THE COMPLEMENT OF $\mathcal{M}(a)$ IN $\mathcal{H}(b)$

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Abstract. Let $b$ be a nonextreme function in the unit ball of $H^\infty$ on the unit disk $\mathbb{D}$ and let $a$ be an outer $H^\infty$ function such that $|a|^2 + |b|^2 = 1$ almost everywhere on $\partial \mathbb{D}$. The sufficient and necessary conditions for the orthogonal complement of $\mathcal{M}(a)$ in $\mathcal{H}(b)$ be finite dimensional has been given by D. Sarason in [9]. Here we describe this space explicitly.

1. Introduction

Let $H^2$ denote the standard Hardy space on the unit disk $\mathbb{D}$ and let $\partial \mathbb{D}$ denote its boundary. For $\varphi \in L^\infty(\partial \mathbb{D})$ the Toeplitz operator on $H^2$ is given by $T_\varphi f = P_+ (\varphi f)$, where $P_+$ is the orthogonal projection of $L^2(\partial \mathbb{D})$ onto $H^2$. For a nonconstant function $b$ in the unit ball of $H^\infty$ the de Branges-Rovnyak space $\mathcal{H}(b)$ is the image of $H^2$ under the operator $(1 - T_b T_b^*)^{1/2}$ with the corresponding range norm. It is known [9, p.10] that $\mathcal{H}(b)$ is a Hilbert space with reproducing kernel

$$k_b^b(z) = \frac{1 - b(w)\overline{b(z)}}{1 - \overline{w}z} \quad (z, w \in \mathbb{D}).$$

Here we are interested in the case when the function $b$ is not an extreme point of the unit ball of $H^\infty$, that is the case when the function $\log(1 - |b|)$ is integrable on $\partial \mathbb{D}$ ([4, p. 138]). Then there exists an outer function $a \in H^\infty$ for which $|a|^2 + |b|^2 = 1$ a.e. on $\partial \mathbb{D}$. Moreover, if we suppose that $a(0) > 0$, then $a$ is uniquely determined, and we say that $(b, a)$ is a pair. Since the function $\frac{1 + b}{1 - b}$ has a positive real part, there exists a positive measure $\mu$ on $\partial \mathbb{D}$ such that

$$\frac{1 + b(z)}{1 - b(z)} = \int_{\partial \mathbb{D}} \frac{1 + e^{-i\theta} z}{1 - e^{-i\theta} z} d\mu(e^{i\theta}) + i \text{Im} \frac{1 + b(0)}{1 - b(0)}, \quad |z| < 1.$$ 

Moreover the function $\left| \frac{a}{1 - b} \right|^2$ is the Radon-Nikodym derivative of the absolutely continuous component of $\mu$ with respect to the normalized Lebesgue measure. So if $f = \frac{a}{1 - b}$, then $f$ is an outer function which belongs to $H^2$. If the measure $\mu$ is absolutely continuous the pair $(b, a)$ is called special. The operator $V_b$ acting on $H^2(\mu)$ (the closure of polynomials

2010 Mathematics Subject Classification. 47B32, 46E22, 30H05.

Key words and phrases. Toeplitz operators, de Branges-Rovnyak spaces, rigid functions, nonextreme functions, kernel functions.
in $L^2(\mu)$) with values in the Branges-Rovnyak space $\mathcal{H}(b)$ is given by

$$
(V_b q)(z) = (1 - b(z)) \int_{\partial \mathbb{D}} \frac{q(e^{i\theta})}{1 - e^{-i\theta} z} \, d\mu(e^{i\theta}).
$$

It is known that $V_b$ is an isometry of $H^2(\mu)$ onto $\mathcal{H}(b)$ ([9], [2]). Furthermore the Toeplitz operators with an unbounded symbols $\varphi \in L^2(\partial \mathbb{D})$ can be defined as unbounded operators on $H^2$ (with the domains containing $H^\infty$) that are continuous operators of $H^2$ into $H(\mathbb{D})$, the space of holomorphic functions on $\mathbb{D}$ with the topology of the locally uniform convergence. Moreover, if $(b, a)$ is a pair and $f = \frac{a}{\overline{b}}$, then the operator $T_{1-b}T_f$ is an isometry of $H^2$ into $\mathcal{H}(b)$. Its range is all of $\mathcal{H}(b)$ if and only if the pair $(b, a)$ is special (see [9], IV-12,13).

If $(b, a)$ is a pair then $\mathcal{M}(a)$ (the range of $T_a H^2 = a H^2$ equipped with the range norm) is contained contractively in $\mathcal{H}(b)$. Moreover, if $(b, a)$ is special, then $\mathcal{M}(a)$ is dense in $\mathcal{H}(b)$ if and only if $f^2$ is a rigid function. Recall that a function $f \in H^1$ is called rigid if no other functions in $H^1$, except for positive scalar multiples of $f$, have the same argument as $f$ a.e. on $\partial \mathbb{D}$.

Let the Toeplitz operator $T_z$, that is, the unilateral shift on $H^2$, be denoted by $S$. It is known that the de Branges-Rovnyak spaces $\mathcal{H}(b)$ are $S^*$-invariant. In the case $b$ is nonextreme the space $\mathcal{H}(b)$ is also invariant under the unilateral shift $S$. Furthermore, in this case, polynomials are dense in $\mathcal{H}(b)$ and $b \in \mathcal{H}(b)$.

Let $\mathcal{H}_0(b)$ denote the orthogonal complement of $\mathcal{M}(a)$ in $\mathcal{H}(b)$. Let $Y$ be the restriction of the shift operator $S$ to $\mathcal{H}(b)$. It is worth to mention here that since the closure of $\mathcal{M}(a)$ in $\mathcal{H}(b)$ is $Y$-invariant, the space $\mathcal{H}_0(b)$ is $Y^*$-invariant. Let $Y_0$ be the compression of $Y$ to the subspace $\mathcal{H}_0(b)$. Characterizations when $\mathcal{H}_0(b)$ has finite dimension are given in Chapter X of [9]. Now we cite some results included therein. It turns out that in this case the space $\mathcal{H}_0(b)$ depends on the spectrum of the restriction of the operator $Y^*$ to $\mathcal{H}_0(b)$ which actually equals $Y_0^*$.

The spectrum of $Y_0$ is contained in the unit circle. We know from [9] that the codimension of $\mathcal{M}(a)$ in $\mathcal{H}(b)$ is $N$ if and only if the operator $Y_0^*$ has eigenvalues $z_1, z_2, \ldots, z_s$ on the unit circle with their algebraic multiplicities $n_1, \ldots, n_s$ and $N = n_1 + n_2 + \cdots + n_s$. Then

$$
\mathcal{H}_0(b) = \bigoplus_{j=1}^s \ker(Y_0^* - z_j)^{n_j}.
$$

For $\lambda \in \partial \mathbb{D}$ let $\mu_\lambda$ be the measure on $\partial \mathbb{D}$ whose Poisson integral is the real part of $\frac{1+\overline{\lambda}b}{1-\overline{\lambda}b}$. If $F_\lambda = \frac{a}{1-\overline{\lambda}b}$, then the Radon-Nikodym derivative of the absolutely continuous component of $\mu_\lambda$ is $|F_\lambda|^2$.

In [9] the following condition for $\mathcal{M}(a)$ to have a finite defect is given.

**Theorem.** Let $N$ be a positive integer, and let $\lambda$ be a point on $\partial \mathbb{D}$ such that the measure $\mu_\lambda$ is absolutely continuous. Then the following conditions are equivalent.

(i) The codimension of $\mathcal{M}(a)$ in $\mathcal{H}(b)$ is $N$. 

(ii) \( F_\lambda = pf \), where \( p \) is a polynomial of degree \( N \) having all of its roots on the unit circle, and \( f \) is a function in \( H^2 \) whose square is rigid.

Our aim is to find explicit description of finite dimensional spaces \( \mathcal{H}_0(b) \).

For \( w \in \mathbb{D} \) and a positive integer \( n \) the function \( \frac{\partial^n k^b_w}{\partial \overline{w}^n} \) is the kernel function in \( \mathcal{H}(b) \) for the functional of evaluation of the \( n \)-th derivative at \( w \), that is, for \( f \in \mathcal{H}(b) \), we have

\[
f^{(n)}(w) = \left\langle f, \frac{\partial^n k^b_w}{\partial \overline{w}^n} \right\rangle.
\]

For \( n = 0, 1, 2, \ldots \) set

\[
v^n_{b,w}(z) = \frac{\partial^n k^b_w}{\partial \overline{w}^n}(z), \quad z, w \in \mathbb{D}.
\]

Our main result is the following.

**Theorem 1.** Assume that a point \( z_0 \in \partial \mathbb{D} \) and for \( \lambda \in \partial \mathbb{D} \setminus \{b(z_0)\} \) the measure \( \mu_\lambda \) is absolutely continuous. If the function \( F_\lambda(1 - \overline{z}_0 z)^{-k-1} \) is in \( H^2 \), then the space \( \text{ker}(Y^* - \overline{z}_0)^k \) is spanned by \( v^0_{b,z_0}, v^1_{b,z_0}, \ldots, v^k_{b,z_0} \) which are the limits of \( v^n_{b,w}, v^1_{b,w}, \ldots, v^k_{b,w} \) as \( w \) tends nontangentially to \( z_0 \).

We mention that this theorem for \( k = 1 \) has been proved in [7].

2. Preliminaries

In this section we collect auxiliary results on the space \( \mathcal{H}(b) \) generated by a nonextreme \( b \) that we will use in our proofs. Let \( X \) denote the restriction of the operator \( S^* \) to \( \mathcal{H}(b) \). Then the adjoint operator \( X^* \) is given by

\[
X^*h = Sh - \left\langle h, S^*b \right\rangle b, \quad \text{(see [2], p. 61), \ [2] Theorem 18.22}).
\]

Moreover, the following formula for the reproducing kernel \( k^b_w \) was given in [8] (see also Theorem 18.21 in [2])

\[
k^b_w = (1 - \overline{w} X^*)^{-1} k^b_0.
\]

Using this formula we derive the following.

**Proposition 1.** If \( \frac{\partial^n k^b_w}{\partial \overline{w}^m} \) are bounded in the norm as \( w \) tends nontangentially to \( z_0 \in \partial \mathbb{D} \), then also the norms of \( \frac{\partial^n k^b_w}{\partial \overline{w}^m} \), \( m = 0, 1, \ldots, n - 1 \), stay bounded as \( w \) tends nontangentially to \( z_0 \).

**Proof.** Formula (4) implies that for \( n = 1, 2, \ldots \)

\[
\frac{\partial^n k^b_w}{\partial \overline{w}^m} = n!(1 - \overline{w} X^*)^{-n-1} X^* k^b_0.
\]

Since

\[
(1 - \overline{w} X^*)^{-n} X^* k^b_0 = (1 - \overline{w} X^*)^{-n-1} X^* k^b_0,
\]
the boundedness of \((1 - \bar{w}X^*)^{-n-1}X^n k_0^b\) implies the boundedness of \((1 - \bar{w}X^*)^{-n}X^n k_0^b\).

To show that
\[
\frac{\partial^{n-1} k_0^b}{\partial \bar{w}^{n-1}} = (n-1)! (1 - \bar{w}X^*)^{-n}X^n k_0^b
\]
is bounded we observe that
\[
X^* \frac{\partial^{n-1} k_0^b}{\partial \bar{w}^{n-1}} = (n-1)! (1 - \bar{w}X^*)^{-n}X^n k_0^b.
\]
It follows from (3) that
\[
\|X^* f\|^2 = \|f\|^2 - |\langle f, S^* b \rangle|^2 \quad \text{(see Corollary 18.23 in [2])}
\]
We know that if \(b\) is nonextreme, then \(b \in \mathcal{H}(b)\). Thus
\[
\left\| \frac{1}{(n-1)!} \frac{\partial^{n-1} k_0^b}{\partial \bar{w}^{n-1}} \right\|^2 = \|(1 - \bar{w}X^*)^{-n}X^n k_0^b\|^2 + |\langle (1 - \bar{w}X^*)^{-n}X^n k_0^b, S^* b \rangle|^2
\]
\[
= \|(1 - \bar{w}X^*)^{-n}X^n k_0^b\|^2 + |\langle (1 - \bar{w}X^*)^{-n}X^n k_0^b, X^n b \rangle|^2
\]
\[
= \|(1 - \bar{w}X^*)^{-n}X^n k_0^b\|^2 + |\langle (1 - \bar{w}X^*)^{-n}X^n k_0^b, b \rangle|^2.
\]

The next proposition was actually stated in [9, pp. 58–59] without proof.

**Proposition 2.** Let \(z_0 \in \partial \mathbb{D}\) and \(b\) be a nonextreme function from the unit ball of \(H^\infty\). If every function in \(\mathcal{H}(b)\) and all of its derivatives up to order \(n\) have nontangential limits at \(z_0\), then also \(b^{(n+1)}\) has a nontangential limit at \(z_0\).

**Proof.** The case \(n = 0\) is contained in [9, VI-4] Assume that every function in \(\mathcal{H}(b)\) and all of its derivatives up to order \(n\) have nontangential limits at \(z_0\). It follows from [9, VI-4] that then the function \(h(z) = \frac{b(z) - b(z_0)}{z - z_0}\) is in \(\mathcal{H}(b)\). Thus there exists the limit
\[
\lim_{z \to z_0} \left( \frac{b(z) - b(z_0)}{z - z_0} \right)^{(n)} = \lambda.
\]
For \(C \geq 1\), let the Stolz domain \(S_C(z_0)\) be defined by
\[
S_C(z_0) = \{ z \in \mathbb{D} : |z - z_0| \leq C(1 - |z|) \},
\]
and for an \(\varepsilon > 0\), put
\[
\alpha_n(\varepsilon) = \sup \left\{ \left( \left( \frac{b(z) - b(z_0)}{z - z_0} \right)^{(n)} \right) - \lambda : z \in S_C(z_0), |z - z_0| < \varepsilon \right\},
\]
then clearly \(\alpha_n(\varepsilon)\) tends to zero with \(\varepsilon\).

Next let \(\gamma_z\) denote the circle with center \(z\) and radius \(\frac{1}{2}(1 - |z|)\). Then for \(z \in S_C(z_0)\), \(\gamma_z\) lies in \(S_C(z_0)\). The Leibniz formula
\[
\left( \frac{b(z) - b(z_0)}{z - z_0} \right)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} (b(z) - b(z_0))^{(n-k)} \frac{(-1)^k k!}{(z - z_0)^{k+1}}
\]
implies that for $\zeta \in \gamma_z$,

$$b^{(n)}(\zeta) = \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k-1}k!(b(\zeta) - b(z_0))^{(n-k)}}{(\zeta - z_0)^k} + \lambda(\zeta - z_0) + \beta(\zeta)(\zeta - z_0),$$

where $|\beta(\zeta)| \leq \alpha_n(|\zeta - z_0|) \leq \alpha_n \left( \frac{3}{2} |z - z_0| \right)$. Using the equality $k\binom{n}{k} = n\binom{n-1}{k}$, we obtain

$$b^{(n)}(\zeta) = \sum_{k=1}^{n} k\binom{n}{k} (b(\zeta) - b(z_0))^{(n-k)} \left( \frac{1}{\zeta - z_0} \right)^{(k-1)} + \lambda(\zeta - z_0) + \beta(\zeta)(\zeta - z_0)$$

$$= \sum_{k=1}^{n} n\binom{n-1}{k-1} (b(\zeta) - b(z_0))^{(n-k)} \left( \frac{1}{\zeta - z_0} \right)^{(k-1)} + \lambda(\zeta - z_0) + \beta(\zeta)(\zeta - z_0)$$

$$= n \left( \frac{b(\zeta) - b(z_0)}{\zeta - z_0} \right)^{(n-1)} + \lambda(\zeta - z_0) + \beta(\zeta)(\zeta - z_0).$$

Finally, since

$$b^{(n+1)}(z) = \frac{1}{2\pi i} \int_{\gamma_z} \frac{b^{(n)}(\zeta)}{(\zeta - z)^2} d\zeta,$$

we get

$$\lim_{z \to z_0} b^{(n+1)}(z) = (n + 1)\lambda.$$

For $\lambda \in \partial \mathbb{D}$ set $W_\lambda = T_{1-z_0} T_{\lambda}$, and recall that if $\mu_\lambda$ is absolutely continuous then $W_\lambda$ is an isometry of $H^2$ onto $\mathcal{H}(b)$. In [9] the structure of finite dimensional spaces $\mathcal{H}_0(b)$ has been studied by means of an operator $A_\lambda$ on $H^2$. Under the assumption that $\mu_\lambda$ is absolutely continuous, $A_\lambda$ intertwines $W_\lambda$ with the operator $Y^*$ i.e.,

$$W_\lambda A_\lambda = Y^* W_\lambda.$$  

The operator $A_\lambda$ is given by

$$A_\lambda = S^* - F_\lambda(0)^{-1}(S^*F_\lambda \otimes 1).$$

Moreover, it has been showed in [9] [9 X-14] that under above assumptions, if for $z_0 \in \partial \mathbb{D}$ the function $\frac{F_\lambda}{(1-z_0 z)^k} \in H^2$, then the kernel of $(A_\lambda - z_0)^k$ is spanned by $(1 - z_0 z)^{-1} F_\lambda, (1 - z_0 z)^{-2} F_\lambda, \ldots, (1 - z_0 z)^{-k} F_\lambda$. It turns out the the inverse statement is also true and we have

**Proposition 3.** Assume that $z_0 \in \partial \mathbb{D}$ and $\lambda \in \partial \mathbb{D}$ is such that $\mu_\lambda$ is absolutely continuous. Then

$$\dim \ker (A_\lambda - z_0)^k = k \iff F_\lambda (1 - z_0 z)^{-k} \in H^2.$$
Proof. In view of [9, X-14] it is enough to show
\[ \dim \ker(A_\lambda - \bar{z}_0)^k = k \implies F_\lambda(1 - \bar{z}_0 z)^{-k} \in H^2. \]

We will show that if \( \dim \ker(A_\lambda - \bar{z}_0)^k = k \), then \( \ker(A_\lambda - \bar{z}_0)^k \) is spanned by \( \frac{z F_\lambda}{(1 - z_0 \bar{z})^2}, \ldots, \frac{z^{k-1} F_\lambda}{(1 - z_0 \bar{z})^k} \). We proceed by induction. The case \( k = 1 \) is proved in [9, X-13]. Suppose that
\[ \ker(A_\lambda - \bar{z}_0)^k = \left\{ \frac{c_0 F_\lambda}{1 - z_0 \bar{z}} + \frac{c_1 z F_\lambda}{(1 - z_0 \bar{z})^2} + \cdots + \frac{c_{k-1} z^{k-1} F_\lambda}{(1 - z_0 \bar{z})^k} : c_0, c_1, \ldots, c_{k-1} \in \mathbb{C} \right\} \]
and \( (A_\lambda - \bar{z}_0) g \in \ker(A_\lambda - \bar{z}_0)^k \). Then
\[ (A_\lambda - \bar{z}_0) g = \frac{c_0 F_\lambda}{1 - z_0 \bar{z}} + \frac{c_1 z F_\lambda}{(1 - z_0 \bar{z})^2} + \cdots + \frac{c_{k-1} z^{k-1} F_\lambda}{(1 - z_0 \bar{z})^k} \]
for some \( c_0, c_1, \ldots, c_{k-1} \). Hence, by (6),
\[ \frac{g(z) - g(0)}{z} - \frac{g(0)}{F_\lambda(0)} \frac{F_\lambda(z) - F_\lambda(0)}{z} - \bar{z}_0 g = \frac{c_0 F_\lambda}{1 - z_0 \bar{z}} + \frac{c_1 z F_\lambda}{(1 - z_0 \bar{z})^2} + \cdots + \frac{c_{k-1} z^{k-1} F_\lambda}{(1 - z_0 \bar{z})^k} \]
or, equivalently,
\[ (1 - z_0 \bar{z}) g = \frac{g(0)}{F_\lambda(0)} F_\lambda + \frac{c_0 z F_\lambda}{1 - z_0 \bar{z}} + \cdots + \frac{c_{k-1} z^{k-1} F_\lambda}{(1 - z_0 \bar{z})^k}. \]

The next lemma will allow us to depict \( \ker(Y^* - \bar{z}_0)^k \) explicitly.

Lemma 1. If \( \mu_\lambda \) is absolutely continuous, then for any positive integer \( k \) and any \( z_0 \in \partial \mathbb{D} \),
\begin{equation}
\ker(Y^* - \bar{z}_0)^k = W_\lambda \ker(A_\lambda - \bar{z}_0)^k.
\end{equation}

Proof. Since \( W_\lambda \) is an isometry of \( H^2 \) onto \( \mathcal{H}(b) \), we have \( W_\lambda^* W_\lambda = \text{id} \) on \( H^2 \). Indeed, for any \( f \in H^2 \),
\[ \langle f, f \rangle_{H^2} = \langle W_\lambda f, W_\lambda f \rangle_{\mathcal{H}(b)} = \langle W_\lambda^* W_\lambda f, f \rangle_{H^2}. \]

Next, if \( g \in \mathcal{H}(b) \) is the image of \( f \in H^2 \) under \( W_\lambda \), then \( W_\lambda^* W_\lambda f = f \) implies \( W_\lambda W_\lambda^* g = g \), i.e. \( W_\lambda W_\lambda^* = \text{id} \) on \( \mathcal{H}(b) \).

Note that (5) implies
\[ W_\lambda (A_\lambda - \bar{z}_0) = (Y^* - \bar{z}_0) W_\lambda. \]
Since \( W_\lambda W_\lambda^* = \text{id} \) on \( \mathcal{H}(b) \) we get
\[ (Y^* - \bar{z}_0) = W_\lambda (A_\lambda - \bar{z}_0) W_\lambda^* \]
and, by iteration,
\begin{equation}
(Y^* - \bar{z}_0)^k = W_\lambda (A_\lambda - \bar{z}_0)^k W_\lambda^*.
\end{equation}

Assume that \( g \in \mathcal{H}(b) \) and \( g = W_\lambda f \) \( (f \in H^2) \) and observe that by (5), \( g = W_\lambda f \in \ker(Y^* - \bar{z}_0)^k \) if and only if
\[ 0 = (Y^* - \bar{z}_0)^k g = W_\lambda (A_\lambda - \bar{z}_0)^k W_\lambda^* W_\lambda f = W_\lambda (A_\lambda - \bar{z}_0)^k f. \]
which means that $f \in \ker(A_\lambda - \bar{z}_0)^k$. □

3. Proof of Theorem 1

In the proof we will use the following result stated in Chapter VII in [9].

Sarason’s Theorem. Assume that a point $z_0 \in \partial \mathbb{D}$ and for $\lambda \in \partial \mathbb{D} \setminus \{b(z_0)\}$ the measure $\mu_\lambda$ is absolutely continuous. Then the following conditions are equivalent.

(i) Each function in $H(b)$ and all of its derivatives up to order $k$ have nontangential limits at $z_0$.
(ii) The function $F_\lambda(1 - \bar{z}_0z)^{-k-1} \in H^2$.
(iii) The functions $\frac{\partial^m k^b_w}{\partial \bar{w}^m}$ are bounded in the norm as $w$ tends nontangentially to $z_0$.

The proof of this theorem for the case when $k = 0$ is given in [9, VI-4]. Since the proof of the general case is only sketched in the cited reference, we include it here for the reader’s convenience.

Proof of Sarason’s Theorem. (i) $\implies$ (iii). Since for $f \in H(b)$

$$f^{(k)}(w) = \left< f, \frac{\partial^k k^b_w}{\partial \bar{w}^k} \right>, \quad w \in \mathbb{D}$$

and the nontangential limit of $f^{(k)}(w)$ at $z_0$ exists, $\sup \{|f^{(k)}(w)|: w \in SC(z_0)\}$ is finite. Let $\varphi_w(f) = f^{(k)}(w)$ be a bounded linear functional on $H(b)$. The Banach-Steinhaus theorem implies that there exists a constant $M > 0$ such that

$$\|\varphi_w\| = \left\| \frac{\partial^k k^b_w}{\partial \bar{w}^k} \right\| \leq M$$

for every $w \in SC(z_0)$.

(iii) $\implies$ (i). We proceed by induction. Assume the implication holds true for $k = 0, 1, \ldots, m - 1$ and suppose that the functions $\frac{\partial^m k^b_w}{\partial \bar{w}^m}$ are bounded in the norm as $w$ tends nontangentially to $z_0$. Then there exists a sequence $\{w_n\} \subset \mathbb{D}$ that converges nontangentially to $z_0$ for which $\left\{ \frac{\partial^m k^b_w}{\partial \bar{w}^m} \right\}$ converges weakly to $h \in H(b)$.

Thus we have

$$h(z) = \langle h, k^b_z \rangle = \lim_{n \to \infty} \left< \frac{\partial^m k^b_{w_n}}{\partial \bar{w}^m}, k^b_z \right> = \lim_{n \to \infty} \frac{\partial^m k^b_{w_n}}{\partial \bar{w}^m}(z).$$

Since

$$\frac{\partial^m k^b_w}{\partial \bar{w}^m}(z) = \sum_{j=0}^{m} \binom{m}{j} \frac{\partial^{m-j}(1 - b(w)b(z))}{\partial \bar{w}^{m-j}} \frac{j!z^j}{(1 - w\bar{z})^{j+1}},$$

(9)
we have

\[(10) \quad h(z) = \lim_{n \to \infty} \frac{\partial^{m} k_{w}^{b}}{\partial w_{n}^{m}}(z) = \lim_{n \to \infty} \frac{(1 - b(w_{n})b(z))m!z^{m}}{(1 - w_{n}z)^{m+1}} + \sum_{j=0}^{m-1} \left( m \right) \lim_{n \to \infty} \frac{-b^{(m-j)}(w_{n})b(z)j!z^{j}}{(1 - w_{n}z)^{j+1}}.\]

Furthermore Proposition 1 implies that the norms \(\|\frac{\partial^{k} k_{w}^{b}}{\partial w^{k}}\|, k = 0, 1, \ldots, m - 1\), stay also bounded as \(w\) tends nontangentially to \(z_{0}\). Hence by induction hypothesis each function in \(\mathcal{H}(b)\) and all of its derivatives up to order \(m - 1\) have nontangential limits at \(z_{0}\). Additionally, by Proposition 2, the derivative of order \(m\) of \(b\) has its nontangential limit at \(z_{0}\). It then follows from (10) that \(\frac{\partial^{m} k_{w}^{b}}{\partial z^{m}}\) converges to \(h\) pointwise as \(w\) tends to \(z_{0}\) nontangentially. Put \(h = \frac{\partial^{m} k_{w}^{b}}{\partial z_{0}^{m}}\) and note that for \(z \in \mathbb{D}\),

\[\lim_{w \to z_{0}} (k_{z}^{b})^{(m)}(w) = \frac{\partial^{m} k_{w}^{b}}{\partial z_{0}^{m}}(z).\]

Indeed,

\[\lim_{w \to z_{0}} (k_{z}^{b})^{(m)}(w) = \lim_{w \to z_{0}} \left( k_{z}^{b}, \frac{\partial^{m} k_{w}^{b}}{\partial w^{m}} \right) = \lim_{w \to z_{0}} \frac{\partial^{m} k_{w}^{b}}{\partial w^{m}}(z) = \frac{\partial^{m} k_{w}^{b}}{\partial z_{0}^{m}}(z) = (k_{z}^{b})^{(m)}(z).\]

This means that for every \(z \in \mathbb{D}\), the derivative \((k_{z}^{b})^{(m)}\) has a nontangential limit at \(z_{0}\). Since the functions \(k_{z}^{b}\) span the space \(\mathcal{H}(b)\) and the norms of \(\frac{\partial^{m} k_{w}^{b}}{\partial w^{m}}\) are bounded as \(w\) tends nontangentially to \(z_{0}\), the desired conclusion follows.

(ii) \(\Rightarrow\) (iii). Assume that the implication holds for \(k = 0, 1, \ldots, m - 1\) and \(F_{\lambda}(1 - \bar{z}_{0}z)^{-m-1} \in H^{2}\). Observe first that if \(F_{\lambda}(1 - \bar{z}_{0}z)^{-m-1} \in H^{2}\), then also \(F_{\lambda}(1 - \bar{z}_{0}z)^{-k-1} \in H^{2}\) for \(k = 0, 1, \ldots, m\).

Let \(V_{\lambda b}\) be defined by (2) with \(b\) replaced by \(\bar{\lambda} b\). Since the pair \((a, \bar{\lambda} b)\) is special, we have

\[V_{\lambda b}(1 - \lambda \bar{b}(w))k_{w} = W_{\lambda}(F_{\lambda}(1 - \lambda \bar{b}(w))k_{w}) = k_{w}, \quad w \in \mathbb{D},\]

(see [3] p.18, [2] Vol.2, p.141)). Since

\[(11) \quad W_{\lambda} \left( \frac{\partial^{m} ((1 - \lambda \bar{b}(w))k_{w})}{\partial \bar{w}^{m}} \right) = \frac{\partial^{m} k_{w}^{b}}{\partial \bar{w}^{m}}.\]

and

\[(12) \quad \frac{\partial^{m} ((1 - \lambda \bar{b}(w))k_{w})}{\partial \bar{w}^{m}} = \sum_{j=0}^{m} \left( m \right) \frac{\partial^{m-j} (1 - \lambda \bar{b}(w))}{\partial \bar{w}^{m-j}} \frac{j!z^{j}}{(1 - w_{j}z)^{j+1}}\]

we see that the preimage of \(\frac{\partial^{m} k_{w}^{b}}{\partial \bar{w}^{m}}\) is a linear combination of the functions \(\frac{F_{\lambda}}{(1 - \bar{w}z)^{j+1}}, j = 0, 1, 2, \ldots, m\), whose coefficients depend on \(\bar{b}(w), \bar{b}'(w), \ldots, \bar{b}^{(m)}(w)\). By the induction hypothesis \(\frac{\partial^{m-j} k_{w}^{b}}{\partial \bar{w}^{m-j}}\) are bounded as \(w\) tends nontangentially to \(z_{0}\). By what we have
already proved, this implies the existence of the limits of \( \frac{\partial^{n}b(w,m)}{\partial w^{m}} \) as \( w \to z_{0} \) nontangentially. Now (11), (12) and the fact that \( W_{\lambda} \) is an isometry imply that the norms of \( \frac{\partial^{n}b(w,m)}{\partial w^{m}} \) stay bounded as \( w \) converges nontangentially to \( z_{0} \).

\((iii) \implies (ii)\). Assume that the implication holds true for \( k = 0, 1, \ldots, m - 1 \) and \( \frac{\partial^{n}b(w,m)}{\partial w^{m}} \) are bounded in the norm as \( w \) tends nontangentially to \( z_{0} \). Since (iii) is equivalent to (i), the nontangential limits of \( \frac{\partial^{n}b(w,m)}{\partial w^{m}} \) as \( w \to z_{0} \) exist. By the induction hypothesis the functions \( \frac{F_{k}}{(1-z_{0}z)^{k+1}}, k = 0, 1, \ldots, m - 1 \) are in \( H^{2} \). Finally, passage to the limit in (11) and (12) as \( w \to z_{0} \) nontangentially and the induction hypothesis show that \( \frac{(1-\lambda b(w,m))F_{k}}{(1-z_{0}z)^{k+1}} \) is in \( H^{2} \). Since \( \lambda \neq b(z_{0}) \) our claim follows.

\(\square\)

**Proof of Theorem 1.** We know from [9, X-14] that for \( k = 0, 1, 2 \ldots \ker(A_{\lambda} - z_{0})^{k+1} \) has dimension \( k + 1 \), and is spanned by the functions \( F_{k}(1-z_{0}z)^{-1}, F_{k}(1-z_{0}z)^{-2}, \ldots, F_{k}(1-z_{0}z)^{-k-1} \). By Lemma 1 the kernel of \( (Y^{*} - z_{0})^{k} \) is spanned by the images of these functions under \( W_{\lambda} \). So, it is enough to show that for \( k = 0, 1, \ldots, W_{\lambda} \left( \frac{F_{k}}{(1-z_{0}z)^{k+1}} \right) \) is a linear combination of \( v_{b,z_{0}}^{0}, v_{b,z_{0}}^{1}, \ldots, v_{b,z_{0}}^{k} \) defined in Introduction. We proceed by induction. The case when \( k = 0 \) has been proved in [7]. Assume that the statement is true for \( k = 0, 1, \ldots, m - 1 \) and \( F_{k}(1-z_{0}z)^{-m-1} \) is in \( H^{2} \). Let \( \{w_{n}\} \subset \mathbb{D} \) be a sequence converging nontangentially to \( z_{0} \). Then

\[
W_{\lambda} \left( \frac{\partial^{m}(1-\lambda b(w_{n}))}{\partial w^{m}} F_{m} \right) = W_{\lambda} \left( \frac{(1-\lambda \overline{b(w_{n})})m!z^{m}F_{m}}{(1-w_{n}z)^{m+1}} \right) - \sum_{j=1}^{m} \binom{m}{j} W_{\lambda} \left( \frac{\lambda b^{(j)}(w_{n})(m-j)!z^{m-j}F_{m}}{(1-w_{n}z)^{m-j+1}} \right) = v_{b,w_{n}}^{m}.
\]

Our assumption implies that \( \lim_{n \to \infty} b^{(j)}(w_{n}) = b^{(j)}(z_{0}), j = 0, \ldots, m \) and the dominated convergence theorem implies that the norm of the each function the operator \( W_{\lambda} \) is acting on converges, as \( n \to \infty \), to a norm of a function in \( H^{2} \) (which is of the form \( \frac{c_{k}F_{k}}{(1-z_{0}z)^{k+1}}, k = 0, 1, \ldots, m \) and \( c_{m} \neq 0 \)). So the passage to the limit yields

\[
W_{\lambda} \left( \frac{(1-\lambda b(z_{0}))m!z^{m}F_{m}}{(1-z_{0}z)^{m+1}} \right) - \sum_{j=1}^{m} \binom{m}{j} W_{\lambda} \left( \frac{\lambda b^{(j)}(z_{0})(m-j)!z^{m-j}F_{m}}{(1-z_{0}z)^{m-j+1}} \right) = v_{b,z_{0}}^{m}.
\]

Since by the induction hypothesis the second term on the left-hand side of the last equality is a linear combination of \( v_{b,z_{0}}^{0}, v_{b,z_{0}}^{1}, \ldots, v_{b,z_{0}}^{m-1} \) our proof is finished.

\(\square\)

**Remarks.** The complement of \( M(a) \) in \( H(b) \) for the case when pairs \( (b,a) \) are rational has been studied for example in [11, 3, 10], and [5]. Also in [6] this space has been described for concrete nonextreme functions \( b \) that are not rational. Analogous result to that stated in Theorem 1 has been obtained in [3] for rational pairs (or their positive powers) where the corresponding point \( z_{0} \in \partial \mathbb{D} \) is a zero of order \( m - 1 \) of the rational
function $a$. One can easily check that in such a case there exists a $\lambda \in \partial \mathbb{D}$ for which $F_{\lambda}(1 - \bar{z}_0 z)^m \in H^2$ and the hypotheses of Theorem 1 are satisfied.

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