Structural Controllability of an NDS With LFT Parameterized Subsystems

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Abstract—This paper studies structural controllability for a networked dynamic system (NDS), in which each subsystem may have different dynamics, and unknown parameters may exist both in subsystem dynamics and in subsystem interconnections. In addition, subsystem parameters are parameterized by a linear fractional transformation. It is proven that controllability keeps to be a generic property for this kind of NDSs. Some necessary and sufficient conditions are then established, respectively, for them to be structurally controllable, to have a fixed uncontrollable mode, and to have a parameter-dependent uncontrollable mode, under the condition that each subsystem interconnection link can take a weight independently. These conditions are scalable, and in their verifications, all arithmetic calculations are performed separately on each subsystem. In addition, these conditions also reveal influences on NDS controllability from subsystem input–output relations, subsystem uncontrollable modes, and subsystem interconnection topology. Based on these observations, the problem of selecting the minimal number of subsystem interconnection links is studied under the requirement of constructing a structurally controllable NDS. A heuristic method is derived with some provable approximation bounds and a low computational complexity.

Index Terms—Matroid intersection, linear fractional transformation (LFT), networked system, structural controllability, topology design.

I. INTRODUCTION

CONTROLLABILITY and observability are now relatively classic concepts in system analysis and synthesis, which are closely related to stabilization, existence of an optimal control, and some other fundamental issues [13], [23], [33]. The past two decades have seen a renewed research interest in controllability and observability with the emergence of complex networks, such as biological transduction networks [18], power networks [25], and social networks [18], [24].

Apart from the extensively adopted concept of controllability proposed by Kalman [13], structural controllability proposed by Lin [17] is also widely utilized in which each entry of system matrices is either fixed to be zero or allowed to take an arbitrary real value. A system with this kind of system matrices is called a structured system, and is said to be structurally controllable, if there exists one numerical realization such that the associated system is controllable. Note that controllability is a generic system property, which means that structural controllability of a structured system guarantees that all its numerical realizations, except for a set of zero Lebesgue measure in the associated parameter space, correspond to a controllable system [10]. A prominent property of structural controllability is that its criteria often have a clear graphical interpretation, which explicitly reveals information flows in a dynamic system, and can usually be easily verified graphically [8], [10], [18]. Recently, structural controllability has also been extended to some nonlinear systems, for example, the driftless bilinear systems [28].

When controllability of a networked dynamic system (NDS) is to be investigated, some challenging issues arise due to nodal dynamics and high dimensions of its global system matrices [9], [34]. Recall that when a criterion for a lumped plant is applied to an NDS with a high dimensional state vector, numerical instability and/or computational prohibitiveness may emerge. Owing to efforts from many researchers, various graphical criteria for (structural) controllability have been obtained [2], [8], [10], [18]. These criteria, mainly focusing on an NDS with each of its subsystems being modeled as a first-order differential equation, give many intuitive and useful insights on how the topology influences performances of an NDS. Recently, extensive attentions have been moved to controllability of a compositional networked system in which each subsystem may have high-order dynamics. For example, controllability is discussed for a network-of-network system via Cartesian product in [4]. In [34]–[36], a general NDS model is adopted in which each subsystem can have different dynamics.

On the other hand, various results have been reported in which more general parameter interdependencies are adopted than those of the structured system model. More precisely, a linear parameterization is adopted in [6], which assumes that each entry of system matrices is an affine function of some free parameters. A so-called “matrix net,” is used in [1]. In [21], Murota introduces a concept called “mixed matrix,” and uses matroid to study structural controllability. Recent related works include graphical interpretations of the conditions of [6] in [16], and leader selection for structured descriptor systems in [5]. The obtained results, which are often expressed through algebraic operations, however, are in general not computationally efficient for a large-scale NDS. Besides, parameter interdepen-
ties adopted in these investigations are not general enough to describe all actual plants, noting that entries in system matrices are usually rational functions of the parameters that govern its movements [10], [23], which include parameters directly describing a physical process, a chemical process, or a biological process, such as mass, temperature, concentration of a chemical element, etc. All these parameters are called a first principle parameter (FPP) in this paper for brevity. An appropriate framework for describing matrices with rational function entries appears to be the linear fractional transformation (LFT) widely adopted in robust control theory [33].

In this paper, we investigate structural controllability of an NDS in which each subsystem can have high-order and heterogeneous dynamics, and unknown parameters are allowed to exist in both subsystem dynamics and in subsystem interconnections. We adopt an LFT to model subsystem parameter interdependencies, which enables describing a large class of plants whose unknown entries in system matrices are rational functions of the plant FPPs, and contains many other descriptions, such as the linear parameterization adopted in [6] and [29] as a special case. At first, necessary and sufficient conditions are established for structural controllability of an LFT parameterized plant under a diagonalization assumption. Compared to the results in [1], [6], [20], and [29], the adopted model is more general in describing relations between system matrices of a plant and its FPPs, and the obtained conditions can be verified in polynomial time. Under the condition that each subsystem interconnection link can take a weight independently, some necessary and sufficient conditions are then derived for the NDS to be structurally controllable. These conditions can be verified efficiently, and the associated arithmetic operations are within each individual subsystem while the associated graphical operations are on the network topology, which makes them scalable and, therefore, attractive for large-scale NDSs. Moreover, these conditions explicitly illustrate how subsystem dynamical and subsystem interconnection topology, jointly influence controllability of an NDS. Based on them, we consider the problem of designing a subsystem interaction topology that minimizes the number of interconnection links under structural controllability restriction. This problem is shown to be NP-hard, and a two-stage algorithm is suggested to approximate it with some provable suboptimality guarantees. Additionally, we discuss the general computational complexity and hardness of structural controllability verification problems with more complicated parameter interdependencies.

The rest of this paper is organized as follows. Section II provides problem statements and some preliminary results. Genericity is established in Section III for the controllability of an NDS with LFT parameterized subsystems. Necessary and sufficient conditions for structural controllability of LFT parameterized plants are presented in Section IV. These conditions are extended to NDSs in Section V. Section VI investigates designs of a minimal subsystem interaction topology to guarantee structural controllability for an NDS, whereas an illustrative example is given in Section VII. Section VIII concludes this paper. Two appendices are included to provide some technical details.

Notations: Given two matrices $M = [M_{11} \ M_{12} ; M_{21} \ M_{22}]$ and $P$ with compatible dimensions, if $I - M_{22} P$ is invertible, a lower LFT is defined as $F(M, P) = M_{11} + M_{12} P(I - M_{22} P)^{-1} M_{21}$.

By $\text{diag}(X_{i \in \mathbb{N}})_{|N|}$, we denote the block diagonal matrix with its $i$th diagonal block being $X_i$, whereas $\text{col}(X_{i \in \mathbb{N}})_{|N|}$ is the matrix stacked by $X_i_{|N|}$ with its $i$th row block being $X_i$. Denoted by $F(\lambda)$, the field of all rational functions of the variable $\lambda$ with real coefficients, and $F(\lambda)^{n_1 \times n_2}$ the set of all $n_1 \times n_2$ matrices with every entry in $F(\lambda)$. $\sigma(A)$ denotes the set of eigenvalues of a square matrix $A$. $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$, and $\mathbb{N}$ denote the real, complex, integral, and nonnegative integral numbers, respectively. Given $n \in \mathbb{N}$, define $[n] = \{1, 2, \ldots, n\}$. For an $n_1 \times n_2$ matrix $M$, $M_{ij}$ or $[M]_{ij}$ denotes its $(i, j)$th entry, and for $J_1 \subseteq [n_1]$ and $J_2 \subseteq [n_2]$, $M_{J_1, J_2}$ denotes the submatrix of $M$ formed by rows indexed by $J_1$ and columns indexed by $J_2$, whereas $M_{J_i}$ denotes the submatrix of $M$ formed by columns indexed by $J_i$. By $\|M\|_0$, we denote the number of nonzero entries in a matrix $M$. Throughout this paper, a set of variables $S$ is said to be algebraically independent, if $S$ do not satisfy any nontrivial polynomial equation with its coefficients in $\mathbb{R}$.

II. Problem Statement and Preliminaries

A. Model for NDSs and Problem Statement

In an actual NDS, it is often the case that its subsystems have high orders and heterogeneous dynamics. To describe dynamics of these NDSs, it appears convenient to utilize the spatially interconnected system model adopted in [15], [34]–[36], in which an NDS $\Sigma$ is constituted of $N$ subsystems, and the dynamics of its $i$th subsystem $\Sigma_i$ is described by

$$
\begin{align*}
\begin{bmatrix}
\dot{x}_i(t) \\
y_i(t)
\end{bmatrix} &= 
\begin{bmatrix}
A_{i1}^{(i)} & A_{i2}^{(i)} & E_{i1}^{(i)} \\
A_{i2}^{(i)} & A_{i3}^{(i)} & E_{i2}^{(i)} \\
C_{i1}^{(i)} & C_{i2}^{(i)} & D_{i1}^{(i)} \\
C_{i3}^{(i)} & C_{i4}^{(i)} & D_{i2}^{(i)}
\end{bmatrix}
\begin{bmatrix}
x_{i1}(t) \\
x_{i2}(t) \\
x_{i3}(t) \\
x_{i4}(t)
\end{bmatrix} +
\begin{bmatrix}
E_{i4}^{(i)} \\
F_{i1}^{(i)} \\
F_{i2}^{(i)} \\
F_{i3}^{(i)}
\end{bmatrix}
\begin{bmatrix}
y_{i1}(t) \\
y_{i2}(t) \\
y_{i3}(t) \\
y_{i4}(t)
\end{bmatrix} \\
\times (I - H_i^{(i)} P^{(i)})^{-1} \\
\begin{bmatrix}
x_i(t) \\
v_i(t) \\
u_i(t)
\end{bmatrix}
\end{align*}
$$

(1)

where $t$ represents the temporal variable, $x_i(t) \in \mathbb{R}^{m_{i1}}$ is the state vector, $u_i(t) \in \mathbb{R}^{m_{i4}}$ is the external input vector, $y_i(t) \in \mathbb{R}^{m_{i3}}$ is the external output vector, $v_i(t)$ and $z_i(t)$ are, respectively, the signal received from other subsystems and the signal sent to other subsystems, which are called the internal input and output vectors, respectively, and whose dimensions can be greater than one.

In the previous model, the system matrices of subsystem $\Sigma_i$ are parameterized by a matrix $P^{(i)}$, which is constituted from the unknown FPPs of this subsystem. Moreover, $E_{i1}^{(i)}, E_{i2}^{(i)},$ and $H_i^{(i)}, j = 1, 2, 3,$ are adopted to represent some known constant matrices that reflect how an FPP influences the subsystem matrices. The matrices with a subscript “0” represent the ingredient of the system matrices that does not vary with its FPPs. This forms an LFT parameterization [33] and is able to describe
system matrices that depend on its FPPs in a rational function way.

In addition, interactions among subsystems are described by
\[ v(t) = \Phi z(t) \]  
(2)
where \( v(t) = \text{col}\{v_i(t)|_{i=1}^N \} \), \( z(t) = \text{col}\{z_i(t)|_{i=1}^N \} \). We call \( \Phi \) the subsystem connection matrix (SCM), which describes the interconnection topology among subsystems. In actual networks, due to communication noises, inaccuracies of parameters describing the interaction channels or variations of spatial distances among subsystems, etc., the weights of the interconnection links may sometimes be hard to know exactly. Hence, while \( \Phi \) is time invariant, it is assumed that only the zero–nonzero patterns of \( \Phi \) are known, that is, the positions of its elements that are fixed to be zero and those that are not constantly equal to zero. A fixed zero element in the SCM \( \Phi \) means that the associated internal output of a subsystem does not directly affect another associated subsystem, while an element whose value is not constantly equal to zero means the contrary. In other words, the SCM \( \Phi \) actually reflects the geometric structure of an NDS. With a little abuse of terminology, an element of the SCM \( \Phi \) not fixed to be zero is also called an FPP of the NDS in this paper.

To illustrate the application significance of the NDS model, consider a mechanical system described in Fig. 1, which consists of \( N \) subsystems constituted from a vehicle, a damper, and a spring. Obviously, the dynamics of each subsystem are determined by the mass of the vehicle, the constants of the springs, and the dampers connected to it, in a rational function manner. Direct algebraic operations show that the system matrices of each subsystem in this plant can have the LFT form described in (1), and dynamics of the whole system can be written as (1)–(2). Details are omitted due to their straightforwardness.

The following assumption is adopted throughout this paper.

**Assumption 1:** The NDS \( \Sigma_i \), as well as each of its subsystem \( \Sigma_i, i \in [N] \), is well posed for almost all feasible values of its FPPs.

Assumption 1 means that for each external input series \( \text{col}\{u_i(t)|_{i=1}^N \} \), both the system states \( \text{col}\{x_i(t)|_{i=1}^N \} \) and the external outputs \( \text{col}\{y_i(t)|_{i=1}^N \} \) are uniquely determined for almost each \( \Phi \) and each \( P^{(i)}, i \in [N] \), having the corresponding prescribed structures. Under this assumption, the main problem discussed in this paper is stated as follows.

**Problem 1:** Assume that all the nonzero entries of \( P^{(1)}, \ldots, P^{(N)} \) and \( \Phi \) are algebraically independent and time invariant, and except \( P^{(i)} \), all the other values of the system matrices are prescribed for each subsystem \( \Sigma_i \) of the NDS \( \Sigma \), \( i \in [N] \). Verify whether or not the NDS \( \Sigma \) is structurally controllable. That is, whether or not there exist at least one feasible value, respectively, for \( \Phi \) and each \( P^{(i)} \) for \( i \in [N] \) with their prescribed structures, such that the corresponding NDS (1)–(2) is controllable.

In the next section, it is shown that if the NDS (1)–(2) is structurally controllable, then for almost all feasible realizations of \( P^{(i)}|_{i=1}^N \) and \( \Phi \), the corresponding NDS is controllable.

**Remark 1:** LFT parameterization is capable of describing almost all rational function matrices [33]. It is worthwhile to mention that we restrict our attentions in this paper to the situation in which each FPP appears only once in \( P^{(i)} \), which can be directly extended to the case in which each FPP has a rank-one coefficient matrix. This adoption is due to the following considerations. First, as discussed in Appendix A, the rank-one case is currently the most possible case for structural controllability verification that one can find deterministic algorithms with a polynomial time complexity or a subexponential time complexity. Second, a large class of traditional actual plants, which may be regarded as a subsystem of an NDS, satisfy this rank-one setting. These plants include many mechanical systems, electrical systems, as well as fluid systems [23], [33]. The assumption that all nonzero entries of \( \Phi \) are algebraically independent might be helpful, under some situations, in understanding the role of subsystem interconnection topology in controllability of NDSs.

**Remark 2:** Many other parametrizations, including the linear parametrization [6], [29], the matrix net [1], and the mixed matrix descriptions [21] are special cases of the LFT parameterization. First, as discussed in Appendix A, the rank-one case is currently the most possible case for structural controllability verification that one can find deterministic algorithms with a polynomial time complexity or a subexponential time complexity. Second, a large class of traditional actual plants, which may be regarded as a subsystem of an NDS, satisfy this rank-one setting. These plants include many mechanical systems, electrical systems, as well as fluid systems [23], [33]. The assumption that all nonzero entries of \( \Phi \) are algebraically independent might be helpful, under some situations, in understanding the role of subsystem interconnection topology in controllability of NDSs.

**B. Equivalent LFT Representation of the NDS**

To investigate structural controllability of the NDS described in (1)–(2), its dynamics is rewritten as another LFT in which all FPPs in the model are included in one matrix. For this purpose, two auxiliary internal input and output vectors are constructed for the \( i \)th subsystem of the NDS (1)–(2), which are denoted, respectively, by \( v^+_i(t) \) and \( z^+_i(t) \), and defined as
\[ z^+_i(t) = [F_{\Phi}^{(i)} F_{2}^{(i)} F_{3}^{(i)}] \text{col}\{x_i(t), v_i(t), u_i(t)\} + H^{(i)} v^+_i(t) \]
\[ v^+_i(t) = P^{(i)} z^+_i(t). \]
(3)
Let \( \tilde{z}_i(t) = \text{col}\{z_i(t), z^+_i(t)\}, \tilde{v}_i(t) = \text{col}\{v_i(t), v^+_i(t)\} \). Denote the dimensions of \( \tilde{v}_i(t) \) and \( \tilde{z}_i(t) \), respectively, by \( m_{\tilde{v}_i} \) and \( m_{\tilde{z}_i} \). Then, it can be straightforwardly shown that when the matrix \( I - H^{(i)} P^{(i)} \) is invertible, the dynamics of subsystem \( \Sigma_i \) can be equivalently expressed in (3) and the following equation:

\[
\begin{bmatrix}
\dot{x}_i(t) \\
\dot{y}_i(t)
\end{bmatrix} =
\begin{bmatrix}
A^{(i)}_{xx} & A^{(i)}_{xy} & B^{(i)}_{xu} & 0 \\
A^{(i)}_{yx} & A^{(i)}_{yy} & B^{(i)}_{yu} & 0 \\
C_{yx}^{(i)} & C_{yy}^{(i)} & D_{yu}^{(i)} & 0
\end{bmatrix}
\begin{bmatrix}
x_i(t) \\
y_i(t) \\
\tilde{v}_i(t) \\
\tilde{z}_i(t)
\end{bmatrix}
\]  
(4)
Given a finite set $E$ and a family $\mathcal{I}$ of subsets of $E$, the pair $(E, \mathcal{I})$ is a matroid if: (1) $\emptyset \in \mathcal{I}$; (2) if $I_1 \in \mathcal{I}$ and $I_2 \subseteq I_1$, then $I_2 \in \mathcal{I}$; (3) if $I_1, I_2 \in \mathcal{I}$ and $|I_1| = |I_2| + 1$, then there is some $x \in I_1 \setminus I_2$ satisfying $I_2 \cup x \in \mathcal{I}$. In this definition, $E$ is called the ground set, and a member of $\mathcal{I}$ is called an independent set. The rank of a matroid $\mathcal{M} = (E, \mathcal{I})$, denoted by $\rho(\mathcal{M})$, is defined as the maximum cardinality of its independent sets. For a matrix $F$, a linear matroid can be defined as $\mathcal{M}(F) = (E, \mathcal{I})$, where $E$ is the set of indices of columns of $F$, $\mathcal{I}$ is the collection of indices of columns of $F$ that are linearly independent. Given two matroids $\mathcal{M}_1$ and $\mathcal{M}_2$ over the same ground set, the matroid intersection $\mathcal{M}_1 \cap \mathcal{M}_2$ is defined as the collection of all common independent sets of $\mathcal{M}_1$ and $\mathcal{M}_2$. The cardinality of the largest independent set in $\mathcal{M}_1 \cap \mathcal{M}_2$, which is also denoted by $\rho(\mathcal{M}_1 \cap \mathcal{M}_2)$ for notation simplicity, can be determined in polynomial time [21]. For two matroids $\mathcal{M}_1 = (I_1, E)$ and $\mathcal{M}_2 = (I_2, E)$, the matroid union $\mathcal{M}_1 \cup \mathcal{M}_2$ is a matroid $(I_3, E)$ such that any $X \subseteq I_3$ can be expressed as $X = Y \cup Z$ with $Y \subseteq I_1$ and $Z \subseteq I_2$. Determining the rank of the union of two linear matroids can also be done in polynomial time.

A set function $f : 2^V \to \mathbb{R}$ is submodular if for all sets $S_1 \subseteq S_2 \subseteq V$ and any element $s \in V \setminus S_2$, it holds that $f(S_1 \cup \{s\}) - f(S_1) \geq f(S_2 \cup \{s\}) - f(S_2)$. A well-known fact is that the rank of a matrix is submodular on any subset of its column vectors [21].

Given a directed graph (digraph) $D$, let $V(D)$ be its vertex set, $E(D)$ its edge set. A path from a vertex $v_1 \in V(D)$ to a vertex $v_n \in V(D)$ is a sequence of edges $(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n)$ with no repeated end vertices, which is denoted by $v_1 \to \cdots \to v_n$. If there is a path from $v_1$ to $v_2$, we say that $v_2$ is reachable from $v_1$. A path from a vertex to itself is called a cycle. Length of a path is the number of its edges. A matching $M$ of a digraph $D$ is a set of edges such that any of its element does not share a common start or end vertex with another element. The size of a matching $M$ is the number of edges contained in $M$. A digraph is strongly connected, if any two of its vertices are reachable from each other. A strongly connected component (SCC) of a digraph is a subgraph that is strongly connected, and is maximal in the sense that no other edges or vertices can be included without breaking the property of being strongly connected. A clique is an undirected graph such that any two of its vertices are adjacent.

Let $S \triangleq \{s_1, \ldots, s_p\}$ be a set of algebraically independent variables, and $G(\lambda; S)$ be a TFWM whose entries are rational functions of $\lambda$ with coefficients being polynomials of $s_i^p_{i=1}$ over $\mathbb{R}$. The generic rank of the TFWM $G(\lambda; S)$ is the maximum rank that $G(\lambda; S)$ can achieve as a function of both $\lambda$ and $s_i^p_{i=1}$. When $s_i^p_{i=1}$ are fixed, $G(\lambda; S)$ is usually abbreviated as $G(\lambda)$, and the maximum rank that $G(\lambda)$ can achieve among all $\lambda$ is sometimes called the normal rank of $G(\lambda)$ [33]. In this paper, given a $n_x \times n_x$ dimensional TFWM $G(\lambda)$ with a full row normal rank (FRNR), we say that $\lambda_0 \in \mathbb{C}$ is a zero of $G(\lambda)$, if rank$(G(\lambda_0)) < n_x$. A TFWM $G(\lambda)$ is said to have a zero depending on $S$, if it has a zero for arbitrarily fixed $s_i^p_{i=1}$ in the corresponding parameter space, while the value of this zero is not independent of $s_i^p_{i=1}$.

C. Some Preliminaries

In this section, we give some preliminaries required in our following derivations.

Lemma 2 (see [11]): Let matrix $M = [M_1, M_2]$ and matrix $M_1^\perp$ be a basis of the left null space of $M_1$. Then, $M$ is of FRR, if and only if $M_1^\perp M_2$ is of FRR.
III. Genericity of the Controllability of the NDS

In this section, genericity is established for well posedness and controllability of the NDS (1)–(2).

Using the symbols $A_{xx}$, $A_{yy}$, $P$, etc., defined in Section II-B, a lumped state-space representation can be obtained for the NDS (1)–(2), given as follows:

$$ \dot{x}(t) = Ax(t) + Bu(t) $$

where $x(t) = \text{col}\{x_i(t)\}_{i=1}^{N}$, $u(t) = \text{col}\{u_i(t)\}_{i=1}^{N}$, and

$$ [A \hspace{20pt} B] = [A_{xx} \hspace{20pt} B_{xu} \hspace{20pt} A_{yy} \hspace{20pt} B_{yu}] = F_l \left( \begin{bmatrix} A_{xx} & A_{xy} \\ A_{yx} & A_{yy} \end{bmatrix} \right), $$

$$ \text{det}(A_{xx} - \lambda I) = 0 $$

in which $F_l(\bullet, \bullet)$ denotes the lower LFT operation defined in Section I. As can be seen, the lumped matrix $[A \hspace{20pt} B]$ also has a form of LFT parameterization. This has been already observed in [34], and is a property of LFT expressions.

Let $s_1, \ldots, s_k$ be the FPPs of $P$ in (6), and denote by $S = (s_1, \ldots, s_k)$. Obviously, $k$ equals the number of nonzero entries in $P$. Let $P$ be the set of admissible matrices of $P$ parameterized by $S$ that make the NDS $\Sigma$ well posed, and $S \subseteq \mathcal{R}^k$, the set of the corresponding admissible values of $S$. To clarify the dependence of matrices $A$ and $B$ on $S$ in (9), in the following $A$ and $B$ are sometimes denoted by $A(S)$ and $B(S)$, respectively.

From the definition of well posedness of a closed-loop system, it can be straightforwardly proven that the subsystem $\Sigma_i$ is well posed if and only if $\text{det}(I - H^{(i)} P^{(i)}) \neq 0$, $i \in [N]$. Moreover, under the condition that each of its subsystems $\Sigma_i |_{s_i=1}$ is well posed, the NDS $\Sigma$ is well posed if and only if $\text{det}(I - A_{xx} P) \neq 0$. Hence, $S$ is open and dense in $\mathcal{R}^k$. With these results, it can be further shown that if there is a particular $P$ such that the NDS $\Sigma$ and each of its subsystems $\Sigma_i |_{s_i=1}$ are well posed, then for almost every $P$ with the same structure, the corresponding system is also well posed.

A property of a parameter dependent system is called generic, if this property holds almost everywhere in the parameter space [10]. Obviously, $s_i |_{s_i=1} = 0$ makes each subsystem $\Sigma_i$ well posed as well as the whole NDS. This implies that the NDS $\Sigma$ is generically well posed, irrespective of the fixed constants in (1).

Proposition 1: Controllability of the NDS (1)–(2) is a generic property.

Proof: Consider the lumped representation of the NDS $\Sigma$ (1)–(2) given in (8)–(9). The controllability matrix of the system (8) is $C(A(S), B(S)) = [B(S) \hspace{20pt} A(S)B(S) \hspace{20pt} \cdots \hspace{20pt} (A(S))^M-1 B(S)]$. Suppose that there exists an $S_0 \in S$ such that the pair $(A(S_0), B(S_0))$ is controllable. This means that there is at least one set of $M_x$ linearly independent column vectors in $C(A(S_0), B(S_0))$. Denote the matrix composed of these vectors by $C(A(S_0), B(S_0))$, where $[J] = M_x$. It can be straightforwardly proven from (9) that there exist two polynomials $d(S)$ and $n(S)$ of the FPPs $S$ such that $\text{det}(\mathcal{C}(A(S), B(S))) = d(S)/n(S)$. Moreover, $n(S)$ can be expressed as $n(S) = \text{det}(I - A_{xx} P)^t$ for some $t \in \mathbb{N}$. As $C(A(S_0), B(S_0))$ is invertible, it further means that the polynomial $d(S)$ is not identically zero. Let $S = \{ S \in S : d(S) = 0 \}$. Then, for any $S \in S \setminus S$, the matrix $C(A(S), B(S))$ is invertible and hence the pair $(A(S), B(S))$ is controllable. Since the set $S$ has zero Lebesgue measure in $S$, Proposition 1 follows.

Noting that both parametric inaccuracies and parametric variations are unavoidable in actual systems. The above-mentioned results imply that rather than numerical values of the FPPs in each subsystem and the SCM $\Phi$, controllability of this NDS is mainly determined by how the system matrices are influenced by these parameters.

According to the well-known PBH test, $(A, B)$ in (8) is controllable if and only if the matrix $[A - \lambda I \hspace{20pt} B]$ is of FRR at each $\lambda \in \mathbb{C}$. In addition, a complex number $\lambda$ that makes $[A - \lambda I \hspace{20pt} B]$ row rank deficient is called an uncontrollable mode [13]. As argued in [1, Lemma 6.2], [19, Lemma 2], [21], and from Lemma 1, if system (8)–(9) parameterized by the FPPs $S$ is structurally uncontrollable (i.e., the associated system is uncontrollable for each fixed $S \in S$), then the determinants of all the $(M_x + M_l) \times (M_x + M_l)$ dimensional submatrices of the MVP $M(\lambda)$ defined in (7), expressed as polynomials of $s_i |_{s_i=1}$ and $\lambda$, share a common divisor. Moreover, this common divisor, expressed as a polynomial of $\lambda$, has at least one root for each fixed $S \in S$, which can be called the uncontrollable mode of system (8)–(9) with respect to (w.r.t.) parameters $S$. Otherwise, this system is structurally controllable [21]. To further characterize structural uncontrollability, uncontrollable modes are divided throughout this paper into the following two classes.

Definition 1: A fixed uncontrollable mode (FUM) of the NDS (1)–(2) is a fixed $\lambda \in \mathbb{C}$ such that the matrix pair $[A(S) - \lambda I \hspace{20pt} B(S)]$ is of FRR for each fixed $S \in S$. In other words, an FUM is an uncontrollable mode of the NDS (1)–(2) that is independent of $S$.

Definition 2: A parameter-dependent uncontrollable mode (PDUM) of the NDS (1)–(2) is an uncontrollable mode that is not independent of $S$; more precisely, it is an uncontrollable mode of the NDS (1)–(2) for each fixed $S \in S$, and across $S \in S$ the set of its values does not belong to any subset of $\mathbb{C}$ with a finite cardinality.

The aforementioned definitions are similar to the notion fixed mode suggested in [32]. According to the continuous dependence of roots of a polynomial on its coefficients, the PDUMs must piecewise continuously depend on $S$. Note that, whether or not a univariate polynomial with coefficients being polynomials of $s_i |_{s_i=1}$ over $\mathbb{R}$ has a root located in a fixed finite set of isolated complex values is a generic property in the parameter space of $S$ [21]. It can be shown that the existence of either an FUM or a PDUM is a generic property of the LFT parameterized system (8)–(9), so is of the NDS (1)–(2) [10, Sec. 6]. Both FUM and PDUM play important roles in the following analysis about the structural controllability of the NDS $\Sigma$.

IV. Structural Controllability of LFT Parameterized Plants

In order to derive computationally feasible conditions for the structural controllability of the NDS described in (1)–(2), we at
first establish a necessary and sufficient condition in this section for a general LFT parameterized matrix pair to be structurally controllable, that is, a matrix pair described in (9) in which the matrices $A_{xx}, A_{xw}, A_{zw}, A_{zw}, B_{zu}$, and $B_{zw}$ are no longer block diagonal. Due to the existence of the inverse of the matrix $I - A_{xx}P$ in the parameterization, some mathematical challenges might arise.

For the sake of derivations, it is assumed that the matrices $A$ and $B$ in (9) have the dimensions of $n \times n$ and $n \times q$, respectively. Moreover, the following assumption is adopted. This assumption is eliminated in the following section for an NDS.

Assumption 2: $P = \text{diag}\{s_1, \ldots, s_k\}$, where the parameters $s_1, \ldots, s_k$ are algebraically independent.

In the following, we derive conditions of the existence of PDMs via a structure analysis of some associated TFM.

Define two TFM, respectively, as

$$G_{zw}(\lambda) = A_{xx}(\lambda I - A_{xx})^{-1}A_{zw} + A_{zw}$$

$$G_{wu}(\lambda) = A_{xx}(\lambda I - A_{xx})^{-1}B_{zu} + B_{zu}.$$  

With the aforementioned definitions, the following Lemma 3 transforms the existence of PDMs into that of zeros of a TFM.

Lemma 3: The system (8)–(9) has a PDM, if and only if the TFM $[G_{zw}(\lambda)P - I \ G_{wu}(\lambda)]$ has a zero depending on $S$.

Proof: Note that when $\lambda \notin \sigma(A_{xz})$, $[A_{xx}(\lambda I - A_{xx})^{-1} I]$ is a basis of the left null space of $\text{col}([A_{xx} - A_{xz}])$. Then by Lemma 2, for a given $\lambda \notin \sigma(A_{xz})$, $M(\lambda)$ in (7) is of FRR, if and only if $[A_{xx}(\lambda I - A_{xx})^{-1} I] [B_{zu}^{-1} - A_{zu}, P] = [G_{zw}(\lambda) I - G_{wu}(\lambda)P]$ is of FRR.

Let $A_{\lambda} = \{\lambda \in C: \lambda \notin \sigma(A_{xz})\}$. Then, $\lambda$ belongs to $A_{\lambda}$ and varying with $S$ is a zero of $M(\lambda)$, if and only if it is a zero of $[G_{zw}(\lambda)P - I \ G_{wu}(\lambda)]$. Notice that the set $\sigma(A_{xz})$ consists of only some isolated elements that do not affect the piecewise continuous dependence of the zeros of $[G_{zw}(\lambda)P - I \ G_{wu}(\lambda)]$ on $S$. By Lemma 1, the result follows.

Recall that the TFM $G_{zw}(\lambda)$ and $G_{wu}(\lambda)$, respectively, have a dimension of $k \times k$ and a dimension of $k \times q$. Construct a digraph $\mathcal{L} = (V, E)$ associated with $[G_{zw}(\lambda) \ G_{wu}(\lambda)]$ as follows. Define the vertex set as $V = U \cup Z$ with $U = \{u_1, \ldots, u_k\}$ and $Z = \{s_1, \ldots, s_k\}$, whereas the edge set as $E = \mathcal{E}_{UZ} \cup \mathcal{E}_{ZZ}$ with $\mathcal{E}_{UZ} = \{(u_i, z_j) : [G_{zw}(\lambda)]_{ji} \neq 0\}$, $\mathcal{E}_{ZZ} = \{(z_i, z_j) : [G_{wu}(\lambda)]_{ji} \neq 0\}$. Here, for $f(\lambda) \in F(\lambda)$, $f(\lambda)$ is not means that $f(\lambda)$ is not identically zero. The digraph constructed in this way is called the auxiliary connection graph (ACG) associated with $[G_{zw}(\lambda) G_{wu}(\lambda)]$. A vertex in $\mathcal{L}$ is input-reachable, if there exists a path from a vertex in $U$ that ends at it. To clarify the dependence of an edge in $\mathcal{L}$ on the variable $\lambda$, edges of $\mathcal{E}_{ZZ}$ are classified by the following definitions.

Definition 3: Given the ACG $\mathcal{L}$ associated with $[G_{zw}(\lambda) G_{wu}(\lambda)]$, an edge $(z_i, z_j) \in \mathcal{E}_{ZZ}$ is a $\lambda$-edge, if $[G_{wu}(\lambda)]_{ji}$ does not take a value independent of the variable $\lambda$; otherwise, $(z_i, z_j) \in \mathcal{E}_{ZZ}$ is a constant-edge. A cycle of $\mathcal{L}$ that contains at least one $\lambda$-edge is a $\lambda$-cycle. An input-unreachable $\lambda$-edge is a $\lambda$-edge such that neither its start vertex nor its end vertex is reachable from $U$. An input-unreachable $\lambda$-cycle (resp. input-unreachable SCC) is a $\lambda$-cycle (resp. SCC) with each of its vertices being input-unreachable.

Based on the aforementioned definitions, the following proposition gives a graph theoretical necessary and sufficient condition for the existence of a PDUM of the system (8)–(9).

Proposition 2: Under Assumption 2, the system (8)–(9) has a PDUM, if and only if there exists an input-unreachable $\lambda$-cycle in the ACG $\mathcal{L}$.

Proposition 2 implies that, under Assumption 2, the existence of a PDUM is only related to the structure of $[G_{zw}(\lambda) G_{wu}(\lambda)]$ and whether or not an entry of $G_{zw}(\lambda)$ depends on the variable $\lambda$, while the interdependencies among different entries of $[G_{zw}(\lambda) G_{wu}(\lambda)]$ do not influence the obtained results.

To prove Proposition 2, the following Lemmas 4–7 are required, whose proofs are deferred to Appendix B.

Lemma 4: Given a digraph $\mathcal{G}$ with $p$ vertices, $p \geq 2$, suppose that there are two distinct cycles $C_1$ and $C_2$ in $\mathcal{G}$ having an equal length $p$. Then, $\forall \epsilon \in E(C_1)$, there exists a cycle in $\mathcal{G}$ with length not exceeding $p - 1$ that contains $\epsilon$.

Lemma 5: Under Assumption 2, let $J = \{j_1, \ldots, j_n\} \subseteq \{1, \ldots, k\}$ and denote $(G_{zw}(\lambda))_{J,J'}$ by $G_{zw}^J(\lambda)$, $\text{diag}\{s_1, \ldots, s_n\}$ by $P_J$. Let $\mathcal{L}^J$ be the subgraph of $\mathcal{L}$ induced by vertices $\{z_{j_1}, \ldots, z_{j_n}\}$. Then, the TFM $G_{zw}^J(\lambda)P_J - I$ has a zero depending on parameters $s_{j_i}$, if and only if $\mathcal{L}^J$ has a $\lambda$-cycle.

Lemma 6: Under Assumption 2, if there does not exist an input-unreachable vertex $z \in Z$ in the digraph $\mathcal{L}$ associated with $[G_{zw}(\lambda) G_{wu}(\lambda)]$, then the obtained matrix after deleting any column from $[G_{zw}(\lambda)P - I \ G_{wu}(\lambda)]$ is of full row generic rank (FRGR, i.e., its generic rank equals its number of rows).

Lemma 7: Under Assumption 2, if the submatrix obtained by deleting any column from $[G_{zw}(\lambda)P - I \ G_{wu}(\lambda)]$ is of FRGR, then $[G_{zw}(\lambda)P - I \ G_{wu}(\lambda)]$ does not have a zero depending on $S$.

Proof of Proposition 2: To prove the if part, suppose that there is an input-unreachable $\lambda$-cycle in the digraph $\mathcal{L}$. Let the vertex set of this $\lambda$-cycle be $Z_s = \{z_{j_1}, \ldots, z_{j_n}\} \subseteq Z$. Moreover, define a set $J$ as $J = \{j_1, \ldots, j_n\}$. Since $Z_s$ are unreachable from $U$, it is clear that there exists a permutation matrix $Q$ such that $[10] QG_{zw}(\lambda)Q^T = [G_{zw}^J(\lambda) G_{wu}^J(\lambda)]$, $QG_{wu}(\lambda) = [0 \ G_{wu}(\lambda)]$, and therefore

$$Q [G_{zw}(\lambda)P - I \ G_{wu}(\lambda)] [Q^T \ 0] [0 \ I] = \begin{bmatrix} G_{zw}^J(\lambda)P_J - I; & 0 \\ G_{wu}^J(\lambda)P_J & 0 \end{bmatrix}$$

where $G_{wu}^J(\lambda) = F(\lambda)^{(k-n_s \times n_s)}G_{wu}^J(\lambda)G_{wu}^J(\lambda) = F(\lambda)^{(k-n_s \times k-n_s)}$, $G_{zw}^J(\lambda) = G_{zw}^J(\lambda)$, and $P_J$ are defined in Lemma 5. By Lemma 5, $G_{zw}^J(\lambda)P_J - I$ has a zero depending on $s_{j_i}$, so does $[G_{zw}(\lambda)P - I \ G_{wu}(\lambda)]$ noting that according to (10) the associated ranks remain invariant. By Lemma 3, this means that the system (8)–(9) has a PDUM.

To prove the only if part, first assume that every vertex in $Z$ is input-reachable in $\mathcal{L}$. Then from Lemmas 6–7, we have that
\[ G_{zz}(\lambda) P - I \quad G_{zu}(\lambda) \] does not have a zero depending on \( S \).
Now suppose that there is an input-unreachable vertex in \( L \),
and denote the set of all input-unreachable vertices by \( \bar{Z}_s \subseteq Z \).
Suppose that there does not exist a \( \lambda \)-cycle in the subgraph of \( L \) induced by \( \bar{Z}_s \) (denoted by \( \bar{L}^2 \)).
Following (10), there is a permutation matrix \( \bar{Q} \) such that
\[
\bar{Q} \left[ G_{zz}(\lambda) P - I \quad G_{zu}(\lambda) \right] \bar{Q}^T = 0 \quad 0
\]
(11)
where \( G_{zz}(\lambda) \in F(\lambda)^{|\bar{Z}_s|\times|\bar{Z}_s|}, G_{zu}(\lambda) \in F(\lambda)^{|\bar{Z}_s|\times|\bar{Z}_s|} \).

Combining Propositions 2 and 3, we have the following theorem.

**Theorem 1:** Under Assumption 2, the LFT parameterized plant (8)–(9) is structurally controllable, if and only if the following two conditions hold simultaneously.

i) There is no input-unreachable \( \lambda \)-cycle in the ACG \( L \).

ii) For each \( \lambda_0 \in \sigma(A_{xx}) \), \( \rho(M_1 \cup M_2(\lambda_0)) = 2k + n \).

**Proof:** The result follows from Propositions 2 and 3, and the analysis on structural uncontrollability in Section III.

Checking whether an ACG \( L \) has an input-unreachable \( \lambda \)-cycle can be done efficiently using SCC decompositions with a complexity of \( O(|V(L)| + |E(L)|) \) [31], which is as follows.

1) Do SCC decompositions on \( L \), and find all the input-unreachable SCCs.

2) If there is a \( \lambda \)-edge in at least one input-unreachable SCC, there will be an input-unreachable \( \lambda \)-cycle in \( L \).

Otherwise, there will be none.

The rationale of the aforementioned procedure lies in that, all vertices of a \( \lambda \)-cycle must belong to the same SCC, as they are reachable from each other. Besides, if there is a \( \lambda \)-edge in an SCC, then a \( \lambda \)-cycle exists, noting that the start and end vertices of this \( \lambda \)-edge are mutually reachable. The matroid union Condition ii) of Theorem 1 enables efficient verification with polynomial time complexity.

In addition, Theorem 1 provides some other information for a structurally uncontrollable system. Particularly, if a plant has a PDUM, there must exist some uncontrollable modes that depend on the FPPs associated with the shortest input-unreachable \( \lambda \)-cycle in the ACG \( L \), which is indicated in the proof of Lemma 5. Moreover, any \( \lambda_0 \) that fails to satisfy Condition ii) of Theorem 1 is an FUM of the plant.

**Remark 4:** A necessary and sufficient condition of structural controllability for a linear parameterized plant was first given in [6]. The condition there has exponential time complexity, and was derived from the decentralized stabilization theory [7], which cannot be directly extended to LFT parameterized plants due to the existence of the matrix inverse \((I - A_{xx} P)^{-1}\). The descriptor system approach in [20] is under the nondimensionality assumption defined therein, which introduces some restrictions on the fixed nonzero constants of the associated matrices. Our derivations for the LFT parameterized plant are mainly based on structure analysis of the associated TFMs, and do not impose any assumptions on the associated constant matrices. Moreover, these results have some physical interpretations when applied to an NDS as shown in the following section.
Illustration of n-ACGs of the example in Section VII. The dotted edges represent $\lambda$-edges, whereas the bold ones represent subsystem links.

V. NETWORK-BASED NECESSARY AND SUFFICIENT CONDITION FOR STRUCTURAL CONTROLLABILITY

In this section, results in Section IV are extended to the NDS (1)–(2) by taking the system matrix structures into account. Note that Assumption 2 is not needed in this section.

For subsystem $\Sigma_i$ in (1) with its augmented system matrices given in (4), define TFMs $G_{in_1}^{\Sigma_i}(\lambda)$ and $G_{in_2}^{\Sigma_i}(\lambda)$, respectively, as:
\[
G_{in_1}^{\Sigma_i}(\lambda) = A_{in_1}^{\Sigma_i}(\lambda I - A_{in_1}^{\Sigma_i})^{-1}A_{in_2}^{\Sigma_i} + A_{in_1}, \quad G_{in_2}^{\Sigma_i}(\lambda) = A_{in_2}^{\Sigma_i}(\lambda I - A_{in_2}^{\Sigma_i})^{-1}B_{in_2}^{\Sigma_i} + B_{in_1}^{\Sigma_i}.
\]
Construct the ACG $T_i$ associated with $[G_{in_1}^{\Sigma_i}(\lambda), G_{in_2}^{\Sigma_i}(\lambda)]$ as $T_i = (U_i \cup V_i \cup Z_i, \mathcal{E}_U, \mathcal{E}_V, \mathcal{E}_Z)$, where $U_i = \{u_{i1}, \ldots, u_{im_i}\}$, $V_i = \{v_{i1}, \ldots, v_{im_i}\}$, and $Z_i = \{z_{i1}, \ldots, z_{im_i}\}$ represent the vertex set of external inputs, internal inputs, and internal outputs of $\Sigma_i$ respectively; $\mathcal{E}_V \cup Z_i$ by connecting these $N$ ACGs $T_i$ from $z_i$ to $v_j$ through the edge set $E_P = \{(zi_p, v_{jp}) : [P_{ij}]_{1 \leq p \leq m_{i1}} \neq 0, i, j \in [N], p \in [m_{i1}], q \in [m_{i2}]\}$ where $P_{ij}$ is the $(i, j)$th block of matrix $P$ in (6). Then, $V_i = \bigcup_{j=1}^{m_{i1}} (U_i \cup Z_i \cup V_j \cup \mathcal{E}_Z)$, $\mathcal{E}_V = \mathcal{E}_P \cup \bigcup_{j=1}^{m_{i1}} (\mathcal{E}_U \cup Z_i \cup \mathcal{E}_Z)$). Afterward, this digraph is called the networked-ACG, and abbreviated as n-ACG. Recall that an entry of $G_{in_1}^{\Sigma_i}(\lambda)$, which depends on the variable $\lambda$, corresponds to a $\lambda$-edge in $T_i$, and definitions of other graph elements in $T_i$ like the $\lambda$-cycle, inherit Definition 3. An illustration of n-ACGs can be found in Fig. 2. Notice that $G_{in_1}^{\Sigma_i}(\lambda)$ and $G_{in_2}^{\Sigma_i}(\lambda)$ are, respectively, the TFMs from the internal inputs $x_i(t)$ and the external inputs $u_i(t)$ to the internal outputs $y_i(t)$ of the augmented system of subsystem $\Sigma_i$, and $P$ has the same structure as the SCM $\Phi$ except for its diagonal blocks. It seems safe to declare that $T_i$ intuitively reflects the information flows over the NDS (1)–(2).

For the NDS (1)–(2) and $P$, to meet Assumption 2, consider the following transformation on its lumped representation (9)
\[
F_1 = \begin{bmatrix} A_{xx} & B_{xx} & A_{xx} & \vdots \\ B_{xx} & B_{xx} & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & P_d \end{bmatrix}, \quad P = UP_dV
\]
where $P_d \triangleq \text{diag}(s_{i1}, \ldots, s_{iN})$, $U$ and $V$ are some constant matrices satisfying $P = UP_dV$. Here, each column of the matrix $U$ and each row of the matrix $V$ is a standard unit basis vector of an Euclidean space. Under the condition that all the FPPs of the NDS are algebraically independent, it can be simply shown that this decomposition always exists.

For the lumped representation in the right-hand side of (14), define TFMs $G_{in_1}^{\Sigma}(\lambda)$ and $G_{in_2}^{\Sigma}(\lambda)$, respectively, as:
\[
G_{in_1}^{\Sigma}(\lambda) = VA_{in_1}(\lambda I - A_{in_1})^{-1}A_{in_2}U + VA_{in_1}U, \quad G_{in_2}^{\Sigma}(\lambda) = VA_{in_2}(\lambda I - A_{in_2})^{-1}B_{in_2} + B_{in_1}.
\]
Denote the ACG associated with $[G_{in_1}^{\Sigma}(\lambda), G_{in_2}^{\Sigma}(\lambda)]$ by $T_\Sigma$. We have the following result, which establishes a relation between $T_\Sigma$ and $T_i$ w.r.t. the existence of an input-unreachable $\lambda$-cycle, thus removing Assumption 2. Its proof is deferred to Appendix B, where the essential idea is to build a relation between graph connectivity and multiplications of the associated matrices. A similar issue is addressed in [28] for bilinear systems by using unit matrices.

Proposition 4: There exists an input-unreachable $\lambda$-cycle in $T_\Sigma$, if and only if there exists an input-unreachable $\lambda$-cycle in the n-ACG $T_i$. In other words, the NDS (1)–(2) has no FUM, if and only if there is no input-unreachable $\lambda$-cycle in $T_\Sigma$.

To derive a computationally efficient criterion for verification of an FUM for NDSs, we adopt some ideas similar to those in [34] and [36]. For each subsystem, define an MVP $M_i(\lambda) = [A_i^{\Sigma}(\lambda) - A_{in_1}^{\Sigma}(\lambda)B_{in_1}^{\Sigma}(\lambda)]$. Suppose that there are $m$ distinct values in $\bigcup_{i=1}^{N} \sigma(A_i^{\Sigma}(\lambda))$, and denote the set constituted from them by $\Lambda = \{\lambda_1, \ldots, \lambda_m\}$. For $i \in [m], j \in [N]$, let $[T_i^{(j)}(\lambda)]_{ij}$ be a matrix constituted by the basis of the left null space of $M_i(\lambda_i)$, and assume that $T_i^{(j)}$ and $Z_i^{(j)}$ have dimensions $m_{ij} \times m_{ij}$ and $m_{ij} \times m_{ij}$, respectively. Notice that $m_{ij} = 0$ if $\lambda_i$ is not a zero of this $M_i(\lambda)$, which is of FRNR. Moreover, let $M_{ir} = \bigcup_{j=1}^{N} m_{irj},$ and $Y_{ir}^{(j)} = T_i^{(j)}A_{in_1}^{\Sigma} + Z_i^{(j)}A_{in_2}^{\Sigma}$. Construct matrices $T_i = \text{diag}(T_i^{(j)}(\lambda)), Y_i = \text{diag}(Y_{ir}^{(j)}(\lambda)), Z_i = \text{diag}(Z_i^{(j)}(\lambda))$. It is clear that $T_i Z_i$ is a basis of the left null space of $\text{col}\{[\lambda_i I - A_{xx} B_{xx}]_{1 \leq i \leq r} - A_{xx} B_{xx}\}$. Let $\lambda_i$ be defined in (7) of FRGR, and if only if $[\lambda_i Z_i]\text{col}\{A_{xx} B_{xx}(P, I_M)\}$ is of FRGR, which is further equivalent to that $[\lambda_i Z_i]\text{col}\{P, I_M\}$ is of FRGR. Hence, we have the following proposition.

Proposition 5: The NDS $\Sigma(1)$–(2) does not have an FUM, if and only if for each $i \in [m], \rho(M(Q_i) \cap M(Q_{Z_i})) = M_{ri},$ where $Q_1 = [P^{T} I_M], Q_{Z_1} = [Y_i Z_i]$. Proof: Notice that for a set of algebraically independent variables $\{s_{11}, s_{12}, t_{11}, \ldots, t_{1m}\}$ where each $t_i$ is nonzero
\[
[Y_i Z_i] \begin{bmatrix} P^{T} I \\ \text{diag}(t_1, \ldots, t_m) \end{bmatrix} = [Y_i Z_i] \begin{bmatrix} \text{diag}(t_1, \ldots, t_m) \\ \text{diag}(t_1, \ldots, t_m) \end{bmatrix},
\]
(15)
It means that the generic rank of $[Y_i Z_i]\text{col}\{P, I_M\}$ equals that of the right-hand side of (15). Denote $\text{diag}(t_1, \ldots, t_m)$ by II. Note that the nonzero entries of $\text{col}\{P^{T} I_M\}$ are algebraically independent. Hence, $M([II^{T} I_M]) = M([P^{T} I_M])$. From the Binet-Cauchy theorem [11, page 28], it can be directly shown that the right-hand side of (15) is of FRGR, if and only if
there exists $J \subseteq [M_r + M_z]$ with $|J| = M_r$, such that $Q_{2i,J}$ is of full column rank, whereas $[I P T \ II]$ is of FRGR. By the definition of matroid intersection, this is equivalent to that $\rho(M(Q_1) \cap M(Q_2)) = M_r$. 

As a result of Propositions 4 and 5, the structural controllability verification of the NDS (1)–(2) can be implemented efficiently. In Proposition 4, the arithmetic operations are imposed on each subsystem in constructing the n-ACG $T_\Sigma$. The existence of an input-unreachable $\lambda$-cycle can be checked using SCC decompositions in time complexity $O(M^2_2)$. In Proposition 5, parameters constituting $Y_i$ and $Z_i$ can be constructed within each subsystem because of their block diagonal structures. When the matroid intersection algorithm is utilized, the only operation that involves arithmetic calculations is the independence oracle call, i.e., verifying whether columns of $Q_1,J$ or $Q_{2i,J}$ for $J \subseteq [M_r + M_z]$ are linearly independent [21]. Denote the time of such an operation by $\tau$. Then, the conditions of Proposition 5 can be tested in polynomial time with complexity $O(\tau M_r(M_r + M_z)^2.5)$ [21]. Note that $Q_{2i}$ composes of two column blocks with each one being block diagonal, and $Q_1$ can be seen structured. Hence, verifying the conditions of both Propositions 4 and 5 involves arithmetic operations only within each subsystem, and graphical operations on the network topology. This property makes our approach attractive when dealing with large-scale systems.

Propositions 4 and 5 also illustrate how subsystem input–output relations and subsystem uncontrollable modes, together with the subsystem interconnection topology, jointly influence network controllability. Note that the n-ACG $T_\Sigma$ somehow reflects the information flows over the NDS, and a $\lambda$-edge means that this channel contains some state information of one associated subsystem. Proposition 4 means that, to guarantee the absence of a PDUM, it is necessary and sufficient to make sure that there does not exist any state involved closed loop that cannot receive signal from the external inputs. Moreover, it can be validated that, if $\lambda_i$ is an uncontrollable mode shared by some subsystems $\Sigma_{j_i}$, [i.e., $A_{xx}^{(j_i)} - \lambda_iI$ is of row rank deficient], then $Z_i$ is not of FRR, $s \in [p]$. From the property of matroid intersection, Proposition 5 implies that, to guarantee that $\lambda_i$ is not an FUM of the NDS, at least $\sum_{s=1}^{p}(m_{rij_s} – \text{rank}(Z^{(j_i)}))$, which is equal to $\sum_{s=1}^{p}(m_{rij_s} – \text{rank}(A_{xx}^{(j_i)} + B^{(j_i)}))$ [11], vertex-disjoint subsystem interconnection links (corresponding to the nonzero entries in $P^{(i)}$), including possible self-loops, should be injected to these augmented subsystems $\Sigma_{j_i}$. These observations are helpful in constructing a controllable NDS.

Propositions 4 and 5 can be combined to derive a simpler criterion for structural controllability of the NDS.

**Corollary 1:** The NDS $\Sigma$ (1)–(2) is structurally controllable, if and only if 
1) there is no input-unreachable $\lambda$-edge in the n-ACG $T_\Sigma$;
2) for each $i \in [n]$, $\rho(M(Q_1) \cap M(Q_2)) = M_r$, where $Q_1$ and $Q_2$ are defined in Proposition 5.

**Proof:** The if part is obvious. To show the only if part, suppose that there exists an input-unreachable $\lambda$-edge $(v_{ip}, z_{iq})$, which is not contained by any cycle in the n-ACG $T_\Sigma$, $i \in [N]$, $p \in [m_{xi}]$, $q \in [m_{zr}]$. Add an edge $(z_{iq}, v_{ip})$ to $T_\Sigma$, then an input-unreachable $\lambda$-cycle emerges. According to Proposition 4, this indicates that the obtained NDS with the addition of edge $(z_{iq}, v_{ip})$ is structurally uncontrollable. It further implies that the original NDS is structurally uncontrollable.

**Remark 5:** Corollary 1 indicates that the existence of an input-unreachable $\lambda$-edge implies the presence of either an FUM or a PDUM. This corollary is significant in proving the performance guarantee of the topology design procedure developed in the following section.

**VI. MINIMAL DESIGN OF SUBSYSTEM INTERCONNECTION TOPOLOGY**

In this section, as an application of the results in Section V, we consider a minimal design of subsystem interconnection topology for an NDS to achieve structural controllability. All proofs of the results in this section are deferred to Appendix B.

Consider the following topology design problem. Given $N$ linear time invariant subsystems $\Sigma_1, \ldots, \Sigma_N$ whose subsystem dynamics are captured in (1) with known FPPs $P^{(i)}_{li} |_{l=1}^N$, find the sparsest structured SCM $\Phi$, such that the constructed NDS is structurally controllable. Denote the NDS by $\Sigma(\Phi)$ when a structured $\Phi$ is designated.

2Note that a nonzero entry of $\Phi$ is associated with an interconnection link between two subsystems, which will be called a subsystem link. For notational simplicity, the system matrices of the subsystem $\Sigma_i$ with a fixed $P^{(i)}$, denoted by $A_{xx}^{(i)}, A_{zz}^{(i)}, B_{zu}^{(i)}, D_{zu}^{(i)}$, etc., are first computed by substituting the value of $P^{(i)}$ into the LFT parameterization in (1), and $m_{zi}$ and $m_{zi}$ denote the dimensions of internal output and internal input vectors of $\Sigma_i$, respectively. All the remaining notations have the same definitions as those in the previous sections. Then, this problem can be formulated as follows.

**Problem 2:** Given $N$ subsystems $\Sigma_1, \ldots, \Sigma_N$,

$$\min_{\Phi \in \{0,1\}^{N^2 \times M_r}} \|\Phi\|_0$$

s.t. $\Sigma(\Phi)$ is structurally controllable. (16)

A similar problem has been investigated in [14] and [37], for a structured system described by graphs. The aforementioned problem is different, noting that the dynamics of subsystems are numerically prescribed and they can be high order and heterogeneous. Applications of this problem include designing interconnection topology for geographically distributed multiagents to achieve consensus [2], designing communication links for geographically distributed sensors for data fusion.

**A. Feasibility and Complexity**

**Lemma 8:** Problem 2 is feasible, if and only if the following three conditions are satisfied simultaneously:

i) for each $i \in [N]$, $(A_{xx}^{(i)}, B_{zu}^{(i)}, A_{zz}^{(i)})$ is controllable;
ii) $M_z \geq \max_{1 \leq l \leq m} M_{ri}$; 

2If $\Sigma(\Phi)$ is structurally controllable, a realization of $\Phi$ making the NDS controllable with probability 1 can be obtained by setting values of the nonzero entries of $\Phi$ randomly from a set of real numbers with a sufficiently large cardinality. This is made clear by Proposition 8 in Appendix A.
iii) there exists at least one integer \( i \in [N] \) such that 
\[ G^{(i)}_{\lambda} (\lambda) \neq 0, \]
for or each \( i \in [N] \), there is no \( \lambda \)-edge in the ACG \( T_i \).

**Proposition 6:** Problem 2 is NP-hard.

### B. Two-Stage Algorithm With Provable Approximation Bounds

Since Problem 2 is NP-hard, we propose a scalable algorithm to approximate it with some provable approximation bounds. This algorithm has two stages.

1) **Stage 1—FUM Elimination:** In this stage, starting from \( N \) disconnected subsystems, we want to select the minimal number of subsystem links such that the obtained NDS \( \Sigma(\Phi) \) does not have an FUM. To clarify the dependence of \( Q_1 \) of Proposition 5 on \( \Phi \), rewrite \( Q_1(\Phi) \triangleq [\Phi^T \, I_M] \). Define a function \( f(\Phi) : \{0, +\}^{M_r \times M_s} \to \mathbb{N} \) as 
\[ f(\Phi) = \sum_{i=1}^{m} \rho(\mathcal{M}((Q_i(\Phi)) \cap \mathcal{M}(Q_{2i}))). \]
Suppose that the feasibility condition is satisfied. By Proposition 5, it suffices to see that the objective of the following Problem 3 is equivalent to that of Stage 1.

**Problem 3:**

\[
\min_{\Phi \in \{0, +\}^{M_r \times M_s}} \| \Phi \|_0 \quad \text{s.t.} \quad f(\Phi) = \sum_{i=1}^{m} M_{ri}. \tag{17}
\]

**Lemma 9:** Problem 3 is NP-hard.

Unfortunately, Problem 3 is NP-hard. Besides, the nonsubmodularity of \( f(\Phi) \) w.r.t. the nonzero elements of \( \Phi \) might even prevent the existence of a nontrivial provable performance guarantee of using a simple greedy algorithm. Here, to seek for an algorithm with a provable performance bound, we propose the following alternative algorithm. This algorithm composes of two steps. The first step is to approximate a lower bound of Problem 3, which can be formulated as a submodular function optimization problem. To this end, let \( J \subseteq [M_r] \) and denote by \( \Omega(J) = \text{diag}\{I_{M_r}, \ldots, I_{M_s}\} \). Define a function \( g(J) \) as 
\[ g(J) = \sum_{i=1}^{m} \rho(\mathcal{M}(\Omega(J)) \cap \mathcal{M}(Q_{2i})). \]
It can be directly proven that \( g(J) = \sum_{i=1}^{m} \text{rank}([Y_i \, Z_i]) \).

Then we introduce another related problem, which is Problem 4, as follows.

**Problem 4:**

\[
\min_{J \subseteq [M_r]} |J| \quad \text{s.t.} \quad g(J) = \sum_{i=1}^{m} M_{ri}. \tag{18}
\]

**Proposition 7:** \( g(J) \) is submodular on \( J \subseteq [M_r] \). Moreover, denote the solution of Problem 4 returned by the greedy algorithm stated as Step 1 in Algorithm 1 by \( J_{\text{grd}} \), and the optimal solution to Problem 3 by \( \Phi^* \). Then, it holds that
\[
\frac{|J_{\text{grd}}|}{\| \Phi^* \|_0} \leq 1 + \log M_{\text{def}} \leq 1 + \log M_s,
\]
in which \( M_{\text{def}} \triangleq \sum_{i=1}^{m} \sum_{j=1}^{N} (m_{xj} - \text{rank}(\mathcal{M}(Q_i(\Phi) \cap \mathcal{M}(Q_{2i})))) \), \( J_{r-1} \) is the return value of the second-to-last iteration of the greedy algorithm.

The second step uses the greedy coloring techniques [31] to restore a feasible solution to Problem 3 from \( J_{\text{grd}} \). Graph coloring is the problem of coloring vertices of a graph such that any two adjacent vertices do not share the same color. Greedy coloring is a heuristic method toward this problem that assigns to a vertex with the smallest available color among all colored vertices not used by its neighbors in a specific order, adding a new color if needed. The adoption of graph coloring is motivated by the following observations. Notice that for a given \( \Phi \), \( \rho(\mathcal{M}(Q_i(\Phi) \cap \mathcal{M}(Q_{2i}))) = M_{ri} \) implies that there are \( M_{ri} \) nonzero entries located in the columns of \( Q_i(\Phi) \). Therefore, the coloring is optimal if the following problem is solved. In fact, it is a packing problem of considering the columns of \( Q_i(\Phi) \) indexed by \( J_i \) for some \( J_i \subseteq [M_r + M_s] \) with \( |J_i| = M_{ri} \).

**Algorithm 1:** Stage 1 of the Topology Design Procedure: Selecting Minimal Subsystem Links to Eliminate FUMs.

**Input:** System matrices of subsystems \( \Sigma_1, \ldots, \Sigma_N \).

**Output:** Approximated \( \Phi \) for Problem 3

**Step 1:** Use a greedy algorithm to approximate Problem 4

1: Calculate \( Y_i, Z_i \), for \( i = 1, \ldots, m \);
2: Initialize \( J \leftarrow \{1, \ldots, M_r\} \), \( J_{\text{grd}} \leftarrow \emptyset \);
3: while \( g(J) < \sum_{i=1}^{m} M_{ri} \) do
4: \[ s \leftarrow a' \in \arg \max_{s \in J_{\text{grd}}} g(J_{\text{grd}} \cup \{a\}) - g(J_{\text{grd}}) \];
5: \( J_{\text{grd}} \leftarrow J_{\text{grd}} \cup \{s\} \);
6: end while

**Step 2:** Use greedy coloring to construct a solution to Problem 3 from \( J_{\text{grd}} \)

7: Find a collection \( \{J_1, \ldots, J_m\} \) of subsets of \( [M_r + M_s] \) satisfying \( |J_i| = M_{ri}, J_i \cap B \subseteq J_{\text{grd}} \), such that \( \text{rank}(Q_{2i}, J_i) = M_{ri}, J_i \cap B \) is maximized per \( i = 1, \ldots, m \), where \( B = \{M_r + 1, \ldots, M_r + M_s\} \);
8: Construct the coloring auxiliary graph \( G(J_1, \ldots, J_m) \);
9: Initialize \( \mathcal{G}_{\text{col}} = G(J_1, \ldots, J_m) \), index \( M_{ri} \) colors as \( 1, \ldots, M_{ri} \), and use them to color \( G(J_1, \ldots, J_m) \) according to the following procedure:
   i) for each vertex \( j \in J_{\text{grd}} \cap B \), color vertex \( j \) by the \( (j - M_r) \)th color;
   ii) for each iteration, do the following operations, until there is no uncolored vertex in \( \mathcal{G}_{\text{col}} \):
      1) among all uncolored vertices, choose the one that is adjacent to the largest number of differently colored vertices, denoted by \( v^* \);
      2) if vertex \( v^* \) has \( M_{ri} \) differently colored neighbors, assign \( k_{\text{max}}^i \) distinct colors to \( v^* \), where \( k_{\text{max}}^i = \max\{M_{ri} : v^* \in J_i, i \in [m]\} \), and remove the edges between \( v^* \) and its neighbors from \( \mathcal{G}_{\text{col}} \); otherwise, assign \( v^* \) a color different from \( v^* ' \)'s colored neighbors, such that the number of already used colors is minimized;
10: Map \( \mathcal{G}_{\text{col}} \) to \( \Phi \), \( \Phi_{ij} = * \) if vertex \( i \) is colored by color \( j \) in \( \mathcal{G}_{\text{col}} \), \( 1 \leq i \leq M_r, 1 \leq j \leq M_{ri} \); the rest entries of \( \Phi \) are zero.

\[ \text{This can be done efficiently in a simple greedy manner.} \]
Theorem 2: The two-stage topology design procedure overall returns an $O(2M_{\max}\log(M_{\text{det}}))$ approximation for Problem 2, where $M_{\max} = \max_{1 \leq i \leq m} M_{ij}$. In Stage 1, Algorithm 1 returns an $O(M_{\max}\log(M_{\text{det}}))$ approximation for Problem 3. In addition, if every vertex of $G_{\text{col}}$ in Substep 10 of Algorithm 1 is colored by only one color, then the approximation factor becomes $O(\log(M_{\text{det}}))$ for Problem 3.

There are two promising features in the aforementioned topology design procedure. First, it has a low computational complexity. To be specific, Step 1 of Stage 1 has complexity $O(NM_{\text{det}})$. Step 2 of Stage 1 incurs at most $O(M_{\max}^3)$ complexity. Stage 2 has $O(M_{\max}M_{\text{det}})$ complexity. Hence, the overall time complexity is $O(M_{\max}^3 + M_{\max}^2)$. Note that we usually have $M_{\max} \ll M_{\text{det}}$ in actual applications [34]. This means that the time complexity of our approach usually increases quadratically with $M_{\max}$ in practice. Second, it has a provable approximation bound, whereas other possible heuristics might not. The approximation bound $2M_{\max}\log(M_{\text{det}})$ might seem loose. However, when the subsystems have smaller dimensions of uncontrollable subspaces that lead to a smaller $M_{\text{det}}$, or have more heterogeneous eigenvalues that leads to a smaller $M_{\max}$, this bound becomes tighter.

VII. ILLUSTRATIVE EXAMPLE

In this section, a numerical example is given to illustrate the results obtained in the previous sections.

Consider an NDS consisting of three subsystems. The system matrices for these subsystems with known FPPs are given at the bottom of this page. From these parameters, the associated SCM has a dimension of $6 \times 5$.

First, consider the SCM $\Phi_a$ with its $(1, 4)^{\text{th}}$, $(2, 3)^{\text{th}}$, $(3, 5)^{\text{th}}$, $(5, 2)^{\text{th}}$ and $(6, 1)^{\text{th}}$ entries being nonzero. The corresponding n-ACG $T_{\Sigma}$ is given in Fig. 2(a). From Fig. 2(a), there is an input-unreachable $\lambda$-cycle in $T_{\Sigma}$, which is $\{v_{12} \rightarrow z_{11} \rightarrow v_{32} \rightarrow z_{32} \rightarrow v_{21} \rightarrow z_{21} \rightarrow v_{12}\}$. Hence, by Proposition 4, the aforementioned NDS has at least one PDUM, which means that it is not controllable under the earlier structured SCM $\Phi_a$ with any specific link weights. This can be validated by some algebraic manipulations.

Second, we show the application of the modified version of Algorithm 1 in selecting the minimal subsystem links to construct an NDS without unstable FUMs. Note that the matrix pair $(A_{ix}^{(i)}, B_{ix}^{(i)})$ has some unstable uncontrollable modes for $i = 2, 3$. This means that subsystems $\Sigma_2$ and $\Sigma_3$ cannot be stabilized by state feedback when isolated. The set of subsystem eigenvalues is $\Lambda = \{1, 0, -1\}$, with unstable mode set being $\Lambda = \{1, 0\}$. Let $\lambda_1 = 1, \lambda_2 = 0$. Applying Algorithm 1 by letting $m = 2$, the coloring auxiliary graph $G(J_1, J_2)$ is given in Fig. 3(a), with its associated colored graph $G_{\text{col}}$ being Fig. 3(b). Accordingly, two subsystem links $(z_{31}, v_{21})$ and $(z_{12}, v_{31})$, as illustrated in Fig. 2(b), are sufficient to eliminate these unstable FUMs. From Fig. 2(b), there does not exist an input-unreachable $\lambda$-cycle. Hence, there is no PDUM in the corresponding NDS.

Since any other FUM if there exists, is stable, for “almost each” numerical $\Phi_0$ with the structure shown in Fig. 2(b), the associated NDS $\Sigma(\Phi_0)$ is stabilizable by a state feedback.

Now suppose that our goal is to find the sparsest SCM to construct a structurally controllable NDS. Continuing the aforementioned procedure, let $\lambda_3 = -1$. Following a similar procedure described earlier, the associated graphs $G(J_1, J_2, J_3)$ and $G_{\text{col}}$...
before and after coloring are given, respectively, in Fig. 3(c) and (d). The corresponding interconnection topology is illustrated in Fig. 2(c), which shows that there does not exist an input-unreachable λ-cycle. Hence, the obtained SCM makes the NDS structurally controllable. Through an exhaustive search, it becomes clear that this interconnection topology is optimal in the sense that it has the minimal number of subsystem links making the associated NDS structurally controllable.

VIII. CONCLUSION

This paper investigates structural controllability of an NDS, in which unknown parameters are allowed to exist in both subsystem dynamics in an LFT parameterized way and in subsystem interconnections, and each subsystem may have high-order, heterogeneous dynamics. Some results are first obtained about structural controllability of an LFT parameterized plant under a diagonalization assumption, which further lead to some necessary and sufficient conditions for the NDS to have an FUM, a PDUM, and to be structurally controllable. These conditions can be verified efficiently, and give some intuitive insights on how the network controllability is influenced by subsystem dynamics, and subsystem interconnection topology. Based on them, a minimal design of subsystem interconnection topology is considered for an NDS to achieve structural controllability. Further research works include studying more generic properties for an NDS with unknown parameters, such as the generic dimension of controllable subspaces, structural controllability with a random switching topology, etc.

APPENDIX A

GENERAL COMPLEXITY AND HARDNESS OF STRUCTURAL CONTROLLABILITY VERIFICATION: HIGH-RANK CASE

In this appendix, we discuss the general computational complexity and hardness of structural controllability verification when the coefficient matrix of a variable in system matrices is not restricted to be rank one. Specifically, for the pair in (9), we set $A_{2r} \equiv 0$ and equivalently write the following $(A, B)$ as in [1]:

$$A = A_0 + \sum_{i=1}^{k} s_i A_i, B = B_0 + \sum_{i=1}^{k} s_i B_i$$

(19)

where $A_i \in \mathbb{R}^{n \times n}$ and $B_i \in \mathbb{R}^{n \times q}$ are constant matrices, and $s_1, \ldots, s_k$ are free parameters.

Definition 4 (RP, [3]): Randomized polynomial time (RP) is the complexity class of problems for which a random algorithm exists with: first, it always runs in polynomial time in the input size; second, if the correct answer is YES, it returns YES with probability at least 1/2; if the correct answer is NO, then it always returns NO.

From Definition 4, the following proposition is obtained.

Proposition 8: Verifying the structural controllability of $(A, B)$ in (19) is RP. More specifically, let $r_{\text{max}} \triangleq \max_{1 \leq i \leq k} \text{rank}(A_i B_i)$, and the set of real numbers $V \subseteq \mathbb{R}$ satisfies $|V| = \min\{2kn_{\text{max}}, 2n^2\}$. If $(A, B)$ is structurally controllable, then, randomly choose an element $(s_1, \ldots, s_k) \in V^k$, where $V^k$ is the set of all $t$-element vectors with each element chosen from $V$, with probability at least 1/2, the obtained numerical $(A, B)$ of (19) is controllable.

Proof: A numerical $(A, B)$ is controllable if and only if the matrix $C$ defined in (20) with dimension $n^2 \times n(q + 1)$ is of FRR [13]

$$C \triangleq \begin{bmatrix} B & I & 0 & 0 & 0 & 0 \\ 0 & -A & B & I & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & -A & B & I \\ 0 & 0 & 0 & 0 & -A & B \end{bmatrix}$$

(20)

If $(A, B)$ in (19) is structurally controllable, there exists at least $n^2$ columns of $C$, the set of whose column indexes is denoted by $K$, such that $\det C_K$ cannot be identically zero. Let $d$ be the total degree of $\det C_K$ (i.e., the highest degree of one monomial). It is obvious that $d \leq n^2$. Notice that also, the degree of $s_i$ in $\det C_K$ is bounded by $\text{rank}(B_i) + (n - 1)\text{rank}(B) \leq nr_{\text{max}}$, thus $d \leq nr_{\text{max}}$. According to [27, Corollary 1], choose randomly an element $(s_1, \ldots, s_k)$ from $V^k$, then under the condition $|V| \geq 2d$, the probability of $\det C_K \neq 0$, denoted by $\Pr$, satisfies

$$\Pr \geq \frac{|V| - d}{|V|^{k-1}} \geq \frac{1}{2}.$$  \hspace{1cm} (21)

Note that verifying whether $(A, B)$ in (19) for a given $(s_1, \ldots, s_k)$ is controllable can be done in polynomial time by testing the rank of the associated controllability matrix. By Definition 4, the result follows.

Proposition 8 quantifies the statement that controllability is a generic property by measuring the cardinality of the param-
eter space and the probability of controllability for a randomly chosen system. On the other hand, under the generally believed conjectures [3] that $RP = P$ and $P \neq NP$, it indicates that verifying the structural controllability of $(A, B)$ of (19) should not be NP-hard, and there should be some derandomized algorithm that can do this efficiently. However, the following theorem reveals the hardness of finding such a deterministic algorithm. In this theorem, the definitions of complexity classes NEXP and $P \setminus \text{poly}$, as well as Permanent, can be referred to, e.g., [3] and [26].

**Theorem 3**: If one can deterministically verify whether the system in (19) is structurally controllable in polynomial time (or even in subexponential time), then either (i) NEXP $\not\subseteq P \setminus \text{poly}$ or (ii) Permanent is not computable by polynomial-sized arithmetic circuits over $\mathbb{Z}$.

Theorem 3 means that finding a deterministic algorithm to verify structural controllability of (19) in polynomial time or even in subexponential time, is at least as hard as proving Statement (i) or (ii) in this theorem, which are two open problems in arithmetic circuits [26]. Although it is generally believed that NEXP $\not\subseteq P \setminus \text{poly}$ and Permanent requires super-polynomial sized circuits [3], it is also commonly agreed that we are far away from proving these lower bounds [26]. Nevertheless, for the case where $[A, B]$ has an arbitrary rank, some efficient random algorithms, or black-box polynomial identity testing algorithms [26], [27], could be adopted.

To prove Theorem 3, we leverage a special problem about arithmetic circuit complexity, called the symbolic determinant identity testing problem (SDIT) [12]. This problem can be stated as follows. For a square matrix $M$ with its entries being either a constant integer or a variable, where each variable could appear more than once, determine whether or not its determinant is identically zero. By reducing the general SDIT to a special instance of structural controllability verification problem, we give the proof as follows. Notice that Theorem 3 is not contradictory to the main results of this paper, as SDIT for rank-one case is deterministically solvable in polynomial time [27], while is more challenging and still open for the general high-rank case.

**Proof of Theorem 3**: Let $B$ be an $n \times n$ matrix, whose entries are either a constant integer or a variable. Let $A = I_n$. It is obvious that $(A, B)$ has the form of (19), and all the coefficient matrices can be obtained in polynomial time. By the PBH test, $(A, B)$ is structurally controllable, if and only if $\det B$ is not identically zero. This means that verifying the structural controllability of $(A, B)$, is at least as hard as the SDIT on $B$. The result then follows from [12, Th. 4.12] and [26, Th. 4.5].

**Appendix B**

**Proofs of Some Technical Results**

**Proof of Lemma 4**: This lemma can simply be derived by listing and comparing the vertex indices of the two distinct cycles. The details are omitted in the interest of space.

**Proof of Lemma 5**: For the ease of notation and without loss of generality, let us consider $J = \{1, \ldots, n_s\}$ and denote $S_J = \{s_1, \ldots, s_{n_s}\}$. Define a weighted multigraph associated with the TFM $G^J_{zz}(\lambda)P^J - I$ as $\mathcal{D}^J = (V^J, E^J, W^J)$, where the vertex set $V^J = \{1, \ldots, n_s\}$, the edge set $E^J = E^J_0 \cup E^J_1$ in which $E^J_0 = \{(j, i) : (G^J_{zz}(\lambda))_{ij} \neq 0\}$ and $E^J_1 = \{(i, i) : i = 1, \ldots, n_s\}$, and the weight set $W^J = \{w(e) : w(e) = \left(G^J_{zz}(\lambda)\right)_{ij}s_j \text{ for } e = (j, i) \in E^J_0, w(e) = -1 \text{ for } e \in E^J_1\}$. Moreover, for an edge $e = (i, j) \in E^J_0$, define $w(e) = \left(G^J_{zz}(\lambda)\right)_{ij}$. From this definition, some vertices may have multiple self-loops, which leads to the term “multigraph.” Let $S_{n_s}$ be the collection of all matchings of $\mathbb{D}^J$ with size $n_s$, and denote the $i$th matching by $C_i$. It is well known that each matching $C_i$ corresponds to a collection of vertex-disjoint cycles that span $\mathbb{D}^J$ [31]. From the definition of determinant [11], we have

$$\det(G^J_{zz}(\lambda)P^J - I) = \sum_{C_i \in S_{n_s}} \text{sgn}(C_i) \prod_{e \in C_i} w(e)$$

where $\text{sgn}(C_i) \in \{-1, 1\}$ is the sign associated with $C_i$.

Notice that a nonzero term $\prod_{e \in C_i} w(e)$ in (22) could either be a monomial of $\{s_1, \ldots, s_{n_s}\}$ depending on $\lambda$, a monomial of $\{s_1, \ldots, s_{n_s}\}$ independent of $\lambda$, or the constant $(-1)^{s_i}$. The necessity is then obvious, since if there does not exist a $\lambda$-cycle in $\mathbb{D}^J$ (therefore, neither in $\mathbb{D}^J$, there exists no term in $\det(G^J_{zz}(\lambda)P^J - I)$ that depends on the variable $\lambda$.

To show the sufficiency, notice that in the digraph $\mathbb{D}^J$, all outgoing edges from a common vertex $i \in V^J$ have weights with the same factor $s_i$ (excluding the self-loop with weight $-1$). This leads to the fact that for two distinct matchings $C_i$ and $C_j$, their associated terms $\prod_{e \in C_i} w(e)$ and $\prod_{e \in C_j} w(e)$ could cancel each other out. Suppose that there exists at least one $\lambda$-cycle in $\mathbb{L}^J$, and so does in $\mathbb{D}^J$ by definition. Denote the $\lambda$-cycle that has the shortest length among all these $\lambda$-cycles in $\mathbb{D}^J$ by $C_{\min} = \{i_1 \rightarrow \cdots \rightarrow i_{k_{\min}} \rightarrow i_1\}$, where $k_{\min}$ is the length of $C_{\min}$. Notice that $C_{\min}$ may not be unique. Denoted by $C_{\min}$, the matching in $\mathbb{D}^J$ constituted by $C_{\min}$ and the rest $n_s - k_{\min}$ self-loops with weights $-1$. We declare that the term associated with $C_{\min}$, $\prod_{j=1}^{k_{\min}} w(\{i_j, i_{j+1}\}) \prod_{j=1}^{k_{\min}} s_{i_j}$, where $k_{\min} + 1$ is defined to be 1 and the sign is ignored, cannot be vanished by other terms in $\det(G^J_{zz}(\lambda)P^J - I)$.

To show this, first observe that there does not exist a set of vertices $Z^J \subseteq \{1, \ldots, n_s\}$ \{i_1, \ldots, i_{k_{\min}}\}$, such that $Z^J \cup \{i_1, \ldots, i_{k_{\min}}\}$ forms a larger cycle containing $C_{\min}$ and corresponds to a monomial of the form $f(\lambda) \prod_{i=1}^{k_{\min}} s_{i_j}$, where $f(\lambda) \in F(\lambda)$. Next, in the subgraph of $\mathbb{D}^J$ induced by \{i_1, \ldots, i_{k_{\min}}\}$, denoted by $\mathbb{D}^J_{C_{\min}}$, suppose that there exists another $\lambda$-cycle $C' = \{i'_1 \rightarrow \cdots \rightarrow i'_{k_{\min}} \rightarrow i'_1\}$, which is distinct from $C_{\min}$. Then, according to Lemma 4, for every edge $e$ in $E(C_{\min}) \cup E(C')$, there exists a cycle in $\mathbb{L}^J_{C_{\min}}$ with length not exceeding $k_{\min} - 1$ that contains $e$. Consequently, a $\lambda$-cycle with length less than $k_{\min}$ emerges in $\mathbb{D}^J$, which is contrary to the shortest length assumption of $C_{\min}$. Therefore, among all the terms $\prod_{e \in C_i} w(e)$, which has the form $f(\lambda) \prod_{i=1}^{k_{\min}} s_{i_j}$, where $f(\lambda) \in F(\lambda)$, there is only one term that corresponds to a matching containing a $\lambda$-edge, whereas all the rest correspond to matchings that consist of only constant-edges (possibly including self-loops with weight $-1$). These terms have the form of $\alpha \prod_{j=1}^{k_{\min}} s_{i_j}$ where $\alpha \in \mathbb{R}$ is constant. Hence, the coefficient of monomial $\prod_{j=1}^{k_{\min}} s_{i_j}$ in $\det(G^J_{zz}(\lambda)P^J - I)$ has the form
The proof follows similar ideas to those of [20, Lemma 5.3]. Suppose that there exists a zero of \([G_{x_2}(\lambda)P - I G_{x_2}(\lambda)]\) depending on \(S\), denoted by \(\lambda^*\). Then, inspired by [20, Lemma 5.3], it suffices to see \(\lambda^*\) as a transcendental element over \(\mathbb{R}\), since \(\lambda^*\) depends on the algebraically independent elements \(s_1, \ldots, s_k\). Choosing the square submatrix \(G_{x_2}(\lambda)P - I\) of \([G_{x_2}(\lambda)P - I G_{x_2}(\lambda)]\), it holds that \(\det(G_{x_2}(\lambda^*)P - I) \equiv 0\) since \(\lambda^*\) is the zero, which means that \(\{\lambda^*\} \cup \{s_1, \ldots, s_k\}\) is algebraically dependent over \(\mathbb{R}\). According to the property of algebraic independence [20, Lemma 2.1], there exists an \(s_t \in \{s_1, \ldots, s_k\}\) such that \(\{\lambda^*\} \cup \{s_1, \ldots, s_t\}\) is algebraically independent over \(\mathbb{R}\) causing a contradiction.

Proof of Proposition 4: Define a digraph associated with \([PG_{x_2}(\lambda) \; PG_{v_2}(\lambda)]\) as \(\tilde{T}_\Sigma = (V \cup U, \tilde{E}_\Sigma \cup E_{\text{Ind}}, \tilde{E}_{\text{inv}})\), where \(U \triangleq \{u_1, \ldots, u_{M_1}\}\), \(V \triangleq \{v_1, \ldots, v_{M_2}\}\), \(E_{\text{Ind}} = \{(v_i, v_j) : [PG_{x_2}(\lambda)]_{ij} \neq 0 \text{ and } [PG_{v_2}(\lambda)]_{ij} \neq 0\}\). Note that all nonzero entries of \(P\) are independent, it turns out that \([PG_{x_2}(\lambda)]_{ij} \neq 0, \text{ if and only if there exists } l \in [M]\) such that \(P_{ji} \neq 0\) and \([PG_{v_2}(\lambda)]_{il} \neq 0\), where \(v_i = v, v = u\). Relabel vertices in \(T_\Sigma\) as \(\bigcup_{i=1}^N u_i \triangleq \{u_1, \ldots, u_{M_1}\}, \bigcup_{i=1}^N v_i \triangleq \{v_1, \ldots, v_{M_2}\}\) according to their correspondences in \(T_\Sigma\). Then, \(T_\Sigma\) and \(\tilde{T}_\Sigma\) are related in the following way: \((v_i, v_j) \in E_{\text{Ind}}\) (resp. \((u_i, u_j) \in E_{\text{Ind}}\)) in \(\tilde{T}_\Sigma\), if and only if there is a triple \((v_i, z_i, v_j)\) for some \(i^*\) such that \((u_i, z_i, v_j) \in E_{\text{Ind}}\) (resp., a triple \((u_i, z_i, v_j)\) with \((u_i, z_i, v_j) \in E_{\Sigma}\)) in \(T_\Sigma\). It can further be validated that there exists an input-unreachable \(\lambda\)-cycle in \(\tilde{T}_\Sigma\), if and only if there exists at least one input-unreachable \(\lambda\)-cycle in \(T_\Sigma\).

Define \(H(\lambda)\) and \(G(\lambda)\) as \(H(\lambda) \triangleq U * G_{\lambda}(\lambda)\), \(G(\lambda) \triangleq V G_{\lambda}(\lambda)\), respectively. Noting that \(P = UP_d \; V_d \oplus \text{diag}\{s_i\}_i=1^k\), it can be seen that \([PG_{x_2}(\lambda)]_{ij} \neq 0\), i.e., \([UP_d \; V G_{\lambda}(\lambda)]_{ij} \neq 0\) (resp. \([PG_{v_2}(\lambda)]_{ij} \neq 0\) (resp. \([PG_{v_2}(\lambda)]_{ij} \neq 0\)) depends on \(\lambda\), if and only if there exists an \(l \in [k]\), such that \(U_{il} \neq 0\) and \([PG_{v_2}(\lambda)]_{il} \neq 0\), where the latter is equivalent to \([G_{\lambda}(\lambda)]_{ij} \neq 0\) (resp. \([G_v(\lambda)]_{ij} \neq 0\)). Similar analysis is valid for \(G_{v_2}(\lambda)\), where it turns out that \([PG_{x_2}(\lambda) \; PG_{v_2}(\lambda)]\) has the same sparsity pattern as \([U \ast G(\lambda) \; U \ast H(\lambda)]\), where the operation \(\ast\) is defined as the same as matrix multiplication except that the involved additions and multiplications between two scalar elements are logic operations OR and AND between binaries, respectively (here binary refers to whether an entry is zero or nonzero). Furthermore, since each column of \(U\) has only one nonzero entry from its definition, no cancellation occurs between two addends in obtaining \(G(\lambda)U\). This means that \([V G_{x_2}(\lambda) \; U \; V G_{v_2}(\lambda)]\) has the same sparsity pattern as \([G(\lambda) \; U \ast H(\lambda)]\).
Denote the set of vertices of $L_\Sigma$ by $W \cup U$ with $W = \{w_1, \ldots, w_\ell\}$ and $U = \{u_1, \ldots, u_M\}$. Suppose that there is a $\lambda$-cycle in $L_\Sigma$, denoted by $C_\lambda = \{w_i \rightarrow w_{i+1} \rightarrow \cdots \rightarrow w_k \rightarrow w_i\}$. Assume that $(w_i, w_j)$ is a $\lambda$-edge without loss of generality. Moreover, suppose that there is a path from $w_i \in U$ to $w_j \in \{w_1, \ldots, w_\ell\}$, and denote such path by $\{w_0 \rightarrow w_1 \rightarrow \cdots \rightarrow w_{\ell}\}$. This means that the $(i_1, i_0)$th entry of $H(\lambda)$, and the $(i_2, i_1)$th, $(i_3, i_2)$th, (i.e., $(i_j, i_{j-1})$th, $(i_j, i_{j-2})$th, $(i_j, k)$th, $(k_j, k_1)$, ..., $(k_j, k_{j-1})$th entries of $G(\lambda) \cup U$ are nonzero. Note that $[G(\lambda) \cup U]_{i,j} \neq 0$, if and only if there exists an $1 \in [M_i]$ such that $[G(\lambda)]_{i,j} \neq 0$ for $i \in [M_i]$ and $U_{i,j} \neq 0$. Hence, there exists a sequence of integers $k_1, \ldots, k_{\ell}$ in $[M_i]$ (possibly with repeated values), such that $[G(\lambda)]_{i_{j+1}, i_{j}} \neq 0$ and $U_{k_{j}, k_{j+1}} \neq 0$ for $j = 1, \ldots, \ell - 1$, $[G(\lambda)]_{i_{j+1}, i_{j}} \neq 0$ and $U_{k_{j}, k_{j+1}} \neq 0$ for $j = 1, \ldots, \ell$, where $i_{\ell+1}$ is defined to be $i_0$, and meanwhile $[G(\lambda)]_{i_{j}, i_{j+1}}$ depends on $\lambda$. Observe that whenever $[G(\lambda)]_{i,j} \neq 0$ and $U_{i,j} \neq 0$, $[U \cup G(\lambda)]_{i,j} \neq 0$. The above indicates that the $(k_2, k_1)$th, $(k_3, k_2)$th, $(k_4, k_3)$th, ..., $(k_\ell, k_{\ell-1})$th entries of $U \cup G(\lambda)$ are nonzeros, where $\ell \in [1,\ldots,L]$. Then, there exists a sequence of edges $(u_0, v_{k_0}), (u_{k_0}, v_{k_1}), (u_{k_1}, v_{k_2}), \ldots$, $(u_{k_{\ell-1}}, v_{k_\ell})$ in $T_\Sigma$ with $(u_{k_j}, v_{k_{j+1}})$ being a $\lambda$-edge. A $\lambda$-cycle containing edge $(u_{k_j}, v_{k_{j+1}})$ can always be found from these edges, and this $\lambda$-cycle is input reachable.

Since every step of the aforementioned analysis is invertible, the direction that a (input reachable) $\lambda$-cycle in $L_\Sigma$ indicates the existence of a (input reachable) $\lambda$-cycle in $L_\Sigma$ follows a similar way. This further leads to Proposition 4.

Proof of Lemma 8: The necessity of i) can be directly derived from [35]. The necessity of ii) can be validated by contradiction: If $M_0 < \max_{1 \leq i \leq n}[M_i]$, $\rho(M(Q_1)) \cap M(Q_2) \leq M_0 < \max_{1 \leq i \leq n}[M_i]$, which never satisfies the condition of Proposition 5. The necessity of iii) is a direct derivation of Condition 1) of Corollary 1 and the definition of $T_\Sigma$.

To show the sufficiency, let $\Phi_{(\Sigma)}$ be the SCM with all of its entries being free parameters. Notice that if i) of Lemma 8 is satisfied, $[Y^{(i)}_1, Z^{(i)}_1]$ is of FRR whenever $m_{i,j} > 0$ by using Lemma 2 inversely, which means that $Q_{2,i}$ is of FRR. Suppose $M_i \geq M_{i+1}$ for all $i \in [n]$. Then, any $M_i$, columns of $Q_{1} \equiv \Phi_{\Sigma}^T I_{M_i}$ are linearly independent. As $Q_{2,i}$ is of FRR, there exists $M_i$, columns of $Q_{2,i}$ that are linearly independent. As a result, $\rho(M(Q_1) \cap M(Q_2)) = M_i$ for all $i \in [n]$. If there is some $i \in [n]$ with $G^{(i)}(\lambda) \neq 0$, it can be validated that $\Phi_{\Sigma}$ is sufficient to make all $\lambda$-edges in the associated n-ACG $T_\Sigma$ input reachable. Therefore, $\Phi_{\Sigma}$ satisfies both conditions of Corollary 1, which is a feasible solution to Problem 2.

Proofs of Proposition 6 and Lemma 9: We put both proofs together as they follow the same argument. The sketch is to find an instance of Problem 2 (resp. Problem 3) that is equivalent to the NP-hard minimal controllability problem in [22]. The latter problem is to determine the minimal controllability problem for a given state transition matrix [22]. Consider an NDS $\Sigma$ with two subsystems $\Sigma_1$ and $\Sigma_2$. The parameters are as follows: For a given $n \in \mathbb{N}$, let $A^{(1)}_{xx} \in \mathbb{R}^{n \times n}$ be a matrix with no repeated eigenvalues whose associated minimal controllability problem is NP-hard, whose construction can be referred to [22, Th. 1]. $A^{(1)}_{xx} = I_n, A^{(2)}_{xx} = 0_{n \times n}, A^{(2)}_{xx} = I_n, B^{(1)}_{xx} = 0_{n \times 1}, B^{(2)}_{xx} = 0_{n \times 1}, A^{(2)}_{xx} = n + 1, A^{(2)}_{xx} = 0_{n \times 1}, A^{(2)}_{xx} = 0_{n \times 1}, B^{(2)}_{xx} = 0_{n \times 1}, B^{(2)}_{xx} = 1$, and the SCM to be determined $\Phi \in \{0, 1\}^{(n+1) \times (n+1)}$. Then, by some algebraic manipulations, it can be validated that, Problem 2 and Problem 3 are both equivalent to the minimal controllability problem associated with $A^{(1)}_{xx}$.

Proof of Proposition 7: Define functions $g_i(\Sigma) \equiv \text{rank}(\{Y_{iJ}, Z_{iJ}\}) \forall i \in [m]$. As $g_i(\Sigma)$ is a rank function on the subset of column vectors of matrix $[Y_i, Z_i]$, $g_i(\Sigma)$ is submodular on $\Sigma \subseteq [\Sigma_m]$. Hence, $g(\Sigma) = \sum_{i=1}^{m} g_i(\Sigma)$ is submodular. Denote the optimal solution to Problem 4 by $J^*$. From [30], it follows that

$$|J_{\text{grd}}| \leq 1 + \log \frac{\sum_{i=1}^{m} M_i - \sum_{i=1}^{m} \text{rank}(Z_i)}{\sum_{i=1}^{m} M_i - g(J_{F=1})}. \quad (23)$$

Let $R_i \subseteq [M_i]$ be the set of indices of nonzero rows of $\Sigma$. Then, it is obvious that $|\Sigma| \geq R_i$. Let $B \equiv \{M_0 + 1, \ldots, M_m + M_e\}$. Let $\Phi$ satisfies $f(\Phi) = \sum_{i=1}^{m} M_i$, it means that $\rho(M(Q_1(\Phi))) \cap M(Q_2) = M_i$, $J_{i} \subseteq [1, M_m]$ is the set of indices of any set of subsystem links to the corresponding NDS. Denote the optimal solution to Problem 4 by $J^*$. From [30], it follows that

$$|J_{\text{grd}}| \leq 1 + \log \frac{\sum_{i=1}^{m} M_i - \sum_{i=1}^{m} \text{rank}(Z_i)}{\sum_{i=1}^{m} M_i - g(J_{F=1})}. \quad (23)$$

By the definition of $Z_{iJ}^{(j)}$, per $i \in [m], j \in [N]$, it holds $m_{i,j} - \text{rank}(Z_{iJ}^{(j)}) = m_{i,j} - \text{rank}(\lambda I - A_{xx} B_{xx}) \leq m_{i,j} - \text{rank}(\lambda I - A_{xx} B_{xx})$. Combined with (23), summation of the aforementioned relations over $[m]$ and $[N]$ leads to the inequalities of Proposition 7.

Proof of Theorem 2: Let $\Phi^*$ denotes the optimal solution to Problem 3, $\Phi^*$ the returned solution by Algorithm 1, and $\Phi_{\text{opt}}$ the optimal solution to Problem 2.

The proof for feasibility of $\Phi^*$ for Problem 3 follows a similar argument to the proof of [38, Th. 5.3], which is omitted due to space consideration. The feasibility of the two-stage algorithm then follows from Propositions 4 and 5.

We then prove these approximation bounds. From (17) in Algorithm 1, it is clear that every vertex of $\Gamma_{\text{col}}$ is colored with no more than $M_{\text{max}}$ colors. Combining Proposition 7, we have $|\Phi^*|_0/|\Phi^*|_0 \leq M_{\text{max}}(\log(M_{\text{col}}) + 1)$. Moreover, if every vertex of $\Gamma_{\text{col}}$ is colored by only one color, then obviously $|\Phi^*|_0/|\Phi^*|_0 \leq \log(M_{\text{col}}) + 1$. For the overall topology design procedure, by Corollary 1, it can be seen that subsystem links in an optimal solution $\Phi_{\text{opt}}$ to Problem 2 can be divided into two subsets. One subset functions as eliminating the FUMs, which is supposed to contain at least $p_{\text{col}}$ subsystem links, and the other functions as eliminating the input-unreachable $\lambda$-edges, which is supposed to contain at least $p_{\text{un}}$ subsystem links (these two subsets may overlap). It turns out that both $p_{\text{col}}$ and $p_{\text{un}}$ do not increase with the addition of any set of subsystem links to the corresponding NDS.
Hence, we have $p_{\text{sp}} + p_{\text{eu}} \leq 2|\Phi_{\text{opt}}| \leq 2|\Phi| \leq p_{\text{sp}}$. On the other hand, given a collection of disconnected subsystems $\Sigma_1, \ldots, \Sigma_M$, it holds that $p_{\text{sp}} \leq p_{\text{eu}}$, recalling that $p_{\text{sp}}$ is the number of input-unreachable source SCCs that contains an $\lambda$-edge in $T_{\Sigma_M}$. Hence, the two-stage algorithm returns $\Phi$ with sparsity $p_{\text{sp}} + |\Phi| \leq p_{\text{eu}} + M_{\text{max}} \log(M_{\text{det}} + 1) p_{\text{sp}} \leq M_{\text{max}} (1 + \log(M_{\text{det}})) (p_{\text{sp}} + p_{\text{eu}}) \leq 2M_{\text{max}} \log(M_{\text{det}}) + 1) |\Phi_{\text{opt}}||\Phi|$. Therefore, the topology design procedure overall has $O(M_{\text{max}} \log(M_{\text{det}}) + 1) \approx \log \log M_{\text{det}}$ approximation.

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