A Plancherel Theorem On a Noncommutative Hypergroup

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Abstract. Let $G$ be a locally compact hypergroup and let $K$ be a compact sub-hypergroup of $G$. $(G, K)$ is a Gelfand pair if $M_c(G//K)$, the algebra of measures with compact support on the double coset $G//K$, is commutative for the convolution. In this paper, assuming that $(G, K)$ is a Gelfand pair, we define and study a Fourier transform on $G$ and then establish a Plancherel theorem for the pair $(G, K)$.

1. Introduction

Hypergroups generalize locally compact groups. They appear when the Banach space of all bounded Radon measures on a locally compact space carries a convolution having all properties of a group convolution except from the fact that the convolution of two point measures is a probability measure with compact support and not necessarily a point measure. The intention was to unify harmonic analysis on duals of compact groups, double coset spaces $G//H$ ($H$ a compact subgroup of a locally compact group $G$), and commutative convolution algebras associated with product linearization formulas of special functions. The notion of hypergroup has been sufficiently studied (see for example [2,4,6,7]). Harmonic analysis and probability theory on commutative hypergroups are well developed meanwhile where many results from group theory remain valid (see [1]). When $G$ is a commutative hypergroup, the convolution algebra $M_c(G)$ consisting of measures with compact support on $G$ is commutative. The typical example of commutative hypergroup is the double coset $G//K$ when $G$ is a locally compact group, $K$ is a compact subgroup of $G$ such that $(G, K)$ is a Gelfand pair. In [4], R. I. Jewett has shown the existence of a positive measure called Plancherel measure on the dual space $\hat{G}$ of a commutative hypergroup $G$. When the hypergroup $G$ is not commutative, it is possible to involve a...
compact sub-hypergroup $K$ of $G$ leading to a commutative subalgebra of $M_c(G)$. In fact, if $K$ is a compact sub-hypergroup of a hypergroup $G$, the pair $(G, K)$ is said to be a Gelfand pair if $M_c(G//K)$ the convolution algebra of measures with compact support on $G//K$ is commutative. The notion of Gelfand pairs for hypergroups is well-known (see [3,8,9]). The goal of this paper is to extend Jewett work’s by obtaining a Plancherel theorem over Gelfand pair associated with non-commutative hypergroup. In the next section, we give notations and setup useful for the remainder of this paper. In section 3, we introduce first the notion of $K$-multiplicative functions and obtain some of their characterizations. Thanks to these results, we establish a one to one correspondence between the space of $K$-multiplicative functions and the dual space of $G$. Then, we define a Fourier tranform on $M_b(G)$, the algebra of bounded measures on $G$ and on $K(G)$, the algebra of continuous functions on $G$ with compact support. Finally, using the fact that $G//K$ is a commutative hypergroup, we prove that there exists a nonnegative measure (the Plancherel measure) on the dual space of $G$.

2. Notations and preliminaries

We use the notations and setup of this section in the rest of the paper without mentioning. Let $G$ be a locally compact space. We denote by:
- $C(G)$ (resp. $M(G)$) the space of continuous complex valued functions (resp. the space of Radon measures) on $G$,
- $C_b(G)$ (resp. $M_b(G)$) the space of bounded continuous functions (resp. the space of bounded Radon measures) on $G$,
- $K(G)$ (resp. $M_c(G)$) the space of continuous functions (resp. the space of Radon measures) with compact support on $G$,
- $C_0(G)$ the space of elements in $C(G)$ which are zero at infinity,
- $c(G)$ the space of compact sub-space of $G$,
- $\delta_x$ the point measure at $x \in G$,
- $\text{spt}(f)$ the support of the function $f$.

Let us notice that the topology on $M(G)$ is the cône topology [4] and the topology on $c(G)$ is the topology of Michael [5].

**Definition 2.1.** $G$ is said to be a **hypergroup** if the following assumptions are satisfied.

(H1) There is a binary operator $*$ named convolution on $M_b(G)$ under which $M_b(G)$ is an associative algebra such that:
\begin{itemize}
  \item[i)] the mapping $(\mu, \nu) \mapsto \mu * \nu$ is continuous from $M_b(G) \times M_b(G)$ in $M_b(G)$.
  \item[ii)] $\forall x, y \in G$, $\delta_x * \delta_y$ is a measure of probability with compact support.
  \item[iii)] the mapping: $(x, y) \mapsto \text{supp}(\delta_x * \delta_y)$ is continuous from $G \times G$ in $c(G)$.
\end{itemize}

(H2) There is a unique element $e$ (called neutral element) in $G$ such that $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x, \forall x \in G$. 

(H3) There is an involutive homeomorphism: \( x \mapsto \pi \) from \( G \) in \( G \), named involution, such that:

i) \((\delta_x * \delta_y)^\pi = \delta_{\pi x} * \delta_{\pi y}, \forall x, y \in G \)

with \( \mu^\pi(f) = \mu(f^\pi) \) where \( f^\pi(x) = f(\pi x), \forall f \in C(G) \) and \( \mu \in M(G) \).

ii) \( \forall x, y, z \in G, z \in \text{supp}(\delta_x * \delta_y) \) if and only if \( x \in \text{supp}(\delta_z * \delta_{\pi z}) \).

The hypergroup \( G \) is commutative if \( \delta_x * \delta_y = \delta_y * \delta_x, \forall x, y \in G \). For \( x, y \in G \), \( x * y \) is the support of \( \delta_x * \delta_y \) and for \( f \in C(G) \),

\[
f(x * y) = (\delta_x * \delta_y)(f) = \int_G f(z) d(\delta_x * \delta_y)(z).
\]

The convolution of two measures \( \mu, \nu \) in \( M_b(G) \) is defined by: \( \forall f \in C(G) \)

\[
(\mu * \nu)(f) = \int_G \int_G (\delta_x * \delta_y)(f)d\mu(x)d\nu(y) = \int_G \int_G f(x * y)d\mu(x)d\nu(y).
\]

For \( \mu \) in \( M_b(G) \), \( \mu^* = (\pi)^\pi \). So \( M_b(G) \) is a *-Banach algebra.

**Definition 2.2.** \( H \subset G \) is a sub-hypergroup of \( G \) if the following conditions are satisfied.

1. \( H \) is non empty and closed in \( G \),
2. \( \forall x \in H, \pi \in H \),
3. \( \forall x, y \in H, \text{supp}(\delta_x * \delta_y) \subset H \).

Let us now consider a hypergroup \( G \) provided with a left Haar measure \( \mu_G \) and \( K \) a compact subhypergroup of \( G \) with a normalized Haar measure \( \omega_K \). Let us put \( M_{\mu_G}(G) \) the space of measures in \( M_b(G) \) which are absolutely continuous with respect to \( \mu_G \). \( M_{\mu_G}(G) \) is a closed self-adjoint ideal in \( M_b(G) \). For \( x \in G \), the double coset of \( x \) with respect to \( K \) is \( K \{ x \} K = \{ k_1 * x * k_2, k_1, k_2 \in K \} \). We write simply \( kxK \) for a double coset and recall that \( KxK = \bigcup_{k_1, k_2 \in K} \text{supp}(\delta_{k_1} * \delta_x * \delta_{k_2}) \). All double coset form a partition of \( G \) and the quotient topology with respect to the corresponding equivalence relation equips the double cosets space \( G//K \) with a locally topology ( [1], page 53). The natural mapping \( p_K : G \rightarrow G//K \) defined by: \( p_K(x) = kxK, x \in G \) is an open surjective continuous mapping. A function \( f \in C(G) \) is said to be invariant by \( K \) or \( K \text{–invariant} \) if \( f(k_1 * x * k_2) = f(x) \) for all \( x \in G \) and for all \( k_1, k_2 \in K \). We denote by \( C^b(G) \), (resp. \( K^b(G) \)) the space of continuous functions (resp. continuous functions with compact support) which are \( K \text{–invariant} \). For \( f \in C^b(G) \), one defines the function \( \tilde{f} \) on \( G//K \) by \( \tilde{f}(kxK) = f(x) \) \( \forall x \in G \). \( \tilde{f} \) is well defined and it is continuous on \( G//K \). Conversely, for all continuous function \( \varphi \) on \( G//K \), the function \( f = \varphi \circ p_K \in C^b(G) \). One has the obvious consequence that the mapping \( f \mapsto \tilde{f} \) sets up a topological isomorphism between the topological vector spaces \( C^b(G) \) and \( C(G//K) \) (see [8, 9]). So, for any \( f \) in \( C^b(G) \), \( f = \tilde{f} \circ p_K \). Otherwise, we consider the \( K \)-projection \( f \mapsto \tilde{f}^b \) (by identifying \( \tilde{f}^b \) and \( \tilde{f}^\pi \)) from \( C(G) \) into \( C(G//K) \) where for \( x \in G \), \( \tilde{f}^b(x) = \int_K \int_K f(k_1 * x * k_2) d\omega_K(k_1) d\omega_K(k_2) \). If \( f \in K(G) \), then \( \tilde{f}^b \in K(G//K) \).

For a measure \( \mu \in M(G) \), one defines \( \mu^b \) by \( \mu^b(f) = \mu(f^b) \) for \( f \in K(G) \). \( \mu \) is said to be \( K \text{–invariant} \) if \( \mu^b = \mu \) and we denote by \( M^b(G) \) the set of all those measures. Considering these properties, one
defines a hypergroup operation on $G//K$ by: $\delta_{KxK} * \delta_{KyK}(\tilde{f}) = \int_K f(x*k*y) d\omega_K(k)$ (see [2, p. 12]). This defines uniquely the convolution $(KxK) * (KyK)$ on $G//K$. The involution is defined by: $KxK = KxK$ and the neutral element is $K$. Let us put $m = \int_G \delta_{KxK} d\mu_G(x)$, $m$ is a left Haar measure on $G//K$. We say that $(G, K)$ is a Gelfand pair if the convolution algebra $M_c(G//K)$ is commutative.

$M_c(G//K)$ is topologically isomorphic to $M^{\#}_c(G)$. Considering the convolution product on $K(G), K(G)$ is a convolution algebra and $K^\#(G)$ is a subalgebra. Thus $(G, K)$ is a Gelfand pair if and only if $K^\#(G)$ is commutative ([3], theorem 3.2.2).

3. Plancherel theorem

Let $G$ be a locally compact hypergroup and let $K$ be a compact sub-hypergroup of $G$. In this section, we assume that $(G, K)$ is a Gelfand pair.

3.1. $K$-multiplicative functions.

Let us put $G^\#_b$ the space of continuous, bounded function $\phi$ on $G$ such that:

1. $\phi$ is $K$-invariant,
2. $\phi(e) = 1$,
3. $\int_K \phi(x*k*y) d\omega_K(k) = \phi(x) \phi(y) \forall x, y \in G$.

Let $\hat{G}$ be the sub-space of $G^\#_b$ containing the elements $\phi$ in $G^\#_b$ such that

$$\phi(x) = \overline{\phi(x)} \forall x \in G.$$ 

$\hat{G}$ is called the dual space of the hypergroup $G$.

Remark 3.1. (1) If $\phi \in \hat{G}$, then $\phi^- \in \hat{G}$.

(2) Equipped with the topology of uniform convergence on compacta, $\hat{G}$ is a locally compact Hausdorff space.

(3) In general, $\hat{G}$ is not a hypergroup.

Definition 3.2. A complex-valued function $\chi$ on $G$ will be called a multiplicative (resp. $K$-multiplicative) function if $\chi$ is continuous and not identically zero, and has the property that:

$$\chi(x*y) = \chi(x) \chi(y) \ (resp. \int_K \chi(x*k*y) d\omega_K(k) = \chi(x) \chi(y)) \forall x, y \in G.$$ 

A multiplicative (resp. $K$-multiplicative) function on $M_b(G)$ is a continuous complex-valued function $F$ not identically zero on $M^\#_b(G)$, and has the property that:

$$F(\mu * \nu) = F(\mu)F(\nu) \ (resp. F(\mu * w_K * \nu) = F(\mu)F(\nu)) \forall \mu, \nu \in M_b(G).$$

For $\chi \in C_b(G)$, not identically zero, let put $F_\chi(\mu) = \int_G \chi d\mu$ for $\mu \in M_b(G)$. 
Proposition 3.3. Let $F$ be a $K$-multiplicative function on $M_b(G)$, then:

i) $F$ is multiplicative on $M^b_\mu(G)$.

ii) $F(w_K) = F(\delta_e) = 1$.

iii) $\forall \mu \in M_b(G)$, $F(\mu^k) = F(\mu)$

iv) $\forall k \in K$, $F(\delta_k) = 1$.

Proof. i) Just remember that $\mu * w_K = \mu, \forall \mu \in M^b_\mu(G)$.

ii) Let $\nu \in M^b_\mu(G)$ such that $F(\nu) \neq 0$.

iii) Let $\mu \in M_b(G)$. Since $\mu^k = w_K * \mu * w_K$, we have

$$F(\mu^k) = F(w_K * \mu * w_K) = F(w_K * \mu * w_K * w_K) = F(w_K * \mu) = F(\delta_e * w_K * \mu) = F(\mu).$$

iv) If $k \in K$, $\delta^k_K = w_K$. Using (ii) and (iii), we have $F(\delta_k) = 1$.

□

Proposition 3.4. Let $\phi \in G^b_\mu$.

i) $F_{\phi}$ is a bounded linear $K$-multiplicative function on $M_b(G)$.

ii) $F_{\phi}$ is not identically zero on $M^b_{\mu_G}(G)$.

Proof. i) That is clear that $F_{\phi}$ is linear and bounded. Let $\mu, \nu \in M_b(G)$. We have

$$F_{\phi}(\mu * w_K * \nu) = \int_G \int_k \int_G \bar{\phi}(x * k * y) d\mu(x) d\nu(k) d\nu(y) = \int_G \int_k \phi(x) d\mu(x) \int_G \phi(y) d\nu(y) = F_{\phi}(\mu) F_{\phi}(\nu).$$

Morever, $F_{\phi}(w_K) = \int_k \bar{\phi}(k) d\nu(k) = 1 \neq 0$.

ii) If $\mu \in M_{\mu_G}(G)$, then $\mu^k = w_K * \mu * w_K \in M^b_{\mu_G}(G)$. Let $f \in K(G)$ with spt($f$) $\subset K$ such that $\int_G f(x) d\mu_G(x) = 1$. $f^k \mu_G \in M^b_{\mu_G}(G) and$

$$F_{\phi}(f^k \mu_G) = F_{\phi}(f \mu_G) = \int_G \bar{\phi}(x) f(x) d\mu_G(x) = \int_K f(x) d\mu_G(x) = 1 \neq 0.$$
\textbf{Theorem 3.5.} 1) Let \( E \) be a multiplicative linear function on \( M^b_{\mu G}(G) \) not identically zero. There exists a unique \( K \)-multiplicative linear function \( F \) on \( M_b(G) \) such that \( F = E \) on \( M^b_{\mu G}(G) \).

2) Let \( F \) be a bounded linear \( K \)-multiplicative function on \( M_b(G) \) not identically zero on \( M^b_{\mu G}(G) \). There exists a unique function \( \phi \) in \( G_{\mu}^b \) such that \( F = F\phi \).

\textbf{Proof.} 1) Let \( \nu \in M^b_{\mu G}(G) \) such that \( E(\nu) \neq 0 \) and put

\[ F(\mu) = \frac{E(\mu^b * \nu)}{E(\nu)}, \text{ for } \mu \in M_b(G) \]

\( F \) is well defined since \( M^b_{\mu G}(G) \) is an ideal in \( M_b(G) \). Let us first see that \( F \) is multiplicative on \( M^b_{\mu G}(G) \). For \( \mu \) and \( \mu' \) in \( M_b(G) \), we have

\[ F(\mu^b * \mu'^b) = \frac{E(\mu^b * \mu'^b * \nu)}{E(\nu)} = \frac{E(\nu * \mu * \mu'^b * \nu)}{E(\nu)} = \frac{E(\nu * \mu * E(\mu'^b * \nu))}{E(\nu)} = \frac{E(\nu * \mu * \mu'^b)}{E(\nu)} F(\mu') \]

Moreover \( F(w_k) = \frac{E(w_k^b * \nu)}{E(\nu)} = \frac{E(\nu)}{E(\nu)} = 1 \). So for \( \mu \) and \( \mu' \) in \( M_b(G) \), we have

\[ F(\mu^b * w_k * \mu'^b) = F(w_k * (w_k * \mu * w_k) * (w_k * \mu^b * w_k) * w_k) \]

\[ = F((w_k * \mu * w_k) * (w_k * \mu^b * w_k)) \]

\[ = F(\mu^b * \mu'^b) \]

\[ = F(\mu) F(\mu'). \]

The uniqueness stems from proposition 3.3.

2) Let \( F \) be a bounded linear \( K \)-multiplicative function on \( M_b(G) \). Let \( \nu \in M^b_{\mu G}(G) \) such that \( F(\nu) \neq 0 \). If \( \mu_1, \mu_2 \in M_b(G) \) then

\[ |F(\mu_1) - F(\mu_2)| = \left| F(\mu_1^b) - F(\mu_2^b) \right| \]

\[ = \left| F(\mu_1^b * \nu) - F(\mu_2^b * \nu) \right| \]

\[ = \left| F((\mu_1 * \nu - \mu_2 * \nu)^b) \right| \]

\[ \leq \frac{\|F\|}{F(\nu)} \|\mu_1 * \nu - \mu_2 * \nu\|. \]
Thus $F$ is positive-continuous by ([4], Theorem 5.6B). By ([4], Theorem 2.2D) there exists a bounded continuous function $h$ on $G$ such that $F(\mu) = \int_G h(x)d\mu(x)$. So $\phi = \overline{h}$.

\[ \square \]

3.2. **Fourier transform on** $M_b(G)$.

**Definition 3.6.** Let $\mu \in M_b(G)$, the Fourier transform of $\mu$ is the map $\widehat{\mu} : \widehat{G} \rightarrow \mathbb{C}$ defined by:

\[ \widehat{\mu}(\phi) = \int_G \phi(x)d\mu(x). \]

**Proposition 3.7.**

i) For $\mu \in M_b(G)$, $\widehat{\mu} \in C_b(\widehat{G})$.

ii) For $\mu \in M_b(G)$, $\widehat{\mu} = \widehat{\mu}^\ast$.

iii) For $\mu \in M_b(G)$, $\widehat{\mu} \in \mathcal{C}_0(\widehat{G})$.

iv) If $\mu \in M_b^0(G)$ and $\nu \in M_b(G)$, then $\widehat{\mu} \ast \nu = \widehat{\mu} \nu$.

**Proof.**

i) We can see that, $\widehat{\mu}(\phi) = \mu(\overline{\phi}) \forall \phi \in \widehat{G}$.

ii) For $\phi \in \widehat{G}$, we have $\widehat{\mu}(\phi) = F_{\overline{\phi}}(\mu)$. So $\widehat{\mu}^\ast(\phi) = F_{\overline{\phi}}(\widehat{\mu}^\ast) = F_{\overline{\phi}}(\mu) = \widehat{\mu}(\phi)$.

iii) This comes from theorem 3.5 and ([4], theorem 6.3G)

iv) Let $\phi$ belongs to $\widehat{G}$, we have

\[
\begin{align*}
\widehat{\mu} \ast \nu(\phi) &= \int_G \phi(x)d\mu \ast \nu(x) \\
&= \int_G \int_G \phi(x \ast y)d\mu(x)d\nu(y) \\
&= \int_G \left[ \int_K \int_K \phi(k_1 \ast x \ast k_2 \ast y)d\omega_K(k_1)d\omega_K(k_2) \right] d\nu(y) \\
&= \int_G \left[ \int_K \int_K \phi(k_1 \ast x \ast k_2 \ast y)d\omega_K(k_2) \right] d\omega_K(k_1)d\mu(x) \right] d\nu(y) \\
&= \int_G \phi^\ast(y) \left[ \int_G \phi^\ast(k_1 \ast x)d\omega_K(k_1) \right] d\mu(x) \right] d\nu(y) \\
&= \int_G \phi^\ast(y) \left[ \int_G \phi^\ast(x)d\mu(x) \right] d\nu(y) \\
&= \int_G \phi^\ast(x)d\mu(x) \int_G \phi^\ast(y)d\nu(y) \\
&= \widehat{\mu}(\phi)\widehat{\nu}(\phi).
\end{align*}
\]

\[ \square \]

**Remark 3.8.** By the definition, the mapping $\mu \mapsto \widehat{\mu}$ from $M_b(G)$ to $C_b(\widehat{G})$ is continuous.

3.3. **Fourier transform on** $G$.

**Definition 3.9.** Let $f \in K^k(G)$, the Fourier transform of $f$ is the map $\widehat{f} : \widehat{G} \rightarrow \mathbb{C}$ defined by:

\[ \widehat{f}(\phi) = \int_G \phi(x)f(x)d\mu_G(x). \]
Proposition 3.10. i) For $f \in \mathcal{K}(G)$, $\hat{f} = \hat{f} \mu_G \in \mathcal{C}_0(\hat{G})$.

ii) If $f \in \mathcal{K}_h(G)$ and $g \in \mathcal{K}(G)$, then $\hat{f} \ast g = \hat{f} g$.

Proof. i) For any $f$ in $\mathcal{K}(G)$, we have

$$
\hat{f}(\phi) = \int_G \phi^-(x) \left( \int_K \int_K f(k_1 \ast x \ast k_2) d\omega_K(k_1) d\omega_K(k_2) \right) d\mu_G(x)
$$

$$
= \int_G f(x) \left( \int_K \int_K \phi^-(k_1 \ast x \ast k_2) d\omega_K(k_1) d\omega_K(k_2) \right) d\mu_G(x)
$$

$$
= \int_G \phi(x) f(x) d\mu_G(x) = \hat{f} \mu_G(\phi) \forall \phi \in \hat{G}
$$

Since $f \mu_G \in M_{\mu_G}(G)$, then $\hat{f} \mu_G \in \mathcal{C}_0(\hat{G})$.

ii) Let $f \in \mathcal{K}_h(G)$ and $g \in \mathcal{K}(G)$. For $\phi \in \hat{G}$, we have

$$
\hat{f} \ast g(\phi) = \int_G \phi^-(x) f \ast g(x) d\mu_G(x)
$$

$$
= \int_G \phi^-(x) \left( \int_G f(x \ast y) g(\varphi) d\mu_G(y) \right) d\mu_G(x)
$$

$$
= \int_G g(\varphi) \left( \int_K \int_K \phi^-(x \ast \varphi)y f(x) d\mu_G(x) d\mu_G(y) \right) d\mu_G(x)
$$

$$
= \int_G g(\varphi) \phi^-(\varphi) d\mu_G(y) \int_K f(x) \int_K \phi^-(k_1 \ast y) d\omega_K(k_1) d\mu_G(x)
$$

$$
= \int_G \phi^-(y) g(y) d\mu_G(y) \int_G \phi^-(x) f(x) d\omega_K(k_1) d\mu_G(x)
$$

$$
= \hat{f}(\phi) \hat{g}(\phi).
$$

We therefore extend the spherical Fourier transform to all $\mathcal{K}(G)$ with $\hat{f} = \hat{f} h$ for any $f \in \mathcal{K}(G)$ and to $L^1(G, \mu_G)$ and $L^2(G, \mu_G)$. We have the following result.

Theorem 3.11. There exists a unique nonnegative measure $\pi$ on $\hat{G}$ such that

$$
\int_G |f(x)|^2 d\mu_G(x) = \int_{\hat{G}} |\hat{f}(\phi)|^2 d\pi(\phi) \text{ for all } f \in L^1(G, \mu_G) \cap L^2(G, \mu_G).
$$

The space $\{ \hat{f} : f \in \mathcal{K}(G) \}$ is dense in $L^2(\hat{G}, \pi)$.

Proof. Considering the space $\hat{G} / \mathcal{K}$ defined by [4], $\hat{\phi} \in \hat{G} / \mathcal{K}$ if and only if $\phi = \hat{\phi} \circ p_K \in \hat{G}$. Let $\hat{\phi}$ belongs to $C_b(\hat{G} / \mathcal{K})$. Let us consider $\varphi : \hat{G} \rightarrow \mathbb{C}$ defined by:

$$
\varphi(\phi) = \overline{\varphi(\hat{\phi})}.
$$
\( \varphi \in C_b(\hat{G}) \) and the mapping
\[
C_b(\hat{G}/\!\!/K) \rightarrow C_b(\hat{G}) \quad \tilde{\varphi} \rightarrow \varphi
\]
is a linear bijection, specifically \( \varphi \in \mathcal{K}(\hat{G}) \iff \tilde{\varphi} \in \mathcal{K}(\hat{G}/\!\!/K) \). By (\ [4\), theorem. 7.3I\), there exist a unique nonnegative measure \( \tilde{\pi} \) on \( \hat{G}/\!\!/K \) such that \( \int_{\hat{G}/\!\!/K} \tilde{\pi}(\tilde{f}) d\tilde{f} = \int_{\hat{G}/\!\!/K} \tilde{f}(KxK) d\pi(KxK) \) for \( \tilde{f} \in L^1(\hat{G}/\!\!/K) \cap L^2(\hat{G}/\!\!/K) \). Let us consider the mapping \( \pi \) defined by \( \pi(\varphi) = \tilde{\pi}(\tilde{\varphi}) \) for \( \varphi \in \mathcal{K}(\hat{G}) \). \( \pi \) is a measure on \( \hat{G} \). Since \( \tilde{\pi} \) is nonnegative, then \( \pi \) is nonnegative. Otherwise, note that \( \tilde{f} = \tilde{\tilde{f}} \) for \( \tilde{f} \in \mathcal{K}^h(\hat{G}) \). Indeed since \( \tilde{f} \in \mathcal{K}^h(\hat{G}) \) then \( \tilde{f} \in \mathcal{K}(\hat{G}/\!\!/K) \) and \( \tilde{f} \in C_b(\hat{G}) \). So \( \tilde{\tilde{f}} \) and \( \tilde{f} \) belong to \( C_b(\hat{G}/\!\!/K) \). For \( \tilde{\varphi} \in \hat{G}/\!\!/K \), we have
\[
\tilde{\tilde{f}} = \tilde{\tilde{f}}(\phi) = \int_{\hat{G}/\!\!/K} \tilde{\phi}(K\phi \!\!\!\!/K) \tilde{f}(KxK) d\pi(KxK)
\]
\[
= \int_{\hat{G}/\!\!/K} \tilde{\phi}^*(KxK) \tilde{f}(KxK) d\pi(KxK)
\]
\[
= \int_G \phi^*(x)f(x) d\mu_G(x)
\]
\[
= \tilde{f}(\phi) = \tilde{f}(\tilde{\varphi})
\]

Let \( \tilde{f} \in \mathcal{K}^h(\hat{G}) \). We have
\[
\int_G \left| \tilde{\tilde{f}}(\phi) \right|^2 d\pi(\phi) = \int_{\hat{G}/\!\!/K} \left| \tilde{\tilde{f}}(\tilde{\phi}) \right|^2 d\tilde{\pi}(\tilde{\phi})
\]
\[
= \int_{\hat{G}/\!\!/K} \left| \tilde{\tilde{f}}(\tilde{\phi}) \right|^2 d\tilde{\pi}(\tilde{\phi})
\]
\[
= \int_{\hat{G}/\!\!/K} \left| \tilde{\tilde{f}}(KxK) \right|^2 d\pi(KxK)
\]
\[
= \int_G \left| f(x) \right|^2 d\mu_G(x).
\]

As \( \tilde{f} = \tilde{\tilde{f}} \) \( \forall f \in \mathcal{K}(G) \) and \( G \) unimodular, we deduce that \( \int_G \left| \tilde{\tilde{f}}(\phi) \right|^2 d\pi(\phi) = \int_G \left| f(x) \right|^2 d\mu_G(x) \) \( \forall f \in \mathcal{K}(G) \). By the continuity of the Fourier transform and by application of the dominated convergence theorem, we conclude that \( \int_G \left| f(x) \right|^2 d\mu_G(x) = \int_G \left| \tilde{\tilde{f}}(\phi) \right|^2 d\pi(\phi) \) for any \( f \) belongs to \( L^1(G, \mu_G) \cap L^2(G, \mu_G) \). Let \( \pi' \) a nonnegative measure on \( \hat{G} \) such that \( \int_G \left| f(x) \right|^2 d\mu_G(x) = \int_G \left| \tilde{\tilde{f}}(\phi) \right|^2 d\pi'(\phi) \) for all \( f \) in \( L^1(G, \mu_G) \cap L^2(G, \mu_G) \). As above but in reverse order \( \pi' \) defines a nonnegative measure \( \tilde{\pi}' \) on \( \hat{G}/\!\!/K \) such that \( \int_{\hat{G}/\!\!/K} \left| \tilde{\tilde{f}}(KxK) \right|^2 d\pi'(KxK) = \int_{\hat{G}/\!\!/K} \left| \tilde{\tilde{f}}(\tilde{\phi}) \right|^2 d\tilde{\pi}'(\tilde{\phi}) \) for \( \tilde{f} \in L^1(\hat{G}/\!\!/K, m) \cap L^2(\hat{G}/\!\!/K, m) \). That is \( \pi' = \tilde{\pi} \) seen the uniqueness of \( \pi \), so \( \pi = \pi' \). Let us put \( \mathcal{F}(\mathcal{K}(G)) = \)
\[ \{ \hat{f} : f \in \mathcal{K}(G) \} \]. Let \( \varphi \in \mathcal{K}(\hat{G}) \) such that \( \langle \hat{f}, \varphi \rangle = \int_{\hat{G}} \hat{f}(\phi) \varphi(\phi) d\pi(\phi) = 0 \forall f \in \mathcal{K}(G) \). We have
\[
\langle \hat{f}, \varphi \rangle = 0 \forall f \in \mathcal{K}(G) \implies \int_{G} \hat{f}(\phi) \varphi(\phi) d\pi(\phi) = 0 \forall f \in \mathcal{K}(G) \\
\implies \int_{\hat{G}} \hat{f}(\phi) \varphi(\phi) d\pi(\phi) = 0 \forall f \in \mathcal{K}(G) \\
\implies \langle \tilde{f}, \tilde{\varphi} \rangle = 0 \forall f \in \mathcal{K}(G) \\
\implies \langle \tilde{f}, \tilde{\varphi} \rangle = 0 \forall \tilde{f} \in \mathcal{K}(G//K) \\
\implies \tilde{\varphi} = 0 \text{ since } \mathcal{F}(\mathcal{K}(G//K)) \text{ is dense in } L^2(\hat{G}/\!\!/K, \tilde{\pi}) \\
\implies \varphi = 0.
\]
So \( (\mathcal{F}(\mathcal{K}(G)))^\perp \cap \mathcal{K}(\hat{G}) = \{0\} \). Since \( \mathcal{K}(\hat{G}) \) is dense in \( L^2(\hat{G}, \pi) \), then \( (\mathcal{F}(\mathcal{K}(G)))^\perp = \{0\} \) and \( \mathcal{F}(\mathcal{K}(G)) \) is dense in \( L^2(\hat{G}, \pi) \).

\[\square\]

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**References**

[1] W.R. Bloom, H. Heyer, Harmonic Analysis of Probability Measures on Hypergroups, de Gruyter Studies in Mathematics 20, Walter de Gruyter, Berlin, 1995.

[2] C.F. Dunkl, The Measure Algebra of a Locally Compact Hypergroup, Trans. Amer. Math. Soc. 179 (1973), 331–348. [https://doi.org/10.1090/s0002-9947-1973-0320635-2](https://doi.org/10.1090/s0002-9947-1973-0320635-2).

[3] B.K. Germain, K. Kinvi, On Gelfand Pair Over Hypergroups, Far East J. Math. 132 (2021), 63–76. [https://doi.org/10.17654/ms132010063](https://doi.org/10.17654/ms132010063).

[4] R.I. Jewett, Spaces With an Abstract Convolution of Measures, Adv. Math. 18 (1975), 1–101. [https://doi.org/10.1016/0001-8708(75)90002-x](https://doi.org/10.1016/0001-8708(75)90002-x).

[5] L. Nachbin, On the Finite Dimensionality of Every Irreducible Unitary Representation of a Compact Group, Proc. Amer. Math. Soc. 12 (1961), 11-12. [https://doi.org/10.1090/s0002-9939-1961-0123197-5](https://doi.org/10.1090/s0002-9939-1961-0123197-5).

[6] K.A. Ross, Centers of Hypergroups, Trans. Amer. Math. Soc. 243 (1978), 251–269. [https://doi.org/10.1090/s0002-9947-1978-0493161-2](https://doi.org/10.1090/s0002-9947-1978-0493161-2).

[7] R. Spector, Mesures Invariantes sur les Hypergroupes, Trans. Amer. Math. Soc. 239 (1978), 147–165. [https://doi.org/10.1090/s0002-9947-1978-0463806-1](https://doi.org/10.1090/s0002-9947-1978-0463806-1).

[8] L. Székelyhidi, Spherical Spectral Synthesis on Hypergroups, Acta Math. Hungar. 163 (2020), 247–275. [https://doi.org/10.1007/s10474-020-01068-9](https://doi.org/10.1007/s10474-020-01068-9).

[9] K. Vati, Gelfand Pairs Over Hypergroup Joins, Acta Math. Hungar. 160 (2019), 101–108. [https://doi.org/10.1007/s10474-019-00946-1](https://doi.org/10.1007/s10474-019-00946-1).