Phase Space Representations and Perturbation Theory for Continuous-time Histories

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Abstract

We consider two technical developments of the formalism of continuous-time histories. First, we provide an explicit description of histories of the simple harmonic oscillator on the classical histories phase space, comparing and contrasting the Q, P and Wigner representations; we conclude that a representation based on coherent states is the most appropriate. Second, we demonstrate a generic method for implementing a perturbative approach for interacting theories in the histories formalism, using the quartic anharmonic oscillator. We make use of the identification of the closed-time path (CTP) generating functional with the decoherence functional to develop a perturbative expansion for the latter up to second order in the coupling constant. We consider both configuration space and phase space histories.

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1 Introduction

The consistent histories approach to quantum mechanics is a framework for the description of individual (closed) quantum systems (see [1],[2],[3],[4]). It provides a reformulation of quantum physics based on histories, namely temporally extended propositions for a physical system. One therefore asks questions about histories of momentum, position, energy and other variables. The probability information for the histories is contained within the decoherence functional, which is a complex valued functional of pairs of histories. In the usual interpretation, its diagonal elements define probabilities within a set of histories (usually coarse-grained), provided a specific consistency condition is satisfied. The basic mathematical objects of histories theory are therefore different from those of the standard formulation, even though in the cases of interest the former can be constructed from the latter.

This paper deals with two specific technical issues of the history formalism that have not been fully developed in the relevant bibliography: i) an explicit description of quantum mechanical histories defined on the classical phase space, and ii) the translation of the usual methods of perturbation theory in the history context. In both issues, our emphasis lies on the construction of the decoherence functional, from which all physical predictions of the theory (probabilities) are derived. These results then allow the translation of common and useful techniques of standard quantum mechanics in the histories framework and provide therefore a tool for addressing problems of a technically more complex nature.

A history is represented by a time-ordered string of projection operators representing propositions about the system. We denote these $\alpha := (\alpha_{t_1}, \alpha_{t_2}, \ldots, \alpha_{t_n})$ with $t_1 < t_2 < \ldots < t_n$. Given a Hamiltonian $H$, and an initial state $\rho_0$, the decoherence functional is defined on pairs of histories as:

$$d(\alpha, \beta) = Tr[C^\dagger_\alpha \rho_0 C_\beta]$$

in which the class operator is given in terms of Heisenberg picture projection operators as $C_\alpha = \alpha_{t_1}(t_1) \alpha_{t_2}(t_2) \ldots \alpha_{t_n}(t_n)$.

Over the last twenty years, the histories formalism has undergone significant developments. For example, the 'history projection operator' (HPO) approach of Isham, Linden and Savvidou [2],[3],[5],[4] (which heavily influences
the current work) has focussed on the temporo-logical structure of histories, whereas Hartle’s ‘generalized quantum mechanics’ [6] and Savvidou’s space-time description of HPO histories [4],[7] (see also [8]) cast quantum theory into a more manifestly covariant form.

In the present work we are interested in the further development of the phase space description of quantum mechanical histories initiated by Anastopoulos [9],[10] and the development of a generic method for the implementation of perturbative techniques in the histories formalism. Central to this work is the identification of the decoherence functional with the closed-time-path (CTP) generating functional (first introduced by Schwinger [11]). From this we gain both a definition of the decoherence functional via a quasi-distribution on the classical histories phase space $\Pi$ (this is the space of all continuous paths on the standard phase space—see [5], Ch. 5), and, as the latter has a well-defined perturbative expansion, a starting point for the description of interacting theories in the histories formalism.

The paper is structured as follows. We briefly present the necessary background material in Section 2, largely to fix notation. In Section 3, we perform an explicit analysis of different phase space representations of the decoherence functional for the case of the simple harmonic oscillator (SHO). We construct a parameterised expression for the phase space distribution, given a generic initial state, at the discrete-time level which allows us to compare the most commonly encountered representations, ie. $Q$, $P$ and Wigner. Concluding that the $P$ representation is ill-defined, we then take the continuous-time limit of the coherent state ($Q$) and Wigner representations. Contrary to the single-time case, we find significant differences in the structure of these representations in the context of histories. In Section 4, we make use of the Fourier transform relationship between the CTP generating functional and the (quasi) distribution to develop, at the continuous-time level, a perturbative expression for the decoherence functional of the quartic anharmonic oscillator (AHO). We construct this expression to second order in the coupling constant in both the configuration space and phase space contexts. In Section 5 we conclude.
2 Background

2.1 Quantum Mechanical Phase Space Representations

Following [12], we associate a c-number function on the classical phase space \( \Gamma \), with an operator \( G(\hat{a}^\dagger, \hat{a}) \) that is a function of the non-commuting operators \( \hat{a}^\dagger \) and \( \hat{a} \), on the Hilbert space \( \mathcal{H} \), according to:

\[
F_\Omega^\Gamma(\alpha, \alpha^*) = \text{Tr}[G(\hat{a}^\dagger, \hat{a})\Delta^\Omega(\alpha, \alpha^*)] 
= \int d^2z \exp\{-z\alpha^* + z^*\alpha\} \Omega(z, z^*) \text{Tr}[G(\hat{a}^\dagger, \hat{a}) \exp\{\hat{a}^\dagger z - \hat{a}z^*\}],\]

(2.1)

where \( \hat{a}^\dagger \) and \( \hat{a} \) are the boson creation and annihilation operators and where \( d^2z \) is the standard Lebesgue measure normalised by \( 2\pi \). \( \Omega(z, z^*) \) represents the linear mapping \( \hat{G} = \Omega\{F\} \) and \( \hat{\Omega}(z, z^*) = [\Omega(-z, -z^*)]^{-1} \). \( \Omega(z, z^*) \) is intimately related to the ordering of non-commuting operators.

The most commonly encountered phase space representations are members of the sub-class of mappings given simply by

\[
\Omega(z, z^*) = \exp\left\{ \frac{s}{2}|z|^2 \right\}.
\]

(2.3)

For \( s = 1, 0, -1 \) we have, respectively, the Q, Wigner and P representations. The phase space representation of the density operator is often referred to as a quasi-probability distribution as it plays an analogous role to a classical probability distribution, ie:

\[
\text{Tr}[\rho G(\hat{a}^\dagger, \hat{a})] \equiv \langle G(\hat{a}^\dagger, \hat{a}) \rangle_{\rho} = \int d^2z F_\rho^\Omega(z, z^*) F^\Omega_G(z, z^*) = \int d^2z F_\rho^\Omega(z, z^*) F^\Omega_G(z, z^*).\]

(2.4)

In [13], Srinivas describes the generalisation of these results so as to write multi-time quantum correlation functions in terms of a multi-time quasi-probability distribution on phase space. The latter is given by:

\[
F_\rho^\Omega(\alpha_1, \alpha_1^*, t_1; \ldots; \alpha_n, \alpha_n^*, t_n) \equiv \text{Tr}[\rho_0 \Delta^\Omega(\alpha_1, \alpha_1^*, t_1) \ldots \Delta^\Omega(\alpha_n, \alpha_n^*, t_n)],
\]

(2.5)

in which the time-evolved representation operator is given by \( \Delta^\Omega(\alpha, \alpha^*, t) = e^{itH} \Delta^\Omega(\alpha, \alpha^*) e^{-itH} \).
2.2 The Relationship Between the CTP Generating Functional and the Decoherence Functional

The reader is referred to [9] for a detailed account of what follows.

The decoherence functional, Eq. (1.1), defined on general operators, can be understood as the expectation value of two strings of operators—one time-ordered and one anti-time-ordered. These correlation functions are generated by the CTP generating functional. Thus, if the history Hilbert space $V$ (defined as a tensor product of copies of the standard Hilbert space), carries a representation of the history Weyl group $U(\xi(\cdot), \chi(\cdot)) = \exp\{iq_\xi + ip_\chi\} \in U$ in which $q_\xi$ and $p_\chi$ are the time-averaged position and momentum operators) then we can define the configuration space CTP generating functional as

$$Z[\xi, \xi'] = d\exp\{iq_\xi + ip_\chi\}.\tag{2.6}$$

Note that the CTP generating functional thus inherits the normalisation condition $Z[0, 0] = 1$.

Denoting for simplicity a phase space path $\gamma \equiv [q(\cdot), p(\cdot)]$, we can formally associate to the decoherence functional a quasi distribution $W^\Omega[\gamma|\gamma']$, on the histories phase space (strictly speaking on $\Pi \times \Pi$) according to

$$d(A, B) = \int D\gamma D\gamma' W^\Omega[\gamma|\gamma']F_A^\Omega[\gamma]F_B^\Omega[\gamma'],\tag{2.7}$$

in which $A$ and $B$ are operators on $V$, and $F_A^\Omega[\gamma]$ is a phase space representation of $A$ defined in analogy to Eq. (2.1).

The phase space CTP generating functional is related to the continuous-time histories phase space quasi-distribution by a (functional) Fourier transform

$$W^\Omega[q(\cdot), p(\cdot)|q'(\cdot), p'(\cdot)] = \int D\xi D\chi D\xi' D\chi' e^{iq\xi + ip\chi - iq'\xi' - ip'\chi'} Z^\Omega[\xi(\cdot), \chi(\cdot)|\xi'(\cdot), \chi'(\cdot)],$$

in which we have used the shorthand $q \cdot \xi \equiv \int dt q(t)\xi(t)$.

The integration over paths in the above expressions is formal. It is properly defined by a consideration of discrete-time histories, the definition of suitable cylinder sets in the space of continuous time histories and extension by continuity to a larger class of phase space paths (see [9] for proof).
3 Phase Space Representations for the SHO

In this section, we explicitly construct and analyse the different phase space representations of the decoherence functional for the case of a single harmonic oscillator, described by the Hamiltonian

\[ \hat{H} = \frac{1}{2} \hat{p}^2 + \frac{1}{2} \omega^2 \hat{q}^2 = \omega \hat{a}^\dagger \hat{a}. \] (3.1)

The key new results of this section are (i) the computation of the discrete-time expression for the distribution, parameterised according to Eq. (2.3), given a generic initial state, (ii) explicit expressions for the Wigner and Q representations at both the discrete-time and continuous-time level, and the infinite time \((t \in \mathbb{R})\) limit of the latter, and (iii) the form of the CTP generating functional that arises in this limit. (This last expression will form the basis of the development of the perturbation theory of the AHO in the next section.)

We employ the following relations to interchange between complex and real coordinates on \(\Gamma\) and its dual

\[ \alpha = \sqrt{\frac{\omega}{2q}} + i \frac{1}{\sqrt{2\omega p}} \] (3.2)

\[ z = -\sqrt{\frac{\omega}{2\chi}} + i \frac{1}{\sqrt{2\omega \xi}}. \] (3.3)

The continuous-time phase space distribution \(W^\Omega[\gamma | \gamma']\) is defined as the limit of the following discrete-time expression [9]

\[ W^\Omega_{n,m}(\alpha_1, \alpha_1^*, t_1; \ldots; \alpha_n, \alpha_n^*, t_n | \alpha'_1, \alpha'_1^*, t'_1; \ldots; \alpha'_m, \alpha'_m^*, t'_m) = Tr(\hat{C}_n \rho_0 \hat{C}_m), \] (3.4)

in which \(\hat{C}_n = \Delta^\Omega_1(\alpha_1, \alpha_1^*, t_1) \ldots \Delta^\Omega_n(\alpha_n, \alpha_n^*, t_n)\). The ‘branches’ are time-ordered so that \(t_1 < t_2 < \ldots < t_n\) and \(t'_1 < t'_2 < \ldots < t'_m\). This is the histories theory generalisation of Eq. (2.5).

As the expressions are somewhat unwieldy in full, we only demonstrate the calculation on one ‘branch’, ie:

\[ W^\Omega_n(\alpha_1, \alpha_1^*, t_1; \ldots; \alpha_n, \alpha_n^*, t_n) \equiv Tr(\rho_0 \hat{C}_n) = \int d^2 z_1 \ldots d^2 z_n e^{-\sum_{i=1}^n (\alpha_i^* z_i - \alpha_i z_i^*)} e^{-\frac{1}{2} \sum_{i=1}^n |z_i|^2} \times \]

\[ Tr[\rho_0 e^{it_1 H} U(z_1, z_1^*) e^{-it_1 H} \ldots e^{it_n H} U(z_n, z_n^*) e^{-it_n H}], \] (3.5)
in which the Weyl operators are given by $U(z, z^*) = \exp\{\hat{a}^\dagger z - \hat{a} z^*\}$.

To maintain generality as to the initial state, it suffices to choose a coherent state as all density matrices can be written as a weighted, diagonal sum of such states. Thus we take $\rho_0 = |\beta\rangle\langle\beta|$, in which $|\beta\rangle := e^{\hat{a}^\dagger \beta - \hat{a} \beta^*}|0\rangle$ for $\beta \in \mathbb{C}$.

Using the composition law

$$U(z, z^*) U(z', z'^*) = e^{\frac{1}{2}(z'^* z - z^* z')} U(z + z', z^* + z'^*)$$

(3.6)

and the time evolution $e^{iHt} U(z, z^*) e^{-iHt} = U(e^{i\omega t} z, e^{-i\omega t} z^*)$, the result is

$$W_n^\Omega(\alpha_1, \alpha_1^*, t_1; \ldots; \alpha_n, \alpha_n^*, t_n) = \exp\left\{\frac{4}{1 + s} \sum_{i,j=1}^n \theta(t_j - t_i) A_{j-i}(s) e^{-i\omega(t_i - t_j)} \alpha_i^* \alpha_j\right\} \times$$

$$\exp\left\{-2\beta^* \beta \sum_{i=1}^n A_i(s) + 2\beta \sum_{i=1}^n \alpha_i^* A_{n+1-i}(s) e^{-i\omega t_i} + 2\beta^* \sum_{i=1}^n \alpha_i A_i(s) e^{i\omega t_i}\right\}$$

(3.7)

in which the coefficients $A_k(s)$ are given by:

$$A_0(s) = -1$$

(3.8)

$$A_1(s) = \frac{1}{1 + s}$$

(3.9)

$$A_{k+1}(s) = \left[1 - 2 \left(\frac{1}{1 + s}\right)\right] A_k(s) \quad (k \geq 1)$$

(3.10)

and in which the step function is given by:

$$\theta(t_i - t_j) = \begin{cases} 
1 & t_i > t_j \\
\frac{1}{2} & t_i = t_j \\
0 & t_i < t_j
\end{cases}$$

(3.11)

The second exponential on the right hand side of Eq. (3.7) contains the boundary terms that arise as the result of our choice of initial state.

The result of the full calculation of Eq. (3.4) contains a similar expression for the other ‘branch’ (with opposite time-ordering) and an expression involving cross-terms between the primed and unprimed quantities. It is given by:
This is the full, discrete-time expression for the phase space distribution associated to the decoherence functional of the SHO with a generic initial state by a general rule of association given by \( \Omega(z, z^*) = \exp \left\{ \frac{4}{1 + s} \sum_{i,j=1}^{n,m} \theta(t_i - t_j) A_{i-j}(s) e^{-i\omega(t_i - t_j)} \alpha_i \alpha_j^* \right\} \times \exp \left\{ \frac{4}{1 + s} \sum_{i,j=1}^{n,m} \theta(t'_j - t'_i) A_{j-i}(s) e^{-i\omega(t'_i - t'_j)} \alpha'_i \alpha'_j^* \right\} \times \exp \left\{ 4 \sum_{i=1}^{n} \sum_{j=1}^{m} A_{n+1-i}(s) A_{m+1-j}(s) e^{-i\omega(t'_j - t'_i)} \alpha_i \alpha_j^* \right\} \times \exp \left\{ -2\beta^* \beta \sum_{i=1}^{n} A_i(s) + 2\beta \sum_{i=1}^{n} \alpha_i A_i(s) e^{-i\omega t_i} + 2\beta^* \sum_{i=1}^{n} \alpha_i A_{n+1-i}(s) e^{i\omega t_i} \right\} \times \exp \left\{ -2\beta^* \beta \sum_{i=1}^{m} A_i(s) + 2\beta \sum_{i=1}^{m} \alpha'_i A_{m+1-i}(s) e^{-i\omega t'_i} + 2\beta^* \sum_{i=1}^{m} \alpha'_i A_i(s) e^{i\omega t'_i} \right\} \times \exp \left\{ 4\beta^* \beta \sum_{i=1}^{n} \sum_{j=1}^{m} A_i(s) A_j(s) - 4\beta \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i A_j^* A_{m+1-j}(s) A_i(s) e^{-i\omega t'_j} - 4\beta^* \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i A_j(s) A_{n+1-i}(s) e^{i\omega t_i} \right\}. \right\} (3.12)

It is clear that the expression Eq. (3.12), is not well-defined for \( s = -1 \), as the first two exponents have \( 1 + s \) in the denominator, which introduces an infinity into the expression for the distribution. Thus we conclude that the P representation is not a good choice for representing quantum mechanical histories on the classical histories phase space. We shall now examine, in turn, the Q representation \( (s = 1) \) and the Wigner representation \( (s = 0) \), and their respective continuum limits.
3.1 The Q representation

The Q representation is given by \( s = 1 \), and thus \( A_k(s) = 0 \) for \( k \geq 2 \). The resulting expression is thus local in time. This is an important property, as it transpires that the phase space decoherence functional satisfies a histories version of the Markov property [10].

Taking \( s = 1 \), Eq. (3.7) becomes

\[
W^Q_n(\alpha_1, \alpha_1^*, t_1; \ldots; \alpha_n, \alpha_n^*, t_n) = \exp \left\{ - \beta^* \beta + \beta \alpha_n^* e^{-i\omega t_n} + \beta^* \alpha_1 e^{i\omega t_1} \right\} 
\]

\[
\times \exp \left\{ - |\alpha_n|^2 + \sum_{i=1}^{n-1} \left( - |\alpha_i|^2 + \alpha_i^* \alpha_{i+1} e^{-i\omega (t_i - t_{i+1})} \right) \right\}. \tag{3.13}
\]

To get the continuous-time limit, we take \( n \) large (so \( t_{i+1} - t_i \equiv \delta t < < 1 \)), define the discrete derivative \( \dot{\alpha}_i = \frac{\alpha_{i+1} - \alpha_i}{\delta t} \), neglect terms of \( O(\delta t^2) \), and then we take \( \delta t \to 0 \). The result is

\[
W^Q[\alpha(\cdot), \alpha^*(\cdot)] = 
\exp \left\{ - |\beta|^2 + \beta \alpha_n(t_n) + \beta^* \alpha_1 - |\alpha_n|^2 + \int_{t_1}^{t_n} dt \left( \alpha^*(t) \dot{\alpha}(t) + i\omega \alpha^*(t) \alpha(t) \right) \right\} \tag{3.14}
\]

The result for the full expression, i.e. the continuum limit of Eq. (3.4) in the Q representation is

\[
W^Q[\alpha(\cdot), \alpha^*(\cdot)|\alpha'(\cdot), \alpha'^*(\cdot)] = \exp \left\{ - |\beta|^2 + \beta \alpha_1 + \beta^* \alpha_1' + \alpha_m^* \alpha_n + 
\int_{t_1}^{t_n} dt \left( \alpha^*(t) \alpha(t) - i\omega \alpha^*(t) \alpha(t) \right) \right\} \tag{3.15}
\]

in which we have also taken \( t_n = t_m' \) as is usual in histories.

In the infinite-time limit, i.e. \([t_1, t_n], [t_1', t_m'] \to [-\infty, \infty]\), the requirement of square-integrability on the space of paths forces the single-time Hilbert spaces at \( t = \pm \infty \) to be one-dimensional, consisting only of the vector \( |0\rangle \). Thus the initial state is the vacuum, i.e. \( \rho_0 = |0\rangle \langle 0| \). If we revert to the \([q(\cdot), p(\cdot)]\) coordinates on \( \Pi \), the above expression, in this limit, becomes simply

\[
W^Q[q(\cdot), p(\cdot)|q'(\cdot), p'(\cdot)] = \exp \left\{ iS[q(t), p(t)] - iS[q'(t), p'(t)] \right\}, \tag{3.16}
\]

where \( S[q(t), p(t)] = \int dt \left[ p(t) \dot{q}(t) - \frac{1}{2} (p(t)^2 + \omega^2 q(t)^2) \right] \) is the phase space action functional.
\section{The Wigner representation}

In the case of the Wigner representation, \( s = 0 \), we get a non-local expression with \( A_{k+1}(s) = -A_k(s) \). For one ‘branch’ we have

\[ W_n^{\text{Wigner}}(\alpha_1, \alpha_1^*; t_1; \ldots; \alpha_n, \alpha_n^*; t_n) = \exp \left\{ 4 \sum_{i,j=1}^{n} \theta(t_j - t_i) (-1)^{i+j+1} e^{-i\omega(t_i - t_j)} \alpha_i^* \alpha_j \right\} \times \exp \left\{ -2 \beta^* \beta \sum_{i=1}^{n} (-1)^{i+1} + 2 \beta \sum_{i=1}^{n} (-1)^{n-i} \alpha_i^* e^{-i\omega t_i} + 2 \beta^* \sum_{i=1}^{n} (-1)^{i+1} \alpha_i e^{i\omega t_i} \right\}. \tag{3.17} \]

The alternating sign makes the calculation of the continuum limit of this expression a little more tricky. However, a similar situation was encountered in \cite{14}, and a solution detailed in Section III.C and Appendix B of that reference. The only difference is that the current expressions contain an explicit time dependence, however this does not complicate the derivation in any significant way. We outline the calculation of the continuum limit of the above expression in the Appendix. The final result is

\[ W^{\text{Wigner}}[\alpha(\cdot), \alpha^*(\cdot) | \alpha'(\cdot), \alpha'^*(\cdot)] = \exp \left\{ \frac{1}{2} |\alpha_1^*|^2 - \frac{1}{2} |\alpha_n|^2 - \alpha_1(t_1)\alpha_n^*(t_n) - \int_{t_1}^{t_n} dt \ (\alpha^*(t)\dot{\alpha}(t) + i\omega\alpha^*(t)\alpha(t)) \right\} \times \exp \left\{ \frac{1}{2} |\alpha_1'|^2 - \frac{1}{2} |\alpha_m'|^2 - \alpha_1'(t_1)\alpha_m'^*(t_m) - \int_{t_1'}^{t_m'} dt \ (\alpha'(t)\dot{\alpha'}(t) - i\omega\alpha'^*(t)\alpha'(t)) \right\} \times \exp \left\{ \alpha_n \alpha_m^* - \alpha_n(t_n)\alpha_1^*(t_1') - \alpha_1(t_1)\alpha_m'^*(t_m') - \alpha_1(t_1)\alpha_1'^*(t_1') \right\} \times \exp \left\{ \beta (\alpha_m'^*(t_m') - \alpha_1'^*(t_1') - \alpha_n^*(t_n) + \alpha_1^*(t_1)) + \beta^* (\alpha_n(t_n) - \alpha_1(t_1) - \alpha_m'(t_m') + \alpha_1'(t_1')) \right\}. \tag{3.18} \]

Thus, in the continuum limit, the classical action reappears along with a collection of boundary terms. These latter are significantly different from those that appear in the Q representation and reflect the symmetric nature of the Wigner representation.

Taking the infinite-time limit again, we find that this result reduces to the
same expression as we obtained for the Q representation, namely

\[
W^{\text{Wigner}}[q(\cdot), p(\cdot) | q'(\cdot), p'(\cdot)] = \exp \left\{ iS[q(t), p(t)] - iS[q'(t), p'(t)] \right\}.
\]  

(3.19)

This is a satisfying result as the classical limits of these expressions should be the same. However, we have compared the Q and Wigner representations in the far more general context of a generic initial state, at both the discrete-time—Eqs. (3.13) and (3.17) respectively—and continuous-time—Eqs. (3.15) and (3.18)—levels.

The discrete-time expressions are the more fundamental as it is at this level that one may define the generic coarse-graining operations that are central to the histories formalism. We have seen the difference in the boundary terms that arise in each representation. But, most importantly, we get a local expression for the Q representation. As we mentioned earlier, this allows us the definition of a Markov property for the distribution and thus the decoherence functional. This is in line with the properties of the wave function (or density matrix) in standard QM, and for this reason we conclude that the Q representation is the most suitable for discussing the phase space structure of the histories formalism.

This analysis complements the discussion in [10] in which it is suggested that a representation based on coherent states is more suitable, and where it was shown that a Markov property for the phase space decoherence functional is a necessary but not sufficient condition for a reconstruction theorem, which regains the standard Hilbert space from the phase space picture.

### 3.3 The phase space CTP generating functional

In this section we compute the phase space CTP generating functional in the infinite time limit. This is given by the Fourier transform of the distribution. It will also form the basis of the perturbative expressions calculated in the next section. Finally, we calculate and briefly discuss the correlation functions that arise.
Wightman and Dyson Green’s functions. A general expression for a mixed $(n, m)$ correlation function is

\[ W^Q[q(\cdot), p(\cdot)|q'(\cdot), p'(\cdot)] = W^{Wigner}[q(\cdot), p(\cdot)|q'(\cdot), p'(\cdot)] = \exp \left\{ iS[q(t), p(t)] - iS[q'(t), p'(t)] \right\} \]  

(3.20)

we drop the superscript in what follows, however, it should be remembered that, in more general situations, the Q representation is the more appropriate. The CTP generating functional will be given by

\[ Z[\xi(\cdot), \chi(\cdot)|\xi'(\cdot), \chi'(\cdot)] = \int DqDpDq'Dp' e^{-2i\Omega(\xi,q')-ip\chi+ipq'\xi'} W[q(\cdot), p(\cdot)|q'(\cdot), p'(\cdot)] \]

(3.21)

\[ = \exp \left\{ -\frac{i}{2} \int dt dt' [\Delta_F(t-t')(\xi'(t') - \chi(t')) + 2(\xi(t) - \chi(t))\Delta^+(t-t')(\xi'(t') - \chi'(t')) - (\xi'(t) - \chi'(t))\Delta_D(t-t')(\xi'(t') - \chi'(t')) \right\} \]

(3.22)

The Green’s functions are

\[ \Delta_F(t-t') = -i\langle 0 | T[q(t)q(t')] | 0 \rangle = \int \frac{dk}{2\pi k^2} \frac{1}{\omega^2 + i\epsilon} e^{-ik(t-t')} \]

(3.23)

\[ \Delta^+(t-t') = i\langle 0 | T[q(t)q(t')] | 0 \rangle = \frac{1}{2\omega} e^{i\omega(t-t')} \]

(3.24)

\[ \Delta_D(t-t') = i\langle 0 | \tilde{T}[q(t)q(t')] | 0 \rangle = \int \frac{dk}{2\pi k^2} \frac{1}{\omega^2 - i\epsilon} e^{-ik(t-t')} \]

(3.25)

where \( \tilde{T} \) indicates anti-time ordering. These are, respectively, the Feynman, Wightman and Dyson Green’s functions.

By writing \( X^a(a = 1, 2) \equiv (q, p) \) and \( J^a(a = 1, 2) \equiv (\xi, \chi) \), we have the general expression for a mixed \((n, m)\) correlation function

\[ G^{n,m}(a_1, t_1; \ldots; a_n, t_n|b_1, t'_1; \ldots; b_m, t'_m) = \]

\[ (-i)^n i^m \frac{\delta^{n+m}}{\delta J^{a_1}(t_1) \ldots \delta J^{a_n}(t_n) \delta J^{b_1}(t'_1) \ldots \delta J^{b_m}(t'_m)} Z[\xi, \chi, |\xi', \chi'|] \]

(3.26)

The two-point functions are readily computed from this expression. Generally, they are as would be expected from standard QM with the exception of
the following

\[ \langle 0 | T[p(t)p(t')] | 0 \rangle = i \partial_t \partial_{t'} \Delta_F(t - t') - i \delta(t - t') \] (3.27)

\[ \langle 0 | \tilde{T}[p(t)p(t')] | 0 \rangle = -i \partial_t \partial_{t'} \Delta_D(t - t') + i \delta(t - t') \] (3.28)

(we have abbreviated \( \partial/\partial t \equiv \partial_t \)). Though we will not discuss these in detail here, these reflect the fact that, in histories theories, we do not have \( p(t) = \dot{q}(t) \) in general. Most relevant to the discussion here is the analysis in [10] in which the difference between velocity and momentum is determined by a ‘random external force’—which arises in the quantum analogue of a stochastic differential equation.

In histories—as opposed to single-time—quantum theory, we can define differentiation with respect to time independently of the dynamical evolution. This leads to the definition of a velocity operator that is independent of, and does not generally commute with the momentum operator [3]. In [15], probabilities for measurements that are extended in time are considered; class operators are constructed that are significantly different for momentum and velocity measurements\(^1\) and the scheme suggests that it may be possible to experimentally distinguish between the two.

4 The Decoherence Functional for the AHO

In this section we develop a perturbative method to extend the formalism of continuous-time histories to include interacting theories. We exemplify the construction in the case of the anharmonic oscillator with a quartic self-interaction, corresponding to a Hamiltonian

\[ H(p, q) = \frac{1}{2} p^2 + \frac{1}{2} \omega^2 q^2 + \frac{\lambda}{4!} q^4. \] (4.1)

The key to this development is the relationship between the decoherence functional and the CTP generating functional described in Section 2.2. Though we have stressed that the discrete-time expressions for the decoherence functional are to be considered the more fundamental, the role played by the CTP generating functional allows us to import the standard mathematical

\(^1\)although they coincide for large coarse-graining
techniques of perturbation theory, for which the underlying discrete-time expressions, and the path integral expressions that arise in the continuous-time limit are well-established. Thus we can, at this stage, work confidently with the functional expressions. Of course to be fully rigorous, one would want to start from the discrete-time expressions - say a discrete-time version of Eq. 4.7 below - but we feel that the results presented here are sufficiently rigorous for the purpose of this paper.

We work both in configuration space and phase space, and the main results of this section are the calculation of a perturbative expression for the distribution up to second order in the coupling constant in each case. This method can readily be extended to encompass more physically realistic situations as the functional techniques that we have adopted are well-defined for initial states other than the vacuum. We also briefly discuss the relationship between the decoherence functional and the CTP effective action.

From here on, a subscript ‘0’ will indicate a quantity referring to the free theory, i.e., for $\lambda = 0$.

### 4.1 The perturbative expansion on configuration space

The construction of the distribution and CTP generating functional for the SHO on configuration space is straightforward as we face no complications with non-commuting operators. The latter is simply derived from consideration of the two-point functions and is given by (see, e.g., [16])

$$Z_0[J, J'] = \exp \left\{ -\frac{i}{2} \int dt dt' \left( J(t) \Delta_F(t - t') J(t') + 2J(t) \Delta^+(t - t') J'(t) - J'(t) \Delta_D(t - t') J'(t') \right) \right\}. \quad (4.2)$$

The Green’s functions were given in Eqs. (3.23)-(3.25), and we note that the Wightman Green’s function is a solution to the SHO equation of motion.

The configuration space distribution will be given by the Fourier transform of this expression, i.e.,

$$d_0[q(\cdot), q'(\cdot)] = \int \mathcal{D}J \mathcal{D}J' e^{iq \cdot J - iq' \cdot J'} Z_0[J, J'] \quad (4.3)$$
and this is readily calculated to be

\[ d_0[q(\cdot), q'(\cdot)] = e^{iS[q] - iS^*[q']} \]  (4.4)

where \( S[q] \) is the classical action calculated on the path \( q(t) \)

\[ S[q] = \frac{1}{2} \int dt (\dot{q}(t)^2 - \omega^2 q(t)^2) = \frac{1}{2} \int dt \Delta(t-t') \Delta(t-t') \]  (4.5)

The complex conjugation reflects the anti-time ordering on the \( q' \) path. This result is exactly as we would expect. It is the decoherence functional defined on a pair of fine-grained configuration space paths (see, eg. [6] where the same result is derived by considering class operators made up of strings of Heisenberg picture projection operators onto regions of configuration space).

Given a family of commuting, self-adjoint operators \( \hat{A}^i \), subsets of \( \Pi_{CS} \)—the space of configuration space histories—will correspond to histories of the observables \( \hat{A}^i \). If we consider two such subsets \( C \) and \( D \), we can formally write the configuration space decoherence functional [10]

\[ d(C, D) = \int \mathcal{D}q \mathcal{D}q' \ d[q(\cdot), q'(\cdot)] \chi_C[q(\cdot)] \chi_D[q'(\cdot)] \]  (4.6)

where \( \chi_C \) is the characteristic function associated with the subset \( C \subset \Pi_{CS} \).

In the case of the AHO with a quartic self-interaction, the CTP generating functional will be given by

\[ Z[J, J'] = \mathcal{N} \exp \left\{ -\frac{i\lambda}{4!} \int dt \left( \frac{\delta^4}{\delta J(t)^4} - \frac{\delta^4}{\delta J'(t)^4} \right) \right\} Z_0[J, J'] \]  (4.7)

The normalisation condition described in Section 2.2 above, i.e., \( Z[0, 0] = 1 \), is equivalent to the cancelling of vacuum diagrams from this series.

The configuration space distribution for the interacting theory is given by the Fourier transform of \( Z[J, J'] \), ie:

\[ d[q(\cdot), q'(\cdot)] = \int \mathcal{D}J \mathcal{D}J' \ e^{iJ \cdot q - iJ' \cdot q'} Z[J, J'] \]  (4.8)

The result is most clearly expressed in an exponential form, ie. \( d[q(\cdot), q'(\cdot)] = e^{iS[q(\cdot), q'(\cdot)]} \). (This is analogous to the situation in standard field theory when
one works with the generating functional for the connected Green’s functions, \( W[J] = -i \ln Z[J] \). After some lengthy, but relatively straightforward calculation, we arrive at the following perturbative expression for \( \tilde{S}[q(\cdot), q'(\cdot)] \):

\[
i\tilde{S}[q(\cdot), q'(\cdot)] = i\tilde{S}_0[q(\cdot), q'(\cdot)] - \frac{i\lambda}{4!} \int dt \left( q(t)^4 - q'(t)^4 \right) + \frac{1}{2} \left( -\frac{i\lambda}{4!} \right)^2 \int dt dt' \left( 192 iq(t)\Delta^+(t-t')^3q'(t') + 144q(t)^2\Delta^+(t-t')^2q'(t')^2 -32iq(t)^3\Delta^+(t-t')q'(t')^3 \right) + O(\lambda^3),
\]

in which \( \tilde{S}_0[q(\cdot), q'(\cdot)] = S[q] - S^*[q'] \). This is the main result of this part of the current work, and in the next section we shall see that the result for the phase space distribution is essentially the same. These expressions can then be used in Eq. (4.6) (for configuration space) and Eq. (2.7) (for phase space), along with suitable coarse-grainings, in order to determine a perturbative expression for the decoherence functional of coarse-grained histories of the AHO.

Writing the decoherence functional in this manner raises the interesting question of how \( \tilde{S}[q, q'] \) is related to the CTP effective action. The latter is defined by a double Legendre transform of the generating functional of connected diagrams \( W[J, J'] = -i \ln Z[J, J'] \)

\[
\Gamma_{CTP}[\bar{q}, \bar{q}'] = W[J, J'] - J \cdot \bar{q} + J' \cdot \bar{q}',
\]

where the sources \((J, J')\) are considered as functionals of the background fields \((\bar{q}, \bar{q}')\), which are, in turn, defined as \( \bar{q} = \frac{\partial W[J, J']}{\partial J} \), \( \bar{q}' = -\frac{\partial W[J, J']}{\partial J'} \) [16].

In [17] it is conjectured that the decoherence functional is defined by the tree-level CTP effective action \( d[q, q'] = e^{\Gamma_{CTP}[\bar{q}, \bar{q}']} \). We can now show that, to \( O(\lambda) \) at least, this is indeed the case, as the CTP effective action for the SHO is given by

\[
\Gamma_{CTP}[q, q'] = \tilde{S}_0(q, q') - \frac{\lambda}{4!} \int dt \left[ q(t)^4 - q'(t)^4 + \Delta_F(0)q(t)^2 + \Delta_D(0)q'(t)^2 \right] + O(\lambda^2)
\]

and we ignore the last two, ‘one-loop’ terms.
4.2 The perturbative expansion on phase space

In Section 3 we discussed in some detail the construction of the phase space distribution for continuous-time histories. In the infinite time limit, the Q and Wigner representations coincided, given by

\[ W_0[q(\cdot), p(\cdot)|q'(\cdot), p'(\cdot)] = \exp \left\{ iS[q(t), p(t)] - iS[q'(t), p'(t)] \right\}. \quad (4.12) \]

The corresponding CTP generating functional was given in Section 3.3. The phase space CTP generating functional for the interacting theory will be given by

\[ Z[\xi(\cdot), \chi(\cdot)|\xi'(\cdot), \chi'(\cdot)] = \mathcal{N} \exp \left\{ -\frac{i\lambda}{4!} \int dt \left( \frac{\delta^4}{\delta \xi(t)^4} - \frac{\delta^4}{\delta \xi'(t)^4} \right) \right\} Z_0[\xi(\cdot), \chi(\cdot)|\xi'(\cdot), \chi'(\cdot)] \quad (4.13) \]

and, once more, the distribution will be given by the Fourier transform of the resulting expression,

\[ W[q(\cdot), p(\cdot)|q'(\cdot), p'(\cdot)] = \int \mathcal{D}\xi \mathcal{D}\chi \mathcal{D}\xi' \mathcal{D}\chi' e^{iq\xi + ip\chi - iq'\xi' - ip'\chi'} Z[\xi(\cdot), \chi(\cdot)|\xi'(\cdot), \chi'(\cdot)]. \quad (4.14) \]

These calculations are greatly simplified if we transform to new coordinates \( u(t) = \xi(t) - \chi(t) \) and \( v(t) = \chi(t) \), and likewise for the primed quantities. Defining

\[ W[q(\cdot), p(\cdot)|q'(\cdot), p'(\cdot)] = e^{i\tilde{S}[q,p|q',p']} \quad (4.15) \]

with:

\[ W_0[q(\cdot), p(\cdot)|q'(\cdot), p'(\cdot)] = e^{i\tilde{S}_0[q,p|q',p']} \equiv e^{iS[q(t),p(t)] - iS[q'(t),p'(t)]}, \quad (4.16) \]

the result, up to second order in \( \lambda \), is given by:

\[ W[q(\cdot), p(\cdot)|q'(\cdot), p'(\cdot)] = \exp \left\{ -\frac{i\lambda}{4!} \int dt \left( q(t)^4 - q'(t)^4 \right) \right\} \]

\[ + \frac{1}{2} \left( -\frac{i\lambda}{4!} \right)^2 \int dt dt' \left( 192i q(t) \Delta^+(t - t')^3 q'(t') + 144 q(t)^2 \Delta^+(t - t')^2 q'(t')^2 \right. \]

\[ - 32i q(t)^3 \Delta^+(t - t')q'(t')^3) + 0(\lambda^3) \right\} W_0[q(\cdot), p(\cdot)|q'(\cdot), p'(\cdot)]. \quad (4.17) \]
We can see that the relationship between the CTP generating functional and the decoherence functional provides a powerful tool for the implementation of perturbative techniques in the histories formalism. We have derived an expression to second order for the phase space distribution. This is important in two ways. First, because we have demonstrated an effective, generic method for dealing with interacting histories theories that can be extended to any system of interest e.g. $\phi^4$ theory, QED. One just starts from the construction of the CTP generating functional. Second, because the result is in terms of a distribution on phase space, we know how to implement coarse-grainings pertaining to a wide class of interesting systems. This could be of immense use, for example, in the discussion of how hydrodynamic variables and their equations of motion emerge from the underlying quantum theory.

5 Conclusion

The aims of this paper have been twofold: to complete the phase space picture of quantum mechanical histories and to demonstrate a method for the implementation of perturbation theory in the histories formalism.

We have studied the simpler, quantum mechanical systems of the simple harmonic oscillator (SHO) the anharmonic oscillator (AHO) with a quartic potential. Apart from the insights gained in the present work, these also act as useful pointers to the form that their field-theoretic analogues may take - the AHO is the natural quantum mechanical analogue of the self-interacting scalar field with a $\phi^4$ potential term. In fact, as we shall show in subsequent work, the results contained herein generalise in a straightforward manner - modulo renormalization - to field theory.

The analysis of the different phase space representations complements the work in [9] and [10]. Although there exists a continuous infinity of maps between phase space $c$-number functions and Hilbert space operators, the most commonly encountered ones are the Wigner representation and a mapping based on coherent states. We have explicitly compared these in the histories formalism, for a general initial state, at both the discrete-time and continuous-time levels for the case of the SHO. Ultimately it is the former that are the most important, as all expressions, (eg., functional integrals
involving the continuous-time expressions) must be understood as a suitable limit of the mathematically well-defined discrete-time expressions. One should emphasise that proper implementation of coarse-graining operations (which is an essential part of the consistent histories programme) relies on the proper discrete-time expression for the histories. We concluded that the Q representation was the most appropriate for phase space histories, chiefly because it allows for a ‘Markov’ property for the distribution.

Finally, we demonstrated the construction of both the configuration space and phase space decoherence functional of the AHO. We made use of the fact that the decoherence functional is related to the closed-time-path (CTP) generating functional via a Fourier transform, and that the latter has a well-defined perturbative expansion.

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Appendix

The continuum limit of the Wigner representation of the decoherence functional

We are calculating the continuum limit of Eq. (3.17). We take ‘$n, m$’ even. The first exponent is rewritten as

\[-2 \sum_{i=1}^{n} |\alpha_i|^2 + 4 \sum_{i=1}^{n-1} \sum_{j=1}^{i} (-1)^{j+1} e^{-i\omega(t_{i+1} - t_{i})} \alpha_{i+1} \alpha_{i+1-j}^* \]

and, following the derivation in [14], this can be shown to equal

\[2 \sum_{i=1,3}^{n-1} [\alpha_{i+1} e^{-i\omega t_{i+1}} - \alpha_i e^{i\omega t_i}] - \alpha_{i+1} e^{i\omega t_{i+1}} (\alpha_{i+1} e^{-i\omega t_{i+1}} - \alpha_{i}^* e^{-i\omega t_i}) - 4 \sum_{i=1,3}^{n-1} (\alpha_{i+1} e^{-i\omega t_{i+1}} - \alpha_i^* e^{-i\omega t_i}) \sum_{j=i+1,3}^{n-2} (\alpha_{j+2} e^{i\omega t_{j+2}} - \alpha_{j+1} e^{i\omega t_{j+1}}).\]
If we now take $n$ large, and thus write $t_{i+1} - t_i = \delta t$, we can rewrite this as

\[
2 \sum_{i=1,3}^{n-1} \left[ \alpha_i^* (\alpha_{i+1} e^{i\omega \delta t} - \alpha_i) - \alpha_{i+1} (\alpha_i^* - \alpha_i^* e^{i\omega \delta t}) \right] - 4 \sum_{i=1,3}^{n-1} e^{-i\omega t_i} (\alpha_{i+1} e^{-i\omega \delta t} - \alpha_i^* \sum_{j=i+1,i+3}^{n-2} e^{i\omega t_{j+1}} (\alpha_{j+2} e^{i\omega \delta t} - \alpha_{j+1})).
\]

Following the same prescription for taking the continuum limit as in Section 3.1, and noting that, since the time steps are in two’s, we will pick up a factor of $1/2$ for each \( \int \), we arrive at

\[
\int_{t_1}^{t_n} dt \left( \alpha^* (t) \dot{\alpha}(t) - \alpha(t) \dot{\alpha}^*(t) + 2i\omega \alpha^*(t) \alpha(t) \right) - \int_{t_1}^{t_n} dt \int_{t_1}^{t_n} dt' e^{-i\omega(t-t')} (\dot{\alpha}^*(t) - i\omega \alpha^*(t)) (\dot{\alpha}(t') + i\omega \alpha(t')).
\]

In the second exponent of Eq. (3.17), ie. the boundary terms, we note that the first term vanishes as we have taken ‘n’ to be even. We can rewrite the second term (and likewise the third term)

\[
2\beta \sum_{i=1}^{n-1} (-1)^{-i} \alpha_i e^{-i\omega t_i} = 2\beta \sum_{i=1,3}^{n-1} (\alpha_{i+1} e^{-i\omega t_{i+1}} - \alpha_i^* e^{-i\omega t_i}).
\]

Thus the second exponent, in the continuum limit is given by

\[
\beta \int_{t_1}^{t_n} dt e^{-i\omega t} (\dot{\alpha}^*(t) - i\omega \alpha^*(t)) - \beta^* \int_{t_1}^{t_n} dt e^{i\omega t} (\dot{\alpha}(t) + i\omega \alpha(t))
\]

We thus arrive at

\[
W_{\text{Wigner}}[\alpha(\cdot), \alpha^*(\cdot)] = \exp \left\{ \int_{t_1}^{t_n} dt \left[ \alpha^* (t) \dot{\alpha}(t) - \alpha(t) \dot{\alpha}^*(t) + 2i\omega \alpha^*(t) \alpha(t) \right] - \int_{t_1}^{t_n} dt \int_{t_1}^{t_n} dt' e^{-i\omega(t-t')} (\dot{\alpha}^*(t) - i\omega \alpha^*(t)) (\dot{\alpha}(t') + i\omega \alpha(t')) \right\} \times \exp \left\{ \beta \int_{t_1}^{t_n} dt e^{-i\omega t} (\dot{\alpha}^*(t) - i\omega \alpha^*(t)) - \beta^* \int_{t_1}^{t_n} dt e^{i\omega t} (\dot{\alpha}(t') + i\omega \alpha(t')) \right\}.
\]

This expression is significantly refined using integration by parts and collecting boundary terms to give

\[
W_{\text{Wigner}}[\alpha(\cdot), \alpha^*(\cdot)] = \exp \left\{ \frac{1}{2} |\alpha_1|^2 - \frac{1}{2} |\alpha_n|^2 - \frac{1}{2} (\alpha_{n}(t_1) \alpha_n(t_n) - \int_{t_1}^{t_n} dt (\alpha(t) \dot{\alpha}^*(t) - i\omega \alpha^*(t) \alpha(t))) \right\}.
\]

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