Discrete Symmetries In Lorentz-Invariant Non-Commutative QED

Katsusada MORITA

Department of Physics, Nagoya University, Nagoya 464-8602, Japan

Abstract

It is pointed out that the usual $\theta$-algebra assumed for non-commuting coordinates is not $P$- and $T$-invariant, unless one formally transforms the non-commutativity parameter $\theta^{\mu\nu}$ in an appropriate way. On the other hand, the Lorentz-covariant DFR algebra, which ‘relativizes’ the $\theta$-algebra by replacing $\theta^{\mu\nu}$ with a second-rank antisymmetric tensor operator $\hat{\theta}^{\mu\nu}$, is $P$- and $T$-invariant. It is then proved that $C$, $P$ and $T$ are separately conserved in Lorentz-invariant Non-Commutative QED.
§1. Introduction

Recently, there have been a lot of works\(^1\) on the non-commutative quantum field theory (NC-QFT), simply called QFT\(^\ast\) in this paper. Their aims are vast in both philosophy and methodology, from string theory connection as initiated by Seiberg and Witten\(^2\) to phenomenological search of Lorentz violation\(^3\) inherent in QFT\(^\ast\). It is, therefore, difficult to put them together in a single catchword, but we consider it worthwhile to pursue the program along this line of thought, which may shed light on the divergence difficulty through modification\(^4\) - \(^9\) of the notion of the space-time structure underlying QFT in spite of the fact that the modification is conceptually radical, putting it as a ‘top-down’ theory which leaves us many challenges ahead, like Lorentz invariance\(^10\) - \(^13\), general covariance\(^14\),\(^15\), unitarity\(^16\), causality\(^17\), analyticity\(^18\), CPT theorem\(^19\) - \(^21\), spin-statistics relation\(^20\),\(^21\), asymptotic conditions and so on.

QFT\(^\ast\) is a QFT on the non-commutative space-time characterized by the \(\theta\)-algebra,

\[
[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu} \tag{1.1}
\]

where \(\hat{x}^\mu (\mu = 0, 1, 2, 3)\) are the hermitian coordinate operators and \((\theta^{\mu\nu})\) is a real antisymmetric constant matrix. Any field in QFT\(^\ast\) becomes an operator, \(\hat{\varphi}(\hat{x})\). In terms of the Weyl symbol \(\varphi(x)\) defined through

\[
\hat{\varphi}(\hat{x}) = \frac{1}{(2\pi)^4} \int d^4k d^4x \varphi(x) e^{-ikx} e^{ik\hat{x}}, \tag{1.2}
\]

QFT\(^\ast\) becomes a nonlocal field theory on the ordinary space-time with the point-wise multiplication of the field variables being replaced by the Moyal *-product,

\[
\frac{1}{(2\pi)^4} \int d^4k e^{ikx} \text{tr}[\varphi_1(\hat{x})\varphi_2(\hat{x}) e^{-ik\hat{x}}] = \varphi_1(x) * \varphi_2(x) = e^{\frac{\lambda}{4} \partial_1 \wedge \partial_2} \varphi_1(x_1) \varphi_2(x_2)|_{x_1 = x_2 = x}, \tag{1.3}
\]

with \(\partial_1 \wedge \partial_2 = \partial_1 \theta^{\mu\nu} \partial_2\mu\) and the normalization \(\text{tr} e^{ik\hat{x}} = (2\pi)^4 \delta^4(k)\). Thus the action defining QFT\(^\ast\) is given by

\[
S = \int d^4x L(\varphi(x), \partial_\mu \varphi(x))_\ast, \tag{1.4}
\]

where the subscript of the Lagrangian indicates that the *-product should be taken for all products of the field variables.

Needless to say, the symmetry of the action (1.4) should also be the symmetry of the \(\theta\)-algebra (1.1) if it is the symmetry of the theory and, conversely, if a transformation leaves the action (1.4)
invariant but is not a symmetry transformation of the $\theta$-algebra, it is not the symmetry of the theory. The internal symmetry is independent of the $\theta$-algebra. This is no longer the case for the external transformations. In particular, the Lorentz symmetry, one of the most fundamental symmetries in (relativistic) QFT, is violated in QFT$_x$. To be more specific, since the $\theta$-algebra is not Lorentz-covariant, it is possible to take the matrix $(\theta^{\mu\nu})$ in a canonical form

$$(\theta^{\mu\nu}) = \begin{pmatrix}
0 & \theta_e & 0 & 0 \\
-\theta_e & 0 & 0 & 0 \\
0 & 0 & 0 & \theta_m \\
0 & 0 & -\theta_m & 0
\end{pmatrix}. \tag{1.5}$$

The symmetry group of the $\theta$-algebra (1.1) is then $O(1, 1) \times SO(2) \bowtie T_4,^{21}$ where $\bowtie$ denotes the semi-direct product. This is only a subgroup of the Poincaré group and thus makes it impossible to classify the asymptotic states in terms of the unitary irreducible representations of the Poincaré group. This implies, for instance, that the tachyonic states to be excluded from the asymptotic states by the spectral condition in QFT may be classified into ‘massive’ states according to the symmetry group $O(1, 1) \times SO(2) \bowtie T_4$, which appear in the intermediate states of a closure relation.\textsuperscript{18, 21} Moreover, Lorentz-covariant fields can not be defined except for the Lorentz scalar field which belongs to the one-dimensional representation of the Lorentz group. We also note that there are two length parameters in (1.4), both of which should be put zero to recover the commutative limit where Lorentz invariance holds true and the renormalization program works. It is natural, however, to suppose from the correspondence principle point of view that there exists only one length parameter which goes to zero in the commutative limit. In fact, Snyder\textsuperscript{4} showed that there exists a Lorentz-invariant NC space-time in which there is a Lorentz scalar fundamental length whose zero-limit reduces the NC space-time to the continuum one.

Existence of a single Lorentz scalar fundamental length $a$ is incorporated into the above scheme if we assume that the non-commutativity parameter $\theta^{\mu\nu}$ is not constant but a second-rank antisymmetric tensor. For we can then simply put

$$\theta^{\mu\nu} = a^2 \bar{\theta}^{\mu\nu}, \tag{1.6}$$

where $a$ has a dimension of length $^*$ and $\bar{\theta}^{\mu\nu}$ is a dimensionless, second-rank antisymmetric tensor. The commutative limit is attained by letting $a \to 0$. One can then no longer put the parameters

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*$^*$ Our introduction of the fundamental length is kinematical in comparison with a dynamical meaning of the NC scale of the order of the Planck length.\textsuperscript{6} There is a long history about the dynamical aspects of the minimum length.\textsuperscript{22}
θ_e and θ_m in (1.4) zero independently to recover the commutative limit. This may raise a difficulty concerning the Hamiltonian formalism of QFT_*, but we remind the readers that the fundamental length, if any, is reconciled with relativity only if the notion of the continuous time-development is abandoned. 23)

It is impossible, however, to regard θ^{µν} in the θ-algebra (1.1) as a c-number tensor if ˆx^µ transforms as a 4-vector. To see this let U(A) be the unitary operator of the Lorentz transformation

\[ \hat{x}'^\mu = U(A)\hat{x}^\mu U^{-1}(A) = \Lambda^\mu_{\nu}\hat{x}^\nu. \]

Sandwiching both sides of (1.1) by the unitary operator U(A) and its inverse we have for a c-number θ^{µν}

\[ [\hat{x}'^\mu, \hat{x}'^\nu] = \Lambda^\mu_{\rho}\Lambda^\nu_{\sigma}\hat{x}^\rho\hat{x}^\sigma = \Lambda^\mu_{\rho}\Lambda^\nu_{\sigma}i\theta^{\rho\sigma} = i\theta^{\mu\nu}. \]

This equation holds true only if θ^{µν} = 0 for \( \Lambda^\mu_{\nu} = \delta^\mu_{\nu} + \omega^\mu_{\nu}, \omega_{\mu\nu} = -\omega_{\nu\mu} \) being not identically vanishing. This reflects a well-known fact of there being no constant antisymmetric second-rank tensor. Consequently, we must look for another NC algebra which naturally provides us with a tensor \( \theta^{\mu\nu} \) and preserves Lorentz invariance of the theory. Such a Lorentz-invariant QFT_\* was formulated by Doplicher, Fredenhagen and Roberts (DFR) \( \text{[6]} \) motivated by the space-time uncertainty relation, and rediscovered by Carlson, Carone and Zobin \( \text{[11]} \) in a search to avoid Lorentz violation in QFT_\*. It is based on the DFR algebra,

\[ \hat{x}^\mu, \hat{x}^\nu = i\hat{\theta}^{\mu\nu}, \quad [\hat{\theta}^{\mu\nu}, \hat{x}^\nu] = 0 = [\hat{\theta}^{\mu\nu}, \hat{\theta}^{\rho\sigma}], \quad \mu, \nu, \rho, \sigma = 0, 1, 2, 3. \quad (1.7) \]

Here \( \hat{\theta}^{\mu\nu} \) is a second-rank antisymmetric tensor operator. Feynman rules of the theory are formulated by Filk \( \text{[7]} \) in an irreducible representation of the DFR algebra. It should be noted \( \text{[13]} \) that the tensor property of the operator \( \hat{\theta}^{\mu\nu} \) is proved by assuming an extra piece in the Lorentz generators, which is to be added to the ordinary orbital angular momentum part, \( \text{[4]} \) and the θ-algebra (1.1) holds true as a ‘weak’ relation,

\[ \langle \theta |[\hat{x}^\mu, \hat{x}^\nu]|\theta \rangle = i\theta^{\mu\nu}\langle \theta |\theta \rangle \quad (1.8) \]

where \( \theta^{\mu\nu} \) is an eigenvalue of the operator \( \hat{\theta}^{\mu\nu} \). Thus \( \theta^{\mu\nu} \) is a second-rank antisymmetric tensor. This makes it possible to define the Lorentz scalar \( a \) through (1.6). \( \text{[12]} \)

Lorentz-invariant action of QFT_\*

\[ \hat{S} = \int d^4x d^6\theta W(\theta)\mathcal{L}(\varphi(x, \theta), \partial_\mu \varphi(x, \theta))_\ast, \quad (1.9) \]
is a simple revision of (1.4) as shown in Ref. 11). Here \( \varphi(x, \theta) \) is the Weyl symbol of the operator \( \hat{\varphi}(\hat{x}, \hat{\theta}) \) defined on the DFR algebra (1.7) and \( W(\theta) \) is a Lorentz-invariant, normalized weight function. The Moyal \( \ast \)-product corresponding to the operator product \( \varphi_1(\hat{x}, \hat{\theta}) \varphi_2(\hat{x}, \hat{\theta}) \) is given by

\[
\frac{1}{(2\pi)^{10}} \int d^4k d^6\sigma e^{ikx+i\sigma\theta} \text{tr}[\varphi_1(\hat{x}, \hat{\theta}) \varphi_2(\hat{x}, \hat{\theta}) e^{-ik\hat{x}-i\sigma\hat{\theta}}] = W(\theta) \varphi_1(x, \theta) \ast \varphi_2(x, \theta) = W(\theta) e^{i\frac{\sigma_\mu \partial_\mu}{2}} \varphi_1(x_1, \theta) \varphi_2(x_2, \theta) \bigg|_{x_1 = x_2 = x},
\]

(1.10)

where \( \sigma \theta \equiv \sigma_{\mu\nu} \theta^{\mu\nu} \) and we have normalized \( \text{tr} e^{i\sigma \hat{\theta}} = \hat{W}(\sigma) \), which is the Fourier component of \( W(\theta) \),

\[
W(\theta) = \frac{1}{(2\pi)^6} \int d^6\sigma \hat{W}(\sigma) e^{-i\sigma\theta}.
\]

Feynman rules that take into account all irreducible representations are yet to be formulated, though a model calculation was attempted in Ref. 12), where the invariant damping factor instead of the oscillating Moyal phase was found in a NC scalar model.

The purpose of the present paper is limited to investigate the discrete symmetries based on the DFR algebra (1.7), since the previous considerations\(^{19) - 21)}\) on the subject are all based on the \( \theta \)-algebra (1.1). The next section discusses the discrete transformations of the \( \theta \)-algebra. The discrete symmetries in QED\(_{\ast} \) are reinvestigated in the section 3 following Ref. 19). The discrete symmetries of the DFR algebra are proved in the section 4. Based on this proof the discrete symmetries in a Lorentz-invariant version\(^{12)}\) of QED\(_{\ast} \) are discussed in the section 5. The last section involves short comments.

§2. Discrete transformations of the \( \theta \)-algebra

Let us define parity \( P \) and time-reversal \( T \) on the operator coordinates by

\[
P \hat{x}^\mu \mathcal{P}^{-1} = \hat{x}_\mu \quad \text{and} \quad T \hat{x}^\mu \mathcal{T}^{-1} = -\hat{x}_\mu,
\]

(2.1)

respectively, where \( \hat{x}_\mu = g_{\mu\nu} \hat{x}^\nu \) with \((g_{\mu\nu}) = (+1, -1, -1, -1)\). Thanks to the Weyl transform (1.2) this in fact induces the transformation, \( x \to x_\mathcal{P} \equiv (x^\mu)_\mathcal{P} = x_\mu \) and \( x \to x_\mathcal{T} \equiv (x^\mu)_\mathcal{T} = -x_\mu \) under \( P \) and \( T \), respectively. Under \( P, T, C \) we have

\[
P[\hat{x}^\mu, \hat{x}^\nu] \mathcal{P}^{-1} = [\hat{x}_\mu, \hat{x}_\nu], \quad T[\hat{x}^\mu, \hat{x}^\nu] \mathcal{T}^{-1} = [-\hat{x}_\mu, -\hat{x}_\nu], \quad C[\hat{x}^\mu, \hat{x}^\nu] \mathcal{C}^{-1} = [\hat{x}^\mu, \hat{x}^\nu].
\]

(2.2)

Since the \( c \)-number \( \theta^{\mu\nu} \) pass through the operators, \( \mathcal{P}, \mathcal{T}, \mathcal{C} \), like the unitary operator of the Lorentz transformations, the \( \theta \)-algebra (1.1) do not respect \( P \) and \( T \) but is \( \mathcal{C} \)-invariant. Nonetheless we may
assume the following transformations, \(^{(2.4)}\)

\[
\theta^{\mu\nu} \rightarrow \theta_{\mu\nu} \text{ (under } P), \theta^{\mu\nu} \rightarrow -\theta_{\mu\nu} \text{ (under } T), \theta^{\mu\nu} \rightarrow \theta^{\mu\nu} \text{ (under } C),
\]

which imply

\[
\theta^{\mu\nu} \rightarrow -\theta^{\mu\nu} \text{ (under } CPT),
\]

to \textit{formally} recover the discrete symmetries of the \(\theta\)-algebra. Remember that time reversal is anti-unitary. It should be noted, however, that the transformations \((2.3)\) and \((2.4)\) except for a trivial case \(C\) can not be derived from the \(\theta\)-algebra itself. Assuming \((2.3)\) and \((2.4)\) in the \(\theta\)-algebra amounts to assuming \(\theta^{\mu\nu}\) to be a \(c\)-number tensor also under the improper Lorentz transformations (with extra minus sign under \(T\)), while we have seen that it is impossible to regard it as a \(c\)-number Lorentz tensor.

In passing we remark that the canonical commutation relations are invariant under the translations, the spatial rotations and the discrete transformations, \(P,T\). In particular, we do not need change the sign of the Planck constant under \(T\) because time reversal transformation is anti-unitary. If time reversal transformation were instead assumed to be unitary for the present purpose only, we would have to change the sign of the Planck constant, \(\hbar \rightarrow -\hbar\) under \(T\). This is possible only if we fudge up Planck ‘operator’

\[
[\hat{x}, \hat{p}] = i\hat{\hbar}
\]

with \(\mathcal{T}\hat{\hbar}\mathcal{T}^{-1} = -\hat{\hbar}\). Let \(|\hbar\rangle\) be an eigenstate of the operator \(\hat{\hbar}\) with the eigenvalue \(\hbar\). Since \(\mathcal{T}\hat{\hbar}|\hbar\rangle = \mathcal{T}\hat{\hbar}\mathcal{T}^{-1}|\hbar\rangle = -\hat{\hbar}|\hbar\rangle = \hat{\hbar}|\hbar\rangle = \hbar \mathcal{T}|\hbar\rangle\), \(\mathcal{T}|\hbar\rangle\) is the eigenstate with the eigenvalue \(-\hbar\), resulting in the required sign change of the Planck constant under \(T\) (assumed to be unitary). Promoting \(\theta^{\mu\nu}\) to an operator \(\hat{\theta}^{\mu\nu}\) does a similar job.

In the next section we accept \((2.3)\) and \((2.4)\) to prove the discrete symmetries of QED\(_4\). It is \textit{necessary}, however, to change the sign of the NC parameter,

\[
\theta^{\mu\nu} \rightarrow -\theta^{\mu\nu} \text{ under } C
\]

\((2.5)\)

to prove \(C\) and \(CPT\) invariance of QED\(_4\) action. This was observed by Sheikh-Jabbari.\(^{(19)}\) We shall use the sign change \((2.5)\) also in the relativistic version.
§3. Discrete symmetries in QED

First of all it would be instructive to reinvestigate the discrete symmetries in QED action. The NC extension of the free Dirac action is given by

\[ S_{D_0} = \int d^4x \bar{\psi}(x) \ast (i\gamma^\mu \partial_\mu - m)\psi(x). \]  

(3.1)

Suppose that the spinor is subject to the \ast-gauge transformation

\[ \psi(x) \rightarrow \hat{g}\psi(x) = U(x) \ast \psi(x), \]  

(3.2)

where \( U(x) \) is assumed to be \ast-unitary:

\[ U(x) \ast U^\dagger(x) = U^\dagger(x) \ast U(x) = 1. \]  

(3.3)

The \ast-gauge invariance of the Dirac action leads to the replacement of the partial derivative with the covariant one,

\[ \partial_\mu \psi(x) \rightarrow D_\mu \psi(x) = \partial_\mu \psi(x) - ieA_\mu(x) \ast \psi(x), \]  

(3.4)

with the transformation law of the NC gauge field

\[ A_\mu(x) \rightarrow \hat{g}A_\mu(x) = U(x) \ast A_\mu(x) \ast U^\dagger(x) + \frac{i}{e}U(x) \ast \partial_\mu U^\dagger(x). \]  

(3.5)

This prescription gives the \ast-gauge-invariant Dirac action, \( S_{D_*} = S_{D_0} + S_{I_*} \), where

\[ S_{I_*} = e\int d^4x \bar{\psi}(x) \ast \gamma^\mu A_\mu(x) \ast \psi(x) \]

\[ = e\int d^4x_1d^4x_2d^4x_3K(x_1; x_2, x_3)\bar{\psi}(x_1)\gamma^\mu A_\mu(x_2)\psi(x_3). \]  

(3.6)

In the last expression we utilized the kernel derived from (1.2) and (1.3),

\[ K(x; x_1, x_2) = \frac{1}{(2\pi)^8}\int d^4p_1d^4p_2e^{ip_1(x-x_1)+ip_2(x-x_2)}e^{-\frac{i}{2}p_1 \wedge p_2}, \]  

(3.7)

with \( p_1 \wedge p_2 \equiv p_1^{\mu}p_2^\nu - p_1^{\nu}p_2^\mu \). It can be shown that it is cyclic, \( K(x; x_1, x_2) = K(x_1; x_2, x) = K(x_2; x, x_1) \).

At this stage we recall the charge conjugation transformation of the spinor \( \psi \) and the (Abelian) gauge field \( A_\mu \),

\[ C\psi(x)C^{-1} = C\bar{\psi}(x), \quad C\bar{\psi}(x)C^{-1} = -\psi(x)C^{-1}, \quad CA_\mu(x)C^{-1} = -A_\mu(x), \]  

(3.8)
respectively, where the charge conjugation matrix $C$ satisfies $C^{-1} \gamma^{\mu} C = -\gamma^{\mu T}$. If we resort to the sign change (2.5), we find that

$$CS_{I*}C^{-1} = -e \int d^4x_1 d^4x_2 d^4x_3 K(x_1; x_2, x_3) \psi(x_1) \gamma^{\mu T} A_\mu(x_2) \bar{\psi}(x_3),$$  \hspace{1cm} (3.9)

where $K(x_1; x_2, x_3) = K(x_1; x_2, x_3)|_{\theta \to -\theta}$. As assumed in Ref. 19, if the fields involved are classical, commuting or anti-commuting at different $x$'s, we may rewrite (3.9) as

$$CS_{I*}C^{-1} = e \int d^4x_1 d^4x_2 d^4x_3 K(x_1; x_2, x_3) \bar{\psi}(x_3) \gamma^{\mu} A_\mu(x_2) \psi(x_1).$$ \hspace{1cm} (3.10)

It follows from (3.7) that $K(x_1; x_2, x_3) = K(x_1; x_3, x_2)$ which equals $K(x_3; x_2, x_1)$ by cyclicity. Consequently, we finally have

$$CS_{I*}C^{-1} = e \int d^4x_1 d^4x_2 d^4x_3 K(x_3; x_2, x_1) \bar{\psi}(x_3) \gamma^{\mu} A_\mu(x_2) \psi(x_1) = S_{I*}. \hspace{1cm} (3.11)$$

There is nothing to prevent us to consider a possibility that the fields neither commute nor anti-commute at different $x$'s. In such a case $C$-invariant $*$-gauge interaction should be the average,

$$S_I = \frac{1}{2} (S_{I*} + CS_{I*}C^{-1}). \hspace{1cm} (3.12)$$

How to obtain (3.12) via gauge principle is solved as follows. Let us write the free NC Dirac action $S_{D_0}$ as the average of (3.1) and

$$S_{D_0} = \int d^4x \psi(x) \bar{\psi}(x) (i \gamma^{\mu T} \partial_\mu + m) \psi(x), \hspace{1cm} (3.13)$$

that is, $S_{D_0} = \frac{1}{2} (S_{D_0*} + S_{D_0*})$. Here we have introduced the anti-Moyal product, denoted $*$,

$$\varphi_1(x) \bar{\varphi}_2(x) \equiv \varphi_2(x) * \varphi_1(x) = \varphi_1(x)e^{-i/2 \gamma^{\mu \nu} \partial_\mu \partial_\nu} \varphi_2(x). \hspace{1cm} (3.14)$$

The $*$-product satisfies the similar relations as those obeyed by the $*$-product. In particular, the $*$-product is associative,

$$(\varphi_1(x) \bar{\varphi}_2(x)) \bar{\varphi}_2(x) = \varphi_1(x) \bar{\varphi}_2(\varphi_2(x) \bar{\varphi}_2(x)) \equiv \varphi_1(x) \bar{\varphi}_2(\bar{\varphi}_2(x) \bar{\varphi}_2(x)). \hspace{1cm} (3.15)$$

It is obvious that one can omit both the symbols $*$ and $*$ upon integration under the same assumption,

$$\int d^4x \varphi_1(x) \bar{\varphi}_2(x) = \int d^4x \varphi_1(x) \bar{\varphi}_2(x) = \int d^4x \varphi_1(x) \varphi_2(x). \hspace{1cm} (3.16)$$

Since the definition (3.14) works only for classical commuting functions, we have to take into account the extra minus sign appearing when exchanging the two spinors, whence (3.13) is equivalent to
(3.1).

Next we introduce the covariant derivative pertinent to the action (3.13). To this end we write
the gauge transformations, (3.2) and (3.5) in an ‘opposite’ but equivalent way,

\[ \psi(x) \rightarrow \hat{g}\psi(x) = \psi(x)\bar{U}(x), \]
\[ A_\mu(x) \rightarrow \hat{g}A_\mu(x) = U_\mu(x)\bar{A}_\mu(x)U(x) + \frac{i}{e} \partial_\mu U_\mu(x)\bar{U}(x). \]  

(3.17)

Note that \( U(x) \) is also \( \bar{\ast} \)-unitary:

\[ U(x)\bar{U}(x) = U_\mu(x)\bar{U}_\mu(x) = 1 \]  

(3.18)

It is important to recognize that the same \( \ast \)-gauge transformation can also be written using the
\( \bar{\ast} \)-product. The \( \ast \)-gauge-invariant action based on (3.13) is then obtained by the replacement

\[ \partial_\mu \bar{\psi}(x) \rightarrow \bar{D}_\mu \bar{\psi}(x) = \partial_\mu \bar{\psi}(x) + ieA_\mu(x)\bar{\psi}(x). \]  

(3.19)

It is easy to prove that

\[ \bar{D}_\mu \bar{\psi}(x) \rightarrow U_\mu(x)\bar{D}_\mu \bar{\psi}(x), \]  

(3.20)

under the \( \bar{\ast} \)-gauge transformation (3.17) with the unitarity (3.18). In total, the NC gauge-invariant
Dirac action is then given by \( S_D = S_{D_0} + S_I \), where

\[ S_I = \frac{e}{2} \int d^4x [\bar{\psi}(x) \ast \gamma^\mu A_\mu(x) \ast \psi(x) - \psi(x) \bar{\psi}(x) \bar{\ast}\gamma^{\mu T} A_\mu(x)\bar{\psi}(x)] \]
\[ = \frac{e}{2} \int d^4x_1d^4x_2d^4x_3 [K(x_1; x_2, x_3)\bar{\psi}(x_1)\gamma^\mu A_\mu(x_2)\psi(x_3) \]
\[ - \bar{K}(x_1; x_2, x_3)\psi(x_1)\gamma^{\mu T} A_\mu(x_2)\bar{\psi}(x_3)]. \]

(3.21)

This is nothing but (3.12) noting (3.9). Similarly, we have \( P, T \) invariance.

Note that, under \( CPT \), \( K \) is changed into \( \bar{K} \) because \( CPT \) is anti-unitary and the sign change
(2.4) is cancelled by (2.5). Remembering \( CPT \) transformation law,

\[ \Theta \psi(x)\Theta^{-1} = -i\gamma_5\gamma_0^T \bar{\psi}(-x), \quad \Theta \bar{\psi}(x)\Theta^{-1} = \psi(-x)i\gamma_5\gamma_0^T, \quad \Theta A_\mu(x)\Theta^{-1} = -A_\mu(-x), \]

(3.22)

with \( \Theta \equiv CPT \), we find that

\[ \Theta S_{I*}\Theta^{-1} = -e \int d^4x_1d^4x_2d^4x_3 \bar{K}(x_1; x_2, x_3)\psi(-x_1)\gamma_0^T \gamma^{\mu T} A_\mu(-x_2) \bar{\psi}(-x_3) \]
\[ = -e \int d^4x_1d^4x_2d^4x_3 \bar{K}(x_1; x_2, x_3)\psi(x_1)\gamma^{\mu T} A_\mu(x_2) \bar{\psi}(x_3) = S_{I*}. \]

(3.23)
For classical fields (commuting or anti-commuting at general \(x\)'s) \(S_I\) = \(S_{I^*}\). This proves CPT invariance of QED\(_*\) action in the fermion sector.

The Maxwell sector is described by the action

\[
S'_M = -\frac{1}{4} \int d^4x F_{\mu\nu}(x) \ast F^{\mu\nu}(x),
\]

where the Maxwell field strength tensor is defined by

\[
F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - ie[A_\mu(x), A_\nu(x)],
\]

with

\[
[A_\mu(x), A_\nu(x)]_\ast \equiv A_\mu(x) \ast A_\nu(x) - A_\nu(x) \ast A_\mu(x)
\]

\[
= \int d^4x_1d^4x_2 K(x; x_1, x_2)[A_\mu(x_1)A_\nu(x_2) - A_\nu(x_1)A_\mu(x_2)].
\]

Under the charge conjugation it goes over to

\[
C[A_\mu(x), A_\nu(x)]_\ast C^{-1} = \int d^4x_1d^4x_2 \tilde{K}(x; x_1, x_2)C[A_\mu(x_1)A_\nu(x_2) - A_\nu(x_1)A_\mu(x_2)]C^{-1}
\]

\[
= \int d^4x_1d^4x_2 \tilde{K}(x; x_1, x_2)[A_\mu(x_1)A_\nu(x_2) - A_\nu(x_1)A_\mu(x_2)]
\]

\[
= A_\mu(x)\ast\ast A_\nu(x) - A_\nu(x)\ast\ast A_\mu(x) \equiv [A_\mu(x), A_\nu(x)]_{\ast\ast}.
\]

Remember \(K \rightarrow \tilde{K}\) under \(C\). Consequently, the field strength does not transform to itself up to sign but is changed into

\[
C F_{\mu\nu}(x) C^{-1} = -G_{\mu\nu}(x)
\]

\[
G_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + ie[A_\mu(x), A_\nu(x)]_{\ast\ast}.
\]

Using the last transformation of (3.17) the field strength \(G_{\mu\nu}(x)\) can be shown to be \(\ast\)-gauge covariant,

\[
G_{\mu\nu}(x) \rightarrow \hat{g} G_{\mu\nu}(x) = U^\dagger(x)\tilde{\ast}G_{\mu\nu}(x)\tilde{\ast}U(x).
\]

For classical fields we can commute two field variables in (3.27) to find \(F_{\mu\nu} = G_{\mu\nu}\) as observed in Ref. 19. For non-classical fields which are noncommuting at general \(x\)’s we have \(F_{\mu\nu} \neq G_{\mu\nu}\). Thus \(C\)-invariant Maxwell\(_*\) action also becomes the average

\[
S_M = -\frac{1}{8} \int d^4x[F_{\mu\nu}(x) \ast F^{\mu\nu}(x) + G_{\mu\nu}(x)\ast G^{\mu\nu}(x)].
\]
Hence, we have shown that
\[ CS_{QED}C^{-1} = S_{QED}, \quad S_{QED} = S_{D} + S_{M}. \] (3.31)

This proves $C$-invariance of QED$_*$. As a final example we consider CPT in the Maxwell sector. Using the last equation of (3.22) and $K \rightarrow \bar{K}$ under CPT, we have
\[ \Theta F_{\mu\nu}(x) \Theta^{-1} = G_{\mu\nu}(-x), \] (3.32)
where $G_{\mu\nu}(x)$ is defined by (3.28). Hence the Maxwell$_*$ action $S_M$ is CPT-invariant. In conjunction with the fermion sector we have proved
\[ \Theta S_{QED} \Theta^{-1} = S_{QED}, \quad S_{QED} = S_{D} + S_{M}. \] (3.33)

Thus QED$_*$ conserves CPT despite of the Lorentz violation.\textsuperscript{19)-21) The proof hinges upon the transformations (2.3) and (2.4) (and the sign change (2.5)) which, however, can not be derived within the framework of the $\theta$-algebra. In this sense the discrete symmetries in QED$_*$ do not match our symmetry criterion in the section 1. We amend these points in the following sections.

§4. Discrete symmetries of the DFR algebra

Although the $\theta$-algebra (1.1) is not invariant under $P$, and $T$, separately, unless the transformations (2.3) and (2.4) are taken into account simultaneously, the DFR algebra (1.7) has the discrete symmetries if we extend the action of the operators, $\mathcal{P}, \mathcal{T}, \mathcal{C}$, to the operators defined on the DFR algebra (1.7) such that
\[ \mathcal{P} \hat{\theta}^{\mu\nu} \mathcal{P}^{-1} = \hat{\theta}^{\mu\nu}, \quad \mathcal{T} \hat{\theta}^{\mu\nu} \mathcal{T}^{-1} = -\hat{\theta}^{\mu\nu}, \]
\[ \mathcal{C} \hat{\theta}^{\mu\nu} \mathcal{C}^{-1} = \hat{\theta}^{\mu\nu}, \quad \Theta \hat{\theta}^{\mu\nu} \Theta^{-1} = -\hat{\theta}^{\mu\nu}. \] (4.1)
Considering the $\theta$-space spanned by the eigenstates of the operator $\hat{\theta}^{\mu\nu}$ as in Ref. 13), (4.1) is equivalent to (2.3) and (2.4). We again emphasize that the non-trivial transformations (2.3) and (2.4) of the components of $\theta^{\mu\nu}$ cannot be obtained within the framework of the $\theta$-algebra. It is associated with the facts that the discrete symmetry operators can be extended to act on the operators defined on the DFR algebra and that $\theta^{\mu\nu}$ is an eigenvalue of the operator $\hat{\theta}^{\mu\nu}$.
§5. Discrete symmetries in Lorentz-invariant QED$_*$

The purpose of this section is to prove the discrete symmetries of Lorentz-invariant QED$_*$. The proof differs from that of QED$_*$ in that both $x$ and $\theta$ become the integration variables in the action (1.9).

Let us start from the Lorentz-invariant free Dirac action

$$\hat{S}_{D0*} = \int d^4x d^6\theta W(\theta) \bar{\psi}(x, \theta) * (i\gamma^\mu \partial_\mu - m) \psi(x, \theta)$$

$$= \int d^6\theta W(\theta) \int d^4x_1 d^4x_2 d^4x_3 K(x_1; x_2, x_3) \bar{\psi}(x_1, \theta) (i\gamma^\mu \partial_\mu - m) \psi(x_2, \theta).$$  \hspace{1cm} (5.1)

An explicit expression in terms of the kernel can be derived from (1.10). Now suppose that the spinor is subject to the $*$-gauge transformation

$$\psi(x, \theta) \rightarrow \hat{g} \psi(x, \theta) = U(x, \theta) \psi(x, \theta),$$

$$\bar{\psi}(x, \theta) \rightarrow \hat{g} \bar{\psi}(x, \theta) = \bar{\psi}(x, \theta) U^\dagger(x, \theta),$$  \hspace{1cm} (5.2)

where $U(x, \theta)$ is assumed to be $*$-unitary:

$$U(x, \theta) U^\dagger(x, \theta) = U^\dagger(x, \theta) U(x, \theta) = 1.$$  \hspace{1cm} (5.3)

The $*$-gauge invariance requires the replacement

$$\partial_\mu \psi(x, \theta) \rightarrow D_\mu \psi(x, \theta) = \partial_\mu \psi(x, \theta) - ie A_\mu(x, \theta) \psi(x, \theta),$$  \hspace{1cm} (5.4)

with the NC gauge field transforming like

$$A_\mu(x, \theta) \rightarrow \hat{g} A_\mu(x, \theta) = U(x, \theta) A_\mu(x, \theta) U^\dagger(x, \theta) + \frac{i}{e} U(x, \theta) \partial_\mu U^\dagger(x, \theta).$$  \hspace{1cm} (5.5)

This prescription gives the $*$-gauge invariant Dirac action, $\hat{S}_{D*} = \hat{S}_{D0*} + \hat{S}_{I*}$, where

$$\hat{S}_{I*} = e \int d^4x d^4\theta W(\theta) \bar{\psi}(x, \theta) * \gamma^\mu A_\mu(x, \theta) \psi(x, \theta)$$

$$= e \int d^6\theta W(\theta) \int d^4x_1 d^4x_2 d^4x_3 K(x_1; x_2, x_3) \bar{\psi}(x_1, \theta) \gamma^\mu A_\mu(x_2, \theta) \psi(x_3, \theta).$$  \hspace{1cm} (5.6)

To investigate $C$-transformation property of this action we recall $C$-transformation\textsuperscript{12)} of the fields in Lorentz-invariant spinor QED$_*$,

$$C \psi(x, \theta) C^{-1} = C \bar{\psi}(x, \theta),$$

$$C \bar{\psi}(x, \theta) C^{-1} = -\psi(x, \theta) C^{-1},$$

$$C A_\mu(x, \theta) C^{-1} = -A_\mu(x, \theta).$$  \hspace{1cm} (5.7)
Since the sign change \((2\cdot5)\) is relevant only for the kernel, \(C\) transformation \((5\cdot7)\) keeps the sign of the argument \(\theta\) of fields as indicated by \((2\cdot3)\). We have using \((2\cdot5)\)

\[
C\hat{S}_{Is}C^{-1} = -\frac{e}{2} \int d^4x \bar{\psi}(x,\theta) \gamma^\mu \gamma^T A_\mu(x,\theta) \bar{\psi}(x,\theta)
\]

\[
= -\frac{e}{2} \int d^6\theta W(\theta) \int d^4x_1 d^4x_2 d^4x_3 \bar{\psi}(x_1,\theta) \gamma^\mu A_\mu(x_2,\theta) \bar{\psi}(x_3,\theta). \tag{5\cdot8}
\]

For classical fields this is further rearranged into

\[
C\hat{S}_{Is}C^{-1} = e \int d^6\theta W(\theta) \int d^4x_1 d^4x_2 d^4x_3 \bar{\psi}(x_3,\theta) \gamma^\mu A_\mu(x_2,\theta) \bar{\psi}(x_1,\theta). \tag{5\cdot9}
\]

Noting that \(\bar{\psi}(x_1; x_2, x_3) = \bar{\psi}(x_1; x_3, x_2) = \bar{\psi}(x_3; x_2, x_1)\) by cyclicity, the right-hand side equals \(\hat{S}_{Is}\):

\[
C\hat{S}_{Is}C^{-1} = \hat{S}_{Is}. \tag{5\cdot10}
\]

For non-classical field we can not make the rearrangement from \((5\cdot8)\) to \((5\cdot9)\) and are unable to arrive at the result \((5\cdot10)\). Hence \(C\)-invariant NC Dirac action in a Lorentz-invariant version is also the average

\[
\hat{S}_D = \frac{1}{2}[(\hat{S}_{D_{Is}} + \hat{S}_{Is}) + C(\hat{S}_{D_{Is}} + \hat{S}_{Is})C^{-1}]. \tag{5\cdot11}
\]

For completeness we explicitly write the third term,

\[
\hat{S}_{D_{Is}} \equiv C\hat{S}_{D_{Is}}C^{-1} = \int d^4x \bar{\psi}(x,\theta) \gamma^\mu \gamma^T \bar{\psi}(x,\theta)
\]

\[
= \int d^6\theta W(\theta) \int d^4x_1 d^4x_2 d^4x_3 \bar{\psi}(x_3,\theta) \gamma^\mu A_\mu(x_2,\theta) \bar{\psi}(x_1,\theta). \tag{5\cdot12}
\]

The reason why the \((5\cdot9)\)-product appears here is the same as in \((3\cdot13)\). Following the procedure in the section 3, *) we write the \((5\cdot2)\)-gauge transformations \((5\cdot5)\) in an ‘opposite’ but equivalent way

\[
\psi(x,\theta) \rightarrow \hat{g}\psi(x,\theta) = \psi(x,\theta) \bar{U}(x,\theta),
\]

\[
\bar{\psi}(x,\theta) \rightarrow \hat{g}\bar{\psi}(x,\theta) = \bar{U}(x,\theta) \bar{\psi}(x,\theta),
\]

\[
A_\mu(x,\theta) \rightarrow \hat{g}A_\mu(x,\theta) = U(\theta) \bar{A}_\mu(x,\theta) \bar{U}(x,\theta) + \frac{i}{e} \partial_\mu \bar{U}(x,\theta) \bar{U}(x,\theta). \tag{5\cdot13}
\]

where \(U(x,\theta)\) is also \((5\cdot9)\)-unitary:

\[
U(x,\theta) \bar{U}(x,\theta) = \bar{U}(x,\theta) U(x,\theta) = 1. \tag{5\cdot14}
\]

*) We may add the argument \(\theta\) to the functions in \((3\cdot14)\), \((3\cdot15)\), and \((3\cdot16)\) with appropriate care.
The ∗-gauge-invariant action is then obtained by the replacement

$$\partial_\mu \bar{\psi}(x,\theta) \to D_\mu \bar{\psi}(x,\theta) = \partial_\mu \bar{\psi}(x,\theta) + ie A_\mu(x,\theta) \bar{\psi}(x,\theta),$$

in (5.12). The covariant derivative $D_\mu$ satisfies

$$\bar{D}_\mu \bar{\psi}(x,\theta) \to U^\dagger(x,\theta) \bar{D}_\mu \bar{\psi}(x,\theta),$$

under the ∗-gauge transformation (5.13). The result yields $\hat{S}_{D_0} = \hat{S}_{D_0^*} + \hat{S}_{I^*}$, where

$$\hat{S}_{I^*} = -e \int d^4x d^6\theta W(\theta) \psi(x,\theta) \bar{\psi}(x,\theta) \gamma^{\mu T} A_\mu(x,\theta) \bar{\psi}(x,\theta) \bar{\psi}(x,\theta).$$

This is nothing but (5.8). Hence if we put $\hat{S}_D = \frac{1}{2}(\hat{S}_{D_0^*} + \hat{S}_{I^*})$, it is now clear that

$$C \hat{S}_D C^{-1} = \hat{S}_D,$$

provided ∗-product ↔ ∗-product, i.e., $K \leftrightarrow \bar{K}$ under $C$.

Before proceeding further we ask ourselves why the sign change (2.5) is necessary in proving $C$ invariance of the action $\hat{S}_D$. Although the free actions (5.1) and (5.12) are the same, the nonlocal Lagrangians corresponding to them are different,

$$\hat{L}_{D_0^*} = \bar{\psi}(x,\theta) * (i\gamma^\mu \partial_\mu - m) \psi(x,\theta)
= \int d^4x_1 d^4x_2 K(x; x_1, x_2) \bar{\psi}(x_1, \theta) (i\gamma^\mu \partial_\mu - m) \psi(x_2, \theta),$$

$$\hat{L}_{D_0^*} = \psi(x,\theta) * (i\gamma^\mu \partial_\mu + m) \bar{\psi}(x,\theta)
= \int d^4x_1 d^4x_2 \bar{K}(x; x_1, x_2) \psi(x_1, \theta) (i\gamma^\mu \partial_\mu + m) \bar{\psi}(x_2, \theta).$$

We are going to compare these nonlocal Lagrangians in connection with $C$. This is in the same spirit as in the commutative theory where the Dirac Lagrangian omitting ∗ in (3.1) is $C$-invariant only up to a total divergence and $C$-invariant free Dirac Lagrangian is the average, *)

$$L_{D_0} = \frac{1}{2} [\bar{\psi}(x)(i\gamma^\mu \partial_\mu - m)\psi(x) + \psi(x)(i\gamma^\mu \partial_\mu + m)\bar{\psi}(x)].$$  

*) This is usually understood when writing the Dirac Lagrangian without the antisymmetrization. We would like to point out, however, that a similar ‘averaging’ involves a nontrivial prescription in the NC setting.
Our procedure to obtain $C$-invariant free Dirac Lagrangian in the NC setting goes through in a similar way. If we transform (5.19) without touching the kernel, we obtain an unpleasant result that $\mathcal{C}\hat{L}_{D_{0s}}\mathcal{C}^{-1}$ is neither equal to $\hat{L}_{D_{0s}}$ itself nor transformed to $\hat{L}_{D_{0s}}$ and similarly for $\mathcal{C}\hat{L}_{D_{0s}}\mathcal{C}^{-1}$. Instead we require as in the commutative case that $\hat{L}_{D_{0s}}$ be transformed to $\hat{L}_{D_{0s}}$ and vice versa under $C$. This requirement is satisfied by assuming that the kernel $K(x; x_1, x_2)$ is changed into the kernel $\hat{K}(x; x_1, x_2)$ and vice versa under $C$. This is seen as follows.

\[
\mathcal{C}\hat{L}_{D_{0s}}\mathcal{C}^{-1} = \int d^4x_1d^4x_2 \bar{K}(x; x_1, x_2) [\psi(x_1, \theta)C^{-1}(i\gamma^\mu \partial_\mu - m)C\bar{\psi}(x_2, \theta)]
\]

\[
= \int d^4x_1d^4x_2 \bar{K}(x; x_1, x_2) \psi(x_1, \theta)(i\gamma^{\mu T} \partial_\mu + m)\bar{\psi}(x_2, \theta) = \hat{L}_{D_{0s}},
\]

\[
\mathcal{C}\hat{L}_{D_{0s}}\mathcal{C}^{-1} = \int d^4x_1d^4x_2 \bar{K}(x; x_1, x_2) [-C\bar{\psi}(x_1, \theta)(i\gamma^{\mu T} \partial_\mu + m)\psi(x_2, \theta)C^{-1}]
\]

\[
= \int d^4x_1d^4x_2 \bar{K}(x; x_1, x_2) \bar{\psi}(x_1, \theta)(i\gamma^\mu \partial_\mu - m)\psi(x_2, \theta) = \hat{L}_{D_{0s}}. \quad (5.21)
\]

The exchange $K \leftrightarrow \bar{K}$ under $C$ corresponds to the sign change (2.5), which resembles an anti-unitary nature, $i \rightarrow -i$ under $T$ since we keep the sign of the argument $\theta$ in the field variables. *)

The Maxwell action is constructed using the field strength tensor

\[
F_{\mu\nu}(x, \theta) = \partial_\mu A_\nu(x, \theta) - \partial_\nu A_\mu(x, \theta) - ie[A_\mu(x, \theta), A_\nu(x, \theta)], \quad (5.22)
\]

where

\[
[A_\mu(x, \theta), A_\nu(x, \theta)]_\star \equiv A_\mu(x, \theta) * A_\nu(x, \theta) - A_\nu(x, \theta) * A_\mu(x, \theta) \quad (5.23)
\]

is the Moyal bracket, so that

\[
\hat{S}'_M = -\frac{1}{4} \int d^4x d^6\theta W(\theta) F_{\mu\nu}(x, \theta) * F^{\mu\nu}(x, \theta). \quad (5.24)
\]

The non-linear term in the field strength may be written as

\[
[A_\mu(x, \theta), A_\nu(x, \theta)]_\star = \int d^4x_1d^4x_2 K(x; x_1, x_2) [A_\mu(x_1, \theta)A_\nu(x_2, \theta) - A_\nu(x_1, \theta)A_\mu(x_2, \theta)]. \quad (5.25)
\]

Under the charge conjugation it goes over to

\[
\mathcal{C}[A_\mu(x, \theta), A_\nu(x, \theta)]_\star \mathcal{C}^{-1} = \int d^4x_1d^4x_2 \bar{K}(x; x_1, x_2) \mathcal{C}[A_\mu(x_1, \theta)A_\nu(x_2, \theta) - A_\nu(x_1, \theta)A_\mu(x_2, \theta)] \mathcal{C}^{-1}
\]

\[
= \int d^4x_1d^4x_2 \bar{K}(x; x_1, x_2) [A_\mu(x_1, \theta)A_\nu(x_2, \theta) - A_\nu(x_1, \theta)A_\mu(x_2, \theta)]
\]

\[
= A_\mu(x, \theta) \hat{\bar{\psi}}A_\nu(x, \theta) - A_\nu(x, \theta) \hat{\bar{\psi}}A_\mu(x, \theta) \equiv [A_\mu(x, \theta), A_\nu(x, \theta)]_\star. \quad (5.26)
\]

*) This is reminiscent of the anti-unitary character of the charge conjugation in the one-particle theory.
Remember $K \to K$ under $C$. Consequently, as in the section 3, the field strength does not transform to itself up to sign but is changed into

$$CF_{\mu \nu}(x, \theta)C^{-1} = -G_{\mu \nu}(x, \theta)$$

$$G_{\mu \nu}(x, \theta) = \partial_\mu A_\nu(x, \theta) - \partial_\nu A_\mu(x, \theta) + ie[A_\mu(x, \theta), A_\nu(x, \theta)]_\theta.$$  \hfill (5.27)

Using the last transformation of (5.13) the field strength $G_{\mu \nu}(x, \theta)$ can be shown to be $\bar{\theta}$-gauge covariant,

$$G_{\mu \nu}(x, \theta) \to \hat{g}G_{\mu \nu}(x, \theta) = U^\dagger(x, \theta)\bar{\theta}G_{\mu \nu}(x, \theta)\bar{\theta}U(x, \theta).$$  \hfill (5.28)

For classical fields we can commute two field variables in (5.26) to find $F_{\mu \nu} = G_{\mu \nu}$ as observed in Ref. 12. For non-classical fields which are noncommuting at general $x$’s we have $F_{\mu \nu} \neq G_{\mu \nu}$. Thus $C$-invariant Maxwell$_\star$ action also becomes the average

$$\hat{S}_M = -\frac{1}{8} \int d^4x d^6\theta W(\theta)[F_{\mu \nu}(x, \theta) \star F^{\mu \nu}(x, \theta) + G_{\mu \nu}(x, \theta)\bar{\theta}G^{\mu \nu}(x, \theta)].$$  \hfill (5.29)

Hence, we have shown that

$$C\hat{S}_{QED}C^{-1} = \hat{S}_{QED}, \quad \hat{S}_{QED} = \hat{S}_D + \hat{S}_M.$$  \hfill (5.30)

This proves $C$-invariance of Lorentz-invariant QED$_\star$.

Next we turn to $P, T$ and $CPT$. We assume that

$$\mathcal{P}\psi(x, \theta)\mathcal{P}^{-1} = \gamma^0 \psi(x_P, \theta_P), \quad \mathcal{P}\bar{\psi}(x, \theta)\mathcal{P}^{-1} = \bar{\psi}(x_P, \theta_P)\gamma^0, \quad \mathcal{P}A_\mu(x, \theta)\mathcal{P}^{-1} = A^\mu(x_P, \theta_P),$$

$$\mathcal{T}\psi(x, \theta)\mathcal{T}^{-1} = R\psi(x_T, \theta_T), \quad \mathcal{T}\bar{\psi}(x, \theta)\mathcal{T}^{-1} = \bar{\psi}(x_T, \theta_T)R^{-1}, \quad \mathcal{T}A_\mu(x, \theta)\mathcal{T}^{-1} = A^\mu(x_T, \theta_T),$$  \hfill (5.31)

with $\theta_P \equiv (\theta^{\mu \nu})_P = \theta_{\mu \nu}, \theta_T \equiv (\theta^{\mu \nu})_T = -\theta_{\mu \nu}$ and $R^{-1}\gamma^{\mu *}R = \gamma_\mu$ with $R = i\gamma_5 C$. The Dirac action $\hat{S}_D = \frac{1}{2}(\hat{S}_{D\gamma} + \hat{S}_{D\iota})$ and the Maxwell$_\star$ action $\hat{S}_M$ are invariant under $P$ and $T$. \hfill *) Hence $P$ and $T$ are separately conserved in Lorentz-invariant QED$_\star$.

The $CPT$ transformation is determined from (5.7) and (5.31) to be

$$\Theta\psi(x, \theta)\Theta^{-1} = -i\gamma_5\gamma^0_\iota \bar{\psi}(-x, -\theta),$$

$$\Theta\bar{\psi}(x, \theta)\Theta^{-1} = \bar{\psi}(-x, -\theta)i\gamma_5\gamma^0_\iota,$$

$$\Theta A_\mu(x, \theta)\Theta^{-1} = -A_\mu(-x, -\theta).$$  \hfill (5.32)

*) We assume that $W(\theta)$ is a function of the invariant $\theta^{\mu \nu}\theta_{\mu \nu}$ only so that $W(\theta_P) = W(\theta_T) = W(\theta)$. 

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We note the difference from the prescription used in (3.23) in that now $K$ is unchanged under CPT because of the simultaneous change $i \rightarrow -i$ and (2.5). The CPT transformation (2.4) is already taken into account in (5.32). The calculation will be reported only in the fermion sector,

$$\Theta \hat{S}_{D*} \Theta^{-1} = -e \int d^6 \theta W(\theta) \int d^4 x_1 d^4 x_2 d^4 x_3 K(x_1; x_2, x_3) \psi(-x_1, -\theta) \gamma^T_0 \gamma^T_0 A_\mu(-x_2, -\theta) \bar{\psi}(-x_3, -\theta)$$

$$= -e \int d^6 \theta W(\theta) \int d^4 x_1 d^4 x_2 d^4 x_3 K(x_1; x_2, x_3) \psi(x_1, \theta) \gamma^\mu T A_\mu(x_2, \theta) \bar{\psi}(x_3, \theta) = \hat{S}_{D*}, \quad (5.33)$$

where we have used the fact that the weight function is even, $W(-\theta) = W(\theta)$. Consequently, we have shown that

$$\Theta \hat{S}_{\text{QED}} \Theta^{-1} = \hat{S}_{\text{QED}}, \quad \hat{S}_{\text{QED}} = \hat{S}_D + \hat{S}_M. \quad (5.34)$$

Lorentz-invariant QED$_*$ is CPT-invariant in accord with our definition of the symmetry described in the section 1.

§6. Conclusions

We have discussed the discrete symmetries in Lorentz-invariant non-commutative field theory based on the DFR algebra. Since anti-particles are an outcome from the marriage of relativity and quantum mechanics, the concept of the charge conjugation can only be defined in relativistic quantum field theory even if the continuum space-time is modified to the non-commutative one. In this respect we differ from the previous discussions$^{19), 20), 24)}$ on the same subject.

The spinor $\psi$ in the previous section is nothing but $\psi_1$ in Ref. 12) with $e \rightarrow 2e$. It was shown there that there are only eight spinors allowed in Lorentz-invariant QED$_*$ compatible with *-gauge transformations. Three of them couple to $A_\mu(x, \theta)$ and the other three to $A'_\mu(x, \theta) = -A_\mu(x, -\theta)$. Both obey the charge quantization condition.$^{25)}$ The neutral spinor among them may represent the neutrinos. There are two more spinors either of which may be identified with the observed charged leptons.

Our discussion is thus restricted to a particular sector of Lorentz-invariant QED$_*$. Nonetheless, it is straightforward to extend it to generic QFT$_*$. At present stage we are unable to find a consistent way of quantization except when the field is independent of $\theta$. This is an important problem in our formalism and we shall come back to it in later communication. For phenomenological purpose$^{11)\, 13)}$ we may use the $\theta$-expansion, that is, a field redefinition which expresses any field with the ‘internal coordinates’ $\theta$ in terms of those without the
‘internal coordinates’. For a generic field $\varphi(x, \theta)$ occurring in the $*$-gauge theory this is accomplished by assuming

$$\varphi(x, \theta) = \varphi^{(0)}(x) + \varphi^{(1)}(x) + \varphi^{(2)}(x) + \cdots,$$

where $\varphi^{(n)}(x)$ is of order $n$ in $\theta, n = 0, 1, 2, \cdots$, and is a function of the lowest-order fields and their derivatives. Consequently, it is only necessary to second quantize the local field $\varphi(x) \equiv \varphi^{(0)}(x)$. One can do this using the Seiberg-Witten map. ²)

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