Spectral analysis of transfer operators associated to Farey fractions

Claudio Bonanno * Sandro Graffi † Stefano Isola ‡

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Abstract

The spectrum of a one-parameter family of signed transfer operators associated to the Farey map is studied in detail. We show that when acting on a suitable Hilbert space of analytic functions they are self-adjoint and exhibit absolutely continuous spectrum and no non-zero point spectrum. Polynomial eigenfunctions when the parameter is a negative half-integer are also discussed.

Keywords: Transfer operators, Farey fractions, spectral theory, period functions, self-reciprocal functions

Riassunto: Analisi spettrale di operatori di trasferimento associati alle frazioni di Farey.

Presentiamo uno studio dettagliato dello spettro di una famiglia ad un parametro di operatori di trasferimento segnati associati alla trasformazione di Farey dell’intervallo unitario in sé. Se fatti agire su un opportuno spazio di Hilbert di funzioni analitiche essi risultano autoaggiunti e con spettro assolutamente continuo (ad eccezione dell’autovalore nullo). Diamo altresì una a classificazione completa delle autofunzioni polinomiali quando il parametro è un semintero negativo.

Scientific Chapter: Mathematical Physics

* Dipartimento di Matematica Applicata, Università di Pisa, via F. Buonarroti 1/c, I-56127 Pisa, Italy, email: <bonanno@mail.dm.unipi.it>
† Dipartimento di Matematica, Università di Bologna, Piazza di Porta S. Donato 5, I-40127 Bologna, Italy, e-mail: <graffi@dm.unibo.it>
‡ Dipartimento di Matematica e Informatica, Università di Camerino, via Madonna delle Carceri, I-62032 Camerino, Italy. e-mail: <stefano.isola@unicam.it>
1 Preliminaires and statement of the main results

Let $F : [0, 1] \to [0, 1]$ be the Farey map defined by

$$F(x) = \begin{cases} \frac{x}{1-x} & \text{if } 0 \leq x \leq \frac{1}{2} \\ \frac{1-x}{x} & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases} \quad (1.1)$$

Its name can be related to the following observation. If we expand $x \in [0, 1]$ in continued fraction, i.e.

$$x = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}} \equiv [a_1, a_2, a_3, \ldots]$$

then

$$x = [a_1, a_2, a_3, \ldots] \mapsto F(x) = [a_1 - 1, a_2, a_3, \ldots] \quad (1.2)$$

with $[0, a_2, a_3, \ldots] \equiv [a_2, a_3, \ldots]$. Differently said, let $\mathcal{F}_n$ be the ascending sequence of irreducible fractions between 0 and 1 constructed inductively in the following way: set first $\mathcal{F}_1 = (0, \frac{1}{1})$, then $\mathcal{F}_n$ is obtained from $\mathcal{F}_{n-1}$ by inserting among each pair of neighbours $\frac{a}{b}$ and $\frac{a'}{b'}$ in $\mathcal{F}_{n-1}$ their Farey sum $\frac{a+a'}{b+b'}$. Thus

$$\mathcal{F}_2 = (0, \frac{1}{1}, \frac{1}{1}) \quad \mathcal{F}_3 = (0, \frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{3}) \quad \mathcal{F}_4 = (0, \frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{3}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4})$$

and so on. The elements of $\mathcal{F}_n$ are called Farey fractions. It is easy to verify that the set of pre-images $\bigcup_{k=0}^{n} F^{-k} \{0\}$ coincides with $\mathcal{F}_n$ for all $n \geq 1$. This implies that $\bigcup_{k=0}^{\infty} F^{-k} \{0\} = \mathbb{Q} \cap [0, 1]$. These two observations are related by the fact that a rational number $\frac{a}{b}$ belongs to $\mathcal{F}_n \setminus \mathcal{F}_{n-1}$ if and only if its continued fraction expansion $\frac{a}{b} = [a_1, a_2, \ldots, a_k]$ with $a_k > 1$ is such that $\sum_{i=1}^{k} a_i = n$.

In this paper we shall study a family of signed generalized transfer operators $\mathcal{P}_q^\pm$ associated to the map $F$, whose action on a function $f(x) : [0, 1] \to \mathbb{C}$ is given by a weighted sum over the values of $f$ on the set $F^{-1}(x)$, namely

$$f(x) \mapsto (\mathcal{P}_q^\pm f)(x) = \left( \frac{1}{x+1} \right)^{2q} \left[ f \left( \frac{x}{x+1} \right) \pm f \left( \frac{1}{x+1} \right) \right] \quad (1.3)$$

where $q$ is a real or complex parameter. The operator $\mathcal{P}_1^+$ is referred to as the Perron-Frobenius operator for the map $F$: its fixed function is the density of an absolutely continuous $F$-invariant measure. In this case one
easily checks that the function $1/x$ has this property. However, since $1/x$ does not belong to $L^1([0,1], dx)$ the statistical properties of the map $F$ have to be described in the framework of infinite ergodic theory [An]. We refer to [Bal] for a general review of transfer operator techniques in dynamical systems theory. Here, one motivation to study signed transfer operators arises from their appearing in dynamical zeta functions such as Selberg and Ruelle’s (see [DEIK], Corollary 3.13, and also [BI]).

Using the Farey fractions, the iterates $P_q^{\pm n}f$ of the above operators can be expressed as suitable sums over the Stern-Brocot tree, the binary tree with root node 1 and whose $n$-th level $L_n$ is given by $L_n = (F_n \setminus F_{n-1}) \cup S(F_n \setminus F_{n-1})$, where $S$ is the map $S: x \rightarrow 1/x$ and such that the elements of $S(F_n \setminus F_{n-1})$ are in reverse order. An important feature of this tree is that each positive rational number appears as a vertex exactly once. The left part of the Stern-Brocot tree (starting from the node $\frac{1}{2}$) is called the Farey tree, with vertex-set $\mathbb{Q} \cap (0,1)$.

An easy generalisation of Proposition 5.9 in [DEIK] yields for all $x \in \mathbb{R}_+$ and $q \in \mathbb{C}$,

$$(P_q^{\pm n}f)(x) = \sum_{\frac{a}{b} \in L_n} \frac{f\left(\frac{n_0(x,a/b)}{ax+b}\right) \pm f\left(\frac{n_1(x,a/b)}{ax+b}\right)}{(ax+b)^{2q}}$$

(1.4)
where $n_0(x, a/b) = \mu x + \nu$ and $n_1(x, a/b) = (a - \mu)x + b - \nu$, for some $0 \leq \mu \leq a$ and $0 \leq \nu \leq b$. In particular $n_0(x, a/b) + n_1(x, a/b) = ax + b$.

In Section 2 we prove

**Theorem 1.1.** For each $q \in (0, \infty)$ there is a Hilbert space of analytic functions $H_q$ on which the operators $\mathcal{P}_q^\pm$ are bounded, self-adjoint and isospectral. Their common spectrum is given by $\{0\} \cup (0,1]$, with $(0,1]$ purely absolutely continuous.

**Remark 1.2.** From thermodynamic formalism it follows that $\mathcal{P}_q^+$ for $q \in (-\infty,1)$, when acting on a suitable Banach space has a leading eigenvalue $\lambda(q) \geq 1$ which is a differentiable and monotonically decreasing function with $\lim_{q \to 1^-} \lambda(q) = 1$ (and $\lambda(q) = 1$ for all $q \geq 1$, see [PS]). From the above theorem we see that corresponding eigenfunction does not belong to the space $H_q$ (for $q = 1$ it is just the invariant density $1/x$). Moreover

$$\lambda(q) = \lim_{n \to \infty} \frac{1}{n} \log(\mathcal{P}_q^{+n}1)(0).$$

Note that by (1.4) we can write

$$(\mathcal{P}_q^{+n}1)(0) = 2 \sum_{\# \in \mathcal{F}_n \setminus \{1\}} b^{-2q}$$

and the above sum is equal to the partition function $Z_{n-1}(2q)$ at (inverse) temperature $2q$ of the number-theoretical spin chain introduced by Andreas Knauf in [Kn].

**Remark 1.3.** One easily checks that the function $f(x) = (1-x)/x$ is an eigenfunction of $\mathcal{P}_q^-$ for $q = 1/2$ and eigenvalue 1. But, again, this function does not belong to $H_q^\pm$.

There are interesting functional symmetries related to the eigenvalue equation for $\mathcal{P}_q^\pm$, which can be rephrased in terms of Hankel transforms. The construction of Section 2 allows for a complete account of the corresponding self-reciprocal functions in $L^2(\mathbb{R}_+)$, discussed in Section 3. Finally, in Section 4 we characterise all polynomial eigenvectors of $\mathcal{P}_q^\pm$ when $q = -k/2$, $k \geq 0$.

### 2 The spectrum of $\mathcal{P}_q^\pm$ for real positive $q$

In this section we give the proof of Theorem 1.1, hence in the sequel we restrict ourselves to the case $q \in (0, \infty)$. The proof of the theorem follows from the results of the following subsections.
2.1 An invariant Hilbert space

In this subsection we introduce a family of Hilbert spaces $\mathcal{H}_q$, where $q \in (0, \infty)$, and give the representation of the operators $\mathcal{P}_q^\pm$ on $\mathcal{H}_q$.

**Definition 2.1.** For $q \in (0, \infty)$ we denote by $\mathcal{H}_q$ the Hilbert space of all complex-valued functions $f$ which can be represented as a generalised Borel transform

$$f(x) = \mathcal{B}_q[\varphi](x) := \frac{1}{x^{2q}} \int_0^\infty e^{-\frac{t}{x}} e^t \varphi(t) \, m_q(dt) \quad \varphi \in L^2(m_q) \quad (2.1)$$

with inner product

$$(f_1, f_2) = \int_0^\infty \varphi_1(t) \overline{\varphi_2(t)} \, m_q(dt) \quad \text{if} \quad f_i = \mathcal{B}_q[\varphi_i] \quad (2.2)$$

and measure $(p = 2q - 1)$

$$m_q(dt) = t^p e^{-t} \, dt \quad (2.3)$$

Function spaces related to that introduced above have been used in [Is], [GI] and [Pre]. In [Is] an explicit connection between the approach presented here and Mayer’s work on the transfer operator for the Gauss map [Ma] is established by means of a suitable operator-valued power series.

**Remark 2.2.** For $q \in \mathbb{C}$, Re $q > 0$, the space $\mathcal{H}_q$ can be regarded as a complex Hilbert space. Setting

$$\chi_p(x) := x^p \quad (p = 2q - 1) \quad (2.4)$$

an alternative representation for $f \in \mathcal{H}_q$ can be obtained by a simple change of variable when $x$ is real and positive:

$$\chi_p \cdot f(x) = \int_0^\infty e^{-s} (\chi_p \cdot \varphi)(sx) \, ds \quad (2.5)$$

Note that a function $f \in \mathcal{H}_q$ is analytic in the disk

$$D_1 = \{ x \in \mathbb{C} : \text{Re} \frac{1}{x} > \frac{1}{2} \} = \{ x \in \mathbb{C} : |x - 1| < 1 \} \quad (2.6)$$

In particular,

$$\chi_p \cdot \varphi(t) = \sum_{n=0}^\infty \frac{a_n}{n!} t^n \quad \Rightarrow \quad (\chi_p \cdot f)(x) = \sum_{n=0}^\infty a_n x^n \quad (2.7)$$

in the sense of formal power series. So the power series of $\chi_p \cdot \varphi$ is obtained Borel transforming that of $\chi_p \cdot f$, in the usual sense. This justify the name of the integral transform $\mathcal{B}_q$.
Remark 2.3. The invariant density $1/x$ for the Farey map, that is the fixed function of $P_1^+$, is the generalised Borel transform (for $q = 1$) of the function $\varphi(t) = 1/t$ which, however, does not belong to $L^2(m_1)$.

Let us now study the Hilbert space $L^2(m_q)$. First of all we notice that the measure $m_q(dt)$ is finite, indeed
\[
\int_0^\infty m_q(dt) = \Gamma(2q) \tag{2.8}
\]
Second, for the linearly independent family of functions $f_n(t) := \frac{t^n}{m}$ ($n \geq 0$) we have
\[
(f_n, f_m) = \frac{\Gamma(n + m + 2q)}{n!m!} \tag{2.9}
\]
This implies that the (generalised) Laguerre polynomials $L^p_n(t)$ ($n \geq 0$, $\Re p > -1$) by
\[
e_n(t) := L^p_n(t) = \sum_{m=0}^n \binom{n + p}{n - m} \frac{(-t)^m}{m!} \tag{2.10}
\]
form a complete orthogonal basis in $L^2(m_q)$, with
\[
(e_n, e_m) = \frac{\Gamma(n + 2q)}{n!} \cdot \delta_{n,m} \tag{2.11}
\]
Moreover, using ([GR], p.850) and (2.11) we get for $m \leq n$
\[
(f_n, e_m) = (-1)^m \frac{\Gamma(n + 2q)}{m!(n-m)!} = (-1)^m \binom{n}{m} \|e_n\|^2
\]
\[
= (-1)^m \frac{\Gamma(n + 2q)}{\Gamma(m + 2q)(n-m)!} \|e_m\|^2
\]
\[
= (-1)^m \binom{n + p}{n - m} \|e_m\|^2 \tag{2.12}
\]
In particular $(f_n, e_n) = (-1)^n\|e_n\|^2$. Also note that $(f_n, e_m) = 0$ for $m > n$.
Comparing to (2.10) we obtain the following result

Lemma 2.4. For each $n \in \mathbb{N}_0$ the numbers
\[
a_{n,m} := \begin{cases} (-1)^m \binom{n + p}{n - m} & \text{if } m \leq n \\ 0 & \text{if } m > n \end{cases}
\]
are the Fourier coefficients of \( f_n \) w.r.t the basis \( (e_m) \), i.e.
\[
    a_{n,m} = \frac{(f_n, e_m)}{\|e_m\|^2}
\]

Moreover
\[
f_n = \sum_{m=0}^{n} a_{n,m} e_m \quad e_n = \sum_{m=0}^{n} a_{n,m} f_m
\]

**Remark 2.5.** In particular, the \((n+1) \times (n+1)\) lower triangular matrix 
\( A_n := (a_{i,j})_{0 \leq i,j \leq n} \) satisfies 
\( A_n^2 = I_{n+1} \). Therefore, the operator \( \Pi_n : L^2(m_q) \to L^2(m_q) \) acting as
\[
    \Pi_n : \sum_{s=0}^{\infty} c_s e_s \mapsto \sum_{s=0}^{\infty} c_s \sum_{r=0}^{n} \frac{(f_r, e_s)}{\|e_s\|^2} f_r = \sum_{r=0}^{n} d_r f_r
\]
with
\[
    d_r := \sum_{s=0}^{r} a_{r,s} c_s \quad \text{or} \quad d^{(n)} = A_n c^{(n)}
\]
where we have set \( c^{(n)} = (c_0, c_1, \ldots, c_n)^T \) and similarly for \( d^{(n)} \), is the orthogonal projection onto the linear subspace spanned by \( (1, t, t^2, \ldots, t^n) \).

Let us now consider the action of the transform \( B_q \) on the functions \( (e_n) \) and \( (f_m) \). We have
\[
    B_q[e_n](x) = \sum_{m=0}^{n} \Gamma(2q + m) \left( \begin{array}{c} n + p \\ n - m \end{array} \right) \frac{(-x)^m}{m!}
\]
\[
    = (n + 1)_p (1 - x)^n
\]
where \( (a)_p := \Gamma(a + p)/\Gamma(a) = a(a + 1) \cdots (a + p - 1) \) is the shifted factorial, and
\[
    B_q[f_m](x) = (n + 1)_p x^n.
\]

The next result describes the action of \( \mathcal{P}_q^\pm \) on the Hilbert space \( \mathcal{H}_q \).

**Proposition 2.6.** For \( q \in (0, \infty) \) the space \( \mathcal{H}_q \) is invariant for \( \mathcal{P}_q^\pm \) and \( \mathcal{P}_q^\pm : \mathcal{H}_q \to \mathcal{H}_q \) are positive operators, isomorphic to self-adjoint compact perturbations of the multiplication operator \( M : L^2(m_q) \to L^2(m_q) \) given by
\[
    (M \varphi)(t) = e^{-t} \varphi(t)
\]
More specifically

\[ \mathcal{P}_q^\pm \mathcal{B}_q [\varphi] = \mathcal{B}_q [P^\pm \varphi] \]

where \( P^\pm = M \pm N \) and \( N : L^2(m_q) \to L^2(m_q) \) is the symmetric integral operator given by

\[ (N\varphi)(t) = \int_0^\infty \frac{J_p(2\sqrt{st})}{(st)^{p/2}} \varphi(s) m_q(ds) \]

where \( J_p \) denotes the Bessel function of order \( p \).

**Proof.** The representation of \( \mathcal{P}_q^\pm \) on \( \mathcal{H}_q \) follows from a direct computation (see [IS, GI]). The positivity amounts to

\[ ((M \pm N)\varphi, \varphi) \geq 0 \quad \forall \varphi \in L^2(m_q), \quad \|\varphi\| = 1 \quad (2.15) \]

and can be checked expanding \( \varphi \) on the basis of (normalised) Laguerre polynomials. Indeed, a calculation using ([GR], pp.849-850) yields

\[ \frac{(Me_n, e_n)}{\|e_n\|^2} = 2^{-2n-2q} \binom{2n+p}{n} \]

and

\[ \frac{(Ne_n, e_n)}{\|e_n\|^2} = 2^{-n-2q} \binom{n+p}{n} \quad 2F_1(-n, n+2q; 2q; 1/2) \]

\[ = 2^{-n-2q} P_n^{(p,0)}(0) \]

where \( P_n^{(a,b)}(x) \) denotes the Jacobi polynomial ([AAR], p.99). Since

\[ P_n^{(p,0)}(0) = (-2)^{-n} \sum_{k=0}^{n} (-1)^k \binom{n+p}{k} \binom{n}{k} \]

and

\[ \binom{2n+p}{n} = \sum_{k=0}^{n} \binom{n+p}{k} \binom{n}{k} \]

we get

\[ \frac{((M \pm N)e_n, e_n)}{\|e_n\|^2} = \frac{1}{2^{2n+2q}} \sum_{k=0}^{n} (1 \pm (-1)^{n-k}) \binom{n+p}{k} \binom{n}{k} \]
and thus (2.15). Finally, $N\varphi$ can be written as $\int_0^\infty k(s,t)\varphi(s)m_q(ds)$ with symmetric kernel

$$k(s,t) = \frac{J_p(2\sqrt{st})}{(st)^{p/2}}$$ (2.16)

From the estimates $J_p(t) \sim 2^{-p} t^p/\Gamma(p+1)$ as $t \to 0^+$ and $J_p(t) = O(t^{-1/2})$ as $t \to \infty$ ([E], vol. II), we see that the kernel $k(s,t)$ is bounded and continuous.

We can now describe the action of $P^\pm$ on $(e_n)$ and $(f_n)$. Applying the integral representation (see [E], Vol. II, p.190)

$$n! e^{-t} L_n^p(t) = \int_0^\infty J_p(2\sqrt{st}) s^p m_q(ds)$$

we get

$$M^{-1}Nf_n = e_n \quad M^{-1}Ne_n = f_n$$ (2.17)

### 2.2 Functional symmetries

Let introduce an isometry which turns out to be useful for the characterisation of eigenfunctions of the operators $P_q^\pm$. Let $J_q$ be the involution defined by

$$(J_qf)(x) := \frac{1}{x^{2q}} f\left(\frac{1}{x}\right)$$ (2.18)

and consider its action on the Hilbert space $\mathcal{H}_q$. We have the following

**Proposition 2.7.** For any $\varphi \in L^2(m_q)$ it holds

$$J_q B_q [\varphi] = B_q [J \varphi]$$ (2.19)

where $J := NM^{-1}$ is a bounded operator in $L^2(m_q)$ with $\|J\| \leq 2\pi$. If moreover $P_q^\pm f = \lambda f$ for some $\lambda \neq 0$ then $f$ satisfies the functional equation

$$J_q f = \pm f$$ (2.20)

**Proof.** The representation of $J_q$ in $\mathcal{H}_q$ is easily checked by first noting that for any $f \in \mathcal{H}_q$ the function $J_q f$ can be written as an ordinary Laplace transform, i.e.

$$f(x) = B_q[\varphi](x) \quad \implies \quad (J_q f)(x) = \int_0^\infty e^{-tx}(\chi_p \cdot \varphi)(t)dt$$ (2.21)
and then using Tricomi’s theorem ([Sue], p.165). Let us prove the bound on $\|J\|$. Adapting formula (33) of [RS], vol. IV, to our situation we get for all $\varphi \in L^2(m_q)$ and $\lambda \in [0,1]$,

$$\|N(M - \lambda)^{-1}\varphi\|^2 \leq \int_0^1 \|N(M - \lambda)^{-1}\varphi\|^2 d\lambda \leq 2\pi \int_{-\infty}^{\infty} \|Ne^{i\tau M}\varphi\|^2 d\tau$$

(2.22)

On the other hand we claim that

$$\int_{-\infty}^{\infty} \|Ne^{i\tau M}\varphi\|^2 d\tau \leq 2\pi \int_0^{\infty} e^{-t} \left( \int_0^{\infty} |J_p(2\sqrt{st})|^2 |\varphi(s)|^2 s^p e^{-s} ds \right) dt$$

(2.23)

To prove (2.23) we write

$$(Ne^{i\tau M}\varphi)(t) = \int_0^{\infty} \frac{J_p(2\sqrt{st})}{(st)^{p/2}} e^{i\tau e^{-s}} \varphi(s) s^p e^{-s} ds$$

so that interchanging the order of integration

$$\|Ne^{i\tau M}\varphi\|^2 = \int_0^{\infty} |G(t,\tau)|^2 e^{-t} dt$$

where we have set

$$G(t,\tau) = \int_0^{\infty} J_p(2\sqrt{st}) e^{i\tau e^{-s}} \varphi(s) s^p e^{-s} ds$$

$$= - \int_0^1 J_p(2\sqrt{-t \ln u}) e^{i\tau u} (-\ln u)^{p/2} \varphi(-\ln u) du$$

Equation (2.23) now follows by applying Fourier-Plancherel theorem:

$$\int_{-\infty}^{\infty} |G(t,\tau)|^2 d\tau = 2\pi \int_0^1 |J_p(2\sqrt{-t \ln u}) \varphi(-\ln u)|^2 (-\ln u)^p du$$

$$= 2\pi \int_0^{\infty} |J_p(2\sqrt{st})|^2 |\varphi(s)|^2 s^p e^{-s} ds$$

Hence, putting together (2.22) and (2.23), we have

$$\|N(M - \lambda)^{-1}\varphi\|^2 \leq 4\pi^2 \int_0^{\infty} e^{-t} \left( \int_0^{\infty} |J_p(2\sqrt{st})|^2 |\varphi(s)|^2 s^p e^{-s} ds \right) dt$$

The right hand side is bounded above by

$$4\pi^2 \|\varphi\|^2 \int_0^{\infty} e^{-t} \sup_{st \geq 0} |J_p(2\sqrt{st})|^2 dt =: 4\pi^2 C \|\varphi\|^2$$

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Using \( \sup_{x \geq 0} |J_p(2\sqrt{x})|^2 = 1 \) we get \( C = 1 \). Therefore

\[
\|N(M - \lambda)^{-1}\|^2 \leq 4\pi^2 \quad \forall \lambda \in [0, 1]
\]

Choosing \( \lambda = 0 \) we get \( \|J\| \leq 2\pi \) as claimed.

To finish the proof, we note that if \( \varphi \in L^2(m_q) \) the functions \( M\varphi \) and \( N\varphi \)
are bounded at infinity. Therefore, if \( f \in \mathcal{H}_q \) satisfies \( P^+_q f = \lambda f \) with \( \lambda \neq 0 \),
then \( f \) extends analytically from the disk \( D_1 \) to the half-plane \( \{ \text{Re } x > 0 \} \). In
addition the expression \( (P^\pm_q f)(x) \) reproduces itself times \( \pm 1 \) if transformed
by substituting \( 1/x \) for \( x \) and dividing through \( x^{2q} \). Hence (2.20) holds.

**Remark 2.8.** Note that (2.18) is only a necessary condition for \( f \) to be an
eigenfunction (with \( \lambda \neq 0 \)). For instance the function \( f(x) = x^{-q} \) (which
does not belong to \( H_q \)) although plainly satisfying (2.18) for all \( q \in (0, \infty) \)
is an eigenfunction of \( P^+_q \) only for \( q = 1 \) (with \( \lambda = 1 \)).

**Remark 2.9.** Applying Proposition 2.7, the eigenvalue equations \( P^\pm_q f = \lambda f \),
with \( \lambda \neq 0 \), can be rewritten as the three-term functional equations,

\[
\lambda f(x) - f(x + 1) = \pm \frac{1}{x^{2q}} f \left( 1 + \frac{1}{x} \right) \tag{2.24}
\]

which for \( \lambda = 1 \) are studied in [Le] and [LeZa].

### 2.3 The spectrum of \( P^\pm \) in \( L^2(m_q) \)

We are now reduced to study the spectrum of the operators \( P^\pm \) in \( L^2(m_q) \).
Let us start studying the operators

\[
Q^\pm = M^{-1} P^\pm = I \pm M^{-1} N \tag{2.25}
\]

We first show that they are bounded in \( L^2(m_q) \).

**Lemma 2.10.** We have \( \|Q^\pm\| \leq 1 + 2\pi \).

**Proof.** The adjoint of the operator \( J = NM^{-1} \) dealt with in the previous
subsection exists and equals \( J^* = M^{-1} N \). A priori it is defined only on
\( D(M^{-1}) \). Recall however that \( J^* \) is continuous if and only if \( J \) is such and
\( \|J^*\| = \|J\| \). The assertion now follows from Proposition 2.7.

Recall now the orthogonal basis of \( L^2(m_q) \) given by \( e_n(t) \) (see (2.11)) and
the independent family of functions \( f_n(t) = \frac{t^n}{n!} \). We introduce the families
of functions

\[
\ell^\pm_n(t) := e_n(t) \pm f_n(t), \quad h^\pm_n(t) := e^{-t}(e_n(t) \pm f_n(t)) \tag{2.26}
\]

and consider the linear manifolds spanned by them.
Proposition 2.11. The linear manifolds $\mathcal{E}^\pm \subset L^2(m,q)$ defined by

$$\mathcal{E}^\pm := \left\{ \sum_{n=0}^{m} c_n h^\pm_n : c_n \in \mathbb{C}, \ 0 \leq n \leq m, \ m \geq 0 \right\} \quad (2.27)$$

have the following properties:

1. they are fixed by the operators $\pm J$, i.e. $\pm J \varphi = \varphi, \ \forall \varphi \in \mathcal{E}^\pm$;

2. their intersection is the trivial subspace, i.e. $\mathcal{E}^+ \cap \mathcal{E}^- = \{0\}$;

3. they are dense, i.e. $\mathcal{E}^\pm \equiv \text{Span} \{h^\pm_n\}_{n \geq 0} = L^2(m,q)$.

Proof. We first use (2.13) and (2.14) to get

$$B_q[h_n^\pm](x) = (n+1)p \frac{1 \pm x^n}{(1+x)^{n+2q}} \quad (2.28)$$

hence $J_q B_q[h_n^\pm](x) = \pm B_q[h_n^\pm](x)$. Now the first property follows upon application of Proposition 2.7.

The second property follows at once from the fact that the operator $J$ is an involution.

Finally, from the proof of Proposition 2.6 and (2.17) one readily gets that $(h_n^\pm, e_n) > 0, \ \forall n \geq 0$. This yields the density of $\mathcal{E}^\pm$ in $L^2(m,q)$.

Let us now consider the functions $(\ell_n^\pm)$. From the definition it follows that the function $\ell_n^{\pm}(t)$ is a polynomial of degree $2k$ for $n = 2k$ and $n = 2k + 1$, $(k \geq 0)$; whereas $\ell_n^{-}(t)$ has degree $2k + 1$ for $n = 2k + 1$ and $n = 2k + 2$, $(k \geq 0)$. Moreover we have $(\ell_n^{\pm}, e_n) = (1 \pm (-1)^n)\|e_n\|^2$ so that

$$(\ell_{2k+1}^+, e_{2k+1}) = (\ell_{2k+2}^-, e_{2k+2}) = 0$$

$$(\ell_{2k}^+, e_{2k}) = 2\|e_{2k}\|^2$$

$$(\ell_{2k+1}^-, e_{2k+1}) = 2\|e_{2k+1}\|^2 \quad (2.29)$$

Proposition 2.12. Let $H^\pm := \text{Span} \{\ell_n^\pm\}_{n \geq 0}$. Then

1. $L^2(m,q) = H^+ \oplus H^-$;

2. $Q^\pm |_{H^\pm} = 2 I$ and $Q^\pm |_{H^\mp} = 0$.

Proof.

1. By the relations (2.29), $H^+$ and $H^-$ do not have non-zero common vectors, thus $H^+ \cap H^- = \{0\}$. Moreover, let $\varphi \in L^2(m,q)$ be such that $\varphi \perp H^+ \oplus H^-$. Since $(\ell_n^\pm, e_n) = (1 \pm (-1)^n)\|e_n\|^2$ we get $\varphi = 0$. 

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2. We recall (2.17),

\[ M^{-1} N f_n = e_n \quad M^{-1} N e_n = f_n \]

From it we get

\[ Q^\pm \ell_n^\pm = 2 \ell_n^\pm \quad \text{and} \quad Q^\pm \ell_n^\mp = 0 \]

For \( \varphi = \sum_{n=0}^m c_n \ell_n^\pm \) we have by linearity \( Q^\pm \varphi = 2 \varphi \) so that \( \|Q^\pm \varphi\| = 2\|\varphi\| \), independently of \( m \). This implies \( Q^\pm \varphi = 2 \varphi \) for all \( \varphi \in H^\pm \). Hence \( Q^\pm H^\pm \subseteq H^\pm \) and \( Q^\pm_{|H^\pm} = 2 I \). In the same way one proves that \( Q^\pm_{|H^\mp} = 0 \).

\[ \square \]

Remark 2.13. From the above it follows that the operators \( Q^\pm \) are bounded in \( L^2(m_q) \) with \( \|Q^\pm\| = 2 \).

The operators \( P^\pm \) are self-adjoint and positive on \( L^2(m_q) \), hence the spectrum is real and positive. Moreover \( \|P^\pm\| \leq \|Q\| \|M\| = 2 \). Hence \( \sigma(P^\pm) \subseteq [0, 2] \). From the previous results we have information on the point spectrum \( \sigma_p(P^\pm) \).

Corollary 2.14. In \( L^2(m_q) \) it holds \( \text{Ker} P^\pm = H^\mp \) and \( \sigma_p(P^\pm) = \{0\} \) with infinite multiplicity.

Proof. We first observe that since \( \text{Ker} M = \{0\} \) we have by Proposition 2.12

\[ \text{Ker} P^\pm = \text{Ker} (MQ^\pm) = \text{Ker} Q^\pm = H^\mp \]

Now suppose that \( P^\pm \varphi = \lambda \varphi \) for some \( 0 < \lambda \leq 2 \) and \( \varphi \neq 0 \). Then \( \varphi \in H^\pm \) and hence \( P^\pm \varphi = MQ^\pm \varphi = 2M \varphi \). Therefore we would have \( (2M - \lambda) \varphi = 0 \) which implies \( \varphi \equiv 0 \).

To discuss the rest of the spectrum, we first characterise in more detail the nature of the perturbation operator \( N \).

Proposition 2.15. For \( \text{Re} q > 0 \) the operator \( N : L^2(m_q) \to L^2(m_q) \) is nuclear (and hence of the trace class). Its spectrum is given by

\[ \sigma(N) = \{0\} \cup \left\{ (-1)^k \alpha^{2(q+k)} \right\}_{k \geq 0} \quad (2.30) \]

where \( \alpha = (\sqrt{5} - 1)/2 \) is the golden mean. Each eigenvalue \( \lambda_k \in \sigma(N) \) is simple and the corresponding (normalised) eigenfunction \( \psi_k \) is given by

\[ \psi_k(t) = \sqrt{\frac{5^q k!}{\Gamma(k + 2q)}} L^p_k(\sqrt{5} t) \exp(-\alpha t) \quad (2.31) \]
Corollary 2.16. For $\text{Re} q > 0$ it holds

$$\text{tr}(N) = \frac{1}{\sqrt{5}} \alpha^p \quad \text{and} \quad \|N\| = \alpha^{2\text{Re} q} < 1$$

Proof of Proposition 2.15. Expanding the kernel of $N$ (see (2.16)) on the basis $(e_n)_{n \geq 0}$, one gets (see [Sze], p.102)

$$J_p \left(\frac{2\sqrt{st}}{(st)^{p/2}}\right) = \sum_{n=0}^{\infty} e_n(s) \frac{e^{-t^n}}{\Gamma(n + 2q)}$$

This yields

$$N\varphi = \sum_{n \geq 0} (\varphi, e_n) g_n$$

where $g_n(t) = Ne_n(t) = e^{-t^n}/n!$. Since

$$\|e_n\| = \sqrt{\frac{\Gamma(n + 2q)}{n!}} \quad \|g_n\| = \sqrt{\frac{\Gamma(2n + 2q)}{n! 3^{n+q}}}$$

we have

$$\sum_n \|e_n\| \|g_n\| < \infty$$

and therefore $N$ is nuclear. To compute the spectrum of $N$ we use the following Hankel transform (see [E], vol. II)

$$\int_0^{\infty} x^{p+\frac{1}{2}} e^{-bx^2} L^p_k(ax)J_p(xy)\sqrt{xy} dx = \frac{(b-a)^k y^{p+\frac{1}{2}}}{2^{p+1} b^{p+k+1}} e^{-\frac{y^2}{4b}} L^p_k \left(\frac{ay^2}{4b(a-b)}\right)$$

which can be recast in terms of the operator $N$ as

$$N \left[ L^p_k(2at) e^{-(2b-1)t}\right] = \frac{(b-a)^k}{2^{2q} b^{2q+k+1}} e^{-\frac{at}{2b}} L^p_k \left(\frac{at}{2b(a-b)}\right)$$

This becomes an eigenvalue equation in $L^2(m_q)$ provided $2b = \alpha^{-1}$ and $2a = \sqrt{5}$. The normalisation constant results from (2.11) noting that

$$\|L^p_k(\sqrt{5} t) \exp (-at)\| = \frac{1}{5^2} \|L^p_k(t)\|$$

This gives the eigenfunctions $\psi_k$, and the proof is complete. \qed

We now put together the previous results. We have seen that for all $q \in (0, \infty)$ the operators $P^\pm = M \pm N$ when acting on $L^2(m_q)$ are self-adjoint and positive with $\|M\| = 1$ and $\|N\| = \alpha^{2q}$. 

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The operator $M$ is spectrally absolutely continuous ([Ka], p.520). Its spectrum, being the essential range of the multiplying function, coincides with $[0, 1]$. This means that in the orthogonal decomposition $L^2(m_q) = H_{ac}(M) \oplus H_s(M)$ of the Hilbert space into the subspace of absolute continuity $H_{ac}(M) = \Pi_{ac}(M)L^2(m_q)$ and that of singularity $H_s(M) = \Pi_s(M)L^2(m_q)$, we have $H_s(M) = 0$ (and thus $\Pi_{ac}(M) = I$).

On the other hand $N_q$ is of the trace class. Therefore, applying the Kato-Rosenblum theorem (see [Ka], p.542, or [RS], vol. III, p.26), it holds

**Proposition 2.17.** The operator $M$ is unitarily equivalent to the spectrally absolutely continuous part of $P^\pm$. Hence on $L^2(m_q)$ we have $\sigma_{ac}(P^\pm) = (0, 1]$.

**Remark 2.18.** Such equivalence is gained by means of the one-parameter family of unitary operators

$$W(\tau) = e^{i\tau P} e^{-i\tau M}, \quad -\infty < \tau < \infty$$

The (strong) limits $W_\pm$ of $W(\tau)$ as $\tau \to \pm \infty$ are called wave operators and $S = W_+^* W_-$ the scattering operator, which is unitary on $L^2(m_q)$ to itself and commutes with $M$. The Kato-Rosenblum theorem says that in this case the wave operators $W_\pm$ exist and are complete, meaning that they are partial isometries with initial domain $L^2(m_q)$ and range $H_{ac}(P) = \Pi_{ac}(P)L^2(m_q)$. Therefore we have $W_+^* W_+ = I$, $W_\pm W_\mp = \Pi_{ac}(P)$ and $PW_\pm = W_\pm M$ (see [RS], vol. III, pp. 17-19).

Putting together Proposition 2.6, Corollary 2.14 and Proposition 2.17 we get Theorem 1.1.

### 3 Digression: self-reciprocal functions in $L^2(\mathbb{R}_+)$

Given a continuous function $\phi$ on $\mathbb{R}_+$ and $q \in \mathbb{C}$, with $\text{Re} \, q > 0$ (or $\text{Re} \, p > -1$), the function $J\phi = N M^{-1} \phi$ considered in Section 2.2 can be viewed as a version of its Hankel transform, i.e.

$$J\phi(t) := \int_0^\infty J_p(2\sqrt{st}) \left(\frac{s}{t}\right)^{p/2} \phi(s) \, ds \quad (3.1)$$

We can also define the conjugate transform $\tilde{J}$ as

$$\tilde{J} := \chi_q J \chi_p^{-1} \quad (3.2)$$
Proof. The first implication is immediate. The second follows from the asymptotic estimates on $J$.

\[ \int_0^\infty J_p(t) \left( \frac{t}{s} \right)^{p/2} \phi(s) \, ds \] (3.3)

From the asymptotic estimates on $J_p(t)$ we see that the conditions on $\phi$ sufficient to give the absolute convergence of the integral (3.1) are $\phi(t) = O(t^{-a})$ as $t \to 0^+$ and $\phi(t) = O(t^{-b})$ as $t \to \infty$ with $a < 2 \Re q$ and $b > \Re q + \frac{1}{2}$. For the integral (3.3) we have the same conditions with $b > \frac{5}{4} - q$ and $a < 1$.

Accordingly, the identity $J_q f = \pm f$ for $f = B_q [\varphi]$ can be rephrased as a self-reciprocity property for the functions $\varphi$ and $\psi := \chi_p \cdot \varphi$, that is

\[ J_q f = \pm f \implies J \varphi = \pm \varphi \quad \text{and} \quad J \psi = \pm \psi \] (3.4)

Lemma 3.1. If $\varphi \in L^2(\mathbb{R}_+)$ then $\varphi \in L^2(m_q) \cap L^2(\mathbb{R}_+)$ provided $\Re p \geq 0$. Conversely, if $\varphi \in L^2(m_q)$ and $J \varphi = \pm \varphi$ then $\varphi \in L^2(\mathbb{R}_+)$. 

Proof. The first implication is immediate. The second follows from the asymptotic estimates on $J_p(t)$. □

Therefore, we shall study self-reciprocal functions in $L^2(\mathbb{R}_+)$. Moreover, by a change of variables the conditions (3.4) can be recast in the form that the function

\[ \phi(t) = 2^{-q+\frac{1}{2}} t^{p+\frac{1}{2}} \varphi \left( \frac{t^2}{2} \right) = 2^{q-\frac{1}{2}} t^{-p+\frac{1}{2}} \psi \left( \frac{t^2}{2} \right) \] (3.5)

satisfies $K \phi = \pm \phi$ where $K$ is the symmetric version of the Hankel transform given by

\[ K \phi(t) := \int_0^\infty J_p(st) \sqrt{s t} \phi(s) \, ds. \] (3.6)

For $\Re p > -1$ the simplest solution of $K \phi = \phi$ is $\phi(t) = \sqrt{2} t^{-\frac{1}{2}}$ which corresponds to $\varphi(t) = t^{-q}$ and $\psi(t) = t^{q-1}$. This solution has been already considered above and does not belong to $L^2(\mathbb{R}_+)$. We refer to [Tit], Chap.9, for an analysis of the equation $K \phi = \phi$ in $L^2(0, \infty)$.

For $a > 0$, let $S_a : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ be given by $(S_a \varphi)(t) := a^q \varphi(at)$. Then $JS_a = S_{1/a} J$. In particular, since $J e^{-t} = e^{-t}$ we have that $a^q e^{-at}$ and $a^{-q} e^{-t/a}$ is a Hankel transform pair for all $a > 0$. Now, the operator $\tilde{J}$ is an adjoint to $J$ in the sense that $< \psi, J \varphi > = < \tilde{J} \psi, \varphi >$ with $< \phi_1, \phi_2 > := \int_0^\infty \phi_1(t) \overline{\phi_2(t)} \, dt$. Whence, the identity

\[ \int_0^\infty a^{-q} e^{-t/a} \psi_1(t) \, dt = \int_0^\infty a^q e^{-at} \psi_2(t) \, dt, \quad a > 0 \] (3.7)
must hold whenever \( \psi_1 \) and \( \psi_2 \) is a pair w.r.t the Hankel transform \( \tilde{J} \). If moreover \( \tilde{\psi}_2 \) is another Hankel transform of \( \psi_1 \) then \( \int_0^\infty e^{-at}(\psi_2 - \tilde{\psi}_2)dt = 0 \) for all \( a > 0 \) so that \( \psi_2 = \tilde{\psi}_2 \) almost everywhere. Therefore the identity (3.7) is a necessary and sufficient condition for \( \psi_1 \) and \( \psi_2 \) to be a pair w.r.t the Hankel transform \( \tilde{J} \). Let moreover

\[
\psi^*(s) := \int_0^\infty \psi(t) t^{s-1} dt \tag{3.8}
\]

be the Mellin transform of \( \psi \). If there are two constants \( a < b \) such that \( \psi(t) = O(t^{-a}) \) as \( t \to 0^+ \) and \( \psi(t) = O(t^{-b}) \) as \( t \to \infty \) then the integral converges for \( s \) in the strip \( a < \text{Re} \ s < b \) and \( \psi^*(s) \) is a holomorphic function in this strip.

**Remark 3.2.** If \( P_q^+ f = \lambda f \) then one easily checks that

\[
\lambda = 1 + \frac{f(1)}{f(0)} \quad \text{and} \quad \frac{\lambda}{2} (\lambda - 1) = f(2) \tag{3.8}
\]

Thus, if \( \lambda \neq 1 \) we have \( f(0) \neq 0 \) and

\[
f(x) \sim f(0) x^{-2q} \quad x \to \infty \tag{3.8}
\]

Therefore, if \( \text{Re} \ q > 0 \) then the Mellin transform \( f^* \) is analytic in the strip \( 0 < \text{Re} \ s < 2 \text{Re} \ q \) and in this region it holds

\[
(J_q f)(x) = f(x) \implies f^*(s) = f^*(2q - s) \tag{3.8}
\]

Now, taking the Mellin transform of both sides in (3.7) we obtain

\[
\Gamma(1 - s) \psi_1^*(s) = \Gamma(s + p) \psi_2^*(1 - p - s) \tag{3.8}
\]

Note that if \( \psi = \chi_p \cdot \varphi \), then \( \psi^*(s) = \varphi^*(s + p) \). Moreover, Mellin transforming (3.5) gives

\[
\phi^*(s) = 2^{\frac{s}{2} - \frac{3}{4}} \varphi^*(\frac{s}{2} + \frac{p}{2} + \frac{1}{4}) = 2^{\frac{s}{2} - \frac{3}{4}} \psi^*(\frac{s}{2} - \frac{p}{2} + \frac{1}{4}) \tag{3.8}
\]

Therefore, if we define the weighted transforms \( \tilde{\varphi}^* \), \( \tilde{\psi}^* \) and \( \tilde{\phi}^* \) as

\[
\tilde{\varphi}^*(s) := \frac{\varphi^*(s)}{\Gamma(s)} \quad \tilde{\psi}^*(s) := \frac{\psi^*(s)}{\Gamma(s + p)} \quad \tilde{\phi}^*(s) := 2^{\frac{s}{2} + 1} \frac{\phi^*(s)}{\Gamma(\frac{s}{2} + \frac{p}{2} + \frac{1}{4})} \tag{3.8}
\]

and taking into account that \( 1 + p - (\frac{s}{2} + \frac{p}{2} + \frac{1}{4}) = \frac{1 - ps}{2} + \frac{p}{2} + \frac{1}{4} \), we have the following result:
Proposition 3.3. The functions \( \varphi, \psi, \phi \in L^2(\mathbb{R}_+) \), related to each other by (3.5), are jointly self-reciprocal, i.e. \( J\varphi = \pm \varphi, \ J\psi = \pm \psi \) and \( K\phi = \pm \phi \), if and only if

\[
\bar{\varphi}^*(s) = \pm \varphi^*(1+p-s), \quad \bar{\psi}^*(s) = \pm \psi^*(1-p-s), \quad \hat{\phi}^*(s) = \pm \hat{\phi}^*(1-s)
\]

The sequences \( h_n^\pm \) introduced in (2.26) were our first example of self-reciprocal functions in \( L^2(\mathbb{R}_+) \), in that \( Jh_n^\pm = \pm h_n^\pm \) for all \( n \geq 0 \). Even more interesting self-reciprocal functions are provided by the conjugate sequences \( \varphi_n, \psi_n \in L^2(\mathbb{R}_+) \), \( n \geq 0 \), defined for \( \text{Re} \ p > -1 \) by

\[
\varphi_n(t) := \sqrt{\frac{2^{n+1} n!}{\Gamma(n+p+1)}} e^{-t^2/2} L_p^n(t^2), \quad \psi_n(t) := (\chi_p \cdot \varphi_n)(t),
\]

and satisfying the condition \( < \varphi_n, \psi_m > = \delta_{n,m} \). They are related to the sequences \( h_n^\pm \) by (see [E], vol. II, p.192)

\[
\varphi_n = (-1)^n \sqrt{\frac{2^{n+1} n!}{\Gamma(n+p+1)}} \sum_{m=0}^{n} \binom{n+p}{n-m} (-2)^m \left( \frac{h_m^+ + h_m^-}{2} \right)
\]

Thus

\[
J \varphi_n = (-1)^n \sqrt{\frac{2^{n+1} n!}{\Gamma(n+p+1)}} \sum_{m=0}^{n} \binom{n+p}{n-m} (-2)^m \left( \frac{h_m^+ - h_m^-}{2} \right)
\]

which, compared to (2.10), yields

\[
J \varphi_n = (-1)^n \varphi_n, \quad \hat{J} \psi_n = (-1)^n \psi_n. \quad (3.10)
\]

Note that

\[
B_q[\varphi_n](x) = (n+1)_p \frac{(1-x)^n}{(1+x)^{n+2q}} \quad (3.11)
\]

so that \( J_q B_q[\varphi_n] = (-1)^n B_q[\varphi_n] \), as expected (compare to (2.28)). Moreover

\[
\bar{\varphi}_n^*(s) = \frac{(p+1)_n}{n!} \binom{2}{s} \binom{-n}{s} \quad (3.12)
\]

which satisfies the functional equation of Proposition 3.3 because of Pfaff’s identity ([AAR], Theorem 2.2.5) which implies

\[
\binom{2}{n} \binom{-n}{c} = (-1)^n \binom{2}{n} \binom{-n}{c-b} \binom{c}{2}.
\]

Finally, the orthonormal family \( \{ \phi_n \} \) of \( L^2(\mathbb{R}_+) \) given by

\[
\phi_n(t) := \sqrt{\frac{2^{n+1} n!}{\Gamma(n+p+1)}} e^{-t^2/2} L_p^\frac{1}{2}(t^2) \quad (3.13)
\]
satisfies
\[ K\phi_n = (-1)^n \phi_n, \quad n \geq 0. \] (3.14)

Thus, the families \( \varphi_n, \psi_n, \phi_n \) furnish a complete characterization of self-reciprocal functions in \( L^2(\mathbb{R}_+) \) for the Hankel transforms \( J, \tilde{J}, K \).

**Remark 3.4.** The functions \( \phi_n \) are also solutions of the differential equation:
\[ \phi''_n - \left( \frac{p^2 - 1/4}{t^2} + t^2 - 4n - 2p - 2 \right) \phi_n = 0 \] (3.15)
as one can check using, e.g., [E], vol. II, p.188. More specifically, the second order differential operator \( H \) given by
\[ H := \frac{1}{2} \left( -\frac{d^2}{dt^2} + \frac{p^2 - 1/4}{t^2} + t^2 \right) \] (3.16)
has for real \( p \geq 1 \) a unique self-adjoint extension on \( C^\infty_0(\mathbb{R}_+) \) which has an integer-spaced spectrum so that
\[ H\phi_n = (2n + p + 1) \phi_n, \quad n \geq 0. \] (3.17)

For \(-1 < p < 1\) there is more than one self-adjoint extension, one of which, however, still satisfies (3.17). Comparing (3.14) and (3.17) one may regard the unitary mapping \( K \) of \( L^2(\mathbb{R}_+) \) onto itself as a hyperdifferential operator of the form \( (2q = p + 1) \)
\[ K = e^{i\pi q} \exp \left(-\frac{i\pi}{2} H \right) \] (3.18)
and acting on a suitable class of analytic functions (see [Bar] and [Wo] for a discussion on this and related correspondences).

**4 Polynomial eigenfunctions of \( P^\pm_q \) for \( q = -k/2 \).**

Although the eigenfunction \( f^{(q)}(x) \) corresponding to the leading eigenvalue \( \lambda(q) \) does not belong to the space \( H_q \) (see Remark 1.2), we shall see that explicit expressions for \( \lambda(q) \) and \( f^{(q)}(x) \) can be obtained when \( q = -k/2 \) with \( k \) a non-negative integer. Note that these values correspond exactly to the simple poles of \( \Gamma(2q) \) and thus, by (2.8), to the \( q \)-values where the

\[ ^1 \text{In quantum mechanics this corresponds to the Schrödinger operator for a two-dimensional isotropic harmonic potential (see [RS], vol. II, p.161).} \]
measure $m_q$ has an infinite mass. On the other hand, for $q = -k/2$ the operators $P_q^\pm$ take the form

$$P_q^\pm f(x) = (x+1)^k \left[ f\left(\frac{x}{x+1}\right) \pm f\left(\frac{1}{x+1}\right) \right]$$

so that they leave invariant the vector space $\oplus_n^{k=0} \mathbb{C} x^n$ of polynomials of degree $\leq k$. In particular we expect $f(-\frac{k}{2})(x)$ is a polynomial of degree $k$ with real coefficients.

To warm up, a direct calculation yields

$$f^{(0)}(x) = 1, \quad \lambda(0) = 2,$$

$$f^{(-\frac{1}{2})}(x) = x + 1, \quad \lambda(-\frac{1}{2}) = 3,$$

$$f^{(-1)}(x) = x^2 + \sqrt{11} x + 1, \quad \lambda(-1) = 5+\sqrt{17},$$

$$f^{(\frac{-3}{2})}(x) = x^3 + 2x^2 + 2x + 1, \quad \lambda(-\frac{3}{2}) = 7,$$

$$f^{(-2)}(x) = x^4 + \sqrt{113} x^3 + 3x^2 + \sqrt{113} x + 1, \quad \lambda(-2) = 11+\sqrt{113}$$

To say more we first need the following result.

**Lemma 4.1.** The $(k+1) \times (k+1)$ real positive matrix $M_k$ defined as

$$M_k(i,j) := \begin{cases} 
(k-i) & \text{if } i < j \\
2 & \text{if } i = j \\
(i) & \text{if } i > j 
\end{cases} \quad (0 \leq i, j \leq k)$$

has the following properties:

1. the symmetry $M_k(i,j) = M_k(k-i,k-j)$ holds for all $0 \leq i, j \leq k$;
2. the sum $S_i$ of the entries in row $i$ equals $S_i = 2^i + 2^{k-i}$;
3. the sum $R_j$ of the entries in column $j$ equals $R_j = \binom{k+2}{j+1}$;
4. if $M_k \Phi = \lambda \Phi$ with $\mathbb{C}^{k+1} \ni \Phi := (b_0, b_1, \cdots, b_k)^T$ and $\lambda \neq 0$ then $\Phi$ is either a palindrome or a skew-palindrome, i.e. $b_i = \pm b_{k-i}$ for $0 \leq i \leq k$;
5. $\sigma(M_k) \subset \mathbb{R}$ for all $k \in \mathbb{N} \cup \{0\}$;
6. $1 \in \sigma(M_k)$ for all $k \in \mathbb{N}$. 

Proof. 1), 2) and 3) follow by direct computation. To prove 4) we write the eigenvalue equation componentwise:

\[ \lambda b_i = \sum_{j=0}^{k} M(i, j) b_j \quad (0 \leq i \leq k) \]

which yields, using the symmetry 1),

\[ \lambda b_{k-i} = \sum_{j=0}^{k} M(k - i, j) b_j = \sum_{j=0}^{k} M(k - i, k - j) b_{k-j} = \sum_{j=0}^{k} M(i, j) b_{k-j} \]

so that \( M_k \Phi = \lambda \Phi \) if and only if \( M_k \Phi' = \lambda \Phi' \) with \( \Phi' := (b_k, b_{k-1}, \ldots, b_0)^T \).

If \( \lambda \) is (geometrically) simple then \( \Phi = \pm \Phi' \) because \( \| \Phi \| = \| \Phi' \| \) where \( \| \| \) is the euclidean norm in \( \mathbb{C}^{k+1} \). In other words, \( \Phi = \pm \Phi' \) is a necessary and sufficient condition for \( \Phi \) and \( \Phi' \) to be linearly dependent. Assume now that \( M_k \Phi = \lambda \Phi \) with \( \lambda \) of geometric (and thus algebraic) multiplicity bigger than 1. If \( \Phi \) and \( \Phi' \) are linearly dependent then we are done. Suppose they are not. Then the vectors \( \Psi_\pm := \Phi \pm \Phi' \) should also be two linearly independent eigenvectors. But this is impossible because \( \Psi_\pm = \pm \Psi_\pm \). This concludes the proof of 4).

As for 5) we observe that \( M_1 \) is symmetric and for each \( k \geq 2 \) it is not difficult to realise that one can construct recursively a positive symmetric \((k+1) \times (k+1)\) matrix \( N_k \) such that the product \( M_k N_k \) is symmetric as well. For instance with \( k = 4 \) one gets

\[
M_4 = \begin{pmatrix}
2 & 4 & 6 & 4 & 1 \\
1 & 2 & 3 & 1 & 0 \\
1 & 2 & 2 & 1 & 0 \\
1 & 3 & 3 & 2 & 1 \\
1 & 4 & 6 & 4 & 2
\end{pmatrix}
\quad N_4 = \begin{pmatrix}
1 & 13 & 1 & 7 & 18 \\
13 & 1 & 3 & 2 & 13 \\
1 & 3 & 3 & 7 & 1 \\
7 & 2 & 7 & 1 & 7 \\
18 & 13 & 1 & 7 & 1
\end{pmatrix}
\]

Then apply Theorem 1 in [DH]. Finally, the vector \( \Phi = (1, 0, \ldots, 0, -1)^T \) always satisfies \( M_k \Phi = \Phi \), which yields 6).

Remark 4.2. If one defines a pseudo-scalar product of \( \Phi = (b_0, b_1, \ldots, b_k)^T \) and \( \Psi = (c_0, c_1, \ldots, c_k)^T \) as \( \langle \Phi, \Psi \rangle := \sum_{i=0}^{k} b_i c_{k-i} \) then the symmetry stated in 1) amounts to \( \langle M_k \Phi, \Psi \rangle = \langle \Phi, M_k^T \Psi \rangle \). Moreover, 4) implies that if \( M_k \Phi = \lambda \Phi \) with \( \lambda \neq 0 \) then \( \langle \Phi, \Phi \rangle = \pm \| \Phi \|^2 \).

Theorem 4.3. Let \( q = -k/2 \), \( k \geq 1 \). The polynomial

\[ f(x) = \sum_{i=0}^{k} \binom{k}{i} b_i x^i \quad (4.1) \]
satisfies $P_q^\pm f = \lambda f$ with $\lambda \neq 0$ if and only if the vector $\Phi = (b_0, b_1, \cdots, b_k)^T$ satisfies $M_k \Phi = \lambda \Phi$ and is either a palindrome (if $P_q^+ f = \lambda f$) or a skew-palindrome (if $P_q^- f = \lambda f$).

**Corollary 4.4.** The eigenvector corresponding to the simple positive maximal eigenvalue $\lambda(-k/2)$ of $M_k$ is always palindromic and we have the bounds:

$$s - 1 + h \leq \lambda \leq S - 1 + \frac{1}{g}$$

where

$$S := \max_i S_i = 2^k + 1 \quad , \quad s := \min_i S_i = \begin{cases} 2^{\frac{k+1}{2}} + 2^{\frac{k-1}{2}}, & k \text{ even} \\ 2^{\frac{k+1}{2}} + 2^k, & k \text{ odd} \end{cases}$$

and

$$h = -s + 2 + \sqrt{s^2 + 4(S - s)} \quad , \quad g = S - 2 + \sqrt{S^2 - 4(S - s)}$$

Proof. Put together the above and [MM], p.155, eq.(9).

**Proof of Theorem 4.3** Setting

$$f(x) = a_k x^k + a_{k-1} x^{k-1} + \cdots + a_1 x + a_0$$ (4.2)

the conditions $J_q f = \pm f$ implies that the sequence of coefficients $a_i$ is either a palindrome or a skew-palindrome, i.e. $a_i = \pm a_{k-i}$, ($0 \leq i \leq k$). Inserting the function $f(x)$ written above into (2.24) with $q = -k/2$ and $k \geq 1$ we get

$$\lambda \sum_{i=0}^k a_i x^i = \sum_{i=0}^k a_i \sum_{j=0}^i \binom{i}{j} (x^j \pm x^{k-j})$$

$$= \sum_{j=0}^k \left[ \sum_{l=j}^k \binom{l}{j} a_l \right] (x^j \pm x^{k-j})$$

$$= \sum_{i=0}^k \left[ \sum_{l=i}^k \binom{l}{i} a_l \pm \sum_{l=k-i}^k \binom{l}{k-i} a_l \right] x^i$$ (4.3)

which in both cases yields

$$\lambda a_i = \sum_{l=0}^i \binom{k-l}{k-i} a_l + \sum_{l=i}^k \binom{l}{i} a_l \quad (0 \leq i \leq k)$$ (4.4)
Defining new coefficients $b_i$ so that

$$a_i = \binom{k}{i} b_i \quad (0 \leq i \leq k) \quad (4.5)$$

and using the identities

$$\binom{k-l}{k-i} \binom{k}{i} = \binom{k}{i} \binom{i}{l} \quad \text{and} \quad \binom{l}{i} \binom{k}{l} = \binom{k}{i} \binom{k-i}{l-i}$$

we see that the above recursion becomes

$$\lambda b_i = \sum_{l=0}^{i} \binom{i}{l} b_l + \sum_{l=1}^{k} \binom{k-i}{l-i} b_l \quad (0 \leq i \leq k) \quad (4.6)$$

and the proof is complete. □

**Example 4.5.** For $k = 4$ we find $\text{sp}(M_4) = \left\{ \frac{11 + \sqrt{113}}{2}, 1, \frac{11 - \sqrt{113}}{2}, -1, -1 \right\}$ and the corresponding eigenvectors are

$$\Phi_1 = \begin{pmatrix} 1 \\ \frac{\sqrt{113} - 1}{16} \\ \frac{1}{2} \\ \frac{\sqrt{113} - 1}{1} \end{pmatrix}, \quad \Phi_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad \Phi_3 = \begin{pmatrix} 1 \\ \frac{\sqrt{113} + 1}{16} \\ \frac{1}{2} \\ \frac{\sqrt{113} + 1}{1} \end{pmatrix},$$

$$\Phi_4 = \begin{pmatrix} 3 \\ 0 \\ -2 \\ 0 \end{pmatrix}, \quad \Phi_5 = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

Therefore the spectrum of $M_4$ yields three eigenfunctions for $\mathcal{P}_{-2}^+$:

$$h_1(x) = x^4 + \frac{\sqrt{113} - 1}{4} x^3 + 3 x^2 + \frac{\sqrt{113} - 1}{4} x + 1$$

$$h_3(x) = x^4 + \frac{\sqrt{113} + 1}{4} x^3 + 3 x^2 + \frac{\sqrt{113} + 1}{4} x + 1$$

and

$$h_4(x) = -3 x^4 + 12 x^2 - 3$$
and two eigenfunctions for $\mathcal{P}_{-2}$:

$$h_2(x) = x^4 - 1$$

and

$$h_5(x) = 4x(1 - x^2).$$

**Remark 4.6.** Eigenvectors of $M_k$ to the eigenvalue 1 are related to the period functions for the modular group (see [CM]). In particular, for $k \in \mathbb{N}$ the eigenvectors $(1, 0, \ldots, 0, -1)^T$ correspond to the fixed functions $x^k - 1$ of $\mathcal{P}_{-k/2}$ which yield the even part of the period functions corresponding to holomorphic Eisenstein forms of weight $k + 2$. The odd parts are computed below in Proposition 4.7. Other linearly independent (skew-palindromic and palindromic) eigenvectors with eigenvalue 1 are expected for $k \geq 10$, as they are related to (even and odd part of) holomorphic cusp forms [A].

For the sake of completeness we end with the following result, a version of which is contained in [CM].

**Proposition 4.7.** Let $B_m$ denote the $m$-th Bernoulli number. For $k \in \mathbb{N} \cup \{0\}$ the function $f_k(x) \in \oplus_{n=-1}^{k+1} \mathbb{C} x^n$ given by

$$f_k(x) := \frac{\zeta(-k)}{2} (1 + x^k) + (-1)^k k! \sum_{-1 \leq n \leq k+1} \frac{B_{n+1} B_{k+1-n}}{(n+1)! (k+1-n)!} x^n$$

satisfies $\mathcal{P}_{-k/2} f_k = f_k$ for $k$ even and $f_k \equiv 0$ for $k$ odd.

Two examples are

$$f_0(x) = \frac{1}{12} \left[ x + \frac{1}{x} - 3 \right]$$

$$f_2(x) = \frac{1}{360} \left[ 5x - \left( x^3 + \frac{1}{x} \right) \right]$$

Note that for $k \geq 1$, the odd parts of the period functions mentioned in Remark 4.6 can be expressed as $\frac{(-1)^k}{k!} (f_k(x) - \frac{\zeta(-k)}{2} (1 + x^k))$ [Za1].

**Proof.** Consider the function $\psi_q(x)$ defined for $\text{Re} \ q > 1$ by

$$\psi_q(x) = \frac{\zeta(2q)}{2} (1 + x^{-2q}) + \sum_{n,m \geq 1} (nx + m)^{-2q}$$

It is shown in [Za2] that the function $\psi_q(x)$ has an analytic extension into the complex $q$-plane with a simple pole at $q = 1$, and the analytic continuation
satisfies (2.24) with the sign + and $\lambda = 1$ for all $q \in \mathbb{C} \setminus \{1\}$. Note that if $\text{Re } q > 1$ then $\psi_q(\infty) = \frac{1}{2} \zeta(2q)$.

The proof then amounts to show that for $q = -k/2$ the analytic extension of the function $\psi_q$ is precisely $f_k$. This is achieved using standard Mellin transform techniques: start from the identity

$$\sum_{n,m \geq 1} (nx + m)^{-2q} = \frac{1}{\Gamma(2q)} \int_0^\infty \sum_{n,m \geq 1} e^{-t(nx+m)t^{2q-1}} dt$$

$$= \frac{1}{\Gamma(2q)} \int_0^\infty \frac{t^{2q-1}}{(e^t - 1)(e^{tx} - 1)} dt$$

Recalling that

$$\frac{1}{e^t - 1} = \sum_{r=-1}^{\infty} \frac{B_{r+1}}{(r + 1)!} t^r = \frac{1}{t} - \frac{1}{2} + \sum_{l=1}^{\infty} \frac{B_{2l}}{(2l)!} t^{2l-1}$$

we get

$$\sum_{n,m \geq 1} (nx + m)^{-2q} = \frac{1}{\Gamma(2q)} \sum_{k=-2}^{\infty} \int_0^\infty c_k(x) t^{k+2q-1} dt$$

with $c_{-2} = 1/x$, $c_{-1} = -\frac{1}{2} (1 + 1/x)$ and for $k \geq 0$

$$c_k = \sum_{-1 \leq n \leq k+1} \frac{B_{n+1} B_{k+1-n}}{(n+1)!(k+1-n)!} x^n$$

Now $\Gamma(2q)$ has simple poles at $2q = -k$, with $k = 0, 1, 2, \ldots$, with residues $(-1)^k/k!$! On the other hand the integral written above has simple poles at $2q = -k$, with $k = -2, -1, 0, 1, \ldots$ the residues being $c_k$. Therefore the analytic continuation of $\sum_{n,m \geq 1} (nx + m)^{-2q}$ has only two simple poles at $q = 1$ and $q = 1/2$ and the claimed expression of $f_k$ follows by taking the limit $2q \to -k$ with $k \in \mathbb{N} \cup \{0\}$. The last assertion is a consequence of the identity $\zeta(-k) = -B_{k+1}/(k + 1)$ for $k$ odd. 

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