One mind, two languages – separate conceptualisations? A case study of students’ bilingual modes for dealing with language-related conceptualisations of fractions

Susanne Prediger a, Taha Kuzu a, Alexander Schüler-Meyer b and Jonas Wagner c

aInstitute for Development and Research in Mathematics Education, TU Dortmund University, Dortmund, Germany; bTU Eindhoven, Eindhoven, The Netherlands; cUniversity Hamburg, Hamburg, Germany

ABSTRACT
International comparisons of languages have shown how mathematical concepts can be expressed and conceptualised in different languages. For multilingual students, these findings raise the question of how the interplay of languages and conceptualisations shape multilingual learning processes. This paper presents an investigation of case studies of bilingual learning processes of Turkish–German speaking students in Grade 7. Investigating the conceptual learning pathways of 14 focus students reveals that multilingual students navigate between different conceptualisations of fractions across both languages. To account for how these multilingual resources can provide opportunities for deepening conceptual understanding of fractions, two bilingual modes are distinguished and empirically substantiated: (1) The complementarity mode, in which code switching occurs for combining different nuances of a conceptualisation, and (2) the connection mode, in which (a) nuances travel across the languages or (b) languages are locally combined in translanguaging practices. Both modes contribute to multiperspective views on the part-whole concept.

Different languages can provide different conceptualisations for mathematical concepts (Barton, 2008). In mathematics education, this observation has led to empirical comparative studies on possible mathematical advantages for speakers of specific languages due to higher or lower proximity to the thinking processes (from Miura, Kim, Chang, & Okamoto, 1988; to Leung, 2017). These comparative studies have raised the awareness of subtle links between language and thinking. However, comparative studies have tended to adopt a monolingual perspective, assuming that each student’s thinking is predominantly shaped by one language. In contrast, for multilingual students, the question is not whether they are acquainted with one or the other conceptualisation, but how the interplay of different languages and conceptualisations shapes their learning processes at the micro-level. From a linguistic perspective, this shift of research focus is important with respect to more recent perspectives on multilingualism that are characterised by dynamic and interconnected multilingual repertoires in multilingual communication.
rather than by separate language proficiencies (House & Rehbein, 2004; Lüdi, 2006; García & Wei, 2014).

In this article, we contribute to fulfilling the need for more research on bilingual learning processes by investigating German–Turkish speaking seventh graders’ learning pathways to the part-whole concept of fractions. The research question is situated at the micro-level of conceptual development processes: How do students in a bilingual teaching intervention adopt and combine language-related conceptualisations of the part-whole concept in or across both languages?

The paper is organised as follows. We first present the theoretical background and empirical starting point. We then refine the research question and present the methodological framework and the empirical results of the qualitative analysis. These results show how 14 focus students relate several conceptualisations to each other in fruitful ways.

**Theoretical and empirical starting points for divergent conceptualisations of fractions in Turkish and German**

This section presents the early thinking about linguistic relativity in general and its impact on comparative mathematics education research. Focusing then on multilingual students, we review theoretical approaches to multilingualism.

**Early discourses on linguistic relativity**

The observation that languages – especially from different language families – can provide different structures and conceptualisations has fuelled controversial academic debates on the so-called linguistic relativity hypothesis since von Humboldt, Sapir and Whorf. It has led to the conjecture that one’s thoughts are affected by one’s spoken language (Lucy, 1996; Gumperz & Levinson, 1996; Pavlenko, 2011). A strong version of the so-called Sapir-Whorf hypothesis considers thinking to be determined by language. This has often been refuted as too narrow (see Pavlenko, 2011, for a historical overview). The weak version of linguistic relativity supposes an impact of language on thinking, without presuming determination, and is more widely accepted (Lucy, 1996).

Pavlenko (2011) emphasises that Sapir and Whorf themselves never promoted the strong version, but mainly referred to the plasticity of human minds and their ability to incorporate new perspectives by learning a second language. She points out that research should not only compare different cultures in monolingual modes, but should also investigate the interaction of language and thought within the minds of multilinguals.

**Different conceptualisations in international comparative studies in mathematics education**

In mathematics education research, the idea of linguistic relativity has been examined in various contexts (e.g. Barton, 2008). It has fuelled several comparative studies that have investigated whether speakers of one language have advantages or disadvantages for their (habitual, not potential) mathematical thinking compared to speakers of other languages (e.g. Miura et al., 1988, as one of the first studies, and Leung, 2017, as one of the most recent). Whereas early studies focused on number names and their influence
on children’s understanding of the place value system (e.g. Miura et al., 1988; Miller, Smith, Zhu, & Zhang, 1995), recent studies have also focused on more global ways of thinking (e.g. Leung, 2017).

Leung (2017) outlines a general difference between analytic and synthetic approaches that seems to impact on students’ conceptualisations, for example, of geometric figures, but also of postal addresses. Whereas Asian analytic conceptualisations proceed from whole geometric figures to more specific components, Western synthetic conceptualisations often proceed from specific pieces to the complete figure (for postal addresses, this means giving the house number before the city). Leung shows empirically that Chinese-speaking children significantly outperform English-speaking children on a test of visual perceptual abilities, especially in visual-motor integration perceptual abilities (Lai & Leung, 2012). Leung (2017) concludes that the analytic Asian conceptualisation might privilege geometrical thinking and argues for further studies on the relationship between conceptualisations and language characteristics.

For fractions, which are the core topic of this article, Bartolini Bussi, Baccaglini-Frank, and Ramploud (2014) have discussed reading orders as relevant language-related differences. European languages read fractions such as ¾ from top to bottom, mentioning the part, 3, first and the whole, 4, afterwards. In contrast, many Asian languages read the denominator first and the numerator afterwards, resulting in a focus first on the whole and then on the part. Bartolini Bussi et al. (2014) hypothesise that this conceptualisation might have benefits for students’ conceptual understanding, as the whole must be split before taking parts.

As these differences for fractions resonate with Leung’s (2017) general distinction between Western synthetic conceptualisation and Asian analytic conceptualisation, we decided to investigate it more thoroughly by comparing German and Turkish expressions for fractions (see Figure 1). In both languages, the symbolically represented mathematical object “fraction” has several meanings, such as ratio, rate and part-whole relationship (Behr, Harel, Post, & Lesh, 1992).

For the part-whole concept, four nuances of conceptualisation were identified by the authors in a theoretical analysis (following Behr et al., 1992; Bartolini Bussi et al., 2014)

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**Figure 1.** Different language-related nuances for the same concept.
and informal observations of monolingual German-speaking and monolingual Turkish-speaking students. These differences concern not only the reading order, but also the articulation of the relation of part and whole:

- The first German expression “Drei von Fünf” (“three of five”) expresses the part (e.g. of a bar), which is coloured, and then the whole in a share concept (marked as “share nuance” in Figure 1). The second German expression “drei 5-tel” (three fifths) provides a slightly different conceptualisation in a quasi-cardinal nuance, which counts the unit fractions as entities (Cramer, Behr, Post, & Lesh, 1997). Both German nuances articulate the part before the whole and hence support the habitual synthetic way of thinking from the part to the whole.

- The first Turkish expression “Beşte üç” (“Five therein three”) begins with the whole and proceeds with the part; this relation is conveyed by the locative suffix “-te” (“therein”), which is used to locate the coloured part in the whole (in this case, the whole bar). We will refer to this conceptualisation as the localising nuance. The second Turkish expression “Beşten üç” (“five thereof three”) refers to a more everyday construction with the ablative case suffix “-ten” (“thereof”). Again, this expression begins with the whole, but this time, the part of a whole is not expressed as if it were located in but as if it were taken out (“y thereof x”), which will be referred to as the taking nuance. In both nuances, the order of articulation corresponds to the analytic way of thinking from the whole to the part.

**Linguistic backgrounds for investigating multilinguals: One mind, two languages, separate conceptualisations?**

The different language-related nuances of the conceptualisations presented above invite us to go beyond doing comparisons involving only monolinguals. Few research studies in mathematics education have followed Pavlenko’s (2011) call to investigate multilingual students, who have more than one language in their mind (e.g. Barwell, 2015). In contrast to monolingual students, multilinguals have to handle all four expressions and all nuances for the part-whole concept. How does this affect their thinking? Do they activate the German-related nuances in German and the Turkish-related nuances in Turkish? These questions are relevant with respect to research that attempts to activate multilingual students’ languages as resources for mathematics learning (Barwell et al., 2016; Planas & Setati-Phakeng, 2014). This work requires some linguistic background on multilingualism, which we provide below.

Bilinguals are not two monolinguals in one mind (Grosjean, 1989). Our article’s title, “One Mind, Two Languages” is borrowed from an insightful anthology (Nicol, 2001) in which various linguists and psycholinguists describe bilingual language processing beyond naive ideas of separate languages. In the introductory article of this anthology, Grosjean (2001) explains his theoretical construct of language modes, between which multilingual speakers change according to contexts and communication partners. The bilingual mode allows them to combine languages and thereby activate relations less available in monolingual modes.

The bilingual mode has mostly been discussed with respect to code switching in a complimentary mode: Many empirical studies have shown that code switching is a frequent
practice in bilingual conversations (see Auer, 2011, for an overview). Early assumptions were based on deficit-oriented views, assuming that individuals use code switching to compensate for lexical deficits. Today, researchers emphasise the situated and purpose-related nature of language use, according to which competent multilingual speakers code switch to combine their social languages in order to add different aspects within one multilingual communication system (Auer, 2011). For multilinguals, codes are complementarily functional not only for compensating deficits. Understanding these patterns can contribute to understanding the ways multilinguals exploit all linguistic resources at their disposal (Lüdi, 2006).

In mathematics learning processes, Moschkovich (2007), Clarkson (2007) and others have identified the bilingual complementarity mode, showing how students make use of code switching to express complementarily mathematical ideas and provide access to alternate meanings and relationships. These findings are in opposition to the frequently articulated preoccupation that bilingual modes might initiate false interference between languages that hinders learning: Will bilingual Turkish–German speaking students be confused by reading fractions in two directions as shown in Figure 1? Although our data show that these kinds of interferences can occasionally occur in the beginning of learning processes (Wagner, Kuzu, Redder, & Prediger, 2018), our study resonates more with other findings (Moschkovich, 2007; Clarkson, 2007; Planas & Setati-Phakeng, 2014) that show opportunities for deepening rather than hindering mathematics learning.

Beyond this bilingual complementary mode, a second facet of the bilingual mode, which we will call the connection mode, has been identified in preliminary studies (Kuzu & Prediger, 2017) that is in line with current research on multilinguals: Beyond code switching, researchers emphasise the holistic character of the multilingual repertoire with its rich mental connections (House & Rehbein, 2004; Lüdi, 2006). This alternative perspective, which rejects the idea of separate language-related conceptualisations, has increasingly been discussed and referred to as “translanguaging practices” (García & Wei, 2014) in linguistic discourse:

Translanguaging differs from the notion of code-switching in that it refers not simply to a shift or a shuttle between two languages, but to the speakers’ construction and use of original and complex interrelated discursive practices that cannot be easily assigned to one or another traditional definition of language, but that make up the speakers’ complete language repertoire. (García & Wei, 2014, p. 22)

Lewis, Jones, and Baker (2012) also describe translanguaging: “both languages are used in a dynamic and functionally integrated manner to organise and mediate mental processes in understanding, speaking, literacy and, not least, learning” (Lewis et al., 2012, p. 641). This relation of translanguaging to holistic, language-synthesising mental processes is crucial for what we call the connection mode as defined below.

Based on these considerations, we expand Grosjean’s (2001) construct of bilingual mode to investigate the interplay of language-related nuances of the fraction concept and multilinguals’ use of languages to mentally construct them:

- **Bilingual complementarity mode:** Used to enrich the conversation for complementary purposes. In mathematics education, this could appear when students combine language-related conceptualisations, each in their original language.
• **Bilingual connection mode**: Translanguaging for synthesising new aspects into an internally shared, mentally integrated system. In mathematics education, this could appear when individuals refer to language-related nuances of conceptualisations in all languages and when they combine aspects of different nuances.

These two versions of the bilingual mode emerged in a preliminary study (Kuzu & Prediger, 2017) and are empirically substantiated in this current article.

**Research question**

Based on the above theoretical and empirical starting points, a second research question must be added to the original one:

(Q1) How do students in a bilingual teaching intervention adopt and combine language-related conceptualisations of the part-whole concept of fractions in or across both languages?

(Q2) When do bilingual modes occur in students’ learning processes when learning about fractions, and, when they do, how can they be described at the micro-level?

**Research context and research design**

**Research context: bilingual teaching intervention on fractions**

The research questions were pursued in a learning-process study that was part of the larger mixed-methods project MuM-Multi (Schüler-Meyer, Prediger, Kuzu, Wessel, & Redder, 2019). The project conducted a bilingual teaching intervention consisting of five sessions of 90 minutes each. It was implemented in 11 groups, each consisting of 3–5 German-Turkish speaking seventh graders (N = 41).

The bilingual teaching intervention aimed at fostering seventh graders’ conceptual understanding of fractions, especially the part-of-a-whole concept and the comparison of fractions. Three main design principles guided its design (Schüler-Meyer et al., 2019; Kuzu, 2019):

• **Design principle of creating rich opportunities for communication and language production**, according to which pushing students’ language output counts as a motor for language learning (Swain, 1985). With respect to the bilingual intervention, the pushed output is extended to encouraging bilingual language production in bilingual modes (Grosjean, 2001) by reducing the monolingual bias through (1) teacher behaviour (the teacher was trained in eliciting and mixing both languages without emphasis on “clean” language use), (2) co-presentation of bilingual worksheets in all cases and (3) explicit positive evaluation of mixed (or Turkish) language use.

• **Design principle of macro-scaffolding** (Gibbons, 2002) by sequencing the conceptual learning trajectory from everyday resources to formal concepts and formal procedures (Freudenthal, 1991; Setati & Adler, 2000), supported by language-learning opportunities progressing from everyday language and meaning-related vocabulary in the academic register and the technical register (Gibbons, 2002). In the bilingual version, some meaning-related and formal Turkish vocabulary were provided by the worksheets and micro-scaffolding interventions by the teacher. As the students were not used to formal
Turkish technical terms, the bilingual material mainly addressed vocabulary from everyday language (e.g. the ablative suffix “-ten” for fractions) and a few formal vocabulary items (e.g. the locative suffix “-te” for fractions).

- **Design principle of relating registers and representations** (Prediger et al., 2016) by systematically moving forwards and backwards between all registers and representations to construct meanings. This principle is enhanced by relating languages in bilingual modes (Auer, 2011; Grosjean, 2001) and the co-presence of all material in German and Turkish. Furthermore, systematic comparison of diverging conceptualisations in Turkish (e.g. fraction from whole to part) and German (from part to whole) are conducted to raise language awareness and to deepen conceptual understanding.

**Data corpus for the learning-process study**

For the study presented here, we selected about 230 minutes of video data out of the larger corpus and sampled 14 focus students. All 41 German-Turkish-speaking students in the bilingual intervention, including the focus students, had low achievement on a fraction test, meaning that they had not succeeded in their regular fraction classroom prior to the intervention. The students had no prior experience of activating their home language, Turkish, in institutionalised mathematics learning situations (Schüler-Meyer et al., 2019). Prior fraction units had been exclusively monolingual in German. Turkish, however, is a viable resource for them, because it is used in everyday life for family and peer group purposes, even in the third generation (Backus, 2013), and includes everyday discussions of fractions.

The qualitative analysis of these 14 students’ learning pathways is focused on seven tasks on the part-whole concept in contexts of comparing fractions. The tasks were chosen as central steps in the learning trajectory towards understanding fractions using the part-whole construct (Figure 1). In these first tasks, which involved visualisations of download bars for a movie from the internet, the students were encouraged to think about units such as gigabytes, as a part of a whole and the whole as the overall size of the movie that is downloaded. The download bar served as a first introduction to the graphical representation of fraction bars. Later, the fraction bars are assembled in the fraction bar board, which allows students to find equivalent fractions (Figure 2).

**Methods for qualitative data analysis**

The transcripts of the complete 230 minutes of video material were analysed qualitatively in two steps with respect to students’ conceptual development across languages (this analytic scheme was developed in Kuzu & Prediger, 2017):

- In Step 1, students’ conceptualisations of fractions were investigated with an analytic tool for fractions adapted from Vergnaud’s (1996) methodology of revealing individual concepts-in-action. After sequencing the transcript, the individual concepts-in-action are extrapolated for each individual articulation of the part-whole concept. By categorising whether the whole and the part are expressed explicitly or implicitly and in which conceptualisation, the individual concepts-in-action are related to the language-related, socially shared conceptualisations from Figure 1 and identified as
German-related synthetic nuances (quasi-cardinal or share nuance) or Turkish-related analytic nuance (localising or taking nuance). In the analytic schemes, the individual concepts-in-action are categorised and presented graphically in fraction bars in order to point out slight deviations from the socially shared conceptualisations.

- In Step 2, all the categorised conceptualisations are additionally coded with respect to the language used by the individual to express them, using red for Turkish and blue for German. The colour codes allow us to investigate the interplay between languages used and language-related nuances in order to study the articulations of bilingual modes.
- In Step 3, the analytic scheme is further condensed in an overview table capturing several cases, which allows the comparison of different cases and thereby the investigation of a global pattern.

In this article, we present three episodes, since their analyses (Steps 1 and 2) show crucial phenomena in a nutshell. They provide answers to research question Q2 and more elaborate articulations of the modes, and give first indications for research question Q1. Q1 is further treated in Step 3 by the broader analysis of all 14 students from a quasi-longitudinal perspective of their learning pathways across all seven tasks.

**Qualitative analysis: three cases and their embedding**

The following three episodes show students’ emergent use of language-related conceptualisations by means of switching and connecting languages. All three episodes took place in the second session of the intervention.

**Episode 1: Ilknur’s and Akasya’s different bilingual modes**

In Episode 1, the girls Ilknur and Akasya work without the teacher on the task of symbolically relating the fractions $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$ and $\frac{5}{6}$ to graphical representations by drawing the shares on the fraction bar board (Figure 2). The transcript starts with Ilknur explaining to Akasya how to draw on the fraction bar board when Akasya calls for help.
Transcript of Episode 1

| Turn/Person | Original | English Translation |
|-------------|----------|---------------------|
| 90 Akasya   | Ich versteh das nicht! | I do not understand that! |
| 91 Ilknur   | Guck, ich zeig dir! Şimdik bak! | Look, I show you! Now look! [points at the 2-element bar on the fraction bar board] |
|             | Şöyle bakıyorsun. Burda ne yazıyor? Ein Halb! [...] O zaman bunu boyuyorsun. | You look this way. What does it say here? One half! [...] Then you have to colour this one here [colours the first piece in the 2-element bar] |
| 92 Akasya   | Niye ama? | But why? |
| 93 Ilknur   | Weil es ein halb ist und das auch ein Halb. | Because it is one half [points at the fraction on the worksheet] and this also one half [points at the first piece in the 2-element bar] |
| 94 Akasya   | Dann kann ich doch zwei Drittel hier kann ich doch auch das hier anmalen, wo ist zwei Drittel hier? | Then I can colour two thirds here [points at the share on the worksheet], I can even colour this here, where is two thirds here? [looks confused at bar board] |
| 95 Ilknur   | Bis zwei Drittel' e bakacaksın | Up to two thirds you have to look [points at ending point of the second piece of the 3-element bar] |
| 96 Akasya   | Ah, zwei Drittel burda! Dann muss ich ja das hier alles anmalen oder was? | Ah, two thirds here! Then I must colour all of this here, or what? |
| 97 Ilknur   | Ja! | Yes! |
| 98 Akasya   | Und hier drei Viertel | And here three fourths [points at the ending point of the third piece in the 4-element bar] |

The analysis of Episode 1 is schematised in Table 1 and explained in the following text.

Ilknur explains how to draw the fractions in the appropriate bars of the fraction bar board (Figure 2). By focussing on the part in “bunu boyuyorsun” (“you colour this one here”) in Turn 91, she starts using the Turkish-related analytic, localising nuance with an implicit prior reference to the whole. When Aksasya expresses her misunderstanding a second time in Turn 92, Ilknur changes to German for further explanations.

When using German, she also changes the nuance: Within the Turkish utterance in Turn 95, Ilknur switches to German (“Up to two thirds”) and then adopts the German-related synthetic quasi-cardinal nuance in order to count two of the three sections of the bar from left to right (see Table 1). From Ilknur’s use of the German preposition “bis” (“up to”) in Turn 95, we can infer that she adopts a quasi-cardinal view of counting. Later she repeats this conceptualisation twice (Turns 99 and 101, not presented here).

Aksasya, however, connects both nuances (the German-related synthetic and Turkish-related analytic) by using the German expression “zwei Drittel” (“two thirds”) in combination with the Turkish “burda” (“here”), taking up the same word Ilknur uses to articulate the Turkish localising nuance in Turn 91, and “das hier anmalen” (“colour all of this...”)
here”) in Turn 96. This also allows her to combine language-related nuances for focussing on different aspects of the part-in-whole concept in the graphical and verbal representations (Prediger et al., 2016).

Summing up, the resumed analysis in Table 1 shows that Ilknur provides a typical example of code-switching in a complementarity mode, characterised by functionally using two languages to express complementary conceptualisations of the Turkish-related localising nuance in Turkish (Turn 91) and the German-related quasi-cardinal nuance in German (Turn 95). Akasya, in contrast, acts in a connection mode, in which she combines two nuances in a multilingual utterance (Turn 96). The connection mode does not determine increased understanding, but occurs together with her emergent understanding of the bar (previously not understood in Turn 90).

**Episode 2: Emir and the travel of conceptualisations from Turkish to German**

The boys Emir and Osman in Episode 2 work on the same task as Ilknur and Akasya. Both immediately draw one half. The transcript starts after the teacher asks them in Turkish where they see the half.
Again, the analysis is schematised in Table 2.

When asked to explain where he sees one half, Emir (who spoke German before) refers to the Turkish-related analytic taking nuance in Turkish (“two thereof one” in Turns 27 and 29), hence adhering to a functional distinction of languages in his immediate answer.

Osman, however, addresses the synthetic German-related share nuance by naming the parts (“two pieces” in Turn 47, “three pieces” in Turn 48) and by referring implicitly to the whole bar, which he chose adequately. In Turn 53, he adds an explicit reference to the whole by explicitly addressing the five-element bar as the whole, in Turkish.

In contrast, Emir keeps his Turkish reading of the order of fractions even when speaking German (Turn 54), but this time in a more localising way, thus addressing the localising nuance. When asked to write down what they noticed, Emir refers to the analytic Turkish-related nuance again, this time in German (Turn 67: “zwei als Ganze und du nimmst eins”).
In summary, the analysis in Table 2 (on the travel of conceptualisations from Turkish to German and indicators for both bilingual modes) shows that Emir's first utterances (Turns 27–29) correspond to the complementarity mode, especially the locative case in Turn 29 ("-de" in "tanesinde"), which carries a Turkish-related nuance. In his later utterances, he activates the connection mode, which is a good example of translanguaging since it is a creative and critical disentanglement of language-use and nuance activation: Emir seems to repeat the Turkish-related nuance in German (Turns 54 and 67), consciously omitting the nuance’s original language affiliation – a phenomenon observable also in other cases (Wagner et al., 2018; Kuzu, 2019).

**Table 2. Analytical scheme for Episode 2.**

| Student | Language use and visualised nuance | Multilingual realisation |
|---------|-----------------------------------|--------------------------|
| Emir (27/29) | T: Two thereof one | Turkish-related analytic taking nuance, expressed in Turkish  
• Reference to the whole explicitly in Turkish (‘iki’)  
• Explicit reference to the coloured part (‘bir’) in Turkish |
| Osman (47-50) | G: Colouring three pieces | German-related synthetic share nuance, expressed in German  
• Implicit reference to the whole bar by its selection  
• Explicit reference to the coloured part in German |
| Osman (53) | GT: Four pieces?  
In five-element bar? | German-related synthetic share nuance, expressed in mixed language  
• Explicit reference to the whole bar in Turkish  
• Explicit reference to the coloured part in German |
| Emir (54) | G: Mm-hmm, yes.  
And in six, you must have five | Turkish-related analytic localising nuance, expressed in German  
• Explicit reference to the whole bar in German  
• Explicit reference to the coloured part in German |
| Emir (67) | G: one twoth, is err, one half is when two is the whole and you take one. | Turkish-related analytic taking nuance, expressed in German  
• Explicit reference to whole bar in German (‘zwei’)  
• Explicit reference to coloured part (‘eins’) in German |

Episode 3: Sevda and Oğuz and the use of a multilingual resource for the same nuance

Episode 3 starts when Sevda and Oğuz have already translated the symbolic fractions 1/2, 2/3, 3/4, etc., into the fraction bar board. The interaction in this group addresses the subtask “Each share may match to multiple bars”. Both find equivalent fractions by identifying equally sized coloured parts of the bars, mostly by comparing their end points.
Transcript of Episode 3

| Turn/Person | Original (Turkish in red, German in blue) | English Translation (from Turkish in red, from German in blue) |
|-------------|------------------------------------------|---------------------------------------------------------------|
| 16 Teacher  | ńok, şimdi ikide bıe uyan başka bi Streifen var mı burda? Also | Ok, now here is another bar matching to two therein one? Well |
| 17-19 Oğuz | Ąh var [...] Bu                                        | eh, there is […] This [deictic reference to coloured part corresponding to 50/100] |
| 22 Teacher  | Hangisi? Nasıl okunuyor o?                            | Which one? How do we read it?                                |
| 23 Oğuz     | Fünfzig hundertstel.                                | Fifty hundredths.                                            |
| 24 Teacher  | Türkçe?                                               | In Turkish?                                                  |
| 25 Oğuz     | Yüzde eili.                                          | Hundred therein fifty.                                       |
| [...]       |                                                        |                                                               |
| 33 Teacher  | Sen Sevda başka bir şey işaretledin mi ikide birde?   | And you, Sevda, did you mark another one for two therein one? |
| 34-36 Sevda | Bura da olyor [...] Sekizde dört.                    | This also works [deictic reference to coloured part of 4/8] […] Eight therein four. |
| [...]       |                                                        |                                                               |
| 40-45 Sevda | Ben burdan hep aşağıya [...] Yani burdan ilk önce bu Striche bakiyoruz [...] Ordoan İşte aşağıya inen çizgilerde | I have, from here down [points at the half of the fraction bars] Well, from here [places her fingertip on ending point of the coloured bar of 1/2] first we look at these lines […] And so, from there to the descending lines [places pen vertically at the ending point of ½, then indicates going along the pen]. |
| [...]       |                                                        |                                                               |
| 66 Teacher  | Mhm Oğuz... Sen ne diyorsun?... Yani niye aynı onlar mesela sayıları farklı birisi ikide bir, birisi dörte iki farklı sayılar, ama niye aynı düşün paylar? | Mhm, Oğuz... What do you say?... Well why are they equal, for example, the numbers are different, one is two therein one and the other one four therein two, two different numbers, but why are the shares equal? |
| 67-69 Oğuz | Also, die teilen sich [...] Zum Beispiel einhalb sind gleich so wie zwei Viertel çünkü burda bi pay kaïyor, ama burda iki pay | Well, they split […] For example, one half is equal to two fourths [points at coloured bar of ½, then to the one of 2/4] because here a share of one is left, and here a share of two [points at coloured bar of ½, then to the one of 2/4]. |

In Turns 17–23, when Oğuz is asked to find shares equal to 1/2, he deictically refers to the coloured part of 50 elements of the 100-element bar (“bu”). When asked to elaborate, he articulates the symbolic fraction 50/100 (Turn 23). In Turns 67–69, he describes the shares of 1/2 and 2/4 by structuring the coloured parts (“They split”) and a tied-together piece (“here a share of one is left and here a share of two”). In sum, Oğuz’s individual notion of the part-in-whole concept seems to attend to the synthetic German-related share nuance, even using the Turkish reading order when asked to translate (Turn 25). Since the Turkish expression follows the teacher’s translation prompt (Turn 24), we assume no change in the mental model. Instead, his notion of nuance stays stable throughout the whole interaction, but different micro-focusses become apparent. Also in earlier (non-printed) parts of the transcripts, a conceptual development with a back-and-forth movement between a discrete and tied-together interpretation of the coloured part of the bar is visible, as in Turns 67–69 (Table 3).
Sevda’s individual interpretations are similar: She also focuses on both the end-points of the coloured bars to find equal shares (Turns 34–36) and the coloured parts, as in her actions she marks the complete part of the coloured bar, not just the end-point. Similar to Öğuz in Turns 17–23, she points at the coloured part of the bar and activates the share nuance, while articulating the fraction 4/8. Although the utterance is in Turkish, we do not assume a different nuance, as the Turkish follows the teacher’s immediate translation prompt (Turn 24). In the subsequent Turns 40–45, she repeats her individual interpretation, this time explicating her strategy of finding equal shares by looking vertically from the end point of the coloured parts of the bars.

Summing up, we conclude that Öğuz activates his multilingual resources in a complementarity mode. In Turns 17–23, he focusses on the coloured part corresponding to 50/100 in Turkish by using deictical means (“There is … this”) and verbalises the fraction 50/100 in German. In this moment, he addresses the graphical and symbolic representation complementarily when articulating the German-related nuance. Turns 67–69 reveal a connection of German and Turkish language resources: Öğuz describes the process of splitting the bars in German (“Well, they split”) and the quantity of pieces in each part in Turkish (“Here a share of one is left and here a share of two”). Thus, he uses both languages to connect aspects of the part-in-whole concept (without referring to regular language-related nuances), which we interpret as subsumable under the connection mode. Again, the connection mode occurs in a moment of deepening understanding.
Sevda activates the connection mode in a different way. To articulate the synthetic German-related share nuance, she uses Turkish. In Turns 34–36, she focuses on the coloured part of the bar, but uses Turkish to point out the part deictically and also uses Turkish for the fraction 4/8. Although her individual notion seems to correspond to the German-related nuance “three from five”, she uses Turkish to describe how she looks vertically “from there”, referring to the end points of the coloured parts of the bars. Thus, her individual notions are strongly affected by the German-related nuance, but her explanations are in Turkish.

Thus, the empirical substantiation of the bilingual connection mode leads us to distinguish two versions: (a) Oğuz locally combines Turkish and German for merging aspects into one individual concept-in-action, whereas (b) Sevda’s thinking is characterised by language-related nuances travelling through the languages.

**Broadening the analysis to more cases: travel of nuances through languages**

All three episodes show phenomena that can also be found in other cases of students and tasks. The complementarity mode, i.e. of functional distinctions of languages for language-related nuances, appeared less often than the two variants of the connection mode: (a) local combinations of languages for the same nuance and (b) nuances travelling through the languages. These patterns reoccur in the broader analysis of Step 3, which zooms out and investigates 14 students’ language use for different nuances across seven tasks, which involves a slightly longer time scale.

In a very condensed form, Table 4 shows the core results of the broader analysis for 10 students, the pattern of four further cases are similar to four of the cases shown here. For each student, the sequence of uttered nuances of the part-whole concept is arranged in four columns according to the socially shared nuances to which the utterance refers. The sequence of rows reflects the chronological order for each student. The transcript turns are denoted in the colours of the languages used. In this condensed representation, the shifts of nuances become visible by the placement along the horizontal axis, and switches in language become visible by the changing colours along the vertical axis.

During the seven tasks, Ilknur, Emir and Ismael address three different nuances, while most other students address two different nuances. Ilknur and Emir start mainly with a functional distinction of languages for language-related nuances in complementarity modes, but then express the nuances in the other language or mixed languages in connection modes. Other students show a similar development: They start mainly with one nuance by using the respective language (or their language of comfort) and later on, the nuance travels to the other language. In the mixed language occurrences, they sometimes locally combine aspects (as Emir does in Turn 67). Both versions of the connection mode make a creative and holistic use of translanguaging instead of solely being bound to one language. The higher frequency of Turkish-related nuances being articulated in German than vice versa is not surprising, given students’ prior experiences in monolingual classrooms.

Of course, these patterns also reveal irregularities: For most students, the initial functional distinction is broken at some point (e.g. Hakan, who starts each nuance in German, or Akasya, in Turkish) but still reflects the main pattern. Deniz and Mediha completely
Table 4. Overview of travel of conceptualisations through languages of 14 students showing transcript lines and language in which the nuances were articulated (German [G] or Turkish [T]).

| Language-related nuances | German-related quasi-cardinal nuance | German-related share nuance | Turkish-related localizing nuance | Turkish-related taking nuance |
|--------------------------|--------------------------------------|-----------------------------|-----------------------------------|-------------------------------|
| Cases                    |                                      | Task 0, #3 (GT)             | Task 0, #10 (T)                   |                               |
| Ilknur                   |                                      | Task 3, #177 (G)            | Task 3, #54 (G)                   | Task 3, #27-29 (T)           |
| (Episode 1)              |                                      | Task 4, #114 (GT)           | Task 3, #67 (G)                   | Task 4, #64 (T)              |
| Akasya                   |                                      | Task 2, #81 (T)             |                                    |                               |
| (Episode 1)              |                                      | Task 2, #107 (G)            |                                    |                               |
| Emir                     |                                      | Task 1, #20 (G)             |                                    | Task 3, #54 (G)              |
| (Episode 2)              |                                      | Task 1, #43 (T)             |                                    | Task 3, #67 (G)              |
| Ismail                   |                                      | Task 1, #20 (G)             |                                    |                               |
| (Episode 2)              |                                      | Task 1, #76 (G)             |                                    |                               |
| similarly: Rükiye, Halim |                                      | Task 2, #15 (G)             |                                    |                               |
| Osman                    |                                      | Task 3, #90 (T)             |                                    | Task 3, #27-29 (T)           |
| (Episode 2)              |                                      | Task 4, #26 (T)             |                                    | Task 3, #67 (G)              |
| Sevda                    |                                      | Task 2, #25 (G)             |                                    |                               |
| (Episode 3)              |                                      | Task 3, #50ff. (GT)         |                                    |                               |
| Öğuz                     |                                      | Task 2, #34 (T)             |                                    |                               |
| (Episode 3)              |                                      | Task 3b, #1 (T)             |                                    |                               |
| similarly: Emrah         |                                      | Task 3b, #4 (T)             |                                    |                               |
| Deniz                    |                                      | Task 3b, #6 (T)             |                                    |                               |
| (further contrasting case)|                                     | Task 4, #11 (G)             |                                    |                               |
| similarly: Medha         |                                      | Task 2, #139 (G)            |                                    |                               |
| Hakan                    |                                      | Task 3a, #21 (GT)           |                                    |                               |
| (further contrasting case)|                                     | Task 3b, #30 (GT)           |                                    |                               |
| Atiye                    |                                      | Task 4, #23 (G)             |                                    |                               |
| (further contrasting case)|                                     | Task 4, #42 (G)             |                                    |                               |
concentrate on German for expressing the meanings of fractions, in line with what they are used to in their regular classrooms. Atiye switches between languages without adding other nuances. All other students activate different conceptualisations across both languages. Hence, both bilingual modes appear, with a higher emphasis on the connection mode.

Discussion and conclusion

The part-whole concept is a good example of how languages can provide different conceptualisations (Bartolini Bussi et al., 2014). The intent of this article was to go beyond cross-cultural comparisons that have monolingual perspectives and to investigate how the different language-related conceptualisations are organised mentally by multilingual students, about which little is known (Pavlenko, 2011). The idea of separate language systems being long ago rejected by multilingualism research (Cummins, 2000; Grosjean, 1989; Lüdi, 2006) led to our research question of how multilingual students make use of intertwined languages and conceptualisations.

By investigating the learning processes of 14 seventh graders (who were acquainted with German-only schooling) on their pathways towards the part-whole concept of fractions, an empirical substantiation of Grosjean’s (2001) distinction of monolingual and bilingual modes was achieved and a further differentiation emerged (see Figure 3). The bilingual complementarity mode was identified in situations in which students use different languages for separate language-related nuances. More often, we identified the bilingual connection mode in which (a) individuals locally combined languages and related aspects or (b) used language-related nuances across languages.

We found only two students who stayed in a monolingual mode, even though the bilingual learning situation was new to them. Of the 12 others, some show instances of adapting the complementarity mode in which the Turkish-related analytic and the German-related synthetic conceptualisations are expressed in their associated language. This use of complementarity seems to enrich the multi-faceted conceptual understanding in the sense emphasised by Pavlenko (2011).

Figure 3. Different perspectives and modes of language-related conceptualisations for monolingual and multilingual students.
Beyond this, we find more moments in a connection mode: Most students activate different conceptualisations and, in the long run, use them also across both languages. The presented case studies and the larger analysis of the data provide evidence that the connection mode in particular can enrich conceptual understanding by synthesising aspects of different conceptualisations into a multi-facetted part-whole concept, which is something we could only sketch in this paper. These observations were confirmed by further and deeper analyses: According to a quantitative study of the same data (Schüler-Meyer, Prediger, Wagner, & Weinert, in press), those students who mixed their languages more frequently showed higher learning gains from pre-test to post-test than those who separated the languages. Further qualitative analyses (Kuzu, 2019; Wagner et al., 2018; Schüler-Meyer et al., 2019) provide in-depth insights into processes of conceptual development emerging in both connection modes, mainly because this multiperspectivity contributed to consolidating the conceptual understanding. We consider this to be a possible explanation for the higher quantitative effects we measured.

Although these findings have been shaped by methodological limitations such as the limited number of focus students, the specific language context (in a school system that up to the present has been monolingual, with a late start of multilingual learning opportunities), and the specific tasks, it is already an interesting contribution to the idea of dynamic and intertwined multilingual repertoires as theorised by Lüdi (2006) and García and Wei (2014). They argue against the view that multilingual students have separate language proficiencies that may work only as complements. This comes back to Sapir-Whorf’s original idea referring to the plasticity of the human mind and its ability to internalise new perspectives via second-language learning (Pavlenko, 2011).

To enrich the academic discourse on translanguaging with the emergent distinctions of bilingual complementary and connection modes, this study should be extended (1) to more students, (2) to students with other language backgrounds and other language contexts and (3) to further mathematics concepts beyond the part-whole concept. In the future, these findings can motivate further studies of the connections of languages as a specific resource of multilingual learners.

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**ORCID**

Susanne Prediger [http://orcid.org/0000-0001-6541-259X](http://orcid.org/0000-0001-6541-259X)
Taha Kuzu [http://orcid.org/0000-0001-7422-0231](http://orcid.org/0000-0001-7422-0231)
Alexander Schüler-Meyer [http://orcid.org/0000-0002-0406-6811](http://orcid.org/0000-0002-0406-6811)
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