On the existence of curves in $K$-trivial threefolds

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October 1998

Abstract

We give a criterion for a continuous family of curves on a nodal $K$-trivial threefold $X_0$ to contribute geometrically rigid curves to a general smoothing of $X_0$. As an application, we prove the existence of geometrically rigid curves of arbitrary degree and explicitly bounded genus on general complete intersection Calabi-Yau threefolds.

Mathematics Subject Classifications (1991): 14C05, 14J32 (Primary); 14C25, 14J28 (Secondary).

Key words: Calabi-Yau threefolds, Hilbert schemes, K3 surfaces, rigid embeddings.

0 Introduction

0.1 Overview Let $X_0$ be a nodal, $K$-trivial threefold, and

$$\begin{array}{ccc}
C & \longrightarrow^- q & X_0 \\
\downarrow^p & & \\
\Lambda & & 
\end{array}$$

a connected, complete, universal family of embeddings of curves in $X$; in other words, $p$ is the universal curve over a component of the Hilbert scheme of curves in $X_0$. Let $X_t$ be a general deformation of $X_0$.

**Question** Does the continuous family $p$ contribute only (geometrically) rigid curves to $X_t$?

0.2 Statements The principal result of this work—Theorem 1.1—is an affirmative answer under the main assumptions that $X_t$ is a family of zero-schemes of regular sections of a locally free sheaf $\mathcal{E}$ on $P$ (an ambient variety where the fibers of $p$ are strongly unobstructed), that $q$ factors through a regular ($h^1(0) = 0$) surface $S \subset X_0$, and that $n - 2 \geq \ell$, where $n$ is the number of nodes of $X_0$ lying in $S$ and $\ell = \dim \Lambda$. 
The central ingredient of the proof is the identification—at least up to extension—in [2] of the sheaf of obstructions on Λ with the sheaf of differentials on Λ with logarithmic poles along n hyperplanes.

The motivating application for this work is Theorem 2.1, the existence of geometrically rigid curves of genus g and degree d in complete intersection Calabi-Yau threefolds, where—with a short list of explicit exceptions—

\[ g \leq n(X_0) - 2 \quad \text{and} \quad d \geq 2g - 3. \]

Here \( n(X_0) \) is the maximal number of ordinary nodes of a general ciCY threefold containing a smooth complete intersection K3 surface.

This result is known for rational curve in quintic threefolds by work of Clemens [1] as refined by Katz [4] and in arbitrary ciCY threefolds by work of Ekedahl-Johnsen-Sommervoll [3]. For elliptic curves it was proved by the author in [5], where the result of [3] were also recovered. Though the final deformation argument given in [3] was of an \textit{ad hoc} nature, the main theorem—to the effect that for general \( X_0 \), the linear system of curves on the K3 surface \( S \) is universal as a family of curves in \( X_0 \)—allows the application of Theorem 1.1 to prove Theorem 2.1 in the present work.

0.3 Acknowledgments The author thanks Madhav Nori for asking the question, János Kollár for his generous support, and especially Herb Clemens, for many hours of stimulating discussion.

0.4 Conventions and Terminology All schemes are of finite type over \( \mathbb{C} \). If \( Z \subset X \) is a closed subscheme, we write \( N_{Z/X} \) for its normal sheaf. We say \( Z \) is \textit{geometrically rigid} in \( X \) if the space of embedded deformation of \( Z \) in \( X \) is zero-dimensional. If, furthermore, this space is reduced, we say that \( Z \) is \textit{infinitesimally rigid} in \( X \). If \( \mathcal{E} \) is a locally free sheaf on a scheme \( X \), and \( s \in \Gamma(X, \mathcal{E}) \) a global section, we let \( Z(s) \) be the zero-scheme of \( s \). If \( Z(s) \hookrightarrow X \) is a regular embedding, there is an exact sequence

\[ 0 \rightarrow N_{Z(s)/X} \rightarrow \mathcal{E}
\]

of locally free \( \mathcal{O}_{Z(s)} \)-modules (see, e.g., [3] Lemma 1.2); we refer to \( \Omega \) as the \textit{excess normal bundle to} \( s \)

1 Deforming curves with nodal threefolds

Let \( P \) be a smooth projective variety, \( \mathcal{E} \) a locally free sheaf of rank \( \dim P - 3 \) on \( P \), and

\[ s_0 \in \Gamma(P, \mathcal{E}) \]

a regular section; set

\[ X_0 = Z(s_0). \]

Let \( S \subset X_0 \) be a surface with

\[ H^1(S, \mathcal{O}_S) = 0. \]
and $\mathcal{L}$ a line bundle on $S$; set

$$\ell := |\mathcal{L}|.$$ 

We make the following assumptions:

(1.0.2) $X_0$ is $K$-trivial.

(1.0.3) The only singularities of $X_0$ which lie in $S$ are the ordinary double points $\{\xi^1, \ldots, \xi^n\}$, and these are distinct from any singularities of $S$ and base-points of $|\mathcal{L}|$. Furthermore,

$$n \geq \ell + 2.$$ 

(1.0.4) For all $C \in |\mathcal{L}|$,

$$H^1(C, N_{C/P}) = 0$$ 

and

$$H^0(C, N_{C/S}) \cong H^0(C, N_{C/X_0}).$$

(1.0.5) There exists $s \in \Gamma(P, \mathcal{E})$ such that $X_t := Z(s_0 + ts)$ is a smoothing of at least one of the $\xi^i$.

1.1 Theorem Under the assumptions (1.0.2)–(1.0.5), the members of $|\mathcal{L}|$ deform to a length $\left(\frac{n-2}{\ell}\right)$ scheme of curves which are geometrically rigid in the general deformation $X_t = Z(s_0 + ts)$ of $X_0$. In particular, $X_t$ contains a geometrically rigid curve which is a deformation of a curve in $|\mathcal{L}|$.

1.2 Proof Let $\Lambda := |\mathcal{L}| \cong \mathbb{P}^\ell$. Then because of (1.0.1), $\Lambda$ is a connected component of the Hilbert scheme of $S$. Now (1.0.4) implies that that $\Lambda$ is likewise a connected component of $\text{Hilb}^{X_0}$, and that $\Lambda$ has a smooth neighborhood $\mathcal{H} \subset \text{Hilb}^P$.

Let

$$C \twoheadrightarrow X_0$$

$$\downarrow p$$

$$\Lambda$$

be the universal curve, and let

$$\mathcal{I} := \text{ideal sheaf of } C \text{ in } \Lambda \times X_0$$

$$\mathcal{J} := \text{ideal sheaf of } C \text{ in } \Lambda \times P.$$ 

Applying the functor

$$F := p_* \circ \text{Hom}_C(\_\_, \mathcal{O}_C)$$

and $\mathcal{L}$ a line bundle on $S$; set

$$\ell := |\mathcal{L}|.$$
to the exact sequence
\[ 0 \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow \mathcal{J}/\mathcal{J}^2 \longrightarrow q^* \mathcal{E} \longrightarrow 0 \]
of conormal sheaves and using the infinitesimal properties of Hilbert schemes
gives the exact sequence
\[ 0 \longrightarrow T_{\Lambda} \longrightarrow T_{\mathcal{H}} \otimes O_{\Lambda} \longrightarrow p^* q^* \mathcal{E} \longrightarrow R^1 F(\mathcal{I}/\mathcal{I}^2) \longrightarrow 0. \]
of $O_{\Lambda}$-modules. Setting
\[ Q := R^1 F(\mathcal{I}/\mathcal{I}^2) \]
we shorten the above to
\[ 0 \longrightarrow N_{\Lambda\setminus \mathcal{H}} \longrightarrow p^* q^* \mathcal{E} \overset{\rho}{\longrightarrow} Q \longrightarrow 0. \]
(1.2.1)

By (1.0.3), the locus of curves in $|\mathcal{L}|$ which pass through the node $\xi^i$ consists of a hyperplane $D_i$. In [2, Theorem 3.3], it was shown that $Q$ is locally free and fits into an exact sequence
\[ 0 \longrightarrow \Omega^1_{\Lambda} \longrightarrow Q \overset{\varepsilon(\cdot)}{\longrightarrow} \bigoplus O_{D_i} \longrightarrow 0. \]
(1.2.2)

Given that $n \geq \ell + 2$ (assumption (1.0.3)), one uses standard exact sequences to compute
\[ \int_{\Lambda} c_{\text{top}}(Q) = \left( n - \frac{2}{\ell} \right) > 0. \]
(1.2.3)

Furthermore,
\[ \text{all non-zero sections of } Q \text{ have isolated zeros.} \]
(1.2.4)

For by (1.2.3), any $\gamma \in \Gamma(\Lambda, Q)$ has zeros. Suppose $\gamma$ vanishes along a curve $C$; then since $C \cdot D^i > 0$ for all $i$, $\varepsilon'(\gamma) = 0$ for all $i$. Since $H^0(\Lambda, \Omega^1_I) = 0$, the claim follows from (1.2.2).

Choose a trivialization of $\mathcal{E}$ on an analytic neighborhood $\Delta^r \ni 0 = \xi^i$ in $P$ such that $s_0(x) = (x_1, \ldots, x_{r-4}, x_{r-3}^2 + \cdots + x_r^2)$. Let $s \in \Gamma(P, \mathcal{E})$ and write $s(x) = (f_1(x), \ldots, f_{r-3}(x))$ in the same coordinates. In the proof of [3, Theorem 3.3], it was shown that the composition
\[ \Gamma(P, \mathcal{E}) \rightarrow \Gamma(\Lambda, p^* q^* \mathcal{E}) \overset{\rho}{\longrightarrow} \Gamma(\Lambda, Q) \overset{\varepsilon}{\longrightarrow} \Gamma(D^i, O^i_D) \]
is given by
\[ s \mapsto f_{r-3}(0). \]

Thus, in light of (1.0.5), there exists a section $s \in \Gamma(P, \mathcal{E})$ such that $\varepsilon(\rho(p^* q^* s)) \neq 0$, so that by (1.2.3) and (1.2.4), $\rho(p^* q^* s)$ has $\left( n - \frac{2}{\ell} \right)$ isolated zeros (counted with multiplicities).
Let $p^P$ and $q^P$ be the projections from the universal family over $H \subset \text{Hilb}^P$. By [3, Theorem 1.5], $p^P_* q^P_* \mathcal{E}$ is locally free, and $\Lambda$ is the zero-scheme of $p^P_* q^P_* s_0$, so that (1.2.1) identifies $Q$ as the excess normal bundle to $p^P_* q^P_* s_0$.

By [5, Theorem 1.5], $p^P_* q^P_* \mathcal{E}$ is locally free, and $\Lambda$ is the zero-scheme of $p^P_* q^P_* s_0$, so that (1.2.1) identifies $Q$ as the excess normal bundle to $p^P_* q^P_* s_0$.

Now by [5, Theorem 1.5] again, the Hilbert scheme of the threefold $X_t := Z(s_0 + ts)$ satisfies

$$\text{Hilb}^{X_t} \cap H = Z(p^P_* q^P_* (s_0 + ts)).$$

The Theorem now follows from the conservation of number and the following lemma. ♣

1.3 Lemma Let $W$ be a smooth variety over $\mathbb{C}$, $\mathcal{E}$ a locally free sheaf on $W$ of rank = dim $W$, and $s_0 \in \Gamma(W, \mathcal{E})$ such that $Z := Z(s_0)$ is smooth. Let $\rho: \mathcal{E} \otimes \mathcal{O}_Z \to \mathcal{M}$ be the excess normal bundle to $s_0$ (see 0.4) and $s \in \Gamma(W, \mathcal{E})$ such that $\rho(s | Z)$ has an isolated zero at $z_0$. Then for general $t$, the section $s_0 + ts$ has an isolated zero in a neighborhood of $z_0$ in $W$.

1.4 Proof By the implicit function theorem, we can choose (analytic) coordinates $(x_1, \ldots, x_n)$ on a neighborhood $U$ of $z_0$ in $W$ and a trivialization

$$\phi: \mathcal{E}|_U \xrightarrow{\sim} \mathcal{O}_U^n$$

such that $\phi(s_0) = (x_1, \ldots, x_c, 0, \ldots, 0)$, where $c = \text{codim}_W Z$. This has the effect of splitting the excess normal sequence, so that if $\phi(s) = (f_1, \ldots, f_n)$,

$$\rho(s | Z \cap U)(x) = (f_{c+1}(x), \ldots, x_{c+1}, \ldots, x_n), \ldots, f_n(0, \ldots, 0, x_{c+1}, \ldots, x_n)).$$

Now for $t \neq 0$, the points of $Z(s_0 + ts)$ satisfy the equations

$$x_1 + tf_1(x) = \cdots = x_c + tf_c(x) = f_{c+1}(x)) = \cdots = f_n(x) = 0.$$

The hypotheses guarantee that when $t = 0$ this system has an isolated solution at $x = z_0$, and so, after shrinking $U$ as necessary, it must have an isolated solution for all $t$ with $|t|$ sufficiently small. ♣

2 Calabi-Yau complete intersections

By a complete intersection of type $(b_1, \ldots, b_{r-e})$ in $\mathbb{P}^r$ we mean a scheme of dimension $e$ whose homogeneous ideal has generators of degrees $b_1 \geq \cdots \geq b_{r-e} \geq 1$. Equivalently, such a scheme is the zero scheme of a regular section of $\bigoplus \mathcal{O}(b_i)$ on $\mathbb{P}^r$. Now by adjunction, there are 5 families of Calabi-Yau complete intersection threefolds: those of types (5), (4,2), (3,3), (3,2,2), and (2,2,2,2).

2.1 Theorem Let $g \geq 0$ and $d \geq 2g - 3$. Then in any of the following cases, the general complete intersection Calabi-Yau threefold of type $(b_i)$ contains a geometrically rigid curve of degree $d$ and genus $g$:
(b_i) = (5): if $g < \min\{d^2/8, 35\}$ and $(d, g) \neq (5, 3)$ and either $d > 2g - 2$ or $d > g + 2$.

(b_i) = (4, 2): if $g < \min\{d^2/8, 31\}$ and $(d, g) \neq (5, 3)$ and either $d > 2g - 2$ or $d > g + 2$.

(b_i) = (3, 3): if $(d, g) = (3, 1)$ or if $g < \min\{d^2/12, 31\}$ and $(d, g) \neq (7, 4)$ and either $d > 2g - 2$ or $d > g + 3$.

(b_i) = (3, 2, 2): if $(d, g) = (3, 1)$ or if $g < \min\{d^2/12, 15\}$ and $(d, g) \neq (7, 4)$ and either $d > 2g - 2$ or $d > g + 3$.

(b_i) = (2, 2, 2, 2): if $(d, g) = (4, 1)$ or if $g < \min\{d^2/16, 9\}$ and $(d, g) \neq (9, 5)$ and either $d > 2g - 2$ or $d > g + 4$.

2.2 Proof By work of Mori [1] (quartic surfaces), Oguiso [2] (rational curves on projective K3’s of arbitrary degree) Knutsen [3] (all genera on K3’s of arbitrary degree) and the author [4] (existence in all genera on complete intersection K3’s), the possible degrees and genera of smooth curves in smooth complex K3 surfaces of arbitrary degree are classified. The most general statement is [5, Theorem 1.1]. For our purposes, the following version is relevant (see [6, Theorem 8.1] and [3, Theorem 3.2]):

(2.2.1) If $m = 2, 3, 4$, there exists a smooth complete intersection K3 surface $S \subset \mathbb{P}^{m+1}$ of degree $2m$ and a smooth curve $C_0 \subset S$ of genus $g$ and degree $d \geq 2g - 2$ such that $\text{Pic} S = \mathbb{Z}C_0 \oplus \mathbb{Z}H$ (where $H$ is the polarizing class on $S$) if and only if

$$g < \frac{d^2}{4m} \quad \text{and} \quad (d, g) \neq (2m + 1, m + 1)$$

or $m = 3$ and $(d, g) = (3, 1)$ or $m = 4$ and $(d, g) = (4, 1)$.

Now given any complete intersection K3 surface $S$ of type $(a_1, \ldots, a_r-2)$ in $\mathbb{P}^r$ (where some of the $a_i$ may be 1), one can construct ciCY threefolds $X_0$ of type $(b_i)$ containing $S$ whenever $b_i \geq a_i$ for $i = 1, \ldots, r - 3$. If the choice of coefficient forms is sufficiently general, $X_0$ has only ordinary double points, the exact number of which is given in the following table (see [3]):

| $(b_i)$ | $(a_j)$ | $n$ | $(b_i)$ | $(a_j)$ | $n$ |
|---------|---------|-----|---------|---------|-----|
| (5)     | (4, 1)  | 16  | (3, 3)  | (3, 2, 1)| 12  |
| (5)     | (3, 2)  | 36  | (3, 3)  | (2, 2, 2)| 32  |
| (4, 2)  | (4, 1, 1)| 4   | (3, 2, 2)| (3, 2, 1)| 6   |
| (4, 2)  | (3, 2, 1)| 18  | (3, 2, 2)| (2, 2, 2)| 16  |
| (4, 2)  | (2, 2, 2)| 32  | (2, 2, 2)| (2, 2, 2)| 8   |

Thus, in all cases of the theorem, we may construct $C_0 \subset S \subset X_0 \subset \mathbb{P}^r$. 
where $C_0$ is a smooth curve of the desired degree $d$ and genus $g$, $S$ is a smooth K3 surface, and $X_0$ is a nodal ciCY threefold of the desired type $(b_i)$, with

\[ n \geq g - 2 \]

nodes. Set $\Lambda := |O_S(C_0)|$, which is of dimension $\ell = g$.

Now a useful property of complete intersection K3 surfaces (see [5, Corollary 1.11]) is that

\[ H^1(C, N_{C/P^r}) = 0 \quad \text{for all} \quad C \in \Lambda \]

if and only if

\[ H^1(C, O(1)) = 0. \]

But $C_0$ is smooth, so (2.2.2) holds whenever $d > 2g - 2$ or when $d \leq 2g - 2$ and $O_C(1)$ is non-special. But in the cases of the theorem, $O_C(1)$ is non-special by Lemma 2.3 below.

Finally, by [5, Theorem 3.5] (and the remark following its proof)

\[ H^0(C, N_{C/S}) \cong H^0(C, N_{S/X_0}) \quad \text{for all} \quad C \in \Lambda. \]

Thus, all hypotheses of Theorem 1.1 are satisfied, so Theorem 2.1 follows.

We conclude with the lemma used in the preceding proof to handle curves of degree $\leq 2g - 2$.

2.3 Lemma Let $S$ be a smooth K3 surface, $H \subset S$ a very ample divisor with $H \cdot H = 2m \geq 4$. Let $C \subset S$ be a smooth curve of genus $g > m + 2$ and assume that $\text{Pic} \, S \cong ZH \oplus ZC$. Then $O_C(H)$ is non-special whenever $C \cdot H >\max\{2g - 4, m + g\}$.

2.4 Proof Set $d := C \cdot H$. By Riemann-Roch for curves, we need only consider the cases $d = 2g - 3$ and $d = 2g - 2$. We remark that given our hypotheses—in particular that $d > m + g$—the divisor $C$ must be ample in $S$. The proof is analogous to the proof of [9, Proposition 6], which in turn relies on the fundamental results of [9].

Now consider the exact sequence

\[ 0 \longrightarrow O(-H) \longrightarrow O(C - H) \longrightarrow O_C(C - H) \longrightarrow 0. \]

Since $h^1(O(-H)) = 0$, it suffices to prove that $H^0(S, O(C - H)) = 0$.

Suppose not and let $D \in |O(C - H)|$. From Serre duality and the exactness of $0 \rightarrow O(-C) \rightarrow O \rightarrow O_C \rightarrow 0$, we have $\chi(O(C)) = g + 1$ so that $C \cdot C = 2g - 2$. Thus,

\[ D \cdot D = C^2 - 2C \cdot H + H^2 = 2m - 2g + 2 < -2. \]

Thus, the genus of $D$ is negative, so that

\[ D = D_1 + D_2 \]
for some effective $D_i$.

Now using that $C$ is ample and that $d \geq 2g - 3$, we have
\[
0 < D_1 \cdot C < D \cdot C = C \cdot C - H \cdot C = 2g - 2 - d \leq 1,
\]
which is a contradiction.

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