On quadruples of Griffiths points

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Abstract. Tabov (Math Mag 68:61–64, 1995) has proved the following theorem: if points $A_1, A_2, A_3, A_4$ are on a circle and a line $l$ passes through the centre of the circle, then four Griffiths points $G_1, G_2, G_3, G_4$ corresponding to pairs $(\Delta_i, l)$ are on a line ($\Delta_i$ denotes the triangle $A_jA_kA_l$, $j,k,l \neq i$). In this paper we present a strong generalisation of the result of Tabov. An analogous property for four arbitrary points $A_1, A_2, A_3, A_4$, is proved, with the help of the computer program “Mathematica”.

Mathematics Subject Classification (2010). 51M04, 51-04.
Keywords. Griffiths point, quadrangle, collinearity.

1. Introduction

Tabov [1] has proved the following theorem: if points $A_1, A_2, A_3, A_4$ are on a circle and a line $l$ passes through the centre of the circle, then four Griffiths points $G_1, G_2, G_3, G_4$ corresponding to pairs $(\Delta_i, l)$ are on a line ($\Delta_i$ denotes the triangle $A_jA_kA_l$, $j,k,l \neq i$).

Explanation. When a point $P$ moves along a line through the circumcenter of a given triangle $\Delta$, the circumcircle of the pedal triangle of $P$ with respect to $\Delta$ passes through a fixed point, called Griffiths point, on the nine-point circle of $\Delta$. The pedal triangle of $P$ with respect to $\Delta$ is the triangle the vertices of which are feet of the perpendiculars from $P$ to the sides of $\Delta$. A very simple construction of the Griffiths point for a pair $(\Delta, l)$ is given in [2]. Namely, we project orthogonally the intersection points of $l$ and the circumcircle of $\Delta$ onto the sides of $\Delta$. The projections of each of these points are collinear and the common point of the two lines is the Griffiths point associated with $(\Delta, l)$.

2. Main results

In this paper we present a much stronger generalisation of the result of Tabov. We consider four arbitrary points $A_1, A_2, A_3, A_4$, no three of them collinear.
By $\Delta_i$ is denoted the triangle $A_jA_kA_l$, $j, k, l \neq i$, $l_i$ is a line passing through the circumcenter of $\Delta_i$, $i = 1, 2, 3, 4$. Finally, $G_i$ is the Griffiths point corresponding to $(\Delta_i, l_i)$, $i = 1, 2, 3, 4$.

**Theorem.** If lines $l_1, l_2, l_3, l_4$ have a common point at infinity (every two of them are parallel), then points $G_1, G_2, G_3, G_4$ are collinear.

**Proof.** As in [1] points $A_1, A_2, A_3, A_4$ are represented by complex numbers $a, b, c, d$, respectively. Without loss of generality, we may assume that points $A_1, A_2, A_3$ are on the circle of centre 0 and radius 1, i.e. $|a| = |b| = |c| = 1$. Similarly, we may assume that lines $l_1, l_2, l_3, l_4$ are parallel to the real axis. Hence [1] $g_4 = \frac{1}{2}(a + b + c - abc)$, where $g_i$ is the complex number representing Griffiths point $G_i$, $i = 1, 2, 3, 4$. It is easy to check that if $A_1, A_2, A_3, A_4$ are on the circle of centre 0 and radius $R$ instead of 1, then

$$g_4 = \frac{1}{2} \left( a + b + c - \frac{abc}{R^2} \right) \quad (2.1)$$

After short calculations we find the number $c_3 = \frac{2(d - c_2)(b - c_2)(d - c_2)}{|b - c_2|^2}$ representing the circumcenter $C_3$ of triangle $A_1A_2A_4$. Now we introduce a new coordinate system by the formula: $z = z' + c_3$. In the new system, according to (2.1),

$$g_3' = \frac{1}{2} \left( a' + b' + c' - \frac{a'b'c'}{|a - c_3|^2} \right).$$

Then in the former coordinate system we have

$$g_3 = \frac{1}{2} \left( a + b + d - c_3 - \frac{(a - c_3)(b - c_3)(d - c_3)}{|a - c_3|^2} \right).$$

In an analogous way we obtain

$$g_2 = \frac{1}{2} \left( a + c + d - c_2 - \frac{(a - c_2)(c - c_2)(d - c_2)}{|a - c_2|^2} \right),$$

$$g_1 = \frac{1}{2} \left( b + c + d - c_1 - \frac{(b - c_1)(c - c_1)(d - c_1)}{|a - c_1|^2} \right).$$

Points $G_2, G_3, G_4$ are collinear iff [1] the equality

$$\frac{g_3 - g_4}{g_2 - g_4} = \frac{g_3 - g_4}{g_2 - g_4} \quad (2.2)$$

holds. In order to prove it, we use the computer program “Mathematica”. The consecutive steps are as follows: First we write the complex numbers $a, b, c, d$ in the form $a = \cos x + i\sin x$, $b = \cos y + i\sin y$, $c = \cos z + i\sin z$, $d = R(\cos u + i\sin u)$. Beginning from now, all formulae are obtained with the help of “Mathematica”.

$$c_3 = \frac{e^{0.5i(x+y)}(1 - R^2)}{-2\cos \frac{x-y}{2} + 2\cos \left( u - \frac{x}{2} - \frac{y}{2} \right)}.$$
Similarly, $c_2 = \frac{e^{0.5i(x+y)}(1 - R^2)}{-2 \cos \frac{x-y}{2} + 2 \cos \left(u - \frac{x}{2} - \frac{y}{2}\right)}$ ($c_2$ represents the circumcenter $C_2$ of triangle $A_1A_3A_4$).

$$g_4 = \frac{1}{2} \left( \cos x + \cos y + \cos z - \cos(x + y + z) + i(\sin x + \sin y + \sin z - \sin(x + y + z)) \right).$$

$$(a - c_2)(c - c_2)(d - c_2) = -\frac{e^{0.5i(4u+x+z)}(e^{ix} - Re^{iu})^2(e^{iz} - Re^{iu})^2(e^{iu} - Re^{ix})(e^{iu} - Re^{iz})}{8 \left( \cos \frac{x-z}{2} - R \cos \left(u - \frac{x}{2} - \frac{z}{2}\right) \right)^3}.$$ 

$$a - c_2 = \cos x + \frac{-2 \cos \frac{x-z}{2} + 2R \cos \left(u - \frac{x}{2} - \frac{z}{2}\right)}{R^2 \cos \frac{x+z}{2}} + \frac{i \times \sin x + \frac{\sin \frac{x+z}{2}}{-2 \cos \frac{x-z}{2} + 2R \cos \left(u - \frac{x}{2} - \frac{z}{2}\right)}} + \frac{R^2 \sin \frac{x+z}{2}}{-2 \cos \frac{x-z}{2} + 2R \cos \left(u - \frac{x}{2} - \frac{z}{2}\right)}.$$ 

$$\text{Abs}^2[a - c_2] = \frac{(1 + R^2 - 2R \cos(u - x))(1 + R^2 - 2R \cos(u - z))}{4 \left( \cos \frac{x-z}{2} - R \cos \left(u - \frac{x}{2} - \frac{z}{2}\right) \right)^2}.$$ 

$$\frac{(a - c_2)(c - c_2)(d - c_2)}{|a - c_2|^2} = \frac{R}{\cos(u + x + z) - i \sin(u + x + z) + \frac{(-1 + R^2) e^{-iu}}{(e^{ix} - Re^{iu})(e^{iz} - Re^{iu})}}.$$ 

$$g_2 = \frac{1}{2} \left( \cos x + \cos z + i \sin x + i \sin z + e^{iu}(e^{iz} - Re^{iu})(Re^{iu} - e^{ix})(-1 + \cos(x + z) + i \sin(x + z)) + \frac{e^{iu}(e^{iz} - Re^{iu})(Re^{iu} - e^{ix})(-1 + \cos(x + z) + i \sin(x + z))}{-e^{i(u+x)} + e^{i(u+z)} + Re^{2iu} + Re^{i(x+z)}} \right).$$

In an analogous way we obtain

$$g_3 = \frac{1}{2} \left( \cos x + \cos y + i \sin x + i \sin y + e^{iu}(e^{iy} - Re^{iu})(Re^{iu} - e^{ix})(-1 + \cos(x + y) + i \sin(x + y)) + \frac{e^{iu}(e^{iy} - Re^{iu})(Re^{iu} - e^{ix})(-1 + \cos(x + y) + i \sin(x + y))}{-e^{i(u+x)} - e^{i(u+y)} + Re^{2iu} + Re^{i(x+y)}} \right).$$

$$L = \frac{g_3 - g_4}{g_2 - g_4} = \frac{\cos \frac{x-z}{2} - R \cos \left(u - \frac{x}{2} - \frac{z}{2}\right) \cos \frac{x+z}{2} \sin \frac{x+y}{2}}{\cos \frac{x-z}{2} - R \cos \left(u - \frac{x}{2} - \frac{y}{2}\right)}.$$ 

Since the above expression is real, the equality $(2.2)$ holds. Obviously, in an identical way we prove that points $G_1, G_2, G_4$ collinear and so on. This ends the proof. □
Remark. As we can observe, using of a computer program to obtain so complicated formulae, was necessary. It should be noticed that the results obtained by transforming symbolic expressions with the help of the program “Mathematica” are quite exact.

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Received: February 4, 2011.
Revised: June 24, 2013.