Nonstationary analytic model of hydrodynamic rolling lubrication

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Abstract. A model of normal oscillations of a roller moving along the surface at a constant speed in the presence of a liquid lubricant layer is considered. The pressure distribution along the lubricant layer is obtained by integrating the Reynolds equation, taking into account both the tangential and normal speed of the roller relative to the support surface. An analytical solution of this equation is constructed by the method of asymptotic expansion in a singular small parameter. The solution contains both regular terms of the expansion in powers of a small parameter and boundary-layer functions that rapidly decay over time.

1. Introduction

The stationary problem of the hydrodynamic contact of a roller moving along a solid surface in the presence of a lubricating layer has been considered in many publications [1–3] and has been rather well studied. The first analytical solution to this problem was found by P. Kapitsa, who obtained the pressure distribution in the layer in the absence of deformations of the contact surfaces [4]. At the same time, many nonstationary aspects of the hydrodynamic contact of the roller with the support surface require further research [5], especially with regard to the analytical presentation of solutions. The importance and relevance of this topic is due to its connection with the problem of modeling roller bearings operating under conditions of variable loads. With such work, there are rapid changes in the gaps between the contacting bodies, which lead to a sharp increase in peak values of pressure in the lubricant layer and increased wear of the friction unit.

The aim of the work is to construct an asymptotic analytical solution for the nonstationary problem of the hydrodynamic contact of a moving roller with a rigid curved surface.

2. Calculate the pressure in the lubricant layer

Suppose a rotating roller moves along a surface in the presence of a liquid lubricant. The scheme of contact interaction of a cylindrical roller with a solid surface coated with a layer of lubricant is illustrated in figure 1. Consider an idealized model of the contact of the roller with the surface at a constant viscosity coefficient in the lubricating layer. In this case, the pressure distribution \(P\) in the layer is determined by the Reynolds equation of the following form:
\[
\frac{\partial}{\partial X} \left( \frac{D^3}{12 \mu} \frac{\partial P}{\partial X} \right) = \frac{\partial}{\partial X} \left( \frac{\partial V}{\partial X} \right) + \frac{\partial D}{\partial T} .
\]

(1)

where \(D\) is the thickness of the lubricating layer, \(\mu\) is the dynamic viscosity of the oil, \(P\) is the pressure in the lubricating layer, \(V\) is the half of sum of the tangential velocities of the surfaces in the contact zone, \(T\) is time and \(X\) is the distance along the contact surface, as shown in figure 1. Here a reference system is used, in which the axis of the roller is at rest, and the surface, respectively, moves in the \(X\) direction.

Assuming that the contact area is small compared to the radius of curvature \(R\), we have the following expression for the thickness of the layer of lubricant [2]:

\[
D = D_m + \left( X - X_m \right)^2 / (2R), \quad R = \frac{R R_2}{R_1 - R_2} ,
\]

(2)

where \(D_m\) is the minimum thickness of the lubricant layer, \(X_m\) is the coordinate of the minimum clearance point, \(R_1\) and \(R_2\) are the radii of curvature of the bearing surface and the roller. In this case, the boundary conditions have the following form [2]:

\[
P(X_1) = P(X_2) = \frac{dP}{dX}(X_2) = 0,
\]

(3)

where \(X_1\) and \(X_2\) are the input and output boundaries of the lubricating layer. Hereafter we assume the frame of reference comoving with the roller axis, and thus \(X_m\) is fixed. To solve the Reynolds equation (1), it is convenient to go to dimensionless variables:

\[
x = \left( X - X_m \right) / (D_m R)^{1/2}, \quad H = D / D_m, \quad q = PD_m^{1.5} / \left( 12 \mu VR^{1/2} \right), \quad \nu = \frac{V}{V} \left( \frac{R}{D_m} \right)^{1/2},
\]

(4)

where \(x\) is the normalized coordinate, \(q\) is the dimensionless pressure, \(\nu\) is the dimensionless vertical velocity. Using the normalizations (4), we transform the original equation (1) to a simpler form:

\[
\frac{\partial}{\partial x} \left( H(x)^3 \frac{\partial q}{\partial x} \right) = \frac{\partial H(x)}{\partial x} + \nu, \quad H(x) = 1 + x^2 / 2 .
\]

(5)

The positions of the input and output boundaries are characterized by dimensionless parameters \(a\) and \(c\). The value of the parameter \(a\) depends on the amount of lubricant. In the case of copious lubrication, it is customary to assume \(a = -\infty\) [3]. Integrating equation (5) with regard to the zero...
boundary condition (3) for the derivative of the pressure function at \( x = c \), we obtain a first order differential equation:

\[
\frac{\partial q}{\partial x} = \frac{H(x) - H(c)}{H(x)} + \frac{v(x-c)}{H(x)} = \frac{(x^2 - c^2)}{(1 + x^2 / 2)} + \frac{v(x-c)}{(1 + x^2 / 2)}.
\]  

(6)

The solution of equation (6) depends on the dimensionless parameter \( \upsilon \), proportional to the normal speed of the roller. Integrating equation (6), we find the pressure distribution in the lubricant layer, taking into account the normal speed of the roller:

\[
q(x) = \frac{1}{2} \left[ \arctan \left( \frac{x}{\sqrt{2}} \right) + \frac{\pi}{2} \right] \frac{\sqrt{2}}{4} \left( 1 - \frac{3}{2} c^2 \right) - (2 + c^2) \frac{x}{(2 + x^2)^2} + \left( \frac{x}{2 + x^2} \right) \left( 1 - \frac{3}{2} c^2 \right) \frac{1}{2} \\
- \upsilon \left( \frac{2}{(x^2 + 2)^2} + \frac{1}{16} \left( \frac{c x}{x^2 + 2} \right) + \frac{1}{32} \frac{c x}{x^2 + 2} + \frac{1}{64} c \arctan \left( \frac{1}{8 \sqrt{2}} \right) \right).
\]  

(7)

The value of the parameter \( c \) is determined by the solution of the equation \( q(c) = 0 \).

The found pressure distribution determines the values of the normalized bearing capacity \( W \), which is a function of the parameter \( \upsilon \):

\[
W(\upsilon) = \int_{a}^{c} q(x, \upsilon) dx.
\]  

(8)

Dependence of the carrying capacity on the speed is very close to linear, and thus we use the following approximation:

\[
W = W(0) - A \upsilon,
\]  

(9)

where the constant coefficients \( W(0) \) and \( A \) are 0.4 and 1.125, respectively. Using the normalization coefficients (4), we transform equation (9) to the dimensional form:

\[
w = W(0) \frac{12 \mu V R}{D_m} - A \frac{12 \mu R^{3/2}}{D_m^{3/2}} V_y.
\]  

(10)

Here, the first term expresses the dependence of the stationary load capacity on the gap at zero normal speed, and the second term takes into account the influence of the normal velocity component. The coefficient before the speed is called as damping coefficient \( \lambda \):

\[
\lambda = A \frac{12 \mu R^{3/2}}{D_m^{3/2}}.
\]  

(11)

Taking into account formulas (9-11) we write the equation of motion of the roller along the normal to the surface:

\[
m \frac{d^2 D_m}{dT^2} + A \frac{12 \mu R^{3/2}}{D_m^{3/2}} \frac{dD_m}{dT} - W(0) \frac{12 \mu V R}{D_m} = -F,
\]  

(12)

where \( m \) is the mass of the roller, \( F \) is the external load.

In the equilibrium situation, the time derivative vanishes, and the minimum layer thickness is found as follows:
\[ D_0 = \frac{12 \mu VRW}{F}. \]  

Taking \( D_0 \) as the base gap, we introduce the dimensionless variables:

\[ D = h D_0, \quad T = t T_0, \quad T_0 = \frac{A(RD_0)^{3/2}}{VW(0)}, \]  

Using the normalization (14), we bring the equation of dynamics to a dimensionless form:

\[ \varepsilon \frac{d^2 h}{dt^2} + \frac{1}{h^{3/2}} \frac{dh}{dt} - \frac{1}{h} + 1 = 0, \quad \varepsilon = \frac{mV^2W(0)^2}{A^2FR}. \]  

The characteristic time of the nonstationary process is determined by the parameter \( T_0 \). At the initial moment, we assume the normal component of velocity to vanish, and the relative minimum thickness of the layer exceeds 1. These condition implies a sharp increase in external load at the initial moment with respect to the equilibrium state. Since the parameter \( \varepsilon \) is small, the equation is “stiff” [7], and we search its asymptotic solution as follows:

\[
\begin{align*}
\frac{h}{H_0(t)} + \frac{H_1(t)}{2H_0^{3/2}} &= \frac{dH_0}{dt} + \frac{1}{H_0} + \frac{1}{H_0} + 1 = 0, \\
\frac{1}{H_0^{3/2}} \frac{dH_0}{dt} + \frac{1}{H_0} &= -\frac{d^2 H_0}{dt^2}, \\
\frac{d^2 \Phi_1}{d\tau^2} + \frac{1}{h(0)^{3/2}} \frac{d\Phi_1}{d\tau} &= 0, \\
\frac{d^2 \Phi_2}{d\tau^2} + \frac{1}{h(0)^{3/2}} \frac{d\Phi_2}{d\tau} &= \frac{3}{2} \frac{\Phi_1}{h(0)} \left[ \frac{1}{\sqrt{h(0)}} \frac{d\Phi_1}{d\tau} + \frac{1}{3} \left( 1 - 3h(0) \right) \right].
\end{align*}
\]

Integrating equation (17), we obtain:

\[ H_0(t) = \left[ e^{\sqrt{\frac{h(0)}{h(0) - 1}} + 1} \right] \left[ e^{\sqrt{\frac{h(0)}{h(0) - 1}} - 1} \right]^{-1/2}, \]

Next, we determine the second term of the regular part of the asymptotics:
\[ H_1 = (1 - H_0) \sqrt{H_0} \left( -\frac{2}{3} H_0^{3/2} + \frac{2}{3} h(0)^{3/2} + \sqrt{H_0} - \sqrt{h(0)} - \frac{1}{2} \ln \frac{\sqrt{H_0} + 1}{\sqrt{H_0} - 1} + \frac{1}{2} \ln \frac{\sqrt{h(0)} + 1}{\sqrt{h(0)} - 1} \right). \] (22)

Integrating equation (19) we find
\[
\frac{d\Phi_1}{d\tau} = C_1 \left( \exp - \frac{\tau}{h(0)^{3/2}} \right). \] (23)

The integration constant \( C_1 \) is determined from the condition of zero initial velocity for a complete asymptotic solution:
\[
C_1 = \left( \frac{1}{h(0)} - 1 \right) h(0)^{3/2}. \] (24)

Substituting the found value into expression (23) and integrating, we find the expression
\[
\Phi_1 = -\left( h(0) - 1 \right) h(0)^{3/2} \exp \left( -\frac{\tau}{h(0)^{3/2}} \right) + C_2. \] (25)

Here the constant \( C_2 \) has to be vanished to satisfy zero boundary condition for increasing \( \tau \). The found boundary layer function gives a nonzero perturbation of order \( \varepsilon \) at the initial moment, which can be compensated by choosing the integration constant \( C_0 \) in the expression for the second term of the regular asymptotics.
\[
C_0 = \left( 1 - H_0 \right) \sqrt{H_0} \frac{h(0)^{3/2} (h(0) - 1)}{(1 - h(0))}. \] (26)

As a result, the final expression for \( H_1 \) takes the form:
\[
H_1 = (H_0 - 1) \sqrt{H_0} \left( -\frac{2}{3} H_0^{3/2} + \frac{2}{3} h(0)^{3/2} - \sqrt{H_0} - \sqrt{h(0)} + \frac{1}{2} \ln \frac{\sqrt{H_0} + 1}{\sqrt{H_0} - 1} - \frac{1}{2} \ln \frac{\sqrt{h(0)} + 1}{\sqrt{h(0)} - 1} \right) + \left( 1 - H_0 \right) \sqrt{H_0} \frac{h(0)^{3/2} (h(0) - 1)}{(1 - h(0))}. \] (27)

The found regular part of the asymptotics gives a nonzero initial velocity of order \( \varepsilon \), to compensate for which a second-order boundary layer function is introduced. To determine this function, we integrate equation (20) with the damping condition as \( \tau \) tends to infinity. As a result, we get the expression:
\[
\frac{d\Phi_2}{d\tau} = \exp \left( -\frac{\tau}{h(0)^{3/2}} \right) \left[ C_3 + A_1 \tau + B_1 h(0)^{3/2} - B_1 h(0)^{3/2} \exp \left( -\frac{\tau}{h(0)^{3/2}} \right) \right], \] (28)

where:
\[
A_1 = \left( 1 - h(0) \right) \left( 2 - \frac{3}{2} h(0) \right), \quad B_1 = \left( 1 - h(0) \right) \left( -\frac{3}{2} h(0) \right). \] (29)

The integration constant \( C_3 \) is determined from the condition of zero initial velocity:
$$C_3 = \frac{\left[ h(0)^2 + 2h(0)(h(0) - 1) \right]}{\exp\left(-\frac{2\tau}{h(0)^{3/2}}\right)} - \left[ A \tau + B h(0)^{3/2} - B h(0)^{3/2} \exp\left(-\frac{\tau}{h(0)^{3/2}}\right) \right]. \quad (30)$$

Integrating equation (28), we obtain the expression of the second order boundary layer function:

$$\Phi_2 = -h(0)^{3/2} \left( C_3 + B h(0)^{3/2} \right) \exp\left(-\frac{\tau}{h(0)^{3/2}}\right) - A h(0)^{3/2} \left( 1 + \frac{\tau}{h(0)^{3/2}} \right) \frac{1}{2} B h(0)^{3/2} \exp\left(-\frac{2\tau}{h(0)^{3/2}}\right). \quad (31)$$

The function found completely satisfies the condition of zero initial velocity and gives a very small perturbation of the initial condition of order $\varepsilon^2$. In principle, it can be compensated by taking into account the further regular term of the asymptotics of order 2. After summing up all the functions of the regular and boundary layer parts of the asymptotics of equation (14), we obtain a complete analytical solution. This solution indicates that after any sudden jump in load, characterized by the dimensionless parameter $h(0)$, the minimum layer thickness tends to a new equilibrium value.

Figure 2(a) shows time variation of the vertical velocity of the roller for the initial condition $h(0) = 2$. One can see that increasing parameter $\varepsilon$ leads to a smoother change in the vertical velocity ($\frac{dh}{dt}$). For very small values of $\varepsilon$, the maximum absolute value of the vertical velocity indicates a strong enhancement, which in turn causes a sharp and large pressure jump.

Knowing the time dependence function of the gap, we determine the time variation of the maximum pressure $q_{\text{max}}$, shown in figure 2(b). The nonstationary relaxation process can be characterized by two time intervals. During the first interval, a very rapid pressure increase is observed. The amplitude and duration of the pressure increase is determined by the parameter $\varepsilon$. Taking smaller parameter $\varepsilon$, one can get faster and stronger enhancement of the pressure in the lubricant layer. The second time interval is related to a smooth pressure decrease to the stationary value, corresponding to the equilibrium state of the roller under a constant load. Also figure 2(b) shows how important it is to take into account non-stationary transients in friction units. With a slow (quasistationary) increase in load $F$, the maximum pressure across the layer increases in proportion to $F^{3/2}$. However, after a sudden jump in load during the transition process, the maximum pressure across the layer quickly increases significantly. For example, with a threefold increase in pressure, the maximum pressure may increase by about 17 times at $\varepsilon = 0.005$.

**Figure 2.** (a) - The vertical velocity and maximum pressure as functions of time for the initial condition $h(0) = 2$ for various $\varepsilon$: 1) $\varepsilon = 0.04$; 2) $\varepsilon = 0.02$; 3) $\varepsilon = 0.01$; 4) $\varepsilon = 0.005$; (b) - Dependence of maximum pressure on time.

### 3. Conclusion

A new analytical asymptotic solution of the problem of non-stationary contact interaction of a roller with a solid surface in the presence of a lubricating layer has been constructed. It is shown that the
The process of pressure relaxation after a sharp increase in load is characterized by two time scales. The first one determines a sharp and significant increase in pressure immediately after a load jump. The second reflects the process of gradual relaxation of pressure to an equilibrium distribution, corresponding to an increased load. The results show the importance of taking into account non-stationary transient processes in friction units. For example, in the event of a sudden load jump of three times, the maximum pressure during a transient process may exceed a stationary pressure peak by more than an order of magnitude. Such significant pressure surges arising in transient nonstationary processes can cause premature runout of the friction unit.

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