A bound on the number of edges in graphs without an even cycle

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Abstract

We show that, for each fixed \( k \), an \( n \)-vertex graph not containing a cycle of length \( 2k \) has at most \( 80 \sqrt{k \log k} \cdot n^{1+1/k} + O(n) \) edges.

Introduction

Let \( \text{ex}(n, F) \) be the largest number of edges in an \( n \)-vertex graph that contains no copy of a fixed graph \( F \). The first systematic study of \( \text{ex}(n, F) \) was started by Turán [16], and now it is a central problem in extremal graph theory (see surveys [14, 9]).

The function \( \text{ex}(n, F) \) exhibits a dichotomy: if \( F \) is not bipartite, then \( \text{ex}(n, F) \) grows quadratically in \( n \), and is fairly well understood. If \( F \) is bipartite, \( \text{ex}(n, F) \) is subquadratic, and for very few \( F \) the order of magnitude is known. The two simplest classes of bipartite graphs are complete bipartite graphs, and cycles of even length. Most of the study of \( \text{ex}(n, F) \) for bipartite \( F \) has been concentrated on these two classes. In this paper, we address the even cycles. For an overview of the status of \( \text{ex}(n, F) \) for complete bipartite graphs see [2]. For a thorough survey on bipartite Turán problems see [8].

The first bound on the problem is due to Erdős[5] who showed that \( \text{ex}(n, C_4) = \Theta(n^{3/2}) \). Thanks to the works of Erdős and Rényi [6], Brown [4, Section 3], and Kövari, Sós and Turán [10] it is now known that

\[
\text{ex}(n, C_4) = (1/2 + o(1))n^{3/2}.
\]

The best current bound for \( \text{ex}(n, C_6) \) for large values of \( n \) is

\[
0.5338n^{4/3} < \text{ex}(n, C_6) \leq 0.6272n^{4/3}
\]
due to Füredi, Naor and Verstraëte [7].

A general bound of \( \text{ex}(n, C_{2k}) \leq \gamma_k n^{1+1/k} \), for some unspecified constant \( \gamma_k \), was asserted by Erdős. The first proof was by Bondy and Simonovits [3], who showed that \( \text{ex}(n, C_{2k}) \leq 20kn^{1+1/k} \) for all sufficiently large \( n \). This was improved by Verstraëte [17] to \( 8(k-1)n^{1+1/k} \) and by Pikhurko [13] to \( (k-1)n^{1+1/k} + O(n) \). The principal result of the present paper is an improvement of these bounds:

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Main Theorem. Suppose $G$ is an $n$-vertex graph that contains no $C_{2k}$, and $n \geq (2k)^{8k^2}$ then
\[
\text{ex}(n, C_{2k}) \leq 80\sqrt{k \log k \cdot n^{1+1/k}} + 10k^2n.
\]

It is our duty to point out that the improvement offered by the Main Theorem is of uncertain value because we still do not know if $\Theta(n^{1+1/k})$ is the correct order of magnitude for $\text{ex}(n, C_{2k})$. Only for $k = 2, 3, 5$ constructions of $C_{2k}$-free graphs with $\Omega(n^{1+1/k})$ edges are known [1, 18, 11, 12]. The first author believes it to be likely that $\text{ex}(n, C_{2k}) = o(n^{1+1/k})$ for all large $k$. We stress again that the situation is completely different for odd cycles, where the value of $\text{ex}(n, C_{2k+1})$ is known exactly for all large $n$ [15].

Proof method and organization of the paper Our proof is inspired by that of Pikhurko [13]. Apart from a couple of lemmas that we quote from [13], the proof is self-contained. However, we advise the reader to at least skim [13] to see the main idea in a simpler setting.

Pikhurko’s proof builds a breadth-first search tree, and then argues that a pair of adjacent levels of the tree cannot contain a $\Theta$-graph\footnote{We recall the definition of a $\Theta$-graph in Section 2}. It is then deduced that each level must be at least $\delta/(k-1)$ times larger than the previous, where $\delta$ is the (minimum) degree. The bound on $\text{ex}(n, C_{2k})$ then follows. The estimate of $\delta/(k-1)$ is sharp when one restricts one’s attention to a pair of levels.

In our proof, we use three adjacent levels. We find a $\Theta$-graph satisfying an extra technical condition that permits an extension of Pikhurko’s argument. Annoyingly, this extension requires a bound on the maximum degree. To achieve such a bound we use a modification of breadth-first search that avoids the high-degree vertices.

What we really prove in this paper is the following:

**Theorem 1.** Suppose $k \geq 4$, and suppose $G$ is a bipartite $n$-vertex graph of minimum degree at least $2d + 5k^2$, where
\[
d \geq \max\left(20\sqrt{k \log k \cdot n^{1/k}}, (2k)^{8k}\right),
\]
then $G$ contains $C_{2k}$.

The Main Theorem follows from Theorem 1 and two well-known facts: every graph contains a bipartite subgraph with half of the edges, and every graph of average degree $d_{\text{avg}}$ contains a subgraph of minimum degree at least $d_{\text{avg}}/2$.

The rest of the paper is organized as follows. We present our modification of breadth-first search in Section 1. In Section 2, which is the heart of the paper, we explain how to find $\Theta$-graphs in triples of consecutive levels. Finally, in Section 3 we assemble the pieces of the proof.

1 Graph exploration

Our aim is to have vertices of degree at most $\Delta d$ for some $k \ll \Delta \ll d^{1/k}$. The particular choice is fairly flexible; we choose to use $\Delta \overset{\text{def}}{=} k^3$. 
Let $G$ be a graph, and let $x$ be any vertex of $G$. We start our exploration with the set $V_0 = \{x\}$, and mark the vertex $x$ as explored. Suppose $V_0, V_1, \ldots, V_{i-1}$ are the sets explored in the 0th, 1st, \ldots, $(i-1)$st steps respectively. We then define $V_i$ as follows:

1. Let $V_i'$ consist of those neighbors of $V_{i-1}$ that have not yet been explored. Let $B_{g_i}$ be the set of those vertices in $V_i'$ that have more than $\Delta d$ unexplored neighbors, and let $S_{m_i} = V_i' \setminus B_{g_i}$.

2. Define
   \[ V_i = \begin{cases} V_i' & \text{if } |B_{g_i}| > \frac{1}{k+1}|V_i'|, \\ S_{m_i} & \text{if } |B_{g_i}| \leq \frac{1}{k+1}|V_i'|. \end{cases} \]

The vertices of $V_i$ are then marked as explored.

We call sets $V_0, V_1, \ldots$ *levels* of $G$. A level $V_i$ is *big* if $|B_{g_i}| > \frac{1}{k+1}|V_i'|$, and is *normal* otherwise.

**Lemma 2.** If $\delta \leq \Delta d$, and $G$ is a bipartite graph of minimum degree $\delta$, then each $v \in V_{i+1}$ has at least $\delta$ neighbors in $V_i \cup V_i'$.

**Proof.** Fix a vertex $v \in V(G)$. We will show, by induction on $i$, that if $v \notin V_1 \cup \cdots \cup V_i$, then $v$ has at least $\delta$ neighbors in $V(G) \setminus (V_i \cup \cdots \cup V_{i-1})$. The base case $i = 1$ is clear. Suppose $i > 1$. If $v \in B_{g_i}$, then $v$ has $\Delta d \geq \delta$ neighbors in the required set. Otherwise, $v$ is not in $V_i'$ and hence has no neighbors in $V_{i-1}$. Hence, $v$ has as many neighbors in $V(G) \setminus (V_i \cup \cdots \cup V_{i-1})$ as in $V(G) \setminus (V_1 \cup \cdots \cup V_{i-2})$, and our claim follows from the induction hypothesis.

If $v \in V_{i+1}$, then the neighbors of $v$ are a subset of $V_1 \cup \cdots \cup V_i \cup V_i'$. Hence, at least $\delta$ of these neighbors lie in $V_i \cup V_{i+1}$.

**Trilayered graphs** A *trilayered graph* with layers $V_1, V_2, V_3$ is a graph $G$ on a vertex set $V_1, V_2, V_3$ such that the only edges in $G$ are between $V_1$ and $V_2$, and between $V_2$ and $V_3$. If $V_1' \subset V_1$, $V_2' \subset V_2$ and $V_3' \subset V_3$, then we denote by $G[V_1', V_2', V_3']$ the trilayered subgraph induced by three sets $V_1', V_2', V_3'$. Any three sets $V_{i-1}, V_i, V_{i+1}$ from the exploration process naturally form a trilayered graph; these graphs and their subgraphs are the only trilayered graphs that appear in this paper.

We say that a trilayered graph has *minimum degree* at least $[A : B, C : D]$ if each vertex in $V_1$ has at least $A$ neighbors in $V_2$, each vertex in $V_2$ has at least $B$ neighbors in $V_1$, each vertex in $V_2$ has at least $C$ neighbors in $V_3$, and each vertex in $V_3$ has at least $D$ neighbors in $V_2$. A schematic drawing of such a graph is on the right.

**2 Θ-graphs**

A *Θ-graph* is a cycle of length at least $2k$ with a chord. We shall use several lemmas from the previous works.

**Lemma 3** (Lemma 2.1 in [13], also Lemma 2 in [17]). Let $F$ be a Θ-graph and $1 \leq l \leq |V(F)| - 1$. Let $V(F) = W \cup Z$ be an arbitrary partition of its vertex set into two non-empty parts such that every path in $F$ of length $l$ that begins in $W$ necessarily ends in $W$. Then $F$ is bipartite with parts $W$ and $Z$. 

3
**Lemma 4** (Lemma 2.2 in [13]). Let \( k \geq 3 \). Any bipartite graph \( H \) of minimum degree at least \( k \) contains a \( \Theta \)-graph.

**Corollary 5.** Let \( k \geq 3 \). Any bipartite graph \( H \) of average degree at least \( 2k \) contains a \( \Theta \)-graph.

For a graph \( G \) and a set \( Y \subset V(G) \) let \( G[Y] \) denote the graph induced on \( Y \). For disjoint \( Y, Z \subset V(G) \) let \( G[Y, Z] \) denote the bipartite subgraph of \( G \) that is induced by the bipartition \( Y \cup Z \).

Suppose \( G \) is a trilayered graph with layers \( V_1, V_2, V_3 \). We say that a \( \Theta \)-graph \( F \subset G \) is well-placed if each vertex of \( V(F) \cap V_2 \) is adjacent to some vertex in \( V_1 \setminus V(F) \).

**Lemma 6.** Suppose \( G \) is a trilayered graph with layers \( V_1, V_2, V_3 \) such that the degree of every vertex in \( V_2 \) is between \( 2d + 5k^2 \) and \( \Delta d \). Suppose \( t \) is a nonnegative integer, and let \( F = \frac{t \cdot e(V_1, V_2)}{20|V_2|} \). Assume that

\[
\begin{align*}
a) & \quad F \geq 2, \\
b) & \quad e(V_1, V_2) \geq 2kF|V_1|, \\
c) & \quad e(V_1, V_2) \geq 8k(t + 1)^2(2\Delta k)^{2k-1}|V_1|, \\
d) & \quad e(V_1, V_2) \geq 8(t/F)^t|k|V_2|, \\
e) & \quad e(V_1, V_2) \geq 20(t + 1)^2|V_2|.
\end{align*}
\]

Then at least one of the following holds:

I) There is a \( \Theta \)-graph in \( G[V_1, V_2] \).

II) There is a well-placed \( \Theta \)-graph in \( G[V_1, V_2, V_3] \).

The proof of Lemma 6 is in two parts: finding trilayered subgraph of large minimum degree (Lemmas 7 and 8), and finding a well-placed \( \Theta \)-graph inside that trilayered graph (Lemma 9).

**Finding a trilayered subgraph of large minimum degree**  The disjoint union of two bipartite graphs shows that a trilayered graph with many edges need not contain a trilayered subgraph of large minimum degree. We show that, in contrast, if a trilayered graph contains no \( \Theta \)-graph between two of its levels, then it must contain a subgraph of large minimum degree:

**Lemma 7.** Let \( a, B, C, D \) be positive real numbers. Suppose \( G \) is a trilayered graph with layers \( V_1, V_2, V_3 \) and the degree of every vertex in \( V_2 \) is at least \( d + 4k^2 + C \). Assume also that

\[
a \cdot e(V_1, V_2) \geq (A + k + 1)|V_1| + B|V_2|.
\]

Then one of the following holds:

I) There is a \( \Theta \)-graph in \( G[V_1, V_2] \).

II) There exist non-empty subsets \( V_1' \subset V_1, V_2' \subset V_2, V_3' \subset V_3 \) such that the induced trilayered subgraph \( G[V_1', V_2', V_3'] \) has minimum degree at least \( [A : B : C : D] \).

III) There is a subset \( \tilde{V}_2 \subset V_2 \) such that \( e(V_1, \tilde{V}_2) \geq (1 - a)e(V_1, V_2) \), and \( |\tilde{V}_2| \leq D|V_3|/d \).
Proof. We suppose that alternative (I) does not hold. Then, by Corollary 5, the average degree of every subgraph of $G[V_1, V_2]$ is at most $2k$.

Consider the process that aims to construct a subgraph satisfying (II). The process starts with $V_1' = V_1$, $V_2' = V_2$, and $V_3' = V_3$, and at each step removes one of the vertices that violate the minimum degree condition on $G[V_1', V_2', V_3']$. The process stops when either no vertices are left, or the minimum degree of $G[V_1', V_2', V_3']$ is at least $[A : B, C : D]$. Since in the latter case we are done, we assume that this process eventually removes every vertex of $G$.

Let $R$ be the vertices of $V_2$ that were removed because at the time of removal they had fewer than $C$ neighbors in $V_3'$. Put

$$E' \overset{\text{def}}{=} \{uv \in E(G) : u \in V_2, v \in V_3, \text{ and } v \text{ was removed before } u\},$$

$$S \overset{\text{def}}{=} \{v \in V_2 : v \text{ has at least } 4k^2 \text{ neighbors in } V_1\}.$$

Note that $|E'| \leq D|V_3|$. We cannot have $|S| \geq |V_1|/k$, for otherwise the average degree of the bipartite graph $G[V_1, S]$ would be at least $\frac{4k^2}{1+1/k} \geq 2k$. So $|S| \leq |V_1|/k$.

The average degree condition on $G[V_1, S]$ implies that

$$e(V_1, S) \leq k(|V_1| + |S|) \leq (k+1)|V_1|.$$ 

Let $u$ be any vertex in $R \setminus S$. Since it is connected to at least $d + C$ vertices of $V_3$, it must be adjacent to at least $d$ edges of $E'$. Thus,

$$|R \setminus S| \leq |E'|/d \leq D|V_3|/d.$$ 

Assume that the conclusion (III) does not hold with $\tilde{V}_2 = R \setminus S$. Then $e(V_1, R \setminus S) < (1-a)e(V_1, V_2)$. Since the total number of edges between $V_1$ and $V_2$ that were removed due to the minimal degree conditions on $V_1$ and $V_2$ is at most $A|V_1|$ and $B|V_2|$ respectively, we conclude that

$$e(V_1, V_2) \leq e(V_1, S) + e(V_1, R \setminus S) + A|V_1| + B|V_2|,$$

$$< (k+1)|V_1| + (1-a)e(V_1, V_2) + A|V_1| + B|V_2|,$$

$$a \cdot e(V_1, V_2) < (A + k + 1)|V_1| + B|V_2|.$$ 

The contradiction completes the proof. \hfill \Box

Remark. The preceding lemma by itself is sufficient to prove the estimate $\max(n, C_{2k}) = O(k^{2/3}n^{1+1/k})$. For that, one chooses approximately $B = k^{2/3}$, $D = k^{1/3}$ and $a = 1/2$. One can then show that when applied to trilayered graphs arising from the exploration process the alternative (III) leads to a subgraph of average degree $2k$. The two remaining alternatives are dealt by Corollary 5 and Lemma 9. However, it is possible to obtain a better bound by iterating the preceding lemma.

Lemma 8. Let $C$ be a positive real number. Suppose $G$ is a trilayered graph with layers $V_1, V_2, V_3$, and the degree of every vertex in $V_2$ is at least $d + 4k^2 + C$. Let $F = \frac{d + 4k^2}{8k|V_3|}$, and assume that $F$ and $e(V_1, V_2)$ satisfy (2). Then one of the following holds:

1) There is a $\Theta$-graph in $G[V_1, V_2]$. 

5
II) There exist numbers $A, B, D$ and non-empty subsets $V_1' \subset V_1, V_2' \subset V_2, V_3' \subset V_3$ such that the induced trilayered subgraph $G[V_1', V_2', V_3']$ has minimum degree at least $[A : B, C : D]$, with the following inequalities that bind $A, B, \text{ and } D$:

$$B \geq 5, \quad (B - 4)D \geq 2k,$$

$$A \geq 2k(\Delta D)^{D-1}. \quad (4)$$

*Proof.* Assume, for the sake of contradiction, that neither (I) nor (II) hold. With hindsight, set $a_j = \frac{1}{t-j+1}$ for $j = 0, \ldots, t-1$. We shall define a sequence of sets $V_2 = V_2^{(0)} \supseteq V_2^{(1)} \supseteq \cdots \supseteq V_2^{(t)}$ inductively. We denote by

$$d_i \overset{\text{def}}{=} \frac{e(V_1, V_2^{(i)})}{|V_2^{(i)}|}$$

the average degree from $V_2^{(i)}$ into $V_1$. The sequence $V_2^{(0)}, V_2^{(1)}, \ldots, V_2^{(t)}$ will be constructed so as to satisfy

$$e(V_1, V_2^{(i+1)}) \geq (1 - a_i)e(V_1, V_2^{(i)}), \quad (5)$$

$$d_{i+1} \geq d_i \cdot Fa_i \prod_{j=0}^{i} (1 - a_j). \quad (6)$$

Note that (5) and the choice of $a_0, \ldots, a_i$ imply that

$$e(V_1, V_2^{(i)}) \geq \frac{1}{t+1} e(V_1, V_2). \quad (7)$$

The sequence starts with $V_2^{(0)} = V_2$. Assume $V_2^{(i)}$ has been defined. We proceed to define $V_2^{(i+1)}$. Put

$$A = a_i e(V_1, V_2^{(i)})/2|V_1| - k - 1,$$

$$B = a_i d_i / 4 + 5,$$

$$D = \min(2k, 8k/a_id_i).$$

With help of (7) and (2c) it is easy to check that the inequalities (4) hold for this choice of constants.

In addition,

$$(A + k + 1)|V_1| + B|V_2^{(i)}| = \frac{3}{4} a_i e(V_1, V_2^{(i)}) + 5|V_2^{(i)}|$$

$$\leq \frac{3}{4} a_i e(V_1, V_2^{(i)}) + \frac{1}{4(t+1)^2} e(V_1, V_2) \quad (2c)$$

$$\leq a_i e(V_1, V_2^{(i)}). \quad (7)$$

So, the condition (3) of Lemma 7 is satisfied for the graph $G[V_1, V_2^{(i)}, V_3]$. By Lemma 7 there is a subset $V_2^{(i+1)} \subset V_2^{(i)}$ satisfying (5) and

$$|V_2^{(i+1)}| \leq D|V_3|/d.$$
Next we show that the set \( V^{(i+1)}_2 \) satisfies inequality (6). Indeed, we have

\[
d_{i+1} = \frac{e(V_1, V^{(i+1)}_2)}{|V_2|} \geq \frac{(1 - a_i)e(V_1, V^{(i)}_2)}{d|V_3|/d} = (1 - a_i)a_i d_i \frac{d}{8k|V_3|} e(V_1, V^{(i)}_2)
\]

\[
\geq (1 - a_i)a_i d_i \frac{d \cdot e(V_1, V_2)}{8k|V_3|} \prod_{j=0}^{i-1} (1 - a_j) = d_i \cdot F a_i \prod_{j=0}^{i} (1 - a_j).
\]

Iterative application of (6) implies

\[
d_i \geq d_0 F^t \prod_{j=0}^{t-1} a_j (1 - a_j)^{t-j} \geq d_0 F^t \prod_{j=0}^{t-1} e^{-1} \frac{1}{t - j + 1} \leq d_0 \frac{(F/e)^t}{(t + 1)!}.
\] (8)

If we have \( |V^{(t)}_2| < |V_1| \), then the average degree of induced subgraph \( G[V_1, V^{(t)}_2] \) is greater than \( e(V_1, V^{(t)}_2)/|V_1| \geq e(V_1, V_2)/(t + 1)|V_1| \geq 2k \), which by Corollary 5 leads to outcome (I).

If \( |V^{(t)}_2| \geq |V_1| \) and \( d_t \geq 4k \), then the average degree of \( G[V_1, V^{(t)}_2] \) is at least \( d_t/2 \geq 2k \), again leading to the outcome (I). So, we may assume that \( d_t < 4k \). Since \( (t + 1)! \leq 2t^t \) we deduce from (8) that

\[
d_0 \leq 4k(t + 1)!(e/F)^t \leq 8k(et/F)^t.
\]

This contradicts (2d), and so the proof is complete. □

Locating well-placed \( \Theta \)-graphs in trilayered graphs We come to the central argument of the paper. It shows how to embed well-placed \( \Theta \)-graphs into trilayered graphs of large minimum degree. Or rather, it shows how to embed well-placed \( \Theta \)-graphs into regular trilayered graphs; the contortions of the previous two lemmas, and the factor of \( \sqrt{\log k} \) in the final bound, come from authors’ inability to deal with irregular graphs.

**Lemma 9.** Let \( A, B, D \) be positive real numbers. Let \( G \) be a trilayered graph with layers \( V_1, V_2, V_3 \) of minimum degree at least \( [A, B, d + k, D] \). Suppose that no vertex in \( V_2 \) has more than \( \Delta d \) neighbors in \( V_3 \). Assume also that

\[
B \geq 5
\]

\[
(B - 4)D \geq 2k - 2
\]

\[
A \geq 2k(D - 1).
\]

Then \( G \) contains a well-placed \( \Theta \)-graph.

**Proof.** Assume, for the sake of contradiction, that \( G \) contains no well-placed \( \Theta \)-graphs. Leaning on this assumption we shall build an arbitrary long path \( P \) of the form
where, for each $i$, vertices $v_i$ and $v_{i+1}$ are joined by a path of length $2D$ that alternates between $V_2$ and $V_3$. Since the graph is finite, this would be a contradiction.

While building the path we maintain the following property:

Every $v \in P \cap V_2$ has at least one neighbor in $V_1 \setminus P$. (*)

We call a path satisfying (*) good.

We construct the path inductively. We begin by picking $v_0$ arbitrarily from $V_1$. Suppose a good path $P = v_0 \leftrightarrow v_1 \leftrightarrow \cdots \leftrightarrow v_{l-1}$ has been constructed, and we wish to find a path extension $v_0 \leftrightarrow v_1 \leftrightarrow \cdots \leftrightarrow v_{l-1} \leftrightarrow v_l$.

For each $i = 1, 2, \ldots, 2D - 1$ we shall define a family $Q_i$ of good paths that satisfy

1. Each path in $Q_i$ is of the form $v_0 \leftrightarrow v_1 \leftrightarrow \cdots \leftrightarrow v_{l-1} \leftrightarrow u$, where $v_{l-1} \leftrightarrow u$ is a path of length $i$ that alternates between $V_2$ and $V_3$. The vertex $u$ is called a terminal of the path. The set of terminals of the paths in $Q_i$ is denoted by $T(Q_i)$.

2. For each $i$, the paths in $Q_i$ have distinct terminals.

3. For odd-numbered indices, we have the inequality

$$|Q_{2i+1}| \geq -3k + A \left( \frac{1}{\Delta} \right)^i \prod_{j \leq i} \left( 1 - \frac{j}{D} \right).$$

4. For even-numbered indices, we have the inequality

$$e(T(Q_{2i}), V_2) \geq d|Q_{2i-1}|.$$

Let

$$t \overset{\text{def}}{=} \left\lceil \frac{B}{2} \right\rceil.$$

We will repeatedly use the following straightforward fact, which we call the small-degree argument: whenever $Q$ is a good path and $u \in V_2$ is adjacent to the terminal of $Q$, then the path $Qu$ is adjacent to fewer than $t$ vertices in $V_1 \cap Q$. Indeed, if vertex $u$ were adjacent to $v_{j_1}, v_{j_2}, \ldots, v_{j_t} \in V_1 \cap Q$, then $v_{j_2} \leftrightarrow u$ (along path $Q$) and the edge $uv_{j_2}$ would form a cycle of total length at least $2D(t-2) + 2 \geq 2D(B/2 - 2) + 2 \overset{(10)}{=} 2k$. As $uv_{j_1}$ is a chord of the cycle, and $u$ is adjacent to $v_{j_1}$ that is not on the cycle, that would contradict the assumption that $G$ contains no well-placed $\Theta$-graph.
The set $Q_1$ consists of all paths of the form $Pu$ for $u \in V_2 \setminus P$. Let us check that the preceding conditions hold for $Q_1$. Vertex $v_{l-1}$ cannot be adjacent to $k$ or more vertices in $P \cap V_2$, for otherwise $G$ would contain a well-placed $\Theta$-graph with a chord through $v_{l-1}$. So, $|Q_1| \geq A-k$. Next, consider any $u \in V_2 \setminus P$ that is a neighbor of $v_{l-1}$. By the small-degree argument vertex $u$ cannot be adjacent to $t$ or more vertices of $P \cap V_1$, and $Pu$ is good.

Suppose $Q_{2i-1}$ has been defined, and we wish to define $Q_{2i}$. Consider an arbitrary path $Q = v_0 \leftrightarrow v_1 \leftrightarrow \cdots \leftrightarrow v_{l-1} \leftrightarrow u \in Q_{2i-1}$. Vertex $u$ cannot have $k$ or more neighbors in $Q \cap V_3$, for otherwise $G$ would contain a well-placed $\Theta$-graph with a chord through $u$. Hence, there are at least $d$ edges of the form $uw$, where $w \in V_3 \setminus Q$. As we vary $u$ we obtain a family of at least $d|Q_{2i-1}|$ paths eligible for inclusion into $Q_{2i}$. We let $Q_{2i}$ consist of any maximal set of such paths with distinct terminals.

Suppose $Q_{2i}$ has been defined, and we wish to define $Q_{2i+1}$. Consider an arbitrary path $Q = v_0 \leftrightarrow v_1 \leftrightarrow \cdots \leftrightarrow v_{l-1} \leftrightarrow u \in Q_{2i}$. An edge $uw$ is called long if $w \in P$, and $w$ is at a distance exceeding $2k$ from $u$ along path $Q$. If $uw$ is a long edge, then from $u$ to $Q$ there is only one edge, namely the edges to the predecessor of $u$ on $Q$, for otherwise there is a well-placed $\Theta$-graph. Also, at most $i$ neighbors of $u$ lie on the path $v_{l-1} \leftrightarrow u$. Since deg $u \geq D$, it follows that at least $(1-i/D)$ deg $u$ short edges from $u$ that miss $v_{l-1} \leftrightarrow u$. Thus there is a set $W$ of at least $(1 - i/D)e(T(Q_{2i}), V_2)$ walks (not necessarily paths!) of the form $v_0 \leftrightarrow v_1 \leftrightarrow \cdots \leftrightarrow v_{l-1} \leftrightarrow uw$ such that $v_{l-1} \leftrightarrow uw$ is a path and $w$ occurs only among the last $2k$ vertices of the walk.

From the maximum degree condition on $V_2$ it follows that walks in $W$ have at least $(1 - i/D)e(T(Q_{2i}), V_2)/\Delta d$ distinct terminals. A walk fails to be a path only if the terminal vertex lies on $P$. However, since the edge $uw$ is short, this can happen for at most $2k$ possible terminals. Hence, there is a $Q_{2i+1} \subset W$ of size $|Q_{2i+1}| \geq (1 - i/D)e(T(Q_{2i}), V_2)/\Delta d - 2k$ that consists of paths with distinct terminals. It remains to check that every path in $Q_{2i+1}$ is good. The only way that $Q = v_0 \leftrightarrow \cdots \leftrightarrow v_{l-1} \leftrightarrow uw \in Q_{2i+1}$ may fail to be good is if $w$ has no neighbors in $V_1 \setminus Q$. By the small-degree argument $w$ has fewer than $t$ neighbors in $V_1$. Since $w$ has at least $B$ neighbors in $V_1$ and $B \geq t+2$, we conclude that $w$ has at least two neighbors in $V_1$ outside the path. Of course, the same is true for every terminal of a path in $Q_{2i+1}$.

Note that $Q_{2D-1}$ is non-empty. Let $Q = v_0 \leftrightarrow \cdots \leftrightarrow v_{l-1} \leftrightarrow u \in Q_{2D-1}$ be an arbitrary path. Note that since $2D-1$ is odd, $u \in V_2$. By the property of terminals of $V_i$ (odd $i$) that we noted in the previous paragraph, there are two vertices in $V_1 \setminus Q$ that are neighbors of $u$. Let $v_l$ be any of them, and let the new path be $Qv_l = v_0 \leftrightarrow \cdots \leftrightarrow v_{l-1} \leftrightarrow uv_l$. This path can fail to be good if there is a vertex $w$ on the path $Q$ that is good in $Q$, but is bad in $Qv_l$. By the small-degree argument, $w$ is adjacent to fewer than $t$ vertices in $Q \cap V_1$ that precede $w$ in $Q$. The same argument applied to the reversal of the path $Qv_l$ shows that $w$ is adjacent to fewer than $t$ vertices in $Q \cap V_1$ that succeeds $w$ in $Q$. Since $2t - 2 < B$, the path $Qv_l$ is good.

Hence, it is possible to build an arbitrarily long path in $G$. This contradicts the finiteness of $G$. □

Lemma 6 follows from Lemmas 8 and 9 by setting $C = d + k$, in view of inequality $4k^2 + k \leq 5k^2$. 

9
3 Proof of Theorem 1

Suppose $G$ has minimum degree of at least $2d + 4k^2 + k$ and contains no $C_{2k}$. Pick a root vertex $x$ arbitrarily, and let $V_0, V_1, \ldots, V_{k-1}$ be the levels obtained from the exploration process in Section 1.

**Lemma 10.** For $1 \leq i \leq k-1$, the graph $G[V_{i-1}, V_i, V_{i+1}]$ contains no well-placed $\Theta$-graph.

**Proof.** The following proof is almost an exact repetition of the proof of Claim 3.1 from [13] (which is also reproduced as Lemma 11 below).

Suppose, for the sake of contradiction, that a well-placed $\Theta$-graph $F \subset G[V_{i-1}, V_i, V_{i+1}]$ exists. Let $Y = V_i \cap V(F)$. Since $F$ is well-placed, for every vertex of $Y$ there is a path of length $i$ to the vertex $x$. The union of these paths forms a tree $T$ with $x$ as a root. Let $y$ be the vertex farthest from $x$ such that every vertex of $Y$ is a $T$-descendant of $y$. Paths that connect $y$ to $Y$ branch at $y$. Pick one such branch, and let $W \subset Y$ be the set of all the $T$-descendants of that branch. Let $Z = V(F) \setminus W$. From $Y \neq V_i \cap V(F)$ it follows that $Z$ is not an independent set of $F$, and so $W \cup Z$ is not a bipartition of $F$.

Let $\ell$ be the distance between $x$ and $y$. We have $\ell < i$ and $2k - 2i + 2\ell < 2k \leq |V(F)|$. By Lemma 3 in $F$ there is a path $P$ of length $2k - 2i + 2\ell$ that starts at some $w \in W$ and ends in $z \in Z$. Since the length of $P$ is even, $z \in Y$. Let $P_w$ and $P_z$ be unique paths in $T$ that connect $y$ to respectively $w$ and $z$. They intersect only at $y$. Each of $P_w$ and $P_z$ has length $i - \ell$. The union of paths $P, P_w, P_z$ forms a $2k$-cycle in $G$.

The same argument (with a different $Y$) also proves the next lemma.

**Lemma 11** (Claim 3.1 in [13]). For $1 \leq i \leq k-1$, neither of $G[V_i]$ and $G[V_i, V_{i+1}]$ contains a bipartite $\Theta$-graph.

The next step is to show that the levels $V_0, V_1, V_2, \ldots$ increase in size. We shall show by induction on $i$ that

$$e(V_i, V_{i+1}) \geq d|V_i|,$$

$$e(V_i, V_{i+1}) \leq 2k|V_{i+1}|,$$

$$e(V_i, V_{i+1}') \leq 2k|V_{i+1}'|,$$

$$|V_{i+1}| \geq (2k)^{-1}d|V_i|,$$

$$|V_{i+1}| \geq \frac{d^2}{400k \log k}|V_{i-1}|.$$

Clearly, these hold for $i = 0$ since each vertex of $V_1$ sends only one edge to $V_0$.

**Proof of (12):** By Lemma 2 the degree of every vertex in $V_i$ is at least $d + 3k + 1$, and so

$$e(V_i, V_{i+1}') \geq |V_i|(d + 3k + 1) - e(V_{i-1}, V_i) \geq (d + k + 1)|V_i|.$$

We next distinguish two cases depending on whether $V_{i+1}$ is big (in the sense of the definition from Section 1). If $V_{i+1}$ is big, then $e(V, V_{i+1}) = e(V, V_{i+1}')$, and (12) follows. If $V_{i+1}$ is normal, then Corollary 5 implies that

$$e(V_i, B_{g_i+1}) \leq k(|V_i| + |B_{g_i+1}|) \leq (k + 1)|V_i|$$
and so
\[ e(V_i, V_{i+1}) = e(V_i, V_{i+1}') - e(V_i, B_{i+1}) \geq d|V_i| \]
implying (12).

\[ \square \]

**Proof of (13):** Consider the graph $G[V_i, V_{i+1}]$. Inequality (12) asserts that the average degree of $V_i$ is at least $d \geq 2k$. If (13) does not hold, then the average degree of $V_{i+1}$ is at least $2k$ as well, contradicting Corollary 5.

\[ \square \]

**Proof for (14):** The argument is the same as for (13) with $G[V_i, V_{i+1}']$ in place of $G[V_i, V_{i+1}]$.

\[ \square \]

**Proof for (15):** This follows from (13) and (12).

\[ \square \]

**Proof of (16) in the case $V_i$ is a normal level:** We assume that (16) does not hold and will derive a contradiction. Consider the trilayered graph $G[V_{i-1}, V_i, V_{i+1}]$. Let $t = 2\log k$. Suppose momentarily that the inequalities (2) in Lemma 6 hold. Then since $V_i$ is normal, the degrees of vertices in $V_i$ are bounded from above by $\Delta d$, and so Lemma 6 applies. However, the lemma’s conclusion contradicts Lemmas 10 and 11. Hence, to prove (16) it suffices to verify inequalities (2a–d) with $F = d \cdot e(V_{i-1}, V_i)/8k|V_{i+1}'|$.

We may assume that
\[ F \geq 2e^2 \log k, \quad (17) \]
and in particular that (2a) holds. Indeed, if (17) were not true, then inequality (12) would imply $|V_{i+1}'| \geq (d^2/16e^2k\log k)|V_{i-1}|$, and thus
\[ |V_{i+1}| \geq (1 - \frac{1}{k})|V_{i+1}'| \geq (d^2/32e^2k\log k)|V_{i-1}|, \]
and so (16) would follow in view of $32e^2 \leq 400$.

Inequality (2b) is implied by (15). Indeed,
\[ e(V_{i-1}, V_i) = 8k|V_{i+1}|F/d \geq 4F|V_i| \geq 2k^{-1}dF|V_{i-1}|, \]
and $d \geq k^2$ by the definition of $d$ from (1).

Inequality (2c) is implied by (1) and (12).

Next, suppose (2d) were not true. Since $F/t \geq e^2$ by (17), we would then conclude
\[ |V_{i+1}| \geq (2k)^{-1}d|V_i| \geq (16k^2)^{-1}(F/et)^t e(V_{i-1}, V_i) \geq (16k^2)^{-1}e^2\log k e(V_{i-1}, V_i) \geq \frac{1}{16}d|V_{i-1}|, \]
and so (16) would follow.

Finally, (2e) is a consequence of (12).

\[ \square \]
Proof of (16) in the case $V_i$ is a big level: We have

$$|V_{i+1}| \geq \frac{1}{2} |V'_{i+1}| \overset{(14)}{=} (4k)^{-1} e(V_i, V'_{i+1}) \geq (4k)^{-1} e(Bg_i, V'_{i+1}) \geq (4k)^{-1} \Delta d|B_{gi}|$$

$$\geq (8k^2)^{-1} \Delta d|V_i| \overset{(15)}{\geq} (16k^3)^{-1} \Delta d|V_{i-1}| = \frac{1}{16} d|V_{i-1}|,$$

and so (16) holds.

We are ready to complete the proof of Theorem 1. If $k$ is even, then $\lfloor k/2 \rfloor$ applications of (16) yield

$$|V_k| \geq \frac{d^k}{(400k \log k)^{k/2}}.$$

If $k$ is odd, then $(k - 1)/2$ applications of (16) yield

$$|V_k| \geq \frac{d^{k-1}}{(400k \log k)^{(k-1)/2}} |V_1| \geq \frac{d^k}{(400k \log k)^{(k-1)/2}}.$$

Either way, since $|V_k| < n$ we conclude that $d < 20 \sqrt{k \log k} \cdot n^{1/k}$.

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12
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