Gersten weight structures for motivic homotopy categories; direct summands of cohomology of function fields, and coniveau spectral sequences

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Abstract

In this paper for any cohomology theory $H$ that can be factored through (the Morel-Voevodsky’s triangulated motivic homotopy category) $SH^{S^1}(k)$ we establish the $SH^{S^1}(k)$-functoriality of coniveau spectral sequences for $H$. We also prove the following interesting result: for any affine essentially smooth semi-local $S$ the Cousin complex for $H^*(S)$ splits; if $H$ also factors through $SH^+(k)$ or $SH^{MGL}(k)$, then this is also true for any primitive $S$. Moreover, the cohomology of such an $S$ is a direct summand of the cohomology of any its open dense subscheme.

In order to prove these facts we construct a triangulated category $\mathcal{D}$ of motivic pro-spectra that contains $SH^{S^1}(k)^c$ as well as certain pro-spectra of function fields over $k$ (together with its $SH(k)$-version $\mathcal{D}^T$, and certain $\mathcal{D}^+$ and $\mathcal{D}^{MGL}$). We decompose the $SH^{S^1}(k)$-spectrum of a smooth variety (in the sense of Postnikov towers) into twisted (pro)spectra of its points. Those belong to the heart of a Gersten weight structure $w$ on $\mathcal{D}$, whereas weight spectral sequences corresponding to $w$ generalize the ’classical’ coniveau ones (to the cohomology of arbitrary objects of $SH^{S^1}(k)^c$). When a cohomological functor is represented by a $Y \in \text{Obj} \ SH^{S^1}(k)$, the corresponding coniveau spectral

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sequences can be expressed in terms of the (homotopy) $t$-truncations of $Y$; this extends to motivic spectra the seminal coniveau spectral sequence computations of Bloch and Ogus. We also prove several (other) direct summand results. In particular, the pro-spectra of function fields (in $\mathcal{D}$ or $\mathcal{D}^T$) contain twisted pro-spectra of their residue fields (for all geometric valuations); hence the same is true for any cohomology of these fields.

The analogues of these results for Voevodsky’s motives (i.e., for $DM_{eff}^{gp}(k)$ instead of $SH^{A^1}(k)$) over countable fields were proved by the author in a previous paper. In the current paper we use model categories (instead of differential graded ones) for the construction of the corresponding $\mathcal{D}$, and apply a new weight structure construction result (for cocompactly cogenerated triangulated categories). We also prove a certain $SH^+(k)$-acyclicity statement for primitive schemes; this result could be interesting for itself.

Contents

1 Preliminaries: $t$-structures, Postnikov towers, and $A^1$-connectivity of spectra
   1.1 Notation and conventions ............................................. 7
   1.2 $t$-structures, Postnikov towers, and idempotent completions . 8
   1.3 On open pairs and pro-schemes ...................................... 11
   1.4 $SH^{A^1}(k)$ and the homotopy $t$-structure on it: reminder . . 12
   1.5 Primitive (pro)schemes: definition ................................. 14

2 Weight structures: reminder and the case of cocompactly cogenerated categories
   2.1 Basic definitions and properties ..................................... 18
   2.2 Weight structures on cocompactly cogenerated categories ....... 19
   2.3 Weight filtrations and spectral sequences ........................... 23
   2.4 The relation with orthogonal $t$-structures .......................... 29

3 Pro-spectra and the Gersten weight structure
   3.1 Pro-spectra: an “axiomatic description” ............................. 32
   3.2 The Gysin distinguished triangle and ‘Gersten’ Postnikov towers for the pro-spectra of pro-schemes ................................. 33
   3.3 The Gersten weight structure: construction and basic properties 37
3.4 Direct summand results for pro-spectra of semi-local schemes

4 On cohomology and coniveau spectral sequences
4.1 Extending cohomology from $SH^S_1(k)^c$ to $\mathcal{D}$
4.2 On cohomology of pro-schemes, and its direct summands
4.3 Coniveau spectral sequences for the cohomology of (pro)spectra
4.4 A duality of motivic spectral categories; comparing spectral sequences
4.5 'Simple' coniveau spectral sequences for the cohomology of Artin-Tate spectra
4.6 On pure extended cohomology theories

5 Conclusion of the proofs: the construction of $\mathcal{D}'$ and $\mathcal{D}^{big}$
5.1 On the levelwise injective model for $SH^S_1(k)^c$: reminder
5.2 $\mathcal{D}'$ and $\mathcal{D}^{big}$: definition and properties

6 The $T$-spectral and $MGl$-module versions of the main results; examples and remarks
6.1 The $T$-spectral Gersten weight structures
6.2 The '$\tau$-positive acyclicity' of primitive schemes
6.3 On the Gersten and Chow weight structures for modules over the algebraic cobordism spectrum
6.4 Our methods and comotives
6.5 On concrete examples of cohomology theories
6.6 Other possibilities for $\mathcal{D}$

Introduction

In [Bon10b] for a countable perfect field $k$ and any cohomology theory $H$ that factors through the Voevodsky’s $DM^eff_{gm}(k)$ it was proven: coniveau spectral sequences (for the cohomology of smooth varieties) can be functorially extended to $DM^eff_{gm}(k)$; the cohomology of an essentially smooth affine semi-local scheme $S/k$ (and more generally, of a primitive pro-scheme) is a retract of the cohomology of any its pro-open dense subscheme. Note that both of these results are far from being obvious; the second one yields that the augmented Cousin complex for the cohomology of an (affine essentially smooth) semi-local (or primitive) $S$ splits. Also, for an $H$ represented by an
object $C$ of $DM_{eff}^c(k)$ (a motivic complex) it was proved that coniveau spectral sequences for $H$ can be 'motivically functorially' expressed in terms of the $t$-truncations of $C$ (with respect to the homotopy $t$-structure on $DM_{eff}^c(k)$); this is an extension of an important result of [BlO94].

The goal of the current paper is to extend these results to arbitrary perfect base fields and to a much wider class of cohomology theories including $K$-theory of various sorts, algebraic, semi-topological, and complex cobordism, and Balmer’s Witt cohomology. We succeed in proving these results via considering certain triangulated categories $\mathcal{D} \supset SH^{S^1}(k)^c$ and $\mathcal{D}^T \supset SH(k)^c$ (as well as $\mathcal{D}^+ \supset SH^+(k)^c$ and $\mathcal{D}^{MGl} \supset SH^{MGl}(k)^c = Ho(MGl - Mod(k))^c$), and introducing certain Gersten weight structures on them. These weight structures are orthogonal to the homotopy $t$-structures for the corresponding motivic categories (via certain nice dualities that we construct). The formalism of weight structures easily yields several functoriality and direct summand results (cf. Theorem 6.1.2(II13–16, III) below). Note that we have several versions of these statements, depending on the choice of a 'motivic' category through which a given cohomology theory $H$ factors. We discuss this matter (together with several 'concrete' examples of cohomology theories) in \S\S 6.5 since all oriented cohomology theories factor through $SH^{MGl}(k)$, we can apply the strongest versions of our results to them. In any case, our direct summand results are much stronger than the universal exactness statement of [CHK97] (see Remark 4.2.2 for more details). Besides, though the setting of ibid. seems to be somewhat more general, the author does not know of any concrete example of a cohomology theory that satisfies the conditions of ibid. and is not representable by an object of $SH^{S^1}(k)$.

We also note that our consideration of (triangulated) pro-spectral categories (together with the purity distinguished triangle for pro-schemes; see Proposition 3.2.2) can be interesting for itself independently from the rest of the paper. It seems that the only triangulated motivic categories of pro-objects considered in the existing literature are the comotivic one of [Bon10b] and the pro-spectral motivic categories studied in the current paper; note in contrast that the category of pro-motives introduced in \S 3.1 of [Deg08a] is not triangulated. Our methods of constructing motivic pro-spectra are far from being original (we mostly use the results of [FaI07] and related papers); yet this has an advantage: they certainly allow the construction of motivic pro-spectra over an arbitrary base scheme $S$.

Another interesting result is our Proposition 6.1.1(6) (on the '$\tau$-positive spectral acyclicity' of primitive schemes).
Besides, in the current paper we study (in detail) weight structures for cocompactly cogenerated triangulated categories (certainly, this can be dualized); this seems to be a rather important new piece of (more or less) abstract homological algebra. Whereas the (first) existence of a weight structure statement in Theorem 2.2.6 is not quite new (and was essentially proved earlier by D. Pauksztello; yet we state it in a more explicit form), several other results are. Note also that our current methods for working with these weight structures are much more interesting than their (motivic ‘bounded’) analogues in [Bon10b] (and so can be used in various settings); see §6.4 below for more remarks on this matter. In particular, it seems that the dual of Theorem 2.2.6 (on weight structures in a compactly generated triangulated category) can be useful for various applications. Also, Proposition 4.6.1 describing pure extended cohomology theories is quite new.

We also mention the relation of our results with certain ones of F. Deglise (see [Deg13b] and [Deg13a]). The methods and results of the papers mentioned (seem to) allow the description of coniveau spectral sequences for \( SH^{S^1}(k) \)-representable theories in terms of the homotopy \( t \)-structure on \( SH^{S^1}(k) \) (see Remark 4.4.2 of [Deg13a] for the \( SH(k) \)-version of this result). Yet the corresponding isomorphisms depend on choices of certain ‘enhancements’ for cohomology; so they only yield the functoriality of these isomorphisms with respect to flat pullbacks and proper pushforwards (note that several previous results on this subject also have similar drawbacks). Besides, the results of Deglise definitely do not yield our direct summands statements.

Lastly, note that we prove (see Proposition 4.3.1(II.3) and Theorem 6.1.2(II.8)); any possible image of any (extended) cohomology of a compact motivic spectrum belonging to \( SH^{S^1}(k)^{t \leq -r} \) or to \( SH(k)^{t \leq -r} \) (i.e., of a spectrum that is \( r - 1 \)-connected in the sense of [Mor03], for any \( r > 0 \)) in the cohomology of a smooth variety is supported in codimension \( \geq r \).

Now we list the contents of the paper. Some more information of this sort can be found at the beginnings of sections (also we list some of the main notation in the end of §1.1).

In §1 we remind some notation (for triangulated categories and related matters); we also introduce pro-schemes (certain projective limits of pointed smooth varieties) and recall certain properties of the \( \mathbb{A}^1 \)-stable homotopy category, the homotopy \( t \)-structure on it, and remind related properties of the corresponding cohomology of semi-local schemes.

In §2 we recall the formalism of weight structures, their relation with (orthogonal) \( t \)-structures, weight filtrations and spectral sequences. We also
prove several new results on weight structures in cocompactly cogenerated triangulated categories, and give a description of pure cohomology for triangulated categories endowed with weight structures.

In §3 we embed $SHS^1(k)$ into a certain triangulated category $\mathcal{D}^{big}$ of pro-spectra. In this section we only give an 'axiomatic' description of pro-spectra; its construction will be described in §5. We use the main properties of $\mathcal{D}^{big}$ for the construction of a certain Gersten weight structure $w$ on $\mathcal{D} \subset \mathcal{D}^{big}$ ($\mathcal{D}$ is the subcategory of $\mathcal{D}^{big}$ cogenerated by $SHS^1(k)^c$; hence it is the largest subcategory of $\mathcal{D}^{big}$ 'detected by the Gersten weight structure'). $w$ possesses several nice properties; in particular, the pro-spectra of function fields and (affine essentially smooth) semi-local schemes over $k$ belong to $\mathcal{D}_{w=0}$ (and 'cogenerate' it), whereas the pro-spectra of arbitrary pro-schemes belong to $\mathcal{D}_{w\geq 0}$. We also easily prove: if $S$ is a semi-local scheme (and $k$ is infinite), $S_0$ is its dense sub-pro-scheme, then $\Sigma^\infty(S_+)$ is a direct summand of $\Sigma^\infty(S_{0+})$; $\Sigma^\infty(\text{Spec } K_+)$ (for a function field $K/k$) contains (as retracts) the pro-spectra of semi-local schemes whose generic point is $\text{Spec } K$, as well as the twisted pro-spectra of residue fields of $K$ (for all geometric valuations). Note that all our arguments easily carry over to the 'stable' setting (i.e., to $SH(k)$ instead of $SHS^1(k)$); we discuss this matter in §6.

§4 is central for this paper. We translate the results of the previous section to cohomology; in particular, we prove that the augmented Cousin complex for the cohomology of a semi-local pro-scheme splits. We also prove that weight spectral sequences and filtrations corresponding to $w$ are canonically isomorphic to the 'usual' coniveau ones (for the cohomology of smooth varieties). On the other hand, the general theory of weight spectral sequences yields that the corresponding spectral sequence $T(H, M)$ converging to $H^*(M)$ (for $M$ being an object of $SHS^1(k)^c$ or $\mathcal{D}$) is $SHS^1(k)^c$-functorial in $M$ starting from $E_2$; this is far from being trivial from the 'classical' definition of coniveau spectral sequences. Next we construct a nice duality $\Phi : \mathcal{D}^{op} \times SHS^1(k) \to \text{Ab}$ (one may say that this is a 'regularization' for the corresponding restriction of the bifunctor $\mathcal{D}^{big}(-,-)$). Since $w$ is orthogonal to $t$ with respect to $\Phi$, our (generalized) coniveau spectral sequences can be expressed in terms of $t$ (starting from $E_2$); this vastly generalizes the corresponding results of Bloch and Ogus. We also explain that for certain varieties and spectra one can choose quite 'economical' versions of weight Postnikov towers and (hence) of generalized coniveau spectral sequences for cohomology. We conclude the section by a description of all pure extended cohomology theories in terms of $Hw$, and prove the equivalence of $Ht$ with the
category of contravariant additive functors \( Hw \to Ab \) converting all products into coproducts.

In §5 we consider certain model categories and construct the categories \( \mathcal{D}' \) and \( \mathcal{D}'^{\text{big}} = \text{Ho}(\mathcal{D}') \) satisfying the properties listed in §3.

In §6 we describe certain possible variations of our methods and results. In particular, we prove the natural analogues of our main results for the triangulated categories of (\( \tau \)-positive) \( T \)-spectra and \( MGl \)-modules. It turns out that for the corresponding pro-spectral categories \( \mathcal{D}^+ \) and \( \mathcal{D}^{MGl} \) the hearts of the corresponding Gersten weight structures contain the pro-spectra of primitive schemes; so we obtain 'more splitting results' than in the \( SH^{St}(k) \)-setting. We also (briefly) compare our methods with the ones of \cite{Bon10b}; the current methods yield that the results of ibid. are also valid in the case when \( k \) is not (necessarily) countable.

Next we explain the consequences of our results for 'concrete' cohomology theories (algebraic, topological, and Hermitian \( K \)-theory, Balmer’s Witt groups, singular, algebraic and complex cobordism, motivic and étale cohomology). We finish the paper by a discussion of some other possible methods for constructing (certain modifications of) our pro-spectral categories.

Our notation below partially follows the one of \cite{Mor03}. All 'concrete' \( t \)-structures considered in this paper are certain versions of (Morel’s) homotopy ones (yet our general notation for \( t \)-structures is quite distinct from the one of ibid., and we introduce it in §1.2).

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1 Preliminaries: \( t \)-structures, Postnikov towers, and \( \mathbb{A}^1 \)-connectivity of spectra

In this section we remind some notation, introduce some new one, and recall some facts on motivic homotopy categories.

In §1.1 we introduce some (mostly, categorical) notation.
In §1.2 we recall the notion of \( t \)-structure (and introduce some notation
for it) and of a Postnikov tower for an object of a triangulated category.

In §1.3 we define the category $\mathcal{OP}$ (of open embeddings in $SmVar$) and $\text{Pro} - \mathcal{OP}$ (certain objects of the latter are called pro-schemes).

In §1.4 we recall some properties of $SH^{S^1}(k)$ and the homotopy $t$-structure on it.

In §1.5 we define primitive (pro)schemes (we will need them in §6).

1.1 Notation and conventions

For categories $C, D$ we write $D \subset C$ if $D$ is a full subcategory of $C$.

For a category $C$, $X, Y \in \text{Obj}C$, we denote by $C(X, Y)$ the set of $C$-morphisms from $X$ to $Y$. We will say that $X$ is a retract of $Y$ if $\text{id}_X$ can be factored through $Y$. Note: if $C$ is triangulated or abelian then $X$ is a retract of $Y$ if and only if $X$ is its direct summand.

For any $D \subset C$ the subcategory $D$ is called Karoubi-closed in $C$ if it contains all retracts of its objects in $C$. We will call the smallest Karoubi-closed subcategory of $C$ containing $D$ the Karoubi-closure of $D$ in $C$; sometimes we will use the same term for the class of objects of the Karoubi-closure of a full subcategory of $C$ (corresponding to some subclass of $\text{Obj}C$).

The Karoubization $\text{Kar}(B)$ (no lower index) of an additive category $B$ is the category of “formal images” of idempotents in $B$ (so $B$ is embedded into an idempotent complete category; it is triangulated if $B$ is). We will say that $B$ is Karoubian if the canonical embedding $B \to \text{Kar}(B)$ is an equivalence, i.e., if any idempotent morphism yields a direct sum decomposition in $B$.

For a category $C$ we denote by $C^{\text{op}}$ its opposite category.

For an additive category $\underline{C}$ an $X \in \text{Obj}\underline{C}$ is called cocomplete if $\underline{C}(\prod_{i \in I} Y_i, X) = \bigoplus_{i \in I} \underline{C}(Y_i, X)$ for any set $I$ and any $Y_i \in \text{Obj}\underline{C}$ (below we will only consider cocomplete objects in categories closed with respect to arbitrary small products). Dually, a compact object of $\underline{C}$ is a cocomplete object of $\underline{C}^{\text{op}}$, i.e., $M$ is compact (in a $\underline{C}$ closed with respect to arbitrary small coproducts) if $\underline{C}(M, -)$ respects coproducts.

For $X, Y \in \text{Obj}\underline{C}$ we will write $X \perp Y$ if $\underline{C}(X, Y) = \{0\}$. For $D, E \subset \text{Obj}\underline{C}$ we will write $D \perp E$ if $X \perp Y$ for all $X \in D$, $Y \in E$. For $D \subset \underline{C}$ we will denote by $D^{\perp}$ the class $\{Y \in \text{Obj}\underline{C} : X \perp Y \ \forall X \in D\}$.

Sometimes we will denote by $D^{\perp}$ the corresponding full subcategory of $\underline{C}$.

Dually, $^{\perp}D$ is the class $\{Y \in \text{Obj}\underline{C} : Y \perp X \ \forall X \in D\}$.
In this paper all complexes will be cohomological, i.e., the degree of all differentials is +1; respectively, we will use cohomological notation for their terms. We will need the following easy observation on the homotopy category of an arbitrary additive category $B$.

**Lemma 1.1.1.** Suppose a complex $M = (M^i) \in \text{Obj}K(B)$ is a $K(B)$-retract of an object of $B$ (i.e., of a complex of the form $\ldots 0 \to 0 \to M' \to 0 \to 0 \to \ldots$ for some $M' \in \text{Obj}B$; we denote it by $M'[0]$). Then $M$ is also a retract of $M^0[0]$.

**Proof.** Suppose that the factorization $M \xrightarrow{f} M'[0] \xrightarrow{g} M$ yields the $K(B)$-retraction mentioned. Then $h = g \circ f$ cannot have non-zero components in non-zero degrees, and we can consider it as a morphism $M \to M^0[0]$ and vice versa. Hence $h$ yields the retraction in question. \hfill $\square$

$\mathcal{C}$ and $\mathcal{D}$ will usually denote some triangulated categories. We will use the term ‘exact functor’ for a functor of triangulated categories (i.e., a functor that preserves the structures of triangulated categories).

A class $D \subset \text{Obj}\mathcal{C}$ will be called extension-closed if $0 \in D$ and for any distinguished triangle $A \to B \to C$ in $\mathcal{C}$ we have: $A, C \in D \implies B \in D$. In particular, an extension-closed $D$ is strict (i.e., contains all objects of $\mathcal{C}$ isomorphic to its elements).

The smallest extension-closed $D$ containing a given $D' \subset \text{Obj}\mathcal{C}$ will be called the extension-closure of $D'$.

We will call the smallest Karoubi-closed triangulated subcategory $\mathcal{D}$ of $\mathcal{C}$ such that $\text{Obj}\mathcal{D}$ contains $D'$ the triangulated subcategory generated by $D'$.

$\mathcal{A}$ will usually denote some abelian category. We will usually assume that $\mathcal{A}$ satisfies AB5, i.e., is closed with respect to all small coproducts, and filtered direct limits of exact sequences in $\mathcal{A}$ are exact.

We will call a covariant additive functor $\mathcal{C} \to \mathcal{A}$ for an abelian $\mathcal{A}$ homological if it converts distinguished triangles into long exact sequences; homological functors $\mathcal{C}^{\text{op}} \to \mathcal{A}$ will be called cohomological when considered as contravariant functors $\mathcal{C} \to \mathcal{A}$.

$H : \mathcal{C} \to \mathcal{A}$ below will always be cohomological.

We will often specify a distinguished triangle by two of its arrows. For $f \in \mathcal{C}(X, Y), X, Y \in \text{Obj}\mathcal{C}$, we will call the third vertex of (any) distinguished triangle $X \xrightarrow{f} Y \to Z$ a cone of $f$. 

9
For a class of objects $C_i \in \text{Obj}_C$, $i \in I$, we will denote by $\langle C_i \rangle$ the smallest strict (see above) full triangulated subcategory containing all $C_i$; for $D \subset C$ we will write $\langle D \rangle$ instead of $\langle C : C \in \text{Obj}_D \rangle$.

We will say that $C_i$ cogenerate $C$ if $C$ is closed with respect to small products and coincides with its smallest strict triangulated subcategory that fulfills this property and contains $C_i$ (in [Bon10b] $C_i$ were called weak cogenerators of $C$).

For additive categories $C, D$ we denote by $\text{AddFun}(C, D)$ the category of additive functors from $C$ to $D$ (we will ignore set-theoretic difficulties here since we will mostly need the categories of functors from those $C$ that are skeletally small).

$\Ab$ is the category of abelian groups.

For a category $C$ and a partially ordered index set $I$ we will call a set $X_i \subset \text{Obj}_C$ a (filtered) projective system if for any $i, j \in I$ there exists some maximum, i.e., an $l \in I$ such that $l \geq i$ and $l \geq j$, and for all $j \geq i$ in $I$ there are fixed morphisms $X_j \to X_i$ satisfying the natural functoriality property.

For such a system we have the natural notion of an inverse limit. Dually, we will call the inverse limit of a system of $X_i \in \text{Obj}_C$ the direct limit of $X_i$ in $C$.

All limits, colimits, and pro-objects in this paper will be filtered ones.

$k$ will be our perfect base field of characteristic $p$; $p$ could be $0$.

We also list some more definitions and the main notation of this paper.

t-structures (together with $H^t$) and Postnikov towers are considered in §1.2; the categories $\mathcal{OP} \subset \text{Pro-OP}$, pro-schemes, and their ‘twists’ $X_+ \langle j \rangle$ are defined in §1.3 (whereas some additional conventions for them are contained in Remark 3.2.1); $\text{SH}^S(k)$ and the $t$-structure $t$ on it, the functor $\Sigma^\infty$, $\Sigma^\infty$, the ‘twists’ $\langle j \rangle$ and $\{ j \}$, semi-local pro-schemes, and normal bundles $N_{X,Z}$ are mentioned in §1.4; primitive (pro)schemes are defined in §1.5; weight structures (together with $C_w \leq i$, $C_w = i$, $C_w \geq i$, $H^w$, $C_{[i,j]}$, and weight complexes), negative subcategories and (positive) weight Postnikov towers are defined in §2.1; coenvelopes and countable homotopy limits are introduces in §2.2; weight filtrations $W^k H^m$, weight spectral sequences $T(H, M)$, and pure cohomology theories are defined in §2.3; (nice) dualities $\Phi$ of triangulated categories and orthogonal weight and $t$-structures are defined in §2.4; the categories $\text{PSH}^S(k)$, $\mathcal{D}'$, and $\mathcal{D}^{bg} = \text{Ho}(\mathcal{D}')$ (together with the extension of $\Sigma^\infty$ to $\text{Pro-OP}$) are introduced in §3.1 (and constructed in §5.2); our main (Gersten) weight structure $w$ (for the category $\mathcal{D}$) is constructed in §3.3; we introduce extended cohomology theories in §4.1; Cousin complexes
$T_H(-)$ are studied in §4.2; cocompact functors $Hw \to A$ and strictly homotopy invariant Nisnevich sheaves are considered in §4.4; $SH(k), \Sigma^\infty, SH^+(k), \eta,$ and $\tau$ are considered in §6.1 (weakly) orientable spectra are studied in §6.2 (and related to the Hopf element $\eta$); $MGl, MGl - \text{Mod}(k), SH^{MGl}(k) = \text{Ho}(MGl - \text{Mod}(k)),$ and $\mathcal{O}^{MGl}$ are considered in §6.3; the category $\mathcal{O}^{\text{mot}}$ of comotives is mentioned in §6.4.

1.2 \textit{t}-structures, Postnikov towers, and idempotent completions

In order to fix the notation, we recall the definition of a $t$-structure.

\textbf{Definition 1.2.1.} A pair of subclasses $\mathcal{C}^t_{\geq 0}, \mathcal{C}^t_{\leq 0} \subset \text{Obj}\mathcal{C}$ for a triangulated category $\mathcal{C}$ will be said to define a $t$-structure $t$ if $(\mathcal{C}^t_{\geq 0}, \mathcal{C}^t_{\leq 0})$ satisfy the following conditions:

(i) $\mathcal{C}^t_{\geq 0}, \mathcal{C}^t_{\leq 0}$ are strict (i.e., contain all objects of $\mathcal{C}$ isomorphic to their elements).

(ii) $\mathcal{C}^t_{\geq 0} \subset \mathcal{C}^t_{\geq 0}[1], \mathcal{C}^t_{\leq 0}[1] \subset \mathcal{C}^t_{\leq 0}$.

(iii) \textbf{Orthogonality.} $\mathcal{C}^t_{\leq 0}[1] \perp \mathcal{C}^t_{\geq 0}$.

(iv) \textbf{$t$-decomposition.} For any $X \in \text{Obj}\mathcal{C}$ there exists a distinguished triangle

\[ A \to X \to B[-1] \to A[1] \tag{1} \]

such that $A \in \mathcal{C}^t_{\leq 0}, B \in \mathcal{C}^t_{\geq 0}$.

We will need some more notation.

\textbf{Definition 1.2.2.} 1. A category $Ht$ whose objects are $\mathcal{C}^t_{=0} = \mathcal{C}^t_{\geq 0} \cap \mathcal{C}^t_{\leq 0}$, $Ht(X, Y) = \mathcal{C}(X, Y)$ for $X, Y \in \mathcal{C}^t_{=0}$, will be called the \textit{heart} of $t$. Recall (cf. Theorem 1.3.6 of [BBD82]) that $Ht$ is abelian (short exact sequences in $Ht$ come from distinguished triangles in $\mathcal{C}$).

2. $\mathcal{C}^t_{=i}$ (resp. $\mathcal{C}^t_{\leq i}$) will denote $\mathcal{C}^t_{\geq 0}[-i]$ (resp. $\mathcal{C}^t_{\leq 0}[-i]$).

\textbf{Remark 1.2.3.} 1. Recall (cf. Lemma IV.4.5 in [GeM03]) that (1) defines additive functors $\mathcal{C} \to \mathcal{C}^t_{\leq 0} : X \to A$ and $\mathcal{C} \to \mathcal{C}^t_{\geq 0} : X \to B$. We will denote $A, B$ by $X^t_{\leq i}$ and $X^t_{\geq 1}$, respectively.

The triangle (1) will be called the \textit{$t$-decomposition of $X$}. If $X = Y[i]$ for some $Y \in \text{Obj}\mathcal{C}, i \in \mathbb{Z}$, then we will denote $A$ by $Y^t_{\leq i}$ (it belongs to $\mathcal{C}^t_{\leq 0}$) and $B$ by $Y^t_{\geq i+1}$ (it belongs to $\mathcal{C}^t_{\geq 0}$), respectively. Objects of the type $Y^t_{\leq i}[j]$ and $Y^t_{\geq i}[j]$ (for $i, j \in \mathbb{Z}$) will be called $t$-\textit{truncations} of $Y$. 

11
2. We denote by $X^{t=i}$ the $i$-th cohomology of $X$ with respect to $t$, that is $(X^{t\leq}t^{\geq})^{i \geq 0}$ (cf. part 10 of §IV.4 of [GeM03]).

3. The following statements are obvious (and well-known): $C^{t \leq 0} = \perp (C^{t \geq 1})$; $C^{t \geq 0} = (C^{t \leq -1}) \perp$.

4. Our conventions for $t$-structures and ‘weights’ (see Remark 2.1.2(3) below) follow the ones of [BBD82]. So, for any $n \in \mathbb{Z}$ ‘our’ $(C^{t \leq n}, C^{t \geq n})$ corresponds to $(C^{t \geq -n}, C^{t \leq -n})$ in the notation of the papers [Mor03], [Fal07], and [Deg13a].

Below we will need the notion of Postnikov tower in a triangulated category several times (cf. §IV.2 of [GeM03]; Postnikov towers are closely related (in an obvious way) to triangulated exact couples of §2.4.1 of [Deg13a]).

**Definition 1.2.4.** Let $C$ be a triangulated category.

1. Let $l \leq m \in \mathbb{Z}$.

We will call a **bounded Postnikov tower** for $M \in \text{Obj}C$ the following data: a sequence of $C$-morphisms $(0 =) Y_l \to Y_{l+1} \to \cdots \to Y_m = M$ along with distinguished triangles

$$Y_{i-1} \to Y_i \to M_i$$

for some $M_i \in \text{Obj}C$; here $l < i \leq m$.

2. An unbounded Postnikov tower for $M$ is a collection of $Y_i$ for $i \in \mathbb{Z}$ that is equipped (for all $i \in \mathbb{Z}$) with: connecting arrows $Y_{i-1} \to Y_i$ (for $i \in \mathbb{Z}$), morphisms $Y_i \to M$ such that all the corresponding triangles commute, and distinguished triangles (2).

In both cases, we will denote $M_{-p}[p]$ by $M^p$; we will call $M^p$ the **factors** of our Postnikov tower.

**Remark 1.2.5.** 1. Composing (and shifting) arrows from triangles (2) for two subsequent $i$ one can construct a complex whose terms are $X^p$ (it is easily seen that this is a complex indeed; see Proposition 2.2.2 of [Bon10a]).

2. Certainly, a bounded Postnikov tower can be easily completed to an unbounded one. For example, one could take $Y_i = 0$ for $i < l$, $Y_i = M$ for $i > m$; then $M^i = 0$ if $i < l$ or $i \geq m$.

### 1.3 On open pairs and pro-schemes

$\text{Var} \supset \text{SmVar} \supset \text{SmPrVar}$ will denote the class of all varieties over $k$, resp. of smooth varieties, resp. of smooth projective varieties. $\text{SmVar}$ is the category of smooth $k$-varieties.
We consider the category $\mathcal{OP}$ of open embeddings of smooth varieties over $k$ (we will denote the object corresponding to an embedding $U \to X$ as $X/U$). We have the obvious (componentwise) disjoint union operation for $\mathcal{OP}$.

For $X \in \text{SmVar}$ we will denote $X \sqcup \text{pt} / \text{pt}$ by $X_+$. We also define certain ("shifted Tate") twist on $\mathcal{OP}$: for $Z = X \setminus U$ we set $X/U(1) = X \times \mathbb{A}^1/(X \times \mathbb{A}^1 \setminus Z \times \{0\})$, so that for any $j > 0$ we have $X/U(j) = (X \times \mathbb{A}^j/(X \times \mathbb{A}^j \setminus Z \times \{0\}))$. This is a functor from $\mathcal{OP}$ into itself. Another functor is $\times Y$ for any fixed $Y \in \text{SmVar}$: $X/U \mapsto X \times Y/U \times Y$; we will be interested in the case $Y = \mathbb{A}^1$.

We will also need the category $\text{Pro}^{-\mathcal{OP}}$ of (filtered) pro-objects of $\mathcal{OP}$. Obviously, any functor $H : \mathcal{OP} \to A$ can be naturally extended to a functor from $\text{Pro}^{-\mathcal{OP}}$ (via direct limits) if $A$ is an abelian category satisfying AB5.

Below we will consider a special sort of objects of $\text{Pro}^{-\mathcal{OP}}$. Those may be called 'disjoint unions of intersections of smooth varieties'. Being more precise, we consider inverse limits of objects of the type $X_i^+$ (for $X_i \in \text{SmVar}$) such that the restriction of the structural morphism $X_i' \to X_i$ (for any $i \geq i'$) to any connected component of $X_i$ is either an open embedding or the projection to the 'extra point' of $X_i^+$. The corresponding object will be denoted by $X^+_+$; we will also write $X = \lim X_i$. Note that one can speak of the connected components of $X$ omitting the extra point (we will call them connected pro-schemes; they are projective limits of smooth connected varieties connected with open dense embeddings); we will write $X = \bigsqcup_{\alpha \in A} X^\alpha$. Note that connected pro-schemes (and their finite disjoint unions) often yield actual $k$-schemes.

We also note that any such union yields a pro-scheme; more generally, for any $j \geq 0$, $X^\alpha = \lim_{\to \alpha \in I_\alpha} X_i^\alpha$, we will consider the object

$$X_+^+(j) = (\bigsqcup_{\alpha \in A} X^\alpha)_+^+(j) = \lim_{\leftarrow B, i_\beta \in I_\beta \forall \beta \in B} (\bigsqcup_{\beta \in B} X_{i_\beta})^+_+(j)$$

for $B$ running through finite subsets of $A$. Here the transition maps go from $X_{i_\beta}(j)$ to $X_{i_\beta}'(j)$ for $i_\beta' \leq i_\beta \in I_i$ or to the 'extra' $\text{pt}(j)$ (if the corresponding $B'$ does not contain $\beta$). We also obtain: $(\bigsqcup X^\alpha)_+^+(j)$ can be presented as the inverse limit of the pro-objects $\bigsqcup_{\beta \in B} X_{i_\beta}^+(j)$.

One can also speak of Zariski points of a pro-schemes $X$. We will say that $X$ is of dimension $\leq d$ if $X_+ = \lim X_i$ for some $X_i$ of dimension at
most $d$. Alternatively, one can check whether the transcendence degrees of all residue fields of $X$ over $k$ are at most $d$.

1.4 $SH^{S^1}(k)$ and the homotopy $t$-structure on it: reminder

We do not give the definition of $SH^{S^1}(k)$ here; we only recall that it can be defined via a chain of functors $H_\ast \Delta^op Pre_\ast \rightarrow H_\ast \Delta^op Shv_\ast \rightarrow \mathcal{H}_k \rightarrow SH^{S^1}(k)$; those are homotopy functors for certain left Quillen functors of closed model categories. Here the (underlying category for) the model category corresponding to $H_\ast \Delta^op Pre_\ast$ (resp. to $H_\ast \Delta^op Shv_\ast$ and $\mathcal{H}_k$) is $\Delta^op Pre_\ast$ (resp. $\Delta^op Shv_\ast$), where $Pre_\ast$ (resp. $Shv_\ast$) denotes the category of presheaves (resp. Nisnevich sheaves) of pointed sets on $SmVar$. We will say more on these model categories and fill in the corresponding details of the proofs in §5.1 below.

For $X/U \in Obj\mathcal{OP}$ we define $\Sigma_\infty(X/U)$ as the image in $SH^{S^1}(k)$ of the discrete pointed simplicial presheaf $SmVar(-,X_+)/SmVar(-,U_+)$, i.e., of the presheaf sending $Y \in SmVar$ into $SmVar(Y,X)/SmVar(Y,U)$ pointed by the ‘image’ of $SmVar(Y,U)$ if the latter is non-empty, and into $SmVar(Y,X)\sqcup pt$ pointed by this point in the opposite case (cf. §2.3.2 of [Deg08b]). Recall also that $SH^{S^1}(k)$ is triangulated monoidal with respect to the operation $\wedge$ compatible with the obvious $\wedge$ for pointed presheaves; it contains the Thom spectrum $T = \Sigma^\infty(\mathbb{A}^1/\mathbb{G}_m)$ (corresponding to the line bundle $\mathbb{A}^1 \rightarrow pt$). We denote the operation $\wedge T^j[-j]$ on $SH^{S^1}(k)$ by $\{j\}$ (for any $j \geq 0$).

We will need the following properties of $SH^{S^1}(k)$.

**Proposition 1.4.1.** 1. The functor $SmVar \rightarrow SH^{S^1}(k): X \mapsto \Sigma^\infty(X_+)$ is exactly the ‘usual’ one considered in §4.2 of [Mor03]. Besides, there is a natural isomorphism $\Sigma^\infty(-\langle j \rangle) \rightarrow \Sigma^\infty(-) \wedge T^j$ for the spectrum $T$ being the one considered in ibid.

2. For any $U \in SmVar$ we have $\Sigma^\infty(U/U) = 0$; for any $X/Y \in Obj\mathcal{OP}$ the functor $\Sigma^\infty$ converts the natural morphism $X/Y \rightarrow X \sqcup U/Y \sqcup U$ into an isomorphism.

3. For any $j \geq 0$ and open embeddings $Z \rightarrow Y \rightarrow X$ of smooth varieties the natural morphisms $\Sigma^\infty(Y/Z\langle j \rangle) \rightarrow \Sigma^\infty(X/Z\langle j \rangle) \rightarrow \Sigma^\infty(X/Y\langle j \rangle)$ can be completed to a distinguished triangle.
4. For any finite set of $X_i \in \text{SmVar}$ the obvious morphisms $(\sqcup X_i)_+(j) \to X_i(j)$ yield an isomorphism $\Sigma^\infty((\sqcup X_i)_+(j)) \cong \prod \Sigma^\infty(X_i(j))$.

5. $\Sigma^\infty$ is homotopy invariant, i.e., for any $X \in \text{SmVar}$ we have $\Sigma^\infty(X_+) \cong \Sigma^\infty(X \times \mathbb{A}^1_+)$.

6. $\Sigma^\infty(-_+)$ converts Nisnevich distinguished squares of smooth varieties into distinguished triangles (see Example 4.1.11 of [Mor03] for more detail).

7. Let $j \geq 0$; let $i : Z \to X$ be a closed embedding of smooth varieties, and denote by $B(X, Z)$ the corresponding deformation to the normal cone variety (see the proof of Theorem 3.2.23 of [MoV99] or §4.1 of [Deg08a]). Then the natural OP-morphisms $X/X \setminus Z(j) \to B(X, Z)/B(X, Z) \setminus Z \times \mathbb{A}^1(j) \text{ and } N_{X, Z}/N_{X, Z} \setminus Z(j) \to B(X, Z) \setminus Z \times \mathbb{A}^1(j)$ (where $N_{X, Z}$ is the normal bundle for $i$) become isomorphisms after we apply $\Sigma^\infty$. 

Proof. 1, 2, 5, and 6 obvious from the well-known properties of $SH^{S^1}(k)$ (see [Mor03]).

3. By assertion 1 we can assume that $j = 0$. In this case it suffices to note that the pair of morphisms in question yields a cofibration sequence in $\Delta^{op}\text{Pre}_*$; cf. also §2.4.1 of [Deg13a].

4. Again, assume that $j = 0$. Obviously, it suffices to prove the statement for the union of two varieties. By assertion 2 in this case it suffices to verify that the distinguished triangle corresponding to $\text{pt} \to \text{pt} \sqcup X_1 \to \text{pt} \sqcup X_1 \sqcup X_2$ (by the previous assertion) splits. The latter is an easy consequence of assertion 2.

7. Again, we can assume that $j = 0$. In this case it suffices to note that the corresponding OP-morphisms map into isomorphisms already in $\mathcal{H}_k$ (this is exactly Proposition 3.2.24 of [MoV99]); all the more we obtain isomorphisms in $SH^{S^1}(k)$.

The following observation relates twists with shifts (somehow).

Remark 1.4.2. 1. In particular, we obtain a distinguished triangle $\Sigma^\infty(\mathbb{G}_m+) \to \Sigma^\infty(A^1_+) \to \Sigma^\infty(A^1/\mathbb{G}_m) \to \Sigma^\infty(\mathbb{G}_m+)[1]$. Similarly, for any $X \in \text{SmVar}$, $j \geq 0$, we get: $\Sigma^\infty(X_+(j + 1)) \cong \text{Cone}(\Sigma^\infty(X_+(j)) \to \Sigma^\infty(X \times \mathbb{G}_m+(j)))[1]$. Besides, $\Sigma^\infty(X_+(j))$ is obviously a retract of $\Sigma^\infty(X \times \mathbb{G}_m+(j))$; hence Cone($\Sigma^\infty(X_+(j)) \to \Sigma^\infty(X \times \mathbb{G}_m+(j))$) is the ’kernel’ of the corresponding projection.
2. Alternatively, for the proof of assertion 4 one can note that disjoint union of pro-schemes yields the bouquet operation in $\Delta^\op Pre_\ast$.

Now we recall the basic properties of Morel’s homotopy $t$-structure on $SH^{S^1}(k)$ (see Theorem 4.3.4 of [Mor03]). We will denote it just by $t$; this is the “main” $t$-structure of this paper. We recall that in ibid. for any $n \in \mathbb{Z}$ the class $SH^{S^1}(k)^{t \leq n}$ was called the one of $1 - n$-connected ($\mathbb{A}^1$-local) spectra (note that our $SH^{S^1}(k)^{t \leq n}$ is $SH^{S^1}(k)^{t \geq -n}$ in the notation of ibid.; see Remark 1.2.3(4)). For an $E \in \text{Obj} SH^{S^1}(k)$ we will denote by $E^n$ (resp. by $E^n_j$) the application of the functor represented by $E[n]$ to $\mathcal{O}P \subset \text{Pro} - \mathcal{O}P$ via $\Sigma^\infty$ (resp. via $\Sigma^\infty \circ (j)$).

**Proposition 1.4.3.** Let $E \in \text{Obj} SH^{S^1}(k)$. Then the following statements are valid.

1. For any $X \in \text{SmVar}$ we have $\Sigma^\infty(X_+) \in SH^{S^1}(k)^{t \leq 0}$.
2. $E \in SH^{S^1}(k)^{t \leq 0}$ if and only if $E^n(X_+) = \{0\}$ for any $X \in \text{SmVar}$, $n < 0$.
3. $E \in SH^{S^1}(k)^{t \leq 0}$ if and only if $E^n(\text{Spec } K) = \{0\}$ for all $n > 0$ and for all function fields $K/k$. If this is the case, we also have $E^{n+j}(\text{Spec } K) = \{0\}$ for any such $K$, $n$, and any $j \geq 0$.
4. For a pro-scheme $S$ and some $i \in \mathbb{Z}$ assume that $E^0(S_+) = \{0\}$ for any $E \in SH^{S^1}(k)^{t \geq i}$. Then we also have $E^{n+j}(S_+) = \{0\}$ for any such $E$ and any $j \geq 0$.
5. If $E \in SH^{S^1}(k)^{t \leq 0}$ then there exist some $E_i$ belonging to the extension-closure of $\{\coprod \Sigma^\infty(S_i)[n_i]\}$ for $S_i \in \text{SmVar}$, $n_i \geq 0$, and a distinguished triangle $\coprod E_i \to E \to \coprod E_i[1]$.

**Proof.** 1. Immediate from Lemma 4.3.3 of ibid.
2. This is the just the definition of $SH^{S^1}(k)^{t \geq 0}$ (see Definition 4.3.1(1) of ibid.).
3. Immediate from Lemmas 4.2.7 and 4.3.11 of ibid.
4. Immediate from loc. cit.
5. Immediate from assertion 2 together with Theorem 12.1 of [KeN13] (here we use the compactness of $\Sigma^\infty X_i[n_i]$ in $SH^{S^1}(k)$).

Below $SH^{S^1}(k)^c$ will denote the triangulated subcategory generated (in the sense described in [11]) by $\{\Sigma^\infty(X_+) : X \in \text{SmVar}\}$. Note that the objects of this subcategory are exactly the compact objects of $SH^{S^1}(k)$.

Now we recall some simple facts and definitions.
Remark 1.4.4. In this paper all semi-local schemes that we consider will be affine essentially smooth ones. Such an $S$ is the semi-localization of a smooth affine variety $V/k$ at a finite collection of Zariski points. Obviously, if $f : V' \to V$ is a finite morphism of smooth varieties (in particular, a closed embedding) then $S \times_V V'$ is (affine essentially smooth) semi-local also. It is well-known that all vector bundles over connected semi-local schemes (in our sense) are trivial.

Certainly, any semi-local scheme yields a pro-scheme (with a finite number of connected components). We will call an $\mathcal{OP}$-disjoint union of an arbitrary set of (affine smooth) semi-local schemes a semi-local pro-scheme.

We will need the following important result (that is essentially well-known also).

**Proposition 1.4.5.** Assume that $k$ is infinite; let $X$ be a semi-local pro-scheme, $E \in SH^{\leq 0}(k)$ Then for any $j \geq 0$, $i > j$, we have $E^j_i(X_+) = \{0\}$ (see the notation of the previous subsection).

**Proof.** Obviously, we can assume that $X$ is connected.

By Proposition [1.4.1](3), $X \mapsto E^j_i(X_+)$ together with $(X, U) \mapsto E^j_i(X/U)$ yields a cohomology theory with supports in the sense of Definition 5.1.1(a) of [CHK97]. The Nisnevich excision for $\Sigma^\infty(-_+)$ yields axiom COH1 of loc. cit., whereas Proposition [1.4.1](5) is precisely axiom COH3 of (§5.3 of) ibid. Hence we can apply Theorem 6.2.1 of ibid.; we obtain an injection $E^{i+j}_j(X_+\langle j \rangle) \to E^{i+j}_{i+j}(X_0\langle j \rangle)$. In order to conclude the proof it suffices to apply Proposition [1.4.1](3,4).

\[\square\]

### 1.5 Primitive (pro)schemes: definition

In \[\S6\] below we will consider the stable motivic homotopy category $SH(k)$ and its 'τ-positive part' $SH^+(k)$. It turns out that in the latter category one can formulate a connectivity result for a class of (pro)schemes wider than that of semi-local ones. So, we will need the following definition.

**Definition 1.5.1.** If $k$ is infinite then a pro-scheme will be called primitive if all of its connected components are affine (essentially smooth) $k$-schemes and their coordinate rings $R_j$ satisfy the following primitivity criterion: for any $n > 0$ every polynomial in $R_\j[X_1, \ldots, X_n]$ whose coefficients generate $R_j$ as an ideal over itself, represents an $R_j$-unit.
If \( k \) is finite, then we will (by an abuse of notation, in this paper) call a pro-scheme primitive whenever it is semi-local in the sense of Remark 1.4.4 (i.e., if all of its connected components are affine essentially smooth semi-local).

**Remark 1.5.2.** Recall that in the case of infinite \( k \) all semi-local \( k \)-algebras satisfy the primitivity criterion (see Example 2.1 of [Wal98]).

We will need the following properties of primitive pro-schemes.

**Proposition 1.5.3.**

1. Assume that a primitive pro-scheme \( S \) is the projective limit of some open subvarieties of a \( V \in \text{SmVar} \). Let \( f : V' \to V \) be a finite morphism of smooth varieties. Then \( S \times_V V' \) is primitive also.

2. Any finitely generated module of constant rank over a primitive ring is free.

**Proof.** Essentially, we have already noted (in Remark 1.4.4) that these statements are valid if \( k \) is finite (since in this case primitive pro-schemes are just semi-local ones by our convention). In the case when \( k \) is infinite (so that our notion of primitivity coincides with the one used by Walker) these assertions are given by Theorems 4.6 and 2.4 of [Wal98], respectively.

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### 2 Weight structures: reminder and the case of cocompactly cogenerated categories

In this section we recall the formalism of weight structures and their relation with (orthogonal) \( t \)-structures. The only new results of this section are Theorem 2.2.6(II,III) and Corollary 2.3.4. The author suspects that the dual to Theorem 2.2.6 (on weight structures in compactly generated triangulated categories) can have interesting applications beyond the scope of the current paper.

#### 2.1 Basic definitions and properties

**Definition 2.1.1.** I For a triangulated category \( C \), a pair of classes \( C_{\leq 0}, C_{\geq 0} \subset \text{Obj} C \) will be said to define a weight structure \( w \) for \( C \) if they satisfy the following conditions:

(i) \( C_{\geq 0}, C_{\leq 0} \) are Karoubi-closed in \( C \) (i.e., contain all \( C \)-retracts of their objects).
(ii) Semi-invariance with respect to translations.
\( C_{w<0} \subset C_{w\leq 0}[1], \ C_{w\geq 0}[1] \subset C_{w\geq 0} \).

(iii) Orthogonality.
\( C_{w\leq 0} \perp C_{w\geq 0}[1] \).

(iv) Weight decompositions.
For any \( M \in \text{Obj}_C \) there exists a distinguished triangle

\[ B \to M \to A \xrightarrow{f} B[1] \tag{3} \]

such that \( A \in C_{w\geq 0}[1], \ B \in C_{w<0} \).

II The category \( Hw \subset C \) whose objects are \( C_{w=0} = C_{w\geq 0} \cap C_{w\leq 0} \);
\( Hw(Z,T) = C(Z,T) \) for \( Z,T \in C_{w=0} \), will be called the heart of \( w \).

III \( C_{w \geq i} \) (resp. \( C_{w \leq i} \), resp. \( C_{w=i} \)) will denote \( C_{w\geq 0}[i] \) (resp. \( C_{w\leq 0}[i] \),
resp. \( C_{w=0}[i] \)).

IV We denote \( C_{w \geq i} \cap C_{w \leq j} \) by \( C_{i,j} \) (so it equals \{0\} for \( i > j \)).

V We will call \( C^b = \bigcup_{i \in \mathbb{Z}} C_{w \leq i} \cap \bigcup_{i \in \mathbb{Z}} C_{w \geq i} \) the class of bounded objects of \( C \). We will say that \( w \) is bounded if \( C^b = \text{Obj}_C \).

Besides, we will call \( \bigcup_{i \in \mathbb{Z}} C_{w \leq i} \) the class of bounded above objects.

VI \( w \) will be called non-degenerate from below (resp. from above) if \( \cap C_{w \leq i} = \{0\} \) (resp. \( \cap C_{w \geq i} = \{0\} \)).

VII Let \( C \) and \( C' \) be triangulated categories endowed with weight structures \( w \) and \( w' \), respectively; let \( F : C \to C' \) be an exact functor.

\( F \) will be called left weight-exact (with respect to \( w, w' \)) if it maps \( C_{w \leq 0} \) into \( C'_{w' \leq 0} \); it will be called right weight-exact if it maps \( C_{w \geq 0} \) to \( C'_{w' \geq 0} \). \( F \) is called weight-exact if it is both left and right weight-exact.

VIII Let \( C \) be a full additive subcategory of a triangulated \( C \).
We will say that \( C \) is negative if \( \text{Obj}_C \perp (\bigcup_{i>0} \text{Obj}(C[i])) \).

IX We will call a Postnikov tower for \( M \) (see Definition 1.2.4) a weight Postnikov tower if all \( Y_j \) are some choices for \( w_{<j} M \). In this case we will call the complex whose terms are \( M^p \) (see Remark 1.2.5) a weight complex for \( M \).

We will call a weight Postnikov tower for \( M \) positive if we take \( w_{\leq r} M = \emptyset \) for all \( r < 0 \) (and so, \( M \in C_{w \geq 0} \)).

Remark 2.1.2. 1. A weight decomposition (of any \( M \in \text{Obj}_C \)) is (almost) never canonical; still we will sometimes denote (any choice of) a pair \((B, A)\) coming from in (3) by \((w_{\leq 0} M, w_{\geq 1} M)\). For an \( l \in \mathbb{Z} \) we denote by \( w_{\leq 1} M \) (resp. \( w_{\geq 1} M \)) a choice of \( w_{\leq 0}(M[l])[l] \) (resp. of \( w_{\geq 1}(M[1-l])[l-1] \)).
2. A simple (and yet useful) example of a weight structure comes from the stupid filtration on the homotopy categories of cohomological complexes $K(B)$ for an arbitrary additive $B$. In this case $K(B)_{w \leq 0}$ (resp. $K(B)_{w \geq 0}$) will be the class of complexes that are homotopy equivalent to complexes concentrated in degrees $\geq 0$ (resp. $\leq 0$). The heart of this weight structure is the Karoubi-closure of $B$ in $K(B)$, in the corresponding category.

3. In the current paper we use the ‘homological convention’ for weight structures; it was previously used in [Heb11], [Wil12], and [Bon13a], whereas in [Bon10a] and in [Bon10b] the ‘cohomological convention’ was used. In the latter convention the roles of $C_{w \leq 0}$ and $C_{w \geq 0}$ are interchanged, i.e., one considers $C_{w \leq 0} = C_{w \geq 0}$ and $C_{w \geq 0} = C_{w \leq 0}$. So, a complex $M \in \text{Obj}_K(B)$ whose only non-zero term is the fifth one has weight $-5$ in the homological convention, and has weight $5$ in the cohomological convention. Thus the conventions differ by ‘signs of weights’; $K(B)_{[i,j]}$ is the class of retracts of complexes concentrated in degrees $[-j,-i]$.

4. In [Bon10a] the axioms of a weight structure also required $C_{w \leq 0}$ and $C_{w \geq 0}$ to be additive. Yet it is not necessary to include this property into the axioms since it follows from the remaining ones; see Proposition 1.3.3(1,2) of ibid.

5. Also, in the current paper we shift the numeration for $Y_i$ (in the definition of weight Postnikov tower) by [1] if compared with [Bon10b].

Now we recall basic properties of weight structures.

**Proposition 2.1.3.** Let $C$ be a triangulated category endowed with a weight structure $w$, $M \in \text{Obj}_C$, $i \in \mathbb{Z}$. Then the following statements are valid.

1. The axiomatics of weight structures is self-dual: for $D = C^{op}$ (so $\text{Obj}_D = \text{Obj}_C$) there exists the (opposite) weight structure $w'$ for which $D_{w' \leq 0} = C_{w \geq 0}$ and $D_{w' \geq 0} = C_{w \leq 0}$.

2. $C_{w \leq i}$, $C_{w \geq i}$, and $C_{w = i}$ are Karoubi-closed and extension-closed in $C$ (and so, additive).

   Besides, if $M \in C_{w < 0}$, then $w_0 M \in C_{w = 0}$ (for any choice of $w_0 M$).

3. If (in $C$) we have a distinguished triangle $A \to B \to C$ for $B \in C_{w = 0}$, $C \in C_{w \geq 1}$, then $A \cong B \bigoplus C[-1]$.

4. For any choice of $w \leq j M$ for $j \in \mathbb{Z}$ there exists a weight Postnikov tower for $M$. Besides, for any weight Postnikov tower we have $\text{Cone}(Y_i \to M) \in C_{w \geq i+1}$; $M^i \in C_{w = 0}$.  

20
5. Conversely, any bounded Postnikov tower (for $M$) with $M^j \in C^{w=0}$ for all $j \in \mathbb{Z}$ is a weight Postnikov tower for it.

6. For $C, w$ as in the previous assertions, weight decompositions are weakly functorial, i.e., any $C$-morphism of objects has a (non-unique) extension to a morphism of (any choices of) their weight decomposition triangles.

7. $C_{w \geq i}$ is closed with respect to all those small products that exist in $C$.

8. $C_{w \geq i} = (C_{w \leq i-1})^\perp$.

9. $w$ induces a bounded weight structure for $C^b$ (i.e., we consider the corresponding full subcategory of $C$ and the classes $C_{w \leq 0} \cap C^b$ and $C_{w \geq 0} \cap C^b$ in it), whose heart equals $H_w$.

10. For any $l \leq m \in \mathbb{Z}$ the class $C_{[l,m]}$ is the smallest Karoubi-closed extension-stable subclass of $\text{Obj}C$ containing $\cup_{l \leq j \leq m} C_{w=j}$.

11. Assume that $M, M' \in C_{w \geq 0}$; suppose that $f \in C(M, M')$ can be extended to a morphism of (some of) their positive Postnikov towers that establishes an isomorphism $M^0 \to M'^0$. Assume also that $M' \in C_{w=0}$. Then $f$ yields a projection of $M$ onto $M'$ (i.e., $M'$ is a retract of $M$ via $f$).

12. $M$ determines its weight complex $t(M)$ up to homotopy equivalence (i.e., up to a $K(H_w)$-isomorphism). In particular, if $M \in C_{w=0}$, then any choice of $t(M)$ is $K(H_w)$-isomorphic to $\cdots \to 0 \to M \to 0 \to \cdots$.

13. If $M_0 \to M_1 \to M_2$ is a distinguished triangle in $C$, then $f$ induces some morphism $t(M_0) \to t(M_1)$ (for any possible choices of these complexes) such that a cone of this morphism yields a weight complex for $M_2$.

14. Let $H_0 : H_w^{\text{op}} \to A$ ($A$ is an arbitrary abelian category) be an additive functor. Choose a weight complex $t(M) = (M^j)$ for each $M \in \text{Obj}C$, and denote by $H(M)$ the zeroth cohomology of the complex $H_0(M^{-j})$. Then $H(-)$ yields a cohomological functor that does not depend on the choices of weight complexes.
15. For $M \in C_{w=0}$ consider some (other) positive weight Postnikov tower for it; let $t(M) = (M^j)$ denote the corresponding weight complex. Then there exists some $N^j \in C_{w=0}$, $j \leq 0$, such that $t(M)$ is $C(Hw)$-isomorphic to $M^0 \oplus (\bigoplus_{j \leq 0} (N^j \overset{id}{\to} N^j)[-j])$ (i.e., $N^j$ is put in degrees $j-1$ and $j$).

16. For $M, M' \in \text{Obj}C$ suppose $f \in C(M, M')$ is compatible with an isomorphism $w \leq i M \to w \leq i M'$. Then $\text{Cone} f \in C_{w \geq i+1}$.

17. If $C$ is Karoubian, then $C_{w \leq i}$, $C_{w \geq i}$, $C_{w = i}$ also are. In particular, this is the case if $C$ is closed with respect to all countable products.

18. Assume that $C$ and $C_{w \leq 0}$ are closed with respect to arbitrary small products. Then the same is true for all $C_{w = j}$, small products of weight decompositions are weight decompositions, and small products of weight Postnikov towers (resp. of weight complexes for some $M_l \in \text{Obj}C$, $l \in L$) are weight Postnikov towers (resp. is a weight complex for $\prod_l M_l$).

19. If $M \in C^b$ then $M \in C_{w \geq i}$ if and only if $C_{w = j} \perp M$ for all $j < i$.

**Proof.** Assertions 1–11 are essentially contained in Theorem 2.2.1 of [Bon10b] (whereas their proofs relied on [Bon10a]; pay attention to Remark 2.1.2(3))!

12. The homotopy equivalence of all possible weight complexes is given by Theorem 3.2.2(II) of [Bon10a] (together with Proposition 3.1.8 of ibid.). It remains to note that for an $M \in C_{w=0}$ one obtains $\cdots \to 0 \to M \to 0 \to \cdots$ (M is on the 0-th position) as a weight complex by setting $(w \leq j M =) Y_j = M$ for $j \geq 0$ and $Y_j = 0$ otherwise.

13. Immediate from Theorem 3.3.1(I) of ibid.

14. Immediate from two previous assertions (see also Remark 3.1.7(2) of ibid.).

15. Assertion 12 yields that that $t(M)$ is $K(Hw)$-isomorphic to the complex $\cdots \to 0 \to M \to 0 \to \cdots$. Hence the corresponding complex $M' = \cdots M^{-2} \to M^{-1} \to M^0 \to M \to 0 \to \cdots$ is zero in $K(Hw)$. Thus it suffices to verify by induction: if a bounded above complex $N = \cdots \to N^{j-2} \to N^{j-1} \overset{d^{j-1}}{\to} N^j \to 0 \to \cdots$ is zero in $K(Hw)$, then $d^{j-1}$ is a projection of $N^{j-1}$ onto a direct summand in $Hw$. Now, $d^{j-1}$ is split by the corresponding component of (any) contracting homotopy for $N$. Hence $\text{Cone} d^{j-1}[-1]$ is the complement of $N^j$ to $N^{j+1}$ in $C$. Since $Hw$ is Karoubi-closed in $C$, we obtain the result.
16. Note first that the octahedral axiom of triangulated categories yields: \( \text{Cone } f \cong \text{Cone}(w_{\geq i+1}M \rightarrow w_{\geq i+1}M') \). Hence the result follows from the extension-closedness of \( \bigcup_{w \geq i+1} \) (see assertion 2).

17. Certainly, a Karoubi-closed class of objects in a Karoubian category is Karoubian. It remains to note that \( C \) is Karoubian if it is closed with respect to countable products; this is Remark 1.6.9 of [Nee01].

18. The first part of the assertion immediately follows from assertion 7. In order to establish the second one it suffices to note that products of distinguished triangles are distinguished; see Remark 1.2.2 of [Nee01]. The third part of the assertion follows immediately.

19. Certainly, for any \( M \in \bigcup_{i \geq 1} C \) we have \( C_{w=j} \perp M \) for all \( j < i \) (by the orthogonality axiom of weight structure). So, we should prove the converse application for a bounded \( M \). By assertion 9 we can assume that \( w \) is bounded. Next, assertion 8 yields: it suffices to prove that \( C_{[l,i-1]} \perp M \) for any \( l < i \). Hence assertion 10 yields the result.

\[ \square \]

Remark 2.1.4. 1. Certainly, in assertion 15 the corresponding \( N^i \) is an \( Hw \)-direct summand both of \( M^i \) and of \( M^{i-1} \) (for any \( i \in \mathbb{Z} \)).

2. A very useful statement (that was applied in several papers) is that any negative generating subcategory \( N \subset C \) (here we assume that \( C \) does not contain any proper strict triangulated subcategories that contain \( N \)) yields a weight structure \( w \) for \( C \) such that \( N \subset Hw \). Yet this is rather a tool for constructing bounded weight structures for 'small' triangulated categories; we will have to prove an alternative existence statement below.

3. Actually, one does not have to pass to the Karoubi-closure in assertion 9. Besides, assertion 19 is valid for any \( M \in \bigcup_{j \in \mathbb{Z}} C_{w \geq j} \). In the proof we could have considered weight Postnikov towers (for objects of \( \bigcup_{j \in \mathbb{Z}} C_{w \geq j} \) \( \cap C_{w \leq i-1} \)).

2.2 Weight structures on cocompactly cogenerated categories

This subsection is mostly dedicated to (the proof of) Theorem 2.2.6. Part I of it was essentially proved in [Pan12] (in the dual form); the dual to this statement can also be found in [Bon13b]. Yet other parts of the Theorem are new.

Our construction of weight structures requires countable 'triangulated' homotopy limits (in general triangulated categories). The latter are dual
to countable homotopy colimits introduced in \[BoN93\] (so in order to prove basic properties of the construction we dualize the corresponding results of ibid. and of \[Nee01\]). We will only apply the results of this subsection to triangulated categories closed with respect to arbitrary small products; so the homotopy limits will always exist.

**Definition 2.2.1.** For a sequence of objects $Y_i$ (starting from some $j \in \mathbb{Z}$) and maps $\phi_i : Y_i \to Y_{i-1}$ we consider the morphism $d : \oplus \text{id}_{Y_i} \oplus (-\phi_i) : D \to D$ (we can define it since its $i$-th component can be easily factorized as the composition $D \to Y_i \oplus Y_{i+1} \to Y_i$). Denote a cone of $d[-1]$ by $Y$. We will write $Y = \lim_{\leftarrow} Y_i$ and call $Y$ the homotopy limit of $Y_i$.

**Remark 2.2.2.** 1. Note that these homotopy limits are not really canonical and functorial in $Y_i$ since the choice of a cone is not canonical, i.e., limits are only defined up to non-canonical isomorphisms.

2. By Lemma 1.7.1 of \[Nee01\] the homotopy limit of $Y_i$ is the same for any subsequence of $Y_i$. In particular, we can discard any (finite) number of first terms in $Y_i$.

3. By Lemma 1.6.6 of \[Nee01\] the homotopy limit of $M \xleftarrow{id_M} M \xleftarrow{id_M} M \xleftarrow{id_M} \ldots$ is $M$. Hence if $\phi_i$ are isomorphisms for $i \gg 0$ and $M_i \cong M$ then $\lim_{\leftarrow} M_i \cong M$.

We also recall the behaviour of homotopy limits under (co)representable functors.

**Lemma 2.2.3.** 1. For any $C \in \text{ObjC}$ we have a natural surjection $C(C, Y) \to \lim_{\leftarrow} C(C, Y_i)$.

2. This map is bijective if all $\phi_i[-1]_* : C(C, Y_{i+1}[-1]) \to C(C, Y_i[-1])$ are surjective for $i \gg 0$.

3. If $C$ is cocompact then $C(Y, C) = \lim_{\rightarrow} C(Y_i, C)$.

Below we will also need a new (simple) piece of homological algebra: the definition of a coenvelope and some of its properties.

**Definition 2.2.4.** For a class $C' \subset \text{ObjC}$ its coenvelope is the smallest subclass of $\text{ObjC}$ that contains $C' \cup \{0\}$, is closed with respect to arbitrary (small) products, and satisfies the following property: for any $\phi_i : Y_i \to Y_{i-1}$ such that $Y_0 \in C$, $\text{Cone}(\phi_i)[-1] \in C$ for all $i \geq 1$, we have $\lim_{\leftarrow i \geq 0} Y_i \in C$ (i.e., $C$ contains all possible cones of the corresponding distinguished triangle; note that those are isomorphic).
Now we verify a simple property of this notion.

**Lemma 2.2.5.** Suppose that for \( C', D \subset \text{Obj}C \), we have \( D \perp C' \). Then for the coenvelope \( C \) of \( C' \) we also have \( D \perp C \).

**Proof.** Since for any \( d \in D \) the functor \( C(d, -) \) converts arbitrary products into products, it suffices to verify (for any \( d \in D \)): if for \( Y_i \) as in Definition 2.2.4 we have \( d \perp Y_0 \), \( d \perp \text{Cone}(\phi_i)[-1] \) for all \( i \geq 1 \), then \( p \perp \lim \rightarrow Y_i \). Now, for any \( i \geq 1 \) we have a long exact sequence

\[
\cdots \rightarrow C(d, Y_i[-1]) \rightarrow C(d, Y_{i-1}[-1]) \rightarrow C(d, \text{Cone}(\phi_i)[-1])(=\{0\}) \rightarrow C(d, Y_i) \rightarrow C(d, Y_{i-1}) \rightarrow \cdots
\]

Hence \( C(d, Y_i[-1]) \) surjects onto \( C(d, Y_{i-1}[-1]) \), whereas the obvious induction yields that \( C(d, Y_j) = \{0\} \) for any \( j \geq 0 \). Then Lemma 2.2.3(2) yields: \( C(d, \lim \rightarrow Y_i) \cong \lim \rightarrow C(d, Y_i) = \{0\} \).

**Theorem 2.2.6.** Let \( C \) be triangulated category that is closed with respect to all small products; let \( C' \subset C \) be a set of cocompact objects such that \( C'[1] \subset C' \). Then for the classes \( C_1 = C'[1] \) and \( C_2 \) being the coenvelope of \( C' \) the following statements are valid.

I They yield a weight structure on \( C \), i.e., there exists a \( w \) such that \( C_{w \geq 0} = C_1 \), \( C_{w \geq 0} = C_2 \).

II Denote by \( C'_w \) the full subcategory of \( C \) whose class of objects is the coenvelope of \( \cup_{i \leq 0} C'[i] \). \( C'_w \) is the subcategory whose objects are \( \perp C' \).

1. \( C' \) and \( C'_w \) are triangulated and closed with respect to all small products.
2. \( C'_1 = C_1 \cap \text{Obj}C'_w \) and \( C_2 \subset \text{Obj}C'_w \) yield a weight structure \( w_{C'_w} \) on \( C'_w \).
3. \( w_{C'_w} \) is non-degenerate from below.
4. \( C'_1 \) is the extension-closure of \( C'_1 \cup C'_w \) in \( C'_w \).
5. \( C'_1 \) is (also) closed with respect to all products.
6. The heart of \( w_{C'_w} \) is closed with respect to all small products; products of weight decompositions in \( C'_w \) are weight decompositions; small products of weight Postnikov towers (resp. of weight complexes \( t(M_i) \) for \( M_i \in \text{Obj}C'_w \)) are weight Postnikov towers (resp. yield \( t(\prod M_i) \)).
7. \( C, C'_w, C'_1, \) and \( C_2 \) are Karoubian.

III For each \( c \in C' \) fix a weight complex \( (c^i) \) (with respect to \( w_{C'_w} \)). Then \( \text{Hw}_{C'_w} \) is equivalent to the Karoubization of the category of all (small) products of \( c^i \) for \( c \) running through all objects \( C', i \in \mathbb{Z} \).
Proof. I Obviously, \((C_1, C_2)\) are Karoubi-closed in \(C\), \(C_1 \subset C_1[1], C_2[1] \subset C_2\). Besides, \(C_1 \perp C_2[1]\) by Lemma 2.2.3.

It remains to verify that any \(M \in \text{Obj} C\) possesses a weight decomposition with respect to \((C_1, C_2)\). We apply (a certain modification of) the method used in the proof of Theorem 4.5.2(I) of [Bon10a] (cf. also the construction of crude cellular towers in §I.3.2 of [Mar83]).

We construct a certain sequence of \(M_k\) for \(k \geq 0\) by induction on \(k\) starting from \(M_0 = M\). Assume that \(M_k\) (for some \(k \geq 0\)) is already constructed. Then we take \(P_k = \prod_{i \in C', f \in \mathcal{C}(c,M_k)} c\) and \(M_{k+1}[1]\) being a cone of the morphism \(\prod_{i \in C', f \in \mathcal{C}(c,M_k)} f : M_k \to P_k\).

Now we ‘assemble’ \(P_k\). The compositions of the morphisms \(h_k : M_{k+1} \to M_k\) given by this construction yields morphisms \(g_i : M_i \to M\) for all \(i \geq 0\). Besides, the octahedral axiom of triangulated categories immediately yields \(\text{Cone}(h_k) \cong P_k\). Now we complete \(g_k\) to distinguished triangles \(M_k \xrightarrow{g_k} M \xrightarrow{a_k} A_k\). The octahedral axiom yields the existence of morphisms \(s_i : A_{i+1} \to A_i\) that are compatible with \(a_i\), such that \(\text{Cone}(a_i) \cong P_{i+1}[1]\) for all \(i \geq 0\).

We consider \(A = \lim_k A_k\); by Lemma 2.2.3(1) \((a_k)\) lift to a certain morphism \(a : M \to A\). We complete \(a\) to a distinguished triangle \(B \xrightarrow{b} M \xrightarrow{g} A \xrightarrow{f} B[1]\). This triangle will be our candidate for a weight decomposition of \(M\).

First we note that \(A_0 = 0\); since \(\text{Cone}(a_i) \cong P_{i+1}[1]\), we have \(A \in C_2\) by the definition of the latter.

It remains to prove that \(B \in C_1[-1]\), i.e., \(B \perp C'\). For \(c \in C'\) we should check that \(\mathcal{C}(B,c) = \{0\}\). The long exact sequence

\[
\cdots \to \mathcal{C}(A,c) \to \mathcal{C}(M,c) \to \mathcal{C}(B,c) \to \mathcal{C}(A[-1],c) \to \mathcal{C}(M[-1],c) \to \cdots
\]

translates this into: \(\mathcal{C}(c,-)(a)\) is surjective and \(\mathcal{C}(-,c)(a[-1])\) is injective. Now, by Lemma 2.2.3(3), \(\mathcal{C}(A,c) \cong \lim_i \mathcal{C}(A_i,c)\) and \(\mathcal{C}(A[-1],c) \cong \lim_i \mathcal{C}(A_i[-1],c)\). Hence the long exact sequences

\[
\cdots \to \mathcal{C}(A_k,c) \to \mathcal{C}(M,c) \to \mathcal{C}(M_k,c) \to \mathcal{C}(A_k[-1],c) \to \mathcal{C}(M[-1],c) \to \cdots
\]

yield: it suffices to verify that \(\lim_k \mathcal{C}(M_k,c) = \{0\}\) (note here that \(h_k\) are compatible with \(s_k\)). Lastly, \(\mathcal{C}(P_k,c)\) surjects onto \(\mathcal{C}(M_k,c)\); hence the group \(\mathcal{C}(M_k,c)\) dies in \(\mathcal{C}(M_{k+1},c)\) for any \(k \geq 0\) and we obtain the result.

II 1. Obvious. Note here: the cocompactness of elements of \(C'\) yields that \(\mathcal{C}_1\) is closed with respect to products; this also yields assertion II.5.
2. Certainly, in order to verify the existence of $w_{C'}$ it suffices to verify that the corresponding weight decompositions exist in $C'$. We check it for an $M \in \text{Obj}C'$ via applying the duals to several results of [Nee01].

Theorem 8.3.3 of ibid. yields that $C'$ satisfies the dual to the Brown representability condition (see Definition 8.2.1 of ibid.). Indeed, $C'$ is cogenerated by $C'$ in the sense described in §1.1 whereas the latter notion is dual to Definition 3.2.9 of ibid.. Hence (the dual to) Theorem 8.4.4 of ibid. yields the existence of an exact functor $F : C \to C'$ that is left adjoint to the embedding of $C'$ into $C$ (the exactness of $F$ is follows from Lemma 5.3.6 of ibid.). Certainly, $F$ is isomorphic to the Verdier localization of $C$ by $C_{\perp}$. Since $C_{\perp}$ is a colocalizing subcategory in $C$ (i.e. satisfies the condition dual to Definition 3.2.6 of ibid.), $F$ also respects all products (see Corollary 3.2.11 of ibid).

Now we apply $F$ to (3). We certainly have $F(M) \cong M$, $F(A) \cong A$. The adjunction for $F$ also yields that $F(B) \in C_1^1$; hence $F(B) \in C_1'$ and we obtain a weight decomposition for $M$ in $C'$.

3. In order to verify that $w_{C'}$ is non-degenerate from below we should verify: if $M \in \text{Obj}C'$ is non-zero then there exist $i \in \mathbb{Z}$ and $c \in C$ such that $C(M,c[i]) \neq 0$. The latter is immediate from the fact that the full subcategory of $C$ whose objects are $\{M[-i], i \in \mathbb{Z}\}$ is triangulated, strict, and closed with respect to all small products.

4. As we have already noted, $F$ projects $C_1$ onto $C_1'$. Besides, the adjunction transformation yields for any $N \in C_1$ a distinguished triangle $N' \to N \to F(N)$ for some $N' \in \text{Obj}C_{\perp}$. On the other hand, $C_1$ certainly contains $C_1'$ and $C_{\perp}$, and is extension-closed.

6. Immediate from the previous assertion by Proposition 2.1.3(18).

7. Immediate from Proposition 2.1.3(17).

III As we have just shown, $Hw_{C'}$ is Karoubian and closed with respect to all products. Since it also contains all $c'$, it contains the Karoubization in question.

Now let $M \in C'_{w_{C'}=0}$. By Lemma 1.1.1 (together with Proposition 2.1.3(15)) we obtain: it suffices to verify the existence of a choice for $t(M) = (M^0)$ such that $M^0$ belongs to the category of products of all $c'$.

Now consider the full subcategory of $C'$ whose objects possess weight complexes all of whose terms are products of certain $c'$. Obviously, this category contains $C'$. Hence Proposition 2.1.3(13) yields that it is triangulated. Moreover, assertion II6 yields that it is closed with respect to all small prod-
ucts. Hence this subcategory coincides with all $\mathcal{C}'$, and so contains $M$. This concludes the proof.

Below we will need two easy methods for producing a new category with a weight structure out of an old one.

Remark 2.2.7. 1. Let $S$ be a set of prime numbers. Then for any triangulated $\mathcal{C}$ the category $\mathcal{C}[S^{-1}]$ with the same objects and morphisms tensored by $\mathbb{Z}[S^{-1}]$ is also triangulated (since this category can be constructed as the Verdier localization of $\mathcal{C}$ by cones of all multiplications by elements of $S$).

Moreover, all the statements of Theorem 2.2.6 will remain true if replace $\mathcal{C}$ by $\mathcal{C}[S^{-1}]$. Indeed, note that $\mathcal{C}' \subset \text{Obj}\mathcal{C}[S^{-1}] = \text{Obj}\mathcal{C}$ yields a set of cocompact cogenerators for $\mathcal{C}[S^{-1}]$, and all our statements and constructions commute with $- [S^{-1}]$; cf. Appendix B of [Lev13].

Besides, if there exists a triangulated $\mathcal{D}$ with a $t$-structure $t$ such that $w_{\mathcal{C}'} \perp t$ with respect to some (nice) duality $\Phi$ (see Definition 2.4.1 below), then $t$ induces a $t$-structure on $\mathcal{D}[S^{-1}]$ in the obvious way, and this $t$-structure is orthogonal to $w_{\mathcal{C}[S^{-1}]}$ with respect to the corresponding duality.

2. Now assume that the category $\mathcal{C}^0 = \langle \mathcal{C}' \rangle_{\mathcal{C}[S^{-1}]}$ decomposes into the direct sum of two (triangulated) subcategories $\mathcal{C}'_1, \mathcal{C}'_2$; denote by $pr'$ the projection of $\mathcal{C}^0$ onto $\mathcal{C}'_1$ (this is an exact functor, and $pr'(M)$ is a retract of $M$ for any $M \in \text{Obj}\mathcal{C}^0$). Then the same argument as the one used in the proof of part II.2 of ibid. yields: $pr'$ extends uniquely to a projection $pr$ of $\mathcal{C}[S^{-1}]$ onto its subcategory $\mathcal{C}'_1$ that is cogenerated by $\mathcal{C}'_1$ (i.e., to the smallest triangulated subcategory of $\mathcal{C}[S^{-1}]$ that contains $\mathcal{C}'_1$ and is closed with respect to all products). Moreover, since $pr$ yields a retraction of each object of $\mathcal{C}[S^{-1}]$, it is $w_{\mathcal{C}[S^{-1}]}$-exact; hence $w_{\mathcal{C}[S^{-1}]}$ restricts to a weight structure on $\mathcal{C}'_1$.

Next, assume that a triangulated $\mathcal{D}$ is compactly generated by a subcategory isomorphic to $\mathcal{C}^0$, and a duality $\Phi : \mathcal{C}[S^{-1}]^{op} \times \mathcal{D} \to \text{Ab}$ is obtained by extending $\mathcal{D}$-representable functors from $\mathcal{C}^0$ to $\mathcal{C}[S^{-1}]$ (cf. Proposition 4.4.1 below). Then this duality factors through the corresponding component $\mathcal{D}'$ of $\mathcal{D}$.

Lastly, note that both of these constructions do not require any ‘models’.

3. Another interesting statement that can be proved in the general context of Theorem 2.2.6 is the description of all pure extended cohomology theories for $\mathcal{C}'$; cf. §4.6 below.
4. The author suspects that dualizing Theorem 2.2.6 would be useful for its application in settings not related to Gersten weight structures. Indeed, $C^{op}$ is closed with respect to arbitrary small coproducts, whereas $C^{c_{op}}$ is compactly generated (and it seems that compactly generated triangulated categories are "more popular" than cocompactly cogenerate ones).

2.3 Weight filtrations and spectral sequences

Till the end of this section $C$ will be endowed with a weight structure $w$; $H : C \to A$ is a cohomological functor. For any $r \in \mathbb{Z}$ denote $H \circ [-r]$ by $H^r$.

First we recall some theory developed in (§2 of) [Bon10a] and [Bon10b].

**Proposition 2.3.1.**

1. For any $m \in \mathbb{Z}$ the object $(W^mH)(M) = \text{Im}(H(w_{\geq m}M) \to H(M)) = \text{Ker}(H(M) \to H(w_{\leq m-1}M))$ does not depend on the choice of $w_{\geq m}M$; it is $C$-functorial in $M$.

We call the filtration of $H(M)$ by $(W^mH)(M)$ its weight filtration.

2. For any $M \in \text{Obj} C$ there is a spectral sequence $T(H, M)$ with $E_1^{pq} = H^q(M^{-p})$. It comes from (any possible) weight Postnikov tower of $M$; so the boundary of $E_1$ is obtained by applying $H^*$ to the corresponding choice of a weight complex for it.

3. $T$ is (canonical and) naturally functorial in $H$ (with respect to exact functors between the target abelian categories) and in $M$ starting from $E_2$.

$T(H, M)$ converges to $H^{p+q}(M)$ if $M$ is bounded. Moreover, the step of filtration given by $(E_l^{m, -l} : l \geq k)$ on $H^m(X)$ equals $(W^kH^m)(M)$ (for any $k, m \in \mathbb{Z}$).

4. $T(H, M)$ satisfies all the properties of the previous assertions (for an arbitrary $M \in \text{Obj} C$) also in the case when $H$ kills $C_{w_{\leq -i}}$ and $C_{w_{\geq i}}$ for some (large enough) $i \in \mathbb{Z}$.

**Proof.** 1. This is (a partial case of) Proposition 2.1.2(2) of ibid.

2.3.4. This is (most of) Theorem 2.4.2 of ibid.

**Remark 2.3.2.**

1. Actually, $(W^mH)(M) = \text{Im}(H(w_{\geq m}M) \to H(M))$ does not depend on the choice of $w_{\geq m}M$ and is functorial in $M$ for any contravariant $H : C \to A$.

2. Certainly, weight spectral sequences can also be constructed for a homological $H$; see Theorem 2.3.2 of ibid.
Now we derive a simple (new) consequence from Proposition 2.3.1.

**Definition 2.3.3.** We will call a cohomological functor \( H : C \to A \) (for an arbitrary abelian \( A \)) a *pure* one if \( H \) kills \( C_{w \leq -1} \) and \( C_{w \geq 1} \).

**Corollary 2.3.4.** Assume that \( C \) is endowed with a weight structure \( w \); let \( A \) be an abelian category. Then the restriction of functors to \( Hw \) yields an equivalence of the "big category" of pure (cohomological) functors \( C \to A \) with the one of additive contravariant functors \( Hw \to A \). The converse correspondence is given by Proposition 2.1.3(14).

**Proof.** The functors given by Proposition 2.1.3(14) are pure by part 12 of loc. cit.; it also yields that the corresponding composition is equivalent to the identity of the big category of pure functors. The converse composition is equivalent to the identity of \( \text{AddFun}(Hw^{\text{op}}, A) \) by Proposition 2.3.1(4).

**Remark 2.3.5.** For any \( C, H \) the functor (see Proposition 2.3.1(2)) \( M \mapsto E_2^{0,0}T(H, M) \) is pure. Indeed, it suffices to verify that it is cohomological. The latter statement follows from the isomorphisms \( E_2^{0,0}T(H, -) \cong \tau_{\leq 0}(\tau_{\geq 0}H) \cong \tau_{\geq 0}(\tau_{\leq 0}H) \) given by Theorem 2.4.2(II) of [Bon10b] (see Remark 4.4.4(3) below and §2.3 of ibid.).

### 2.4 The relation with orthogonal \( t \)-structures

Let \( C, D \) be triangulated categories. We consider certain pairings \( C^{\text{op}} \times D \to A \). In the following definition we consider a general \( A \), yet below we will mainly need \( A = \text{Ab} \).

**Definition 2.4.1.** 1. We will call a (covariant) bi-functor \( \Phi : C^{\text{op}} \times D \to A \) a *duality* if it is bi-additive, homological with respect to both arguments, and is equipped with a (bi)natural bi-additive transformation \( \Phi(A[1], Y[1]) \cong \Phi(A[1], Y[1]) \).

2. We will say that \( \Phi \) is *nice* if for any distinguished triangles \( T = A \overset{f}{\to} B \overset{m}{\to} C \to A[1] \) in \( C \) and \( X \overset{f}{\to} Y \overset{g}{\to} Z \overset{h}{\to} X[1] \) in \( D \) we have: the natural
morphism $p$:

$$
\text{Ker}(\Phi(A, X) \bigoplus \Phi(B, Y) \bigoplus \Phi(C, Z))
\longrightarrow
\begin{pmatrix}
\Phi(A, -)(f) & -\Phi(-, Y)(l) & 0 \\
0 & g(B) & -\Phi(-, Z)(m) \\
-\Phi(-, X)(-1)(n) & 0 & \Phi(C, -)(h)
\end{pmatrix}
$$

is epimorphic.

3. Suppose $\mathcal{C}$ is endowed with a weight structure $w$, $\mathcal{D}$ is endowed with a $t$-structure $t$. Then we will say that $w$ is (left) orthogonal to $t$ with respect to $\Phi$ if the following orthogonality condition is fulfilled: $\Phi(X, Y) = 0$ if $X \in \mathcal{C}_w \geq 0$ and $Y \in \mathcal{D}_t \geq 1$ or if $X \in \mathcal{C}_w \leq 0$ and $Y \in \mathcal{D}_t \leq -1$.

'Natural' dualities are nice; we will justify this thesis now.

**Proposition 2.4.2.** 1. If $A = Ab$, $F : \mathcal{D} \to \mathcal{C}$ is an exact functor, then $\Phi(X, Y) = \mathcal{C}(X, F(Y))$ is nice.

2. For triangulated categories $\mathcal{D}$, $\mathcal{C}' \subset \mathcal{C}$, $\mathcal{C}'$ is skeletally small, and $A$ satisfying AB5, let $\Phi' : \mathcal{C}'^{\text{op}} \times \mathcal{D} \to A$ be a duality. For any $Y \in \text{Obj} \mathcal{D}$ we extend the functor $\Phi'(-, Y)$ from $\mathcal{C}'$ to $\mathcal{C}$ by the method of Proposition 4.1.1 below; we denote the functor obtained by $\Phi(-, Y)$. Then the corresponding bi-functor $\Phi$ is a duality $(\mathcal{C}'^{\text{op}} \times \mathcal{D} \to A)$. It is nice whenever $\Phi'$ is.

**Proof.** 1. It suffices to note that the niceness restriction is a generalization of the axiom (TR3) of triangulated categories (any commutative square can be completed to a morphism of distinguished triangles) to the setting of dualities of triangulated categories.

2. This is Proposition 2.5.6(3) of ibid. \qed

Now we describe the relation of weight spectral sequences with orthogonal structures.

**Proposition 2.4.3.** Assume that $w$ on $\mathcal{C}$ and $t$ on $\mathcal{D}$ are orthogonal with respect to a nice duality $\Phi$; $M \in \text{Obj} \mathcal{C}$, $Y \in \text{Obj} \mathcal{D}$.

Consider the spectral sequence $S$ coming from the following exact couple: $E_2^{pq}(S) = \Phi(M, Y^{t=\varrho}[p - 1])$, $E_2^{pq}(S) = \Phi(M, Y^{t=\varrho}[p])$ (we start $S$ from $E_2$).
This spectral sequence is naturally isomorphic to \( T(H, M) \) for \( H : N \mapsto \Phi(N, Y) \) if we consider the latter as starting from \( E_2 \).

\[ \text{Proof.} \] This is (a part of) Theorem 2.6.1 of ibid. \( \square \)

3 Pro-spectra and the Gersten weight structure

We embed \( SH^{S_1}(k) \) into a certain triangulated motivic homotopy category \( \mathcal{D}^{big} \); we will call it objects \textit{pro-spectra}. Below we will need several properties of \( \mathcal{D}^{big} \); yet our arguments usually will not rely on its explicit definition. For this reason, in §3.1 we only list the main properties of \( \mathcal{D}^{big} \) and of related categories. We use them in §3.2 for the construction of certain Gysin distinguished triangles and Postnikov towers for the pro-spectra of pro-schemes.

Next we construct (in §3.3) certain \textit{Gersten weight structures} on \( \mathcal{D}^{big} \supset \mathcal{D} \). \( \mathcal{D} \) is the subcategory of \( \mathcal{D}^{big} \) cogenerated by \( SH^{S_1}(k)^c \); one may say that it is the largest subcategory of \( \mathcal{D}^{big} \) 'detected by the Gersten weight structure'. The (common) heart of these Gersten weight structures is 'cogenerated' (via products and direct summands) by certain twists of pro-spectra of functions fields, whereas the pro-spectra of arbitrary pro-schemes belong to \( \mathcal{D}_{w \geq 0} \). It follows immediately that the Postnikov tower \( Po(X) \) provided by Proposition 3.2.4 is a weight Postnikov tower with respect to \( w \).

If \( k \) is infinite, then \( HW \) also contains the pro-spectra of semi-local pro-schemes over \( k \). Using this fact, in §3.4 we prove: if \( S \) is a semi-local pro-scheme (and \( k \) is infinite), \( S_0 \) is its dense sub-pro-scheme, then \( \Sigma^{\infty}(S_+) \) is a direct summand of \( \Sigma^{\infty}(S_{0+}) \); \( \Sigma^{\infty}(\text{Spec } K_+) \) (for a function field \( K/k \)) contains (as retracts) the pro-spectra of (affine essentially smooth) semi-local schemes whose generic point is \( \text{Spec } K \), as well as the twisted pro-spectra of residue fields of \( K \) (for all geometric valuations). It follows that the 'Gersten' weight complex of any semi-local pro-scheme splits.

3.1 Pro-spectra: an 'axiomatic description'

In §5.2 below we will construct a triangulated category \( \mathcal{D}^{big} \) as the homotopy category of a certain stable model category \( \mathcal{D}' \). In this section we will only describe certain properties of the categories \( \mathcal{D}^{big} \) and \( \mathcal{D}' \) that we will apply somewhat 'axiomatically'.

32
Let $PSH^{S^1}(k)$ denote the model for $SH^{S^1}(k)$ constructed in [Jar00] (we will say more on it in §5.1 below).

**Proposition 3.1.1.**

1. There exists a commutative diagram of categories and functors

$$
\xymatrix{
\mathcal{O} \mathcal{P} \ar[r]^{P\Sigma^\infty} & PSH^{S^1}(k) \ar[d]^{\text{Ho}_{SH^{S^1}(k)}} \ar[r]^c & \mathcal{D}' \ar[d]^{\text{Ho}_D} \\
SH^{S^1}(k) \ar[r]^{\text{Ho}(c)} & \mathcal{D}^{\text{big}}
}
$$

such that $\text{Ho}_{SH^{S^1}(k)} \circ P\Sigma^\infty = \Sigma^\infty$.

2. $\mathcal{D}'$ is closed with respect to all small filtered limits; $\mathcal{D}^{\text{big}}$ is closed with respect to all small products.

3. $\text{Ho}(c)$ is a full exact embedding; it yields a family of cocompact cogenerators for $\mathcal{D}^{\text{big}}$.

We will often write $SH^{S^1}(k) \subset \mathcal{D}^{\text{big}}$ (without mentioning the embedding functor).

4. For any projective system $X_i \in \text{Obj}\mathcal{D}'$, $C \in \text{Obj}SH^{S^1}(k)$ we have:

$$
\mathcal{D}^{\text{big}}(\text{Ho}_D(\lim_{i} X_i), C) \cong \lim_{i \in I} \mathcal{D}^{\text{big}}(\text{Ho}_D(X_i), C).
$$

5. Extend $\Sigma^\infty$ to $\text{Pro}-\mathcal{O} \mathcal{P}$ via the rule $\Sigma^\infty(\lim_i P_i) = \text{Ho}_D(\lim_i (c \circ P\Sigma^\infty(P_i)))$.

Then for any for any $j \geq 0$ and for any compatible system of open embeddings $Z_i \to Y_i \to X_i$ the natural morphisms $\Sigma^\infty((\lim_i Y_i/\lim_i Z_i(j))) \to \Sigma^\infty((\lim_i X_i/\lim_i Z_i(j))) \to \Sigma^\infty((X_i/\lim_i Y_i(j)))$ extend to a distinguished triangle.

We will usually call the objects of $\mathcal{D}^{\text{big}}$ pro-spectra.

We describe some consequences of the 'axioms' listed.

**Corollary 3.1.2.**

1. For a $\mathcal{D}^{\text{big}}$-morphism $f : X \to Y$ we have: $f$ is an isomorphism if and only if $\mathcal{D}^{\text{big}}(-, C)(f)$ is for any $C \in \text{Obj}SH^{S^1}(k)$.

2. In particular, for any projective system $I$ and any compatible system of morphisms $f_i \in \mathcal{D}'(X_i, Y_i)$ we have: $\text{Ho}_D(\lim_i f_i)$ is an isomorphism if and only if $\lim_i (\mathcal{D}^{\text{big}}(-, C)(\text{Ho}_D(f_i))$ is an isomorphism for any $C \in \text{Obj}SH^{S^1}(k)$. 

33
3. Let \( X = \sqcup X^\alpha \) be a decomposition of a pro-scheme. Then for any \( j \geq 0 \) we have: \( \Sigma^\infty(X_+(j)) \) is naturally isomorphic to \( \prod \Sigma^\infty(X^\alpha_+(j)) \).

Proof. 1. For \( Z = \text{Cone } f \) it suffices to note: by Proposition 3.1.1(3), we have \( Z = 0 \) if and only if \( Z \perp SH^{st}(k) \).

2. It suffices to combine the previous assertion with part 4 of loc. cit.

3. The natural \( \text{Pro-OP} \)-morphisms \( X \to X^\alpha \) yield a canonical morphism \( \Sigma^\infty(X_+(j)) \to \prod \Sigma^\infty(X^\alpha_+(j)) \). We have to check that this is an isomorphism. To this end (by assertion 1) we can fix a \( C \in \text{Obj}SH^{st}(k) \) and verify for \( H = \mathcal{D}^{big}(-, C) \) that \( H(\Sigma^\infty(X_+(j))) \cong H(\prod \Sigma^\infty(X^\alpha_+(j))) \).

Next, the cocompactness of \( C \) in \( \mathcal{D}^{big} \) yields that \( H(\prod \Sigma^\infty(X^\alpha_+(j))) \cong \bigoplus H(\Sigma^\infty(X^\alpha_+(j))) \). It remains to recall that \( \sqcup X^\alpha_+(j) \) (for \( \alpha \in A \)) can be presented as the inverse limit of \( \sqcup \alpha \in \beta X^\alpha_+(j) \) for \( \beta \) running through finite subsets of \( A \) (in \( \text{Pro-OP} \); see §1.3); hence Proposition 3.1.1(4) yields the result.

\[ \square \]

3.2 The Gysin distinguished triangle and 'Gersten' Postnikov towers for the pro-spectra of pro-schemes

We make some more remarks on pro-schemes.

Remark 3.2.1. For pro-schemes \( U = \lim_i U_i \) and \( V = \lim_j V_j \) we will call an element of \( \text{Pro-OP}(U_+, V_+) \) a closed (resp. open) embedding if it can be obtained as the limit of closed (resp. open) embeddings of pointed varieties (so, we consider only pro-smooth sub-pro-schemes of pro-schemes). We define a general pro-embedding \( U \to V \) similarly; we will say that \( U \) is of codimension \( c \) in \( V \) if \( U_i \) is of codimension \( c \) in \( V_i \) for any \( i \); in particular, we will use this convention for defining the codimension of Zariski points. Similarly, we will say that an inverse limit of open embeddings such that all complements have codimensions \( \geq c \) is an open embedding of pro-schemes with complement of codimension \( \geq c \). Certainly, the complement of a closed sub-pro-scheme of \( V \) is always an open sub-pro-scheme. Also, the generic point of a connected pro-scheme is its open sub-pro-scheme.

Now we define normal bundles for closed embeddings of pro-schemes. For an embedding of a connected pro-scheme \( X = \lim X_i \) a normal bundle is an element of the direct limit of the sets of isomorphism classes of vector bundles over \( X_i \). If a pro-scheme \( X \) is actually a scheme, then any closed embedding of \( X \) into \( Y \) does yield a (normal) vector bundle over it; this is a
projective module over the coordinate ring of \( X \) if the latter is affine. Lastly, if \( X \) is connected then the rank of this module is the codimension of \( X \) in the corresponding component of \( Y \), where the codimension of \( X = \lim \downarrow X_i \) in \( X \) equals the one of any \( X_i \) that is connected.

Moreover, these observations are compatible with the isomorphisms given by Proposition 1.4.1(7). Besides, we can also pass to inverse limits for distinguished triangles given by part 3 of loc. cit. if the connecting morphisms come from \( \mathcal{O}P \); see Proposition 3.1.1(5).

**Proposition 3.2.2.** Let \( Z, X \) be pro-schemes, \( Z = \bigsqcup Z^\alpha \) (\( Z^\alpha \) are connected) be a closed sub-pro-scheme of \( X \). Then the following statements are fulfilled.

1. For any \( j \geq 0 \) the natural morphism \( \Sigma^\infty(X \setminus Z_+) \rightarrow \Sigma^\infty(X_+\langle j \rangle) \) extends to a distinguished triangle (in \( D^{big}(\mathcal{O}) \)): \( \Sigma^\infty(X \setminus Z_+) \rightarrow \Sigma^\infty(X_+\langle j \rangle) \rightarrow \prod \Sigma^\infty(N_{X,Z^\alpha}/N_{X,Z^\alpha} \setminus Z^\alpha\langle j \rangle) \).

2. Assume that all \( Z^\alpha \) come from semi-local \( k \)-schemes (or just from \( k \)-schemes such that all vector bundles over them are trivial), and that \( Z \) (and so, all \( Z^\alpha \)) are of codimension \( c \) in \( X \). Then the latter product converts into \( \prod \Sigma^\infty(Z^\alpha_+\langle j + c \rangle) \).

**Proof.**

1. As we have just noted, we can pass to the inverse limits of distinguished triangles given by Proposition 1.4.1(3). It remains to note that Corollary 3.1.2 enables us to rewrite the third vertex of this triangle in the form desired.

2. If the normal bundles over all \( Z^\alpha \) are trivial, then certainly \( N_{X,Z^\alpha}/N_{X,Z^\alpha} \setminus Z^\alpha\langle j \rangle \cong Z^\alpha_+\langle j + c \rangle \) (in \( \text{Pro} - \mathcal{O}P \)). It suffices to note that this is the case for semi-local \( Z^\alpha \) (see Remark 1.4.1).

**Remark 3.2.3.**

1. It seems that instead of Corollary 3.1.2 (which relies on \( SH^{St}(k) \)-cogenerators) we could have used a formal model category argument in the proof.

2. The isomorphism of the second assertion is (usually) not canonical.

Now we will construct a certain Postnikov tower \( Po(M) \) for \( M \) being the (twisted) pro-spectrum of a pro-scheme \( Z \) that will be related to the coniveau spectral sequences for (the cohomology of) \( Z \). Note that we consider the general case of an arbitrary pro-scheme \( Z \) (since in this paper pro-schemes play an important role); yet this case is not much distinct from the (partial) case of \( Z \in SmVar \).
Proposition 3.2.4. Let $Z = \sqcup Z^\alpha$ be a pro-scheme of dimension $\leq d$; for all $i \geq 0$ we denote by $Z^i$ the set of points of $Z$ of codimension $i$.

Then any $j \geq 0$ there exists a Postnikov tower for $M = \Sigma^\infty(Z_+(j))$ such that $l = j - 1$, $m = d + j$, $M_{i+j} \cong \prod_{z \in Z^i} \Sigma^\infty(z_+(j+i))$ for $0 \leq i \leq d$.

Proof. Since any product of distinguished triangles is distinguished (see Remark 1.2.2 of [Nee01]), we can assume $Z$ to be connected.

We consider a projective system $L$ whose elements are sequences of closed subschemes $\emptyset = Z_{d+1} \subset Z_d \subset Z_{d-1} \subset \cdots \subset Z_0$. Here $Z_0 \in SmVar$, $Z_i \in Var$ for all $i > 0$, $Z$ is (pro)-open in $Z_0$, $Z_0$ is connected; for all $i > 0$ we have: (all irreducible components of) all $Z_i$ are everywhere of codimension $\geq i$ in $Z_0$; $Z_{i+1}$ contains the singular locus of $Z_i$ (for $i \leq d$). The ordering in $L$ is given by open embeddings of varieties $U_i = Z_0 \setminus Z_i$ for $i > 0$. For $\lambda \in L$ we will denote the corresponding sequence by $\emptyset = Z^{\lambda}_{d+1} \subset Z^{\lambda}_d \subset Z^{\lambda}_{d-1} \subset \cdots \subset Z^{\lambda}_0$.

Now, for any $i \geq 0$ the limit $\lim_{\lambda \in L \downarrow \lambda} Z^{\lambda}_i \setminus Z^{\lambda}_{i+1}$ equals $\sqcup_{z \in Z^i} z$. Hence the previous proposition yields a distinguished triangle $\Sigma^\infty(\lim(Z^{\lambda}_0 \setminus Z^{\lambda}_{i+1} + j)) \to \Sigma^\infty(\lim(Z^{\lambda}_0 \setminus Z^{\lambda}_{i+1} + j)) \to \prod_{z \in Z^i} \Sigma^\infty(z_+(j+i))$. So, setting $Y_{i+j} = \Sigma^\infty(\lim(Z^{\lambda}_0 \setminus Z^{\lambda}_{i+1} + j))$ for $-1 \leq i \leq d$, one obtains a tower as desired.

Remark 3.2.5. 1. The same reasoning also yields an unbounded positive Postnikov tower for $M = \Sigma^\infty(Z_+)$ in the case when $Z$ is an arbitrary pro-scheme.

2. Certainly, if we shift a Postnikov tower for $\Sigma^\infty(Z_+(j))$ by $[c]$ for some $c \in \mathbb{Z}$, we obtain a Postnikov tower for $\Sigma^\infty(Z_+(j))[c]$. Yet if we want it to be a weight Postnikov tower with respect to the Gersten weight structure that we will construct below, we will also have to shift the indices by $c$.

Now we prepare to the construction of the Gersten weight structure.

Proposition 3.2.6. Let $X$ be a pro-scheme, $j \geq 0$. Then $\Sigma^\infty(X_+(j))$ belongs to the coenvelope of $\Sigma^\infty(U_+(j))[i]$ for $U \in SmVar$, $i \geq 0$.

Proof. By Corollary 3.1.2.3 it suffices to verify the statement for the connected components of $X$ (and for $j$ being fixed). Hence it is sufficient to prove that the result is valid for all pro-schemes of dimension $\leq d$ by induction on $d$.

So, for some $d > 0$ we can assume that $\Sigma^\infty(V_+(j))$ belongs to the coenvelope of $\Sigma^\infty(U_+(j))[i]$ if $V$ is any pro-scheme of dimension $\leq d - 1$. It suffices to verify that the same is true for a fixed connected $X$ of dimension $d$. 

36
Next, using Proposition 3.2.4 (together with Remark 1.4.2(1)) one can easily see that $X$ can be replaced by its generic point, i.e., it suffices to verify the statement for $X$ being the generic point of some smooth (connected) $Z$ of dimension $d$.

Now we apply Proposition 3.2.4. We obtain distinguished triangles (see the notation of Definition 1.2.4) $Y_{i+j-1} \rightarrow Y_{i+j} \rightarrow \prod_{z \in Z} \Sigma^\infty(\langle j+i \rangle)$ for $0 \leq i \leq d$. Now, $Y_{d+j} = \Sigma^\infty(Z_+(\langle j \rangle))$ belongs to the coenvelope in question, and the same is true for $\prod_{z \in Z} \Sigma^\infty(\langle z+(j+i) \rangle)$ for all $0 < i \leq d$ by the inductive assumption. Hence $Y_j = \Sigma^\infty(X_+(\langle j \rangle))$ also belongs to this coenvelope.

\[ \qed \]

### 3.3 The Gersten weight structure: construction and basic properties

Now we describe the main weight structure of this paper.

We apply Theorem 2.2.6 in the case $C = D^{big}$, $C' = \{ \Sigma^\infty(X_+[i]) \}$ for $X \in SmVar$, $i \geq 0$. Denote the corresponding category $C'$ by $\mathcal{D}$; note that the full embedding $Ho(c) : SH^{S^1}(k) \rightarrow \mathcal{D}^{big}$ restricts to an embedding $SH^{S^1}(k)^c \rightarrow \mathcal{D}$ (we will just write $SH^{S^1}(k)^c \subset \mathcal{D}$). We obtain a weight structure $w$ on $\mathcal{D}$. We will call it the Gersten weight structure, since it is closely connected with Gersten resolutions of cohomology (cf. Remark 4.4.4(2) below). By default, below $w$ will denote the Gersten weight structure.

Now we can easily prove several important properties of this structure.

**Theorem 3.3.1.** The following statements are valid.

1. $\mathcal{D}_{w \geq 0}$ is the coenvelope of $C' = \{ \Sigma^\infty(X_+[i]) \}$ for $X \in SmVar$, $i \geq 0$, $\mathcal{D}_{w \leq 0} = ^{C'}[1]$.

2. $\mathcal{D}$, $\mathcal{D}_{w \leq 0}$, and $\mathcal{D}_{w \geq 0}$ are closed with respect to all small products (both in $\mathcal{D}$ and in $\mathcal{D}^{big}$).

3. Small products of weight decompositions in $\mathcal{D}$ are weight decompositions, and small products of weight Postnikov towers (resp. of weight complexes $t(M_i)$ for $M_i \in Obj \mathcal{D}$) are weight Postnikov towers (resp. a choice for $t(\prod M_i)$).

4. $\Sigma^\infty(X_+)(\langle j \rangle) \in \mathcal{D}_{w \geq j}$ for any pro-scheme $X$ and any $j \geq 0$.  

37
5. If $X$ is the spectrum of a function field over $k$, or if $X$ is semi-local and $k$ is infinite then $\Sigma^\infty(X_+(j)) \in D_{w=j}$.

6. For any pro-scheme $X$ and any $j \geq 0$ the Postnikov tower for $\Sigma^\infty(X_+(j))$ given by Proposition 3.2.4 is a weight Postnikov tower for it. In particular, if $X$ is of dimension $\leq d$, then $\Sigma^\infty(X_+(j)) \in D_{[j, j+d]}$.

7. $H_w$ is equivalent to the Karoubization of the category of all $\prod \Sigma^\infty(\Spec K_i^+\{j_i\})$ for $K_i$ being function fields over $k$, $j_i \geq 0$.

8. All objects of $SH^S_1(k)^c$ are bounded in $D$ (recall that we assume $SH^S_1(k)^c$ to be a subcategory of $D$).

9. We have $\text{Obj} SH^S_1(k)^c \cap SH^S_1(k)^{t \leq 0} = \text{Obj} SH^S_1(k)^c \cap D_{w \geq 0}$.

10. If $f : U \to X$ is an open embedding of pro-schemes such that the complement is of codimension $\geq i$ in $X$ (see Remark 3.2.1), then $\Sigma^\infty(X/U_+(j)) \in D_{w \geq i+j}$.

11. $\{\Sigma^\infty(U_+^j)\}$ for $U \in \text{SmVar}$ cogenerate $D$; $w$ is non-degenerate from below.

**Proof.**

1. This is exactly the description of $w_{c^*}$ given by Theorem 2.2.6.
2. The first part of the assertion is given by part II.1 of loc. cit., the second one is given by part II.5 of loc. cit., the third one is contained in Proposition 2.1.3(7).
3. Immediate from Theorem 2.2.6(II.6).
4. For any $U \in \text{SmVar}$ we have $\Sigma^\infty(U_+(j)) \in D_{w \geq j}$ by Remark 1.4.2(1). Hence the statement follows immediately from Proposition 3.2.6 (recall the definition of coenvelope).
5. It remains to verify that $\Sigma^\infty(X_+(j)) \in D_{w \leq j}$. By the definition of the latter class, for any $U \in \text{SmVar}$, $i > j$ we should verify that $\Sigma^\infty(X_+(j)) \bot \Sigma^\infty(U_+)[i]$.

Now, we have $\Sigma^\infty(U_+)[i] \in SH^S_1(k)^{t \leq -i}$ by Proposition 1.4.3(1). Next, Proposition 3.1.1(4) yields that for $E = \Sigma^\infty(U_+)$ we have $D(\Sigma^\infty(X_+(j)), \Sigma^\infty(U_+)[i]) \cong E^j(X)$ (in the notation used in Proposition 1.4.5). Hence loc. cit. yields the result in the case of an infinite $k$; if $k$ is finite (and $X$ is the spectrum of a function field) then one should apply Proposition 1.4.3(3).
6. By the previous assertion, for this tower we obtain $M^i \in D_{w=0}$ (see Definition 1.2.3(2)). Hence Proposition 2.1.3(5) yields the result.
7. Immediate from the previous assertion by Theorem 2.2.6(III).
8. Immediate from the previous two assertions.
9. We have $D_{w \leq -1} \perp D_{w \geq 0}$. In particular, if $E \in D_{w \geq 0}$ then assertion 5 yields: for any function field $K/k$ we have $\Sigma^\infty(\text{Spec } K_+)[{-n}] \perp E$ for any $n > 0$. If $E$ also belongs to $\text{Obj} SH^{S^1}(k)^c$, this translates into $E^n(\text{Spec } K_+) = 0$ for all $n > 0$ (see Proposition 3.1.1(4)). Hence $E \in SH^{S^1}(k)^{t \leq 0}$ (see Proposition 1.4.3(3)).

Now we prove the converse implication. If $E \in SH^{S^1}(k)^{t \leq 0} \cap \text{Obj} SH^{S^1}(k)^c \cap \text{Obj} SH^{S^1}(k)^c$ then $E_j^{n+j}(\text{Spec } K_+) = \{0\}$ for all $n > 0$, $j \geq 0$, and function fields over $k$ (see Proposition 1.4.3(3)). Certainly this translates into $\Sigma^\infty(\text{Spec } K_+)[{-n - j}] \perp E$. Since $E$ is cocompact in $D$, it follows (by assertion 7) that $Hw[{-n}] \perp E$. Hence Proposition 2.1.3(19) (together with the previous assertion) yields that $E \in D_{w \geq 0}$.

10. Since $f$ induces a bijection of the corresponding sets of points of codimension $< i$, it suffices to combine assertion 6 with part 16 of Proposition 2.1.3.

11. Immediate from Theorem 2.2.6(II.3).

Remark 3.3.2. 1. Describing weight decompositions for arbitrary objects of $SH^{S^1}(k)^c \subset D$ explicitly seems to be difficult. Still, one can say something about these weight decompositions and weight complexes using their functoriality properties. In particular, knowing weight complexes for $X, Y \in \text{Obj} SH^{S^1}(k)^c$ (or just $\text{Obj} D$) and $f \in D(X, Y)$ one can describe the weight complex of $\text{Cone}(f)$ up to a homotopy equivalence as the corresponding cone. Besides, let $X \to Y \to Z$ be a distinguished triangle (in $D$). Then for any choice of $(X_{w \leq 0}, X_{w \geq 1})$ and $(Z_{w \leq 0}, Z_{w \geq 1})$ there exists a choice of $(Y_{w \leq 0}, Y_{w \geq 1})$ such that there exist distinguished triangles $X_{w \leq 0} \to Y_{w \leq 0} \to Z_{w \leq 0}$ and $X_{w \geq 1} \to Y_{w \geq 1} \to Z_{w \geq 1}$; see Lemma 1.5.4 of [Bon10a].

2. The author suspects that $w$ is also non-degenerate from above. In any case, we will mostly be interested in bounded objects.

3. Certainly, we could also have considered the Gersten weight structure on the whole $D^{\text{big}}$. Yet this does not seem to make much sense, since this weight structure will not 'detect' the objects of the corresponding 'extra summand' $C_1$; cf. Theorem 2.2.6(II.4).

Besides, the proof of Theorem 2.2.6(II.2) provides us with a weight-exact 'projection' $F : D^{\text{big}} \to D$. In particular, for $E \in \text{Obj} SH^{S^1}(k)$ we have $E \in D_{w \geq 0}^{\text{big}}$ if and only if $F(E) \in D_{w \geq 0}$. Moreover, if $E \in D_{w \geq 0}^{\text{big}} \cap \text{Obj} SH^{S^1}(k)$ then for any function field $K/k$ and $n > 0$ we have $E^n(\text{Spec } K_+) = \{0\}$; hence
3.4 Direct summand results for pro-spectra of semi-local schemes

Theorem 3.3.1 easily implies the following interesting result.

**Theorem 3.4.1.** 1. Assume that $k$ is infinite; let $S$ be a semi-local pro-scheme and $S_0$ be its dense sub-pro-scheme. Then $\Sigma^\infty(S_+)$ is a direct summand of $\Sigma^\infty(S_0+)$.  

2. Suppose moreover that $S_0 = S \setminus T$, where $T$ is a closed sub-pro-scheme of $S$. Then we have $\Sigma^\infty(S_0+) \cong \Sigma^\infty(S_+) \oplus \Sigma^\infty(N_{S,T}/N_{S,T \setminus T})[-1]$.  

3. Assume in addition that $T$ is of codimension $j$ (everywhere) in $S$. Then this decomposition takes the form $\Sigma^\infty(S_0+) \cong \Sigma^\infty(S_+) \oplus \Sigma^\infty(T_+(j))[-1]$.

**Proof.** We can assume that $S$ and $S_0$ are connected.

1. By Theorem 3.3.1(1), we have: $\Sigma^\infty(S_0+), \Sigma^\infty(S_+) \in D_{w \geq 0}$; $\Sigma^\infty(\text{Spec } k(S)_+)$ could be assumed to be the zeroth term of their weight complexes for a choice of weight complexes compatible with some positive Postnikov weight towers for them; the embedding $S_0 \to S$ is compatible with $id_{\Sigma^\infty(\text{Spec } k(S)_+)}$ (since we have a commutative triangle $\text{Spec } k(S) \to S_0 \to S$ of pro-schemes). Hence Proposition 2.1.3[11] yields the result.

2. By Proposition 3.2.2(1) we have a distinguished triangle $\Sigma^\infty(S_0+) \to \Sigma^\infty(S_+) \to \Sigma^\infty(N_{S,T}/N_{S,T \setminus T})[-1]$. By Theorem 3.3.1[15] we have $\Sigma^\infty(S_0+) \in D_{w \geq 0}$, $\Sigma^\infty(S_+) \in D_{w = 0}$, $\Sigma^\infty(N_{S,T}/N_{S,T \setminus T}) \in D_{w \geq 1}$. Hence Proposition 2.1.3[3] yields the result.

3. By Remark 1.4.4, $T$ is semi-local itself. Hence it suffices to combine the previous assertion with Proposition 3.2.2[2].

**Corollary 3.4.2.** I Assume that $k$ is infinite.

1. Let $S$ be a connected (affine essentially smooth) semi-local scheme; let $S_0$ be its generic point. Then $\Sigma^\infty(S_+)$ is a retract of $\Sigma^\infty(S_0+)$.  

2. Let $K$ be a function field over $k$; let $K'$ be the residue field for a geometric valuation $v$ of $K$ of rank $r$. Then $\Sigma^\infty(\text{Spec } K'_+)\{r\}$ is a retract of $\Sigma^\infty(\text{Spec } K_+)$.  

If $H_w$ is equivalent to the Karoubization of the category of all $\prod \Sigma^\infty(\text{Spec } K'_i)$ for $K_i$ being function fields over $k$.  

40
Proof. I.1. This is just a partial case of part 1 of the theorem.

2. Obviously, it suffices to prove the statement in the case $r = 1$. Next, $K$ is the function field of some normal projective variety over $k$. Hence there exists a $U \in SmVar$ such that: $k(U) = K$, $v$ yields a non-empty closed subscheme of $U$ of codimension 1 everywhere. It easily follows that there exists a pro-scheme $S$ (i.e., an inverse limit of smooth varieties) whose only points are the spectra of $K$ and $K'$. So, $S$ is local affine; hence it is semi-local.

By part 2 of the previous theorem, we have

$$\Sigma^\infty (\text{Spec } K_+) = \Sigma^\infty (S_+) \bigoplus \Sigma^\infty (\text{Spec } K'_+) \{1\};$$

this concludes the proof.

II If $k$ is infinite, assertion I.2 yields that we can get rid of the twists mentioned in the (very similar) Theorem [3.3.1(7)].

In $k$ is finite, one should apply the fact that $\Sigma^\infty (\text{Spec } K'_+) \{r\}$ is a retract of $\Sigma^\infty (\text{Spec } k(G^r_{et}(K)))$ instead (whereas the statement mentioned can be easily established using the method of the proof of Theorem [3.4.1(1)].

Remark 3.4.3.

1. Note that we do not construct any explicit splitting morphisms in the decompositions above. Probably, one cannot choose any canonical splittings here (in the general case); so there is no (automatic) compatibility for any pair of related decompositions. Respectively, though the pro-spectra coming from function fields contain tons of direct summands, there seems to be no general way to decompose them into indecomposable summands.

2. Yet Proposition [3.2.2] easily yields that $\Sigma^\infty (\text{Spec } k(t)_+) \cong \Sigma^\infty (\text{pt}_+) \bigoplus \prod \Sigma^\infty (z_+) \{1\};$ here $z$ runs through all closed points of $A^1$ (considered as a scheme over $k$; note here that $\Sigma^\infty (A^1_+) \cong \Sigma^\infty (\text{pt}_+)$).

Now we recall that any Postnikov weight tower for an $M \in Obj D$ defines an $Hw$-complex that is well defined up to homotopy. It turns out that the augmented Gersten weight complex for a semi-local pro-scheme splits.

Proposition 3.4.4. For a semi-local pro-scheme $S$ over an infinite $k$ consider the Postnikov tower of $M = \Sigma^\infty (S_+)$ given by Proposition [3.2.2]; denote the corresponding complex by $t(M) = M^i$. Then there exists some $N^i \in C_{w=0}$, $i \leq 0$, such that $t(M)$ is $C(Hw)$-isomorphic to $M^0 \bigoplus (\bigoplus_{i \leq 0} (N^i \rightarrow N^i)[i])$. 41
Proof. Theorem 3.3.1(6) yields that $t(M)$ is a weight complex for $M$. Hence the result is given by Proposition 2.1.3(5).

Below we will use this statement in order to deduce a similar result for cohomology.

4 On cohomology and coniveau spectral sequences

This is the central section of the paper.

In §4.1 we describe (following H. Krause) a natural method for extending cohomological functors from a triangulated (small) $C' \subset C$ to $C$. This method is compatible with the usual definition of cohomology for pro-schemes.

In §4.2 we (easily) translate the results of the previous section to cohomology; in particular, the cohomology of (the spectrum of) a function field $K/k$ contains direct summands corresponding to the cohomology of semi-local (affine essentially smooth) schemes whose generic point is $K$, as well as twisted cohomology of residue fields of $K$. Note: here the cohomology of pro-schemes mentioned is calculated in the 'usual' way.

In §4.3 we consider weight spectral sequences corresponding to (the Gersten weight structure) $w$. We observe that these spectral sequences generalize naturally the classical coniveau spectral sequences. Besides, for a fixed $H : \mathcal{D} \to A$ our (generalized) coniveau spectral sequence converging to $H^\ast(M)$ (where $M$ could be any object of $SH^{S_1}(k)^c$ or a bounded object of $\mathcal{D}$) is $\mathcal{D}$-functorial in $M$ (in particular, it is $SH^{S_1}(k)^c$-functorial if restricted to $SH^{S_1}(k)^c$); this fact is non-trivial even when restricted to the spectra of smooth varieties. Besides, we prove: the image of any extended cohomology of a compact motivic spectrum belonging to $SH^{S_1}(k)^{l\leq -r}$ (with respect to any $SH^{S_1}(k)^c$-morphism) in the cohomology of a smooth variety is supported in codimension $\geq r$ (for any $r > 0$).

In §4.4 we construct a nice duality $\Phi : \mathcal{D}^{op} \times SH^{S_1}(k) \to Ab$: we prove that $w$ is orthogonal to $t$ with respect to $\Phi$. It follows that our generalized coniveau spectral sequences can be expressed in terms of $t$ (starting from $E_2$); this vastly generalizes the corresponding seminal result of [BK94].

In §4.5 we note that for certain varieties and spectra one can choose quite 'economical' versions of weight Postnikov towers and (hence) of the (generalized) coniveau spectral sequences for cohomology.
In §4.6 we prove that all pure extended cohomological functors $D \to A$ come (via the correspondence provided by Corollary 2.3.4) from those (contravariant additive) functors $Hw \to A$ that convert all products into coproducts. As a consequence, we prove (if $k$ is infinite) that this result (applied for $A = Ab$) yields a complete description of the heart of $t$ (i.e., of the category of strictly homotopy invariant Nisnevich sheaves on $SmVar$).

4.1 Extending cohomology from $SH^{S^1}(k)^c$ to $D$

Certainly, we would like to apply the results of the previous sections to the cohomology of pro-spectra. The problem is that cohomology is 'usually' defined on $SH^{S^1}(k)$ (or on $SH^{S^1}(k)^c$). So we describe (and apply) a general method for extending cohomological functors from a full triangulated $C' \subset C$ to $C$ (after H. Krause). Its advantage is that it yields functors that are 'continuous' with respect to inverse limits in $D$.

The construction requires $C'$ to be skeletally small, i.e., there should exist a subset (not just a subclass!) $D \subset ObjC'$ such that any object of $C'$ is isomorphic to some element of $D$; this is certainly true for $SH^{S^1}(k)^c$. Since the distinction between small and skeletally small categories will not affect our arguments and results, we will ignore it in the rest of the paper.

Recall that for an abelian category $A$ and any small $C'$ the category $AddFun(C'^{op}, A)$ is abelian also; complexes in it are exact if and only if they are exact when applied to any object of $C'$, and the same is true for coproducts.

Now suppose that $A$ satisfies AB5.

**Proposition 4.1.1.** I Let $A$ be fixed; consider an $H' \in AddFun(C'^{op}, A)$.

1. One can construct an extension of $H'$ to an additive functor $H : C \to A$ (i.e., the restriction of $H$ to $C'$ is equal to $H'$). It is cohomological if and only if $H$ is. The correspondence $H' \mapsto H$ given by this construction is functorial and additive in the obvious sense.

2. Moreover, suppose that in $C$ we have a projective system $X_l, l \in L$, equipped with a compatible system of morphisms $X \to X_l$, such that the latter system for any $Y \in ObjC'$ induces an isomorphism $C(X, Y) \cong \lim_{\to}C(X_l, Y)$. Then we have $H(X) \cong \lim_{\to}H(X_l)$.

II Let $X \in ObjC'$ be fixed.

1. For a family of $X_l \in ObjC'$ and $f_l \in C(X, X_l)$ assume that $(f_l)$ induce a surjection $\bigoplus C(X_l, -) \to C(X, -)$ in $AddFun(C', Ab)$. Then $f_l$ also yield
a surjection $\bigoplus H'(X_i) \to H(X)$.

Moreover, such a set of $(X_i, f_i)$ exists for any $X \in \text{ObjC}$.

2. Let $F' \xrightarrow{f'} G' \xrightarrow{g'} H'$ be a (three-term) complex in $\text{AddFun}\left(\mathcal{C'}^{\text{op}}, \mathcal{A}\right)$ that is exact in the middle; suppose that $H'$ is cohomological. Then the complex $F \xrightarrow{f} G \xrightarrow{g} H$ (here $F, G, H, f, g$ are the corresponding extensions) is exact in the middle also.

3. If $H' \cong \coprod H'_i$ in $\text{AddFun}\left(\mathcal{C'}^{\text{op}}, \mathcal{A}\right)$, then $H(X) \cong \coprod H_i(X)$ for the corresponding extension functors $H_i$.

III Apply the previous assertions for $\mathcal{C} = \mathcal{D}$, $\mathcal{C}' = \text{SH}^S(k)^c$ (note that $\text{SH}^S(k)^c$ is skeletally small). Then the extension of $H' : \text{SH}^S(k)^c \to \mathcal{A}$ to $H : \mathcal{D} \to \mathcal{A}$ satisfies the following properties.

1. $H$ converts those inverse limits in $\mathcal{D}'$ that are mapped by $\text{Ho}(c)$ inside $\mathcal{D}$ into the corresponding direct limits in $\mathcal{A}$.

2. $H$ converts products in $\mathcal{D}$ into coproducts in $\mathcal{A}$.

3. $H$ can be characterized (up to a canonical isomorphism) as the only extension of $H'$ that satisfies the previous two assumptions.

4. $H$ converts countable homotopy limits in $\mathcal{D}$ into the corresponding direct limits in $\mathcal{A}$.

Proof. I, II: this is an easy application of the results of [Kra00]; see Proposition 1.2.1 of [Bon10b].

III These statements are easy consequences of assertion I.2.

In order to obtain assertion III.1 this result should be combined with Proposition 3.1.1[4]. Next, part 3 of loc. cit. (the cocompactness of objects of $\text{SH}^S(k)^c$ in $\mathcal{D}$) immediately yields assertion III.2.

Assertion III.3 is given by (the dual to) Lemma 2.3 of [Kra00] (note that $\text{SH}^S(k)^c$ cogenerates $\mathcal{D}$; see Theorem 3.3.1[11]).

In order to obtain assertion III.4 one should apply assertion III.2 and recall Lemma 2.2.3[3].

Remark 4.1.2. 1. In the setting of assertion III we will call $H$ an extended cohomology theory. Note that assertion III.3 yields a complete characterization of extended theories.

2. As a partial case of assertion III.1 we obtain for any $j \geq 0$ and any pro-scheme $X = \lim X_i$: $H(\Sigma^\infty(X_+(j))) \cong \lim H(\Sigma^\infty(X_{i+}(j)))$. This is certainly compatible with the usual way of extending cohomology from varieties to their inverse limits; so all of the results below can be applied to the 'classical' definitions of $K$-theory, algebraic cobordism, motivic cohomology, etc. of

44
semi-local schemes. For example, the value of an extended theory at the spectrum of an (essentially smooth) local (resp. Henselian) ring will be the corresponding Zariski (resp. Nisnevich) residue.

Also, recall that $H$ coincides with $H'$ on $SH^S(k)^c$ (see assertion I.1); hence we obtain ‘classical’ values of cohomology for all compact objects of $SH^S(k)$ also.

3. We will construct $\mathcal{D}'$ in §5.2 below as the category of (filtered) pro-objects of $PSH^S(k)$. Hence (by part I.2 of the Proposition) all extended cohomology theories factor through the category $Pro^{-}SH^S(k)$ (that is certainly not triangulated). Thus the pairing $\Phi : \mathcal{D}' \times SH^S(k)$ that we will construct in §4.4 below factors through $(Pro^{-}SH^S(k))^{op} \times SH^S(k)$.

4. Certainly, the direct summand results of Theorem 3.4.1 also yield similar statements in $Pro^{-}SH^S(k)^c \subset Pro^{-}SH^S(k)$.

4.2 On cohomology of pro-schemes, and its direct summands

We easily prove that the results of the previous section easily imply similar assertions for extended cohomology theories.

Proposition 4.2.1. Suppose $k$ is infinite.

Let $S$ be a semi-local pro-scheme; let $H : \mathcal{D} \to A$ be an extended cohomology theory.

1. Let $S_0$ be a dense sub-pro-scheme of $S$. Then $H(\Sigma^\infty(S_+))$ is a direct summand of $H(\Sigma^\infty(S_+))$.

2. Suppose moreover that $S_0 = S \setminus Z$, where $Z$ is a closed subscheme of $S$ of codimension $j > 0$. Then we have $H(\Sigma^\infty(S_+)) \cong H(\Sigma^\infty(S_+)) \oplus H(\Sigma^\infty(Z_+(j))[-1])$.

3. Suppose $S$ is connected and $S_0$ is the generic point of $S$. Then $H(\Sigma^\infty(S_+))$ is a retract of $H(\Sigma^\infty(S_+))$ in $A$.

4. Let $K$ be a function field over $k$. Let $K'$ be the residue field for a geometric valuation $v$ of $K$ of rank $j$. Then $H(\Sigma^\infty(Spec K'_+\{j\}))$ is a retract of $H(\Sigma^\infty(Spec K_+))$ in $A$.

5. Consider the Postnikov tower of $M = \Sigma^\infty(S_+)$ given by Proposition 3.2.4; denote the corresponding complex by $t(M) = M^i$. Denote by $T_H(S)$ the Cousin complex $(H(M^{-i}))$. Then there exist some $A^i \in \text{Obj}_A$ for $i \geq 0$ such that $T_H(S)$ is $C(A)$-isomorphic to $H(S) \oplus A^0 \to A^0 \oplus A^1 \to A^1 \oplus A^2 \to \ldots$. 

45
Proof. 1. Immediate from Theorem 3.4.1(1).
2. Immediate from Theorem 3.4.1(2).
3. Immediate from Corollary 3.4.2(1.1).
4. Immediate from Corollary 3.4.2(1.2).
5. Immediate from Proposition 3.4.4.

Remark 4.2.2. Certainly, assertion 5 for \( H \) being a ‘motivic homotopy’ theory is stronger than the universal exactness Theorem 6.2.1 of [CHK97]. A caution: the definition of the universal exactness given in ibid. is not quite correct; see [Sch04].

4.3 Coniveau spectral sequences for the cohomology of (pro)spectra

Let \( H : \mathcal{D} \to \mathbb{A} \) be a cohomological functor, \( M \in \text{Obj} \mathcal{D} \).

**Proposition 4.3.1.** I.1. Any choice of a weight spectral sequence \( T(H, M) \) (see Proposition 2.3.1) corresponding to the Gersten weight structure \( w \) is canonical and \( \mathcal{D} \)-functorial in \( X \) starting from \( E_2 \).

2. \( T(H, M) \) converges to \( H(M) \) if \( M \) is bounded with respect to \( w \).

3. Let \( H \) be an extended theory (see Remark 4.1.2), \( M = \Sigma^\infty(Z_+) \) for a \( Z \in \text{SmVar} \). Then any choice of \( T(H, M) \) starting from \( E_2 \) is canonically isomorphic to the classical coniveau spectral sequence (converging to the \( H \)-cohomology of \( Z \); see §1 of [CHK97]). In particular, the corresponding filtration is the coniveau one.

II. Let \( M' \in \text{Obj} SH^{S^1}(k)^c \cap SH^{S^1}(k)^{r \leq -r} \) for some \( r \in \mathbb{Z} \). Then the following statements are valid.

1. \( H(M') = (W^r(H))(M') \) (see Remark 2.3.2(1)).
2. For any \( g \in \text{D}(M, M') \) we have \( \text{Im}(H(g)) \subset (W^r(H))(M) \).
3. Now let \( H \) be an extended theory, \( M = \Sigma^\infty(Z_+) \) for a \( Z \in \text{SmVar} \). Then for any \( g \) as above the object \( \text{Im}(H(g)) \) is supported in codimension \( r \), i.e., there exists an open \( U \subset Z \) such that \( Z \setminus U \) is of codimension \( \geq r \) in \( Z \) and \( \text{Im}(H(g)) \) is killed by the restriction morphism \( H(\Sigma^\infty(Z_+)) \to H(\Sigma^\infty(U_+)) \).

**Proof.** I.1. This is just a partial case of Proposition 2.3.1

2. Immediate since \( w \) is bounded; see part 3 of loc. cit.

46
3. Recall that in Proposition 3.2.4 a Postnikov tower \( P^o(M) \) for \( M \) was obtained from certain ‘geometric’ Postnikov towers by passing to the inverse limit in \( \mathcal{D}' \).

Now, in §3 of [Deg13b] two exact couples (for the \( H \)-cohomology of \( Z \)) were constructed. One of them was obtained by applying \( H \) to our geometric towers and then passing to the inductive limit (in \( A \)). Moreover, it was shown that this couple yields the same (coniveau) spectral sequence as the other one mentioned in loc. cit. (see §2.1 of ibid.; cf. also Remark 2.4.1 of [Bon10a]), whereas the latter couple coincides with the one considered in §1.2 of [CHK97]. Furthermore, Remark 4.1.2(2) yields that the limit mentioned is (naturally) isomorphic to the spectral sequence obtained via \( H \) from \( P^o(M) \).

Next, since \( P^o(M) \) is a weight Postnikov tower for \( M \), the latter spectral sequence is one of the possible choices for \( T(H, M) \).

Lastly, assertion 1 yields that all other possible \( T(H, M) \) (they depend on the choice of a weight Postnikov tower for \( X \)) starting from \( E_2 \) are also canonically isomorphic to the classical coniveau spectral sequence mentioned.

II.1. By Theorem 3.3.1(9) we have \( M' \in \mathcal{D}_{w \geq r} \). Hence we can take \( w \geq r, M' = M' \), and the result is immediate from Remark 2.3.2(1).

2. Immediate from the \( \mathcal{D} \)-functoriality of our weight filtration (given by loc. cit.) together with the previous assertion.

3. This is an easy consequence of the previous assertion together with assertion I.3 (recall the usual definition of the coniveau filtration).

\[ \square \]

Remark 4.3.2. 1. Assertion II.3 together with its motivic analogue (see §6.4 below) could be quite actual for the study of ‘classical’ motives. Note that one can apply it for \( M' \) being a cone of some morphism \( \Sigma^\infty(Z'_+ \to \Sigma^\infty(Z_+ \text{ for some } Z \in SmVar (here Z could be a point), whereas \( H \) can be the \( i \)-th cohomology for some ‘standard’ cohomology theory and an \( i \in \mathbb{Z} \).

Besides, one can easily prove loc. cit. for any (not necessarily additive) functor \( \mathcal{D} \to A \) that converts homotopy limits into direct limits and sends zero morphisms into zero maps.

Certainly, the statement is interesting only if \( r > 0 \).

2. Assertion I.3 of the proposition yields a good reason to call (any choice of) \( T(H, M) \) a *generalized coniveau spectral sequence* (for a general \( H, A \), and \( M \in Obj \mathcal{D} \)); this will also distinguish (this version of) \( T \) from weight spectral sequences corresponding to other weight structures. We will give more justification for this term in Remark 4.4.4 below. So, the corresponding
filtration could be called the (generalized) coniveau filtration (for a general $M$).

3. Actually, in order to obtain a coniveau spectral sequence for $(H, Z)$ using the recipe of [Deg08a] and [CHK97] it is not sufficient to compute just the cohomology of (the spectra of smooth) varieties. One also needs to apply $H$ to certain objects of $\mathcal{OP}$ in order to compute the $E_1$-terms of the exact couple, whereas the connecting morphisms of the couple come from the natural comparison morphisms between relative cohomology and the cohomology of varieties (see §1.1 and Definition 5.1.1(a) of ibid.). So, for those 'classical' cohomology theories for which all of this information has an 'independent' definition, one should check whether it can be 'factored through $\mathcal{SH}$'. This seems to be true for all of 'well-known' theories. For $K$-theory this fact is given by Corollary 1.3.6 of [PPR09]; for étale cohomology the proof is easy; one could use an argument from the proof of Theorem 4.1 of [Deg13b]. On the other hand, in order to calculate the coniveau filtration it suffices to know the restriction of $H$ to the (spectra of smooth) varieties; so this does not require any of this complicated extra information. Besides, our (pretty standard) arguments yield that for any $i \in \mathbb{Z}$, $j \geq 0$ and any cohomology theory satisfying axioms 5.1.1(a), COH1, and COH3 of [CHK97] (which is certainly the case for all of the examples we are interested in) the $E_1$-terms of the 'standard' coniveau spectral sequences are isomorphic to our ones.

4.4 A duality of motivic spectral categories; comparing spectral sequences

In order to apply the formalism of orthogonal structures we need the following statement.

**Proposition 4.4.1.** For each $M \in \text{Obj} SH^{S^1}(k)$ consider the (cohomological) functor $H_M : \mathcal{D} \to \text{Ab}$ obtained by extending $\mathcal{D}(-, M)$ via Proposition 4.1.1(III).

Then the following statements are valid.

1. The pairing $\Phi : \mathcal{D}^{\text{op}} \times SH^{S^1}(k) \to \text{Ab} : \Phi(X, M) = H_M(X)$ is a nice duality of triangulated categories.

2. For any $X \in \text{Obj} \mathcal{D}$ the functor $\Phi(X, -)$ converts coproducts into coproducts.
II The Gersten weight structure \( w \) on \( \mathcal{D} \) is orthogonal to the homotopy \( t \)-structure \( t \) on \( SH^{S^1}(k) \).

III For any \( M \in \text{Obj} SH^{S^1}(k) \) the functor \( \Phi(-, M) \) converts filtered inverse limits in \( \mathcal{D}' \) into direct limits in \( Ab \).

Proof. I.1. Immediate from Proposition 2.4.2.
I.2. Immediate from Proposition 4.1.1(III.2).

II We should check: \( \Phi(X, Y) = \{0\} \) if either (1) \( X \in \mathcal{D}_{w \geq 0} \) and \( Y \in SH^{S^1}(k)^{t \geq 1} \) or if (2) \( X \in \mathcal{D}_{w \leq 0} \) and \( Y \in SH^{S^1}(k)^{t \leq -1} \).

We verify the orthogonality in question in the setting (1). Proposition 4.1.1(III.4) yields: it suffices to present \( X \) as a countable homotopy limit of certain \( X_i \in \text{Obj} \mathcal{D} \) such that \( \Phi(X_i, Y) = \{0\} \). Since \( \Phi(-, Y) \) is a cohomological functor, Theorem 3.3.1(1) yields: it suffices to verify that \( \Phi(\prod_{i} \Sigma^\infty (S_i)[n_i], Y) = \{0\} \) for any family of \( S_i \in \text{SmVar}, n_i \geq 0 \). Next, Proposition 4.1.1(III.2) yields that it suffices to verify this statement for a single \( S_i \). In this case the assertion is immediate from the definition of \( t \) (see Proposition 1.4.3(2)) and the fact that the restriction of \( \Phi(-, Y) \) to \( SH^{S^1}(k)^c \) (by definition) is just \( SH^{S^1}(k)(-, Y) \).

Lastly we verify the orthogonality assertion for setting (2). Since \( \Phi(X, -) \) is homological and respects coproducts, Proposition 1.4.3(5) yields: it suffices to verify that \( \Phi(X, \Sigma^\infty (S_i)[n_i]) = \{0\} \) for any \( S_i \in \text{SmVar}, n_i \geq 1 \). The latter follows from Proposition 4.1.1(II.1) together with the orthogonality axiom for \( w \).

III Immediate from Proposition 4.1.1(III.1).

Remark 4.4.2. Suppose we have an inductive system \( M_i \in \text{Obj} SH^{S^1}(k), i \in I \), connected by a compatible family of morphisms with some \( M \in SH^{S^1}(k) \) such that: for any \( Z \in \text{Obj} SH^{S^1}(k) \) we have \( SH^{S^1}(k)(Z, M_i) \cong \lim_{\rightarrow} SH^{S^1}(k)(Z, M_i) \) (via these morphisms \( M_i \to M \)). In such a situation it is reasonable to call \( M \) the homotopy colimit of \( M_i \).

Then Proposition 4.1.1(I.2) yields: \( X \in \text{Obj} \mathcal{D} \) we have \( \Phi(X, M) = \lim_{\rightarrow} \Phi(X, M_i) \). So, one may say that all objects of \( \mathcal{D} \) are 'compact with respect to \( \Phi \)', whereas part 3 of the proposition yields that all objects of \( SH^{S^1}(k) \) are 'cocompact with respect to \( \Phi \). Note that both of these statement would not be true if we would have taken \( \Phi(X, M) = \mathcal{D}^{big}(X, M) \).

Now we relate generalized coniveau spectral sequences with the homotopy \( t \)-structure (in \( SH^{S^1}(k) \)). This is a vast extension of the seminal results of
Corollary 4.4.3. If $H$ is represented by a $Y \in \text{Obj} SH^{S^1}(k)$ (via our $\Phi$) then for a motivic (pro)spectrum $M$ our generalized coniveau spectral sequence $T(H, M)$ starting from $E_2$ can be naturally and functorially expressed in terms of the cohomology of $M$ with coefficients in the $t$-truncations of $Y$ (as in Proposition 2.4.3).

Proof. Immediate from Proposition 4.4.1.

Remark 4.4.4. 1. Our comparison statement is true for the $Y$-cohomology of an arbitrary $M \in \text{Obj} SH^{S^1}(k)^c$; this extends to $SH^{S^1}(k)^c$ Theorem 4.1 of [Deg13b] (whereas the latter essentially extends the results of §6 of [BLO94]). We obtain one more reason to call $T$ (in this case) a generalized coniveau spectral sequence for (the cohomology of) motivic spectra.

Note also that the methods of Deglise do not (seem to) yield the $SH^{S^1}(k)^c$-functoriality of the isomorphism in question.

2. If $Y \in SH^{S^1}(k)^{t=0}$ (i.e., it is a strictly $A^1$-invariant Nisnevich sheaf with transfers; see Lemma 4.3.7 of [Mor03] or the proof of Corollary 4.6.3 below), then $E_2(T)$ yields the Gersten resolution for the sheaf $H(-,+)$ (when $X$ varies); this is why we called $w$ the Gersten weight structure.

3. In §2.3–2.5 of [Bon10b] for any weight structure $w$ on a triangulated category $C$, and a cohomological $H : C \to A$ certain virtual $t$-truncations were studied: for any $M \in \text{Obj} C$, $i \in \mathbb{Z}$, one considers $\tau \leq i H(M) = \text{Im}(H(w_{\geq -i-1}) \to H(w_{\geq -i}))$, $\tau \geq i H(M) = \text{Im}(H(w_{\leq -i}) \to H(w_{\leq -i+1}))$ (the connecting morphisms here are uniquely determined by the choices of the corresponding weight decompositions). Virtual $t$-truncations have several nice properties; in particular, we obtain cohomological functors $C \to A$. The name is justified by the following isomorphisms of functors from $\mathcal{D}$ to $\mathcal{A}$: $\Phi(-, (t \leq i Y)[-i]) \cong \tau_{\leq i}(\Phi(-, Y))$ and $\Phi(-, (t \geq i Y)[-i]) \cong \tau_{\geq i}(\Phi(-, Y))$; see Proposition 2.5.4(1) of [Bon10b].

4. A related observation: virtual $t$-truncations of arbitrary extended cohomology theories are also extended (see Theorem 3.3.1[3] and Remark 4.1.2(1)).
4.5 ’Simple’ coniveau spectral sequences for the cohomology of Artin-Tate spectra

The formalism of weight structures yields much flexibility for the calculation of $T(H, M)$ (that certainly yield spectral sequences that are canonically isomorphic starting from $E_2$). Usually none of these spectral sequences are ’simple’; yet for certain $M \in Obj SH^S(k)^c \subset Obj \mathfrak{D}$ there exist very ’economic’ weight Postnikov towers.

Indeed, assume that $M$ is an Artin-Tate spectrum, i.e., there exist finite extensions $k_i$ of $k$ such that $M$ belongs to the triangulated subcategory of $SH^S(k)^c$ generated by $\Sigma^\infty(Spec k_{i+}(j_i))$ (for $j_i \geq 0$; certainly, the number of $k_i$ can be assumed to be finite here). We note that there exist a weight structure for Artin-Tate spectra whose heart consists of (finite!) direct sums of objects of this sort (and their retracts; see Remark 2.1.4 (2)), whereas a weight Postnikov tower with respect to this weight structure is certainly also a Gersten weight structure (by Proposition 2.1.3(5)).

One can construct a vast family of varieties that yield Artin-Tate spectra. To this end it is useful to note that this category contains $\Sigma^\infty(Spec k_{i+}(j_i)), \Sigma^\infty(G_m^+), \Sigma^\infty(\mathbb{P}^n_+)$ for any $n \geq 0$, and it is a tensor triangulated subcategory of $SH^S(k)^c$ (whereas the tensor product corresponds to the product of varieties). Besides, if one replaces $SH^S(k)$ by $Ho(MGl-\text{Mod}(k))$ here (see §6.3 below) then can apply the ’usual’ Gysin distinguished triangle in order to find pro-schemes whose pro-spectra are Artin-Tate ones.

Lastly, one can detect Artin-Tate spectra using the weight complex functor (cf. Proposition 2.1.3(12)); see Corollary 8.1.2 of [Bon10a].

4.6 On pure extended cohomology theories

Corollary 2.3.4 gives a certain description of the ’big category’ of pure functors $\mathfrak{D} \to A$. Yet it is certainly more interesting to analyse the category of pure extended functors (when $A$ is an AB5-category) instead since these functors essentially yield a full subcategory of $\text{AddFun}(SH^S(k)^c, A)$.

It turns out: it is quite easy to distinguish pure extended functors from all other pure ones; the corresponding restriction is somewhat similar to the one for Brown representability.

**Proposition 4.6.1.** Suppose $A$ satisfies AB5.
Then restricting pure (see Definition 2.3.3) extended functors $\mathcal{O} \to A$ to $\mathcal{H} w$ yields an equivalence of the category of these functors with the category of those additive contravariant functors from $\mathcal{H} w$ to $A$ that satisfy the following cocompactness condition: they convert all (small) $\mathcal{H} w$-products into $A$-coproducts.

Proof. We only have to check that the restriction of the equivalence provided by Corollary 2.3.4 to pure extended functors yields the equivalence of this category with the one of cocompact functors from $\mathcal{H} w$ to $Ab$.

By Proposition 4.1.1(III.2), extended functors convert $\mathcal{H} w$-products into coproducts. Moreover, part III.3 of loc. cit. yields: it remains to prove that the functor obtained from a cocompact $H_0 : \mathcal{H} w^{op} \to A$ (by the method given by Proposition 2.1.3(14)) yields a functor that converts $\mathcal{O}$-products into $A$-coproducts. The latter is immediate from Theorem 3.3.1(3).

Remark 4.6.2. Note that the virtual $t$-cohomology of any extended cohomological functor is pure extended; cf. Remarks 4.4.4(3,4) and 2.3.5.

Applying our proposition we obtain a certain description of the heart of $t$, i.e., of the category of strictly homotopy invariant sheaves (see Definition 4.3.5 of [Mor03]).

Corollary 4.6.3. Suppose $k$ is infinite.

Then the restriction of $\Phi : \mathcal{O}^{op} \times SH^{Si}(k) \to Ab$ to $\mathcal{H} w^{op} \times Ht$ yields an equivalence of $Ht$ with the category of cocompact (additive contravariant) functors $\mathcal{H} w \to Ab$.

Proof. The previous proposition yields: the restriction of $\Phi$ mentioned gives an equivalence of the category of cocompact functors $\mathcal{H} w \to Ab$ with the one of pure extended functors from $\mathcal{O}$ to $Ab$. Applying Proposition 4.1.1(III.3) we obtain: it suffices to prove that $SH^{Si}(k)(-, -)$ fully embeds $Ht$ into the category of cohomological functors from $SH^{Si}(k)^c$ to $Ab$, and $Ht$-representable functors are exactly those ones whose extensions to $\mathcal{O}$ are pure. The first statement mentioned is an immediate consequence of Lemma 4.3.7(2) of [Mor03] that embeds $Ht$ into presheaves of abelian groups on $SmVar$. Next, Proposition 4.4.1 yields that objects of $Ht$ define pure functors on $\mathcal{O}$ (via $\Phi$) indeed. Besides, by Lemma 4.3.7(2) of [Mor03] we get: it suffices to check that the restriction of an $Ht$-representable theory $H'$ from $SH^{Si}(k)^c$ to the spectra of smooth varieties yields a strictly homotopy invariant sheaf.

52
N after sheafification, i.e., that \( H_{Nis}^i(N) \) are homotopy invariant functors \( SmVar^{op} \to Ab \) (cf. Proposition 1.4.15). Loc. cit. yields: it suffices to check that \( H_{Nis}^i(N)(X) \cong H^i(\Sigma^\infty(X_+)[-i]) \) for \( X \in SmVar \). Certainly, the latter fact would be obvious if we knew that \( H^i(\Sigma^\infty(-)) \) satisfies Nisnevich descent (i.e., it converts Nisnevich distinguished squares into long exact sequences; cf. part 5 of loc. cit.) and that for the corresponding extended \( H \) and any essentially smooth Henselian scheme \( Y/k \), \( i \neq 0 \), we have \( H(\Sigma^\infty(Y_+)[-i]) = 0 \). The first of these statements is given by loc. cit., whereas the second one is immediate from Proposition 1.2.1(3).

\[ \square \]

## 5 Conclusion of the proofs: the construction of \( \mathcal{D}' \) and \( \mathcal{D}^{big} \)

In this section we construct the categories \( \mathcal{D}' \) and \( \mathcal{D}^{big} \), and prove that they satisfy the properties listed in Proposition 3.1.1.

### 5.1 On the levelwise injective model for \( SH^S(k) \): reminder

First we recall some properties of (certain) injective model structures for the categories mentioned in §1.4. In this paper all the model categories will have functorial factorizations of morphisms.

**Proposition 5.1.1.** There exist proper simplicial model structures for the categories \( \Delta^{op} Pre^\bullet, \Delta^{op} Shv^\bullet \), for \( \Delta^{op} Shv^\bullet \) that is equal to \( \Delta^{op} Shv^\bullet \) as an 'abstract' category, and for a certain category \( PSH^S(k) \) whose homotopy category is \( SH^S(k) \) satisfying the following properties.

1. The cofibrations in \( \Delta^{op} Pre^\bullet \) and \( \Delta^{op} Shv^\bullet \) are exactly the (levelwise) injections; hence all objects of these categories are cofibrant.

2. The natural comparison functors \( \Delta^{op} Pre^\bullet \to \Delta^{op} Shv^\bullet \to \Delta^{op} Shv^\bullet \to PSH^S(k) \) are left Quillen functors.

3. For any \( j \geq 0 \) and open embeddings \( Z \to Y \to X \) the diagram corresponding to \( Y/Z(j) \to X/Z(j) \to X/Y(j) \) (cf. Proposition 1.4.13) in \( \Delta^{op} Shv^\bullet \) yields a cofibration sequence.

**Proof.** 1. This is just a part of (Jardine’s) definition of the injective model structures for these categories.
2. This statement is well-known to experts; it can be deduced from the results of [MoV99] and [Mor03]; see also Theorem 2.9 of [Jar00].

3. For $j = 0$ we obviously obtain a cofibration sequence (for the injective model structure) already in $\Delta^{op}\text{Pre}_\bullet$ (as we have already said above). In the general case $Y/Z(j) \to X/Z(j)$ also yields an injection of discrete simplicial (pre)sheaves; hence it suffices to calculate the corresponding quotient sheaf. The isomorphism in question can be easily obtained by induction using Zariski descent for sheaves (that yields: the morphism of presheaves corresponding to $X/U \times X'/U' \to X' \times X/U \times U'$ becomes an isomorphism after Zariski sheafification). Note also that the natural analogue of the latter statement in $\Delta^{op}\text{Shv}'_\bullet$ (that is quite sufficient for our purposes) is mentioned in §2.3.2 of [Deg08b].

5.2 $\mathcal{D}'$ and $\mathcal{D}'_{big}$: definition and properties

First we define a certain (stable) model category $\mathcal{D}'$ using the results of [FaI07].

As a category it will be just the category of (filtered) pro-objects of $\text{PSH}^{S^1}(k)$ (see §5 of ibid.). We endow it with the strict model structure; see §5.1 of ibid. (so, weak equivalences and cofibrations are essential levelwise weak equivalence and cofibrations of pro-objects). An important observation here is that this model structure is a partial case of a $t$-model structure in the sense of §6 of ibid. if one takes the following 'degenerate' $t$-structure $t': \text{SH}^{S^1}(k)^{t \geq 0} = \text{ObjSH}^{S^1}(k)$, $\text{SH}^{S^1}(k)^{t \leq 0} = \{0\}$. Indeed (see Remark 6.4 of ibid.), $t'$ is a $t$-model structure in the sense of Definition 4.1 of ibid.: one has a functorial factorization of any morphism $f \in \text{SH}^{S^1}(k)(X, Y)$ (for $X, Y \in \text{ObjSH}^{S^1}(k)$) as $f \circ \text{id}_X$; note that $\text{id}_X$ is an $n$-equivalence and $f$ is a co-$n$-equivalence (in the sense of Definition 3.2 of ibid.) for any $n \in \mathbb{Z}$ (pay attention to Remark [1.2.3(4)]).

Now we list some basic properties of $\mathcal{D}'$ and its homotopy category $\mathcal{D}'_{big}$ (that are partial cases of the corresponding general statements). We will denote the pro-object corresponding to a projective system $X_i$ by $(X_i)$. Note that $(X_i)$ is exactly the (inverse) limit of the system $X_i$ in $\mathcal{D}'$ (by the definition of morphisms in this category).

**Proposition 5.2.1.** Let $X_i, Y_i, Z_i, i \in I$, be projective systems in $\text{PSH}^{S^1}(k)$. Then the following statements are valid.

54
1. $\mathcal{D}'$ is a proper stable simplicial model category.

2. If some morphisms $X_i \to Y_i$ for all $i \in I$ yield a compatible system of cofibrations (resp. of weak equivalences; resp. some couples of morphisms $X_i \to Y_i \to Z_i$ yield a compatible system of cofibration sequences) then the corresponding morphism $(X_i) \to (Y_i)$ is a cofibration also (resp. a weak equivalence; resp. the couple of morphisms $(X_i) \to (Y_i) \to (Z_i)$ is a cofibration sequence).

3. The natural embedding $c : PSH^{S^1}(k) \to \mathcal{D}'$ is a left Quillen functor; it also respects weak equivalences and fibrations.

4. For any $M \in \text{Obj} PSH^{S^1}(k)$ we have: $D_{big}((X_i), c(M)) \cong \varprojlim SH^{S^1}(k)(X_i, M)$.

In particular, the functor $Ho(c) : SH^{S^1}(k) \to D_{big}$ is a full embedding.

5. All objects of $Ho(c)(SH^{S^1}(k))$ are cocompact in $D_{big}$.

6. $Ho(c)(SH^{S^1}(k))$ cogenerates $D_{big}$.

Proof. 1. Theorems 6.3 and 6.13 of ibid. yield everything except the existence of functorial factorizations for morphisms in $\mathcal{D}'$ (see of [Hov99]). The existence of functorial factorizations is given by Theorem 1.2 of [BaS13] (see the text following Remark 1.4 of ibid.).

2. The first two parts of the assertion are contained in the definition of the strict $t$-structure. The last part follows immediately (since $\mathcal{D}'$ is proper).

3. The first part of the assertion is given by Lemma 8.1 (or §5.1) of [Fal07]. The second part is immediate from the description of weak equivalences in $\mathcal{D}'$ given in loc. cit.

4. Immediate from Corollary 8.7 of ibid.

5. We should verify that $D_{big}(\prod_{i \in I} Y_i, X) = \bigoplus_{i \in I} D_{big}(Y_i, X)$ for $Y_i$ being fibrant objects of $\mathcal{D}'$, $X \in Ho(c)(\text{Obj} SH^{S^1}(k))$. Now (see Example 1.3.11 of [Hov99]) the product of $Y_i$ in $D_{big}$ comes from their product in $\mathcal{D}'$. Certainly, if $Y_i = (Y_{ij})$ then their product in $\mathcal{D}'$ can be presented by the projective system of all $\prod_{i \in J} Y_{ij}$ for $J \subset I$ (i.e., we take at most one $Y_{ij}$ for each $i \in I$ and consider the corresponding index category; note here that products are partial cases of inverse limits). Hence the statement follows from the previous assertion.

Alternatively, one can combine the argument dual to the one in the proof of Theorem 7.4.3 of [Hov99] with the fact that $c(f)$ for $f$ running through all fibrations in $PSH^{S^1}(k)$ yield a set of generating fibrations for $\mathcal{D}'$ (see Theorem 6.1 of [Cho06]).

6. Loc. cit. also yields that $\mathcal{D}'$ admits a non-functorial version of the generalized cosmall object argument with respect to $c(f)$. Hence we can
apply the dual of the argument used in the proof of Theorem 7.3.1 of [Hov99].

Now we prove Proposition 3.1.1. Assertions 1 and 2 of loc. cit. are given by our constructions. Assertion 3 is given by parts 4 and 5 of the previous proposition. Assertion 4 also follows from part 4 of our proposition. Lastly, assertion 5 is immediate from the combination of its part 2 with Proposition 5.1.1(3).

6 The $T$-spectral and $MGl$-module versions of the main results; examples and remarks

In this section we describe certain variations of the methods and results of the previous ones, and discuss several examples of cohomology theories. We will be somewhat sketchy sometimes.

In §6.1 we prove that the (natural analogue of the) Gersten weight structure can be constructed for the stable motivic category of $T$-spectra also; the obvious analogues of the properties of $w$ established above can be proved without any difficulty. Moreover, the heart of the version of this weight structure for the 'τ-positive' part $SH^+(k)$ of $SH(k)[\frac{1}{2}]$ contains the corresponding spectra of all primitive (pro)schemes; hence in the corresponding pro-spectral category $D^+$ all the nice 'direct summand' properties of semi-local pro-schemes (proved above) have their 'primitive' analogues. These results rely on the 'τ-positive' acyclicity of primitive schemes that we prove in §6.2.

In §6.3 we verify that all our results can be carried over to the triangulated category of modules over the motivic cobordism spectrum (and the corresponding pro-category). $Ho(MGl − Mod(k))$ also supports a certain Chow weight structure (if char $k = 0$). In §6.4 we briefly compare our methods with the ones of [Bon10b], and note that the results of ibid. are also valid in the case when $k$ is not countable.

In §6.5 we discuss the consequences of our results for 'concrete' cohomology theories (algebraic, topological, and Hermitian $K$-theory, Balmer’s Witt groups, singular, algebraic and complex cobordism, étale and motivic cohomology).

In §6.6 we mention certain alternative methods for constructing $D'$ and $D$. 56
6.1 The $T$-spectral Gersten weight structures

In §5 of [Mor03] the stable model category of $\mathbb{P}^1$-motivic spectra was considered. By Remark 5.1.10 of ibid., this category is naturally Quillen equivalent to the (similarly defined) model category of $T$-spectra, where $T$ corresponds to $\text{pt}(\langle 1 \rangle)$. We denote the latter category by $PSH^T(k)$; its homotopy category will be denoted by $SH(k)$. Our constructions and result can be carried over from $PSH^S(k)$, $SH^S(k)$ (and other (pro)spectral categories) to this setting; we will say more on this in Theorem 6.1.2 below. In order to prove loc. cit. we start from formulating the $T$-spectral analogue of Propositions 1.4.1 and 1.4.3.

To this end we recall that $PSH^T(k)$ is (also) equipped with a left Quillen functor from $\Delta^{op}\text{Shv}'\cdot$; hence one can define the natural analogues $\Sigma^\infty_T$ and $\Sigma^\infty$ of $\Sigma^\infty$ and $\Sigma^\infty$, respectively. Besides, $PSH^T(k)$ and $SH(k)$ are equipped with natural functors $\wedge^T$. The main distinction of $T$-spectra from $S^1$-ones is that $\wedge^T$ is a (right) Quillen auto-equivalence of $PSH^T(k)$; so we will assume it to be invertible on $SH(k)$. Similarly to §1.4, we will denote the operation $\wedge^T_j[-j]$ by $\{j\}$ (for any $j \in \mathbb{Z}$).

Besides, $SH(k)$ is (also) endowed with a certain homotopy $t$-structure, which we will denote by $t^T$. It is defined (see Theorem 5.2.3 of ibid.) via the functors $\pi_n(-)_m$ for $n, m \in \mathbb{Z}$; those send $E \in SH(k)$ to the Nisnevich sheafification of the presheaf $U \mapsto E^{-m}_n(U) = SH(k)(\Sigma^\infty_T(U) \wedge T^{-n}[m], E)$ (see Remark 5.1.3 and Definition 5.1.12 of ibid). Similarly to §1.4 we extend $E^{-m}_n(-)$ to all inverse limits of objects of the type $X_+$ in $\text{Pro} - \mathcal{O}P$.

So, one easily obtains (most of) the connectivity properties of $SH(k)$ listed below (paying attention to Remark 1.2.3(4)). Moreover, it turns out that the class of $T$-spectra enjoying nice ‘connectivity’ properties is wider than the one coming from semi-local pro-schemes.

Proposition 6.1.1. The following statements are valid.

1. The functor $-\{j\}$ is $t$-exact with respect to $t^T$ for any $j \in \mathbb{Z}$.

2. For any $X \in \text{SmVar}$ we have: $\Sigma^\infty_T(X_+) \in SH(k)^{t^T \leq 0}$.

3. For $E \in \text{Obj} SH(k)$ we have: $E \in SH(k)^{t^T \geq 0}$ if and only if $E^{n+j}_j(X_+) = 0$ for any $X \in \text{SmVar}$, $n < 0$.

4. For $E \in \text{Obj} SH(k)$ we have: $E \in SH(k)^{t^T \leq 0}$ if and only if $E^{n+j}_j(\text{Spec } K_+) = 0$ for any $n > 0$, $j \in \mathbb{Z}$, and a function field $K/k$. 57
5. Suppose $k$ is infinite and $S$ is semi-local; then for any $E \in SH(k)^{t^T \leq 0}$, $i > j$ we have: $E^i_j(S) = \{0\}$.

6. Suppose that $S$ is primitive (see Definition 1.5.1); consider the projection $pr^+$ of $SH(k)[\frac{1}{2}]$ onto its subcategory $SH^+(k)$ of ‘$\tau$-positive’ objects (see the proof of Theorem 6.1.2(I.2) below for more details on this functor). Then for any $E \in Obj SH(k)[\frac{1}{2}]^{t^T \leq 0}$ (for the natural definition of $t^T$ for $SH(k)[\frac{1}{2}]$), $i > j$, we have: $(pr^+(E))^i_j(S) = \{0\}$.

Proof. 1. Immediate from the definition of $t^T$ (Definition 5.2.1 of ibid.).

2. See Example 5.2.2 of ibid.

3. Since $t^T$ is defined in terms of the corresponding Nisnevich sheaves, we should verify whether $E^{n+j}_j(S) = 0$ for all $n < 0$, $S$ being an essentially smooth Henselian scheme. Hence the assertion follows from the previous one (together with the orthogonality axiom of $t$-structures).

4. Applying the argument used in the proof of the previous assertion we get: it suffices to verify for a sheaf of the type $\pi_n(E)_m$ that it is 0 if it vanishes at all function fields over $k$. Now, all $\pi_n(E)_m$ are strictly homotopy invariant (see Remark 5.1.13 of ibid.). Hence the statement follows from Lemma 3.3.6 of ibid.

5. The statement follows from the previous assertions via the method used in the proof of Proposition 1.4.5.

6. See the next subsection. 

Now we note that minor modifications of the methods used for the study of $SH^{S^1}(k)$ yield the following results.

Theorem 6.1.2. I.1. There exist a natural analogue $\mathcal{D}^T$ of $\mathcal{D}$ that is closed with respect to all small products, and is equipped with an exact auto-equivalence $- \wedge T$ compatible with the $SH(k)$-version of this functor. $\mathcal{D}^T$ canonically contains the triangulated subcategory $SH(k)^c \subset SH(k)$ generated (in the sense described in §1.1) by $\Sigma^\infty_T(X)_+ \wedge T^j$ for $X \in SmVar$, $j \in \mathbb{Z}$, whereas $\Sigma^\infty_T$ extends to a functor $Pro-OP \to \mathcal{D}^T$ (that we also denote by $\Sigma^\infty_T$).

2. $t^T$ induces a $t$-structure $t^+$ on $SH^+(k)$.

3. The functor $pr^+$ yields a natural exact projection functor $pr^+_\mathcal{D}^T : \mathcal{D}^T[\frac{1}{2}] \to \mathcal{D}^+$ (via the method described in Remark 2.2.7(2)).

II The following statements are valid.
1. There exists a weight structure \( w^T \) on \( \mathcal{D}^T \) that is non-degenerate from below such that \( \mathcal{D}^T_{w^T \geq 0} \) is the coenvelope of \( C^T_1 = \{ \Sigma^\infty_T(X_+)[i\{j\}] \} \) for \( X \in \text{SmVar}, \ i \geq 0, \ j \in \mathbb{Z}, \mathcal{D}^T_{w^T \leq 0} = \perp(C^T_1[1]). \)

2. \( \{j\} \) is \( w^T \)-exact for any \( j \in \mathbb{Z}. \)

3. \( Hw^T \) is equivalent to the Karoubization of the category of all \( \prod \Sigma^\infty_T(\text{Spec} K_i + [j_i]) \) for \( K_i \) running through function fields over \( k, \ j_i \in \mathbb{Z}. \)

4. If \( f : U \to X \) is an open embedding of pro-schemes such that the complement is of codimension \( \geq i \) in \( X \) (see Remark 3.2.1), then \( \Sigma^\infty_T(X/U) \in \mathcal{D}^T_{w^T \geq i}. \)

5. For a pro-scheme \( X \) consider the Postnikov tower for \( \Sigma^\infty_T(X_+) \) given by the natural \( T \)-spectral analogue of the construction in Proposition 3.2.4. Then this Postnikov tower is a weight one. In particular, if \( X \) is of dimension \( \leq d, \) then \( \Sigma^\infty_T(X_+) \in \mathcal{D}^T_{[0,d]} \).

6. For any cohomological functor \( H' : SH(k)^c \to A (A \text{ satisfies AB5}) \) we consider its extension to a functor \( H : \mathcal{D}^T \to A \) via the method mentioned in Proposition 4.1.1.

   In particular, we do so for the functor \( SH(k)(-, M) \) (from \( SH(k)^c \) to \( Ab \)) for all \( M \in \text{Obj} SH(k). \) Then the collection of these functors yields a nice duality \( \Phi_T : (\mathcal{D}^T)^{op} \times SH(k) \to Ab \) such that \( w^T \perp \Phi_T t^T. \)

7. For any cohomological functor \( H : \mathcal{D}^T \to A \) the weight spectral sequence \( T = T(M,H) \) (for \( M \in \text{Obj} \mathcal{D}^T \)) corresponding to \( w^T \) is functorial in \( H \) and is \( \mathcal{D}^T \)-functorial in \( M \) starting from \( E_2. \) In the case when \( M = \Sigma^\infty_T(X_+), \ X \in \text{SmVar}, \) one can choose \( T(H,M) \) to be the ‘standard’ coniveau spectral sequence (starting from \( E_1; \) certainly, \( T(H,M) \) does not depend on any choices starting from \( E_2; \) cf. 4.2).

   We will call such a \( T(H,M) \) a generalized coniveau spectral sequence (as we also did in loc. cit.).

8. For any \( r \in \mathbb{Z} \) we have: \( \text{Obj} SH(k)^c \cap SH(k)^{t^T \leq -r} = \text{Obj} SH(k)^c \cap \mathcal{D}^T_{w^T \geq r}. \)

   Besides, for any \( M' \in \text{Obj} SH(k)^c \cap SH(k)^{t^T \leq -r}, \ M = \Sigma^\infty_T(Z) \) (for a \( Z \in \text{SmVar} \)), \( g \in \mathcal{D}^T(M,M'), \) and an extended \( H : \mathcal{D}^T \to A \) we
have: \( \text{Im}(H(g)) \) is supported at codimension \( \geq r \) in \( Z \) (cf. Proposition 4.3.1(II.3)).

9. For any \( N \in \text{Obj}_{\text{SH}}(k) \) the generalized coniveau spectral sequences for the functor \( \Phi_T(-, N) \) can be \( \mathcal{D}^T \)-functorially expressed in terms of the \( t^T \)-truncations of \( N \) starting from \( E_2 \) (cf. Proposition 2.4.3).

10. The category of pure extended cohomological functors from \( \mathcal{D}^T \) to an abelian \( \mathbb{A} \) (satisfying AB5) is naturally equivalent to the category of those contravariant additive functors \( Hw^T \to \mathbb{A} \) that convert all \( Hw^T \)-products into coproducts.

11. If \( k \) is infinite then the category of those contravariant additive functors \( Hw^T \to \text{Ab} \) that convert all \( Hw^T \)-products into coproducts is also equivalent to the heart of \( t^T \).

12. If \( k \) is infinite and \( X \) is semi-local, then \( \Sigma^\infty_T(X_+) \in \mathcal{D}^T_{w^r=0} \).

13. If \( k \) is infinite and \( S \) is a connected semi-local pro-scheme, \( S_0 \) is its dense sub-pro-scheme, then \( \Sigma^\infty_T(S_+) \) is a direct summand of \( \Sigma^\infty_T(S_{0+}) \). Moreover, if \( S_0 = S \setminus Z \), where \( Z \) is a closed sub-pro-scheme of \( S \), then \( \Sigma^\infty_T(S_{0+}) \cong \Sigma^\infty_T(S_+) \bigoplus \Sigma^\infty_T(N_SZ/N_{S,Z} \setminus Z)[-1] \). If \( Z \) is of codimension \( j \) (everywhere) in \( S \), then this decomposition takes the form \( \Sigma^\infty_T(S_{0+}) \cong \Sigma^\infty_T(S_+) \bigoplus \Sigma^\infty_T(Z_+) \wedge T^j[-1] \).

14. Suppose \( k \) is infinite. Let \( K \) be a function field over \( k \); let \( K' \) be the residue field for a geometric valuation \( v \) of \( K \) of rank \( r \). Then \( \Sigma^\infty_T(\text{Spec } K') \{r\} \) is a retract of \( \Sigma^\infty_T(\text{Spec } K_+) \).

15. For an infinite \( k \) and an extended cohomological functor \( H \) (see assertion 10) the natural analogues of the previous two assertions hold (for the \( H \)-cohomology of the corresponding pro-schemes).

16. Suppose \( k \) is infinite and \( S \) is a semi-local pro-scheme; consider the Postnikov tower of \( M = \Sigma^\infty_T(S_+) \) given by assertion 13 and denote the corresponding complex by \( t(M) = M^i \). Denote by \( T_H(S) \) the complex \( (H(M^{-i})) \). Then there exist some \( A^i \in \text{Obj} \mathbb{A} \) for \( i \geq 0 \) such that \( T_H(S) \) is \( C(\mathbb{A}) \)-isomorphic to \( H(S) \bigoplus A^0 \to A^0 \bigoplus A^1 \to A^1 \bigoplus A^2 \to \ldots \).

III The natural analogues of the assertions of part II hold if we replace \( \mathcal{D}^T \) by \( \mathcal{D}^+ \), \( \Sigma^\infty_T \) by \( \Sigma^\infty_+ = \text{pr}^+_T \circ \Sigma^\infty_T[1/2] \), \( C^r_T \) by \( C_+ = \text{pr}^+_T(C^r_T) \), \( w^T \) by \( w^+ \).
the corresponding $w^+$, $SH(k)^c$ by $SH^+(k)^c = pr^+(SH(k)^c[\frac{1}{2}])$, $\Phi_T$ by the corresponding $\Phi_+$, and $t^T$ by $t^+$.

Moreover, in the analogues of assertions [11]--[16] it is not necessary to assume that $k$ is infinite; if it is infinite, then it suffices to assume that $S$ is primitive (in all of these assertions expect [11] and [14]).

Proof. I.1. The main distinction of this assertion from its $SH^{S^1}(k)$-analogue is that we want to extend $\wedge T$ to an invertible exact functor on $D_T$. This is easy since (as we have already said) $\wedge T$ possesses a 'nice lift' to $P SH_T(k)^{[\frac{1}{2}]}$.

2. Recall that $pr^+$ is the natural projection of the category $SH(k)^{[\frac{1}{2}]}$ (obtained by the method described in Remark 2.2.7(1); cf. also Appendix B of [Lev13]) onto the first summand in the Morel’s decomposition $SH(k)^{[\frac{1}{2}]} \cong SH^+(k) \oplus SH^-(k)$ (see the text preceding Lemma 6.7 of ibid.). So, $pr^+$ is an idempotent endofunctor of $SH(k)^{[\frac{1}{2}]}$, and for any $M \in Obj SH(k)$ the object $pr^+(M)$ is a canonical direct summand of $M_{SH(k)^{[\frac{1}{2}]}}$. The assertion follows immediately.

3. Remark 2.2.7(1,2) yields the result.

II Again, we can use the $SH(k)$-analogues of the methods used in the previous sections (together with Proposition 6.1.1). The main distinction is given by the $-\{j\}$-stability of the corresponding $C_T'$ (for any $j \in \mathbb{Z}$); since $-\{j\}$ is an automorphism of $D_T$, it is $w^T$-exact.

III Again, Remark 2.2.7 yields the result. Indeed, all the distinguished triangles, cocompact objects, and morphism group calculations we require 'come from' $SH(k)$ and $D_T$, whereas the duality $D^+ \times SH(k)^{[\frac{1}{2}]} \rightarrow Ab$ induced by $\Phi$ restricts to a duality $\Phi^+ : D^+ \times SH^+(k) \rightarrow Ab$. Note here: $\Phi^+$ can be used (in particular) for the calculation of $D^+(\Sigma_{T,+}^{\infty}(M), N)$ for any $M \in Obj Pro -OP$, $N \in Obj SH^+(k)^c \subset Obj D^+$. 

\[ \square \]

Remark 6.1.3. 1. We could have considered some bispectral model for $SH(k)$; this would have given a natural left Quillen connection functor $PSH^{S^1}(k) \rightarrow PSH_T^+(k)$ that would certainly yield exact functors $SH^{S^1}(k) \rightarrow SH(k)$ and $D \rightarrow D_T$. This would have allowed carrying over some of the properties from $S^1$-(pro)spectra to $T$-ones directly.

We will also apply this observation in §6.5 below.

2. We note that $SH^+(k) = SH(k)^{[\frac{1}{2}]}$ if $-1$ is a sum of squares in $k$. Indeed, in the case $\text{char } k > 0$ this fact is given by Lemma 6.8 of [Lev13]. For
This statement can be easily extracted from the proof of Lemma 6.7 of ibid.; we will give the detail in the next subsection.

3. We conjecture that in the case when $\cdot 1$ is a sum of squares in $k$, the natural analogues of Proposition 6.1.1(6) and (thus) of Theorem 6.1.2(III) hold for $SH(k)$ instead of $SH^+(k)$.

### 6.2 The $\tau$-positive acyclicity’ of primitive schemes

Now we prove Proposition 6.1.1(6). We reduce it to the ‘weakly orientable spectral acyclity of primitive schemes’.

We recall that the heart of $t^T$ is the category of homotopy modules (see Definition 5.2.4 of [Mor03] or Definition 1.2.2 of [Deg13a]). This category possesses an exact faithful forgetful functor to the abelian category of $\mathbb{Z}$-graded Nisnevich sheaves: $M \mapsto (M_n)$; here $M_n = \pi_0(M)_n$ in the notation considered in the beginning of the previous subsection. Besides, any object $M$ of $Ht^T$ is (functorially) equipped with a system of morphisms $\eta = (\eta_n : M_n \to M_{n-1})$ (for $n$ running over $\mathbb{Z}$); see §6.2 of [Mor03] and Definition 1.2.7 of [Deg13a]. A homotopy module is called orientable if all of these $\eta_n$ are zero.

Certainly, any object of $Ht^+$ also yields a homotopy module (that is $\mathbb{Z}[\frac{1}{2}]$-linear).

Now we prove an interesting lemma that we will apply below in the case when $S$ is primitive.

**Proposition 6.2.1.** Assume that a pro-scheme $S$ possesses the following property: for any $F$ being a homotopy invariant (Nisnevich) sheaf with transfers in the sense of §3.1 of [Voe00a] we have $H^i_{Nis}(S, F) = \{0\}$ for all $i > 0$. Then the following statements are valid.

1. Suppose $E$ belongs to $SH(k)^{t^T \leq 0}$ (resp. to $SH^+(k)^{t^+ \leq 0}$); assume also that $E^{t^T = m}$ (resp. $E^{t^+ = m}$) is orientable for any $m < 0$ (i.e., that $E$ is weakly orientable in the sense of §4.2 of [Deg13a]). Then for any $i > j \in \mathbb{Z}$ we have $E^i_j(S) = \{0\}$.

2. In particular, this is true for any $E \in SH^+(k)^{t^+ \leq 0}$.

**Proof.** Obviously, we can assume that $S$ is connected of some dimension $d$. Then $E^i_j(S) = \{0\}$ for any $E \in SH(k)^{t^T \leq -d-1}$ (resp. for $E \in SH^+(k)^{t^+ \leq -d-1}$; immediate from Theorem 6.1.2(II.5.6)). Hence the natural spectral sequence $(E^{t^T = 0})^{j+p}_j(S_+) \Rightarrow E^j_j(S_+)$ (resp. its $t^+$-analogue) converges. Thus we can assume that $E = M[r]$ for some $r \geq 0$, $M \in Obj Ht^T$ (resp. in $M \in Obj Ht^+$).
1. By Remark 1.2.4 of [Deg13a] for \( M = (M_i) \) we have: 
\[ E_i^j(S_+) = H^{r+i-j}(S, M_j) \]
we define \( H^*(\cdot, M_j) \) on pro-schemes via the method of §1.4.1. Next, Theorem 1.3.4 of ibid. implies that \( M_j \) is a homotopy invariant sheaf with transfers, and we obtain the result.

2. Certainly, it suffices to verify that all homotopy modules for \( H_t^+ \) are orientable. Now, recall (see §6 of [Lev13]) that the graded ring \( K_*^{MW}(k) \) (functorially) acts on \( \bigoplus M_j \), whereas for any \( M \in \text{Obj}SH^+(k) \) (by definition) the corresponding action of \( \tau = id + \eta \circ [-1] \) is identical. Rewriting the relation 4 in Definition 6.3.1 of [Mor03] as \( \eta \tau + \eta = 0 \) we obtain the result.

Next we finish the proof of Remark 6.1.3(2). Since \(-1\) is a sum of squares, it is well known that the Witt ring \( W(k) \) is annihilated by \( 2^N \) for some \( N \geq 0 \). It follows that \( 2^N \eta = 0 \) in \( K_*^{MW}(k) \). Indeed, this statement is given by Lemma 6.7 of [Lev13] in the case \( \text{char} \ k > 0 \), whereas for \( \text{char} \ k = 0 \) one should apply the argument used in the proof of loc. cit. (i.e., use the natural injection of groups \( W(k) \eta \to K_*^{MW}(k) \)).

It remains to verify the 'motivic acyclicity' of primitive schemes.

**Proposition 6.2.2.** Let \( S \) be a connected primitive pro-scheme, let \( S_0 \) be its generic point; assume that \( F : SmVar \to Ab \) is a homotopy invariant Nisnevich sheaf with transfers (in the sense of [Voe00a]).

Then \( H^i(F, S) = \{0\} \) for any \( i > 0 \).

**Proof.** In the case of an infinite \( k \) our assertion follows from Theorem 4.19 of [Wal98]. If \( k \) is finite, then \( S_0 \) is semi-local (by our convention); so we may apply Corollary 4.18 of [Voe00b] instead.

**Remark 6.2.3.** 1. So, we proved the statement in question via 'decomposing' \( E \in \text{Obj}SH(k) \) into its \( t^F \)-components, and using its weak orientability (in the sense of [Deg13a]). If one wants to study the 2-torsion part of \( SH(k) \) (also), it could make sense to look for a filtration of \( E \in \text{Obj}SH(k) \) such that its 'factors' (in the sense of Postnikov towers) are weakly orientable.

There is an important and well-known candidate for such a filtration (for effective objects of \( SH(k) \)): this is the slice filtration (as defined by Voevodsky). Note here that the slices are modules over the Eilenberg-Maclane spectrum \( MZ \) (that represents motivic cohomology; see Theorem 3.2 of [Pel09]); hence they are weakly orientable by Lemma 4.2.2 of [Deg13a]. Besides, passing to slices preserves \( SH(k)^{t \leq 0} \); see Lemma 4.3 of [Lev12]. The main problem
with this method is the convergence question for the corresponding spectral sequence; it is only known (by Theorem 4 of [Lev13]) if \( k \) is of finite cohomological dimension. This is rather restrictive; yet possibly the proof of loc. cit. (for 2-torsion spectra) could be modified in order to yield the result when \(-1\) is a sum of squares in \( k \) (at least, when \( \text{char} \ k \neq 2 \)).

2. One could try to apply the ‘slice method’ in order to extend Theorem 6.1.2(III) to \( \mathcal{D} \) (or to \( \mathcal{D}[\frac{1}{p}] \) if \( p > 0 \); hence also to cohomology theories that factor through \( \text{SH}^S(k)^c \) or through \( \text{SH}^S(k)^c[\frac{1}{p}] \), respectively) by applying the results of [Lev10]. Yet the author does not know of any conditions that would ensure the convergence of the slice spectral sequence for elements of \( \text{SH}^S(k)^c \).

3. It is a very interesting question whether one can prove (some version of) the \( \text{SH}(k) \)-acyclity of primitive schemes ‘directly’, i.e., using a version of Voevodsky’s split standard triple argument (as in the proof of Theorem 4.19 of [Wal98]). Possibly this can be done using Voevodsky’s framed correspondences.

6.3 On the Gersten and Chow weight structures for modules over the algebraic cobordism spectrum

As we have seen in the previous subsection, passing from \( \text{SH}(k) \) to \( \text{SH}^+(k) \) enforces certain ‘orientability’ on \( T \)-spectra. Another (probably, more well-known) method for introducing ‘orientations’ is to consider modules over the \( T \)-ring spectrum \( \text{MGl} \) representing the motivic cobordism spectrum (see Theorem 1.0.1 of [PPR08]). Recall that \( K \)-theory, motivic and étale cohomology, and (certainly) algebraic cobordism are orientable in this sense; so that the spectra representing them are \( \text{MGl} \)-modules and we can apply the results of this subsection to these cohomology theories (see §6.5 below for some more details).

Now we verify that our results can be applied to the corresponding category \( \text{MGl} - \text{Mod}(k) \) (of left \( \text{MGl} \)-modules in \( \text{PSH}^T(k) \)). Similarly to Proposition 38 of [OsR08], one can verify that \( \text{MGl} - \text{Mod}(k) \) is endowed with a proper stable model structure such the ‘free module functor’ \( \text{PSH}^S(k) \rightarrow \text{MGl} - \text{Mod}(k) \) \( (M \mapsto \text{MGl} \wedge M) \) is a left Quillen one. Actually, it seems that some effort is needed in order to guarantee the left properness of \( \text{MGl} - \text{Mod}(k) \); its right properness is obvious. We denote \( \text{Ho}(\text{MGl} - \text{Mod}(k)) \) by \( \text{SH}^{\text{MGl}}(k) \).
Next, we consider the category $\text{Pro} - MGl - \text{Mod}(k)$; the corresponding category $\mathfrak{D}^{MGl} \subset \text{Ho}(\text{Pro} - MGl - \text{Mod}(k))$ will be triangulated and will contain the naturally defined 'MGl-module pro-spectra' for all pro-schemes. The natural analogues of all the assertions of Proposition 3.1.1 will be valid.

Hence in order to ensure the existence (and nice properties) of the Gersten weight structure for $D^{MGl}$ (cf. Theorem 6.1.2(III)) it suffices to verify the 'MGl-module acyclicity' of primitive schemes. Certainly, the right adjoint to the free $MGl$-module functor is the natural forgetful functor. Hence one should verify: for a primitive pro-scheme $S$ and any $X \in \text{SmVar}$, $E = \Sigma^{\infty}(X_+) \land MGl$ we have $(E)^i_j(S_+) = \{0\}$ if $i > j \in \mathbb{Z}$. As shown in the previous subsection, to this end it suffices to check that $E$ belongs to $SH(k)^{T \leq 0}$ and is weakly orientable. The first statement is immediate from the $t$-non positivity of $\Sigma^{\infty}(X_+)$ (see Proposition 6.1.2) and of $MGl$ (see Corollary 3.9 of [Hoy13]), and the fact that $SH(k)^{T \leq 0}$ is $\land$-closed (see §1.2.3 of [Deg13a]). The weak orientability is given by Corollary 4.1.7 of ibid.

Remark 6.3.1. 1. One of the advantages of $MGl$-modules is that the corresponding spectra of Thom spaces of vector bundles only depend on the bases of the bundles and on their dimension. Hence if $U = P \cup D_i$, where $P$ and all intersections of $D_i$ are products of $\mathbb{G}_m$, $\mathbb{A}^1$, projective spaces, and (the spectra of) finite extensions of $k$, then the $MGl$-spectrum of $U$ is an Artin-Tate one, and one can write 'simple' coniveau spectral sequences for the cohomology of $U$ (see §4.5; cf. also the proof of Theorem 4.27 of [LeW13] and Proposition 6.5.1 of [Bon09]).

2. Mark Hoyis has kindly written down the proof the following statement (see [Hoy12]): if char $k = 0$, then $MGl^{2n+i}_n(X) = \{0\}$ for any $i \geq 0$ and smooth projective $X$. Applying the Poincare duality in $SH(k)$ (see Theorem 2.2 of [Rio05]), we obtain the $SH^{MGl}(k)$-negativity (see Definition 2.1.1) of the category of $MGl$-module spectra of smooth projective $k$-varieties (together with their $T$-twists). Thus these twists will belong to the heart of the weight structure 'cogenerated' by them in $\mathfrak{D}^{MGl}$ (see Theorem 2.2.6 and Theorem 4.5.2 (I.2) of [Bon10a]). We obtain a certain Chow weight structure on $\mathfrak{D}^{MGl}$ (that could also be defined on $SH^{MGl}(k)$ itself; both of these structures restrict to a bounded weight structure on the subcategory of compact objects of $SH^{MGl}(k)$). Recall (from Remark 2.4.3(2) and §6.6 of [Bon10a]) that Chow weight structures yield a mighty tool for generalizing and studying weight filtrations and weight spectral sequences a-la Deligne.

Possibly, the author will treat this weight structure in a separate paper.
(someday).

3. We also obtain the existence of a certain Chow $t$-structure $t_{Chow}$ on $SH^{MGl}(k)$ (by Theorem 4.5.2(I.1) of [Bon10a]; cf. also §7.1 of ibid.) such that the corresponding Chow weight structure is adjacent (i.e., orthogonal with respect to $SH^{MGl}(k)(-, -)$) to $w_{Chow,SH^{MGl}(k)}$. In particular, it follows that the heart of $t_{Chow}$ is isomorphic to the category of additive contravariant functors from the category of ‘Chow-cobordism’ motives to abelian groups (see Theorem 4.5.2(II.2) of ibid). Besides, the Chow weight structure on $D^{MGl}$ is also orthogonal to $t_{Chow}$ with respect to the corresponding $\Phi_{MGl}$.

4. Note that there cannot exist any Chow weight structure on $SH(k)^c \subset D^T$ (whose heart will contain $\Sigma_\infty^T(P_+)$ for all smooth projective $P/k$) since $\eta$ yields a non-zero morphism from $\Sigma_\infty^T(P_+) \to \Sigma_\infty^T(pt_+)[1]$. The author does not know whether the Chow weight structure can be defined on $SH^+(k)^c \subset D^+$. This depends on the vanishing of the $\tau$-positive parts of certain ‘$T$-negative’ stable homotopy groups of spheres over all (finitely generated) extensions of $k$. On the other hand, $SH^+(k) \otimes \mathbb{Q}$ is isomorphic to the ‘big’ version $DM(k) \otimes \mathbb{Q}$ of the category of Voevodsky’s motives with rational coefficients (by a result of Morel; see Theorem 16.2.13 of [CiD09]); hence the Chow weight structure exists on this category and also on $D^+ \otimes \mathbb{Q}$.

6.4 Our methods and comotives

In [Bon10b] a certain category of comotives that we will here denote by $D^{mot}$ was constructed; this was a certain ‘completion’ of Voevodsky’s $DM^{eff}_{gm}(k)$. The ‘axiomatics’ of $D^{mot}$ was quite similar to Proposition 3.1.1 of the current paper (actually, no analogue of $D^{big}$ was considered in ibid.; yet $D^{mot}$ satisfies all the properties needed for the application of natural analogues of the arguments above). Another distinction of ibid. from this paper is that the construction of $D^{mot}$ used differential graded categories (and was somewhat similar to the ‘spectral version’ of $D$ that we will mention in the next subsection); the formalism of model categories was not applied directly (yet essentially a model for $D^{mot}$ was considered).

A serious disadvantage of the methods of [Bon10b] is that Gysin distinguished triangles were constructed using a ‘purely triangulated’ argument. This yielded the ‘comotic’ analogue of Proposition 3.2.2 for countable inverse limits of smooth varieties (with respect to open embeddings) only. Also, the Gersten weight structure in ibid. was constructed via a direct verification of the negativity of the category of (Tate twists) of comotives for primitive
schemes. This negativity check relied on the computation of certain higher inverse limits, that the author was only able to do for a countable \( k \) (since in this case \( \lim^i = 0 \) for \( i > 1 \)). Also, this method allowed to construct the Gersten weight structure only on the subcategory of \( D^{mot} \) generated by the corresponding version of \( HW \). Hence the corresponding computation of the heart of \( w \) was 'automatic', whereas our one (see Theorem 3.3.1(7) and Theorem 2.2.6(III)) is really non-trivial (and the author was quite amazed to find out that the result is parallel to that of [Bon10b]). Similarly, the verification of the fact that \( w \perp t \) requires more effort in our setting (see Proposition 4.4.1(II)) than the proof of Proposition 4.5.1(2) of ibid.; hence the current version of the proof is 'much more conceptual'. Also, it was more difficult to establish the 'duality' between \( HW \) and \( Ht \) given by Corollary 4.6.3 than the corresponding comotivic fact (since strict homotopy invariance seems to be a more mysterious restriction on a Nisnevich sheaf than the existence of transfers). Note also that we give a complete description of all pure extended cohomology theories (in Proposition 4.6.1 whereas no analogue of this result was mentioned in ibid.).

So, the methods of the current paper can be easily applied to (various versions) of comotives (moreover, the formalism of differential graded categories gives somewhat more flexibility for this setting than the model category one, though the latter one can also be applied). Thus we obtain the existence of the Gersten weight structure and the 'splitting' properties for the corresponding cohomology of primitive pro-schemes (cf. Theorem 6.1.2(III)) over an arbitrary perfect field \( k \). Note also that the methods of §6.2 of [Bon10b] work over base fields of arbitrary cardinality; so we obtain the corresponding analogue of the "Brown representability" Corollary 4.6.3 for this setting.

Besides, the differential graded formalism yield the possibility of constructing dualities of comotives with certain triangulated categories of \( A \)-linear motives' (with \( A \neq Ab \)) via the connecting functor \( SH(k) \rightarrow DM(k) \) (the latter is the 'big' category of Voevodsky’s motives; see Example 2.2.6 of [Deg13a] or Remark 6.3.1(4)); this yields the orthogonality of the Gersten weight structures with the corresponding 'homotopy' \( t \)-structures on these \( A \)-motives.

Lastly, note that \( D^{mot} \) supports a weight structure that extends the Chow weight structure of \( DM^{eff}_{gm}(k) \) (yet one has to invert \( char k \) in the coefficient ring if it is positive); cf. Remark 6.3.1(2), §6.6 of [Bon10a], and [Bon11a].
6.5 On concrete examples of cohomology theories

We recall that there exist natural exact connecting functors $\text{SH}^S(k) \to \text{SH}(k) \to \text{SH}^{\text{MGl}}(k) \to \text{DM}(k)$ (see Remark 6.1.3(1)) and also $\text{SH}(k) \to \text{SH}^+(k)$. So, one may say that there are 'more' (cohomological) functors that factor through $\text{SH}^S(k)^c$ than those that factor through $\text{SH}(k)^c$ or other motivic categories. Still, author does not know of many examples of cohomological functors on $\text{SH}^S(k)^c$ that do not factor through $\text{SH}(k)^c$. Besides, if $f : \mathbb{C} \to D$ is one of the comparison functors mentioned, then the knowledge that $H^* : \mathbb{C} \to A$ factors through $D$ yields that the corresponding generalized coniveau spectral sequences and filtrations are $D$-functorial, which is certainly stronger than $\mathbb{C}$-functoriality. So, it always makes sense to factor $H^*$ through a 'more structured' motivic category (if possible).

Thus, for cohomology theories that are $\text{SH}^S(k)$-representable one can apply the results of §4, whereas to $\text{SH}(k)$-representable theories it makes (more) sense to apply Theorem 6.1.2 (see Remark 6.1.3(1)). Recall that $\text{SH}(k)$-representable theories include Hermitian K-theory and Balmer’s Witt groups (when $p \neq 2$; see Theorems 5.5 and 5.8 of [Hor05], respectively). Besides, any embedding $\sigma : k \to \mathbb{C}$ yields the corresponding $\text{SH}(k)$-representable semi-topological $K$-theory and semi-topological cobordism (see Theorem 1.0.3 of [KrP13]); whereas the natural comparison functor $\text{Re}_B^\sigma$ from $\text{SH}(k)$ to the 'topological' stable homotopy category $\text{SH}$ (see [Ayo10]) yields that all 'topologically stable' cohomology theories (including complex $K$-theory and complex cobordism) also factor through $\text{SH}(k)$. So, for all of these theories one has the $\text{SH}(k)$-functoriality of (generalized) coniveau filtrations and spectral sequences, as well as several direct summand results for semi-local pro-schemes (if $k$ is infinite).

Next we note that the adjunction $\text{SH}(k) \rightleftarrows \text{SH}^{\text{MGl}}(k)$ yields: any functor $\text{SH}(k)$-representable by a 'strict' $\text{MGl}$-module (i.e., by a one coming from $\text{MGl}-\text{Mod}(k)$) also factors through $\text{SH}^{\text{MGl}}(k)$. In particular, this is the case for all cohomology theories that come from orientable ring spectra in $\text{SH}(k)$ (see Theorem 1.0.1 of [PPR08]). Hence we obtain the $\text{SH}^{\text{MGl}}(k)$-functoriality of the corresponding (generalized) coniveau filtrations and coniveau spectral sequences, and direct summand results for primitive pro-schemes (without assuming that $k$ is infinite). Orientable cohomology theories include algebraic cobordism and algebraic $K$-theory (see Example 1.2.3 of ibid.); if we fix an embedding of $k$ into $\mathbb{C}$, we also obtain 'algebraic' orientability of complex $K$-theory, Morava $K$-theories, and complex cobordism (cf. Example 12.2.3(3))
of \cite{Chu09}.

Moreover, recall that motivic, morphic (see Theorem 5.1 of \cite{Hub00}), etale, singular, and 'mixed' (see §2.3 of \cite{Hub00}) cohomological functors factor through \(DM(k)\). Note here that the targets of (certain versions of) the latter three cohomology theories are 'richer' than \(Ab\); this certainly makes our direct summand results (cf. Theorem 6.1.2(II)–III) more interesting. So, we obtain 'motivic' functoriality of the corresponding filtrations and spectral sequences, and certain direct summand results for the cohomology of all primitive pro-schemes.

Lastly, we note that all the triangulated motivic categories we consider are tensor ones; hence for any \(Y \in SmVar\) the cohomology theory \(H^*_Y : X \mapsto H^*(X \times Y)\) factors through a given motivic category whenever \(H^*\) does. Hence, we can apply all the statements established above to the corresponding 'modifications' of the cohomology theories mentioned.

### 6.6 Other possibilities for \(\mathcal{D}\)

The author doubts that all possible \(\mathcal{D}^{big}\) are isomorphic (at least, if we do not add some additional 'axioms' to Proposition 3.1.1). Surely, there exist distinct models \(\mathcal{D}'\) for pro-spectra (possibly, not all of them are Quillen equivalent), and they could possess quite different model-theoretic properties. Luckily, the choices for these categories do not affect the cohomology of pro-schemes and objects of \(SH^{S^1}(k)^c\) (see Remark 4.1.2(2)).

We mention an alternative method for constructing \(\mathcal{D}\) and a model for it (avoiding \(\mathcal{D}^{big}\)). To this end one should note that \(PSH^{S^1}(k)\) can be turned into a spectral model category in the sense of \cite{ScS03}. Next one can choose cofibrant fibrant replacements for all isomorphism classes of objects of \(SH^{S^1}(k)^c\) (they will form a set that we will denote by \(P\)). Then Definition 3.9.1 of ibid. yields the full spectral category \(E(P)\) of \(P\) and (after dualizing) a functor \(\text{Hom}(-, P)\) from \(PSH^{S^1}(k)\) to the right spectral modules over \(E(P)\). It seems that the dual to the argument used in the proof of Theorem 3.9.3(i) yields: \(\text{Hom}(-, P)\) is a left Quillen functor; its target is a stable model category; the restriction of \(\text{Ho}(\text{Hom}(-, P))\) to \(SH^{S^1}(k)^c\) embeds it as a full subcategory of cocompact cogenerators into the corresponding category \(\mathcal{D}\). In particular, this methods allows to avoid the construction of \(\mathcal{D}^{big}\). Yet it seems that this method of constructing and studying \(\mathcal{D}\) requires more effort than the 'pro-object' one (that we have used above).
Remark 6.6.1. 1. Besides, the author suspects that $\mathcal{D}$ can be obtained via a Bousfield localization of $\mathcal{D}'$. It seems that 'classical' statements on Bousfield localizations cannot be applied here; possibly class-combinatorial model categories could help.

2. Lastly, note that our constructions of $\mathcal{D}'$ and $\mathcal{D}^{big}$ can be carried over for an arbitrary $k$; the perfectness of $k$ is only required in order to ensure that residue fields of $k$-varieties are essentially smooth over $k$ (so that we can apply the Morel-Voevodsky purity theorem).

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