Solution to a problem of Katona on counting cliques of weighted graphs

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Abstract

A subset $I$ of the vertex set $V(G)$ of a graph $G$ is called a $k$-clique independent set of $G$ if no $k$ vertices in $I$ form a $k$-clique of $G$. An independent set is a 2-clique independent set. Let $\pi_k(G)$ denote the number of $k$-cliques of $G$. For a function $w : V(G) \to \{0, 1, 2, \ldots\}$, let $G(w)$ be the graph obtained from $G$ by replacing each vertex $v$ by a $w(v)$-clique $K^v$ and making each vertex of $K^v$ adjacent to each vertex of $K^u$ for each edge $\{u, v\}$ of $G$. For an integer $m \geq 1$, consider any $w$ with $\sum_{v \in V(G)} w(v) = m$. For $U \subseteq V(G)$, we say that $w$ is uniform on $U$ if $w(v) = 0$ for each $v \in V(G) \setminus U$ and, for each $u \in U$, $w(u) = \lfloor m/|U| \rfloor$ or $w(u) = \lceil m/|U| \rceil$. Katona asked if $\pi_k(G(w))$ is smallest when $w$ is uniform on a largest $k$-clique independent set of $G$. He placed particular emphasis on the Sperner graph $B_n$, given by $V(B_n) = \{X : X \subseteq \{1, \ldots, n\}\}$ and $E(B_n) = \\{\{X, Y\} : X \subset Y \in V(B_n)\}$. He provided an affirmative answer for $k = 2$ (and any $G$). We determine graphs for which the answer is negative for every $k \geq 3$. These include $B_n$ for $n \geq 2$. Generalizing Sperner’s Theorem and a recent result of Qian, Engel and Xu, we show that $\pi_k(B_n(w))$ is smallest when $w$ is uniform on a largest independent set of $B_n$. We also show that the same holds for complete multipartite graphs and chordal graphs. We show that this is not true of every graph, using a deep result of Bohman on triangle-free graphs.

1 Introduction

Let $\mathbb{N}$ denote set of positive integers, and let $\mathbb{N}_0$ denote $\{0\} \cup \mathbb{N}$. For $n \in \mathbb{N}_0$, let $[n]$ denote the $n$-set $\{i \in \mathbb{N} : i \leq n\}$ (note that $[0] = \emptyset$). For a set $X$, let $2^X$ denote the power set of $X (\{A : A \subseteq X\})$ and, for $k \in \mathbb{N}_0$, let $\binom{X}{k}$ denote $\{A \in 2^X : |A| = k\}$.

A family $\mathcal{A}$ of sets is called an antichain or a Sperner family if $A \nsubseteq B$ for every $A, B \in \mathcal{A}$ with $A \neq B$. A cornerstone in extremal set theory is Sperner’s Theorem [12], which bounds the size of an antichain.

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Theorem 1 (Sperner’s Theorem [12]). If $\mathcal{A} \subseteq 2^{[n]}$ and $\mathcal{A}$ is an antichain, then

$$|\mathcal{A}| \leq \binom{n}{\frac{n}{2}}.$$ 

Moreover, equality holds if and only if $\mathcal{A} = \binom{[n]}{\frac{n}{2}}$ or $\mathcal{A} = \binom{[n]}{\frac{n}{2}}$.

There have been many generalizations, extensions and variants of Theorem 1; see, for example, [4, 5, 6, 9]. Of particular relevance to this paper is a generalization due to Qian, Engel and Xu [11] in which repetition of sets is allowed. A multifamily is a pair $(\mathcal{F}, q)$ such that $\mathcal{F}$ is a family and $q$ is a function with domain $\mathcal{F}$ and codomain $\mathbb{N}_0$. A multifamily can be viewed as a family $\mathcal{F}$ such that, for each $F \in \mathcal{F}$, $F$ appears $q(F)$ times. Let

$$\theta(\mathcal{F}, q) = \sum_{F \in \mathcal{F}} \left( \frac{q(F)}{2} \right) + \sum_{F, F' \in \mathcal{F}, F \subseteq F'} q(F)q(F').$$

Theorem 2 (Qian, Engel and Xu [11]). For any $n \geq 1$ and $m \geq 1$, the minimum of $\theta(\mathcal{F}, q)$ over all multifamilies $(\mathcal{F}, q)$ with $\mathcal{F} \subseteq 2^{[n]}$ and $\sum_{F \in \mathcal{F}} q(F) = m$ is attained if $\mathcal{F} \in \{ \binom{[n]}{\lfloor n/2 \rfloor}, \binom{[n]}{\lceil n/2 \rceil} \}$ and $q(F) \in \{ |m/|\mathcal{F}| |, m/|\mathcal{F}| \}$ for each $F \in \mathcal{F}$.

Qian, Engel and Xu actually proved that the result holds for the quantity $\theta(\mathcal{F}, q) + \sum_{F \in \mathcal{F}} \left( \frac{q(F)}{2} \right)$ rather than $\theta(\mathcal{F}, q)$ [11, Theorem 1.1], but Theorem 2 follows from this and Theorem 1.

Recently, Katona [8] obtained a far-reaching generalization of Theorem 2. To be able to state his result, we require a number of definitions.

As usual, we denote the vertex set and the edge set of a graph $G$ by $V(G)$ and $E(G)$, respectively. We take any graph $G$ to be simple, that is, $G = (V(G), E(G))$ with $E(G) \subseteq \binom{V(G)}{2}$. We may represent any edge $(u, v)$ by $uv$. The open neighbourhood $N_G(v)$ of a vertex $v$ of $G$ is the set of neighbours of $v$, that is, $N_G(v) = \{ u \in V(G) : uv \in E(G) \}$. The closed neighbourhood $N_G[v]$ of $v$ is the set $N_G(v) \cup \{ v \}$. For $X \subseteq V(G)$, $G[X]$ denotes the subgraph of $G$ induced by $X$, that is, $G[X] = (X, E(G) \cap \binom{X}{2})$.

For a graph $G$ and a (weight) function $w : V(G) \to \mathbb{N}_0$, let $G(w)$ be the graph obtained from $G$ by replacing each vertex $v$ by a $w(v)$-clique $K^u$ and making each vertex of $K^u$ adjacent to each vertex of $K^v$ for each edge $uv$ of $G$. More formally, $G(w)$ is given by

$$V(G(w)) = \{(v, i) : v \in V(G), i \in [w(v)]\}$$

and

$$E(G(w)) = \bigcup_{v \in V(G)} \left( \{(v, i) : i \in [w(v)]\} \right) \cup \bigcup_{uv \in E(G)} \{(u, i)(v, j) : i \in [w(u)], j \in [w(v)]\}.$$ 

We call $w$ an $m$-weighting of $G$, where $m = \sum_{v \in V(G)} w(v)$. For $k \geq 1$, let $K_k(G)$ denote the set of vertex sets of $k$-cliques ($k$-vertex complete subgraphs) of $G$, and let $\pi_k(G)$ denote
the number of $k$-cliques of $G$. Thus, $\pi_k(G) = |K_k(G)|$. Observe that the number of edges of $G(w)$ is

$$
\pi_2(G(w)) = \sum_{v \in V(G)} \left(\frac{w(v)}{2}\right) + \sum_{uv \in E(G)} w(u)w(v),
$$

and for any $k \geq 1$,

$$
\pi_k(G(w)) = \sum_{i=1}^{k} \sum_{\{v_1, \ldots, v_i\} \in K_i(G)} \sum_{k_1 + \cdots + k_i = k} \frac{i!}{k_1! \cdots k_i!} \prod_{j=1}^{i} \binom{w(v_j)}{k_j}.
$$

Given an $m$-weighting $w$ of $G$ and a subset $U$ of $V(G)$, we say that $w$ is uniform on $U$ if $w(v) = 0$ for each $v \in V(G) \setminus U$ and, for each $u \in U$, $w(u) = \lfloor m/|U| \rfloor$ or $w(u) = \lceil m/|U| \rceil$. If $U$ is an independent set of $G$ (that is, $uv \notin E(G)$ for every $u, v \in U$) of maximum size and $w$ is uniform on $U$, then, as in [8], $w$ is said to be uniform-$\alpha$.

For $n \geq 1$, let $B_n$ be the Sperner graph given by $V(B_n) = 2^n$ and $E(G) = \{XY : X \subseteq Y \in 2^n\}$. Note that Sperner’s Theorem gives us the size of a largest independent set of $B_n$. Theorem 2 may thus be restated as follows.

**Theorem 3.** If $n \geq 1$, $m \geq 1$, $w$ and $w'$ are $m$-weightings of $B_n$, and $w'$ is uniform-$\alpha$, then $\pi_2(B_n(w')) \leq \pi_2(B_n(w))$.

Rather surprisingly, Katona showed that Theorem 3 can be generalized to arbitrary graphs.

**Theorem 4 (Katona [8]).** If $m \geq 1$, $w$ and $w'$ are $m$-weightings of a graph $G$, and $w'$ is uniform-$\alpha$, then $\pi_2(G(w')) \leq \pi_2(G(w))$.

He then asked the general question below. For a graph $G$ and $k \geq 1$, we call a subset $I$ of $V(G)$ a $k$-clique independent set of $G$ if no $k$-element subset of $I$ is a member of $K_k(G)$ (that is, $\binom{I}{k} \cap K_k(G) = \emptyset$). Let the size of a largest $k$-clique independent set of $G$ be denoted by $\alpha_k(G)$ and called the $k$-clique independence number of $G$.

**Problem 1 ([8] Problem 3).** Is it true that if $m \geq 1$, $k \geq 2$, $w$ and $w'$ are $m$-weightings of a graph $G$, and $w'$ is uniform on a largest $k$-clique independent set of $G$, then $\pi_k(G(w')) \leq \pi_k(G(w))$?

Theorem 4 provides a positive answer to Problem 1 for $k = 2$. Unfortunately, if $k \geq 3$, one cannot hope for a positive answer to Problem 1, as shown in the following counterexample and in Section 4 (see Remarks 1 and 2).

**Counterexample 1.** Let $G$ be the 3-vertex path ($\{a, b, c\}$, $\{ab, bc\}$), let $k \geq 3$, let $w'$ be the $3k$-weighting of $G$ that is uniform on $V(G)$, and let $w$ be the $3k$-weighting of $G$ defined by $w(a) = w'(a) + w'(b)$, $w(b) = 0$, and $w(c) = w'(c)$. Note that $V(G)$ is the largest $k$-clique independent set of $G$. We have

$$
\pi_k(G(w')) = \binom{w'(a) + w'(b)}{k} + \binom{w'(b) + w'(c)}{k} - \binom{w'(b)}{k} = 2\binom{2k}{k} - \binom{k}{k},
$$
\[ \pi_k(G(w)) = \binom{w'(a) + w'(b)}{k} + \binom{w'(c)}{k} = \binom{2k}{k} + \binom{k}{k}, \]

and hence \( \pi_k(G(w)) < \pi_k(G(w')). \)

As our primary contribution, we completely address Problem 1 for Sperner graphs, thereby answering another question of Katona [8, Problem 3].

**Theorem 5.** If \( n \geq 1, m \geq 1, k \geq 2, w \) and \( w' \) are \( m \)-weightings of \( B_n \), and \( w' \) is uniform on \( \binom{[n]}{\lfloor n/2 \rfloor} \), then \( \pi_k(B_n(w')) \leq \pi_k(B_n(w)) \).

Theorem 5 generalizes the inequalities in Theorems 1 and 3. Indeed, let \( A \subseteq 2^{[n]} \) such that \( A \) is an antichain. Let \( w_1 \) and \( w_2 \) be \( \left( \binom{n}{\lfloor n/2 \rfloor} + 1 \right) \)-weightings of \( B_n \) such that \( w_1 \) is uniform on \( A \) and \( w_2 \) is uniform on \( \binom{[n]}{\lfloor n/2 \rfloor} \). Note that \( \pi_2(B_n(w_2)) = 1 \). Thus, by Theorem 5, \( \pi_2(B_n(w_1)) \geq 1 \). The inequality in Theorem 1 follows. Now, by Theorem 1, \( w' \) is uniform-\( \alpha \). By Theorem 3 for \( k = 2 \), Theorem 5 follows.

**Remark 1.** In [8, Problem 3], Katona placed particular emphasis on solving Problem 1 for \( G = B_n \). Theorem 5 already tells us how to minimize \( \pi_k(B_n(w)) \). In Section 4, we show that, furthermore, the answer to Problem 1 for \( G = B_n \) is negative if \( n \geq 2, k \geq 3, \) and \( m \geq k\alpha_k(B_n) \).

We also address Problem 1 for complete multipartite graphs and chordal graphs.

If \( I_1, \ldots, I_r \) are pairwise disjoint non-empty sets and \( G \) is the graph with \( V(G) = \bigcup_{i=1}^{r} I_i \) and \( E(G) = \bigcup_{(i,j) \in \binom{[r]}{2}} \{xy: x \in I_i, y \in I_j \} \), then \( G \) is called a complete multipartite graph, and \( I_1, \ldots, I_r \) are called the maximal partite sets of \( G \).

**Theorem 6.** If \( m \geq 1, k \geq 2, w \) and \( w' \) are \( m \)-weightings of a complete multipartite graph \( G \), and \( w' \) is uniform-\( \alpha \), then \( \pi_k(G(w')) \leq \pi_k(G(w)) \).

**Remark 2.** In Section 4, we show that, furthermore, if \( G \) is a complete multipartite graph with a maximal partite set \( I \) that is larger than the others, \( k \geq 3, \) and \( m \geq k\alpha_k(G) \), then the answer to Problem 1 is negative. Note that Counterexample 1 is the special case where the maximal partite sets of \( G \) are \( \{a, c\} \) and \( \{b\} \).

Let \( K(G) \) denote the set of vertex sets of cliques of \( G \); that is, \( K(G) = \bigcup_{k=1}^{\lfloor V(G) \rfloor} K_k(G) \). A graph \( G \) is said to be chordal if for some sequence \( v_1, \ldots, v_n \) such that \( V(G) = \{v_1, \ldots, v_n\} \) and \( n = |V(G)|, N_G[v_i] \setminus \{v_j: j \in [i-1]\} \in K(G) \) for each \( i \in [n] \).

**Theorem 7.** If \( m \geq 1, k \geq 2, w \) and \( w' \) are \( m \)-weightings of a chordal graph \( G \), and \( w' \) is uniform-\( \alpha \), then \( \pi_k(G(w')) \leq \pi_k(G(w)) \).

In view of Theorems 5, 7, one may wonder whether the minimum of \( \pi_k(G(w)) \) is always attained when \( w \) is uniform-\( \alpha \). We show this to be false for \( k = 3 \), using a result on triangle-free graphs (graphs containing no 3-clique) due to Bohman [2].
Theorem 8. There exist a graph $G$, a positive integer $m$, and an $m$-weighting $w$ of $G$ such that $\pi_3(G(w)) < \pi_3(G(w'))$ for every uniform-$\alpha$ $m$-weighting $w'$ of $G$.

We propose the following conjecture.

Conjecture 1. If $m \geq 1$, $k \geq 2$, and $G$ is a graph, then for some $k$-clique independent set $I$ of $G$ and some $m$-weighting $w'$ of $G$ that is uniform on $I$, $\pi_k(G(w')) \leq \pi_k(G(w))$ for any $m$-weighting $w$ of $G$.

The paper is organized as follows. Section 2 contains basic tools that are used in the proofs of Theorems 5–7. In Section 3, we establish a general weight shifting lemma from which Theorems 5–7 follow, and we prove Theorems 6 and 7. In Section 4, we prove our main results, given by Theorems 5 and 8 and the claims in Remarks 4 and 2.

2 Basic lemmas

The following known fact is very useful, and we prove it for completeness.

Lemma 1. If $n \geq 1$, $m \geq 1$, $k \geq 2$, $w$ and $w'$ are $m$-weightings of an $n$-vertex graph $G$ with no edges, and $w'$ is uniform-$\alpha$, then $\pi_k(G(w')) \leq \pi_k(G(w))$.

Remark 3. Let $k$, $G$, and $w'$ be as in Lemma 1. Thus, $w'(v) \in \{\lfloor m/n \rfloor, \lceil m/n \rceil\}$ for each $v \in V(G)$. By the division algorithm, $m = \lceil m/n \rceil n + r$ for some $r \in \{0\} \cup [n-1]$. If $r = 0$, then $\lfloor m/n \rfloor = m/n = \lceil m/n \rceil$. Suppose $r \neq 0$. Then, $m/n > \lfloor m/n \rfloor = \lceil m/n \rceil - 1$. Since $\sum_{v \in V(G)} w'(v) = m$, we obtain $\lfloor |\{v \in V(G) : w'(v) = \lfloor m/n \rfloor\}| \rfloor = n - r$ and $\lfloor |\{v \in V(G) : w'(v) = \lceil m/n \rceil\}| \rfloor = r$. Therefore, if $w_1$ and $w_2$ are uniform-$\alpha$ $m$-weightings of $G$, then $\pi_k(G(w_1)) = \pi_k(G(w_2))$.

Proof of Lemma 7. Let $v_1, v_2 \in V(G)$ such that $w(v_1) = \min\{w(v) : v \in V(G)\}$ and $w(v_2) = \max\{w(v) : v \in V(G)\}$. Since $\sum_{v \in V(G)} w(v) = m$, we have $w(v_1) \leq m/n$ and $w(v_2) \geq m/n$. Since $w(v) \in \mathbb{N}_0$ for each $v \in V(G)$, $w(v_1) \leq \lfloor m/n \rfloor$ and $w(v_2) \geq \lfloor m/n \rfloor$. If $w(v_1) = \lfloor m/n \rfloor$ and $w(v_2) = \lceil m/n \rceil$, then $w$ is uniform-$\alpha$, so $\pi_k(G(w')) = \pi_k(G(w))$ by Remark 3. Suppose $w(v_1) \neq \lfloor m/n \rfloor$ or $w(v_2) \neq \lceil m/n \rceil$. Then, $w(v_1) \leq \lfloor m/n \rfloor - 1$ or $w(v_2) \geq \lceil m/n \rceil + 1$.

Suppose $w(v_1) \leq \lfloor m/n \rfloor - 1$. Let $w_1$ be the $m$-weighting of $G$ such that $w_1(v_1) = w(v_1) + 1$, $w_1(v_2) = w(v_2) - 1$, and $w_1(v) = w(v)$ for each $v \in V(G) \setminus \{v_1, v_2\}$. We have

$$\pi_k(G(w)) - \pi_k(G(w_1)) = \left(\frac{w(v_2)}{k}\right) - \left(\frac{w(v_2) - 1}{k}\right) + \left(\frac{w(v_1)}{k}\right) - \left(\frac{w(v_1) + 1}{k}\right)$$

$$= \left(\frac{w(v_2) - 1}{k - 1}\right) - \left(\frac{w(v_1)}{k - 1}\right) \geq 0$$

as $w(v_2) - 1 \geq w(v_1)$ and $k - 1 \geq 1$. Thus, $\pi_k(G(w)) \geq \pi_k(G(w_1))$. We apply this procedure until we obtain an $m$-weighting $w_p$ of $G$ such that $\min\{w_p(v) : v \in V(G)\} = 0$.
\[ m/n \]. We have \( \pi_k(G(w)) \geq \pi_k(G(w_p)) \). Let \( v_{p,1}, v_{p,2} \in V(G) \) such that \( w_p(v_{p,1}) = [m/n] \) and \( w_p(v_{p,2}) = \max\{w_p(v) : v \in V(G)\} \). If \( w_p(v_{p,2}) = [m/n] \), then \( w_p \) is uniform-\( \alpha \), so \( \pi_k(G(w')) = \pi_k(G(w_p)) \) by Remark 3. Suppose \( w_p(v_{p,2}) \geq [m/n] \). Since \( w_p(v_{p,2}) \geq [m/n] \), we obtain \( w_p(v_{p,2}) \geq [m/n] + 1 \). Since \( \sum_{u \in V(G)} w_p(v) = m, m/n > [m/n] = w_p(u) \) for some \( u \in V(G) \setminus \{v, v_{p,2}\} \). Let \( w_{p+1} \) be the \( m \)-weighting of \( G \) such that \( w_{p+1}(u) = w_p(u) + 1, w_{p+1}(v_{p,2}) = w_p(v_{p,2}) - 1 \), and \( w_{p+1}(v) = w_p(v) \) for each \( v \in V(G) \setminus \{u, v_{p,2}\} \). As in (1), we obtain \( \pi_k(G(w_{p+1})) \geq \pi_k(G(w_p)) \). We apply this procedure until we obtain an \( m \)-weighting \( w_q \) of \( G \) such that \( \max\{w_q(v) : v \in V(G)\} = [m/n] \). Since \( w_q(v_{p,1}) = [m/n] = \min\{w_q(v) : v \in V(G)\} \), \( w_q \) is uniform-\( \alpha \), so \( \pi_k(G(w')) = \pi_k(G(w_q)) \) by Remark 3.

If \( w(v) \geq [m/n] + 1 \), then \( \pi_k(G(w')) \leq \pi_k(G(w)) \) by a similar argument.

Lemma 2. If \( n \geq 1 \), \( m \geq 1 \), \( k \geq 2 \), \( w \) and \( w' \) are \( m \)-weightings of an \( n \)-vertex graph \( G \), \( I \) and \( I' \) are independent sets of \( G \) with \( |I| \leq |I'| \), \( w(v) = 0 \) for each \( v \in V(G) \setminus I \), and \( w' \) is uniform on \( I' \), then \( \pi_k(G(w')) \leq \pi_k(G(w)) \).

Proof. Let \( u_1, \ldots, u_r \) be the distinct vertices in \( I \). Let \( v_1, \ldots, v_s \) be the distinct vertices in \( I' \). Then, \( r \leq s \). Let \( H = G[I'] \). Let \( w_H \) be the \( m \)-weighting of \( H \) such that \( w_H(v_i) = w(u_i) \) for each \( i \in [r] \) and \( w_H(v_j) = 0 \) for each \( j \in [s] \setminus [r] \). Let \( w'_H \) be the uniform-\( \alpha \) \( m \)-weighting of \( H \) such that \( w'_H(v_i) = w'(v_i) \) for each \( i \in [s] \). By Lemma 1, \( \pi_k(H(w'_H)) \leq \pi_k(H(w_H)) \). Since \( \pi_k(H(w'_H)) = \pi_k(G(w')) \) and \( \pi_k(H(w_H)) = \pi_k(G(w)) \), the result follows.

3 A weight shifting lemma

In the proof of Theorem 4, Katona defined the following weight shifting operation along an edge. For a graph \( G \), an \( m \)-weighting \( w \) of \( G \), and an edge \( ab \) of \( G \), let \( w_{ab} \) be the \( m \)-weighting of \( G \) given by

\[
 w_{ab}(v) = \begin{cases} 
 0 & \text{if } v = a, \\
 w(b) + w(a) & \text{if } v = b, \\
 w(v) & \text{otherwise} 
\end{cases}
\]

for each \( v \in V(G) \). It was proved in [8] that \( \pi_2(G(w_{ab})) \leq \pi_2(G(w)) \) or \( \pi_2(G(w_{ab})) \leq \pi_2(G(w)) \). Thus, by applying the weight shift operation repeatedly, one arrives at an \( m \)-weighting \( w' \) of \( G \) such that \( \pi_2(G(w')) \leq \pi_2(G(w)) \) and \( w' \) is non-zero only on an independent set. Theorem 4 follows from this.

Remark 4. Unfortunately, for \( k \geq 3 \), Katona’s shifting technique does not always decrease \( \pi_k(G(w)) \) or leave it unchanged. For instance, for \( k = 3 \), consider the illustration in Figure 4. If we start with the \( m \)-weighting in Figure 4(a), then each shift produces a larger number of triangles, as demonstrated in Figures 4(b) and 4(c).

In view of the remark above, new ideas would therefore be needed to address Problem 1. Still, instead of shifting along one edge at a time, is shifting simultaneously along many edges at a time conceivable? In our main lemma, we show that this is indeed the case, provided a number of conditions are satisfied.
Lemma 3. If \( m \geq 1, k \geq 2, w \) is an \( m \)-weighting of a graph \( G, A = \{a_1, \ldots, a_r\} \) and \( B = \{b_1, \ldots, b_r\} \) are disjoint \( r \)-element subsets of \( V(G) \) such that

1. \( a_ib_i \in E(G) \) for each \( i \in [r] \),
2. \( B \) is an independent set of \( G \), and
3. \( N_G(b_i) \setminus (A \cup \{v \in V(G) : w(v) = 0\}) \subseteq N_G(a_i) \) for each \( i \in [r] \),

and \( w' \) is the \( m \)-weighting of \( G \) given by

\[
w'(v) = \begin{cases} 
0 & \text{if } v \in A, \\
w(b_i) + w(a_i) & \text{if } v = b_i \text{ for some } i \in [r], \\
w(v) & \text{otherwise}
\end{cases}
\]

for each \( v \in V(G) \), then \( \pi_k(G(w')) \leq \pi_k(G(w)) \).

Proof. Our strategy is to associate a unique \( k \)-clique of \( G(w) \) to each \( k \)-clique of \( G(w') \). More precisely, we construct an injective function from \( \mathcal{K}_k(G(w')) \) to \( \mathcal{K}_k(G(w)) \). This gives us \( \pi_k(G(w')) \leq \pi_k(G(w)) \).

Let \( \phi : V(G(w')) \to V(G(w)) \) such that, for each \((v, i) \in V(G(w'))\),

\[
\phi((v, i)) = \begin{cases} 
(a_j, i - w(b_j)) & \text{if } v = b_j \text{ and } i \in [w(b_j) + w(a_j)] \setminus [w(b_j)], \\
(v, i) & \text{otherwise.}
\end{cases}
\]

Note that \( \phi \) is bijective. We may abbreviate \( \phi((v, i)) \) to \( \phi(v, i) \).

Consider any \( X \in \mathcal{K}_k(G(w')) \). Thus, \( G(w')[X] \) is a \( k \)-clique of \( G(w') \). Let \((v_1, y_1), \ldots, (v_k, y_k)\) be the vertices in \( X \). Let \( \phi(X) \) denote \( \{\phi(v_1, y_1), \ldots, \phi(v_k, y_k)\} \). Since \( \phi \) is injective, \( |\phi(X)| = k \). We show that \( \phi(X) \in \mathcal{K}_k(G(w)) \), that is, \( \phi(v_i, y_i)\phi(v_j, y_j) \in E(G(w)) \) for every \( i, j \in [k] \) with \( i \neq j \). If \( \phi(v_i, y_i) = (v_i, y_i) \) and \( \phi(v_j, y_j) = (v_j, y_j) \), then \( y_i \in [w(v_i)] \), \( y_j \in [w(v_j)] \), and, since \( (v_i, y_i)(v_j, y_j) \in E(G(w')) \), \( (v_i, y_i)(v_j, y_j) \in E(G(w)) \). Suppose \( \phi(v_i, y_i) \neq (v_i, y_i) \) or \( \phi(v_j, y_j) \neq (v_j, y_j) \). We may assume that \( \phi(v_i, y_i) \neq (v_i, y_i) \). Thus, \( v_i = b_p \) for some \( p \in [r] \), \( y_i \notin [w(v_i)] \), and \( \phi(v_i, y_i) = (a_p, y) \) for some \( y \in [w(a_p)] \). If \( v_j = b_p \), then \( \phi(v_j, y_j) \in \{X \times [w(a_p)] \cup (\{b_p\} \times [w(b_p)]) \).
Suppose \( \Phi \) be the distinct vertices in \( W \) for each \( v \in B \). Since \( (v, j) \in V(G(w')) \), we have \( w'(v) \neq 0 \), so \( v \notin A \). Since \( v \notin A \cup B \), we have \( \phi(v, j) = (v, j), y_j \in [w(v)] \), and hence \( w(v) \neq 0 \). Given that \( N_G(B) \setminus (A \cup \{ v \in V(G) : w(v) = 0 \}) \subseteq N_G(B) \), \( v \in N_G(B) \) (as \( b_v = v \in E(G) \)). Thus, since \( \phi(v, j) = (u, y)(v, j), \phi(v, j)(v, j) \in E(G) \).

Let \( \Phi : \mathcal{K}_k(G(w')) \rightarrow \mathcal{K}_k(G(w)) \) such that \( \Phi(X) = \phi(X) \) for each \( X \in \mathcal{K}_k(G(w')) \). Suppose \( \Phi(X_1) = \Phi(X_2) \) for some \( X_1, X_2 \in \mathcal{K}_k(G(w')) \). Let \( (v_1, y_1), \ldots, (v_k, y_k) \) be the members of \( X_2 \). For each \( i \in [k] \), let \( (v'_i, y'_i) \) be the member of \( X_1 \) such that \( \phi(v'_i, y'_i) = \phi(v_i, y_i) \). Since \( \phi \) is injective, \( (v'_i, y'_i) = (v_i, y_i) \) for each \( i \in [k] \). Thus, \( X_1 = X_2 \). Therefore, \( \Phi \) is injective, and hence the size of its domain \( \mathcal{K}_k(G(w')) \) is at most the size of its codomain \( \mathcal{K}_k(G(w)) \).

We remark that the condition in Lemma 3 that \( B \) is an independent set can be generalized to condition 2 in the next lemma, using the same argument.

**Lemma 4.** If \( m \geq 1, k \geq 2, w \) is an \( m \)-weighting of a graph \( G \), \( A = \{a_1, \ldots, a_r\} \) and \( B = \{b_1, \ldots, b_r\} \) are disjoint \( r \)-element subsets of \( V(G) \) such that

1. \( a_i b_i \in E(G) \) for each \( i \in [r] \),
2. \( a_i a_j, a_j b_i \in E(G) \) for every \( i, j \in [r] \) with \( b_ib_j \in E(G) \), and
3. \( N_G(b_i) \setminus (A \cup \{ v \in V(G) : w(v) = 0 \}) \subseteq N_G(a_i) \) for each \( i \in [r] \),

and \( w' \) is the \( m \)-weighting of \( G \) given by

\[
w'(v) = \begin{cases} 
0 & \text{if } v \in A, \\
w(b_i) + w(a_i) & \text{if } v = b_i \text{ for some } i \in [r], \\
w(v) & \text{otherwise}
\end{cases}
\]

for each \( v \in V(G) \), then \( \pi_k(G(w')) \leq \pi_k(G(w)) \).

We now provide the proofs of Theorems 6 and 7 immediately demonstrating the applicability of Lemma 3. The lemma also has a crucial role in the proof of Theorem 8.

**Proof of Theorem 6.** Let \( I_1, \ldots, I_r \) be the distinct maximal partite sets of \( G \) such that \( |I_1| \geq \cdots \geq |I_r| \). Note that \( I_1, \ldots, I_r \) are independent sets of \( G \). For each \( j \in [r] \), let \( n_j = |I_j| \) and let \( a_{j, 1}, \ldots, a_{j, n_j} \) be the distinct vertices in \( I_j \).

Let \( W \) be the set of \( m \)-weightings of \( G \). For each \( w \in W \), let \( h(w) = \max\{j \in [r] : w(v) \neq 0 \text{ for some } v \in I_j\} \). Let \( W^* = \{ w^* \in W : \pi_k(G(w^*)) \leq \pi_k(G(w)) \text{ for each } w \in W \} \). Let \( w_0 \in W^* \) such that \( h(w_0) \leq h(w) \) for each \( w \in W^* \).

Suppose \( h(w_0) > 1 \). Let \( s = h(w_0) \). Let \( J = \{a_{s-1,1}, \ldots, a_{s-1,n_s}\} \). Since \( J \subseteq I_{s-1} \), \( J \) is an independent set of \( G \). Since \( G \) is a complete multipartite graph, \( a_{s,i} a_{s-1, i} \in E(G) \) and \( N_G(a_{s-1, i}) \setminus I_s \subseteq N_G(a_{s, i}) \) for each \( i \in [n_s] \). Let \( w_1 \) be the \( m \)-weighting of \( G \) such that \( w_1(a_{s,i}) = 0 \) for each \( i \in [n_s], w_1(a_{s-1, i}) = w_0(a_{s-1, i}) + w_0(a_{s, i}) \) for each \( i \in [n_s] \), and
\[ w_1(v) = w_0(v) \text{ for each } v \in V(G) \setminus (I_s \cup J). \] By Lemma 3 \( \pi_k(G(w_1)) \leq \pi_k(G(w_0)) \). We have \( h(w_1) = h(w_0) - 1 \), contradicting the choice of \( w_0 \). Therefore, \( s = 1 \).

We have shown that \( w_0(v) = 0 \) for each \( v \in V(G) \setminus I_1 \). By Lemma 2 \( \pi_k(G(w')) \leq \pi_k(G(w_0)) \). Consequently, \( w' \in W^* \).

**Proof of Theorem 7** We use induction on \( |V(G)| + |E(G)| \). Let \( n = |V(G)| \). The result is trivial if \( n = 1 \) (the base case). Suppose \( n > 2 \). If \( E(G) = \emptyset \), then the result is given by Lemma 1. Suppose \( E(G) \neq \emptyset \). Since \( G \) is chordal, there exists a sequence \( v_1, \ldots, v_n \) such that \( V(G) = \{v_1, \ldots, v_n\} \) and \( N_G[v_i] \setminus \{v_j : j \in [i-1]\} \in K(G) \) for each \( i \in [n] \).

Suppose \( N_G(v_1) = \emptyset \). Let \( G_1 = G[V(G) \setminus \{v_1\}] \) and \( m_1 = m - w(v_1) \). Let \( w_1 \) be the \( m_1 \)-weighting of \( G_1 \) such that \( w_1(v) = w_0(v) \) for each \( v \in V(G_1) \). Let \( w'_1 \) be a uniform-\( \alpha \)-\( m_1 \)-weighting of \( G_1 \). Then, for some largest independent set \( I_1 \) of \( G_1 \), \( w'_1(v) = 0 \) for each \( v \in V(G_1) \setminus I_1 \) and, for each \( u \in I_1 \), \( w'_1(u) = [m_1/|I_1|] \) or \( w'_1(u) = [m_1/|I_1|] \). By the induction hypothesis, \( \pi_k(G_1(w'_1)) \leq \pi_k(G_1(w_1)) \). Let \( w'' \) be the \( m' \)-weighting of \( G \) such that \( w''(v_1) = w(v_1) \) and \( w''(v) = w'_1(v) \) for each \( v \in V(G_1) \). We have \( \pi_k(G(w'')) = (\pi_k(G_1(w'_1)) + \pi_k(G_1(w_1))) \leq (\pi_k(G_1(w'_1)) + \pi_k(G_1(w_1))) = \pi_k(G(w)) \). Since \( N_G(v_1) = \emptyset \), \( \{v_1\} \cup I_1 \) is an independent set of \( G \), so \( \pi_k(G(w')) \leq \pi_k(G(w'')) \) by Lemma 2. Thus, \( \pi_k(G(w')) \leq \pi_k(G(w)) \).

Now suppose \( N_G(v_1) \neq \emptyset \). Let \( h = \min \{i \in [n] \setminus \{1\} : v_i \in N_G(v_1)\} \). Let \( w^* \) be the \( m \)-weighting of \( G \) such that \( w^*(v_1) = w(v_1) + w(v_h) \), \( w^*(v_h) = 0 \), and \( w^*(v) = w(v) \) for each \( v \in V(G) \setminus \{v_1, v_h\} \). By Lemma 3 \( \pi_k(G(w^*)) \leq \pi_k(G(w)) \). Let \( G_2 = G[V(G) \setminus \{v_h\}] \). Let \( w' \) be the \( m \)-weighting of \( G_2 \) such that \( w'_2(v) = w^*(v) \) for each \( v \in V(G_2) \). Let \( w'_2 \) be a uniform-\( \alpha \)-\( m_2 \)-weighting of \( G_2 \). By the induction hypothesis, \( \pi_k(G_2(w'_2)) \leq \pi_k(G_2(w_2)) \). Let \( w'' \) be the \( m \)-weighting of \( G \) such that \( w''(v_h) = 0 \) and \( w''(v) = w'_2(v) \) for each \( v \in V(G_2) \). By Lemma 2 \( \pi_k(G(w'')) \leq \pi_k(G(w'')) \). Since \( \pi_k(G(w'')) = \pi_k(G_2(w'_2)) \leq \pi_k(G_2(w_2)) = \pi_k(G(w^*)) \leq \pi_k(G(w))^* \), it follows that \( \pi_k(G(w')) \leq \pi_k(G(w)) \).

**4 Proofs of the main results**

In the proof of Theorem 5 we use the following well-known consequence of the König–Hall Theorem [10, 7]; see, for example, [3] page 7, Corollary 4.

**Lemma 5.** Let \( r \) and \( n \) be integers such that \( 0 \leq r \leq n \). If \( r < n/2 \), then there exists an injection \( f : \binom{n}{r} \to \binom{n}{r+1} \) such that \( A \subset f(A) \) for each \( A \in \binom{n}{r} \). If \( r > n/2 \), then there exists an injection \( f : \binom{n}{r} \to \binom{n}{r-1} \) such that \( f(A) \subset A \) for each \( A \in \binom{n}{r} \).

**Proof of Theorem 5** Let \( W \) be the set of \( m \)-weightings of \( B_n \). For any \( w \in W \), let \( g(w) = \min \{i \in [0] \cup [n] : w(v) \neq 0 \text{ for some } v \in \binom{n}{i}\} \) and \( g'(w) = \max \{i \in [0] \cup [n] : w(v) \neq 0 \text{ for some } v \in \binom{n}{i}\} \), and let

\[
\begin{align*}
h(w) &= \min \{[n/2], g(w)\} \quad \text{and} \quad h'(w) = \max \{[n/2], g'(w)\}.
\end{align*}
\]
Let $W^* = \{ w^* \in W : \pi_k(G(w^*)) \leq \pi_k(G(w)) \text{ for each } w \in W \}$. Let $w_0 \in W^*$ such that $h'(w_0) - h(w_0) \leq h'(w) - h(w)$ for each $w \in W^*$. It suffices to show that $\pi_k(B_n(w')) \leq \pi_k(B_n(w_0))$.

Suppose $h(w_0) < [n/2]$. Let $a_1, \ldots, a_r$ be the distinct members of $\binom{[n]}{h(w_0)}$. By Lemma 5, there exist $r$ distinct members $b_1, \ldots, b_r$ of $\binom{[n]}{h(w_0)+1}$ such that $a_1 b_1, \ldots, a_r b_r \in E(B_n)$. Let $I = \{ a_1, \ldots, a_r \}$ and $J = \{ b_1, \ldots, b_r \}$. For any $i \in [r]$ and $c \in V(B_n)$ with $b_i \subset c$, we have $a_i \subset c$ as $a_i \subset b_i$. Thus, $N_{B_n}(a_i) \subset I \cup \{ v \in V(B_n) : w_0(v) = 0 \} \subset N_{B_n}(a_i)$ for each $i \in [r]$. Let $w_1$ be the $m$-weighting of $B_n$ such that $w_1(a_i) = 0$ for each $i \in [r]$, $w_1(b_i) = w_0(b_i) + w_0(a_i)$ for each $i \in [r]$, and $w_1(v) = w_0(v)$ for each $v \in V(B_n) \setminus (I \cup J)$. By Lemma 3, $\pi_k(B_n(w_1)) \leq \pi_k(B_n(w_0))$. We have $h'(w_1) - h(w_1) = h'(w_0) - h(w_0) - 1$, contradicting the choice of $w_0$. Therefore, $h(w_0) = [n/2]$.

Suppose $h'(w_0) > [n/2]$. Let $a_1, \ldots, a_r$ be the distinct members of $\binom{[n]}{h(w_0)+1}$. By Lemma 5, there exist $r$ distinct members $b_1, \ldots, b_r$ of $\binom{[n]}{h(w_0)}$ such that $a_1 b_1, \ldots, a_r b_r \in E(B_n)$. Let $I = \{ a_1, \ldots, a_r \}$ and $J = \{ b_1, \ldots, b_r \}$. For any $i \in [r]$ and $c \in V(B_n)$ with $c \subset b_i$, we have $c \subset a_i$ as $b_i \subset a_i$. Thus, $N_{B_n}(b_i) \subset I \cup \{ v \in V(B_n) : w_0(v) = 0 \} \subset N_{B_n}(a_i)$ for each $i \in [r]$. Let $w_1$ be the $m$-weighting of $B_n$ such that $w_1(a_i) = 0$ for each $i \in [r]$, $w_1(b_i) = w_0(b_i) + w_0(a_i)$ for each $i \in [r]$, and $w_1(v) = w_0(v)$ for each $v \in V(B_n) \setminus (I \cup J)$. By Lemma 3, $\pi_k(B_n(w_1)) \leq \pi_k(B_n(w_0))$. We have $h'(w_1) - h(w_1) = h'(w_0) - h(w_0) - 1$, contradicting the choice of $w_0$. Therefore, $h(w_0) = [n/2]$.

We have shown that $w_0(v) = 0$ for each $v \in V(B_n) \setminus \binom{[n]}{[n/2]}$. By Lemma 2, $\pi_k(B_n(w')) \leq \pi_k(B_n(w_0))$. Consequently, $w' \in W^*$.

We now show that, as stated in Remark 1, the answer to Problem 2 for $G = B_n$ is negative if $n \geq 2$, $k \geq 3$, and $m \geq k \alpha_k(B_n)$.

**Theorem 9.** If $n \geq 2$, $k \geq 3$, $m \geq k \alpha_k(B_n)$, $w$ and $w'$ are $m$-weightings of $B_n$, $w$ is uniform-$\alpha$, and $w'$ is uniform on a largest $k$-clique independent set of $G$, then $\pi_k(G(w)) < \pi_k(G(w'))$.

**Proof.** We build on the proof of Theorem 5. Let $w_0$ be an $m$-weighting of $B_n$ that is uniform on a largest $k$-clique independent set $I$ of $B_n$. For a contradiction, suppose $w_0 \in W^*$. Since $m \geq k \alpha_k(B_n) = k |I|$, $w_0(v) \geq k$ for each $v \in I$. Let $M = \binom{[n]}{|I|}$ and $M' = \binom{[n]}{[n/2]-1}$. Since $k \geq 3$ and $M \cup M'$ is a 3-clique independent set of $B_n$, $|I| \geq |M| + |M'|$. Thus, $I \notin M$, and hence $h(w_0) < h'(w_0)$.

Suppose $h(w_0) < [n/2]$. Let $p = h(w_0) - [n/2]$. Thus, $p \geq 0$. If $p > 0$, then, as shown in the proof of Theorem 5, we can define a sequence $w_1, \ldots, w_p$ of $m$-weightings of $B_n$ such that, for each $i \in [p]$, $h'(w_i) = h'(w_{i-1}) - 1$, $\pi_k(B_n(w_i)) \leq \pi_k(B_n(w_{i-1}))$, and hence $w_i \in W^*$. Note that $h'(w_p) = [n/2]$ and $h(w_p) = h(w_0)$. Let $q = p + [n/2] - h(w_0)$. Since $h(w_0) \leq [n/2] - 1$, $q \geq p + 1$. As shown in the proof of Theorem 5, we can define a sequence $w_{p+1}, \ldots, w_q$ of $m$-weightings of $B_n$ such that, for each $i \in [q-p]$, $h(w_{p+i}) = h(w_{p+i-1}) + 1$,
\[ h'(w_{p+i}) = h'(w_p) = \lceil n/2 \rceil, \quad \pi_k(B_n(w_{p+i})) \leq \pi_k(B_n(w_{p+i-1})), \] and hence \( w_{p+i} \in W^* \). Since \( h(w_q) = \lceil n/2 \rceil = h'(w_q) \),

\[ w_q(v) = 0 \text{ for each } v \in V(B_n) \setminus M. \tag{2} \]

Suppose \( w_q(u) = 0 \) for some \( u \in M \). Since \( m \geq k|I| > k|M|, w_q(u') > k \) for some \( u' \in M \). Let \( w'_q \) be the \( m \)-weighting of \( B_n \) such that \( w'_q(u) = 1, w'_q(u') = w_q(u') - 1, \) and \( w'_q(v) = w_q(v) \) for each \( v \in V(B_n) \setminus \{u, u'\} \). This gives us \( \pi_k(B_n(w'_q)) < \pi_k(B_n(w_q)) \), contradicting \( w_q \in W^* \). Thus, \( w_q(v) > 0 \) for each \( v \in M \). Note that, since \( w_0(v) \geq k \) for each \( v \in I \), and \( w_0(v) = 0 \) for each \( v \in V(B_n) \setminus I \), we have

\[ w_i(v) \geq k \text{ for each } i \in [q] \text{ and each } v \in V(B_n) \text{ such that } w_i(v) \neq 0. \]

Let \( a_1, \ldots, a_r \) be the distinct members of \( \{v \in M' : w_{q-1}(v) \neq 0\} \). There exist \( r \) distinct elements \( b_1, \ldots, b_r \) of \( M \) such that \( a_i b_i \in E(B_n) \) and \( w_q(b_i) = w_{q-1}(b_i) + w_{q-1}(a_i) \geq 2k \) for each \( i \in [r] \), and we have \( w_q(a_i) = 0 \) and \( w_{q-1}(a_i) \geq k \) for each \( i \in [r], w_q(v) = w_{q-1}(v) \geq k \) for each \( v \in M \setminus \{b_1, \ldots, b_r\} \), and \( w_q(v) = w_{q-1}(v) = 0 \) for each \( v \in V(B_n) \setminus (M \cup \{a_1, \ldots, a_r\}) \). Let \( X = \{a_1, \ldots, a_r\}, Y = \{b_1, \ldots, b_r\}, \) and \( Y' = \bigcup_{i=1}^r (N_{B_n}(a_i) \cap M) \). Thus, \( Y \subseteq Y' \). Each member of \( X \) is a set of size \( \lceil n/2 \rceil - 1 \) and has exactly \( n - (\lceil n/2 \rceil - 1) \) supersets in \( M \), and each member of \( M \) has exactly \( \lceil n/2 \rceil \) subsets in \( M' \). We have

\[ (n - \lceil n/2 \rceil + 1)r = \sum_{x \in X} |N_{B_n}(x) \cap M| = \{xy \in E(B_n) : x \in X, y \in Y'\} \]

\[ = \sum_{y \in Y'} |N_{B_n}(y) \cap X| \leq \lceil n/2 \rceil |Y'|, \]

so \( r \leq |Y'| \), and equality holds only if \( n \) is odd. Suppose \( r < |Y'| \). Then, \( y \notin Y \) for some \( y \in Y' \). Now \( y \in N_{B_n}(a_j) \) for some \( j \in [r] \), so \( a_j y \in E(G) \) and \( y \neq b_j \). Let \( S = \{A \in K_k(B_n(w_{q-1})) : A \cap \{(a_i, i) : i \in [w_{q-1}(a_i)]\} \neq \emptyset \neq A \cap \{(y, i) : i \in [w_{q-1}(y)]\}\}. \) Then, \( S \neq \emptyset \) as \( k \geq 3, w_{q-1}(a_j) \geq k, \) and \( w_{q-1}(y) \geq k \). We have

\[ \pi_k(B_n(w_{q-1})) \geq \sum_{v \in M \setminus Y} \left( \frac{w_{q-1}(v)}{k} \right) + \sum_{i=1}^r \left( \frac{w_{q-1}(a_i) + w_{q-1}(b_i)}{k} \right) + |S| = \pi_k(B_n(w_q)) + |S|, \]

contradicting \( w_{q-1} \in W^* \). Therefore, \( r = |Y'| \), and hence \( Y' = Y \) and \( n \) is odd. Thus, no member of \( X \) is a subset of a member of \( M \setminus Y \), and hence \( X \cup (M \setminus Y) \) is an antichain of size \( |M| \). By Theorem \( \square \) \( X \cup (M \setminus Y) \) is \( M \) or \( M' \). Since \( \emptyset \neq X \subseteq M' \), we obtain \( X = M' \) and \( Y = M \). Suppose \( w_{q-1}(b_i) > 0 \) for some \( i \in [r] \). Since \( n \geq 2, n \) is odd, and \( X = M' \), we have \( a_j b_i \in E(B_n) \) for some \( j \in [r] \setminus \{i\} \). Let \( y = b_i \) and define \( S \) as above. As in \((3)\), we obtain \( \pi_k(B_n(w_{q-1})) \geq \pi_k(B_n(w_q)) + |S| > \pi_k(B_n(w_q)) \), contradicting \( w_{q-1} \in W^* \). Thus, since \( M = Y = \{b_1, \ldots, b_r\}, w_{q-1}(v) = 0 \) for each \( v \in V(B_n) \setminus M' \). This implies that \( \max\{i \in \{0\} \cup [n] : w_q(v) \neq 0 \} \) for some \( v \in \binom{[n]}{1} \) \leq \lceil n/2 \rceil - 1, \) so \( p = 0 \) and, since \( |I| > |M'| \), we have \( h(w_0) \leq \lceil n/2 \rceil - 2, q \geq 2, \) and \( h(w_{q-2}) = \lceil n/2 \rceil - 2 \). Let \( M'' = \binom{[n]}{\lceil n/2 \rceil - 2} \). We apply the argument starting after \((2)\) for \( M', M'', w_{q-1}, \) and \( w_{q-2} \).
instead of $M$, $M'$, $w_q$, and $w_{q-1}$, respectively, and the inequality corresponding to $r \leq |Y'|$ that we obtain is strict. Similarly to the above, this contradicts $w_{q-2} \in W^*$.

Therefore, we must have $h(w_0) \geq \lceil n/2 \rceil$. For any $v \in V(B_n)$, let $v' = [n] \setminus v$. Let $I' = \{ v' : v \in I \}$, and let $w'_0$ be the $m$-weighting of $B_n$ such that $w'_0(v) = w_0(v')$ for each $v \in V(B_n)$. For any $u, v \in V(B_n)$, $u' \subseteq v'$ if and only if $v \subseteq u$, so $u'v' \in E(B_n)$ if and only if $uv \in E(B_n)$. Thus, $\pi_k(B_n(w_0')) = \pi_k(B_n(w_0))$. Recall that $h(w_0) < h'(w_0)$. We have $h(w_0') = n - h'(w_0) < \lfloor n/2 \rfloor$ as $h(w_0) \geq \lceil n/2 \rceil$. By applying the argument for $I$ to $I'$, we obtain a contradiction.

Therefore, $w_0 \notin W^*$. By Theorem 5, the result follows. 

We next show that, as stated in Remark 2, the answer to Problem 1 is also negative if $G$, $k$, and $m$ are as in Remark 2.

**Theorem 10.** If $I_1, I_2, \ldots, I_r$ are the distinct maximal partite sets of a complete multipartite graph $G$, $|I_1| > |I_2| \geq \cdots \geq |I_r|$, $k \geq 3$, $m \geq k\alpha_k(G)$, and $w$ and $w'$ are $m$-weightings of $G$ such that $w$ is uniform-$\alpha$, and $w'$ is uniform on a largest $k$-clique independent set of $G$, then $\pi_k(G(w)) < \pi_k(G(w'))$.

**Proof.** We build on the proof of Theorem 6. We use the same idea in the proof of Theorem 6 which is to shift weights from one end to a largest independent set and shift weights from the other end to a second largest independent set. The proof for the current setting is similar but simpler.

Let $M = I_1$ and $M' = I_2$. For each $w \in W$, let $h(w)$ be as in the proof of Theorem 6 and let $h'(w) = \min\{ j \in [r] : w(v) \neq 0 \text{ for some } v \in I_j \}$. Let $w_0$ be an $m$-weighting of $G$ that is uniform on a largest $k$-clique independent set $I$ of $G$. Then, $|I| \geq |M| + |M'|$, so $h'(w_0) < h(w_0)$. Since $m \geq k\alpha_k(G) = k|I|$, $w_0(v) \geq k$ for each $v \in I$. For a contradiction, suppose $w_0 \in W^*$.

Let $p = h'(w_0) - 1$. If $p > 0$, then, as shown in the proof of Theorem 6, we can define a sequence $w_1, \ldots, w_p$ of $m$-weightings of $G$ such that, for each $i \in [p]$, $h'(w_i) = h'(w_{i-1}) - 1$, $\pi_k(G(w_i)) \leq \pi_k(G(w_{i-1}))$, and hence $w_i \in W^*$. Note that $h'(w_p) = 1$ and $h(w_p) = h(w_0) > 1$. Let $q = p + h(w_0) - 1$. Since $h(w_0) \geq 2$, $q \geq p + 1$. We can define a sequence $w_{p+1}, \ldots, w_q$ of $m$-weightings of $G$ such that, for each $i \in [q - p]$, $h(w_{p+i}) = h(w_{p+i-1}) - 1$, $h'(w_{p+i}) = h'(w_p) = 1$, $\pi_k(G(w_{p+i})) \leq \pi_k(G(w_{p+i-1}))$, and hence $w_{p+i} \in W^*$. Since $h(w_q) = 1 = h'(w_q)$, $w_q(v) = 0$ for each $v \in V(G) \setminus M$.

As in the proof of Theorem 6, $w_q(v) > 0$ for each $v \in M$, and $w_i(v) \geq k$ for each $i \in [q]$ and each $v \in V(G)$ such that $w_i(v) \neq 0$.

Let $a_1, \ldots, a_s$ be the distinct members of $\{ v \in M' : w_{q-1}(v) \neq 0 \}$. Let $a_{1,1}, \ldots, a_{1,n_1}, \ldots, a_{r,1}, \ldots, a_{r,n_r}$ be as in the proof of Theorem 6. For each $i \in [s]$, $a_i = a_{j_i}$, for some $j_i \in [n_2]$. For each $i \in [s]$, let $b_i = a_{1,j_i}$. We have $w_q(b_i) = w_{q-1}(b_i) + w_{q-1}(a_i) \geq 2k$ for each $i \in [s]$. Let $Y = \{ b_1, \ldots, b_s \}$. Let $y = a_{1,n_1}$. Since $n_1 > n_2$, $y \notin Y$. As in the proof of Theorem 6,
we obtain $\pi_k(G(w_{q-1})) > \pi_k(G(w_q))$ (as $y \in M \subseteq N_G(x)$ for each $x \in M'$), contradicting $w_{q-1} \in W^*$.

Therefore, $w_0 \notin W^*$. By Theorem 6 the result follows. □

**Proof of Theorem 8** For any $n \in \mathbb{N}$, let $G_n$ be a triangle-free $n$-vertex graph and let $I_n$ be a largest independent set of $G_n$. Let $\alpha_n = |I_n|$, let $w_n$ be a $3n$-weighting of $G_n$ that is uniform on $V(G_n)$ (so $w(v) = 3$ for each $v \in V(G_n)$), and let $w'_n$ be a $3n$-weighting of $G_n$ that is uniform on $I_n$. It suffices to show that $\pi_3(G_n(w_n)) < \pi_3(G_n(w'_n))$ for some $n \in \mathbb{N}$. We have

$$\pi_3(G_n(w_n)) = \left(\frac{3}{3}\right)n + 2\left(\frac{3}{1}\right)\left(\frac{3}{2}\right)|E(G_n)| = n + 18|E(G_n)|$$

and

$$\pi_3(G_n(w'_n)) \geq \left(\frac{3n/\alpha_n}{3}\right)\alpha_n.$$  

Bohman [2] showed that we can choose the sequence $\{G_n\}_{n \in \mathbb{N}}$ so that $\alpha_n = O(\sqrt{n \log n})$ and the maximum degree $\Delta(G_n)$ (max $\{|N_{G_n}(v)|: v \in V(G_n)\}$) of $G_n$ satisfies $\Delta(G_n) = \Theta(\sqrt{n \log n})$ (see [1, Theorem 2.5]). Since $I_n$ is maximal, $V(G_n) = I_n \cup \bigcup_{v \in I_n} N_{G_n}(v)$, so

$$n = |I_n \cup \bigcup_{v \in I_n} N_{G_n}(v)| \leq |I_n| + \sum_{v \in I_n} |N_{G_n}(v)| \leq (1 + \Delta(G_n))\alpha_n,$$

and hence $\alpha_n \geq n/(1 + \Delta(G_n))$. Thus, for $n$ sufficiently large, we have

$$\alpha_n \geq \frac{n}{a \sqrt{n \log n}} \quad \text{and} \quad \frac{1}{\alpha_n} \geq \frac{1}{b \sqrt{n \log n}}$$

for some positive real numbers $a$ and $b$, and hence

$$\pi_3(G_n(w'_n)) \geq \left(\frac{3n/b \sqrt{n \log n}}{3}\right)\frac{n}{a \sqrt{n \log n}}.$$  

Since $\left(\frac{n}{a}\right) \sim p^2/6$, we obtain

$$\left(\frac{3n/b \sqrt{n \log n}}{3}\right) = \Theta\left(\frac{n^{3/2}}{(\log n)^{3/2}}\right),$$

so $\pi_3(G_n(w'_n)) = \Omega\left(n^2/(\log n)^2\right)$. By the handshaking lemma, $2|E(G_n)| \leq n\Delta(G_n)$, so

$$|E(G_n)| = O\left(\frac{n}{2 \sqrt{n \log n}}\right) = O\left(n^{3/2} \sqrt{\log n}\right).$$

Thus, $\pi_3(G_n(w_n)) = O\left(n^{3/2} \sqrt{\log n}\right)$. Since $n^{3/2} \sqrt{\log n} = o(n^2/(\log n)^2)$, $\pi_3(G_n(w_n)) < \pi_3(G_n(w'_n))$ for $n$ sufficiently large. □
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References

[1] N. Alon, S. Ben–Shimon and M. Krivelevich, A note on regular Ramsey graphs, J. Graph Theory 64 (2010), 244–249.
[2] T. Bohman, The triangle-free process, Adv. Math. 221 (2009), 1653–1677.
[3] B. Bollobás, Combinatorics: set systems, hypergraphs, families of vectors and combinatorial probability, Cambridge University Press, Cambridge, 1986.
[4] S. Dass, W. Gan and B. Sudakov, Sperner’s theorem and a problem of Erdős, Katona and Kleitman, Discrete Appl. Math. 24 (2015), 585–608.
[5] A.P. Dove, J.R. Griggs, R.J. Kang and J.S. Sereni, Supersaturation in the Boolean lattice, Integers 14A (2014), 1–7.
[6] P. Erdős, On a lemma of Littlewood and Offord, Bull. Amer. Math. Soc. 51 (1945), 898–902.
[7] P. Hall, On representatives of subsets, J. London Math. Soc. 10 (1935), 26–30.
[8] G.O.H. Katona, A generalization of the independence number, Discrete Appl. Math. 321 (2022), 1–3.
[9] D. Kleitman, A conjecture of Erdős–Katona on commensurable pairs among subsets of an n-set, in: Theory of Graphs: Proc. Colloq. Tihany, 1966, pp. 215–218.
[10] D. König, Gráfok és mátrixok, Mat. Fiz. Lapok 38 (1931), 116–119.
[11] J. Qian, K. Engel and W. Xu, A generalization of Sperner’s theorem and an application to graph orientations, Discrete Appl. Math. 157 (2009), 2170–2176.
[12] E. Sperner, Ein Satz über Untermengen einer endlichen Menge, Math. Z. 27 (1928), 544–548.