The generalized Solow model

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Abstract.
A modification of the known mathematical macroeconomics model of macroeconomics are studied in which a delay factor is presumed. This led to the replacement of the ordinary differential equation, which cannot exhibit periodic cycles on the equations with a deviating argument (functional-differential equations). It was possible to show the existence of periodic solutions that can and are intended to describe the periodic cycles in the market economy.

The mathematical portion is based on the application of the modern theory of dynamical systems with an infinite-dimensional space of initial conditions. This will allow us to apply the Andronov-Hopf Theorem for equations with a deviating argument in such a form that the parameters of the cycles are located. We will also apply the well-known Krylov-Bogolyubov algorithm that is extended to infinite-dimensional dynamical systems that is used and reduces the problem to the analysis of the finite-dimensional system of ordinary differential equations-the normal Poincare-Dulac form.

1. Introduction
In macroeconomics, there are a large number of mathematical models that are traditionally considered classical and basic. Traditionally among them, a mathematical model proposed by R.M. Solow [1,2]. This model is usually called the Solow model or the Solow-Swan model. It is designed to describe the dynamics of the labor-power ratio of labor (see also [3-8])

\[
p' = -\alpha p + \beta p^k,
\]

where \( \alpha, \beta \) are positive constants, \( p = p(\tau) \) is the value of funds at the time \( \tau \). After replacements (normalizations)

\[
\tau = \gamma_0 t, \; p = \gamma_1 x, \; \gamma_0 \alpha = 1, \; \beta \gamma_0 \gamma_1^{k-1} = 1, \; \gamma_0 = \frac{1}{\alpha}, \; \gamma_1 = \left( \frac{\alpha}{\beta} \right)^{1/(k-1)}, \; \beta, \gamma_0, \gamma_1 > 0.
\]

Equation (1.1) acquires the following form

\[
\dot{x} = -x + x^k, \; x = x(t).
\]

The first terms on the right-hand side of equations (1.1), (1.2) are responsible for the depreciation of the funds. The second terms are proportional to the investments. Next we will consider equation (1.1) for the normalized value of the funds \( x(t) > 0 \), via Equation (1.2).
Observe that Equation (1.2) has a unique positive equilibrium state \( x(t) = 1 \) if \( k \neq 1 \). Then for \( k \in (0, 1) \), this equilibrium state is globally asymptotically stable (all solutions of the Equation (1.2) with positive initial conditions converge to a given equilibrium position).

In this particular model, it is usually assumed that \( k \in (0, 1) \) is a stable economic equilibrium \( x = 1 \). Notice that the positive equilibrium state \( x = 1 \) is unstable, when \( k \in (1, \infty) \). Equation (1.1) cannot have stationary solutions with the exception of the equilibrium states. Hence this mathematical model does not quite adequately describe the real economic processes, for which, of course, periods of ups and downs are the characteristic.

In this paper, we will show that the assumption of the delay factor in the Solow Model leads to a meaningful change in the dynamics of the solutions and makes it possible to isolate the range of the parameters of the problem under which the periodic cycles exist. It is possible that the delay is introduced into both terms, which produces the following equation

\[
\dot{x} = -y + y^k, \tag{1.3}
\]

where as previously \( y = x(t - h) \). Naturally we see that \( h > 0 \).

If we supplement equations (1.3) with the initial condition

\[
x(t) = \varphi(t), \tag{1.4}
\]

where the given function \( x(t) \in C[-h, 0] \) to the space of continuous functions defined on \([-h, 0]\), then we obtain Cauchy problem: (1.3), (1.4). This problem generates a local semiflow.

Equation (1.3) has an equilibrium state \( x(t) = 1 \). In the next section, we will address the stability question. Equation (1.3) can be rewritten in the following form

\[
\begin{align*}
\dot{u} &= -v + kv + \frac{k(k-1)}{2}v^2 + \frac{k(k-1)(k-2)}{6}v^3 + o(v^3), \\
\end{align*}
\]

after replacement which is acquired from Equation (1.3) we get

\[
x = 1 + u, y = 1 + v, u = u(t), v = v(t) = u(t-h),
\]

and applying the Taylors Formula. Notice that all three equations have a zero equilibrium state corresponding to the equilibrium states \( x = 1 \) of the original equation (1.3). Now we will consider the equation (1.5) for which the structure of the neighborhood of the zero equilibrium state will be studied.

2. Analysis of linearized equation

We will first address the stability question of the zero equilibrium state of Equation (1.5). To study this question, the analysis of the linearized Equation (1.5) will be necessary. In this case we obtain the following equation

\[
\dot{u} = -v + kv. \tag{2.1}
\]

It is well known [9] that stability of solutions of the linear differential equation (2.1) can be reduced to an analysis of the characteristic equation

\[
\lambda = (k-1) \exp(-\lambda h). \tag{2.2}
\]

Observe when \( h = 0 \) then we have the root \( \lambda = k - 1 < 0 \). To find the positive \( \min h = H \) for which roots with \( Re\lambda = 0 \) emerge in Equation (2.2). The case when \( \lambda = 0 \) is not possible for any \( h \).
Consequently, the critical case in the stability problem is possible. For the corresponding value $h$, the stability spectrum (the set of roots of the characteristic equation (2.2)) contains the pair of purely imaginary roots $\pm i\sigma$, where $\sigma > 0$.

The corresponding pairs $(h, \sigma)$ are defined as solutions of the system

$$(k - 1) \cos \sigma h = 0, \sigma = (1 - k) \sin \sigma h (k \neq 0).$$

Therefore, we finally obtain a system for determining $H$ and $\sigma$ already of the following form

$$\cos \sigma h = 0, \sigma h = (1 - k) \sin \sigma h$$

or after relabeling $\omega = \sigma h$, the system (2.3) can be written in the form

$$\cos \omega = 0, (1 - k) \sin \omega = \omega.$$  \hspace{1cm} (2.4)

Note that system (2.4) has the following set of solutions

$$\omega_m = \frac{\pi}{2} + \pi m, h_m = \frac{\omega_m}{(1 - k) \sin \omega_m}, m = 0, \pm 1, \pm 2, \ldots$$

The smallest $H = \min\{h_m\} > 0$ is possible if $\omega = \frac{\pi}{2}, H = \frac{\pi}{2(1 - k)}, \sigma = 1 - k$. Therefore for $H = \frac{\pi}{2(1 - k)}$, the stability spectrum contains a pair of pure imaginary roots $\lambda_{1,2} = \pm i\sigma$ and for the remaining $\lambda_k$ the inequality $\text{Re} \lambda_k < 0$ is valid.

Now let $h = H(1 + \gamma \varepsilon)$. Then from the characteristic equation (2.2) we obtain $\lambda_0' = d\lambda(\varepsilon)_{|\varepsilon=0} = \lambda_0' + i\sigma_0'$, and the equality $\lambda_0' = -i\sigma H \lambda_0' + H \sigma^2 \gamma$, which allows us to determine $\lambda_0'$. In our case we acquire the following equality: $\lambda_0' = 2\pi(1 - k)\gamma/(4 + \pi^2)$. Note that if we choose the constant $\gamma = 1$, then the inequality $\lambda_0' > 0$ will hold true. In this case, as $h$ increases, the roots $\pm i\sigma$ go to the right half-plane of the complex plane. Consequently, the zero solution of the auxiliary equation (2.1) loses its stability; in particular, the existence of a stable cycle is possible. The last question will be studied in the next section.

3. Periodic Solutions

Our intents of this section is to study the existence and stability of cycles of Equation (1.5). In Equation (1.5) we set:

$$t = \frac{h(\varepsilon)}{H}, \quad h(\varepsilon) = H(1 + \gamma \varepsilon), \quad \gamma \in R, \varepsilon \in (0, \varepsilon_0), 0 < \varepsilon_0 < 1, H = \frac{\pi}{2(1 - k)}.$$  \hspace{1cm} (3.1)

Notice that $\Theta$ is a new time. As a result Equation (1.5) is rewritten in the form

$$u' = (1 + \gamma \varepsilon)[-v + kv + \frac{k(k - 1)}{2}v^2 + \frac{k(k - 1)(k - 2)}{6}v^3 + o(v^3)],$$  \hspace{1cm} (3.1)

where $v = u(\Theta - H)$. In the neighborhood of the zero equilibrium state Equation (3.1) has a two-dimensional smooth invariant manifold $M_2(\varepsilon)$ [10]. In this case, all solutions of Equation (3.1) approach it with the velocity of the exponent with time if their initial conditions are small. The dynamics of the solutions of Equation (3.1) is restored after analyzing the system of two ordinary differential equations - normal form (NF). In the assumed case, NF can be written in complex form [6-8]

$$z' = (\alpha + i\beta)z + (d + ic)|z|^2 + O(\varepsilon),$$  \hspace{1cm} (3.2)
where $\alpha, \beta, d, c \in R$. These coefficients can be written out in an explicit form, which will be done below after the implementation of the algorithm for constructing the defining equation, which is commonly called NF. These coefficients depend on the parameters of Equation (3.1). In our case, these are $k$ and $H$. For this purpose, recently it has been customary to use the adaptation of the Krylov-Bogolyubov algorithm to infinite-dimensional dynamical systems [6-8]. In the NF (3.2) $z = z(s)$ is a complex function, and $s = \varepsilon \Theta, \varepsilon \in (0, \varepsilon_0)$. If we assume that the first Lyapunov value $d$ is non-zero a priori ($d \neq 0$), then the solution of Equations (3.1) with initial conditions in a small neighborhood of the zero solution can be expediently sought in the following form [6-8]

$$u(\Theta, \varepsilon) = \varepsilon^{1/2}u_1(\Theta, z) + \varepsilon u_2(\Theta, z) + \varepsilon^{3/2}u_3(\Theta, z) + O(\varepsilon^2), \quad (3.3)$$

where $u_1(\Theta, z) = z(s) \exp(i\sigma\Theta) + \varepsilon z(s) \exp(-i\sigma\Theta), \ z(s)$ is one of the solutions of the NF. The sufficiently smooth functions $u_2(\Theta, z), u_3(\Theta, z)$ with respect to the variable $\Theta$ have period $2\pi/\sigma$ and

$$M_{\pm}(u_m) = \frac{\sigma}{2\pi} \int_0^{2\pi/\sigma} u_m(\Theta, z) \exp(\pm i\sigma\Theta)d\Theta = 0, \ m = 2, 3.$$ 

We substitute the sum (3.3) into Equation (3.1) and equate the coefficients for the powers $\varepsilon, \varepsilon^{3/2}$. Hence we obtain two linear nonhomogeneous delayed differential equations. Thus, to determine $u_2(\Theta, z)$, we obtain the following equation

$$\frac{\partial u_2}{\partial \Theta} + (1 - k)v_2 = \Phi_2(\Theta, z), \quad (3.4)$$

where $\Phi_2(\Theta, z) = \frac{k(k - 1)}{2}v_1^2, v_2 = u_2(\Theta - H, z), v_1(\Theta) = u_1(\Theta - H)$. Deriving the equation for $u_3(\Theta, z)$, one can note that $\frac{d}{d\Theta} \psi(\Theta, s) = \frac{\partial \psi}{\partial \Theta} + \frac{\partial \psi}{\partial s} \varepsilon$. Therefore, we obtain the equation

$$\frac{\partial u_3}{\partial \Theta} + v_3 - kv_3 = \Phi_3(\Theta, z), \quad (3.5)$$

$$\Phi_3(\Theta, z) = k(k - 1)v_1v_2 + \frac{k(k - 1)(k - 2)}{6}v_1^3 + \gamma(k - 1)v_1 -$$

$$- \varepsilon (\frac{\pi}{2} \exp(i\sigma\Theta) + \varepsilon (\frac{\pi}{2} \exp(-i\sigma\Theta) - \varepsilon' \exp(i\sigma\Theta) - \varepsilon' \exp(-i\sigma\Theta)) \varepsilon.$$

Comment. Linear Differential Equation with a deviating argument

$$\frac{du}{d\Theta} + (1 - k)u(t - H) = \Phi(\Theta), \ H = \frac{\pi}{2(1 - k)},$$

where $\Phi(\Theta)$ is a $2\pi/\sigma$ periodic function, has periodic solutions with the same period if

$$M_{\pm}(\Phi(\Theta)) = \frac{\sigma}{2\pi} \int_0^{2\pi/\sigma} \Phi(\Theta) \exp(\pm i\sigma\Theta)d\Theta = 0.$$ 

The equalities of $M_{\pm}(u) = 0$ single one such solution.

Therefore, in the case of Equation (3.4), when

$$\Phi_2(\Theta, z) = \frac{k(1 - k)}{2} [z^2 \exp(2i\sigma\Theta) - 2|z|^2 + \pi^2 \exp(-2i\sigma\Theta)],$$
the solution of this equation should be selected in the following form

\[ u_2(\Theta, z) = \eta_2 z^2 \exp(2i\sigma\Theta) + \eta_0|z|^2 + \eta_2 z^2 \exp(-2i\sigma\Theta), \]

where \( \eta_2 \in C, \ \eta_0 \in R. \) After substituting \( u_2(\Theta, z) \) in the selected form in the corresponding equation we determine that \( \eta_0 = -k, \ \eta_2 = -\frac{k}{10}(1 + 2i). \) We now analyze the nonhomogeneous differential equation (3.5). From the solvability conditions for this equation in the class \( 2\pi/\sigma \) of periodic functions, we obtain the following equation for \( z(s) \)

\[ z' = (\alpha + i\beta)z + (d + ic)|z|^2, \quad (3.6) \]

In our case we get

\[ \alpha = 2\pi\gamma\frac{1 - k}{4 + \pi^2}, \ \beta = 4\pi\frac{1 - k}{4 + \pi^2}, \ d = -\frac{4k(1-k)}{5(4 + \pi^2)}[(3\pi/2 - 1)k + 5\pi], \ c = -\frac{4k(1-k)}{5(4 + \pi^2)}[5 + (3 + \pi/2)k]. \]

We emphasize that Equation (3.6) is a “shorter” version of the NF. Note that for all \( k < 0 \) under the assumption of the first Lyapunov value \( d < 0. \)

**Lemma.** Equation (3.6) has a periodic solution

\[ z(s) = \rho \exp(i\nu s), \]

where \( \rho = \sqrt{\gamma} \frac{\pi}{2k}[(\frac{3\pi}{2} - 1)k + \frac{5\pi}{2}]^{-1}, \ \nu = \beta + c\rho^2, \) if \( \gamma > 0. \) Moreover, this periodic solution is stable while the zero solution is unstable. For \( \gamma < 0 \) this equation has no non-trivial periodic solutions and the zero solution of the NF is asymptotically stable.

Now by substituting \( z(s) = \rho \exp(i\nu s) \) into Equation (3.6) leads to a system of algebraic equations for determining \( \rho \) and \( \nu \)

\[ \alpha \rho + d\rho^3 = 0, \ \nu = \beta + c\rho^2. \]

The first equation of the given system has a non-trivial solution \( \rho > 0, \) if \( \alpha \) and \( d \) of different signs and only the zero solution if the signs \( \alpha, d \) coincide. The stability analysis of a non-trivial periodic solution is described by the standard scheme.

Now we set \( z = \rho \exp(i\nu s)(1 + w). \) The linearized version for the vector-valued function \( \eta = (w_1, w_2) (w = w_1 + iw_2) \) has the following form

\[ \eta' = B\eta, B = \begin{pmatrix} dp^2 & 0 \\ d\rho^2 & 0 \end{pmatrix}. \]

One eigenvalue of the matrix is \( \lambda_1 = d\rho^2 < 0, \) if \( d < 0 (\alpha > 0) \) and this is an eigenvalue of \( \lambda_1 > 0, \) if \( d > 0 (\alpha < 0) \), and the second eigenvalue of the matrix is \( \lambda_2 = 0. \) The Andronov-Witt theorem implies the validity of the lemmas assertion. The results from [10-13] imply the validity of the assertion.

**Theorem.** There exists \( \varepsilon_0 > 0, \) such that for all \( \varepsilon \in (0, \varepsilon_0) \) Equation (3.1) for \( \gamma > 0 \) has a stable (orbitally asymptotically stable) limit cycle corresponding to the NF cycle with the following asymptotic formula

\[ u_2(\Theta, \varepsilon) = \varepsilon^{1/2}[\rho \exp(i(\sigma + \varepsilon \nu)\Theta + i\varphi) + \exp(-i(\sigma + \varepsilon \nu)\Theta - i\varphi)] + +\varepsilon \rho^2[\eta_2 \exp(2i(\sigma + \varepsilon \nu)\Theta + 2i\varphi) + \eta_0 + \eta_2 \exp(-2i(\sigma + \varepsilon \nu)\Theta - 2i\varphi) + o(\varepsilon)], \]

where \( \varphi \in R \) and the constants \( \rho, \sigma, \nu, \eta_0, \eta_2 \) were indicated earlier.
These substitutions allow us to transfer the results of the theorem 1 to Equation (1.5). Now let \( h = H(1 + \varepsilon) \). Then the periodic solution \( u_*(\Theta, \varepsilon) \) corresponds to the periodic solution of Equation (1.5)

\[
x_*(t, \varepsilon) = 1 + u_*(\frac{t}{1 + \varepsilon}, \varepsilon) (\gamma = 1).
\]

Naturally, the solution \( x_*(t, \varepsilon) \) is stable in the sense of A.M. Lyapunov.

Also notice that the periodic solutions depend on the selection of the parameter \( k \). For instance when \( k = 1 - \delta \) and \( \delta \) is sufficiently small, we obtain periodic solutions with long periods that can be interpreted as ”long waves ” N.D. Kondratiev (see also [14]).

4. Conclusions
In this paper we have shown that the delay effects can significantly change the dynamics of solutions in the classical Solow model. At least in the presented form of the correspondingly modified Solow Equation may appear to fluctuate prices, which is typical for pricing within a market economy.

A similar effect occurs if a delay is introduced into another classical model of macroeconomics - the ”supply-demand” model (market model). In this classical version, there are no oscillatory solutions but the assumption of the delay showed that under certain variations periodic cycles may exist. The selection of the parameters of the problem has a modified version of the equation that already has a periodic cycle ([7]).

In macroeconomics, a similar situation is reproduced for the one that occurred in mathematical ecology. If we consider the well-known Verhulst Equation

\[
\dot{N} = \alpha N(1 - N),
\]

then of course it does not exhibit periodic solutions. Notice that \( \alpha > 0, N = N(t) \) resembles the number of species. This is a significant contribution by Hutchinson [15], who proposed to study the following equation

\[
\dot{N} = \alpha N(1 - N(t - h)), \ h > 0,
\]

which is already capable of describing fluctuations in the population size in a single-species bio-sinopsis.

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5. References
[1] Solow R M 1956 The Quarterly Journal of Economics 70 65
[2] Swan T W 1956 Economic Record 32 334
[3] Zhang W B 1983 Synergetic Economics: Time and Change in Nonlinear Economics (Berlin-New-York: Springer-Verlag)
[4] Puu T 1997 Nonlinear Economic Dynamics (Berlin-New-York: Springer-Verlag)
[5] Ferrara M, Guerini L and Mavilla R 2013 Applied Mathematical Sciences 7 4249
[6] Kulikov A N and Kulikov D A 2015 Taurida Journal of Computer Science Theory and Mathematics 2 87
[7] Kulikov D A 2017 IOP Conf. Series: Journal of Physics: Conf. Series 788 6 p.
[8] Kulikov D A 2018 Zhurnal SVMO 20 225
[9] Hale J 1977 Theory of functional differential equations (Berlin-New-York: Springer-Verlag)
[10] Kulikov 1976 Studies of stability and the theory of oscillation 67
[11] Kolesov Yu S 1974 Vestnik Yaroslavl. univ. 7 3
[12] Kulikov Yu S and Svirita D 1976 Differential equations and applications 16 41
[13] Kolesov A Yu, Kulikov A N and Rozov N Kh 2003 Differ. Equ. 39 614
[14] Kondratiev N D 1993 Special opinion. Book 2 (Moscow: Nauka(in Russian))
[15] Hutchinson G E 1948 Annals of the New York Academy of Sciences 50 221