Explicit Sequential Equilibria in Linear Quadratic Games with Arbitrary Number of Exchangeable Players: A Non-Standard Riccati Equation

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Abstract—In this paper, we investigate a class of nonzero-sum dynamic stochastic games consisting of an arbitrary number of exchangeable players. The players have linear dynamics with quadratic cost functions and are coupled in both dynamics and cost through the empirical mean and covariance matrix of the states and actions. We study cooperative and non-cooperative games under three information structures: perfect sharing, aggregate sharing, and no sharing. For perfect and aggregate sharing information structures, we use a gauge transformation to solve for the best-response equations of players and identify a necessary and sufficient condition under which a unique subgame perfect Nash equilibrium exists. The equilibrium is proved to be fair and linear in the local state and aggregate state. The corresponding gains are obtained by solving a novel non-standard Riccati equation, whose dimension is independent of the number of players, thus making the solution scalable. Two approximate sequential Nash solutions are also proposed for the case of no-sharing information structure. To the best of our knowledge, this paper is the first to propose a unified framework to obtain the exact sequential Nash solution in closed form including both cooperative and non-cooperative linear quadratic games with an arbitrary number of exchangeable players. In addition, the main results are extended to the discounted and time-averaged infinite-horizon cost functions. Furthermore, the obtained results can naturally be generalized to the case with multiple exchangeable sub-populations. Finally, a numerical example is provided to demonstrate the efficacy of the obtained results.

I. Motivation

The history of nonzero-sum linear quadratic (LQ) games can be traced back to work of Case [1] and Starr and Ho [2], [3] over five decades ago. Early results demonstrated that finding the Nash solution with perfect information structure can be much more difficult than solving the LQ optimal control problem. For instance, a two-player game with strictly convex cost functions may be not playable (i.e., the game may not admit a solution) [4]; may admit uncountably many solutions (where each solution leads to a different set of pay-offs) [5]; may admit a unique Nash solution in a stochastic game but admit infinitely many solutions in its deterministic form [5], or may have a unique Nash solution for an arbitrary finite horizon but have no solution, a unique solution, or infinitely many solutions in the infinite horizon [6]. For a stochastic LQ game, it is well known that the necessary and sufficient condition for the Nash solution to exist uniquely is equivalent to the existence and uniqueness condition for the solution of a particular set of coupled Riccati equations [5]. For a deterministic LQ game, however, it is more difficult to establish an existence condition for a unique Nash solution because the above-mentioned coupled Riccati equations do not necessarily identify all the possible Nash solutions. The interested reader is referred to [7], [8] for some counterexamples where the coupled Riccati equations do not have a solution but the deterministic game admits one, and to [5] for a counterexample where the coupled Riccati equations have a unique solution but the deterministic game admits uncountably many solutions. In addition, as the number of players increases, the curse of dimensionality exacerbates the above challenges from a computational viewpoint. On the other hand, due to limited communication resources in practice, it may not be feasible to implement the perfect information structure, especially when the number of players is large. In such a case, one may need to consider “imperfect” structures with players having private information, making it conceptually difficult for players to reach an agreement. Finding the Nash solution in LQ games under imperfect information structure involves non-convex optimization, which in general destroys the linearity of the solution. For instance, it is shown in [9] via a counterexample that nonlinear strategies can outperform linear strategies. As a result, it has been a long-standing challenge in game theory to identify games that possess tractable solutions irrespective of the number of players. The reader is referred to [10]–[12] for more details on the theory and application of LQ games.

Due to difficulties outlined in the previous paragraph, mean-field LQ games were developed for a class of exchangeable players to provide an approximate tractable Nash solution when the number of players is asymptotically large [13]–[16]. In the simplest formulation of mean-field LQ games, the players are coupled in the dynamics and cost through the empirical mean of states (i.e., aggregate state). The key idea lies in the fact that the effect of a single player on other players is negligible when the number of players goes to infinity. This observation simplifies the infinite-population game to a constrained optimal control problem where: (a) a generic player computes its best-response strategy by solving an optimal tracking problem described by a backward Hamilton-
Jacobi-Bellman equation with respect to some hypothesized reference trajectory, and (b) for the reference trajectory to be admissible, it is constrained to be equal to the mean-field (the aggregate state of the infinite-population game) whose dynamics is expressed by a forward advection or Fokker-Planck-Kolmogorov equation. This is a fixed-point requirement, and sufficient conditions can be imposed to guarantee the existence of a unique solution [14]. It is then shown that this solution is an approximate Nash equilibrium for the finite-population game. On the other hand, when players use identical state feedback strategies and share the same cost function, the mean-field converges to the expectation of the generic player’s state due to the strong law of large numbers. This transforms the above two-body optimization problem into a one-body optimization problem of McKean-Vlasov type, where the expectation of the state appears in the dynamics and cost function. The resultant problem is known as the mean-field-type LQ control problem. The solution of this problem is obtained by forward-backward equations in [17] and by backward equations (Riccati equations) in [18], [19]. It is demonstrated in [20] that the solution of mean-field type control does not necessarily coincide with that of the mean-field game. In contrast to the above results that are only relevant when the number of players is large, the authors in [21], [22] consider a finite-population mean-field LQ game wherein players are decoupled in dynamics. When attention is restricted to Gaussian random variables with identical and affine stationary feedback strategies, the finite-population game reduces to a constrained optimal control problem where the existence of a Nash solution is conditioned on the solution of a coupled algebraic Riccati-Sylvester equation. For more details on the theory and applications of mean-field coupled models, the reader is referred to [23] and references therein.

In the context of this paper, we consider LQ games with an arbitrary number of exchangeable players. Players are called exchangeable if their dynamics and cost functions remain invariant under arbitrary indexing. By a simple algebraic analysis, one can show that any LQ game with exchangeable players can be formulated as a LQ game wherein the players are coupled in the dynamics and cost through the empirical mean as well as the empirical covariance matrix of the states and actions of players. This formulation encompasses mean-field LQ games and mean-field-type LQ control models, simultaneously. In light of the above observation, we consider three information structures: perfect sharing, aggregate sharing and no sharing, the precise structure of which we now detail. In the first information structure, every player has access to the joint state vector; in the second one, every player has only access to its local state and the aggregate state. In the third one, every player has only access to its local state. In addition, note that the mean-field equilibrium, formulated in terms of coupled forward-backward equations, is not necessarily a “sequential” equilibrium [24] or “trembling-hand” equilibrium [25] in the sense that it does not consider the off-equilibrium-path events occurring at the macroscopic level. More precisely, players use fixed-point calculations to guess the trajectory of the mean-field before the game starts, but there is no guarantee that they will pursue the initially agreed-upon trajectory (belief) at every stage of the game if an unexpected event changes their belief about the mean-field, for example, in the presence of a common mistake or a new piece of information.

The remainder of this paper is organized as follows. In Section II the problem is formulated and the main contributions of the paper are outlined. In Section III the subgame perfect Nash solution under perfect and aggregate sharing information structures is identified, and in Section IV two approximate sequential Nash solutions under no-sharing information structure are proposed. Two special games are presented in Sections VI and VII where the above solutions are computed by solving two standard Riccati equations. In Section VIII a brief discussion on possible generalizations of the presented results is provided and in Section IX a numerical example is given to validate the theoretical findings. Finally in Section X the main results are summarized and conclusions are drawn.

II. PROBLEM FORMULATION

Throughout this paper, $\mathbb{N}$ and $\mathbb{R}$ denote natural and real numbers, respectively. For any $n \in \mathbb{N}$, the short-hand notations $x_{1:n}$ and $\mathbb{N}_n$ are used to denote the vector $(x_1, \ldots, x_n)$ and the set $\{1, \ldots, n\}$, respectively. For any vectors $x$, $y$, and $z$, $\text{vec}(x, y, z)$ represents the vector $[x^\top, y^\top, z^\top]^\top$. Given any square matrices $A$, $B$, and $C$, $\text{diag}(A, B, C)$ is the block diagonal matrix with matrices $A$, $B$ and $C$ on its main diagonal. Let $A$ be an $n \times n$ block matrix; then $A^{i,j}$ refers to the block matrix located at the $i$-th row and the $j$-th column, $i, j \in \mathbb{N}_n$. Given a set $X = \{x_1, \ldots, x_n\}$, $X^{-1}$ is the set without its $i$-th component, $i \in \mathbb{N}_n$. $\text{Tr}()$ is the trace of a matrix, $\text{Cov}(\cdot)$ is the covariance matrix of a random vector and $E[\cdot]$ is the expectation of an event.
Consider a finite-horizon dynamic game with \( n \in \mathbb{N} \) rational players that have linear dynamics and quadratic cost functions. In order to have a meaningful game, the number of players \( n \) is assumed to be greater than 1 in the sequel. Let \( x_i^t \in \mathbb{R}^{d_x} \) and \( u_i^t \in \mathbb{R}^{d_u} \), \( d_x, d_u \in \mathbb{N} \), be the state and action of player \( i \in \mathbb{N}_n \) at time \( t \in \mathbb{N}_T \), where \( T \in \mathbb{N} \) denotes the game horizon. Denote by \( \mathbf{x}_t = \text{vec}(x_1^t, \ldots, x_n^t) \) and \( \mathbf{u}_t = \text{vec}(u_1^t, \ldots, u_n^t) \) the stacked state and the stacked action at time \( t \in \mathbb{N}_T \), respectively. Let the random initial states be such that \( \text{Cov}(\mathbf{x}_1) \) is a bounded matrix. Let also the empirical mean of the states and actions be defined as follows:

\[
\bar{\mathbf{x}}_t := \frac{1}{n} \sum_{i=1}^n x_i^t, \quad \bar{\mathbf{u}}_t := \frac{1}{n} \sum_{i=1}^n u_i^t.
\] (1)

The state of player \( i \in \mathbb{N}_n \) at time \( t \in \mathbb{N}_T \) evolves as follows:

\[
x_{i, t+1} = A_i x_i^t + B_i u_i^t + D_i \bar{x}_t + E_i \bar{u}_t + w_i^t,
\] (2)

where \( A_i, B_i, D_i \) and \( E_i \) are time-varying matrices of appropriate dimensions, and \( w_i^t \) is the zero-mean local noise of player \( i \) at time \( t \) with a positive semidefinite covariance matrix, i.e., \( \mathbb{E}[w_i^t w_i^t] = 0_{d_x \times d_x} \) and \( \text{Cov}(w_i^t) \geq 0_{d_x \times d_x} \). Denote by \( \mathbf{w}_t = \text{vec}(w_1^t, \ldots, w_n^t) \) the stacked noise vector at time \( t \in \mathbb{N}_T \), and assume that its covariance matrix \( \text{Cov}(\mathbf{w}_t) \) is bounded. Note that the boundedness of the covariance matrices can be ensured by a realistic assumption in practice. The random variables \( (x_1, w_1, \ldots, w_T) \) are defined on a common probability space and are mutually independent. The running cost function of player \( i \in \mathbb{N}_n \) at time \( t \in \mathbb{N}_T \) is assumed to be quadratic in local states and actions as well as in empirical mean and covariance matrix of states and actions as follows:

\[
c_i^t(\mathbf{x}_t, \mathbf{u}_t) = (x_i^t)^\top Q_i x_i^t + 2(x_i^t)^\top S_i^x \bar{x}_t + (\bar{x}_t)^\top P_i^x \bar{x}_t + (u_i^t)^\top R_i u_i^t + 2(u_i^t)^\top S_i^u \bar{u}_t + (\bar{u}_t)^\top P_i^u \bar{u}_t
\]

\[+ \frac{1}{n} \sum_{i=1}^n (x_i^t)^\top G_i^x x_i^t + \frac{1}{n} \sum_{i=1}^n (u_i^t)^\top G_i^u u_i^t,
\] (3)

where \( Q_t, R_t, S_t^x, S_t^u, P_t^x, P_t^u, G_t^x \) and \( G_t^u \) are symmetric matrices of appropriate dimensions.

**Remark 1.** It is to be noted that cost function (3) is the most general representation of a generic LQ game with exchangeable players. The intention is to present a unified framework for numerous cases of mean-field models such as mean-field games, mean-field-type games and mean-field control, which have been studied separately within the context of LQ games. Also, we are particularly interested in the so-called extended mean-field games, where the effect of aggregate action is accounted for (this type of games has applications in financial models). Importantly, we strive to give an exact but scalable version of the solution to the finite-population version of these games. In addition, the covariance term in (3) is of particular interest in applications such as robust and quantile sensitive games, where high variance is not desirable.

It is also possible to consider cross terms between states and actions in (3), however, since such an extension is trivial, it is not considered here for simplicity of presentation. It is worth highlighting that any exchangeable function can be equivalently expressed as a function of empirical distribution [26].

**Definition 1.** For ease of reference, the empirical mean of states will hereafter be referred to as the **aggregate state**.

In this paper, we consider three information structures. The first one is called **Perfect Sharing (PS)**, where every player has access to the stacked state, i.e., the control action of player \( i \in \mathbb{N}_n \) at time \( t \in \mathbb{N}_T \) is given by:

\[
u_i^t = g_i^t(\mathbf{x}_{1:t}), \quad \text{(PS)}
\]

where \( g_i^t : \mathbb{R}^{nd_x} \to \mathbb{R}^{d_u} \) is a measurable function with respect to the \( \sigma \)-algebra generated by \( \{x_1, w_1, \ldots, w_t\} \). The second information structure is called **Aggregate Sharing (AGS)**, and requires that every player observes its local state as well as the aggregate state, i.e., the control action of player \( i \in \mathbb{N}_n \) at time \( t \in \mathbb{N}_T \) is described by:

\[
u_i^t = g_i^t(x_{1:t}, \bar{x}_{1:t}), \quad \text{(AGS)}
\]

where \( g_i^t : \mathbb{R}^{2nd_x} \to \mathbb{R}^{d_u} \) with the corresponding measurability properties. In practice, there are various applications in which aggregate sharing is plausible. For example, it is a common practice in the stock markets to provide players (e.g., buyers, sellers and brokers) with statistical data on the total amount of shares, trades and exchanges. Also, in a smart grid, an independent service operator may collect and broadcast the aggregate demand and supply. It is also possible to compute the aggregate state, without a central authority, by means of consensus algorithms (similar to the case where each agent in a swarm computes the aggregate state in a distributed manner by local interactions with its neighbors [27], [28]). The third information structure is called **No Sharing (NS)**, where each player needs nothing more than its local state, i.e.,

\[
u_i^t = g_i^t(x_i^t), \quad \text{(NS)}
\]

where \( g_i^t : \mathbb{R}^{d_x} \to \mathbb{R}^{d_u} \) is the control law of player \( i \in \mathbb{N}_n \) at time \( t \in \mathbb{N}_T \). When the number of players is relatively large and sharing the aggregate state is infeasible, NS information structure is more practical as it requires no communication among players, except at the beginning of the control horizon. In general, AGS and NS information structures are appropriate for security-based applications and situations wherein players are weary of sharing their private states.

In the sequel, it is assumed that the admissible set of control actions are square integrable for both finite and infinite-horizon cases. In addition, we refer to \( g_i^1_{1:T} \) as the **strategy of player** \( i \in \mathbb{N}_n \) and to \( \mathcal{G} := \{g_1^1_{1:T}, \ldots, g_n^1_{1:T}\} \) as the **strategy of the game**, and to \( \mathcal{G}_{PS} \) and \( \mathcal{G}_{AGS} \) as the set of PS and AGS strategies, respectively.

**A. Problem statement**

Define the following cost function for any player \( i \in \mathbb{N}_n \):

\[
J_i^n(g) := \mathbb{E}^k \left[ \sum_{t=1}^T c_i^t(x_t, u_t) \right],
\] (4)

where the above expectation is taken with respect to the probability measures that are induced on the random variables by the choice of strategy \( g \). We formulate the following three problems.
**Problem 1.** Find an admissible sequential Nash strategy $g^*$ under perfect sharing information structure, if it exists, such that for every player $i \in \mathbb{N}_n$ and any strategy $g_{1:T} \in \mathcal{G}_{PS}$,

$$J^i_n(g_{1:T}, y_{1:T}) \leq J^i_n(g^{*,i}_{1:T}, y^{*,i}_{1:T}).$$

**Problem 2.** Find an admissible sequential Nash strategy $g^*$ under aggregate sharing information structure, if it exists, such that for every player $i \in \mathbb{N}_n$ and any strategy $g_{1:T} \in \mathcal{G}_{AGS}$,

$$J^i_n(g_{1:T}, y_{1:T}) \leq J^i_n(g^{*,i}_{1:T}, y^{*,i}_{1:T}).$$

**Problem 3.** Find an approximate admissible sequential Nash strategy $\hat{g}$ under no-sharing information structure, if it exists, such that for every player $i \in \mathbb{N}_n$:

$$|J^i_n(g_{1:T}, y_{1:T}) - J^i_n(g^{*,i}_{1:T}, y^{*,i}_{1:T})| \leq \varepsilon(n), \quad (5)$$

where $\lim_{n \to \infty} \varepsilon(n) = 0$.

In this paper, we consider both the team setting where players share the same objective function, and the non-cooperative game setting, where the cost functions of players are individualized and distinct.

**B. Main challenges and contributions**

To solve Problem 1, one may use a formulation based on the standard backward induction, leading to an investigation of the solution of $n$ coupled matrix Riccati equations [10], which is a difficult task in general. In addition, if a solution exists, its numerical computation suffers from the curse of dimensionality, as matrices in Problem 1 are fully dense. To solve Problem 2 for a finite number of players, neither standard nor mean-field LQ games [14] can be directly employed. This is because the former generally requires some additional information which is normally not available, while the latter becomes a viable alternative only for a sufficiently large number of players. To solve Problem 3 on the other hand, one can use a mean-field LQ game approach [14] but because of the considered objective functions, the problem is not necessarily reducible to the classical mean-field game with a tracking cost function, in particular because of empirical actions in the cost function of players. In addition, similarly to [13], [14], the mean-field game approach tends to overshadow the role of the number of players in the solution by sending directly the number of players to infinity and then evaluating the quality of the approximation.

The main contributions of this paper are outlined below.

1) We propose a systematic approach to solve LQ games with an arbitrary number of exchangeable players by introducing a gauge transformation. More precisely, we identify a necessary and sufficient condition for the existence of a unique subgame perfect Nash equilibrium for Problems 1 and 2 where the effect of an individual player on others is not necessarily negligible. In addition, we obtain the equilibrium in an explicit scalable form, in particular, one where the complexity of calculations is independent of the number of players, and such that each player’s control law is linear in the local state and aggregate state (Theorems 1 and Corollary 1). The above-mentioned condition along with the Nash equilibrium is described by a novel non-standard Riccati equation.

2) We propose two no-sharing strategies for Problem 3 in Theorems 5 and 4 with different prices of information; see Remark 4 and the numerical example presented in Section IX for more detail on the differences between the two strategies. In addition, to illustrate the effect of the number of players on the Nash solution, we present two counterexamples wherein the finite-population game admits a unique solution while the infinite-population game has no solution, and vice versa. For the special case of a social welfare function, however, we prove that under PS and AGS information structures, the solution is insensitive to the number of players. In this case, the two proposed no-sharing approximate strategies are identical and independent of the number of players.

3) We extend our main results to the infinite-horizon case, where the cost functions in Problems 1 and 3 are replaced by discounted and time-averaged cost functions (Theorems 5 and 6).

4) We study two special games wherein the proposed non-standard Riccati equation reduces to two standard Riccati equations (Theorems 3 and 11). In these special games, the existence and uniqueness condition can be simplified, as noted in Remarks 6 and 8. We also compare our findings with the existing results in mean-field game and mean-field-type control. Furthermore, we show that the Nash and team-optimal solutions are not necessarily the same in LQ games with identical cost functions; see a counterexample in [10] on the difference between local convexity (Nash solution) and global convexity (team-optimal solution).

5) We shed some light on the non-uniqueness feature of the solution in deterministic models as well as in infinite-population games. In addition, we present a unified framework to study similarities and differences between diverse formulations of mean-field games.

6) We also briefly discuss a few generalizations of the proposed non-standard Riccati equation in Section VIII including partially exchangeable games wherein the population of players can be partitioned into a few disjoint sub-populations with exchangeable players. In particular, since the proposed methodology does not rely on the negligible effect of a single player, it is applicable to non-trivial cases such as major-minor and common-noise problems with the advantage that no additional complication arises.

For ease of reference, a summary of assumptions used in this paper is outlined in Table I.

**III. MAIN RESULTS FOR PROBLEMS 1 AND 2**

In this section, we present solutions for Problems 1 and 2.

A. Solution of Problem 1
**Finite horizon conditions**

A 1. Necessary and sufficient condition for the existence of a unique solution

A 2. Independent random variables for NS information structure

A 3. Asymptotic stability condition for the finite-model NS strategy

A 4. Asymptotic stability condition for the infinite-model NS strategy

**Additional conditions in infinite horizon**

A 5. Asymptotic stability condition for the non-standard Riccati equation

A 6. Asymptotic stability condition for the proposed NS strategies

| TABLE I |
| A SUMMARY OF ASSUMPTIONS USED IN THIS PAPER. |

| Sufficient conditions |
| (A 7. and A 8.) Sufficient conditions for A 1. and A 5. in Case I |
| (A 9. and A 10.) Sufficient conditions for A 1. and A 5. in cooperative game |
| (A 11. and A 12.) Sufficient conditions for A 1. and A 5. in Case II |

Due to the orthogonality induced by the gauge transformation, i.e., \( \sum_{i=1}^{n}(\tilde{\xi}_t^i)\top G_t^i \tilde{\xi}_t = 0 \) and \( \sum_{i=1}^{n}(\tilde{u}_t^i)G_t^i\top \tilde{u}_t^i = 0 \), the per-step cost of any player \( i \in \mathbb{N}_n \) at time \( t \in \mathbb{N}_T \), given by (3), can be expressed in terms of the new variables as follows:

\[
c_i^j(X_t, \tilde{u}_t) := \left[ \begin{array}{c} \tilde{x}_t^i \\ \tilde{x}_t \end{array} \right] \top Q_t \left[ \begin{array}{c} \tilde{x}_t^i \\ \tilde{x}_t \end{array} \right] + \left[ \begin{array}{c} \tilde{u}_t^i \\ \tilde{u}_t \end{array} \right] \top \tilde{R}_t \left[ \begin{array}{c} \tilde{u}_t^i \\ \tilde{u}_t \end{array} \right] + \frac{1}{n} \left( \sum_{j \neq i}^{n}(\tilde{d}_t^j)\top G_t^j(\tilde{x}_t^j) \right) - \frac{1}{n-1} \left( \tilde{x}_t^i \right)\top G_t^i(\tilde{x}_t^i), \tag{11} \right.
\]

where

\[
Q_t := \left[ \begin{array}{c} \tilde{Q}_t + \frac{1}{n-1}G_t^i \tilde{Q}_t + S_t^e \\ \tilde{Q}_t + 2S_t^e + P_t + G_t^e \end{array} \right], \\
\tilde{R}_t := \left[ \begin{array}{c} \tilde{R}_t + \frac{1}{n-1}G_t^i \tilde{R}_t + 2S_t^u + P_t + G_t^u \end{array} \right].
\]

Note that the number of players \( n \) is more than one; hence, the denominators in (11) are non-zero. We define two non-standard Riccati equations which will help formulate the Nash solution in terms of the modified state equations and costs given above. For any \( t \in \mathbb{N}_T \), define matrices \( \tilde{M}^n_{1:T} \) backward in time as follows:

\[
\tilde{M}_t^n := \tilde{Q}_t + \tilde{A}_t^n \tilde{M}_{t+1}^{n,1} \tilde{A}_t^n + (\tilde{F}_t^n)\top B_t^n \tilde{M}_{t+1}^{n,1} \tilde{B}_t^n + \tilde{A}_t^n \tilde{M}_{t+1}^{n,1} B_t^n F_t^n + (\tilde{F}_t^n)\top (\tilde{R}_t + \tilde{B}_t^n \tilde{M}_{t+1}^{n,1} \tilde{B}_t^n) F_t^n, \\
\tilde{M}_{T+1}^{n} = \mathbf{0}_{2d_x \times 2d_x}, \tag{12} \right.
\]

where \( F_t^n := \text{diag}(\tilde{F}_t^n) = \text{diag}(\tilde{K}_t^n, \tilde{K}_t^n) \) and matrices \( \tilde{F}_t^n, \tilde{F}_t^n, \tilde{K}_t^n \) are given by:

\[
\tilde{F}_t^n = (1 - \frac{1}{n}) \left[ R_t + \frac{1}{n-1}G_t^i + B_t^n \tilde{M}_{t+1}^{n,1} B_t^n \right] + \frac{1}{n} \left[ R_t + S_t^u + (B_t + E_t)\top \tilde{M}_{t+1}^{n,2} B_t^n \right], \\
\tilde{F}_t^n = (1 - \frac{1}{n}) \left[ R_t + S_t^u + B_t^n \tilde{M}_{t+1}^{n,1} (B_t + E_t) \right] + \frac{1}{n} \left[ R_t + 2S_t^u + P_t + G_t^e + (B_t + E_t)\top \tilde{M}_{t+1}^{n,2} (B_t + E_t) \right], \\
\tilde{K}_t^n = -(1 - \frac{1}{n}) \left[ B_t^n \tilde{M}_{t+1}^{n,1} A_t^n \right] - \frac{1}{n} \left[ (B_t + E_t)\top \tilde{M}_{t+1}^{n,2} A_t^n \right] + \frac{1}{n} \left[ (B_t + E_t)\top \tilde{M}_{t+1}^{n,2} (A_t + D_t) \right]. \tag{13} \right.
\]

In addition, for any \( t \in \mathbb{N}_T \), define matrices \( \tilde{M}_t^n \) backward in time as follows:

\[
\tilde{M}_t^n := G_t^i + A_t^n \tilde{M}_{t+1}^{n,1} A_t^n + (\tilde{F}_t^n)\top B_t^n \tilde{M}_{t+1}^{n,1} A_t^n + A_t^n \tilde{M}_{t+1}^{n,1} B_t^n (\tilde{F}_t^n)\top \tilde{K}_t^n + (\tilde{F}_t^n)\top \tilde{K}_t^n, \\
\tilde{M}_{T+1}^{n} = \mathbf{0}_{2d_x \times 2d_x}. \tag{14} \right.
\]

**Remark 2.** It is to be noted that equations (12) and (14) are symmetric, yet non-standard, Riccati equations that always admit a solution if matrices \( F_t^n \) and \( F_t^n \) are invertible.
Let scalars $\ell_{i,t}^{n}$ be defined backward in time for any $t \in \mathbb{N}_T$ and $i \in \mathbb{N}_n$:

$$
\ell_{i,t}^{n} := \ell_{i,t+1}^{n} + \text{Tr} \left( E \left[ \tilde{w}_i(T) \tilde{w}_i(T) \right] M_{t+1}^{n} \right) + \frac{1}{n} \sum_{i=1}^{n} \text{Tr} \left( E \left[ \tilde{w}_i(T) \tilde{w}_i(T) \right] M_{t+1}^{n} \right) - \frac{1}{n-1} \text{Tr} \left( E \left[ \tilde{w}_i(T) \tilde{w}_i(T) \right] M_{t+1}^{n} \right),
$$

$$
\ell_{i,t+1}^{n} = 0.
$$

We now introduce a necessary and sufficient condition for the existence of a unique subgame perfect Nash solution, which can be easily verified by recursively using equations (12) and (13).

**Assumption 1.** For any $t \in \mathbb{N}_T$, matrix $(1 - \frac{1}{n}) \tilde{F}_t^{n} + \frac{1}{n} \tilde{F}_t^{n}$ is positive definite, and matrices $\tilde{F}_t^{n}$ and $\tilde{E}_t^{n}$ are invertible.

The above assumption is not restrictive, as it will be shown in Sections V and VII that it holds under standard sufficient conditions for mean-field-type control and mean-field games.

**Theorem 1.** Let Assumption 7 hold. There exists a unique subgame Nash equilibrium for Problem 7 such that for any player $i \in \mathbb{N}_n$ at any time $t \in \mathbb{N}_T$,

$$
u_{i,t} = (\tilde{F}_t^{n})^{-1} \tilde{K}_t^{n,i} + (\tilde{F}_t^{n})^{-1} \tilde{K}_t^{n} \tilde{x}_t,
$$

where the above gains are obtained from (13). In addition, the optimal cost of player $i$ is given by:

$$
J_{n,i} := \text{Tr} \left( E \left[ \tilde{x}_i(T) \tilde{x}_i(T) \right] \tilde{M}_t^{n} \right) + \ell_{i,t}^{n} + \frac{1}{n} \sum_{i=1}^{n} \text{Tr} \left( E \left[ \tilde{x}_i(T) \tilde{x}_i(T) \right] \tilde{M}_t^{n} \right) - \frac{1}{n-1} \text{Tr} \left( E \left[ \tilde{x}_i(T) \tilde{x}_i(T) \right] \tilde{M}_t^{n} \right).
$$

**Proof.** For any $t \in \mathbb{N}_T$, denote by $u_{i,-i}$ and $g_{i,-i}$ respectively the joint (stacked) action and control law of all players but player $i \in \mathbb{N}_n$. Fix the strategies of all players but player $i$, i.e. $g_{i,-i}$, and consider an arbitrary trajectory (sample path) $u_{-i}$. Note that when $u_{i,T}^{n}$ is fixed, the only control action in $u_{i}$ is $u_{i,t}$, $\forall t \in \mathbb{N}_T$. Hence, one can write the following dynamic program to identify the best response of player $i$:

$$
V_{i,t}^{n}(\tilde{x}_t) = \min_{u_{i,t}} \left( \mathbb{E} \left[ \mathbb{E} \left[ \mathbb{E} \left[ \tilde{v}_{i,t} \right| \tilde{x}_t \right| \tilde{x}_t \right] \right] \right| \tilde{x}_t, u_{i,t}),
$$

where $V_{i,t+1}(\tilde{x}_{t+1}) = 0$. Notice that the value function $V_{i,t}^{n}(\tilde{x}_t)$ depends on the actions of other players, i.e. $u_{i,-i}$, in general, which naturally leads to a fixed-point argument. However, we will show in the sequel that there is only one consistent solution $V_{i,t}^{n}(\tilde{x}_t)$ for player $i$ to construct a Nash equilibrium, and that solution is independent of $u_{i,-i}$. To this end, we prove by backward induction that the value function of player $i$ at time $t$ takes the following form:

$$
V_{i,t}^{n}(\tilde{x}_t) = \left[ \begin{array}{c} \tilde{x}_t^T \\ \tilde{x}_t^T \end{array} \right] \tilde{M}_t^{n} \left[ \begin{array}{c} \tilde{x}_t^T \\ \tilde{x}_t^T \end{array} \right] + \ell_{i,t}^{n} + \frac{1}{n} \sum_{j \neq i} (\tilde{x}_t^T \tilde{M}_t^{n,j}(\tilde{x}_t)) - \frac{1}{n-1} (\tilde{x}_t^T \tilde{M}_t^{n}(\tilde{x}_t)).
$$

It is straightforward to verify that (18) holds at $t = T + 1$ due to the boundary conditions. Suppose that (18) holds at $t + 1$:

$$
V_{i,t+1}(\tilde{x}_{t+1}) = \left[ \begin{array}{c} \tilde{x}_{t+1}^T \\ \tilde{x}_{t+1}^T \end{array} \right] \tilde{M}_{t+1}^{n} \left[ \begin{array}{c} \tilde{x}_{t+1}^T \\ \tilde{x}_{t+1}^T \end{array} \right] + \ell_{i,t+1}^{n} + \frac{1}{n} \sum_{j \neq i} (\tilde{x}_{t+1}^T \tilde{M}_{t+1}^{n,j}(\tilde{x}_{t+1})) - \frac{1}{n-1} (\tilde{x}_{t+1}^T \tilde{M}_{t+1}(\tilde{x}_{t+1})).
$$

It is desired now to show that (18) holds at time $t$ as well. From (11), (17), and (19), it follows that:

$$
V_{i,t}^{n}(\tilde{x}_t) = \min_{u_{i,t}} \left( \mathbb{E} \left[ \mathbb{E} \left[ \mathbb{E} \left[ \tilde{v}_{i,t} \right| \tilde{x}_t \right| \tilde{x}_t \right] \right] \right| \tilde{x}_t, u_{i,t}) + \frac{1}{n} \sum_{j \neq i} (\tilde{x}_{t}^T \tilde{M}_{t}^{n,j}(\tilde{x}_{t})) - \frac{1}{n-1} (\tilde{x}_{t}^T \tilde{M}_{t}(\tilde{x}_{t})).
$$

Incorporating (8) and (10) in the above equation yields:

$$
V_{i,t}^{n}(\tilde{x}_t) = \min_{u_{i,t}} \left( \mathbb{E} \left[ \mathbb{E} \left[ \mathbb{E} \left[ \tilde{v}_{i,t} \right| \tilde{x}_t \right| \tilde{x}_t \right] \right] \right| \tilde{x}_t, u_{i,t}) + \frac{1}{n} \sum_{j \neq i} (\tilde{x}_{t}^T \tilde{M}_{t}^{n,j}(\tilde{x}_{t})) - \frac{1}{n-1} (\tilde{x}_{t}^T \tilde{M}_{t}(\tilde{x}_{t})).
$$

In order to find a minimizer $u_{i,t}$, one can differentiate with respect to $u_{i,t}$ and set the derivative to zero. From (7) and after eliminating the terms depending on local noises (since they are independent from actions and have zero mean), one arrives at:

$$
\left[ \begin{array}{c} \frac{1}{n} \sum_{j \neq i} (\tilde{x}_{t}^T \tilde{M}_{t}^{n,j}(\tilde{x}_{t})) - \frac{1}{n-1} (\tilde{x}_{t}^T \tilde{M}_{t}(\tilde{x}_{t})) \\ \frac{1}{n} \sum_{j \neq i} (\tilde{x}_{t}^T \tilde{M}_{t}^{n,j}(\tilde{x}_{t})) - \frac{1}{n-1} (\tilde{x}_{t}^T \tilde{M}_{t}(\tilde{x}_{t})) \end{array} \right] = \left[ \begin{array}{c} \tilde{x}_{t}^T \tilde{M}_{t}^{n} \tilde{A}_{t} \tilde{x}_{t} \\ \tilde{x}_{t}^T \tilde{M}_{t}^{n} \tilde{A}_{t} \tilde{x}_{t} \end{array} \right] + \left[ \begin{array}{c} \tilde{x}_{t}^T \tilde{M}_{t}^{n} \tilde{B}_{t} \\ \tilde{x}_{t}^T \tilde{M}_{t}^{n} \tilde{B}_{t} \end{array} \right].
$$

In conclusion, we have that $u_{i,t}$ is a Nash equilibrium.
Due to the linear dependence introduced by the gauge transformation, i.e., \( \sum_{i=1}^{n} x^i_t = 0_{d_x \times 1} \) and \( \sum_{i=1}^{n} \bar{u}^i_t = 0_{d_u \times 1} \), equation (21) is simplified as:

\[
\begin{bmatrix}
1 - \frac{1}{n} \n
\end{bmatrix}^T (\bar{R}_t + \bar{B}_t^T \bar{M}_{t+1}^n \bar{B}_t) \begin{bmatrix}
\bar{u}^i_t \\
\bar{u}_i_t
\end{bmatrix} = - \begin{bmatrix}
1 - \frac{1}{n} \n
\end{bmatrix}^T A_t^n \bar{M}_{t+1}^n \bar{B}_t \begin{bmatrix}
x^i_t \\
\bar{x}_t
\end{bmatrix}. \tag{22}
\]

Equation (22) can be rewritten in terms of \( \bar{F}_t^n, \bar{F}_t^n, \bar{K}_t^n \) and \( \bar{K}_t^n \), given by (13), as follows:

\[
\bar{F}_t^n \bar{u}^i_t + \bar{F}_t^n \bar{u}_i_t = \bar{K}_t^n \bar{x}_t + \bar{K}_t^n \bar{x}_t. \tag{23}
\]

It is important to notice that equation (23) holds irrespective of the actions of other players, and it admits only one solution across all players under Assumption 1. More precisely, by averaging (23) over all players and upon noting that \( \sum_{i=1}^{n} x^i_t = 0_{d_x \times 1} \) and \( \sum_{i=1}^{n} \bar{u}^i_t = 0_{d_u \times 1} \), the following equalities are obtained:

\[
\bar{u}^i_t = (\bar{F}_t^n)^{-1} \bar{K}_t^n \bar{x}_t, \quad \bar{u}_t = (\bar{F}_t^n)^{-1} \bar{K}_t^n \bar{x}_t. \tag{24}
\]

What is left to show is that the above solution is a minimizer regardless of the actions of other players. To do this, equation (23) can be expressed in terms of the actions of other players as follows:

\[
n(1 - \frac{1}{n}) \bar{F}_t^n + \frac{1}{n} \bar{F}_t^n u^i_t + (- \frac{1}{n} \bar{F}_t^n + \frac{1}{n} \bar{F}_t^n) \sum_{j \neq i} u^j_t =
\]

\[
((1 - \frac{1}{n}) \bar{K}_t^n + \frac{1}{n} \bar{K}_t^n) x^i_t + (- \frac{1}{n} \bar{K}_t^n + \frac{1}{n} \bar{K}_t^n) \sum_{j \neq i} x^j_t. \tag{25}
\]

Since the control laws of all other players, i.e. \( g_i^{-1} \), at time \( t \) are fixed and the action \( u^i_t \) has no effect on its past (i.e. \( x_1:t \)), one can conclude that \( \sum_{j \neq i} u_j^i = \sum_{j \neq i} u^j_t(x_{1:t}) \) is independent of action \( u^i_t \). Therefore, action \( u^i_t \) in equation (25) is the unique minimizer and the best-response action of player \( i \) due to the fact that matrix \( \left( \frac{1}{n} \bar{F}_t^n + \frac{1}{n} \bar{F}_t^n \right) \) is positive definite according to Assumption 1 guaranteeing strict convexity.

The last step in the induction is to incorporate (24) into (20) in order to retrieve (18). The proof follows from (24) and the fact that \( x^i_t = \bar{x}^i_t + \bar{x}_t \) and \( u^i_t = \bar{u}^i_t + \bar{u}_t \).

B. Solution of Problem 2

**Corollary 1.** Let Assumption 1 hold. There exists a unique subgame perfect Nash equilibrium for Problem 2 such that for any player \( i \in \mathbb{N}_n \) at time \( t \in \mathbb{N}_T \):

\[
u^i_t = (\bar{F}_t^n)^{-1} \bar{K}_t^n x^i_t + ((\bar{F}_t^n)^{-1} \bar{K}_t^n - (\bar{F}_t^n)^{-1} \bar{K}_t^n) \bar{x}_t,
\]

where the gain matrices in the above equation can be calculated from (13).

**Proof.** Since the solution of Problem 1 is unique under Assumption 1 and is implementable under the AGS information structure, it implies that the solution of Problem 2 must coincide with that of Problem 1 because \( G_{AGS} \subseteq G_{PS} \).

**Remark 3.** Theorem 1 and Corollary 1 hold even if the initial states and local noises are non-exchangeable, non-Gaussian and fully correlated across players. This is an immediate consequence of the proof of Theorem 1 which shows that the subgame perfect equilibrium does not depend on the probability distribution of the initial states and driving noises as long as they are white (i.e., independent over time). In addition, it is to be noted that the computational complexity of the solution does not depend on the number of players because the dimension of the non-standard Riccati equation (12) is independent of the number of players.

According to the Nash equilibrium presented in Corollary 1 at every stage of the game, players make their decisions based on their current local states (private information) and the aggregate state (public information), irrespective of what has happened in the past, i.e., the players are not penalized or rewarded for their past actions. However, this does not generally hold for NS information structure because players need to construct an error-prone prediction of the unobserved aggregate state; see Section V for more details.

C. Informationally non-unique solutions

For stochastic models, the subgame perfect Nash equilibrium presented in Theorem 1 and Corollary 1 is also the unique Nash solution, according to [5] Theorem 5. In addition, it is the unique open-loop Nash equilibrium for deterministic models, according to [8, Theorem 3]; however, it is not necessarily the only Nash equilibrium. In particular, the stochastic game does not admit a Nash solution if Assumption 1 is violated while the deterministic game may still admit a Nash solution. To demonstrate this, we borrow some remarks from [5] to explain the non-uniqueness of the solution, also called *informationally non-unique* solutions in [33]. Consider a two-player game (i.e., \( n = 2 \)), where the Nash strategy is linear in the states of players. When there is no uncertainty (deterministic case), the state of player 1 can be represented by the previous states of players 1 and 2 in many different ways. Each representation may lead to a different optimization problem for player 2, resulting in a distinct solution. On the other hand, when the model is stochastic, there is only one unique representation of the solution, which is the closed-loop memoryless representation in Theorem 1 and Corollary 1.

The above informational non-uniqueness feature is useful in explaining some of the non-uniqueness results in the infinite-population game. Thus, because of the asymptotically negligible effect of individuals, one can express the infinite-population game as a two-player game between a generic player and an aggregate massive player. Since the dynamics of the mass is deterministic, its state (the mean-field) can be represented in different ways based on its previous states, where each representation may lead to a different best-response equation at the generic player, resulting in non-unique solutions. Consequently, the infinite-population game can have a unique sequential equilibrium but admit uncountably many non-sequential ones, or have no sequential equilibrium but still admit a Nash equilibrium. An immediate implication of the above discussion is that mean-field equilibrium does not necessarily coincide with the subgame perfect Nash equilibrium. For classical models with a tracking cost function, however, it can
be shown that Assumption ¹ holds, implying that the classical mean-field equilibrium coincides with the subgame perfect Nash equilibrium; see Subsection VII-A for more details.

IV. MAIN RESULTS FOR PROBLEM ³

So far, we have assumed that the aggregate state can be shared among players; however, this is not always feasible, specially when the number of players is large. In this case, the dynamic programming decomposition proposed in the previous section does not work because the optimization problem from each player’s point of view is no longer Markovian. In other words, players do not have access to a sufficient statistic of the past history of states and actions of all players. To this end, we propose two approximate NS solutions for Problem ³ and to determine the quality of each approximation, we define a measure, called price of information, in a slightly different manner than [34].

Definition 2 (Price of information). The price of information of an NS strategy $\tilde{g}$ is defined as the optimality gap given by the left-hand side of inequality (5).

To distinguish between games under AGS and NS information structures, let $y_i^t \in \mathbb{R}^{d_x}$ and $v_i^t \in \mathbb{R}^{d_v}$, respectively, denote the state and action of player $i \in \mathbb{N}_n$ at time $t \in \mathbb{N}_T$ under NS information structure. Denote by $z_i^n := \mathbb{E}[x_i]$, the expectation of the aggregate state $\bar{x}_i$ under AGS structure. It evolves deterministically in time under the linear strategy (15) as follows:

$$z_{i+1}^n = (A_i + D_i + (B_i + E_i)(\tilde{F}_i^t)^{-1}\tilde{K}_i^t)z_i^n.$$ 

Let $\tilde{y}_t := \frac{1}{n}\sum_{i=1}^n y_i^t$ and $\tilde{v}_t := \frac{1}{n}\sum_{i=1}^n v_i^t$, $t \in \mathbb{N}_T$. Let also the initial state of player $i$ be $y_1^t = x_1^t$. At time $t \in \mathbb{N}_T$, the state of player $i$ evolves according to (2), as follows:

$$y_{i+1}^t = A_i y_i^t + B_i v_i^t + D_i \tilde{y}_t + E_i \tilde{v}_t + u_i^t,$$  

where the proposed NS control action of player $i$ is given by:

$$v_i^t = (\tilde{F}_i^t)^{-1}\tilde{K}_i^t y_i^t + ((\tilde{F}_i^t)^{-1}\tilde{K}_i^t - (\tilde{F}_i^t)^{-1}\tilde{K}_i^t)z_i^n.$$  

Similarly to ³, define the following variables for every player $i \in \mathbb{N}_n$ at time $t \in \mathbb{N}_T$,

$$\bar{y}_t := y_t - \bar{y}_t, \quad \bar{v}_t := v_t - \bar{v}_t.$$

Lemma 1. Let Assumption ¹ hold. At any $t \in \mathbb{N}_T$, $\bar{x}_t = \bar{y}_t$ and $\bar{u}_t = \bar{v}_t$.

Proof. The proof follows from backward induction. Initially, $\bar{x}_1 = \bar{y}_1 = \bar{y}_1$ and $\bar{u}_1 = (\tilde{F}_1^t)^{-1}\tilde{K}_1\bar{y}_1 = \bar{v}_1$. Suppose $\bar{x}_t = \bar{y}_t$ and $\bar{u}_t = \bar{v}_t$ at time $t$. Then, it results from (2) and (26) that:

$$\bar{x}_{t+1} = A_t \bar{x}_t + B_t \bar{u}_t + \bar{w}_t = A_t \bar{y}_t + B_t \bar{v}_t + \bar{w}_t = \bar{y}_{t+1}.$$  

In addition, from (15), (27) and (28), one arrives at:

$$\bar{u}_{t+1} = (\tilde{F}_t^t)^{-1}\tilde{K}_t^t \bar{x}_{t+1} = (\tilde{F}_t^t)^{-1}\tilde{K}_t^t \bar{y}_{t+1} = \bar{v}_{t+1}.$$  

Define now the following relative errors at time $t \in \mathbb{N}_T$:

$$e_t^1 := \bar{x}_t - \bar{y}_t, \quad e_t := \bar{y}_t - z_t^n, \quad \zeta_t := \bar{x}_t - z_t^n.$$  

Lemma 2. Let Assumption ¹ hold. For every player $i \in \mathbb{N}_n$, the relative errors defined in (29) evolve linearly in time, i.e.,

$$\begin{bmatrix} e_{t+1}^i \\ e_{t+1} \\ \zeta_{t+1} \\ \bar{w}_t \\
\end{bmatrix} = \tilde{A}_t^i \begin{bmatrix} e_{t}^i \\ e_{t} \\ \zeta_{t} \\ \bar{w}_t \\
\end{bmatrix} + \begin{bmatrix} \bar{w}_t \\
\end{bmatrix},$$  

where $\tilde{A}_t^i := \text{diag}(A_i + B_i(\tilde{F}_i^t)^{-1}\tilde{K}_i^t, A_i + D_i + (B_i + E_i)(\tilde{F}_i^t)^{-1}\tilde{K}_i^t).$

Proof. The proof follows from (2), (3), (15), (27) and (29).

Lemma 3. Let Assumption ¹ hold. Then:

$$\mathbb{E}[e_{t+1}^i] = \mathbb{E}[\zeta_{t+1}] = \mathbb{E}[\bar{w}_t] = 0_{d_x \times 1}, \quad \forall t \in \mathbb{N}_T.$$  

Proof. At time $t = 1$, $\bar{x}_1 = \bar{y}_1$; therefore, $\mathbb{E}[e_{1+1}^i] = \mathbb{E}[\zeta_{1+1}] = \mathbb{E}[\bar{w}_t] = 0_{d_x \times 1}$. For any time $t > 1$, the lemma holds because $\mathbb{E}[\bar{w}_t] = 0_{d_x \times 1}$ and the dynamics of the relative errors are linear, according to Lemma 2.

Let $g^*$ and $\tilde{g}$ denote the strategies proposed in (15) and (27), respectively. For any player $i$, define $\Delta J_n^i$ as the price of information (optimality gap) as follows:

$$\Delta J_n^i := J_n^i(\tilde{g}) - J_n^i(g^*), \quad \forall i \in \mathbb{N}_n.$$  

Lemma 4. Let Assumption ¹ hold. For any player $i \in \mathbb{N}_n$,

$$\Delta J_n^i = \mathbb{E}\left[\sum_{t=1}^T [e_t^i \cdot e_t + \zeta_t^i \cdot \bar{Q}_t^i[e_t^i \cdot e_t + \zeta_t^i]]\right],$$  

where matrix $\bar{Q}_t^i, t \in \mathbb{N}_T$, is given by:

$$\bar{Q}_t^{1,1} := 0_{d_x \times d_x}, \quad \bar{Q}_t^{1,2} := \bar{Q}_t^{1,3} := \bar{Q}_t^{2,1} := \bar{Q}_t^{2,2} := \bar{Q}_t^{2,3} := 0_{d_x \times d_x},$$  

$$\bar{Q}_t^{1,1} := \bar{Q}_t^{1,2} := \bar{Q}_t^{1,3} := \bar{Q}_t^{2,1} := \bar{Q}_t^{2,2} := \bar{Q}_t^{2,3} := \bar{Q}_t^{3,2} := \bar{Q}_t^{3,3} := 0_{d_x \times d_x},$$  

where $\bar{Q}_t^i$ is given by:

$$\begin{bmatrix} \text{Cov}(x_t^1 - \bar{x}_t) & \mathbb{E}[(x_t^1 - \bar{x}_t)x_t^1] & \mathbb{E}[(x_t^1 - \bar{x}_t)x_t^1] \\
\mathbb{E}[(x_t^1 - \bar{x}_t)x_t^1] & \text{Cov}(\bar{x}_1) & \mathbb{E}[(\bar{x}_1 - \bar{w}_1)x_1] \\
\mathbb{E}[(x_t^1 - \bar{x}_t)x_t^1] & \mathbb{E}[(\bar{x}_1 - \bar{w}_1)x_1] & \text{Cov}(\bar{x}_1) \end{bmatrix}.$$  

Proof. The proof is presented in Appendix A.

Denote by $H^x$ and $H^w$, $t \in \mathbb{N}_T$, the following covariance matrices:

$$H^x_t = \begin{bmatrix} \text{Cov}(x_t^1 - \bar{x}_1) & \mathbb{E}[(x_t^1 - \bar{x}_1)x_t^1] & \mathbb{E}[(x_t^1 - \bar{x}_1)x_t^1] \\
\mathbb{E}[(x_t^1 - \bar{x}_1)x_t^1] & \text{Cov}(\bar{x}_1) & \mathbb{E}[(\bar{x}_1 - \bar{w}_1)x_1] \\
\mathbb{E}[(x_t^1 - \bar{x}_1)x_t^1] & \mathbb{E}[(\bar{x}_1 - \bar{w}_1)x_1] & \text{Cov}(\bar{x}_1) \end{bmatrix},$$  

$$H^w_t = \begin{bmatrix} \text{Cov}(w_t^1 - \bar{w}_1) & \mathbb{E}[(w_t^1 - \bar{w}_1)\bar{w}_1^1] & \mathbb{E}[(w_t^1 - \bar{w}_1)\bar{w}_1^1] \\
\mathbb{E}[(w_t^1 - \bar{w}_1)\bar{w}_1^1] & \text{Cov}(\bar{w}_1) & \mathbb{E}[(\bar{w}_1 - \bar{w}_1)\bar{w}_1^1] \\
\mathbb{E}[(w_t^1 - \bar{w}_1)\bar{w}_1^1] & \mathbb{E}[(\bar{w}_1 - \bar{w}_1)\bar{w}_1^1] & \text{Cov}(\bar{w}_1) \end{bmatrix}.$$  

(31)
**Theorem 2.** Let Assumption 1 hold. The price of information of the NS strategy (27) can be described by a Lyapunov equation as follows:

\[ \Delta J^n_t = \text{Tr}(H^n \tilde{M}^n_t) + \sum_{i=1}^{T-1} \text{Tr}(H^n_i \tilde{M}^n_{t+i}), \]

(32)

where \( \tilde{M}^n_{T+1} = 0_{d_x \times d_x} \) and \( \tilde{M}^n_t \) is the solution of the following Lyapunov equation:

\[ \tilde{M}^n_t = \tilde{A}^n_t \tilde{M}^n_{t+1} \tilde{A}^n_t + \tilde{Q}^n_t, \quad \forall t \in \mathbb{N}_T. \]

(33)

**Proof.** From Lemma 3, \( \Delta J^n_t \) is a quadratic function of the relative errors, and from Lemma 2, the relative errors have linear dynamics. Therefore, this may be viewed as an uncontrollable linear quadratic system where the total expected cost can be expressed in terms of the covariance matrices of the initial states and local noises (i.e., \( H^n_x \) and \( H^n_w \)), and the Lyapunov equation (33). \( \blacksquare \)

**Lemma 5.** At any time \( t \in \mathbb{N}_T \), \( \tilde{M}^{1,1}_t = 0_{d_x \times d_x} \).

**Proof.** From Lemma 2 and Theorem 2, it follows that Lemma 5 holds at the terminal time, i.e., \( \tilde{M}^{1,1}_T = \tilde{Q}^{1,1}_T = 0_{d_x \times d_x} \). It can be inductively shown that Lemma 5 holds at time \( t \in \mathbb{N}_T \), according to equation (33) and Lemma 2. \( \tilde{M}^{1,1}_t = (\tilde{A}^{n,1}_t)^\dagger \tilde{M}^{n,1,1}_t \tilde{A}^{1,1}_t + \tilde{Q}^{1,1}_t = 0_{d_x \times d_x} \). \( \blacksquare \)

**A. Finite-model solution for Problem 3**

It is desired now to show that the proposed strategy, given by (27), is a solution for Problem 3. The following assumptions are imposed on the model.

**Assumption 2.** The initial local states \( (x^n_1, \ldots, x^n_T) \) are independent random variables with identical mean \( \mu_x \in \mathbb{R}^{d_x} \) (that is bounded and independent of \( n \)). In addition, local noises \( (w^n_1, \ldots, w^n_T) \), \( t \in \mathbb{N}_T \), are independent random variables. Furthermore, covariance matrices \( \text{Cov}(x^n_i) \) and \( \text{Cov}(w^n_i) \) are bounded and independent of \( n \) for any \( i \in \mathbb{N}_n \) and \( t \in \mathbb{N}_T \).

The initial states as well as local noises are not necessarily identically distributed.

**Assumption 3.** There exist matrices \( \tilde{A}_t \) and \( \tilde{Q}_t \) such that for every \( n \geq n_0 \), \( n_0 \in \mathbb{N} \), matrices \( \tilde{A}^n_t \) and \( \tilde{Q}^n_t \) are uniformly bounded with respect to \( n \), i.e., \( \tilde{A}^n_t \leq \tilde{A}_t \) and \( \tilde{Q}^n_t \leq \tilde{Q}_t \), where \( \leq \) denotes element-wise inequality.

**Theorem 3.** Let Assumptions 1 and 2 hold for any \( n \geq n_0, n_0 \in \mathbb{N} \). The price of information \( \Delta J^n_t \), given by (30), converges to zero at the rate \( 1/n \) for every player \( i \in \mathbb{N}_n \), i.e.,

\[ \Delta J^n_t \in O\left(\frac{1}{n}\right), \quad \forall n \geq n_0. \]

Consequently, NS strategy (27) constitutes one solution for Problem 3.

**Proof.** From Assumption 2, one has:

\[ \text{Cov}(\tilde{x}_1) = \frac{1}{n^2} \sum_{i=1}^{n} \text{Cov}(x_i) \leq \frac{1}{n} e^x, \]

\[ \text{Cov}(\tilde{w}_t) = \frac{1}{n^2} \sum_{i=1}^{n} \text{Cov}(w_i^t) \leq \frac{1}{n} e^w, \]

(34)

where \( e^x \) and \( e^w \), \( t \in \mathbb{N}_T \), are some upper bounds on the covariance matrices of the initial states and local noises (that do not depend on \( n \)). In addition, for any \( i \in \mathbb{N}_n \),

\[ E[\tilde{x}_1^i - \tilde{x}_1] = E[\tilde{x}_1^i \tilde{x}_1^T] - E[\tilde{x}_1 \tilde{x}_1^T] \]

\[ = \frac{1}{n} E[x_1^i (x_1^i)^\dagger] + \frac{n-1}{n} \mu_x \mu_x^\dagger - \text{Cov}(\tilde{x}_1) - \mu_x \mu_x^\dagger \]

\[ = \frac{1}{n} \text{Cov}(x_1^i) - \text{Cov}(\tilde{x}_1). \]

Similarly, at any \( t \in \mathbb{N}_T \),

\[ E[(\tilde{w}_t - \tilde{w}_t)^T] = \frac{1}{n} \text{Cov}(w_t^i) - \text{Cov}(\tilde{w}_t). \]

(35)

It is implied from (34)–(35) that all block matrices of the covariance matrices \( H^n_x \) and \( H^n_w \), except the ones in the first row and first column, decay to zero at the rate \( 1/n \). On the other hand, the block matrices in the first row and first column of matrices \( H^n_x \) and \( H^n_w \) have no effect on \( \Delta J^n_t \), according to equation (32) and Lemma 5. Consequently, one can conclude that \( \Delta J^n_t \), described by (32), converges to zero at the rate \( 1/n \) because matrices \( \tilde{M}^{1,1}_t \) are uniformly bounded with respect to \( n \) according to Assumption 3 and equation (33). \( \blacksquare \)

**B. Infinite-model solution for Problem 3**

In general, the proposed NS strategy (27) depends on the number of players \( n \). Alternatively, one can replace matrices \( F^n_i, F^n_i, \tilde{K}^n_i, \tilde{K}^n_i \), \( t \in \mathbb{N}_T \), in the strategy with their infinite-population limits (that do not depend on \( n \)) and obtain an NS strategy that is independent of \( n \). To this end, we impose the following assumption on the model.

**Assumption 4.** All matrices in the player dynamics (2) and per-step cost (3) are independent of the number of players \( n \).

Let \( \tilde{F}^\infty_t, \tilde{F}^\infty_t, \tilde{K}^\infty_t \) and \( \tilde{K}^\infty_t \), \( t \in \mathbb{N}_T \), denote the matrices given by (13), when \( n \) is set to \( \infty \). Denote by \( z^\infty_t = E[\tilde{x}_t], t \in \mathbb{N}_T \), the expectation of the aggregate state \( \tilde{x}_t \) of the infinite population evolving linearly as follows:

\[ \tilde{z}^{\infty}_{t+1} = (\tilde{A}_t + D_t + (B_t + E_t)(\tilde{F}^\infty_t)^{-1}\tilde{K}^\infty_t)z^\infty_t \]

Then, the NS control action of player \( i \in \mathbb{N}_n \) in the finitestate population game is provided by:

\[ v^i_t = (\tilde{F}^\infty_t)^{-1}\tilde{K}^\infty_t y^i_t + ((\tilde{F}^\infty_t)^{-1}\tilde{K}^\infty_t - (\tilde{F}^\infty_t)^{-1}\tilde{K}^\infty_t)z^\infty_t. \]

(36)

**Proposition 1.** Let Assumption 1 hold for the infinite-population model, i.e., when \( n \) is set to \( \infty \). Under Assumptions 2 and 4, the performance of the NS strategy (36), denoted by \( g^\infty \), converges to that of the PS strategy (15) as the number of players tends to infinity, i.e.,

\[ \lim_{n \to \infty} \left| J^i_n(\tilde{g}^\infty) - J^i_n(\hat{g}^\infty) \right| = 0, \quad i \in \mathbb{N}_n. \]
Proof. The proof follows from the fact that $J^i_n(\bar{g}_\infty)$ is a bounded and continuous function with respect to $n$ and strategy $36$ coincides with the PS strategy in the infinite-population model, due to the strong law of large numbers under Assumption 2. In particular, $J^i_n(\bar{g}_\infty)$ can be described by a polynomial regression in terms of $\mu_x$, $\text{Cov}(x_1^i)$ and $\text{Cov}(w_i)$, $t \in \mathbb{N}$, because: (a) the state dynamics are linear under strategy $\bar{g}_\infty$; (b) local noises are zero-mean; (c) per-step cost $f^i$ is a quadratic function of the states under strategy $\bar{g}_\infty$ and (d) the initial states as well as local noises are independent, meaning that $\text{Cov}(\bar{x}_1) = \frac{1}{n^2} \sum_{i=1}^n \text{Cov}(x_1^i)$ and $\text{Cov}(\bar{w}_i) = \frac{1}{n} \sum_{i=1}^n \text{Cov}(w_i)$. Therefore, $J^i_n(\bar{g}_\infty)$ is bounded and continuous with respect to $n$ since the resultant weighing matrices as well as covariance matrices are all bounded and continuous with respect to $n$, in view of Assumptions 2 and 4.

From Proposition 1 one can establish a bound for the price of information of the NS strategy $36$ as follows.

**Theorem 4.** Let Assumptions 1, 2 hold for any $n \geq n_0$, $n_0 \in \mathbb{N}$. The price of information of the NS strategy $36$, denoted by $\bar{g}_\infty$, converges to zero as the number of players tends to infinity, i.e.,

$$\lim_{n \to \infty} |J^i_n(\bar{g}_\infty) - J^i_n(\bar{g}_\infty)| = 0, \quad i \in \mathbb{N}.$$  \hspace{1cm} (37)

Subsequently, strategy $36$ is a solution of Problem 2.

Proof. From the triangle inequality, it follows that:

$$|J^i_n(\bar{g}_\infty) - J^i_n(\bar{g}_\infty)| \leq |J^i_n(\bar{g}_\infty) - J^i_n(\bar{g}_\infty)| + |J^i_n(\bar{g}_\infty) - J^i_n(\bar{g}_\infty)|.$$  \hspace{1cm} (37)

The first term of the right-hand side of (37) converges to zero as $n$ goes to infinity because $J^i_n$ is a bounded and continuous function with respect to $n$ according to (16) under Assumptions 1, 2. More precisely, it follows from Assumptions 1, 2 and 4 that matrices $(\hat{F}^n)^{-1}K^n$ and $(\hat{F}^n)^{-1}K^n$ are uniformly bounded with respect to $n$ at any time $t \in \mathbb{N}$. Hence, matrices $\hat{M}^n_t$ and $\hat{M}^n_t$, given by (12) and (14), are also uniformly bounded and continuous with respect to $n$. Consequently, $J^i_n$ in (16) is a bounded and continuous function with respect to $n$ under Assumption 2, i.e. $\lim_{n \to \infty} J^i_n = J^i_\infty = J^i(\bar{g}_\infty)$. The second term follows from Proposition 1.

**Remark 4.** Although NS strategies $27$ and $36$ converge to the same (unique) Nash solution as $n \to \infty$, they have subtle differences. More precisely, strategy $27$ takes the number of players into account, potentially leading to a smaller price of information compared to that of strategy $36$. For example when the game is deterministic and $\bar{z}_1$ is known, strategy $27$ becomes the optimal Nash solution while strategy $36$ remains an approximate Nash solution, in general. On the other hand, strategy $36$ is computed based on the infinite-population model, which simplifies the analysis, as $n$ is set to infinity.

Note that NS strategies $27$ and $36$ are not in the form of forward-backward equations and their numerical computation requires no fixed-point condition; their complexity increases only linearly with respect to the game horizon $T$.

**V. Infinite Horizon**

In this section, we extend our main results to the infinite horizon case. To this end, it is assumed that the model described in Section 1 is time-homogeneous, and hence, substep $t$ is dropped from the model parameters. Given any discount factor $\beta \in (0, 1)$, define the following discounted cost function for player $i \in \mathbb{N}$:

$$J^{i,\beta}_n(g) = (1 - \beta)\mathbb{E}\left[\sum_{t=1}^\infty \beta^{t-1} c^i(x_t, u_t)\right],$$

where the per-step cost function is given by (1). In addition, define the following time-averaged cost function:

$$J^{i,\beta}_n(g) = \lim_{T \to \infty} \frac{1}{T} \mathbb{E}\left[\sum_{t=1}^T c^i(x_t, u_t)\right].$$

By a simple change of variables $\hat{x}_t := \sqrt{\beta}x_t^i$ and $\hat{u}_t := \sqrt{\beta}u_t^i$, $i \in \mathbb{N}$, $t \in \mathbb{N}$, any discounted cost function can be formulated as an undiscounted cost function, where matrices in player dynamics (2) are modified as follows:

$$\hat{A} := \sqrt{\beta}A, \quad \hat{B} := \sqrt{\beta}B, \quad \hat{D} := \sqrt{\beta}D, \quad \hat{E} := \sqrt{\beta}E.$$  \hspace{1cm} (38)

In the sequel, we focus our attention on the time-averaged cost function for simplicity of exposition; however, the results obtained can be easily extended to discounted cost functions by the resultant change of system parameters (38), similar to Corollary 2.

In general, it is not straightforward to derive conditions under which the solutions of equations (12) and (14) converge to a unique bounded limit as $T \to \infty$. In Assumption 5 we consider any such condition that induces this property, and in Sections VI and VII we provide two sufficient conditions under which Assumption 5 is satisfied.

**Assumption 5.** The backward ordinary difference equations (12) and (14) admit unique bounded solutions as the horizon $T$ goes to infinity. More precisely, $\lim_{T \to \infty} M^n_t =: M^n$ and $\lim_{T \to \infty} \hat{M}^n_t =: \hat{M}^n$, where $M^n$ and $\hat{M}^n$ are the solutions of the algebraic form of equations (12) and (14), respectively.

**A. Solutions of Problems 1 and 2**

**Theorem 5.** Let Assumptions 1, 5 hold. There exists a stationary subgame perfect Nash equilibrium for Problems 1 and 2 with a time-averaged cost function such that for any player $i \in \mathbb{N}$ at time $t \in \mathbb{N}$,

$$u_t^i = (\hat{F}^n)^{-1}K^n x_t^i + ((\hat{F}^n)^{-1}K^n - (\hat{F}^n)^{-1}K^n)x_1^i,$$  \hspace{1cm} (39)

where gain matrices $\hat{F}^n$, $\hat{F}^n$, $\hat{K}^n$ and $\hat{K}^n$ in the above equation are obtained from the algebraic form of equations (12) and (13), where the subscript $t$ is omitted.

Proof. The proof follows from Theorem 1 and Corollary 1 and Assumptions 1, 5. Denote by $\Sigma_i^1 := E[\bar{w}_t^i(\bar{w}_t^i)^\top]$ and $\Sigma_i^2 := E[\bar{u}_t^i(\bar{u}_t^i)^\top]$, $t \in \mathbb{N}$, the time-homogeneous covariance matrices associated with player
Therefore, the optimal cost is given by:
\[
V_i^i(x_1) = \left[ \frac{\hat{x}_1}{1} \right]^T M_{n}^{\bar{n}} \left[ \frac{\hat{x}_1}{1} \right] + \ell_i^{1,n} + \frac{1}{n} \left( \sum_{j \neq i} \left( \frac{\hat{x}_1}{1} \right)^T M_{n}^{\bar{n}} \left( \frac{\hat{x}_1}{1} \right) - \frac{1}{n-1} \left( \frac{\hat{x}_1}{1} \right)^T M_{n}^{\bar{n}} \left( \frac{\hat{x}_1}{1} \right) \right),
\]
where
\[
\ell_i^{1,n} = \sum_{t=1}^T \text{Tr} \left( \Sigma^i M_{n}^{t+1} \right) + \frac{1}{n} \sum_{i=1}^n \text{Tr} \left( \Sigma^i M_{n}^{t+1} \right) - \frac{1}{n-1} \text{Tr} \left( \Sigma^i M_{n}^{t+1} \right). \quad (40)
\]
Define \( \Phi^{\bar{n}}_t := M_{T+t-2}^{\bar{n}} \) and \( \Phi^i_t := M_{T+t-2}^{n} \) for any \( t \in \mathbb{N}_{T+1} \), where \( \Phi^{\bar{n}}_t = M_{T+t-1}^{\bar{n}} = 0 \) for all \( t \neq 1,2, \ldots, T \) and \( \Phi^i_t = M_{T+t-1}^{n} = 0 \).
Rewrite equation (40) as follows:
\[
\ell_i^{1,n} = \sum_{t=1}^T \text{Tr} \left( \Sigma^i \Phi^{\bar{n}}_t \right) + \frac{1}{n} \sum_{i=1}^n \text{Tr} \left( \Sigma^i \Phi^i_t \right) - \frac{1}{n-1} \text{Tr} \left( \Sigma^i \Phi^i_t \right).
\]
Under Assumption 5, the following equalities hold:
\[
\lim_{T \to \infty} \Phi^{\bar{n}}_{T+1} = \lim_{T \to \infty} M_{n}^{T} = \bar{M}, \quad (41)
\]
\[
\lim_{T \to \infty} \Phi^i_{T} = \lim_{T \to \infty} M_{n}^{T} = \bar{M}. \quad (42)
\]
Therefore, the optimal cost is given by:
\[
\mathbb{E}[V_i^i(x_1)] = \lim_{T \to \infty} \frac{1}{T} \left( \text{Tr} \left( \mathbb{E} \left[ \frac{\hat{x}_1}{1} \left( \frac{\hat{x}_1}{1} \right)^T \right] M_{n}^{T} \right) + \frac{1}{n} \sum_{i=1}^n \text{Tr} \left( \mathbb{E} \left[ \frac{\hat{x}_1}{1} \left( \frac{\hat{x}_1}{1} \right)^T \right] M_{n}^{T} \right) \right)
+ \frac{1}{n} \sum_{i=1}^n \text{Tr} \left( \mathbb{E} \left[ \frac{\hat{x}_1}{1} \left( \frac{\hat{x}_1}{1} \right)^T \right] M_{n}^{T} \right) - \frac{1}{n-1} \text{Tr} \left( \Sigma^i M_{n}^{T} \right).
\]
\[
= \text{Tr} \left( \Sigma^i \bar{M} \right) + \frac{1}{n} \sum_{i=1}^n \text{Tr} \left( \Sigma^i \bar{M} \right) - \frac{1}{n-1} \text{Tr} \left( \Sigma^i \bar{M} \right). \quad (43)
\]
According to (13), the control law at any time \( t \in \mathbb{N}_T \) is described by:
\[
\bar{F}_n = (1 - \frac{1}{n}) \left[ R + \frac{1}{n-1} G^n + B^n \Phi_{T-t+1}^{\bar{n}} \right] + \frac{1}{n} \left[ R + S^n + (B+E)^T \Phi_{T-t+1}^{\bar{n}} \right],
\]
\[
\tilde{F}_n = (1 - \frac{1}{n}) \left[ R + S^n + B^n \Phi_{T-t+1}^{n} \right] + \frac{1}{n} \left[ R + 2S^n + P^n + G^n + (B+E)^T \Phi_{T-t+1}^{n} \right],
\]
\[
\tilde{K}_n = -\left( 1 - \frac{1}{n} \right) \left[ B^n \Phi_{T-t+1}^{n}, 1 \right] - \frac{1}{n} \left( (B+E) \Phi_{T-t+1}^{n}, 2 \right],
\]
\[
\tilde{K}_n = -\left( 1 - \frac{1}{n} \right) \left[ B^n \Phi_{T-t+1}^{n}, 1 \right] - \frac{1}{n} \left( (B+E) \Phi_{T-t+1}^{n}, 2 \right]. \quad (44)
\]
From (41), (42) and (44), one can conclude that there exists a stationary strategy \( \bar{M} \) and \( \tilde{M} \), which is Nash for \( T \to \infty \).

### B. Finite-model solution for Problem 3

Similarly to strategy (27), define the following cost function for any player \( i \in \mathbb{N}_n \) based on the solution of Theorem 5:
\[
v_i^t = (\bar{F}^n)^{-1} \tilde{K}^n y_i^t + ((\bar{F}^n)^{-1} \tilde{K}^n - (\bar{F}^n)^{-1} \tilde{K}^n) z_i^t, \quad t \in \mathbb{N},
\]
where \( y_i^t \) is the state of player \( i \) at time \( t \), and for any \( t \in \mathbb{N},
\[
z_i^{t+1} = (A + D + (B+E)(\bar{F}^n)^{-1} \tilde{K}^n) z_i^t,
\]
with the initial condition \( z_i^0 = \mathbb{E}[^{1}x]. \)
To ensure that the propagation of the error induced by the imperfection of NS information structure is bounded, the following assumption is imposed on the model.

**Assumption 6.** Let the conditions of Assumption 5 hold, and in addition, assume matrix \( A^\bar{n} \) in Lemma 3 is Hurwitz.

Two easily verifiable sufficient conditions for the above assumption are presented in part III of Theorems 9 and 11. Let \( g^\bar{n} \) and \( g^i \) denote the strategies proposed in (39) and (45), respectively. For any \( i \in \mathbb{N}_n \), define \( \Delta J_i^{n,1} \) as the price of information for player \( i \in \mathbb{N}_n \) as follows:
\[
\Delta J_i^{n,1} := J_i^{n,1}(g^\bar{n}) - J_i^{n,1}(g^i), \quad i \in \mathbb{N}_n.
\]

**Theorem 6.** Let Assumptions 7-9 and 10 hold for any \( n \geq n_0, n_0 \in \mathbb{N} \). The price of information \( \Delta J_i^{n,1} \), given by (46), converges to zero at the rate \( 1/n \) for every \( i \in \mathbb{N}_n \), i.e.,
\[
\Delta J_i^{n,1} = \text{Tr}(H^n \tilde{M}^n) = \mathcal{O}\left( \frac{1}{n} \right), \quad \forall n \geq n_0.
\]

where \( \tilde{M}^n \) is the solution of the algebraic Lyapunov equation:
\[
\tilde{M}^n = (\tilde{A}^n)^T \tilde{M}^n \tilde{A}^n + \tilde{Q}^n, \quad \forall n \geq n_0.
\]

Consequently, NS strategy (45) is a solution for Problem 3 with time-averaged cost function.

**Proof.** The proof follows along the same lines of the proofs of Theorems 1 and 3 with \( T \to \infty \). In particular, the optimality gap between the time-averaged cost functions of strategies (39) and (45) is given by Theorem 6 as follows:
\[
\Delta J_i^{n,1} := \lim_{T \to \infty} \frac{1}{T} \left( \text{Tr}(H^n \tilde{M}^n) + \sum_{t=1}^{T-1} \text{Tr}(H^n \tilde{M}^n \tilde{F}^{n+1}_{t+1}) \right). \quad (49)
\]
From Assumption 6, the Lyapunov equation (32) converges to the algebraic Lyapunov equation (45). Therefore, one arrives at the equality relation in (47) from (48) and (49). Similarly to the proof of Theorem 3, all block matrices of the covariance matrix \( H^n \) in (31) decay to zero at the rate \( 1/n \) from (34) and (35), except the block matrix in the first row and first column. However, the effect of this non-zero block matrix is canceled out by the corresponding block matrix of \( \tilde{M}^n \), that is zero according to Lemma 5. Therefore, it follows from equation (48) and Assumption 6 that \( \Delta J_i^{n,1} \) is uniformly bounded in \( n \) due to uniform boundedness of the matrices \( \tilde{A}^n \) and \( \tilde{Q}^n \), according to Assumption 3. Hence, it results from (47) that \( \Delta J_i^{n,1} \) converges to zero at the rate \( 1/n \).
C. Infinite-model solution for Problem 3

Similarly to strategy \( (36), \) define an alternative NS strategy for any \( i \in \mathbb{N}_0 \) and \( t \in \mathbb{N} \) in the finite-population game based on the infinite-population model of Theorem 5 as follows:

\[ v_i^t = (\bar{F}^\infty)^{-1}\bar{K}^\infty y_i^t + ((\bar{F}^\infty)^{-1}\bar{K}^\infty - (\bar{F}^\infty)^{-1}\bar{K}^\infty)z_i^t, \] (50)

where \( y_i^t \) is the state of player \( i \) at time \( t \), and

\[ z_i^{t+1} = (A + D + (B + E)(\bar{F}^\infty)^{-1}\bar{K}^\infty)z_i^t, \quad \forall t \in \mathbb{N}, \] (51)

with the initial condition \( z_i^0 = E[x_1] \).

**Lemma 6.** Let Assumptions 7 and \( \mathcal{A} \) hold for every \( n \geq n_0, n_0 \in \mathbb{N} \). Under Assumptions 2 and 3 the states of players in the finite-population model under strategy \( (50) \) are bounded in time, i.e., strategy \( (50) \) is a stable strategy.

**Proof.** It follows from \( (25), (50) \) and \( (51) \) that:

\[ \begin{bmatrix} y_i^{t+1} - y_i^t \\ y_i - z_i^t \end{bmatrix} = \bar{A}^\infty \begin{bmatrix} y_i^t - y_i^t \\ y_i - z_i^t \end{bmatrix} + \begin{bmatrix} u_i^t \\ w_i \end{bmatrix}, \]

(52)

where \( \bar{A}^\infty = \text{diag}(A + B(\bar{F}^\infty)^{-1}\bar{K}^\infty, A + D + (B + E)(\bar{F}^\infty)^{-1}\bar{K}^\infty) \). Since matrix \( \bar{A}^\infty \) is assumed to be Hurwitz in Assumption 6 when \( n = \infty \), states \( y_i^t \) in \( (52) \), \( t \in \mathbb{N}, i \in \mathbb{N}_0 \), are bounded.

**Theorem 7.** Let Assumptions 3 and 4 hold for any \( n \geq n_0, n_0 \in \mathbb{N} \). The price of information of the NS strategy \( (50) \), denoted by \( \hat{g}_{\infty}^n \), converges to zero as the number of players tends to infinity, i.e.,

\[ \lim_{n \to \infty} |J_n^{1,1}(\hat{g}_{\infty}^n) - J_n^{1,1}(g_{\infty}^n)| = 0, \quad \forall i \in \mathbb{N}_n, \]

where \( J_n^{1,1}(g_{\infty}^n) \) is the optimal time-averaged performance of player \( i \) under AGS strategy \( (39) \). Subsequently, strategy \( (50) \) is a solution of Problem 3.

**Proof.** The proof follows along the same lines of the proof of Theorem 4. From the triangle inequality:

\[ |J_n^{1,1}(\hat{g}_{\infty}^n) - J_n^{1,1}(g_{\infty}^n)| \leq |J_n^{1,1}(g_{\infty}^n) - J_n^{1,1}(g_{\infty}^n)| + |J_n^{1,1}(g_{\infty}^n) - J_n^{1,1}(g_{\infty}^n)|. \] (53)

The first term in the right-hand side of \( (53) \) converges to zero as \( n \) goes to infinity because \( J_n^{1,1}(g_{\infty}^n) \), given by \( (43) \), is a bounded and continuous function with respect to \( n \) under Assumptions 3 and 4. In particular, matrices \( (\bar{F}^n)^{-1}\bar{K}^n \) and \( (\bar{F}^n)^{-1}\bar{K}^n \) are uniformly bounded with respect to \( n \); thus, matrices \( \bar{M}_n \) and \( \bar{M}_n \) are also bounded and continuous with respect to \( n \). In other words, \( J_n^{1,1}(g_{\infty}^n) \) in \( (43) \) is a bounded and continuous function with respect to \( n \) under Assumption 2, i.e., \( \lim_{n \to \infty} J_n^{1,1}(g_{\infty}^n) = J_\infty^{1,1}(g_{\infty}^n) \). To prove that the second term in the right-hand side of \( (53) \) converges to zero as well, one can show that \( J_n^{1,1}(g_{\infty}^n) \) is a bounded and continuous function of \( n \), similarly to the proof of Proposition 1 on noting that \( J_n^{1,1}(g_{\infty}^n) \) is also bounded in time for any \( n \geq n_0 \), according to Lemma 6.

**Remark 5.** Theorems 5 and 7 imply that under Assumption 6 it does not matter if \( n \) tends to infinity first or \( T \); regardless of which one approaches infinity faster, the resultant asymptotic solutions are identical.

**Corollary 2.** Given any discount factor \( \beta \in (0, 1) \), Theorems 3 and 4 hold for Problems 2 and 5 with discounted cost function, where matrices \( A, B, D \) and \( E \) are replaced by matrices \( \sqrt{\beta}A, \sqrt{\beta}B, \sqrt{\beta}D \) and \( \sqrt{\beta}E \), respectively.

**Proof.** The proof follows directly from the change of variables \( (39) \) and the proofs of Theorems 3 and 4. In particular, for Theorems 6 and 7, the optimality gap between the discounted cost function of the AGS and NS strategies under Assumption 6 is given by:

\[ \Delta J_n^{\beta,1} := (1 - \beta) \left( \text{Tr}(H^2\hat{M}^n) + \sum_{t=1}^{\infty} \beta^{t-1} \text{Tr}(H^{\infty}M^n) \right) \]

(54)

The convergence results can also be shown by a derivation similar to the proofs of Theorems 6 and 7.

VI. THE FIRST SPECIAL CASE OF THE NON-STANDARD RICCATI EQUATION

So far, we have presented the solutions of Problems 1 and 3 based on the solution of the non-standard Riccati equation \( (12) \). In this section and the next one, we provide two interesting special models in which the non-standard Riccati equation \( (12) \) reduces to two standard Riccati equations. For ease of display, superscript \( n \) is dropped from the matrices in the Riccati equations.

**Assumption 7.** Let \( S_\ell^T = -Q_\ell \) and \( S_\nu^T = -R_\nu \), \( \forall \ell \in \mathbb{N}_T \). In addition, let matrices \( Q_\ell + \frac{1}{n-1}G_\ell^T \) and \( P_\ell^T + G_\ell^T - Q_\ell \) be positive semi-definite and matrices \( R_\ell + \frac{1}{n-1}G_\ell \) and \( P_\ell + G_\ell - R_\ell \) positive definite, for all \( \ell \in \mathbb{N}_T \).

**Lemma 7.** Let Assumption 7 hold. Then, \( \bar{M}_n^{1,2} = \bar{M}_n^{2,1} = 0_{d_\times d_\times} \), for any \( n \in \mathbb{N}_T \).

**Proof.** Since \( \bar{M}_n^{1,2} = \bar{M}_n^{2,1} = 0_{d_\times d_\times} \), one can use backward induction to show that matrices \( \bar{M}_n^{1,2} \) and \( \bar{M}_n^{2,1} \), given by \( (12) \), are zero at any \( \ell \in \mathbb{N}_T \) under Assumption 7.

**Theorem 8.** The following statements are true for the finite-horizon cost:

(I) Under Assumption 7, the non-standard Riccati equation \( (12) \) is split into two standard Riccati equations:

\[ \bar{M}_n^{1,1} = Q_\ell + \frac{1}{n-1}G_\ell + A_\ell^T \bar{M}_n^{1,1} + A_\ell \bar{M}_n^{1,1} B_\ell \]

\[ \times \left( R_\ell + \frac{1}{n-1}G_\ell + B_\ell^T \bar{M}_n^{1,1} B_\ell \right) - B_\ell^T \bar{M}_n^{1,1} A_\ell, \]

\[ \bar{M}_n^{2,2} = P_\ell^T + G_\ell^T - Q_\ell + (A_\ell + D_\ell)^T \bar{M}_n^{2,2} (A_\ell + D_\ell) \]

\[ - (A_\ell + D_\ell)^T \bar{M}_n^{2,2} B_\ell (B_\ell + E_\ell)^T \]

\[ \times \left( P_\ell^T + G_\ell^T - R_\ell + (B_\ell + E_\ell)^T (B_\ell + E_\ell) \right) \]

(54)

**Proof.** The decomposition of the non-standard Riccati equation \( (12) \) follows from Assumption 7, Lemma 7 and equations \( (12) \) and \( (13) \). Furthermore, Assumption 7 implies As-
Assumption 1 because matrices $\bar{F}_t^n$ and $\bar{F}_t^n$, $\forall t \in \mathbb{N}_T$, are positive definite under Assumption 7.

**Assumption 8.** Suppose $(A, B)$ and $((A + D), (B + E))$ are stabilizable, and $(A, Q + \frac{1}{n-1}G^x)^{1/2}$ and $(A + D), (P^x + G^x - Q)^{1/2}$ are detectable.

**Theorem 9.** The following statements are true for the time-averaged infinite-horizon cost function:

(I) Assumptions 7 and 8 together imply Assumptions 1 and 5.

(II) Under Assumptions 7 and 8, the algebraic form of the non-standard Riccati equation decomposes into two standard algebraic Riccati equations as follows:

$$
\tilde{M}^{1,1} = Q + \frac{1}{n-1}G^x + A^T \tilde{M}^{1,1} A - A^T \tilde{M}^{1,1} B \\
\times (R + \frac{1}{n-1}G^n + B^T \tilde{M}^{1,1} B)^{-1} B^T \tilde{M}^{1,1} A,
$$

$$
\tilde{M}^{2,2} = P^x + G^x - Q + (A + D)^\top \tilde{M}^{2,2} (A + D) - (A + D)^\top \tilde{M}^{2,2} (B + E) \left( P^x + G^x - R + (B + E)^\top \right)
\times \tilde{M}^{2,2} (B + E) \right)^{-1} (B + E)^\top \tilde{M}^{2,2} (A + D).
$$

(III) Under Assumptions 7 and 8, Assumption 6 reduces to the assumption that the matrix:

$$
A + D - (B + E) \left( R + \frac{1}{n-1}G^n + B^T \tilde{M}^{1,1} B \right)^{-1} B^T \tilde{M}^{1,1} A,
$$

is Hurwitz. Note that this condition holds when the player dynamics are decoupled (i.e., matrices $D$ and $E$ are zero).

**Proof.** The proofs of parts (I) and (II) follow from the decomposition in Theorem 8 and the result of Lemma 7 under Assumptions 7 and 8, which state that the solution of equation (12) converges to the solution of the above decoupled standard algebraic Riccati equations as the time horizon goes to infinity. In addition, it can be shown that the matrix $(\tilde{M}^{1,1} - \frac{1}{n-1} \tilde{M}_t)$ evolves according to a standard Riccati equation with matrices $(A, B, Q, R)$. Subsequently, it follows that under Assumption 8, the solution of equation (14), which is $\tilde{M}_t$, converges to a uniquely bounded matrix as the time horizon goes to infinity.

The proof of part (III) follows from Assumption 8 and the fact that matrices $A + B \left( F^n \right)^{-1} \tilde{K}^n$ and $A + D + (B + E) \left( F^n \right)^{-1} \tilde{K}^n$ are Hurwitz because they are solutions of the standard algebraic Riccati equations under Assumption 8.

Therefore, Assumption 6 reduces to the assumption that matrix $A + D + (B + E) \left( F^n \right)^{-1} \tilde{K}^n$ is Hurwitz.

**Counterexample 1.** It is possible to construct an example where the finite-population game admits a unique solution while the infinite-population game admits no solution at all. To this end, consider the following scalars: $Q_t < 0$, $R_t > 0$, $S_t^x = -Q_t$, $S_t^n = -R_t$, $P_t^x = Q_t$, $P_t^n = G_t^n = R_t$, and $G_t^n = -100Q_t$, $\forall t \in \mathbb{N}_T$. With this set of parameters, the solution of the first Riccati equation in Theorem 8 (i.e., $\tilde{M}^{1,1}_t$) is positive for relatively small $n$ but it becomes negative for sufficiently large $n$, because $Q_t$ in (54) becomes dominant. Therefore, matrix $\bar{F}_t^n$ in the infinite-population game becomes negative, which violates the necessary condition in Assumption 1 for the existence of a Nash solution (note that $\bar{F}_t^n = \left[ R_t + B_t^T \tilde{M}^{1,1}_{t+1} B_t \right]$). As a result, the infinite-population game in this example admits no solution while the finite-population game admits a unique solution for small $n$.

**Proposition 2.** Let Assumptions 3 and 7 hold. Then, Assumption 3 holds for any $n_0 \geq 2$.

**Proof.** The proof follows directly from the fact that matrices $(\bar{F}^n_{t})^{-1} \tilde{K}_t^{n}$ and $(\bar{F}^n_{t})^{-1} \tilde{K}_t^{n}$ are uniformly bounded with respect to any $n \geq 2$ under Assumption 3 according to equation (13). Lemma 7 and considering the decoupled Riccati equations in Theorems 8 and 9.

**Remark 6.** From Theorem 8, it follows that Theorems 1 and 3 can be presented by decoupled standard algebraic Riccati equations under Assumption 7. Similarly, from Theorem 9, it results that Theorems 5 and 7 and Corollary 2 can be expressed by two decoupled standard algebraic Riccati equations under Assumptions 7 and 8 and a Hurwitz condition.

A. Cooperative game (social welfare function)

Now, consider a special case of the above class where all players have an identical cost function, i.e., matrices $Q_t$, $S_t^x$, $R_t$, and $S_t^n$, $\forall t \in \mathbb{N}_T$, are zero. In such a case, the standard Riccati equations are independent of the number of players.

**Assumption 9.** Let matrices $Q_t$, $S_t^x$, $R_t$, and $S_t^n$, $\forall t \in \mathbb{N}_T$. Let also matrices $G_t^n$ and $P_t^n + G_t^n$ be positive semi-definite and matrices $G_t^n$ and $P_t^n + G_t^n$ be positive definite.

**Corollary 3.** When all players have an identical cost function, the following statements are true for the finite-horizon case:

(I) Assumption 8 implies Assumption 7.

(II) Under Assumption 9, the non-standard Riccati equation (12) is split into two standard Riccati equations:

$$
\tilde{M}^{1,1}_t = G_t^n + A^T \tilde{M}^{1,1}_{t+1} A_t - A^T \tilde{M}^{1,1}_{t+1} B_t \\
\times (G_t^n + B_t^T \tilde{M}^{1,1}_{t+1} B_t)^{-1} B_t^T \tilde{M}^{1,1}_{t+1} A_t,
$$

$$
\tilde{M}^{2,2}_t = P_t^n + G_t^n + (A_t + D_t)^T \tilde{M}^{2,2}_{t+1} (A_t + D_t) - (A_t + D_t)^T \tilde{M}^{2,2}_{t+1} (B_t + E_t) \left( P_t^n + G_t^n + (B_t + E_t)^\top \right)
\times \tilde{M}^{2,2}_{t+1} (B_t + E_t) \right)^{-1} (B_t + E_t)^\top \tilde{M}^{2,2}_{t+1} (A_t + D_t).
$$

**Proof.** The proof follows from Theorem 8 after multiplying both sides of the first Riccati equation by $(n - 1)$, and redefining $\tilde{M}^{1,1}_t := (n - 1) \tilde{M}^{1,1}_{t+1}$, $\forall t \in \mathbb{N}_T$.

**Assumption 10.** Suppose $(A, B)$ and $((A + D), (B + E))$ are stabilizable, and $(A, G^x^{1/2})$ and $((A + D), (P^x + G^x)^{1/2})$ are detectable.

**Corollary 4.** When all players have an identical cost function, the following statements are true for the time-averaged infinite-horizon cost function:

(I) Assumptions 9 and 10 together imply Assumptions 7 and 8.
(II) Under Assumptions 2 and 10, the algebraic form of the non-standard Riccati equation decomposes into two standard algebraic Riccati equations as follows:
\[
\begin{align*}
\dot{M}^{1,1} = G^* + A^T\dot{M}^{1,1} - A^T\dot{M}^{1,1}\ B \\
\times (G^u + B^T\dot{M}^{1,1} B)^{-1} B^T\dot{M}^{1,1} A, \\
\dot{M}^{2,2} = P^* + G^x + (A + D)^\top\dot{M}^{2,2}(A + D) - \\
(\hat{A} + D)^\top\dot{M}^{2,2}(B)\ (P^u + G^u + (B + E)^\top) \\
\times \dot{M}^{2,2}(B + E)^{-1} (B + E)^\top\dot{M}^{2,2}(A + D).
\end{align*}
\]

(III) Under Assumptions 9 and 10, Assumption 9 reduces to the condition that
\[
A + D - (B + E)\left(G^u + B^T\dot{M}^{1,1} B\right)^{-1} B^T\dot{M}^{1,1} A,
\]
is Hurwitz. Note that this condition holds when the player dynamics are decoupled (i.e., matrices \(D\) and \(E\) are zero).

**Proof.** The proof follows is similar to the proof of Theorem 9 considering the decomposition in Corollary 5.

**Remark 7.** When the cost function is social welfare function, the finite-population and infinite-population NS strategies in Theorems 3 and 4, respectively, are identical because in this case the finite-population and infinite-population gains are identical and independent of the number of players.

**B. Relation to the existing literature**

The results of Subsection 6A can be related to the existing results in three areas: mean-field-type control, social optima mean-field games and mean-field teams.

In mean-field-type control, it is well known that the solution is linear in the local state, and the expectation of the state of a generic player, and that the corresponding gains are obtained by solving standard Riccati equations as in Corollaries 3 and 4, 18, 19. Mean-field-type control can be regarded as a special case with the social welfare function, wherein the information structure is AGS and the number of players is infinite. It is to be noted that the solution of mean-field-type control is not necessarily implementable under NS information structure because for instance, an extra stability assumption is required for the case when the dynamics of players are coupled (see part (II) of Corollary 4).

In social optima mean-field games with decoupled dynamics, an approximate Nash solution is determined by employing mean-field game and person-by-person approaches. This solution is described in terms of two coupled forward and backward ordinary differential equations. For the case of homogeneous players, it is straightforward to show that the continuous-time counterparts of the proposed decoupled algebraic Riccati equations in Corollary 4 solve the coupled ordinary differential equations in 36, and simplify the existence and uniqueness conditions.

On the other hand, the solution of mean-field teams coincides with the Nash solution presented in Corollaries 3 and 4, 30, 31, 37. This implies that the cooperative LQ game and LQ team with an arbitrary number of exchangeable players have identical solutions under AGS information structure. For more recent developments on mean-field teams and deep teams, the reader is referred to 32, 38–40.

**VII. The second special case of the non-standard Riccati equation**

In this section, we present a special case for the infinite-population game.

**Assumption 11.** For any \(t \in \mathbb{N}_T\), let matrices \(D_t, E_t, S_t^i, \ P^u_t\) and \(G^u_t\) be zero, and matrices \(Q_t\) and \(Q_t + S_t^2\) be positive semi-definite and matrix \(R_t\) be positive definite.

For the parameters given in Assumption 11, matrices in (13) take a special structure for any number of players \(n \in \mathbb{N}\), as described below:
\[
\begin{align*}
\dot{F}^n_t &= R_t + B^T_t((1 - \frac{1}{n})\dot{M}^{1,1}_{t+1} + \frac{1}{n}\dot{M}^{2,2}_{t+1})B_t, \\
\dot{K}^n_t &= -B^T_t((1 - \frac{1}{n})\dot{M}^{1,1}_{t+1} + \frac{1}{n}\dot{M}^{2,2}_{t+1})A_t,
\end{align*}
\]
(55)

**Theorem 10.** The following statements are true for the finite-horizon case of an infinite-population game where \(n = \infty\):

(I) Assumption 7 implies Assumption 4.

(II) Under Assumption 11, the non-standard Riccati equation reduces to two standard Riccati equations as follows:
\[
\begin{align*}
\dot{M}^{1,1}_t &= Q_t + A_t^\top\dot{M}^{1,1}_{t+1}A_t - A_t^\top\dot{M}^{1,1}_{t+1}B_t(R_t + B_t^\top\dot{M}^{1,1}_{t+1}B_t)^{-1}B_t^\top\dot{M}^{1,1}_{t+1}A_t, \\
\dot{M}^{2,2}_t &= Q_t + S_t^2 + A_t^\top\dot{M}^{2,2}_{t+1}A_t - A_t^\top\dot{M}^{2,2}_{t+1}B_t(R_t + B_t^\top\dot{M}^{2,2}_{t+1}B_t)^{-1}B_t^\top\dot{M}^{2,2}_{t+1}A_t.
\end{align*}
\]

**Proof.** For any \(t \in \mathbb{N}_T\), it results from (55) that
\[
\begin{align*}
\dot{F}^\infty_t &= R_t + B^T_t\dot{M}^{1,1}_{t+1}B_t, & \quad \dot{K}^\infty_t &= -B^T_t\dot{M}^{1,1}_{t+1}A_t, \\
\dot{F}^\infty_t &= R_t + B^T_t\dot{M}^{2,2}_{t+1}B_t, & \quad \dot{K}^\infty_t &= -B^T_t\dot{M}^{2,2}_{t+1}A_t.
\end{align*}
\]
(56)

Note that matrix \(\dot{M}^{2,2}_{t}\), \(t \in \mathbb{N}_T\) does not play a role in determining the solution when \(n\) is set to infinity; hence, it can be discarded. The proof is now completed by incorporating equation (56) and Assumption 11 into equation (12), and on noting that Assumption 11 implies Assumption 1 as \(\dot{F}^\infty_t\) and \(\dot{F}^\infty_t\) are positive definite.

**Assumption 12.** Suppose \((A, B)\) is stabilizable, and the pairs \((A, Q^{1/2})\) and \((A, (Q + S^2)^{1/2})\) are detectable.

**Theorem 11.** The following statements are true for the time-averaged infinite-horizon cost function of an infinite-population game where \(n = \infty\):

(I) Assumptions 7 and 12 together imply Assumptions 7 and 5.
(II) Under Assumptions [7] and [2] the algebraic form of the non-standard Riccati equation reduces to two standard algebraic Riccati equations as follows:

\[
\dot{M}_{1;1} = Q + A^T \dot{M}_{1;1} A - A \dot{M}_{1;1} B (R + B^T \dot{M}_{1;1} B)^{-1} B^T \dot{M}_{1;1} A,
\]

\[
\dot{M}_{2;1} = Q + S^2 + A^T \dot{M}_{2;1} A - A \dot{M}_{2;1} B (R + B^T \dot{M}_{2;1} B)^{-1} B^T \dot{M}_{2;1} A.
\]

(III) Assumptions [7] and [2] together imply Assumption 6.

Proof. The proof follows from the decomposition in Theorem 10 and from Assumptions 11 and 12 on noting that \( \dot{M}_{1;1}^2 \) and \( \dot{M}_1 \) vanish from feedback gains (13) and value function (12), respectively, when \( n \) is set to infinity. In addition, Assumption 6 holds due to the fact that matrices \( A + B (F_{\infty})^{-1} \hat{K}_{\infty} \) and \( A + D + (B + E) (F_{\infty})^{-1} \hat{K}_{\infty} \) are Hurwitz under Assumption 12, i.e., they are stable solutions of the above standard algebraic Riccati equations. 

Proposition 3. Let Assumptions 4 and 7 hold. There exists a sufficiently large \( n_0 \in \mathbb{N} \) for which Assumption 6 holds.

Proof. The proof follows directly from (55) and the fact that for a sufficiently large \( n \), matrices in (55) under Assumption 11 converge to a limit, that is independent of \( n \), as \( n \to \infty \).

Remark 8. Under Assumption 1, Theorems 1 and 4 can be presented by two decoupled standard Riccati equations, according to Theorem 10. Similarly, under Assumptions 11 and 12, Theorems 3 and 4, and Corollary 2 can be expressed by two decoupled standard algebraic Riccati equations, according to Theorem 11.

Counterexample 2. We use this special case to illustrate that the infinite-population game may have a unique solution while the finite-population game may not. Let \( P_t^x \) be a large negative definite matrix for every \( t \in \mathbb{N} \), and choose other matrices according to Assumption 11. In this case, \( F_t \) in (13) becomes a large negative definite matrix violating the positive definiteness condition in Assumption 11. On the other hand, according to Theorem 10, the existence and uniqueness of the solution of the infinite-population game is independent of matrix \( P_t^x \), because the effect of \( F_t \) in Assumption 1 vanishes at the rate \( 1/n \) as \( n \to \infty \). This means that although the Nash solution does not exist in the finite-population game in this case, an approximate Nash solution may exist (which is the Nash solution of the infinite-population game according to Theorems 3 and 10).

A. Relation to existing literature

In the infinite-population non-cooperative game with decoupled player dynamics, approximate Nash solutions for discounted and time-averaged cost functions are proposed in [14] and [41], respectively. The solutions are presented in terms of coupled forward and backward ordinary differential equations. It can be shown that the continuous-time counterparts of the proposed decoupled algebraic Riccati equations in Theorem 11 are consistent with the coupled ordinary differential equations and simplify the existence and uniqueness conditions in [14], [41]. A similar relationship between forward-backward equations of mean-field games and Riccati equations has recently been established in [42] in terms of one symmetric and one non-symmetric Riccati equation, which may be viewed as a special case of our non-standard Riccati equation with PS information structure, where matrices \( E_t, S_t^u, P_t^u, G_t^u \) and \( G_t^n \) are zero, \( n = \infty \), and \( Q_t + 2S_t^x + P_t^x \) is a positive definite matrix taking the special form: \( (I - \Gamma_t)^T Q_t (I - \Gamma_t) \) for a given matrix \( \Gamma_t \). It is to be noted that for NS information structure, an additional stability condition (similar to Assumption 6), apart from the boundedness condition of Riccati equations in Assumption 5 is required to ensure that the propagation of the error associated with the imperfection of NS information structure remains bounded in the finite-population game.

Remark 9. For the case when \( n = \infty \) and the player dynamics are decoupled, the solution of the cooperative game is equal to that of the competitive model if \( Q_t = G_t^c, R_t = G_t^c, S_t^c = P_t^c, \) and \( P_t^c = 0_{d_c \times d_c}, \forall t \in \mathbb{N} \), according to Corollary 4 and Theorem 11. In such a case, the best selfish action is equivalent to the best selfless action (a similar finding is reported in [36]). Note that if \( P_t^c \neq S_t^c \), then the above equivalence may not hold because although the solution of the competitive model does not depend on \( P_t^c \), that of the cooperative game does.

VIII. Generalizations

A salient feature of the proposed methodology is the fact that it can be naturally extended to other variants of LQ games such as zero-sum, risk-sensitive, partially exchangeable games with cooperative and non-cooperative sub-populations. In addition, its unified framework can shed light on the similarities and differences between mean-field games and mean-field-type games. This is useful in providing an explicit closed-form solution for the existing results in mean-field models. For example, major-minor mean-field model [43] may be viewed as a special case of the partially exchangeable models, where the size of the major sub-population is one. In what follows, we briefly discuss the extension of our results to partially exchangeable players.

Consider a partially exchangeable game with \( k \in \mathbb{N} \) disjoint sub-populations, where each sub-population consists of \( n_k \in \mathbb{N} \) exchangeable players. The players are coupled in dynamics and cost through the empirical mean and covariance of the states and actions of the players in each sub-population. Using the proposed gauge transformation, the resultant game can be reduced to a standard \( k \)-player game wherein the coupled standard matrix Riccati equations [10] are replaced by coupled non-standard matrix Riccati equations (similar to the one introduced in this paper).

IX. A Numerical Example

In this section, we present a numerical example to illustrate the difference between the two NS strategies proposed in Theorems 3 and 4.
Example 1. Consider a game described in Section II with the following numerical parameters: $A_i = 1$, $B_i = 1$, $D_i = 0$, $E_t = 0$, $Q_t = 1$, $S_t^f = -0.5$, $P_t^e = 5$, $R_t = 5$, $S_t^u = P^u_t = G^u_t = G^f_t = 0$, $\mu_x = 10$, $\text{Cov}(z_t^1) = 2$, $\text{Cov}(w_t^i) = 1$ and $T = 50$ for every $t \in \mathbb{N}_T$ and $i \in \mathbb{N}_n$, where the probability distributions of the initial states and local noises are i.i.d. and Gaussian. It is shown in Figure 1 that the finite-population NS strategy proposed in Theorem 3 which takes the number of players into account, converges to the optimal solution faster than the infinite-population NS strategy proposed in Theorem 4 as the number of players goes to infinity.

X. Conclusions

In this paper, we studied cooperative and non-cooperative linear quadratic games with an arbitrary number of index-invariant players under three different information structures, and obtained exact and approximate sequential Nash solutions by deriving a novel non-standard Riccati equation. The key idea was to use a gauge transformation to induce some orthogonality and linear dependence among variables in order to arrive at a low-dimensional solution. In addition, we established several convergence results and characterized the role of the number of players in cooperative and non-cooperative games. We generalized our main results to the discounted and time-averaged infinite-horizon cost functions and investigated two special cases wherein the non-standard Riccati equation, used to solve the problem, reduces to two decoupled standard Riccati equations.

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It follows from (3), (4) and (30) that
\[
\Delta J^*_n = E \left[ \sum_{t=1}^T \left[ \bar{q}_t^i \bar{y}_t^i \right]^\top \bar{Q}_t \left[ \bar{q}_t^i \bar{y}_t^i \right] + \left[ \bar{v}_t^i \bar{v}_t^i \right]^\top \bar{R}_t \left[ \bar{v}_t^i \bar{v}_t^i \right] \right] 
+ \frac{1}{n} \left( \sum_{j \neq i} (\bar{y}_t^j)^\top G^*_t (\bar{y}_t^j) \right) 
+ \frac{1}{n} \left( \sum_{j \neq i} (\bar{v}_t^j)^\top G^*_t (\bar{v}_t^j) \right) 
- \left[ \bar{x}_t^i \bar{x}_t^i \right]^\top \bar{Q}_t \left[ \bar{x}_t^i \bar{x}_t^i \right] 
- \left[ \bar{u}_t^i \bar{u}_t^i \right]^\top \bar{R}_t \left[ \bar{u}_t^i \bar{u}_t^i \right] 
- \frac{1}{n} \left( \sum_{j \neq i} (\bar{y}_t^j)^\top G^*_t (\bar{y}_t^j) \right) 
- \frac{1}{n} \left( \sum_{j \neq i} (\bar{v}_t^j)^\top G^*_t (\bar{v}_t^j) \right) 
\]
\[
\left( \bar{e}_t^i + z_t^n \right) \bar{Q}_t \left[ \bar{e}_t^i + z_t^n \right] 
- \left[ \bar{e}_t^i + z_t^n \right]^\top \bar{Q}_t \left[ \bar{e}_t^i + z_t^n \right] 
\]
\[
\frac{1}{n} \left( \tilde{F}_t^n \right)^{-1} \tilde{K}_t^n \bar{e}_t^i 
- \left[ \left( \tilde{F}_t^n \right)^{-1} \tilde{K}_t^n \bar{e}_t^i \right]^\top \tilde{R}_t 
\times \left[ \left( \tilde{F}_t^n \right)^{-1} \tilde{K}_t^n \bar{e}_t^i \right] 
\times \left[ \left( \tilde{F}_t^n \right)^{-1} \tilde{K}_t^n \bar{e}_t^i \right]^\top \tilde{R}_t 
\times \left[ \left( \tilde{F}_t^n \right)^{-1} \tilde{K}_t^n \bar{e}_t^i \right] 
\]
\[
\left[ \left( \tilde{F}_t^n \right)^{-1} \tilde{K}_t^n \bar{e}_t^i \right]^\top \tilde{R}_t \left[ \left( \tilde{F}_t^n \right)^{-1} \tilde{K}_t^n \bar{e}_t^i \right] 
\times \left[ \left( \tilde{F}_t^n \right)^{-1} \tilde{K}_t^n \bar{e}_t^i \right] 
\]
where (a) follows from Lemma 1 and equations (15), (27) and (29), and (b) is a consequence of Lemma 3 on noting that $z_t^n$ is deterministic. The proof is now complete from the definition of $\bar{Q}_t^n$. 

APPENDIX A

PROOF OF LEMMA 4

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