Transformations of the Hypergeometric $4F_3$ with One Unit Shift: A Group Theoretic Study

Dmitrii Karp $^{1,*}$ and Elena Prilepkina $^{2,3}$

$^1$ Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City 700000, Vietnam
$^2$ School of Economics and Management, Far Eastern Federal University, Vladivostok 690950, Russia; pril-elena@yandex.ru
$^3$ Institute of Applied Mathematics, Far Eastern Branch of the Russian Academy of Sciences, Vladivostok 690041, Russia

* Correspondence: dmitriibkarp@tdtu.edu.vn

Received: 2 October 2020; Accepted: 2 November 2020; Published: 5 November 2020

Abstract: We study the group of transformations of $4F_3$ hypergeometric functions evaluated at unity with one unit shift in parameters. We reveal the general form of this family of transformations and its group property. Next, we use explicitly known transformations to generate a subgroup whose structure is then thoroughly studied. Using some known results for $3F_2$ transformation groups, we show that this subgroup is isomorphic to the direct product of the symmetric group of degree 5 and 5-dimensional integer lattice. We investigate the relation between two-term $4F_3$ transformations from our group and three-term $3F_2$ transformations and present a method for computing the coefficients of the contiguous relations for $3F_2$ functions evaluated at unity. We further furnish a class of summation formulas associated with the elements of our group. In the appendix to this paper, we give a collection of Wolfram Mathematica routines facilitating the group calculations.

Keywords: generalized hypergeometric function; hypergeometric transformations; transformation groups; symmetric group

1. Introduction and Preliminaries

Groups comprising transformation of the generalized hypergeometric functions that preserve their value at unity can be traced back to Kummer’s formula ([1], Corollary 3.3.5), see (2) below. These groups play an important role in mathematical physics. In particular, the group theoretic properties of hypergeometric transformations constitute the key ingredient of a succinct description of the symmetries of Clebsh-Gordon’s and Wigner’s $3-j$, $6-j$ and $9-j$ coefficients from the angular momentum theory [2–5]. The Karlsson-Minton summation formula for the generalized hypergeometric function with integral parameter differences (IPD) was largely motivated by a computation of a Feymann’s path integral. Furthermore, IPD hypergeometric functions appear in calculation of several integrals in high energy field theories and statistical physics [6]. See also introduction and references in [7] for further applications in mathematical physics and relation to Coxeter groups.

The generalized hypergeometric function ([1], Section 2.1.2), ([8], Chapter 16) is defined by the series

$$p+1F_p \left( \begin{array} {c} a_1, \ldots, a_{p+1} \\ b_1, \ldots, b_p \end{array} ; z \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_{p+1})_n}{n!(b_1)_n \cdots (b_p)_n} z^n$$  (1)
whenever it converges. When evaluated at the unit argument, \( z = 1 \), it represents a function of \( 2p + 1 \) complex parameters with obvious symmetry with respect to separate permutation of the \( p + 1 \) top and the \( p \) bottom parameters. As the above series diverges at \( z = 1 \) if the parametric excess satisfies \( \Re \left( \sum_{k=1}^{p} (b_k - a_k) - a_{p+1} \right) < 0 \), the first problem that arises is to construct an analytic continuation to the values of parameters in this domain. For \( 3F_2 \) function this problem is partially solved by the transformation (\([1]\), Corollary 3.3.5)

\[
3F_2 \left( \begin{array}{c} a, b, c \\ d, e \end{array} \right) = \frac{\Gamma(e)\Gamma(d + e - a - b - c)}{\Gamma(e - c)\Gamma(d + e - b - a)} \cdot 3F_2 \left( \begin{array}{c} d - b, d - a, c \\ d, d + e - b - a \end{array} \right)
\]

(2)
discovered by Kummer in 1836. In the above formula we have omitted the argument 1 from the notation of the hypergeometric series and this convention will be adopted throughout the paper. The series the right hand side of (2) converges when \( \Re(e - c) > 0 \) so that we get the analytic continuation to this domain. An important aspect of the above formula is that it can be applied to itself directly or after permuting some of the top and/or bottom parameters. This leads to a family of transformations which can be studied by group theoretic methods. A notable member of this family is Thomae’s (1879) transformation (\([1]\), Corollary 3.3.6)

\[
3F_2 \left( \begin{array}{c} a, b, c \\ d, e \end{array} \right) = \frac{\Gamma(d)\Gamma(e)\Gamma(s)}{\Gamma(e - c)\Gamma(s + b)\Gamma(s + a)\Gamma(s + b)} \cdot 3F_2 \left( \begin{array}{c} d - c, e - c, s \\ s + a, s + b \end{array} \right)
\]

(3)
where \( s = d + e - a - b - c \), which gave the name to the whole family of \( 3F_2 \) transformations generated by the algorithm described above. In an important work [9] the authors undertook a detailed group theoretic study of Thomae’s transformations as well as transformations for the terminating \( 4F_3 \) series and Bailey’s three-term relations for \( 3F_2 \). In particular, they have shown (\([9]\), Theorem 3.2) that the function

\[
f(x, y, z, u, v) = \frac{3F_2 \left( \begin{array}{c} x + u + v, y + u + v, z + u + v \\ x + y + z + 2u + v, x + y + z + u + 2v \end{array} \right)}{\Gamma(x + y + z + 2u + v)\Gamma(x + y + z + u + 2v)\Gamma(x + y + z)},
\]

(4)
is invariant with respect to the entire symmetric group \( P_5 \) acting on its 5 arguments (note that another, simpler version of this symmetry is given by (\([2]\), Equation (7)). This symmetry was, in fact, first observed by Hardy in his 1940 lectures (\([10]\), Notes on Lecture VII). The work [9] initiated the whole stream of papers on group-theoretic interpretations of hypergeometric and \( q \)-hypergeometric transformations. See, for instance, Refs. \([2,4,7,11-14]\) and references therein.

We note in passing that the analytic continuation problem for general \( p \) was solved by Nørlund [15] and Olsson [16] with later rediscovery by Bühring [17] without resorting to group-theoretic methods. More recently, Kim, Rathie and Paris derived (\([18]\), p. 116) the following transformation

\[
4F_3 \left( \begin{array}{c} a, b, c, f + 1 \\ d, e, f \end{array} \right) = \frac{\Gamma(e)\Gamma(\psi)}{\Gamma(e - c)\Gamma(\psi + c)} \cdot 4F_3 \left( \begin{array}{c} d - a - 1, d - b - 1, c, \eta + 1 \\ d, d + e - a - b - 1, \eta \end{array} \right)
\]

(5)
with \( \psi = d + e - a - b - c - 1 \) and

\[
\eta = \frac{(d - a - 1)(d - b - 1)f}{ab + (d - a - b - 1)f}.
\]
This transformation can be iterated, but it is not immediately obvious what is the general form of the transformations obtained by such iterations. In our recent paper ([19], p. 14, above Theorem 2) we found another identity of a similar flavor which can be viewed as a generalization of (2):

$$4F_3 \left( a, b, c, f + 1 \atop d, e, f \right) = \frac{(\psi f - c(d - a - b))\Gamma(c)\Gamma(\psi)}{f\Gamma(e + d - a - b)\Gamma(e - c)} \cdot 4F_3 \left( d - a, d - b, c, \xi + 1 \atop d, e + d - a - b, \xi \right),$$

where \( \xi = f + (d - a - b)(f - c)/(e - c - 1) \). The main purpose of this paper is to present a general form of the family of transformations of which the above two identities are particular cases, demonstrate that this family forms a group and analyze the structure of the subgroup generated by explicitly known transformations (5)–(8). Before we delve into this analysis let us now record two more transformations that this family forms a group and analyze the structure of the subgroup generated by explicitly known transformations (5)–(8). Before we delve into this analysis let us now record two more transformations obtained by such iterations. In our recent paper ([19], p. 14, above Theorem 2) we found another identity of a similar flavor which can be viewed as a generalization of (2):

$$4F_3 \left( a, b, c, f + 1 \atop d, e, f \right) = \frac{(\psi f + bc)\Gamma(\psi)\Gamma(d)\Gamma(e)}{f\Gamma(a)\Gamma(\psi + b + 1)\Gamma(\psi + c + 1)} \cdot 4F_3 \left( \psi, d - a, e - a, \zeta + 1 \atop d + e - a - c, d + e - a - b, \zeta \right),$$

where \( \zeta = \psi + bc/f, \psi = d + e - a - b - c - 1; \) and

$$4F_3 \left( a, b, c, f + 1 \atop d, e, f \right) = \frac{(abc + f d\psi)\Gamma(\psi)\Gamma(e)}{f d\Gamma(e - a)\Gamma(\psi + a + 1)} \cdot 4F_3 \left( a, d - b, d - c, \nu + 1 \atop d + 1, \psi + a + 1, \nu \right),$$

where \( \nu = (abc + f d\psi)/(bc + f \psi) \).

Please note that each \( 4F_3 \) function containing a parameter pair \( \begin{bmatrix} f + 1 \\ f \end{bmatrix} \) can be decomposed into a sum of two \( 3F_2 \) functions (and we will demonstrate that there are numerous different decompositions of this type). Hence, each of the identities (5)–(8) can be written as a four-term relation for \( 3F_2 \). However, it will be seen from the subsequent considerations that, in fact, all such relations reduce to three or even two terms, and, moreover, the structure seems to be more transparent if we keep the \( 4F_3 \) function as the basic building block of our analysis. It will be revealed that the group structure of our transformations is closely related to that of the Thomae group generated by two-term transformations (2) and (3) and with contiguous three terms relations for \( 3F_2 \). We believe that our subgroup generated by (5)–(8) covers all possible two-term transformations for \( 4F_3 \) with one unit shift (more precisely all transformations of the form (10) below), but we were unable to prove this claim and leave it as a conjecture.

The paper is organized as follows. In the following section we give a general form of the transformations exemplified above and prove that they form a group. We further demonstrate that this group is isomorphic to a subgroup of \( \text{SL}(\mathbb{Z}) \) (integer matrices with unit determinant). In Section 3, we give a comprehensive analysis of the structure of the subgroup generated by the transformations (5)–(8) by showing that it is isomorphic to a direct product of the symmetric group \( P_3 \) and the integer lattice \( \mathbb{Z}^5 \). In Section 4 we explore the relation between our transformations and three-term relations for \( 3F_2 \) hypergeometric function. In particular, we show that the contiguous relations for \( 3F_2 \) functions studied recently in [20] can also be computed from the elements of our group. Section 5 provides a method of deducing summation formulas for \( 4F_3 \) with non-linearly restricted parameters while Section 6 contains the proof of Lemma 1. Finally, the Appendix A contains explicit forms of some key elements of our subgroup and several Wolfram Mathematica® routines facilitating the group calculations.
2. The Group Structure of the Unit Shift $4F_3$ Transformations

Inspecting the $4F_3$ transformations presented in Section 1 we see that they share a common structure that we will present below. To this end, let $r = (a, b, c, d, e, 1)^T$ be the column vector and define

$$F(r, f) = 4F_3\left(\begin{array}{c} a, b, c, f + 1 \\ d, e, f \end{array}\right).$$

(9)

All transformations found in Section 1 have the following general form

$$F(r, f) = C(r, f)F(Dr, \eta),$$

(10)

where $D$ is a unit determinant $6 \times 6$ matrix with integer entries and the bottom row $(0, 0, 0, 0, 0, 1)$;

$$\eta = \frac{\varepsilon f + \lambda(r)}{a(r) f + \beta(r)},$$

(11)

where $\varepsilon \in \{0, 1\}$, $\lambda(r)$, $a(r)$ and $\beta(r)$ are rational functions of the arguments $a, b, c, d, e$ (some of them may vanish identically, but $\lambda = 1$ if $\varepsilon = 0$). The coefficient $C(r, f)$ has the form

$$C(r, f) = \frac{N(r) f + P(r)}{K(r) f + L(r)},$$

(12)

where $N(r)$, $P(r)$, $K(r)$, $L(r)$ are (possibly vanishing) functions of $\Gamma$-type by which we mean ratios of products of gamma functions whose arguments are integer linear combinations of the components of $(a, b, c, d, e, 1)$. When $N(r) \neq 0$ we will additionally require that the ratio $P(r)/N(r)$ be a rational function of parameters. In fact, this last requirements is redundant, but in order to avoid it the following claim is needed: the ratio $F_2(r)/F_1(r)$ with $F_i$, $i = 1, 2$, defined in (14), is not a function of gamma type for general parameters. We were unable to find a proof of this claim in the literature although it seems to be generally accepted to be true.

Formula (10) defines a transformation $T$ characterized by the matrix $D$ and the functions $C(r, f)$, $\eta = \eta(r, f)$. Two such transformations $T_1$, $T_2$ will be considered equal if $D_1 = D_2$, $C_1(r, f) \equiv C_2(r, f)$ and $\eta_1(r, f) \equiv \eta_2(r, f)$.

According to the elementary relation $(f + 1)_n = (f)_n(1 + n/f)$, we have

$$F(r, f) = F_1(r) + \frac{1}{f} F_2(r),$$

(13)

where

$$F_1(r) = 3F_2\left(\begin{array}{c} a, b, c \\ d, e \end{array}\right), \quad F_2(r) = \frac{abc}{de} 3F_2\left(\begin{array}{c} a + 1, b + 1, c + 1 \\ d + 1, e + 1 \end{array}\right).$$

(14)

It is not immediately obvious if the composition of two transformations (10) with $\eta$ and $C$ having the forms (11) and (12), respectively, should have the same form. The following theorem shows that it is indeed the case and these transformations form a group.

**Theorem 1.** Each transformation (10) necessarily has the form

$$F(r, f) = M(r) \frac{\varepsilon f + \lambda(r)}{f} F(Dr, \eta), \quad \text{where} \quad \eta = \frac{\varepsilon f + \lambda(r)}{a(r) f + \beta(r)},$$

(15)
where \( M(r) \) is a function of \( \Gamma \)-type, \( \varepsilon \in \{0, 1\} \), \( \lambda(r), \alpha(r), \beta(r) \) are rational functions of the arguments \( a, b, c, d, e \) (possibly vanishing but with \( \lambda = 1 \) if \( \varepsilon = 0 \)).

The collection \( T \) of transformations (15) forms a group with respect to composition. More explicitly, if \( T_1, T_2 \in T \) with parameters indexed correspondingly, then \( T = T_2 \circ T_1 \) is given by

(I) If \( \varepsilon_1 \varepsilon_2 + \alpha_1(r)\lambda_2(D_1r) \neq 0 \), then \( \varepsilon = 1, M(r) = M_1(r)M_2(D_1r)(\varepsilon_1 \varepsilon_2 + \alpha_1(r)\lambda_2(D_1r)) \),

\[
\begin{align*}
\lambda(r) &= \frac{\varepsilon_2 \lambda_1(r) + \lambda_2(D_1r)\beta_1(r)}{\varepsilon_1 \varepsilon_2 + \alpha_1(r)\lambda_2(D_1r)}, \\
\alpha(r) &= \frac{\varepsilon_1 \alpha_2(D_1r) + \alpha_1(r)\beta_2(D_1r)}{\varepsilon_1 \varepsilon_2 + \alpha_1(r)\lambda_2(D_1r)}, \\
\beta(r) &= \frac{\lambda_1(r)\alpha_2(D_1r) + \beta_1(r)\beta_2(D_1r)}{\varepsilon_1 \varepsilon_2 + \alpha_1(r)\lambda_2(D_1r)}, \\
D &= D_2D_1.
\end{align*}
\]

(II) If \( \varepsilon_1 \varepsilon_2 + \alpha_1(r)\lambda_2(D_1r) = 0 \), then \( \varepsilon = 0, M(r) = M_1(r)M_2(D_1r)(\varepsilon_2 \lambda_1(r) + \lambda_2(D_1r)\beta_1(r)) \),

\[
\begin{align*}
\lambda(r) &= 1, \\
\alpha(r) &= \frac{\varepsilon_1 \lambda_2(D_1r) + \beta_2(D_1r)}{\varepsilon_2 \lambda_1(r) + \lambda_2(D_1r)\beta_1(r)}, \\
\beta(r) &= \frac{\lambda_1(r)\alpha_2(D_1r) + \beta_1(r)\beta_2(D_1r)}{\varepsilon_2 \lambda_1(r) + \lambda_2(D_1r)\beta_1(r)}, \\
D &= D_2D_1.
\end{align*}
\]

Each \( T \in T \) of the form (15) has an inverse \( T^{-1} \) determined by the parameters \( \hat{\varepsilon}, \hat{\lambda}(r), \hat{\alpha}(r), \hat{\beta}(r), \hat{D} \) given by:

(III) If \( \beta(r) \neq 0 \), then \( \hat{\varepsilon} = 1 \) and

\[
\begin{align*}
\hat{\lambda}(r) &= -\frac{\lambda(D^{-1}r)}{\beta(D^{-1}r)}, \\
\hat{\alpha}(r) &= -\frac{\alpha(D^{-1}r)}{\beta(D^{-1}r)}, \\
\hat{\beta}(r) &= \frac{\varepsilon}{\beta(D^{-1}r)}, \\
\hat{\lambda}(r) &= \frac{\beta(D^{-1}r)}{\alpha(D^{-1}r)}, \\
\hat{D} &= D^{-1}.
\end{align*}
\]

(IV) If \( \beta(r) = 0 \), then \( \hat{\varepsilon} = 0 \) and

\[
\begin{align*}
\hat{\lambda}(r) &= \frac{1}{M(D^{-1}r)\alpha(D^{-1}r)}, \\
\hat{\alpha}(r) &= 1, \\
\hat{\beta}(r) &= \frac{\alpha(D^{-1}r)}{\lambda(D^{-1}r)}, \\
\hat{\lambda}(r) &= \frac{\alpha(D^{-1}r)}{\lambda(D^{-1}r)}, \\
\hat{D} &= D^{-1}.
\end{align*}
\]

Proof of Theorem 1. We start by showing that the form of the coefficient \( C(r, f) = (Nf + P) / (Kf + L) \) defined in (12) is restricted to

\[
C(r, f) = M + W/f,
\]

where \( M = M(r), W = W(r) \) are some functions of \( \Gamma \)-type, possibly one of them vanishing. It follows from (12) and (13) that transformation (10) is equivalent to

\[
\frac{F_1(r)f + F_2(r)}{f} = \frac{(Nf + P)(F_1(Dr)\eta + F_2(Dr))}{(Kf + L)\eta},
\]

where \( N = N(r), P = P(r), K = K(r), L = L(r) \). Solving this equation we get

\[
\eta = \frac{f(fN + P)F_2(Dr)}{LF_2(r) + fKF_2(r) - f^2NF_1(Dr) - fPF_1(Dr) + f^2KF_1(r) + fLF_1(r)}.
\]
In order that \( \eta \) had the form (11) the following identity must hold
\[
(fN + P)F_2(Dr)(\alpha f + \beta) = (\epsilon f + \lambda)(LF_2(r) + fKF_2(r) - f^2NF_1(Dr) - fPF_1(Dr) + f^2KF_1(r) + fLF_1(r)). \tag{18}
\]
The free term of the cubic on the right hand side equals \( \lambda LF_2(r) \) while it vanishes on the left hand side, so that \( \lambda L = 0 \). If \( L = 0 \) we obtain (16). Otherwise, if \( \lambda = 0 \) identity (18) takes the form
\[
(fN + P)F_2(Dr)(\alpha f + \beta) = LF_2(r) + fKF_2(r) - f^2NF_1(Dr) - fPF_1(Dr) + f^2KF_1(r) + fLF_1(r). \tag{19}
\]
If \( N = 0 \), then \( K = 0 \) and we again arrive at (16). If \( N \neq 0 \) the value \( f = -P/N \) must be a root of the quadratic on the right hand side of (19). In other words, we must have
\[
LF_2(r) - \frac{P}{N}KF_2(r) - \frac{P^2}{N^2}NF_1(Dr) + \frac{P}{N}PF_1(Dr) + \frac{P^2}{N^2}KF_1(r) - \frac{P}{N}LF_1(r) = 0
\]
or
\[
\left( L - \frac{P}{N}K \right) \left( F_2(r) - \frac{P}{N}F_1(r) \right) = 0.
\]
Equality \( L = PK/N \) again leads to (16). The equality \( F_2(r) = PF_1(r)/N \) is impossible for rational \( P/N \), as demonstrated by Ebisu and Iwasaki in ([20], Theorem 1.1) which proves our claim (16). If \( P/N \) is a function of gamma type then so is \( F_2(r)/F_1(r) \) which would contradic the claim made before the theorem, but as we could not find a proof of this claim we explicitly prohibit this situation in the definition of \( C(r,f) \).

Substituting \( (Nf + P)/(Kf + L) \) by \( M + W/f \) in (17) we can now express \( \eta \) as follows:
\[
\eta = -\frac{(Mf + W)F_2(Dr)}{(MF_1(Dr) - F_1(r))f + F_1(Dr)W - F_2(r)}. \tag{20}
\]
Next suppose \( M \neq 0 \). Then \( C(r,f) = M(\epsilon f + W/M)/f \) with \( \epsilon = 1 \). Comparison of (20) with (11) yields \( W/M = \lambda \) which proves that the transformation (10) must have the form (15). Moreover,
\[
\alpha = -\frac{McF_1(Dr) - F_1(r)}{MF_2(Dr)}, \quad \beta = -\frac{F_1(Dr)\lambda M - F_2(r)}{MF_2(Dr)}.
\]
These equalities can be rewritten as the system
\[
\begin{cases}
F_1(r) = M(\epsilon F_1(Dr) + \alpha F_2(Dr)), \\
F_2(r) = M(\lambda F_1(Dr) + \beta F_2(Dr)).
\end{cases} \tag{21}
\]
Suppose now that \( M = 0, W \neq 0 \). Then \( C(r,f) = W(\epsilon f + \lambda)/f \) with \( \epsilon = 0, \lambda = 1 \). From (20) we have
\[
\eta = \frac{\epsilon f + \lambda}{\alpha f + \beta},
\]
where again \( \epsilon = 0, \lambda = 1, \) and \( \alpha = F_1(r)/(W\alpha F_2(Dr)), \quad \beta = -(WF_1(Dr) - F_2(r))/(WF_2(Dr)) \) or
\[
\begin{cases}
F_1(r) = W\alpha F_2(Dr) = W(\epsilon F_1(Dr) + \alpha F_2(Dr)), \\
F_2(r) = W(F_1(Dr) + \beta F_2(Dr)) = W(\lambda F_1(Dr) + \beta F_2(Dr)).
\end{cases} \tag{22}
\]
Renaming \( W \) into \( M \) we have thus proved that the transformation again has the form (15) and the system (21) is satisfied. The computation of composition is straightforward:

\[
T = T_2 \circ T_1 \iff T : F(r, f) = M_1(r)\frac{\varepsilon_1 f + \lambda_1(r)}{f}F(D_1r, \eta_1)
\]

\[
= M_1(r)M_2(D_1r)\frac{\varepsilon_1 f + \lambda_1(r)}{f}F(D_2D_1r, \eta_2)
\]

\[
= M_1(r)M_2(D_1r)\frac{\varepsilon_1 f + \lambda_1(r)}{f}F(D_2D_1r, \eta_2)
\]

If \( \varepsilon_1 \varepsilon_2 + \lambda_1(r) \lambda_2(D_1r) \neq 0 \), we can divide by this quantity leading to case (I). If it vanishes we get case (II). Given a transformation \( T \in \mathcal{T} \) of the from (15) it is rather straightforward to compute its inverse. We omit the details. \( \square \)

**Remark 1.** Theorem 1 implies that each transformation \( t \in \mathcal{T} \) is uniquely characterized by the collection \( \{ \varepsilon, M(r), \lambda(r), a(r), \beta(r), D \} \), where \( \varepsilon \in \{0, 1\} \), \( M(r) \) is a function of gamma type, \( \lambda(r) \), \( a(r) \) and \( \beta(r) \) are rational functions of parameters \( a, b, c, d, e \) and \( D \) is 6 × 6 unit determinant integer matrix with bottom row \((0, \ldots, 0, 1) \). We will express this fact by writing \( T \sim \{ \varepsilon, M(r), \lambda(r), a(r), \beta(r), D \} \). Occasionally, we will omit the dependence on \( r \) in the notation of the functions \( M(r), \lambda(r), a(r), \beta(r) \) for brevity.

Please note that for \( \varepsilon = 1 \) and non-vanishing \( a, \beta \) and \( \lambda \) the system (21) takes the form of \( _4F_3 \rightarrow _3F_2 \) reduction formulas

\[
\begin{cases}
F(Dr, a(r)^{-1}) = M(r)^{-1}F_1(r), \\
F(Dr, \lambda(r) / \beta(r)) = (M(r)\lambda(r))^{-1}F_2(r).
\end{cases}
\]

Next, we clarify the structure of the group \( \mathcal{T} \) further. The composition rule involves all the parameters \( M(r), \lambda(r), a(r), \beta(r) \) and \( D \). The following theorem implies that the matrix \( D \) determines all other parameters uniquely. Denote by \( \overline{\text{SL}}(n, \mathbb{Z}) \) the subgroup of the special linear group \( \text{SL}(n, \mathbb{Z}) \) of \( n \times n \) integer matrices with unit determinant comprising matrices whose last row has the form \((0, \ldots, 0, 1) \).

**Theorem 2.** The mapping \( T \sim \{ \varepsilon, M(r), \lambda(r), a(r), \beta(r), D_T \} \rightarrow D_T \) is isomorphism, so that the group \( (\mathcal{T}, \circ) \) is isomorphic to a subgroup of \( \overline{\text{SL}}(6, \mathbb{Z}) \) which we denote by \((D_T, \cdot)\).

**Proof of Theorem 2.** One direction is clear: each transformation \( T \in \mathcal{T} \) by construction defines a matrix \( D_T \in \overline{\text{SL}}(6, \mathbb{Z}) \) and the composition rule (I), (II) in Theorem 1 involves the product of matrices. Hence, to establish our claim it remains to prove that the kernel of the homomorphism \( T \rightarrow D_T \) is trivial. Assume the opposite: there exists a transformation \( T \in \mathcal{T} \) with the identity matrix \( D = I \) and non-trivial parameters \( \varepsilon, M, \lambda, a, \beta \). The system (21) then takes the form

\[
\begin{cases}
(1 - Me)F_1(r) = MaF_2(r), \\
M\lambda F_1(r) = (1 - M\beta)F_2(r).
\end{cases}
\]

If \( a = \lambda = 0 \) we get \( M = \varepsilon = 1 \) from the first equation and \( \beta = 1 \) from the second equation, which amounts to the trivial identity transformation. We will show that all other cases are impossible. Indeed, Ebisu and Iwasaki demonstrated in ([20], Theorem 1.1) that the functions \( F_1(r) \) and \( F_2(r) \) are linearly independent.
over the field of rational functions of parameters. If \( \alpha = 0 \) and \( \lambda \neq 0 \), then \( M = e = 1 \) from the first equation and \( F_1(r)/F_2(r) = (1 - \beta)/\lambda \) from the second equation contradicting linear independence. Similarly, if \( \alpha \neq 0 \) and \( \lambda = 0 \), then \( M = 1/\beta \) from the second equation, so that \( F_2(r)/F_1(r) = (1 - \beta)/\lambda \) is rational from the first equation leading again to contradiction. Finally, if both \( \alpha \neq 0 \) and \( \lambda \neq 0 \) we arrive at the identities

\[
\frac{F_2(r)}{F_1(r)} = 1 - M e = \frac{\lambda M}{1 - M \beta} \Rightarrow (1 - M \beta)(1 - M e) = \alpha \lambda M^2.
\]

Linear independence of the functions \( F_1(r) \), \( F_2(r) \) over rational functions implies that the function \( M = M(r) \) must be a ratio of products of gamma functions irreducible to a rational function. On the other hand, by the ultimate equality \( M(r) \) solves the quadratic equation with rational coefficients:

\[
M = M(r) = \mu(r) \pm \sqrt{\nu(r)}
\]

with rational \( \mu, \nu \). It is easy to see that this is not possible as \( \Gamma \) is meromorphic with infinite number of poles and no branch points, while \( \mu(r) \pm \sqrt{\nu(r)} \) may only have a finite number of poles and zeros and has branch points. \( \square \)

3. The Subgroup of \( T \) Generated by Known Transformations

We can now rewrite the transformations (5)–(8) in the standard form (15). Denote by \( \psi = d + e - a - b - c - 1 \) the parametric excess of the function on the left hand side of (15). Identity (7) is determined by the following set of parameters

\[
M_1 = \frac{\Gamma(\psi + 1) \Gamma(d) \Gamma(e)}{\Gamma(a) \Gamma(d + e - a - c) \Gamma(d + e - a - b)}, \quad \varepsilon_1 = 1, \quad \lambda_1 = \frac{bc}{\psi},
\]

\[
D_1 = \begin{bmatrix}
-1 & -1 & -1 & 1 & 1 & -1 \\
-1 & 0 & 0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 \\
-1 & 0 & -1 & 1 & 1 & 0 \\
-1 & -1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}, \quad \alpha_1 = \frac{1}{\psi}, \quad \beta_1 = 0.
\]

(24a)

We will call this transformation \( T_1 \).

The standard parameters of transformation (6) are given by

\[
M_2 = \frac{\Gamma(e) \Gamma(\psi + 1)}{\Gamma(e + d - a - b) \Gamma(e - c)}, \quad \varepsilon_2 = 1, \quad \lambda_2 = \frac{c(-d + a + b)}{\psi},
\]

\[
D_2 = \begin{bmatrix}
-1 & 0 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
-1 & -1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}, \quad \alpha_2 = 0, \quad \beta_2 = \frac{e - c - 1}{\psi}.
\]

(25a)

We will call this transformation \( T_2 \).

The standard parameters of transformation (8) are given by
We will call this transformation \( T_3 \).

Finally, transformation (5) in the standard form (15) is parameterized by

\[
M_4 = \frac{\Gamma(\psi)\Gamma(\epsilon)}{\Gamma(\epsilon - c)\Gamma(\psi + c)}, \quad \epsilon_4 = 1, \quad \lambda_4 = 0, \quad \alpha_4 = \frac{d - a - b - 1}{(d - a - 1)(d - b - 1)},
\]

\[
D_4 = \begin{bmatrix}
-1 & 0 & 0 & 1 & 0 & -1 \\
0 & -1 & 0 & 1 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
-1 & -1 & 0 & 1 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}, \quad \beta_4 = \frac{ab}{(d - a - 1)(d - b - 1)}.
\]  

We will call this transformation \( T_4 \). It is easy to see that it is of order 2, i.e., \( T_4^2 = I \).

The four transformations \( T_1, T_2, T_3, T_4 \) (or, equivalently, (5)–(8)) combined with permutations of the upper and lower parameters generate a subgroup of \( \mathcal{T} \) which we will call \( \hat{T} \). Isomorphism established in Theorem 2 induces an isomorphism between \( \hat{T} \) and a subgroup of \( \text{SL}(6, \mathbb{Z}) \) which we denote by \( D_\hat{T} \).

A complete characterization of \( \hat{T} \) and \( D_\hat{T} \) will follow. Before we turn to it, we remark that to our belief, the complete group \( \mathcal{T} \) contains no elements other than those in \( \hat{T} \). We were unable, however, to prove this claim. Let us thus state it as a conjecture.

**Conjecture.** The subgroup \( \hat{T} \) generated by the transformations (24)–(27) coincides with the entire group \( \mathcal{T} \) of all transformations of the form (10) or, equivalently, of the form (15).

Denote by \( S_j, j = 1, \ldots, 5 \), the transformation shifting the \( j \)-th component of the parameter vector \( \mathbf{r} \) by +1, i.e., \( S_j \) is characterized by the matrix \( \hat{S}_j \) such that \( \hat{S}_1 \mathbf{r} = (a + 1, b, c, d, e, 1) \), \( \hat{S}_2 \mathbf{r} = (a, b + 1, c, d, e, 1) \), etc. It is not a priori obvious that such transformations should exist among the elements of \( \hat{T} \). The following theorem shows that it is indeed the case.

**Theorem 3.** The group \( \hat{T} \) contains the transformations \( S_j, j = 1, \ldots, 5 \).

**Proof of Theorem 3.** Due to permutation symmetry it is clearly sufficient to display the transformations \( S_1 \) and \( S_4 \). We will need the inverse of the transformation \( T_1 \) defined in (24). Using Theorem 1 we calculate

\[
\begin{align*}
\binom{a, b, c, f + 1}{d, e, f} &= M_1 \binom{d + e - a - b - c - 1, d - a - 1, e - a - 1, \eta_1 + 1}{d + e - a - c - 1, d + e - a - b - 1, \eta_1},
\end{align*}
\]  

where

\[
M_1 = \frac{\Gamma(\eta_1 + 1)\Gamma(\epsilon)}{\Gamma(\eta_1 + 1 - d - e - a - b - c - 1)\Gamma(d + e - a - c - 1)},
\]
\[ N_1 = \frac{\Gamma(d)\Gamma(e)\Gamma(\psi)}{\Gamma(p+b)\Gamma(p+c)\Gamma(a)}, \quad \hat{\theta}_1 = 0, \quad \hat{\lambda}_1 = 1, \quad \hat{\alpha}_1 = \frac{1}{(d-a-1)(e-a-1)}, \]

\[ \hat{\beta}_1 = \frac{-a}{(d-a-1)(e-a-1)}, \quad \text{so that} \quad \hat{\eta}_1 = \frac{(d-a-1)(e-a-1)}{f-a}. \]

Next, exchanging the roles of \( d' + e - a - b - 1 \) and \( d \) and the roles of \( d - a - 1 \) and \( c \) in (5) or, equivalently, post-composing \( T_4 \) with permutation (13) (45) we will obtain a transformation that we call \( \hat{T}_4 \). Then \( \hat{T}_4 \circ \hat{T}_4 \) takes the form

\[
{}_{4}F_{3}\left(\begin{array}{c} a, b, c, f + 1 \\ d, e, f \end{array}\right) = \frac{\Gamma(d)\Gamma(e)\Gamma(\psi)}{\Gamma(b+\psi)\Gamma(c+\psi)\Gamma(a+1)}{}_{4}F_{3}\left(\begin{array}{c} \psi - 1, e - a - 1, d - d - 1, \hat{\eta}_4 + 1 \\ c + \psi, b + \psi, \hat{\eta}_4 \end{array}\right)
\]

with

\[ \hat{\eta}_4 = \frac{(\psi - 1)(e - a - 1)(d - a - 1)f}{abc + (1 + 2a + a^2 - bc - d - ad - e - ae + de)f}. \]

Applying \( T_1^{-1} \) to the right hand side of (29) we obtain the transformation \( S_1 \):

\[
{}_{4}F_{3}\left(\begin{array}{c} a, b, c, f + 1 \\ d, e, f \end{array}\right) = \frac{e^f + \lambda}{f}{}_{4}F_{3}\left(\begin{array}{c} a + 1, b, c, \eta + 1 \\ d, e, \eta \end{array}\right),
\]

where \( \epsilon = 1 \), and

\[ M = 1 - \frac{bc}{(d-a-1)(e-a-1)}, \quad \lambda = \frac{abc}{a^2 - bc + (d-1)(e-1) - a(d+e-2)}, \]

\[ \alpha = \frac{d + e - a - b - c - 2}{a^2 - bc + (d-1)(e-1) - a(d+e-2)}, \quad \beta = -\frac{a(d + e - a - b - c - 2)}{a^2 - bc + (d-1)(e-1) - a(d+e-2)}. \]

According to (15) we thus obtain the following expression for \( \eta \)

\[ \eta = \frac{abc + (1 + 2a + a^2 - bc - d - ad - e - ae + de)f}{a(2 + a + b + c - d - e) - (2 + a + b + c - d - e)f}. \]

Application of the transformation \( T_3 \) given by (26) to itself yields \( T_3 \circ T_3 \) in the form:

\[
{}_{4}F_{3}\left(\begin{array}{c} a, b, c, f + 1 \\ d, e, f \end{array}\right) = \frac{a(d - b)(d - c)(bc + f\psi) + (d + 1)(e - a)(abc + f\psi)}{f(d + 1)e\psi}{}_{4}F_{3}\left(\begin{array}{c} a, b + 1, c + 1, \hat{\eta}_3 + 1 \\ d + 2, e + 1, \hat{\eta}_3 \end{array}\right),
\]

where

\[ \hat{\eta}_3 = \frac{a(d - b)(d - c)(bc + f\psi) + (d + 1)(e - a)(abc + f\psi)}{(d - b)(d - c)(bc + f\psi) + (e - a)(abc + f\psi)}. \]

On the other hand, using (28) we compute \( T_1^{-2} \) as follows:

\[
{}_{4}F_{3}\left(\begin{array}{c} a, b, c, f + 1 \\ d, e, f \end{array}\right) = \frac{(d - 1)(e - 1)(f - a)}{f(d - a - 1)(e - a - 1)}{}_{4}F_{3}\left(\begin{array}{c} a, b - 1, c - 1, \hat{\eta}_1' + 1 \\ d - 1, e - 1, \hat{\eta}_1' \end{array}\right)
\]

\[ \hat{\eta}_1' = \frac{(d-a-1)(e-a-1)}{f-a}. \]
with
\[ \eta_1' = \frac{(b - 1)(c - 1)(f - a)}{(d - a - 1)(e - a - 1) - \varphi(f - a)}. \]

Comparing these formulas we see that the composition \( T_1^{-2} \circ T_3^2 \) gives the transformation \( S_4 \) shifting \( d \rightarrow d + 1 \) while \( a, b, c, e \) remain intact:
\[
4F_3 \left( \begin{array}{c} a, b, c, f + 1 \\ d, e, f \end{array} \right) = \frac{f + \lambda}{f} 4F_3 \left( \begin{array}{c} a, b, c, \eta + 1 \\ d + 1, e, \eta \end{array} \right),
\]
so that \( \epsilon = 1, M = 1 \),
\[
\lambda = \frac{abc}{d(d + e - a - b - c - 1)}, \quad \alpha = \frac{1}{d}, \quad \beta = \frac{(b - d)(c - d) + a(b + c - d)}{d(d + e - a - b - c - 1)}, \quad \eta = \frac{\epsilon f + \lambda}{\alpha f + \beta}.
\]

Each transformation \( S_j, j = 1, \ldots, 5 \), obviously generates a subgroup of \( \hat{T} \) isomorphic to \( \mathbb{Z} \)—the additive group of integers. Hence, in the parlance of group theory, the above theorem can be restated and enhanced as follows.

**Corollary 1.** The group \( \hat{T} \) contains a subgroup \( S \) isomorphic to the 5-dimensional integer lattice \( \mathbb{Z}^5 \). Furthermore, this subgroup is normal.

**Proof of Corollary 1.** By the previous theorem we only need to prove normality. Denote by \( S \) the subgroup of the matrix group \( D_\mathbb{Z} \) generated by the shift matrices \( \hat{S}_j, j = 1, \ldots, 5 \). Clearly, \( S \) comprises \( 6 \times 6 \) matrices whose principal \( 5 \times 5 \) sub-matrix equals the identity matrix \( I_5 \), the 6-th row is \((0, \ldots, 0, 1)\) and the 6-th column is \((k_1, \ldots, k_5, 1)\) for some \( k_i \in \mathbb{Z} \). As all elements of \( D_\mathbb{Z} \) have integer entries and the bottom row \((0, \ldots, 0, 1)\) it is easy to see that for any shift matrix \( S \in S \) and any matrix \( D \in D_\mathbb{Z} \) both products \( DS \) and \( SD \) have the principal \( 5 \times 5 \) sub-matrix equal to that of \( D \) and the last column of the form \((k_1, \ldots, k_5, 1)\) for some \( k_i \in \mathbb{Z} \). Running over all elements of \( S \) while keeping \( D \) fixed we see that the left and right conjugacy classes of the element \( D \) with respect to \( S \) coincide.

The above corollary implies that we can take the factor group \( D_\mathbb{Z}/S \). Each element in \( D_\mathbb{Z}/S \) is a conjugacy class containing a representative with the last column \((0, \ldots, 0, 1)^T\). Next, we note that the principal \( 5 \times 5 \) sub-matrix of the matrix \( D_2 \) from (25b) of the transformation (6) is equal to that of the Kummer’s transformation (2). This transformation together with the permutation group \( P_3 \times P_2 \) representing the obvious invariance with respect to separate permutations of the upper and lower parameters generate the entire group of Thomae transformations [9]. Next, comparing the principal \( 5 \times 5 \) sub-matrices of the further generators \( D_1, D_3, D_4 \) with the matrices of the Thomae transformations found, for instance in ([4], Appendix 1), we see that all of them occur among the elements of the group of the Thomae transformations. Hence, it remains to apply Theorem 3.2 from [9] asserting that the group of the Thomae transformations is isomorphic the 120-element symmetric group \( P_5 \) of permutations on five symbols. Isomorphism is given by a linear change of variables seen in (4). Hence, our final result is the following theorem.

**Theorem 4.** The group \( \hat{T} \) is isomorphic to \( P_5 \times \mathbb{Z}^5 \).
As the entire group of the Thomae transformations for $3F_2$ can be generated by the identity (2) and the permutation group $P_3 \times P_2$, the above theorem implies that our entire group $\mathcal{T}$ can be generated by the identity (6) (transformation $T_2$) and the top parameter shift transformation $S_1$ together with the obvious symmetries $P_3 \times P_2$. For example, the bottom parameter shift transformation can be obtained as follows:

$$(d - c, e - c, \psi, \psi + a, \psi + b) \xrightarrow{T_2^2} (a, b, c, d, e) \xrightarrow{S_1^{-1}} (a + 1, b, c - 1, d, e) \xrightarrow{T_2} (d - c + 1, e - c + 1, \psi, \psi + a + 1, \psi + b).$$

Comparing the first and the last terms in this chain we see that we got the bottom parameter shift transformation $S_4$ using only $T_2$ and top shift transformations $S_1$, $S_2$, $S_3$ obtained from $S_1$ by permuting top parameters.

Theorem 4 further implies that there is a straightforward algorithm for computing any transformation from the group $\mathcal{T}$. Details are given in the Appendix A to this paper.

4. Related $3F_2$ Transformation

The proof of Theorem 1 shows that each transformation $T \in \mathcal{T}$ is associated with the system (21) of two $3F_2$ transformations. This system leads immediately to the following corollary.

**Proposition 1.** Each transformation $T \sim \{\epsilon, M(r), \lambda(r), a(r), \beta(r)\} \in \mathcal{T}$ induces a transformation for the ratio

$$\Psi(r) := \frac{F_2(r)}{F_1(r)} = \frac{abc}{de} 3F_2 \left( \begin{array}{c}
    a + 1, b + 1, c + 1 \\
    d + 1, e + 1
  \end{array} \right) = \frac{d}{dx} 3F_2 \left( \begin{array}{c}
    a, b, c \\
    d, e
  \end{array} \right) \bigg|_{x = 1}$$

of the form

$$\Psi(r) = \frac{\beta(r) \Psi(Dr) + \lambda(r)}{\alpha(r) \Psi(Dr) + \epsilon}.$$

Next, we observe that any two elements of $\mathcal{T}$ generate a three-term relation for $3F_2$.

**Proposition 2.** For any two transformations from the group $T$: $T_1 \sim \{\epsilon_1, M_1(r), \lambda_1(r), a_1(r), \beta_1(r), D_1\}$ and $T_2 \sim \{\epsilon_2, M_2(r), \lambda_2(r), a_2(r), \beta_2(r), D_2\}$ satisfying the condition $a_2 \beta_1 - a_1 \beta_2 \neq 0$, the following identities hold

$$F_1(r) = M_1 \frac{a_2 \beta_1 \epsilon_1 - a_1 \beta_2 \lambda_1}{a_2 \beta_1 - a_1 \beta_2} F_1(D_1 r) + M_2 \frac{a_2 \lambda_2 - a_1 \beta_2 \epsilon_2}{a_2 \beta_1 - a_1 \beta_2} F_1(D_2 r), \quad (32)$$

(the dependence on $r$ is omitted for brevity) and

$$F_2(r) = M_1 \frac{\beta_1 \beta_2 \epsilon_1 - a_1 \beta_2 \lambda_1}{a_2 \beta_1 - a_1 \beta_2} F_1(D_1 r) + M_2 \frac{a_2 \beta_1 \lambda_2 - a_1 \beta_2 \epsilon_2}{a_2 \beta_1 - a_1 \beta_2} F_1(D_2 r), \quad (33)$$

where as before, $F_1(r) = 3F_2 \left( \begin{array}{c}
    a, b, c \\
    d, e
  \end{array} \right)$, $F_2(r) = (abc)/(de) 3F_2 \left( \begin{array}{c}
    a + 1, b + 1, c + 1 \\
    d + 1, e + 1
  \end{array} \right)$.

**Proof of Proposition 2.** Solving (21) for each transformation we, in particular, get the system of equations:
works as follows: first step is to calculate transformations
\[ T \] 

An important subclass of these transformations are pure shifts (the principal 5 × 5 matrix to this end we simply iterate transformations λ parameters. To calculate the resulting parameters, respectively, combining them with the necessary permutations of the upper and lower rows, we provide a collection of Mathematica routines in the Appendix A to this paper. Our algorithm works as follows: first step is to calculate transformations \( T_1, T_2 \in T \) associated with the matrices

\[
D_1 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & k_1 \\
0 & 1 & 0 & 0 & 0 & k_2 \\
0 & 0 & 1 & 0 & 0 & k_3 \\
0 & 0 & 0 & 1 & 0 & k_4 \\
0 & 0 & 0 & 0 & 1 & k_5
\end{pmatrix}, \quad D_2 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & m_1 \\
0 & 1 & 0 & 0 & 0 & m_2 \\
0 & 0 & 1 & 0 & 0 & m_3 \\
0 & 0 & 0 & 1 & 0 & m_4 \\
0 & 0 & 0 & 0 & 1 & m_5
\end{pmatrix}.
\]

To this end we simply iterate transformations \( S^+, S^- \) realizing the shifts by ±1 of the first and forth parameters, respectively, combining them with the necessary permutations of the upper and lower parameters. To calculate the resulting λ, a and β the composition rule from Theorem 1 is used with the help of Mathematica routine. Then it remains to apply formula (32). For example, we get:

\[
3F_2 \left( \frac{a, b, c}{d, e} \right) = \frac{d + e - a - b - c - 1}{e} 3F_2 \left( \frac{a + 1, b + 1, c + 1}{d + 1, e + 1} \right) + \frac{(a - d)(d - b)(d - c)}{d(d + 1)e} 3F_2 \left( \frac{a + 1, b + 1, c + 1}{d + 2, e + 1} \right). \]

An important subclass of these transformations are pure shifts (the principal 5 × 5 submatrices of \( D_1, D_2 \) are identity matrices). This subclass comprises the so-called contiguous relations, studied recently in detail in [20]. In particular, Theorem 1.1 from [20] claims the existence of the unique rational functions \( u(r), v(r) \) such that

\[
3F_2 \left( \frac{a, b, c}{d, e} \right) = u(r) 3F_2 \left( \frac{a + k_1 b + k_2 c + k_1}{d + k_4 e + k_5} \right) + v(r) 3F_2 \left( \frac{a + m_1 b + m_2 c + m_3}{d + m_4 e + m_5} \right)
\]

for any two distinct non-zero integer vectors \( (k_1, k_2, k_3, k_4, k_5), (m_1, m_2, m_3, m_4, m_5) \). Furthermore, Ebisu and Iwasaki presented a rather explicit algorithm in [20] for computing the functions \( u(r), v(r) \) for given shifts. Proposition 2 furnishes an alternative method for computing these functions. For its realization we provide a collection of Mathematica routines in the Appendix A to this paper. Our algorithm works as follows: first step is to calculate transformations \( T_1, T_2 \in T \) associated with the matrices
Please note that identity (34) is obtained from (36) by an application of a Thomae relation to the first term on the right hand side. In a similar fashion, contiguous relations and Thomae transformations generate all three-term relations from Proposition 2, induced by the elements of the the group \( \hat{\mathcal{T}} \). We note that the relations covered by Proposition 2 are different from the three-term relations for \( 3F_2 \) summarized by Bailey in ([21], Section 3.7) and studied from group-theoretic viewpoint in ([9], Section IV). This can be seen for example by comparing the matrices ([9], Equation (2.6c)) with the matrices \( D \) associated with \( \hat{\mathcal{T}} \).

The system (21) follows from the representation (13) of \( 4F_3 \) with one unit shift as a linear combination of two \( 3F_2 \) functions. However, Formula (13) is just one example of such decomposition. The two propositions below give many more ways to expand the \( 4F_3 \) with unit shift into linear combination of \( 3F_2 \). Proposition 3 is proved directly in terms of hypergeometric series manipulations as its results will be used below in Section 6 to prove Lemma 1 used to generate the group \( \hat{\mathcal{T}} \).

**Proposition 3.** The following identities hold true:

\[
3F_2\left(a, b, c; d, e\right) + \gamma 3F_2\left(\alpha - 1, b, c; d, e\right) = (\gamma + 1)4F_3\left(\alpha - 1, b, c, \xi + 1; d, e, \xi\right),
\]

where \( \xi = (\gamma + 1)(\alpha - 1) \);

\[
3F_2\left(a, b, c; d, e\right) + \gamma 3F_2\left(\alpha + 1, b, c; d + 1, e\right) = (\gamma + 1)4F_3\left(a, b, c, \nu + 1; d + 1, e, \nu\right),
\]

where \( \nu = (\gamma + 1)d/\gamma d + \alpha \); and

\[
3F_2\left(a, b, c; d, e\right) + \gamma 3F_2\left(a, b + 1, c + 1; d + 1, e + 1\right) = 4F_3\left(a - 1, b, c, \lambda + 1; d, e, \lambda\right),
\]

where \( \lambda = (\alpha - 1)bc/(bc + \gamma de) \).

**Proof of Proposition 3.** We have

\[
3F_2\left(a, b, c; d, e\right) + \gamma 3F_2\left(\alpha - 1, b, c; d, e\right) = 1 + \gamma + \sum_{n=1}^{\infty} \frac{(\alpha)_n(b)_n(c)_n + \gamma(\alpha - 1)_n(b)_n(c)_n}{(d)_n(e)_n n!} = (1 + \gamma) \left(1 + \sum_{n=1}^{\infty} \frac{(\alpha - 1)_n(b)_n(c)_n}{(d)_n(e)_n n!} \left(1 + \frac{n}{(\alpha - 1)(\gamma + 1)}\right)\right) = (\gamma + 1)4F_3\left(a - 1, b, c, \xi + 1; d, e, \xi\right),
\]

where \( \xi = (\gamma + 1)(\alpha - 1) \) and we used \((\alpha)_n = (\alpha - 1)_n(1 + n/(\alpha - 1))\). Next,

\[
3F_2\left(a, b, c; d, e\right) + \gamma 3F_2\left(\alpha + 1, b, c; d + 1, e\right) = 1 + \gamma + \sum_{n=1}^{\infty} \frac{(\alpha)_n(b)_n(c)_n}{(d + 1)_n(e)_n n!} \left(1 + \frac{n}{d} + \gamma + \frac{\gamma n}{\alpha}\right) = (\gamma + 1)4F_3\left(a, b, c, \nu + 1; d + 1, e, \nu\right),
\]

where \( \nu = (\gamma + 1)d/\gamma d + \alpha \) and we used \((\alpha + 1)_n = (\alpha)_n(1 + n/\alpha)\).

Finally, using the obvious identities \((b)_n = b(b + 1)_n - 1\) and \((\alpha)_n = (\alpha - 1)_{n+1}/(\alpha - 1)\) we get
where \( \lambda = (\alpha - 1)bc / (bc + \gamma de) \). \( \square \)

Other ways to represent \( 4F_3 \) with one unit shift as a linear combination of \( 3F_2 \) are found by substituting (32) and (33) into (13). This is done in the following proposition.

**Proposition 4.** Any two transformations from the group \( T : T_1 \sim \{ e_1, M_1(r), \lambda_1(r), \alpha_1(r), \beta_1(r), D_1 \} \) and \( T_2 \sim \{ e_2, M_2(r), \lambda_2(r), \alpha_2(r), \beta_2(r), D_2 \} \) satisfying the condition \( \alpha_2 \beta_1 - \alpha_1 \beta_2 \neq 0 \) (for brevity we omit the dependence on \( r \) in the parameters) induce the decomposition

\[
4F_3 \left( \frac{a, b, c, f + 1}{d, e, f} \right) = M_1 \frac{\beta_1 \alpha_1}{\alpha_2 \beta_1 - \alpha_1 \beta_2} \left( a_2 + \frac{\beta_2}{f} \right) F_1(D_1 r) + M_2 \frac{\alpha_2 \lambda_2 - \beta_2 \alpha_2}{\alpha_2 \beta_1 - \alpha_1 \beta_2} \left( a_1 + \frac{\beta_1}{f} \right) F_1(D_2 r),
\]

where \( F_1(r) = 3F_2 \left( \frac{a, b, c}{d, e} \right) \).

Let us exemplify (40) with the following two decompositions:

\[
4F_3 \left( \frac{a, b, c, f + 1}{d, e, f} \right) = \left( \frac{d + e - a - b - c - 1}{e} + \frac{abc}{def} \right) 3F_2 \left( \frac{a + 1, b + 1, c + 1}{d + 1, e + 1} \right) + \frac{(a - d)(d - b)(d - c)}{ed(1 + d)} 3F_2 \left( \frac{a + 1, b + 1, c + 1}{d + 2, e + 1} \right)
\]

and

\[
4F_3 \left( \frac{a, b, c, f + 1}{d, e, f} \right) = A 3F_2 \left( \frac{a + 1, b, c}{d, e} \right) + B 3F_2 \left( \frac{a + 1, b + 1, c + 1}{d + 2, e + 1} \right),
\]

where

\[
A = 1 + \frac{bc(f - a)}{f(b(d - c) - d(e - a - c - 1))}, \quad B = \frac{bc(a - d)(b - d)(c - d)(f - a)}{def(1 + d)(b(c - d) + d(d + e - a - c - 1))}.
\]
5. Summation Formulas

In ([22], Equation (45)) we established the following summation formula

\[
{4F_3}\left(\begin{array}{c} a, b, c, f + 1 \\ d, e, f \end{array}\right) = \frac{\Gamma(d)\Gamma(e)}{\Gamma(a + 1)\Gamma(b + 1)\Gamma(c + 1)},
\]

valid if

\[
e_1(d, e) - e_1(a, b, c) = 2 \quad \text{and} \quad f = \frac{e_3(a, b, c)}{e_2(a, b, c) - e_2(1 - d, 1 - e)},
\]

where \(e_k(\cdot)\) denotes the \(k\)-th elementary symmetric polynomial. Now, if we apply any transformation of the form (15) and impose the above restrictions on the parameters on the right hand side, we obtain

\[
{4F_3}\left(\begin{array}{c} a, b, c, f + 1 \\ d, e, f \end{array}\right) = M(r)\frac{\epsilon_f + \lambda(r)}{f} F(q, \eta) = \frac{M(r)(\epsilon_f + \lambda(r))\Gamma(q_4)\Gamma(q_5)}{f\Gamma(q_1 + 1)\Gamma(q_2 + 1)\Gamma(q_3 + 1)},
\]

where \((q_1, q_2, q_3, q_4, q_5, 1) = Dr\), and the conditions \(e_1(q_4, q_5) - e_1(q_1, q_2, q_3) = 2\) and

\[
\eta = \frac{\epsilon_f + \lambda(r)}{\alpha(r)f + \beta(r)} = \frac{e_3(q_1, q_2, q_3)}{e_2(q_1, q_2, q_3) - e_2(1 - q_4, 1 - q_5)}
\]

must hold. Expressing \(f\) they are equivalent to

\[
e_1(q_4, q_5) - e_1(q_1, q_2, q_3) = 2 \quad \text{and} \quad f = \frac{\lambda(r)(e_2(q_1, q_2, q_3) - e_2(1 - q_4, 1 - q_5))}{\alpha(r)e_3(q_1, q_2, q_3) - \epsilon(e_2(q_1, q_2, q_3) - e_2(1 - q_4, 1 - q_5))}. \tag{42b}
\]

As \(q_i = q_i(a, b, c, d, e), i = 1, \ldots, 5\), are linear functions we arrive at the following proposition:

**Proposition 5.** Each transformation \(T \in \mathcal{T}\) as characterized by the collection \(\{\epsilon, M(r), \lambda(r), \alpha(r), \beta(r), D\}\) corresponds to a summation formula (42a) valid under restrictions (42b) with \((q_1, \ldots, q_5, 1) = Dr\).

We will illustrate Proposition 5 by applying it to transformation (25). First condition in (42b) becomes \(e = c + 2\). In view of this condition formula (42a) takes the form

\[
{4F_3}\left(\begin{array}{c} a, b, c, f + 1 \\ d, e, f \end{array}\right) = \frac{(c + 1)\Gamma(d)\Gamma(d - a - b + 2)(f\psi + c(a + b - d))}{\Gamma(d - a + 1)\Gamma(d - b + 1)f\psi},
\]

where \(\psi = d - a - b + 1\) and, by the second condition in (42b),

\[
f = -\frac{c(a + b - d)}{\psi} + \frac{(d - a)(d - b)c}{\psi((d - a)(d - b + c) + (d - b)c + (d - 1)(a + b - d - c - 1))}.
\]

Further examples will be given in [23].

6. Proof of Lemma 1

Write identity (13) in expanded form

\[
{4F_3}\left(\begin{array}{c} a, b, c, f + 1 \\ d, e, f \end{array}\right) = 3F_2\left(\begin{array}{c} a, b, c \\ d, e \end{array}\right) + \frac{abc}{fde} 3F_2\left(\begin{array}{c} a + 1, b + 1, c + 1 \\ d + 1, e + 1 \end{array}\right). \tag{43}
\]
Applying Thomae’s transformation (3) to both \(3F_2\) functions on the right hand side, we get (\(\psi = d + e - a - b - c - 1\)):

\[
4F_3\left(\frac{a, b, c, f + 1}{d, e, f}\right) = \frac{\Gamma(\psi + 1)\Gamma(d)\Gamma(e)}{\Gamma(a)\Gamma(\psi + b + 1)\Gamma(\psi + c + 1)} \times \left[3F_2\left(\frac{\psi + 1, d - a, e - a}{\psi + b + 1, \psi + c + 1}\right) + \frac{bc}{f\psi}3F_2\left(\frac{\psi, d - a, e - a}{\psi + b + 1, \psi + c + 1}\right)\right].
\]

Now we employ Proposition 3. Application of Formula (37) to the linear combination in brackets yields

\[
4F_3\left(\frac{a, b, c, f + 1}{d, e, f}\right) = \frac{(f\psi + bc)\Gamma(\psi)\Gamma(d)\Gamma(e)}{f\Gamma(a)\Gamma(\psi + b + 1)\Gamma(\psi + c + 1)}4F_3\left(\frac{\psi, d - a, e - a, \eta + 1}{\psi + b + 1, \psi + c + 1, \eta}\right),
\]

where \(\eta = \psi + bc/f\). This proves transformation given by (7).

In a similar fashion, if we apply the Kummer transformation (2) to \(3F_2\) on the right hand side of (43) we get:

\[
4F_3\left(\frac{a, b, c, f + 1}{d, e, f}\right) = \frac{\Gamma(\psi + 1)\Gamma(d)}{\Gamma(d - a)\Gamma(\psi + a + 1)} \left[3F_2\left(\frac{a, e - b, e - c}{\psi, \psi + a + 1}\right) + \frac{abc}{f\psi}3F_2\left(\frac{a + 1, e - b, e - c}{\psi + 1, \psi + a + 1}\right)\right].
\]

Applying the relation (38) to the linear combination in brackets we then obtain

\[
4F_3\left(\frac{a, b, c, f + 1}{d, e, f}\right) = \frac{(abc + f\psi)\Gamma(\psi)\Gamma(d)}{fe\Gamma(d - a)\Gamma(\psi + a + 1)}4F_3\left(\frac{a, e - b, e - c, \lambda + 1}{\psi + 1, \psi + a + 1, \lambda}\right),
\]

where

\[
\lambda = \frac{abc + f\psi}{bc + f\psi}.
\]

This proves transformation (8).

**Author Contributions:** The authors contributed equally to this work. Both authors have read and agreed to the published version of the manuscript.

**Funding:** The second author was funded by the Ministry of Science and Higher Education of the Russian Federation (supplementary agreement No. 075-02-2020-1482-1 of 21 April 2020).

**Conflicts of Interest:** The authors declare no conflict of interest.

**Appendix A**

In this appendix we will display the explicit form of the main building blocks needed for calculating the elements of the group \(T\). Just as it stands for Thomae’s transformations ([4], Appendix 1), we have ten different identities with zero shifts. They are obtained as follows: permuting \(a \leftrightarrow b\) and \(a \leftrightarrow c\) in Formula (7) we get three transformations, while \(a \leftrightarrow b, a \leftrightarrow c\) and \(d \leftrightarrow e\) in (6) leads to six more transformations. Adding the identity transformation we arrive at ten “Thomae-like” zero-shift transformations for \(4F_3\) containing the parameter pair \(\left[\frac{f + 1}{f}\right]\). The entire 120 element subgroup of
“Thomae-like” zero-shift transformations is obtained by the obvious 12 permutations of three top and two bottom parameters on the right hand side of each of the ten transformations described above.

All further transformations are obtained by consecutive application of the four shifting transformations \( S^+ \), \( S^- \) and permutations of top and bottom parameters to the 120 transformations described above. Transformation \( S^+ \) shifting the top parameter \( a \) by +1 (denoted by \( S_1 \) in Section 3) is given by (30). Combining parameters it can be written as:

\[
\mathbf{4F}_3 \left( \frac{a, b, c, f + 1}{d, e, f} \right) = \left( \frac{bc}{(d - a - 1)(e - a - 1)} \right) \left( \frac{1}{1 + \frac{\lambda}{\psi F}} \right) \mathbf{4F}_3 \left( \frac{a + 1, b, c, \eta + 1}{d, e, \eta} \right),
\]

where

\[
\lambda = \frac{a b c}{a(2 + a - d - e) - b c + (d - 1)(e - 1)}, \quad \eta = \frac{abc + ((a + 1)(a + 1 - d - e) - b c + d e) f}{(a - f)(2 + a + b + c - d - e)}.
\]

Its inverse \( S^- \) is given by:

\[
\mathbf{4F}_3 \left( \frac{a, b, c, f + 1}{d, e, f} \right) = \left( \frac{bc}{\psi f} \right) \mathbf{4F}_3 \left( \frac{a - 1, b, c, \eta + 1}{d, e, \eta} \right),
\]

where

\[
\eta = \frac{(a - 1)(b c + \psi f)}{a(2 + b c - a) + b c - d e + \psi f}.
\]

The transformation \( S_+ \) shifting the bottom parameter \( d \) by +1 (denoted by \( S_4 \) in Section 3) is given by (31). It can be written more compactly as

\[
\mathbf{4F}_3 \left( \frac{a, b, c, f + 1}{d, e, f} \right) = \frac{abc + \psi d f}{\psi d f} \mathbf{4F}_3 \left( \frac{a, b, c, \eta + 1}{d + 1, e, \eta} \right),
\]

where \( \psi = e + d - a - b - c - 1 \) and

\[
\eta = \frac{abc + \psi d f}{d (d - a - b - c) + a b + a c + b c + \psi f}.
\]

Finally, its inverse transformation \( S_- \) shifting a bottom parameter by \(-1\) has the form

\[
\mathbf{4F}_3 \left( \frac{a, b, c, f + 1}{d, e, f} \right) = \frac{[\{(d - b - 1)(d - c - 1) - a(d - b - c - 1)\}f - abc](d - 1)}{(d - a - 1)(d - b - 1)(d - c - 1)f} \mathbf{4F}_3 \left( \frac{a, b, c, \eta + 1}{d - 1, e, \eta} \right),
\]

where

\[
\eta = \frac{abc + \{(1 - d)(d - a - b - c - 1) - ab + a c + b c\} f}{(d + e - a - b - c - 2)(f - d + 1)}.
\]

In the remaining part of the Appendix we present several Wolfram Mathematica® routines intended for dealing with the group \( T \) together with an example of their use. Listing A1 contains the function CMPS\([T_1, T_2]\) that takes as input two transformations \( T_1, T_2 \) and computes their composition \( T_2 \circ T_1 \). The form in which the parameters \( \epsilon_i, M_i, \lambda_i, \alpha_i, \beta_i \) and \( D_i, i = 1, 2, \) should be supplied can be seen from the example in Listing A5. Similarly, Listing A2 contains the function INV\([T]\) that computes the inverse of a given transformation \( T \). The output provided by CMPS and INV can be printed in an easily readable
form using the function $PRN[T]$ given in Listing A3. The same Listing A3 contains the function $INPT[T]$ that converts the output form of the functions CMPS and INV into the input form of the same functions, so that further compositions or inverses could be computed from such output. For numerical verification of the outputs of CMPS and INV the function $RHS[T]$ presented in Listing A4 converts these outputs into an expression that can be evaluated by the Mathematica function $N[...]$ after the parameters have been assigned some numerical values, see an example at the end of Listing A5.

Listing A1. Composition.

```
CMPS[T1_, T2_] := Module[{eps1 = T1[[1]], M1 = T1[[2]], lam1 = T1[[3]], alpha1 = T1[[4]],
    beta1 = T1[[5]], D1 = T1[[6]], eps2 = T2[[1]], M2 = T2[[2]], lam2 = T2[[3]], alpha2 = T2[[4]], beta2 = T2[[5]],
    D2 = T2[[6]], R = {{a}, {b}, {c}, {d}, {e}, {1}}},
    RR = Flatten[Drop[R, {6}]],
    If [ Simplify[eps1 + eps2 + alpha1*RR + lam2*Flatten[Drop[D1.R, {6}]]] == 0,
        {0, FullSimplify[10*RR + lam2*Flatten[Drop[D1.R, {6}]] + beta1*RR]],
        Simplify[eps1 + alpha1*RR + lam2*Flatten[Drop[D1.R, {6}]]]
        + beta1*RR + lam2*Flatten[Drop[D1.R, {6}]] + beta1*RR, Simplify[alpha1*RR + lam2*Flatten[Drop[D1.R, {6}]]]
        + beta1*RR + lam2*Flatten[Drop[D1.R, {6}]]] == 0,
        {eps1 + eps2 + alpha1*RR + lam2*Flatten[Drop[D1.R, {6}]]],
        Simplify[eps1 + eps2 + alpha1*RR + lam2*Flatten[Drop[D1.R, {6}]]])
    ]]
```

Listing A2. Inversion.

```
INV[T1_] := Module[{eps = TT[[1]], M = TT[[2]], lam = TT[[3]], alpha = TT[[4]], beta = TT[[5]], D = TT[[6]],
    R = {{a}, {b}, {c}, {d}, {e}, {1}}},
    RR = Flatten[Drop[R, {6}]],
    If [ Simplify[beta*RR] == 0,
        {0, FullSimplify[Inverse[D]*R, D, R]}
        + lam*Flatten[Drop[Inverse[D], R, 6]]],
        Simplify[Inverse[D] + lam*Flatten[Drop[Inverse[D], R, 6]]]
        + lam*Flatten[Drop[Inverse[D], R, 6]],
        Simplify[Inverse[D] + lam*Flatten[Drop[Inverse[D], R, 6]]] + eps*lam*Flatten[Drop[Inverse[D], R, 6]]
        + lam*Flatten[Drop[Inverse[D], R, 6]],
        Simplify[Inverse[D] + lam*Flatten[Drop[Inverse[D], R, 6]]] + eps*lam*Flatten[Drop[Inverse[D], R, 6]]
        + lam*Flatten[Drop[Inverse[D], R, 6]],
        Simplify[Inverse[D] + lam*Flatten[Drop[Inverse[D], R, 6]]] + eps*lam*Flatten[Drop[Inverse[D], R, 6]]
        + lam*Flatten[Drop[Inverse[D], R, 6]]]]
```

Listing A3. Conversion into input form and printing.

```
exprToFunction[expr_, vars_] := ToExpression[ToString[FullForm[expr]].MapIndexed[#1 -> Slot @@ #2 &, vars]] <> "&";
```

```
expToFunction[expr_, vars_] := ToExpression[ToString[FullForm[expr]].MapIndexed[#1 -> Slot @@ #2 &, vars]] <> "&";
```

```
expToFunction[expr_, vars_] := ToExpression[ToString[FullForm[expr]].MapIndexed[#1 -> Slot @@ #2 &, vars]] <> "&";
```

Listing A4. Conversion into computable form.

```
RHS[T1_] := Simplify[TT[[2]] + (TT[[1]] + TT[[3]])/f, HypoGeometricPFQ[Join[Flatten[Drop[TT[[6]], {a}, {b}, {c}, {d}, {e}, {1}], {1}, {6}], {1}, {6}], {1}, {6}] + ETA[TT] + 1],
    Join[Flatten[Drop[TT[[6]], {a}, {b}, {c}, {d}, {e}, {1}], {1}, {6}], {1}, {6}] + ETA[TT]]];
```
Listing A5. Example of use.

(* Definition of the first transformation *)
eps1=1; M1[a_,b_,c_,d_,e_]:=Gamma[d+e-a-b-c]/Gamma[d]/Gamma[e]/Gamma[d+e-a-c]/Gamma[d+e-a-b];
lam1[a_,b_,c_,d_,e_]:=b/(d+e-a-b-c-1); alpha1[a_,b_,c_,d_,e_]:=1/(d+e-a-b-c-1);

(* Definition of the second transformation *)
eps2=1; M2[a_,b_,c_,d_,e_]:=Gamma[d+e-a-b-c]/Gamma[e]/Gamma[e-c]/Gamma[d+e-a-b-c-1];
lam2[a_,b_,c_,d_,e_]:=(a+b-1)/Gamma[a+b-e-d]; alpha2[a_,b_,c_,d_,e_]:=(a+b-1)/(a+b-e-d);

(* composition T2T1 *)
T1T2=CMPS[{eps1, M1, lam1, alpha1, beta1, D1}, {eps2, M2, lam2, alpha2, beta2, D2}];

(* Inverse of T1 *)
T1INV = INV[{eps1, M1, lam1, alpha1, beta1, D1}];

(* Printing the parameters of T2T1 *)
PRN[T1T2]
epsilon=1
M=-(a c+(1+b-d) e)Gamma[d]/(e Gamma[d+e-a-b-c-1])/(Gamma[-a-b-c-d+e])
Lambda=-((a b c)/(a c+(1+b-d)e))
alpha=(1+b-d)/(a c+e+b e-d)

(* Numerical verification of the transformation NEW using RHS[...] *)
a=1+2/3; b=-13/17+2; c=3/7; d=5/11; e=5+44/17; f=12/13;
In[51]:= N[HypergeometricPFQ[{a, b, c, f+1}, {d, e, f}, 1], 15]
Out[51]= 2.22268615827388
In[52]:= N[RHS[NEW], 15]
Out[52]= 2.22268615827388

References
1. Andrews, G.E.; Askey, R.; Roy, R. Special Functions; Cambridge University Press: Cambridge, UK, 1999.
2. Krattenthaler, C.; Srinivasa Rao, K. On group theoretical aspects, hypergeometric transformations and symmetries of angular momentum coefficients. In Symmetries in Science XI; Kluwer Acad. Publ.: Dordrecht, The Netherlands, 2005; pp. 355–375.
3. Rao, K.S. Hypergeometric series and Quantum Theory of Angular Momentum. In Selected Topics in Special Functions; Agarwal, R.P., Manocha, H.L., Srinivasa Rao, K., Eds.; Allied Publishers Ltd.: New Delhi, India, 2001; pp. 93–134.
4. Rao, K.S.; Doebner, H.D.; Natterman, P. Generalized hypergeometric series and the symmetries of 3 – j and 6 – j coefficients. In Number Theoretic Methods. Developments in Mathematics; Kanemitsu, S., Jia, C., Eds.; Springer: Boston, MA, USA, 2002; Volume 8, pp. 381–403
5. Rao, K.S.; Lakshminarayanan, V. Generalized Hypergeometric Functions, Transformations and Group Theoretical Aspects; IOP Science: Bristol, UK, 2018.
6. Shpot, M.A.; Srivastava, H.M. The Clausenian hypergeometric function $\text{}_3F_2$ with unit argument and negative integral parameter differences. *Appl. Math. Comput.* 2015, 259, 819–827.

7. Formichella, M.; Green, R.M.; Stade, E. Coxeter group actions on $\text{}_4F_3(1)$ hypergeometric series. *Ramanujan J.* 2011, 24, 93–128. [CrossRef]

8. Olver, F.W.J.; Lozier, D.W.; Boisvert, R.F.; Clark, C.W. (Eds.) *NIST Handbook of Mathematical Functions*; Cambridge University Press: Cambridge, UK, 2010.

9. Beyer, W.A.; Louck, J.D.; Stein, P.R. Group theoretical basis of some identities for the generalized hypergeometric series. *J. Math. Phys.* 1987, 28, 497–508. [CrossRef]

10. Hardy, G.H. Ramanujan: Twelve Lectures on Subjects Suggested by His Life and Work; AMS Chelsea Pub.: Providence, RI, USA, 1999; p. 111.

11. Green, R.M.; Mishev, I.D.; Stade, E. Coxeter group actions and limits of hypergeometric series. *Ramanujan J.* 2020. [CrossRef]

12. Mishev, I.D. Coxeter group actions on Saalschützian $\text{}_4F_3(1)$ series and very-well-poised $\text{}_7F_6(1)$ series. *J. Math. Anal. Appl.* 2012, 385, 1119–1133. [CrossRef]

13. Rao, K.S.; Van der Jeugt, J.; Raynal, J.; Jagannathan, R.; Rajeswari, V. Group theoretical basis for the terminating $\text{}_3F_2(1)$ series. *J. Phys. A Math. Gen.* 1992, 25, 861–876. [CrossRef]

14. Van der Jeugt, J.; Rao, K.S. Invariance groups of transformations of basic hypergeometric series. *J. Math. Phys.* 1999, 40, 6692–6700. [CrossRef]

15. Nørlund, N.E. Hypergeometric functions. *Acta Math.* 1955, 94, 289–349. [CrossRef]

16. Olsson, P.O.M. Analytic continuation of higher-order hypergeometric functions. *J. Math. Phys.* 1966, 7, 702–710. [CrossRef]

17. Bühring, W. Generalized hypergeometric functions at unit argument. *Proc. Am. Math. Soc.* 1992, 114, 145–153. [CrossRef]

18. Kim, Y.S.; Rathie, A.K.; Paris, R.B. On two Thomae-type transformations for hypergeometric series with integral parameter differences. *Math. Commun.* 2014, 19, 111–118.

19. Karp, D.B.; Prilepkina, E.G. Beyond the beta integral method: transformation formulas for hypergeometric functions via Meijer’s $G$ function. *arXiv* 2019, arXiv:1912.11266.

20. Ebisu, A.; Iwasaki, K. Three-term relations for $\text{}_3F_2(1)$. *J. Math. Anal. Appl.* 2018, 463, 593–610. [CrossRef]

21. Bailey, W.N. *Generalized Hypergeometric Series*; Stecherthafner Service Agency: New York, NY, USA; London, UK, 1964; Reprinted from: Cambridge Tracts in Mathematics and Mathematical Physics, 1935, Volume 32.

22. Karp, D.B.; Prilepkina, E.G. Degenerate Miller-Paris transformations. *Results Math.* 2019, 74. [CrossRef]

23. Çetinkaya, A.; Karp, D. Summation formulas for some hypergeometric and some digamma series. *Commun. Korean Math. Soc.* 2020, in preparation.

Publisher’s Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.

© 2020 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0/).