Particle acceleration in relativistic turbulence: a theoretical appraisal

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We discuss the physics of stochastic particle acceleration in relativistic MHD turbulence, combining numerical simulations of test-particle acceleration in synthetic wave turbulence spectra with detailed analytical estimates. In particular, we study particle acceleration in wave-like isotropic fast mode turbulence, in Alfvén and slow Goldreich-Sridhar type wave turbulence (properly accounting for local anisotropy effects), including resonance broadening due to wave decay and pitch-angle randomization. At high particle rigidities, the contributions of those three modes to acceleration are comparable to within an order of magnitude, as a combination of several effects (partial disappearance of transit-time damping for fast modes, increased scattering rate for Alfvén and slow modes due to resonance broadening). Additionally, we provide analytical arguments regarding acceleration beyond the regime of MHD wave turbulence, addressing the issue of non-resonant acceleration in a turbulence comprised of structures rather than waves, as well as the issue of acceleration in small-scale parallel electric fields. Finally, we compare our results to the existing literature and provide ready-to-use formulas for applications to high-energy astrophysical phenomenology.

I. INTRODUCTION

The physical mechanisms that govern energy dissipation and conversion in astrophysical sources and the acceleration of particles, from sub-relativistic momenta to the most extreme observed cosmic-ray energies, are longstanding problems in space plasma physics and high-energy astrophysics. Magnetized turbulence plays a central role in this field, whether directly or indirectly. At the least, it provides the essential scattering agent at the core of diffusive shock acceleration [1, 2], and reconnection itself is intimately linked to turbulence, either because turbulence generates reconnection, e.g. [3], or the converse [4]. Electromagnetic turbulence indeed provides an efficient source of particle acceleration, following Fermi’s original idea [5] that a particle interacting with randomly moving magnetic mirrors of typical velocity dispersion $\beta_m c$ gains energy in a stochastic manner at a rate $\propto \beta_m^2 c^2$. Such stochastic acceleration has been invoked to explain high-energy emission in a variety of astrophysical environments, from solar system plasmas to the remote high-energy Universe, e.g., impulsive solar flares [6], the galactic center [7], accretion disks [8], pulsar wind nebulae [9], galaxy clusters [10], active galactic nuclei [11], gamma-ray bursts [12], etc.

Astrophysical collisionless turbulence is commonly described as an energy cascade spanning orders of magnitude in length scales, from a large stirring scale down to the small wavelengths of dissipative physics, with, in general, most of the fluctuation power in velocity and electromagnetic fields being carried by the larger scales. The usual order-of-magnitude timescale for particle acceleration reads $t_{\text{acc}} \sim t_{\text{scatt}}/\beta_m^2$, where $t_{\text{scatt}}$ represents the scattering timescale, i.e., the time needed for the particle to start diffusing in the turbulence. Hence, for the purpose of accelerating particles to high energies in cosmic plasmas, which is the main theme of this paper, one is interested in particles interacting with large-scale modes of a fast-moving turbulence spectrum: large-scale, because the scattering timescale is a growing function of energy, and fast-moving, because of the scaling of the acceleration timescale. Here, we will thus be interested in the physics of particle interactions with relativistic ideal magnetohydrodynamic (MHD) turbulence, the ideal Ohm’s law being applicable on the scales of interest.

How particles gain energy in a turbulent setting can be addressed using a variety of theoretical or numerical tools. Quasilinear theory, e.g. [13], provides an analytical estimate of the various diffusion tensor components to first order in the spectrum of electromagnetic fluctuations, which are commonly described as a sum of linear eigenmodes of the plasma. In the ideal MHD approximation, those are the incompressible Alfvén modes (thereafter indexed with A) as well as the fast and slow magnetosonic modes (indexed by F and S), while entropy modes, pure density perturbations advected with the medium, do not play any role. Recent non-linear extensions allow to include part of the perturbative expansion in turbulent fluctuations by introducing corrections to the particle trajectory, e.g. [14] and references therein. An important outcome of quasilinear calculations is the existence of resonant particle-wave interactions which provide possibly fast scattering rates, hence short acceleration timescales. However, whether collisionless turbulence can be realistically described as a sum of waves is a long-standing debate, e.g. [15] for a recent appraisal. Furthermore, the intrinsic anisotropic nature of modern MHD turbulence theories [16] prohibits particle-wave resonances [17]. Hence, non-resonant phenomena, corresponding to the interaction of particles
with non-trivial velocity structures, are likely to play a role, and our investigation will confirm this point of view.

On the numerical front, the physics of transport and acceleration can be probed (i) by following test-particles in a synthetic turbulence generated from a sum of plane waves, e.g., [18–21], (ii) by following test-particles in full 3D MHD simulations [22, 23], or (iii), more recently, from 3D particle-in-cell (PIC) simulations [24–28]. While the latter method offers a fully kinetic picture of the collisionless turbulence, from plasma length scales upward, thus allowing, in particular, a self-consistent treatment of the early injection and acceleration stages, MHD simulations provide a useful representation of the largest length scales with a potentially large dynamic range. Although the first method (i) remains subject to the criticism of describing the turbulence as a superposition of linear waves, it allows to relate in a direct way the theoretical predictions for acceleration to the assumptions of the model and to probe effects beyond quasilinear theory. As such, it offers an interesting tool to interpret the results of more evolved simulations, that remain costly to achieve a reasonable dynamic range, and possibly sensitive to some degree to how the turbulence is initialized.

Hence, the above analytical and numerical methods nicely complement each other and we adopt this stance in the present paper. More specifically, we combine theoretical arguments borrowed from quasilinear and extended quasilinear theories as applied to modern turbulence theories to derive predictions for acceleration in relativistic MHD wave turbulence, which we compare to test-particle simulations. With respect to previous work on this topic, our work improves on Ref. [19], which considered an isotropic bath of Alfvén waves and neglected resonance broadening effects; it improves on [29], which considered isotropic turbulence of pure fast, slow and Alfvén waves, by taking into account the effect of anisotropy and mode decay in the turbulence spectrum; it also improves on Ref. [21] by paying attention to the alignment of small-scale modes to the local large-scale magnetic field, which has dramatic consequences on the acceleration physics, as we show; we also improve on the previous semi-analytical studies of Refs.[13, 17] by considering the relativistic regime of wave phase velocities. Finally, we also provide original theoretical estimates for turbulent acceleration in the absence of resonant wave-particle interactions.

Our paper is laid out as follows. Section II addresses the case of a pure fast mode turbulence; Sec. III that of a pure slow mode turbulence and Sec. IV that of pure Alfven turbulence. For each, we compare analytical predictions to numerical simulations. In Sec. V, we discuss some aspects of stochastic acceleration outside the realm of ideal MHD wave turbulence, i.e., we provide estimates for non-resonant acceleration and characterize the influence of small-scale violations of Ohm’s law. Our numerical and analytical results are brought together in Sec. VI and compared to recent ab initio simulations of turbulent acceleration. Finally, we provide a summary in Sec. VII.

II. FAST MODE TURBULENCE

Simulations of sub-relativistic compressible MHD turbulence seem to indicate that Alfvén and slow magnetosonic modes are the main contributors to the kinetic and/or magnetic energy spectra, with little contribution from fast magnetosonic modes [16, 30, 31], although recent work [32] indicates that this very result depends on how turbulence is driven at the outer scale. In any case, this does not exclude that fast modes can influence particle acceleration, depending on their scattering efficiency. Besides, simulations by [33] suggest that in the relativistic regime, fast modes are strongly coupled to Alfvénic modes and maintain a substantial fraction of energy, which furthermore tends to increase with magnetization.

Given the variety of numerical setups used to simulate MHD turbulence, the size of the parameter space to be probed (e.g., the $\beta$–parameter of the plasma, the magnetization, the strength of the turbulence etc.) and the large uncertainty on the scaling laws obtained from simulations or observations, it is still not clear which of the available phenomenology, e.g. [16, 34–37], appears best suited to describe the properties of the cascade for these different modes. Here, we rely on the general result that emerges from theoretical considerations and simulations in the sub-relativistic limit, which defines the cascade of fast modes as isotropic, with a power spectrum index $\sim 1.5 - 1.7$, and an anisotropic cascade of Alfvén modes and (passive) slow modes, à la Goldreich-Sridhar.

A. Theoretical predictions

For fast modes, we introduce the power spectrum:

$$S_k^F = \frac{\eta}{1 - \eta} B_0^2 \frac{(1 - q^2)}{8\pi k^3_{\text{min}}} \left(\frac{k}{k_{\text{min}}}\right)^{-q^2 - 2},$$

(1)

properly normalized over the range of wave vectors $k \in [k_{\text{min}}, k_{\text{max}}]$ with $k_{\text{max}} \gg k_{\text{min}}$, to the amplitude of the magnetic perturbations contributed by fast modes: $2 \int d^3k S_k^F = \eta q / (1 - \eta) B_0^2 = \eta / (1 - \eta) B_0^2$, where the last equality hold for pure fast mode turbulence and the factor 2 accounts for the summation over positive and negative frequencies. Here, $\eta = \langle \delta B^2 \rangle / \langle B_0^2 \rangle$ characterizes the relative magnitude of the turbulent magnetic energy density.
The polarization of fast modes in special relativistic MHD is given by, e.g. [37]

\[ \omega_{F} = \pm \frac{k}{\sqrt{2}} \left\{ \beta_{0}^{2} + \beta_{A}^{2} \beta_{s}^{2} \mu_{k}^{2} + \left[ (\beta_{0}^{2} + \beta_{A}^{2} \beta_{s}^{2} \mu_{k}^{2})^{2} - 4 \beta_{A}^{2} \mu_{s}^{2} \mu_{k}^{2} \right]^{1/2} \right\}^{1/2}, \]

(2)

\[ \delta B_{F}^{k} = \delta b_{k} \left( -\frac{k_{\parallel}}{k_{\perp}} + \frac{k_{\perp}}{k_{\parallel}} \right), \]

(3)

\[ \delta u_{F}^{k} = \frac{\omega_{F}}{k} \frac{\delta b_{k}}{B_{0}} \left( \frac{k_{\perp}}{k_{\parallel}} + \frac{\beta_{s}^{2} k_{\perp}}{\omega_{F}^{2} - \beta_{s}^{2} k_{\parallel}^{2}} k_{\parallel} \right), \]

(4)

\[ \delta E_{F}^{k} = \frac{\omega_{F}}{k} \frac{\delta b_{k}}{B_{0} k_{\parallel}}, \]

(5)

where \( \beta_{A}, \beta_{0} \) and \( \beta_{F} = \left( \beta_{A}^{2} + \beta_{s}^{2} - \beta_{A}^{2} \beta_{s}^{2} \right)^{1/2} \) denote the Alfvén speed, the sound speed and the fast speed, respectively. The wave vector \( k \), with modulus \( k \), is decomposed over a parallel and a perpendicular component with respect to \( B_{0} \): \( k_{\parallel} = k_{\parallel} B_{0}/B_{0}, k_{\perp} = k - k_{\parallel}, \) while \( \mu_{k} = k_{\parallel}/k \). The (complex) perturbation amplitude \( \delta b_{k} \) is related to the power spectrum through: \( S_{F}^{k} = \langle |\delta b_{k}|^{2} \rangle \).

In our analytical predictions, we use the following simplified formula for the mode pulsation: \( \omega_{F} \approx \pm \beta_{0} k \), which provides a good approximation over the range of \( \mu_{k} \in [-1,1] \). These analytical predictions are obtained through quasilinear theory, which evaluates the second-order moments \( \langle \Delta \mu^{2} \rangle \) and \( \langle \Delta p^{2} \rangle \) by direct integration over the electromagnetic fields that the particle experiences over a trajectory which is described by the zeroth-order gyration around the background magnetic field, see [13] for details. Everywhere in this paper, \( \mu \) represents the pitch-angle cosine of the particle with respect to the background magnetic field and \( \mathbf{p} \) its momentum; hence \( \mu = p_{\perp}/p \).

A detailed application of quasilinear theory then provides the following form for the diffusion coefficients:

\[ D_{\mu \mu}^{F} = \frac{\Omega^{2} \pi^{2} (1 - \mu^{2})}{2 B_{0}^{2}} \sum_{n=-\infty}^{+\infty} \int_{k_{\min}}^{k_{\max}} dk \left[ \frac{\int_{-1}^{1} dp_{\perp} K_{n+1}(z_{\perp}) - J_{n-1}(z_{\perp})}{\pi} \right]^{2} \]

\[ \times \omega_{F}^{2} \left( \frac{k_{\mu} \mathcal{R}_{k} \left[ J_{n+1}(z_{\perp}) - J_{n-1}(z_{\perp}) \right]}{1 + \alpha \gamma_{F}^{2}} \right), \]

(6)

\[ D_{pp}^{F} = \frac{\Omega^{2} \pi^{2} \beta_{0}^{2} (1 - \mu^{2})}{2 B_{0}^{2}} \sum_{n=-\infty}^{+\infty} \int_{k_{\min}}^{k_{\max}} dk \left[ \frac{\int_{-1}^{1} dp_{\perp} R_{k} \left[ J_{n+1}(z_{\perp}) - J_{n-1}(z_{\perp}) \right]}{\pi} \right]^{2} \]

\[ \times \omega_{F}^{2} \left( 1 + \alpha \gamma_{F}^{2} \right) S_{F}^{k}, \]

(7)

with the following notations: the sum over \( \pm \) sums over positive and negative real frequencies \( \omega_{F} \), while that over \( n \) runs over the harmonics of the gyrofrequency; \( \gamma_{d} > 0 \) accounts for the possible finite lifetime of modes, i.e. the full mode pulsation \( \omega = \omega_{F} - i \gamma_{d} |\omega_{F}| ; \Omega = c/r_{\parallel} = eB_{0}/p \) and \( z_{\perp} = k_{\perp} \Omega^{-1} \sqrt{1 - \mu^{2}} \).

Finally, Eqs. (6) and (7) can be averaged over \( \mu \) to obtain the quasilinear predictions for the diffusion coefficients for an isotropic population of particles, \( D_{\mu \mu} \) and \( D_{pp} \), linked by the relation

\[ D_{pp} \sim p^{2} \beta_{F}^{2} D_{\mu \mu}. \]

(8)

Below, we provide the theoretical scalings for \( D_{\mu \mu} \); those for \( D_{pp} \) can be directly obtained through the above relation.

The resonance function \( R_{k} \) characterizes the interaction between the particle and a given mode. Various forms for this function in standard and extended quasilinear theories, which account for resonance broadening through wave decay and partial randomization of the particle pitch-angle cosine, are introduced and discussed in App. A. By wave decay, it is meant that modes are assigned a finite lifetime, which effectively implies a temporal correlation time of finite extent. Here, we do not include this effect for fast modes, but we will comment on its possible influence. By contrast, the partial randomization of the pitch-angle cosine of the particle in a turbulent bath is guaranteed, because the direction of the total magnetic field does not coincide exactly with that of \( B_{0} \), hence \( \mu \), which is defined relative to the latter, is effectively a random quantity, see App. A for details.

Neglecting this effect for the time being, and considering furthermore \( \gamma_{d} \to 0 \), we have \( R_{k} = \delta (k_{\mu} \mu - \omega_{F} + n \Omega) \), with \( n \in \mathbb{Z} \), bringing out the infinite discrete set of resonances of standard quasilinear theory. The \( n = 0 \) resonance describes the Landau, also called transit-time damping (TTD) resonance, which results from the interaction of the particle with the compressive modes moving along the background magnetic field [13, 38]. The \( n \neq 0 \) resonances are described as Landau-synchrotron, or gyroresonances, between the particle motion around the background field and the motion of the mode along the field.

For the above Dirac resonance function, we derive the following quasilinear scalings in the limit \( r_{\parallel} k_{\min} \ll 1 \) \( (r_{\parallel} = c/\Omega \) the particle gyroradius):

\[ D_{\mu \mu}^{F-\text{TTD}} \sim (1 - \beta_{F})^{\alpha} \eta (r_{\parallel} k_{\min})^{\alpha - 2} k_{\min}, \]

\[ D_{pp}^{F-\text{Gyr}} \sim 0.1 \eta (r_{\parallel} k_{\min})^{\alpha - 2} k_{\min}, \]

(9)

(10)

for the quasilinear TTD and gyroresonant contributions of the pitch-angle diffusion coefficient, where \( \alpha \approx 4 \) is a \( (\beta_{A}-\text{dependent}) \) effective exponent. This scaling reveals that the transit-time damping contribution is strongly suppressed in the relativistic limit \( \beta_{F} \to 1 \). This is easily understood from the resonance condition: \( k_{\mu} \mu - \beta_{F} k = 0 \), which requires \( \mu > \beta_{F}/k_{\parallel} \), with \( \beta_{F} \approx 1 \), i.e. the parallel phase velocity becomes superluminal, thereby preventing any phase resonance.
Accounting for resonance broadening restores part of the transit-time damping contribution in the relativistic limit. To see this, consider the resonance broadening that results from pitch-angle randomization, with \( (\mu k_0^2)^{1/2} \sim \eta^{1/4} \), as discussed in App. A, for strongly magnetized particles, meaning \( r_g k_{\text{min}} \ll 1 \). Then \( R_k \propto \exp\left[-(\mu k\eta^{1/4} - \beta_F)^2/2\eta^{1/2}\right] \). At large values of \( \beta_F \), more specifically \( \beta_F > \eta^{1/4} \) where the term in the exponential cannot vanish, the resonance function can be approximated by its value at \( \mu k = 1 \) and a width \( \beta_F/\eta^{1/4} \) in \( \mu k \). One then obtains an estimate for \( D_{\mu\mu}^{\text{TDD}} \), that is comparable to, albeit slightly smaller, than \( D_{\mu\mu}^{\text{Gyr}} \) at \( \beta_F \sim 1 \).

For \( \beta_F < \eta^{1/4} \), the resonance \( k_{\mu k}(\mu) - \beta_F k = 0 \) can be met at values \( |\mu_k| < 1 \), hence one recovers the standard quasi-linear result, in which the transit-time damping contribution exceeds the gyroresonant ones by about 1.5 order of magnitude.

For particles of large rigidity, meaning \( r_g k_{\text{min}} \gg 1 \), the scattering is dominated by gyroresonances because \( D_{\mu\mu}^{\text{TDD}} \sim \eta (r_g k_{\text{min}})^{-3} k_{\text{min}} \) while \( D_{\mu\mu}^{\text{Gyr}} \sim \eta (r_g k_{\text{min}})^{-2} k_{\text{min}} \). The latter scaling is typical of a particle of gyroradius \( r_g \) interacting with a turbulence on coherence scales \( k_{\text{min}}^{-1} \ll r_g \).

### B. Numerical simulations and discussion

Appendix B presents in detail our numerical procedure for the Monte Carlo simulations of test-particle transport and acceleration in a given synthetic turbulence, described as a sum of linear waves. It appears important to recall here that the electric field is calculated using the ideal Ohm’s law, see Eq. (B4), rather than as the sum of the \( \delta E_k^F \) modes detailed above, in order to avoid the emergence of non-MHD effects.

Besides those parameters defining the turbulence, \( q_F \), \( \beta_A \), \( \beta_s \) and \( \eta \), the main (dimensionless) parameter that pilots the interaction of particles with waves is \( r_g k_{\text{min}} \), with \( r_g \) larger than the smallest wavelength of the turbulence \( (r_g > k_{\text{max}}^{-1}) \), unless specified otherwise.

To limit the size of parameter space, we restrict ourselves to the fiducial values: \( q_F = 5/3, \beta_s \approx 10^{-2}\beta_A, \eta = 0.3 \). The value of \( \beta_s \) exerts no influence on our results as long as \( \beta_s \ll \beta_A \), since \( \beta_F \approx \beta_A \) in this limit. The quasilinear predictions for the diffusion coefficients directly scale with \( \eta \), and we have checked that this scaling holds in our numerical simulations.

We first show in Fig. 1 the time evolution of various quantities for \( \beta_A = 0.7 \) and \( r_g k_{\text{min}} \approx 10^{-3} \). This figure illustrates several noteworthy points about the acceleration of particles in a relativistic turbulence. The variance of the momentum distribution function \( \sigma^2_p \) can be described, at early times, by a coherent oscillating pattern of energy gains and losses as particles gyrate around the background magnetic field and collect the influence of the perturbations. This modulation remains coherent over the coherence time of the random force that the particles suffer. Eventually, the stochastic build-up of the perturbations leads to the decorrelation of the pitch-angle, after a time \( \sim t_{\text{scatt}} \), or \( \sim 0.3 k_{\text{min}}^{-1} \) in Fig. 1. Particle acceleration occurs soon after, since the expected acceleration timescale \( t_{\text{acc}} \sim t_{\text{scatt}}/\beta_A^2 \). Once acceleration takes place, the behavior of \( \sigma^2_p/2t \) becomes strongly super-diffusive, because the diffusion coefficient \( D_{pp} \) is a growing function of energy. In detail, quasilinear theory predicts \( D_{pp} \propto p^w \), and the solution of

![FIG. 1. Time evolution of various quantities for an ensemble of 1000 test-particles interacting with a fast mode turbulence at \( \beta_A = 0.7, \eta = 0.3 \), with initially, \( r_g k_{\text{min}} \approx 10^{-3} \). For each panel, from top to bottom: perpendicular mean square displacement; parallel mean square displacement; pitch-angle correlation function, pitch-angle mean square displacement, momentum distribution variance; finally, mean Lorentz factor normalized to its initial value. The red dashed lines show the fits. The orange dash-dotted line shows an estimation of the scattering timescale deduced from the fit of \( (\mu k_0) \) while the magenta dotted one indicates the acceleration timescale defined as \( (\Gamma) (t_{\text{acc}}) = 2\Gamma_0 \). The cyclotron frequency is written \( \Omega_0 = eB_0/m \).](image-url)
the corresponding Fokker-Planck equation can be shown to exhibit in this case a momentum evolutionary law
\[
\langle p^2 \rangle^{1/2} \sim \langle p \rangle \propto \Delta t^{1/(2-\alpha)} \propto \Delta t^{3/2}.
\]

Hence, once acceleration starts to operate, the particle momentum increases at a fast rate until it reaches the point where \( r_g k_{\text{min}} \approx 1 \), at \( t \sim 100 k_{\text{min}}^{-1} \) in Fig. 1. At larger values of the rigidity, quasilinear theory now predicts \( D_{pp} \propto p^\beta \), which directly stems from the lack of resonances for particles with gyroradius outside the inertial range of turbulence. Accordingly, at \( r_g k_{\text{min}} \gg 1 \), the particle experiences a turbulence that it sees as small-scale on its gyroradius scale, hence \( t_{\text{acc}} \propto p^2 \), guaranteeing that \( t_{\text{acc}} \propto p^2 \), in agreement with \( D_{pp} = p^2/(2t_{\text{acc}}) \propto p^0 \). The constancy of \( D_{pp} \) then indicates that \( \sigma_p^2/(2t) \) remains fixed at this plateau value, and one recovers normal diffusion of the momentum.

Interestingly, as the momentum evolves fast in the acceleration process, the spatial diffusion coefficient, in particular the parallel one, also becomes strongly super-diffusive. We note here that quasilinear theory cannot describe this stage, because one of its intrinsic assumption is that particles moves on unperturbed orbits at constant energy, which is clearly not the case here. This super-diffusive regime has important consequences for the maximum energy of accelerated particles, because it allows them to escape faster than on a naive diffusive timescale; this is discussed in more detail in Sec. VI.

We now compare the quasilinear predictions for the pitch-angle averaged diffusion coefficients \( D_{\mu}^p \) and \( D_{pp}^p \) over a range of rigidities and mode velocities, in Fig. 2 and 3. In our simulations, particles are injected with random initial directions, hence random initial pitch-angles. These diffusion coefficients are thus understood as averaged over \( \mu \). The details of the procedure used to derive them are given in Appendix B; in short, \( D_{\mu} \) is estimated as the slope of the linear fit of \( \langle [\Delta \mu(t)]^2 \rangle \) while \( D_{pp} \) is determined by a linear fit of \( \sigma_p^2(t) \), as indicated by the dashed horizontal lines in the illustrative Fig. 1. Estimations of the acceleration and scattering timescales are also indicated as vertical lines, with \( t_{\text{acc}} \) taken as the time at which the average \( \langle p \rangle \) over the population of Monte Carlo particles is twice the initial value, and \( t_{\text{scatt}} \) is defined as the decorrelation time of the pitch-angle cosine, inferred through an exponential fit of \( \langle \mu(\Delta t)\mu(0) \rangle \). This procedure necessarily introduces a degree of arbitrariness, however, the scalings of the diffusion coefficients with respect to rigidity \( r_g k_{\text{min}} \) and mode velocity \( \beta_A \) (and with turbulence strength parameter \( \eta \)) are respected.

As discussed above, the transit-time damping contribution largely dominates the gyroresonant contributions at low \( \beta_A \), but disappears, at least partially, in the relativistic regime \( \beta_A \rightarrow 1 \). This feature is illustrated in the top panel of Fig. 3, which shows the (theoretical) relative contributions of TTD and gyro-resonances at two extreme values of the mode velocity, \( \beta_A = 0.01 \) and \( \beta_A = 0.9 \). In standard quasilinear theory, the TTD contribution has almost completely disappeared at \( \beta_A = 0.9 \).

Here, we have chosen to present only the theoretical predictions corresponding to standard quasilinear theory, i.e. not including any resonance broadening effect. Our simulations indeed consider fast modes of an infinite lifetime, and our theoretical calculations reveal that resonance broadening through pitch-angle randomization does not affect much diffusion in fast mode turbulence, although it will in the case of slow mode anisotropic turbulence. More precisely, its effect is negligible at values \( \beta_A \lesssim 0.1 \), because the TTD resonance is then strongly dominant; at values \( \beta_A \gtrsim 0.7 \), it increases the standard QLT predictions by a factor not larger than \( \sim 2 \), due to a partial restoration of the otherwise vanishing TTD contribution.

Overall, our numerical results for \( D_{\mu} \), and for \( D_{pp} \) at \( \beta_A < 0.7 \), appear in reasonable agreement with the theoretical expectations of quasilinear theory, while the predictions for \( D_{pp} \) at \( \beta_A \gtrsim 0.7 \) lie below the Monte Carlo results by about an order of magnitude. The actual discrepancy is, however, reduced once the following effects have been accounted for. Firstly, resonance broadening increases the theoretical predictions by a factor as large as 2 for \( \beta_A = 0.7 \), somewhat smaller than 2 for \( \beta_A = 0.9 \). Secondly, the motional electric field carried by the turbulence, \( \langle \delta E^2 \rangle^{1/2} \approx O(\eta^{1/2} \beta_A) \), imparts a net energy gain to the particles that are initialized with a momentum \( p_0 \) in the laboratory frame, as they first encounter the

![FIG. 2. Comparison of the theoretical QLT predictions for the \( \mu \)-averaged pitch-angle cosine diffusion coefficient (colored lines, for various values of \( \beta_A \) as indicated) with values extracted from test-particle Monte Carlo simulations (symbols) as a function of the initial rigidity. The turbulence amplitude is \( \eta = 0.3 \). See text for details.](image)
turbulence in our Monte Carlo simulations. This energy gain is akin to a first-order Fermi process, and it becomes significant in the relativistic regime: \( \Delta p/p \simeq \gamma_u^2 - 1 \), with \( \gamma_u \simeq (1 + \eta u^2) \) the typical Lorentz factor of the turbulence frame (that in which the motional electric field vanishes), i.e., a factor \( \simeq 1.1 \) for \( \beta_A = 0.9, 0.3 \) for \( \beta_A = 0.7 \): the values measured in the simulations are 0.7 for \( \beta_A = 0.9 \) and 0.4 for \( \beta_A = 0.7 \). Consequently, the (Monte Carlo) data points of Fig. 2 and 3 should be shifted to the right by \( 1 + \Delta p/p \) for a proper comparison with the theoretical predictions. Since \( D_{pp} \propto p^{5/3} \), this amounts to shifting these data points downward, by a factor 2.4 for \( \beta_A = 0.9 \), 1.8 for \( \beta_A = 0.7 \). The actual discrepancy is thus more of the order of a factor \( \sim 3 - 4 \).

Furthermore, we note that the measurement of the diffusion coefficients in the limit \( \beta_A \to 1 \) becomes somewhat intricate, because the diffusion plateau over which the estimate of \( \langle \Delta u^2 \rangle / (2 \Delta t) \) or \( \langle \Delta p^2 \rangle / (2 \Delta t) \) can be obtained, is of limited extent in the highly relativistic regime. We exemplify this in App. B for the case of Alfvén turbulence. We also note that, if we were to measure the momentum diffusion coefficient as \( p^2/t_{acc} \), where \( t_{acc} \) is defined as the timescale at which the particle has doubled its energy, our Monte Carlo estimates would be reduced by a factor of the order of 3 at large \( \beta_A \) (with values at small \( \beta_A \) being weakly affected), thus reducing significantly the mismatch, see the discussion in App. B.

Therefore, the apparent discrepancy between the theoretical predictions and the numerical data at large \( \beta_F \) is not as important as it seems to be and it is difficult, at the present stage, to conclude on its significance. It likely results from the measurement procedure, but it may also point to a rate of energy gain that exceeds the predictions by a factor of a few.

III. SLOW MODE TURBULENCE

A. Theoretical predictions

From a formal standpoint, the analysis for slow modes is the same as for fast modes. The expressions for the perturbation polarizations, Eqs. (3)-(5) remain valid, one has simply to replace \( \omega_F \) with

\[
\omega_S = \pm \frac{k}{\sqrt{2}} \left\{ \beta_F^2 + \beta_S^2 \beta_s^2 \mu_k^2 \right. \\
\left. - \left[ (\beta_F^2 + \beta_S^2 \beta_s^2 \mu_k^2)^2 - 4 \beta_A^2 \beta_s^2 \mu_k^2 \right]^{1/2} \right\}^{1/2}
\]

(11)

Likewise, the diffusion coefficients adopt the same form as before, Eqs. (6) and (7), except that we now adopt a Goldreich-Sridhar power spectrum,

\[
S_k^{\text{sg}} = \frac{\eta}{1 - \eta} D_0^2 \frac{(q_S - 1)(3q_S - 5)}{16\pi k_{\min}^{4}} \left( \frac{k_{\perp}}{k_{\min}} \right)^{-q_S - 1} \times g \left[ \frac{k_{\perp}}{k_{\min}^{2/3}} \right]^{1/3},
\]

(12)

with \( g(x) \) a smooth function, peaking at \( -1 \leq x \leq 1 \); for the above normalization, we have chosen \( g(x) = \Theta(1 - |x|) \). This function imposes the anisotropic scaling between parallel and perpendicular modes. To simplify our analytical calculations, an approximate dispersion relation is used again, \( \omega_S \approx \pm \beta_S k_{\parallel} \).

We now derive the predicted analytical scalings, starting with the case of standard quasilinear theory, without resonance broadening. We first recall that even though slow modes carry compressive perturbations, the TTD contribution is null in standard quasilinear theory as the linear resonance condition reads \( \pm k_{\parallel} \beta_S - k_{\parallel} \mu = 0 \) and is virtually never met. Using Eqs. (6) and (7) and inserting the above power-spectrum (with \( q_S = 7/3 \)), one derives the following scalings for gyroresonant interactions at small rigidities, i.e., \( r_g k_{\min} \ll 1 \),

\[
D_{\mu \mu}^{\text{SGyr}} \sim 0.1 \eta (r_g k_{\min})^{3/2} k_{\min},
\]

(13)

and \( D_{\mu \mu}^{\text{SGyr}} \sim 0.1 \eta (r_g k_{\min})^{-2} k_{\min} \) as usual, in the high-energy limit \( r_g k_{\min} \gg 1 \). The strong scaling \( D_{\mu \mu}^{\text{SGyr}} \propto (r_g k_{\min})^{3/2} \) stems from the anisotropy of the Goldreich-Sridhar spectrum [17], since \( k_{\perp} \gg k_{\parallel} \) implies that resonant particles with \( r_g k_{\parallel} \sim 1 \) will cross many uncorrelated fluctuations in the perpendicular direction.
We now turn to the more physically motivated case of damped modes. The condition of critical balance underlying Goldreich-Sridhar model implies that the timescale of nonlinear interactions coincides with the linear propagation timescale, accordingly we set \( \gamma_d = 1 \) in numerical applications. In the presence of time decorrelation, the resonance function takes on a Breit-Wigner form, see Eq. (A2) with \( \omega = \omega_S \). A direct calculation of the integrals then provides the following scalings in the low rigidity limit \( r_gk_{\text{min}} \ll 1 \):

\[
D^{S-\text{TDD}}_{\mu\mu} \approx 0.1\eta\gamma_d\beta_S \left[ 1 - 3 \ln (r_gk_{\text{min}}) \right] k_{\text{min}}, \quad (14)
\]

\[
D^{S-\text{Gyr}}_{\mu\mu} \approx 0.1\eta\gamma_d\beta_S k_{\text{min}}. \quad (15)
\]

In agreement with previous studies carried out in the non-relativistic limit [17, 23, 40], we find that the resonance broadening allows TTD interactions, which now dominate the transport, and provide for an additional contribution to the pitch-angle of the particle with respect to the direction of the local mean field, which is the total field at the given point averaged over scales larger than the particle gyroradius. In order to account for this effect, we artificially reduce the amplitude of the turbulence down to low values, \( \eta \sim 0.01 \), which guarantees that everywhere, the field line direction does not depart from that of \( B_0 \) by an angle larger than \( \delta B/B_0 \sim \eta^{1/2} \). Then, provided that the eddy anisotropy verifies \( k_{\parallel}/k_{\perp} \gtrsim \delta B/B_0 \), this eddy can be considered as aligned with respect to local mean field \( B \), even if in practice, it is aligned along \( B_0 \). The interest of setting the parallel direction along \( B_0 \) if, of course, to preserve a standard Fourier decomposition throughout space for our numerical simulations. Since particle-wave resonances require \( k_{\parallel}/r_g \sim 1 \), with \( k_{\parallel}/k_{\parallel} \sim k_{\perp}^{-1/2} k_{\text{min}}^{-1/2} \) (critical balance), the above constraints now imply \( r_gk_{\text{min}} \gtrsim \eta \).

The above setup with \( \eta = 0.01 \) thus allows us to probe the physics of particle acceleration in a realistic anisotropic Goldreich-Sridhar like configuration for particles with \( r_gk_{\text{min}} \gtrsim 0.01 \). Since the analytical scalings predict \( D_{\mu\mu} \propto \eta \) and \( D_{\mu\mu} \propto \eta \), we may, in turn, extrapolate these scalings to the regime of larger \( \eta \) provided they match the numerical simulations. To limit computational cost, we consider a hot plasma with \( \beta_S = 1/\sqrt{3} \) and \( \beta_A = 0.5 \).

Figures 4 and 5 compare the generalized quasilinear predictions for the pitch-angle averaged diffusion coefficients described in Sec. IIIA to the values derived from the Monte-Carlo simulations. For dynamical turbulence \( (\gamma_d = 1) \), we find good agreement between the simulations and the semi-analytical predictions, with \( D_{\mu\mu} \sim \text{cst} \) up to a logarithmic correction \( \ln (r_gk_{\text{min}}) \).

For undamped modes \( (\gamma_d = 0) \) however, the scattering time in Monte Carlo simulations remains comparable to that for damped modes, and several orders of magnitude larger than the standard quasilinear estimates, which furthermore predict \( D_{\mu\mu} \propto (r_gk_{\text{min}})^{3/2} \). The momentum diffusion coefficient \( D_{\mu\mu} \) reveals a similar discrepancy. These results are however in satisfactory agree-

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1 [17], who uses an exponential decay model for the turbulence decorrelation similar to ours, obtains the same logarithmic dependence for the diffusion coefficients, our numerical values are however larger by about an order of magnitude.
ment with our theoretical predictions given in Eq. (16), which include resonance broadening by pitch-angle randomization. Although this effect lies beyond quasilinear theory, which considers unperturbed orbits, such partial randomization is inherent to our numerical Monte Carlo simulations, which follow the trajectories of the particles in the exact electromagnetic field configuration. Our test-particle simulations also indicate that the diffusion coefficient depends on the pitch-angle cosine, being more pronounced at values $\mu \simeq \pm 0.5$ than at $\mu \simeq \pm 1$, as expected from the broadening of the TTD resonance described above. This suggests that our theoretical scaling given in Eq. (16) captures the physics that the test-particle simulations reproduce.

Note the scale in the diffusion coefficients, in particular that of $D_{\mu\mu}$, which suggests $t_{\text{scatt}} \sim 10^2 k_{\text{min}}^{-1}$, at $r g k_{\text{min}} \ll 1$. This low value of $D_{\mu\mu}$ (equivalently, this large value of $t_{\text{scatt}}$) is a direct consequence of our choice $\eta = 0.01$, since $D_{\mu\mu} \propto \eta$ (hence $t_{\text{scatt}} \propto \eta^{-1}$), see Eqs. (13), (14) and (16). Hence, extrapolating to $\eta \sim 1$ predicts a scattering timescale of the order of $k_{\text{min}}^{-1}$. When adapting the values of $D_{pp}$, one must also keep in mind that those shown in

\begin{eqnarray*}
\omega_A &=& \pm \beta_A k_\parallel, \\
\delta B_k^A &=& b_k \frac{B_0 \times k_\perp}{B_0 k_\perp}, \\
\delta u_k^A &=& \mp \beta_A \frac{\delta B_k^A}{B_0}, \\
\delta E_k^A &=& \mp \beta_A \delta b_k \frac{k_\perp}{k_\perp},
\end{eqnarray*}

\begin{equation}
D_{\mu\mu} = \frac{\Omega^2 \pi^2 (1 - \mu^2)}{2 B_0^2} \sum_{n=\pm \infty} \sum_{n=-\infty}^{+\infty} \int dk_\perp dk_\parallel k_\perp ^2 \\
\times \mathcal{R}_k [J_{n+1}(z_\perp) + J_{n-1}(z_\perp)]^2 k_\parallel ^{-2} \\
\times [(k_\parallel - \mu \omega_A)^2 + \mu^2 \gamma_0^2 \omega_A^2] S_k^A,
\end{equation}

IV. ALFVEN MODE TURBULENCE

A. Theoretical predictions

We use the same Goldreich-Sridhar power spectrum as for slow modes, Eq. (12), and repeat the generalized quasilinear calculations of Sec.III A with dispersion and polarization relations appropriate for Alfvén modes.

Fig. 5 assume $\eta = 0.01$ and that $D_{pp} \propto \eta$. 

FIG. 4. Quasilinear predictions for the $\mu$-averaged pitch-angle cosine diffusion coefficient in a Goldreich-Sridhar like turbulence of slow modes (lines), and corresponding values extracted from test-particle Monte Carlo simulations (symbols), plotted as a function of the initial rigidity. The legends have the following meanings: “$\delta \omega$-broadening” incorporates resonance broadening associated to the finite lifetime of the modes (with $\gamma_0 = 1$); “$\delta \mu$-broadening” takes into account the partial randomization of the pitch-angle, while “linear” corresponds to the standard QUT prediction without resonance broadening. The turbulence amplitude is here set to $\eta = 0.01$. See text for details.

FIG. 5. Same as Fig. 4 for the momentum diffusion coefficient. For comparison, the prediction for the diffusion coefficient in an isotropic turbulence of fast modes at same $\beta_A$, $\beta_s$ (corresponding to $\beta_p \simeq 0.68$) and $\eta$ is overlaid in gray/dot-dashed.
The expressions differ from those derived for magnetosonic modes through the relative sign of the Bessel functions, which incidentally ensures that the \( n = 0 \) term is null \((J_{-1} = -J_1)\), as well as through the dispersion relation and power spectrum of the modes (which differs from that of fast modes), of course. But besides the fact that there are no TTD interactions, as a result of the absence of compressible magnetic perturbations, the general remarks made for slow modes in Sec. III A remain true.

For the idealized case of undamped waves, \( \gamma_d = 0 \), we derive the following theoretical scaling for \( r_g k_{\text{min}} \ll 1 \):

\[
D_{\mu \mu}^{A, \text{Gyr}} \sim 0.1 \eta (r_g k_{\text{min}})^{3/2} k_{\text{min}},
\]

similar to that obtained for slow mode waves in the same case. At high rigidities, \( r_g k_{\text{min}} \gg 1 \), \( D_{\mu \mu}^{Gyr} \sim 0.1 \eta (r_g k_{\text{min}})^{-2} k_{\text{min}} \), as always.

If one now accounts for a finite lifetime of the waves, we obtain, for \( r_g k_{\text{min}} \ll 1 \):

\[
D_{\mu \mu}^{A, \text{Gyr}} \sim 0.1 \eta \gamma_d \beta_A \left[ 1 - 2.7 \ln (r_g k_{\text{min}}) \right] k_{\text{min}},
\]

and that for \( r_g k_{\text{min}} \gg 1 \) remains unchanged.

By contrast with slow modes, the resonance broadening that results from the partial randomization of the pitch-angle does not play any significant role here, for two essential reasons. Firstly, Alfvén waves do not possess a magnetic perturbation parallel to the mean magnetic field, so that the broadening is of relative order \( \eta^{1/2} \), instead of \( \eta^{1/4} \) for magnetosonic modes. Consequently, the resonance remains narrow if \( \eta \ll 1 \). Secondly, as discussed in the case of slow modes, this source of broadening does not affect as strongly gyroresonant interactions as it does enhance TTD contributions. For this reason, we do not anticipate any effect from the partial randomization of the pitch-angle, and our numerical simulations will confirm this result.

B. Numerical simulations and discussion

In our Monte Carlo simulations of Alfvén turbulence, the Goldreich-Sridhar local anisotropy is enforced using the same technique as for slow modes: we artificially reduce the amplitude of the turbulence down to a low level, \( \eta = 0.01 \), which allows to probe gyroresonant interactions down to \( r_g k_{\text{min}} \approx 0.01 \).

Figures 6 and 7 compare the values of the diffusion coefficients obtained from numerical estimations of Eq. (21) and (22) to those derived from our Monte Carlo simulations for the fiducial parameters \( \eta = 10^{-2} \), \( \beta_A = 0.9 \).

![FIG. 6. Quasilinear predictions for the \( \mu \)-averaged pitch-angle cosine diffusion coefficient in a Goldreich-Sridhar like turbulence of Alfvén modes, with (“\( \delta \omega \)-broadening”) and without (“linear”) wave decay (lines) and corresponding values extracted from test-particle Monte Carlo simulations (symbols), plotted as a function of the initial rigidity. The turbulence amplitude is here set to \( \eta = 0.01 \). See text for details.](image)

![FIG. 7. Same as Fig. 6 for the momentum diffusion coefficient. For comparison, the prediction for the diffusion coefficient in an isotropic turbulence of fast modes at same \( \beta_A \), \( \beta_s \) (corresponding to \( \beta_p \approx 0.9 \)) and \( \eta \) is overlaid in gray/dot-dashed.](image)
In these figures, the blue symbols (respectively the blue dashed lines) correspond to the Monte Carlo simulation results (respectively the quasilinear predictions) for undamped waves. They are found to agree fairly well one with the other, thus confirming the theoretical scalings of this (idealized) Goldreich-Sridhar phenomenology. We note, in particular, the difference with respect to slow mode waves, for which a non-negligible scattering frequency was observed in our test-particle Monte Carlo simulations for this case, because of the broadening of the TTD contribution by pitch-angle randomization.

For damped modes, there is reasonable agreement for \( r_g k_{\text{min}} \approx 0.04, 0.1 \) but less so for our lowest point at \( r_g k_{\text{min}} \approx 0.01 \), for which the simulations indicate diffusion coefficients smaller than the expectations by a factor of the order of a few. However, the uncertainties on the values of \( D_{\mu\nu} \) and \( D_{pp} \) that we extract from the simulations are larger here than for other cases, as discussed in some detail in App. B. In particular, if we were to use the pitch-angle correlation timescale as a proxy for \( 1/D_{\mu\nu} \), we would infer from our Monte Carlo simulations diffusion coefficients a factor \( \approx 4 \) larger. The data point at \( r_g k_{\text{min}} \approx 0.01 \) would thus fit well the expectations, but that at \( r_g k_{\text{min}} \approx 0.1 \) would exceed them by a factor \( \approx 4 \). It is thus difficult, at this stage, to conclude on the significance of this slight departure.

Additional data points at lower rigidities would be needed to study this issue further, but such simulations come at a high numerical cost. Figure 6 reveals that, to observe the scattering of particles with \( r_g k_{\text{min}} = 0.01 \) at \( \eta = 0.01 \), it is necessary to integrate over \( \sim 10^5 \) (initial) gyrotimes, hence many more elementary time steps, since one needs to sample properly the gyrotyme. Descending in rigidity by one order of magnitude, down to \( r_g k_{\text{min}} = 0.001 \), would require setting \( \eta = 0.001 \), hence a total integration time ten times larger, or a hundred times larger when measured in terms of gyrotime. Such simulations appear prohibitive at the present time and we note that, to our knowledge, our Monte Carlo simulations are the first to reveal the behavior of particle transport and acceleration in a prescribed Goldreich-Sridhar anisotropic turbulence.

Finally, we note here as well that the values of \( D_{\mu\nu} \) and \( D_{pp} \) take comparatively low values because of our choice \( \eta = 0.01 \). To extrapolate these values to larger values of \( \eta \), one must keep in mind that both scale linearly with \( \eta \), see Eqs. (23) and (24). In particular, for \( \eta \sim 1 \), one expects a scattering timescale of the order of \( k_{\text{min}}^{-1} \) at \( r_g k_{\text{min}} \ll 1 \).

V. BEYOND MHD WAVE TURBULENCE

A. Non-resonant turbulent acceleration

So far, we have discussed particle acceleration through resonant (or quasi-resonant) interactions with MHD waves. Yet, it is not obvious that such linear eigenmodes of the plasma provide a faithful description of actual modes in strongly interacting turbulence. In this Section, we address some salient features of the non-resonant acceleration of a particle in a (relativistic) turbulent bath.

On a general level, this physics can be described as the interaction of a particle with a fluctuating electric field that is directly associated through ideal Ohm’s law to the fluctuating velocity field of the plasma. Particles then gain energy as they experience the compressive, shearing, accelerating and vortical motions of the medium, e.g. [41–43].

To characterize the acceleration rate, one needs to specify the statistics of the velocity field and how the particle experience the resulting electric fields, i.e., how it is transported across the cells of coherence of the turbulence. At large rigidities, meaning \( r_g k_{\text{min}} \gg 1 \), particles cross a coherence cell in a near-ballistic manner, hence the scattering timescale can be directly expressed as \( t_{\text{scatt}} \approx \eta^{-1} k_{\text{min}}^{-1} (r_g k_{\text{min}})^2 \). For magnetized particles, i.e., \( r_g k_{\text{min}} \ll 1 \), the problem is however much more complex and lies beyond the scope of the present study. Here, we rather aim to discuss how a given scattering timescale \( t_{\text{scatt}} \) impacts the acceleration process. This scattering timescale \( t_{\text{scatt}} \) may be smaller or larger than \( k_{\text{min}}^{-1} \), with important consequences for the physics of acceleration.

A key point, indeed, is that the particle experiences in a different way the modes on large scales \( k^{-1} \), i.e., such that \( t_{\text{scatt}} \lesssim k^{-1} \), and the small-scale modes for which \( k^{-1} \lesssim t_{\text{scatt}} \) [43]. Specifically, for a turbulent spectrum strongly peaked on a single scale \( k_0^{-1} \), the diffusion coefficient obeys

\[
D_{pp} \approx \frac{p^2}{K^2} \langle (\partial u)^2 \rangle_K \frac{t_{\text{scatt}}}{K} \quad (t_{\text{scatt}} K \ll 1)
\]

\[
\approx \frac{p^2}{K^2} \langle (\partial u)^2 \rangle_K \frac{t_{\text{scatt}}}{K^2} \quad (t_{\text{scatt}} K \gg 1) \quad (25)
\]

In the above formula, the quantity \( \langle (\partial u)^2 \rangle_K \) represents, in a symbolic way, the contribution of the compressive, shearing, accelerating and vortical motions on scale \( k_0^{-1} \), to the acceleration process. In detail, the decomposition of the random 4-velocity fluid in terms of acceleration (characterized by a 3-vector \( a \)), compressive (characterized by a compression scalar \( \theta \)), shearing (3-tensor \( \sigma^{ij} \)) and vortical (3-tensor \( \omega^{ij} \)) motions leads to:

\[
\partial_t \delta u_i = \delta^0_{\alpha} a_i + \delta^3_{\alpha} \left( \frac{1}{3} \theta \delta_{ij} + \sigma_{ji} + \omega_{ji} \right).
\]  

(26)

with: \( \theta = \nabla \cdot u \), \( \sigma_{ij} = (\partial_i u_j + \partial_j u_i)/2 - \frac{1}{2} \delta_{ij} \theta \) and \( \omega_{ij} = (\partial_i u_j - \partial_j u_i)/2 \). \( i, j \in \{1, 2, 3\} \) and \( \alpha \in \{0, 1, 2, 3\} \). In the sub-relativistic limit \( \langle \delta u^2 \rangle \ll 1 \) (\( \delta u \) is here understood as a four-velocity), only the compressive and shearing terms contribute to leading order. In the highly relativistic limit, however, all terms contribute roughly equally, see [43] for details.

We now generalize these results, in particular Eq. (25), to the case of a broadband spectrum of 4-velocity fluctuations extending over the range \([k_{\text{min}}, k_{\text{max}}]\), characterized
by a 1D spectral index $1 < q_u < 2$. We note that Ref. [42] has studied a similar problem, using a different formalism for turbulent shear acceleration in the sub-relativistic limit.

We therefore write

$$\langle \delta u^2 \rangle = \int_{k_{\text{min}}}^{k_{\text{max}}} d \ln k \left( \langle |\delta u_k|^2 \rangle_k \right),$$

(27)

with $\langle |\delta u_k|^2 \rangle_k \propto k^{-1-q_u}$ the typical 4-velocity perturbation on scale $k^{-1}$. We also approximate:

$$\langle (\partial \nu)^2 \rangle_k \equiv \alpha_k k^2 \langle |\delta u_k|^2 \rangle_k,$$

(28)

with $\alpha_k$ a factor of the order of unity that depends on the properties of the turbulence. For instance, Alfvén waves do not contain a compressive term, while magnetosonic waves do.

If $k_{\text{max}}^{-1} \ll t_{\text{scatt}} \ll k_{\text{min}}^{-1}$, one can split the turbulence in a large-scale and a small-scale cascade around $t_{\text{scatt}}$, so that the diffusion coefficient receives two contributions (of comparable magnitude):

$$D_{\text{pp}} |_{k^{-1} \gg t_{\text{scatt}}} \simeq p^2 t_{\text{scatt}} \left( \langle \delta u^2 \rangle \right) k_{\text{min}}^2$$

$$\times \int_{k_{\text{scatt}}}^{k_{\text{min}}} d \ln k \left( \frac{k}{k_{\text{min}}} \right)^{3-q_u}$$

$$\approx p^2 \left( t_{\text{scatt}} k_{\text{min}} \right)^{q_u-2} \left( \langle \delta u^2 \rangle \right) k_{\text{min}},$$

$$D_{\text{pp}} |_{k^{-1} \ll t_{\text{scatt}}} \simeq p^2 t_{\text{scatt}}^{-1} \left( \langle \delta u^2 \rangle \right)$$

$$\times \int_{k_{\text{scatt}}}^{k_{\text{min}}} d \ln k \left( \frac{k}{k_{\text{min}}} \right)^{1-q_u}$$

$$\approx p^2 \left( t_{\text{scatt}} k_{\text{min}} \right)^{q_u-2} \left( \langle \delta u^2 \rangle \right) k_{\text{min}}.$$

(29)

Interestingly, the total contribution scales as

$$D_{\text{pp}} \propto p^{2-\epsilon} \langle \delta u^2 \rangle,$$

(30)

with $\epsilon$ positive but small compared to unity, because $t_{\text{scatt}}$ usually has a mild dependence on $p$, while $q_u - 2$ is negative and small in magnitude. Using for instance the quasilinear scaling for isotropic turbulence, $t_{\text{scatt}} \propto (r_g k_{\text{min}})^{2-q_u}$, we find $D_{\text{pp}} \propto p^2 (r_g k_{\text{min}})^{(2-q_u)(2-q_u)}$. One may furthermore note that the above result, Eq. (29), departs from the naive scaling $t_{\text{acc}} = p^2 / D_{\text{pp}} \approx t_{\text{scatt}} / \langle \delta u^2 \rangle$, because the particle is now sensitive to the detailed structure of the turbulent power spectrum. Indeed, one now obtains:

$$t_{\text{acc}} \approx k_{\text{min}}^{-1} \left( t_{\text{scatt}} k_{\text{min}}^{2-q_u} / \langle \delta u^2 \rangle \right).$$

(31)

If $t_{\text{scatt}} \gg k_{\text{min}}^{-1}$, however, the above integration procedure now gives

$$D_{\text{pp}} |_{k^{-1} \ll t_{\text{scatt}}} \approx p^2 \left( \langle \delta u^2 \rangle / t_{\text{scatt}} \right).$$

(32)

In particular, for $r_g k_{\text{min}} \gg 1$, we have $t_{\text{scatt}} \gg k_{\text{min}}^{-1}$ and $t_{\text{scatt}} \propto r_g^2$, hence we recover $D_{\text{pp}} \propto p^1$.

### B. Non-MHD electric fields

Ideal Ohm’s law $\mathbf{E} = -\mathbf{\beta} \times \mathbf{B}$ prevents the existence of electric fields aligned with the local magnetic field and guarantees that there exists at every point a reference frame in which the electric field can be screened out (the local plasma rest frame).

In the presence of a parallel electric field, acceleration could proceed at a much faster rate, unhindered by the magnetic field. On the largest physical scales, however, it is believed that the ideal Ohm’s law provides a satisfactory approximation and our simulations have implemented this constraint. Ideal MHD rather breaks down on small length scales, generating parallel electric fields in reconnecting current sheets [44], or more generally, because of inertial and kinetic effects, e.g. [45]. Of course, if the typical current sheet width is $l_\perp$, then particles with gyroradius $r_g \gg l_\perp$ stream through the sheet with small deflection and energy gain, hence such reconnection processes govern the small-scale physics of dissipation, heating, and injection, but not acceleration to high energies.

Our aim, here, is to quantify the influence of these small-scale violations of Ohm’s law on the acceleration process. We first characterize the statistics of these electric field fluctuations through a power spectrum, assuming that they extend on spatial scales $> k_{\text{max}}^{-1}$ but peak on $k_{\text{max}}^{-1}$ with, correspondingly, a 1D index $q_E < 1$ ($q_E$ can possibly take negative values). In Appendix C, we calculate the relevant index from the corrections to Ohm’s law for Alfvén, fast and slow wave turbulence respectively, for pair and electron-proton plasmas.

The general problem can be brought back to the one-dimensional analogue that describes the evolution of the particle momentum through

$$\frac{dp}{dt} = q \mu \delta E_\parallel,$$

(33)

where $\delta E_\parallel$ is understood here as a random field, and $\mu$ represents the effective (random) velocity of the particle, characterized by a time correlation function of step $t_{\text{scatt}}$. Note that this scattering timescale may differ from that introduced in the previous sections, as it may be affected by the small-scale electric field itself.

Equation (33) describes a diffusion process, because over a time interval $\Delta t$, $\langle \Delta p \rangle = 0$ but $\langle \Delta p^2 \rangle \neq 0$. Explicitly the correlation functions, we adopt

$$\langle \mu(t) \mu(0) \rangle = \mu(0)^2 \exp \left[ \frac{-t}{4 t_{\text{scatt}}^2} \right],$$

(34)

for the velocity. The scattering timescale verifies $t_{\text{scatt}} = \int_0^{+\infty} dt (\mu(t) \mu(0))/\mu(0)^2$, as it should.

In the spirit of the previous sections, we decompose the electric field fluctuations over scales $k^{-1}$ as a power-law spectrum and we further assume that at each such scale, the coherence length of the corresponding mode is also $k^{-1}$. This generalized decomposition in wavepackets
mimics the decomposition of the damped modes of the Alfvén and slow modes. Hence, we use
\[
\langle \delta E_{\|}(t_1)\delta E_{\|}(t_2) \rangle = \alpha_E \langle \delta E^2_{\|} \rangle \int dk \, k^{-q_E} \exp[-k|t_1-t_2|].
\] (35)
The exponential term involves a correlation time \(k^{-1}\), which can be seen as the turn-over timescale of the parallel electric field structure in the turbulence. Note that, using linear or squared expressions in the exponentials, either for Eq. (34) or for Eq. (35), would modify our final results by a factor of the order of unity only. The prefactor \(\alpha_E = |1-q_E| k^{-q_E-1}_{max}\) provides the correct normalization to the equal-time (equal-position) amplitude \(\langle \delta E^2_{\|} \rangle\).

We thus derive the momentum diffusion coefficient as
\[
\langle \Delta p^2 \rangle = 2\Delta t \, q^2 (\delta E^2_{\|} \alpha_E \mu_0/2)^2 \, t_{scatt} \times \int dk \, k^{-q_E} \, e^{k^2 t_{scatt}^2/\pi} \text{Erfc} \left[ \frac{k t_{scatt}}{\sqrt{\pi}} \right].
\] (36)
The complementary error function is here defined as: \(\text{Erfc}(x) = (2/\sqrt{\pi}) \int_x^{+\infty} dt \, e^{-t^2}\). Integrating this contribution over \(k\) then gives the approximate diffusion coefficient \(D_{pp}^{\delta E_{\|}} = \langle \Delta p^2 \rangle / 2\Delta t\):
\[
D_{pp}^{\delta E_{\|}} \approx q^2 \langle \delta E^2_{\|} \rangle t_{scatt} \quad (t_{scatt} \ll k^{-1}_{max})
\]
\[
\approx q^2 \langle \delta E^2_{\|} \rangle \frac{(t_{scatt}k_{max})^{q_E}}{k_{max}} \quad (t_{scatt} \gg k^{-1}_{max}, 0 < q_E)
\]
\[
\approx q^2 \langle \delta E^2_{\|} \rangle k_{max}^{-1} \quad (t_{scatt} \gg k^{-1}_{max}, q_E < 0)
\] (37)

This diffusion coefficient thus increases up as the scattering timescale, until the particle starts to see these fluctuations as small scales, meaning \(t_{scatt} \gg k^{-1}_{max}\), at which point it may either level off, if \(q_E < 0\) (corresponding to a spectrum that is sharply peaked at \(k_{max}\)), or increase as \(q_E^{scatt}\), if the long-wavelength fluctuations at \(k^{-1} > t_{scatt}\) retain enough power, meaning \(0 < q_E < 1\). We note that, in Eq. (37), the first approximation indeed describes the diffusion coefficient of a particle changing direction every \(t_{scatt}\) while traveling in a roughly coherent electric field, while the third approximation describes how a particle gains energy by traveling ballistically over many incoherent patches of parallel electric field of typical scale \(k^{-1}_{max}\). In the intermediate limit, the particle feels the structure of the power spectrum of electric field fluctuations. For reference, we derive in App. C, \(q_E \approx -7/3, -5/3, -1\) respectively for isotropic fast mode, Goldreich-Sridhar slow mode and Goldreich-Sridhar Alfvén mode turbulence, from the deviations to the ideal Ohm’s law in a pair plasma.

Provided \(q_E^{scatt}\) scales less fast with energy than the power \(\sim 2\), acceleration by the large-scale MHD turbulence, as described in the previous sections, eventually takes over at some energy which may be easily calculated from the above expressions, once \(t_{scatt}(p)\), \(q_E\) and \(\langle \delta E^2_{\|} \rangle\) are specified. To ease the comparison, we note that the units in which we express \(D_{pp}^{\delta E_{\|}}\) in Figs. 3, 5 and 7 are \(q^2 B_0^2/(k_{min}c)\), while those for \(D_{pp}^{\delta E_{\|}}\) are \(q^2 \langle \delta E^2_{\|} \rangle/(k_{max}c)\).

VI. DISCUSSION

The physics of particle acceleration in a turbulent setting is governed by a variety of effects, depending on whether the cascade can be described as isotropic or not, whether it can be approximated as linear waves or not, whether one assumes ideal MHD to hold or not. In this Section, we recap and bring together the results obtained in the previous sections and compare them to the results of recent first-principles extensive numerical simulations of turbulent acceleration.

A. Comparison between modes, general results

In an actual turbulent setting, one expects a mixed contribution from various modes. As mentioned earlier, numerical MHD simulations of sub-relativistic turbulence generally point to the dominance of Alfvén and slow modes, with a minor contribution from fast magnetosonic modes [16, 30, 31]. This result however appears to depend on how the turbulence is driven at the outer scale [32], and the phenomenology may differ in the relativistic regime [33]. We can nevertheless combine our various results, adopting the notations \(q_E\), \(q_A\) and \(q_S\) for the respective contributions of the various modes to the total magnetic energy density.

Our numerical simulations for acceleration in fast, Alfvén and slow mode turbulence provide the following scalings in the inertial range \(r_gk_{min} < 1\):
\[
D_{pp}^F \approx 1.4q_E \left( r_gk_{min} \right)^{q_E} \left( \beta_A/0.9 \right)^2,
\]
\[
D_{pp}^A \approx 0.25q_A \left( r_gk_{min} \right)^{q_A} \left[ 1 - 2.7\ln (r_gk_{min}) \right] \left( \beta_A/0.9 \right)^3,
\]
\[
D_{pp}^S \approx 0.04q_S \left( r_gk_{min} \right)^{q_S} \left[ 1 - 1.8\ln (r_gk_{min}) \right] \left( \beta_S/0.5 \right)^2,
\] (38)

where all diffusion coefficients are here written in units of \(m^2\Omega_0^2/k_{min}\), with \(\Omega_0 = eB_0/m\) the cyclotron frequency. Note that \(m^2\Omega_0^2/k_{min} = p_{conf}^2k_{min}\), with \(p_{conf}\) the confinement momentum such that \(r_g(p_{conf})k_{min} = 1\). We rely here on the (more realistic) picture of damped modes for Alfvén and slow modes, and we have used \(q_E = 3\), \(q_A = 4\) and \(q_S = 7/3\) in our simulations. The wavenumber \(k_{min}\) is related to the outer scale of the turbulence through: \(k_{min} = 2\pi/I_{max}\). Note that the effective slow mode phase speed \(\beta_S\) cannot be realistically larger than \(\sim 0.5\) because it is bounded by the sound speed, which is itself bounded by \(1/\sqrt{3} \approx 0.58\) in a relativistically hot plasma. Note also the scaling \(\propto \beta_A^2\) for Alfvén modes,
to be contrasted with the naive square law. Here, one extra power comes from $\Im \omega \propto \Re \omega$, which controls the scattering through gyroresonance broadening. We have omitted this contribution for slow modes, since our analysis has revealed that pitch-angle randomization provides an equally strong source of scattering. Finally, in the region $r_g k_{\text{min}} \gg 1$, these scalings are each continued into a constant value $D_{pp} \propto \eta m^2 k_{\text{min}}$.

The acceleration timescale is conveniently written as:

$$t_{\text{acc}} = (r_g k_{\text{min}})^2 (D_{pp} / m^2 \Omega_0^2 k_{\text{min}})^{-1} k_{\text{min}}^{-1}. \tag{39}$$

In Sec. V A, we have also discussed the physics of acceleration of particles in a more generic turbulent setting, which is not described as a sum of linear waves but whose velocity field is decomposed into a sum of compressive, shearing, vortical and accelerating motions. Although the scattering timescale cannot be predicted on general grounds in such a situation, we have found that, for $t_{\text{scatt}} \lesssim k_{\text{min}}^{-1}$, $D_{pp} \sim \langle \delta u^2 \rangle k_{\text{min}}$ in terms of the power spectrum index of the 4-velocity fluctuations, $q_u$. We have argued there that, since $t_{\text{scatt}}$ is generically a mild (increasing) function of $p$, and since $2 - q_u$ is small compared to unity, $D_{pp}$ can be written in the general form:

$$D_{pp} \sim \rho^2 \langle \delta u^2 \rangle k_{\text{min}}, \tag{40}$$

i.e., $D_{pp} \sim \langle \delta u^2 \rangle (r_g k_{\text{min}})^2$ in units of $m^2 \Omega_0^2 / k_{\text{min}}$. The above estimate is valid up to a correction of the order $(r_g k_{\text{min}})^{-\epsilon}$, with $\epsilon$ significantly smaller than unity. Note that the amplitude of the turbulence, $\eta$, is here included in the 4-velocity fluctuation amplitude $\langle \delta u^2 \rangle$.

These different contributions are brought together in Fig. 8 for the case of a relativistically hot plasma with magnetization of order unity for $\eta_{F,S,A} = \eta = 0.3$. Following the above confrontation between our theoretical estimations and the results from the simulations, one can assume that in this figure, $D_{pp}$ at the lowest rigidities is probably slightly underestimated for fast modes and overestimated for slow and Alfvén waves.

This figure reveals that the contributions of fast and Alfvén modes are roughly comparable at large rigidities, $r_g k_{\text{min}} \gtrsim 0.1$, but that the fast mode contribution dominates at lower rigidities due to its softer dependence on $r_g$. Recall however that this dependence scales directly with $q_r$, and if $q_r \sim 2$, the fast mode would no longer dominate over Alfvén modes.

In a relativistically hot, magnetized plasma, the phase speed of the slow modes can reach mildly relativistic values, hence slow mode acceleration becomes truly efficient as well. In particular, we find that, at equal (relativistic) phase velocities, the diffusion coefficient in slow modes is only a factor of a few below that of Alfvén modes. This hierarchy differs strongly from what is observed in the sub-relativistic regime, namely a strong dominance of fast modes over Alfvén modes, with a negligible contribution from slow modes. This arises as a combination of several effects, notably the partial disappearance of transit-time damping for fast modes, and the partial restoration of gyroresonances for Alfvén modes due to the finite lifetime of the modes.

We also observe that our predictions for non-resonant acceleration, i.e. acceleration in a turbulence whose spectrum is composed of structures rather than linear waves, also match the above resonant values, at least in the range of momenta considered, at the upper end of the inertial range.

An interesting outcome of our test-particle Monte Carlo simulations is to reveal that, as the particle gains energy through its stochastic interactions, it undergoes super-diffusive spatial transport along the parallel direction. This has consequences for the maximal energy of acceleration, in particular whether a given particle can reach the confinement energy $r_g \sim R_n$, with $R_n$ the size of the source, or not. Assuming $R_n \sim L_{\text{max}}$, we plot in

![Fig. 8. Comparison of the predicted momentum diffusion coefficients for various turbulence modes for a relativistically hot plasma ($\beta_0 \approx 0.58$) with $\beta_A = 0.5$ and $\eta = 0.3$ (corresponding to $\delta B / B \approx 0.7$). For fast modes this corresponds to $\beta_F \approx 0.68$, while for slow modes, $\beta_S \approx 0.45$. In the case of Alfvén and slow modes, we considered damped modes with $|\Im \omega| = |\Re \omega|$ (i.e. $\gamma_d = 1$). The dashed line shows the theoretical prediction $D_{pp} \sim \eta \rho^2 k_{\text{min}}$, which corresponds to non-resonant acceleration (Sec. V A and see text for details), and $D_{pp} \approx \eta m^2 \Omega_0^2 / k_{\text{min}}$ at $r_g k_{\text{min}} \gg 1$. In dotted line, we indicate the Bohm scaling for comparison, $D_{pp} \sim r_g k_{\text{min}} m^2 \Omega_0^2 / k_{\text{min}}$.](image)
expect such effects to be absent. Such simulations thus provide a perfect experimental benchmark for our results. These simulations find that acceleration proceeds in two stages. The first is rapid and likely associated to particle energization in current sheets, through e.g. reconnection-type acceleration, as discussed in these studies. The second is slower and presumably proceeds through stochastic acceleration in a turbulence that mostly obeys the ideal Ohm’s law. In particular, Ref. [27] derives a diffusion coefficient $D_{pp} \propto p^{2/3}$ in the first stage, but $D_{pp} \propto p^2$ in the second, see also [28]. Our discussion of the previous sections agrees well with such a picture. If the small-scale physics can be described in terms of acceleration in parallel electric fields whose power lies on scale $k_{max}$, we have found that $D_{pp} \propto t_{scatt}$ for $t_{scatt} \ll k_{max}^{-1}$, while $D_{pp} \propto t_{qE}^{qq}$ for $t_{scatt} \gg k_{max}^{-1}$ if $0 < q_E < 1$. Consider for instance the latter possibility: the scaling of Ref. [27] is then recovered if $t_{scatt} \propto p^{2/(3q_E)}$. Alternatively, if $t_{scatt} \propto p^2$, describing the limit in which the particle at such energies is sensitive to the electric field structures on scales $k_{max}$ but not to the large-scale MHD turbulence, then $q_E = 1/3$ would explain the observed scaling. A detailed study of the statistics of the small-scale parallel electric fields and a careful follow-up of the particle momentum in the corresponding PIC simulations would allow to test such hypotheses.

Regarding the acceleration on larger scales, the main result of Refs. [27, 28] is $t_{acc} \sim L_{max}/(\delta u^2)$. This agrees well with the scalings given previously, since it corresponds to $D_{pp} \sim 0.16 \eta \beta m (r_{kmin})^2$ in units of $m^2 \Omega_B^2/k_{min}$, with $\beta_m$ the relevant mode velocity, for direct comparison to Eqs. (38) and (40). In particular, it broadly agrees with acceleration in Alfven and/or slow mode turbulence, accounting for wave decay, or with non-resonant acceleration in a generic turbulence. In order to better understand which of these description better applies, one would need to perform a mode decomposition of the turbulence as in Ref. [16], and to track the scaling of $t_{scatt}$ with energy in these PIC simulations.

### B. Comparison to recent numerical results

Recently, several groups have provided the first particle-in-cell (PIC) numerical simulations of relativistic turbulence [25–28]. The merit of such simulations is to provide a self-consistent description of the nonlinear interaction between the thermal plasma, the non-thermal population and the electromagnetic fields. The main disadvantages of such simulations are, of course, their high numerical cost, and their need to resolve (finely enough) the skin depth scale of the plasma. The numerical cost has to be balanced against the number of particles involved, less particles per cell meaning a higher numerical noise. Resolving the skin depth scale allows to properly model the small-scale dissipative physics, which is missing in MHD simulations, but it also means that the maximal energy at acceleration is limited by the dynamic range of the simulation, just as the acceleration mechanism may be affected by small-scale physical effects which would be absent on the large scales of astrophysical sources.

Nevertheless, the dynamical range of the recent simulations shown in Refs. [26–28] is so large that one may
analytical scalings for the scattering ($t_{\text{scatt}}$) and acceleration ($t_{\text{acc}}$) timescales. We have notably included the effect of a finite mode lifetime, as expected for strong turbulence, and discussed in some detail the resonance broadening associated to the partial randomization of the pitch-angle of the particle. We have compared these predictions to dedicated Monte Carlo simulations of test-particle acceleration in synthetic wave turbulence. We have paid attention to the notion of local anisotropy, which is inherent to the Goldreich-Sridhar phenomenology of anisotropic turbulence. Our Monte Carlo simulations naturally account for the partial randomization of the pitch-angle and they include, where necessary, the finite lifetime of turbulent modes. The satisfactory agreement that we reach between those analytical scalings and the numerical simulations suggests that our analytical predictions capture the salient effects of particle acceleration in a prescribed wave turbulence.

We notably observe the following effects in a relativistic turbulent setting:

1. For an isotropic cascade of fast magnetosonic modes the transit-time damping contribution is progressively reduced as $\beta_p$ increases, because the longitudinal phase velocity of the wave then becomes superluminal. Even though resonance broadening effects preserve part of the transit-time damping contribution, most of the acceleration occurs through linear gyroresonant interactions. The diffusion coefficient accordingly scales as $D^{\perp}_{pp} \sim \beta_P^2 \eta p^2 k_{\min} (r_g k_{\min})^{q-2}$ for $r_g k_{\min} \lesssim 1$, with $q$ the 1D index of the turbulence spectrum and $k_{\min} = L_{\max} / (2\pi)$ in terms of the outer scale $L_{\max}$.

2. For an anisotropic, Goldreich-Sridhar like cascade of slow mode waves, the transit-time damping contribution provides the dominant contribution to scattering and acceleration, thanks to the resonance broadening implied by the partial randomization of the pitch-angle. Gyroresonant-like interactions are negligible. The diffusion coefficient accordingly scales as $D^{\perp}_{pp} \sim \beta_P^2 \eta p^2 k_{\min}$ for $r_g k_{\min} \lesssim 1$, up to a logarithmic correction.

3. For an anisotropic, Goldreich-Sridhar like cascade of Alfvén waves, the resonance broadening associated to the finite lifetime of the modes restores gyroresonant-like interactions, that would otherwise be inefficient. The partial randomization of the pitch-angle does not play any significant role. The diffusion coefficient accordingly scales to $D^{\perp}_{pp} \sim \beta_A^2 \eta p^2 (r_g k_{\min})^{q-2}$ for $r_g k_{\min} \lesssim 1$, with $q$ the 1D index of the turbulence spectrum and $k_{\min} = L_{\max} / (2\pi)$ in terms of the outer scale $L_{\max}$.

4. At large rigidity, $r_g k_{\min} \gtrsim 1$, the particle effectively sees small-scale turbulence, hence $D^{\perp}_{pp} \sim \beta^2 \eta m^2 \Omega q_{\min}^{-1} \propto p^0$ for all three types of modes, with $\beta$ the relevant phase velocity.

5. Overall, we find that, in a relativistic setting, if all three types of modes share a similar fraction of the turbulent energy, they give roughly comparable contributions to the acceleration of particles, for momenta not far below the top of the inertial range where $r_g \sim k_{\min}^{-1}$. At lower rigidities, $r_g \ll k_{\min}^{-1}$, the contribution of fast modes becomes dominant, because of its softer scaling with $r_g$.

6. We have also provided general arguments concerning the acceleration of particles in a turbulence that cannot be described as a bath of linear waves, but rather as a combination of compressive, shearing, vortical and accelerating fluid motions obeying the ideal Ohm’s law. We have shown, notably, that if the scattering timescale of particles is such that $t_{\text{scatt}} \lesssim k_{\min}^{-1}$ (as for all three types of modes above at $r_g \ll k_{\min}$), then the diffusion coefficient $D^{\perp}_{pp} \sim \langle \delta u^2 \rangle p^2 k_{\min}^{-1}$ up to a correction factor $(r_{\text{scatt}} k_{\min})^{q-2}$ that depends weakly on energy. Here, $q_u$ represents the index of the 4-velocity 1D turbulent spectrum, and $\langle \delta u^2 \rangle$ its amplitude.

7. We have also discussed the possible contributions of violations of ideal Ohm’s law, showing that they peak on the small scales $\sim k_{\max}$ of the turbulent cascade, and we have characterized their magnitude. We notably find that, if the 1D power spectrum index $q_E$ of the parallel electric field component verifies $0 < q_E < 1$ and $t_{\text{scatt}} \gg k_{\max}^{-1}$, then such small-scale effects provide a contribution $D^{\perp}_{pp} \sim q^2 u_{\parallel}^2 k_{\max}^{-1} (r_{\text{scatt}} k_{\max})^{q_E}$ which may dominate the acceleration at very small rigidities.

8. We have compared our results to recent ab initio simulations of turbulence using kinetic particle-in-cell simulations and shown that the above generally agree with the observed results.

Finally, we provide ready-to-use analytical scalings for applications to high-energy astrophysical phenomenology in Sec. VI.

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Appendix A: Resonance broadening in quasilinear theory

The theoretical estimates of the diffusion coefficients, Eqs. (6) and (21) for pitch-angle diffusion, Eqs. (7) and (22) for momentum diffusion, involve resonance functions $R_k$ that characterize the interaction between particles and waves. At a formal level, this resonance function derives from the time-integrated Fourier transform of the propagator that connects the position of the particle at different times in the turbulent bath, e.g. [46]. Accounting for gyromotion around the background magnetic field, this resonance function is expressed as

$$R_k = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\tau e^{i(k||\mu - \omega + n\Omega)\tau}.$$  \hspace{1cm} (A1)

In standard quasilinear theory, with $\omega \in \mathbb{R}$ and $\mu$ the initial pitch-angle cosine of the particle, this resonance becomes a Dirac-function generating the transit-time damping and the infinite harmonic series of gyrosynchrotron resonances, as explained in the text.

It has long been appreciated, however, that these idealized resonances are actually broadened to some degree by various physical effects. In our case, the two major causes of broadening are the finite lifetime of the turbulent modes and the partial randomization of the pitch-angle of the particle. For the sake of the argument, we treat each case separately.

If the linear eigenmodes are assigned a finite lifetime, the mode frequency can be written $\omega = \Re \omega - i\gamma_d |R\omega|$, with $\gamma_d > 0$. The resonance function then takes on a Breit-Wigner form,

$$R_k = \frac{1}{\pi} \frac{\gamma_d |R\omega|}{(k||\mu - \Re \omega + n\Omega)^2 + \gamma_d^2 |R\omega|^2},$$  \hspace{1cm} (A2)

whose finite width is directly governed by $\gamma_d |R\omega|$.

The presence of net magnetic field fluctuations implies that the pitch-angle of the particle is modulated at all times by a random quantity, which scales as some power of the fluctuation. This consequently broadens the resonance of particles with waves and permits, in particular, transit time damping to occur in slow mode turbulence over a broad range of particle pitch-angles, see Sec. III. This effect has been introduced in the subrelativistic regime in a number of studies, starting with [47]; see e.g. [29, 40] for recent implementations. Here, we provide a detailed discussion of this effect in the relativistic limit, and emphasize the differences with respect to these previous studies.

If the pitch-angle cosine $\mu$ of the particle becomes a random quantity, the ballistic propagator $\langle e^{i(k||\mu - \omega + n\Omega)\Delta t} \rangle$ can be approximated, to second-order in the cumulant expansion of $\mu$, by

$$\langle e^{i(k||\mu - \omega + n\Omega)\Delta t} \rangle \simeq \exp \left[ i \left( k||\mu - \omega + n\Omega \right) \Delta t \right.$$

$$\left. - \frac{1}{2} k||^2 \langle \Delta \mu^2 \rangle \Delta t^2 \right].$$

Consequently, the resonance function becomes

$$R_k = \frac{1}{\left( 2\pi k||^2 \langle \Delta \mu^2 \rangle \right)^{1/2}} \exp \left[ - \frac{\left( k||\langle \mu \rangle - \omega + n\Omega \right)^2}{2k||^2 \langle \Delta \mu^2 \rangle} \right],$$  \hspace{1cm} (A3)

and it is entirely characterized by $\langle \mu \rangle$ and $\langle \Delta \mu^2 \rangle$.

For strongly magnetized particles, meaning $r_s k_{\min} \ll 1$, these moments can be estimated using the conservation laws of the first adiabatic invariant and of the energy, which guarantee that $(1 - \pi^2)/\bar{B}$ is a conserved quantity along the trajectory. In this expression, $\pi$ represents the pitch-angle cosine as measured relative to the direction of the unperturbed field, by an amount $\delta B^2_{\|}/2B_0^2 \sim \eta/(1 - \eta)$.

Let us first consider the case $\eta \ll 1$, as used in Sec. III and IV to build an effective model of the anisotropic Goldreich-Sridhar phenomenology, so that $\bar{\mu} \approx \mu$. Then, as the particle travels from one coherence cell of the turbulence, on scale $k_{\min}^{-1}$, to another, its pitch-angle cosine evolves from $\mu'$ to $\mu''$, and to order $\eta$,

$$\mu'' \approx \mu'' - (1 - \mu^2) \left( \Delta B_{\|}/B_0 + \frac{1}{2} \Delta \delta B^2_{\perp}/B_0^2 \right),$$  \hspace{1cm} (A4)

where $\Delta B_{\|} = \delta B''_{\|} - \delta B''_{\perp}$ represents the change in the parallel random magnetic field component, $\Delta B_{\perp} = \delta B''_{\perp} - \delta B''_{\perp}$ that in the perpendicular component. One important observation is that for slow modes, $\Delta B_{\|}/B_0 \sim |\eta/(1 - \eta)|^{1/2} \sqrt{2/3}$ while for Alfvén modes, $\Delta B_{\|}/B_0 = B_{\|}/B_0 = 0$. For simplicity, we rewrite the second term in the rhs of Eq. (A4) as $- (1 - \mu^2) \delta$ in the following, with $\delta \approx \eta^{1/2}$ for slow modes, $\delta \approx \eta$ for Alfvén modes. To go further, it proves convenient to split the pitch-angle domain according to the sign, using here the symmetry $\mu \leftrightarrow -\mu$ of our estimates. We thus focus here on $\mu > 0$. Moreover we distinguish escaping particles with initial pitch angle cosine larger than $\delta^{1/2}$ and trapped ones.

The latter case corresponds to particles undergoing mirror reflections. The pitch-angle cosine of such particles becomes randomized with $\langle \mu \rangle \sim \delta^{1/2}$ and rms value $(\Delta \mu^2)^{1/2} \sim \delta^{1/2}$, as indicated by Eq. (A4). At values of $\delta$ not far below unity, this scaling holds for most particles.

For particles inside the loss cone, meaning $\mu_0 \gtrsim \delta^{1/2}$, the pitch-angle cosine remains confined around the initial value, $\langle \mu \rangle \approx |\mu_0|$, to within $\Delta \mu \gtrsim \delta/|\mu_0|$, since Eq. (A4) then gives $\mu'' \approx \mu' - (1 - \mu^2) \Delta \delta B_{\|}/(2\mu' B_0)$ to lowest order.

As a consequence, the resonance is significantly narrower for particles with $\mu_0 \gtrsim \delta^{1/2}$ than for those with $\mu_0 \lesssim \delta^{1/2}$, hence the pitch-angle averaged diffusion coefficients are strongly dominated by particles in the former range. We have checked this numerically as follows. We have first performed a numerical study of a stochastic system that follows the evolution of the pitch-angle co-
sine through Eq. (A4) step by step, accounting for mirroring whenever it occurs, in order to derive accurate estimates of \( \langle \mu \rangle \) and \( \langle \Delta \mu^2 \rangle \). We have then incorporated these estimates in the resonance function and computed the diffusion coefficients in various regimes of interest and compared the obtained values.

In light of these studies, we find that the broadening of the resonance can be modeled, to the lowest order of accuracy, by assuming \( \langle \mu \rangle \simeq \delta^{1/2} \) and \( \langle \Delta \mu^2 \rangle^{1/2} \simeq \delta^{1/2} \). For comparison, Ref. [40] uses \( \langle \mu \rangle = \mu_0 \) and \( \langle \Delta \mu^2 \rangle \sim \eta^{1/4} \sim \delta^{1/2} \) for all \( \mu_0 \), while Ref. [29] rather finds \( \Delta \mu \sim \eta^{1/4} \sim \delta^{1/2} \). Which regime applies depends on \( \eta \) and \( \mu_0 \), as explained above. Recall also that, for magnetoacoustic modes, \( \delta^{1/2} \sim \eta^{1/4} \), hence the amount of resonance broadening is significant, even in low-amplitude turbulence.

Appendix B: Numerical Simulations

1. Field prescription

The magnetic field is described as the sum of a static and uniform magnetic field \( B_0 \) and a turbulent component \( \delta B \), expressed as the superposition of \( N_k \) MHD eigenmodes (pure fast, slow or Alfvén modes) with wave vector moduli between \( k_{\min} \) and \( k_{\max} \), \( k_{\max} \) is chosen so that \( k_{\max}^{-1} \sim 100 \) per decade (which is rather conservative).

a. Isotropic fast mode turbulence - runs F

For pure isotropic fast mode turbulence simulations, we draw \( N_k \) waves vectors with equally log-spaced norms and random directions and build the magnetic perturbations at the coordinates \((t, r)\) in the plasma rest frame as

\[
\delta B(t, r) = \sum_{i=1}^{N_k} \delta B_{k_i}^F \cos(k_i \cdot r - \omega_i t + \phi_i), \tag{B1}
\]

where \( \omega_i \) and \( \delta B_{k_i}^F \) are obtained by injecting the wave vector \( k_i \) in the dispersion and polarization relations Eqs. (2), (3) and \( \phi_i \) is the random phase. The vectors are normalized according to the chosen spectral scaling and level of turbulence, namely for a Kolmogorov spectrum

\[
\delta k_i^2 = \frac{2\eta}{1 - \eta} \rho_0^2 \left( \frac{k_i}{k_{\min}} \right)^{-\frac{2}{3}} \left[ \sum_{j=1}^{N_k} \left( \frac{k_j}{k_{\min}} \right)^{-\frac{2}{3}} \right]^{-\frac{1}{2}}. \tag{B2}
\]

In the same fashion, we compute from Eq. (4) the four-velocity perturbation\(^2\) as

\[
\delta u(t, r) = \sum_{i=1}^{N_k} \delta u_{k_i}^F \cos(k_i \cdot r - \omega_i t + \phi_i), \tag{B3}
\]

so that the electric field can be built according to the ideal MHD Ohm’s law

\[
\delta E(t, r) = -\frac{\delta u(t, r)}{\sqrt{1 + \delta u^2(t, r)}} \times [B_0 + \delta B(t, r)]. \tag{B4}
\]

b. Anisotropic slow and Alfvén mode turbulence

For slow and Alfvén waves, we enforce approximately the Goldreich-Sridhar scaling. The difficulty resides in that we need \( k_i \) to compute \( \delta B_{k_i}^S \), but the direction of \( \delta B \) depends on the direction of the local magnetic field that we are trying to build, so a recursive procedure would be required. For simplicity, we assume that the direction of the local field can be approximated to that of \( B_0 \). This approximation is reasonable if the perturbation is small enough at the scales of interest, that is, \( \delta B / B \ll k_\parallel / k \sim (k_\parallel / k_{\min})^{-1/2} \) which is increasingly constraining as we move to smaller scales. We therefore limit the dynamic range of these simulations and only consider \( r_g k_{\min} \gtrsim 10^{-2} \). For this limiting case and at scales close to the resonance condition \( k_\parallel \sim r_g^{-1} \), we thus should have \( \delta B / B \lesssim 0.1 \) or \( \eta \lesssim 10^{-2} \), which is the value that we adopt.

Undamped waves For undamped waves, the procedure to construct the perturbed fields is essentially the same as in Sec. B 1 a. \( N_k \) wave vectors are drawn with evenly log-spaced perpendicular components between \( \sqrt{2/3} k_{\min} \) and \( k_{\max} \) along random directions in the plane perpendicular to \( B_0 \) (in line with the remarks of the precedent paragraph), while the parallel components are defined as in [21],

\[
k_i^\parallel = \pm \frac{\sqrt{3}}{2} k_i^{2/3} k_{\min}^{1/3}. \tag{B5}
\]

For slow modes, we then take in combination with Eqs. (11), (3) and (4)

\[
\delta B(t, r) = \sum_{i=1}^{N_k} \delta B_{k_i}^S \cos(k_i \cdot r - \omega_i t + \phi_i), \tag{B6}
\]

\[
\delta u(t, r) = \sum_{i=1}^{N_k} \delta u_{k_i}^S \cos(k_i \cdot r - \omega_i t + \phi_i), \tag{B7}
\]

\(^2\) For linear modes, there is no distinction between the three-velocity and the spatial part of the four-velocity but for finite amplitude modes, we treat \( \delta u \) as a four-velocity to ensure that \( |\delta \mathbf{E}| \) defined by Eq. (B4) remains smaller than \( |\delta \mathbf{B}| \).
with amplitudes scaling as
\[ \delta b_{ij}^2 = \frac{2\eta}{1-\eta} B_0^2 \left( \frac{k_{i \perp}}{k_{\min \perp}} \right)^{-\frac{2}{3}} \left[ \sum_{j=1}^{N_k} \left( \frac{k_{j \perp}}{k_{\min \perp}} \right)^{-\frac{2}{3}} \right]^{-1} \tag{B8} \]

and construct the electric field as previously explained. A similar method is employed for Alfvén modes.

Damped waves For simulations with modes of finite time correlation, along the same lines as [48], we add, compared to the previously described procedure, some pulsation spreading around the solution \( \omega_{k, i} \) of the linear dispersion relation Eq. (11):
\[ \delta B(t, r) = \sum_{i=1}^{N_k} \sum_{j=1}^{N_k} \delta B_{ij}^B \cos(k_i \cdot r - \omega_{ij} t + \phi_{ij}), \tag{B9} \]
\[ \delta u(t, r) = \sum_{i=1}^{N_k} \sum_{j=1}^{N_k} \delta u_{ij}^B \cos(k_i \cdot r - \omega_{ij} t + \phi_{ij}), \tag{B10} \]

using the polarization given by Eqs. (3) and (4) for \( k_i \), the corresponding \( \omega_{k, i} \), and the magnitude
\[ \delta b_{ij}^2 = \frac{2\eta}{1-\eta} B_0^2 \Delta \chi_{ij} \left( \frac{k_{i \perp}}{k_{\min \perp}} \right)^{-\frac{2}{3}} \times \left[ \sum_{m,n} \Delta m, \chi_{mn} \left( \frac{k_{m \perp}}{k_{\min \perp}} \right)^{-\frac{2}{3}} \right]^{-1}, \tag{B11} \]

where \( \Delta \) is the constant logarithmic spacing of \( \omega_{ij} \) at fixed \( i \) and \( \chi_{ij} \) is the function describing the decorrelation of modes away from \( \omega_{k, i} \), and whose exact form is model-dependent. To make contact with our theoretical calculations, we choose
\[ \chi_{ij} = \frac{\gamma_d |\omega_{k, i}|}{\pi} \frac{|\omega_{ij}|}{\gamma_d \omega_{k, i}^2 + (\omega_{ij} - \omega_{k, i})^2}, \tag{B12} \]

which is the Fourier transform of \( t \mapsto e^{i \omega_{k, i} t - \gamma_d |\omega_{k, i}| t} \), and corresponds to the "Nonlinear Anisotropic Dynamical Turbulence" model of [48]. The pulsation bandwidth is set to \( 6 |\omega_{k, i}| \) and sampled by \( N_\omega = 16 \) modes (tests with \( N_\omega = 64 \) give identical results).

2. Measurements

For a given physical setup and particle rigidity, \( \sim 1000 \) particles are tracked, each injected with the same energy in different turbulence realizations and with random initial pitch-angles. The Bulirsch-Stoer integrator [49] is used to evolve the positions \( r \) and 4-velocities \( \Gamma \beta \) in the plasma rest frame, according to the equations of motion
\[ \frac{d\mathbf{r}}{dt} = \beta, \]
\[ \frac{d\Gamma \beta}{dt} = \frac{q}{m} (\delta \mathbf{E} + \beta \times \mathbf{B}), \tag{B13} \]

where \( q \) and \( m \) are the electric charge and the rest mass of the particle and \( \mathbf{B} = \mathbf{B}_0 + \delta \mathbf{B} \) is the total magnetic field, constructed according to the procedure presented above.

a. Pitch-angle diffusion coefficient

The pitch-angle diffusion coefficient is estimated by computing the mean square pitch-angle displacement \( \langle \Delta \mu(t) \rangle^2/2t \), where \( \langle \ldots \rangle \) denotes the average over the particles, i.e. over the pitch-angles and the turbulence realizations. We fit \( \langle \Delta \mu(t) \rangle^2 \) using least squares to a linear function at early times (but still \( \gtrsim r_g \) to omit the ballistic regime) and identify \( D_{\mu \mu} \) to one half of the slope of the fitting function. One limitation is that irrespective of the energy dependence of the diffusion coefficients, \( \langle \Delta \mu(t) \rangle^2/2t \) eventually displays a sub-diffusive behaviour as \( t \) increases due to the bounded nature of \( \Delta \mu(t) \) and a well defined plateau cannot always be observed.

These results were cross-checked against another estimation relying on the pitch-angle correlation function \( \langle \mu(t) \mu(0) \rangle \), where the data is fitted against a decaying exponential model \( \propto \exp(-t/t_{\text{scatt}}) \) and \( D_{\mu \mu} \) is estimated as \( 1/t_{\text{scatt}} \). Most results were found to differ by a factor of a few\(^3\) (see for instance Fig. 10), although for a few simulations (namely that of fast modes with \( \beta_F \gtrsim 0.7 \), \( r_g k_{\min} \gtrsim 1 \)) the data are not well fitted by an exponential model. We also note that this method requires longer computation times than when using the running displacement.

b. Momentum diffusion coefficient and acceleration timescale

In a similar fashion, the pitch-angle averaged momentum diffusion coefficient is evaluated from the dispersion of the energy distribution with \( D_{pp} \) set to the slope of the linear fit of \( \sigma_p^2(t)/2 \) where \( \sigma_p \) is the standard deviation of the distribution of \( p \) at early times. For sub-relativistic simulations, this is equivalent to using the mean square displacement around the initial value \( \langle \Delta p(t) \rangle^2/2t \). For relativistic setups, particles are subject to a quick initial increase of energy due to some first order Fermi acceleration, hence using \( \langle \Delta p(t) \rangle^2/2t \) would lead to overestimate \( D_{pp} \). This initial acceleration moreover implies that we do not measure the diffusion coefficient for the initial \( r_g k_{\min} \) but rather a slightly larger value (≈ 1.4 larger for \( \beta_A = 0.7 \) and ≈ 1.7 larger for \( \beta_A = 0.9 \)). Finally, we note that the energy distribution is not always

\(^3\)In details, when derived from the correlation function \( D_{\mu \mu} \) is found to be ≈ 3 times larger for fast modes, ≈ 4 times larger for Alfvén modes, ≈ equal for damped slow modes, ≈ 1.7 times smaller for undamped slow modes.
well described by a Gaussian [e.g. 21, 29]. In particular, for undamped slow modes turbulence in which there are two different populations of particles (those sensitive to resonance broadening and those which are not), the distributions display large wings (see for instance Fig. 11) and the above described procedure leads to overestimate the acceleration efficiency.

In light of this last remark, the pitch-angle cosine averaged acceleration timescale is not derived from \( D_{pp} \) but defined as the time when the average Lorentz factor has doubled, \( \langle \hat{T} (t_{acc}) \rangle = 2 \Gamma_0 \). Both approaches lead to similar results for most simulations anyway.\(^4\)

\(^4\) In details, for \( r_g k_{min} < 1 \), \( t_{acc} D_{pp}/2p^2 \approx 0.6 \) for fast modes at \( \beta_\Lambda = 10^{-2} \), \( \approx 0.8 \) at \( \beta_\Lambda = 10^{-1} \), \( \approx 2.5 \) at \( \beta_\Lambda \geq 0.7 \); \( \approx 0.4 \) for undamped and damped Alfvén modes; \( \approx 4 \) and \( \approx 0.4 \) for undamped and damped slow modes. For \( r_g k_{min} = 1 \), \( 1 \lesssim t_{acc} D_{pp}/2p^2 \lesssim 1.5 \).

**FIG. 10.** From top to bottom, pitch-angle correlation function; pitch-angle mean square displacement; standard deviation of the distribution of \( p \), ratio of the mean Lorentz factor to its initial value, as a function of time, for the simulation of damped Alfvén modes at the smallest rigidity. The red dashed lines indicates the results of the fits discussed in the text.

**FIG. 11.** Histograms of \( p \) at \( t \approx 10^3 r_g \) for particles injected in isotropic fast mode turbulence (top) and undamped slow mode turbulence (bottom). The data at times within \( \pm 1\% \) of each other were used to increase the number of events (5120). The red dashed lines depict the normal distributions corresponding to the mean and variance values of the data.

### Appendix C: Power spectrum of electric field fluctuations in wave turbulence

In a pair plasma, inertia effects appear in the Ohm’s law and generate the following extra electric field component [50]:

\[
\delta E^x \sim \frac{\partial}{\partial t} \left( \frac{w}{2\pi^2 n^2} \delta j \right) \quad (C1)
\]

emphasizing that \( \delta E^x \) is here measured in the plasma rest frame; \( w \) represents the enthalpy density, \( n \) the number density and \( \delta j \) the total current density. In Fourier space, relating \( \delta j \) to \( \delta B \) to lowest order in \( \omega/k \), we obtain for this extra electric field component:

\[
\delta E^x_k = \kappa \frac{\omega}{4\pi^2 n} k \times \delta B_k \quad (C2)
\]

where \( \kappa = \left[ 1 + \frac{T}{m} \frac{\rho}{\Gamma - 1} \right] \), \( \Gamma \) denotes the adiabatic index and \( T \) the temperature. The scaling \( \delta E^x \propto \omega k \delta B_k \) confirms that the parallel electric field power is maximum on the smallest length scales of the turbulent cascade.

Although each component of this small-scale electric field is orthogonal to the corresponding magnetic field component in Fourier space, this is no longer the case once the sum over \( k \) is performed. More specifically, \( \langle \delta E^x \cdot B_0 \rangle = 0 \) and \( \langle \delta E^x \cdot \delta B \rangle = 0 \) for all modes, but \( \langle (\delta E^x \cdot B_0)^2 \rangle \neq 0 \) and \( \langle (\delta E^x \cdot \delta B)^2 \rangle \neq 0 \) for Alfvén polarization, while \( \langle (\delta E^x \cdot B_0)^2 \rangle = 0 \) but \( \langle (\delta E^x \cdot \delta B)^2 \rangle \neq 0 \) for magnetosonic modes.
Using the polarization of each mode and the corresponding spectrum described earlier, one can calculate the rms parallel electric field \( \delta E_\parallel = \left( (\delta E_\times \cdot B)^2 \right)^{1/2}/B \), recalling that \( B = B_0 + \delta B \).

For an isotropic cascade of fast modes, we derive
\[
\delta E_{F}^\parallel \simeq \frac{\kappa}{4 \pi \sqrt{30}} \beta_F \eta \frac{k_\text{max} k_\text{min}^{3/2}}{\omega_p^2} B. \tag{C3}
\]

For a Goldreich-Sridhar cascade of slow modes,
\[
\delta E_{S}^\parallel \simeq \frac{\kappa}{16\pi} \beta_S \eta \left( \frac{k_\text{max} k_\text{min}^{2/3}}{\omega_p^2} \right) B. \tag{C4}
\]

Finally, for a Goldreich-Sridhar cascade of Alfvén modes, we obtain
\[
\delta E_{A}^\parallel \simeq \frac{\kappa}{8\pi} \beta_A \frac{k_\text{max} k_\text{min}}{\omega_p^2} \times B \left[ \sqrt{\frac{7}{3}} \eta + \sqrt{\eta(1-\eta)} \left( \frac{k_\text{max}}{k_\text{min}} \right)^{1/3} \right]. \tag{C5}
\]
The second term in brackets corresponds to the component parallel to \( B_0 \).

In the case of electron-ion plasmas, the above corrections are completed by the Biermann battery term, which provides the additional electric field
\[
\delta E_k^\times = ik \frac{\delta p_{ek}}{n} \tag{C6}
\]
in terms of the electron pressure fluctuation \( \delta p_{ek} \). Using \( \delta p_{ek} \sim \hat{\Gamma} \frac{e^2}{m_p} k \cdot \delta h / \omega_c \), as applied to linear magnetosonic modes, we find here as well that the power of \( \delta E_\parallel \) peaks at small scales. Its contribution is dominated by the product \( \delta E_\times \cdot B_0 \), which gives
\[
\delta E_F^\parallel \simeq \frac{\Gamma}{\sqrt{6}} \frac{T_e}{m_p} \frac{k_\text{max}^{2/3} k_\text{min}^{1/3}}{\omega_c} \sqrt{\eta(1-\eta)} B \tag{C7}
\]
for fast modes, and
\[
\delta E_S^\parallel \simeq \frac{\Gamma^2}{2\sqrt{2} \rho_c} \frac{T_e}{m_p} \frac{k_\text{max}^{1/3} k_\text{min}^{2/3}}{\omega_c} \sqrt{\eta(1-\eta)} B \times \left[ 2\sqrt{2} + \sqrt{\frac{\eta}{1-\eta}} \left( \frac{k_\text{max}}{k_\text{min}} \right)^{1/3} \right] \tag{C8}
\]
for slow modes, in terms of \( \omega_c = qB/(m_p c) \) the proton cyclotron frequency.

The various dependencies on \( k_\text{max} \) of the above estimates provide the scale dependence of the corresponding electric field, i.e. if the spectral index of the (one-dimensional) power spectrum of \( \delta E_\parallel \) is \( q_E \), meaning \( k^2 \langle |\delta E_k|^2 \rangle \propto k^{-q_E} \), then \( \delta E_\parallel \propto \langle k_{\text{max}}^{1-q_E} \rangle / 2 \), assuming \( q_E < 1 \). In detail, Eq. (C3) for F modes leads to \( q_E = -7/3 \), Eq. (C4) for S modes leads to \( q_E = -5/3 \), and Eq. (C5) for A modes leads to \( q_E = -1 \). Equivalently, the parallel electric field rms strength on a given scale \( k \) scales as \( \delta E_\parallel^k \propto k^{1-q_E}/2 \), to be contrasted with a typical \( \delta E_\times^k \propto k^{-1/3} \) for the MHD component. Consequently, the parallel electric field energy may represent a non-negligible contribution to the total electromagnetic energy density, but in all cases studied above, its power is strongly suppressed on the largest scales.

[1] R. Blandford and D. Eichler, Phys. Rep. 154, 1 (1987).
[2] M. A. Malkov and L. O. Drury, Reports on Progress in Physics 64, 429 (2001).
[3] A. Lazarian, L. Vlahos, G. Kowal, H. Yan, A. Beresnyak, and E. M. de Gouveia Dal Pino, Sp. Sc. Rev. 173, 557 (2012), arXiv:1211.0008 [astro-ph.SR]; G. Kowal, D. A. Falceta-Gonçalves, A. Lazarian, and E. T. Vishniac, Astrophys. J. 838, 91 (2017), arXiv:1611.03914 [astro-ph.GA].
[4] J. F. Drake, M. Swisdak, H. Che, and M. A. Shay, Nature 443, 553 (2006); M. Hoshino, Phys. Rev. Lett. 108, 135003 (2012), arXiv:1201.0837; J. T. Dahlin, J. F. Drake, and M. Swisdak, Physics of Plasmas 23, 120704 (2016); F. Guo, X. Li, W. Daughton, H. Li, Y.-H. Liu, W. Yan, D. Ma, and P. Kilian, arXiv e-prints, arXiv:1901.08308 (2019), arXiv:1901.08308 [astro-ph.HE].
[5] E. Fermi, Phys. Rev. 75, 1169 (1949).
[6] J. A. Miller, T. N. Larosa, and R. L. Moore, Astrophys. J. 461, 445 (1996); V. Petrosian and S. Liu, Astrophys. J. 610, 550 (2004), arXiv:astro-ph/0401585 [astro-ph]; R. Selkowitz and E. G. Blackman, Month. Not. Roy. Astron. Soc. 354, 870 (2004), arXiv:astro-ph/0407504 [astro-ph]; N. Bian, A. G. Emslie, and E. P. Kontar, Astrophys. J. 754, 103 (2012), arXiv:1206.0472 [astro-ph.SR].
[7] S. Liu, V. Petrosian, and F. Melia, Astrophys. J. 611, L101 (2004), astro-ph/0403487; S. Liu, F. Melia, V. Petrosian, and M. Fatuzzo, Astrophys. J. 647, 1099 (2006), arXiv:astro-ph/0603137 [astro-ph].
[8] E. Quataert, Astrophys. J. 500, 978 (1999), arXiv:astro-ph/9710127 [astro-ph]; B. D. G. Chandran, F. Foucart, and A. Tchekhovskoy, Journal of Plasma Physics 84, 905840310 (2018), arXiv:1707.06216 [astro-ph.HE]; S. S. Kimura, K. Tomida, and K. Murase, Month. Not. Roy. Astron. Soc. 485, 163 (2019), arXiv:1812.03901 [astro-ph.HE].
[9] M. Lemoine, Journal of Plasma Physics 82, 635820401 (2016), arXiv:1607.01543 [astro-ph.HE]; S. Xu, N. Klinger, O. Kargaltsev, and B. Zhang, Astrophys. J. 872, 10 (2019), arXiv:1812.10827 [astro-ph.HE].
[10] G. Brunetti and A. Lazarian, Month. Not. Roy. Astron.
[43] M. Lemoine, Phys. Rev. D 99, 083006 (2019), arXiv:1903.05917 [astro-ph.HE].
[44] N. F. Loureiro and S. Boldyrev, Phys. Rev. Lett. 118, 245101 (2017); A. Mallet, A. A. Schekochihin, and B. D. G. Chandran, Month. Not. Roy. Astron. Soc. 468, 4862 (2017), arXiv:1612.07604 [astro-ph.SR].
[45] N. H. Bian, E. P. Kontar, and J. C. Brown, Astron. Astrophys. 519, A114 (2010), arXiv:1006.2662 [astro-ph.SR].
[46] T. H. Dupree, Physics of Fluids 9, 1773 (1966).
[47] H. J. Voelk, Reviews of Geophysics and Space Physics 13, 547 (1975).
[48] M. Hussein and A. Shalchi, Astrophys. J. 817, 136 (2016).
[49] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, Numerical Recipes in Fortran 90 (2Nd Ed.): The Art of Parallel Scientific Computing (Cambridge University Press, New York, NY, USA, 1996).
[50] M. Gedalin, Phys. Rev. Lett. 76, 3340 (1996).