Abstract. Discrete derived categories were introduced by Vossieck [35] and classified by Bobiński, Geiß, Skowroński [8]. In this article, we describe the homomorphism hammocks and autoequivalences on these categories. We classify silting objects and bounded t-structures.

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Introduction

In this article, we study the bounded derived categories of finite-dimensional algebras that are discrete in the sense of Vossieck [35]. Informally speaking, discrete derived categories can be thought of as having structure intermediate in complexity between the derived categories of hereditary algebras of finite representation type and those of tame type. Note, however, that the algebras with discrete derived categories are not hereditary. We defer the precise definition until the beginning of the next section.

The study of the structure of discrete derived categories was begun by Bobiński, Geiß and Skowroński in [8], who obtained a canonical form for the derived equivalence class of algebras with a given discrete derived category; see Figure 1. This canonical form is parametrised by integers $n \geq r \geq 1$ and $m > 0$, and the corresponding algebra denoted by $\Lambda(r, n, m)$. We restrict our attention to the case $n > r$, which ensures finite global dimension. In [8], the authors also determined the components of the Auslander-Reiten (AR) quiver of derived-discrete algebras and computed the suspension functor.

The structure exhibited in [8] is remarkably simple, which brings us to our principal motivation for studying these categories. Discrete derived categories are sufficiently simple to make explicit computation highly accessible but also non-trivial enough to manifest interesting behaviour. In particular, sitting naturally inside these categories are the (higher) cluster categories of type $A_\infty$ studied in [18] and [19], and an abundance of spherelike objects in the sense of [17]. Moreover, their structure is highly reminiscent of that of the bounded derived categories of cluster-tilted algebras of type $\tilde{A}_n$ studied in [1].
suggesting approaches developed here to understand discrete derived categories are likely to find applications more widely in the study of derived categories of gentle algebras.

Understanding homological properties of algebras means understanding the structure of their derived categories. We investigate several key aspects of the structure of discrete derived categories: the structure of homomorphism spaces, the autoequivalence groups of the categories, and the t-structures and co-t-structures inside discrete derived categories.

The basis of our work is giving a combinatorial description via AR quivers of which indecomposable objects admit non-trivial homomorphism spaces between them, so called ‘Hom-hammocks’. As a byproduct, we get the following interesting property of these categories: the dimensions of the homomorphism spaces between indecomposable objects have a common bound. In fact, in Theorem 5.1 we show there are unique homomorphisms, up to scalars, whenever \( r > 1 \), and in the exceptional case \( r = 1 \), the common dimension bound is 2. We believe this property holds independent interest and warrants further investigation. See [15] for a different approach to measuring the ‘smallness’ of discrete derived categories.

In Theorem 4.6 we explicitly describe the group of autoequivalences. For this, we introduce a generalisation of spherical twist functors arising from cycles of exceptional objects. The action of these twists on the AR components of \( \Lambda(r, n, m) \) is a useful tool, which is employed here and will also be used in the sequel.

In Section 6 we address the classification of bounded t-structures and co-t-structures in \( D^b(\Lambda(r, n, m)) \), which are important in understanding the cohomology theories occurring in triangulated categories, and have recently become a focus of intense research as the principal ingredients in the study of Bridgeland stability conditions [11], and their co-t-structure analogues [23]. Further investigation into the properties of (co-)t-structures and the stability manifolds is the subject of the sequel.

We study the (co-)t-structures indirectly via certain generating sets: silting subcategories, which behave like the projective objects of hearts and generalise tilting objects. In general, one cannot get all bounded t-structures in this way, but in Proposition 6.1 we show that the heart of each bounded t-structure in \( D^b(\Lambda) \) is equivalent to \( \text{mod}(\Gamma) \), where \( \Gamma \) is a finite-dimensional algebra of finite representation type. The upshot is that using some correspondences of König and Yang [26], classifying silting objects is enough to classify all bounded (co-)t-structures. We show that \( D^b(\Lambda(r, n, m)) \) admits a semi-orthogonal decomposition into \( D^b(kA_n+m-1) \) and the thick subcategory generated by an exceptional object. Using Aihara and Iyama’s silting reduction [1], we classify the silting objects in Theorem 6.18. We finish with an explicit example of \( \Lambda(2, 3, 1) \) in Section 7.

Acknowledgments: We are grateful to Aslak Bakke Buan, Martin Kalck, Henning Krause, and Dong Yang. Much of this paper was prepared while all three authors were Scientific Assistants at Leibniz Universität Hannover. We thankfully acknowledge Leibniz Universität Hannover for financial support and our colleagues at the Institut für Algebra, Zahlentheorie und Diskrete Mathematik and at the Institut für Algebraische Geometrie for their kind hospitality and for creating an excellent research environment.

1. DISCRETE DERIVED CATEGORIES AND THEIR AR-QUIVER

We always work over a fixed algebraically closed field \( k \). All modules will be finite dimensional right modules.

1.1. DISCRETE DERIVED CATEGORIES. We are interested in \( k \)-linear, Hom-finite triangulated categories which are small in a certain sense. One precise definition of such smallness is given by Vossieck [35]; here we present a slight generalisation of his notion: a derived
category (or, more generally and intrinsically, a Hom-finite triangulated category with a t-structure) $\text{D}$ is discrete, if for every map $v: \mathbb{Z} \to K_0(\text{D})$ there are only finitely many isomorphism classes of objects $D \in \text{D}$ with $[H^i(D)] = v(i) \in K_0(\text{D})$ for all $i \in \mathbb{Z}$.

Let us elaborate on the connection to [35]: Vossieck speaks of positive dimension vectors $v \in K_0(\text{D})(\mathbb{Z})$ which he can do since he has $\text{D} = \text{D}^b(\text{A})$ for a finite dimensional algebra, so $K_0(\text{A}) \cong \mathbb{Z}^r$. In our slight generalisation of his notion, we cannot do so, but for finite dimensional algebras the new notion gives back the old one: if $v$ is negative somewhere, there will be no objects of that dimension vector whatsoever. For the same reason, we don’t have to assume that $v$ has finite support: if it doesn’t, the set of objects of that class is empty.

Obviously, derived categories of path algebras of type ADE Dynkin quivers are examples of discrete categories. Furthermore, [35] shows that the bounded derived category of a finite dimensional algebra $\text{A}$, which is not of finite representation type, is discrete if and only if $\text{A}$ is Morita equivalent to the bound quiver algebra of a gentle quiver with exactly one cycle having different numbers of clockwise and anticlockwise orientations.

Furthermore, in [8], Bobiński, Geiß and Skowroński give a classification of such algebras. More precisely, for $\text{A}$ connected and not of Dynkin type, the derived category $\text{D}^b(\text{A})$ is discrete if and only if $\text{A}$ is derived equivalent to the path algebra $\Lambda(\text{r}, \text{n}, \text{m})$ for the quiver with relations given in Figure 1 and some values of $\text{r}, \text{n}, \text{m}$.

1.2. The AR quiver of $\text{D}^b(\Lambda(\text{r}, \text{n}, \text{m}))$. The algebra $\Lambda(\text{r}, \text{n}, \text{m})$ has finite global dimension if and only if $\text{n} > \text{r}$. In the following, we always make this assumption. Therefore the derived category $\text{D}^b(\Lambda(\text{r}, \text{n}, \text{m}))$ enjoys duality in the form

\[ \text{Hom}(A, B) = \text{Hom}(B, \Sigma \tau A)^* \]

functorially in $A, B \in \text{D}^b(\Lambda(\text{r}, \text{n}, \text{m}))$. In other words, $S := \Sigma \tau$ is a Serre functor for $\text{D}^b(\Lambda(\text{r}, \text{n}, \text{m}))$. We will use both notations, depending on the context. Some general properties of $\text{D}^b(\Lambda(\text{r}, \text{n}, \text{m}))$ are: this triangulated category is algebraic, Hom-finite, Krull–Schmidt and indecomposable; see Appendix A.1.1 for details.

We collect together some more special properties of $\text{D}^b(\Lambda(\text{r}, \text{n}, \text{m}))$ which will be crucial throughout the paper; the reference is [8]. By [8 Theorem B], the AR quiver of $\text{D}^b(\Lambda(\text{r}, \text{n}, \text{m}))$ has precisely $3r$ components; these and the subcategories of objects of the same type are denoted by

\[ \mathcal{X}^0, \ldots, \mathcal{X}^{r-1}, \quad \mathcal{Y}^0, \ldots, \mathcal{Y}^{r-1}, \quad \mathcal{Z}^0, \ldots, \mathcal{Z}^{r-1}; \]

of these, the $\mathcal{X}$ and $\mathcal{Y}$ components are of type $\mathbb{Z}\Lambda_\infty$, whereas the $\mathcal{Z}$ components are of type $\mathbb{Z}\Lambda_\infty\infty$. It will be convenient to have notation for the subcategories generated by
indecomposables of the same type:

\[ \mathcal{X} := \text{add} \bigcup_i \mathcal{X}^i, \quad \mathcal{Y} := \text{add} \bigcup_i \mathcal{Y}^i, \quad \mathcal{Z} := \text{add} \bigcup_i \mathcal{Z}^i. \]

For each \( k = 0, \ldots, r - 1 \), we label the indecomposable objects in \( \mathcal{X}^k, \mathcal{Y}^k, \mathcal{Z}^k \) as follows:

\( X_{ij}^k \in \mathcal{X}^k \) with \( i, j \in \mathbb{Z}, j \geq i \);
\( Y_{ij}^k \in \mathcal{Y}^k \) with \( i, j \in \mathbb{Z}, i \geq j \);
\( Z_{ij}^k \in \mathcal{Z}^k \) with \( i, j \in \mathbb{Z} \).

**Properties 1.1.** This labelling is chosen in such a way that the following properties hold:

1. Irreducible morphisms go from an object with coordinate \((i, j)\) to objects \((i + 1, j)\) and \((i, j + 1)\) in the same component (when they exist).

2. The AR translate of an object with coordinate \((i, j)\) is the object with coordinate \((i - 1, j - 1)\) in the same component, i.e. \( \tau X_{ij}^k = X_{i-1,j-1}^{k+1} \) etc.

3. The suspension of indecomposable objects is given below, with \( k = 0, \ldots, r - 2 \):

\[
\begin{align*}
\Sigma X_{ij}^k &= X_{ij}^{k+1}, & \Sigma X_{ij}^{-1} &= X_{i+r+m,j+r+m}^0, \\
\Sigma Y_{ij}^k &= Y_{ij}^{k+1}, & \Sigma Y_{ij}^{-1} &= Y_{i+r-n,j+r-n}^0, \\
\Sigma Z_{ij}^k &= Z_{ij}^{k+1}, & \Sigma Z_{ij}^{-1} &= Z_{i+r+m,j+r-n}^0.
\end{align*}
\]

In particular, \( \Sigma^n |_{\mathcal{X}} = \tau^{-m-n} \) and \( \Sigma^n |_{\mathcal{Y}} = \tau^{-n-r} \).

4. There are distinguished triangles, for any \( i, j, d \in \mathbb{Z} \) with \( d \geq 0 \):

\[
\begin{align*}
X_{i,i+d}^k &\rightarrow Z_{ij}^k \rightarrow Z_{i+d+1,j}^k \rightarrow \Sigma X_{i,i+d}^k, \\
Y_{j+d,j}^k &\rightarrow Z_{ij}^k \rightarrow Z_{i+j+d+1}^k \rightarrow \Sigma Y_{j+d,j}^k.
\end{align*}
\]

5. There are chains of non-zero morphisms for any \( i \in \mathbb{Z} \) and \( k = 0, \ldots, r - 1 \):

\[
\begin{align*}
X_{ii}^k &\rightarrow X_{i,i+1}^k \rightarrow \cdots \rightarrow Z_{i,i-1}^k \rightarrow Z_{ii}^k \rightarrow Z_{i,i+1}^k \rightarrow \cdots \rightarrow \Sigma X_{i+1,i-1}^k \rightarrow \Sigma X_{i,i-1}^k \rightarrow \Sigma X_{i-1,i-1}^k, \\
Y_{ii}^k &\rightarrow Y_{i+1,i}^k \rightarrow \cdots \rightarrow Z_{i,i-1}^k \rightarrow Z_{ii}^k \rightarrow Z_{i+1,i}^k \rightarrow \cdots \rightarrow \Sigma Y_{i+1,i-1}^k \rightarrow \Sigma Y_{i,i-1}^k \rightarrow \Sigma Y_{i-1,i-1}^k.
\end{align*}
\]

Later, we will often use the ‘height’ of indecomposable objects in \( \mathcal{X} \) or \( \mathcal{Y} \) components.

For \( X_{ij}^k \in \text{ind}(\mathcal{X}^k) \), we set \( h(X_{ij}^k) = j - i \) and call it the height of \( X_{ij}^k \) in the component \( \mathcal{X}^k \). Similarly, for \( Y_{ij}^k \in \text{ind}(\mathcal{Y}^k) \), we set \( h(Y_{ij}^k) = i - j \) and call it the height of \( Y_{ij}^k \) in the component \( \mathcal{Y}^k \). The *mouth* of an \( \mathcal{X} \) or \( \mathcal{Y} \) component consists of all objects of height 0.

### 2. Hom spaces: Hammocks

In this section, for a fixed indecomposable object \( A \in D^b(\Lambda) \) we compute the so-called ‘Hom-hammock’ of \( A \), i.e. the set of indecomposables \( B \in D^b(\Lambda) \) with \( \text{Hom}(A, B) \neq 0. \)
By duality, this also gives the outgoing Hom-hammocks: $\text{Hom}(-, A) = \text{Hom}(S^{-1}A, -)^*$. Therefore we generally refrain from listing the Hom(-, A) hammocks explicitly.

The precise description of the hammocks is slightly technical. However, the result is quite simple, and the following schematic indicates the hammocks $\text{Hom}(X, -) \neq 0$ and $\text{Hom}(Z, -) \neq 0$ for indecomposables $X \in \mathcal{X}$ and $Z \in \mathcal{Z}$:

![Diagram showing hammocks](image)

2.1. Hammocks from the mouth. We start with a description of the Hom-hammocks of objects at the mouths of all $\mathbb{Z}A_{\infty}$ components. The proof relies on Happel’s equivalence of $\mathcal{D}^b(\Lambda(r, n, m))$ with the stable module category of the repetitive algebra of $\Lambda(r, n, m)$. The well-known theory of string (and band) modules, which we summarise in Appendix B, provides a useful tool to understand the indecomposable objects and homomorphisms between them.

To make our statements of Hom-hammocks more readable, we employ the language of rays and corays. Let $V = V_{i,j}$ be an indecomposable object of $\mathcal{D}^b(\Lambda(r, n, m))$ with coordinates $(i, j)$. Recall the conventions that $j \geq i$ if $V \in \mathcal{X}$ whereas $i \geq j$ if $V \in \mathcal{Y}$. Denoting the AR component of $V$ by $\mathcal{C}$ and its objects by $V_{a,b}$, we define the rays and corays from, to, and through $V$ by

- $\text{ray}_+(V_{i,j}) := \{V_{i,j+l} \in \mathcal{C} \mid l \in \mathbb{N}\}$,
- $\text{ray}_-(V_{i,j}) := \{V_{i,j-l} \in \mathcal{C} \mid l \in \mathbb{N}\}$,
- $\text{ray}_{\pm}(V_{i,j}) := \{V_{i,j+l} \in \mathcal{C} \mid l \in \mathbb{Z}\}$,
- $\text{coray}_+(V_{i,j}) := \{V_{i,j+l} \in \mathcal{C} \mid l \in \mathbb{N}\}$,
- $\text{coray}_-(V_{i,j}) := \{V_{i,j-l} \in \mathcal{C} \mid l \in \mathbb{N}\}$,
- $\text{coray}_{\pm}(V_{i,j}) := \{V_{i,j+l} \in \mathcal{C} \mid l \in \mathbb{Z}\}$.

Note that, because of the orientation of the components, the ray of an indecomposable $X^k \in \mathcal{X}^k$ at the mouth consists of indecomposables in $\mathcal{X}^k$ reached by arrows going out of $X^k_{ii}$, while in the $\mathcal{Y}$ components the ray of $Y^k$ contains objects which have arrows going in to it.

For the next statement, whose proof is deferred to Lemma [3, 4], recall that the Serre functor is given by suspension and AR translation: $S = \Sigma\tau$. Also, rays and corays commute with these three functors.

**Lemma 2.1.** Let $A \in \text{ind}(\mathcal{D}^b(\Lambda(r, n, m)))$ with $r > 1$ and let $i, k \in \mathbb{Z}$, $0 \leq k < r$. Then

- $\text{Hom}(X^k_{ii}, A) = \mathbb{K}$ if $A \in \text{ray}_+(X^k_{ii}) \cup \text{coray}_-(S X^k_{ii}) \cup \text{ray}_+(Z^k_{ii})$,
- $\text{Hom}(Y^k_{ii}, A) = \mathbb{K}$ if $A \in \text{coray}_+(Y^k_{ii}) \cup \text{ray}_-(S Y^k_{ii}) \cup \text{coray}_+(Z^k_{ii})$,
- $\text{Hom}(A, X^k_{ii}) = \mathbb{K}$ if $A \in \text{coray}_-(X^k_{ii}) \cup \text{ray}_+(S^{-1}X^k_{ii}) \cup \text{ray}_+(S^{-1}Z^k_{ii})$,
- $\text{Hom}(A, Y^k_{ii}) = \mathbb{K}$ if $A \in \text{ray}_-(Y^k_{ii}) \cup \text{coray}_+(S^{-1}Y^k_{ii}) \cup \text{coray}_-(S^{-1}Z^k_{ii})$

and in all other cases the Hom spaces are zero. For $r = 1$ the Hom-spaces are as above, except $\text{Hom}(X^0_{ii}, \text{id}(X^0_{i,i+m})) = \mathbb{K}^2$.

2.2. Hom-hammocks for objects in $\mathcal{X}$ components. Assume $A = X^k_{ij} \in \text{ind}(\mathcal{X}^k)$. In order to describe the various Hom-hammocks conveniently, we set

- $A_0 := X^k_{jj}$ to be the intersection of the coray through $A$ with the mouth of $\mathcal{X}^k$,
- $\partial A := X^k_{ii}$ to be the intersection of the ray through $A$ with the mouth of $\mathcal{X}^k$. 


By definition, $A_0$ and $0A$ have height 0. If $A$ sits at the mouth, then $A = A_0 = 0A$.

We introduce notation for line segments in the AR quiver: given two indecomposable objects $A, B \in D^b(\Lambda(r,n,m))$ which lie on a ray or coray (in particular, sit in the same component), then the finite set consisting of these two objects and all indecomposables lying between them on the (co-)ray is denoted by $\overline{AB}$. Finally, before we state the proposition, recall our convention that $X^r = X^0$ and note that $0(SA) = \Sigma(r(0A))$.

**Proposition 2.2 (Hammocks Hom($\mathcal{X}_k$, $\mathcal{X}$)).** Let $A = X^k_0 \in \text{ind}(\mathcal{X}_k)$ and assume $r > 1$. For any indecomposable object $B \in \text{ind}(D^b(\Lambda))$ the following cases apply:

- $B \in \mathcal{X}_k$: then $\text{Hom}(A, B) \neq 0 \iff B \in \text{ray}_+(\overline{AA_0})$;
- $B \in \mathcal{X}_k^{k+1}$: then $\text{Hom}(A, B) \neq 0 \iff B \in \text{coray}_+(0(SA), SA)$;
- $B \in \mathcal{Z}_k$: then $\text{Hom}(A, B) \neq 0 \iff B \in \text{ray}_+(\overline{Z_{ji}^k})$

and $\text{Hom}(A, B) = 0$ for all other $B \in \text{ind}(D^b(\Lambda))$.

For $r = 1$, these results still hold, except that the $\mathcal{X}$-clauses are replaced by

- $B \in \mathcal{X}_0$: then $\text{Hom}(A, B) \neq 0 \iff B \in \text{ray}_+(\overline{AA_0}) \cup \text{coray}_+(0(\tau^{-m}A), \tau^{-m}A)$.

**Proof.** The main tool in the proof of this, and the following propositions, will be induction on the height of $A$ — the induction base step is proved in Lemma 2.1 which gives the hammocks for indecomposables of height 0. We give a careful exposition for the first claim, and for $r > 1$. The $r = 1$ case will be treated in Proposition 5.2.

**Case $B \in \mathcal{X}_k$:** For any indecomposable object $A \in \mathcal{X}_k$, write $R(A)$ for the subset of $\mathcal{X}_k$ specified in the statement, i.e. bounded by the rays out of $A$ and $A_0$, and the line segment $\overline{AA_0}$. The existence of non-zero homomorphisms $A \to B$ for objects $B \in R(A)$ follows directly from the properties of the AR quiver.

For the vanishing statement, we proceed by induction on the height of $A$. If $A$ sits on the mouth of $\mathcal{X}_k$, then Lemma 2.1 states indeed that the $\text{Hom}(A, B) \neq 0$ if and only if $B$ is in the ray of $A$. Note that $R(A)$ is precisely $\text{ray}_+(A)$ in this case.

Now let $A \in \mathcal{X}_k$ be any object of height $h := h(A) > 0$. We consider the diamond in the AR mesh which has $A$ as the top vertex, and the corresponding AR triangle $A' \to A \oplus C \to A'' \to \Sigma A'$, where $h(A') = h(A'') = h - 1$ and $h(C) = h - 2$. (If $h = 1$, we are in the degenerate case with $C = 0$.) It is clear from the definitions that $A_0 = A''_0 = 0C_0$ and there are inclusions $R(A'') \subset R(A) \subset R(A') \cup R(A'')$. We start with an
object $B \in \mathcal{X}^k$ such that $B \notin R(A') \cup R(A'')$. By the induction hypothesis, we know that $R(A'), R(C)$ and $R(A'')$ are the Hom-hammocks in $\mathcal{X}^k$ for $A', C, A''$, respectively. Since $B$ is contained in none of them, we see that $\text{Hom}(A', B) = \text{Hom}(C, B) = \text{Hom}(A'', B) = 0$. Applying $\text{Hom}(-, B)$ to the given AR triangle shows $\text{Hom}(A, B) = 0$.

It remains to show that $\text{Hom}(A, D) = 0$ for objects $D \in (R(A') \cup R(A'')) \setminus R(A)$ which can be seen to be the line segment $\overline{A'A_0}$. Again we work up from the mouth: $\text{Hom}(A, A_0') = 0$ and $\text{Hom}(A, \tau A_0') = 0$ by Lemma 2.1 as before. The extension $D_1$ given by $\tau A_0' \to D_1 \to A_0' \to \Sigma \tau A_0'$ is the indecomposable object of height 1 on $\overline{A'A_0}$. Applying $\text{Hom}(A, -)$ to this triangle, we find $\text{Hom}(A, D_1) = 0$, as required. The same reasoning works for the objects of heights 2, \ldots, $h - 1$ on the segment.

**Case $B \in \mathcal{X}^{k+1}$:** We start by showing the existence of non-zero homomorphisms to indecomposable objects in the desired region. For any $B$ in this region, it follows directly from the properties of the AR quiver that there is a non-zero homomorphism from $B$ to $\mathcal{S}A$. However by Serre duality we see that $\text{Hom}(A, B) = \text{Hom}(B, \mathcal{S}A) \neq 0$ as required. The statement that $\text{Hom}(A, B) = 0$ for all other $B \in \mathcal{X}^{k+1}$ can be proved by an induction argument which is analogous to the one given in the first case above.

**Case $B \in \mathcal{Z}^k$:** For any indecomposable object $A = X_{ij}^k \in \mathcal{X}^k$, write $V(A)$ for the region in $\mathcal{Z}^k$ specified in the statement, i.e. the region bounded by the rays through $Z_{ij}^k$ and $Z_{ji}^k$. We start by proving that $\text{Hom}(A, B) \neq 0$ for $B \in V(A)$. The first chain of morphisms in Properties 1.1(5), implies that $\text{Hom}(A, B) \neq 0$ for any $B \in \text{ray}_{\pm}(Z_{ii}^k)$. For any other $B' = Z_{i+s,t}^k \in V(A)$, so $t \in \mathbb{Z}$ and $s \in \{1, \ldots, h(A) = j - i\}$, we consider the special triangle $X_{i,i+s-1}^k \to B \to B' \to \Sigma X_{i,i+s-1}^k$ from Properties 1.1(4), where $B = Z_{it}^k \in \text{ray}_{\pm}(Z_{ii}^k)$. Applying $\text{Hom}(A, -)$ leaves us with the exact sequence

$$\text{Hom}(A, X_{i,i+s-1}^k) \to \text{Hom}(A, B) \to \text{Hom}(A, B') \to \text{Hom}(A, \Sigma X_{i,i+s-1}^k).$$

By looking at the Hom-hammocks in the $\mathcal{X}$-components that we already know, we see that the left-hand term vanishes as $X_{i,i+s-1}^k$ is on the same ray as $A$ but has strictly lower height. Similarly, we observe that the right-hand term of the sequence vanishes: $0 = \text{Hom}(X_{i,s+1}^k, \tau A) = \text{Hom}(A, \Sigma X_{i,i+s-1}^k)$. Hence $\text{Hom}(A, B') = \text{Hom}(A, B) \neq 0$.

For the Hom-vanishing part of the statement, we again use induction on the height $h := h(A) \geq 0$. For $h = 0$, Lemma 2.1 gives $V(A) = \text{ray}_{\pm}(Z_{ii}^k)$. For $h > 0$, as before we consider the AR mesh which has $A$ as its top vertex: $A' \to A \oplus C \to A'' \to \Sigma A'$. For any $Z \in \text{ind}(\mathcal{Z}^k)$, we apply $\text{Hom}(-, Z)$ to this triangle and find that $\text{Hom}(A, Z) \neq 0$ implies $\text{Hom}(A', Z) \neq 0$ or $\text{Hom}(A'', Z) \neq 0$. Therefore $\text{Hom}(A, B) = 0$ for all $B \notin V(A') \cup V(A'') = V(A)$, where the final equality is clear from the definitions.

**Remaining cases:** These comprise vanishing statements for entire AR components, namely $\text{Hom}(X^k, X^j) = 0$ for $j \neq k, k + 1$, and $\text{Hom}(X^k, Y^j) = 0$ for any $j$, and $\text{Hom}(X^k, Z^j) = 0$ for $j \neq k$. All of those follow at once from Lemma 2.1 with no non-zero maps from $A$ to the mouths of the specified components of type $\mathcal{X}$ and $\mathcal{Y}$, $\text{Hom}$ vanishing can be seen using induction on height and considering a square in the AR mesh. The vanishing to the $\mathcal{Z}^k$ components with $k \neq j$ follows similarly. \hfill $\square$

### 2.3. Hom-hammocks for objects in $\mathcal{Y}$ components.

Assume $A = Y_{ij}^k \in \text{ind}(\mathcal{Y}^k)$. This case is similar to the one above. Put

$^0A := Y_{ii}^k$ to be the intersection of the coray through $A$ with the mouth of $\mathcal{X}^k$, and $A^0 := Y_{jj}^k$ to be the intersection of the ray through $A$ with the mouth of $\mathcal{X}^k$. 7
Proposition 2.3 (Hammocks Hom(\(\mathcal{Y}^k, -\)). Let \(A = \mathcal{Y}^k_{ij} \in \text{ind}(\mathcal{Y}^k)\) and assume \(r > 1\).

For any indecomposable object \(B \in \text{ind}(\text{D}^b(\Lambda))\) the following cases apply:

- \(B \in \mathcal{Y}^k\): then \(\text{Hom}(A, B) \neq 0 \iff B \in \text{coray}_+(AA^0)\);
- \(B \in \mathcal{Y}^{k+1}\): then \(\text{Hom}(A, B) \neq 0 \iff B \in \text{ray}_-(0(SA), SA)\);
- \(B \in \mathcal{Z}^k\): then \(\text{Hom}(A, B) \neq 0 \iff B \in \text{coray}_+(Z^k_{ij}Z^k_{ij})\)

and \(\text{Hom}(A, B) = 0\) for all other \(B \in \text{ind}(\text{D}^b(\Lambda))\).

For \(r = 1\), these results still hold, except that the \(\mathcal{Y}\)-clauses are replaced by

- \(B \in \mathcal{Y}^0\): then \(\text{Hom}(A, B) \neq 0 \iff B \in \text{coray}_+(AA^0) \cup \text{ray}_-(0(\tau A), \tau A)\).

Proof. These statements are analogous to those of Proposition 2.2.

In this case, the vanishing statement for full components consists of \(\text{Hom}(\mathcal{Y}^k, \mathcal{X}^j) = 0\) for any \(j\), and \(\text{Hom}(\mathcal{Y}^k, \mathcal{Y}^j) = 0\) for \(j \neq k, k + 1\), and \(\text{Hom}(\mathcal{Y}^k, \mathcal{Z}^j) = 0\) for \(j \neq k\).

2.4. \textbf{Hom-hammocks for objects in} \(\mathcal{Z}\) \textbf{components}. Let \(A = Z^k_{ij} \in \text{ind}(\mathcal{Z}^k)\). By Lemma 2.1, we know that the following objects are well defined:

- \(A_0 := \) the unique object at the mouth of an \(\mathcal{X}\) component for which \(\text{Hom}(A, A_0) \neq 0\),
- \(A^0 := \) the unique object at the mouth of a \(\mathcal{Y}\) component for which \(\text{Hom}(A, A^0) \neq 0\).

In fact, \(A_0 \in \mathcal{X}^{k+1}\) and \(A^0 \in \mathcal{Y}^{k+1}\).

Proposition 2.4 (Hammocks Hom(\(\mathcal{Z}^k, -\)). Let \(A = Z^k_{ij} \in \text{ind}(\mathcal{Z}^k)\) and assume \(r > 1\).

For any indecomposable object \(B \in \text{ind}(\text{D}^b(\Lambda))\) the following cases apply:

- \(B \in \mathcal{X}^{k+1}\): then \(\text{Hom}(A, B) \neq 0 \iff B \in \text{ray}_+(\text{coray}_-(A_0))\);
- \(B \in \mathcal{Y}^{k+1}\): then \(\text{Hom}(A, B) \neq 0 \iff B \in \text{ray}_-(\text{coray}_+(A^0))\);
- \(B \in \mathcal{Z}^k\): then \(\text{Hom}(A, B) \neq 0 \iff B \in \text{ray}_+(\text{coray}_+(A))\);
- \(B \in \mathcal{Z}^{k+1}\): then \(\text{Hom}(A, B) \neq 0 \iff B \in \text{ray}_-(\text{coray}_-(SA))\)

and \(\text{Hom}(A, B) = 0\) for all other \(B \in \text{ind}(\text{D}^b(\Lambda))\).

For \(r = 1\), these results still hold, with the \(\mathcal{Z}\)-clauses replaced by

- \(B \in \mathcal{Z}^0\): then \(\text{Hom}(A, B) \neq 0 \iff B \in \text{ray}_+(\text{coray}_+(A)) \cup \text{ray}_-(\text{coray}_-(SA))\).

Proof. The cases \(B \in \mathcal{X}^{k+1}\) and \(B \in \mathcal{Y}^{k+1}\) follow by Serre duality from Proposition 2.2 and Proposition 2.3 respectively.
Thus let $B = Z^l_{ab} \in Z^l$ be an indecomposable object in a $Z$ component. There are two special distinguished triangles associated with $B$; see Section 1:

\[
\begin{array}{cccc}
\_0B & \rightarrow & B & \rightarrow & B' \rightarrow \Sigma_0B \\
\downarrow & & \downarrow & & \downarrow \\
X_{aa}^l & \rightarrow & Z_{ab}^l & \rightarrow & Z_{a+1,b}^l \rightarrow \Sigma X_{aa}^l \\
\end{array}
\quad \text{and} \quad
\begin{array}{cccc}
0B & \rightarrow & B & \rightarrow & B'' \rightarrow \Sigma_0B \\
\downarrow & & \downarrow & & \downarrow \\
Y_{bb}^l & \rightarrow & Z_{ab}^l & \rightarrow & Z_{a,b+1}^l \rightarrow \Sigma Y_{bb}^l \\
\end{array}
\]

where $0B = X_{aa}^l$ is the unique object at the mouth of a $X$ component with $\text{Hom}(0B, B) = 0$ and similarly $0B = Y_{bb}^l$ is unique at a $Y$ mouth with $\text{Hom}(0B, B) = 0$. We get two exact sequences by applying $\text{Hom}(A, -)$:

\[
\begin{align*}
\text{Hom}(A, 0B) & \rightarrow \text{Hom}(A, B) \rightarrow \text{Hom}(A, B') \rightarrow \text{Hom}(A, \Sigma_0B), \\
\text{Hom}(A, 0B) & \rightarrow \text{Hom}(A, B) \rightarrow \text{Hom}(A, B'') \rightarrow \text{Hom}(A, \Sigma_0B).
\end{align*}
\]

Case $l \neq k, k + 1$: In this case $\text{Hom}(A, B) = \text{Hom}(A, B') = \text{Hom}(A, B'')$ follows from the above triangles via these exact sequences and Lemma 2.1. But this implies $\text{Hom}(A, Z)$ for all $Z \in Z^l$ and in particular $\text{Hom}(A, B) = \text{Hom}(A, \Sigma_0B)$ for all $c \in \mathbb{Z}$. It follows that $\text{Hom}(A, B) = 0$ as $\Lambda(r, n, m)$ has finite global dimension.

Case $l = k$: Again, we first show that the dimension function $\text{Hom}(A, -)$ is constant on certain regions of $Z^k$. In particular, we have

(1) \quad $\text{Hom}(A, B) = \text{Hom}(A, B')$ for $B \notin \text{ray}_\pm(SA) \cup \text{ray}_\pm(\tau A)$;

(2) \quad $\text{Hom}(A, B) = \text{Hom}(A, B'')$ for $B \notin \text{coray}_\pm(SA) \cup \text{coray}_\pm(\tau A)$.

Half of the first equality follows through the chain of equivalences

\[
\text{Hom}(A, 0B) \neq 0 \iff A \in \text{ray}_\pm(S^{-1}Z^k_{aa}) \iff A \in \text{ray}_\pm(S^{-1}B) \iff B \in \text{ray}_\pm(SA).
\]

Likewise one obtains $\text{Hom}(A, \Sigma_0B) \neq 0 \iff B \in \text{ray}_\pm(\tau A)$, giving the first equality. Using the other triangle, the second equality is analogous.

The component $Z^k$ is divided by $\text{ray}_\pm(\tau A)$ and $\text{coray}_\pm(\tau A)$ into four regions:

$\mathcal{U}$: The upwards-open region including $\text{ray}_\pm(\tau A) \setminus \{\tau A\}$ but excluding $\text{coray}_\pm(\tau A)$;

$\mathcal{L}$: The left-open region including $\text{ray}_\pm(\tau A) \cup \text{coray}_\pm(\tau A)$;

$\mathcal{D}$: The downwards-open region including $\text{coray}_\pm(\tau A) \setminus \{\tau A\}$ but excluding $\text{ray}_\pm(\tau A)$;

$\mathcal{R}$: The right-open region excluding $\text{ray}_\pm(\tau A) \cup \text{coray}_\pm(\tau A)$.

Using (1) above coupled with the fact that $\mathcal{U}$ contains infinitely many objects $\Sigma^{-rc} A$ with $c \in \mathbb{N}$, shows by the finite global dimension of $\Lambda(r, n, m)$ that no objects in $\mathcal{U}$ admit non-trivial morphisms from $A$. Using (2) and analogous reasoning shows that no objects in $\mathcal{D}$ admit non-trivial morphisms from $A$. Non-existence of non-trivial morphisms from $A$ to objects in $\mathcal{L}$ follows as soon as $\text{Hom}(A, \tau A) = \text{Hom}(1(A, A) = 0$ by using (2) above. The existence of the stalk complex of a projective module in the $Z$ component, Lemma 3.6, coupled with the transitivity of the action of the automorphism group of $D^b(\Lambda(r, n, m))$ on the $Z$ component, which is proved in Section 4 using only Lemma 2.1 shows that $\text{Hom}(1(A, A) = 0$ for all $A \in Z$.

Finally, $\mathcal{R} = \text{coray}_+(\text{ray}_-(A))$ is the non-vanishing hammock simply by $\text{Hom}(A, A) \neq 0$ and using either (1) or (2).

Case $l = k + 1$: This is analogous to the previous case.

\[\square\]

Remark 2.5. In the case that $r > 1$, Propositions 2.2, 2.3 and 2.4 say that each component of the AR quiver of $D^b(\Lambda(r, n, m))$ is standard, i.e. that there are no morphisms in the infinite radical. Note that the components are not standard when $r = 1$.  

9
In this purely categorical section, we consider an abstract source of autoequivalences coming from exceptional cycles. These generalise the tubular mutations from [28] as well as spherical twists. In fact, a quite general and categorical construction has been given in [32]. However, for our purposes this is still a little bit too special, as the Serre functor will act with different degree shifts on the objects in our exceptional cycles. We also give a quick proof using spanning classes.

Let $D$ be a $k$-linear, Hom-finite algebraic triangulated category. Assume that $D$ has a Serre functor $S$ and is indecomposable; see Appendix A.1 for these notions. Recall that an object $E \in D$ is called exceptional if $\text{Hom}^*(E, E) = k \cdot \text{id}_E$. For any object $A \in D$ we define the functor

$$F_A : D \to D, \quad X \mapsto \text{Hom}^*(A, X) \otimes A$$

and note that there is a canonical evaluation morphism $F_A \to \text{id}$ of functors. Also note that for two objects $A_1, A_2 \in D$ there is a common evaluation morphism $F_{A_1} \oplus F_{A_2} \to \text{id}$. In fact, for any sequence of objects $A_\bullet = (A_1, \ldots, A_n)$, we define the associated twist functor $T_{A_\bullet}$ as the cone of the evaluation morphism — this gives a well-defined, exact functor by our assumption that $D$ is algebraic:

$$F_{A_\bullet} \to \text{id}_D \to T_{A_\bullet} \to \Sigma F_{A_\bullet}, \quad \text{with} \quad F_{A_\bullet} := F_{A_1} \oplus \cdots \oplus F_{A_n}$$

These functors behave well under equivalences:

**Lemma 3.1.** Let $\varphi : D \to D'$ be a triangle equivalence of algebraic $k$-linear triangulated categories induced from a dg functor, and let $A_\bullet = (A_1, \ldots, A_n)$ be any sequence of objects. Then there are functor isomorphisms $F_{\varphi(A_\bullet)} = \varphi F_{A_\bullet} \varphi^{-1}$ and $T_{\varphi(A_\bullet)} = \varphi T_{A_\bullet} \varphi^{-1}$.

**Proof.** This follows the standard argument for spherical twists: For $F_{A_\bullet}$ we have

$$\varphi F_{A_\bullet} \varphi^{-1} = \bigoplus_i \text{Hom}^*(A_i, \varphi^{-1}(-)) \otimes \varphi(A_i) = \bigoplus_i \text{Hom}^*(\varphi(A_i), -) \otimes \varphi(A_i) = F_{\varphi(A_\bullet)}.$$

Conjugating the evaluation functor morphism $F_{A_\bullet} \to \text{id}$ with $\varphi$, we find that $\varphi T_{A_\bullet} \varphi^{-1}$ is the cone of the conjugated evaluation functor morphism $F_{\varphi(A_\bullet)} \to \text{id}$ which is the evaluation morphism for $\varphi(A_\bullet)$. Hence that cone is $T_{\varphi(A_\bullet)}$. \hfill $\square$

**Definition.** A sequence $(E_1, \ldots, E_n)$ of objects of $D$ is called an exceptional $n$-cycle if

1. every $E_i$ is an exceptional object,
2. there are integers $k_i$ such that $S(E_i) \cong \Sigma^{k_i} E_{i+1}$ for all $i$ (where $E_{n+1} := E_1$),
3. $\text{Hom}^*(E_i, E_j) = 0$ unless $i = j$ or $j = i + 1$.

This definition assumes $n \geq 2$ but a single object $E$ should be considered an ‘exceptional 1-cycle’ (a linguistic oxymoron) if $E$ is a spherical object, i.e. there is an integer $k$ with $S(E) \cong \Sigma^k E$ and $\text{Hom}^*(E, E) = k \oplus \Sigma^{-k} k$. In this light, the above definition, and statement and proof of Theorem 3.4 are generalisations of the treatment of spherical objects and their twist functors as in [21, §8].

In an exceptional cycle, the only non-trivial morphisms among the $E_i$ apart from the identities are given by $\alpha_i : E_i \to \Sigma^{k_i} E_{i+1}$. This explains the terminology: the subsequence $(E_1, \ldots, E_{n-1})$ is an honest exceptional sequence, but the full set $(E_1, \ldots, E_n)$ is not — the morphism $\alpha_n : E_n \to \Sigma^{k_n} E_1$ prevents it from being one, and instead creates a cycle.

**Remark 3.2.** All objects in an exceptional $n$-cycle are fractional Calabi–Yau: since $S(E_i) \cong \Sigma^{k_i} E_{i+1}$ for all $i$, applying the Serre functor $n$ times yields $S^n(E_i) \cong \Sigma^k E_i$, where $k := k_1 + \cdots + k_n$. Thus the Calabi–Yau dimension of each object in the cycle is $k/n$. 

Example 3.3. We mention that this severely restricts the existence of exceptional $n$-cycles of geometric origin: Let $X$ be a smooth, projective variety over $k$ of dimension $d$ and let $D := D^b(coh X)$ be its bounded derived category. The Serre functor of $D$ is given by $S(-) = \Sigma^d(-) \otimes \omega_X$ and in particular, is given by an autoequivalence of the standard heart followed by an iterated suspension. If $E_\ast$ is any exceptional $n$-cycle in $D$, we find $S^\ast(E_\ast) = \Sigma^{dn}E_\ast \otimes \omega_X^\ast \cong \Sigma^dE_\ast$, hence $k = k_1 + \cdots + k_n = dn$ and $E_\ast \otimes \omega_X^\ast \cong E_\ast$. If furthermore the exceptional $n$-cycle $E_\ast$ consists of sheaves, then this forces $k_i = d$ to be maximal for all $i$, as non-zero extensions among sheaves can only exist in degrees between 0 and $d$. However, $S^\ast E_i = \Sigma^dE_i \otimes \omega_X \cong \Sigma^dE_{i+1}$ implies $E_{i+1} \cong E_i \otimes \omega_X$ for all $i$.

As an example, let $X$ be an Enriques surface. Its structure sheaf $\mathcal{O}_X$ is exceptional, and the canonical bundle $\omega_X$ has minimal order 2. In particular, $(\mathcal{O}_X, \omega_X)$ forms an exceptional 2-cycle and, by the next theorem, gives rise to an autoequivalence of $D^b(X)$.

**Theorem 3.4.** Let $E_\ast = (E_1, \ldots, E_n)$ be an exceptional $n$-cycle in $D$. Then the twist functor $T_{E_\ast}$ is an autoequivalence of $D$.

**Proof.** We define two classes of objects of $D$ by $E := \{S^lE_i \mid l \in \mathbb{Z}, i = 1, \ldots, n\}$ and $\Omega := E \cup E^\perp$. Note that $E$ and hence $\Omega$ are closed under suspensions and cosuspensions. It is a simple and standard fact that $\Omega$ is a spanning class for $D$, i.e. $\Omega^\perp = 0$ and $\perp \Omega = 0$; the latter equality depends on the existence of a Serre functor for $D$. Note that spanning classes are often called ‘(weak) generating sets’ in the literature.

**Step 1:** We start by computing $T_{E_\ast}$ on the objects $E_i$ and the maps $\alpha_i$. For notational simplicity, we will treat $E_1$ and $\alpha_1 : E_1 \to \Sigma k_1 E_2$. It follows immediately from the definition of exceptional cycle that $F_{E_\ast}(E_1) = E_1 \oplus \Sigma^{-k_1}E_n$. The cone of the evaluation morphism is easily seen to sit in the following triangle

$$E_1 \oplus \Sigma^{-k_1}E_n \xrightarrow{(id, \Sigma^{-k_1} \alpha_n)} E_1 \xrightarrow{0} \Sigma^{1-k_1}E_n \xrightarrow{(-\Sigma^{-k_1} \alpha_n, id)} \Sigma E_1 \oplus \Sigma^{1-k_1}E_n,$$

so that $T_{E_\ast}(E_1) = \Sigma^{1-k_1}E_n$. The zero morphism in the middle follows trivially from $\text{Hom}(E_1, \Sigma^{-k_1}E_n) = 0$ which always holds by assumption unless $n = 2$ and $k_1 = 1 - k_2$.

The third map is indeed the one specified above; this can be formally checked with the octahedral axiom, or one can use the vanishing of the composition of two adjacent maps in a triangle. It is straightforward to check the special case $n = 2$ and $k_1 + k_2 = 1$.

Likewise, we find $F_{E_\ast}(E_2) = \Sigma^{-k_1}E_1 \oplus E_2$ and $T_{E_\ast}(E_2) = \Sigma^{1-k_1}E_1$. Now consider the following diagram of distinguished triangles, where the vertical maps are induced by $\alpha_1$:

$$
\begin{array}{ccc}
E_1 \oplus \Sigma^{-k_1}E_n & \xrightarrow{(id, \Sigma^{-k_1} \alpha_n)} & E_1 \\
| & \downarrow{\alpha_1} & \downarrow{T(\alpha_1)} \\
E_1 \oplus \Sigma^{k_1}E_2 & \xrightarrow{(\alpha_1, id)} & \Sigma^{k_1}E_2 \\
| & \downarrow{(id, 0)} & \downarrow{\Sigma \alpha_1} \\
E_1 \oplus \Sigma^{k_1}E_2 & \xrightarrow{(id, \Sigma \alpha_1)} & \Sigma E_1 \oplus \Sigma^{1-k_1}E_2 \\
\end{array}
$$

Hence, the commutativity of the right-hand square forces $T_{E_\ast}(\alpha_1) = -\Sigma^{1-k_1} \alpha_n$.

**Step 2:** The above computation shows that the functor $T_{E_\ast}$ is fully faithful when restricted to $E$. It is also obvious from the construction of the twist that $T_{E_\ast}$ is the identity when restricted to $E^\perp$.

Let $E_i \in E$ and $X \in E^\perp$. Then $\text{Hom}^\ast(E_i, X) = 0$ and also $\text{Hom}^\ast(T_{E_\ast}(E_i), T_{E_\ast}(X)) = \text{Hom}^\ast((\Sigma^{1-k_1}E_{i-1}), X) = 0$. Finally, we use Serre duality and the defining property of $E_\ast$ to see that

$$\text{Hom}^\ast(X, E_i) = \text{Hom}^\ast(X, \Sigma^{-k_1}S(E_{i-1})) \cong \text{Hom}^\ast(E_{i-1}, \Sigma^{k_1}X)^* = 0.$$
Combining all these statements, we deduce that $T_{E_*}$ is fully faithful when restricted to the spanning class $\Omega$, hence bona fide fully faithful by general theory; see e.g. [21, Proposition 1.49]. Note that $T_{E_*}$ has left and right adjoints as the identity and $F_{E_*}$ do.

**Step 3:** With $T_{E_*}$ fully faithful, the defining property of Serre functors gives a canonical map of functors $S^{-1}T_{E_*}S \to \text{id}$ which can be spelt out in the following diagram:

$$
\begin{array}{c}
\bigoplus_i \text{Hom}^\bullet(E_i, S(-)) \otimes S^{-1}(E_i) \\
\downarrow \\
\bigoplus_i \text{Hom}^\bullet(E_i, -) \otimes E_i \\
\downarrow \\
\text{id} \\
\text{id} \\
\bigoplus_i \hom(E_i) \otimes \hom(E_i) \\
\downarrow \\
T_{E_*} \\
\end{array}
$$

It is easy to check that the left-hand vertical arrow is an isomorphism whenever we plug in objects from $\Omega$: both vector spaces are zero for objects from $E_i$, and the claim follows from $S(E_i) \cong \Sigma^k_i E_{i+1}$ for objects $E_i$. Hence $T_{E_*}$ commutes with the Serre functor on $\Omega$, and so by more general theory is essentially surjective; see [21, Corollary 1.56], this is the place where we need the assumption that $D$ is indecomposable.

**Remark 3.5.** We point out that the twist $T_{E_*}$ defined above is an instance of a spherical functor $S_\bullet$, given by the following data:

$$
S: D^b(k^n) \to D, \quad k_i \mapsto E_i,
$$

$$
R: D \to D^b(k^n), \quad X \mapsto (\text{Hom}^\bullet(E_1, X)) \oplus \cdots \oplus (\text{Hom}^\bullet(E_n, X))
$$

where $D^b(k^n) = \bigoplus_n D^b(k)$ is a decomposable category. It is easy to see that $R$ is right adjoint to $S$ and that $T_{E_*}$ coincides with the cone of the adjunction morphism $SR \to \text{id}$.

One condition for $S$ to be a spherical functor is that the cone of $\text{id} \to RS$ should be an autoequivalence of $D^b(k^n)$. This amounts to precisely the computation of $T_{E_*}$ on the $E_i$ and $\alpha_i$ carried out above.

4. **Autoequivalence groups of discrete derived categories**

We now use the general machinery of the previous section to show that categories $D^b(\Lambda(r, n, m))$ possess two very interesting and useful autoequivalences. We will denote these by $T_X$ and $T_Y$ and prove some crucial properties: they commute with each other, act transitively on the indecomposables of each $Z^k$ component and provide a factorisation of the Auslander–Reiten translation: $T_X T_Y = \tau^{-1}$. Moreover, $T_X$ acts trivially on $Y$ and $T_Y$ acts trivially on $X$; see Proposition 4.3 and Corollary 4.4 for the precise assertions. We then give an explicit description of the group of autoequivalences of $D^b(\Lambda(r, n, m))$ in Theorem 4.6.

The category $D = D^b(\Lambda(r, n, m))$ with $n > r$ is Hom-finite, indecomposable, algebraic and has Serre duality (see Appendix A.1). Therefore we can apply the results of the previous section to $D$.

Our first observation is that every sequence of $m + r$ consecutive objects at the mouth of $X^0$ is an exceptional $(m + r)$-cycle; likewise, every sequence of $n - r$ consecutive objects at the mouth of $Y^0$ is an exceptional $(n - r)$-cycle, by which we mean a $(r + 1)$-spherical object in case $n - r = 1$. For the moment, we specify two concrete sequences:

$$
E_* = (E_1, \ldots, E_{m+r}) := (X^0_{m+r,m+r}, \ldots, X^0_{11}), \quad \text{i.e. } E_i = X^0_{m+r+1-i,m+r+1-i},
$$

$$
F_* = (F_1, \ldots, F_{n-r}) := (Y^0_{n-r,n-r}, \ldots, Y^0_{11}), \quad \text{i.e. } F_i = Y^0_{n-r+1-i,n-r+1-i}.
$$

**Lemma 4.1.** $E_*$ forms an exceptional $(m + r)$-cycle in $D$ with $k_* = (1, \ldots, 1, 1 - r)$, and $F_*$ forms an exceptional $(n - r)$-cycle in $D$ with $k_* = (1, \ldots, 1, 1 + r)$. 


Proof. The object $X_{i1}^0$ is exceptional by Lemma 2.1, hence any object at the mouth $X_{ii}^0 = \tau_1^{-i}(X_{11}^0)$ is. This point also gives the second condition of exceptional cycles: for $i = 1, \ldots, m + r - 1,$ we have $SE_i = \Sigma \tau X_{m+r+1-i,m+r+1-i}^{0} = \Sigma X_{m+r-i,m+r-i}^{0} = \Sigma E_{i+1}$ and at the boundary step we have $SF_{m+r} = \Sigma \tau X_{11}^{0} = \Sigma X_{00}^{0} = \Sigma 1-r X_{m+r,m+r}^{0} = \Sigma 1-r E_{1},$ where we freely make use of the results stated in Section 1. Hence the degree shifts of the above proof works as well: $\text{Hom}(X_{ii}^0, X_{ii}^0) = \Sigma \tau Y_{11}^0 = \Sigma Y_{00}^0 = \Sigma 1+r X_{n-r,n-r}^0 = \Sigma 1+r F_1.$

The same reasoning works for $Y,$ now with the boundary step degree computation $SF_{n-r} = \Sigma \tau Y_{11}^0 = \Sigma Y_{00}^0 = \Sigma 1+r X_{n-r,n-r}^0 = \Sigma 1+r F_1.$

However, the actual choice of exceptional cycle is not relevant as the following easy lemma shows. It also allows us to write $T_X$ instead of $T_{E_i}$ and $T_Y$ instead of $T_{F_i}.$

Lemma 4.2. Any two exceptional cycles $E_i, E_i'$ at the mouths of $\mathcal{X}$ components differ by suspensions and AR translations, and the associated twist functors coincide: $T_{E_i} = T_{E_i'}.$

Proof. A suitable iterated suspension will move $E_i'$ into the $\mathcal{X}$ component that $E_i$ inhabits, and two exceptional cycles at the mouth of the same AR component obviously differ by some power of the AR translation. Thus we can write $E_i' = \Sigma a \tau b E_i$ for some $a, b \in \mathbb{Z}$. We point out that the suspension and the AR translation commute with all autoequivalences (it is a general and easy fact that the Serre functor does). Finally, we have $T_{E_i'} = T_{\Sigma a \tau b E_i} = \Sigma a \tau b T_{E_i} \Sigma^{-a} \tau^{-b} = T_{E_i},$ using Lemma 3.1.

Proposition 4.3. The twist functors $T_X$ and $T_Y$ act as follows on $D^b(\Lambda),$ where $k = 0, \ldots, r - 1$ and $i, j \in \mathbb{Z}$:

$$T_X|_{\mathcal{X}} = \tau^{-1}, \quad T_X|_Y = \text{id}, \quad T_X(Z_{ij}^k) = Z_{i+1,j}^k,$$

$$T_Y|_{\mathcal{X}} = \text{id}, \quad T_Y|_Y = \tau^{-1}, \quad T_Y(Z_{ij}^k) = Z_{i,j+1}^k.$$
The following technical lemma about the additive closures of the $\mathcal{X}$ and $\mathcal{Y}$ components will be used later on, but is also interesting in its own right. Using the twist functors, the proof is easy.

**Lemma 4.5.** Each of $\mathcal{X}$ and $\mathcal{Y}$ is a thick triangulated subcategory of $\mathcal{D}$.  

**Proof.** The proof of Proposition 4.3 contains the fact $\text{thick}(E) = \mathcal{Y}$. Perpendicular subcategories are always closed under extensions and direct summands; since $\text{thick}(E)$ is by construction a triangulated subcategory, the orthogonal complement $\mathcal{Y}$ is triangulated as well. 

Our results enable us to compute the group of autoequivalences of $\mathcal{D}^b(\Lambda(r,n,m))$. For $\Lambda(1,2,0)$, König and Yang [20, Lemma 9.3] showed $\text{Aut}(\mathcal{D}^b(\Lambda(1,2,0))) \cong \mathbb{Z}^2 \times \mathbb{k}^*$. 

**Theorem 4.6.** The group of autoequivalences of $\mathcal{D}^b(\Lambda(r,n,m))$ is generated by $T_\mathcal{X}$, $T_\mathcal{Y}$, $\Sigma$ and common scaling of arrows, subject to the relations 

\[ [T_\mathcal{X}, T_\mathcal{Y}] = 1, \quad [\Sigma, T_\mathcal{X}] = 1, \quad [\Sigma, T_\mathcal{Y}] = 1, \quad \Sigma^r = T_\mathcal{X}^{m+r} T_\mathcal{Y}^{-n}, \]

and $\text{Aut}(\mathcal{D}^b(\Lambda(r,n,m))) \cong \mathbb{Z}^2 \times \mathbb{Z}/\ell \times \mathbb{k}^*$ as abstract groups, where $\ell := \gcd(r,m,n)$.  

**Proof.** The suspension commutes with all triangle functors. The remaining relations stated in the theorem follow from Proposition 4.3.

Let $G := \langle \Sigma, T_\mathcal{X}, T_\mathcal{Y} \rangle$ be the subgroup of $\text{Aut}(\mathcal{D}^b(\Lambda))$ generated by the suspension and twist functors. This is an abelian group with three generators and one relation $\Sigma^r = T_\mathcal{X}^{m+r} T_\mathcal{Y}^{-n}$. By elementary algebra, there is an isomorphism of groups $G \cong \mathbb{Z}^2 \times \mathbb{Z}/\ell$ with $\ell = \gcd(r,n,m) = \gcd(r,m+r,r-n)$. Furthermore, $G$ is a normal subgroup: the suspension $\Sigma$ commutes with all exact endofunctors, and for the twists this is the statement of Lemma 3.1.

The inclusion $G \subset \text{Aut}(\mathcal{D}^b(\Lambda))$ has a section: choosing an object $Z \in \text{ind}(\mathcal{Z})$, we map $\sigma : \text{Aut}(\mathcal{D}^b(\Lambda)) \to G$, $\varphi \mapsto T_\mathcal{X}^\varphi T_\mathcal{Y}^{\varphi} \Sigma^a$ such that $T_\mathcal{X}^\varphi T_\mathcal{Y}^{\varphi} \Sigma^a \varphi(Z) = Z$. This is possible since $G$ acts transitively on $\text{ind}(\mathcal{Z})$ and autoequivalences preserve the types of AR components.

Therefore $\text{Aut}(\mathcal{D}^b(\Lambda)) \cong G \times \ker(\sigma)$. Let now $\varphi \in \ker(\sigma)$. Then $\varphi$ fixes all objects of $\mathcal{Z}$, hence by Properties 1.1(4) all objects of $\mathcal{D}^b(\Lambda)$. Thus $\varphi$ is induced by an automorphism $\varphi : \Lambda \to \Lambda$ fixing all idempotents $e_i$. As the underlying graph of $\Lambda(r,n,m)$ is simply-laced, $\varphi$ can only act by scaling arrows. It is a well-known fact that inner automorphisms induce autoequivalences of $\text{mod}(\Lambda(r,n,m))$ and $\mathcal{D}^b(\Lambda(r,n,m))$ which are isomorphic to the identity; see [30, §3].

We are left to classify the outer automorphisms of $\Lambda(r,n,m)$. Scaling of arrows leads to a subgroup $(\mathbb{k}^*)^{m+n}$ of $\text{Aut}(\Lambda(r,n,m))$. However, choosing an indecomposable idempotent $e$ (i.e. a vertex) together with a scalar $\lambda \in \mathbb{k}^*$ produces a unit $u = 1_\Lambda + (\lambda - 1)e$, and hence an inner automorphism $c_u \in \text{Aut}(\Lambda)$. It is easy to check that $c_u(\alpha) = \frac{1}{\lambda} \alpha$ if $\alpha$ ends at $e$, and $c_u(\alpha) = \lambda \alpha$ if $\alpha$ starts at $e$, and $c_u(\alpha) = \alpha$ otherwise. The form of the quiver of $\Lambda(r,n,m)$ shows that an $(n+m-1)$-subtorus of the subgroup $(\mathbb{k}^*)^{m+n}$ of arrowscaling automorphisms consists of inner automorphisms. Furthermore, the automorphism scaling all arrows simultaneously by the same number is easily seen not to be inner, hence, $\ker(\sigma) = \mathbb{k}^*$. 

5. **Hom spaces: dimension bounds and graded structure**

In this section, we prove a strong result about $\mathcal{D}^b(\Lambda) := \mathcal{D}^b(\Lambda(r,n,m))$ which says that the dimensions of homomorphism spaces between indecomposable objects have a common bound. We also present the endomorphism complexes in Lemma 5.3.
5.1. Hom space dimension bounds. The bounds are given in the following theorem; for more precise information in case \( r = 1 \) see Proposition 5.2.

**Theorem 5.1.** Let \( A, B \) be indecomposable objects of \( D^b(A(r, n, m)) \) where \( n > r \). If \( r \geq 2 \), then \( \dim \text{Hom}(A, B) \leq 1 \) and if \( r = 1 \), then \( \dim \text{Hom}(A, B) \leq 2 \).

**Proof.** Our strategy for establishing the dimension bound follows that of the proofs of \( \text{Hom} \)-hammocks. Let \( A, B \in \text{ind}(D^b(A(r, n, m))) \) and assume \( r > 1 \). In this proof, we use the abbreviated notation \( \text{hom} = \dim \text{Hom} \); see Appendix A. We want to show \( \text{hom}(A, B) \leq 1 \) by considering the various components separately.

**Case** \( A \in X^k \) or \( Y^k \): Consider first \( A, B \in X^k \) and perform induction on the height of \( A \). If \( A = A_0 \) sits at the mouth, then \( \text{hom}(A, B) \leq 1 \) by Lemma 2.1. For \( A \) higher up, and assuming \( \text{hom}(A, B) \neq 0 \), which means \( B \in \text{ray}_+(A_0) \), we consider the triangle

\[
A' \xrightarrow{f} A \xrightarrow{g} A'' \xrightarrow{\Sigma A'}
\]

defined by \( A' := 0_A \) and \( A'' := \text{cone}(f) \) if \( B \in \text{ray}_+(A) \), and \( A' := \text{coray}_+(A) \cap \text{ray}_-(B) \) and \( A' := \text{cocone}(g) \) otherwise, where by abuse of notation the intersection means the indecomposable additive generator of the specified subcategory.

Using hammocks from Proposition 2.2, we see that \( \text{Hom}(A'', B) = 0 \) if \( B \in \text{ray}_+(A) \) and \( \text{Hom}(A', B) = 0 \) otherwise. Thus from the exact sequence

\[
\text{Hom}(\Sigma^{-1}A', B) \rightarrow \text{Hom}(A'', B) \rightarrow \text{Hom}(A, B) \rightarrow \text{Hom}(A', B) \rightarrow \text{Hom}(\Sigma A'', B)
\]

we derive \( \text{hom}(A, B) = \text{hom}(A'', B) \) if \( B \in \text{ray}_+(A) \) and else \( \text{hom}(A, B) = \text{hom}(A', B) \).

The induction hypothesis then gives \( \text{hom}(A, B) \leq 1 \).

The subcase \( B \in X^{k+1} \) follows from the above by Serre duality.

Furthermore, the above argument applies without change to \( B \in Z^k \) — with \( \text{ray}_+(A) \subset Z^k \) understood to mean the subset of indecomposables of \( Z^k \) admitting non-zero morphisms from \( A \) (these form a ray in \( Z^k \)) and similarly \( \text{ray}_-(B) \subset X^k \), and application of Proposition 2.3. An obvious modification, which we leave to the reader, extends the argument to \( B \in Z^{k+1} \). The statements for \( A \in Y \) are completely analogous.

**Case** \( A \in Z^k \): In light of Serre duality, we don’t need to deal with \( B \in X \) or \( B \in Y \). Therefore we turn to \( B \in Z \). However, we already know from the proof of Proposition 2.4 that the dimensions in the two non-vanishing regions \( \text{ray}_+(\text{coray}_+(A)) \) and \( \text{ray}_-(\text{coray}_-(SA)) \) are constant. Since the \( Z \) components contain the simple \( S(0) \) and the twist functors together with the suspension act transitively on \( Z \), it is clear that \( \text{hom}(A, A) = \text{hom}(A, SA)^+ = 1 \). This completes the proof. \( \square \)

**Proposition 5.2.** Let \( r = 1 \) and \( X, A \in \text{ind}(X) \). Then

\[
\text{hom}(X, A) = 2 \iff A \in \text{ray}_+(XX_0) \cap \text{coray}_-(0(SX, SX)).
\]

**Proof.** The argument is similar to the computation of the Hom-hammocks in the \( Z \) components from Section 2. We proceed in several steps.

**Step 1:** For any \( A \in \text{ind}(X) \) of height 0 the claim follows from Lemma 2.1. Otherwise we consider the AR mesh which has \( A \) at the top, and let \( A' \) and \( A'' \) be the two indecomposables of height \( h(A) - 1 \). There are two triangles:

\[
\text{(ray)} \quad 0A \rightarrow A \rightarrow A'' \rightarrow \Sigma(0A) = 0\Sigma A,
\]

\[
\text{(coray)} \quad A' \rightarrow A \rightarrow A_0 \rightarrow \Sigma A',
\]
where, as before, \(0_A\) and \(A_0\) are the unique indecomposable objects on the mouth which are contained in respectively \(\text{ray}_-(A)\) and \(\text{coray}_+(A)\). Applying the functor \(\text{Hom}(X, -)\) to both triangles we obtain two exact sequences:

\[
\begin{align*}
(3) & \quad \text{Hom}(X, 0_A) \to \text{Hom}(X, A) \xrightarrow{\varphi} \text{Hom}(X, A') \xrightarrow{\psi} \text{Hom}(X, \Sigma_{0}A), \\
(4) & \quad \text{Hom}(X, \Sigma^{-1}A_0) \xrightarrow{\mu} \text{Hom}(X, A') \to \text{Hom}(X, A) \xrightarrow{\delta} \text{Hom}(X, A_0).
\end{align*}
\]

Since \(0_A\) and \(A_0\) lie on the mouth of the component, Lemma 2.1 implies that the outer terms have dimension at most 2. Using the fact that \(X_0\) and \(\mathcal{O}S\mathcal{X}\) are the only objects of the Hom-hammock from \(X\) lying on the mouth, Lemma 2.1 actually yields:

\[
\begin{align*}
\text{hom}(X, 0_A) > 0 & \iff A \in \text{ray}_+(0_A) \cup \text{ray}_+(\mathcal{O}S\mathcal{X}), \\
\text{hom}(X, \Sigma_0A) > 0 & \iff A \in \text{ray}_+(\Sigma^{-1}X_0) \cup \text{ray}_+(\Sigma^{-1}0\mathcal{S}\mathcal{X}), \\
\text{hom}(X, \Sigma^{-1}A_0) > 0 & \iff A \in \text{coray}_-(\Sigma X_0) \cup \text{coray}_-(\Sigma_0\mathcal{S}\mathcal{X}), \\
\text{hom}(X, A_0) > 0 & \iff A \in \text{coray}_-(X_0) \cup \text{coray}_-(\mathcal{O}S\mathcal{X}).
\end{align*}
\]

The spaces are 2-dimensional precisely when \(A\) belongs to the intersections of the (co)rays on the right-hand side, which can only happen when \(\mathcal{O}S\mathcal{X} = X_0\). The set of rays and corays listed above divide the component into regions. In this proof, each region is considered to be closed below and open above.

**Step 2:** The function \(\text{hom}(X, -)\) is constant on each region, and changes by at most 1 when crossing a (co)ray if \(\mathcal{O}S\mathcal{X} \neq X_0\), and by at most 2 otherwise.

The first claim is clear from exact sequences (3) and (4). We show the second claim for rays; for corays the argument is similar. We get \(\text{hom}(X, A) \leq \text{hom}(X, A') + \text{hom}(X, 0_A)\) from sequence (3). This yields the stated upper bound for \(\text{hom}(X, A)\), as \(\text{hom}(X, 0_A) \leq 1\) when \(\mathcal{O}S\mathcal{X} \neq X_0\) and \(\text{hom}(X, 0_A) \leq 2\) otherwise. For the lower bound, instead observe that \(\text{hom}(X, A') \leq \text{hom}(X, \Sigma_0A) + \text{hom}(X, A),\) again from sequence (3).

**Step 3:** \(\psi = 0\) unless \(A \in \text{ray}_+(\Sigma^{-1}0\mathcal{S}\mathcal{X})\) and \(\mu = 0\) unless \(A \in \text{coray}_-(\Sigma X_0)\).

If \(A \notin \text{ray}_+(\Sigma^{-1}X_0) \cup \text{ray}_+(\Sigma^{-1}0\mathcal{S}\mathcal{X})\) then \(\text{hom}(X, \Sigma_0A) = 0\) and so \(\psi = 0\) trivially. Therefore, we just need to consider \(A \in \text{ray}_+(\Sigma^{-1}X_0)\) but \(A \notin \text{ray}_+(\Sigma^{-1}0\mathcal{S}\mathcal{X})\), and in this case \(\text{hom}(X, \Sigma_0A) = 1\). It is clear that the maps going down the coray from \(X\) to \(X_0\) span a 1-dimensional subspace of \(\text{Hom}(X, \Sigma_0A)\), which therefore is the whole space. Using properties of the \(\mathcal{Z}A\) mesh, the composition of such maps with a map along \(\text{ray}_+(X_0)\) from \(X_0\) to \(\Sigma A\) defines a non-zero element in \(\text{Hom}(X, \Sigma A)\). Thus the map \(\text{Hom}(X, \Sigma_0A) \to \text{Hom}(X, \Sigma A)\) in the sequence (3) is injective and it follows that \(\psi = 0\). The proof of the second statement is similar: here we use the chain of morphisms in Properties 1.1.5 to show that the map \(\text{Hom}(X, \Sigma^{-1}A) \to \text{Hom}(X, \Sigma^{-1}A_0)\) in the sequence (4) is surjective.

**Step 4:** If \(\text{ray}_+(\Sigma^{-1}X_0)\) (or \(\text{coray}_-(\Sigma_0\mathcal{S}\mathcal{X})\), respectively) does not coincide with one of the other three (co)rays, then crossing it does not affect the value of \(\text{hom}(X, -)\).

Suppose \(\text{ray}_+(\Sigma^{-1}X_0) \ni A\) doesn’t coincide with \(\text{ray}_+(X_0)\), \(\text{ray}_+(0\mathcal{S}\mathcal{X})\) or \(\text{ray}_+(\Sigma^{-1}0\mathcal{S}\mathcal{X})\). Thus \(\text{hom}(X, 0_A) = 0\), and from Step 3 the map \(\psi = 0\), hence \(\text{Hom}(X, A) = \text{Hom}(X, A')\). Similarly, suppose \(A \in \text{coray}_-(\Sigma_0\mathcal{S}\mathcal{X})\) and this doesn’t coincide with any of the other corays. Then \(\text{hom}(X, A_0) = 0\) and \(\mu = 0\) and again the claim follows.

**Step 5:** There are three possible configurations of rays and corays determining the regions where \(\text{hom}(X, -)\) is constant.

It follows from Step 4 that it suffices to consider the remaining rays and corays,

\[
\text{ray}_+(\Sigma^{-1}0\mathcal{S}\mathcal{X}), \text{ray}_+(\Sigma_0\mathcal{S}\mathcal{X}), \text{ray}_+(X_0) \text{ and } \text{coray}_-(\Sigma X_0), \text{coray}_-(\Sigma_0\mathcal{S}\mathcal{X}), \text{coray}_-(X_0),
\]

...
for determining the regional constants \( \text{hom}(X, -) \). Note that these are precisely the rays and corays required to bound the regions \( \text{ray}_+(XX_0) \) and \( \text{coray}_-(0SX, SX) \) of the statement of the proposition. Considering their relative positions on the mouth, \( \Sigma^{-1}0SX \) is always furthest to the left and \( \Sigma X_0 \) is furthest to the right, while \( 0SX \) can lie to the left, or to the right, or coincide with \( X_0 \), depending on the height of \( X \). We consider now the case where \( 0SX \) is to the left of \( X_0 \). We label the regions in the following diagram by letters A–M (this is the order in which we treat them, and the subscripts indicate the claimed \( \text{hom}(X, -) \) for the region):

\[
\begin{array}{cccccccc}
A & B & C & D & E & F & G & H \\
\hline
I & J & K & L & M & \Sigma^{-1}0SX & 0SX & \Sigma X_0
\end{array}
\]

First we note that regions A–E all contain part of the mouth and so \( \text{hom}(X, -) = 0 \) here. Looking at the maps from \( X \) that exist in the AR component we see that \( \text{hom}(X, -) \geq 1 \) on regions H, I, K and L; and on F, G, J and K using Serre duality. However regions F–I are reached by crossing a single ray or coray from one of the regions A–E. By Step 2 we thus get \( \text{hom}(X, -) = 1 \) on regions F–I.

Now look at the element \( A \in \text{ray}_+(S_0X) \cap \text{coray}_-(X_0) \); this is the object of minimal height in region K. We can see that \( A \in \text{coray}_+(X) \) and the map down the coray from \( X \) to \( A_0 \), factors through the map from \( A \) to \( A_0 \). Therefore the map \( \delta \) in the second exact sequence \([1]\) is non-zero. It is clear that \( A \notin \text{coray}_-(\Sigma X_0) \) so \( \mu = 0 \) by Step 3 above. We deduce from sequence \([1]\) that \( \text{hom}(X, A) > \text{hom}(X, A') \), so \( \text{hom}(X, A) > 1 \) since \( A' \) is in region G. Since \( A \) is an object in region K, which can be reached from region D by crossing just two rays, Step 2 now gives \( \text{hom}(X, -) = 2 \) on region K.

In the same vein, consider \( A \in \text{ray}_+(S_0X) \cap \text{coray}_-(\Sigma X_0) \), the object of minimal height in region L. Observe that \( A'' \in \text{ray}_+(\tau^{-1}S_0X) \cap \text{coray}_-(\Sigma X_0) = \text{add} \Sigma X \) from which we can see that the map to \( \text{Hom}(X, 0A) \) in \([3]\) is surjective. Now \( A \notin \text{ray}_+(\Sigma^{-1}0SX) \), so \( \psi = 0 \) by Step 3 and hence \( \text{hom}(X, A) = \text{hom}(X, A'') \). With \( A'' \) in region I where we already know \( \text{hom}(X, A'') = 1 \), we get \( \text{hom}(X, -) = 1 \) on region L.

Finally we now take up \( A \in \text{ray}_+(\Sigma^{-1}S_0X) \cap \text{coray}_-(\Sigma X_0) \), the object of minimal height in region M. It is clear that \( A \notin \text{ray}_+(S_0X) \cup \text{ray}_+(X_0) \), so \( \text{hom}(X, 0A) = 0 \). A short calculation shows \( A'' \in \text{ray}_+(X) \), and again using the chain of morphisms in Properties \([1,1,5]\), we see that there is a map \( X \to \Sigma_0A = S_0X \) factoring through \( A'' \). Looking at the sequence \([3]\) it follows that \( \text{hom}(X, A) < \text{hom}(X, A'') = 1 \) since \( A'' \) is in region L. Therefore, \( \text{hom}(X, -) = 0 \) on region M. For region J, we see that since it is sandwiched between regions K and M, \( \text{hom}(X, -) = 1 \) here.

This deals with the case that \( 0SX \) lies to the left of \( X_0 \). If instead it lies to the right, analogous reasoning applies. Finally, if \( 0SX = X_0 \), matters are simpler: in that case, the regions C and F–I all vanish.

5.2. **Graded endomorphism algebras.** In this section we use the Hom-hammocks and universal hom space dimension bounds to recover some results of Bobiński on the graded endomorphism algebras of algebras with discrete derived categories; see \([7, \text{Section 4}]\). Our
approach is somewhat different, so we provide proofs for the convenience of the reader. Using these descriptions we give a coarse classification of the homological properties of the indecomposable objects of discrete derived categories.

In order to conveniently write down the endomorphism complexes, we define four functions \( \delta^+_X, \delta^-_X, \delta^+_Y, \delta^-_Y : \mathbb{N} \to \mathbb{N} \) by

\[
\delta^+_X(h) := \left[ \frac{h}{m + r} \right], \quad \delta^-_X(h) := \left[ \frac{h + 1}{m + r} \right], \quad \delta^+_Y(h) := \left[ \frac{h + 1}{n - r} \right], \quad \delta^-_Y(h) := \left[ \frac{h}{n - r} \right].
\]

We write \( \delta^\pm(A) \) to mean \( \delta^+_X(h(A)) \) or \( \delta^+_Y(h(A)) \) for \( A \in \text{ind}(\mathcal{X}) \) or \( A \in \text{ind}(\mathcal{Y}) \), respectively.

**Lemma 5.3.** The endomorphism complexes of \( A \in \text{ind}(\mathcal{X}) \) and \( B \in \text{ind}(\mathcal{Y}) \) are

\[
\text{Hom}^\bullet(A, A) = \bigoplus_{l=0}^{\delta^+(A)} \Sigma^{-lr}k \oplus \bigoplus_{l=1}^{\delta^-(A)} \Sigma^{lr-1}k \quad \text{and} \quad \text{Hom}^\bullet(B, B) = \bigoplus_{l=0}^{\delta^+(B)} \Sigma^{-lr}k \oplus \bigoplus_{l=1}^{\delta^-(B)} \Sigma^{-lr-1}k.
\]

In words, the functions \( \delta^+ \) and \( \delta^- \) determine the ranges of self-extensions of positive and negative degree, respectively. We point out that the result holds for all \( r \geq 1 \).

**Proof.** Let \( A \in \text{ind}(\mathcal{X}) \), assuming \( r > 1 \). Suspending if necessary, we may suppose that \( A = X^0_0 \). We are looking for all \( d \in \mathbb{Z} \) with \( \text{Hom}^d(A, A) = \text{Hom}(A, \Sigma^dA) \neq 0 \). By Proposition 2.2, this is only possible for either \( d = 0 \) or \( d \equiv 1 \) modulo \( r \).

We start with the first possibility: \( d = lr \) for some \( l \in \mathbb{Z} \). By Properties 1.1(3) and (2),

\[
\Sigma^{lr}A = \tau^{-(m+r)}A = X^0_{i+l(m+r), j+l(m+r)}
\]

which is an indecomposable object in \( \mathcal{X}^0 \) sharing its height \( h = j - i \) with \( A \). Again using Proposition 2.2, we can reformulate the claim as follows:

\[
\text{Hom}^{lr}(A, A) \neq 0 \iff \Sigma^{lr}A \in \text{ray}_+(\mathcal{A}A_0)
\]

\[
\iff i \leq i + l(m + r) \leq i + h \iff 0 \leq l(m + r) \leq h
\]

\[
\iff 0 \leq l \leq \delta^+_X(h) = \delta^+(A),
\]

where the set of \( h + 1 \) objects in \( \ast \) are precisely the objects in \( \text{ray}_+(\mathcal{A}A_0) \) of height \( h \). We now turn to the other possibility, \( d = 1 + lr \) for some \( l \in \mathbb{Z} \). Here we get

\[
\text{Hom}^{1+lr}(A, A) \neq 0 \iff \Sigma^{1+lr}A \in \text{ray}_+(\mathcal{S}A, \mathcal{S}A)
\]

\[
\iff \Sigma^{1+lr}A = X^1_{i+l(m+r), j+l(m+r)} \in \{ \tau^h\mathcal{S}A, \ldots, \mathcal{S}A \} = \{ X^1_{i-h-1,j-1}, \ldots, X^1_{i-1,j-1} \}
\]

\[
\iff i - h - 1 \leq i + l(m + r) \leq i - 1 \iff -h - 1 \leq l(m + r) \leq -1
\]

\[
\iff 1 \leq -l \leq \delta^-_X(h) = \delta^-(A).
\]

As we know from Theorem 5.1 all Hom spaces have dimension 1 when \( r > 1 \), these two computations give

\[
\text{Hom}^\bullet(A, A) = \bigoplus_{l \in \mathbb{Z}} \Sigma^{-l} \text{Hom}(A, \Sigma^lA) = \bigoplus_{l=0}^{\delta^+(A)} \Sigma^{-lr}k \oplus \bigoplus_{l=1}^{\delta^-(A)} \Sigma^{-lr-1}k.
\]

For \( r = 1 \) and \( A = X^0_{ij} \in \text{ind}(\mathcal{X}) \), by Proposition 2.2, the hammock \( \text{Hom}(A, -) \neq 0 \) is \( \text{ray}_+(\mathcal{A}A_0) \cup \text{coray}_-(\mathcal{S}A, \mathcal{S}A) \). We treat each part separately:

\[
\Sigma^lA = \tau^{-(m+1)}X^0_{ij} = X^0_{i+l(m+1), j+l(m+1)} \in \text{ray}_+(\mathcal{A}A_0)
\]

\[
\iff 0 \leq l(m + 1) \leq h \iff 0 \leq l \leq \delta^+(h)
\]

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and, noting \( SA = X_{i+m,j+m}^0 \),
\[
\Sigma^l A \in \text{coray}_r(SA, SA)
\]
\[\iff m - h \leq l(m + 1) \leq m \iff 0 \leq -l \leq \left[ \frac{h - m}{m + 1} \right] = 1 + \delta^-(h).\]

The last inequality translates to the same degree range as in the statement of the lemma—note the index shift by 1. The claim for the endomorphism complex of \( B \in \text{ind}(\mathcal{Y}) \) is proved in the same way; now using \( h = i - j \), \( \Sigma' = \tau^{n-r} \) and the hammocks specified by Proposition 2.3. \( \square \)

5.3. Coarse classification of objects. Our previous results allow us to give a crude grouping of the indecomposable objects of \( \text{D}^b(\Lambda(r, n, m)) \). In the \( \mathcal{X} \) and \( \mathcal{Y} \) components, the distinction depends on the height of an object, i.e. the distance from the mouth; see page 4. Recall that an object \( D \) of a \( k \)-linear Hom-finite triangulated category \( D \) is exceptional if \( \text{hom}^*(D, D) = 1 \), then necessarily \( \text{Hom}^*(D, D) = k \); see Appendix A.7 and \( D \) is called spherelike if \( \text{hom}^*(D, D) = 2 \), then necessarily \( \text{Hom}^*(D, D) = k \oplus \Sigma^{-1}k \) for some \( d \in \mathbb{Z} \) and \( D \) is called \( d \)-spherelike; see [17] for details. Assuming \( D \) has a Serre functor \( S \), a \( d \)-spherelike object \( D \) is called \( d \)-spherical if \( S(D) = \Sigma^dD \); see [21 §8].

**Proposition 5.4.** Each object \( A \in \text{ind}(\text{D}^b(\Lambda(r, n, m))) \) is of exactly one type below:
- Exceptional if \( A \in \mathcal{Z} \), or \( A \in \mathcal{X} \) with \( h(A) < m + r - 1 \), or \( A \in \mathcal{Y} \) with \( h(A) < n - r - 1 \).
- \((1 - r)\)-spherelike if \( A \in \mathcal{X} \) with \( h(A) = m + r - 1 \).
- \((1 + r)\)-spherelike if \( A \in \mathcal{Y} \) with \( h(A) = n - r - 1 \).
- \( \dim \text{Hom}^*(A, A) \geq 3 \) with \( \text{Hom}^{<0}(A, A) \neq 0 \) else.

**Remark 5.5.** In fact, the direct sum \( E_1 \oplus E_2 \) of two exceptional objects \( E_1 \) and \( E_2 \) with \( \text{Hom}^*(E_1, E_2) = \text{Hom}^*(E_2, E_1) = 0 \) is a \( 0 \)-spherelike object. Examples for \( r > 1 \) are given by taking \( E_1 \in \mathcal{X} \) and \( E_2 \in \mathcal{Y} \) at the mouths. The theory of spherelike objects also applies in this degenerate case, but is less interesting [17 Appendix].

**Proof.** We know from Lemma B.6 that the projective module \( P(n - r) \in \mathcal{Z} \). This is an exceptional object by Proposition 2.3. As the autoequivalence group acts transitively on \( \text{ind}(\mathcal{Z}) \) by Corollary 4.4, every indecomposable object of \( \mathcal{Z} \) is exceptional. The remaining parts of the proposition all follow from Lemma 5.3. We only give the argument for \( A \in \text{ind}(\mathcal{X}) \), as the one for indecomposable objects of \( \mathcal{Y} \) runs entirely parallel.

Observing the trivial inequalities \( 0 \leq \delta^+(A) \leq \delta^-(A) \), we see that \( A \) is exceptional if and only if \( 1 = \dim \text{Hom}^*(A, A) = 1 + \delta^+(A) + \delta^-(A) \). In turn, this happens precisely if \( \delta^-(A) = 0 \), which means \( h < m + r - 1 \).

Similarly, \( A \) is spherelike if and only if \( 2 = \dim \text{Hom}^*(A, A) = 1 + \delta^+(A) + \delta^-(A) \) which is equivalent to \( \delta^+(A) = 0 \) and \( \delta^-(A) = 0 \). The only solution of these equations is \( h = m + r - 1 \). Furthermore, in this case the endomorphism complex is \( \text{Hom}^*(A, A) = k \oplus \Sigma^{-1}k \), so that \( A \) is indeed \((1 - r)\)-spherelike. \( \square \)

**Corollary 5.6.** Spherical objects exist in \( \text{D}^b(\Lambda(r, n, m)) \) only if \( m = 0 \), \( r = 1 \) or \( n - r = 1 \). More precisely, \( A \in \text{ind}(\text{D}^b(\Lambda(r, n, m))) \) is
- \( 0 \)-spherical if and only if \( m = 0 \), \( r = 1 \) and \( A \) sits at an \( \mathcal{X} \)-mouth;
- \( n \)-spherical if and only if \( n = r + 1 \) and \( A \) sits at an \( \mathcal{Y} \)-mouth.

**Proof.** The only candidates for spherical objects are the spherelike objects listed in Proposition 5.4. Start with \( A \in \mathcal{X} \) with \( h(A) = m + r - 1 \). Then \( A \) is spherical if and only if \( SA = \Sigma^{-r}A \). By \( S = \Sigma X \) and \( \Sigma^{-r} = \Sigma^{m+r} \) (Properties [11][3]), this is equivalent to
Lemma 6.3. Let \( \text{ind} \) that every indecomposable object of height less than \( \text{ind} \) (co)suspension, — has a negative self extension. Such objects can’t lie in the heart and so again, up to \( H \) one suspension can sit in the heart \( \text{ind} \) that all objects of \( H \) objects. Then we show the stronger statement that any object 

Proof. We use the fact that there can be no negative extensions between objects in the heart \( H \). This certainly implies that ascending and descending chains of inclusions have to become stationary, and hence that \( H \) is a finite length category.

6. Reduction to Dynkin type \( A \) and classification results

Two keys for understanding the homological properties of algebras are t-structures and co-t-structures, especially bounded ones. The main theorem of [26], cited in the appendix as Theorem [A.6], states that for finite-dimensional algebras, bounded co-t-structures are in bijection with silting objects, which are in turn in bijection with bounded t-structures whose heart is a length category; see Appendices [A.5] and [A.6] for a more detailed overview.

It turns out, however, that any bounded t-structure in \( D^b(\Lambda(r,n,m)) \) has length heart, and hence to classify both bounded t-structures and bounded co-t-structures it is sufficient to classify silting objects in \( D^b(\Lambda(r,n,m)) \). This is the main goal of this section. In the first part, we prove that any bounded t-structure in \( D^b(\Lambda(r,n,m)) \) is length, then we obtain a semi-orthogonal decomposition \( D^b(\Lambda(r,n,m)) = \langle D^b(k\Lambda_{n+1}), Z \rangle \), for some trivial thick subcategory \( Z \), and use this to bootstrap Keller-Vossieck’s classification of silting objects in the bounded derived categories of path algebras of Dynkin type \( A \) to get a classification of silting objects in discrete derived categories.

6.1. All hearts in \( D^b(\Lambda(r,n,m)) \) are length. The main result of this section is:

Proposition 6.1. Any heart of a t-structure of a discrete derived category has only a finite number of indecomposable objects up to isomorphism, and is a length category.

We prove each statement in the following two lemmas. The first is a generalisation of the corresponding statement for the algebra \( \Lambda(1,2,0) \) proved in [26]; the second is a general statement about Hom-finite abelian categories.

Lemma 6.2. Any heart of a t-structure of a discrete derived category has a finite number of indecomposable objects up to isomorphism.

Proof. We use the fact that there can be no negative extensions between objects in the heart \( H \) of a t-structure \( (X,Y) \). Suppose \( H \) contains an indecomposable \( Z \in \text{ind}(Z) \). Then any other indecomposable object in \( H \) must lie outside the hammocks \( \text{Hom}^{-\infty}(Z,-) \neq 0 \) and \( \text{Hom}^{\infty}(-,Z) \neq 0 \). Looking at the complement of these Hom-hammocks, it is clear that all objects of \( \text{ind}(H) \cap Z \) must be (co)suspensions of a finite set of objects. At most one suspension can sit in the heart \( H \); hence \( \text{ind}(H) \cap Z \) is finite.

In the \( \mathcal{X} \) and \( \mathcal{Y} \) components we use a similar argument: by Proposition [5.4], any object \( X_{i,j} \) which is sufficiently high up in an \( \mathcal{X} \) component — here \( j - i \geq r + m - 1 \) will do — has a negative self extension. Such objects can’t lie in the heart and so again, up to (co)suspension, \( \text{ind}(H) \cap \mathcal{X} \) is finite.

Similarly, for any object \( Y = Y_{i,j} \) sufficiently far down in an \( \mathcal{Y} \) component — here \( i - j \geq n - r - 1 \) suffices — the hammocks \( \text{Hom}^{-\infty}(Y,-) \neq 0 \) or \( \text{Hom}^{\infty}(-,Y) \neq 0 \) contain every indecomposable object of height less than \( r - n \). The same argument then shows that \( \text{ind}(H) \cap \mathcal{Y} \) is finite as well. \( \square \)

Lemma 6.3. Let \( H \) be a Hom-finite abelian category with finitely many indecomposable objects. Then \( H \) is a finite length category.

Proof. We show the stronger statement that any object \( A \in H \) has only finitely many subobjects. This certainly implies that ascending and descending chains of inclusions have to become stationary, and hence that \( H \) is a finite length category.
Suppose that there is an infinite set of pairwise non-isomorphic subobjects of \( A \). Writing each of these subobjects as a direct sum of (the finitely many) indecomposables, we find that there is a \( U \in \text{ind}(H) \) such that \( U, U^{\oplus 2}, \ldots \) occur as summands of these subobjects, hence \( U, U^{\oplus 2}, \ldots \subset A \). However, the number \( \text{hom}(U, A) < \infty \) is a bound on the number of copies of \( U \) which can occur in \( A \) — a contradiction.

\[ \text{Remark 6.4.} \] Proposition \([6.1]\) means that the heart of each bounded t-structure in \( \mathbb{D}^b(\Lambda(r, n, m)) \) is equivalent to \( \text{mod}(\Gamma) \), where \( \Gamma \) is a finite-dimensional algebra of finite representation type.

Knowing this, we can now turn our attention solely to classifying the silting objects. The first step in our approach is to decompose \( \mathbb{D}^b(\Lambda(r, n, m)) \) into a semi-orthogonal decomposition, one of whose orthogonal subcategories is the bounded derived category of a path algebra of Dynkin type \( A \).

\[ \text{6.2. A semi-orthogonal decomposition: reduction to Dynkin type } A. \] We start by showing that the derived categories of derived-discrete algebras always arise as extensions of derived categories of path algebras of type \( A \) by a single exceptional object.

\[ \text{Proposition 6.5.} \] Let \( Z \in \text{ind}(\mathcal{Z}) \) and \( Z = \text{thick}_{\mathbb{D}^b(\Lambda)}(\mathcal{Z}) \). Then \( Z^\perp \simeq \mathbb{D}^b(kA_{n+m-1}) \) and there is a semi-orthogonal decomposition \( \mathbb{D}^b(\Lambda(r, n, m)) = \langle \mathbb{D}^b(kA_{n+m-1}), Z \rangle \). In particular, \( Z \) is functorially finite in \( \mathbb{D}^b(\Lambda(r, n, m)) \). Moreover, \( \mathbb{D}^b(\Lambda(r, n, m)) \) has a full exceptional sequence.

\[ \text{Proof.} \] By Proposition \([5.4]\), the object \( Z \) is exceptional. This implies, on general grounds, that the thick hull of \( Z \) just consists of sums, summands and (co)suspensions: \( Z = \text{add}(\Sigma_i Z \mid i \in \mathbb{Z}) \) and that \( Z \) is an admissible subcategory of \( \mathbb{D}^b(\Lambda) \); for this last claim see \([9\text{ Theorem 3.2}]\). Furthermore \( \mathbb{D}^b(\Lambda) = \langle Z^\perp, Z \rangle \) is the standard semi-orthogonal decomposition for an exceptional object; see Appendix \([A.1]\) for details. Another way to see this: inspect the Hom-hammocks of Section \([2]\) and apply Lemma \([A.1]\) to find that the subcategory \( Z \) is functorially finite in \( \mathbb{D}^b(\Lambda) \). Applying \([25\text{ Proposition 1.3}]\) gives the admissibility of \( Z \).

Lemma \([B.6]\) places the indecomposable projective \( P(n-r) \) in the \( Z \) component of \( \mathbb{D}^b(\Lambda) \). Using the transitive action of the autoequivalence group on \( \text{ind}(\mathcal{Z}) \), see Corollary \([4.4]\) we thus can assume, without loss of generality, that \( Z = P(n-r) = e_{n-r}\Lambda \). There is a full embedding \( \iota: \mathbb{D}^b(\Lambda/\Lambda e_{n-r}\Lambda) \to \mathbb{D}^b(\Lambda) \) with essential image \( \text{thick}_{\mathbb{D}^b(\Lambda)}(e_{n-r}\Lambda)^\perp = Z^\perp \); see, for example, \([2\text{ Lemma 3.4}]\). Inspecting the Gabriel quiver of \( \Lambda/\Lambda e_{n-r}\Lambda \), we see that this quiver satisfies the criteria of \([5\text{ Theorem}]\). For the convenience of the reader, we list those criteria which are relevant for our case, where we have specialised the conditions of \([5]\) to bound quivers:

\[ (\alpha_1) \] The underlying graph is a tree.
\[ (\alpha_3) \] All relations are zero-relations of length two.
\[ (\alpha_4) \] Each vertex has at most four neighbours.
\[ (\alpha_6) \] A vertex with three neighbours sits in a full subgraph of the form: \[ ----- \] with at least four vertices and zero-relations as shown.
\[ (\alpha_7) \] There is no full subgraph of the type \[ ---- \] with at least four vertices and zero-relations as shown.

Therefore \( \Lambda/\Lambda e_{n-r}\Lambda \) is an iterated tilted algebra of type \( A_{n+m-1} \). It is well known that this implies \( \mathbb{D}^b(\Lambda/\Lambda e_{n-r}\Lambda) \simeq \mathbb{D}^b(kA_{n+m-1}) \); see \([16]\). Combining these pieces, we get \( Z^\perp \simeq \mathbb{D}^b(kA_{n+m-1}) \). The final claim about \( \mathbb{D}^b(\Lambda) \) having a full exceptional sequence follows at once from the fact that \( \mathbb{D}^b(kA_{n+m-1}) \) has one. \( \Box \)
Remark 6.6. The subcategory of type $\mathcal{D}^b(kA_{n+m-1})$ can be explicitly identified in $\mathcal{D}^b(\Lambda(r,n,m))$; see Figure [4]. The choice of right orthogonal to $Z$ was arbitrary, since Serre duality provides an equivalence $Z \rightarrow Z^\perp$, $X \mapsto S(X)$. We mention in passing that the thick subcategory $Z$ is equivalent to $\mathcal{D}^b(\mod(kA_1))$.

The silting objects of $\mathcal{D}^b(\mod(kA_{n+m-1}))$ are well understood from work of Keller and Vossieck in [25]. We shall now use the technique of silting reduction from [1] to bootstrap Keller and Vossieck’s classification to discrete derived categories.

6.3. Silting reduction. The main technical tool in the classification is the following result of Aihara and Iyama in [1]:

**Theorem 6.7 (Silting reduction [1] Theorem 2.37).** Let $\mathcal{D}$ be a Krull-Schmidt triangulated category, $U \subset \mathcal{D}$ a thick, contravariantly finite subcategory and $F: \mathcal{D} \rightarrow \mathcal{D}/U$ the canonical functor. Then for any silting subcategory $N$ of $U$, there is an injective map

$$\{\text{silting subcategories } M \text{ of } \mathcal{D} \mid N \subseteq M\} \rightarrow \{\text{silting subcategories of } \mathcal{D}/U\}, \quad M \mapsto F(M).$$

If $U$ is functorially finite in $\mathcal{D}$, then the map is bijective.

We are working towards an explicit description of the inverse map $G$ in Proposition [6.15]. The subcategory $B := \text{susp} \sum N$ is the ‘co-aise’ of a co-t-structure and thus covariantly finite in $U$. Putting this together with $U$ being functorially finite in $\mathcal{D}$ it gives rise to a co-t-structure $(A, B)$, where $A := \perp B$. Now let $K$ be a silting subcategory of $U^\perp$ and consider the approximation triangle of $K \in K$ with respect to the co-t-structure $(A,B)$,

$$A_K \rightarrow K \rightarrow B_K \rightarrow \Sigma A_K$$

with $A_K \in A$ and $B_K \in B$. Aihara and Iyama show in [1] in their proof of Theorem 6.7 that $G(K) := \text{add}(N \cup \{A_K \mid K \in K\})$ is a silting subcategory of $\mathcal{D}$.

**Definition.** Assume the notation and hypotheses of Theorem 6.7 above. Given a silting subcategory $N$ of $U$, by abuse of notation we shall write $G_N$ for the map $G_N: U^\perp \rightarrow \mathcal{D}$, which for $V \in U^\perp$, is defined by

$$G_N(V) \rightarrow V \xrightarrow{J_V} B_V \rightarrow \Sigma G_N(V),$$

\[\text{Figure 4. Above: } D^b(kA_6) \cong \text{thick}(Z)^\perp \hookrightarrow D^b(\Lambda(2,5,2)).\]

Remaining components $X^1 = \Sigma X^0, Y^1 = \Sigma Y^0, Z^1 = \Sigma Z^0$ not shown.

Below: AR quiver of $D^b(kA_6)$ with its $D^b(\Lambda(2,5,2))$ pieces.
Corollary 6.9. Any silting subcategory of $D$ contains an indecomposable object from the $Z$ components.

Proof of lemma. By Lemma 4.5, the additive closure of the $X$ components of $D$ is a thick subcategory of $D$, and likewise for the additive closure of the $Y$ components. Furthermore, these two subcategories are fully orthogonal by Propositions 2.2 and 2.3, so that their sum is a thick subcategory of $D$ as well. Therefore we cannot have $M \subset X \oplus Y$ as that would force $D = \text{thick}(M) = X \oplus Y$, a contradiction. □

Theorem 6.7 coupled with Proposition 6.5 tells us that all silting objects in $D$ containing $Z$ can be obtained by lifting silting objects in $Z \approx D(kA_{n+m-1})$ back up to $D$. In other words, any silting object in $D$ can be described by a pair $(Z, M')$ consisting of an indecomposable object $Z \in Z$ and a silting object $M' \in Z \approx D(kA_{n+m-1})$.

We now make a brief expository digression explaining Keller and Vossieck’s classification of silting subcategories of $D$-mod($kA_1$), from which the silting subcategories of $D$ can be ‘glued’.

6.4. Classification of silting objects in Dynkin type $A$. Consider the following diagram of the AR quiver of $D(kA_1)$ with coordinates $(g, h)$ with $g \in Z$ and $h \in \{1, \ldots, t\}$.

```
... (1,2) (0,1) (0,2) (1,1) (2,2) (3,1)...
|       |       |       |       |       |
(1,3) (0,3) (1,3) (2,3) (3,3) (3,3)...
```

Following [23], a quiver $Q = (Q_0, Q_1)$ is called an $A_t$-quiver if $|Q_0| = t$, its underlying graph is a tree, and $Q_1$ decomposes into a disjoint union $Q_1 = Q_\alpha \cup Q_\beta$ such that at any vertex at most one arrow from $Q_\alpha$ ends, at most one arrow from $Q_\alpha$ starts, at most one arrow from $Q_\beta$ ends and at most one arrow from $Q_\beta$ starts.

For a vertex $x$, define maps $s_\alpha, e_\alpha, s_\beta, e_\beta : Q_0 \to \mathbb{N}$ by

$$s_\alpha(x) := \# \{ y \in Q_0 \mid \text{the shortest walk from } x \text{ to } y \text{ starts with an arrow in } Q_\alpha \};$$
$$e_\alpha(x) := \# \{ y \in Q_0 \mid \text{the shortest walk from } y \text{ to } x \text{ ends with an arrow in } Q_\alpha \}.$$  

The functions $s_\beta$ and $e_\beta$ are defined analogously. With these maps, there is precisely one map $\varphi_Q := (g_Q(x), h_Q(x)) : Q_0 \to (ZA_1)_0$, where $g_Q$ and $h_Q$ correspond to the coordinates in the AR quiver of $D(kA_1)$, such that $h_Q(x) = 1 + e_\alpha(x) + s_\beta(x)$ and $g_Q(y) = g_Q(x)$ for
each arrow $x \to y$ in $Q_\alpha$, and $g_Q(y) = g_Q(x) + c_\alpha(x) + s_\alpha(x) + 1$ for each arrow $x \to y$ in $Q_\beta$, and finally normalised by $\min_{x \in Q_0}\{g_Q(x)\} = 0$.

By abuse of notation we identify the object $T_Q := \varphi_Q(Q_0)$ with the direct sum of the indecomposables lying at the corresponding coordinates. This map gives rise to the following classification result.

**Theorem 6.10** ([25], Section 4). The assignment $Q \mapsto T_Q$ induces a bijection from isomorphism classes of $A_t$-quivers and tilting objects $T$ in $D^b(kA_t)$ which satisfy the condition $\min\{g(U) \mid U \text{ is an indecomposable summand of } T\} = 0$.

Note that in Dynkin type $A_t$, the summands of any tilting object $T = \bigoplus_{i=1}^t T_i$ can be re-ordered to give a strong, full exceptional collection $\{T_1, \ldots, T_t\}$, see [25, Section 5.2].

We now have the following classification of silting objects in $D^b(A)$.

**Theorem 6.11** ([25], Theorem 5.3). Let $T = \bigoplus_{i=1}^t T_i$ be a tilting object in $D^b(kA_t)$ whose summands have been ordered to form an exceptional collection. Let $p: \{1, \ldots, t\} \to \mathbb{N}$ be a weakly increasing function. Then $M = \bigoplus_{i=1}^t M_i$, where $M_i := \Sigma^{p(i)} T_i$, is a silting object in $D^b(kA_t)$. Moreover, all silting objects of $D^b(kA_t)$ occur in this way.

The machinery above is rather technical, so we give a quick example of the classification of tilting (and hence silting) objects in $D^b(kA_3)$.

**Example 6.12** (Classification of tilting objects in $D^b(kA_3)$). When $t = 3$, up to isomorphism there are the following possible $A_3$-quivers:

$$
\begin{align*}
1 & \xrightarrow{\alpha} 2 \xrightarrow{\alpha} 3, \quad 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3, \quad 1 \xrightarrow{\alpha} 2 \xleftarrow{\beta} 3, \\
1 & \xleftarrow{\beta} 2 \xrightarrow{\beta} 3, \quad 1 \xrightarrow{\beta} 2 \xrightarrow{\beta} 3, \quad 1 \xrightarrow{\beta} 2 \xleftarrow{\alpha} 3.
\end{align*}
$$

Computing the $\varphi_Q$ for each of the above quivers gives the following, where each 3-tuple denotes $(\varphi_Q(1), \varphi_Q(2), \varphi_Q(3))$:

$$
\begin{align*}
((0, 1), (0, 2), (0, 3)), \quad ((0, 1), (0, 3), (2, 1)), \quad ((1, 1), (1, 2), (0, 3)), \\
((0, 3), (0, 2), (1, 1)), \quad ((0, 3), (1, 2), (2, 1)), \quad ((0, 3), (2, 1), (2, 3)).
\end{align*}
$$

We indicate the corresponding tilting objects in the following sketch:

```
\begin{center}
\begin{tikzpicture}
\foreach \x in {1,...,3}
\foreach \y in {1,...,3}
\node at (\x,\y) {$\cdot$};
\end{tikzpicture}
\end{center}
```

In each sketch the triangle depicts the standard heart for the quiver $1 \to 2 \to 3$ whose indecomposable projectives have coordinates $(0, i)$ for $i = 1, 2, 3$.

From this one sees there are twelve tilting objects in $D^b(\text{mod}(kA_3))$ up to suspension:

$$
\begin{align*}
P(1) \oplus P(2) \oplus P(3), \quad P(1) \oplus P(3) \oplus S(3), \quad P(2) \oplus P(3) \oplus S(2), \\
P(3) \oplus I(2) \oplus S(2), \quad P(3) \oplus I(2) \oplus S(3), \quad P(3) \oplus S(3) \oplus \Sigma S(2), \\
S(2) \oplus I(2) \oplus \Sigma P(2), \quad S(2) \oplus \Sigma P(1) \oplus \Sigma P(3), \quad I(2) \oplus \Sigma P(1) \oplus S(3), \\
\Sigma P(3) \oplus \Sigma P(2) \oplus S(3), \quad S(3) \oplus \Sigma P(2) \oplus \Sigma S(2), \quad S(3) \oplus \Sigma S(2) \oplus \Sigma^2 P(1).
\end{align*}
$$
6.5. Classification of silting objects for derived-discrete algebras. Silting objects in \( \mathcal{D}^b(\Lambda) \) correspond to pairs \((Z, M')\), where \( Z \in \text{ind}(\mathcal{I}) \) and \( M' \) is a silting object of \( \mathcal{I}^\perp \simeq \mathcal{D}^b(k \mathcal{A}_{n+m-1}) \). However, a silting object in \( \mathcal{D}^b(\Lambda) \) may have more than one indecomposable summand in the \( \mathcal{I} \) components. Thus, using silting reduction, we will obtain multiple descriptions of the same object. To rectify this problem, we classify silting objects for which \( Z \in \text{ind}(\mathcal{I}) \) is minimal with respect to a total order on \( \text{ind}(\mathcal{I}) \) defined as follows. Let \( Z' \in \text{ind}(\mathcal{I}^i) \) and \( Z'' \in \text{ind}(\mathcal{I}^j) \) and define

\[
Z' \preceq Z'' \iff \begin{cases} \text{ray}(\Sigma^{j-1}Z') \leq \text{ray}(Z'') & \text{if } i < j; \\ \text{ray}(\tau^{-1}\Sigma^{j-1}Z') \leq \text{ray}(Z'') & \text{if } i > j; \\ \text{coray}(Z') \leq \text{coray}(Z'') & \text{if } i = j \text{ and ray}(Z') = \text{ray}(Z''); \\ \text{ray}(Z') < \text{ray}(Z'') & \text{otherwise}, \end{cases}
\]

where \( \text{ray}(Z''') \leq \text{ray}(Z''') \) if and only if \( i \leq k \) and \( \text{coray}(Z''') \leq \text{coray}(Z''') \) if and only if \( j \leq l \). Equivalently, for \( Z \in \text{ind}(\mathcal{I}^i) \), the total order is defined by successor sets,

\[
\{ \tilde{Z} \in \text{ind}(\mathcal{I}) | Z \preceq \tilde{Z} \} = \text{ray}_+ (Z) \cup \text{ray}_+ (\text{coray}_+(\tau^{-1}Z)) \cup \text{ray}_+ (\text{coray}_+(\Sigma^{[i+1\ldots r-1]}Z)) \cup \text{ray}_- (\text{coray}_+(\tau^{-1}\Sigma^{[0\ldots i-1]}Z)).
\]

**Lemma 6.13.** The relation \( \preceq \) defined above defines a total order on indecomposables in the \( \mathcal{I} \) components.

**Proof.** Anti-symmetry: Suppose \( Z \preceq Z' \) and \( Z' \preceq Z \) with \( Z \in \text{ind}(\mathcal{I}^i) \) and \( Z' \in \text{ind}(\mathcal{I}^j) \). If \( i = j \), then anti-symmetry is clear. For a contradiction, suppose \( i < j \). Then \( \text{ray}(\Sigma^{j-i}Z) \leq \text{ray}(Z') \) and \( \text{ray}(\tau^{-1}\Sigma^{j-i}Z') \leq \text{ray}(Z) \). In particular, it follows that \( \text{ray}(\tau^{-1}Z') \leq \text{ray}(\Sigma^{j-i}Z) \leq \text{ray}(Z') \), which is a contradiction, since \( \text{ray}(\tau^{-1}Z') > \text{ray}(Z') \). The same argument works when \( i > j \).

Transitivity: Suppose \( Z \preceq Z' \) and \( Z' \preceq Z'' \) with \( Z \in \text{ind}(\mathcal{I}^i) \), \( Z' \in \text{ind}(\mathcal{I}^j) \) and \( Z'' \in \text{ind}(\mathcal{I}^k) \). One simply analyses the different possibilities for \( i, j, k \). We do the case \( i > j \) and \( j < k \); the rest are similar. The first inequality means that \( \text{ray}(\sigma^{j-i}Z) \leq \text{ray}(Z') \) and the second inequality means that \( \text{ray}(\Sigma^{k-j}Z') \leq \text{ray}(Z'') \). There are two subcases: first assume \( i \leq k \). In this case, apply \( \tau \Sigma^{j-i} \) to the condition arising from the first inequality and combine this with the second inequality to get \( \text{ray}(\Sigma^{j-i}Z') \leq \text{ray}(\tau \Sigma^{k-j}Z') \). Then it follows that \( \text{ray}(\Sigma^{j-i}Z') \leq \text{ray}(Z') \). Finally, assume \( i > k \). The argument is the same but now apply \( \tau \Sigma^{j-i} \) to the condition arising from the first inequality and combine with the second inequality to get \( \text{ray}(\Sigma^{j-i}Z') \leq \text{ray}(\tau \Sigma^{k-j}Z') \).

Totality: Suppose \( Z \in \text{ind}(\mathcal{I}^i) \) and \( Z' \in \text{ind}(\mathcal{I}^j) \). If \( i = j \) then it is clear that either \( Z \preceq Z' \) or \( Z' \preceq Z \). Now suppose \( i < j \). If \( \text{ray}(\Sigma^{j-i}Z) \leq \text{ray}(Z') \) then \( Z \preceq Z' \) and we are done, so suppose that \( \text{ray}(\Sigma^{j-i}Z) > \text{ray}(Z') \). Then it follows that \( \text{ray}(\Sigma^{j-i}Z') < \text{ray}(Z) \), in which case, because \( \tau^{-1} \) increases the index of the ray by 1, one gets \( \text{ray}(\tau^{-1}\Sigma^{j-i}Z') \leq \text{ray}(Z) \) and hence \( Z' \preceq Z \). A similar argument holds in the case \( i > j \). Thus, \( \preceq \) is indeed a total order.

In order to make the next definition, we need the following lemma.

**Lemma 6.14.** If \( U \in \text{ind}(\mathcal{I}^\perp) \) then \( G_{Z_0}(U) \) is also indecomposable.

**Proof.** Let \( U \in \text{ind}(\mathcal{I}^\perp) \). The Hom-hammocks for the \( \mathcal{I} \) components, Proposition 2.4, imply that \( U \) admits non-trivial morphisms to at most two suspensions of \( Z_0 \).

Suppose \( U \) admits a non-trivial morphism to precisely one suspension of \( Z_0 \). Completing this morphism to a distinguished triangle yields one of the triangles listed in Properties 1.14. In particular, the cocone of the morphism, which defines \( G_{Z_0}(U) \), is indecomposable.
Now suppose $U$ admits non-trivial morphisms to precisely two suspensions of $Z_0$. Again, examining the Hom-hammocks incident with $Z$ shows that this can only occur when $U$ and the two suspensions of $Z_0$ sit in a mesh in one of the $Z$ components. In this case, the cocone can be read off directly from the AR quiver; in particular, it is indecomposable. □

We now ensure we identify each silting subcategory of $M$ of $D^b(\Lambda)$ as precisely one pair $(Z_0, M')$, with $M'$ a silting object of $Z_0 \simeq D^b(kA_{n+m-1})$ by insisting that $Z_0 \subseteq Z$ for each $Z \in \text{ind}(Z) \cap \text{add} M'$. Recall the map $G$ from page 22 and define

$$Z_0^\perp := \text{add}(U \in \text{ind}(Z_0) \mid G_{Z_0}(U) \in Z \text{ and } G_{Z_0}(U) \prec Z_0),$$

considered as an additive subcategory.

With the identification of $D^b(kA_{n+m-1})$ in $D^b(\Lambda(r, n, m))$ of Remark 6.6, we can give an explicit description of the additive subcategory $Z_0^\perp$. We first explicitly compute the map $G_{Z_0}: \text{ind}(Z_0) \to D^b(\Lambda(r, n, m))$ on objects, in the case $Z_0 = Z_0^0,0$.

**Proposition 6.15.** Suppose $Z_0 = Z_0^0,0$ and let $Z = \text{thick}(Z_0)$. Then $G_{Z_0}(U) = U$ except that $G_{Z_0}(\Sigma'U) = \Sigma'G_{Z_0}(U)$ for the following pairs of $U$ and $i$:

\[
\begin{array}{ccc}
U: & X^i_{1,0} & \cdots & X^i_{1,r+m-1} & Y^i_{1,0} & \cdots & Y^i_{n-r-1,0} & i \geq 0 \\
G_{Z_0}(U): & Z^i_{1,0} & \cdots & Z^i_{1,r+m-1,0} & Z^i_{0,1} & \cdots & Z^i_{0,n-r-1} \\
\end{array}
\]

\[
\begin{array}{ccc}
U: & Z^i_{0,0} & \cdots & Z^i_{0,1} & Z^i_{r-m,n-r} & \cdots & Z^i_{r,n-r} & i \geq 0 \\
G_{Z_0}(U): & Y^i_{-1,0} & \cdots & Y^i_{-1,1} & X^i_{1,r+m-1,0} & \cdots & X^i_{1,r+m-1,r+m-1} & Z^i_{r,m+0} \\
\end{array}
\]

\[
\begin{array}{ccc}
U: & Z^i_{-r,m,0} & \cdots & Z^i_{-1,0} & 0 \leq i < r \\
G_{Z_0}(U): & X^i_{-r-m-1,0} & \cdots & X^i_{-1,0} \\
\end{array}
\]

**Proof.** The ‘co-aisle’ of the colocal structure $(A, B)$ with respect to which the function $G$ is defined is given by $B = \text{susp} \Sigma Z_0 = \text{add}\{\Sigma'Z_0 \mid i \geq 1\}$. Using Proposition 2.4, one can easily compute $A := \text{add} B$. If $U \in A$, then $G_{Z_0}(U) = U$, so examining $A \cap Z_0^\perp$ gives the list of exceptions above. One now computes the cocones $G_{Z_0}(U)$ directly using the triangles described in Lemma 6.14 above. □

We can now use Proposition 6.15 to describe the additive subcategory $Z_0^\perp$ explicitly. By the standard heart of $D^b(kA_{n+m-1})$, we mean the one corresponding to the module category of the path algebra $\Gamma = kA_{n+m-1}$ with the linear orientation $1 \leftarrow 2 \leftarrow \cdots \leftarrow n + m - 1$. We take as cohomological degree zero in $D^b(\Gamma)$ the standard heart containing the unique indecomposable objects in $Z_0^0 \cap Z$ admitting non-zero morphisms to $Z_0$.

**Corollary 6.16.** With the conventions described above, the additive subcategory $Z_0^\perp$ is

$$Z_0^\perp = \text{add}\{\Sigma i A \mid i \leq -r\} \cup \text{add}\{\Sigma i B \mid 1 - r \leq i < 0\} \cup \text{add}(C),$$

where the sets of indecomposables $A, B$ and $C$ are defined as follows:

- $A := \{X \in \text{mod}(\Gamma) \mid \text{Hom}_r(P_{r+m}, X) \neq 0\}$;
- $B := A \cap \{X \in \text{mod}(\Gamma) \mid \text{Hom}_r(P_{r+m+1}, X) \neq 0\}$ (empty when $n - r = 1$);
- $C := \{P(r + m - 1), \ldots, P(n + m - 1)\}$ (empty when $n - r = 1$).

To illustrate Corollary 6.16, we sketch the additive subcategory $Z_0^\perp$ in the case of $\Lambda(2,5,2)$ and $Z_0 = Z_0^0,0$ below.
unless the object lies in homological degree 0, 1 or 2, there is not sufficient intersection

Consider the additive subcategory $T$ category $T$ consists of the thick subcategory $Z$ $\subseteq \text{ind}(Z)$, and write $Z = \text{thick}_{D^b(\Lambda)}(Z_0)$. Then there is a bijection between

1. Silting subcategories $M$ of $D^b(\Lambda)$ with $Z_0 \subseteq M$ and $Z_0 \preceq \text{ind}(Z) \cap M$.
2. Silting subcategories $N$ of $Z^\perp$ with $N \cap Z^\perp = \emptyset$.

**Theorem 6.18.** In $D^b(\Lambda)$ there are bijections between

1. Pairs $(Z_0, N)$ where $Z_0 \in \text{ind}(Z)$ and $N$ is a silting subcategory of $D^b(kA_{m+n-1})$ containing no objects in the additive subcategory $Z^\perp$.
2. Silting subcategories of $D^b(\Lambda(r, n, m))$.
3. Bounded t-structures in $D^b(\Lambda(r, n, m))$.
4. Bounded co-t-structures in $D^b(\Lambda(r, n, m))$.

**7. A detailed example: $\Lambda(2, 3, 1)$**

In this section we examine the algebra $\Lambda(2, 3, 1)$ in detail. Let $Z_0 = Z_{0,0}^0$ and, as usual, write $Z = \text{thick}(Z_0)$. Take the convention for homological degree as in Corollary 6.16. With this convention, we have the following identification of indecomposable objects in $Z^\perp$ and $D^b(kA_3)$:

$$
\begin{array}{c}
Z_{0,-1}^0 & P(3) \\
X_{0,1}^0 & Z_{1,-1}^0 & \mapsto & P(2) & I(2) \\
X_{0,0}^0 & X_{1,1}^0 & Z_{2,-1}^0 & P(1) & S(2) & S(3)
\end{array}
$$

Using Corollary 6.16, Theorem 6.11 and the explicit calculation of the tilting objects, up to suspension, in Example 6.12, we compute the twelve families of silting objects in $D^b(kA_3)$ that lift to silting objects in $D^b(\Lambda(2, 3, 1))$ containing $Z_{0,0}^0$ as the minimal indecomposable summand in the $Z$ components. The results of this computation are presented in Table 1.

We make the following observation regarding tilting objects in $D^b(\Lambda(2, 3, 1))$.

**Proposition 7.1.** Let $\Lambda = \Lambda(2, 3, 1)$, $Z \in \text{ind}(Z)$, and put $Z = \text{thick}(Z)$ and $F_Z: D^b(\Lambda) \to Z^\perp \cong D^b(kA_3)$, as usual. Then:

1. There are precisely six tilting objects in $D^b(\Lambda)$ containing $Z$ as a summand.
2. If $T \in D^b(\Lambda)$ is a tilting object containing $Z$ as a summand then $F_Z(T)$ is a tilting object in $Z^\perp$.

**Proof.** The proof is a direct computation. Without loss of generality, we may set $Z = Z_{0,0}^0$. Consider the additive subcategory $T := (\bigcap_{n \neq 0}^\perp (\Sigma^n Z)) \cap (\bigcap_{n \neq 0}^\perp (\Sigma^n Z^\perp)) \cap Z^\perp$. The subcategory $T$ consists of the thick subcategory $Z^\perp \cap Z \cong D^b(k)$, which has just one indecomposable object in each homological degree, together with finitely many indecomposables in homological degrees 0, 1 and 2.

Examining the Hom-hammocks from each of the indecomposables in $Z^\perp \cap Z$ shows that unless the object lies in homological degree 0, 1 or 2, there is not sufficient intersection...
We indicate the corresponding bounded t-structure in Figure 5 on the next page.

from a silting object \( M \)

We finish with an explicit example of a t-structure in object \( Z \).

Example 7.3. We take the silting object \( \Lambda = \Lambda(2, 3, 1) \) and bounded t-structure containing \( Z_{0,0}^0 \) as the \( \preceq \)-minimal summand in \( Z \).

with \( T \) to give rise to a tilting object. Thus we must form tilting objects from only finitely many indecomposables. A detailed analysis of the Hom-hammocks of these finitely many indecomposables gives rise to the six tilting objects obtained from \( Z_{0,0}^0 \) and the following objects:

\[
\begin{align*}
Z_{-1,0}^0 \oplus X_{1,2,-2}^0 \oplus X_{0,0}^0, & \quad X_{1,-2,-2}^1 \oplus X_{1,-1,-1}^1 \oplus X_{0,0}^0, \quad X_{1,-1}^1 \oplus X_{2,-1}^1 \oplus X_{0,0}^0, \\
X_{1,-1}^1 \oplus X_{0,0}^0 \oplus X_{0,1}^0, & \quad X_{1,-1}^1 \oplus X_{0,1}^0 \oplus X_{1,1}^1, \quad X_{1,-1}^1 \oplus X_{1,1}^1 \oplus Z_{1,0}^0.
\end{align*}
\]

The second claim can be directly computed. \( \square \)

Our computations lead us to state the following conjecture:

Conjecture 7.2. Let \( \Lambda = \Lambda(r, n, m) \), \( Z \in \text{ind}(Z) \), write \( Z = \text{thick}(Z) \) and \( F_Z : D^b(\Lambda) \to Z^\perp \simeq D^b(kA_{n+m-1}) \), as usual. Then:

1. There are finitely many tilting objects in \( D^b(\Lambda) \) containing \( Z \) as a summand.
2. If \( T \in D^b(\Lambda) \) is a tilting object containing \( Z \) as a summand then \( F_Z(T) \) is a tilting object in \( Z^\perp \).

We finish with an explicit example of a t-structure in \( D^b(\Lambda(2, 3, 1)) \). Recall how to obtain from a silting object \( M \) a bounded t-structure \( (X_M, Y_M) \) and bounded co-t-structure \( (A_M, B_M) \):

\[
X_M := (\Sigma^{<0} M)^\perp = \text{susp } M \quad \text{and} \quad Y_M := (\Sigma^{>0} M)^\perp, \\
A_M := \frac{1}{2} (\Sigma^{<0} M) = \text{cosusp } \Sigma^{-1} M \quad \text{and} \quad B_M := (\Sigma^{<0} M)^\perp = \text{susp } M.
\]

Example 7.3. We take the silting object \( N = \Sigma^{-2} S(2) \oplus P(1) \oplus \Sigma^3 P(3) \in D^b(kA_3) \) and as before set \( Z_0 = Z_{0,0}^0 \) and \( Z = \text{thick}(Z_0) \). As explained above, \( N \) corresponds to the object \( M' = \Sigma^{-2} X_{1,1}^0 \oplus X_{0,0}^0 \oplus \Sigma^3 Z_{0,-1}^0 = X_{1,-2,-2}^0 \oplus X_{0,0}^0 \oplus Z_{1,0}^0 \in Z^\perp \). By Proposition 6.15, \( M' \) lifts under \( G_{Z_0} \) to the silting object \( M = Z_{0,0}^0 \oplus X_{1,-2,-2}^0 \oplus X_{0,0}^0 \oplus Z_{1,0}^0 \in D^b(\Lambda(2, 3, 1)) \).

We indicate the corresponding bounded t-structure in Figure 5 on the next page.
Black squares (■) are the four summands of $M$, with $Z_0$ the top one in $Z^0$. Squares ■ mark positive suspensions of these $(\Sigma^{>0}M)$ and squares □ mark negative suspensions $(\Sigma^{<0}M)$. The coaisle $Y_M = (\Sigma^{\geq 0}M)^\perp$ is indicated by the light background (●), and the aisle $X_M = (\Sigma^{<0}M)^\perp$ by the dark background (●). The corresponding co-t-structure $(A_M, B_M)$ is right adjacent in the sense of [10] to the t-structure $(X_M, Y_M)$, i.e. $B_M = X_M$ and $A_M := \perp B_M = \perp \text{susp } M = \perp (\Sigma^{\geq 0}M)$.

Figure 5. A t-structure in $D^b(\Lambda(2, 3, 1))$. 

Appendix A. Notation, Terminology and Basic Notions

In this section we collect some notation and basic terminology, which is mostly standard. We always work over an algebraically closed field $k$ and denote the dual of a vector space $V$ by $V^*$. Throughout, $D$ will be a $k$-linear triangulated category with suspension (otherwise know as shift or translation) functor $\Sigma: D \to D$.

For two objects $A, B \in D$, we use the shorthand $\text{Hom}^i(A, B) = \text{Hom}(A, \Sigma^iB)$ resembling Ext spaces in abelian categories, and $\text{hom}(A, B) = \dim \text{Hom}(A, B)$ for dimensions.
of homomorphism spaces. We write

\[ \text{Hom}^\geq 0(A, B) = \bigoplus_{i \geq 0} \text{Hom}(A, \Sigma^i B) \quad \text{and} \quad \text{Hom}^\bullet(A, B) = \bigoplus_{i \in \mathbb{Z}} \Sigma^{-i} \text{Hom}(A, \Sigma^i B) \]

for aggregated homomorphism spaces (and similarly for obvious variants) and for the homomorphism complex, a complex of vector spaces with zero differential.

A.1. Properties of triangulated categories and their subcategories. A \( k \)-linear triangulated category \( D \) is said to be

**algebraic**: if \( D \) arises as the homotopy category of a \( k \)-linear differential graded category; see [24]. Examples are bounded derived categories of \( k \)-linear abelian categories.

**Hom-finite**: if \( \dim \text{Hom}(D_1, D_2) < \infty \) for all objects \( D_1, D_2 \in D \). The bounded derived category \( D^b(\Lambda) \) of any finite-dimensional \( k \)-algebra \( \Lambda \) is Hom-finite.

**Krull–Schmidt**: if every object of \( D \) is isomorphic to a finite direct sum of objects all of whose endomorphism rings are local. In this case, the direct sum decomposition is unique up to isomorphism. Bounded derived categories of \( k \)-linear Hom-finite abelian categories are Krull–Schmidt.

**indecomposable**: if for every decomposition \( D \cong D_1 \oplus D_2 \) with triangulated categories \( D_1 \) and \( D_2 \) either \( D_1 \cong 0 \) or \( D_2 \cong 0 \). The derived category of a finite-dimensional algebra is indecomposable if the associated Gabriel quiver is connected.

in possession of **Serre duality**: if there is an equivalence \( S : D \Rightarrow D \) with \( \text{Hom}(D_1, D_2) \cong \text{Hom}(D_2, S D_1)^* \), bifunctorially in \( D_1, D_2 \in D \). Such an autoequivalence is canonical and unique, if it exists, and called the **Serre functor** of \( D \).

The existence of a Serre functor is equivalent to the existence of Auslander–Reiten triangles; see [30, §I.2]. If \( \Lambda \) is a finite-dimensional \( k \)-algebra, then \( D^b(\Lambda) \) has Serre duality if and only if \( \Lambda \) has finite global dimension; in this case, the Serre functor is given by the suspended Auslander–Reiten translation: \( S = \Sigma r \).

We conclude that \( D^b(\Lambda(r, n, m)) \) is algebraic, Hom-finite, Krull–Schmidt and indecomposable for all choices of \( r, n, m \). It has Serre duality if and only if \( n > r \), which we always assume in this article.

A.2. Subcategories of triangulated categories. Let \( C \) be a collection of objects of \( D \), regarded as a full subcategory. We recall the following terminology:

\( C^\perp \), the right orthogonal to \( C \), the full subcategory of \( D \in D \) with \( \text{Hom}(C, D) = 0 \),

\( C \perp \), the left orthogonal to \( C \), the full subcategory of \( D \in D \) with \( \text{Hom}(D, C) = 0 \).

If \( C \) is closed under suspensions and cosuspensions, then \( C^\perp \) and \( C \perp \) are triangulated subcategories of \( D \).

**thick**(\( C \)), the thick subcategory generated by \( C \), the smallest triangulated subcategory of \( D \) containing \( C \) which is also closed under taking direct summands.

\( \text{susp}(C) \), the (co-)suspended subcategory generated by \( C \), the smallest full subcategory of \( D \) containing \( C \) which is closed under (co-)suspension, extensions and taking direct summands.

\( \text{cosusp}(C) \), direct summands.

\( \text{add}(C) \), the additive subcategory of \( D \) containing \( C \), the smallest full subcategory of \( D \) containing \( C \) which is closed under finite coproducts and direct summands.

\( \text{ind}(C) \), the set of indecomposable objects of \( C \), up to isomorphism.

\( \langle C \rangle \), the smallest full subcategory of \( D \) containing \( C \) that is closed under extensions, i.e. if \( C'' \rightarrow C \rightarrow C''' \rightarrow \Sigma C'' \) is a triangle with \( C'', C''' \in \mathcal{C} \) then \( C \in \mathcal{C} \).
The ordered extension closure of a pair of subcategories \((C_1, C_2)\) of \(D\) is defined as:

\[ C_1 \ast C_2 := \operatorname{add}\{D \in D \mid C_1 \to D \to C_2 \to \Sigma C_1 \text{ for } C_1 \in C_2 \text{ and } C_2 \in C_2\} \]

This operation is associative and \(C\) is extension closed in \(D\) if and only if \(C \ast C \subseteq C\).

A.3. Approximations and adjoints. For this section only, suppose \(D\) is an additive category and \(C\) a full subcategory of \(D\).

Recall that \(C\) is called right admissible in \(D\) if the inclusion functor \(C \hookrightarrow D\) admits a right adjoint. Analogously for left admissible. A subcategory \(C\) is called admissible if it is both left and right admissible.

Often, one does not need admissibility but only approximate admissibility. A right \(C\)-approximation of an object \(D \in D\) is a morphism \(C \to D\) with \(C \in C\) such that the induced maps \(\operatorname{Hom}(C', C) \to \operatorname{Hom}(C', D)\) are surjective for all \(C' \in C\). A morphism \(f: C \to D\) is called a minimal right \(C\)-approximation if \(fg = f\) is only possible for isomorphisms \(g: C \to C\). Dually for (minimal) left \(C\)-approximations.

We say \(C\) is

- contravariantly finite in \(D\) if all objects of \(D\) have right \(C\)-approximations;
- covariantly finite in \(D\) if all objects of \(D\) have left \(C\)-approximations;
- functorially finite in \(D\) if it is contravariantly finite and covariantly finite in \(D\).

Sometimes, right \(C\)-approximations are called \(C\)-precovers and left \(C\)-approximations are called \(C\)-preenvelopes. If for all \(D \in D\) the induced map \(\operatorname{Hom}(C', C) \to \operatorname{Hom}(C', D)\) above were bijective instead of surjective, then \(C\) would be even right admissible. In this sense, the morphism \(C \to D\) ‘approximates’ the right adjoint to the inclusion functor.

This relationship is even stronger for Krull-Schmidt triangulated categories \(D\): by [25, Proposition 1.3], a suspended subcategory \(C\) of \(D\) is contravariantly finite if and only if \(C\) is right admissible in \(D\); dually for covariantly finite cosuspended subcategories. Thus, a thick subcategory \(C\) of \(D\) is functorially finite if and only if it is admissible.

Functorial finiteness can often be deduced from \(\operatorname{Hom}\)-finiteness. More precisely:

**Lemma A.1.** Let \(D\) be a \(\operatorname{Hom}\)-finite, Krull-Schmidt (not necessarily triangulated) category with a subcategory \(C\). If for any \(C \in C\) the set \(H_C := \{D \in \operatorname{ind}(D) \mid \operatorname{Hom}(C, D) \neq 0\}\) is finite then \(C\) is covariantly finite in \(D\). Dually, if for any \(C \in C\) the set \(H^C := \{D \in \operatorname{ind}(D) \mid \operatorname{Hom}(D, C) \neq 0\}\) is finite then \(C\) is contravariantly finite in \(D\).

**Proof.** This is immediate: for \(C \in C\) the morphism \(\bigoplus_{D \in H_C} D \otimes \operatorname{Hom}(D, C) \to C\) is a (not necessarily minimal) left \(C\)-approximation of \(D\). Dually for contravariant finiteness. \(\square\)

A.4. Silting subcategories. Silting objects are a generalisation of tilting objects, which were introduced in [25]. However, we follow the terminology of [1].

Let \(M\) be a subcategory of a triangulated category \(D\).

- \(M\) is called a partial silting subcategory if \(\operatorname{Hom}^{>0}(M, M) = 0\).
- \(M\) is called a silting subcategory if it is partial silting and \(\text{thick}_D(M) = D\).
- An object \(D \in D\) is called a silting object if \(\operatorname{add}(D)\) is a silting subcategory.
- Two silting objects \(D, D' \in D\) are equivalent if and only if \(\operatorname{add}(D) = \operatorname{add}(D')\).

There is an obvious connection between silting objects and silting subcategories:

**Lemma A.2.** Let \(D\) be a \(\operatorname{Hom}\)-finite, Krull–Schmidt triangulated category. Then \(D\) has a silting object if and only if \(D\) has a silting subcategory and \(K(D)\) is free of finite rank.

We record the following easy observations:

**Lemma A.3.** A silting subcategory is extension-closed.
Lemma A.4. If \( D \) is a Hom-finite, Krull-Schmidt triangulated category with a silting object, then any subcategory \( N \) of a silting subcategory \( M \) is functorially finite in \( M \).

Proof. The existence of a silting object implies that \( M \) and \( N \) are each additively generated by finitely many objects. Now apply Lemma A.1. \( \Box \)

A.5. Torsion pairs, t-structures and co-t-structures. We assume again that \( D \) is a \( k \)-linear triangulated category. A pair \((X,Y)\) of full subcategories closed under direct summands is called a torsion pair if \( \text{Hom}(X,Y) = 0 \) and \( D = X \oplus Y; \) see [22].

Both \( X \) and \( Y \) are then extension closed. By definition, for every \( D \in D \) there is a triangle \( X \rightarrow D \rightarrow Y \rightarrow \Sigma X \) with \( X \in D \) and \( Y \in Y \). The map \( X \rightarrow D \) is a right \( X \)-approximation and \( D \rightarrow Y \) is a left \( Y \)-approximation, i.e. \( X \) is contravariantly finite and \( Y \) is covariantly finite in \( D \). The triangle is called the approximation triangle of \( D \). By abuse of terminology, we shall call \( X \) the aisle and \( Y \) the co-aisle of the torsion pair. The abuse arises as this terminology is normally reserved for the case that \((X,Y)\) is a t-structure (see below).

The torsion pair \((X,Y)\) will be called bounded if \( \bigcup_{i \in \mathbb{Z}} \Sigma^i X = \bigcup_{i \in \mathbb{Z}} \Sigma^i Y = D \). Torsion pairs appear in three important guises, namely \((X,Y)\) is called a

- t-structure \([6]\) if \( \Sigma X \subseteq X \) \((\iff \Sigma^{-1} Y \subseteq Y)\);
- co-t-structure \([29]\) if \( \Sigma^{-1} X \subseteq X \) \((\iff \Sigma Y \subseteq Y)\);
- stable t-structure (also semi-orthogonal decomposition) \([10]\) if \( \Sigma X = X \) \((\iff \Sigma Y = Y)\).

For historical reasons, when the terminology ‘semi-orthogonal decomposition’ is used the torsion pair is often written as \((Y,X)\).

If \((X,Y)\) is a t-structure then its heart \( H = X \cap \Sigma Y \) is an abelian subcategory of \( D \). A bounded t-structure is determined by its heart via \( X = \text{susp} H \) and \( Y = \text{cosusp} \Sigma^{-1} H \).

If \((X,Y)\) is a co-t-structure then its co-heart \( M = X \cap \Sigma^{-1} Y \) is a partial silting subcategory of \( D \). Note that, if \( M \) is abelian then it is semisimple. A co-t-structure is bounded if and only if \( M \) is a silting subcategory. Moreover, a bounded co-t-structure is determined by its co-heart via (see \([11, \text{Proposition } 2.23]\))

\[
X = \text{cosusp } M = \bigcup_{l \geq 0} \Sigma^{-l} M \ast \Sigma^{-l+1} M \ast \cdots \ast M, \quad \text{and, } Y = \text{susp } M = \bigcup_{l \geq 0} M \ast \Sigma M \ast \cdots \ast \Sigma^l M.
\]

Remark A.5. If \((X,Y)\) is a t-structure then the approximation triangle is functorial and called the truncation triangle, with \( X \rightarrow D \) being a right minimal \( X \)-approximation called the right truncation and \( D \rightarrow Y \) a left minimal \( Y \)-approximation called the left truncation of \( D \). Another way to express this functoriality is: the inclusion \( X \hookrightarrow D \) has a right adjoint (given by \( D \mapsto X \)) and \( Y \hookrightarrow D \) has a left adjoint. In particular, truncations are minimal approximations. We mention that ‘t-structure’ is an abbreviation for ‘truncation structure’.

A.6. König-Yang correspondences. The notions of silting subcategories, t-structures and co-t-structures for finite dimensional \( k \)-algebras are related by the following correspondences of König and Yang. Before we state them, recall an abelian category \( A \) is called a length category if it is both artinian and noetherian.

Theorem A.6 ([26, Theorem 6.1]). Let \( \Lambda \) be a finite dimensional \( k \)-algebra. There are bijections between

(i) equivalence classes of silting objects in \( K^b(\text{proj}(\Lambda)) \),
(ii) bounded t-structures in \( D^b(\text{mod}(\Lambda)) \) whose heart is a length category,
(iii) bounded co-t-structures in \( K^b(\text{proj}(\Lambda)) \).
A.7. Exceptional sequences and semi-orthogonal decompositions. The notion of semi-orthogonal decomposition $D = \langle C_1, C_2 \rangle$ is synonymous with that of a stable t-structure $(C_2, C_1)$, see [3], and leads to equivalences $C_1 \cong D/C_2$ and $C_2 = D/C_1$. An admissible subcategory $C \subset D$ produces two semi-orthogonal decompositions $D = \langle C, \perp C \rangle = \langle C^-, C^+ \rangle$.

An object $E$ of a $k$-linear triangulated category $D$ is exceptional if $\text{Hom}(E, E) = k$ and $\text{Hom}^\geq(E, E) = 0$, i.e. $E$ has the smallest possible derived endomorphism ring. Exceptional objects are characterised by the following property (which is used in the text): $\text{thick}_D^0(E) = \text{add}(\Sigma^i E \mid i \in \mathbb{Z})$. If thick$(E)$ is admissible then an exceptional object $E$ leads to semi-orthogonal decompositions $D = \langle \text{thick}(E)^-, \text{thick}(E) \rangle$.

An exceptional sequence in $D$ is a tuple $(E_1, \ldots, E_t)$ of exceptional objects such that $\text{Hom}^*(E_i, E_j) = 0$ for all $i > j$. The sequence is full if $\text{thick}_D(E_1, \ldots, E_t) = D$ and strong if $\text{Hom}^*(E_i, E_j) = \text{Hom}(E_i, E_j)$, i.e. all homomorphisms occur in degree zero. A full, strong exceptional sequence $(E_1, \ldots, E_t)$ gives rise to the tilting object $E_1 \oplus \cdots \oplus E_t$. Similarly, a full exceptional sequence $(E_1, \ldots, E_t)$ with $\text{Hom}^>0(E_i, E_j) = 0$ for all $i, j$ gives rise to a silting object.

### APPENDIX B. THE REPETITIVE ALGEBRA AND STRING MODULES

For a finite-dimensional algebra $\Lambda$, Happel showed in [16] that there is a full embedding $F : \text{mod}(\Lambda) \to D^b(\Lambda)$, where $\text{mod}(\Lambda)$ denotes the stable module category of the repetitive algebra $\Lambda$, and $F$ is called the Happel functor. In the case that $\Lambda$ is a gentle algebra, the repetitive algebra is special biserial (see [33]) and there is a convenient description of the indecomposable objects of $\text{mod}(\Lambda)$ using string and band modules; see [12]. The algebras $\Lambda(r, n, m)$ we study in this paper are gentle, so this machinery applies.

#### B.1. The repetitive algebra

The notion of a repetitive algebra was introduced by Hughes and Waschbüsch in [20]. The standard references are [20, 31, 33]. The relations for $\Lambda(r, n, m)$ are also recalled in [8]. The following summary is based on [33].

Let $Q = (Q_0, Q_1)$ be a finite, connected quiver with vertices $Q_0$ and arrows $Q_1$. A path $p$ in $Q$ is a sequence of arrows $p = a_1a_2 \cdots a_t$ with $s(a_{i+1}) = e(a_i)$ for $1 \leq i < t$. The start of $p$, $s(p) = s(a_1)$ and the end of $p$, $e(p) = e(a_t)$. The path $p$ is said to have length $t$. Note there is a trivial path of length 0, $e_v$, corresponding to each vertex $v \in Q_0$. The concatenation $p_1p_2$ of paths $p_1$ and $p_2$ is defined if and only if $e(p_1) = s(p_2)$. A path $q$ is called a subpath of a path $p$ if $p = p_1qp_2$ for some (not necessarily non-trivial) paths $p_1$ and $p_2$. Write $P_a$ for the set of paths of $Q$. A relation for $Q$ is a non-zero linear combination of paths of length at least 2 which have the same starting points and end points. A zero-relation is a relation of the form $p$ (sometimes written $p = 0$). A commutativity relation is a relation of the form $p = q$.

Now let $\rho$ be a set of zero- and commutativity relations for $Q$ and consider the path algebra arising from the bound quiver $\Lambda := kQ/\langle \rho \rangle$. Two paths $p_1$ and $p_2$ in $Q$ are equivalent if $p_1 = p'vp''$ and $p_2 = p''vp'$, where $v - w$ or $w - v$ is a commutativity relation in $\rho$. Note that this generates an equivalence relation on $P_a$; we denote the equivalence class of a path $p$ by $\bar{p}$. A path $p$ in $Q$ is called a path in $(Q, \rho)$ if for each $p' \in \bar{p}$, $p'$ does not have a subpath belonging to $\rho$. A path $a_1 \cdots a_n$ is called maximal if $ba_1 \cdots a_n$ and $a_1 \cdots a_n c$ are not paths in $(Q, \rho)$ for each $b$ and $c$ such that $e(b) = s(a_1)$ and $e(a_n) = s(c)$.

The repetitive algebra $\hat{\Lambda} := k\hat{Q}/\langle \hat{\rho} \rangle$, where $\hat{Q} = (\hat{Q}_0, \hat{Q}_1)$ is specified by:

- the vertex set is given by $\hat{Q}_0 := \mathbb{Z} \times Q_0$;
- for each arrow $a : x \to y$ in $Q_1$ there is an arrow $(i, a) : (i, x) \to (i, y)$ in $\hat{Q}_1$.
• for each maximal path $p$ in $(Q, \rho)$, there is a connecting arrow $\hat{p} : (i, y) \to (i+1, x)$ in $\hat{Q}$, where $s(p) = x$ and $e(p) = y$.

If $p$ is a path in $Q$, the corresponding path in $(i, Q)$ is denoted $(i, p)$. Let $p = p_1p_2$ be a maximal path in $(Q, \rho)$. Then the path $(i, p_2)(i, \hat{p})(i + 1, p_1)$ is called a full path in $\hat{Q}$.

We now define the relations:

• $\hat{p}$ inherits the relations from $\rho$, i.e. for paths $p$, $p_1$ and $p_2$ in $Q$, if $p \in \rho$ (resp. $p_1 - p_2 \in \rho$) then $(i, p) \in \hat{p}$ (resp. $(i, p_1) - (i, p_2) \in \hat{p}$) for all $i \in \mathbb{Z}$.

• Let $p$ be a path that contains a connecting arrow. If $p$ is not a subpath of a full path then $p \in \hat{p}$.

• Let $p = p_1p_2p_3$ and $q = q_1q_2q_3$ be maximal paths in $(Q, \rho)$ with $p_2 = q_2$. Then $(i, p_3)(i, \hat{p})(i + 1, p_1) - (i, q_3)(i, \hat{q})(i + 1, q_1) \in \hat{p}$ for all $i \in \mathbb{Z}$.

Denote the set of paths in $(\hat{Q}, \hat{\rho})$ by $\hat{P}_a$.

It is sometimes easier to view $\hat{Q}$ as a type of $\mathbb{Z}$-graded quiver, where the inherited arrows have degree zero and the connecting arrows have degree one. For the algebra $\hat{\Lambda}(r, n, m)$, the quiver $\hat{Q}(r, n, m)$ is shown in the figure below, where the inherited arrows $a_i$, $b_i$, $c_i$ have degree zero and $x_i$, $y$ have degree one.

### B.2. String modules.

If $\Lambda = kQ/\langle \rho \rangle$ is gentle, then its repetitive algebra $\hat{\Lambda} = k\hat{Q}/\langle \hat{\rho} \rangle$ is special-biserial and the indecomposable $\hat{\Lambda}$-modules can be described explicitly using the formalism of string and band modules. In the case of the algebras $\Lambda(r, n, m)$ of interest in this paper, only string modules occur as non-trivial indecomposable modules in the stable module category $\text{mod}(\hat{\Lambda}(r, n, m))$. Thus we describe only the strings, omitting any further reference to bands.

For each arrow $a \in \hat{Q}_1$, introduce a formal inverse $\bar{a}$ with $s(\bar{a}) = e(a)$ and $e(\bar{a}) = s(a)$. For a path $p = a_1 \cdots a_n$ the inverse path $\bar{p} = \bar{a}_n \cdots \bar{a}_1$.

A walk $w$ of length $l > 0$ in $(\hat{Q}, \hat{\rho})$ is a sequence $w = w_1 \cdots w_l$, satisfying the usual concatenation requirements, where each $w_i$ is either an arrow or an inverse arrow. Formal inverses of walks are defined in the obvious way. Starting and ending vertices of walks and their inverses are defined analogously to those for paths.

A walk is called a string if it contains neither subwalks of the form $\bar{a}a$ or $\overline{a}a$ for some $a \in \hat{Q}_1$, nor a subwalk $v$ such that $v \in \hat{\rho}$ or $v \in \hat{\rho}$. Denote the set of strings of $(\hat{Q}, \hat{\rho})$ by $\text{St}$. Modulo the equivalence relation $w \sim \hat{w}$, the strings form an indexing set for the so-called string modules of $\hat{\Lambda}$; see [12] for precise details on how to pass to a representation-theoretic description of the modules.

Let $(i, x)$ be a vertex in $\hat{Q}$. From [14], there is a linear order on strings $w$ and $v$ in $(\hat{Q}, \hat{\rho})$ such that $e(w) = e(v) = (i, x)$: namely,

$$v < w \iff \begin{cases} \text{either } w = w'v, \text{ with } w' \in \hat{Q}_1; \\ \text{or } v = v'w, \text{ with } v' \in \hat{Q}_1^{-1}; \\ \text{or } v = v'c, w = w'c \text{ with } w' \in \hat{Q}_1 \text{ and } v' \in \hat{Q}_1^{-1}. \end{cases}$$
Let \( w \) be a string. We define the successor \( w[1] \) to be the minimal string such that \( e(w[1]) = e(w) \) and \( w < w[1] \) in the linear order on strings ending at \( e(w) \). Similarly, define \([1]w\) to be the minimal string such that \( s([1]w) = s(w) \) and \( w < [1]w \) in the linear order on strings starting at \( s(w) \). The predecessor operations are defined analogously and denoted by \( w[-1] \) and \([-1]w\), respectively. For \( n \in \mathbb{Z} \), we denote by \( w[n] = w[1] \cdots [1] \) the \( n \)-times repeated sucessor operation; similarly for \([n]w\). Note that, applying the machinery of Butler and Ringel in \[12\], couched in the language of ‘starting/ending at peaks/deeps’, the strings \( w[1] \) and \([1]w\) sit in the following AR mesh:

\[
\begin{array}{c}
    w[1] \\
    \downarrow & \downarrow \\
    w & [1]w[1] = [1](w[1]) = ([1]w)[1]. \\
\end{array}
\]

**Example B.1.** Consider \( \Lambda(2, 3, 1) \), where we relabel the arrows in the figure on page 34 as \( a = a_1 \), \( b = b_0 \), \( c = b_1 \), \( d = c_2 \), \( x = x_2 \) and \( y = y \) to avoid cumbersome subscripts. We have the following linear order on the strings ending at vertex \((0, 0)\):

\[
    x\bar{b}xc\bar{b} < x\bar{c}xb < x\bar{c}b < b_{1(0, 0)} < \bar{b}\bar{a}yd_0 < \bar{a}yd_0 < \bar{b}\bar{a}\bar{y}d\bar{y}d_0 < \bar{a}\bar{y}d\bar{a}yd_0,
\]

where we write \( \bar{b}_0 = (0, \bar{b}) \) for short (similarly with \( d_0 \)), \( 1_{(0,0)} \) denotes the trivial string at vertex \((0, 0)\), and we have only indicated the degree of final arrow/inverse arrow; the others can be deduced from this. Thus, for example, \( \bar{a}yd_0[1] = \bar{b}\bar{a}\bar{y}d\bar{a}yd_0 \) and the mesh starting at \( 1_{(0,0)} \) is

\[
\begin{array}{c}
    \bar{b}\bar{a}yd_0 \\
    \downarrow & \downarrow \\
    1_{(0,0)} & \bar{b}\bar{a}\bar{y}d\bar{a}yd_0. \\
\end{array}
\]

**B.3. Maps between string modules.** It is straightforward to compute the maps between string modules. This was first observed in \[13\] and later generalised in \[27\]. We follow the neat exposition given in \[34, Section 2\].

For a string \( w \), define the set of factor strings, \( \text{Fac}(w) \), to be the set of decompositions \( w = def \) with \( d, e, f \in \text{St} \), where \( d = d_1 \cdots d_n \) and \( f = f_1 \cdots f_m \), in which we require \( d \) to be trivial or \( d_n \in Q_1^{-1} \) and \( f \) to be trivial or \( f_1 \in Q_1 \). Similarly, the set of substrings, \( \text{Sub}(w) \), is the set of decompositions in which we require \( d \) to be trivial or \( d_n \in Q_1 \) and \( f \) to be trivial or \( f_1 \in Q_1^{-1} \). A pair \( ((d_1, e_1, f_1), (d_2, e_2, f_2)) \in \text{Fac}(w) \times \text{Sub}(w) \) is called admissible if \( e_1 = e_2 \) or \( e_1 = \bar{e}_2 \). Then the main results of \[13\] and \[27\] assert:

**Proposition B.2.** Let \( v, w \in \text{St} \) and suppose \( M_v \) and \( M_w \) are their corresponding string modules. Then \( \text{hom}(M_v, M_w) = \# \{ \text{admissible pairs in } \text{Fac}(v) \times \text{Sub}(w) \} \).

**B.4. Strings and maps for derived-discrete algebras.** Here we list some pertinent facts about strings and string modules for discrete derived categories from \[8\], and establish some additional routine but useful properties. First we start with a lemma, which is collection of facts from \[8\].

**Lemma B.3** (\[8\]). Denote the simple modules of \( \Lambda(r, n, m) \) by \( S(i) \) for \(-m \leq i < n\). In the coordinate system introduced in Properties \[1.7\], \( Z^0_{0,0} = S(0) \). Then:
(i) If $m > 0$ then $S(-1) = X^1_{0,0}$; in particular there is a simple module on the mouth of the $X$ component.

(ii) If $r < n$ then $S(n-r)$ lies on the mouth of the $Y$ component.

The embedding $\text{mod}(\Lambda(r,n,m)) \hookrightarrow \text{mod}(\hat{\Lambda}(r,n,m))$ maps simple modules $S(i) \hookrightarrow S(0,i)$; the latter corresponds to the trivial string $1_{(0,i)}$. Since morphisms to and from a simple module cannot factor through projective modules, it follows that $\text{Hom}(S(0,i),X) = \text{Hom}(S(0,i),X)$ and $\text{Hom}(X,S(0,i)) = \text{Hom}(X,S(0,i))$ for all $X \in \text{mod}(\Lambda(r,n,m))$.

**Lemma B.4.** Let $A \in \text{ind}(D^b(\Lambda(r,n,m)))$ with $r > 1$ and let $i,k \in \mathbb{Z}$, $0 \leq k < r$. Then

$$\text{Hom}(X^k_i,A) = k$$

if $A \in \text{ray}_+(X^k_i) \cup \text{coray}_-(SX^k_i) \cup \text{ray}_-(Z^k_i)$,

$$\text{Hom}(Y^k_i,A) = k$$

if $A \in \text{coray}_+(Y^k_i) \cup \text{ray}_-(SY^k_i) \cup \text{coray}_+(Z^k_i)$,

and in all other cases the Hom spaces are zero. For $r = 1$ the Hom-spaces are as above, except $\text{Hom}(X^0_i,X^0_{i+m}) = k^2$.

**Proof.** First, note that the other two statements of Lemma 2.1 follow from these by Serre duality. We prove the statement in two cases, $m > 0$ and $m = 0$.

**Case $m > 0$:** Since the action of $\tau$ and $\Sigma$ together is transitive on the set of objects at the mouths of the $X$ components, by Lemma B.3 we may assume that $X^k_0 = S(0,1)$. By Proposition B.2 there is a morphism $S(0,-1)$ to an indecomposable object $X$ if and only if the string $w$ corresponding to $X$ admits a substring string decomposition $w = def \in \text{Sub}(w)$ such that $e = 1_{(0,-1)}$ or $d = 1_{(0,-1)}$. The unique direct arrow ending at $(0,1)$ is $a_{(0,-2)}$ when $m > 1$ and $y_{1}$ when $m = 1$. Likewise, the unique inverse arrow starting at $(0,-1)$ is $a_{(0,-2)}$ when $m > 1$ and $y_{-1}$ when $m = 1$. Hence, strings $w$ with substring decompositions $w = def$, with $e = 1^\pm_{(0,-1)}$ either end with the direct arrow $a_{(0,-2)}$ if $m > 1$ or $y_{-1}$ if $m = 1$, or start with the inverse arrow $a_{(0,-2)}$ if $m > 1$ or $y_{-1}$ if $m = 1$, or is precisely the trivial string $1^0_{(0,-1)}$. Moving along a ray or coray corresponds to carrying out the successor operation $[1]$. Thus, the strings corresponding to objects on the same ray all end at the same vertex and those on the same coray all start at the same vertex. The chain of morphisms in Properties 1.1(5) corresponds to the linearly ordered set of strings ending at $1_{(0,-1)}$. Thus, the ‘extended ray’ of Properties 1.1(5) consists of precisely the strings with substring decompositions $def$ with $e = 1^\pm_{(0,-1)}$. Reading this off gives $\text{ray}_+(S(0,1))$ and $\text{coray}_-(SS(0,1))$ in the $X$ components, and $\text{ray}_+(Z^1_{0,0})$ in the $Z$ component.

The Hom-hammock of objects admitting morphisms from $S(0,n-r)$, which is in the $Y$ component for any $m$, can be obtained in an analogous fashion.

**Case $m = 0$:** We shall use an embedding $D^b(\Lambda(r,n,0)) \hookrightarrow D^b(\Lambda(r,n,1))$. By [8 Lemma 3.1], the (stalk complex of the) indecomposable projective $P(-1)$ lies on the mouth of the $X$ component. Applying Lemma A.1 to $P = \text{thick}_{D^b(\Lambda(r,n,1))}(P(-1))$ means that $P^\perp \simeq D^b(\Lambda(r,n,0))$ as in the proof of Proposition 6.5. We can now use the case $m > 0$ to compute the Hom-hammocks in $P^\perp$ to give the result. \hfill \square

**Remark B.5.** In the ‘extended ray’ of strings ending at $1_{(0,-1)}$, to obtain the part of this linearly ordered set corresponding to $\text{coray}_-(SS(0,1)))$, we consider the inverse strings of those ending at $1_{(0,-1)}$ with the direct arrow $a_{(0,-2)}$ (for $m > 1$) or $y_{-1}$ (for $m = 1$). We thus obtain strings starting with the corresponding inverse arrow, which gives the coray.

In the classification of silting objects for derived-discrete algebras, we need to explicitly locate the indecomposable projective $\Lambda$-module $P(n-r)$ in the AR quiver of $D^b(\Lambda)$. 36
Lemma B.6. The projective $P(n - r) \in \mathcal{Z}$ in the AR quiver of $D^b(\Lambda(r, n, m))$.

Proof. The simple module $S(n - r + 1) \in \text{mod}(\Lambda)$ corresponds to the trivial string $1_{(0, n-r+1)}$. One can show by direct computation that for any $n \in \mathbb{Z}$ both $[n]1_{(0, n-r+1)}$ and $1_{(0, n-r+1)}[n]$ exist. This means that $1_{(0, n-r+1)}$ sits in a $\mathbb{Z}A^\infty_\infty$-component of the AR quiver, for otherwise, eventually one of $[n]1_{(0, n-r+1)}$ or $1_{(0, n-r+1)}[n]$ would not be defined. The projective $P(n - r)$ is represented by the string $b_{n-r}$, which is given by $1_{(0, n-r+1)}[-1]$, and hence lies in the same component as $S(n - r + 1)$, i.e. $P(n - r) \in \mathcal{Z}$.

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Email: broomhead@math.uni-hannover.de, david.pauksztello@manchester.ac.uk, david.ploog@uni-due.de