Quantum algorithms for Josephson networks

Jens Siewert\textsuperscript{1,2} and Rosario Fazio\textsuperscript{1}
\textsuperscript{1} Dipartimento di Metodologie Fisiche e Chimiche (DMFCI), Università di Catania, viale A.Doria 6, I-95125 Catania, Italy
\textsuperscript{2} Istituto Nazionale per la Fisica della Materia (INFM), Unità di Catania, Italy
\textsuperscript{2} Institut für Theoretische Physik, Universität Regensburg, D-93040 Regensburg, Germany

(March 22, 2022)

Abstract

We analyze possible implementations of quantum algorithms in a system of (macroscopic) Josephson charge qubits. System layout and parameters to realize the Deutsch algorithm with up to three qubits are provided. Special attention is paid to the necessity of entangled states in the various implementations. Further, we demonstrate explicitly that the gates to implement the Bernstein-Vazirani algorithm can be realized by using a system of uncoupled qubits.
Quantum information processing has initiated an impressive research activity throughout the scientific community during the past decade. The interest in quantum computation, in particular, is stimulated by the discovery of quantum algorithms \[1,2\] which can outperform their classical counterparts in solving problems of significant practical relevance.

In recent years, considerable progress has been made in the field of “quantum hardware”, *i.e.* in the search for physical systems appropriate for the implementation of quantum computation. It is equally important in the development of quantum computing to experimentally realize complete quantum algorithms. To date, this has been achieved in liquid-state NMR \[3–10\], in atomic physics \[11\] and by optical interferometry \[12\]. A solid-state implementation of Grover’s algorithm has recently been proposed in Ref. \[13\].

The quest for large scale integrability has stimulated an increasing interest in superconducting nanocircuits \[14–18\] as possible candidates for the implementation of a quantum computer. The recent experimental breakthrough for Josephson qubits \[19–21\] is the first important step towards a solid-state quantum computer.

Naturally the question arises whether, at the present technological level, it is possible to implement also quantum algorithms in these systems. Here we concentrate on charge qubits \[14–16\] and show how the Deutsch algorithm \[22–24\] and the Bernstein-Vazirani algorithm \[27\] can be run on a Josephson quantum computer. We analyze the experiment by Nakamura et al. \[15\] in terms of quantum interferometry \[24\] and show that it corresponds to the implementation of the one-qubit version of Deutsch’s algorithm. By generalizing this idea we show how the N-qubit Deutsch algorithm, with \(N \leq 3\), can be implemented. Entanglement is required only for \(N \geq 3\) \[25\]. Finally we show explicitely that the Bernstein-Vazirani algorithm \[27\] can be implemented using uncoupled qubits (for arbitrary \(N\)). Therefore it can be realized by means of the setup of Ref. \[13\].

Consider the subset of Boolean functions \(f : \{0,1\}^N \rightarrow \{0,1\}\) with the property that \(f\) is either constant or balanced (that is, it has an equal number of 0 outputs as 1s). The Deutsch-Jozsa algorithm \[22–24\] determines whether a function \(f\) is constant or balanced. With a classical algorithm, this problem would, in the worst case, require \(2^{N-1} + 1\) evaluations of \(f\) whereas the quantum algorithm solves it with a single evaluation by means of the following steps (here we focus on the refined version by Collins et al. \[24\], see also Fig. 1).

(i) All qubits are prepared in the initial state \(|0\rangle\), therefore the N-qubit register is in the state \(|00\ldots0\rangle\).

(ii) Perform an N-qubit Hadamard transformation \(\mathcal{H}\)

\[
|x\rangle \xrightarrow{\mathcal{H}} \sum_{y \in \{0,1\}^N} (-1)^{x \cdot y} |y\rangle, \quad (x \in \{0,1\}^N), \tag{1}
\]

where \(x \cdot y = (x_1 \wedge y_1) \oplus \ldots \oplus (x_N \wedge y_N)\) is the scalar product modulo two. This is equivalent to performing a one-bit Hadamard transformation to each qubit individually.

(iii) Apply the \(f\)-controlled phase shift \(U_f \[24,25\]

\[
|x\rangle \xrightarrow{U_f} (-1)^{f(x)} |x\rangle, \quad (x \in \{0,1\}^N). \tag{2}
\]

Note that we will use the convention \(f(00\ldots0) = 0\).

(iv) Perform another Hadamard transformation \(\mathcal{H}\).

(v) Measure the final state of the register. If the result is \(|00\ldots0\rangle\) the function \(f\) is
constant; if, however, the amplitude \( a_{|00...0\rangle} \) of the state \(|00...0\rangle\) is zero the function \( f \) is balanced. This is because

\[
a_{|00...0\rangle} = \frac{1}{2^N} \sum_{x \in \{0,1\}^N} (-1)^{f(x)}. \tag{3}
\]

Entanglement is not needed for the one-bit and two-bit case \(^{25}\) while it is necessary for the exponential speedup for a higher number of qubits \(^{26}\).

In order to implement the algorithm we have to show that each individual step (preparation of the state, gate operations, measurement) can be realized. It is well known how to prepare and to measure the states in Josephson charge qubits \(^{14,16,19}\). Our task is to demonstrate that the gate operations corresponding to all possible functions \( f \) can be performed with a single device. An important aspect of our proposal is that the gate operations are represented in a basis of superpositions of charge states.

**One-qubit and two-qubit Deutsch algorithm** - In the one-bit case there is one constant function and one balanced function \( f \) (due to our choice \( f(0) = 0 \), see above). The gate \( U_f \) implementing the constant function is the one-qubit identity operator \( I_1 \). The balanced function can be represented (with respect to the computational basis) by \( \sigma_z \) where \( \sigma_i \) denote the Pauli matrices.

The sequence of steps \( (i)-(v) \) can be carried out with a Josephson qubit. A Josephson charge qubit \(^{14,16,19}\) is a Cooper-pair box (see Fig. 3a) which is characterized by two energy scales, the charging energy \( E_{\text{ch}} = (2e)^2/(2C) \) (here \( C \) is the total capacitance of the island) and the Josephson energy \( E_J \ll E_{\text{ch}} \) of the tunnel junction. At low temperatures \( T \ll E_J \) only the two charges states with 0 and 1 excess Cooper pair on the island are important and the system behaves as a two-level system with the Hilbert space \( \{|0\rangle, |1\rangle\} \) and the one-qubit Hamiltonian

\[
H_{1q} = \left( E_{\text{ch}}/2 \right) (2n_x - 1) \sigma_z - \left( E_J/2 \right) \sigma_x. \tag{4}
\]

Here \( n_x = C_gV_g/(2e) \) is the offset charge which can be controlled by the gate voltage.

The one-bit version of the Deutsch algorithm is already realized in the experiment by Nakamura et al (see Figs. 1 and 2). First the system is prepared in a symmetric superposition of the states. The Rabi oscillation in the experiment corresponds to the action of the controlled phase shift \( U_f \). Finally the system is measured. Note, however, an important difference between the experiment and the steps \( (i) - (v) \): In the usual representation of Deutsch’s algorithm the gate \( U_f \) acts on the same basis states which then are measured; the Hadamard transformation in the experiment produces the appropriate superpositions. In contrast to this, the measurement in the experiment is done in the charge basis while the “gate” acts in the basis \( \{|+,|-\} \) which is related to the charge basis by a Hadamard transformation. By identifying \( U_f \leftrightarrow \exp \left( i(E_J t/2\hbar)\sigma_z \right) \) the different functions \( f \) can be implemented by choosing the time \( t \) appropriately (see Table I).

This observation suggests the possibility to implement the Deutsch algorithm in a setup of more than one charge qubit by performing the same sequence of gate voltage pulses as in the experiment of Ref. \(^{19}\). The gates should operate at the degeneracy point \( n_x^{(j)} = 1/2 \) of the charge qubits. What we need is to find the proper parameters and operation times to obtain all possible gates \( U_f \).
For two qubits this realization is obvious as the two-qubit algorithm can be implemented by using two uncoupled qubits \[25\]. The gates \(\sigma_z^{(1)} \otimes I_1^{(2)}, I_1^{(1)} \otimes \sigma_z^{(2)}\), \(\sigma_z^{(1)} \otimes \sigma_z^{(2)}\) implementing the balanced functions (the upper index denotes the qubit number) and \(I_1^{(1)} \otimes I_1^{(2)}\) for the constant function can be realized in complete analogy to the one-qubit algorithm.

Three-qubit Deutsch algorithm - The realization of the three-qubit version of the algorithm is more difficult. Apart from the constant function 35 balanced functions need to be implemented. Moreover, the three-qubit algorithm involves gates \(U_f\) which produce entangled final states.

The goal is to proceed along the same lines as above, that is, preparation of the state \(|000\rangle\), sudden sweep of \(n_x^{(j)}\) etc. The action of the gates \(U_f\) takes place in the basis \(|+++\rangle, |++-\rangle, \ldots, |---\rangle\}. In order to find efficient ways for the implementation we first analyze the functions \(f\) and the corresponding gates \(U_f\).

Apart from the constant function and its gate \(I_1^{(1)} \otimes I_1^{(2)} \otimes I_1^{(3)}\) there are 7 balanced functions for which the gates are separable: \(\sigma_z^{(1)} \otimes I_1^{(2)} \otimes I_1^{(3)}, I_1^{(1)} \otimes \sigma_z^{(2)} \otimes I_1^{(3)}, \ldots, \sigma_z^{(1)} \otimes \sigma_z^{(2)} \otimes \sigma_z^{(3)}\). Further there are 12 balanced functions for which the gates factorize into a one-qubit part and a two-qubit part as in example I) below. The other gates entangle all three qubits and can be divided into two classes (see example II) and III)). There are 12 gates of class II) and 4 gates of class III).

I) \[\frac{1}{2} \left( I_1^{(1)} \otimes I_1^{(2)} + \sigma_z^{(1)} \otimes I_1^{(2)} - I_1^{(1)} \otimes \sigma_z^{(2)} + \sigma_z^{(1)} \otimes \sigma_z^{(2)} \right) \otimes \sigma_z^{(3)}\]

II) \[\frac{1}{2} \left( \sigma_z^{(1)} \otimes I_1^{(2)} \otimes I_1^{(3)} - I_1^{(1)} \otimes I_1^{(2)} \otimes \sigma_z^{(3)} + \sigma_z^{(1)} \otimes \sigma_z^{(2)} \otimes I_1^{(3)} + I_1^{(1)} \otimes \sigma_z^{(2)} \otimes \sigma_z^{(3)} \right)\]

III) \[\frac{1}{2} \left( \sigma_z^{(1)} \otimes I_1^{(2)} \otimes I_1^{(3)} - I_1^{(1)} \otimes \sigma_z^{(2)} \otimes I_1^{(3)} + I_1^{(1)} \otimes I_1^{(2)} \otimes \sigma_z^{(3)} + \sigma_z^{(1)} \otimes \sigma_z^{(2)} \otimes \sigma_z^{(3)} \right)\]

All separable qubit operations can be carried out in analogy with the one-qubit case above. In the following we discuss how the entangling gate operations can be achieved. For the realization of these gates a coupling of tunable strength between the qubits is required.

There are various ways to couple charge qubits \[11,14,15,18\]. Here we investigate coupling via Josephson junctions \[28\]. Each qubit island is coupled to its nearest neighbor using a symmetric SQUID (see Fig. 3b).

Assuming that both the \(j\)-th qubit and the \(j\)-th coupling junction are tunable by local fluxes \(\Phi^{(j)}, \Phi_K^{(j)}\) the Hamiltonian for the \(N\)-qubit system at the degeneracy point \(n_x^{(j)} = 1/2\) reads

\[H_{Nq} = \sum_{j=1}^{N} \left\{ H_{1q}^{(j)} (\Phi^{(j)}) + E_K^{(j)} \sigma_z^{(j)} \sigma_z^{(j+1)} \right.\]
\[-\left. (1/2)J_K^{(j)} (\Phi^{(j)}) \left[ \sigma_z^{(j)} \sigma_z^{(j+1)} + \text{h.c.} \right] \right\} \]

Here \(J_K^{(j)}\) is the Josephson energy of the \(j\)-th coupling SQUID and \(\sigma_\pm = (\sigma_x \pm i\sigma_y)/2\).

For small coupling capacitance \(C_K^{(j)} \ll C^{(j)}\) we have \(E_K^{(j)} = (C_K^{(j)}/C^{(j)}) E_{ch}^{(j)}/2\). We will assume that \(E_K^{(j)}\) is negligible. Since in practice the capacitive coupling is always present it is necessary to have \(J_K^{(j)} (\Phi = 0) \gg 4E_K^{(j)}\). Then the dynamics of the system approximates the ideal dynamics sufficiently well.

Consider now the first and the second qubit coupled by \(J_K^{(1)}\). By choosing, e.g., \(-E_J^{(1)} = E_J^{(2)} = \pm J_K^{(1)} = J\) and the operation time \(t \approx 0.97(2\pi/J)\) we obtain an operation similar
to a swap gate for the qubits 1 and 2 for which we introduce the notation (in the basis \{ |++\rangle, \ldots, |--\rangle \}_{12})

\[ [\pm 12] := \begin{pmatrix} 0 & 0 & 0 & \pm i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \pm i & 0 & 0 & 0 \end{pmatrix}. \]

(6)

By denoting one-bit phase shifts for the \( j \)-th qubit

\[ [\pm j] := \begin{pmatrix} 1 & 0 \\ 0 & \pm i \end{pmatrix}, \]

(7)

we can write a sequence of operations which gives the two-bit entangling gate in example \( I \) above:

\[ [+1][-2] \rightarrow [-12] \rightarrow \sigma_z^{(3)}. \]

(8)

After suddenly sweeping \( n_x^{(1)} \) and \( n_x^{(2)} \) to the degeneracy, first the one-bit phase shifts are performed while \( J_{K}^{(1)} = 0 \). Then \( J_{K}^{(1)} \) is switched on suddenly in order to do the two-bit rotation. The \( \sigma_z^{(3)} \) rotation can be done at any moment since the third qubit is decoupled from the other two. Finally the \( n_x^{(j)} \) are swept back suddenly and the register is measured.

There are numerous ways to represent the three-bit entangling gates. At least two different two-bit rotations need to be applied. During the second two-bit rotation the third qubit has to be “halted”. This can be done by switching off both the \( E_J \) and the \( J_K \) which couple to this qubit. A possible sequence for example \( II \) is

\[ [+1][-2] \rightarrow [+13] \rightarrow [-12], \]

(9)

and for example \( III \)

\[ \sigma_z^{(1)} \sigma_z^{(2)} \sigma_z^{(3)} \rightarrow [+12] \rightarrow [+23] \rightarrow [+12]. \]

(10)

The complete set of entangling gates can be obtained from the sequences (8) - (10) by cyclic permutations of qubit numbers (and appropriate sign changes), thereby paying attention that the parameter settings are compatible for both one-bit and two-bit operations. We note that universality in quantum computation implies that given some two-bit gate, it is possible to realize any algorithm. This procedure, however, in general requires much longer sequences of one-bit and two-bit gates. Our proposal allows the implementation of simple algorithms following the scheme given in Fig. 1 and is amenable of an experimental verification with present-day technology. It is interesting to note that the completely entangling gates of class \( II \) and \( III \) can be realized approximately with a single three-qubit operation. In Table II we list the parameters for the various implementations including estimates for the accuracy of the respective operation.

**The Bernstein-Vazirani algorithm**\[(27,24)\] - It is analogous to the Deutsch algorithm described in the beginning, with the difference that the function \( f \) has the form \( f = a \cdot x \oplus b \), \((a, x \in \{0, 1\}^N, b \in \{0, 1\}) \). By measuring the register after running the algorithm once
(the gate in step (iii) is denoted by $U_a$) one obtains the number $a$ in binary representation which classically would require $N$ function calls. The fact that there is no entanglement in the Bernstein-Vazirani algorithm has been observed in Ref. [29]. Here we demonstrate explicitely that the algorithm can be obtained by applying only one-qubit operations.

The gate $U_a$ can be rewritten as a product of single-qubit gates (using the definition $(\sigma_k)^0 := I_1$)

$$U_a = (-1)^b \prod_{j=1}^{N} (\sigma_z^{(j)})^{a_j}$$

(11)

where $a_j$ denotes the $j$-th digit of $a$ in binary representation. Apart from the global phase $(-1)^b$ this is the part of the Deutsch algorithm with completely separable gates. As the algorithm starts with a product state, no entanglement is involved at any step. (We note that one can rewrite the action of the whole algorithm $H U_a H$ as $\prod_j (\sigma_z^{(j)})^{a_j}$ which trivially gives the result.) It is therefore possible to realize the Bernstein-Vazirani algorithm with Josephson networks in complete analogy with the implementation for the one-qubit and two-qubit Deutsch algorithm.

ACKNOWLEDGMENTS

The authors would like to thank L. Amico, D.V. Averin, I. Chuang, P. Delsing, G. Falci, J.B. Majer, Y. Makhlin, A. Osterloh, G.M. Palma, F. Plastina, C. Urbina, V. Vedral and C. v.d. Wal for stimulating discussions. This work was supported in part by the EC-TMR, IST-Squbit and INFM-PRA-SSQI.
REFERENCES

[1] P.W. Shor, in *Proc. 35th Ann. Symp. on Foundation of Computer Science*, (IEEE Computer Society, Los Alamos, CA, 1994), p. 124.
[2] L.K. Grover, Phys. Rev. Lett. **79**, 325 (1997).
[3] N.A. Gershenfeld and I.L. Chuang, Science, **275**, 350 (1997).
[4] D.G. Cory *et al.*, Proc. Nat. Acad. Sci. USA **94**, 1634 (1997).
[5] I.L. Chuang *et al.*, Nature **393**, 143 (1998).
[6] J.A. Jones, M. Mosca, and R.H. Hansen, Nature **393**, 344 (1998).
[7] N. Linden, H. Bariat, and R. Freeman, Chem. Phys. Lett. **296**, 61 (1998).
[8] K. Dorai, Arvind, and A. Kumar, Phys. Rev. A **61**, 42306 (2000).
[9] Arvind, K. Dorai, and A. Kumar, LANL e-print quant-ph/9909067 (1999).
[10] D. Collins *et al.*, Phys. Rev. A **62**, 22304 (2000).
[11] J. Ahn, T.C. Weinacht, and P.H. Bucksbaum, Science **287**, 463 (2000).
[12] P.G. Kwiat *et al.*, LANL e-print quant-ph/9905080 (1999).
[13] M.N. Leuenberger and D. Loss, Nature **410**, 789 (2000).
[14] A. Shnirman, G. Schöen and Z. Hermon, Phys. Rev. Lett. **79**, 2371 (1997).
[15] D.A. Averin, Sol. State Comm. **105**, 659 (1998).
[16] Y. Makhlin, G. Schöen and A. Shnirman, Nature **398**, 305 (1999).
[17] J.E. Mooij *et al.*, Science **285**, 1036 (1999).
[18] G. Falci *et al.*, Nature **407**, 355 (2000).
[19] Y. Nakamura, Yu.A. Pashkin, J.S. Tsai, Nature **398**, 786 (1999).
[20] J.R. Friedman *et al.*, Nature **406**, 43 (2000).
[21] C. v.d. Wal *et al.*, Science **290**, 773 (2000).
[22] D. Deutsch, Proc. R. Soc. London A **400**, 97 (1985).
[23] D. Deutsch and R. Jozsa, Proc. R. Soc. London A **439**, 553 (1992).
[24] R. Cleve *et al.*, Proc. R. Soc. London A **454**, 339 (1998).
[25] D. Collins, K.W. Kim, and W.C. Holton, Phys. Rev. A. **58**, R1633 (1998).
[26] H. Azuma, S. Bose, and V. Vedral, LANL e-print quant-ph/0102029 (2001).
[27] E. Bernstein and U. Vazirani, in *Proc. 25th Ann. ACM Symp. on the Theory of Computing*, New York, ACM Press 1993.
[28] J. Siewert *et al.*, J. Low Temp. Phys. **118**, 795 (2000).
[29] D.A. Meyer, LANL e-prints quant-ph/0004092, quant-ph/0007070 (2000).
### TABLES

| \( f \)     | gate \( U_f \) | time \( t \)  |
|------------|----------------|---------------|
| constant   | \( I_1 \)      | \( 2\pi \hbar /E_J \) |
| balanced   | \( \sigma_z \)  | \( \pi \hbar /E_J \) |

**TABLE I.**

| gate | implementation | \( E_j^{(1)} \) | \( E_j^{(2)} \) | \( E_J \) | \( J_K^{(1)} \) | \( J_K^{(2)} \) | \( J_K^{(3)} \) | time \( t/(2\pi \hbar /J) \) | \( a(0|0\rangle)(E_J^{(j)} = 0) \) | \( a(0|0\rangle)(E_J^{(j)} = J/40) \) |
|------|----------------|----------------|----------------|--------|----------------|----------------|----------------|------------------|----------------|------------------|
| II   | sequence (9)   | -J             | J              | J      | -J             | 0              | J              | 0.97 (2bit op.) | 2 \( \cdot 10^{-3} \) | 2 \( \cdot 10^{-4} \) |
| II   | single operation | -J/2           | 0              | J/2    | J              | J              | 0              | 0.80             | 7 \( \cdot 10^{-5} \) | 2 \( \cdot 10^{-2} \) |
| III  | sequence (10)  | J              | -J             | J      | J              | J              | 0              | 0.97 (2bit op.) | 3 \( \cdot 10^{-4} \) | 6 \( \cdot 10^{-3} \) |
| III  | single operation | J/2            | -J/2           | 0.83J  | 0              | J              | J              | 1.19             | < \( 10^{-5} \) | 2 \( \cdot 10^{-3} \) |

**TABLE II.** Parameters for various realizations of the gates \( II \) and \( III \). The coefficient \( a(0|0\rangle) \) is a measure for the fidelity of the operation (for an ideal operation \( a(0|0\rangle) = 0 \)). The operation time for the sequences refers to the time needed for the two-qubit rotations. Single-qubit rotations are assumed to be perfect.
FIG. 1. The sequence of operations to perform the Deutsch algorithm on a register of $N$ qubits. According to Ref. [24] it can be interpreted in terms of quantum interferometry. The first Hadamard transformation produces a superposition of all possible states. Thus, with the application of the $f$-controlled gate $U_f$ the outcome of $f$ for all possible arguments is evaluated at the same time. The second Hadamard transformation brings all computational paths together.

FIG. 2. Schematic representation of Nakamura’s experiment [19]. The qubit is prepared in the ground state $|0\rangle$. After suddenly sweeping the gate voltage the system starts Rabi oscillations between the eigenstates of the new Hamiltonian $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$. After the time $t$ the gate voltage is swept back suddenly which freezes the final state; then the qubit is measured.

FIG. 3. a) A charge qubit. The Josephson energy of the junction can be controlled by the magnetic flux $\Phi$: $E_J(\Phi) = 2E_J \cos(\pi\Phi/\Phi_0)$, where $E_J$ is the Josephson energy of the junctions of the symmetric SQUID and $\Phi_0 = h/(2e)$ [16]. Typical parameters are $E_J \sim 30\mu eV$ and $E_{ch} \sim 500\mu eV$. b) A possible realization of coupled charge qubits.
prepare state $|00...0\rangle$ → $H$ → $U_f$ → $H$ → measure register
