The Effect Race in Fine-Grained Concurrency

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Abstract
Most existed work require knowledge about the effect of program instructions (or statements) to analyze and verify algorithms. In this paper, by revealing some findings on executions of object programs, we define two basic concepts – effect equivalence relation and effect race relation. Further, we show three effect theorems about the race and histories. The core result is that the effect race relation is the accurate relation to capture the internal steps, of which precedence orders are the reason to cause chaotic histories. In addition, the concept – linearization points – widely used in the object verification, is defined formally as the typical effect race relation. These results provide a clear basis for analyzing intricate fine-grained executions. We conduct a lot of experiments on real object algorithms to show the accuracy and efficiency of these definitions in practice. A simple quantitative analysis method for these algorithms is also proposed.

Keywords Concurrent data structures, Branching Bisimulation, Verification, Effects, Linearizability

1 Introduction
1.1 Overview
Most highly-optimized concurrent data structures (also called concurrent objects) are designed by using fine-grained synchronization techniques (e.g., CAS, coupling-locks), which involve intricate interleavings. The main correctness conditions of concurrent objects, e.g., sequential consistency [?] and linearizability [10], are defined on the coarse-grained notion of the history – a finite sequence of call and return actions. These actions as the interactions with clients are called visible actions, and all the internal program instructions are regarded as silence actions. We found that these visible actions on the history have the following main features:

(1) visible actions are acquired from executions.
(2) no visible actions get access to the shared state.

When a shared mutable state is modified by an internal instruction, the term “effect” is utilized by programmers and verifiers to express the impact of the state change on the outside world. Due to the race condition, the shared states in the fine-grained program can be modified in many different execution orders. This brings a lot of disordered and unexpected histories. Algorithm analysis starts from the histories. How to correctly understand the effect of state changes on visible actions is the crux for the verification. For example, linearization points (LPs) [8] are the typical instructions to represent the effect of method calls and have being used as the main means to prove the linearizability (e.g., [11, 21]). However, locating LPs is a bottleneck when conducing a proof. The existed work can only give the informal descriptions of effects of LPs (e.g., future-dependent LPs or helping [14]) in terms of the observed phenomenon on individual algorithms.

In this paper, we provide some formal basis for analyzing the effect of concurrent programs. By reflection on LPs, we first reveal that the ordinary trace equivalence cannot precisely perceive the effect of LPs, and branching potentials play a vital role to determine effects. This motivates us to define the effect equivalence relation on states based on the max-trace equivalence [17].

The core definition in the work is the effect race relation, a binary relation on internal steps. Fine-grained concurrency involves a lot of race that access the shared state. Effect theorems in the paper reveal that the effect race relation is the accurate relation to capture the internal steps, of which different execution orders are the reason to cause disorder histories. This finding provides a clear clue to analyze complex fine-grained algorithms.

We also find and define a neat structure, called effect structure, which is a subset of executions of object programs. Effect structures establish a connection between the effect of each non-stutter step and executions. It is shown that each step relating the states that are not effect equivalent is critical and represents a race with another non-stutter step in the effect structure. The existence of effect structures assures the results of the Effect theorems.

It needs to be emphasized that these results have no concern with the implementation details, but simply rely on two conditions: (i) visible actions in a system satisfy the above features (1) and (2), and (ii) the stuttering property guaranteed by the effect equivalence relation. So these results are suitable for the general algorithm.

We further formalize LPs by means of the effect race relation, which allows understanding LPs more clearly.

Since the max-trace equivalence is equivalent to the branching bisimilar [17], in practice, the effect equivalence relation can be computed efficiently by the branching bisimulation equivalence. We have conducted experiments on various
well-known concurrent data structures. Experimental results validate the accuracy and efficiency of these definitions on analyzing real algorithms.

1.2 Trace equivalence and effect equivalence

The subtlety of fine-grained executions on the effect can be illustrated using the classic Herlihy and Wing queue algorithm [10], shown in Figure 1. The queue’s representation is an indexed array AR with back as the index to denote the next unused slot in AR. Each slot is initialized to a value null, and back is initialized to 1. The queue has two methods, Enq and Deq. An Enq execution contains two steps: first gets a copy of back and increases back; then stores an element at AR[i]. A Deq execution visits AR in ascending order, starting from index 1 and ending at back − 1. If Deq finds a non-null value at slot i, it will return the value of AR[i], otherwise it tries the next slot. If no element is found, then Deq will restart. Each execution step of a method call is atomic and is interleaved with steps of other concurrent method calls.

```plaintext
E0 Enq(x:T) { 
E1 (i, back):=(back, back+1); /* increment */
E2 AR[i]:=x; /* store */
E3 return
E4 }

D0 Deq() { 
D1 while true do {
D2 range := back;
D3 for (0 < i < range ) do {
D4 (x, AR[i]):=(AR[i], null); /* swap */
D5 if (x != null) then return (x)
D6 } } 
```

**Figure 1.** Herlihy and Wing queue.

The behaviour of the concurrent system can be modeled as a labeled transition system. It is common to understand an algorithm by observing possible (finite) executions at each state (e.g., [3]). For the Herlihy and Wing queue example, consider a system of three threads t1, t2 and t3, with t1 executing Enq(a), t2 executing Deq and t3 executing Enq(b) concurrently. A part of the transition graph generated from the system is depicted in Figure 2, where s0 is the initial state, and each step on executions is labeled with the corresponding instructions (i.e., E1 or D1). The call and return actions of the Enq method (i.e., E0 and E3), of a thread t are denoted by t.Enq(v) and t.ret respectively (similar notions for the Deq). All internal computation steps are invisible, and denoted by τ. The states marked with o have some additional τ transitions which are irrelevant to the discussions and hence omitted.

An interesting step is s −→ r with instruction τ(t2,E2). It is the LP for method call t2.Enq(b), and takes effect to change the empty queue by storing b at AR[2] successfully. The effect of the LP is witnessed by the return action t2.ret(b) on r4 −→ r5 transition. However, for the LP s −→ r, traces cannot distinguish the effects of s and r. By omitting states and τ transitions on executions, it is not difficult to see that s and r have the same set of traces. We use −→ to denote a sequence of τ transitions. First, every trace of r is a trace of s. The other direction of inclusion can be seen by the following executions from s, that is, the trace of s below

\[
\begin{align*}
\tau &\rightarrow s \xrightarrow{t1.ret} s_1 \xrightarrow{t2.ret(a)} s_2 \xrightarrow{t1.ret} s_3 \xrightarrow{t2.ret(b)} s_4 \xrightarrow{t1.ret} s_5 \\
\tau &\rightarrow s \xrightarrow{t1.ret} s_2 \xrightarrow{t2.ret(b)} s_3 \xrightarrow{t1.ret} s_4 \xrightarrow{t2.ret(a)} s_5 \xrightarrow{t1.ret} s_6
\end{align*}
\]

can be matched, by the following traces from r

\[
\begin{align*}
\tau &\rightarrow r \xrightarrow{t1.ret} r_1 \xrightarrow{t2.ret(b)} r_2 \xrightarrow{t1.ret} r_3 \\
\tau &\rightarrow r \xrightarrow{t1.ret} r_4 \xrightarrow{t3.ret(a)} r_5 \xrightarrow{t1.ret} r_6 \\
\tau &\rightarrow r \xrightarrow{t1.ret} r_7 \xrightarrow{t2.ret(a)} r_8 \xrightarrow{t1.ret} r_9 \xrightarrow{t3.ret} r_{10} \xrightarrow{t1.ret} r_{11}
\end{align*}
\]

This is a well-known phenomenon in concurrency: s and r have the same trace set, but after t1.ret, the trace set of s3 cannot be matched by any trace sets at r3 or r6. Thus, different effects of s and r are captured by branches, that is, the trace set of s3, r3 and r6 on the subsequent executions from s and r, respectively. Therefore, branching potentials play a vital role in determining the effect of the fine-grained implementation. This inspires us to characterize the effect equivalence relation based on branching potentials.

**Organizations.** Section 2 briefly reviews object systems and histories. Section 3 defines the effect equivalence relation. Section 4 presents the effect race relation and effect structures. Section 5 provides three effect theorems. Section 6 shows branching bisimulation. Section 7 analyzes the effect of real algorithms. Section 8 defines linearization points. Section 9 presents a simple quantitative analysis for algorithms. Section 10 concludes.

2 Object Systems and Histories

2.1 Object Systems

The behaviors of a concurrent object can be adequately described as a labeled transition system. We assume there is a language for describing concurrent algorithms, and the language is equipped with an operational semantics to generate labeled transition systems as defined below. To generate an object’s behaviour, we use the most general clients [15?], which repeatedly invoke an object’s methods in any order and with any possible parameters.

In the context, “object systems” refer to either the transition systems or the program texts. Let m(n) denote method m with parameter n. For simplicity, all methods will take one parameter and return an integer value.

**Definition 2.1 (Labeled transition systems for concurrent objects).** A labeled transition system Λ is a quadruple (s → s, A, s0) where

- S is the set of states,
- A = \{(t.call1(m(n)), t.ret(n’), m), (t, r) | t ∈ {1…k}, where k is the number of threads\} is the set of actions.
- → ⊆ S × A × S is the transition relation,
- s0 ∈ S is the initial state.
We shall write $s \xrightarrow{a} s'$ to abbreviate $(s, a, s') \in \rightarrow$.

When analysing the behaviours of a concurrent object, we are interested in the interactions (i.e., call and return) between the object and its clients, while the internal instructions of the object are considered invisible and modeled by silence action $\tau$.

We write $s \xrightarrow{r} s'$ to mean $s \xrightarrow{(t, r)} s'$ for some $t$. A path $\rho(s)$ starting at a state $s$ of an object system is a finite or infinite sequence $s \xrightarrow{a_1} s_1 \xrightarrow{a_2} s_2 \xrightarrow{\ldots} \cdots$. An execution is a path starting from the initial state, which represents an entire computation of the object system. A trace of state $s$ is a sequence of visible actions obtained from a path of $s$ by omitting states and invisible actions.

2.2 Histories
A history is a finite execution traces consisting of call and return actions, to model the behavior of concurrent objects.

A history is sequential if (1) it starts with a method call, (2) call actions and return actions alternate in the history, (3) each return matches immediately the previous call. A sequential history is legal if it respects the sequential specification of the object. If $H$ is a history and $\tau$ a thread, then the projection of $H$ on $\tau$, written $H|\tau$, is called the subhistory of $H$ on $\tau$. An operation $e$ is a pair which consists of an invocation event $(t, \text{call}, m(n))$ and the matching response event $(t, \text{ret}(n'), m)$. We shall use $e.\text{call}$ and $e.\text{ret}$ to denote, respectively, the invocation and response events of an operation $e$. The operation ordering in $H$ can be formally described using an irreflexive partial order $\prec_H$ by requiring that $(e, e') \in \prec_H$ if $e.\text{ret}$ precedes $e'.\text{call}$ in $H$. Operations that are not related by $\prec_H$ are said to be concurrent (or overlapping). If $H$ is sequential then $\prec_H$ is a total order.

The key idea behind the correctness conditions of concurrent objects (e.g., linearizability) is to compare concurrent histories to legal sequential histories. We show the linearizability relation on histories [10, 23].

3 Effect Equivalence
This section explains the motivation on the effect equivalence from the perspective of executions, and then formalize it by using the max-trace equivalence, and finally define stutter steps in object systems.

3.1 Motivation on the Effect Equivalence
In the object implementation, whether a step takes effect is the crux to form a completed history. Therefore, if a step $s \rightarrow r$ is stutter for the object execution, then $s$ and $r$ should have the same effect. The effect change of a step is related to the change of object states, which can be captured by observing visible actions in the system. Intuitively, a step $s \rightarrow r$ keeps the same effect implies that, for any path $\rho(s)$ from $s$, there exists an path $\rho(r)$ from $r$ such that $\rho(r)$ can ’match’ $\rho(s)$. Informally, the ’match’ implies that

1. $s$ and $r$ are trace equivalent; 
2. $\rho(s)$ and $\rho(r)$ are effect stutter equivalent.

The Herlihy and Wing queue example shows the importance of branching potentials for the effect of method calls. We now show, to assure the stutter equivalence w.r.t. the effect change, it is necessary to consider traces of each intermediate state of executions.
Example 1. In Figure 3, it is easy to see states 1 and 2 are trace equivalent. But 1 \rightarrow 3 and 2 \rightarrow 4 \rightarrow 3 are not stutter equivalent w.r.t. the effect change, since the trace set of intermediate state 4 is neither equivalent to the trace set of 1, nor to the trace set of 3. Therefore, the path 2 \rightarrow 4 \rightarrow 3 cannot match 1 \rightarrow 3.

![Figure 3. The effect of intermediate state 4.](image)

The above example shows that the effect equivalence relies on the traces of intermediate states over a path, and further, for these intermediate states, their effects also depend on each intermediate state over the subsequent executions from that state. Thus, the effect equivalence relation should be defined in an inductive way. We show the intuitive idea as follows.

Let $T^1(s)$ denote the trace set of state $s$. If $s$ and $r$ are effect equivalent, then (i) $s$ and $r$ are trace equivalent, i.e., $T^1(s) = T^1(r)$. (ii) any path $\rho(s)$ from $s$ should be effect stutter equivalent to path $\rho(r)$ from $r$, and vice versa. To guarantee it, we compare the effect equivalence of each intermediate state over $\rho(s)$ and $\rho(r)$.

Let $\rho(s) = s_0 \xrightarrow{a_1} s_1 \xrightarrow{a_2} s_2 \xrightarrow{a_3} \cdots \xrightarrow{a_m} s_n$ and $\rho(r) = r_0 \xrightarrow{a_1} r_1 \xrightarrow{a_2} r_2 \xrightarrow{a_3} \cdots \xrightarrow{a_m} r_m$. First, we compute the ordinary traces of each $s_i$ and $r_i$, and form sequence $\rho_1 = (T^1(s), a_1, T^1(s_1), \cdots)$ obtained from $\rho(s)$, and $\rho_2 = (T^1(r), a_1, T^1(r_1), \cdots)$ obtained from $\rho(r)$. Thus, $\rho_1$ is required to be stutter equivalent to $\rho_2$. Let $T^1(s)$ (resp. $T^1(r)$) denote $\rho_1$ (resp. $\rho_2$). The effect equivalence of $s_i$ and $r_i$ will influence the effect equivalence of starting states $s$ and $r$. To guarantee the effect equivalence of $s_1, s_2, r_1, r_2, \cdots$, we not only see the ordinary trace sets of $s_1, s_2, r_1, r_2, \cdots$, but also the trace equivalence of the intermediate states over paths from these states on $\rho(s)$ and $\rho(r)$. Thus, it forms $T^2(s_1)$ and $T^2(r_1)$ for $s_1, s_2, r_1, r_2, \cdots$. To guarantee the effect stutter equivalence, the sequences $(T^2(s), a_1, T^2(s_1), a_2, \cdots)$, denoted by $T^2(s)$, and $(T^2(r), a_1, T^2(r_1), a_2, \cdots)$, denoted by $T^2(r)$, should be stutter equivalent. The process proceeds until $T^k(s) = T^k(r)$ for any $k$. Therefore, the trace sets of each state on the paths from $s$ and $r$ will influence the effect equivalence of the starting states $s$ and $r$. We continue the process, until for any $k$, $T^k(s) = T^k(r)$.

3.2 $K$-traces

From the above discussions, in order to arrive at an adequate notion of state equivalence to reflect the execution effect, we need to consider not only the traces from $s$ and $r$, but also the traces from each intermediate state that lies on the paths from them. This motivates the effect equivalence to be defined in an inductive way, which coincides with the max-trace equivalence [17]. For $k \in \mathcal{N}$ and state $s$, let $T^k(s)$ denote the $k$-trace set of $s$.

**Definition 3.1 ([17]).** The notions of $k$-traces and $k$-trace sets of a system $\Delta$ are defined as follows:

1. $T^k(s)$ is the set of all $k$’-traces of $s$, for $k < k$.
2. A $k$-trace of a state $s_0$ is obtained from a sequence $(T^k(s_0), a_1, T^k(s_1), a_2, \cdots, a_n, T^k(s_n))$ such that $\Delta$ has a path $s_0 \xrightarrow{a_1} s_1 \xrightarrow{a_2} s_2 \xrightarrow{a_3} \cdots s_n$ by replacing all subsequences $(T^k(s_i), a_{i+1}, T^k(s_{i+1}), a_{i+2}, \cdots, a_{i+k}, T^k(s_{i+k}))$ with $a_{i+1} \equiv a_{i+2} \equiv \cdots \equiv a_{i+k} \equiv r$, and $T^k(s_i) = T^k(s_{i+k}) = \cdots = T^k(s_k)$.

Two states $r$ and $s$ are $k$-trace equivalent, written $r \equiv_k s$, if $T^k(r) = T^k(s)$; They are max-trace equivalent, written $r \equiv s$, if $r \equiv_k s$ for all $k$.

It is straightforward to see that $\equiv_k$ and $\equiv$ are equivalence relations by definition. $T^0(s) = \emptyset$ for every state $s$, and $T^1(s)$ is just the set of the ordinary traces from $s$; $T^2(s)$ includes all the 0-trace and 1-trace of $s$, and so on for $T^3(s)$, $\cdots$, $T^k(s)$, which keeps track of more trace information of intermediate states during the execution from $s$. Also if $k' < k$ then $T^{k'}(s) \subseteq T^k(s)$. Note that $r \equiv_k s$ implies $r \equiv_{k+1} s$ for any $k' < k$. From this it follows that, for any object system, there exists a $k$ such that $r \equiv_k s$ iff $r \equiv_{k+1} s$. The smallest such a $k$ is called the cap of the system.

Example 2. In Figure 2, $T^2(s)$ and $T^2(r)$ are computed as follows.

\[
T^2(s) = \{(T^1(s), r, T^1(s_1), r, T^1(s_2), r, \text{ret}, T^1(s_3), \cdots), (T^1(s), r, T^1(r_1), r, T^1(r_2), r, \text{ret}, T^1(r_3), \cdots), (T^1(r), r, T^1(r_1), r, T^1(r_2), r, \text{ret}, T^1(r_3), \cdots)\}
\]

Since $T^1(s_3) \neq T^1(r_3) \neq T^1(r_3)$, it follows $T^2(s) \neq T^2(r)$.

3.3 Effect equivalence and stutter steps

We define the effect equivalence relation based on $\equiv$.

**Definition 3.2.** Let $\Delta$ be an object system. States $s$ and $r$ in $\Delta$ are effect equivalent if and only if $s \equiv r$.

**Definition 3.3.** Let $\Delta = (S, \rightarrow, \mathcal{A}, s_0)$ be an object system. There are some notations.

- An **effect step** is a path $s_1 \xrightarrow{r} \cdots \xrightarrow{r} s_n \xrightarrow{a} r$ with $s_1 \equiv \cdots \equiv s_n \equiv r$ (with $n \geq 1, a \in \mathcal{A}$), denoted by ES($s_1, a, r$). Sometimes $s \xrightarrow{r} r$ with $s \equiv r$ is denoted by $s \equiv r$.

- An **effect state** is an effect equivalence class $[s]_\equiv$ of $s$, which is defined by $[s]_\equiv = \{s' \mid s \equiv s', s' \in S\}$.

- Let $E(\sigma)$ be the set of effect states on path $\sigma$.

As we mentioned, the effect equivalence relation should satisfy (i) the trace equivalence and (ii) the stutter equivalence w.r.t. the effect. We define the effect stutter equivalence on paths and prove the result.

**Definition 3.4.** Let $\sigma$ and $\rho$ be paths.
1. They are $k$-trace stutter equivalent, if the $k$-trace obtained from $\sigma$ is the same as the $k$-trace obtained from $\rho$.
2. They are effect stutter equivalent, denoted by $\sigma \equiv \rho$, if they are $k$-trace equivalent for any $k > 0$.

**Theorem 3.5.** For a step $s \xrightarrow{r} r$, $s \equiv r$ iff for any path $\sigma(s)$, there exists path $\sigma(r)$, such that $\sigma(s) \equiv \sigma(r)$.

By Theorem 3.5, the stutter step in object systems can be precisely captured by the relation $\equiv$.

**Definition 3.6.** Let $\Delta$ be an object system. A internal transition $s \xrightarrow{r} r$ is a stutter step in the system $\Delta$, if and only if $s \equiv r$.

**Lemma 3.7.** Let $\rho$ be a $\tau$-loop. Any steps on $\rho$ are stutter steps.

Let’s look back on HW queue again. In Figure 2, the non-stutter steps are colored red. (Note that when adding more operations, more effect steps, e.g., $r_1 \rightarrow r_2$ are exposed.) We can see $s_3 \rightarrow s_3$ labeled with $t_3.E_2$ and $s_3 \rightarrow s_3$ labeled with $t_2.D_4$ are effect steps. Executions $\rho : 0 \Rightarrow s_0$ and $\sigma : 0 \Rightarrow r_0$ have the same trace to reach $s_4$ and $r_1$ respectively, and $\tau$-paths after $s_4$ and $r_1$ have different effect states. So we can conclude that different return actions on $\rho$ and $\sigma$ is caused by the executions of $t_3.E_2$ and $t_2.D_4$ from $s_3$.

4 The Relation of Effect Race

Like the concept of the data race, the effect race is also the basic concept in concurrent programs, on which almost all the program analysis and verification implicitly depend. Based on the effect equivalence relation, we define the effect race relation $\ll$ on internal transitions.

**4.1 A preliminary result**

We first show a preliminary result about the relation of effect steps and executions. Let $\sigma(s, s')$ denote a path from $s$ to $s'$ on $\sigma$ if $s \neq s'$; or a single s, if $s = s'$. The theorem shows that: for any $s \not\equiv r$, there are $\sigma(s, s')$ and $\rho(r, r')$ that pass along different effect states, but the same trace to reach $s'$ and $r'$ such that $s' \not\equiv r'$.

**Theorem 4.1.** Let $\Delta = (S, \rightarrow, \mathcal{A}, s_0)$ be an object system. For any states $s, r \in S$, if $s \not\equiv r$, then there exist paths $\sigma(s, s')$ from $s$ to $s'$ and $\sigma(r, r')$ from $r$ to $r'$ satisfying:

1. $\sigma(s, s')$ and $\sigma(r, r')$ have the same trace;
2. $E(\sigma(s, s')) \cap E(\sigma(r, r')) = \emptyset$;
3. $s' \not\equiv r'$.

**Proof:** Let $k$ be the cap of $\Delta$. There exists $l$ with $1 \leq l \leq k$ such that $s \not\equiv r$, but $s \equiv_l r$. Base. If $l = 1$, let $s_n = s$ and $r_m = r$, these results are straightforward. Induction. For $l \geq 2$, suppose that for any states $s'$ and $r'$ with $s' \not\equiv_l r'$, the results 1-3 hold. Because $s \not\equiv r$, there exits a $(l - 1)$-trace $\rho(s)$ from $s$ such that $\rho(s) \not\equiv T^l(r)$, or there exists a $(l - 1)$-trace $\rho(r)$ from $r$ such that $\rho(r) \not\equiv T^l(s)$. Suppose $\rho(s) \not\equiv T^l(r)$. Let $(l - 1)$-trace $\rho(s) = (T^{l-1}(s), a_1, T^{l-1}(s_1), \cdots)$ with $a_1 \in \mathcal{A}$ and $s \not\equiv_{l-1} s_1$. Because $s \not\equiv_l r$ with $l \geq 2$, we have $s \equiv_1 r$. Therefore, there must exist $(l - 1)$-trace $\rho(r) = (T^{l-1}(r), b_1, T^{l-1}(r_1), \cdots)$ such that $a_1 = b_1$. Because $\rho(s) \not\equiv T^l(r)$, it follows $T^{l-1}(s_1) \not\equiv T^{l-1}(r_1)$, that is, $s_1 \not\equiv_{l-1} r_1$. By hypothesis, it is easy to see that the results 1-3 hold. The case $\rho(r) \not\equiv T^l(s)$ can be proved similarly.

Theorem 4.1 is a general result regarding the relation of $s \not\equiv r$ and executions. In fact, not all of execution fragments satisfying Theorem 4.1 have a meaningful connection with the effect of $s \rightarrow r$. In Figure 4 (1) and (2), suppose $s \rightarrow r$, $s \rightarrow s'$ and $s \rightarrow s_1 \rightarrow s_2$. It is easy to see $r$ from $s$ (blue lines) and $\sigma$ from $r$ (red lines) satisfying Theorem 4.1. But the branch consisting of $\sigma$ and $\rho$ do not identify $s \not\equiv r$. In diagram (1), since there are no intermediate states along the paths $s \rightarrow s'$ and $r \rightarrow s'$, by Definition 3.1, $s \equiv r$. In diagram (2), there are two pathes $s \rightarrow s_1 \rightarrow s_2$ and $r \rightarrow s_2$, where $s_1$ is an intermediate state. But there is no branch from $s_1$ to make $s \not\equiv s_1 \not\equiv s_2$. So, $s \equiv r$ w.r.t. $\sigma$ and $\rho$.

Hence, it is not proper to understand effect steps from the entire visible actions. We need a precise relation to reveal the meaning of effect steps for the real programs.

**4.2 Effect race**

**Definition 4.2.** Let $a$ be a action in object systems. Two effect steps $ES(s, \tau, r)$ and $ES(s, a, s')$ are a branch unit, if $s' \not\equiv r$.

**Definition 4.3.** An internal effect step $ES(s, \tau, r)$ is identified by effect step $ES(s, a, q)$, where $a$ is any action in object systems, if

1. $ES(s, \tau, r)$ and $ES(s, a, q)$ are a branch unit; and
2. for any $ES(r, a, l)$, $l \not\equiv q$.

In Figure 4 (1), $s \rightarrow r$ is not identified by $s \rightarrow s'$.  

**Lemma 4.4.** For each internal effect step $\alpha$, there exists an effect step $\beta$, such that $\alpha$ is identified by $\beta$.

Let $s \rightarrow r$ denote zero or more $\tau$-steps, and $\alpha \rightarrow r$ denote $s \rightarrow s_1 \rightarrow \cdots \rightarrow r$. 

**Definition 4.5.** Let $\Delta = (S, \rightarrow, \mathcal{A}, s_0)$ be an object system, and $ES(s, \tau, r)$ be identified by $ES(s, a, q)$. 

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Manuscript,
1. ES(s, τ, r) is independent of ES(s, a, q) w.r.t. effect state [l]_\|, if (i) there exist r → l_1 and q → l_2 such that l_1, l_2 ∈ [l]_\| for any (a ∈ \mathcal{A}) or (ii) there is q → l_2 such that l_2 = r for a = τ.

2. ES(s, τ, r) is dependent of ES(s, a, q) w.r.t. [l]_\|, if it is not independent of ES(s, a, q) w.r.t. [l]_\|.

**Definition 4.6** (The effect race relation). Let \( \Phi \) be a set of effect steps and \( \ll \subseteq \Phi \times \Phi \) be the effect race relation. For effect steps \( \alpha, \beta \in \Phi, \alpha \ll \beta \) if \( \alpha \) is dependent of \( \beta \) w.r.t. any effect states.

**Theorem 4.7.** Let \( \alpha \) and \( \beta \) be internal effect steps. The following properties of \( \ll \) hold:

1. symmetric: \( \alpha \ll \beta \) implies \( \beta \ll \alpha \).
2. irreflexive: \( \ll \not\equiv \alpha \).
3. non-transitive: \( \alpha \ll \beta \) and \( \beta \ll \mu \) does not imply \( \alpha \ll \mu \).

**Example 3.** In Figure 2, \( s \rightarrow r \) is independent of \( s \rightarrow s_2 \) w.r.t. \([r_2]\), and \( s_3 \rightarrow r_3 \ll s_3 \rightarrow s_4 \).

As we mentioned in Section 1, for the object program, call and return actions do not access the shared object state. So the effect race is only related to internal steps.

**Lemma 4.8.** Let \( \Delta = (S, \rightarrow, A, S_0) \) be an object system, and \( ES(s, \tau, r) \) is identified by \( ES(s, a, q) \)

1. If \( a \) is a visible action, then \( ES(s, \tau, r) \) is independent of \( ES(s, a, q) \) w.r.t. some effect state.
2. If \( ES(s, \tau, r) \ll ES(s, a, q) \), then \( a = \tau \).

**Theorem 4.9.** For each effect step \( \alpha \), there must exist an effect step \( \beta \) such that

1. either \( \alpha \) is independent of \( \beta \);
2. or \( \alpha \ll \beta \).

The relation \( \ll \) is defined on general internal executions, where an effect step is a path consisting of several stuffer steps \( l_1 \rightarrow l_2 \) and a step \( l_n \rightarrow l \). By the stutter equivalence in Definition 3.4, in fact, each effect step \( \alpha \) is stutter equivalent to an internal transition \( s \rightarrow r \). Therefore, for \( \alpha \ll \beta \), there exist non-stutter steps \( s \rightarrow r \) and \( s \rightarrow l \) such that \( s \rightarrow r \ll s \rightarrow l \).

**Lemma 4.10.** For \( \ll \in \Delta \), there exists \( s \rightarrow_{\neq} r \) and \( s \rightarrow_{\neq} l \) such that \( s \rightarrow_{\neq} r \ll s \rightarrow_{\neq} l \) in \( \Delta \).

**Definition 4.11.** Let \( \alpha \) and \( \beta \) be labeled with instructions \( c_1 \) and \( c_2 \) respectively. If \( \alpha \ll \beta \), then \( c_1 \) and \( c_2 \) are effect race instructions from \( s \), denoted by \( c_1 \ll_s c_2 \).

### 4.3 Effect structures and critical steps

A neat structure, called effect structure, is defined. We show that each effect step has at least one effect structure that is responsible for recognizing its effect. For convenience sake, we give the following notations.

1. \( ES(s, \tau, r)^\alpha \) denotes that execution \( \sigma \) passes through \( ES(s, \tau, r) \), and the two notions \( ES(s, \tau, r)^\alpha \) and \( ES(s, \tau, q)^\beta \) mean that \( \sigma \) and \( \rho \) share the same prefix from initial \( s_0 \) to \( s \).
2. \( \sigma' = (s'/s) \) is an execution, which has the same states of \( \sigma \) except replacing the state \( s \) in \( \sigma \) by \( s' \).

**Definition 4.12** (race structures). Let \( \Delta \) be an object system. If \( ES(s_1, \tau, r_1) \ll ES(s_1, \tau, s_1 + 1) \), the set of executions \( \sigma \) and \( \rho \) with \( ES(s_1, \tau, r_1)^\sigma \) and \( ES(s_1, \tau, s_1 + 1)^\rho \) in \( \Delta \) is called the race structure of the relation \( \ll \), denoted by \( RACE\mathcal{S}TR(s_1, r_1, s_1 + 1) \).

**Figure 5.** Effect structures.

In Figure 5 (1), we have \( s_1 \rightarrow r_1 \ll s_1 \rightarrow s_2 \), so the set of \( \sigma \) and \( \rho \) is the race structure \( RACE\mathcal{S}TR(s_1, r_1, s_2) \). The race structure is also called the effect structure, denoted by \( ESTR(s_1, r_1) \).

Further, in Figure 5 (2), we have \( s_1 \rightarrow r_2 \ll s_1 \rightarrow s_4 \). Let \( \sigma, \rho \in RACE\mathcal{S}TR(s_3, r_3, s_4) \), where \( \sigma = s_0 \rightarrow s_2 \rightarrow s_3 \rightarrow r_2 \rightarrow s_4 \) and \( \rho = s_0 \rightarrow s_2 \rightarrow s_3 \rightarrow s_1 \rightarrow s_4 \). Let \( \sigma' = (r_2/s_3) \), that is, \( \sigma' = s_0 \rightarrow s_2 \rightarrow r_2 \rightarrow s_3 \rightarrow s_4 \). If \( s_2 \rightarrow r_2 \) is independent of \( s_2 \rightarrow s_3 \) w.r.t. \([r_3]_\| \), then the set \( \{ \sigma' \} \cup ESTR(s_3, r_3) \) is the effect structure \( ESTR(s_2, r_2) \). So \( ESTR(s_1, r_1) \) can be defined. Obviously \( ESTR(s_3, r_3) \subseteq ESTR(s_2, r_2) \subseteq ESTR(s_1, r_1) \).

**Definition 4.13** (effect structures). Let \( \Delta \) be an object system. An effect structure of \( ES(s_1, \tau, r_1) \), denoted by \( ESTR(s_1, r_1) \), is a set of executions, which is defined as:

1. If there exists \( ES(s_1, \tau, r_1) \ll ES(s_1, \tau, s_1 + 1) \), then \( ESTR(s_1, r_1) = RACE\mathcal{S}TR(s_1, r_1, s_1 + 1) \).
2. Let \( \rho, \sigma \in ESTR(s_1, r_1) \) with \( ES(s_1, \tau, r_1)^\rho \) and \( ES(s_1, \tau, r_1)^\sigma \) and \( ESTR(s_1, \tau, s_1 + 1)^\rho \) and \( ESTR(s_1, \tau, s_1 + 1)^\sigma \), if \( ES(s_1, \tau, r_1) \) is independent of \( ES(s_1, \tau, s_1 + 1) \) w.r.t. \([r_1]_\| \), then \( ESTR(s_1, r_1) \) is a set of executions.

An important property of \( ESTR(s, r) \) is that, \( s \not\ll r \) decided in the entire system \( \Delta \) can be precisely decided in \( ESTR(s, r) \). If the set of executions in \( \Delta \) is a subset of the set of executions in \( \Delta' \), then we denote \( \Delta \subseteq \Delta' \). The "precisely" means that: for any larger systems \( \Delta' \subseteq \Delta \), which has the same effect states as \( ESTR(s, r) \), but with more transitions, there still has \( s \not\ll r \) in \( \Delta' \). This shows that \( s \not\ll r \) always holds, which do not need other branch and extra intermediate effect states outside \( ESTR(s, r) \) to decide. Note that effect states and related transitions in \( ESTR(s, r) \) and \( \Delta' \) come from \( \Delta \).
Theorem 4.14. States $s \not\equiv r$ in $\text{Estr}(s, r)$.

Theorem 4.15. Let $\Delta$ be an object system. For $\Delta'$ with $\text{Estr}(s, r) \subseteq \Delta' \subseteq \Delta$, if $\Delta'$ has the same effect states as $\text{Estr}(s, r)$, then $s \not\equiv r$ in $\Delta'$.

An effect step may have more than one effect structures depending on how many branch units associated with the step. We show the existence of the effect structure for each effect step. The result can be proved by Theorem 4.9 and Definition 4.13.

Lemma 4.16. Let $\Delta$ be a finite object system. For each $s_1 \xrightarrow{\tau \neq} r_1$ in $\Delta$, there exists $\text{Estr}(s_1, r_1)$.

Now the precise connection between effect steps and executions is established by means of effect structure $\text{Estr}(s, r)$. The following theorem improves the results of Theorem 4.1 by restricting executions to $\text{Estr}(s, r)$.

Theorem 4.17. Let $\Delta$ be a finite system. For each steps $s \xrightarrow{\tau \neq} r$, there are executions $\sigma$ and $\rho$ and states $s'$ and $r'$ such that

1. $\sigma, \rho \in \text{Estr}(s, r)$;
2. $\rho(s, s')$ and $\sigma(r, r')$ have the same race;
3. $E(\rho(s, s')) \cap E(\sigma(r, r')) = \emptyset$;
4. $s' \not\equiv r'$.

By Theorem 4.17, it is easy to see that each $s \xrightarrow{\tau} r$ in $\text{Estr}(s, r)$ has the potential to represent a race with another non-stutter step in $\text{Estr}(s, r)$ to cause different traces on $\sigma$ and $\rho$. Such the race can be seen immediately when $\text{Estr}(s, r)$ is a race structure.

Definition 4.18 (critical steps). A $\tau$-transition $s \xrightarrow{\tau} r$ is called a critical step of an object system, if $s \not\equiv r$.

Let $R$ be a race structure of $\Delta$, and $\text{RaceStr}(\Delta) = \bigcup R$ denote all the race structures in $\Delta$.

Theorem 4.19. Let $\Delta$ be a finite object system and $\Phi_\Delta$ be the set of critical steps in $\Delta$.

$$\Phi_\Delta = \bigcup \{s \xrightarrow{\tau} r \mid \text{Estr}(s, r) \ni R, R \in \text{RaceStr}(\Delta)\}$$

Therefore, all critical steps in a system can be found based on the race structure. On the other hand, finding the race structure can fall back on each critical step.

5 Three Effect Theorems

In the following, we consider any completed concurrent histories with the same method calls but different return actions. These concurrent histories are the main concerns for the verification.

Definition 5.1. Let $H_1$ and $H_2$ be completed concurrent histories with the same call actions. If they have different return actions, then $H_1 \neq H_2$; otherwise, $H_1 = H_2$.

In an object system $\Delta$, there may have many interleaved executions such that their visible actions are the same as $H_1$ or $H_2$. We focus on all the executions $\sigma$ in $\Delta$ such that $H(\sigma) = H_1$ or $H(\sigma) = H_2$. These interleaved executions constitute a subsystem of $\Delta$ relevant with the visible actions of $H_1$ and $H_2$, denoted by $\Delta(H_1H_2)$.

Definition 5.2. Let $\Delta$ be an object system and $H_1 \neq H_2$. The subsystem $\Delta(H_1H_2) = \{\sigma \mid \sigma$ is an execution of $\Delta$ s.t. $H(\sigma) = H_1$ or $H(\sigma) = H_2\}$.

Lemma 5.3. Let $\Delta$ be a finite system. For each $\sigma \in \text{Estr}(s, r)$, there is $\sigma' \in \text{RaceStr}(\Delta)$ s.t. $H(\sigma) = H(\sigma')$.

Proof: By Definitions 4.13 and 4.5 and Theorem 4.17.

In the following, we give three effect theorems about the race structures and histories.

We first show that: for two completed concurrent histories $H_1 \neq H_2$, $H_1$ and $H_2$ are enumerable by means of the effect race relation in system $\Delta(H_1H_2)$.

**Effect Theorem 1:**

Theorem 5.4. If $H_1 \neq H_2$, then there exist a race structure $R$ in $\Delta(H_1H_2)$ and $\sigma, \rho \in R$ such that $H(\sigma) = H_1$ and $H(\rho) = H_2$.

Proof: Since $H_1 \neq H_2$, there are different return actions $t, \text{ret}(a)$ on $H_1$ and $t, \text{ret}(b)$ on $H_2$ for the same method call $t$. Therefore, there is a $\tau$-step $s \xrightarrow{\tau} r$ such that $s \not\equiv r$ is recognized by $t, \text{ret}(a)$ and $t, \text{ret}(b)$. By Theorem 4.16, there is a $\text{Estr}(s, r)$ in $\Delta(H_1H_2)$. By Theorem 4.17 and Lemma 5.3, there are $\sigma, \rho \in R$ such that $H(\sigma) = H_1$ and $H(\rho) = H_2$.

In $\Delta(H_1H_2)$, there may have more than one race structures according to the event orders on histories. Each race structure is associated with two internal steps $\alpha$ and $\beta$ such that they satisfy $\alpha \ll \beta$. Therefore, different return actions of $H_1$ and $H_2$ are in essence caused by the race on precedence orders of the steps $\alpha$ and $\beta$.

To validate the application of Theorem 5.4 in infinite systems, we show that: the effect race relation in a small system also holds in larger systems with more method calls. Therefore, the effect race relation in finite systems are the sound basis to analyze algorithm and prove the correctness (e.g., linearizability) of infinite systems.

**Lemma 5.5.** Let $\Delta'$ be an object system, and $\Delta(H_1H_2) \subseteq \Delta'$. If $s \xrightarrow{\tau} r$ in $\Delta(H_1H_2)$, then $s \xrightarrow{\tau} r$ in $\Delta'$.

Proof: Because $s \xrightarrow{\tau} r$ in $\Delta(H_1H_2)$, by Definition 3.1, there is an effect step $s \xrightarrow{\tau} s_1$ on an execution $\sigma$ with $H(\sigma) = H_1$ (or $H_2$), such that for any path $r \rightarrow r_1$ in $\Delta(H_1H_2)$, $s_1 \not\equiv r_1$. Suppose $s \equiv r$ in $\Delta'$. Then there exists $r \rightarrow l$ on $\rho ' \not\equiv \Delta(H_1H_2)$ such that $l \equiv s_1$ in $\Delta'$. By the stutter equivalence in Theorem 3.5, $H(\rho') = H(\sigma)$. Therefore, $\rho' \in \Delta(H_1H_2)$, which is a contradiction.
Effect Theorem II:

Theorem 6.2. Let Δ(Δ₁,Δ₂) ⊆ Δ'.

\[ c₁ ≼ c₂ \text{ in } Δ(Δ₁,Δ₂) \iff c₁ ≼ c₂ \text{ in } Δ' \]

Proof: By Lemma 5.5.

For a system Δ, let \( H(Δ) = \{ H \mid \exists H'. H \neq H' \text{ in } Δ \} \) be a set of different concurrent histories in Δ. Although a fine-grained program involves a large number of disordered concurrent histories, we show that: for the entire object program Λ, completed concurrent histories in \( H(Δ) \) are enumerable in race structure \( \text{RACEStr}(Λ) \).

Effect Theorem III:

Theorem 5.7. Let Δ be an object system.

\[ H(Δ) \subseteq \{ H(σ) \mid σ \in \text{RACEStr}(Λ) \} \]

Proof: By Theorem 5.4 and Theorem 5.6.

Therefore, the effect race relation is the accurate relation to capture the internal steps, of which preceding orders cause disorder concurrent histories. Verifying histories in the entire system can thus be transformed to verify the simple race structure \( \text{RACEStr}(Δ) \).

6 Branching Bisimulation

Branching bisimulation [17] refines Milner’s weak bisimulation [1] by requiring two related states that should preserve not only their own branching structure but also the branching potentials of all intermediate states that are passed through. It has been shown that branching bisimulation is an equivalent characterization of the max-trace equivalence. Thus, we can provide an efficient way to compute the effect race relation of algorithms.

6.1 Branching bisimulation for concurrent objects

Definition 6.1. Let \( Δ = (S, →, A, s₀) \) be an object system. A symmetric relation \( R \) on \( S \) is a branching bisimulation if for all \((s₁, s₂) \in R\), the following holds:

1. if \( s₁ →^a s₁' \) where \( a \) is a visible action, then there exists \( s₂' \) such that \( s₂ →^a s₂' \) and \((s₁', s₂') \in R\).

2. if \( s₁ → s₁' \), then either \((s₁', s₂) \in R\), or there exist \( l₁, \ldots, l_i \) and \( (s₁', s₂') \in R\), such that \( s₂ →^a l₁ →^a \cdots →^a l_i \leftrightarrow s₁' \) and \((s₁', s₂') \in R\).

Let \( \approx \) be the largest branching bisimulation. Then \( \approx \) is an equivalence relation.

Theorem 6.2. [17] For any states \( s \) and \( r \) in an object system, \( s \equiv r \) if and only if \( s \approx r \).

For finite state systems, branching bisimulation can be computed in polynomial time [24, 29].

6.2 Quotient Object Systems

Given an object system \( Δ = (S, →, A, s₀) \), for any \( s \in S \), let \( [s]_≈ \) be the equivalence class of \( s \) under \( ≈ \), and \( S/≈ = \{ [s]_≈ \mid s \in S \} \) the set of the equivalence classes under \( ≈ \).

Definition 6.3 (Quotient transition system). For an object system \( Δ = (S, →, A, s₀) \), the quotient transition system \( Δ/≈ \) is defined as:\n\( Δ/≈ = (S/≈, →, A, [s₀]_≈) \), where the transition relation \( →/≈ \) is generated by the following rules:

\[
(1) [s]_≈ →^a [s']_≈ \quad (a ≠ τ) \\
(2) [s]_≈ →^τ [s']_≈ \\
\]

Theorem 6.4. For a \( τ \) path \( s₁ →^τ \cdots →^τ s_n →^τ r \) in \( Δ \), it is an effect step if and only if \( [s_n]_≈ →^τ [r]_≈ \) is a transition in \( Δ/≈ \), where \( s₁, \ldots, s_n \in [s_n]_≈ \).

7 Effects of Concrete Algorithms

This section takes the real object algorithms as examples to show that complicated executions and their effects, which are informally described and used in the existed work, can be precisely captured by the effect equivalence relation and the effect race relation.

7.1 CCAS

CCAS in Figure 6 is a simplified RDCSS [11] and contains complicated executions. Instead of returning true or false in conventional cas, the cas in CCAS returns the old value of the shared variable \( a \). To update the value of \( a \), the thread first constructs a descriptor with its id \( cid \) and the expected old value \( o \) and the new value \( n \). The element in a can be either a value or a descriptor. If the thread read a descriptor by the cas operation, this means another thread has registered itself first and a help method called Complete is performed to help that thread to finish updating the value. Furthermore, a global variable \( f\text{lag} \) can also influence the success of Complete. Initially, \( f\text{lag} := \text{true} \) and \( a := 1 \). We consider an object system \( Δ \) involving the following four concurrent method calls: \( t₁.C\text{CAS}(1, 2) \), \( t₂.C\text{CAS}(2, 3) \), \( t₃.C\text{CAS}(2, 5) \) and \( t₄.C\text{SetFlag}(f\text{alse}) \). Since an object system enumerates all the possible interleaved steps of each thread at each state, the state space is exponential increase.

C₁ CCAS(o, n) { C₁₁ Complete(d) { C₁₂ local r, d; C₁₃ d := cons(cid, o, n); C₁₄ r := cas(cid, o, d); C₁₅ while (IsDesc(r)) { C₁₆ Complete(r); C₁₇ r := cas(cid, o, d); C₁₈ } C₁₉ if (r = o) Complete(d); C₂₀ return r; } C₁₁.F1 SetFlag(b); }
The generated system is shown in Table 1, where the state spaces of $\Delta$ and $\Delta^\#_s$ are 4382 and 330 respectively; the total number of $r$-transitions in $\Delta$ and $\Delta^\#_s$ are 8218 and 220 respectively.

The corresponding instructions of critical steps in $\Delta$ are also recoded. Each critical step is an essential state transformation annotated in the proof [11, 14].

**Effect equivalence of CCAS**

We first apply the effect equivalence relation to analyzing CCAS. The main feature of CCAS is the helping, of which the effect is informally described in many verification work (e.g., [7]). We show effect equivalence relation $\equiv$ precisely captures the implicit meaning of helping.

![Diagram](image)

**Figure 7.** (1) Effect equivalence relation of CCAS; (2) Transitions in the quotient.

The effect equivalence class of states in the entire system $\Delta$ has been computed. Figure 7 (1) presents executions of threads $t_1$ and $t_2$, and marks the state number and equivalence class numbers of each state\(^1\), e.g., the equivalence class of state 50 is [229]. From the equivalence class, it is clear which steps are stutter, and which steps are critical. In particular, the states of the following $r$-steps:

\[
83 \xrightarrow{t_1.C_{13}} 136, \quad 152 \xrightarrow{t_1.C_{13}} 230, \quad 152 \xrightarrow{t_2.C_{13}} 257
\]


correspond to the same effect states [228] and [43], implying $t_2.C_{13}$ and $t_1.C_{13}$ on these transitions take the same effect. The analysis of executions in Figure 7 (1) can be equivalently transformed to analyzing the quotient in Figure 7 (2), where

\[
t_1.C_{13} \text{ and } t_2.C_{13} \text{ (and } C_{15}) \text{ share the same transition. This clearly shows thread } t_2 \text{ helps thread } t_1 \text{ complete the method call before completing its own method call.}
\]

**Effect race of CCAS**

We now see the effect race relations of CCAS. The quotient is partly shown in Figure 8, where the corresponding instructions of critical steps are labeled at each step. There are the following effect race relation:

\[
t_1.C_{13} \preceq_{187} t_4.F_1 \quad t_1.C_{15} \preceq_{91} t_2.C_{17}
\]

One race is about shared variable $flag$, where different orders of reading $flag$ by $t_1.C_{13}$ and updating $flag$ by $t_4.F_1$ will result in different return actions. The other race is about changing variable $a$, where the effect race of $t_1.C_{15}$ and $t_2.C_{17}$ appear at the state [91] where $flag$ has been assigned to false. If $t_2.C_{17}$ takes effect, then $t_2$ helps $t_1$ complete the method call and keep the old value of $a$ unchanged. Otherwise, if $t_1.C_{15}$ takes effect first, then $t_1$ updates $a$ to the new value 2 since $t_1$ reads $flag := true$ earlier.

The quotient lets us quickly find the effect race relation. This example confirms Effect Theorem I that effect race instructions are accurate to cause all different visible actions. By Effect Theorem II, these effect race relations in Figure 8 is valid for analyzing larger systems.

7.2 The MS lock-free queue

A fine-grained program may involve many instructions to modify the shared state. We take MS queue [6] to show that not all the instructions access to the shared variable have the potential to generate the effect race relation. Figure 9 shows the implementation of the methods `enq` and `deq` of the MS queue. The queue’s representation a linked-list, where `head` and `tail` refer to the first and the last node respectively.
01 Enq(v) { 16 Deq() { 02 local x, t, s, b; 03 x: = new_node(v); 04 while (true) { 05 t: = Tail; s: = t.next; 06 if (t: = Tail) { 07 if (s: = null) { 08 b: = cast(&Tail, t, s); 09 if (b) { 10 cas(&Tail, t, s); 11 return EMPTY; 12 } else cas(&Tail, t, s); 13 } } 20 \begin{align*} 
\text{Figure 9. MS lock-free queue: enqueue and dequeue} 
\end{align*}

Let \( \Delta \) be an object system including 2 threads invoking methods for 3 times. The state spaces of \( \Delta \) and \( \Delta \mathcal{E} \), and instructions of critical steps are shown in Table 2, from which we can see that all the steps labeled with Tail are stutter steps. Therefore, instead of manually analyzing intricate executions, we can compute which instructions access to the shared state are critical.

| MS queue | #states | #t | Instructions of critical steps |
|----------|---------|---|--------------------------------|
| \( \Delta \) | 49038   | 72950 | Line\_enq and Line\_deq: when cas = true; |
| \( \Delta \mathcal{E} \) | 863    | 448  | Line\_deq: when h = Head, Line\_enq |

Table 2. Instructions of critical steps for MS queue.

### 8 Linearization Points

An informal description of LPs [8] is shown as follows: every method call on an execution appears to take effect instantly at some time point between its invocation and its response, behaving as defined by the sequential definition. Such the point, corresponding to the execution of an instruction, is referred to as the LP of the method call. In the following, we give a precise definition of LPs.

Let \( e_1 \) be an operation associated with thread \( t_1 \), and \( e_{i,m} \) denote the invoked method in \( e \). For convenience, each operation in an execution has a different name. Let \( S \) be a sequential history including operations \( e_0 < e_1 < \cdots < e_n \). The partial order of the invocation and response events and \( \tau \)-step \( \alpha \) on an execution \( \alpha \) can be denoted as \( e.gc \ll \alpha < e \cdot \text{ret} < \alpha \cdot \cdots \).

**Definition 8.1.** Let \( \Delta \) be an object system, and \( S_1 \) and \( S_2 \) legal sequential histories including operations \( e_0 < e_1 < \cdots < e_l-1 < e_l \) and \( e_0 < e_2 < \cdots < e_{l-1} < e_l \). If there exist executions \( \alpha \) and \( \rho \) in \( \Delta \), and steps \( \alpha \) labeled with \( t_1 \) and \( \beta \) with \( t_{i+1} \cdot c' \) such that

1. \( H(\alpha) \subseteq \text{fin} S_1 \) and \( H(\rho) \subseteq \text{fin} S_2 \);  
2. \( \alpha \ll \beta \);  
3. \( e_0 \cdot \text{ret} < \alpha \) and \( e_0 \cdot \text{ret} < \beta \) \((0 \leq k \leq l-1)\).  

then, instruction \( c \) is an LP for \( e_{i,m} \) on \( H(\alpha) \).  

By the symmetry of \( \ll \), the instruction labeled on \( \beta \) is an LP for \( e_{i+1,m} \) on \( H(\rho) \). It is easy to see that the effect race of \( \alpha \) and \( \beta \) represents the race on completing operations \( e_i \) and \( e_{i+1} \) on \( \sigma \) and \( \rho \), behaving as defined in the specification. Let \( c_t \) be the number of critical steps associated with thread \( t \).

**Definition 8.2.** Let \( \Delta \) be an object system and \( m \) be an object method of \( \Delta \).

1. The LP of \( m \) is non-fixed, if there is a \( (t, call, m) \) in \( \Delta \), such that between the \((t, call, m)\) and the matched \((t, ret, m)\), \( c_t > 1 \).

2. The LP of method \( m \) is fixed, if it is not non-fixed for any object system \( \Delta' \) with \( \Delta \subseteq \Delta' \).

In practice, to see the LP of a method call, a larger system is needed to reveal all the effect race relations of the method call. For example, in Figure 2, when threads invoke 2 times of methods, the effect race relation \( t_3.E_2 \ll \_s, t_1.E_2 \) from 5 is exposed. It is easy to see \( E_2 \) is the LP for Enq(b) on the execution \( s_0 \Rightarrow s \Rightarrow r \Rightarrow r_6 \). Other algorithms can also be analyzed by Definition 8.1.

### 9 A Quantitative Analysis of Fine-Grained Algorithms

Understanding the fine-grained algorithm is difficult due to a lot of intricate interleavings. Instead of manual analysis, the critical steps of an algorithm can be computed. The critical steps-rate, shorted as C-rate, is given as follows:

\[
\text{C-rate} = \frac{\text{the number of } \tau\text{-transitions in } \Delta \mathcal{E}}{\text{the number of } \tau\text{-transitions in } \Delta}
\]

In general, for different algorithms with the same parameters of method calls, more critical steps an algorithm has, more complicated interleavings the algorithm involves. In the section, we give a quantitative analysis for different fine-grained implementation in terms of critical steps. All experiments are conducted on a server which is equipped with a 4 × 12-core AMD CPU @ 2.1 GHz and 192 GB memory under 64-bit Debian 7.6.

#### 9.1 Critical steps of Herlihy and Wing queue

The branching bisimulation quotient of the HW queue in Figure 2 is shown in Table 3, where the object system in Figure 2 has 292 states and 670 transitions (among them 368 \( \tau \)-transitions); the quotient system has only 52 states and 116 transitions (among them 28 \( \tau \)-transitions). The C-rate is 7.6% (28 out of 368), which implies only a small portion of the invisible steps in the original system are responsible for the effect of executions, while the remaining 92.4% are stutter hence can be abstracted away, as is done in the quotient system.

Table 4 summaries the \#-transitions in the quotient. There are 24 \( \tau \)-transitions \( [s] = \Rightarrow [r] \), that \( s \) and \( r \) are not 1-trace equivalent, which correspond to the instructions \( E_1 \) (4), \( E_2 \) (16) and \( D_4 \) (4); and 4 \( \tau \)-transitions \( [s] = \Rightarrow [r] \), that \( s \) and \( r \)
are not 2-trace equivalent but 1-trace equivalent, which are labeled with $E_2$. Any transition labeled with $D_2$ are stutter steps.

| System $\Delta$ | States | Total trans. | $\# \tau$-trans. |
|------------------|--------|--------------|------------------|
| Quo. $\Delta$   | 52     | 116          | 28               |
| C-rate           | -      | -            | 7.6%             |

Table 3. The C-rate of the HW queue in Figure 2.

There are multiple proof-based techniques has been developed based on rely-guarantee reasoning (e.g., [11, 12, 14, 28, 30]) or simulation methods (e.g., [3, 19–21]) to verify concurrent objects. These techniques often involve identifying LPs and their auxiliary variables to construct the state function [31]. However, although these work are applicable to a wide range of popular non-blocking algorithms (e.g., [9, 10, 27]), they lack a formal basis for understanding fine-grained concurrency. Due to the intricate executions, analyzing the fine-grained interleavings puzzles verifiers when conducting a proof.

For the HM list, the critical steps of methods Add and Rem are labeled by successful cas, which implies an item are successfully added to or removed from the list. However the critical steps of unsuccessful Add and Rem contains two instructions, which are in the while-loops of Find. Which steps labeled with these instruction are critical steps depends on the concrete execution. The Add and Rem of lazy list have the similar analysis as HM list, except it has more steps labeled with Lock, which makes the number of critical steps of lazy list is larger than that of HM list (11385). Method Contains of the lazy list also has two instructions. More than one instructions in a method implies the method has non-deterministic effects, making verification more difficult.

A comparison of different finite instances of these algorithms are summarized in Table 7. For most algorithms, the C-rates are less than 2%. Because the quotient abstracts away all $\tau$-transitions irrelevant to the execution effect, it shows that the enormous state space can be obtained based on quotients. Furthermore, from Table 7, we can see that if there are more threads with more operations, the C-rates will become less and less for scalable concurrent data structures.

| #-Op | Objects | $\# r$ in $\Delta$ | $\# r$ in $\Delta\neq$ | Time(s) | C-rate |
|------|---------|-------------------|------------------------|---------|--------|
| 2-3  | MS      | 72950             | 448                    | 0.23    | 0.61%  |
| 2-3  | DGLM    | 62328             | 448                    | 0.27    | 0.72%  |
| 3-2  | HW      | 128727            | 4062                   | 0.48    | 3.2%   |
| 3-2  | HM list | 1007592           | 11385                  | 1.71    | 1.1%   |
| 3-2  | lazy list | 2607504          | 15297                  | 5.92    | 0.59%  |
| 3-2  | opt. list | 2670636       | 9843                   | 4.38    | 0.37%  |
| 4-1  | CCAS    | 2296             | 115                    | 0.07    | 5.0%   |
| 2-2  | HP      | 25366            | 108                    | 0.11    | 0.43%  |

Table 5. State space and C-rates of systems in Table 6.
Various weak bisimulation [1, 17, 18, 34] have been proposed in process algebra. Nothing but branching (or stuttering) bisimulation satisfies the stuttering equivalence that is an important condition in our paper to apply bisimulation to analyzing concurrent programs.

__Conclusions__. This paper attempts to provide a formal and efficient basis for analyzing fine-grained algorithms. Two basic concepts – the effect equivalence relation and effect race relation – are defined to precisely capture various phenomena of effects in concurrent programs, which are obscure and intricate for programmers and verifiers. A lot of inter-leavings with instructions access to the shared states make understanding fine-grained algorithms difficult. Effect theorems reveal that chaotic concurrent histories are in essence caused by the internal steps satisfying the effect race relation. This validates the accuracy and application of the effect race relation in practice, which provides verifiers a clear clue to analyze complex algorithms. Further, linearization points are characterized by the effect race relation. We have conducted a lot experiments to show the efficiency of these definitions for analyzing real fine-grained concurrent programs.

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| #Th-Op | Algorithms | The corresponding instructions of critical steps for various algorithms |
|--------|-----------|------------------------------------------------------------------|
| 2-3    | MS/DGLM queue [6] | Enq: \( b := \text{cas}(&t.nex,t, s, x) \) when cas is true. Deq: (1) \( s := h.nex; (2) \text{if} (h == \text{Head}) \) when it is true; (3) \( b := \text{cas}(&\text{Head}, h, s) \) when cas is true. |
| 3-2    | HW queue [10] | Enq: (1) \( (i, \text{back}) := (\text{back}, \text{back} + 1); (2) \text{AR}[i] := x \). Deq: \( \text{AR}[i] := (\text{AR}[i], \text{null}) \), when \( x = \text{AR}[i](i == \text{null}) \) or \( x = \text{AR}[i](i == \text{null}) \). |
| 3-2    | HM list [8] | Add: \( \text{pred.next}.\text{cas}(\text{node}, \text{false}, \text{false}) \) when cas is true. Rem: \( \text{prev.next}.\text{cas}(\text{succ}, \text{succ}, \text{false}, \text{true}) \) when cas is true. Find: (1) \( \text{curr} := \text{pred.next}.\text{getReference}(); (2) \text{succ} := \text{curr.next}.\text{getmarked}, \text{marked} == \text{false} \). |
| 3-2    | Lazy list [9] | Add: \( \text{pred.next} := \text{node} \); Rem: \( \text{curr.marked} := \text{true} \); Look: \( \text{node}\text{marked} := \text{false} \). |
| 3-2    | Opt. list [8] | Add: \( \text{lock} \); Rem: \( \text{lock} \); Look: \( \text{node} \). |
| 4-1    | CCAS [7] | CCAS: (1) \( r := \text{cas}(\&a, o, d); (2) b := \text{flag} \); (3) \( \text{cas}(\&a, d, d, n) \) and (4) \( \text{cas}(\&a, d, d, o) \) when cas succeeds. SetFlag: \( \text{flag} := b \). |
| 2-2    | HP(Treiber) [5] | Pop: (1) \( \text{cas}(\&\text{Top}, \text{old}, x) \) when cas succeeds; (2) \( \text{old} := \text{Top} \); Push: \( \text{cas}(\&\text{Top}, \text{old}, x) \) when cas succeeds |

**Table 6.** The instructions of critical steps in fine-grained algorithms computed by the branching bisimulation quotient.

| MS    | DGLM | HW   | Opt  | Lazy | HML | HP  |
|-------|------|------|------|------|-----|-----|
| 2     | 1.2% | 1.3% | 4.7% | 1.06%| 1.5%| 3.4%| 0.43%|
| 3     | 0.61%| 0.72%| 3.8% | 1.08%| 1.4%| 2.7%| 0.21%|
| 4     | 0.32%| 0.39%| 3.1% | 1.08%| 1.1%| 1.9%| 0.07%|
| 5     | 0.16%| 0.20%| 2.0% | 1.08%| 0.94%| 1.5%| 0.01%|
| 3     | 0.04%| 0.06%| 1.47%| 0.30%| 0.36%| 0.5%| 0.03%*|

**Table 7.** The C-rate in different concrete algorithms.

cause chaotic histories. Since the effect equivalence relation in finite systems can be computed by the branching bisimilar in the polynomial time, these results can be used efficiently in practice.

Model-based verification work of the fine-grained concurrency have also been proposed in e.g., [15, 16, 22, 23, 25, 26]. These work can verify and debug linearizability of finite systems automatically. But how to correct understanding non-blocking algorithms is still obscure for these verification work, some of which also involves on manually annotated LPs (e.g., [22]). Understanding and debugging finite concurrent systems can facilitate proofs of infinite systems [32, 33]. Effect theorem II in our paper shows that the effect race relations of a small system still hold in larger system. This implies that essential effect relations exposed on a small system are the sound base for analyzing infinite system. This paper does not discuss how to select the smallest finite system to reveal all the essential effect relations for the inductive proof of infinite systems. This will be the future work.
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