The Charm Quark’s Mass

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1. INTRODUCTION

Nonperturbative lattice calculations of the hadron spectrum provide a connection between experimentally measured masses and the couplings of the (lattice) QCD Lagrangian. By convention, however, the \( \overline{\text{MS}} \) couplings \( \bar{\alpha}(M_Z) \) and \( \bar{m}(\mu) \), used in phenomenology, are usually quoted. The two sets of definitions can be related to each other in perturbation theory. For example,

\[
\bar{m}(\mu) = M(\alpha) \left[ 1 + \frac{\alpha(q^*)}{4\pi} \left( C_0 + \gamma_0 \ln \mu^2 a^2 \right) \right],
\]

where \( \gamma_0 = 4 \). The lattice mass \( M \) and, by implication, \( C_0 = C_0(M) \) are specified below. Eq. (1) omits higher orders in the gauge coupling \( \alpha \) and power-law artifacts.

This paper determines the charm quark’s mass, \( \bar{m}_{\text{ch}} \), from quenched calculations of the \( \bar{c}\bar{c} \) spectrum. To anticipate the main sources of uncertainty, let us recall recent determinations the average of the up and down quarks’ masses \[ \bar{m}_u \approx \bar{m}_d \approx (\bar{m}_{\text{ch}}/3)^{1.0} \pm 0.1 \bar{m}_{\text{ch}} \].

The three largest uncertainties stem from, in descending order, the quenched approximation, the extrapolation to the continuum limit (even with the clover action), and perturbation theory. Each of these takes on a different guise for charm, however.

The error in a coupling from quenching can be partly explained by noting that couplings run differently in the quenched approximation. One can account for this effect by running the couplings down to typical mesonic momenta with \( n_f = 0 \) and then back up to a high scale with \( n_f \neq 0 \). But \( \bar{m}(\mu) \) does not run for \( \mu < m \), so quenching should not affect \( \bar{m}(\mu) \) much.

One might expect lattice spacing errors to be worse for charmonium than for light mesons, since \( 0.4 \leq m_{0,\text{ch}}a \leq 1 \) on our lattices. One might expect lattice spacing errors to be worst for charmonium than for light mesons.

That leaves perturbation theory as the source of the largest uncertainty. To make the most of the one-loop approximation, the only order available, we use results for \( m_0a \neq 0 \). Furthermore, we try to reduce the effect of truncating at one loop by choosing \( \alpha(q^*) \) in Eq. (1) to absorb logarithms from higher orders.

2. CUT-OFF EFFECTS

In a heavy-quark system, such as charmonium, typical three-momenta are only a few hundred MeV, suggesting that worrisome lattice artifacts are of order \( (m_0a)^\ast \). On the other hand, it is well-known that actions for Wilson fermions approach the static limit as \( m_0a \to \infty \), showing that higher-dimensional operators are suppressed by a factor of order \( 1/(m_0a)^\ast \). The lattice Hamiltonian (defined by the transfer matrix) clarifies the middle ground, \( m_0a \approx 1 \). One finds

\[
\bar{H}_{\text{lat}} = \bar{H}_{\text{cont}} + \delta \bar{H}.
\]

Contributions to the artifact \( \delta \bar{H} \) take the form

\[
\langle a\delta \bar{H}_{n}^{[l]} \rangle \sim g^2 b_{l}^{[n]}(m_0a) \bar{p}a s_{n+1},
\]
where \( p \) is a few hundred MeV, and \( s_0 > 0 \). The function \( g_0^{[l]} \) is bounded \( \tilde{g} \). It is safe to replace it by a number of order unity, and thus the effect is about the same size for splittings of charmonium as for masses of light-quark hadrons.

Eq. (3) applies only if the hopping parameter \( \kappa \) is adjusted until the meson’s kinetic mass
\[
M_2 := (\partial^2 E/\partial p^2)_{p=0}^{-1}
\]
equals the meson’s physical mass. When \( M^0 \neq 0 \), the rest mass \( M_1 := E(0) \) is smaller. Nevertheless, the splittings of meson rest masses are accurate up to Eq. (3). In particular, the spin-averaged binding energy
\[
B_{1a} := (M_{1QQ})^{MC} - 2(M_{1Q}a)^{PT},
\]
where \( M_{1QQ} \) is the spin average of mesons’ rest masses, has relative errors of order \( \min(p^2 a^2, v^2) \). (When the quark’s rest mass \( M_{1Q} \) is computed to finite order in perturbation theory, \( B_{1a} \) suffers perturbative errors as well.)

To determine \( \bar{m}_{ch} \), we rely, therefore, on the following Monte Carlo calculations: We define the lattice spacing \( a \) in physical units from \( \Delta M = M_{ch} - \frac{3}{2}(M_{\eta_c} + 3M_{J/\psi}) \). We then obtain the quark mass either from the spin-averaged binding energy \( B_1 \) of the 1S states, or from their spin-averaged kinetic mass \( M_{2QQ} \).

3. PERTURBATION THEORY

3.1. When \( m_{0a} \neq 0 \)

If the Monte Carlo has \( m_{0a} \neq 0 \) it is necessary to take \( m_{0a} \neq 0 \) when deriving Eq. (4). Although \( C_0 \) remains bounded \( \tilde{g} \), its value can change significantly for nonzero \( m_{0a} \).

Eq. (4) is obtained by computing the quark’s pole mass in lattice and in \( \overline{\text{MS}} \) perturbation theory. Because the lattice breaks Euclidean invariance, several “masses” \( (M_1, M_2, \text{etc}) \) describe the pole. One would like to pick a pole mass without dire lattice artifacts. We use two methods. In the first, we take the binding energy and set
\[
m_{\text{pole}} = \frac{1}{2}(M_{QQ}^{\text{expt}} - B_1)
\]
with \( B_{1a} \) from Eq. (3) and \( a \) from \( \Delta M \). In the second method, we use the quark’s kinetic mass, but reduce uncertainty in tuning \( \kappa \) by taking a from the meson’s kinetic mass:
\[
m_{\text{pole}} = (M_{2QQ}a)^{PT} \frac{M_{QQ}^{\text{expt}}}{(M_{2QQ}a)^{MC}}.
\]

When \( B_{1a} \) and \( M_{2QQ}a \) are expanded in perturbation theory, Eqs. (5) and (6) can be matched to the expansion of \( m_{\text{pole}} \) in \( \overline{\text{MS}} \). The manipulations at one loop define \( M \) and \( C_0 \) in Eq. (3).

One needs, therefore, the loop corrections to the quark’s rest and kinetic masses. From formulas \( \tilde{g} \) for \( M_1 \) and \( M_2 \), to all orders in \( g_0^2 \) and in \( m_{0a} \), one can expand
\[
M_1 = \sum_{l=0} g_0^{2l} M_1^{[l]}.
\]
One finds \( M_1^{[0]} = \log(1 + M_0) \), where \( M_0 = 1/2 \kappa - 1/2 \kappa_{\text{crit}} \). Refs. \( \tilde{g} \) show results for \( M_1^{[1]} \).

The kinetic mass has further loop corrections, so let \( Z_{M_2}(M_1) = M_2/m_2(M_1) \). The function \( m_2(M) \) is chosen so that \( Z_{M_2}^{[0]} = 1 \), but it is evaluated at the all-orders \( M_1 \). With this definition \( Z_{M_2}(0) = 1 \), to all orders in \( g_0^2 \). Also, \( Z_{M_2}^{[1]} \) is tadpole-free. It is small \( (0 \geq Z_{M_2}^{[1]} > -0.1 \) and hardly depends on the clover coupling \( c_{\text{SW}} \).

3.2. Choosing \( \alpha(q^*) \)

With only the one-loop approximation at hand, the right-hand side of Eq. (8) is sensitive to the choice of scheme for \( \alpha \) and its scale \( q^* \). Since Eq. (8) is the combination of lattice and \( \overline{\text{MS}} \) perturbation theory, the original series must be expressed in a common scheme and the scales must be run to a common one. Here we use the scales suggested in Refs. \( \tilde{g} \), primarily for \( \alpha_V \), but also for \( \bar{\alpha} \).

For dimensional regulators Ref. (10) prescribes
\[
\ln q_0^2 / \mu^2 = \Gamma^*/I,
\]
where \( I \) is derived from the Feynman diagram for \( I \) by replacing gluon propagators by
\[
q^{-2} \rightarrow q^{-2} \left[ \ln(q^2 / \mu^2) - b'_f / \beta_0^f \right].
\]
The constant depends on the scheme: for \( \bar{\alpha} \), \( b'_f / \beta_0^f = 5/3 \); for \( \alpha_V \), \( b'_f / \beta_0^f = 0 \).

Similarly, for the lattice Ref. (10) prescribes
\[
\ln q_{\text{LM}}^2 = \Gamma^*/I,
\]
where $I^*$ now comes from the replacement
\[
\hat{q}^{-2} \mapsto \hat{q}^{-2} \ln(q^2 a^2).
\] (12)

With no constant, this prescription is for the coupling defined in Ref. [10], which coincides with $\alpha_V$ through next-to-leading order.

When combining the series to form Eq. (1), one can combine $q^2_{\mathrm{BLM}}$ and $q^2_{\mathrm{LM}}$ in the usual way,
\[
\ln(q^2 a^2) = (I^*_{\mathrm{lat}} - I^*_{\mathrm{cont}})/(I^*_{\mathrm{lat}} - I^*_{\mathrm{cont}})
\] (13)

provided the constants used to define the $I^*$s are compatible. (Otherwise the final $q^*$ has problems as $m a, m/\mu \rightarrow 0$.) Most straightforward, we find, is to use $\alpha_V$ and to extract $\bar{m}_{\mathrm{ch}}(m_{\mathrm{ch}})$ directly from Eq. (1). The resulting $q^*$s are a few GeV but somewhat $a$ dependent.

4. RESULTS

We have computed the charmonium spectrum for $(\beta, c_{\mathrm{SW}}) = (5.5, 1.69), (5.7, 1.57), (5.9, 1.50), \text{ and } (6.1, 1.40)$ [12]. Our (preliminary) results for $\bar{m}_{\mathrm{ch}}(m_{\mathrm{ch}})$ with tadpole-improved perturbation theory are plotted against $a^2$ in Fig. 1. The error bar is dominated by the unknown two-loop correction to Eq. (1), estimated to be twice the square of the one-loop term. When the analysis is repeated without tadpole improvement, but still choosing $\alpha(q^*)$ as in Sect. [2], the data change negligibly. The subdominant uncertainty is from the Monte Carlo statistics of $M_{2QQ}$.

Extrapolating the average of the two methods linearly in $a^2$ yields
\[
\bar{m}_{\mathrm{ch}}(m_{\mathrm{ch}}) = 1.33 \pm 0.08 \text{ GeV}.
\] (14)

The error bar now incorporates uncertainty in the extrapolation, e.g., extrapolating linearly in $a$. Note that the quoted result neither explicitly corrects for, nor assigns an error to, quenching, because $\bar{m}(\mu)$ does not run when $\mu < m$.

A 6% uncertainty for the charm quark’s mass is twice the 3% quoted for the top quark’s mass from collider experiments. Alas, without two-loop (or nonperturbative) matching for $m_{\mathrm{top}} a \neq 0$, top standards will be impossible to achieve.

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