GLOBAL FIELD DYNAMICS AND COSMOLOGICAL STRUCTURE
FORMATION

Ruth Durrer

Institut für Theoretische Physik, Universität Zürich,
Winterthurerstr. 190, CH-8057 Zürich, SWITZERLAND

Abstract. In this contribution we discuss gravitational effects of global scalar fields and, especially, of global topological defects. We first give an introduction to the dynamics of global fields and the formation of defects. Next we investigate the induced gravitational fields, first in a flat background and then in the expanding universe. In flat space, we explicitly calculate the gravitational fields of exact global monopole and global texture solutions and discuss the motion of photons and massive particles in these geometries. We also show that slowly moving particles and the energy of photons are not affected in static scalar field configurations with vanishing potential energy. In expanding space, we explore the possibility that global topological defects from a phase transition in the very early universe may have seeded inhomogeneities in the energy distribution which yielded the observed large scale structure in the Universe, the sheets of galaxies, clusters, voids ... . We outline numerical simulations which have been performed to tackle this problem and briefly discuss their results.

1e-mail: durrer@physik.unizh.ch
1. INTRODUCTION

During this meeting we have learned about phase transitions, the formation of topological defects during phase transitions, and the dynamics which usually leads to certain scaling laws for the density of defects and their correlation length. In most of the previous talks gravitation has been disregarded; it was unimportant for the examples under consideration. In these lectures we want to discuss the gravitational interaction of global defects with matter and radiation.

As we shall see, the gravitational coupling strength of topological defects is of the order of

\[ \mu = G T_c^2 = \left( T_c / m_{pl} \right)^2, \]

where \( T_c \) is the symmetry breaking temperature and \( m_{pl} = 1/\sqrt{G} \sim 10^{19} GeV \sim 10^{82} K \) is the Planck mass \( (\hbar = c = k_{Boltzmann} = 1 \) throughout). For gravitation to become important, the symmetry breaking scale thus cannot be too far below the Planck scale. In the electroweak phase transition, e.g., with \( T_c \sim 100 GeV, \mu \sim 10^{-34} \) and gravity can be ignored completely.

We shall see later, that topological defects might be responsible for cosmological structure formation, if they form during a phase transition at \( T_c \sim 10^{16} GeV \), \( \mu \sim 10^{-6} \). This energy coincides roughly with GUT scale. Certainly, this energy scale can never be probed directly by accelerators or any present day astrophysical events like supernovae. There are thus justified doubts if we will ever have a detailed picture of the physics taking place at these energies. On the other hand, if the ideas explored here turn out to be correct, and topological defects due to a phase transition in the very early universe have triggered cosmological structure formation, then the galaxy distribution in the universe and the anisotropies in the cosmic microwave background (CMB) may be relics of physics at GUT scales!

Since the generation and evolution of defects is quite a generic feature, we hope that the main results will not depend very sensitively on the detailed physical model. In that sense, I think, the scenarios discussed later in these lectures should be regarded as a kind of toy models which we hope are capable of capturing the main features, but we should not expect them to make predictions / verify observations to much better than within a factor of two.

In the next section we introduce some generalities on the dynamics of global fields and defect formation. We mention some important results from homotopy theory, present the \( \sigma \)-model approximation for the dynamics and discuss Derrick’s theorem. In Section 3, we study gravitational effects of global fields in flat spacetime. We calculate the gravitational influence of a global monopole and a global texture on matter and radiation. In Section 4, we investigate global defects in expanding space and especially the possibility that they may seed the formation of large scale structure in the universe. We shortly discuss cosmological perturbation theory, the Harrison Zel’dovich spectrum and numerical simulations of structure formation. We conclude in Section 5.
2. GLOBAL FIELD DYNAMICS AND DEFECT FORMATION

We consider a scalar field (order parameter) $\phi$ with Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_{\mu} \phi \cdot \partial^{\mu} \phi - V(\phi) ,$$

(2)

$\phi \in \mathcal{V}$, where $\mathcal{V}$ is a finite dimensional vector space and $\cdot$ denotes a scalar product in $\mathcal{V}$. Here "scalar field" does not refer to the number of components of the field $\phi$ but to the transformation of $\phi$ under rotations of physical space: Be $R$ a rotation, then $(D(R)\phi)(x, t) = \phi(R^{-1}x, t)$.

If we quantize the field $\phi$ at finite temperature, we can take into account the interactions of the $\phi$ particles with the thermal bath by replacing $V$ by an effective potential $V_T$. The precise form of $V_T$ depends on $V$ and on the interactions of $\phi$ with other particles, fermions, gauge bosons ... . In the simplest situation with only the scalar field $\phi \in \mathbb{R}^N$ and

$$V = \frac{1}{8} \lambda(\phi^2 - \eta^2)^2 ,$$

(3)

one finds in one loop approximation the high temperature corrections

$$V_T = \frac{1}{8} \lambda(\phi^2 - \eta^2)^2 - n\frac{\pi^2}{90} T^4 + \frac{\lambda(3\phi^2 - \eta^2)^2}{48} T^2 + \mathcal{O}(T) + V(T=0) ,$$

(4)

where $n$ denotes the number of helicity states of the $\phi$ field. At high temperatures, $T^2 \gg \eta^2$, the only minimum of $V_T$ is the field value $\phi = 0$. As the temperature drops below the critical temperature $T_c = 2\eta$, additional minima at $\langle \phi \rangle^2 = \eta^2(1 - T^2/T_c^2)$ develop and the vacuum manifold $\mathcal{N}_T$ (space of minima of $V_T$) becomes an $(N-1)$–sphere.

More generally, we assume $\mathcal{L}$ to be invariant under the action of some compact Lie group $G$ on $\mathcal{V}$, which leaves only $\phi = 0$ invariant. If the vacuum manifold consists only of the invariant element $\phi = 0$, the symmetry is unbroken. Since $V_T$ is temperature dependent, at some other temperature, the vacuum manifold, $\mathcal{N}_T$ may become non trivial and contain an element $\phi_o \neq 0$. Since $V_T$ is invariant under $G$, the whole orbit $\{g\phi_o | g \in G\}$ then belongs to $\mathcal{N}_T$. If the symmetry group $G$ is maximal, $\mathcal{N}_T$ consists just of the orbit of $\phi_o$, which is given by $G/H$, where $H \subset G$ denotes the invariance group of $\phi_o$, $H = \{h \in G | h\phi_o = \phi_o\}$. The symmetry $G$ is then spontaneously broken to the remaining symmetry group $H$.

Even though the opposite case can also occur [1], we shall generally assume that the symmetry is restored at high temperatures, $T > T_c$ and spontaneously broken at lower temperatures $T < T_c$. If the temperature then falls below the critical temperature, $T_c$, and if $\mathcal{N}_T$ is topologically non trivial, defects of dimension $d$ in spacetime can form via the Kibble mechanism [2]. The collection of the different topological defects possible in four spacetime dimensions is presented in table 1.

Since the reader is probably quite familiar with the appearance of domain walls, strings and monopoles, let me just briefly explain textures: We consider a
Table 1. Topological defects in four dimensional spacetime

| Homotopy $\pi_n$, dimension in spacetime= d=4-1-n | appearance |
|-----------------------------------------------|-------------|
| $\pi_0(\mathcal{N}) \neq 0$ N disconnected     | walls form  | $d=3$          |
| $\pi_1(\mathcal{N}) \neq 0$ N contains strings form | lines in space | $d=2$          |
| $\pi_2(\mathcal{N}) \neq 0$ N contains monopoles form | points in space | $d=1$          |
| $\pi_3(\mathcal{N}) \neq 0$ N contains textures form | events in spacetime | $d=0$          |

Field configuration, $\phi$ which is asymptotically constant (as it has to be if we require the field to have finite energy). At fixed time $t$, $\phi$ can then be regarded as map from compactified space $\mathbb{R}^3 = \mathbb{R}^3 \cup \{\infty\}$, with

$$\phi(\infty) = \lim_{|x| \to \infty} \phi(x).$$

Since $\mathbb{R}^3$ is topologically equivalent to $S^3$, we can now regard a vacuum configuration, $\phi$, as a map from $S^3$ into $\mathcal{N}$. If the image, $\phi(S^3)$ (which is of course topologically again $S^3$) is not contractable in $\mathcal{N}$, the configuration cannot evolve into the trivial one, $\phi = \text{constant}$, without leaving the vacuum manifold. Such a configuration is called texture. If $\phi$ has finite energy, Derricks theorem tells us that it will shrink and eventually evolve into the trivial configuration, leaving the vacuum manifold at some spacetime event (with extension of the order the inverse symmetry breaking scale). This event is the texture singularity.

We now want to state a few facts from homotopy theory which are commonly used throughout. Proofs and further information can be found in [3, 4, 5].

In contrast to homology groups, there exists no general algorithm to calculate homotopy groups $\pi_n$ for $n > 1$. Although a lot of research has been carried out, not even all homotopy groups of the sphere are known!

The following results from homotopy theory of Lie groups are often useful: Since every compact connected Lie group is a product of some of the following groups:

$$U(n), SO(n), SU(n), Sp(n), Spin(n), G_2, F_4, E_6, E_7 \text{ or } E_8,$$

it is sufficient to discuss these groups. Here $Sp(n)$ denotes the symplectic group of dimension $2n$, $Spin(n)$ denotes the spin group of dimension $n$, i.e., the universal covering group of $SO(n)$ and $G_2 ... E_8$ are the exception groups. Be $G$ one of the above groups, then

$$\pi_1(U(n)) = \mathbb{Z}, \quad \pi_1(SO(n)) = \mathbb{Z}_2 \quad \text{and} \quad \pi_1(G) = 0 \quad \text{for all others},$$

$$\pi_2(G) = 0,$$
\[ \pi_3(G) = \mathbf{Z} \quad \text{if} \quad G \neq SO(4) \quad \text{and} \quad \pi_3(SO(4)) = \mathbf{Z} \oplus \mathbf{Z}. \]

\[ \pi_k(U(n)) = 0 \quad \text{for all} \quad k > 1. \]

Corresponding identities for direct products follow from

\[ \pi_k(G_1 \times G_2) = \pi_k(G_1) \oplus \pi_k(G_2). \]

The main tool to determine \( \pi_n \) are exact sequences.

**Definition:** Be \( A_n \) a sequence of sets and \( \varphi_n \) a mapping from \( A_n \) to \( A_{n+1} \). The sequence

... \( A_n \xrightarrow{\varphi_n} A_{n+1} \xrightarrow{\varphi_{n+1}} A_{n+2} \) ...

is called exact if

\[ \ker(\varphi_{n+1}) = \text{im}(\varphi_n). \]

**Theorem:** For a subgroup \( H \subset G \) the sequence

... \( \pi_n(H) \xrightarrow{j} \pi_n(G) \xrightarrow{i} \pi_n(G/H) \xrightarrow{\partial} \pi_{n-1}(H) \xrightarrow{j} \pi_{n-1}(G) \)...

is exact. Here \( j \) and \( i \) denote the trivial inclusion map and \( \partial \) is the boundary map. Since \( \pi_2(G) = 0 \), we thus obtain

\[ \pi_2(G/H) = \pi_1(H) \quad \text{for all simply connected groups} \quad G; \]

i.e. for all groups with \( \pi_1(G) = 0 \).

### 2.1. The \( \sigma \)– model approximation

If the system under consideration is at a temperature \( T \) much below the critical temperature, \( T \ll T_c \), it becomes more and more improbable for the field \( \phi \) to leave the vacuum manifold. \( \phi \) will leave the vacuum manifold only if it would otherwise be forced to gradients of order \( (\nabla \phi)^2 \sim \lambda \phi^2 \eta^2 \), thus only over length scales of order \( l = 1/(\sqrt{\lambda} \eta) \equiv m^{-1}_\phi \) (\( l \) is the transversal extension of the defects). If we are willing to loose the information of the precise field configuration over these tiny regions (for GUT scale phase transitions \( l \sim 10^{-30}\text{cm} \) as compared to cosmic distances of the order of \( 1\text{Mpc} \sim 10^{24}\text{cm} !!) it seems well justified to fix \( \phi \) to the vacuum manifold \( \mathcal{N} \). Instead of discussing the field equation from (2),

\[ \Box \phi + \frac{\partial V}{\partial \phi} = 0, \]

we require \( \phi \in \mathcal{N} \). \( \mathcal{N} \subset \mathcal{V} \) is a Riemannian submanifold with the induced scalar product. The remaining field equation \( \Box \phi = 0 \) then just demands that

\[ \phi : \mathcal{M} \to \mathcal{N}. \]
is a harmonic map from spacetime $\mathcal{M}$ into $\mathcal{N}$. There exists a waste mathematical literature on harmonic maps and their singularities which might be useful for us and should be explored [6].

The topological defects we are interested in are singularities of these maps. When the gradients of $\phi$ become very large, like, e.g., towards the center of a global monopole, the field leaves the vacuum manifold and assumes non vanishing potential energy. If $\phi \in \mathcal{N}$ is enforced, a singularity develops by topological reasons.

In the physics literature harmonic maps are known as $\sigma$–models. They were originally introduced because of their similarities with non Abelian gauge theories (the corresponding field equations also contain non–linear gradient terms). The action of a $\sigma$–model is given by

$$ S_{\sigma} = \int_{\mathcal{M}} g^{\mu\nu} \partial_{\mu} \phi^A \partial_{\nu} \phi^B h_{AB}(\phi) \sqrt{|g|} d^4x , $$

(6)

where $h_{AB}$ denotes the metric on $\mathcal{N}$ and $g_{\mu\nu}$ is the metric of spacetime. Let us consider once more the $O(N)$ example:

$$ V = \frac{1}{8} \lambda (\phi^2 - \eta^2)^2 , \; \phi \in \mathbb{R}^N . $$

We now fix $\phi$ to lay in the vacuum manifold, $S^{N-1}$ with radius $\eta$, by introducing a Lagrange multiplier.

$$ L_{\sigma} = \partial_{\mu} \phi \cdot \partial^{\mu} \phi - \alpha (\phi^2 - \eta^2) . $$

Variation w.r.t $\phi$ yields

$$ \Box \phi + 2 \alpha \phi = 0 . $$

(7)

We multiply (6) with $\phi$ to obtain $\alpha = -\phi \cdot \Box \phi / (2 \eta^2)$. Inserting this in (6), we obtain the field equation

$$ \Box \phi - (\phi \cdot \Box \phi) \phi = 0 . $$

(8)

In other words, the projection of $\Box \phi$ onto the hyperplane tangent to the sphere has to vanish, i.e., $\Box \phi = 0$ on $\mathcal{N}$. In terms of the dimensionless variable $\beta = \phi / \eta$, (8) reads

$$ \Box \beta - (\beta \cdot \Box \beta) \beta = 0 , $$

(9)

which shows that the $\sigma$–model is scale free.
2.1.1. Analytic flat space solutions

A global string along the z–axis is described by the field configuration $\phi \in \mathbb{R}^2$:

\[
\phi(x, y, z) = \eta e^\rho = \eta (\cos \varphi, \sin \varphi), \quad \sigma–\text{model} \quad (10)
\]
\[
\phi(x, y, z) = f_S(\rho)\eta (\cos \varphi, \sin \varphi), \quad \text{full field eqn.} \quad (11)
\]

where $f_S$ satisfies

\[
f_S'' + \frac{1}{\rho} f_S' - \frac{1}{\rho^2} f_S + \frac{\lambda}{2} (f_S^2 - 1) f_S = 0, \quad (12)
\]

with boundary conditions $f_S(0) = 0$ and $f_S(\infty) = 1$. Here $\rho$ is the cylindrical radius and $\varphi$ the polar angle, $(x, y) = \rho (\cos \varphi, \sin \varphi)$.

A spherically symmetric, static global monopole is described by the field configuration $\phi \in \mathbb{R}^3$ with

\[
\phi(x, y, z) = \eta e^r = \eta (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta), \quad \sigma–\text{model} \quad (13)
\]
\[
\phi(x, y, z) = f_M(r)\eta (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta), \quad \text{full field eqn.} \quad (14)
\]

where $f_M$ satisfies

\[
f_M'' + \frac{2}{r} f_M' - \frac{2}{r^2} f_M + \frac{\lambda}{2} (f_M^2 - 1) f_M = 0, \quad (15)
\]

with boundary conditions $f_M(0) = 0$ and $f_M(\infty) = 1$. The equations for $f_S$ and $f_M$ can only be solved numerically.

A spherically symmetric global texture is described by the field configuration $\phi \in \mathbb{R}^4$ with

\[
\phi = \eta (\sin \chi \sin \theta \cos \varphi, \sin \chi \sin \theta \sin \varphi, \sin \chi \cos \theta, \cos \chi).
\quad (16)
\]

With the ansatz $\chi = \chi(r, t)$, the $\sigma$–model field equation (8) leads to

\[
(-\partial_t^2 + \partial_r^2 + \frac{2}{r} \partial_r)\chi = \frac{\sin(2\chi)}{r^2}.
\]

Since the $\sigma$–model is scale invariant, we further require $\chi = \chi(y)$ with $y = t/r$. In terms of this scaling variable, the equation of motion for $\chi$ reduces to the ordinary differential equation

\[
(y^2 - 1)\chi'' = \sin(2\chi), \quad (17)
\]

with exact solutions

\[
\chi(y) = 2 \arctg(\pm y) \pm n\pi.
\]
Figure 1. The function $\chi$ is shown for $t < 0$, solid curve, and for $t > 0$, dashed curve. $\chi$ goes from 0 to $\pi$ for negative times, i.e. the configuration winds once around $S^3$, and from $\pi$ to $\frac{3}{2}\pi$ and back to $\pi$ for positive times, i.e., no winding.

These solutions were originally found by Turok and Spergel. To obtain a solution which winds around the three sphere at negative times and collapses at $t = 0$, we patch together $\chi$ as follows:

$$\chi(y) = \begin{cases} 2\arctg(y) + \pi, & -\infty \leq y \leq 1 \\ 2\arctg(1/y) + \pi, & 1 \leq y \leq \infty \end{cases} \quad (18)$$

The behavior of $\chi$ as function of $r$ for positive and negative times is shown in Fig. 1. The kink at $r = t$ for positive times is due to the singularity of the $\sigma$–model solution at $r = t = 0$. It would be softened in a solution of the full field equations. Physically, this kink represents the wake of Goldstone bosons in which the massive mode at $r = t = 0$ has decayed and which now travels out with the speed of light. One easily sees that the energy of these three configurations diverges. For a large ball of radius $R$ one finds

$$E_{\text{string}}(R) = \int T^0_0(z, \rho) = \frac{1}{2} \int (\nabla \phi)^2 \propto R \log(R\eta) \ , \quad (19)$$

$$E_{\text{monopole}}(R) = \int T^0_0(r) = \frac{1}{2} \int (\nabla \phi)^2 \propto R \ , \quad (20)$$

$$E_{\text{texture}}(R) = \int T^0_0(r) = \frac{1}{2} \int [\dot{\phi}^2 + (\nabla \phi)^2] \propto R \text{ for } R >> t \ . \quad (21)$$

Before we go on to discuss the gravitational effects of these solutions, let me briefly note some thoughts concerning Derrick’s theorem.
2.2. Derrick’s theorem

Since it is so simple and beautiful, let me state the theorem with proof. 

**Theorem:** In $d = 3$ dimensions there are no non trivial static finite energy solutions for a scalar field whose potential energy is bounded from below.

**Proof:** For static configurations, the variation of the action can be replaced by the variation of the energy, $E$.

$$E = \int \left[ \frac{1}{2} (\nabla \phi)^2 + V(\phi) \right] d^3 x = I_1 + I_2 ,$$

with

$$I_1 = \int \frac{1}{2} (\nabla \phi)^2 d^3 x \quad \text{and} \quad I_2 = \int V(\phi) d^3 x .$$

Without loss of generality, we may assume $V \geq 0$ (otherwise, consider $E - V_{\text{min}}$). Then $I_1 > 0$ and $I_2 \geq 0$. We assume now $\phi(x)$ be a non trivial solution and consider the scaled configuration $\phi_\lambda(x) = \lambda^{-1} \phi(\lambda^{-1} x)$. For the scaled configuration we have

$$I_1(\lambda) \equiv I_1(\phi_\lambda) = \lambda I_1 \quad \text{and} \quad I_2(\lambda) \equiv I_2(\phi_\lambda) = \lambda^3 I_2 .$$

Therefore

$$\partial_\lambda E|_{\lambda=1} = I_1 + 3I_2 > 0 .$$

This contradicts our assumption of $\phi$ being a solution. 

From this we can immediately conclude that our solutions for global strings and monopoles discussed before must have infinite energy. But also the time dependent texture solution has infinite energy (21).

Perivolaropoulos has put forward the following argument: In the cosmological context we should truncate the energy at some large radius $R$, the horizon distance or the distance to the next defect. Then the variation of the scaled energy yields

$$\partial_\lambda E|_{\lambda=1} = I_1 + 3I_2 - R \partial_R (I_1 + I_2) ,$$

which, due to the negative term, can vanish. The second variation shows that a configuration with vanishing first variation does represent a minimum of the truncated energy and thus is stable against shrinking and expansion.

But of course this argument does not explain the existence of the string and monopole solutions considered previously. Furthermore, the argument would also allow for stable static texture solution (with infinite energy). There have been some analytical and numerical arguments, that it is the winding condition that renders the textures unstable. For winding number $n > 0.5$ textures tend to shrink and for $n < 0.5$ they tend to expand. Nevertheless, in my opinion, a clear understanding of the numerical finding that there exist stable static (infinite energy) string and monopole solutions, but probably no stable static texture solution is still missing.
3. GRAVITATIONAL EFFECTS OF SCALAR FIELDS IN FLAT SPACETIME

3.1. Generalities

The energy momentum tensor of a scalar field configuration in the $\sigma$-model approximation is given by

$$T_{\mu\nu}^{(\phi)} = \partial_{\mu}\phi \cdot \partial_{\nu}\phi - \frac{1}{2}g_{\mu\nu}\partial_{\lambda}\phi \cdot \partial^{\lambda}\phi .$$  \hspace{1cm} (22)

We set

$$\rho = T_{0}^{0} = \frac{1}{2}(\dot{\phi}^2 + (\nabla\phi)^2)$$  \hspace{1cm} (23)

$$p = \frac{1}{3}T_{i}^{i} = \frac{1}{6}(\dot{\phi}^2 - (\nabla\phi)^2)$$  \hspace{1cm} (24)

$$\pi_{ij} = T_{ij} - g_{ij}p = \partial_{i}\phi\partial_{j}\phi - \frac{1}{3}g_{ij}(\nabla\phi)^2 .$$  \hspace{1cm} (25)

For static global field configurations $\rho + 3p = 0$. This indicates that static global field configurations, like an infinite straight string or a hedgehog monopole, do not gravitationally attract nonrelativistic particles.

To discuss the gravitational effects of test particles and radiation in general, we have in principle to solve Einstein's equations,

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}^{(\phi)} ,$$  \hspace{1cm} (26)

and investigate the geodesics in the resulting geometry. For a typical field coherence length $l$, we have $8\pi G T_{\mu\nu} \sim 8\pi G \eta^2/l^2$. For a GUT phase transition this is of the order of $10^{-5}/l^2 - 10^{-4}/l^2$. The induced changes of the metric will thus be small, of order $10^{-5} - 10^{-4}$, and we can treat gravity in first order perturbation theory. I.e., we insert in eqn. (26) the unperturbed, flat spacetime, energy momentum tensor and equate it to the Einstein tensor $G_{\mu\nu}$ obtained from first order corrections to the flat metric (or, in the cosmological context to the Friedmann Robertson Walker metric).

3.2. Spherically symmetric field configurations

For the sake of simplicity, we now restrict ourselves to spherically symmetric configurations. In first order perturbation theory, the metric can then be parametrized by

$$g = -(1 - 2\Psi)dt^2 + (1 - 2\Phi)\delta_{ij}dx^i dx^j .$$  \hspace{1cm} (27)

The linearized Einstein equations yield

$$-\triangle \Phi = 4\pi G \rho$$  \hspace{1cm} (28)

$$-\triangle (\Phi - \Psi) = 8\pi G \triangle \Pi , \text{ where } \partial_{i}\partial_{j}\Pi - \frac{1}{3}\triangle \Pi = \pi_{ij} .$$  \hspace{1cm} (29)
(In the spherically symmetric case it is always possible to find such an anisotropy potential $\Pi$.) For ordinary matter, $\rho \gg \pi_{ij}$, and thus $\Phi = -\Psi$. $\Psi$ is the relativistic analog to the Newtonian gravitational potential, and slowly moving matter only couples to $\Psi$. Using the equation of motion (8) for $\phi$, one can show that for static configurations

$$\triangle \Pi = \frac{1}{4}(\nabla \phi)^2 = \frac{1}{2}\rho .$$

Eqn. (28) and (29) then yield $\Psi = 0$. This shows again that nonrelativistic matter is not affected by static global field configurations.

It is easy to calculate the connection coefficients (Christoffel symbols) from Ansatz (27). Inserting them into the geodesic equation for a photon moving with four velocity

$$n = (1, n) + \delta n , \quad p = En ,$$

one obtains in first order perturbation theory

$$\delta n^0 = -2\Psi|_i^f + \int_i^f (\dot{\Phi} - \dot{\Psi})d\lambda$$
$$\delta n_i = \int_i^f \partial_i(\Phi - \Psi)d\lambda ,$$

where the integrals are performed along the unperturbed photon trajectory. The meaning of these quantities is the following: We consider an emitter/observer of a light ray moving according to the velocity field

$$u = (1 + \Psi, v) \text{ with } v^2 \ll 1 .$$

The 0–component of $u$ is determined by the condition $u^2 = -1$. The energy shift of a photon relative to emitter and observer is generally given by $\delta E = (p \cdot u)(f) - (p \cdot u)(i)$. In our situation this yields

$$\delta E/E = \delta(u \cdot n) = n \cdot v|_i^f + \Psi|_i^f - \int_i^f (\dot{\Phi} - \dot{\Psi})d\lambda .$$

The first term on the right hand side of (33) is the usual, special relativistic Doppler term. The second term is due to the difference of the gravitational potential at the position of the emitter and observer, and the third term is a path dependent contribution due to the change of the gravitational potentials during the passage of the photons. Since for static scalar fields $\Psi = \dot{\Psi} = \dot{\Phi} = 0$, the gravitational contributions to $\delta E$ vanish in the static situation.

Eqn. (24) is related to light deflection. Be $e$ the radial unit vector. The deflection of a light ray emitted at position $i$ and observed at $f$ is then given by

$$\alpha = \delta(n \cdot e)(i) - \delta(n \cdot e)(f) = -\int_i^f \partial_i(\Phi - \Psi)e_i d\lambda .$$
For the gravitational field from ordinary matter ($\Phi = -\Psi$), we recover the old result by Einstein (with the correct factor of 2).

For a slowly moving massive particle in a weak spherically symmetric gravitational field, we make the ansatz

$$ u = (1, 0) + \delta u . $$

From the geodesic equation, we then obtain

$$ \delta u^i = - \int \partial_i \Psi dt . $$

Since $\Psi$ vanishes in the static case, we find that slowly moving particles are not affected by static field configurations.

Taking into account also (33), we thus have proven the following

**Theorem:** In static scalar field configurations with negligible potential energy, the gravitational redshift of photons and the gravitational acceleration of slowly moving particles vanish.

The gravitational field of static configurations thus affects matter and radiation only by deflection of relativistic particles.

### 3.3. Two examples

To be somewhat more specific, we now want to insert into (33) and (34) the global monopole and global texture solutions obtained in the last section.

**Global monopole:** From the linearized Einstein equations we find for the hedgehog monopole solution [15]

$$ \Psi = 0 , \quad \Phi = -8\pi G\eta^2 \ln (r/l) \equiv -\epsilon \ln (r/l) , \quad (37) $$

where we have set $\epsilon \equiv 8\pi G\eta^2$ and $l$ is an arbitrary constant of integration. We consider a light ray passing the monopole with impact parameter $b$. Its unperturbed trajectory is given by $x(\lambda) = \lambda n + be$. Since the configuration is static, $\delta E$ vanishes. Inserting (37) in (34) yields the deflection angle

$$ \alpha_M = - \int \partial_i \Phi e^i d\lambda = \epsilon \int_{-\infty}^{\infty} \frac{b}{\lambda^2 + b^2} d\lambda = \epsilon \pi . \quad (38) $$

This result was originally found by different methods by Barriola and Vilenkin [16].

**Global Texture:** For the texture solution (18) we obtain [18]

$$ \Psi = \frac{\epsilon}{2} \ln \left( \frac{t^2 + r^2}{t^2} \right) , \quad \Phi = -\frac{\epsilon}{2} \ln \left( \frac{t^2 + r^2}{l^2} \right) . \quad (39) $$
For a light ray passing the texture with impact parameter $b$ at impact time $\tau$ ($t = \tau + \lambda$, $r^2 = b^2 + \lambda^2$), we find the deflection angle

$$\alpha_T = -\int_i^f (\Phi - \Psi)_i e^i d\lambda$$

$$\approx e \int_{-\infty}^{\infty} \frac{b}{b^2 + 2\lambda^2 + 2\lambda \tau + \tau^2} d\lambda$$

$$= \epsilon \pi \frac{b}{\sqrt{2b^2 + \tau^2}}.$$  \hspace{0.5cm} (40)

This result was first obtained in [17].

To calculate the energy shift of a photon passing the texture, we have to 'renormalize' the result obtained from naively inserting (39) in (33). Due to the unphysical infinite energy of solution (18), the energy shift contains a divergent logarithmic term which we neglect in the $\approx$ sign in eqn. (41). (In [15], this renormalization is discussed in some detail.)

$$\frac{\delta E}{E} = \Psi_i^f + \int_i^f (\dot{\Phi} - \dot{\Psi}) d\lambda \approx \epsilon \pi \frac{\tau}{\sqrt{\tau^2 + 2b^2}}.$$  \hspace{0.5cm} (41)

This result was first obtained in [7].

The interesting difference between the results for monopoles and texture is due to the time dependence of the latter. This, first of all, yields a non-vanishing energy shift for the texture. An observer receiving photons from behind a collapsing texture sees them first redshifted (if they pass the texture before collapse) and then blueshifted (see Fig. 2). An observer in perfect alignment with a background quasar and a global monopole sees the quasar image as Einstein ring with fixed opening angle. In the case of a global texture, the Einstein ring opens up some time before texture collapse, reaches a maximum opening angle of the same order of magnitude as in the monopole case and then shrinks back to a point [17, 15].

The gravitational field of our global texture solution (18) also accelerates slowly moving particles. Inserting (39) in (36) leads to the well-known result [7, 18]

$$v(f) - v(i) = -\epsilon \pi e_r.$$  \hspace{0.5cm} (42)

Slowly moving particles around a collapsing texture thus acquire a net infall velocity of amplitude $\epsilon \pi$.

4. GLOBAL DEFECTS AS SEEDS FOR COSMOLOGICAL STRUCTURE FORMATION

Many observational results, like Hubble expansion, primordial nucleosynthesis, the isotropy and the thermal spectrum of the cosmic microwave background, confirm the idea that on large scales the Universe is homogeneous and isotropic. On large
Figure 2. The temperature fluctuation, $\Delta T/T$, induced by a spherically symmetric collapsing texture as function of the impact time of the observer. The solid line shows the result in expanding space, the dashed line is the flat space result. The collapse time of the texture is $t_c \approx 20$ (in arbitrary units). The difference of the two curves is due to the existence of horizons in expanding space: Photons, which pass the texture long before or after the collapse are not influenced in expanding spacetime, but acquire the maximum energy shift in flat spacetime.

On smaller scales, the observable Universe is thus well approximated by a Friedmann universe, which evolved from a very hot thermal state, the big bang, by adiabatic expansion.

On smaller scales, clearly, the Universe is lumpy. Laborious mapping of the 3d galaxy distribution has shown that this clumpiness persists on scales up to $(30 - 50)h^{-1}\text{Mpc}$. The galaxies themselves are arranged in relatively thin sheets surrounding seemingly empty voids of diameters up to $50h^{-1}\text{Mpc}$. ($1\text{Mpc} \approx 3.2 \times 10^6\text{ly} \approx 3.1 \times 10^{24}\text{cm}$) (see Fig. 3).

With the help of the Cosmic Background Explorer (COBE) satellite, anisotropies have been found also in the cosmic microwave background (CMB) which are on the level $\sqrt{\left\langle \left(\frac{\Delta T}{T}\right)^2\right\rangle}(\theta) \approx 10^{-5}$, on all angular scales $\theta > 7^\circ$.

These findings support the old idea of Lifshitz that the cosmic structures might have formed by gravitational instability from small initial fluctuations.

Cosmological perturbation theory shows, that perturbations in the radiation field can not grow substantially. Therefore, $\Delta T/T$ yields the amplitude of initial
fluctuations \((\delta \rho/\rho)_{in} \sim 3\Delta T/T\). On the other hand, perturbations in pressureless matter \((p \ll \rho, \text{cosmic dust})\) grow roughly by a factor \(a_0/a_{eq} = z_{eq} + 1\), where \(a\) denotes the scale factor of the universe, a subscript \(0\) denotes present time and \(eq\) denotes the time of equal matter and radiation density. If the matter content of the universe is given by baryons only, \(z_{eq} \leq 10^3\), and our naive estimates lead to perturbations which are roughly by a factor 10 too small to yield the observed structures. However, if we assume that the universe is dominated by dark matter leading to critical density, \(\Omega = 1\), we have \(z_{eq} \sim 10^4\), of the correct order of magnitude to lead to the nonlinear clustering observed today.

There remains one basic ingredient to the gravitational instability picture: How did the small initial perturbations of order \(10^{-5} - 10^{-4}\) emerge? Presently two mechanisms are primarily investigated:

- Quantum fluctuations ‘frozen in’ as classical perturbations of the energy density after an epoch of inflation.
- Topological defects from a phase transition in the early universe.

In this workshop, we concentrate on the second possibility. We have seen in the last section that topological defects yield gravitational perturbations of the order of \(8\pi G\eta^2 \equiv \epsilon\). To obtain \(\epsilon = 10^{-5} - 10^{-4}\), we need a GUT scale phase transition, \(\eta \sim 10^{16} GeV\).
4.1. Scaling

Let us now assume that on large scales the Universe can be described by a Friedmann universe with vanishing spatial curvature, $\Omega = 1$. The metric of spacetime can then be given by

$$ ds^2 = a^2(-dt^2 + \delta_{ij}dx^i dx^j) . $$

Here $a$ is the cosmic scale factor and $t$ is conformal time. It is related to the cosmic time, $t_{\text{cos}}$, which has elapsed since the big bang by

$$ t_{\text{cos}}(t) = \int_0^t a(t') dt' $$

(see also contribution by T.W.B. Kibble). To be relevant for structure formation, topological defects must make up an approximately constant fraction of order $\epsilon$ of the total energy density of the universe. In the cosmological context, we then say that the defects obey scaling. Let us estimate the energy density of global defects, neglecting the potential energy:

$$ a^2 \langle \rho_{\text{def}} \rangle = \frac{1}{2} \langle (\nabla \phi)^2 \rangle + \frac{1}{2} \langle (\partial_t \phi)^2 \rangle \sim \eta^2 / t^2 , $$

where we have assumed that $\phi$ changes typically over a horizon scale. On the other hand, from the Friedmann equation (assuming the scale factor $a$ to obey a power law), we have

$$ a^2 \rho = \frac{3}{8\pi G} (\dot{a}/a)^2 \sim \frac{1}{8\pi G t^2} , $$

so that $\langle \rho_{\text{def}} \rangle / \rho \sim \epsilon$.

From this result one might conclude that all global defects obey scaling. But the above argument is somewhat too simplistic as the case of global strings shows: Let us approximate the energy of a global string inside one horizon volume by the corresponding energy of a straight cosmic string in flat space:

$$ E(t) = 2\pi \int_\eta^{t_{\text{cos}}} dz r dr (\eta^2 / r^2) = 2\pi \eta^2 t_{\text{cos}} \log(t_{\text{cos}} \eta) , $$

and thus

$$ a^2 \langle \rho_{\text{str}} \rangle \sim \eta^2 / t^2 \log(at\eta) . $$

In the case of strings, we thus obtain a logarithmic correction term which, for a GUT scale phase transition, amounts to a factor of approximately 150 today. For higher $O(N)$ defects like monopoles, textures and $O(N)$ models with $N > 4$, the scaling behavior becomes clean (see Fig. 4).

In the case of local defects, only cosmic strings obey scaling. Monopoles stop interacting soon after formation and then scale like massive particles:

$$ n_M \sim 1 / t_c^3 (a_c/a)^3 , \quad \rho_M = m_M n_M \propto a^{-3} . $$
Figure 4. The scaling behavior of $(\rho + 3p)a^2$ found numerically in $(128)^3$ simulations is shown for four different $O(N)$ models. Time is given in units of the grid spacing $\Delta x$. For comparison, the dashed line $\propto 1/t^2$ is shown. For $N > 3$ scaling is very clean until $t \approx 80$, where finite size effects can become important.
The universe at GUT scale is radiation dominated, \( \rho \propto a^{-4} \). Therefore, soon after the phase transition \( \rho_M \gg \rho \) leading to

\[
\Omega_M(t_0)h^2 \sim 10^{14}(T_c/10^{15}\text{GeV})^3(m_M/10^{16}\text{GeV}) \gg \Omega_0h^2 .
\]

This is the famous monopole problem in cosmology [22]. The reason why this represents a serious problem is the following: Imagine some simple, compact grand unified group \( G \), like \( SU(5) \), breaking (in one or several steps) to \( H = SU(3) \times SU(2) \times U(1) \). The existence of monopoles at the end is then determined by the exact sequence

\[
\pi_2(G) \rightarrow \pi_2(G/H) \rightarrow \pi_1(H) \rightarrow \pi_1(G) .
\]

Since \( \pi_2(G) = 0 \), and for a simple group also \( \pi_1(G) = 0 \) we find

\[
\pi_2(G/H) = \pi_1(H) = \mathbb{Z} .
\]

Monopoles thus always form. By the analogous sequence for \( \pi_1(G/H) \),

\[
0 = \pi_1(G) \rightarrow \pi_1(G/H) \rightarrow \pi_0(H) = 0 ,
\]

we conclude that no strings form. The monopoles are thus not connected by strings and are stable.

This is a beautiful example showing that observations of the present universe can lead to predictions about high energy physics and cosmology at GUT scale. The most simple GUT scenario is not compatible with standard cosmology. One either has to invoke a period of inflation or change the GUT idea [22].

4.2. Cosmological perturbation theory

So far, we have only seen that the orders of magnitude come out reasonable for structure formation with topological defects from a GUT scale phase transition. We would like to obtain more precise results. We want to simulate the evolution of defects, calculate the gravitational fields they produce, which in turn affect the distribution of matter and radiation. We want to calculate the induced anisotropies in the cosmic radiation field and in the matter distribution, \( (\Delta T/T)(t_0, \mathbf{x}, \mathbf{n}) \) and \( (\delta \rho/\rho)(t_0, \mathbf{x}) \), \( v_{pec}(t_0, \mathbf{x}) \).

An important tool for this calculation is cosmological perturbation theory. We do not develop it here, but just mention the basic equations which determine our problem. For more details see, e.g., [15].

- The equation of motion for the scalar field:

\[
\Box \phi + \frac{\partial V}{\partial \phi} = 0 \text{ (potential model), or}\]

\[
\Box \phi - \phi \frac{(\phi \cdot \Box \phi)}{\eta^2} = 0 \text{ (sigma model)} .
\]
• The perturbation of the energy momentum tensor:

\[ \delta T_{\mu\nu} = T_{\mu\nu}(\phi) + \delta T_{\mu\nu}^{\text{matter}}. \]  
\hspace{1cm} (48)

• The linearized Einstein equations:

\[ \delta T_{\mu\nu} = \delta G_{\mu\nu}. \]  
\hspace{1cm} (49)

• The equations of motion linearized about the Friedmann background:
  - The Liouville equation for photons

\[ p^\mu \partial_\mu f + \Gamma^i_{\alpha\beta} p^\alpha p^\beta \frac{\partial f}{\partial p^i} = 0. \]  
\hspace{1cm} (50)
  - The cold dark matter equation of motion, \( p = 0 \),

\[ T^{\mu\nu}_{;\nu} = 0. \]  
\hspace{1cm} (51)

For a perturbed Planck distribution, the Liouville equation can be cast into a perturbation equation for the temperature only: \[\text{[15, 23]}\]. Be \( \mathbf{x} \) the observer position and \( \mathbf{n} \) the direction of observation. If we set \( T(\mathbf{x}, \mathbf{n}) = \bar{T}(1 + m(\mathbf{x}, \mathbf{n})) \), the perturbation equation corresponding to (50) can be expressed as

\[ (\partial_t + n^i \partial_i)\chi = -3n^i \partial_i E_{ij} - n^k n^j \epsilon_{ikl} \partial_l B_{ij}, \]  
\hspace{1cm} (52)

where

\[ \chi = \Delta m + \text{(monopole term + dipole term)}, \]

and \( E_{ij}, B_{ij} \) denote the electric and magnetic parts of the Weyl tensor. If we are only interested in the spherical harmonic amplitudes \( a_{lm} \) of \( m(\mathbf{n}) \) for harmonics higher than the dipole, \( l \geq 2 \), it is thus sufficient to determine \( \Delta^{-1} \chi \). For a fixed observer position a monopole term can not be distinguished from the background temperature and a dipole term can be attributed to the peculiar velocity of the observer. Therefore, monopole and dipole terms anyway do not contain information on the temperature fluctuations.

From (51) we obtain a perturbation equation for the energy density perturbations, \( D \) of the dark matter.

\[ \ddot{D} + \left( \frac{\dot{a}}{a} \right) \dot{D} - 4\pi G a^2 \rho_{dm} D = 4\pi G a^2 \dot{\phi}^2. \]  
\hspace{1cm} (53)

From \( \chi \), we can obtain \( m = \delta T / T \) by inverse Laplacian. The first information to be compared with observations are the power spectra or, correspondingly, the auto–correlation functions of \( \delta T / T \) and \( D \). We expand \( \delta T / T \) in spherical harmonics:

\[ (\delta T / T)(t_0, \mathbf{x}, \mathbf{n}) = \sum_{l,m} a_{lm}(\mathbf{x}) Y_{lm}(\mathbf{n}). \]
The power spectrum of \( \delta T/T \) is then given by
\[
c_l = \frac{1}{(2l+1)n_x} \sum_{m,x} |a_{lm}(x)|^2 ,
\] (54)
where \( n_x \) is the number of observer positions \( x \) averaged over, and \( 2l+1 \) is the number of values \( -l \leq m \leq l \). One easily finds the temperature correlation function \[24\]
\[
\langle (\delta T/T)(n)(\delta T/T)(n') \rangle_{n,n'=\cos \theta} = \frac{1}{4\pi} \sum_{l} (2l+1)c_l P_l(\cos \theta) .
\] (55)

\( P_l \) denotes the \( l \)th Legendre polynomial and \( \langle \rangle \) indicates averaging over positions and over all directions \( n, n' \) with relative angle \( \theta \).

The power spectrum of dark matter perturbations (called ‘structure function’ in condensed matter physics) is the Fourier transform of the correlation function. Indicating Fourier transforms by a tilde, we have

\[
P(k) \equiv |\tilde{D}(k)|^2 = \tilde{C}(k) ,
\]
(56)

\[
C(r) = \langle D(x)D(x+nr) \rangle_{x,n}
\]
is the correlation function \[25\].

### 4.3. The Harrison Zel’dovich spectrum

Let us assume that the only scale in the structure formation problem is the horizon scale. Then we expect the variance of the mass perturbation on this scale to be a constant, \( A \), independent of time \[26, 27\]:

\[
A = \langle |\delta M/M|^2 \rangle_{2\pi/k=t} \approx k^3 |P(k, t = 2\pi/k)| .
\] (57)

Once the perturbations ‘enter the horizon’, \( k > 2\pi/t \), their behavior depends on the expansion law of the background spacetime. From a simple analysis of linear perturbation theory one finds that perturbations cannot grow if spacetime is radiation dominated (Mézanos effect, \[28\]), and they grow proportional to the scale factor \( a \) if spacetime is matter dominated. Let us denote by \( t_{eq}, a_{eq} \) the conformal time and scale factor of the universe at the time when the energy density of radiation equals that of matter. During the matter dominated regime the scale factor grows like \( a(t) \propto t^2 \). Defining \( a_k = a(t = 2\pi/k) \), we obtain on scales which are subhorizon today (\( k > 2\pi/t_0 \))

\[
\tilde{D}(k, t_0) \approx \begin{cases} 
Ak^{-3/2}(a_0/a_k) = Ak^{1/2}(t_0/2\pi)^2, & k < 2\pi/t_{eq} \\
Ak^{-3/2}(a_0/a_{eq}) = Ak^{-3/2}z_{eq}, & k > 2\pi/t_{eq} 
\end{cases} .
\] (58)

The the Harrison Zel’dovich spectrum can thus be approximated roughly by the form

\[
P(k) = |D(k)|^2 \approx \frac{A_k t_0^2}{1 + (k/k_{eq})^2} ,
\] (59)
Figure 5. The points are the IRAS redshift space spectrum with $\Omega = 1$. The box indicates the power spectrum inferred from the COBE DMR results with spectral index $n = 1$. The solid line is the spectrum of a standard CDM scenario with $\Omega = 1$, normalized to the real space variance of IRAS galaxies, $\sigma_8 = 0.7$ (this figure is taken from Fisher et al. [29]).

Correspondingly, one can show that for a scale invariant spectrum (57), the microwave background fluctuations behave like

$$c_l \propto \frac{1}{l(l+1)} , \quad c_l = \frac{5c_2}{l(l+1)} .$$  

(60)

Numerical simulations and analytical arguments show that structure formation by global topological defects leads to an approximately scale invariant spectrum of perturbations.

4.4. Numerical Simulations
4.4.1. The scalar field: The equation of motion of a scalar field in expanding space is given by

\[ g^{\mu\nu} \nabla_\mu \nabla_\nu \phi + \frac{\partial V}{\partial \phi} = 0. \]  

(61)

Defining \( \beta = \phi/\eta \) and \( m = \sqrt{\lambda \eta} \), (61) yields for our \( O(N) \) models in a Friedmann universe

\[ \partial_t^2 \beta + 2(\dot{a}/a) \partial_t \beta - \nabla^2 \beta = \frac{1}{2} a^2 m^2 (\beta^2 - 1) \beta. \]  

(62)

This equation as it stands is not tractable numerically in the regime which is interesting for large scale structure formation. The two scales in the problem are the horizon scale \( t \) and the inverse symmetry breaking scale, the comoving scale \((am)^{-1}\). At recombination, e.g., these scales differ by a factor of about \( 10^{53} \) and can thus not both be resolved numerically.

There are two approximations to treat the scalar field numerically. As we shall see, they are complementary and thus the fact that both approximations agree with each other within about 10\% is reassuring. The first possibility is to replace \((am)^{-1}\) by \( w \), the smallest scale which can be resolved in a given simulation, typically twice the grid spacing, \( w \sim 2\Delta x \). The time dependence of \((am)^{-1}\) which results in a steepening of the potential is mimicked by an additional damping term, \( 2(\dot{a}/a) \rightarrow \gamma \dot{a}/a \) with \( \gamma \sim 3 \) [30]. Numerical tests have shown, that this procedure, which usually is implemented by a modified staggered leap frog scheme [31], is not very sensitive on the values of \( \gamma \) and \( w \) chosen. With this method we have replaced the growing comoving mass \( am \) by the largest mass which our code can resolve. For a, say \((256)^3\) grid which simulates the evolution of the scalar field until today, we have \( 256 \Delta x \sim t_0 \sim 4 \times 10^{17} \) sec, so that \( w \sim 2 \times 10^{15} \) sec, i.e., \( am \sim \eta/z_{rec} \sim 10^{13} GeV \) is replaced by about \( w^{-1} = 10^{-40} GeV! \)

We believe this mimics the behavior of the field, since the actual mass of the scalar field is irrelevant as long as it is much larger than the typical kinetic and gradient energies associated with the field which are of the order the inverse horizon scale. Therefore, as soon as the horizon scale is substantially larger than \( \Delta x \), the code should mimic the true field evolution on scales larger then \( w \). But, to my knowledge, there exists no rigorous mathematical approximation scheme leading to the above treatment of the scalar field which would then also yield the optimal choice for \( \gamma \).

Alternatively, we can treat the scalar field in the \( \sigma \)–model approximation. This approach is opposite to the one outlined above in which the scalar field mass is much too small, since the \( \sigma \)–model corresponds to setting the scalar field mass infinity.

The \( \sigma \)–model equation of motion cannot be treated numerically with a leap frog scheme, since it contains non–linear time derivatives. In this case, a second order accurate integration scheme has been developed by varying the discretized action with respect to the field [32].
Initially, the field $\phi$ itself and/or the velocities $\dot{\phi}$ are laid down randomly on the grid points. The initial time, $t_{in}$ is chosen to be the grid size, $t_{in} = \Delta x$, so that the field at different grid points should not be correlated. The configuration is then evolved in time with one of the approximation schemes discussed above.

The two different approaches have been extensively tested, and good agreement has been found on scales larger than about 2 – 3 grid sizes [33, 34]. This is very encouraging, especially since the two treatments are complementary: In the $\sigma$–model, we let the scalar field mass $m$ go to infinity. In the potential approach, we replace it by $\sim 1/\Delta x \sim 1/\Delta x \sim 1/100 ly \sim 10^{-34}$ GeV.

The integration of the scalar field equation is numerically the hardest part of the problem, since it involves the solution of a nonlinear partial differential equation.

4.4.2. The gravitational perturbations: Once $\beta(x, t)$ is known, we can calculate the energy momentum tensor

$$T^{(\phi)}_{\mu\nu} = \eta^2 \left[ (\partial_\mu \beta \cdot \partial_\nu \beta) - \frac{1}{2} g_{\mu\nu} (\partial_\lambda \beta \cdot \partial_\lambda \beta) \right].$$

(63)

From eqn. (53), we can further determine the perturbation of the dark matter energy momentum tensor. The perturbed Einstein equations then yield an algebraic equation for the electric part of the Weyl tensor, $E_{ij}$, and an equation of motion for the magnetic part of the Weyl tensor, $B_{ij}$ (see [23]). The magnetic contributions, which consist of vector and tensor perturbations only, usually amount to about (10 – 20) % of the electric contributions which are a combination of scalar and vector perturbations of the gravitational field, see Fig. 6.

4.4.3. The perturbations of the cosmic background radiation and the dark matter: Using eqn. (52), we finally obtain

$$\frac{\delta T}{T} = \Delta^{-1} \chi$$

$$= \int_{t_i}^{t} \left\{ n^i \Delta^{-1} (\partial^i E_{ij}) + n^k n^j \Delta^{-1} (\epsilon_{ikl} \partial^l B_{ij}) \right\} (t', x - n(t - t'), n) dt'.$$ 

From eqn. (53), we can calculate $D$ and the power spectrum $P(k) = |D(k)|^2$. Patching together simulations with different physical grid size, we can enlarge the range of comoving wave numbers $k$ covered, see Fig. 7.

By the decomposition into spherical harmonics,

$$a_{lm}(x) = \int_{S^2} (\delta T/T)(t_0, x, n) Y_{lm}(n) d\Omega,$$

(64)

and (54), we determine the $c_l$’s. Since the spectrum is close to scale invariant, it is entirely determined by the quadrupole moment, $Q$,

$$Q = \frac{4\pi}{5} c_2 \approx \alpha \epsilon = Q_{COBE} = (0.6 \pm 0.1) \times 10^{-5}.$$

(65)
Figure 6. The amplitude of the electric and magnetic source terms to the photon equation if motion are shown as a function of wavenumber $k$ in arbitrary scale. For small wavelength (large scales) the magnetic part contributes about 1/4, decaying to roughly 1/10 on small scales. The quantities graphed are $E = \frac{1}{3} \sum_i (\partial^i E_{ij})^2$, $B = \frac{1}{3} \sum_{ij} (\epsilon_{ijk} \partial^k B_{ki})^2$. $\bar{B}$ is represented by the solid line.

Figure 7. The final dark matter spectrum of density fluctuations for 3 texture simulations with different physical grid sizes patched together.
The value of $\alpha$ above can be obtained by numerical simulations\cite{35, 36, 32, 23} and is typically of the order $0.1 \leq \alpha \leq 1$. For textures one finds\cite{23}

$$\epsilon = 8\pi G \eta^2 = (2.2 \pm 1) \times 10^{-5}. \quad (66)$$

The shape of the dark matter spectrum is again approximately determined by scale invariance. The integral of the dark matter perturbation spectrum over scales larger than $R$, determines the mass variation, $\sigma^2(R)$ over these scales. The comparison of this dark matter mass variation with the observed variation of the galaxy distribution yields a scale dependent bias factor, $b(R)$. The bias factors obtained this way are of the order $b \sim 2 - 4$, which is somewhat larger than expected\cite{37}. The global defect models normalized to the COBE results for the microwave background fluctuations probably yield somewhat too small perturbations in the dark matter. Nevertheless, the uncertainties concerning the bias factor and the nonlinear physics going into the calculation of the bias factor seem to me to leave room for doubts. It would be more convincing to rule out the the scenarios from the completely linear determination of the microwave background fluctuations alone. So far, only the gravitational interaction of the radiation field with perturbations has been taken into account. To calculate $\delta T/T(\theta)$ on scales, $\theta < 2^\circ$, which enter the horizon before recombination, when baryons and photons still are a tightly coupled fluid, the baryon photon interaction has to be taken into account and the recombination process has to be modeled. For pure CDM without scalar field this calculation has been performed on different levels of accuracy\cite{38, 39, 40}. For global defect induced fluctuations, intermediate and small scale anisotropies have only been approximated in the case when the universe is reionized at some early redshift, $z > 100$, and baryons and photons are coupled again via Thompson scattering. In this case, photon diffusion severely damps fluctuations on scales smaller than the horizon at $z \sim 100$, i.e., on all scales smaller than about $5^\circ$\cite{41, 42, 43}. One of the missing pieces in the global defect scenarios is thus a detailed calculation of the microwave background spectrum on angular scales $\theta < 2^\circ$ or $l > 100$ for a non reionized universe.

On the other hand, the CMB fluctuations, are not determined by the spectrum $c_l$ alone. The spectrum just yields the two point correlation function which determines the fluctuations only if they are Gaussian distributed. In general the $a_{lm}$ also yield non zero higher correlation functions. The skewness $S$ and the kurtosis, $K$ of the distribution of $2^\circ \times 2^\circ$ pixels are found to be\cite{36}

$$S = -4 \pm 2.3$$

$$K = 32 \pm 29.$$

The deviation from Gaussian distribution is also shown in Fig. 8. These higher order correlations are an important mean to distinguish models with global defects from models with initial fluctuations from an inflationary epoch which usually yield Gaussian fluctuations.
Figure 8. The pixel distribution of $\Delta T/T$ in a synthesized map of microwave background fluctuations for the texture scenario of structure formation. The dashed line shows a Gaussian with the same width and the same number of pixels. The negative skewness and the positive kurtosis are clearly visible.

5. CONCLUSIONS

We have discussed gravitational interaction of global scalar fields with matter and radiation. We have found that static global field configurations with vanishing potential energy do not affect slowly moving particles and do not redshift photons.

Topological defects which form during phase transitions in the early universe can have important cosmological consequences. For gravitational interactions of the defects with the cosmic matter and radiation to be relevant, the defects must form due to a phase transition at GUT scale. In this case they may even seed the formation of cosmological large scale structure. Even though it is not yet clear if defect induced structure formation scenarios do work out in detail, up today they remain an intriguing alternative to initial fluctuations from inflation since they also yield a scale invariant spectrum of perturbations.

Acknowledgement: I’m grateful to my collaborator Zhi–Hong Zhou, who helped me in preparing some of the figures for these proceedings. I have learned a lot from discussions with many of the participants, especially Robert Brandenberger, Pedro Ferreira, Tom Kibble, Andrew Little, Leandros Perivolaropoulos, Tomislav Prokopec, Paul Shellard and Neil Turok. Finally, I want to express special thanks to Ann Davis, the organizer of this lively and stimulating meeting.
REFERENCES

[1] Langaker and Pi, *Phys. Rev. Lett.* **45**, 1 (1980).
[2] T.W.B. Kibble, *Phys. Rep.* **67**, 183 (1980), see also these proceedings.
[3] C. Nash and S. Sen, *Topology and Geometry for Physicists*, Academic Press, London (1983).
[4] S. Iyanaga and Y. Kawada, *Encyclopedic Dictionary of Mathematics*, Mathematical Society of Japan, MIT Press, Cambridge, MA (1977).
[5] S.T. Hu, *Homotopy Theory*, Academic Press, London (1959).
[6] J. Eells and L. Lemaire, *Bull. London Math. Soc.* **10**, 1 (1978);
   *ibid.* **20**, 385 (1980);
   R.T. Smith *Proc. Amer. Math. Soc.* **47**, 229 (1975).
[7] N. Turok and D.N. Spergel, *Phys. Rev. Lett.* **64**, 2736 (1990).
[8] G. Derrick, *J. Math. Phys.* **5**, 1252 (1964).
[9] L. Perivolaropoulos, *Nucl. Phys.* B375, 665 (1992).
[10] S. Åminneborg, *Nucl. Phys.* B388, 521 (1992).
[11] R.A. Leese and T. Prokopec, *Phys. Rev. D44*, 3749 (1991).
[12] J. Borill, E. Copeland and A. Liddle, *Phys. Lett.* B258, 310 (1991).
[13] T. Prokopec, A. Sornborger and R. Brandenberger, *Phys. Rev. D45*, 1971 (1992).
[14] L. Perivolaropoulos, *Phys. Rev. D46*, 1858 (1992).
[15] R. Durrer, *Fund. Cosmic Physics* **15**, 209 (1994).
[16] M. Barriola and A. Vilenkin, *Phys. Rev. Lett.* **63**, 341 (1989).
[17] R. Durrer, M. Heusler, P. Jetzer and N. Straumann, *Nucl. Phys.* B368, 527 (1992).
[18] R. Durrer, *Phys. Rev. D42*, 2533 (1990).
[19] M.J. Geller and J.P. Huchra, *Science* **246**, 897 (1989).
[20] G.F. Smoot et al., *Astrophys. J.* **396**, L1 (1992);
   E.L. Wright et al., *Astrophys. J.* **396**, L13 (1992).
[21] E.M. Lifshitz, *Soviet J. Phys. (JETP)* **10**, 116 (1946).
[22] E. Kolb and M. Turner, *The Early Universe*, Addison Wesley, Redwood City (1990).
[23] R. Durrer and Z.-H. Zhou, *Phys. Rev. Lett.*, submitted (1994).
[24] T. Padmanabhan, *Structure Formation in the Universe*, Cambridge University Press (1993).
[25] G. Efstathiou in: *Physics of the Early Universe*, Scottish Universities Summer Schools in Physics, ed. J.A. Peacock, A.F. Heavens and A.T. Davies (1990).
[26] E.R. Harrison, *Phys. Rev. D1*, 2726 (1970).
[27] Ya. Zel’dovich, *Mon. Not. Roy. ast. Soc.* **160**, 1p (1972).
[28] P. Mézáros, *Astron. Astrophys.* **37**, 225 (1974).
[29] K.B. Fisher, M. Davis, M.A. Strauss, A. Yahil and J.P. Huchra, *Astrophys. J.* **402**, 42 (1993).

[30] W. Press, D. Spergel and B. Ryden, *Astrophys. J.* **347**, 590 (1989).

[31] W.H. Press, B.P. Flannery S.A. Teucholsky and W.T. Vetterling, *Numerical Recipes*, Cambridge University Press (1990).

[32] U. Pen, D.N. Spergel and N. Turok, *Phys. Rev. D* **49**, 692 (1994).

[33] J. Borill, E.J. Copeland, A.R. Liddle, A. Stebbins and S. Veeraraghavan, *Phys. Rev. D* **50**, 2469 (1994), and references therein.

[34] J. Borill, *Phys. Rev. D* **50**, 3676 (1994).

[35] D. Bennett and S. Rhie *Astrophys. J.* **406**, L7 (1993).

[36] R. Durrer, A. Howard and Z.–H- Zhou, *Phys. Rev. D* **49**, 681 (1994).

[37] N. Katz, L. Hernquist and D.H. Weinberg, *Astrophys. J.* **399**, L109 (1992); R. Cen and J. Ostriker, ibid., L113 (1992).

[38] N. Sugiyama and N. Gouda, *Prog. Theor. Phys.* **88**, 803 (1992).

[39] W. Hu and N. Sugiyama, *Astrophys. J. Lett.*, submitted (1994).

[40] U. Seljak, *Astrophys. J. Lett.*, submitted (1994).

[41] R. Durrer, *J. Infrared Phys. Technol.* **35**, 83 (1994).

[42] D. Coulson, P. Ferreira, D. Graham and N. Turok, *Nature* **368**, 27 (1994).
