On reduced models for superstrings on $AdS_n \times S^n$

M. Grigoriev\textsuperscript{a}\textsuperscript{1} and A.A. Tseytlin\textsuperscript{b,a}\textsuperscript{2}

\textsuperscript{a} Tamm Theory Department, Lebedev Physical Institute, Leninsky 53, Moscow 119991, Russia
\textsuperscript{b} Blackett Laboratory, Imperial College, London SW7 2AZ, U.K.

Abstract

We review the Pohlmeyer reduction procedure of the superstring sigma model on $AdS_n \times S^n$ leading to a gauged WZW model with an integrable potential coupled to 2d fermions. In particular, we consider the case of the Green-Schwarz superstring on $AdS_3 \times S^3$ supported by RR flux. The bosonic part of the reduced model is given by the sum of the complex sine-Gordon Lagrangian and its sinh-Gordon counterpart. We determine the corresponding fermionic part and discuss possible existence of hidden 2d supersymmetry in the reduced action. We also elaborate on some general aspects of the Pohlmeyer reduction applied to the $AdS_5 \times S^5$ superstring.

1 Introduction

Further progress in understanding AdS/CFT correspondence requires solving the superstring theory on $AdS_5 \times S^5$. Being an essentially nonlinear theory (IIB Green-Schwarz superstring on $PSU(2,2|4)/SO(4,1|4) \times SO(5)$ supercoset \cite{1}) this theory is difficult to quantize directly. By analogy with the flat space GS superstring one can try to utilize an appropriate version of a light-cone gauge, but that does not simplify the action and, in contrast to the flat space case, breaks 2d Lorentz invariance. The lack of 2d Lorentz invariance makes it hard to apply directly the known results and methods of 2d integrable field theory. In particular, the $S$-matrix of scattering of string fluctuations in a light-cone gauge is not 2d Lorentz invariant and constraints on it are a priori unclear.

An alternative approach \cite{3, 2} is to use a version of the Pohlmeyer “reduction” \cite{4} which allows one to reformulate the theory in terms of physical degrees of freedom only. It is based on writing the equations of motion in terms of the coset currents, solving explicitly the Virasoro constraints by introducing a new set of fundamental variables algebraically related to the currents and then reconstructing the action for the new independent variables. Remarkable features of the Pohlmeyer-reformulated theory for the GS $AdS_5 \times S^5$ model are the explicit 2d Lorentz invariance and the standard kinetic term for the fermions. As in the purely bosonic case \cite{5}, the $AdS_5 \times S^5$ Pohlmeyer reduction preserves the integrable structure – the reduced theory is an integrable deformation of a gauged WZW model by an extra potential term, i.e. a special case of non-abelian Toda theory. In addition, it contains fermionic
terms and thus resemble a 2d supersymmetric generalization of the gauged WZW model. In an appropriate free-theory limit the reduced action coincides with the pp-wave action for 8+8 massive degrees of freedom \[6\] (which in turn generalizes the flat space light cone gauge action).

The hope is that this reduced theory for the \(AdS_5 \times S^5\) superstring should be the starting point for its quantization. There are still a number of open problems at the classical level (the interpretation of conserved charges, choice of vacuum, fixing the residual gauge symmetry, existence of world sheet supersymmetry, etc.) remain to be explored further. This suggests to study first simpler low-dimensional analogs, i.e. \(AdS_n \times S^n\) GS models with \(n = 2, 3\). In the \(AdS_2 \times S^2\) case the reduced theory happens to be very simple and can be identified with the \(N = 2\) 2d supersymmetric extension of the sine-Gordon model \[2\].

Here we shall address the next non-trivial case of the \(AdS_3 \times S^3\) superstring. The corresponding GS superstring action \[7, 8\] is slightly different in the structure from that in the \(AdS_5 \times S^5\) and \(AdS_2 \times S^2\) cases. As a result, the reduction scheme used in \[2\] requires some modification. Since the bosonic part of the \(AdS_3 \times S^3\) sigma model is a principal chiral model defined on the group space \(G = SU(1, 1) \times SU(2)\), this requires to understand how to do the Pohlmeyer reduction in the case where the target space is a group manifold.

The Pohlmeyer reduction of the \(F/G\) coset sigma model is based on using the \(G\) gauge symmetry. One can formally describe the principal chiral model also as a coset one by representing \(G\) as a symmetric space \(G \times G/G\) where the denominator subgroup is embedded diagonally (see also \[9\]). The Pohlmeyer reduced theory can then be identified with the \(G/H\) gauged WZW model with a potential, with \(H\) being a subgroup corresponding to the Cartan subalgebra \(h\) of \(g\).

Below we shall show how this procedure can be applied to the GS superstring on \(AdS_3 \times S^3\). Compared to the \(AdS_5 \times S^5\) case in \[2\] the only nontrivial ingredient is the explicit realization of the \(Z_4\) grading of the \(psu(1, 1|2) \oplus psu(1, 1|2)\) superalgebra.

In Section \[2\] we shall give an algebraic construction of the Pohlmeyer reduction for a principal chiral model.

In Section \[3\] we shall explicitly identify the \(Z_4\) grading on \(psu(1, 1|2) \oplus psu(1, 1|2)\) superalgebra and perform the Pohlmeyer reduction of the corresponding superstring sigma model. The main motivation is to see if the resulting reduced sigma model has \(N = 2\) 2d supersymmetry as we found earlier in the \(AdS_2 \times S^2\) case. The conjectured presence of world sheet supersymmetry in the \(AdS_3 \times S^3\) and also \(AdS_5 \times S^5\) cases would be quite surprising since it is absent in the original Green-Schwarz action in which fermions are 2d scalars and have an unusual kinetic term. Unfortunately, the presence of 2d supersymmetry is not apparent in the reduced \(AdS_3 \times S^3\) action we derive below.

In Section \[4\] we shall make some general comments on the reduced model: its relation to original model, conserved charges, vacuum configuration and perturbative expansion near it.

2 Pohlmeyer reduction for strings on a group manifold

The principal chiral model (PCM) for a simple group \(G\) can be represented as a coset sigma model for

\[
\frac{F}{G} = \frac{G \times G}{G},
\]

(2.1)
where \( \bar{G} \cong G \) is a subgroup of \( G \times G \). In general, we can represent elements of \( F = G \times G \) as pairs \((g_1, g_2)\). The denominator subgroup \( \bar{G} \) is chosen to be the twisted diagonal subgroup, i.e. the subgroup of \((g, \bar{\chi}(g))\), where \( \bar{\chi} \) is an automorphism of \( G \) compatible with the invariant bilinear form \( \text{Tr} \) on Lie algebra \( \mathfrak{g} \).\(^3\) The standard example is when \( \bar{\chi} \) is an identity so that \( \bar{G} \) is embedded diagonally. Another useful choice of \( \bar{\chi} \) is when \( G \) is defined in a matrix representation so that the transposition \( ^t \) is an anti-automorphism of \( G \) (i.e. \( a^t \) belongs to \( G \) for any \( a \in G \) and \( (ab)^t = b^t a^t \)): then one can set \( \bar{\chi}(a) = (a^t)^{-1} \).

Let the pair \((a, b)\) with \( a, b \in \mathfrak{g} \) denote an element of Lie algebra \( \mathfrak{f} \) of \( F \). The invariant bilinear form on \( \mathfrak{g} \) induces that on \( \mathfrak{f} = \mathfrak{g} \oplus \mathfrak{g} \). Then subalgebra \( \bar{\mathfrak{g}} \subset \mathfrak{f} \) which is the Lie algebra of \( \bar{G} \) is isomorphic to \( \mathfrak{g} \) and is formed by \((a, \chi(a))\) where \( \chi \) is the Lie algebra automorphism induced by \( \bar{\chi} \). Because \( \chi \) is compatible with the \( \text{Tr} \), i.e. \( \text{Tr}(\chi(a)\chi(b)) = \text{Tr}(ab) \), the orthogonal complement \( \mathfrak{p} \) of \( \mathfrak{g} \) in \( \mathfrak{f} \) is formed by elements \((a, -\chi(a))\). Homogeneous space \( (2.1) \) is, in fact, a symmetric space:

\[
[\bar{\mathfrak{g}}, \bar{\mathfrak{g}}] \subset \bar{\mathfrak{g}}, \quad [\bar{\mathfrak{g}}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \bar{\mathfrak{g}}, \quad \mathfrak{f} = \bar{\mathfrak{g}} \oplus \mathfrak{p}.
\]

In particular, in the case where \( \bar{\chi}(g) = (g^t)^{-1} \) the corresponding Lie algebra automorphism is \( \chi(a) = -a^t \); the subalgebra \( \bar{\mathfrak{g}} \) is then formed by \((a, -a^t)\) while \( \mathfrak{p} \) is formed by \((a, a^t)\).

The \( F/G \) coset sigma model is defined by the Lagrangian \((f \in F)\)

\[
L = -\frac{1}{2} \text{Tr}(P_a P^a), \quad P_a = (f^{-1} \partial_a f)_{\mathfrak{p}}.
\]

In the above case of \( (2.1) \) it is equivalent to the standard principal chiral field model. Indeed, using the gauge freedom one can always set \((g_1, g_2) = (g, 1)\). In this gauge the above Lagrangian becomes

\[
L = -\frac{1}{8} \text{Tr}(g^{-1} \partial_a g g^{-1} \partial^a g).
\]

Having identified the PCM as a special coset model one can attempt to perform its Pohlmeyer reduction. One option is to treat it as a classical 2d field theory and use the conformal symmetry to fix the components of the stress tensor \( T_{++} = \mu^2 \), \( T_{--} = \mu^2 \). Another one is to consider strings on \( G \times \mathbb{R}_t \); then the conditions \( T_{++} = \mu^2 \), \( T_{--} = \mu^2 \) will emerge as the Virasoro constraints in the conformal gauge supplemented by the \( t = \mu T \) condition fixing the residual conformal diffeomorphisms.

The Pohlmeyer reduction (see, e.g., [2] for an exposition of the general scheme) ammounts to using the \( \bar{G} \) gauge freedom to fix one component of \( P_a \)\(^4\)

\[
P_+ = \mu T,
\]

where \( T \) is a particular element of \( \mathfrak{p} \), i.e. \( T = (t, -\chi(t)), t \in \mathfrak{g} \). One can then parametrize \( P_- \) as

\[
P_- = \mu \bar{g}^{-1} T \bar{g},
\]

\(^3\)Although the reduced theory does not depend on \( \bar{\chi} \) in the bosonic case, the formulation of GS supercoset sigma model requires nontrivial \( \bar{\chi} \). That is why we keep here \( \bar{\chi} \) for generality.

\(^4\)This construction of the reduced model is not unique in the case when the coset space \( F/\bar{G} \) has rank bigger than one, i.e. \( \text{rank}(F) - \text{rank}(G) = 2, 3, \ldots \) [10]. Since the case of our prime interest \((AdS_n \times S^n)\) is based on rank one cosets, here we shall discuss only this “canonical” choice. Let us note, however, that in order to apply the Pohlmeyer type reduction to the PCM with \( G \) of rank \( > 1 \) one also needs to fix values of other Casimirs besides \( \text{Tr}(P_+ P_+) \) and \( \text{Tr}(P_- P_-) \). This more general reduction procedure [10] may be useful in studying special solutions of such models.
where \( \bar{g} \) is a new field taking values in \( \bar{G} \), i.e. having the form \((g, \bar{\chi}(g))\). Then the original \( \bar{G} \) gauge symmetry \((f \to kf, \ k \in \bar{G})\) is broken to the \( \bar{H} \) gauge symmetry, where \( \bar{H} \subset \bar{G} \) is a subgroup of elements preserving \( T \): the corresponding subalgebra \( \bar{h} \) is the centralizer of \( T \) in \( \bar{g} \).

In addition, the introduction of \( \bar{g} \) brings in the new gauge symmetry: \( \bar{g} \) and \( \bar{h}\bar{g} \) with \( \bar{h} \in \bar{H} \) represent the same \( P_\pm \). The resulting (on-shell) formulation should thus have \( \bar{H} \times \bar{H} \) gauge symmetry, \( \bar{g} \to \bar{h}\bar{g}\bar{h}' \).

For a compact group \( G \) one can assume \( t \) in \( T = (t, -\chi(t)) \) to be a nonvanishing element of the Cartan subalgebra of \( g \). The centraliser of \( t \) in \( g \) is the Cartan subalgebra \( h \). The centralizer of \( T \) in \( \bar{g} \) is then the same Cartan subalgebra embedded (twisted diagonally) into \( \bar{g} \cong g \).

In addition to the field \( \bar{g} \) in (2.6) one finds also the 2d gauge field components \( \bar{A}_\pm \) and \( \bar{A}_\mp \) taking values in \( \bar{h} \) and transforming under the gauge groups (we shall assume the standard vector gauging here) – they emerge from the other components of the current \( f^{-1}df \). After a partial gauge fixing the Pohlmeyer-reduced system for the PCM is then represented by \( G/\bar{H} \) gauged WZW model with a potential. For a given automorphism \( \chi \), the fields \( \bar{g} = (g, \bar{\chi}(g)) \) and \( \bar{A}_\pm = (A_\pm, \chi(A_\pm)) \) are uniquely determined by their first components; it is useful to describe the reduced model in terms of \( g \in G \) and \( A_\pm \in h \) (the action does not depend on \( \chi \)):

\[
L_r = -\frac{1}{2} \text{Tr}(g^{-1}\partial_- g g^{-1}\partial_+ g) + \text{WZ term} - \mu^2 \text{Tr}(g^{-1}tg t) + \text{Tr} (- A_+ \partial_- g g^{-1} + A_- g^{-1} \partial_+ g + g^{-1} A_+ g A_- - A_+ A_-).
\]

Here \( A_\pm \) take values in the Cartan subalgebra \( h \subset g \) and \( t \in g \) is a fixed element of \( h \). The corresponding model is the “homogeneous sine-Gordon” model [11] that was studied in the literature [12].

The first nontrivial example is given by \( G = SO(3) \). In this case the reduced Lagrangian (2.7) leads to the complex sine-Gordon (CSG) model after eliminating the auxiliary fields \( A_\pm \). The CSG model [4] is known to be the reduced theory for the coset \( S^3 = SO(4)/SO(3) \). The equivalence is obvious if one uses the representation \( so(4) \cong so(3) \oplus so(3) \).

Let us note that the PCM for \( G = SO(3) \) subject to the Virasoro constraints (i.e. the reduced model for strings on \( S^3 \times \mathbb{R}^1 \)) also admits an alternative Fadeev-Reshetikhin reduction [13]. The FR theory is formulated in terms of two unit 3-vectors or 4 independent variables (related locally to the original current components) and is described by a first-order action. It thus has the same number of degrees of freedom (two in a second-derivative form) as in the CSG model. However, in contrast to CSG, the FR model is not explicitly 2d Lorentz invariant. The CSG and the FR models which are both related to the same PCM equations of motion with the Virasoro constraints imposed should then be related by a (nonlocal) field redefinition[3]

In the next section we shall consider the reduced model for the superstring on \( AdS_3 \times S^3 \). The bosonic part of the \( AdS_3 \times S^3 \) superstring sigma model [8] is the direct sum of the coset models of the type (2.7), i.e. \( AdS_3 \times S^3 \cong SU(1,1) \times SU(2) \) can be represented as a coset (2.7) with \( G = SU(1,1) \times SU(2) \).

---

[3] It might be possible to consider the FR and CSG models as originating from two different gauges of the \( G \times G \) coset sigma model.
3 Superstring theory on $AdS_3 \times S^3$

The Green-Schwarz superstring on $AdS_3 \times S^3$ supported by RR 3-form flux can be formulated as a coset model for the supercoset $[7, 8]$

\[
\frac{PSU(1,1|2) \times PSU(1,1|2)}{SU(2) \times SU(1,1)}
\] (3.1)

The superalgebra $psu(1, 1|2)$ of $PSU(1, 1|2)$ is represented by $(2|2) \times (2|2)$ traceless supermatrices satisfying an appropriate reality condition; the quotient is over the central subalgebra generated by the unit matrix (for details see, e.g., [2]). This algebra (as well as $psu(2, 2|4)$ and its higher-dimensional analogs) admits a $Z_4$-grading [14]. This grading appears to be extremely useful in studying such sigma-models and their Pohlmeyer-type reductions. In particular, the formulation of superstrings on $AdS_2 \times S^2$ or $AdS_5 \times S^5$ is most convenient in terms of $Z_4$-decomposition of the algebra-valued currents.

3.1 $Z_4$ grading of the superalgebra

In the present case we need a $Z_4$ decomposition of the superalgebra $\hat{f} = psu(1, 1|2) \oplus psu(1, 1|2)$. The grading we are interested in is different from the one induced by the standard grading on each term in the sum: the one we are looking for mixes the two terms.

To identify the required grading in terms of matrix representation let us consider first the bosonic part given by a direct sum of two copies of $su(1,1) \oplus su(2)$. The degree zero component is formed by elements of the form $(a, -a^t)$ with $a \in su(1,1) \oplus su(2)$ while the degree 2 component is formed by $(a, a^t)$. These two components are orthogonal to each other and satisfy

\[
[f_0, f_0] \subset f_0, \quad [f_0, f_2] \subset f_2, \quad [f_2, f_2] \subset f_0,
\] (3.2)

so that they can be identified with the even-degree components of the $Z_4$-decomposition. Moreover, the degree zero component is obviously isomorphic to $su(1,1) \oplus su(2)$, i.e. to the denominator of the coset (3.1).

To extend the grading to the fermionic components it is useful to consider first the grading of the complexified algebra $\hat{f}_C = psl_C(2|2) \oplus psl_C(2|2)$ and to represent its elements by $8 \times 8$ block-diagonal matrices of the form

\[
\begin{pmatrix}
    a & \alpha & 0 & 0 \\
    \beta & b & 0 & 0 \\
    0 & 0 & c & \gamma \\
    0 & 0 & \delta & d
\end{pmatrix},
\] (3.3)

Here $a, c, b, d$ are $2 \times 2$ bosonic matrices from $sl(2)$; $\alpha, \beta, \gamma, \delta$ are complex fermionic matrices. The antiautomorphism determining the $Z_4$ structure is given by

\[
M^\Omega = -K^{-1}M^{st}K, \quad K = \begin{pmatrix}
    0 & K \\
    K & 0
\end{pmatrix}, \quad K = \begin{pmatrix}
    1 & 0 \\
    0 & 1
\end{pmatrix},
\] (3.4)
where $1$ is the unit $2 \times 2$ matrix and $s^t$ denotes the transposition of the supermatrices. More explicitly, one has
\[
\begin{pmatrix} a & \alpha & 0 & 0 \\ \beta & b & 0 & 0 \\ 0 & 0 & c & \gamma \\ 0 & 0 & \delta & d \end{pmatrix} \Omega = - \begin{pmatrix} c^t & -\delta^t & 0 & 0 \\ -\gamma^t & d^t & 0 & 0 \\ 0 & 0 & a^t & -\beta^t \\ 0 & 0 & \alpha^t & b^t \end{pmatrix}
\] (3.5)

The $Z_4$ components $f_k^c$ are then identified as the eigenspaces of $\Omega$, i.e. $M^\Omega = i^k M$ for $M \in f_k^c$ so that $f^c = f_k^c \oplus f_1^c \oplus f_2^c \oplus f_3^c$.

To obtain $psu(1, 1|2) \oplus psu(1, 1|2)$ one needs to impose the reality condition $M^* = -M$ where $*$ is an antilinear antiautomorphism defined as
\[
\left( \begin{array}{cc} a & \alpha \\ \beta & b \end{array} \right)^* = \left( \begin{array}{cc} \Sigma a^\dagger \Sigma & -i \Sigma \beta^\dagger \\ -i \alpha^\dagger \Sigma & b^\dagger \end{array} \right), \quad \Sigma = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right),
\] (3.6)

and analogously for the second copy of $psl(2|2)$. Here $^\dagger$ denotes the ordinary hermitian conjugation.

In terms of the components the reality condition reads as
\[
\Sigma a^\dagger \Sigma = -a, \quad b^\dagger = -b, \quad i\Sigma \beta^\dagger = \alpha, \quad i\Sigma \delta^\dagger = \gamma,
\] (3.7)

along with the same conditions for the components of the second copy of $sl(2|2)$ (i.e. for $c, d, \gamma, \delta$).

It turns out that the $Z_4$ decomposition of $psl_c(2|2) \oplus psl_c(2|2)$ is compatible with the above reality condition in the sense that if $M \in f_k^c$, i.e. $M^\Omega = i^k M$ then $M^* \in f_k^c$. This implies that $\Omega$ induces the $Z_4$ decomposition of $\hat{f} = psu(1, 1|2) \oplus psu(1, 1|2)$
\[
\hat{f} = \hat{f}_0 \oplus \hat{f}_1 \oplus \hat{f}_2 \oplus \hat{f}_3, \quad \hat{f}_i, \hat{f}_j \subset \hat{f}_{i+j \text{mod } 4}.
\] (3.8)

The subspace $\hat{f}_k$ is given by the intersection of $\hat{f} \subset f^c$ and $f_k^c \subset f^c$.

Once the $Z_4$-grading is identified, the construction of the superstring sigma model coincides with that for $AdS_5 \times S^5$ and $AdS_3 \times S^2$ cases in [14]. The Lagrangian is written in terms of the $Z_4$-components of the current $J_\pm = \hat{f}^{-1} \partial_\pm \hat{f}$
\[
J_+ = A_+ + P_+ + Q_1+ + Q_2+, \quad A_+ \in \hat{f}_0, \quad Q_1+ \in \hat{f}_1, \quad P_+ \in \hat{f}_2, \quad Q_2+ \in \hat{f}_3.
\] (3.9)

Explicitly, in the conformal gauge ($\text{STr}$ is the supertrace)
\[
L_{\text{GS}} = \text{STr} \left[ P_+ P_- + \frac{1}{2} (Q_1+ Q_2- - Q_1- Q_2+) \right].
\] (3.10)

The Virasoro constraints are $\text{STr}(P_+ P_-) = 0$ and $\text{STr}(P_- P_-) = 0$. The GS action (before conformal gauge fixing) is invariant under the fermionic $\kappa$-symmetry. This invariance can be partially fixed by the following gauge condition [2]:
\[
Q_1- = 0, \quad Q_2+ = 0.
\] (3.11)
3.2 Pohlmeyer reduction

Given a superstring action written in terms of the $Z_4$ components of the currents, all the remaining steps of the Pohlmeyer reduction are the same as in the $AdS_5 \times S^5$ or $AdS_2 \times S^2$ cases discussed in [2]. Here we give a short review of the procedure concentrating on the subtleties of the $AdS_3 \times S^3$ case.

The Pohlmeyer reduction is performed in terms of the $Z_4$-components $A, Q_1, P, Q_2$ of the current $J_{\pm} = f^{-1} \partial_{\pm} f$. The components $Q_{1-}$ and $Q_{2+}$ are set to zero as partial $\kappa$-symmetry gauge fixing. Using the $f_0$-gauge symmetry one can assume that $P_+ = p_+ T$ where $p_+ = p_+(\sigma)$ is some scalar function and $T$ is a fixed element of $\mathfrak{f}_2$. In the case at hand there are inequivalent choices of $T$. One can, for instance, take $T = (t, -t^t)$ with $t|_{su(2)} = 0$ or, alternatively, $t|_{su(1,1)} = 0$. These choices are clearly inequivalent. The “nondegenerate” choice we are going to utilize is the one where both $su(1,1)$ and $su(2,2)$ parts are nonvanishing. Namely, we take (cf. [2])

$$T = \text{diag}(t, t^t), \quad t = \frac{i}{2} \text{diag}(1, -1, 1, -1).$$

(3.12)

Note that in this matrix representation $t$ coincides with $T$ used in the $AdS_2 \times S^2$ case in [2].

The choice of $T$ in (3.12) induces the decomposition $\hat{f} = \hat{f}^\parallel \oplus \hat{f}^\perp$ in each of the two $\text{psu}(1,1|2)$ sectors. More precisely,

$$\hat{f} = \hat{f}^\parallel \oplus \hat{f}^\perp, \quad P^\parallel \zeta^\parallel = \zeta^\parallel, \quad P^\parallel \chi^\perp = 0,$$

$$\zeta^\parallel \in \hat{f}^\parallel, \quad \chi^\perp \in \hat{f}^\perp, \quad P^\parallel \equiv -[T, [T, \cdot]].$$

(3.13)

(3.14)

Note that $\hat{f}^\parallel = \text{Im}(\text{ad}(T))$ and $\hat{f}^\perp = \text{ker}(\text{ad}(T))$; moreover, for $\zeta^\parallel \in \hat{f}^\parallel$ one also has $\{T, \zeta^\parallel\} = 0$.

In each $\text{psu}(1,1|2)$ sector the decomposition $\hat{f} = \hat{f}^\perp \oplus \hat{f}^\parallel$ is identical to that in the $AdS_2 \times S^2$ case in [2]. However, the choice of the subspaces

$$\hat{f}_0^\parallel = \mathfrak{h}, \quad \hat{f}_0^\perp = \mathfrak{m}, \quad \hat{f}^\parallel_{1,3}, \quad \hat{f}^\perp_{1,3}$$

(3.15)

here is different from [2] as $Z_4$-grading is defined in a different way and mixes the two $\text{psu}(1,1|2)$ sectors. In particular, the subalgebra $\mathfrak{h} = \hat{f}_0^\parallel$ is two dimensional, $\mathfrak{h} \cong u(1) \oplus u(1)$, and a useful choice of its basis is

$$h^{(A)} = \text{diag}(i, -i, 0, 0, -i, i, 0, 0), \quad h^{(S)} = \text{diag}(0, 0, i, -i, 0, 0, -i, i).$$

(3.16)

Next, one uses the Virasoro constraint $\text{STr}(P_+ P_+) = 0$ and the residual conformal invariance to set $p_+ = \mu$ for some constant $\mu$ so that $P_+ = \mu T$. Introducing the $\mathcal{G}$-valued field $\bar{g}$ ($\mathcal{G} \subset \mathcal{F}$ is a subgroup corresponding to the subalgebra $\hat{f}_0^\parallel \cong su(1,1) \oplus su(2)$, i.e. $\mathcal{G} \equiv SU(1,1) \times SU(2)$) one solves the equation of motion $\partial_+ P_+ + [A_+, P_-] = 0$ and the Virasoro constraint $\text{STr}(P_- P_-) = 0$ by

$$P_- = \mu \bar{g}^{-1} T \bar{g},$$

(3.17)

where one again used the remaining conformal transformation freedom. Finally, solving the remaining equations of motion by choosing

$$A_- = \bar{A}_-, \quad A_+ = \bar{g}^{-1} \partial_+ \bar{g} + \bar{g}^{-1} \bar{A}_+ \bar{g}$$

(3.18)
one ends up with only the Maurer-Cartan equation imposed on the \( \hat{f} \)-connection \( J \) parametrised in terms of the new fields: \( \hat{g} \)-valued field \( \hat{g} \), the \( \hat{h} = \hat{f}_0^\perp \)-valued fields \( \hat{A}_+, \hat{A}_- \) and the fermionic fields \( Q_1, Q_2 \). In this parametrisation the Maurer-Cartan equation is invariant under the \( \hat{H} \times \hat{H} \) local symmetry (recall that in our case \( \hat{H} \approx U(1) \times U(1) \) is the Lie group whose Lie algebra is \( \hat{h} = \hat{f}_0^\perp \)).

Finally, one uses the residual kappa-invariance to set to zero the components \( Q^\perp_{1+} \) and \( (\hat{g} Q_{2-} \hat{g}^{-1})^\perp \).

The remaining components of the fermionic currents are parametrized in terms of the new fermionic fields \( \Psi_1, \Psi_2 \) taking values in \( \hat{f}_1 \) and \( \hat{f}_3 \) respectively:

\[
Q^\perp_{1+} = \sqrt{\mu} \Psi_1, \quad (\hat{g} Q_{2-} \hat{g}^{-1})^\perp = \sqrt{\mu} \Psi_2. \tag{3.19}
\]

Using \( \hat{H} \times \hat{H} \) local symmetry one can satisfy the following constraints:

\[
\tau(\hat{A}_+) = (\hat{g}^{-1} \partial_+ \hat{g} + \hat{g}^{-1} \hat{A}_+ \hat{g})_h - \frac{1}{2} [[T, \Psi_1], \Psi_1],
\]

\[
\hat{A}_- = (\hat{g} \partial_- \hat{g}^{-1} + \hat{g} \tau(\hat{A}_-) \hat{g}^{-1})_h - \frac{1}{2} [[T, \Psi_2], \Psi_2]. \tag{3.20}
\]

where \( \tau \) is an automorphism of \( \hat{h} \) which is assumed to preserve the inner product (i.e. the trace). This automorphism is introduced for generality to make the resulting theory having a nonsingular expansion around the natural vacuum \( \hat{g} = 1 \). This can be achieved by choosing \( \tau(\hat{A}) = -\hat{A} \) which is an automorphism of \( u(1) \oplus u(1) \) (this will correspond to axial instead of vector gauging). The residual gauge transformations (i.e. the transformations preserving the Maurer-Cartan equations and the constraints (3.20)) read as:

\[
\hat{g} \rightarrow h^{-1} \hat{g} \hat{\tau}(h), \quad \hat{A}_+ \rightarrow h^{-1} \hat{A}_+ h + h^{-1} \partial_+ h, \quad \hat{A}_- \rightarrow h^{-1} \hat{A}_- h + h^{-1} \partial_- h,
\]

\[
\Psi_1 \rightarrow \hat{\tau}(h)^{-1} \Psi_1 \hat{\tau}(h), \quad \Psi_2 \rightarrow h^{-1} \Psi_2 h. \tag{3.22}
\]

The Maurer-Cartan equations and the constraints (3.20) can then be obtained from the following local Lagrangian\(^8\)

\[
L_{\text{tot}} = L_{\text{WZW}} + \mu^2 \text{STr}(\hat{g}^{-1} T \hat{g} T) + \text{STr} \left( \Psi_2 T \hat{D}_+ \Psi_2' + \Psi_1 T \hat{D}_- \Psi_1' \right) + \mu \text{STr} \left( \hat{g}^{-1} \Psi_2 \hat{g} \Psi_1 \right), \tag{3.23}
\]

where

\[
\hat{D}_+ \Psi_2 = \partial_+ \Psi_2 + [\hat{A}_+, \Psi_2], \quad \hat{D}_- \Psi_1 = \partial_- \Psi_1 + [\tau(\hat{A}_-), \Psi_1]. \tag{3.24}
\]

and \( \Psi_1, \Psi_2 \) are constrained by the condition that they anticommute with \( T \) (i.e. take values in \( \hat{f}_{1,3}^\| \)). 

\( L_{\text{WZW}} \) which depends only on the bosonic fields is given explicitly by

\[
L_{\text{WZW}} = \frac{1}{2} \text{STr}(\hat{g}^{-1} \partial_+ \hat{g} \hat{g}^{-1} \partial_- \hat{g}) + \text{WZ-term}
\]

\[
+ \text{STr}(\hat{A}_+ \partial_- \hat{g} \hat{g}^{-1} - \tau(\hat{A}_-) \hat{g}^{-1} \partial_+ \hat{g} - \hat{g}^{-1} \hat{A}_+ \hat{g} \tau(\hat{A}_-) + \hat{A}_+ \hat{A}_-). \tag{3.25}
\]

Here the supertrace in the bosonic terms accounts for the relative minus sign in the contributions of the \( S^3 \) and \( AdS_3 \) parts (leading to the correct final signs).

\(^8\)The equations similar to those contained in the Maurer-Cartan equations appeared in a different context in \([19]\) and are formally invariant under a 2d supersymmetry. However, besides these equations the Lagrangian (3.23) leads also to the constraints (3.20) that are not, in general, invariant under the supersymmetry transformations (cf. \([2]\)).
3.3 Reduced Lagrangian in terms of independent degrees of freedom

Similarly to the purely bosonic case, the Lagrangian \( (3.23) \) of the reduced model can be usefully parametrized in terms of the bosonic fields taking values in one copy of \( psu(1,1|2) \) only. Namely, let \( g \) be an \( SU(1,1) \times SU(2) \)-valued field, \( A_\pm \) the \( u(1) \oplus u(1) \)-valued gauge fields, and \( \Psi'_1, \Psi'_2 \) take values in the fermionic part of the “parallel” subspace of the first \( psu(1,1|2) \), i.e.

\[
g = \left( \begin{array}{cc} g_A & 0 \\ 0 & g_S \end{array} \right), \quad A_\pm = \left( \begin{array}{cc} A^A_\pm & 0 \\ 0 & A^S_\pm \end{array} \right), \quad \Psi'_{1,2} = \left( \begin{array}{cc} 0 \\ i\psi'^\dagger_{1,2} \Sigma \\ 0 \end{array} \right). \quad (3.26)
\]

Here \( A \) and \( S \) refer to the \( AdS \) and the sphere parts, i.e. \( g_A \) and \( g_S \) are in the fundamental representations of \( SU(1,1) \) and \( SU(2) \) respectively, \( A^A_\pm = a^A_\pm \text{diag}(i\Sigma,0) \), \( A^S_\pm = a^S_\pm \text{diag}(0,i\Sigma) \) and \( \psi_1, \psi_2 \) are antidiagonal complex fermionic matrices. Recall that \( \Sigma = \text{diag}(1,-1) \) and \( t = \frac{1}{2} \text{diag}(\Sigma,\Sigma) \).

More explicitly, let us choose the following basis in \( su(1,1) \) and \( su(2) \) in terms of the Pauli matrices: \( \hat{R}_1 = \sigma_1, \hat{R}_2 = i\sigma_3, \hat{R}_3 = \sigma_2 \), and \( R_1 = i\sigma_1, R_2 = i\sigma_3, R_3 = i\sigma_2 \) (see Appendix A for details). To simplify the presentation let us first consider the case of \( \tau = 1 \). One can parametrize the group valued field \( g \) in terms of the Euler angles \( \phi, \chi \) and \( \varphi, \theta \) as

\[
g_A = \exp \left( \frac{1}{2} \chi \hat{R}_2 \right) \exp \left( \phi \hat{R}_1 \right) \exp \left( \frac{1}{2} \chi \hat{R}_2 \right), \quad g_S = \exp \left( \frac{1}{2} \theta \hat{R}_2 \right) \exp \left( \varphi \hat{R}_1 \right) \exp \left( \frac{1}{2} \theta \hat{R}_2 \right). \quad (3.27)
\]

Explicitly,

\[
g_A = \left( \begin{array}{cc} e^{i\chi} \cosh \phi & \sinh \phi \\ \sinh \phi & e^{-i\chi} \cosh \phi \end{array} \right), \quad g_S = \left( \begin{array}{cc} e^{i\theta} \cos \varphi & i \sin \varphi \\ i \sin \varphi & e^{-i\theta} \cos \varphi \end{array} \right). \quad (3.28)
\]

One can then solve for the gauge fields using their equations

\[
A_+ = (\hat{A}_+)_{\mathfrak{g}}, \quad \hat{A}_+ = g^{-1} \partial_+ g + g^{-1} A_+ g - \frac{1}{2} [[t, \Psi'_1], \Psi'_1], \quad (3.29)
\]

\[
A_- = (\hat{A}_-)_g, \quad \hat{A}_- = g \partial_- g^{-1} + g A_- g^{-1} - \frac{1}{2} [[t, \Psi'_2], \Psi'_2], \quad (3.30)
\]

following from the Lagrangian \( (3.23) \) with \( \tau = 1 \). The fermionic terms entering the constraints give

\[
\frac{1}{2} [[t, \Psi'_1], \Psi'_1] = (\alpha \beta - \gamma \delta)(\hat{R}_2 - R_2), \quad \frac{1}{2} [[t, \Psi'_2], \Psi'_2] = (\lambda \nu - \rho \sigma)(\hat{R}_2 - R_2), \quad (3.31)
\]

where we have introduced the real components of the fermions in \( (3.26) \) as

\[
\psi_1 = \left( \begin{array}{cc} 0 \\ \gamma + i\delta \alpha + i\beta \\ 0 \end{array} \right), \quad \psi_2 = \left( \begin{array}{cc} 0 \\ \rho + i\sigma \lambda + i\nu \\ 0 \end{array} \right). \quad (3.32)
\]

One then finds

\[
A^A_+ = \frac{\partial_+ \chi (1 + \cosh 2\phi) - 2(\alpha \beta - \gamma \delta)}{2(1 - \cosh 2\phi)} \hat{R}_2, \quad A^S_+ = \frac{\partial_+ \theta (1 + \cos 2\varphi) + 2(\alpha \beta - \gamma \delta)}{2(1 - \cos 2\varphi)} \hat{R}_2, \quad (3.33)
\]

and similar expressions for \( A_- \) with \( \partial_+ \chi \to -\partial_- \chi \) and \( \alpha \beta - \gamma \delta \to \lambda \nu - \rho \sigma \).
Using the equations of motion for $A_{\pm}$ one can write the reduced Lagrangian in the form

\[ L_{\text{tot}} = \frac{1}{2} \TR \left( g_A^{-1} \partial_+ g_A^{-1} \partial_- g_A \right) - \frac{1}{2} \TR \left( g_S^{-1} \partial_+ g_S^{-1} \partial_- g_S \right) + \text{potential} \]

\[ + \text{fermionic kinetic term} + \text{fermionic interaction term} \tag{3.34} \]

\[ + \TR \left( A_+^A [g_A, \psi_1] (\partial_- g_A g_A^{-1} + \frac{1}{2} [t, \psi'_2], \psi'_2)_{\text{AdS}} - A_+^S [g_S, \psi_1] (\partial_- g_S g_S^{-1} + \frac{1}{2} [t, \psi'_2], \psi'_2)_{\text{S}} \right). \]

The bosonic part of the Lagrangian that comes from the WZW and potential terms (i.e. terms not involving $A_{\pm}$) is

\[ L_1 = \partial_+ \varphi \partial_- \varphi + \frac{1}{2} (1 + \cos 2 \varphi) \partial_+ \theta \partial_- \theta \]

\[ + \partial_+ \phi \partial_- \phi - \frac{1}{2} (1 + \cos 2 \phi) \partial_+ \chi \partial_- \chi + \frac{\mu^2}{2} (\cos 2 \varphi - \cos 2 \phi). \tag{3.35} \]

The fermionic interaction term is found to be $\text{STr} (g^{-1} \psi'_2 g^\dagger \psi'_1) = -2\text{Im} [\TR (g_A^{-1} \psi_2 g \psi_1^\dagger \Sigma)]$ and together with the fermionic kinetic terms they give

\[ L_2 = \alpha \partial_+ \alpha + \beta \partial_- \beta + \gamma \partial_+ \gamma + \delta \partial_- \delta + \lambda \partial_+ \lambda + \nu \partial_+ \nu + \rho \partial_+ \rho + \sigma \partial_+ \sigma \]

\[ -2\mu \left( \sinh \phi \sin \varphi (\lambda \gamma + \nu \delta - \rho \alpha - \sigma \beta) + \cosh \phi \cos \varphi [\cos (\chi + \theta)(\rho \delta - \sigma \gamma) \right. \]

\[ - \lambda \beta + \nu \alpha) - \sin (\chi + \theta)(\lambda \alpha + \nu \beta + \rho \gamma + \sigma \delta)] \right). \tag{3.36} \]

Finally, the terms that originate from the elimination of $A_{\pm}$ (third line of (3.34) are

\[ L_3 = \frac{[\partial_+ \chi (1 + \cosh 2 \phi) - 2(\alpha \beta - \gamma \delta)] [\partial_- \chi (1 + \cosh 2 \phi) + 2(\lambda \nu - \rho \sigma)]}{2(\cosh 2 \phi - 1)} \]

\[ + \frac{[\partial_+ \theta (1 + \cosh 2 \varphi) + 2(\alpha \beta - \gamma \delta)] [\partial_- \theta (1 + \cosh 2 \varphi) - 2(\lambda \nu - \rho \sigma)]}{2(1 - \cos 2 \varphi)}. \tag{3.37} \]

Then the Lagrangian (3.34) becomes

\[ L_{\text{tot}} = L_1 + L_2 + L_3 \equiv L_B + L_F. \tag{3.38} \]

The purely bosonic terms in $L_1$ and $L_3$ combine into the direct sum of the CSG action and its “hyperbolic” counterpart which is the reduced Lagrangian for the bosonic string in $AdS_3 \times S^3$:

\[ L_B = \partial_+ \varphi \partial_- \varphi + \cot^2 \varphi \partial_+ \theta \partial_- \theta + \partial_+ \phi \partial_- \phi + \coth^2 \phi \partial_+ \chi \partial_- \chi + \frac{\mu^2}{2} (\cos 2 \varphi - \cosh 2 \phi), \tag{3.39} \]

while the fermionic ones give:

\[ L_F = L_2 - \cot^2 \varphi [\partial_+ \theta (\lambda \nu - \rho \sigma) - \partial_- \theta (\alpha \beta - \gamma \delta)] + \coth^2 \phi [\partial_+ \chi (\lambda \nu - \rho \sigma) - \partial_- \chi (\alpha \beta - \gamma \delta)] \]

\[ - (\alpha \beta - \gamma \delta)(\lambda \nu - \rho \sigma) \left[ \frac{1}{\sin^2 \varphi} + \frac{1}{\sinh^2 \phi} \right]. \tag{3.40} \]
For the Lagrangian $L_{tot}$ the point $\varphi = \phi = 0$ which is a minimum of the potential is a singular point of the kinetic-term. At the same time the regular point of the kinetic term $\varphi = \pi/2$, $\phi = i\pi/2$ is a maximum of the potential. One can by-pass this complication as in the purely bosonic case – by using the axial gauged gWZW theory instead of the vector gauged one.

To find the axial gauging analog of the above reduced Lagrangian (3.38) we are to take $\tau(a) = -a$, $a \in \mathfrak{h}$ in (3.23). Using this asymmetric gauge also affects the parametrization of the group element: now one is to use

$$g = \hat{\tau}(g_2)g_1g_2,$$

(3.41)

leading to (cf. (3.27))

$$g_A = \exp\left(-\frac{1}{2} R_2\right) \exp (\phi R_1) \exp\left(\frac{1}{2} R_2\right), \quad g_S = \exp\left(-\frac{1}{2} \theta R_2\right) \exp (\varphi R_1) \exp\left(\frac{1}{2} \theta R_2\right).$$

(3.42)

One can then redo the same steps as above and get the corresponding Lagrangian in terms of the physical degrees of freedom only. Details of this are given in the Appendix A. It turns out that similarly to the purely bosonic CSG case the resulting Lagrangian can be obtained directly from the vector-gauged $L_{tot}$ by an appropriate “analytic continuation”. Namely, transforming the variables according to

$$\varphi \rightarrow \varphi + \frac{\pi}{2}, \quad \phi \rightarrow \phi + \frac{i\pi}{2}, \quad \theta \rightarrow -\theta, \quad \chi \rightarrow -\chi,$$

(3.43)

and redefining the coupling as $\mu \rightarrow -i\mu$ one gets the resulting “dual” Lagrangian

$$L_{tot}^{axial} = \partial_+ \varphi \partial_\varphi + \tan^2 \varphi \partial_+ \theta \partial_\theta + \partial_+ \phi \partial_\phi + \tanh^2 \phi \partial_+ \chi \partial_\chi + \frac{\mu^2}{2} (\cos 2\varphi - \cosh 2\phi)$$

$$+ \alpha \partial_+ \alpha + \beta \partial_+ \beta + \gamma \partial_+ \gamma + \delta \partial_+ \delta + \lambda \partial_+ \lambda + \nu \partial_+ \nu + \rho \partial_+ \rho + \sigma \partial_+ \sigma$$

$$+ \tan^2 \varphi \left[ \partial_+ \theta (\lambda \nu - \rho \sigma) - \partial_\theta (\alpha \beta - \gamma \delta) \right] - \tanh^2 \phi \left[ \partial_+ \chi (\lambda \nu - \rho \sigma) - \partial_\chi (\alpha \beta - \gamma \delta) \right]$$

$$- (\alpha \beta - \gamma \delta) (\lambda \nu - \rho \sigma) \left[ \frac{1}{\cos^2 \varphi} - \frac{1}{\cosh^2 \phi} \right] - 2\mu \left( \cosh \phi \cos \varphi (\lambda \gamma + \nu \delta - \rho \alpha - \sigma \beta) \right.$$

$$+ \cosh \phi \cos \varphi \left[ \cos (\chi + \theta) (-\rho \delta + \sigma \gamma + \lambda \beta - \nu \alpha) - \sin (\chi + \theta) (\lambda \alpha + \nu \beta + \rho \gamma + \sigma \delta) \right] \right).$$

(3.44)

Note that in order to obtain this Lagrangian directly from (3.23) with $\tau(a) = -a$ one also needs to redefine the fermions as follows: $\alpha \rightarrow -\delta, \delta \rightarrow \alpha, \beta \rightarrow \gamma, \gamma \rightarrow -\beta$.

Since (as follows from (3.40) and (3.44)) we may identify the fermions $\alpha, \beta, \gamma, \delta$ and $\lambda, \nu, \rho, \sigma$ with 2d Majorana-Weyl spinors, a natural question then is if the total reduced Lagrangian has a 2d supersymmetry, i.e. if it can be interpreted as a supersymmetric extension of (3.39). This is indeed possible for a consistent truncation of $L_{tot}$ (for definiteness let us consider (3.38)) found by setting $\chi = \theta = 0$, $\lambda = \gamma = \sigma = \beta = 0$ which produces the reduced Lagrangian for the $AdS_2 \times S^2$ superstring [2]:

$$L_{trunc.} = \partial_+ \varphi \partial_\varphi + \partial_+ \phi \partial_\phi + \frac{\mu^2}{2} (\cos 2\varphi - \cosh 2\phi) + \alpha \partial_+ \alpha + \delta \partial_+ \delta$$

$$+ \nu \partial_+ \nu + \rho \partial_+ \rho - 2\mu \left[ \cosh \phi \cos \varphi (\nu \alpha + \rho \delta) + \sinh \phi \sin \varphi (\nu \delta - \rho \alpha) \right].$$

(3.45)

---

This point is still a regular expansion point for the corresponding Hamiltonian, assuming the momenta of $\theta$ and $\chi$ are constant in the vacuum.
This Lagrangian is equivalent [2] to the $N = 2$ supersymmetric sine-Gordon Lagrangian\cite{15}: 

$$
L = \partial_+ \Phi \partial_- \Phi^* - |W'(\Phi)|^2 + \psi_L^* \partial_+ \psi_L + \psi_R^* \partial_- \psi_R + \left[ W''(\Phi) \psi_L \psi_R + W''(\Phi^*) \psi_L^* \psi_R^* \right],
$$

(3.46)

where 

$$
\Phi = \varphi + i \phi, \quad \psi_L = \nu + i \rho, \quad \psi_R = -\alpha + i \delta, \quad W = \mu \cos \Phi.
$$

At the same time, both the CSG model and its “hyperbolic” analog admit $N = 2$ supersymmetric extensions \cite{16} based on interpreting $\xi \equiv \ln \cos \varphi + i \theta$ and $\eta \equiv \ln \cosh \phi + i \chi$ as complex scalar components of chiral superfields and using that 

$$
d\varphi^2 + \cot^2 \varphi d\theta^2 = \frac{\partial^2 K}{\partial \xi \partial \xi^*} d\xi d\xi^*, \quad d\phi^2 + \coth^2 \phi d\chi^2 = \frac{\partial^2 K'}{\partial \eta \partial \eta^*} d\eta d\eta^*.
$$

Then $K$ and $K'$ are the corresponding Kahler potentials, while the two superpotentials are $\mu e^\xi$ and $\mu e^\eta$. The resulting $N = 2$ supersymmetric Lagrangian is, however, a direct sum of the two decoupled $N = 2$ theories and thus cannot be equivalent to the above $L_{\text{tot}}$ (in particular, it does not admit the above $N = 2$ SG truncation (3.45)).

To show that $L_{\text{tot}}$ \cite{3,4} or (3.44) has $N = 2$ supersymmetry one may try to use non-standard types of $N = 2$ superfields (see, e.g., \cite{17}). While the sigma-model part of (3.39) admits straightforward $N = 1$ supersymmetrization, incorporating the potential terms appears to be non-trivial (cf. \cite{18} and refs. there). The existence of 2d supersymmetry of the reduced Lagrangian $L_{\text{tot}}$ thus remains an open problem.

### 4 Comments on Pohlmeyer reduction of strings on $AdS_n \times S^n$

In this section we shall make few general comments clarifying some aspects of Pohlmeyer reduction of strings on $AdS_n \times S^n$ spaces and extending the discussion in [2].

#### 4.1 Relation to Pohlmeyer reduction in the pure $AdS_n$ case

Considering strings moving on $AdS_n \times S^n$ we have assumed that the conformal gauge (Virasoro) condition $T_{\pm\pm}^{AdS} + T_{\pm\pm}^{S} = 0$ is satisfied by $T_{\pm\pm}^{S} = \mu^2$, $T_{\pm\pm}^{AdS} = -\mu^2$. Indeed, if strings move on a sphere their stress tensor must be positive and by residual conformal transformation can be made constant. However, there is a special subclass of strings which are localised on the sphere and move only in $AdS_n$; then we should have $T_{\pm\pm}^{S} = 0$, $T_{\pm\pm}^{AdS} = 0$. In the context of string theory in $AdS_n \times S^n$ this special case should be viewed as a limit $\mu \to 0$ of the general case\cite{10}. Still, since in the non-compact $AdS_n$ case the condition $T_{\pm\pm}^{AdS} = 0$ has, in general, nontrivial solutions, one can formally study how the Pohlmeyer reduction should be implemented in this case. Earlier discussions of this pure $AdS_n$ reduction appeared in \cite{20,21} and we shall explain their relation to our approach.

Let us start with the simplest case of $AdS_2 = F/G = SO(2, 1)/SO(1, 1)$ and use the standard matrix representation for $SO(2, 1)$ by $3 \times 3$ orthogonal matrices with the subgroup $SO(1, 1)$ embedded\footnote{In the case of $AdS_n \times S^n$ the standard and natural choice of the expansion point or vacuum is the BMN one, i.e. the geodesic $t = \mu \tau$, $\psi = \mu \tau$, implying a non-zero value for $\mu$.}
there exists such diagonally (the signature choice is \((- - +))\). The Lie algebras are denoted by \(\mathfrak{f} = \text{so}(2, 1)\) and \(\mathfrak{g} = \text{so}(1, 1)\). The orthogonal decomposition \(\mathfrak{f} = \mathfrak{p} \oplus \mathfrak{g}\) induces the decomposition \(J = \mathcal{P} + \mathcal{A}\) of the \(\mathfrak{f}\)-current \(J = f^{-1}df, \ f \in F = \text{SO}(2, 1)\). The Virasoro constraints

\[
\text{Tr}(P_+P_+) = \text{Tr}(P_-P_-) = 0
\]

(4.1)

imply that \(P_\pm\) are proportional to \(T_+\) or to \(T_-\) given by

\[
T_+ = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad T_- = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.
\]

(4.2)

Note that these two choices are gauge inequivalent, i.e. \(T_+ \neq g^{-1}T_-g\) for any \(g \in G\).

Let us now consider two options: (i) both \(P_+\) and \(P_-\) are proportional to \(T_+\) (or \(T_-\)); (ii) \(P_+\) is proportional to \(T_+\) and \(P_-\) to \(T_-\). In the first case the dynamics is trivial. Indeed, the \(\mathfrak{g}\)-component of the MC equation takes the form \(\partial_-\mathcal{A}_+ - \partial_+\mathcal{A}_- = 0\) ([\(P_-, P_+\]) vanishes due to the assumption that both components are proportional to \(T_+\)). This implies that \(\mathcal{A}_\pm\) can be set to zero by a gauge transformation. The remaining equations of motion take the form \(\partial_-P_+ = 0, \ \partial_+P_- = 0\) and can be satisfied by making appropriate conformal transformations.

In the second case

\[
P_+ = p_+T_+, \quad P_- = p_-T_- \tag{4.3}
\]

and by a gauge transformation one can set \(p_+ = m = \text{const.}\) Parametrizing \(P_- = p_-T\) as \(P_- = me^{2\phi}T_-\) where \(\phi\) is a new field we find that the Virasoro constraints and part of the equations of motion are thus solved by

\[
P_+ = mT_+, \quad P_- = me^{2\phi}T_- , \quad \mathcal{A}_+ = -\partial_+\phi R_1 , \quad \mathcal{A}_- = 0 , \quad R_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{4.4}
\]

where \(R_1\) is an element of \(\mathfrak{g}\). Note that \([R_1, T_+] = \pm T_+\) and \([T_-, T_+] = -2R_1\), i.e. \(R_1, T_\pm\) form the \(sl(2) \approx \text{so}(2, 1)\) algebra. The only remaining equation is the \(\mathfrak{g}\)-component of the Maurer-Cartan one which gives

\[
\partial_-\partial_+\phi + m^2 e^{2\phi} = 0 , \tag{4.5}
\]

i.e. the Liouville equation. It follows from

\[
L = \partial_+\phi\partial_-\phi - m^2 e^{2\phi} , \tag{4.6}
\]

which is thus the Lagrangian of the corresponding reduced theory. Note that \(m\) here can be set to any fixed value by a shift of \(\phi\) (the reduced theory has residual conformal invariance).

The point we would like to make is that this model can be viewed as a limit of the Pohlmeyer-reduced model for strings on \(\text{AdS}_2 \times S^1\) of the type discussed in the previous sections. Indeed, in this

\[
\text{However, if one replaces } SO(2, 1)/SO(1, 1) \text{ with the coset of slightly larger groups, namely, } O(2, 1)/O(1, 1) \text{ then there exists such } g \text{ that } T_+ = g^{-1}T_-g, \text{ e.g., } g = \text{diag}(1, 1, -1) \text{ with } \det g = -1 \text{ so that } g \text{ does not belong to } SO(2, 1).\]
case choosing the conformal gauge and fixing the residual conformal freedom by choosing the angle of $S^1$ as $\psi = \mu \tau$ the reduced theory is described by the sinh-Gordon Lagrangian

$$L = \partial_+ \phi \partial_- \phi - \frac{\mu^2}{2} \cosh 2\phi .$$

(4.7)

Introducing $\phi = \varphi + \ln \mu$ we get

$$L = \partial_+ \phi \partial_- \phi - \frac{\mu^2}{4} (\mu^{-2} e^{2\phi} + \mu^2 e^{-2\phi}) .$$

(4.8)

Then taking the limit $\mu \to 0$ we get precisely the Liouville Lagrangian \((4.6)\) (with $m = \frac{1}{2}$). This is just a manifestation of the fact that solutions where string moves only in $AdS_2$ can be obtained as a limit of solutions where it moves also along $S^1$.

Starting with string theory on 3-dimensional space $AdS_2 \times S^1$ one finds the reduced Lagrangian by completely fixing the reparametrization freedom and it thus contains just 3-2=1 physical degree of freedom. At the same time, while string theory on $AdS_2$ should have no dynamical (transverse) degrees of freedom, this is an apparent contradiction with the reduced Lagrangian \((4.6)\) depending on one field $\phi$. The resolution of this puzzle is that the corresponding Liouville action is still invariant under the conformal diffeomorphisms which in present case are remnants of the original reparametrization freedom and should thus be treated as a gauge symmetry. Fixing this symmetry should leave no dynamical degrees of freedom.

Analogous considerations can be also applied to the reduced model for strings on $AdS_3$. Starting from the reduced model for strings on $AdS_3 \times S^1$ described by the Lagrangian

$$L = \partial_+ \phi \partial_- \phi + \tanh^2 \phi \partial_+ \theta \partial_- \theta - \frac{\mu^2}{2} \cosh 2\phi ,$$

(4.9)

the equations of motion are

$$\partial_+ \partial_- \phi - \frac{\sinh \phi}{\cosh^2 \phi} \partial_+ \theta \partial_- \theta + \frac{1}{2} \mu^2 \sinh 2\phi = 0 ,$$

(4.10)

$$\partial_+ (\tanh^2 \phi \partial_- \theta) + \partial_- (\tanh^2 \phi \partial_+ \theta) = 0 .$$

(4.11)

Writing them in terms of the rescaled variables $\phi' = \phi + \log \mu$ and $\theta' = 2\sqrt{2} \mu \theta$ and taking the limit $\mu \to 0$ we get

$$\partial_+ \partial_- \phi' + \frac{1}{2} \mu^2 e^{2\phi'} \partial_+ \theta' \partial_- \theta' - \frac{1}{2} \mu^{-2} e^{-2\phi'} = 0 ,$$

$$\partial_+ (\partial_- \theta') + \partial_- (\partial_+ \theta') = 0 .$$

(4.12)

The second equation can be solved as $\theta' = \zeta_+ (\sigma^+) + \zeta_- (\sigma^-)$. In terms of $\phi = \phi' - \frac{1}{4} \ln (\partial_+ \theta' \partial_- \theta')$ the first equation takes the form $\partial_+ \partial_- \phi + \sqrt{\partial_+ \zeta_+ \partial_- \zeta_-} \sinh 2\phi = 0$, which can be put into a simpler sinh-Gordon form

$$\partial_+ \partial_- \phi + \sinh 2\phi = 0$$

(4.13)

by a $\zeta_{\pm}$-dependent conformal reparametrization of the worldsheet coordinates. This then agrees with the result of the earlier discussion \([20, 21]\) of the Pohlmeyer reduction of the $AdS_3$ sigma model (starting with the equations of motion in the formulation in terms of embedding coordinates). Note
that the $\mu \to 0$ limit of the $AdS_3 \times S^1$ theory we have used was taken at the level of the equations of motion. It cannot be directly implemented at the Lagrangian level starting with the Lagrangian of the hyperbolic CSG model (the reduced model for strings on $AdS_3 \times S^1$) but it may be possible to take it at the level of the extended gWZW action containing additional gauge fields.\footnote{To get a smooth limit at the action level one should presumably incorporate more fields, going back to the gWZW formulation of the reduced theory for $AdS_3 \times S^1$.}

Let us now comment on the general case of the coset $F/G = SO(2, n - 1)/SO(1, n - 1)$. We shall use the standard matrix representation and assume that the signature is $(- - + \cdots +)$. The subspace $p = \mathfrak{f} \ominus \mathfrak{g}$ is then represented by elements with nonvanishing first raw and first column. Let $e_i$ ($i = 0, \ldots, n - 1$) be the standard orthonormal basis in $p \ominus \mathfrak{g}$ with $\text{Tr}(e_0 e_0) = 1$ and $\text{Tr}(e_i e_i) = -1$ for $i > 0$. The current components $P_\pm$ then decompose as $P_\pm = e_i e_i$. By making a $G$-gauge transformation one can always satisfy the Virasoro constraint $\text{Tr}(P_+ P_+) = 0$ by (here we set an arbitrary mass scale $m$ that one can put in front of $T$ to 1)

$$P_+ = T, \quad T = \begin{pmatrix} 0 & 1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} .$$

(4.14)

Here $T$ is an obvious generalization of $T_+$ in the $AdS_2$ case (see (4.2)), but unlike the $AdS_2$ case in higher dimensions there are no inequivalent choices for $T$: analogs of $T_+$ and $T_-$ are related by gauge transformations.

In the gauge where $P_+ = T$ the equations of motion $\partial_- P_+ + [A_-, P_+] = 0$ can be solved for $A_-$ as $A_- = A_-$, where $A_-$ is an arbitrary $\mathfrak{h}$ valued field, with $\mathfrak{h} \subset \mathfrak{g}$ being a centralizer of $T$ in $\mathfrak{g}$. Let $G_0 \cong SO(n - 1)$ be diagonally embedded into $G$. By $G_0$ transformation one can always set $P_-$ = 0 for $i > 1$. The Virasoro constraint $\text{Tr}(P_- P_-) = 0$ then implies $(P_0^1)^2 - (P_1^0)^2 = 0$, i.e. $P_0^1 = \pm P_1^0$ (and by specializing the $G_0$ transformation one can also set $P_1^0 = P_0^0$). This allows one to use the following parametrization

$$P_- = e^\phi g^{-1} T g , \quad g \in G_0 ,$$

(4.15)

where $g$ and $\phi$ are the new variables.\footnote{In the $AdS_2$ case $G_0$ was trivial so that the field $g$ was not present.} The general solution to the equation $\partial_+ P_- + [A_+, P_-] = 0$, considered as a condition on $A_+$, is

$$A_+ = g^{-1} \partial_+ g + g^{-1} A_+ g - \partial_+ \phi \ g^{-1} R_1 g , \quad g \in G_0 ,$$

(4.16)

where $R_1$ is a basic element of the subgroup (which is an obvious generalization of $R_1$ in the $AdS_2$ case). The only remaining equation is the $\mathfrak{g}$-component of the MC equation that gives

$$D_-(g^{-1} \partial_+ g + g^{-1} A_+ g - \partial_+ \phi \ g^{-1} R_1 g) - D_+ A_- = -e^\phi [g^{-1} T g, T] .$$

(4.17)

Note that contrary to the standard Pohlmeyer reduction here we did not fix the residual conformal symmetry: eq. (4.17) is conformally invariant with $g$ transforming as a scalar and $\phi$ as a Liouville field.
It remains to be understood in general how to find a Lagrangian from which (4.17) may follow. For that we may need to fix the residual conformal symmetry; that may also help to explain the relation to other reduced Lagrangians in the literature [20, 21]. For example, in the AdS\(_3\) case we are left with two independent fields \(\phi\) and \(g\) while in [20, 21] one finds \(\partial_+ \partial_\alpha - e^\alpha - we^{-\alpha} = 0, \partial_+ u = 0, \partial_- v = 0\) and after solving for \(u, v\) (which essentially fixes the conformal symmetry) and redefining \(\alpha\) one ends up with sinh-Gordon equation for a single dynamical field. The analogous step of solving for \(g\) in AdS\(_3\) example can be done in our case (cf. the above discussion) but generalization to higher dimensional cases remains to be worked out.

Let us stress again that in the context of the AdS\(_5\) \(\times\) S\(_5\) theory it is natural to view the subsector of the pure AdS\(_5\) solutions as a \(\mu \to 0\) limit of the general string motions as described by the reduced theory of [2].

### 4.2 The vacuum and perturbative expansion of the reduced model

Going back to the reduced Lagrangian of a \(F/G\) bosonic coset model which is similar to (2.7) one needs to choose an \(H\)-gauge to isolate the physical degrees of freedom. One option is to impose \(A_+ = A_- = 0\) which is possible at the level of the equations of motion [2]. In this gauge one gets the equation of non-abelian Toda theory with \(g = 1\) as a natural vacuum point. The expansion near this point leads to massive excitation spectrum with \(\mu\) playing the role of a fiducial mass scale which appears due to spontaneous breaking of the residual conformal invariance by the condition like \(t = \mu\tau\) in the \(R_v \times F/G\) case. A drawback of this “on-shell” approach is that the equations in the \(A_+ = A_- = 0\) gauge do not in general follow from a local Lagrangian for the remaining independent degrees of freedom.

If instead one imposes the gauge on the group element \(g\) and then integrates out \(A_\pm\) one, in general, gets a sigma model with target space metric which is singular at the natural vacuum point \(g = 1\), so that the perturbative expansion near this point appears to be not well defined. This is due to the fact that the term \(A_+ A_+ - g^{-1} A_+ g A_-\) in the vector-gauged WZW model is degenerate at \(g = 1\). In the case when \(\mathfrak{h}\) is Abelian this problem can be cured by using the automorphism \(\tau(A) = -A\) as we did in the AdS\(_3\) \(\times\) S\(^3\) case. However, already for \(S^n\) or AdS\(_n\) with \(n \geq 4\) the gauge algebra \(\mathfrak{h}\) is nonabelian and this modification does not help.

One may try a more general modification of the Lagrangian as in the asymmetrically gauged WZW model [23]. Namely, one may choose the gauge groups acting from the left and the right to be different embeddings of \(H\) into \(G\). However, this generalisation does not seem to be relevant in the Pohlmeyer reduction context as the left and the right gauge groups are determined by the choice of the fixed elements \(T_+\) and \(T_-\) from \(p = \mathfrak{t} \oplus \mathfrak{g}\) which define \(P_+ = \mu T_+\) and \(P_- = \mu g^{-1} T_- g\) that solve the Virasoro conditions. In fact, for a rank 1 coset all such choices are equivalent and, moreover, \(T_-\) can be made equal to \(T_+\) by an appropriate redefinition of the field \(g\).

An alternative to the “on-shell” gauge on \(A_+\) or the “off-shell” gauge on \(g\) is an intermediate choice: to treat \(g\) and \(A_\pm\) on an equal footing, expand near \(g = 1, A_\pm = 0\) point and impose a gauge on some combination of fluctuations of \(g\) and \(A_\pm\). That may lead to a non-degenerate perturbation theory but it is not clear a priori if all of the resulting modes are then massive. A closely related possibility is to parametrize \(A_+ = h^{-1} \partial_+ h, A_- = h^{-1} \partial_- h\) and then replace the gWZW part of the action by a difference of the two WZW actions \(I(h^{-1} g h') - I(h^{-1} h')\). One would then need to decide how gauge-
fix (and redefine) the fields \(g, h, \tilde{h}'\) to make the expansion near \(g = 1\) regular. Further discussion of this will appear in [22].

More generally, one may consider an expansion near a non-trivial background of the reduced model that corresponds to some solitonic solution of the original string model. For example, a general constant solution of the complex sine-Gordon theory corresponds to a rigid string solution on \(R_t \times S^3\) and expanding near it leads to a non-degenerate (and UV finite) perturbation theory [22]. More generally, vacuum solutions with constant Lagrange multiplier for the embedding coordinates (or constant value of the field that enters the potential of the reduced model for strings on \(R \times S^n\) or \(AdS_n \times S^1\) or \(AdS_n \times S^m\)) correspond to rigid circular strings with several angular momenta constructed in [24].

### 4.3 Relation between solutions of the reduced and the original model

Let us now discuss in which sense the classical dynamics of the reduced model determines the dynamics of the original theory for strings on \(R_t \times S^3\). A natural dynamical variable of the original model is a group element \(f \in F\). The equations of motion and the constraints are expressed in terms of the current \(J_\pm = f^{-1} \partial_\pm f\) that automatically satisfies the Maurer–Cartan equation. They read

\[
D_+ P_- = 0, \quad D_- P_+ = 0, \quad -\frac{1}{2} \text{Tr}(P_+ P_+) = \mu^2, \quad -\frac{1}{2} \text{Tr}(P_- P_-) = \mu^2. \quad (4.18)
\]

The Pohlmeyer reduction procedure amounts to imposing first a particular \(G\)-gauge condition (i.e. the reduction gauge) [5]. In this gauge the components of the current \(J\) are expressed in terms of the new fields \(g, A_\pm\) satisfying the equations of motion of the reduced system

\[
J_+ = g^{-1} A_+ g + g^{-1} \partial_+ g + \mu T, \quad J_- = A_- + \mu g^{-1} T g. \quad (4.19)
\]

Since the equations of motion of the reduced system are essentially the MC equations for \(J\) (parametrized by \(g, A_\pm\)) one can reconstruct the configuration \(f(\sigma^+, \sigma^-)\) of the original model in terms of a solution \((g, A_\pm)\) of the reduced system or \(J_\pm(\sigma^+, \sigma^-)\) by solving the auxiliary linear problem:

\[
f^{-1} \partial_+ f = g^{-1} A_+ g + g^{-1} \partial_+ g + \mu T, \quad f^{-1} \partial_- f = A_- + \mu g^{-1} T g. \quad (4.20)
\]

This system has a unique solution for any initial data \(f|_{\sigma^+ = \sigma^-} = f_0\) \((f_0 \in F)\) specified at a given point on the world sheet.

---

14 An example is \((E, S; J)\) circular string: it is stretched along a circle in \(AdS_3\) and a circle in \(S^1\). It has as its charges the energy \(E\) and the spin \(S\) in \(AdS_3\) and the spin \(J\) in \(S^1\). Less trivial solitonic solutions of the reduced models correspond to more complicated “inhomogeneous” string solutions. One example is the “giant magnon” (on an infinite line) in \(R \times S^2\) that was constructed in [25] from the sine-Gordon soliton (see also [26]). It can be viewed [27] as a special case of an infinite spin limit of a folded \((J_1, J_2)\) string on \(R \times S^3\) where \(J_1\) and \(E\) are taken to infinity. For regular closed string the folded \((J_1, J_2)\) string on \(R \times S^3\) [28] originates from a regular soliton of (complex) sine-Gordon model. Same remark applies to folded string in \(AdS_3 \times S^1\) [29] and spiky string [30] that correspond to solitons of the sinh-Gordon model [21].

15 In the case of strings on \(AdS_n \times S^m\) one needs also to use the conformal transformations in order to impose \(T_{++} = \pm \mu^2, \ T_{--} = \pm \mu^2\) in the \(AdS_n\) or \(S^n\) sectors together with the Virasoro constraints. The same also applies to the Pohlmeyer reduction of the \(F/G\) coset sigma model (in contrast to the reduction of strings on \(F/G \times R_t\) where \(T_{++} = \mu^2, \ T_{--} = \mu^2\) are just the Virasoro constraints in the conformal gauge supplemented by the condition \(t = \mu t\) fixing the residual conformal diffeomorphisms).
The group $F$ naturally acts from the left on the initial data. This action induces the left action of $F$ on the space of solutions to (4.20). In this way one recovers the global $F$ symmetry present in the original model but not seen in the reduced model, i.e. in the formulation in terms of the currents (which are the invariants of the global left $F$-action). To summarize, any solution to the original system is equivalent to a solution of (4.20) for an appropriate choice of the solution $J_{\pm}(\sigma^+, \sigma^-)$ of the reduced system and an initial condition $f_0$. The equivalence means that they are related by a $G$-gauge transformation and a conformal reparametrization.

### 4.4 Conserved charges

While the original global $F$ symmetry is not visible in the reduced model formulated in terms of the currents, one can still classify the solutions (and thus states) of the reduced model by values of (higher) Casimir operators which are also invariant under $F$.

Indeed, let us consider the counterpart of the natural vacuum solution of the reduced system $g = 1$, $A_{\pm} = 0$ in the original string model on $R_t \times F/G$. Here we shall use the on-shell gauge $A_{\pm} = 0$. Then eq. (4.20) takes the form

$$f^{-1}\partial_+ f = \mu T, \quad f^{-1}\partial_- f = \mu T,$$

and it can be formally solved by

$$f = e^{\mu(\sigma^+ + \sigma^-)} f_0 = e^{\mu \tau} f_0.$$  \hspace{1cm} (4.22)

The $f$-valued conserved (Noether) current corresponding to the global left action of the group $F$ on the coset $F/G$ has the form (see, e.g., [31])

$$j^a = f (f^{-1}\partial_a f)_p f^{-1},$$  \hspace{1cm} (4.23)

as one can see from the fact that the equations of motion for the coset model can be written as $\partial_a j^a = 0$. Evaluating the corresponding conserved charge on the above solution one gets:

$$M = \int d\sigma \, j_\tau = \mu \int d\sigma \, f_0 T f_0^{-1},$$  \hspace{1cm} (4.24)

where we have used that $j_\tau = (f^{-1}\partial_\tau f)_p = T$. Assuming the space direction $\sigma$ to be compact $(0 < \sigma \leq 2\pi)$ one gets a non-zero value for the quadratic Casimir

$$K \equiv -\frac{1}{2} \text{Tr} (MM) = (2\pi \mu)^2.$$  \hspace{1cm} (4.25)

Here we have used the convention $\text{Tr}(TT) = -2$.

For more general solutions it is nontrivial to find the explicit values of the Casimir operators. For example, let us consider the $S^n$ model described by the embedding coordinates: $L = \partial_+ X_1 \partial_- X_1$, $X^2 = 1$. Then the $F = SO(n+1)$ symmetry leads to the Noether currents conserved on the equations of motion

$$(j_{iak})_a = X_i \partial_a X_k - X_k \partial_a X_i, \quad \partial_a j^a_{iak} = 0.$$  \hspace{1cm} (4.26)
The corresponding charges and the quadratic Casimir of $SO(n + 1)$

\[ M_{ik} = \int d\sigma (j_{ik})_\tau, \quad K = \frac{1}{2} M_{ik} M_{ik} \]  

(4.27)

are then conserved in $\tau$. Explicitly,

\[ (j_{ik})_\tau (j_{ij})_\tau = 2 \partial_\tau X_i \partial_\tau X_i = -2 \text{Tr}(P_\tau P_\tau) \]

Since in the vacuum of the reduced model $g = 1$ we have $P_+ = \mu T$, $P_- = \mu g^{-1} T g = \mu T$ then $P_\tau = P_+ + P_- = 2 \mu T$, $P_\sigma = 0$ so that again $(j_{ik})_\tau (j_{ij})_\tau = 8 \mu^2$. But in general

\[ K = \frac{1}{2} \int d\sigma (j_{ik})_\tau (\sigma) \int d\sigma' (j_{ik})_\tau (\sigma') \]  

(4.28)

so that it is not clear if $K$ is non-zero and is related to $\mu$ unless $(j_{ik})_\tau$ is constant in $\sigma$. The reduced theory thus does not tell us much about the charges of the original theory before we actually solve the linear problem for $X_i$ or, equivalently, for $f$.

Acknowledgments

We are grateful to G. Arutyunov, U. Lindstrom, L. Miramontes, G. Papadopoulos and especially R. Roiban for useful discussions. A.A.T. is grateful to the organizers of the International conference on progress of string theory and quantum field theory at the Osaka City University in December 2007 for their kind hospitality. The work of M.G. was partially supported by the Dynasty foundation, RFBR grant 07-01-00523, and grant LSS-4401.2006.2.

A  Details of computation of $AdS_3 \times S^3$ reduced Lagrangian

We shall use the following basis in $su(1, 1)$ and $su(2)$

\[
\begin{align*}
\bar{R}_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \bar{R}_2 &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, & \bar{R}_3 &= \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \\
R_1 &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, & R_2 &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, & R_3 &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\end{align*}
\]

(A.1) (A.2)

The parametrization of $g$ in terms of the Euler angles reads

\[
\begin{align*}
g_A &= (g_A)_2 (g_A)_1 (g_A)_2, & (g_A)_1 &= \exp(\phi \bar{R}_1), & (g_A)_2 &= \exp\left(\frac{1}{2} \chi \bar{R}_2\right), \\
g_s &= (g_s)_2 (g_s)_1 (g_s)_2, & (g_s)_1 &= \exp(\varphi R_1), & (g_s)_2 &= \exp\left(\frac{1}{2} \theta R_2\right),
\end{align*}
\]

(A.3) (A.4)

or more explicitly:

\[
\begin{align*}
(g_A)_1 &= \begin{pmatrix} \cosh \phi & \sinh \phi \\ \sinh \phi & \cosh \phi \end{pmatrix}, & (g_A)_2 &= \begin{pmatrix} \exp\left(\frac{1}{2} i \chi\right) & 0 \\ 0 & \exp\left(-\frac{1}{2} i \chi\right) \end{pmatrix}, \\
(g_s)_1 &= \begin{pmatrix} \cos \varphi & i \sin \varphi \\ i \sin \varphi & \cos \varphi \end{pmatrix}, & (g_s)_2 &= \begin{pmatrix} \exp\left(\frac{1}{2} i \theta\right) & 0 \\ 0 & \exp\left(-\frac{1}{2} i \theta\right) \end{pmatrix}.
\end{align*}
\]

(A.5) (A.6)
In order to solve (3.29) (3.30) for \( A_+, A_- \), let us note the following useful relations:

\[
\begin{align*}
(g_A^{-1} \partial_+ g_A)_b &= \partial_+ \chi \frac{1 + \cosh 2\phi}{2} \bar{R}_2, \\
(g_s^{-1} \partial_+ g_s)_b &= \partial_+ \theta \frac{1 + \cos 2\varphi}{2} R_2, \\
(g_A^{-1} A_+ g_A)_b &= \cosh 2\phi A_+^A, \\
(g_s^{-1} A_+ g_s)_b &= \cos 2\varphi A_+^S,
\end{align*}
\] (A.7)

where we have used \((g_A)^1 R_2(g_A)_1 = \cosh 2\phi \bar{R}_2 + \sinh 2\phi \bar{R}_3, (g_s)^1 R_2(g_s)_1 = \cos 2\varphi R_2 + \sin 2\varphi R_3, (g_A)^2 R_1(g_A)_2 = \cosh \chi \bar{R}_1 - \sin \chi \bar{R}_3, (g_s)^2 R_1(g_s)_2 = \cos \theta R_1 - \sin \theta R_3, (g_A)^1 R_2(g_A)_1(g_A)_2 = \cosh 2\phi \bar{R}_2 + \sinh 2\phi \sin \chi \bar{R}_1 + \sinh 2\phi \cos \chi \bar{R}_3, (g_s)^1 R_2(g_s)_1(g_s)_2 = \cos 2\varphi R_2 + \sin 2\varphi \sin \theta R_1 + \sin 2\varphi \cos \theta R_3, (g_A)^1 R_2(g_A)_1(g_A)_2 = \cosh 2\phi \bar{R}_2 + \sinh 2\phi \sin \chi \bar{R}_1 + \sinh 2\phi \cos \chi \bar{R}_3, (g_s)^1 R_2(g_s)_1(g_s)_2 = \cos 2\varphi R_2 + \sin 2\varphi \sin \theta R_1 + \sin 2\varphi \cos \theta R_3.

Parametrizing the fermions according to (3.32) one arrives at (3.31) and then gets the explicit solution for \( A_+, A_- \).

In computing the third line of (3.34) the following relations are useful

\[
(\partial_- g_A g_A^{-1})_b = \frac{1 + \cosh 2\phi}{2} \partial_- \chi \bar{R}_2, \\
(\partial_- g_s g_s^{-1})_b = \frac{1 + \cos 2\varphi}{2} \partial_- \theta R_2.
\] (A.9)

Let us note also that the fermionic interaction term entering the Lagrangian can be computed in terms of \( 2 \times 2 \) matrices using the observation that the two contributions to the supertrace of \( g^{-1} \Psi_2 g \Psi_1 \) are complex conjugates of one another so that \( S \text{Tr}(g^{-1} \Psi_2 g \Psi_1') = -2 \text{Im}[\text{Tr}(g_A^{-1} \Psi_2 g s \Psi_1')] \).

Let us also give some details on direct computation of the reduced Lagrangian in the axial gauging case. In terms of the Euler angles the parametrization of the group element reads as

\[
g_A = \begin{pmatrix}
\cosh \phi & e^{-i\chi} \sinh \phi \\
e^{i\chi} \sinh \phi & \cosh \phi
\end{pmatrix}, \\
g_s = \begin{pmatrix}
\cos \varphi & ie^{-i\theta} \sin \varphi \\
i e^{i\theta} \sin \varphi & \cos \varphi
\end{pmatrix}.
\] (A.10)

One can then solve for the gauge fields using their equations

\[
\begin{align*}
-A_+ &= (\tilde{A}_+)_b, \\
\tilde{A}_+ &\equiv g^{-1} \partial_+ g + g^{-1} A_+ g - \frac{1}{2} [[t, \Psi_1'], \Psi_1'], \\
A_- &= (\tilde{A}_-)_b, \\
\tilde{A}_- &\equiv g \partial_- g^{-1} - g A_- g^{-1} - \frac{1}{2} [[t, \Psi_2'], \Psi_2'].
\end{align*}
\] (A.11)

following from the Lagrangian (3.23) with \( \tau(a) = -a \). Similarly to the previous case one finds

\[
A_+^S = -\frac{\partial_+ \theta (1 - \cosh 2\varphi) + 2(\alpha \beta - \gamma \delta)}{2(1 + \cosh 2\varphi)} \bar{R}_2, \\
A_+^A = -\frac{\partial_+ \chi (1 - \cosh 2\phi) - 2(\alpha \beta - \gamma \delta)}{2(1 + \cosh 2\phi)} \bar{R}_2,
\]

and also the expressions for \( A_- \) with \( \partial_+ \chi \rightarrow -\partial_- \chi \) and \( \alpha \beta - \gamma \delta \rightarrow \lambda \nu - \rho \sigma \). Note also the following useful relations:

\[
(\partial_- g_A g_A^{-1})_b = -\frac{1 - \cosh 2\phi}{2} \partial_- \chi \bar{R}_2, \\
(\partial_- g_s g_s^{-1})_b = -\frac{1 - \cos 2\varphi}{2} \partial_- \theta R_2.
\] (A.13)
Eliminating $A_{\pm}$ as in the vector gauge case one finds the bosonic part of the Lagrangian that comes from the WZW and potential terms

$$L_{axial}^1 = \partial_+ \varphi \partial_- \varphi + \frac{1}{2} (1 - \cos 2 \varphi) \partial_+ \theta \partial_- \theta$$

$$+ \partial_+ \phi \partial_- \phi - \frac{1}{2} (1 - \cosh 2 \phi) \partial_+ \chi \partial_- \chi + \frac{\mu^2}{2} (\cos 2 \varphi - \cosh 2 \phi).$$  
(A.14)

The fermionic interaction term together with the fermionic kinetic terms give

$$L_{axial}^2 = \alpha \partial_- \alpha + \beta \partial_- \beta + \gamma \partial_- \gamma + \delta \partial_- \delta + \lambda \partial_+ \lambda + \nu \partial_+ \nu + \rho \partial_+ \rho + \sigma \partial_+ \sigma$$

$$- 2 \mu \left( \cosh \phi \cos \varphi (-\lambda \beta + \nu \alpha + \rho \delta - \delta \gamma) + \sin \phi \sin \varphi \left[ \cos (\chi + \theta) (-\rho \alpha - \sigma \beta + \lambda \gamma + \nu \delta) - \sin (\chi + \theta) (-\rho \beta + \sigma \alpha - \lambda \delta + \nu \gamma) \right] \right).$$  
(A.15)

Finally, the terms that originate from the elimination of $A_{\pm}$ are

$$L_{axial}^3 = - \left[ \partial_+ \chi (1 - \cosh 2 \phi) - 2(\alpha \beta - \gamma \delta) \right] \left[ \partial_- \chi (1 - \cosh 2 \phi) - 2(\lambda \nu - \rho \sigma) \right]$$

$$\frac{2(1 + \cosh 2 \phi)}{(1 + \cos 2 \phi)} + \left[ \partial_+ \theta (1 - \cos 2 \phi) + 2(\alpha \beta - \gamma \delta) \right] \left[ \partial_- \theta (1 - \cos 2 \phi) + 2(\lambda \nu - \rho \sigma) \right] \frac{2(1 + \cos 2 \phi)}{(1 + \cos 2 \phi)}.$$  
(A.16)

Then $L_{axial}^{tot} = L_{axial}^1 + L_{axial}^2 + L_{axial}^3$. The purely bosonic terms in $L_{axial}^1$ and $L_{axial}^3$ combine into the direct sum of the CSG action and its “hyperbolic” counterpart

$$L_{B}^{axial} = \partial_+ \varphi \partial_- \varphi + \tan^2 \varphi \partial_+ \theta \partial_- \theta + \partial_+ \phi \partial_- \phi + \tanh^2 \phi \partial_+ \chi \partial_- \chi + \frac{\mu^2}{2} (\cos 2 \varphi - \cosh 2 \phi),$$  
(A.17)

while the fermionic terms give

$$L_{F}^{axial} = L_{B}^{axial} + \tan^2(\varphi) [\partial_+ \theta (\lambda \nu - \rho \sigma) + \partial_- \theta (\alpha \beta - \gamma \delta)]$$

$$- \tanh^2(\phi) [\partial_+ \chi (\lambda \nu - \rho \sigma) + \partial_- \chi (\alpha \beta - \gamma \delta)] + (\alpha \beta - \gamma \delta) (\lambda \nu - \rho \sigma) \left[ \frac{1}{\cos^2 \varphi} - \frac{1}{\cosh^2 \phi} \right].$$  
(A.18)

Redefining the fermions according to $\alpha \to \delta, \beta \to -\alpha, \gamma \to -\gamma, \gamma \to \beta$ and combining all of the terms together one indeed gets Lagrangian (3.44).

References

[1] R. R. Metsaev and A. A. Tseytlin, “Type IIB superstring action in AdS(5) x S(5) background,” Nucl. Phys. B 533, 109 (1998) [arXiv:hep-th/9805028].

[2] M. Grigoriev and A. A. Tseytlin, “Pohlmeyer reduction of AdS5 x S5 superstring sigma model,” Nucl. Phys. B 800, 450 (2008) [arXiv:0711.0155 [hep-th]].

[3] A. Mikhailov and S. Schafer-Nameki, “Sine-Gordon-like action for the Superstring in AdS(5) x S(5),” JHEP 0805, 075 (2008) [arXiv:0711.0195 [hep-th]].
[4] K. Pohlmeyer, “Integrable Hamiltonian Systems And Interactions Through Quadratic Constraints,” Commun. Math. Phys. 46, 207 (1976).

[5] I. Bakas, Q. H. Park and H. J. Shin, “Lagrangian Formulation of Symmetric Space sine-Gordon Models,” Phys. Lett. B 372, 45 (1996) [arXiv:hep-th/9512030].

[6] R. R. Metsaev, “Type IIB Green-Schwarz superstring in plane wave Ramond-Ramond background,” Nucl. Phys. B 625, 70 (2002) [arXiv:hep-th/0112044]. D. E. Berenstein, J. M. Maldacena and H. S. Nastase, “Strings in flat space and pp waves from N = 4 super Yang Mills,” JHEP 0204, 013 (2002) [arXiv:hep-th/0202021].

[7] J. Rahmfeld and A. Rajaraman, “The GS string action on AdS(3) x S(3) with Ramond-Ramond charge,” Phys. Rev. D 60, 064014 (1999) [hep-th/9809164].

[8] I. Pesando, “The GS type IIB superstring action on AdS(3) x S(3) x T4,” JHEP 9902, 007 (1999) [hep-th/9809145]. J. Park and S. J. Rey, “Green-Schwarz superstring on AdS(3) x S(3),” JHEP 9901, 001 (1999) [hep-th/9812062]. N. Berkovits, C. Vafa and E. Witten, “Conformal field theory of AdS background with Ramond-Ramond flux,” JHEP 9903, 018 (1999) [hep-th/9902098]. R. R. Metsaev and A. A. Tseytlin, “Superparticle and superstring in AdS(3) x S(3) Ramond-Ramond background in light-cone gauge,” J. Math. Phys. 42, 2987 (2001) [hep-th/0011191].

[9] J. L. Miramontes, “T-duality in massive integrable field theories: The homogeneous and complex sine-Gordon models,” Nucl. Phys. B 702 (2004) 419 [arXiv:hep-th/0408119].

[10] L. Miramontes, private communication (February, 2008).

[11] C. R. Fernandez-Pousa, M. V. Gallas, T. J. Hollowood and J. L. Miramontes, “The symmetric space and homogeneous sine-Gordon theories,” Nucl. Phys. B 484, 609 (1997) [hep-th/9606032].

[12] O. A. Castro-Alvaredo, “Bootstrap methods in 1+1 dimensional quantum field theories: The homogeneous sine-Gordon models,” hep-th/0109212.

[13] L. D. Faddeev and N. Y. Reshetikhin, “Integrability Of The Principal Chiral Field Model In (1+1)-Dimension,” Annals Phys. 167 (1986) 227. T. Klose and K. Zarembo, “Bethe ansatz in stringy sigma models,” J. Stat. Mech. 0605, P006 (2006) [arXiv:hep-th/0603039].

[14] N. Berkovits, M. Bershadsky, T. Hauer, S. Zhukov and B. Zwiebach, “Superstring theory on AdS(2) x S(2) as a coset supermanifold,” Nucl. Phys. B 567 (2000) 61 [hep-th/9907200].

[15] K. I. Kobayashi and T. Uematsu, “N=2 supersymmetric Sine-Gordon theory and conservation laws,” Phys. Lett. B 264, 107 (1991).

[16] E. Napolitano and S. Sciuto, “The N=2 Supersymmetric Generalization Of The Complex Sine-Gordon Model,” Phys. Lett. B 113, 43 (1982).
[17] M. Rocek, C. h. Ahn, K. Schoutens and A. Sevrin, “Superspace WZW models and black holes,” arXiv:hep-th/9110035. S. J. J. Gates and W. Merrell, “D=2 N=(2,2) Semi Chiral Vector Multiplet,” JHEP 0710, 035 (2007) [arXiv:0705.3207]. W. Merrell, L. A. P. Zayas and D. Vaman, “Gauged (2,2) Sigma Models and Generalized Kahler Geometry,” JHEP 0712, 039 (2007) [arXiv:hep-th/0610116].

[18] W. Machin and G. Papadopoulos, “Supersymmetric gauge theories, vortices and equivariant cohomology,” Class. Quant. Grav. 20, 1233 (2003) [arXiv:hep-th/0208076].

[19] H. Aratyn, J. F. Gomes and A. H. Zimerman, “Supersymmetry and the KdV equations for integrable hierarchies with a half-integer gradation,” Nucl. Phys. B 676 (2004) 537. [arXiv:hep-th/0309099]. J. L. Gervais and M. V. Savelev, “Higher Grading Generalizations Of The Toda Systems,” Nucl. Phys. B 453 (1995) 449 [arXiv:hep-th/9505047].

[20] H. J. De Vega and N. G. Sanchez, “Exact Integrability Of Strings In D-Dimensional De Sitter Space-Time,” Phys. Rev. D 47, 3394 (1993). H. J. de Vega, A. V. Mikhailov and N. G. Sanchez, “Exact string solutions in (2+1)-dimensional de Sitter space-time,” Theor. Math. Phys. 94, 166 (1993) [Teor. Mat. Fiz. 94N2, 232 (1993)] [arXiv:hep-th/9209047]. F. Combes, H. J. de Vega, A. V. Mikhailov and N. G. Sanchez, “Multistring solutions by soliton methods in de Sitter space-time,” Phys. Rev. D 50, 2754 (1994) [arXiv:hep-th/9310073]. A. L. Larsen and N. G. Sanchez, “Sinh-Gordon, Cosh-Gordon and Liouville Equations for Strings and Multi-Strings in Constant Curvature Spacetimes,” Phys. Rev. D 54, 2801 (1996) [arXiv:hep-th/9603049].

[21] A. Jevicki, K. Jin, C. Kalousios and A. Volovich, “Generating AdS String Solutions,” JHEP 0803, 032 (2008) [arXiv:0712.1193 [hep-th]]. A. Jevicki and K. Jin, “Solitons and AdS String Solutions,” arXiv:0804.0412 [hep-th].

[22] R. Roiban and A.A. Tseytlin, to appear.

[23] T. Quella and V. Schomerus, “Asymmetric cosets,” JHEP 0302, 030 (2003) [arXiv:hep-th/0212119].

[24] G. Arutyunov, J. Russo and A. A. Tseytlin, “Spinning strings in AdS(5) x S5: New integrable system relations,” Phys. Rev. D 69 (2004) 086009 [arXiv:hep-th/0311004].

[25] D. M. Hofman and J. M. Maldacena, “Giant magnons,” J. Phys. A 39, 13095 (2006) [hep-th/0604135].

[26] H. Y. Chen, N. Dorey and K. Okamura, “Dyonic giant magnons,” JHEP 0609, 024 (2006) [hep-th/0605155].

[27] J. A. Minahan, A. Tirziu and A. A. Tseytlin, “Infinite spin limit of semiclassical string states,” JHEP 0608, 049 (2006) [arXiv:hep-th/0606145].

[28] S. Frolov and A. A. Tseytlin, “Rotating string solutions: AdS/CFT duality in non-supersymmetric sectors,” Phys. Lett. B 570, 96 (2003) [arXiv:hep-th/0306143].
[29] S. Frolov and A. A. Tseytlin, “Semiclassical quantization of rotating superstring in AdS(5) x S(5),” JHEP 0206, 007 (2002) [arXiv:hep-th/0204226].

[30] M. Kruczenski, “Spiky strings and single trace operators in gauge theories,” JHEP 0508, 014 (2005) [arXiv:hep-th/0410226].

[31] H. Eichenherr and M. Forger, “More About Nonlinear Sigma Models On Symmetric Spaces,” Nucl. Phys. B 164, 528 (1980) [Erratum-ibid. B 282, 745 (1987)].