A note on the quasiconvex Jensen divergences and the quasiconvex Bregman divergences derived thereof

Frank Nielsen∗ Gaëtan Hadjeres†

∗Sony Computer Science Laboratories Inc., Tokyo, Japan
†Sony Computer Science Laboratories, Paris, France

Abstract

We first introduce the class of strictly quasiconvex and strictly quasiconcave Jensen divergences which are oriented (asymmetric) distances, and study some of their properties. We then define the strictly quasiconvex Bregman divergences as the limit case of scaled and skewed quasiconvex Jensen divergences, and report a simple closed-form formula which shows that these divergences are only pseudo-divergences at countably many inflection points of the generators. To remedy this problem, we propose the δ-averaged quasiconvex Bregman divergences which integrate the pseudo-divergences over a small neighborhood in order obtain a proper divergence. The formula of δ-averaged quasiconvex Bregman divergences extend even to non-differentiable strictly quasiconvex generators. These quasiconvex Bregman divergences between distinct elements have the property to always have one orientation finite while the other orientation is infinite. We show that these quasiconvex Bregman divergences can also be interpreted as limit cases of generalized skewed Jensen divergences with respect to comparative convexity by using power means. Finally, we illustrate how these quasiconvex Bregman divergences naturally appear as equivalent divergences for the Kullback-Leibler divergences between probability densities belonging to a same parametric family of distributions with nested supports.

Keywords: oriented forward and reverse distances, Jensen divergence, Bregman divergence, quasiconvexity, inflection points, comparative convexity, power means, nested densities.

1 Introduction, motivation, and contributions

A dissimilarity $D(O,O')$ is a measure of the deviation of an object $O'$ from a reference object $O$ (i.e., $D_O(O') := D(O,O')$) which satisfies the following two basic properties:

Non-negativity. $D(O,O') \geq 0, \forall O, O'$

Law of the indiscernibles. $D(O,O') = 0$ if and only if $O = O'$.

In other words, a dissimilarity $D(O,O')$ satisfies $D(O,O') \geq 0$ with equality if and only if $O = O'$.

A pseudo-dissimilarity is a measure of deviation for which the non-negativity property holds but

---

*E-mail: Frank.Nielsen@acm.org, Web: [https://FrankNielsen.github.io/](https://FrankNielsen.github.io/)
†E-mail: Gaetan.Hadjeres@sony.com
not necessarily the law of the indiscernibles [31]. The objects can be vectors, probability distributions, random variables, strings, graphs, etc. In general, a dissimilarity may not be symmetric, i.e., potentially we may have $D(O, O') \neq D(O', O)$. In that case, the dissimilarity is said to be \textit{oriented}, and we consider the following two reference orientations of the dissimilarity: the \textit{forward ordinary dissimilarity} $D(O : O')$ and its associated \textit{reverse dissimilarity} $D^r(O : O') := D(O' : O)$. Notice that we used the ':' notation instead of the comma delimiter ',' between the dissimilarity arguments to emphasize that the dissimilarity may be asymmetric. In the literature, a dissimilarity is also commonly called a \textit{divergence} [3] although several additional meanings may be associated to this term like a dissimilarity between probability distributions instead of vectors (e.g., the Kullback-Leibler divergence [12] in information theory) or like a notion of smoothness (e.g., a $C^3$ contrast function in information geometry [3]). A dissimilarity may also be loosely called a \textit{distance} although this may convey to mathematicians in some contexts the additional notion of a dissimilarity satisfying the metric axioms (non-negativity, law of the indiscernibles, symmetry and triangular inequality).

The \textit{Bregman divergences} [10, 9] were introduced in operations research, and are widely used nowadays in machine learning and information sciences. For a strictly convex and smooth generator $F$, called the \textit{Bregman generator}, we define the corresponding Bregman divergence between parameter vectors $\theta$ and $\theta'$ as:

$$B_F(\theta : \theta') = F(\theta) - F(\theta') - (\theta - \theta')^\top \nabla F(\theta').$$ (1)

Bregman divergences are always finite, and generalize many common distances [5], including the Kullback-Leibler (KL) divergence and the squared Euclidean and Mahalanobis distances. Furthermore, the KL divergence between two probability densities belonging to a same exponential family [6, 5] amount to a \textit{reverse Bregman divergence} between the corresponding parameters when setting the Bregman generator to be the cumulant function of the exponential family [4]. Moreover, a bijection between regular exponential families [6] and the so-called class of “regular Bregman divergences” was reported in [5] and used for learning statistical mixtures showing that the expectation-maximization algorithm is equivalent to a Bregman soft clustering algorithm. Bregman divergences have been extended to many non-vector data types like matrix arguments [32] or functional arguments [16].

In this note, we consider defining the notion of Jensen divergences [27] for strictly quasiconvex or strictly quasiconcave generators, and the induced notion of Bregman divergences. We term them \textit{quasiconvex Bregman divergences} (and omit to prefix it by 'strictly' for sake of brevity). We then establish a connection between the KL divergence between parametric families of densities with nested supports and these quasiconvex Bregman divergences.

We summarize our main contributions as follows:

- By using quasiconvex generators instead of convex generators, we define the skewed quasiconvex Jensen divergences (Definition 1) and derived thereof quasiconvex Bregman divergences (Definition 3 and Theorem 1). The quasiconvex Bregman divergences turn out to be only pseudo-divergences at inflection points of the generator. Since this happens only at countably many points, we still loosely call them quasiconvex Bregman divergences. We can also integrate the quasiconvex Bregman (pseudo-)divergence over a small neighborhood and obtain a $\delta$-averaged quasiconvex Bregman divergence in [11, 2]. The $\delta$-averaged quasiconvex Bregman divergence are also well-defined for strictly quasiconvex but not differentiable generators. Quasiconvex Bregman divergences between distinct parameters always have one orientation finite while the other one evaluates to infinity.
We show that quasiconvex Jensen divergences and quasiconvex Bregman divergences can be reinterpreted as generalized Jensen and Bregman divergences with comparative convexity [25, 30] using power means in the limit case (§2.3 and §2.3).

We exhibit some parametric families of probability distributions with strictly nested supports such that the Kullback-Leibler divergences between them amount to equivalent quasiconvex Bregman divergences (§4).

The paper is organized as follows: Section 2 defines the quasiconvex and quasiconcave difference distances by analogy to Jensen difference distances [34, 27], study some of their properties, and show how to obtain them as generalized Jensen divergences [30] obtained from comparative convexity using power means. Henceforth their name: quasiconvex Jensen divergences. When the generator is quasilinear instead of quasiconvex, we call them quasilinear Jensen divergences. We then define the quasiconvex Bregman divergences in §3 as limit cases of scaled and skewed quasiconvex Jensen divergences, and report a closed-form formula which highlights the fact that one orientation of the distance is always finite while the other one is always infinite (for divergences between distinct elements). Since the quasiconvex Bregman divergences are only pseudo-divergences at inflection points, we define the δ-averaged quasiconvex Bregman divergences in §3.2. We also recover the formula by taking the limit case of power means Bregman divergences that were introduced using comparative convexity [30].

In §4, we consider the problem of finding parametric family of probability distributions for which the Kullback-Leibler divergence amount to a quasiconvex Bregman divergence. We illustrate one example showing that nested supports of the densities ensure the property of having one orientation finite while the other one is infinite. Finally, §5 concludes this note and hints at applications perspectives of these quasiconvex Bregman divergences, including flat and hierarchical clustering.

2 Divergences based on inequality gaps of quasiconvex or quasiconcave generators

2.1 Quasiconvex and quasiconcave difference dissimilarities

In this work, a divergence or distance $D(\theta : \theta')$ refers to a dissimilarity such that $D(\theta : \theta') \geq 0$ with equality iff. $\theta = \theta'$. A pseudo-divergence or pseudo-distance only satisfies the non-negativity property but not necessarily the law of the indiscernibles of the dissimilarities.

Consider a function $Q : \Theta \subset \mathbb{R}^D \to \mathbb{R}$ which satisfies the following “Jensen-type” inequality [8] for any $\alpha \in (0, 1)$:

$$Q((\theta \theta')_\alpha) < \max\{Q(\theta), Q(\theta')\}, \quad \theta \neq \theta' \in \Theta \subset \mathbb{R},$$

(2)

where $(\theta \theta')_\alpha := (1 - \alpha)\theta + \alpha\theta'$ denotes the weighted linear interpolation of $\theta$ with $\theta'$, and $\Theta$ the parameter space. Function $Q$ is said strictly quasiconvex [17, 7, 33, 8] as it relaxes the strict convexity inequality:

$$Q((\theta \theta')_\alpha) < (1 - \alpha)Q(\theta) + \alpha Q(\theta') \leq \max\{Q(\theta), Q(\theta')\}.$$  

(3)

Let $Q$ denote the space of such strictly quasiconvex real-valued function, and let $C$ denote the space of strictly convex functions. We have $C \subset Q$: Any strictly convex function or any strictly increasing function is quasiconvex, but not necessarily the converse: Some examples of quasiconvex functions
Figure 1: The first three functions (from left to right) are quasiconvex because any level set is convex, but the last function is not quasiconvex because the dotted line intersects the function in four points (and therefore the level set is not convex). The first function is convex, the second function is quasiconvex but not convex (a chord may intersect the function in more than two points), the third function is monotonous and here concave (quasilinear).

which are not convex are $Q(\theta) = \sqrt{\theta}$, $Q(\theta) = \theta^3$, $Q(\theta, \theta') = \log(\theta^2 + (\theta')^2)$, etc. Decreasing and then increasing functions are quasiconvex but may not be necessarily smooth. Some concave functions like $Q(\theta) = \log \theta$ are quasiconvex. The sum of quasiconvex functions are not necessarily quasiconvex. In the same spirit that function convexity can be reduced to set convexity via the epigraph representation of the function, a function $Q$ is quasiconvex if the level set $L_\alpha := \{ x : Q(x) \leq \alpha \}$ is (set) convex for all $\alpha \in \mathbb{R}$. When $Q$ is univariate, a quasiconvex function is also commonly called unimodal (i.e., decreasing and then increasing function). Thus a multivariate quasiconvex function can be characterized as being unimodal along each line of its domain. Figure 1 displays some examples of quasiconvex functions with one function that fails to be quasiconvex. Notice that strictly monotonic functions which are both strictly quasiconvex and strictly quasiconcave are termed strictly quasilinear. The ceil function $\text{ceil}(\theta) = \inf\{ z \in \mathbb{Z} : z \geq \theta \}$ is an example of quasilinear function (idem for the floor function). Another example, are the linear fractional functions $Q_{a,b,c,d}(\theta) = \frac{a^\top \theta + b}{c^\top \theta + d}$ which are quasilinear functions on the domain $\Theta = \{ \theta : c^\top \theta + d > 0 \}$. We denote by $\mathcal{L} \subset Q$ the set of strictly quasilinear functions, and by $\mathcal{H}$ the set of strictly quasiconcave functions.

**Definition 1 (Quasiconvex difference distance)** The quasiconvex difference distance (or qcvx distance for short) for $\alpha \in (0,1)$ is defined as the inequality difference gap of Eq. 2

$$q_{\text{cvx}}^\alpha J_Q(\theta : \theta') := \max\{Q(\theta), Q(\theta')\} - Q((\theta \theta')_\alpha) \geq 0,$$

$$= \max\{Q(\theta), Q(\theta')\} - Q((1 - \alpha)\theta + \alpha\theta')).$$  \hspace{1cm} (4)

By definition, the quasiconvex difference distance is a dissimilarity satisfying $q_{\text{cvx}}^\alpha J_Q(\theta : \theta') = 0$ iff. $\theta = \theta'$ when the generator $Q$ is strictly quasiconvex (see Eq. 3).

**Remark 1** Notice that we could also have defined a log-ratio gap as a dissimilarity:

$$q_{\text{cvx}}^\alpha JL_Q(\theta : \theta') := -\log \left( \frac{Q((\theta \theta')_\alpha)}{\max\{Q(\theta), Q(\theta')\}} \right).$$  \hspace{1cm} (6)

However, in that case we should have required the extra condition that the generator does not vanish in the domain, i.e., $Q(\theta) \neq 0$ for any $\theta \in \Theta$.  

4
**Property 1** Let \( a > 0 \) and \( b \in \mathbb{R} \), and define \( Q_{a,b}(\theta) = aQ(\theta) + b \). Functions \( Q_{a,b} \) are quasiconvex, and \( \text{qcvx} J_{Q_{a,b}}^\alpha(\theta : \theta') = a \text{qcvx} J_{Q}^\alpha(\theta : \theta') \).

Similarly, we can characterize a strictly quasiconcave real-valued function \( H \in \mathcal{H} : \Theta \subset \mathbb{R}^D \rightarrow \mathbb{R} \) by the following inequality for \( \alpha \in (0, 1) \):

\[
H((\theta \theta')_a) > \min \{H(\theta), H(\theta')\}, \quad \theta \neq \theta' \in \Theta \subset \mathbb{R}^D.
\]

This allows one to define the quasiconcave difference distance (or qccv distance for short):

**Definition 2 (Quasiconcave difference distance)** For \( Q \) a quasiconcave function and \( \alpha \in (0, 1) \), we define the quasiconcave distance as:

\[
\text{qccv} J_H^\alpha(\theta : \theta') := H((\theta \theta')_a) - \min \{H(\theta), H(\theta')\},
\]

\[
= H((1 - \alpha)\theta + \alpha\theta') - \min \{H(\theta), H(\theta')\}.
\]

Similarly, we have \( \text{qccv} J_{H_{a,b}}^\alpha(\theta : \theta') = a \text{qccv} J_{H}^\alpha(\theta : \theta') \) for \( a > 0 \) and \( b \in \mathbb{R} \).

Now, observe that for any \( a, b \in \mathbb{R} \), we have\(^1\) \( \min \{a, b\} = -\max \{-a, -b\} \) (or equivalently \( \max \{a, b\} = -\min \{-a, -b\} \)). Thus it follows the following identity:

**Property 2** A quasiconcave difference distance with quasiconcave generator \( H \) is equivalent to a quasiconvex difference distance for the quasiconvex generator \( Q = -H \):

\[
\text{qccv} J_{H}^\alpha(\theta : \theta') = \text{qcvx} J_{-H}^\alpha(\theta : \theta'), \quad \text{qcvx} J_{Q}^\alpha(\theta : \theta') = \text{qccv} J_{-Q}^\alpha(\theta : \theta').
\]

**Proof.**

\[
\text{qccv} J_{H}^\alpha(\theta : \theta') = H((\theta \theta')_a) - \min \{H(\theta), H(\theta')\},
\]

\[
= \max \{-H(\theta), -H(\theta')\} - (-H((\theta \theta')_a)),
\]

\[
= \text{qcvx} J_{-H}^\alpha(\theta : \theta').
\]

Therefore, we consider without loss of generality quasiconvex difference distances in the reminder.

### 2.2 Relationship of quasiconvex difference distances with Jensen difference distances

Since for any \( a, b \in \mathbb{R} \), we have \( \max \{a, b\} = \frac{a + b}{2} + \frac{1}{2}|b - a| \), \( \min \{a, b\} = \frac{a + b}{2} - \frac{1}{2}|b - a| \) and \( \max(a, b) - \min(a, b) = |b - a| \), we can rewrite Eq. 3 to get

\[
\text{qcvx} J_{Q}^\alpha(\theta : \theta') = \frac{Q(\theta) + Q(\theta')}{2} + \frac{1}{2}|Q(\theta) - Q(\theta')| - Q((\theta \theta')_a),
\]

\[
= eJ_{Q}^\alpha(\theta : \theta') + \frac{1}{2}|Q(\theta) - Q(\theta')| + Q(\theta) \left( \frac{\alpha - 1}{2} \right) + Q(\theta') \left( \frac{1}{2} - \alpha \right),
\]

\[\text{Indeed, } \max \{a, b\} = \frac{a + b}{2} + \frac{1}{2}|b - a| = -\left( \frac{a - b}{2} - \frac{1}{2}|b - a| \right) = \frac{a - b}{2} - \frac{1}{2}\left| b - a \right| = -\min \{-a, -b\}.\]
where
\[ eJ_Q^\alpha(\theta, \theta') := (Q(\theta)Q(\theta'))_\alpha - Q((\theta\theta')_\alpha), \]  \hfill (16)
is called the extended Jensen divergence, a Jensen-type divergence extended to quasiconvex generators instead of ordinary convex generators.

**Property 3 (Upperbounded the extended Jensen divergence by \( qcvxJ_Q^\alpha \))** We have:
\[ eJ_Q^\alpha(\theta : \theta') \leq qcvxJ_Q^\alpha(\theta : \theta') \] \hfill (17)
since \((Q(\theta)Q(\theta'))_\alpha \leq \max\{Q(\theta), Q(\theta')\}\). In particular, when \(Q = F\) is strictly convex, we have \(0 \leq J_F^\alpha(\theta : \theta') \leq qcvxJ_F^\alpha(\theta : \theta')\).

Notice that \(eJ_Q^\alpha(\theta, \theta') \geq 0\) when \(Q\) is strictly convex, but may be negative when only quasiconvex. For example, \(Q(\theta) = \log \theta\) is a quasiconvex and concave function, and therefore \(eJ_Q^\alpha(\theta, \theta') \leq 0\).

When \(\alpha = \frac{1}{2}\), we get the following identity:

**Property 4 (Regularization of extended Jensen divergences)**
\[ qcvxJ_Q(\theta : \theta') = \frac{Q(\theta) + Q(\theta')}{2} + \frac{1}{2}|Q(\theta) - Q(\theta')| - Q\left(\frac{\theta + \theta'}{2}\right), \] \hfill (18)
\[ = eJ_Q(\theta, \theta') + \frac{1}{2}|Q(\theta) - Q(\theta')|, \] \hfill (19)
where
\[ eJ_Q(\theta, \theta') := \frac{Q(\theta) + Q(\theta')}{2} - Q\left(\frac{\theta + \theta'}{2}\right), \] \hfill (20)
is an extension of the Jensen divergence \([11, 34]\) to a quasiconvex generator \(Q\).

Thus when the generator is convex, we can interpret the quasiconvex divergence as a \(\ell_1\)-regularization of the ordinary Jensen divergence. When the generator \(Q\) is not convex, beware that \(eJ_Q(\theta, \theta')\) may be negative but we always have \(eJ_Q(\theta, \theta') \geq -\frac{1}{2}|Q(\theta) - Q(\theta')|\).

Similarly, when the generator \(H\) is strictly quasiconcave, we rewrite the quasiconvex difference distance as
\[ qcvxJ_H(\theta : \theta') = H\left(\frac{\theta + \theta'}{2}\right) - H(\theta) + H(\theta') + \frac{1}{2}|H(\theta) - H(\theta')|, \] \hfill (21)
\[ = eJ_{-H}(\theta, \theta') + \frac{1}{2}|H(\theta) - H(\theta')|. \] \hfill (22)

### 2.3 Quasiconvex difference distances: The viewpoint of comparative convexity

In \([30]\), a generalization of the skewed Jensen divergences with respect to comparative convexity \([25]\) is obtained using a pair of weighted means. A *mean* between two reals \(x\) and \(y\) belonging to an interval \(I \subset \mathbb{R}\) is a bivariate function \(M(x, y)\) such that
\[ \min\{x, y\} \leq M(x, y) \leq \max\{x, y\}. \] \hfill (23)
That is, a mean satisfies the in-betweeness property (see [25], p. 328). A weighted mean $M_\alpha$ for $\alpha \in [0,1]$ can always be built from a mean by using the dyadic expansion of real numbers, see [25].

Consider two weighted means $M_\alpha$ and $N_\alpha$.

A function $F$ is said $(M,N)$ convex iff:

$$N_\alpha(F(\theta), F(\theta')) \geq F(M_\alpha(\theta, \theta')), \; \theta, \theta' \in \Theta.$$  \hfill (24)

We recover the ordinary convexity when $M_\alpha = N_\alpha = A_\alpha$, where $A_\alpha(x,y) = (1-\alpha)x + \alpha y$ is the weighted arithmetic mean.

We can define the $\alpha$-skewed $(M,N)$-Jensen divergence as:

$$J_{F,\alpha}^{M,N}(\theta : \theta') := N_\alpha(F(\theta), F(\theta')) - F(M_\alpha(\theta, \theta')).$$  \hfill (25)

By definition, $J_{F,\alpha}^{M,N}(\theta : \theta') \geq 0$ when $F$ is a $(M,N)$-strictly convex function.

A quasi-arithmetic mean [25] is defined for a continuous strictly increasing function $f : I \subset \mathbb{R} \to J \subset \mathbb{R}$ as:

$$M_f(p,q) := f^{-1} \left( \frac{f(p) + f(q)}{2} \right).$$  \hfill (26)

These quasi-arithmetic means are also called Kolmogorov-Nagumo-de Finetti means [21, 24, 13]. Without loss of generality, we assume strictly increasing functions instead of monotonic functions since $M_{-f} = M_f$. By choosing $f(x) = x$, $f(x) = \log x$ or $f(x) = \frac{1}{x}$, we recover the Pythagorean arithmetic, geometric, and harmonic means, respectively.

Now, consider the family of power means for $x, y > 0$:

$$P_0(x,y) := \sqrt{xy}, \quad P_\delta(x,y) := (\frac{x^\delta + y^\delta}{2})^{\frac{1}{\delta}}, \; \delta \neq 0.$$  \hfill (27)

These means fall in the class of quasi-arithmetic means obtained for $f_\delta(x) = x^\delta$ for $\delta \neq 0$ with $I = J = (0, \infty)$, and include in the limit cases the maximum and minimum values: $\lim_{\delta \to +\infty} P_\delta(a,b) = \max\{a, b\}$ and $\lim_{\delta \to -\infty} P_\delta(a,b) = \min\{a, b\}$.

The power mean Jensen divergence [30] is defined as a special case of the $(M,N)$-Jensen divergence by:

$$J_{F}^{P_\delta}(\theta : \theta') := J_{F}^{A,P_\delta}(\theta : \theta') = P_\delta(F(\theta), F(\theta')) - F(\theta\theta')_\alpha,$$  \hfill (28)

for a $(A, P_\delta)$ strictly convex generator $F$.

Let us now observe that the quasiconvex difference distance is a limit case of power mean Jensen divergences:

**Property 5** $(qcvx J_Q$ as a limit case of power mean Jensen divergences) We have

$$qcvx J_Q(\theta : \theta') = \lim_{\delta \to \infty} J_{F}^{P_\delta}(\theta : \theta').$$  \hfill (29)

Notice that a strictly quasiconvex function $Q$ is interpreted as a $(A, \max)$-strictly convex function in comparative convexity, a limit case of $(A, P_\delta)$-convexity. From now on, we term the quasiconvex difference distance the quasiconvex Jensen divergence.
3 Bregman divergences for quasiconvex generators

3.1 Quasiconvex Bregman divergences as limit cases of quasiconvex Jensen divergences

Recall that for a strictly quasiconvex generator $Q$, define the $\alpha$-skewed quasiconvex distance $qcvx_{Q}(\theta : \theta')$ as

$$qcvx_{Q}(\theta : \theta') := \max \{Q(\theta), Q(\theta')\} - Q((\theta\theta')_{\alpha}).$$

(30)

We have

$$qcvx_{Q}(\theta : \theta') \geq 0,$$

(31)

with equality if and only if $\theta = \theta'$. Notice that we do not require smoothness\footnote{19} of $Q$, and $qcvx_{Q} = qcvx_{\frac{1}{2}}$ is symmetric. For an asymmetric divergence $D(\theta : \theta')$, denote $D^{r}(\theta : \theta') = D(\theta' : \theta)$ the reverse divergence.

By analogy to Bregman divergences\footnote{5} being interpreted as limit cases of scaled and skewed Jensen divergences\footnote{37, 27}:

$$\lim_{\alpha \to 1} -\frac{1}{\alpha(1-\alpha)} J_{F}^{\alpha}(\theta : \theta') = B_{F}(\theta : \theta'),$$

(32)

$$\lim_{\alpha \to 0^{+}} \frac{1}{\alpha(1-\alpha)} J_{F}^{\alpha}(\theta : \theta') = B_{F}^{r}(\theta : \theta') = B_{F}(\theta' : \theta).$$

(33)

Let us define the following divergence:

**Definition 3 (Quasiconvex Bregman pseudo-divergence)** For a strictly quasiconvex generator $Q \in Q$, we define the quasiconvex Bregman pseudo-divergence as

$$qcvx_{B}Q(\theta : \theta') := \lim_{\alpha \to 1} -\frac{1}{\alpha(1-\alpha)} qcvx_{Q}^{\alpha}(\theta : \theta').$$

(34)

As it will be shown below, we get only a pseudo-divergence in the limit case.

**Theorem 1 (Formula for the quasiconvex Bregman pseudo-divergence)** For a strictly quasiconvex and differentiable generator $Q$, the quasiconvex Bregman pseudo-divergence is

$$qcvx_{B}Q(\theta : \theta') = \begin{cases} 
-(\theta - \theta')^{\top} \nabla Q(\theta') & \text{if } Q(\theta) \leq Q(\theta') \\
+\infty & \text{otherwise (i.e., } Q(\theta) > Q(\theta').) 
\end{cases}$$

(35)

**Proof.** By definition, we have

$$qcvx_{B}Q(\theta : \theta') = \lim_{\alpha \to 1} -\frac{1}{\alpha(1-\alpha)} (\max \{Q(\theta), Q(\theta')\} - Q((\theta\theta')_{\alpha})).$$

Applying a first-order Taylor expansion to $Q((\theta\theta')_{\alpha})$, we get

$$Q((\theta\theta')_{\alpha}) \simeq_{\alpha \to 1} Q(\theta') - (1-\alpha)(\theta - \theta')^{\top} \nabla Q(\theta').$$

(36)

Thus we have

$$qcvx_{B}Q(\theta : \theta') = \lim_{\alpha \to 1} -\frac{1}{\alpha(1-\alpha)} \left( \max \{Q(\theta), Q(\theta')\} - Q(\theta') - (1-\alpha)(\theta - \theta')^{\top} \nabla Q(\theta') \right).$$

(37)

Consider the following two cases:
Figure 2: An example of a strictly quasiconvex function $Q$ with (countably) many inflection points (at locations $\theta_i$'s) for which the derivative vanishes $Q'(\theta_i) = 0$ and the second derivative $Q''$ changes sign at the $\theta_i$'s.

- Case $\max\{Q(\theta), Q(\theta')\} = Q(\theta')$: That is, $Q(\theta') \geq Q(\theta)$. Then it follows that
  \[
  \text{qcvx} B_Q(\theta : \theta') = \lim_{\alpha \to 1^-} \frac{1}{\alpha(1-\alpha)} \left( -(1-\alpha)(\theta - \theta')^\top \nabla Q(\theta') \right),
  \]
  (38)
  \[
  = -(\theta - \theta')^\top \nabla Q(\theta').
  \]
  (39)

- Case $\max\{Q(\theta), Q(\theta')\} = Q(\theta)$: That is, $Q(\theta) \geq Q(\theta')$. Then we have
  \[
  \text{qcvx} B_Q(\theta : \theta') = \lim_{\alpha \to 1^-} \frac{1}{\alpha(1-\alpha)} \left( Q(\theta) - Q(\theta') - (1-\alpha)(\theta - \theta')^\top \nabla Q(\theta') \right).
  \]
  We have $\lim_{\alpha \to 1^-} Q(\theta) - Q(\theta') - (1-\alpha)(\theta - \theta')^\top \nabla Q(\theta') = Q(\theta) - Q(\theta') = \Delta_Q(\theta : \theta')$ that is finite and different from 0 when $\theta \neq \theta'$, and therefore $\lim_{\alpha \to 1^-} \frac{1}{\alpha(1-\alpha)} \Delta_Q(\theta : \theta') = +\infty$.

Let us now prove the axiom of non-negativity and disprove the law of the indiscernibles at inflection points for the quasiconvex Bregman pseudo-divergences.

- Law of the indiscernibles: Clearly, $\text{qcvx} B_Q(\theta : \theta) = 0$ for all $\theta \in \Theta$. So consider $\theta \neq \theta'$, and $\text{qcvx} B_Q(\theta : \theta') = -\nabla Q(\theta')^\top (\theta - \theta') = 0$ for $Q(\theta') \geq Q(\theta)$. It is enough to consider the 1D case, by considering the divergence restricted to the line passing through $\theta$ and $\theta'$ intersected by the domain $\Theta$. We may have countably many inflection points $\theta'$ for which $Q'(\theta') = 0$. At those inflection points, we may find $\theta \neq \theta'$ such that $\text{qcvx} B_Q(\theta : \theta') = 0$. Thus the quasiconvex Bregman divergence does not satisfy the law of the indiscernibles. Figure 2 displays an example of such a quasiconvex function with a few inflection points.

For example, consider the strictly quasiconvex generator $Q(x) = x^3$, with $\theta < 0$ and $\theta' = 0$. We have:
  \[
  \text{qcvx} J_Q^\alpha(\theta : \theta') = \max\{Q(\theta), Q(\theta')\} - Q((1-\alpha)\theta + \alpha\theta') = -(1-\alpha)^3 \theta^3 > 0.
  \]
  (40)

Defining the corresponding quasiconvex Bregman divergence by taking the limit of scaled quasiconvex Jensen divergence yields
Thus the quasiconvex Bregman divergence is only a pseudo-divergence at countably many inflection points. Section 3.2 will overcome this problem by introducing the δ-averaged quasiconvex Bregman divergence.

- Non-negativity follows from a classic theorem of quasiconvex analysis which reports a first-order condition for a function to be quasiconvex. A $C^1$ function $Q : \Theta \subset \mathbb{R}^D \to \mathbb{R}$ is quasiconvex iff. the following property holds (see Theorem 21.14 of [35] and §3.4.3 of [37]):

$$Q(\theta') \geq Q(\theta) \Rightarrow \nabla Q(\theta')(\theta' - \theta') \leq 0.$$  

(42)

That is equivalent to $\nabla Q(\theta')^\top (\theta - \theta') \leq 0$ or $\text{qcvx} B_{Q}(\theta : \theta') = -\nabla Q(\theta')^\top (\theta - \theta') \geq 0$.

Notice that when $Q = F$ is strictly convex and differentiable, then the property also follows from the non-negativity of the corresponding Bregman divergence $B_F(\theta : \theta') \geq 0$ and $F(\theta') \geq F(\theta)$:

$$F(\theta) - F(\theta') - (\theta - \theta')^\top \nabla F(\theta') \geq 0,$$

$$- (\theta - \theta')^\top \nabla F(\theta') \geq F(\theta') - F(\theta) \geq 0.$$  

(43)\hspace{1cm} (44)

Notice that $-(\theta - \theta')^\top \nabla Q(\theta') = (\theta' - \theta)^\top \nabla Q(\theta') \geq 0$ when $Q(\theta) \leq Q(\theta')$. Figure 3 illustrates the quasiconvex Bregman divergence for a strictly quasiconvex generator which is strictly concave and has no inflection point.

An interesting property is that if $\text{qcvx} B_Q(\theta : \theta') < \infty$ for $\theta \neq \theta'$ then necessarily $\text{qcvx} B_Q(\theta' : \theta) = \infty$, and vice-versa (when both parameters are not at inflection points). The forward $\text{qcvx} B_Q$ and reverse $\text{qcvx} B_Q$ quasiconvex Bregman pseudo-divergences are both finite only when $Q(\theta) = Q(\theta')$ and then we have $\text{qcvx} B_Q(\theta : \theta) = 0$ or when one parameter is an inflection point.

Moreover, we have the following decomposition for a quasiconvex function $Q \in Q$:

$$e_{B_Q}(\theta : \theta') = Q(\theta) - Q(\theta') + \text{qcvx} B_Q(\theta : \theta'),$$  

(45)

when $Q(\theta) \leq Q(\theta')$, where $e_{B_Q}$ stands for the extended Bregman divergence, i.e., the Bregman divergence extended to a quasiconvex generator.

**Remark 2 (Separability/non-separability of generators and divergences)** When the $D$-dimensional generator $Q$ is separable, i.e., $Q(\theta) = \sum_{i=1}^D Q_i(\theta_i)$ where $\theta = (\theta_1, \ldots, \theta_D)$ and the

\footnote{By analogy to a classic second-order condition for a strictly convex and differentiable function $F$ to be convex: To have its Hessian $\nabla^2$ positive-definite (Alexandrov’s theorem). Similarly, the first-order condition for convexity of a function states that a differentiable function $F$ with convex domain is convex iff. $F'(\theta) \geq F'(\theta') + (\theta - \theta')^\top \nabla F'(\theta)$ from which we recover the Bregman divergence: $B_F(\theta : \theta') = F(\theta) - F(\theta') - (\theta - \theta')^\top \nabla F(\theta') \geq 0.$}
Figure 3: Illustration of the quasiconvex Bregman divergence for a strictly quasilinear function $Q$ chosen to be concave (e.g. logarithmic type).

$Q_i$’s are differentiable and quasiconvex univariate functions, the quasiconvex Bregman divergence rewrites as

$$
\text{qcvx} B_Q(\theta : \theta') = \begin{cases} 
- \sum_{i=1}^{D} (\theta_i - \theta'_i) Q_i'(\theta'_i) & \text{if } Q(\theta) \leq Q(\theta') \\
\infty & \text{otherwise (} Q(\theta) > Q(\theta') \)).
\end{cases}
$$

Notice that the condition for the quasiconvex Bregman divergence to be infinite is $Q(\theta) > Q(\theta')$, and not that there exists one index $i \in \{1, \ldots, D\}$ such that $Q_i(\theta_i) > Q_i(\theta'_i)$. Thus, we have $\text{qcvx} B_Q(\theta : \theta') \neq \sum_{i=1}^{D} \text{qcvx} B_{Q_i}(\theta_i : \theta'_i)$. This is to contrast with Bregman divergences for which the separability of the generator $F(\theta) = \sum_{i=1}^{D} F_i(\theta_i)$ yields the separability of the divergence: $B_F(\theta : \theta') = \sum_{i=1}^{D} B_{F_i}(\theta_i : \theta'_i)$.

### 3.2 The $\delta$-averaged quasiconvex Bregman divergence

We shall overcome the problem of indiscernability for quasiconvex Bregman pseudo-divergences:

$$
\text{qcvx} B_Q(\theta : \theta') = (\theta' - \theta) Q'(\theta') \quad \text{for } Q(\theta') \geq Q(\theta).
$$

Since the number of inflection points is at most countable for a strictly quasiconvex generator $Q$, the function $\theta \mapsto \text{qcvx} B_Q(\theta : \theta')$ can only be identically zero on a set of null measure. We propose to integrate over a neighborhood of the parameters to obtain a strictly positive divergence when $\theta' \neq \theta$.

Given a prescribed parameter $\delta \neq 0$, we introduce the $\delta$-averaged quasiconvex Bregman divergence $\text{qcvx} B_Q^\delta$ via the following definition:

$$
\text{qcvx} B_Q^\delta(\theta, \theta') := \frac{1}{\delta} \int_{0}^{\delta} \text{qcvx} B_Q(\theta + u : \theta' + u)du.
$$

Choosing $\delta$ to be a strictly positive multiple of $\theta' - \theta$ ensures that this integral is always finite since $Q(\theta' + u) \geq Q(\theta + u)$ for $u \in I(0, \delta)$, where $I(a, b) := \{ta + (1-t)b, \ t \in [0,1] \}$ denotes the interval with endpoints $a$ and $b$. 

11
We now prove this claim. For all \( u \in I(0, \delta) \), we have \( \theta' \in I(\theta, \theta' + u) \) so that
\[
Q(\theta') < \max \{ Q(\theta), Q(\theta' + u) \} = Q(\theta' + u) \quad \text{since} \quad Q(\theta) \leq Q(\theta').
\]
Similarly, \( \theta + u \in I(\theta, \theta') \) or \( \theta + u \in I(\theta', \theta' + u) \). In the first case, if \( \theta + u \in I(\theta, \theta') \) we have
\[
Q(\theta + u) < \max \{ Q(\theta), Q(\theta' + u) \} \leq Q(\theta' + u).
\]
In the second case, \( \theta + u \in I(\theta', \theta' + u) \), and we obtain
\[
Q(\theta + u) < \max \{ Q(\theta'), Q(\theta' + u) \} \leq Q(\theta' + u),
\]
proving the claim.

By construction, this \( \delta \)-averaged quasiconvex Bregman divergence now satisfies the law of the indiscernables.

When \( Q \) is differentiable, we obtain:
\[
B^\delta_Q(\theta, \theta') := \frac{1}{\delta} \int_0^\delta (\theta' - \theta) Q'(\theta' + u) du = (\theta' - \theta) \left( \frac{Q(\theta' + \delta) - Q(\theta')}{\delta} \right). \tag{49}
\]
We note that the rhs. of (49) can also serve as the definition of the \( B^\delta_Q \) divergences, even when the strictly quasiconvex function \( Q \) is not differentiable. This motivates us to introduce the next definition, where we now denote by \( \delta > 0 \) the positive ratio between \( \delta \) and \( \theta' - \theta \) of the preceding section.

**Definition 4 (\( \delta \)-averaged quasiconvex Bregman divergence)** For a prescribed \( \delta > 0 \) and a strictly quasiconvex generator \( Q \) not necessarily differentiable, the \( \delta \)-averaged quasiconvex Bregman divergence is defined by
\[
B^\delta_Q(\theta, \theta') := \begin{cases} 
\frac{1}{\delta} (Q(\theta' + \delta(\theta' - \theta)) - Q(\theta')) & \text{if } Q(\theta') \geq Q(\theta) \\
+\infty & \text{otherwise}
\end{cases} \tag{50}
\]
Let us report some examples of \( \delta \)-averaged quasiconvex Bregman divergences:

- \( Q(x) = x \).
  \[
  B^\delta_Q(\theta : \theta') = \frac{(1 + \delta)\theta' - \delta \theta - \theta'}{\delta} = \theta' - \theta,
  \]
  when \( \theta' \geq \theta \), or \( +\infty \) otherwise.

- \( Q(x) = x^2 \).
  \[
  B^\delta_Q(\theta : \theta') = 2\theta'(\theta' - \theta) + \delta(\theta' - \theta)^2,
  \]
  when \( |\theta'| \geq |\theta| \), or \( +\infty \) otherwise.

- \( Q(x) = x^3 \).
  \[
  B^\delta_Q(\theta : \theta') = 3\theta'^2(\theta' - \theta) + 3\theta'\delta(\theta' - \theta)^2 + \delta^2(\theta' - \theta)^3,
  \]
  when \( \theta' \geq \theta \), or \( +\infty \) otherwise. At the inflection point \( \theta' = 0 \), we now have
  \[
  B^\delta_Q(\theta : \theta') = -\delta^2 \theta^3 > 0 \quad \forall \theta < 0.
  \]
3.3 Quasiconvex Bregman divergences as limit cases of power mean Bregman divergences

For sake of simplicity, consider scalar divergences below. In [30], the \((M,N)\)-Bregman divergence is defined as the limit case:

\[
B_{F}^{M,N}(p : q) = \lim_{\alpha \rightarrow 1^{-}} \frac{1}{\alpha(1 - \alpha)} J_{F,\alpha}^{M,N}(p : q) = \lim_{\alpha \rightarrow 1^{-}} \frac{1}{\alpha(1 - \alpha)} (N_{\alpha}(F(p), F(q)) - F(M_{\alpha}(p,q))).
\]  

(51)

In particular, the univariate power mean Bregman divergences are obtained by taking the power means, yielding the following formula:

\[
B_{F}^{\delta_{1},\delta_{2}}(p : q) = \frac{F^{\delta_{2}}(p) - F^{\delta_{2}}(q)}{\delta_{2}F^{\delta_{2}-1}(q)} - \frac{p^{\delta_{1}} - q^{\delta_{1}}}{\delta_{1}q^{\delta_{1}-1}} F'(q).
\]  

(52)

Let \(\delta_{2} = r\) and \(\delta_{1} = 1\). Then we get the subfamily of \(r\)-power Bregman divergences:

\[
B_{F}^{r}(\theta : \theta') = \frac{F^{r}(\theta) - F^{r}((\theta')}{rF^{r-1}((\theta'))} - (\theta - \theta')F'(\theta'),
\]  

(53)

\[
= \frac{F^{r}(\theta)}{rF^{r-1}((\theta'))} - \frac{F((\theta'})}{\theta'} - (\theta - \theta')F'(\theta').
\]  

(54)

In Eq. 54, when \(F(\theta) > F(\theta')\) then we have \(\lim_{r \rightarrow \infty} B_{F}^{r}(\theta : \theta') = \infty\) since \(\left(\frac{F^{r}(\theta)}{F^{r}(\theta')}\right)\) diverges. Otherwise \(qcvx B_{F}^{r}(\theta : \theta') = \lim_{r \rightarrow \infty} B_{F}^{r}(\theta : \theta') = -(\theta - \theta')F'(\theta')\) since \(\lim_{r \rightarrow \infty} \frac{F(\theta')}{r} = 0\) (because \(|F(\theta')| < \infty\).

When \(r \rightarrow \infty\), the power mean operator \(P_{r}\) tends to the maximum operator: \(\lim_{r \rightarrow \infty} P_{r}(a,b) = \max\{a,b\}\), and the \((A,P_{\theta})\)-Bregman divergence tends to the quasiconvex Bregman pseudo-divergence.

3.4 Some illustrating examples of quasiconvex Bregman divergences

We concisely report two univariate quasiconvex scalar Bregman divergences:

- For \(Q(\theta) = \theta\) with \(\theta \in \mathbb{R}\), we have

\[
qcvx J_{Q}^{3}(\theta : \theta') = \max\{\theta, \theta'\} - (1 - \alpha)\theta + \alpha\theta'.
\]

We consider the two cases for calculating the limit \(qcvx B_{Q}(\theta : \theta') = \lim_{\alpha \rightarrow 1^{-}} \frac{1}{\alpha(1 - \alpha)} qcvx J_{Q}^{\alpha}(\theta : \theta')\):

- When \(\theta' \geq \theta\):

\[
\lim_{\alpha \rightarrow 1^{-}} \frac{1}{\alpha(1 - \alpha)} qcvx J_{Q}^{\alpha}(\theta : \theta') = \lim_{\alpha \rightarrow 1^{-}} \frac{1}{\alpha(1 - \alpha)}((- (1 - \alpha) \theta + (1 - \alpha)\theta') = \theta' - \theta \geq 0.
\]

- When \(\theta > \theta'\):

\[
\lim_{\alpha \rightarrow 1^{-}} \frac{1}{\alpha(1 - \alpha)} qcvx J_{Q}^{\alpha}(\theta : \theta') = \lim_{\alpha \rightarrow 1^{-}} \frac{1}{\alpha(1 - \alpha)}(\theta - (1 - \alpha)\theta - \alpha) = \lim_{\alpha \rightarrow 1^{-}} \frac{1}{1 - \alpha} (\theta - \theta') = +\infty.
\]
Thus we have the following quasiconvex Bregman divergence: $q^\text{cvx}B_Q(\theta : \theta') = \theta' - \theta$ for $\theta' \geq \theta$ and $+\infty$ when $\theta' < \theta$.

- When $Q(\theta) = \log \theta$, we have $Q'(\theta) = \frac{1}{\theta}$ and $q^\text{cvx}B_Q(\theta : \theta') = 1 - \frac{\theta}{\theta'}$ for $\log \theta' \geq \log \theta$ (i.e., $\theta' \geq \theta$) and $+\infty$ when $\theta' < \theta$.

- For $Q(\theta) = \sqrt{\theta}$ and $\theta \in \Theta = (0, \infty)$, we have $Q'(\theta) = \frac{1}{2\sqrt{\theta}}$ and $q^\text{cvx}B_Q(\theta : \theta') = \frac{1}{2} \left( \sqrt{\theta'} - \frac{\theta}{\sqrt{\theta'}} \right)$ for $\sqrt{\theta'} \geq \sqrt{\theta}$ (i.e., $\theta' \geq \theta$), and $+\infty$ when $\theta' < \theta$.

4 Statistical divergences, parametric families of distributions and equivalent parameter divergences

Consider a probability space $(\mathcal{X}, \mathcal{F}, \mu)$ with $\mathcal{X}$, $\mathcal{F}$, and $\mu$ denoting the sample space, the $\sigma$-algebra and the positive measure, respectively. The most celebrated statistical divergence between two densities $p_\theta \ll \mu$ and $p_{\theta'} \ll \mu$ absolutely continuous with respect to a measure $\mu$ is the Kullback-Leibler (KL) divergence (also called relative entropy [12]), defined by:

$$\text{KL}[p : q] = \begin{cases} \int_{x \in \mathcal{X}} p(x) \log \frac{p(x)}{q(x)} \, d\mu(x), & \text{supp}(p) \subset \text{supp}(q), \\ +\infty, & \text{supp}(p) \not\subset \text{supp}(q). \end{cases}$$

(55)

where $\text{supp}(p) = \{ x \in \mathbb{R} : p(x) > 0 \}$ denotes the support of a distribution $p(x)$, and $\log \frac{0}{0} = 0$ by convention. Thus the KL divergence is said unbounded in general.

In general, a statistical divergence between densities belonging to the same parametric family $\mathcal{P} = \{ p_\theta \}_\theta$ of mutually absolutely continuous densities is equivalent to a corresponding parameter divergence $B$:

$$B(\theta : \theta') := D[p_\theta : p_{\theta'}].$$

(56)

For example, when $\mathcal{P} = \{ p_\theta(x) = \exp(x^\top \theta - F(\theta)) d\mu(x) \}_\theta$ is an exponential family [23, 6, 5] on a probability space $(\mathcal{X}, \mathcal{F}, \mu)$, then the Kullback-Leibler divergence between two densities of the exponential family (e.g., two Gaussians distributions belonging to the Gaussian exponential family) amount to a reverse Bregman divergence [5] for the Bregman generator set to the cumulant function $F(\theta) = \log \int \exp(x^\top \theta) d\mu(x)$:

$$\text{KL}[p_\theta : p_{\theta'}] = B(\theta : \theta') = B_{F^\top}(\theta : \theta') = B_F(\theta' : \theta).$$

(57)

Banerjee et al. [5] proved a bijection between regular natural exponential families and so-called regular Bregman divergences. Note that since the Csiszár’s $f$-divergence [2, 3] (including the KL divergence) is invariant to one-to-one smooth mapping $m(x)$ of the sample space $x$, the same Bregman divergence equivalent to the KL divergence can be obtained for different exponential families where $y = m(x)$. For example, the KL divergence between two normal distributions or two “equivalent” log-normal distributions is the same (using the mapping $y = \log x$). This can be also noticed by the matching of their cumulant function: $F_{\text{normal}}(\theta) = F_{\text{lognormal}}(\theta)$.

Quasiconvex Bregman divergences have the interesting property to be finite for one orientation and infinite for the other orientation. Thus to find an example of parametric family of distributions

\footnote{The Jensen-Shannon divergence [26] is a particular symmetrization of the KL divergence which is always bounded, and may accept densities with different supports.}
which the KL divergence amount to a quasiconvex Bregman divergence, we shall consider parametric distributions with nested supports (or nested densities), so that one orientation of the KL divergence will be finite while the other is will be equal to infinity.

For example, consider the family of univariate uniform densities ($D = 1$):

$$p_{\theta}(x) = 1_{0 < x < e^{\theta}} e^{-\theta},$$

(58)

where $1_A$ denotes the indicator function of $A$. We have $\text{supp}(p_{\theta'}) \subset \text{supp}(p_{\theta})$ for $0 < \theta' \leq \theta$. Then we have

$$\text{KL}[p_{\theta} : p_{\theta'}] = \begin{cases} \theta' - \theta = \text{qcvx} \times B_{Q}(\theta : \theta') & 0 < \theta \leq \theta', \\ +\infty & \theta' > \theta. \end{cases},$$

(59)

for $Q(\omega) = \omega$.

Notice that the family $P = \{p_{\theta}\}$ is not an exponential family since the family has not a fixed support. A truncated exponential family with fixed truncation parameters yields an exponential family which may neither be regular nor steep (e.g., the singly truncated normal distributions [14]).

Now, consider the parametric family $\{q_{\theta}\}$ of nested densities:

$$q_{\theta}(x) = 1_{0 < x < e^{\theta}} x^{\frac{\alpha-1}{\alpha}} e^{-\theta x},$$

(60)

for a prescribed $\alpha > 1$. After a short calculation (or using a computer algebra system as reported in Appendix A), we find that

$$\text{KL}[q_{\theta} : q_{\theta'}] = \begin{cases} \alpha(\theta' - \theta) = \text{qcvx} \times B_{Q}(\theta : \theta') & \theta' \geq \theta > 0, \\ +\infty & \theta' < \theta. \end{cases},$$

(61)

for $Q(\omega) = \omega$. Thus we have built several parametric families of nested densities that up to a scaling factor yields the same quasiconvex Bregman divergence.

For parametric densities belonging to the same exponential family, it is known that the Bhattacharyya distance amount to a Jensen divergence [27]. For an exponential family $p_{\theta}(x) = \exp(\theta^\top x - F(\theta))d\mu(x)$ with cumulant function $F$, the cross-entropy between two densities [28] is

$$h(p_{\theta} : p_{\theta'}) = \int -p_{\theta}(x) \log p_{\theta'}(x) d\mu(x) = F(\theta') - (\theta')^\top \nabla F(\theta),$$

(62)

and the entropy is

$$h(p_{\theta}) = h(p_{\theta} : p_{\theta}) = F(\theta) - \theta^\top \nabla F(\theta).$$

(63)

Since $\text{KL}(p_{\theta} : p_{\theta'}) = B_{F}(\theta' : \theta) = F(\theta') - F(\theta) - (\theta' - \theta)^\top \nabla F(\theta)$, when $F(\theta') \leq F(\theta)$, we have $-(\theta' - \theta)^\top \nabla F(\theta) = \text{qcvx} \times B_{F}(\theta' : \theta)$, and it follows that

$$\text{qcvx} \times B_{F}(\theta' : \theta) = \text{KL}(p_{\theta} : p_{\theta'}) + F(\theta) - F(\theta'), \quad F(\theta') \leq F(\theta).$$

(64)

The Wasserstein distance between two nested univariate distributions has been studied in [22] with applications to Bayesian statistics to study the influence of the prior distribution in the posterior distribution in the finite sample size setting.
First-order condition

\[ F(\theta) \geq F(\theta^\prime) + (\theta - \theta^\prime)^\top \nabla F(\theta^\prime) \]
Divergence when \( F \) strictly convex and differentiable

Pseudo-divergence/condition for divergence

\[ B_F(\theta : \theta^\prime) = F(\theta) - F(\theta^\prime) + (\theta - \theta^\prime)^\top \nabla F(\theta^\prime) \]

| Convexity of \( F \) | Quasiconvexity of \( Q \) |
|------------------------|--------------------------|
| \( F(\theta) \geq F(\theta^\prime) + (\theta - \theta^\prime)^\top \nabla F(\theta^\prime) \) | \( Q(\theta) \leq Q(\theta^\prime) \Rightarrow (\theta - \theta^\prime)^\top \nabla Q(\theta^\prime) \leq 0 \) |
| \( B_F(\theta : \theta^\prime) = F(\theta) - F(\theta^\prime) + (\theta - \theta^\prime)^\top \nabla F(\theta^\prime) \) | \( - (\theta - \theta^\prime)^\top \nabla Q(\theta^\prime) \) if \( Q(\theta) \leq Q(\theta^\prime) \) |
| \[ + \infty \] otherwise. | \[ + \infty \] otherwise. |

Divergence when \( Q \) strictly quasiconvex with no inflection point

Table 1: Bregman divergence and Bregman quasidivergence with their relationship to first-order convexity and quasiconvexity.

5 Conclusion and perspectives

We have introduced novel families of distortions between vector parameters: The quasiconvex Jensen divergences and the quasiconvex Bregman divergences. We showed that the quasiconvex Jensen divergences measuring the difference gaps of the quasiconvex inequalities can be interpreted as an \( \ell_1 \)-regularized ordinary Jensen divergence. We noticed that any quasiconcave Jensen divergence amounts to an equivalent quasiconvex Jensen divergence for the negative generator. We then derived the quasiconvex Bregman pseudo-divergences as limit cases of scaled and skewed quasiconvex Jensen divergences for strictly quasiconvex generators. The quasiconvex Bregman pseudo-divergences is a pseudo-divergence only at countably many inflection points of the generators. We thus propose to define the \( \delta \)-averaged quasiconvex Bregman divergences by integrating the pseudo-divergence over a small neighborhood. This yields a formula (Eq. 50) that can be used as the definition of the quasiconvex Bregman divergence even for non-differentiable strictly quasiconvex generators. We also showed how to derive again the result of the quasiconvex Bregman pseudo-divergences using comparative convexity using the limit case of power means. A key property of the quasiconvex Bregman divergences between distinct elements is that they are necessarily finite on one orientation and infinite for the opposite orientation. Finally, we showed how some of these quasiconvex Bregman divergences can be obtained from the Kullback-Leibler divergence between densities belonging to the same parametric family of distributions with nested support. We can retrieve the Bregman pseudo-divergences and quasiconvex Bregman pseudo-divergences from first-order convexity and quasiconvexity conditions, as illustrated in Table 1. Additional conditions on the generators ensure that the pseudo-divergences are divergences and satisfy the law of the indiscernibles (i.e., strict convexity and differentiability for Bregman divergences and strict quasiconvexity without inflection points for the quasiconvex Bregman divergences).

In future work, we shall consider applications of these novel divergences like clustering: We note that the generic \( k \)-means++ probabilistic seeding analysis reported in [29] does not apply because of the forward/reverse infinite property of these quasiconvex Bregman divergences. We may consider discrete \( k \)-means, \( k \)-center (with the minimum enclosing ball obtained from quasiconvex programming [15, 20, 18, 1] when \( k = 1 \), and quasiconvex Bregman hierarchical clustering [36].

A Calculations using a computer algebra system

Using the computer algebra system MAXIMA\(^4\) we report the calculation of the KL divergence for nested densities.

\(^4\)Freely downloadable at [http://maxima.sourceforge.net/](http://maxima.sourceforge.net/)
assume(alpha>1);
assume(theta>0);
p(x,theta):=alpha*(x**(alpha-1))/(exp(theta*alpha));
integrate(p(x,theta),x,0,exp(theta));
assume(thetap>theta);
/* KL divergence */
integrate(p(x,theta)*log(p(x,theta)/p(x,thetap)),x,0,exp(theta));

References

[1] Akshay Agrawal and Stephen Boyd. Disciplined quasiconvex programming. arXiv preprint arXiv:1905.00562, 2019.

[2] Syed Mumtaz Ali and Samuel D Silvey. A general class of coefficients of divergence of one distribution from another. Journal of the Royal Statistical Society: Series B (Methodological), 28(1):131–142, 1966.

[3] Shun-ichi Amari. Information geometry and its applications, volume 194. Springer, 2016.

[4] Katy S Azoury and Manfred K Warmuth. Relative loss bounds for on-line density estimation with the exponential family of distributions. Machine Learning, 43(3):211–246, 2001.

[5] Arindam Banerjee, Srujana Merugu, Inderjit S Dhillon, and Joydeep Ghosh. Clustering with Bregman divergences. Journal of machine learning research, 6(Oct):1705–1749, 2005.

[6] Ole Barndorff-Nielsen. Information and exponential families in statistical theory. John Wiley & Sons, 2014.

[7] Bernard Bereanu. Quasi-convexity, strictly quasi-convexity and pseudo-convexity of composite objective functions. Revue française d’automatique informatique recherche opérationnelle. Mathématique, 6(R1):15–26, 1972.

[8] Stephen Boyd and Lieven Vandenberghe. Convex optimization. Cambridge university press, 2004.

[9] Lev M. Bregman. The relaxation method of finding the common point of convex sets and its application to the solution of problems in convex programming. USSR computational mathematics and mathematical physics, 7(3):200–217, 1967.

[10] Lev Meerovich Brègman. Finding the common point of convex sets by the method of successive projection. Doklady Akademii Nauk, 162(3):487–490, 1965. in Russian.

[11] Jacob Burbea and C Radhakrishna Rao. Entropy differential metric, distance and divergence measures in probability spaces: A unified approach. Journal of Multivariate Analysis, 12(4):575–596, 1982.

[12] Thomas M Cover and Joy A Thomas. Elements of information theory. John Wiley & Sons, 2012.

[13] Bruno De Finetti. Sul concetto di media. Istituto italiano degli attuari, 3:369–396, 1931.
[14] Joan Del Castillo. The singly truncated normal distribution: a non-steep exponential family. *Annals of the Institute of Statistical Mathematics*, 46(1):57–66, 1994.

[15] David Eppstein. Quasiconvex programming. *Combinatorial and Computational Geometry*, 52(287-331):3, 2005.

[16] Bela A Frigyik, Santosh Srivastava, and Maya R Gupta. Functional Bregman divergence. In *2008 IEEE International Symposium on Information Theory*, pages 1681–1685. IEEE, 2008.

[17] Harvey J Greenberg and William P Pierskalla. A review of quasi-convex functions. *Operations research*, 19(7):1553–1570, 1971.

[18] Elad Hazan, Kfir Levy, and Shai Shalev-Shwartz. Beyond convexity: Stochastic quasi-convex optimization. In *Advances in Neural Information Processing Systems*, pages 1594–1602, 2015.

[19] Rishabh Iyer and Jeff A Bilmes. Submodular-Bregman and the Lovász-Bregman divergences with applications. In *Advances in Neural Information Processing Systems*, pages 2933–2941, 2012.

[20] Qifa Ke and Takeo Kanade. Quasiconvex optimization for robust geometric reconstruction. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 29(10):1834–1847, 2007.

[21] Andrey Nikolaevich Kolmogorov. Sur la notion de moyenne. *Acad. Naz. Lincei Mem. Cl. Sci. His. Mat. Natur. Sez.*, 12:388–391, 1930.

[22] Christophe Ley, Gesine Reinert, Yvik Swan, et al. Distances between nested densities and a measure of the impact of the prior in Bayesian statistics. *The Annals of Applied Probability*, 27(1):216–241, 2017.

[23] Weiwen Miao and Marjorie Hahn. Existence of maximum likelihood estimates for multidimensional exponential families. *Scandinavian journal of statistics*, 24(3):371–386, 1997.

[24] Mitio Nagumo. Über eine Klasse der Mittelwerte. *Japanese journal of mathematics: Transactions and abstracts*, 7(0):71–79, 1930.

[25] Constantin P Niculescu and Lars-Erik Persson. *Convex Functions and Their Applications: A Contemporary Approach*. Springer, 2018. second edition.

[26] Frank Nielsen. On the Jensen–Shannon symmetrization of distances relying on abstract means. *Entropy*, 21(5):485, 2019.

[27] Frank Nielsen and Sylvain Boltz. The Burbea-Rao and Bhattacharyya centroids. *IEEE Transactions on Information Theory*, 57(8):5455–5466, 2011.

[28] Frank Nielsen and Richard Nock. Entropies and cross-entropies of exponential families. In *2010 IEEE International Conference on Image Processing*, pages 3621–3624. IEEE, 2010.

[29] Frank Nielsen and Richard Nock. Total Jensen divergences: definition, properties and clustering. In *2015 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, pages 2016–2020. IEEE, 2015.
[30] Frank Nielsen and Richard Nock. Generalizing skew Jensen divergences and Bregman divergences with comparative convexity. *IEEE Signal Process. Lett.*, 24(8):1123–1127, 2017.

[31] Frank Nielsen, Ke Sun, and Stéphane Marchand-Maillet. On Hölder projective divergences. *Entropy*, 19(3):122, 2017.

[32] Richard Nock, Brice Magdalou, Eric Briys, and Frank Nielsen. Mining matrix data with Bregman matrix divergences for portfolio selection. In *Matrix Information Geometry*, pages 373–402. Springer, 2013.

[33] Jean-Paul Penot. Glimpses upon quasiconvex analysis. In *ESAIM Proceedings*, volume 20, pages 170–194. EDP Sciences, 2007.

[34] C Rao and T Nayak. Cross entropy, dissimilarity measures, and characterizations of quadratic entropy. *IEEE Transactions on Information Theory*, 31(5):589–593, 1985.

[35] Carl P Simon, Lawrence Blume, et al. *Mathematics for economists*, volume 7. Norton New York, 1994.

[36] Matus Telgarsky and Sanjoy Dasgupta. Agglomerative Bregman clustering. In *Proceedings of the 29th International Conference on International Conference on Machine Learning*, pages 1011–1018. Omnipress, 2012.

[37] Jun Zhang. Divergence function, duality, and convex analysis. *Neural Computation*, 16(1):159–195, 2004.