Classification of black holes in three dimensional spacetime by the $W_{1+\infty}$ symmetry

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The BMS symmetry and related near horizon symmetry play important role in holography in asymptotically flat spacetimes. They may also be crucial for the solution of information paradox. We still don’t fully understand those infinite-dimensional symmetry. On the other hand, $W_{1+\infty}$ symmetry is the quantum version of area-preserving diffeomorphism of the plane. It can be used to classify the quantum Hall effect universality classes. In this paper, we will show that the form symmetries can be obtained form the $W_{1+\infty}$ symmetry. With the help of $W_{1+\infty}$, the black holes in three dimensional spacetime can be classified just like the quantum hall effect. This gives another evidence to support our claim that “black hole as topological insulator”.

INTRODUCTION

The Bondi-Metzner-Sachs (BMS) group \[1–3\] play an important role in holography in asymptotically flat spacetimes \[4–8\]. The corresponding algebra is an infinite-dimensional extension of the translation part of the Poincare algebra. Near the black hole event horizon, there exist a similar infinite-dimensional symmetries \[9–13\], which may be used to solving the information paradox.

Recently the author claimed that “black hole can be considered as a kind of topological insulator” \[14, 15\], and two evidences were given to support this claim. The first evidence comes from the black hole “membrane paradigm” \[16, 17\], which says that the horizon of black hole behaves like an electrical conductor. On the other hand, the vacuum inside can be considered as an insulator. The second evidence comes from the fact that the horizon of black hole can support massless modes \[13, 18\]. Those are two key properties of topological insulator. Since the infinite-dimensional symmetries appears in the black hole side, it should also appears in the topological insulator side. Actually it does!

In 2+1 dimensional spacetime, the topological insulator (also called quantum spin Hall state) \[19, 20\] can be realized as a bilayer integer quantum Hall (IQH) system with opposite $T-$ symmetry \[21\]. For quantum Hall effect, a fundamental property is that the electrons form an incompressible fluid \[22, 23\], that is, the droplets of incompressible fluid have constant area. The small deformations of droplet at constant area can be generated by reparametrizations of the coordinates of the plane with unit Jacobian, i.e. the area-preserving diffeomorphisms \[24–27\]. Those diffeomorphisms form an infinite-dimensional group, and the relevant algebra is called \(w_\infty\) algebra. We follow the symbols of Ref. \[27\] and set \(eB = 1\). The algebra element \(W(\vec{k})\) satisfy the following algebra,

\[
[W(\vec{k}), W(\vec{k}')] = -i(\vec{k} \times \vec{k}')W(\vec{k} + \vec{k}'),
\]

where \(\vec{k} \times \vec{k}' = k_1k_2' - k_2k_1'\).

A quantum version of this algebra is called \(W_{1+\infty}\) algebra (without central extension), and satisfies the following algebra

\[
[W(\vec{k}), W(\vec{k}')] = -2i \sin\left(\frac{\vec{k} \times \vec{k}'}{2}\right)W(\vec{k} + \vec{k}').
\]  

NEAR HORIZON SYMMETRY FROM \(W_{1+\infty}\) SYMMETRY

In this section we consider the near horizon symmetry in three-dimensional spacetime from the \(W_{1+\infty}\) symmetry. The algebra of near horizon symmetry is given in Ref. \[12\] and we just summarize the main results. The near horizon symmetry algebra is a semidirect sum of the Witt algebra generated by \(Y_n\) with an abelian current \(T_n\), and the commutation relation read

\[
[T_m, T_n] = 0, \\
[Y_m, T_n] = -nT_{m+n}, \\
[Y_m, Y_n] = (m - n)Y_{m+n}.
\]

The \(T_n\) generates a supertranslation and \(Y_n\) generates a superrotation. Through the Sugawara construction one can get the \(bms_3\) algebra.
In the following we will show that the near horizon symmetry algebra $\mathfrak{w}_{1+\infty}$ is a subalgebra of the $W_{1+\infty}$. The generators of the $W_{1+\infty}$ algebra $L^{(n)}_m$ satisfying the following relation \[27\],

\[
\begin{align*}
[L_m^{(0)}, L_n^{(0)}] &= 0, \\
[L_m^{(1)}, L_n^{(0)}] &= -nL_m^{(0)}, \\
[L_m^{(1)}, L_n^{(1)}] &= (m-n)L_m^{(1)}, \\
[L_m^{(2)}, L_n^{(0)}] &= -2nL_m^{(1)}, \\
&\quad \ldots \ .
\end{align*}
\]

(4)

It is easy to show that, after the identification $T_m = L_m^{(0)}$, $Y_m = L_m^{(1)}$, one get exactly the algebra \[3\]. They form a closed subalgebra of $W_{1+\infty}$. The physical meaning of those generator is also clear: since the $L^{(n)}_m$ is a component of the conformal spin-$(n+1)$ field, so $T_m$ relates to spin-1 field and $Y_m$ spin-2 field.

This $W_{1+\infty}$ has no central extension term, but one can also consider the central extension $\tilde{W}_{1+\infty}$ which has Kac-Moody algebra as subalgebra. The new algebra has the following commutation relation \[27\],

\[
\begin{align*}
[\tilde{L}_m^{(0)}, \tilde{L}_n^{(0)}] &= n\delta_{n+m,0}, \\
[\tilde{L}_m^{(1)}, \tilde{L}_n^{(0)}] &= -n\tilde{L}_m^{(0)}, \\
[\tilde{L}_m^{(1)}, \tilde{L}_n^{(1)}] &= (m-n)\tilde{L}_m^{(1)} + \frac{1}{12}n(n^2 - 1)\delta_{n+m,0}, \\
&\quad \ldots \ .
\end{align*}
\]

(5)

It contains an abelian Kac-Moody algebra and $c = 1$ Virasoro algebra as a subalgebra.

Interesting this subalgebra also appears in Ref. \[28\], where a set of asymptotically AdS$_3$ boundary conditions were found. But the central charge and the level of the Kac-Moody algebra are different.

**CLASSIFICATION OF BLACK HOLE BY THE $W_{1+\infty}$ SYMMETRY**

The $W_{1+\infty}$ symmetry was used to classify the universality classes of quantum incompressible fluids \[26\]. These classes are specified by the kinematical data of the electrical charge $Q$ and the spin $J$ of the quasi-particles in quantum Hall fluid. They are the eigenvalues of $L_0^{(0)}$ and $L_0^{(1)}$, i.e.

\[
L_0^{(0)}|Q> = e|Q>, \quad L_0^{(1)}|Q> = J|Q> .
\]

(6)

All unitary, irreducible, highest-weight representation of the $W_{1+\infty}$ have been found \[29, 30\]. They exist for positive integer central charge $c = m = 1, 2, \cdots$ and are labeled by a $m$–component highest-weight vector $\vec{r} = \{r_1, \cdots, r_m\}$.

For generic filling fractions of quantum Hall fluid, the spectrum is given by

\[
Q = \sum_{i,j=1}^{m} K_{ij}^{-1}n_j, \quad \frac{e}{\pi} = 2J = \sum_{i,j=1}^{m} n_i K_{ij}^{-1}n_j,
\]

(7)

where $K_{ij}$ is a $m \times m$ symmetric, integral-valued matrix with $K_{ii}$ odd, and $n_j$ integers represent the number of vortices (quasi-particles) created in the $j – th$ component of the fluid. For quantum spin Hall effect (topological insulator in three-dimensional spacetime), the relevant K-matrice is $K = diag(1, -1)$ \[21\].
It was claimed that the black hole can be considered as a kind of topological insulator [14, 15]. So, in three dimensional spacetime, black holes can be considered as quantum spin Hall fluid, and we can classify the black holes with the help of $W_{1+\infty}$. The K-matric is $K = \text{diag}(1, -1)$ and $c = m = 2$. The kinematical data $(E, J)$ will be the eigenvalues of $L_0^{(i)} = T_0$ and $L_0^{(1)} = Y_0$, which relate to the mass and angular momentum of the black hole. We denote the highest-weight state as $|E, J >$ with

$$L_0^{(0)}|E, J > = E|E, J >, \quad L_0^{(1)}|E, J > = J|E, J >. \quad (8)$$

Consider the BTZ black hole as an example. The eigenvalue of $T_0, Y_0$ were given [12]

$$T_0 = \frac{\kappa r_+}{4G}, \quad Y_0 = \frac{r_+ - r_-}{4Gl}, \quad (9)$$

where $\kappa = \frac{r_+^2 - r_-^2}{4r_+}$ is the surface gravity and $l$ the AdS radius. Since $m = 2$ is fixed, the states will be specified by two integer $(n_1, n_2)$. To get those two dimensionless number, one must choose a unit mass $E_0$ for $E$, just the same as the unit electrical charge $e$ for $Q$. Denote $nE_0 = \frac{\kappa r_+}{4G}$, and similar to (7), we can get

$$n = n_1 - n_2 = \frac{\kappa r_+}{4GE_0}, \quad J = \frac{1}{2}(n_1^2 - n_2^2) = \frac{r_+ - r_-}{4Gl}. \quad (10)$$

It is easy to get the solution

$$n_1 = \frac{E_0 r_-}{\kappa l} + \frac{r_+ \kappa}{8GE_0}, \quad n_1 = \frac{E_0 r_-}{\kappa l} - \frac{r_+ \kappa}{8GE_0}. \quad (11)$$

A natural choice of $E_0$ is $E_0 = 1/G$, which gives

$$n_1 = \frac{r_-}{\kappa Gl} + \frac{r_+ \kappa}{8} = \frac{1J}{2G\sqrt{M^2l^2 - J^2}} + \frac{G\sqrt{M^2l^2 - J^2}}{l},$$

$$n_2 = \frac{r_-}{\kappa Gl} - \frac{r_+ \kappa}{8} = \frac{1J}{2G\sqrt{M^2l^2 - J^2}} - \frac{G\sqrt{M^2l^2 - J^2}}{l}. \quad (12)$$

For non-rotating BTZ black hole, $J = 0$ gives

$$n_1 = -n_2 = MG = \frac{M}{M_{PL}}, \quad (13)$$

where $M_{PL}$ is the Planck mass.

In $W_{1+\infty}$ algebra, all operators $L_0^{(i)}$ are simultaneously diagonal and assign other quantum numbers to the quasi-particle [26]. In quantum Hall fluid, those quantum numbers measure the radial moments of the charge distribution of a quasi-particle. In black hole physics, those quantum numbers can be considered as ‘W-hairs’ and maybe crucial for information paradox [31].

**CONCLUSION**

In this paper, we consider the relation of the near horizon symmetry of black hole in three dimensional spacetime with the $W_{1+\infty}$ symmetry of quantum Hall effect. It is found that the former is a subalgebra of the latter. The $W_{1+\infty}$ algebra is a quantum version of area-preserving diffeomorphism algebra, which is a dynamical symmetry of quantum Hall liquid. This is because the quantum Hall droplet is incompressible fluid, so has constant area. If we consider
black hole as quantum hall droplet, the origin of the near horizon symmetry is very clear: it is the sub-symmetry of the area-preserving diffeomorphism. With the help of the $W_{1+\infty}$ we can also give a classification of black holes. They are specified by two integer $(n_1, n_2)$ which are functions of the black hole mass and angular momentum $(M, J)$. Now let’s pay attention to four dimensional spacetime. Due to the above discussion, the near horizon symmetry algebra of black hole can be considered as sub-algebra of the volume-preserving diffeomorphism. The representation of this algebra may also give a classification of black holes in four dimensional spacetime and the ‘hairs’ used for solving information paradox.

The central extension of the supertranslation algebra is an abelian Kac-Moody algebra. This algebra can also be get from another approach. It is well known that, $(2 + 1)$–dimensional general relativity with $\Lambda = -\frac{1}{l^2}$ can be cast into $SO(2, 1) \times SO(2, 1)$ Chern-Simons theory. On a manifold with a boundary, the Chern-Simons theory reduces to a chiral Wess-Zumino-Novikov-Witten(WZNW) theory with $SO(2, 1)$ Kac-Moody algebra on the boundary. If the boundary is chosen to be the horizon of a black hole, the $SO(2, 1)$ Kac-Moody algebra reduces to $SO(1, 1)$ Kac-Moody algebra.

$W_{1+\infty}$ is also used to solve the information paradox, since it has infinite set of quantum numbers, i.e. ‘$W$-hair’. Due to our result, the near horizon symmetry algebra is a subalgebra of $W_{1+\infty}$, the ‘$W$-hair’ can be considered as generalized ‘soft hair’.

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