Some quantum measurements with three outcomes can reveal nonclassicality where all two-outcome measurements fail

H. Chau Nguyen* and Otfried Gühne†
Naturwissenschaftlich-Technische Fakultät, Universität Siegen, Walter-Flex-Straße 3, 57068 Siegen, Germany
(Dated: January 13, 2020)

Measurements serve as the intermediate communication layer between the quantum world and our classical perception. So, the question which measurements efficiently extract information from quantum systems is of central interest. Using quantum steering as a nonclassical phenomenon, we show that there are instances, where the results of all two-outcome measurements can be explained in a classical manner, while the results of some three-outcome measurements cannot. This points at the important role of the number of outcomes in revealing the nonclassicality hidden in a quantum system. Moreover, our methods allow to improve the understanding of quantum correlations by delivering novel criteria for quantum steering and improved ways to construct local hidden variable models.

Introduction.— It is widely believed that, at the fundamental level, our world behaves according to the laws of quantum mechanics, although we can only perceive it classically [1]. In fact, realizing the hidden potential of quantum mechanical systems in information processing has ignited the burst of quantum information and quantum computation during the last years [2]. To transfer the quantum mechanical concepts to that of our familiar classicality, quantum measurements are required [3]. The question how to use quantum measurements to interact efficiently with quantum mechanical systems is thus of central interest in quantum information theory [3].

In 1964, Bell found that measurements performed locally on a bipartite quantum system can yield results which cannot be explained with a classical intuition based on the assumptions of locality and realism [4, 5]. This phenomenon manifests itself as the violation of Bell inequalities, and a famous example of such an inequality is the Clauser-Horne-Shimony-Holt (CHSH) inequality, designed for two parties with two measurements, having two outcomes each. Not all entangled states violate the CHSH inequality [6], and one may wonder whether the usage of measurements with more outcomes helps in observing nonclassical behaviors. Is there a quantum state for which the infinite set of all possible two-outcome measurements does not lead to nonclassical effects, but some three-outcome measurements lead to a Bell inequality violation? This question has not been answered despite decades of research, arguably due to the complex structure of Bell correlations.

There are, however, other nonclassical correlations in quantum mechanics besides the violation of Bell inequalities. An important one is captured by the notion of quantum steering [7, 8]. This phenomenon goes back to Schrödinger’s observation that in the Einstein-Podolsky-Rosen argument, one party (typically called Alice) can steer the state of the other party (called Bob) by making suitable measurements [9]. The modern formulation of this effect has been given by Wiseman and coworkers [10] and since then it was found to be connected to many subjects in quantum information processing. For instance, it has been shown that the measurements made by Alice have to be incompatible, implying that commuting measurements as in classical physics are not suitable [11, 12]. Furthermore, the theory of quantum steering has turned out to be useful to solve long standing open problems concerning Bell inequalities, e.g., the construction of states having a positive partial transpose, but violating a Bell inequality [13, 14].

The goal of this paper is twofold. First, we will show that for some quantum states a finite number of measurements of three outcomes can reveal quantum steering, while the infinite set of all measurements with two outcomes cannot. This proves that the number of measurement outcomes can be important to the question whether nonclassical effects can be observed or not. We note that in recent works it has been demonstrated that the correlations of certain multi-outcome measurements cannot be explained by assuming that all of these measurements

* chau.nguyen@uni-siegen.de
† offried.guehne@uni-siegen.de
themselves have only two effective outcomes [15–17]. But this does not concern the fundamental limitation of the whole infinite set of two-outcome measurements as comparison to those with more outcomes in revealing quantum correlations.

Second, the methods developed in this paper allow one to advance the theory of quantum steering in several directions. In particular, we derive novel criteria for steerability and unsteerability, and present significantly improved local hidden variable models for so-called Werner states, which show that they do not violate any Bell inequality, even if the most general measurements are considered [18].

Quantum steering.— Consider the situation where Alice and Bob share a bipartite quantum state $\rho$ and Alice performs a measurement (denoted by $x$) with $n$ outcomes. This is generally described by a collection of $n$ positive operators, $\{E_a(x)\}_{a=1}^{n}$, $E_a(x) \geq 0$, normalized by $\sum_{a=1}^{n} E_a(x) = 1$, which form a so-called positive operator valued measure (POVM). Bob’s system is then found in the ensemble of conditional states $\{\rho_{a|x} = \text{Tr}[\rho(E_a(x) \otimes 1)]\}$. It has been noted early that by choosing different measurements, Alice’s can steer Bob’s system to ensembles that are intuitively ‘incompatible’ with each other, such as pure eigenstates of noncommutative observables, conflicting with our intuition of classical locality [4, 9]. However, it was not until 2007 that this naive notion of ‘incompatibility’ gained a precise definition. Wiseman et al. [10] pointed out that incompatible ensembles in general mean that they cannot be derived from a single collection of states, called a local hidden state (LHS) ensemble. An LHS ensemble is simply a distribution $\mu$ on Bob’s pure states $|\lambda\rangle$. The different ensembles $\{\text{Tr}[\rho(E_a(x) \otimes 1)]\}$ corresponding to different measurement choices $x$ can be derived from the single LHS ensemble $\mu$ if one can reach any conditional state $\text{Tr}[\rho(E_a(x) \otimes 1)]$ from the states $|\lambda\rangle$ via classical postprocessing. That means that there are probabilities $G_a(x)|\lambda\rangle$ such that

$$\text{Tr}[\rho(E_a(x) \otimes 1)] = \int \text{d}\mu(\lambda) G_a(x)(\lambda) |\lambda\rangle \langle \lambda|, \quad (1)$$

where the integration is taken over Bob’s pure states. If this is the case, one says that $\rho$ admits an LHS model, or in short, $\rho$ is unsteerable. The postprocessing functions $G_a(x)(\lambda)$ are called Alice’s response functions. Being probabilities, the response functions $G_a(x)(\lambda)$ are constrained by $0 \leq G_a(x)(\lambda) \leq 1$, $\sum_{a=1}^{n} G_a(x)(\lambda) = 1$. If such an LHS model does not exist, one says that $\rho$ is steerable [10].

The role of measurements.— Crucially for our purpose, Alice’s steering abilities depend on the set of measurements $M$ she can potentially make. This allows one to quantify how much steering the measurements of a class $\mathcal{M}$ reveal for a state $\rho$. Specifically, we define the steering critical radius $R_{\mathcal{M}}(\rho)$ to be the maximum of the mixing parameter $\eta$ such that $\rho_{\eta} = (1-\eta) (\mathbb{I}_A \otimes \rho_B) / d_A$ is unsteerable with measurements in $\mathcal{M}$,

$$R_{\mathcal{M}}(\rho) = \max\{ \eta \geq 0 : \rho_{\eta} \text{ is unsteerable w.r.t. } \mathcal{M} \}. \quad (2)$$

Here $\mathbb{I}_A$ denotes the identity operator acting on system $A$ and $\rho_B$ denotes the reduced state of system $B$, $\rho_B = \text{Tr}_A(\rho)$. Geometrically $1 - R_{\mathcal{M}}(\rho)$ measures the distance from $\rho$ to the surface separating steerable/unsteerable states (with measurements in $\mathcal{M}$) relatively to the noisy and unsteerable state $(\mathbb{I}_A \otimes \rho_B)/d_A$, see also Fig. 1. We have deliberately used the same name critical radius as in Ref. [19] as it can be shown to reduce to the same definition for two-qubit systems, where the critical radius measures the inscribed radius of certain convex object that naturally emerges in the context of quantum steering. In a similar fashion, we define $S(\rho)$ to be the maximum mixing parameter $\eta$ such that $\rho_{\eta}$ becomes separable, i.e., it can be written as a convex combination of product states [6].

The structure of measurements.— The set of POVMs has a nested structure: measurements with $n$ outcomes are naturally a subset of that of measurements with $n+1$ outcomes. Measurements with two outcomes, so-called dichotomic measurements, are the most elementary, and also among the most often measurements that are performed routinely in experiments. Measurements whose effects $E_a$ are rank-1 projections will be referred to as projective measurements which are the standard measurements occurring in textbooks.

For $\mathcal{M}$ being the set of POVMs of $n$ outcomes, or projective measurements, we simply denote the critical radii by $R_n$, and $R_{\text{POVM}}$, respectively. Since any POVM can be written as a mixture of POVMs with at most $d^2$ outcomes, measurements with $n > d^2$ outcomes do not bring any more steerability to Alice [18, 20]. So we can also denote $R_{\text{POVM}} = R_{d^2}$. Because measurements with $n$ outcomes form a subset of that with $n+1$ outcomes, and projective measurements form a subset of measurements with $d_A$ outcomes, the critical radii organize in the following sequence

$$R_{\text{POVM}} \uparrow R_2 \geq \cdots \geq R_{d_A} \geq \cdots \geq R_{d_A^2} = R_{\text{POVM}}, \quad (3)$$

which is valid for any state. Fig. 1 illustrates this sequence geometrically.

Although difficult to compute, already in their early paper, Wiseman et al. [10] remarked that $R_{\text{POVM}}$ can be computed for the Werner states and the isotropic states. More recently, it has been shown that $R_2$ can also be computed for arbitrary two-qubit states [19, 21–23]. Further, numerical evidences suggested that for two-qubit states, the chain in fact collapses to a single value $R_2 = R_{\text{POVM}} = R_3 = R_4$ [19].

Here we report a practically closed formula for $R_2$ for the high-dimensional isotropic states and Werner states and show that $R_2 > R_{\text{POVM}} \geq R_{\text{POVM}}$ for systems other than qubits. This is in particular true for dimension
d = 3: $R_2 > R_3$ for the three-dimensional isotropic and Werner states. Since by replacing the infinite set of 3-POVMs by a finite subset of measurements, one can approach $R_3$ (from above) as close as possible. So, there exists a finite set of measurements of three outcomes which gives a smaller critical radius than $R_2$. These three-outcome measurements can then reveal nonclassicality, where all two-outcome measurements cannot.

Werner states and the isotropic states.— Recall that the fully antisymmetric state of dimension $d \times d$ is defined by $W^d = 2\pi_+/(d^2 - d)$, where $\pi_+$ is the projection onto the antisymmetric subspace of $\mathbb{C}^d \otimes \mathbb{C}^d$, spanned by vectors of the type $|ij\rangle - |ji\rangle$ [6]. The Werner state at mixing probability $\eta$ is then defined by mixing this projection with the white noise, $W^d_\eta = \eta W^d + (1-\eta)(\mathbb{I}/d) \otimes (\mathbb{I}/d)$. This is in line with the notation introduced before Eq. (2), as we have $\text{Tr}(A W^d) = 1/d$.

By construction, the Werner states are symmetric under application of the same local unitary operation $U \in U(d)$ on both parties, namely, $W^d_\eta = (U \otimes U)W^d_\eta(U^\dagger \otimes U^\dagger)$ [6]. It has been shown that Werner states are separable if and only if $\eta \geq 1/(d+1)$ [6], which can be written in the above notation as $S(W^d) = 1/(d+1)$. Werner states are unsteerable with projective measurements if and only if $\eta \geq 1 - 1/d$ [6, 10], thus $R_{\text{POVM}}(W^d) = 1 - 1/d$.

To define the isotropic states, one first considers the maximally entangled state on $\mathbb{C}^d \otimes \mathbb{C}^d$, defined by $S^d = |\phi_+\rangle\langle\phi_+|$, where $|\phi_+\rangle = 1/\sqrt{d} \sum_{k=1}^d |k\rangle \otimes |k\rangle$. The isotropic state at mixing probability $\eta$ is then $S^d_\eta = \eta S^d + (1-\eta)(\mathbb{I}/d) \otimes (\mathbb{I}/d)$. The isotropic state also has a symmetry under local unitaries $U \in U(d)$, as $S^d_\eta = (U \otimes U^*)S^d_\eta(U^\dagger \otimes (U^*)^\dagger)$, where $U^*$ stands for the complex conjugate of $U$ [25]. It is well-known that $S(S^d) = 1/(d+1)$ [25], and $R_{\text{POVM}}(S^d) = (H_d - 1)/(d-1)$, where $H_d = 1 + 1/2 + \cdots + 1/d$ [10].

The uniform distribution as LHS ensemble.— When writing down an LHS model as in Eq. (2) for Werner states or isotropic states, it is known [10, 26] that one can restrict the attention to a probability distribution which is the uniform distribution according to the Haar measure, denoted by $\omega$, over Bob’s Bloch sphere. It is easily see from the argument given in Ref. [26] that this remains true also if the measurements are limited to generalised ones of any fixed number of outcomes.

To proceed, we consider the set of conditional states Alice can simulate using this distribution $\omega$, which is given by

$$\mathcal{K}(\omega) = \left\{ K = \int d\omega(\lambda) g(\lambda) |\lambda\rangle\langle\lambda| : 0 \leq g(\lambda) \leq 1 \right\}. \quad (4)$$

The set $\mathcal{K}(\omega)$ is known as the capacity of $\omega$ [19, 26]. In higher-dimensional spaces, $\mathcal{K}(\omega)$ has complicated structure and no complete characterization of its geometry is known. However, we will see that even a partial information of $\mathcal{K}(\omega)$ will be sufficient to characterize quantum steering of Werner states and isotropic states.

Dichotomic measurements. Each dichotomic measurement is completely characterized by one of its two effects, say $M$, since the other is $\mathbb{I} - M$. It follows directly from the definition of quantum steering that Werner states and isotropic states are unsteerable if and only if the corresponding conditional state $\text{Tr}[\rho(M \otimes \mathbb{I})]$ is inside $\mathcal{K}(\omega)$ for all measurement effects $M$ on Alice’s side.

Let us have a closer look at the set of measurement effects on Alice’s side, $\{ M : 0 \leq M \leq \mathbb{I} \}$. This is a convex set, of which the extreme points are precisely the projection operators. These can be organized in hyperplanes corresponding to different ranks of the projections. It is then natural to introduce finer subsets of 2-POVMs whose two effects are projections and the lower rank is $r$. Accordingly, we use $R^d_2(M)$ to denote the steering critical radius corresponding to this subset of measurements. We then have

$$R_2 = \min_{r=1,\ldots,[d/2]} R^d_2,$$

where $[d/2]$ is the maximal integer not greater than $d/2$.

Reducing the dimension and main result.— The following observation is crucial to computing $R^d_2$: For Werner states and isotropic states, a conditional state of Bob’s system corresponding to a projection $P$ on Alice’s side belongs to a special two-dimensional plane spanned by the projection itself and the identity operator, span$\{P, \mathbb{I}\}$. This is easily verified by direct computation of the conditional states in these cases. Consequently, instead of considering the general capacity $\mathcal{K}(\omega)$, we can consider its cross-section with these two-dimensional subspaces and the original high-dimensional problem is now reduced to a two-dimensional one. Fortunately, in these two-dimensional spaces, the cross-section with $\mathcal{K}(\omega)$ can be computed exactly. The formulae are somewhat cumbersome, but can be explicitly given; see Appendix A and B. To find the critical radii $R^d_2$ of the fully antisymmetric state and the maximally entangled state, we simply identify the critical mixing probability threshold at which Bob’s conditional states corresponding to a projection of rank $r$ is at the border of this cross-section; for the details, see Appendix B and C.
The remaining step is the discrete minimization of $R_d^2$ with respect to the rank $r$ of the projection in Eq. (5). We find that for both Werner states and isotropic states, $R_d^2$ is always minimal at $r = 1$ for all dimensions $d \leq 10^5$ and conjecture that this holds in general. In other words, among dichotomic measurements, those with a rank-1 effects are conjectured to be most useful for quantum steering. This eventually leads to the steering critical radius

$$R_d(W^d) = (d - 1)^2[1 - (1 - 1/d)^{1/(d-1)}]$$

for Werner states, and

$$R_d(S^d) = 1 - d^{-1/(d-1)}$$

for isotropic states. These critical radii are presented in Fig. 2 together with other known thresholds for these two families of states.

As an example, for the system of two qutrits, $d = 3$, we find for the Werner state $R_d(W^3) = 4(1 - \sqrt{2/3}) \approx 0.734$, which is strictly larger than $R_{PV}^2(W^3) = 2/3 \approx 0.667$, and for the isotropic state $R_d(S^3) = 1 - 1/\sqrt{3} \approx 0.423$, which is also strictly larger than $R_{PV}^2(S^3) = 5/12 \approx 0.417$. These are thus explicit examples that quantum steering revealed by dichotomic measurements is strictly weaker than quantum steering with measurements having three outcomes.

Steering with arbitrary POVMs.— As long as quantum steerability is concerned, it follows from Ref. [18] that without loss of generality, one can assume that Alice’s measurements consist of $d^2$ rank-1 effects, $E = (E_1, E_2, \cdots, E_{d^2})$ with $E_a = \alpha_a P_a$, where $P_a$ are rank-1 projections, $0 \leq \alpha_a \leq 1$, and $\sum_a \alpha_a = d$. Let us consider the Werner state $W^d_\eta$. For outcome $a$ of Alice’s measurement, Bob’s system is steered to $\text{Tr}_A(W_\eta^d E_a \otimes \mathbb{I}) = \alpha_a \text{Tr}_A[W_\eta^d P_a \otimes \mathbb{I}]$. One sees that apart from the multiplication factor $\alpha_a$, the conditional states are essentially that of $n = d^2$ dichotomic measurements ($P_a, \mathbb{I} - P_a$). But even if the state is unsteerable with dichotomic measurements and the explicit response functions are given, it is not possible to directly combine them to form a response function for the general POVM $E$, which requires the normalization for the response function as probabilities, $\sum_a \alpha_a = 1$. To achieve the normalization, one has to soften the response functions for the dichotomic measurements in a suitable way. Barrett was the first who used this idea to construct an LHS model with POVMs for certain entangled Werner states [18]. As it turns out, his construction is in fact most suitable when the two parties are correlated, such as when they share an isotropic state. For the Werner states, the two parties are however anticorrelated. We therefore propose the following response function for the Werner state,

$$G_a(\lambda) = \alpha_a \langle \lambda | \frac{1}{d} - \frac{P_a}{d-1} | \lambda \rangle \Theta(1/d - \langle \lambda | P_a | \lambda \rangle)$$

$$+ \frac{\alpha_a}{d} \left[ 1 - \sum_{b=1}^n \alpha_b \langle \lambda | \frac{1}{d} - \frac{P_b}{d-1} | \lambda \rangle \Theta(1/d - \langle \lambda | P_b | \lambda \rangle) \right].$$

The physical intuition for this response function and detailed calculation are discussed in Appendix D. With this, direct computation gives

$$R_{PV}^2(W^d) \geq \frac{1 + (d - 1)^{d+1}d^{-d}}{d + 1}.$$  \hspace{1cm} (9)

Fig. 2 shows that this significantly improves the bound given by the original Barrett construction, in particular it remains finite as $d$ tends to infinity, $\lim_{d \to \infty} R_{PV}^2(W^d) = 1/e$, where $e$ is the Euler’s natural constant. Note that the existence of our LHS model proves that in the considered parameter range the Werner states do not violate any Bell inequality.

Steering criteria for general states.— We now show that our methods for highly symmetric states can be used to analyse steerability of generic high-dimensional states, where Bob’s reduced state is of full rank. In this case, because steerability is invariant under local filtering on Bob’s side [27–29], we can assume Bob’s reduced state to be maximally mixed (by applying an appropriate filter transformation on his side).

Then, one can use the fact that the steerability from Alice to Bob is non-increasing under local channels on Alice’s side [30]. Given two states $\rho$ and $\tau$, each with Bob’s reduced state maximally mixed, we define $D(\rho, \tau) = \max\{\eta \geq 0 : \rho_\eta = (E \otimes \mathbb{I})[\tau]\}$, where $E$ is a channel on Alice’s side, $\mathbb{I}$ is the identity channel, and $\rho_\eta$ is a state affected by noise as used in Eq. (2). Slightly extending the result of [30], it directly follows that given an unsteerable state $\tau$, i.e., $R_\eta(\tau) \geq 1$, then

$$R_\eta(\rho) \geq D(\rho, \tau).$$  \hspace{1cm} (10)

Given $\tau$, the computation of $D(\rho, \tau)$ is a standard optimization over the channel $E$, which can be done using semidefinite programming [31]. By choosing $\tau$ to be an unsteerable Werner state, or an unsteerable isotropic state, Eq. (10) gives a lower bound for $R_\eta(\rho)$ and consequently a way to prove the steerability of a generic high-dimensional state, which is an open problem which appears in various situations [7, 32, 33].

Interestingly, one can also turn the logic of Eq. (10) around and prove steerability. In this case, one chooses $\rho$ to be a state of which $R_\eta(\rho)$ is known, e.g., a Werner state or an isotropic state, then $D(\rho, \tau) > R_\eta(\rho)$ implies that $R_\eta(\tau) < 1$, which proves the steerability of $\tau$.

Another way to prove steerability for general states uses the symmetry of the Werner and isotropic states. It is easy to see that the critical radius do not decrease under averaging the state with random local unitaries. Thus by twirling a state $\rho$ to a Werner state or an isotropic state [6, 25], of which the critical radius is known, we find

$$R_\eta(\rho) \leq \min \left\{ \frac{(d + 1)R_\eta(W^d)}{1 - dF_W}, \frac{(d^2 - 1)R_\eta(S^d)}{d^2 - F_S} \right\},$$  \hspace{1cm} (11)

where $F_S = \text{Tr}(S^d \rho)$ and $F_W = \text{Tr}(W^d \rho)$, with $F^d$ being the swap operator between two systems of dimension $d$. 
Eventually, we obtain dichotomic measurements can easily be simulated, while three-outcome measurements cannot. Moreover, we provided novel criteria for the steerability and unsteerability of general quantum states. Especially the presented LHS model for Werner states improves the known models drastically. These results will be useful for the applications of steering in information processing, such as quantum key distribution in asymmetric scenarios [34], or the characterization of joint measureability [7].

Acknowledgement. — We thank Sébastien Designolle, Matthias Kleinman, M. Toan Nguyen, Jiangwei Shang, Baker Travis and Roope Uola for inspiring discussions. This work was supported by the DFG and the ERC (Consolidator Grant 683107/TempoQ).

Appendix A: Integration over the high dimensional Bloch sphere

We will frequently have to work with integrals over the high dimensional Bloch sphere (i.e., the set of pure states). Here we describe how that can be done, following Refs. [6, 18] with small modifications.

Specifically, we work with the Hilbert space of dimension $d$. Let $Q$ be a projection of rank $k$, we are interested in the following integration

$$a_n(k, t) = \int \omega(\lambda) \langle \lambda | Q | \lambda \rangle^n \Theta((\langle \lambda | Q | \lambda \rangle - t),$$

where $\Theta$ is Heaviside’s step function and $\omega$ denotes the Haar measure over the pure states. Note that although the projection $Q$ appears in the integral on the right-hand side, we will see that the left-hand side only depends on its rank $k$, which justifies the notation $a_n(k, t)$.

We choose the basis $\{|i\rangle\}_{i=1}^d$ such that $Q = \sum_{i=1}^{k} |i\rangle\langle i|$. The pure state can be written as $|\lambda\rangle = \sum_{i=1}^{d} r_i e^{i\theta_i} |i\rangle$. The Haar measure thus can be formally written as

$$d\omega(\lambda) = \frac{1}{2} \prod_{i=1}^{d} r_i dr_i d\theta_i \delta(\sum_{i=1}^{d} r_i^2 - 1).$$

The range of $r_i$ is $[0, +\infty)$ and the range of $\theta_i$ is $[0, 2\pi)$. The normalisation factor $Z$ can be found by

$$Z = \prod_{i=1}^{d} \int_{0}^{+\infty} r_i dr_i \int_{0}^{2\pi} d\theta_i \delta(\sum_{i=1}^{d} r_i^2 - 1).$$

Now note that the integrands in (A1) and (A3) do not depend on the phase $\theta_i$, thus the integration over the phase $\theta_i$ can be carried out directly. Moreover, the integrals over $r_i$ can be simplified by changing the variable $u_i = r_i^2$. Eventually, we obtain

$$a_n(k, t) = \frac{I_n(Q, t)}{I_0(Q, 0)},$$

with

$$I_n(Q, t) = \int du \delta(1 - \sum_{i=1}^{d} u_i) \Theta(\sum_{i=1}^{k} u_i - t)(\sum_{i=1}^{k} u_i)^n,$$

where $du = du_1 du_2 \ldots du_d$ and the integral is taken over the whole range $[0, +\infty)$ of $u_i$.

Let

$$s_p(\xi) = \int dx_1 dx_2 \ldots dx_p \delta(\xi - x_1 - x_2 - \ldots - x_p).$$

Then by rescaling the integral variable, one can easily show that

$$s_p(\xi) = s_p(1)\xi^{p-1}.$$
Note that $s_p(1)$ is simply the area of the $p - 1$ probability simplex, which still carries a $\delta$-function.
With this notation, we then can integrate out $u_{k+1}, u_{k+2}, \ldots, u_d$ in (A5) to get
\[
I_n(Q,t) = s_{d-k}(1) \int du_1 \ldots du_k \Theta(\sum_{i=1}^k u_i - t)(1 - \sum_{i=1}^k u_i)^{d-k-1}(\sum_{i=1}^k u_i)^n. \tag{A8}
\]
To carry out this integral, we write
\[
I_n(Q,t) = s_{d-k}(1) \int dx \int du_1 \ldots du_k \Theta(x(t) - (1 - x)^{d-k-1}x^n) \delta(x - u_1 - u_2 - \ldots - u_k) \tag{A9}
\]
Upon changing the integral order, we have
\[
I_n(Q,t) = s_{d-k}(1) \int dx \left( \int du_1 \ldots du_k \Theta(x(t) - (1 - x)^{d-k-1}x^n) \delta(x - u_1 - u_2 - \ldots - u_k) \right) \tag{A10}
\]
\[
= s_{d-k}(1) s_k(1) \int dx (1 - x)^{d-k-1}x^{n+k-1+n} \tag{A11}
\]
\[
= s_{d-k}(1) s_k(1) \beta(1 - t, d - k, k + n), \tag{A12}
\]
where $\beta(a,b) = \int_0^z d\xi \xi^{a-1}(1 - \xi)^{b-1}$ is Euler’s incomplete $\beta$-function. So
\[
a_n(k,t) = \frac{\beta(1 - t, d - k, k + n)}{\beta(d - k, k)}, \tag{A13}
\]
where $\beta(a,b) = \beta(1, a, b)$ is Euler’s complete $\beta$-function. As we remarked in the paragraph following (A1), $a_n(k,t)$
only depends on the rank $k$ of the projection $Q$.

**Appendix B: The canonical cross-sections of the capacity of the uniform distribution**

Generally, it has been shown [22, 26] that the extreme points of $\mathcal{K}(\omega)$ are of the form
\[
K(Z) = \int d\omega(\lambda) \Theta(\langle \lambda | Z | \lambda \rangle) |\lambda \rangle \langle \lambda |,
\tag{B1}
\]
with varying operator $Z$. In particular, let us consider a special family of these extreme points where $Z = Q - t \mathbb{1}$,
where $Q$ is a (fixed) projection of rank $k$ and varying $t$,
\[
K(Q,t) = \int d\omega(\lambda) \Theta(\langle \lambda | Q | \lambda \rangle - t) |\lambda \rangle \langle \lambda |. \tag{B2}
\]

Let us now show that $K(Q,t)$ is in the span of $\{ \mathbb{1}, Q \}$. While this can be done directly by inspection, a more elegant
argument makes use of the concepts of von Neumann algebras [35]. Since $Q$ is a projection, the span of $\{ \mathbb{1}, Q \}$ is also
the von Neumann algebra generated by $\mathbb{1}_Z$ and $Q$. To show that $K(Q,t)$ is in the algebra, we show that it commutes
with all unitaries in the commutant of the span of $\{ \mathbb{1}, Q \}$ [35]. That is, let $U$ be an unitary operator that commutes
with $Q$, we want to show that $U$ also commute with $K(Q,t)$. Indeed,
\[
UK(Q,t)U^\dagger = \int d\omega(\lambda) \Theta(\langle \lambda | Q | \lambda \rangle - t)U|\lambda \rangle \langle \lambda |U^\dagger. \tag{B3}
\]

Upon transforming $|\lambda'\rangle = U|\lambda \rangle$ and noting that the Haar measure is invariant under this transformation, and that
$\langle \lambda' | UQU^\dagger | \lambda' \rangle = \langle \lambda | Q | \lambda \rangle)$ since $U$ commutes with $Q$, we obtain an identical formula as equation (B2) for $K(Q,t)$.

Being in the span of $\{ \mathbb{1}, Q \}$, $K(Q,t)$ is characterised by two parameters $\text{Tr}[K(Q,t)] = a_0(k,t)$ and $\text{Tr}[QK(Q,t)] = a_1(k,t)$, with
\[
a_0(k,t) = \frac{\beta(1 - t, d - k, k)}{\beta(d - k, k)}, \tag{B4}
\]
\[
a_1(k,t) = \frac{\beta(1 - t, d - k, k + 1)}{\beta(d - k, k)}, \tag{B5}
\]
as defined in Eq. (A1) and Eq. (A13).

As $t$ varying from 0 to 1, $K(Q,t)$ draws a curve starting at $\mathbb{1}$ and ending at 0 in the plane spanned by $\{ \mathbb{1}, Q \}$. As a consequence, this forms a half of the boundary of the cross-section of $\mathcal{K}(\omega)$ in this plane. The other half of boundary
of the cross-section is formed by $K(\mathbb{1} - Q,t)$ for $t$ varying from 0 to 1.
Appendix C: Critical radii for dichotomic measurements

1. Werner states

Recall that the fully antisymmetric state of dimension $d \times d$ is defined by

$$W^d = \frac{2\pi_-}{d(d-1)},$$

where $\pi_-$ is the projection onto the antisymmetric subspace of $\mathbb{C}^d \otimes \mathbb{C}^d$. The Werner state acting in dimension $d$ is obtained as a convex combination of the fully antisymmetric state $W^d$ with the maximally mixed state,

$$W^d_\eta = \eta W^d + (1-\eta) \frac{1}{d} \otimes \frac{1}{d},$$

and $\eta$ is referred to as the mixing parameter.

Suppose Alice makes a dichotomic measurement $E = (P, \mathbb{1} - P)$, where $P$ is a projection of rank $r$. For the outcome $P$, Bob’s system is steered to

$$\text{Tr}_A(W^d_\eta P \otimes \mathbb{1}) = \eta \frac{1-P}{d(d-1)} + \left[\frac{r-1}{d-1} + (1-\eta) \frac{r}{d} \right] \frac{1}{d},$$

where $r = \text{rank}(P)$.

Observe that this steering outcome belongs to the plane spanned by $\mathbb{1}$ and $\mathbb{1} - P$. We consider the cross-section of the capacity of the uniform distribution $\mathcal{K}(\omega)$ in the corresponding plane, i.e., $K(\mathbb{1} - P, t)$, with the border described by equation (B5). We are interested in whether the conditional state (C3) is inside this cross-section. The condition for this to happen can be easily derived by identifying the critical value $\eta_c$ for the mixing parameter such that the conditional state (C3) is on the border of the capacity (B5), which is given by

$$a_0(d-r, t_c) = \frac{r}{d},$$

$$a_1(d-r, t_c) = \eta_c \frac{d-r}{d(d-1)} + \left[\frac{r-1}{d} + (1-\eta_c) \frac{r}{d} \right] \left(1 - \frac{r}{d}\right).$$

Solving $t_c$ from equation (C4), one can compute $\eta_c$ from equation (C5). Recall that $\eta_c$ is in fact precisely the definition of the critical radii, $R^c_\eta(W^d) = \eta_c$.

One can derive a more explicit formula for $R^c_\eta(W^d)$. Indeed, from equation (C5), we find

$$\eta_c = \frac{d^2(d-1)}{r(d-r)} \left[ a_1(d-r, t_c) + \frac{r^2}{d^2} - \frac{r}{d} \right].$$

Upon using the definition of $a_n(k, t)$ in equation (A13), the recursive relation for the incomplete $\beta$-function [36, page 263],

$$\beta(z, a, b + 1) = \frac{b}{a+b} \beta(z, a, b) + \frac{1}{a+b} z^a (1-z)^b,$$

and the definition of the complete $\beta$-function in terms of the $\Gamma$-function [36, page 259],

$$\beta(a, b) = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)},$$

one arrives at

$$R^c_\eta(W^d) = \frac{(d-1) \Gamma(d+1)}{\Gamma(r+1) \Gamma(d-r+1)} (1-t_c)^r t_c^{d-r}.$$

Although not given in a closed form for arbitrary $r$, $R^c_\eta(W^d)$ can be easily computed in a computer. For all $d \leq 10^5$, we compute $R^c_\eta(W^d)$ and find that it is always minimised at $r = 1$. Thus in all these cases we can identify $R_2$ with $R^c_2$. For $r = 1$, equation (C4) can be solved explicitly for $t_c$, and we arrive at

$$R_2(W^d) = (d-1)^2 [1 - (1-1/d)^{1/(d-1)}].$$
2. Isotropic states

Recall that the maximally entangled state on $\mathbb{C}^d \otimes \mathbb{C}^d$ is defined by

$$S^d = |\psi_+\rangle\langle\psi_+|,$$  \hspace{1cm}  \text{(C11)}

where $|\psi_+\rangle = \frac{1}{\sqrt{d}} \sum_{k=1}^{d} |k\rangle \otimes |k\rangle$ for certain basis $|k\rangle$. The isotropic state is then defined by

$$S^d_\eta = \eta S^d_0 + (1 - \eta) \mathbb{I}_d \otimes \mathbb{I}_d. \hspace{1cm} \text{(C12)}$$

The computation of $R_{r2}(S^d)$ follows similar steps as that for the Werner state. There is a remarkable difference, though. For the isotropic state, the steering outcome at Bob’s side corresponding to the projection outcome $P$ at Alice’s side,

$$\text{Tr}_A[S^d_\eta (P \otimes \mathbb{I})] = \frac{r}{d} \left[ \eta \bar{P} + (1 - \eta) \frac{\mathbb{I}}{d} \right],$$ \hspace{1cm}  \text{(C13)}

belongs to the canonical cross-section of the capacity of the uniform distribution indicated by $K(\bar{P}, t)$. Here $\bar{P}$ denotes the complex conjugate of $P$. Thus here we need to consider the cross-section of $K(\omega)$ with the plan spanned by $\bar{P}$ and $\mathbb{I}$, in contrast to the case for the Werner states.

Following the same steps in Section C1, we proceed by identifying the critical value $\eta_c$ for the mixing parameter such that the conditional state (C13) is on the border of the capacity (B5), which is given by

$$a_0(r, t_c) = \frac{r}{d},$$ \hspace{1cm}  \text{(C14)}

$$a_1(r, t_c) = \frac{r^2}{d^2 - 1}. \hspace{1cm} \text{(C15)}$$

Solving $t_c$ from equation (C14), one can compute $\eta_c$ from equation (C15). Again, $\eta_c$ is in fact precisely the definition of the critical radii, $R_{r2}(S^d) = \eta_c$.

An explicit formula for $R_{r2}(S^d)$ can also be derived. From equation (C15), we find

$$\eta_c = \frac{a_1(r, t_c) - r^2/d^2}{r/d(1 - r/d)}. \hspace{1cm} \text{(C16)}$$

Then using the definition of $a_n(k, t)$ in equation (A13), the recursive relation (C7) and the relation between $\beta$-function and $\Gamma$-function (C8), one obtains

$$R_{r2}(S^d) = \frac{\Gamma(d + 1)}{\Gamma(d - r + 1)\Gamma(r + 1)}(1 - t_c)^d - r t_c. \hspace{1cm} \text{(C17)}$$

Note the difference with the equation (C9) for the Werner state. For all $d \leq 10^5$, we again find that $R_{r2}$ is minimised at $r = 1$. We thus have for all $d \leq 10^5$,

$$R_2(S^d) = 1 - d^{-1/(d-1)}. \hspace{1cm} \text{(C18)}$$

Appendix D: Barrett’s model for the Werner states

In the following, we present the details of the derivation of the bound

$$R_{\text{POVM}}(W^d) \geq \frac{1 + (d - 1)^{d+1}d^{-d}}{d + 1}. \hspace{1cm} \text{(D1)}$$

This bound is the critical mixing parameter $\eta_c$ such that

$$\text{Tr}(W^d_{\eta_c} E_\omega \otimes \mathbb{I}) = \int d\omega(\lambda)G_\omega(\lambda)|\lambda\rangle\langle\lambda|,$$ \hspace{1cm}  \text{(D2)}

where
with the response function

\[ G_a(\lambda) = \alpha_a \langle \lambda \vert \frac{1 - P_a}{d-1} \vert \lambda \rangle \Theta(1/d - \langle \lambda \vert P_a \vert \lambda \rangle) + \frac{\alpha_a}{d} \left( 1 - \sum_{b=1}^{d^2} \alpha_b \langle \lambda \vert \frac{1 - P_b}{d-1} \vert \lambda \rangle \Theta(1/d - \langle \lambda \vert P_b \vert \lambda \rangle) \right). \tag{D3} \]

Recall from the main text that \( E_a = \alpha_a P_a \), where \( P_a \) are rank-1 projections. One can recognise that the first term in this response function is, up to a prefactor, given by the response functions for dichotomic measurements \( \Theta(1/d - \langle \lambda \vert P_a \vert \lambda \rangle) \). The second term is constructed such that the response function is automatically normalised, \( \sum_{a=1}^{d^2} G_a(\lambda) = 1 \). It is easy to show that the function is positive, thus is a valid response function.

We need to compute the operator on the right hand side of equation (D2). To do this, we note

\[ \int d\omega(\lambda) G_a(\lambda) \langle \lambda \vert \lambda \rangle = \alpha_a X_a + \frac{\alpha_a}{d} \left( 1 - \sum_{b=1}^{d^2} \alpha_b X_b \right), \tag{D4} \]

where

\[ X_a = \int d\omega(\lambda) \frac{1}{d-1} \langle \lambda \vert Q_a \vert \lambda \rangle \Theta(\langle \lambda \vert Q_a \vert \lambda \rangle - (1 - 1/d)) \langle \lambda \vert \lambda \rangle, \tag{D5} \]

where \( Q_a = \mathbb{1}_B - P_a \). We again can show that \( X_a \) is in the span of \( \{ \mathbb{1}, Q_a \} \), which can be characterised by

\[ \text{Tr}(X_a) = \frac{1}{d-1} a_1(d - 1,1 - 1/d), \tag{D6} \]

\[ \text{Tr}(X_a Q_a) = \frac{1}{d-1} a_2(d - 1,1 - 1/d). \tag{D7} \]

The critical value of \( \eta_c \) where this construction of local hidden state model works is then

\[ \eta_c = \frac{d^2}{d-1} a_2(d - 1,1 - 1/d) - da_1(d - 1,1 - 1/d). \tag{D8} \]

With the explicit expressions of \( a_2(d - 1,1 - 1/d) \) and \( a_1(d - 1,1 - 1/d) \) one obtains equation (D1).

[1] W. H. Zurek, “Decoherence, einselection, and the quantum origins of the classical,” Rev. Mod. Phys. 75, 715–775 (2003).
[2] M. A. Nielsen and I. L. Chuang, Quantum computation and quantum information (Cambridge University Press, 2010).
[3] M. Schlosshauer, Decoherence and the Quantum-to- Classical Transition (Springer, 2007).
[4] A. Einstein, B. Podolsky, and N. Rosen, “Can quantum-mechanical description of physical reality be considered complete,” Phys. Rev. 47, 777 (1935).
[5] J. S. Bell, “On the Einstein-Podolsky-Rosen paradox,” Physics 1, 195 (1964).
[6] R. F. Werner, “Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model,” Phys. Rev. A 40, 4277 (1989).
[7] R. Uola, A. C. S. Costa, H. C. Nguyen, and O. Gühne, “Quantum steering,” arXiv:1903.06663 (2019).
[8] D. Cavalcanti and P. Skrzypczyk, “Quantum steering: a review with focus on semidefinite programming,” Rep. Prog. Phys. 80, 024001 (2016).
[9] E. Schrödinger, “Discussion of probability relations between separated systems,” Proc. Cambridge Philos. Soc. 31, 555 (1935).
[10] H. M. Wiseman, S. J. Jones, and A. C. Doherty, “Steering, entanglement, nonlocality, and the Einstein-Podolsky-Rosen paradox,” Phys. Rev. Lett. 98, 140402 (2007).
[11] R. Uola, C. Budroni, O. Gühne, and J.-P. Pellonpää, “A one-to-one mapping between steering and joint measurability problems,” Phys. Rev. Lett. 115, 230402 (2015).
[12] M. T. Quintino, T. Vértesi, and N. Brunner, “Joint measurability, Einstein-Podolsky-Rosen steering, and Bell nonlocality,” Phys. Rev. Lett. 113, 160402 (2014).
[13] T. Moroder, O. Gittsovich, M. Huber, and O. Gühne, “Steering bound entangled states: A counterexample to the stronger Peres conjecture,” Phys. Rev. Lett. 113, 050404 (2014).
[14] T. Vértesi and N. Brunner, “Disproving the Peres conjecture by showing Bell nonlocality from bound entanglement,” Nat. Commun. 5, 1–5 (2014).
[15] M. Kleinmann and A. Cabello, “Quantum correlations are stronger than all nonsignaling correlations produced by n-outcome measurements,” Phys. Rev. Lett. 117, 150401 (2016).
[16] M. Kleinmann, T. Vértesi, and A. Cabello, “Proposed experiment to test fundamentally binary theories,” Phys. Rev. A 96, 032104 (2017).
X.-M. Hu, B.-H. Liu, Y. Guo, G.-Y. Xiang, Y.-F. Huang, C.-F. Li, G.-C. Guo, M. Kleinmann, T. Vértesi, and A. Cabello, “Observation of stronger-than-binary correlations with entangled photonic qutrits,” Phys. Rev. Lett. 120, 180402 (2018).

J. Barrett, “Nonsequential positive-operator-valued measurements on entangled mixed states do not always violate a bell inequality,” Phys. Rev. A 65, 042302 (2002).

H. C. Nguyen, H. V. Nguyen, and O. Gühne, “Geometry of Einstein–Podolsky–Rosen correlations,” Phys. Rev. Lett. 112, 240401 (2019).

G. M. D’Ariano, P. Lo Presti, and P. Perinotti, “Classical randomness in quantum measurements,” J. Phys. A: Math. Gen. 38, 5979 (2005).

S. Jevtic, M. J. W. Hall, M. R. Anderson, M. Zwierz, and H. M. Wiseman, “Einstein-Podolsky-Rosen steering and the steering ellipsoid,” J. Opt. Soc. Am. B 32, A40 (2015).

H. C. Nguyen and T. Vu, “Nonseparability and steerability of two-qubit states from the geometry of steering outcomes,” Phys. Rev. A 94, 012114 (2016).

H. C. Nguyen and T. Vu, “Necessary and sufficient condition for steerability of two-qubit states by the geometry of steering outcomes,” Europhys. Lett. 115, 10003 (2016).

M. L. Almeida, S. Pironio, J. Barrett, G. Tóth, and A. Acín, “Noise robustness of the nonlocality of entangled quantum states,” Phys. Rev. Lett. 99, 040403 (2007).

M. Horodecki and P. Horodecki, “Reduction criterion of separability and limits for a class of distillation protocols,” Phys. Rev. A 59, 4206 (1999).

H. C. Nguyen, A. Milhe, T. Vu, and S. Jevtic, “Quantum steering with positive operator valued measures,” J. Phys. A 51, 355302 (2018).

R. Uola, T. Moroder, and O. Gühne, “Joint measurability of generalized measurements implies classicality,” Phys. Rev. Lett. 113, 160403 (2014).

M. T. Quintino, T. Vértesi, D. Cavalcanti, R. Augusiak, M. Demianowicz, A. Acín, and N. Brunner, “Inequivalence of entanglement, steering, and Bell nonlocality for general measurements,” Phys. Rev. A 92, 032107 (2015).

R. Gallego and L. Aolita, “Resource theory of steering,” Phys. Rev. X 5, 041008 (2015).

T. J. Baker, S. Wollmann, G. J. Pryde, and H. M. Wiseman, “Necessary conditions for steerability of two qubits, from consideration of local operations,” arXiv:1906.04693 (2019).

S. Boyd and L. Vandenberghe, Convex Optimization (Cambridge University Press, 2004).

R. Augusiak, M. Demianowicz, and A. Acín, “Local hidden-variable models for entangled quantum states,” J. Phys. A: Math. Theor. 47, 424002 (2014).

S. Brierley, M. Navascues, and T. Vértesi, “Convex separation from convex optimization for large-scale problems,” arXiv:1609.05011 (2016).

C. Branciard, E. G. Cavalcanti, S. P. Walborn, V. Scarani, and H. M. Wiseman, “One-sided device-independent quantum key distribution: Security, feasibility, and the connection with steering,” Phys. Rev. A 85, 010301 (2012).

A. Connes, Noncommutative Geometry (Academic Press, 1994).

M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables (Dover, 1964).