Invariants in electromagnetic and gravitational adjoint fields

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The paper discusses the impact of adjoint fields on the conservation laws in the gravitational field and electromagnetic field, by means of the characteristics of octonions. When the adjoint field can not be neglected, it will cause the predictions to departure slightly from the conservation laws, which include mass continuity equation, charge continuity equation, and conservation of spin. The adjoint field of electromagnetic field has an effect on conservation of mass, and that of gravitational field on conservation of charge. The inferences explain how the adjoint field influences some conservation laws in the gravitational field and electromagnetic field.

PACS numbers: 03.50.De; 04.50.-h; 06.30.Dr; 11.80.Cr.

Keywords: conservation law; invariant; mass density; charge density; dark matter; octonion.

I. INTRODUCTION

The algebra of quaternions can be used to describe the scalar invariants and some conservation laws in the gravitational field. The algebra of octonions can be used to demonstrate the scalar invariants in the case for electromagnetic field and gravitational field, including conservation of mass and conservation of energy. The results are only dealt with quaternion operator, but octonion operator. In the octonion space, the operator should be extended from the quaternion operator to the octonion operator.

Making use of the octonion operator, the gravitational field demonstrated by the octonion operator will generate an adjoint field. The source of adjoint field includes the adjoint mass and field strength have the influence may influence the distribution of electric charge. However, when the adjoint mass is combined with the ordinary mass to become a sort of special particle, therefore we can measure its various characteristics, and have following relation equation.

\[ R_i I_i = r_i i_0 \circ 0 \]

where, \( r_0 = v_0 t; \) \( v_0 \) is the speed of light-like, \( t \) denotes the time; \( R_0 = V_0 T \); \( V_0 \) is the speed of light-like, \( T \) denotes a time-like physical quantity; \( \circ \) denotes the octonion multiplication. \( i = 0, 1, 2, 3 \); \( j = 1, 2, 3 \); \( i_0 = 1 \).

We may consider directly the quaternion space as the two-dimensional complex space, and the octonion space as the four-dimensional complex space.

In some special cases, the adjoint mass is combined with the ordinary mass to become a sort of special particle. Therefore we can measure its various characteristics, and have following relation equation.

\[ V_i I_i = v_i i_0 \circ 0 \]

where, \( d_i \) and \( D_i \) are all real.

When the coordinate system is transformed into the other, the physical quantity \( D \) will be transformed into the new octonion \( D' = (d'_0, d'_1, d'_2, d'_3, D'_0, D'_1, D'_2, D'_3) \).

\[ D' = K^* \circ D \circ K \]

where, \( K \) is the octonion, and \( K^* \circ K = 1 \); \( \circ \) denotes the conjugate of octonion; \( \circ \) is the octonion multiplication.

The octonion \( D \) satisfies the following equations.

\[ d_0 = d'_0 \]

In the above equation, the scalar part \( d_0 \) is preserved during the octonion coordinates are transforming. Some scalar invariants of electromagnetic field will be obtained from this characteristics of the octonion.
TABLE I: The octonion multiplication table.

| 1 | i₁ | i₂ | i₃ | I₀ | I₁ | I₂ | I₃ |
|---|----|----|----|----|----|----|----|
| 1 | 1  | i₁ | i₂ | i₃ | I₀ | I₁ | I₂ | I₃ |
| i₁| i₁|−1 | i₁|−i₁ | I₁|−I₀|−I₃| I₂ |
| i₂| i₂|−i₃| 1 | i₁ | I₂| I₁|−I₀| I₃ |
| i₃| i₃| i₂|−i₁|−1 | I₃|−I₂| I₁|−I₀|

The linear momentum density $S_{gg} = m\mathcal{V}_g$ is the source of gravitational field, and its adjoint linear momentum density $S_{ge} = \tilde{m}\mathcal{V}_g \circ I_0$ is that of adjoint field. They combine together to become the field source $S_g$.

$$\mu S_g = - (\mathcal{B}_g \circ v_0 + \mathcal{S}_g) \circ \mathcal{B}_g = \mu_{gg} S_{gg} + \mu_{ge} S_{ge} - \mathcal{B}_g \circ \mathcal{B}_g / v_0$$ (14)

where, $\tilde{m}$ is the adjoint mass density; $\mu_{gg}$ and $\mu_{ge}$ are the coefficients; $\ast$ denotes the conjugate of octonion.

The adjoint field energy density.

$$\mathcal{B}_g \circ \mathcal{B}_g / (2\mu_{gg})$$ (15)

The above means that the adjoint field energy makes a contribution to the gravitational mass.

III. GRAVITATIONAL FIELD

The gravitational field and its adjoint field both can be demonstrated by the quaternions, although they are quite different from each other.

A. Invariants in gravitational field

1. Potential and strength

The gravitational field potential is

$$\mathcal{A}_g = \Sigma(a_i i_k) \ (8)$$

The strength $\mathcal{B}_g = \Sigma(b_{gi} i_k) + \Sigma(B_{gi} I_k)$ consists of the gravitational strength $\mathcal{B}_{gg}$ and adjoint field strength $\mathcal{B}_{ge}$.

$$\mathcal{B}_g = \Diamond \circ \mathcal{A}_g = \mathcal{B}_{gg} + \mathcal{B}_{ge} \ (9)$$

where, $\mathcal{B}_{gg} = \Sigma(b_{gi} i_k)$, $\mathcal{B}_{ge} = \Sigma(B_{gi} I_k)$; $b_{gi}$ and $B_{gi}$ are all real; $\Diamond = \Sigma i_k(\partial/\partial r_i) + \Sigma I_k(\partial/\partial R_i)$; $\partial_i = \partial/\partial r_i$.

In the above equation, we choose the following gauge conditions to simplify succeeding calculation.

$$\partial a_0 / \partial r_0 - \Sigma(\partial a_j / \partial r_j) = 0$$

$$\partial a_0 / \partial R_0 + \Sigma(\partial a_j / \partial R_j) = 0$$

The gravitational strength $\mathcal{B}_{gg}$ in Eq.(9) includes two parts, $\mathcal{B}_g = (g_{g01}, g_{g02}, g_{g03})$ and $\mathcal{B}_g = (g_{g23}, g_{g31}, g_{g12})$.

$$g_{g} / v_0 = i_1(\partial_0 a_0 + \partial_1 a_0) + i_2(\partial_0 a_2 + \partial_2 a_0) + i_3(\partial_0 a_3 + \partial_3 a_0) \ (10)$$

$$b_{g} = i_1(\partial_2 a_3 - \partial_3 a_2) + i_2(\partial_3 a_1 - \partial_1 a_3) + i_3(\partial_1 a_2 - \partial_2 a_1) \ (11)$$

while the adjoint field strength $\mathcal{B}_{ge}$ involves two parts, $\mathcal{E}_g = (E_{g01}, E_{g02}, E_{g03})$ and $\mathcal{B}_g = (E_{g23}, E_{g31}, E_{g12})$.

$$E_{g} / v_0 = I_1(\partial_0 a_0 - \partial_1 a_0) + I_2(\partial_2 a_0 - \partial_0 a_2) + I_3(\partial_0 a_3 - \partial_3 a_0) \ (12)$$

$$B_{g} = I_1(\partial_2 a_3 - \partial_3 a_2) + I_2(\partial_3 a_1 - \partial_1 a_3) + I_3(\partial_1 a_2 - \partial_2 a_1) \ (13)$$

2. Conservation of mass

In the gravitational field and its adjoint field, the linear momentum density $\mathcal{P} = \mu \mathcal{S}_g / \mu_{gg}$ is written as

$$\mathcal{P} = \tilde{m} v_0 + \Sigma(m v_j i_j) + \Sigma(M_j V_i i_k \circ I_0) \ (16)$$

where, $\tilde{m} = m - (\mathcal{B}_g \circ \mathcal{B}_g / v_0^2)$ ; $\mu_{gg} = m \mu_{ge} / \mu_{gg}$.

The above means that the gravitational mass density $\tilde{m}$ is changed with either the gravitational strength or the adjoint field strength in the gravitational field and its adjoint field.

From Eq.(6), we have one linear momentum density, $\mathcal{P}'(p_{0}', p_{1}', p_{2}', p_{3}', p_{f}', P_{1}', P_{2}', P_{3}')$, when the octonion coordinate system is rotated. And we obtain the invariant equation from Eqs.(7) and (16).

$$\tilde{m} v_0 = \tilde{m}' v_0$$ (17)

Under Eqs.(3), (7), and (17), we find the gravitational mass density $\tilde{m}$ remains unchanged.

$$\tilde{m} = \tilde{m}'$$ (18)

The above means that if we choose the definitions of velocity and linear momentum, the inertial mass density and gravitational mass density will keep unchanged respectively, under the coordinate transformation in Eq.(6) in the gravitational field and its adjoint field.

3. Mass continuity equation

In the gravitational field and its adjoint field, the applied force density $\mathcal{F} = \Sigma(f_i i_k) + \Sigma(F_i I_k)$ is defined from the linear momentum density $\mathcal{P}$ in Eq.(16).

$$\mathcal{F} = v_0 (\mathcal{B}_g / v_0 + \Diamond)^* \circ \mathcal{P} \ (19)$$

where, the scalar $f_0 = v_0 \Sigma(\partial p_i / \partial r_i) + v_0 \Sigma(\partial P_i / \partial R_i) + \Sigma(b_{gj} p_j + B_{gj} P_j)$.
When the coordinate system rotates, we have the new force density $\mathbf{F}'(f_0', f_{1}', f_{2}', f_{3}', f_{4}', f_{5}', f_{6}')$.

By Eq.(7), we have

$$f_0 = f_0'$$  (20)

When $f_0' = 0$ in the above, we have the conservation of mass in the case for coexistence of the gravitational field and its adjoint field.

$$\Sigma\{\partial(p_i + P_i)/\partial r_i\} + \Sigma(b_{gj}p_j + B_{gj}P_j)/v_0 = 0$$  (21)

If the $b_{gi} = B_{gi} = 0$, the above will be reduced to the following equation.

$$\partial(m + M_g)/\partial t + \Sigma\{\partial(p_j + P_j)/\partial r_j\} = 0$$  (22)

further, if there is not adjoint field, we have

$$\partial m/\partial t + \Sigma(\partial p_j/\partial r_j) = 0$$  (23)

The above states that the adjoint field strength, adjoint mass, and gravitational strength have a tiny influence on the conservation of mass in the gravitational field and its adjoint field, although the $\Sigma(b_{gj}p_j + B_{gj}P_j)/v_0$ and $\Sigma(\partial P_i/\partial r_i)$ both are usually very tiny when the fields are weak. In case of we choose the definitions of the applied force and velocity in the gravitational field and adjoint field, the conservation of mass will be the invariant under the octonion transformation in Eq.(6).

4. Conservation of spin

The angular momentum density $\mathbf{L} = \Sigma(l_i \mathbf{i}_i) + \Sigma(L_i \mathbf{I}_i)$ is defined from the radius vector $\mathbf{R}$, physical quantity $\mathbf{X}$, and linear momentum density $\mathbf{P}$ in the octonion space.

$$\mathbf{L} = (\mathbf{R} + k_{rx} \mathbf{X}) \circ \mathbf{P}$$  (24)

where, $l_0$ is considered as the spin angular momentum density in the gravitational field and adjoint field: $l_0 = (r_0 + k_{rx}x_0)p_0 - \Sigma\{(r_j + k_{rx}x_j)p_j\} - \Sigma\{(R_i + k_{rx}X_i)P_i\}$; $k_{rx}$ is the coefficient.

When the octonion coordinate system rotates, we have the new angular momentum density $\mathbf{L}' = \Sigma(l_i' \mathbf{i}_i' + L_i' \mathbf{I}_i')$. Under the octonion coordinate transformation, the spin density remains unchanged from Eq.(7).

$$l_0 = l_0'$$  (25)

The above means the adjoint field, space, time, and strength have an influence on orbital angular momentum and spin angular momentum. The spin angular momentum density $l_0$ will change with time in the gravitational field and adjoint field, although $l_0$ is one invariant under the octonion transformation.

5. Conservation of energy

The total energy density $\mathcal{W} = \Sigma(w_i \mathbf{i}_i) + \Sigma(W_i \mathbf{I}_i)$ is defined from the angular momentum density $\mathbf{L}$.

$$\mathcal{W} = v_0(\mathbf{B}_g/v_0 + \mathbf{\hat{c}}) \circ \mathbf{L}$$  (26)

where, the scalar part $w_0 = v_0\partial l_0'/\partial r_0 - v_0\Sigma(\partial l_j'/\partial r_j) - v_0\Sigma(\partial l_i'/\partial r_i) - \Sigma(b_{gj}l_j + B_{gj}L_j)$.

When the coordinate system rotates, we have the new energy density $\mathcal{W}' = \Sigma(w_i' \mathbf{i}_i' + W_i' \mathbf{I}_i')$. Under the octonion transformation, the scalar part of total energy density is the energy density and remains unchanged by Eq.(7).

$$w_0 = w_0'$$  (27)

In some special cases, the right side is equal to zero. We obtain the conservation of spin angular momentum.

$$\partial l_0'/\partial r_0 - \Sigma(\partial l_j'/\partial r_j) - \Sigma(\partial l_i'/\partial r_i) = 0$$  (28)

If the last term is neglected, the above is reduced to

$$\partial l_0'/\partial r_0 - \Sigma(\partial l_j'/\partial r_j) = 0$$  (29)

further, if there is not adjoint field, we have

$$\partial l_0'/\partial r_0 - \Sigma(\partial l_i'/\partial r_i) = 0$$  (30)

The above means the energy density $w_0$ is variable in the case for coexistence of the gravitational field and adjoint field, because the adjoint mass, velocity, and strength have the influence on the angular momentum density. While the scalar $w_0$ is the invariant under the octonion transformation from Eqs.(7) and (25).

6. Conservation of power

In the gravitational field with adjoint field, the external power density $\mathcal{N}$ can be defined from the total energy density $\mathcal{W}$ in Eq.(26).

$$\mathcal{N} = v_0(\mathbf{B}_g/v_0 + \mathbf{\hat{c}})^* \circ \mathcal{W}$$  (31)

| Definition | Invariant | Meaning |
|------------|-----------|---------|
| $\mathbf{R}$ | $r_0 = r_0'$ | Galilean invariant |
| $\mathbf{V}$ | $v_0 = v_0'$ | Invariable speed of light |
| $\mathbf{A}$ | $a_0 = a_0'$ | Invariable scalar potential |
| $\mathbf{B}$ | $b_0 = b_0'$ | Invariable gauge |
| $\mathbf{F}$ | $p_0 = p_0'$ | Invariable mass density |
| $\mathbf{P}$ | $f_0 = f_0'$ | Conservation of mass |
| $\mathbf{L}$ | $l_0 = l_0'$ | Invariable spin density |
| $\mathbf{W}$ | $w_0 = w_0'$ | Invariable energy density |
| $\mathbf{N}$ | $n_0 = n_0'$ | Conservation of energy |
where, the external power density $N$ includes the power density in the gravitational field and adjoint field. The external power density can be rewritten as follows.

$$N = n_0 + \Sigma(n_i j_i) + \Sigma(N_i I_i)$$  \hspace{1cm} (32)

where, the scalar $n_0 = v_0 \Sigma(\partial w_i/\partial r_i) + v_0 \Sigma(\partial W_i/\partial R_i) + \Sigma(B_{ij} w_j + B_{ij} f_j)$. When the coordinate system rotates, we have the new external power density $\tilde{N} = \Sigma(n_i' j_i' + N_i' I_i')$. Under the octonion coordinate transformation, the scalar part of external power density is the power density and remains unchanged by Eq.(7).

$$n_0 = n_0'$$  \hspace{1cm} (33)

In a special case, the right side is equal to zero. And then, we obtain the conservation of energy.

$$\Sigma \{ \partial(w_i + W_i)/\partial r_i \} + \Sigma(b_{ij} w_j + B_{ij} f_j)/v_0 = 0$$  \hspace{1cm} (34)

If the last term is neglected, the above is reduced to

$$\Sigma(\partial w_i/\partial r_i) + \Sigma(\partial W_i/\partial r_i) = 0$$  \hspace{1cm} (35)

further, if the last term is equal to zero, we have

$$\Sigma(\partial w_i/\partial r_i) = 0$$  \hspace{1cm} (36)

The above means that the power density $n_0$ will be variable in the case for coexistence of the gravitational field and its adjoint field, although the $n_0$ is the scalar invariant under the octonion transformation. And the adjoint mass, strength, and torque density etc. have a few influence on the energy continuity equation in the gravitational field and the adjoint field.

B. Invariants in gravitational adjoint field

1. Conservation of adjoint mass

In the adjoint field, a new physical quantity $P_g = P \circ I_0$ can be defined from Eq.(16).

$$P_g = M_g V_0 + \Sigma(M_g V_j i_j) - \{ \bar{m} v_0 I_0 + \Sigma(m v_j I_j) \}$$  \hspace{1cm} (37)

By Eq.(6), we have the linear momentum density, $P_g' = \Sigma(P'_i i'_i - p'_i I_i)$, when the octonion coordinate system is rotated. Under the octonion coordinate transformation, the scalar part of $P_g$ remains unchanged.

$$M_g V_0 = M'_g V'_0$$  \hspace{1cm} (38)

With Eqs.(3), (7), and the above, we obtain the conservation of adjoint mass as follows. And $M_g$ is the scalar invariant, which is in direct proportion to the adjoint mass density $\bar{m}$.

$$M_g = M'_g$$  \hspace{1cm} (39)

The above means that if we emphasize the definitions of velocity and linear momentum, the adjoint mass density will remain the same, under the coordinate transformation in the adjoint field and gravitational field.

2. Continuity equation of adjoint mass

In the octonion space, a new physical quantity $F_g = F \circ I_0$ can be defined from Eq.(19).

$$F_g = F_0 + \Sigma(F_i i_j) - \Sigma(f_i I_i)$$  \hspace{1cm} (40)

where, the scalar $F_0 = v_0 \Sigma(\partial P_i/\partial r_i) + v_0 \Sigma(\partial p_i/\partial R_i) + \Sigma(b_{ij} P_j - B_{ij} p_j)$. When the coordinate system rotates, we have the octonion applied force density $\tilde{F}_g = \Sigma(F'_i i'_j - f'_i I_i')$. Under the coordinate transformation, the scalar part of $F_g$ remains unchanged.

$$F_0 = F'_0$$  \hspace{1cm} (41)

When the right side is equal to zero in the above, we have the continuity equation of adjoint mass in the case for coexistence of the adjoint field and gravitational field.

$$\Sigma \{ \partial(P_i - p_i)/\partial r_i \} + \Sigma(b_{ij} P_j - B_{ij} p_j)/v_0 = 0$$  \hspace{1cm} (42)

If the last term is neglected, the above is reduced to

$$\Sigma(\partial P_i/\partial r_i) - \Sigma(\partial p_i/\partial r_i) = 0$$  \hspace{1cm} (43)

further, if the last term is equal to zero, we have

$$\Sigma(\partial P_i/\partial r_i) = 0$$  \hspace{1cm} (44)

The above states that the gravitational strength and adjoint strength have an influence on continuity equation of adjoint mass, although $\Sigma(b_{ij} P_j - B_{ij} p_j)/v_0$ is usually very tiny when field are weak. The continuity equation of adjoint mass is the invariant under the octonion coordinate transformation.

Comparing Eq.(20) with Eq.(41), we find that the mass continuity equation Eq.(21) and continuity equation of adjoint mass Eq.(42) can’t be effective at the same time. That means that some invariants will not be effective simultaneously in the gravitational field and adjoint field.

3. Conservation of adjoint spin

In the octonion space, a new physical quantity $L_g = L \circ I_0$ can be defined from Eq.(24).

$$L_g = L_0 + \Sigma(L_i i_j) - \Sigma(l_i I_i)$$  \hspace{1cm} (45)

where, $L_0 = (r_0 + k_{rx} x_0) P_0 - \Sigma \{ (r_j + k_{rx} x_j) P_j \} + \Sigma \{ (R_i + k_{rx} X_i) p_i \}$.

When the octonion coordinate system rotates, we have the angular momentum density $L_g' = \Sigma(L'_i i'_j - l'_i I_i')$ from Eq.(6). Under the coordinate transformation, the scalar part of $L_g$ deduces the conservation of adjoint spin.

$$L_0 = L'_0$$  \hspace{1cm} (46)

The above means that the adjoint spin density $L_0$ is an invariant in the case for coexistence of the gravitational field and its adjoint field, under the octonion coordinate transformation.
TABLE III: The definitions and adjoint invariants of physical quantities in the gravitational field with its adjoint field in the octonion space.

| definition | invariant conservation |
|------------|------------------------|
| $\mathbb{R} \circ L_0'$ | $R_0 = R_0'$ invariable scalar |
| $\mathbb{V} \circ L_0'$ | $V_0 = V_0'$ invariable speed of light – like |
| $\mathbb{X} \circ L_0'$ | $X_0 = X_0'$ invariable scalar |
| $\mathbb{A} \circ L_0'$ | $A_0 = A_0'$ invariable adjoint scalar potential |
| $\mathbb{B} \circ L_0'$ | $B_0 = B_0'$ invariable adjoint field gauge |
| $\mathbb{F} \circ L_0'$ | $F_0 = F_0'$ conservation of adjoint mass |
| $\mathbb{L} \circ L_0'$ | $L_0 = L_0'$ continuity equation of adjoint mass |
| $\mathbb{W} \circ L_0'$ | $W_0 = W_0'$ conservation of adjoint energy |
| $\mathbb{N} \circ L_0'$ | $N_0 = N_0'$ conservation of adjoint power |

4. Conservation of adjoint energy

In the octonion space, a new physical quantity $\mathbb{W}_g = \mathbb{W} \circ L_0'$ can be defined from Eq.(26).

$$\mathbb{W}_g = W_0 + \Sigma(W_j i_j) - \Sigma(w_i I_i)$$

where, the scalar part $W_0 = v_0 \partial L_0/\partial r_0 - v_0 \Sigma(\partial L_j/\partial r_j) + v_0 \Sigma(\partial l_i/\partial R_i) - \Sigma(b_{ij} L_j - B_{ij} l_j)$.

When the octonion coordinate system rotates, we have the energy density $\mathbb{N}_g = \Sigma(n_i' I_i)$ from Eq.(6). Under the coordinate transformation, the scalar part of $\mathbb{N}_g$ remains unchanged. And then we obtain the conservation of adjoint power as follows.

$$N_0 = N_0'$$

When the right side is equal to zero in the above, we have the continuity equation of adjoint energy in the case for coexistence of the adjoint field and gravitational field.

$$\Sigma(\partial(W_i - w_i)/\partial r_i) + \Sigma(b_{ij} W_j - B_{ij} w_j)/v_0 = 0$$

If the last term is neglected, the above is reduced to

$$\Sigma(\partial W_i/\partial r_i) - \Sigma(\partial w_i/\partial r_i) = 0$$

further, if the last term is equal to zero, we have

$$\Sigma(\partial W_i/\partial r_i) = 0$$

The above means the adjoint power density $N_0$ is the invariant in the case for coexistence of the gravitational field and adjoint field, under the octonion coordinate transformation.

IV. ELECTROMAGNETIC FIELD

Making use of the octonion operator, the electromagnetic field demonstrated by the octonion operator will also generate an adjoint field. The source of adjoint field includes the adjoint charge and adjoint electric current. Similarly, the adjoint charge and its movement can not be observed by usual experiments. However, when the adjoint charge is combined with the ordinary charge to become the charged particles, their movements will be accompanied by some mechanical or electric effects. And this kind of adjoint charge may be considered as one kind of candidate for dark matter [7].

The electromagnetic field and its adjoint field both can be demonstrated by the quaternions also, although they are quite different from each other indeed.

With the invariant property of octonions, we find that the adjoint charge, adjoint mass, velocity curl, and field strength have the influence on some conservation laws in the electromagnetic field, under the octonion coordinate transformation.

In some cases, the adjoint charge is combined with the ordinary charge to become one sort of particle. Further, the ordinary mass $m$, adjoint mass $\bar{m}$, ordinary charge $q$, and adjoint charge $\bar{q}$ can be combined together to become another sort of particle. Therefore we can measure their various characteristics, and have following relation.

$$R_i I_i = r_i \bar{i}_i \circ I_0 \ ; \ V_i I_i = v_i \bar{i}_i \circ I_0 .$$
A. Invariants in electromagnetic field

1. Potential and strength

The electromagnetic field potential is

\[ A_e = \Sigma(A_{i1} I_1) \]  

(57)

The electromagnetic potential are combined with the gravitational potential to become the field potential \( \hat{A} = k_0 g + k_1 A_e \), with \( k_1 \) being the coefficient.

The field strength \( \mathbb{E} = \Sigma(b_{i1} I_1) + \Sigma(B_{i1} I_1) \) consists of the gravitational strength \( \mathbb{E}_g \) and the electromagnetic strength \( \mathbb{E}_e \), with \( k_0 \) being the coefficient.

\[ \mathbb{E} = \mathbb{E}_e + \mathbb{E}_g \]  

(58)

The strength \( \mathbb{E}_e = \Sigma(b_{i1} I_1) + \Sigma(B_{i1} I_1) \) consists of the electromagnetic strength \( \mathbb{E}_{eg} \) and adjoint strength \( \mathbb{E}_{ee} \).

\[ \mathbb{E}_e = \mathbb{E}_e' + \mathbb{E}_e'' \]  

(59)

where, \( \mathbb{E}_{ee} = \Sigma(b_{i1} I_1) \); \( \mathbb{E}_{eg} = \Sigma(B_{i1} I_1) \); \( \mathbb{E}_{ee} \) and \( \mathbb{E}_{eg} \) are all real.

In the above equation, we choose the following gauge conditions to simplify succeeding calculation.

\[ \partial A_0 / \partial r_0 - \Sigma(\partial A_1 / \partial r_j) = 0 \]

\[ \partial A_0 / \partial r_0 + \Sigma(\partial A_j / \partial r_j) = 0 \]

The adjoint field strength \( \mathbb{E}_{ee} \) in Eq.(59) includes two parts, \( \mathbb{g}_e = (g_001, g_002, g_003) \) and \( \mathbb{b}_e = (b_23, b_31, b_12) \).

\[ \mathbb{g}_e / v_0 = \mathbb{i}_1 (\partial_t A_1 - \partial_1 A_0) + \mathbb{i}_2 (\partial_0 A_2 - \partial_2 A_0) + \mathbb{i}_3 (\partial_3 A_3 - \partial_3 A_0) \]

(60)

\[ \mathbb{b}_e = \mathbb{i}_1 (\partial_t A_2 - \partial_2 A_3) + \mathbb{i}_2 (\partial_1 A_3 - \partial_3 A_1) + \mathbb{i}_3 (\partial_1 A_1 - \partial_1 A_2) \]

(61)

simultaneously, the electromagnetic field strength \( \mathbb{E}_{eg} \) involves two components, \( \mathbb{E}_e = (B_{e01}, B_{e02}, B_{e03}) \) and \( \mathbb{B}_e = (B_{e23}, B_{e31}, B_{e12}) \).

\[ \mathbb{E}_e / v_0 = \mathbb{I}_1 (\partial_t A_1 + \partial_1 A_0) + \mathbb{I}_2 (\partial_0 A_2 + \partial_2 A_0) + \mathbb{I}_3 (\partial_3 A_3 + \partial_3 A_0) \]

(62)

\[ \mathbb{B}_e = \mathbb{I}_1 (\partial_t A_2 - \partial_2 A_3) + \mathbb{I}_2 (\partial_1 A_3 - \partial_3 A_1) + \mathbb{I}_3 (\partial_1 A_1 - \partial_1 A_2) \]

(63)

The electric current density \( S_{eg} = qV_g \circ \mathbb{I}_0 \) is the source of electromagnetic field, and its adjoint electric current density \( S_{ee} = qV_g \) is that of adjoint field. They combine together to become the field source \( S_e \).

In the octonion space, the electromagnetic source \( S_e \) can be combined with gravitational source \( S_g \) to become the source \( S \).

\[ \mu S = - (\mathbb{B} / v_0 + \mathbb{g}^\star) \circ \mathbb{B} \]

\[ = \mu_{gg} S_{gg} + \mu_{ee} S_{ee} - \mathbb{B}^\star \circ \mathbb{B} / v_0 \]

\[ + k_b (\mu_{ee} S_{ee} + \mu_{eg} S_{eg}) \]

(64)

where, \( k_0^2 = \mu_{gg} / \mu_{eg} \), \( \mu_{gg}, \mu_{ee}, \mu_{ee}, \) and \( \mu_{eg} \) are the coefficients.

The \( \mathbb{B}^\star \circ \mathbb{B} / (2\mu_{gg}) \) is the field energy density.

\[ \mathbb{B}^\star \circ \mathbb{B} / \mu_{gg} = \mathbb{B}^\star \circ \mathbb{B} / (2\mu_{gg}) \]

(65)

The above means that the electromagnetic field and its adjoint field make a contribution to the gravitational mass also in the octonion space.

2. Conservation of mass

In the electromagnetic field, gravitational field and their adjoint fields, the linear momentum density \( P = \mu S / \mu_{gg} \) is written as

\[ P = \hat{m} v_0 + \Sigma(m v_i I_1) + \Sigma(M_q V_i \circ \mathbb{I}_0) \]

\[ + \Sigma(M_q V_i \circ \mathbb{I}_0) + \Sigma(M_{v_1} \circ \mathbb{I}_0) \]

(66)

where, \( \hat{m} = m - (\mathbb{B}^\star \circ \mathbb{B} / \mu_{gg}) / v_0^2 \); \( M_q = q k_{bb} \mu_{gg} / \mu_{gg} \)

\[ M_e = \hat{q} k_{bb} \mu_{ee} / \mu_{gg} \]

The above means that the gravitational mass density \( \hat{m} \) is changed with all four kinds of field strengths in the electromagnetic field, gravitational field, and their two kinds of adjoint fields.

From Eq.(6), we have one linear momentum density, \( P^0(p_0, p^1, p^2, p^3, P^1, P^2, P^3) \), when the octonion coordinate system is rotated. And we obtain the invariant equation from Eqs.(7) and (66).

\[ (\hat{m} + M_e) v_0 = (\hat{m}' + M'_e) v_0' \]

(67)

Under Eqs.(3), (7), and (67), we find the gravitational mass density \( \hat{m} + M_e \) remains unchanged.

\[ \hat{m} + M_e = \hat{m}' + M'_e \]

(68)

The above means that if we choose the definitions of velocity and linear momentum, the inertial mass density \( (m + M_e) \) and gravitational mass density \( \hat{m} + M_e \) will keep unchanged respectively, under the octonion coordinate transformation in Eq.(6) in the electromagnetic field, gravitational field, and their adjoint fields.

3. Mass continuity equation

In the electromagnetic field, gravitational field, and their adjoint fields, the applied force density \( F = \Sigma(f_i I_1) + \Sigma(F_i I_1) \) is defined from the linear momentum density \( P = \Sigma(p_i I_1) + \Sigma(P_i I_1) \) in Eq.(66).

\[ F = v_0 (\mathbb{B} / v_0 + \mathbb{g}^\star) \circ \mathbb{B} \]

(69)

where, the scalar \( f_0 = v_0 \Sigma(\partial p_i / \partial r_i) + \Sigma(\partial P_i / \partial R_i) + \Sigma(b_{gg} p_1 + B_{gg} P_1 + k_b B_{ee} P_3) \)

When the coordinate system rotates, we have the new force density \( F' = (f_0', f_1', f_2', f_3', F_0', F_1', F_2', F_3') \).
By Eq.(7), we have
\[ f_0 = f'_0 \] (70)

When \( f'_0 = 0 \) in the above, we have the conservation of mass in the case for coexistence of the electromagnetic field, gravitational field, and their adjoint fields.

\[
\Sigma \{ \partial(p_i + P_i)/\partial r_i \} + \Sigma(b_{gi}p_j + B_{gi}P_j)/v_0 + \Sigma(b_{gi}b_{ci}p_j + B_{ci}B_{cj})/v_0 = 0
\] (71)

If the \( b_{gi} = B_{gi} = b_{ci} = B_{ci} = 0 \), the above will be reduced to the following equation.

\[
\Sigma(\partial p_j/\partial r_j) + \Sigma(\partial P_j/\partial r_j) = 0
\] (72)

Further, if the last term can be neglected, we have
\[
\Sigma(\partial p_i/\partial r_i) = 0
\] (73)

In case of we choose the definitions of applied force and velocity in the electromagnetic field, gravitational field, and their adjoint fields, the conservation of mass will be the invariant under the octonion transformation in Eq.(6). The above states also that four kinds of field strengths, adjoint mass, and adjoint charge have a tiny influence on conservation of mass, although the impact is usually very small when the fields are weak.

4. Conservation of spin

The angular momentum density \( L = \Sigma(l_i \cdot \hat{i} + \Sigma(L_i \cdot I_i) \) is defined from the radius vector \( \mathbb{R} \), physical quantity \( \mathbb{X} \), and linear momentum density \( \mathbb{P} \) in the octonion space.

\[
L = (\mathbb{R} + k_{rx} \mathbb{X}) \circ \mathbb{P}
\] (74)

where, \( l_0 \) is considered as the spin angular momentum density in the gravitational field and adjoint field; \( l_0 = (r_0 + k_{rx} x_0) - \Sigma \{ (r_j + k_{rx} x_j) p_j \} - \Sigma \{ (R_i + k_{rx} X_i) P_i \}; k_{rx} \) is the coefficient.

When the octonion coordinate system rotates, we have the new angular momentum density \( L' = \Sigma(l'_i \cdot \hat{i} + L'_i \cdot I'_i) \). Under the octonion coordinate transformation, the spin density remains unchanged from Eq.(7).

\[
l_0 = l'_0
\] (75)

The above means that the space, time, adjoint fields, electromagnetic field, and gravitational field have small influence on orbital angular momentum and spin angular momentum. The spin angular momentum density \( l_0 \) will change with time, although \( l_0 \) is an invariant under the octonion transformation.

5. Conservation of energy

The total energy density \( \mathbb{W} = \Sigma(w_i \cdot \hat{i} + \Sigma(W_i \cdot I_i) \) is defined from the angular momentum density \( L \).

\[
\mathbb{W} = v_0(\mathbb{B}/v_0 + \hat{\mathbb{O}}) \circ \mathbb{L}
\] (76)

where, the scalar part \( w_0 = v_0 \partial l_0/\partial r_0 - v_0 \Sigma(\partial l_i/\partial r_i) - v_0 \Sigma(\partial L_i/\partial r_i) - \Sigma(b_{gi}l_j + B_{gi}L_j + k_b b_{cj}l_j + k_b B_{cj}L_j) \).

When the coordinate system rotates, we have the new energy density \( \mathbb{W}' = \Sigma(w'_i \cdot \hat{i} + W'_i \cdot I'_i) \). Under the octonion transformation, the scalar part of total energy density is the energy density and remains unchanged by Eq.(7).

\[
w_0 = w'_0
\] (77)

In some special cases, the right side is equal to zero. We obtain the conservation of spin angular momentum.

\[
-\Sigma(b_{gi}l_j + B_{gi}L_j + k_b b_{cj}l_j + k_b B_{cj}L_j)/v_0 + \partial l_0/\partial r_0 - \Sigma(\partial l_i/\partial r_i) - \Sigma(\partial L_i/\partial r_i) = 0
\] (78)

If the first term is zero, the above is reduced to

\[
\partial l_0/\partial r_0 - \Sigma(\partial l_i/\partial r_i) = 0
\] (79)

further, if last term is zero, we have

\[
\partial l_0/\partial r_0 = \Sigma(\partial l_i/\partial r_i) = 0
\] (80)

The above means the energy density \( w_0 \) is variable with time in the case for coexistence of the electromagnetic field, gravitational field, and their adjoint fields, because the adjoint mass, adjoint charge, velocity, and strength have the influence on the angular momentum density. While the scalar \( w_0 \) is the invariant under the octonion transformation from Eq.(7).

6. Conservation of power

In the electromagnetic field and gravitational field with their adjoint fields, the external power density \( \mathbb{N} = \Sigma(n_i \cdot \hat{i}) + \Sigma(N_i \cdot I_i) \) can be defined from the total energy density \( \mathbb{W} \).

\[
\mathbb{N} = v_0(\mathbb{B}/v_0 + \hat{\mathbb{O}})^* \circ \mathbb{W}
\] (81)

where, the scalar \( n_0 = v_0 \Sigma(\partial w_i/\partial r_i) + v_0 \Sigma(\partial W_i/\partial r_i) + \Sigma(b_{gi}w_j + B_{gi}W_j + k_b b_{cj}w_j + k_b B_{cj}W_j) \).

### TABLE IV: The definitions and the mechanics invariants of electromagnetic field and gravitational field with their adjoint fields in the octonion space.

| definition | invariant | meaning |
|------------|-----------|---------|
| \( \mathbb{R} \) | \( r_0 = r'_0 \) | Galilean invariant |
| \( V \) | \( v_0 = v'_0 \) | invariable speed of light |
| \( A \) | \( a_0 = a'_0 \) | invariable scalar potential |
| \( B \) | \( b_0 = b'_0 \) | invariable gauge |
| \( P \) | \( p_0 = p'_0 \) | invariable mass density |
| \( F \) | \( f_0 = f'_0 \) | conservation of mass |
| \( L \) | \( l_0 = l'_0 \) | invariable spin density |
| \( \mathbb{W} \) | \( w_0 = w'_0 \) | invariable energy density |
| \( \mathbb{N} \) | \( n_0 = n'_0 \) | conservation of energy |
When the coordinate system rotates, we have the new external power density $N^i = \Sigma(n_i^i n_i' + N_i I_i')$. Under the octonion coordinate transformation, the scalar part of external power density is the power density and remains unchanged by Eq.(7).

$$n_0 = n_0'$$  \hspace{1cm} (82)

In a special case, the right side is equal to zero. And then, we obtain the conservation of energy.

$$\Sigma \{ \partial(w_i + W_i)/\partial r_i \} + \Sigma(b_{ij} w_j + B_{ij} W_j)/v_0 + \Sigma k(b_{ij} w_j + B_{ij} W_j)/v_0 = 0$$  \hspace{1cm} (83)

If last two terms are zeros, the above is reduced to

$$\Sigma(\partial w_i/\partial r_i) + \Sigma(\partial W_i/\partial r_i) = 0$$  \hspace{1cm} (84)

further, if last term is equal to zero, we have

$$\Sigma(\partial w_i/\partial r_i) = 0$$  \hspace{1cm} (85)

The above means that the power density $n_0$ will be variable in the case for coexistence of the electromagnetic field, gravitational field, and their adjoint fields, although the $n_0$ is the invariant under the octonion transformation. And the adjoint mass, adjoint charge, field strength, and torque density etc. have a few influence on the energy continuity equation in the octonion space.

B. Invariants in electromagnetic adjoint field

1. Conservation of charge

In the adjoint field, a new physical quantity $F_q = P \circ I_0'$ can be defined from Eq.(66).

$$F_q = P_0 + \Sigma(P_j I_j) - \{p_0 I_0 + \Sigma(p_j I_j)\}$$  \hspace{1cm} (86)

where, $P_0 = (M_g + M_q)V_0$ .

By Eq.(6), we have the linear momentum density, $P_q' = \Sigma(P'_j I'_j - p'_j I_j)$, when the octonion coordinate system is rotated. Under the octonion coordinate transformation, the scalar part of $P_q'$ remains unchanged.

$$(M_g + M_q)V_0 = (M'_g + M'_q)V_0'$$  \hspace{1cm} (87)

By means of Eqs.(3), (7), and the above, we obtain the conservation of charge as follows. And $(M_g + M_q)$ is a scalar invariant, which is a function of the adjoint mass density $\tilde{m}$ and ordinary charge $q$ .

$$M_g + M_q = M'_g + M'_q$$  \hspace{1cm} (88)

The above means if we emphasize definitions of velocity and linear momentum, the charge density $(M_g + M_q)$ will remain the same, under the coordinate transformation in the electromagnetic field, gravitational field, and their adjoint fields.

2. Charge continuity equation

In the octonion space, a new physical quantity $F_q = F \circ I_0'$ can be defined from Eq.(69).

$$F_q = F_0 + \Sigma(F_j I_j) - \Sigma(f_i I_i)$$  \hspace{1cm} (89)

where, the scalar $F_0 = v_0 \Sigma(\partial P_i/\partial r_i) - v_0 \Sigma(\partial p_i/\partial r_i) + \Sigma(b_{ij} P_j - B_{ij} p_j) + k(b_{ij} P_j - B_{ij} p_j) - k_0 B_{ij} p_j$.

When the octonion coordinate system rotates, we have the applied force density $F_q' = \Sigma(F'_j I'_j - f'_i I_i)$ . Under the coordinate transformation, the scalar part of $F_q'$ remains unchanged.

$$F_0 = F_0'$$  \hspace{1cm} (90)

When the right side is equal to zero, we have the charge continuity equation in the case for coexistence of the gravitational field, electromagnetic field, and adjoint fields.

$$\Sigma(\partial(P_i - p_i)/\partial r_i) + \Sigma(b_{ij} P_j - B_{ij} P_j)/v_0 + \Sigma k(b_{ij} P_j - B_{ij} p_j) = 0$$  \hspace{1cm} (91)

If last two terms are zeros, the above is reduced to

$$\Sigma(\partial P_i/\partial r_i) - \Sigma(\partial p_i/\partial r_i) = 0$$  \hspace{1cm} (92)

further, if the last term is equal to zero, we have

$$\Sigma(\partial P_i/\partial r_i) = 0$$  \hspace{1cm} (93)

The above states that the electromagnetic strength, gravitational strength, and adjoint field strengths have an effect on the charge continuity equation, although the impact is usually very tiny when fields are weak. And the charge continuity equation is an invariant under the octonion coordinate transformation.

3. Conservation of spin magnetic moment

In the octonion space, a new physical quantity $L_q = L \circ I_0'$ can be defined from Eq.(74).

$$L_q = L_0 + \Sigma(L_j I_j) - \Sigma(l_i I_i)$$  \hspace{1cm} (94)

where, $L_0 = (r_0 + k_x x_0)P_0 - \Sigma \{ (x_j + k_x x_j) P_j \} + \Sigma \{ (x_0 + k_x x_0) p_0 \}$ .

When the octonion coordinate system rotates, we have the angular momentum density $L_q' = \Sigma(L'_j I'_j - l'_i I_i)$ from Eq.(6). Under the octonion coordinate transformation, the scalar part of $L_q$ deduces the conservation of spin magnetic moment.

$$L_0 = L_0'$$  \hspace{1cm} (95)

The above states the spin magnetic moment density $L_0$ is an invariant in the case for coexistence of the electromagnetic field, gravitational field and their adjoint fields, under the octonion coordinate transformation.
further, if the last term is equal to zero, we have
under the octonion coordinate transformation.

When the octonion coordinate system rotates, we have
the scalar part of
When the right side is equal to zero in the above, we have
the conservation of power-like as follows.

When the right side is equal to zero in the above, we have
the continuity equation of the electromagnetic field, gravitational
field and their adjoint fields.

If last two terms are zeros, the above is reduced to

The above means that the power-like density $N_0$ is
one scalar invariant in the case for coexistence of the
electromagnetic field, gravitational field and their adjoint
fields, under the octonion coordinate transformation.

V. CONCLUSIONS

In the octonion space, the gravitational field described
by the octonion operator will generate one adjoint field.
Similarly, the electromagnetic field will be accompanied
by its adjoint field. These two sorts of adjoint fields will
impact the scalar invariants and conservation laws in the
electromagnetic field and gravitational field.

In the gravitational field with its gravitational adjoint
field, the gravitational mass density is changed with the
gravitational strength and adjoint field strength. And the
mass continuity equation will be changed with the field
strength, velocity, and adjoint mass. From the definitions
of the angular momentum and velocity, the spin density,
energy density, and power density will be variable for the
influence of the adjoint field potential and adjoint field
strength. While the spin continuity equation and energy
continuity equation will be changed with the impact of
the velocity and adjoint field strength etc.

The gravitational mass density will be variable in the case
for coexistence of the gravitational field and electromagnetic
field with their adjoint fields. The gravitational
mass density is changed with the gravitational strength,
electromagnetic strength, gravitational adjoint field, and
electromagnetic adjoint field. And the mass continuity
equation will be changed with the electromagnetic field
strength, velocity, and adjoint charge. The spin density,
energy density, and power density all will be variable for the influence of the electromagnetic adjoint field, adjoint field strength, and adjoint charge. Meanwhile the spin continuity equation and energy continuity equation will be changed with the impact of the adjoint charge and electromagnetic adjoint field strength etc.

In the gravitational field and electromagnetic field with their adjoint fields, there exist some electric invariants, which are associated with ordinary charge and adjoint mass. These electric invariants will be variable for the influence of the field potential and field strength. And some conservation laws and scalar invariants can not be effective simultaneously.

It should be noted that the study for some scalar invariants of electromagnetic and gravitational adjoint fields examined only one simple case with very weak field strength and low velocity in the gravitational field and electromagnetic field with their adjoint fields. Despite its preliminary character, this study can clearly indicate the field strength and adjoint fields of the gravitational field and electromagnetic field have the limited influence on the scalar invariants. For the future studies, the related investigation will concentrate on only the predictions of scalar invariants in the strong adjoint field strength with high velocity in gravitational field and electromagnetic field with their adjoint fields.

Acknowledgments

This project was supported partially by the National Natural Science Foundation of China under grant number 60677039, Science & Technology Department of Fujian Province of China under grant number 2005HZ1020 and 2006H0092, and Xiamen Science & Technology Bureau of China under grant number 3502Z20055011.

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