A BLOCH-WIGNER EXACT SEQUENCE OVER LOCAL RINGS

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Abstract. In this article we extend the Bloch-Wigner exact sequence over local rings, where their residue fields have more than nine elements. Moreover, we prove Van der Kallen’s theorem on the presentation of the second $K$-group of local rings such that their residue fields have more than four elements. Note that Van der Kallen proved this result when the residue fields have more than five elements. Although we prove our results over local rings, all our proofs also work over semilocal rings where all their residue fields have similar properties as the residue field of local rings.

Introduction

The Bloch-Wigner exact sequence studies the second and the third $K$-groups of a field. On the one hand, it gives Matsumoto’s theorem on the presentation of the second $K$-group and on the other hand it gives a precise description of the indecomposable part of the third $K$-group of the field.

This result, proved by Bloch and Wigner independently and in somewhat different form, in one of its early forms gives the exact sequence

$$0 \to \mathbb{Q}/\mathbb{Z} \to H_3(\text{SL}_2(k), \mathbb{Z}) \to p(k) \to k^\times \wedge k^\times \to H_2(\text{SL}_2(k), \mathbb{Z}) \to 0,$$

where $k$ is an algebraically closed field of characteristic zero [1], [3]. Here $p(k)$ is the pre-Bloch group of $k$, which is the free abelian group generated by the symbols $[a], a \in k^\times - \{1\}$ up to defining relations

$$[a] - [b] + \begin{bmatrix} b \\ a \end{bmatrix} - \left[ \frac{1 - a^{-1}}{1 - b^{-1}} \right] + \left[ \frac{1 - a}{1 - b} \right]$$

and the map $p(k) \to k^\times \wedge k^\times$ is given by $[a] \mapsto a \wedge (1 - a)$. Moreover, the map $k^\times \wedge k^\times = H_2(\text{SL}_2(k), \mathbb{Z}) \to H_2(\text{SL}_2(k), \mathbb{Z})$ is induced by the map $k^\times \to \text{SL}_2(k)$, $a \mapsto \text{diag}(a, a^{-1})$. For description of other maps involved in the above sequence see [3, App. A].

Using the homology stability theorem for the general or special linear groups of $k$, one can prove (see [14] or [9]) that

$$K_2(k) \simeq H_2(\text{SL}_2(k), \mathbb{Z}) \quad \text{and} \quad K_3^{\text{ind}}(k) \simeq H_3(\text{SL}_2(k), \mathbb{Z}).$$

Note that $K_3^{\text{ind}}(k)$ is the indecomposable part of the third $K$-group of $k$, which is the cokernel of the natural map from the third Milnor $K$-group
$K_3^M(k)$ to the third $K$-group $K_3(k)$. Therefore the Bloch-Wigner exact sequence finds the following form

$$0 \to \mathbb{Q}/\mathbb{Z} \to K^\text{ind}_3(k) \to \mathfrak{p}(k) \to k^\times \wedge k^\times \to K_2(k) \to 0.$$ 

This exact sequence had many important applications and was the source of many deep ideas in algebraic $K$-theory. Thus it was very important to generalize it to a wider class of rings. In a remarkable paper, Suslin has generalized this exact sequence to all infinite fields. In fact, he showed that for any infinite field $F$ we have the Bloch-Wigner exact sequence

$$0 \to \text{Tor}_1^\mathbb{Z}(\mu(F), \mu(F)) \to K^\text{ind}_3(F) \to \mathfrak{p}(F) \to (F^\times \otimes \mathbb{Z} F^\times)_\sigma \to K_2(F) \to 0,$$

where the group $\text{Tor}_1^\mathbb{Z}(\mu(F), \mu(F))$ is the unique nontrivial extension of $\text{Tor}_1^\mathbb{Z}(\mu(F), \mu(F))$ by $\mathbb{Z}/2$ and $(F^\times \otimes \mathbb{Z} F^\times)_\sigma := F^\times \otimes \mathbb{Z} F^\times/(a \otimes b + b \otimes a : a, b \in F^\times)$. In [6] Hutchinson showed that the above exact sequence also holds over finite fields with more than three elements. His proof is different than Suslin’s original proof and only works for finite fields.

This article should be seen as a sequel to [10], [12], where a Bloch-Wigner exact sequence has been proved over rings with many units, e.g. local or semilocal rings whose residue fields are infinite. In this article, we will extend the Bloch-Wigner exact sequence over local rings, where their residue fields have more than nine elements. But before proving this, we prove Van der Kallen’s generalization of Matsumoto’s theorem on the presentation of the second $K$-group of local rings.

More precisely, first we prove that if $R$ is a local ring with maximal ideal $\mathfrak{m}_R$ and $R/\mathfrak{m}_R$ has more than four elements, then we have the exact sequence

$$\mathfrak{p}(R) \to (R^\times \otimes \mathbb{Z} R^\times)_\sigma \to K_2(R) \to 0.$$ 

This immediately implies that $K_2(R) \simeq K_2^M(R)$ (Proposition 3.6), where for us

$$K_2^M(R) := R^\times \otimes \mathbb{Z} R^\times/(a \otimes (1 - a), b \otimes (-b) : a, 1 - a, b \in R^\times).$$

Note that when $R/\mathfrak{m}_R$ has more than five elements, the term $b \otimes (-b)$ can be removed from the definition of $K_2^M(R)$ (Lemma 1.5).

Furthermore, we prove a Bloch-Wigner exact sequence over local rings. Let $|R/\mathfrak{m}_R| > 9$ and $|R/\mathfrak{m}_R| \neq 16, 32$. If $R$ is a domain or is an algebra over a field we may only assume that $|R/\mathfrak{m}_R| > 9$. Then we will prove that we have the exact sequence

$$T_R \to K^\text{ind}_3(R) \to \mathfrak{p}(R) \to (R^\times \otimes \mathbb{Z} R^\times)_\sigma \to K_2(R) \to 0,$$

where $T_R$ sits in the short exact sequence

$$0 \to \text{Tor}_1^\mathbb{Z}(\mu(R), \mu(R))_\sigma \to T_R \to H_1(\Sigma_2, \mu_2 \otimes \mathbb{Z} \mu_2)(R) \to 0.$$ 

Moreover, if there is a homomorphism $R \to F$, $F$ a field, such that the map $\mu(R) \to \mu(F)$ is injective, e.g. $R$ is a domain, then we have the exact sequence

$$0 \to \text{Tor}_1^\mathbb{Z}(\mu(R), \mu(R)) \to K^\text{ind}_3(R) \to \mathfrak{p}(R) \to (R^\times \otimes \mathbb{Z} R^\times)_\sigma \to K_2(R) \to 0,$$
where the composition $\text{Tor}_1^R(\mu(R), \mu(R)) \to \text{Tor}_1^R(\mu(R), \mu(R))^\sim \to K^\text{ind}_3(R)$
is induced by the map $\mu(R) \to \text{SL}_2(R)$, $\xi \mapsto \text{diag}(\xi, \xi^{-1})$.

We also prove Hutchinson’s Bloch-Wigner exact sequence over finite fields [6]. Let $F$ be a finite field with at least four elements. Then we prove that if $F \neq \mathbb{F}_4, \mathbb{F}_8$, then we have the exact sequence

$$0 \to \text{Tor}_1^Z(\mu(F), \mu(F))^\sim \to H_3(\text{SL}_2(F))_{F^\times} \to B(F) \to 0,$$

and if $F = \mathbb{F}_4$ or $\mathbb{F}_8$, then we have the exact sequence

$$0 \to \text{Tor}_1^Z(\mu(F), \mu(F)) \oplus \mathbb{Z}/2 \to H_3(\text{SL}_2(F))_{F^\times} \to B(F) \to 0,$$

where $B(F)$ is the Bloch group of $F$, i.e. $B(F) := \ker(p(F) \to (F^\times \otimes \mathbb{Z}F^\times)_{\sigma})$. Moreover, we show that if $F \neq \mathbb{F}_4, \mathbb{F}_8$, then $H_3(\text{SL}_2(F))_{F^\times} \simeq K^\text{ind}_3(F)$ and if $F = \mathbb{F}_4$ or $\mathbb{F}_8$, then $H_3(\text{SL}_2(F))_{F^\times} \simeq K^\text{ind}_3(F) \oplus \mathbb{Z}/2$ and thus we have the Bloch-Wigner exact sequence

$$0 \to \text{Tor}_1^Z(\mu(F), \mu(F))^\sim \to K^\text{ind}_3(F) \to B(F) \to 0.$$

For the proof of the above results we need certain strong homology stability results for the second and the third homology of general linear groups. Although homology stability results with sharp stability bound is well-known for general linear groups of local rings with infinite residue fields [13], we couldn’t find such a results over local rings with finite residue fields. In Sections 3 and 4 we will prove certain stability results that are good enough for our main applications (Theorems 3.2, 4.6).

It is worth to mention that almost all the results of this article are valid if we replace the local ring $R$ with a semilocal ring such that all its residue fields has the same property that $R/m_R$ has.

In this paper we shall assume throughout that $R$ is a commutative local ring with maximal ideal $m_R$ unless explicitly stated to the contrary. Moreover by the homology group $H_n(G)$ we will mean the homology of the group $G$ with integral coefficients, i.e. $H_n(G) := H_n(G, \mathbb{Z})$.

1. THE MAIN SPECTRAL SEQUENCE

Let $R$ be a commutative local ring with maximal ideal $m_R$. Let $C_i(R^2)$ be the free abelian group generated by the set of all $(l+1)$-tuples $(v_0, \ldots, v_l)$, where every $v_i \in R^2$ is a basis of a direct summand of $R^2$ and any two disjoint vectors $v_i, v_j$ are a basis of $R^2$. We consider $C_i(R^2)$ as a left $GL_2(R)$-module in a natural way. If necessary, we convert this action to a right action by the definition $m.g := g^{-1}m$. Let us define the $l$-th differential operator

$$\partial_l : C_i(R^2) \to C_{i-1}(R^2), \quad l \geq 1,$$

as an alternating sum of face operators which throws away the $i$-th component of generators. Hence we have the complex

$$C_\bullet(R^2) : \cdots \to C_2(R^2) \xrightarrow{\partial_2} C_1(R^2) \xrightarrow{\partial_1} C_0(R^2) \to 0.$$

Let $\partial_{-1} = \epsilon : C_0(R^2) \to \mathbb{Z}$ be defined by $\sum_i n_i(v_{0,i}) \mapsto \sum_i n_i$. It is easy to see that $C_1(R^2) \to C_0(R^2) \to \mathbb{Z} \to 0$ is exact and thus $H_0(C_\bullet(R^2)) = \mathbb{Z}$. 

BLOCH-WIGNER THEOREM

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Let $G$ be a group and $L_\bullet$ a complex of left $G$-modules:

$$L_\bullet: \cdots \to L_2 \to L_1 \to L_0 \to 0.$$ 

The $n$-th homology of $G$ with coefficient in $L_\bullet$, denoted by $H_n(G, L_\bullet)$, is defined as the $n$-th homology of the total complex of the double complex $C_\bullet(G) \otimes_G L_\bullet$, where $C_\bullet(G) \to \mathbb{Z}$ is the standard resolution of $G$. This double complex induces two spectral sequences

$$E_2^{p,q}(G) = H_p(G, H_q(L_\bullet)) \Rightarrow H_{p+q}(G, L_\bullet),$$

and

$$E_1^{p,q}(G) = H_q(G, L_p) \Rightarrow H_{p+q}(G, L_\bullet),$$

(see [2, §5, Chap. VII]).

**Lemma 1.1.** Let the complex $L_\bullet$ be exact for $1 \leq i \leq n$ and $M = H_0(L_\bullet)$. Then $H_i(G, L_\bullet) \simeq H_i(G, M)$ for $0 \leq i \leq n$.

**Proof.** This follows from an easy analysis of the spectral sequence $E_2^{p,q}(G)$.

For simplicity, in the rest of this section we will assume

$$\infty := \langle e_1 \rangle, \quad 0 := \langle e_2 \rangle, \quad 1 := \langle e_1 + e_2 \rangle, \quad b^{-1} := \langle e_1 + be_2 \rangle, \quad b \in R^\times.$$ 

Since $GL_2(R)$ acts transitively on the sets of generators of $C_i(R^2)$ for $i = 0, 1, 2$, by the Shapiro lemma we have

$$E_0^{0,q} \simeq H_q(\text{Stab}_{GL_2(R)}(\infty)) = H_q(B_2),$$

$$E_1^{1,q} \simeq H_q(\text{Stab}_{GL_2(R)}(\infty, 0)) = H_q(T_2),$$

$$E_2^{2,q} \simeq H_q(\text{Stab}_{GL_2(R)}(\infty, 0, 1)) = H_q(R^\times),$$

where $B_2 := \begin{pmatrix} R^\times & R \\ 0 & R^\times \end{pmatrix}$ and $T_2 := \begin{pmatrix} R^\times & 0 \\ 0 & R^\times \end{pmatrix} \simeq R^\times \times R^\times$. Moreover, the orbits of the action of $GL_2(R)$ on $C_3(R^2)$ and $C_4(R^2)$ are given by the frames

...
From the extension 0

\begin{proof}

Since the Hochschild-Serre spectral sequence and so by Lemma 1.3. If \( \mathfrak{m}_R \neq 2 \), then \( H_1(B_2) \simeq H_1(T_2) \), \( H_2(B_2) \simeq H_2(T_2) \oplus H_2(R)_{R^\times} \) and \( H_3(B_2) \simeq H_3(T_2) \oplus A_3 \), where \( A_3 \) sits in the exact sequence

\[ H_2(T_2, H_2(R)) \to H_3(R)_{R^\times} \to A_3 \to H_1(T_2, H_2(R)) \to 0. \]

Here the action of \( R^\times \) on \( H_1(R) \) is induced by the natural action of \( R^\times \) on \( R \), i.e. \( a.r := ar \).

\begin{proof}

From the extension \( 0 \to R \to B_2 \to T_2 \to 1 \), we obtain the Lyndon-Hochschild-Serre spectral sequence

\[ E^2_{r,s} = H_r(T_2, H_s(R)) \Rightarrow H_{r+s}(B_2). \]

Since \( R/\mathfrak{m}_R \) has at least three elements, there is \( a \in R^\times \) such that \( a-1 \in R^\times \). Then

\[ H_0(T_2, H_1(R)) = H_0(T_2, R) = R/(a-1 : a \in R^\times) = 0, \]

and so by Lemma 1.4 below (for \( \varphi : T_2 \to R^\times, (a, b) \mapsto ab^{-1} \)), for any \( r \geq 0 \), we have \( E^2_{r,1} = H_r(T_2, H_1(R)) = 0 \). Now by an easy analysis of the above spectral sequence we obtain the desired results. \qedhere
\end{proof}

\begin{lemma} \text{(Suslin.)} Let \( G \) be an abelian group, \( A \) a commutative ring, \( M \) an \( A \)-module and \( \varphi : G \to A^\times \) a homomorphism of groups which turns \( A \) and \( M \) into \( G \)-modules. If \( H_0(G, A) = 0 \), then for any \( n \geq 0 \), \( H_n(G, M) = 0 \).

\end{lemma}

\begin{proof}

The pre-Bloch group \( p(R) \) of a commutative ring \( R \) is the quotient of the free abelian group \( Q(R) \) generated by symbols \( \{a\} \), \( a, 1-a \in R^\times \), by the subgroup generated by elements of the form

\[ [a] - [b] + \left[ \frac{b}{a} \right] - \left[ \frac{1-a^{-1}}{1-b^{-1}} \right] + \left[ \frac{1-a}{1-b} \right], \]

for any \( a, b \in R^\times \). This group has a natural action of \( R^\times \), and for \( a, b \in R^\times \) and \( r \in R^\times \), we have

\[ p(ra) = p(r) p(a) \]

if \( r \neq 0 \). The action of \( R^\times \) on the cohomology groups of the extension \( 0 \to R \to B_2 \to T_2 \to 1 \) is induced by the natural action of \( R^\times \) on \( R \), i.e. \( a.r := ar \).

The proof is then completed by the exact sequence

\[ 0 \to H_0(T_2, H_1(R)) \to H_0(T_2, H_2(R)) \to H_1(T_2, H_2(R)) \to 0, \]

and the fact that the cohomology ring of \( T_2 \) is free over \( R^\times \).
\end{proof}
where \(a, 1 - a, b, 1 - b, a - b \in R^\times\). Let the map \(\lambda : Q(R) \to R^\times \otimes R^\times\) be defined by \([a] \mapsto a \otimes (1 - a)\). Then by a direct computation we have

\[
\lambda([a] - [b] + \left[\frac{b}{a}\right] - \left[\frac{1 - a^{-1}}{1 - b^{-1}}\right] + \left[\frac{1 - a}{1 - b}\right]) = a \otimes \left(\frac{1 - a}{1 - b}\right) + \left(\frac{1 - a}{1 - b}\right) \otimes a.
\]

Let

\[(R^\times \otimes R^\times)_\sigma := R^\times \otimes R^\times / \langle c \otimes d + c \otimes b : c, d \in R^\times\rangle.
\]

We denote the elements of \(p(R)\) and \((R^\times \otimes R^\times)_\sigma\) represented by \([a]\) and \(a \otimes b\) again by \([a]\) and \(a \otimes b\), respectively. Thus we have the well-defined map, denoted again by \(\lambda\),

\[
\lambda : p(R) \to (R^\times \otimes R^\times)_\sigma, \quad [a] \mapsto a \otimes (1 - a).
\]

The kernel of \(\lambda\) is called the Bloch group of \(R\) and is denoted by \(B(R)\). Thus we obtain the exact sequence

\[0 \to B(R) \to p(R) \to (R^\times \otimes R^\times)_\sigma \to K_2^{MS}(R) \to 0,
\]

where

\[K_2^{MS}(R) := R^\times \otimes R^\times / \langle a \otimes (1 - a), b \otimes c + c \otimes b : a, b, c \in R^\times\rangle.
\]

The \(n\)-th Milnor \(K\)-group of a commutative ring \(R\) is defined as the abelian group \(K_n^M(R)\) generated by symbols \(\{a_1, \ldots, a_n\}, a_i \in R^\times, i = 1, \ldots, n\), subject to the following relations

(i) \(\{a_1, \ldots, a_i a'_i, \ldots, a_n\} = \{a_1, \ldots, a_i, \ldots, a_n\} + \{a_1, \ldots, a'_i, \ldots, a_n\}, \) any \(i\),

(ii) \(\{a_1, \ldots, a_n\} = 0\) if there exist \(i, j, i \neq j\), such that \(a_i + a_j = 0\) or \(1\).

Clearly we have the anti-commutative product map

\[K_n^M(R) \otimes \mathbb{Z} K^M_m(R) \to K^M_{m+n}(R),
\]

\[\{a_1, \ldots, a_m\} \otimes \{b_1, \ldots, b_n\} \mapsto \{a_1, \ldots, a_m, b_1, \ldots, b_n\}.
\]

Since in \(R^\times \otimes R^\times\) we have \(b \otimes c + c \otimes b = bc \otimes (bc) - b \otimes (-bc) - b \otimes (-b) - c \otimes (-c)\), the natural map \(K_2^{MS}(R) \to K_2^M(R)\) is surjective.

**Lemma 1.5.** (i) Let \(R\) be either a field or a local ring with \(|R/m_R| \neq 2\). Then \(K_2^M(R) \cong K_2^{MS}(R)\).

(ii) Let \(R\) be either a field or a local ring with \(|R/m_R| > 2\). Then

\[K_2^M(R) \cong R^\times \otimes R^\times / \langle a \otimes (1 - a) : a, 1 - a \in R^\times\rangle.
\]

In particular, \(K_n^M(R) \cong \bigotimes_{\mathbb{Z}} R^\times / T\), where \(T\) is the subgroup generated by the elements \(a_1 \otimes \cdots \otimes a_n\) such that there are \(i, j, i \neq j\), with \(a_i + a_j = 1\).

**Proof.** (i) We denote the element of \(K_2^{MS}(R)\) represented by \(a \otimes b\), by \(\{a, b\}_{MS}\). Thus it is sufficient to prove that for any \(a \in R^\times\), \(\{a, -a\}_{MS} = 0\). If \(1 - a \in R^\times\), then \(-a = (1 - a)/(1 - a^{-1})\) and so

\[\{a, -a\}_{MS} = \{a, 1 - a\}_{MS} - \{a, 1 - a^{-1}\}_{MS} = \{a^{-1} - a^{-1}\}_{MS} = 0.
\]
This covers the case of fields. Let $R$ be local and $1 - a \notin R^\times$. Since $|R/m_R| \neq 2$, there is always $c \in R^\times$ such that $c, 1 - c, 1 - ac \in R^\times$. Then by the above argument $\{c, -c\}_{MS} = \{ac, -ac\}_{MS} = 0$ and thus

$$\{a, -a\}_{MS} = \{ac, -ac\}_{MS} - \{c, c\}_{MS} - \{a, c\}_{MS} - \{c, a\}_{MS} = 0.$$  

(ii) This can be done as Steps 1-5 in the proof of [18, Theorem 8.4]. □

Now let $|R/m_R| \neq 2$. Then by Lemmas 1.2 and 1.1, for $n = 0, 1, 2$ we have

$$H_n(GL_2(R), G_\bullet(R^2)) \simeq H_n(GL_2(R)).$$

If $|R/m_R| \neq 2, 3$, then we also have the above isomorphism for $n = 3$.

Now we study the differentials of the spectral sequence $E^1_{p,q}$ for small values of $p$ and $q$. It is not difficult to see that

$$d_{1,q}^1 = H_q(\sigma) - H_q(\sigma'),$$

where $\sigma, \sigma' : T_2 \rightarrow B_2$ are given by $\sigma(a, b) = \text{diag}(b, a)$ and $\sigma'(a, b) = \text{diag}(a, b)$. Moreover,

$$d_{2,q}^1 = H_q(\Delta),$$

where $\Delta : R^\times \rightarrow T_2$ is the diagonal map $a \mapsto (a, a)$. Hence $d_{2,q}^1$ always is injective. On the other hand, by a direct computation one can show that for any $z \in H_q(R^\times)$,

$$d_{3,q}^1(z.p(a)) = 0$$

and

$$d_{1,q}^1(z.q(a, b)) = z_.(p(a) - p(b) + p(b/a) - p(1/a - 1/b) + p(1/a - 1/b)).$$

Putting all these together, the $E^2$-terms of our spectral sequence look as follow:

* $H_3(T_2)_{\sigma} \oplus A_3$ $H_3(T_2)^{\sigma}/H_3(R^\times)$ $0$

$H_2(T_2)_{\sigma} \oplus H_2(R^\times)_{R^\times}$ $(R^\times \otimes R^\times)^{\sigma}$ $0$ $H_2(R^\times) \otimes p(R)$

$R^\times$ $0$ $0$ $R^\times \otimes p(R)$ *

$\mathbb{Z}$ $0$ $0$ $p(R)$ *

Note that $H_2(T_2)_{\sigma} \simeq H_2(R^\times) \oplus (R^\times \otimes R^\times)_{\sigma}$.

For an arbitrary group $G$, let $C_n(G) \rightarrow \mathbb{Z}$ and $B_n(G) \rightarrow \mathbb{Z}$ denote the (left) standard and the (left) bar resolution of $G$, respectively. We turn $C_n(G)$ and $B_n(G)$ into a right $G$-module in usual way. Note that the map

$$C_n(G) \rightarrow B_n(G), \quad (g_0, \ldots, g_n) \mapsto [g_0g_1^{-1}, g_1g_2^{-1}, \ldots, g_{n-2}g_{n-1}^{-1}, g_{n-1}g_n^{-1}]$$

induces the identity map of $H_n(G)$. For simplicity the element of $H_n(G)$ represented by $\sum m[g_1, \ldots, g_n]$ again is denoted by $\sum m[g_1, \ldots, g_n]$. 
Lemma 1.6. Let \( H \) be a subgroup of \( G \). Let \( \theta : G/H \to G \) be any (set theoretic) section of the natural map (of sets) \( \pi : G \to G/H \), \( g \mapsto gH \). For \( g \in G \), let \( \overline{g} := (\theta \circ \pi(g))^{-1}g \). Then the map \( \mathcal{C}_n(G) \to \mathcal{C}_n(H) \) given by \( (g_0, \ldots, g_n) \mapsto (\overline{g_0}, \ldots, \overline{g_n}) \) induces an \( H \)-morphism of the standard complexes \( \mathcal{C}_\bullet(G) \to \mathcal{C}_\bullet(H) \) and for any \( n \), the homomorphism  
\[
H_n(H) = H_n(\mathcal{C}_\bullet(G)_H) \to H_n(\mathcal{C}_\bullet(H)_H) = H_n(H)
\]
coincides with the identity map \( \text{id}_{H_n(H)} \).

Proof. This is easy to prove. In fact this map can be seen as the inverse of the identity homomorphism  
\[
H_n(H) = H_n(\mathcal{C}_\bullet(H)_H) \to H_n(\mathcal{C}_\bullet(G)_H) = H_n(H)
\]
induced by the inclusion of \( H \) in \( G \). \( \square \)

For any \( n \)-tuple \( (g_1, g_2, \ldots, g_n) \) of pairwise commuting elements of \( G \), let  
\[
c(g_1, g_2, \ldots, g_n) := \sum_{\sigma \in \Sigma_n} \text{sign}(\sigma)[g_{\sigma(1)}]g_{\sigma(2)}\cdots g_{\sigma(n)} \in H_n(G),
\]
where \( \Sigma_n \) is the symmetric group of degree \( n \). In fact, \( c(g_1, g_2, \ldots, g_n) \) is the image of \( g_1 \wedge \cdots \wedge g_n \) under the composition  
\[
\bigwedge^n \mathbb{Z} A \to H_n(A) \to H_n(G),
\]
where \( A \) is the abelian subgroup of \( G \) generated by \( g_1, \ldots, g_n \) and the first map is the Pontryagin product.

Lemma 1.7. The differential map  
\[
d^3_{3,0} : \mathfrak{p}(R) \to H_2(R^\times) \oplus (R^\times \otimes R^\times)_\sigma \oplus H_2(R)_{R^\times}
\]
is given by  
\[
d^3_{3,0}([a]) = \left( a \wedge (1 - a), -a \otimes (1 - a), -2c(a, 1 - a) \right).
\]

Proof. This is a long and tedious calculation. Here we argue as in [11]. For simplicity let \( F_i := \mathcal{C}_i(\text{GL}_2(R)), \mathcal{C}_i := \mathcal{C}_i(R^2), \text{GL}_2 = \text{GL}_2(R) \) and consider the following commutative diagram  
\[
\begin{array}{ccccccccc}
F_2 \otimes_{\text{GL}_2} C_3 & \longrightarrow & F_2 \otimes_{\text{GL}_2} C_2 & \longrightarrow & F_2 \otimes_{\text{GL}_2} C_1 & \longrightarrow & F_2 \otimes_{\text{GL}_2} C_0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
F_1 \otimes_{\text{GL}_2} C_3 & \longrightarrow & F_1 \otimes_{\text{GL}_2} C_2 & \longrightarrow & F_1 \otimes_{\text{GL}_2} C_1 & \longrightarrow & F_1 \otimes_{\text{GL}_2} C_0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
F_0 \otimes_{\text{GL}_2} C_3 & \longrightarrow & F_0 \otimes_{\text{GL}_2} C_2 & \longrightarrow & F_0 \otimes_{\text{GL}_2} C_1 & \longrightarrow & F_0 \otimes_{\text{GL}_2} C_0.
\end{array}
\]
The element \([a] \in \mathfrak{p}(R)\) comes from \( x_a := (1) \otimes (\infty, 0, 1, a^{-1}) \in F_0 \otimes_{\text{GL}_2} C_3 \), and \((1) \otimes \partial_3(\infty, 0, 1, a^{-1}) = \left[ (g_1) - (g_2) + (g_3) - (1) \right] \otimes (\infty, 0, 1) \in F_0 \otimes_{\text{GL}_2} C_2 \), where  
\[
g_1 = \begin{pmatrix} 0 & 1 \\ a & 1 \end{pmatrix}, \ g_2 = \begin{pmatrix} 1 - a & a \\ 0 & a \end{pmatrix}, \ g_3 = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}.
\]
If \( y_a := \left[(g_2, g_1) - (g_3, 1)\right](\infty, 0, 1) \in F_1 \otimes_{\text{GL}_2} C_2 \), then \( \delta_1 \otimes \text{id}_{C_2}(y_a) = x_a \).

Now \( z_a := (\text{id}_{F_1} \otimes \partial_2)(y_a) = \left[(g_2, g_1) - (g_3, 1)\right] \otimes \partial_2(\infty, 0, 1) \in F_1 \otimes_{\text{GL}_2} C_1 \), is equal to

\[
\left[(g_2g_1, g_1^2) - (g_3g_1, g_1) - (g_2^2, g_1g_2) + (g_3g_2, g_2) + (g_2, g_1) - (g_3, 1)\right] \otimes (\infty, 0).
\]

Since \( a^{-1}g_2g_1 = g_1^2g_3^{-1}(a^{-1}g_2^2g_3^2, g_1g_2) = (g_3g_2, g_3g_1)(a^{-1}(1 - a) 0 0 a) \), by a direct calculation one can see that \( \delta_2(u_a) \otimes (\infty, 0) = z_a \), where

\[
u_a = + (g_3g_1, g_2, g_1) - (g_3g_2, g_3g_1, g_2) - (a^{-1}g_2^2g_3^2, g_2, g_2g_1) + (a^{-1}g_2^2g_3^2, a^{-1}g_3^2, 1) - (a^{-1}g_2^2g_3^2, a^{-1}g_3^2, 1).
\]

Here by \( ag, a \in R^\times \) and \( g \in \text{GL}_2 \), we mean \( \left[\begin{array}{cc} a & 0 \\ 0 & a \end{array}\right] g \). Note that \( d_2^2([a]) \in E_{0, 2}^2 = H_2(B_2)/\text{im}(d_1^2) \) is represented by

\[
u_a \otimes \partial_1(\infty, 0) = (u_aw - u_a) \otimes (\infty) \in F_2 \otimes_{\text{GL}_2} C_0,
\]

where \( w = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right] \). On the chain level the isomorphism

\[
H_2(\text{GL}_2(R), C_0(R^2)) \cong H_2(B_2)
\]

is given by \( F_* \otimes_{\text{GL}_2} C_0 \to F_* \otimes_{B_2} Z, y \otimes (\infty) \to y \otimes 1 \). Let \( C_\bullet(B_2) \to Z \) be the standard resolution of \( Z \) over \( B_2 \). By Lemma 1.6, an augmented preserving chain map of \( B_2 \)-resolutions

\[
F_* = C_\bullet(\text{GL}_2(R)) \to C_\bullet(B_2)
\]

is obtained as follows: The map \( s : \text{GL}_2/B_2 \to \text{GL}_2 \) given by

\[
s(gB) := \left\{ \begin{array}{ll} 1 & \text{if } g(\infty) = \infty \\ w & \text{if } g(\infty) = 0 \\ \left[\begin{array}{cc} 1 & 0 \\ b & 1 \end{array}\right] & \text{if } g(\infty) = b^{-1}, \end{array}\right.
\]

is a (set-theoretic) section of the canonical projection \( \pi : \text{GL}_2 \to \text{GL}_2/B_2 \). Now if \( \overline{g} := (s \circ \pi(g))^{-1} g \), then we have

\[
\overline{g} = \left\{ \begin{array}{ll} g & \text{if } g(\infty) = \infty \\ wg & \text{if } g(\infty) = 0 \\ \left[\begin{array}{cc} 1 & 0 \\ -b & 1 \end{array}\right] g & \text{if } g(\infty) = b^{-1}. \end{array}\right.
\]

Thus on the chain level the map

\[
F_n \otimes_{\text{GL}_2} C_0 \to C_n(B_2) \otimes_{B_2} Z, (g_0, \ldots, g_n) \otimes (\infty) \mapsto (\overline{g_0}, \ldots, \overline{g_n}) \otimes 1,
\]
induces the homomorphism (1.1).

Hence by a direct computation we see that under the map (1.1) the element $d_{3,0}^2([a]) \in H_2(B_2) = H_2(B_4(B_2))$ is represented by the element $\mathbf{X}_a \in B_2(B_2)_B$, where

\[
\begin{align*}
\mathbf{X}_a &= + \left[ \begin{pmatrix} a^{-1} & a^{-1} \\ 0 & a \end{pmatrix} \right] \left( a^{-1} \right) - \left[ \begin{pmatrix} -a & a + 1 \\ 0 & a^{-1} \end{pmatrix} \right] \left( -1 \ a + 1 \right) \\
- \left[ \begin{pmatrix} a & -a^{-1} \\ 0 & 1 \end{pmatrix} \right] \left( a^{-1} \ a \right) + \left[ \begin{pmatrix} -a^{-1} & a + 1 \\ 0 & a \end{pmatrix} \right] \left( -a \ a + 1 \right) \\
- \left[ \begin{pmatrix} a & 1 & -a^{-2} \\ 0 & a^{-1} \end{pmatrix} \right] \left( 1 \ 1 \right) + \left[ \begin{pmatrix} a^{-1} & 1 & -a^{-2} \\ 0 & a \end{pmatrix} \right] \left( -1 \ a + 1 \right) \\
+ \left[ \begin{pmatrix} a & 1 & -a^{-2} \\ 0 & a^{-1} \end{pmatrix} \right] \left( 0 \ a^{-1} \ a \right) - \left[ \begin{pmatrix} a^{-1} & 1 & -a^{-2} \\ 0 & a \end{pmatrix} \right] \left( -1 \ a + 1 \right).
\end{align*}
\]

Let $Y_a \in H_2(T_2)_a$ and $Z_a \in H_2(N_2)_{R^2}$ be

\[
Y_a = + \left[ \begin{pmatrix} a^{-1} & a \end{pmatrix} \right] \left( a, 1 \right) - \left[ \begin{pmatrix} -a & a^{-1} \end{pmatrix} \right] \left( -1, a \right) - \left[ \begin{pmatrix} a, 1 \end{pmatrix} \right] \left( a^{-1}, a \right)
+ \left[ \begin{pmatrix} -a^{-1} & a^2 \end{pmatrix} \right] \left( -a, a^{-1} \right) - \left[ \begin{pmatrix} a, a^{-1} \end{pmatrix} \right] \left( 1, a \right) + \left[ \begin{pmatrix} a^{-1} & a \end{pmatrix} \right] \left( -1, a \right)
+ \left[ \begin{pmatrix} a, a^{-1} \end{pmatrix} \right] \left( a^{-1}, -a^{-1}(a-1)^2 \right) - \left[ \begin{pmatrix} a^{-1} & a \end{pmatrix} \right] \left( a^{-2}(a-1)^2, 1 \right)
- \left[ \begin{pmatrix} a^{-1} & -a^{-1}(a-1)^2 \end{pmatrix} \right] \left( a, a^{-1} \right) + \left[ \begin{pmatrix} a^{-2}(a-1)^2 & 1 \end{pmatrix} \right] \left( a^{-1}, a \right)
+ \left[ \begin{pmatrix} 1 & a \end{pmatrix} \right] \left( a, 1 \right) - \left[ \begin{pmatrix} a & 0 \end{pmatrix} \right] \left( a^{-1}, 0 \right).
\]
\[ Z_a = + \left[ \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \right] \left[ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right] - \left[ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right] \left[ \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \right] \\
+ \left[ \begin{pmatrix} 1 & a^{-2} \\ 0 & 1 \end{pmatrix} \right] \left[ \begin{pmatrix} 1 & a^{-2} \\ 1 & 1 \end{pmatrix} \right] - \left[ \begin{pmatrix} 1 & a^{-2} \\ 0 & 1 \end{pmatrix} \right] \left[ \begin{pmatrix} 1 & -a^{-2} \\ 0 & 1 \end{pmatrix} \right] \\
+ \left[ \begin{pmatrix} 1 & a(a+1) \\ 0 & 1 \end{pmatrix} \right] \left[ \begin{pmatrix} 1 & -a(a+1) \\ 1 & 1 \end{pmatrix} \right] - \left[ \begin{pmatrix} 1 & a(a+1) \\ 0 & 1 \end{pmatrix} \right] \left[ \begin{pmatrix} 1 & a(a+1) \\ 0 & 1 \end{pmatrix} \right] \\
+ \left[ \begin{pmatrix} 1 & a^{-1} \\ 0 & 1 \end{pmatrix} \right] \left[ \begin{pmatrix} 1 & a^{-1} \\ 1 & 1 \end{pmatrix} \right] - \left[ \begin{pmatrix} 1 & a^{-1} \\ 0 & 1 \end{pmatrix} \right] \left[ \begin{pmatrix} 1 & -a^{-1} \\ 0 & 1 \end{pmatrix} \right] \\
+ \left[ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right] \left[ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right] - \left[ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right] \left[ \begin{pmatrix} a & 0 \\ 1 & 1 \end{pmatrix} \right] \\
+ \left[ \begin{pmatrix} 1 & -a(a-1)^{-1} \\ 0 & a \end{pmatrix} \right] \left[ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right] - \left[ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right] \left[ \begin{pmatrix} 1 & -a(a-1)^{-1} \\ 1 & 1 \end{pmatrix} \right]. \]

By a direct computation, one sees that in \( \mathcal{B}_2(B_2) \mathcal{B}_2 \) we have

\[ X_a = Y_a - Z_a + \delta_3(W_a), \]

where \( W_a \) is the following element of \( \mathcal{B}_3(B_2) \mathcal{B}_2 \cong \mathbb{Z} \otimes_{\mathcal{B}_2} \mathcal{B}_3(B_2) \):

\[ W_a = -\left[ \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \right] \left[ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right] \left[ \begin{pmatrix} a & -1 \\ 0 & 1 \end{pmatrix} \right] - \left[ \begin{pmatrix} 1 & a^{-1}(a+1) \\ 0 & 1 \end{pmatrix} \right] \left[ \begin{pmatrix} 1 & a^{-1}(a+1) \\ 1 & 1 \end{pmatrix} \right] \left[ \begin{pmatrix} -1 & 0 \\ 0 & a \end{pmatrix} \right] \\
+ \left[ \begin{pmatrix} -a & 0 \\ 0 & a \end{pmatrix} \right] \left[ \begin{pmatrix} 1 & a^{-2}(a+1) \\ 1 & 1 \end{pmatrix} \right] \left[ \begin{pmatrix} -1 & a+1 \\ 0 & a \end{pmatrix} \right] + \left[ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right] \left[ \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \right] \left[ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right] \\
+ \left[ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right] \left[ \begin{pmatrix} 1 & -a^{-2} \\ 0 & 1 \end{pmatrix} \right] \left[ \begin{pmatrix} a^{-1} & a^{-1} \\ 1 & 1 \end{pmatrix} \right] - \left[ \begin{pmatrix} 1 & a^{-2} \\ 0 & 1 \end{pmatrix} \right] \left[ \begin{pmatrix} 1 & a^{-2} \\ 0 & 1 \end{pmatrix} \right] \left[ \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \right] \\
- \left[ \begin{pmatrix} -a^{-1} & 0 \\ 0 & a \end{pmatrix} \right] \left[ \begin{pmatrix} 1 & a(a+1) \\ 0 & 1 \end{pmatrix} \right] \left[ \begin{pmatrix} -a & a+1 \\ 0 & a \end{pmatrix} \right] - \left[ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right] \left[ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right] \left[ \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \right] \\
+ \left[ \begin{pmatrix} 1 & a(a+1) \\ 0 & 1 \end{pmatrix} \right] \left[ \begin{pmatrix} a & a+1 \\ 0 & 1 \end{pmatrix} \right] \left[ \begin{pmatrix} -a & 0 \\ 0 & a \end{pmatrix} \right] + \left[ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right] \left[ \begin{pmatrix} a & a-1 \\ 0 & 1 \end{pmatrix} \right] \left[ \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \right] \\
+ \left[ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right] \left[ \begin{pmatrix} 1 & a^{-1}(1-a^{-2}) \\ 0 & 1 \end{pmatrix} \right] \left[ \begin{pmatrix} -1 & 0 \\ 0 & a \end{pmatrix} \right] - \left[ \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \right] \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \left[ \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \right] \\
+ \left[ \begin{pmatrix} 1 & a^{-1}(1-a^{-2}) \\ 0 & 1 \end{pmatrix} \right] \left[ \begin{pmatrix} 1 & a^{-1} \\ 0 & 1 \end{pmatrix} \right] \left[ \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \right] - \left[ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right] \left[ \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \right] \left[ \begin{pmatrix} 0 & 1 \\ 0 & a \end{pmatrix} \right] \\
- \left[ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right] \left[ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right] \left[ \begin{pmatrix} -a^{-2} & 0 \\ 1 & 1 \end{pmatrix} \right] - \left[ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right] \left[ \begin{pmatrix} a & a^{-1} \\ 0 & 1 \end{pmatrix} \right] \left[ \begin{pmatrix} -1 & a+1 \\ 0 & a \end{pmatrix} \right] \\
- \left[ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right] \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \left[ \begin{pmatrix} 1 & a^{-2} \\ 0 & 1 \end{pmatrix} \right] - \left[ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right] \left[ \begin{pmatrix} a & a^{-1} \\ 0 & 1 \end{pmatrix} \right] \left[ \begin{pmatrix} -1 & a+1 \\ 0 & a \end{pmatrix} \right]. \]
\[
+ \left[ \begin{array}{c}
1 & a(1 - a^{-2}) \\
0 & 1 \\
\end{array} \right] \left[ \begin{array}{c}
1 & a^{-1}(a + 1) \\
0 & 1 \\
\end{array} \right] \left[ \begin{array}{c}
-1 & 0 \\
0 & a \\
\end{array} \right] + \left[ \begin{array}{c}
a^{-1} & 0 \\
0 & a \\
\end{array} \right] \left[ \begin{array}{c}
1 & a \\
0 & 1 \\
\end{array} \right] \left[ \begin{array}{c}
a & 0 \\
0 & 1 \\
\end{array} \right] \\
\left[ \begin{array}{c}
1 & a^{-1} \\
0 & a^{-1} \\
\end{array} \right] \left[ \begin{array}{c}
1 & -a^{-1} \\
0 & 1 \\
\end{array} \right] \left[ \begin{array}{c}
a^{-1} & 0 \\
0 & a^{-1}(a - 1)^{2} \\
\end{array} \right] + \left[ \begin{array}{c}
a & 0 \\
0 & 1 \\
\end{array} \right] \left[ \begin{array}{c}
a & 0 \\
0 & 1 \\
\end{array} \right] \left[ \begin{array}{c}
1 & -a^{-1} \\
0 & 1 \\
\end{array} \right] \\
\left[ \begin{array}{c}
1 & a \\
0 & 1 \\
\end{array} \right] \left[ \begin{array}{c}
a^{-1} \\
0 \\
\end{array} \right] \left[ \begin{array}{c}
a^{-1} \\
0 \\
\end{array} \right] \left[ \begin{array}{c}
0 & 1 \\
0 & a \\
\end{array} \right] \\
\left[ \begin{array}{c}
a^{-1} \\
0 \\
\end{array} \right] \left[ \begin{array}{c}
0 & 1 \\
0 & a \\
\end{array} \right] \left[ \begin{array}{c}
a & 0 \\
0 & 1 \\
\end{array} \right] \left[ \begin{array}{c}
a & 0 \\
0 & 1 \\
\end{array} \right] \left[ \begin{array}{c}
a & 0 \\
0 & 1 \\
\end{array} \right] + \left[ \begin{array}{c}
1 & a^{-1} \\
0 & a^{-1} \\
\end{array} \right] \left[ \begin{array}{c}
a^{-1} \\
0 \\
\end{array} \right] \left[ \begin{array}{c}
1 & a^{-1} \\
0 & a^{-1} \\
\end{array} \right] + \left[ \begin{array}{c}
0 & 1 \\
0 & a \\
\end{array} \right] \left[ \begin{array}{c}
0 & 1 \\
0 & a \\
\end{array} \right] \left[ \begin{array}{c}
1 & a \\
0 & a \\
\end{array} \right] \\
\left[ \begin{array}{c}
1 & a^{-1} \\
0 & a^{-1} \\
\end{array} \right] \left[ \begin{array}{c}
1 & -a^{-1} \\
0 & a^{-1} \\
\end{array} \right] \left[ \begin{array}{c}
a^{-1} \\
0 \\
\end{array} \right] \left[ \begin{array}{c}
a^{-1} \\
0 \\
\end{array} \right] \left[ \begin{array}{c}
0 & 1 \\
0 & a \\
\end{array} \right] + \left[ \begin{array}{c}
0 & 1 \\
0 & a \\
\end{array} \right] \left[ \begin{array}{c}
1 & a^{-1} \\
0 & a^{-1} \\
\end{array} \right] \left[ \begin{array}{c}
0 & 1 \\
0 & a \\
\end{array} \right] \left[ \begin{array}{c}
1 & a \\
0 & a \\
\end{array} \right] \\
\left[ \begin{array}{c}
1 & a^{-1} \\
0 & a^{-1} \\
\end{array} \right] \left[ \begin{array}{c}
1 & a^{-1} \\
0 & a^{-1} \\
\end{array} \right] \left[ \begin{array}{c}
0 & 1 \\
0 & a \\
\end{array} \right] + \left[ \begin{array}{c}
0 & 1 \\
0 & a \\
\end{array} \right] \left[ \begin{array}{c}
0 & 1 \\
0 & a \\
\end{array} \right] \left[ \begin{array}{c}
0 & 1 \\
0 & a \\
\end{array} \right] + \left[ \begin{array}{c}
0 & 1 \\
0 & a \\
\end{array} \right] \left[ \begin{array}{c}
0 & 1 \\
0 & a \\
\end{array} \right] \left[ \begin{array}{c}
0 & 1 \\
0 & a \\
\end{array} \right] \\
\left[ \begin{array}{c}
1 & a^{-1} \\
0 & a^{-1} \\
\end{array} \right] \left[ \begin{array}{c}
1 & a^{-1} \\
0 & a^{-1} \\
\end{array} \right] \left[ \begin{array}{c}
-1 & 0 \\
0 & a \\
\end{array} \right] + \left[ \begin{array}{c}
-1 & 0 \\
0 & a \\
\end{array} \right] \left[ \begin{array}{c}
1 & -a(a + 1) \\
-1 & a \\
\end{array} \right] \left[ \begin{array}{c}
-1 & 0 \\
0 & a \\
\end{array} \right] + \left[ \begin{array}{c}
1 & a(a + 1) \\
0 & a^{-1} \\
\end{array} \right] \left[ \begin{array}{c}
-1 & 0 \\
0 & a \\
\end{array} \right] + \left[ \begin{array}{c}
1 & a(a + 1) \\
0 & a^{-1} \\
\end{array} \right] \left[ \begin{array}{c}
1 & -a(a + 1) \\
1 & a \\
\end{array} \right] + \left[ \begin{array}{c}
1 & a(a + 1) \\
0 & a^{-1} \\
\end{array} \right] \left[ \begin{array}{c}
1 & -a(a + 1) \\
1 & a \\
\end{array} \right] + \left[ \begin{array}{c}
1 & a(a + 1) \\
0 & a^{-1} \\
\end{array} \right] \left[ \begin{array}{c}
1 & -a(a + 1) \\
1 & a \\
\end{array} \right] + \left[ \begin{array}{c}
1 & a(a + 1) \\
0 & a^{-1} \\
\end{array} \right] \left[ \begin{array}{c}
1 & -a(a + 1) \\
1 & a \\
\end{array} \right] + \left[ \begin{array}{c}
1 & a(a + 1) \\
0 & a^{-1} \\
\end{array} \right] \left[ \begin{array}{c}
1 & -a(a + 1) \\
1 & a \\
\end{array} \right]
\]
Using the fact that

\[- \left[ \begin{pmatrix} 1 & a + 1 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} -1 \\ 0 \end{pmatrix} \begin{pmatrix} -a \\ 0 \end{pmatrix} - \begin{pmatrix} -a \\ 0 \end{pmatrix} \begin{pmatrix} 0 & a \end{pmatrix} \begin{pmatrix} 1 & \end{pmatrix} \begin{pmatrix} 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} a^{-1} (a + 1) \right] - \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} -a \\ 0 \end{pmatrix} \begin{pmatrix} 0 & a \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} a^{-1} (a + 1) \right] - \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} a^{-1} (a + 1) \right] - \left[ \begin{pmatrix} 1 & a^{-1} (a + 1) \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} a^{-1} (a + 1) \right].

Now it is easy to see that

\[\delta_3((a^{-1}, a) (-1, a) (-a, a^{-1})) = + [(-1, a) (-a, a^{-1})] + [(a^{-1}, a) (a, 1)] - [(-a^{-1}, a^2) (-a, a^{-1})] - [(a^{-1}, a) (-1, a)],\]

we see that

\[Y_a = +c((-1, a), (-a, a^{-1})) + 2c((a^{-1}, a), (a, 1)) + c((1, a), (a^{-1}, a)) + c((a^{-2} (a^{-1} - 1), a^{-1}, a) + c((a^{-1} (a^{-1} - 1), (a^{-1} - 1, a)) + c((1, a), (a^{-1}, a^{-1} (a^{-1} - 1)))
\[= +c((1, a), (1 - a, 1)) - c((a, 1), (1 - 1, a)) \in H_2(T_2)_{\sigma}.\]

Also we have

\[Z_a = +c\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} a + 1, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} + c\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} a^{-2}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}\right)
\[+ c\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} a + 1, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} + c\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} a, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}\right)
\[+ c\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} a + 1, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} + c\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} a^{-2}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}\right).
\]

Now it is easy to see that \(d_{3,0}^3([a]) = Y_a + Z_a\) corresponds to the element \((a \land (1 - a), -a \otimes (1 - a), -2c(a, 1 - a)) \in H_2(R^x) \otimes (R^x \otimes R^x)_{\sigma} \otimes H_2(R)_{R^x}.\]

**Corollary 1.8.** (i) Let \(R/m_R\) has at least three elements. Then

\[H_2(GL_2(R)) \cong H_2(GL_1(R)) \oplus ((R^x \otimes R^x)_{\sigma} \oplus H_2(R)_{R^x}) / K,\]

where \(K = (a \otimes (1 - a), 2c(a, 1 - a)) : a, 1 - a \in R^x).\)

(ii) If \(R/m_R\) has at least four elements, then

\[H_2(SL_2(R))_{R^x} \cong ((R^x \otimes R^x)_{\sigma} \oplus H_2(R)_{R^x}) / K.\]
Proof. (i) By an easy analysis of the above spectral sequence and Lemma 1.7, one sees that

\[ H_2(\text{GL}_2(R)) \simeq \left( H_2(T_2) \oplus H_2(R)_{R^\times} \right) / \text{im}(d_{3,0}^3) \]

\[ = \left( H_2(R^\times) \oplus (R^\times \otimes R^\times)_{\sigma} \oplus H_2(R)_{R^\times} \right) / L, \]

where \( L = \langle (a \land (1 - a), -a \otimes (1 - a), -2c(a, 1 - a)) \mid a, 1 - a \in R^\times \rangle \). Let \( T := \left( H_2(R^\times) \oplus (R^\times \otimes R^\times)_{\sigma} \oplus H_2(R)_{R^\times} \right) / L \). From the maps

\[ H_2(\text{GL}_1(R)) \to T, \quad x \mapsto (x, 0, 0) + L, \]

\[ T \to H_2(\text{GL}_1(R)), \quad (x, c \otimes d, z) + L \to x + c \land d, \]

\[ \left( (R^\times \otimes R^\times)_{\sigma} \oplus H_2(R)_{R^\times} \right) / K \to T, \quad (a \otimes b, z) + K \mapsto (a \land b, -a \otimes b, z) + L, \]

\[ T \to \left( (R^\times \otimes R^\times)_{\sigma} \oplus H_2(R)_{R^\times} \right) / K, \quad (x, c \otimes d, z) + L \to (-c \otimes d, z) + K, \]

we obtain the isomorphism \( T \simeq H_2(\text{GL}_1(R)) \oplus \left( (R^\times \otimes R^\times)_{\sigma} \oplus H_2(R)_{R^\times} \right) / K \).

(ii) Since \( |R/\mathfrak{m}_R| \geq 4 \), \( H_1(\text{SL}_2(R)) = 0 \). Then from the corresponding Lyndon-Hochschild-Serre spectral sequence of the extension

\[ 1 \to \text{SL}_2(R) \to \text{GL}_2(R) \to R^\times \to 1, \]

it is easy to show that \( H_2(\text{GL}_2(R)) \simeq H_2(\text{GL}_1(R)) \oplus H_2(\text{SL}_2(R))_{R^\times} \). Now the claim follows from (i).

\[ \square \]

2. Homology of affine groups

Lemma 1.3 shows that the groups \( H_i(R^\times, H_m(R^n, \mathbb{Z})) \) are important in the study of the homology of affine groups. They already have been studied by Nesterenko and Suslin [16], [13] over local rings with infinite residue field and by Hutchinson [6], [7] over a large class of local rings.

Proposition 2.1. Let \( R \) be a local ring. If \( R/\mathfrak{m}_R \) is finite of order \( p^d \), we suppose that \( 1 \leq m < (p - 1)d \).

(i) If \( m = 1 \) or \( m = 2 \), then for any \( r \geq 0 \), \( H_r(R^\times, H_m(R^n, \mathbb{Z})) = 0 \).

(ii) For any prime field \( k \) and any \( r \geq 0 \), \( H_r(R^\times, H_m(R^n, k)) = 0 \).

(iii) If \( R \) is a domain or an algebra over a field, then for any \( r \geq 0 \), \( H_r(R^\times, H_m(R^n, \mathbb{Z})) = 0 \).

Proof. Part (i) follows directly from Lemma [7, Lemma 3.17]. The proof of (ii) is similar to the proof of [13, Proposition 1.10] or [16, Proposition 1.7]. For (iii) see [7, Lemma 3.18]. In fact, Hutchinson proved (iii) for local domains, but his arguments also works for local algebras over fields. \( \square \)

Let \( G_m(R) \) be a subgroup of \( \text{GL}_m(R) \) and \( G_n(R) \) a subgroup of \( \text{GL}_n(R) \) and assume that either \( R^\times I_m \subseteq G_m(R) \) or \( R^\times I_n \subseteq G_n(R) \). Let \( M(R) \) be a free submodule of \( M_{m,n}(R) \) such that \( G_m(R)M(R) = M(R) = M(R)G_n(R) \).

Then \( A_{m,n}(R) := \left( \begin{array}{cc} G_m(R) & M(R) \\ 0 & G_n(R) \end{array} \right) \) is a subgroup of the affine group

\[ \text{Aff}_{m,n}(R) := \left( \begin{array}{cc} \text{GL}_m(R) & M_{m,n}(R) \\ 0 & \text{GL}_n(R) \end{array} \right). \]
Proposition 2.2. Let $R$ be a local ring. If $R/m_R$ is finite, we assume that it is of order $p^d$. Let the natural homomorphism

$$
\phi_q : H_q(G_m(R) \times G_n(R)) \to H_q(A_{m,n}(R))
$$

be induced by the inclusion $G_m(R) \times G_n(R) \to A_{m,n}(R)$.

(i) For $0 \leq q \leq 2$, $\phi_q$ is an isomorphism if $q < p-1d$. So $\phi_0$ always is an isomorphism, $\phi_1$ is an isomorphism if $|R/m_R| \neq 2$, $\phi_2$ is an isomorphism if $|R/m_R| \neq 2, 3, 4$.

(ii) If $R$ is a domain or an algebra over a field, then $\phi_q$ is an isomorphism for $0 \leq q < (p-1)d$.

(iii) The map $\phi_q$ is an isomorphism for $0 \leq q < (p-1)d-2$. In particular, if $R/m_R$ is infinite, then $\phi_q$ is an isomorphism for any $q$.

Proof. This can be done as the proof of [16, Theorem 1.9]. For (iii) we also need the next lemma.

Lemma 2.3. Let $f : H \to G$ be a homomorphism of groups and let $n$ be a positive integer. If $H_i(f) : H_i(H, k) \to H_i(G, k)$ is an isomorphism for any $0 \leq i \leq n$, and any prime field $k$, then $H_i(f) : H_i(H) \to H_i(G)$ is an isomorphism for any $0 \leq i \leq n-2$.

Proof. Let $p$ be a prime. Then from the long exact sequence of the homology of $H$ and $G$ applied to $0 \to \mathbb{Z}/p^{d-1} \to \mathbb{Z}/p^d \to \mathbb{Z}/p \to 0$ and by induction on $d$, we see that $H_i(H, \mathbb{Z}/p^d) \to H_i(G, \mathbb{Z}/p^d)$ is an isomorphism for any $0 \leq i \leq n-1$ and any $d \geq 1$. This together with the fact that $Q/\mathbb{Z} \cong \bigoplus_{p \text{ prime}} \lim \mathbb{Z}/p^d$, imply that $H_i(H, Q/\mathbb{Z}) \to H_i(G, Q/\mathbb{Z})$ is an isomorphism for any $0 \leq i \leq n-1$. Now by applying a similar method to the exact sequence $0 \to \mathbb{Z} \to Q \to Q/\mathbb{Z} \to 0$, we see that $H_i(H, \mathbb{Z}) \to H_i(G, \mathbb{Z})$ is an isomorphism for any $0 \leq i \leq n-2$.

Example 2.4. Let $B_2(\mathbb{F}_q)$ and $T_2(\mathbb{F}_q)$ denote $B_2$ and $T_2$ over the finite field $R = \mathbb{F}_q$. If $q \neq 2, 3, 4, 8$, then by Proposition 2.2, $H_i(B_2(\mathbb{F}_q)) \simeq H_i(T_2(\mathbb{F}_q))$ for $0 \leq i \leq 3$. In this example we will discuss these groups when $q = 2, 3, 4, 8$.

Note that

$$
H_2(\mathbb{F}_q) \simeq \bigwedge^2 \mathbb{F}_q \quad \text{and} \quad H_3(\mathbb{F}_q) \simeq \bigwedge^3 \mathbb{F}_q \oplus (\mathbb{F}_q \otimes \mathbb{F}_q)^{-\sigma},
$$

where $\sigma(a \otimes b) = b \otimes a$ (see [17, Lemma 5.5], [6, p. 38]).

(i) $R = \mathbb{F}_2$: In this case $B_2(\mathbb{F}_2) \simeq \mathbb{Z}/2$ and $T_2(\mathbb{F}_2) = \{1\}$. Thus

$$
H_1(B_2(\mathbb{F}_2)) \simeq H_1(T_2(\mathbb{F}_2)) \oplus \mathbb{Z}/2, \quad H_2(B_2(\mathbb{F}_2)) \simeq H_2(T_2(\mathbb{F}_2))
$$

and

$$
H_3(B_2(\mathbb{F}_2)) \simeq H_3(T_2(\mathbb{F}_2)) \oplus \mathbb{Z}/2.
$$

(ii) $R = \mathbb{F}_3$: Since $H_2(\mathbb{F}_3) = 0$, for any $r \geq 0$ we have $E^2_{r2} = 0$, where this spectral sequence was discussed in the proof of Lemma 1.3. Moreover, $H_3(\mathbb{F}_3) \simeq (\mathbb{F}_3 \otimes \mathbb{F}_3)^{-\sigma} = \{0, 1 \otimes 1, 1 \otimes 2\}$ and by a direct computation we see that the action of $\mathbb{F}_3^\times$ on $H_3(\mathbb{F}_3)$ is trivial (note that $\mathbb{F}_3^\times$ acts diagonally.
on \((\mathbb{F}_3 \otimes \mathbb{F}_3)^{-\sigma}\). Thus \(H_3(\mathbb{F}_3)_{\mathbb{F}_3^\times} \simeq \mathbb{Z}/3\). Now from the spectral sequence \(E^2_{r,s}\) we get

\[
H_1(B_2(\mathbb{F}_3)) \simeq H_1(T_2(\mathbb{F}_3)), \quad H_2(B_2(\mathbb{F}_3)) \simeq H_2(T_2(\mathbb{F}_3))
\]

and

\[
H_3(B_2(\mathbb{F}_3)) \simeq H_3(T_2(\mathbb{F}_3)) \oplus \mathbb{Z}/3.
\]

(iii) \(R = \mathbb{F}_4\): Clearly \(H_2(\mathbb{F}_4) \simeq \bigwedge^2_2 \mathbb{F}_4 \simeq \mathbb{Z}/2\) and thus \(H_2(\mathbb{F}_4)_{\mathbb{F}_4^\times} \simeq \mathbb{Z}/2\). These show that the action of \(T_2(\mathbb{F}_4)\) on \(H_2(\mathbb{F}_4)\) is trivial. Thus using the Universal Coefficient Theorem one can show that \(E^2_{1,2}\) and \(E^2_{2,2}\) are trivial. By applying the Künneth formula to \((\mathbb{F}_4,+) \simeq \mathbb{Z}/2 \oplus \mathbb{Z}/2\), one sees that \(H_3(\mathbb{F}_4)\) has 8 elements. Thus if \(\mathbb{F}_4 = \{0, 1, \alpha, \alpha + 1 \mid \alpha^2 = \alpha + 1\}\), then

\[
H_3(\mathbb{F}_4) \simeq (\mathbb{F}_4 \otimes \mathbb{F}_4)^{-\sigma} = \langle 1 \otimes 1, 1 \otimes \alpha, 1 \otimes \alpha + \alpha \otimes 1 \rangle
\]

\[
= \{0, 1 \otimes 1, \alpha \otimes \alpha, \alpha \otimes 1, 1 \otimes \alpha + 1 \otimes \alpha, \alpha \otimes \alpha + 1 \otimes \alpha + 1, 1 \otimes 1 + \alpha \otimes \alpha + 1 \otimes \alpha + 1\}.
\]

Now by a direct calculation we have \(H_3(\mathbb{F}_4)_{\mathbb{F}_4^\times} \simeq (\mathbb{F}_4 \otimes \mathbb{F}_4)^{-\sigma}_{\mathbb{F}_4^\times} \simeq \mathbb{Z}/2\). Finally from the spectral sequence \(E^2_{r,s}\) we get the isomorphisms

\[
H_1(B_2(\mathbb{F}_4)) \simeq H_1(T_2(\mathbb{F}_4)), \quad H_2(B_2(\mathbb{F}_4)) \simeq H_2(T_2(\mathbb{F}_4)) \oplus \mathbb{Z}/2
\]

and

\[
H_3(B_2(\mathbb{F}_4)) \simeq H_3(T_2(\mathbb{F}_4)) \oplus \mathbb{Z}/2.
\]

(iv) \(R = \mathbb{F}_8\): Here we need to compute \(H_3(\mathbb{F}_8)_{\mathbb{F}_8^\times}\). We have

\[
H_3(\mathbb{F}_8) \simeq \bigwedge^3_2 \mathbb{F}_8 \oplus (\mathbb{F}_8 \otimes \mathbb{F}_8)^{-\sigma} \simeq \mathbb{Z}/2 \oplus (\mathbb{F}_8 \otimes \mathbb{F}_8)^{-\sigma}.
\]

Again using the Künneth formula one sees that \(H_3(\mathbb{F}_8)\) has \(2^7\) elements. Thus \((\mathbb{F}_8 \otimes \mathbb{F}_8)^{-\sigma}\) has \(2^6\) elements. If we assume

\[
\mathbb{F}_8 = \{0, 1, \alpha, \alpha + 1, \alpha^2, \alpha^2 + 1, \alpha^2 + \alpha, \alpha^2 + \alpha + 1 \mid \alpha^3 = \alpha + 1\},
\]

then

\[
(\mathbb{F}_8 \otimes \mathbb{F}_8)^{-\sigma} = \langle 1 \otimes 1, \alpha \otimes \alpha, \alpha^2 \otimes \alpha^2, 1 \otimes \alpha + \alpha \otimes 1, 1 \otimes \alpha^2 + \alpha^2 \otimes 1, \alpha^2 \otimes \alpha + \alpha \otimes \alpha^2 \rangle.
\]

Now by a direct computation one sees that \((\mathbb{F}_8 \otimes \mathbb{F}_8)^{-\sigma}_{\mathbb{F}_8^\times} = 0\). Therefore \(H_3(\mathbb{F}_8)_{\mathbb{F}_8^\times} \simeq \mathbb{Z}/2\) and hence

\[
H_1(B_2(\mathbb{F}_8)) \simeq H_1(T_2(\mathbb{F}_8)), \quad H_2(B_2(\mathbb{F}_8)) \simeq H_2(T_2(\mathbb{F}_8))
\]

and

\[
H_3(B_2(\mathbb{F}_8)) \simeq H_3(T_2(\mathbb{F}_8)) \oplus \mathbb{Z}/2.
\]
Example 2.5. Let \( R = \mathbb{F}_q \) be the finite field with \( q \) elements. If \( q \geq 5 \), then by Corollary 1.8 and Proposition 2.1 we have \( H_2(\text{SL}_2(\mathbb{F}_q)) = 0 \). If \( q = 4 \), then \( H_2(\mathbb{F}_4) \simeq \mathbb{Z}/2 \) and \( K^M_2(\mathbb{F}_4) = 0 \). Thus by Corollary 1.8, we have

\[
H_2(\text{SL}_2(\mathbb{F}_4)) \simeq H_2(\mathbb{F}_4) \simeq \mathbb{Z}/2.
\]

If \( q = 3 \), then \( H_2(\mathbb{F}_3) = H_2(\mathbb{F}_3^2) = 0 \) and \( K^M_2(\mathbb{F}_3) = 0 \). Now by Corollary 1.8, \( H_2(\text{GL}_2(\mathbb{F}_3)) = 0 \). If \( q = 2 \), then \( \text{GL}_2(\mathbb{F}_2) = \text{SL}_2(\mathbb{F}_2) \simeq \Sigma_3 \). By looking at the associated Lyndon-Hochschild-Serre spectral sequence \( E^n_{p,q} \) of the extension

\[
1 \to A_3 \to \Sigma_3 \to \Sigma_3/A_3 \to 1,
\]

for any pair \( (p,q) = (p,2s) \) or \( (p,q) = (2r,0) \), we have \( E^n_{p,q} = 0 \). Since the action of \( \Sigma_3/A_3 \) on \( A_3 \) is non-trivial, \( E^n_{0,1} = H_0(\Sigma_3/A_3,A_3) = 0 \). Thus by Lemma 2.6 below, \( E^n_{p,1} = H_p(\Sigma_3/A_3,A_3) = 0 \) for all \( p \geq 0 \). An easy analysis of the above spectral sequence implies that \( H_1(\text{SL}_2(\mathbb{F}_2)) \simeq \mathbb{Z}/2 \), \( H_2(\text{SL}_2(\mathbb{F}_2)) = 0 \). Therefore,

\[
H_2(\text{GL}_2(\mathbb{F}_q)) = \begin{cases} 
0 & \text{if } q \neq 4 \\
\mathbb{Z}/2 & \text{if } q = 4.
\end{cases}
\]

Lemma 2.6. Let \( G \) be an abelian group and \( M \) a finitely generated \( G \)-module such that \( H_0(G,M) = 0 \). Then for any \( n \geq 0 \), \( H_n(G,M) = 0 \).

Proof. See [4, p. 11–12].

For a subgroup \( H \) of a group \( G \) and a \( G \)-module \( M \), the natural map \( H_n(H,M) \to H_n(G,M) \) is called the corestriction map and is denoted by

\[
\text{cor}_H^G : H_n(H,M) \to H_n(G,M).
\]

When the index of \( H \) in \( G \) is finite, i.e. \( |G:H| \leq \infty \), for any \( n \geq 0 \) there is a restriction map, called transfer map,

\[
\text{res}_H^G : H_n(G,M) \to H_n(H,M),
\]

such that

\[
\text{cor}_H^G \circ \text{res}_H^G = [G:H] \text{id}_{H_n(G,M)}.
\]

[2, Proposition 9.5, Chap. III].

In case \( G \) is finite, by putting \( H = \{1\} \), one sees that \( H_n(G,M) \) is annihilated by \( |G| \) for all \( n > 0 \) [2, Corollary 10.2, Chap. III]. It is well known that when \( M \) is a finitely generated \( G \)-module, then \( H_n(G,M) \) is finite for all \( n > 0 \).

Let \( G \) be a finite group. For a \( g \in G \), let \( H_n(H) \to H_n(gHg^{-1}) \), \( z \mapsto g.z \), be induced by the natural map \( H \to gHg^{-1} \). We say \( z \in H_n(H) \) is \( g \)-invariant if

\[
\text{res}^H_{H \cap gHg^{-1}}(z) = \text{res}^{gHg^{-1}}_{H \cap gHg^{-1}}(g.z).
\]

Let

\[
\text{inv}_G(H_n(H)) := \{ z \in H_n(H) \mid z \text{ is } g \text{-invariant for all } g \in G \}.
\]
If $H$ is a $p$-Sylow subgroup of $G$, then one can show that

$$H_n(G)_{(p)} \simeq \operatorname{inv}_G(H_n(H)),$$

where $H_n(G)_{(p)}$ is the $p$-primary component of $H_n(G)$ [2, Theorem 10.3, Chap. III]. Moreover, if $H$ is normal in $G$, then

$$H_n(G)_{(p)} \simeq \operatorname{inv}_G(H_n(H)) \simeq H_n(H)_{G/H}.$$

For $g \in G$ and $z \in H_n(H)$ the condition $\operatorname{res}^H_{H \cap gHg^{-1}}(z) = \operatorname{res}^g_{H \cap gHg^{-1}}(g.z)$ is trivially satisfied if $H \cap gHg^{-1} = 1$. Thus to determine $\operatorname{inv}_G(H_n(H))$ for a $p$-Sylow subgroup $H$, it is enough to consider only the set $\operatorname{Conj}(G, H)$ of those elements $g$ for which $H \cap gHg^{-1} \neq 1$:

$$\operatorname{Conj}(G, H) := \{ g \in G \mid H \cap gHg^{-1} \neq 1 \}.$$

**Example 2.7.** Let $F = \mathbb{F}_p^m$, where $p$ is a prime. Let $N_2(F) = \begin{pmatrix} 1 & F \\ 0 & 1 \end{pmatrix} \simeq F$.

Then $N_2(F)$ is a $p$-Sylow subgroup of $GL_2(F)$. It is easy to see that $A \in \operatorname{Conj}(GL_2(F), N_2(F))$ if and only if $A \in B_2(F)$. Thus

$$\operatorname{Conj}(GL_2(F), N_2(F)) = B_2(F) = \operatorname{Conj}(B_2(F), N_2(F)),$$

and hence

$$H_3(GL_2(F))_{(p)} \simeq \operatorname{inv}_{GL_2(F)}(H_3(N_2(F)))$$

$$= \operatorname{inv}_{B_2(F)}(H_3(N_2(F))) \simeq H_3(B_2(F))_{(p)}.$$

Now by Example 2.4, we have

$$H_3(GL_2(\mathbb{F}_p^m))_{(p)} = \begin{cases} \mathbb{Z}/p & \text{if } p^m = 2, 3, 4, 8 \\ 0 & \text{otherwise}. \end{cases}$$

### 3. Stability for the Second Homology Group and the Second K-group

The homology stability results have many important application in algebraic $K$-theory. In this article the homology stability for the second and the third homology of the general linear group play very important roles in proving our main results.

Let $C'_l(R^n)$ be the free abelian group with a basis consisting of $(l + 1)$-tuples $(w_0, \ldots, w_l)$, where every $\min\{l + 1, n\}$ of $w_i \in R^n$ are basis of a free direct summand of $R^n$ and consider it as $GL_n(R)$-module in a natural way. Let us define the differential operators $\partial_l^i : C'_l(R^n) \to C'_{l-1}(R^n)$ similar to those in the complex $C'_*(R^2)$. So we have the complex of $GL_n(R)$-modules:

$$C'_n(R^n) : \cdots \to C'_{n-1}(R^n) \to C'_{n-2}(R^n) \to \cdots \to C'_0(R^n) \to C'_{-1}(R^n) \to 0,$$

where $C'_{-1}(R^n) = \mathbb{Z}$.

**Lemma 3.1.** The complex $C'_*(R^n)$ is exact for $-1 \leq i \leq n - 2$.

**Proof.** This follows from [19, §2, Theorem].
Now as before, if the residue field of \( R \) is finite, we assume that \(|R/\mathfrak{m}_R| = p^d\). Let \( s < (p-1)d - 2 \). If \( R \) is a domain or an algebra over a field, we may only assume that \( s < (p-1)d \).

Let \( L'_i := C'_{i-1}(R^n) \) and let \( C_\bullet(GL_n(R)) \to \mathbb{Z} \) be the standard resolution of \( GL_n(R) \). From the double complex \( C_\bullet(GL_n(R)) \otimes_{GL_n(R)} L'_\bullet \) we obtain the first quadrant spectral sequence

\[
E^1_{r,s}(n) = H_s(GL_n(R), L'_r) \Rightarrow H_{r+s}(GL_n(R), L'_\bullet),
\]

(see Section 1). It follows from Lemmas 3.1 and 1.1 that for \( 0 \leq m \leq n-1 \),

\[
H_m(GL_n(R), L'_\bullet) = 0.
\]

Let \( \sigma_r := (e_1, \ldots, e_r) \in L'_r, 1 \leq r \leq n \). Then by the Shapiro lemma we have

\[
E^1_{r,s}(n) := H_s(GL_n(R), L'_r) \simeq H_s(\text{Stab}_{GL_n(R)}(\sigma_r)),
\]

where \( \text{Stab}_{GL_n(R)}(\sigma_r) \). Then by Proposition 2.2 we have

\[
E^1_{r,s}(n) = H_s(\text{Stab}_{GL_n(R)}(\sigma_r)) \simeq H_s(GL_{n-r}(R)).
\]

Moreover, it is not difficult to see that for \( 1 \leq r \leq n \), the differential

\[
d^1_{r,s}(n) : H_s(GL_{n-r}(R)) \to H_s(GL_{n-r+1}(R))
\]

is defined as

\[
d^1_{r,s}(n) = \sum_{i=1}^r (-1)^{i+1} H_s(\text{inc}) = \begin{cases} H_s(\text{inc}) & \text{if } r \text{ is odd} \\ 0 & \text{if } r \text{ is even}, \end{cases}
\]

where \( \text{inc} : GL_{n-r}(R) \to GL_{n-r+1}(R) \) is the inclusion map [19, § 4], [13, Lemma 2.4].

**Theorem 3.2.** Let \( R \) be a local ring. If \( R/\mathfrak{m}_R \) is finite we assume that it has \( p^d \) elements.

(i) If \(|R/\mathfrak{m}_R| \neq 2, 3, 4\), then

\[
H_2(GL_2(R)) \Rightarrow H_2(GL_3(R)) \Rightarrow \cdots.
\]

(ii) If \( n < (p-1)d \) and \( R \) is a domain or an algebra over a field, then

\[
H_n(GL_n(R)) \Rightarrow H_n(GL_{n+1}(R)) \Rightarrow \cdots.
\]

(iii) If \( n < (p-1)d - 2 \), then

\[
H_n(GL_n(R)) \Rightarrow H_n(GL_{n+1}(R)) \Rightarrow \cdots.
\]

**Proof.** Here one can argue as in [13, pp. 127–128]. \( \square \)

**Remark 3.3.** An unpublished work of Quillen shows that if \( F \) is any field with more than two elements, then for any \( n \) we have the homology stability

\[
H_n(GL_n(F)) \Rightarrow H_n(GL_{n+1}(F)) \Rightarrow \cdots.
\]

It is nice and important to know that whether such a result is true when \( F \) is replaced by a local ring \( R \) such that \( R/\mathfrak{m}_R \) has more than two elements.
Now we will show that when \(|R/m_R| \neq 2, 3, 4\), then the surjective map 
\(H_2(GL_2(R)) \to H_2(GL_3(R))\) is injective.

Let \(\hat{C}_l(R^3)\), \(l \geq 0\), be the free abelian group with a basis consisting of 
\((l + 1)\)-tuple \((\langle v_0 \rangle, \ldots, \langle v_l \rangle)\), where every \(\min\{l + 1, 2\}\) \(v_i \in R^n\) are basis
of a free direct summand of \(R^m\). We define the differential \(\partial_l : \hat{C}_l(R^3) \to \hat{C}_{l-1}(R^3)\), \(l \geq 0\), similar to differentials of \(C_\ast(R^2)\) and we construct the complex
\[
\hat{C}_\ast(R^3) : \cdots \to \hat{C}_2(R^3) \xrightarrow{\partial_2} \hat{C}_1(R^3) \xrightarrow{\partial_1} \hat{C}_0(R^3) \xrightarrow{\partial_0} \hat{C}_{-1}(R^3) = \mathbb{Z} \to 0
\]
in usual way.

**Lemma 3.4.** The complex \(\hat{C}_\ast(R^3)\) is exact for \(-1 \leq i < |\mathbb{P}^2(R/m_R)| - 1\). In particular, for any local ring \(R\), \(\hat{C}_\ast(R^3)\) is exact for \(-1 \leq i < 6\).

**Proof.** The proof of this lemma is similar to the proof of Lemma 1.2 given in [7]. Let \(k := R/m_R\) and consider \(\mathbb{P}^2(k) = \{\mathfrak{p} \mid \mathfrak{p} \in k^2 - \{0\}\}\). For any finite subset \(S\) of \(\mathbb{P}^2(k)\), let \(D_l(S)\) be the subgroup of \(\hat{C}_l(R^3)\) generated by the

\[
\hat{C}_*(R^3) : \ldots \to \hat{C}_2(R^3) \xrightarrow{\partial_2} \hat{C}_1(R^3) \xrightarrow{\partial_1} \hat{C}_0(R^3) \xrightarrow{\partial_0} \hat{C}_{-1}(R^3) = \mathbb{Z} \to 0
\]

Thus if \((\langle v_0 \rangle, \ldots, \langle v_l \rangle)\) is a generator of \(\hat{C}_l(R^3)\) and \(x \notin \{\mathfrak{p}_0, \ldots, \mathfrak{p}_l\}\), then

\[
\hat{\partial}_{l+1} \circ s_x((v_0), \ldots, (v_l)) = (\langle v_0 \rangle, \ldots, \langle v_l \rangle) - s_x \circ \hat{\partial}_l((v_0), \ldots, (v_l)).
\]

Suppose \(x_i \in \mathbb{P}^2(k), 0 \leq i \leq l\), are disjoints and choose \(v_{x_i} \in R^3\) such that \(\langle \mathfrak{p}_{x_i} \rangle = x_i\).

Let \(z = z' + z'' \in \ker(\hat{\partial}_{l-1})\), where \(z'\) is generated by terms that belong to \(D_{l-1}\{x_0\}\) and \(z''\) is generated by terms which do not belong to \(D_{l-1}\{x_0\}\). Then

\[
\hat{\partial}_l \circ s_{x_0}(z) = \hat{\partial}_l \circ s_{x_0}(z'')
\]

\[
= z'' + s_{x_0} \circ \hat{\partial}_{l-1}(z'')
\]

\[
= z + s_{x_0} \circ \hat{\partial}_{l-1}(z) - z' = s_{x_0} \circ \hat{\partial}_{l-1}(z')
\]

\[
= z - z' - s_{x_0} \circ \hat{\partial}_{l-1}(z').
\]

Clearly \(-z' = s_{x_0} \circ \hat{\partial}_{l-1}(z') \in D_{l-1}\{x_0\}\). Thus

\[
(\hat{\partial}_l \circ s_{x_0} - \text{id}_{\hat{C}_{l-1}(R^3)})(z) = z_0,
\]

where \(z_0 \in D_{l-1}\{x_0\}\) and clearly \(z_0 \in \ker(\hat{\partial}_{l-1})\). In a similar way we have

\[(\hat{\partial}_l \circ s_{x_1} - \text{id}_{\hat{C}_{l-1}(R^3)})(z_0) = z_1,\]

where \(z_1 \in D_{l-1}\{x_0, x_1\}\) and \(z_1 \in \ker(\hat{\partial}_{l-1})\).
Repeating this process we get
\[ z_l = (\hat{\partial}_l \circ s_{x_l} - \text{id}_{\hat{C}_{l-1}(R^3)}) \circ \cdots \circ (\hat{\partial}_1 \circ s_{x_0} - \text{id}_{\hat{C}_{l-1}(R^3)})(z), \]
where \( z_l \in D_{l-1}(\{x_0, \ldots, x_l\}) = 0 \) and thus \( z_l = 0 \). From the above formula we have \( \hat{\partial}_l(-1)^l z = 0 \) for some \( y \) and therefore \( \hat{\partial}_l((-1)^l y) = z \).

The last claim follows from the fact that \( |\mathbb{P}^2(k)| \geq |\mathbb{P}^2(\mathbb{F}_2)| = 7 \). \( \square \)

Set \( \hat{L}_l := \hat{C}_{l-1}(R^3) \) for \( l \geq 0 \) and consider the complex
\[ \hat{L}_\bullet : \cdots \to \hat{L}_2 \to \hat{L}_1 \to \hat{L}_0 \to 0. \]

Then we have the first quadrant spectral sequence
\[ \hat{E}^1_{r,s}(n) = H_s(\text{GL}_3(R), \hat{L}_r) \Rightarrow H_{r+s}(\text{GL}_3(R), \hat{L}_\bullet). \]

By Lemmas 3.4 and 1.1, for \( 0 \leq m \leq 6 \), we have \( H_m(\text{GL}_3(R), \hat{L}_\bullet) = 0 \).

Moreover, by Proposition 2.2,
\[ \hat{E}^1_{r,s} = \begin{cases} H_s(\mathbb{R}^r \times \text{GL}_3(R)) & \text{if } 0 \leq r \leq 2 \\ H_s(\text{GL}_3(R), \hat{C}_{r-1}(R^3)) & \text{if } r \geq 3. \end{cases} \]

**Lemma 3.5.** (i) Let \( |R/\mathfrak{m}_R| \neq 2, 3, 4 \). Then the complex
\[ H_2(\mathbb{R}^2 \times \text{GL}_1(R)) \xrightarrow{\alpha_{-\text{inc}}} H_2(\mathbb{R}^2 \times \text{GL}_2(R)) \xrightarrow{\text{inc}} H_2(\text{GL}_3(R)) \to 0 \]
is exact, where \( \alpha(a,b,c) = (b,a,c) \).

(ii) Let \( |R/\mathfrak{m}_R| \neq 2, 3, 4, 5, 8, 9, 16, 32 \). If \( R \) is a domain or an algebra over a field, we only may assume that \( |R/\mathfrak{m}_R| \neq 2, 3, 4, 8 \). Then we have the exact sequence
\[ H_3(\mathbb{R}^2 \times \text{GL}_1(R)) \xrightarrow{\alpha_{-\text{inc}}} H_3(\mathbb{R}^2 \times \text{GL}_2(R)) \xrightarrow{\text{inc}} H_3(\text{GL}_3(R)) \to 0. \]

**Proof.** This can be proved as [9, Corollary 3.5]. In fact the claims follow from analysis of the above spectral sequence as done in [9, § 3]. \( \square \)

**Proposition 3.6.** Let \( R \) be a local ring.

(i) If \( |R/\mathfrak{m}_R| \neq 2, 3, 4 \), then we have the homology stability
\[ H_2(\text{GL}_2(R)) \xrightarrow{\simeq} H_2(\text{GL}_3(R)) \xrightarrow{\simeq} H_2(\text{GL}_4(R)) \xrightarrow{\simeq} \cdots. \]

(ii) If \( |R/\mathfrak{m}_R| \neq 2, 3, 4 \), then \( K_2^M(R) \approx K_2(R) \) and the natural map \( K_2^M(R) \to H_2(\text{GL}_2(R)) \) is given by \( \{a,b\} \mapsto c\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}\right) \).

(iii) (Van der Kallen) If \( |R/\mathfrak{m}_R| > 5 \), then
\[ K_2(R) \approx K_2^M(R) \approx (\mathbb{R}^\times \otimes \mathbb{R}^\times)/(a \otimes (1 - a) : a, 1 - a \in \mathbb{R}^\times). \]

**Proof.** (i) By Proposition 3.2, we must prove that the map \( H_2(\text{GL}_2(R)) \to H_2(\text{GL}_3(R)) \) is injective. By Lemma 3.5, we have the exact sequence
\[ H_2(\mathbb{R}^2 \times \text{GL}_1(R)) \xrightarrow{\alpha_{-\text{inc}}} H_2(\mathbb{R}^2 \times \text{GL}_2(R)) \xrightarrow{\text{inc}} H_2(\text{GL}_3(R)) \to 0 \]
and by the Universal Coefficient Theorem,

\[ H_2(R^\times \times GL_2(R)) \simeq H_2(R^\times) \oplus H_2(GL_2(R)) \oplus R^\times \otimes H_1(GL_2(R)), \]

\[ H_2(R^\times \times GL_1(R)) \simeq H_2(R^\times) \oplus H_2(GL_1(R)) \oplus R^\times \oplus R^\times \]

\[ \oplus R^\times \otimes H_1(GL_1(R)) \oplus R^\times \otimes H_1(GL_1(R)) \]

If \( x \in \ker (H_2(GL_2(R)) \to H_2(GL_3(R))) \), then

\[ (0, x, 0) \in \ker (\text{inc}_* : H_2(R^\times \times GL_2(R) \to H_2(GL_3(R))). \]

Thus there is

\[ z = (c(a_1, b_1), c(a_2, b_2), x', a \otimes b, c \otimes d, e \otimes f) \in H_2(R^\times \times GL_1(R)) \]

such that \( (\alpha_* - \text{inc}_*)(z) = (0, x, 0) \) and so

\[ 0 = -c(a_1, b_1) + c(a_2, b_2), \]

\[ x = +c(\text{diag}(a_1, 1), \text{diag}(b_1, 1)) - c(\text{diag}(a_2, 1), \text{diag}(b_2, 1)) \]

\[ + c(\text{diag}(c, 1), \text{diag}(1, d)) - c(\text{diag}(e, 1), \text{diag}(1, f)), \]

\[ 0 = -b \otimes a - a \otimes b - c \otimes d + e \otimes f. \]

All these imply that

\[ x = -c(\text{diag}(b, 1), \text{diag}(1, a)) - c(\text{diag}(a, 1), \text{diag}(1, b)) = 0. \]

(ii) Since \( |R/\mathfrak{m}_R| \geq 5 \), Proposition 2.1 implies that \( H_2(R)_{R^\times} = 0. \) Thus by Corollary 1.8 and Lemma 1.5 we have

\[ H_2(GL_2(R)) \simeq H_2(GL_1(R)) \oplus K_2^M(R). \]

Now by the homology stability result from (i), we have

\[ K_2^M(R) \simeq H_2(GL_2(R))/H_2(GL_1(R)) \]

\[ \simeq H_2(GL(R))/H_2(GL_1(R)) \]

\[ \simeq H_2(SL(R)) \]

\[ = K_2(R). \]

Moreover, clearly \( K_2^M(R) \to H_2(GL_2(R)) \) coincides with the composition

\[ K_2^M(R) \to H_2(T_2) \to H_2(GL_2(R)) \]

and by the proof of Corollary 1.8, this is given by

\[ \{a, b\} \mapsto c\left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} \right) - c\left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} \right) \]

\[ = c\left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} \right). \]

(iii) The last claim follows from (ii) and Lemma 1.5. \( \square \)
Example 3.7. Since $H_2(\text{GL}_2(\mathbb{F}_4)) \simeq \mathbb{Z}/2$ (see Example 2.5) and

$$H_2(\text{GL}(\mathbb{F}_4)) \simeq H_2(\text{GL}_1(\mathbb{F}_4)) \oplus K_2(\mathbb{F}_4) = 0,$$

the natural map $H_2(\text{GL}_2(\mathbb{F}_4)) \to H_2(\text{GL}(\mathbb{F}_4))$ is surjective but not injective. But $K_2^M(\mathbb{F}_4) = K_2(\mathbb{F}_4) = 0$.

Remark 3.8. (i) Dennis has proved that for any local ring $R$ the natural map $K_2^M(R) \to K_2(R)$ is surjective [15, Theorem 2.5]. His result also is true for any semilocal ring which has at most one residue field with two elements.

(ii) It is a result of Van der Kallen that $K_2^M(R) \to K_2(R)$ is an isomorphism for any local ring such that its residue field has more than five elements. In fact, he proved this for a larger class of rings called 5-fold stable rings [18, 5.2].

4. Stability for the Third Homology Group and the Third $K$-Group

To prove the Bloch-Wigner exact sequence we will need to show that the map $H_3(\text{GL}_3(R)) \to H_3(\text{GL}_4(R))$ is an isomorphism. We prove this when the residue field of $R$ has sufficient elements.

Let $S = \{w_1, \ldots, w_m\}$ be a subset of $R^n$. We say that $(v_1, \ldots, v_l)$ is in general position with $S$ if for all $a_i \in R$, $i = 1, \ldots, l + m$ and only $n$ many of them non-vanishing,

$$a_1v_1 + \cdots + a_lv_l + a_{l+1}v_1 + \cdots + a_{l+m}v_m = 0$$

implies that $a_i = 0$ for all $i = 1, \ldots, l$.

Definition 4.1. Let $c_n(R)$ be the minimum of the numbers $|S|$ where $S$ is a finite subset of $R^n$ such that there is no further element in $R^n$ which is in general position to $S$.

Remark 4.2. We believe that $c_n(R)$ is equal to the largest natural number such that there is a finite subset $T$ of $R^n$ such that the element of $T$ are in general position. This can be seen by direct calculations for local rings with small residue fields.

Example 4.3. If $v_1, \ldots, v_n, v_{n+1}, \ldots, v_l \in R^n$ are in general position, then by multiplying the above vectors with a suitable invertible matrix we may assume that $v_1 = e_1, \ldots, v_n = e_n$. Thus $v_{n+1} = a_1e_1 + \cdots + a_ne_n$, where for any $i$, $a_i \neq 0$. Replacing $e_i$ with $a_ie_i$ for $1 \leq i \leq n$ and multiplying them with a suitable invertible matrix, we may assume that $v_1 = e_1, \ldots, v_n = e_n, v_{n+1} = e_1 + \cdots + e_n$. Moreover, in a similar way we may assume that $v_{n+j} = e_1 + a_{2j}e_2 + \cdots + a_ne_n$ for $2 \leq j \leq l - n$.

(i) The elements of the set $\{e_1, e_2, e_1 + e_2, e_1 + a_ie_2 | i = 1, \ldots, m\} \subseteq R^2$ are in general position if and only if $a_i, 1 - a_i, a_i - a_j \in R^\times$, where $i \neq j$. Thus

$$c_2(R) = |\mathbb{P}^1(R/m_R)| = |R/m_R| + 1.$$
(ii) The above argument shows that \( c_n(R) \geq n + 1 \). If \( |R/m_R| \leq n \), then clearly \( c_n(R) = n + 1 \).

(iii) If \( |R/m_R| = \infty \), then it is not difficult to show that for any finite set \( S \) of elements in general position of \( R^n \), one can always find many vectors that are in general position with \( S \). Therefore \( c_n(R) = \infty \).

(iv) The elements of the set

\[
\{ e_1, e_2, e_3, e_1 + e_2 + e_3, e_1 + a_i e_2 + b_i e_3 \mid i = 1, \ldots, m \} \subseteq R^3
\]

are in general position if and only if

\[
a_i, 1-a_i, b_i, 1-b_i, a_i-a_j, b_i-b_j, a_i-b_i, x_{ij} := a_i b_j - a_j b_i, x_{ij} - x_{ik} + x_{jk} \in R^\times,
\]

where \( i, j, k \) are distinct. By direct computations we get

\[
c_3(R) = \begin{cases}
4 & \text{if } |R/m_R| = 2, 3 \\
6 & \text{if } |R/m_R| = 4, 5 \\
8 & \text{if } |R/m_R| = 7, 8, 9
\end{cases}
\]

\[
c_3(R) \geq 9 \quad \text{if } |R/m_R| > 9.
\]

(v) By direct computations one can show that

\[
c_4(R) = \begin{cases}
5 & \text{if } |R/m_R| = 2, 3, 4 \\
8 & \text{if } |R/m_R| = 5
\end{cases}
\]

\[
c_4(R) \geq 9 \quad \text{if } |R/m_R| \geq 7.
\]

Note that in \( \mathbb{F}_4 \), the elements of the set \( \{ e_1, e_2, e_3, e_4, e_1 + e_2 + e_3 + e_4, e_1 + 2e_2 + 3e_3 + 4e_4, e_1 + 5e_2 + 6e_3 + 2e_4, e_1 + 3e_2 + 4e_3 + 5e_4, e_1 + 6e_2 + 2e_3 + 3e_4 \} \)

are in general position.

For a non-negative integer \( l \), let \( \tilde{C}_l(R^n) \) be the free abelian group with a basis consisting of \( (l + 1) \)-tuples \( (v_0, \ldots, v_l) \), where \( v_i \in R^n - \{ 0 \} \) and for any \( k \leq l + 1 \), the submodule of \( R^n \) generated by any \( k \) elements of the set \( \{ v_0, \ldots, v_l \} \) is a free summand of \( R^n \). We define the differential operators \( \partial_l : \tilde{C}_l(R^n) \rightarrow \tilde{C}_{l-1}(R^n) \), \( l \geq 0 \), in the usual way and consider \( \tilde{C}_l(R^n) \) as a left \( \text{GL}_n(R) \)-module in a natural way. Note that \( \tilde{C}_{-1}(R^n) := \mathbb{Z} \). So we have the complex of \( \text{GL}_n(R) \)-modules:

\[
\tilde{C}_\bullet(R^n) : \cdots \rightarrow \tilde{C}_2(R^n) \rightarrow \tilde{C}_1(R^n) \rightarrow \tilde{C}_0(R^n) \rightarrow \tilde{C}_{-1}(R^n) \rightarrow 0.
\]

For a finite subset \( S \) of \( R^n \), let \( \tilde{C}_\bullet(R^n, S) \) be the subcomplex of \( \tilde{C}_\bullet(R^n) \), where \( \tilde{C}_l(R^n, S) \) is the free abelian group generated by terms \( (v_0, \ldots, v_l) \) which are in general position with \( S \). Note that \( \tilde{C}_{-1}(R^n, S) = \mathbb{Z} \) and \( \tilde{C}_\bullet(R^n, \emptyset) = \tilde{C}_\bullet(R^n) \).
Lemma 4.4. For a finite subset $S$ of $R^n$, the complex $\tilde{C}_\bullet(R^n, S)$ is exact for $-1 \leq m \leq (c_n(R) - |S| - 3)/2$. In particular, $\tilde{C}_\bullet(R^n)$ is exact for $-1 \leq m \leq (c_n(R) - 3)/2$. If $R$ is a field, then $\tilde{C}_\bullet(R^n)$ is exact.

Proof. The first part can be proved similar to [8, Theorem 3]. The proof is by induction on $m$. If $m = -1$, then from $-1 \leq (c_n(R) - |S| - 3)/2$, we obtain $|S| < c_n(R)$. So there is $v \in R^n$ in general position with $S$. Thus $\hat{c}(n(v)) = n$. This shows that $\tilde{C}_0(R^n, S) \rightarrow \tilde{C}_{-1}(R^n, S) = \mathbb{Z}$ is surjective. Now let $m = 0$. Then $|S| + 2 < c_n(R)$. Let $z \in \ker(\hat{c})$. We may assume that $z = (v_0) - (v'_0)$. Set $S' := S \cup \{v_0, v'_0\}$. Since $|S'| = |S| + 2 < c_n(R)$, there is a $w \in R^n$ which is in general position with $S'$. Thus $y := (w, v_0) - (w, v'_0) \in \tilde{C}_1(R^n, S)$ and $\partial_1(y) = x$.

Now let the claim is true for all numbers $< m$. Fix a nonzero vector $v \in R^n$ which is in general position with $S$.

For $z = \sum_j l_j(v_{0,j}, \ldots, v_{m,j}) \in \tilde{C}_m(R^n, S)$, let $I(z)$, $-1 \leq I(z) \leq m$, be the greatest natural number such that for any $j$, $(v_{0,j}, \ldots, v_{I(z),j})$ is in general position with $S_j := S \cup \{v_j\} \cup \{v_{I(z)+1,j}, \ldots, v_{m,j}\}$.

Now suppose that $z \in \ker(\tilde{\partial}_m)$. We want to show that $z \in \text{im}(\tilde{\partial}_{m+1})$. First we show that we may assume that $I(z) \geq 0$. Let $I(z) = -1$ and consider the set $S_j := S \cup \{v_j\} \cup \{v_{0,j}, \ldots, v_{m,j}\}$. Since $|S_j| = |S| + m + 2 < c_n(R)$ for any $j$, there is $w_j \in R^n$ which is in general position with $S_j$. Now if

$$z_1 := z - \tilde{\partial}_{m+1} \left( \sum_j l_j(w_j, v_{0,j}, \ldots, v_{m,j}) \right),$$

then $\tilde{\partial}_m(z_1) = 0$ and $I(z_1) \geq 0$. Thus we may assume that $I(z) \geq 0$. Moreover, we want to show that we may assume that $I(z) = m$. So let $-1 < I(z) < m$ and write $z = z' + z''$, where $z'$ contains those terms $y_j := (v_{0,j}, \ldots, v_{m,j})$ of $z$ such that $I(y_j) = I(z)$ and $z''$ contains those terms $y_j$ of $z$ such that $I(y_j) > I(z)$. We may write $z = \sum_k s_k x_k + z''$ such that $s_k \in \tilde{C}_{I(z)}(R^n)$ and $x_k \in \tilde{C}_{m-I(z)-1}(R^n)$. Note that the terms $x_k$ are disjoint. Thus

$$\sum_k \tilde{\partial}_{I(z)}(s_k)x_k + (-1)^{I(z)} s_k \tilde{\partial}_{m-I(z)-1}(x_k) + \tilde{\partial}_m(z'') = 0.$$ 

We show that the only terms $y_k'$ in the left hand side of the above formula with $I(y_k') < I(z)$ are the terms of $\sum_k \tilde{\partial}_{I(z)}(s_k)x_k$. Clearly the terms of $\tilde{\partial}_m(z'')$ and $s_k \tilde{\partial}_{m-I(z)-1}(x_k)$ do not have this property. Now assume that $y_k' := (v_{0,k}, \ldots, v_{I(z),k}, v_{I(z)+1,k}, \ldots, v_{m,k})$, $0 \leq i \leq I(z)$, is a term of $\tilde{\partial}_{I(z)}(s_k)x_k$. If $I(y_k') = I(z)$, then the vector $v_{I(z)+1,k}$ is in general position with $S \cup \{v\} \cup \{v_{I(z)+2,k}, \ldots, v_{m,k}\}$. But
$(v_0,k,\ldots,v_{I(z),k})$ is in general position with $S \cup \{v\} \cup \{v_{I(z)+1,k},\ldots,v_{m,k}\}$. This implies that $(v_0,k,\ldots,v_{I(z),k},v_{I(z)+1,k})$ should be in general position with the set $S \cup \{v\} \cup \{v_{I(z)+2,k},\ldots,v_{m,k}\}$, which is not possible, because $(v_0,k,\ldots,v_{m,k})$ is a term of $z'$.  

Since $x_k$’s are disjoint, we have $\partial I(z)(s_k) = 0$. If $x_k = (w_{1,k},\ldots,w_{m-I(z),k})$, then clearly the terms of $s_k$ are in general position with the set $S_k^{\prime\prime\prime} := S \cup \{v\} \cup \{w_{1,k},\ldots,w_{m-I(z),k}\}$. Thus $s_k \in \tilde{C}_{I(z)}(R^n, S_k^{\prime\prime\prime})$. Moreover, we have

$$I(z) \leq \frac{c_n(R) - (|S| + m - I(z) + 1)}{2}.$$  

So by induction, there is $s_k' \in \tilde{C}_{I(z)+1}(R^n, S_k^{\prime\prime\prime})$ such that $\partial I(z)+1(s_k') = s_k$, for any $k$. Now if we put $z_1 := z - \sum_k \partial I(z)+1(s_k'x_k)$, then $z_1 \in \ker(\partial_m)$ and

$$I(z_1) = I(z'') + (-1)^{I(z)+1} \sum_k s_k' \tilde{\partial}_{m-I(z)-1}(x_k) > I(z).$$

By continuing this process we may assume that $I(z) = m$. This means that the vector $v$ is in general position with the set $S \cup \{v_{0,k},\ldots,v_{m,k}\}$ for all $k$. Thus if $Z := \sum_k l_k(v, v_{0,k},\ldots,v_{m,k})$, then $Z \in \tilde{C}_{m+1}(R^n, S)$ and we have $\tilde{\partial}_{m+1}(Z) = z$.

If $R = F$ is a field, then the proof of the claim is very easy. Let $z_1 = \sum_j l_j(v, v_{0,j},\ldots,v_{m,j}) \in \tilde{C}_{m}(F^n)$ and let $v$ be any nonzero vector of $F^n$. Then $Z_1 := \sum_j l_j(v, v_{0,j},\ldots,v_{m,j}) \in \tilde{C}_{m+1}(F^n)$. Now if $z_1 \in \ker(\partial_m)$, then $\tilde{\partial}_{m+1}(Z_1) = z_1$. 

**Remark 4.5.** One can show that in general $\tilde{C}_{\bullet}(R^n)$ is exact for $-1 \leq m \leq n - 2$. But we need the exactness of this complex for $m \leq n - 1$ (at least when $n = 3$). As we have seen in the above lemma this is easy when $R$ is a field. This is probably true for any local ring, but I do not know how to resolve this problem without considering $c_n(R)$. Once this is done, we may remove the condition about $c_n(R)$ from the following theorem.

**Theorem 4.6.** Let $R$ be a local ring. If $R/m_R$ is finite we assume that $|R/m_R| = p^d$.

(i) If $R$ is a domain or an algebra over a field and $n \leq \min\{(p-1)d - 1, (c_{n+1}(R) - 3)/2\}$, then

$$H_n(\GL_n(R)) \cong H_n(\GL_{n+1}(R)) \cong H_n(\GL_{n+2}(R)) \cong \cdots.$$  

If $R$ is a field we may only assume that $n \leq (p-1)d - 1$.

(ii) If $n \leq \min\{(p-1)d - 3, (c_{n+1}(R) - 3)/2\}$, then

$$H_n(\GL_n(R)) \cong H_n(\GL_{n+1}(R)) \cong H_n(\GL_{n+2}(R)) \cong \cdots.$$  

**Proof.** This can be done as the proof of Theorem 1 in Subsection 2.2 of [5].
Proposition 4.7. Let $|R/m_R| \neq 2, 3, 4, 5, 8, 9, 16, 32$. If $R$ is a domain or an algebra over a field, we may only assume that $|R/m_R| \neq 2, 3, 4, 5, 8$. Then we have the homology stability

$$H_3(\text{GL}_3(R)) \xrightarrow{\cong} H_3(\text{GL}_4(R)) \xrightarrow{\cong} H_3(\text{GL}_5(R)) \xrightarrow{\cong} \cdots.$$ 

Furthermore, $\text{K}_{3 \text{rd}}^0(R) \cong H_3(\text{SL}(R))/T$, were $T$ is generated by the elements $\mathfrak{c}(\text{diag}(a, 1, a^{-1}), \text{diag}(b, b^{-1}, 1), \text{diag}(c, 1, c^{-1})), a, b, c \in R^\times$.

Proof. The homology stability result follows from Theorem 4.6 and Example 4.3(v). The second claim can be proved as [12, Corollary 2.4, §2]. Here we need the homology stability to generalize [12, Proposition 2.1]. \qed

5. Third homology of general linear groups of rank 2 and 3

Let $R$ be a local ring such that its residue field has at least four elements. Then $H_n(\text{GL}_2(R), C^*(R^2)) \cong H_n(\text{GL}_2(R))$ for $0 \leq n \leq 3$. The spectral sequence $E^1_{p,q}$ gives a filtration

$$0 = F_{-1} H_3(\text{GL}_2(R)) \subseteq \cdots \subseteq F_3 H_3(\text{GL}_2(R)) = H_3(\text{GL}_2(R)),$$

such that $E_{i,3-i}^\infty \simeq F_i H_3(\text{GL}_2(R))/F_{i-1} H_3(\text{GL}_2(R))$. By Lemma 1.7, we have

$$B(R) \simeq E_{3,0}^\infty \simeq H_3(\text{GL}_2(R))/F_2 H_3(\text{GL}_2(R)).$$

Since $E_{2,1}^\infty = 0$, $F_1 H_3(\text{GL}_2(R)) = F_2 H_3(\text{GL}_2(R))$. Moreover,

$$E_{0,3}^\infty \simeq F_0 H_3(\text{GL}_2(R)) = \text{im}(H_3(B_2) \to H_3(\text{GL}_2(R))).$$

The next lemma studies the map

$$(R^\times \otimes R^\times)_{\sigma}^\sigma = E_{1,2}^2 \to E_{1,2}^\infty = F_2 H_3(\text{GL}_2(R))/H_3(B_2) \subseteq H_3(\text{GL}_2(R))/H_3(B_2).$$

Lemma 5.1. Let $u \in (R^\times \otimes R^\times)_{\sigma}^\sigma \subseteq H_2(T_2)^\sigma$ and $h \in B_2(T_2)_{T_2}$ a representing cycle for $u$. Let $\tau$ be the automorphism of $B_\bullet(T_2)_{T_2}$ induced by $\sigma$ and let $\tau(h) - h = \partial_3^T(b), b \in B_3(T_2)_{T_2}$. Then the image of $u$ under the map

$$(R^\times \otimes R^\times)_{\sigma}^\sigma \to H_3(\text{GL}_2(R))/H_3(B_2)$$

coincides with the homology class of the cycle $b - \rho_s(h)$, where $s := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and

$$\rho_s([g_1|g_2]) := [s|sg_1s^{-1}|sg_2s^{-1}] - [g_1|s|sg_2s^{-1}] + [g_1|g_2|s].$$

Proof. See [17, Lemma 2.5]. \qed

Let $\text{GM}_2(R)$ denotes the group of monomial matrices in $\text{GL}_2(R)$ and consider the extension

$$1 \to T_2 \to \text{GM}_2(R) \to \Sigma_2 \to 1,$$

where $\Sigma_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$. We often think of $\Sigma_2$ as the symmetric group of order two $\{1, \sigma\}$. Note that $\text{GM}_2(R) = T_2 \rtimes \Sigma_2$ and the action of
σ on T2 is given by σ(a, b) = (b, a). From this extension we obtain the first quadrant spectral sequence

\[ E^{2}_{p,q} = H_p(\Sigma_2, H_q(T_2)) \Rightarrow H_{p+q}(\text{GM}_2(R)). \]

This spectral sequence gives us a filtration

\[ 0 = F_{-1}H_3(\text{GM}_2(R)) \subseteq \ldots \subseteq F_3H_3(\text{GM}_2(R)) = H_3(\text{GM}_2(R)), \]

such that

\[
\begin{align*}
E^{\infty}_{3,0} &\simeq F_0H_3(\text{GM}_2(R)) = H_3(T_2)^\sigma, \\
E^{\infty}_{1,2} &\simeq E^{3}_{1,2} \simeq F_1H_3(\text{GM}_2(R))/F_0H_3(\text{GM}_2(R)), \\
E^{\infty}_{2,1} &\simeq F_2H_3(\text{GM}_2(R))/F_1H_3(\text{GM}_2(R)) = 0, \\
E^{\infty}_{3,0} &\simeq H_3(\text{GM}_2(R))/F_2H_3(\text{GM}_2(R)).
\end{align*}
\]

From the natural inclusion Σ_2 ⊆ GM_2(R), one easily sees that the composition \( H_3(\Sigma_2) \to H_3(\text{GM}_2(R)) \to H_3(\Sigma_2) \) coincides with the identity map. Thus \( E^{\infty}_{3,0} \simeq H_3(\Sigma_2) \). Now the above relations imply the following isomorphisms

\[
H_3(\text{GM}_2(R)) \simeq H_3(\Sigma_2),
\]

(5.1)

\[
E^{\infty}_{2,1} \simeq F_2H_3(\text{GM}_2(R))/H_3(T_2).
\]

(5.2)

The next lemma gives an explicit description of the composition

\[
H_3(T_2)^\sigma \longrightarrow E^{\infty}_{1,2} \xrightarrow{\sim} F_2H_3(\text{GM}_2(R))/H_3(T_2) \subseteq H_3(\text{GM}_2(R))/H_3(T_2).
\]

**Lemma 5.2.** Let \( u \in (R^\times \otimes R^\times)^\sigma \subseteq H_2T_2 \) and \( h \in B_3(T_2)T_2 \) a representing cycle for \( u \). Let \( \tau \) be the automorphism of \( B_\bullet(T_2)T_2 \) induced by \( \sigma \) and let \( \tau(h) - h = \partial_3T_2(b) \), \( b \in B_3(T_2)T_2 \). Then the image of \( u \) under the map

\[
(R^\times \otimes R^\times)^\sigma \to H_3(\text{GM}_2(R))/H_3(T_2)
\]

coincides with the homology class of the cycle \( b - \rho_s(h) \), where \( \rho_s \) is defined in Lemma 5.1.

**Proof.** See [12, Lemma 4.3].

For more details about the spectral sequence \( E^i_{p,q} \) see [12, Section 4]. Since the residue field of \( R \) has at least four elements, \( H_1(C_\bullet(R^2)) = 0 \) for \(-1 \leq i \leq 3\). From this we obtain a natural map

\[
\varphi : H_3(\text{GL}_2(R)) \to H_3(C_\bullet(R^2)_{\text{GL}_2(R)}) = \mathfrak{p}(R).
\]

**Theorem 5.3.** Let \(|R/\mathfrak{m}_R| \neq 2, 3, 4, 5, 8, 9, 16, 32\). If \( R \) is a domain or an algebra over a field, we may only assume that \(|R/\mathfrak{m}_R| \neq 2, 3, 4, 8\). Then we have the exact sequence

\[
H_3(\text{GM}_2(R)) \to H_3(\text{GL}_2(R)) \xrightarrow{\varphi} B(R) \to 0.
\]
Proof. By Proposition 2.2, \( H_i(B_2) \cong H_i(T_2) \) for \( 0 \leq i \leq 3 \). Now consider the natural map \( H_3(\text{GM}_2(R)) \rightarrow H_3(\text{GL}_2(R)) \). It is clear from the filtrations of \( H_3(\text{GM}_2(R)) \) and \( H_3(\text{GL}_2(R)) \) that \( F_0H_3(\text{GM}_2(R)) \) maps onto \( F_0H_3(\text{GL}_2(R)) \). This fact together with Lemmas 5.2 and 5.1 imply that \( F_0H_3(\text{GM}_2(R)) \) maps onto \( F_2H_3(\text{GL}_2(R)) \). Thus

\[
\text{im}(H_3(\text{GM}_2(R))) = F_2H_3(\text{GL}_2(R)) + \text{im}H_3(\Sigma_2).
\]

The matrix \( s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) is conjugate to the matrix \( \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \in B_2 \). Hence

\[
\text{im}(H_3(\Sigma_2) \rightarrow H_3(\text{GL}_2(R))) \subseteq \text{im}(H_3(B_2) \rightarrow H_3(\text{GL}_2(R))) = F_0H_3(\text{GL}_2(R)) \subseteq F_2H_3(\text{GL}_2(R)).
\]

This completes the proof of the theorem. \( \square \)

**Theorem 5.4.** Let \( |R/m_R| \neq 2, 3, 4, 5, 8, 9, 16, 32 \). If \( R \) is a domain or an algebra over a field, we only may assume that \( |R/m_R| \neq 2, 3, 4, 8 \). Then the kernel of \( \text{inc}_*: H_3(\text{GL}_2(R)) \rightarrow H_3(\text{GL}_3(R)) \) consists of elements of the form

\[
\sum a \otimes \{b, c\} + b \otimes \{a, c\} = 0 \in R^x \otimes K_2^M(R).
\]

In particular, \( \ker(\text{inc}_*) \subseteq R^x \cup H_2(\text{GL}_1(R)) \subseteq H_3(\text{GL}_2(R)) \), where the cup product is induced by the inclusion \( \text{inc} : R^x \times \text{GL}_1(R) \rightarrow \text{GL}_2(R) \). Moreover \( \ker(\text{inc}_*) \) is a 2-torsion group.

**Proof.** This can be proved as [11, Theorem 3.1] using Lemma 3.5(ii). \( \square \)

Let \( C_l(R^n) \) be the free abelian groups with a basis consisting of \( (l + 1) \)-tuples \( \langle w_0, \ldots, w_l \rangle \), where every \( \min\{l + 1, n\} \) of \( w_i \in R^n \) are basis of a free direct summand of \( R^n \). Let us define differentials \( \partial_l : C_l(R^n) \rightarrow C_{l-1}(R^n) \), \( l \geq 0 \), in usual way. So we have the complex of \( \text{GL}_n(R) \)-modules:

\[
C_\bullet(R^n) : \cdots \rightarrow C_2(R^n) \rightarrow C_1(R^n) \rightarrow C_0(R^n) \rightarrow C_{-1}(R^n) = \mathbb{Z} \rightarrow 0.
\]

**Lemma 5.5.** The complex \( C_\bullet(R^n) \) is exact for \( -1 \leq i \leq (c_n(R) - 3)/2 \).

**Proof.** This can be proved as Lemma 4.4. \( \square \)

In particular, this implies that if \( |R/m_R| > 9 \), then \( C_\bullet(R^3) \) is exact for \( -1 \leq i \leq 3 \) (see Example 4.3). Now as in [17, §3] we can construct a map

\[
\rho : H_3(\text{GL}_3(R)) \rightarrow \mathfrak{p}(R)
\]

such that the composition \( H_3(\text{GL}_2(R)) \rightarrow H_3(\text{GL}_3(R)) \xrightarrow{\rho} \mathfrak{p}(R) \) coincides with the map \( \varphi : H_3(\text{GL}_2(R)) \rightarrow \mathfrak{p}(R) \) constructed in above.

**Theorem 5.6.** Let \( R \) be a local ring such that \( |R/m_R| > 9 \) and \( |R/m_R| \neq 16, 32 \). If \( R \) is a domain or is an algebra over a field we may only assume that \( |R/m_R| > 9 \). Then we have the exact sequence

\[
H_3(\text{GM}_2(R)) \oplus H_3(T_3) \rightarrow H_3(\text{GL}_3(R)) \xrightarrow{\rho} B(R) \rightarrow 0.
\]
Proof. Let \( T_3 := R^3 = R^\times \times R^\times \times R^\times \) embeds diagonally in \( \text{GL}_3(R) \). Then by Lemma 3.5(ii) it is easy to show that \( H_3(\text{GL}_3(R)) \) is generated by the images of \( H_3(\text{GL}_2(R)) \) and \( R^\times \times R^\times \times R^\times = \text{im}((R^\times)^{\otimes 3} \to H_3(\text{GL}_3(R))\). Now one can use Theorem 5.3 to prove the claim as it is done in [17, §3]. \( \square \)

6. A Bloch-Wigner exact sequence

Now we are ready to formulate the main theorem of this article.

Theorem 6.1 (Bloch-Wigner exact sequence). Let \( R \) be a local ring such that \(|R/m_R| > 9\) and \(|R/m_R| \neq 16, 32\). If \( R \) is a domain or is an algebra over a field we may only assume that \(|R/m_R| > 9\). Then we have the exact sequence

\[ T_R \to K_3^{\text{ind}}(R) \to B(R) \to 0, \]

where \( T_R \) sits in the short exact sequence

\[ 0 \to \text{Tor}_1^Z(\mu(R), \mu(R))_\sigma \to T_R \to H_1(\Sigma_2, \mu_{2\infty}(R) \otimes \mathbb{Z} \mu_{2\infty}(R)) \to 0. \]

Moreover, if there is a homomorphism \( R \to F \), \( F \) a field, such that the map \( \mu(R) \to \mu(F) \) is injective, then we have the exact sequence

\[ 0 \to \text{Tor}_1^Z(\mu(R), \mu(R))^{\sim} \to K_3^{\text{ind}}(R) \to B(R) \to 0, \]

where the composition

\[ \text{Tor}_1^Z(\mu(R), \mu(R)) \to \text{Tor}_1^Z(\mu(R), \mu(R))^{\sim} \to K_3^{\text{ind}}(R) \]

is induced by the map \( \mu(R) \to \text{SL}_2(R), \xi \mapsto \text{diag}(\xi, \xi^{-1}) \).

Proof. This can be done as the proof of [12, Theorem 5.1] \( \square \)

Note that in the above theorem, \( T_R \) can be defined as

\[ T_R := F_2H_3(\text{GM}_2(R))/H_3(T_2)_\sigma = F_2H_3(\text{GM}_2(R))/H_3(T_2), \]

and we have the following lemma.

Lemma 6.2. Let \( R \) be any commutative ring. Then \( T_R \) sits in the short exact sequence

\[ 0 \to \text{Tor}_1^Z(\mu(R), \mu(R))_\sigma \to T_R \to H_1(\Sigma_2, \mu_{2\infty}(R) \otimes \mu_{2\infty}(R)) \to 0. \]

Moreover, if we have a homomorphism \( R \to F \), \( F \) a field, such that \( \mu(R) \to \mu(F) \) is injective, then \( T_R \cong \text{Tor}_1^Z(\mu(R), \mu(R))^{\sim} \).

Proof. See [12, Section 4, Lemma 4.4, Corollary 4.5]. \( \square \)

Corollary 6.3. Let \( R \) be a discrete valuation ring with field of fraction \( K \) and residue field \( F = R/m_R \) with more than nine elements. Suppose that either \( \text{char}(K) = \text{char}(F) \) or \( F \) is finite. Then the natural map \( B(R) \to B(K) \) is an isomorphism.

Proof. By [7, Theorem 2.1] \( K_3^{\text{ind}}(R) \cong K_3^{\text{ind}}(K) \). Thus the claim follows from the Bloch-Wigner exact sequence for \( R \) and \( K \), obtained from Theorem 6.1. Note that here \( \mu(R) = \mu(K) \). \( \square \)
If \( F \) is a finite field such that \(|F| > 9\), then by Theorem 6.1 we have the Bloch-Wigner exact sequence

\[
0 \to \text{Tor}_1^\mathbb{Z}(\mu(F), \mu(F)) \to K_3^\text{ind}(F) \to B(F) \to 0.
\]

But by a direct approach we can prove a better result with easier arguments.

**Proposition 6.4.** Let \( F \) be a finite field with at least four elements.

(i) If \( F \neq \mathbb{F}_4, \mathbb{F}_8 \), then we have the exact sequence\[
0 \to \text{Tor}_1^\mathbb{Z}(\mu(F), \mu(F)) \to H_3(\text{SL}_2(F))_{F^\times} \to B(F) \to 0.
\]

(ii) If \( F \neq \mathbb{F}_4, \mathbb{F}_8 \), then \( H_3(\text{SL}_2(F))_{F^\times} \simeq K_3^\text{ind}(F) \) and if \( F = \mathbb{F}_4 \) or \( \mathbb{F}_8 \) then \( H_3(\text{SL}_2(F))_{F^\times} \simeq K_3^\text{ind}(F) \oplus \mathbb{Z}/2 \). Moreover, we have the Bloch-Wigner exact sequence

\[
0 \to \text{Tor}_1^\mathbb{Z}(\mu(F), \mu(F)) \to K_3^\text{ind}(F) \to B(F) \to 0.
\]

**Proof.** Since \( F^\times \) is a finite cyclic group, \( H_2(\text{GL}_1(F)) = 0 \). If \(|F| > 4\), we have \( H_1(\text{SL}_2(F)) = 0 \), \( H_2(\text{SL}_2(F))_{F^\times} = 0 \) (see Example 2.5). Hence by Lemma 2.6, \( H_r(F^\times, H_s(\text{SL}_2(F))) = 0 \) for all \( r \geq 0 \) and for \( s = 1, 2 \). If \( F = \mathbb{F}_4 \), we have \( H_1(\text{SL}_2(F)) = 0 \) and \( H_2(\text{SL}_2(F)) = \mathbb{Z}/2 \). Since \( \mathbb{F}_4^\times = \mathbb{F}_4^{\times 2} \), the action of \( F^\times \) on \( H_s(\text{SL}_2(F)) \) is trivial. These together with the universal coefficient theorem imply that \( H_r(F^\times, H_s(\text{SL}_2(F))) = 0 \) for any \( r \geq 1 \) and \( s = 1, 2 \). Now by an easy analysis of the corresponding Lyndon-Hochschild-Serre spectral sequence of the extension

\[1 \to \text{SL}_2(F) \to \text{GL}_2(F) \to F^\times \to 1,\]

we obtain the isomorphism

\[H_3(\text{SL}_2(F))_{F^\times} \simeq H_3(\text{GL}_2(F))/H_3(\text{GL}_1(F)).\]

(i) First let \( F \neq \mathbb{F}_4, \mathbb{F}_8 \). From the proof of Theorem 5.3 we see that we have the exact sequence

\[F_2H_3(\text{GM}_2(F)) \to H_3(\text{GL}_2(F)) \to B(F) \to 0.\]

Let \( M := H_3(F^\times) \oplus H_3(F^\times) \subseteq H_3(T_2) \). From the commutative diagram

\[
\begin{array}{ccc}
M & \to & M \\
\downarrow & & \downarrow \\
F_2H_3(\text{GM}_2(F)) & \to & H_3(\text{GL}_2(F)) \to B(F) \to 0,
\end{array}
\]

we obtain the exact sequence \( T_F \to H_3(\text{SL}_2(F))_{F^\times} \to B(F) \to 0 \). By Lemma 6.2, \( T_F \simeq \text{Tor}_1^\mathbb{Z}(\mu(F), \mu(F)) \). To finish the proof of the proposition we have to prove that the map

\[\text{Tor}_1^\mathbb{Z}(\mu(F), \mu(F)) \to H_3(\text{SL}_2(F))_{F^\times}\]
is injective. Since
\[
\text{Tor}_1^{\mathbb{Z}}(\mu(F), \mu(F)) \to \text{Tor}_1^{\mathbb{Z}}(\mu(F), \mu(F)) = \text{Tor}_1^{\mathbb{Z}}(\mu(F), \mu(F))
\]
is injective, it is sufficient to prove that
\[
\text{Tor}_1^{\mathbb{Z}}(\mu(F), \mu(F)) \to H_3(\SL_2(F))
\]
is injective. Since \(H_3(\SL_2(F)) \simeq K_3^{\text{ind}}(F)\) \([10, \text{Corollary 5.4}], [9, \text{Proposition 6.4}]\) it is sufficient to prove that
\[
\text{Tor}_1^{\mathbb{Z}}(\mu(F), \mu(F)) \to K_3^{\text{ind}}(F)
\]
is injective. But this has been proved by Suslin \([17, \text{Theorem 5.2}]\), thus the result follows.

Now let \(F = \mathbb{F}_4\) or \(\mathbb{F}_8\). Then \(H_3(B_2(F)) \simeq H_3(T_2(F)) \oplus \mathbb{Z}/2\) (see Example 2.4). Now from the filtration of \(H_3(\GL_2(F))\) induced by the spectral sequence \(E^1_{p,q}\), we obtain the exact sequences
\[
0 \to F_2 H_3(\GL_2(F)) \to H_3(\GL_2(F)) \to B(F) \to 0,
\]
\[
H_3(B_2(F)) \to F_2 H_3(\GL_2(F)) \to E^2_{2,1} \to 0,
\]
which imply the exact sequences
\[
0 \to F_2 H_3(\GL_2(F))/M \to H_3(\SL_2(F))_{F^\times} \to B(F) \to 0,
\]
\[
\text{Tor}_1^{\mathbb{Z}}(\mu(F), \mu(F)) \oplus \mathbb{Z}/2 \to F_2 H_3(\GL_2(F))/M \to E^2_{2,1} \to 0.
\]
By Lemmas 5.1 and 5.2, \(E^2_{2,1} \to E^2_{2,1}\) is surjective. But
\[
E^2_{2,1} \simeq H_1(\Sigma, \mu_2 \otimes \mu_2^*) = 0.
\]
Thus we have the exact sequence
\[
\text{Tor}_1^{\mathbb{Z}}(\mu(F), \mu(F)) \oplus \mathbb{Z}/2 \to H_3(\SL_2(F))_{F^\times} \to B(F) \to 0.
\]
Similar to the above, we can show that the natural homomorphism
\[
\text{Tor}_1^{\mathbb{Z}}(\mu(F), \mu(F)) \to H_3(\SL_2(F))_{F^\times}
\]
is injective. The map \(\mathbb{Z}/2 \to H_3(\SL_2(\mathbb{F}_8))_{\mathbb{F}_8^\times}\) is the composition map
\[
\mathbb{Z}/2 \simeq \bigwedge^3_{\mathbb{Z}} \mathbb{F}_8^\times \to H_3(\mathbb{F}_8) \simeq H_3(N(\mathbb{F}_8)) \to H_3(\SL_2(\mathbb{F}_8))_{\mathbb{F}_8^\times}
\]
and the map \(\mathbb{Z}/2 \to H_3(\SL_2(\mathbb{F}_4))_{\mathbb{F}_4^\times}\) is the composition map
\[
\mathbb{Z}/2 \simeq (\mathbb{F}_4^\times \otimes \mathbb{F}_4^\times)^{-\sigma} \simeq H_3(\mathbb{F}_4)_{\mathbb{F}_4^\times} \simeq H_3(N(\mathbb{F}_4)) \to H_3(\SL_2(\mathbb{F}_4))_{\mathbb{F}_4^\times}
\]
which both maps are injective by Example 2.7. This completes the proof of (i).

(ii) Let \(F \neq \mathbb{F}_4, \mathbb{F}_8\). Then by Theorem 5.4, the kernel of the map \(\text{inc}_x : H_3(\GL_2(F)) \to H_3(\GL_2(F))\) consists of elements of the form \(x = \sum c(\text{diag}(a, 1), \text{diag}(1, b), \text{diag}(c, c^{-1}))\) provided that
\[
\sum a \otimes \{b, c\} + b \otimes \{a, c\} = 0 \in F^\times \otimes K_2^M(F).
\]
Since \(x \in \text{im}(H_2(F^\times) \otimes F^\times \to H_3(\GL_2(F)))\) and \(H_2(F^\times) = 0\), the kernel of \(\text{inc}_x\) is trivial. On the other hand from the exact sequence of Lemma 3.5(ii),
Now let $F = \mathbb{F}_4$, or $F = \mathbb{F}_8$. Since $F^\times = F^\times_2$, we have $H_3(\text{SL}_2(F))_{F^\times} = H_3(\text{SL}_2(F))$. But by a direct calculation one can show that $H_3(\text{SL}_2(\mathbb{F}_4)) \simeq \mathbb{Z}/30$ and $H_3(\text{SL}_2(\mathbb{F}_8)) \simeq \mathbb{Z}/126$. Moreover, by Quillen’s calculation of the $K$-groups of finite fields, we have $K_3^{\text{ind}}(\mathbb{F}_q) = K_3(\mathbb{F}_q) \simeq \mathbb{Z}/(q^2 - 1)$. Thus $K_3^{\text{ind}}(\mathbb{F}_4) \simeq \mathbb{Z}/15$ and $K_3^{\text{ind}}(\mathbb{F}_8) \simeq \mathbb{Z}/63$. Therefore $H_3(\text{SL}_2(F))_{F^\times} \simeq K_3^{\text{ind}}(\mathbb{F}_q) \oplus \mathbb{Z}/2$. (See Example 6.6 below for another proof of this part.) Finally, the Bloch-Wigner exact sequence follows from (i).

□

**Corollary 6.5.** The Bloch group of a finite field $\mathbb{F}_q$ with at least four elements is

$$B(\mathbb{F}_q) \simeq \begin{cases} \mathbb{Z}/(q + 1) & \text{if } q \text{ is even} \\ \mathbb{Z}/(q^2 - 1) & \text{if } q \text{ is odd.} \end{cases}$$

**Proof.** This follows from the Bloch-Wigner exact sequence for finite fields proved in the previous proposition and the fact that

$$\text{Tor}_1^\mathbb{Z}(\mu(\mathbb{F}_q), \mu(\mathbb{F}_q)) \simeq \mathbb{F}_q^\times \simeq \mathbb{Z}/(q - 1).$$

Note that $K_3^{\text{ind}}(\mathbb{F}_q) = K_3(\mathbb{F}_q) \simeq \mathbb{Z}/(q^2 - 1)$. Moreover, if $q$ is even then $\text{Tor}_1^\mathbb{Z}(\mu(\mathbb{F}_q), \mu(\mathbb{F}_q)) \simeq \mathbb{Z}/(q - 1)$ and if $q$ is odd then $\text{Tor}_1^\mathbb{Z}(\mu(\mathbb{F}_q), \mu(\mathbb{F}_q)) \simeq \mathbb{Z}/2(q - 1)$.

□

**Example 6.6.** Let $F$ be any finite field. We already have proved that

$$H_3(\text{SL}_2(F))_{F^\times} \simeq H_3(\text{GL}_2(F))/H_3(\text{GL}_1(F)),$$

(see the proof of Proposition 6.4). In fact we proved this for $|F| \geq 4$. But the same isomorphism can be proved easily for $|F| = 2, 3$.

If $\text{char}(F) = p > 0$, then by Example 2.7, we have

$$(H_3(\text{SL}_2(F))_{F^\times})_{(p)} \simeq \begin{cases} \mathbb{Z}/p & \text{if } |F| = 2, 3, 4, 8 \\ 0 & \text{otherwise}. \end{cases}$$

This implies that

$$H_3(\text{SL}_2(F))_{F^\times} \simeq \begin{cases} H_3(\text{SL}_2(F), \mathbb{Z}[1/p])_{F^\times} \oplus \mathbb{Z}/p & \text{if } |F| = 2, 3, 4, 8 \\ H_3(\text{SL}_2(F), \mathbb{Z}[1/p])_{F^\times} & \text{otherwise.} \end{cases}$$
But Hutchinson has proved that for any finite field $F$, the action of $F^\times$ on $H_3(\text{SL}_2(F), \mathbb{Z}[1/p])$ is trivial and $H_3(\text{SL}_2(F), \mathbb{Z}[1/p]) \simeq K_3^{\text{ind}}(F)$ [6, Lemma 3.8, Corollary 3.9]. These imply that

$$H_3(\text{SL}_2(F))_{F^\times} \simeq \begin{cases} K_3^{\text{ind}}(F) \oplus \mathbb{Z}/p & \text{if } |F| = 2, 3, 4, 8 \\ K_3^{\text{ind}}(F) & \text{otherwise.} \end{cases}$$

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