HERMITE-HADAMARD TYPE INEQUALITIES FOR HARMONICAL \((h_1, h_2)\)-CONVEX INTERVAL-VALUED FUNCTIONS

RUONAN LIU AND RUN XU*

School of Mathematical Sciences
Qufu Normal University
Qufu 273165, Shandong, China

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Abstract. We introduce the concept of interval harmonical \((h_1, h_2)\)-convex functions, establish some new Hermite-Hadamard type inequalities on interval Riemann integrable functions, and generalize the results of Noor et al. 2015 and Zhao Dafang et al. 2019.

1. Introduction. The theory of interval analysis has a long history, but this field did not appear until the 1950s. Significant research results, interval and interval value function was originally introduced by Moore in his work \[18\] numerical analysis. In this past 50 years, it has been applied in various fields, such as: interval differential equation \[9\], aeroelasticity \[16\], automatic error analysis \[22\], computer graphic \[24\], neural network output optimization \[6\], and so on.

We all know that convexity of functions plays an important role in mathematics, economics, probability theory, optimal control theory, inequality also plays an important role in mathematics and other scientific fields. In the research of various scholars, the concept of function convexity is based on inequality. Because of the importance of the Hermite-Hadamard inequality, many scholars have studied the Hermite-Hadamard inequality for convex functions. The following is the classical Hermite-Hadamard inequality:

\[
f \left( \frac{m + n}{2} \right) \leq \frac{1}{n - m} \int_{m}^{n} f(t)dt \leq \frac{f(m) + f(n)}{2},
\]

where \( f : I \subseteq R \to R \) be a convex function on the interval \( I \) of real numbers and \( m, n \in I \) with \( m < n \). Because of the versatility of the Hermite-Hadamard inequality, many scholars have extended inequalities on classical convexity to \( s \)-convex \([10, 26]\), \( p \)-convex \([13]\), \( h \)-convex \([25, 8, 14, 23]\) and \((h_1, h_2)\)-convex \([2]\), etc. In 2014, İşcan also introduced the concept of harmonic convexity and established some Hermite-Hadamard inequalities for this kind of function in \[11\]. In 2015, Noor et  

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* Corresponding author: Run Xu.
al. introduced the concept of harmonic \( h \)-convex functions and established some related Hermite-Hadamard inequalities in [19]:

Let \( f : I \rightarrow R \) be a harmonically \( h \)-convex function, where \( m, n \in I \) with \( m < n \). If \( f \in L[m, n] \), then

\[
\frac{1}{2h(\frac{1}{2})} f \left( \frac{2mn}{m+n} \right) \leq \frac{mn}{n-m} \int_m^n f(x) \frac{dx}{x^2} \leq [f(m) + f(n)] \int_0^1 h(t) dt. \tag{2}
\]

In recent years, Zhao Dafang et al. extend Hermite-Hamamard inequality based on \( h \)-convex functions to interval \( h \)-convex functions [28], interval harmonic \( h \)-convex functions [27] and interval \((h_1, h_2)\)-convex functions [1] in combination with interval analysis, they obtained many new results. In addition, readers interested in inequalities on interval-valued functions and harmonic convex functions may refer to reference [3, 4, 5, 21, 12, 15, 17, 20].

Here are some of the results from [28], [27] and [1]:

\( R_1 \) [28]: Let \( f : [m, n] \rightarrow R^+_1 \) be an interval-valued function such that \( f(t) = \llbracket f(t), f(t) \rrbracket \) and \( f \in IR_{[m, n]} \), \( h : [0, 1] \rightarrow R \) be a non-negative function and \( h \left( \frac{1}{2} \right) \neq 0 \). If \( f \in SX(h, [m, n], R^+_1) \), then

\[
\frac{1}{2h(\frac{1}{2})} f \left( \frac{m+n}{2} \right) \geq \frac{1}{n-m} \int_m^n f(x) \frac{dx}{x^2} \geq [f(m) + f(n)] \int_0^1 h(t) dt. \tag{3}
\]

\( R_2 \) [27]: Let \( f : K_h \rightarrow R^+_1 \) be an interval-valued function with \( m < n \) and \( m, n \in K_h, f \in IR_{[m, n]} \), and let \( h : [0, 1] \rightarrow (0, \infty) \) be a continuous function. If \( f \in SX(h, K_h, R^+_1) \), then

\[
\frac{1}{2h(\frac{1}{2})} f \left( \frac{2mn}{m+n} \right) \geq \frac{mn}{n-m} \int_m^n f(x) \frac{dx}{x^2} \geq [f(m) + f(n)] \int_0^1 h(t) dt. \tag{4}
\]

\( R_3 \) [1]: Let \( f : [m, n] \rightarrow R^+_1, h_1, h_2 : [0, 1] \rightarrow R^+ \) and \( h \left( \frac{1}{2}, \frac{1}{2} \right) \neq 0 \). If \( f \in SX((h_1, h_2), [m, n], R^+_1) \) and \( f \in IR_{[m, n]} \), then

\[
\frac{1}{2H(\frac{1}{2}, \frac{1}{2})} f \left( \frac{m+n}{2} \right) \geq \frac{1}{n-m} \int_m^n f(t) dt \geq [f(m) + f(n)] \int_0^1 H(x, 1-x) dx. \tag{5}
\]

This paper introduces the concept of harmonic \((h_1, h_2)\)-convex interval valued functions. On the basis of [1], we study Hermite-Hadamard inequalities on harmonic \((h_1, h_2)\)-convex interval valued functions, our results further extend some known inequalities in [19], [27].

The arrangement of this paper is as follows: In section 2, the definitions and properties of interval valued functions and harmonic convex functions are given for the preparation work. In section 3, we obtain Hermite-Hadamard inequalities on some new function classes by using harmonic \((h_1, h_2)\)-convex interval-valued functions. In section 4, the conclusion is given.

2. Preliminaries. We begin by recalling some basic definitions, notation and properties, which are used throughout the paper. Let’s say \( I \) denote the space of all intervals of \( R \), \( [m] \in I \) is defined as:

\[
[m] = [m, m] = \{ x \in R \mid m \leq x \leq m \}, m, m \in R,
\]
where \([m]\) is a bounded interval of real numbers. When \(m\) and \(\bar{m}\) are equal, the interval \([m]\) is said to be degenerated. We call \([m]\) positive if \(m > 0\) or negative if \(\bar{m} < 0\). We denote by \(R_I\) the set of all intervals of \(R\), and use \(R^+\) and \(R^-\) to represent the sets of all positive intervals and negative intervals.

The inclusion \(\subseteq\) is defined by
\[
[m] \subseteq [n] \iff [m, \bar{m}] \subseteq [n, \bar{n}] \iff m \leq n, \bar{m} \leq \bar{n}.
\]

For an arbitrary real number \(\lambda\) and \([m]\), the interval \(\lambda[m]\) is given by
\[
\lambda[m, \bar{m}] = \begin{cases} 
\lambda m/\lambda\bar{m} & \text{if } \lambda > 0, \\
\{0\} & \text{if } \lambda = 0, \\
\lambda m, \lambda\bar{m} & \text{if } \lambda < 0.
\end{cases}
\]

For \([m] = [m, \bar{m}]\), and \([n] = [n, \bar{n}]\), the four arithmetic operators are defined by
\[
\begin{align*}
[m] + [n] &= [\min\{m, n\}, \max\{m, n\}], \\
[m] - [n] &= [\min\{m, n\}, \max\{m, \bar{n}\}], \\
[m] \cdot [n] &= \{\min\{mn, m\bar{n}, m\bar{n}, mn\}, \max\{mn, m\bar{n}, \bar{m}n\}\}, \\
[m]/[n] &= \{\min\{m/n, m/n, m/n, m/n\}, \max\{m/n, \bar{m}/\bar{n}, \bar{m}/\bar{n}\}\},
\end{align*}
\]

where \(0 \not\in [n, \bar{n}]\). For interval \([m, \bar{m}]\) and \([n, \bar{n}]\), the Hausdorff distance is defined by
\[
d([m, \bar{m}], [n, \bar{n}]) = \max\{|m - n|, |ar{m} - \bar{n}|\}.
\]

It is well known that \((R, d)\) is a complete metric space.

In [7], Dinghas gives the definition of Riemannian integrability of interval valued functions. We are going to use \(IR_{([m, n])}\) for the set of all Riemannian integrable interval valued functions, and \(R_{([m, n])}\) for the set of all Riemannian integrable real functions.

**Definition 2.1** [7] Let \(f : [m, n] \to R_I\) be such that \(f(t) = [\underline{f}(t), \overline{f}(t)]\) for each \(t \in [m, n]\), and \(\underline{f}, \overline{f} \in R_{([m, n])}\). Then we say that \(f \in IR_{([m, n])}\) and denote
\[
\int_m^n f(t)dt = \left[ \int_m^n \underline{f}(t)dt, \int_m^n \overline{f}(t)dt \right].
\]

**Definition 2.2** [1] Let \(h_1, h_2 : [0, 1] \subseteq I \to R^+\) such that \(h_1, h_2 \neq 0\), \(f : I \to R^+_I\) is called an interval \((h_1, h_2)\)-convex function, if
\[
h_1(x)h_2(1-x)f(m) + h_1(1-x)h_2(x)f(n) \leq f(xm + (1-x)n)
\]
for all \(m, n \in I\) and \(x \in [0, 1]\). If the set inclusion is reversed, then \(f\) is said to be \((h_1, h_2)\)-concave interval-valued function. The set of all \((h_1, h_2)\)-convex and \((h_1, h_2)\)-concave interval-valued functions are denoted by \(SX((h_1, h_2), I, R^+_I)\) and \(SV((h_1, h_2), I, R^+_I)\).

**Definition 2.3** Let \(h_1, h_2 : [0, 1] \subseteq I \to R^+\) such that \(h_1, h_2 \neq 0\), \(f : I \to R^+_I\) is called a harmonical \((h_1, h_2)\)-convex function, if
\[
h_1(x)h_2(1-x)f(m) + h_1(1-x)h_2(x)f(n) \geq f\left( \frac{mn}{xm + (1-x)n} \right)
\]
for all \(m, n \in I\), and \(x \in [0, 1]\).
Define 2.4 Let \( h_1, h_2 : [0, 1] \subseteq I \to R^+ \) such that \( h_1, h_2 \neq 0, \ f : I \to R^+_I \) is called a harmonical \((h_1, h_2)\)-convex interval-valued function, if
\[
h_1(x)h_2(1-x)f(m) + h_1(1-x)h_2(x)f(n) \subseteq f \left( \frac{mn}{xm + (1-x)n} \right)
\]
for all \( m, n \in I \), and \( t \in [0, 1] \). If the set inclusion is reversed, then \( f \) is said to be a harmonical \((h_1, h_2)\)-concave interval-valued function. The set of all harmonical \((h_1, h_2)\)-convex and harmonical \((h_1, h_2)\)-concave interval-valued functions are denoted by \( SHX((h_1, h_2), I, R^+_I) \) and \( SHV((h_1, h_2), I, R^+_I) \).

3. Main results.  Some new results of Hermite-Hadamard inequality on harmonical \((h_1, h_2)\)-convex interval-valued functions are given below. In what follows, let \( H(x, y) = h_1(x)h_2(y) \) for \( x, y \in [0, 1] \).

Theorem 3.1 Let \( h_1, h_2 : [0, 1] \subseteq I \to R^+ \) such that \( H \left( \frac{1}{2}, \frac{1}{2} \right) \neq 0, \ f : [m, n] \to R^+_I \) and \( f \in IR_{(m,n)} \). If \( f \in SHX((h_1, h_2), [m, n], R^+_I) \), then
\[
\frac{1}{2H \left( \frac{1}{2}, \frac{1}{2} \right)} f \left( \frac{2mn}{m+n} \right) \geq \frac{mn}{n-m} \int_m^n f(t) \frac{dt}{t^2} \geq [f(m)+f(n)] \int_0^1 H(x, 1-x)dx. \quad (10)
\]
Proof. If \( f \in SHX((h_1, h_2), I, R^+_I) \), then
\[
h_1 \left( \frac{1}{2} \right) h_2 \left( \frac{1}{2} \right) f(a) + h_1 \left( \frac{1}{2} \right) h_2 \left( \frac{1}{2} \right) f(b) \subseteq f \left( \frac{2ab}{a+b} \right).
\]
Let
\[
a = \frac{mn}{xm + (1-x)n}, \quad b = \frac{mn}{(1-x)m + xn}.
\]
Then
\[
H \left( \frac{1}{2}, \frac{1}{2} \right) \left[ f \left( \frac{mn}{xm + (1-x)n} \right) + f \left( \frac{mn}{(1-x)m + xn} \right) \right] \subseteq f \left( \frac{2mn}{m+n} \right). \quad (11)
\]
Divide both sides of equation (11) by \( H \left( \frac{1}{2}, \frac{1}{2} \right) \) and integrate over \([0, 1]\), we have
\[
\frac{1}{H \left( \frac{1}{2}, \frac{1}{2} \right)} \int_0^1 f \left( \frac{2mn}{m+n} \right) dx = \frac{1}{H \left( \frac{1}{2}, \frac{1}{2} \right)} \left[ \int_0^1 f \left( \frac{2mn}{m+n} \right) dx, \int_0^1 \mathcal{F} \left( \frac{2mn}{m+n} \right) dx \right]
\]
\[
= \frac{1}{H \left( \frac{1}{2}, \frac{1}{2} \right)} f \left( \frac{2mn}{m+n} \right)
\]
\[
\geq \int_0^1 \left[ f \left( \frac{mn}{xm + (1-x)n} \right) + f \left( \frac{mn}{(1-x)m + xn} \right) \right] dx
\]
\[
= \left[ \int_0^1 f \left( \frac{mn}{xm + (1-x)n} \right) + f \left( \frac{mn}{(1-x)m + xn} \right) \right] dx,
\]
\[
\int_0^1 \left[ \mathcal{F} \left( \frac{mn}{xm + (1-x)n} \right) + \mathcal{F} \left( \frac{mn}{(1-x)m + xn} \right) \right] dx
\]
\[
= \left[ \frac{2mn}{n-m} \int_m^n \frac{f(t)}{t^2} dt, \frac{2mn}{n-m} \int_m^n \frac{\mathcal{F}(t)}{t^2} dt \right]
\]
\[
= \frac{2mn}{n-m} \int_m^n \frac{f(t)}{t^2} dt.
\]
Multiply both sides by \( \frac{1}{2} \), then the first inclusion relation of equation (10) is proved.
By hypothesis, we have
\[ f \left( \frac{mn}{xm + (1 - x)n} \right) \geq h_1(x)h_2(1 - x)f(m) + \geq h_1(1 - x)h_2(x)f(n), \]
\[ f \left( \frac{mn}{(1 - x)m + xn} \right) \geq h_1(1 - x)h_2(x)f(m) + \geq h_1(x)h_2(1 - x)f(n). \]

Add these two and integrate over \([0,1]\), we have
\[ \int_0^1 \left[ f \left( \frac{mn}{xm + (1 - x)n} \right) + f \left( \frac{mn}{(1 - x)m + xn} \right) \right] \, dx \]
\[ \geq \int_0^1 h_1(x)h_2(1 - x)[f(m) + f(n)] \, dx + \int_0^1 h_1(1 - x)h_2(x)[f(m) + f(n)] \, dx \]
\[ = 2[f(m) + f(n)] \int_0^1 H(x, 1 - x) \, dx. \]

Multiply both sides by \(\frac{1}{2}\), then (10) is proved. \(\square\)

**Corollary 3.1** Let \(h_1, h_2 : [0, 1] \subseteq I \rightarrow R^+\) such that \(H \left( \frac{1}{2}, \frac{1}{2} \right) \neq 0\), \(f : [m, n] \rightarrow R^+_1\) and \(f \in SHV((h_1, h_2), I, R^+_1)\), then
\[ \frac{1}{2H \left( \frac{1}{2}, \frac{1}{2} \right)} f \left( \frac{2mn}{m + n} \right) \leq \frac{mn}{n - m} \int_m^n f(t) \, dt \leq [f(m) + f(n)] \int_0^1 H(x, 1 - x) \, dx. \]

**Remark 3.1** If \(f = \overline{f}\), then Theorem 3.1 reduces to harmonical \((h_1, h_2)\)-convex function:
\[ \frac{1}{2H \left( \frac{1}{2}, \frac{1}{2} \right)} f \left( \frac{2mn}{m + n} \right) \leq \frac{mn}{n - m} \int_m^n f(t) \, dt \leq [f(m) + f(n)] \int_0^1 H(x, 1 - x) \, dx. \]

If \(H(x, y) \equiv h_1(x)\), then Theorem 3.1 reduces to harmonical \(h\)-convex interval-valued function.
([27] Theorem 1):
\[ \frac{1}{2h \left( \frac{1}{2} \right)} f \left( \frac{2mn}{m + n} \right) \geq \frac{mn}{n - m} \int_m^n f(t) \, dt \geq [f(m) + f(n)] \int_0^1 h(x) \, dx. \]

If \(H(x, y) \equiv h_1(x)\), and \(f = \overline{f}\), then Theorem 3.1 reduces to harmonical \(h\)-convex function.
([19] Theorem 3.2):
\[ \frac{1}{2h \left( \frac{1}{2} \right)} f \left( \frac{2mn}{m + n} \right) \leq \frac{mn}{n - m} \int_m^n f(t) \, dt \leq [f(m) + f(n)] \int_0^1 h(x) \, dx. \]

If \(h_1(x) = x^s\), then Theorem 3.1 reduces to harmonical \(s\)-convex interval-valued function.
([27] Remark 1 (3)):
\[ 2^{s - 1} f \left( \frac{2mn}{m + n} \right) \geq \frac{mn}{n - m} \int_m^n \frac{f(t)}{t^2} \, dt \geq \frac{1}{s + 1} [f(m) + f(n)]. \]
If \( h_1(x) = h_2(x) \equiv 1 \), then Theorem 3.1 reduces to harmonical \( p \)-convex interval-valued function.

([27] Remark 1 (5)):

\[
\frac{1}{2} f \left( \frac{2mn}{m+n} \right) \geq \frac{mn}{n-m} \int_{m}^{n} \frac{f(t)}{t^2} dt \geq [f(m) + f(n)]. \tag{17}
\]

**Theorem 3.2** Let \( h_1, h_2 : [0,1] \subseteq I \rightarrow R^+ \) such that \( H \left( \frac{1}{2}, \frac{1}{2} \right) \neq 0, f : [m,n] \rightarrow R^+_I \) and \( f \in \text{IR}_{[m,n]} \). If \( f \in \text{SHX}((h_1, h_2), [m,n], R^+_I) \), then

\[
\frac{1}{4H^2(\frac{1}{2}, \frac{1}{2})} f \left( \frac{2mn}{m+n} \right) \geq \frac{mn}{n-m} \int_{m}^{n} \frac{f(t)}{t^2} dt \geq \Delta_1 \geq \frac{mn}{n-m} \int_{m}^{n} \frac{f(t)}{t^2} dt \geq \Delta_2 \geq \left[ \frac{1}{2} + H \left( \frac{1}{2}, \frac{1}{2} \right) \right] [f(m) + f(n)] \int_{0}^{1} H(x,1-x)dx,
\]

where

\[
\Delta_1 = \frac{1}{4H^2(\frac{1}{2}, \frac{1}{2})} \left[ f \left( \frac{4mn}{m+3n} \right) + f \left( \frac{4mn}{3m+n} \right) \right],
\]

\[
\Delta_2 = \left[ \frac{f(m) + f(n)}{2} + f \left( \frac{2mn}{m+n} \right) \right] \int_{0}^{1} H(x,1-x)dx.
\]

**Proof.** Since \( f \in \text{SHX}((h_1, h_2), [m,n], R^+_I) \), for \([m, \frac{2mn}{m+n}]\), we have

\[
 h_1 \left( \frac{1}{2} \right) h_2 \left( \frac{1}{2} \right) f(a) + h_1 \left( \frac{1}{2} \right) h_2 \left( \frac{1}{2} \right) f(b) \leq f \left( \frac{2ab}{a+b} \right).
\]

Let

\[
a = \frac{m \frac{2mn}{m+n}}{xm + (1-x) \frac{2mn}{m+n}}, \quad b = \frac{m \frac{2mn}{m+n}}{(1-x)m + x \frac{2mn}{m+n}}.
\]

Then

\[
H \left( \frac{1}{2}, \frac{1}{2} \right) \left[ f \left( \frac{m\frac{2mn}{m+n}}{xm + (1-x) \frac{2mn}{m+n}} \right) + f \left( \frac{m\frac{2mn}{m+n}}{(1-x)m + x \frac{2mn}{m+n}} \right) \right] \leq f \left( \frac{4mn}{m+3n} \right).
\]

Integrating both sides of the above inequality over \([0,1]\), we have

\[
f \left( \frac{4mn}{m+3n} \right) \geq H \left( \frac{1}{2}, \frac{1}{2} \right) \left[ \int_{0}^{1} f \left( \frac{m\frac{2mn}{m+n}}{xm + (1-x) \frac{2mn}{m+n}} \right) + f \left( \frac{m\frac{2mn}{m+n}}{(1-x)m + x \frac{2mn}{m+n}} \right) dx, \right.
\]

\[
\left. \int_{0}^{1} f \left( \frac{m\frac{2mn}{m+n}}{xm + (1-x) \frac{2mn}{m+n}} \right) + f \left( \frac{m\frac{2mn}{m+n}}{(1-x)m + x \frac{2mn}{m+n}} \right) dx \right] \geq H \left( \frac{1}{2}, \frac{1}{2} \right) \frac{4mn}{n-m} \int_{m}^{n} \frac{f(t)}{t^2} dt \int_{m}^{n} \frac{f(t)}{t^2} dt
\]

\[
= H \left( \frac{1}{2}, \frac{1}{2} \right) \frac{4mn}{n-m} \int_{m}^{n} \frac{f(t)}{t^2} dt.
\]
Similarly, for $[\frac{m+n}{2}, n]$, let

$$a = \frac{2mn}{mn+n}, \quad b = \frac{2mn}{(1-x)n + x(n+mn)}. \quad (12)$$

Then

$$H\left(\frac{1}{2}\frac{1}{2}\right) \left[f\left(\frac{2mn}{mn+n}\right) + f\left(\frac{2mn}{(1-x)n + x(n+mn)}\right)\right] \subseteq f\left(\frac{4mn}{3m+n}\right). \quad (13)$$

In the above way, we have

$$f\left(\frac{4mn}{3m+n}\right) \supseteq H\left(\frac{1}{2}\frac{1}{2}\right) \frac{4mn}{n-m} \int_{m}^{n} f(t) \frac{1}{t^2} dt. \quad (20)$$

Hence

$$\Delta_1 = \frac{1}{4H\left(\frac{1}{2}\frac{1}{2}\right)} \left[f\left(\frac{4mn}{m+3n}\right) + f\left(\frac{4mn}{3m+n}\right)\right] \supseteq \frac{mn}{n-m} \int_{m}^{n} f(t) \frac{1}{t^2} dt. \quad (21)$$

Consequently, we get

$$\frac{1}{4H^2\left(\frac{1}{2}\frac{1}{2}\right)} f\left(\frac{2mn}{m+n}\right)$$

$$= \frac{1}{4H^2\left(\frac{1}{2}\frac{1}{2}\right)} f\left(\frac{2mn}{m+3n} + \frac{4mn}{3m+n}\right)$$

$$\supseteq \frac{1}{4H^2\left(\frac{1}{2}\frac{1}{2}\right)} \left[H\left(\frac{1}{2}\frac{1}{2}\right) f\left(\frac{4mn}{m+3n}\right) + H\left(\frac{1}{2}\frac{1}{2}\right) f\left(\frac{4mn}{3m+n}\right)\right]$$

$$= \Delta_1$$

$$\supseteq \frac{mn}{n-m} \int_{m}^{n} f(t) \frac{1}{t^2} dt.$$
Corollary 3.2 Let \( h_1, h_2 : [0, 1] \subseteq I \to R^+ \) such that \( H \left( \frac{1}{2}, \frac{1}{2} \right) \neq 0, f : [m, n] \to R^+ \) and \( f \in \text{IR}_f((m, n)) \). If \( f \in \text{SHV}((h_1, h_2), I, R^+_f) \), then

\[
\frac{1}{4H^2 \left( \frac{1}{2}, \frac{1}{2} \right)} \int f \left( \frac{2mn}{m+n} \right) \leq \Delta_1 \leq \frac{mn}{n-m} \int f(t) dt \int_0^t \frac{dt}{t^2} \leq \Delta_2 \subseteq \left[ \frac{1}{2} + H \left( \frac{1}{2}, \frac{1}{2} \right) \right] \int H(x, 1-x) dx.
\]

(21)

Remark 3.2 If \( f = f_1 \), then Theorem 3.2 reduces to harmonical \((h_1, h_2)\)-convex function:

\[
\frac{1}{4H^2 \left( \frac{1}{2}, \frac{1}{2} \right)} \int f \left( \frac{2mn}{m+n} \right) \leq \Delta_1 \leq \frac{mn}{n-m} \int f(t) dt \int_0^t \frac{dt}{t^2} \leq \Delta_2 \subseteq \left[ \frac{1}{2} + H \left( \frac{1}{2}, \frac{1}{2} \right) \right] \int H(x, 1-x) dx.
\]

(22)

If \( H(x, y) \equiv h_1(x) \), then Theorem 3.2 reduces to harmonical \( h \)-convex interval-valued function.

(27) Theorem 2:

\[
\frac{1}{4h^2 \left( \frac{1}{2} \right)} \int f \left( \frac{2mn}{m+n} \right) \geq \frac{1}{4h \left( \frac{1}{2} \right)} \left[ f \left( \frac{3m+n}{4} \right) + f \left( \frac{m+3n}{4} \right) \right] \geq \frac{mn}{n-m} \int f(t) dt \int_0^t \frac{dt}{t^2} \geq \int h(x) dx \geq \left[ \frac{1}{2} + h \left( \frac{1}{2} \right) \right] \int H(x, 1-x) dx.
\]

(23)

If \( h_1(x) = x^s \), then Theorem 3.2 reduces to harmonical \( s \)-convex interval-valued function.

(27) Remark 2:

\[
4^{s-1} f \left( \frac{2mn}{m+n} \right) \geq 2^{s-2} \left[ f \left( \frac{3m+n}{4} \right) + f \left( \frac{m+3n}{4} \right) \right] \geq \frac{mn}{n-m} \int f(t) dt \int_0^t \frac{dt}{t^2} \geq \frac{1}{s+1} \left[ f(m) + f(n) + f \left( \frac{m+n}{2} \right) \right]
\]

(24)

If \( h_1(x) = h_2(x) \equiv 1 \), then Theorem 3.2 reduces to harmonical \( p \)-convex interval-valued function.

(27) Remark 2:

\[
\frac{1}{4} f \left( \frac{2mn}{m+n} \right) \geq \frac{1}{4} \left[ f \left( \frac{3m+n}{4} \right) + f \left( \frac{m+3n}{4} \right) \right] \geq \frac{mn}{n-m} \int f(t) dt \int_0^t \frac{dt}{t^2} \geq \frac{3}{2} \left[ f(m) + f(n) \right]
\]

(25)
Theorem 3.3 Let \( h_1, h_2 : [0, 1] \subseteq I \rightarrow R^+ \), \( f, g : [m, n] \rightarrow R^+ \) and \( f g \in IR_{([m, n])} \). If \( f, g \in SHX((h_1, h_2), [m, n], R^+) \), then

\[
\frac{mn}{n - m} \int_m^n \frac{f(t)g(t)}{t^2} dt \supseteq M(m, n) \int_0^1 H^2(x, 1 - x) dx + N(m, n) \int_0^1 H(x, x) H(1 - x, 1 - x) dx,
\]

(26)

where

\[
M(m, n) = f(m)g(m) + f(n)g(n), \quad N(m, n) = f(m)g(n) + f(n)g(m).
\]

Proof. If \( f, g \in SHX((h_1, h_2), [m, n], R^+) \), one has

\[
f \left( \frac{mn}{xm + (1 - x)n} \right) \supseteq h_1(x)h_2(1 - x)f(m) + h_1(1 - x)h_2(x)f(n),
\]

\[
g \left( \frac{mn}{xm + (1 - x)n} \right) \supseteq h_1(x)h_2(1 - x)g(m) + h_1(1 - x)h_2(x)g(n).
\]

The above two equation are multiplied, then

\[
f \left( \frac{mn}{xm + (1 - x)n} \right) g \left( \frac{mn}{xm + (1 - x)n} \right) \supseteq H^2(x, 1 - x)f(m)g(m) + H^2(1 - x, x)f(n)g(n) + H(x, x)H(1 - x, 1 - x)[f(m)g(n) + f(n)g(m)].
\]

Integrating both sides of the above inequality over \([0, 1]\), we get

\[
\int_0^1 f \left( \frac{mn}{xm + (1 - x)n} \right) g \left( \frac{mn}{xm + (1 - x)n} \right) dx = \left[ \int_0^1 f \left( \frac{mn}{xm + (1 - x)n} \right) g \left( \frac{mn}{xm + (1 - x)n} \right) dx, \int_0^1 f \left( \frac{mn}{xm + (1 - x)n} \right) g \left( \frac{mn}{xm + (1 - x)n} \right) dx \right] = \left[ \frac{mn}{n - m} \int_m^n \frac{f(t)g(t)}{t^2} dt, \frac{mn}{n - m} \int_m^n \frac{f(t)g(t)}{t^2} dt \right] = \frac{mn}{n - m} \int_m^n \frac{f(t)g(t)}{t^2} dt \supseteq \int_0^1 H^2(x, 1 - x)[f(m)g(m) + f(n)g(n)] dx + \int_0^1 H(x, x)H(1 - x, 1 - x)[f(m)g(n) + f(n)g(m)] dx = M(m, n) \int_0^1 H^2(x, 1 - x) dx + N(m, n) \int_0^1 H(x, x)H(1 - x, 1 - x) dx.
\]

This completes the proof. \( \square \)
Corollary 3.3 Let \( h_1, h_2 : [0, 1] \subseteq I \rightarrow R^+ \), \( f, g : [m,n] \rightarrow R^+ \) and \( fg \in IR([m,n]) \). If \( f, g \in SHV((h_1, h_2), I, R^+) \), then

\[
\frac{mn}{n-m} \int^n_m \frac{f(t)g(t)}{t^2} dt \subseteq M(m,n) \int^1_0 H^2(x, 1-x)dx + N(m,n) \int^1_0 H(x,x)H(1-x, 1-x)dx,
\]

where

\[
M(m,n) = f(m)g(m) + f(n)g(n), \quad N(m,n) = f(m)g(n) + f(n)g(m).
\]

Remark 3.3 If \( f = \overline{f} \), then Theorem 3.3 reduces to harmonical \((h_1, h_2)-\)convex function:

\[
\frac{mn}{n-m} \int^n_m \frac{f(t)g(t)}{t^2} dt \leq M(m,n) \int^1_0 H^2(x, 1-x)dx + N(m,n) \int^1_0 H(x,x)H(1-x, 1-x)dx.
\]

If \( H(x,y) = h_1(x) \), then Theorem 3.3 reduces to harmonical \( h-\)convex interval-valued function. ([27] Theorem 3):

\[
\frac{mn}{n-m} \int^n_m \frac{f(t)g(t)}{t^2} dt \supseteq M(m,n) + N(m,n).
\]

If \( h_1(x) = x^s \), then Theorem 3.3 reduces to harmonical \( s-\)convex interval-valued function. ([27] Remark 3):

If \( h_1(x) = h_2(x) \equiv 1 \), then Theorem 3.3 reduces to harmonical \( p-\)convex interval-valued function. ([27] Remark 3):
Theorem 3.4 Let \( h_1, h_2 : [0, 1] \to \mathbb{R}^+ \) such that \( H \left( \frac{1}{2}, \frac{1}{2} \right) \neq 0 \), \( f, g : [m, n] \to R^+ \) and \( f, g \in IR_{(m,n)} \). If \( f, g \in SHX((h_1, h_2), [m, n], R^+_I) \), then

\[
\frac{1}{2H^2 \left( \frac{1}{2}, \frac{1}{2} \right)} f \left( \frac{2mn}{m+n} \right) g \left( \frac{2mn}{m+n} \right) \geq \frac{mn}{n-m} \int_{m}^{n} \frac{f(t)g(t)}{t^2} dt \\
+ M(m, n) \int_{0}^{1} H(x, x) H(1-x, 1-x) dx \\
+ N(m, n) \int_{0}^{1} H^2(x, 1-x) dx,
\]

where

\[
M(m, n) = f(m)g(m) + f(n)g(n), \quad N(m, n) = f(m)g(n) + f(n)g(m).
\]

Proof. By hypothesis, one has

\[
f \left( \frac{2mn}{m+n} \right) \geq H \left( \frac{1}{2}, \frac{1}{2} \right) f \left( \frac{mn}{xm + (1-x)n} \right) + H \left( \frac{1}{2}, \frac{1}{2} \right) f \left( \frac{mn}{(1-x)m + x} \right),
\]

\[
g \left( \frac{2mn}{m+n} \right) \geq H \left( \frac{1}{2}, \frac{1}{2} \right) g \left( \frac{mn}{xm + (1-x)n} \right) + H \left( \frac{1}{2}, \frac{1}{2} \right) g \left( \frac{mn}{(1-x)m + x} \right).
\]

Then

\[
f \left( \frac{2mn}{m+n} \right) g \left( \frac{2mn}{m+n} \right) \geq H^2 \left( \frac{1}{2}, \frac{1}{2} \right) \left[ f \left( \frac{mn}{xm + (1-x)n} \right) g \left( \frac{mn}{xm + (1-x)n} \right) \\
+ f \left( \frac{mn}{(1-x)m + x} \right) g \left( \frac{mn}{(1-x)m + x} \right) \right] \\
+ H^2 \left( \frac{1}{2}, \frac{1}{2} \right) \left[ f \left( \frac{mn}{xm + (1-x)n} \right) g \left( \frac{mn}{xm + (1-x)n} \right) \\
+ f \left( \frac{mn}{(1-x)m + x} \right) g \left( \frac{mn}{(1-x)m + x} \right) \right] \\
\geq H^2 \left( \frac{1}{2}, \frac{1}{2} \right) \left[ f \left( \frac{mn}{xm + (1-x)n} \right) g \left( \frac{mn}{xm + (1-x)n} \right) \\
+ f \left( \frac{mn}{(1-x)m + x} \right) g \left( \frac{mn}{(1-x)m + x} \right) \right] \\
+ H^2 \left( \frac{1}{2}, \frac{1}{2} \right) \left[ (H(x, 1-x)f(m) + H(1-x, x)f(n)) (H(1-x, x)g(m) + H(x, 1-x)g(n)) \\
+ (H(1-x, x)f(m) + H(x, 1-x)f(n)) (H(x, 1-x)g(m) + H(1-x, x)g(n)) \right]
\]

\[
= H^2 \left( \frac{1}{2}, \frac{1}{2} \right) \left[ f \left( \frac{mn}{xm + (1-x)n} \right) g \left( \frac{mn}{xm + (1-x)n} \right) \\
+ f \left( \frac{mn}{(1-x)m + x} \right) g \left( \frac{mn}{(1-x)m + x} \right) \right] \\
+ H^2 \left( \frac{1}{2}, \frac{1}{2} \right) [2H(x, x) H(1-x, 1-x)(f(m)g(m) + f(n)g(n)) \\
+ (H^2(x, 1-x) + H^2(1-x, x))(f(m)g(n) + f(n)g(m))].
\]
Integrating both sides of the above inequality over $[0, 1]$, we get
\[
\int_0^1 f \left( \frac{2mn}{m+n} \right) g \left( \frac{2mn}{m+n} \right) dx
= \left[ \int_0^1 f \left( \frac{2mn}{m+n} \right) g \left( \frac{2mn}{m+n} \right) dx \right] \int_0^1 \bar{f} \left( \frac{2mn}{m+n} \right) \bar{g} \left( \frac{2mn}{m+n} \right) dx
\]
\[
= f \left( \frac{2mn}{m+n} \right) g \left( \frac{2mn}{m+n} \right)
\geq H^2 \left( \frac{1}{2}, \frac{1}{2} \right) \left[ \frac{2mn}{n-m} \int_m^n \frac{f(t)g(t)}{t^2} dt \right]
+ 2H^2 \left( \frac{1}{2}, \frac{1}{2} \right) M(m,n) \int_0^1 H(x,x)H(1-x,1-x)dx
+ 2H^2 \left( \frac{1}{2}, \frac{1}{2} \right) N(m,n) \int_0^1 H^2(x,1-x)dx.
\]
Multiply both sides by $\frac{1}{2H^2 \left( \frac{1}{2}, \frac{1}{2} \right)}$, equation (33) can be obtained.

This completes the proof. \(\Box\)

**Corollary 3.4** Let $h_1, h_2 : [0, 1] \subseteq I \rightarrow R^+$ such that $H \left( \frac{1}{2}, \frac{1}{2} \right) \neq 0$, $f, g : [m, n] \rightarrow R^+$ and $f, g \in IR((h_1, h_2), I, R^+)$, then
\[
\frac{1}{2H^2 \left( \frac{1}{2}, \frac{1}{2} \right)} f \left( \frac{2mn}{m+n} \right) g \left( \frac{2mn}{m+n} \right) \leq \frac{mn}{n-m} \int_m^n \frac{f(t)g(t)}{t^2} dt
+ M(m,n) \int_0^1 H(x,x)H(1-x,1-x)dx \quad (34)
+ N(m,n) \int_0^1 H^2(x,1-x)dx,
\]
where
\[
M(m,n) = f(m)g(m) + f(n)g(n), \quad N(m,n) = f(m)g(n) + f(n)g(m).
\]

**Remark 3.4** If $f = \bar{f}$, then Theorem 3.4 reduces to harmonical $(h_1, h_2)$–convex function:
\[
\frac{1}{2H^2 \left( \frac{1}{2}, \frac{1}{2} \right)} f \left( \frac{2mn}{m+n} \right) g \left( \frac{2mn}{m+n} \right) \leq \frac{mn}{n-m} \int_m^n \frac{f(t)g(t)}{t^2} dt
+ M(m,n) \int_0^1 H(x,x)H(1-x,1-x)dx \quad (35)
+ N(m,n) \int_0^1 H^2(x,1-x)dx.
\]
If $H(x,y) \equiv h_1(x)$, then Theorem 3.4 reduces to harmonical $h$–convex interval-valued function.
\( \left( \frac{1}{2H^2 \left( \frac{1}{2}, \frac{1}{2} \right)} \right)^f \left( \frac{2mn}{m+n} \right) g \left( \frac{2mn}{m+n} \right) \geq \frac{mn}{n-m} \int_{m}^{n} \frac{f(t)g(t)}{t^2} dt \\
+ \frac{n-m}{n} \int_{m}^{n} \frac{f(t)g(t)}{t^2} dt \\
+ M(m,n) \int_{0}^{1} h(x)h(1-x)dx \\
+ N(m,n) \int_{0}^{1} h^2(x)dx. \) (36)

If \( h_1(x) = x^s \), then Theorem 3.4 reduces to harmonical \( s \)-convex interval-valued function.

\( \left( \frac{1}{2H^2 \left( \frac{1}{2}, \frac{1}{2} \right)} \right)^f \left( \frac{2mn}{m+n} \right) g \left( \frac{2mn}{m+n} \right) \geq \frac{mn}{n-m} \int_{m}^{n} \frac{f(t)g(t)}{t^2} dt \\
+ B(s+1,s+1)M(m,n) \\
+ \frac{1}{2s+1}N(m,n). \) (37)

If \( h_1(x) = h_2(x) \equiv 1 \), then Theorem 3.4 reduces to harmonical \( p \)-convex interval-valued function.

\( \left( \frac{1}{2H^2 \left( \frac{1}{2}, \frac{1}{2} \right)} \right)^f \left( \frac{2mn}{m+n} \right) g \left( \frac{2mn}{m+n} \right) \geq \frac{mn}{n-m} \int_{m}^{n} \frac{f(t)g(t)}{t^2} dt \\
+ M(m,n) + N(m,n). \) (38)

4. Conclusions. We introduced the concepts of “harmonical \((h_1, h_2)\)-convex functions” and “harmonic \((h_1, h_2)\)-convex interval valued functions”, and prove some new Hermite-Hadamard inequalities, which are compared with previous results. Our results are helpful to the development of interval calculus inequalities. As a future research direction, we intend to study the Hermite-Hadamard inequality for fractional integration on harmonical \((h_1, h_2)\)-convex interval-valued functions.

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Author’s contributions. RN.L carried out the main results and completed the corresponding proof. RX participated in the proof and help to complete introduction. All authors read and approved the final manuscript.

Data availability statement. We declare that the data and material in the paper can be used publicly.
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E-mail address: 1085385126@qq.com
E-mail address: xurun2005@163.com