sigma > \frac{4}{7}$, the following estimates hold:

$$\left\| B^{-1/2} \hat{R}_f \right\|_{L^8} \lesssim \left\langle t \right\rangle^{-\frac{2N+1}{4N}} P_1 + \langle |\xi_0|^{4N} t \rangle^{-\frac{2N+2}{4N}} P_2,$$

$$\left\| (x)^{-\sigma} B^{1/2} \hat{R}_f \right\|_{L^4} \lesssim \langle t \rangle^{-\frac{1}{2}} P_3 + \langle |\xi_0|^{4N} t \rangle^{-\frac{1}{2}} P_4,$$

where $P_1, P_2, P_3, P_4$ are defined by (6.3), (6.4), (6.5) and (6.6) respectively.

Proof. By Lemma 2.1, Lemma 6.1 and the corollary thereafter, we have

$$\left\| B^{-1/2} \hat{R}_f \right\|_{L^8} \lesssim \int_0^t \left\| e^{-iB(t-s)} B^{-1/2} \partial f \hat{R} \right\|_{L^8} ds$$

$$\lesssim \int_0^t \min\{|t-s|^{-9/8}, |t-s|^{-3/8}\} \| B^{-1/2} \partial f \hat{R} \|_{W^{2,8/7}} ds$$

$$\lesssim \int_0^t \min\{|t-s|^{-9/8}, |t-s|^{-3/8}\} \left( \langle s \rangle^{-\frac{2N+1}{4N}} \langle |\xi_0|^{4N} s \rangle^{-\frac{1}{2N}} P_1 + \langle |\xi_0|^{4N} s \rangle^{-\frac{2N+2}{4N}} P_2 \right) ds$$

$$\lesssim \langle t \rangle^{-\frac{2N+1}{4N}} \langle |\xi_0|^{4N} t \rangle^{-\frac{1}{2N}} P_1 + \langle |\xi_0|^{4N} t \rangle^{-\frac{2N+2}{4N}} P_2,$$

where we can choose

$$P_1 = [\xi]^{2} \frac{A_f}{4N} + [\xi]^{2} \frac{|\xi_0|^{-1} C_f}{4N} + [\xi] \frac{A_f C_f}{4N} + [\xi] \frac{A_f D_f}{4N} + |\xi_0|^{4/3} A_f^{5/3}, \quad (6.3)$$

$$P_2 = [\xi]^{2N+3} \frac{B_f}{4N} + [\xi]^{2} \frac{B_f}{4N} + [\xi] \frac{B_f C_f}{4N} + [\xi] \frac{B_f D_f}{4N} + |\xi_0|^{4/3} B_f^{5/3}. \quad (6.4)$$

Similarly, by Lemma 2.3, Lemma 6.3 and the corollary thereafter, we have

$$\left\| (x)^{-\sigma} B^{1/2} \hat{R}_f \right\|_{L^4} \lesssim \int_0^t \left\| (x)^{-\sigma} e^{-iB(t-s)} B^{1/2} \partial f \hat{R} \right\|_{L^4} ds$$

$$\lesssim \sum_{d \in \{1,2,3,5\}} \int_0^t \min\{|t-s|^{-5/4}, |t-s|^{-1/2}\} \| (x)^{\sigma} B^{1/2} \partial f \hat{R}_d \|_{W^{1,4/3}} ds$$

$$+ \int_0^t |t-s|^{-1/2} \| B^{1/2} \partial f \hat{R}_4 \|_{W^{1,4/3}} ds$$

$$\lesssim \langle t \rangle^{-\frac{1}{2}} P_3 + \langle |\xi_0|^{4N} t \rangle^{-\frac{1}{2}} P_4,$$

where we choose

$$P_3 = [\xi]^{2} \frac{A_f}{4N} + [\xi]^{2} \frac{A_f C_f}{4N} + [\xi] \frac{A_f D_f}{4N} + |\xi_0| A_f^2, \quad (6.5)$$

$$P_4 = [\xi]^{2N+3} \frac{B_f}{4N} + [\xi]^{2} \frac{B_f D_f}{4N} + [\xi] \frac{B_f C_f}{4N} + [\xi] \frac{B_f D_f}{4N} + \langle |\xi_0|^{4N} \rangle^{-9} B_f^2. \quad (6.6)$$

We finally come to the following error estimates:
Proposition 6.6. For $\sigma > \frac{5}{2}$, the following estimates hold:

\[
\| (x)^{-\sigma} B^{-1/2} R_f \|_{L^2} \lesssim \langle t \rangle^{-\frac{2N+1}{4N}} \langle \xi_0 |4Nt \rangle^{-\frac{1}{4N}} \left( \| B^{-1/2} f (0) \|_{X} + \| \xi_0 |2N+1 + P_1 \rangle + \langle \xi_0 |^4Nt \rangle^{-\frac{2N+2}{4N}} P_2, \right) \tag{6.7}
\]

\[
| R_\xi | \lesssim \langle t \rangle^{-\frac{4N+2}{4N}} P_5 + \langle t \rangle^{-\frac{2N+1}{4N}} \langle \xi_0 |^4Nt \rangle^{-\frac{2N+1}{4N}} P_6 + \langle \xi_0 |^4Nt \rangle^{-\frac{4N+2}{4N}} P_7, \tag{6.8}
\]

where $P_5, P_6, P_7$ are defined in (6.9), (6.10) and (6.11) respectively.

For the sake of convenience we first prove

Lemma 6.7.

\[
| \partial_\xi R \| \lesssim \langle t \rangle^{-\frac{4N+2}{4N}} \left( \| x \| A_f^2 + A_f^3 \right) + \langle \xi_0 |^4Nt \rangle^{-\frac{4N+2}{4N}} \left( \| \xi |^{4N+2} + \| \xi |^{4N+2} (A_f + B_f) + \| x \| B_f^2 + B_f^3 \right). \]

Proof. By using Leibnitz rule and explicit formula of $R(iii.0 - iii.5)$, we have

\[
\begin{align*}
| \partial_\xi R_0 | & \lesssim \| \xi \|^{4N+3} \lesssim \langle \xi_0 |^4Nt \rangle^{-\frac{4N+3}{4N}}, \\
| \partial_\xi R_1 | & \lesssim \| \xi \|^{4N+2} \| B^{-1/2} f \|_{L^8} \lesssim \langle \xi_0 |^4Nt \rangle^{-\frac{6N+4}{4N}} \xi^{1N+2} (A_f + B_f), \\
| \partial_\xi R_2 | & \lesssim \| \xi \| \langle t \rangle^{-\frac{4N+2}{4N}} \| x \| A_f^2 + \langle \xi_0 |^4Nt \rangle^{-\frac{4N+2}{4N}} \| x \| B_f^2, \\
| \partial_\xi R_3 | & \lesssim \langle \xi \| ^4Nt \rangle^{-\frac{3N+4}{4N}} A_f^3 + \langle \xi_0 |^4Nt \rangle^{-\frac{2N+4}{4N}} B_f^3, \\
| \partial_\xi R_4 | & \lesssim \langle \xi \| ^4Nt \rangle^{-\frac{1N+4}{4N}} A_f^4 + \langle \xi_0 |^4Nt \rangle^{-\frac{1N+4}{4N}} B_f^4, \\
| \partial_\xi R_5 | & \lesssim \| \xi \|^{M+} \lesssim \langle \xi_0 |^4Nt \rangle^{-\frac{M+}{4N}} \| \xi \|^{M+}.
\end{align*}
\]

Hence, we get desired estimate of $\partial_\xi R$.

Proof of Proposition 6.6. By Hölder’s inequality,

\[
\begin{align*}
\| (x)^{-\sigma} B^{-1/2} R_f \|_{L^2} & \lesssim \| e^{-iBt} B^{-1/2} f (0) \|_{L^8} + \left\| \int_0^t e^{-iB(t-s)} B^{-1/2} \partial_\xi R ds \right\|_{L^8} \\
& + \sum_{(\mu,\nu) \in M_1} | \eta' (0) \tilde{\eta}'' (0) | \left\| (x)^{-\sigma} e^{-iBt} (B - \omega (\nu - \mu) - i0)^{-1} B^{-1/2} \Phi_{\mu\nu} \right\|_{L^2} \\
& + \sum_{(\mu,\nu) \in M_1} \int_0^t \left\| (x)^{-\sigma} e^{-iB(t-s)} (B - \omega (\nu - \mu) - i0)^{-1} B^{-1/2} \Phi_{\mu\nu} \right\|_{L^2} \left| \frac{d}{ds} (\eta'' \tilde{\eta''}) \right| ds \\
& + \sum_{(\mu,\nu) \in M \setminus M_1} \int_0^t \left\| (x)^{-\sigma} e^{-iB(t-s)} B^{-1/2} \Phi_{\mu\nu} \right\|_{L^2} | \eta'' \tilde{\eta''} | ds \\
& := I + II + III + IV + V.
\end{align*}
\]

In the proof of Proposition 5.2 we have showed that

\[
I = \| e^{-iBt} B^{-1/2} f (0) \|_{L^8} \lesssim \langle t \rangle^{-9/8} \| B^{-1/2} f (0) \|_{X}.
\]

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By Proposition 6.5,
\[ II = \left\| B^{-1/2} \mathcal{R}_f \right\|_{L^8} \lesssim \langle t \rangle^{-\frac{2N+1}{4N}} \mathcal{P}_1 + \langle |\xi_0|^4 N t \rangle^{-\frac{2N+3}{4N}} \mathcal{P}_2. \]
By singular resolvents estimates Lemma 2.4, we have
\[ III \lesssim \langle t \rangle^{-6/5} |\xi_0|^{2N+1}. \]
For the term \( IV \), since \( |\frac{d^4}{ds^4} (\eta^\mu \bar{\eta}^\mu) | \lesssim |\eta|^{2N} |\bar{\eta}| \), and
\[
|\hat{\eta}| = |\hat{\xi} + i\omega \xi| \leq |\partial_\xi Z_0| + \langle |\partial_\xi G, f \rangle \rangle + |\langle \partial_\xi \tilde{G}, \bar{f} \rangle \rangle + |\partial_\xi \mathcal{R}| 
\lesssim \langle t \rangle^{-\frac{4N+2}{4N}} \left( |\xi| \frac{1}{4N} A_f^2 + A_f^3 \right) + \langle |\xi_0|^4 N t \rangle^{-\frac{2N+3}{4N}} \left( |\xi| \frac{1}{4N} B_f^2 + B_f^3 \right),
\]
it holds that
\[
IV \lesssim \int_0^t \langle t - s \rangle^{-6/5} |\xi_0|^{4N} s \left[ |\xi|^{2N} \frac{1}{4N} \left( |\xi| \frac{1}{4N} A_f^2 + A_f^3 \right) \right] ds 
+ \int_0^t \langle t - s \rangle^{-6/5} \langle |\xi_0|^4 N s \rangle^{-\frac{2N+3}{4N}} |\xi|^{2N} \frac{1}{4N} \left( |\xi| \frac{1}{4N} B_f^2 + B_f^3 \right) ds 
\lesssim \langle t \rangle^{-\frac{4N+2}{4N}} |\xi|^{2N} \frac{1}{4N} \left( |\xi| \frac{1}{4N} A_f^2 + A_f^3 \right) + \langle |\xi_0|^4 N t \rangle^{-\frac{2N+3}{4N}} |\xi|^{2N} \frac{1}{4N} \left( |\xi| \frac{1}{4N} B_f^2 + B_f^3 \right).
\]
Similarly,
\[
V \lesssim \int_0^t \langle t - s \rangle^{-6/5} |\xi_0|^{4N} s^{-\frac{2N+3}{4N}} ds [|\xi|^{2N+3} \frac{1}{4N}].
\]
Combining the above estimates and recall the formula of \( \mathcal{P}_1, \mathcal{P}_2 \), we get desired estimates for \( \mathcal{R}_f \).
Then using formula (6.6) of \( \mathcal{R}_\xi \), also combining the estimates of \( \mathcal{R}_f \) and Lemma 6.7, we get estimates of \( \mathcal{R}_\xi \):
\[
|\mathcal{R}_\xi| \lesssim \langle |\xi_0|^4 N t \rangle^{-\frac{4N+1}{4N}} |\xi|^{4N+3} \frac{1}{4N} + \langle |\xi_0|^4 N t \rangle^{-\frac{1}{2}} |\xi|^{2N} \frac{1}{4N} \langle \langle x \rangle^{-\sigma} B^{-1/2} \mathcal{R}_f \rangle \|_{L^2} + |\partial_\xi \mathcal{R}| 
\lesssim \langle t \rangle^{-\frac{4N+2}{4N}} \mathcal{P}_5 + \langle t \rangle^{-\frac{2N+1}{4N}} |\xi_0|^4 N t \langle |\xi_0|^4 N t \rangle^{-\frac{2N+3}{4N}} \mathcal{P}_6 + \langle |\xi_0|^4 N t \rangle^{-\frac{2N+3}{4N}} \mathcal{P}_7,
\]
where
\[
\mathcal{P}_5 = \left[ |\xi| \frac{1}{4N} A_f^2 + A_f^3 \right], \quad \mathcal{P}_6 = \left[ |\xi|^{2N} \frac{1}{4N} \left( \| B^{-1/2} f(0) \|_{X} + |\xi_0|^{2N+1} + \mathcal{P}_1 \right) \right], \quad \mathcal{P}_7 = \left[ |\xi|^{2N} \frac{1}{4N} \mathcal{P}_2 + \left[ |\xi|^{4N+2} (A_f + B_f) + \left[ |\xi| \frac{1}{4N} B_f^2 + B_f^3 \right. \right. \right.
\]
7 Proof of Theorem 1.2

Now we shall prove Theorem 1.2 by bootstrap arguments. Denote the original variables to be \((\xi', f') = T_2N(\xi, f)\), then by (1.4) the initial data \((\xi_0', f_0'(0))\) satisfies

\[
\|B^{-1/2} f'(0)\|_X \lesssim |\xi_0'| \lesssim \epsilon.
\]

By Theorem 3.2 (ii), it is equivalent to

\[
\|B^{-1/2} f(0)\|_X \lesssim |\xi_0| \lesssim \epsilon.
\]

Then by energy conservation, we have

\[
|\xi(t)| + \|B^{1/2} f\|_{L^2} \lesssim |\xi(0)| + \|B^{1/2} f(0)\|_{L^2} \leq (1 + C_0)|\xi_0|.
\]

(7.1)

Proposition 7.1. Assume \([\xi]_X(T) \lesssim |\xi_0|\), then (6.1) and (6.2) hold with

\[
\begin{align*}
A_f(T) & \lesssim |\xi_0|, \\
B_f(T) & \lesssim |\xi_0|^{2N+1}, \\
C_f(T) & \lesssim |\xi_0|, \\
D_f(T) & \lesssim |\xi_0|^{2N+1},
\end{align*}
\]

consequently,

\[
|R_\xi| \lesssim \langle t \rangle^{-\frac{4N+2}{4N}} |\xi_0|^3 + \langle t \rangle^{-\frac{2N+1}{4N}} \langle |\xi_0|^4N t \rangle^{\frac{2N+1}{4N}} |\xi_0|^{2N+1} + \langle |\xi_0|^4N t \rangle^{-\frac{4N+2}{4N}} |\xi_0|^{4N+3}.
\]

Proof. The proof consists of a bootstrap argument using Proposition 5.2 and 6.5. First, by (7.1), the result (6.1) and (6.2) hold for small \(t < 1\). Then, assume this is true for \(0 < t < t^* < T\) with

\[
\begin{align*}
A_f(T) & \leq K|\xi_0|, \\
B_f(T) & \leq K|\xi_0|^{2N+1}, \\
C_f(T) & \leq K|\xi_0|, \\
D_f(T) & \leq K|\xi_0|^{2N+1},
\end{align*}
\]

where \(K\) is a large constant to be chosen. By \([\xi]_X(T) \lesssim |\xi_0|\) and Proposition 5.5 5.8 and 5.9 hold for \(0 < t < t^*\) with

\[
\begin{align*}
P_1 \lesssim |\xi_0|^2 A_f + |\xi_0|^2 C_f + |\xi_0|^2 A_f C_f + |\xi_0|^2 A_f D_f + |\xi_0|^4/3 A_f^{5/3} & \lesssim K^2 |\xi_0|^2, \\
P_2 \lesssim |\xi_0|^{2N+3} + |\xi_0|^2 B_f + |\xi_0|^2 D_f + |\xi_0|^2 B_f^2 + |\xi_0|^2 B_f C_f + |\xi_0|^2 B_f D_f + |\xi_0|^{4/3} B_f^{5/3} & \lesssim K^2 |\xi_0|^{2N+3}, \\
P_3 \lesssim |\xi_0|^2 A_f + |\xi_0|^2 C_f + |\xi_0|^2 A_f C_f + |\xi_0|^2 A_f D_f + |\xi_0|^2 A_f^2 & \lesssim K^2 |\xi_0|^3, \\
P_4 \lesssim |\xi_0|^{2N+3} + |\xi_0|^2 B_f + |\xi_0|^2 D_f + |\xi_0|^2 B_f^2 + |\xi_0|^2 B_f C_f + |\xi_0|^2 B_f D_f + |\xi_0|^{2(2N-1)} B_f^2 & \lesssim K^2 |\xi_0|^{2N+3}.
\end{align*}
\]
By Proposition 5.2 we have
\[ \|B^{-1/2}f\|_{L^8} \lesssim \langle t \rangle^{-\frac{2N+1}{4N}} (|\xi_0| + K^2|\xi_0|^2) + \langle |\xi_0|^{4N}t \rangle^{-\frac{2N+1}{4N}} (|\xi_0|^{2N+1} + K^2|\xi_0|^{2N+3}), \]
\[ \|\langle x \rangle^{-\sigma} B^{1/2}f\|_{L^4} \lesssim \langle t \rangle^{-\frac{1}{2}} (|\xi_0| + K^2|\xi_0|^3) + \langle |\xi_0|^{4N}t \rangle^{-\frac{1}{2}} (|\xi_0|^{2N+1} + K^2|\xi_0|^{2N+3}). \]

By choosing $K$ large and $|\xi_0|$ small we complete the bootstrap argument. In addition, we have
\[ \mathcal{P}_5 = [\xi] \frac{A_f^2 + A_j^3}{A_f} \lesssim |\xi_0|^3, \quad (7.2) \]
\[ \mathcal{P}_6 = [\xi]^{3N} \left( \|B^{-1/2}f(0)\|_X + |\xi_0|^{2N+1} + \mathcal{P}_1 \right) \lesssim |\xi_0|^{2N+1}, \quad (7.3) \]
\[ \mathcal{P}_7 = [\xi]^{3N} \mathcal{P}_2 + [\xi]^{4N+2} (A_f + B_f) + [\xi]^{4N} B_f^2 + B_j^3 \lesssim |\xi_0|^{4N+3}, \quad (7.4) \]

by Proposition 5.6 we get estimates of $|\mathcal{R}_\xi|$.

\[ \Box \]

Proof of Theorem 7.2 We use bootstrap argument again to show that
\[ [\xi]^{\frac{1}{4N}} (T^*) \leq 8C^*|\xi_0| \quad (7.5) \]
holds for all $T > 0$, where $C^* = [1 + (4N\gamma)^{-1}]^{\frac{1}{4N}} (1 + C_0)$ is an absolute constant. Note that by energy conservation, (7.5) holds for $T \leq |\xi_0|^{-4N(1-\delta_0)}$. For $T \geq |\xi_0|^{-4N(1-\delta_0)}$, we assume that
\[ [\xi]^{\frac{1}{4N}} (T^*) \leq 8C^*|\xi_0| \quad (7.6) \]
for some $|\xi_0|^{-4N(1-\delta_0)} \leq T^* < T$. Then, by Proposition 7.1, we have
\[ |\mathcal{R}_\xi| \lesssim \langle |\xi_0|^{4N} t \rangle^{-\frac{2N+2}{4N}} (|\xi_0|^{4N+2})^{(1-\delta_0)} + |\xi_0|^{4N+2})^{(1-\delta_0/2)} + |\xi_0|^{4N+3} \]
for $0 \leq t \leq T^*$. Choosing $\delta_0 < \frac{1}{4N+2}$, we have $|\mathcal{R}_\xi| \lesssim \langle |\xi_0|^{4N} t \rangle^{-\frac{2N+2}{4N}} |\xi_0|^{4N+1+\delta_1}$ for some $\delta_1 > 0$. Hence, we can apply Theorem 5.1 at $t_0 = |\xi_0|^{-4N(1-\delta_0)}$ with $Q_0 \approx |\xi_0|^{4N+1+\delta_1}$. Note that
\[ |r(t_0) - r(0)| \leq \int_0^{t_0} \left( 4N\gamma r^2(t) + 4N|\mathcal{R}_\xi| r^{\frac{4N}{4N+1}}(t) \right) dt \]
\[ \leq C \int_0^{t_0} \left( \frac{|\xi_0|^{8N}}{r^2} + |\xi_0|^{4N+1+\delta_1} |\xi_0|^{4N-1} \right) dt \]
\[ \leq C |\xi_0|^{4N(1+\delta_0)} \]
\[ \leq \frac{1}{2} r(0), \]
we have $\frac{1}{2} |\xi_0|^{4N} \leq |\xi(t_0)|^{4N} \leq \frac{3}{2} |\xi_0|^{4N}$. Therefore, we get
\[ |\xi(t)|^{4N} \leq 4(1 + 4N\gamma |\xi_0|^{4N} t)^{-1} \left( |\xi_0|^{4N} + \frac{C(C^*)|\xi_0|^{4N+\delta_1/2}}{(1 + 4N\gamma |\xi_0|^{4N} t)^{\delta_1/2}} \right) \]
\[ \leq 5 (C^*)^{4N} (1 + |\xi_0|^{4N} t)^{-1} |\xi_0|^{4N}. \]

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The last inequality holds by using
\[(1 + 4N \gamma |\xi_0|^{4N}t)^{-1} \leq (C^*)^{4N} (1 + |\xi_0|^{4N}t)^{-1}\]
and choosing $|\xi_0|$ to be sufficiently small such that $C(C^*)|\xi_0|^{\delta_1/2} \leq 1$.

Next, we shall prove the lower bound of $|\xi_t|$. We have proved that $|\xi(t)|^{4N} \approx |\xi_0|^{4N} \approx (1 + |\xi_0|^{4N}t)^{-1} |\xi_0|^{4N}$ for $0 \leq T \leq |\xi_0|^{-4N(1-\delta_0)}$. For $T \geq |\xi_0|^{-4N(1-\delta_0)}$, by Proposition 7.1 we have
\[|\mathcal{R}_\xi| \lesssim (|\xi_0|^{4N}t)^{-\frac{N+1}{4N}} |\xi_0|^{4N+1+\delta_1},\]
thus using Lemma 5.1 we also obtain $|\xi(t)|^{4N} \gtrsim (1 + |\xi_0|^{4N}t)^{-1} |\xi_0|^{4N}$. Combining above estimates we obtain
\[|\xi(t)| \approx \frac{|\xi_0|}{(1 + |\xi_0|^{4N}t)^{\frac{1}{4N}}}.\]

By Proposition 7.1 we also have
\[\|B^{-1/2}f\|_{L^s} \lesssim \langle t \rangle^{-\frac{2N+1}{4N}} |\xi_0| + \langle |\xi_0|^{4N}t \rangle^{-\frac{2N+1}{4N}} |\xi_0|^{2N+1}.\]

By the property of normal form transformation, for the original variables $\{\xi', f'\} = T_{2N}(\xi, f)$, we have
\[|\xi'(t)| \approx \frac{|\xi'_0|}{(1 + |\xi'_0|^{4N}t)^{\frac{1}{4N}}},\]
and
\[\|B^{-1/2}f'\|_{L^s} \lesssim \langle t \rangle^{-\frac{2N+1}{4N}} |\xi'_0| + \langle |\xi'_0|^{4N}t \rangle^{-\frac{2N+1}{4N}} |\xi'_0|^{2N+1}.\]

Write $\xi'(t) = \rho(t)e^{-i(\omega t + \theta(t))}$, then $q(t) = \sqrt{\frac{2}{\omega}} \text{Re} \xi(t) = \sqrt{\frac{2}{\omega}} \rho(t) \cos(\omega t + \theta(t))$, and
\[u(t, x) = R(t) \cos(\omega t + \theta(t)) \varphi(x) + \eta(t, x)\]
with $R(t) = \sqrt{\frac{2}{\omega}} \rho(t)$. Hence, the above estimates are equivalent to
\[|R(t)| \approx \frac{|R(0)|}{(1 + |R(0)|^{4N}t)^{\frac{1}{4N}}},\]
and
\[\|\eta(t)\|_{L^s} \lesssim \frac{|R(0)|}{(1 + t)^{\frac{2N+1}{4N}}} + \frac{|R(0)|^3}{(1 + |R(0)|^{4N}t)^{\frac{3}{4N}}}.\]

In addition, since
\[\hat{\xi} + i\omega \xi' = \hat{\rho e^{-i(\omega t + \theta)}} - i\hat{\theta} \xi',\]
where $|\xi' + i\omega \xi'| = O(|t|^{-\frac{3}{4N}})$, $|\rho| = O(|t|^{-\frac{3}{4N}})$, we get $|\hat{\theta}| = O(|t|^{-\frac{1}{4N}})$, hence $\theta(t) = O(|t|^{-\frac{1}{4N}})$, which completes the proof of Theorem 1.2.

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A Convolution Estimates

The following lemma deals with the decay rate of the convolution of two functions, which is used frequently in the asymptotic analysis in Section 5 and Section 6.

Lemma A.1. Let $0 < \delta < 1$ and $0 < f(t) \lesssim \min\{|t|^{-(1+\delta)}, |t|^{-(1-\delta)}\}$, then

$$\int_0^t f(t-s)\langle|\xi_0| S t\rangle^{-\alpha} ds \lesssim \langle|\xi_0| t\rangle^{-\alpha}, \quad \forall \ 0 \leq \alpha \leq 1 + \delta. \quad (A.1)$$

Proof. Note that $f$ is integrable on the whole line, thus it suffices to consider $t \geq 2|\xi_0|^{-4N}$. Then we have

$$\int_0^t f(t-s)\langle|\xi_0| S t\rangle^{-\alpha} ds \leq \int_0^{|\xi_0|^{-4N}} f(t-s)\langle|\xi_0| S t\rangle^{-\alpha} ds + \int_{|\xi_0|^{-4N}}^{t/2} f(t-s)\langle|\xi_0| S t\rangle^{-\alpha} ds + \int_{t/2}^t f(t-s)\langle|\xi_0| S t\rangle^{-\alpha} ds \triangleq I + II + III,$$

where

$$I \lesssim t^{-(1+\delta)}|\xi_0|^{-4N} \lesssim \langle|\xi_0| t\rangle^{-(1+\delta)}.$$

$$II \lesssim t^{-(1+\delta)} \int_{|\xi_0|^{-4N}}^{t/2} \langle|\xi_0| S t\rangle^{-\alpha} ds \lesssim \begin{cases} t^{-(1+\delta)}|\xi_0|^{-4N\alpha} t^{1-\alpha} \lesssim \langle|\xi_0| t\rangle^{-\alpha} & \text{if } \alpha < 1, \\ t^{-(1+\delta)}|\xi_0|^{-4N\alpha}|\xi_0|^{4N(\alpha-1)} \lesssim \langle|\xi_0| t\rangle^{-(1+\delta)} & \text{if } \alpha > 1, \end{cases}$$

$$III \lesssim \langle|\xi_0| t\rangle^{-\alpha} \int_{t/2}^t f(t-s) ds \lesssim \langle|\xi_0| t\rangle^{-\alpha}.$$

Combining the above estimates, we obtain the desired decay estimate. \hfill \square

Similarly, we can also prove the following lemma:

Lemma A.2. For $\alpha > 1$, we have

$$\int_0^t (t-s)^{-1/2}\langle|\xi_0| S t\rangle^{-\alpha} ds \lesssim |\xi_0|^{-2N} \langle|\xi_0| t\rangle^{-1/2}. \quad (A.2)$$

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