The Aoki phase for \( N_f = 2 \) Wilson fermions revisited

A. Sternbeck\(^a\), E.-M. Ilgenfritz\(^a\), W. Kerler\(^a\), M. Müller-Preussker\(^a\) and H. Stüben\(^b\)

\(^a\)Institut für Physik, Humboldt-Universität zu Berlin, D-12489 Berlin, Germany
\(^b\)Konrad-Zuse-Zentrum für Informationstechnik Berlin, D-14195 Berlin, Germany

We report on a numerical reinvestigation of the Aoki phase in full lattice QCD with two flavors of unimproved Wilson fermions. For zero temperature the Aoki phase can be confirmed at inverse coupling \( \beta = 4.0 \) and \( \beta = 4.3 \), but not at \( \beta = 4.6 \) and \( \beta = 5.0 \). At non-zero temperature the Aoki phase was found to exist also at \( \beta = 4.6 \).

1. The Aoki phase

In studies of chiral symmetry breaking it is desirable to start from an (almost) chirally invariant formulation of lattice fermions. At present this is best realized by using fermions satisfying the Ginsparg-Wilson relation, e.g. using the overlap operator, or by working with the domain wall framework [1]. In such approaches Wilson fermions are still important as an input.

For the Wilson-Dirac operator (which breaks chiral invariance explicitly) in \( N_f = 2 \) lattice QCD Aoki [2] has predicted that in a certain parameter range there is a phase in which parity and flavor symmetry are both spontaneously broken. It is separated from an unbroken phase by a line of second order phase transition on which the pion states are expected to become massless [3]. In Fig. 1 the proposed phase diagram is sketched in the \((\beta, \kappa)\)-plane.

In a recent paper [4] it has been pointed out that the features of the Aoki phase are of relevance for locality and for restoration of chiral invariance in quenched and full QCD with Ginsparg-Wilson and domain wall fermions. Accordingly, the region of the Aoki phase has to be avoided in such investigations in order not to spoil physical reliability.

In the light of this it is important to have precise information about the region where the Aoki phase really occurs. In the past it has been questioned [5] whether the Aoki phase exists also at larger \( \beta \)-values, i.e. towards the continuum. However, previous investigations of this issue did not lead to a unique answer. Therefore we have addressed this problem again [6].

Here we report on results showing that most likely the Aoki phase does not exist for \( \beta \geq 4.6 \) at zero temperature. Furthermore, we present data for finite temperature indicating such broken phase still at \( \beta = 4.6 \) and connecting it to the finite temperature phase transition.

![Figure 1. Phase structure proposed by Aoki et al. The shaded region B is the phase where flavor and parity are spontaneously broken. Both symmetries are conserved in regions A.](image-url)

2. Simulation details

To investigate the Aoki phase we have simulated lattice QCD with two flavors of unimproved Wilson fermions using the Hybrid Monte Carlo algorithm. An explicitly symmetry-breaking source...
term was added to the fermion matrix $M_W$,

$$M(h) = M_W + hi\gamma_5\tau^3,$$

(1)

since without the added term the order parameter $\langle \bar{\psi}i\gamma_5\tau^3\psi \rangle$ would always be zero on a finite lattice. Then $\langle \bar{\psi}i\gamma_5\tau^3\psi \rangle$ was measured varying the lattice size $V$ and the (non-zero) $h$-values. The order parameter $\langle \bar{\psi}i\gamma_5\tau^3\psi \rangle_{h=0}$ monitoring spontaneous symmetry breaking is obtained by taking the double limit in the following order

$$\langle \bar{\psi}i\gamma_5\tau^3\psi \rangle_{h=0} = \lim_{h\to0} \lim_{V\to\infty} \langle \bar{\psi}i\gamma_5\tau^3\psi \rangle.$$

(2)

For zero temperature the simulations were performed on lattices ranging from $4^4$ to $12^4$ at $\beta$-values 4.0, 4.3, 4.6, and 5.0, with $\kappa$ and $h$ in the intervals $0.15 \leq \kappa \leq 0.28$ and $0.003 \leq h \leq 0.04$, respectively. For each $\beta$ and each lattice size (starting on $6^4$) the physically relevant values of $\kappa$ were determined first, namely the region where $\langle \bar{\psi}i\gamma_5\tau^3\psi \rangle$ has a peak at finite $h$.

This is shown on the left hand sides of Figs. 2 and 3. One sees that for $\beta = 4.3$ and $\beta = 4.6$ the crucial values are near $\kappa = 0.21$ and $\kappa = 0.1981$, respectively. At these values we have increased the lattice sizes until the measurements for different sizes agreed within errors. In this way we achieved that we can treat our largest lattices as infinitely large.

3. The extrapolation $h \to 0$

As described in Ref. [6] a reasonable ansatz to extrapolate the data to $h = 0$ is

$$f(h) = A + Bh^C + \ldots.$$  

(3)

With $C$ as a free parameter it describes the data well. Indeed, the parameter of interest, $A$, is then robust against the introduction of correction terms linear and quadratic in $h$.

At $\beta = 4.3$ the parameter $\langle \bar{\psi}i\gamma_5\tau^3\psi \rangle$ has a non-vanishing extrapolation, while at $\beta = 4.6$ the best fit gives $A = 0$. Combined with similar simulation results at $\beta = 4.0$ and $\beta = 5.0$ (see [6]) we confirm the Aoki phase at $(\beta, \kappa) = (4.0, 0.22)$ and $(4.3, 0.21)$. The fit parameters $B$ and $C$ turn out to agree for different $\beta$ within errors (the values are $B \approx 1$ and $C \approx 0.65$). At $\beta = 4.6$ the ansatz
Figure 4. The area of the phase diagram studied in Ref. [6]. Squares indicate that $\langle \bar{\psi} i\gamma_5 \tau^3 \psi \rangle_{h=0}$ is finite, diamonds indicate that $\langle \bar{\psi} i\gamma_5 \tau^3 \psi \rangle_{h=0}$ vanishes, and stars denote points where $\langle \bar{\psi} i\gamma_5 \tau^3 \psi \rangle_{h=0} \neq 0$ is uncertain. The lines sketch the limits of the Aoki phase. The data indicate that the point $(\beta, \kappa) = (4.0, 0.215)$ (circle) is very close to the lower critical line [6].

4. Outlook for finite temperature

We have also performed simulations for the finite-temperature case. In contrast to zero temperature we find the Aoki phase at $\beta = 4.6$ as shown in Fig. 5. One can see that the Polyakov loop (open symbols) steeply rises near $\kappa = 0.19705$ where $\langle \bar{\psi} i\gamma_5 \tau^3 \psi \rangle$ (filled symbols) extrapolates to a finite value. This means that the finite-temperature phase transition occurs close to the Aoki phase. We were not able to detect a separation of this transition from the Aoki phase. This is in contrast to Ref. [7] where a scenario with separated transition lines is proposed.

Figure 5. The parameter $\langle \bar{\psi} i\gamma_5 \tau^3 \psi \rangle$ and the Polyakov loop as function of $\kappa$ on a $8^3 \times 4$ and a $10^3 \times 4$ lattice at $h = 0.005$ and $\beta = 4.6$ (left, the lines are drawn to guide the eye). At $\kappa = 0.19705$ we obtained $\langle \bar{\psi} i\gamma_5 \tau^3 \psi \rangle \neq 0$ from an extrapolation explained in the text (right).

Acknowledgements

All simulations were done on the Cray T3E at Konrad-Zuse-Zentrum für Informationstechnik Berlin. A. S. would like to thank the DFG-funded graduate school GK 271 for support.

References

[1] For a review see e.g. F. Niedermayer, Nucl. Phys. Proc. Suppl. 73, 105 (1998).
[2] S. Aoki, Phys. Rev. D30, 2653 (1984).
[3] S. Aoki, Phys. Rev. Lett. 57, 3136 (1986).
[4] M. Golterman, Y. Shamir, hep-lat/0306002; Y. Shamir, talk at Lattice 2003.
[5] For a discussion see S. Sharpe, R. Singleton, Jr., Phys. Rev. D58, 074501 (1998).
[6] E.-M. Ilgenfritz, W. Kerler, M. Müller-Preussker, A. Sternbeck, H. Stüben, hep-lat/0309057.
[7] S. Aoki, Nucl. Phys. Proc. Suppl. 60A, 206 (1998), and references given there.