Retrospective on Quantization

Christian Frønsdal
Department of Physics and Astronomy, University of California,
Los Angeles, CA 90024-1547, USA (fronsdal@physics.ucla.edu)

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Abstract. Quantization is still a central problem of modern physics. One example of an unsolved problem is the quantization of Nambu mechanics. After a brief comment on the role of Harrison cohomology, this review concentrates on the central problem of quantization of QCD and, more generally, quark confinement seen as a problem of quantization. Several suggestions are made, some of them rather extravagant.

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1. Quantization

A well known text book on Quantum Mechanics offers, in an early chapter, a Universal Quantization Paradigm. After the usual familiar applications one finds, towards the end of the book, a chapter dealing with the rigid rotator. Needless to say, the “universal” paradigm has been forgotten.

I remember trying to explain to a colleague, more than 25 years ago, what we were trying to understand about quantization. He would not agree that there was a problem. Of course, we were all aware of the connection between quantization and the theory of group representations; this was explained, in principle, already by von Neumann, in his proof of the uniqueness of the unitary representation of the Heisenberg group. But few people anticipated the greatly expanded role that has come to be played by the Heisenberg group in modern mathematics, or the much more sophisticated applications of group representation theory found in modern physics.

Moshé had come to realize that the essence of quantization is deformation theory, almost any deformation of a classical structure amounts to quantization [8]. Perhaps the most radical example turned up during his last attack on the problem of quantization of Nambu mechanics [5]. This problem leads to a search for an abelian deformation of the ordinary algebra of functions. It turns out that such deformations are governed by Harrison cohomology, just as non-abelian deformations are described within de Rham cohomology. But Harrison cohomology is trivial on smooth manifolds [22]. This is an interesting difficulty, but a greater obstacle is the fact that few mathematicians have taken up the study of this subject, so that no instructive examples of Harrison
cohomology are available \cite{19, 1, 17}. The following example may therefore be of some interest.

Consider the algebraic variety \( A \) (a cone, or degenerate conic) associated with conformal field theory in \( n - 1 \) real, Euclidean dimensions,

\[
A = \mathbb{R}^{n+1} / \sum_{i=1,...,n} x_i^2 - y^2 = 0.
\]

Let \( W \) be the commutative \( \mathbb{C} \)-algebra of polynomials in the coordinates \( x_1, ..., x_n, y \),

\[
W = \mathbb{C}[x_1, ..., x_n, y],
\]

and let \( W(A) \) be the quotient algebra

\[
W(A) = W / \sum_{i=1,...,n} x_i^2 - y^2 = 0
\]

of functions on \( A \). This algebra admits non trivial Harrison cohomology. We exhibit a particular, non trivial cocycle and use it to construct an abelian deformation of the algebra of functions on \( A \).

Every \( f \in W(A) \) has a unique decomposition,

\[
f(x,y) = f_1(x) + y f_2(x),
\]

with \( f_1, f_2 \) being polynomials in \( x_1, ..., x_n \). Define a deformed product, denoted \(*\), by

\[
f * g = fg + \lambda C(f,g), \quad C(f,g) := f_2 g_2, \quad f, g \in W(A).
\]

That \( C \) is a cocycle is readily verified; that is,

\[
dC(f,g,h) := fC(g,h) - C(fg,h) + C(f,g)h - C(f,g)h = 0.
\]

It follows that, to first order in \( \lambda \) (regarded as a formal variable) the deformed product is associative. (Actually, it is associative to all orders in \( \lambda \).) That it is not trivial can be seen by the observation that, after fixing \( \lambda \) in \( \mathbb{R} \), it is isomorphic to

\[
\mathbb{C}[x_1, ..., x_n, y] / \sum_{i=1,...,n} x_i^2 - y^2 + \lambda = 0,
\]

an algebra of functions on the nondegenerate conic \( \mathbb{R}^{n+1} / \sum_{i=1,...,n} x_i^2 - y^2 + \lambda = 0 \).

I suggest that Harrison cohomology may have interesting applications in quantum field theory, especially in connection with certain anomalies, and perhaps in the context of operator product expansions.
2. Color

The success of the concept of color in strong interaction physics goes far beyond solving the problem for which it was invented, that of allowing quarks to evade the constraint that connects spin and statistics [23, 16]. It is a pity, nevertheless, that this rather primitive idea of color has displaced the far more interesting concept of parastatistics. Wigner’s point of view, that led him to invent parastatistics, is curiously close to the more modern concept of current algebra [25]. Evidently, the observables are the currents. In conventional local quantum field theory they are bilinears in the enveloping algebra of a Heisenberg algebra; their algebraic structure is that of $Sp(\infty)$, and if the linears and the unit are included we obtain the super Lie algebra $OSp(\infty)$. Naturally, one is interested in representations of this algebra, more particularly in those of highest weight, realized in Fock spaces. The familiar ones are induced from one-dimensional representations of the subalgebra that is generated by $1$, $a_i$ and $a_i^*a_j + a_j^*a_i$, $i, j = 1, 2, \ldots$, of the form

$$a_i|0\rangle = 0, \quad (a_i^*a_j + a_j^*a_i)|0\rangle = \lambda \delta_{ij}|0\rangle.$$

The representation, and with it the statistics, is completely fixed by the number $\lambda$. This is a simple but fundamental fact that deserves to be mentioned in any text that aims to explain Bose-Einstein quantization. The simplest case is $\lambda = 1$; in this case one easily verifies that the vector

$$(a_i^*a_j - a_j^*a_i)|0\rangle$$

has zero norm. To get a Hilbert space, one must project out the subspace that contains all null vectors, hence the postulate that creation operators commute is actually redundant, as is the statement that the many-particle states are symmetric. Higher, integer values of $\lambda$ lead to parastatistics [18, 9].

Interest in parastatistics had another aim in the context of strong interactions, the problem of confinement [18]. Taken in a very general sense, the problem is to construct a theory that contains particles that do not appear as asymptotic states. This is very easy to do in a nonrelativistic context, but very difficult to reconcile with relativity and the precepts of local field theory. Let us talk about this extremely important subject.

The hypothetical theory that I want to discuss contains field operators that have some properties in common with the quark fields of QCD. Namely, they have the same formal transformation properties under various groups, and they appear in perturbation theory in much the same way. But the crucial properties are these:

1. The theory contains no asymptotic quarks and
2. the asymptotic states are (in some sense) systems consisting of several quarks.
Here are some of the ways that one can fantasize about these objects.

A. The quarks are much like ordinary particles, but they are prevented from separating from each other because they are tied together with rubber bands or flux tubes. This is the most “physical” model. The most basic properties of the theory are very different from the theory of free quarks, and the true nature of the theory is not revealed by perturbation theory.

B. The quarks are singletons; quantum fields that are not locally observable except when produced in pairs [10]. Ordinary massless fields are singleton currents, quantization relies on a generalized version of the Heisenberg algebra. The quark fields are permitted to be non-local since they are not locally observable. Indeed, it seems a pity, when one tries to deal with unobservable fields, not to take advantage of this freedom to step outside the narrow framework of local field theory. Singleton field theory provides an example of topological gauge fields that can be transformed away locally; possibly there are other ways to introduce quark fields that are artifacts of a gauge symmetry. The problem is to relate them to hadrons.

C. Perturbative local field theory is somewhat at a loss when it comes to incorporating unstable particles. It is possible, however, to concoct a real Lagrangian that describes particles with complex masses, that I shall call quirks, interacting with more conventional fields, and such that the Fourier transforms of the propagators are analytic near the real axis. If $\psi(p)$ is the Fourier transform of such a field operator, then the norm of the asymptotic state associated with it is given by the discontinuity of the generalized function $G(p^2)$ defined by

$$\langle 0 | \psi(p) \bar{\psi}(q) | 0 \rangle = \delta(p - q)G(p^2)$$

across the real axis, and this vanishes for quirks. Hence states that contain an isolated quirk is null. States that contain two or more quirks do not have zero norm; or rather, whether they do or do not depends on the Feynman rules that one invents for them. More about this below.

D. Recall that the scattering amplitudes of string theory, invented by Veneziano [24], describe the scattering of spinless particles that turned out to be tachyons. String theory came of age when, with the aid of supersymmetry, the means were found to decouple this unphysical mode from the physical states. Yet all string states are constructed as states of two or more of these objects. The similarity to the philosophy of QCD is striking. The “constituents” of string theory are bosons, those of QCD are fermions. What is wanted is a QCD without physical quarks, or a string theory with constituents that have some of the properties of quarks. This topic will also be developed below.

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1. Take note, however, that a resolution of the ills of QCD will be worth a million dollars US only if found strictly within the conventional framework, unlikely as this is.
3. Quirks

The dominant modern attitude to elementary particle physics is based on the idea of scattering matrix, a mapping of a Hilbert space of initial on an identical Hilbert space of final states. The salient point about unstable particles is that they do not appear among these states. Consider a fictitious unstable particle – a “quirk” – and let us try to associate a fundamental field operator to it. If all interactions are switched off, then the unstable particle presumably becomes stable, and in that limit it must appear in the Hilbert space of asymptotic states; at least, that is the conventional point of view. But this would imply an important discontinuity in the structure of the space of asymptotic states, and consequently a discontinuity of the values of physical observables, as functions of the coupling strengths; the very concept of analytic perturbations becomes meaningless.

The suggestion that we offer here is as follows. Let us incorporate the unstable nature of quirks into the point of departure of perturbation theory. In the free theory there are stable particles represented by free fields, but the free Lagrangian also contains fundamental noninteracting quirk fields. Once we understand this free field theory, then interactions can be introduced, and now there is a hope of constructing an analytic perturbation theory. Procedures such as this are familiar in the context of quantum mechanics, where the first step in setting up a perturbation scheme consists of splitting the Hamiltonian into two parts. Two considerations govern the splitting; that is, the choice of the unperturbed Hamiltonian and \textit{ipso facto} the nature of the unperturbed system: (1) The free theory should be exactly solvable, and (2) the perturbation should be analytic. (Example: a Schrödinger atom with an additional potential proportional to \(1/r^2\); it is necessary to include this term in the free Hamiltonian. Another example: Dollard’s treatment of the infrared problem [3].) The use of counterterms in renormalization theory can be described in the same way. And here the analogy is a close one, for the width of the resonance may well be renormalized in perturbation theory. The shift of the vacuum in the Standard Model, to the minimum of the Higgs potential, offers another close analogy, for the non-vanishing vacuum expectation value redefines the free theory by incorporating into it a feature of the interaction, to render the perturbation analytic (or at least, more likely to be analytic).

Let \(\langle 0 |\) be the unique vacuum state, and let \(\psi\) denote a free Dirac-type quirk field operator. We anticipate a perturbation theory in which the calculation of S-matrix elements reduces to the evaluation of vacuum expectation values of products of field operators. Suppose that

\[
\psi(f)|0\rangle \neq 0, \quad \psi(f) = \int f(x) \psi(x),
\]

then there is no alternative to accepting the fact that this state must have a place in the theory. But there is no need to jump to the conclusion that it
belong to the Hilbert space of asymptotic states. Taking a cue from gauge theories, we suppose that fundamental field operators act in a larger space, possibly indefinite, and that the physical Hilbert space is a quotient space (or a quotient of a semi-definite subspace) by a subspace of null states (states of vanishing norm). In particular, to avoid the appearance of quirks in the asymptotic states, we must suppose that the asymptotic state associated with $\psi(f)|0\rangle$ have zero norm.

Let us further assume the conventional construction of the inner product, in terms of $\psi(f)$ and a hermitian conjugate field operator $\bar{\psi}(f)$, namely, the norm of this asymptotic state is determined by a discontinuity of the matrix element $\langle 0| \psi(f) \bar{\psi}(f) |0\rangle$. Unitarity of the theory demands that this discontinuity be represented as

$$\sum_n \langle 0| \bar{\psi}(f)|n\rangle \langle n|\psi(f) |0\rangle,$$

where the sum runs over a set of asymptotic states. Since the quirk field operator does not create such states, the norm must be zero:

$$\text{Discontinuity } \langle 0| \psi(f) \bar{\psi}(f) |0\rangle = 0.$$

This is our main premise: that a state created from the vacuum by the quantum field operator associated with an unstable particle have zero norm. Let us emphasize that this refers to the action of one fundamental, unstable field operator; we shall see that some states containing two unstable particles do not have zero norm. This is a minor miracle that reflects the fact that, strictly speaking, quantum field operators are not operators (as we know, for example, by the existence of anomalies).

4. The quirk Lagrangian

We construct an example of a Lagrangian field theory of noninteracting quirks. Since the Lagrangian must be real we need two spinor field, $\psi_1$ and $\psi_2$. The following vacuum expectation values for quirk fields incorporates all expected properties,

$$\langle 0|\psi_1(x)\bar{\psi}_2(x')|0\rangle = \int dp \, e^{ip(x-x')} \frac{1}{p - m + i\lambda},$$

$$\langle 0|\psi_2(x)\bar{\psi}_1(x')|0\rangle = \int dp \, e^{ip(x-x')} \frac{1}{p - m - i\lambda},$$

where $m$ and $\lambda$ are real parameters that characterize the quirk. (The integration runs over $\mathbb{R}^4$.) Namely, (1) it has the usual hermiticity property for Dirac fields, (2) the Fourier transform can be continued to a function analytic
near the real axis, with no discontinuity, and consequently the norm of the
asymptotic state associated with $\psi(x)|0\rangle$ is zero, (3) the matrix elements fall
off exponentially for large values of $|x - x'|^2$.

This function will take the place, for quirk fields, of the usual vacuum
expectation value of the time ordered product. To complete the Wick rules
we need to define the vacuum expectation values of products of the funda-
mental fields; we may assume that this will involve symmetrization, and it is
immediately clear that there may be an opportunity for getting around the
usual connection between spin and statistics.

A real Lagrangian that is consistent with these rules is,

$$L = \bar{\psi}_1 (p - m + i\lambda) \psi_2 - \bar{\psi}_1 J_1 - \bar{\psi}_2 J_2 + h.c.$$ .

A change of basis leads to

$$L = \bar{\psi}_+(p - m) \psi_+ - \bar{\psi}_-(p - m) \psi_- + \bar{\psi}_+ \psi_+ + \bar{\psi}_- \psi_- - (\bar{\psi}_+ J_+ + \psi_- J_- + h.c.).$$

The $\lambda$-term can be introduced via a Yukawa interaction with a scalar field
that is given a non-vanishing vacuum expectation value. Of course, the $m$-
term can be interpreted in the same way, by the usual Higgs mechanism.
It is perhaps significant that both sectors are unstable without the vacuum
expectation value.

The idea of associating certain exotic particles with null states has been
advanced in other contexts as well, as we have mentioned already. The most
obvious apparent difficulty would seem to be the argument that “if $\psi|0\rangle$
has zero norm, then surely $\psi\psi|0\rangle$ must have zero norm as well”. This would
be true if the operator products were not singular. The norm of a 2-quirk
state; for example, of the asymptotic state associated with $\psi_1(f)\psi_2(f')|0\rangle$, is
determined by the discontinuity of

$$\int dk \frac{1}{k - m + i\lambda} \frac{1}{p - k - m - i\lambda} f(k)f'(p - k). \quad (4.1)$$

Ordinary Feynman rules, applicable to stable particles, are evaluated by con-
version to complex contour integrals, and deformation of the contours of
integration over internal energies, here $k_0$. The location of the singularities, in
the second and fourth quadrants, authorizes the rotation of the contour from
the real axis to the imaginary axis. (More precisely, this is true for real $\vec{k}$
and for $p$ real and spacelike.) In (4.1) we have an integrand with poles in all four
quadrants of the complex $k_0$ plane, and rotation of the contour is not possible.
When $\lambda$ is not zero, and if the $k_0$ contour follows the real axix, then this
function has no discontinuity for any real value of $p$. Instead we shall define
the two-point quirk correlation function by (4.1), but with the $k_0$ integration
following the imaginary axis. The usual treatment of this integral leads to an
expression that is analytic in $p$ except for singularities of a parameter-integral
of the form

\[ \int_0^1 d\alpha \left[ \alpha^2 p^2 - \alpha (p^2 - 4i\lambda m) + (m - i\lambda)^2 \right]^{-2}. \]

Vary \( p^2 \) along the real axis, from left to right. The two poles start at \( \alpha = 0, 1 \), move into the upper half plane until they meet at the point \((1 + im/\lambda)/2 \) when \( p^2 = -4\lambda^2 \). One of them goes to infinity as \( p^2 \) passes the origin and returns in the lower half plane, while the other crosses the real axis at \( 1/2 \) when \( p^2 = 2m^2 - 2\lambda^2 \). The contour of integration finally gets pinched at at \((1 - i\lambda/m)/2 \) when \( p^2 \) reaches the value \( 4m^2 \). Result: there is branch point at \( p^2 = (2m)^2 \), independently of the value of \( \lambda \).

Thus we learn that the asymptotic state associated with \( \bar{\psi}_1 \bar{\psi}_2 |0\rangle \) does not have zero norm, in spite of the fact that the states \( \bar{\psi}_1 |0\rangle \) and \( \bar{\psi}_2 |0\rangle \) are null states. There are physical, propagating states with masses above \( 2m \), independently of the value of \( \lambda \).

5. Dynamics of string theory

Now I want to tell you a very ancient story, as briefly as I can. The story begins with the invention of the Dirac equation [3],

\[ (p \cdot \gamma - m)\psi(x) = 0. \]

Recall that \( \psi \) is a field on space time, taking values in a finite dimensional Lorentz module. Solutions exist only for timelike momenta, this is of primary importance to the interpretation. Dirac’s equation applies to electrons; for a while it was thought to apply to protons as well. Of course it does, in a very approximate sense. But to keep to the historical order let me next recall the very interesting work of Majorana [20].

Majorana may have had the same motivation as Dirac, but he was less ready to accept the presence of solutions with negative energy. He asked if it is possible for the operator \( \gamma_0 \) to have a strictly positive spectrum. Of course, relativistic invariance demands that the field take values in a Lorentz module, and the answer found by Majorana was that yes, it is possible if this module is unitary and hence infinite dimensional. Majorana’s work was ignored.

I come now to the applicability of Dirac’s equation to the proton. As the experimental probing of proton structure advanced, it became clear that it was not a simple Dirac particle. The structure was described in terms of elastic form factors. These form factors decrease with increasing momentum transfer, and this is not consonant with Dirac’s equation. More generally, any particle described by a Dirac-type equation, where the field lives in a finite dimensional Lorentz module, has polynomial form factors. In contrast, the
form factor of a Majorana particle does decrease with increasing momentum transfer; in fact, it looks very much like the proton form factor found experimentally.

For this reason, theories of the Majorana type became, briefly, quite popular. One knows what is wrong with these infinite component field theories, but let me first go over some of their good points. Perhaps the most surprising fact is that some theories of this type have a very sensible physical interpretation, as two-particle systems [11]. In fact, one of the useful applications is a marked improvement of the Bethe-Salpeter method for dealing with bound states in quantum field theory [13]. The non-relativistic limit is instructive, it is a formulation of the theory of the Schrödinger hydrogen atom. Of course, here we shall not see a Lorentz module, but a Galilei module. What is important is that we are describing a system with a complicated internal structure. I shall limit myself to discussing just one aspect of the theory, the structure of perturbation theory.

Consider the scattering of photons from an atom [12]. The Feynman diagram represents a sequence of events taking place: incoming photon and atom, absorption of the photon, propagation of the atom in an excited state, emission of a photon, outgoing photon and atom. The amplitude for the process is a parallel sequence: a vertex factor \( \exp(i \vec{k} \cdot \vec{r}) \), a propagator \((E - H_0)^{-1}\), a vertex factor \(\exp(-i \vec{k}' \cdot \vec{r})\). The propagator depends on the energy \(E\) of the intermediate state, but it is not sensitive to the momentum. This is because the momentum of a very heavy object is not observable. But we may, if we wish, define the Hamiltonian of an atom with momentum \(\vec{k}\) by setting

\[
H_{\vec{k}} := \exp(i \vec{k} \cdot \vec{r})H_0\exp(-i \vec{k} \cdot \vec{r}).
\]

The Schrödinger equation now involves \(\vec{k}\) and so does the wave function. In this entirely equivalent formulation of the problem ... and this is the nonrelativistic approximation to Majorana’s theory ... the amplitude is constructed in terms that are those of relativistic local field theory, namely: Incoming photon with momentum \(\vec{k}\) and atom with momentum \(\vec{p}\), (no exponential factor), propagator for an atom with momentum \(\vec{k} + \vec{p}\), (no exponential factor), outgoing photon with momentum \(\vec{k}'\) and atom with momentum \(\vec{p} + \vec{k} - \vec{k}'\).

String theory took over the main ideas of infinite component field theories. The spark of the string theory revolution came from the realization that pole dominated scattering amplitudes behave better when there is an infinite sequence of poles. But the known infinite component field theories suffered from the ubiquitous space like solutions, about which more below; that is, bad behaviour with respect to crossing symmetry. The additional ingredient of string theory is duality: one contrives to construct an amplitude that is at the same time a sum over pole contributions in the direct channel and a sum over pole terms in the crossed channel. This was accomplished by Veneziano [24].
In the beginning there was just Veneziano’s amplitude, and the multiparticle generalization. Then it was shown that these amplitudes could be factorized, just like the amplitude that we have been talking about, into a sequence of propagators separated by exponential momentum factors \([14]\). But what followed created a huge gulf between string theory and Majorana theory. Fubini and Veneziano \([15]\), in a very beautiful paper, succeeded in doing exactly the opposite of what I just did for the hydrogen atom. Instead of getting rid of the vertex factors, they managed to eliminate all the propagators, thereby creating vertex algebras and two-dimensional quantum field theory!

This result is truly wonderful, but something was lost. The well-known Lagrangian string theories in high dimensions describe the dynamics of elementary boson and fermion fields. But the dynamical fields of string theory are the vertex operators, exponentials of these elementary fields; scattering amplitudes are constructed entirely out of vertex operators. There may be something like a string field operator, directly related to vertex operators, and we do have some hints about the dynamics, as we shall see.

So far I have stressed only the attractive features of Majorana’s infinite dimensional version of Dirac’s equation. The bad news is that it possesses solutions with space-like momenta. Similar equations, studied intensively during the late sixties, including relativistic versions of the hydrogen atom, all shared this devastating feature. There was a time when theories were said to be implausible to the degree that they predict the existence of new particles. The Dirac equation predicted only the positron, and that was not too much, but the Majorana equation and its generalization predicted infinite numbers of excited particle states. Actually, this feature should be counted an advantage, especially after the advent of dual models and the realization that it was responsible for the good properties of form factors. The irony is that the predicted excited states turned out to be too few!

Most models have energy spectra labelled by an integer, \(n\) say. The degeneracy of the \(n\)th level, in theories of the Majorana type, are polynomial in \(n\). In dual models the degeneracy grows exponentially. This may be an essential characteristic of theories that avoid spacelike solutions. The degree of degeneracy associated with the Veneziano amplitude is known from the work of Fubini and Veneziano \([15]\) on factorization of this amplitude. In this work there appears a propagator, namely

\[
(L_0 - p^2)^{-1}, \quad L_0 = -2\left(\sum n a_n^* a_n - 1\right),
\]

where \(L_0\) is the operator that was later interpreted as the most important operator of the Virasoro algebra, \(\{a_n\}\) is an infinite set of oscillator operators and \(p\) is the momentum of the state. Let \(|0\rangle\) be the vacuum associated with all these oscillators, then the domain of \(L_0\) is the space \([a_1^*, a_2^*, \ldots] \otimes |0\rangle\), and the mass\(^2\) spectrum of the model is the spectrum of \(L_0\). In other words, the
states satisfy the equation
\[(L_0 - p^2)\Psi = 0, \ \Psi \in [[a_1^*, a_2^*, ...]] \otimes |0\rangle.\]

Though second order, this equation is very much of the Majorana type, though the internal space is bigger than anything that Majorana or his emulators had dreamed of.

This equation is a big part of a dynamical theory of strings. It certainly does not contain everything, for there is one (only one!) spacelike solution, and plenty of states with negative norm. Some of the states of negative norm are eliminated from the Veneziano amplitude by the exponential factors associated with the vertices. Fubini and Veneziano showed that the amplitude could be understood entirely in terms of vertex operators, after absorbing the propagators into the vertex operators. My inclination is to try the opposite tack, to improve the propagator by incorporating into it the role of the vertex factors. This should facilitate the task of constructing the interaction.

A promising attempt to formulate an action principle for quantum string dynamics was made by Witten 15 years ago [25]. This work seems to be applicable only to open strings. I believe that this work should be taken up again, but not with the too modest aim of incorporating closed strings. If the result is to have any bearing on the outstanding open problem of our time, then it is not enough just to look for an extension of Witten’s treatment of the Veneziano model; what is needed is a very nontrivial generalization.

6. Summary

Quantization still offers challenging mathematical problems, but the biggest problem of all is to make sense of QCD. There are hints that string theory, more precisely the Veneziano model, may point in the right direction. The suppression of the tachyon in super string theory suggests that quarks (and gluons) may be the suppressed constituents of a more elaborate version of the Veneziano model. Steps towards such a generalization could include the following small steps:

1. Incorporate the Fadde’ev-Popov ghosts into the vertex operators.
2. Construct the vertex operators for the superstring.
3. Construct the vertex operators on an AdS background.
4. Replace the outer, scalar fields in the amplitude by scalar superfields. (Note that there are no supersymmetric tachyons?)
5. Combine (3) and (4). Possibly, the first two may have been accomplished already [2, 3]. The third, without ghosts and without supersymmetry, is easy. For the 4-point function the result is \(A(s,t)\Psi(x_1, x_2, x_3, x_4)\), evaluated at \(x_1 = x_2, x_3 = x_4\), where \(A\) is Veneziano’s function, \(s = (\nabla_1 + \nabla_2)^2, \ t = (\nabla_2 + \nabla_4)^2\) are covariant d’Alembertians and \(\Psi\) is a product of tachyon wave functions. The fourth suggestion seems to lead to a spectrum \(m \ll n\) (instead of \(m^2 \ll n\)). Finally
the fifth one may include an interesting possibility involving singletons and massless particles with all spins.

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References
1. Barr, M., "A cohomology theory for commutative algebras. I, II", Proc. Amer. Math. Soc. 16 (1965), 1379–1384.
2. Berkovits, Nathan; Vafa, Cumrun. “N = 4 topological strings”. Nuclear Phys. B 433 (1995) 123–180 (hep-th/9407190).
3. Dirac, P.A.M., Proc. Roy. Soc. A 167 (1938), 148.
4. Dolan, Louise; Witten, Edward. “Vertex operators for AdS3 background with Ramond-Ramond flux”. J. High Energy Phys. 1999, no. 11, Paper 3, 21pp. (hep-th/9910205).
5. Dito, G.; Flato, M.; Sternheimer, D. and Takhtajan, L. “Deformation quantization and Nambu mechanics”, Comm. Math. Phys. 183 (1997), 1–22 (hep-th/9602016).
6. Dollard, John D. “On the definition of scattering subspaces in nonrelativistic quantum mechanics. J. Mathematical Phys. 18 (1977), 229–232.
7. Faddeev, L.D., “Operator anomaly for the Gauss law”, Phys.Lett. B 145 (1984) 81–84.
8. Flato, M., “Deformation view of physical theories”, Czech.J.Phys. B 32 (1982), 472–475.
9. Flato, M. and Fronsdal, C., “Parastatistics, highest weight osp(N,∞) modules, singleton statistics and confinement”, J.Phys.Geom. (1989), 293–309.
10. Flato, M. and Fronsdal, C., “Quarks or Singletons”, Phys.Lett. B 172 (1986) 412–416.
11. Fronsdal, C., “Infinite multiplets and the hydrogen atom”, Phys. Rev. 156 (1967), 1665–1677.
12. Fronsdal, C., “Compton scattering from bound electrons”, Phys. Rev. 179 (1969), 1513–1517.
13. Fronsdal, C. and Huff, R.W., “Two-Body Problem in Quantum Field Theory”, Phys.Rev. D3 (1971), 1689.
14. Fubini, S., Gordon, D.and Veneziano, G., “A General Treatment of Factorization in Dual Resonance Models”, Phys.Lett. 29B (1969) 679–82.
15. Fubini, S. and Veneziano, G., “Duality in operator formalism”, Nuovo Cim. 67A (1970), 29–47.
16. Gell-Mann, M., “Current topics in particle physics”, in Proceedings of the thirteenth International Conference on High-Energy Physics. Berkeley 1966.
17. Gerstenhaber, M. and Schack, S. D., “A Hodge type decomposition for commutative algebra cohomology”, J. Pure and Appl. Alg. 48 (1987) 229-247. “Algebraic cohomology and deformation theory. Deformation theory of algebras and structures and applications,” (II Ciocci, 1986). 11–264, NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci., 247, Kluwer Acad. Publ., Dordrecht, 1988.
18. Greenberg, O. W., “Spin and unitary-spin independence in a paraquark model of baryons and mesons”, Phys.Rev.Lett. 13 (1964), 1100-1102.
19. Harrison, D. K., “Commutative Algebras and Cohomology”, Trans. Amer. Math. Soc. 104 (1962), 191–204.
20. Majorana, H., Nuovo Cimento 9 (1932), 335.
21. Ohnuki, Y., and Kamefuchi, S., “Wave functions of identical particles”, Ann. Phys. 57 (1969) 337–358.
22. Pinczon, Georges. “On the equivalence between continuous and differential deformation theories. Lett. Math. Phys. 39 (1997), 143–156.
23. Tavkhelidze, N., “Higher Symmetries and Composite Models of Elementary Particles”, Proceedings of the Conference on High-Energy Physics and Elementary Particles, International Centre for Theoreticle Physics, Trieste 1965.
24. Veneziano, G., “Construction of a Crossing-symmetric, Regge-Behaved Amplitude for Linearly Rising Trajectories”, Nuovo Cimento 57A (1968) 190.
25. Wigner, E. P., “Do the equations of motion determine the quantum mechanical commutation relations?”, Phys. Rev. 77, (1950), 711–712.
26. Witten, Edward. “Noncommutative geometry and string theory”, Nuclear Phys. B268 (1986), 253–294.
