ON CHUDNOVSKY-RAMANUJAN TYPE FORMULAE

IMIN CHEN AND GLEB GLEBOV

Abstract. In a well-known 1914 paper, Ramanujan gave a number of rapidly converging series for $1/\pi$ which are derived using modular functions of higher level. D. V. and G. V. Chudnovsky in their 1988 paper derived an analogous series representing $1/\pi$ using the modular function $J$ of level 1, which results in highly convergent series for $1/\pi$, often used in practice. In this paper, we explain the method of D. V. and G. V. Chudnovsky in the context of elliptic curves, modular curves, and the Picard-Fuchs differential equation. In doing so, we also generalize their method to produce formulae which are valid around any singular point of the Picard-Fuchs differential equation. Applying the method to the family of elliptic curves parameterized by the absolute Klein invariant $J$ of level 1, we determine all Chudnovsky-Ramanujan type formulae which are valid around one of the three singular points: $0, 1, \infty$.

Contents

1. Introduction 2
2. Preliminaries 3
2.1. Singular values 3
2.2. The hypergeometric function 3
2.3. Algebraic identities 4
3. Periods and families of elliptic curves 5
3.1. Hypergeometric representations of $\Omega_1$ 5
3.2. Quasi-period relations 10
3.3. Complex multiplication period relations 11
4. $J = \infty$ case 13
4.1. Examples 15
5. $J = 1$ case 16
5.1. Proof of Theorem 1.1 18
5.2. Examples 18
6. $J = 0$ case 19
6.1. Proof of Theorem 1.2 20
6.2. Examples 20
7. Further work 22
References 23

Date: September 2016.

2010 Mathematics Subject Classification. Primary: 11Y60; Secondary: 14H52, 14K20, 33C05.

Key words and phrases. elliptic curves; elliptic functions; elliptic integrals; Dedekind eta function; Eisenstein series; hypergeometric function; $j$-invariant; Picard-Fuchs differential equation.

This work was supported by an NSERC Discovery Grant and SFU VPR Bridging Grant.
1. Introduction

The formula
\[ 12 \sum_{n=0}^{\infty} (-1)^n \frac{545140134n + 13591409}{(640320)^{n+1/2}} \frac{(6n)!}{(3n)!n!^3} = \frac{1}{\pi} \]
has been discovered by D. V. and G. V. Chudnovsky [6], [7]. It has been used in practice for a record breaking computation of the digits of $\pi$. Ramanujan [14] was the first to give examples of such formulae.

The derivation of formulae like (1) has traditionally involved specialized knowledge of identities of classical functions and modular functions (see, for instance, [4], [5]). In the case of D. V. and G. V. Chudnovsky [6], (1) is derived by proving the following precursor formula (see Theorem 4.2) valid for imaginary quadratic $\tau$ in a certain simply-connected domain $C_{1/J, \infty}$ of $\mathbb{C}$,
\[ \frac{a \sqrt{J}}{\pi \sqrt{d} \sqrt{J - 1}} = F^2 \frac{1 - s_2(\tau)}{6} - J \frac{d}{dJ} F^2, \]
where $F = 2F_1(1/12, 5/12; 1/12; J)$, $J = J(\tau)$, and
\[ \tau = \frac{-b + \sqrt{-d}}{2a}, \]
for integers $a, b, c$ such that $a > 0$ and $-d = b^2 - 4ac$. Here, $J = j/12^3$ is the absolute Klein invariant and $j$ is the classical $j$-invariant. Evaluation at a class number one singular value $\tau$ and then using Clausen's identity to simplify the series yields (1).

The purpose of this paper is to explain the method of D. V. and G. V. Chudnovsky [6] in detail and in a way which links it to the modern theory of elliptic curves and modular curves. In fact, we generalize the method to work for monodromy around any elliptic point, in addition to the cusps. In doing so, we give a fairly transparent but broad framework which in principle suggests a systematic way to tabulate such formulae for genus zero congruence groups commensurable with $\text{SL}_2(\mathbb{Z})$, or at least those which are triangular. A feature of this method is that it no longer requires specialized knowledge of classical functions and identities, but rather an explicit form of the Picard-Fuchs differential equation and Kummer's method [11] to determine its hypergeometric solutions [2].

Even in the case of $J$ of level 1, the method gives new formulae (no longer for $1/\pi$, but related constants), corresponding to monodromy around $J = 0$ and $J = 1$, which are derived in this paper. In particular, we prove the following Chudnovsky-Ramanujan type formulae:

**Theorem 1.1.** If $\tau$ is as in (3) and lies in $C_{(J-1)/J,i}$, then
\[ \frac{\tau + i}{2\pi \alpha^2 \sqrt{3} \sqrt{1 - J}} \left( a \frac{\tau + i}{\sqrt{-d}} - 1 \right) = F^2 \frac{1 - s_2(\tau)}{6} - J \frac{d}{dJ} F^2, \]
where $F = 2F_1(1/12, 5/12; 1/2; (J-1)/J)$, $J = J(\tau)$, $\alpha = 2i\eta(i)^2$, and the principal branch of the square root is used.

**Theorem 1.2.** If $\tau$ is as in (3) and lies in $C_{J/(J-1),\rho}$, then
\[ -\frac{\tau - \rho}{2\pi \alpha^2 \sqrt{3} (1 - J)^{1/3}} \left( a \frac{\tau - \rho}{\sqrt{-d}} - 1 \right) = F^2 \left[ \frac{J}{6(1 - J)} + \frac{s_2(\tau)}{6} \right] + J \frac{d}{dJ} F^2, \]
where \( F = {}_2F_1(1/12,7/12;2/3;J/(J - 1)) \), \( J = J(\tau) \), \( \alpha = i\eta(\rho)^2\sqrt{3} \), and the principal branch of the square and cube root is used.

The simply-connected domains \( C_{1/J,\infty}, C_{(J-1)/J,i}, C_{J/(J-1),\rho} \) are defined and depicted in Section 3.1. In Section 5.2 and 6.2 we list all possible identities obtained by evaluating the above formulae at class number one singular values \( \tau \).

2. Preliminaries

In this section, we recall some basic definitions and facts we need later in the paper.

2.1. Singular values. It is known from the theory of complex multiplication of elliptic curves that \( j(\tau) \) and \( J(\tau) \) are rational for \( \tau = \sqrt{-N} \) whenever \( N = 1, 2, 3, 4, 7 \) \([8, pp. 237-238]\).

| \( N \) | \( j(\tau) \) | \( J(\tau) \) |
|---|---|---|
| 1 | 12 \(^4\) | 1 |
| 2 | 20 \(^3\) | 5\(^3\)/3\(^3\) |
| 3 | 2 \cdot 30 \(^3\) | 5\(^3\)/2\(^2\) |
| 4 | 66 \(^3\) | 11\(^3\)/2\(^3\) |
| 7 | 255 \(^3\) | 5\(^3\)17\(^3\)/2\(^6\) |

Table 1: Special values of \( j(\tau) \) and \( J(\tau) \) at \( \tau = \sqrt{-N} \)

Moreover, \( j(\tau) \) and \( J(\tau) \) are also rational for \( \tau = \frac{-1 + \sqrt{-N}}{2} \) if \( N = 3, 7, 11, 19, 27, 43, 67, 163 \) \([8, pp. 237-238]\).

| \( N \) | \( j(\tau) \) | \( J(\tau) \) |
|---|---|---|
| 3 | 0 | 0 |
| 7 | -15 \(^3\) | -5\(^3\)/4\(^3\) |
| 11 | -32 \(^3\) | -8\(^3\)/3\(^3\) |
| 19 | -96 \(^3\) | -8\(^3\) |
| 27 | -3 \cdot 160 \(^3\) | -40\(^3\)/3\(^2\) |
| 43 | -960 \(^3\) | -80\(^3\) |
| 67 | -5280 \(^3\) | -440\(^3\) |
| 163 | -640320 \(^3\) | -53360\(^3\) |

Table 2: Special values of \( j(\tau) \) and \( J(\tau) \) at \( \tau = \frac{-1 + \sqrt{-N}}{2} \)

2.2. The hypergeometric function.

Definition. The Pochhammer symbol is defined by

\[(\alpha)_n = \alpha(\alpha + 1)(\alpha + 2)\cdots(\alpha + n - 1),\]

where \( n \) is a positive integer and \( (\alpha)_0 = 1 \).

Remark. It is easily shown that \((\alpha)_n = \Gamma(\alpha + n)/\Gamma(\alpha)\), where \( \Gamma(z) \) is the gamma function.
We have Lemma 2.1. quantities as functions of \( \tau \) its radius of convergence, where it converges absolutely. Hence it is seen that

\[
\frac{d}{dz} f(z) = ab c 2F1(a + 1, b + 1; c + 1; z).
\]

2.3. Algebraic identities. In what follows, we will often not specify the branch of the \( n \)-th root function during intermediate calculations. This means that any formulae stated will only be valid up to some \( n \)-th root of unity, where \( n \leq 6 \).

For instance, we will use the following algebraic identity, which is valid up to a 6-th root of unity dependent on the quantities \( g_2, g_3, J, \) or on \( \tau \) if we regard these quantities as functions of \( \tau \).

**Lemma 2.1.** We have

\[
\frac{3g_3}{2g_2} \frac{\sqrt{J}}{\sqrt{J^2 - 1}} = \frac{(J \Delta)^{1/6}}{\sqrt{12}}.
\]

**Proof.** Note that up to a 6-th root of unity we have

\[
\frac{3g_3}{2g_2} (j \Delta)^{-1/6} \frac{\sqrt{J}}{\sqrt{J^2 - 1}} = \frac{3g_3}{2g_2} (12^3 g_2^3)^{-1/6} \sqrt{\frac{12^3 g_2^3 / \Delta}{12^3 g_2^3 / \Delta - 12^3}}
\]

\[
= \frac{3g_3}{2g_2} \frac{12g_2}{\sqrt{(12^3 g_2^3 / \Delta - 12^3) \Delta}}
\]

\[
= \frac{3g_3}{2g_2} \frac{12g_2}{\sqrt{12^3 g_2^3 / \Delta - 12^3}}
\]

\[
= \frac{3g_3}{2g_2} \frac{12g_2}{\sqrt{12^3 g_2^3 / \Delta - 12^3}}
\]

But \( \Delta = g_2^3 - 27 g_3^2 \), so

\[
\frac{3g_3}{4 \sqrt{3g_2^3 - 3\Delta}} = \frac{3g_3}{4 \sqrt{3g_2^3 - 3\Delta}}
\]

which simplifies to \( 1/12 \). \( \Box \)

In Theorem 1.1, 1.2, 4.2 we have exact equalities with the principal branch of the \( n \)-th root function involved. This was established by first showing that we have an identity \( f(\tau) = \epsilon(\tau) g(\tau) \) on some connected subset \( U \) of \( \mathbb{C} \), where \( f(\tau) \) and \( g(\tau) \) are continuous on \( U \), \( \epsilon(\tau) \) is an \( n \)-th root of unity, and \( n \) does not depend on \( \tau \in U \). Thus, \( f(\tau)/g(\tau) \) is (or can be extended to) a continuous function on a connected set \( U \) to a finite subset with the induced discrete topology, and is hence constant.
Thus, if there is one \( \tau_0 \in U \) such that \( f(\tau_0)/g(\tau_0) = 1 \), then in fact \( \epsilon(\tau) = 1 \) on all of \( U \).

Using Ramanujan’s differential equations [13, p. 142] (see also [15]) for \( E_4(\tau) \) and \( E_6(\tau) \), namely,

\[
q \frac{d}{dq} E_4(\tau) = \frac{E_2(\tau)E_4(\tau) - E_6(\tau)}{3} \quad \text{and} \quad q \frac{d}{dq} E_6(\tau) = \frac{E_2(\tau)E_6(\tau) - E_4(\tau)^2}{2},
\]

where \( q = e^{2\pi i \tau} \), and the identity,

\[
J(\tau) = \frac{E_4(\tau)^3}{E_4(\tau)^3 - E_6(\tau)^2},
\]

we obtain

\[
\frac{J'}{J} = -2\pi i \frac{E_6(\tau)}{E_4(\tau)},
\]

where \( J' = \frac{dJ}{d\tau} \).

**Remark.** We remark that the equations in (5) can be viewed as determining the effect of Ramanujan’s \( \theta \)-operator on the ring of modular forms of level 1 [16].

3. Periods and families of elliptic curves

For \( u \neq 0 \), the map \( \varphi_u : (x, y) \mapsto (u^2x, u^3y) \) gives an isomorphism between the elliptic curves

\[
E_1 : y^2 = 4x^3 - g_2x - g_3 \quad \text{and} \quad E_2 : y^2 = 4x^3 - \tilde{g}_2x - \tilde{g}_3,
\]

where

\[
\tilde{g}_2 = u^4g_2 \quad \text{and} \quad \tilde{g}_3 = u^6g_3,
\]

that is, \( \varphi_u : E_1 \cong \tilde{E}_2 \).

For every elliptic curve \( E \) over \( \mathbb{C} \), there is a lattice \( \Lambda \subseteq \mathbb{C} \) such that \( \iota_\Lambda : \mathbb{C}/\Lambda \cong E(\mathbb{C}) \), with \( \iota_\Lambda : z \mapsto [\wp(z, \Lambda) : \wp'(z, \Lambda) : 1] \), is an isomorphism of the Riemann surfaces [17, Chapter VI, Section 5, Proposition 5.2]. Furthermore, the diagram

\[
\begin{array}{ccc}
\mathbb{C}/\Lambda & \xrightarrow{\iota_\Lambda} & E_\Lambda(\mathbb{C}) \\
\downarrow{\sim} & & \downarrow{\varphi_u} \\
\mathbb{C}/u^{-1}\Lambda & \xrightarrow{\iota_{u^{-1}}\Lambda} & E_{u^{-1}}(\mathbb{C})
\end{array}
\]

commutes so that the isomorphism \( \varphi_u \) corresponds to scaling \( \Lambda \) by \( u^{-1} \).

With this in mind, we can compare three families of elliptic curves (together with their (normalized) discriminants and associated lattices). Consider

\[
E : y^2 = 4x^3 - g_2x - g_3, \quad \Delta(E) = \Delta = g_2^3 - 27g_3^2, \quad \Lambda(E) = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2.
\]

If \( u = \omega_1 \), then \( E \) is isomorphic to

\[
E_\tau : y^2 = 4x^3 - g_2(\tau)x - g_3(\tau), \quad \Delta(E_\tau) = \Delta(\tau) = g_2(\tau)^3 - 27g_3(\tau)^2, \quad \Lambda_\tau = \mathbb{Z} + \mathbb{Z}\tau.
\]

If \( u = \Delta^{-1/2} \), then \( E \) is isomorphic to

\[
\tilde{E} : y^2 = 4x^3 - \gamma_2x - \gamma_3, \quad \Delta(\tilde{E}) = 1, \quad \Lambda(\tilde{E}) = \mathbb{Z}+\mathbb{Z}\tilde{\omega}_1 + \mathbb{Z}\tilde{\omega}_2,
\]

where

\[
\gamma_2 = \Delta^{-1/3}g_2 = J^{1/3} \quad \text{and} \quad \gamma_3 = \Delta^{-1/2}g_3 = \sqrt{\frac{J-1}{27}}.
\]
If \( u = \sqrt{g_2/g_3} \), then \( E \) is isomorphic to
\[
E_J : y^2 = 4x^3 + g(x + 1), \quad \Delta(E_J) = \Delta(J) = \frac{3^9J^2}{16(1 - J)^2}, \quad \Lambda(E_J) = \mathbb{Z}\Omega_1 + \mathbb{Z}\Omega_2,
\]
where
\[
g = \frac{g_3^3}{g_5} = \frac{27J}{1 - J}.
\]
Alternatively, \( \bar{E} \cong E_J \) if
\[
u = \Delta^{1/12} \sqrt{\frac{g_2}{g_3}} = J^{1/6} \left( \frac{27}{J - 1} \right)^{1/4}.
\]
Also, notice that \( E_\tau \) is isomorphic to \( E_J \) with \( u = \sqrt{g_2(\tau)/g_3(\tau)} \).

It is known that \( \Omega_1, \Omega_2 \) satisfy the Picard-Fuchs differential equation [9, p. 34]
\[
d\Omega + \frac{1}{J} d\Omega + \frac{31J - 4}{144J^2(1 - J)^2} \Omega = 0.
\]
(7)

For completeness, we describe below more precisely what this means.

Let \( E : y^2 = 4x^3 - g_2x - g_3 \) be an elliptic curve over \( \mathbb{C} \). By the uniformization theorem, there exists a unique lattice \( \Lambda \subseteq \mathbb{C} \) such that \( g_2 = g_2(\Lambda) \) and \( g_3 = g_3(\Lambda) \).

The set of points \( E(\mathbb{C}) \) is a complex Lie group which is topologically a torus. Let \( H_1(E(\mathbb{C}), \mathbb{Z}) \) denote the first homology group of the topological space \( E(\mathbb{C}) \). It is known that \( H_1(E(\mathbb{C}), \mathbb{Z}) \cong \mathbb{Z}^2 \), where simple loops \( \alpha \) and \( \beta \) in \( E(\mathbb{C}) \) can be taken as a \( \mathbb{Z} \)-basis as depicted in [17] Chapter VI, Section 1, Figure 6.5.

Let \( X(2)(\mathbb{C}) \) denote the modular curve of level 2 corresponding to the Legendre family of elliptic curves given by \( y^2 = x(x - 1)(x - \lambda) \). The natural \( S_3 \)-covering \( X(2)(\mathbb{C}) \cong \mathbb{P}^1(\mathbb{C}) \rightarrow X(1)(\mathbb{C}) \cong \mathbb{P}^1(\mathbb{C}) \) is unramified outside of \( \{0, 1, \infty\} \).

Defining \( \Omega_1 = \Omega_1(J) \) and \( \Omega_2 = \Omega_2(J) \) as continuous functions of \( J \) amounts to choosing a \( \mathbb{Z} \)-basis \( \{\gamma_1(J), \gamma_2(J)\} \) for \( H_1(E_J, \mathbb{Z}) \), where \( \gamma_1(J) \) and \( \gamma_2(J) \) vary continuously for \( J \) in some open subset of \( X(1)(\mathbb{C}) \setminus \{0, 1, \infty\} \). Once we have this, we can then define
\[
\Omega_k = \Omega_k(J) = \int_{\gamma_k(J)} \frac{dx}{y}, \quad H_k = H_k(J) = \int_{\gamma_k(J)} \frac{x}{y} dx,
\]
and proceed to show that these functions can be differentiated with respect to \( J \).

Let \( U \) be a simply-connected open subset of \( X(1)(\mathbb{C}) \setminus \{0, 1, \infty\} \) and let \( J_0 \in U \).

Let \( \sigma \) be a path in \( U \) such that \( \sigma(0) = J_0 \) and \( \sigma(1) = J \). By the path-lifting lemma, there exists a unique path \( \bar{\sigma} \) in \( X(2)(\mathbb{C}) \) lifting \( \sigma \), where \( \bar{\sigma}(0) \) corresponds to a fixed initial choice of labeling of the distinct roots \( e_0(J_0), e_1(J_0), e_\lambda(J_0) \) of the cubic \( 4x^3 + g(J_0)(x + 1) \). Hence, for \( J \in U \), we have a well-defined labeling of the distinct roots \( e_0(J), e_1(J), e_\lambda(J) \) of the cubic \( 4x^3 + g(J)(x + 1) \), which vary continuously for \( J \in U \).

Let \( \alpha_1(J) \) be a simple loop which encircles \( e_0(J) \) and \( e_1(J) \), and does not pass through the other root. Let \( \alpha_2(J) \) be a simple loop which encircles \( e_1(J) \) and \( e_\lambda(J) \), and does not pass through the other root. Then \( \{\alpha_1(J), \alpha_2(J)\} \) forms a \( \mathbb{Z} \)-basis for \( H_1(E_J, \mathbb{Z}) \) as depicted in [17] Chapter VI, Section 1, Figure 6.5, where \( \alpha_1(J) \) and \( \alpha_2(J) \) vary continuously for \( J \in U \).
For later purposes, we can also modify $\alpha_1(J)$ and $\alpha_2(J)$ in the following way so that we can apply Theorem \ref{thm:main}. Let $z_0 = \alpha_2(J_0)/\alpha_1(J_0)$ and $q_0 \in \text{SL}_2(\mathbb{Z})$ be such that $q_0(z_0)$ lies in the closure of the standard fundamental domain

$$\mathcal{F} = \{ \tau \in \mathbb{H} : |\text{Re}(\tau)| < 1/2, |\tau| > 1 \}$$

for $\text{SL}_2(\mathbb{Z})$, and define

$$\begin{pmatrix} \gamma_2(J) \\ \gamma_1(J) \end{pmatrix} = q_0 \begin{pmatrix} \alpha_2(J) \\ \alpha_1(J) \end{pmatrix}$$

for all $J \in U$.

3.1. **Hypergeometric representations of $\Omega_1$.** Let $\zeta(z, \Lambda)$ be the Weierstrass $\zeta$-function attached to the lattice $\Lambda \subseteq \mathbb{C}$. The quasi-period $\eta(\omega, \Lambda)$ is defined by

$$\eta(\omega, \Lambda) = \zeta(z + \omega, \Lambda) - \zeta(z, \Lambda)$$

for $\omega \in \Lambda$. The dependence on $\Lambda$ is often suppressed if there is no confusion. Since

$$\zeta(\lambda z, \lambda \Lambda) = \lambda^{-1} \zeta(z, \Lambda),$$

(see, for instance, \cite{12} Chapter 18, Section 1) it follows that

$$\eta(\lambda \omega, \lambda \Lambda) = \lambda^{-1} \eta(\omega, \Lambda).$$

Define

$$\eta_k = \eta(\omega_k, \Lambda(E)), \quad \bar{\eta}_k = \eta(\bar{\omega}_k, \Lambda(\bar{E})), \quad H_k = \eta(\Omega_k, \Lambda(E_J)).$$

From the homotheties relating the lattices $\Lambda(E), \Lambda(\bar{E}), \Lambda(E_J)$, we have

$$\eta_k \Delta^{-1/12} = \bar{\eta}_k,$$

$$\eta_k \sqrt{\frac{g_2}{g_3}} = H_k.$$}

Under the map of Riemann surfaces

$$\iota_{\Lambda(E_J)} : \mathbb{C} \rightarrow \mathbb{C}/\Lambda(E_J) \rightarrow E_{\Lambda(E_J)}(\mathbb{C}),$$

given by $\iota_{\Lambda(E_J)}(z) = [\varphi(z, \Lambda(E_J)) : \varphi'(z, \Lambda(E_J)) : 1]$, the differential $-dx/y$ pulls back to the differential $d\zeta$ on $\mathbb{C}$. It follows that

$$H_k = \eta(\Omega_k, \Lambda(E_J))$$

$$= \zeta(z + \Omega_k, \Lambda(E_J)) - \zeta(z, \Lambda(E_J))$$

$$= \int_z^{z + \Omega_k} d\zeta(s, \Lambda(E_J))$$

$$= -\int_{\gamma_k(J)} x \, dx,$$

so the definition of $H_k$ above coincides with the definition given in the previous section.

| Elliptic curve | Periods | Quasi-periods |
|---------------|---------|---------------|
| $E$           | $(\omega_1, \omega_2)$ | $(\eta_1, \eta_2)$ |
| $E_\tau$     | $(1, \tau)$ | $(\eta_1 \omega_1, \eta_2 \omega_1)$ |
| $\tilde{E}$  | $(\tilde{\omega}_1, \tilde{\omega}_2) = (\omega_1 \Delta^{1/12}, \omega_2 \Delta^{1/12})$ | $(\tilde{\eta}_1, \tilde{\eta}_2) = (\eta_1 \Delta^{-1/12}, \eta_2 \Delta^{-1/12})$ |
| $E_J$         | $(\Omega_1, \Omega_2) = (\omega_1 \sqrt{g_3/g_2}, \omega_2 \sqrt{g_3/g_2})$ | $(H_1, H_2) = (\eta_1 \sqrt{g_3/g_2}, \eta_2 \sqrt{g_2/g_3})$ |

Table 3: Summary: Periods and Quasi-periods
Lemma 3.1. Let $F$ be the standard fundamental domain for $\text{SL}_2(\mathbb{Z})$ given by $F = \{ \tau \in \mathcal{H} : |\text{Re}(\tau)| < 1/2, |\tau| > 1 \}$. Let $C_\nu = \{ \tau \in F : |\nu(J)| < 1 \}$, where $\nu(J)$ be one of $1/J$, $(J-1)/J$, $J/(J-1)$. Then $C_\nu$ is an open set in $\mathcal{H}$ which is a union of at most two simply-connected components.

Proof. Let $\mathcal{H}^* = \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$. The function $J$ gives a complex analytic isomorphism of Riemann surfaces $J : \text{SL}_2(\mathbb{Z}) \setminus \mathcal{H}^* \to \mathbb{P}^1(\mathbb{C})$, a fortiori, a homeomorphism of topological spaces. This still holds for $\nu(J)$ as it is a Möbius transformation. Furthermore, the restriction of $\nu(J)$ to $C_\nu$ gives a homeomorphism between $C_\nu$ and $\{ \nu(J) \in \mathbb{P}^1(\mathbb{C}) : |\nu(J)| < 1 \} \setminus \partial F$, which is a union of at most two connected simply connected components. Here, $\partial F$ denotes the boundary of $F$. □

Let $C_{\nu(J),z}$ denote the connected component of $C_{\nu(J)}$ with $z$ lying in the closure of $C_{\nu(J)}$. The connected components $C_{1/J,\infty}$, $C_{(J-1)/J,i}$, $C_{J/(J-1),\rho}$ are depicted in the following plots, where $x = \text{Re}\tau$ and $y = \text{Im}\tau$. Note that the regions are open, unbounded, and extend vertically towards $\infty$.

![Figure 1. $C_{1/J,\infty} = \{ \tau : |1/J| < 1, x^2 + y^2 > 1 \}$](image-url)
We now recall some results from [2] which are obtained by Kummer’s method of determining the solutions of the Picard-Fuchs differential equation (7). Each such solution results in a formula for the period $\Omega_1$ in terms of hypergeometric functions.

**Theorem 3.2** ($J = \infty$ case). Suppose $\tau = \omega_2/\omega_1$ is in the connected component $C_{1/J, \infty}$ of the open set $|J| > 1$. Then

$$\bar{\omega}_1 = \omega_1 \Delta^{1/12} = \frac{2\pi}{\sqrt{2\sqrt{3}}} J^{-1/12} {}_2F_1 \left( \frac{1}{12}, \frac{5}{12}; 1; \frac{1}{J} \right),$$

where $J = J(\tau)$. 
Proof. This follows from [2] p. 255, (14)], \( \Delta(\tau) = \omega_1^2 \Delta \), and the classical identity
\( \Delta(\tau) = (2\pi)^{12} \eta(\tau)^{24} \). □

The other three identities in [2] p. 255, (14)-(15)] can be obtained from the
above identity using Euler’s and Pfaff’s hypergeometric transformations.

**Theorem 3.3** (\( J = 1 \) case). Suppose \( \tau = \omega_2/\omega_1 \) is in the connected component
\( C_{(J-1)/J,i} \) of the open set \(|(J-1)/J| < 1 \). Then
\[
\tilde{\omega}_1 = \omega_1 \Delta^{1/12} = \frac{2\pi \alpha}{\tau + i} J^{-1/12} 2F_1 \left( \begin{array}{c} \frac{1}{12}, \frac{5}{12}, \frac{1}{2}; \frac{J-1}{J} \end{array} \right),
\]
where \( J = J(\tau) \) and \( \alpha = 2i\eta(i)^2 \).

**Proof.** This follows from [2] p. 253, (10)], \( \Delta(\tau) = \omega_1^2 \Delta \), and the classical identity
\( \Delta(\tau) = (2\pi)^{12} \eta(\tau)^{24} \). □

In [2] p. 253, (10)-(11)], there are five other identities convergent near \( J = 1 \). The ones involving \( 1-J \) do not converge at the class number one singular values \( \tau \).
The remaining identity involving \( 2F_1(7/12,11/12;3/2;(J-1)/J) \) yields one further
example of Chudnovsky-Ramanujan type formulae for level 1 using the method of
this paper.

**Theorem 3.4** (\( J = 0 \) case). Suppose \( \tau = \omega_2/\omega_1 \) is in the connected component
\( C_{J/(J-1),\rho} \) of the open set \(|J/(J-1)| < 1 \). Then
\[
\tilde{\omega}_1 = \omega_1 \Delta^{1/12} = \frac{2\pi \alpha}{\tau - \rho} (1-J)^{-1/12} 2F_1 \left( \begin{array}{c} \frac{1}{12}, \frac{7}{12}, \frac{2}{3}; \frac{J}{J-1} \end{array} \right),
\]
where \( J = J(\tau) \) and \( \alpha = i\eta(\rho)^2 \sqrt{3} \).

**Proof.** This follows from [2] p. 254, (12)], \( \Delta(\tau) = \omega_1^2 \Delta \), and the classical identity
\( \Delta(\tau) = (2\pi)^{12} \eta(\tau)^{24} \). □

In [2] pp. 254-255, (12)-(13)], there are five other identities convergent near \( J = 0 \). The ones involving \( J \) do not converge at the class number one singular values \( \tau \).
The remaining identity involving \( 2F_1(5/12,11/12;4/3;J/(J-1)) \) yields one further
example of Chudnovsky-Ramanujan type formulae for level 1 using the method of
this paper.

3.2. **Quasi-period relations.** The following result appears in [7] (4.5)], but we
provide additional details of its proof.

**Theorem 3.5.** For \( k = 1,2 \), we have
\[
\tilde{\eta}_k = -2\sqrt{3} J^{2/3} \sqrt{J-1} d\tilde{\omega}_k \, dJ.
\]

**Proof.** Let \( \omega = \omega_k, \eta = \eta_k, \Omega = \Omega_k, H = H_k \), with \( k = 1,2 \). From [3] p. 34], we
have the following differential relation:
\[
36J(J-1) \frac{d\Omega}{dJ} = 3(2+J)\Omega - 2(J-1)H,
\]
which gives
\[
H = \frac{3}{2} \frac{(J+2)\Omega - 12J(J-1)\frac{d\Omega}{dJ}}{J-1} = \frac{3}{2} \frac{J+2}{J-1} \Omega - 18J \frac{d\Omega}{dJ}.
\]
Recall that $\Omega = \omega \sqrt{g_3/g_2}$, $H = \eta \sqrt{g_2/g_3}$ and $\bar{\omega} = \omega \Delta^{1/12}$, $\bar{\eta} = \eta \Delta^{-1/12}$. Since
\[
\Delta^{-1/12} \sqrt{\frac{g_3}{g_2}} = J^{-1/6} \left( \frac{27}{J-1} \right)^{-1/4} \quad \text{and} \quad \Delta^{1/12} \sqrt{\frac{g_2}{g_3}} = J^{1/6} \left( \frac{27}{J-1} \right)^{1/4},
\]
it is plain that
\[
\Omega = \bar{\omega} J^{-1/6} \left( \frac{27}{J-1} \right)^{-1/4} \quad \text{and} \quad H = \bar{\eta} J^{1/6} \left( \frac{27}{J-1} \right)^{1/4},
\]
and
\[
\frac{d\Omega}{dJ} = \frac{3^{1/4}(J+2)}{36J^{5/6}(J-1)^{3/4}} \bar{\omega} + \frac{3^{1/4}(J-1)^{1/4} d\bar{\omega}}{3J^{1/6} dJ}.
\]

Substituting these into (8) gives
\[
\bar{\eta} \left( \frac{27}{J-1} \right)^{1/4} = 3 \frac{J + 2}{2} \bar{\omega} J^{-1/3} \left( \frac{27}{J-1} \right)^{-1/4}
\]
\[
- 18J^{5/6} \left[ \frac{3^{1/4}(J+2)}{36J^{7/6}(J-1)^{3/4}} \bar{\omega} + \frac{3^{1/4}(J-1)^{1/4} d\bar{\omega}}{3J^{1/6} dJ} \right]
\]
\[
= -18J^{5/6} \frac{3^{1/4}(J-1)^{1/4} d\bar{\omega}}{3J^{1/6} dJ}
\]
Solving for $\bar{\eta}$ we obtain
\[
\bar{\eta} = -2 \sqrt{3} J^{2/3} \sqrt{J-1} \frac{d\bar{\omega}}{dJ}.
\]

3.3. Complex multiplication period relations. Let
\[
s_2(\tau) = \frac{E_4(\tau)}{E_6(\tau)} \left( E_2(\tau) - \frac{3}{\pi \text{Im} \tau} \right).
\]
It is known that $s_2(\tau)$ is rational at $\tau = \sqrt{-N}$ for $N = 2, 3, 4, 7$ and at $\tau = -\frac{1 + \sqrt{-N}}{2}$ for $N = 7, 11, 19, 27, 43, 67, 163$ [1] Lemma 4.1. Below are tables giving these rational values.

| $N$ | $s_2(\tau)$ |
|-----|-------------|
| 2   | 5/14        |
| 3   | 5/11        |
| 4   | 11/21       |
| 7   | 85/133      |

Table 4: Special values of $s_2(\tau)$ at $\tau = \sqrt{-N}$
Table 5: Special values of $s_2(\tau)$ at $\tau = \frac{-1 + \sqrt{-N}}{2}$

| $N$ | $s_2(\tau)$ |
|-----|-------------|
| 7   | 5/21        |
| 11  | 32/77       |
| 19  | 32/57       |
| 27  | 160/253     |
| 43  | 640/903     |
| 67  | 33440/43617 |
| 163 | 77265280/90856689 |

From $g_2(\tau) = \frac{4\pi^4E_4(\tau)}{3}$ and $g_3(\tau) = \frac{8\pi^6E_6(\tau)}{27}$, and the fact that $g_2(\tau) = \omega_1^4g_2$ and $g_3(\tau) = \omega_1^6g_3$, we obtain:

$$
3g_3 \frac{s_2(\tau)}{2g_2} = \frac{\pi^2}{3\omega_1^2} \left[ E_2(\tau) - \frac{3}{\pi \text{Im} \tau} \right].
$$

**Theorem 3.6.** We have

$$
2\omega_1\eta_1 \text{Im} \tau - \omega_1^2 \left[ 2 \text{Im} \tau \frac{3g_3}{2g_2} s_2(\tau) \right] = 2\pi i.
$$

**Proof.** From the evident identity

$$
2\omega_1\eta_1 \text{Im} \tau - \left[ 2\omega_1\eta_1 - \frac{2\pi}{\text{Im} \tau} \right] \text{Im} \tau = 2\pi,
$$

we obtain

$$
2\omega_1\eta_1 \text{Im} \tau - \frac{2\pi^2 \text{Im} \tau}{3} \left[ \frac{3\omega_1\eta_1}{\pi^2} - \frac{3}{\pi \text{Im} \tau} \right] = 2\pi.
$$

From [10, p. 298] (or [12, Chapter 18, Section 3]), we have

$$
E_2(\tau) = \frac{3\omega_1\eta_1}{\pi^2}.
$$

Hence

$$
2\omega_1\eta_1 \text{Im} \tau - \frac{2\pi^2 \text{Im} \tau}{3} \left[ E_2(\tau) - \frac{3}{\pi \text{Im} \tau} \right] = 2\pi.
$$

Using (9), and multiplying through by $i$, completes the proof.

**Remark.** If $\tau$ is as in (3), then

$$
\frac{\omega_1\eta_1 \sqrt{-d}}{a} - \omega_1^2 \left[ \frac{\sqrt{-d}}{a} \frac{3g_3}{2g_2} s_2(\tau) \right] = 2\pi i.
$$

However, it will deem convenient to rewrite this as

$$
(10) \quad \frac{\omega_1}{2\pi} \left[ \eta_1 - \omega_1^3 \frac{3g_3}{2g_2} s_2(\tau) \right] = \frac{a}{\sqrt{d}}.
$$
4. J = ∞ case

In this section, we derive (2) and thereby obtain (1) and formulae like it.

Lemma 4.1. We have

\[(11) \quad \left(\frac{1}{6}\right)_n^6 \left(\frac{5}{6}\right)_n^6 \left(\frac{1}{2}\right)_n^6 = 12^{-3n} (6n)! \quad \frac{(3n)!}{(3n)!}.\]

Proof. Notice that

\[\left(\frac{p}{q}\right)_n = q^{-n} \prod_{k=1}^{n} (qk + p - q), \quad p, q \in \mathbb{N}\]

follows directly from the definition of \((a)_n\). Therefore,

\[\left(\frac{1}{6}\right)_n^6 \left(\frac{5}{6}\right)_n^6 \left(\frac{3}{6}\right)_n^6 = 6^{-3n} \prod_{k=1}^{n} (6k - 5)(6k - 3)(6k - 1)\]

\[= \frac{3 \cdot 5 \cdot 7 \cdots (6n - 1)}{6^{3n}}\]

\[= 6^{-3n} (6n)! \quad \frac{2 \cdot 4 \cdot 6 \cdots 6n}{(3n)!}\]

and the proof is complete. \[\square\]

The following is a slight generalization of what appears in [7 (4.7)].

Theorem 4.2. If \(\tau\) is as in (3) and lies in \(C_{1/J, \infty}\), then

\[(12) \quad \frac{\sqrt{J}}{a \sqrt{d}} = F^{-2} \left(1 - \frac{s_2(\tau)}{6}\right) - \frac{J \frac{d}{dJ} F^2}{dJ} F^2,\]

where \(F = \text{\emph{F}}_1 (1/12, 5/12; 1; 1/J)\) and the principal branch of the square root is used.

Proof. Let \(F = \text{\emph{F}}_1 (1/12, 5/12; 1; 1/J)\). By Theorem 3.2,

\[\omega_1 = \frac{2\pi}{12^{1/4}} J^{-1/12} F \quad \text{and} \quad \Omega_1 = \frac{2\pi}{12^{1/4}} (J\Delta)^{-1/12} F,\]

so

\[\frac{d\omega_1}{dJ} = \frac{2\pi}{12^{1/4}} \left( J^{-1/12} \frac{dF}{dJ} + F \frac{d}{dJ} J^{1/12} \right) \]

\[= \frac{2\pi}{12^{1/4}} \left( J^{-1/12} \frac{dF}{dJ} - \frac{F}{12 J^{13/12}} \right) \]

\[= \frac{2\pi}{12^{1/4}} J^{-1/12} \left( F - \frac{dF}{dJ} \right).\]
Further, from Theorem 3.5,
\[ \eta_1 = -2\sqrt{3}J^{2/3}\sqrt{J - \frac{d\omega_1}{dJ}} \Delta^{1/12} \]
\[ = \frac{4\sqrt{3}}{\sqrt{12}} J^{7/12} \sqrt{J - 1} \left( \frac{F}{12} - J \frac{dF}{dJ} \right) \Delta^{1/12} \]
\[ = \frac{4\sqrt{3}}{\sqrt{12}} J^{5/12} \sqrt{J - 1} \left( \frac{F}{12} - J \frac{dF}{dJ} \right) \Delta^{1/12}. \]

Substituting the above expressions for \( \omega_1 \) and \( \eta_1 \) into (10) gives
\[ \left( J\Delta \right)^{-1/12} \frac{2\sqrt{3}}{\sqrt{12}} J^{5/12} \sqrt{J - 1} \left( \frac{F}{12} - J \frac{dF}{dJ} \right) \Delta^{1/12} \]
\[ - \left( J\Delta \right)^{-1/12} \frac{3g_3}{2g_2} \frac{\sqrt{J}}{\sqrt{J - 1}} s_2(\tau) \frac{a}{2\pi \sqrt{d}} \]
or, what is the same thing,
\[ \frac{\sqrt{J - 1}}{\sqrt{J}} F \left[ \frac{F}{12} - J \frac{dF}{dJ} \right] \]
\[ - \left( J\Delta \right)^{-1/6} \frac{3g_3}{2g_2} \frac{\sqrt{J}}{\sqrt{J - 1}} s_2(\tau) \frac{a}{2\pi \sqrt{d}} \]

Using Lemma 2.1 leads to the desired result. □

Remark. To justify that the principal branch of the square root makes the formula valid, we verified the formula at \( \tau = \sqrt{-2} \).

The following is a slight generalization of what appears in [6, (1.4)].

**Theorem 4.3.** If \( \tau \) is as in [9] and lies in \( C_{\rho, \infty} \), then
\[ \frac{a}{\pi \sqrt{d}} \frac{\sqrt{J}}{\sqrt{J - 12}} = \sum_{n=0}^{\infty} \left( \frac{1 - s_2(\tau)}{6} - n \right) \frac{(6n)!}{(3n)! n!^3} J^{n/2}. \]

**Proof.** Using Clausen’s formula [11, p. 116]
\[ \frac{\sqrt{J - 1}}{\sqrt{J}} F \left[ \frac{F}{12} - J \frac{dF}{dJ} \right] \]
we arrive at
\[ \frac{\sqrt{J - 1}}{\sqrt{J}} F \left[ \frac{F}{12} - J \frac{dF}{dJ} \right] \]
\[ - \left( J\Delta \right)^{-1/6} \frac{3g_3}{2g_2} \frac{\sqrt{J}}{\sqrt{J - 1}} s_2(\tau) \frac{a}{2\pi \sqrt{d}} \]

On account of (11) we have
\[ \frac{d}{dz} F \left( \frac{1}{12}, \frac{5}{12}; 1; \frac{1}{z} \right) \]
\[ = \sum_{n=0}^{\infty} \frac{(6n)!}{(3n)! n!^3} \frac{z^n}{12^{3n}} \]

hence
\[ \frac{d}{dz} F \left( \frac{1}{12}, \frac{5}{12}; 1; \frac{1}{z} \right) \]
\[ = \sum_{n=0}^{\infty} \frac{(6n)!}{(3n)! n!^3} \frac{n}{12^{3n}} z^n \]

Changing \( z \) into \( 1/z \) yields this
\[ \frac{d}{dz} F \left( \frac{1}{12}, \frac{5}{12}; 1; \frac{1}{z} \right) \]
\[ = \sum_{n=0}^{\infty} \frac{(6n)!}{(3n)! n!^3} \frac{n}{12^{3n}} z^{-n}. \]
Upon setting $z = J = j/12^3$, and utilizing (12), we obtain the desired result. □

4.1. Examples. The values $\tau = \sqrt{-N}$, where $N = 2, 3, 4, 7$, are such that $J(\tau)$ is rational, satisfy (3), and lie in $C_{1/1,\infty}$. Hence, the formula above holds whenever $\tau = \sqrt{-N}, N = 2, 3, 4, 7$. We state all the possible identities.

$\tau = \sqrt{-2}$:

$$\frac{5\sqrt{5}}{28\pi} = \sum_{n=0}^{\infty} \left( \frac{3}{28} + n \right) \frac{(6n)!}{(3n)!n!^3} 20^{-3n}.$$

$\tau = \sqrt{-3}$:

$$\frac{5\sqrt{15}}{66\pi} = \sum_{n=0}^{\infty} \left( \frac{1}{11} + n \right) \frac{(6n)!}{(3n)!n!^3} 2^{-n} 30^{-3n}.$$

$\tau = \sqrt{-4}$:

$$\frac{11\sqrt{33}}{252\pi} = \sum_{n=0}^{\infty} \left( \frac{5}{63} + n \right) \frac{(6n)!}{(3n)!n!^3} 66^{-3n}.$$

$\tau = \sqrt{-7}$:

$$\frac{8\sqrt{255}}{7182\pi} = \sum_{n=0}^{\infty} \left( \frac{8}{133} + n \right) \frac{(6n)!}{(3n)!n!^3} 225^{-3n}.$$

The values $\tau = \frac{1+\sqrt{-N}}{2}$, where $N = 2, 3, 4, 7, 11, 19, 27, 43, 67, 163$, are such that $J(\tau)$ is rational, satisfy (3), and lie in $C_{1/1,\infty}$. So the theorem above holds for $\tau = \frac{1+\sqrt{-N}}{2}, N = 2, 3, 4, 7, 11, 19, 27, 43, 67, 163$. We give all the possible formulae.

$\tau = \frac{1+\sqrt{-2}}{2}$:

$$\frac{5\sqrt{15}}{63\pi} = \sum_{n=0}^{\infty} (-1)^n \left( \frac{8}{63} + n \right) \frac{(6n)!}{(3n)!n!^3} 15^{-3n}.$$

$\tau = \frac{1+\sqrt{-11}}{2}$:

$$\frac{16\sqrt{7}}{77\pi} = \sum_{n=0}^{\infty} (-1)^n \left( \frac{15}{154} + n \right) \frac{(6n)!}{(3n)!n!^3} 32^{-3n}.$$

$\tau = \frac{1+\sqrt{-19}}{2}$:

$$\frac{16\sqrt{6}}{171\pi} = \sum_{n=0}^{\infty} (-1)^n \left( \frac{25}{342} + n \right) \frac{(6n)!}{(3n)!n!^3} 96^{-3n}.$$

$\tau = \frac{1+\sqrt{-27}}{2}$:

$$\frac{8\sqrt{30}}{227\pi} = \sum_{n=0}^{\infty} (-1)^n \left( \frac{31}{506} + n \right) \frac{(6n)!}{(3n)!n!^3} 3^{-n} 160^{-3n}.$$

$\tau = \frac{1+\sqrt{-33}}{2}$:

$$\frac{32\sqrt{15}}{8127\pi} = \sum_{n=0}^{\infty} (-1)^n \left( \frac{263}{5418} + n \right) \frac{(6n)!}{(3n)!n!^3} 960^{-3n}.$$
Theorem 5.2. \( \tau = \frac{1+\sqrt{-67}}{2} \):
\[
\frac{880\sqrt{330}}{130851\pi} = \sum_{n=0}^{\infty} (-1)^n \left( \frac{10177}{261702} + n \right) \frac{(6n)!}{(3n)!n!^3} 5280^{-3n}.
\]
\[
\tau = \frac{1+\sqrt{-163}}{2}.
\]
\[
\frac{213440\sqrt{10005}}{272570067\pi} = \sum_{n=0}^{\infty} (-1)^n \left( \frac{13591409}{545140134} + n \right) \frac{(6n)!}{(3n)!n!^3} 640320^{-3n}.
\]

5. \(J = 1\) Case

In this section we derive a formula analogous to (2) which results from the hypergeometric representation of \(\Omega_1\) in Theorem 3.3. We begin with the following proposition.

Proposition 5.1. We have
\[
\frac{E_4(\tau)}{E_6(\tau)} = \frac{2\pi^2}{9\omega_1^2} \tau^{1/3} \frac{\sqrt{27}}{\sqrt{J-1}}. \quad \tilde{\omega}_1 = \omega_1 \Delta^{1/12}.
\]
Proof. Since
\[
g_2(\tau) = \frac{4\pi^4 E_4(\tau)}{3} \quad \text{and} \quad g_3(\tau) = \frac{8\pi^6 E_6(\tau)}{27},
\]
we get
\[
\frac{E_4(\tau)}{E_6(\tau)} = \frac{2\pi^2}{9} \frac{g_2(\tau)}{g_3(\tau)}.
\]
But \(g_2(\tau) = \omega_1^4 g_2\) and \(g_3(\tau) = \omega_1^6 g_3\), so
\[
\frac{E_4(\tau)}{E_6(\tau)} = \frac{2\pi^2}{9\omega_1^2} \frac{g_2}{g_3}.
\]
Lastly, it follows from Lemma 2.1 that
\[
\frac{g_2}{g_3} = \frac{J^{1/3}}{\Delta^{1/6}} \frac{\sqrt{27}}{\sqrt{J-1}}
\]
so the proposed identity follows at once since \(\tilde{\omega}_1 = \omega_1 \Delta^{1/12}\). \(\square\)

Theorem 5.2. If \(\tau\) is as in 3 and lies in \(C_{(J-1)/J,i}\), then
\[
\frac{i}{\pi} \left[ \frac{a(\tau + i)^2}{2i\alpha^2\sqrt{3d}} \frac{\sqrt{J}}{\sqrt{J-1}} - \frac{F^2}{\tau + i} \frac{E_4(\tau)}{E_0(\tau)} \right] = \frac{F^2}{2} \frac{1 - s_2(\tau)}{6} - \frac{J}{J} \frac{d}{dJ} F^2,
\]
where \(F = {}_2F_1(1/12, 5/12; 1/2; (J-1)/J)\) and \(\alpha = 2i\eta(i)^2\).

Proof. Let \(F = {}_2F_1(1/12, 5/12; 1/2; (J-1)/J)\). Recall from Theorem 3.3 that
\[
\tilde{\omega}_1 = J^{-1/12} \frac{m}{\tau_0} F,
\]
where \(m = 2\pi\alpha\) and \(\tau_0 = \tau + i\). If \(J' = \frac{dJ}{d\tau} \neq 0\), then locally \(\tau\) is the inverse of \(J\) and \(\frac{d\tau}{dJ} = 1/J'\) by the inverse function theorem. Hence,
\[
\frac{d\tilde{\omega}_1}{dJ} = J^{-1/12} \frac{m}{\tau_0} \frac{dF}{d\tau} + F \frac{d}{dJ} \left( \frac{m}{\tau_0} J^{-1/12} \right)
\begin{align*}
= & J^{-1/12} \frac{m}{\tau_0} \left[ \frac{dF}{dJ} - \left( \frac{1}{12 J} + \frac{1}{\tau_0 J} \right) F \right].
\end{align*}
\]
From Theorem 3.5

\[ \eta_1 = -2\sqrt{3}J^{7/12}\sqrt{J} - \frac{m}{\tau_0} \left( \frac{dF}{dJ} - \left( \frac{1}{12J} + \frac{1}{\tau_0 J'} \right) F \right) \Delta^{1/12}. \]

Using (10) we find that

\[ \frac{a}{\sqrt{d}} = -\frac{mF}{2\pi \tau_0 (J \Delta)^{1/12}} \left[ 2\sqrt{3}J^{7/12}\sqrt{J} - \frac{m}{\tau_0} \left( \frac{dF}{dJ} - \left( \frac{1}{12J} + \frac{1}{\tau_0 J'} \right) F \right) \Delta^{1/12} \right. \]

\[ + \left. \frac{mF}{\tau_0 (J \Delta)^{1/12}} \frac{3g_3}{2g_2} s_2(\tau) \right]. \]

Consequently,

\[ \frac{2\pi a\tau_0^2}{m^2 \sqrt{d}} = -J^{-1/12} F \left[ 2\sqrt{3}J^{7/12}\sqrt{J} - \frac{m}{\tau_0} \left( \frac{dF}{dJ} - \left( \frac{1}{12J} + \frac{1}{\tau_0 J'} \right) F \right) \right. \]

\[ + \left. \frac{F \sqrt{J - 1}}{J^{1/12} \Delta^{1/6}} \frac{3g_3}{2g_2} s_2(\tau) \right]. \]

Because it is easily seen from Lemma 2.1 that

\[ \frac{3g_3}{2g_2} \Delta^{-1/6} = \frac{\sqrt{J - 1}}{2\sqrt{3}J^{1/3}}. \]

Equivalently,

\[ \frac{2\pi a\tau_0^2}{m^2 \sqrt{3d} \sqrt{J - 1}} = F^2 \left[ \frac{1}{6} + \frac{2J}{\tau_0 J'} - \frac{s_2(\tau)}{6} \right] - 2JFdF. \]

However,

\[ 2JFdF = J \frac{dF}{dJ} F^2, \]

so

\[ \frac{2\pi a\tau_0^2}{m^2 \sqrt{3d} \sqrt{J - 1}} = F^2 \left[ \frac{2J}{\tau_0 J'} - \frac{1}{6} - \frac{s_2(\tau)}{6} \right] - J \frac{d}{dJ} F^2. \]

Hence, using (3) we obtain

\[ \frac{2\pi a\tau_0^2}{m^2 \sqrt{3d} \sqrt{J - 1}} = \frac{i}{\pi} \left( \frac{F^2 E_4(\tau)}{\tau_0 E_6(\tau)} \right) = F^2 \left[ \frac{1}{6} - \frac{s_2(\tau)}{6} \right] - J \frac{d}{dJ} F^2. \]

Since \( m = 2\pi \alpha \) we have

\[ \frac{i}{\pi} \left( \frac{a\tau_0^2}{2i\alpha^2 \sqrt{3d} \sqrt{J - 1}} \right) \left( \frac{F^2 E_4(\tau)}{\tau_0 E_6(\tau)} \right) = F^2 \left[ \frac{1}{6} - \frac{s_2(\tau)}{6} \right] - J \frac{d}{dJ} F^2. \]

\[ \square \]

**Remark.** The above derivation uses (13), which in turn is valid when \( J' \neq 0 \). This holds for \( \tau \in C_{(J-1)/J,i} \) aside for isolated points. Hence, the above identity holds for all \( \tau \in C_{(J-1)/J,i} \).
5.1. **Proof of Theorem 1.1.** Using Proposition 5.1 and Theorem 5.2 we obtain

\[
\frac{i}{\pi} \left[ a(\tau + i)^2 \sqrt{J} - \frac{F^2}{\tau + i 9\omega_1^2} \frac{2\pi^2}{f^{1/3} \sqrt{J - 1}} \right] = F^2 \frac{1 - s_2(\tau)}{6} - J \frac{d}{dJ} F^2.
\]

We see from Theorem 3.3 that

\[
\tilde{\omega}_1^2 = \frac{4\pi^2 \alpha^2}{(\tau + i)^2 J^{1/6}} F^2,
\]

so

\[
\frac{i}{\pi} \left[ a(\tau + i)^2 \sqrt{J} - \frac{\tau + i \sqrt{27J}}{18\alpha^2 \sqrt{J - 1}} \right] = F^2 \frac{1 - s_2(\tau)}{6} - J \frac{d}{dJ} F^2
\]

or

\[
\frac{i(\tau + i)}{\sqrt{J - 1}} \left( a(\tau + i) \sqrt{3} - \frac{\sqrt{3}}{6\alpha^2} \right) = F^2 \frac{1 - s_2(\tau)}{6} - J \frac{d}{dJ} F^2,
\]

which simplifies to

\[
\frac{\tau + i}{2\pi \alpha^2 \sqrt{J - 1}} \left( a \frac{\tau + i}{\sqrt{-d} - 1} \right) = F^2 \frac{1 - s_2(\tau)}{6} - J \frac{d}{dJ} F^2.
\]

**Remark.** To justify that the principal branch of the square root makes the formula valid, we verified the formula at \(\tau = \sqrt{-2}\).

5.2. **Examples.** From (4), we have

\[
\frac{d}{dJ} \left( a, b; \frac{c}{J - 1} \right) = 2 \frac{ab}{c} \frac{F_1 \left( a, b; c; \frac{J - 1}{J} \right)}{F_1 \left( a + 1, b + 1; c + 1; \frac{J - 1}{J} \right)}.
\]

The values \(\tau = \sqrt{-N}\), where \(N = 2, 3, 4, 7\), are such that \(J(\tau)\) is rational, satisfy \(\Re(J_{-1} / J_i)\), and lie in \(C_{(J_{-1})/J_i}\). Thus, the above theorem holds for \(\tau = \sqrt{-N}\), \(N = 2, 3, 4, 7\). We give all the possible identities.

\(\tau = \sqrt{-2}\):

\[
\frac{5(1 + \sqrt{2})(-2 + \sqrt{2})\sqrt{2} \sqrt{3} \sqrt{5}}{1344 \pi \eta(i)^4} = 3 \frac{28^2 F_1 \left( \frac{1}{12}, \frac{5}{12}, \frac{1}{2}; \frac{98}{125} \right)^2}{100^2 F_1 \left( \frac{1}{12}, \frac{5}{12}, \frac{1}{2}; \frac{98}{125} \right) 2 F_1 \left( \frac{13}{12}, \frac{17}{12}, \frac{3}{2}; \frac{98}{125} \right)}.
\]

\(\tau = \sqrt{-3}\):

\[
\frac{5(1 + \sqrt{3})(-3 + \sqrt{3})\sqrt{3} \sqrt{5}}{1584 \pi \eta(i)^4} = \frac{1}{11^2 F_1 \left( \frac{1}{12}, \frac{5}{12}, \frac{1}{2}; \frac{121}{125} \right)^2}{\frac{1}{227^2 F_1 \left( \frac{1}{12}, \frac{5}{12}, \frac{1}{2}; \frac{121}{125} \right) 2 F_1 \left( \frac{13}{12}, \frac{17}{12}, \frac{3}{2}; \frac{121}{125} \right)}.
\]
\[
\tau = \sqrt{-4}: \\
- \frac{11\sqrt{11}}{672\pi \eta(i)^4} = \\
\frac{5}{63} \binom{1}{12} \binom{5}{12} \binom{1}{2} \binom{3^3 \eta^2}{11^3}^2 \\
- \frac{10}{3^4 \pi^4} \binom{1}{12} \binom{5}{12} \binom{1}{2} \binom{3^3 \eta^2}{11^3}^2 \binom{13}{12} \binom{17}{12} \binom{3}{2} \binom{3^3 \eta^2}{11^3}^2.
\]

\[
\tau = \sqrt{-7}: \\
\frac{85(1 + \sqrt{7})(-7 + \sqrt{7})\sqrt{7}\sqrt{85}}{2^{13}3^2 \pi^2 \eta(i)^4} = \\
\frac{8}{13^3} \binom{1}{12} \binom{5}{12} \binom{1}{2} \binom{3^5 \cdot 7 \cdot 19^2}{5^3 17^3}^2 \\
- \frac{16}{3^2 5^3 17^2} \binom{1}{12} \binom{5}{12} \binom{1}{2} \binom{3^5 \cdot 7 \cdot 19^2}{5^3 17^3}^2 \binom{13}{12} \binom{17}{12} \binom{3}{2} \binom{3^5 \cdot 7 \cdot 19^2}{5^3 17^3}.
\]

6. \(J = 0\) case

In this section we derive a formula analogous to (2) which results from the hypergeometric representation of \(\Omega_1\) in Theorem 3.4.

**Theorem 6.1.** If \(\tau\) is as in (3) and lies in \(C_{J/(J-1), \rho}\), then

\[
\frac{i}{\pi} \left[ \frac{\alpha(\tau - \rho)^2}{2a^2 \sqrt{3d} \ (1-J)^{1/3}} + \frac{F^2}{\tau - \rho} \ E_4(\tau) \right] = \frac{F^2}{6(1-J) + \frac{s_2(\tau)}{6}} + J \frac{d}{dJ} F^2,
\]

where \(F = 2 F_1(1/12, 7/12; 2/3; J/(J-1))\) and \(\alpha = \eta(\rho)^2 \sqrt{3}\).

**Proof.** Let \(F = 2 F_1(1/12, 7/12; 2/3; J/(J-1))\). Recall from Theorem 3.4 that

\[
\tilde{w}_1 = m \frac{(1-J)^{-1/12}}{\tau_0} F,
\]

where \(m = 2\pi \alpha \) and \(\tau_0 = \tau - \rho\). If \(J' = \frac{dJ}{dT} \neq 0\), then locally \(\tau\) is the inverse of \(J\) and \(\frac{dJ}{dT} = 1/J'\) by the inverse function theorem. Thus,

\[
\frac{d\tilde{w}_1}{dJ} = (1-J)^{-1/12} m \frac{dF}{\tau_0} + F \frac{d}{dJ} \left( \frac{m}{\tau_0} (1-J)^{-1/12} \right) = (1-J)^{-1/12} m \frac{dF}{\tau_0} \left( \frac{1}{12(1-J)} - \frac{1}{\tau_0 J'} \right) F.
\]

Using Theorem 5.3 we get

\[
\eta_1 = - \frac{2\sqrt{3}J^{2/3} \sqrt{J - 1}}{(1-J)^{1/12}} m \frac{dF}{\tau_0} \left( \frac{1}{12(1-J)} - \frac{1}{\tau_0 J'} \right) F \Delta^{1/12}.
\]

By virtue of (10) we obtain

\[
a \sqrt{d} = - \frac{m F}{2\pi \tau_0 (1-J)^{1/12}} \left[ 2\sqrt{3}J^{2/3} \sqrt{J - 1} m \frac{dF}{\tau_0} \left( \frac{1}{12(1-J)} - \frac{1}{\tau_0 J'} \right) F \right] \Delta^{1/12} + \frac{m F}{\tau_0 (1-J)^{1/12}} \frac{3g_3}{2g_2} s_2(\tau).
\]
or
\[
\frac{2\pi\alpha\tau_0^2}{m^2\sqrt{d}} = -\frac{F}{(1-J)^{1/6}} \left[ 2\sqrt{3}J^{2/3}\sqrt{J-1} \left[ \frac{dF}{dJ} + \left( \frac{1}{12(1-J)} - \frac{1}{\tau_0 J'} \right) F \right] 
+ \frac{F}{\Delta^{1/6}} \frac{3g_3}{2g_2} \frac{s_2(\tau)}{J} \right].
\]
Using Lemma 2.1 we get
\[
\frac{3g_3}{2g_2} \Delta^{-1/6} = \frac{\sqrt{J} - 1}{2\sqrt{3}J^{1/3}},
\]
which yields
\[
\frac{2\pi\alpha\tau_0^2}{m^2\sqrt{d}} \frac{J^{1/3}}{(1-J)^{1/6}} = -2\frac{J F}{\tau_0 J'} \left[ \frac{dF}{dJ} + \left( \frac{1}{12(1-J)} - \frac{1}{\tau_0 J'} \right) F \right] - \frac{s_2(\tau)}{6} F^2.
\]
Therefore,
\[
-\frac{2\pi\alpha\tau_0^2}{m^2\sqrt{3d}} \frac{J^{1/3}}{(1-J)^{1/6}} + \frac{2J}{\tau_0 J'} \frac{F^2}{6(1-J) + \frac{s_2(\tau)}{6}} = J \frac{d}{dJ} F^2
\]
as
\[
2J F \frac{dF}{dJ} = J \frac{d}{dJ} F^2.
\]
Using (10) we obtain
\[
-\frac{2\pi\alpha\tau_0^2}{m^2\sqrt{3d}} \frac{J^{1/3}}{(1-J)^{1/6}} + \frac{i}{\pi} \frac{F^2}{\tau_0 E_6(\tau)} = F^2 \left[ \frac{J}{6(1-J) + \frac{s_2(\tau)}{6}} \right] + J \frac{d}{dJ} F^2,
\]
that is,
\[
\frac{i}{\pi} \left[ \frac{\alpha\tau_0^2}{2\alpha^2\sqrt{3d}} \frac{J^{1/3}}{(1-J)^{1/3}} + \frac{F^2}{\tau_0 E_6(\tau)} \right] = F^2 \left[ \frac{J}{6(1-J) + \frac{s_2(\tau)}{6}} \right] + J \frac{d}{dJ} F^2.
\]

6.1. Proof of Theorem 1.2 Using Proposition 5.1 and Theorem 6.1 we obtain
\[
\frac{i}{\pi} \left[ \frac{a(\tau - \overline{\tau})^2}{2\alpha^2\sqrt{3d}} \frac{J^{1/3}}{(1-J)^{1/3}} + \frac{F^2}{\tau - \overline{\tau}} \frac{4\alpha^2}{9\alpha^2\sqrt{3}} \sqrt{\frac{J}{1-J}} \right] = F^2 \left[ \frac{J}{6(1-J) + \frac{s_2(\tau)}{6}} \right] + J \frac{d}{dJ} F^2.
\]
We see from Theorem 3.4 that
\[
\overline{\omega}_1^2 = \frac{4\alpha^2}{(\tau - \overline{\tau})^2(1-J)^{1/6}} F^2,
\]
so
\[
\frac{i}{\pi} \left[ \frac{a(\tau - \overline{\tau})^2}{2\alpha^2\sqrt{3d}} \frac{(\tau - \overline{\tau})^2}{6\alpha^2\sqrt{-3}} \frac{J^{1/3}}{(1-J)^{1/3}} \right] = F^2 \left[ \frac{J}{6(1-J) + \frac{s_2(\tau)}{6}} \right] + J \frac{d}{dJ} F^2;
\]
which simplifies to
\[
-\frac{\tau - \overline{\tau}}{2\pi\alpha^2\sqrt{3d}} \frac{J^{1/3}}{(1-J)^{1/3}} \left[ \frac{a(\tau - \overline{\tau})^2}{6\alpha^2\sqrt{-d}} \right] = F^2 \left[ \frac{J}{6(1-J) + \frac{s_2(\tau)}{6}} \right] + J \frac{d}{dJ} F^2.
\]
Remark. To justify that the principal branch of the square and cube root makes the formula valid, we verified the formula at $\tau = -\frac{1 + \sqrt{-N}}{2}$.

6.2. Examples. From (4), we get

$$
\frac{d}{dJ} F_1(a, b; c; \frac{J}{J-1})^2 = - \frac{2ab}{c(J-1)^2} F_1(a, b; c; \frac{J}{J-1}) 2F_1(a+1, b+1; c+1; \frac{J}{J-1}).
$$

The values $\tau = -\frac{1 + \sqrt{-N}}{2}$, where $N = 7, 11, 19, 27, 43, 67, 163$, are such that $J(\tau)$ is rational, satisfy (3) and lie in $\mathbb{C}$. So the theorem above holds if $\tau = -\frac{1 + \sqrt{-N}}{2}$. We state all the possible identities.

$$
\tau = -\frac{1 + \sqrt{-7}}{2}:
$$

$$
\frac{3 - \sqrt{-3}}{\pi \eta(\rho)^4} \frac{\tau^{1/6}}{756} = \frac{5}{693} = 5
\begin{align*}
&- \frac{40}{567} 2F_1 \left( \frac{1}{12}, \frac{7}{12}, \frac{3}{189} \right)^2 \\
&+ \frac{500}{15309} 2F_1 \left( \frac{1}{12}, \frac{7}{12}, \frac{3}{189} \right) \\
&\times 2F_1 \left( \frac{13}{12}, \frac{19}{12}, \frac{5}{125} \right).
\end{align*}
$$

$$
\tau = -\frac{1 + \sqrt{-11}}{2}:
$$

$$
\frac{3 - \sqrt{-3}}{\pi \eta(\rho)^4} \frac{\tau^{1/3}}{693} = \frac{4}{693} = 4
\begin{align*}
&- \frac{48}{539} 2F_1 \left( \frac{1}{12}, \frac{7}{12}, \frac{512}{3}, \frac{539}{} \right)^2 \\
&+ \frac{288}{41503} 2F_1 \left( \frac{1}{12}, \frac{7}{12}, \frac{512}{3}, \frac{539}{} \right) \\
&\times 2F_1 \left( \frac{13}{12}, \frac{19}{12}, \frac{5}{12} \right).
\end{align*}
$$

$$
\tau = -\frac{1 + \sqrt{-19}}{2}:
$$

$$
\frac{3 - \sqrt{-3}}{\pi \eta(\rho)^4} \frac{\tau^{1/6}}{513} = \frac{8}{513} = 8
\begin{align*}
&- \frac{112}{1539} 2F_1 \left( \frac{1}{12}, \frac{7}{12}, \frac{512}{3}, \frac{513}{} \right)^2 \\
&+ \frac{224}{1789507} 2F_1 \left( \frac{1}{12}, \frac{7}{12}, \frac{512}{3}, \frac{513}{} \right) \\
&\times 2F_1 \left( \frac{13}{12}, \frac{19}{12}, \frac{5}{12} \right).
\end{align*}
$$
\( \tau = \frac{1 + \sqrt{-27}}{2} \):

\[
3 - \sqrt{3} \frac{27^{1/6}}{\eta(\rho)^{1/3}} \frac{20}{6831} = - \frac{3920}{6409} {\text{2F1}} \left( \frac{1}{12}, \frac{7}{12}; \frac{5}{3}; \frac{64000}{6409} \right)^2 \\
+ \frac{84000}{4097152081} {\text{2F1}} \left( \frac{1}{12}, \frac{7}{12}; \frac{2}{3}; \frac{64000}{6409} \right) \\
\times {\text{2F1}} \left( \frac{13}{12}, \frac{19}{12}; \frac{5}{3}; \frac{64000}{6409} \right).
\]

\( \tau = \frac{1 + \sqrt{-43}}{2} \):

\[
3 - \sqrt{3} \frac{43^{1/6}}{\eta(\rho)^{1/3}} \frac{200}{24381} = - \frac{74560}{1536003} {\text{2F1}} \left( \frac{1}{12}, \frac{7}{12}; \frac{2}{3}; \frac{512000}{512001} \right)^2 \\
+ \frac{32000}{112347867429} {\text{2F1}} \left( \frac{1}{12}, \frac{7}{12}; \frac{2}{3}; \frac{512000}{512001} \right) \\
\times {\text{2F1}} \left( \frac{13}{12}, \frac{19}{12}; \frac{5}{3}; \frac{512000}{512001} \right).
\]

\( \tau = \frac{1 + \sqrt{-67}}{2} \):

\[
3 - \sqrt{3} \frac{67^{1/6}}{\eta(\rho)^{1/3}} \frac{1700}{392553} = - \frac{9937840}{25552003} {\text{2F1}} \left( \frac{1}{12}, \frac{7}{12}; \frac{2}{3}; \frac{85184000}{85184001} \right)^2 \\
+ \frac{5324000}{3109848868443429} {\text{2F1}} \left( \frac{1}{12}, \frac{7}{12}; \frac{2}{3}; \frac{85184000}{85184001} \right) \\
\times {\text{2F1}} \left( \frac{13}{12}, \frac{19}{12}; \frac{5}{3}; \frac{85184000}{85184001} \right).
\]

\( \tau = \frac{1 + \sqrt{-163}}{2} \):

\[
3 - \sqrt{3} \frac{163^{1/6}}{\eta(\rho)^{1/3}} \frac{533600}{817710201} = - \frac{11363838226240}{455794119168003} {\text{2F1}} \left( \frac{1}{12}, \frac{7}{12}; \frac{2}{3}; \frac{151931373056000}{151931373056001} \right)^2 \\
+ \frac{9495710816000}{989278010372078456086619429} {\text{2F1}} \left( \frac{1}{12}, \frac{7}{12}; \frac{2}{3}; \frac{151931373056000}{151931373056001} \right) \\
\times {\text{2F1}} \left( \frac{13}{12}, \frac{19}{12}; \frac{5}{3}; \frac{151931373056000}{151931373056001} \right).
\]

7. **Further work**

It is natural to apply the method of this paper to systematically derive Chudnovsky-Ramananjan type formulae for other families of elliptic curves, which we hope to do.
in our future work. For the interested reader, it is perhaps instructive to briefly discuss another example to give a sense of the generality of this method.

Consider the Legendre family of elliptic curves given by \( y^2 = x(x-1)(x-\lambda) \). The Picard-Fuchs differential equation for this family is well known and given by

\[
\lambda(1-\lambda) \frac{d^2 \Omega}{d\lambda^2} + (1-2\lambda) \frac{d\Omega}{d\lambda} - \frac{\Omega}{4} = 0,
\]

which is a hypergeometric differential equation with parameters \( a = 1/2, b = 1/2, c = 1 \), and it has three regular singular points: 0, 1, \( \infty \).

Kummer’s method yields six (distinct) hypergeometric solutions of the form

\[
\lambda^\alpha (1-\lambda)^\beta \, _2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \nu(\lambda)\right),
\]

where \( \nu(\lambda) \) is one of

\[
\lambda, \quad 1-\lambda, \quad \frac{1}{\lambda}, \quad \frac{1}{1-\lambda}, \quad \frac{\lambda}{\lambda-1}, \quad \frac{\lambda-1}{\lambda}.
\]

In particular, \( \alpha = 0 \) and \( \beta = 0 \) if \( \nu(\lambda) = \lambda, 1-\lambda \); \( \alpha = -1/2 \) and \( \beta = 0 \) if \( \nu(\lambda) = 1/\lambda, (\lambda-1)/\lambda \); \( \alpha = 0 \) and \( \beta = -1/2 \) if \( \nu(\lambda) = 1/(1-\lambda), \lambda/(\lambda-1) \). Each of these solutions will be valid near one of the singular points 0, 1, \( \infty \) and will give rise to a hypergeometric representation of \( \Omega_1 \) in terms of \( \lambda \). Applying the method used in this paper with \( \lambda \) in place of \( J \), one can derive Chudnovsky-Ramanujan type formulae corresponding to each hypergeometric representation of \( \Omega_1 \), which will be valid near one of the cusps 0, 1, \( \infty \).

In fact, according to [7], some of Ramanujan’s original formulae in [14] are derived using the hypergeometric representations of the periods of the Legendre family. So this case was considered earlier than the level 1 case studied by D. V. and G. V. Chudnovsky. It would be interesting to do a complete determination using the method of this paper.

References

[1] G. E. Andrews, R. Askey, and R. Roy, *Special Functions*, Cambridge University Press, Cambridge, 1999.
[2] N. Archinard, Exceptional sets of hypergeometric series, *J. Number Theory* 101 (2003) 244-269.
[3] B. C. Berndt, *Ramanujan’s Notebooks*, Part IV, Springer-Verlag, New York, 1994.
[4] J. Borwein and P. Borwein, Pi and the AGM, Wiley, New York, 1987.
[5] H. H. Chan and H. Verrill, The Apéry numbers, the Almkvist-Zudilin numbers and new series for \( 1/\pi \), *Math. Res. Lett.* 16 (2009) 405-420.
[6] D. V. Chudnovsky and G. V. Chudnovsky, Approximation and complex multiplication according to Ramanujan, *Ramanujan Revisited*, G. E. Andrews, R. A. Askey, B. C. Berndt, K. G. Ramanathan, and R. A. Rankin, eds., Academic Press, Boston, 1988, 375-472.
[7] D. V. Chudnovsky and G. V. Chudnovsky, Use of computer algebra for Diophantine and differential equations, *Computer algebra*, Lecture Notes in Pure and Appl. Math. 113, D. V. Chudnovsky and R. D. Jenks, eds., Dekker, New York, 1989, 1-81.
[8] D. Cox, *Primes of the Form \( x^2 + ny^2 \) : Fermat, Class Field Theory, and Complex Multiplication*, 2nd ed., Wiley, New York, 1989.
[9] R. Fricke and F. Klein, *Vorlesungen über die Theorie der elliptischen Modulfunktionen*, Teubner, Leipzig, 1890.
[10] A. G. Greenhill, *The Applications of Elliptic Functions*, Macmillan and Co., New York, 1892.
[11] E. E. Kummer, Über die hypergeometrische Reihe, *J. Reine Angew. Math.* 15 (1836) 39-83, 127-172.
[12] S. Lang, *Elliptic Functions*, 2nd ed., Springer, New York, 1987.
[13] S. Ramanujan, *Collected Papers*, Cambridge University Press, Cambridge, 1927.
[14] S. Ramanujan, Modular equations and approximations to $\pi$, *Quart. J. Math. (Oxford)* 45 (1914) 350-372.
[15] S. Ramanujan, On certain arithmetical functions, *Trans. Cambridge Phil. Soc.* 22 (1916) 159-184.
[16] J.-P. Serre, Congruences et formes modulaire (d’après H. P. F. Swinnerton-Dyer), *Séminaire Bourbaki, 24e année (1971/1972), Exp. No. 416*, Lecture Notes in Math. 317, Springer, Berlin, 1973, 319-338.
[17] J. H. Silverman, *The Arithmetic of Elliptic Curves*, Springer-Verlag, New York, 1985.

IMIN CHEN, DEPARTMENT OF MATHEMATICS, SIMON FRASER UNIVERSITY, BURNABY, BRITISH COLUMBIA, CANADA
E-mail address: ichen@sfu.ca

GLEB GLEBOV, DEPARTMENT OF MATHEMATICS, SIMON FRASER UNIVERSITY, BURNABY, BRITISH COLUMBIA, CANADA
E-mail address: gglebov@sfu.ca