A fractional supersymmetric oscillator
and its coherent states

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Abstract
We review some basic elements on \(k\)-fermions, which are objects interpolating between bosons and fermions. In particular, we define \(k\)-fermionic coherent states and study some of their properties. The decomposition of a \(Q\)-uon into a boson and a \(k\)-fermion leads to a definition of fractional supercoherent states. Such states involve bosonic coherent states and \(k\)-fermionic coherent states. We construct an Hamiltonian which generalizes the ordinary (or \(Z_2\)-graded) supersymmetric oscillator Hamiltonian. Our Hamiltonian describes a fractional (or \(Z_k\)-graded) supersymmetric oscillator for which the fractional supercoherent states are coherent states.

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1 Introduction

In the last ten years, a considerable attention has been paid to fractional supersymmetric (and para-supersymmetric) quantum mechanics [1-17]. Fractional supersymmetry corresponds to a $Z_k$-grading and can be realized in terms of generalized Grassmann variables [1,3,18,19] or, in the spirit of $q$-deformations of the oscillator algebra [20-24], in terms of para-fermionic variables [1-17] or $k$-fermionic variables [25,26]. On the other hand, $q$-deformed Glauber coherent states have been studied in several physical contexts [27-31].

It is the aim of this paper to construct the Hamiltonian for one of the simplest fractional supersymmetric system, viz., the $Z_k$-graded supersymmetric oscillator. The construction of the Hamiltonian and its coherent states lies on the decomposition of a $Q$-ion into a boson and a $k$-fermion.

2 On $k$-Fermionic Operators

2.1 The $k$-fermionic algebra $F_k$

Let us start by defining what we shall refer to a $k$-fermionic algebra $F_k$. The algebra $F_k$ is spanned by five operators $f_-, f_+, f_+^+, f_-^+$ and $N$ through the following relations classified in three types.

(i) The $[f_-, f_+, N]$-type:

\[ f_- f_+ - q f_+ f_- = 1 \]
\[ N f_- - f_- N = -f_- , \quad N f_+ - f_+ N = +f_+ \]
\[ (f_-)^k = (f_+)^k = 0 \]

(ii) The $[f_+^+, f_-^+, N]$-type:

\[ f_+^+ f_-^+ - q f_-^+ f_+^+ = 1 \]
\[ N f_+^+ - f_+^+ N = -f_+^+, \quad N f_-^+ - f_-^+ N = +f_-^+ \]
\[ (f_+^+)^k = (f_-^+)^k = 0 \]

(iii) The $[f_-, f_+, f_+^+, f_-^+]$-type:

\[ f_- f_+^+ - q^{-\frac{1}{2}} f_+^+ f_- = 0 , \quad f_+ f_-^+ - q^{+\frac{1}{2}} f_-^+ f_+ = 0 \]

where the number

\[ q := \exp\left(\frac{2\pi i}{k}\right) , \quad k \in \mathbb{N} \setminus \{0, 1\} \]

is a root of unity and $\bar{q}$ stands for the complex conjugate of $q$. The couple $(f_-, f_+^+)$ of annihilation operators is connected to the couple $(f_+, f_-^+)$ of creation operators via the Hermitean conjugation relations

\[ f_+^+ = (f_+)^\dagger, \quad f_-^+ = (f_-)^\dagger \]
and $N$ is an Hermitean operator. It is clear that the case $k = 2$ corresponds to fermions and the case $k \to \infty$ to bosons. In the two latter cases, we can take $f_- \equiv f_+^\dagger$ and $f_+ \equiv f_-^\dagger$. In the other cases, the consideration of the two couples $(f_-, f_+^\dagger)$ and $(f_+, f_-^\dagger)$ is absolutely necessary. In the case where $k$ is arbitrary, we shall speak of $k$-fermions.

### 2.2 A representation of $F_k$

A $k$-dimensional representation of $F_k$, on a $k$-dimensional Hilbert space spanned by the orthonormal set $\{|n\rangle : n = 0, 1, \cdots, k - 1\}$, is easily obtained from

\[
\begin{align*}
f_-|n\rangle &= \left(\left[n + s - \frac{1}{2}\right]_q\right)^{\frac{1}{2}}|n - 1\rangle \quad \text{with} \quad f_-|0\rangle = 0 \\
f_+|n\rangle &= \left(\left[n + s + \frac{1}{2}\right]_q\right)^{\frac{1}{2}}|n + 1\rangle \quad \text{with} \quad f_+|k - 1\rangle = 0 \\
f_+^\dagger|n\rangle &= \left(\left[n + s - \frac{1}{2}\right]_q\right)^{\frac{1}{2}}|n - 1\rangle \quad \text{with} \quad f_+^\dagger|0\rangle = 0 \\
f_-^\dagger|n\rangle &= \left(\left[n + s + \frac{1}{2}\right]_q\right)^{\frac{1}{2}}|n + 1\rangle \quad \text{with} \quad f_-^\dagger|k - 1\rangle = 0
\end{align*}
\]

and

\[N|n\rangle = n|n\rangle\]

The notation is as follows: we have $s := \frac{1}{2}$ and the factorials $[n]_q!$ and $[n]_{\bar{q}}!$ are defined by

\[ [n]_p! := [1]_p[2]_p \cdots [n]_p \quad \text{for} \quad n \in \mathbb{N}^* \quad \text{and} \quad [0]_p! := 1 \]

with

\[ [x]_p := \frac{1 - p^x}{1 - p} \quad \text{for} \quad x \in \mathbb{R} \]

where $p = q, \bar{q}$. Note that the action of $f_-, f_+, f_+^\dagger$ and $f_-^\dagger$ on $|n\rangle$ is reminiscent of finite quantum mechanics discussed by many authors in various domains (for example, see [32] and [33]).

### 2.3 A Grassmanian realization of $F_k$

It is possible to find a realization of the operators $f_-, f_+, f_+^\dagger$ and $f_-^\dagger$ in terms of Grassmann variables $(\theta, \bar{\theta})$ and their $q$- and $\bar{q}$-derivatives ($\partial_\theta, \partial_{\bar{\theta}}$). We take Grassmann variables $\theta$ and $\bar{\theta}$ such that $\theta^k = \bar{\theta}^k = 0 \ [1,3,18,19]$. The sets $\{1, \theta, \cdots, \theta^{k-1}\}$ and $\{1, \bar{\theta}, \cdots, \bar{\theta}^{k-1}\}$ span the same Grassmann algebra $\Sigma_k$. The $q$- and $\bar{q}$-derivatives are formally defined by

\[
\partial_\theta f(\theta) := \frac{f(q\theta) - f(\theta)}{(q - 1)\theta}
\]
\[
\frac{\partial_{\bar{q}} g(\bar{\theta})}{(\bar{q} - 1)\bar{\theta}} := \frac{g(q\bar{\theta}) - g(\bar{\theta})}{(q - 1)\theta}
\]

Therefore, by taking
\[
f_+ = \theta, \quad f_- = \partial_{\theta}, \quad f^+_\bar{\theta} = \bar{\theta}, \quad f^+_{\bar{\theta}} = \partial_{\bar{\theta}}
\]
we have
\[
\partial_{\theta}\theta - q\theta\partial_{\theta} = 1, \quad (\partial_{\theta})^k = \theta^k = 0 \\
\partial_{\bar{\theta}}\bar{\theta} - q\bar{\theta}\partial_{\bar{\theta}} = 1, \quad (\partial_{\bar{\theta}})^k = \bar{\theta}^k = 0 \\
\partial_{\theta}\partial_{\bar{\theta}} - q^{1/2}\partial_{\theta}\partial_{\bar{\theta}} = 0, \quad \theta\bar{\theta} - q^{1/2}\theta\bar{\theta} = 0
\]

Following Majid and Rodríguez-Plaza \[19\], we define the integration process
\[
\int d\theta \theta^n = \int d\bar{\theta} \bar{\theta}^n := 0 \quad \text{for} \quad n = 0, 1, \ldots, k - 2
\]
and
\[
\int d\theta \theta^{k-1} = \int d\bar{\theta} \bar{\theta}^{k-1} := 1
\]
which gives the Berezin integration for the particular case \( k = 2 \).

3 Coherent States

3.1 The \( k \)-fermionic coherent states

We now define the states
\[
|\theta\rangle := \sum_{n=0}^{k-1} \frac{\theta^n}{[n]_q^{1/2}} |n\rangle
\]
and
\[
|\bar{\theta}\rangle := \sum_{n=0}^{k-1} \frac{\bar{\theta}^n}{[n]_{\bar{q}}^{1/2}} |n\rangle
\]
as (finite) linear combinations of the eigenvectors \(|n\rangle\) of the operator \( N \). The states \(|\theta\rangle\) and \(|\bar{\theta}\rangle\) are \( k \)-fermionic coherent states in the sense that they satisfy the eigenvalue equations
\[
f_- |\theta\rangle = \theta |\theta\rangle, \quad f^+_\bar{\theta} |\bar{\theta}\rangle = \bar{\theta} |\bar{\theta}\rangle
\]
Similarly, we define
\[
\langle\theta| := \sum_{n=0}^{k-1} \langle n| \frac{\theta^n}{[n]_q^{1/2}} \]
and
\[
\langle\bar{\theta}| := \sum_{n=0}^{k-1} \langle n| \frac{\bar{\theta}^n}{[n]_{\bar{q}}^{1/2}}
\]
which are the dual states of the coherent states $|\theta\rangle$ and $|\bar{\theta}\rangle$, respectively.

Note that, in the spirit of the works by Wang, Kuang and Zeng [31], it is also possible to define even and odd $k$-fermionic coherent states, which are eigenstates of $(f_-)^2$ and $(f_+)^2$, and more generally $m$-components of $k$-fermionic coherent states, which are eigenstates of $(f_-)^m$ and $(f_+)^m$.

We easily get the three following results.

**Result 1.** We have

$$(\theta'|\theta) = e_p(\bar{\theta}\theta), \quad (\bar{\theta}'|\bar{\theta}) = e_p(\theta\bar{\theta}')$$

where the $p$-deformed exponential $e_p$ is defined by

$$e_p : X \mapsto e_p(X) := \sum_{n=0}^{k-1} \frac{X^n}{[n]_p!}$$

with $p = q, \bar{q}$.

**Result 2.** We have the over-completeness property

$$\int \int d\theta |\theta\rangle \mu(\theta, \bar{\theta}) (\theta|d\bar{\theta} = \int \int d\bar{\theta} |\bar{\theta}\rangle \mu(\bar{\theta}, \theta) (\bar{\theta}|d\theta = 1$$

where $\mu$ is defined via

$$\mu(\theta, \bar{\theta}) := \sum_{n=0}^{k-1} ([n]_q!/[n]_{\bar{q}}!)^{1/2} \theta^{k-1-n} \bar{\theta}^{k-1-n}$$

and the integrals have to be understood in terms of the above-mentioned integration process.

**Result 3.** By defining the coherence factor $g^{(m)}$, of order $m$, as

$$g^{(m)} := \frac{(\theta|\left(f_+\right)^m (f_-)^m |\theta)}{(\theta|f_+ f_- |\theta)^m}$$

we obtain

$$|g^{(m)}| = \begin{cases} 0 & \text{for } m > k - 1 \\ 1 & \text{for } m \leq k - 1 \end{cases}$$

and we conclude that a $k$-fermionic state cannot be occupied by more than $k - 1$ identical $k$-fermions, a statement that induces a generalized Pauli exclusion principle.

### 3.2 On the $Q$-uon → boson + $k$-fermion decomposition

We know that a pair of $Q$-uons (with $Q$ generic) can give rise to a pair of bosons and a pair of $q$-uons (with $q$ a root of unity) by making use of a limiting procedure where $Q \rightarrow q$ [34]. This is quite well-known in the case of the Macfarlane [21] (and
Biedenharn [22]) Q-rons. The limiting procedure can be adapted to the case of the Arik and Coon [20] Q-rons in the following way. We begin with a pair of Q-rons $(a_-, a_+)$ satisfying

$$a_- a_+ - Q a_+ a_- = 1$$

where $Q$ is generic. We assume that

$$Q \to q := \exp\left(\frac{2\pi i}{k}\right), \quad k \in \mathbb{N} \setminus \{0, 1\}$$

and then we take

$$(a_\pm)^k = 0$$

If we define

$$b_\pm := \lim_{Q \to q} \frac{(a_\pm)^k}{(|k| q!)^{1/2}}$$

we get the result

$$b_- b_+ - b_+ b_- = 1$$

so that the pair $(b_-, b_+)$ is a pair of ordinary bosons. We redefine $a_\pm$ as $f_\pm$ for $Q = q$. Therefore, we also have a pair of $k$-fermions $(f_-, f_+)$ satisfying

$$f_- f_+ - q f_+ f_- = 1$$

It can be proved that the $b$’s commute with the $f$’s. As a conclusion, we have generated the set $\{b_-, b_+, f_-, f_+\}$ from the set $\{a_-, a_+\}$. Indeed, the decomposition $\{a_-, a_+\} \to \{b_-, b_+, f_-, f_+\}$ corresponds to the $Z$-line $\leftrightarrow (z, \theta)$-superspace isomorphism described by Dunne et al. [34] and Mansour et al. [35].

### 3.3 Fractional supercoherent states

We may question: what happens to an ordinary $Q$-deformed coherent state

$$|Z\rangle := \sum_{n=0}^{\infty} \frac{Z^n}{(n|Q|)^{1/2}} |n\rangle$$

(with $Q$ generic and $Z$ a complex number) when $Q$ goes to a root of unity? By using the just described decomposition $Z \leftrightarrow (z, \theta)$, it is possible to show that the limit

$$|z, \theta\rangle := \lim_{Z \to (z, \theta)} \lim_{Q \to q} |Z\rangle$$

can be written

$$|z, \theta\rangle = |z\rangle \otimes |\theta\rangle$$

where

$$|z\rangle := \sum_{r=0}^{\infty} \frac{z^r}{\sqrt{r!}} |r\rangle$$
is an ordinary bosonic coherent state ($z$ is a bosonic complex variable) and
\[ |\theta\rangle := \sum_{s=0}^{k-1} \frac{\theta^s}{([s]_q!)^{\frac{1}{2}}} |s\rangle \]
is a $k$-fermionic coherent state as introduced in subsection 3.1 ($\theta$ is a $k$-fermionic Grassmann variable in $\Sigma_k$). The state
\[ |z, \theta\rangle = \sum_{r=0}^{\infty} \frac{z^r}{\sqrt{r!}} |r\rangle \otimes \sum_{s=0}^{k-1} \frac{\theta^s}{([s]_q!)^{\frac{1}{2}}} |s\rangle \]
shall be called a fractional supercoherent state.

We note that the state $|z, \theta\rangle$ is an eigenstate of the operator $b_- f_-$ with the eigenvalue $z\theta$. Furthermore, we can generate the state $|z, \theta\rangle$ from the vacuum state $|0\rangle \otimes |0\rangle \equiv |r = 0\rangle \otimes |s = 0\rangle$
owing to the operator
\[ D_q(z, \theta) := \exp(zb_+) e_q(\theta f_+) \]
As a matter of fact, we have
\[ |z, \theta\rangle = D_q(z, \theta) |0\rangle \otimes |0\rangle \]
and thus the operator $D_q(z, \theta)$ plays the rôle of a displacement or dilation operator.

It is interesting to mention that, for fixed $k$, each fractional supercoherent state is a linear combination of the Vourdas [32] coherent states on the Riemann surface $R_k = \mathbb{C}^*/Z_k$ with coefficients in the Grassmann algebra $\Sigma_k$ spanned by $\{1, \theta, \cdots, \theta^{k-1}\}$. We note in passing that the Vourdas coherent states
\[ |z, k, s\rangle := \sum_{r=0}^{\infty} \frac{z^{kr}}{\sqrt{r!}} |kr + s\rangle, \quad s = 0, 1, \cdots, k - 1 \]
correspond to a $Z_k$-grading of the Fock space associated to an ordinary harmonic oscillator. In terms of the latter states, the fractional supercoherent state $|z^k, \theta\rangle$ can be developed, modulo the correspondence $|kr + s\rangle \leftrightarrow |r\rangle \otimes |s\rangle$, as
\[ |z^k, \theta\rangle = |z, k, 0\rangle + \theta |z, k, 1\rangle + \cdots + \frac{\theta^{k-1}}{([k-1]_q!)^{\frac{1}{2}}} |z, k, k - 1\rangle \]
with coefficients in $\Sigma_k$. 

6
4 A Fractional Supersymmetric Oscillator

4.1 Preliminaries

At this stage, a legitimate question arises: what is the Hamiltonian having the fractional supercoherent states $|z, \theta \rangle$ as coherent states? An immediate answer can be obtained in the case $k = 2$. In this case, we have

$$|z, \theta \rangle = \sum_{r=0}^{\infty} \frac{z^r}{\sqrt{r!}} |r\rangle \otimes |0\rangle + \theta \sum_{r=0}^{\infty} \frac{z^r}{\sqrt{r!}} |r\rangle \otimes |1\rangle$$

which turns out to be a supercoherent state for an ordinary supersymmetric oscillator [36]. Such a supersymmetric oscillator corresponds to a $Z_2$-grading. Since the fractional supercoherent state $|z, \theta \rangle$ corresponds to a $Z_k$-grading, we foresee that the Hamiltonian we are looking for is the one for a fractional supersymmetric oscillator corresponding to a $Z_k$-grading. We now proceed to the construction of this Hamiltonian.

4.2 A $Z_k$-graded supersymmetric oscillator

Our basic ingredients consist of a pair of ordinary bosons $(b_-, b_+)$ and a pair of $k$-fermions $(f_-, f_+)$. The $f$’s satisfy $q$-commutation relations and the $b$’s usual commutation relations (see above). In addition, the $f$’s commute with the $b$’s. Indeed, the pairs $(b_-, b_+)$ and $(f_-, f_+)$ may be considered as originating from a pair of $Q$-uons $(a_-, a_+)$ through the isomorphism between the braided line and the one-dimensional superspace.

Let us define the operators $X_-$ and $X_+$ by

$$X_- := b_- \left[ f_- + \frac{(f_+)^{k-1}}{[k-1]_q!} \right]$$

$$X_+ := b_+ \left[ f_- + \frac{(f_+)^{k-1}}{[k-1]_q!} \right]^{k-1}$$

and the Klein operator $K$ by

$$K := f_- f_+ - f_+ f_-$$

which reduces to the Witten operator for $k = 2$. It is a simple matter of calculation to check that $X_-, X_+$ and $K$ satisfy

$$X_- X_+ - X_+ X_- = 1$$

$$KX_+ - qX_+ K = 0, \quad KX_+ - \bar{q}X_- K = 0$$

$$K^k = 1$$
plus some ordinary commutation relations with the operator $M := X_+ X_-$, namely

\[
MX_+ - X_- M = -X_-, \quad MX_+ - X_+ M = +X_+ \\
MK - KM = 0
\]

The operators $X_-, X_+, K$ and $M$ thus generate an extended Weyl-Heisenberg algebra. We note that the form of the commutation relation $[X_-, X_+] = 1$ is the same as for the ordinary Weyl-Heisenberg algebra; it differs from the one used by Plyushchay [12] and generalized by Quesne and Vansteenkiste [16].

The next step is to introduce the $k$ projection operators

\[
\Pi_i := \frac{1}{k} \sum_{s=0}^{k-1} q^{si} K^s, \quad i = 0, 1, \ldots, k - 1
\]

for the cyclic group $Z_k$. We are thus in a position to define the two supercharges

\[
Q_- := X_- (1 - \Pi_{k-1}), \quad Q_+ := X_+ (1 - \Pi_0)
\]

among $k$ possible definitions. We easily verify that the $Q$’s satisfy the nilpotency relations

\[
(Q_-)^k = (Q_+)^k = 0
\]

Following the technique developed by Rubakov and Spiridonov in their work on para-fermions [1] (see also [6]), we introduce an Hamiltonian $H$ by means of the defining relation

\[
(Q_-)^{k-1} Q_+ + (Q_-)^{k-2} Q_+ Q_- + \cdots + Q_+ Q_- (Q_-)^{k-1} = (Q_-)^{k-2} H
\]

This leads to the Hamiltonian

\[
H = X_- X_+ \Pi_1 + \sum_{\ell=2}^{k-1} (X_+ X_- - \ell + 1)(\Pi_0 + \Pi_1 + \cdots + \Pi_{k-\ell-1}) \\
+ \sum_{\ell=2}^{k-1} \ell (X_- X_+ + \frac{\ell - 1}{2}) \Pi_\ell + X_+ X_- (1 - \Pi_{k-1})
\]

which can be seen to satisfy the commutation relation

\[
HQ_\pm - Q_\pm H = 0
\]

and thus the two supercharges $Q_-$ and $Q_+$ can be regarded as constants of motion.
4.3 Some examples

4.3.1 Example 1

As a first example, we take $k = 2$, i.e., $q = -1$. Then, we have

\[ X_- := b_-(f_- + f_+) \]
\[ X_+ := b_(f_- + f_+) \]

and

\[ K := f_-f_+ - f_+f_- \]

where $(b_-, b_+)$ are ordinary bosons and $(f_-, f_+)$ ordinary fermions. The operators $X_-, X_+$ and $K$ satisfy

\[
X_-X_+ - X_+X_- = 1 \\
KX_+ + X_+K = 0, \quad KX_- + X_-K = 0 \\
K^2 = 1
\]

which reflect bosonic and fermionic degrees of freedom. The projection operators

\[ \Pi_0 := \frac{1}{2}(1 + K) \]
\[ \Pi_1 := \frac{1}{2}(1 - K) \]

are here simple chirality operators and the supercharges

\[ Q_- := X_-\Pi_0 \]
\[ Q_+ := X_+\Pi_1 \]

have the property

\[(Q_-)^2 = (Q_+)^2 = 0\]

The Hamiltonian $H$ assumes the form

\[ H = X_+X_-\Pi_0 + X_-X_+\Pi_1 \]

which can be rewritten as

\[ H = Q_-Q_+ + Q_+Q_- \]

It is clear that $H$ commutes with $Q_-$ and $Q_+$. In terms of boson and fermion operators, we have

\[ Q_- = f_+b_- \]
\[ Q_+ = f_-b_+ \]

and

\[ H = b_+b_- + f_+f_- \]
so that $H$ corresponds to the ordinary (or $\mathbb{Z}_2$-graded) supersymmetric oscillator whose energy spectrum $E$ is

$$E = 1 \oplus 2 \oplus 2 \oplus \cdots$$

with equally spaced levels, the ground state being a singlet (denoted by 1) and all the excited states being doublets (denoted by 2). Finally, note that the fractional supercoherent state $|z, \theta\rangle$ with $k = 2$ is a coherent state for the Hamiltonian $H$ [36].

### 4.3.2 Example 2

We continue with $k = 3$, i.e.,

$$q = \exp \left( \frac{2\pi i}{3} \right)$$

In this case, we take

$$X_- := b_- \left[ f_- + \frac{(f_+)^2}{[2]^2} \right]$$

$$X_+ := b_+ \left[ f_- + \frac{(f_+)^2}{[2]^2} \right]$$

and

$$K := f_- f_+ - f_+ f_-$$

where $(b_-, b_+)$ are ordinary bosons and $(f_-, f_+)$ are 3-fermions. We hence have

$$X_- X_+ - X_+ X_- = 1$$

$$K X_+ - q X_+ K = 0, \quad K X_- - \frac{1}{q} X_- K = 0$$

$$K^3 = 1$$

Our general definitions can be specialized to

$$\Pi_0 := \frac{1}{3} (1 + q^3 K + q^3 K^2)$$

$$\Pi_1 := \frac{1}{3} (1 + q K + q^2 K^2)$$

$$\Pi_2 := \frac{1}{3} (1 + q^2 K + q^1 K^2)$$

for the projection operators and to

$$Q_- := X_- (\Pi_0 + \Pi_1)$$

$$Q_+ := X_+ (\Pi_1 + \Pi_2)$$
for the supercharges with the property

\[(Q_-)^3 = (Q_+)^3 = 0\]

By introducing the Hamiltonian \(H\) via

\[
(Q_-)^2 Q_+ + Q_- Q_+ Q_- + Q_+ (Q_-)^2 = Q_- H
\]

we obtain

\[
H = (2X_+X_- - 1) \Pi_0 + (2X_+X_- + 1) \Pi_1 + (2X_+X_- + 3) \Pi_2
\]

We can check that \(H\) commutes with \(Q_-\) and \(Q_+\). The energy spectrum of \(H\) reads

\[E = 1 \oplus 2 \oplus 3 \oplus 3 \oplus \cdots\]

It contains equally spaced levels with a nondegenerate ground state (denoted as 1), a doubly degenerate first excited state (denoted as 2) and a sequel of triply degenerate excited states (denoted as 3).

### 4.4 Concluding Remarks

The work presented in this talk constitutes a further step towards fractional supersymmetric (or para-supersymmetric) quantum mechanics with \(N = 2\) supercharges. We have given a realization of an extended Weyl-Heisenberg algebra in terms of \(k\)-fermionic and purely bosonic operators. This algebra may be considered as originating from the decomposition of a \(Q\)-uon algebra into a \(k\)-fermionic algebra and a purely bosonic algebra. As a basic paradigm, we have investigated a fractional (or \(Z_k\)-graded) supersymmetric oscillator associated to our version of the Weyl-Heisenberg algebra.

The \(Z_k\)-graded oscillator described in the present paper share some properties with the \(C_\lambda\)-extended harmonic oscillator introduced by Quesne and Vansteenkiste and discussed at this symposium [16]. The main difference between the work in Ref. [16] and our work lies in the fact that the Weyl-Heisenberg algebra used by Quesne and Vansteenkiste involves the commutation relation

\[
X_-X_+ - X_+X_- = \sum_{s=0}^{k-1} c_s K^s
\]

(where \(c_s \in \mathbb{C}\) with \(c_0 = 1\)) instead of \(X_-X_+ - X_+X_- = 1\). The Weyl-Heisenberg algebra considered in [16] generalizes the one used by Plyushchay [12] in its study of the Calogero-Vasiliev system and corresponding to \(k = 2\), viz.,

\[
X_-X_+ - X_+X_- = 1 + cK
\]

(where \(c \in \mathbb{C}\)). In fact, it is also possible to find a realization of the extended Weyl-Heisenberg algebra worked out by Plyushchay [12] and by Quesne and Vansteenkiste.
in terms of $k$-fermionic operators (or Grassmann variables) and bosonic operators (or complex variables); this will be the subject of a future work.

Our approach is also concerned with the coherent states associated to the $\mathbb{Z}_k$-graded supersymmetric oscillator. We have conjectured that these coherent states can be obtained from the decomposition of $Q$-deformed coherent states into $k$-fermionic coherent states and purely bosonic coherent states. On the basis of the results obtained in Ref. [36], this conjecture is certainly true when $k = 2$. In the case $k \geq 3$, we still have to prove it. In this respect, it would be useful as suggested by Solomon [37], to examine the time evolution operator associated to the general Hamiltonian $H$ derived in this paper. Along this vein, it is a challenge to understand the connection between $k$-fermions [25] which are of central importance in our approach and anyons [38,39] which are also objects interpolating between fermions and bosons.

The aspects of supersymmetry touched upon in this work concern a small part of supersymmetry (namely, some aspects of supersymmetric quantum mechanics). Nevertheless, it may be worth to close with a few words on the present status of supersymmetry in physics. From an optimistic point of view, supersymmetry is certainly a very appealing concept that proved to be useful in mathematical physics and that is promising in particle physics. However, from a pessimistic point of view, we have to realize that there is still no experimental evidence for supersymmetry except perhaps some hints in the so-called fractional Hall effect and in nuclear spectroscopy. (It is claimed from time to time that supersymmetry exists in atomic nuclei, see for instance Ref. [40]. In our opinion, supersymmetry in atomic nuclei simply relates the structure of odd-odd nuclei to even-even and odd-$A$ systems rather than describes a symmetry between bosons and fermions.) No evidence for supersymmetry exists in fundamental particle physics: no superparticle has been discovered yet and the limits on the Higgs mass afforded by supersymmetry have not led to the Higgs particle. The short-term hopes for the discovery of supersymmetry in high energy physics are concentrated on analyses of the last $e^+ e^-$ experiments of LEP at CERN in 2000 and the forthcoming $p\bar{p}$ experiments of DØ and CDF at the Tevatron collider (Fermi-Lab). If nothing is found at CERN and Fermi-Lab, we shall have to wait the start of the pp experiments of LHC at CERN in 2005 (or more). We hope that supersymmetry will not remain solely an elegant theoretical concept around the unification of internal and Lorentzian symmetries and that it will receive some experimental confirmation in a not too distant future.

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References

[1] V.A. Rubakov, V.P. Spiridonov, Mod. Phys. Lett. A, 3 (1988) 1337.
[2] J. Beckers, N. Debergh, Mod. Phys. Lett. A, 4 (1989) 1209 ; Nucl. Phys. B, 340 (1990) 767.
[3] A.T. Filippov, A.P. Isaev, A.B. Kurdikov, Mod. Phys. Lett. A, 7 (1992) 2129 ; Int. J. Mod. Phys. A, 8 (1993) 4973.
[4] S. Durand, Mod. Phys. Lett. A, 7 (1992) 2905 ; Phys. Lett. B, 312 (1993) 115 ; Mod. Phys. Lett. B, 8 (1993) 2323.
[5] R. Kerner, J. Math. Phys., 33 (1992) 403.
[6] A. Khare, J. Phys. A: Math. Gen., 25 (1992) L749 ; J. Math. Phys., 34 (1993) 1274.
[7] N. Debergh, J. Math. Phys., 34 (1992) 1270 ; J. Phys. A: Math. Gen., 26 (1993) 7219 ; 27 (1994) L213.
[8] J.L. Matheus-Valle, M.A. R.-Monteiro, Mod. Phys. Lett. A, 7 (1992) 3023 ; Phys. Lett. B, 300 (1993) 66 ; L.P. Collatto, J.L. Matheus-Valle, J. Math. Phys., 37 (1996) 6121.
[9] N. Mohammedi, Mod. Phys. Lett. A, 10 (1995) 1287.
[10] D. Bonatsos, P. Kolokotronis, C. Daskaloyannis, Mod. Phys. Lett. A, 10 (1995) 2197.
[11] J.A. de Azcárraga, A.J. Macfarlane, J. Math. Phys., 37 (1996) 1115.
[12] M.S. Plyushchay, Ann. Phys. (N.Y.), 245 (1996) 339 ; Mod. Phys. Lett. A, 11 (1996) 397.
[13] N. Fleury, M. Rausch de Traubenberg, Mod. Phys. Lett. A, 11 (1996) 899 ; M. Rausch de Traubenberg, M.J. Slupinski, Mod. Phys. Lett. A, 12 (1997) 3051 ; preprint: hep-th/9904126.
[14] A. Mostafazadeh, Int. J. Mod. Phys. A, 11 (1996) 2957.
[15] M. Daoud, Y. Hassouni, Prog. Theor. Phys., 97 (1997) 1033.
[16] C. Quesne, N. Vansteenkiste, Phys. Lett. A, 240 (1998) 21 ; see also this volume.
[17] H. Ahmedov, Ö.Ö. Dayi, preprint: math.QA/9903093 ; see also this volume.
[18] A. Le Clair, C. Vafa, Nucl. Phys. B, 401 (1993) 413 ; see also: C. Ahn, D. Bernard, A. Le Clair, Nucl. Phys. B, 346 (1990) 409.
[19] S. Majid, M.J. Rodríguez-Plaza, J. Math. Phys., 35 (1994) 3753.
[20] M. Arik, D.D. Coon, J. Math. Phys., 17 (1976) 524.
[21] A.J. Macfarlane, J. Phys. A: Math. Gen., 22 (1989) 4581.
[22] L.C. Biedenharn, J. Phys. A: Math. Gen., 22 (1989) L873.
[23] C.-P. Sun, H.-C. Fu, J. Phys. A: Math. Gen., 22 (1989) L983.
[24] A.I. Solomon, Phys. Lett. A, 196 (1994) 29.
[25] M. Daoud, Y. Hassouni, M. Kibler, The $k$-fermions as objects interpolating between fermions and bosons, in: Symmetries in Science X, eds. B. Gruber and M. Ramek (Plenum Press, New York, 1998), pp. 63-77; Yad. Fiz. 61 (1998) 1935.

[26] M. Kibler, M. Daoud, An alternative basis for the Wigner-Racah algebra of the group SU(2), to be published in Turkish J. Phys. (2000); preprint: physics/9712034.

[27] M. Chaichian, D. Ellinas, P.P. Kulish, Phys. Rev. Lett., 65 (1990) 980.

[28] J. Katriel, A.I. Solomon, J. Phys. A: Math. Gen., 24 (1991) 2093.

[29] R.J. McDermott, A.I. Solomon, J. Phys. A: Math. Gen., 27 (1994) L15; 27 (1994) 2037.

[30] V.I. Man'ko, G. Marmo, E.C.G. Sudarshan, F. Zaccaria, Phys. Scripta, 55 (1997) 520.

[31] F.B. Wang, L.M. Kuang, Phys. Lett. A, 169 (1992) 225; L.M. Kuang, F.B. Wang, Phys. Lett. A, 173 (1993) 221; Le-Man Kuang, Fa-Bo Wang, Gao-Jian Zeng, Phys. Lett. A, 176 (1993) 1.

[32] A. Vourdas, Phys. Rev. A, 41 (1990) 1653; 43 (1991) 1564; A. Vourdas, C. Bendjaballah, Phys. Rev. A, 47 (1993) 3523; A. Vourdas, J. Phys. A: Math. Gen., 29 (1996) 4275.

[33] D. Ellinas, J. Mod. Optics, 38 (1991) 2393; Phys. Rev. A, 45 (1992) 3358.

[34] R.S. Dunne, A.J. Macfarlane, J.A. de Azcárraga, J.C. Pérez Bueno, Phys. Lett. B, 387 (1996) 294; Czech. J. Phys., 46 (1996) 1145; 46 (1996) 1235.

[35] M. Mansour, M. Daoud, Y. Hassouni, Phys. Lett. B, 454 (1999) 281.

[36] Y. Bérubé-Lauzière, V. Hussin, J. Phys. A: Math. Gen., 26 (1993) 6271.

[37] A.I. Solomon (private communication to M.K.).

[38] J.M. Leinaas, J. Myrheim, Nuovo Cimento B, 37 (1977) 1.

[39] G.A. Goldin, R. Menikoff, D.H. Sharp, J. Math. Phys., 21 (1980) 650; 22 (1981) 1664; G.A. Goldin, D.H. Sharp, Phys. Rev. Lett., 76 (1996) 1183.

[40] A. Metz, J. Jolie, G. Graw, R. Hertenberger, J. Gröger, C. Günther, N. Warr, Y. Eisermann, Phys. Rev. Lett., 83 (1999) 1542.