SYMMETRIES AND CONSERVATION LAWS OF HAMILTONIAN SYSTEMS

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Abstract. In this paper we study the infinitesimal symmetries, Newtonoid vector fields, infinitesimal Noether symmetries and conservation laws of Hamiltonian systems. Using the dynamical covariant derivative and Jacobi endomorphism on the cotangent bundle we find the invariant equations of infinitesimal symmetries and Newtonoid vector fields and prove that the canonical nonlinear connection induced by a regular Hamiltonian can be determined by these symmetries. Finally, an example from optimal control theory is given.

MSC2010: 37J15, 53C05, 70H33, 70H05

Keywords: infinitesimal symmetries, dynamical covariant derivative, Jacobi endomorphism, Hamiltonian vector field.

1. Introduction

The notion of symmetry plays a very important role in all field theories being related with conservation laws by Noether type theorems. The use of the symmetries of a system in the description of its dynamical evolution has a long history and goes back to the classical mechanics (see for instance [1],[2],[20]). Also, the Lagrangian and Hamiltonian formalisms are fundamental concepts in physics, differential equations or optimal control and in most of cases the study starts with a variational problem formulated for a regular Lagrangian on the tangent bundle $TM$ over the manifold $M$ and very often the whole set of problems is transferred on the dual space $T^*M$, endowed with a Hamiltonian function, via Legendre transformation.

The present paper contains some contributions to the study of symmetries of Hamiltonian systems and shows how the well-known local symmetries of Lagrangian systems emerge in Hamiltonian formulation. The tangent bundle has a canonical tangent structure $J$ and together with a semispray $S$ (system of second order differential equation-SODE) induce a nonlinear connection that describes the geometry of the system [10],[14]. These structures lead to the notions of Jacobi endomorphism and dynamical covariant derivative introduced by J. Carinena and E. Martinez (see [8],[21]) which have been used in the study of symmetries for SODE in [7]. But, the existence of a symplectic structure on the tangent bundle depends on a Lagrangian function on $TM$. We have to remark that some type of symmetries on the tangent bundle as dynamical symmetries, Lie symmetries, Cartan (Noether) symmetries (see e.g. [1],[12],[10],[17],[28],[29],[31]) and Newtonoid vector field ([7],[19]) have been studied in a lot of papers, where the main geometric structures are the semispray, the symplectic structure $\omega_L$ induced by a regular Lagrangian $L$ and the energy $E_L$. Contrary, the cotangent bundle is endowed with a canonical symplectic structure and does not have a canonical tangent structure or something similar with a semispray. However, the existence of a pseudo-metric structure or a regular
Hamiltonian on $T^*M$ permit us to define an adapted tangent structure $J$ and a regular vector field $\rho$ which induce a nonlinear connection [24]. These geometrical structures permit us to introduce the Jacobi endomorphism and dynamical covariant derivative on $T^*M$ (see [26],[27]) which will be used in this paper in order to find the invariant equations of the infinitesimal symmetries of Hamiltonian systems. In fact, this work contains the ideas proposed by the author in [27]. Different types of symmetries and conservation laws for Hamiltonian systems can be found, for example, in [3],[9],[15],[18],[23],[30].

The paper is organized as follows. In the second section the preliminary geometrical structures on the cotangent bundle are recalled (see for instance [22],[24],[25],[26],[27],[33] and references therein). We introduce the Berwald linear connection $\mathcal{D}$ on $T^*M$ induced by a nonlinear connection $\mathcal{N}$ and study its properties in subsection 2.1. We show that this linear connection is compatible with the horizontal and vertical projectors, adapted tangent structure and complex structure. Also, we find its action on the local Berwald basis. Moreover, we prove that in the case of the horizontal $J$-regular vector field $\rho$, the Berwald linear connection coincides with the dynamical covariant derivative, that is $\mathcal{D}_\rho = \nabla$. Consequently, in this case, the integral curves of a $J$-regular vector field are the geodesics of the Berwald linear connection.

In the third section we investigate the symmetries of Hamiltonian systems on the cotangent bundle. First, for a regular Hamiltonian on $T^*M$, we introduce an integrable adapted tangent structure $J_H$, a regular vector field $\rho_H$, which is the Hamiltonian vector field, and find the coefficients of the canonical nonlinear connection. Moreover, we give the expression of the Jacobi endomorphism, which depends only on the regular Hamiltonian and find the action of the dynamical covariant derivative on the local Berwald basis. Next, using the Hamiltonian vector field, canonical symplectic structure and adapted tangent structure, we study the infinitesimal symmetries, natural infinitesimal symmetries, Newtonoid vector field, infinitesimal Noether symmetries and conservation laws of Hamiltonian systems. Also, using the dynamical covariant derivative and Jacobi endomorphism on $T^*M$, we find the invariant equations of the infinitesimal symmetries and Newtonoid vector field and prove that these symmetries determine the canonical nonlinear connection. Moreover, we show when one of these symmetries will imply the others and that there is a one to one correspondence between the exact infinitesimal Noether symmetry and conservation laws. Finally, an example from optimal control theory is given.

2. Geometrical structures on the cotangent bundle

Let $M$ be a differentiable, $n$-dimensional manifold and $(T^*M, \tau, M)$ its cotangent bundle. If the local coordinates on $\tau^{-1}(U)$ are denoted $(x^i, p_i)$, $(i = 1, n)$ then the natural basis on $T^*M$ is $(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial p_i})$ and $(dx^i, dp_i)$ is the dual natural basis. The following geometric objects [33]

$$C^* = p_i \frac{\partial}{\partial p_i}, \quad \theta = p_i dx^i, \quad \omega = d\theta = dp_i \wedge dx^i,$$

have the following properties:

1° $C^*$ is a vertical vector field, globally defined on $T^*M$, which is called the Liouville-Hamilton vector field.

2° The 1-form $\theta$ is globally defined on $T^*M$ and is called the Liouville 1-form.
The 2-form $\omega$ is the canonical symplectic structure. If $L$ and $K$ are $(1, 1)$-type tensor field then the Frölicher-Nijenhuis bracket $[L, K]$ is the vector valued 2-form $\Omega$

\[
[L, K](X, Y) = [LX, KY] + [KX, LY] + (LK + KL)[X, Y] - L[X, KY] - K[X, LY] - L[KX, Y] - K[LX, Y].
\]

and the Nijenhuis tensor of $L$ is given by

\[
N_L(X, Y) = \frac{1}{2}[L, L] = [LX, LY] + L^2[X, Y] - L[X, LY] - L[LX, Y].
\]

For a vector field $X$ in $\mathcal{X}(M)$ the Frölicher-Nijenhuis bracket $[X, L] = \mathcal{L}_X L$ is the $(1, 1)$-type tensor field on $M$ given by $\mathcal{L}_X L = \mathcal{L}_X \circ L - L \circ \mathcal{L}_X$, where $\mathcal{L}_X$ is the usual Lie derivative. On the cotangent bundle $T^*M$ there exists the integrable vertical distribution $V_\tau T^*M$, $u \in T^*M$ generated locally by the basis $\{\frac{\partial}{\partial p_i}\}_{i=1}^n$. A nonlinear connection on $T^*M$ is defined by an almost product structure $\mathcal{N}$ (i.e. a morphism $\mathcal{N} : \mathcal{X}(T^*M) \to \mathcal{X}(T^*M)$ with $\mathcal{N}^2 = \text{Id}$) such that $V^T T^*M = \text{Ker}(\text{Id} + \mathcal{N})$. If $\mathcal{N}$ is a nonlinear connection then $HT^*M = \text{Ker}(\text{Id} - \mathcal{N})$ is the horizontal distribution associated to $\mathcal{N}$, which is supplementary to the vertical distribution, that is $TT^*M = V^T T^*M \oplus HT^*M$. If $\mathcal{N}$ is a nonlinear connection then on the every domain of the local chart $\tau^{-1}(U)$, the adapted basis of the horizontal distribution $HT^*M$ is

\[
\mathcal{N}_{ij}(x, p) = \frac{\partial}{\partial x^i} + \mathcal{N}_{ij} \frac{\partial}{\partial p_j},
\]

where $\mathcal{N}_{ij}(x, p)$ are the coefficients of the nonlinear connection. The dual adapted basis is $\delta p_i = dp_i - \mathcal{N}_{ij} dx^j$. The system of vector fields $\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta p_i}\right)$ defines the local Berwald basis on $T^*M$. A nonlinear connection induces the horizontal and vertical projectors given by

\[
h = \frac{1}{2}(\text{Id} + \mathcal{N}), \quad v = \frac{1}{2}(\text{Id} - \mathcal{N}), \quad h = \frac{\delta}{\delta x^i} \otimes dx^i, \quad v = \frac{\partial}{\partial p_i} \otimes \delta p_i,
\]

which satisfy the properties $h^2 = h$, $v^2 = v$, $hv = vh = 0$, $h + v = \text{Id}$, $h - v = \mathcal{N}$. The nonlinear connection $\mathcal{N}$ on $T^*M$ is called symmetric if $\omega(hX, hY) = 0$, for $X, Y \in \mathcal{X}(T^*M)$, that is $\mathcal{N}_{ij} = \mathcal{N}_{ji}$. The following equations hold

\[
(1) \quad \begin{bmatrix} \frac{\delta}{\delta x^i} & \frac{\delta}{\delta x^j} \\ \frac{\delta}{\partial p_k} & \frac{\delta}{\partial p_l} \end{bmatrix} = R_{ijk} \frac{\partial}{\partial p_k}, \quad \begin{bmatrix} \frac{\delta}{\delta x^i} & \frac{\delta}{\partial p_j} \\ \frac{\delta}{\partial p_k} & \frac{\delta}{\partial p_l} \end{bmatrix} = -N_{ikr} \frac{\partial}{\partial p_j} - N_{ir} \frac{\partial}{\partial p_j}, \quad \begin{bmatrix} \frac{\partial}{\partial p_i} & \frac{\partial}{\partial p_j} \end{bmatrix} = 0,
\]

(2) \quad \begin{bmatrix} \frac{\delta}{\delta x^i} & \frac{\delta}{\delta x^j} \\ \frac{\delta}{\partial p_k} & \frac{\delta}{\partial p_l} \end{bmatrix} = R_{ijk} \frac{\partial}{\partial p_k} + \frac{\delta N_{jk}}{\delta x^i} - \frac{\delta N_{ik}}{\delta x^j}.
\]

The curvature of the nonlinear connection $\mathcal{N}$ on $T^*M$ is given by $\Omega = -\frac{1}{2}[h, h]$, where $h$ is the horizontal projector and $\frac{1}{2}[h, h]$ is the Nijenhuis tensor of $h$. In local coordinates $\Omega = -\frac{1}{2}R_{ijk} dx^i \wedge dx^j \otimes \frac{\delta}{\partial p_k}$, where $R_{ijk}$ is given by (2). An almost tangent structure on $T^*M$ is a morphism $\mathcal{J} : \mathcal{X}(T^*M) \to \mathcal{X}(T^*M)$ of rank $n$ such that $\mathcal{J}^2 = 0$. The almost tangent structure is called adapted if $\text{Im} \mathcal{J} = \text{Ker} \mathcal{J} = V^T T^*M$ (see [24]). The following properties hold

\[
\mathcal{J} h = \mathcal{J}, \quad h \mathcal{J} = 0, \quad \mathcal{J} v = 0, \quad v \mathcal{J} = \mathcal{J}.
\]
Locally, an adapted almost tangent structure has the form

\[ \mathcal{J} = t_{ij}dx^i \otimes \frac{\partial}{\partial p_j}, \]

where \( t_{ij}(x, p) \) is a tensor field of rank \( n \). The existence of a nonlinear connection on \( T^*M \) is equivalent with the conditions \( N^*\mathcal{J} = -\mathcal{J}, \mathcal{J}N = \mathcal{J} \). The adapted almost tangent structure \( \mathcal{J} \) is integrable if and only if \( \frac{\partial t_{ij}}{\partial p_k} = \frac{\partial t_{ik}}{\partial p_j} \), where \( t_{ij}t^{jk} = \delta^i_j \). Also, \( \mathcal{J} \) is called symmetric if \( \omega(\mathcal{J}X, Y) = \omega(\mathcal{J}Y, X) \), which locally is equivalent with the symmetry of the tensor \( t_{ij}(x, p) \). From [24], [27] we have that a vector field \( \rho \in \mathcal{X}(T^*M) \) is called \( \mathcal{J} \)-regular if it satisfies the equation

\[ \mathcal{J}[\rho, JX] = -JX, \quad \forall X \in \mathcal{X}(T^*M). \]

Locally, a vector field on \( T^*M \) given in local coordinates by

\[ \rho = \xi^i(x, p)\frac{\partial}{\partial x^i} + \chi_i(x, p)\frac{\partial}{\partial p_i}, \]

is \( \mathcal{J} \)-regular if and only if \( t^{ij} = \delta^i_j \frac{\partial}{\partial p_i} \), where \( t_{ij}t^{jk} = \delta^i_j \). For a \( \mathcal{J} \)-regular vector field \( \rho \) and an arbitrary nonlinear connection \( N \) with induced \( (h, v) \) projectors, we consider the vertically valued \((1, 1)\)-type tensor field on \( T^*M \backslash \{0\} \) given by [27]

\[ \Phi = v \circ L_{\rho}h, \]

which is called the Jacobi endomorphism. In local coordinates we obtain

\[ L_{\rho} \frac{\delta}{\delta x^i} = -\frac{\delta \xi^i}{\delta x^j} \frac{\delta}{\delta x^j} + R_{jk} \frac{\partial}{\partial p_k}, \quad L_{\rho} \frac{\partial}{\partial p_j} = -t^{ji} \frac{\delta}{\delta x^i} + \left( t^{ji}N_{ik} - \frac{\partial \chi_k}{\partial p_j} \right) \frac{\partial}{\partial p_k}. \]

Locally, the Jacobi endomorphism has the form

\[ \Phi = \mathcal{R}_{ij}dx^i \otimes \frac{\partial}{\partial p_j}, \quad \mathcal{R}_{jk} = \frac{\delta \xi^i}{\delta x^j}N_{ik} - \frac{\delta \chi_k}{\delta x^j} + \rho(N_{jk}). \]

We can also recover the Jacobi endomorphism \( \Phi \) from the curvature tensor \( \Omega \) through the formula \( \Phi = \iota_{\rho}\Omega + v \circ L_{\rho}h \). Moreover, if \( \rho \) is a horizontal \( \mathcal{J} \)-regular vector field then \( \rho = h\rho, v\rho = 0 \) and \( \Phi = \iota_{\rho}\Omega \). Locally, it results

\[ \rho = \xi^i \frac{\delta}{\delta x^i}, \quad \chi_i = \xi^kN_{ki}, \quad \mathcal{R}_{ij} = R_{kij}\xi^k, \]

which show us the relation between the Jacobi endomorphism given by (6) and curvature tensor from (2). For a given \( \mathcal{J} \)-regular vector field \( \rho \) on \( T^*M \) the Lie derivative \( L_{\rho} \) defines a tensor derivation on \( T^*M \), but does not preserve some of the geometric structure as adapted tangent structure or nonlinear connection. Next, using a nonlinear connection, we introduce a tensor derivation on \( T^*M \), called the dynamical covariant derivative, that preserves some geometric structures (see e.g. [7], [21], [32] for the tangent bundle case). Using [27] we set:

**Definition 1.** A map \( \nabla : \mathcal{T}(T^*M \backslash \{0\}) \to \mathcal{T}(T^*M \backslash \{0\}) \) is said to be a tensor derivation on \( T^*M \backslash \{0\} \) if the following conditions are satisfied:

i) \( \nabla \) is \( \mathbb{R} \)-linear,

ii) \( \nabla \) is type preserving, i.e. \( \nabla(T^*_s(T^*M \backslash \{0\})) \subset T^*_s(T^*M \backslash \{0\}) \), for each \( (r, s) \in \mathbb{N} \times \mathbb{N} \),

iii) \( \nabla \) obeys the Leibnitz rule \( \nabla(T \otimes S) = \nabla T \otimes S + T \otimes \nabla S \),

iv) \( \nabla \) commutes with any contractions.
For a $\mathcal{J}$-regular vector field $\rho$ and an arbitrary nonlinear connection $\mathcal{N}$ with induced $(h,v)$ projectors, we consider the map $\nabla : \mathcal{X}(T^*M \setminus \{0\}) \to \mathcal{X}(T^*M \setminus \{0\})$ given by

$$\nabla = h \circ \mathcal{L}_\rho \circ h + v \circ \mathcal{L}_\rho \circ v,$$

which is called the dynamical covariant derivative with respect to $\rho$ and the nonlinear connection $\mathcal{N}$. By setting $\nabla f = \rho(f)$, for $f \in C^\infty(T^*M \setminus \{0\})$ using the Leibnitz rule and the requirement that $\nabla$ commutes with any contraction, we can extend the action of $\nabla$ to arbitrary tensor fields on $T^*M \setminus \{0\}$ (see [27]). By direct computation we obtain $\nabla h = \nabla v = 0$ and the action of $\nabla$ on the Berwald basis:

$$\nabla \frac{\partial}{\partial p_j} = \left( t^{ij} \mathcal{N}_{ik} - \frac{\partial \chi_k}{\partial p_j} \right) \frac{\partial}{\partial p_k}, \quad \nabla \delta p_j = - \left( t^{ij} \mathcal{N}_{ij} - \frac{\partial \chi_j}{\partial p_k} \right) \delta p_k.$$

The following results hold [27]

$$h \circ \mathcal{L}_\rho \circ \mathcal{J} = -h, \quad \mathcal{J} \circ \mathcal{L}_\rho \circ v = -v,$$

$$\nabla \mathcal{J} = \mathcal{L}_\rho \mathcal{J} + h - v, \quad \nabla \mathcal{J} = \left( \rho(t_{ij}) + t_{ijk} \frac{\partial \xi_j}{\partial x^i} - t_{ik} \frac{\partial \chi_j}{\partial p_k} + 2 \mathcal{N}_{ij} \right) dx^i \otimes \frac{\partial}{\partial p_j}.$$

Given an adapted tangent structure $\mathcal{J}$ and a $\mathcal{J}$-regular vector field $\rho$, then the compatibility condition $\nabla \mathcal{J} = 0$ fix the canonical nonlinear connection with $h, v$ projectors

$$h = \frac{1}{2} \left( \text{Id} - \mathcal{L}_\rho \mathcal{J} \right), \quad v = \frac{1}{2} \left( \text{Id} + \mathcal{L}_\rho \mathcal{J} \right).$$

The $(1,1)$-type tensor field

$$\mathcal{N} = -\mathcal{L}_\rho \mathcal{J},$$

is the almost product structure which will be used in the following. The local coefficients are given by [24]

$$\mathcal{N}_{ij} = \frac{1}{2} \left( t_{ik} \frac{\partial \chi_j}{\partial p_k} - t_{kj} \frac{\partial \xi_j}{\partial x^i} - \rho(t_{ij}) \right).$$

The almost complex structure has the form $\mathbb{F} = h \circ \mathcal{L}_\rho h - \mathcal{J}$ and in local coordinates we have $\mathbb{F} = t^{ij} \frac{\partial}{\partial x^i} \otimes \delta p_j - t_{ij} \frac{\partial}{\partial p_i} \otimes dx^j$. The dynamical covariant derivative has in this case the properties [27]

$$\nabla \mathcal{J} = 0, \quad \nabla \mathbb{F} = 0.$$

Moreover, if $\rho$ is a horizontal $\mathcal{J}$-regular vector field then $\nabla \rho = 0$.

2.1. Berwald linear connection on the cotangent bundle. Next, we introduce the Berwald linear connection induced by a nonlinear connection and prove that in the case of horizontal $\mathcal{J}$-regular vector field it coincides with the dynamical covariant derivative. This connection was introduced on the tangent bundle by L. Berwald in [4] and studied later in [11], [22] and [3].

**Definition 2.** The Berwald linear connection on the cotangent bundle is given by

$$\mathcal{D} : \mathcal{X}(T^*M \setminus \{0\}) \times \mathcal{X}(T^*M \setminus \{0\}) \to \mathcal{X}(T^*M \setminus \{0\}),$$

$$\mathcal{D}_X Y = v[hX,vY] + h[vX,hY] + \mathcal{J}[vX,(\mathbb{F} + \mathcal{J})Y] + (\mathbb{F} + \mathcal{J})[hX,\mathcal{J}Y].$$
Because all the structures from the right hand side of (14) are additive, it results that \( \mathcal{D} \) is also additive, with respect to both arguments. Next, we prove that \( D_{Y}X = fD_{X}Y, \forall f \in C^\infty(T^*M) \) by straightforward computation, using the relations \( vh = hv = Jv = 0 \) and \( (\mathcal{F} + J)h = 0 \). Indeed, for the first term from \( D_{Y}X \) we have \( v[fhX, vY] = v(f[hX, vY] - (vY)(f)hX) = f[vhX, vY] \). In order to prove the relation \( D_{Y}fY = X(f)Y + fD_{X}Y \) we remark that \( D_{Y}fY = fD_{X}Y + (hX)(f)v^{2}Y + (vX)(f)h^{2}Y + (vX)(f)(\mathcal{F} + J)(Y) + (hX)(f)(\mathcal{F} + J)(JY) \). But \( v^{2} = v, h^{2} = h, J(\mathcal{F} + J) = h \) (see \( [27] \)) and it results \( D_{Y}fY = fD_{X}Y + (hX)(f)vY + (vX)(f)hY + (vX)(f)vY + (hX)(f)hY = fD_{X}Y + (hX)(f)Y + (vX)(f)Y = fD_{X}Y + X(f)Y \) which prove that \( \mathcal{D} \) is a linear connection.

**Proposition 1.** The Berwald linear connection has the following properties

\[
\mathcal{D}h = 0, \quad \mathcal{D}v = 0, \quad \mathcal{D}J = 0, \quad \mathcal{D}\mathcal{F} = 0.
\]

**Proof.** Using the properties of the vertical and horizontal projectors we obtain \( \mathcal{D}_{X}vY = v[hX, vY] + J[vX, (\mathcal{F} + J)Y] \) and \( v(\mathcal{D}_{X}Y) = v[hX, vY] + J[vX, (\mathcal{F} + J)Y] \) which yields \( \mathcal{D}v = 0 \). Also, \( \mathcal{D}_{X}hY = h[vX, hY] + (\mathcal{F} + J)[hX, JY] = h(\mathcal{D}_{X}Y) \) and it results \( \mathcal{D}h = 0 \). Moreover, \( \mathcal{D}_{X}JY = v[hX, JY] + J[vX, hY] + J(\mathcal{D}_{X}Y) = J[vX, hY] + v(hX, JY) + J(\mathcal{D}_{X}Y) \) and we obtain \( \mathcal{D}J = 0 \). \( \mathcal{D}_{X}\mathcal{F}Y = v[hX, JY] + h[vX, (\mathcal{F} + J)Y] + J[vX, JY] + (\mathcal{F} + J)[hX, vY] \) and \( \mathcal{F}(\mathcal{D}_{X}Y) = (\mathcal{F} + J)[hX, vY] - J[vX, hY] + h[vX, (\mathcal{F} + J)Y] - v[hX, JY] = \mathcal{D}_{X}\mathcal{F}Y \) which yields \( \mathcal{D}\mathcal{F} = 0 \).

It results that the Berwald connection preserves both horizontal and vertical vector fields. Locally, we have the following formulas

\[
\begin{align*}
\mathcal{D}_{x^r} & \frac{\delta}{\delta x^j} = t^{ks} \left( \frac{\delta_{jk}}{\delta x^r} - t^{r}_{j} \frac{\partial N_{ik}}{\partial p_{r}} \right) \frac{\delta}{\delta x^s}, & \mathcal{D}_{x^r} & \frac{\partial}{\partial p_{j}} = - \frac{\partial N_{ir}}{\partial p_{j}} \frac{\partial}{\partial p_{r}}, \\
\mathcal{D}_{x^r} & \frac{\delta}{\delta p_{j}} = 0, & \mathcal{D}_{x^r} & \frac{\partial}{\partial p_{j}} = t^{ks} \frac{\partial N_{ik}}{\partial p_{j}} \frac{\partial}{\partial p_{s}}.
\end{align*}
\]

We can see that the dynamical covariant derivative has the same properties and this leads to the next result.

**Theorem 1.** If \( \rho \) is a horizontal \( J \)-regular vector field then the following equality holds

\[
\nabla = \mathcal{D}_{\rho}.
\]

**Proof.** If \( \rho \) is a horizontal \( J \)-regular vector field then \( \rho = h\rho \) and \( v\rho = 0 \) which implies

\[
\mathcal{D}_{\rho}Y = v[\rho, vY] + (\mathcal{F} + J)[\rho, JY].
\]

But \( \nabla Y = h[\rho, hY] + v[\rho, vY] \) and we will prove that \( h[\rho, hY] = (\mathcal{F} + J)[\rho, JY] \) using the computation in local coordinates. Let us consider \( Y = X^{i}(x, p)\frac{\partial}{\partial x^{i}} + Y_{j}(x, p)\frac{\partial}{\partial p_{j}} \) and using (1) we get

\[
\begin{align*}
[\rho, hY] & = \left[ \xi^{i} \frac{\delta}{\delta x^{i}}, X^{j} \frac{\delta}{\delta x^{j}} \right] = \xi^{i} X^{j} R_{rijk} \frac{\partial}{\partial p_{k}} + \xi^{i} \frac{\delta X^{j}}{\delta x^{i}} \frac{\delta}{\delta x^{j}} \frac{\delta}{\delta x^{i}} - X^{j} \frac{\delta \xi^{i}}{\delta x^{i}} \frac{\delta}{\delta x^{j}} = \left[ \xi^{i} \frac{\delta X^{j}}{\delta x^{i}} - X^{j} \frac{\delta \xi^{i}}{\delta x^{i}} \right] \frac{\delta}{\delta x^{j}}, \\
h[\rho, hY] & = \left( \xi^{i} \frac{\delta X^{j}}{\delta x^{i}} - X^{j} \frac{\delta \xi^{i}}{\delta x^{i}} \right) \frac{\delta}{\delta x^{j}}.
\end{align*}
\]
Introducing the expression of the canonical nonlinear connection in the case of horizontal $J$-regular vector field given by

$$N_{ij} = -t_{kj} \frac{\partial \xi^k}{\partial x^i} - \xi_k \frac{\delta t_{ij}}{\delta x^k} + \xi^l t_{lk} \frac{\partial N_{lj}}{\partial p_k},$$

we obtain

$$\hbar[\rho, hY] = \left( \xi^i \frac{\delta X^j}{\delta x^i} + X^k t_{kl} \frac{\partial}{\partial p_l} \right) \frac{\delta}{\delta x^j} = \xi^i X^k t_{kl} \frac{\partial N_{ij}}{\partial p_l} \frac{\partial}{\partial p_j} + \xi^i \frac{\delta t_{ij}}{\delta x^i} \frac{\partial N_{lj}}{\partial p_l} \frac{\partial}{\partial p_j} \frac{\partial}{\partial x^i},$$

and using that $(F + J) \left( \frac{\delta}{\delta s} \right) = 0$, $(F + J) \left( \frac{\partial}{\partial p_i} \right) = t^{ik} \frac{\delta}{\delta x^k}$ we obtain

$$(F + J)[\rho, JY] = \left( \xi^i \frac{\delta X^j}{\delta x^i} + X^k t_{kl} \frac{\delta t_{ij}}{\delta x^i} - \xi^l t_{lk} \frac{\partial N_{ij}}{\partial p_l} \right) \frac{\delta}{\delta x^j},$$

which ends the proof. □

Moreover, $\nabla \rho = D_{\rho} \rho = 0$ and it results that the integral curves of $\rho$ are geodesics of the Berwald linear connection.

### 3. Symmetries of Hamiltonian systems

A Hamilton space $\mathbb{H}$ is a pair $(M, H)$ where $M$ is a differentiable, $n$-dimensional manifolds and $H$ is a function on $T^*M$ with the properties:

1° $H : (x, p) \in T^*M \rightarrow H(x, p) \in \mathbb{R}$ is differentiable on $T^*M$ and continue on the null section of the projection $\tau : T^*M \rightarrow M$.

2° The Hessian of $H$ with respect to $p_i$ is nondegenerate

$$g^{ij} = \frac{\partial^2 H}{\partial p_i \partial p_j}, \quad \text{rank} \|g^{ij}(x, p))\| = n \text{ on } T^*M \backslash \{0\}.$$  

3° The tensor field $g^{ij}(x, p)$ has constant signature on $T^*M \backslash \{0\}$. The triple $(T^*M, \omega, H)$ is called a Hamiltonian system.

The Hamiltonian $H$ on $T^*M$ induces a pseudo-Riemannian metric $g_{ij}$ with $g_{ij}g^{jk} = \delta^k_i$ and $g^{jk}$ given by (15) on $VT^*M$. It induces a unique adapted tangent structure denoted

$$J_H = g_{ij} dx^i \otimes \frac{\partial}{\partial p_j}.$$

A $J$-regular vector field induced by the regular Hamiltonian $H$ has the form

$$\rho_H = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial x^i} + \lambda t \frac{\partial}{\partial p_t}.$$

There exists a unique Hamiltonian vector field $\rho_H \in \mathcal{X}(T^*M)$ which is a $J$-regular vector field such that $i_{\rho_H} \omega = -dH$, given by

$$\rho_H = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i} \frac{\partial}{\partial p_i}.$$
The symmetric nonlinear connection
\[ N = -\mathcal{L}_{\rho H} \mathcal{J}_H, \]
has the coefficients (see [22], [25])
\[ N_{ij} = \frac{1}{2} \left( \{g_{ij}, H\} - \left( g_{ik} \frac{\partial^2 H}{\partial p_k \partial x^j} + g_{jk} \frac{\partial^2 H}{\partial p_k \partial x^i} \right) \right), \]
where the Poisson bracket is
\[ \{g_{ij}, H\} = \frac{\partial g_{ij}}{\partial p_k} \frac{\partial H}{\partial x^k} - \frac{\partial H}{\partial p_k} \frac{\partial g_{ij}}{\partial x^k}, \]
and is called the canonical nonlinear connection of the Hamilton space \((M, H)\), which is a metric nonlinear connection, that is \(\nabla g = 0\) (see [26]). In this case, the coefficients of the Jacobi endomorphism have the form
\[ R_{jk} = \frac{\partial^2 H}{\partial p_i \partial x^j} N_{ik} + \frac{\partial^2 H}{\partial p_i \partial x^k} N_{ji} + N_{jl} N_{ik} g^{li} + \frac{\partial^2 H}{\partial x^j \partial x^k} + \rho_H (N_{jk}), \]
and the action of the dynamical covariant derivative on the Berwald basis is given by
\[ \nabla \frac{\delta}{\delta x^j} = h \left[ \rho_H, \frac{\delta}{\delta x^j} \right] = - \left( \frac{\partial^2 H}{\partial p_i \partial x^j} + N_{jk} g^{ki} \right) \frac{\delta}{\delta x^i}, \]
\[ \nabla \frac{\partial}{\partial p_j} = v \left[ \rho_H, \frac{\partial}{\partial p_j} \right] = \left( \frac{\partial^2 H}{\partial p_j \partial x^i} + N_{jk} g^{ki} \right) \frac{\partial}{\partial p_i}. \]

In the following, we study the symmetries of Hamiltonian systems (see also [30]) on the cotangent bundle using the Hamiltonian vector field and the adapted tangent structure.

**Definition 3.** A vector field \( X \in \mathcal{X}(T^* M) \) is an infinitesimal symmetry of Hamiltonian vector field if \([\rho_H, X] = 0\).

If we consider \( X = X^i(x, p) \frac{\partial}{\partial x^i} + Y_i(x, p) \frac{\partial}{\partial p_i} \), then an infinitesimal symmetry is given by the equations
\[ X \left( \frac{\partial H}{\partial p_i} \right) = \rho_H (X^i), \quad X \left( \frac{\partial H}{\partial x^i} \right) + \rho_H (Y^i) = 0, \]
and the first relation leads to
\[ Y_k = g_{ki} \left( \rho_H (X^i) - X^j \frac{\partial^2 H}{\partial p_j \partial x^i} \right). \]

**Definition 4.** A vector field \( \tilde{Z} \in \mathcal{X}(M) \) is said to be a natural infinitesimal symmetry if its complete lift to \( T^* M \) is an infinitesimal symmetry, that is \([\rho_H, \tilde{Z}^{C*}] = 0\).

We know that, for \( \tilde{Z} = \tilde{Z}^i(x) \frac{\partial}{\partial x^i} \), the complete lift on \( T^* M \) is given by [33]
\[ \tilde{Z}^{C*} = \tilde{Z}^i(x) \frac{\partial}{\partial x^i} + p_j \frac{\partial \tilde{Z}^j}{\partial x^i} \frac{\partial}{\partial p_i}, \]
and a natural infinitesimal symmetry is characterized by the equations
\[ \tilde{Z}^{C*} \left( \frac{\partial H}{\partial p_k} \right) = \frac{\partial H}{\partial p_k} \frac{\partial \tilde{Z}^k}{\partial x^i}. \]
Next, we introduce the Newtonoid vector field on $T^*M$ (see [23], [7] for tangent bundle case) which helps us to find the canonical nonlinear connection induced by a regular Hamiltonian.

**Definition 5.** A vector field $X \in \mathcal{X}(T^*M)$ is called a Newtonoid vector field if

$$J_H \left[ \rho_H, X \right] = 0.$$ 

In local coordinates we obtain

$$g^{ij} \left( X \left( \frac{\partial H}{\partial p_i} \right) - \rho_H \left( X^i \right) \right) \frac{\partial}{\partial p_j} = 0,$$

and using that \( \text{rank} \|g^{ij}(x,p)\| = n \) it results the equation

$$X \left( \frac{\partial H}{\partial p_i} \right) = \rho_H \left( X^i \right),$$

which leads to the expression of a Newtonoid vector field

$$X = X^i \frac{\partial}{\partial x^i} + g_{ki} \left( \rho_H \left( X^i \right) - X^j \frac{\partial^2 H}{\partial p_i \partial x^j} \right) \frac{\partial}{\partial p_k}.$$ 

We remark that $X$ is an infinitesimal symmetry if and only if it is a Newtonoid vector field and satisfies the equation

$$X \left( \frac{\partial H}{\partial x^i} \right) + \rho_H \left( Y^i \right) = 0.$$ 

The set of Newtonoid vector fields is given by

$$\mathcal{X}_{\rho_H} = \text{Ker} \left( J_H \circ \mathcal{L}_{\rho_H} \right) = \text{Im} \left( Id + J_H \circ \mathcal{L}_{\rho_H} \right).$$

In the following, we will use the dynamical covariant derivative and Jacobi endomorphism in order to find the invariant equations of Newtonoid vector field and infinitesimal symmetries. Let $\rho_H$ be the Hamiltonian vector field, $\mathcal{N}$ an arbitrary nonlinear connection with induced $v, h$ projectors and $\nabla$ the induced dynamical covariant derivative. We set:

**Proposition 2.** A vector field $X \in \mathcal{X}(T^*M)$ is a Newtonoid vector field if and only if

$$v(X) = J_H(\nabla X).$$

**Proof.** We have the relation (10) $J_H \circ \nabla = J_H \circ \mathcal{L}_{\rho_H} + v$ and it results $J_H[\rho_H, X] = 0$ if and only if $v(X) = J_H(\nabla X)$. \qed

**Proposition 3.** A vector field $X \in \mathcal{X}(T^*M)$ is a infinitesimal symmetry if and only if $X$ is a Newtonoid vector field and satisfies the equation

$$\nabla (J_H \nabla X) + \Phi(X) = 0.$$ 

**Proof.** A vector field $X \in \mathcal{X}(T^*M)$ is an infinitesimal symmetry if and only if $h[\rho_H, X] = 0$ and $v[\rho_H, X] = 0$. Composing by $J_H$ we obtain $J_H h[\rho_H, X] = J_H[\rho_H, X] = 0$ which means that $X$ is a Newtonoid vector field. Also,

$$v[\rho_H, X] = v[\rho_H, vX] + v[\rho_H, hX] = \nabla (vX) + \Phi(X) = \nabla (J_H(\nabla X)) + \Phi(X),$$

which ends the proof. \qed
For \( f \in C^\infty(T^*M) \) and \( X \in \mathcal{X}(T^*M) \) we define the product
\[
f * X = (Id + \mathcal{J}_H \circ \mathcal{L}_{\rho_H})(fX) = fX + f\mathcal{J}_H[\rho_H, X] + \rho_H(f)\mathcal{J}_H X,
\]
and it result that a vector field \( X \) is a Newtonoid if and only if
\[
X = X^i(x, p) \cdot \frac{\partial}{\partial x^i}.
\]

Also, if \( X \in \mathcal{X}_{\rho_H} \) then \( f * X = fX + \rho_H(f)\mathcal{J}_H X \) (see [21], [22] for the case of tangent bundle). Next theorem proves that the canonical nonlinear connection induced by a regular Hamiltonian can be determined by symmetries.

**Theorem 2.** Let us consider the Hamiltonian vector field \( \rho_H \), an arbitrary nonlinear connection \( \mathcal{N} \) and \( \nabla \) the dynamical covariant derivative. The following conditions are equivalent:

i) \( \nabla \) restricts to \( \nabla : \mathcal{X}_{\rho_H} \rightarrow \mathcal{X}_{\rho_H} \) satisfies the Leibnitz rule with respect to the * product.

ii) \( \nabla \mathcal{J}_H = 0 \)

iii) \( \mathcal{L}_{\rho_H} \mathcal{J}_H + \mathcal{N} = 0 \),

iv) \( \mathcal{N}_{ij} = \frac{1}{2} \left( \{g_{ij}, H\} - (g_{ik}\frac{\partial H}{\partial p_k} + g_{jk}\frac{\partial H}{\partial p_k}) \right) \).

**Proof.** For ii) \( \Rightarrow \) i) let us consider \( X \in \mathcal{X}_{\rho_H} \) and using (21) we get \( vX = \mathcal{J}_H(\nabla X) \). Applying \( \nabla \) to both sides, we obtain \( \nabla (vX) = \nabla (\mathcal{J}_H \nabla X) \) which yields \( (\nabla v)X + v(\nabla X) = (\nabla \mathcal{J}_H)(\nabla X) + \mathcal{J}_H \nabla (\nabla X) \). Using the relations \( \nabla v = 0 \), \( \nabla \mathcal{J}_H = 0 \) it results \( v(\nabla X) = \mathcal{J}_H \nabla (\nabla X) \) which implies \( \nabla X \in \mathcal{X}_{\rho_H} \). For \( X \in \mathcal{X}_{\rho_H} \) we obtain
\[
\nabla (fX) = \nabla (fX + \rho_H(f)\mathcal{J}_H X) = \rho_H(f)X + f\nabla X + \rho_H(f)\mathcal{J}_H X + \rho_H(f)\mathcal{J}_H X.
\]

But \( \nabla (\mathcal{J}_H X) = (\nabla \mathcal{J}_H)X + \mathcal{J}_H (\nabla X) \) and from \( \nabla \mathcal{J}_H = 0 \) it results \( \nabla (\mathcal{J}_H X) = \mathcal{J}_H (\nabla X) \) which leads to \( \nabla (fX) = \nabla fX + f\nabla X \).

For i) \( \Rightarrow \) ii) we prove that \( \nabla \mathcal{J}_H \) vanishes on the set \( \mathcal{X}_{\rho_H} \cup \mathcal{X}^v(T^*M \setminus \{0\}) \) which is a set of generators for \( \mathcal{X}(T^*M \setminus \{0\}) \). For \( X \in \mathcal{X}^v(T^*M \setminus \{0\}) \) we have \( \mathcal{J}_H X = 0 \) and \( \mathcal{J}_H (\nabla X) = 0 \) which lead to \( \nabla \mathcal{J}_H (X) = \nabla (\mathcal{J}_H X) - \mathcal{J}_H (\nabla X) = 0 \). Next, if \( X \in \mathcal{X}_{\rho_H} \) then from \( \nabla (fX) = \nabla fX + f\nabla X \) it results \( \rho_H(f)\nabla (\mathcal{J}_H X) = \rho_H(f)\mathcal{J}_H \nabla X \), which implies \( \rho_H(f)\mathcal{J}_H X = 0 \) for an arbitrary function \( f \in C^\infty(T^*M \setminus \{0\}) \) and arbitrary vector field \( X \in \mathcal{X}_{\rho_H} \). Therefore \( \nabla \mathcal{J}_H = 0 \) which ends the proof. The equivalence of ii), iii), iv) results from (11).

Considering the canonical nonlinear connection \( \mathcal{N} = -\mathcal{L}_{\rho_H} \mathcal{J}_H \), we get the following results.

**Proposition 4.** A vector field \( X \in \mathcal{X}(T^*M) \) is a infinitesimal symmetry if and only if \( X \) is a Newtonoid vector field and satisfies the equation
\[
\nabla^2 \mathcal{J}_H X + \Phi(X) = 0,
\]
which locally yields
\[
\nabla^2 g_{ij} \cdot X^i + R_{ij} X^i = 0.
\]

**Proof.** If \( \mathcal{N} \) is the canonical nonlinear connection, then \( \nabla \mathcal{J}_H = 0 \) and using \( \nabla \mathcal{J}_H = \nabla \circ \mathcal{J}_H - \mathcal{J}_H \circ \nabla \) from (22) it results (23). Also, we obtain that the local components of the vertical vector field (23) are (24).
Definition 6. a) An infinitesimal Noether symmetry of the Hamiltonian $H$ is a vector field $X \in \mathcal{X}(T^*M)$ such that
\[ \mathcal{L}_X \omega = 0, \quad \mathcal{L}_X H = 0. \]

b) A vector field $\tilde{X} \in \mathcal{X}(M)$ is said to be an invariant vector field for the Hamiltonian $H$ if $\tilde{X}^{C^*(H)} = 0$.

c) A function $f \in C^\infty(M)$ is a constant of motion (or a conservation law) for the Hamiltonian $H$ if $\mathcal{L}_{\rho_H} f = 0$.

Proposition 5. Every infinitesimal Noether symmetry is an infinitesimal symmetry.

Proof. From the symplectic equation $i_{\rho_H} \omega = -dH$, applying the Lie derivative in both sides, it results
\[ \mathcal{L}_X (i_{\rho_H} \omega) = -\mathcal{L}_X dH = -d\mathcal{L}_X H = 0. \]
Also, from the formula $i_{[X, \rho_H]} = \mathcal{L}_X \circ i_{\rho_H} - i_{\rho_H} \circ \mathcal{L}_X$ we obtain
\[ \mathcal{L}_X (i_{\rho_H} \omega) = i_{[X, \rho_H]} \omega + i_{\rho_H} \mathcal{L}_X \omega = i_{[X, \rho_H]} \omega, \]
which leads to $i_{[X, \rho_H]} \omega = 0$ and we get $[X, \rho_H] = 0$. ⊓ ⊔

Proposition 6. If $\tilde{X}$ is a vector field on $M$ such that $\mathcal{L}_{\tilde{X}^c} \theta$ is closed and $d(\tilde{X}^c H) = 0$, then $\tilde{X}$ is a natural infinitesimal symmetry.

Proof. We have
\[ i_{[\tilde{X}^c, \rho_H]} \omega = \mathcal{L}_{\tilde{X}^c} (i_{\rho_H} \omega) - i_{\rho_H} (\mathcal{L}_{\tilde{X}^c} \omega) = -\mathcal{L}_{\tilde{X}^c} dH - i_{\rho_H} (\mathcal{L}_{\tilde{X}^c} d\theta) = -d\mathcal{L}_{\tilde{X}^c} H - i_{\rho_H} d(\mathcal{L}_{\tilde{X}^c} \theta) = -d(\tilde{X}^c H) = 0, \]
because $d(\mathcal{L}_{\tilde{X}^c} \theta) = 0$. ⊓ ⊔

Proposition 7. The Hamiltonian vector field $\rho_H$ is an infinitesimal Noether symmetry.

Proof. Using the skew symmetry of the symplectic 2-form $\omega$, it results
\[ 0 = i_{\rho_H} \omega(\rho_H) = -dH(\rho_H) = \rho_H(H) = \mathcal{L}_{\rho_H} H. \]
Also, from $d\omega = 0$ we get
\[ \mathcal{L}_{\rho_H} \omega = di_{\rho_H} \omega + i_{\rho_H} d\omega = -dH = 0. \]
Since Lie and exterior derivatives commute, we obtain for an infinitesimal Noether symmetry
\[ d\mathcal{L}_X \theta = \mathcal{L}_X d\theta = \mathcal{L}_X \omega = 0. \]
It results that the 1-form $\mathcal{L}_X \theta$ is a closed 1-form and consequently $\mathcal{L}_{\rho_H} \theta$ is closed.

Definition 7. An infinitesimal Noether symmetry $X \in \mathcal{X}(T^*M)$ is said to be an exact infinitesimal Noether symmetry if the 1-form $\mathcal{L}_X \theta$ is exact.

The next result proves that there is a one to one correspondence between the exact infinitesimal Noether symmetry and conservation laws. Also, if $X$ is an exact infinitesimal Noether symmetry, then there is a function $f \in C^\infty(M)$ such that $\mathcal{L}_X \theta = df$. 

Theorem 3. If $X$ is an exact infinitesimal Noether symmetry, then $f - \theta(X)$ is a conservation law for the Hamiltonian $H$. Conversely, if $f \in C^\infty(M)$ is a conservation law for $H$, then $X \in \mathcal{X}(\mathbb{T}^*M \setminus \{0\})$ the unique solution of the equation $i_X \omega = -df$ is an exact infinitesimal Noether symmetry.

Proof. We have $\rho_H(f - \theta(X)) = d(f - \theta(X))(\rho_H) = (\mathcal{L}_X \theta - di_X(\theta)) (\rho_H) = i_X d\theta(\rho_H) = i_X \omega(\rho_H) = -i_{\rho_H} \omega(X) = dH(X) = 0$, and it results that $f - \theta(X)$ is a conservation law for the dynamics associated to the regular Hamiltonian $\tilde{H}$. Conversely, if $X$ is a conservation law for the Hamiltonian $H$ then its complete lift $\tilde{X}_C^*$ is an exact infinitesimal Noether symmetry. Moreover, the function $\theta(\tilde{X})$ is a conservation law for the Hamiltonian $H$.

Proof. We have that $\mathcal{L}_{\tilde{X}_C,\theta} H = \tilde{X}_C^*(H) = 0$. Next, we prove that $\mathcal{L}_{\tilde{X}_C,\theta} = 0$ using the computation in local coordinates.

\[
(\mathcal{L}_{\tilde{X}_C,\theta}) \left( \frac{\partial}{\partial x^i} \right) = \tilde{X}_C^* \left( \theta \left( \frac{\partial}{\partial x^i} \right) \right) - \theta \left[ \tilde{X}_C^*, \frac{\partial}{\partial x^i} \right] = \\
\quad = \tilde{X}_C^* (p_i) - \theta \left( -\frac{\partial \tilde{X}_j}{\partial x^i} \frac{\partial}{\partial x^j} + \frac{\partial^2 \tilde{X}_j}{\partial x^i \partial x^j} p_j \frac{\partial}{\partial p_k} \right) \\
\quad = -\frac{\partial \tilde{X}_j}{\partial x^i} p_j + \frac{\partial \tilde{X}_j}{\partial x^i} p_j = 0,
\]

\[
(\mathcal{L}_{\tilde{X}_C,\theta}) \left( \frac{\partial}{\partial p_i} \right) = \tilde{X}_C^* \left( \theta \left( \frac{\partial}{\partial p_i} \right) \right) - \theta \left[ \tilde{X}_C^*, \frac{\partial}{\partial p_i} \right] = -\theta \left( \frac{\partial \tilde{X}_j}{\partial x^i} \frac{\partial}{\partial p_j} \right) = 0.
\]

It results that $0 = d\mathcal{L}_{\tilde{X}_C,\theta} = \mathcal{L}_{\tilde{X}_C,\theta} = \mathcal{L}_{\tilde{X}_C,\omega}$ and $\tilde{X}_C^*$ is an exact infinitesimal Noether symmetry. Using Proposition 3.5 we have that $\tilde{X}_C^*$ is an infinitesimal symmetry and consequently, $\tilde{X}$ is a natural infinitesimal symmetry. Moreover, according to Theorem 3.8 for $f = 0$ it results that

\[
\theta \left( \tilde{X}_C^* \right) = p_i \tilde{X}^i,
\]

is a conservation law for the Hamiltonian $H$.

3.1. Example. Let us consider the following distributional system in $\mathbb{R}^2$ (driftless control affine system):

\[
\begin{cases}
\dot{x}^1 = u^1 + u^2 x^1 \\
\dot{x}^2 = u^2
\end{cases}
\]

Let $x_0$ and $x_1$ be two points in $\mathbb{R}^2$. An optimal control problem consists of finding the trajectories of our control system which connect $x_0$ and $x_1$ and minimizing the Lagrangian

\[
\min \int_0^T L(u(t))dt, \quad L(u) = \frac{1}{2} ((u^1)^2 + (u^2)^2), \quad x(0) = x_0, \quad x(T) = x_1.
\]
where \( \dot{x}^i = \frac{dx^i}{dt} \) and \( u^1, u^2 \) are control variables. Using the Pontryagin Maximum Principle, we find the Hamiltonian function on the cotangent bundle \( T^*\mathbb{R}^2 \) in the form

\[
H(x, p, u) = p_i \dot{x}^i - L = p_1 (u^1 + u^2 x^1) + p_2 u^2 - \frac{1}{2} ((u^1)^2 + (u^2)^2),
\]

with the condition \( \frac{\partial H}{\partial u} = 0 \), which leads to \( u^1 = p_1, u^2 = p_1 x^1 + p_2 \). We obtain

\[
H(x, p) = \frac{1}{2} (p_1^2 (1 + (x^1)^2) + 2p_1 p_2 x^1 + p_2^2) = \frac{1}{2} (p_1 + (p_1 x^1 + p_2)^2),
\]

and it results

\[
\frac{\partial H}{\partial p_1} = p_1 (1 + (x^1)^2) + p_2 x^1, \quad \frac{\partial H}{\partial p_2} = p_1 x^1 + p_2, \quad \frac{\partial H}{\partial x^i} = p_i^2 x^1 + p_1 p_2, \quad \frac{\partial H}{\partial x^2} = 0.
\]

The Hamilton’s equations lead to the following system of differential equations

\[
\begin{aligned}
\dot{x}^1 &= p_1 (1 + (x^1)^2) + p_2 x^1, \\
\dot{x}^2 &= p_1 x^1 + p_2, \\
\dot{p}_1 &= -p_1 (p_1 x^1 + p_2), \\
\dot{p}_2 &= 0 \Rightarrow p_2 = ct.
\end{aligned}
\]

The Hessian matrix of \( H \) with respect to \( p \) is

\[
g^{ij}(x) = \frac{\partial^2 H}{\partial p_i \partial p_j} = \begin{pmatrix} 1 + (x^1)^2 & x^1 \\ x^1 & 1 \end{pmatrix}, \quad i, j = 1, 2
\]

and it results that \( H \) is regular \((\text{rank} \|g^{ij}(x, p)\| = 2)\) and its inverse matrix has the form

\[
g_{ij}(x) = \begin{pmatrix} 1 & -x^1 \\ -x^1 & 1 + (x^1)^2 \end{pmatrix}.
\]

The adapted tangent structure is given by

\[
\mathcal{J}_H = dx^1 \otimes \frac{\partial}{\partial p_1} - x^1 dx^1 \otimes \frac{\partial}{\partial p_2} - x^1 dx^2 \otimes \frac{\partial}{\partial p_1} + (1 + (x^1)^2) dx^2 \otimes \frac{\partial}{\partial p_2}.
\]

The \( \mathcal{J}_H \)-regular vector field is the Hamiltonian vector field (16)

\[
\rho_H = \left( p_1 (1 + (x^1)^2) + p_2 x^1 \right) \frac{\partial}{\partial x^1} + \left( p_1 x^1 + p_2 \right) \frac{\partial}{\partial x^2} - \left( p_1^2 x^1 + p_1 p_2 \right) \frac{\partial}{\partial p_1}.
\]

and from Proposition 3.7 is an infinitesimal Noether symmetry for the dynamics induced by the regular Hamiltonian \( H \). Moreover, if \( X = p_2 \frac{\partial}{\partial x^2} \) then \([\rho_H, X] = 0\) and it results that \( X \) is an infinitesimal symmetry for the Hamiltonian vector field \( \rho_H \).

The local coefficients of the canonical nonlinear connection (17) have the following form

\[
\begin{aligned}
\mathcal{N}_{11} &= - (p_1 x^1 + p_2), \quad \mathcal{N}_{22} = -x^1 \left( p_1 (1 + (x^1)^2) + p_2 x^1 \right), \\
\mathcal{N}_{12} &= \mathcal{N}_{21} = x^1 \left( p_1 x^1 + p_2 \right).
\end{aligned}
\]

By straightforward computation we obtain that

\[
\rho_H = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial x^i} + \frac{\partial H}{\partial p_i} \mathcal{N}_{ij} \frac{\partial}{\partial p_j} = \frac{\partial H}{\partial p_i} \mathcal{N}_{ij} \frac{\partial}{\partial p_j} = \frac{\partial H}{\partial p_i} \left. \frac{\partial}{\partial x^i} \right|_{\delta},
\]

\[
\mathcal{N}_{ij} \frac{\partial}{\partial p_j} = \delta^i_1.
\]
and it results that the Hamiltonian vector field is a horizontal $J_H$-regular vector field. In this case we obtain that the Jacobi endomorphism (18) is given by

$$R_{ij} = R_{kij} \frac{\partial H}{\partial p_k},$$

where $R_{kij}$ are the local coefficients from (2) of the curvature of the canonical nonlinear connection with nonzero components

$$R_{121} = 2p_1x^1 + p_2 = -R_{211},$$
$$R_{212} = p_1 + 2p_1(x^1)^2 + p_2x^1 = -R_{122}.$$

Also, $\nabla \rho_H = D_{\rho_H} \rho_H = 0$ and it results that the integral curves of horizontal Hamiltonian vector field $\rho_H$ are geodesics of the Berwald linear connection.

Conclusions and further developments.

The main purpose of this work is to study the symmetries of Hamiltonian systems on the cotangent bundle using the same methods as in the study of the symmetries for second order differential equations on the tangent bundle. The role of the canonical tangent structure and the semispray on $TM$ is taken by the adapted tangent structure and the regular vector field on $T^*M$, which can be defined in the presence of a regular Hamiltonian. However, the cotangent bundle has a canonical symplectic structure, which can be found on the tangent bundle only in the presence of a Lagrangian function. Also, we find the invariant equations of some type of symmetries on $T^*M$ using the Jacobi endomorphism and dynamical covariant derivative. Moreover, in the case of the horizontal regular vector field, in particular the Hamiltonian vector field, we prove that the dynamical covariant derivative coincides with Berwald linear connection. It results that the integral curves of the horizontal Hamiltonian vector field are the geodesics of the Berwald linear connection. We find the relations between infinitesimal symmetries, natural infinitesimal symmetries, Newtonoid vector field, infinitesimal Noether symmetries and conservation laws on $T^*M$ and show when one of them will imply the others. In the last part of the paper an examples from optimal control theory is given. As further developments, we can use the dynamical covariant derivative and Jacobi endomorphism in the study of symmetries for $k$-symplectic Hamiltonian systems.

References

[1] Abraham, R., Marsden, J., *Foundation of mechanics*, Benjamin, New-York, 1978.
[2] Arnold, V.I., *Mathematical methods of classical mechanics*, Graduate Texts in Mathematics, Springer, vol. 60, 1989.
[3] Balseiro, P., Sansonetto, N., *A geometric characterization of certain first integrals for nonholonomic systems with symmetries*, SIGMA, 12 (2016), 018.
[4] Berwald, L., *On system of second order ODE’s whose integral curves are topologically equivalent to the system of straight lines*, Annals of Mathematics, 48 (1947), 93-215.
[5] Bucătaru, I., Miron, R., *Finsler-Lagrange geometry: applications to dynamical systems*, Ed. Romanian Academy, 2007.
[6] Bucătaru, I., *Linear connection for systems of higher order differential equations*, Houston Journal of Mathematics, 32, no.1 (2005), 315-332.
[7] Bucătaru, I., Constantinescu, O., Dahl M.F., *A geometric setting for systems of ordinary differential equations*, Int. J. Geom. Methods Mod. Phys., vol. 8, no.6, 12019 (2011).
[8] Cariñena, J. F., Martinez, E., *Generalized Jacobi equation and inverse problem in classical mechanics*, in ”Group Theoretical Methods in Physics”, Proc. 18th Int. Colloquim 1990, Moskow, vol. II Nova Science Publishers, 1991, New York.
[9] Castellani, L., *Symmetries in constrained Hamiltonian systems*, Annals of Physics, vol.143, 2 (1982), 357-371.
[10] Crampin, M., *Tangent bundle geometry for Lagrangian dynamics*, J. Phys. A: Math. Gen. 16 (1983), 3755-3772.
[11] Crampin, M., Martínez E., Sarlet, W., *Linear connections for system of second-order ordinary differential equations*, Ann. Inst. Henry Poincare, 65 no.2 (1996), 223-249.
[12] Crampin, M., Pirani, F.A.E., *Applicable differential geometry*, Cambridge University Press (1986).
[13] Frölicher A., Nijenhuis A., *Theory of vector-valued differential forms*, Nederl. Akad. Wetensch. Proc. Ser. A. 59 (1956), 338-359.
[14] Grifone, J., *Structure presque tangente et connections I*, Ann. Inst. Fourier, 22, no.1 (1972), 287-334.
[15] Janyska, J., *Remarks on infinitesimal symmetries of geometrical structures of the classical phase space of general relativistic test particle*, Int. J. Geom. Methods Mod. Phys., 12, 1560020 (2015).
[16] de León, M., Rodrigues, P.R., *Method of differential geometry in analytical mechanics*, North-Holland Publishing Co. Amsterdam, 1989.
[17] de León, M., Martín de Diego, D., Santamaría Merino, A., *Symmetries in classical field theories*, Int. J. Geom. Meth. Mod. Phys., 5 (2004), 651-710.
[18] Marle, C. M., *Symmetries of Hamiltonian systems on symplectic and Poisson manifolds*, in: Similarity and Symmetry Methods, vol. 73, series Lecture Notes in Applied and Computational Mechanics, 185-269.
[19] G. Marmo, N. Mukunda, *Symmetries and constant of the motion in the Lagrangian formalism on $TQ$: beyond point transformations*, Nuovo Cim. B, 92 (1986), 1-12.
[20] Marsden, J.E., Ratiu, T., *Introduction to mechanics and symmetry*, Springer, 2013.
[21] Martínez, E., Carinena J.F., Sarlet W., *Derivations of differential forms along the tangent bundle projection II*, Diff. Geom. Appl. 3, no.1 (1993), 1-29.
[22] Miron, R., Hrimiuc, D., Shimada, H., Sabău, S., *The geometry of Hamilton and Lagrange spaces*, Springer, vol. 118, (2001).
[23] Mukhanov, V., Wipf, A., *On the symmetries of Hamiltonian systems*, Int. J. Mod. Phys. A, 10, no. 04 (1995), 579-610.
[24] Oproiu, V., *Regular vector fields and connections on cotangent bundles*, An. Stiint. Univ. A.I.Cuza, Iasi, S.1. Math., 37, no.1, (1991), 87-104.
[25] Popescu, L., *A note on nonlinear connections on the cotangent bundle*, Carpathian J. Math., 25, no.2 (2009), 203-214.
[26] Popescu, L., Criveanu R., *A note on metric nonlinear connections on the cotangent bundle*, Carpathian J. Math., 27, no. 2 (2011), 261-268.
[27] Popescu, L., *Geometrical structures on the cotangent bundle*, Int. J. Geom. Methods Mod. Phys., vol. 13, no. 5 (2016) 1650071.
[28] Prince, G., *Toward a classification of dynamical symmetries in classical mechanics*, Bull. Austral. Math. Soc., vol. 27 (1983), 53-71.
[29] Prince, G., *A complete classification of dynamical symmetries in classical mechanics*, Bull. Austral. Math. Soc., vol. 32 (1985), 299-308.
[30] Puta, M., *Hamiltonian mechanical systems and geometric quantization*, Springer, vol. 260, (1993).
[31] Sarlet, W., *Adjoint symmetries of second-order differential equations and generalizations*, Differential geometry and its applications (Brno, 1989), World Sci. Publ., (1990) 412-421.
[32] Sarlet, W., *Linear connections along the tangent bundle projection*, In: Variations, Geometry and Physics, (eds. O. Krupkova, D. Saunders), Nova Science Publishers (2008).
[33] Yano, K., Ishihara, S., *Tangent and cotangent bundles*, M. Dekker Inc., New-York, (1973).

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