ON THE SIMPLICITY OF THE TENSOR PRODUCT OF TWO
SIMPLE MODULES OF QUANTUM AFFINE ALGEBRAS

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Abstract. Lapid and Mínguez gave a criterion for the irreducibility of the parabolic induction $\sigma \times \pi$, where $\sigma$ is a ladder representation and $\pi$ is an arbitrary irreducible representation of the general linear group over a non-archimedean field. Through quantum affine Schur-Weyl duality, when $k$ is large enough, this gives a criterion for the irreducibility of the tensor product of a snake module and any simple representation of the quantum affine algebra $U_q(\hat{sl}_k)$.

The goal of this paper is to add conditions to their criterion such that it works for any $k \geq 1$. We prove the criterion in the case where both modules are snake modules or one of them is a fundamental module at an extremity node and the other is any simple module. We also defined a similar criterion in the Grassmannian cluster algebra $\mathbb{C}[\text{Gr}(k, n, \sim)]$, and show that for any $k \geq 1$, two ladders are compatible if and only if the corresponding tableaux satisfy the criterion. This generalizes Leclerc and Zelevinsky’s result that two Plücker coordinates are compatible if and only if they are weakly separated.

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1. Introduction

Let $\mathfrak{g}$ be a simple Lie algebra, $q \in \mathbb{C}^\times$ be not a root of unity and consider $U_q(\mathfrak{g})$, the corresponding quantum affine algebra. The quantum affine algebra being a Hopf algebra, its category $\mathcal{C}$ of finite-dimensional representations is a tensor category. The simple finite-dimensional representations were classified by Chari-Pressley [CP94], they
are highest \((l-)\)-weight representations, with weight in a certain monoid. However, some important questions about the category \(\mathcal{C}\) remain open. For example, it is not known, in general, whether the tensor product \(L(M) \otimes L(N)\) of two irreducible finite-dimensional representations is also irreducible. Additionally, for questions related to the cluster algebra structure of the Grothendieck ring \(K_0(\mathcal{C})\) (see [HL10, HL16]), it is important to know which irreducible representations \(L\) are real, i.e. its tensor square \(L(M) \otimes L(M)\) is irreducible.

In theory, one can use the \(q\)-characters, a refinement of the characters introduced by Frenkel-Reshetikhin [FR99], to answer this question. Indeed, given two simple \(U_q(\mathfrak{g})\)-modules \(L(M), L(N)\), their tensor product is simple if and only if their \(q\)-characters satisfy \(\chi_q(L(M) \otimes L(N)) = \chi_q(L(M)) \cdot \chi_q(L(N)) = \chi_q(L(MN))\). In practice, there is no general formula for computing the \(q\)-characters of all simple representations. There exists an algorithm established by Frenkel-Mukhin [FM01] to compute the \(q\)-character \(\chi_q(L(M))\) starting with the highest weight \(M\), but it is known to fail in some cases.

In this paper, we focus on the case of a type \(A\) quantum affine algebra, that is \(\mathfrak{g} = \mathfrak{sl}_k\). By quantum affine Schur Weyl duality (see [CP96], and the review [Gur21a]), when \(k > N\), the representation theory of the quantum affine algebra \(U_q(\widehat{\mathfrak{sl}}_k)\) is equivalent to that of the general linear group over a non-archimedean field \(GL_N(F)\), and this equivalence is monoidal: the tensor product is translated into the parabolic induction.

In [LM16], Lapid-Mínguez considered a certain class of representations called ladders, whose equivalence under the quantum affine Schur Weyl duality are studied under the name snake modules (see [MY12]). They established a combinatorial criterion for the irreducibility of the parabolic induction \(\sigma \times \pi\), where \(\sigma\) is a ladder representation and \(\pi\) is an arbitrary irreducible representation of \(GL_N(F)\). Thus, through quantum affine Schur-Weyl duality, when \(k\) is large enough, Lapid and Mínguez’s criterion gives a criterion of the simplicity of the tensor product of a snake module and any simple module of the quantum affine algebra \(U_q(\widehat{\mathfrak{sl}}_k)\). The goal of this paper is to add conditions to Lapid and Mínguez’s criterion to obtain a criterion which works for any \(k \geq 1\).

Our main result is the following.

**Theorem 1.1** (Theorem 6.10, Proposition 6.11). Let \(L(M), L(M')\) be simple \(U_q(\widehat{\mathfrak{sl}}_k)\)-modules. Assume either \(L(M')\) is a fundamental representation at an extremal node, or both \(L(M)\) and \(L(M')\) are snake modules. Then the tensor product \(L(M) \otimes L(M')\) is irreducible if and only if the highest weights \(M, M'\) satisfy two combinatorial conditions \(\text{LC}_k(\mathbf{m}_M, \mathbf{m}_M')\) and \(\text{LC}_k(\mathbf{m}_{M'}, \mathbf{m}_M)\).

In order to prove this result, we can still make use of the quantum affine Schur-Weyl duality. More precisely, we use the equivalences of categories between the category \(\mathcal{C}\) of finite-dimensional \(U_q(\widehat{\mathfrak{sl}}_k)\)-modules and the categories \(\mathcal{C}^{GN}\) of finite-length \(GL_N(F)\) representations, for \(k > N\). We prove an isomorphism of rings between the Grothendieck ring of \(\mathcal{C}\) and a quotient of the direct sum of the Grothendieck rings \(\mathcal{C}^{GN}\) (Proposition 3.11).
An important ingredient of the proof comes from the results of Maxim Gurevich [Gur21b], who gave an algorithm to compute the decomposition of a tensor product of any two ladder representations into irreducible representations.

Additionally, Hernandez and Leclerc [HL10] showed that in the case of a quantum affine algebra of type $A$, there is an isomorphism between the cluster algebra structure of the Grothendieck ring of certain subcategories $\mathcal{C}_\ell$ of $\mathcal{C}$ and some Grassmannian cluster algebras. More precisely, there is an isomorphism between $K_0(\mathcal{C}_\ell)$ and $\mathbb{C}[\text{Gr}(k,n,\sim)]$, where $n = k + \ell + 1$ and $\mathbb{C}[\text{Gr}(k,n,\sim)]$ is the quotient of $\mathbb{C}[\text{Gr}(k,n)]$ by the ideal generated by $P_{i,i+1,...,i+k-1}$, $i \in [n-k+1]$, where $P_{i,i+1,...,i+k-1}$ is a Plücker coordinate.

It is shown in [CDFL20] that the dual canonical basis of $\mathbb{C}[\text{Gr}(k,n,\sim)]$ is parametrized by some rectangular semistandard Young tableaux. In particular, cluster monomials in $\mathbb{C}[\text{Gr}(k,n,\sim)]$ are of the form $\text{ch}(T)$, where $T$ is such a tableau. Two cluster variables $\text{ch}(T)$, $\text{ch}(T')$ are called compatible if they appear in the same cluster, and this implies that $\text{ch}(T)\text{ch}(T') = \text{ch}(T \cup T')$. It is conjectured in [CDFL20] that $\text{ch}(T)\text{ch}(T') = \text{ch}(T \cup T')$ is equivalent to the condition that $\text{ch}(T)$, $\text{ch}(T')$ are compatible.

The result of Theorem 1.1 translates to the context of the Grassmannian cluster algebra. We obtain a combinatorial criterion on the ladders tableaux $T, T'$ equivalent to the condition $\text{ch}(T)\text{ch}(T') = \text{ch}(T \cup T')$ (Corollary 6.12). In particular, it generalizes a result of Leclerc-Zelevinsky [LZ98] stating that any two Plücker coordinates are compatible if and only if they are weakly separated.

Note that by adopting the approach of [LM20], our results can also be applied to quantum affine superalgebras. Then $L(M)$ should be a skew representation while $L(M')$ needs to be replaced by any simple module that is a subquotient of tensor products of evaluation vector representations.

The paper is organized as follows. We start by recalling results on representations of quantum affine algebras and $p$-adic groups in Section 2. Then, Section 3 is dedicated to the quantum affine Schur-Weyl duality, and how these representations are related. We established the isomorphism of Grothendieck rings that will be used later on. We consider the context of the Grassmannian cluster algebra in Section 4, and compare the combinatorial data coming from multisegments and tableaux. In Section 5, we establish an analog of the Zelevinsky classification [Zel80], in terms of representations of the type $A$ quantum affine algebra. Some of the presented results are known, which but have either not been written explicitly, or not in comparison with their $p$-adic equivalent. The results are proven intrinsically, and not usual quantum affine Schur-Weyl duality. We describe the conjectured criterion for the simplicity of the tensor product of two $U_q(\widehat{\mathfrak{sl}_k})$-modules in Section 6, and obtain the main results. Finally, Section 7 contains the proofs of the two main intermediate results.

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**Notations.** For convenience of the reader, we collect key notation here.

- $U_q(\mathfrak{g})$ the quantum affine algebra for a complex simple Lie algebra $\mathfrak{g}$, $I$ the set of vertices of its Dynkin diagram, Section 2.1.
- $\hat{I} = \{(i, s) \in I \times \mathbb{Z} \mid i + s - 1 \in 2\mathbb{Z}\}$, $\mathcal{P}$ the free abelian group in formal variables $Y_{i,s}^{\pm 1}$, $(i, s) \in \hat{I}$, $\mathcal{P}^+$ the submonoid of $\mathcal{P}$ generated by $Y_{i,s}$, $(i, s) \in \hat{I}$, Section 2.2.
- $C$ the category of finite dimensional representations of $U_q(\mathfrak{g})$, $\mathcal{C}^\mathbb{Z}$ a subcategory of $\mathcal{C}$ and $\mathcal{R}_\mathfrak{g}$ its Grothendieck ring, $\mathcal{R}_k = \mathcal{R}_{\mathfrak{sl}_k}$, Section 2.2.
- $L(M)$ the simple $U_q(\mathfrak{g})$-module with highest $\ell$-weight $M$ and $\chi_q(M) = \chi_q(L(M))$ its $q$-character, Section 2.2.
- $\mathcal{C}$ the category of complex, smooth representations of $U_q(\mathfrak{g})$, $\mathcal{C}^\mathbb{Z}$ a certain subcategory of $\mathcal{C}$ and $\mathcal{C}_k$ a certain quotient of $\mathcal{C}^\mathbb{Z}$, $\mathcal{R}$ and $\mathcal{R}_k$ the respective Grothendieck rings of $\mathcal{C}^\mathbb{Z}$ and $\mathcal{C}_k$, Sections 2.3, 3.
- For $\Delta = [c, d]_\rho$, $b(\Delta) = \rho^e_{\rho^d}$, $e(\Delta) = \rho^d_{\rho^d}$, $\hat{\Delta} = [c - 1, d - 1]_\rho$, $\rho = [c + 1, d]_\rho$, $\nu_\rho$ is the character $\nu_\rho(g) = \det(g)^{\nu_\rho}$, $s_\rho$ is a certain element in $\mathbb{R}_{>0}$, Section 2.3.
- Left aligned order $\leq_0$ on segments, right aligned order $\leq_e$ on segments, precede order $<$ on segments, $k$-precede order on segments $<_k$, Sections 2.3, 4.5.
- Irr is the set of irreducible representations in $\mathcal{C}$ up to equivalence. $\text{Irr}_k$ is the set of irreducible representations in $\mathcal{C}_k$ up to equivalence, Sections 2.3, 3.
- $\text{Gr}(k, n)$ the Grassmannian of $k$-planes in $\mathbb{C}^n$ and $\mathbb{C}[\text{Gr}(k, n)]$ its homogeneous coordinate ring; $\mathbb{C}[\text{Gr}(k, n, \sim)]$ the quotient of $\mathbb{C}[\text{Gr}(k, n)]$ by the Plücker coordinates with column set a consecutive interval; $P_{i_1, \ldots, i_n} \in \mathbb{C}[\text{Gr}(k, n)]$ a Plücker coordinate, Section 4.2.
- $\text{SSYT}(k, [n])$ the monoid of rectangular semistandard Young tableaux with $k$ rows and with entries in $[n]$; $\text{SSYT}(k, [n], \sim)$ the monoid of $\sim$-equivalence classes, Section 4.2.
- $\text{ch}(T)$ is the dual canonical basis element in $\mathbb{C}[\text{Gr}(k, n, \sim)]$ corresponding to a tableau $T \in \text{SSYT}(k, [n], \sim)$, Section 4.2.
- $\sim$-matching, $\rightarrow$-matching function, best $\rightarrow$-matching, Section 6.2.
- $X_{m,n}$, $X_{m,n}^{(k)}$, $X_{T,T'}$, are certain sets defined in Section 6.3.
- $\text{LC}(m, n)$, $\text{LC}_k(m, n)$, $\text{LC}(T, T')$ are certain conditions defined in Section 6.3.
• $\text{LI}(\mathbf{m}, \mathbf{n})$, $\text{RI}(\mathbf{m}, \mathbf{n})$, $\text{LI}_k(\mathbf{m}, \mathbf{n})$, $\text{RI}_k(\mathbf{m}, \mathbf{n})$ are certain conditions defined in Section 6.1.

2. Quantum affine algebras and $p$-adic groups

2.1. Quantum affine algebras. Let $\mathfrak{g}$ be a simple Lie algebra of simply laced type, and $I = \{1, \ldots, \ell\}$ be the vertices of its Dynkin diagram. Let $\hat{\mathfrak{g}}$ be the associated affine Lie algebra and $C = (C_{ij})_{0 \leq i, j \leq \ell}$ the Cartan matrix of $\mathfrak{g}$. Fix $q \in \mathbb{C}^*$ which is not a root of unity.

The $q$-numbers, $q$-factorials and $q$-binomial coefficients are defined as follows:

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_q! = [h]_q \cdots [1]_q, \quad \binom{m}{n}_q = \frac{[m]_q!}{[n]_q! [m-n]_q!}.$$  

The quantum affine algebra $U_q(\hat{\mathfrak{g}})$ in Drinfeld’s realization [Dd87] is the $\mathbb{C}$-algebra with generators $x_i^\pm$, $k_i^{\pm 1}$ ($i = 0, \ldots, \ell$) and relations:

$$k_i k_i^{-1} = k_i^{-1} k_i = 1, \quad k_i k_j = k_j k_i, \quad k_i x_j^\pm k_i^{-1} = q^{\varepsilon_j} C_{ij} x_j^\pm, \quad [x_i^+, x_j^-] = \delta_{ij} \frac{k_i^{1} - k_i^{-1}}{q - q^{-1}}, \quad \sum_{r=0}^{1-C_{ij}} \frac{(-1)^r (1-C_{ij})}{r!} q^r (x_i^\pm)^r x_j^\pm (x_i^\pm)^{1-C_{ij} - r} = 0.$$  

The algebra $U_q(\hat{\mathfrak{g}})$ has a structure of a Hopf algebra with the comultiplication $\Delta$ and antipode $S$ given on the generators by the formulas:

$$\Delta(k_i) = k_i \otimes k_i, \quad \Delta(x_i^+) = x_i^+ \otimes 1 + 1 \otimes x_i^+, \quad \Delta(x_i^-) = x_i^- \otimes k_i^{-1} + 1 \otimes x_i^-, \quad S(k_i^{\pm 1}) = k_i^{\mp 1}, \quad S(x_i^+) = -x_i^+ k_i, \quad S(x_i^-) = -k_i^{-1} x_i^-.$$  

2.2. Finite dimensional modules of quantum affine algebras. In this section, we recall the standard facts about finite-dimensional $U_q(\hat{\mathfrak{g}})$-modules and their $q$-characters, see [CP94, CP95, FR99].

Let $\mathcal{C}$ be the category of finite-dimensional $U_q(\hat{\mathfrak{g}})$-modules (of type 1). Chari-Pressley [CP94] have shown that irreducible representations in this category are highest weight modules, and are labeled by Drinfeld polynomials. Here, we write these highest weights as dominant monomials in $\mathbb{Z}[Y_{i,a}| i \in I, a \in \mathbb{C}^*]$, $M = Y_{i_1,a_1} Y_{i_2,a_2} \cdots Y_{i_r,a_r}$. For $M$ such a dominant monomial, let $L(M)$ be the simple module of highest weight $M$. For $i \in I$, $a \in \mathbb{C}^*$, the simple module $L(Y_{i,a})$ is called a fundamental representation.

Let $\hat{I} = \{(i, s) \in I \times \mathbb{Z} | i + s - 1 \in 2\mathbb{Z}\}$\(^1\). For all quantum parameters $a \in \mathbb{C}^*$, let $\mathcal{P}_a^+$ be the dominants monomials of the form $Y_{i_1,a_1} Y_{i_2,a_2}^2 \cdots Y_{i_r,a_r}^r$, for $(i, j, s) \in \hat{I}$. For $a_1, \ldots, a_r$ such that $a_i/a_j \notin q^{2\mathbb{Z}}$ for $i \neq j$, and for $M_i \in \mathcal{P}_{a_i}^+$, the tensor product

$$L(M_1) \otimes L(M_2) \otimes \cdots \otimes L(M_r)$$

\(^1\)It corresponds to the choice of height function $\xi_i = i - 1$, see [HL10]
is irreducible. Thus we can restrict ourselves to the study of the subcategory of representations in \( \mathcal{C} \) whose composition factors have highest weight in one \( \mathcal{P}_a \) (for any choice of \( a \)).

From now on, for \((i, s) \in \hat{I} \), let us write \( Y_{i,s} \) for \( Y_{i,q^s} \). Denote by \( \mathcal{P} \) the free abelian group generated by \( Y_{i,s}^* \), \((i, s) \in \hat{I} \), and let \( \mathcal{P}^+ \) the submonoid of \( \mathcal{P} \) generated by \( Y_{i,s} \), \((i, s) \in \hat{I} \). Let \( \mathcal{R}_q \) be the Grothendieck ring of the category \( \mathcal{C}_q \), it is freely generated, as a commutative ring, by the images of the fundamental modules \( L(Y_{i,s}) \), \((i, s) \in \hat{I} \). For \( \mathfrak{g} = \mathfrak{sl}_k \), we use the notation \( \mathcal{R}_k := \mathcal{R}_{\mathfrak{sl}_k} \).

Let \( \mathbb{Z}\mathcal{P} = \mathbb{Z}[Y_{i,s}^* \mid (i, s) \in \hat{I}] \) be the group ring of \( \mathcal{P} \). Frenkel-Reshetikhin [FR99] have introduced the \( q \)-character, it is an injective ring morphism

\[ \chi_q : \mathcal{R}_q \to \mathbb{Z}\mathcal{P}. \]

For a \( U_q(\mathfrak{g}) \)-module \( V \), \( \chi_q(V) \) encodes the decomposition of \( V \) into common generalized eigenspaces for the action of a large commutative subalgebra of \( U_q(\mathfrak{g}) \) (the loop-Cartan subalgebra).

**Example 2.1.** For \( \mathfrak{g} = \mathfrak{sl}_2 \), the \( q \)-character of the fundamental representation \( L(Y_{1,0}) \) is

\[ \chi_q(L(Y_{1,0})) = Y_{1,0} + Y_{1,-1}^{-1}. \]

### 2.3. Representations of \( p \)-adic groups.

We recall certain results about representations of \( p \)-adic groups, [BZ77, Zel80, LM18].

Let \( F \) be a non-archimedean local field with a normalized absolute value \( |\cdot| \). For any reductive group \( G \) over \( F \), let \( \mathcal{C}(G) \) be the category of complex, smooth representations of \( G(F) \) of finite length and let \( \text{Irr} G \) be the set of irreducible objects of \( \mathcal{C} \) up to equivalence. Let \( G_N = GL_N(F), \ N = 0, 1, 2, \ldots \). For \( \pi_i \in \mathcal{C}(G_N^i), i = 1, 2 \), denote by \( \pi_1 \times \pi_2 \in \mathcal{C}(G_{N_1+N_2}) \) the representation which is parabolically induced from \( \pi_1 \otimes \pi_2 \). The parabolic induction endows the category \( \oplus_{N \geq 0} \mathcal{C}(G_N) \) with the structure of a tensor category.

Denote by \( \mathcal{R}_N^G \) (resp. \( \mathcal{R}^G \)) the Grothendieck ring of \( \mathcal{C}(G_N) \) (resp. \( \mathcal{C} = \oplus_{N \geq 0} \mathcal{C}(G_N) \)). Then \( \mathcal{R}^G = \oplus_{N \geq 0} \mathcal{R}_N^G \) is a commutative graded ring under \( \times \). Denote \( \text{Irr} G_N \) and denote by \( \text{Irr}_c \subset \text{Irr} \) the subset of supercuspidal representations of \( G_N, N > 0 \). For \( \pi \in \mathcal{C}(G_N) \), we denote \( \deg(\pi) = N \).

To every supercuspidal representation \( \rho \) of \( GL_m(F) \), one can associate an unramified character \( \nu_\rho \) of \( F^* \) of the form \( \nu_\rho = |\cdot|^s \rho \), where \( s_\rho \in \mathbb{R}_{\geq 0} \) with the property that for any supercuspidal representation \( \rho' \) of \( GL_{m'}(F) \), \( m' \in \mathbb{Z}_{\geq 1} \), the induced representation \( \rho \times \rho' \) is reducible if and only if \( m' = m \), and either \( \rho' = \rho \nu_\rho \) or \( \rho = \rho' \nu_\rho \).

For \( \rho \in \text{Irr}_c \), we denote \( \overrightarrow{\rho} = \rho \nu_\rho \), \( \overleftarrow{\rho} = \rho \nu_\rho^{-1} \). A segment is a finite nonempty subset of \( \text{Irr}_c \) of the form \( \Delta = \{ \rho_1, \ldots, \rho_r \} \), where \( \rho_{i+1} = \overrightarrow{\rho_i} \), \( i \in [r-1] \). We denote \( \delta(\Delta) = \rho_1, e(\Delta) = \rho_r \), and \( \deg(\Delta) = \sum_{i=1}^{r} \rho_i \) ( \( \deg(\Delta) \) is called the degree of \( \Delta \)). Usually we write \( \Delta \) as \( \Delta = [b(\Delta), e(\Delta)] \). For a segment \( \Delta = \{ \rho_1, \ldots, \rho_r \} \), we denote \( Z(\Delta) = \text{soc}(\rho_1 \times \cdots \times \rho_r) \in \text{Irr}_c(\deg \Delta). \)
where \( \text{soc}(\pi) \) denotes the socle of \( \pi \), i.e., the largest semisimple subrepresentation of \( \pi \). We use the convention that \( Z(\emptyset) = 1 \).

A multisegment is a formal finite sum \( \mathbf{m} = \sum_{i=1}^{m} \Delta_i \) of segments. Let \( \text{Mult} \) denote the resulting commutative monoid of multisegments. Denote \( \deg \mathbf{m} = \sum_{i=1}^{m} \deg(\Delta_i) \).

For a segment \( \Delta = \{\rho_1, \ldots, \rho_r\} \), denote
\[
\vec{\Delta} = (\vec{\rho}_1, \ldots, \vec{\rho}_r), \quad \hat{\Delta} = (\hat{\rho}_1, \ldots, \hat{\rho}_r).
\]

For two segments \( \Delta, \Delta' \), say that \( \Delta \) precedes \( \Delta' \) (denoted by \( \Delta \prec \Delta' \)) if
\[
b(\Delta) + 1 \leq b(\Delta') \leq e(\Delta) + 1 \leq e(\Delta'). \tag{2.1}
\]
If either \( \Delta \prec \Delta' \) or \( \Delta' \prec \Delta \), we say that \( \Delta \) and \( \Delta' \) are linked.

Let \( \mathbf{m} = \sum_{i=1}^{m} \Delta_i \) be a multisegment. We say that \( \mathbf{m} \) is ordered if \( \Delta_i \neq \Delta_j \) for all \( i < j \).

For an ordered multisegment \( \mathbf{m} = \sum_{i=1}^{m} \Delta_i \), we denote
\[
\zeta(\mathbf{m}) = Z(\Delta_1) \times \cdots \times Z(\Delta_m) \in \mathcal{C}(G_{\deg \mathbf{m}}),
\]
and \( Z(\mathbf{m}) = \text{soc}(\zeta(\mathbf{m})) \in \text{Irr} G_{\deg \mathbf{m}} \). The map
\[
\text{Mult} \to \text{Irr}, \quad \mathbf{m} \mapsto Z(\mathbf{m})
\]
is a bijection (the Zelevinsky Classification), see [BZ77, Zel80].

For a supercuspidal representation \( \rho \), we write a segment \( \{\rho^i : i \in [a, b]_\rho\} \) as \([a, b]_\rho \) or \([a, b] \), \( a, b \in \mathbb{Z}, a \leq b \). For \( \Delta = [c, d]_\rho \), denote \( b(\Delta) = \rho^c, e(\Delta) = \rho^d, \hat{\Delta} = [c-1, d-1]_\rho, \).

For any \( \pi \in \text{Irr} \), there exist supercuspidal representations \( \rho_1, \ldots, \rho_r \), uniquely determined up to permutation, such that \( \pi \) is a subrepresentation of \( \rho_1 \times \cdots \times \rho_r \). Denote by \( \text{supp}(\pi) = \{\rho_1, \ldots, \rho_r\} \) (not multiset) and it is called supercuspidal support of \( \pi \) [LM16].

For a supercuspidal representation \( \rho \), denote \( \text{Irr}_\rho = \{\pi \in \text{Irr} : \text{supp}(\pi) \subseteq Z_\rho\} \), where \( Z_\rho = \{\rho^a : a \in \mathbb{Z}\} \) is the cuspidal line of \( \rho \). If the lines \( \rho_1, \rho_2, \ldots, \rho_r \) are distinct, then for \( \pi_i \in \text{Irr}_{\rho_i} \), the representation
\[
\pi_1 \times \pi_2 \times \cdots \times \pi_r
\]
is irreducible. Thus in practice, it is enough to consider representations with support inside a single supercuspidal line. We denote by \( \mathcal{C}^Z \) the Serre subcategory of \( \bigoplus_{N \geq 0} \mathcal{C}(G_N) \) of representations whose supercuspidal support is a fixed line \( Z_\rho \) (since these categories are all equivalent, the choice of the supercuspidal representation \( \rho \) does not matter), its irreducible representations are given by \( (\text{Irr}_\rho =) \text{Irr}^Z \), which are indexed by the elements of \( \text{Mult} \). Let \( \mathcal{R} \) be the Grothendieck ring of this category, it is freely generated, as a commutative ring, by the images of the \( Z(\Delta) \), for all \( \Delta \in \text{Seg} \).

There are two orders on segments in a multisegment: the left and right aligned orders \( \leq_b \) and \( \leq_e \). For segments \( \Delta, \Delta' \), we have \( \Delta \leq_b \Delta' \) if \( b(\Delta) \leq b(\Delta') \), and either \( b(\Delta) < b(\Delta') \) or \( e(\Delta) \leq e(\Delta') \). We have \( \Delta \leq_e \Delta' \) if \( e(\Delta) \leq e(\Delta') \), and either \( e(\Delta) < e(\Delta') \) or \( b(\Delta) \leq b(\Delta') \).
Note that if the segments $\Delta_i$ of a multisegment $m$ are ordered decreasingly for either the left or right aligned order, then the multisegment $m$ is ordered. Thus any multisegment can be ordered.

3. Quantum Affine Schur-Weyl duality

The correspondence between finite dimensional representations of quantum affine algebras of type $A$ and finite length representations of $p$-adic general linear groups can be formalized using quantum affine Schur-Weyl duality.

3.1. The functor $F$. The quantum Schur-Weyl duality was introduced by Jimbo in [Jim86]. Let $V = \mathbb{C}^k$ be the natural representation of the quantum algebra $U_q(\mathfrak{sl}_k)$. Then for all $N \geq 1$, there exists a left action of the Hecke algebra $H_N(q^2)$ on $V \otimes^N$ which commutes with the action of $U_q(\mathfrak{sl}_k)$. For $M$ a right $H_N(q^2)$-module, one can equip $M \otimes_{H_N(q^2)} V \otimes^N$ with a left $U_q(\mathfrak{sl}_k)$-module structure. Moreover, if $N < k$, the functor

$$M \rightarrow M \otimes_{H_N(q^2)} V \otimes^N$$  \hspace{1cm} (3.1)

is an equivalence of categories between the category of finite-dimensional $H_N(q^2)$-modules and the category of finite-dimensional $U_q(\mathfrak{sl}_k)$-modules of level $N$.

In [CP96] Chari-Pressley extended this result to the affine case. They showed that if $M$ is a right module for the affine Hecke algebra $\hat{H}_n(q^2)$, then the image of (3.1) can be endowed with a left $U_q(\hat{\mathfrak{sl}}_k)$-module structure. Thus defining a functor between the category of finite-dimensional modules for the affine Hecke algebra $\hat{H}_N(q^2)$ the category $\mathcal{C}_{\hat{\mathfrak{sl}}_k}^N$ of finite-dimensional $U_q(\hat{\mathfrak{sl}}_k)$-modules of level $N$. If $N < k$, this functor is again an equivalence of categories.

On the other hand, by a result of Heiermann [Hei11] (see also [Bor76] [Cas80], [BK98], [SS12] or the survey paper [Gur21a]), there exists an equivalence of categories between smooth, finite-length representations of $GL_N(F)$ in a simple Bernstein block, and the category of finite-dimensional right $\hat{H}_n(q^2)$-modules.

Thus, for $k > N$, after reduction to cuspidal lines in the $p$-adic case, and restriction of the quantum parameters for representations of the quantum affine algebra, we obtain equivalences of categories:

$$\mathcal{F}_{k,N} : \mathcal{C}^Z(G_N) \sim \mathcal{C}_{\hat{\mathfrak{sl}}_k}^{N,Z}. \hspace{1cm} (3.2)$$

By summing over $N$, we get a functor $\mathcal{F}$:

$$\mathcal{F} : \mathcal{C}^Z \rightarrow \bigoplus_{N \geq 0} \mathcal{C}_{\hat{\mathfrak{sl}}_k}^{N,Z} =: \mathcal{C}_k. \hspace{1cm} (3.3)$$

We recall some properties of the functor $\mathcal{F}$. 
Proposition 3.1. (1) The functor $\mathcal{F}$ is a tensor functor between the tensor category $\mathcal{C}^\mathbb{Z}$ and the tensor category $\mathcal{C}_k$. That is, for any pair of representations $\pi_1, \pi_2$,

$\mathcal{F}(\pi_1 \times \pi_2) \cong \mathcal{F}(\pi_1) \otimes \mathcal{F}(\pi_2)$.

(2) The functor $\mathcal{F}$ is exact.

(3) The functor $\mathcal{F}$ send simple modules to simple modules. For a multisegment $m \in \text{Mult}$, $\mathcal{F}(Z(m)) \cong L(M_m)$, where the factors in the monomial are obtained from the segments of $m$ by the following correspondence:

$$\begin{cases} [a, b] & \mapsto Y_{b-a+1, a-b}, \\ \left[\frac{i-p}{2}, \frac{i-p-1}{2}\right] & \mapsto Y_{i,p} \end{cases}$$

(3.4)

In particular, for $a \in \mathbb{Z}$, the image of the segment $[a, a]$ is a fundamental representation

$\mathcal{F}(Z([a, a])) \cong L(Y_{1, -2a})$.

3.2. Quotient and localization of category. Note the following result, which can be translated directly from [KKK18]:

Proposition 3.2 ([KKK18, Proposition 4.3.1, Theorem 4.3.3]). Let $[a, b]$ be a segment of length $\ell = b - a + 1$. Then we have

$$\mathcal{F}(Z([a, b])) \cong \begin{cases} L(Y_{\ell,-a-b}) & \text{if } 1 \leq \ell \leq k - 1, \\ \mathbb{C} & \text{if } \ell = 0 \text{ or } \ell = k, \\ 0 & \text{if } \ell > k. \end{cases}$$

(3.5)

More generally, if $m = \Delta_1 + \cdots + \Delta_N$ is a multisegment and $\ell_1, \ldots, \ell_N$ are the corresponding lengths of the segments, then

- if $\ell_i > k$ for at least one $1 \leq i \leq N$, then $\mathcal{F}(Z(m)) \cong 0$,
- if $\ell_i \leq k$ for all $1 \leq i \leq N$, then $\mathcal{F}(Z(m))$ is a simple module.

Following [KKK18, Section 4], we introduce a quotient of the category $\mathcal{C}^\mathbb{Z}$ in order to kill all simple modules which are send to 0 by the functor $\mathcal{F}$.

Let $\mathcal{S}_k$ be the smallest Serre subcategory of $\mathcal{C}^\mathbb{Z}$ such that

1. $\mathcal{S}_k$ contains $Z([a, a + k])$, for all $a \in \mathbb{Z}$,
2. for all $\pi_1 \in \mathcal{S}$ and $\pi_2 \in \mathcal{C}^\mathbb{Z}$, then $\pi_1 \times \pi_2$ and $\pi_2 \times \pi_1$ are in $\mathcal{S}_k$.

Consider the quotient category $\mathcal{C}^\mathbb{Z}/\mathcal{S}_k$, and let $Q : \mathcal{C}^\mathbb{Z} \to \mathcal{C}^\mathbb{Z}/\mathcal{S}_k$ the canonical functor. The category $\mathcal{C}^\mathbb{Z}/\mathcal{S}_k$ is an abelian tensor category. Since the functor $\mathcal{F}$ send $\mathcal{S}_k$ to 0, it factors through $Q$ and we obtain an exact tensor functor:

$$\mathcal{F}' : \mathcal{C}^\mathbb{Z}/\mathcal{S}_k \to \mathcal{C}_k.$$  

(3.6)

We have a description of the simple objects in $\mathcal{C}^\mathbb{Z}/\mathcal{S}_k$. 


Proposition 3.3 ([KKK18, Proposition 4.4.1]). There is a bijection:

\[
\begin{align*}
\{ \text{simple objects} & \text{ in } C^Z/S_k \} \leftrightarrow \{ \text{multisegments } \Delta_1 + \cdots + \Delta_N \} \\
& \text{ with } \ell(\Delta_i) \leq k, \forall 1 \leq i \leq N
\end{align*}
\]

\[Q(Z(\Delta_1 + \cdots + \Delta_N)) \leftrightarrow \Delta_1 + \cdots + \Delta_N.\]

Note from Proposition 3.2, that for all \(a \in \mathbb{Z}\), the images of the \(\mathcal{F}'(Z([a, a + k - 1]))\) are isomorphic to the trivial representation. Thus following [KKK18] we can localize \(C^Z/S_k\) to reflect this fact.

For all \(a \in \mathbb{Z}\), let \(Z_a = Z([a, a + k - 1])\). Then the \((Z_a)_{a \in \mathbb{Z}}\) form a commuting family of central objects in the quotient category \(C^Z/S_k\), in the sense of [KKK18, Appendix A]. Let \(\mathcal{C}'_k\) be the localization \((C^Z/S_k)[Z_a^{-1} \mid a \in \mathbb{Z}]\) by this commuting family. Let \(\mathcal{C}_k\) denote the tensor category \((C^Z/S_k)[Z_a \simeq 1 \mid a \in \mathbb{Z}]\), as in [KKK18, A.7]. There is an exact monoidal functor \(\Omega:\)

\[\Omega : C^Z/S_k \rightarrow C_k.\]

Let us denote by \(\overline{Q}\) the composition of the functors \(Q\) and \(\Omega:\)

\[\overline{Q} : C^Z \rightarrow C^Z/S_k \rightarrow \Omega : C_k.\]

Recall some properties of the functor \(\overline{Q}:\)

Proposition 3.4 ([KKK18, Theorem B.1.1.-B.1.2., Lemma A.7.1.-A.7.2.]). The functor \(\overline{Q}\) satisfies the following:

\begin{enumerate}
\item \(\overline{Q}\) is exact,
\item every exact sequence in \(\mathcal{C}_k\) is isomorphic under \(\overline{Q}\) to the image of an exact sequence in \(C^Z\),
\item if \(\pi \in S_k\), then \(\overline{Q}(\pi) \simeq 0\),
\item for all \(a \in \mathbb{Z}\), \(\overline{Q}(Z_a) \simeq 1\), the trivial representation,
\item simple objects in \(\mathcal{C}_k\) are isomorphic to the images under \(\overline{Q}\) of simple objects in \(C^Z\) which are not in \(S_k\).
\end{enumerate}

More precisely, by combining (3),(4) and (5),

\[\overline{Q}(Z(\Delta_1 + \cdots + \Delta_N)) \leftrightarrow \Delta_1 + \cdots + \Delta_N.\]

As \(k \in \mathbb{Z}_{>0}\) is fixed, we will denote by the images under \(\overline{Q}\) by \(\overline{\pi}\), for \(\pi\) a representation in \(C^Z\), or \(\overline{Z}(\mathbf{m}) = \overline{Z(\mathbf{m})}\), for \(\mathbf{m}\) a multisegment.

We have the following.
Theorem 3.5 ([KKK18, Theorem 4.6.5, Proposition A.7.3]). The functor $\mathcal{F} : \mathcal{C}^Z/\mathcal{S}_k \to \mathcal{C}_k$ factors through $\mathcal{C}_k$, leading to an exact functor:

$$\tilde{\mathcal{F}} : \mathcal{C}_k \to \mathcal{C}_k.$$  \hfill (3.8)

Based on similar results between representations of quantum affine algebras and representations of quiver Hecke algebras (see [Fuj22]), we formulate the following:

Conjecture 3.6. The functor $\tilde{\mathcal{F}}$ is an equivalence of categories.

3.3. Grothendieck ring approach. From now on, let us drop the superscript $Z$ for clarity of notations. For $N < k$, $\mathcal{F}_{k,N}$ being an exact functor, the equivalence of categories (3.2) induces an isomorphism of Grothendieck groups:

$$\varphi_{k,N} : \mathcal{R}_N^G \xrightarrow{\sim} K_0(\mathcal{C}_{sk}^N).$$  \hfill (3.9)

As the tensor structures are compatible, summing over $N$ and taking $k$ to infinity, we get an isomorphism of rings:

$$\varphi : \mathcal{R} \xrightarrow{\sim} K_0(\text{Rep}(U_q(\hat{\mathfrak{sl}}_{\infty}))).$$  \hfill (3.10)

Let us make a note on the structure of $K_0(\text{Rep}(U_q(\hat{\mathfrak{sl}}_{\infty})))$. The quantum infinite rank cluster algebra $U_q(\hat{\mathfrak{sl}}_{\infty})$ is a well-defined generalized Kac-Moody algebra, but whose representations have been less studied than that of the finite-rank quantum affine algebra (see for example [KS10, KS15]). Here, we take as definition for the ring $K_0(\text{Rep}(U_q(\hat{\mathfrak{sl}}_{\infty})))$ the colimit cluster algebra obtained from the cluster algebra structures of the Grothendieck rings $\mathcal{R}_k$.

Indeed, Hernandez-Leclerc have introduced a cluster algebra structure on the Grothendieck rings of some subcategories of $\mathcal{C}_k$ in [HL10], then of the “half” category $\mathcal{C}_k^-$ in [HL16]. More recently, Kashiwara-Kim-Oh-Park [KKOP22] have given a cluster algebra structure on the Grothendieck ring $\mathcal{R}_k$ of the full category $\mathcal{C}_k$. For $k \geq 1$, the corresponding cluster algebra $\mathcal{A}(k)$ is given by an initial seed for which the quiver is a $\infty \times (k - 1)$-grid with alternating arrows. For $(m,i) \in \mathbb{Z}_{\geq 1} \times I$, the corresponding initial cluster variable is the class of a Kirillov-Reshetikhin module $[W_{m,i,m}^{(i)}]$, where

$$W_{m,i}^{(i)} = L(Y_{i,r}Y_{i,r+2} \cdots Y_{i,r+2m-2}).$$

These cluster algebras are embedded into each other, up to freezing the boundary vertices (see Example below).

Example 3.7. Here we see how because of the inclusion of quivers, the cluster algebra $\mathcal{A}(4)$ is a sub-cluster algebra of $\mathcal{A}(5)$:
Note that the initial cluster seeds considered here are full sub-seed of one another, and connected only by coefficients, in the sense of \cite{GG18}. Thus, as in the same work, one can define the colimit $\mathcal{A}(\infty)$ of these cluster algebras, its initial quiver is then an infinite grid on the quarter plane, with alternating arrows. Let us take here this cluster algebra $\mathcal{A}(\infty)$ as the definition for the ring $K_0(\text{Rep}(U_q(\mathfrak{sl}_\infty)))$.

### 3.4. Quotients of Grothendieck rings

In this section, we make analog considerations as in section 3.2, in terms of Grothendieck rings.

For $k \in \mathbb{Z}_{\geq 1}$, let $\mathcal{I}_k$ be the ideal of the Grothendieck ring $\mathcal{R}$ generated by the classes of the segments representations $[Z([a, a + m])]$, with $a \in \mathbb{Z}$ and $m \geq k$, and by the $[Z_a] - 1$, for $a \in \mathbb{Z}$. Then let $\mathcal{R}_k$ be the quotient:

$$\mathcal{R}_k := \mathcal{R}/\mathcal{I}_k.$$  

**Remark 3.8.** It is expected, in line with Conjecture 3.6, that $\mathcal{R}_k$ is the Grothendieck ring of the category $\mathcal{C}_k$.

The following provides some information on the quotient ring $\mathcal{R}_k$.

**Proposition 3.9.** For $\mathbf{m} = \Delta_1 + \Delta_2 + \cdots + \Delta_N$ a multisegment, its image in the quotient ring $\mathcal{R}_k$ is

$$[Z(\mathbf{m})]_k = \begin{cases} 
0 & \text{if } \mathbf{m} \notin \text{Mult}_{k+1}, \\
\frac{Z(\overline{\mathbf{m}^k})}{Z(\overline{\mathbf{m}})} & \text{else,}
\end{cases}$$  

(3.11)

where $\overline{\mathbf{m}}^k = \{ \Delta_i \in \mathbf{m} \mid \ell(\Delta_i) < k \}$.

This can be proven outside the context of the quantum affine Schur-Weyl duality, using know properties on $\mathcal{C}^\mathbb{Z}$. We first need the following technical lemma.

**Lemma 3.10.** Let $\begin{cases} a_1 \leq a_2 \leq \cdots \leq a_N, \\
b_1 \geq b_2 \geq \cdots \geq b_N
\end{cases}$ be two $N$-tuples of integers. Let $\sigma, \sigma'$ be permutations in $\mathcal{S}_N$, such that $\sigma$ satisfies $\sigma(i) < \sigma(i + 1)$ whenever $a_i = a_{i+1}$ and $\sigma(i)^{-1} < \sigma(i + 1)^{-1}$ whenever $b_i = b_{i+1}$.
Then if $\sigma' \leq \sigma$ (for the Bruhat order),
\[
\max \{ b_{\sigma'(i)} - a_i \mid i \in [N] \} \geq \max \{ b_{\sigma(i)} - a_i \mid i \in [N] \},
\]
with equality if and only if $\sigma'(i_0) = \sigma(i_0)$, for all $i_0$ such that $b_{\sigma(i_0)} - a_{i_0} = \max \{ b_{\sigma(i)} - a_i \}$.

**Proof.** If $\sigma = \text{id}$, then the result is trivial. Otherwise, consider $i < j$ such that $\sigma(i) > \sigma(j)$, and let $\sigma' = \sigma \cdot (ij)$. Then $b_{\sigma'(i)} - a_i = b_{\sigma(j)} - a_i > b_{\sigma(j)} - a_j$, as $a_j > a_i$ by the condition on $\sigma$. On the other hand, $b_{\sigma'(i)} - a_i > b_{\sigma'(i)} - a_i$, as $b_{\sigma'(i)} > b_{\sigma'(i)}$, by the condition on $\sigma$. Thus
\[
b_{\sigma'(i)} - a_i > \max \{ b_{\sigma(i)} - a_i, b_{\sigma(j)} - a_j \}.
\]

Similarly, we have $b_{\sigma'(j)} - a_j < \min \{ b_{\sigma(i)} - a_i, b_{\sigma(j)} - a_j \}$.

Thus if neither $i$ nor $j$ are indexes where the maximal difference is reached,
\[
\max \{ b_{\sigma'(i)} - a_i \mid i \in [N] \} = \max \{ b_{\sigma(i)} - a_i \mid i \in [N] \}.
\]

Otherwise,
\[
\max \{ b_{\sigma'(i)} - a_i \mid i \in [N] \} > \max \{ b_{\sigma(i)} - a_i \mid i \in [N] \}.
\]

The general result follows by transitivity of the Bruhat order. $\square$

We now prove Proposition 3.9.

**Proof.** As in [LM18], consider two $N$-tuples of integers $a_1 \leq a_2 \leq \cdots \leq a_N$ and $b_1 \geq b_2 \geq \cdots \geq b_N$, and define $\sigma \in S_N$ such that
\[
m = [a_{\sigma^{-1}(1)}, b_1] + [a_{\sigma^{-1}(2)}, b_2] + \cdots + [a_{\sigma^{-1}(N)}, b_N],
\]
and satisfying $\sigma(i) < \sigma(i + 1)$ whenever $a_i = a_{i+1}$ and $\sigma(i)^{-1} < \sigma(i + 1)^{-1}$ whenever $b_i = b_{i+1}$.

Then by [LM18, Corollary 10.1]
\[
Z(m) = \sum_{\sigma' \leq \sigma} \text{sgn}(\sigma') P_{\sigma', \sigma} \prod_{i=1}^{N} Z([a_{\sigma'(i)}, b_i]) \in \mathcal{R}, \quad (3.12)
\]
where $P_{\sigma', \sigma}$ denotes the Kazhdan–Lusztig polynomial with respect to $S_N$.

Suppose now there exists $i \in [N]$ such that $b_i - a_{\sigma^{-1}(i)} \geq k$ (that is $\ell(\Delta_i) \geq k + 1$). Then by Lemma 3.10, for all $\sigma' \leq \sigma$, there exists $j \in [N]$ such that $b_j - a_{\sigma'^{-1}(j)} \geq k$. Hence by (3.12), $[Z(m)]_k = 0$.

Suppose now that $\max\{e(\Delta_i) - b(\Delta_i)\} < k$ and let $J = \{i \in [N] \mid e(\Delta_i) - b(\Delta_i) = k - 1\}$. Then from Lemma 3.10, in the image of (3.12) in $\mathcal{R}_k$, the terms such that $\sigma'(i) \neq \sigma(i)$ for at least one $i \in J$ are sent to 0. The result is
\[
[Z(m)]_k = \sum_{\bar{\sigma}' \leq \bar{\sigma} \in S_{[N] \setminus J}} P_{\bar{\sigma}', \bar{\sigma}} \prod_{i \in [N] \setminus J} Z([a_{\bar{\sigma}'^{-1}(i)}, b_i]),
\]
where $\bar{\sigma}, \bar{\sigma}'$ denote the "flattening" of the permutations $\sigma, \sigma'$, when one has removed all $i \in J$. 
From [BW03, Lemma 17], we know that for \( \sigma' \in \mathcal{S}_N \) such that \( \sigma'(i) = \sigma(i) \) for all \( i \in J \), then \( \sigma' \leq \sigma \) if and only if \( \tilde{\sigma}' \leq \tilde{\sigma} \). If we consider \( \Delta(\sigma', \sigma) = \{ i \in [N] \mid \exists j, \sigma' < \sigma'(ij) \leq \sigma \} \), then from Lemma 3.10, \( J \cap \Delta(\sigma', \sigma) = \emptyset \). Thus, using [BW03, Lemma 39], we know that \( P_{\sigma', \sigma} = P_{\sigma', \sigma} \). Thus \( [Z(m)]_k = [Z(m')]_k \). □

Now consider the quotient ring:

\[
K_0(\text{Rep}(U_q(\hat{\mathfrak{sl}}_\infty))/\varphi(\mathcal{I}_k)).
\]

Note that \( \varphi(\mathcal{I}_k) \) is the ideal of \( K_0(\text{Rep}(U_q(\hat{\mathfrak{sl}}_\infty))) \) generated by the classes \( [L(Y_{i,r})] \), with \( i \geq k + 1, r \in \mathbb{Z} \) and \( [L(Y_{k,r})] - 1 \).

**Proposition 3.11.** We have the following isomorphism of rings:

\[
\mathcal{R}_k \cong \mathcal{R}_k \quad (= K_0(\text{Rep}(U_q(\hat{\mathfrak{sl}}_\infty)))).
\]

**Proof.** The ring isomorphism \( \varphi \) induces an isomorphism \( \tilde{\varphi} \) between the quotient rings:

\[
\tilde{\varphi} : \mathcal{R}_k \cong K_0(\text{Rep}(U_q(\hat{\mathfrak{sl}}_\infty))/\varphi(\mathcal{I}_k)).
\]

Thus, all is there left to prove is that \( K_0(\text{Rep}(U_q(\hat{\mathfrak{sl}}_\infty))/\varphi(\mathcal{I}_k)) \cong \mathcal{R}_k \).

Using Proposition 3.9, we know that all \( \mathcal{I}_k \) contains all classes \( [Z(m)] \), where the multisegment \( m \) contains at least one segment of length \( \geq k + 1 \). Thus \( \varphi(\mathcal{I}_k) \) contains all classes \( [L(M)] \) with at least one factor \( Y_{i,r} \) of \( M \) such that \( i \geq k + 1 \).

We now use the cluster algebra structure of \( K_0(\text{Rep}(U_q(\hat{\mathfrak{sl}}_\infty))) \), as in Section 3.3. Recall that the cluster algebra \( \mathcal{A}(\infty) \) is built on a \( \mathbb{Z}_2 \)-quiver, with initial variables certain classes of Kirillov-Reshetikhin modules \( [W_{m,r}^{(i)}] \). We can see that the initial cluster variables \( [W_{m,r}^{(i)}] \), with \( i \geq k + 1 \) are sent to 0 is the quotient, similarly the classes \( [W_{m,r}^{(k)}] \) are sent to 1. Thus, as rings, the quotient \( \mathcal{A}(\infty)/\varphi(\mathcal{I}_k) \) is isomorphic to \( \mathcal{A}(k) \). Hence,

\[
K_0(\text{Rep}(U_q(\hat{\mathfrak{sl}}_\infty))/\varphi(\mathcal{I}_k)) \cong \mathcal{R}_k,
\]

which concludes the proof. □

### 4. Dominant monomials, tableaux, and multisegments

In this section, we summarize the correspondences between the combinatorial data associated to the ring \( \mathcal{R}_k \) and Grassmannians cluster algebras.

**4.1. Dominant monomials and multisegments.** According to results in Section 3, under the equivalence of categories, multisegments and dominant monomials are identified via the following correspondence between segments and fundamental monomials:

\[
\begin{align*}
\text{Seg}_k = \{ [a,b] \mid b - a + 1 < k \} & \cong \mathcal{P}^+ = \{ Y_{i,p} \mid (i,p) \in \hat{I} \}, \\
[a,b] & \mapsto Y_{b-a+1,a-b}, \\
[\frac{1-i-p}{2}, \frac{i-p-1}{2}] & \leftrightarrow Y_{i,p}
\end{align*}
\] (4.1)
We denote this correspondence by $m \mapsto M_m$ and $M \mapsto m_M$ accordingly.

**Remark 4.1.** The correspondence between dominant monomials and multisegments we use here is different from the one used in [CDFL20]. The one used in [CDFL20] is given by $[a,b] \mapsto Y_{b-a+1,a+b-1}$. Accordingly, the correspondence between dominant monomials and semistandard Young tableaux used in this paper is also different from the one used in [CDFL20].

### 4.2. Grassmannian cluster algebras and semistandard Young tableaux

Let us recall some facts about semistandard Young tableaux and their relation to the Grassmannian cluster algebra.

A **semistandard Young tableau** is a Young tableau with weakly increasing rows and strictly increasing columns. For $k, n \in \mathbb{Z}_{\geq 1}$, we denote by $\text{SSYT}(k, [n])$ the set of rectangular semistandard Young tableaux with $k$ rows and $n$ entries in $[n]$ (with arbitrarily many columns). The empty tableau is denoted by $\emptyset$.

For $S, T \in \text{SSYT}(k, [n])$, let $S \cup T$ be the row-increasing tableau whose $i$th row is the union of the $i$th rows of $S$ and $T$ (as multisets), [CDFL20].

We call $S$ a factor of $T$, and write $S \subset T$, if the $i$th row of $S$ is contained in that of $T$ (as multisets), for $i \in [k]$. In this case, we define $\frac{T}{S} = S^{-1}T = TS^{-1}$ to be the row-increasing tableau whose $i$th row is obtained by removing that of of $S$ from that of $T$ (as multisets), for $i \in [k]$.

A tableau $T \in \text{SSYT}(k, [n])$ is **trivial** if each entry of $T$ is one less than the entry below it.

For any $T \in \text{SSYT}(k, [n])$, we denote by $T_{\text{red}} \subset T$ the semistandard tableau obtained by removing a maximal trivial factor from $T$. For trivial $T$, one has $T_{\text{red}} = \emptyset$.

Let “$\sim$” be the equivalence relation on $S, T \in \text{SSYT}(k, [n])$ defined by: $S \sim T$ if and only if $S_{\text{red}} = T_{\text{red}}$. We denote by $\text{SSYT}(k, [n], \sim)$ the set of $\sim$-equivalence classes.

Denote by $\text{Gr}(k, n)$ the Grassmannian of $k$-planes in $\mathbb{C}^n$ and $\mathbb{C}[\text{Gr}(k, n)]$ its homogeneous coordinate ring. Define $\mathbb{C}[\text{Gr}(k, n, \sim)]$ to be the quotient of $\mathbb{C}[\text{Gr}(k, n)]$ by the ideal generated by $P_{i_1, \ldots, i_n} - 1$ with $\{i_1, \ldots, i_n\}$ being a consecutive interval, and $P_{i_1, \ldots, i_n} \in \mathbb{C}[\text{Gr}(k, n)]$ being the Plücker coordinate.

In [CDFL20], it is shown that the elements in the dual canonical basis of $\mathbb{C}[\text{Gr}(k, n, \sim)]$ are in bijection with semistandard Young tableaux in $\text{SSYT}(k, [n], \sim)$. The elements in the dual canonical basis of $\mathbb{C}[\text{Gr}(k, n, \sim)]$ are in bijection with simple modules in a certain category of finite dimensional $U_q(\mathfrak{sl}_k)$-modules [HL10].

A one-column tableau is called a **fundamental tableau** if its content is $[i, i+k] \setminus \{r\}$ for $r \in \{i+1, \ldots, i+k-1\}$. A tableau $T$ is said to have **small gaps** if each of its columns is a fundamental tableau. Then any tableau in $\text{SSYT}(k, [n])$ is $\sim$-equivalent to a unique small gap semistandard tableau.
By [CDFL20, Theorem 5.8], for every $T \in \text{SSYT}(k, \lbrack n \rbrack)$, the corresponding element $\text{ch}(T)$ in the dual canonical basis of $\mathbb{C}[[\text{Gr}(k,n,\sim)]]$ is given by

$$\text{ch}(T) = \sum_{u \in S_m} (-1)^{f(uw_T)} p_{uw_0,w_Tw_0}(1) P_{u,T} \in \mathbb{C}[[\text{Gr}(k,n,\sim)]]$$

(4.2)

where $m$ is the number of columns of $T'$, $T'$ is the small gap tableau such that $T \sim T'$, $P_{u,T}$ is some monomial of Plücker coordinates, $w_T$ is some permutation in $S_k$, $p_{u,v}(q)$ is a Kazhdan-Lusztig polynomial.

### 4.3. Tableaux and multisegments

For $a, b \in \mathbb{Z}$ with $a \leq b < a+k-1$, denote by $T_{a,b}$ the fundamental tableau with entries $\{1-a,2-a,\ldots,k-a+1\}\setminus\{k-b\}$. Denote by $\text{Fund}(k, \lbrack n \rbrack)$ the set of fundamental tableaux with $k$ rows and with entries in $\lbrack n \rbrack$. From [CDFL20], the tableau $T_{a,b}$ corresponds to the segment $[a,b]$. We have:

$$\text{Seg}_k := \{(a,b) \in \text{Seg}_k \mid k + 1 - n \leq a \leq 0\} \simeq \text{Fund}(k, \lbrack n \rbrack),$$

$$\iff T_{a,b},$$

$$[1-i,k-r] \leftrightarrow T_{\{i+1,\ldots,i+k-1\}\setminus\{r\}}.$$  

(4.3)

Any tableau $T$ can be decomposed uniquely into the union of fundamental tableaux. We define the multisegment $m_T$ corresponding to $T$ as the sum of the segments corresponding to the fundamental tableaux in the decomposition. For a multisegment $m$, we define $T_m$ to be the union of the fundamental tableaux corresponding to the segments in $m$.

By the correspondence between dominant monomials and multisegments in Section 4.1, we have a correspondence between dominant monomials and tableaux induced by the following correspondence:

$$\{Y_{i,p} \mid (i,p) \in \hat{I}, 1 \leq i + p \leq 2n - 2k\} \simeq \text{Fund}(k, \lbrack n \rbrack),$$

$$\iff T_{\frac{1+i}{2},\frac{p-1}{2}},$$

$$Y_{k+r-i,r-k+i-1} \leftrightarrow T_{\{i+1,\ldots,i+k-1\}\setminus\{r\}}.$$  

(4.4)

We denote the tableau corresponding to a dominant monomial $M$ by $T_M$ and denote by $M_T$ the dominant monomial corresponding to a tableau $T$.

For a one-column tableau $T$ with entries $i_1,\ldots,i_k$, we call the numbers in $\lbrack i_1,i_k \rbrack \setminus \{i_1,\ldots,i_k\}$ the missing numbers.

For a one-column tableau $T$, we define two sequences $i_T, j_T$ as follows. If the set of missing numbers of $T$ is empty, then $i_T, j_T$ are empty. If the set of missing numbers of $T$ is not empty, then $j_T$ is the increasing sequence with entries being the missing numbers of $T$ and $i_T$ is $(i_1,\ldots,i_1+r-1)$, where $i_1$ is the first entry of $T$ and $r$ is the size of the set of missing numbers of $T$.

For a tableau $T$ with columns $T_1,\ldots,T_m$, we define $i_T = i_{T_1} \cdots i_{T_m}, j_T = j_{T_1} \cdots j_{T_m}$ to be the concatenation of sequences.

A multisegment $m = \sum_{i=1}^m [a_i,b_i]$ is called regular if $a_i \neq a_j$ and $b_i \neq b_j$, for all $i \neq j$. A multisegment $m = \sum_{i=1}^m [a_i,b_i]$ is said to be a ladder if $a_1 < \ldots < a_m$ and $b_1 < \ldots < b_m$. 
The following is clear.

**Lemma 4.2.** For a tableau $T$, $m_T$ is regular if and only if the numbers in $i_T$ are all different and the numbers in $j_T$ are all different.

The multisegment $m_T$ is a ladder if and only if the numbers in $j_T$ are all less than the numbers in $j_{T_{r+1}}$ for every $r = 1, \ldots, m - 1$, where $T_1, \ldots, T_m$ are columns of $T$.

When $m_T$ is a ladder, we call $T$ and $\text{ch}(T)$ a ladder or a snake. The $U_q(\mathfrak{sl}_k)$-module $L(M_m)$ corresponding to $m_T$ is a snake module, as introduced in [MY12].

4.4. Weakly separated and unlinked.

**Definition 4.3** ([LZ98]). Given two $k$-subsets $I$ and $J$ of $\{1, \ldots, n\}$, denote by $\min(J)$ the minimal element in $J$ and by $\max(I)$ the maximal element in $I$, we write $I < J$ if $\max(I) < \min(J)$. The sets $I$ and $J$ are called weakly separated if at least one of the following two conditions holds:

1. $J - I$ can be partitioned into a disjoint union $J - I = J' \sqcup J''$ so that $J' < I - J < J''$;
2. $I - J$ can be partitioned into a disjoint union $I - J = I' \sqcup I''$ so that $I' < J - I < I''$.

Recall that in Section 4.3, we denote $T_{a,b}$ to be the one-column tableau with entries $1 - a, 2 - a, \ldots, k - b - 1, k - b + 1, \ldots, k - a + 1$.

We have the following easy criterion to determine if two one column tableaux with small gaps are weakly separated.

**Lemma 4.4.** Let $T, T'$ be one column tableaux with small gaps. Write $T = T_{a,b}$ and $T = T'_{c,d}$, and let $I_T, I_{T'}$ be the corresponding $k$-subsets of $\{n\}$. Then $T$ and $T'$ are not weakly separated if and only if

$$\begin{cases}
  k - b \in I_{T'}, \\
  k - d \in I_T,
\end{cases} \quad \text{and} \quad (a - c)(b - d) > 0.$$

**Proof.** For the direct implication, suppose without loss of generality that $a \leq c$. Then let us suppose first that $b = d$. If $a = c$ then $T = T'$ and $T, T'$ are weakly separated (w.s.). Thus $a < c$, then

$$I_T \setminus I_{T'} = \{k - c + 2, k - c + 3, \ldots, k - a + 1\},$$

and

$$I_{T'} \setminus I_T = \{1 - c, 2 - c, \ldots, -a\}.$$ 

Thus $T, T'$ are w.s.

Now suppose that $k - b \notin \{1 - c, 2 - c, \ldots, k - c + 1\}$. As $k - b \geq 1 - a \geq 1 - c$, we have $k - b > k - c + 1$. We consider two cases: $k - d < 1 - a$ and $k - d \geq 1 - a$.
In both cases, $T$ and $T'$ are w.s.

By a similar reasoning, we can prove that if $k - d \notin \{1 - a, 2 - a, \ldots, k - a + 1\}$, then $T$ and $T'$ are w.s.

For the reverse implication, suppose $k - b \in I_T'$, $k - d \in I_T$ and $(a - c)(b - d) > 0$. We can assume that $a < c$. We have $k - b \leq k - c + 1$ and then the configuration is the following.

\[
\begin{array}{c}
I_T \quad I_{T'}' \quad I_{T'} \quad I_T' \quad I_T' \quad I_T
\end{array}
\]

\[
\begin{array}{c}
1 - c \quad 1 - a \quad k - b \quad k - a + 1
\end{array}
\]

\[
\begin{array}{c}
k - d \quad k - c + 1
\end{array}
\]

We see that $T$ and $T'$ are not weakly separated.

The following result gives an explicit correspondence between two one column tableaux with small gaps not being weakly separated, and their corresponding segment being linked.

**Proposition 4.5.** Let $T, T'$ be one column tableaux with small gaps. Write $T = T_{a,b}$ and $T = T'_{c,d}$, and let $I_T, I_{T'}$ be the corresponding $k$-subsets of $[n]$. Then the $k$-subsets $I_T$ and $I_{T'}$ are not weakly separated if and only if the corresponding segments $[a,b]$ and $[c,d]$ are linked and

\[ k > \max (d - a, b - c). \]

**Proof.** Suppose without loss of generality that $a \leq c$.

Let us suppose that $[a,b] \succ [c,d]$ and $k > d - a$. As $a + 1 \leq c \leq b + 1 \leq d$, then $1 - c, k - b \in J \setminus I$, $k - d, k - a + 1 \in I \setminus J$, and

\[ 1 - c < k - d < k - b < k - a + 1. \]

Therefore $I$ and $J$ are not weakly separated.

Suppose now that $I_T$ and $I_{T'}$ are not weakly separated. From Lemma 4.4, $k - d \in [1 - a, k - a + 1] \setminus \{k - b\}$, thus $k > d - a$. Similarly, $k - b \in [1 - c, k - c + 1] \setminus \{k - d\}$ implies $c \leq b + 1$. Additionally, as $(a - c)(b - d) > 0$, we deduce that in fact $a < c$ and $d < d$. Therefore $a + 1 \leq c \leq b + 1 \leq d$, and $[a,b]$ and $[c,d]$ are linked.

**Remark 4.6.** We need the condition that $k$ is sufficient large in Proposition 4.5. For example, in the case of $k = 6$, the multisegments $[-1,2] + [-4,-1]$ correspond to the $k$-subsets $\{2,3,5,6,7,8\}, \{5,6,8,9,10,11\}$. The multisegments are linked but the $k$-subsets are weakly separated.

In the case of $k = 7$, we have that the multisegments $[-1,2] + [-4,-1]$ correspond to the $k$-subsets $\{2,3,4,6,7,8,9\}, \{5,6,7,9,10,11,12\}$. The multisegments are linked and the $k$-subsets are not weakly separated.
4.5. **Zelevinsky classification in the category** $\mathcal{C}_k$. Inspired by Proposition 4.5, we introduce the following.

**Definition 4.7.** If $\Delta, \Delta'$ are two segments of length $< k$. We say that $\Delta$ $k$-precedes $\Delta'$ if $\Delta < \Delta'$ and $e(\Delta') - b(\Delta) < k$. We denote it by $\Delta \prec_k \Delta'$. If either $\Delta \prec_k \Delta'$ or $\Delta' \prec_k \Delta$ we say that $\Delta$ and $\Delta'$ are $k$-linked.

**Lemma 4.8.** For $\Delta, \Delta'$ segments in $\text{Mult}_k$, the product $\overline{Z}(\Delta) \times \overline{Z}(\Delta')$ is simple in $\mathcal{C}_k$ if and only if $\Delta$ and $\Delta'$ are $k$-unlinked.

**Proof.** If $\Delta < \Delta'$, then $\overline{Z}(\Delta) \times \overline{Z}(\Delta')$ is of length 2, and we have the following short exact sequence (see [LM16, Lemma A.9])

$$0 \to \overline{Z}(\Delta \cup \Delta') \times \overline{Z}(\Delta \cap \Delta') \to \overline{Z}(\Delta) \times \overline{Z}(\Delta') \to \overline{Z}(\Delta + \Delta') \to 0. \quad (4.5)$$

If moreover $\Delta \prec_k \Delta'$, then $\overline{Z}(\Delta \cup \Delta') \times \overline{Z}(\Delta \cap \Delta')$ is non-zero, thus $\overline{Z}(\Delta \cup \Delta') \times \overline{Z}(\Delta \cap \Delta')$ is a proper subobject of $\overline{Z}(\Delta) \times \overline{Z}(\Delta')$, which is not simple.

If $e(\Delta') - b(\Delta) \geq k$, then the length of $\Delta \cup \Delta'$ is $\geq k + 1$ and thus $\overline{Z}(\Delta \cup \Delta') = 0$. The image of Equation (4.5) in $\mathcal{C}_k$ gives that $\overline{Z}(\Delta) \times \overline{Z}(\Delta')$ is simple and isomorphic to $\overline{Z}(\Delta + \Delta')$.

If $\Delta$ and $\Delta'$ are unlinked, then $\overline{Z}(\Delta) \times \overline{Z}(\Delta')$ is irreducible and thus $\overline{Z}(\Delta) \times \overline{Z}(\Delta')$ is simple. $\square$

For $\mathbf{m} = \Delta_1 + \cdots + \Delta_N \in \text{Mult}_k$ an ordered multisegment, let us write

$$\zeta(\mathbf{m}) := \overline{Z}(\Delta_1) \times \cdots \times \overline{Z}(\Delta_N).$$

The following in the key element of Zelevinsky’s classification, it generalizes Lemma 4.8.

**Proposition 4.9.** For $\mathbf{m} = \Delta_1 + \Delta_2 + \cdots + \Delta_N \in \text{Mult}_k$, $\zeta(\mathbf{m})$ is simple if and only if the segments $\Delta_i$ are pairwise $k$-unlinked.

We use the following intermediate result.$^2$

**Lemma 4.10 ([Zel80, Section 7]).** The terms in the Jordan-Hölder series of $\overline{Z}(\Delta_1) \times \cdots \times \overline{Z}(\Delta_N)$ are of the form $\overline{Z}(\mathbf{n})$, where $\mathbf{n}$ is obtained from $\mathbf{m}$ by a sequence of operations where a pair of linked segments $\{\Delta, \Delta'\}$ are replaced by $\{\Delta \cap \Delta', \Delta \cup \Delta'\}$.

**Proof.** If the segments $\Delta_i$ are pairwise $k$-unlinked, for all pair of possibly linked segments $\Delta_i, \Delta_j$, the union $\Delta_i \cup \Delta_j$ is of length $\geq k + 1$. Thus by Lemma 4.10, all terms in the Jordan-Hölder series of $\overline{Z}(\Delta_1) \times \cdots \times \overline{Z}(\Delta_N)$ are sent to 0 by $\mathcal{Q}$ except $\overline{Z}(\mathbf{m})$. Hence $\zeta(\mathbf{m})$ is simple and isomorphic to $\overline{Z}(\mathbf{m})$.

Conversely, if $\zeta(\mathbf{m})$ is simple, then all terms in the Jordan-Hölder series of $\overline{Z}(\Delta_1) \times \cdots \times \overline{Z}(\Delta_N)$ are sent to 0 by $\mathcal{Q}$. We deduce that the segments of $\mathbf{m}$ are pairwise $k$-unlinked. $\square$

$^2$In [Zel80], the proof is given in terms of Langlands classification but it is actually valid for both classifications.
Corollary 4.11 (k-Zelevinsky classification). For \( m = \Delta_1 + \Delta_2 + \cdots + \Delta_N \in \text{Mult}_k \) such that \( \Delta_i \not\prec_k \Delta_j \) for all \( i < j \),

\[
\mathcal{Z}(m) = \text{soc}(\mathcal{Z}(\Delta_1) \times \mathcal{Z}(\Delta_2) \times \cdots \times \mathcal{Z}(\Delta_N)) \quad \text{in } \mathcal{C}_k.
\]

The following is a consequence of the k-Zelevinsky classification.

Lemma 4.12. Write \( m = \sum_i \Delta_i \) and \( n = \sum_j \Delta'_j \). If for all \( i, j \), \( \Delta_i \not\prec_k \Delta'_j \), then \( \mathcal{Z}(m + n) = \text{soc}(\mathcal{Z}(m) \times \mathcal{Z}(n)) = \text{cos}(\mathcal{Z}(n) \times \mathcal{Z}(m)) \) in \( \mathcal{C}_k \).

5. Quantum Affine Zelevinsky Classification

5.1. Tensor product of fundamental representations. Using a combinatorial result of Nakajima [Nak03], we can write a closed formula for the \( q \)-character of a fundamental representation of the quantum affine algebra \( U_q(\hat{\mathfrak{g}}_k) \).

Let \( \hat{I} = \{ (i, p) \mid 1 \leq i \leq k-1, i-p \in 2\mathbb{Z} + 1 \} \).

Proposition 5.1 ([Nak03, Proposition 4.6]). Let \( (i, p) \in \hat{I} \), then the \( q \)-character of the fundamental representation \( L(Y_{i,p}) \) can be written

\[
\chi_q(L(Y_{i,p})) = \sum_{1 \leq j_1 < \cdots < j_i \leq k} \prod_{m=1}^i Y_{j_m,p+i+j_m-2m} (Y_{j_m-1,p+i+j_m-2m+1})^{-1}.
\]

(5.1)

Lemma 5.2. Let \( (i, p) \in \hat{I} \), the only monomials appearing in \( \chi_q(L(Y_{i,p})) \) with exactly one negative power of \( Y \) have the form

\[
Y_{q,p+i-q} (Y_{q+r,p+i+r-q})^{-1} Y_{i+r,p+r},
\]

(5.2)

where \( \left\{ \begin{array}{l} 0 \leq q \leq i - 1, \\ 1 \leq r \leq k - i. \end{array} \right. \)

Proof. Fix \( 1 \leq j_1 < \cdots < j_i \leq k \), and let \( m \) be the corresponding monomial of \( \chi_q(L(Y_{i,p})) \) in (5.1). We note that if \( j_{m+1} = j_m + 1 \), then the negative powered \( Y \)-variable in the \( (m+1) \)th factor of \( m \) will cancel out with the positive powered \( Y \)-variable in the \( m \)th factor of \( m \). Hence, the monomial \( m \) has exactly one negative powered \( Y \)-variable if and only if the tuple \( (j_1, j_2, \cdots, j_i) \) is formed of two disconnected intervals of integers, thus if and only if it has the form

\[
(1, 2, \ldots, q, q+r+1, q+r+2, \ldots, r+i),
\]

with \( 0 \leq q \leq i - 1, \ r \leq 1 \) and \( r+i \leq k \). In that case, the monomial is the one from (5.2). \( \square \)

We also make use of the following result, proven algebraically in [Kas02], and geometrically in [VV02], and the more precise statement given in [Cha02].
Theorem 5.3 ([Kas02, Cha02, VV02]). Let \((i, p), (j, s) \in \hat{I}\). If \(p - s \notin \mathcal{S}\), where
\[
\mathcal{S} := \{2m - i - j + 2 \mid 1 \leq i, j \leq m \leq k, m \leq i + j - 1\},
\]
then the tensor product \(L(Y_{j,s}) \otimes L(Y_{i,p})\) is cyclic and generated by the tensor product of the highest weight vectors.

In particular, if \(p_1 \geq p_2 \geq \cdots \geq p_N\) (resp. \(p_1 \leq p_2 \leq \cdots \leq p_N\)) then tensor product \(L(Y_{i_1,p_1}) \otimes L(Y_{i_2,p_2}) \otimes \cdots \otimes L(Y_{i_N,p_N})\) is cyclic (resp. cocyclic), and generated by the tensor product of the highest weight vectors (resp. any non zero submodule contains the tensor product of the highest weight vectors).

Proposition 5.4. Let \((i, p), (j, s) \in \hat{I}\) such that \(p \leq s\). The tensor product \(L(Y_{j,s}) \otimes L(Y_{i,p})\) is reducible if and only if
\[
p + i + 2 \leq j + s \leq 2k + p - i, \tag{5.3}
\]
\[
-p - i \leq j - s \leq i - p - 2. \tag{5.4}
\]
Moreover in that case, the tensor product is of length 2 and we have the following short exact sequence
\[
0 \to L(Y_{q, p+i-q} Y_{i+r, p+r}) \to L(Y_{j,s}) \otimes L(Y_{i,p}) \to L(Y_{j,s}Y_{i,p}) \to 0, \tag{5.5}
\]
where \(q = \frac{1}{2}(i + j + p - s)\), \(r = \frac{1}{2}(-i + j - p + s)\).

Proof. Let us consider the dominant monomials appearing in the product of \(q\)-characters \(\chi_q(L(Y_{j,s})) \chi_q(L(Y_{i,p}))\). One of these dominant monomials is naturally the product of the highest weights \(Y_{j,s} Y_{i,p}\). As \(p \leq s\), the only other possible dominant monomials are of the form \(Y_{j,s} M\), with \(M\) a monomial of \(\chi_q(L(Y_{i,p}))\) with exactly one negative powered \(Y\)-variable. From Lemma 5.2, we know that these \(M\) have the form
\[
M = Y_{q, p+i-q} (Y_{q+r, p+i-r-q})^{-1} Y_{i+r, p+r}, \quad \text{with} \quad \begin{cases} 0 \leq q \leq i - 1, \\ 1 \leq r \leq k - i \end{cases}. \tag{5.6}
\]
Thus \(Y_{j,s} M\) is dominant if and only if we have \(Y_{j,s} = Y_{q+r, p+i-r-q}\), or equivalently
\[
j = q + r,
\]
\[
s = p + i + r - q.
\]

Suppose now that the tensor product \(L(Y_{j,s}) \otimes L(Y_{i,p})\) is reducible, then necessarily one of the \(Y_{j,s} M\) is dominant. Thus \((j, s)\) satisfy the inequalities of (5.6), which are equivalent to the inequalities (5.3) and (5.4).

Conversely, suppose now that the inequalities (5.3), (5.4) are satisfied. In particular, one has
\[
s - p \geq |i - j| + 2,
\]
and thus $L(Y_{j,s}Y_{i,p})$ is a snake module, and using [MY12, Theorem 6.1], we know $Y_{j,s}Y_{i,p}$ is its unique dominant monomial.

Let us write $j = q + r$ and $s = p + i + r - q$, then $(q, r)$ satisfy the inequalities in (5.6). Necessarily, the corresponding monomial $Y_{j,s}M$ is dominant; it is also the only one of this form. Thus we have the following relation between the $q$-characters

$$
\chi_q(L(Y_{j,s}))\chi_q(L(Y_{i,p})) = \chi_q(L(Y_{j,s}Y_{i,p})) + \chi_q(L(Y_{q,p+i-q}Y_{i+r,p+r})).
$$

(5.7)

In particular, $\chi_q(L(Y_{j,s}Y_{i,p})) \neq \chi_q(L(Y_{j,s}))\chi_q(L(Y_{i,p}))$ and the tensor product $L(Y_{j,s}) \otimes L(Y_{i,p})$ is reducible.

Moreover, from Theorem 5.3, as $s \geq p$, $L(Y_{j,s} \otimes Y_{i,p})$ is a cyclic module, generated by the tensor product of the highest weight vectors of its factors. Hence it has a unique maximal submodule and a unique simple quotient. From the highest weight, its unique simple quotient is isomorphic to $L(Y_{j,s} Y_{i,p})$. Thus by (5.7), its unique maximal proper submodule has the $q$-character of $L(Y_{q,p+i-q} Y_{i+r,p+r})$. Since $L(Y_{q,p+i-q} Y_{i+r,p+r})$ is simple, it is isomorphic to the unique maximal proper submodule of $L(Y_{j,s} \otimes Y_{i,p})$. In conclusion we have the short exact sequence (5.5).

**Theorem 5.1.** Let $(i,p), (j,s) \in \hat{I}$. The tensor product $L(Y_{i,p}) \otimes L(Y_{j,s})$ is irreducible if and only if the one column tableaux corresponding to $Y_{i,p}$ and $Y_{j,s}$ are weakly separated.

**Proof.** Recall that the segments corresponding to $Y_{i,p}$ and $Y_{j,s}$ are respectively

$$
[a, b] := \left[\frac{1 - i - p}{2}, \frac{i - p - 1}{2}\right] \quad \text{and} \quad [c, d] := \left[\frac{1 - j - s}{2}, \frac{j - s - 1}{2}\right].
$$

Suppose first that $p = s$. On the one side, both inequalities (5.3) and (5.4) cannot be satisfied, and thus from Proposition 5.4, the tensor product $L(Y_{i,p}) \otimes L(Y_{j,s})$ is irreducible.

On the other side, the segments $[a, b]$ and $[c, d]$ have the same mid point $-p$ and thus cannot be linked. Using Proposition 4.5, we deduce that necessarily the one column tableaux corresponding to $Y_{i,p}$ and $Y_{j,s}$ are weakly separated, which proves the result in this case.

Assume now without loss of generality that $p < s$. From Proposition 4.5, the one column tableaux corresponding to $Y_{i,p}$ and $Y_{j,s}$ are not weakly separated if and only if the segments $[c, d] < [a, b]$ and $k > b - c$.

One can easily see that the four inequalities

$$
c + 1 \leq a \leq d + 1 \leq b, \quad \text{and} \quad k > b - c
$$

are equivalent to the four inequalities of (5.3),(5.4) (knowing that $j + s$ and $p - i$ have the same parity). Hence the desired equivalence is obtained through Proposition 5.4.

**5.2. Fundamental representations as socles.** In the Zelevinsky classification, simple modules corresponding to segments are defined as the socles of an inductive product of characters. We prove the quantum affine analog of this construction: fundamental
representations of the quantum affine algebra can be defined as the socle of a tensor product of fundamental representations at node 1 (of the form $L(Y_{1,k})$).

**Proposition 5.5.** For all $(i, p) \in \hat{I}$, $L(Y_{i,p})$ is isomorphic to the unique maximal submodule of

$$L(Y_{1,p+i-1}) \otimes L(Y_{1,p+i-3}) \otimes \cdots \otimes L(Y_{1,p-i+1}).$$

(5.8)

*Proof.* Let $W$ be the tensor product (5.8), $W := \bigotimes_{m=1}^{i} L(Y_{i,p+i-2m+1})$. From Theorem 5.3, we know that $W$ is a cyclic module, generated by the tensor product of the highest weight vectors of its factors, hence it has a unique maximal submodule and a unique simple quotient.

Let us show by induction on $i$ that its unique maximal submodule is $L(Y_{i,p})$. The result is trivial for $i = 1$.

By induction, the unique maximal submodule of $L(Y_{1,p+i-1}) \otimes L(Y_{1,p+i-3}) \otimes \cdots \otimes L(Y_{1,p-i+1})$ is $L(Y_{1,p+i-1})$. Consider $L(Y_{i-1,p+1}) \otimes L(Y_{1,p-i+1})$. By Proposition 5.4, this tensor product is of length 2 and its unique maximal submodule is $L(Y_{i,p})$. Thus $L(Y_{i,p})$ is the unique maximal submodule of $W$. 

\[ \square \]

5.3 Simple representations as socles. More generally, all simple representations of the quantum affine algebra $U_q(\hat{sl}_k)$ can be obtained as the socles of some tensor product of fundamental representations.

**Proposition 5.6.** Let $m = Y_{i_1,p_1} Y_{i_2,p_2} \cdots Y_{i_N,p_N}$, such that for all $1 \leq j \leq N$, $(i_j, p_j) \in \hat{I}$ and $p_1 \leq p_2 \leq \cdots \leq p_N$, then $L(m)$ is isomorphic to the unique irreducible submodule of the standard module

$$M(m) := L(Y_{i_1,p_1}) \otimes L(Y_{i_2,p_2}) \otimes \cdots \otimes L(Y_{i_N,p_N}).$$

(5.9)

*Proof.* From Theorem 5.3, the standard module $M(m)$ is cocyclic, and every non zero submodule of $M(m)$ contains the tensor product of the highest weight vectors.

Let $S$ be the submodule of $M(m)$ generated by the tensor product of the highest weights vectors. Then $S$ is a simple $\ell$-highest weight module of highest weight $m$. Thus $S$ is isomorphic to $L(m)$. Moreover, $S$ is clearly the unique irreducible submodule of $M(m)$. 

\[ \square \]

**Remark 5.7.** In the Zelevinsky classification, the standard modules corresponding to a multisegment $m = \Delta_1 + \cdots + \Delta_N$ is defined as the parabolic induction $Z(\Delta_1) \times \cdots \times Z(\Delta_N)$, where the $\Delta_i$ are ordered such that for all $i < j$, $\Delta_i$ does not precede $\Delta_j$. The same condition is applied here. Indeed, let $(i, p), (j, s)$ satisfy the conditions of the first part of Theorem 5.3, and let $[a, b]$ and $[c, d]$ be the segments corresponding respectively to $L(Y_{j,s})$ and $L(Y_{i,p})$. Then $2(d-a) = i+j+s-p-2$. As $s-p \notin S$ by hypothesis, we deduce that either $c \leq a$, $d \leq b$ or $b+1 < c$, which would mean that $[a, b]$ does not precede $[c, d]$. 

or \( k \leq d - a \), which would mean that the corresponding one column tableaux are weakly separated (see Proposition 4.5).

Moreover, the short exact sequence (5.5) is the analog of the following known short exact sequence: if \( \Delta \prec \Delta' \), then
\[
0 \to Z(\Delta \cap \Delta') \times Z(\Delta \cup \Delta') \to Z(\Delta) \times Z(\Delta') \to Z(\Delta + \Delta') \to 0.
\]

6. A Criterion for the Simplicity of Tensor Products

6.1. Condition for irreducibility. Following [LM16], for \( \pi = Z(m), \sigma = Z(n) \), irreducible objects in \( \mathcal{C}_k \), denote by \( \text{LI}_k(\pi, \sigma) \) the condition \( Z(m + n) = \text{soc}(\pi \times \sigma) \) in \( \mathcal{C}_k \), and denote by \( \text{RI}_k(\pi, \sigma) \) the condition \( Z(m + n) = \text{cos}(\pi \times \sigma) \) in \( \mathcal{C}_k \). We also use the notations from [LM16]. For \( \pi = Z(m), \sigma = Z(n) \), irreducible objects in \( \mathcal{C}^Z \), denote by \( \text{LI}(\pi, \sigma) \) the condition \( Z(m + n) = \text{soc}(\pi \times \sigma) \) in \( \mathcal{C}^Z \), and denote by \( \text{RI}(\pi, \sigma) \) the condition \( Z(m + n) = \text{cos}(\pi \times \sigma) \) in \( \mathcal{C}^Z \).

Lemma 6.1 ([LM16, Lemma 4.2]). For \( \pi = Z(m), \sigma = Z(n) \in \text{Irr}_k, \pi \times \sigma \) is irreducible if and only if \( \text{LI}_k(\pi, \sigma) \) and \( \text{RI}_k(\pi, \sigma) \).

The following is a consequence of Proposition 3.4.

Proposition 6.2. For \( \pi = Z(m), \sigma = Z(n) \), irreducible representations in \( \mathcal{C}^Z \), such that all segments in \( m \) and \( n \) have length inferior to \( k \), then we have the implication
\[
\text{LI}(\pi, \sigma) \Rightarrow \text{LI}_k(\pi, \sigma).
\]

Remark 6.3. Lemma 4.12 translates as the following: if \( m = \sum_i \Delta_i \) and \( n = \sum_j \Delta'_j \) satisfy \( \Delta_i <_k \Delta'_j \), for all \( i, j \), then \( \text{LI}_k(Z(m), Z(n)) \) holds.

6.2. Matching functions. Following [LM16], let \( X, Y \) be finite sets and \( \sim \) a relation between \( X \) and \( Y \). An injective function \( f : X \to Y \) satisfying \( f(x) \sim x \) for all \( x \in X \) is called a \( \sim \)-matching function between \( X \) and \( Y \). An injective function \( f \) from a subset of \( X \) to \( Y \) satisfying \( f(x) \sim x \) for all \( x \) in the domain of \( f \) is called a \( \sim \)-matching between \( X \) and \( Y \). Suppose that \( X \) and \( Y \) are totally ordered with respect to \( \leq_X \) and \( \leq_Y \) respectively. Define a \( \sim \)-matching \( f \) between \( X \) and \( Y \) and its domain \( I \) recursively by \( x \in I \) if and only if there is \( y \in Y \setminus f(I \cap X_x) \) such that \( y \sim x \) in which case \( f(x) = \min\{y \in Y \setminus f(I \cap X_x) : y \sim x\} \). This \( \sim \)-matching is called a best \( \sim \)-matching between \( X \) and \( Y \).

For two multisegments \( m = \sum_{i=1}^N \Delta_i, n = \sum_{i=1}^{N'} \Delta'_i \in \text{Mult} \), and two sets \( X, Y \subset [N] \times [N'] \), define a relation \( \sim_{m,n} \) (or simply \( \sim \) if there is no confusion) between \( Y \) and \( X \) as follows:
\[
(i_2, j_2) \sim (i_1, j_1) \text{ if and only if either } i_1 = i_2, \Delta'_{j_2} < \Delta'_{j_1} \text{ or } j_1 = j_2, \Delta_i < \Delta_{i_2}.
\]

For two multisegments \( m = \sum_{i=1}^N \Delta_i, n = \sum_{i=1}^{N'} \Delta'_i \in \text{Mult}_k \), and two sets \( X, Y \subset [N] \times [N'] \), define a relation \( \sim_k(m,n) \) (or simply \( \sim_k \) if there is no confusion) between \( Y \) and \( X \) as follows:
\[
(i_2, j_2) \sim_k (i_1, j_1) \text{ if and only if either } i_1 = i_2, \Delta'_{j_2} <_k \Delta'_{j_1} \text{ or } j_1 = j_2, \Delta_i <_k \Delta_{i_2}.
\]
6.3. **The combinatorial conditions** $\text{LC}(m, m')$, $\text{LC}_k(m, m')$ and $\text{LC}(T, T')$. Denote by $<$ the lexicographical order on the set of one-column tableaux with gap weight $\leq 1$. For example, in SSYT(3, [5]),

\[
\begin{array}{ccccccc}
2 & 2 & 3 & 3 & 4 & 5 & 5 \\
1 & 4 & 4 & 1 & 1 & 5 & 5
\end{array}
\]

**Remark 6.4.** The lexicographical order on one-column tableaux with small gaps does not correspond to either the left or right aligned order introduced in Section 2.3. However, if the segments in a multisegments $m$ are ordered such that the corresponding one column tableaux are in increasing lexicographical order, then the multisegment $m$ is ordered.

In particular, if two segments $\Delta, \Delta'$ are such that $T_\Delta \leq T_{\Delta'}$ and $\Delta \leq_b \Delta'$ (or $\Delta \leq_e \Delta'$), then $\Delta$ and $\Delta'$ are unlinked.

For two multisegments $m, m'$, Lapid and Minguez [LM16] defined

\[
X_{m,m'} = \{(i, j) : \Delta_i < \Delta'_j\},
\]

\[
Y_{m,m'} = \{(i, j) : \Delta_i \prec \Delta'_j\}.
\]

They [LM16] introduced a condition $\text{LC}(m, m')$: there is an injective map $f : X_{m,m'} \to Y_{m,m'}$, $f(i, j) = (i', j')$, such that for any $(i, j) \in X_{m,m'}$, either $i = i'$, $\Delta'_j < \Delta'_j$ or $j = j'$, $\Delta_i < \Delta'_j$.

We consider the analogs for the quotient ring of the sets used by Lapid and Minguez [LM16] to establish a combinatorial criterion to determine the irreducibly of a product of representations. For $k \in \mathbb{Z}_{\geq 1}$ and two multisegments $m = \sum_i \Delta_i$, $m' = \sum_j \Delta'_j$,

\[
X^{(k)}_{m,m'} = \{(i, j) : \Delta_i \prec_k \Delta'_j\},
\]

\[
Y^{(k)}_{m,m'} = \{(i, j) : \Delta_i \prec_k \Delta'_j\}.
\]

We denote by $\text{LC}_k(m, m')$ the following condition: there is an injective map $f : X^{(k)}_{m,m'} \to Y^{(k)}_{m,m'}$, $f(i, j) = (i', j')$, such that for any $(i, j) \in X^{(k)}_{m,m'}$, either $i = i'$, $\Delta'_j \prec_k \Delta'_j$ or $j = j'$ and $\Delta_i \prec_k \Delta'_j$.

Let $T, T' \in \text{SSYT}(k, [n])$ with decomposition into unions of one-column small gap tableaux $T = T_1 \cup \cdots \cup T_r$, $T' = T'_1 \cup \cdots \cup T'_{r'}$, where the factors are written in the lexicographical order $T_i < T_{i+1}$, $T'_i < T'_{i+1}$.

Denote by $\text{pr}(T)$ the promotion of $T$ (cf. [Sch72]) and

\[
X_{T,T'} = \{(i, j) : T_i < T'_j \text{ are not weakly separated, } \min T'_j < \min T_i \},
\]

\[
Y_{T,T'} = \{(i, j) : \text{pr}(T_i), T'_j \text{ are not weakly separated, } \min T'_j \leq \min T_i \}.
\]

We denote by $\text{LC}(T, T')$ the following condition: there is an injective map $f : X_{T,T'} \to Y_{T,T'}$, $f(i, j) = (i', j')$, such that for any $(i, j) \in X_{T,T'}$, either $i = i'$, $T'_j$ and $T'_j$ are not
weakly separated, and \( \min T'_j < \min T'_j \) or \( j = j' \), \( T_i \) and \( T'_r \) are not weakly separated, and \( \min T'_r < \min T_i \).

**Lemma 6.5.** For \( T, T' \in \text{SSYT}(k, [n]) \), and \( m = m_T, m' = m_{T'} \) the corresponding multisegments. Then

\[
X_{T,T'} = X_{m,m'}^{(k)},
\]

\[
Y_{T,T'} = Y_{m,m'}^{(k)}.
\]

**Proof.** Let us write the decompositions into one-column small gap tableaux \( T = T_1 \cup \cdots \cup T_r \), \( T' = T'_1 \cup \cdots \cup T'_{r'} \) and let \( \Delta_1, \Delta_2, \ldots, \Delta_r, \Delta'_1, \Delta'_2, \ldots, \Delta'_{r'} \) be the corresponding segments.

Then the equality \( X_{T,T'} = X_{m,m'}^{(k)} \) is a direct consequence of Proposition 4.5.

Let \( (i, j) \in Y_{T,T'} \). If \( \text{pr}(T_i) = T_i + 1 \), where all the values are increased by one, then as before using Proposition 4.5 we deduce that \( (i, j) \in Y_{m,m'}^{(k)} \).

If the maximal value of \( T_i \) is \( n \), then \( \Delta_i = [a, b] \) with \( a = k - n + 1 \), and the entries of \( \text{pr}(T_i) \) are \( I = \{1\} \cup \{2 - a, n\} \setminus \{k - b + 1\} \). Write \( \Delta'_j = [c, d] \), by assumption \( \min(T'_j) \leq \min(T_i) \) so \( a \leq c \). If \( b > d \), then the configuration of segments is the following.

\[
\begin{array}{cccccccc}
I & J & 1 & 1 - c & 2 - a & k - b + 1 & k - d & k - c + 1 \\
I \setminus J & & & & & & & \\
J \setminus I' & & & & & & & \\
\end{array}
\]

Which is in contradiction with the fact that \( \text{pr}(T_i) \) and \( T'_j \) are not weakly separated, thus \( b \leq d \). Now suppose, \( b < c \). Then, as \( k - c + 1 \leq k - b \), all the values in \( J \setminus I \) are between 1 and \( 2 - a \) and \( \text{pr}(T_i) \) and \( T'_j \) are weakly separated: a contradiction. Thus we have \( a \leq c \leq b \leq d, \)

\([a - 1, b - 1] < [c, d], \) and \( (i, j) \in Y_{m,m'}^{(k)} \).

Conversely, let \( (i, j) \in Y_{m,m'}^{(k)}. \) If \( b(\Delta_i) > k - n + 1 \), then \( \Delta_i \) is the segment corresponding to the tableau \( \text{pr}(T_i) = T_i + 1 \), and we deduce that \( (i, j) \in Y_{T,T'}. \) Otherwise, the segment \( \Delta_i \) has no corresponding tableau. However, if we write \( \Delta_i = [a, b] \) and \( \Delta'_j = [c, d] \), then the condition \( \Delta_i < \Delta'_j \) is equivalent to \( a \leq c \leq b \leq d \). Thus, the values \( k - d, n \) appear in the tableau \( \text{pr}(T_i) \) but not \( T'_j \) and the values \( 1 - c, k - b + 1 \) appear in the tableau \( T'_j \) but not \( \text{pr}(T_i) \). As \( 1 - c < k - d < k - b + 1 < n \), we deduce that \( \text{pr}(T_i) \) and \( T'_j \) are weakly separated. Thus \( (i, j) \in Y_{T,T'}. \)

Using Proposition 4.5 and Lemma 6.5, is is clear that for \( T, T' \) tableaux in \( \text{SSYT}(k, [n]) \) and \( m_T, m_{T'} \) the corresponding multisegments, then

\[
\text{LC}(T, T') \iff \text{LC}_k(m_T, m_{T'}).
\]
Lemma 6.6. Let \( m = \sum_i \Delta_i \) and \( m' = \sum_j \Delta'_j \) be multisegments, and fix \( k \in \mathbb{Z}_{\geq 1} \). Then we have the implications:

\[
\begin{align*}
LC_{k+1}(m, m') & \Rightarrow LC_k(m, m'), \\
LC(m, m') & \Rightarrow LC_k(m, m').
\end{align*}
\] (6.5) (6.6)

Proof. Suppose \( LC(m, m') \) and let \( f \) be the matching function between the sets \( X_{m,m'} \) and \( Y_{m,m'} \). Let \( \tilde{f} \) be the restriction of \( f \) to the subset \( X_{m,m'}^{(k)} \). Then we can see that the image of \( \tilde{f} \) is contained in \( Y_{m,m'}^{(k)} \), and \( \tilde{f} \) defines a matching:

\[
\tilde{f} : X_{m,m'}^{(k)} \to Y_{m,m'}^{(k)}.
\]

Indeed, let \((i, j) \in X_{m,m'}^{(k)} \) and \( \tilde{f}(i, j) = (i', j') \) be its image. If \( i = i' \), then by definition of \( \tilde{f} \), \( \Delta_{j'} < \Delta_{j} \), and thus \( e(\Delta_{j'}) \leq e(\Delta_{j}) - 1 \). Then \( e(\Delta_{j'}) - b(\Delta_i) \leq e(\Delta_{j'}) - 1 - b(\Delta_i) < k - 1 \), as \((i, j) \in X_{m,m'}^{(k)} \). Thus \((i, j') \in Y_{m,m'}^{(k)} \). Moreover, as \( \sum \Delta_i < \Delta_{j'} \), we also have \( e(\Delta_{j'}) - b(\Delta_{j'}) < k \). If \( j = j' \), we show similarly that \((i', j) \in Y_{m,m'}^{(k)} \) and that \( e(\Delta_{i'}) - b(\Delta_i) < k \).

The implication (6.5) can be proven in a similar way, as it is clear that \( X_{m,m'}^{(k)} \subseteq X_{m,m'}^{(k+1)} \) and \( Y_{m,m'}^{(k)} \subseteq Y_{m,m'}^{(k+1)} \). \( \square \)

6.4. Main result. The following is the analog in our context of [LM16, Proposition 6.20].

Proposition 6.7. Let \( n \in \text{Mult}_k \), and \( a \in \mathbb{Z} \), then

1. The condition \( LC_k([a,a], n) \) is equivalent to \( LI_k(Z([a,a]), Z(n)) \).
2. The condition \( RC_k([a,a], n) \) is equivalent to \( RI_k(Z([a,a]), Z(n)) \).

We conjecture the following generalization to Proposition 6.7.

Conjecture 6.8. Proposition 6.7 is also true if we replace \([a,a]\) by any ladder \( m \).

Proposition 6.7 will be proven in Section 7.1.

Proposition 6.9. Let \( m, n \in \text{Mult}_k \) be ladders, the following conditions are equivalent:

1. \( LC_k(m, n) \) and \( LC_k(n, m) \),
2. \( [\overline{Z}(m) \times \overline{Z}(n)] = [\overline{Z}(m + n)] \in \text{R}_k \).

Proposition 6.9 will be proven in Section 7.2. We can now prove the main result.

Theorem 6.10. Let \( L(M), L(M') \) be simple snake \( U_q(sl_k) \)-modules. Then the following conditions are equivalent:

1. \( LC_k(m_M, m_{M'}) \) and \( LC_k(m_{M'}, m_M) \),
2. the tensor product \( L(M) \otimes L(M') \) is irreducible.
Proof. If both \( L(M) \) and \( L(M') \) are snake modules, then from Proposition 6.9, (1) is equivalent to the condition \( \left[ Z(m_M) \times Z(m_{M'}) \right] = \left[ Z(m_M + m_{M'}) \right] \in R_k \). Using Proposition 3.11, the latter is equivalent to the condition \( \left[ L(M) \otimes L(M') \right] = \left[ L(MM') \right] \in R_k \), which can also be translated thanks to the \( q \)-character into \( \chi_q(L(M))\chi_q(L(M')) = \chi_q(L(MM')) \). Finally, this condition is equivalent to (2).

**Proposition 6.11.** Theorem 6.10 is also true if \( L(M') \) is a fundamental representation at an extremal node \( L(Y_{1,r}) \) or \( L(Y_{k-1,r}) \), and \( L(M) \) any simple module.

**Proof.** If \( L(M') \) is the fundamental representation \( L(Y_{1,-2a}) \), and \( L(M) \) is any simple module, then using Proposition 6.7, the condition (1) of Theorem 6.10 is equivalent to both the conditions \( LL_k(Z([a,a]), Z(m_M)) \) and \( RL_k(Z([a,a]), Z(m_M)) \) being satisfied. From Lemma 6.1, these conditions are equivalent to \( Z([a,a]) \times Z(m_M) \) being irreducible in \( C_k \), and thus to \( \left[ Z([a,a]) \times Z(m_M) \right] = \left[ Z([a,a] + m_M) \right] \in R_k \). With the same reasoning as above, this condition is equivalent to (2) of Theorem 6.10.

The result carries to the case of the fundamental representation \( L(Y_{k-1,r}) \) as it is dual to a fundamental module by symmetry of the Dynkin diagram (see [FM01]). Note that, as for the segments of length 1, if \( m = [a, a + k - 2] \) and \( n \) is any multisegment, then \( X_{m,n} = X_{m,n}^{(k)} \) and \( Y_{m,n} = Y_{m,n}^{(k)} \), and the proof of Proposition 6.7 carries to that case.

**Corollary 6.12.** Let \( T, T' \in SSYT(k, [n]) \) be tableaux corresponding to ladders. Then the following conditions are equivalent:

1. \( LC(T, T') \) and \( LC(T', T) \).
2. \( \text{ch}(T)\text{ch}(T') = \text{ch}(T \cup T') \in SSYT(k, [n], \sim) \).

**Proof.** Thanks to (6.4), (1) is equivalent to both \( LC_k(m_T, m_{T'}) \) and \( LC_k(m_{T'}, m_T) \). These are equivalent by Theorem 6.10 to the tensor product \( L(M_T) \otimes L(M_{T'}) \) being irreducible, or also \( \chi_q(L(M_T))\chi_q(L(M_{T'})) = \chi_q(L(M_T M_{T'})) \). The latter is equivalent to (2) by Theorem 3.17 and Proposition 3.26 in [CDFL20].

Considering the results of [LM18], we expect the following.

**Conjecture 6.13.** Theorem 6.10 is true if either \( L(M) \) or \( L(M') \) is a snake module and other is an arbitrary simple module.

**Remark 6.14.** We expect Corollary 6.12 to be not only in \( \mathbb{C}[Gr(k,n,\sim)] \) but also in \( \mathbb{C}[Gr(k,n)] \).

Any \( T \in SSYT(k, [n]) \) can be written as \( T = T' \cup T'' \), where \( T' \) has small gaps and \( T'' \) is a fraction of two trivial tableaux. In Section 5.2 of [CDFL20], it is conjectured that \( \text{ch}(T) = P_T^\ast \text{ch}(T) \) is an element in \( \mathbb{C}[Gr(k,n)] \), where \( P_T^\ast \) is the Laurent monomial in trivial frozen Plücker coordinates corresponding to \( T'' \).

Thus we expect that condition (1) in Corollary 6.12 should also be equivalent to

\[ \text{ch}(T)\text{ch}(T) = \text{ch}(T \cup T') \in \mathbb{C}[Gr(k,n)]. \]
We give two examples to explain Theorem 6.10.

**Example 6.15.** Let $m = [-4, -1]$, $m' = [-1, 2]$. Then $X_{m,m'} = Y_{m,m'} = \{(1, 1)\}$, $X_{m',m} = Y_{m',m} = \emptyset$. The condition $LC(m, m')$ is not satisfied and the condition $LC(m', m)$ is satisfied. Therefore the representation $Z(m) \times Z(m')$ in $C$ is not irreducible.

Denote by $T_m^{(k)}$ the tableau in SSYT$(k, [m])$ ($n$ is a large enough integer). Since the maximum length of segments in $m$, $m'$ is 4, we take $k \geq 5$.

For $k \geq 5$, $LC(T_m^{(k)}, T_m^{(k)})$ is satisfied. For $k = 5, 6$, $LC(T_m^{(k)}, T_m^{(k)})$ is satisfied. For $k \geq 7$, $LC(T_m^{(k)}, T_m^{(k)})$ is not satisfied.

For example, in the case of $k = 5$, we have

\[
T_m^{(k)} = \begin{array}{cccc}
5 & 7 & \emptyset \\
4 & 6 & 5 \\
3 & 9 & 8 \\
2 & 10 & 7 \\
1 & 11 & 12 \\
\end{array}
\]

and $X_{T_m^{(k)}, T_m^{(k)}} = Y_{T_m^{(k)}, T_m^{(k)}} = X_{T_m^{(k)}, T_m^{(k)}} = Y_{T_m^{(k)}, T_m^{(k)}} = \emptyset$. Therefore $LC(T_m^{(k)}, T_m^{(k)})$ and $LC(T_m^{(k)}, T_m^{(k)})$ are satisfied.

In the case of $k = 7$, we have

\[
T_m^{(k)} = \begin{array}{cccc}
5 & 6 & \emptyset \\
4 & 7 & 3 \\
3 & 9 & 6 \\
2 & 10 & 7 \\
1 & 11 & 9 \\
\end{array}
\]

and $X_{T_m^{(k)}, T_m^{(k)}} = \{(1, 1)\}$, $Y_{T_m^{(k)}, T_m^{(k)}} = \emptyset$, $X_{T_m^{(k)}, T_m^{(k)}} = Y_{T_m^{(k)}, T_m^{(k)}} = \emptyset$. Therefore $LC(T_m^{(k)}, T_m^{(k)})$ is not satisfied and $LC(T_m^{(k)}, T_m^{(k)})$ is satisfied.

The $U_q(\mathfrak{sl}_k)$-modules ($k \geq 5$) correspond to $m = [-4, -1]$ and $m' = [-1, 2]$ are $L(Y_{4,5})$ and $L(Y_{4,-1})$, respectively. Therefore for $k = 5, 6$, the $U_q(\mathfrak{sl}_k)$ module $L(Y_{4,5}) \otimes L(Y_{4,-1})$ is simple. For $k \geq 7$, the $U_q(\mathfrak{sl}_k)$ module $L(Y_{4,5}) \otimes L(Y_{4,-1})$ is not simple. In the case of $k = 7$, we have short exact sequence

\[
0 \rightarrow L(Y_{1,2}) \rightarrow L(Y_{4,5}) \otimes L(Y_{4,-1}) \rightarrow L(Y_{4,5}Y_{4,-1}) \rightarrow 0.
\]

In the case of $k \geq 8$, we have short exact sequence

\[
0 \rightarrow L(Y_{1,2}) \otimes L(Y_{7,2}) \rightarrow L(Y_{4,5}) \otimes L(Y_{4,-1}) \rightarrow L(Y_{4,5}Y_{4,-1}) \rightarrow 0.
\]

**Example 6.16.** Let

\[
m = [-4, -3] + [-5, -4],
\]

\[
n = [0, 1] + [-1, 0] + [-2, -2] + [-2, -1] + [-3, -3] + [-3, -3] + [-5, -4].
\]
Then the corresponding monomials are
\[ M_m = Y_{2,7}Y_{2,9}, \quad M_n = Y_{2,1}Y_{2,1}Y_{2,3}Y_{1,6}Y_{2,9}. \]

In \( \mathcal{R} \), we have
\[ [Z(m)][Z(n)] = [Z(m + n)] + [Z(m')], \]
where \( m' = [0, 1] + [-1, 0] + [-4, -1] + [-2, -2] + [-5, -3] + [-3, -3] + [-5, -4] \).

For \( k = 3 \), we have that
\[ \chi_q(L(M_m))\chi_q(L(M_n)) = \chi_q(L(M_mM_n)), \]
and
\[ \text{ch}(T^{(k)}_m)\text{ch}(T^{(k)}_n) = \text{ch}(\begin{array}{cccc}
5 & 8 & 9 \\
1 & 3 & 6 \\
4 & 5 & 8 \\
\end{array}) = \text{ch}(\begin{array}{cccc}
1 & 5 & 8 & 5 \\
6 & 7 & 9 & 9 \\
\end{array}) = \text{ch}(T^{(k)}_m \cup T^{(k)}_n). \]

The tensor product \( L(M_m) \otimes L(M_n) \) is simple. Now we check the conditions \( \text{LC}(T^{(k)}_m, T^{(k)}_n) \) and \( \text{LC}(T^{(k)}_n, T^{(k)}_m) \). We have
\[
T^{(k)}_m = \begin{array}{ccc}
5 & 6 & 7 \\
8 & 9 & \end{array} \quad T^{(k)}_n = \begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & 0 \\
6 & 7 & 9 \\
\end{array}
\]

\[ X_{T^{(k)}_m, T^{(k)}_n} = \{(1, 3), (2, 5), (2, 6)\}, \quad Y_{T^{(k)}_m, T^{(k)}_n} = \{(1, 5), (1, 6), (2, 7)\}. \]

There is a matching function from \( X_{T^{(k)}_m, T^{(k)}_n} \) to \( Y_{T^{(k)}_m, T^{(k)}_n} \): \( (1, 3) \mapsto (1, 5), (2, 5) \mapsto (2, 7), (2, 6) \mapsto (1, 6) \). Therefore \( \text{LC}(T^{(k)}_m, T^{(k)}_n) \) is satisfied.

We have
\[ X_{T^{(k)}_n, T^{(k)}_m} = \{(7, 1)\}, \quad Y_{T^{(k)}_n, T^{(k)}_m} = \{(7, 2)\}. \]

There is a matching function from \( X_{T^{(k)}_n, T^{(k)}_m} \) to \( Y_{T^{(k)}_n, T^{(k)}_m} \): \( (7, 1) \mapsto (7, 2) \). Therefore \( \text{LC}(T^{(k)}_n, T^{(k)}_m) \) is satisfied.

For \( k = 4 \), we have that
\[ \chi_q(L(M_m))\chi_q(L(M_n)) = \chi_q(L(M_mM_n)) + \chi_q(L(Y_{2,1}Y_{2,1}Y_{1,4}Y_{3,8}Y_{1,6}Y_{2,9})). \]
and
\[
\text{ch}(T_m^{(k)}) \text{ch}(T_n^{(k)}) = \text{ch}(\begin{array}{cccc}
5 & 1 & 3 & 0 \\
6 & 2 & 4 & 7 \\
9 & 5 & 6 & 9 \\
10 & 8 & 8 & 10 \end{array}) + \text{ch}(\begin{array}{cccc}
1 & 6 & 0 & 1 \\
2 & 5 & 4 & 3 \\
3 & 9 & 9 & 5 \\
8 & 8 & 10 & 10 \end{array}) \neq \text{ch}(T_m \cup T_n^{(k)}).
\]

Note that \(5\) corresponds to the trivial module.

The tensor product \(L(M_m) \otimes L(M_n)\) is not simple. Now we check the conditions \(\text{LC}(T_m^{(k)}, T_n^{(k)})\) and \(\text{LC}(T_n^{(k)}, T_m^{(k)})\). We have:

\[
\begin{array}{cccc}
T_m^{(k)} & -5 & 6 \\
& 6 & 7 \\
& 8 & 9 \\
& 9 & 10 \end{array}
\quad \begin{array}{cccc}
T_n^{(k)} & -1 & 2 \\
& 2 & 3 \\
& 4 & 5 \\
& 5 & 6 \\
& 6 & 7 \\
& 7 & 8 \\
& 8 & 9 \\
& 9 & 10 \end{array}
\]

\[
X_{T_m^{(k)}, T_n^{(k)}} = \{(1, 3), (1, 4), (2, 5), (2, 6)\}, \quad Y_{T_m^{(k)}, T_n^{(k)}} = \{(1, 5), (1, 6), (2, 7)\}.
\]

There is no matching function from \(X_{T_m^{(k)}, T_n^{(k)}}\) to \(Y_{T_m^{(k)}, T_n^{(k)}}\). Therefore \(\text{LC}(T_m^{(k)}, T_n^{(k)})\) is not satisfied.

We have
\[
X_{T_n^{(k)}, T_m^{(k)}} = \{(7, 1)\}, \quad Y_{T_n^{(k)}, T_m^{(k)}} = \{(7, 1), (7, 2)\}.
\]

There is a matching function from \(X_{T_n^{(k)}, T_m^{(k)}}\) to \(Y_{T_n^{(k)}, T_m^{(k)}}\): \(7, 1 \mapsto (7, 2)\). Therefore \(\text{LC}(T_n^{(k)}, T_m^{(k)})\) is satisfied.

For \(k \geq 5\), we have that
\[
\chi_q(L(M_m)\chi_q(L(M_n)) = \chi_q(L(M_m M_n)) + \chi_q(L(Y_{2,-1} Y_{2,1} Y_{4,5} Y_{1,4} Y_{3,8} Y_{1,6} Y_{2,9})).
\]

The tensor product \(L(M_m) \otimes L(M_n)\) is not simple. We have:

\[
X_{T_m^{(k)}, T_n^{(k)}} = \{(1, 3), (1, 4), (2, 5), (2, 6)\}, \quad Y_{T_m^{(k)}, T_n^{(k)}} = \{(1, 5), (1, 6), (2, 7)\}.
\]

There is no matching function from \(X_{T_m^{(k)}, T_n^{(k)}}\) to \(Y_{T_m^{(k)}, T_n^{(k)}}\). Therefore \(\text{LC}(T_m^{(k)}, T_n^{(k)})\) is not satisfied.

We have
\[
X_{T_n^{(k)}, T_m^{(k)}} = \{(7, 1)\}, \quad Y_{T_n^{(k)}, T_m^{(k)}} = \{(7, 1), (7, 2)\}.
\]
There is a matching function from $X_{t_n^{(k)}, t_m^{(k)}}$ to $Y_{t_n^{(k)}, t_m^{(k)}}$: $(7, 1) \mapsto (7, 2)$. Therefore $\text{LC}(T_n^{(k)}, T_m^{(k)})$ is satisfied.

7. Proofs of main results

In this section, we mostly drop the subscripts and write $m, n$, as we only consider multisegments.

7.1. Proof of Proposition 6.7. Suppose here that $m$ is of length 1 and write $m = [a, a]$. Recall that $k \in \mathbb{Z}_{> 0}$ is fixed.

Remark 7.1. Note that in this case, the corresponding tableaux and monomial are $T = T_{a,a}$ ($T_{a,b}$ is defined in Section 4.3) and $M_T = Y_{1,-2a}$.

Let $n$ be any multisegment classifying an irreducible object in $C_k$, then $n = \Delta_1 + \cdots + \Delta_N$, where all $\Delta_i$ have length $e(\Delta_i) - b(\Delta_i) + 1 < k$.

As $X_{m,n} \subset \{1\} \times [N]$, by abuse of notations, we consider $X_{m,n}$ as a subset of $[N]$. Then

$$X_{m,n} = \{ i \in [N] | b(\Delta_i) = a + 1 \}, \quad X_{m,n} = \{ i \in [N] | b(\Delta_i) = a + 1, e(\Delta_i) < k + a \}.$$ 

It is clear that $X_{m,n}^{(k)} \subset X_{m,n}$. Suppose that $i \in X_{m,n}^{(k)}$. Then $b(\Delta_i) = a + 1$ for each $i \in [N]$. By the condition on the length of the segments in $n$, we have for each $i \in [N]$, $e(\Delta_i) - b(\Delta_i) + 1 < k$. Therefore $e(\Delta_i) - (a + 1) + 1 < k$ and hence $e(\Delta_i) < k + a$. Thus $i \in X_{m,n}^{(k)}$. It follows that

$$X_{m,n} = X_{m,n}^{(k)}.$$ 

Similarly,

$$Y_{m,n} = Y_{m,n}^{(k)}.$$ 

Thus is this case, the condition $\text{LC}(m, n)$ is equivalent to the condition $\text{LC}_k(m, n)$.

We need to prove the following result, which combined with Proposition 6.2 proves Proposition 6.7 in this case.

Proposition 7.2. For $\rho = Z([a,a])$ and $\pi = Z(n)$, where $n \in \text{Irr}_k$,

$$\text{Li}_k(\rho, \pi) \Rightarrow \text{Li}(\rho, \pi).$$

We use the following result, stated in [LM16, Theorem 5.11], based on results appearing in [M09, Theorem 7.5] and [Jan07, Theorem 2.2.1].

Proposition 7.3. Let $f$ be the best matching between $X_{[a,a],n}$ and $Y_{[a,a],n}$. Then

1. $\text{soc}(\rho \times Z(n)) = Z(n + [a,a])$ if and only if $f$ is a function from $X_{[a,a],n}$ to $Y_{[a,a],n}$.
2. If $f$ is a not function from $X_{[a,a],n}$ to $Y_{[a,a],n}$ and $i \in X_{[a,a],n}$ is the minimal index which does not belong to the domain of $f$, then

$$\text{soc}(\rho \times Z(n)) = Z(n - \Delta_i + [a,e(\Delta_i)]).$$
Proof. From Proposition 7.3, if the condition LI(ρ, π) is not satisfied, then soc(ρ × Z(n)) = Z(n − Δ + [a, e(Δ)])]. However, for i ∈ X[a, b], e(Δ_i) < k + a so the segment [a, e(Δ_i)] has length strictly lower than k + 1. Hence the image under \( Q \) of Z(n − Δ_i + [a, e(Δ_i)]) is not satisfied. Thus the condition LI_k(\( Q \), \( \Pi \)) is also not satisfied. □

**Example 7.4.** (1) Let \( k \geq 3 \), \( ρ = Z([0, 0]) \), \( n = Δ_1 + Δ_2 \), \( Δ = [1, 2] \), \( Δ_2 = [0, 1] \). Then \( X_{\rho, n}^{(k)} = \{1\} \), \( Y_{\rho, n}^{(k)} = \{2\} \). The best \( ∼_k \)-matching \( f \) between \( X_{\rho, n}^{(k)} \) and \( Y_{\rho, n}^{(k)} \) is given by \( f(1) = 2 \). The domain of \( f \) is \( X_{\rho, n}^{(k)} \) and hence \( f \) is a function from \( X_{\rho, n}^{(k)} \) to \( Y_{\rho, n}^{(k)} \). We have that soc(\( ρ × Z(n) \)) = \( Z([0, 0] + n) \). In the language of quantum affine algebra, this means for \( U_q(\mathfrak{sl}_k) \)-modules \( L(Y_{1,0}) \) and \( L(Y_{2,3}Y_{2,1}) \), the socle of \( L(Y_{1,0}) ⊕ L(Y_{2,3}Y_{2,1}) \) is \( L(Y_{1,0}Y_{2,3}Y_{2,1}) \).

(2) Let \( k = 4 \), \( ρ = Z([0, 0]) \), \( n = Δ_1 + Δ_2 \), \( Δ = [2, 3] \), \( Δ_2 = [1, 2] \). Then \( X_{\rho, n}^{(k)} = \{2\} \), \( Y_{\rho, n}^{(k)} = ∅ \). The domain of the best \( ∼_k \)-matching \( f \) between \( X_{\rho, n}^{(k)} \) and \( Y_{\rho, n}^{(k)} \) is ∅. Therefore \( f \) is not a function from \( X_{\rho, n}^{(k)} \) to \( Y_{\rho, n}^{(k)} \). We have that soc(\( ρ × Z(n) \)) = \( Z([2, 3] + [0, 2]) \) in \( C_k \). In the language of quantum affine algebra, this means for \( U_q(\mathfrak{sl}_k) \)-modules \( L(Y_{1,0}) \) and \( L(Y_{2,5}Y_{2,3}) \), the socle of \( L(Y_{1,0}) ⊕ L(Y_{2,5}Y_{2,3}) \) is \( L(Y_{2,5}Y_{3,2}) \).

Let \( k = 3 \), \( ρ = Z([0, 0]) \), \( n = Δ_1 + Δ_2 \), \( Δ = [2, 3] \), \( Δ_2 = [1, 2] \). Then \( X_{\rho, n}^{(k)} = \{2\} \), \( Y_{\rho, n}^{(k)} = ∅ \). The domain of the best \( ∼_k \)-matching \( f \) between \( X_{\rho, n}^{(k)} \) and \( Y_{\rho, n}^{(k)} \) is ∅. Therefore \( f \) is not a function from \( X_{\rho, n}^{(k)} \) to \( Y_{\rho, n}^{(k)} \). We have that soc(\( ρ × Z(n) \)) = \( Z([2, 3] + [0, 2]) \) in \( C_k \). In the language of quantum affine algebra, this means for \( U_q(\mathfrak{sl}_k) \)-modules \( L(Y_{1,0}) \) and \( L(Y_{2,5}Y_{2,3}) \), the socle of \( L(Y_{1,0}) ⊕ L(Y_{2,5}Y_{2,3}) \) is \( L(Y_{2,5}) \).

### 7.2. Proof of Proposition 6.9.

First we recall results in [Gur19], [Gur21b]. For \( λ = (λ_1, \ldots, λ_m), μ = (μ_1, \ldots, μ_m) ∈ \mathbb{Z}^m \), denote

\[
\mathbf{m}(λ, μ) = \sum_{i=1}^{m} [λ_i, μ_i].
\]

For \( x \) a permutation in \( S_m \) such that \( λ_i ≤ μ_{x(i)} + 1 \), for all \( 1 ≤ i ≤ m \),

\[
\mathbf{m}_x = \mathbf{m}_x(λ, μ) = \sum_{i=1}^{m} [λ_i, μ_{x(i)}].
\] (7.1)

Let \( \mathbf{m}, \mathbf{n} \) be multisegments and let \( λ = (λ_1 ≤ \ldots ≤ λ_m), μ = (μ_1 ≤ \ldots ≤ μ_m) ∈ \mathbb{Z}^m \) be the tuples of integers which are the beginning and ending of the multisegments in \( \mathbf{m} + \mathbf{n} \), respectively.

For \( x ∈ S_m \) and a segment \( Δ ∈ \mathbf{m}_x(λ, μ) \), define a sequence

\[
\text{Seq}(\mathbf{m}, \mathbf{n}, Δ) = ((Δ′_1, Δ_{i_1}, n′_1), (Δ′_2, Δ_{i_2}, n′_2), \ldots, (Δ′_r, Δ_{i_r}, n′_r))
\]
as follows.
• First we order the segments in \( m + n = \sum_i \Delta_i \) such that \( \Delta_i \preceq_b \Delta_{i+1} \).
• Step 1. Take the smallest \( i_1 \) such that the support of \( \Delta_{i_1} \) intersects the support of \( \Delta \), if several segments start at the same point, take the longest one. Let \( \Delta'_1 \) be the sub-segment of \( \Delta_{i_1} \) whose support is the intersection of the supports of \( \Delta_{i_1} \) and \( \Delta \). If \( \Delta_{i_1} \) is a segment of \( m \), then \( n'_1 = m \). Otherwise, \( n'_1 = n \).
• Step 2. Now consider the smallest \( i_2 \) such that the support of \( \Delta_{i_2} \) intersects the support of \( \Delta \setminus \Delta'_1 \), and \( \Delta_{i_2} \) and \( \Delta_{i_1} \) (the \( \Delta_{i_1} \) in previous step) are neither both in \( m \) nor both in \( n \). Let \( \Delta'_2 \) be the sub-segment of \( \Delta_{i_2} \) whose support is the intersection of the supports of \( \Delta_{i_2} \), \( \Delta \setminus \Delta'_1 \). If \( \Delta_{i_2} \) is a segment of \( m \), then \( n'_2 = m \). Otherwise, \( n'_2 = n \).
• Continue this procedure, at Step \( j \), take the smallest \( i_j \) such that the support of \( \Delta_{i_j} \) intersects the support of \( \Delta \setminus (\cup_{s=1}^{j-1} \Delta'_s) \), and \( \Delta_{i_j} \) and \( \Delta_{i_{j-1}} \) (the \( \Delta_{i_{j-1}} \) in previous step) are neither both in \( m \) nor both in \( n \). Let \( \Delta'_j \) be the sub-segment of \( \Delta_{i_j} \) whose support is the intersection of supports of \( \Delta_{i_j} \), \( \Delta \setminus (\cup_{s=1}^{j-1} \Delta'_s) \). If \( \Delta_{i_j} \) is a segment of \( m \), then \( n'_j = m \). Otherwise, \( n'_j = n \).
• Continue this procedure until at some step \( j \), there is no \( \Delta_{i_j} \) whose support intersects the support of \( \Delta \setminus (\cup_{s=1}^{j-1} \Delta'_s) \), and \( \Delta_{i_j} \) and \( \Delta_{i_{j-1}} \) (the \( \Delta_{i_{j-1}} \) in previous step) are neither both in \( m \) nor both in \( n \).

The multisegments \( m, n \) are said to be \emph{tiled} by \( \Delta \) if the union of the supports of \( \Delta'_j \), \( j = 1, \ldots, r \), is the support of \( \Delta \), and the end points of \( \Delta'_j \) and \( \Delta_{i_j} \) are the same.

We say that removing \( \text{Seq}(m, n, \Delta) \) from \( m, n \) is to remove \( \Delta'_j \) from \( \Delta_{i_j} \) for each \( j = 1, \ldots, r \). After removing, we obtain two multisegments \( m', n' \) whose segments are sub-segments of \( m, n \) respectively.

**Example 7.5.** Let \( m = [-6, -1] + [-2, 3] + [-1, 4], n = [-4, 1] + [0, 2] \). Then \( m + n = \sum_i \Delta_i = [-6, -1] + [-4, 1] + [-2, 3] + [-1, 4] + [0, 2] \) (ordered using \( \preceq_b \)). For \( \Delta = [-4, 3], \)

\[
\text{Seq}(m, n, \Delta) = ((([-4, 1], \Delta_2, n), ([2, 3], \Delta_3, m))).
\]

In this case, the multisegments \( m, n \) are tiled by \( \Delta \).

For \( \Delta = [-6, 2], \)

\[
\text{Seq}(m, n, \Delta) = ((([-6, -1], \Delta_1, m), ([0, 1], \Delta_2, n), ([2, 2], \Delta_3, m))).
\]

In this case, the multisegments \( m, n \) are not tiled by \( \Delta \) since the end point of \([2, 2]\) and \( \Delta_3 = [-2, 3] \) are different.

Now, write the segments of \( m_x = \sum_{i=1}^r \Delta_i \) such that \( \Delta_i \preceq_b \Delta_{i+1} \). Using the terminology of Gurevich [Gur21b], we say that \( m_x \) \emph{tiles} \( (m, n) \) if the following procedure stops only after step \( r \).

• Step 1. If the multisegments \( m, n \) are tiled by \( \Delta_1 \), then let \( m_1, n_1 \) be the multisegments obtained by removing \( \text{Seq}(m, n, \Delta_1) \) from \( m, n \).
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• Step 2. If the multisegments \( \mathbf{m}_1, \mathbf{n}_1 \) are tiled by \( \Delta_2 \), then let \( \mathbf{m}_2, \mathbf{n}_2 \) be the multisegments obtained by removing \( \text{Seq}(\mathbf{m}_1, \mathbf{n}_1, \Delta_2) \) from \( \mathbf{m}_1, \mathbf{n}_1 \).

• Step \( j \). Continue this procedure, if the multisegments \( \mathbf{m}_{j-1}, \mathbf{n}_{j-1} \) are tiled by \( \Delta_j \), then let \( \mathbf{m}_j, \mathbf{n}_j \) be the multisegments obtained by removing \( \text{Seq}(\mathbf{m}_{j-1}, \mathbf{n}_{j-1}, \Delta_j) \) from \( \mathbf{m}_{j-1}, \mathbf{n}_{j-1} \).

The following is clear from the construction of the tiling sequences.

**Lemma 7.6.** Suppose \( \mathbf{m}_x \) tiles \( (\mathbf{m}, \mathbf{n}) \), and let \( \Delta_i, \Delta_j \in \mathbf{m}_x \) be such that \( \Delta_i \prec \Delta_j \), and the tiling sequences of \( \Delta_i \) and \( \Delta_j \) start by segments of \( \mathbf{m} \) and \( \mathbf{n} \) (resp. \( \mathbf{n} \) and \( \mathbf{m} \)). Let \( \mathbf{m}_{x'} \) be the multisegment obtained from \( \mathbf{m}_x \) by replacing \( \Delta_i + \Delta_j \) by \((\Delta_i \cup \Delta_j) + (\Delta_i \cap \Delta_j)\). Then \( \mathbf{m}_{x'} \) still tiles \( (\mathbf{m}, \mathbf{n}) \).

For \( \lambda = (\lambda_1 \leq \ldots \leq \lambda_m), \mu = (\mu_1 \leq \ldots \leq \mu_m) \in \mathbb{Z}^m \), let \( S_{\lambda, \mu}^m \) be the double-coset of permutations inside \( S_m \) up to the following equivalence

\[
x \sim x' \iff \mathbf{m}_x(\lambda, \mu) = \mathbf{m}_{x'}(\lambda, \mu).
\]  
(7.2)

**Theorem 7.7** ([Gur19, Theorem 1.2], [Gur21b, Theorem 1.1]). Let \( \mathbf{m}, \mathbf{n} \) be ladders in \( \text{Mult}_k \) and let \( \lambda = (\lambda_1 \leq \ldots \leq \lambda_m), \mu = (\mu_1 \leq \ldots \leq \mu_m) \in \mathbb{Z}^m \) be the tuples of integers which are the beginning and ending of \( \mathbf{m} + \mathbf{n} \) respectively. Then we have

\[
[Z(\mathbf{m}) \times Z(\mathbf{n})] = \sum_{x \in S(\mathbf{m}, \mathbf{n})} [Z(\mathbf{m}_x)] \in \mathcal{R},
\]  
(7.3)

where \( S(\mathbf{m}, \mathbf{n}) \subset S_{\lambda, \mu}^m \) is formed of permutations \( x \in S_m \) such that \( \lambda_i \leq \mu_{x(i)} + 1, \forall i \), \( \mathbf{m}_x \) tiles \( \mathbf{m}, \mathbf{n} \) and the longest representative \( x' \) of \( x \) in \( S_{\lambda, \mu}^m \) is 321-avoiding.

Therefore in \( \mathcal{R}_k \), for ladders \( \mathbf{m}, \mathbf{n} \) in \( \text{Mult}_k \), we have

\[
[Z(\mathbf{m}) \times Z(\mathbf{n})] = \sum_{x \in S(\mathbf{m}, \mathbf{n})} [Z(\mathbf{m}_x)] \in \mathcal{R}_k.
\]  
(7.4)

**Remark 7.8.** In (7.3), all terms corresponding to permutations \( x \in S_m \) for which there is \( i \) such that \( \lambda_i > \mu_{x(i)} + 1 \) are sent to 0. In (7.4), additionally, all terms corresponding to permutations \( x \in S_m \) for which there is \( i \) such that \( \mu_{x(i)} > \lambda_i + k - 1 \) are sent to 0.

For \( \mathbf{m}, \mathbf{n} \in \text{Mult}_k \) ladders, write \( \mathbf{m} = \Delta_1 + \Delta_2 + \cdots + \Delta_N \), and \( \mathbf{n} = \Delta'_1 + \Delta'_2 + \cdots + \Delta'_{N'} \), in their left aligned form \((b(\Delta_i) < b(\Delta_{i+1})) \) and \((b(\Delta'_i) < b(\Delta'_{i+1})) \). Let \( X = X_{\mathbf{m}, \mathbf{n}}^{(k)} \) and \( Y = Y_{\mathbf{m}, \mathbf{n}}^{(k)} \), and denote by \( NC_k(\mathbf{m}, \mathbf{n}) \) the condition that there exists indices \( 1 \leq i \leq N, 1 \leq j \leq N' \), and \( m \geq 0 \) such that

- \((i + l, j + l) \in X, \) for all \( 0 \leq l \leq m, \)
- \((i, j - 1) \) and \((i + m + 1, j + m) \) are not in \( Y, \)
- \((i + l + 1, j + l) \in Y \setminus X, \) for all \( 0 \leq l \leq m - 1. \)

Note that \((i, j) \in Y \setminus X \) implies that \( b(\Delta_i) = b(\Delta'_j) \) or \( e(\Delta_i) = e(\Delta'_j) \).

The following is easy an modification from [LM16, Lemma 6.21]:
Lemma 7.9. For two ladders \( \mathbf{m}, \mathbf{n} \in \text{Mult}_k \), the negation of \( \text{LC}_k(\mathbf{m}, \mathbf{n}) \) is equivalent to \( \text{NC}_k(\mathbf{m}, \mathbf{n}) \).

We prove Proposition 6.9, via the following equivalent result.

Proposition 7.10. For ladders \( \mathbf{m}, \mathbf{n} \) in \( \text{Mult}_k \), the conditions \( \text{LC}_k(\mathbf{m}, \mathbf{n}) \) and \( \text{LC}_k(\mathbf{n}, \mathbf{m}) \) hold if and only if there is no \( \mathbf{m}_x \) appearing on the right hand side of (7.4) such that \( \mathbf{m}_x \neq \mathbf{m} + \mathbf{n} \) and \( \overline{\mathbf{m}}_x \) is not 0.

Proof. We write \( \mathbf{m} = \sum_i \Delta_i^{(m)} \), \( \mathbf{n} = \sum_i \Delta_i^{(n)} \) using the order \( \leq_b \). Let \( \lambda = (\lambda_1 \leq \ldots \leq \lambda_m), \mu = (\mu_1 \leq \ldots \leq \mu_m) \in \mathbb{Z}^m \) be the tuples of integers which are the beginning and ending of \( \mathbf{m} + \mathbf{n} \) respectively. We write \( \mathbf{m} + \mathbf{n} = \sum_j \Delta_j \) using the order \( \leq_b \). We have \( \mathbf{m} + \mathbf{n} = \mathbf{m}_x(\lambda, \mu) \) for some \( x \in S_m \).

\( (\Rightarrow) \) Suppose, without loss of generality, that the condition \( \text{LC}_k(\mathbf{m}, \mathbf{n}) \) does not hold. From Lemma 7.9, we know that the condition \( \text{NC}_k(\mathbf{m}, \mathbf{n}) \) holds. Let \( 1 \leq i \leq N, 1 \leq j \leq N', \) and \( m \geq 0 \) be the data associated to the condition \( \text{NC}_k(\mathbf{m}, \mathbf{n}) \).

Let \( \mathbf{m}' \) be the segment obtained from \( \mathbf{m} + \mathbf{n} \) by replacing, for all \( 0 \leq l \leq m \), the segments \( \Delta_{i+l}^{(m)} \), \( \Delta_{j+l}^{(n)} \) by \( \left( \Delta_{i+l}^{(m)} \cup \Delta_{j+l}^{(n)} \right), \left( \Delta_{i+l}^{(m)} \cap \Delta_{j+l}^{(n)} \right) \). Since \( \Delta_{i+l}^{(m)} \preceq \Delta_{j+l}^{(n)} \), the resulting multisegment has the same set of extremities as \( \mathbf{m} + \mathbf{n} \), thus there exists \( x' \in S_m \) such that \( \mathbf{m}' = \mathbf{m}_{x'}(\lambda, \mu) \). Moreover, \([\mathbb{Z}(\mathbf{m}_{x'})]\neq 0 \in \mathcal{R}_k\). Let us show that \( \mathbf{m}_{x'} \) tiles \( \mathbf{m} + \mathbf{n} \) and that the longest representative of \( x' \) in \( S_m^{(\lambda, \mu)} \) is 321-avoiding.

Suppose \( x \in S_m^{(\lambda, \mu)} \) is chosen as the longest representative for the equivalence relation (7.2), and let \( i_0 < i_1 < \cdots < i_m < j_0 < j_1 < \cdots < j_m \) be such that for all \( 1 \leq l \leq m \), \( \Delta_{i+l}^{(m)} = \Delta_{i_l} = [\lambda_{i_l}, \mu_{x(i_l)}] \) and \( \Delta_{j+l}^{(n)} = \Delta_{j_l} = [\lambda_{j_l}, \mu_{x(j_l)}] \) respectively \(^3\). Note that \( i_l < j_l \) for all \( l \). Then

\[
x' = x \prod_{l=1}^{m} (i_l, j_l),
\]

where \((i_l, j_l)\) denotes the transposition. Note that the product in (7.5) is commutative.

Since \( x \) is clearly 321-avoiding, any 321-pattern in \( x' \) must involve at least one of the modified indices. Moreover, the 321-pattern cannot appear before \( i_0 \), because that would mean there is \( p < q < i_0 \) such that \( x'(p) = x(p) \geq x'(q) = x(q) \geq x'(i_0) = x(j_0) > x(i_0) \), which is impossible since \( \mathbf{m} \) and \( \mathbf{n} \) are ladders. Similarly, the 321-pattern cannot appear after \( j_m \).

Next, assume that a 321-pattern occurs involving the positions \( i_0 \) and \( j_0 \):

\[
\begin{align*}
\Delta_{i_0} \cap \Delta_{j_0} & \quad \text{(a)} \\
\Delta_{i_0} \cup \Delta_{j_0} & \quad \text{(b)}
\end{align*}
\]

\(^3\)The indices \( j - 1 \) and \( i + m + 1 \) do not necessarily exist for the condition \( \text{NC}_k(\mathbf{m}, \mathbf{n}) \) to be satisfied.
If in case (a) the blue segment belongs to \( n \), then \( j_{-1} \) exists and by hypothesis \((i_0, j_{-1}) \not\in Y\), a contradiction. If in case (a) the middle segment belongs to \( m \), and is not modified \((m = 1)\), then \((i_1, j_0) \not\in Y\) gives the contradiction. In case (b), the red segment necessarily belongs to \( n \) (since \( m \) is a ladder), and the contradiction comes again from \((i_0, j_{-1}) \not\in Y\).

By a similar reasoning, we can see that a 321-pattern cannot appear after \( i_m, j_m \) involving the two of them.

Now consider what happens in the middle of the transformed block:

We can see that in both blue and red cases, no 321-pattern can appear: the longest permutations corresponding to the modified multisegments are 3124 and 2413, respectively.

The fact that \( m \) and \( n \) are ladders prevents 321-patterns from appearing between non consecutive pairs of segments.

Finally, we can see that \( m, n \) tiles \( m, n \), by successive application of Lemma 7.6.

(⇒) Suppose we have a 321-avoiding permutation \( x' \in S^{(\lambda, \mu)} \) such that \( m, n \neq m + n, 0 \) and \( m, n \) tiles \( m + n \).

Since \( m, n \neq m + n \), there is at least one segment in \( m, n \) whose sequence Seq in the algorithm above is of length \( \geq 2 \). Take \( \Delta \) to be the minimal such segment (for the order \( \leq \)). Replace \( m \) and \( n \) by the multisegments obtained after removing all the segments before \( \Delta \) in \( m, n \), as they are all equal to segments of \( m + n \), and denote the resulting multisegments by \( m \) and \( n \) again (in other words, we can assume that \( \Delta \) is the first segment of \( m + n \)). Now, let us assume without loss of generality that \( \text{Seq}(m, n, \Delta) \) starts as follows

\[
\text{Seq}(m, n, \Delta) = ((\Delta^l_1, \Delta^{m}_i, m), (\Delta^l_2, \Delta^{(n)}_j, n), \ldots).
\]

As above, we use the notations \( X = X^{(k)} \) and \( Y = Y^{(k)} \). As \( m, n \) tiles \( m + n \), necessarily \( \Delta^l_1 = \Delta^{(m)}_i \) and \( b(\Delta) = b(\Delta^{(m)}_i) \). Moreover, \( b(\Delta^{(n)}_j) > b(\Delta^{(m)}_i) \), by definition of \( \Delta^{(m)}_i \), and \([e(\Delta^{(m)}_i) + 1, e(\Delta)] \cap \Delta^{(n)}_j \neq \emptyset \), thus \( e(\Delta^{(n)}_j) \geq e(\Delta^{(m)}_i) + 1 \). Next, we distinct 2 cases:

(A) \[
\Delta^{(m)}_i \Delta^{(n)}_j \Delta
\]

or \( \Delta^{(m)}_i \Delta^{(m)}_j \Delta \)

(B)
In the case (A), \( \Delta_{x}^{(m)} \prec_{k} \Delta_{y}^{(n)} \), thus \((i, j) \in X \). In the case (B), there is another segment \( \Delta_{x}^{(m)} \) in \( m \) such that \( \Delta_{y}^{(m)} \prec_{k} \Delta_{x}^{(n)} \), thus \((i', j) \in X \). Note that in the example above, we have \( \Delta_{x}^{(m)} \prec_{k} \Delta_{y}^{(m)} \), but it is not necessarily the case, there can be several segments between \( \Delta_{x}^{(m)} \) and \( \Delta_{y}^{(m)} \). In any case, we have a pair \((i, j) \in X \).

Suppose \((i, j - 1) \in Y \), then either \( b(\Delta_{x}^{(m)}) = b(\Delta_{y}^{(n)}) \) or \( e(\Delta_{x}^{(m)}) = e(\Delta_{y}^{(n)}) \). In both cases, it would imply a 321-pattern in the longest representative of \( x' \). For example:

There exist another segment \( \Delta' \in m' \), which starts at \( b(\Delta_{x}^{(n)}) \), and necessarily at least another segment after both \( \Delta, \Delta' \) which ends before (the end point of \( \Delta_{y}^{(n)} \) must be reached).

The others cases all similarly lead to a 321-pattern in \( x' \), a contradiction.

Let \( m \geq 0 \) such that for all \( 0 \leq l \leq m-1 \), \((i+l+1, j+l) \in Y \setminus X \) and \((i+m+1, j+m) \notin Y \setminus X \). If \((i+m+1, j+m) \notin Y \), then the condition \( \text{NC}_{k}(m, n) \) is satisfied, which concludes the proof.

If \((i+m+1, j+m) \in X \), then we replace \((i, j) \) by \((i', j') = (i+m+1, j+m) \in X \). We have \((i'-1, j'-1) \in Y \setminus X \). If \( b(\Delta_{x}^{(m)}) = b(\Delta_{y}^{(n)}) \), then \( b(\Delta_{x}^{(m)}) > b(\Delta_{y}^{(n)}) \) and \((i', j'-1) \notin Y \). Similarly, if \( e(\Delta_{x}^{(m)}) = e(\Delta_{y}^{(n)}) \), then \( e(\Delta_{x}^{(m)}) > e(\Delta_{y}^{(n)}) \) and \((i', j'-1) \notin Y \). We continue this process until we get to \((i+m+1, j+m) \notin Y \) (or there is no segment \( i+m+1 \)). As a result, the condition \( \text{NC}_{k}(m, n) \) is satisfied, which negates \( \text{LC}_{k}(m, n) \).

We give the following examples to explain Proposition 7.10.

**Example 7.11.** Let \( m = [-6, -1] + [-2, 3] + [-1, 4] \), \( n = [-4, 1] + [0, 2] \) and let \( k = 15 \). We have that \( \lambda = (-6, -4, -2, -1, 0) \), \( \mu = (-1, 1, 2, 3, 4) \). Since the stabilizers \( S_{\lambda} \) and \( S_{\mu} \) are both trivial, we have that for each \( x \in S_{m} \), the only one element \( x' \in S_{m} \) such that \( m' = m \) is \( x' = x \). For \( x = 31245 = s_{1}s_{2} \in S_{5} \), \( m_{x} = [-6, 2] + [-4, -1] + [-2, 1] + [-1, 3] + [0, 4] \). Here \( x = 31245 \) is the one-line notation of a permutation. The permutation \( x \) is 321-avoiding and \( \overline{m}_{x} \) is not 0. We have

\[
\text{Seq}(m, n, [-6, 2]) = ([-6, -1], [-6, -1], m), ([0, 1], [-4, 1], n), ([2, 2], [-2, 3], m)).
\]

Since the end points of \([2, 2], [-2, 3] \) are different, the segment \([-6, 2] \) does not tile \( m, n \). Therefore \( m_{x} \) does not tile \( m, n \). Hence \( [\overline{Z}(m_{x})] \) does not appear on the right hand side of (7.4).
For $x = 24153 = s_4s_2s_3s_1 \in S_5$, $m_x = [-6, 1] + [-4, 3] + [-2, -1] + [-1, 4] + [0, 2]$. The permutation $x$ is $321$-avoiding and $\overline{m}_x$ is not $0$. We have

\[\text{Seq}(m, n, [-6, 1]) = \{([-6, -1], [-6, -1], m), ([0, 1], [-4, 1], n)\},\]
\[\text{Seq}(m_1, n_1, [-4, 3]) = \{([-4, -1], [-4, -1], n), ([0, 3], [-2, 3], m)\},\]
\[\text{Seq}(m_2, n_2, [-2, -1]) = \{([-2, -1], [-2, -1], m)\},\]
\[\text{Seq}(m_3, n_3, [-1, 4]) = \{([-1, 4], [-1, 4], m)\},\]
\[\text{Seq}(m_4, n_4, [0, 2]) = \{([0, 2], [0, -2], n)\}.
\]

Therefore $m_x$ tiles $m, n$. Hence $[Z(m_x)]$ appears on the right hand side of (7.4).

By checking all $x \in S_5$, we obtain that

\[\begin{align*}
[Z(m)][Z(n)] &= Z([-6, -1] + [-4, 1] + [-2, 3] + [-1, 4] + [0, 2]) \\
&\quad + Z([-6, -1] + [-4, 3] + [-2, 1] + [-1, 4] + [0, 2]) \\
&\quad + Z([-6, 1] + [-4, 1] + [-2, 3] + [-1, 4] + [0, 2]) \\
&\quad + Z([-6, 3] + [-4, -1] + [-2, 1] + [-1, 4] + [0, 2]) \\
&\quad + Z([-6, 1] + [-4, 3] + [-2, 1] + [-1, 4] + [0, 2]).
\end{align*}\]

Now we check the condition $LC_k(m, n)$ and $LC_k(n, m)$. We have that

\[X_{m,n}^{(k)} = \{(1, 1), (1, 2)\}, \quad Y_{m,n}^{(k)} = \{(1, 1)\}.
\]

The size of $Y_{m,n}^{(k)}$ is less than the size of $X_{m,n}^{(k)}$. Therefore there is no injective matching function from $X_{m,n}^{(k)}$ to $Y_{m,n}^{(k)}$. The condition $LC_k(m, n)$ is not satisfied.

We have that

\[X_{n,m}^{(k)} = \{(1, 2), (1, 3)\}, \quad Y_{n,m}^{(k)} = \{(1, 2), (1, 3)\}.
\]

The third segment of $m$ does not $k$-precede the second segment of $m$. Therefore there is no injective matching function from $X_{n,m}^{(k)}$ to $Y_{n,m}^{(k)}$. The condition $LC_k(n, m)$ is not satisfied.

In the case of $k = 7$, $[Z(m)][Z(n)] = [Z(m+n)]$. Now we check the condition $LC_k(m, n)$ and $LC_k(n, m)$. We have that

\[X_{m,n}^{(k)} = Y_{m,n}^{(k)} = \emptyset.
\]

Therefore there is an injective matching function from $X_{m,n}^{(k)}$ to $Y_{m,n}^{(k)}$. The condition $LC_k(m, n)$ is satisfied.

We have that

\[X_{n,m}^{(k)} = Y_{n,m}^{(k)} = \emptyset.
\]
Therefore there is an injective matching function from $X_{n,m}^{(k)}$ to $Y_{n,m}^{(k)}$. The condition $\text{LC}_k(n,m)$ is satisfied.

**Example 7.12.** Let $m = [-9, -4] + [-8, -2] + [-5, 0]$, $n = [-10, -3] + [-7, -2] + [-6, -1]$ and let $k \geq 9$. Then

\[
[\mathbb{Z}(m)]\mathbb{Z}(n)] \cdot \mathbb{Z}(m) = \mathbb{Z}([-5, 0] + [-6, -1] + [-7, -2] + [-8, -2] + [-9, -4] + [-10, -3])
\]

\[
+ \mathbb{Z}([-5, 0] + [-6, -1] + [-7, -2] + [-8, -3] + [-9, -4] + [-10, -2])
\]

\[
+ \mathbb{Z}([-5, 1] + [-6, 0] + [-7, -2] + [-8, -2] + [-9, -4] + [-10, -3])
\]

\[
+ \mathbb{Z}([-5, 0] + [-6, -2] + [-7, -4] + [-8, -1] + [-9, -2] + [-10, -3])
\]

\[
+ \mathbb{Z}([-5, 0] + [-6, 0] + [-7, -2] + [-8, -3] + [-9, -4] + [-10, -2])
\]

\[
+ \mathbb{Z}([-5, -2] + [-6, 0] + [-7, -4] + [-8, -1] + [-9, -2] + [-10, -3])
\]

\[
+ \mathbb{Z}([-5, -1] + [-6, -2] + [-7, -4] + [-8, 0] + [-9, -2] + [-10, -3])
\]

For each $k \geq 9$, $\mathbb{Z}(m) \times \mathbb{Z}(n)$ is not irreducible. Now we check the condition $\text{LC}_k(m,n)$ and $\text{LC}_k(n,m)$. We have that

\[
X_{m,n}^{(k)} = \{(1, 2), (1, 3), (2, 3)\}, \ Y_{m,n}^{(k)} = \{(1, 2), (1, 3), (2, 2), (2, 3)\}
\]

Therefore there is no injective matching function from $X_{m,n}^{(k)}$ to $Y_{m,n}^{(k)}$. The condition $\text{LC}_k(m,n)$ is not satisfied.

We have that

\[
X_{n,m}^{(k)} = \{(1, 2), (1, 3), (2, 3), (3, 3)\}, \ Y_{n,m}^{(k)} = \{(1, 2), (1, 3), (2, 3), (3, 3)\}
\]

Consider $(1, 2)$ in $X_{n,m}^{(k)}$. The only element in $Y_{n,m}^{(k)}$ which is possibly match $(1, 2) \in X_{n,m}^{(k)}$ is $(1, 3)$. But $\Delta_3^{(m)} = [-5, 0]$ does not $k$-precede $\Delta_2^{(m)} = [-8, -2]$. Therefore $(1, 2) \in X_{n,m}^{(k)}$ does not match any element in $Y_{n,m}^{(k)}$. Thus there is no injective matching function from $X_{n,m}^{(k)}$ to $Y_{n,m}^{(k)}$. The condition $\text{LC}_k(n,m)$ is not satisfied.

The authors declare that they have no conflict of interest.

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