Complete population transfer in 4-level system via Pythagorean triple coupling

Haim Suchowski\textsuperscript{1,*} and Dmitry B. Uskov\textsuperscript{2,†}

\textsuperscript{1}Department of Physics of Complex Systems, Weizmann Institute of Science, Rehovot 76100, Israel
\textsuperscript{2}Department of Physics and Engineering Physics, Tulane University, New Orleans, LA 70118, and Hearne Institute for Theoretical Physics at Louisiana State University, Baton Rouge LA, 70806
\textsuperscript{3}Kavli Institute for Theoretical Physics, Santa Barbara, CA 93106, USA

(Dated: November 16, 2009)

We describe a relation between the requirement of complete population transfer in a four-mode system and the generating function of Pythagorean triples from number theory. We show that complete population transfer will occur if ratios between coupling coefficients exactly match one of the Pythagorean triples \((a, b, c) \subset \mathbb{Z}, c^2 = a^2 + b^2\). For a four-level ladder system this relation takes a simple form \((V_{12}, V_{23}, V_{34}) \sim (c, b, a)\), where coefficients \(V_{ij}\) describe the coupling between modes. We find that the structure of the evolution operator and the period of complete population transfer are determined by two distinct frequencies. A combination of these frequencies provides a generalization of the two-mode Rabi frequency for a four-mode system.

Revealing hidden mathematical structures behind physical phenomena is of great importance, especially when qualitative dynamical properties of a quantum system reduce to a basic relation from the number theory. Here we show how the Pythagorean triple, which is the set of three integer numbers \((a, b, c)\), satisfying the Pythagorean equation \(a^2 + b^2 = c^2\), is found to play a significant role in the dynamics of a four-mode system.

Describing the evolution of a general multi-mode system and finding conditions for complete population transfer from one mode to another is a subject of extensive research for a variety of classical and quantum systems. Coherent manipulation of population of states in atomic and molecular quantum systems \cite{1}, spin control in nuclear magnetic resonances \cite{2}, quantum information processing \cite{3, 4}, and directional optical waveguide technology \cite{5} are only a few examples where complete population transfer is desired. Here we study the four-mode dynamics, which is of particular importance for the quantum information processing technology, where two-qubit quantum logic gates serve as elementary building blocks for designing fully functional scalable devices \cite{2, 3, 4}.

In general, solutions of dynamic coupled equations are difficult to analyze, and even for the simplest case of a two-level system, realized by a spin-$\frac{1}{2}$ particle or a two-level atomic system, only a handful of analytical solutions are known in the literature. The simplest one, known as the Rabi solution \cite{1}, describes a two-level system with a constant coupling. For this solution, complete population transfer between two states occurs only when the frequency of an external driving field is in resonance with the energy difference between the modes. The period of complete population transfer, called the Rabi flopping time, is inversely proportional to the strength of the

FIG. 1: Finite level realizations of the two- and four- mode systems of coupled equations (a) A four-level system with periodic nearest neighbor coupling, known as diamond shape structure. (b) A four-level system with \(V_{14} = 0\), known as a ladder type structure. (c) A simple two-level system with coupling coefficient \(V_{e0}\) and the detuning of \(\Delta\).

mode coupling. A geometric visualization of a two-level system by representing its state by a point within the Bloch sphere, plays important role in developing a clear intuitive understanding of the two-mode dynamics \cite{5, 6}. Our approach helps to extend this geometric approach to four-level systems.

The task of finding schemes for complete population transfer between selected states becomes increasingly difficult in multi-mode coupled systems. There are group-theoretical methods which provide a rigorous tool of how to determine whether a system is \textit{wavefunction controllable}, i.e. when any initial state in a quantum system can be transferred into an arbitrary final state \cite{10, 11}, however these methods are nonconstructive and do not provide a general recipe on how to implement complete population transfer scheme for a concrete system. So far, there is only a limited number of systematic methods which can provide this goal. The schemes exploiting adiabatic evolution are known to be able to achieve this goal asymptotically, however these methods require strong pair-wise sequence of coupling pulses as well as

\textsuperscript{*}Electronic address: haim.suchowski@weizmann.ac.il
\textsuperscript{†}Electronic address: uskov@tulane.edu
very long dynamical time \[12, 13, 14, 15\]. Few solutions of the complete population transfer problem, which requires a set of coupling coefficients to satisfy some special relations, were found for N-mode systems \[16, 17\].

In the present paper we describe a new analytical solution for a four coupled mode system with nearest-neighbor coupling. We show a clear similarity in the structure of this four-level solution and the structure of the Rabi two-level solution, originating from a common geometric character of both solutions. We exploit the fact that a four-dimensional Hilbert space can be represented as a tensor product of two two-dimensional Hilbert spaces. For the four-level nearest-neighbor coupling system the relevance of this tensor-product structure is revealed in its full simplicity when the Hamiltonion is rewritten in the basis of the Bell states. To do a qualitative physical analysis of the resulting set of two-level equations we use a transformation known as the Hopf map \[18, 19\]. The final equations for the evolution of two separate pairs of modes \(|\psi_1\rangle \leftrightarrow |\psi_3\rangle\) and \(|\psi_2\rangle \leftrightarrow |\psi_4\rangle\) provide natural generalization of the Rabi two-mode solution for the four-level system. Finally we derive an equation for the time required for complete population transfer to occur.

The requirement of complete population transfer in a four-level system imposes certain analytical relations on the coupling coefficients and we find that these relations have special algebraic character: they are identical to a formula, which generates all primitive Pythagorean triples (PPTs). Such a triple is a set of three real numbers \(a, b, c\) which do not possess a common factor and satisfy the equation \(a^2 + b^2 = c^2\). For instance, \((3; 4; 5)\) and \((5; 12; 13)\) are primitive triples, whereas \((6; 8; 10)\) is not PPT. Note also, that in spite of the fact that a set of numbers \((1, 1, \sqrt{2})\) satisfy the Pythagorean relation, these numbers are not a Pythagorean triple. Through the centuries finding a formula to generate these triplets has intrigued both amateur and professional mathematicians. The first general solution was given by Euclid in his Elements \[20\]. It states that for any pair \((p, q)\) of positive odd integers with \(p > q\), the triple \((a, b, c) = (p^2 - q^2, 2pq, p^2 + q^2)\) is Pythagorean. Other types of generating functions of PPTs can be found elsewhere \[21, 22, 23\].

We start with a nearest-neighbor coupling four-mode Hamiltonian, when each level is coupled to two neighboring levels, as in the diamond-shaped structure, shown in Fig. 1:

\[
\hat{H} = \begin{pmatrix}
0 & V_{12} & 0 & V_{14} \\
V_{12} & 0 & V_{23} & 0 \\
0 & V_{23} & 0 & V_{34} \\
V_{14} & 0 & V_{34} & 0
\end{pmatrix}.
\]  

As an example of physical realization of this system, we can consider a laser-field driven four-level atom. Then in the rotating wave approximation the coupling coefficients are defined as \(V_{ij} = \mu_{ij} \epsilon(t)/(\hbar)\). Here \(\epsilon(t)\) is the field amplitude and \(\mu_{ij}\) are dipole matrix elements between nearest-neighbor levels \(i\) and \(j\). A general case of periodic nearest-neighbor coupling (diamond-type structure), is schematically shown in Fig. 1(a), and particular case of \(V_{14} = 0\), describing a ladder-type coupling, is presented in Fig. 1(b).

We rewrite Hamiltonian \(\hat{H}\) in the Bell basis using the unitary transformation

\[
\hat{W} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}.
\]

The wave-functions are transformed as \(|\psi_n^B\rangle = \hat{W} |\psi_n\rangle\), and the Hamiltonian becomes \(\hat{H}_W = \hat{W}^\dagger \hat{H} \hat{W}\). In this basis two distinct \(su(2)\) subalgebras, formally associated with qubits 1 and 2, can be identified. We rewrite the Hamiltonian as a linear combination of Pauli matrices \(\sigma_{(x,y,z)}\) acting on qubits 1 and 2:

\[
\begin{align*}
\hat{H}_W &= \hat{h}^{(1)} \otimes \hat{f}^{(2)} + \hat{f}^{(1)} \otimes \hat{h}^{(2)}, \\
\hat{h}^{(1)} &= \frac{(V_{12} + V_{14})}{2} \hat{\sigma}_z + \frac{(V_{23} - V_{12})}{2} \hat{\sigma}_x, \\
\hat{h}^{(2)} &= \frac{(V_{23} + V_{34})}{2} \hat{\sigma}_z - \frac{(V_{23} - V_{14})}{2} \hat{\sigma}_x.
\end{align*}
\]

In the language of Lie group theory Hamiltonian \(\hat{H}_W\) generates a subgroup \(SU(2) \otimes SU(2)/\mathbb{Z}_2\) of the full \(SU(4)\) group. The dynamic problem factorizes into two separate problems for two \(SU(2)\) unitary operators acting on two qubits, thereby geometric tools for visualization of resulting solutions are readily available. By using an algebraic property of local transformations, we can represent the action of an \(SU(2) \times SU(2)/\mathbb{Z}_2\) operator on a four-dimensional state vector as left and right multiplication by two \(2 \times 2\) \(SU(2)\) matrices, acting on a \(2 \times 2\) complex matrix. The latter represents an element of the four-dimensional Hilbert space \[24\].

Thus we rewrite the equations for the evolution of amplitudes \(a_n(t)\) of the states \(|\psi_n\rangle, n \in \{1, 2, 3, 4\}\) in a form of two rotations: \(u_1\) acts from the left, and \(u_2\) acts from the right:

\[
\hat{A}(t) = u_1(t) \hat{A}(t = 0) u_2^T(t).
\]

Here \(\hat{A}(t)\) contains the information about four amplitudes \(a_{1,2,3,4}\):

\[
\hat{A}(t) = a_1(t) \hat{I} + a_2(t) \hat{\sigma}_x + ia_3(t) \hat{\sigma}_y + a_4(t) \hat{\sigma}_z,
\]

and the operators \(u_1\) and \(u_2\) are local rotations of qubits 1 and 2 correspondingly

\[
\begin{align*}
u_1(t) &= \exp \left\{ \frac{it}{2} (V_{12} - V_{34}) \hat{\sigma}_x + (V_{23} + V_{14}) \hat{\sigma}_z \right\}, \\
u_2(t) &= \exp \left\{ \frac{it}{2} (V_{12} + V_{34}) \hat{\sigma}_x - (V_{23} - V_{14}) \hat{\sigma}_z \right\}.
\end{align*}
\]

It immediately follows from this relation that if the system is initialized in the ground state \(|\psi_1\rangle\) so that
Two-mode dynamics

$SU(a)$ $SU(b)$

Nearest-neighbor four mode dynamics

$\tau$ $\Omega$

metric fashion. Suppose that we chose to rotate the basis which allows us to solve the problem in an elegant geometric fashion. Suppose that we chose to rotate the basis in the space spanned by vectors $|\psi_1\rangle$ and $|\psi_3\rangle$. Such rotation apparently will have no effect on the evolution of the states $|\psi_1\rangle$ and $|\psi_3\rangle$. We can represent this transformation as a phase multiplication acting on two complex vectors $(V_{12} + iV_{14}) \rightarrow (V_{12} + iV_{14}) e^{i\theta}$ and $(V_{23} + iV_{34}) \rightarrow (V_{23} + iV_{34}) e^{i\theta}$. The invariance of the amplitudes $a_1(t)$ and $a_3(t)$ under such a transformation means that the amplitudes $a_1$ and $a_3$ are determined not by the full set of coupling coefficients $\{V_{12}, V_{23}, V_{34}, V_{14}\} \in R^4$, but by an element of the quotient space $R^2 \times R^2/\text{SO}(2)$, described by a special algebraic transformation, known as the Hopf $S^3 \rightarrow S^2$ projective map [18, 19]. In physics, the Hopf map is commonly associated with the Bloch Sphere representation of a pure state. In our problem, the map takes the 4-dimensional $V_{ij}$ space to a 3-dimensional $\xi_n$ space,

\[
\xi_0 = \frac{1}{2} (V_{12}^2 + V_{14}^2 + V_{23}^2 + V_{34}^2), \\
\xi_1 = V_{12} V_{23} + V_{14} V_{34}, \\
\xi_2 = V_{12} V_{34} - V_{23} V_{14}, \\
\xi_3 = \frac{1}{2} (V_{12}^2 + V_{14}^2 - V_{23}^2 - V_{34}^2).
\]

(6)

The new coordinates $\xi_n$ satisfy the equation $\xi_3^2 - \xi_1^2 - \xi_2^2 = 0$, which is the equation for a 3-dimensional cone embedded in four-dimensional Euclidean space $\mathbb{R}^4$. Physical meaning of this set of parameters will be clarified when we use them to derive the final solution for the amplitudes $a_n(t)$.

Time dependent amplitudes of the modes $|\psi_1\rangle$ and $|\psi_3\rangle$ can be expressed directly from equation (4) using coupling coefficients $V_{ij}$, however by using the projective coordinates $\{\xi_1, \xi_2, \xi_3\}$, algebraic expressions for these amplitudes can be significantly compactified. As it immediately follows from equation (4),

\[
a_3(t) = -\frac{\xi_1}{\sqrt{\xi_1^2 + \xi_3^2}} \sin (V_L t) \sin (V_R t), \\
a_1(t) = \cos (V_L t) \cos (V_R t) - \frac{\xi_3}{\sqrt{\xi_1^2 + \xi_3^2}} \sin (V_L t) \sin (V_R t).
\]

Here $V_{L,R} \equiv \sqrt{\frac{1}{2}(\xi_0 \mp \xi_2)}$ are the left and right frequencies, respectively. Explicit formulas for $V_L$ and $V_R$ are given in Table I.

The form of the exact solution given by equation (7) is similar to the form or the Rabi solution for a two-level system, as demonstrated in Table I. In the two-level case, shown in Fig. 1(c), we use notations $a_0$ and $a_e$ for the amplitudes of the ground and excited states, while detuning was denoted as $\Delta = \omega_e - \omega_g - \omega_{laser}$. As can be seen from Table I, instead of one frequency $V$ for the two-mode dynamics, the four-mode system is characterized by two generalized frequencies $V_L$ and $V_R$, and instead of single sine and cosine functions in the two mode case,
we see that the four-mode dynamics is determined by the product of two sine and two cosine functions. The pre-factors of the sine function in both cases share a similar structure, being formally equal to $x$- and $z$- components of unit vectors. One can see that there is a very close analogy between variables $\xi_{1,2,3}$ and components of the torus vector for two-mode dynamics [1].

Now we have all the necessary equations to solve the problem of complete population transfer. To realize complete population transfer from the state $|\psi_1\rangle$ to the state $|\psi_3\rangle$ at time $t = \tau$ one has to set the $\xi_3$ variable equal to zero, in the same fashion as the detuning $\Delta$ in the two-mode case has to equal zero (the requirement of on resonant interaction). Next, complete population transfer will occur only when dynamic angles $V_1t$ and $V_Rt$, simultaneously complete a $\pi$-phase rotation, i.e. when $V_L = \frac{\pi}{2\tau} (2m_1 + 1) \equiv \frac{\pi}{2\tau} p$ and $V_R = \frac{\pi}{2\tau} (2m_2 + 1) \equiv \frac{\pi}{2\tau} q$. After some trivial algebra, we derive the following solution:

$$\left(\xi_0, \xi_1, \xi_2\right) = \frac{\pi^2}{2\tau^2} \left(\frac{p^2 + q^2}{2}, \frac{p^2 + q^2}{2}\right) = \frac{\omega^2}{2} \left(c, a, b\right). \quad (8)$$

This solution exactly matches the definition of the generating function of primitive Pythagorean triples [20]. For the nearest-neighbor 4-mode coupling problem complete population transfer can occur between two nonadjacent states $|\psi_1\rangle$ and $|\psi_3\rangle$ (or $|\psi_2\rangle$ and $|\psi_4\rangle$) only when the ratio $(\xi_0 : \xi_1 : \xi_2)$ is equal to ratio of a Pythagorean triple. The equation for coefficients $(V_{12}, V_{23}, V_{34}, V_{14})$, can be obtained from equation (8) by inverting the transformation (6). For the special case of latter-type coupling, where $V_{14} = 0$, this solution becomes $\xi_0 = V_{12}^2$, $\xi_1 = V_{23}V_{12}$, $\xi_2 = V_{34}V_{12}$ such that relation between the nearest-neighbor coupling coefficients and the Pythagorean triple takes a simple form of a proportion $(V_{12}; V_{23}; V_{34}) \sim (c, a, b)$.

We tested our theoretical prediction by performing numerical simulations on the dynamics of a four-level ladder transitions in $Rb_5$: $5S_{1/2} \leftrightarrow 5P_{3/2} \leftrightarrow 4D_{3/2} \leftrightarrow 4F_{5/2}$, with resonant CW interaction of $780.2nm$, $1.529\mu m$, $1.344\mu m$, respectively. The coupling coefficients were chosen to satisfy the simplest Pythagorean triple ratio $(V_{12} : V_{23} : V_{34}) \sim (5 : 3 : 4)$. As seen in Fig. 2, numerical results are in complete agreement with the analytical solution and confirm that there is periodic population transfer between states $|\psi_1\rangle$ and $|\psi_3\rangle$. The time period for complete population transfer is given by

$$\tau = \frac{\pi}{\Omega_P} = \frac{\pi}{\sqrt{V_L^2 + V_R^2}} = \frac{\pi}{\sqrt{2\xi_0}}. \quad (9)$$

Here we denoted the transition time required to achieve population transfer as $\Omega_P$, analogous with the Rabi frequency for a two-level dynamics. Note that this parameter scales as the absolute value of the torque vector, similar with the two-mode case.

In conclusion, we identified a new scheme for complete population transfer in a four-mode systems. We observed very close connection between the structure of solution for the nearest-neighbor coupling four-level system and the generating function of primitive Pythagorean triples. This solution can be used not only for time-dependent problems, but also for some problems of spatial propagation of light pulses, such as coupling between directional waveguides. We expect that similar solutions, revealing deeper link with the number theory, can be found for 6- and 8-level systems. The present method, describing the four-level dynamics, can be generalized to include more complex exactly solvable two-level models. This work is in progress.

This research was supported by ISF and by the NSF under Grants PHY-0545390 and in part by the National Science Foundation under Grant No. PHY05-51164 for the Calvi Institute for Theoretical Physics, UCSB. One of us (HS) is grateful to the Azrieli Foundation for financial support.

[1] L. D. Allen and J. H. Eberly (Wiley, New York, 1975).
[2] J. Keeler (John Wiley and Sons, 2005).
[3] M. A. Nielsen and I. L. Chung (Cambridge University Press, Cambridge, 2000).
[4] A. Sorensen and M. Klaus (1999).
[5] A. Yariv, Phys. Rev. 70, 460 (1973).
[6] A. R. Rau and D. Uskov, Phys. Rev. A 61, 032301 (2000).
[7] D. Uskov and A. R. Rau, Phys. Rev. A 78, 022331 (2008).
[8] M. Born and E. Wolf (Cambridge University Press, 1999).
[9] F. Bloch, Phys. Rev. 70, 460 (1946).
[10] H. Rabitz and G. Turinici, Chemical Physics 267, 1 (2001).
[11] D. Tannor (University Science Books, 2007).
[12] J. R. Kuklinski, U. Gaubatz, F. T. Hioe, and K. Bergmann, Phys. Rev. A 40, 6741 (1989).
[13] V. S. Malinovsky and D. J. Tannor (1998).
[14] N. V. Vitanov, B. W. Shore, and K. Bergmann, Eur. Phys. J. D 4, 29 (1998).
[15] Y. N. Demkov and V. I. Osherov (1968).
[16] B. Shore and K. Cook, Phys. Rev. (1979).
[17] B. Shore, K. Bergmann, A. Kuhn, S. Sciemann, J. Oreg, and J. H. Eberly (1992).
[18] H. Hopf, Mathematische Annalen 104, 637665 (1931).
[19] D. W. Lyons, Mathematics Magazine 76, 87 (2003).
[20] http://aleph0.clarku.edu/~djoyce/java/elements/elements.html (2007).
[21] F. J. M. Barning, Math. Centrum Amsterdam Afd. Zuiver Wisk p. 37 (1963).
[22] P. J. Arpaia, Mathematics Magazine 44, 26 (1971).
[23] D. McCullough and E. Wade, The College Mathematics Journal 34, 107 (2003).
[24] Diter, Chemical Physics 167, 2 (2007).
[25] J. Kocik, Adv. appl. Clifford alg. 17, 71-93 (2006).