REAL HYPERSURFACES IN THE COMPLEX QUADRIC WITH REEB PARALLEL STRUCTURE JACOBI OPERATOR

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Abstract. In this paper, we first introduce the full express of the Riemannian curvature tensor of a real hypersurface $M$ in complex quadric $Q^m$ from the equation of Gauss. Next we derive a formula for the structure Jacobi operator $R_{\xi}$ and its derivative under the Levi-Civita connection of $M$. We give a complete classification of Hopf real hypersurfaces with Reeb parallel structure Jacobi operator, $\nabla_\xi R_{\xi} = 0$, in the complex quadric $Q^m$, $m \geq 3$.

1. Introduction

For Hermitian symmetric space of compact type different from the above ones, we can give the example of complex quadric $Q^m = SO_{m+2}/SO_m SO_2$, which is a complex hypersurface in the complex projective space $\mathbb{C}P^{m+1}$ (see Romero [23], [24], Smyth [25], Suh [28], [29]). The complex quadric can also be regarded as a kind of real Grassmann manifolds of compact type with rank 2 (see Besse [1], Helgason [6], and Knap [12]). Accordingly, the complex quadric $Q^m$ admits two important geometric structures, a complex conjugation structure $A$ and a Kähler structure $J$, which anti-commute with each other, that is, $AJ = -JA$. Then for $m \geq 2$ the triple $(Q^m, J, g)$ is a Hermitian symmetric space of compact type with rank 2 and its maximal sectional curvature is equal to 4 (see Kobayashi and Nomizu [14], Reckziegel [22]).

In addition to the complex structure $J$ there is another distinguished geometric structure on $Q^m$, namely a parallel rank two vector bundle $\mathfrak{A}$ which contains an $S^1$-bundle of real structures, that is, complex conjugations $A$ on the tangent spaces of $Q^m$. The set is denoted by $\mathfrak{A} = \{ A_\lambda \mid \lambda \in S^1 \subset \mathbb{C} \}$, $[z] \in Q^m$, and it is the set of all complex conjugations defined on $Q^m$. Then $\mathfrak{A}$ becomes a parallel rank 2-subbundle of End $T_{[z]}Q^m$, $[z] \in Q^m$. This geometric structure determines a maximal $\mathfrak{A}$-invariant subbundle $\mathcal{Q}$ of the tangent bundle $TM$ of a real hypersurface $M$ in $Q^m$. Here the notion of parallel vector bundle $\mathfrak{A}$ means that $(\nabla_X A)Y = q(X)JAY$ for any vector fields $X$ and $Y$ on $Q^m$, where $\nabla$ and $q$ denote a connection and a certain 1-form defined on $T_{[z]}Q^m$, $[z] \in Q^m$ respectively (see Smyth [25]).

Recall that a nonzero tangent vector $W \in T_{[z]}Q^m$ is called singular if it is tangent to more than one maximal flat in $Q^m$. There are two types of singular tangent vectors for the complex hyperbolic quadric $Q^m$:

\[ \text{Key words: Reeb parallel structure Jacobi operator, singular normal vector field, } \mathfrak{A}\text{-isotropic, } \mathfrak{A}\text{-principal, Kähler structure, complex conjugation, complex quadric.} \]

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• If there exists a conjugation $A \in \mathfrak{A}$ such that $W \in V(A) = \{X \in T_{z|}Q^m \mid AX = X\}$, then $W$ is singular. Such a singular tangent vector is called $\mathfrak{A}$-principal.

• If there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $Z_1, Z_2 \in V(A)$ such that $W/||W|| = (Z_1 + JZ_2)/\sqrt{2}$, then $W$ is singular. Such a singular tangent vector is called $\mathfrak{A}$-isotropic, where $V(A) = \{X \in T_{z|}Q^m \mid AX = X\}$ and $JV(A) = \{X \in T_{z|}Q^m \mid AX = -X\}$ are the $(+1)$-eigenspace and $(-1)$-eigenspace for the involution $A$ on $T_{z|}Q^m$, $[z] \in Q^m$.

On the other hand, Okumura [17] proved that the Reeb flow on a real hypersurface in $\mathbb{C}P^m = SU_{m+1}/S(U_1U_m)$ is isometric if and only if $M$ is an open part of a tube around a totally geodesic $\mathbb{C}P^k$ in $\mathbb{C}P^m$ for some $k \in \{0, \ldots, m - 1\}$. For the complex 2-plane Grassmannian $G_2(\mathbb{C}^{m+2}) = SU_{m+2}/S(U_2U_m)$ a classification was obtained by Berndt and Suh [2]. The Reeb flow on a real hypersurface in $G_2(\mathbb{C}^{m+2})$ is isometric if and only if $M$ is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$. For the complex quadric $Q^m = SO_{m+2}/SO_2SO_m$, Berndt and Suh [3] have obtained the following result:

**Theorem A.** Let $M$ be a real hypersurface in the complex quadric $Q^m$, $m \geq 3$. Then the Reeb flow on $M$ is isometric if and only if $m$ is even, say $m = 2k$, and $M$ is an open part of a tube around a totally geodesic $\mathbb{C}P^k$ in $Q^{2k}$.

For the complex hyperbolic space $\mathbb{C}H^m$ a classification was obtained by Montiel and Romero [16]. They proved that the Reeb flow on a real hypersurface in $\mathbb{C}H^m$ is isometric if and only if $M$ is an open part of a tube around a totally geodesic $\mathbb{C}H^k$ in $\mathbb{C}H^m$ for some $k \in \{0, \ldots, m - 1\}$. The classification problems related to the Reeb parallel shape operator, parallel Ricci tensor, and harmonic curvature for real hypersurfaces in the complex quadric $Q^m$ were recently given in Suh [26], [28] and [29] respectively.

The notion of isometric Reeb flow was introduced by Hutching and Taubes [7] and the geometric construction of horospheres in a non-compact manifold of negative curvature was mainly discussed in the book due to Eberlein [5].

On the other hand, Jacobi fields along geodesics of a given Riemannian manifold $(\tilde{M}, g)$ satisfy a well known differential equation. This equation naturally inspires the so-called Jacobi operator. That is, if $R$ denotes the curvature operator of $\tilde{M}$, and $X$ is a tangent vector field to $\tilde{M}$, then the Jacobi operator $R_X \in \text{End}(T_{z|}\tilde{M})$ with respect to $X$ at $z \in \tilde{M}$, defined by $(R_X Y)(z) = (R(Y, X)X)(z)$ for any $Y \in T_{z|}\tilde{M}$, becomes a self adjoint endomorphism of the tangent bundle $TM$ of $M$. Thus, each tangent vector field $X$ to $\tilde{M}$ provides a Jacobi operator $R_X$ with respect to $X$. In particular, for the Reeb vector field $\xi$, the Jacobi operator $R_\xi$ is said to be the *structure Jacobi operator*.

Actually, many geometers have considered the fact that a real hypersurface $M$ in Kähler manifolds has *parallel structure Jacobi operator* (or Reeb parallel structure Jacobi operator, respectively), that is, $\nabla_X R_\xi = 0$ (or $\nabla_\xi R_\xi = 0$, respectively) for any tangent vector field $X$ on $M$. Recently Ki, Pérez, Santos and Suh [10] have investigated the Reeb parallel structure Jacobi operator in the complex space form $M^m(c)$, $c \neq 0$, and have used it to study some principal curvatures for a tube over a totally geodesic submanifold. In particular, Pérez, Jeong and Suh [19] have investigated real hypersurfaces $M$ in $G_2(\mathbb{C}^{m+2})$ with parallel structure Jacobi operator, that is, $\nabla_X R_\xi = 0$ for any tangent vector field $X$ on $M$. Jeong, Suh and Woo [9] and Pérez and Santos [20] have generalized such a notion to the recurrent structure Jacobi operator, that is, $(\nabla_X R_\xi)Y = \beta(X)R_\xi Y$ for a certain
1-form $\beta$ and any vector fields $X, Y$ on $M$ in $G_2(\mathbb{C}^{m+2})$ or $\mathbb{CP}^m$. In [8], Jeong, Lee, and Suh have considered a Hopf real hypersurface with Codazzi type of structure Jacobi operator, $(\nabla_X R_\xi) Y = (\nabla_Y R_\xi) X$, in $G_2(\mathbb{C}^{m+2})$. Moreover, Pérez, Santos and Suh [21] have further investigated the property of the Lie $\xi$-parallel structure Jacobi operator in complex projective space $\mathbb{CP}^m$, that is, $\mathcal{L}_\xi R_\xi = 0$.

Motivated by these results, in this paper we want to give a classification of Hopf real hypersurfaces in $Q^m$ with non-vanishing geodesic Reeb flow and Reeb parallel structure Jacobi operator, that is, $\nabla_\xi R_\xi = 0$. Here a real hypersurface $M$ is said to be Hopf if the Reeb vector field $\xi$ of $M$ is principal by the shape operator $S$, that is, $S\xi = g(S\xi, \xi)\xi = \alpha\xi$. In particular, if the Reeb curvature function $\alpha = g(S\xi, \xi)$ identically vanishes, we say that $M$ has a vanishing geodesic Reeb flow. Otherwise, $M$ has a non-vanishing geodesic Reeb flow.

Under these background and motivation, first we prove the following:

**Theorem 1.** There does not exist any Hopf real hypersurface in the complex quadric $Q^m$, $m \geq 3$, with Reeb parallel structure Jacobi operator and $\mathfrak{A}$-principal singular normal vector field, provided with non-vanishing geodesic Reeb flow.

Now let us consider a Hopf real hypersurface with $\mathfrak{A}$-isotropic singular normal vector field $\nu$ in $Q^m$. Then by virtue of Theorem A we can give a complete classification of Hopf real hypersurfaces in $Q^m$ with Reeb parallel structure Jacobi operator as follows:

**Theorem 2.** Let $M$ be a Hopf real hypersurface in the complex quadric $Q^m$, $m \geq 3$, with Reeb parallel structure Jacobi operator and non-vanishing geodesic Reeb flow. If $M$ has the $\mathfrak{A}$-isotropic singular normal vector field in $Q^m$, then $M$ is locally congruent to a tube around the totally geodesic $\mathbb{CP}^k$ in $Q^{2k}$, where $m = 2k$, and $r \in (0, \frac{\pi}{4}) \cup (\frac{\pi}{4}, \frac{\pi}{2})$.

2. The complex quadric

For more background to this section we refer to [11], [14], [22], [26], [27], [29] and [32]. The complex quadric $Q^m$ is the complex hypersurface in $\mathbb{CP}^{m+1}$ which is defined by the equation $z_1^2 + \cdots + z_{m+2}^2 = 0$, where $z_1, \ldots, z_{m+2}$ are homogeneous coordinates on $\mathbb{CP}^{m+1}$. We equip $Q^m$ with the Riemannian metric which is induced from the Fubini Study metric on $\mathbb{CP}^{m+1}$ with constant holomorphic sectional curvature 4. The Kähler structure on $\mathbb{CP}^{m+1}$ induces canonically a Kähler structure $(J, g)$ on the complex quadric. For a nonzero vector $z \in \mathbb{C}^{m+2}$ we denote by $[z]$ the complex span of $z$, that is, $[z] = \mathbb{C}z = \{\lambda z | \lambda \in S^1 \subset \mathbb{C}\}$. Note that by definition $[z]$ is a point in $\mathbb{CP}^{m+1}$. For each $[z] \in Q^m \subset \mathbb{CP}^{m+1}$ we identify $T_{[z]}Q^m$ with the orthogonal complement $\mathbb{C}^{m+2} \ominus \mathbb{C}z$ of $\mathbb{C}z$ in $\mathbb{C}^{m+2}$ (see Kobayashi and Nomizu [14]). The tangent space $T_{[z]}Q^m$ can then be identified canonically with the orthogonal complement $\mathbb{C}^{m+2} \ominus (\mathbb{C}z \oplus \mathbb{C}\rho)$ of $\mathbb{C}z \oplus \mathbb{C}\rho$ in $\mathbb{C}^{m+2}$, where $\rho \in \nu_{[z]}Q^m$ is a normal vector of $Q^m$ in $\mathbb{CP}^{m+1}$ at the point $[z]$.

The complex projective space $\mathbb{CP}^{m+1}$ is a Hermitian symmetric space of the special unitary group $SU_{m+2}$, namely $\mathbb{CP}^{m+1} = SU_{m+2} / SU_m U_1$. We denote by $o = [0, \ldots, 0, 1] \in \mathbb{CP}^{m+1}$ the fixed point of the action of the stabilizer $SU_m U_1$. The special orthogonal group $SO_{m+2} \subset SU_{m+2}$ acts on $\mathbb{CP}^{m+1}$ with cohomogeneity one. The orbit containing $o$ is a totally geodesic real projective space $\mathbb{RP}^{m+1} \subset \mathbb{CP}^{m+1}$. The second singular orbit
of this action is the complex quadric $Q^m = SO_{m+2}/SO_m SO_2$. This homogeneous space model leads to the geometric interpretation of the complex quadric $Q^m$ as the Grassmann manifold $G^+_2(\mathbb{R}^{m+2})$ of oriented 2-planes in $\mathbb{R}^{m+2}$. It also gives a model of $Q^m$ as a Hermitian symmetric space of rank 2. The complex quadric $Q^1$ is isometric to a sphere $S^2$ with constant curvature, and $Q^2$ is isometric to the Riemannian product of two 2-spheres with constant curvature. For this reason we will assume $m \geq 3$ from now on.

For a unit normal vector $\rho$ of $Q^m$ at a point $[z] \in Q^m$ we denote by $A = A_\rho$ the shape operator of $Q^m$ in $\mathbb{C}P^{m+1}$ with respect to $\rho$. The shape operator is an involution on the tangent space $T_{[z]}Q^m$ and

$$T_{[z]}Q^m = V(A_\rho) \oplus JV(A_\rho),$$

where $V(A_\rho)$ is the $(+1)$-eigenspace and $JV(A_\rho)$ is the $(-1)$-eigenspace of $A_\rho$. Geometrically this means that the shape operator $A_\rho$ defines a real structure on the complex vector space $T_{[z]}Q^m$, or equivalently, is a complex conjugation on $T_{[z]}Q^m$. Since the real codimension of $Q^m$ in $\mathbb{C}P^{m+1}$ is 2, this induces an $S^1$-subbundle $\mathfrak{A}$ of the endomorphism bundle $\text{End}(TQ^m)$ consisting of complex conjugations. There is a geometric interpretation of these conjugations. The complex quadric $Q^m$ can be viewed as the complexification of the $m$-dimensional sphere $S^m$. Through each point $[z] \in Q^m$ there exists a one-parameter family of real forms of $Q^m$ which are isometric to the sphere $S^m$. These real forms are congruent to each other under action of the center $SO_2$ of the isotropy subgroup of $SO_{m+2}$ at $[z]$. The isometric reflection of $Q^m$ in such a real form $S^m$ is an isometry, and the differential at $[z]$ of such a reflection is a conjugation on $T_{[z]}Q^m$. In this way the family $\mathfrak{A}$ of conjugations on $T_{[z]}Q^m$ corresponds to the family of real forms $S^m$ of $Q^m$ containing $[z]$, and the subspaces $V(A)$ in $T_{[z]}Q^m$ correspond to the tangent spaces $T_{[z]}S^m$ of the real forms $S^m$ of $Q^m$.

The Gauss equation for $Q^m \subset \mathbb{C}P^{m+1}$ implies that the Riemannian curvature tensor $\bar{R}$ of $Q^m$ can be described in terms of the complex structure $J$ and the complex conjugations $A \in \mathfrak{A}$:

$$\bar{R}(X, Y)Z = g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX - g(JX, Z)JY$$

$$- 2g(JX, Y)JZ + g(AY, Z)AX$$

$$- g(AX, Z)AY + g(JAY, Z)JAX - g(JAX, Z)JAY. \quad (2.1)$$

It is well known that for every unit tangent vector $U \in T_{[z]}Q^m$ there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $Z_1, Z_2 \in V(A)$ such that

$$U = \cos(t)Z_1 + \sin(t)JZ_2$$

for some $t \in [0, \pi/4]$ (see [22]). The singular tangent vectors correspond to the values $t = 0$ and $t = \pi/4$. If $0 < t < \pi/4$ then the unique maximal flat containing $U$ is $\mathbb{R}Z_1 \oplus \mathbb{R}JZ_2$. Later we will need the eigenvalues and eigenspaces of the Jacobi operator $\bar{R}_U = \bar{R}(\cdot, U)U$ for a singular unit tangent vector $U$.

1. If $U$ is an $\mathfrak{A}$-principal singular unit tangent vector with respect to $A \in \mathfrak{A}$, then the eigenvalues of $\bar{R}_U$ are 0 and 2 and the corresponding eigenspaces are $\mathbb{R}U \oplus J(V(A) \oplus \mathbb{R}U)$ and $(V(A) \oplus \mathbb{R}U) \oplus \mathbb{R}JU$, respectively.

2. If $U$ is an $\mathfrak{A}$-isotropic singular unit tangent vector with respect to $A \in \mathfrak{A}$ and $X, Y \in V(A)$, then the eigenvalues of $\bar{R}_U$ are 0, 1 and 4 and the corresponding eigenspaces are $\mathbb{R}U \oplus \mathbb{C}(JZ_1 + Z_2)$, $T_{[z]}Q^m \ominus (\mathbb{C}Z_1 \oplus \mathbb{C}Z_2)$ and $\mathbb{R}JU$, respectively.
3. Real hypersurfaces in $Q^m$

Let $M$ be a real hypersurface in $Q^m$ and denote by $(\phi, \xi, \eta, g)$ the induced almost contact metric structure. By using the Gauss and Wingarten formulas the left-hand side of (2.1) becomes

$$\bar{R}(X, Y)Z = R(X, Y)Z - g(SY, Z)SX + g(SX, Z)SY$$

$$+ \{ g((\nabla_X S)Y, Z) - g((\nabla_Y S)X, Z) \} N,$$

where $R$ and $S$ denote the Riemannian curvature tensor and the shape operator of $M$ in $Q^m$, respectively. Taking tangent and normal components of (2.1) respectively, we obtain

$$g(R(X, Y)Z, W) - g(SY, Z)g(SX, W) + g(SX, Z)g(SY, W)$$

$$= g(Y, Z)g(X, W) - g(X, Z)g(Y, W) + g(JY, Z)g(JX, W)$$

$$- g(JX, Z)g(JY, W) - 2g(JX, Y)g(JZ, W) + g(AY, Z)g(AX, W)$$

$$- g(AX, Z)g(AY, W) + g(JAY, Z)g(JAX, W) - g(JAX, Z)g(JAY, W),$$

and

$$g((\nabla_X S)Y, Z) - g((\nabla_Y S)X, Z)$$

$$= \eta(X)g(JY, Z) - \eta(Y)g(JX, Z) - 2\eta(Z)g(JX, Y)$$

$$+ g(JY, Z)g(AX, N) - g(AX, Z)g(AY, N)$$

$$+ \eta(AX)g(JAY, Z) - \eta(AY)g(JAX, Z)$$

where $X, Y, Z$ and $W$ are tangent vector fields of $M$.

Note that $JX = \phi X + \eta(X) N$ and $JN = -\xi$, where $\phi X$ is the tangential component of $JX$ and $N$ is a (local) unit normal vector field of $M$. The tangent bundle $TM$ of $M$ splits orthogonally into $TM = C \oplus \mathbb{R} \xi$, where $C = \ker \eta$ is the maximal complex subbundle of $TM$. The structure tensor field $\phi$ restricted to $C$ coincides with the complex structure $J$ restricted to $C$, and $\phi \xi = 0$. Moreover, since the complex quadric $Q^m$ has also a real structure $A$, we decompose $AX$ into its tangential and normal components for a fixed $A \in \mathfrak{a}_{[2]}$ and $X \in T_{[2]} M$:

$$AX = BX + \rho(X) N$$

(3.3)

where $BX$ is the tangential component of $AX$ and

$$\rho(X) = g(AX, N) = g(X, AN) = g(X, A\xi) = g(JX, A\xi).$$

From these notations, the equations (3.1) and (3.2) can be written as

$$R(X, Y)Z - g(SY, Z)SX + g(SX, Z)SY$$

$$= g(Y, Z)X - g(X, Z)Y + g(JY, Z)\phi X - g(JX, Z)\phi Y - 2g(JX, Y)\phi Z$$

$$+ g(JY, Z)BX - g(AX, Z)BY + g(JAY, Z)\phi BX$$

$$- g(JAY, Z)\rho(X)\xi - g(JAX, Z)\phi BY + g(JAX, Z)\rho(Y)\xi$$

and

$$(\nabla_X S)Y - (\nabla_Y S)X$$

$$= \eta(X)\phi Y - \eta(Y)\phi X - 2g(JX, Y)\xi$$

$$+ g(AX, N)BY - g(AY, N)BX + \eta(AX)\phi BY$$

$$- \eta(AX)\rho(Y)\xi - \eta(AY)\phi BX + \eta(AY)\rho(X)\xi,$$
which are called the equations of Gauss and Codazzi, respectively. Moreover, from (3.1) the Ricci tensor \( \text{Ric} \) of \( M \) is given by
\[
\text{Ric}X = (2m - 1)X - 3\eta(X)\xi + g(A\xi, \xi)BX - g(A\xi, N)\phi A\xi \\
+ g(AX, \xi)A\xi + hSX - S^2X,
\]
where \( h = \text{Tr}S \).

As mentioned in section 2 since the normal vector field \( N \) belongs to \( T_{[z]}Q^m, [z] \in M \), we can choose \( A \in \mathfrak{A}_{[z]} \) such that
\[
N = \cos(t)Z_1 + \sin(t)JZ_2
\]
for some orthonormal vectors \( Z_1, Z_2 \in V(A) \) and \( 0 \leq t \leq \frac{\pi}{4} \) (see Proposition 3 in [22]). Note that \( t \) is a function on \( M \). If \( t = 0 \), then \( N = Z_1 \in V(A) \), therefore we see that \( N \) becomes the \( \mathfrak{A} \)-principal singular tangent vector field. On the other hand, if \( t = \frac{\pi}{4} \), then \( N = \frac{1}{\sqrt{2}}(Z_1 + JZ_2) \). That is, \( N \) is to be the \( \mathfrak{A} \)-isotropic singular tangent vector field. In addition, since \( \xi = -JN \), we have
\[
\begin{align*}
\xi &= \sin(t)Z_2 - \cos(t)JZ_1, \\
AN &= \cos(t)Z_1 - \sin(t)JZ_2, \\
A\xi &= \sin(t)Z_2 + \cos(t)JZ_1.
\end{align*}
\]
This implies \( g(\xi, AN) = 0 \) and \( g(A\xi, \xi) = -g(AN, N) = -\cos(2t) \) on \( M \). At each point \([z] \in M \) we define the maximal \( \mathfrak{A} \)-invariant subspace of \( T_{[z]}M, [z] \in M \) as follows:
\[
\mathcal{Q}_{[z]} = \{ X \in T_{[z]}M \mid AX \in T_{[z]}M \text{ for all } A \in \mathfrak{A}_{[z]} \}.
\]
It is known that \( N_{[z]} \) is \( \mathfrak{A} \)-principal, then \( \mathcal{Q}_{[z]} = \mathcal{C}_{[z]} \) (see [20]).

We now assume that \( M \) is a Hopf hypersurface in the complex quadric \( Q^m \). Then the shape operator \( S \) of \( M \) in \( Q^m \) satisfies \( S\xi = \alpha \xi \) with the Reeb function \( \alpha = g(S\xi, \xi) \) on \( M \). By virtue of the Codazzi equation, we obtain the following lemma.

**Lemma 3.1** ([31]). Let \( M \) be a Hopf hypersurface in \( Q^m, m \geq 3 \). Then we obtain
\[
X\alpha = (\xi\alpha)\eta(X) + 2g(A\xi, \xi)g(X, AN) \quad (3.6)
\]
and
\[
2g(S\phi SX, Y) - \alpha g((\phi S + S\phi)X, Y) - 2g(\phi X, Y) \\
+ g(X, AN)g(Y, A\xi) - g(Y, AN)g(X, A\xi) \\
- g(X, A\xi)g(JY, A\xi) + g(Y, A\xi)g(JX, A\xi) \\
- 2g(X, AN)g(\xi, A\xi)\eta(Y) + 2g(Y, AN)g(\xi, A\xi)\eta(X) = 0 \quad (3.7)
\]
for any tangent vector fields \( X \) and \( Y \) on \( M \).

**Remark 3.2.** By virtue of (3.6) we know if \( M \) has vanishing geodesic Reeb flow (or constant Reeb curvature, respectively), then the normal vector \( N \) is singular. In fact, under this assumption (3.6) becomes \( g(A\xi, \xi)g(X, AN) = 0 \) for any tangent vector field \( X \) on \( M \). Since \( g(A\xi, \xi) = -\cos(2t) \), the case of \( g(A\xi, \xi) = 0 \) implies that \( N \) is \( \mathfrak{A} \)-isotropic. Besides, if \( g(A\xi, \xi) \neq 0 \), that is, \( g(AN, X) = 0 \) for all \( X \in TM \), then
\[
AN = \sum_{i=1}^{2m} g(AN, e_i)e_i + g(AN, N)N = g(AN, N)N,
\]
which implies that \( N = A^2 N = g(AN, N)AN \). Taking an inner product with \( N \), it follows \( g(AN, N) = \pm 1 \). Since \( g(AN, N) = \cos(2t) \) where \( t \in [0, \frac{\pi}{4}) \), we obtain \( AN = N \). Hence \( N \) should be \( \mathfrak{A} \)-principal.

**Lemma 3.3** (26). Let \( M \) be a Hopf hypersurface in \( Q^m \) such that the normal vector field \( N \) is \( \mathfrak{A} \)-principal everywhere. Then \( \alpha \) is constant. Moreover, if \( X \in \mathcal{C} \) is a principal curvature vector of \( M \) with principal curvature \( \lambda \), then \( 2\lambda \neq \alpha \) and its corresponding vector \( \phi X \) is a principal curvature vector of \( M \) with principal curvature \( \frac{\alpha \lambda + 2}{2\lambda - \alpha} \).

**Lemma 3.4** (26). Let \( M \) be a Hopf hypersurface in \( Q^m \), \( m \geq 3 \), such that the normal vector field \( N \) is \( \mathfrak{A} \)-isotropic everywhere. Then \( \alpha \) is constant.

If the normal vector \( N \) is \( \mathfrak{A} \)-isotropic, then we obtain

\[
g(A\xi, N) = g(A\xi, \xi) = g(AN, N) = 0
\]

from (3.5) and the notation of \( N \). Taking the covariant derivative of \( g(AN, N) = 0 \) along the direction of any \( X \in T_{[z]}M, [z] \in M \), it becomes

\[
0 = X(g(AN, N)) = g(\nabla_X(AN), N) + g(AN, \nabla_X N)
= g((\nabla_X A)N + A(\nabla_X N), N) + g(AN, \nabla_X N)
= g(q(X)JAN - ASX, N) - g(AN, SX)
= -2g(ASX, N),
\]

where we have used the covariant derivative of the complex structure \( A \), that is, \((\nabla_X A)Y = q(X)JAY \) and the formula of Weingarten. Then the above formula gives \( SAN = 0 \), because \( AN \) becomes a tangent vector field on \( M \) for \( \mathfrak{A} \)-isotropic unit normal vector field \( N \).

On the other hand, by differentiating \( g(A\xi, N) = 0 \) and using the formula of Gauss, we have:

\[
0 = g(\nabla_X (A\xi), N) + g(A\xi, \nabla_X N)
= g((\nabla_X A)\xi + A(\nabla_X \xi), N) + g(A\xi, \nabla_X N)
= g((\nabla_X A)\xi, N) + g(\nabla_X \xi + \sigma(X, \xi), AN) + g(A\xi, \nabla_X N)
= g(q(X)JA\xi, N) + g(\phi SX + g(SX, \xi)N, AN) - g(A\xi, SX)
= -2g(A\xi, SX),
\]

where \( \sigma \) is the second fundamental form of \( M \) and \( \phi AN = JAN = -AJN = A\xi \). By \( g(A\xi, N) = 0 \), the vector field \( A\xi \) becomes a tangent vector field on \( M \) with \( \mathfrak{A} \)-isotropic unit normal vector field \( N \). Then the above formula gives \( SA\xi = 0 \).

Moreover, when the normal vector \( N \) is \( \mathfrak{A} \)-isotropic, the tangent vector space \( T_{[z]}M, [z] \in M \), is decomposed

\[
T_{[z]}M = [\xi] \oplus [A\xi, AN] \oplus \mathcal{Q},
\]

where \( \mathcal{C} \oplus \mathcal{Q} = \mathcal{Q}^\perp = \text{Span}[A\xi, AN] \). From the equation (3.7), we obtain

\[
(2\lambda - \alpha)S\phi X = (\alpha \lambda + 2)\phi X
\]

for some principal curvature vector \( X \in \mathcal{Q} \subset T_{[z]}M \) such that \( SX = \lambda X \). If \( 2\lambda - \alpha = 0 \) (i.e. \( \lambda = \frac{\alpha}{2} \)), then \( \alpha \lambda + 2 = \frac{\alpha^2 + 4}{2} = 0 \), which makes a contradiction. Hence we obtain:
Lemma 3.5. Let $M$ be a Hopf hypersurface in $Q^m$ such that the normal vector field $N$ is $\mathfrak{X}$-isotropic. Then $SA\xi = 0$ and $SAN = 0$. Moreover, if $X \in Q$ is a principal curvature vector of $M$ with principal curvature $\lambda$, then $2\lambda \neq \alpha$ and its corresponding vector $\phi X$ is a principal curvature vector of $M$ with principal curvature $\frac{\alpha + 2}{\lambda - \alpha}$.

On the other hand, from the property of $g(A\xi, N) = 0$ on a real hypersurface $M$ in $Q^m$ we see that the non-zero vector field $A\xi$ is tangent to $M$. Hence by Gauss formula it induces

$$\nabla_X(A\xi) = \nabla_X(A\xi) - \sigma(X, A\xi)$$

for any $X \in TM$. From $AN = AJ\xi = -JA\xi$ and $JA\xi = \phi A\xi + \eta(A\xi)N$, it gives us

\[
\begin{aligned}
\text{Tangent Part : } \nabla_X(A\xi) &= q(X)\phi A\xi + B\phi S\xi \quad - g(S\xi, A\xi)N \\
\text{Normal Part : } q(X)g(A\xi, \xi) &= -g(AN, \nabla_X\xi) + g(S\xi, A\xi)g(A\xi, \xi) + g(S\xi, A\xi)
\end{aligned}
\]

In particular, if $M$ is Hopf, then the second equation in (3.8) becomes

$$g(\xi)g(A\xi, \xi) = 2\alpha g(A\xi, \xi).$$

(3.9)

4. Proof of Theorem 1

- Reeb parallel structure Jacobi operator with $\mathfrak{X}$-principal normal -

Let us $M$ be a real hypersurface in the complex quadric $Q^m$, $m \geq 3$, with Reeb parallel structure Jacobi operator, that is,

$$(\nabla_\xi R_\xi)Y = 0$$

(*)

for all tangent vector fields $Y$ of $M$.

As mentioned in section 1 the structure Jacobi operator $R_\xi \in \text{End}(TM)$ with respect to the unit tangent vector field $\xi \in TM$ is induced from the curvature tensor $R$ of $M$ given in section 3 as follows: for any tangent vector fields $Y$, $Z \in TM$

$$g(R_\xi Y, Z) = g(R(Y, \xi)\xi, Z)$$

$$= g(Y, Z) - \eta(Y)\eta(Z) + g(A\xi, \xi)g(AY, Z) - g(Y, A\xi)g(A\xi, Z)$$

$$- g(AY, N)g(AN, Z) + \alpha g(SY, Z) - \alpha^2 \eta(Y)\eta(Z),$$

where we have used $J\xi = N$, $JA = -AJ$, and $g(A\xi, N) = 0$.

Remark. For any tangent vector field $X$ of $M$ the vector field $AX$ belongs to $TQ^m$, that is, $AX = BX + \rho(X)N \in TM \oplus (TM)^\perp = TQ^m$. Therefore, from 14 the structure Jacobi operator of $M$ is given by

$$R_\xi Y = Y - \eta(Y)\xi + g(A\xi, \xi)BY - g(A\xi, Y)A\xi$$

$$- g(\phi A\xi, Y)\phi A\xi + \alpha SY - \alpha^2 \eta(Y)\xi.$$  

(4.1)

Here we have used that $A\xi = B\xi \in TM$ (i.e. $\rho(\xi) = g(AN, \xi) = 0$) and $AN = AJ\xi = -JA\xi = -\phi A\xi - \eta(A\xi)N$. 


Taking the covariant derivative of (4.1) along the direction of $X \in TM$, then we have

$$(\nabla_X R_\xi)Y = -g(Y, \nabla_X \xi)\eta(Y) \nabla_X \xi + g(\nabla_X (A\xi), \xi)BY + g(A\xi, \nabla_X \xi)BY$$

$$+ g(A\xi, \xi)(\nabla_X BY) - g(\nabla_X (A\xi), Y)A\xi - g(A\xi, Y)\nabla_X (A\xi)$$

$$- g((\nabla_X \phi)A\xi, Y)\phi A\xi + g(\nabla_X (A\xi), \phi Y)\phi A\xi$$

$$- g(\phi A\xi, Y)(\nabla_X \phi)A\xi - g(\phi A\xi, Y)\phi(\nabla_X (A\xi))$$

$$+ (X\alpha)SY + \alpha(\nabla_X S)Y - 2\alpha(X\alpha)\eta(Y)\xi - \alpha^2 g(Y, \nabla_X \xi)\xi - \alpha^2 \eta(Y) \nabla_X \xi$$

$$= -g(Y, \phi SX)\xi - \eta(Y)\phi SX + g(\phi SX, \xi)BY + g(A\xi, \phi SX)BY$$

$$+ g(A\xi, \xi)\left\{ q(X)JAY + g(SX, Y)AN - q(X)g(AY, \xi)N \right\}$$

$$+ g(A\xi, \xi)\left\{ g(SX, Y)g(A\xi, \xi)N + g(AN, Y)SX \right\}$$

$$- \left\{ (q(X) - \alpha \eta(X))g(\phi A\xi, Y) + g(\phi SX, Y) \right\} A\xi$$

$$- g(A\xi, Y)\left\{ (q(X) - \alpha \eta(X))\phi A\xi + B\phi SX \right\}$$

$$- \left\{ g(A\xi, \xi)g(SX, Y) - g(SX, A\xi)\eta(Y) \right\} \phi A\xi + (q(X) - \alpha \eta(X))g(A\xi, Y)\phi A\xi$$

$$- \left\{ (q(X) - \alpha \eta(X))g(A\xi, \xi)\eta(Y) - g(B\phi SX, \phi Y) \right\} \phi A\xi$$

$$- g(\phi A\xi, Y)\left\{ \eta(A\xi)SX - g(SX, A\xi)\xi \right\}$$

$$+ g(\phi A\xi, Y)\left\{ (q(X) - \alpha \eta(X))A\xi - g(A\xi, \xi)\xi - \phi B\phi SX \right\}$$

$$+ (X\alpha)SY + \alpha(\nabla_X S)Y - 2\alpha(X\alpha)\eta(Y)\xi - \alpha^2 g(Y, \phi SX)\xi - \alpha^2 \eta(Y)\phi SX,$$

where we have used (3.8) and

$$(\nabla_X B)Y = \nabla_X (BY) - B(\nabla_X Y)$$

$$= \tilde{\nabla}_X (BY) - \sigma(X, BY) - B(\nabla_X Y)$$

$$= \tilde{\nabla}_X (AY - g(AY, N)N) - g(SX, BY)N - B(\nabla_X Y)$$

$$= (\nabla_X A)Y + A(\tilde{\nabla}_X Y) - g(\nabla_X (AY), N)N - g(AY, \tilde{\nabla}_X N)N$$

$$- g(AY, N)\nabla_X N - g(SX, BY)N - B(\nabla_X Y)$$

$$= q(X)JAY + A(\nabla_X Y) + g(SX, Y)AN - q(X)g(JAY, N)N$$

$$- g(\nabla_X Y, AN)N - g(SX, Y)g(AN, N)N$$

$$+ g(AY, N)SX + g(AY, N)SX - g(SX, BY)N - B(\nabla_X Y)$$

$$= q(X)JAY + g(SX, Y)AN - q(X)g(AY, \xi)N$$

$$+ g(SX, Y)g(A\xi, \xi)N + g(AY, N)SX.$$ 

Since $M$ is a Hopf real hypersurface in $Q^m$ with Reeb parallel structure Jacobi operator, it yields that

$$g(A\xi, \xi)\left\{ q(\xi)JAY + \alpha \eta(Y)AN - q(\xi)g(AY, \xi)N \right\} \quad (4.2)$$
\[ g(A\xi, \xi) \{ \alpha \eta(Y)g(A\xi, \xi)N + \alpha g(AN, Y)\xi \} \]
\[ - (q(\xi) - \alpha)g(A\xi, \xi)\eta(Y)\phi A\xi - g(\phi A\xi, Y)g(\xi, A\xi)(q(\xi) - \alpha)\xi \]
\[ + (\xi \alpha)SY + \alpha(\nabla_\xi S)Y - 2\alpha(\xi \alpha)\eta(Y)\xi = 0. \]

From now on, we assume that \( M \) is a Hopf real hypersurface with non-vanishing geodesic Reeb flow and with Reeb parallel structure Jacobi operator in the complex quadric \( Q^m \), \( m \geq 3 \). In addition, we suppose that the normal vector field \( N \) of \( M \) is \( A \)-principal. Then this assumption gives us

\[ AN = N \quad \text{and} \quad A\xi = -\xi \]

from (3.5). So it follows that \( AY \in TM \) for all \( Y \in TM \), that is, \( g(AY, N) = g(Y, AN) = 0 \). Moreover, taking the derivative to \( AN = N \) with respect to the Levi-Civita connection \( \bar{\nabla} \) of \( Q^m \) and using (3.8), we get

\[ ASY = SY - 2\alpha(Y)\xi, \quad (4.3) \]

together with \( (\nabla_Y A)X = q(Y)JAX \) and \( \nabla_Y N = -SY \).

From these properties, the equation (4.2) can be rearranged as follows.

\[ 0 = (\nabla_\xi R_\xi)Y \]
\[ = -q(\xi)JAY - q(\xi)\eta(Y)N + (\xi \alpha)SY + \alpha(\nabla_\xi S)Y - 2\alpha(\xi \alpha)\eta(Y)\xi \quad (4.4) \]

In addition, from (3.9) we know \( q(\xi) = 2\alpha \). By Lemma 3.3 and our assumption, the Reeb curvature function \( \alpha \) is non-zero constant on \( M \). So (4.4) reduces to the following

\[ (\nabla_\xi S)Y = 2\phi AY, \quad (4.5) \]

together with \( JAY = \phi AY + \eta(AY)N = \phi AY - \eta(Y)N \).

On the other hand, by using the equation of Codazzi in section (3.6), we have

\[ g((\nabla_\xi S)Y - (\nabla_Y S)\xi, Z) = g(\phi Y, Z) - g(AY, N)g(A\xi, Z) \]
\[ + g(\xi, A\xi)g(JAY, Z) + g(\xi, AY)g(AN, Z) \]
\[ = g(\phi Y, Z) - g(\phi AY, Z). \]

Since \( M \) is Hopf and Lemma 3.3, it leads to

\[ (\nabla_\xi S)Y = (\nabla_Y S)\xi + \phi Y - \phi AY \]
\[ = \alpha \phi SY - S\phi SY + \phi Y - \phi AY. \]

From this, together with (4.5), it follows that

\[ \alpha \phi SY - S\phi SY + \phi Y = 3\phi AY. \quad (4.6) \]

By virtue of Lemma 3.1 for the \( \mathfrak{A} \)-principal unit normal vector field, we obtain

\[ 2S\phi SY = \alpha(S\phi + \phi S)Y + 2\phi Y. \quad (4.7) \]

Therefore, (4.6) can be written as

\[ \alpha(\phi S - S\phi)Y = 6\phi AY. \quad (4.8) \]

Inserting \( Y = SX \) for \( X \in \mathfrak{C} \) into (4.8) and taking the structure tensor \( \phi \) leads to

\[ \alpha S^2X + \alpha \phi S\phi SX = 6ASX, \]
where \( C = \ker \eta \) denotes the maximal complex subbundle of \( TM \), which is defined by a distribution \( C = \{ X \in T[z]M \mid \eta(X) = 0 \} \) in \( T[z]M \), \( z \in M \). By using (4.3) and (4.7) this equation gives us

\[
\alpha^2 \phi S \phi X = -2\alpha S^2 X + \alpha^2 SX + 2\alpha X + 12SX
\]

(4.9)

for all \( X \in C \).

On the other hand, in this subsection we have assumed that the normal vector field \( N \) of \( M \) is \( \mathfrak{A} \)-principal. It follows that \( AY \in TM \) for all \( Y \in TM \). From this, the anti-commuting property with respect to \( J \) and \( A \) implies \( \phi AX = -A\phi X \). Hence (4.8) can be expressed as

\[
\alpha(\phi S - S\phi)Y = -6A\phi Y.
\]

(4.10)

Putting \( Y = \phi X \) into (4.10), it gives

\[
\alpha \phi S \phi X = -\alpha SX + 6AX
\]

for all \( X \in C \). Inserting this into (4.9) gives

\[
3\alpha AX + \alpha S^2X - \alpha^2 SX - \alpha X - 6SX = 0.
\]

(4.11)

Taking the complex conjugate \( A \) to (4.11) and using (4.3) again, we get

\[
3\alpha X + \alpha S^2X - \alpha^2 SX - \alpha AX - 6SX = 0,
\]

(4.12)

for all \( X \in C \). Summing up (4.11) and (4.12), gives \( AX = X \) for all \( X \in C \). This gives a contradiction. In fact, it is well known that the trace of the real structure \( A \) is zero, that is, \( \text{Tr}A = 0 \) (see Lemma 1 in [25]). For an orthonormal basis \( \{ e_1, e_2, \ldots, e_{2m-2}, e_{2m-1} = \xi, e_{2m} = N \} \) for \( TQ^m \), where \( e_j \in C \) \( (j = 1, 2, \ldots, 2m - 2) \), the trace of \( A \) is given by

\[
\text{Tr}A = \sum_{i=1}^{2m} g(Ae_i, e_i)
\]

\[
= g(AN, N) + g(A\xi, \xi) + \sum_{i=1}^{2m-2} g(Ae_i, e_i)
\]

\[
= 2m - 2.
\]

It implies that \( m = 1 \). But we now consider for the case \( m \geq 3 \).

Consequently, this completes the proof that there does not exists a Hopf real hypersurface \( (\alpha \neq 0) \) in complex quadrics \( Q^m \), \( m \geq 3 \), with Reeb parallel structure Jacobi operator and \( \mathfrak{A} \)-principal normal vector field.

5. Proof of Theorem 2
- Reeb parallel structure Jacobi operator with \( \mathfrak{A} \)-isotropic normal -

In this section, we assume that the unit normal vector field \( N \) is \( \mathfrak{A} \)-isotropic and \( M \) is a real hypersurface in complex quadric \( Q^m \) with non-vanishing geodesic Reeb flow and with Reeb parallel structure Jacobi operator. Then the normal vector field \( N \) can be written as

\[
N = \frac{1}{\sqrt{2}}(Z_1 + JZ_2)
\]
for some orthonormal vectors $Z_1, Z_2 \in V(A)$, where $V(A)$ denotes a $(+1)$-eigenspace of the complex conjugation $A \in \mathfrak{a}$. Then it follows that

$$AN = \frac{1}{\sqrt{2}}(Z_1 - JZ_2), \quad AJN = -\frac{1}{\sqrt{2}}(JZ_1 + Z_2), \quad JN = \frac{1}{\sqrt{2}}(JZ_1 - Z_2).$$

Then it gives that

$$g(\xi, A\xi) = g(JN, AJN) = 0, \quad g(\xi, AN) = 0 \quad \text{and} \quad g(AN, N) = 0,$$

which means that both vector fields $AN$ and $A\xi$ are tangent to $M$. From this and Lemma $3.4$, we see that the shape operator $S$ of $M$ becomes to be Reeb parallel, that is, $(\nabla_{\xi} S)Y = 0$ for all tangent vector field $Y$ on $M$.

On the other hand, from the Codazzi equation (3.2) we obtain

$$(\nabla_{\xi} S)Y = (\nabla_Y S)\xi + \phi Y - g(AY, N)A\xi + g(A\xi, Y)AN$$

$$= (Y \alpha)\xi + \alpha \phi SY - S\phi SY + \phi Y + g(A\xi, Y)AN - g(AN, Y)A\xi$$

$$= \frac{\alpha}{2}(\phi S - S\phi)Y,$$

where the third equality holds from Lemmas $3.1$ and $3.4$. From this and $M$ has non-vanishing geodesic Reeb flow, we see that $M$ has isometric Reeb flow, that is, $S\phi = \phi S$.

Consequently, we obtain:

**Proposition 5.1.** Let $M$ be a real hypersurface with non-vanishing geodesic Reeb flow in the complex quadrics $Q^m$, $m \geq 3$. If the unit normal vector field $N$ of $M$ is $\mathfrak{a}$-isotropic and the structure Jacobi operator $R_{\xi}$ of $M$ is Reeb parallel, then the shape operator $S$ of $M$ satisfies the property of Reeb parallelism. Moreover, it means that the Reeb flow of $M$ is isometric.

**Theorem B.** Let $M$ be a real hypersurface of the complex quadric $Q^m$, $m \geq 3$. The Reeb flow on $M$ is isometric if and only if $m$ is even, say $m = 2k$, and $M$ is an open part of a tube around a totally geodesic $\mathbb{C}P^k$ in $Q^{2k}$.

Then by virtue of Theorem B, we assert: if $M$ is a real hypersurface in $Q^m$, $m \geq 3$, with the assumptions given in Proposition $5.1$, then $M$ is an open part of $(\mathcal{T}_A)$. Here the model space $(\mathcal{T}_A)$ is a tube over a totally geodesic complex projective space $\mathbb{C}P^k$ in $Q^{2k}$, $m = 2k$.

From now on, let us check the converse problem, that is, the model space $(\mathcal{T}_A)$ satisfies the all assumptions stated in Proposition $5.1$. In order to do this, we first introduce one proposition given in [26].

**Proposition A.** Let $(\mathcal{T}_A)$ be the tube of radius $0 < r < \frac{\pi}{2}$ around the totally geodesic $\mathbb{C}P^k$ in $Q^{2k}$. Then the following statements hold:

(i) $(\mathcal{T}_A)$ is a Hopf hypersurface.

(ii) Every unit normal vector $N$ of $(\mathcal{T}_A)$ is $\mathfrak{a}$-isotropic and therefore can be written in the form $N = (Z_1 + JZ_2)/\sqrt{2}$ with some orthonormal vectors $Z_1, Z_2 \in V(A)$ and $A \in \mathfrak{a}$.

(iii) $(\mathcal{T}_A)$ has four distinct constant principal curvatures and the property that the shape operator leaves invariant the maximal complex subbundle $\mathcal{C}$ of $T(\mathcal{T}_A)$ are $J$-invariant. The principal curvatures and corresponding principal curvature spaces of $(\mathcal{T}_A)$ are as follows.
| principal curvature | eigenspace       | multiplicity |
|---------------------|------------------|--------------|
| 0                   | $\mathbb{C}(JZ_1 + Z_2)$ | 2            |
| $-\tan(r)$          | $W_1$            | $2k - 2$     |
| $\cot(r)$           | $W_2$            | $2k - 2$     |
| $2 \cot(2r)$        | $RJN$            | 1            |

(iv) Each of the two focal sets of $(T_A)$ is a totally geodesic $\mathbb{C}P^k \subset Q^{2k}$.
(v) $S\dot{\phi} = \phi S$ (isometric Reeb flow).
(vi) $(T_A)$ is a homogeneous hypersurface of $Q^{2k}$. More precisely, it is an orbit of the $U_{k+1}$-action on $Q^{2k}$ isomorphic to $U_{k+1}/U_{k-1}U_1$, an $S^{2k-1}$-bundle over $\mathbb{C}P^k$.

By virtue of (i) and (ii) in Proposition A, $(T_A)$ is a Hopf real hypersurface with \(\mathfrak{A}\)-isotropic normal vector $N$ in $Q^m$. Moreover, the structure Jacobi operator $R_\xi$ of $(T_A)$ should be Reeb parallel, because of $\alpha = 2 \cot(2r) \neq 0$, $0 < r < \frac{\pi}{2}$.

References

[1] A. L. Besse, *Einstein Manifolds*, Springer-Verlag, 2008.
[2] J. Berndt and Y.J. Suh, *Real hypersurfaces with isometric Reeb flow in complex two-plane Grassmannians*, Monatsh. Math. 137 (2002), 87–98.
[3] J. Berndt and Y.J. Suh, *Real hypersurfaces with isometric Reeb flow in complex quadrics*, Internat. J. Math. 24 (2013), 1350050, 18 pp.
[4] J. Berndt and Y.J. Suh, *Contact hypersurfaces in Kähler manifolds*, Proceedings of the American Math. Soc. 143 (2015), 2637-2649.
[5] P.B. Eberlein, *Geometry of nonpositively curved manifolds*, University of Chicago Press, Chicago, IL, 1996.
[6] S. Helgason, *Differential geometry, Lie groups and symmetric spaces*, Graduate Studies in Math., Amer. Math. Soc., 34, 2001.
[7] M. Hutching and C.H. Taubes, *The Weinstein conjecture for stable Hamiltonian structures*, Geom. Topol. 13 (2009), 901–941.
[8] I. Jeong, H. Lee and Y.J. Suh, *Real hypersurfaces in complex two-plane Grassmannians whose structure Jacobi operator is of Codazzi type*, Acta Math. Hungar. 125 (2009), 141–160.
[9] I. Jeong, Y.J. Suh and C. Woo, *Real hypersurfaces in complex two-plane Grassmannians with recurrent structure Jacobi operator*, Real and Complex Submanifolds, Springer Proc. Math. Stat. 106, Springer, Tokyo, 2014, 267–278.
[10] U-H. Ki, J.D. Pérez, F.G. Santos and Y.J. Suh, *Real hypersurfaces in complex space forms with $\xi$-parallel Ricci tensor and structure Jacobi operator*, J. Korean Math. Soc. 44 (2007), 307–326.
[11] S. Klein, *Totally geodesic submanifolds in the complex quadric*, Differential Geom. Appl. 26 (2008), 79-96.
[12] A. W. Knapp, *Lie Groups Beyond an Introduction (2nd Ed.)*, Progress in Mathematics, Birkhäuser Boston, 2002.
[13] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry, Vol. I*, Reprint of the 1963 original, Wiley Classics Library, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1996.
[14] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry, Vol. II*, A Wiley-Interscience Publication, John Wiley & Sons Inc., New York, 1996.
[15] H. Lee and Y.J. Suh, *Real hypersurfaces with recurrent normal Jacobi operator in the complex quadric* (accepted in J. Geom. Phys.).
[16] S. Montiel and A. Romero, *On some real hypersurfaces of a complex hyperbolic space*, Geom. Dedicata 20 (1986), 245–261.
[17] M. Okumura, *On some real hypersurfaces of a complex projective space*, Trans. Amer. Math. Soc. 212 (1975), 355–364.
[18] J.D. Pérez, *Commutativity of Cho and structure Jacobi operators of a real hypersurface in a complex projective space*, Ann. Mat. Pura Appl. 194 (2015), no. 6, 1781-1794.

[19] J.D. Pérez, I. Jeong and Y.J. Suh, *Real hypersurfaces in complex two-plane Grassmannian with parallel structure Jacobi operator*, Acta. Math. Hungar. 22 (2009), 173-186.

[20] J.D. Pérez and F.G. Santos, *Real hypersurfaces in complex projective space with recurrent structure Jacobi operator*, Differential Geom. Appl. 26 (2008), 218-223.

[21] J.D. Pérez, F.G. Santos and Y.J. Suh, *Real hypersurfaces in complex projective space whose structure Jacobi operator is Lie ξ-parallel*, Diff. Geom. Appl. 22 (2005), 181-188.

[22] H. Reckziegel, *On the geometry of the complex quadric*, Geometry and topology of submanifolds, VIII (Brussels, 1995/Nordfjordeid, 1995), 302-315, World Sci. Publ., River Edge, NJ, 1996.

[23] A. Romero, *Some examples of indefinite complete complex Einstein hypersurfaces not locally symmetric*, Proc. Amer. Math. Soc. 98 (1986), 283-286.

[24] A. Romero, *On a certain class of complex Einstein hypersurfaces in indefinite complex space forms*, Math. Z. 192 (1986), 627-635.

[25] B. Smyth, *Differential geometry of complex hypersurfaces*, Ann. of Math. 85 (1967), 246-266.

[26] Y.J. Suh, *Real hypersurfaces in the complex quadric with Reeb parallel shape operator*, Internat. J. Math. 25 (2014), no. 6, 1450059 (17pages).

[27] Y.J. Suh, *Real hypersurfaces in the complex quadric with Reeb invariant shape operator*, Differential Geom. Appl. 38 (2015), 10-21.

[28] Y.J. Suh, *Real hypersurfaces in the complex quadric with parallel Ricci tensor*, Adv. in Math. 281 (2015), 886-905.

[29] Y.J. Suh, *Real hypersurfaces in the complex quadric with harmonic curvature*, J. Math. Pures Appl. 106 (2016), no. 3, 393-410.

[30] Y.J. Suh, *Real hypersurfaces in the complex quadric with parallel normal Jacobi operator*, Math. Nachr. 290 (2017), 442-451.

[31] Y.J. Suh, *Real hypersurfaces in the complex quadric with parallel structure Jacobi operator*, Differential Geom. Appl. 51 (2017), 33-48.

[32] Y.J. Suh and D.H. Hwang, *Real hypersurfaces in the complex quadric with commuting Ricci tensor*, Sci. China Math. 59 (2016), no. 11, 2185-2198.

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