Stochastic Gauss-Newton Algorithms for Nonconvex Compositional Optimization

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Abstract

We develop two new stochastic Gauss-Newton algorithms for solving a class of stochastic nonconvex compositional optimization problems frequently arising in practice. We consider both the expectation and finite-sum settings under standard assumptions. We use both classical stochastic and SARAH estimators for approximating function values and Jacobians. In the expectation case, we establish \( O(\varepsilon^{-2}) \) iteration complexity to achieve a stationary point in expectation and estimate the total number of stochastic oracle calls for both function values and its Jacobian, where \( \varepsilon \) is a desired accuracy. In the finite sum case, we also estimate the same iteration complexity and the total oracle calls with high probability. To our best knowledge, this is the first time such global stochastic oracle complexity is established for stochastic Gauss-Newton methods. We illustrate our theoretical results via numerical examples on both synthetic and real datasets.

1 Introduction

We consider the following stochastic compositional nonconvex optimization problem:

\[
\min_{x \in \mathbb{R}^p} \left\{ \Psi(x) := \phi(F(x)) \equiv \phi\left( E_\xi [F(x, \xi)] \right) \right\},
\]

where \( F: \mathbb{R}^p \times \Omega \to \mathbb{R}^q \) is stochastic function defined on a probability space \((\Omega, \mathbb{P})\), and \( \phi: \mathbb{R}^q \to \mathbb{R} \cup \{+\infty\} \) is a proper, closed, and convex, but not necessarily smooth.

As a special case, if \( \Omega \) is finite, i.e. \( \Omega := \{\xi_1, \cdots, \xi_n\} \) and \( \mathbb{P}(\xi = \xi_i) = p_i > 0 \) for \( i \in [n] := \{1, \cdots, n\} \) and \( \sum_{i=1}^n p_i = 1 \), then by introducing \( F_i(x) := n p_i F(x, \xi_i) \), \( F(x) \) can be written into the following finite-sum:

\[
F(x) := \frac{1}{n} \sum_{i=1}^n F_i(x),
\]

and (1) reduces to:

\[
\min_{x \in \mathbb{R}^p} \left\{ \Psi(x) := \phi(F(x)) \equiv \phi\left( \frac{1}{n} \sum_{i=1}^n F_i(x) \right) \right\}.
\]

This expression can also be viewed as a stochastic average approximation of \( F(x) := E_\xi [F(x, \xi)] \) in (1). Note that the setting (1) is completely different from the one \( \min_x \{ \Psi(x) := E_\xi [\phi(F(x, \xi), \xi)] \} \) in [6, 5, 8]. Using Fenchel conjugate, we can also write (1) into a nonconvex-concave saddle-point problem:

\[
\min_{x \in \mathbb{R}^p} \max_{y \in \mathbb{R}^q} \left\{ \langle F(x), y \rangle - \phi^*(y) \equiv \phi\left( \frac{1}{n} \sum_{i=1}^n F_i(x) \right) \right\},
\]

where \( \phi^* \) is the Fenchel conjugate of \( \phi \). This is another approach to solve (1) or (2).

Problem (1) or its special form (2) covers various applications in different domains (both deterministic and stochastic) such as penalty methods for constrained optimization, parameter estimation, nonlinear least-squares,
The optimal value of (1) is bounded from below, i.e. \( \Psi^* > -\infty \). The function \( \phi \) is \( M_\phi \)-Lipschitz continuous, and \( F \) is \( L_F \)-average smooth, i.e., there exists two finite and nonnegative constants \( M_\phi \) and \( L_F \) such that

\[
\left| \phi(u) - \phi(v) \right| \leq M_\phi \| u - v \|, \forall u, v \in \mathbb{R}^q,
\]

\[
\mathbb{E}_\xi \left[ \| F'(x, \xi) - F'(y, \xi) \|_2^2 \right] \leq L_F^2 \| x - y \|^2, \forall x, y.
\]

For the finite-sum case (2), we impose a stronger assumption that \( \| F_i'(x) - F_i'(y) \| \leq \sigma_F \) and \( \| F_i'(x) - F_i'(y) \| \leq \sigma_D \) for all \( x, y \in \mathbb{R}^p \) and all \( i \in [n] \). Here, we use spectral norm for the Jacobian.

**Assumption 1.2.** There exist \( \sigma_F, \sigma_D \in [0, +\infty) \) such that the variance of \( F \) and \( F' \) is uniformly bounded, i.e., \( \mathbb{E}_\xi \left[ \| F'(x, \xi) - F(x, \xi) \|_2^2 \right] \leq \sigma_F^2 \) and \( \mathbb{E}_\xi \left[ \| F'(x, \xi) - F'(x, \xi) \|_2^2 \right] \leq \sigma_D^2 \), respectively. In the finite sum case (2), we again impose stronger conditions \( \| F_i(x) - F(x) \| \leq \sigma_F \) and \( \| F_i'(x) - F'(x) \| \leq \sigma_D \) for all \( x \in \mathbb{R}^p \) and for all \( i \in [n] \).

Assumptions 1.1 and 1.2 are standard and cover a wide class of models in practice as opposed to existing works. The stronger assumptions imposed on (2) allow us to develop adaptive subsampling schemes later.

**Related work:** Problem (1) or (2) has been widely studied in the literature under both deterministic (including the finite-sum (2) and \( n = 1 \)) and stochastic settings, see, e.g. [7, 8, 15, 20, 27, 30]. If \( q = 1 \) and \( \phi(u) = u \), then (1) reduces to the standard stochastic optimization model studied in, e.g. [11, 22]. In the deterministic setting, the common method to solve (1) is the Gauss-Newton (GN) scheme. This method has been studied in several papers, including [7, 8, 15, 20, 27, 30]. In such settings, GN only requires Assumption 1.1 to have global convergence guarantees [7, 20].

In the stochastic setting of the form (1), [30, 31] proposed stochastic compositional gradient descent methods to solve more general forms than (1), but they required a set of stronger assumptions than Assumptions 1.1 and 1.2 including the smoothness of \( \phi \). These methods eventually belong to gradient-based class. Other works in this direction include [16, 17, 33, 34, 32, 32], which also rely on similar approaches. Together with algorithms, convergence guarantees, stochastic oracle complexity bounds have also been estimated. For instance, [30] estimates \( O(\varepsilon^{-\delta}) \) oracle complexity for solving (1), while it is improved to \( O(\varepsilon^{-4.5}) \) in [31]. Recent works such as [35] further improve the complexity to \( O(\varepsilon^{-1}) \). However, these methods are completely different from GN and require much stronger assumptions, including the smoothness of \( \phi \) and \( F \).

One main challenge to design algorithms for (1) is the bias of stochastic estimators. Some researchers have tried to remedy this issue by proposing more sophisticated sampling schemes, see, e.g. [8]. Other works relies on biased estimators but using variance reduction techniques, e.g. [35].

**Challenges:** The stochastic formulation (1) creates several challenges for developing numerical methods. First, it is often nonconvex. Many papers consider special cases when \( \Psi \) is convex. This only holds if \( \phi \) is convex and \( F \) is linear, or \( \phi \) is convex and monotone and \( F \) is convex or concave. Clearly, such a setting is almost unrealistic or very limited. One can assume weak convexity of \( \Psi \) and add a regularizer to make the resulting problem convex but this completely changes the model. Second, \( \phi \) is often nonsmooth such as norm or gauge functions. This prevents the use of gradient-based methods. Third, even when both \( \phi \) and \( F \) are smooth, to guarantee Lipschitz continuity of \( \nabla \Psi \), it requires simultaneously \( F, F', \phi, \) and \( \phi' \) to be Lipschitz continuous. This condition is very restrictive and often requires additional bounded constraints or bounded domain assumption. Otherwise, it fails to hold even for bilinear functions. Finally, in stochastic settings, it is very challenging to form unbiased estimate for gradients or subgradients of \( \Psi \), making classical stochastic-based method inapplicable.
Our approach and contribution: Our main motivation is to overcome the above challenges by following a different approach. We extend the GN method from the deterministic setting [13, 20] to the stochastic setting [1]. Our methods can be viewed as inexact variants of GN using stochastic estimators for both function values \( F(x) \) and Jacobian \( F'(x) \). This approach allows us to cover a wide class of [1], while only requires standard assumptions as Assumptions 1.1 and 1.2. Our contribution can be summarized as follows:

(a) We develop an inexact GN framework to solve [1] and [2] using inexact estimations of \( F \) and its Jacobian \( F' \). This framework is independent of approximation schemes for generating approximate estimators. We characterize approximate stationary points of [1] and [2] via proximal gradient mappings. Then, we prove global convergence guarantee of our method to a stationary point under appropriate inexact computation.

(b) We analyze stochastic oracle complexity of our GN algorithm when mini-batch stochastic estimators are used. We separate our analysis into two cases. The first variant is to solve [1], where we obtain convergence guarantee in expectation. The second variant is to solve [2], where we use adaptive mini-batches and obtain convergence guarantee with high probability.

(c) We also provide oracle complexity of this algorithm when mini-batch SARAH estimators in [21] are used for both [1] and [2]. Under an additional and mild assumption, this estimator significantly improves the oracle complexity compared to the mini-batch stochastic one.

We believe that our methods are the first ones achieving global convergence rates and stochastic oracle complexity for solving [1] and [2] under standard assumptions. It is completely different from existing works such as [30, 31, 16, 34, 33, 35], where we only use Assumptions 1.1 and 1.2 while not imposing any special structure on \( \phi \) and \( F \), including smoothness. When using SARAH estimators, we impose the Lipschitz continuity of \( F \) to achieve better oracle complexity. This additional assumption is still much weaker than the ones used in existing works. However, without this assumption, our GN scheme with SARAH estimators still converges (see Remark 4.1).

Paper outline: The rest of this paper is organized as follows. Section 2 recalls some mathematical tools. Section 3 develops an inexact GN framework. Sections 4 analyzes convergence and complexity of the two stochastic variants of our GN framework using different stochastic estimators. Numerical examples are given in Section 5. All the proofs are deferred to the appendix.

2 Background and Mathematical Tools

We first characterize the optimality condition of [1]. Next, we recall the prox-linear mapping of the compositional function \( \Psi(x) := \phi(F(x)) \) and its properties.

Basic notation: We work with Euclidean spaces \( \mathbb{R}^p \) and \( \mathbb{R}^q \). Given a convex set \( X \), \( \text{dist}(u, X) := \inf_{x \in X} \|u - x\| \) denotes the Euclidean distance from \( u \) to \( X \). For a convex function \( f \), we denote \( \partial f \) its subdifferential, \( \nabla f \) its gradient, and \( f^* \) its Fenchel conjugate. For a smooth function \( F : \mathbb{R}^p \to \mathbb{R}^q \), \( F'(\cdot) \) denotes its Jacobian. For vectors, we use Euclidean norms, while for matrices, we use spectral norms, i.e. \( \|X\| := \sigma_{\max}(X) \). \( \lfloor \cdot \rfloor \) stands for number rounding.

2.1 Exact and Approximate Stationary Points

The optimality condition of [1] can be written as

\[
0 \in \partial \Psi(x^*) \equiv F'(x^*)^\top \partial \phi(F(x^*)),
\]

or equivalently

\[
\text{dist}(0, \partial \Psi(x^*)) = 0.
\]

Any \( x^* \) satisfying (5) is called a stationary point of [1] or [2].

Since \( \phi \) is convex, let \( \phi^* \) be the Fenchel conjugate of \( \phi \) and \( y^* \in \partial \phi(F(x^*)) \), then (5) can be rewritten as

\[
0 = F'(x^*)^\top y^* \quad \text{and} \quad 0 \in -F(x^*) + \partial \phi^*(y^*).
\]

Now, if we define

\[
\mathcal{E}(x, y) := \|F'(x)^\top y\| + \text{dist}(0, -F(x) + \partial \phi^*(y)),
\]

then the optimality condition (5) of [1] or [2] becomes

\[
\mathcal{E}(x^*, y^*) = 0.
\]
Note that once a stationary point \( x^* \) is available, we can compute \( y^* \) as any element \( y^* \in \partial \phi(F(x^*)) \) of \( \phi \circ F \).

In practice, we can only find an approximate stationary point \( \tilde{x} \) and its dual \( \tilde{y} \) such that \( (\tilde{x}, \tilde{y}) \) approximates \( (x^*, y^*) \) of (1) or (2) up to a given accuracy \( \varepsilon \geq 0 \) as follows:

**Definition 2.1.** Given \( \varepsilon > 0, \tilde{x}, \in \mathbb{R}^p \) is said to be an \( \varepsilon \)-stationary point of (1) if there exists \( \tilde{y} \in \mathbb{R}^q \) such that

\[
\mathcal{E}(\tilde{x}, \tilde{y}) \leq \varepsilon,
\]

where \( \mathcal{E}(\cdot) \) is defined by (7). This condition may holds in expectation, where \( \mathbb{E} [\cdot] \) is taken over all the randomness generated by the problem and the stochastic algorithm, or with high probability \( 1 - \delta \), which will be specified later.

### 2.2 Prox-Linear Operator and Its Properties

**(a) Prox-linear operator:** Since we assume that the Jacobian \( F'(\cdot) \) of \( F \) is Lipschitz continuous with a Lipschitz constant \( L_F \in (0, +\infty) \), and \( \phi \) is \( M_{\phi} \)-Lipschitz continuous as in Assumption 1.1, we have (see Appendix A):

\[
\phi(F(z)) \leq \phi(F(x) + F'(x)(z - x)) + \frac{M_{\phi} L_F}{2} \|z - x\|^2,
\]

for all \( z, x \in \mathbb{R}^p \). Given \( x \in \mathbb{R}^p \), let \( \bar{F}(x) \approx F(x) \) and \( \bar{J}(x) \approx F'(x) \) be approximation of \( F(x) \) and its Jacobian \( F'(x) \) (deterministic or stochastic), respectively. We consider the following approximate prox-linear model:

\[
\bar{T}_M(x) := \arg\min_{z \in \mathbb{R}^p} \left\{ \bar{Q}_M(z; x) := \phi(\bar{F}(x) + \bar{J}(x)(z - x)) + \frac{M}{2} \|z - x\|^2 \right\},
\]

where \( M > 0 \) is a given constant. As usual, if \( \bar{F}(x) = F(x) \) and \( \bar{J}(x) = F'(x) \), then

\[
T_M(x) := \arg\min_{z \in \mathbb{R}^p} \left\{ Q_M(z; x) := \phi(F(x) + F'(x)(z - x)) + \frac{M}{2} \|z - x\|^2 \right\},
\]

is the exact prox-linear operator of \( \Psi \). In this context, we also call \( \bar{T}_M(\cdot) \) an approximate prox-linear operator of \( \Psi \).

The optimality condition of \( \bar{T}_M(\cdot) \) becomes

\[
0 \in \bar{J}(x)^\top \partial \phi(\bar{F}(x) + \bar{J}(x)(\bar{T}_M(x) - x)) + M(\bar{T}_M(x) - x).
\]

**(b) Prox-gradient mapping:** We also define the prox-gradient mapping and its approximation, respectively as

\[
\begin{align*}
G_M(x) & := M(x - T_M(x)), \\
\bar{G}_M(x) & := M(x - \bar{T}_M(x)).
\end{align*}
\]

Clearly if \( \|G_M(x)\| = 0 \), then \( x = T_M(x) \) and \( x \) is a stationary point of (1). In our context, we can only compute \( \bar{G}_M(x) \) as an approximation of \( G_M(x) \).

**(c) Characterizing approximate stationary points:** The following lemma bounds the optimality error \( \mathcal{E}(\cdot) \) defined by (7) via the approximate prox-gradient mapping \( \bar{G}_M(x) \).

**Lemma 2.1.** Let \( \bar{T}_M(x) \) be computed by (11) and \( \bar{G}_M(x) \) be defined by (14). Then, \( \mathcal{E}(\bar{T}_M(x), y) \) of (1) defined by (7) with \( y \in \partial \phi(F(\bar{T}_M(x))) \) is bounded by

\[
\mathcal{E}(\bar{T}_M(x), y) := \text{dist} \left( 0, -F'(\bar{T}_M(x)) + \partial \phi^*(y) \right) + \|F'(\bar{T}_M(x))^\top y\|
\leq \left( 1 + \frac{M_{\phi} L_F}{M} \right) \|\bar{G}_M(x)\| + \frac{(1 + L_F)}{2M} \|\bar{G}_M(x)\|^2
+ \|F(x) - F(x)\| + \frac{1}{2} \|\bar{J}(x) - F'(x)\|^2.
\]
From Lemma 2.1 we can see that if we use exact oracles $\bar{F}(x) = F(x)$ and $\bar{J}(x) = F'(x)$, then
\[
\mathcal{E}(T_M(x), y) \leq \left(1 + \frac{M\phi L_F}{M}\right)\|G_M(x)\| + \frac{(1 + L_F)}{2M^2}\|G_M(x)\|^2.
\]
Furthermore, from (15), if we can guarantee $\|\bar{F}(x) - F(x)\| \leq \mathcal{O}(\varepsilon)$, $\|\bar{J}(x) - F'(x)\| \leq \mathcal{O}(\sqrt{\varepsilon})$, and $\|G_M(x)\| \leq \mathcal{O}(\varepsilon)$, then
\[
\mathcal{E}(\tilde{T}_M(x), y) \leq \mathcal{O}(\varepsilon),
\]
which shows that $\tilde{T}_M(x)$ is a $\mathcal{O}(\varepsilon)$-stationary point of (1) in the sense of Definition 2.1. Our goal is to approximate $F$ and $F'$ and compute $G_M(x)$ to guarantee these conditions.

3 Inexact Gauss-Newton Framework

In this section, we develop a conceptual inexact Gauss-Newton (IGN) framework for solving (1) and (2).

3.1 Descent Property and Approximate Conditions

Lemma 3.1 provides a key bound regarding (11), which will be used for convergence analysis of our algorithms.

**Lemma 3.1.** Let Assumption 1.1 hold and $\tilde{T}_M(x)$ be computed by (11). Then, for any $\beta_d > 0$, we also have
\[
\phi(F(\tilde{T}_M(x))) \leq \phi(F(x)) + 2L_\phi\|F(x) - \tilde{F}(x)\| + M_\phi\|F'(x) - \tilde{J}(x)\|\|x - \tilde{T}_M(x)\|
\]
\[
- \frac{(2M - M_\phi L_F)}{2}\|\tilde{T}_M(x) - x\|^2,
\]
\[
\leq \phi(F(x)) + 2L_\phi\|F(x) - \tilde{F}(x)\| + \frac{M_\phi}{2\beta_d}\|F'(x) - \tilde{J}(x)\|_F^2,
\]
\[
- \frac{(2M - M_\phi L_F - \beta_d L_\phi)}{2}\|\tilde{T}_M(x) - x\|^2.
\]

Since we approximate both $F$ and its Jacobian $F'$ in our prox-linear model (11), we assume that this approximation satisfies one of the following two conditions:

- **Condition 1:** Given a tolerance $\varepsilon > 0$ and $M > \frac{1}{2}M_\phi(L_F + \beta_d)$, at each iterate $x_t \in \mathbb{R}^p$, it holds that
  \[
  \begin{aligned}
  &\|\tilde{F}(x_t) - F(x_t)\| \leq \frac{C_g\varepsilon^2}{16M_\phi M^2}, \\
  &\|\tilde{J}(x_t) - F'(x_t)\| \leq \frac{\sqrt{2}\varepsilon C_g}{M\sqrt{2M_\phi}},
  \end{aligned}
  \]
  where $C_g := 2M - M_\phi (L_F + \beta_d) > 0$.

- **Condition 2:** Given $C_f > 0$, $C_\phi > 0$, and $\beta_d > 0$, let $C_g := 2M - M_\phi (L_F + \beta_d)$ and $C_a := M_\phi (2\sqrt{C_f} + \frac{C_\phi}{2\beta_d})$, where $M > 0$ is chosen such that $C_g > C_a$. For $x_0 \in \mathbb{R}^p$, we assume that
  \[
  \begin{aligned}
  &\|\tilde{F}(x_0) - F(x_0)\| \leq \frac{(C_g - C_a)^2}{16M_\phi M^2}, \\
  &\|\tilde{J}(x_0) - F'(x_0)\| \leq \frac{\sqrt{2}\varepsilon (C_g - C_a)}{M\sqrt{2M_\phi}},
  \end{aligned}
  \]
  while, for any iterate $x_t \in \mathbb{R}^p (t \geq 1)$, we assume that
  \[
  \begin{aligned}
  &\|\tilde{F}(x_t) - F(x_t)\| \leq \sqrt{C_f} \|x_t - x_{t-1}\|^2, \\
  &\|\tilde{J}(x_t) - F'(x_t)\| \leq \sqrt{C_d} \|x_t - x_{t-1}\|.
  \end{aligned}
  \]

The condition (17) assumes that both $\tilde{F}$ and $\tilde{J}$ should respectively well approximate $F$ and $F'$ up to a given accuracy $\varepsilon$. Here, the function value $F$ must have higher accuracy than its Jacobian $F'$. The condition (19) is adaptive, which depends on the norm $\|x_t - x_{t-1}\|$ of the iterates $x_t$ and $x_{t-1}$. This condition is less conservative than (17).

3.2 The Inexact Gauss-Newton Algorithm

We first develop a conceptual stochastic Gauss-Newton method as described in Algorithm 1.
Consequently, the total number of iterations $T$ to achieve
\[ \min_{0 \leq t \leq T} \| \tilde{G}_M(x_t) \| \leq \varepsilon \]
is at most
\[ T := \left\lfloor \frac{4M^2 \| \Psi(x_0) - \Psi^* \|}{\varepsilon^2} \right\rfloor = O \left( \frac{1}{\varepsilon^2} \right), \]
where $D := C_g$ for (a) and $D := C_g - C_a$ for (b).

**Remark 3.1.** The condition $\min_{0 \leq t \leq T} \| \tilde{G}_M(x_t) \| \leq \varepsilon$ shows that $\liminf_{t \to \infty, \varepsilon \downarrow 0} \| \tilde{G}_M(x_t) \| = 0$. That is there is subsequence $x_{t_k}$ of $\{x_t\}$ such that $\| \tilde{G}_M(x_{t_k}) \| \to 0$ as $k \to +\infty$ and $\varepsilon \to 0$.

### 4 Stochastic Gauss-Newton Methods

#### 4.1 SGN with Mini-Batch Stochastic Estimators

As a natural instance of Algorithm 1 we propose to approximate $F(x_t)$ and $F'(x_t)$ in Algorithm 1 by mini-batch stochastic estimators as:

\[
\begin{align*}
\tilde{F}(x_t) &:= \frac{1}{b_t} \sum_{i \in \mathcal{B}_t} F(x_t, \xi_i), \\
\tilde{J}(x_t) &:= \frac{1}{b_t} \sum_{i \in \mathcal{B}_t} F'(x_t, \xi_i),
\end{align*}
\]

where the mini-batches $\mathcal{B}_t$ and $\tilde{\mathcal{B}}_t$ are not necessarily independent, $b_t := |\mathcal{B}_t|$, and $\hat{b}_t := |\tilde{\mathcal{B}}_t|$. Using (22) we prove our first result in expectation on stochastic oracle complexity of Algorithm 1 for solving (1).

**Theorem 4.1.** Suppose that Assumptions 1.1 and 1.2 hold for 1. Let $\tilde{F}_t$ and $\tilde{J}_t$ defined by (22) be mini-batch stochastic estimators of $F(x_t)$ and $F'(x_t)$, respectively. Let $\{x_t\}$ be generated by Algorithm 1 (called SGN) to solve (1). Assume that $b_t$ and $\hat{b}_t$ in (22) are chosen as

\[
\begin{align*}
b_t &:= \left[ \frac{256M^2 \sigma^2}{C_g \varepsilon^4} \right] = O \left( \frac{\sigma^2}{\varepsilon^2} \right), \\
\hat{b}_t &:= \left[ \frac{2M^2 \sigma \beta_c \varepsilon^2}{\beta_c \varepsilon^2} \right] = O \left( \frac{\sigma^2}{\varepsilon^2} \right),
\end{align*}
\]
for some constant $C_f > 0$ and $C_d > 0$. Furthermore, let $\hat{x}_T$ be chosen uniformly randomly in $\{x_i\}_{t=0}^T$ as the output of Algorithm 1 after $T$ iterations. Then

$$E \left[ \|\tilde{G}_M(\hat{x}_T)\|^2 \right] = \frac{1}{(T+1)} \sum_{t=0}^T E \left[ \|\tilde{G}_M(x_t)\|^2 \right] \leq \frac{2M^2 \|\Psi(x_0) - \Psi^*\| + \varepsilon^2}{2}, \quad (24)$$

where $C_g := 2M - M_0(L_F + \beta_d)$ with $M > \frac{1}{2}M_0(L_F + \beta_d)$. Moreover, the number $T_f$ of function evaluations $F(x_t, \xi)$ and the number $T_d$ of Jacobian evaluations $F'(x_t, \xi)$ to achieve $E \left[ \|\tilde{G}_M(\hat{x}_T)\|^2 \right] \leq \varepsilon^2$ do not exceed

$$\begin{align*}
T_f & := \frac{1024M^6 \sigma_F^2 [\Psi(x_0) - \Psi^*]}{C_g \varepsilon^2} = O \left( \frac{\sigma_F^2}{\varepsilon^6} \right), \\
T_d & := \frac{8M^4 M_0 \sigma_D^2 [\Psi(x_0) - \Psi^*]}{\beta_d C_g \varepsilon^4} = O \left( \frac{\sigma_D^2}{\varepsilon^4} \right). 
\end{align*} \quad (25)$$

Note that this result still holds for (2) since it is a special case of (1). Now, we derive the convergence result for the stochastic Gauss-Newton methods described in Algorithm 1 under Assumptions 1.1 and 1.2.

**Theorem 4.2.** Suppose that Assumptions 1.1 and 1.2 hold for (2). Let $\overline{F}_i$ and $\overline{J}_i$ defined by (22) be mini-batch stochastic estimators to approximate $F(x_t)$ and $F'(x_t)$, respectively. Let $\{x_i\}$ be generated by Algorithm 1 for solving (2). Assume that $b_i$ and $b_t$ in (22) are chosen such that $b_i := \min \{ n, \hat{b}_i \}$ and $b_t := \min \{ n, \hat{b}_t \}$ for $t \geq 0$, with

$$\begin{align*}
\hat{b}_0 & := \left[ \frac{32M_0^2 \sigma_F (48\sigma_F M_0 M^2 + (C_g - C_a)\varepsilon^2)}{3(C_g - C_a)^2 \varepsilon^4} \right] \cdot \log \left( \frac{p+1}{\delta} \right), \\
\hat{b}_0 & := \left[ \frac{4M \sqrt{2M_0 \sigma_D} \left( \frac{3M \sqrt{2M_0 \sigma_D} + \beta_d (C_g - C_a)\varepsilon}{\beta_d (C_g - C_a)\varepsilon^2} \right)}{3C_g \| x_t - x_{t-1} - 1 \|^2} \cdot \log \left( \frac{p+q}{\delta} \right) \right], \\
b_t & := \left[ \frac{6\sigma_F^2 + 2\sigma_F \sqrt{C_f} \| x_t - x_{t-1} \|^2}{3C_f \| x_t - x_{t-1} \|^2} \right] \cdot \log \left( \frac{p+1}{\delta} \right) \quad (t \geq 1), \\
b_t & := \left[ \frac{6\sigma_D^2 + 2\sigma_D \sqrt{C_d} \| x_t - x_{t-1} \|^2}{3C_d \| x_t - x_{t-1} \|^2} \right] \cdot \log \left( \frac{p+q}{\delta} \right) \quad (t \geq 1),
\end{align*}$$

for $\delta \in (0, 1)$, and $C_f$ and $C_g$ given in **Condition 2.** Then, with probability at least $1 - \delta$, the bound (21) in Theorem 3.1 still holds. Moreover, the total number $T_f$ of function evaluations $F_i(x_t)$ and the total number $T_d$ of Jacobian evaluations $F_i'(x_t)$ to guarantee $\min_{0 \leq t \leq T} \|\tilde{G}_M(x_t)\| \leq \varepsilon$ do not exceed

$$\begin{align*}
T_f & := O \left( \frac{\sigma_F^2 [\Psi(x_0) - \Psi^*]}{\varepsilon^6} \cdot \log \left( \frac{p + 1}{\delta} \right) \right), \\
T_d & := O \left( \frac{\sigma_D^2 [\Psi(x_0) - \Psi^*]}{\varepsilon^4} \cdot \log \left( \frac{p + q}{\delta} \right) \right). \quad (27)
\end{align*}$$

To the best of our knowledge, the oracle complexity bounds stated in Theorems 4.1 and 4.2 are the first result for the stochastic Gauss-Newton methods described in Algorithm 1 under Assumptions 1.1 and 1.2. Whereas there exist several methods for solving (1), these algorithms are either not in the form of GN schemes as ours or rely on a different set of assumptions. For instance, [8, 9] considered a different model and used stochastic subgradient methods, while [33, 35] directly applied a variance reduction gradient descent method and required a stronger set of assumptions.

### 4.2 SGN with SARAH Estimators

Algorithm 1 with mini-batch stochastic estimators (22) has high oracle complexity bounds when $\varepsilon$ is sufficiently small, especially for the function evaluations $F'(\cdot, \xi)$. We attempt to reduce this complexity by exploiting a biased estimator called SARAH in [21] in this subsection.
Then, generated by Algorithm 2 to solve Assumption 4.1. Theorem 4.3. Suppose that Assumptions 1.1 and 1.2, and 4.1 are satisfied for then, the following theorem states the convergence and oracle complexity bounds of Algorithm 2.

\[
\begin{align*}
\tilde{F}_t &:= \tilde{F}_{t-1} + \frac{1}{m} \sum_{\xi_i \in B_t} (F(x_t, \xi_i) - F(x_{t-1}, \xi_i)), \\
J_t &:= J_{t-1} + \frac{1}{m} \sum_{\xi_i \in B_t} (F'(x_t, \xi_i) - F'(x_{t-1}, \xi_i)),
\end{align*}
\]

where the snapshots \(\tilde{F}_0\) and \(J_0\) are given, and \(B_t\) and \(\hat{B}_t\) are two mini-batches of size \(b_t := |B_t|\) and \(\hat{b}_t := |\hat{B}_t|\). Using both the standard stochastic estimators \((22)\) and these SARAH estimators \((28)\), we modify Algorithm 1 to obtain the following two-loop variant as in Algorithm 2.

**Algorithm 2 (SGN with SARAH estimators (SGN2))**

1. **Initialization:** Choose \(\bar{x}^0 \in \mathbb{R}^p\) and \(M > 0\).
2. For \(s := 1, \cdots, S\) do
3. Generate mini-batches \(B_s\) (size \(b_s\)) and \(\hat{B}_s\) (size \(\hat{b}_s\)).
4. Evaluate \(F_0(s)\) and \(J_0(s)\) at \(x_0(s) := \bar{x}^{s-1}\) from \((22)\).
5. Update \(x_1(s) := T_M(x_0(s))\) based on \((11)\).
6. **Inner Loop:** For \(t := 1, \cdots, m\) do
7. Generate mini-batches \(B_t(s)\) and \(\hat{B}_t(s)\).
8. Evaluate \(F_t(s)\) and \(J_t(s)\) from \((28)\).
9. Update \(x_{t+1}(s) := T_M(x_t(s))\) based on \((11)\).
10. **End of Inner Loop**
11. Set \(\bar{x}^s := x_{m+1}(s)\).
12. **End For**

In Algorithm 2 every outer iteration \(s\), we take a snapshot \(\bar{x}^s\) using \((22)\). Then, we run Algorithm 2 up to \(m\) iterations in the inner loop \(t\) but using SARAH estimators \((28)\).

Let us first prove convergence and oracle complexity estimates in expectation of Algorithm 2 for solving \((1)\). However, we require an additional assumption for this case:

**Assumption 4.1.** The function \(F\) is \(M_F\)-average Lipschitz continuous, i.e. \(\mathbb{E}_\xi \left[ \|F(x, \xi) - F(y, \xi)\|^2 \right] \leq M_F^2 \|x - y\|^2\) for all \(x, y \in \mathbb{R}^p\).

Though Assumption 4.1 is relatively strong, it has been used in several models, including neural network training under a bounded weight assumption.

Given a tolerance \(\varepsilon > 0\) and a constant \(C > 0\), we first choose \(M > 0\), \(\delta_d > 0\), and two constants \(\gamma_1 > 0\) and \(\gamma_2 > 0\) such that

\[
\left\{ \begin{array}{l}
\theta_F := 2M - M_0 (L_F + 6d) - \gamma_1 M_F^2 - \gamma_2 L_F^2 > 0 \\
m := \left[ \frac{\Psi(x^0) - \Psi^*}{\theta_F C \varepsilon} \right].
\end{array} \right.
\]

Next, we choose the mini-batch sizes of \(B_s\), \(\hat{B}_s\), \(B_t(s)\), and \(\hat{B}_t(s)\), respectively as follows:

\[
\left\{ \begin{array}{l}
b_s := \frac{2CM_0^2 \sigma_F^2}{\theta_F C \varepsilon} \\
\hat{b}_s := \frac{4CM_0 \sigma_F^2}{\theta_F \delta_d} \\
b_t(s) := \frac{8M_0^2 (m+1-t)}{\theta_F \gamma_1 \varepsilon} \\
\hat{b}_t(s) := \frac{M_0 (m+1-t)}{\gamma_2 \delta_d}.
\end{array} \right.
\]

Then, the following theorem states the convergence and oracle complexity bounds of Algorithm 2.

**Theorem 4.3.** Suppose that Assumptions 1.1 and 1.2 and 4.1 are satisfied for \((1)\). Let \(\{x_t(s)\}_t=0\rightarrow m\) be generated by Algorithm 2 to solve \((1)\). Let \(\theta_F\) and \(m\) be chosen by \((29)\), and the mini-batches \(b_s\), \(\hat{b}_s\), \(b_t(s)\), and \(\hat{b}_t(s)\) be set as in \((30)\). Assume that the output \(\bar{x}_T\) of Algorithm 2 is chosen uniformly randomly in \(\{x_t(s)\}_t=0\rightarrow m\). Then:
The total number of iterations $T$ to obtain $\mathbb{E}\left[\|\tilde{G}_M(x_t)\|^2\right] \leq \varepsilon^2$ is at most

$$
T := S(m + 1) = O\left(\frac{8M^2 \left[\Psi(\bar{x}^0) - \Psi^*\right]}{\theta_F \varepsilon^2}\right) = O\left(\frac{1}{\varepsilon^2}\right).
$$

Moreover, the total stochastic oracle calls $T_f$ and $T_d$ for evaluating stochastic estimators of $F(x_t, \zeta)$ and its Jacobian $F'(x_t, \zeta)$, respectively do not exceed:

$$
\begin{align*}
T_f & := \mathcal{O}\left(\frac{M^4 M_s^2 \left[\Psi(\bar{x}^0) - \Psi^*\right]}{\theta_F^2 \varepsilon^2}\right), \\
T_d & := \mathcal{O}\left(\frac{M^2 M_s \left[\Psi(\bar{x}^0) - \Psi^*\right]}{\theta_F \varepsilon^3}\right).
\end{align*}
$$

Finally, we show that $x_t$ computed by our methods is indeed an approximate stationary point of (1) or (2).

**Corollary 4.1.** If $x_t$ satisfies $\|\tilde{G}_M(x_t)\| \leq \varepsilon$ for given $\varepsilon > 0$, then under either **Condition 1** or **Condition 2**, and for any $y_t \in \partial \phi(F(x_t+1))$, we have $\mathcal{E}(x_{t+1}, y_t) \leq O(\varepsilon)$. Consequently, $x_{t+1}$ is a $O(\varepsilon)$-stationary point of (1) or (2).

**Proof.** From Lemma 2.1, we have

$$
\mathcal{E}(x_{t+1}, y_t) \leq \left(1 + \frac{M_s L_F}{M}\right) \|\tilde{G}_M(x_t)\| + \frac{1}{2} \|\tilde{J}_t - F'(x_t)\|^2 + \frac{(1 + L_F)}{2M^2} \|\tilde{G}_M(x_t)\|^2 + M_s \|\tilde{F}_t - F(x_t)\|.
$$

Under either **Condition 1** or **Condition 2**, we have $\|\tilde{F}_t - F(x_t)\| \leq O(\varepsilon^2)$ and $\|\tilde{J}_t - F'(x_t)\|^2 \leq O(\varepsilon^2)$. Hence, if $\|\tilde{G}_M(x_t)\| \leq \varepsilon$, then using these three bounds into the last estimate, one can show that $\mathcal{E}(x_{t+1}, y_t) \leq O(\varepsilon)$. Consequently, $x_{t+1}$ is a $O(\varepsilon)$-stationary point of (1) or (2).

**Remark 4.1 (Algorithm 2 without Assumption 4.1).** We claim that Algorithm 2 still converges without Assumption 4.1. However, its oracle complexity remains $O\left(\sigma_T^2 \varepsilon^{-6}\right)$ for $F$ and $O\left(\sigma_T^2 \varepsilon^{-4}\right)$ for $F'$ as in Algorithm 1. We therefore omit the proof of this statement.

Another main step of both Algorithms 1 and 2 is to compute $\tilde{T}_M(x_t)$. We will provide different routines in Sup. Doc. E to efficiently compute $\tilde{T}_M(x_t)$.

### 4.3 Extension to The Regularization Setting (3)

It is straightforward to extend our methods to handle a regularizer $g$ as in (3). If $g$ is nonsmooth and convex, then we can modify (11) as follows:

$$
\tilde{T}_M(x_t) := \arg\min_{z \in \mathbb{R}^p} \left\{ \tilde{Q}_M(z; x_t) := \phi(\tilde{F}(x_t) + \tilde{J}(x_t)(z - x_t)) + g(z) + \frac{M}{2} \|z - x_t\|^2 \right\}.
$$

Then, we obtain variants of Algorithms 1 and 2 for solving (3), where our theoretical guarantees in this paper remain preserved. This subproblem can be efficiently solved by a primal-dual method as presented in Appendix F. If $g$ is $L_g$-smooth, then we can replace $g$ in (11) by its quadratic surrogate $g(x_t) + \langle \nabla g(x_t), z - x_t \rangle + \frac{L_g}{2} \|z - x_t\|^2$.

### 5 Numerical Experiments

We conduct two numerical experiments to evaluate the performance of Algorithm 1 (SGN) and Algorithm 2 (SGN2). Further details of our experiments are in Appendix F.
5.1 Stochastic Nonlinear Equations

We consider a nonlinear equation: \( \mathbb{E}_\xi [\mathbf{F}(x, \xi)] = 0 \) as the expectation of a stochastic function \( \mathbf{F} : \mathbb{R}^p \times \Omega \rightarrow \mathbb{R}^q \).

This equation can be viewed as a natural extension of nonlinear equations from a deterministic setting to a stochastic setting, including stochastic dynamic systems and PDEs. It can also present as the first-order optimality condition \( \mathbb{E}_\xi [\nabla \mathbf{G}(x, \xi)] = 0 \) of a stochastic optimization problem \( \min_{x} \mathbb{E}_\xi [\mathbf{G}(x, \xi)] \). Moreover, it can be considered as a special case of stochastic variational inequality in the literature, see, e.g., [23].

Instead of directly solving \( \mathbb{E}_\xi [\mathbf{F}(x, \xi)] = 0 \), we can formulate it into the following minimization problem:

\[
\min_{x \in \mathbb{R}^p} \left\{ \Psi(x) := \|\mathbb{E}_\xi [\mathbf{F}(x, \xi)]\| \right\},
\]

where \( \mathbf{F}(x, \xi) := (\mathbf{F}_1(x, \xi), \mathbf{F}_2(x, \xi), \ldots, \mathbf{F}_q(x, \xi))^\top \) such that \( \mathbf{F}_i : \mathbb{R}^p \rightarrow \mathbb{R} \) is the expectation of \( \mathbf{F}_i(\cdot, \xi) \), i.e., \( \mathbf{F}_i(x) := \mathbb{E}_\xi [\mathbf{F}_i(x, \xi)] \) for \( i = 1, \ldots, q \), and \( \|\cdot\| \) is a given norm (e.g., \( \ell_2 \) or \( \ell_1 \)-norm).

Assume that we take average approximation of \( \mathbb{E}_\xi [\mathbf{F}(x, \xi)] \) to obtain a finite sum \( F(x) = \frac{1}{n} \sum_{i=1}^{n} F_i(x) \) for sufficiently large \( n \). In the following experiments, we choose \( q = 4 \), and for \( i = 1, \ldots, n \), we choose the component \( F_i \) as:

\[
F_i(x) := \begin{bmatrix}
(1 - \tanh(y_i(a_i^\top x + b_i))) \\
(1 - \frac{1}{1 + \exp(-y_i(a_i^\top x + b_i))})^2 \\
\ln(1 + e^{-y_i(a_i^\top x + b_i)}) - \ln(1 + e^{-y_i(a_i^\top x + b_i) - 1}) \\
\ln(1 + (y_i(a_i^\top x + b_i) - 1)^2)
\end{bmatrix}
\]

where \( a_i \) is the \( i \)-th row of an input matrix \( \mathbf{A} \in \mathbb{R}^{n \times p} \), and \( y \in \{-1, 1\}^n \) and \( b \in \mathbb{R}^n \) are two input vectors. These functions were used in binary classification involving nonconvex losses, e.g., [37]. Since they are nonnegative, (34) can be considered as simultaneously solving the binary classification with 4 different losses (see Appendix F).

We implement both Algorithms 1 (SGN) and 2 (SGN2) to solve (34). We also compare them with the baseline using the full samples instead of calculating \( \tilde{J} \) and \( \tilde{F} \) as in [22] and [28]. We call it the deterministic GN scheme (GN).

**Experiment setup:** We test three algorithms on one synthetic and two standard datasets: w8a and covtype from LIBSVM. To get sufficiently large number of samples, we upsample these two datasets by bootstrapping to obtain \( n = 10^6 \) samples. For the synthetic dataset, we generate \( \mathbf{A} \), \( y \), and \( b \) as described in Appendix F.

![Figure 1: The performance of 3 algorithms on the synthetic data.](https://www.csie.ntu.edu.tw/~cjlin/libsvm/)

To find appropriate batch sizes for \( \tilde{J} \) and \( \tilde{F} \), we perform a grid search over the mini-batch sizes of \( \left\{ \frac{n}{100}, \frac{n}{100}, \frac{n}{100}, \frac{n}{100}, \frac{n}{100}, \frac{n}{100}, \frac{n}{100}, \frac{n}{10}, \frac{n}{2}, n \right\} \) to estimate the best ones. We obtain \( b_t = \tilde{b}_t^{(s)} := 10^4 \) for \( \tilde{J} \) in both SGN and SGN2, while \( b_t := 10^6 \) in SGN and \( b_t^{(s)} := 10^5 \) in SGN2 for \( \tilde{F} \).

We experiment on two different instances of (34) using either \( \phi(\cdot) = \|\cdot\|_1 \) or \( \phi(\cdot) = \|\cdot\|_2 \). The performance of three algorithms is shown in Figure 1 for the synthetic dataset. This figure depicts the relative objective residuals \( \frac{\phi(x^\ast) - \phi^\ast}{\phi(x^\ast)} \) over the number of used samples where \( \phi^\ast \) is obtained by running GN for a really long
time. In both cases, SGN2 works best while SGN is still much better than the baseline GN in terms of sample efficiency. However, due to the choice of mini-batches, both SGN and SGN2 are saturated at a certain level of accuracy $\varepsilon$ as indicated in Theorems 4.1 and 4.3, while SGN2 still achieves lower residuals than SGN.

For the w8a and covtype datasets, we observe similar behavior as shown in Figure 3, where SGN2 is more efficient than SGN, and both SGN schemes outperform GN. Nevertheless, SGN does not highly outperform the baseline GN as in the synthetic dataset.

### 5.2 Optimization Involving Expectation Constraints

We consider the following optimization problem:

$$\min_{x \in \mathbb{R}^p} \left\{ g(x) \text{ s.t. } \mathbb{E}_\xi [F(x, \xi)] \leq 0 \right\},$$

where $g : \mathbb{R}^p \rightarrow \mathbb{R} \cup \{+\infty\}$ is a convex function, possibly nonsmooth, and $F : \mathbb{R}^p \times \Omega \rightarrow \mathbb{R}^q$ is a smooth stochastic function. This problem has various applications such as optimization with conditional value at risk (CVaR) constraints and metric learning [14] among others. Let us consider an exact penalty formulation of (35) as

$$\min_{x \in \mathbb{R}^p} \left\{ \Psi(x) := g(x) + \phi(\mathbb{E}_\xi [F(x, \xi)]) \right\},$$

where $\phi(u) := \rho \sum_{i=1}^q |u_i|$ with $|u_i| := \max\{0, u_i\}$ and $\rho > 0$ is a given penalty parameter. Clearly, (36) coincides with (35), an extension of (35).

We evaluate 3 algorithms on the asset allocation problem [24] as an instance of (35):

$$\begin{cases}
\min_{x \in \mathbb{R}^p, \tau \in [\xi, \bar{\tau}]} -c^\top z + \phi \left( \tau + \frac{1}{\rho} \sum_{i=1}^n [-\xi_i^\top z - \tau]\right)
\text{s.t. } z \in \Delta_p := \{ \hat{z} \in \mathbb{R}_+^q | \sum_{i=1}^q \hat{z}_i = 1 \}.
\end{cases}$$

To apply our methods, we smooth $[u]_+$ by $u + \frac{\sqrt{u^2+\gamma^2} - \gamma}{2}$ for a sufficiently small $\gamma > 0$. If we introduce $x := (z, \tau)$, $F_i(x) := \tau + \frac{1}{\rho} \left( \sqrt{(\xi_i^\top z + \tau)^2 + \gamma^2 - \xi_i^\top z - \tau - \gamma} \right)$ for $i = 1, \ldots, n$, and $g(x) = -c^\top z + \delta_{\Delta_p}(x)$, then we can reformulate the smoothed approximation of (37) into (35), where $\delta_{\Delta_p \times [\xi, \bar{\tau}]}$ is the indicator of $\Delta_p \times [\xi, \bar{\tau}]$. Note that $F_i$ is Lipschitz continuous with the Lipschitz constant $L_{F_i} := \|\xi_i\|_2 / 2\rho$. In our experiments, we choose $[\xi, \bar{\tau}]$ to be $[0, 1]$, $\beta := 0.1$, and $\gamma := 10^{-3}$. We were experimenting different $\rho$ and $M$, and eventually set $\rho := 5$ and $M := 5$.

We test three algorithms: GN, SGN, and SGN2 on both synthetic and real datasets. On one hand, we follow the code from [13] to generate the data with $n \in \{5 \times 10^4, 10^5\}$ and $p \in \{200, 300, 500\}$. On the other hand, we obtain real datasets of US stock prices for 889 or 865 types of stocks as described, e.g., [24] then bootstrap them to obtain different datasets of sizes $n = 5 \times 10^4$ and $n = 10^5$. The details of data generation and additional results are given in Appendix F.
This section provides the full proof of technical results in Section 2. Let us first recall the bound (10). The proof of this bound can be found, e.g., in [20]. However, for completeness, we prove it here.

The performance of three algorithms on these datasets is depicted in Figure 3. SGN is still much better than GN in both experiments while SGN2 is the best among three.

Numerical results have confirmed the advantages of SGN and SGN2 which well align with our theoretical analysis.

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A Appendix: Proof of Technical Results in Section 2

This section provides the full proof of technical results in Section 2. Let us first recall the bound (10). The proof of this bound can be found, e.g., in [20]. However, for completeness, we prove it here.

The proof of (10). Since $F'$ is $L_F$-Lipschitz continuous with a Lipschitz constant $L_F$, we have $\|F(y) - F(x) - F'(x)(y-x)\| \leq \frac{L_F}{2} \|y - x\|^2$ for any $x, y \in \mathbb{R}^p$. On the other hand, since $\phi$ is $M_\phi$-Lipschitz continuous, we have $\phi(u) \leq \phi(v) + M_\phi \|u - v\|$ for any $u, v \in \mathbb{R}^q$. Hence, we have

$$\phi(F(y)) \leq \phi(F(x) + F'(x)(y-x)) + M_\phi \|F(y) - F(x) - F'(x)(y-x)\| \leq \phi(F(x) + F'(x)(y-x)) + \frac{M_\phi L_F}{2} \|y - x\|^2,$$

which proves (10).

A.1 The Proof of Lemma 2.1: Approximate Optimality Condition

Lemma 2.1. Suppose that Assumption 1.1 holds. Let $\tilde{T}_M(x)$ be computed by (11) and $\tilde{G}_M(x)$ be defined by (14). Then, $\mathcal{E}(\tilde{T}_M(x), y)$ of (1) or (2) defined by (7) with $y \in \partial \phi(F(\tilde{T}_M(x)))$ is bounded by

$$\mathcal{E}(\tilde{T}_M(x), y) := \text{dist}\left(0, -F(\tilde{T}_M(x)) + \partial \phi^*(y)\right) + \|F'(\tilde{T}_M(x))^\top y\|$$

$$\leq \left(1 + \frac{M_\phi L_F}{M}ight) \|\tilde{G}_M(x)\| + \frac{(1 + L_F)}{2M^2} \|\tilde{G}_M(x)\|^2 + \|F(x) - F(x)\| + \frac{1}{2} \|\tilde{J}(x) - S_F(x)\|^2. \tag{15}$$

Proof. First, we can rewrite (13) as follows:

$$\begin{cases}
    r_F(x) = F'(\tilde{T}_M(x))^\top y \\
    r_D(x) \in -F(\tilde{T}_M(x)) + \partial \phi^*(y). 
\end{cases} \tag{38}$$

Figure 3: The performance of three algorithms on two datasets.
where

\[
\begin{aligned}
   r_F(x) &:= M(x - \tilde{T}_M(x)) + (F'(\tilde{T}_M(x)) - \tilde{J}(x))^\top y \\
   r_D(x) &:= \tilde{F}(x) + \tilde{J}(x)(\tilde{T}_M(x) - x) - F(\tilde{T}_M(x)).
\end{aligned}
\]

(39)

Since \( y \in \partial \phi(\tilde{F}(x) + \tilde{J}(x)(\tilde{T}_M(x) - x)) \) and \( \phi \) is \( M_\phi \)-Lipschitz continuous, we can bound \( y \) as \( \|y\| \leq M_\phi \). Now, we need to bound \( r_F \) as follows:

\[
\|r_F(x)\| = \|M(x - \tilde{T}_M(x)) + (F'(\tilde{T}_M(x)) - \tilde{J}(x))^\top y\|
\leq M\|x - \tilde{T}_M(x)\| + \|F'(\tilde{T}_M(x)) - \tilde{J}(x)\|_F\|y\| + \|F'(x) - \tilde{J}(x)\|_F\|y\|
\leq \|\tilde{G}_M(x)\| + M_\phi\|F'(\tilde{T}_M(x)) - \tilde{J}(x)\|_F + M_\phi\|F'(x) - \tilde{J}(x)\|
\leq (1 + \frac{M_\phi L_\phi}{M})\|\tilde{G}_M(x)\| + M_\phi\|F'(x) - \tilde{J}(x)\|.
\]

Similarly, we can also bound \( r_D \) as

\[
\|r_D(x)\| = \|\tilde{F}(x) + \tilde{J}(x)(\tilde{T}_M(x) - x) - F(\tilde{T}_M(x))\|
\leq \|\tilde{F}(x) - F(x)\| + \|\tilde{F}(x) + F'(\tilde{T}_M(x) - x) - F(\tilde{T}_M(x))\| + ||\tilde{J}(x) - F'(x)||_F||\tilde{T}_M(x) - x||
\leq \|\tilde{F}(x) - F(x)\| + \frac{1}{M}\|\tilde{T}_M(x) - x\|^2 + \frac{1}{2}\|F'(x) - \tilde{J}(x)\|^2 + \frac{1}{2}\|\tilde{T}_M(x) - x\|^2
\leq \|\tilde{F}(x) - F(x)\| + \frac{1}{2}\|F'(x) - \tilde{J}(x)\|^2 + \|\tilde{F}(x) - F(x)\| + \frac{1}{2}\|F'(x) - \tilde{J}(x)\|^2.
\]

Combining these bounds, we can show that

\[
\mathcal{E}(\tilde{T}_M(x), y) := \|F'(\tilde{T}_M(x))^\top y\| + \text{dist}\left(0, -F(\tilde{T}_M(x)) + \partial \phi^*(y)\right)
\leq \left(1 + \frac{M_\phi L_\phi}{M}\right)\|\tilde{G}_M(x)\| + \|\tilde{G}_M(x)\|^2 + \|\tilde{F}(x) - F(x)\| + \frac{1}{2}\|F'(x) - \tilde{J}(x)\|^2,
\]

which is exactly \( \|y\| \) \( \square \).

## B Appendix: The Proof of Technical Results in Section 3

This appendix provides the full proof of technical results in Section 3 on convergence of the inexact Gauss-Newton framework, Algorithm I.

### B.1 The Proof of Lemma 3.1: Descent Property

**Lemma 3.1** Let Assumption 3.1 hold, \( \tilde{T}_M(x) \) be computed by (11), and \( \tilde{G}_M(x) := M(x - \tilde{T}_M(x)) \) be the prox-gradient mapping of \( F \). Then, for any \( \tilde{z} \in \mathbb{R}^p \), we have

\[
\phi(\tilde{F}(x) + \tilde{J}(x)(\tilde{T}_M(x) - x)) \leq \phi(\tilde{F}(x) + \tilde{J}(x)(z - x)) - (\tilde{G}_M(x), z - x) - \frac{1}{M}\|\tilde{G}_M(x)\|^2.
\]

(40)

For any \( \beta_\delta > 0 \), we also have

\[
\phi(\tilde{F}(\tilde{T}_M(x))) \leq \phi(F(x)) + 2L_\phi\|F(x) - \tilde{F}(x)\| + M_\phi\|F'(x) - \tilde{J}(x)\|\|x - \tilde{T}_M(x)\| - \frac{(2M - M_\phi L_\phi)}{2}\|\tilde{T}_M(x) - x\|^2
\leq \phi(F(x)) + 2L_\phi\|F(x) - \tilde{F}(x)\| + \frac{M_\phi}{\beta_\delta^2}\|F'(x) - \tilde{J}(x)\|^2 + \frac{(2M - M_\phi L_\phi - \beta_\delta L_\phi)}{2M^2}\|\tilde{G}_M(x)\|^2.
\]

(16)

The optimality condition (13) can be written as

\[
\tilde{J}(x)^\top y = M(x - \tilde{T}_M(x)) \quad \text{and} \quad y \in \partial \phi(\tilde{F}(x) + \tilde{J}(x)(\tilde{T}_M(x) - x)).
\]
\begin{align*}
\phi(\tilde{F}(x) + \tilde{J}(x)(z - x)) &\geq \phi(\tilde{F}(x) + \tilde{J}(x)(\tilde{T}_M(x) - x)) + (y, \tilde{F}(x) + \tilde{J}(x)(z - x) - (\tilde{F}(x) + \tilde{J}(x)(\tilde{T}_M(x) - x))) \\
&\geq \phi(\tilde{F}(x) + \tilde{J}(x)(\tilde{T}_M(x) - x)) + (\tilde{J}(x)^\top y, z - \tilde{T}_M(x)) \\
&= \phi(\tilde{F}(x) + \tilde{J}(x)(\tilde{T}_M(x) - x)) + M(z - \tilde{T}_M(x), x - \tilde{T}_M(x)) \\
&= \phi(\tilde{F}(x) + \tilde{J}(x)(\tilde{T}_M(x) - x)) + M(x - \tilde{T}_M(x), z - x) + M\|x - \tilde{T}_M(x)\|^2 \\
&= \phi(\tilde{F}(x) + \tilde{J}(x)(\tilde{T}_M(x) - x)) + (\tilde{G}_M(x), z - x) + \frac{1}{M}\|\tilde{G}_M(x)\|^2,
\end{align*}

which implies \cite{40}.

Now, combining \cite{10} and \cite{40}, we can show that
\begin{align*}
\phi(F(\tilde{T}_M(x))) &\leq \phi(F(x) + F'(x)(\tilde{T}_M(x) - x)) + \frac{M_\phi M L_F}{2}\|\tilde{T}_M(x) - x\|^2 \\
&\leq \phi(\tilde{F}(x) + \tilde{J}(x)(\tilde{T}_M(x) - x)) + \frac{M_\phi M L_F}{2}\|\tilde{T}_M(x) - x\|^2 \\
&\quad + |\phi(F(x) + F'(x)(\tilde{T}_M(x) - x)) - \phi(\tilde{F}(x) + \tilde{J}(x)(\tilde{T}_M(x) - x))| \\
&\leq \phi(\tilde{F}(x) + \tilde{J}(x)(z - x)) - M(x - \tilde{T}_M(x), z - x) - \frac{2M_\phi M L_F}{2}\|\tilde{T}_M(x) - x\|^2 \\
&\quad + M_\phi\|F(x) - \tilde{F}(x) + [F'(x) - \tilde{J}(x)](\tilde{T}_M(x) - x)\| \\
&\leq \phi(F(x)) - \frac{2M_\phi M L_F}{2}\|\tilde{T}_M(x) - x\|^2 + M_\phi\|F(x) - \tilde{F}(x) - \tilde{J}(x)(z - x)\| \\
&\quad - M(x - \tilde{T}_M(x), z - x) + M_\phi\|F'(x) - \tilde{J}(x))(\tilde{T}_M(x) - x)\|.
\end{align*}

Now, substituting \( z = x \) into this estimate, we obtain
\begin{align*}
\phi(F(\tilde{T}_M(x))) &\leq \phi(F(x)) - \frac{2M_\phi M L_F}{2}\|\tilde{T}_M(x) - x\|^2 + 2M_\phi\|F(x) - \tilde{F}(x)\| \\
&\quad + M_\phi\|F'(x) - \tilde{J}(x))(\tilde{T}_M(x) - x)\|. 
\end{align*}
(41)

Using the Cauchy-Schwarz inequality, we have
\begin{equation}
||(F'(x) - \tilde{J}(x))(\tilde{T}_M(x) - x)|| \leq \|F'(x) - \tilde{J}(x)\|\|\tilde{T}_M(x) - x\|.
\end{equation}
(42)

Next, applying Young’s inequality to the right hand side of (42), for any \( \beta_d > 0 \), we obtain
\begin{equation}
||(F'(x) - \tilde{J}(x))(\tilde{T}_M(x) - x)|| \leq \|F'(x) - \tilde{J}(x)\|\|\tilde{T}_M(x) - x\| \leq \frac{1}{2\beta_d}\|F'(x) - \tilde{J}(x)\|^2 + \frac{\beta_d}{2}\|\tilde{T}_M(x) - x\|^2.
\end{equation}
(43)

Finally, plugging (43) into (41), we have
\begin{align*}
\phi(F(\tilde{T}_M(x))) &\leq \phi(F(x)) - \frac{2M_\phi M L_F}{2}\|\tilde{T}_M(x) - x\|^2 + 2L_\phi\|F(x) - \tilde{F}(x)\| + M_\phi\|F'(x) - \tilde{J}(x))(\tilde{T}_M(x) - x)\| \\
&\leq \phi(F(x)) - \frac{2M_\phi M L_F - \beta_d L_\phi}{2}\|\tilde{T}_M(x) - x\|^2 + 2L_\phi\|F(x) - \tilde{F}(x)\| + \frac{M_\phi}{2\beta_d}\|F'(x) - \tilde{J}(x)\|^2,
\end{align*}
for any \( \beta_d > 0 \), which exactly implies \cite{16}.

\section*{B.2 The Proof of Theorem 3.1: Convergence Rate of Algorithm 1}

\textbf{Theorem 3.1} Assume that Assumptions 1.1 and 1.2 are satisfied. Let \( \{x_t\} \) be generated by Algorithm 2 to solve either (1) or (2). Then, the following statements hold:
(a) If \( (17) \) holds for some \( \varepsilon \geq 0 \), then
\begin{equation}
\min_{0 \leq t \leq T} \|\tilde{G}_M(x_t)\|^2 = \frac{1}{(T + 1)} \sum_{t=0}^{T} \|\tilde{G}_M(x_t)\|^2 \leq \frac{2M^2 \|\Psi(x_0) - \Psi^*\|}{C_g(T + 1)} + \varepsilon^2, 
\end{equation}
(20)
where \( C_g := 2M - M_\phi(L_F + \beta_d) \) for \( M > \frac{1}{2}M_\phi(L_F + \beta_d) \).
\( \text{(b) If } \) (18) and (19) hold for given \( C_g > C_a > 0 \), then
\[
\min_{0 \leq t \leq T} \| \tilde{G}_f(x_t) \|^2 = \frac{1}{(T+1)} \sum_{t=0}^{T} \| \tilde{G}_f(x_t) \|^2 \leq \frac{2M^2 [\Psi(x_0) - \Psi^*]}{(C_g - C_a)(T+1)} + \frac{\varepsilon^2}{2}. \tag{21} \]
Consequently, the total number of iterations \( T \) to achieve \( \min_{0 \leq t \leq T} \| \tilde{G}_f(x_t) \| \leq \varepsilon \) is at most
\[
T := \left[ \frac{4M^2 [\Psi(x_0) - \Psi^*]}{D\varepsilon^2} \right] = O \left( \frac{[\Psi(x_0) - \Psi^*]}{\varepsilon^2} \right),
\]
where \( D := C_g \) for (a) and \( D := C_g - C_a \) for (b).

**Proof.** Using the second inequality of (16) with \( x := x_t \) and \( T_f(x) = x_{t+1} \), we have
\[
\phi(F(x_{t+1})) \leq \phi(F(x_t)) - \frac{(2M - M_\phi(L_F + \beta_d))}{2} \| x_{t+1} - x_t \|^2 + 2M_\phi \| F(x_t) - \bar{F}_t \| + \frac{M_\phi \| F'(x_t) - \tilde{J}_t \|^2}{2\beta_d}. \tag{44}
\]
(a) If \( 17 \) holds for some \( \varepsilon \geq 0 \), then using \( 17 \) into \( 44 \), we have
\[
\phi(F(x_{t+1})) \leq \phi(F(x_t)) - \frac{C_g}{2} \| x_{t+1} - x_t \|^2 + 2M_\phi \cdot \frac{C_g \varepsilon^2}{16M_\phi M^2} + \frac{M_\phi \beta_d \varepsilon^2}{4M_\phi M^2},
\]
where \( C_g := 2M - M_\phi(L_F + \beta_d) > 0 \). Since \( \Psi(x) = \phi(F(x)) \), the last estimate leads to
\[
\Psi(x_{t+1}) \leq \Psi(x_t) - \frac{C_g}{2} \| x_{t+1} - x_t \|^2 + \frac{C_g \varepsilon^2}{4M^2}.
\]
By induction, \( \tilde{G}_f(x_t) := M_f(x_t - T_f(x_t)) \), and \( \Psi(x_{t+1}) \geq \Psi^* \), we can show that
\[
\frac{1}{M^2(T+1)} \sum_{t=0}^{T} \| \tilde{G}_f(x_t) \|^2 = \frac{1}{T+1} \sum_{t=0}^{T} \| x_{t+1} - x_t \|^2 \leq \frac{2[\Psi(x_0) - \Psi^*]}{C_g(T+1)} + \frac{\varepsilon^2}{2M^2}. \tag{45}
\]
which leads to (20).

(b) If (18) and (19) are used, then from (44) and (19), we have
\[
\phi(F(x_{t+1})) \leq \phi(F(x_t)) - \frac{C_g}{2} \| x_{t+1} - x_t \|^2 + \frac{C_g \varepsilon^2}{2\beta_d}, \forall \ v \geq 1.
\]
where \( C_g := 2M - M_\phi(L_F - \beta_d)M_\phi \) and \( C_a := 2M_\phi \sqrt{\beta_d} + \frac{M_\phi C_d}{2\beta_d} \). For \( t = 0 \), it follows from (44) and (18) that
\[
\phi(F(x_1)) \leq \phi(F(x_0)) - \frac{C_g}{2} \| x_1 - x_0 \|^2 + \frac{(C_g - C_a)\varepsilon^2}{4M^2}.
\]
Now, note that \( \Psi(x) = \phi(F(x)) \), the last two estimates respectively become
\[
\Psi(x_{t+1}) \leq \Psi(x_t) - \frac{C_g}{2} \| x_{t+1} - x_t \|^2 + \frac{C_g \varepsilon^2}{2\beta_d}, \forall \ v \geq 1,
\]
and for \( t = 0 \), it holds that
\[
\Psi(x_1) \leq \Psi(x_0) - \frac{C_g}{2} \| x_1 - x_0 \|^2 + \frac{(C_g - C_a)\varepsilon^2}{4M^2}.
\]
By induction and \( \Psi^* \leq \Psi(x_{t+1}) \), this estimate leads to
\[
\Psi^* \leq \Psi(x_{t+1}) \leq \Psi(x_0) - \frac{(C_g - C_a)}{2} \sum_{t=0}^{T} \| x_{t+1} - x_t \|^2 + \frac{(C_g - C_a)\varepsilon^2}{4M^2}.
\]
Since \( C_g > C_a \), the last inequality implies
\[
\frac{1}{M^2(T+1)} \sum_{t=0}^{T} \| \tilde{G}_f(x_t) \|^2 = \frac{1}{T+1} \sum_{t=0}^{T} \| x_{t+1} - x_t \|^2 \leq \frac{2[\Psi(x_0) - \Psi^*]}{(C_g - C_a)(T+1)} + \frac{\varepsilon^2}{4M^2},
\]
which leads to (21). The last statement of this theorem is a direct consequence of either (20) or (21), and
\[
\min_{0 \leq t \leq T} \| \tilde{G}_f(x_t) \|^2 \leq \frac{1}{(T+1)} \sum_{t=0}^{T} \| \tilde{G}_f(x_t) \|^2.
\]
Appendix: High Probability Inequalities and Variance Bounds

Since our methods are stochastic, we recall some mathematical tools from high probability and concentration theory, as well as variance bounds that will be used for our analysis. First, we need the following lemmas to estimate sample complexity of our algorithms.

**Lemma C.1** (Matrix Bernstein inequality \cite{matrix_bernstein}(Theorem 1.6)). Let $X_1, X_2, \ldots, X_n$ be independent random matrices in $\mathbb{R}^{p_1 \times p_2}$. Assume that $\mathbb{E}[X_i] = 0$ and $\|X_i\| \leq R$ a.s. for $i = 1, \ldots, n$ and given $R > 0$, where $\| \cdot \|$ is the spectral norm. Define $\sigma_X^2 := \max \{ \| \sum_{i=1}^n \mathbb{E} \{ [X_i X_i^T] \} \|, \| \sum_{i=1}^n \mathbb{E} \{ X_i^T X_i \} \| \}$. Then, for any $\epsilon > 0$ we have

$$\text{Prob} \left( \left\| \sum_{i=1}^n X_i \right\| \geq \epsilon \right) \leq (p_1 + p_2) \exp \left( -\frac{3\epsilon^2}{6\sigma_X^2 + 2R\epsilon} \right).$$

As a consequence, if $\sigma_X^2 \leq \bar{\sigma}_X^2$ for a given $\bar{\sigma}_X^2 > 0$, then

$$\text{Prob} \left( \left\| \sum_{i=1}^n X_i \right\| \leq \epsilon \right) \geq 1 - (p_1 + p_2) \exp \left( -\frac{3\epsilon^2}{6\bar{\sigma}_X^2 + 2R\epsilon} \right).$$

**Lemma C.2** (\cite{cataly}). Let $\tilde{F}(x_t)$ and $\tilde{J}(x_t)$ be the mini-batch stochastic estimators of $F(x_t)$ and $F'(x_t)$ defined by \cite{cataly}, respectively, and $\mathcal{F}_t := \sigma(x_0, x_1, \ldots, x_{t-1})$ be the $\sigma$-field generated by $\{x_0, x_1, \ldots, x_{t-1}\}$. Then, these are unbiased estimators, i.e., $\mathbb{E} \left[ \tilde{F}(x_t) | \mathcal{F}_t \right] = F(x_t)$ and $\mathbb{E} \left[ \tilde{J}(x_t) | \mathcal{F}_t \right] = F'(x_t)$. Moreover, under Assumption 1.2 we have

$$\mathbb{E} \left[ \| \tilde{F}(x_t) - F(x_t) \|^2 | \mathcal{F}_t \right] \leq \frac{\sigma_{\tilde{F}}^2}{b_t} \quad \text{and} \quad \mathbb{E} \left[ \| \tilde{J}(x_t) - F'(x_t) \|^2 | \mathcal{F}_t \right] \leq \frac{\sigma_{\tilde{D}}^2}{b_t}. \quad (46)$$

**Lemma C.3** (\cite{cataly, matrix_bernstein}). Let $\tilde{F}_t$ and $\tilde{J}_t$ be the mini-batch SARAH estimators of $F(x_t)$ and $F'(x_t)$, respectively defined by \cite{cataly}, and $\mathcal{F}_t := \sigma(x_0, x_1, \ldots, x_{t-1})$ be the $\sigma$-field generated by $\{x_0, x_1, \ldots, x_{t-1}\}$. Then, we have the following estimate

$$\mathbb{E} \left[ \| \tilde{F}_t - F(x_t) \|^2 | \mathcal{F}_t \right] = \| \tilde{F}_{t-1} - F(x_{t-1}) \|^2 + \rho_t \mathbb{E}_\xi \left[ \| F(x_t, \xi) - F(x_{t-1}, \xi) \|^2 \right] - \rho_t \| F(x_t) - F(x_{t-1}) \|^2, \quad (47)$$

where $\rho_t := \frac{n - b_t}{(n-1)b_t}$ if $F(x) := \frac{1}{n} \sum_{i=1}^n F_i(x)$, and $\rho_t := \frac{1}{b_t}$, otherwise, i.e., $F(x) = \mathbb{E}_\xi \{ F(x, \xi) \}$.

Similarly, we also have

$$\mathbb{E} \left[ \| \tilde{J}_t - F'(x_t) \|^2 | \mathcal{F}_t \right] = \| \tilde{J}_{t-1} - F'(x_{t-1}) \|^2 + \rho_t \mathbb{E}_\xi \left[ \| F'(x_t, \xi) - F'(x_{t-1}, \xi) \|^2 \right] - \rho_t \| F'(x_t) - F'(x_{t-1}) \|^2, \quad (48)$$

where $\hat{\rho}_t := \frac{n - b_t}{(n-1)b_t}$ if $F(x) := \frac{1}{n} \sum_{i=1}^n F_i(x)$, and $\hat{\rho}_t := \frac{1}{b_t}$, otherwise, i.e., $F(x) = \mathbb{E}_\xi \{ F(x, \xi) \}$.

**Appendix: The Proof of Technical Results in Section 4**

This appendix provides the full proof of technical results in Section 4 on our stochastic Gauss-Newton methods.

**D.1 The Proof of Theorem 4.1** Convergence of SGN for solving \cite{sgn}

**Theorem 4.1**. Suppose that Assumptions 1.1 and 1.2 hold for \cite{sgn}. Let $\tilde{F}_t$ and $\tilde{J}_t$ defined by \cite{cataly} be mini-batch stochastic estimators of $F(x_t)$ and $F'(x_t)$, respectively. Let $\{x_t\}$ be generated by Algorithm 1 (called SGN) to solve \cite{sgn}. Assume that $b_t$ and $\bar{b}_t$ in \cite{cataly} are chosen as

$$\begin{align*}
\left\{ \begin{array}{l}
\begin{aligned}
b_t &:= \frac{256M^2_2 \Lambda^4}{\bar{\sigma}_{\tilde{F}}^4} = \mathcal{O} \left( \frac{\sigma_{\tilde{F}}^2}{\varepsilon^4} \right) \\
\bar{b}_t &:= \frac{2M_0 M^2 \bar{\sigma}_{\tilde{F}}^2}{\beta \bar{b} \delta \varepsilon^2} = \mathcal{O} \left( \frac{\sigma_{\tilde{F}}^2}{\varepsilon^2} \right).
\end{aligned}
\end{array} \right.
\end{align*}$$

(23)
Furthermore, let \( \hat{x}_T \) be chosen uniformly randomly in \( \{x_t\}_{t=0}^T \) as the output of Algorithm \( 1 \) after \( T \) iterations. Then

\[
E \left[ \| \hat{G}_M(\hat{x}_T) \|^2 \right] = \frac{1}{(T + 1)} \sum_{t=0}^T E \left[ \| \hat{G}_M(x_t) \|^2 \right] \leq \frac{2M^2 \| \Psi(x_0) - \Psi^* \|}{C_g(T + 1)} + \frac{\varepsilon^2}{2},
\]

(24)

where \( C_g := 2M - M\phi(L_F + \beta_d) \) with \( M > \frac{1}{2} M\phi(L_F + \beta_d) \). Moreover, the total number \( \mathcal{T}_f \) of function evaluations \( \mathbf{F}(x_t, \xi) \) and the total number \( \mathcal{T}_d \) of Jacobian evaluations \( \mathbf{F}'(x_t, \xi) \) to achieve \( E \left[ \| \hat{G}_M(\hat{x}_T) \|^2 \right] \leq \varepsilon^2 \) do not exceed

\[
\begin{align*}
\mathcal{T}_f &:= \frac{1024 M^6 M^2 \sigma_F^2 [\Psi(x_0) - \Psi^*]}{C_g \varepsilon^6} = O \left( \frac{\sigma_F^2}{\varepsilon^6} \right), \\
\mathcal{T}_d &:= \frac{8 M^4 M^2 \sigma_D^2 [\Psi(x_0) - \Psi^*]}{\beta_d C_g \varepsilon^4} = O \left( \frac{\sigma_D^2}{\varepsilon^4} \right).
\end{align*}
\]

(25)

Proof. Let \( \mathcal{F}_t := \sigma(x_0, x_1, \ldots, x_{t-1}) \) be the \( \sigma \)-field generated by \( \{x_0, x_1, \ldots, x_{t-1}\} \). By repeating a similar proof as of (20), but taking the full expectation overall the randomness with \( E [ \cdot ] = E [ E [ \cdot ] \mid \mathcal{F}_{t+1} ] \), we have

\[
\frac{1}{(T + 1)} \sum_{t=0}^T E \left[ \| \hat{G}_M(x_t) \|^2 \right] \leq \frac{2M^2 \| \Psi(x_0) - \Psi^* \|}{C_g(T + 1)} + \frac{\varepsilon^2}{2},
\]

(49)

where \( C_g := 2M - M\phi(L_F + \beta_d) \) with \( M > \frac{1}{2} M\phi(L_F + \beta_d) \). Moreover, by the choice of \( \hat{x}_T \), we have

\[
E \left[ \| \hat{G}_M(\hat{x}_T) \|^2 \right] = \frac{1}{(T + 1)} \sum_{t=0}^T E \left[ \| \hat{G}_M(x_t) \|^2 \right].
\]

Combining this relation and (49), we prove (24).

Next, by Lemma C.2 to guarantee the condition (17) in expectation, i.e.:

\[
E \left[ \| \bar{F}(x_t) - F(x_t) \|^2 \mid \mathcal{F}_t \right] \leq \frac{C_g^2 \varepsilon^4}{256 M^2 \phi M^4},
\]

\[
E \left[ \| \bar{J}(x_t) - F'(x_t) \|^2 \mid \mathcal{F}_t \right] \leq \frac{\beta_d C_g \varepsilon^2}{2M^2 M^2},
\]

we have to choose \( \frac{\sigma_F^2}{b_t} \leq \frac{C_g^2 \varepsilon^4}{256 M^2 M^4} \) and \( \frac{\sigma_D^2}{b_t} \leq \frac{\beta_d C_g \varepsilon^2}{2M^2 M^2} \), which respectively lead to

\[
b_t \geq \frac{256 M^2 \sigma_F^2}{C_g^2 \varepsilon^4} \quad \text{and} \quad b_t \geq \frac{2M^2 \sigma_D^2}{\beta_d C_g \varepsilon^2}.
\]

By rounding to the nearest integer, we obtain (23). Using (20), we can see that since \( E \left[ \| \hat{G}_M(\hat{x}_T) \|^2 \right] = \frac{1}{(T + 1)} \sum_{t=0}^T E \left[ \| \hat{G}_M(x_t) \|^2 \right] \), to guarantee \( E \left[ \| \hat{G}_M(\hat{x}_T) \|^2 \right] \leq \varepsilon^2 \), we impose \( \frac{2M^2 \| \Psi(x_0) - \Psi^* \|}{C_g (T + 1)} \leq \frac{\varepsilon^2}{2} \), which leads to \( T := \left\lceil \frac{4M^2 \| \Psi(x_0) - \Psi^* \|}{C_g \varepsilon^2} \right\rceil \). Hence, the total number \( \mathcal{T}_f \) of function evaluations \( \mathbf{F}(\cdot) \) can be bounded by

\[
\mathcal{T}_f := T b_t \left[ \frac{1024 M^6 M^2 \sigma_F^2 [\Psi(x_0) - \Psi^*]}{C_g \varepsilon^6} \right] = O \left( \frac{\sigma_F^2}{\varepsilon^6} \right).
\]

Similarly, the total number \( \mathcal{T}_d \) of Jacobian evaluations \( \mathbf{F}'(\cdot) \) can be bounded by

\[
\mathcal{T}_d := T b_t \left[ \frac{8 M^4 M^2 \sigma_D^2 [\Psi(x_0) - \Psi^*]}{\beta_d C_g \varepsilon^4} \right] = O \left( \frac{\sigma_D^2}{\varepsilon^4} \right).
\]

These two last estimates prove (25).

D.2 The Proof of Theorem 4.2: Convergence of SGN for Solving $\|F\|
abla$

Theorem 4.2. Suppose that Assumptions 1.1 and 1.2 hold for $\|F\|\nabla$. Let $\tilde{F}_i$ and $\tilde{J}_i$ defined by (22) be mini-batch stochastic estimators to approximate $F(x_i)$ and $F'(x_i)$, respectively. Let $\{x_i\}$ be generated by Algorithm 2 for solving $\|F\|\nabla$. Assume that $b_i$ and $b_{\tilde{t}}$ in (22) are chosen such that $b_i := \min\{n, b_{\tilde{t}}\}$ and $b_{\tilde{t}} := \min\{n, b_i\}$ for $t \geq 0$, where

$$
\begin{align*}
\hat{b}_0 &:= \frac{32M_\phi M^2 \sigma_F [48 \sigma_F M_\phi M^2 + (C_g - C_a)\varepsilon^2]}{3(C_g - C_a)^2 \varepsilon^4} \cdot \log \left(\frac{p + 1}{\delta}\right) \\
\hat{b}_t &:= \frac{4M \sqrt{2M_\phi \sigma_D} \left(3M \sqrt{2M_\phi \sigma_D} + \sqrt{\beta_d (C_g - C_a)\varepsilon}\right)}{\beta_d (C_g - C_a)\varepsilon^2} \cdot \log \left(\frac{p + q}{\delta}\right) \\
\hat{b}_{\tilde{t}} &:= \frac{\left(6\sigma_F^2 + 2\sigma_F \sqrt{C_f} \|x_t - x_{t-1}\|^2\right)}{3C_f^2 \|x_t - x_{t-1}\|^4} \cdot \log \left(\frac{p + 1}{\delta}\right) \\
\hat{b}_{\tilde{t}} &:= \frac{\left(6\sigma_D^2 + 2\sigma_D \sqrt{C_d} \|x_t - x_{t-1}\|\right)}{3C_d \|x_t - x_{t-1}\|^2} \cdot \log \left(\frac{p + q}{\delta}\right)
\end{align*}
$$

for $\delta \in (0, 1)$, and $C_f$ and $C_d$ given in Condition 2. Then, with probability at least $1 - \delta$, the bound (21) in Theorem 3.3 still holds. Moreover, the total number $T_f$ of function evaluations $F_i(x_i)$ and the total number $T_{\tilde{t}}$ of Jacobian evaluations $F'_i(x_i)$ to guarantee $\min_{0 \leq t \leq T} \|G_M(x_t)\| \leq \varepsilon$ do not exceed

$$
\begin{align*}
T_f &:= \mathcal{O} \left(\frac{\sigma_F^2 [\Psi(x_0) - \Psi^*]}{\varepsilon^6} \cdot \log \left(\frac{p + 1}{\delta}\right)\right), \\
T_{\tilde{t}} &:= \mathcal{O} \left(\frac{\sigma_D^2 [\Psi(x_0) - \Psi^*]}{\varepsilon^4} \cdot \log \left(\frac{p + q}{\delta}\right)\right).
\end{align*}
$$

Proof. We first use Lemma C.1 to estimate the total number of samples for $F(x_i)$ and $F'(x_i)$. Let $F_i := \sigma(x_0, x_1, \cdots, x_{t-1})$ be the $\sigma$-field generated by $\{x_0, x_1, \cdots, x_{t-1}\}$. We define $X_i := F_i(x_i) - F(x_i) \in \mathbb{R}^p$ for $i \in \mathcal{B}_t$. Conditioned on $F_i$, due to the choice of $\mathcal{B}_t$, $\{X_i\}_{i \in \mathcal{B}_t}$ are independent vector-valued random variables and $\mathbb{E}[X_i] = 0$. Moreover, by Assumption 1.2 we have $\|F_i(x) - F(x)\| \leq \sigma_F$ for all $i \in [n]$. This implies that $\|X_i\| \leq \sigma_F$ a.s. and $\mathbb{E}\left[\|X_i\|^2\right] \leq \sigma_F^2$. Hence, the conditions of Lemma C.1 hold. In addition, we have

$$\sigma_X^2 := \max \left\{\left\|\sum_{i \in \mathcal{B}_t} \mathbb{E}[X_i X_i^\top]\right\|, \left\|\sum_{i \in \mathcal{B}_t} \mathbb{E}[X_i^\top X_i]\right\|\right\} \leq \sum_{i \in \mathcal{B}_t} \mathbb{E}\left[\|X_i\|^2\right] \leq b_{\tilde{t}} \sigma_F^2.
$$

Since $\tilde{F}_i := \frac{1}{b_{\tilde{t}}} \sum_{i \in \mathcal{B}_t} F_i(x_i)$, by Lemma C.1 we have

$$\text{Prob} \left(\|\tilde{F}_i - F(x_i)\| \leq \varepsilon\right) = \text{Prob} \left(\left\|\sum_{i \in \mathcal{B}_t} X_i\right\| \leq b_{\tilde{t}} \varepsilon\right) \geq 1 - (p + 1) \exp \left(-\frac{3b_{\tilde{t}}^2 \varepsilon^2}{6b_{\tilde{t}} \sigma_F + 2\sigma_F b_{\tilde{t}} \varepsilon}\right) = 1 - (p + 1) \exp \left(-\frac{3b_{\tilde{t}}^2 \varepsilon^2}{6b_{\tilde{t}} \sigma_F + 2\sigma_F b_{\tilde{t}} \varepsilon}\right).
$$

Let us choose $\delta \in (0, 1)$ such that $\delta \geq (p + 1) \exp \left(-\frac{3b_{\tilde{t}}^2 \varepsilon^2}{6b_{\tilde{t}} \sigma_F + 2\sigma_F b_{\tilde{t}} \varepsilon}\right)$ and $\delta \leq 1$, then $\text{Prob} \left(\|\tilde{F}_i - F(x_i)\| \leq \varepsilon\right) \geq 1 - \delta$. Hence, we have $b_{\tilde{t}} \geq \frac{(6\sigma_F^2 + 2\sigma_F \sqrt{C_f} \|x_t - x_{t-1}\|^2)}{3C_f^2 \|x_t - x_{t-1}\|^4} \cdot \log \left(\frac{p + 1}{\delta}\right)$.

To guarantee the first condition of (18), we choose $\varepsilon := \frac{(C_g - C_a)\varepsilon^2}{16M_\phi M^2}$. Then, the condition on $b_0$ leads to $b_0 \geq \frac{32M_\phi M^2 \sigma_F [48 \sigma_F M_\phi M^2 + (C_g - C_a)\varepsilon^2]}{3(C_g - C_a)\varepsilon^4} \cdot \log \left(\frac{p + 1}{\delta}\right)$. To guarantee the first condition of (19), we choose
Therefore, we can even bound these two last estimates prove (27).

Next, we estimate a sample size for \( \tilde{T} \). Let us define \( Y_i := F'_i(x_i) - F'(x_i) \). Then, similar to the proof above of \( X_i \) for \( F \), we have \( \tilde{T}_i - F'(x_i) = \frac{1}{b_t} \sum_{i \in B_t} (F'_i(x_i) - F'(x_i)) = \frac{1}{b_t} \sum_{i \in B_t} Y_i \). Under Assumption 1.2 the sequence \( \{Y_i\} \) satisfies all conditions of Lemma C.1. Hence, we obtain

\[
\Pr \left( \| \tilde{T}_i - F'(x_i) \| \leq \epsilon \right) \geq 1 - (p + q) \exp \left( \frac{-3b_t \epsilon^2}{6\sigma_D^2 + 2\sigma_D \epsilon} \right).
\]

Hence, we can choose \( \hat{b}_t \geq \frac{6\sigma_D^2 + 2\sigma_D \epsilon}{3\epsilon^2} \cdot \log \left( \frac{p + q}{\delta} \right) \). From the second condition of (18), if we choose \( \epsilon := \frac{\sqrt{\beta_d(C_g - C_u) \epsilon}}{M \sqrt{2M_{\sigma D}}} \), then we have \( \hat{b}_t \geq \frac{4M \sqrt{2M_{\sigma D}} (3M \sqrt{2M_{\sigma D} + \sqrt{\beta_d(C_g - C_u) \epsilon}})^2}{\beta_d(C_g - C_u) \epsilon^2} \cdot \log \left( \frac{p + q}{\delta} \right) \). From the second condition of (19), if we choose \( \epsilon := \sqrt{\beta_d} \| x_t - x_{t-1} \| \), then we have \( \hat{b}_t \geq \frac{(6\sigma_D^2 + 2\sigma_D \sqrt{\beta_d} \| x_t - x_{t-1} \|) \epsilon}{3\epsilon^2 \| x_t - x_{t-1} \|} \cdot \log \left( \frac{p + q}{\delta} \right) \). Rounding \( \hat{b}_t \), we obtain

\[
\hat{b}_0 := \frac{4M \sqrt{2M_{\sigma D}} (3M \sqrt{2M_{\sigma D} + \sqrt{\beta_d(C_g - C_u) \epsilon}})^2}{\beta_d(C_g - C_u) \epsilon^2} \cdot \log \left( \frac{p + q}{\delta} \right),
\]

\[
\hat{b}_t := \frac{(6\sigma_D^2 + 2\sigma_D \sqrt{\beta_d} \| x_t - x_{t-1} \|) \epsilon}{3\epsilon^2 \| x_t - x_{t-1} \|} \cdot \log \left( \frac{p + q}{\delta} \right), \quad t \geq 1.
\]

Since \( \hat{b}_t \leq n \) for all \( t \geq 0 \), combining these conditions, we obtain \( \hat{b}_t := \min \{ n, \hat{b}_t \} \) for \( t \geq 0 \), which proves the second part of (26).

For \( t \geq 1 \), we have \( \| \tilde{G}_M(x_{t-1}) \| = M \| x_t - x_{t-1} \| > \epsilon \). Otherwise, the algorithm has been terminated. Therefore, we can even bound \( b_t \) and \( \hat{b}_t \) as

\[
b_t \leq \frac{2M^2 \sigma_F (3M^2 \sigma_F + \sqrt{C_f \epsilon^2})}{3C_{\epsilon} \epsilon^2} \cdot \log \left( \frac{p + q}{\delta} \right)
\]

and

\[
\hat{b}_t \leq \frac{M (6\sigma_D^2 + 2\sigma_D \sqrt{\beta_d} \| x_t - x_{t-1} \|) \epsilon}{3\epsilon^2 \| x_t - x_{t-1} \|} \cdot \log \left( \frac{p + q}{\delta} \right).
\]

From (21), since \( \min_{0 \leq t \leq T} \| \tilde{G}_M(\tilde{x}_T) \| \leq \frac{1}{(T+1)} \sum_{t=0}^T \| G_M(x_t) \| \), to guarantee \( \min_{0 \leq t \leq T} \| \tilde{G}_M(\tilde{x}_T) \| \leq \epsilon \), we impose

\[
\frac{2M^2 \| \Psi(x_0) - \Psi^* \|}{(C_g - C_u)(T+1)} \leq \frac{\epsilon^2}{2}.
\]

which leads to \( T := \frac{4M^2 \| \Psi(x_0) - \Psi^* \|}{(C_g - C_u)^2} \). Hence, the total number \( T_f \) of function evaluations \( F(\cdot) \) can be bounded by

\[
T_f := b_0 + (T - 1)b_t \leq \frac{32M_{\sigma F} M^2 \sigma_F (48\sigma_F M_{\sigma F} + (C_g - C_u) \epsilon^2)}{(C_g - C_u) \epsilon^2} + \frac{8M^4 \sigma_F (3M^2 \sigma_F + \sqrt{C_f \epsilon^2}) \| \Psi(x_0) - \Psi^* \|}{3C_{\epsilon} (C_g - C_u) \epsilon^2} \cdot \log \left( \frac{p + q}{\delta} \right).
\]

Similarly, the total number \( T_d \) of Jacobian evaluations \( F'(\cdot) \) can be bounded by

\[
T_d := \hat{b}_0 + (T - 1)\hat{b}_t \leq \frac{4M \sqrt{2M_{\sigma D}} (3M \sqrt{2M_{\sigma D} + \sqrt{\beta_d(C_g - C_u) \epsilon}})^2}{\beta_d(C_g - C_u) \epsilon^2} \cdot \log \left( \frac{p + q}{\delta} \right).
\]

These two last estimates prove (27).
Then:
Suppose that Assumptions 1.1 and 1.2, and 4.1 are satisfied for \( \{x_t^{(s)}\}_{t=1}^{S} \) be generated by Algorithm 2 to solve 1. Let \( \theta_F \) and \( m \) be chosen by (29), and the mini-batches \( b_s, \hat{b}_{s}, \hat{b}_{s}^{(s)}, \) and \( \hat{b}_{s}^{(s)} \) be set as in (30). Assume that the output \( \hat{x}_T \) of Algorithm 2 is chosen uniformly randomly in \( \{x_t^{(s)}\}_{t=1}^{S} \).

Then:

(a) The following bound holds

\[
E \left[ \| \tilde{G}_M(\tilde{x}_T) \|^2 \right] = \frac{1}{S(m+1)} \sum_{s=1}^{S} \sum_{t=0}^{m} E \left[ \| \tilde{G}_M(x_t) \|^2 \right] \leq \varepsilon^2. \tag{31}
\]

(b) The total number of iterations \( T \) to obtain \( E \left[ \| \tilde{G}_M(x_t) \|^2 \right] \leq \varepsilon^2 \) is at most

\[
T := S(m+1) = \frac{8M^2 \left( \Psi(\bar{0}) - \Psi^* \right)}{\theta_F \varepsilon^2} = O \left( \frac{1}{\varepsilon^2} \right).
\]

Moreover, the total stochastic oracle calls \( T_f \) and \( T_d \) for evaluating stochastic estimators \( F(x_t, \xi) \) of \( F \) and its Jacobian \( F'(x_t, \xi) \), respectively do not exceed:

\[
\begin{aligned}
T_f &:= O \left( \frac{M^4 M_{\phi}^2 [\Psi(\bar{0}) - \Psi^*]}{\theta_F^2 \varepsilon^5} \right), \\
T_d &:= O \left( \frac{M^2 M_{\phi} [\Psi(\bar{0}) - \Psi^*]}{\theta_F \varepsilon^3} \right).
\end{aligned} \tag{32}
\]

Proof. We first analyze the inner loop. Using (16) with \( x := x_t^{(s)} \) and \( T_M(x) = x_{t+1}^{(s)} \), and then taking the expectation conditioned on \( F_{t+1}^{(s)} := \sigma(x_0^{(s)}, x_1^{(s)}, \ldots, x_t^{(s)}) \), we have

\[
E \left[ \phi(F(x_{t+1}^{(s)})) \mid F_{t+1}^{(s)} \right] \leq \phi(F(x_t^{(s)})) - \frac{12M - M_{\phi}(L_F + \beta_d)}{2} E \left[ \| x_{t+1}^{(s)} - x_t^{(s)} \|^2 \mid F_{t+1}^{(s)} \right] + \frac{L_{\phi}}{\xi_t} E \left[ \| F(x_t^{(s)}) - \tilde{F}(x_t^{(s)}) \|^2 \mid F_{t+1}^{(s)} \right] + M_{\phi} \xi_t^2,
\]

for any \( \xi_t^* > 0 \), where we use \( 2ab \leq a^2 + b^2 \) and the Jensen inequality \( \left( E \left[ \| F(x_t^{(s)}) - \tilde{F}(x_t^{(s)}) \|^2 \mid F_{t+1}^{(s)} \right] \right)^2 \leq E \left[ \| F(x_t^{(s)}) - \tilde{F}(x_t^{(s)}) \|^2 \mid F_{t+1}^{(s)} \right] \) in the second line. Taking the full expectation both sides of the last inequality, and noting that \( \Psi(x) = \phi(F(x)) \), we obtain

\[
E \left[ \Psi(x_{t+1}^{(s)}) \right] \leq E \left[ \Psi(x_t^{(s)}) \right] - C_F E \left[ \| x_{t+1}^{(s)} - x_t^{(s)} \|^2 \right] + \frac{L_{\phi}}{\xi_t} E \left[ \| F(x_t^{(s)}) - \tilde{F}(x_t^{(s)}) \|^2 \right] + M_{\phi} \xi_t^2,
\]

where \( C_F := 2M - M_{\phi}(L_F + \beta_d) > 0 \), and \( \beta_d > 0 \) and \( \xi_t^* > 0 \) are given.

Next, from Lemma C.3 using the Lipschitz continuity of \( F' \) in Assumption 1.2 we have

\[
E \left[ \| J_{t+1}^{(s)} - F'(x_t^{(s)}) \|^2 \right] \leq E \left[ \| J_{t+1}^{(s)} - F'(x_{t-1}^{(s)}) \|^2 \right] + \frac{L_{\phi}}{b_t} E \left[ \| x_{t+1}^{(s)} - x_t^{(s)} \|^2 \right]. \tag{51}
\]

Similarly, using Lemma C.3 we also have

\[
E \left[ \| \tilde{F}_t^{(s)} - F(x_t^{(s)}) \|^2 \mid F_{t+1}^{(s)} \right] \leq \| \tilde{F}_t^{(s)} - F(x_{t-1}^{(s)}) \|^2 + \frac{1}{b_t} E \left[ \| F(x_t, \xi) - F(x_{t-1}, \xi) \|^2 \right].
\]

Taking the full expectation both sides of this inequality, and using Assumption 4.1 we obtain

\[
E \left[ \| \tilde{F}_t^{(s)} - F(x_t^{(s)}) \|^2 \right] \leq E \left[ \| \tilde{F}_t^{(s)} - F(x_{t-1}^{(s)}) \|^2 \right] + M_{\phi} \xi_t^2 E \left[ \| x_{t-1}^{(s)} - x_t^{(s)} \|^2 \right]. \tag{52}
\]
Let us define a Lyapunov function as
\[
\mathcal{L}(x_t^{(s)}) := \mathbb{E} \left[ \Psi(x_t^{(s)}) \right] + \frac{a_t^s}{2} \mathbb{E} \left[ \| \tilde{F}_t^{(s)} - F(x_t^{(s)}) \|^2 \right] + \frac{c_t^s}{2} \mathbb{E} \left[ \| \tilde{J}_t^{(s)} - F'(x_t^{(s)}) \|^2 \right],
\]
for some \( a_t^s > 0 \) and \( c_t^s > 0 \).

Combining (50), (51), and (52), and then using the definition of \( \mathcal{L} \) in (53), we have
\[
\mathcal{L}(x_{t+1}^{(s)}) = \mathbb{E} \left[ \Psi(x_t^{(s)}) \right] + \frac{a_{t+1}^s}{2} \mathbb{E} \left[ \| \tilde{F}_{t+1}^{(s)} - F(x_{t+1}^{(s)}) \|^2 \right] + \frac{c_{t+1}^s}{2} \mathbb{E} \left[ \| \tilde{J}_{t+1}^{(s)} - F'(x_{t+1}^{(s)}) \|^2 \right]
\leq \mathbb{E} \left[ \Psi(x_t^{(s)}) \right] - \left[ \frac{C_0}{2} - \frac{M^2 a_{t+1}^s}{2b_{t+1}} - \frac{L^2 c_{t+1}^s}{2b_{t+1}} \right] \mathbb{E} \left[ \| x_{t+1}^{(s)} - x_t^{(s)} \|^2 \right] + M_0 \xi_t^s
\hfill (54)
\]
\[
+ \left( \frac{a_{t+1}^s}{2} + \frac{L_0}{s\xi_t^s} \right) \mathbb{E} \left[ \| F(x_t^{(s)}) - \tilde{F}(x_t^{(s)}) \|^2 \right] + \left( \frac{c_{t+1}^s}{2} + \frac{M_0}{2s^2} \right) \mathbb{E} \left[ \| F'(x_t^{(s)}) - \tilde{J}(x_t^{(s)}) \|^2 \right].
\]

If we assume that
\[
a_t^s \geq a_{t+1}^s + \frac{M_0}{\xi_t^s}, \quad \text{ and } \quad c_t^s \geq c_{t+1}^s + \frac{M_0}{\beta d},
\]
then, from (54), we have
\[
\mathcal{L}(x_t^{(s)}) \leq \mathcal{L}(x_{t+1}^{(s)}) - \frac{\rho_{t+1}^s}{2} \mathbb{E} \left[ \| x_{t+1}^{(s)} - x_t^{(s)} \|^2 \right] + M_0 \xi_t^s,
\]
where \( \rho_{t+1}^s := C_0 - \frac{M^2 a_{t+1}^s}{b_{t+1}} - \frac{L^2 c_{t+1}^s}{b_{t+1}}. \)

Let us first fix \( \xi_t^s := \xi > 0 \). Next, we choose \( a_t^s := (m + 1 - t) \frac{M_0}{\xi} \) and \( c_t^s := (m + 1 - t) \frac{M_0}{\beta d} \). Clearly, \( a_{m+1}^s = a_m^s = 0 \) and they both satisfy the condition (55). Then, we choose \( b_t^s := \frac{1}{\gamma_0} a_t^s = (m + 1 - t) \frac{M_0}{\gamma d} \) and \( b_{t+1}^s := \frac{1}{\gamma_2} c_t^s = (m + 1 - t) \frac{M_0}{\gamma \delta d} \) for some \( \gamma_0 > 0 \) and \( \gamma_2 > 0 \). In this case, we have \( \rho_t^s = C_0 - M^2 \gamma_1 - L^2 \gamma_2 \equiv \rho > 0 \) by appropriately choosing \( \gamma_1 \) and \( \gamma_2 \). Consequently, (56) reduces to
\[
\mathcal{L}(x_t^{(s)}) \leq \mathcal{L}(x_{t+1}^{(s)}) - \frac{\rho}{2} \mathbb{E} \left[ \| x_{t+1}^{(s)} - x_t^{(s)} \|^2 \right] + M_0 \xi_t^s.
\]

Summing up this inequality from \( t = 0 \) to \( t = m \), we obtain
\[
\frac{\rho}{2} \sum_{t=0}^{m} \mathbb{E} \left[ \| x_{t+1}^{(s)} - x_t^{(s)} \|^2 \right] \leq \mathcal{L}(x_0^{(s)}) - \mathcal{L}(x_{m+1}^{(s)}) + (m + 1)M_0 \xi_t^s.
\]

Using the fact that \( \tilde{x}^{s-1} = x_0^{(s)} \) and \( \tilde{x}^s = x_{m+1}^{(s)} \), we have
\[
\frac{\rho}{2} \sum_{t=0}^{m} \mathbb{E} \left[ \| x_{t+1}^{(s)} - x_t^{(s)} \|^2 \right] \leq \mathcal{L}(\tilde{x}^{s-1}) - \mathcal{L}(\tilde{x}^s) + (m + 1)M_0 \xi_t^s.
\]

Summing up this inequality from \( s = 1 \) to \( S \) and multiplying the result by \( \frac{2}{\rho S(m+1)} \), we obtain
\[
\frac{1}{S(m+1)} \sum_{s=1}^{S} \sum_{t=0}^{m} \mathbb{E} \left[ \| x_{t+1}^{(s)} - x_t^{(s)} \|^2 \right] \leq \frac{2 \left[ \mathcal{L}(\tilde{x}^0) - \mathcal{L}(\tilde{x}^S) \right]}{\rho S(m+1)} + \frac{2 M_0 \xi_t^s}{\rho}.
\]

Since \( \mathcal{L}(\tilde{x}^0) = \Psi(\tilde{x}^0) + \frac{(m+1)M_0}{2\xi} \mathbb{E} \left[ \| \tilde{F}_0 - F(\tilde{x}^0) \|^2 \right] + \frac{(m+1)M_0}{2\beta d} \mathbb{E} \left[ \| \tilde{J}_0 - F'(\tilde{x}^0) \|^2 \right] \) and \( \mathcal{L}(\tilde{x}^S) = \mathbb{E} \left[ \Psi(\tilde{x}^S) \right] \geq \Phi^* \), we obtain from (57) that
\[
\frac{1}{S(m+1)} \sum_{s=1}^{S} \sum_{t=0}^{m} \mathbb{E} \left[ \| x_{t+1}^{(s)} - x_t^{(s)} \|^2 \right] \leq \frac{2 \left[ \Psi(\tilde{x}^0) - \Psi^* \right]}{\rho S(m+1)} + \frac{M_0 \xi_t^s}{\rho S(m+1)} \mathbb{E} \left[ \| \tilde{F}_0 - F(\tilde{x}^0) \|^2 \right] + \frac{2 M_0 \xi_t^s}{\rho}.
\]
Note that $\mathbb{E} \left[ \| \widetilde{F}_0 - F(\tilde{x}_0) \|^2 \right] \leq \frac{\sigma_F^2}{\delta}$ and $\mathbb{E} \left[ \| \widetilde{J}_0 - F'(\tilde{x}_0) \|^2 \right] \leq \frac{\sigma_F^2}{\delta}$ due to the choice of $b = b_0 > 0$ and $\tilde{b} = \tilde{b}_0 > 0$ at Step 1 of Algorithm 2. Hence, we can further bound (58) as

$$\frac{1}{S(m + 1)} \sum_{s=1}^{S} \sum_{t=0}^{m} \mathbb{E} \left[ \| x_{t+1}^{(s)} - x_t^{(s)} \|^2 \right] \leq \frac{2 \left[ \mathbb{E} \left( \tilde{x}_0 \right) - \mathbb{E} \left( \tilde{x}_t \right) \right]^2}{\rho S (m + 1)} + \frac{M_0 \sigma_F^2}{\xi \rho S b} + \frac{M_0 \sigma_D^2}{\rho \beta_d S b} + \frac{2M_0 \xi}{\rho}. \quad (59)$$

Since $\| \tilde{G}_M(x_t^{(s)}) \| = M \| x_{t+1}^{(s)} - x_t^{(s)} \|$, to guarantee $\frac{1}{S(m + 1)} \sum_{s=1}^{S} \sum_{t=0}^{m} \mathbb{E} \left[ \| \tilde{G}_M(x_t^{(s)}) \|^2 \right] \leq \varepsilon^2$ for a given tolerance $\varepsilon > 0$, we need to set

$$\frac{2 \left[ \mathbb{E} \left( \tilde{x}_0 \right) - \mathbb{E} \left( \tilde{x}_t \right) \right]^2}{\rho S (m + 1)} + \frac{M_0 \sigma_F^2}{\xi \rho S b} + \frac{M_0 \sigma_D^2}{\rho \beta_d S b} + \frac{2M_0 \xi}{\rho} = \frac{\varepsilon^2}{4M^2}.$$ 

Let us break this condition into

$$\frac{2 \left[ \mathbb{E} \left( \tilde{x}_0 \right) - \mathbb{E} \left( \tilde{x}_t \right) \right]^2}{\rho S (m + 1)} = \frac{\varepsilon^2}{4M^2} \quad \text{and} \quad \frac{M_0 \sigma_F^2}{\xi \rho S b} = \frac{2M_0 \xi}{\rho} = \frac{\varepsilon^2}{4M^2}.$$ 

Hence, we can choose $\xi := \frac{\varepsilon^2}{8M^2 \rho S b}$, $\tilde{b} := \frac{4M_0 \sigma_F^2}{\rho \beta_d S b}$, $b := \frac{2M_0 \sigma_D^2}{\rho \gamma S b}$, and $S(m + 1) = \frac{8M^2 \varepsilon^2}{\rho \varepsilon^2} \left( \mathbb{E} \left( \tilde{x}_0 \right) - \mathbb{E} \left( \tilde{x}_t \right) \right)^2$.

Now, let us choose $m + 1 := \bar{C} \varepsilon$ for some constant $\bar{C} > 0$. Then, we can estimate the total number $T_f$ of function evaluations $F(x_t^{(s)}, \xi)$ as follows:

$$T_f := \sum_{s=1}^{S} b_s + \sum_{s=1}^{S} \sum_{t=0}^{m} \tilde{b}_t^{(s)} = Sb + \frac{M_0}{\gamma d_2} \sum_{s=1}^{S} \sum_{t=0}^{m} (m + 1 - t)$$

$$= \frac{2M_0 \sigma_F^2}{\rho \gamma d^{2} \varepsilon^2} + \frac{8M^2 \sigma_F^2}{\gamma d^{2} \varepsilon^2} \cdot \frac{S(m + 1)(m + 2)}{2}$$

$$= \frac{2M_0 \sigma_F^2}{\rho \gamma d^{2} \varepsilon^2} + \frac{8M^2 \sigma_F^2}{\gamma d^{2} \varepsilon^2} \cdot \frac{S \left( \mathbb{E} \left( \tilde{x}_0 \right) - \mathbb{E} \left( \tilde{x}_t \right) \right)^2}{\rho \varepsilon^2} \cdot \frac{\bar{C} \varepsilon}{2\varepsilon}.$$ 

$$= O \left( \frac{M^4 \sigma_F^2 \left( \mathbb{E} \left( \tilde{x}_0 \right) - \mathbb{E} \left( \tilde{x}_t \right) \right)^2}{\rho \varepsilon^2} \right).$$

Similarly, the total number $T_d$ of Jacobian evaluations $F'(x_t^{(s)}, \xi)$ can be bounded as

$$T_d := \sum_{s=1}^{S} \tilde{b}_s + \sum_{s=1}^{S} \sum_{t=0}^{m} \tilde{b}_t^{(s)} = Sb + \frac{M_0}{\beta d_2} \sum_{s=1}^{S} \sum_{t=0}^{m} (m + 1 - t)$$

$$\leq \frac{4M_0 \sigma_D^2}{\rho \gamma d \varepsilon^2} + \frac{8M_2 \sigma_D^2}{\beta d \gamma \varepsilon^2} \cdot \frac{S \left( \mathbb{E} \left( \tilde{x}_0 \right) - \mathbb{E} \left( \tilde{x}_t \right) \right)^2}{\rho \varepsilon^2} \cdot \frac{\bar{C} \varepsilon}{2\varepsilon}.$$ 

$$= O \left( \frac{M^2 \sigma_F^2 \left( \mathbb{E} \left( \tilde{x}_0 \right) - \mathbb{E} \left( \tilde{x}_t \right) \right)^2}{\rho \varepsilon^2} \right).$$

Hence, we have proved (32).}

### D.4 The proof of Theorem D.1: Convergence of Algorithm 2 for (2)

Although Theorem 1 significantly improves stochastic oracle complexity compared to Theorem 4.1, it requires additional assumptions. Assumption 4.1 is usually used in compositional models such as neural network and parameter estimation. However, we still attempt to establish a convergence and complexity result without Assumption 4.1 for Algorithm 2 to solve (2).

**Theorem D.1.** Suppose that Assumptions 1.1 and 1.2 are satisfied for (2). Let $\{ x_t^{(s)} \}_{s=1}^{S}$ be generated by Algorithm 2 to solve (2). Let the mini-batches $b_s$, $b_t^{(s)}$, $\tilde{b}_s$, and $\tilde{b}_t^{(s)}$ be set as follows:

$$\begin{align*}
    b_s &:= O \left( \frac{\sigma_F^2}{\rho} \cdot \log \left( \frac{p+1}{\delta} \right) \right) \\
    b_t^{(s)} &:= O \left( \frac{\sigma_F^2}{\rho} \cdot \log \left( \frac{p+1}{\delta} \right) \right) \\
    \tilde{b}_s &:= O \left( \frac{\sigma_D^2}{\rho} \cdot \log \left( \frac{p+1}{\delta} \right) \right) \\
    \tilde{b}_t^{(s)} &:= O \left( m^2 \cdot \log \left( \frac{p+1}{\delta} \right) \right).
\end{align*}$$

(60)
Then, with probability at least $1 - \delta$:

(a) The following bound holds

$$\frac{1}{S(m + 1)} \sum_{s=1}^{S} \sum_{t=0}^{m} \|\tilde{G}_M(x_t)\|^2 \leq O(\epsilon^2).$$

(b) The total number of iterations $T$ to achieve

$$\min_{0 \leq t \leq m, 1 \leq s \leq S} \|\tilde{G}_M(x_t^{(s)})\| \leq \epsilon$$

is at most $T := S(m + 1) = O\left(\frac{\|\Psi(\hat{x}^0) - \Psi^*\|}{\epsilon^2}\right)$. Moreover, the total stochastic oracle calls $T_f$ and $T_d$ to approximate $F$ and its Jacobian $F'$, respectively do not exceed

$$\begin{cases} T_f := O\left(\frac{\|\hat{x}^0\| + \frac{1}{\epsilon}}{\epsilon} \cdot \log \left(\frac{\epsilon + 1}{\epsilon}\right)\right) \\
T_d := O\left(\frac{\|\hat{x}^0\| + \frac{1}{\epsilon}}{\epsilon} \cdot \log \left(\frac{\epsilon + 1}{\epsilon}\right)\right).
\end{cases}$$

**Remark D.1.** Although we do not gain an improvement on the worst-case oracle complexity through Theorem D.1, we observe in our experiment that Algorithm 1.2 highly outperforms SGN. We believe that there is an artifact in our proof of Theorem D.1.

**Proof.** We first analyze the inner loop of Algorithm 2. For simplicity of notation, we drop the superscript ($^*$) in the following derivations until it is recalled. We first verify the conditions (19) if we use the SARAH estimators (28) for $F'(x_t)$ and $F(x_t)$. Let $F_t := \sigma(x_0, x_1, \cdots, x_t)$ be the $\sigma$-field generated by $\{x_0, x_1, \cdots, x_t\}$. We define $X_t := F'_t(x_t) - F'_t(x_{t-1}) - [F'(x_t) - F'(x_{t-1})]$. Then, clearly, conditioned on $F_t$, we have $\{X_t \mid F_{t-1}\}$ is mutually independent and $E[X_t \mid F_{t-1}] = 0$. Moreover, by Assumption 1.2, we have

$$\|X_t\| = \|F'_t(x_t) - F'_t(x_{t-1}) - [F'(x_t) - F'(x_{t-1})]\| \leq \|F'_t(x_t) - F'_t(x_{t-1})\| + \|F'(x_t) - F'(x_{t-1})\| \leq 2L_F \|x_t - x_{t-1}\| := \hat{\delta}_t.$$

We consider $Z_t := \frac{1}{b_t} \sum_{i \in B_t} [F'_i(x_t) - F'_i(x_{t-1}) - F'(x_t) + F'(x_{t-1})] = \frac{1}{b_t} \sum_{i \in B_t} X_i$. We have

$$\sigma_X^2 := \max \left\{ \left\| \sum_{i \in B_t} E[X_i X_i^T \mid F_{t-1}] \right\|, \left\| \sum_{i \in B_t} E[X_i^T X_i \mid F_{t-1}] \right\| \right\} \leq \sum_{i \in B_t} E[\|X_i\|^2 \mid F_{t-1}] \leq \hat{b}_t \hat{\delta}_t^2.$$

For any $\hat{\epsilon} > 0$, we can apply Lemma C.1 to obtain the following bound

$$\text{Prob}(\|Z_t\| \leq \hat{\epsilon} \hat{\delta}_t) = \text{Prob}\left(\|\sum_{i \in B_t} X_i\| \leq \hat{b}_t \hat{\epsilon} \hat{\delta}_t\right) \geq 1 - (p + q) \exp \left(-\frac{3\hat{b}_t^2 \hat{\epsilon}^2 \hat{\delta}_t^2}{6\hat{b}_t \hat{\delta}_t^2 + 2p\hat{\epsilon} \hat{\delta}_t}\right) \geq 1 - (p + q) \exp \left(-\frac{3\hat{b}_t^2 \hat{\epsilon}^2}{6\hat{\delta}_t^2}\right).$$

Hence, if we choose $\delta \geq (p + q) \exp \left(-\frac{3\hat{b}_t^2 \hat{\epsilon}^2}{6\hat{\delta}_t^2}\right)$ and $\delta \leq 1$, we obtain $\text{Prob}(\|Z_t\| \leq \hat{\epsilon} \hat{\delta}_t) \geq 1 - \delta$ for all $t \geq 0$. The condition in $\hat{b}_t$ leads to

$$\hat{b}_t \geq \frac{6 + 2\hat{\epsilon}}{3\hat{\delta}_t^2} \cdot \log \left(\frac{p + q}{\delta}\right).$$

By the update (28), we have $\tilde{J}_t - F(x_t) = [\tilde{J}_{t-1} - F'(x_{t-1})] + \frac{1}{b_t} \sum_{i \in B_t} [F'_i(x_t) - F'_i(x_{t-1}) - F'(x_t) + F'(x_{t-1})] = [\tilde{J}_{t-1} - F'(x_{t-1})] + Z_t$. Hence, by the triangle inequality, we get

$$\|\tilde{J}_t - F'(x_t)\| = \|\tilde{J}_0 - F'(x_0) + \sum_{j=1}^{t} Z_j\| \leq \|\tilde{J}_0 - F'(x_0)\| + \sum_{j=1}^{t} \|Z_j\|.$$
On the other hand, by the update (22) of \( \tilde{J}_0 \) as \( \tilde{J}_0 := \frac{1}{t} \sum_{i \in B} F_i(x_{t0}), \) where \( \tilde{b} := \hat{b}_s \) and \( \tilde{B} := \tilde{B}_s, \) with a similar proof as of Theorem 4.2 we can show that if we choose \( \tilde{b} \geq \frac{6\sigma^2 + 2\sigma_D \epsilon_0}{3\sigma_0^2} \log \left( \frac{p + q}{\delta} \right) \) then

\[
\Pr(\|\tilde{J}_0 - F'(x_{t0})\| \leq \epsilon_0) \geq 1 - \delta.
\]

This inequality implies

\[
\|J_t - F'(x_t)\|^2 \leq 2\epsilon_t^2 + 8L_F^2\epsilon_t^2 t \sum_{s=1}^{t} \|x_s - x_{s-1}\|^2.
\] (61)

Our next step is to estimate the \( \|\tilde{F}_t - F(x_t)\| \). We define \( Y_t := F_t(x_t) - F_t(x_{t-1}) - [F_t(x_t) - F(x_{t-1})] \) and \( U_j := F_j(x_{t-1}) - F_j(x_{t-1}) - [F_j(x_{t-1}) - F(x_{t-1})] \) for \( j \in [n] \). In this case, \( \{Y_t\}_{t \in \mathcal{B}_i} \) is mutually independent and \( \mathbb{E}[Y_t] = 0 \). We also have

\[
\|Y_t\| = \left\| F_t(x_t) - F_t(x_{t-1}) - \frac{1}{n} \sum_{j=1}^{n}[F_j(x_{t}) - F_j(x_{t-1})] \right\|
\]

\[
= \left\| F_t(x_t) - F_t(x_{t-1}) - F_t'(x_{t-1})(x_t - x_{t-1}) + \frac{1}{n} \sum_{j=1}^{n}[F'_j(x_{t-1}) - F_j'(x_{t-1})](x_t - x_{t-1}) \right\|
\]

\[
\leq \frac{1}{n} \left\| \sum_{j \neq i} U_{t,j} \right\| + \left\| F_j'(x_{t-1}) - F'(x_{t-1}) \right\| (x_t - x_{t-1})
\]

\[
\leq \frac{1}{n} \sum_{j \neq i} \left\| U_j \right\| + \frac{2-\delta}{n} \left\| U_t \right\| + \left\| F_j'(x_{t-1}) - F'(x_{t-1}) \right\| \left\| x_t - x_{t-1} \right\|
\]

\[
\leq \frac{(n-1)L_F}{n} \left\| x_t - x_{t-1} \right\|^2 + \sigma_D \left\| x_t - x_{t-1} \right\|.
\]

Now, we consider \( W_t := \frac{1}{b_t} \sum_{i \in \mathcal{B}_i} Y_t = \frac{1}{b_t} \sum_{i \in \mathcal{B}_i} [F_i(x_t) - F_i(x_{t-1}) - F(x_t) + F(x_{t-1})] \). For any \( \epsilon > 0 \), we can apply Lemma C.1 to obtain the following bound

\[
\Pr(\|W_t\| \leq \epsilon \sigma_t) = \Pr(\|\sum_{i \in \mathcal{B}_i} Y_{ti} \| \leq \epsilon b_t \sigma_t) \geq 1 - (p + 1) \exp \left( -\frac{3b_t^2 \epsilon^2}{6b_t \sigma_t^2 + 2\sigma_D \epsilon b_t \sigma_t} \right)
\]

\[
= 1 - (p + 1) \exp \left( -\frac{3b_t^2 \epsilon^2}{6b_t \sigma_t^2 + 2\epsilon b_t \sigma_t} \right).
\]

Hence, if we choose \( \delta \geq (p + 1) \exp \left( -\frac{3b_t^2 \epsilon^2}{6b_t \sigma_t^2 + 2\sigma_D \epsilon b_t \sigma_t} \right) \) and \( \delta \leq 1 \), then we obtain \( \Pr(\|W_t\| \leq \epsilon \sigma_t) \geq 1 - \delta \) for all \( t \geq 0 \). The condition in \( b_t \) leads to \( b_t \geq \frac{6\delta + 2\sigma_D \epsilon}{6\epsilon \sigma_t} \log \left( \frac{p + 1}{\delta} \right) \).

Note that since \( \tilde{F}_0 := \frac{1}{t} \sum_{i \in B} F_i(x_{t0}) \) is updated by (22), to guarantee

\[
\Pr(\|\tilde{F}_0 - F(x_{t0})\| \leq \epsilon_0) \geq 1 - \delta,
\]

we choose the mini-batch size \( b \geq \frac{6\delta + 2\sigma_D \epsilon}{3\epsilon} \log \left( \frac{p + 1}{\delta} \right) \).

By the update of \( \tilde{F}_t \) from (28), we have \( \tilde{F}_t - F(x_t) = [\tilde{F}_{t-1} - F(x_{t-1})] + \frac{1}{b_t} \left[ F_t(x_t) - F_t(x_{t-1}) - F(x_t) + F(x_{t-1}) \right] + W_t \). Hence, by induction, it implies that \( \tilde{F}_t - F(x_t) = [\tilde{F}_0 - F(x_0)] + \sum_{s=1}^{t} W_s \), which leads to

\[
\|\tilde{F}_t - F(x_t)\| \leq \|\tilde{F}_0 - F(x_0)\| + \sum_{s=1}^{t} \|W_s\| \leq \epsilon_0 + \epsilon \sum_{s=1}^{t} \left[ L_F \left\| x_s - x_{s-1} \right\|^2 + \sigma_D \left\| x_s - x_{s-1} \right\| \right].
\] (62)
Now, we analyze the inner loop of \( t = 0 \) to \( m \). Using (16) with \( x := x^{(s)}_t \) and \( \bar{T}_M(x) = x^{(s)}_{t+1} \), we have
\[
\phi(F(x^{(s)}_{t+1})) \leq \phi(F(x^{(s)}_t)) - \frac{C_d}{2} \|x^{(s)}_{t+1} - x^{(s)}_t\|^2 + 2M \phi(F(x^{(s)}_t) - F(x^{(s)}_{t+1})) + \frac{M \phi}{2 \beta_d} \|F'(x^{(s)}_t) - \bar{F}(x^{(s)}_{t+1})\|^2,
\]
where \( C_d := 2M - M \phi (L_F + \beta_d) > 0 \) and \( \beta_d > 0 \) is given. Combining (63), (61), and (62), we have
\[
\phi(F(x^{(s)}_{t+1})) \leq \phi(F(x^{(s)}_t)) - \frac{C_d}{2} \|x^{(s)}_{t+1} - x^{(s)}_t\|^2 + 2M \phi \left[ \epsilon_0 + L_F \epsilon \sum_{t=1}^t \|x^{(s)}_j - x^{(s)}_{j-1}\|^2 \right] + \frac{L_F}{\beta_d} \left[ 2 \epsilon_0^2 + 8L_F \epsilon^2 (\sum_{t=1}^t \|x^{(s)}_j - x^{(s)}_{j-1}\|^2) \right] + 2M \phi \sigma D \epsilon \sum_{t=0}^t \|x^{(s)}_j - x^{(s)}_{j-1}\|^2.
\]
Summing up this inequality from \( t = 0 \) to \( t = m \), we obtain
\[
\phi(F(x^{(s)}_{m+1})) \leq \phi(F(x^{(s)}_0)) - \frac{C_d}{4} \sum_{t=0}^m \|x^{(s)}_{t+1} - x^{(s)}_t\|^2 + 2M \phi (m+1) \epsilon_0 + \frac{M \phi (m+1) \epsilon m^2}{2 \gamma} + \frac{M \phi (m+1) \epsilon_0^2}{\beta_d} + T_m,
\]
where \( T_m \) is defined as
\[
T_m := 2M \phi L_F \epsilon \sum_{t=0}^m \sum_{j=1}^t \|x^{(s)}_j - x^{(s)}_{j-1}\|^2 + 2M \phi \sigma D \epsilon \sum_{t=0}^m \sum_{j=1}^t \|x^{(s)}_j - x^{(s)}_{j-1}\|^2 + \frac{4L_F \epsilon^2}{\beta_d} \sum_{t=0}^m \sum_{j=1}^t \sum_{j=0}^t \|x^{(s)}_j - x^{(s)}_{j-1}\|^2 - \frac{C_d}{4} \sum_{t=0}^m \|x^{(s)}_{t+1} - x^{(s)}_t\|^2.
\]
Let \( u_{t-1} := \|x^{(s)}_t - x^{(s)}_{t-1}\| \). Then, we can rewrite \( T_m \) as
\[
T_m = 2M \phi L_F \epsilon \left[ u_0^2 + (u_0^2 + u_1^2) + \cdots + (u_0^2 + u_1^2 + u_{m-1}^2) \right] + 2M \phi \sigma D \epsilon [u_0 + (u_0 + u_1) + \cdots + (u_0 + u_1 + u_{m-1})] + 4L_F \epsilon^2 [u_0^2 + 2(u_0^2 + u_1^2) + \cdots + m(u_0^2 + u_1^2 + \cdots + u_{m-1}^2)] - \frac{C_d}{4} [u_0^2 + u_1^2 + \cdots + u_m^2]
\]
\[
= 2M \phi L_F \epsilon m + 4L_F \epsilon^2 m(m+1) - \frac{C_d}{4} u_0^2 + \left[ 2M \phi L_F \epsilon (m-1) + 4L_F \epsilon^2 m(m-1) - \frac{C_d}{4} \right] u_1^2 + \cdots + \left[ 2M \phi L_F \epsilon + 4L_F \epsilon^2 m(m-1) - \frac{C_d}{4} \right] u_m^2 + \cdots + \frac{M \phi \sigma D \epsilon m^2}{\gamma} [u_0^2 + u_1^2 + \cdots + u_{m-1}^2] + M \phi \sigma D \epsilon \gamma m^2
\]
\[
\leq 2M \phi L_F \epsilon m + 4L_F \epsilon^2 m(m+1) - \frac{C_d}{4} u_0^2 + \left[ 2M \phi L_F \epsilon (m-1) + 4L_F \epsilon^2 m(m-1) - \frac{C_d}{4} \right] u_1^2 + \cdots + \left[ 2M \phi L_F \epsilon + 4L_F \epsilon^2 m(m-1) - \frac{C_d}{4} \right] u_m^2 + \cdots + \frac{M \phi \sigma D \epsilon m}{\gamma} + 4L_F \epsilon^2 m(m+1) - \frac{C_d}{4} u_0^2 + \left[ 2M \phi L_F \epsilon + 4L_F \epsilon^2 m(m-1) - \frac{C_d}{4} \right] u_1^2 + \cdots + \left[ 2M \phi L_F \epsilon + 4L_F \epsilon^2 m(m-1) - \frac{C_d}{4} \right] u_m^2 + \cdots + \frac{M \phi \sigma D \epsilon m}{\gamma} + 4L_F \epsilon^2 m(m+1) - \frac{C_d}{4} u_0^2 + \left[ 2M \phi L_F \epsilon + 4L_F \epsilon^2 m(m-1) - \frac{C_d}{4} \right] u_1^2 + \cdots + \left[ 2M \phi L_F \epsilon + 4L_F \epsilon^2 m(m-1) - \frac{C_d}{4} \right] u_m^2 + \cdots + \frac{M \phi \sigma D \epsilon m}{\gamma}.
\]
If we impose the following condition
\[
M \phi \left( 2L_F + \frac{\sigma D}{\gamma} \right) \epsilon m + \frac{2L_F^3 \epsilon^2 m(m+1)}{\beta_d} \leq \frac{C_d}{4},
\]
then...
then \( T_m^s \leq M_0 \sigma_D \epsilon \gamma m^2 \).

Under this condition, \([64]\) reduces to

\[
\phi(F(\bar{x}^s)) \leq \phi(F(\bar{x}^{s-1})) - \frac{C_2}{4} \sum_{t=0}^{m} \|x_t^{(s)} - x_t^{(s)}\|^2 + 2M_0 (m + 1) \epsilon_0 + \frac{M_0 (m + 1) \epsilon_0^2}{\beta_d} + M_0 \sigma_D \epsilon \gamma m^2.
\]

Summing up this inequality from \( s = 1 \) to \( S = s \) and rearranging the result, we obtain

\[
\frac{1}{S(m + 1)} \sum_{s=1}^{S} \sum_{t=0}^{m} \|x_t^{(s)} - x_t^{(s)}\|^2 \leq \frac{4}{C_g (m + 1) S} \left[ \phi(F(\bar{x}^0)) - \phi(F(\bar{x}^S)) \right] + \frac{4M_0}{C_g} \left( 2\epsilon_0 + \sigma_D m \gamma + \frac{\epsilon_0^2}{\beta_d} \right).
\]

Using \( \phi(F(\bar{x}^S)) \geq \Psi^* \) and \( \Psi(x) = \phi(F(x)) \), we obtain from the last inequality that

\[
\frac{1}{S(m + 1)} \sum_{s=1}^{S} \sum_{t=0}^{m} \|\tilde{G}_M(x_t^{(s)})\|^2 \leq \frac{4M^2}{C_g (m + 1) S} \left[ \Psi(\bar{x}^0) - \Psi^* \right] + \frac{M^2 M_0}{C_g} \left( 2C_0 + \sigma_D C_1 + \frac{\tilde{C}_0}{\beta_d} \right) \epsilon^2.
\]

Clearly, if we choose \( \epsilon_0 := C_0 \epsilon^2 \), \( \epsilon := \frac{C_1 \epsilon}{\gamma} \), and \( \epsilon^2 := \frac{\tilde{C}_1}{m(m + 1)} \), for some positive constant \( C_0 \), \( C_1 \), \( \tilde{C}_0 \), and \( \tilde{C}_1 \), then we obtain from the last estimate that

\[
\frac{1}{S(m + 1)} \sum_{s=1}^{S} \sum_{t=0}^{m} \|\tilde{G}_M(x_t^{(s)})\|^2 \leq \frac{4M^2}{C_g (m + 1) S} \left[ \Psi(\bar{x}^0) - \Psi^* \right] + \frac{M^2 M_0}{C_g} \left( 2C_0 + \sigma_D C_1 + \frac{\tilde{C}_0}{\beta_d} \right) \epsilon^2,
\]

where we use the fact that \( \tilde{G}_M(x_t^{(s)}) = M(x_t^{(s)} - x_t^{(s)}) \). Now, assume that the condition \([65]\) is tight. Using the choice of accuracies, we obtain

\[
M_0 \left( 2L_F + \frac{\sigma_D}{\gamma} \right) \frac{C_1 \epsilon^2}{\gamma} + \frac{2L \tilde{C}_1}{\beta_d} = \frac{C_0}{4}.
\]

If we choose \( \gamma := \epsilon \), then this condition becomes \( 2M_0 (L_F \epsilon + \sigma_D) C_1 + \frac{2L \tilde{C}_1}{\beta_d} = \frac{C_0}{4} \) and \( \epsilon := \frac{C_1 \epsilon}{\gamma} \).

Now, with the choice of \( \epsilon_0 \), \( \epsilon \), \( \tilde{\epsilon} \), and \( \tilde{\epsilon} \) as above, we can set the mini-batch sizes as follows:

\[
\begin{align*}
  b_s &:= \left[ \frac{6 \sigma_D^3 + 2 \sigma_D C_0 \epsilon^2}{3C_0^2 \epsilon^2} \cdot \log \left( \frac{p+1}{\delta} \right) \right] = \mathcal{O} \left( \frac{\sigma_D^3}{\epsilon^2} \cdot \log \left( \frac{p+1}{\delta} \right) \right), \\
b_t^{(s)} &:= \left[ \frac{m [6m + 2 \sigma_D C_1 \epsilon]}{3C_1 \epsilon^2} \cdot \log \left( \frac{p+1}{\delta} \right) \right] = \mathcal{O} \left( \frac{m^2}{\epsilon^2} \cdot \log \left( \frac{p+1}{\delta} \right) \right), \\
\tilde{b}_s &:= \left[ \frac{6 \sigma_D^3 + 2 \sigma_D \sqrt{C_1} \epsilon}{3C_1 \epsilon^2} \cdot \log \left( \frac{p+q}{\delta} \right) \right] = \mathcal{O} \left( \frac{\sigma_D^3}{\epsilon^2} \cdot \log \left( \frac{p+q}{\delta} \right) \right), \\
\tilde{b}_t^{(s)} &:= \left[ \frac{\sqrt{m(m+1)} [6 \sqrt{m(m+1)} + 2 \sqrt{C_1} \epsilon]}{3C_1 \epsilon^2} \cdot \log \left( \frac{p+q}{\delta} \right) \right] = \mathcal{O} \left( m^2 \cdot \log \left( \frac{p+q}{\delta} \right) \right).
\end{align*}
\]

Since \( S(m + 1) = \frac{8M^2 \Psi(\bar{x}^0) - \Psi^*}{C_g \epsilon^2} \), if we choose \( m := \frac{C_0}{\epsilon} \), then \( S = \frac{8M^2 \Psi(\bar{x}^0) - \Psi^*}{C_g \epsilon^2} \). The total complexity is

\[
T_f := \sum_{s=1}^{S} b_s + \sum_{s=1}^{S} \sum_{t=0}^{m} b_t^{(s)} = \frac{6 \sigma_D^3 + 2 \sigma_D C_0 \epsilon^2}{3C_0^2 \epsilon^2} \cdot \log \left( \frac{p+1}{\delta} \right) + \frac{S(m+1)[6m + 2 \sigma_D C_1 \epsilon]}{3C_1 \epsilon^2} \cdot \log \left( \frac{p+1}{\delta} \right)
\]

\[
= \mathcal{O} \left( \frac{\sigma_D^3}{\epsilon^2} \cdot \log \left( \frac{p+1}{\delta} \right) \right) + \mathcal{O} \left( \frac{\Psi(\bar{x}^0) - \Psi^*}{\epsilon^2} \cdot \log \left( \frac{p+1}{\delta} \right) \right),
\]

\[
T_d := \sum_{s=1}^{S} \tilde{b}_s + \sum_{s=1}^{S} \sum_{t=0}^{m} \tilde{b}_t^{(s)} = \frac{6 \sigma_D^3 + 2 \sigma_D \sqrt{C_1} \epsilon}{3C_1 \epsilon^2} \cdot \log \left( \frac{p+q}{\delta} \right) + \frac{S(m+1) \sqrt{m(m+1)} [6 \sqrt{m(m+1)} + 2 \sqrt{C_1} \epsilon]}{3C_1 \epsilon^2} \cdot \log \left( \frac{p+q}{\delta} \right)
\]

\[
= \mathcal{O} \left( \frac{\sigma_D^3}{\epsilon^2} \cdot \log \left( \frac{p+q}{\delta} \right) \right) + \mathcal{O} \left( \frac{\Psi(\bar{x}^0) - \Psi^*}{\epsilon^2} \cdot \log \left( \frac{p+q}{\delta} \right) \right).
\]

This proves our theorem.
E Appendix: Subroutines for Computing GN Search Directions

One main step of SGN methods is to compute the Gauss-Newton direction by solving the subproblem (11). We can write this problem as follows
\[ \min_{d \in \mathbb{R}^m} \left\{ \phi(\bar{F}_t + \bar{J}_t d) + \bar{g}(d) + \frac{M}{2} \|d\|_2^2 \right\}, \] (68)
where \( \bar{F}_t \approx F(x_t) \), \( \bar{J}_t \approx F'(x_t) \), \( d := x - x_t \), \( \phi \) is convex, \( \bar{g}(d) := g(x_t + d) \), and \( M > 0 \) is given. This is a basic convex problem, and we can apply different methods to solve it. Here, we describe two methods for solving (68).

E.1 Accelerated Dual Proximal-Gradient Method
For accelerated dual proximal-gradient method, we consider the case \( \bar{g}(d) = 0 \) for simplicity. Using Fenchel’s conjugate of \( \phi \), we can write \( \phi(\bar{F}_t + \bar{J}_t d) = \max \left\{ \langle \bar{F}_t + \bar{J}_t d, u \rangle - \phi^*(u) \right\} \). Assume that strong duality holds for (68), then using this expression, we can write it as
\[ \min_d \max_u \left\{ \langle \bar{F}_t + \bar{J}_t d, u \rangle - \phi^*(u) + \frac{M}{2} \|d\|_2^2 \right\} \iff \max_u \left\{ \min_d \left\{ \langle \bar{F}_t + \bar{J}_t d, u \rangle + \frac{M}{2} \|d\|_2^2 \right\} - \phi^*(u) \right\}. \]
Solving the inner problem \( \min_d \left\{ \langle \bar{F}_t + \bar{J}_t d, u \rangle + \frac{M}{2} \|d\|_2^2 \right\} \), we obtain \( d^*(u) := -\frac{1}{M} \bar{J}_t^\top u \). Substituting it into the objective, we eventually get the dual problem as follows:
\[ \min_u \left\{ \frac{1}{2M} \|\bar{J}_t^\top u\|_2^2 - \langle \bar{F}_t, u \rangle + \phi^*(u) \right\}. \] (69)
We can solve this problem by an accelerated proximal-gradient method [2,19], which is described as follows:

**Algorithm 3 (Accelerated Dual Proximal-Gradient (ADPG))**

1: **Initialization:** Choose \( u_0 \in \mathbb{R}^m \). Set \( \tau_0 := 1 \) and \( \hat{u}_0 := u_0 \). Evaluate \( L := \frac{1}{M} \|\bar{J}_t \| \).
2: **For** \( k := 0, \cdots, k_{\text{max}} \) **do**
3: \( u_{k+1} := \text{prox}_{(1/L)\phi^*} \left( \hat{u}_k - \frac{1}{L} \left( \frac{1}{M} \bar{J}_t^\top \hat{u}_k - \bar{F}_t \right) \right) \).
4: \( \tau_{k+1} := \frac{1 + \sqrt{1 + 4\tau_k^2}}{2} \).
5: \( \hat{u}_{k+1} := u_{k+1} + \left( \frac{\tau_k - 1}{\tau_{k+1}} \right) (u_{k+1} - u_k) \).
6: **End For**
7: **Output:** Reconstruct \( d^* := -\frac{1}{M} \bar{J}_t^\top u_k \) as an approximate solution of (68).

Note that in Algorithm 3 we use the proximal operator \( \text{prox}_{\lambda \phi^*} \) of \( \phi^* \). However, by Moreau’s identity, \( \text{prox}_{\lambda \phi^*}(v) + \lambda \text{prox}_{\phi/\lambda}(v/\lambda) = v \), we can again use the proximal operator \( \text{prox}_{\phi/\lambda} \) of \( \phi \).

E.2 Primal-Dual First-Order Methods
We can apply any primal-dual algorithm from the literature [11,14,10,12,28,26] to solve (68). Here, we describe the Chambolle-Pock’s primal-dual method [3] to solve (68).

Let us define \( \hat{\phi}(z) := \phi(z + F_k) \) and \( \hat{\psi}(d) := \bar{g}(d) + \frac{M}{2} \|d\|_2^2 \). Since (68) is strongly convex with the strong convexity parameter \( \mu_{\bar{\psi}} := M \), we can apply the strongly convex primal-dual variant as follows:

Choose \( \sigma_0 > 0 \) and \( \tau_0 > 0 \) such that \( \sigma_0 \tau_0 \leq \frac{1}{\|J_t\|} \). For example, we can choose \( \sigma_0 = \tau_0 = \frac{1}{\|J_t\|} \), or we choose \( \sigma_0 > 0 \) first, and choose \( \tau_0 := \frac{1}{\sigma_0 \|J_t\|} \). Choose \( d_0 \in \mathbb{R}^p \) and \( u_0 \in \mathbb{R}^m \) and set \( \bar{d}_0 := d_0 \). Then, at each iteration \( k \geq 0 \), we update
\[
\begin{align*}
    u_{k+1} & := \text{prox}_{\sigma_k \hat{\phi}^*} \left( u_k + \sigma_k \bar{J}_t \bar{d}_k \right) \\
    d_{k+1} & := \text{prox}_{\tau_k \hat{\psi}} \left( d_k - \tau_k \bar{J}_t^\top u_{k+1} \right) \\
    \theta_k & := 1/\sqrt{1 + 2M\tau_k}, \quad \tau_{k+1} := \theta_k \tau_k, \quad \sigma_{k+1} := \sigma_k / \theta_k \\
    \bar{d}_{k+1} & := d_{k+1} + \theta_k (d_{k+1} - d_k). \quad \text{(70)}
\end{align*}
\]
Alternatively to the Accelerated Dual Proximal-Gradient and the primal-dual methods, we can also apply the alternating direction method of multipliers (ADMM) to solve (65). However, this method requires to solve a linear system, that may not scale well when the dimension \( p \) is large.

### F Appendix: Details of The Experiments in Section 5

In this appendix, we provide the details of our experiments in Section 5 including modeling, data generating routines, and experiment configurations. We also provide more experiments for the second example. The experiment is implemented in Python 3.6 running on a Macbook Pro with 2.3 GHz Quad-Core and 8 GB RAM.

#### F.1 Stochastic Nonlinear Equations

Our goal is to solve the following expectation nonlinear equation as described in Subsection 5.1:

\[
F(x) = 0, \quad \text{where} \quad F(x) := \mathbb{E}_\xi[F(x, \xi)]. \tag{71}
\]

Here, \( F \) is a stochastic vector function from \( \mathbb{R}^p \times \Omega \rightarrow \mathbb{R}^q \). As discussed in the main text, (71) covers the first-order optimality condition \( \mathbb{E}_\xi[\nabla_x G(x, \xi)] = 0 \) of a stochastic optimization problem \( \min_x \mathbb{E}_\xi[G(x, \xi)] \) as a special case. More generally, it also covers the KKT condition of a stochastic optimization problem with equality constraints. However, these problems may not have stationary point, which leads to an inconsistency of (71). As a remedy, we can instead consider

\[
\min_x \{ \Psi(x) := \|\mathbb{E}_\xi[F(x, \xi)] \| \}, \tag{72}
\]

for a given norm \( \| \cdot \| \) (e.g., \( \ell_1 \)-norm or \( \ell_2 \)-norm). Problem (71) also covers the expectation formulation of stochastic nonlinear equations such as stochastic ODEs or PDEs.

In our experiment from Subsection 5.1, we only consider one instance of (72) by choosing \( q = 4 \) and \( F_i (i = 1, \cdots, n) \) as

\[
F_i(x) := \begin{bmatrix}
(1 - tanh(y_i(a_i^T x + b_i))) \\
\left(1 - \frac{1}{1 + \exp(-y_i(a_i^T x + b_i))}\right)^2 \\
\ln(1 + e^{-y_i(a_i^T x + b_i)}) - \ln(1 + e^{-y_i(a_i^T x + b_i) - 1}) \\
\ln(1 + (y_i(a_i^T x + b_i) - 1)^2)
\end{bmatrix}, \tag{73}
\]

where \( a_i \) is the \( i \)-row of an input matrix \( A \in \mathbb{R}^{n \times p} \), and \( y \in \{1, -1\}^n \) is a vector of labels, and \( b \in \mathbb{R}^n \) is a bias vector in binary classification. Note that the binary classification problem with nonconvex loss has been widely studied in the literature, including [37], where one aims at solving:

\[
\min_{x \in \mathbb{R}^p} \left\{ H(x) := \frac{1}{n} \sum_{i=1}^{n} \ell(y_i(a_i^T x + b_i)) \right\}, \tag{74}
\]

for a given loss function \( \ell \). If \( \ell \) is nonnegative, then instead of solving (74), we can solve \( \min_x \|H(x)\| \). If we have \( q \) different losses \( \ell_j \) for \( j = 1, \cdots, q \) and we want to solve \( q \) problems of the form (74) for different losses simultaneously, then we can formulate such a problem into (72) to have \( \min_x \|H(x)\| \), where \( H(x) := (H_1(x), H_2(x), \cdots, H_q(x))^\top \). Since we use different losses, under the formulation (72), we can view it as a kind of model selection task for binary classification.

**Datasets:** We test three algorithms: GN, SGN, and SGN2 on both synthetic and two real datasets: \texttt{w8a} and \texttt{covtype} from LIBSVM. For synthetic datasets, we generate matrices \( A_j \) randomly from a Gaussian distribution with mean 0 and unit variance. The vector \( b \) is uniformly generated in the range \([-1, 1]\), and \( y \) is also generated randomly as \( y := \text{sign}(s^2) \) for a standard random Gaussian vector \( s^2 \) (if \( s^2 = 0, \ y = 1 \)). For the \texttt{w8a} and \texttt{covtype} datasets, to get sufficiently large number of samples, we upsample them by bootstrapping to obtain \( n = 10^6 \) samples.

**Parameter configuration:** We can easily check that \( F \) defined by (73) satisfies Assumption 1.1 and Assumption 2. However, we do not accurately estimate the Lipschitz constant of \( F' \) since it depends on the dataset. We were instead experimenting with different choices of the parameter \( M \) and \( \rho \), and eventually fix
\( \rho := 1 \) and \( M := 1 \) for our tests. We also choose the mini-batch sizes for both \( \tilde{F} \) and \( \tilde{J} \) in SGN and SGN2 by sweeping over the set of \( \{ 10^2, 10^3, 10^4, 10^5 \} \) to estimate the best ones. We eventually obtain \( b_t := 10^4 \) in SGN and \( \hat{b}_{t}^{(s)} := 10^4 \) in SGN2 for \( \tilde{J} \), while \( b_t := 10^6 \) in SGN and \( \hat{b}_{t}^{(s)} := 10^5 \) in SGN2 for \( \tilde{F} \), which seem working well.

### F.2 Optimization Involving Expectation Constraints

We consider an optimization problem involving expectation constraints as described in (35). As mentioned, this problem has various applications in different fields, including optimization with conditional value at risk (CVaR) constraints and metric learning [14].

Instead of solving the constrained setting (35), we consider its exact penalty formulation (36):

\[
\min_{x \in \mathbb{R}^p} \left\{ \Psi(x) := g(x) + \phi(\mathbb{E}_\xi [F(x, \xi)]) \right\},
\]

where \( \phi(u) := \rho \sum_{i=1}^n [u_i]_+ \) with \([u]_+ := \max \{0, u\}\) is a penalty function, and \( \rho > 0 \) is a given penalty parameter. It is well-known that under mild conditions and \( \rho \) sufficiently large (e.g., \( \rho > \|y^*\|_\star \)), the dual norm of the optimal Lagrange multiplier \( y^* \), if \( x^* \) is a stationary point of (36) and it is feasible to (35), then it is also a stationary point of (35).

As a concrete instance of (35), we solve the following asset allocation problem studied in [24, 14]:

\[
\begin{align*}
\min_{\mathbf{z} \in \mathbb{R}^p, \tau \in [\bar{\tau}, \tilde{\tau}]} & \quad -c^\top \mathbf{z} \\
\text{s.t.} & \tau + \frac{1}{\beta n} \sum_{i=1}^n [-\xi_i^\top \mathbf{z} - \tau]_+ \leq 0, \\
& \mathbf{z} \in \Delta_p := \{ \mathbf{z} \in \mathbb{R}^p_p \mid \sum_{i=1}^p \mathbf{z}_i = 1 \}.
\end{align*}
\]

Here, \( \Delta_p \) denotes the standard simplex in \( \mathbb{R}^p \), and \([\bar{\tau}, \tilde{\tau}]\) is a given range of \( \tau \). The exact penalty formulation of (75) is given by (37):

\[
\min_{\mathbf{z} \in \Delta_p, \tau \in [\bar{\tau}, \tilde{\tau}]} \left\{ -c^\top \mathbf{z} + \phi \left( \tau + \frac{1}{\beta n} \sum_{i=1}^n [-\xi_i^\top \mathbf{z} - \tau]_+ \right) \right\},
\]

where \( \phi(u) := \rho [u]_+ \) with given \( \rho > 0 \). However, since \([\xi_i^\top \mathbf{z} - \tau]_+ \) is nonsmooth, we smooth it by \( \sqrt{(\xi_i^\top \mathbf{z} + \tau)^2 + \gamma^2 - \gamma - \xi_i^\top \mathbf{z} - \tau} \) for sufficiently small \( \gamma > 0 \). Hence, (37) can be approximated by

\[
\min_{\mathbf{z} \in \Delta_p, \tau \in [\bar{\tau}, \tilde{\tau}]} \left\{ -c^\top \mathbf{z} + \phi \left( \tau + \frac{1}{\beta n} \sum_{i=1}^n \left[ \sqrt{(\xi_i^\top \mathbf{z} + \tau)^2 + \gamma^2 - \gamma - \xi_i^\top \mathbf{z} - \tau} \right] \right) \right\}.
\]

If we introduce \( x := (z, \tau) \), \( F_i(x) := \tau + \frac{1}{2\beta} \left[ \sqrt{(\xi_i^\top \mathbf{z} + \tau)^2 + \gamma^2 - \gamma - \xi_i^\top \mathbf{z} - \tau} \right] \) for \( i = 1, \cdots, n \), and \( g(x) = -c^\top \mathbf{z} + \delta_{\Delta_p \times [\bar{\tau}, \tilde{\tau}]}(x) \), where \( \delta_x \) is the indicator of \( X \), then we can reformulate (76) into (3). It is obvious to check that \( F_i \) is Lipschitz continuous with \( M_{F_i} := 1 + \frac{\|\xi_i\|_2 + 1}{\beta \gamma} \) and its gradient \( \nabla F_i \) is also Lipschitz continuous with \( L_{F_i} := \frac{\|\xi_i\|_2^2}{2\beta \gamma} \). Hence, Assumptions 1.1 and 4.1 hold.

**Datasets:** We consider both synthetic and US stock datasets. For the synthetic datasets, we follow the code from [13] to generate the data with \( n \in \{5 \times 10^4, 10^5\} \) and \( p \in \{200, 300, 500\} \). We obtain real datasets of US stock prices for 889 or 865 types of stocks as described, e.g., [25] then bootstrap them to obtain different datasets of sizes \( n = 5 \times 10^4 \) and \( n = 10^5 \).

**Parameter setting:** We fix the smoothness parameter \( \gamma := 10^{-3} \) and choose the range \([\bar{\tau}, \tilde{\tau}]\) to be \([0, 1]\). The parameter \( \beta := 0.1 \) as discussed in [14]. Note that we do not use the theoretical values for \( M \) as in our theory since that value is obtained in the worst-case. We were instead experimenting different values for the penalty parameter \( \rho \) and \( M \), and eventually get \( \rho := 5 \) and \( M := 5 \) as default values for this example.

**Experiment setup:** We implement our algorithms: SGN and SGN2, and also a baseline variant, the deterministic GN scheme (i.e., we exactly evaluate \( F \) and its Jacobian using the full batches) as in the first example. Similar to the first example, we sweep over the same set of possible mini-batch sizes, and eventually get \( \hat{b}_{t} := 10^3 \) in SGN and \( \hat{b}_{t}^{(s)} := 500 \) in SGN2 for \( \tilde{J} \), while \( \hat{b}_{t} := 10^4 \) in SGN and \( \hat{b}_{t}^{(s)} := 5 \times 10^3 \) in SGN2 for \( \tilde{F} \), respectively.
**Additional Experiments:** We run three algorithms: GN, SGN, and SGN2 with 3 synthetic datasets, where the first one was reported in Figure 3 of the main text. For other two datasets, the results are shown in Figure 4.

![Figure 4: The performance of three algorithms on two synthetic datasets.](image)

Clearly, SGN2 is the best, while SGN still outperform GN in these two datasets. We believe that this experiment confirms our theoretical results presented in the main text.

Now, if we use three different US Stock datasets with different sizes, then the performance of three algorithms is revealed in Figure 5 where the same convergence behavior is again observed.

![Figure 5: The performance of three algorithms on three different US stock datasets.](image)
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