The Isomorphism Problem for Cyclic Algebras and an Application

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ABSTRACT
The isomorphism problem means to decide if two given finite-dimensional simple algebras with center $K$ are $K$-isomorphic and, if so, to construct a $K$-isomorphism between them. Applications lie in computational aspects of representation theory, algebraic geometry and Brauer group theory.

The paper presents an algorithm for cyclic algebras that reduces the isomorphism problem to field theory and thus provides a solution if certain field theoretic problems including norm equations can be solved (this is satisfied over number fields). As an application, we can compute all automorphisms of any given cyclic algebra over a number field. A detailed example is provided which leads to the construction of an explicit noncrossed product division algebra.

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I.1.2 [Symbolic and Algebraic Manipulation]: Algorithms—Algebraic algorithms

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Theory, Design

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Introduction
Let $K$ be a field and let $A_1$ and $A_2$ be two finite-dimensional central-simple $K$-algebras ($A_i$ has no proper two-sided ideal and the center is $K$). The isomorphism problem for $A_1$ and $A_2$ means the problem to decide whether $A_1$ and $A_2$ are $K$-isomorphic and, if so, to construct a $K$-isomorphism between them. We assume that $A_1$ and $A_2$ have equal dimension, for otherwise they are trivially non-isomorphic.

The special case when $A_2$ is the full matrix ring over $K$ is called the splitting problem for $A_1$. We shall call a $K$-isomorphism $A_1 \rightarrow M_n(K)$ a splitting of $A_1$.

There are several applications. For instance, to compute the irreducible representations over $K$ of a finite group $G$ with $|G|$ not divisible by $\text{char } K$, one can decompose the semisimple group ring $KG$ into its simple components (see §15.6 or Haile’s proof [6] for an algorithm) and then solve the splitting problem for each component. As another example, finding $K$-rational points on a Brauer-Severi variety $V$ is equivalent to the splitting problem for the central-simple $K$-algebra associated with $V$ (cf. [4]). The splitting problem also occurs if one wants to compute orthogonal idempotent generators in central-simple algebras. In this paper we pursue an application that is motivated by explicit algebra constructions.

Because various constructions make use of automorphisms of simple algebras that are nontrivial on the center, we study (in section 4) the problem of extending an automorphism of the field $K$ to an automorphism of the central-simple $K$-algebra $A$, and show that it reduces to the isomorphism problem.

The present paper solves the isomorphism problem for algebras that are presented as cyclic algebras. A cyclic algebra is a central-simple algebra that contains a maximal subfield (i.e. a subfield with maximal degree) which is cyclic over the center. We say an algebra is presented as a cyclic algebra if a cyclic maximal subfield is explicitly given. For example, this can be a presentation by structure constants plus an explicit generator of a cyclic maximal subfield. Having only the theoretical information that the algebra is cyclic, e.g. if the center is a global field ([11 Thm. 32.20]), is not sufficient. There is no algorithm available that can produce cyclic maximal subfields of central-simple algebras except for certain small degrees. Indeed, for quaternion algebras this is trivial, and for cubic algebras one finds a cyclic maximal subfield simply by proceeding along the lines of Wedderburn’s proof [10 §15.6] or Haile’s proof [6] that every division algebra of degree three is cyclic.

The algorithms of this paper work by reducing the isomorphism problem to norm equations. Norm equations are in general hard to solve, but algorithms are known over number fields (e.g. Simon [12]) and of course over finite fields. Using

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computer algebra systems like KASH [3] and MAGMA [2] for those norm equations, our algorithms actually become applicable over number fields (and finite fields).

1. PRELIMINARIES

Let $K$ be a field. Unless stated otherwise, all algebras are finite-dimensional $K$-algebras. The tensor product $\otimes$ and the isomorphism $\cong$ without subscripts mean tensor product and isomorphism over $K$, respectively. A $K$-algebra $A$ is called central-simple if $A$ has no proper two-sided ideals and its center $Z(A)$ is $K$. The reader is assumed to be familiar with the basic theory of central-simple algebras as in the textbook sources Pierce [10] or Reiner [11]. A few relevant terms are briefly recalled in the sequel.

Let $A$ be a central-simple $K$-algebra. The degree of $A$, denoted $\deg A$, is the square root of the dimension $\dim A$ (the dimension is always a square). The algebra $A$ is called split if $A \cong M_n(K)$ where $n = \deg A$. The opposite algebra $A^\circ$ of $A$ is the $K$-space $A$ with multiplication redefined by $a \circ b := ba$. If $B$ is another central-simple $K$-algebra of degree $n$ then

$$A \cong B \iff A \otimes B^\circ \cong M_{n^2}(K).$$

A maximal subfield of $A$ is a commutative subfield $L \subseteq A$ with $[L : K] = \deg A$. The algebra $A$ is called a crossed product if $A$ contains a maximal subfield that is Galois over $K$. Moreover, $A$ is called cyclic (resp. bicyclic) if $A$ contains a maximal subfield that is cyclic (resp. bicyclic) over $K$.

1.1 Cyclic Algebras

Suppose $A$ is cyclic of degree $n$ and let $L$ be a maximal subfield cyclic over $K$ with $\Gal(L/K) \cong \langle \sigma \rangle$. By the Skolem-Noether Theorem there is an element $v \in A^*$ such that the inner automorphism $\text{Inn}(v) : x \mapsto xv^{-1}$ of $A$ satisfies $\text{Inn}(v)|_L = \sigma$. For any such $v$ we have $v^n \in K$ and

$$A = \bigoplus_{i=0}^{n-1} Lv^i$$

as a $K$-space. Setting $a = v^n$ we denote this algebra by

$$A = (L/K, \sigma, a, v)$$

(in the literature $v$ is usually omitted from the notation). For any integer $k$ relatively prime to $n$,

$$(L/K, \sigma, a, v) = (L/K, \sigma^k, a^k, v^k).$$

We have

$$(L/K, \sigma, a, v) \cong M_n(K) \iff a \in N_{L/K}(L)$$

and, more generally,

$$(L/K, \sigma, a, v) \cong (L/K, \sigma, b, v) \iff a/b \in N_{L/K}(L).$$

1.2 Generalized Cyclic Algebras

Suppose $A$ contains a subfield $L$ that is cyclic over $K$ but not necessarily a maximal subfield. Then $A$ is called a generalized cyclic algebra (cf. generalized crossed products in Kursev-Yanchevskii [9] or Tignol [13]). Let $\sigma$ generate $\Gal(L/K)$, let $[L : K] = n_0$ and let $B$ denote the centralizer $Z_A(L)$ of $L$ in $A$. By the Skolem-Noether Theorem there is an element $v \in A^*$ such that $\text{Inn}(v)|_L = \sigma$. For any such $v$ we have $v^{n_0} \in B$ and

$$A = \bigoplus_{i=0}^{n_0-1} Bu^i$$

as a $K$-space. Since $B$ is the centralizer of $L$, we have $vBu^{-1} = B$. Setting $\tilde{\sigma} := \text{Inn}(v)|_B$ and $a := v^{n_0}$ we write

$$A = (B/K, \tilde{\sigma}, a, v).$$

Conversely, for any extension $\tilde{\sigma}$ of $\sigma$ to $B$ there are $v \in A^*$ and $a \in B^*$ such that (4) holds. If $\tilde{\sigma}$ is fixed then

$$(B/K, \tilde{\sigma}, a, v) \cong (B/K, \tilde{\sigma}, b, w) \iff a/b \in N_{L/K}(L)$$

(5)

(a/b lies in $K$ if $a$ and $b$ arise in this way).

1.3 Bicyclic Algebras

We will use for bicyclic algebras the notation that was introduced in Amitsur-Saltman [1] for the more general "abelian crossed products" (see also Jacobson [8] §4.6, pp.174)). Suppose $A$ is cyclic and let $F$ be a maximal subfield bicyclic over $K$. Let $G := \Gal(F/K) = G_1 \times G_2$, $G_i = \langle \sigma_i \rangle$ and $|G_i| = n_i$. We denote by $F_1$ the fixed field of $\sigma_1$ and by $N_1$ the norm map of the extension $F/F_1$. Clearly, $F/F_1$ and $F_{3-i}/K$ each have group $G_i$. By the Skolem-Noether Theorem there are elements $z_1, z_2 \in A^*$ such that $\text{Inn}(z_i)|_F = \sigma_i$ for $i = 1, 2$. For any such $z_1, z_2$ we have

$$A = \bigoplus_{i=0}^{n_1-1} \bigoplus_{j=0}^{n_2-1} Fz_1^iz_2^j$$

as a $K$-space. With the action of $z_i$ on $F$ being fixed, the algebra $A$ is determined up to isomorphism by the elements

$$b_1 := z_1^{n_1}, \quad b_2 := z_2^{n_2}, \quad u := z_2z_1^{-1}z_2^{-1}.$$ 

We write

$$A = (F/K, z, u, b)$$

where $z = (z_1, z_2)$ and $b = (b_1, b_2)$. The elements $b_1, b_2$ and $u \in F^*$ satisfy the relations

$$N_1(u) = \frac{\sigma_2(b_2)}{b_1}, \quad N_2(u) = \frac{b_2}{\sigma_1(b_2)}, \quad b_i \in F_1$$

(cf. [1] Lemma 1.2), and $b_1$ are also sufficient for given elements $b_1, b_2, u \in F^*$ to define a bicyclic algebra (cf. [1] Theorem 1.3]). If $u = 1$ then (6) imply $b_1, b_2 \in K$ and we have a canonical isomorphism

$$(F/K, z, 1, b) \cong (F_2/K, \sigma_1, b_1, z_1) \cong (F/K, \sigma_2, b_2, z_2)$$

(7)

that identifies $F$ with $F_2 \otimes F_1$ and the $z_i$ on both sides, respectively. In particular, we conclude that $(F/K, z, 1, 1)$ is split.

THEOREM 1. $A = (F/K, z, u, b)$ is split if and only if there are $x_1, x_2 \in F^*$ such that

$$N_1(x_1) = b_1, \quad N_2(x_2) = b_2, \quad \frac{\sigma_2(x_1)}{x_1} \frac{x_2}{\sigma_1(x_2)} = u.$$ (8)

PROOF. Since $(F/K, z, 1, 1) \cong M_n(K)$ the statement is a special case of [1] Theorem 1.4].

2. THE SPLITTING PROBLEM

Let $A$ be a central-simple $K$-algebra of degree $n$.

Splitting Problem. Decide whether $A$ is split and, if so, compute a splitting of $A$, i.e. a $K$-isomorphism $A \to M_n(K)$.
For cyclic algebras the splitting problem quite obviously reduces to the solution of a norm equation. The point of this section is to show the same for bicyclic algebras. However, we start with the details of the cyclic case.

**Algorithm 1** (Splitting of cyclic algebra). Let a cyclic algebra $A = (L/K, \sigma, a, v)$ of degree $n$ be given. The splitting problem for $A$ is solved as follows.

1. Fix a $K$-embedding $\psi : L \rightarrow M_n(K)$.
2. Compute $X \in M_n(K)$ such that $\operatorname{lnn}(X)|_{\psi(L)} = \psi \sigma \psi^{-1}$ and set $b := X^n$. Then we have $b \in K$.
3. Solve the norm equation $N_{L/K}(x) = a/b$ for $x \in L$. If there is no solution then $A$ is not split, otherwise $x$ is extended to a splitting $A \rightarrow M_n(K)$ by mapping $v$ to $\psi(x)X$.

**Proof.** Step 1 amounts to computing the minimal polynomial over $K$ of a primitive element of $L$. The matrix $X$ in step 2 exists by the Skolem-Noether Theorem (cf. §K) and its computation is a linear problem. Moreover, we have $b = X^n \in K$ and $M_n(K) = (\psi(L)/K, \psi \sigma \psi^{-1}, b, X)$. Step 3: By $\text{[1]}$, $A$ is split if and only if $a/b \in M_{L/K}(L)$. If $x \in L$ is a solution to the equation $N_{L/K}(x) = a/b$ then $\psi(x)X^n = \psi(N_{L/K}(x))b = a$. This shows that mapping $v$ to $\psi(x)X$ indeed defines an extension of $\psi$ to $A$. □

**Remark 1.** Algorithm $\text{[1]}$ can be used to compute a splitting of the bicyclic algebra $(F/K, z, u, b)$. Indeed, we take the canonical isomorphism

$$(F/K, z, u, b) \rightarrow (F_2/K, \sigma_1, 1, z_1) \otimes (F_1/K, \sigma_2, 1, z_2)$$

from $\text{[1]}$, compute splittings $(F_{2-i}/K, \sigma_{1}, 1, z_1) \rightarrow M_n(K)$ with Algorithm $\text{[1]}$ and finally compose with an isomorphism $M_n(K) \otimes M_n(K) \rightarrow M_{2n}(K)$.

Now let $A = (F/K, z, u, b)$ be an arbitrary bicyclic algebra. By Theorem $\text{[1]}$ $A$ is split if and only if the system of equations $\text{[3]}$ has a solution. In fact, any solution $(x_1, x_2)$ to $\text{[3]}$ defines an isomorphism

$$(F/K, z, u, b) \rightarrow (F/K, w, 1, 1), \quad z_1 \rightarrow x_1 w_1$$

(cf. the proof of $\text{[1]}$ Theorem 1.4). Together with Remark $\text{[1]}$ we get a splitting of $A$. It remains to solve the system $\text{[3]}$, and this covers the rest of the section.

**Lemma 1** (Bicyclic Hilbert 90). If $x_1, x_2 \in F^*$ satisfy

$$N_1(x_1) = 1, \quad N_2(x_2) = 1 \quad \text{and} \quad \frac{\sigma_2(x_1)}{x_1} \cdot \frac{x_2}{\sigma_1(x_2)} = 1$$

then there is $y \in F^*$ with $x_i = \frac{\sigma_i(y)}{y}$ for $i = 1, 2$.

**Proof.** In fact a stronger statement holds: if elements $x_1, \ldots, x_r \in F^*$ satisfy $N_i(x_i) = 1$ and $x_i = \frac{\sigma_i(y)}{y}$ for all $1 \leq i, j \leq r$ then $y \in F^*$ exists with $x_i = \frac{\sigma_i(y)}{y}$ for all $1 \leq i \leq r$ (cf. $\text{[3]}$ Proposition 4.630, p.179]). However, we give a shorter proof for $r = 2$.

Let $x_1 = \frac{\sigma_1(y)}{y_1}$ with $y_1 \in F^*$, by Hilbert’s Theorem 90. Then

$$\frac{\sigma_1(x_2)}{x_2} = \frac{\sigma_2(x_1)}{x_1} \cdot \frac{\sigma_2(y_1)}{\sigma_1(y_1)} \cdot \frac{\sigma_1(x_2)}{x_2} \cdot \frac{x_2}{\sigma_1(x_2)} = \frac{\sigma_1(x_2)}{x_2} \cdot \frac{x_2}{\sigma_1(x_2)} = \frac{\sigma_1(y_2)}{y_2}$$

and hence $x_2 = \frac{\sigma_2(y_1)}{y_1}$ for some $c \in F_1^*$. It follows $N_2(c) = 1$.

Let $c = \frac{\sigma_2(y_2)}{y_2}$ with $y_2 \in F_2^*$, by Hilbert’s Theorem 90. Defining $y := y_1 y_2$ we get $x_i = \frac{\sigma_i(y)}{y}$ for $i = 1, 2$. □

**Proposition 1.** Suppose $\text{[3]}$ has a solution $(x_1, x_2)$. Then the set of all solutions is

$$S := \{(x_1, x_2) \mid x_1 \in F^*, x_2 = \frac{\sigma_1(y_2)}{y_2} \}.$$

In particular, for any $x_1, x_2 \in F^*$ with $N_1(x_1') = b_1$ there is $x_2' \in F^*$ such that $(x_1', x_2') \in S$.

**Proof.** An easy calculation shows that any $(x_1', x_2') \in S$ solves $\text{[3]}$. For the converse apply Lemma $\text{[3]}$. The second statement is another application of Hilbert’s Theorem 90. □

**Algorithm 2** (Solution to $\text{[3]}$). If a solution to $\text{[3]}$ exists then one is found performing the following steps. Conversely, if all steps have a solution then the resulting $(x_1, x_2)$ is a solution to $\text{[3]}$.

1. Solve $N_1(x_1) = b_1$ for $x_1 \in F^*$.
2. Solve

$$\frac{\sigma_1(x_2')}{x_2'} = \frac{u x_1}{\sigma_2(x_1)} \quad \text{for} \quad x_2' \in F^*.$$

3. Solve $N_2(x_2'') = b_2 N_2(x_2')$ for $x_2'' \in F_2^*$.
4. Define $x_2 := x_2'' x_2'$.

**Proof.** A straightforward calculation verifies that any $(x_1, x_2)$ computed by these steps is a solution to $\text{[3]}$. Conversely, suppose $\text{[3]}$ has a solution and show each step has a solution. Step 1 is obvious. Step 2 has a solution by Hilbert’s Theorem 90 because, using $\text{[3]}$,

$$N_1(u x_1/\sigma_2(x_1)) = N_1(u) b_1/\sigma_2(b_1) = 1.$$

Step 3: Since $\text{[3]}$ has a solution, Proposition $\text{[3]}$ shows the existence of an element $x_2 \in F^*$ with $(x_1, x_2) \in S$. Setting $x_2'' := x_2 x_2'$, $\text{[3]}$ implies $\sigma_1(x_2')/x_2'' = 1$ and $N_2(x_2'') = b_2 N_2(x_2')$. This shows that step 3 has a solution. □

Algorithm $\text{[2]}$ reduces the splitting of a bicyclic algebra to two consecutive (not simultaneous) norm equations; the rest (step 2) is linear algebra. Note that the first norm equation (step 1) lives in the larger fields $F_2/F_1$ whereas the second one (step 3) lives in $F_1/F_2$. Algorithms for norm equations over number fields can be found in Simon $\text{[22]}$ and the references cited therein.

**Remark.** The splitting problem has two parts: first, to decide whether the algebra is split, and second, to compute a splitting. One might be tempted to think that it suffices for the first part to decide the solvability of norm equations, and that solutions are required only for the second part. Indeed, this is true for cyclic algebras because only one norm equation appears. For bicyclic algebras, however, two norm equations occur in Algorithm $\text{[2]}$ and the second one is built from a solution to the first. Thus, at least the first norm equation has to be solved explicitly for any part of the splitting problem.
3. THE ISOMORPHISM PROBLEM

Let \( A_1 \) and \( A_2 \) be central-simple \( K \)-algebras of degree \( n \).

**Isomorphism Problem.** Decide whether \( A_1 \) and \( A_2 \) are \( K \)-isomorphic and, if so, compute a \( K \)-isomorphism between them.

We show for general \( A_1 \) and \( A_2 \) how the isomorphism problem reduces to the splitting problem (see Algorithm \( \text{Algorithm 3} \) below). This is due to the equivalence \( \text{[1]} \). When specializing thereafter to cyclic algebras, the isomorphism problem eventually reduces to norm equations.

**Remark 2.** A \( K \)-algebra isomorphism \( \varphi : A_1 \otimes A_2^\prime \to C \) is equivalent to a pair \((\varphi_1, \varphi_2)\) where \( \varphi_1 : A_1 \to C \) is a \( K \)-embedding and \( \varphi_2 : A_2 \to C \) is a \( K \)-anti-embedding such that \( \varphi_1(A_1) \) is the centralizer of \( \varphi_2(A_2) \) in \( C \). Of course, one obtains \( \varphi_1, \varphi_2 \) from \( \varphi \) by composing \( \varphi \) with the canonical embedding \( \varepsilon_1 : A_1 \to A_1 \otimes A_2^\prime \) and canonical anti-embedding \( \varepsilon_2 : A_2 \to A_1 \otimes A_2^\prime \), respectively.

**Algorithm 3.** Suppose a \( K \)-isomorphism \( \varphi : A_1 \otimes A_2^\prime \to M_n,\sigma(K) \) is given in the form of a pair \((\varphi_1, \varphi_2)\) as in Remark 2. Then \( K \)-isomorphisms \( \chi : A_1 \to A_2 \) and \( \chi' : A_2 \to A_1 \) are computed as follows.

1. Fix a \( K \)-basis of \( A_1 \) and with respect to this basis identify \( M_n,\sigma(K) \) with \( \text{End}_K(A_1) \).
2. Compute \( X \in M_n,\sigma(K) \) such that \( \text{Inn}(X) \circ \varphi_1 \) is the left-regular representation of \( A_1 \).
3. Set \( \varphi_2 := \text{Inn}(X) \circ \varphi_2 \). Then \( \varphi_2(A_2) = \rho(A_1), \) where \( \rho \) is the right-regular representation of \( A_1 \).
4. Define \( \chi := \varphi_2^{-1} \circ \rho \) and \( \chi' := \rho^{-1} \circ \varphi_2 \).

**Proof.** The left-regular representation

\[
\lambda : A_1 \to \text{End}_K(A_1), \quad a \mapsto \lambda_a, \quad \lambda_a(x) := ax
\]

is a \( K \)-algebra embedding and the right-regular representation

\[
\rho : A_1 \to \text{End}_K(A_1), \quad a \mapsto \rho_a, \quad \rho_a(x) := xa
\]

is a \( K \)-algebra anti-embedding. The matrix \( X \) in step 2 exists by the Skolem-Noether Theorem and its computation is a linear problem. We get

\[
\varphi_2(A_2) = Z_{M_n,\sigma(K)}(X \varphi_1(A_1)X^{-1}) = Z_{M_n,\sigma(K)}(\lambda(A_1)) = \rho(A_1).
\]

Since \( \rho \) and \( \varphi_2 \) are both anti, \( \chi \) and \( \chi' \) as defined in step 4 are isomorphisms.

**Remark.** The reduction of the isomorphism problem for \( A_1 \) and \( A_2 \) to the splitting problem for \( A_1 \otimes A_2^\prime \) as in Algorithm 3 is accompanied by an increase in the algebra degree from \( n \) to \( n^2 \).

Now we turn to the case when \( A_1 \) and \( A_2 \) are both cyclic, say

\[
A_1 = (L_1/K, \sigma_1, a_1, v_1) \quad \text{and} \quad A_2 = (L_2/K, \sigma_2, a_2, v_2).
\]

We distinguish two special cases and the general case.

**Special Case 1.** \( L_1 \cong_K L_2 \). Let \( \chi : L_1 \to L_2 \) be a \( K \)-isomorphism. Obviously, \( \chi \sigma_1 \chi^{-1} = \sigma_2 \) for some \( i \) relatively prime to \( n \). Replacing \( v_2 \) with \( v_2' \) we can assume by \( \text{[2]} \) that \( \chi \sigma_1 \chi^{-1} = \sigma_2 \). Then, by \( \text{[3]} \),

\[
A_1 \cong A_2 \iff a_1/a_2 \in N_{L_2/K}(L_2).
\]

This equivalence is “constructive”: if \( x \in L_2 \) is an element with \( N_{L_2/K}(x) = a_1/a_2 \), then mapping \( v_1 \mapsto xv_2 \) extends \( \chi \) to a \( K \)-isomorphism \( A_1 \to A_2 \). The isomorphism problem for \( A_1 \) and \( A_2 \) is thus reduced to finding a solution to a norm equation in the field extension \( L_2/K \).

**Special Case 2.** \( L_1/K \) and \( L_2/K \) are linearly disjoint, i.e. \( F := L_1 \otimes L_2 \) is a field. By \( \text{[4]} \), \( A_1 \otimes A_2^\prime \) is isomorphic to the bicyclic algebra \( C := (F/K, \{1, b\}) \) with \( z = (z_1, z_2) \) and \( b = (a_1, a_2') \). (Here, we regard \( \sigma_1, \sigma_2 \) as automorphisms of \( F \)). Indeed, the canonical isomorphism \( \varphi : A_1 \otimes A_2^\prime \to C \) maps \( \lambda v_1 \otimes 1 \mapsto \lambda z_1' \) for all \( \lambda \in L_1 \) and \( 1 \otimes \lambda v_2' \mapsto \lambda z_2' \) for all \( \lambda \in L_2 \). Thus, Algorithm 3 reduces the isomorphism problem for \( A_1 \) and \( A_2 \) to the splitting problem for \( C \). Since \( C \) is bicyclic, this further reduces to norm equations by \( \text{[2]} \).

**Remark.** If the degree \( n \) is prime then we are in one of the special cases.

**General Case.** Regarding \( L_1, L_2 \) as subfields of some common overfield, we set \( L_0 := L_1 \cap L_2 \) and consider the centralizers \( B_1 := Z_{A_1}(L_0) \) and \( B_2 := Z_{A_2}(L_0) \). If \( A_1 \cong A_2 \) there is by the Skolem-Noether Theorem a \( K \)-isomorphism \( \chi : A_1 \to A_2 \) with \( \chi(L_0) = L_0 \) and \( \chi|_{L_0} = \text{id}_{L_0} \). For any such \( \chi \) we have \( \chi(B_1) = B_2 \), i.e. the restriction \( \chi|_{B_1} \) is an \( L_0 \)-isomorphism \( B_1 \to B_2 \).

We therefore solve the general case by starting with the isomorphism problem for \( B_1 \) and \( B_2 \) over \( L_0 \) (which is special case 2). If it has no solution then \( A_1 \not\cong A_2 \). Assume otherwise and suppose an \( L_0 \)-isomorphism \( \chi_0 : B_1 \to B_2 \) is computed. We identify \( B_1 \) and \( B_2 \) under \( \chi_0 \) and simply write \( B \) for both of them. By \( \text{[5]} \), there are elements \( w_1 \in A_1^*, w_2 \in A_2^* \) and \( b_1, b_2 \in B^* \) such that

\[
A_1 = (B/K, \tilde{\sigma}, b_1, w_1) \quad \text{and} \quad A_2 = (B/K, \tilde{\sigma}, b_2, w_2)
\]

with the same \( \tilde{\sigma} \) for both algebras. Since \( w_1, w_2 \) are predicted by the Skolem-Noether Theorem, their computation is a linear problem. We proceed as in special case 1 but for generalized cyclic algebras. By \( \text{[6]} \),

\[
A_1 \cong A_2 \iff b_1/b_2 \in N_{L_0/K}(L_0).
\]

This equivalence is “constructive”: if \( x \in L_0 \) is an element with \( N_{L_0/K}(x) = b_1/b_2 \), then mapping \( v_1 \mapsto xv_2 \) defines a \( K \)-isomorphism \( A_1 \to A_2 \). Thus, also the general case is reduced to norm equations.

4. EXTENDING FIELD AUTOMORPHISMS TO SIMPLE ALGEBRAS

Let \( A \) be a central-simple \( K \)-algebra and let \( \sigma \) be an automorphism of \( K \) of finite order.

**Extension Problem.** Decide whether \( \sigma \) extends to an automorphism of \( A \) and, if so, compute an extension.

It is convenient to reformulate this problem using the algebra \( \varphi^{-1}A \) which is obtained from \( A \) by redefining the \( K \)-action as \( \lambda a := \sigma^{-1}(\lambda)a \) for all \( \lambda \in K \). Then \( \sigma \) extends to \( A \)
if and only if $A$ and $\sigma^{-1} A$ are isomorphic as $K$-algebras. In fact, any $K$-algebra isomorphism $\sigma^{-1} A \to A$ becomes an extension of $\sigma$ after identifying $\sigma^{-1} A$ as a ring with $A$. The extension problem is therefore just a special case of the isomorphism problem.

If $A$ is cyclic then $\sigma^{-1} A$ is also cyclic, hence, by the results of the preceding sections, the extension problem for cyclic algebras reduces to norm equations.

Remark 3. Let $A = (L/K, \tau, a, v)$ and identify $\sigma^{-1} A$ as a ring with $A$. For any extension of $\sigma$ to $L$ (call it also $\sigma$), we have

$$\sigma^{-1} A = (\sigma L/K, \sigma \tau \sigma^{-1}, \sigma a, v).$$

The ring identity map $A \to \sigma^{-1} A$ is defined by $\sigma : L \to \sigma L$ and $v \mapsto v$.

We finish with a detailed example for the solution of the extension problem. Let $K$ be the cubic number field

$$K = \mathbb{Q}(\alpha), \quad \text{Irr}(\alpha, \mathbb{Q}) = x^3 + x^2 - 2x - 1,$$

doctoral real subfield of the $7$-th cyclotomic field. We have $\text{Gal}(K/\mathbb{Q}) = \langle \sigma \rangle$ with

$$\sigma(\alpha) = -\alpha^2 - \alpha + 1.$$

Let $L$ be the cubic extension

$$L = K(\theta), \quad \text{Irr}(\theta, K) = x^3 + (\alpha - 2)x^2 + (-\alpha - 1)x + 1,$$

which is cyclic and has $\text{Gal}(L/K) = \langle \tau \rangle$ with

$$\tau(\theta) = -\theta^2 + (-\alpha + 1)\theta + 2.$$

We will solve the extension problem for $\sigma$ and the cyclic division algebra

$$D = (L/K, \tau, a, v), \quad a = 2(\alpha^2 - \alpha - 2).$$

In order to do so we solve the isomorphism problem for $D$ and $\sigma^{-1} D$. According to Remark [1] we have

$$\sigma^{-1} D = (\sigma L/K, \sigma \tau \sigma^{-1}, \sigma a, v),$$

where $\sigma L = K(\eta)$ with

$$\text{Irr}(\eta, K) = x^3 + (-\alpha^2 - \alpha - 1)x^2 + (\alpha^2 + \alpha - 2)x + 1,$$

$$\sigma \tau \sigma^{-1}(\eta) = -\eta^2 + (\alpha^2 + \alpha)\eta + 2 \quad \text{and} \quad \sigma(a) = 2(\alpha^2 + 2a - 1).$$

Of course, $\text{Irr}(\eta, K)$ is obtained from $\text{Irr}(\theta, K)$ by applying $\sigma$ to each coefficient.

Since $L_1/K$ and $L_2/K$ are linearly disjoint, we proceed as in special case 2 and consider the bicyclic algebra

$$C := (F/K, z, 1, b)$$

with $F = L \otimes \sigma L$, $\sigma_1 = \tau \otimes \text{id}$, $\sigma_2 = \text{id} \otimes \sigma \tau \sigma^{-1}$, $F_1 = \sigma L$, $F_2 = L, z = (z_1, z_2)$ and $b = (a, \sigma a^{-1})$. The canonical isomorphism $D \otimes \sigma^{-1} D \to C$ maps $\lambda v' \otimes 1 \mapsto \lambda z_1^2$ and $1 \otimes \sigma(\lambda)v' \mapsto z_1^2 \sigma(\lambda)$ for all $\lambda \in L$. We compute a splitting of $C$ following [2].

First apply Algorithm 2 Step 1. The computer algebra system Magma [2] gives

$$\begin{align*}
\chi_1 &= \frac{1}{5} (-(7\alpha^2 + 9\alpha + 4) + 2(-2\alpha^2 + 6\alpha - 1)\eta + (\alpha^2 - 6\alpha + 4)\eta^2 \\
&\quad + (14\alpha^2 - 12\alpha - 9)\eta + (20\alpha^2 - 7\alpha - 14)\eta^2 + (-12\alpha^2 + 3\alpha + 12)\eta^3 + \\
&\quad + (-3\alpha^2 + 3\alpha + 2)\eta^2 + (-7\alpha^2 - \alpha + 5)\eta^2 + (4\alpha^2 - 5)\eta^3 \eta^2).
\end{align*}$$

Step 2. As a solution to a linear equation system one finds

$$\begin{align*}
\chi_2 &= \frac{1}{5} (61\alpha^2 - 261\alpha - 173) + (194\alpha^2 + 428\alpha + 132)\eta + (-78\alpha^2 - 98\alpha - 21)\eta^2 \\
&\quad + (-19\alpha^2 - 120\alpha + 8)\eta^3 + (287\alpha^2 - 266\alpha - 196)\eta^4 + (-92\alpha^2 + 1300 + 45)\eta^5 \\
&\quad + (7\alpha^2 + 70\alpha + 14)\eta^6 + (-202\alpha^2 - 179\alpha - 24)(\eta^2 + (63\alpha^2 + 28\alpha + 7)\eta^3 \eta^2).
\end{align*}$$

Step 3. Exceptionally in this example, the element

$$b_2N_2(x_2') = \frac{1}{50} (-1601\alpha^2 + 693\alpha + 609)$$

lies in $K$. We compute as a cubic root :

$$x_2' = \frac{1}{14} (-19\alpha^2 - \alpha - 6).$$

Step 4. Finally, we get

$$\begin{align*}
x_2 &= \frac{1}{14} ((-8\alpha^2 - 3a + 10) + (-2\alpha^2 - 13a - 1))\eta + (2\alpha^2 + 6a - 6)\eta^2 \\
&\quad + (-3a^2 + 5a + 2)\eta^3 + (6a^2 - 10a + 10)\eta^4 + (-2a^2 - a + 1)\eta^5 + \\
&\quad (2\alpha^2 - \alpha - 6)\eta^6 + (a^2 + 10a - 3)\eta^7 + (-a^2 - 3a + 3)\eta^8.
\end{align*}$$

This completes Algorithm 2. On a 32 bit machine with an 800 MHz processor the computation time for the norm equation in step 1 was less than ten minutes and was negligible for steps 2–4.

At this point we have shown that $C$ is split, hence $\sigma$ extends to $D$. We continue to compute an extension of $\sigma$. Since $(x_1, x_2)$ is a solution to [2], we obtain an isomorphism $\varphi : C \to M_2(K)$ as it is described after Algorithm 2. Using Algorithm 3 we get the $K$-isomorphism $\chi' : \sigma^{-1} D \to D$ defined by

$$\chi'(\alpha) := \frac{1}{473} (1, \theta, \theta^2),$$

$$\text{(3033x - 154x - 276, 314x^2 - 218x - 326, -48x^2 + 151x + 157, -390x^2 + 708x - 855, 46x^2 - 238x - 430, -397x^2 - 270x + 275, 1, 1)} \quad \text{and}$$

$$\chi'(\alpha) = (1, \theta, \theta^2) \begin{pmatrix} 0 & a^2 + a & 0 \\ 0 & -\alpha - 1 & 0 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} (a^2 + a) + (-\alpha + 1)\theta - \theta^2 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}.$$

The resulting extension $\tilde{\sigma}$ of $\sigma$ is

$$\tilde{\sigma} : D \to D, \quad \theta \mapsto \chi'(\theta), \quad v \mapsto \chi'(v).$$

Remark. The example is taken from the paper [2] which uses $D$ and $\tilde{\sigma}$ to construct a noncrossed product division algebra in the form of the twisted Laurent series ring $D(x; \tilde{\sigma})$, i.e. the ring of all formal series $\sum_{i \geq 0} d_x x^i, d \in \mathbb{Z}$, with multiplication of monomials $dx^i \cdot dx^j = d\delta^j(i)d^{i+j}$. The ring $D(x; \tilde{\sigma})$ is a division algebra of degree 9 over the power series field $\mathbb{Q}(t)$. As shown in [2], $D$ does not contain a maximal subfield that is Galois over $\mathbb{Q}$, and this property implies that $D(x; \tilde{\sigma})$ is a noncrossed product. The above computation explains how the $\tilde{\sigma}$ given in [2] was found. (Note that $D$ in [2] is accidentally defined with $-a$ instead of $a$. This sign error is corrected in the version at arXiv:math/0703038.)

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