Induction and Mutually Obstructing Equilibria

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A unified, consistent and simple view of the Faraday law of induction is presented, which consists of two points: discriminating the lab- from the rest-frame electric field and understanding it is the impossibility for both fields to vanish simultaneously, which generates and maintains the circular current. A number of illustrative examples are considered, including a mechano-electric pendulum to exhibit periodic and reversible conversion between electrical and mechanical energy.

I. INTRODUCTION

The Faraday law of induction equates the change of the magnetic flux \( \phi \) to the sum of potential drops along a wire loop,

\[
\frac{d}{dt} \phi = \oint \vec{A} \cdot d\vec{l} = - \oint \vec{E} \cdot d\vec{l} = - \sum U_i. \tag{1}
\]

It contains two effects: The first concerns a changing field at constant area, \( \frac{d}{dt} \phi = \oint (\frac{d}{dt} \vec{B}) \cdot d\vec{A} \), and is obtained by integrating the Maxwell equation, \( \frac{d}{dt} \vec{B} = - \nabla \times \vec{E} \).

The second effect is given by changing the area of a conducting loop at a static (and frequently uniform) field, \( \frac{d}{dt} \phi = \oint \vec{B} \cdot (\vec{v} \times d\vec{l}) = \vec{B} \cdot \frac{d}{dt} \vec{A} \). This is a little harder to grasp: Since \( \frac{d}{dt} \vec{B} = 0 \), an integration of the electric field around the loop is zero, \( \oint \nabla \times \vec{E} \cdot d\vec{A} = \oint \vec{E} \cdot d\vec{l} = 0 \), and it appears surprising at first that a current should nevertheless flow. The prevalent explanation is: The electrons in the moving section of the loop are subject to the Lorentz force, finite even if the electric field vanishes, \( \vec{F} = e(\vec{E} + \vec{v} \times \vec{B}) = e\vec{v} \times \vec{B} \). It is their response to, and the resultant motion along, \( \vec{F} \) that gives rise to the current \( I \), see Fig 1. More quantitatively, observing that \( \vec{v} \times \vec{B} \) is the force per unit charge, same as the electric field \( \vec{E} \), one concludes that both should also otherwise be similar. So integrating \( \vec{v} \times \vec{B} \) along the loop yields an “induced” electric potential \( U_{\text{ind}} = \oint \vec{v} \times \vec{B} \cdot d\vec{l} \). And it is only natural to employ \( U_{\text{ind}} \) in the Ohm law,

\[
RI = U_{\text{ind}} = \oint \vec{v} \times \vec{B} \cdot d\vec{l}, \tag{2}
\]

with \( R \) the total electric resistance of the loop, and \( I \) the current. Because this is reminiscent of batteries, \( U_{\text{ind}} \) is also referred to as an electromotive force.

II. OBJECTIONS

When first encountering the Faraday’s law, students are frequently sensitized by their teachers to the disconcerting fact that this beautifully simple law lacks a unified understanding, as one needs both the Lorentz force and one of the Maxwell equations to derive it. In his famous lectures, Feynman succinctly described, and lamented, a general sense of resignation:

We know of no other place in physics where such a simple and accurate general principle requires for its real understanding an analysis in terms of two different phenomena. Usually such a beautiful generalization is found to stem from a single deep underlining principle. Nevertheless, in this case there does not appear to be any such profound implication. We have to understand the “rule” as the combined effect of two quite separate phenomena.

Two objections to the above view of induction are also worthy of some attention. First, there are two different Lorentz forces: The macroscopic one, \( \vec{j} \times \langle \vec{B} \rangle \), given in terms of the averaged field, expresses the directly verifiable force on a current carrying wire. The force on an electron, \( e(\vec{E} + \vec{v}_e \times \vec{B}) \), on the other hand, is a microscopic formula, given in terms of the electron velocity \( \vec{v}_e \) and the microscopic field \( \vec{B} \). (Only in this paragraph does \( \vec{B} \) denote the microscopic field. It is always the macroscopic, coarse-grained field otherwise.) Electrons in conductors have a broad velocity distribution, and they are exposed to strongly varying fields. Therefore, it appears strikingly bold to assert that the average magnetic force per unit charge, \( \langle \vec{v}_e \times \vec{B} \rangle \), is simply \( \vec{v} \times \langle \vec{B} \rangle \), the velocity of the wire times the average \( B \)-field, as we did above.

It is instructive to reflect upon the Hall Effect in this context. If evaluated in the same naive fashion, employing the Lorentz force in a classical free electron model, the result is notoriously unreliable and rarely agrees well with experiments. In fact, even the sign may be wrong — then it is referred to as the anomalous Hall effect. So why is the Faraday law universally accurate?

Second, even disregarding these doubts, the above derivation appears to contain a logical error. Its two
resulting formulas are, for \( \vec{B} \) uniform,

\[
\vec{A} \cdot \frac{\partial}{\partial t} \vec{B} = -\sum U_i, \quad \vec{B} \cdot \frac{\partial}{\partial t} \vec{A} = -U^\text{ind}. \quad (3)
\]

We may of course add both equations, obtaining \( \vec{A} \cdot \frac{\partial}{\partial t} \vec{B} + \vec{B} \cdot \frac{\partial}{\partial t} \vec{A} = 0 \) on the left, with \(-U^\text{ind}\) included in \(-\sum U_i\) on the right, as one usually does. But we must not forget that each formula remains valid on its own. If the field is static, \( \frac{\partial}{\partial t} \vec{B} = 0 \), the sum of potentials vanishes, \( \sum U_i = 0 \), irrespective whether \( \vec{B} \cdot \frac{\partial}{\partial t} \vec{A} \) vanishes or not. If there is only one resistive element in the circuit, reducing \( \sum U_i \) to a single voltage drop \( U_R \), then both this voltage and the current will always vanish, \( I = U_R/R = 0 \). In other words, even if \( \vec{B} \cdot \frac{\partial}{\partial t} \vec{A} = -U^\text{ind} \) is finite, we must not write \( RI = U^\text{ind} \), as in Eq (2), to account for Faraday’s observation, as this clearly violates the Maxwell equation.

### III. TWO ELECTRIC EQUILIBRIA

The consideration below avoids all these difficulties and inconsistencies. We start by introducing the electric field \( \vec{E}^0 \) of the conductor’s local rest-frame. It is related to the lab-frame field \( \vec{E} \) as

\[
\vec{E}^0 = \vec{E} + \vec{v} \times \vec{B}. \quad (4)
\]

(Only terms to first order in \( v/c \) are included in this paper.) Note that \( \vec{v} \) is the macroscopic velocity of the medium – an unambiguous, directly observable quantity. Rest-frame fields are important, because conductors strive to reduce them. As long as \( \vec{E}^0 \) is finite, there is a current \( \vec{j} = \sigma \vec{E}^0 \), which redistributes the charge to relax \( \vec{E}^0 \) to zero. Only then is the conductor in equilibrium.

One could not possibly substitute \( \vec{E} \) for \( \vec{E}^0 \) in these statements, because \( \vec{E} \) depends on the observer’s frame that can be changed at will, while the conductor’s equilibrium is an unambiguous fact, independent of observers.

A metallic object at rest is in equilibrium if \( \vec{E} = 0 \); if it moves with the velocity \( \vec{v} \), we have \( \vec{E}^0 = 0 \) instead, so the lab-frame field is finite, \( \vec{E} = -\vec{v} \times \vec{B} \). In any configuration such as in Fig. 1 that offers two inequivalent paths, “frustration” sets in, as the moving section strives to establish a finite potential difference, \(-\int \vec{v} \times \vec{B} \cdot d\vec{\ell}\), by charge separation, while the stationary part attempts to eliminate it: The incompatibility of both equilibria, working hard to obstruct each other, is what gives rise to a current that flows as long as \( \vec{v} \) is finite.

A limiting case is easy to see: If the resistance of the sliding bar is much larger than that of the stationary arc, see Fig. 1, the latter is much better able to maintain equilibrium, \( \vec{E} \approx 0 \), so the current \( \vec{j} = \sigma \vec{E}^0 \) is approximately \( \sigma \vec{v} \times \vec{B} \) – same as obtained above using the Lorentz force. For a general quantitative account, we follow Landau and Lifshitz to integrate the Maxwell equation in the form \( \frac{\partial}{\partial t} \vec{B} = -\nabla \times (\vec{E}^0 - \vec{v} \times \vec{B}) \), arriving at

\[
\int d\vec{A} \cdot \frac{\partial}{\partial t} \vec{B} + \oint \vec{B} \cdot (\vec{v} \times d\vec{\ell}) = -\oint \vec{E}^0 \cdot d\vec{\ell}. \quad (5)
\]

Identifying the conductor’s velocity \( \vec{v} \) with that the area \( \vec{A} \) changes, the two terms on the left may be combined as \( \frac{\partial}{\partial t} \oint \vec{B} \cdot d\vec{A} \), and the result is the Faraday’s law, properly given in terms of the rest-frame potential drops \( U_i^0 \),

\[
\frac{\partial}{\partial t} \phi = \frac{\partial}{\partial t} \oint \vec{B} \cdot d\vec{A} = -\oint \vec{E}^0 \cdot d\vec{\ell} = -\sum U_i^0. \quad (6)
\]

We now revisit Fig. 1 to analyze it as two inequivalent paths characterized by two resistors, see Fig. 2. Clearly, Eq (6) simply states

\[
\frac{\partial}{\partial t} \phi = -(U_1 + U_2) = -(R_1 + R_2)I, \quad (7)
\]

as the result of the general case. Note that the potential remains constant, \( \vec{E}^0 \equiv 0 \), between the resistors, and there is no need for an electromotive force.

### IV. A MECHANO-ELECTRIC PENDULUM

A worthwhile variation contains all three elements: resistor, coil and capacitance. With \( L \) denoting the inductivity, \( C \) the capacitance, and \( \omega \) the frequency, they are

\[
U^0 = -i\omega L I, \quad U^0 = I/(-i\omega C). \quad (8)
\]

Inserting them in Eq (6), assuming a uniform and constant \( \vec{B} \), denoting the relevant length of the moving wire as \( \ell \), see Fig. 3 we find

\[
[i\omega L - R + (i\omega C)^{-1}] I = \vec{B} \cdot (\vec{v} \times \vec{\ell}). \quad (9)
\]

The motion of the wire (of mass \( M \)) is subject to the Lorentz force. For uniform \( \vec{v} \), it is given as

\[
M \frac{d^2 \vec{v}}{dt} = \int \vec{j} \times \vec{B} dV = \vec{\ell} \times \vec{B} I. \quad (10)
\]

Combining the last two equations, we arrive at a mechano-electric pendulum,

\[
\left( \omega^2 + \frac{i\omega R}{L} - \omega_0^2 \right) I = 0, \quad \omega_0 = \pm \sqrt{\frac{1}{CL} + \frac{B^2}{ML}}. \quad (11)
\]

Since both the capacitance and the moving wire contribute to the restoring force, the wire alone would suffice to form a pendulum with the inductance. Clearly, the numbers are such that a conveniently observable resonance of around 1 Hz seems possible – if the sliding resistance can be sufficiently reduced.

![Figure 2](image-url)
bars to be identical, with resistance $R$ contribution from the horizontal wire. The result is again a frustration-induced current, each respectively characterized by the effective resistance, and the four field-free ones working to eliminate it.

To obtain the breaking force, we insert this expression for the current $I$ in Eq. (10),

$$M \frac{d^2 \vec{v}}{dt^2} = -(4\ell^2 B^2 / 3R) \vec{v},$$

with $3RM/(4\ell^2 B^2)$ being the relaxation time – which of course will change after another vertical wire moves into the field. If one of the two horizontal wires is lacking, no current at all will flow, and no breaking takes place – because there is no inequivalent paths. Each portion of the wire net is happily in equilibrium by itself, without the need or possibility to obstruct that of the other portion.

Similar circumstances reign when a solid metal plate enters a magnetic field, because the value of the effective $R_2$ and $R_1$ decreases with the width of its region. At the beginning, the field-exposed region is narrow, and $R_2$ is the dominating one. Toward the end, when the field-free region is narrow, $R_1$ becomes large and is the one limiting the current. The largest current flows, and maximal breaking by the Lorentz force occurs, when the plate is half in. No current at all flows when one of the two regions ceases to exist. If a metal comb enters the field, each tooth is in effect an electrically independent entity. Due to their narrow width, the resistance is always large in comparison, so the current, the Lorentz force and the breaking are always much smaller.

The next example is the eddy-current break, a piece of metal moving with $\vec{v}$, with only part of the metal exposed to a magnetic field. Equilibrium is given by $\vec{E} = 0$ outside the field-exposed region, and by $\vec{E} = -\vec{v} \times \vec{B}$ inside it. Any deviation from these values churns up a current $\vec{j}$ to re-establish them – obviously quite the same dilemma as before. The result is again a frustration-induced current, which dissipates the kinetic energy of the moving plate effectively. Constant magnetic field and charge density imply the field equations:

$$\nabla \times \vec{E} = 0, \quad \nabla \cdot \vec{j} = \sigma \nabla \cdot (\vec{E} + \vec{v} \times \vec{B}) = 0,$$

and the boundary conditions: $\Delta E_0 = 0$, $\Delta E_n = -\vec{v} \times \vec{B}$. These have been solved assuming constant $\vec{B}$, $\vec{v}$, and a circular field-exposed region. The result is a dipolar current field, with equal effective resistance, $R_2 = R_1$.

V. BULK CONDUCTORS

The validity of the Faraday law, Eq. (4), is confined to wire loops, because the conductor’s velocity $\vec{v}$ was identified with the rate the area $A$ changes, or $A = v\ell$. Yet the circumstance of mutually obstructing electric equilibria, giving rise to currents, occurs under rather more general conditions – including especially bulk metal. Consider first the wire grid of Fig. 4 moving with the velocity $\vec{v}$ from a region without field into one with a finite magnetic field. One may of course apply Eq. (6) for all possible loops of this grid, though it is much simpler to map it onto Fig. 2, considering two inequivalent paths, the two field-exposed bars striving to establish a potential difference, and the four field-free ones working to eliminate it. Each is respectively characterized by the effective resistance, $R_2 = R/2$ and $R_1 = R/4$, if we take all vertical bars to be identical, with resistance $R$, and neglect the contribution from the horizontal wire. The result is again Eq. (7), or

$$-(R/2 + R/4) I = \dot{\vec{B}} \cdot (\vec{v} \times \vec{v}),$$

and

VI. FEYNMAN’S ROCKING CONTACT

Finally, we discuss Feynman’s rocking contact, which consists of two metal plates with slightly curved edges, such that their contact is only in one point, see the dotted lines of Fig. 5. It was presented in his lectures as an example for circumstances in which the Faraday’s law does not hold, with a hint that it could be analyzed with the Lorentz force. However, in any bulk geometry such as the one here, there is little reason why the electron velocity $\vec{v}_e$ should in any way resemble $\vec{v}$ of the metal. So the Lorentz force cannot be too useful. And it is the rest-frame electric field that will again do the trick.
FIG. 5: Rocking contacts, as indicated by the dotted lines, between two plates with curved edges—though the physics of the two spherical plates should be rather similar.

FIG. 6: Two metal wheels oscillating in the presence of a $B$-field generates an alternating current.

To understand the rocking contact, first consider a metal wheel rotating with the angular velocity $\omega$, in the presence of a field $\vec{B} \parallel \vec{\omega}$. Being in equilibrium, $\vec{E}^0 = 0$, implies $\vec{E} = -\vec{v} \times \vec{B}$ with $\vec{v} = \vec{\omega} \times \vec{r}_\perp$, or $\vec{E} = -\vec{r}_\perp (\vec{\omega} \cdot \vec{B})$, with the constant electric density, $\rho = \vec{\nabla} \cdot \vec{E} = -2\vec{\omega} \cdot \vec{B}$. Assuming the wheel is neutral, there is a total surface charge of $2V \vec{\omega} \cdot \vec{B}$ at the rim ($V$ denotes the volume of the wheel). Reversing the rotation also reverses the charges.

Next consider Fig. 6 depicting two metal wheels rotating in place, so the runner in between moves. The runner has two conducting surfaces separated by an insulating sheet. Connecting the two surfaces will give rise to a one-shot current, which neutralizes the opposite surface charges of the wheels—no frustration here. Oscillating the runner generates an alternating current.

Now reconsider Fig. 5 depicting either two wheels rocking against each other, or if we take away the metal behind the dotted lines, the rocking contact of 6. The geometry is slightly more complex, because there is also a contribution from the translational velocity. But the basic analysis of the last example remains valid. Since the rotating parts are already in contact to each other, little if any current will travel via the wires.

VII. SUMMARY

We summarize. Starting from the fact that an electric equilibrium is given only if the electric field of the local rest frame vanishes, or $\vec{E} = -\vec{v} \times \vec{B}$, many phenomena concerning metal parts moving in the presence of a magnetic field are shown to become easily understandable, and accessible for fully or semi-quantitative analysis. The Faraday Law is seen as a special case of two mutually obstructing electric equilibria, as a result of which a circular electric current is maintained.

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1 See for instance the classic introductory textbook by D. Halliday, R. Resnick, J. Walker, *Fundamentals of Physics* (John Wiley, 2000).
2 R.P. Feynman, R.B. Leighton, M. Sands, *The Feynman Lectures on Physics II* (Addison Wesley, 1970) §17-1.
3 L.D. Landau and E.M. Lifshitz, *Electrodynamics of Continuous Media* (Pergamon Press, 1984), §63.
4 It is assumed that the two inductive elements, say because the fields point in orthogonal directions, do not couple.
5 P.J. Salzman, J.R. Burke, S.M. Lea, Am. J. Phys. 69(5), 586-590, 2001
6 Fig. 17-3 of ref. 2, *The Feynman Lectures on Physics II*. 