SOME OPIAL-TYPE INEQUALITIES APPLICABLE TO DIFFERENTIAL EQUATIONS INVOLVING IMPULSES

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ABSTRACT. The purpose of this paper is to obtain Opial-type inequalities that are useful to study various qualitative properties of certain differential equations involving impulses. After we obtain some Opial-type inequalities, we apply our results to certain differential equations involving impulses.

1. INTRODUCTION

Opial-type inequalities are very useful to study various qualitative properties of differential equations. For a good reference of the work on such inequalities together with various applications, we recommend the monograph [1]. In this paper we obtain some Opial-type inequalities that involve Stieltjes derivatives which are applicable to differential equations with impulses. Differential equations involving impulses arise in various real world phenomena, we refer to the monograph [8].

2. PRELIMINARIES

To obtain our results in this paper we need some preliminaries.

Let $\mathbb{R}$ be the set of all real numbers. Assume that $[a, b] \subset \mathbb{R}$ is a bounded interval. A function $f : [a, b] \rightarrow \mathbb{R}$ is called regulated on $[a, b]$ if both

$$f(s+) = \lim_{\eta \to 0^+} f(s + \eta), \text{ and } f(t-) = \lim_{\eta \to 0^+} f(t - \eta)$$

exist for every point $s \in [a, b], t \in (a, b)$, respectively. Let $G([a, b])$ be the set of all regulated functions on $[a, b]$. For $f \in G([a, b])$ we define $f(a-) = f(a), f(b+) = f(b)$.

For convenience we define

$$\Delta^+ f(s) = f(s+) - f(s), \quad \Delta^- f(s) = f(s) - f(s-) \quad \text{and} \quad \Delta f(s) = f(s+) - f(s-).$$

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Remark 2.1. Let $f \in G([a, b])$. Since both $f(s^+)$ and $f(s^-)$ exist for every $s \in [a, b]$ it is obvious that $f$ is bounded on $[a, b]$, and since $f$ is the uniform limit of step functions, $f$ is Borel measurable (see [3, Theorem 3.1]).

For a closed interval $I = [c, d]$, we define $f(I) = f(d) - f(c)$.

A tagged interval $(\tau, [c, d])$ in $[a, b]$ consists of an interval $[c, d] \subset [a, b]$ and a point $\tau \in [c, d]$.

Let $I_i = [c_i, d_i] \subset [a, b], i = 1, ..., m$. We say that the intervals $I_i$ are pairwise non-overlapping if

$$\text{int}(I_i) \cap \text{int}(I_j) = \emptyset$$

for $i \neq j$ where $\text{int}(I)$ denotes the interior of an interval $I$.

A finite collection $\{(\tau_i, I_i) : i = 1, 2, ..., m\}$ of pairwise non-overlapping tagged intervals is called a tagged partition of $[a, b]$ if $\cup_{i=1}^{m} I_i = [a, b]$. A positive function $\delta$ on $[a, b]$ is called a gauge on $[a, b]$.

From now on we use notation $\overline{1,m} = 1, ..., m$.

Definition 2.2 ([6, 9]). Let $\delta$ be a gauge on $[a, b]$. A tagged partition $P = \{(\tau_i, I_i) : t_{i-1} < t_i, i = \overline{1,m}\}$ of $[a, b]$ is said to be $\delta$-fine if for every $i = \overline{1,m}$ we have

$$\tau_i \in [t_{i-1}, t_i] \subset (\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i)).$$

Moreover if a $\delta$-fine partition $P$ satisfies the implications

$$\tau_i = t_{i-1} \Rightarrow i = 1, \quad \tau_i = t_i \Rightarrow i = m,$$

then it is called a $\delta^*$-fine partition of $[a, b]$.

The following lemma implies that for a gauge $\delta$ on $[a, b]$ there exists a $\delta^*$-fine partition of $[a, b]$. This also implies the existence of a $\delta$-fine partition of $[a, b]$.

Lemma 2.3 ([6]). Let $\delta$ be a gauge on $[a, b]$ and a dense subset $\Omega \subset (a, b)$ be given. Then there exists a $\delta^*$-fine partition $P = \{(\tau_i, [t_{i-1}, t_i]) : i = \overline{1,m}\}$ of $[a, b]$ such that $t_i \in \Omega$ for $i = \overline{1,m} - 1$.

We now give a formal definition of two types of the Kurzweil integrals.

Definition 2.4 ([6, 9]). Assume that $f, g : [a, b] \rightarrow \mathbb{R}$ are given. We say that $f \, dg$ is Kurzweil integrable (or shortly, K-integrable) on $[a, b]$ and $v \in \mathbb{R}$ is its integral if for every $\varepsilon > 0$ there exists a gauge $\delta$ on $[a, b]$ such that

$$\left| \sum_{i=1}^{m} f(\tau_i) g(I_i) - v \right| \leq \varepsilon,$$
provided \( P = \{(\tau_i, I_i) : i = 1, m\} \) is a \( \delta \)-fine tagged partition of \([a, b]\). In this case we define \( v = \int_a^b f(s) \, dg(s) \) (or, shortly, \( v = \int_a^b f \, dg \)).

If, in the above definition, \( \delta \)-fine is replaced by \( \delta^* \)-fine, then we say that \( f \, dg \) is Kurzweil* integrable (or, shortly, \( K^* \)-integrable) on \([a, b]\) and we define \( v = (K^*) \int_a^b f \, dg \).

**Remark 2.5.** By the above definition it is obvious that \( K \)-integrability implies \( K^* \)-integrability.

The following results are needed in this paper. For other properties of the \( K \)-integrals, see, e.g., [2, 7, 9, 10].

In this paper \( BV([a, b]) \) denotes the set of all functions that are of bounded variation on \([a, b]\).

**Theorem 2.6 ([11, 2.15. Theorem]).** Assume that \( f \in G([a, b]) \) and \( g \in BV([a, b]) \). Then both \( f \, dg \) and \( g \, df \) are \( K \)-integrable on \([a, b]\) and

\[
\int_a^b f \, dg + \int_a^b g \, df = f(b)g(b) - f(a)g(a) + \sum_{t \in [a, b]} [\Delta^- f(t) \Delta^- g(t) - \Delta^+ f(t) \Delta^+ g(t)].
\]

**Remark 2.7.** In the above theorem, the sum \( \sum_{t \in [a, b]} [\Delta^- f(t) \Delta^- g(t) - \Delta^+ f(t) \Delta^+ g(t)] \) is actually a countable sum because every regulated function has only countable discontinuities.

**Theorem 2.8 ([10, p. 40, 4.25. Theorem]).** Let \( h \in BV([a, b]) \), \( g : [a, b] \rightarrow \mathbb{R} \) and \( f : [a, b] \rightarrow \mathbb{R} \). If the integral \( \int_a^b g \, dh \) exists and \( f \) is bounded on \([a, b]\), then the integral \( \int_a^b f(s) \, d[\int_a^s g(v) \, dh(v)] \) exists if and only if the integral \( \int_a^b f(s)g(s) \, dh(s) \) exists and in this case we have

\[
\int_a^b f(s) \, d[\int_a^s g(v) \, dh(v)] = \int_a^b f(s)g(s) \, dh(s).
\]

**Theorem 2.9 ([10, p. 34, 4.13. Corollary]).** Assume that \( f \in G([a, b]) \) and \( g \in BV([a, b]) \). Then we have for every \( t \in [a, b] \)

\[
\lim_{n \to 0^+} \int_a^{t+\eta} f(s) \, dg(s) = \int_a^t f(s) \, dg(s) \pm f(t) \Delta^\pm g(t).
\]

The following result is the Hölder’s inequality for \( K \)-integral. In this paper we frequently use this inequality.
Theorem 2.10. (Hölder’s inequality) Assume that \( f, g \in G([a, b]) \) and \( h \) is a nondecreasing function defined on \([a, b]\). Let \( p > 1, \frac{1}{p} + \frac{1}{q} = 1 \). Then we have

\[
\int_a^b |fg| \, dh \leq \left( \int_a^b |f|^p \, dh \right)^{\frac{1}{p}} \left( \int_a^b |g|^q \, dh \right)^{\frac{1}{q}}.
\]

(2.1)

Proof. The proof of this theorem is very similar to the proof of the classical Hölder’s inequality. So we omit the proof.

\[\square\]

3. Stieltjes Derivatives

In this section we state the results in [4, 5] that are essential to obtain our main results.

Throughout this section, we assume that \( f \in G([a, b]) \) and \( g \) is a nondecreasing function on \([a, b]\).

We say that the function \( g \) is not locally constant at \( t \in (a, b) \) if there exists \( \eta > 0 \) such that \( g \) is not constant on \((t - \varepsilon, t + \varepsilon)\) for every \( 0 < \varepsilon < \eta \). We also say that the function \( g \) is not locally constant at \( a \) and \( b \), respectively if there exist \( \eta, \eta^* > 0 \) such that \( g \) is not constant on \([a, a + \varepsilon), (b - \varepsilon^*, b]\) respectively, for every \( \varepsilon \in (0, \eta), \varepsilon^* \in (0, \eta^*) \).

Definition 3.1 ([4]). If \( g \) is not locally constant at \( t \in (a, b) \), we define

\[
\frac{df}{dg}(t) = \lim_{\eta, \delta \to 0^+} \frac{f(t + \eta) - f(t - \delta)}{g(t + \eta) - g(t - \delta)},
\]

provided that the limit exists.

If \( g \) is not locally constant at \( t = a \) and \( t = b \) respectively, we define

\[
\frac{df}{dg}(a) = \lim_{\eta \to 0^+} \frac{f(a + \eta) - f(a)}{g(a + \eta) - g(a)}, \quad \frac{df}{dg}(b) = \lim_{\delta \to 0^+} \frac{f(b) - f(b - \delta)}{g(b) - g(b - \delta)},
\]

respectively, provided that the limits exist. Frequently we use \( f'_g(t) \) instead of \( \frac{df}{dg}(t) \).

Remark 3.2. It is obvious that if \( g \) is not continuous at \( t \) then \( f'_g(t) \) exists. Thus if \( f'_g(t) \) does not exist then \( g \) is continuous at \( t \). \( f'_g(t) \) is called a Stieltjes derivative of \( f \) with respect to \( g \).

Theorem 3.3 ([4]). Assume that if \( g \) is not locally constant at \( t \in [a, b] \). If \( f \) is continuous at \( t \) or \( g \) is not continuous at \( t \), then we have
\[
\frac{d}{dg(t)} \int_a^t f(s) \, dg(s) = f(t).
\]

K*-integrals recover Stieltjes derivatives.

**Theorem 3.4** ([4]). Assume that if \( g \) is constant on some neighborhood of \( t \) then there is a neighborhood of \( t \) where both \( f \) and \( g \) are constant. Suppose that \( f'_g(t) \) exists at every \( t \in [a, b] - \{c_1, c_2, \ldots\} \), where \( f \) is continuous at every \( t \in \{c_1, c_2, \ldots\} \). Then we have

\[
(K^*) \int_a^b f'_g(s) \, dg(s) = f(b) - f(a).
\]

**Lemma 3.5** ([4]). Assume that if \( g \) is constant on some neighborhood of \( t \) then there is a neighborhood of \( t \) such that both \( f_1 \) and \( f_2 \) are constant there. If both \( \frac{df_1(t)}{dg(t)} \) and \( \frac{df_2(t)}{dg(t)} \) exist and \( f_1, f_2 \in G([a, b]) \), then we have

\[
\frac{d[f_1(t)f_2(t)]}{dg(t)} = \frac{df_1(t)}{dg(t)} f_2(t^+) + f_1(t^-) \frac{df_2(t)}{dg(t)}.
\]

Similarly to the Riemann integral we have the following integration by parts formula.

**Theorem 3.6.** (Integration by Parts) Assume that functions \( f, g, h \in G([a, b]) \) are all left-continuous and \( h \) is nondecreasing. Suppose that both \( f'_h(t) \) and \( g'_h(t) \) exist for every \( t \in [a, b] \) and \( f'_h, g'_h \in G([a, b]) \). Then we have

\[
(3.1) \quad \int_a^b f'_h g \, dh = f(b)g(b) - f(a)g(a) - \int_a^b f g'_h \, dh + \sum_{a \leq t \leq b} [\Delta^- f(t)\Delta^- g(t) - \Delta^+ f(t)\Delta^+ g(t)].
\]

**Proof.** By Theorem 2.8 and Theorem 3.4 we have

\[
\int_a^b f \, dg = \int_a^b f(s) \left[ \int_s^b g'_h \, dh \right] = \int_a^b f g'_h \, dh.
\]

So by Theorem 2.6 we get
\[
\int_a^b f'_h g \, dh = \int_a^b g(s) \, d \left[ \int_a^s f'_h \, dh \right] = \int_a^b g \, df
\]

\[
= f(b)g(b) - f(a)g(a) - \int_a^b f \, dg + \sum_{a \leq t \leq b} [\Delta^- f(t)\Delta^- g(t) - \Delta^+ f(t)\Delta^+ g(t)]
\]

\[
= f(b)g(b) - f(a)g(a) - \int_a^b fg' \, dh + \sum_{a \leq t \leq b} [\Delta^- f(t)\Delta^- g(t) - \Delta^+ f(t)\Delta^+ g(t)].
\]

This completes the proof. \qed

Let

\[ a \leq t_1 < t_2 < \cdots < t_m < b. \]

The Heaviside function \( H_\tau : \mathbb{R} \rightarrow \{0, 1\} \) is defined by

\[
H_\tau(t) = \begin{cases} 
0, & \text{if } t \leq \tau \\
1, & \text{if } t > \tau.
\end{cases}
\]

Using the Heaviside function \( H_\tau \), we define function \( \phi : [a, b] \rightarrow \mathbb{R} \) by

\[
(3.2) \quad \phi(t) = t + \sum_{k=1}^m H_{t_k}(t), \quad t \in [a, b].
\]

**Remark 3.7.** It is obvious that the function \( \phi \) is strictly increasing and of bounded variation on \([a, b] \), and left-continuous on \([a, b] \).

**Lemma 3.8** ([5]). Assume that \( f \in G([a, b]) \) and \( f'(t) \) exists for \( t \neq t_k, k = 1, m \)

Then we have

(a) \quad \int_{t_k}^t f'_\phi(s) \, ds = f(t) - f(t_k^-), \quad f'_\phi(t_k) = f(t_k^+) - f(t_k^-),

(b) \quad \int_a^t f \, d\phi = \int_a^t f(s) \, ds + \sum_{a \leq t_k < t} f(t_k).

4. **Opial-type Integral Inequalities involving Stieltjes Derivatives**

In this section we obtain some Opial-type integral inequalities involving Stieltjes derivatives. The Opial-type inequalities have many interesting applications in the theory of differential equations (see, e.g., [1]).

Throughout this paper we always assume that

\[ a \leq t_1 < t_2 < \cdots < t_m < b, \]
and that a function \( \alpha : [a, b] \rightarrow \mathbb{R} \) is strictly increasing on \([a, b]\), and continuous at \( t \neq t_k \), and \( \Delta \alpha(t_k) \neq 0 \), for every \( k = 1, m \).

**Remark 4.1.** Note that strictly increasing implies nondecreasing, and a nondecreasing function is regulated.

Let \( PC([a, b]) = \{ u \in G([a, b]) : u \) is continuous at every \( t \neq t_k, k = 1, m \}. \)

From now on we always assume that \( u, u'_\alpha \in PC([a, b]) \), and we define

\[
u_+(t) = u(t+), \quad u_-(t) = u(t-), \quad \forall t \in [a, b].
\]

The following result is an Opial-type inequality with Stieltjes derivatives.

**Theorem 4.2.** Assume that \( u(a) = u(b) = 0 \). If both \( u \) and \( \alpha \) are left-continuous on \([a, b]\), then we have

\[
\int_a^b (|u| + |u_+|)|u'_\alpha| \, d\alpha \leq K \int_a^b (u'_\alpha)^2 \, d\alpha,
\]

where \( K = \inf_{h \in [a,b]} \max \{ \alpha(h) - \alpha(a), \alpha(b) - \alpha(h) \} \).

**Proof.** Let for \( t \in [a, b] \),

\[
y(t) = \int_a^t |u'_\alpha| \, d\alpha, \quad z(t) = \int_t^b |u'_\alpha| \, d\alpha.
\]

By Theorem 2.9, the functions, \( y \) and \( z \) are left-continuous on \([a, b]\). Also by Theorem 3.3, we have

\[
y'_\alpha(t) = |u'_\alpha(t)| = -z'_\alpha(t)
\]

and we have by Theorem 3.4 and \( u(a) = u(b) = 0 \)

\[
|u(t)| \leq y(t), \quad |u(t)| \leq z(t),
\]

for \( t \in [a, b] \). So by Theorem 3.4, Lemma 3.5, and using Hölder’s inequality, we get

\[
\int_a^h (|u| + |u_+|)|u'_\alpha| \, d\alpha \leq \int_a^h (y + y_+)y'_\alpha \, d\alpha = \int_a^h (y^2)' \, d\alpha
\]

\[
= y^2(h) = [\alpha(h) - \alpha(a)] \int_a^h (u'_\alpha)^2 \, d\alpha,
\]

and similarly we obtain

\[
\int_h^b (|u| + |u_+|)|u'_\alpha| \, d\alpha \leq -\int_h^b (z + z_+)z'_\alpha \, d\alpha = -\int_h^b (z^2)' \, d\alpha = z^2(h)
\]

\[
\leq [\alpha(b) - \alpha(h)] \int_h^b (u'_\alpha)^2 \, d\alpha.
\]
So we have
\[
\int_a^b (|u| + |u_+|) |u'_\alpha| \, d\alpha = \int_a^h (|u| + |u_+|) |u'_\alpha| \, d\alpha + \int_h^b (|u| + |u_+|) |u'_\alpha| \, d\alpha \\
\leq [\alpha(h) - \alpha(a)] \int_a^h (u'_\alpha)^2 \, d\alpha + [\alpha(b) - \alpha(h)] \int_h^b (u'_\alpha)^2 \, d\alpha \\
\leq K_\alpha \int_a^b (u'_\alpha)^2 \, d\alpha.
\]
The proof is complete. □

A slightly more general result is as follows.

**Theorem 4.3.** Assume that \( u(b) = 0 \). If both \( u \) and \( \alpha \) are left-continuous on \([a, b]\), then we have
\[
(4.3) \quad \int_a^b (|u| + |u_+|) |u'_\alpha| \, d\alpha \leq [\alpha(b) - \alpha(a)] \int_a^b (u'_\alpha)^2 \, d\alpha.
\]

**Proof.** From (4.2) we have
\[
\int_a^b (|u| + |u_+|) |u'_\alpha| \, d\alpha \leq [\alpha(b) - \alpha(h)] \int_h^b (u'_\alpha)^2 \, d\alpha \leq [\alpha(b) - \alpha(a)] \int_a^b (u'_\alpha)^2 \, d\alpha.
\]
So we get
\[
\int_a^b (|u| + |u_+|) |u'_\alpha| \, d\alpha \leq [\alpha(b) - \alpha(a)] \int_a^b (u'_\alpha)^2 \, d\alpha.
\]
This gives (4.3). The proof is complete. □

More generally we have the following result.

**Theorem 4.4.** Let \( p \geq 0, q \geq 1, r \geq 0, m \geq 1 \) be real numbers and let \( f \in PC([a, b]) \) be a positive function on \([a, b]\) with \( \inf_{s \in [a, b]} f(s) > 0 \). Assume that both functions \( u \) and \( \alpha \) are left-continuous on \([a, b]\). If \( u(b) = 0 \), then we have
\[
(4.4) \quad \int_a^b f \, |u|^{m(p+q)} |u'_\alpha|^{mr} \, d\alpha \leq [(p + q + r)^m I(m, f)]^{p+q} \int_a^b f \, |u,v'|^{m(p+q+r)} \, d\alpha,
\]
where \( I(m, f) = \int_a^b f \gamma \, d\alpha \), \( \gamma(t) = \left[ f_s^b (f) \gamma(t) \right]^{m-1} \) for \( m \neq 1 \), and \( \gamma(t) = \left[ \inf_{s \in [t,b]} f(s) \right]^{-1} \) for \( m = 1 \).

**Proof.** Let for \( t \in [a, b] \),
\[
z(t) = \int_t^b |u'_\alpha| \, d\alpha.
\]
Then by Theorem 3.4, \(|u(t)| \leq z(t)\) and by Theorem 2.9, \(z\) is left-continuous, and non-increasing on \([a, b]\).

If \(t \neq t_k, k = 1, m\), then by Theorem 3.3, \(z'_\alpha(t)\) exists, and by Theorem 2.9, \(z\) is continuous at \(t\). Using the Mean Value Theorem and by the definition of the Stieltjes derivatives, if \(z\) is not locally constant at \(t\), then we have,

\[
(z^{p+q})'_\alpha(t) = \lim_{\delta, \eta \to 0^+} \frac{z^{p+q}(t + \eta) - z^{p+q}(t - \delta)}{\alpha(t + \eta) - \alpha(t - \delta)} - z(t - \delta)
\]

Then by hypotheses, if \(\alpha\) is non-increasing on \([a, b]\), \(z\) is left-continuous, and \(z'_\alpha = -|u'_\alpha| \leq 0\), and by the Mean Value Theorem, and by the definition of the Stieltjes derivatives, we have,

\[
(z^{p+q})'_\alpha(t_k) = [z^{p+q}(t_k) - z^{p+q}(t_k-)]/[\alpha(t_k+) - \alpha(t_k-)]
\]

Thus we have

\[
-(z^{p+q})'_\alpha(t) \leq -(p + q)z^{p+q-1}(t)z'_\alpha(t), \quad \forall t \in [a, b].
\]

Let \(\beta(t) = \int_t^t f\ d\alpha\). Then by hypotheses, \(\beta\) is strictly increasing on \([a, b]\).

Since

\[
(z^{p+q})'_\beta(t) = \lim_{\delta, \eta \to 0^+} \frac{z^{p+q}(t + \eta) - z^{p+q}(t - \delta)}{\beta(t + \eta) - \beta(t - \delta)} - \beta(t - \delta)
\]

we have by Theorem 3.4 and (4.5), since \(z(b) = 0\) and \(z'_\alpha \leq 0\),

\[
z^{p+q}(t) = -\int_t^t (z^{p+q})'_\beta d\beta = -\int_t^t f^{-1}(z^{p+q})'_\alpha d\beta
\]

\[
\leq (p + q)\int_t^t f^{-1}z^{p+q-1}(-z'_\alpha) d\beta = (p + q)\int_t^t f^{-1}z^{p+q-1}|z'_\alpha| d\beta.
\]

Using Hölder’s inequality with indices \(m, \frac{m}{m-1}\), we have

\[
z^{m(p+q)}(t) \leq (p + q)^m \gamma(t)\int_a^b z^{m(p+q-1)}|z'_\alpha|^m d\beta, \quad \forall t \in [a, b].
\]
Integrating (4.6) on \([a, b]\) and using Hölder’s inequality with indices \(q, \frac{q}{q-1}\), and considering \(\int_a^b f^\gamma \, d\beta = \int_a^b f\, d\alpha\) by Theorem 2.8, we get

\[
(4.7) \quad \int_a^b z^{m(p+q)} \, d\beta \\
\quad \leq (p + q)^m I(m, f) \int_a^b \left( z^{\frac{mp}{q}} |z'_\alpha|^m \right) \cdot z^{m(p+q-1) - \frac{mp}{q}} \, d\beta \\
\quad \leq (p + q)^m I(m, f) \left( \int_a^b z^{mp} |z'_\alpha|^m \, d\beta \right) \left( \int_a^b z^{m(p+q)} \, d\beta \right)^{\frac{q-1}{q}}.
\]

If \(\int_a^b z^{m(p+q)} \, d\beta = 0\), then

\[
(4.8) \quad \int_a^b z^{m(p+q)} \, d\beta \leq [(p + q)^m I(m, f)]^q \int_a^b z^{mp} |z'_\alpha|^m \, d\beta
\]

is obviously true, otherwise, dividing both sides of (4.7) by \(\left( \int_a^b z^{m(p+q)} \, d\beta \right)^{\frac{q-1}{q}}\) and then taking the \(q\)th power on both sides of the resulting inequality we get also (4.8).

Using the Hölder’s inequality with indices \(\frac{p+q}{r}, \frac{p+q}{q}\), we have, by (4.8),

\[
(4.9) \quad \int_a^b z^{m(p+q)} |z'_\alpha|^{mr} \, d\beta \\
\quad = \int_a^b \left[ z^{m(p/(q+r))} |z'_\alpha|^{mr} \cdot z^{m(p+q)-m(p/(q+r))} \right] \, d\beta \\
\quad \leq \left[ \int_a^b z^{mp} |z'_\alpha|^{m(q+r)} \, d\beta \right]^{r/(q+r)} \left[ \int_a^b z^{m(p+q+r)} \, d\beta \right]^{q/(q+r)} \\
\quad \leq \left[ \int_a^b z^{mp} |z'_\alpha|^{m(q+r)} \, d\beta \right]^{\frac{r}{r+q}} \left[ (p + q + r)^m I(m, f) \right]^{q+q} \int_a^b z^{mp} |z'_\alpha|^{m(q+r)} \, d\beta \\
\quad = [(p + q + r)^m I(m, f)]^q \int_a^b z^{mp} |z'_\alpha|^{m(q+r)} \, d\beta.
\]

Using Hölder’s inequality with indices \(\frac{q+r}{p}, \frac{q+r}{q}\), we get by (4.9)

\[
(4.10) \quad \int_a^b z^{m(p+q)} |z'_\alpha|^{mr} \, d\beta \leq [(p + q + r)^m I(m, f)]^q \int_a^b z^{mp} |z'_\alpha|^{m(q+r)} \, d\beta, \\
\quad \leq [(p + q + r)^m I(m, f)]^q \int_a^b \left[ z^{mp} |z'_\alpha|^{m(r/(p+q))} \right] \cdot \left[ |z'_\alpha|^{m(q+r)-m(rp/(p+q))} \right] \, d\beta.
\]
\[ \int_{a}^{b} z^{m(p+q)} |z'|^{mr} \, d\beta \]

Then the inequality

\[ \int_{a}^{b} z^{m(p+q)} |z'|^{mr} \, d\beta \leq [(p + q + r)^m I(m, f)]^\frac{p}{p+q} \int_{a}^{b} |z'|^{m(p+q+r)} \, d\beta \]

is obviously true, otherwise, dividing both sides of (4.10) by \( \int_{a}^{b} z^{m(p+q)} |z'|^{mr} \, d\beta \) and then taking the \( \frac{p}{p+q} \)th power on both sides of the resulting inequality we get also (4.11). Since \( |u| \leq z \) and \( |u'|_{\alpha} = |z'|_{\alpha} \) we have

\[ \int_{a}^{b} f |u|^{m(p+q)} |u'|_{\alpha}^{mr} \, d\alpha = \int_{a}^{b} |u|^{m(p+q)} |u'|_{\alpha}^{mr} \, d\beta \leq \int_{a}^{b} z^{m(p+q)} |z'|^{mr} \, d\beta, \]  

by (4.11)

\[ \leq [(p + q + r)^m I(m, f)]^\frac{p}{p+q} \int_{a}^{b} |z'|^{m(p+q+r)} \, d\beta \]

This gives (4.4). The proof is complete.

5. Some Applications to certain Differential Equations involving Impulses

In this section we always assume that both functions \( u \) and \( u' \) are left-continuous on \( [a, b] \), and that \( \alpha = \phi \) (see (3.2)). Consider the following impulsive differential equation: for \( k = \overline{1, m} \),

\[ u'' + q(t)u = 0, \quad t \neq t_k, \]

\[ \Delta u'(t_k) = a_k u'(t_k), \]

\[ \Delta u(t_k) = b_k u'(t_k), \quad b_k \neq 0, \]

where \( q_1 \in PC([a, b]) \). Now we define

\[ u'''_{\alpha}(t) = (u')'_{\alpha}(t). \]

Since by Lemma 3.8 for \( k = \overline{1, m} \)

\[ u'''_{\alpha}(t) = \begin{cases} 
  u''(t), & t \neq t_k \\
  \Delta u'(t_k), & t = t_k,
\end{cases} \]

the equation (5.1) implies the following equation:

\[ u'''_{\alpha} + p(t)u' + q(t)u = 0, \]
If the function \( u \) satisfy the equation (5.1) and \( c \in [a, b] \), then we have

\[
(5.3) \quad p(t) = \begin{cases} 
0, & t \neq t_k \\
-a_k, & t = t_k,
\end{cases} \quad q(t) = \begin{cases} 
q_1(t), & t \neq t_k \\
0, & t = t_k, \quad k = 1, m.
\end{cases}
\]

We need the following result.

**Lemma 5.1.** If the function \( u \) satisfies the equation (5.1) and \( c \in [a, b] \), then we have

\[
(5.4) \quad \int_a^c |u'| \, d\alpha = \int_a^c |u'_\alpha'| \, d\alpha + \sum_{a \leq t_k < c} (1 - |b_k|) |u(t_k)u'(t_k)|,
\]

\[
(5.5) \quad \sum_{a \leq t_k < c} |b_k| |u(t_k)u'(t_k)| \leq \int_a^c |u'_\alpha'| \, d\alpha,
\]

\[
(5.6) \quad \int_a^c u'u'_\alpha \, d\alpha = \int_a^c |u'_\alpha|^2 \, d\alpha + \sum_{a \leq t_k < c} (b_k - b_k^2) |u'(t_k)|^2,
\]

\[
(5.7) \quad \sum_{a \leq t_k < c} b_k^2 |u'(t_k)|^2 \leq \int_a^c |u'_\alpha|^2 \, d\alpha.
\]

**Proof.** In the proof, we frequently use Lemma 3.8, \( u'(t) = u'_\alpha(t) \), \( t \neq t_k \), \( \Delta u'(t_k) = u''(t_k) = a_k u'(t_k) \), and \( \Delta u(t_k) = u'_\alpha(t_k) = b_k u'(t_k) \), \( k = 1, m \).

\[
\int_a^c |u'| \, d\alpha = \int_a^c |u(s)u'(s)| \, ds + \sum_{a \leq t_k < c} |u(t_k)u'(t_k)|
= \int_a^c |u(s)u'_\alpha(s)| \, ds + \sum_{a \leq t_k < c} |u(t_k)u'(t_k)|
= \int_a^c |u(s)u'_\alpha(s)| \, ds + \sum_{a \leq t_k < c} |u(t_k)u'_\alpha(t_k)| - \sum_{a \leq t_k < c} |u(t_k)u'(t_k)| + \sum_{a \leq t_k < c} |u(t_k)u'(t_k)|
= \int_a^c |u'_\alpha| \, d\alpha - \sum_{a \leq t_k < c} |u(t_k)u'_\alpha(t_k)| + \sum_{a \leq t_k < c} |u(t_k)u'(t_k)|
= \int_a^c |u'_\alpha| \, d\alpha - \sum_{a \leq t_k < c} b_k |u(t_k)u'(t_k)| + \sum_{a \leq t_k < c} |u(t_k)u'(t_k)|
= \int_a^c |u'_\alpha| \, d\alpha + \sum_{a \leq t_k < c} (1 - |b_k|) |u(t_k)u'(t_k)|.
\]
This gives (5.4). And
\[
\sum_{a \leq t_k < c} |u(t_k)u'(t_k)| \leq \int_a^c |u(s)u'(s)| \, ds + \sum_{a \leq t_k < c} |u(t_k)u'(t_k)|
\]
\[
= \int_a^c |u(s)u'_a(s)| \, ds + \sum_{a \leq t_k < c} |u(t_k)u'_a(t_k)| - \sum_{a \leq t_k < c} |u(t_k)u'_a(t_k)| + \sum_{a \leq t_k < c} |u(t_k)u'(t_k)|
\]
\[
= \int_a^c |uu'_a| \, d\alpha - \sum_{a \leq t_k < c} b_k|u(t_k)u'(t_k)| + \sum_{a \leq t_k < c} |u(t_k)u'(t_k)|.
\]
This gives (5.5). And
\[
\int_a^c u'u'_a \, d\alpha = \int_a^c u'(s)u'_a(s) \, ds + \sum_{a \leq t_k < c} u'(t_k)u'_a(t_k)
\]
\[
= \int_a^c |u'_a|^2 \, ds + \sum_{a \leq t_k < c} u'(t_k)u'_a(t_k)
\]
\[
= \int_a^c |u'_a|^2 \, ds + \sum_{a \leq t_k < c} |u'_a(t_k)|^2 - \sum_{a \leq t_k < c} |u'_a(t_k)|^2 + \sum_{a \leq t_k < c} u'(t_k)u'_a(t_k)
\]
\[
= \int_a^c |u'_a|^2 \, d\alpha - \sum_{a \leq t_k < c} b_k^2|u'(t_k)|^2 + \sum_{a \leq t_k < c} b_k|u'(t_k)|^2
\]
\[
= \int_a^c |u'_a|^2 \, d\alpha + \sum_{a \leq t_k < c} (b_k - b_k^2)|u'(t_k)|^2.
\]
This gives (5.6). Also
\[
\sum_{a \leq t_k < c} |u'(t_k)|^2 \leq \int_a^c |u'(s)|^2 \, ds + \sum_{a \leq t_k < c} |u'(t_k)|^2
\]
\[
= \int_a^c |u'_a(s)|^2 \, ds + \sum_{a \leq t_k < c} |u'(t_k)|^2
\]
\[
= \int_a^c |u'_a(s)|^2 \, ds + \sum_{a \leq t_k < c} |u'_a(t_k)|^2 - \sum_{a \leq t_k < c} |u'_a(t_k)|^2 + \sum_{a \leq t_k < c} |u'(t_k)|^2
\]
\[
= \int_a^c |u'_a|^2 \, d\alpha - \sum_{a \leq t_k < c} |u'_a(t_k)|^2 + \sum_{a \leq t_k < c} |u'(t_k)|^2
\]
\[
= \int_a^c |u'_a|^2 \, d\alpha - \sum_{a \leq t_k < c} b_k^2|u'(t_k)|^2 + \sum_{a \leq t_k < c} |u'(t_k)|^2.
\]
This gives (5.7). The proof is complete.
**Theorem 5.2.** Assume that $u$ satisfies the equation (5.1) and $u'(a) = 0, u(a) \neq 0$.
If we have

\[
1 > \alpha(b) - \alpha(a) \left[ \max_{a \leq s \leq b} |Q(s)| + \max_{a \leq t_k \leq b} \left| a_k \right| \left( 1 + \max_{a \leq t_k \leq b} \frac{|1 - b_k|}{|b_k|} \right) \right]
+ \max_{a \leq t_k \leq b} \left| \frac{1 - b_k + a_k}{b_k} \right|
\]

where $Q(t) = \int_a^t q \, d\alpha$, then $u(t) \neq 0$ for every $t \in [a, b]$.

**Proof.** Assume that there is a number $c \in (a, b)$ with $u(c) = 0$. Then multiplying both sides of (5.2) by $u$ and integrating we have

\[
\int_a^c uu'' \, d\alpha + \int_a^c pu' \, d\alpha + \int_a^c qu^2 \, d\alpha = 0.
\]

Using Theorem 3.3, Lemma 3.5 and Theorem 3.6, and $u(c) = Q(a) = 0$, we get, since, by Theorem 2.9 and Remark 3.7, $Q$ is left-continuous on $[a, b]$, and $\Delta \alpha(t_k) = \Delta^+ \alpha(t_k) = 1, q(t_k) = 0, k = \frac{1}{m}$,

\[
\int_a^c qu^2 \, d\alpha = \int_a^c Q' u^2 \, d\alpha
= [Qu^2]_a^c - \int_a^c Q(u^2)' \, d\alpha - \sum_{a \leq t_k < c} \Delta^+ Q(t_k) \Delta^+ u^2(t_k), \quad \text{since} \quad \Delta^- Q(t_k) = 0,
= - \int_a^c Q(u^2)' \, d\alpha - \sum_{a \leq t_k < c} q(t_k)(u^2)'(t_k), \quad \text{by Theorem 2.9}
= - \int_a^c Q(u + u+)u'_\alpha \, d\alpha.
\]

Since both $u$ and $u'$ are left-continuous

\[
\Delta^+ u'(t_k) = \Delta u'(t_k) = a_k u'(t_k),
\Delta^+ u(t_k) = \Delta u(t_k) = b_k u'(t_k).
\]

By Lemma 3.8 and Lemma 5.1, we get, since $u(c) = u'(a) = 0$,

\[
\int_a^c uu'' \, d\alpha = \int_a^c u(u')' \, d\alpha
\]
By (5.9), (5.10) and (5.11), we have

\[ (5.12) \]

\[
[u'']_a^c - \int_a^c u'_a u' \, d\alpha - \sum_{a \leq t_k < c} \Delta^+ u(t_k) \Delta^+ u'(t_k), \quad \text{since} \quad \Delta^- u(t_k) = 0
\]

\[ = - \int_a^c u'_a u' \, d\alpha - \sum_{a \leq t_k < c} a_k b_k |u'(t_k)|^2
\]

\[ = - \int_a^c |u'_a|^2 \, d\alpha - \sum_{a \leq t_k < c} (b_k - b_k^2) |u'(t_k)|^2 - \sum_{a \leq t_k < c} a_k b_k |u'(t_k)|^2
\]

\[ = - \int_a^c |u'_a|^2 \, d\alpha - \sum_{a \leq t_k < c} b_k (1 - b_k + a_k) |u'(t_k)|^2.
\]

By (5.9), (5.10) and (5.11), we have

\[
\int_a^c (u'_a)^2 \, d\alpha + \int_a^c Q(u + u_+) u'_a \, d\alpha
\]

\[ - \int_a^c p uu' \, d\alpha + \sum_{a \leq t_k < c} b_k (1 - b_k + a_k) |u'(t_k)|^2 = 0.
\]

Hence by Theorem 4.3 and Lemma 5.1, we get

\[
(5.12) \quad \int_a^c (u'_a)^2 \, d\alpha \leq \int_a^c |Q|||u| + |u_+||u'_a| \, d\alpha + \int_a^c |p| uu' \, d\alpha
\]

\[ + \sum_{a \leq t_k < c} |b_k||1 - b_k + a_k||u'(t_k)|^2
\]

\[ \leq \max_{a \leq s \leq c} |Q(s)||\alpha(c) - \alpha(a)| \int_a^c (u'_a)^2 \, d\alpha + \max_{a \leq t_k < c} \left|p(t_k)\right| \int_a^c |u u'| \, d\alpha
\]

\[ + \sum_{a \leq t_k < c} \left|1 - b_k + a_k\right| b_k^2 |u'(t_k)|^2
\]

\[ \leq \max_{a \leq s \leq c} |Q(s)||\alpha(c) - \alpha(a)| \int_a^c (u'_a)^2 \, d\alpha + \max_{a \leq t_k < c} \left|b_k\right| \int_a^c |u u'| \, d\alpha
\]

\[ + \max_{a \leq t_k \leq c} \left|1 - b_k + a_k\right| \sum_{a \leq t_k < c} b_k^2 |u'(t_k)|^2
\]

\[ \leq \max_{a \leq s \leq c} |Q(s)||\alpha(c) - \alpha(a)| \int_a^c (u'_a)^2 \, d\alpha
\]

\[ + \max_{a \leq t_k \leq c} \left|b_k\right| \left( \int_a^c |u u'_a| \, d\alpha + \sum_{a \leq t_k < c} |1 - |b_k||u(t_k) u'(t_k)|\right)
\]

\[ + \max_{a \leq t_k \leq c} \left|1 - b_k + a_k\right| \int_a^c (u'_a)^2 \, d\alpha
\]
\[
\leq \max_{a \leq s \leq c} |Q(s)| \|\alpha(c) - \alpha(a)\| \int_a^c (u'_a)^2 \, d\alpha \\
+ \max_{a \leq t_k \leq c} |a_k| \left( \int_a^c |uu'_a| \, d\alpha + \sum_{a \leq t_k < c} \frac{1 - |b_k|}{|b_k|} |b_k| |u(t_k)u'(t_k)| \right) \\
+ \max_{a \leq t_k \leq c} \frac{1 - b_k + a_k}{|b_k|} \int_a^c (u'_a)^2 \, d\alpha \\
\leq \max_{a \leq s \leq c} |Q(s)| \|\alpha(c) - \alpha(a)\| \int_a^c (u'_a)^2 \, d\alpha \\
+ \max_{a \leq t_k \leq c} |a_k| \left( \int_a^c |uu'_a| \, d\alpha + \max_{a \leq t_k \leq c} \frac{1 - |b_k|}{|b_k|} \int_a^c |uu'_a| \, d\alpha \right) \\
+ \max_{a \leq t_k \leq c} \frac{1 - b_k + a_k}{|b_k|} \int_a^c (u'_a)^2 \, d\alpha \\
\leq \max_{a \leq s \leq c} |Q(s)| \|\alpha(c) - \alpha(a)\| \int_a^c (u'_a)^2 \, d\alpha \\
+ \max_{a \leq t_k \leq c} |a_k| |\alpha(c) - \alpha(a)| \left( 1 + \max_{a \leq t_k \leq c} \frac{1 - |b_k|}{|b_k|} \right) \int_a^c (u'_a)^2 \, d\alpha \\
+ \max_{a \leq t_k \leq c} \frac{1 - b_k + a_k}{|b_k|} \int_a^c (u'_a)^2 \, d\alpha.
\]

If
\[
0 = \int_a^c (u'_a)^2 \, d\alpha = \int_a^c (u'_a)^2(s) \, ds + \sum_{a \leq t_k < c} (u'_a)^2(t_k),
\]
then, since \(0 = \int_a^c (u'_a)^2(s) \, ds = \int_a^c (u')^2(s) \, ds, u'(t) = 0, \forall t \in [a, b] - \{t_k : t_k < c\}\)
and \(u'(t_k) = u(t_{k+}) - u(t_k) = 0\). This implies that \(u\) is a constant on \([a, c]\). So
\(u(c) = u(a) \neq 0\). But this is a contradiction to \(u(c) = 0\). Hence we conclude that
\(\int_a^c (u'_a)^2 \, d\alpha > 0\).

In (5.12), canceling \(\int_a^c (u'_a)^2 \, d\alpha\), we get a contradiction to (5.8). This completes
the proof. \(\square\)

In the following result we apply Theorem 4.4.

**Theorem 5.3.** Let \(q \in PC([a, b])\) and let \(\alpha = \phi\) (see (3.2)). If \(u \in PC([a, b])\) is
left-continuous and a nontrivial solution of the following equation:
\[
(u'_a)^m + \frac{q(t)u^{m+1}}{1 + |u| + (u')^2} = 0, \quad u(b) = 0, \quad (m = 1, 3, 5, \ldots),
\]
then we have
\[
1 \leq I(m, 1) \max_{a \leq s \leq b} |q(s)|.
\]
Proof. Substituting \( f \equiv 1, p = 0, q = 1, r = 0 \) into Theorem 4.4, then we have
\[
\int_a^b |u|^{-m} \, d\alpha \leq I(m, 1) \int_a^b |u'|^{-m} \, d\alpha.
\]
So we have
\[
\int_a^b |u'|^{-m} \, d\alpha \leq \int_a^b \frac{|q||u|^{-m+1}}{1 + |u| + (u')^2} \, d\alpha \leq \int_a^b |q||u|^{-m} \, d\alpha
\]
\[
\leq I(m, 1) \max_{a \leq s \leq b} |q(s)| \int_a^b |u'|^{-m} \, d\alpha.
\]
Canceling \( \int_a^b |u'|^{-m} \, d\alpha \), we get (5.13). \( \square \)

REFERENCES

1. R.P. Agarwal & P.Y.H. Pang: Opial inequalities with applications in differential and difference equations. Kluwer Academic Publishers, Dordrecht, 1995.
2. R. Henstock: Lectures on the theory of integration. World Scientific, Singapore, 1988.
3. C.S. Höning: Volterra Stieltjes-integral equations. North Holland and American Elsevier, Mathematics Studies 16, Amsterdam and New York, 1973.
4. Y.J. Kim: Stieltjes derivatives and its applications to integral inequalities of Stieltjes type. J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. 18 (2011), no. 1, 63-78.
5. ______: Stieltjes derivative method for integral inequalities with impulses. J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. 21 (2014), no. 1, 61-75.
6. P. Krejčí & J. Kurzweil: A nonexistence result for the Kurzweil integral. Math. Bohem. 127 (2002), 571-580.
7. W.F. Pfeffer: The Riemann approach to integration: local geometric theory. Cambridge Tracts in Mathematics 109, Cambridge University Press, 1993.
8. A.M. Samoilenko & N.A. Perestyuk: Impulsive differential equations. World Scientific, Singapore, 1995.
9. Š. Schwabik: Generalized ordinary differential equations. World Scientific, Singapore, 1992.
10. Š. Schwabik, M. Tvrdý & O. Vejvoda: Differential and integral equations: boundary value problems and adjoints. Academia and D. Reidel, Praha and Dordrecht, 1979.
11. M. Tvrdý: Regulated functions and the Perron-Stieltjes integral. Časopis pešť. mat. 114 (1989), no. 2, 187-209.

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