NONCOMMUTATIVE NOVIKOV ALGEBRAS

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Abstract. The class of Novikov algebras is a popular object of study among classical nonassociative algebras. The generic example of a Novikov algebra may be obtained from a differential associative and commutative algebra. We consider a more general class of linear algebras which may be obtained in the same way from not necessarily commutative associative algebras with a derivation.

Keywords: Novikov algebra, derivation, associative algebra

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1. Introduction

The variety of Novikov algebras is one of the most important classes of non-associative algebras. The definition of a Novikov algebra goes back to the paper [1] devoted to the study of Poisson brackets of hydrodynamic type, though it appeared earlier in [11] as a tool for constructing Hamiltonian operators in formal variational calculus. The term “Novikov algebra” was proposed in [18].

Let us briefly show how the axioms of Novikov algebras appear in [1]. We state the corresponding construction in a “coordinate-free” form. Suppose \( V \) is a non-associative algebra over a field \( \mathbb{k} \) with a bilinear product \( \circ \), and \( A \) is an associative and commutative algebra with a derivation \( d \). Consider the space \( A \otimes V \) equipped with a bilinear operation \([·, ·]\) given by

\[
[a \otimes v, b \otimes w] = d(a)b \otimes (v \circ w) - d(b)a \otimes (w \circ v),
\]

where \( a, b \in A, v, w \in V \). Then \([·, ·]\) is a Lie bracket if the algebra \((V, \circ)\) meets the following relations for all \( u, v, w \in V \):

\[
(u \circ v) \circ w = (u \circ w) \circ v,
\]

\[
(u, v, w)_{\circ} = (v, u, w)_{\circ},
\]

where \((x, y, z)_{\circ} = (x \circ y) \circ z - x \circ (y \circ z)\). These two identities define the variety of Novikov algebras.

In terms of operads [12] this construction may be interpreted as follows. Suppose \( \mathcal{N} \) is the binary quadratic operad generated by \( \mathcal{N}(2) \cong \mathbb{k}S_2 \) (one non-symmetric binary operation) relative to the defining relations representing all identities of degree 3 that hold for the operation \( a \ast b = d(a)b \) on all (associative) commutative algebras with a derivation. Then the operad Nov

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of Novikov algebras is isomorphic to $\mathcal{N}^a$, the Koszul dual to $\mathcal{N}$. This is a well-known fact (see, e.g., [10]) that $\text{Nov}' \simeq \text{Nov}^{op}$, the anti-isomorphic operad. Thus every commutative algebra $A$ with a derivation $d$ is indeed a Novikov algebra relative to the operation

$$(1) \quad a \circ b = ad(b),$$

$a, b \in A$.

The structure theory of Novikov algebras started with [25], where it was shown that a finite-dimensional simple Novikov algebra over an algebraically closed field of zero characteristic is 1-dimensional. Further development of the structure and representation theory of Novikov algebras was obtained in [19, 20, 23, 24].

A significant progress in the combinatorial study of Novikov algebras was achieved in [8], where a monomial basis of the free Novikov algebra was found. It turned out that the free Novikov algebra $\text{Nov}(X)$ generated by a set $X$ embeds into the free commutative differential algebra $\text{ComDer}(X; d)$ relative to the operation (1).

In [4], the Gröbner–Shirshov bases theory for Novikov algebras was developed. The constructions and proofs of [4] are essentially based on the results of [8], they lead to a more general result (a “Poincaré–Birkhoff–Witt Theorem”): every Novikov algebra may be embedded into an appropriate commutative differential algebra relative to the operation (1).

Recent advances in the combinatorial theory of Novikov algebras include the study of algebraic dependence [7], nilpotence and solvability [22, 26].

In this paper, we consider non-commutative analogues of Novikov algebras whose structure theory is richer even over an algebraically closed field of zero characteristic. These systems originated in [16], they were examples of derived algebras in [13]. Namely, assume $A$ is an associative (but not necessarily commutative) algebra with a derivation $d$. Introduce two new operations $\prec$ and $\succ$ on the space $A$ as follows:

$$a \prec b = ad(b), \quad a \succ b = d(a)b,$$

for $a, b \in A$. In [16], two identities of degree 3 that hold for such operations $\prec, \succ$ were found. It was shown in [13] that there are no more independent identities.

We denote the corresponding variety of algebras with two operations (as well as the binary quadratic operad governing this variety) by $DA$s, the class of derived associative algebras which is also natural to call noncommutative Novikov algebras.

We will prove that every $DA$s-algebra embeds into an appropriate associative differential algebra, which is a non-commutative analogue of the “PBW Theorem” of [4], see also [14]. In this paper we present two proofs of the
latter statement. The first one is based on the general theory of derived algebras from [13] which substantially depends on the fundamental results of [8] and [4]. The second one is completely independent and based on the combinatorial Diamond Lemma for associative algebras. One cannot use the standard Gröbner–Shirshov bases method (see, e.g., [2, 3]) for such a proof since the compositions obtained highly depend on the multiplication table in a particular $D_{As}$-algebra. (So the PBW Theorem in its “strong” sense [17] does not hold here, as well as for Novikov algebras.) However, we develop a “weight-restricted” Gröbner–Shirshov bases approach based on the consideration on an appropriate rewriting system (see, e.g., [15]). In this way, we may get the desired embedding without calculating the complete Gröbner–Shirshov basis of the universal associative differential envelope of a $D_{As}$-algebra.

2. Derived varieties

Throughout the paper, we will use the following notations. An algebra is a linear space $A$ equipped with a family $I$ of bilinear operations (“multiplications”) $(x, y) \mapsto x \cdot_i y$, $i \in I$. A derivation of such an algebra is a linear operator $d : A \to A$ such that $d(x \cdot_i y) = d(x) \cdot_i y + x \cdot_i d(y)$ for all $a, y \in A$, $i \in I$.

A variety is a class of algebras with a given family of operations which is closed with respect to subalgebras, direct products, and homomorphic images. By the classical Birkhoff Theorem, a variety is defined by a family of identities. If Var is a variety of algebras and $X$ is a nonempty set then $\text{Var}(X)$ stands for the free $\text{Var}$-algebra generated by $X$.

Denote $\text{VarDer}$ the variety of $\text{Var}$-algebras equipped with a derivation. Then $\text{VarDer}(X; d)$ is the free algebra in $\text{VarDer}$ generated by $X$ (here $d$ denotes the derivation). It is clear that $\text{VarDer}(X; d)$ is isomorphic as a $\text{Var}$-algebra to $\text{Var}(X^{(\omega)})$, where

$$X^{(\omega)} = \bigcup_{n \geq 0} X^{(n)}, \quad X^{(n)} = d^n(X).$$

Suppose Var is a variety of algebras defined by multi-linear identities. So are the classical examples: the variety As of associative algebras, the variety Com of associative and commutative algebras. The same symbol Var will be used for the (symmetric) operad governing the variety Var (see, e.g., [12]).

Given $A \in \text{VarDer}$, a $\text{Var}$-algebra with a derivation $d$, define new operations

$$x \prec_i y = x \cdot_i d(y), \quad x \succ_i y = d(x) \cdot_i y, \quad x, y \in A,$n for all $i \in I$. The same linear space $A$ equipped with the family of operations $\prec_i, \succ_i, i \in I$, is denoted $A^{(d)}$. Let $D\text{Var}$ stand for the variety generated by all systems $A^{(d)}$ obtained in this way from all $A \in \text{VarDer}$. The defining
identities of $D\text{Var}$ are exactly those that hold for the operations $\prec_i, \succ_i$ on all differential Var-algebras.

For example, if $\text{Var} = \text{Com}$ then we obviously have $x \succ y = y \prec x$. If we denote $x \prec y$ by $xy$ then the following identities hold on $D\text{Com}$:

\begin{align*}
(2) & \quad (xy)z - x(yz) = (yx)z - y(xz), \\
(3) & \quad (xy)z = (xz)y,
\end{align*}

These are exactly the defining relations of Novikov algebras, and it follows from [8] that there are no more independent identities that hold in $D\text{Com}$. Thus, $D\text{Com} = \text{Nov}$.

In general, it was shown in [13] that the operad governing $D\text{Var}$ coincides with the Manin white product of operads (see [12]) $\text{Var}$ and $\text{Nov}$. If $\text{Var}$ is quadratic (i.e., there is a basis of defining identities of degree $\leq 3$) then finding the identities of $D\text{Var}$ (i.e., the defining relations of the corresponding operad) is a standard linear algebra problem. In particular (see [13]), if $\text{Var} = \text{As}$ then $D\text{As}$ is the class of linear spaces equipped with two bilinear operations $\prec$ and $\succ$ such that

\begin{align*}
(4) & \quad x \succ (y \prec z) = (x \succ y) \prec z, \\
(5) & \quad (x \prec y) \succ z - x \succ (y \succ z) = x \prec (y \succ z) - (x \prec y) \prec z.
\end{align*}

The identities (4), (5) first appeared in [16] where it was pointed out that they hold on associative algebras with a derivation. It follows from [13] that there are no more independent multilinear identities that hold on $D\text{As}$.

It is straightforward to derive the following identity as a corollary of (4) and (5):

\begin{align*}
(6) & \quad (x \prec y, z, b)\prec = (x, y \succ z, b)\prec,
\end{align*}

where $(a, b, c)\prec = (a \prec b) \prec c - a \prec (b \prec c)$. In a similar way,

\begin{align*}
(7) & \quad (x, y \prec z, b)\succ = (x, y, z \succ b)\succ
\end{align*}

holds in every $D\text{As}$-algebra.

**Example 1.** Let $A = M_n(\mathbb{C})$, $n \geq 3$, and let $d$ be the inner derivation $d(x) = [a, x]$, where $a = \text{diag} (a_1, \ldots, a_n)$. If $a_1, \ldots, a_n \in \mathbb{C}$ are pairwise different then $A^{(d)}$ is a simple $D\text{As}$-algebra.

Indeed, all matrix unities $e_{ij}$ for $i \neq j$ belong to the image of $d$. An ideal of $A^{(d)}$ is closed with respect to multiplication by $d(x)$, $x \in A$. If an ideal $J$ of $A^{(d)}$ contains a nonzero matrix $y$ with $y_{ij} \neq 0$ then $y_{ij}e_{il} = e_{il}y_{jl} \in J$ for every $l \neq i, j$. Hence, the diagonal matrix unities along with the identity matrix belong to $J$, and thus $J = M_n(\mathbb{C})$.

**Lemma 1.** For every set $X$, the free $D\text{As}$-algebra generated by $X$ is a subalgebra of $\text{AsDer}(X; d)^{(d)}$. 
Proof. Without loss of generality we may suppose $X$ is an infinite set. The identity map $x \mapsto x$, $x \in X$, extends to a homomorphism $\tau : \mathrm{DAs}(X) \to \mathrm{AsDer}(X; d)^{(d)}$.

If $\tau(f) = 0$ for a (homogeneous) $f \in \mathrm{DAs}(X)$ then, obviously, the relation $f = 0$ holds in $A^{(d)}$ identically for every $A \in \mathrm{AsDer}$. If $f$ is a multilinear element of $\mathrm{DAs}(X)$ then $f = 0$ by [13], since it follows from [11], (5). Otherwise, consider the complete linearization $Lf = Lf(x_1, \ldots, x_n)$ of $f$. Split $Lf$ into summands with a fixed order of elements:

$$Lf = \sum_{\sigma \in S_n} f_\sigma(x_{\sigma(1)}, \ldots, x_{\sigma(n)}),$$

where $f_\sigma(y_1, \ldots, y_n)$ is a linear combination of terms obtained from the word $y_1 \ldots y_n$ by some bracketing with operations $\prec, \succ$. For example, if $f = x \prec x$ then $Lf = x_1 \prec x_2 + x_2 \prec x_1$, $f_{\mathrm{id}} = x_1 \prec x_2$, $f_{(12)} = x_2 \prec x_1$. Since $\tau$ does not change the order of variables $x_1, \ldots, x_n$, each of the multilinear summands $f_\sigma$ belongs to the kernel of $\tau$. Hence, $f_\sigma = 0$ in $\mathrm{DAs}(X)$ for every $\sigma \in S_n$. As $f$ may be obtained from $f_{\mathrm{id}}$ by substitution of variables (converse to what was done for linearization), we have $f = 0$ in $\mathrm{DAs}(X)$. Therefore, $\tau$ is injective.

In this paper we will prove that not only free, but every $\mathrm{DAs}$-algebra actually embeds into an appropriate $\mathrm{AsDer}$-algebra.

Remark 1. The Koszul dual operad $\mathrm{DAs}^!$ has the following nilpotence property. It is generated by two binary operations $\vdash$ and $\lhd$ modulo the relations

$$(x_1 \vdash x_2) \vdash x_3 = 0, \quad (x_1 \vdash (x_2 \vdash x_3)) = 0,$$

$$(x_1 \vdash x_2) \vdash x_3 = (((x_1 \vdash x_2) \vdash x_3) = (x_1 \vdash (x_2 \vdash x_3)) = (x_1 \vdash (x_2 \vdash x_3)),$$

$$(x_1 \vdash (x_2 \vdash x_3)) = ((x_1 \vdash x_2) \vdash x_3).$$

It is not hard to find the Gröbner basis of the corresponding nonsymmetric operad (as in [6], see also [5]):

$$(x \vdash (x \lhd x)), \quad (x \lhd (x \vdash x)), \quad (x \vdash (x \lhd (x \vdash x))), \quad (x \lhd (x \vdash (x \lhd x)))$$

$$(x \rhd (x \vdash x)) - (x \lhd (x \vdash x)), \quad (x \lhd (x \rhd (x \vdash x))) - (x \lhd x(x \rhd x)),$$

$$(x \vdash (x \rhd x)) - (x \lhd x(x \rhd x)), \quad (x \rhd (x \vdash x)) - (x \lhd (x \rhd x)).$$

Hence, $\mathrm{DAs}^!(n) = 0$ for $n \geq 4$. In particular, one may derive that the operad $\mathrm{DAs}$ is not Koszul in the same way as it was done for Nov in [10].

3. Free noncommutative Novikov algebra

In this section, we describe the image of the free $\mathrm{DAs}$-algebra generated by a set $X$ in the free associative differential algebra and introduce the notions
which will be used in the sequel. Suppose the set of generators $X$ is equipped with a linear order.

By Lemma $\footnote{DAs\langle X \rangle \subset As\text{Der}\langle X; d \rangle \simeq As\langle X \cup X' \cup X'' \cup \ldots \rangle}$, where $x^{(n)} = d^n(x)$ for $x \in X$. A monomial $u \in As\text{Der}\langle X; d \rangle$ may be uniquely written in the following form:

$$u = x_1^{(m_1)} \cdots x_k^{(m_k)}, \quad x_i \in X, \quad n_i \in \mathbb{Z}_+.$$  

Let the potential of $u$ be the following polynomial in a formal variable $t$:

$$pt(u) = \sum_{j \geq 0} N_j t^j \in \mathbb{Z}[t],$$

where $N_j$ is the number of indices $i \in \{1, \ldots, k\}$ such that $n_i = j$. For example,

$$pt(x'x''xx'x''x'(3)x''x') = t^3 + 2t^2 + 3t + 1.$$  

Introduce the following order on monomials in $As\text{Der}\langle X; d \rangle$. For two such monomials $u, v$, let $u \ll v$ if either the leading coefficient of $pt(v) - pt(u)$ is positive or $pt(u) = pt(v)$ but $u < v$ lexicographically in $(\mathbb{Z}_+ \times X)^*$, assuming that a monomial of the form (8) is identified with the word

$$(n_1, x_1) \cdots (n_k, x_k) \in (\mathbb{Z}_+ \times X)^*,$$

and the pairs in $\mathbb{Z}_+ \times X$ are compared lexicographically.

For a polynomial $f \in As\text{Der}\langle X; d \rangle$, let $\bar{f}$ stand for the principal (leading) monomial of $f$ with respect to the order $\ll$.

Let us note the following properties of the potential.

**Proposition 1.** (i) The length $|u|$ of $u$ is equal to $pt(u)|_{t=1}$; (ii) For every monomials $u, v \in As\text{Der}\langle X; d \rangle$ we have $pt(uv) = pt(u) + pt(v)$; (iii) The order $\ll$ is a monomial one, i.e., $u_1 \ll u_2$ implies $u_1v \ll u_2v$ and $vu_1 \ll vu_2$ for every monomial $v$.

**Proof.** Statements (i) and (ii) are obvious. In particular, if $pt(u) = pt(v)$ then $u$ and $v$ have equal length. To get (iii), note (ii) and the well-known monomiality of the lexicographic order. \hfill $\square$

Define the weight of a differential monomial $u \in As\text{Der}\langle X; d \rangle$ as follows:

$$\text{wt}(u) = \left. \frac{d}{dt} \frac{pt(u)}{t} \right|_{t=1} = -N_0 + N_2 + 2N_3 + \ldots,$$

for $pt(u) = N_0 + tN_1 + t^2N_2 + \ldots$. Obviously, $pt(uv) = pt(u) + pt(v)$.

**Proposition 2.** If $\text{wt}(u) = -m < 0$ then $u = u_1 \ldots u_m$ with $\text{wt}(u_i) = -1$.

**Proof.** Let $u$ be of the form (8), $\text{wt}(u) = -m = n_1 + \cdots + n_k - k$. For $m = 1$, there is nothing to prove. If $m > 1$ then there exists a prefix $u_1$ of $u$ such that $\text{wt}(u_1) = -1$. Indeed, if $n_1 = 0$ then $u_1$ is of length 1. If $n_1 > 0$ then
consider all prefixes of \( u \) consecutively. Adding a single letter to a word may decrease its weight maximum by 1. Since the total weight of \( u \) is negative and the first letter is of nonnegative weight, there should exist a prefix of weight \(-1\). Obvious induction completes the proof. \( \square \)

Note that if \( u = u_1 \ldots u_m \), \( \text{wt}(u_i) = -1 \), and \( u \) contains a letter of nonnegative weight (i.e., a derivative of \( x \)) then this letter belongs to a subword \( u_i \) of length \( > 1 \).

**Proposition 3.** If \( u \) is of the form \( \langle X \rangle \), \( \text{wt}(u) = -1 \) and there exist \( i < j \) such that \( n_i, n_j > 0 \), then \( u \) contains a proper subword \( v \) of length \( |v| > 1 \) and \( \text{wt}(v) = -1 \).

**Proof.** Let us split \( u \) into two subwords \( u = w_1w_2 \) in such a way that \( x_i^{(n_i)} \) appears in \( w_1 \) and \( x_j^{(n_j)} \) in \( w_2 \). Then either of these words \( w_1, w_2 \) should have a negative weight since \( \text{wt}(w_1) + \text{wt}(w_2) = \text{wt}(u) = -1 \). If \( \text{wt}(w_1) < 0 \) then by Proposition 2, \( w_1 \) splits into subwords of weight \(-1\), and the subword \( v \) with \( x_i^{(n_i)} \) should be of length \( > 1 \). The case \( \text{wt}(w_2) < 0 \) is similar. \( \square \)

Recall the homomorphism \( \tau : DAs(X) \rightarrow \text{AsDer}(X; d)^{(d)} \) from the proof of Lemma 1. \( \tau(x) = x, \tau(u < v) = \tau(u)\tau(d(\tau(v))), \tau(u > v) = d(\tau(u))\tau(v). \)

This is an injective map.

**Theorem 1.** The image of \( \tau \) coincides with the linear span of differential monomials \( u \) with \( \text{wt}(u) = -1 \).

**Proof.** It is obvious from the definition of \( \tau \) that its image consists of wt-homogeneous differential polynomials of weight \(-1\). It is enough to note that for every monomial \( u \in \text{AsDer}(X; d) \) the polynomial \( d(u) \) is weight-homogeneous and \( \text{wt}(d(u)) = \text{wt}(u) + 1 \).

Conversely, let \( u \) be a monomial of the form \( \langle X \rangle \) with \( \text{wt}(u) = -1 \). Let us show by induction on \( |u| = k \) that \( u \in \text{Im} \tau \).

For \( |u| = 1 \) the claim is obvious. Assume \( |u| = k > 1 \) and the statement is true for all monomials of length \( < k \).

Suppose there are more than one \( n_i > 0 \) in \( u \). Then by Proposition 3 there exists a proper subword \( v \) in \( u \) such that \( |v| > 1, \text{wt}(v) = -1 \). By induction, \( v = \tau(g) \) for some \( g \in DAs(X) \). Consider \( u = u_1vu_2 \) as a differential word in the alphabet \( X \cup \{v\} \): in this extended alphabet, the length of \( u \) is smaller than \( k \). Thus by the induction assumption there exists \( f \in DAs(X \cup \{v\}) \) such that \( \tau(f) = u \in \text{AsDer}(X \cup \{v\}; d) \). It remains to replace in \( f \) the variable \( v \) with \( g \) to get a desired pre-image of \( u \) in \( DAs(X) \).

Suppose there is only one \( n_i > 0 \), i.e.,

\[
u = x_1 \ldots x_i x_i^{(k-1)} x_{i+1} \ldots x_k.\]
Consider
\[ w = x_1 \prec (x_2 \prec \cdots \prec (x_{i-1} \prec (\cdots ((x_i \succ x_{i+1}) \succ x_{i+2}) \succ \cdots \succ x_k)) \cdots) . \]
Then all monomials in \( u - \tau(w) \) are of weight \(-1\) and have potentials of degree \(< k - 1\), thus contain more than one derivative. Therefore, \( u - \tau(w) \) belongs to the image of \( \tau \) and so is \( u \). □

**Corollary 1.** The elements \( \tau^{-1}(u), \text{wt}(u) = -1 \), form a linear basis of \( \text{DAs}(X) \).

At the level of operads (i.e., for multilinear components of \( \text{DAs}(X) \)), we obtain the following

**Corollary 2.** For every \( n \geq 1 \) we have \( \dim \text{DAs}(n) = n! \dim \text{Nov}(n) \), so the Manin white product of operads \( \text{As} \circ \text{Nov} \) coincides with the Hadamard product \( \text{As} \otimes \text{Nov} \).

An intriguing open problem is to describe all those binary quadratic operads \( \text{Var} \) for which \( \text{Var} \circ \text{Nov} \) coincides with \( \text{Var} \otimes \text{Nov} \).

### 4. Embedding into differential algebras

Let \( D \) be a \( \text{DAs} \)-algebra, i.e., a linear space equipped with two bilinear operations \( \prec, \succ \) satisfying (1), (5). In this section we will show that there exists an associative differential algebra \( A \in \text{AsDer} \) with a derivation \( d \) such that \( D \subseteq A(d) \). This is a noncommutative analogue of the “PBW Theorem” [1] for Novikov algebras.

We present here two proofs of this statement. The first one is based on Corollary [1] and thus depends on the fundamental result of [8] on the free Novikov algebra. The second proof is completely independent of the last result.

**Proposition 4.** Let \( X \) be a set and let \( J \) be an ideal in \( \text{DAs}(X) \). The latter is embedded into \( \text{AsDer}(X; d) \). Denote by \( I \) the (differential) ideal in \( \text{AsDer}(X; d) \) generated by \( J \). Then \( I \cap \text{DAs}(X) = J \).

**Proof.** It is enough to show \( I \cap \text{DAs}(X) \subseteq J \) since the inverse embedding is obvious. Suppose \( f \in I \) is weight-homogeneous: all monomials in \( f \) are of weight \(-1\). This is a necessary and sufficient condition for \( f \in \text{DAs}(X) \). As \( f \in I \), it can be presented in the form
\[ f = \sum_{i=1}^{k} F_i(g_i, x_1, \ldots, x_n), \quad g_i \in J, \]
where each \( F_i(t_i, x_1, \ldots, x_n) \) is an associative differential polynomial in the variables \( X \cup \{ t_i \} \). Hence there exists \( F(t_1, \ldots, t_k, x_1, \ldots, x_n) \in \text{As}(\hat{X}(\omega)) \),
\[ \hat{X} = X \cup \{ t_1, \ldots, t_k \}, \text{ such that } \]
\[ f = F(t_1, \ldots, t_k, x_1, \ldots, x_n) |_{t_i = g_i, i = 1, \ldots, k}. \]

Since \( \text{wt}(f) = -1 \) and \( \text{wt}(g_i) = -1 \) for all \( i = 1, \ldots, k \), the polynomial \( F \) is also weight-homogeneous, \( \text{wt}(F) = -1 \). By Theorem 1, \( F \in \text{DAs}(\hat{X}) \), so \( f = F|_{t_i = g_i} \in J \).

**Theorem 2.** For every noncommutative Novikov algebra \( D \in \text{DAs} \) there exists an associative algebra \( A \) with a derivation \( d \) such that \( D \subseteq A^{(d)} \).

**Proof.** Let \( D \cong \text{DAs}(X) / J \) for some set \( X \) and ideal \( J \). Construct the differential ideal \( I \) of \( \text{AsDer}(X; d) \) generated by \( J \). Then \( A = \text{AsDer}(X; d) / I \) is a differential associative algebra, and the kernel of the map
\[ \text{DAs}(X) \xrightarrow{\subseteq} \text{AsDer}(X; d) \to \text{AsDer}(X; d) / I \]
coincides with \( I \cap \text{DAs}(X) \) which is \( J \) by Proposition 4. This map is a homomorphism of DAs-algebras. Therefore, \( D \) embeds into \( A^{(d)} \).

**Alternative proof of Theorem 2** Let \( X \) be a basis of \( D \), then \( x \prec y \) and \( x \succ y \) for \( x, y \in X \) are linear forms in \( X \).

Consider \( F = \text{AsDer}(X; d) = \text{As}(X^{(\omega)}) \), where \( X^{(\omega)} = X \cup X' \cup X'' \cup \ldots \), and the ideal \( I \) of \( F \) generated by all derivatives of
\[ xy' - x \prec y, \quad x'y - x \succ y, \quad x, y \in X. \]

Then \( F / I \) is the universal enveloping associative differential algebra of \( D \). The problem is to show that \( D \) embeds into \( F / I \), i.e., \( I \cap \mathbb{k}X = 0 \).

Suppose \( X \) is linearly ordered, then the words in the alphabet \( X^{(\omega)} \) are monomially ordered with respect to the order \( \ll \) described in Proposition 4. The explicit set of generators of \( I \) as of an ideal in the free associative algebra \( F \) is given by
\[ R_n(x, y) = xy^{(n)} + \sum_{s=1}^{n-1} \binom{n-1}{s} x^{(s)} y^{(n-s)} - (x \prec y)^{(n-1)}, \]
\[ L_n(x, y) = x^{(n)} y + \sum_{s=1}^{n-1} \binom{n-1}{s} x^{(n-s)} y^{(s)} - (x \succ y)^{(n-1)}, \]
where \( x, y \in X, \ n \geq 1 \). Note that \( R_n(x, y) \) and \( L_n(x, y) \) are weight-homogeneous, all monomials are of weight \( n - 2 \). According to the order \( \ll \), the principal parts of these relations are \( \hat{R}_n(x, y) = xy^{(n)}, \hat{L}_n(x, y) = x^{(n)}y \).

Unfortunately, we cannot apply the standard Gröbner–Shirshov bases technique since the principal parts of compositions depend on the particular form of \( x \prec y \) and \( x \succ y \). For example, the composition of intersection (see, e.g., [2]) of \( R_1(x, y) = xy' - x \prec y \) and \( L_1(y, z) = y'z - y \succ z \) is \( x \prec y)z - x(y \succ z) \), so the choice of a principal part is unclear in general.
However, we can draw the desired conclusion if we turn to the Diamond Lemma for rewriting systems [15], which lies in the foundation of Gröbner–Shirshov bases theories.

Consider the rewriting system \( \mathcal{G} \) corresponding to the generators \( X^{(\omega)} \) and rewriting rules \( R_n(x, y), L_n(x, y) \). This is an oriented graph with vertices \( F \).

Two polynomials \( f \) and \( g \) are connected by an edge \( f \to g \) if \( g \) is obtained from \( f \) by eliminating a subword of the form \( xy^{(n)} \) or \( x^{(n)}y \) in a monomial of \( f \) by means of \( R_n(x, y) \) or \( L_n(x, y) \), respectively. For example,

\[
yz''x \to yz'x' - y(z > x)' \to (y < z)x' - y(z > x)',
\]
\[
yz''x \to y'z'x - (y < z)'x \to y'(z > x) - (y < z)'x
\]
are edges if \( \mathcal{G} \). A polynomial \( f \in F \) belongs to \( I \) if and only if the vertex \( f \) is connected with 0 by a (non-oriented) path in the graph \( \mathcal{G} \).

Since the relations (9), (10) are weight-homogeneous, it is enough to consider only weight-homogeneous vertices in \( F \).

Let us state a series of properties of the oriented graph \( \mathcal{G} \).

**Lemma 2.** Let \( f \in F \) be a weight-homogeneous polynomial, \( \text{wt}(f) = -1 \). Then there exists a chain (oriented path)

\[
f \to \cdots \to t,
\]
where \( t \in kX \).

We will denote a chain from \( f \) to \( g \) by

\[
f \to^* g.
\]

**Proof.** Let \( \text{pt}(f) \) be the maximal potential of monomials in \( f \). If \( \text{pt}(f) = 1 \) then \( f \) itself is a linear form in \( X \). Assume the statement is true for all polynomials \( g \) such that \( \text{pt}(g) < \text{pt}(f) \). Note that the principal parts of \( R_n(x, y) \) and \( L_n(x, y) \) have the potentials greater (even by degree) than all other monomials in these relations. Hence, given \( f \) as in the statement with \( \text{pt}(f) > 1 \), there exists a chain \( f \to f_1 \to \cdots \to f_k \) where \( \text{pt}(f_k) < \text{pt}(f) \): one has to apply rewriting rules to all monomials of maximal potential in \( f \).

By induction, \( f_k \to^* t \in kX \), so the same is true for \( f \). \( \square \)

**Lemma 3.** Let \( f \in F \) be a weight-homogeneous polynomial, \( \text{wt}(f) = -m \), \( m > 1 \). Then there exists a chain

\[
f \to^* t \in kX^m.
\]

**Proof.** It is enough to prove the statement for \( f = w \), where \( w \) is a monomial in \( X^{(\omega)} \), \( \text{wt}(w) = -m \). By Proposition 2 \( w \) can be represented as \( w = u_1 \ldots u_m \), \( \text{wt}(u_i) = -1 \). Lemma 2 completes the proof: if there exist chains \( u_i \to^* t_i \in kX \) then

\[
u_1u_2 \ldots u_m \to^* t_1u_2 \ldots u_m \to^* \cdots \to^* t_1 \ldots t_m \in kX^m.
\]
Lemma 4. Let $f \in F$ be a weight-homogeneous polynomial, $\text{wt}(f) = 0$. Then there exists a chain

$$f \rightarrow^* t \in \mathbb{k}(X')^*.$$  

Proof. It is enough to consider the case when $f$ is a monomial $w$ in the alphabet $X^{(\omega)}$ of weight zero.

Proceed by induction on the potential of $w$. If $\text{pt}(w) = Nt$ then there is nothing to prove. Assume $\deg \text{pt}(w) > 1$ and the statement holds for all monomials $u$ such that $\text{wt}(u) = 0$ and $\text{pt}(u) < \text{pt}(w)$.

Choose a letter $z^{(n)}$ in $w$ ($z \in X$) with maximal $n = \deg \text{pt}(w)$. Then $w = w_1z^{(n)}w_2$, $\text{wt}(w_1) + \text{wt}(w_2) = -n + 1 < 0$. Hence, at least one of $w_1$, $w_2$ has a negative weight. Suppose $\text{wt}(w_2) = -k < 0$. Then by Lemmas 2, 3 there exists a chain $w_2 \rightarrow^* \sum_i \alpha_i t_{i_1} \ldots t_{i_k}, t_{ji} \in X$. So $w$ may be reduced to a linear combination of $w_1z^{(n)}t_{i_1} \ldots t_{i_k}$, the latter reduces by $L_n(z, t_1)$ to a polynomial $g$ of smaller potential, and $g \rightarrow^* t \in \mathbb{k}(X')^*$ by induction. □

Denote by $G_m$ the subgraph of $G$ spanned by all weight-homogeneous vertices of weight $m$. Since the relations $R_n(x, y)$ and $L_n(x, y)$ are weight-homogeneous, we may conclude that if $f, g \in G_m$ are connected in $G$ (by a non-oriented path) then they are connected in $G_m$, i.e., there exists such a path through vertices of weight $m$. In particular, $G_{-1}$ is a rewriting system containing all linear polynomials $\mathbb{k}X$.

Lemma 5. Let $f \in F$ be a weight-homogeneous polynomial, $\text{wt}(f) = m > 0$. Then there exists a chain

$$f \rightarrow^* t \in \mathbb{k}(X^{(\omega)} \setminus X)^*.$$  

Proof. Assume there exists a terminal vertex $t$ such that $f \rightarrow^* t$ but $t$ contains a letter from $X = X^{(0)}$ in at least one of its monomials. Since the overall weight is positive, this monomial should contain also letters of the form $y^{(n)}$, $n > 0$, and hence there should be an occurrence of a subword $xy^{(n)}$ or $y^{(n)}x$. This contradicts terminality of $t$. □

Recall that a rewriting system is said to be confluent if for every connected $f, g$ there exists a vertex $h$ and chains $f \rightarrow^* h$ and $g \rightarrow^* h$.

A critical pair in a rewriting system is a pair of edges $w \rightarrow f_1$, $w \rightarrow f_2$ starting in a vertex $w$ (ambiguity). Such a pair is called convergent if there exist a vertex $h$ and two chains $f_1 \rightarrow^* h$, $f_2 \rightarrow^* h$. According to the Diamond Lemma, a rewriting system is confluent if and only if all its critical pairs are convergent.

Proposition 5. The rewriting system $G_{-1}$ is confluent.
Proof. For the rewriting system \( G \) (or \( G_{-1} \)) it is enough to consider the following ambiguities:

\[
w = w_1 xy^{(n)} zw_2 \quad \text{or} \quad w = w_1 x^{(n)} y z^{(m)} w_2, \quad x, y, z \in X.
\]

The critical pairs consist of edges which are defined by relations \( R_n(x, y) \), \( L_n(y, z) \) or \( L_m(x, y) \), \( R_m(y, z) \), respectively.

Let us prove the lemma by induction on the potential of vertices. Namely, for a polynomial \( p \in \mathbb{Z}_+[t] \), denote by \( G^p_{-1} \) the subgraph of \( G_{-1} \) spanned by all those vertices \( f \) of \( G_{-1} \) for which \( pt(u) \leq p \) for all monomials \( u \) that occur in \( f \) (one has to assume the zero vertex belongs to \( G^p_{-1} \)). For example, the vertices of the graph \( G^1_{-1} \) are \( kX \), and this graph has no edges (hence, it is a confluent rewriting system).

**Lemma 6.** Two vertices \( f \) and \( g \) are connected by a non-oriented path in \( G^p_{-1} \) if and only if

\[
f - g = \sum_{i=1}^{n} \alpha_i u_i s_i v_i, \quad u_i, v_i \in (X^{(w)})^*, \quad \alpha_i \in k,
\]

where \( s_i \) are of the form (9) or (10), and \( pt(u_i s_i v_i) \leq p \).

**Proof.** The “only if” part is obvious. To prove the “if” part, proceed by induction on the number \( n \) of summands in the right-hand side of (11). For \( n = 1 \), suppose the monomial \( u_1 s_1 v_1 \) appears if \( g \) with a coefficient \( \gamma \in k \) (possibly zero). If \( \alpha_1 + \gamma = 0 \) then there is an edge \( g \rightarrow f \) in \( G^1_{-1} \). If \( \gamma = 0 \) then there is an edge \( f \rightarrow g \). If \( \gamma \neq 0 \) and \( \alpha_1 + \gamma \neq 0 \) then there is a vertex \( h \) in \( G^1_{-1} \) such that \( f \rightarrow h \leftarrow g \). The induction step is obvious. \( \square \)

Let us fix a polynomial \( q \in \mathbb{Z}_+[t] \) and assume \( G^p_{-1} \) is confluent for all \( p < q \). The purpose is to show that \( G^q_{-1} \) is also confluent. To that end, check the Diamond Lemma condition (convergence of critical pairs).

**Case 1:** Consider the ambiguity \( w = w_1 xy^{(n)} zw_2 \), \( wt(w) = -1 \), and a critical pair \( w \rightarrow f_1, w \rightarrow f_2 \), where

\[
f_1 = w_1 \left( (x \prec y)^{n-1} - \sum_{s=1}^{n-1} \left( \frac{n-1}{s} \right) x^{(s)} y^{(n-s)} \right) zw_2,
\]

\[
f_2 = w_1 x \left( (y \succ z)^{n-1} - \sum_{s=1}^{n-1} \left( \frac{n-1}{s} \right) y^{(n-s)} z^{(s)} \right) w_2.
\]

First, assume \( n > 1 \). Then we may apply a series of rewriting rules \( L_{n-1}(x \prec y, z) \) to \( f_1 \) (that is, we apply \( L_{n-1}(t, z) \) for each letter \( t \in X \) in
the linear form \(x < y\). For \(f_2\), apply \(R_{n-1}(x, y \succ z)\). As a result, we obtain

\[
f_1 \rightarrow^* f_{11} = w_1((x < y) \succ z)^{(n-2)} - \sum_{j=1}^{n-2} \binom{n-2}{j} (x < y)^{(n-1-j)} z^{(j)}
\]

\[
- \sum_{s=1}^{n-1} \binom{n-1}{s} x^{(s)} y^{(n-s)} z w_2
\]

\[
f_2 \rightarrow^* f_{21} = w_1((x < (y \succ z))^{(n-2)} - \sum_{j=1}^{n-2} \binom{n-2}{j} x^{(j)} (y \succ z)^{(n-1-j)}
\]

\[
- \sum_{s=1}^{n-1} \binom{n-1}{s} x y^{(n-s)} z^{(s)} w_2
\]

\[
= w_1((x < y) \succ z)^{(n-2)} - (x \succ (y \succ z))^{(n-2)} + ((x < y) < z)^{(n-2)}
\]

\[
- \sum_{j=1}^{n-2} \binom{n-2}{j} x^{(j)} (y \succ z)^{(n-1-j)} - \sum_{s=1}^{n-1} \binom{n-1}{s} x y^{(n-s)} z^{(s)} w_2,
\]

the latter equality follows from \(\text{(5)}\). Note that \(f_{11}\) and \(f_{21}\) have smaller potential than \(q = pt(w)\). Moreover, \(f_{11} - f_{21}\) may be presented in the form \(\text{(III)}\), where all summands have potentials smaller than \(q\). Indeed, \(f_{11} - f_{21} = w_1 g w_2\), where

\[
g = - \sum_{j \geq 1} \binom{n-2}{j} (x < y)^{(n-1-j)} z^{(j)} - \sum_{s \geq 1} \binom{n-1}{s} x^{(s)} y^{(n-s)} z
\]

\[
- ((x < y) < z)^{(n-2)} + \sum_{j \geq 1} \binom{n-2}{j} x^{(j)} (y \succ z)^{(n-1-j)}
\]

\[
+ \sum_{s \geq 1} \binom{n-1}{s} x y^{(n-s)} z^{(s)} + (x \succ (y \succ z))^{(n-2)}.
\]

Let us rewrite the terms \((x \succ (y \succ z))^{(n-2)}\) and \(((x < y) < z)^{(n-2)}\) back into the product of derivatives:

\[
g_1 = - \sum_{j \geq 1} \binom{n-1}{j} (x < y)^{(n-1-j)} z^{(j)} + \sum_{j \geq 1} \binom{n-1}{j} x^{(j)} (y \succ z)^{(n-1-j)}
\]

\[
+ \sum_{s \geq 1} \binom{n-1}{s} x y^{(n-s)} z^{(s)} - \sum_{s \geq 1} \binom{n-1}{s} x^{(s)} y^{(n-s)} z.
\]
This rewriting involves only summands of smaller potential than $xy^{(n)}z$. Hence, $g$ and $g_1$ are connected by a non-oriented path in the graph $G_{n-3}^r$, where $r < \text{pt}(xy^{(n)}z)$. Similarly, $g_1$ is connected to zero in the same graph. Hence, in the difference of these two expressions the associators cancel each other by (6). Hence, there exist chains $f_{11} \to^* h$ and $f_{21} \to^* h$ for an appropriate vertex $h$. Therefore, the critical pair $f_1 \leftrightarrow w \to f_2$ is convergent.

Next, suppose $n = 1$. Since $w = w_1 xy' zw_2$ is of weight $-1$, we have $\text{wt}(w_1) + \text{wt}(w_2) = 1$, i.e., at least one of $w_1$, $w_2$ has a positive weight.

Assume $\text{wt}(w_2) \geq 1$. By Lemma [5] there exists a chain $w_2 \to^* g$, where all monomials in $g$ are of the form $a'_1 \ldots a'_r b^{(m)} v$, $m \geq 2$, $r \geq 0$, $a_i, b \in X$, $v \in (X^{(ω)})^*$. Then there exist chains in $G_{-1}^p$, $p < q = \text{wt}(w)$, reducing $f_1$ and $f_2$ to linear combinations of monomials

$$w_1(x \prec y) za'_1 \ldots a'_r b^{(m)} v, \quad w_1(x \succ z) za'_1 \ldots a'_r b^{(m)} v,$$

respectively. Suppose $r > 0$.

Note that there exists a chain in $G_{-1}$ of the form

$$xyb^{(m)} \to^* -(x, y, b)^{(m-2)} + \sum_{(s, t) \neq (0, 0)} \binom{m - 2}{s, t} x^{(s)} y^{(t)} b^{(m-s-t)}.$$

Hence,

$$(x \prec y) za'_1 \ldots a'_r b^{(m)} \to^* (x \prec y) za'_1 \ldots a'_r b^{(m)}$$

$$\rightarrow^* -((x \prec y), z_r, b)^{(m-2)} + \text{terms of smaller potential},$$

$$x(y \succ z) za'_1 \ldots a'_r b^{(m)} \to^* x(y \succ z) za'_1 \ldots a'_r b^{(m)}$$

$$\rightarrow^* -(x, (y \succ z), b)^{(m-2)} + \text{terms of smaller potential},$$

where $z_r = ((\ldots (z \prec a_1) \prec \ldots) \prec a_r)$. Note that $(y \succ z)_r = y \succ z_r$ by (4). Hence, in the difference of these two expressions the associators cancel each other by (4).

Therefore, there exist vertices $g_1$, $g_2$ in $G_{-1}^p$, $p < q$, such that $f_1 \to^* g_1$, $f_2 \to^* g_2$, and if we expand all remaining operations $\prec$ and $\succ$ in the expression for $g_1 - g_2$ then we obtain a differential polynomial of smaller potential than $\text{pt}(w)$. By a straightforward computation one can verify that this polynomial is zero. Hence, there exists a vertex $h$ in $G_{-1}^p$ such that $g_1 \to^* h$, $g_2 \to^* h$, and thus the critical pair $f_1 \leftrightarrow w \to f_2$ is convergent.

The case when $\text{wt}(w_1) \geq 1$ is completely similar, one has to use (7) instead of (4).
Case 2: Consider the ambiguity \( w = w_1 x^{(n)} y z^{(m)} w_2 \), \( \text{wt}(w) = -1 \), and a critical pair \( w \to f_1, w \to f_2 \), where

\[
f_1 = w_1 \left( (x \succ y)^{(n-1)} - \sum_{s=1}^{n-1} \left( \frac{n-1}{s} \right) x^{(n-s)} y^{(s)} \right) z^{(m)} w_2,
\]

\[
f_2 = w_1 x^{(n)} \left( (y \prec z)^{(m-1)} - \sum_{s=1}^{m-1} \left( \frac{m-1}{s} \right) y^{(s)} z^{(m-s)} \right) w_2.
\]

Since \( \text{wt}(w) = -1 \), we have \( \text{wt}(w_1) + \text{wt}(w_2) = -n - m + 2 \).

First, assume \( n, m > 1 \). Then \( \text{wt}(w_1) + \text{wt}(w_2) = -n - m + 2 < 0 \), so at least one of \( w_1 \) or \( w_2 \) has a negative weight. Suppose, for example, that \( \text{wt}(w_2) < 0 \). Then we may replace \( w_2 \) with its successor in \( G_{-1} \) from Lemma 3. Thus we have two chains in \( G^p_{-1}, p < q = \text{pt}(w) \): \( f_1 \to^* g_1, f_2 \to^* g_2 \), where each monomial in \( g_i \) is obtained from \( f_i \) by means of replacing \( w_2 \) with a linear combination of the form \( \sum_j \alpha_j a_j v_j, a_j \in X \).

Apply \( L_m(z, a_j) \) to \( g_1 \) and \( L_{m-1}(y \succ z, a_j) \) to \( g_2 \). Such a rewriting decreases the maximal potential of these polynomials and thus we may "invert" one of the initial edges, corresponding to \( L_n(x, y) \):

\[
f_1 \to^* g_1 \leftrightarrow \sum_j \alpha_j w_1 \left( x^{(n)} y (z \succ a_j)^{(m-1)} - \sum_{s=1}^{m-1} \left( \frac{m-1}{s} \right) x^{(n)} y z^{(m-s)} a_j^{(s)} \right)
\]

\[
\to^* h_1 := \sum_j \alpha_j w_1 \left( x^{(n)} (y \prec (z \succ a_j))^{(m-2)} - \sum_{t \geq 1} \left( \frac{m-2}{t} \right) x^{(n)} y^{(t)} (z \succ a_j)^{(m-1-t)} \right) - \sum_{s=1}^{m-1} \left( \frac{m-1}{s} \right) x^{(n)} y z^{(m-s)} a_j^{(s)} v_j
\]

Here \( \leftrightarrow \) denotes the fact that two vertices are connected by a non-oriented path in \( G^p_{-1}, p < q \).

Similarly,

\[
f_2 \to^* g_2 \leftrightarrow h_2 := \sum_j \alpha_j w_1 \left( x^{(n)} ((y \prec z) \succ a_j)^{(m-2)} - \sum_{t \geq 1} \left( \frac{m-2}{t} \right) x^{(n)} (y \prec z)^{(m-1-t)} a_j^{(t)} \right) - \sum_{s=1}^{m-1} \left( \frac{m-1}{s} \right) x^{(n)} y^{(s)} z^{(m-s)} a_j^{(s)} v_j
\]

It remains to calculate \( h_1 - h_2 \), apply (5) to the \( (m - 2) \)th derivative of \( y \prec (z \succ a_j) - (y \prec z) \succ a_j = (y \prec z) \prec a_j - y \succ (z \succ a_j) \), and then expand back all \( \prec, \succ \) (this expansion corresponds to a moving through non-oriented paths in \( G^p_{-1}, p < q \), since the terms of maximal potential cancel each other in \( h_1 - h_2 \) due to (5)). The polynomial obtained in this way from \( h_1 - h_2 \) is zero, so \( h_1, h_2 \) are connected by a non-oriented path in \( G^p_{-1} \) with
p < q, and the same is true for \( f_1, f_2 \). The inductive assumption for \( G_{p-1} \) implies the critical pair \( f_1 \leftarrow w \rightarrow f_2 \) is convergent.

Next, assume \( n = m = 1 \). Then \( f_1 = w_1(x \succ y)z'w_2 \rightarrow^* h := w_1((x \succ y) \prec z)w_2 \) and, similarly, \( f_2 \rightarrow^* h \) by (4). Hence, the critical pair \( f_1 \leftarrow w \rightarrow f_2 \) is convergent in this case.

Finally, if either \( n = 1, m > 1 \) or \( n > 1, m = 1 \) then one may proceed as in the previous two subcases of Case 2 to prove that the critical pair \( f_1 \leftarrow w \rightarrow f_2 \) is convergent.

Let us complete the proof of Theorem 2. If \( f \in \mathbb{k}X \cap I \) then \( f \) is connected with 0 in \( G \) and thus in \( G_{-1} \). Since \( G_{-1} \) is confluent there should exist chains starting at 0 and at \( f \) finishing in a single vertex \( h \). But both \( f \) and 0 are terminal vertices: there are no edges starting at them. Hence, \( f = 0 \), which proves \( \mathbb{k}X \cap I = 0 \).

Remark 2. Our new proof of Theorem 2 may be adjusted for commutative algebras. In this way one may get an alternative proof of the embedding Theorem in [4] which is independent from the fundamental result of [8] on free Novikov algebras.

Corollary 3. Let \( M \) be an arbitrary nonassociative algebra with one operation \((a, b) \mapsto ab\). Then there exists \( A \in DAs \) such that \( M \subseteq (A, \prec) \). The same is true for \((A, \succ)\).

Proof. One has to follow the lines of the alternative proof of Theorem 2 and construct the universal enveloping associative differential algebra for \( M \) as a quotient of \( F = As(X^{(\omega)}) \) modulo the ideal generated by the relations \( R_n(x, y) \) from (9), \( n \geq 1, x, y \in X \). Here \( X \) is a basis of \( M \) and in \( R_n(x, y) \) one should replace \( \prec \) with the product in \( M \). There are no ambiguities (compositions) of relations (9), so this is a Gröbner–Shirshov basis in the free associative algebra \( F \) and \( M \) embeds into the corresponding quotient.

It is well known that if \((V, \circ)\) is a Novikov algebra then its commutator algebra \( V^{(-)} \) constructed on the same linear space with respect to new operation \([x, y] = x \circ y - y \circ x\) is a Lie algebra. Every Lie algebra \( V^{(-)} \) obtained in this way meets the identity

\[
\sum_{\sigma \in S_4} (-1)^\sigma [x_{\sigma(1)}, [x_{\sigma(2)}, [x_{\sigma(3)}, x_{\sigma(4)}]]] = 0.
\]

(It was found in [9] and re-discovered in [21].) It is an open problem to find the complete list of independent Lie identities that hold on all commutator Novikov algebras. In the noncommutative case (for DAs-algebras), we may find the answer.
Corollary 4. For $A \in DAs$, let $A^{(-)}$ stand for the same linear space equipped with the operation $[x, y] = x \prec y - y \succ x$. Then there are no identities that hold on all $A^{(-)}$ for $A \in DAs$.

Proof. As in the previous statements, use the monomial order $\ll$ from Proposition 1. Then for every nonassociative algebra $M$ with a linear basis $X$ one may construct its universal enveloping associative differential algebra as a quotient of $As\langle X^{(\omega)} \rangle$ modulo the ideal generated by all derivatives of

$$xy' - y'x - xy, \quad x, y \in X.$$ 

The principal parts of these derivatives are of the form $y^{(n)}x$, $n \geq 1$, and there are no ambiguities.

Hence, every nonassociative algebra embeds into $A^{(-)}$ for an appropriate $A \in DAs$. □

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