COFINALITY OF NORMAL IDEALS ON $P_\kappa(\lambda)$  

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Abstract
Given an ordinal $\delta \leq \lambda$ and a cardinal $\theta \leq \kappa$, an ideal $J$ on $P_\kappa(\lambda)$ is said to be $[\delta]^{<\theta}$-normal if given $B_e \in J$ for $e \in P_\theta(\delta)$, the set of all $a \in P_\kappa(\lambda)$ such that $a \in B_e$ for some $e \in P_\theta(\delta) \cap a$ lies in $J$. We give necessary and sufficient conditions for the existence of such ideals and describe the least one, denoted by $NS_{\kappa,\lambda}^{[\delta]^{<\theta}}$. We compute the cofinality of $NS_{\kappa,\lambda}^{[\delta]^{<\theta}}$.

0. Introduction
Given a regular uncountable cardinal $\kappa$ and a cardinal $\lambda \geq \kappa$, an ideal on $P_\kappa(\lambda)$ is said to be normal if it is closed under diagonal unions of $\lambda$ many of its members. Building on work of Jech [J] and Menas [Me], Carr [C] described the least such ideal, usually denoted by $NS_{\kappa,\lambda}$. Numerous variations on the original notion of normality have been considered over the years. We are interested in two of these variants. First, there is a notion called ‘strong normality’ which has been rather extensively studied (see e.g. [CP], [F], [M1], [CLP]). The definition involves diagonal unions of length $\lambda^{<\kappa}$. [CLP] gives necessary and sufficient conditions for the existence of strongly normal ideals and describes the least such ideal when there is one. As the terminology implies, every strongly normal ideal is normal. The other notion is that of $\delta$-normality for an ordinal $\delta \leq \lambda$. An ideal on $P_\kappa(\lambda)$ is called $\delta$-normal if it is closed under diagonal unions of length $\delta$. Thus $\lambda$-normality is the same as normality. $\delta$-normality has been studied by Abe [A] who gave a description of the smallest $\delta$-normal ideal on $P_\kappa(\lambda)$.

We introduce a more general concept, that of $[\delta]^{<\theta}$-normality, where $\delta$ is, as above, an ordinal with $\delta \leq \lambda$, and $\theta$ is a cardinal with $\theta \leq \kappa$. The definition is similar to that of strong normality, with this difference that our diagonal unions are indexed by $[\delta]^{<\theta}$. So $[\lambda]^{<\kappa}$-normality is identical with strong normality, whereas $[\delta]^{<2}$-normality is the same as $\delta$-normality.

We give necessary and sufficient conditions for the existence of $[\delta]^{<\theta}$-normal ideals on $P_\kappa(\lambda)$ and describe the least such ideal, which we denote by $NS_{\kappa,\lambda}^{[\delta]^{<\theta}}$.

$[\lambda]^{<\theta}$-normality (for $\theta$ a regular infinite cardinal less than $\kappa$) has been independently considered by Džamonja [D]. In particular, Claim 2.9 and Corollary 2.13 of [D] provide alternative descriptions of $NS_{\kappa,\lambda}^{[\lambda]^{<\theta}}$. 
Given an ideal $J$, its cofinality $\text{cof}(J)$ is its least number of generators, i.e. the least size of any subcollection $X$ of $J$ such that every member of the ideal is included in some element of $X$. We determine the cofinality of $NS^{[\delta]^<\theta}_{\kappa,\lambda}$. Its computation involves a multidimensional version of the dominating number $\mathfrak{d}_\kappa$, which is no surprise, as Landver [Lemma 1.16 in [L]) proved that the cofinality of the minimal normal ideal on $\kappa$ is $\mathfrak{d}_\kappa$.

Part of the paper is concerned with the problem of comparing the various ideals that are considered. Given two pairs $(\delta, \theta)$ and $(\delta', \theta')$, we investigate whether $NS^{[\delta]^<\theta}_{\kappa,\lambda}$ and $NS^{[\delta']^<\theta'}_{\kappa,\lambda}$ are equal, and, more generally, whether one of the two ideals is a restriction of the other (there is more about this in [MPéS]). It is for instance shown that $NS^{[\delta]^<\theta}_{\kappa,\lambda} = NS^{[\delta]^{<\theta}}_{\kappa,\lambda} |A$ for some $A$.

Section 1 collects basic definitions and facts concerning ideals on $P_\kappa(\lambda)$. This is standard material except for Proposition 1.4. In Section 2 we introduce the property of $[\delta]^<\theta$-normality and state necessary and sufficient conditions for the existence of a $[\delta]^<\theta$-normal ideal on $P_\kappa(\lambda)$. The discussion is very much like the one regarding the existence of a strongly normal ideal, and arguments are routine. We briefly consider various weaker properties (compare Proposition 2.3 ((iii) and (iv)), Proposition 2.6 (ii) and Corollary 2.8 (ii) with Proposition 2.5 (ii)) and characterize the ideals that satisfy them.

In Sections 3 and 4 we show that we could without loss of generality assume that $\theta$ is an infinite cardinal and $\delta$ is either a cardinal less than $\kappa$, or a multiple of $\kappa$. We describe the smallest $[\delta]^<\theta$-normal ideal on $P_\kappa(\lambda)$, denoted by $NS^{[\delta]^<\theta}_{\kappa,\lambda}$. Section 5 is concerned with the case that $\theta$ is a limit cardinal. It is proved that if $\delta \geq \kappa$ and $\theta$ is a singular strong limit cardinal, then every $[\delta]^<\theta$-normal ideal on $P_\kappa(\lambda)$ is $[\delta]^<\theta^+$-normal. Sections 6 and 7 deal with the question of the existence of an ordered pair $(\delta', \theta') \neq (\delta, \theta)$ such that $\delta' \leq \delta$, $\theta' \leq \theta$ and $NS^{[\delta]^{<\theta}}_{\kappa,\lambda} = NS^{[\delta']^{<\theta'}}_{\kappa,\lambda} |A$ for some $A$.

In Section 8 we introduce a three-cardinal version, denoted by $\mathfrak{d}^\mu_{\kappa,\lambda}$, of the dominating number $\mathfrak{d}_\kappa$. There are many identities involving the $\mathfrak{d}^\mu_{\kappa,\lambda}$'s, and we present some of them.

The cofinality of $NS^{[\delta]^<\theta}_{\kappa,\lambda}$ is computed in Section 9.
1. Ideals

**Definition.** Given a set $A$ and a cardinal $\tau$, we put $P_\tau(A) = [A]^{<\tau} = \{a \subseteq A : |a| < \tau\}$.

Throughout the section $\rho$ will denote an infinite cardinal and $\mu$ a cardinal with $\mu \geq \rho$.

The section presents some basic material concerning ideals on $P_\rho(\mu)$. Let us start by re-calling some definitions.

**Definition.** We set $\hat{a} = \{b \in P_\rho(\mu) : a \subseteq b\}$ for every $a \in P_\rho(\mu)$.

**Definition.** $I_{\rho,\mu}$ is the collection of all $B \subseteq P_\rho(\mu)$ such that $B \cap \hat{a} = \emptyset$ for some $a \in P_\rho(\mu)$.

**Definition.** By an ideal on $P_\rho(\mu)$, we mean a collection $K$ of subsets of $P_\rho(\mu)$ such that

1. $P(B) \subseteq K$ for all $B \in K$,
2. $\bigcup Y \in K$ for all $Y \subseteq K$ with $0 < |Y| < \text{cf}(\rho)$,
3. $I_{\rho,\mu} \subseteq K$, and
4. $P_\rho(\mu) \notin K$.

**Definition.** Two ideals $I,J$ on $P_\rho(\mu)$ cohere if $I \cup J \subseteq K$ for some ideal $K$ on $P_\rho(\mu)$.

The following is easily verified.

**Proposition 1.1.** $I_{\rho,\mu}$ is an ideal on $P_\rho(\mu)$.

**Definition.** Let $K$ be an ideal on $P_\rho(\mu)$.

We set $K^+ = P(P_\rho(\mu)) - K$ and $K^* = \{B \subseteq P_\rho(\mu) : P_\rho(\mu) - B \in K\}$.

$\text{non}(K)$ is the least cardinality of any $A \subseteq P_\rho(\mu)$ with $A \in K^+$.

$\text{cof}(K)$ is the least cardinality of any $S \subseteq K$ with $K = \bigcup_{B \in S} P(B)$.

The following is well-known.
**Proposition 1.2.** Let \( K \) be an ideal on \( P_\rho(\mu) \). Then \( \text{non}(K) \leq \text{cof}(K) \).

*Proof.* Let \( S \subseteq K \) be such that \( K = \bigcup_{B \in S} P(B) \). Pick \( a_B \in P_\rho(\mu) - B \) for \( B \in S \). Then \( \{a_B : B \in S\} \in K^+ \).

\[ \square \]

**Definition.** We put \( u(\rho, \mu) = \text{non}(I_\rho, \mu) \).

**Proposition 1.3.**

(0) \( \mu \leq u(\rho, \mu) \).

(1) \( \text{cf}(\rho) \leq \text{cf}(u(\rho, \mu)) \).

*Proof.*

(0) : Given \( A \in I^+_\rho, \mu \), we have \( \mu = \bigcup A \) and therefore \( \mu \leq \rho \cdot |A| \). This proves the desired inequality if \( \mu > \rho \). Given \( B \subseteq P_\rho(\mu) \) with \( |B| < \rho \), pick \( \alpha_b \in \rho - b \) for \( b \in B \). Then \( \{\alpha_d : d \in B\} \not\subseteq b \) for all \( b \in B \), and consequently \( B \in I_\rho, \rho \). Hence \( u(\rho, \rho) \geq \rho \).

(1) : Use the fact that \( P_\rho(\mu) \) is closed under unions of less than \( \text{cf}(\rho) \) many of its members. \[ \square \]

The following result will be used in Section 8.

**Proposition 1.4.** Let \( K \) be an ideal on \( P_\rho(\mu) \). Further let \( A \in K^+ \), and set \( \chi = \min\{|A \cap C| : C \in K^*\} \). Assume that \( \text{cof}(K) \leq \chi \). Then \( \chi \) is the largest cardinal \( \tau \) such that there exists a partition of \( A \) into \( \tau \) sets in \( K^+ \).

*Proof.* Pick \( C \in K^* \) with \( |A \cap C| = \chi \), and set \( D = A \cap C \).

Let us first suppose that there exists \( g : A \rightarrow \chi^+ \) such that \( g^{-1}(\{\alpha\}) \in K^+ \) for all \( \alpha \in \chi^+ \). Then \( \{C \cap g^{-1}(\{\alpha\}) : \alpha \in \chi^+\} \) is a partition of \( D \) into \( \chi^+ \) pieces in \( K^+ \), which contradicts the fact that \( |D| = \chi \).
Let us now show that there exists a partition of $A$ into $\chi$ sets in $K^+$. Select a bijection $j : \chi \times \chi \to \chi$, and let $B_\beta$ for $\beta < \chi$ be such that $K = \bigcup_{\beta < \chi} P(B_\beta)$. Define $H_\xi$ for $\xi < \chi$ so that $H_{j(\alpha, \beta)} = B_\beta$ for every $(\alpha, \beta) \in \chi \times \chi$. Notice that given $\beta, \eta < \chi$, there is $\xi \geq \eta$ with $H_\xi = B_\beta$. Now construct $a_\xi^\eta$ for $\eta \leq \xi < \chi$ so that

(0) $\{a_\xi^{\xi'} : \gamma \leq \xi'\} \cap \{a_\xi^\eta : \eta \leq \xi\} = \emptyset$ for $\xi' < \xi < \chi$.

(1) $a_\xi^\eta \neq a_\xi^\eta'$ for $\eta' < \eta \leq \xi < \chi$.

(2) $a_\xi^\eta \in D - (H_\xi \cup B_\eta)$ for $\eta \leq \xi < \chi$.

Set $A_\eta = \{a_\xi^\eta : \eta \leq \xi < \chi\}$ for $\eta < \chi$. Then the following hold:

(i) $|A_\eta| = \chi$.

(ii) $A_\eta \in K^+ \cap P(A)$.

(iii) $A_\eta \cap B_\eta = \emptyset$.

(iv) $A_\eta \cap A_{\eta'} = \emptyset$ for $\eta' < \eta$.

\[ \square \]

**Corollary 1.5.** There exist $A_e \in I_{\rho, \mu}^+ \cap P(\hat{e})$ for $e \in P_\rho(\mu)$ such that (a) $|A_e| = \mu^{< \rho}$ for every $e \in P_\rho(\mu)$, and (b) $A_e \cap A_{e'} = \emptyset$ for all $e, e' \in P_\rho(\mu)$ with $e \neq e'$.

**Proof.** By the proof of Proposition 1.4. \[ \square \]

**Definition.** Given an ideal $K$ on $P_\rho(\mu)$, we put $K|A = \{B \subseteq P_\rho(\mu) : B \cap A \in K\}$ for every $A \in K^+$.

**Proposition 1.6.** Let $K$ be an ideal on $P_\rho(\mu)$, and $A \in K^+$. Then $K|A$ is an ideal on $P_\rho(\mu)$. Moreover, $K \subseteq K|A$ and $cof(K|A) \leq cof(K)$.

**Proof.** Use the fact that for every $B \subseteq P_\rho(\mu)$, $B \in K|A$ if and only if $B \subseteq E \cup (P_\rho(\mu) - A)$ for some $E \in K$. \[ \square \]

We will make use of the following observation.
**Proposition 1.7.** Let $I, J, K$ be three ideals on $P_\rho(\mu)$ such that $I \subseteq J \subseteq K$. Assume that there exists $A \in I^+$ such that $K = I|A$. Then $J|A = I|A$.

**Proof.** Notice that $A \in J^+$ since $A \in K^*$. For each $B \subseteq P_\rho(\mu)$, we have

$$B \in I|A \Rightarrow B \in J|A$$

and

$$B \not\in I|A \Rightarrow B \cap A \not\in I|A \Rightarrow B \not\in K \Rightarrow B \not\in J|A.$$

$\square$

2. $[\delta]^{<\theta}$-normality

Throughout the remainder of the paper $\kappa$ denotes a regular infinite cardinal, $\lambda$ a cardinal with $\lambda \geq \kappa$, $\theta$ a cardinal with $2 \leq \theta \leq \kappa$, and $\delta$ an ordinal with $1 \leq \delta \leq \lambda$.

We set $\overline{\theta} = \theta$ if $\theta < \kappa$, or $\theta = \kappa$ and $\kappa$ is a limit cardinal, and $\overline{\theta} = \nu$ if $\theta = \kappa = \nu^+$.

Throughout the remainder of the paper $J$ denotes a fixed ideal on $P_\kappa(\lambda)$.

In this section we introduce the notion of $[\delta]^{<\theta}$-normal ideal on $P_\kappa(\lambda)$ and describe necessary and sufficient conditions for the existence of such ideals. Let us start with a few definitions.

**Definition.** Given $X_e \subseteq P_\kappa(\lambda)$ for $e \in P_\theta(\delta)$, we let

$$\Delta_{e \in P_\theta(\delta)} X_e = \bigcap_{e \in P_\theta(\delta)} (X_e \cup \{a \in P_\kappa(\lambda) : e \not\in P|a \cap \theta(a)\})$$

and

$$\nabla_{e \in P_\theta(\delta)} X_e = \bigcup_{e \in P_\theta(\delta)} (X_e \cap \{a \in P_\kappa(\lambda) : e \in P|a \cap \theta(a)\}).$$

**Definition.** We define $\nabla^{[\delta]^{<\theta}} J \subseteq P(P_\kappa(\lambda))$ by $B \in \nabla^{[\delta]^{<\theta}} J$ if and only if there are $B_e \in J$ for $e \in P_\theta(\delta)$ such that

$$B \subseteq \{a \in P_\kappa(\lambda) : a \cap \theta = \emptyset\} \cup \left( \bigcup_{e \in P_\theta(\delta)} B_e \right).$$
Lemma 2.1.

(0) \( J \subseteq \nabla^{[\delta] < \theta} J \).

(1) \( \cup Y \in \nabla^{[\delta] < \theta} J \) for all \( Y \in P_\kappa(J) - \{\emptyset\} \).

(2) Assume that \( \delta' \) is an ordinal with \( \delta \leq \delta' \leq \lambda \), \( \theta' \) is a cardinal with \( \theta \leq \theta' \leq \kappa \), and 
\( J' \) is an ideal on \( P_\kappa(\lambda) \) with \( J \subseteq J' \). Then \( \nabla^{[\delta] < \theta} J \subseteq \nabla^{[\delta'] < \theta'} J' \).

Proof.

(0) : It suffices to observe that \( B \subseteq \{ a \in P_\kappa(\lambda) : a \cap \theta = \emptyset \} \cup ( \bigcup_{e \in P_\theta(\delta)} \nabla^{X_e^{\alpha}} B ) \) for every \( B \in J \).

(1) : Use the fact that if \( X_e^{\alpha} \subseteq P_\kappa(\lambda) \) for \( e \in P_\theta(\delta) \) and \( \alpha < \rho \), and \( \rho \) is a cardinal with \( \rho > 0 \), then \( \bigcup_{\alpha < \rho \in P_\theta(\delta)} \nabla^{X_e^{\alpha}} = \nabla^{\bigcup_{\alpha < \rho} X_e^{\alpha}} = \nabla^{\bigcup_{\alpha < \rho} X_e^{\alpha}} \).

(2) : Use (0),(1) and the fact that \( \nabla^{X_e^{\alpha}} B_e \in \nabla^{[\delta'] < \theta'} J' \) whenever \( B_e \in J \) for \( e \in P_\theta(\delta) \).

\( \square \)

Proposition 2.2.

(0) \( \nabla^{[\delta] < \theta} J = \nabla^{[\delta] < \theta} J \).

(1) If \( |P_\theta(\delta)| < \kappa \), then \( J = \nabla^{[\delta] < \theta} J \).

Proof.

(0) : Assume that \( \theta = \kappa = \nu^+ \). Then clearly, \( P(\widehat{\nu}) \cap \nabla^{[\delta] < \kappa} J = P(\widehat{\nu}) \cap \nabla^{[\delta] < \kappa} J \). Hence 
\( \nabla^{[\delta] < \kappa} J = \nabla^{[\delta] < \kappa} J \) by Lemma 2.1 ((0) and (1)).

(1) : Use Lemma 2.1 (0).

\( \square \)

Definition. Given \( A \subseteq P_\kappa(\lambda) \), \( f : A \longrightarrow P_\theta(\delta) \) is \( P_\theta(\delta) \)-regressive if \( f(a) \in P_{|a \cap \theta|}(a) \) for all \( a \in A \) with \( a \cap \theta \neq \emptyset \).
PROPOSITION 2.3. The following are equivalent:

(i) $P_\kappa(\lambda) \not\in \nabla^{[\delta]}<\theta J$.

(ii) $\nabla^{[\delta]}<\theta J$ is an ideal on $P_\kappa(\lambda)$.

(iii) $\sum_{e \in P_\theta(\delta)} C_e \in J^+$ whenever $C_e \in J^*$ for $e \in P_\theta(\delta)$.

(iv) $\sum_{e \in P_\theta(\delta)} C_e \in I_{\kappa,\lambda}^+$ whenever $C_e \in J^*$ for $e \in P_\theta(\delta)$.

(v) For every $P_\theta(\delta)$-regressive $f : P_\kappa(\lambda) \to P_\theta(\delta)$, there is $D \in J^+$ such that $f$ is constant on $D$.

Proof.

(i) $\to$ (ii) : By Lemma 2.1 ((0) and (1)).

(ii) $\to$ (iii) : Use Lemma 2.1 (0) and the fact that $\sum_{e \in P_\theta(\delta)} C_e \in P_\kappa(\lambda) - \nabla e \in P_\theta(\delta)$ whenever $C_e \subseteq P_\kappa(\lambda)$ for $e \in P_\theta(\delta)$.

(iii) $\to$ (iv) : Trivial.

(iv) $\to$ (v) : Use the fact that $\sum_{e \in P_\theta(\delta)} (P_\kappa(\lambda) - f^{-1}\{e\}) \in I_{\kappa,\lambda}$ for every $P_\theta(\delta)$-regressive $f : P_\kappa(\lambda) \to P_\theta(\delta)$.

(v) $\to$ (i) : Assume that there are $B_e \in J$ for $e \in P_\theta(\delta)$ such that

$\{a \in P_\kappa(\lambda) : a \cap \theta \neq \emptyset\} \subseteq \sum_{e \in P_\theta(\delta)} B_e$.

Then there is a $P_\theta(\delta)$-regressive $f : P_\kappa(\lambda) \to P_\theta(\delta)$ with the property that $a \in B_{f(a)}$ for all $a \in P_\kappa(\lambda)$ with $a \cap \theta \neq \emptyset$. Clearly, $f^{-1}\{e\} \in J$ for every $e \in P_\theta(\delta)$.

□

Definition. $J$ is $[\delta]^{<\theta}$-normal if $J = \nabla^{[\delta]}<\theta J$. 
**Proposition 2.4.** Let $\delta'$ be an ordinal with $1 \leq \delta' \leq \delta$, and $\theta'$ a cardinal with $2 \leq \theta' \leq \theta$. Then every $[\delta]^{<\theta}$-normal ideal on $P_\kappa(\lambda)$ is $[\delta']^{<\theta'}$-normal.

*Proof.* By Lemma 2.1(2). \hfill \Box

**Proposition 2.5.** The following are equivalent:

(i) $J$ is $[\delta]^{<\theta}$-normal.

(ii) $\Delta_{e \in P_\theta(\delta)} C_e \subseteq J^*$ whenever $C_e \subseteq J^*$ for $e \in P_\theta(\delta)$.

(iii) $P_\kappa(\lambda) \not\subseteq \nabla^{[\delta]^{<\theta}} (J|A)$ for $A \in J^+$.

(iv) Given $A \subseteq J^+$ and a $P_\theta(\delta)$-regressive $f : A \rightarrow P_\theta(\delta)$, there is $D \subseteq J^+ \cap P(A)$ such that $f$ is constant on $D$.

*Proof.*

(i) $\iff$ (ii): Use Lemma 2.1 (0).

(iii) $\iff$ (iv): By Proposition 2.3 ((i)$\iff$(v)) and Lemma 2.1 (0).

(iii) $\rightarrow$ (ii): Use Proposition 2.3 ((i)$\rightarrow$(iii)).

(ii) $\rightarrow$ (iii): Use Proposition 2.3 ((iv)$\rightarrow$(i)) and the fact that

$$A \cap \Delta_{e \in P_\theta(\delta)} X_e = A \cap \Delta_{e \in P_\theta(\delta)} ((P_\kappa(\lambda) - A) \cup X_e)$$

whenever $A \subseteq P_\kappa(\lambda)$ and $X_e \subseteq P_\kappa(\lambda)$ for $e \in P_\theta(\delta)$. \hfill \Box

Proposition 2.6 ((i)$\iff$(iii)) shows that the $[\delta]^{<\theta}$-normality of $J$ can be seen as a global property which corresponds to the local property “$P_\kappa(\lambda) \not\subseteq \nabla^{[\delta]^{<\theta}} J$”. Let us next briefly consider a weaker (see Corollary 2.7) local property. The corresponding global property will be dealt with in Corollary 2.8.
**Proposition 2.6.** Assume $P_\kappa(\lambda) \not\subseteq \nabla^{[\delta]<\theta} I_{\kappa,\lambda}$. Then the following are equivalent:

(i) $J$ and $\nabla^{[\delta]<\theta} I_{\kappa,\lambda}$ cohere.

(ii) $\bigtriangleup e \in P_\theta(\delta)$ whenever $C_e \in I^+_{\kappa,\lambda}$ for $e \in P_\theta(\delta)$.

(iii) Given $A \in J^*$ and a $P_\theta(\delta)$-regressive $f : A \to P_\theta(\delta)$, there is $D \in I^+_{\kappa,\lambda} \cap P(A)$ such that $f$ is constant on $D$.

**Proof.**

(i) $\rightarrow$ (ii) : Straightforward.

(ii) $\rightarrow$ (iii) : Let $A \in J^*$ and $f : A \to P_\theta(\delta)$ with the property that $f^{-1}(\{e\}) \in I_{\kappa,\lambda}$ for $e \in P_\theta(\delta)$. Then $f(a) \not\subseteq P_{|a\cap\theta|}(a)$ for all $a \in A \cap \bigtriangleup e \in P_\theta(\delta) (P_\kappa(\lambda) - f^{-1}(\{e\}))$.

(iii) $\rightarrow$ (i) : Assume that (iii) holds. Given $B_e \in I_{\kappa,\lambda}$ for $e \in P_\theta(\delta)$, define $f : \nabla e B_e \to P_\theta(\delta)$ so that for every $a \in \nabla e B_e$, $f(a) \in P_{|a\cap\theta|}(a)$ and $a \in B_f(a)$. Then $f$ is $P_\theta(\delta)$-regressive. Moreover, $f^{-1}(\{e\}) \in I_{\kappa,\lambda}$ for every $e \in P_\theta(\delta)$.

It follows that $\nabla e B_e \not\subseteq J^*$. Hence, setting $K = \{B \cup E : B \in J$ and $E \in \nabla^{[\delta]<\theta} I_{\kappa,\lambda}\}$, we have that $K$ is an ideal on $P_\kappa(\lambda)$ with $J \cup (\nabla^{[\delta]<\theta} I_{\kappa,\lambda}) \subseteq K$. □

**Corollary 2.7.** If $P_\kappa(\lambda) \not\subseteq \nabla^{[\delta]<\theta} J$, then $J$ and $\nabla^{[\delta]<\theta} I_{\kappa,\lambda}$ cohere.

**Proof.** By Lemma 2.1 (2), Proposition 2.3 ((i) $\rightarrow$ (iii)) and Proposition 2.6 ((ii) $\rightarrow$ (i)). □

**Corollary 2.8.** Assume $P_\kappa(\lambda) \not\subseteq \nabla^{[\delta]<\theta} I_{\kappa,\lambda}$. Then the following are equivalent:

(i) $J \cap A$ and $\nabla^{[\delta]<\theta} I_{\kappa,\lambda}$ cohere for every $A \in J^+$.

(ii) $\bigtriangleup e \in P_\theta(\delta)$ whenever $C_e \in I^+_{\kappa,\lambda}$ for $e \in P_\theta(\delta)$.

(iii) Given $A \in J^+$ and a $P_\theta(\delta)$-regressive $f : A \to P_\theta(\delta)$, there is $D \in I^+_{\kappa,\lambda} \cap P(A)$ such that $f$ is constant on $D$.

(iv) $\nabla^{[\delta]<\theta} I_{\kappa,\lambda} \subseteq J$.
We will now show that \([\delta]^{<2}\text{-}normality\) is the same as \(\delta\text{-}normality\) (which was studied by Abe in [A]). Let us first recall the following definitions.

**Definition.** Given \(X_\alpha \subseteq P_\kappa(\lambda)\) for \(\alpha < \delta\), we set
\[
\Delta_{\alpha<\delta} X_\alpha = \bigcap_{\alpha<\delta} (X_\alpha \cup (P_\kappa(\lambda) - \{\alpha\}))
\]
and
\[
\nabla_{\alpha<\delta} X_\alpha = \bigcup_{\alpha<\delta} (X_\alpha \cap \{\alpha\}).
\]

**Definition.** Given \(K \subseteq P(P_\kappa(\lambda))\), we define \(\nabla^\delta K \subseteq P(P_\kappa(\lambda))\) by: \(B \in \nabla^\delta K\) if and only if there are \(B_\alpha \in K\) for \(\alpha < \delta\) such that \(B \subseteq (P_\kappa(\lambda) - \{0\}) \cup \nabla_{\alpha<\delta} B_\alpha\).

**Definition.** \(J\) is \(\delta\)-normal if \(J = \nabla^\delta J\).

**Proposition 2.9.** \(J\) is \(\delta\)-normal if and only if \(J\) is \([\delta]^{<2}\)-normal.

**Proof.** The result easily follows from the following two remarks:

1) Let \(X_\alpha \subseteq P_\kappa(\lambda)\) for \(\alpha < \delta\). Define \(Y_e\) for \(e \in P_2(\delta)\) by: \(Y_{\{\alpha\}} = X_\alpha\) for \(\alpha \in \delta\), and \(Y_\emptyset = \emptyset\). Then \((P_\kappa(\lambda) - \hat{2}) \cup \nabla_{\alpha<\delta} X_\alpha = (P_\kappa(\lambda) - \hat{2}) \cup \nabla_{e \in P_2(\delta)} Y_e\).

2) Let \(X_e \subseteq P_\kappa(\lambda)\) for \(e \in P_2(\delta)\). Define \(Y_\alpha\) for \(\alpha < \delta\) by \(Y_\alpha = X_{\{\alpha\}}\). Then \((P_\kappa(\lambda) - \hat{2}) \cup X_\emptyset \cup \nabla_{\alpha<\delta} Y_\alpha = (P_\kappa(\lambda) - \hat{2}) \cup \nabla_{e \in P_2(\delta)} X_e\).

We finally turn to the question of existence of \([\delta]^{<\theta}\)-normal ideals. Let us first deal with the degenerate case \(\kappa = \omega\).

**Proposition 2.10.** Assume \(\kappa = \omega\). Then there exists a \([\delta]^{<\theta}\)-normal ideal on \(P_\kappa(\lambda)\) if and only if \(\delta < \omega\).
Proof. The right-to-left implication is immediate from Proposition 2.2 (1). For the reverse implication, observe that $P_\omega(\lambda) = (P_\omega(\lambda) - \hat{2}) \cup \nabla \setminus B_e$, where $B_\emptyset = \emptyset$ and $\mathcal{B}_n = \{ a \in P_\omega(\lambda) : a \cap \omega = n \}$ for $n \in \omega$. Hence $P_\omega(\lambda) \in \nabla^{[\omega] \prec 2} I_{\omega, \lambda}$ by Lemma 2.1 ((0) and (1)). If $\delta \geq \omega$, then $P_\omega(\lambda) \in \nabla^{[\delta] \prec \theta} I_{\delta, \lambda}$ by Lemma 2.1 (2), and therefore $J$ is not $[\delta]^{<\theta}$-normal.

We will now look for sufficient conditions for the existence of $[\delta]^{<\theta}$-normal ideals on $P_\kappa(\lambda)$ in the case $\kappa > \omega$. We will use the following key lemma.

**Lemma 2.11.**

(0) Assume $\theta \cdot \aleph_0 < \kappa$ and $|P_\theta(\mu)| < \kappa$ for every cardinal $\mu < \kappa$. Then $P_\kappa(\lambda) \not\in \nabla^{[\lambda]^{<\theta}} I_{\kappa, \lambda}$.

(1) ([M1]) Assume that $\kappa$ is Mahlo. Then $P_\kappa(\lambda) \not\in \nabla^{[\lambda]^{<\kappa}} I_{\kappa, \lambda}$.

Proof.

(0) : Let $b_e \in P_\kappa(\lambda)$ for $e \in P_\theta(\lambda)$, and fix $a \in P_\kappa(\lambda)$. Set $\rho = \theta \cdot \aleph_0$ if $\theta \cdot \aleph_0$ is regular, and $\rho = (\theta \cdot \aleph_0)^+$ otherwise. Now define $x_\alpha \in P_\kappa(\lambda)$ for $\alpha < \rho$ so that

(i) $x_0 = a \cup \theta$.

(ii) If $\alpha > 0$, then $\bigcup_{\beta < \alpha} x_\beta \subseteq x_\alpha$ and $x_\alpha \subseteq \bigcap \{ \widetilde{b}_e : e \in P_\theta \bigcup_{\beta < \alpha} x_\beta \}$.

Set $x = \bigcup_{\alpha < \rho} x_\alpha$. Given $e \in P_{|x \cap \theta|}(\lambda)$, there is $\beta < \rho$ with $e \in P_\theta(x_\beta)$. Then $b_e \subseteq x_{\beta+1} \subseteq x$.

Thus $\widetilde{a} \cap \bigcup_{e \in P_\theta(\lambda)} \widetilde{b}_e \neq \emptyset$. Hence $P_\kappa(\lambda) \not\in \nabla^{[\lambda]^{<\theta}} I_{\kappa, \lambda}$ by Proposition 2.3 ((iv) $\rightarrow$ (i)).

(1) : Let $b_e \in P_\kappa(\lambda)$ for $e \in P_\kappa(\lambda)$, and fix $a \in P_\kappa(\lambda)$. Define $x_\alpha \in P_\kappa(\lambda)$ and $\gamma_\alpha \in \kappa$ for $\alpha < \kappa$ so that

(i) $\gamma_\alpha = \cup(x_\alpha \cap \kappa)$.

(ii) $x_0 = a$.

(iii) $x_{\alpha+1} \subseteq \widetilde{x_\alpha} \cap \{ \widetilde{\gamma_\alpha + 1} \} \cap \bigcap_{e \subseteq x_\alpha} \widetilde{b}_e$. 

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(iv) $x_\alpha = \bigcup_{\beta < \alpha} x_\beta$ if $\alpha$ is an infinite limit ordinal $> 0$.

There is a regular infinite cardinal $\tau$ such that $\gamma_\tau = \tau$. Then $x_\tau \in \hat{\alpha} \cap \Delta \sum_{e \in P_\kappa(\lambda)} \hat{b}_e$. Hence $P_\kappa(\lambda) \notin \nabla^\kappa I_{\kappa,\lambda}$ by Proposition 2.3 ((iv) $\rightarrow$ (i)).

Definition. For $f : P_\theta(\delta) \rightarrow P_\kappa(\lambda)$, $C^\kappa_{f,\lambda}$ denotes the set of all $a \in P_\kappa(\lambda)$ such that $a \cap \theta \neq \emptyset$ and $f(e) \subseteq a$ for every $e \in P_{|a \cap \delta|}(a \cap \delta)$.

The following is straightforward.

**Lemma 2.12.** Given $B \subseteq P_\kappa(\lambda)$, $B \in \nabla^{[\delta, \theta]} I_{\kappa,\lambda}$ if and only if $B \cap C^\kappa_{f,\lambda} = \emptyset$ for some $f : P_\theta(\delta) \rightarrow P_\kappa(\lambda)$.

**Lemma 2.13.** Assume that $\delta \geq \kappa$ and either $\theta = \kappa$ and $\kappa$ is Mahlo, or $3 \leq \theta$, $\theta \cdot \aleph_0 < \kappa$ and $|P_\theta(\mu)| < \kappa$ for every cardinal $\mu < \kappa$. Then $\nabla^{[\delta, \theta]} I_{\kappa,\lambda}$ is a $[\delta, \theta]$-normal ideal on $P_\kappa(\lambda)$.

**Proof.** $\nabla^{[\delta, \theta]} I_{\kappa,\lambda}$ is an ideal on $P_\kappa(\lambda)$ by Lemma 2.11, Lemma 2.1 (2) and Proposition 2.3 ((i) $\rightarrow$ (ii)).

Assume $\theta \geq \omega$. Given $g_b : P_\theta(\delta) \rightarrow P_\kappa(\lambda)$ for $b \in P_\theta(\delta)$, define $f : P_\theta(\delta) \rightarrow P_\kappa(\lambda)$ by

$$f(e) = \bigcup_{b, c \in P_\theta(\delta)} g_b(c).$$

Then $\hat{\omega} \cap C^\kappa_{f,\lambda} \subseteq \Delta \sum_{b \in P_\theta(\delta)} C^\kappa_{g_b,\lambda}$. Hence $\nabla^{[\delta, \theta]} I_{\kappa,\lambda}$ is $[\delta, \theta]$-normal by Lemma 2.12.

Now assume $3 \leq \theta < \omega$. Select a bijection $j : P_\theta(\delta) \rightarrow P_\delta(\delta)$. Given $g_b : P_\theta(\delta) \rightarrow P_\kappa(\lambda)$ for $b \in P_\theta(\delta)$, define $f : P_\theta(\delta) \rightarrow P_\kappa(\lambda)$ by

$$f(e) = \bigcup \{g_b(c) : b, c \in P_\theta(\delta) \text{ and } j(b) \cup j(c) \subseteq e\}.$$

Then $\hat{\delta} \cap C^\kappa_{j,\lambda} \cap C^\kappa_{f,\lambda} \subseteq \Delta \sum_{b \in P_\theta(\delta)} C^\kappa_{g_b,\lambda}$. Hence $\nabla^{[\delta, \theta]} I_{\kappa,\lambda}$ is $[\delta, \theta]$-normal by Lemma 2.12. □

**Lemma 2.14.** Assume $J$ is $[\delta, \theta]$-normal. Then $J$ is $[\delta, \theta]$-normal.
Proof. If $\overline{\theta} \geq 3$, $J = \nabla^{[\delta]} < \overline{\theta} J = \nabla^{[\delta]} < \theta J$ by Proposition 2.2 (0). If $\overline{\theta} < 3$, $J \subseteq \nabla^{[\delta]} < \theta J \subseteq \nabla^{[\delta]} < 3 J \subseteq J$ by Lemma 2.1 ((0) and (2)). □

It remains to show that our sufficient conditions are also necessary ones.

**Lemma 2.15.** Assume $P_\kappa(\lambda) \not\in \nabla^{[\delta]} < \theta I_{\kappa,\lambda}$, and let $\mu, \tau$ be two cardinals such that $\mu < \kappa \cap (\delta + 1)$ and $0 < \tau < \theta^+ \cap \kappa$. Then $|P_\tau(\mu)| < \kappa$.

Proof. Suppose otherwise, and pick a one-to-one $j : \kappa \rightarrow P_\tau(\mu)$. Define $f : \overline{\mu \cup \tau} \rightarrow P_\tau(\mu)$ by $f(a) = j(\cup(a \cap \kappa))$. Then $f$ is $P_\theta(\delta)$-regressive, which contradicts Proposition 2.3 ((i) $\rightarrow$ (v)). □

**Lemma 2.16.**

(0) Assume that $\delta \geq \kappa > \omega$ and $\delta$ is a limit ordinal. Then

$$\{a \in P_\kappa(\lambda) : \cup(a \cap \delta) \text{ is a limit ordinal and } \cup(a \cap \delta) \not\in a \cap \delta \} \in (\nabla^{[\delta]} < 2 I_{\kappa,\lambda})^*.$$  

(1) Assume $\delta \geq \kappa > \omega$. Then the set of all $a \in P_\kappa(\lambda)$ such that $cf(\cup(a \cap \eta)) < |a \cap \overline{\theta}|$ for some limit ordinal $\eta$ with $\kappa \leq \eta \leq \delta$ and $cf(\eta) \geq \overline{\theta}$ lies in $\nabla^{[\delta]} < \delta I_{\kappa,\lambda}$.

(2) Assume $\kappa > \omega$, and let $C$ be a closed unbounded subset of $\kappa$. Then

$$\{a \in P_\kappa(\lambda) : a \cap \kappa \in C\} \in (\nabla^{[\kappa]} < 2 I_{\kappa,\lambda})^*.$$  

Proof. Use Lemma 2.11 (0) and Proposition 2.3. □

**Lemma 2.17.** Assume that $\kappa$ is an uncountable limit cardinal and $P_\kappa(\lambda) \not\in \nabla^{[\kappa]} < \kappa I_{\kappa,\lambda}$. Then $\kappa$ is Mahlo.

Proof. By Lemma 2.1 (3) and Lemma 2.16. □
Our study of the case \( \kappa > \omega \) culminates in the following

**Proposition 2.18.**

(0) Assume that \( \kappa > \omega \). Further assume that \( \delta < \kappa \), or \( \theta < \kappa \), or \( \kappa \) is not a limit cardinal. Then there exists a \([\delta]^{<\theta}_\kappa\) normal ideal on \( P_\kappa(\lambda) \) if and only if \( |P_\theta(\mu)| < \kappa \) for every cardinal \( \mu < \kappa \cap (\delta + 1) \).

(1) Assume that \( \delta \geq \kappa > \omega \), \( \theta = \kappa \) and \( \kappa \) is a limit cardinal. Then there exists a \([\delta]^{<\theta}_\kappa\) normal ideal on \( P_\kappa(\lambda) \) if and only if \( \kappa \) is Mahlo.

*Proof.*

(0) : Let us first assume that there exists a \([\delta]^{<\theta}_\kappa\) normal ideal on \( P_\kappa(\lambda) \). Then \( P_\kappa(\lambda) \notin \nabla[\delta]^{<\theta}_\kappa I_{\kappa,\lambda} \) by Lemma 2.1 (2). Notice that if \( \delta < \kappa \), \( \theta = \kappa \) and \( \kappa \) is a limit cardinal, then setting \( \tau = |\delta|^+ \), we have that \( \tau < \theta^+ \cap \kappa \) and \( P_\theta(|\delta|) = P_\tau(|\delta|) \). Hence by Lemma 2.15, \( |P_\theta(\mu)| < \kappa \) for every cardinal \( \mu < \kappa \cap (\delta + 1) \).

Conversely, assume that \( |P_\theta(\mu)| < \kappa \) for every cardinal \( \mu < \kappa \cap (\delta + 1) \). If \( \delta < \kappa \), then \( |P_{\theta^+}(\delta)| < \kappa \), and therefore \( I_{\kappa,\lambda} \) is \([\delta]^{<\theta^+}_\kappa\) normal by Proposition 2.2 (1). If \( \delta \geq \kappa \), then \( \theta < \kappa \), and consequently \( \nabla[\delta]^{<\theta^+}_\kappa I_{\kappa,\lambda} \) is a \([\delta]^{<\theta^+}_\kappa\) normal ideal on \( P_\kappa(\lambda) \) by Lemma 2.13. Thus by Lemma 2.14 there exists a \([\delta]^{<\theta}_\kappa\) normal ideal on \( P_\kappa(\lambda) \).

(1) : If \( \kappa \) is Mahlo, then \( \nabla[\delta]^{<\theta}_\kappa I_{\kappa,\lambda} \) is a \([\delta]^{<\theta}_\kappa\) normal ideal on \( P_\kappa(\lambda) \) by Lemma 2.13. Conversely, if there exists a \([\delta]^{<\theta}_\kappa\) normal ideal on \( P_\kappa(\lambda) \), then \( \kappa \) is Mahlo by Lemma 2.1 (2) and Lemma 2.17. \( \square \)

**Corollary 2.19.** There exists a \([\delta]^{<\theta}_\kappa\) normal ideal on \( P_\kappa(\lambda) \) if and only if there exists a \([\delta \cap \kappa]^{<\theta^+}_\kappa\) normal ideal on \( P_\kappa(\kappa) \).

*Proof.* By Propositions 2.10 and 2.18. \( \square \)
Corollary 2.20. Assume that $\delta < \kappa$ and there exists a $[\delta]^\theta$ - normal ideal on $P_\kappa(\lambda)$.

Then every ideal on $P_\kappa(\lambda)$ is $[\delta]^\theta$ - normal.

Proof. By Propositions 2.2 and 2.18(0).

The following (see e.g. [EHMaR]) is due independently to Hajnal and Shelah.

Lemma 2.21. Let $\mu$ be an infinite cardinal. Then $\mu^\rho$ assumes only finitely many values for $\rho$ with $2^\rho < \mu$.

Lemma 2.22. Let $\mu, \chi$ be two infinite cardinals such that $2^{<\chi} \leq \mu$. Then $(\mu^{<\chi})^{<\chi} = \mu^{<\chi}$.

Proof. If there exists a cardinal $\tau < \chi$ such that $2^\tau = \mu$, then $\mu^{<\chi} = (2^\tau)^{<\chi} = 2^{<\chi} = \mu$.

Otherwise, there exists by Lemma 2.21 a cardinal $\rho < \chi$ such that $\mu^{<\chi} = \mu^\rho$. Then $(\mu^{<\chi})^{<\chi} = (\mu^\rho)^{<\chi} = \mu^{<\chi}$.

Proposition 2.23. Assume that there exists a $[\kappa]^{<\theta}$ - normal ideal on $P_\kappa(\lambda)$. Then (a) ([M3]) $\kappa^{<\overline{\theta}} = \kappa$, and (b) $(\mu^{<\overline{\theta}})^{<\overline{\theta}} = \mu^{<\overline{\theta}}$ for every cardinal $\mu > \kappa$.

Proof. (b) follows from Lemma 2.22 since by Proposition 2.18 $2^{<\overline{\theta}} \leq \kappa$.

3. $NS^{[\delta]^{<\theta}}_{\kappa,\lambda}$

In this section we describe the smallest $[\delta]^{<\theta}$-normal ideal on $P_\kappa(\lambda)$. We will need the following which shows that we could without loss of generality assume $\theta$ to be an infinite cardinal.
Lemma 3.1. Assume $J$ is $[\delta]^<\theta$-normal. Then $J$ is $[\delta]^<\theta \cdot \aleph_0$-normal.

Proof. We can assume that $\theta < \omega$ since otherwise the result is trivial. The desired conclusion is immediate from Proposition 2.2 (1) in case $\delta < \omega$. Now assume $\delta \geq \omega$. We have $\kappa > \omega$ by Proposition 2.10. Fix $A \in J^+$ and a $P_\omega(\delta)$-regressive $f : A \to P_\omega(\delta)$. We define a $P_\theta(\delta)$-regressive $g : A \cap \overset{\sim}{\omega} \to P_\theta(\delta)$ by $g(a) = \{|f(a)|\}$. By Proposition 2.5 ((i) $\to$ (iv)), there are $C \in J^+ \cap P(A \cap \overset{\sim}{\omega})$ and $n \in \omega$ such that $g$ is identically $n$ on $C$. If $n = 0$, $f$ is clearly constant on $C$. Otherwise, select a bijection $j_a : n \to f(a)$ for each $a \in C$. Using Proposition 2.5 ((i) $\to$ (iv)), define $C_k \in J^+$ for $k \leq n$ and $h_i : C_i \to P_\theta(\delta)$ for $i < n$ so that

1. $C_0 = C$.
2. $C_{i+1} \subseteq C_i$.
3. $h_i(a) = \{j_a(i)\}$.
4. $h_i$ is constant on $C_{i+1}$.

Then $f$ is constant on $C_n$. Hence $J$ is $[\delta]^<\omega$-normal by Proposition 2.5 ((iv) $\to$ (i)). □

Proposition 3.2. If there exists a $[\delta]^<\theta$-normal ideal on $P_\kappa(\lambda)$, then the smallest such ideal is $\nabla [\delta]^<\theta \cdot 3 I_{\kappa,\lambda}$.

Proof. Assume that there exists a $[\delta]^<\theta$-normal ideal on $P_\kappa(\lambda)$. Then $\nabla [\delta]^<\theta \cdot 3 I_{\kappa,\lambda} \subseteq K$ for every $[\delta]^<\theta$-normal ideal $K$ on $P_\kappa(\lambda)$ by Lemmas 3.1 and 2.1 (2). Moreover, $\nabla [\delta]^<\theta \cdot 3 I_{\kappa,\lambda}$ is itself a $[\delta]^<\theta$-normal ideal on $P_\kappa(\lambda)$ by the proofs of Propositions 2.10 and 2.18. □

Definition. Assuming the existence of a $[\delta]^<\theta$-normal ideal on $P_\kappa(\lambda)$, we set $NS_{\kappa,\lambda}^{[\delta]^<\theta} = \nabla [\delta]^<\theta \cdot 3 I_{\kappa,\lambda}$.

Proposition 3.3. Let $\delta'$ be an ordinal with $1 \leq \delta' \leq \delta$, and $\theta'$ be a cardinal with $2 \leq \theta' \leq \theta$. Then $NS_{\kappa,\lambda}^{[\delta']^<\theta'} \subseteq NS_{\kappa,\lambda}^{[\delta]^<\theta}$.

Proof. By Proposition 2.4. □
Proposition 3.4. \(\mathcal{NS}_{\kappa,\lambda}^{[\delta]^{<\theta}} = \mathcal{NS}_{\kappa,\lambda}^{[\delta]^{<\theta} \cdot \aleph_0} = \mathcal{NS}_{\kappa,\lambda}^{[\delta]^{<\theta}}\).

Proof. We have \(\mathcal{NS}_{\kappa,\lambda}^{[\delta]^{<\theta}} \subseteq \mathcal{NS}_{\kappa,\lambda}^{[\delta]^{<\theta}} \subseteq \mathcal{NS}_{\kappa,\lambda}^{[\delta]^{<\theta} \cdot \aleph_0} \subseteq \mathcal{NS}_{\kappa,\lambda}^{[\delta]^{<\theta}}\) by Lemma 3.1 and Propositions 3.2, 2.2 (0) and 3.3.

\[\Box\]

Proposition 3.5. \(\mathcal{NS}_{\kappa,\lambda}^{[\delta]^{<\theta}} = I_{\kappa,\lambda}\) if \(\delta < \kappa\).

Proof. By Corollary 2.20.

Definition. We put \(\mathcal{NS}_{\kappa,\lambda}^{\delta} = \mathcal{NS}_{\kappa,\lambda}^{[\delta]^{<\theta}}\).

It follows from Propositions 2.9 and 3.2 that \(\mathcal{NS}_{\kappa,\lambda}^{\delta}\) is the smallest \(\delta\)-normal ideal on \(P_\kappa(\lambda)\).

We will conform to usage and denote \(\mathcal{NS}_{\kappa,\lambda}^{\lambda}\) by \(\mathcal{NS}_{\kappa,\lambda}\).

The following is due to Abe [A].

Proposition 3.6. Assume \(\kappa \leq \delta < \kappa^+\). Then \(\mathcal{NS}_{\kappa,\lambda}^{\delta} = \nabla^{[\delta]^{<\theta}} I_{\kappa,\lambda}\).

Proof. Let us first prove the assertion for \(\delta = \kappa\). Given \(f_b : P_2(\kappa) \to P_\kappa(\lambda)\) for \(b \in P_2(\kappa)\), define \(f : P_2(\kappa) \to P_\kappa(\lambda)\) by \(f(e) = \bigcup_{b \in P_2((\cup e) + 1)} \bigcup_{c \in P_2((\cup e) + 1)} f_b(c)\). Then \(C_{\kappa,\lambda}^{\delta} \subseteq \bigcup_{b \in P_2(\kappa)} C_{f_b}^{\kappa,\lambda}\). Hence \(\nabla^{[\kappa]^{<\theta}} I_{\kappa,\lambda}\) is \([\kappa]^{<\theta}\)-normal by Lemma 2.12 and Proposition 2.5 ((ii) \to (i)). It follows that \(\mathcal{NS}_{\kappa,\lambda}^{\kappa} = \nabla^{[\kappa]^{<\theta}} I_{\kappa,\lambda}\) by Proposition 3.2 and Lemma 2.1 (2).

Now assume \(\kappa < \delta < \kappa^+\). By Propositions 2.18 (0) and 4.4 (below), there exists \(A \in (\nabla^{[\delta]^{<\theta}} I_{\kappa,\lambda})^*\) such that \(\mathcal{NS}_{\kappa,\lambda}^{[\delta]^{<\theta}} = \mathcal{NS}_{\kappa,\lambda}^{[\kappa]^{<\theta}} | A\). Then by Lemma 2.1 (2),
\[\nabla^{[\delta]^{<\theta}} I_{\kappa,\lambda} \subseteq \mathcal{NS}_{\kappa,\lambda}^{[\delta]^{<\theta}} = (\nabla^{[\kappa]^{<\theta}} I_{\kappa,\lambda}) | A \subseteq \nabla^{[\delta]^{<\theta}} I_{\kappa,\lambda}.\]

\[\Box\]

Abe [A] also showed that for \(\delta \geq \kappa^+\), \(\mathcal{NS}_{\kappa,\lambda}^{\delta} \setminus \nabla^{[\delta]^{<\theta}} I_{\kappa,\lambda} \neq \phi\) (in fact \(\nabla^{[\kappa]^{<\theta}}(\nabla^{[\kappa^+]^{<\theta}} I_{\kappa,\lambda}) \setminus \nabla^{[\delta]^{<\theta}} I_{\kappa,\lambda} \neq \phi)\).
By Lemma 2.12, \( NS_{\kappa,\lambda}^{[\theta] < \theta} \) is the set of all \( B \subseteq P_\kappa(\lambda) \) such that \( B \cap C_f^{\kappa,\lambda} = \emptyset \) for some \( f : P_{\theta,3}(\delta) \rightarrow P_\kappa(\lambda) \). The following generalizes a well-known (see Lemma 1.13 in [Me] and Proposition 1.4 in [M2]) characterization of \( NS_{\kappa,\lambda} \).

**Proposition 3.7.** Assume \( \delta \geq \kappa \). Then given \( B \subseteq P_\kappa(\lambda), B \in NS_{\kappa,\lambda}^{[\theta] < \theta} \) if and only if \( B \cap \{ a \in C_f^{\kappa,\lambda} : a \cap \kappa \in \kappa \} = \emptyset \) for some \( g : P_{\theta,3}(\delta) \rightarrow P_3(\lambda) \).

**Proof.** Set \( \tau = 2 \) if \( \theta < \omega \) and \( \delta < \kappa^+ \), \( \tau = 3 \) if \( \theta < \omega \) and \( \delta \geq \kappa^+ \), and \( \tau = \theta \) if \( \theta \geq \omega \). Then by Lemma 2.12 and Propositions 3.4 and 3.6, it suffices to show that for every \( f : P_\tau(\delta) \rightarrow P_\kappa(\lambda) \), there exists \( g : P_{\theta,3}(\delta) \rightarrow P_3(\lambda) \) with the property that \( \{ a \in C_f^{\kappa,\lambda} : a \cap \kappa \in \kappa \} \subseteq C_f^{\kappa,\lambda} \). Thus fix \( f : P_\tau(\delta) \rightarrow P_\kappa(\lambda) \). Pick a bijection \( j_e : |f(e)| \rightarrow f(e) \) for each \( e \in P_\tau(\delta) \).

Let us first assume that \( \theta \geq \omega \). Define \( h : P_\tau(\delta) \rightarrow \kappa \) by

\[
  h(e) = \omega \cup (\{(\cup(e \cap \kappa)) + 1\) + |f(e)|).
\]

We define \( k : P_\tau(\delta) \rightarrow \lambda \) as follows. Given \( e \in P_\tau(\delta) \), set \( \alpha = \cup(e \cap \kappa) \). We put \( k(e) = 0 \) if \( \alpha \notin e \). Assuming now that \( \alpha \in e \), put \( c = e - \{\alpha\} \) and \( \xi = \cup(c \cap \kappa) \), and let \( \beta \) denote the unique ordinal \( \zeta \) such that \( \alpha = (\xi + 1) + \zeta \). We put \( k(e) = j_c(\beta) \) if \( \beta \in |f(e)| \) and \( k(e) = 0 \) otherwise. Finally define \( g : P_\tau(\delta) \rightarrow P_3(\lambda) \) by \( g(e) = \{h(e), k(e)\} \). Now fix \( a \in C_f^{\kappa,\lambda} \) with \( a \cap \kappa \in \kappa \), and \( c \in P_{|a \cap \tau|}(a \cap \delta) \). Put \( \xi = \cup(c \cap \kappa) \). Given \( \beta \in |f(c)| \), set \( e = c \cup \{\xi + 1 + \beta\} \). Since \( h(c) \subseteq a \), we have \( \omega \subseteq a \) and \( (\xi + 1) + \beta \in a \), and therefore \( e \in P_{|a \cap \tau|}(a \cap \delta) \). Hence \( j_e(\beta) \in a \), since clearly \( k(e) = j_c(\beta) \). Thus \( f(e) \subseteq a \).

Let us next assume that \( \theta < \omega \) and \( \delta \geq \kappa^+ \). Select a bijection \( h : P_3(\delta) \rightarrow \delta - \kappa \). Define \( k : P_3(\delta) \rightarrow \lambda \) so that (a) \( k(\emptyset) = 2 \), and (b) given \( e \in P_3(\delta) \), \( k(\{h(e)\}) = |f(e)| \) and for all \( \beta \in |f(e)| \), \( k(\{\beta, h(e)\}) = j_e(\beta) \). Then define \( g : P_{\theta,3}(\delta) \rightarrow P_3(\lambda) \) so that \( g(e) = \{h(e), k(e)\} \) for all \( e \in P_3(\delta) \). It is readily checked that \( g \) is as desired.
Finally, assume that $\overline{\theta} < \omega$ and $\delta < \kappa^+$. Define $h : P_2(\delta) \rightarrow \kappa$ by:

(i) $h(\emptyset) = 2 + |f(\emptyset)|$.

(ii) $h(\{\alpha\}) = (\alpha + 1) + |f(\{\alpha\})|$ for $\alpha \in \kappa$.

(iii) $h(\{\alpha\}) = |f(\{\alpha\})|$ for $\alpha \in \delta - \kappa$.

Then define $k : P_3(\delta) \rightarrow \lambda$ so that

(0) $k(\{\beta\}) = j_0(\beta)$ whenever $\beta \in |f(\emptyset)|$.

(1) $k(\{\alpha, (\alpha + 1) + \beta\}) = j_{\{\alpha\}}(\beta)$ whenever $\alpha \in \kappa$ and $\beta \in |f(\{\alpha\})|$.  

(2) $k(\{\alpha, \beta\}) = j_{\{\alpha\}}(\beta)$ whenever $\alpha \in \delta - \kappa$ and $\beta \in |f(\{\alpha\})|$.

Finally define $g : P_{\overline{\beta},3}(\delta) \rightarrow P_3(\lambda)$ so that $g(e) = \{h(e), k(e)\}$ if $e \in P_2(\delta)$, and $g(e) = \{k(e)\}$ if $e \in P_3(\delta) - P_2(\delta)$. Then $g$ is as desired.

4. Variations of $\delta$

This section is concerned with the case when $\delta$ is not a cardinal.

Throughout the section it is assumed that $\delta \geq \kappa$.

Our first remark is that we do not lose generality by assuming that $\delta = \kappa \alpha$ for some ordinal $\alpha > 0$. Lemma 4.1 and Proposition 4.2 generalize results of Abe [A].

**Lemma 4.1.** Assume that $\delta = \kappa \alpha$ for some ordinal $\alpha > 0$, and $J$ is $[\delta]^{< \theta}$-normal. Then $J$ is $[\delta + \xi]^{< \theta}$-normal for every $\xi < \kappa$.

**Proof.** Fix $\xi < \kappa$. Since $\xi + \kappa \alpha = \kappa \alpha$, we can define $j : \kappa \alpha + \xi \rightarrow \kappa \alpha$ by $j(\beta) = \xi + \beta$ for $\beta < \kappa \alpha$, and $j(\kappa \alpha + \gamma) = \gamma$ for $\gamma < \xi$. Set

$$C = \hat{\xi} \cap \{a \in P_\kappa(\lambda) : (\forall \beta \in a \cap \kappa \alpha) \ j(\beta) \in a\}.$$
Then clearly \( C \in (NS^{[\delta]<\theta}_{\kappa,\lambda})^* \). Now given \( A \in J^+ \) and a \( P_\theta(\delta + \xi) \)-regressive \( f : A \rightarrow P_\theta(\delta + \xi) \), define \( g : A \cap C \rightarrow P_\theta(\delta) \) by \( g(a) = j[f(a)] \). Since \( A \cap C \in J^+ \) by Proposition 3.2, and \( g \) is \( P_\theta(\delta) \)-regressive, we have by Proposition 2.5 ((i)\( \rightarrow \) (iv)) that \( g \) is constant on some \( D \in J^+ \). Then \( f \) is constant on \( D \). Hence \( J \) is \([\delta + \xi]<\theta\)-normal by Proposition 2.5 ((iv)\( \rightarrow \) (i)).

**Proposition 4.2.** Assume that \( \delta = \kappa \alpha \) for some ordinal \( \alpha > 0 \). Then

\[
\begin{align*}
(a) \quad & NS^{[\delta]<\theta}_{\kappa,\lambda} = NS^{[\delta+\xi]<\theta}_{\kappa,\lambda} \text{ for every } \xi < \kappa. \\
(b) \quad & NS^{[\delta+\kappa]<2}_{\kappa,\lambda} - NS^{[\delta]<\theta}_{\kappa,\lambda} \neq \emptyset.
\end{align*}
\]

**Proof.**

(a) : By Lemma 4.1, Propositions 3.2 and 3.3.

(b) : Select \( f : [\delta + \kappa]<^3 \rightarrow P_\kappa(\lambda) \) so that \( f([\beta]) = \{\beta + 1\} \) for every \( \beta \in \delta + \kappa \). Given \( g : P_{\overline{\Theta},3}(\delta) \rightarrow P_\kappa(\lambda) \), pick \( a \in C^\kappa_{\overline{\Theta}} \) and \( \gamma \in (\delta + \kappa) - \delta \) with \( \gamma \geq \cup(a \cap (\delta + \kappa)) \). Then \( a \cup \{\gamma\} \in C^\kappa_{\overline{\Theta}} - C_f^{\kappa,\lambda} \). Hence \( P_\kappa(\lambda) - C_f^{\kappa,\lambda} \in NS^{[\delta+\kappa]<2}_{\kappa,\lambda} - NS^{[\delta]<\theta}_{\kappa,\lambda} \) by Lemma 2.12.

**Lemma 4.3.** The following are equivalent :

\[
\begin{align*}
(i) \quad & J \text{ is } [\delta]<\theta \text{- normal.} \\
(ii) \quad & \nabla^\delta I_{\kappa,\lambda} \subseteq J \text{ and } J \text{ is } [|\delta|]<\theta \text{- normal.}
\end{align*}
\]

**Proof.** (i) \( \rightarrow \) (ii) : By Lemma 2.1 (2).

(ii) \( \rightarrow \) (i) : Select a bijection \( j : \delta \rightarrow |\delta| \) and set

\[
D = P_\kappa(\lambda) - \bigcup_{\alpha < \delta} (P_\kappa(\lambda) - \{j(\alpha)\}).
\]

Then \( D \) lies in \((\nabla^\delta I_{\kappa,\lambda})^* \) and so in \( J^* \). Now fix \( A \in J^+ \) and a \( P_{\overline{\Theta},3}(\delta) \)-regressive \( f : A \rightarrow P_{\overline{\Theta},3}(\delta) \). Define \( g : A \cap D \rightarrow P_{\overline{\Theta},3} (|\delta|) \) by \( g(a) = j[f(a)] \). Since \( g \) is \( P_{\overline{\Theta},3}(|\delta|) \)-regressive, we can find \( C \in J^+ \cap P(A \cap D) \) and \( u \in P_{\overline{\Theta},3}(|\delta|) \) so that \( g(a) = u \) for all \( a \in C \). Then \( f \) takes the constant value \( j^{-1}(u) \) on \( C \). 

\( \square \)
Let us remark in passing that Lemma 4.3 can be combined with a result of [M4] to show that $J$ is $[\delta]^\theta$-normal if and only if it is $\delta$-normal and $(\mu, |\delta|)$-distributive for every infinite cardinal $\mu < \theta$.

**Proposition 4.4.** $NS^{[\delta]^\theta}_{\kappa, \lambda} = NS^\gamma_{\kappa, \lambda} | D$ for some $D \in (\nabla^\delta I_{\kappa, \lambda})^*$.

*Proof.* By the proof of Lemma 4.3. □

Using Cantor’s normal form for the base $|\delta|$, one easily obtains the following.

**Proposition 4.5.** Assume that $\gamma < \delta \leq \gamma^\gamma$, where $\gamma = |\delta|$. Then $NS^\delta_{\kappa, \lambda} = NS^\gamma_{\kappa, \lambda} | A$, where $A$ is the set of all $a \in P_\kappa(\lambda)$ with the following property: Suppose that $1 \leq p < \omega$, $\gamma > \eta_1 > \ldots > \eta_p$ and $\gamma > \xi_i \geq 1$ for $1 \leq i \leq p$. Then $\alpha \in a$ if and only if $\{\eta_1, \xi_1, \ldots, \eta_p, \xi_p\} \subseteq a$.

Thus for example $NS^{\kappa^+\kappa}_{\kappa, \lambda} = NS^\kappa_{\kappa, \lambda} | A$, where $A$ is the set of all $a \in P_\kappa(\lambda)$ such that

$$a - \kappa = \{\kappa + \alpha : \alpha \in a \cap \kappa\},$$

and $NS^{\kappa^2}_{\kappa, \lambda} = NS^\kappa_{\kappa, \lambda} | B$, where $B$ is the set of all $a \in P_\kappa(\lambda)$ such that

$$a - \kappa = \{\kappa \beta + \alpha : \alpha, \beta \in a \cap \kappa \text{ and } \beta \geq 1\}.$$

5. Variations of $\theta$

**Proposition 5.1.** Assume that $\delta \geq \kappa$ and $\overline{\theta} \cdot \aleph_0$ is a regular cardinal, and let $\theta'$ be a cardinal such that $\theta' \leq \kappa$ and $\overline{\theta} \cdot \aleph_0 < \overline{\theta'}$. Then $NS^{[\delta]^\theta'}_{\kappa, \lambda} - NS^{[\delta]^\theta}_{\kappa, \lambda} \neq \emptyset$ (and therefore $NS^{[\delta]^\theta}_{\kappa, \lambda} \neq NS^{[\delta]^\theta'}_{\kappa, \lambda}$).

*Proof.* Given $f : P_{\overline{\theta} \cdot 3}(\delta) \to P_\kappa(\lambda)$, we use Proposition 2.18 (0) to define $a_\alpha \in P_\kappa(\lambda)$ and $\gamma_\alpha \in \kappa$ for $\alpha < \overline{\theta} \cdot \aleph_0$ as follows:
\( a_0 = \overline{\theta} \cdot 3. \)

(ii) \( \gamma_\alpha = \cup(a_\alpha \cap \kappa). \)

(iii) \( a_{\alpha+1} = a_\alpha \cup (\gamma_\alpha + 1) \cup (\bigcup f[P_{\overline{\theta},3}(a_\alpha \cap \delta)]). \)

(iv) \( a_\alpha = \bigcup_{\beta < \alpha} a_\beta \) if \( \alpha \) is an infinite limit ordinal.

Put \( a = \bigcup_{\alpha < \overline{\theta} \cdot \aleph_0} a_\alpha. \) Then \( a \in C_\overline{\theta}^{\kappa,\lambda} \) and \( cf(\cup(\alpha \cap \kappa)) = \overline{\theta} \cdot \aleph_0. \) Hence
\[
\{ a \in P_\kappa(\lambda) : cf(\cup(\alpha \cap \kappa)) = \overline{\theta} \cdot \aleph_0 \} \in (NS^{[\delta]^{<\theta}}_{\kappa,\lambda})^+ \]
by Lemma 2.12. It remains to observe that
\[
\{ a \in P_\kappa(\lambda) : cf(\cup(\alpha \cap \kappa)) > \overline{\theta} \cdot \aleph_0 \} \in (\nabla^{[\kappa]^{<\overline{\theta}}} I_{\kappa,\lambda})^* \]
by Lemma 2.16 (1).

We will see that the conclusion of Proposition 5.1 may fail if \( \overline{\theta} \cdot \aleph_0 \) (and hence \( \theta \)) is a singular cardinal. The remainder of the section is concerned with the case when \( \theta \) is a limit cardinal.

The following is immediate from Proposition 2.18 (0).

**Proposition 5.2.** Suppose that \( \theta \) is a limit cardinal with \( \theta < \kappa. \) Then the following are equivalent:

(i) There exists a \([\delta]^{<\theta}\)-normal ideal on \( P_\kappa(\lambda). \)

(ii) For each cardinal \( \rho \) with \( 2 \leq \rho < \theta, \) there exists a \([\delta]^{<\rho}\)-normal ideal on \( P_\kappa(\lambda). \)

Notice that if \( \theta = \kappa \) and \( \kappa \) is an inaccessible cardinal that is not Mahlo, then by Proposition 2.18, (ii) holds but (i) does not.

**Proposition 5.3.** Assume that \( \delta \geq \kappa \) and \( \theta \) is a limit cardinal. Then the following are equivalent:

(i) \( J \) is \([\delta]^{<\theta}\)-normal.

(ii) \( J \) is \([\delta]^{<\rho}\)-normal for every cardinal \( \rho \) with \( 2 \leq \rho < \theta. \)
Proof. (i) → (ii) : By Lemma 2.1 (2).

(ii) → (i) : By Proposition 2.5, it suffices to show that if \( A \in J^+ \) and \( f : A \to P_\theta(\delta) \) is \( P_\theta(\delta) \)-regressive, then \( f \upharpoonright D \) is \( P_\rho(\delta) \)-regressive for some \( D \in J^+ \cap P(A) \) and some cardinal \( \rho \) with \( 2 \leq \rho < \theta \). This is clear if \( \theta < \kappa \). Assuming \( \theta = \kappa \), put \( B = \{ a \in A : a \cap \kappa \in \kappa \} \).

Then \( |f(a)| \in a \cap \kappa \) for every \( a \in B \) with \( a \cap \kappa \neq \emptyset \). It remains to observe that by Lemmas 2.1 (2) and 2.16 (2), \( J \) is \( [\kappa]^{2^\kappa} \)-normal and \( B \cap \hat{2} \in J^+ \).

We have the following corresponding characterization of \( NS^{[\delta]^{<\theta}}_{\kappa,\lambda} \).

**Proposition 5.4.** Assume that \( \delta \geq \kappa \) and \( \theta \) is a limit cardinal. Then

\[
NS^{[\delta]^{<\theta}}_{\kappa,\lambda} = \nabla^\theta( \bigcup_{2 \leq \rho < \theta} NS^{[\delta]^{<\rho}}_{\kappa,\lambda} ).
\]

Proof. We have \( \nabla^\theta( \bigcup_{2 \leq \rho < \theta} NS^{[\delta]^{<\rho}}_{\kappa,\lambda} ) \subseteq \bigcup_{2 \leq \rho < \theta} NS^{[\delta]^{<\rho}}_{\kappa,\lambda} \subseteq NS^{[\delta]^{<\theta}}_{\kappa,\lambda} \) by Proposition 2.9 and Lemma 2.1 (2). Now fix \( B \in NS^{[\delta]^{<\theta}}_{\kappa,\lambda} \). Then by Lemma 2.12, there is \( f : P_\theta(\delta) \rightarrow P_\kappa(\lambda) \) such that \( B \cap C_{\kappa,\lambda} = \emptyset \). Set \( f_\rho = f \upharpoonright P_\rho(\delta) \) for each cardinal \( \rho \) with \( 2 \leq \rho < \theta \). Let us define \( D \subseteq P_\kappa(\lambda) \) by:

\[
D = \{ a \in P_\kappa(\lambda) : a \cap \kappa \text{ is an infinite limit cardinal} \}
\]

otherwise. Then \( D \in (NS^{[\delta]^{<\theta}}_{\kappa,\lambda})^* \) by Lemmas 2.16 (2) and 2.1 (2). Let \( A \) be the set of all \( a \in D \) such that \( a \in C_{f_\rho}^{\kappa,\lambda} \) for every cardinal \( \rho \) with \( 2 \leq \rho < \theta \cap (a \cap \kappa) \). Then clearly, \( A \subseteq C_{f_\rho}^{\kappa,\lambda} \) and \( P_\kappa(\lambda) - A \in \nabla^\theta( \bigcup_{2 \leq \rho < \theta} NS^{[\delta]^{<\rho}}_{\kappa,\lambda} ) \). Hence, \( B \in \nabla^\theta( \bigcup_{2 \leq \rho < \theta} NS^{[\delta]^{<\rho}}_{\kappa,\lambda} ) \).  

\( \square \)

Now we focus on the case when \( \theta \) is a singular cardinal.

**Proposition 5.5.** Assume that there exists a \( [\delta]^{<\theta} \)-normal ideal on \( P_\kappa(\lambda) \), \( \theta \) is a singular cardinal, and either \( \delta \geq 2^{<\theta} \), or \( \delta \geq \theta \) and \( cf(\theta^{<\theta}) \neq cf(\theta) \). Then there exists a \( [\delta]^{<\theta^+} \)-normal ideal on \( P_\kappa(\lambda) \).
Proof. Notice that by Proposition 2.18 (0), $2^{<\theta} \leq \theta^{<\theta} < \kappa$. First suppose that $\theta \leq \delta < 2^{<\theta}$ and $cf(\theta^{<\theta}) \neq cf(\theta)$. Then there is a cardinal $\tau < \theta$ such that $\theta^{<\theta} = \theta^\tau$. We get

$$|\delta|^\theta \leq (2^\theta)^\theta = \theta^{cf(\theta)} = \theta^{\tau \cdot cf(\theta)} = \theta^{<\theta},$$

so the desired conclusion follows from Proposition 2.18 (0). Now suppose $\delta \geq 2^{<\theta}$. Let $\mu$ be a cardinal with $2^{<\theta} \leq \mu < \kappa \cap (\delta + 1)$. Then by Lemma 2.22 and Proposition 2.18 (0),

$$\mu^\theta = (\mu^{<\theta})^{<\theta} = \mu^{<\theta} < \kappa,$$

From this together with Proposition 2.18 (0), we get the desired conclusion. □

Observe that if $\theta$ is a singular cardinal with $cf(\theta^{<\theta}) = cf(\theta)$, then for $\delta = \theta$ and $\kappa = (\theta^{<\theta})^+$, (a) there is a $[\delta]^{<\theta}$-normal ideal on $P_\kappa(\lambda)$, but (b) there is no $[\delta]^{<\theta^+}$-normal ideal on $P_\kappa(\lambda)$ (since $\theta^{\theta} = (\theta^{<\theta})^{cf(\theta)} \geq \kappa$).

Let us recall the following definition.

**Definition.** Given cardinals $\sigma, \nu, \rho$ with $2 \leq \sigma$ and $\sigma \cdot \aleph_0 \leq \nu$, $cov(\rho, \nu, \nu, \sigma)$ is the least cardinal $\mu$ for which one can find $X \subseteq P_\nu(\rho)$ such that $|X| = \mu$ and for every $c \in P_\nu(\rho)$, there is $d \in P_\sigma(X)$ with $c \subseteq \cup d$.

**Lemma 5.6.** ([S1]) Let $\sigma, \nu, \rho$ be three cardinals such that $2 \leq \sigma$ and $\sigma \cdot \aleph_0 \leq \nu < \rho$. Then the following hold:

(i) $cov(\rho, \nu, \nu, \sigma) \geq \rho$.

(ii) $cov(\rho^+, \nu, \nu, \sigma) = \rho^+ \cdot cov(\rho, \nu, \nu, \sigma)$.

(iii) If $cf(\rho) < \sigma$, then $cov(\rho, \nu, \nu, \sigma) = \bigcup_{\nu < \mu < \rho} cov(\mu, \nu, \nu, \sigma)$.

**Proposition 5.7.** Assume that $\theta$ is a singular cardinal, $\delta \geq \kappa$ and there is a cardinal $\sigma$ such that $2 \leq \sigma < \theta$ and $cov(\ | \delta |, \theta, \sigma) = | \delta |$. Then every $[\delta]^{<\theta}$-normal ideal on $P_\kappa(\lambda)$ is $[\delta]^{<\theta^+}$-normal.
Proof. Assume $J$ is $[\delta]^{<\theta}$-normal. Since by Proposition 2.18 (0) $2^{<\theta} < \delta$, we can find $x_\xi \in P_\theta(\delta)$ for $\xi \in \delta$, and $f : P_\theta(\delta) \rightarrow P_\sigma(\delta)$ so that $c = \bigcup_{\xi \in f(c)} x_\xi$ for every $c \in P_\theta(\delta)$.

Now fix $A_e \in J^*$ for $e \in P_{\theta^+}(\delta)$. Put $B_d = A_e \cup \bigcup_{\xi \in d} x_\xi$ for $d \in P_\theta(\delta)$. Set $C = \Delta_{c \in P_\theta(\delta)} \bigcup_{c \in P_\theta(\delta)} f(c)$, $D = \Delta_{d \in P_\theta(\delta)} B_d$ and $E = C \cap D \cap \hat{\theta}$. Then $E \in J^*$ by Proposition 2.5. Let $a \in E$ and $e \in P_{\theta^+}(a \cap \delta)$ be given. Select $c_\zeta \in P_\theta(\delta)$ for $\zeta < cf(\theta)$ so that $e = \bigcup_{\zeta < cf(\theta)} c_\zeta$. For each $\zeta < cf(\theta)$, we get $c_\zeta \in P_{\theta^+}(a \cap \delta)$ and therefore $f(c_\zeta) \subseteq a$. So setting $d = \bigcup_{\zeta < cf(\theta)} f(c_\zeta)$, we have $d \in P_{\theta^+}(a \cap \delta)$ and consequently $a \in B_d$. Notice that $B_d = A_e$, since

$$\bigcup_{\xi \in d} x_\xi = \bigcup_{\zeta < cf(\theta)} \bigcup_{\xi \in f(c_\zeta)} x_\xi = \bigcup_{\zeta < cf(\theta)} c_\zeta = e.$$  

Thus $E \subseteq \Delta_{e \in P_{\theta^+}(\delta)} A_e$, and therefore $\Delta_{e \in P_{\theta^+}(\delta)} A_e \in J^*$. Hence by Proposition 2.5, $J$ is $[\delta]^{<\theta^+}$-normal. \hfill \Box

**Corollary 5.8.** Assume that $\theta$ is a singular cardinal and $\delta \geq \kappa$. Assume further that either $\theta$ is a strong limit cardinal, or $\delta < \kappa^+$. Then every $[\delta]^{<\theta}$-normal ideal on $P_\kappa(\lambda)$ is $[\delta]^{<\theta^+}$-normal (and hence $\text{NS}_{\kappa,\lambda}^{[\delta]^{<\theta}} = \text{NS}_{\kappa,\lambda}^{[\delta]^{<\theta^+}}$).

**Proof.** The result follows from Proposition 5.7 and the following remarks: (a) If $\theta$ is a singular strong limit cardinal, then by a result of Shelah [S2], for every cardinal $\rho > \theta$, there is a cardinal $\sigma$ such that $2 \leq \sigma < \theta$ and $\text{cov} (\rho, \theta, \theta, \sigma) = \rho$. (b) If $n < \omega$, then by Proposition 2.23 $\text{cov} (\kappa^{+n}, \theta, \theta, 2) = (\kappa^{+n})^{<\theta} = \kappa^{+n}$. (c) Using Lemma 5.6, it is easy to show by induction that if $\omega \leq \gamma < \theta$, then $\text{cov} (\kappa^{+\gamma}, \theta, \theta, |\gamma|^+) = \kappa^{+\gamma}$. \hfill \Box

6. The case $\kappa \leq \delta < \kappa^{+\theta}$

**Definition.** $E_{\kappa,\lambda}$ denotes the set of all $a \in P_\kappa(\lambda)$ such that $a \cap \kappa \neq \phi$ and $a \cap \kappa = \bigcup (a \cap \kappa)$. 

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**Proposition 6.1.** ([M3]) Assuming the existence of a \([\kappa]^{<\theta}\)-normal ideal on \(P_\kappa(\lambda)\), the following are equivalent:

(i) \(J\) is \([\kappa]^{<\theta}\)-normal.

(ii) \(J\) is \(\kappa\)-normal and \(\{a \in E_{\kappa,\lambda} : cf(a \cap \kappa) \geq \bigcup (a \cap \theta)\} \in J^*\).

We will show that this result can be generalized.

**Definition.** Let \(\rho\) be a cardinal with \(\kappa \leq \rho\), and \(\beta\) be an ordinal with \(1 \leq \beta < \kappa\). Then \(A^{\rho,\beta}_{\kappa,\lambda}\) denotes the set of all \(a \in P_\kappa(\lambda)\) such that (a) \(\alpha + 1 \in a\) for every \(\alpha \in a \cap (\rho + \beta - \rho)\), and (b) \(\rho^+ \gamma \in a\) for every \(\gamma < \beta\).

Thus if \(a \in A^{\rho,\beta}_{\kappa,\lambda}\) and \(\gamma < \beta\), then \(\bigcup (a \cap \rho^+(\gamma + 1))\) is a limit ordinal that is strictly greater than \(\rho^+ \gamma\) and does not belong to \(a\).

**Proposition 6.2.** Assume that \(\delta = \rho^+ \beta\), where \(\rho\) is a cardinal with \(\kappa \leq \rho\), and \(\beta\) an ordinal with \(1 \leq \beta < \theta\). Then the following are equivalent:

(i) \(J\) is \([\delta]^{<\theta}\)-normal.

(ii) \(J\) is \([\delta]^{<|\beta|+}\)-normal and \([\rho]^{<\theta}\)-normal, and the set of all \(a \in A^{\rho,\beta}_{\kappa,\lambda}\) such that \(cf(\bigcup (a \cap \rho^+(\alpha + 1))) \geq \bigcup (a \cap \theta)\) for every \(\alpha < \beta\) lies in \(J^*\).

**Proof.**

(i) \(\rightarrow\) (ii) : By Lemma 2.16 (1) and Propositions 2.4 and 6.1.

(ii) \(\rightarrow\) (i) : By Proposition 5.3 it suffices to prove the result for \(\theta < \kappa\). We can also assume that \(|\beta|^{+} < \theta\) (since otherwise the result is trivial) and (by Lemma 3.1) that \(\theta\) is an infinite cardinal.

For \(\gamma \in \delta - \rho\), select a bijection \(\tilde{\gamma} : \gamma \rightarrow |\gamma|\). Let \(B\) be the set of all \(a \in A^{\rho,\beta}_{\kappa,\lambda}\) such that (**) \(\theta \subseteq a\), \((***) \) \(cf(\bigcup (a \cap \rho^+(\alpha + 1))) \geq \theta\) for all \(\alpha < \beta\), and (****) \(\tilde{\gamma}(\xi) \in a\) whenever \(\gamma \in a \cap (\delta - \rho)\) and \(\xi \in a \cap \gamma\). Notice that \(B \in J^*\). For \(a \in B\) and \(\alpha < \beta\), select \(z^a_\alpha \subseteq a \cap (\kappa^+(\alpha + 1) - \kappa^+\alpha)\) so that o.t. \((z^a_\alpha) = cf(\bigcup (a \cap \kappa^+(\alpha + 1)))\) and \(\bigcup z^a_\alpha = \bigcup (a \cap \kappa^+(\alpha + 1))\).

Now fix \(C \in J^+\) and a \(P_\theta(\delta)\)-regressive \(F : C \rightarrow P_\theta(\delta)\). Set \(D = C \cap B\). For \(a \in D\) and \(1 \leq \eta \leq \beta\), define \(k^a_\eta : P_\theta(a \cap \rho^+\eta) \rightarrow P_{|\eta|+}(a \cap \rho^+\eta)\) as follows:
(0) \( k_1^\alpha(e) = \{ \gamma \} \), where \( \gamma \) is the least \( \zeta \in z_0^\alpha \) such that \( e \leq \zeta \).

(1) If \( e \neq \rho^{+\eta} \neq \phi \), \( k_{n+1}^\alpha(e) = \{ \gamma \} \cup k_\eta^\alpha(\gamma[e]) \), where \( \gamma \) is the least \( \zeta \in z_0^\alpha \) such that \( e \leq \zeta \).

Otherwise \( k_{n+1}^\alpha(e) = k_\xi^\alpha(e) \), where \( \xi \) is the least \( \chi \geq 1 \) such that \( e \leq \rho^{+\chi} \).

(2) Suppose that \( \eta \) is a limit ordinal. If \( \eta = \kappa^{+\eta} \), \( k_\eta^\alpha(e) = \bigcup_{\alpha < \eta} k_{\alpha+1}^\alpha(e \cap \rho^{+(\alpha+1)}) \).

Otherwise \( k_\eta^\alpha(e) = k_\xi^\alpha(e) \), where \( \xi \) is the least \( \chi \geq 1 \) such that \( e \leq \rho^{+\chi} \).

Let \( a \in D \). For \( 1 \leq \xi \leq \beta \), let \( \Phi_\xi \) assert that given \( \zeta \in e \in P_0(a \cap \rho^{+\xi}) \), there are \( n \in \omega \) and \( \gamma_0, \ldots, \gamma_n \in k_\xi^\alpha(e) \) such that \( \zeta \in \gamma_0, (\gamma_j \circ \cdots \circ \gamma_0)(\zeta) \in \gamma_{j+1} \) for \( j = 0, \ldots, n - 1 \), and \( (\gamma_n \circ \cdots \circ \gamma_0)(\zeta) \in a \cap \rho \). Let us show by induction that \( \Phi_\xi \) holds. Given \( \zeta \in e \in P_0(a \cap \rho^{+\xi}) \), let \( k_\xi^\alpha(e) = \{ \gamma \} \). Then \( e \subseteq \gamma \) and \( \gamma(\zeta) \in a \cap \rho \). Thus \( \Phi_1 \) holds. Next suppose that \( 1 < \alpha \leq \beta \) and \( \Phi_\xi \) holds for \( 1 \leq \xi < \alpha \). Let \( \zeta \in e \in P_0(a \cap \rho^{+\alpha}) \), where \( e - \rho^{+\xi} \neq \phi \) for every \( \xi < \alpha \).

Define \( \xi, \gamma_0 \), and \( e' \) as follows:

(a) If \( \alpha \) is a limit ordinal, \( \xi = \alpha \) such that \( \zeta \in \rho^{+(\alpha+1)} \). Otherwise \( \xi + 1 = \alpha \).

(b) \( \gamma_0 \in z_\xi^\alpha \cap k_\xi^{\alpha+1}(e \cap \rho^{+(\xi+1)}) \).

(c) \( e \cap \rho^{+(\xi+1)} \subseteq \gamma_0 \).

(d) \( e' = \tilde{\gamma}_0[e \cap \rho^{+(\xi+1)}] \).

(e) \( k_\xi^\alpha(e') \subseteq k_\xi^{\alpha+1}(e \cap \rho^{+(\xi+1)}) \).

Then \( \xi < \alpha \) and \( \zeta \in \gamma_0 \in z_\xi^\alpha \cap k_\alpha^\alpha(e) \). Moreover \( \tilde{\gamma}_0(\zeta) \in e' \in P_0(a \cap \rho^{+\xi}) \) and \( k_\xi(e') \subseteq k_\alpha^\alpha(e) \). If \( \xi = 0 \), then \( \tilde{\gamma}_0(\zeta) \in a \cap \rho \). Otherwise, there are \( \gamma_1, \ldots, \gamma_n \in k_\xi^\alpha(e') \), where \( 1 \leq n < \omega \), such that \( \tilde{\gamma}_0(\zeta) \in \gamma_1, (\gamma_j \circ \cdots \circ \gamma_1)(\tilde{\gamma}_0(\zeta)) \in \gamma_{j+1} \) for \( j = 1, \ldots, n - 1 \), and \( (\gamma_n \circ \cdots \circ \gamma_1)(\tilde{\gamma}_0(\zeta)) \in a \cap \rho \). So \( \Phi_\alpha \) holds.

Define \( G : D \rightarrow P_{|\beta|+}(\delta) \) by \( G(a) = k_\beta^\alpha(F(a)) \). Since \( G \) is \( P_{|\beta|+}(\delta) \)-regressive, there are \( T \in J^+ \cap P(D) \) and \( x \in P_{|\beta|+}(\delta) \) such that \( G \) takes the constant value \( x \) on \( T \). For \( a \in T \) and \( \zeta \in F(a) \), pick \( \chi_\xi^\alpha \in a \cap \rho \) so that there exist \( n \in \omega \) and \( \gamma_0, \ldots, \gamma_n \in x \) such that \( \zeta \in \gamma_0, (\gamma_j \circ \cdots \circ \gamma_0)(\zeta) \in \gamma_{j+1} \) for \( j = 0, \ldots, n - 1 \), and \( (\gamma_n \circ \cdots \circ \gamma_0)(\zeta) = \chi_\xi^\alpha \). Now
define $H : T \rightarrow P_\theta(\rho)$ by $H(a) = \{ \chi^a_\zeta : \zeta \in F(a) \}$. Since $H$ is $P_\theta(\rho)$-regressive, we can find $W \in J^+ \cap P(T)$ and $y \in P_\theta(\rho)$ so that $H$ takes the constant value $y$ on $W$. Let $d$ be the set of all $\zeta \in \delta$ for which one can find $n \in \omega$ and $\gamma_0, \ldots, \gamma_n \in x$ so that $\zeta \in \gamma_0$, $(\tilde{\gamma}_j \circ \ldots \circ \tilde{\gamma}_0)(\zeta) \in \gamma_{j+1}$ for $j = 0, \ldots, n - 1$ and $(\tilde{\gamma}_n \circ \ldots \circ \tilde{\gamma}_0)(\zeta) \in y$. Then $|d| < \theta$ and $F[W] \subseteq P_\theta(d)$. Since $|P_\theta(d)| < \kappa$ by Proposition 2.18 (0), there are $Z \in J^+ \cap P(W)$ and $v \in P_\theta(d)$ such that $F$ takes the constant value $v$ on $Z$. □

**Corollary 6.3.** Assume that $|\delta| = \kappa^n$, where $n < \omega$. Then $NS^{[\delta]^{< \theta}}_{\kappa, \lambda} = NS^{[\delta]^{< \theta}}_{\kappa, \lambda} | C$, where $C$ is the set of all $a \in P_\kappa(\lambda)$ such that $cf(\cup (a \cap \kappa^m)) \geq \cup (a \cap \theta)$ for every $m \leq n$.

*Proof.* By Lemma 4.3 and Propositions 6.1 and 6.2. □

**Corollary 6.4.** Assume that $|\delta| = \kappa^\beta$, where $\omega \leq \beta < \theta$. Then $NS^{[\delta]^{< \theta}}_{\kappa, \lambda} = NS^{[\delta]^{< \beta}}_{\kappa, \lambda} \ | C$, where $C$ is the set of all $a \in P_\kappa(\lambda)$ such that (a) $cf(\cup (a \cap \kappa)) \geq \cup (a \cap \theta)$, and (b) $cf(\cup (a \cap \kappa^{(\alpha+1)})) \geq \cup (a \cap \theta)$ for every $\alpha < \beta$.

*Proof.* By Lemma 4.3 and Propositions 6.1 and 6.2. □

So for example for $\kappa > \omega_2$ and $\lambda = \kappa^\omega$, $NS^{[\lambda]^{< \omega_2}}_{\kappa, \lambda} = NS^{[\lambda]^{< \omega_1}}_{\kappa, \lambda} \ | C$, where $C$ is the set of all $a \in P_\kappa(\lambda)$ such that $cf(\cup (a \cap \kappa^n)) \geq \omega_2$ for every $n < \omega$. We will see later (see Corollary 9.6) that if $\lambda^\omega = 2^\lambda$, then $NS^{[\lambda]^{< \omega_2}}_{\kappa, \lambda} | A \neq NS^{[\lambda]^{< \omega_1}}_{\kappa, \lambda} | A$ for all $A$.

7. $NS^{[\delta]^{< \theta}}_{\kappa, \lambda} | A$

In this section we continue to investigate whether given $\delta' \geq \delta$ and $\theta' \geq \theta$ with $(\delta', \theta') \neq (\delta, \theta)$, it is possible to find $A$ such that $NS^{[\delta']^{< \theta'}}_{\kappa, \lambda} = NS^{[\delta]^{< \theta}}_{\kappa, \lambda} | A$. The following is obvious.
**Lemma 7.1.** Let $\delta'$ be an ordinal with $\delta \leq \delta' \leq \lambda$, and $\theta'$ be a cardinal with $\theta \leq \theta' \leq \kappa$.

Then the following are equivalent:

(i) There exists $A \in (\mathcal{NS}_{[\delta]^{<\theta}})^+$ such that $\mathcal{NS}_{[\delta']^{<\theta'}}_{C_{f}} = \mathcal{NS}_{[\delta]^{<\theta}} \mid A$.

(ii) There is $f : P_{\mathcal{NS}_{3}}(\delta') \to P_{\kappa}(\lambda)$ such that for every $h : P_{\mathcal{NS}_{3}}(\delta') \to P_{\kappa}(\lambda)$, one can find $k : P_{\mathcal{NS}_{3}}(\delta) \to P_{\kappa}(\lambda)$ with $C_{f}^{\kappa,\lambda} \cap C_{k}^{\kappa,\lambda} \subseteq C_{h}^{\kappa,\lambda}$.

We start with a positive result.

**Lemma 7.2.** Let $\delta'$ be an ordinal with $\delta \leq \delta' \leq \lambda$, and $\theta'$ be a cardinal with $\theta \leq \theta' \leq \kappa$.

Assume that $\delta \geq \kappa$ and $|\delta|^{\varnothing} = |\delta'|^{\varnothing}$. Then $\mathcal{NS}_{[\delta]^{<\theta}} = \mathcal{NS}_{[\delta]^{<\theta}} \mid A$ for some $A \in (\varnothing)^{<\theta}I_{\kappa,\lambda}^*$. 

**Proof.** Select a bijection $j : P_{\mathcal{NS}_{3}}(\delta') \to P_{\mathcal{NS}_{3}}(\delta)$ with $j(\phi) = \phi$, and let $i$ denote its inverse.

Define $f : P_{\mathcal{NS}_{3}}(\delta') \to P_{\kappa}(\lambda)$ by: $f(b) = (\varnothing \cdot 3) \cup j(b)$ if $\varnothing < \kappa$, and $f(b) = |j(b)|^+ \cup j(b)$ otherwise. Then $C_{f}^{\kappa,\lambda} \in (\varnothing)^{<\theta}I_{\kappa,\lambda}^*$ by Lemma 2.12. Now given $h : P_{\mathcal{NS}_{3}}(\delta') \to P_{\kappa}(\lambda)$, define $k : P_{\mathcal{NS}_{3}}(\delta) \to P_{\kappa}(\lambda)$ so that

(i) $k(e) = (h \circ i)(e)$ whenever $e \in P_{\mathcal{NS}_{3}}(\delta)$;

(ii) If $\varnothing = 2$, then $k(\{\alpha, \beta\}) = h(i(\{\alpha\}) \cup i(\{\beta\}))$ whenever $\alpha$ and $\beta$ are two distinct members of $\delta$.

It is readily checked that $C_{f}^{\kappa,\lambda} \cap C_{k}^{\kappa,\lambda} \subseteq C_{h}^{\kappa,\lambda}$. Hence $\mathcal{NS}_{[\delta]^{<\theta}} = \mathcal{NS}_{[\delta]^{<\theta}} \mid C_{f}^{\kappa,\lambda}$.

**Lemma 7.3.** Assume that there exists a $[\kappa]^{<\theta}$-normal ideal on $P_{\kappa}(\lambda)$. Let $\nu$ be a cardinal with $\nu > \kappa$, and $\sigma$ be the least cardinal $\tau$ with $\tau^{<\varnothing} \geq \nu$. Then (a) $\sigma \geq \kappa$, (b) $\mu^{<\varnothing} < \sigma$ for every cardinal $\mu < \sigma$, (c) $\sigma^{<\varnothing} = \nu^{<\varnothing}$, and (d) $\sigma^{<\varnothing} = \sigma$ if $\text{cf}(\sigma) \geq \varnothing$, and $\sigma^{<\varnothing} = \sigma^{\text{cf}(\sigma)}$ otherwise.

**Proof.** Proposition 2.23 tells us that $\kappa^{\varnothing} = \kappa$, so $\sigma > \kappa$. Moreover given a cardinal $\mu$ with $\kappa < \mu < \sigma$, we have $\mu^{<\varnothing} < \sigma$ since otherwise by Proposition 2.23
Proposition 7.4. Assume \( \delta \geq \kappa \), and let \( \tau \) the least cardinal \( \tau' \) such that \( \tau' \leq \delta' \leq \kappa, \) where \( \delta' \) is an ordinal with \( \kappa \leq \delta' \leq \lambda, \) and \( \theta' \) a cardinal with \( 2 \leq \theta' \leq \kappa. \) Then \( NS_{\kappa,\lambda}^{[\delta]^{<\theta}} \big| C = NS_{\kappa,\lambda}^{[\theta']^{<\delta}} \big| C \) for some \( C \in (\nabla[\delta]^{<\theta}\Gamma_{\kappa,\lambda})^* \), where \( \delta'' = \delta \cup \delta' \) and \( \theta'' = \theta \cup \theta'. \)

Proof. By Lemma 7.2 we can find \( A, B \in (\nabla[\theta']^{<\delta'}\Gamma_{\kappa,\lambda})^* \) so that \( NS_{\kappa,\lambda}^{[\delta]^{<\theta}} \big| A = NS_{\kappa,\lambda}^{[\theta']^{<\delta'}} \big| B. \) Then \( C = A \cap B \) is as desired. \( \square \)

We will now describe some situations when \( \delta \leq \delta', \theta \leq \theta', |\delta|^{<\theta} < |\delta'|^{<\theta} \) and there is no \( A \) such that \( NS_{\kappa,\lambda}^{[\delta]^{<\theta}} \big| A = NS_{\kappa,\lambda}^{[\delta']^{<\theta'}} \big| A, \) thus providing partial converses to Lemma 7.2.

Definition. Assume \( \kappa < \kappa. \) Then for \( f : P_{\theta,\beta}(\delta) \rightarrow P_{\kappa}(\lambda) \) and \( X \subseteq \lambda, \) we define \( \Gamma_f(X) \) as follows. Let \( \rho = \kappa \cdot \aleph_0 \) if \( \kappa \cdot \aleph_0 \) is a regular cardinal, and \( \rho = (\kappa \cdot \aleph_0)^+ \) otherwise. Define \( X_\alpha \subseteq \lambda \) for \( \alpha < \rho \) by:

1. \( X_0 = X. \)
2. \( X_{\alpha+1} = X_\alpha \cup ( \cup f[P_{\theta,\beta}(X_\alpha \cap \delta)] ). \)
3. \( X_\alpha = \bigcup_{\beta < \alpha} X_\beta \) if \( \alpha \) is an infinite limit ordinal.

Now let \( \Gamma_f(X) = \bigcup_{\alpha < \rho} X_\alpha. \)
Notice that

\[ \Gamma_f(X) = \bigcap \{ Y : X \subseteq Y \subseteq \lambda \ \text{and} \ (\forall e \in P_{\mathfrak{P}}(Y \cap \delta)) f(e) \subseteq Y \}. \]

**Definition.** Let \( \delta' \) be an ordinal with \( \delta \leq \delta' \leq \lambda \), and \( \theta' \) be a cardinal with \( \theta \leq \theta' \leq \kappa \). Given \( f : P_{\mathfrak{P}}(\delta') \rightarrow P_\kappa(\lambda) \) and \( k : P_{\mathfrak{P}}(\delta) \rightarrow P_\kappa(\lambda) \), we define \( u(f,k) : P_{\mathfrak{P}}(\delta') \rightarrow P_\kappa(\lambda) \) by \( (u(f,k))(e) = f(e) \cup k(e) \) if \( e \in P_{\mathfrak{P}}(\delta') \), and \( (u(f,k))(e) = f(e) \) otherwise.

Notice that if \( \mathfrak{F} < \kappa \) and there exists a \( [\delta']^{\theta'} \)-normal ideal on \( P_\kappa(\lambda) \), then \( \Gamma_{u(f,k)}(a) \in C^\kappa \cap C^\kappa \) for every \( a \in P_\kappa(\lambda) \) with \( \mathfrak{F} \cdot 3 \subseteq a \).

**Proposition 7.6.** Let \( \delta' \) be an ordinal with \( \kappa \cup \delta \leq \delta' \leq \lambda \), and \( \theta' \) be a cardinal with \( \theta \leq \theta' \leq \kappa \). Assume that \( |\delta|^{<\mathfrak{F}} < |\delta'|^{<\mathfrak{F}} < \lambda \). Then \( NS_{\kappa,\lambda}^{[\delta']^{<\theta'}} \neq NS_{\kappa,\lambda}^{[\delta]^{<\theta}} |A| \) for all \( A \in (NS_{\kappa,\lambda}^{[\delta]^{<\theta}})^+ \).

**Proof.** Let \( f : P_{\mathfrak{P}}(\delta') \rightarrow P_\kappa(\lambda) \). Set \( \nu = \kappa \cup (|\delta|^{<\mathfrak{F}})^+ \) and select a one-to-one \( i : \nu \rightarrow P_{\mathfrak{P}}(\delta') \) and a one-to-one \( j : \nu \rightarrow \lambda - (\nu \cup \delta \cup (\cup \text{rang}(f))) \). Define \( h : P_{\mathfrak{P}}(\delta') \rightarrow P_\kappa(\lambda) \) so that \( h(i(\xi)) = \{ j(\xi) \} \) for every \( \xi \in \nu \). Now let \( k : P_{\mathfrak{P}}(\delta) \rightarrow P_\kappa(\lambda) \).

Pick \( \xi \in \nu \) so that \( j(\xi) \notin \cup \text{ran}(k) \).

First assume \( \mathfrak{F} < \kappa \). We set \( b = \Gamma_{u(f,k)}((\mathfrak{F} \cdot 3) \cup i(\xi)) \). Then \( b \notin C^\kappa \lambda \) since \( j(\xi) \notin b \). Next assume \( \mathfrak{F} = \kappa \). We define \( d_\beta \in P_\kappa(\lambda) \) and \( \gamma_\beta \in \kappa \) for \( \beta < \kappa \) as follows:

1. \( \gamma_0 = 0 \cup (i(\xi) \cup |i(\xi)|)^+ \) if \( \mathfrak{F} = \kappa \) and \( d_0 = (\mathfrak{F} \cdot 3) \cup i(\xi) \cup |i(\xi)|)^+ \) otherwise.
2. \( \gamma_\beta = \cup (d_\beta \cap \kappa) \).
3. \( \gamma_{\beta+1} = \gamma_\beta + 1 \cup (\cup \{ (u(f,k))(e) : e \in P_{\mathfrak{P}}(d_\beta \cap \kappa) \}) \).
4. \( d_\beta = \bigcup_{\zeta < \beta} d_\xi \) if \( \beta \) is an infinite limit ordinal.

Select a regular infinite cardinal \( \tau < \kappa \) so that (a) \( \gamma_\tau = \tau \), and (b) \( \mathfrak{F} \leq \tau \) if \( \mathfrak{F} < \kappa \).

Then \( d_\tau \in C^\kappa \cap C^\kappa \). Moreover \( i(\xi) \in P_{\mathfrak{P}}(d_\tau \cap \delta') \) and \( j(\xi) \notin d_\tau \), so \( d_\tau \notin C^\kappa \square \)
**Proposition 7.7.** Let $\mu$ be a cardinal with $\kappa \leq \mu < \lambda$. Assume that either $\lambda$ is a regular cardinal, or $u(\mu^+, \lambda) = \lambda$. Then $\text{NS}_{\kappa, \lambda} \neq \text{NS}_{\kappa, \lambda}^{\mu}$ for every $A \in (\text{NS}_{\kappa, \lambda}^{\mu})^+$. 

**Proof.** Let us first deal with the case when $\lambda$ is regular. Fix $f : P_3(\lambda) \rightarrow P_\kappa(\lambda)$. Let $C$ be the set of all $\beta \in \lambda$ such that $f(e) \subseteq \beta$ for every $e \in P_3(\beta)$. Notice that $C$ is a closed unbounded set. Define $h : P_2(\lambda) \rightarrow P_2(\lambda)$ so that $h(\{\xi\}) = \{\beta_\xi\}$, where $\beta_\xi$ is the least element $\beta$ of $C$ such that $\beta > 3 \cup \xi$. Now given $k : P_3(\mu) \rightarrow P_\kappa(\lambda)$, select $\xi \in \lambda$ so that $\cup \text{ran}(k) \subseteq \xi$. Setting $b = \Gamma_{u(f, k)}(3 \cup \{\xi\})$, we have $b \notin C_h^{\kappa, \lambda}$ since $h(\{\xi\}) \not\subseteq b$. 

Next assume that $\lambda$ is a singular cardinal and $u(\mu^+, \lambda) = \lambda$. Fix $f : P_3(\lambda) \rightarrow P_\kappa(\lambda)$. Select a one-to-one $j : \lambda \rightarrow P_\mu^+(\lambda)$ so that $\text{ran}(j) \in I_{\mu^+, \lambda}$. Define $h : P_2(\lambda) \rightarrow P_2(\lambda)$ so that $h(\{\xi\}) = \{\beta_\xi\}$, where $\beta_\xi$ is the least element $\beta$ of $C$ such that $\beta \notin \Gamma_f(\{\xi\} \cup j(\xi))$. Now given $k : P_3(\mu) \rightarrow P_\kappa(\lambda)$, select $\xi \in \lambda$ so that $3 \cup (\cup \text{ran}(k)) \subseteq j(\xi)$. Set $b = \Gamma_{u(f, k)}(3 \cup \{\xi\})$. Then $b \subseteq \Gamma_f(\{\xi\} \cup j(\xi))$ and therefore $b \notin C_h^{\kappa, \lambda}$. \[\square\]

**Proposition 7.8.** Let $\sigma$ be a cardinal such that (a) $\kappa < \sigma \leq \lambda$, and letting $\theta = (\text{cf}(\sigma))^+$, (b) $\theta < \kappa$, (c) $\sigma^{<\theta} \geq \lambda$, (d) $\mu^{<\theta} < \sigma$ for every cardinal $\mu < \sigma$, and (e) $u(\sigma, \lambda) \leq \lambda^{<\theta}$. Further let $\nu$ be a cardinal with $\kappa \leq \nu < \sigma$. Then $\text{NS}_{\kappa, \lambda}^{[\sigma]^{<\theta}} \neq \text{NS}_{\kappa, \lambda}^{[\nu]^{<\theta}}$ for every $A \in (\text{NS}_{\kappa, \lambda}^{[\nu]^{<\theta}})^+$. 

**Proof.** Fix $f : P_\theta(\sigma) \rightarrow P_\kappa(\lambda)$. Select $A \in I^+_{\sigma, \lambda}$ so that $A \subseteq \{a \in P_\sigma(\lambda) : \kappa \subseteq a\}$ and $|A| \leq \lambda^{<\theta}$. From Lemma 7.3 we get $\lambda^{<\theta} = \sigma^{<\theta}$. So we can find a one-to-one $j : A \rightarrow P_\theta(\sigma)$. Notice that if $a \in A$, then setting $\mu = |a \cup j(a)|$, we have $|\Gamma_f(a \cup j(a))| \leq \mu^{<\theta}$ since by Proposition 2.23 $(\mu^{<\theta})^{<\theta} = \mu^{<\theta}$. Define $h : P_\mu(\sigma) \rightarrow P_2(\lambda)$ so that for every $a \in A$, $h(j(a)) = \{\xi_a\}$, where $\xi_a$ is the least element of the set $\lambda - \Gamma_f(a \cup j(a))$. Now given $k : P_\theta(\nu) \rightarrow P_\kappa(\lambda)$, pick $a \in A$ so that $\cup \text{ran}(k) \subseteq a$, and put $b = \Gamma_{u(f, k)}(\theta \cup j(a))$. Then $h(j(a)) \not\subseteq b$ since $b \subseteq \Gamma_f(a \cup j(a))$, hence $b \notin C_h^{\kappa, \lambda}$. \[\square\]

**Corollary 7.9.** Assume that $\theta = (\text{cf}(\lambda))^+$, $\theta < \kappa$ and $\mu^{<\theta} < \lambda$ for every cardinal $\mu < \lambda$. Then for every cardinal $\nu$ with $\kappa \leq \nu < \lambda$, and every $A \in (\text{NS}_{\kappa, \lambda}^{[\nu]^{<\theta}})^+$, $\text{NS}_{\kappa, \lambda}^{[\lambda]^{<\theta}} \neq \text{NS}_{\kappa, \lambda}^{[\nu]^{<\theta}} |A$. 

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8. Dominating numbers

Throughout this section $\mu$ will denote a cardinal with $\mu > 0$.

The dominating numbers we will consider now are three-dimensional generalizations of the well-known cardinal invariant $\mathfrak{d}$. The connection with the notion of $[\delta]^{<\theta}$-normality will be established in the next section.

We will see that our numbers admit several equivalent definitions. It is convenient to give the following ‘unofficial’ definition first.

**Definition.** $\delta^\mu_{\kappa,\lambda}$ is the smallest cardinality of any $F \subseteq {}^\mu P_\kappa(\lambda)$ such that for every $g \in {}^\mu P_\kappa(\lambda)$, there is $f \in F$ with $|\{\alpha \in \mu : g(\alpha) \not\subseteq f(\alpha)\}| < \mu$.

The following two propositions will be very useful.

**Proposition 8.1.** $\delta^\mu_{\kappa,\lambda} \geq u(\kappa, \lambda)$.

**Proof.** Given $F \subseteq {}^\mu P_\kappa(\lambda)$ with $|F| < u(\kappa, \lambda)$, it is easy to define $g \in {}^\mu P_\kappa(\lambda)$ so that $g(\alpha) \not\subseteq f(\alpha)$ for all $\alpha \in \mu$ and $f \in F$. □

**Proposition 8.2.** $\text{cf}(\delta^\mu_{\kappa,\lambda}) > \mu$.

**Proof.** We can assume that $\mu \geq \omega$, since the result is immediate from Propositions 8.1 and 1.3 (0) if $\mu < \omega$. Select a bijection $j : \mu \times \mu \rightarrow \mu$. Suppose toward a contradiction that there are $F_\gamma \subseteq {}^\mu P_\kappa(\lambda)$ for $\gamma < \mu$ such that a) $|F_\gamma| < \delta^\mu_{\kappa,\lambda}$ for all $\gamma < \mu$, b) $F_\gamma \cap F_\delta = \emptyset$ whenever $\gamma, \delta < \mu$ are such that $\gamma \neq \delta$, and c) for every $g \in {}^\mu P_\kappa(\lambda)$, there is $f \in \bigcup_{\gamma < \mu} F_\gamma$ with $|\{\alpha < \mu : g(\alpha) \not\subseteq f(\alpha)\}| < \mu$. For each $\gamma < \mu$, there is $g_\gamma \in {}^\mu P_\kappa(\lambda)$ such that $|\{\alpha < \mu : g_\gamma(\alpha) \not\subseteq f(j(\gamma, \alpha))\}| = \mu$ for every $f \in F_\gamma$. Define $h \in {}^\mu P_\kappa(\lambda)$ by $h(j(\gamma, \alpha)) = g_\gamma(\alpha)$. There are $\gamma < \mu$ and $f \in F_\gamma$ such that $|\{\beta < \mu : h(\beta) \not\subseteq f(\beta)\}| < \mu$. Then $|\{\alpha < \mu : h(j(\gamma, \alpha)) \not\subseteq f(j(\gamma, \alpha))\}| < \mu$, a contradiction. □
Definition. \( F \subseteq \mathcal{P}_\kappa(\lambda) \) is \( \mathcal{P}_\kappa(\lambda) \)-dominating if for every \( g \in \mathcal{P}_\kappa(\lambda) \), there is \( f \in F \) such that \( g(\alpha) \subseteq f(\alpha) \) for all \( \alpha < \mu \).

The ‘official’ definition of our three-cardinal version of the dominating number \( \nu \) reads as follows.

Definition. \( \nu_{\kappa, \lambda}^\mu \) is the least cardinality of any \( \mathcal{P}_\kappa(\lambda) \)-dominating \( F \subseteq \mathcal{P}_\kappa(\lambda) \).

Let us first observe that \( \nu_{\kappa, \lambda}^\mu \) is a familiar quantity in case \( \mu < \kappa \).

\textbf{Proposition 8.3.} Assume \( \mu < \kappa \). Then \( \nu_{\kappa, \lambda}^\mu = u(\kappa, \lambda) \).

\textit{Proof.} Since clearly \( \nu_{\kappa, \lambda}^\mu \geq \delta_{\kappa, \lambda}^\mu \), we get \( \nu_{\kappa, \lambda}^\mu \geq u(\kappa, \lambda) \) by Proposition 8.1. For the reverse inequality, observe that given \( g \in \mathcal{P}_\kappa(\lambda) \), we have \( g(\alpha) \subseteq \bigcup \text{ran}(g) \) for all \( \alpha < \mu \).

\textbf{Proposition 8.4.} \( \nu_{\kappa, \lambda}^\mu = \delta_{\kappa, \lambda}^\mu \).

\textit{Proof.} It is immediate that \( \nu_{\kappa, \lambda}^\mu \geq \delta_{\kappa, \lambda}^\mu \). If \( \mu < \kappa \), the reverse inequality follows from Propositions 8.1 and 8.3.

Now assume \( \mu \geq \kappa \). Select a bijection \( j : \mu \times \mu \to \mu \), and let \( F \subseteq \mathcal{P}_\kappa(\lambda) \) be such that for every \( g \in \mathcal{P}_\kappa(\lambda) \), there is \( f \in F \) with \( |\{ \alpha < \mu : g(\alpha) \not\subseteq f(\alpha) \}| < \mu \).

For \( f \in F \) and \( \beta < \mu \), define \( f_\beta \in \mathcal{P}_\kappa(\lambda) \) by \( f_\beta(\xi) = f(j(\beta, \xi)) \). Notice that \( |\{ f_\beta : \beta < \mu \text{ and } f \in F \}| \leq |F| \) by Proposition 8.2. Given \( h \in \mathcal{P}_\kappa(\lambda) \), define \( g \in \mathcal{P}_\kappa(\lambda) \) by \( g(j(\beta, \xi)) = h(\xi) \). Pick \( f \in F \) with \( |\{ \alpha < \mu : g(\alpha) \not\subseteq f(\alpha) \}| < \mu \). There exists \( \beta < \mu \) such that

\[ \{ \alpha < \mu : g(\alpha) \not\subseteq f(\alpha) \} \cap \{ j(\beta, \xi) : \xi < \mu \} = \emptyset. \]

Then

\[ h(\xi) = g(j(\beta, \xi)) \subseteq f(j(\beta, \xi)) = f_\beta(\xi) \]

for every \( \xi < \mu \).

The following will be repeatedly used.
Corollary 8.5. \( \mathfrak{d}^\mu_{\kappa,\lambda} \geq \lambda \).

Proof. By Propositions 8.4, 8.1 and 1.3 (0).

Let us consider another variation on the definition of \( \mathfrak{d}^\mu_{\kappa,\lambda} \).

Definition. \( \Delta^\mu_{\kappa,\lambda} \) is the least cardinality of any \( F \subseteq \mu P_\kappa(\lambda) \) with the property that for every \( g \in \mu \lambda \), there is \( f \in F \) such that \( g(\alpha) \in f(\alpha) \) for all \( \alpha \in \mu \).

Proposition 8.6. \( \Delta^\mu_{\kappa,\lambda} \leq \mathfrak{d}^\mu_{\kappa,\lambda} \leq \Delta^\mu_{\kappa,\text{\tau}} \), where \( \tau = \kappa \) if \( \kappa \) is a limit cardinal, and \( \tau = \nu \) if \( \kappa = \nu^+ \).

Proof. It is immediate that \( \Delta^\mu_{\kappa,\lambda} \leq \mathfrak{d}^\mu_{\kappa,\lambda} \). Let us show the other inequality. Select a bijection \( j_a : |a| \rightarrow a \) for each \( a \in P_\kappa(\lambda) \). Let \( F \subseteq (\mu \times \tau) \kappa \) be such that for every \( g \in (\mu \times \tau) \lambda \), there is \( f \in F \) with the property that \( g(\gamma, \xi) \in f(\gamma, \xi) \) for every \( (\gamma, \xi) \in \mu \times \tau \).

For \( f \in F \), define \( k_f \in \mu P_\kappa(\lambda) \) by \( k_f(\gamma) = \bigcup \{ f(\gamma, 1 + \xi) : \xi < \xi(\gamma) \} \). Given \( h \in \mu P_\kappa(\lambda) \), define \( g \in (\mu \times \tau) \lambda \) by: (i) \( g(\gamma, 0) = |h(\gamma)| \), and (ii) \( g(\gamma, 1 + \xi) = j_h(\gamma)(\xi) \) if \( \xi < g(\gamma, 0) \), and \( g(\gamma, 1 + \xi) = 0 \) otherwise. There is \( f \in F \) such that \( g(\gamma, \xi) \in f(\gamma, \xi) \) for all \( (\gamma, \xi) \in \mu \times \tau \). We have that \( h(\gamma) \subseteq k_f(\gamma) \) for every \( \gamma \in \mu \). Hence \( \mathfrak{d}^\mu_{\kappa,\lambda} \leq |F| \).

We will now see that \( \mathfrak{d}^\mu_{\kappa,\lambda} \) is easy to compute if \( \lambda \) is large with respect to \( \mu \).

Lemma 8.7.

(0) Assume \( \mu < \kappa \). Then \( \lambda^{<\kappa} = \mathfrak{d}^\mu_{\kappa,\lambda} \cdot 2^{<\kappa} \).

(1) Assume \( \mu \geq \kappa \). Then \( \lambda^\mu = \mathfrak{d}^\mu_{\kappa,\lambda} \cdot 2^\mu \).

Proof.

(0) : It is well-known (see [DoM]) that \( \lambda^{<\kappa} = u(\kappa, \lambda) \cdot 2^{<\kappa} \). So the result follows from Proposition 8.3.

(1) : \( \lambda^\mu = |\mu P_\kappa(\lambda)| \leq \mathfrak{d}^\mu_{\kappa,\lambda} \cdot |\mu (2^{<\kappa})| \leq |\mu P_\kappa(\lambda)| \).
Proposition 8.8.

(0) Assume $\mu < \kappa$ and $\lambda \geq 2^{<\kappa}$. Then $\vartheta_{\kappa,\lambda}^{\mu} = \lambda^{<\kappa}$.

(1) Assume $\mu \geq \kappa$ and $\lambda \geq 2^\mu$. Then $\vartheta_{\kappa,\lambda}^{\mu} = \lambda^\mu$.

Proof. By Lemma 8.7 and Corollary 8.5. □

Proposition 8.9. Assume GCH. Then

a) $\vartheta_{\kappa,\lambda}^{\mu} = \mu^+$ if $\mu \geq \lambda$.

b) $\vartheta_{\kappa,\lambda}^{\mu} = \lambda^+$ if $\mu < \lambda$ and $\mu^+ \cdot \kappa > cf(\lambda)$.

c) $\vartheta_{\kappa,\lambda}^{\mu} = \lambda$ if $\mu^+ \cdot \kappa \leq cf(\lambda)$.

Proof. a) : By Propositions 8.2 and 8.4 and Lemma 8.7 (1).

b) and c) : By Proposition 8.8. □

Notice that $\vartheta_{\kappa,\lambda}^{\mu} \geq \lambda$ and $cf(\vartheta_{\kappa,\lambda}^{\mu}) \geq \mu^+ \cdot \kappa$ by Corollary 8.5 and Propositions 8.2, 8.4, 8.3 and 1.3 (1). Thus Proposition 8.9 shows that $\vartheta_{\kappa,\lambda}^{\mu}$ assumes its least possible value under GCH. Let us now show that $\kappa$-c.c. forcing preserves this minimal value in case $\kappa > \omega$.

Proposition 8.10. Assume $\kappa > \omega$, and let $(P, \prec)$ be a $\kappa$-c.c. notion of forcing. Then $(\vartheta_{\kappa,\lambda}^{||\mu||})^V \leq (\vartheta_{\kappa,\lambda}^{\mu})^V$.

Proof. Let $G$ be $P$-generic over $V$. Given an ordinal $\xi$ and $f : \xi \rightarrow \lambda$ in $V[G]$, there is by Lemma 6.8 in chapter VII of [K], $F : \xi \rightarrow P_\kappa(\lambda)$ in $V$ with the property that $f(\alpha) \in F(\alpha)$ for every $\alpha < \xi$. It immediately follows that $(\Delta_{\kappa,\lambda}^{||\mu||})^{V[G]} \leq (\vartheta_{\kappa,\lambda}^{\mu})^V$, which by Proposition 8.6 gives $(\vartheta_{\kappa,\lambda}^{||\mu||})^{V[G]} \leq (\vartheta_{\kappa,\lambda}^{\mu})^V$ if $\mu \geq \kappa$.

Now assume $\mu < \kappa$. Then $(\vartheta_{\kappa,\lambda}^{||\mu||})^{V[G]} = (u(\kappa, \lambda))^{V[G]}$ and $(\vartheta_{\kappa,\lambda}^{\mu})^{V[G]} = (u(\kappa, \lambda))^V$ by Proposition 8.3. In $V$, let $A \in I_{\kappa,\lambda}^+$. In $V[G]$, let $b \in P_\kappa(\lambda)$, and select a bijection $j : |b| \rightarrow b$. There exists $F : |b| \rightarrow P_\kappa(\lambda)$ in $V$ such that $j(\alpha) \in F(\alpha)$ for all $\alpha < |b|$. Pick $a \in A$ with

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∪ ran(F) ⊆ a. Then b ⊆ a. Thus it is still true in V[G] that A ∈ I^+_{κ,λ}. It follows that 
(u(κ, λ))^V[G] ≤ (u(κ, λ))^V.

We will present a few identities and inequalities that can be used to evaluate \( \mathfrak{d}_{κ,λ}^μ \) in the absence of GCH. The following is immediate.

**Lemma 8.11.** Let \( τ \) and \( ν \) be cardinals such that \( τ ≥ λ \) and \( ν ≥ µ \). Then \( \mathfrak{d}_{κ,τ}^ν ≥ \mathfrak{d}_{κ,λ}^µ \).

**Proposition 8.12.** Assume \( λ > κ \) and \( cf(λ) ≥ κ \cdot µ^+ \). Then \( \mathfrak{d}_{κ,λ}^µ = λ \cdot \bigcup_{κ ≤ ρ < λ} \mathfrak{d}_{κ,ρ}^µ \).

**Proof.** ≤ : Observe that \( µP_κ(λ) = \bigcup_{κ ≤ ρ < λ} µP_κ(α) \).

≥ : By Corollary 8.5 and Lemma 8.11. □

We will use the following two-cardinal version of \( \mathfrak{d} \).

**Definition.** \( \mathfrak{d}_λ^µ \) is the least cardinality of any \( F ⊆ µλ \) with the property that for every \( g ∈ µλ \), there is \( f ∈ F \) such that \( g(α) ≤ f(α) \) for every \( α < µ \).

We put \( \mathfrak{d}_κ = \mathfrak{d}_κ^κ \).

Thus \( \mathfrak{d} = \mathfrak{d}_ω \).

**Lemma 8.13.** Assume \( cf(λ) ≥ κ \). Then \( Δ_{κ,λ}^µ ≥ \mathfrak{d}_λ^µ \).

**Proof.** Let \( F ⊆ µP_κ(λ) \) be such that for every \( g ∈ µλ \), there is \( f ∈ F \) with the property that \( g(α) ∈ f(α) \) for all \( α < µ \). Given \( g ∈ µλ \), select \( f ∈ F \) so that \( g(α) ∈ f(α) \) for all \( α < µ \). Then \( g(α) ≤ ∪ f(α) \) for every \( α < µ \). □
Proposition 8.14. \( \mathfrak{d}_{\kappa, \kappa}^\mu = \mathfrak{d}_\kappa^\mu. \)

Proof. We have \( \mathfrak{d}_{\kappa, \kappa}^\mu \geq \mathfrak{d}_\kappa^\mu \) by Lemma 8.13. Now let \( F \subseteq \mu \kappa \) be such that for every \( g \in \mu \kappa \), there is \( f \in F \) with the property that \( g(\alpha) \leq f(\alpha) \) for every \( \alpha \in \mu \). Given \( h \in \mu P_\kappa(\kappa) \), select \( f \in F \) so that \( \cup h(\alpha) < f(\alpha) \) for all \( \alpha \in \mu \). Then \( h(\alpha) \subseteq f(\alpha) \) for every \( \alpha \in \mu \). Hence \( \mathfrak{d}_{\kappa, \kappa}^\mu \leq \mathfrak{d}_\kappa^\mu. \)

The following basic observation is very fruitful.

Proposition 8.15.

\[
\begin{align*}
(0) \quad \mathfrak{d}_{\kappa, \lambda}^\mu & \leq \mathfrak{d}_{\kappa, \rho}^\mu \cdot \mathfrak{d}_{\rho, \lambda}^\mu \leq \mathfrak{d}_{\kappa, \lambda}^{\mu, \rho} \text{ for every cardinal } \rho \text{ with } \kappa \leq \rho < \lambda, \\
(1) \quad \mathfrak{d}_{\kappa, \lambda}^\mu & \leq \mathfrak{d}_{\kappa, \rho}^\mu \cdot \mathfrak{d}_{\rho, \lambda}^\mu \leq \mathfrak{d}_{\kappa, \lambda}^{\mu, \rho} \text{ for every regular cardinal } \rho \text{ with } \kappa \leq \rho \leq \lambda. 
\end{align*}
\]

Proof. Fix a cardinal \( \rho \) with \( \kappa \leq \rho \leq \lambda \), and let \( \tau \) be a regular cardinal with \( \rho \leq \tau \leq \lambda \cap \rho^+ \).

Pick a bijection \( j_a : |a| \rightarrow a \) for each \( a \in P_\tau(\lambda) \).

Let us first show that \( \mathfrak{d}_{\kappa, \lambda}^\mu \leq \mathfrak{d}_{\kappa, \rho}^\mu \cdot \mathfrak{d}_{\rho, \lambda}^\mu \). Select a \( \mu P_\kappa(\rho) \)-dominating \( F \subseteq \mu P_\kappa(\rho) \) and a \( \mu P_\tau(\lambda) \)-dominating \( G \subseteq \mu P_\tau(\lambda) \). Define \( \varphi : F \times G \rightarrow \mu P_\kappa(\lambda) \) by

\[
(\varphi(f, g))(\alpha) = j_{g(\alpha)}[f(\alpha) \cap |g(\alpha)|].
\]

We claim that \( \text{ran}(\varphi) \) is \( \mu P_\kappa(\lambda) \)-dominating. Thus let \( r \in \mu P_\kappa(\lambda) \). Pick \( g \in G \) so that \( r(\alpha) \subseteq g(\alpha) \) for all \( \alpha < \mu \). Then pick \( f \in F \) so that \( j_{g(\alpha)}^{-1}(r(\alpha)) \subseteq f(\alpha) \) for every \( \alpha < \mu \).

Then \( r(\alpha) \subseteq (\varphi(f, g))(\alpha) \) for all \( \alpha < \mu \), which proves our claim.

Let us next show that \( \mathfrak{d}_{\kappa, \rho}^\mu \cdot \mathfrak{d}_{\rho, \lambda}^\mu \leq \mathfrak{d}_{\kappa, \lambda}^{\mu, \rho} \). We have \( \mathfrak{d}_{\kappa, \rho}^\mu \leq \mathfrak{d}_{\kappa, \lambda}^{\mu, \rho} \) by Lemma 8.11. Now let \( H \subseteq (\mu \times \rho)P_\kappa(\lambda) \) be such that for every \( p \in (\mu \times \rho)P_\kappa(\lambda) \), there is \( h \in H \) with the property that \( p(\alpha, \beta) \subseteq h(\alpha, \beta) \) for every \( (\alpha, \beta) \in \mu \times \rho \). Given \( q \in \mu P_\tau(\lambda) \), pick \( h \in H \) so that \( \{j_{q(\alpha)}(\beta)\} \subseteq h(\alpha, \beta) \) whenever \( \alpha \in \mu \) and \( \beta \in |q(\alpha)| \). If \( \tau = \rho^+ \), then \( q(\alpha) \subseteq \bigcup_{\beta \in \rho} h(\alpha, \beta) \), and we can conclude that \( \mathfrak{d}_{\tau, \lambda}^\mu \leq \mathfrak{d}_{\kappa, \lambda}^{\mu, \rho} \). Now assume \( \tau = \rho \), and let \( K \subseteq \mu \tau \) be such that for every \( i \in \mu \tau \), there is \( k \in K \) with the property that \( i(\alpha) \leq k(\alpha) \) for all \( \alpha < \mu \). Then...
there is $k \in K$ such that $|q(\alpha)| \leq k(\alpha)$ for every $\alpha < \mu$. We have $q(\alpha) \subseteq \bigcup_{\beta \in k(\alpha)} h(\alpha, \beta)$ for all $\alpha < \mu$. Thus $\mathcal{A}^\mu_{\gamma, \lambda} \leq \mathcal{A}^\mu_{\gamma, \lambda} \cdot \mathcal{A}_\gamma$, which gives $\mathcal{A}^\mu_{\gamma, \lambda} \leq \mathcal{A}^\mu_{\gamma, \lambda}$, since $\mathcal{A}_\gamma \leq \mathcal{A}^\mu_{\gamma, \tau} \leq \mathcal{A}^\mu_{\gamma, \lambda}$ by Lemmas 8.11 and 8.13 and Proposition 8.6. □

**Corollary 8.16.** Let $\sigma$ and $\chi$ be uncountable cardinals such that $\sigma \leq \mu \cap \chi$ and $\operatorname{cf}(\sigma) = \omega$. Then there is a regular infinite cardinal $\tau < \sigma$ such that $\mathcal{A}^\mu_{\tau, \lambda} = \mathcal{A}^\mu_{\tau, \lambda}$ for every regular cardinal $\rho$ with $\tau \leq \rho < \sigma$.

**Proof.** Pick regular infinite cardinals $\sigma_0 < \sigma_1 < \sigma_2 < \ldots$ with $\sigma = \bigcup \sigma_n$. Then by Proposition 8.15 (1), $\mathcal{A}^\mu_{\sigma_0, \lambda} \geq \mathcal{A}^\mu_{\sigma_1, \lambda} \geq \mathcal{A}^\mu_{\sigma_2, \lambda} \geq \ldots$, and $\mathcal{A}^\mu_{\sigma_n, \lambda} \geq \mathcal{A}^\mu_{\sigma_n, \lambda} \geq \mathcal{A}^\mu_{\sigma_{n+1}, \lambda}$ whenever $n \in \omega$ and $\rho$ is a regular cardinal with $\sigma_n \leq \rho \leq \sigma_{n+1}$. Hence there exists $k \in \omega$ such that $\mathcal{A}^\mu_{\sigma_k, \lambda} = \mathcal{A}^\mu_{\sigma_k, \lambda}$ for every regular cardinal $\rho$ with $\sigma_k \leq \rho < \sigma$. □

**Corollary 8.17.** Assume $\kappa \leq \mu < \lambda$. Then $\mathcal{A}^\mu_{\kappa, \lambda} = \mathcal{A}^\mu_{\kappa, \mu} \cdot u(\mu^+, \lambda)$.

**Proof.** By Propositions 8.15 (0) and 8.3. □

Proposition 8.3 and Corollary 8.17 show that for $\mu \leq \lambda$, the value of $\mathcal{A}^\mu_{\kappa, \lambda}$ is determined by the values taken by $\mathcal{A}^\mu_{\kappa, \tau}$ and $u(\tau, \lambda)$ when $\tau$ ranges from $\kappa$ to $\lambda$.

Let us next consider the relationship between $\mathcal{A}^\mu_{\kappa, \lambda}$ and $\mathcal{A}^\mu_{\kappa, \lambda^+}$.

**Proposition 8.18.**

1. $\mathcal{A}^\mu_{\kappa, \lambda+n} = \mathcal{A}^\mu_{\kappa, \lambda} \cdot (\prod_{i=1}^n \mathcal{A}^\mu_{\lambda^+, i})$ for each $n \in \omega - \{0\}$.

2. Assume $\mu \leq \lambda$. Then $\mathcal{A}^\mu_{\kappa, \lambda+n} = \mathcal{A}^\mu_{\kappa, \lambda} \cdot \lambda^{+n}$ for every $n \in \omega$.

**Proof.**

1. We get $\mathcal{A}^\mu_{\kappa, \lambda^+} = \mathcal{A}^\mu_{\kappa, \lambda} \cdot \mathcal{A}^\mu_{\lambda^+}$ by Propositions 8.15 (0), 8.14 and 8.6 and Lemmas 8.13 and 8.11. The desired result is then obtained by induction.

1. The result follows from (0) and Propositions 8.3 and 8.14 if $n > 0$, and from Corollary 8.5 otherwise. □
Corollary 8.19.

(0) $\vartheta_{\kappa,\kappa+}^\mu = \prod_{i=0}^n \vartheta_{\kappa+i}^\mu$ for every $n \in \omega$.

(1) $\vartheta_{\kappa,\kappa+}^\kappa = \vartheta_{\kappa} \cdot \kappa^+$ for every $n \in \omega$.

(2) $\vartheta_{\kappa,\lambda}^\lambda = \vartheta_{\kappa,\lambda}^\lambda$.

Proof.

(0) : By Propositions 8.18 (0) and 8.14.

(1) : By Propositions 8.18 (1) and 8.14.

(2) : By Propositions 8.18 (1), 8.4 and 8.2. □

Lemma 8.20. $\vartheta_{\kappa,\lambda}^\mu \leq \vartheta_{\omega,\mu(\kappa,\lambda)}^\mu$.

Proof. If $\mu \geq \kappa$, then $\vartheta_{\omega,u(\kappa,\lambda)}^\mu \geq \vartheta_{\kappa,u(\kappa,\lambda)}^\mu \geq \vartheta_{\kappa,\lambda}^\mu$ by Propositions 8.15 (1) and 1.3 (0) and Lemma 8.11. If $\mu < \kappa$, then $\vartheta_{\omega,u(\kappa,\lambda)}^\mu \geq u(\kappa,\lambda) = \vartheta_{\kappa,\lambda}^\mu$ by Corollary 8.5 and Proposition 8.3. □

Proposition 8.21. Assume $u(\kappa,\lambda) = \lambda$. Then $\vartheta_{\omega,\lambda}^\mu = \vartheta_{\omega,\kappa}^\mu \cdot \vartheta_{\kappa,\lambda}^\mu$.

Proof. By Lemmas 8.20 and 8.11 and Proposition 8.15 (1). □

Notice that $\vartheta_{\omega,\lambda}^1 = \vartheta_{\omega,\kappa}^1 \cdot \vartheta_{\kappa,\lambda}^1$ if and only if $u(\kappa,\lambda) = \lambda$.

Let us now deal with the computation of $\vartheta_{\kappa,\lambda}^\mu < \eta$.

Proposition 8.22.

(0) $\vartheta_{\kappa,\lambda}^\mu < \eta = \vartheta_{\kappa,\lambda}^\mu < \eta < \kappa$.

(1) $\vartheta_{\kappa,\lambda}^\mu < \eta < \eta \cdot m_{\eta,\lambda}^\mu$ for every regular cardinal $\eta$ with $\kappa \leq \eta \leq \lambda$.

(2) $\vartheta_{\kappa,\lambda}^\mu < \eta < \eta^+ = \lambda$.

(3) $\vartheta_{\kappa,\lambda}^\mu < \eta < \eta < \eta^+ = \lambda$.
Proof. (0), (1) and (2) : Let \( \eta \) be an uncountable cardinal \( \leq \lambda \). Let us assume that \( \eta < \lambda \) if \( \kappa \leq \eta \) and \( \eta \) is singular. We define \( \rho \) and \( \tau \) by :

(i) \( \rho = \kappa \) and \( \tau = \kappa^{< \eta} \) if \( \eta < \kappa \).

(ii) \( \rho = \eta \) and \( \tau = 2^{< \eta} \) if \( \kappa \leq \eta \) and \( \eta \) is regular.

(iii) \( \rho = \eta^+ \) and \( \tau = \eta^{< \eta} \) if \( \kappa \leq \eta \) and \( \eta \) is singular.

Let \( F \subseteq \mu P_\rho(\lambda) \) be \( \mu P_\rho(\lambda) \)-dominating, and \( K \subseteq \mu P_\kappa(\tau) \) be \( \mu P_\kappa(\tau) \)-dominating. Fix a bijection \( j : \lambda^{< \eta} \rightarrow P_\kappa(\lambda) \). For \( f \in F \) and \( \alpha \in \mu \), select a one-to-one \( i_{f,\alpha} : j^{-1}(P_\kappa(f(\alpha))) \rightarrow \tau \). Given \( h \in \mu P_\kappa(\lambda^{< \eta}) \), pick \( f \in F \) so that \( \cup(j[h(\alpha)]) \subseteq f(\alpha) \) for every \( \alpha \in \mu \). Then pick \( k \in K \) so that \( i_{f,\alpha}[h(\alpha)] \subseteq k(\alpha) \) for all \( \alpha \in \mu \). Then \( h(\alpha) \subseteq i_{f,\alpha}^{-1}(k(\alpha)) \) for every \( \alpha \in \mu \). Hence \( d^{\mu}_{\kappa,\lambda^{< \eta}} \leq d^{\mu}_{\kappa,\tau} \cdot d^{\mu}_{\rho,\lambda} \).

Since \( \tau \leq \lambda^{< \eta} \), we have \( d^{\mu}_{\kappa,\lambda^{< \eta}} \geq d^{\mu}_{\kappa,\tau} \) by Lemma 8.11. If \( \rho = \kappa \), \( d^{\mu}_{\kappa,\lambda^{< \eta}} \geq d^{\mu}_{\rho,\lambda} \) by Lemma 8.11. If \( \rho = \lambda \), \( d^{\mu}_{\kappa,\lambda^{< \eta}} \geq d^{\mu}_{\rho,\lambda} \) by Lemmas 8.11 and 8.13 and Propositions 8.6 and 8.14. If \( \kappa < \rho < \lambda \cap \mu^+ \), \( d^{\mu}_{\kappa,\lambda^{< \eta}} = d^{\mu}_{\rho,\lambda^{< \eta}} \geq d^{\mu}_{\rho,\lambda} \) by Proposition 8.15 (1) and Lemma 8.11. Finally, if \( \rho = \eta \) and \( \rho > \mu \), \( d^{\mu}_{\kappa,\lambda^{< \eta}} \geq \lambda^{< \eta} \geq u(\rho, \lambda) = d^{\mu}_{\rho,\lambda} \) by Corollary 8.5 and Proposition 8.3. Thus if it is not the case that \( \mu \cdot \kappa < \rho = \eta^+ < \lambda \), then \( d^{\mu}_{\kappa,\lambda^{< \eta}} \geq d^{\mu}_{\kappa,\tau} \cdot d^{\mu}_{\rho,\lambda} \).

(3) : Let \( \eta \) be a regular cardinal with \( \kappa \leq \eta < \lambda \). Assume \( \eta \leq \mu \). We get \( d^{\mu}_{\kappa,\lambda^{< \eta}} = d^{\mu}_{\kappa,\lambda^{< \eta}} \cdot d^{\mu}_{\eta,\lambda} \) by (1). Moreover, \( d^{\mu}_{\eta,\lambda} = d^{\mu}_{\eta,\mu} \cdot d^{\mu}_{\eta^{< \mu},\lambda} \) by Proposition 8.15 (0), and \( d^{\mu}_{\kappa,\lambda^{< \eta}} \geq d^{\mu}_{\kappa,\eta} \geq d^{\mu}_{\kappa,\eta} \geq d^{\mu}_{\kappa,\eta} = d^{\mu}_{\eta,\eta} \) by Lemmas 8.11 and 8.13 and Propositions 8.6 and 8.14. It follows that \( d^{\mu}_{\kappa,\lambda^{< \eta}} = d^{\mu}_{\kappa,\lambda^{< \eta}} \cdot d^{\mu}_{\eta^{< \mu},\lambda} \).

Assume now \( \eta^+ = \lambda \) and \( \eta > \mu \). Since \( (\eta^+)^{< \eta} = \eta^{< \eta} \cdot \eta^+ \) and \( \eta^{< \eta} = 2^{< \eta} \), we have by Lemma 8.11 that \( d^{\mu}_{\kappa,\lambda^{< \eta}} = d^{\mu}_{\kappa,\lambda^{< \eta}} \cdot d^{\mu}_{\kappa,\eta^+} = d^{\mu}_{\kappa,\eta^+} \cdot d^{\mu}_{\kappa,\eta^+} \). The desired conclusion now follows from the following three observations : a) \( d^{\mu}_{\kappa,\eta^+} = d^{\mu}_{\kappa,\eta^{< \mu}} \cdot \eta^+ \) by Proposition 8.18 (1).

b) \( d^{\mu}_{\kappa,\eta^+} \cdot d^{\mu}_{\kappa,\eta} = d^{\mu}_{\kappa,\eta^+} \) by Lemma 8.11. c) \( \eta^+ = d^{\mu}_{\eta^+,\lambda} \) by Propositions 8.3 and 1.3 (0). \( \square \)
Let us make the following remark concerning Proposition 8.2 (3). Assume GCH, and let $\eta$ be a cardinal such that $\kappa \cdot \mu < \eta = cf(\lambda) < \lambda$. Then $d_{\kappa,\lambda}^{\mu} \neq d_{\kappa,2\eta}^{\mu} \cdot d_{\eta^{+},\lambda}^{\mu}$, since by Proposition 8.9 $d_{\kappa,\lambda}^{\mu} = \lambda$ and $d_{\eta^{+},\lambda}^{\mu} = \lambda^{+}$.

**Corollary 8.23.** Let $n \in \omega$ be such that $\omega^{n} \leq \lambda$, and assume that $\mu \geq \omega$ if $n = 0$. Then

$$d_{\omega^{n},\lambda^{n}}^{\mu} = d_{\omega^{n},\lambda^{n} \cdot 2^{n}}^{\mu}.$$  

**Proof.** The result follows from Proposition 8.22 (0) if $n \geq 2$, and from Proposition 8.22 (1) if $n = 1$. Let us now turn to the case $n = 0$. If $\lambda = \omega$, the result is trivial. If $\lambda > \omega$, we get

$$d_{\omega,\lambda}^{\mu} = d_{\omega,2^{\omega}}^{\mu} \cdot d_{\omega,\lambda}^{\mu} = d_{\omega,2^{\omega}}^{\mu} \cdot d_{\omega,\omega}^{\mu} \cdot d_{\omega,\lambda}^{\mu} = d_{\omega,\omega}^{\mu} \cdot d_{\omega,\lambda}^{\mu} = d_{\omega,\lambda \cdot 2^{\omega}}^{\mu}$$

by Propositions 8.22 (1) and 8.15 (0) and Lemma 8.11. \[\square\]

Notice that if $n = 0$ and $\mu < \omega$, then $d_{\omega^{n},\lambda^{n}}^{\mu} \neq d_{\omega^{n},\lambda^{n} \cdot 2^{n}}^{\mu}$ for some values of $\lambda$, since $d_{\omega,\lambda}^{\mu} = \lambda^{n}$ and $d_{\omega,\lambda \cdot 2^{n}}^{\mu} = \lambda \cdot 2^{n}$ by Propositions 8.3 and 1.3 (0).

**Corollary 8.24.** If $\lambda \geq 2^{\kappa}$, then $d_{\kappa,\lambda}^{\mu} = d_{\kappa,\lambda}^{\kappa}$.  

**Proof.** By Proposition 8.22 (1) and Lemma 8.11. \[\square\]

**Corollary 8.25.** Let $\sigma$ be an infinite cardinal such that $cf(\sigma) < \kappa$ and $\kappa^{cf(\sigma)} < \sigma < \lambda \leq \sigma^{cf(\sigma)}$. Then $d_{\kappa,\lambda}^{\mu} = d_{\kappa,\sigma}^{\mu}$.  

**Proof.** If $(cf(\sigma))^{+} < \kappa$, then by Lemma 8.11 and Proposition 8.22 (0),

$$d_{\kappa,\lambda}^{\mu} \leq d_{\kappa,\sigma}^{\mu} = d_{\kappa,\kappa^{cf(\sigma)}}^{\mu} = d_{\kappa,\sigma}^{\mu} \leq d_{\kappa,\lambda}^{\mu}.$$  

If $(cf(\sigma))^{+} = \kappa$, then by Lemma 8.11 and Proposition 8.22 (1),

$$d_{\kappa,\lambda}^{\mu} \leq d_{\kappa,\sigma}^{\mu} = d_{\kappa,\kappa^{cf(\sigma)}}^{\mu} \cdot d_{\kappa,\sigma}^{\mu} = d_{\kappa,\sigma}^{\mu} \leq d_{\kappa,\lambda}^{\mu}.$$ \[\square\]

We finally investigate $d_{\kappa,\lambda}^{\mu^{<\rho}}$. 

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**Proposition 8.26.** Let \( \rho \) be an infinite cardinal with \( \rho \leq \mu \). Then \( \mathfrak{d}^{\mu<\rho}_{\kappa,\lambda} \) is the least cardinality of any \( F \subseteq (P_{\rho}(\mu))P_{\kappa}(\lambda) \) with the property that for any \( g \in (P_{\rho}(\mu))P_{\kappa}(\lambda) \), there is \( f \in F \) with \( \{d \in P_{\rho}(\mu) : g(d) \subseteq f(d)\} \in I^*_{\rho,\mu} \).

*Proof.* Let \( F \subseteq (P_{\rho}(\mu))P_{\kappa}(\lambda) \) be such that for every \( g \in (P_{\rho}(\mu))P_{\kappa}(\lambda) \), there is \( f \in F \) with the property that \( \{d \in P_{\rho}(\mu) : g(d) \subseteq f(d)\} \in I^*_{\rho,\mu} \). By Corollary 1.5, there are \( A_e \in \widehat{\mathcal{E}} \cap I^+_{\rho,\mu} \) for \( e \in P_{\rho}(\mu) \) such that (a) \( |A_e| = \mu^{<\rho} \) for every \( e \in P_{\rho}(\mu) \), (b) \( A_e \cap A_{e'} = \emptyset \) whenever \( e, e' \in P_{\rho}(\mu) \) are such that \( e \neq e' \), and (c) \( \bigcup_{e \in P_{\rho}(\mu)} A_e = P_{\rho}(\mu) \).

Select a bijection \( j_e : A_e \rightarrow P_{\rho}(\mu) \) for each \( e \in P_{\rho}(\mu) \). Given \( h \in (P_{\rho}(\mu))P_{\kappa}(\lambda) \), define \( g \in (P_{\rho}(\mu))P_{\kappa}(\lambda) \) so that \( g(d) = h(j_e(d)) \) whenever \( d \in A_e \). Pick \( f \in F \) and \( e \in P_{\rho}(\mu) \) so that \( \widehat{e} \subseteq \{d \in P_{\rho}(\mu) : g(d) \subseteq f(d)\} \). Then \( h(j_e(d)) \subseteq (f \circ j_{e'}^{-1})(j_e(d)) \) for every \( d \in A_e \).

Thus \( \mathfrak{d}^{\mu<\rho}_{\kappa,\lambda} \leq |F| \cdot \mu^{<\rho} \), and therefore \( \mathfrak{d}^{\mu<\rho}_{\kappa,\lambda} \leq |F| \) by Propositions 8.2 and 8.4. \( \square \)

**Proposition 8.27.** Let \( \rho \) be an infinite cardinal such that \( \rho \leq \mu \) and \( 2^\tau < \kappa \) for every cardinal \( \tau < \rho \). Then \( \mathfrak{d}^{\mu<\rho}_{\kappa,\lambda} = \mathfrak{d}^{\mu<\rho}_{\kappa,\lambda} \).

*Proof.* We have \( \mathfrak{d}^{\mu<\rho}_{\kappa,\lambda} \leq \mathfrak{d}^{\mu<\rho}_{\kappa,\lambda} \) by Lemma 8.11. For the other inequality, fix \( A \in I^+_{\rho,\mu} \) and \( F \subseteq A^P_{\kappa}(\lambda) \) with the property that for every \( g \in A^P_{\kappa}(\lambda) \), there is \( f \in F \) such that \( g(a) \subseteq f(a) \) for all \( a \in A \). For \( f \in F \), define \( f' \in (P_{\rho}(\mu))P_{\kappa}(\lambda) \) as follows: given \( b \in P_{\rho}(\mu) \), pick \( a \in A \) with \( b \subseteq a \), and set \( f'(b) = f(a) \). Now given \( h \in (P_{\rho}(\mu))P_{\kappa}(\lambda) \), define \( g \in A^P_{\kappa}(\lambda) \) by \( g(a) = \bigcup_{b \subseteq a} h(b) \). Select \( f \in F \) so that \( g(a) \subseteq f(a) \) for all \( a \in A \). Then \( h(b) \subseteq f'(b) \) for all \( b \in P_{\rho}(\mu) \). \( \square \)

9. \( \text{cof}(J) \)

This section is devoted to the computation of \( \text{cof}(\text{NS}_{\kappa,\lambda}^{[\delta]^{<\theta}}) \).

**Lemma 9.1.** Assume \( \nabla^{[\delta]^{<\theta}} I_{\kappa,\lambda} \subseteq J \). Then \( \text{cof}(J) \geq \mathfrak{d}_{\kappa,\lambda}^{[\delta]} \).

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Proof. Fix $S \subseteq J$ with $J = \bigcup_{B \in S} P(B)$. For $B \in S$, define $h_B : P_{\overline{g}}(\delta) \rightarrow P_\kappa(\lambda) - B$ so that $e \in P_{\overline{g} \cap h_B(e)}(h_B(e))$ for all $e \in P_{\overline{g}}(\delta)$. Given $g : P_{\overline{g}}(\delta) \rightarrow P_\kappa(\lambda)$, there is $B \in S$ with $P_\kappa(\lambda) - B \subseteq \Delta_{e \in P_{\overline{g}}(\delta)} g(e)$ by Proposition 2.2 (0) and Corollary 2.8 ((iv) $\rightarrow$ (ii)). Then $g(e) \subseteq h_B(e)$ for every $e \in P_{\overline{g}}(\delta)$.

\begin{proposition}
cof\((NS_{\kappa,\lambda}^{\delta,\theta} | A) = d_{\overline{g}(\delta)}^{[\P(\delta)])} for every A \in (NS_{\kappa,\lambda}^{\delta,\theta})^+.
\end{proposition}

\begin{proof}
Let us first observe that if $f : P_{\overline{g}}(\delta) \rightarrow P_\kappa(\lambda)$ and $g : P_{\overline{g}}(\delta) \rightarrow P_\kappa(\lambda)$ are such that $f(e) \subseteq g(e)$ for all $e \in P_{\overline{g}}(\delta)$, then $C^\kappa_{\lambda} \subseteq C^\kappa_{\lambda}$. Hence cof\((NS_{\kappa,\lambda}^{\delta,\theta} \subseteq d_{\overline{g}(\delta)}^{[\P(\delta)])} by Lemma 2.12. So given $A \in (NS_{\kappa,\lambda}^{\delta,\theta})^+$, we have cof\((NS_{\kappa,\lambda}^{\delta,\theta} | A) \leq d_{\overline{g}(\delta)}^{[\P(\delta)])} by Proposition 1.6. The reverse inequality holds by Lemma 9.1 since $NS_{\kappa,\lambda}^{\delta,\theta} | A$ is $[\delta,\theta)$-normal.
\end{proof}

The following is well-known.

\begin{corollary}
cof\((I_{\kappa,\lambda} | A) = u(\kappa, \lambda) for every A \in I_{\kappa,\lambda}^+.
\end{corollary}

\begin{proof}
We have $I_{\kappa,\lambda} = NS_{\kappa,\lambda}^{[\theta,\lambda]}$ by Propositions 3.5, 2.10 and 2.18 (0). So the result follows from Propositions 9.2 and 8.3.
\end{proof}

It follows from Proposition 3.5 and Corollary 9.3 that cof\((NS_{\kappa,\lambda}^{\delta,\theta} | A) = u(\kappa, \lambda) for all A \in (NS_{\kappa,\lambda}^{\delta,\theta})^+ if \delta < \kappa. For \delta \geq \kappa we have the following.

\begin{corollary}
Assume \delta \geq \kappa. Then cof\((NS_{\kappa,\lambda}^{\delta,\theta} | A) = d_{\overline{g}(\delta)}^{[\P(\delta)])} for every A \in (NS_{\kappa,\lambda}^{\delta,\theta})^+.
\end{corollary}

\begin{proof}
By Propositions 9.2, 8.26 and 2.18.
\end{proof}

Under GCH, we obtain the following values.
**Corollary 9.5.** Assume that the GCH holds and \( \delta \geq \kappa \), and let \( A = (\text{NS}_{\kappa, \lambda}^{[\delta, \theta)} + \)\). Then

a) \( \text{cof}(\text{NS}_{\kappa, \lambda}^{[\delta, \theta)} \mid A) = \lambda^{++} \) if \( \delta = \lambda \) and \( \text{cf}(\lambda) < \theta \).

b) \( \text{cof}(\text{NS}_{\kappa, \lambda}^{[\delta, \theta)} \mid A) = \lambda^{+} \) if \( \text{cf}(\lambda) \leq |\delta|^{< \theta} < \lambda \), or \( \lambda \leq |\delta|^{< \theta} \) and \( \text{cf}(\lambda) \geq \theta \).

c) \( \text{cof}(\text{NS}_{\kappa, \lambda}^{[\delta, \theta)} \mid A) = \lambda \) if \( |\delta|^{< \theta} < \text{cf}(\lambda) \).

**Proof.** By Propositions 8.9 and 9.2. \( \square \)

**Corollary 9.6.** Let \( \delta' \) be an ordinal with \( \kappa \leq \delta' \leq \lambda \), and \( \theta' \) be a cardinal with \( 2 \leq \theta' \leq \kappa \).

Assume that either \( \lambda^{[\delta']^{< \theta}} \leq |\delta|^{< \theta} \), or \( \lambda^{[\delta']^{< \theta}} = \lambda \) and \( \text{cf}(\lambda) \leq |\delta|^{< \theta} \). Then there is no \( A \in (\text{NS}_{\kappa, \lambda}^{[\delta, \theta)}) + (\text{NS}_{\kappa, \lambda}^{[\delta, \theta)}) + \) such that \( \text{NS}_{\kappa, \lambda}^{[\delta, \theta)} \mid A = \text{NS}_{\kappa, \lambda}^{[\delta, \theta)} \mid A \).

**Proof.** It suffices to observe that by Propositions 8.2 and 8.4, (a) if \( \lambda^{[\delta']^{< \theta}} \leq |\delta|^{< \theta} \), then \( \delta_{\kappa, \lambda}^{[\delta']^{< \theta}} \leq \lambda^{[\delta']^{< \theta}} \leq |\delta|^{< \theta} < \delta_{\kappa, \lambda}^{[\delta']^{< \theta}} \); and (b) if \( \lambda^{[\delta']^{< \theta}} = \lambda \) and \( \text{cf}(\lambda) \leq |\delta|^{< \theta} \), then

\[ \delta_{\kappa, \lambda}^{[\delta']^{< \theta}} \leq \lambda^{[\delta']^{< \theta}} = \lambda < \delta_{\kappa, \lambda}^{[\delta']^{< \theta}}. \]

\( \square \)

**Corollary 9.7.** Assume \( \delta \geq \kappa \). Then

\[ \text{cof}(\text{NS}_{\kappa, \lambda}^{[\delta, \theta)}) = \text{cof}(\text{NS}_{\kappa, \lambda}^{[\delta, \theta)} \cdot \text{cov}(\lambda, (|\delta|^{< \theta})^{+}, (|\delta|^{< \theta})^{+}, 2). \]

**Proof.** If \( \theta < \kappa \), then by Propositions 8.22 (0) and 2.23, \( \delta_{\kappa, \lambda}^{[\delta, \theta]} = \delta_{\kappa, \lambda}^{[\delta, \theta]} = \delta_{\kappa, \lambda}^{[\delta, \theta]} \).

If \( \theta = \kappa \), then by Propositions 8.22 (1) and 2.18 (1) and Lemma 8.11, \( \delta_{\kappa, \lambda}^{[\delta, \theta]} = \delta_{\kappa, \lambda}^{[\delta, \theta]} \cdot \delta_{\kappa, \lambda}^{[\delta, \theta]} = \delta_{\kappa, \lambda}^{[\delta, \theta]} \cdot \delta_{\kappa, \lambda}^{[\delta, \theta]} \). In any case we have \( \delta_{\kappa, \lambda}^{[\delta, \theta]} = \delta_{\kappa, \lambda}^{[\delta, \theta]} \). Hence if \( |\delta|^{< \theta} < \lambda \), we can infer from Corollary 8.17 that

\[ \delta_{\kappa, \lambda}^{[\delta, \theta]} = \delta_{\kappa, \lambda}^{[\delta, \theta]} \cdot u((|\delta|^{< \theta})^{+}, \lambda) = \delta_{\kappa, \lambda}^{[\delta, \theta]} \cdot \text{cov}(\lambda, (|\delta|^{< \theta})^{+}, (|\delta|^{< \theta})^{+}, 2). \]

If \( |\delta|^{< \theta} \geq \lambda \), Lemma 8.11 tells us that \( \delta_{\kappa, \lambda}^{[\delta, \theta]} \leq \delta_{\kappa, \lambda}^{[\delta, \theta]} \leq \delta_{\kappa, \lambda}^{[\delta, \theta]} \), so
\[ \delta_{\kappa, \lambda}^{[\delta, \theta]} = \delta_{\kappa, \lambda}^{[\delta, \theta]} = \delta_{\kappa, \lambda}^{[\delta, \theta]} \cdot \text{cov}(\lambda, (|\delta|^{< \theta})^{+}, (|\delta|^{< \theta})^{+}, 2). \]

\( \square \)
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