Integer Quantum Hall Transition and Random $SU(N)$ Rotation

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We reduce the problem of integer quantum Hall transition to a random rotation of an $N$-dimensional vector by an $su(N)$ algebra, where only $N$ specially selected generators of the algebra are nonzero. The group-theoretical structure revealed in this way allows us to obtain a new series of conservation laws for the equation describing the electron density evolution in the lowest Landau level. Not having the full system of wave operators, the magnetic length $\ell$ is much smaller than the size $L$ of the system and particles can be adequately localized. In a certain sense, $G(x,t)$ is the probability of transition of a quantum particle from the origin to point $x$ in the course of time $t$. This statement would be exactly true if we allowed the particle to travel over all the quantum states, i.e., not only those belonging to the lowest Landau level. Not having the full system of wave functions does not allow us to localize the particle at a distance smaller than the magnetic length $\ell = \hbar c/(eB)$. For example, the initial condition for the $G$ function have the form $G(x,0) = A \exp(-x^2/2\ell^2)$, with some normalization constant $A$. However, for strong enough magnetic fields, the magnetic length $\ell$ is much smaller than the size of the system and particles can be adequately localized.

The projected density operators obey the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \hat{\rho} = [\hat{H}, \hat{\rho}],$$

whose Hamiltonian is just the random potential $V(x)$ projected onto the lowest Landau level, $\hat{H} = \int d^2x \ V(x) \hat{\rho}(x)$. The correlation function $G$ satisfies an analogous equation, which due to remarkably simple commutation relations between the density operators $\hat{\rho}$, can be written in the Fourier space ($k$ is a two-dimensional vector, $k = (k_1, k_2)$) [3]:

$$i\hbar \frac{\partial}{\partial t} G(k,t) = \int d^2q \ 2i \sin \left( \frac{l^2}{2} k \times q \right) V(k - q) \times \exp \left[ -\frac{l^2}{2} (k^2 - k \cdot q) \right] G(q,t).$$

For a simple, “from the first principles,” derivation of this equation we refer the reader to [3]. All the information about the quantum Hall transition (QHT) is contained in this equation, which we use as a starting point in the present work. Unfortunately, it is not obvious how to solve this equation.

In this letter, we are going to demonstrate the hidden group-theoretical structure of Eq. (3). We show that it admits a series of conserved integrals of motion that have not been known before. We reduce the problem of QHT to a finite-dimensional problem of a random rotation by banded $su(N)$ matrices with the band width $\sim \sqrt{N}$. We also show that when the width of the band is much smaller than $\sqrt{N}$, such a model describes classical percolation.

Let us start with the classical limit $l \to 0$. In this case, Eq. (2) is simplified considerably,

$$\frac{\partial}{\partial t} G(x,t) = \epsilon_{ij} \partial_i V(x) \partial_j G(x,t).$$

A formal solution to (3) can be written at once [3]. It is $G(x,t) = \delta(x - x(t))$ where $\frac{dx_i}{dt} = \epsilon_{ij} \partial_j V(x)$, the equations which were thoroughly studied in [4]. From here it is easy to deduce that we are trying to describe a particle which percolates along the equipotential lines of a random potential. Obviously that does not capture the physics of quantum Hall transition.

The reason why a solution to (3) was so easy to write down is contained in its infinitely many integrals of motion. Indeed, it is not hard to check that any expression

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of the form
\[ I_{mn} = \int d^2x \ G^m V^n \]  
(4)
is conserved along the solutions of (3). The conservation of \( I_{10} \) and \( I_{11} \) is simply a consequence of the conservation of the probability and energy. However, higher order integrals have a less obvious meaning. Now, in a quite remarkable way, these integrals of motion do not get destroyed as one goes back to the original equations (\( \mathfrak{3} \)), at least for integer \( m \) and \( n \). They only get slightly modified.

To see that, let us put the particle on a torus. The wave functions of the lowest Landau level can be written explicitly in the Landau gauge,
\[ \psi_\alpha(x, y) = \sum_m \exp \left( 2\pi i (x + iy) (Nm + \alpha) \right) - \left( \frac{(Nm + \alpha)^2}{N} \right) e^{-\pi N x^2}. \]
(5)

Here \( 0 \leq x, y \leq 1, \) \( N \) denotes the number of flux quanta through the torus and \( \alpha \) goes from \( 0 \) to \( N - 1 \), labeling the \( N \) states on the torus in the lowest Landau level. These wave functions describe the electron localized along a narrow strip around the line \( x = \alpha/N \). Notice that the magnetic length \( l \) is now automatically chosen in the form
\[ l^2 = \frac{1}{2\pi N}. \]
(6)

It is now a matter of simple calculations to project the density operators \( \hat{\rho}(k_1, k_2) \) onto the lowest Landau level on the torus. In these notations, \( k_1 \) and \( k_2 \) are integer numbers. As a result, the density operators are now \( N \times N \) matrices in the basis of states (\( \mathfrak{2} \)) which can be written in the following form.

Consider a pair of unitary unimodular \( N \times N \) matrices:
\[ h = \begin{pmatrix} 0 & 1 & \ldots & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & \ldots & \ldots & \ldots & 0 \end{pmatrix}, \]
(7)
\[ f = \text{diag}(1, \epsilon, \ldots, \epsilon^{N-1}), \]
(8)
where \( h \) is a cyclic permutation matrix, and \( \epsilon = \exp(2\pi i/N) \). These matrices have the following properties: \( hf = \epsilon fh, \) \( h^N = f^N = 1 \). Now introduce the matrices
\[ L(k_1, k_2) = \epsilon^{k_1 k_2}/2 f^{k_1} h^{k_2}, \]
(9)
where \( (k_1, k_2) \neq (0, 0) \). The matrices \( L(k_1, k_2) \) are periodic in \( k_1 \) and \( k_2 \) with period \( N \) up to coefficient \( \pm 1 \). They can be chosen as a basis for the \( su(N) \) algebra.

The product and the commutator of two such operators have the following form,
\[ L_q L_p = \exp \left( \frac{\pi i N q \times p}{N} \right) L_{q+p}, \]
(10)
\[ [L_q, L_p] = 2i \sin \left( \frac{\pi i N q \times p}{N} \right) L_{q+p}. \]
(11)

One can easily write down the explicit expressions for the matrix elements of these matrices:
\[ (L(k_1, k_2))_{\alpha, \beta} = \epsilon^{k_1 k_2}/2 + k_1 (\alpha - 1) \delta_{\alpha, \beta} \mod N. \]
(12)
Then the density operator can be expressed in terms of these matrices as
\[ \hat{\rho}(k_1, k_2) = \exp \left( -\frac{k_1^2 + k_2^2}{2N \pi} \right) L(k_1, k_2). \]
(13)

To be specific, (\( \mathfrak{13} \)) gives the density operator in the Schrödinger representation, as opposed to (\( \mathfrak{1} \)) where the Heisenberg representation was assumed. From this point on, we will understand \( \hat{\rho} \) as a time-independent density operator.

It is not difficult to check that the density operator written in this form indeed satisfies the commutation relations discussed, for example, in (\( \mathfrak{2} \)) with the magnetic length chosen according to (\( \mathfrak{3} \)).

The projected Hamiltonian takes the form
\[ \hat{H} = \sum_{k_1, k_2} V(-k_1, -k_2) \rho(k_1, k_2). \]
(14)

Since the Fourier components of the potential \( V(x) \) are random, we see that the Hamiltonian is none other than a random \( su(N) \) matrix. This completes the projection to the lowest Landau level on the torus. At this point we can easily rederive (\( \mathfrak{2} \)) simply by writing \( \hat{\rho} \) as a matrix and commuting it with the Hamiltonian \( \hat{H} \) as prescribed in (\( \mathfrak{2} \)) using the explicit matrix definitions (\( \mathfrak{13} \)). Finally, let us introduce the matrix \( \hat{G} = \sum_{k_1, k_2} G(k, t) \hat{\rho}(k_1, k_2) \). Here as a consequence of choosing \( k_1 \) and \( k_2 \) on the torus as integer numbers, we denote \( k = 2\pi(k_1, k_2) \). As a consequence of (\( \mathfrak{2} \)), it satisfies the equation identical to (\( \mathfrak{1} \)),
\[ i\hbar \frac{\partial}{\partial t} \hat{G} = [\hat{H}, \hat{G}]. \]
(15)

Through all these manipulations we succeeded in reducing the original problem of a particle on an infinite plane in a random potential to a problem with a finite number of degrees of freedom. That allows us immediately to write down the generalizations of the integrals of motion (\( \mathfrak{3} \)). Indeed, since a trace of a commutator of finite matrices is equal to zero,
\[ \hat{I}_{nm} = \text{Tr} \left( \hat{G}^m \hat{H}^n \right) \]
(16)
is conserved. These integrals can be expressed in terms of $G(k, t)$ and $V(k)$ using explicit expressions for the matrices $\hat{\rho}$.

The first few of the integrals in (10) coincide with their “classical” counterparts in (4). For example, $I_{10}$ and $I_{11}$ are equal to $I_{10}$ and $I_{11}$ and are still the probability and energy conservation, respectively. Higher order integrals of motion become increasingly more complicated. For example,

$$I_{30} = \sum_{k, s, r} g(k) g(s) g(r - k - s)$$

$$\times \exp \left[ \frac{\pi i}{N} (k \times s + k \times r + s \times r) + \pi i N r_1 r_2 \right],$$  

(17)

where $k = 2\pi(k_1, k_2)$, $s = 2\pi(s_1, s_2)$, $r = 2\pi N(r_1, r_2)$, $g(k) = G(k) \exp[-\pi (k_1^2 + k_2^2)/(2N)]$, and the summation is going over integer $k_1$, $k_2$, $s_1$, $s_2$, $r_1$, and $r_2$.

Despite being rather complicated, this expression reduces to its classical counterpart, $I_{30} = \int d^2x G^3(x)$, in the limit $t \to 0$ ($N \to \infty$). It is of course possible to show, after some algebra, that it is indeed conserved under the time evolution (12). A simple mathematical reason lies behind this: in the limit $N \to \infty$, the operators $\hat{\rho}$ become the generators of the group of volume preserving diffeomorphisms on a torus (12). Such a group represents motion of an incompressible fluid, which is precisely the meaning of (4).

2. We are going to demonstrate now that all the features of QHT are preserved in this picture. Consider a particle placed in one of the states (4). That means, it is localized along the $x$ direction and is extended along the $y$ direction. Wait some time $t$ and measure the average square of the displacement of the particle along the $x$ direction,

$$\langle x^2(t) \rangle \propto \frac{1}{N^2} \sum_{\alpha, \beta = 1}^{N} (e^{\pi i \hat{H} t})_{\alpha \beta}^2 (\alpha - \beta)^2,$$

(18)

where $(.)_{\alpha \beta}$ denotes the matrix element of the matrix inside the brackets. We expect that, according to (4), at large enough times no matter how large the finite size of the system is reached, the deviation $\langle x^2(t) \rangle$ behaves as

$$\langle x^2 \rangle \sim t^{1-1/(2\nu)}.$$  

(19)

The critical exponent $\nu$ is to be determined, and is believed to be close to $7/3$.

The Hamiltonian (4) appears to be a linear combination of $su(N)$ matrices with random coefficients. However, if we were indeed a random $su(N)$ matrix with probability distribution invariant under $SU(N)$ rotations, it would lead to a particle instantaneously hopping all over the torus. In fact, such a Hamiltonian would be unphysical. As a consequence of the strong exponential suppression of high Fourier modes in the density operators (13), the $SU(N)$ rotation invariant Hamiltonian corresponds to the random potential $V(x)$ varying considerably at distances much smaller than the magnetic length. In fact, in the continuum limit it would lead to the potential having Fourier harmonics growing exponentially at large $k$, as follows from the analysis of (13).

A more physical setting involves the potential whose Fourier harmonics at least do not grow at large $k$. While it is possible to choose a particular random $V(x)$ satisfying this property and compute the Hamiltonian (4), it is not necessary to do so. Instead, we can just point out that the exponential in (13) strongly suppresses high Fourier harmonics of the random potential, such that $k_1, k_2 > n = \sqrt{2N/\pi}$. Therefore, we can simply choose the Hamiltonian to be

$$\hat{H} = \sum_{k_1, k_2 = 0}^{n} v_{(k_1, k_2)} L_{(k_1, k_2)},$$

(20)

where $v_{(k_1, k_2)}$ are random independent variables with equal mean square values $\langle v^2 \rangle$. The crucial part of (20) is that the Hamiltonian is a linear combination of only $N$ generators of the $su(N)$ algebra, out of total $N^2$ generators, with random coefficients. From the explicit form (12) of matrices $L_{(k_1, k_2)}$ we see that the random matrix $\hat{H}$ has nonzero elements only on the diagonal strip of the width $\sim N^{1/2}$. Within the strip, only the matrix elements no farther than $\sim N^{1/2}$ from each other along a given diagonal are correlated with each other. Such banded random matrix theory, which reproduces the physics of the QHT, was considered in a similar context in (6), although the Hamiltonian (20) was not introduced there.

This framework provides a very convenient setting for numerical calculation of $\nu$. Below we present the numerical simulations for matrix $1000 \times 1000$. The Hamiltonian has been chosen in the form (20). The random coefficients $v_{(k_1, k_2)}$ have thus been generated in the square $|k_1|, |k_2| < n = \sqrt{2N/\pi}$, and the real and imaginary parts of each Fourier mode $v_{(k_1, k_2)} = v_{(k_1, k_2)}^* \sim \sqrt{2N/\pi}$ were chosen randomly, independently, and uniformly from the interval $[-0.5, 0.5]$. Instead of calculating the matrix exponent in (13) directly, we have simulated the equation

$$\dot{Z}_\alpha = i \hat{H}_{\alpha \beta} Z_\beta,$$

(21)

where $Z_\alpha$ is a wave function in the representation of the states on the torus. It is easy to check that in this representation,

$$g(k_1, k_2) = Z_{\alpha}^* \left[ L_{(k_1, k_2)} \right]_{\alpha \beta} Z_{\beta},$$

(22)

where $g(k)$ is introduced in (17).

After the random potential $\hat{H}$ had been generated, and the initial distribution had been chosen in the form

$$Z_{\alpha} = \begin{cases} 1, & \text{if } \alpha = \alpha_0, \\ 0, & \text{otherwise}, \end{cases}$$

(23)
the dispersion \( \langle x^2(t) \rangle = \sum_{\alpha} (\alpha - \alpha_0)^2 |Z_\alpha|^2 \) was calculated as a function of time. The same calculation was performed for all the initial positions of the particle, \( \alpha_0 = 1,\ldots,N \), and the average was taken over all such realizations. The result is shown in Fig. (1). As we see, we closely reproduce the universally accepted value of \( \nu \) without much difficulty.

3. As was already discussed, a naive \( l \to 0 \) limit leads to the percolation picture of QHT. However, \( l \to 0 \) is the same as \( N \to \infty \). It should indeed be possible to reproduce the percolation behavior if the scale of the random potential is much larger than the magnetic length \( l \), but much smaller than the size of the system. In other words, we need to keep only such modes in (14) for which \( 1 \ll n \ll N^{1/2} \). The percolating behavior holds until the width of the diagonal strip of the matrix \( \exp(iHt) \) becomes equal to \( N^{1/2} \). After that, the regime changes to QHT. The square of this width is given by (13), and therefore grows as \( t^{1-1/(2\nu^*)} \), with the percolation \( \nu^* \) equal to \( 4/3 \). The crossover time (mixing time) is thus equal, in physical units, to \( V_0 t_m / \hbar \sim (N/n^2)^{2\nu^*}/(2\nu^*-1) \sim \left( l_0/\ell \right)^{2\nu^*}/(2\nu^*-1) \), where \( l_0 \) is the scale of the random potential and \( V_0 \) is its typical amplitude. The percolation regime is valid for \( t < t_m \).

Our simulations of Eq. (21) show that for \( n \leq (2N/\pi)^{1/2} \), the behavior of \( x^2 \) coincides with (13) at \( t \to \infty \), but follows some intermediate asymptotics before that. We expect these asymptotics to be identical to those describing steady percolating flow of an incompressible two-dimensional fluid, i.e., \( \nu^* = 4/3 \). This seems to agree with our numerics, but more extensive simulations are required [8]. In the other limit, \( n \gg \sqrt{2N/\pi} \), the behavior becomes diffusive, \( \langle x^2 \rangle \sim t \).

In real experiments, the random potential length is often much larger than the magnetic length \( l \). It would be interesting to devise an experiment which would probe the intermediate percolation asymptotics, perhaps by looking at finite frequency conductivity.

In conclusion, we have presented a model that reduces the problem of quantum Hall transition to a finite-dimensional problem of random rotation by \( su(N) \) matrices. As a consequence, the equation (2) describing quantum Hall transition, admits a series of integrals of motion. The random rotation matrix \( H \) is quite arbitrary, except for the dependence on the parameter \( N \) through the exponential cutoff in (13). This rather crucial dependence amounts to vanishing of all the matrix elements but those belonging to the diagonal strip of the width \( n \sim N^{1/2} \). Within this framework, the quantum Hall transition exponent \( \nu \) can easily be evaluated numerically. Attempts to change \( n \) independently of \( N \) lead to other models, e.g. those describing diffusion and percolation in a steady incompressible 2D flow. That leads to a prediction that in very smooth potentials quantum Hall transition should exhibit some properties of percolation.

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