PRINCIPAL VALUES FOR RIESZ TRANSFORMS AND RECTIFIABILITY

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Abstract. Let $E \subset \mathbb{R}^d$ with $\mathcal{H}^n(E) < \infty$, where $\mathcal{H}^n$ stands for the $n$-dimensional Hausdorff measure. In this paper we prove that $E$ is $n$-rectifiable if and only if the limit
$$\lim_{\varepsilon \to 0} \int_{y \in E : |x-y| > \varepsilon} \frac{x - y}{|x-y|^{n+1}} d\mathcal{H}^n(y)$$
exists $\mathcal{H}^n$-almost everywhere in $E$. To prove this result we obtain precise estimates from above and from below for the $L^2$ norm of the $n$-dimensional Riesz transforms on Lipschitz graphs.

1. Introduction

Given $x \in \mathbb{R}^d$, $x \neq 0$, we consider the signed Riesz kernel $K(x) = \frac{x}{|x|^{n+1}}$, for an integer such that $0 < n \leq d$. Observe that $K$ is a vectorial kernel. The $n$-dimensional Riesz transform of a finite Borel measure $\mu$ on $\mathbb{R}^d$ is defined by
$$R^n \mu(x) = \int K(x-y) \, d\mu(y), \quad x \notin \text{supp}(\mu).$$
Notice that the integral above may fail to be absolutely convergent for $x \in \text{supp}(\mu)$. For this reason one considers the $\varepsilon$-truncated $n$-dimensional Riesz transform, for $\varepsilon > 0$:
$$R^n_\varepsilon \mu(x) = \int_{|x-y| > \varepsilon} K(x-y) \, d\mu(y), \quad x \in \mathbb{R}^d.$$
The principal values are denoted by
$$\text{p.v.} R^n \mu(x) = \lim_{\varepsilon \to 0} R^n_\varepsilon \mu(x),$$
whenever the limit exists.

One says that a subset $E \subset \mathbb{R}^d$ is $n$-rectifiable if there exists a countable family of $n$-dimensional $C^1$ submanifolds $\{M_i\}_{i \geq 1}$ such that
$$\mathcal{H}^n \left( E \setminus \bigcup_i M_i \right) = 0,$$
where $\mathcal{H}^n$ stands for the $n$-dimensional Hausdorff measure.
In this paper we are interested in the relationship between rectifiability and Riesz transforms. One of our main results is the following.

**Theorem 1.1.** Let $E \subset \mathbb{R}^d$ with $\mathcal{H}^n(E) < \infty$. Then $E$ is $n$-rectifiable if, and only if, the principal value $\text{p.v.} R^n(\mathcal{H}^n_{|E})(x)$ exists for $\mathcal{H}^n$-almost every $x \in E$.

In fact, the “only if” part of the theorem (rectifiability implies existence of principal values) was well known (see [MP], for example). On the other hand, under the additional assumption that

\begin{equation}
\liminf_{r \to 0} \frac{\mathcal{H}^n(B(x,r) \cap E)}{r^n} > 0 \quad \mathcal{H}^n\text{-a.e. } x \in E,
\end{equation}

Mattila and Preiss proved [MP] that if the principal value $\text{p.v.} R^n(\mathcal{H}^n_{|E})(x)$ exists $\mathcal{H}^n$-almost everywhere in $E$, then $E$ is rectifiable. Getting rid of the hypothesis (1.1) was an open problem raised by authors in [MP].

Let us also remark that in the particular case $n = 1$, Theorem 1.1 was previously proved in [To1] (and in [Ma2] under the assumption (1.1)) using the relationship between the Cauchy transform and curvature of measures (for more information on this curvature, see [Me] and [MeV], for example). In higher dimensions the curvature method does not work (see [Fa]) and new techniques are required.

We do not know if Theorem 1.1 holds if one replaces the assumption on the existence of principal values for the Riesz transforms by

\[ \sup_{\varepsilon > 0} |R^n_{\varepsilon}(\mathcal{H}^n_{|E})(x)| < \infty \quad \mathcal{H}^n\text{-a.e. } x \in E. \]

That this is the case for $n = 1$ was shown in [To1] using curvature. However, for $n > 1$ this is an open problem that looks very difficult (probably, as difficult as proving that the $L^2$ boundedness of Riesz transforms with respect to $\mathcal{H}^n_{|E}$ implies the $n$-rectifiability of $E$).

Given a Borel measure $\mu$ on $\mathbb{R}^d$, its upper and lower $n$-dimensional densities are defined, respectively, by

\[
\Theta^{n,\ast}_{\mu}(x) = \limsup_{r \to 0} \frac{\mu(B(x,r))}{r^n}, \quad \Theta^\ast_{\mu}(x) = \liminf_{r \to 0} \frac{\mu(B(x,r))}{r^n}.
\]

So (1.1) means that the lower $n$-dimensional densities with respect to $\mathcal{H}^n_{|E}$ is positive $\mathcal{H}^n$-a.e. in $E$. We recall that if $\mathcal{H}^n(E) < \infty$, then

\[ 0 < \Theta^\ast_{\mathcal{H}^n_{|E}}(x) < \infty \quad \mathcal{H}^n\text{-a.e. } x \in E. \]

However there are sets $E$ with $0 < \mathcal{H}^n(E) < \infty$ such that the lower density $\Theta^\ast_{\mathcal{H}^n_{|E}}(x)$ vanishes for every $x \in E$ (see [Ma1] Chapter 6), for example).

Theorem 1.1 is a particular case of the following somewhat stronger result.

**Theorem 1.2.** Let $\mu$ be a finite Borel measure on $\mathbb{R}^d$. Let $E \subset \mathbb{R}^d$ be such that for all $x \in E$ we have

\[ 0 < \Theta^{n,\ast}_{\mu}(x) < \infty \quad \text{and} \quad \exists \text{p.v.} R^n\mu(x). \]
Then $E$ is $n$-rectifiable.

Our arguments to prove Theorems 1.1 and 1.2 are very different from the ones in [MP] and [Ma], which are based on the use of tangent measures. A fundamental step in our proof consists in obtaining precise $L^2$ estimates of Riesz transforms on Lipschitz graphs. In a sense, these $L^2$ estimates play a role analogous to curvature of measures in [To1]. Loosely speaking, the second step of the proof consists of using these $L^2$ estimates to construct a Lipschitz graph containing a suitable piece of $E$, by arguments more or less similar to the ones in [L2].

To describe in detail the $L^2$ estimates mentioned above we need to introduce some additional terminology. We denote the projection

$$(x_1, \ldots, x_n, \ldots, x_d) \mapsto (x_1, \ldots, x_n, 0, \ldots, 0)$$

by $\Pi$, and we set $\Pi^\perp = I - \Pi$. We also denote

$$R^{n,\perp}_\mu(x) = \Pi^\perp(R^n_\mu(x)) \quad \text{and} \quad R^{n,\perp}_\varepsilon(x) = \Pi^\perp(R^n_\varepsilon(x)).$$

That is to say, $R^{n,\perp}_\mu(x)$ and $R^{n,\perp}_\varepsilon(x)$ are made up of the components of $R^n_\mu(x)$ and $R^n_\varepsilon(x)$ orthogonal to $\mathbb{R}^n$, respectively (we are identifying $\mathbb{R}^n$ with $\mathbb{R}^n \times \{(0, \ldots, 0)\}$).

**Theorem 1.3.** Consider the $n$-dimensional Lipschitz graph $\Gamma := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^{d-n} : y = A(x)\}$, and let $d\mu(z) = g(z) dH^n_\Gamma(z)$, where $g(\cdot)$ is a function such that $C_1^{-1} \leq g(z) \leq C_1$ for all $z \in \Gamma$. Suppose that $A$ has compact support. If $\|g - 1\|_2 \leq C_2 \|\nabla A\|_2$ and $\|\nabla A\|_\infty \leq \varepsilon_0$, with $0 < \varepsilon_0 < 1$ small enough (depending on $C_2$), then we have

$$\|\text{p.v.} R^{n,\perp}_\mu \|_{L^2(\mu)} \approx \|\text{p.v.} R^n_\mu \|_{L^2(\mu)} \approx \|\nabla A\|_2.$$

Let us remark that the existence of the principal values $\text{p.v.} R^n_\mu \mu$ a.e. under the assumptions of the theorem is a well know fact. If we take $g(x) \equiv 1$, we obtain:

**Corollary 1.4.** Consider the $n$-dimensional Lipschitz graph $\Gamma := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^{d-n} : y = A(x)\}$, and let $\mu = H^n_\Gamma$. Suppose that $A$ has compact support. If $\|\nabla A\|_\infty \leq \varepsilon_0$, with $0 < \varepsilon_0 \leq 1$ small enough, then

$$\|\text{p.v.} R^{n,\perp}_\mu \|_{L^2(\mu)} \approx \|\text{p.v.} R^n_\mu \|_{L^2(\mu)} \approx \|\nabla A\|_2.$$

The upper estimate $\|\text{p.v.} R^n_\mu \|_{L^2(\mu)} \leq \|\nabla A\|_2$ is an easy consequence of some of the results from [Da] and [Go] and also holds replacing $\varepsilon_0$ by any big constant (see Lemma 3.1 in Section 3 for more details). The lower estimate $\|\text{p.v.} R^{n,\perp}_\mu \|_{L^2(\mu)} \geq \|\nabla A\|_2$ is more difficult. To prove it we use a Fourier type estimate as well as the quasiothogonality techniques developed in [Go]. In particular, the coefficients $\alpha(Q)$ (see Section 2 for the definition) introduced in that paper are an important tool for the proof.

We remark that we do not know if the inequalities $\|\text{p.v.} R^n_\mu \|_{L^2(\mu)} \geq C_3^{-1}\|\nabla A\|_2$ or $\|\text{p.v.} R^{n,\perp}_\mu \|_{L^2(\mu)} \geq C_3^{-1}\|\nabla A\|_2$ in Theorem 1.3 or Corollary...
1.4 hold assuming $\| \nabla A \|_\infty \leq C_4$ instead of $\| \nabla A \|_\infty \leq \varepsilon_0$, with $C_4$ arbitrarily large and $C_3$ possibly depending on $C_4$.

Obtaining lower estimates for the $L^2$ norm of $n$-dimensional Riesz transforms in $\mathbb{R}^d$ is also important for other problems, such as the characterization of removable singularities for bounded analytic functions (for $n = 1$) and Lipschitz harmonic functions (for $n \geq 1$). For instance, in [Ma1], in order to characterize some Cantor sets which are removable for Lipschitz harmonic functions in $\mathbb{R}^{n+1}$ first one needs to get a lower estimate of the norm $\| \text{p.v.} R^n_{\mu} \|_{L^2(\mu)}$, where $\mu$ is the natural probability measure supported on the given Cantor set. Analogous results for bilipschitz images of Cantor sets are obtained in [GPT]. See also [ENV] for other recent results which involve lower estimates of $L^2$ norms of Riesz transforms, and [Da2], [To2], [Vo] for other questions on removability of singularities of bounded analytic functions and Lipschitz harmonic functions.

The plan of the paper is the following. In Section 2 we introduce some preliminary notation and state some results that will be needed in the rest of the paper. Sections 3-6 are devoted to the proof of Theorem 1.3, while Theorem 1.2 is proved in Sections 7-10 by arguments inspired in part by the corona type constructions of [Lé2] and [DS1].

2. Preliminaries

As usual, in the paper the letter ‘$C$’ stands for an absolute constant which may change its value at different occurrences. On the other hand, constants with subscripts, such as $C_1$, retain its value at different occurrences. The notation $A \lesssim B$ means that there is a positive absolute constant $C$ such that $A \leq CB$. Also, $A \approx B$ is equivalent to $A \lesssim B \lesssim A$.

An open ball with center $x$ and radius $r$ is denoted by $B(x,r)$. If we want to remark that this is an $n$-dimensional ball, we write $B_n(x,r)$. Given $0 < n \leq d$, we say that a Borel measure $\mu$ on $\mathbb{R}^d$ is $n$-dimensional Ahlfors-David regular, or simply AD regular, if there exist some constants $C_0$ and $C_1$ such that $C_0^{-1}r^n \leq \mu(B(x,r)) \leq C_1r^n$ for all $x \in \text{supp}(\mu)$, $0 < r \leq \text{diam}(\text{supp}(\mu))$. It is not difficult to see that such a measure $\mu$ must be of the form $d\mu = \rho dH^n_{\text{supp}(\mu)}$, where $\rho$ is some positive function bounded from above and from below.
Given $E \subset \mathbb{C}$ and a cube $Q \subset \mathbb{R}^d$, we set

$$
\beta_E(Q) = \inf_L \left\{ \frac{1}{\ell(Q)^n} \int_{\partial Q} \left( \frac{\text{dist}(y, L)}{\ell(Q)} \right)^p \, d\mu(y) \right\}^{1/p},
$$

where the infimum is taken over all $n$-planes $L$ in $\mathbb{R}^d$. The $L^p$ version of $\beta$ is the following,

$$
\beta_{p,\mu}(Q) = \inf_L \left\{ \frac{1}{\ell(Q)^n} \int_{\partial Q} \left( \frac{\text{dist}(y, L)}{\ell(Q)} \right)^p \, d\mu(y) \right\}^{1/p},
$$

where the infimum is taken over all $n$-planes in $\mathbb{R}^d$ again. In our paper we will have $E = \text{supp}(\mu)$ and, to simplify notation, we will write $\beta$ (or $\beta_\infty$) and $\beta_p$ instead of $\beta_E$ and $\beta_{p,\mu}$. The definition of $\beta_p(B)$ for a ball $B$ is analogous to the one of $\beta_p(Q)$ for a cube $Q$.

**Remark 2.1.** Consider the $n$-dimensional Lipschitz graph $\Gamma := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^{d-n} : y = A(x)\}$, and let $d\mu(z) = d\mathcal{H}^n_\Gamma(z)$. Suppose that $\|\nabla A\|_\infty \leq C_5$. By [Do, Theorem 6], we have

$$
\|\nabla A\|_2^2 \approx \sum_{Q \in \mathcal{D}} \beta_1(Q)^2 \mu(Q) \approx \sum_{Q \in \mathcal{D}} \beta_2(Q)^2 \mu(Q),
$$

with constants depending only on $C_5$.

Given a set $A \subset \mathbb{R}^d$ and two Borel measures $\sigma$, $\nu$ on $\mathbb{R}^d$, we set

$$
\text{dist}_A(\sigma, \nu) := \sup \left\{ \left| \int f \, d\sigma - \int f \, d\nu \right| : \text{Lip}(f) \leq 1, \text{supp}(f) \subset A \right\}.
$$

Given a Borel measure $\mu$ on $\mathbb{R}^d$ and a cube $Q$ which intersects $\text{supp}(\mu)$, we consider the closed ball $B_Q := \overline{B}(z_Q, 3\text{diam}(Q))$, where $z_Q$ and $\text{diam}(Q)$ stand for the center and diameter of $Q$, respectively. Then we define

$$
\alpha_\mu^n(Q) := \frac{1}{\ell(Q)^{n+1}} \inf_{c \geq 0, L} \text{dist}_{B_Q}(\mu, c\mathcal{H}^n_{\{L\}}),
$$

where the infimum is taken over all the constants $c \geq 0$ and all the $n$-planes $L$. For convenience, if $Q$ does not intersect $\text{supp}(\mu)$, we set $\alpha_\mu^n(Q) = 0$. To simplify notation, sometimes we will also write $\alpha(Q)$ instead of $\alpha_\mu^n(Q)$.

We denote by $c_Q$ and $L_Q$ the constant and the $n$-plane that minimize $\text{dist}_{B_Q}(\mu, L_Q)$ (it is easy to check that this minimum is attained). We also write $\mathcal{L}_Q := c_Q\mathcal{H}^n_{\{L_Q\}}$, so that

$$
\alpha_\mu^n(Q) = \frac{1}{\ell(Q)^{n+1}} \text{dist}_{B_Q}(\mu, c_Q\mathcal{H}^n_{\{L_Q\}}) = \frac{1}{\ell(Q)^{n+1}} \text{dist}_{B_Q}(\mu, L_Q).
$$

Let us remark that $c_Q$ and $L_Q$ (and so $\mathcal{L}_Q$) may be not unique. Moreover, we may (and will) assume that $L_Q \cap B_Q \neq \emptyset$.

Recall that when $\mu$ is AD regular, one can construct some kind of dyadic lattice of cubes adapted to the measure $\mu$. The cubes from this lattice are not true cubes, although they play the role of dyadic cubes with respect to
μ, in a sense. See [Da1] Appendix 1, for example. The definitions of β_p(Q) and α(Q) are the same as above for this type of “cubes”.

In [To3] it is shown that β_1(Q) ≤ Cα(Q) when μ is an AD regular n-dimensional measure and Q is a cube of the dyadic lattice associated to μ. The opposite inequality is false, in general.

We denote

\[ δ_μ^n(x, r) = \frac{μ(B(x, r))}{r^n}, \]

and if B = B(x, r), we set δ_μ^n(B) = δ_μ^n(x, r). Sometimes, to simplify notation we will write δ(x, r) instead of δ_μ^n(x, r).

3. Auxiliary lemmas for the proof of Theorem 1.3

3.1. More notation and definitions. Throughout Sections 3-6, μ stands for the measure described in the assumptions of Theorem 1.3. That is, μ = g Η^n, where Γ is the Lipschitz graph \{(x, y) ∈ R^d : y = A(x)\}. Observe that μ is AD regular.

Recall that Π is the projection \((x_1, \ldots, x_n, \ldots, x_d) \mapsto (x_1, \ldots, x_n)\). We denote \(x_0 = Π(x) = (x_1, \ldots, x_n)\) (we identify \(x_0 ∈ R^n\) with \((x_0, 0, \ldots, 0) ∈ R^d\)) and, also, \(x^⊥ = Π^⊥(x) = (x_{n+1}, \ldots, x_d)\).

In the particular case of a Lipschitz graph and μ as above, the construction of the dyadic lattice \(D\) associated to μ is very simple: let \(D_0\) be the lattice of the usual dyadic cubes of \(R^n\). A subset \(Q \subset Γ\) is a cube from \(D\) if and only if it is of the form

\[ Q = Π^{-1}(Q_0) \cap Γ \]

for some \(Q_0 ∈ D_0\). If \(ℓ(Q_0) = 2^{-j}\) (where \(ℓ(\cdot)\) stands for side length), we set \(ℓ(Q) = 2^{-j}\) and \(Q ∈ D_j\). If \(z_{Q_0}\) is the center of \(Q_0\), then we say that \(Π^{-1}(z_{Q_0}) \cap Γ\) is the center of \(Q\). The definition of \(λQ_0\), for \(λ > 0\), is analogous.

Let ψ be a non negative radial \(C^∞\) function such that \(χ_{B(0,1/8)} ≤ ψ ≤ χ_{B(0,1/4)}\). For each \(j ∈ Z\), set \(ψ_j(x) := ψ(2^j x)\) and \(ϕ_j := ψ_j - ψ_{j+1}\), so that each function \(ϕ_j\) is non negative and supported on \(B(0, 2^{-j+2}) \setminus B(0, 2^{-j-4})\), and moreover we have

\[ \sum_{j ∈ Z} ϕ_j(x) = 1 \quad \text{for all } x ∈ R^d \setminus \{0\}. \]

We need to consider the following vectorial kernels:

\[ K_j(x) = ϕ_j(x_0) \frac{x}{|x|^{n+1}} \quad j ∈ Z, \]  

(3.1)

and

\[ \tilde{K}_j(x) = ϕ_j(x_0) \frac{x}{|x_0|^{n+1}} \quad j ∈ Z, \]  

(3.2)

for \(x ∈ R^d\). The operators associated to \(K_j\) and \(\tilde{K}_j\) are, respectively,

\[ R_jμ(x) = \int K_j(x - y) dμ(y), \quad \tilde{R}_jμ(x) = \int \tilde{K}_j(x - y) dμ(y). \]
Notice that, formally,

\[ R_\mu(x) = \sum_{j \in \mathbb{Z}} R_j \mu(x). \]

Moreover, abusing notation sometimes we will write \( R_\mu \) instead of p.v.\( R_\mu \). When \( \mu \) is like in Theorem 1.3 this does not cause any trouble, since the \( \mu \)-a.e. existence of principal values is a well known result.

Let us remark that, perhaps it would be more natural to replace \( \varphi_j(x) \) by \( \varphi_j(x_0) \) in the definitions of the kernels \( K_j \) and \( \tilde{K}_j \) (like in [To3]). However, for some of the calculations below the definitions above are more convenient (although the choice of \( \varphi_j(x) \) instead of \( \varphi_j(x_0) \) would also work with minor modifications and some additional work).

We also denote by \( K^i_j(x) \) and \( \tilde{K}^i_j(x) \) the \( i \)-th component of \( K_j(x) \) and \( \tilde{K}_j(x) \) respectively, and we set

\[ K^i_j(x) = \varphi_j(x_0) \frac{x^i}{|x|^{n+1}} \]

and

\[ \tilde{K}^i_j(x) = \varphi_j(x_0) \frac{x^i}{|x_0|^{n+1}}, \]

and we denote by \( R^i_j \) and \( \tilde{R}^i_j \) the corresponding operators with kernels \( K^i_j \) and \( \tilde{K}^i_j \).

### 3.2. The upper estimate for the \( L^2 \) norm of Riesz transforms.

**Lemma 3.1.** Consider the \( n \)-dimensional Lipschitz graph \( \Gamma := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^{d-n} : y = A(x)\} \). Suppose that \( \|
abla A\|_\infty \leq C_6 \) and let \( d\mu(z) = g(z) d\mathcal{H}^n_\Gamma(z) \), where \( g(\cdot) \) is a function such that \( C_1^{-1} \leq g(z) \leq C_1 \) for all \( z \in \Gamma \). Then we have

\[ \| \text{p.v.} R_\mu \|_{L^2(\mu)} \lesssim \| \nabla A \|_2 + \| g - 1 \|_2, \]

with constants depending on \( C_6 \) and \( C_1 \).

**Proof.** By Theorem 1.2 in [To3] we have

\[ \| \text{p.v.} R_\mu \|_{L^2(\mu)}^2 \lesssim \sum_{Q \in \mathcal{D}} \alpha(Q)^2 \mu(Q), \]

and by Theorem 1.1 and Remark 4.1 in the same paper,

\[ \sum_{Q \in \mathcal{D}} \alpha(Q)^2 \mu(Q) \lesssim \sum_{Q \in \mathcal{D}} \beta_1(Q)^2 \mu(Q) + \| g - 1 \|_2^2. \]

By [Do] Theorem 6 we have

\[ \sum_{Q \in \mathcal{D}} \beta_1(Q)^2 \mu(Q) \approx \| \nabla A \|_2^2, \]

and so the lemma follows. \( \square \)
3.3. Auxiliary lemmas for the lower estimate. In the following lemma we collect a pair of trivial estimates. The easy proof is left for the reader.

Lemma 3.2. Denote $\delta = 2^{-j}$. For all $x \in \mathbb{R}^d$ and all $1 \leq i \leq n + 1$, we have

$$|K_j^i(x)| \lesssim \frac{|x_i| |x|^2}{\delta^{n+3}} \chi_{A(0, \delta/3,3\delta)};$$

and

$$|\nabla K_j^i(x)| \lesssim \frac{1}{\delta^{n+3}} \chi_{A(0, \delta/3,3\delta)}.$$

Notice that

$$||x| - |x_0|| \leq \frac{|x\perp|}{|x|}.$$ From this estimate and easy calculations, one gets

Lemma 3.3. Denote $\delta = 2^{-j}$. For $x \in \mathbb{R}^d$ such that $|x| \approx |x_0|$, and for $1 \leq i \leq d$, we have

$$|K_j^i(x) - \tilde{K}_j^i(x)| \lesssim \frac{|x_i| |x\perp|^2}{\delta^{n+3}} \chi_{A(0, \delta/16, \delta)};$$

and

$$|\nabla (K_j^i - \tilde{K}_j^i)(x)| \lesssim \frac{|x\perp|^2}{\delta^{n+3}} \chi_{A(0, \delta/16, \delta)}.$$ The proof is left for the reader again.

Lemma 3.4. For all $j \in \mathbb{Z}$ and all $Q \in \mathcal{D}_j$, we have

$$\int_Q |R_j\mu|^2 \, d\mu \lesssim \left[ \beta_2(Q)^2 + \alpha(Q)^2 \right] \mu(Q).$$

Also, if $D_Q$ is the line that minimizes $\beta_1(Q)$ and

$$\beta_\infty(Q) \leq \varepsilon_2 \quad \text{and} \quad \sin \angle(D_0, D_Q) \leq \varepsilon_2,$$

with $\varepsilon_2$ small enough, then

$$\int_Q |R_j\mu - \tilde{R}_j\mu|^2 \, d\mu \lesssim \varepsilon_2^4 \left[ \beta_2(Q)^2 + \alpha(Q)^2 \right] \mu(Q).$$

Proof. The estimate (3.3) has been proved in [To3, Lemma 5.1]. The inequality (3.3) has a quite similar proof. For completeness, we show the detailed arguments. Consider the kernel $D_j = K_j - \tilde{K}_j$, and let $T_j$ be the operator associated to $D_j$.

Denote by $D_Q$ the line that minimizes $\beta_1(Q)$ and let $L_Q$ the one that minimizes $\alpha(Q)$. From the fact that $\beta_1(Q) \lesssim \alpha(Q)$ it easily follows that

$$\text{dist}_H(L_Q \cap B_Q, D_Q \cap B_Q) \lesssim \alpha(Q) \ell(Q),$$

where $\text{dist}_H$ stands for Hausdorff distance. Take $x \in Q \subset \Gamma$. Consider the orthogonal projection $x'$ of $x$ onto $D_Q$. Since we are assuming that $\beta_\infty(Q)$ is very small we have $|x - x'| \ll \text{diam}(Q)$ and then $\text{supp}(D_j(x' - \cdot)) \subset B_Q$. 


First we will estimate \( T_j \mu(x') \). Let \( U \) be a thin tubular neighborhood of \( D_Q \cap B_Q \) of width \( \leq C \varepsilon_2 \text{diam}(Q) \) containing \( \text{supp}(\mu) \cap B_Q \) and denote \( f(y) = D_j(x' - y) \). Notice that for \( y \in U \cap \text{supp}(D_j(x' - y)) \) we have \(|x' - y| \approx |x_0 - y_0|\), and so by Lemma 3.3 for these \( y \)'s,

\[
|\nabla f(y)| = |\nabla D_j(x' - y)| \lesssim \frac{|x' - y|}{\ell(Q)^{n+3}}.
\]

We have

\[
|x' - y| \lesssim \ell(Q) \left( \beta_\infty(Q) + \sin \angle(D_0, D_Q) \right),
\]

where \( D_0 \) stands for the \( n \)-plane \( D_0 = \mathbb{R}^n \times \{0, \ldots, 0\} \), and so we get

\[
(3.6) \quad |\nabla f(y)| \lesssim \frac{\varepsilon_2^2}{\ell(Q)^{n+1}}.
\]

We extend \( f|_{U \cap B_Q} \) to a function \( \tilde{f} \) supported on \( B_Q \) with \( ||\nabla \tilde{f}||_\infty \lesssim \varepsilon_2^2/\ell(Q)^{n+1} \). Since \( K_j(\cdot) \) is odd and \( x' \in D_Q \), we have \( \int D_j(x' - y) \, d\mathcal{H}_1^n|_{D_Q}(y) = 0 \), and so

\[
\left| \int D_j(x' - y) \, d\mu(y) \right| = \left| \int D_j(x' - y) \, d\mu(y) - c_Q \int D_j(x' - y) \, d\mathcal{H}_1^n|_{D_Q}(y) \right|
\]

\[
= \left| \int \tilde{f}(y) \, d\mu(y) - c_Q \int \tilde{f}(y) \, d\mathcal{H}_1^n|_{D_Q}(y) \right|
\]

\[
\lesssim \frac{\varepsilon_2^2}{\ell(Q)^{n+1}} \text{dist}_B(\mu, c_Q \mathcal{H}_1^n|_{D_Q}).
\]

In these estimates \( c_Q \) stands for the constant minimizing the definition of \( \alpha(Q) \). By the definition of \( \alpha(Q) \) and \( \varepsilon_2 \) one easily gets

\[
\text{dist}_B(\mu, c_Q \mathcal{H}_1^n|_{D_Q}) \lesssim \alpha(Q) \ell(Q)^{n+1}.
\]

Thus, \( |T_j \mu(x')| \lesssim \varepsilon_2^2 \alpha(Q) \).

Now we turn our attention to \( T_j \mu(x) \). We have

\[
|T_j \mu(x) - T_j \mu(x')| \lesssim |x - x'| \sup_{\xi \in [x, x']} |\nabla T_j \mu(\xi)|.
\]

By an estimate analogous to (3.6) we have

\[
|\nabla T_j \mu(\xi)| \leq \int |\nabla D_j(\xi - y)| \, d\mu(y) \lesssim \frac{\varepsilon_2^2}{\ell(Q)},
\]

since \( |\xi - y| \approx |\xi_0 - y_0| \) for \( \xi \in [x, x'] \) and \( y \in \text{supp}(\mu) \cap \text{supp}(D_j(\xi - \cdot)) \).

Therefore,

\[
|T_j \mu(x) - T_j \mu(x')| \lesssim \frac{\varepsilon_2^2 \text{dist}(x, D_Q)}{\ell(Q)},
\]

and so

\[
|T_j \mu(x)| \lesssim \frac{\varepsilon_2^2 \text{dist}(x, D_Q)}{\ell(Q)} + |T_j \mu(x')| \lesssim \frac{\varepsilon_2^2 \left( \text{dist}(x, D_Q) / \ell(Q) \right) + \alpha(Q)}{\ell(Q)}.
\]

The lemma is a direct consequence of this estimate. \( \square \)
From the preceding result we get the following.

**Lemma 3.5.** For \( j \in \mathbb{Z} \), let us denote

\[
\beta_{2,j}(\Gamma)^2 := \sum_{Q \in D_j} \beta_2(Q)^2 \mu(Q) \quad \text{and} \quad \alpha_j(\Gamma)^2 := \sum_{Q \in D_j} \alpha(Q)^2 \mu(Q).
\]

Suppose that

\[
\beta_\infty(Q) \leq \varepsilon_2 \quad \text{and} \quad \sin \angle(D_0, D_Q) \leq \varepsilon_2,
\]

where \( D_Q \) is the line that minimizes \( \beta_1(Q) \) and \( \varepsilon_2 \) is small enough. We have

\[
|\langle R_j^+ \mu, R_k^1 \mu \rangle - \langle \tilde{R}_j^+ \mu, \tilde{R}_k^1 \mu \rangle| \lesssim \varepsilon_2^2 (\beta_{2,j}(\Gamma) + \alpha_j(\Gamma)) \left( \beta_{2,k}(\Gamma) + \alpha_k(\Gamma) \right).
\]

**Proof.** We set

\[
|\langle R_j^+ \mu, R_k^1 \mu \rangle - \langle \tilde{R}_j^+ \mu, \tilde{R}_k^1 \mu \rangle| \\
\leq \left| \langle R_j^+ \mu - \tilde{R}_j^+ \mu, R_k^1 \mu \rangle \right| + \left| \langle \tilde{R}_j^+ \mu, R_k^1 \mu - \tilde{R}_k^1 \mu \rangle \right| \\
\leq \| R_j^1 \mu - \tilde{R}_j^1 \mu \|_2 \| R_k^1 \mu \|_2 + \| \tilde{R}_j^1 \mu - R_k^1 \mu \|_2 \| R_k^1 \mu - \tilde{R}_k^1 \mu \|_2 \\
+ \| R_j^1 \mu \|_2 \| R_k^1 \mu - \tilde{R}_k^1 \mu \|_2.
\]

If we plug the estimates (3.3) and (3.4) into the preceding inequality, the lemma follows. \( \square \)

4. **The key Fourier estimate**

Consider the image measure \( \sigma := \Pi_{\#} \mu \) on \( \mathbb{R}^n \) and set

\[
H_j(x,y_0) = \varphi_j(x-y_0) \frac{A(x_0) - A(y_0)}{|x_0 - y_0|^{n+1}}.
\]

We have

\[
(4.1) \quad \langle \tilde{R}_j^+ \mu, \tilde{R}_k^1 \mu \rangle = \iiint \tilde{K}_j^+ (x,y) \tilde{K}_k^1 (x,z) d\mu(x) d\mu(y) d\mu(z) \\
= \iiint H_j(x,y_0) H_k(x,y_0) d\sigma(x_0) d\sigma(y_0) d\sigma(z_0) =: I_0
\]

Below we will calculate \( I_0 \) using the Fourier transform in the special case in which \( \sigma \) coincides with the Lebesgue \( n \)-dimensional measure on \( \mathbb{R}^n \). This will allow us to prove Theorem 1.3 in this particular situation. The full theorem will follow easily from this case.

**Lemma 4.1.** Let us denote \( \delta = 2^{-j}, \varepsilon = 2^{-k} \), and assume \( \delta \leq \varepsilon \). We have

\[
(4.2) \quad 0 \leq \iiint_{(\mathbb{R}^n)^3} H_j(x,y) H_k(x,z) dxdydz \approx \delta \varepsilon \int_{|\xi| \leq 1/\delta} |\hat{A}(\xi)|^2 |\xi|^4 d\xi \\
+ \frac{\delta}{\varepsilon} \int_{1/\varepsilon \leq |\xi| \leq 1/\delta} |\hat{A}(\xi)|^2 |\xi|^2 d\xi \\
+ \frac{1}{\delta \varepsilon} \int_{|\xi| \geq 1/\delta} |\hat{A}(\xi)|^2 d\xi.
\]
Proof. For $x \in \mathbb{R}^n$, we denote $\eta(x) = \varphi(x)/|x|^{n+1}$. Notice that
\[
\frac{\varphi_j(x)}{|x|^{n+1}} = \frac{1}{\delta^{n+1}} \eta\left(\frac{x}{\delta}\right) =: \frac{1}{\delta} \eta_\delta(x),
\]
and analogously for $\varphi_k(x)/|x|^{n+1}$. By the change of variables $y = x + s$, $z = x + t$, and by Plancherel the triple integral on the left hand side of (4.2) equals
\[
I_0 := \int \int \int \left( \varphi_j(x - y) \frac{A(x) - A(y)}{|x - y|^{n+1}} \right) \left( \varphi_k(x - z) \frac{A(x) - A(z)}{|x - z|^{n+1}} \right) dxdydz
\]
\[
= \frac{1}{\delta \varepsilon} \int \int \int \eta_\delta(s)(A(x) - A(x + s)) \eta_\varepsilon(t)(A(x) - A(x + t)) dxdsdt
\]
\[
= \frac{1}{\delta} \int \int \int |\hat{A}(\xi)|^2 \left( 1 - e^{-2\pi i \xi s} \right) \eta_\delta(s) \left( 1 - e^{-2\pi i \xi t} \right) \eta_\varepsilon(t) d\xi ds dt.
\]

By Fubini, taking Fourier transform (for the $s$ and $t$ variables), we get
\[
I_0 = \frac{1}{\delta \varepsilon} \int \int \int |\hat{A}(\xi)|^2 \left( \hat{\eta}(0) - \hat{\eta}(\delta \xi) \right) \left( \hat{\eta}(0) - \hat{\eta}(\varepsilon \xi) \right) d\xi.
\]

Let
\[
f_\delta(\xi) := \frac{1}{\delta} \left( \hat{\eta}(0) - \hat{\eta}(\delta \xi) \right).
\]

It is easy to check that $f_\delta(\xi)$ is real and positive for $\xi \neq 0$. Moreover, using that $\hat{\eta}$ is radial and $\hat{\eta} \in \mathcal{S}$, we get $f_\delta(\xi) \approx C\delta |\xi|^2$ as $\delta \to 0$, and $f_\delta(\xi) \approx C/\delta$ as $|\xi| \to \infty$. So we infer that
\[
f_\delta(\xi) \approx \delta |\xi|^2 \quad \text{if} \quad |\xi| \leq \frac{1}{\delta}, \quad \text{and} \quad f_\delta(\xi) \approx \frac{1}{\delta} \quad \text{if} \quad |\xi| \geq \frac{1}{\delta}.
\]

Analogous estimates hold for the corresponding function $f_\varepsilon(\xi)$. Therefore,
\[
(4.3) \quad I_0 \approx \delta \varepsilon \int_{|\xi| \leq 1/\varepsilon} |\hat{A}(\xi)|^2 |\xi|^4 d\xi + \delta \varepsilon \int_{1/\varepsilon \leq |\xi| \leq 1/\delta} |\hat{A}(\xi)|^2 |\xi|^2 d\xi
\]
\[
+ \frac{1}{\delta \varepsilon} \int_{|\xi| \geq 1/\delta} |\hat{A}(\xi)|^2 d\xi.
\]

\[\square\]

5. Proof of Theorem 1.3 in the particular case $d\sigma \equiv dx$

We will need the following result from [To3] (it is not stated explicitly there, although it is proved in the paper):

\[\text{This follows from the fact that}
\]
\[
\hat{\eta}(0) = \int \eta(s) ds > \int \cos(2\pi \xi s) \eta(s) ds = \tilde{\eta}(\xi)
\]
\[\text{for all} \ \xi \neq 0, \ \text{since} \ \eta \text{is a non negative radial function from} \ S.\]
Theorem 5.1. Let $\mu$ be an $n$-dimensional AD regular measure. For any positive integer $N_0$, we have

$$
\sum_{j,k:|j-k|>N_0} |\langle R_j^\perp \mu, R_k^\perp \mu \rangle| \leq C 2^{-N_0/4} \sum_{Q \in \mathcal{D}} \alpha(Q)^2 \mu(Q).
$$

Moreover, under the assumptions of Theorem 1.3, if $\Pi^# \mu = \rho(x) \, dx$, we have

$$
\sum_{Q \in \mathcal{D}} \alpha(Q)^2 \mu(Q) \lesssim \sum_{Q \in \mathcal{D}} \beta_1(Q)^2 \mu(Q) + \|\rho - 1\|_2^2.
$$

Let us remark that in [To3] the preceding result has been proved with $\varphi_j(x)$ replacing $\varphi_j(x_0)$ in the definition of the kernel $K_j$ in (3.1). However, it is easy to check that all the estimates of [To3] work with the slightly different definition in (3.1) when $\mu$ is supported on a Lipschitz graph.

Proof of Theorem 1.3 in the particular case $d\sigma \equiv dx$. By Lemma 3.1 we only need to prove the lower estimate $\|p.v. R^\perp \mu\|_{L^2(\mu)} \gtrsim \|\nabla A\|_2$. We set

$$
\|R^\perp \mu\|_{L^2(\mu)}^2 = \sum_{j,k:|j-k|\leq N_0} \langle R_j^\perp \mu, R_k^\perp \mu \rangle + \sum_{j,k:|j-k|>N_0} \langle R_j^\perp \mu, R_k^\perp \mu \rangle =: S_1 + S_2.
$$

In this identity $R^\perp \mu$ can be understood either as the principal value or as an $L^2(\mu)$ limit. We will show that if $\varepsilon_0$ is small enough, then

$$
S_1 \approx \sum_{Q \in \mathcal{D}} \beta_2(Q)^2 \mu(Q)
$$

(with constants depending on $N_0$), while $|S_2| \leq S_1/2$. The theorem follows from these estimates.

The inequality

$$
S_1 \lesssim \sum_{Q \in \mathcal{D}} \beta_2(Q)^2 \mu(Q)
$$

is a direct consequence of (5.3), (5.2), and the fact that $\rho \equiv 1$. Now we consider the converse estimate. We denote

$$
T_j f(x) = \int_{\mathbb{R}^n} H_j(x,y) f(y) dy.
$$

By (4.1) we have

$$
\langle \tilde{R}_j^\perp \mu, \tilde{R}_k^\perp \mu \rangle = \langle T_j 1, T_k 1 \rangle_{\mathbb{R}^n}.
$$

Then we set

$$
\langle R_j^\perp \mu, R_k^\perp \mu \rangle = \langle T_j 1, T_k 1 \rangle_{\mathbb{R}^n} + \left( \langle R_j^\perp \mu, R_k^\perp \mu \rangle - \langle \tilde{R}_j^\perp \mu, \tilde{R}_k^\perp \mu \rangle \right) =: \langle T_j 1, T_k 1 \rangle_{\mathbb{R}^n} + E_{j,k}.
$$
By Lemma 4.1, since $\langle T_j \mu, T_k \mu \rangle_{\mathbb{R}^n} \geq 0$, we have

$$
\sum_{j, k : |j - k| \leq N_0} \langle T_j 1, T_k 1 \rangle_{\mathbb{R}^n} \geq \sum_{j \in \mathbb{Z}} \|T_j 1\|_2^2 \geq \sum_{j \in \mathbb{Z}} \int_{2^{j-1} \leq |\xi| \leq 2^{j+1}} |\hat{A}(\xi)|^2 |\xi|^2 d\xi \\
\approx \|\nabla A\|_2^2 \approx \sum_{Q \in \mathcal{D}} \beta(Q)^2 \mu(Q).
$$

(5.3)

We consider now the terms $E_{j,k}$. Since $\|\nabla A\|_\infty \leq \varepsilon_0$, we infer that $\beta(Q) \lesssim \varepsilon_0$, and then from Lemma 3.5 if $\varepsilon_0$ is small enough we deduce

$$
\sum_{j, k : |j - k| \leq N_0} |E_{j,k}| \lesssim \varepsilon_0^2 \sum_{j, k : |j - k| \leq N_0} (\beta_{2,j}(\Gamma) + \alpha_j(\Gamma)) (\beta_{2,k}(\Gamma) + \alpha_k(\Gamma)) \\
\lesssim N_0 \varepsilon_0^2 \sum_{Q \in \mathcal{D}} (\alpha(Q)^2 + \beta(Q)^2) \mu(Q).
$$

From (5.2) we obtain

$$
\sum_{j, k : |j - k| \leq N_0} |E_{j,k}| \lesssim N_0 \varepsilon_0^2 \sum_{Q \in \mathcal{D}} \beta(Q)^2 \mu(Q).
$$

(5.4)

By the estimates (5.3) and (5.4), if $\varepsilon_0$ is small enough (for a given $N_0$), we infer that

$$
S_1 \gtrsim \sum_{Q \in \mathcal{D}} \beta(Q)^2 \mu(Q).
$$

(5.5)

Finally we turn our attention to $S_2$. By Theorem 5.1 we have

$$
|S_2| \lesssim 2^{-N_0/4} \sum_{Q \in \mathcal{D}} \beta_1(Q)^2 \mu(Q).
$$

Therefore, by (5.5), $S_2 \leq C 2^{-N_0/4} S_1 \leq S_1/2$ if $N_0$ is big enough. We are done. \qed

6. PROOF OF THEOREM 1.3 IN FULL GENERALITY

**Lemma 6.1.** Consider the $n$-dimensional Lipschitz graph $\Gamma := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^{d-n} : y = A(x)\}$, with $\|\nabla A\|_\infty \leq C_8$, and let $\mu$ be supported on $\Gamma$ such that $d\Pi_{\#}(\mu) = dx$. Then $R^\perp_\mu$ is bounded in $L^2(\mu)$ with

$$
\|R^\perp_\mu\|_{L^2(\mu), L^2(\mu)} \leq C_9 \|\nabla A\|_\infty,
$$

with $C_9$ depending only on $C_8$.

**Proof.** We think that this is essentially known. However, for completeness we give some details of the proof. Consider the kernel

$$
K(x, y) = \frac{A(x) - A(y)}{(|x - y|^2 + |A(x) - A(y)|^2)^{(n+1)/2}},
$$

where $A$ is the Lipschitz extension of $A$, and $\Pi_{\#}(\mu)$ is the pushforward measure of $\mu$ on $\mathbb{R}^d$. Then $R^\perp_\mu(x) = \int_{\mathbb{R}^d} K(x, y) d\Pi_{\#}(\mu)(y)$.

$$
\|R^\perp_\mu\|_{L^2(\mu), L^2(\mu)} \leq \|K\|_{L^2(\mu)} \leq C_9 \|\nabla A\|_\infty,
$$

where $C_9$ depends only on $C_8$. \qed
and the associated Calderón-Zygmund operator

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y) f(x) \, dx,$$

for $f \in L^2(\mathbb{R}^n)$. When $n = 1 = d - 1$, we have the expansion

$$K(x, y) = \sum_{j=1}^{\infty} (-1)^j \frac{(A(x) - A(y))^{2j-1}}{|x - y|^{2j}} = \sum_{j=1}^{\infty} K_j(x, y),$$

and the corresponding associated operators are the Calderón commutators $C_j$. It is well known that

$$\|C_j\|_{2,2} \leq C^{2j} \|\nabla A\|_{2j-1}$$

(see [Da1, p.50], for example), and so if $\|\nabla A\|_{\infty}$ is small enough the lemma follows.

For other $n$’s and $d$’s the result also holds. For example, it can be deduced from [To3]: if $A$ is supported on a cube $Q$, then we have

$$\|R^\perp \mu\|_2 \lesssim \|\nabla A\|_2 \leq \|\nabla A\|_\infty \mu(Q)^{1/2}.$$

By a localization argument, one can prove that for any cube $P$,

$$\|R^\perp (\chi_P \mu)\|_2 \lesssim \|\nabla A\|_\infty \mu(P)^{1/2},$$

and then by the $T1$ theorem the lemma follows (taking into account that the Calderón-Zygmund constants involved in the kernel $K(x, y)$ are bounded above by $\|\nabla A\|_{\infty}$ too).

**Remark 6.2.** Consider the function $\tilde{A} : \mathbb{R}^n \to \mathbb{R}^d$ given by $\tilde{A}(x) = (x, A(x))$, where $A$ is the Lipschitz function that defines the Lipschitz graph $\Gamma$. Notice that the density function $\rho(x)$ such that $\Pi_# \mu = \rho(x) \, dx$ is given by

$$\rho(x) = g(x) \cdot J \tilde{A}(x),$$

where $J \tilde{A}(x)$ stands for the $n$-dimensional Jacobian of $\tilde{A}$. Recall that

$$J \tilde{A}(x) = \left( \sum_B (\det B)^2 \right)^{1/2},$$

where the sum runs over all the $n \times n$ submatrices $B$ of $D \tilde{A}(x)$, the differential map of $\tilde{A}$ at $x$ (see [Mo, p. 24], for example). Then it is easy to check that

$$(J \tilde{A}(x))^2 = 1 + e(x),$$

with

$$|e(x)| \lesssim \sup_{i,j} |\partial_i A_j(x)|^2$$

(in fact, $e(x) = \sum_{i,j} (\partial_i A_j(x))^2 + \ldots$, where “…” stands for some terms which involve higher order products of derivatives of $A$). So we also have

$$J \tilde{A}(x) = 1 + e_0(x),$$
with
\[ |e_0(x)| \lesssim \sup_{i,j} |\partial_i A_j(x)|^2. \]
As a consequence,
\[ |\rho(x) - 1| = |g(x)(1 + e_0(x)) - 1| \leq |g(x) - 1| + C|e_0(x)|. \]
Observe that \( \|e_0\|_\infty \lesssim \|\nabla A\|_\infty \leq \varepsilon_0^2 \) and \( \|e_0\|_2 \lesssim \|\nabla A\|_\infty \|\nabla A\|_2 \). Then the assumptions of Theorem 1.3 ensure that
\[ (6.1) \quad \|\rho - 1\|_2 \leq \|g - 1\|_2 + C\|e_0\|_2 \leq C\|\nabla A\|_2. \]

**Proof of Theorem 1.3** Recall that we only need to prove the lower estimate \( \|p.v. R^\perp \mu\|_{L^2(\mu)} \gtrsim \|\nabla A\|_2 \). Consider the measure \( \mu_0 \) supported on \( \Gamma \) such that \( \Pi_{\#}\mu_0 = dx \). Recall that
\[ (6.2) \quad \|R^\perp \mu_0\|_{L^2(\mu_0)} \approx \|\nabla A\|_2. \]
Since
\[ \Pi_{\#}\mu = g(x) J\tilde{A}(x) \, dx =: \rho(x) \, dx, \quad x \in \mathbb{R}^n, \]
it turns out that
\[ \rho(\Pi(x)) \, d\mu_0(x) = d\mu(x), \quad x \in \mathbb{R}^d. \]
We denote \( h(x) = \rho(\Pi(x)) \), and so we have
\[ d\mu(x) - d\mu_0(x) = (h(x) - 1) \, d\mu_0(x), \]
with \( \|h - 1\|_{L^2(\mu_0)} \lesssim \|\nabla A\|_2 \), by (6.1). So, from Lemma 6.1 we deduce
\[ \left| \|R^\perp \mu\|_{L^2(\mu_0)} - \|R^\perp \mu_0\|_{L^2(\mu_0)} \right| \leq \|R^\perp (h - 1) \, d\mu_0\|_{L^2(\mu_0)} \]
\[ \lesssim \|\nabla A\|_\infty \|h - 1\|_{L^2(\mu_0)} \leq \varepsilon_0 \|\nabla A\|_2. \]
If \( \varepsilon_0 \) is small enough, from (6.2) we infer that
\[ \|R^\perp \mu\|_{L^2(\mu_0)} \approx \|R^\perp \mu_0\|_{L^2(\mu_0)} \approx \|\nabla A\|_2, \]
which implies that
\[ \|R^\perp \mu\|_{L^2(\mu)} \approx \|\nabla A\|_2, \]
since \( g(x) \approx h(x) \approx 1 \) for all \( x \).

7. **The Main Lemma for the proof of Theorem 1.2**

This and the remaining sections are devoted to the proof of Theorem 1.2. For \( \varepsilon > 0 \) we denote
\[ \tilde{R}_\varepsilon \mu(x) = \int \frac{x - y}{(|x - y|^2 + \varepsilon^2)^{(n+1)/2}} \, d\mu(y), \]
and also
\[ \hat{R}_\varepsilon \mu(x) = \int \psi(\varepsilon^{-1}(x - y)) \frac{x - y}{|x - y|^{n+1}} \, d\mu(y), \]
where $\psi$ is a $\mathcal{C}^\infty$ radial function such that $\chi_{\mathbb{R}^d \setminus B(0,1)} \leq \psi \leq \chi_{\mathbb{R}^d \setminus B(0,1/2)}$. We also set
\[
\tilde{R}_{\varepsilon_1, \varepsilon_2}\mu(x) = \tilde{R}_{\varepsilon_1}\mu(x) - \tilde{R}_{\varepsilon_2}\mu(x),
\]
and
\[
\tilde{R}_{\varepsilon_1, \varepsilon_2}\mu(x) = \tilde{R}_{\varepsilon_1}\mu(x) - \tilde{R}_{\varepsilon_2}\mu(x).
\]
It is easy to check that if $p \nu. R\mu(x)$ exists for some $x \in \mathbb{R}^d$, then
\[
\lim_{\varepsilon \to 0} \tilde{R}_\varepsilon\mu(x) = \lim_{\varepsilon \to 0} \tilde{R}_\varepsilon\mu(x) = \lim_{\varepsilon \to 0} R\mu(x).
\]
(Hint: write $\tilde{R}_\varepsilon\mu(x)$ and $\tilde{R}_\varepsilon\mu(x)$ as a convex combination of $R\varepsilon\mu(x)$, $\varepsilon > 0$.
We also denote $c_n = \mathcal{L}^n(B_n(0,1))$, where $\mathcal{L}^n$ stands for the $n$-dimensional Lebesgue measure.

**Main Lemma 7.1.** Let $\mu$ be a finite Borel measure on $\mathbb{R}^d$. Let $B_0 = B(x_0, r_0)$ be a closed ball such that there exists a compact subset $F \subset 10B_0$, with $x_0 \in F$, which satisfies

(a) $\mu(8B_0) = c_n 8^n r_0^n$ and $\mu(10B_0 \setminus F) \leq \delta_1\mu(B_0)$,
(b) $\mu(B(x, r)) \leq M_1 r^n$ for all $x \in F, r > 0$, and $\mu(B(x, r)) \leq c_n(1 + \delta_1)r^n$ for all $x \in F$ and $0 < r \leq 100 r_0$,
(c) $\|R\mu\|_{L^2(\mu|F), L^2(\mu|F)} \leq M_2$,
(d) $|R_{\varepsilon_1, \varepsilon_2}\mu(x)| + |R_{\varepsilon_1, \varepsilon_2}\mu(x)| \leq \delta_2$ for all $x \in F$ and $0 < \varepsilon_1 < \varepsilon_2 \leq \delta_2^{-1} r_0$.

If $\delta_1, \delta_2$ are small enough, with $\delta_1 = \delta_1(M_2)$ and $\delta_2 = \delta_2(M_1, M_2)$, then there exists an $n$-dimensional Lipschitz graph $\Gamma$ such that
\[
\mu(\Gamma \cap F \cap B_0) \geq \frac{9}{10} c_n r_0^n.
\]

Let us remark that the Lipschitz constant of the graph $\Gamma$ depends on the constants $M_1, M_2$ and $\delta_1, \delta_2$, and tends to 0 as $\delta_1 + \delta_2 \to 0$, for fixed $M_1, M_2$.

**Proof of Theorem 1.2 using Main Lemma 7.1**. Consider an arbitrary subset $E \subset E$. Given $\delta > 0$, for each $i \in \mathbb{Z}$ set
\[
E_i = \{x \in \tilde{E} : (1 + \delta)^i \leq \Theta^* \mu(x) < (1 + \delta)^{i+1}\},
\]
so that $\mu(\tilde{E} \setminus \bigcup_i E_i) = 0$. For $j \geq 1$, denote
\[
E_{i,j} = \{x \in E_i : \delta_1 r^n(x, r) \leq (1 + \delta)^{i+2} \text{ if } 0 < r \leq 1/j\},
\]
Notice that for all $x \in E_{i,j}$, we have
\[
\mu(B(x, r)) \leq M_{i,j} r^n \quad \text{for all } r > 0 \text{ and some fixed } M_{i,j}.
\]
From the fact that $R\mu(x) < \infty$ on $E$, arguing as in [To1], we can split each set $E_{i,j}$ as
\[
E_{i,j} = \bigcup_{k \geq 1} E_{i,j,k},
\]
so that, for each $k$,

$$
\| R_{\mu \mid E_{i,j,k}} \|_{L^2(\mu \mid E_{i,j,k}), L^2(\mu \mid E_{i,j,k})} \leq k.
$$

Given any constant $\varepsilon_0 > 0$, for each $m \geq 1$ we set

$$
E_{i,j,k,m} = \left\{ x \in E_{i,j,k} : \sup_{0 < \varepsilon_1, \varepsilon_2 \leq 1/m} (| \tilde{R}_{\varepsilon_1,\varepsilon_2} \mu (x) | + | \hat{R}_{\varepsilon_1,\varepsilon_2} \mu (x) |) \leq \varepsilon_0 \right\}.
$$

It is clear that

$$
\tilde{E} = \bigcup_{i,j,k,m} E_{i,j,k,m}.
$$

Consider $\tilde{E}_{i,j,k,m} \subset E_{i,j,k,m}$ such that $\tilde{E}_{i,j,k,m} \cap \tilde{E}_{i',j',k',m'} = \emptyset$ if $(i,j,k,m) \neq (i',j',k',m')$ and we still have

$$
\tilde{E} = \bigcup_{i,j,k,m} \tilde{E}_{i,j,k,m}.
$$

For each density point $x$ of $\tilde{E}_{i,j,k,m}$ consider a ball $B_x = B(x, r_x)$ with radius

$$
0 < r_x \leq \min (1/(100j), \varepsilon_0/m)
$$

such that

$$
\mu (B_x \setminus \tilde{E}_{i,j,k,m}) \leq \delta \mu (\tilde{E}_{i,j,k,m})
$$

and

$$
(1 + \delta)^{i-1} \leq \delta^n \mu (x, r_x) \leq (1 + \delta)^{i+2}.
$$

If we take $\delta$ and $\varepsilon_0$ small enough, we set $F := \tilde{E}_{i,j,k,m}$, and we apply Main Lemma 7.1 to the measure \( \frac{\varepsilon \gamma^n}{\mu (B_x)} \mu \) and to the ball $B_0 = \frac{1}{8} B_x$, we infer the existence of a Lipschitz graph such as the one described in the Main Lemma. If we consider a Vitali type covering with a family of disjoint balls $B_x$, we deduce that there exists a rectifiable subset $F_{i,j,k,m} \subset \tilde{E}_{i,j,k,m}$ with

$$
\mu (F_{i,j,k,m}) \geq \frac{9}{10} \mu (\tilde{E}_{i,j,k,m}).
$$

We set $\tilde{F} := \bigcup_{i,j,k,m} F_{i,j,k,m}$, and then we have

$$
\mu (\tilde{F}) \geq \frac{9}{80} \mu (\tilde{E}).
$$

It is easy to check that this implies that $E$ is rectifiable. \( \square \)

The remaining sections of the paper are devoted to the proof of Main Lemma 7.1.

8. Flatness of $\mu$ when the Riesz transforms are small

We set

$$
P(x, \varepsilon) = \int \frac{\varepsilon}{(|x - y|^2 + \varepsilon^2)^{(n+1)/2}} d\mu (y)
$$

and

$$
P_2(x, \varepsilon) = \int \frac{\varepsilon^3}{(|y|^2 + \varepsilon^2)^{(n+3)/2}} d\mu (y).
$$
Lemma 8.1. Let \( \mu \) be a Borel measure on \( \mathbb{R}^d \). Consider \( \varepsilon > 0 \) and \( x \in \mathbb{R}^d \) such that \( |x| \leq \varepsilon/4 \). We have

\[
\tilde{R}_\varepsilon \mu(x) - \tilde{R}_\varepsilon \mu(0) = T(x) + E(x),
\]

with

\[
T(x) = \int \frac{(|y|^2 + \varepsilon^2) x - (n + 1)(x \cdot y)y}{(|y|^2 + \varepsilon^2)(n+3/2)} d\mu(y),
\]

and

\[
|E(x)| \leq C_{10} \frac{|x|^2}{\varepsilon^2} P(0, \varepsilon).
\]

Proof. The arguments are analogous to the ones of Lemma 5.1 in [To4] for the Cauchy transform. We will show the details for completeness.

The Taylor expansion of the function \( 1/(s + \varepsilon^2)^{(n+1)/2} \) at \( s_0 \) is

\[
\frac{1}{(s + \varepsilon^2)^{(n+1)/2}} = \frac{1}{(s_0 + \varepsilon^2)^{(n+1)/2}} - \frac{n + 1}{2(s_0 + \varepsilon^2)^{(n+3)/2}} (s - s_0) + \frac{(n + 1)(n + 3)}{8(\xi + \varepsilon^2)^{(n+5)/2}} (s - s_0)^2,
\]

where \( \xi \in [s_0, s] \). If we set \( s_0 = |y|^2 \), \( s = |x - y|^2 \), and we multiply by \( x - y \), we obtain

\[
\frac{x - y}{(|x - y|^2 + \varepsilon^2)^{(n+1)/2}} = \frac{x - y}{(|y|^2 + \varepsilon^2)^{(n+1)/2}} - \frac{n + 1}{2(|y|^2 + \varepsilon^2)^{(n+3)/2}} (|x|^2 - 2x \cdot y) + \frac{(n + 1)(n + 3)}{8(\xi_{x,y} + \varepsilon^2)^{(n+5)/2}} (|x|^2 - 2x \cdot y)^2,
\]

where \( \xi_{x,y} \in [|y|^2, |x - y|^2] \). If we integrate with respect to \( d\mu(y) \), we get

\[
\tilde{R}_\varepsilon \mu(x) = \tilde{R}_\varepsilon \mu(0) + T(x) + E(x),
\]

with

\[
E(x) = \frac{n + 1}{2} \int \frac{|x|^2 (x - y) + 2(x \cdot y)x}{(|y|^2 + \varepsilon^2)^{(n+3)/2}} d\mu(y)
+ \int \frac{(n + 1)(n + 3)(x - y)}{8(\xi_{x,y} + \varepsilon^2)^{(n+5)/2}} (|x|^2 - 2x \cdot y)^2 d\mu(y) =: E_1(x) + E_2(x).
\]

To estimate \( E_1(x) \), from \( |x| \leq \varepsilon/4 \) and \( ||x|^2(x - y) + 2(x \cdot y)x| \leq C|x|^2(|y| + \varepsilon) \)
we deduce

\[
|E_1(x)| \lesssim \int \frac{|x|^2}{(|y|^2 + \varepsilon^2)^{(n+2)/2}} d\mu(y) \leq \frac{|x|^2}{\varepsilon^2} P(0, \varepsilon).
\]
For $E_2(x)$ we take into account that $\xi_{x,y} + \varepsilon^2 \approx |y|^2 + \varepsilon^2$ and, again, that $|x| \leq \varepsilon/4$. Then,

$$|E_2(x)| \lesssim |x|^2 \int \frac{(|x| + |y|)^3}{(|y|^2 + \varepsilon^2)^{(n+5)/2}} \, d\mu(y) \lesssim |x|^2 \int \frac{1}{(|y|^2 + \varepsilon^2)^{(n+2)/2}} \, d\mu(y) \leq \frac{|x|^2}{\varepsilon^2} P(0, \varepsilon).$$

We will need the following result. See [Lé1, Lemma 2.8] for the proof, for example.

**Lemma 8.2.** Let $\mu$ be a Borel measure on $\mathbb{R}^d$. Suppose that $\mu(B(x,r)) \leq r^n$ for all $x \in \mathbb{R}^d$. Let $B(y,t)$ be a ball such that $\delta(y,t) \geq C_{11}^{-1}$. Then there are $n+1$ balls $\Delta_0, \ldots, \Delta_n$ centered at $\text{supp}(\mu) \cap B(y,t)$ with radius $t/C_{12}$ such that $\delta(B_i) \geq C_{13}^{-1}$ and for all $(x_0, \ldots, x_n) \in \Delta_0 \times \ldots \times \Delta_n$ we have

$$(8.2) \quad \text{vol}^n((x_0, \ldots, x_n)) \geq \frac{t^n}{C_{14}},$$

where $\text{vol}^n((x_0, \ldots, x_n))$ denotes the $n$-volume of the $n$-simplex with vertices $x_0, \ldots, x_n$.

The arguments for the following lemma are very similar to the ones of [To3, Lemma 7.4]. We will show again the detailed proof for the sake of completeness.

**Lemma 8.3.** Let $B(y,t)$ and let $x_0, \ldots, x_n \in B(y,t)$ satisfy (8.2). Then any point $x_{n+1} \in B(y,3t)$ satisfies

$$\text{dist}(x_{n+1}, L) \lesssim \frac{\varepsilon}{P_2(x_0, \varepsilon)} \sum_{j=1}^{n+1} |\bar{R}_\varepsilon \mu(x_j) - \bar{R}_\varepsilon \mu(x_0)| + \frac{P(x_0, \varepsilon)}{P_2(x_0, \varepsilon)} \frac{t^2}{\varepsilon},$$

where $L$ is the $n$-plane passing through $x_0, \ldots, x_n$.

**Proof.** We only have to consider the case $\varepsilon > t$ and moreover, without loss of generality, we assume that $x_0 = 0$. We denote by $z$ the orthogonal projection of $x_{n+1}$ onto $L$. Then by Lemma 8.1, we have

$$|T(x_j)| \lesssim |\bar{R}_\varepsilon \mu(x_j) - \bar{R}_\varepsilon \mu(x_0)| + \frac{t^2}{\varepsilon^2} P(0, \varepsilon).$$

for $j = 1, \ldots, n + 1$. Let $e_1, \ldots, e_n$ be an orthonormal basis of $L$, and set $e_{n+1} = (x_{n+1} - z)/|x_{n+1} - z|$ (we suppose that $x_{n+1} \not\in L$), so that $e_{n+1}$ is a unitary vector orthogonal to $L$. Since the points $x_j, j = 1, \ldots, n$ are linearly independent with “good constants” (i.e. they satisfy (8.2)) we get

$$|T(e_j)| \lesssim \frac{1}{t} \sum_{j=1}^{n} |T(x_j)| \lesssim \sum_{j=1}^{n} |\bar{R}_\varepsilon \mu(x_j) - \bar{R}_\varepsilon \mu(x_0)| + \frac{t}{\varepsilon^2} P(0, \varepsilon).$$
for $i = 1, \ldots, n$. Also, since $z \in L$ and $|z| \leq t$, we have $|T(z)| \leq \sum_{j=1}^{n} |T(x_j)|$, and so by (8.3),
\[ |T(e_{n+1})| = \frac{1}{\text{dist}(x_{n+1}, L)} |T(z-x_{n+1})| \lesssim \frac{1}{\text{dist}(x_{n+1}, L)} \left( \sum_{j=1}^{n+1} |\tilde{R}_\varepsilon \mu(x_j) - \tilde{R}_\varepsilon \mu(x_0)| + \frac{t^2}{\varepsilon^2} P(0, \varepsilon) \right). \]

Therefore,
\[ (8.4) \quad \sum_{j=1}^{n+1} T(e_j) \cdot e_j \lesssim \frac{1}{\text{dist}(x_{n+1}, L)} \left( \sum_{j=1}^{n+1} |\tilde{R}_\varepsilon \mu(x_j) - \tilde{R}_\varepsilon \mu(x_0)| + \frac{t^2}{\varepsilon^2} P(0, \varepsilon) \right). \]

On the other hand, from the definition of $T$ in (8.1), if we denote $y(i) = y \cdot e_i$, we get
\[ (8.5) \quad \sum_{j=1}^{n+1} T(e_j) \cdot e_j = \int \frac{(n+1)(|y|^2 + \varepsilon^2) - (n+1) \sum_{j=1}^{n+1} y_j^2}{(|y|^2 + \varepsilon^2)^{(n+3)/2}} d\mu(y) \]
\[ = (n+1) \int \frac{\varepsilon^2 + \sum_{j=n+2}^{d} y_j^2}{(|y|^2 + \varepsilon^2)^{(n+3)/2}} d\mu(y) \geq (n+1) \frac{\varepsilon P_2(0, \varepsilon)}{\varepsilon}. \]

The lemma follows from (8.4) and (8.5). \qed

**Lemma 8.4.** Let $\mu$ be a Borel measure on $\mathbb{R}^d$ and $B(x,r)$ such that
\[ \mu(B(x,r)) \geq C_{15}^{-1} r^n, \quad \mu(B(x,t)) \leq M t^n \text{ for all } t \geq r. \]

Then there exists $r_1$ with $r \leq r_1 \leq C_{16} r$, with $C_{16}$ depending on $C_{15}$ and $M$, such that
\[ P(x,r_1) \leq 2^{n+4} \delta(x,r_1) \quad \text{and} \quad \delta(x,r_1) \geq \delta(x,r). \]

**Proof.** To simplify notation we set $a = 2^{n+4}$. The lemma follows from the following:

**Claim.** Under the assumptions of the lemma, either $P(x,r) \leq a \delta(x,r)$ or there exists some $t$ with $r \leq t \leq C_{18} r$ (with $C_{18}$ depending on $C_{15}$ and $M$) such that $\delta(x,t) \geq 8 \delta(x,r)$.

Suppose that the above statement holds. If $P(x,r) > a \delta(x,r)$, then there exists $s_1$ with $r < s_1 \leq C_{18} r$ such that $\delta(x,s_1) \geq 8 \delta(x,r)$.

By repeated application of the claim, we deduce that either there exists a sequence $s_1, s_2, s_3, \ldots, s_m$ such that
\[ (8.6) \quad \delta(x,s_m) \geq 8 \delta(x,s_{m-1}) \geq \ldots \geq 8^{m-1} \delta(x,s_1) \geq 8^m C_{15}^{-1}, \]
or
\[ (8.7) \quad \text{there exists some } s_j, 1 \leq j \leq m-1, \text{ such that } P(x,s_j) \leq a \delta(x,s_j). \]
The statement (8.6) fails for \( m \) big enough since \( \delta(x, s_j) \leq M \) for all \( j \). Thus (8) holds for some \( j \) big enough, and so the lemma follows by choosing the minimal such \( j \).

To prove the claim we set

\[
P(x, r) = \left( \int_{|x-y| \leq r} + \sum_{k \geq 1} \int_{2^{k-1}r < |x-y| \leq 2^k r} \right) \left( \frac{r}{(|x-y|^2 + r^2)^{(n+1)/2}} \right) d\mu(y)
\]

\[
\leq \frac{\mu(B(x, r))}{r^n} + \sum_{k \geq 1} \frac{r}{(2^{k-1}r)^{n+1}} \mu(B(x, 2^k r))
\]

\[
\leq \delta(x, r) + \sum_{k=1}^{N} 2^{n+1-k} \delta(x, 2^k r) + M \sum_{k \geq N+1} 2^{n+1-k}
\]

\[
= \delta(x, r) + \sum_{k=1}^{N} 2^{n+1-k} \delta(x, 2^k r) + M 2^{n+1-N}.
\]

Since \( P(x, r) \geq a \delta(x, r) \) we infer that

\[
(a-1) \delta(x, r) \leq \sum_{k=1}^{N} 2^{n-k} \delta(x, 2^k r) + M 2^{n+1-N}.
\]

For \( N \) big enough we have \( M 2^{n+1-N} \leq C_{15}^{-1} \leq \delta(x, r) \), and so

\[
(a-2) \delta(x, r) \leq 2^{n+1} \sum_{k=1}^{N} 2^{-k} \delta(x, 2^k r),
\]

which implies that there exists some \( k \in [1, N] \) such that

\[
\delta(x, 2^k r) \geq 2^{-n-1}(a-2) \delta(x, r) \geq 8 \delta(x, r)
\]

(recall that \( a = 2^{n+4} \)).

\[
\Box
\]

**Lemma 8.5.** Let \( \mu \) be a Borel measure on \( \mathbb{R}^d \), \( F \subset \mathbb{R}^d \) and \( B = B(x, r) \) such that

\[
|\tilde{R}_\varepsilon \mu(y) - \tilde{R}_\varepsilon \mu(z)| \leq \delta \quad \text{for all } y, z \in F \cap 3B \text{ and } r \leq \varepsilon \leq \delta^{-1}r,
\]

and

\[
\mu(F \cap B) \geq C_{15}^{-1} r^n, \quad \mu(B(x, t)) \leq Mt^n \text{ for all } t \geq r.
\]

Then we have

\[
\beta_{\infty,F}(B) \leq \varepsilon_1,
\]

with \( \varepsilon_1 \) depending on \( C_{15}, \delta, M \), and \( \varepsilon_1 \to 0 \) as \( \delta \to 0 \) for each fixed \( C_{15}, M \).

**Proof.** Let \( \Delta_0, \ldots, \Delta_n \) be balls of radius \( t \) like the ones in Lemma 8.2 with \( C^{-1}r \leq t \leq r \) and \( \mu(F \cap \Delta_i) \geq t \) (we apply Lemma 8.2 to \( \mu_{F \cap B} \)). Consider \( z_i \in F \cap \Delta_i \) for each \( i = 0, \ldots, n \). Given any \( \ell \) with \( r \leq \ell \leq \delta^{-1}r \), by (8.8) and Lemma 8.3 for any \( y \in F \cap 3B \) we have

\[
(8.9) \quad \text{dist}(y, L) \leq C \frac{\ell}{P_2(x, \ell)} \delta + C \frac{P(x, \ell)}{P_2(x, \ell)} \frac{r^2}{\ell},
\]
where \( L \) is the \( n \)-plane passing through \( z_0, \ldots, z_n \).

Given \( \varepsilon_1 > 0 \), take \( s \geq r \) such that
\[
\frac{C r^2}{s} \leq \frac{\varepsilon_1 r}{2}.
\]
Notice that \( \delta(x, s) \geq C(\varepsilon_1) \delta(x, r) \). By Lemma 8.4 we can choose \( \ell \geq s \) such that \( s \leq \ell \leq C_{16}s \) (with \( C_{16} \) depending on \( \varepsilon_1 \)) and
\[
P(x, \ell) \leq 2^{n+4} \delta(x, \ell) \quad \text{and} \quad P_2(x, \ell) \geq C^{-1} \delta(x, \ell) \geq C(\varepsilon_1)^{-1} \delta(x, r).
\]
Moreover, if \( \delta \) is small enough then we also have \( \ell \leq \delta^{-1}r \), so that \( (8.9) \) holds, and then we deduce that
\[
dist(y, L) \preccurlyeq C(\varepsilon_1) \delta \ell + \frac{\varepsilon_1 r}{2} \leq C\varepsilon_1 r,
\]
if \( \delta \ll C(\varepsilon_1)^{-1} \).

\[\square\]

9. Construction of the Lipschitz graph for the proof of Main Lemma 7.1

9.1. Léger’s theorem. To construct the Lipschitz graph \( \Gamma \) we will follow quite closely the arguments of [Lé2]. Recall that in this paper the author proves that if \( E \subset \mathbb{R}^d \) has finite length and finite curvature, then \( E \) is rectifiable (i.e., \( 1 \)-rectifiable). A more precise result is the following (see [Lé2, Proposition 1.1]):

**Theorem 9.1.** For any constant \( C_{17} \geq 10 \), there exists a number \( \eta > 0 \) such that if \( \sigma \) is a Borel measure on \( \mathbb{R}^d \) verifying
- \( \sigma(B(0, 2)) \geq 1 \), \( \text{supp}\sigma \subset B(0, 2) \),
- for any ball \( B \), \( \sigma(B) \leq C_{19} \text{diam}(B) \),
- \( c^2(\sigma) \leq \eta \),
then there exists a Lipschitz graph \( \Gamma \) such that \( \sigma(\Gamma) \geq \frac{99}{100} \sigma(\mathbb{R}^n) \).

Let us remark that, although Léger’s theorem is a 1-dimensional result, it easily generalizes to higher dimensions, as the author claims in [Lé2].

Instead of an estimate on the curvature of \( \mu \), to prove the Main Lemma 7.1 we will use \( L^2(\mu) \) estimates of Riesz transforms (by means of Theorem 1.3).

9.2. The stopping regions for the construction of the Lipschitz graph. In the rest of the paper we assume that \( \mu, B_0 \) and \( F \) satisfy the assumptions of Main Lemma 7.1. Notice that by Lemma 8.5 we know that there exists some \( n \)-plane \( D_0 \) such that
\[
dist(x, D_0) \leq C \delta r_0 \quad \text{for all } x \in F.
\]
Without loss of generality we will assume that \( D_0 = \mathbb{R}^n \times \{(0, \ldots, 0)\} \equiv \mathbb{R}^n \).

As stated above, to construct the Lipschitz graph, we follow very closely the arguments from [Lé2]. First we need to define a family of stopping time
regions, which are the same as the ones defined in [Lé2 Subsection 3.1]. Given positive constants \(\delta_0, \varepsilon, \alpha\) to be fixed below, we set

\[
S_{\text{total}} = \left\{ (x, t) \in (F \cap B_0) \times (0, 8r_0), \begin{array}{l}
(i) \quad \delta_F(x, t) \geq \frac{1}{2}\delta_0 \\
(ii) \quad \beta_{1,F}(x, t) < 2\varepsilon \\
(iii) \quad \exists D_{x,t} \text{ s.t. } \left\{ \beta_{1,F}^{D_{x,t}}(x, t) \leq 2\varepsilon, \text{ and } \mathcal{L}(D_{x,t}, D_0) \leq \alpha \right\} \right\}.
\]

In the definition above to simplify notation we have denoted \(\delta_F(x, r) \equiv \delta_{\mu_F}(x, r)\) and \(\beta_{1,F}(x, r) \equiv \beta_{1,\mu_F}(B(x, r))\). Also, \(D_{x,t}\) are \(n\)-planes depending on \(x\) and \(t\) and

\[
\beta_{1,F}^{D_{x,t}}(x, t) = \frac{1}{t^n} \int_{y \in F : |x-y| \leq \varepsilon t} \frac{\text{dist}(y, D_{x,t})}{t} d\mu(y).
\]

Let us remark that \(\delta_0, \varepsilon, \alpha\) will be chosen so that \(0 < \varepsilon \ll \alpha \ll \delta_0 \ll 1\).

For \(x \in F \cap B_0\) we set

\[
(9.1) \quad h(x) = \sup \left\{ t > 0 : \exists y \in F, \exists \tau, \frac{t}{3} \geq \tau \geq \frac{t}{4}, x \in B(y, \frac{\tau}{3}), (y, \tau) \notin S_{\text{total}} \right\},
\]

and

\[
S = \left\{ (x, t) \in S_{\text{total}} : t \geq h(x) \right\}.
\]

Notice that if \((x, t) \in S\), then \((x, t') \in S\) for \(t' > t\).

Now we consider the following partition of \(F \cap B_0\):

\[
Z = \left\{ x \in F \cap B_0 : h(x) = 0 \right\},
\]

\[
F_1 = \left\{ x \in F \cap B_0 \setminus Z : \exists y \in F, \exists \tau \in \left[ \frac{h(x)}{5}, \frac{h(x)}{2} \right], x \in B(y, \frac{\tau}{2}), \delta(y, \tau) \leq \delta_0 \right\},
\]

\[
F_2 = \left\{ x \in F \cap B_0 \setminus (Z \cup F_1) : \exists y \in F, \exists \tau \in \left[ \frac{h(x)}{5}, \frac{h(x)}{2} \right], x \in B(y, \frac{\tau}{2}), \beta_{1,F}(y, \tau) \geq \varepsilon \right\},
\]

\[
F_3 = \left\{ x \in F \cap B_0 \setminus (Z \cup F_1 \cup F_2) : \exists y \in F, \exists \tau \in \left[ \frac{h(x)}{5}, \frac{h(x)}{2} \right], x \in B(y, \frac{\tau}{2}), \mathcal{L}(D_{y,\tau}, D_0) \geq \frac{2}{3} \alpha \right\}.
\]

**Remark 9.2.** It is easy to check that if \(x \in F_3\), then for \(h(x) \leq t \leq 100h(x)\) we have \(\mathcal{L}(D_{x,h(x)}, D_0) \geq \alpha/2\), due to the fact that \(\varepsilon \ll \alpha\). See [Lé2 Remark 3.3].

The only difference between the definitions above and the ones in [Lé2 Subsection 3.1] is that we work with \(n\)-dimensional densities, \(\beta\)'s, and planes, while in [Lé2] the dimension is \(n = 1\).

**9.3.** \(F_2\) is void.

**Lemma 9.3.** If \(\delta_2\) is small enough in Main Lemma 7.1, then \(F_2\) is void. Moreover, \(\beta_{\infty,F}(x, r) \leq \varepsilon^2\) for all \(x \in F\) and \(r > 3h(x)\).
Proof. By definition, since $r > 3h(x)$, then $(x, r) \in \Sigma_{\text{total}}$, and then $\delta_F(x, r) \geq \delta_0$. We set $s := M_1 r_0 / \delta_2$. For $y \in F$ with $|x - y| \leq 3r$ and $0 < \tau \leq r_0 / \delta_2$ we have

$$|\tilde{R}_\tau \mu(x) - \tilde{R}_\tau \mu(y)| \leq |\tilde{R}_{\tau, s} \mu(x)| + |\tilde{R}_{\tau, s} \mu(y)| + |\tilde{R}_s \mu(x) - \tilde{R}_s \mu(y)|$$

(notice that $\tau < s$). By the smoothness of the kernel of $\tilde{R}_s$ and the assumption (b) in Main Lemma 7.1, it is easy to check that

$$|\tilde{R}_{\tau, s} \mu(x)| + |\tilde{R}_{\tau, s} \mu(y)| \leq M_1 |x - y| / s \lesssim M_1 r_0 / s = \delta_2.$$ 

Also, by (d) in Main Lemma 7.1 since $s \leq r_0 / \delta_2^2$ (for $\delta_2$ small enough), we have

$$|\tilde{R}_{\tau, s} \mu(x)| + |\tilde{R}_{\tau, s} \mu(y)| \leq 2 \delta_2.$$ 

Therefore,

$$|\tilde{R}_\tau \mu(x) - \tilde{R}_\tau \mu(y)| \leq C \delta_2$$

$0 < \tau \leq r_0 / \delta_2$, and so from Lemma 8.5 we derive $\beta_{\infty, F}(x, r) \leq \varepsilon^2$ for all $x \in F$ and $r \geq 2h(x)$, assuming $\delta_2$ small enough (notice that $\delta_2$ may depend on $\delta_0$). In particular, this implies that $F_2$ is void.

Let us remark that we have preferred to maintain the definition of $F_2$ in the preceding subsection in order to keep the analogy with the construction in [Le2], although here $F_2$ turns out to be void.

9.4. The Lipschitz graph and the size of $F_1$. For $x \in \mathbb{R}^d$ we set

$$d(x) = \inf_{(X, t) \in S} (|X - x| + t),$$

and for $p \in D_0$,

$$D(p) = \inf_{x \in \Pi^{-1}(p)} d(x) = \inf_{(X, t) \in S} (|\Pi(X) - p| + t).$$

Notice that $d$ and $D$ are 1-Lipschitz functions. Moreover, $h(x) \geq d(x)$ for $x \in F \cap B_0$, and

$$Z = \{x \in F \cap B_0 : d(x) = 0\}.$$ 

Observe also that $d(\cdot)$ is defined on $\mathbb{R}^d$, and not only on $F \cap B_0$. Moreover, $d(x) \geq r_0$ if $x \notin 2B_0$, since $(X, t) \in S$ implies that $X \in F \cap B_0$.

The construction of the Lipschitz graph $\Gamma$ is basically the same as the one in [Le2]. The only difference is that in our case the dimension is $n > 1$. So, we have:

Lemma 9.4. There exists a Lipschitz function $A : \mathbb{R}^n \to \mathbb{R}^{d-n}$ supported on $\Pi(3B_0)$ with $\|\nabla A\|_{\infty} \leq C \alpha$ such that if we set $\tilde{A}(p) = (p, A(p))$ for $p \in \mathbb{R}^n$ and

$$\tilde{F} = \{x \in F : \text{dist}(x, \tilde{A}(\Pi(x))) \leq \varepsilon^{1/2} d(x)\},$$

then we have

$$\mu(F \setminus \tilde{F}) \leq C \varepsilon^{1/2} \mu(F).$$
Moreover,

\[ |\nabla^2 A(p)| \leq \frac{C\varepsilon}{D(p)}, \quad p \in \mathbb{R}^n. \]

See Lemma 3.13 and Proposition 3.8 of [Le2] for the details.

Notice that if \( x \notin 2B_0 \), then \( d(x) > r_0 \), and taking into account that \( \beta_{\infty,F}(10B_0) \leq \varepsilon^2 \), it turns out that \( F \setminus 2B_0 \subset \tilde{F} \) (recall also that \( F \subset 10B_0 \)).

To tell the truth, the Lipschitz graph that is constructed in [Le2] needs not to be supported on \( \Pi(3B_0) \), however it is not difficult to show that if one has a Lipschitz graph \( A_0 \) satisfying the assumptions above except the one on the support, then one can take \( A = A_0\eta \) where \( \eta : \mathbb{R}^n \to \mathbb{R} \) is a \( C^\infty \) function such that \( \chi_{\Pi(2B_0)}\eta \leq \chi_{\Pi(3B_0)} \).

**Remark 9.5.** To prove Main Lemma 7.1 we will show that if parameters \( \delta_0, \alpha \) and \( \varepsilon \) are chosen small enough, then \( \mu(\tilde{F} \cap B_0) \geq \frac{99}{100}c_n\varepsilon^2 \) (see Lemma 10.5) and the sets \( F_1 \) and \( F_3 \) are much smaller that \( \mu(\tilde{F} \cap B_0) \). By the preceding construction and definitions, we have \( \tilde{F} \cap B_0 \setminus (F_1 \cup F_2 \cup F_3) \subset \Gamma \).

Arguing as in [Le2] Proposition 3.19], if \( \delta_0 \) and \( \varepsilon \) are small enough, we get

**Lemma 9.6.**

\[ \mu(F_1) \leq 10^{-6}\mu(F \cap B_0). \]

### 9.5. A technical lemma.

The following is a technical result that will be used below.

**Lemma 9.7.** If \( x \in F \) and \( y \in \mathbb{R}^d \) satisfy \( \Pi(x) = \Pi(y) \), then

\[ d(x) \lesssim d(y), \]

and so

\[ d(x) \approx D(\Pi(x)). \]

**Proof.** The second assertion is a straightforward consequence of the first one. So we only have to prove that \( d(x) \lesssim d(y) \). Set \( \ell = |x - y| \). We distinguish several cases:

- If \( \ell \leq d(x)/2 \), since \( d(\cdot) \) is 1-Lipschitz, it follows that \( |d(x) - d(y)| \leq d(x)/2 \), and so \( d(x) \approx d(y) \).

- Suppose that \( d(x)/2 < \ell \leq r_0 \) and that \( d(y) \leq d(x)/8 \). By the definition of \( d(x) \) it turns out that there exists some \( (X, \ell) \in S \) such that

\[ |X - x| + \ell \leq 2d(x) < 4\ell. \]

Notice that we also have \( (X, 6\ell) \in S \) (because \( \ell \leq r_0 \)), and thus

\[ \beta_{\infty,F}(X, 6\ell) \leq \varepsilon^2 \]

by Lemma 9.3. If \( D_{X,6\ell} \) stands for the \( n \)-plane that minimizes \( \beta_{\infty,F}(X, 6\ell) \), since \( \lambda(D_0, D_{X,6\ell}) \ll 1 \) and \( \Pi(x) = \Pi(y) \),

\[ |x - y| \leq 2[\text{dist}(x, D_{X,6\ell}) + \text{dist}(y, D_{X,6\ell})] \leq 12\varepsilon^2\ell + 2\text{dist}(y, D_{X,6\ell}), \]

and

\[ |x - y| \leq 12\varepsilon^2\ell. \]

Hence, \( d(x) \lesssim 12\varepsilon^2\ell \).

That is, we also proved \( d(x) \lesssim d(y) \), since \( d(y) \leq d(x)/8 \) and \( \ell \leq d(x)/2 \).
since \( x \in F \).

By the definition of \( d(y) \), there exists \((Y, u) \in S\) such that

\[
|Y - y| + u \leq 2d(y) \leq \frac{d(x)}{4} \leq \frac{\ell}{2}.
\]

Since

\[
|Y - X| \leq |Y - y| + |y - x| + |x - X| \leq \ell + \ell + 4\ell = 6\ell
\]

and \( Y \in F \), we also have

\[
\text{dist}(Y, D_{X, 6\ell}) \leq 6\varepsilon^2 \ell,
\]

by Lemma 9.7 again. Therefore, by (9.4),

\[
\text{dist}(y, D_{X, 6\ell}) \leq |y - Y| + \text{dist}(Y, D_{X, 6\ell}) \leq \frac{\ell}{2} + 12\varepsilon^2 \ell.
\]

Thus by (9.3),

\[
|x - y| \leq \frac{\ell}{2} + 24\varepsilon^2 \ell < \ell
\]

if \( \varepsilon \) is small enough, which is a contradiction.

\[\bullet\] Suppose now that \( \ell > r_0 \). Since \( F \subset 10B_0, \beta_{\infty, F}(10B_0) \ll 1, \Pi(x) = \Pi(y), \) and \( |x - y| \geq r_0 \), by geometric arguments it easily follows that \( \text{dist}(y, F) \gtrsim r_0 \). This implies that \( d(y) \gtrsim r_0 \) by the definition of \( d(y) \), and so \( d(y) \gtrsim d(x) \).

\[\square\]

10. The proof that \( F_3 \) is small

10.1. The strategy. For \( x \in \mathbb{R}^d \), we set

\[
\ell(x) := \frac{1}{10} D(\Pi(x)).
\]

Also, for any measure \( \sigma \) we denote

\[
\mathcal{R}_{\ell(x), r_0}^{+} \sigma(x) := \hat{\mathcal{R}}_{\ell(x), r_0}^{+} \sigma(x) - \hat{\mathcal{R}}_{r_0}^{+} \sigma(x).
\]

For simplicity we have preferred the notation \( \mathcal{R}_{\ell(x), r_0}^{+} \sigma(x) \) instead of \( \hat{\mathcal{R}}_{\ell(x), r_0}^{+} \sigma(x) \), although the latter seems more natural.

Roughly speaking, the arguments to show that \( F_3 \) cannot be too big are the following:

\( F_3 \) big \( \Rightarrow \|\nabla A\|_2 \) big \( \Rightarrow \|\mathcal{R}^{+}_{\ell(x), r_0} \mathcal{H}^p_{1[1]}\|_{L^2(\Gamma)} \) big

\[
\Rightarrow \|\mathcal{R}^{+}_{\ell(x), r_0} \mathcal{H}^p_{1[1\cap 5B_0]}\|_{L^2(\Gamma \cap 4B_0)} \text{ big}
\]

\[
\Rightarrow \|\mathcal{R}^{+}_{\ell(x), r_0} \mathcal{H}^p_{1\cap \mu[F]}\|_{L^2(\Gamma \cap 4B_0)} \text{ big} \Rightarrow \|\mathcal{R}^{+}_{\ell(x), r_0} \mu\|_{L^2(\mu[F])} \text{ big},
\]

which contradicts the assumptions of Main Lemma 7.1.

Let us explain some more details. The fact that \( \|\nabla A\|_2 \) must be big if \( F_3 \) is big follows from the definition of \( F_3 \). Loosely speaking, if \( x \in F_3 \), then the approximating Lipschitz graph has slope \( \gtrsim \alpha \) near \( x \), by construction. As a consequence, we should expect \( \|\nabla A\|_2 \gtrsim \alpha \mu(F_3)^{1/2} \) (or a similar inequality) to hold.
The implication
\[ \| \nabla A \|_2 \text{ big } \Rightarrow \| R^1 \mathcal{H}^n \|_{L^2(\Gamma)} \text{ big} \]
is a direct consequence of Theorem 11.3. Finally, the implications
\[ \| R^1 \mathcal{H}^n \|_{L^2(\Gamma)} \text{ big } \Rightarrow \cdots \Rightarrow \| R^1_{\partial(\cdot),r} \mu \|_{L^2(\mu,F)} \text{ big} \]
follow, basically, by approximation. For these arguments to work one has to control the “errors” in this approximation. In particular, the errors must be smaller than \( C\alpha \mu(F_3)^{1/2} \). A key point here is that these errors depend mostly on the parameter \( \varepsilon \) in the definition of \( F_2 \) and we have chosen \( \varepsilon \ll \alpha \).

10.2. The implication \( F_3 \text{ big } \Rightarrow \| \nabla A \|_2 \text{ big} \).

Lemma 10.1. We have
\[ \mu(F_3) \leq C\alpha^{-2} \| \nabla A \|_2^2 + C\varepsilon^{1/2} \mu(F). \]

Proof. For a fixed \( x \in F_3 \), consider the ball \( B = B(x,r) \), with \( r = 2h(x) \) (recall that \( h(x) \) was defined in (9.11)). Suppose that \( \mu(B \cap \tilde{F}) \geq \mu(B \cap F)/2 \).

By Lemma 10.2 there are \( n + 1 \) balls \( \Delta_0, \ldots, \Delta_n \) with radius \( t/C_{12} \) such that \( \mu(\tilde{F} \cap \Delta_i) \geq C(\delta)^{-1}r^n \) and for all \( (x_0, \ldots, x_n) \in \Delta_0 \times \cdots \times \Delta_n \) we have
\[ \text{vol}_n((x_0, \ldots, x_n)) \geq C^{-1}r^n. \]

By Remark 9.2 we have \( \angle(D_{x,r},D_0) \geq \alpha/2 \). Then, it is easy to check that
\[ m_B(\{A - m_B(A)\}) \geq C^{-1} \alpha r, \]

since, for each \( i \), \( \Delta_i \cap \tilde{F} \) is very close to the graph of \( A \) and also very close to \( D_{x,r} \), and moreover \( \varepsilon^{1/2} \ll \alpha \). As a consequence, by Poincaré inequality,
\[ m_B(\nabla A) \geq C^{-1} \frac{m_B(\{A - m_B(A)\})}{r} \geq C^{-1} \alpha. \]

Thus, for this ball we have
\[ \| \chi_B \nabla A \|_2^2 \geq C^{-1} \alpha^2 r^n. \]

Take now a Besicovitch covering of \( F_3 \) with balls \( B_i = B(x_i, r_i) \) as above (i.e. \( x_i \in F_3 \) and \( r_i = 2h(x_i) \)). Denote by \( I_1 \) the collection of balls \( B_i \) such that \( \mu(B_i \cap \tilde{F}) \geq \mu(B_i \cap F)/2 \). We have
\[ \alpha^2 \sum_{i \in I_1} \mu(B_i \cap F) \leq C \sum_{i \in I_1} \| \chi_{B_i} \nabla A \|_2^2 \leq C \| \nabla A \|_2^2. \]

For the balls \( B_i \) in the other collection, that we denote by \( I_2 \), we have \( \mu(B_i \cap \tilde{F}) < \mu(B_i \cap F)/2 \). Thus,
\[ \mu(B_i \cap F \setminus \tilde{F}) \geq \frac{1}{2} \mu(B_i \cap F), \quad i \in I_2. \]

So we get
\[ \sum_{i \in I_2} \mu(B_i \cap F) \leq 2 \sum_i \mu(B_i \cap F \setminus \tilde{F}) \leq C \mu(F \setminus \tilde{F}) \leq C \varepsilon^{1/2} \mu(F). \]

The lemma follows from (10.1) and (10.2). \( \square \)
10.3. The implication $\|\nabla A\|_2$ big $\Rightarrow \|R^+ (\mathcal{H}_n^m)\|_{L^2(\Gamma)}$ big. This is a direct consequence of Corollary 1.4. Indeed, recall that we showed that

$$\tag{10.3} \|R^+ (\mathcal{H}_n^m)\|_{L^2(\Gamma)} \approx \|\nabla A\|_2,$$

assuming that $\|\nabla A\|_\infty$ is small enough, which is true in our construction if $\alpha \ll 1$.

10.4. The implication $\|R^+ (\mathcal{H}_n^m)\|_{L^2(\Gamma)}$ big $\Rightarrow \|R^+_{II(\cdot), r_0} (\mathcal{H}_n^m \cap \Gamma \cap 4B_0)\|_{L^2(\Gamma \cap 4B_0)}$ big.

**Lemma 10.2.**

$$\|R^+ (\mathcal{H}_n^m)\|_{L^2(\Gamma)} \leq \|R^+_{II(\cdot), r_0} (\mathcal{H}_n^m \cap \Gamma \cap 4B_0)\|_{L^2(\Gamma \cap 4B_0)} + C \alpha^2 r_0^{n/2}.$$

**Proof.** Recall that $\text{supp}(A) \subset 3B_0$. We set

$$\|R^+ (\mathcal{H}_n^m)\|_{L^2(\Gamma)} \leq \|\chi_{4B_0} R^+ (\mathcal{H}_n^m \cap \Gamma \cap 4B_0)\|_{L^2(\Gamma)} + \|\chi_{4B_0} R^+ (\mathcal{H}_n^m \cap \Gamma \cap 4B_0)\|_{L^2(\Gamma)} = I + II + III.$$

Let us see that the terms $II$ and $III$ are small. We consider first $II$. Given $x \in 4B_0$, we have

$$|R^+ (\mathcal{H}_n^m \cap \Gamma \cap 4B_0)(x)| \leq \int \frac{|x^+ - y^+|}{|x - y|^{n+1}} d\mathcal{H}^n(y) = \int \frac{\text{dist}(x, D_0)}{|x - y|^{n+1}} d\mathcal{H}^n(y) \lesssim \frac{\text{dist}(x, D_0)}{r_0}.$$

If we square and integrate the last estimate on $4B_0$, we get

$$II^2 \lesssim \beta_2 \Gamma (2B_0)^2 r_0^n \lesssim \varepsilon^2 r_0^n.$$

To estimate the term $III$ we take $x \in \Gamma \setminus 4B_0 = D_0 \setminus 4B_0$ (so $x^+ = 0$), and we set

$$|R^+ (\mathcal{H}_n^m)(x)| \leq \int \frac{\text{dist}(y, D_0)}{|x - y|^{n+1}} d\mathcal{H}^n(y) = \int \frac{\text{dist}(y, D_0)}{|x - y|^{n+1}} d\mathcal{H}^n(y) \approx \frac{1}{(r_0 + |x - x_0|)^{n+1}} \int \text{dist}(y, D_0) d\mathcal{H}^n(y) \leq \beta_1 \Gamma (2B_0) \frac{r_0^{n+1}}{(r_0 + |x - x_0|)^{n+1}}.$$

Squaring and integrating on $D_0 \setminus 4B_0$, we obtain

$$III^2 \lesssim \beta_1 \Gamma (2B_0)^2 r_0^n \leq \varepsilon^2 r_0^n.$$

To deal with the term $I$, given $x \in \Gamma \cap 4B_0$, we set

$$|R^+ (\mathcal{H}_n^m \cap \Gamma \cap 5B_0)(x)| \leq |R^+_{II(\cdot), r_0} (\mathcal{H}_n^m \cap \Gamma \cap 5B_0)(x)| + |R^+_{II(\cdot), r_0} (\mathcal{H}_n^m \cap \Gamma \cap 5B_0)(x)| + |R^+_{III, r_0} (\mathcal{H}_n^m \cap \Gamma \cap 5B_0)(x)|.$$

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We consider first the term \( |\hat{R}^+_{r_0}(\mathcal{H}^n_{\Gamma\cap 5B_0})(x)| \), for \( x \in \Gamma \cap 4B_0):$

\[
|\hat{R}^+_{r_0}(\mathcal{H}^n_{\Gamma\cap 5B_0})(x)| \leq \int_{y \in \Gamma\cap 5B_0; |y-x| > r_0/2} \frac{|x^+ - y^+|}{|x - y|^{n+1}} d\mathcal{H}^n_\Gamma(y) \\
\lesssim \int_{y \in \Gamma\cap 5B_0} \frac{\text{dist}(x, D_0) + \text{dist}(y, D_0)}{r_0^{n+1}} d\mathcal{H}^n_\Gamma(y) \lesssim \beta_{\infty, 1}(2B_0).
\]

So we get

\[
(10.4) \quad \|\chi_{4B_0} \hat{R}^+_{r_0}(\mathcal{H}^n_{\Gamma\cap 5B_0})\|_{L^2(\Gamma)}^2 \lesssim \beta_{\infty, 1}(2B_0)^2 r_0^n \lesssim \varepsilon^2 r_0^n.
\]

To estimate \( |R^+_{0, \ell(x)}(\mathcal{H}^n_{\Gamma\cap 5B_0})(x)| \) we will use the smoothness of \( \Gamma \) on the stopping cubes. That is, we will use the estimate \((9.2)\). Notice first that

\[
R^+_{0, \ell(x)}(\mathcal{H}^n_{\Gamma\cap 5B_0})(x) = R^+_{0, \ell(x)}(\mathcal{H}^n_\Gamma)(x)
\]

for \( x \in 4B_0 \), since \( \ell(x) < r_0 \). So if we set \( x = \tilde{A}(p), y = \tilde{A}(q) \), with \( p, q \in \mathbb{R}^n \), we have

\[
(10.5) \quad R^+_{0, \ell(x)}(\mathcal{H}^n_{\Gamma\cap 5B_0})(x) = \int \left( 1 - \psi \left( \frac{\tilde{A}(p) - \tilde{A}(q)}{D(p)/10} \right) \right) \frac{A(p) - A(q)}{|A(p) - A(q)|^{n+1}} J(\tilde{A})(q)dq,
\]

where \( dq \) stands for the \( n \)-dimensional Lebesgue measure. We denote by \( S(x) \) the integral on the right hand side of \((10.5)\), and we set

\[
S(x) = \int \left( 1 - \psi \left( \frac{p - q}{D(p)/10} \right) \right) \frac{A(p) - A(q)}{|A(p) - A(q)|^{n+1}} dq \\
+ \int \left( \psi \left( \frac{p - q}{D(p)/10} \right) - \psi \left( \frac{\tilde{A}(p) - \tilde{A}(q)}{D(p)/10} \right) \right) \frac{A(p) - A(q)}{|A(p) - A(q)|^{n+1}} dq \\
+ \int \left( 1 - \psi \left( \frac{\tilde{A}(p) - \tilde{A}(q)}{D(p)/10} \right) \right) \frac{A(p) - A(q)}{|A(p) - A(q)|^{n+1}} (J(\tilde{A})(q) - 1) dq
\]

\[
= S_1(x) + S_2(x) + S_3(x).
\]

Recall that by Remark \((6.2)\) we have

\[
\|J(\tilde{A}) - 1\|_2 \lesssim \|\nabla A\|_\infty \|\nabla A\|_2.
\]

So, by the \( L^2 \) boundedness of Riesz transforms on Lipschitz graphs we get

\[
(10.6) \quad \|S_3\|_2 \lesssim \|\nabla A\|_\infty \|\nabla A\|_2.
\]

To deal with \( S_2 \) notice that

\[
(10.7) \quad |\tilde{A}(p) - \tilde{A}(q)| - |p - q| \leq |A(p) - A(q)| \leq C\alpha |p - q| \leq \frac{1}{2} |p - q|,
\]

and so

\[
\frac{1}{2} |p - q| \leq |\tilde{A}(p) - \tilde{A}(q)| \leq 2|p - q|.
\]
Since \( \psi(z) = 0 \) if \( |z| \leq 1/2 \) and \( \psi(z) = 1 \) if \( |z| \geq 1 \), we deduce that
\[
\psi\left(\frac{p - q}{D(p)/10}\right) - \psi\left(\frac{\tilde{A}(p) - \tilde{A}(q)}{D(p)/10}\right) = 0
\]
if \( |p - q| \leq D(p)/40 \) or \( |p - q| \geq D(p)/5 \). Moreover, from the mean value theorem and (10.7),
\[
\left| \psi\left(\frac{p - q}{D(p)/10}\right) - \psi\left(\frac{\tilde{A}(p) - \tilde{A}(q)}{D(p)/10}\right) \right| \leq C|p - q|/D(p).
\]
Thus,
\[
|S_2(x)| \lesssim \int_{D(p)/40 \leq |p - q| \leq D(p)/5} \frac{\alpha|p - q|}{D(p)} \frac{|A(p) - A(q)|}{|p - q|^{n+1}} dq
\]
\[
\lesssim \frac{\alpha^2}{D(p)} \int_{D(p)/40 \leq |p - q| \leq D(p)/5} \frac{1}{|p - q|^{n-1}} dq \lesssim \alpha^2.
\]
Therefore,
\[
(10.8) \quad ||S_2||_2 \lesssim \alpha^2 r_0^{n/2}.
\]
We are left with the term \( S_1(x) \). By Taylor’s formula, we have
\[
\frac{A(p) - A(q)}{(|p - q|^2 + |A(p) - A(q)|^2)^{(n+1)/2}} = \sum_{k=0}^{\infty} (-1)^k \frac{(n + 2k - 1)!!}{2^k k!} \frac{|A(p) - A(q)|^{2k}}{|p - q|^{n+2k+1}} (A(p) - A(q)).
\]
The series is uniformly convergent since \( |A(p) - A(q)|/|p - q| \leq C\alpha \ll 1 \). Notice that the integrand in \( S_1 \) vanishes if \( |p - q| > D(p)/10 \). On the other hand, by Taylor’s formula and (11.2) we also have
\[
A(p) - A(q) = \nabla A(p)(p - q) + E(p, q),
\]
with
\[
(10.9) \quad |E(p, q)| \leq C \sup_{z \in B(p, D(p)/10)} |\nabla^2 A(z)| |p - q|^2
\]
\[
\leq \varepsilon |p - q|^2 \sup_{z \in B(p, D(p)/10)} \frac{1}{D(z)} \lesssim \frac{\varepsilon |p - q|^2}{D(p)},
\]
since \( D(\cdot) \) is 1-Lipschitz. Then it turns out that
\[
(A(p) - A(q)) |A(p) - A(q)|^{2k} = \nabla A(p)(p - q) |\nabla A(p)(p - q)|^{2k} + E_k(p, q),
\]
with\(^2\)
\[
|E_k(p, q)| \leq C \varepsilon 2^{-k} |p - q|^{2k+2}/D(p).
\]
\(^2\)For this estimate we take into account that \( ||\nabla A||_{\infty} + \varepsilon \leq 1/4 \) and we use the fact that \( (a + b)^m = a^m + c \), with \( |c| \leq 2^k |b| \max(|a|, |b|)^{m-1} \).
We have
\[ S_1(x) = \sum_{k=0}^{\infty} (-1)^k \frac{(n+2k-1)!!}{2^k k!} \]
\[ \times \int \left( 1 - \psi \left( \frac{p-q}{D(p)/10} \right) \right) \frac{\nabla A(p)(p-q) \nabla A(p)(p-q)^{2k}}{|p-q|^{n+2k+1}} dq \]
\[ + \sum_{k=0}^{\infty} (-1)^k \frac{(n+2k-1)!!}{2^k k!} \int_{|p-q| \leq D(p)/10} \frac{E_k(p,q)}{|p-q|^{n+2k+1}} dq. \]

Notice that the first sum on the right side above vanishes because each integral in the sum equals zero by the antisymmetry of the integrand. We obtain
\[ |S_1(x)| \leq \sum_{k=0}^{\infty} \frac{(n+2k-1)!!}{2^k k!} \int_{|p-q| \leq D(p)/10} \frac{|E_k(p,q)|}{|p-q|^{n+2k+1}} dq \]
\[ \lesssim \sum_{k=0}^{\infty} \frac{(n+2k-1)!!}{4^k k!} \int_{|p-q| \leq D(p)/10} \frac{\varepsilon}{D(p)|p-q|^{n+1}} dq \]
\[ \approx \sum_{k=0}^{\infty} \frac{(n+2k-1)!!}{4^k k!} \varepsilon \approx \varepsilon. \]

From (10.5), (10.6), (10.8), and (10.10) we deduce that
\[ \| \chi_{B_0} R_{\ell,0}^1(x) (H_{\Gamma \cap S_0}^0) \|_{L^2(\Gamma)} \lesssim \| \nabla A \|_\infty \| \nabla A \|_2 + \alpha^{2\epsilon_0} \lesssim \alpha^{2\epsilon_0} \]
because \( \alpha^2 \ll \varepsilon \) and \( \| \nabla A \|_2 \lesssim \alpha^{\epsilon_0} \), since \( A \) is supported on \( \Pi(3B_0) \).

Therefore, by (10.4),
\[ I \leq C(\varepsilon + \alpha^{\epsilon_0}) \lesssim \alpha^2 \]

The lemma follows from the preceding estimate and the ones obtained above for the terms \( II \) and \( III \).

\[ \square \]

10.5. The implication \( \| R_{\ell,0}^1(x_0,0) (H_{\Gamma \cap S_0}^0) \|_{L^2(\Gamma \cap S_0)} \) big \( \Rightarrow \| R_{\ell,0}^1(x_0,0) \mu \tilde{F} \|_{L^2(\Gamma)} \) big. This implication is one of the most delicate steps of the proof that \( F_3 \) is a small set. Let \( \varphi : \mathbb{R}^n \to \mathbb{R} \) be a smooth radially non increasing function with \( \| \varphi \|_1 = 1 \) such that \( \text{supp}(\varphi) \subset B_0(0,1) \) and \( \varphi \) equals 1 on \( B_0(0,\epsilon_0) \) for some \( 0 < \epsilon_0 < 1 \) which may depend on \( n \). As usual, for \( t > 0 \) we denote,
\[ \varphi_t(x) = \frac{1}{t^n} \varphi \left( \frac{x}{t} \right), \quad x \in \mathbb{R}^n. \]

Then we consider the function \( g : \mathbb{R}^n \to \mathbb{R} \) given by
\[ g(x) = \varphi_{\epsilon^{1/4}D(x)} \ast \| \mu_{\| \tilde{F} \|}(x). \]

We will show below that \( g(x) \) is very close to the measure \( dx \) on \( B(x_0,6r_0) \), in a sense.

First we need the following preliminary result:
Lemma 10.3. For all $x, y \in \mathbb{R}^n$,

$$|\varphi_{\varepsilon^{1/4}D(x)}(x - y) - \varphi_{\varepsilon^{1/4}D(y)}(x - y)| \lesssim \frac{\varepsilon^{1/4}}{(\varepsilon^{1/4}D(y))^n} \chi_{B(0,C\varepsilon^{1/4}D(y))}(x - y).$$

Proof. For any $z \in \mathbb{R}^n$ and $s, t > 0$ with $s \approx t$,

$$|\varphi_s(z) - \varphi_t(z)| \leq \frac{1}{s^n} \left|\varphi\left(\frac{z}{s}\right) + \frac{1}{t^n} \left|\varphi\left(\frac{z}{t}\right) - \varphi\left(\frac{z}{s}\right)\right|\right| \leq \frac{C|s - t|}{s^{n+1}} \varphi\left(\frac{z}{s}\right) + \frac{C}{t^{n+1}} \left|\frac{z}{s} - \frac{z}{t}\right| \leq \frac{C|s - t|}{s^{n+1}},$$

since we may assume that $|z| \lesssim s$. As a consequence,

$$|\varphi_s(z) - \varphi_t(z)| \leq \frac{C|s - t|}{s^{n+1}} \chi_{B(0,C\varepsilon^{1/4}D(y))}(z).$$

We set $s = \varepsilon^{1/4}D(y)$ and $t = \varepsilon^{1/4}D(x)$. Notice that $\varphi_{\varepsilon^{1/4}D(x)}(x - y) \neq 0$ implies that $|x - y| \leq \varepsilon^{1/4}D(x)$, and then it turns out that $D(x) \approx D(y)$. Of course, the same happens if $\varphi_{\varepsilon^{1/4}D(y)}(x - y) \neq 0$. In both cases we have

$$\frac{|s - t|}{s} = \frac{|D(x) - D(y)|}{D(y)} \leq \frac{|x - y|}{D(y)} \lesssim \varepsilon^{1/4}.$$

Therefore,

$$|\varphi_{\varepsilon^{1/4}D(x)}(x - y) - \varphi_{\varepsilon^{1/4}D(y)}(x - y)| \lesssim \frac{\varepsilon^{1/4}}{(\varepsilon^{1/4}D(y))^n} \chi_{B(0,C\varepsilon^{1/4}D(y))}(x - y).$$

\[\square\]

Lemma 10.4. Let $\nu$ be a Borel measure on $\mathbb{R}^n$ such that

$$\nu(B(x, r)) \leq r^n \quad \text{for all } x \in \text{supp}(\nu) \text{ and } r \geq \eta r_0.$$

For any $\delta > 0$, if $\eta > 0$ is small enough, we have

$$\nu(B(x, r)) \leq (1 + \delta)r^n \quad \text{for all } x \in \mathbb{R}^n \text{ and } r \geq r_0.$$

Moreover, $\eta$ only depends on $\delta$.

Proof. Given a ball $B(x, r)$ with $r \geq r_0$ we can consider a family of disjoint balls $B_i$ contained in $B(x, r)$ centered at points in $\text{supp}(\nu)$ with radii $r_i \geq \eta^{1/2}r$, such that

$$\mathcal{L}^n(B(x, r) \setminus \bigcup_i B_i) \leq \frac{\delta}{2} r^n,$$

assuming that $\eta$ is small enough. Then, (10.11)

$$\nu(B(x, r)) = \sum_i \nu(B_i) + \nu\left(B(x, r) \setminus \bigcup_i B_i\right) \leq r^n + \nu\left(B(x, r) \setminus \bigcup_i B_i\right).$$
We consider now a Besicovitch covering of \( \text{supp}(\nu) \cap B(x, r) \setminus \bigcup_i B_i \) with balls \( B_j' \) centered at points in \( \text{supp}(\nu) \cap B(x, r) \setminus \bigcup_i B_i \) with radii \( r_j' = \eta r \) for all \( j \). Then we have

\[
\nu \left( B(x, r) \setminus \bigcup_i B_i \right) \leq \sum_j \nu(B_j') \leq \sum_j (r_j')^n \leq C L^n \left( U_{\eta r} \left( B(x, r) \setminus \bigcup_i B_i \right) \right),
\]

where \( U_{\eta r}(A) \) denotes the \( \eta r \)-neighborhood of \( A \). We have

\[
U_{\eta r} \left( B(x, r) \setminus \bigcup_i B_i \right) \subset \left( B(x, r) \setminus \bigcup_i B_i \right) \cup U_{\eta r}(\partial B(x, r)) \cup \bigcup_i U_{\eta r}(\partial B_i),
\]

and so

\[
L^n \left( U_{\eta r} \left( B(x, r) \setminus \bigcup_i B_i \right) \right) \leq \frac{\delta}{2} r^n + C \eta r^n + C \sum_i r_i^{n-1}(\eta r).
\]

Since \( \eta r \leq \eta^{1/2} r_i \) for all \( i \), we get

\[
L^n \left( U_{\eta r} \left( B(x, r) \setminus \bigcup_i B_i \right) \right) \leq \left( \frac{\delta}{2} + C \eta \right) r^n + C \eta^{1/2} \sum_i r_i^n
\]

\[
\leq \left( \frac{\delta}{2} + C \eta + C \eta^{1/2} \right) r^n \leq \left( \frac{\delta}{2} + C \eta^{1/2} \right) r^n.
\]

From (10.11) and (10.11) we infer that

\[
\nu(B(x, r)) \leq \left( 1 + \frac{\delta}{2} + C \eta^{1/2} \right) r^n,
\]

and the lemma follows if \( \eta \) is small enough. \( \square \)

In next lemma we show that \( g \) is very close to the function identically 1 on \( 8B_0 \). We also prove that \( \mu(\tilde{F} \cap B_0) \) is big, which was already mentioned in Remark 9.5.

**Lemma 10.5.** If \( \varepsilon \) has been chosen small enough and \( \delta_1 \leq \omega^2 \) (where \( \delta_1 \) is the constant from (a) and (b) in Main Lemma 7.1), then we have

\[
\Pi_\#(\mu|_{\tilde{F}})(B(p, r)) \leq c_n(1 + \omega^2) r \quad \text{for all } p \in \mathbb{R}^n \text{ and } r \geq \varepsilon^{1/2} D(p),
\]

(10.14)

\[
0 \leq g(p) \leq 1 + C_20 r^2 \quad \text{for all } p \in \mathbb{R}^n,
\]

(10.15)

\[
\|\chi_{8B_0}(g - 1)\|_1 \leq C \omega^2 r_0^n,
\]

and

\[
\|\chi_{8B_0}(g - 1)\|_2 \leq C \omega r_0^{n/2}.
\]

Also,

\[
\mu(\tilde{F} \cap B_0) \geq \frac{99}{100} c_n r_0^n.
\]

(10.17)
Moreover, supp(ψ) \subset U_{C\varepsilon} \supseteq \Pi^{-1}(B_n(p, r)) \cap \bar{F} (recall that \Pi^{-1}(B_n(p, r)) is an n-dimensional ball in \mathbb{R}^n), we have \beta_{\infty, F}(x, t) \leq C\varepsilon \text{ for } t \geq D(p), \text{ we infer that there exists some n-plane } L \text{ such that}

\Pi^{-1}(B_n(p, r)) \cap \bar{F} \subset U_{C\varepsilon} \subset \Pi^{-1}(B_n(p, r)) \cap \bar{F} (recall that \varepsilon \leq \varepsilon_0^1 D(p).

Further, by construction, the n-plane L satisfies \angle(L, \mathbb{R}^n) \leq C\alpha. \text{ All together, this implies that there exists some ball } B(z, R) \subset \mathbb{R}^d, \text{ with}

\[ R \leq (1 + C \sin(\alpha)^2)^{1/2}r + C\varepsilon^{1/2}r \leq (1 + C\alpha^2 + C\varepsilon^{1/2})r, \]

such that

\[ \Pi^{-1}(B_n(p, r)) \cap \bar{F} \subset B(z, R). \]

If \( p \in \Pi(L) \), then we may take \( z \in \bar{F} \), and so by the assumption (b) in the Main Lemma

\[ \Pi \# \mu_{\bar{F}}(B_n(p, r)) \leq \mu(B(z, R)) \leq C_n(1 + \delta_1)(1 + C\alpha^2 + C\varepsilon^{1/2})^n r^n \]

\[ \leq C_n(1 + \delta_1 + C_2\alpha^2)r^n \]  

(10.18)

for \( r \geq \varepsilon^{1/2}D(p) \) (recall that \( \varepsilon^{1/2} \ll \alpha^2 \)).

Consider now the case \( p \notin \Pi(L) \). Suppose that \( \varepsilon^{1/4}D(p) \leq r \leq D(p) \) and let \( \nu = \Pi \# \mu_{\bar{F} \cap B(p, D(p)/10)} \). By (10.18),

\[ \nu(B(z, r)) \leq C_n(1 + \delta_1 + C_2\alpha^2)r^n \]

for all \( z \in \text{supp}(\nu) \) and \( r \geq \varepsilon^{1/2}D(z) \approx \varepsilon^{1/2}D(p) \). From Lemma 10.4 we deduce that

\[ \nu(B(p, r)) \leq C_n(1 + \delta_1 + 2C_2\alpha^2)r^n \]

if \( r \geq \epsilon_0^{1/4}D(p) \) and \( \varepsilon \) is small enough (recall that \( \epsilon_0 \) was defined at the beginning of the current subsection).

To prove (10.14) for a given \( p \in \mathbb{R}^n \), let \( \psi : \mathbb{R} \to \mathbb{R} \) be such that \( \psi(|q|) = \varphi_{\varepsilon^{1/4}D(p)}(q) \) and denote \( \sigma = \Pi \# \mu_{\bar{F}} \). We have

\[ g(p) = \int \psi(|p - q|) d\sigma(p) = -\int_0^\infty \int_{|p| - q|}^\infty \psi'(r) dr d\sigma(p) \]

\[ = -\int_0^\infty \Pi \# \mu_{\bar{F}}(B_n(p, r)) \psi'(r) dr. \]

(10.19)

Notice that

\[ \sigma(B_n(p, r)) = \Pi \# \mu_{\bar{F}}(B_n(p, r)) = \mu(\Pi^{-1}(B_n(p, r)) \cap \bar{F}). \]

Moreover, \text{supp}(\psi') \subset [\epsilon_0^{1/4}D(p), \varepsilon^{1/4}D(p)], \text{ and so}

\[ g(p) = -\int_{\epsilon_0^{1/4}D(p)}^{\varepsilon^{1/4}D(p)} \mu(\Pi^{-1}(B_n(p, r)) \cap \bar{F}) \psi'(r) dr. \]

Thus,

\[ |g(p)| \leq C_n(1 + \delta_1 + 2C_2\alpha^2) \int_{\epsilon_0^{1/4}D(p)}^{\varepsilon^{1/4}D(p)} r^n |\psi'(r)| \, dr = 1 + \delta_1 + 2C_2\alpha^2, \]
and (10.14) follows. Now we turn our attention to (10.15). First we will show that

\begin{equation}
\int_{B_n(x_0,8r_0)} g(p) dp \geq (1 - C \varepsilon^{1/4}) \mathcal{L}^n(8B_0 \cap \mathbb{R}^n).
\end{equation}

Since \( D(p) \leq 9r_0 \) for all \( p \in \Pi(8B_0) \), we have

\begin{equation}
\int_{B_n(x_0,(8+9\varepsilon^{1/4})r_0)} g(p) dp = \int_{B_n(x_0,(8+9\varepsilon^{1/4})r_0)} \varphi_{\varepsilon^{1/4}D(p) * \sigma(p)} dp
\end{equation}

\begin{align*}
&= \int_{p \in B_n(x_0,(8+9\varepsilon^{1/4})r_0)} \int_{\varphi_{\varepsilon^{1/4}D(p)}(p-q)} d\sigma(q) dp \\
&\geq \int_{q \in B_n(x_0,8r_0)} \int_{\varphi_{\varepsilon^{1/4}D(p)}(p-q)} dp d\sigma(q).
\end{align*}

Recall now that by Lemma 10.3

\[ |\varphi_{\varepsilon^{1/4}D(p)}(p-q) - \varphi_{\varepsilon^{1/4}D(q)}(p-q)| \lesssim \frac{\varepsilon^{1/4}}{(\varepsilon^{1/4}D(q))^n} \chi_{B(q,C\varepsilon^{1/4}D(q))}(p). \]

From this inequality and (10.21) we get

\begin{equation}
\int_{B_n(x_0,(8+9\varepsilon^{1/4})r_0)} g(p) dp \geq \int_{q \in B_n(x_0,8r_0)} \int_{\varphi_{\varepsilon^{1/4}D(q)}(p-q)} dp d\sigma(q)
\end{equation}

\begin{align*}
&\quad - \int_{q \in B_n(x_0,8r_0)} \frac{\varepsilon^{1/4} \mathcal{L}^n(B(q,C\varepsilon^{1/4}D(q)))}{(\varepsilon^{1/4}D(q))^n} d\sigma(q) \\
&= (1 - C \varepsilon^{1/4}) \sigma(B_n(x_0,8r_0)) \\
&\geq (1 - C \varepsilon^{1/4}) \mathcal{L}^n(B_n(x_0,8r_0)).
\end{align*}

The last inequality follows from the assumption (a) of Main Lemma 7.1 and the fact that \( \mu(F \setminus \tilde{F}) \lesssim \varepsilon^{1/2} \mu(F) \). Inequality (10.20) is a consequence of (10.22) and the estimate \( \|g\|_\infty \leq 2 \) by (10.14).

The estimate (10.15) is a direct consequence of (10.14) and (10.20):

\begin{align*}
\int_{\Pi(8B_0)} |(1 + C_2 \alpha^2) - g(p)| dp &= \int_{\Pi(8B_0)} ((1 + C_2 \alpha^2) - g(p)) dp \\
&= (1 + C_2 \alpha^2) \mathcal{L}^n(\Pi(8B_0)) - \int_{\Pi(8B_0)} g(p) dp \\
&\leq (C_2 \alpha^2 + C \varepsilon^{1/4}) \mathcal{L}^n(\Pi(8B_0)).
\end{align*}

Thus,

\[ \int_{\Pi(8B_0)} |1 - g(p)| dp \leq (2C_2 \alpha^2 + C \varepsilon^{1/4}) \mathcal{L}^n(\Pi(8B_0)), \]

and so we get (10.15) if \( \varepsilon \) is small enough.
On the other hand, (10.16) is a direct consequence of (10.15):

\[ \int_{\Pi(8B_0)} |1 - g(p)|^2 dp \leq (1 + \|g\|_\infty) \int_{\Pi(8B_0)} |1 - g(p)| dp \leq C \alpha^2 r_0^n. \]

Finally we deal with (10.17): if we argue as in (10.21) and (10.22), with \( B_n(x_0, 8r_0) \setminus B_n(x_0, \frac{999}{1000}r_0) \) instead of \( B_n(x_0, 8r_0) \), we get

\[ \sigma(\Pi(8B_0) \setminus \Pi(\frac{999}{1000}B_0)) \leq \int_{B_n(x_0, (8+9\varepsilon^{1/4})r_0) \setminus B_n(x_0, (1-9\varepsilon^{1/4})\frac{999}{1000}r_0)} g(p) dp + C \varepsilon^{1/4} r_0^n \]

\[ \leq c_n (1 + C\alpha^2)(8^n - \frac{999}{1000})r_0^n + C \varepsilon^{1/4} r_0^n. \]

Since \( \mu(\widetilde{F} \cap B_0) \geq \sigma(\Pi(\frac{999}{1000}B_0)) \) if \( \beta_{\infty, F}(B_0) \) is small enough, we have

\[ \mu(\widetilde{F} \cap B_0) \geq \sigma(\Pi(8B_0)) - \sigma(\Pi(8B_0) \setminus \Pi(\frac{999}{1000}B_0)) \]

\[ \geq c_n 8^n r_0^n - C \varepsilon^{1/2} r_0^n - c_n (1 + C\alpha^2)(8^n - \frac{999}{1000})r_0^n - C \varepsilon^{1/4} r_0^n \]

\[ \geq \frac{99}{100} c_n r_0^n, \]

if \( \alpha \) and \( \varepsilon \) are small enough.

Recall that \( \Pi \) stands for the orthogonal projection of \( \mathbb{R}^d \) onto \( D_0 \equiv \mathbb{R}^n \), and \( \sigma = \Pi_{\#} \mu |_{\widetilde{F}} \). We also denote by \( P \) the projection from \( \mathbb{R}^d \) onto \( \Gamma \) which is orthogonal to \( D_0 \equiv \mathbb{R}^n \). Moreover, for \( x \in \Gamma \) we set

\[ h(x) = \frac{g(\Pi(x))}{JA(\Pi(x))}, \]

so that \( h(x) d\mathcal{H}^n_\Gamma(x) \) is the image measure of \( g(x) dx \) by \( P \).

**Lemma 10.6.** If \( f : \mathbb{R}^d \to \mathbb{R} \) is a function with \( \text{supp}(f) \subset 5B_0 \), then we have

\[ (10.23) \]

\[ \left| \int_{P(5B_0)} f(x) h(x) d\mathcal{H}^n(x) - \int_{5B_0 \cap \widetilde{F}} f(x) d\mu(x) \right| \]

\[ \leq \int_{p \in \Pi(6B_0)} \frac{C}{(\varepsilon^{1/4} D(q))^{n/2}} |f(\widetilde{A}(p)) - f(\widetilde{A}(q))| d\sigma(q) dp \]

\[ + \int_{p \in \Pi(6B_0)} f(\widetilde{A}(p)) b(p) dp \]

\[ + \int_{5B_0 \cap \widetilde{F}} |f(P(x)) - f(x)| d\mu(x), \]

where \( b(p) \) is some function satisfying \( \|b\|_\infty \lesssim \varepsilon^{1/4} \).
Proof. We have
\[
\int_{P(5B_0)} f h d\mathcal{H}^n - \int_{5B_0 \cap F} f d\mu
= \left( \int_{\Pi(5B_0)} f(\tilde{A}(p)) g(p) \, dp - \int_{\Pi(5B_0)} f(\tilde{A}(p)) \, d\Pi_\# \mu|_{\tilde{F}}(p) \right)
+ \left( \int_{5B_0 \cap F} f(P(x)) \, d\mu(x) - \int_{5B_0 \cap F} f(x) \, d\mu(x) \right) =: S + T.
\]
For this identity we took into account that
\[
\int_{\Pi(5B_0)} f(\tilde{A}(p)) \, d\Pi_\# \mu|_{\tilde{F}}(p) = \int_{P(5B_0)} f(x) \, dP_\# \mu|_{\tilde{F}}(x) = \int_{5B_0 \cap F} f(P(x)) \, d\mu(x).
\]
To estimate the term \(S\) we recall that
\[
g(p) = \varphi_{\varepsilon^{1/4}D(p)} * \Pi_\#(\mu|_{\tilde{F}})(p) = \varphi_{\varepsilon^{1/4}D(p)} * \sigma(p),
\]
and so
\[
S = \int_{p \in \Pi(5B_0)} f(\tilde{A}(p)) \varphi_{\varepsilon^{1/4}D(p)}(p - q) \, d\sigma(q) \, dp - \int_{q \in \Pi(6B_0)} f(\tilde{A}(q)) \, d\sigma(q)
= \int_{p \in \Pi(6B_0)} \left[ f(\tilde{A}(p)) - f(\tilde{A}(q)) \right] \varphi_{\varepsilon^{1/4}D(q)}(p - q) \, d\sigma(q) \, dp
+ \int_{p \in \Pi(6B_0)} f(\tilde{A}(p)) \left[ \varphi_{\varepsilon^{1/4}D(p)}(p - q) - \varphi_{\varepsilon^{1/4}D(q)}(p - q) \right] \, d\sigma(q) \, dp
=: S_1 + S_2,
\]
since
\[
\int_{p \in \Pi(6B_0)} \varphi_{\varepsilon^{1/4}D(q)}(p - q) \, dp = 1 \quad \text{for } q \in \text{supp}(f \circ \tilde{A}).
\]
Clearly, we have
\[
|S_1| \lesssim \int_{p \in \Pi(6B_0)} \frac{1}{|p - q| \leq \varepsilon^{1/4}D(q)} \left| f(\tilde{A}(p)) - f(\tilde{A}(q)) \right| \, d\sigma(q) \, dp.
\]
To deal with \(S_2\) we denote
\[
b(p) = \int \left[ \varphi_{\varepsilon^{1/4}D(p)}(p - q) - \varphi_{\varepsilon^{1/4}D(q)}(p - q) \right] \, d\sigma(q).
\]
By Lemma \[10.3\]
\[
|b(p)| \lesssim \frac{\varepsilon^{1/4} \sigma(B(p, C\varepsilon^{1/4}D(q)))}{(\varepsilon^{1/4}D(q))^n} \lesssim \varepsilon^{1/4}.
\]
Concerning the term \(T\), we have
\[
|T| \leq \int_{5B_0 \cap F} \left| f(P(x)) - f(x) \right| \, d\mu(x).
\]
\[\square\]
Lemma 10.7.
\[ \| R_{(\ell, \cdot)}^+ (\mu_{[F \cap 5B_0]}) - R_{(\ell, \cdot)}^+ (h \, \mathcal{H}_n^{\infty} | \Gamma \cap 5B_0) \|_{L^2(\Gamma \cap 4B_0)} \lesssim \varepsilon^{1/4} r_0^{n/2}. \]

Proof. For any \( x \in \Gamma \cap 4B_0 \) we have
\[
B(x) := R_{(\ell, \cdot)}^+ (\mu_{[F \cap 5B_0]}) (x) - R_{(\ell, \cdot)}^+ (h \, \mathcal{H}_n^{\infty} | \Gamma \cap 5B_0) (x)
= \int_{y \in F \cap 5B_0} K_{(\ell, \cdot), r_0} (x - y) \, d\mu(y) - \int_{y \in P(5B_0)} K_{(\ell, \cdot), r_0} (x - y) \, h(y) \, d\mathcal{H}^n(y).
\]
To estimate \( B(x) \) we apply Lemma 10.6 with \( f(y) = K_{(\ell, \cdot), r_0} (x - y), \) for each fixed \( x \in \Gamma \cap 4B_0. \) To simplify notation, we set \( F(x, p) := K_{(\ell, \cdot), r_0} (x - \tilde{A}(p)). \) The first term on the right side of (10.23) for this choice of \( f \) is
\[
B_1(x) := \int_{p \in \Pi(6B_0), \, |p-q| \leq \varepsilon^{1/4} D(q)} \frac{C}{(\varepsilon^{1/4} D(q))^n} \left| F(x, p) - F(x, q) \right| \, d\sigma(q) \, dp.
\]
For \( p, q \) satisfying \( |p - q| \leq \varepsilon^{1/4} D(p) \lesssim \varepsilon^{1/4} D(q) \) we have
\[
\left| F(x, p) - F(x, q) \right| \lesssim \frac{\varepsilon^{1/4} D(q)}{(\varepsilon^{1/4} D(q))^n + \ell(x)^{n+1}}.
\]
Moreover,
\[
D(q) \lesssim 10 \ell(x) + |D(q) - 10 \ell(x)| \lesssim 10 \ell(x) + 10 |\Pi(x) - q| \lesssim \ell(x) + |x - \tilde{A}(q)|,
\]
and \( D(\Pi(x)) = 10 \ell(x) \) and \( D \) is 1-Lipschitz. So we get
\[
\left| F(x, p) - F(x, q) \right| \lesssim \frac{\varepsilon^{1/4} D(q)}{(D(q) + |x - \tilde{A}(q)|)^{n+1}}.
\]
Thus,
\[
|B_1(x)| \lesssim \int_{q \in \Pi(6B_0)} \frac{\varepsilon^{1/4} D(q)}{(D(q) + |x - \tilde{A}(q)|)^{n+1}} \, d\sigma(q)
\lesssim \int_{z \in F \cap 7B_0} \frac{\varepsilon^{1/4} \ell(z)}{(\ell(z) + |x - z|)^{n+1}} \, d\mu(z).
\]
Consider now the operator
\[
(10.24) \quad Sf(x) = \int_{z \in F \cap 7B_0} \frac{\varepsilon^{1/4} \ell(z)}{(\ell(z) + |x - z|)^{n+1}} f(z) \, d\mu(z), \quad x \in \Gamma \cap 4B_0.
\]
It is easy to check that its adjoint satisfies
\[
|S^* f(z)| \lesssim \varepsilon^{1/4} M f(z),
\]
where \( M \) stands for the Hardy-Littlewood maximal operator
\[
M f(z) = \sup_{r > 0} \frac{1}{r^n} \int_{B(z, r) \cap \Gamma \cap 4B_0} |f| \, d\mathcal{H}^n.
\]
which is bounded from $L^2(\Gamma \cap 4B_0)$ into $L^2(\mu_{|\tilde{F} \cap 7B_0})$, and so
\[
\|S^*\|_{L^2(\Gamma \cap 4B_0), L^2(\mu_{|\tilde{F} \cap 7B_0})} \lesssim \varepsilon^{1/4}.
\]
Therefore, $S : L^2(\mu_{|\tilde{F} \cap 7B_0}) \to L^2(\Gamma \cap 4B_0)$ is bounded with norm $\lesssim \varepsilon^{1/4}$ and then
\[
\int_{x \in \Gamma \cap 4B_0} \left( \int_{y \in \tilde{F} \cap 7B_0} \frac{\varepsilon^{1/4} d(y)}{d(y) + |y - x|^{n+1}} d\mu(y) \right)^2 d\mathcal{H}^n(x) \lesssim \varepsilon^{1/2} r_0^n.
\]
Thus,
\[
\|B_1\|_{L^2(\Gamma \cap 4B_0)} \lesssim \varepsilon^{1/4} r_0^{n/2}.
\]
We deal now with the second term on the right hand side of (10.23), with $f(y) = K_{\ell(x),r_0}^+(x - y)$:
\[
B_2(x) := \left| \int_{x \in \Pi(6B_0)} K_{\ell(x),r_0}^+(x - \tilde{A}(p)) b(p) \, d\mu(p) \right|.
\]
By the $L^2$-boundedness of Riesz transforms on $L^2(\Gamma)$ and the fact that $\|b\|_\infty \lesssim \varepsilon^{1/4}$, we get
\[
\|B_2\|_{L^2(\Gamma \cap 4B_0)} \lesssim \|\chi_{\Pi(6B_0)} b\|_2 \lesssim \varepsilon^{1/4} r_0^{n/2}.
\]
Finally we deal with the third term on the right side of (10.23):
\[
B_3(x) = \int_{5B_0 \cap \tilde{F}} |K_{\ell(x),r_0}^+(x - P(y)) - K_{\ell(x),r_0}^+(x - y)| \, d\mu(x).
\]
Since
\[
|\nabla K_{\ell(x),r_0}^+(z)| \lesssim \frac{1}{(\ell(x) + |z|)^{n+1}}
\]
for all $z \in \mathbb{R}^d$, and $|y - P(y)| \leq C \text{dist}(y, \Gamma) \leq C \varepsilon^{1/2} d(y)$ for $y \in \tilde{F}$, we deduce that
\[
|K_{\ell(x),r_0}^+(x - y) - K_{\ell(x),r_0}^+(x - P(y))| \lesssim \frac{|y - P(y)|}{(\ell(x) + |y - x|)^{n+1}} \lesssim \frac{\varepsilon^{1/2} d(y)}{(\ell(x) + |y - x|)^{n+1}}.
\]
Therefore,
\[
|R_{\ell(x),r_0}^+ (\mu_{|\tilde{F} \cap 5B_0}) (x) - R_{\ell(x),r_0}^+(P_{\#} \mu_{|\tilde{F} \cap 5B_0})(x)| \lesssim \int_{y \in \tilde{F} \cap 5B_0} \frac{\varepsilon^{1/2} d(y)}{(\ell(x) + |y - x|)^{n+1}} \, d\mu(y).
\]
Recall that $\ell(x) = 10D(\Pi(x))$, and since $D(\cdot)$ is 1-Lipschitz,
\[
D(\Pi(y)) \lesssim D(\Pi(x)) + |y - x| = 10 \ell(x) + |x - y|.
\]
For $y \in F$, by Lemma 9.17, we infer that $d(y) \approx D(\Pi(y))$, and so
\[
d(y) \lesssim \ell(x) + |x - y|.
\]
Lemma 10.8. We have

\[ \|B_3\|_{L^2(\Gamma \cap 4B_0)} \lesssim \int_{x \in \Gamma \cap 4B_0} \left( \int_{y \in \tilde{F} \cap 5B_0} \frac{\varepsilon^{1/2}d(y)}{(d(y) + |y - x|)^{n+1}} \right) d\mu(y). \]

Let \( T : L^2(\mu_{\tilde{F} \cap 5B_0}) \to L^2(\Gamma \cap 4B_0) \) be the following operator

\[ Tf(x) = \int_{y \in \tilde{F} \cap 5B_0} \frac{\varepsilon^{1/2}d(y)}{(d(y) + |y - x|)^{n+1}} d\mu(y). \]

Arguing as in the case of the operator \( S \) from (10.23), it is easy to check that \( T : L^2(\mu_{\tilde{F} \cap 5B_0}) \to L^2(\Gamma \cap 4B_0) \) is bounded with norm \( \lesssim \varepsilon^{1/2} \) and then

\[ \int_{x \in \Gamma \cap 4B_0} \left( \int_{y \in \tilde{F} \cap 5B_0} \frac{\varepsilon^{1/2}d(y)}{(d(y) + |y - x|)^{n+1}} d\mu(y) \right)^2 d\mathcal{H}^n(x) \lesssim \varepsilon \mathcal{H}^n(\Gamma \cap 4B_0) \lesssim \varepsilon r_0^n. \]

Thus we obtain

\[ \|B_3\|_{L^2(\Gamma \cap 4B_0)} \lesssim \varepsilon^{1/2} r_0^{n/2}. \]

If we add the estimates obtained for \( B_1, B_2 \) and \( B_3 \), the lemma follows. \( \square \)

**Lemma 10.8.** We have

(10.26)

\[ \|R^{1+}_{\ell(\cdot),r_0}(h \, d\mathcal{H}^n_{\Gamma \cap 5B_0}) - R^{1+}_{\ell(\cdot),r_0} \mathcal{H}^n_{\Gamma \cap 5B_0}\|_{L^2(\Gamma \cap 4B_0)} \lesssim \alpha^2 \|\nabla A\|^2 + \alpha^2 r_0^{n/2}. \]

Let us remark that, for the arguments in Lemma 10.11 below, it is important that the last term on the right side of (10.26) is \( \alpha^2 r_0^{n/2} \) instead of \( \alpha r_0^{n/2} \), say.

**Proof.** By Lemma 6.1, we have

\[ \|R^{1+}_{\ell(\cdot),r_0}\|_{L^2(\Gamma \cap 4B_0),L^2(\Gamma \cap 4B_0)} \lesssim \|\nabla A\|_{\infty}. \]

Thus

\[ \|R^{1+}_{\ell(\cdot),r_0}(h \, d\mathcal{H}^n_{\Gamma \cap 5B_0}) - R^{1+}_{\ell(\cdot),r_0} \mathcal{H}^n_{\Gamma \cap 5B_0}\|_{L^2(\Gamma \cap 4B_0)} \lesssim \|\nabla A\|_{\infty} \|h - 1\|_{L^2(\Gamma \cap 6B_0)'.} \]

On the other hand, writing \( p = \Pi(x) \) we have

\[ |h(x) - 1| = \left| \frac{g(p)}{J(A)(p)} - 1 \right| \leq \left| \frac{g(p)}{J(A)(p)} - g(p) \right| + |g(p) - 1|. \]

Recalling that \( \|J(\tilde{A}) - 1\|_2 \leq \|\nabla A\|_{\infty} \|\nabla A\|_2 \) and \( \|\chi_{\Pi(2B_0)}(g - 1)\|_2 \lesssim \alpha r_0^{n/2} \), we get

\[ \|h - 1\|_{L^2(\Gamma \cap 6B_0')} \lesssim \alpha \|\nabla A\|_2 + \alpha r_0^{n/2}, \]

and thus

\[ \|R^{1+}_{\ell(\cdot),r_0}(h \, d\mathcal{H}^n_{\Gamma \cap 5B_0}) - R^{1+}_{\ell(\cdot),r_0} \mathcal{H}^n_{\Gamma \cap 5B_0}\|_{L^2(\Gamma \cap 4B_0)} \lesssim \alpha^2 \|\nabla A\|_2 + \alpha^2 r_0^{n/2}. \]

\( \square \)
10.6. The implication $\| R_{\ell^{(\cdot), r_0} \mu_{\mathcal{F}}(\cdot)} \|_{L^2(\Gamma)} \bigg \|_{L^2(\mu_{\mathcal{F}}) \bigg} \bigg)$ big. Recall that on $4B_0$, the image measure of $P_{\#} \mu_{\mathcal{F}}$ by $\Pi$ coincides with $\sigma$ and that $h d\mathcal{H}_1^n \bigg|_{\Gamma \cap 5B_0} = P_{\#} (g(x) \, dx)$, with $g(x) = (\varphi_{\varepsilon^{1/4}} \rho(x) \ast \sigma)(x)$. We denote $G_1 = \{ p \in \Pi(8B_0) : g(p) > 1/2 \}$, and $G_0 = P(G_1)$.

**Lemma 10.9.** We have

$$\mathcal{H}^n(\Gamma \cap 6B_0 \setminus G_0) \lesssim \alpha^2 r_0^n.$$  

**Proof.** By (10.15) we have

$$\int_{\Pi(8B_0)} |g - 1| \, dx \leq C \alpha^2 r_0^n.$$  

Thus,

$$\mathcal{L}^n(\Pi(8B_0) \setminus G_1) \leq \mathcal{L}^n \{ p \in \Pi(8B_0) : |g(p) - 1| > 1/2 \} \leq 2 \int_{\Pi(8B_0)} |g - 1| \, dp \leq C \alpha^2 r_0^n.$$  

It is clear that then we also have

$$\mathcal{H}^n(\Gamma \cap 6B_0 \setminus P(G_1)) \lesssim \alpha^2 r_0^n.$$  

\[ \square \]

**Lemma 10.10.**

$$\| R_{\ell^{(\cdot), r_0} (\mu_{\mathcal{F} \cap 5B_0})} \|_{L^2(\Gamma \cap 4B_0 \cap G_0)} \lesssim \| R_{\ell^{(\cdot), r_0} (\mu_{\mathcal{F} \cap 5B_0})} \|_{L^2(\mu_{\mathcal{F} \cap 4B_0})} + \varepsilon^{1/8} r_0^n/2.$$  

**Proof.** We denote $f(x) = R_{\ell^{(\cdot), r_0} (\mu_{\mathcal{F} \cap 5B_0})}(x)$. Since $h(x) > 1/3$ on $G_0$, we have

$$\| f \|_{L^2(\Gamma \cap 4B_0 \cap G_0)}^2 \leq 3 \int_{\Gamma \cap 4B_0} |f|^2 \, h \, d\mathcal{H}^n$$  

$$\leq 3 \int_{\Gamma \cap 4B_0} |f|^2 \, h \, d\mathcal{H}^n - \int_{\mathcal{F} \cap 4B_0} |f|^2 \, d\mu + 3 \int_{\mathcal{F} \cap 4B_0} |f|^2 \, d\mu.$$  

To prove the lemma it is enough to show that

$$(10.27) \quad I := \int_{\Gamma \cap 6B_0} |f|^2 \, h \, d\mathcal{H}^n - \int_{\mathcal{F} \cap 6B_0} |f|^2 \, d\mu \lesssim \varepsilon^{1/8} r_0^n/2.$$  

To this end we will use Lemma [10.7] with $|f|^2$ instead of $f$, and with $6B_0$ replacing $5B_0$, and $7B_0$ replacing $6B_0$. Notice that $\text{supp}(f) \subset 6B_0$. It is
clear that Lemma 10.6 also holds in this situation. So we have
\[
I \leq C \int_{p \in \Pi(7B_0)} \frac{1}{(\varepsilon^{1/4} D(q))^\pi} \left| f(\tilde{A}(p)) \right|^2 - \left| f(\tilde{A}(q)) \right|^2 \, d\sigma(q) \, dp \\
+ \left| f(\tilde{A}(p)) \right|^2 b(p) \, dp + \int_{7B_0 \cap \tilde{F}} \left| f(P(x)) \right|^2 - \left| f(x) \right|^2 \, d\mu(x) \\
=: C I_1 + I_2 + I_3,
\]
where \( b(p) \) is some function satisfying \( \|b\|_\infty \lesssim \varepsilon^{1/4} \).

First we estimate \( I_1 \). Setting
\[
(10.28) \quad \left| f(\tilde{A}(p)) \right|^2 - \left| f(\tilde{A}(q)) \right|^2 \leq \left| f(\tilde{A}(p)) - f(\tilde{A}(q)) \right| \times (\left| f(\tilde{A}(p)) \right| + \left| f(\tilde{A}(q)) \right|)
\]
and applying Cauchy-Schwartz, we get
\[
(10.29) \quad I_1 \leq \left( \int_{p \in \Pi(7B_0)} \frac{1}{(\varepsilon^{1/4} D(q))^\pi} \left| f(\tilde{A}(p)) - f(\tilde{A}(q)) \right|^2 d\sigma(q) \, dp \right)^{1/2} \\
\times \left( \int_{p \in \Pi(7B_0)} \frac{1}{(\varepsilon^{1/4} D(q))^\pi} (\left| f(\tilde{A}(p)) \right| + \left| f(\tilde{A}(q)) \right|)^2 d\sigma(q) \, dp \right)^{1/2} \\
=: I_{1,1}^{1/2} \times I_{1,2}^{1/2}.
\]
To estimate \( I_{1,1} \) notice that if \( |p - q| \leq \varepsilon^{1/4} D(q) \), then
\[
(10.30) \quad \left| f(\tilde{A}(p)) - f(\tilde{A}(q)) \right| = \left| R_{\xi(\cdot),r_0 \mu_5 B_0 \cap \tilde{F}}(\tilde{A}(p)) - R_{\xi(\cdot),r_0 \mu_5 B_0 \cap \tilde{F}}(\tilde{A}(q)) \right| \lesssim \varepsilon^{1/4}.
\]
For this inequality notice \( D(p) \approx D(q) \) because \( |p - q| \leq \varepsilon^{1/4} D(q) \), and recall also that \( \ell(x) = 10D(\Pi(x)) \). We leave the details for the reader. Therefore,
\[
I_{1,1} = \int_{p \in \Pi(7B_0)} \frac{1}{(\varepsilon^{1/4} D(q))^\pi} \left| f(\tilde{A}(p)) - f(\tilde{A}(q)) \right|^2 d\sigma(q) \, dp \\
\lesssim \varepsilon^{1/2} \int_{p \in \Pi(7B_0)} \frac{1}{(\varepsilon^{1/4} D(q))^\pi} d\sigma(q) \, dp \lesssim \varepsilon^{1/2} r_0^n.
\]
To deal with \( I_{1,2} \) we set
\[
(10.31) \quad I_{1,2}^{1/2} \leq \left( \int_{p \in \Pi(7B_0)} \frac{1}{(\varepsilon^{1/4} D(q))^\pi} \left| R_{\xi(\cdot),r_0 \mu_5 B_0 \cap \tilde{F}}(\tilde{A}(p)) \right|^2 d\sigma(q) \, dp \right)^{1/2} \\
+ \left( \int_{p \in \Pi(7B_0)} \frac{1}{(\varepsilon^{1/4} D(q))^\pi} \left| R_{\xi(\cdot),r_0 \mu_5 B_0 \cap \tilde{F}}(\tilde{A}(q)) \right|^2 d\sigma(q) \, dp \right)^{1/2} \\
=: I_{1,2,a}^{1/2} + I_{1,2,b}^{1/2}.
\]
Concerning $I_{1,2,a}$, we have

\[
I_{1,2,a}^{1/2} \lesssim \left( \int_{p \in \Pi(7B_0)} |R_{\ell(\cdot),r_0}^1 \mu|_{5B_0 \cap \bar{F}}(\widetilde{A}(p))|^2 \, dp \right)^{1/2} \lesssim r_0^{n/2},
\]

by the $L^2$ boundedness of Riesz transforms from $L^2(\mu_{\bar{F}})$ into $L^2(\Gamma)$. For the last integral in the right side of (10.31) we take into account that $D(p) \approx D(q)$, and then we get

\[
(10.32)
\]

\[
I_{1,2,b}^{1/2} \lesssim \left( \int_{q \in \Pi(7.5B_0)} |R_{\ell(\cdot),r_0}^1 \mu|_{5B_0 \cap \bar{F}}(\widetilde{A}(q))|^2 \, d\sigma(q) \right)^{1/2}
\]

\[
= C \left( \int_{q \in \Pi(7.5B_0)} |f(\widetilde{A}(q))|^2 \, d\sigma(q) \right)^{1/2} \leq C \left( \int_{y \in 8B_0} |f(P(y))|^2 \, d\mu_\bar{F}(y) \right)^{1/2}
\]

\[
\leq C \left( \int_{y \in 8B_0} |f(P(y)) - f(y)|^2 \, d\mu_\bar{F}(y) \right)^{1/2} + C \left( \int_{y \in 8B_0} |f(y)|^2 \, d\mu_\bar{F}(y) \right)^{1/2}.
\]

Using the $L^2(\mu_{\bar{F}})$ boundedness of Riesz transforms, the last integral is $\lesssim r_0^n$. For the first one we argue as in (10.30): given $y \in \bar{F}$, we have $|y - P(y)| \lesssim \varepsilon^{1/2} d(y) \approx \varepsilon^{1/2} \ell(y)$, and then it easily follows that

\[
|f(P(y)) - f(y)| = |R_{\ell(\cdot),r_0}^1 \mu|_{5B_0 \cap \bar{F}}(P(y)) - R_{\ell(\cdot),r_0}^1 \mu|_{5B_0 \cap \bar{F}}(y)| \lesssim \varepsilon^{1/2}.
\]

Therefore, the first term on the right side of (10.32) is bounded above by $\varepsilon^{1/2} r_0^{n/2}$, and so $I_{1,2,b}^{1/2} \lesssim r_0^{n/2}$, and thus $I_{1,2}^{1/2} \lesssim r_0^{n/2}$. Recalling that $I_{1,1} \leq \varepsilon^{1/2} r_0^n$, we deduce that

\[
I_1 \lesssim \varepsilon^{1/4} r_0^n.
\]

To estimate the integral $I_2$ we use the fact that $\|b\|_\infty \lesssim \varepsilon^{1/4}$, and so

\[
I_2 \lesssim \varepsilon^{1/4} \int_{x \in \Pi(7B_0)} |f(\widetilde{A}(p))|^2 \, dp.
\]

The last integral is similar to $I_{1,2,a}$, and thus we have

\[
I_2 \lesssim \varepsilon^{1/4} r_0^n.
\]

To deal with $I_3$ we argue as in (10.28), and similarly to (10.29), we infer that

\[
(10.33) \quad I_3 \leq \left( \int_{x \in 7B_0 \cap \bar{F}} |f(P(x)) - f(x)|^2 \, d\mu(x) \right)^{1/2}
\]

\[
\times \left( \int_{x \in 7B_0 \cap \bar{F}} |f(P(x))| + |f(x)|^2 \, d\mu(x) \right)^{1/2} =: I_{3,1}^{1/2} \times I_{3,2}^{1/2}.
\]
The integral $I_{3,1}$ is similar to the first one on right side of (10.32), and so we have $I_{3,1} \lesssim \varepsilon^{1/4} r_0^n$. For $I_{3,2}$ we set

$$I_{3,2} \leq \left( \int_{x \in 7B_0 \cap F} |f(x)|^2 \, d\mu(x) \right)^{1/2} + \left( \int_{x \in 7B_0 \cap F} |f(P(x))|^2 \, d\mu(x) \right)^{1/2}.$$ 

The first term on the right side is bounded above by $C r_0^{n/2}$, by the $L^2(\mu_{\bar{F}})$ boundedness of Riesz transforms, and for the second one we write

$$S := \left( \int_{x \in 7B_0 \cap F} |f(P(x))|^2 \, d\mu(x) \right)^{1/2} \leq \int_{x \in 7B_0 \cap F} |f(x)|^2 \, d\mu(x) \right)^{1/2} + \int_{x \in 7B_0 \cap F} |f(P(x)) - f(x)|^2 \, d\mu(x) \right)^{1/2}.$$

As above, the first term satisfies $\lesssim r_0^{n/2}$, and the second one coincides with $I_{3,1}$, and so we have $S \lesssim r_0^{n/2}$. Thus, $I_{3,2} \lesssim r_0^{n/2}$, and then $I_3 \lesssim \varepsilon^{1/4} r_0^n$.

If we gather the estimates obtained for $I_1$, $I_2$ and $I_3$, (10.27) follows and we are done.

10.7. The proof that $F_3$ is small.

**Lemma 10.11.** We have

$$\mu(F_3) \leq \alpha^{1/2} \mu(F).$$

**Proof.** We will use all the results obtained Subsections 10.2-10.6. From (10.3) and Lemma 10.2, we deduce

$$\|\nabla A\|_2 \lesssim \|R^+_{\ell(\cdot), r_0} (H_{1 \cap S B_0}^n)\|_{L^2(\Gamma \cap 4B_0)} + C \alpha^{2} r_0^{n/2}. \tag{10.34}$$

By Lemma 10.9 and since $R^+_{\ell(\cdot), r_0}$ is bounded in $L^4(\Gamma)$ with norm $\lesssim \|\nabla A\|_\infty \lesssim \alpha$, we deduce

$$\|R^+_{\ell(\cdot), r_0} (H_{1 \cap S B_0}^n)\|_{L^2(\Gamma \cap 4B_0 \setminus G_0)} \lesssim \|R^+_{\ell(\cdot), r_0} (H_{1 \cap S B_0}^n)\|_{L^4(\Gamma \cap 4B_0) \setminus G_0)}^{1/2} \lesssim \alpha^{2} \mathcal{H}^n(\Gamma \cap 4B_0 \setminus G_0)^{1/2} \lesssim \alpha^{3} r_0^n.$$

From this inequality and (10.34) we derive

$$\|\nabla A\|_2 \lesssim \|R^+_{\ell(\cdot), r_0} (H_{1 \cap S B_0}^n)\|_{L^2(\Gamma \cap 4B_0 \cap G_0)} + \alpha^{3/2} \mu(F)^{1/2},$$

since $\alpha \ll 1$. This estimate and Lemmas 10.7 and 10.8 imply that

$$\|\nabla A\|_2 \lesssim \|R^+_{\ell(\cdot), r_0} \mu_{\bar{F} \cap 5B_0}\|_{L^2(\Gamma \cap 4B_0 \cap G_0)} + \alpha^{2} \|\nabla A\|_2 + (\alpha^{3/2} + \alpha^2 + \varepsilon^{1/4}) \mu(F)^{1/2}.$$ 

Thus, if $\alpha$ is small enough and $\varepsilon^{1/4} \leq \alpha^{3/2}$, we get

$$\|\nabla A\|_2 \lesssim \|R^+_{\ell(\cdot), r_0} \mu_{\bar{F} \cap 5B_0}\|_{L^2(\Gamma \cap 4B_0 \cap G_0)} + \alpha^{3/2} \mu(F)^{1/2}.$$
Together with Lemma 10.1, this implies that
\begin{equation}
\mu(F_3) \lesssim \alpha^{-2} \|\nabla A\|_2^{1/2} \varepsilon^{1/2} \mu(F) \lesssim \alpha^{-2} \|R^{\perp}_{\ell,\cdot} \mu_{|\tilde{F} \cap 5B_0}\|_{L^2(\Gamma \cap 4B_0 \cap G_0)} + \alpha \mu(F).
\end{equation}
Recall that by Lemma 10.10, we have
\begin{equation}
\|R^{\perp}_{\ell,\cdot} \mu_{|\tilde{F} \cap 5B_0}\|_{L^2(\Gamma \cap 4B_0 \cap G_0)} \lesssim \|R^{\perp}_{\ell,\cdot} \mu_{|\tilde{F} \cap 5B_0}\|_{L^2(\mu_{|\tilde{F} \cap 4B_0})} + \varepsilon^{1/8} r_0^{n/2}.
\end{equation}
From this estimate and (10.35) we deduce that
\begin{equation}
\mu(F_3) \lesssim \alpha^{-2} \|R^{\perp}_{\ell,\cdot} \mu_{|\tilde{F} \cap 5B_0}\|_{L^2(\mu_{|\tilde{F} \cap 4B_0})} + \alpha \mu(F)
\end{equation}
(assuming always \(\varepsilon \ll \alpha \ll 1\)).

Now we denote
\[B_1 = \{ x \in \tilde{F} : R_* \mu_{|5B_0} \cdot \tilde{F}(x) > \varepsilon^{1/4} \}.
\]
By the boundedness of Riesz transforms from \(M(\mathbb{R}^d)\) (the space of finite Borel measures on \(\mathbb{R}^d\)) into \(L^{1,\infty}(\mu_{|\tilde{F}})\), we get
\[\mu(B_1) \lesssim \frac{\mu(5B_0 \setminus \tilde{F})}{\varepsilon^{1/4}} \lesssim \frac{\varepsilon^{1/2} \mu(\tilde{F})}{\varepsilon^{1/4}} = \varepsilon^{1/4} \mu(\tilde{F}).
\]
By Cauchy-Schwartz and the \(L^2(\mu_{|\tilde{F}})\) boundedness of Riesz transforms we get
\[\|R^{\perp}_{\ell,\cdot} \mu_{|\tilde{F} \cap 5B_0}\|_{L^2(\mu_{|B_1})}^2 \leq \|R^{\perp}_{\ell,\cdot} \mu_{|\tilde{F} \cap 5B_0}\|_{L^2(\mu_{|B_1})}^2 \mu(B_1)^{1/2} \lesssim \varepsilon^{1/8} \mu(F).
\]
On the other hand, from (10.36) we infer that
\[\|R^{\perp}_{\ell,\cdot} \mu_{|\tilde{F} \cap 5B_0}\|_{L^2(\mu_{|\tilde{F} \cap 4B_0})}^2 \geq C^{-1} \alpha^2 \left[ \mu(F_3) - C \alpha \mu(F) \right].
\]
Suppose that \(\mu(F_3) > \alpha^{1/2} \mu(F)\). Then \(\mu(F_3) - C \alpha \mu(F) \gtrsim \alpha^{1/2} \mu(F)\), and by the preceding estimates we get
\[\|R^{\perp}_{\ell,\cdot} \mu_{|\tilde{F} \cap 5B_0}\|_{L^2(\mu_{|\tilde{F} \cap 4B_0})} \geq C^{-1} \alpha^{5/2} \mu(F) \geq \frac{1}{2} \|R^{\perp}_{\ell,\cdot} \mu_{|\tilde{F} \cap 5B_0}\|_{L^2(\mu_{|B_1})}^2,
\]
because \(\varepsilon^{1/8} \ll \alpha^{5/2}\). Therefore,
\[\|R^{\perp}_{\ell,\cdot} \mu_{|\tilde{F} \cap 5B_0}\|_{L^2(\mu_{|\tilde{F} \cap 4B_0} \setminus B_1)}^2 = \|R^{\perp}_{\ell,\cdot} \mu_{|\tilde{F} \cap 5B_0}\|_{L^2(\mu_{|\tilde{F} \cap 4B_0})}^2 - \|R^{\perp}_{\ell,\cdot} \mu_{|\tilde{F} \cap 5B_0}\|_{L^2(\mu_{|B_1})}^2 \geq \frac{1}{2} \|R^{\perp}_{\ell,\cdot} \mu_{|\tilde{F} \cap 5B_0}\|_{L^2(\mu_{|\tilde{F} \cap 4B_0} \setminus B_1)}^2 \gtrsim \alpha^{5/2} \mu(F).
\]
Since \(R_* \mu_{|5B_0} \cdot \tilde{F}(x) \leq \varepsilon^{1/4}\) on \(\tilde{F} \setminus B_1\), we have
\[\|R^{\perp}_{\ell,\cdot} \mu_{|5B_0} \cdot \tilde{F}\|_{L^2(\mu_{|\tilde{F} \cap 4B_0} \setminus B_1)} \leq \varepsilon^{1/2} \mu(4B_0).
\]
Thus we deduce that
\[\|R^{\perp}_{\ell,\cdot} \mu_{|5B_0} \cdot \tilde{F}\|_{L^2(\mu_{|\tilde{F} \cap 4B_0} \setminus B_1)} \gtrsim (C^{-1} \alpha^{5/2} - C \varepsilon^{1/2}) \mu(F) \gtrsim \alpha^{5/2} \mu(F).
\]
Since $R_{\ell, r_0}^\perp(\mu|_{5B_0})(x) = R_{\ell, r_0}^\perp(\mu)(x)$ for any $x \in 4B_0$, we have
\[\|R_{\ell, r_0}^\perp(\mu)(x)\|_{L^2(\mu|_{\tilde{\mathcal{E}} \cap 4B_0 \setminus B_1})} \gtrsim \alpha^{5/2} \mu(F),\]
which contradicts the assumption (d) in Main Lemma 7.1. □

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