Limitations on the use of the Heisenberg picture

J.D. Franson and R.A. Brewster
University of Maryland Baltimore County, Baltimore, MD 21250 USA

The Schrödinger and Heisenberg pictures are equivalent formulations of quantum mechanics in the sense that they give the same expectation value for any operator. We consider a sequence of two or more unitary transformations and show that the Heisenberg operator produced after the first transformation cannot be viewed as the input to the second transformation. The experimental consequences of this are illustrated by several examples in quantum optics.

The Schrödinger and Heisenberg formulations of quantum mechanics are physically equivalent [1-5]. The Heisenberg picture is often useful for calculating the time evolution of complicated systems, while the Schrödinger picture sometimes provides the most straightforward way to understand the fundamental properties of a system. Here we consider a sequence of two or more unitary transformations and show that the Heisenberg operator produced after the first transformation cannot be viewed as the input to the second transformation. This raises some fundamental questions as to the physical meaning of the Heisenberg representation of the electric field operator, for example. In addition, we will discuss several examples of experiments in quantum optics where an inappropriate use of the Heisenberg picture can produce misleading or incorrect results.

The situation of interest is illustrated in Fig. 1, where a quantum system undergoes two successive unitary transformations \( \hat{U} \) and \( \hat{V} \). Consider an observable operator \( \hat{F} \) in the Heisenberg picture that takes on the forms \( \hat{F}_0 \), \( \hat{F}_1 \), and \( \hat{F}_2 \) as illustrated. We show that the output \( \hat{F}_1 \) of the first transformation \( \hat{U} \) cannot be viewed as the input to the second transformation \( \hat{V} \).

Here \( \hat{H}_0 \) is the time-independent Hamiltonian responsible for transformation \( \hat{U} \), \( t \) is the evolution time, and \( \hbar \) is Planck’s constant divided by \( 2\pi \). A similar expression describes transformation \( \hat{V} \).

Since the output of transformation \( \hat{U} \) forms the input to transformation \( \hat{V} \), one might assume that the final form of the operator \( \hat{F} \) is given by

\[
\hat{F}_2 = \hat{V}^\dagger \hat{F}_1 \hat{V} = \hat{V}^\dagger (\hat{U}^\dagger \hat{F}_0 \hat{U}) \hat{V}.
\]

That is not the case, however, as can be seen by introducing the total transformation \( \hat{T} \) given by

\[
\hat{T} = \hat{V} \hat{U}.
\]

In analogy with Eq. (1), \( \hat{F}_2 \) is actually given by

\[
\hat{F}_2 = \hat{T}^\dagger \hat{F}_0 \hat{T} = (\hat{U}^\dagger \hat{V}^\dagger) \hat{F}_0 (\hat{V} \hat{U}).
\]

It can be seen from Eq. (5) that the operators \( \hat{V} \) and \( \hat{V}^\dagger \) corresponding to the second transformation are applied adjacent to the initial Heisenberg operator \( \hat{F}_0 \). That is not equivalent to Eq. (3), where it was assumed that \( \hat{F}_1 \) must form the input to the second transformation.
Eqs. (3) through (5) show that the Heisenberg operator produced after the first transformation cannot be viewed as the input to the second transformation, which raises some fundamental questions as to the physical meaning of the operator \( \hat{F}_1 \). By construction, \( \hat{F}_1 \) will give the same expectation value as would be obtained using the Schrödinger picture, but it is not suitable for predicting the results of a measurement made after a second transformation \( \hat{V} \).

Nevertheless, Heisenberg operators are widely used to characterize the output of an experimental device even though a second transformation could be applied later on [6-8]. We will now illustrate some of the limitations in that approach by considering the decoherence produced by a beam splitter, which is one of the most widely used devices in quantum optics.

Consider a quantum state of light \( |\psi_0\rangle \) that is incident on a beam splitter in the input path labelled \( A \) in Fig. 2. The other input path labeled \( B \) will be assumed to be in the vacuum state containing no photons. The corresponding output modes will be denoted by \( A' \) and \( B' \). If the reflection coefficient \( R \) of the beam splitter is nonzero, then part of the vacuum fluctuations in mode \( B \) will be coupled into output mode \( A' \). This can be described by a quantum noise operator \( \hat{N} \) as discussed in more detail below. In addition, the beam splitter couples part of the amplitude of the input state \( |\psi_0\rangle \) into output path \( B' \), which leaves some amount of “which-path” information in the environment. We will show that the usual Heisenberg-picture treatment of a beam splitter provides a correct description of the quantum noise \( \hat{N} \), but it does not describe the additional decoherence due to the which-path information left in the environment.

![Fig. 2. A quantum state \( |\psi_0\rangle \) is incident on a beam splitter in mode \( A \). Vacuum fluctuation noise is coupled from mode \( B \) into the output mode \( A' \), while which-path information is coupled into the environment in mode \( B' \). The Heisenberg operator in Eq. (6) does not include the effects of the which-path information.](image)

In order to analyze this situation in more detail, we will denote the photon annihilation operator in path \( \hat{a}_A \) by \( \hat{\tilde{a}}_A \), while the corresponding operator in the other input mode will be denoted by \( \hat{\tilde{b}}_B \). The corresponding operators in the two output modes will be denoted by \( \hat{\tilde{a}}_A' \) and \( \hat{\tilde{b}}_B' \). It can be shown that the unitary transformation \( \hat{U} \) for a beam splitter has the property that

\[
\hat{x}_1 = \hat{U}^\dagger \hat{x}_0 \hat{U} = \sqrt{1-R^2} \hat{x}_0 - R \hat{x}_0 = T_r \hat{x}_0 + \hat{N}.
\]

Here \( \hat{x}_0 = (\hat{a}_A + \hat{\tilde{a}}_A')/2 \) is one of the quadratures (phase components) of the input electric field in mode \( A \), while \( \hat{x}_0 = (\hat{\tilde{b}}_B + \hat{\tilde{b}}_B')/2i \) is the orthogonal quadrature in input mode \( B \) [6]. The factor of \( T_r = \sqrt{1-R^2} \) corresponds to the transmission coefficient of the beam splitter, while we have defined the noise operator \( \hat{N} \) by \( \hat{N} = -R \hat{x}_0 \). Eq. (6) suggests that the only effect of a beam splitter is to attenuate the amplitude of the field by a factor of \( T_r \) while adding quantum noise \( \hat{N} \).

We will focus our attention on the case in which \( R << 1 \). In that limit, \( T_r \rightarrow 1 \) while \( \hat{N} \rightarrow 0 \) and \( \hat{x}_1 \rightarrow \hat{x}_0 \).

In the Heisenberg picture, the output of the beam splitter in Eq. (6) appears to be the same as the input for small values of the reflectivity. Although this may seem intuitively correct, only the expectation value of \( \hat{x}_1 \) is unchanged by the beam splitter and other properties of the output state can be very different from the input, as we will now show.

In order to see this, we will analyze the amount of interference that can occur between the two components of a Schrödinger cat state \( |\psi_0\rangle \) [9, 10] after it has passed through a beam splitter as illustrated in Fig. 3. The cat state of interest is defined by

\[
|\psi_0\rangle = c_n \left( |\alpha_0\rangle + e^{-i\phi} |\alpha_0\rangle \right),
\]

where \( c_n \) is a suitable normalizing constant, \( \phi \) is a phase shift, and \( \alpha_0 \) is a complex parameter. A coherent state \( |\alpha\rangle \) is defined [11, 12] as usual by

\[
|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle,
\]

where \( |n\rangle \) is a number state of the electromagnetic field containing \( n \) photons. The initial cat state corresponds to a superposition of two coherent states with different phases,
as illustrated in phase space in Fig. 4, where we have assumed for simplicity that $\alpha_0 = i|\alpha_0|$.

![Diagram of apparatus to measure quantum interference](image)

**Fig. 3.** Apparatus to measure the amount of quantum interference between the two components of a Schrödinger cat state $|\psi_0\rangle$ after it passes through a beam splitter with the vacuum state $|\text{vac}\rangle$ in the other input port. A single photon $\gamma$ passes through an interferometer shown by the red (dashed) lines. A Kerr medium $K$ (along with a constant bias phase shift not shown) will produce a phase shift of $\phi^\pm$ depending on the path taken by the photon. $D_1$ is a single-photon detector while $D_2$ is a homodyne detector that measures the phase of the field. The results are post-selected on a single photon detected in $D_1$ with a $90^\circ$ phase measured in $D_2$. The phase shift $\phi^\pm$ causes two components of the initial cat state to overlap and produce quantum interference that depends on the single-photon phase shift $\theta$, as illustrated in Fig. 4.

![Diagram of interference](image)

**Fig. 4.** Interference of the two components of a cat state produced by the apparatus shown in Fig. 3. The two components of the initial cat state shown in red (solid circles) are displaced by an angle $\phi$ from the imaginary axis in phase space. The Kerr medium in Fig. 3 produces a phase shift of $\phi^\pm$ as illustrated by the blue arrows, which displaces the cat state components to the locations indicated by the light blue (dashed) circles. The overlapping components labelled $I$ produce quantum interference, while the non-overlapping components labelled with an $X$ are eliminated by the post-selection process described in Fig. 3.

Interference between the two initial components of the cat state can be produced using the interferometer arrangement shown on the right-hand side of Fig. 3 [13]. Here a single photon $\gamma$ propagates through an interferometer that contains a Kerr medium $K$ in one of the two paths. Depending on which path the single photon takes, the Kerr medium will apply a phase shift of $\phi^\pm$ to the cat state. This will produce an overlap of the two components of the cat state at a phase of $\pi/2$, along with two other non-overlapping probability amplitudes as illustrated in Fig. 4. A homodyne detector is used to measure the phase of the output field, and we only accept (post-select) those events in which the homodyne detector measured a final phase of $\pi/2$. This post-selection process eliminates the contributions from the non-overlapping probability amplitudes, while quantum interference between the two overlapping probability amplitudes will produce a $\cos^2(\theta)$ dependence of the interference pattern. Here $\theta$ is a single-photon phase shift inserted into one of the paths through the interferometer.

The visibility of the interference between the two components of the cat state can be analyzed in the Schrödinger picture using the Husimi-Kano Q-function [14-16] defined by

$$ Q(\alpha^\prime, \beta^\prime) = \frac{1}{\pi} \langle \alpha^\prime | (\beta^\prime) \hat{\rho} | \beta^\prime \rangle | \alpha^\prime \rangle. \tag{9} $$

Here $|\alpha^\prime\rangle$ and $|\beta^\prime\rangle$ denote arbitrary coherent states in modes $A'$ and $B'$ of the beam splitter while $\hat{\rho}$ is the density operator for the system. The unitary transformation $\hat{U}$ produced by the beam splitter can be written [17] in the factored form

$$ \hat{U} = e^{i R \hat{b}^\dagger \hat{b}} |T_r \rangle \langle T_r | \hat{a}^\dagger \hat{b} \hat{b}^\dagger \hat{a}^\dagger |T_r \rangle \langle T_r |, \tag{10} $$

while the transformation $\hat{V}$ produced by the single-photon interferometer and Kerr cell is given [13] by

$$ \hat{V} = \frac{1}{2} \left( e^{i \theta} \hat{\Phi}_+ + \hat{\Phi}_- \right). \tag{11} $$

Here the operators $\hat{\Phi}_\pm = e^{\pm i \phi \hat{n}}$ shift the phase of the field by $\pm \phi$, where $\hat{n}$ is the photon number operator. It should be noted that $\hat{V}$ is not unitary in this example, as will be discussed below.

The visibility $v$ of the quantum interference pattern can be calculated in the Schrödinger picture using Eqs. (9) through (11), as described in the appendix. The result is that

$$ v = e^{-2R^2 \sin^2(\phi)|\alpha_0|^2}. \tag{12} $$

It can be seen from Eq. (12) that the visibility will be exponentially small for arbitrarily small values of $R$,
provided that the product $R|\alpha_0|\sin(\theta)|$ is larger than 1. Since $\hat{N} \rightarrow 0$ for $R \ll 1$, this shows that the which-path information left in mode $B$ of the beam splitter can seriously degrade the visibility even when the quantum noise $\hat{N}$ is negligible. It is well-known that cat states are very sensitive to photon loss [18, 19], and these results show that the origin of this decoherence is unrelated to the quantum noise operator $\hat{N}$.

The exponential decrease in the visibility in Eq. (12) is inconsistent with the Heisenberg operator in Eq. (6), where the input and output fields appear to be the same for $R \ll 1$. Although Eq. (6) gives the correct expectation value, it does not describe the entanglement of the field with other modes and it cannot be used to predict the results of subsequent transformations or measurements. As a result, the Heisenberg picture gives a very limited description of the field after a unitary transformation $\hat{U}$.

A further difficulty in the use of the Heisenberg picture can be seen if we assume that the second transformation $\hat{V}$ is not unitary as in Eq. (11). Non-unitary transformations occur in quantum optics when post-selection or heralding techniques [20-25] are used, and they also arise from the non-Hermitian Hamiltonians that are currently of widespread interest in other fields [26-28]. If $\hat{V}$ and thus $\hat{T}$ are not unitary, then Eq. (2) no longer holds but we could potentially use Eq. (5) to define the final output of the process in the Heisenberg picture as $\hat{F}_2 = \hat{T}^{\dagger} \hat{F}_0 \hat{T}$. The corresponding expectation value is

$$\langle \hat{F} \rangle = \langle \psi_0 | \hat{T}^{\dagger} \hat{F}_0 \hat{T} | \psi_0 \rangle = \langle \hat{T} \psi_0 | \hat{F}_0 | \hat{T} \psi_0 \rangle. \quad (13)$$

The right-hand side of Eq. (13) can be recognized as the expectation value of $\hat{F}$ in the Schrödinger picture, where $|\hat{T} \psi_0\rangle$ is the final state of the system. Thus this definition of $\hat{F}_2$ does give the correct expectation value.

However, the square of $\hat{F}_2$ does not give the correct expectation value in the Heisenberg picture:

$$\langle \hat{F}^2 \rangle_H = \langle \psi_0 | \hat{F}_2^2 | \psi_0 \rangle = \langle \psi_0 | (\hat{T}^{\dagger} \hat{F}_0 \hat{T}) (\hat{T}^{\dagger} \hat{F}_0 \hat{T}) | \psi_0 \rangle \quad (14)$$

In contrast, the expectation value of $\hat{F}_2^2$ in the Schrödinger picture is given by

$$\langle \hat{F}^2 \rangle_S = \langle \psi_0 | (\hat{T}^{\dagger} \hat{F}_0 \hat{T}) | \psi_0 \rangle. \quad (15)$$

If $\hat{V}$ and thus $\hat{T}$ are not unitary, then $\hat{T}^{\dagger} \hat{T} \neq \hat{I}$ and Eqs. (14) and (15) are not equivalent. Thus $\langle \hat{F}^2 \rangle_H \neq \langle \hat{F}^2 \rangle_S$, and the use of the Heisenberg picture would give the wrong expectation value for $\hat{F}_2^2$. This can lead to incorrect conclusions regarding the variance of the output field produced by an optical device, for example [13].

Eq. (14) suggests that the Heisenberg picture may not be appropriate for non-unitary transformations where Eq. (2) does not hold. The time evolution of $\hat{F}$ and $\hat{F}^2$ can be calculated separately using the Heisenberg equation [29], but the resulting operator for $\hat{F}^2$ is not the square of the operator $\hat{F}$ in that case, which raises some questions as to the physical meaning of the Heisenberg operators for non-unitary transformations. The Heisenberg picture has also been used to describe open quantum systems [30-32]. In any event, our main point is that the use of the Heisenberg picture to characterize the output of a unitary device, such as a beam splitter, can also give an incomplete description of the properties of a system. As another example of this, we recently showed that the well-known linear input/output relation for an optical parametric amplifier [6] in the Heisenberg picture includes the effects of quantum noise but it does not include the decoherence due to which-path information and entanglement with the environment [13].

In summary, the Heisenberg and Schrödinger pictures are physically equivalent in the sense that they give the same expectation values. But the Heisenberg operators produced after an initial transformation cannot be viewed as the input to a subsequent transformation. This raises some fundamental questions regarding the physical meaning of the Heisenberg operator for the electric field, for example. One might argue in retrospect that these results should be apparent, but informal discussions with many experts in quantum mechanics invariably led to an initial agreement with Eq. (3).

As a practical matter, Heisenberg operators are often used to describe the output of an optical device such as an amplifier [6]. Our results show that the use of Heisenberg operators may not describe the true nature of an output state, such as its entanglement with other modes. As we have shown for the simple case of a beam splitter, the use of Heisenberg operators may suggest that the output of a device is essentially the same as the input when they are actually very different. Although the Heisenberg picture is very useful, it should be used with caution since the Heisenberg operator produced by one transformation cannot be used to predict the results of a subsequent transformation.

**ACKNOWLEDGEMENTS**

We would like to acknowledge many valuable discussions with Todd Pittman. This work was supported in part by the National Science Foundation under grant # PHY-1802472.
Here we describe the calculation of the visibility $v$ in Eq. (12) of the main text in more detail. If we include the second input to the beam splitter, then the initial state $|\psi_\alpha\rangle$ of the system in the Schrödinger picture is

$$|\psi_\alpha\rangle = c_n \left( |e^{i\theta} \alpha_0\rangle_A + e^{-i\theta} \alpha_0\rangle_A \right) \otimes |0\rangle_B,$$

where $A$ and $B$ label the two input modes of the beam splitter as in Fig. 2. Here $|0\rangle_B$ denotes the vacuum state in

polarizing beam splitters, Phys. Rev. A 64, 062311 (2001).

22. T.B. Pittman, M.J. Fitch, B.C. Jacobs, and J.D. Franson, Experimental controlled-NOT logic gate for single photons in the coincidence basis, Phys. Rev. A 68, 032316 (2003).

23. T.C. Ralph and A.P. Lund, Proc. of 9th International Conf. on Quantum Communication Measurement and Computing, edited by A. Lvovsky (AIP, New York) p. 155 (2009).

24. M. Micuda, I. Straka, M. Mikova, M. Dusek, N.J. Cerf, J. Fiurasek, and M. Jezek, Phys. Rev. Lett. 109, 180503 (2012).

25. R.A. Brewster, I.C. Nodurft, T.B. Pittman, and J.D. Franson, Noiseless attenuation using an optical parametric amplifier, Phys. Rev. A 96, 042307 (2017).

26. T.E. Lee, Anomalous edge state in a non-Hermitian lattice, Phys. Rev. Lett 116, 133903 (2016).

27. H. Shen, B. Zhen, and L. Fu, Topological band theory for non-Hermitian Hamiltonians, Phys. Rev. Lett 120, 146402 (2018).

28. Z. Gong, Y. Ashida, K. Kawabata, T. Takasan, S. Higashikawa, and M. Ueda, Topological phases of non-Hermitian systems, Phys. Rev. X 8, 031079 (2018).

29. For example, see R. Jozsa, Complex weak values in quantum measurement, Phys. Rev. A 76, 044103 (2007).

30. N. Gisin, Time correlations and Heisenberg picture in the quantum state diffusion model of open systems, J. Mod. Optics 40, 2313 (1993).

31. H.P. Breuer, B. Kappeler, and F. Petruccione, Heisenberg picture operators in the stochastic wave function approach to open quantum systems, Eur. Phys. J. D1, 9 (1998); arXiv:quant-ph/9807080.

32. S.R. Clark, J. Prior, M.J. Hartmann, D. Jaksch, and M.B. Plenio, Exact matrix product solutions in the Heisenberg picture of an open quantum spin chain, New J. Phys. 12, 025005 (2010).
mode B and a coherent state $|\alpha\rangle$ is defined in the text.

Equation (A1) corresponds to a pure state (a Schrödinger cat), whose initial density operator $\hat{\rho}$ can be written in the form [13]

$$\hat{\rho} = \hat{\rho}_{++} + \hat{\rho}_{+-} + \hat{\rho}_{-+} + \hat{\rho}_{--}. \quad (A2)$$

Here the total density operator $\hat{\rho}$ has been written as the sum of four terms defined by

$$\hat{\rho}_{++} = c_n^2 |\alpha\rangle \langle \alpha|_A \otimes |0\rangle_B$$

$$\hat{\rho}_{+-} = c_n^2 |\alpha\rangle \langle -\alpha|_A \otimes |0\rangle_B$$

$$\hat{\rho}_{-+} = c_n^2 |\alpha\rangle \langle \alpha|_A \otimes |0\rangle_B$$

$$\hat{\rho}_{--} = c_n^2 |\alpha\rangle \langle -\alpha|_A \otimes |0\rangle_B.$$

Our goal is to calculate the visibility of the quantum interference in the apparatus shown in Fig. 3 of the main text. This can be done using the two mode Q-function of equation (9). Since the Q-function is a linear function of the density operator, it can be written as

$$Q(\alpha', \beta') = Q_+(\alpha', \beta') + Q_-(\alpha', \beta') + Q_{+-}(\alpha', \beta') + Q_{-+}(\alpha', \beta'), \quad (A4)$$

where $Q_+(\alpha', \beta') = \langle \alpha' | \langle \beta' | \hat{\rho}_{++} | \beta' \rangle | \alpha' \rangle / \pi^2$, for example.

Eqs. (A2) through (A4) give the initial forms of $\hat{\rho}$ and $Q(\alpha', \beta')$. The final form of $Q(\alpha', \beta')$ can be found by applying the transformation $\hat{V} \hat{U}$ to $\hat{\rho}$. As shown in Fig. 4, we post-select on those situations where the phase shift from the Kerr medium cancels the phase shift in the original cat state component to give a net phase shift of zero. Therefore, we need only keep the relevant term of the operator $\hat{V}$ for each term in the Q-function. For example,

$$Q_+(\alpha', \beta') = e^{-i\theta} \langle \alpha' | \langle \beta' | \hat{\Phi}_- \hat{U} \hat{\rho}_{++} \hat{U}^\dagger \hat{\Phi}_+ | \beta' \rangle | \alpha' \rangle_A \quad (A5)$$

Inserting Eq. (10) from the main text in Eq. (A9) gives

$$Q_+(\alpha', \beta') = e^{-i\theta} \langle \alpha' | \langle \beta' | \hat{\Phi}_- \hat{U} \hat{\rho}_{++} \hat{U}^\dagger \hat{\Phi}_+ | \beta' \rangle | \alpha' \rangle_A \quad (A5)$$

where $P_{\text{max}}$ and $P_{\text{min}}$ refer to the maximum and minimum counting rates. It can be shown using Eqs. (A4) through Eq. (A6) that

$$v = \frac{1}{c_n^2} \left| \int Q_+(\alpha', \beta') d^2\alpha' d^2\beta' \right|, \quad (A7)$$

provided that there was negligible overlap between the two components of the original cat state. Thus, we need only focus on $Q_+(\alpha, \beta)$. Using Eq. (A3) in Eq. (A5) gives

$$Q_+(\alpha, \beta) = c_n^2 e^{-i\theta} \left\{ e^{i\phi} \alpha' \right\}_A \left\{ \beta' \right\}_B \left\{ 0 \right\}_B \left\{ \alpha \right\}_A \left\{ \beta \right\}_B$$

$$\times \left\{ e^{-i\phi} \alpha \right\}_A \left\{ \beta \right\}_B \left\{ 0 \right\}_B \left\{ \alpha' \right\}_A \left\{ \beta' \right\}_B.$$
Using the standard formula for the inner product of two coherent states [11, 12] in Eq. (A11) gives

\[ f_+ = e^{-\frac{1}{2} \left| \langle \beta' | \beta' \rangle - \langle \alpha' | \alpha' \rangle \right|^2} e^{i \alpha' a_+ e^{R} + i \beta' a_+ e^{R}} \tag{A12} \]

with a similar expression for \( f_- \). Inserting these values for \( f_+ \) and \( f_- \) into Eq. (A10) gives

\[ Q_{++}(\alpha', \beta') = \frac{c_n^2 e^{-i \theta}}{\pi} e^{-\frac{1}{2} \left| \langle \beta' | \beta' \rangle - \langle \alpha' | \alpha' \rangle \right|^2} e^{i R (\alpha' + a_0 \alpha') - i R (\beta' + a_0 \beta')} \tag{A13} \]

Using Eq. (A13) in Eq. (A7) and performing the integral gives the visibility as

\[ v = e^{-2 R^2 \sin^2 \phi |a_0|^2} \tag{A14} \]

which agrees with Eq. (12) of the main text.

We note that there exist simpler methods to arrive at Eq. (14), but we chose to use the Q-function in order to allow a comparison with previous results for an optical parametric amplifier [13].