NOTE ON THE TURÁN NUMBER OF THE LINEAR 3-GRAPH $C_{13}$

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Abstract. Let the crown $C_{13}$ be the linear 3-graph on 9 vertices \{a, b, c, d, e, f, g, h, i\} with edges

$$E = \{\{a, b, c\}, \{a, d, e\}, \{b, f, g\}, \{c, h, i\}\}.$$ 

Proving a conjecture of Gyárfás et. al., we show that for any crown-free linear 3-graph $G$ on $n$ vertices, its number of edges satisfy

$$|E(G)| \leq \frac{3(n - s)}{2}$$

where $s$ is the number of vertices in $G$ with degree at least 6. This result, combined with previous work, essentially completes the determination of linear Turán number for linear 3-graphs with at most 4 edges.

1. Introduction

A linear 3-graph $G = (V, E)$ consists of a finite set of vertices $V = V(G)$ and a collection $E = E(G)$ of 3-element subsets of $V$ (edges), such that any two edges in $E$ share at most one vertex. If $H$ and $F$ are linear 3-graphs, then $H$ is $F$-free if it contains no copy of $F$. For a linear 3-graph $F$, and a positive integer $n$, the linear Turán number $\text{ex}(n, F)$ is the maximum number of edges in any $F$-free linear 3-graph on $n$ vertices.

Let the crown $C_{13}$ be the linear 3-graph on 9 vertices \{a, b, c, d, e, f, g, h, i\} with edges

$$E = \{\{a, b, c\}, \{a, d, e\}, \{b, f, g\}, \{c, h, i\}\}.$$ 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{crown.png}
\caption{The crown $C_{13}$.}
\end{figure}

The study of $\text{ex}(n, C_{13})$ was initiated by Gyárfás, Ruszinkó and Sárközy in [3], where they showed the bounds

$$6\lfloor \frac{n - 3}{4} \rfloor + \epsilon \leq \text{ex}(n, C_{13}) \leq 2n.$$ 

where $\epsilon = 0$ if $n - 3 \equiv 0, 1 \pmod{4}$, $\epsilon = 1$ if $n - 3 \equiv 2 \pmod{4}$, and $\epsilon = 3$ if $n - 3 \equiv 3 \pmod{4}$. In [4], Gyárfás et. al. showed that every linear 3-graph with minimum degree 4 contains a crown. They also proposed some
ideas to obtain the exact bounds. Very recently, Fletcher showed in [2] the improved upper bound
\[
ex(n, C_{13}) < \frac{5}{3} n.
\]
In this paper, we show that the lower bound in [3] is essentially tight, thus resolving a conjecture in [1]. In fact, we show the following stronger result.

**Theorem 1.1.** Let \( G \) be any crown-free linear 3-graph on \( n \) vertices. Then its number of edges satisfies
\[
|E(G)| \leq \frac{3(n - s)}{2},
\]
where \( s \) is the number of vertices in \( G \) with degree at least 6.

Furthermore, we show that when \( s \) is small, the upper bound can be improved.

**Theorem 1.2.** Let \( G \) be any crown-free linear 3-graph on \( n \) vertices, and let \( s \) be the number of vertices in \( G \) with degree at least 6. If \( s \leq 2 \), then the number of edges satisfies
\[
|E(G)| \leq \frac{10(n - s)}{7}.
\]

Combining the two theorems above, we immediately conclude that the lower bound in [3] is exact when \( n \equiv 3 \mod 4 \) and \( n \geq 63 \).

**Corollary 1.3.** If \( n \geq 63 \), then
\[
ex(n, C_{13}) \leq \frac{3(n - 3)}{2}.
\]

The paper is structured as follows. In Section 2 and Section 3 we present the main innovative inequality and prove our main theorems, quotient a technical and familiar lemma that we prove in Section 4.

2. Proof of Theorem 1.1

Let \( G \) be any linear 3-graph. For each \( v \in V(G) \), let \( d(v) \) be the degree of \( v \), which is the number of edges in \( E(G) \) that contains \( v \). For each edge \( e \in E(G) \) and positive integers \( a \geq b \geq c \), we write \( D(e) \geq (a, b, c) \) if we can write \( e = \{x, y, z\} \) such that \( d(x) \geq a \), \( d(y) \geq b \) and \( d(z) \geq c \).

Suppose the contrary. Let \( G \) be the smallest linear 3-graph such that \( G \) has greater than \( 3(n - s)/2 \) edges. For each \( v \in V(G) \), let \( \chi(v) = 1 \) if \( d(v) \leq 5 \), and \( \chi(v) = 0 \) otherwise.

Our key innovation is the following observation
\[
\sum_{e \in E(G)} \sum_{v \in V(G), v \in e} \frac{\chi(v)}{d(v)} = \sum_{v \in V(G)} \sum_{e \in E(G), v \in e} \frac{\chi(v)}{d(v)} = \sum_{v \in V(G)} \chi(v) = n - s.
\]
As \( |E(G)| > 3(n - s)/2 \), we conclude that there exists an edge \( e = \{x, y, z\} \) such that
\[
\frac{\chi(x)}{d(x)} + \frac{\chi(y)}{d(y)} + \frac{\chi(z)}{d(z)} < \frac{2}{3}.
\]
Without loss of generality, assume \( d(x) \leq d(y) \leq d(z) \). First we note that \( d(x) \geq 2 \) and \( d(y) \geq 4 \), as otherwise \([1]\) would be violated. If \( d(z) \geq 6 \), then we can easily find a \( C_{13} \) by choosing an edge \( e_1 \neq e \) adjacent to \( x \), choosing an edge \( e_2 \) adjacent to \( y \) that does not share a vertex with \( e_1 \), and finally choosing an edge \( e_3 \) adjacent to \( z \) that does not share a vertex with \( e_1 \) and \( e_2 \), contradiction. Therefore, we have \( d(z) \leq 5 \), and \([1]\) implies that \( D(e) \geq (5, 5, 4) \).
We use the following lemma to handle the case $D(e) \geq \langle 5,5,4 \rangle$. As the lemma is quite straightforward using the techniques in [1], [2] and [3], we delay the lengthy proof to Section [4].

**Lemma 2.1.** Let $G$ be a crown-free graph and $e = \{x,y,z\} \in E(G)$ satisfy $D(e) \geq \langle 5,5,4 \rangle$. Then, the vertex set of all vertices sharing an edge with $\{x,y,z\}$,

$$S = \bigcup_{f \in E(G), f \cap \{x,y,z\} \neq \emptyset} f,$$

contains exactly 11 vertices and all vertices in $S$ have degree at most 5. The set of edges that contains at least one vertex in $S$,

$$E_S = \{ f : f \in E(G), f \cap S \neq \emptyset \},$$

contains at most 13 edges, and all elements of $E_S$ are subsets of $S$. In other words, the subgraph $G[S]$ is a connected component of $G$.

Let $G - S$ be the graph obtained by deleting the vertices $S$ and the edges in $E_S$. By the lemma, the graph $G - S$ has $n' = n - 11$ vertices and at least $|E(G)| - 13$ edges. Furthermore, the number of vertices in $G - S$ of degree at least 6 is exactly $s$. Therefore, we conclude that

$$|E(G - S)| \geq |E(G)| - 13 > \frac{3(n - s)}{2} - 13 > \frac{3(n' - s)}{2}$$

contradicting the assumption that $G$ is the smallest counterexample to Theorem [1.1]. So we have shown Theorem [1.1].

### 3. Proof of Theorem [1.2]

We use the same notations as Section [2].

Suppose the contrary. Let $G$ be the smallest linear 3-graph such that $G$ has at most 2 vertices with degree at least 6 and has greater than $10(n - s)/7$ edges.

For each $e \in E(G)$ and $v \in e$, we define a weight $\chi(v, e)$ as follows: let $\chi(v, e) = 1$ if $d(v) = 1, 2, 4, 5$, and $\chi(v, e) = 0$ if $d(v) \geq 6$. If $d(v) = 3$, let $\chi(v, e) = 0.05$ if there exists at least one vertex in $e$ with degree at least 6, and $\chi(v, e) = 0.9$ otherwise.

Since $s \leq 2$, we have

$$\sum_{e \in E(G)} \sum_{v \in V(G), v \in e} \frac{\chi(v, e)}{d(v)} = \sum_{v \in V(G)} \sum_{e \in E(G), v \in e} \frac{\chi(v, e)}{d(v)} \leq n - s.$$

As $|E(G)| > 10(n - s)/7$, we conclude that there exists an edge $e = \{x, y, z\}$ such that

$$(2) \quad \frac{\chi(x, e)}{d(x)} + \frac{\chi(y, e)}{d(y)} + \frac{\chi(z, e)}{d(z)} < \frac{7}{10}.$$  

Without loss of generality, assume $d(x) \leq d(y) \leq d(z)$. First we note that $d(x) \geq 2$, as otherwise (2) would be violated. Then note that if $d(y) \leq 3$, no matter $d(z)$ is greater than 6 or not (2) would also be violated, thus $d(y) \geq 4$.

The rest of the proof proceeds exactly the same as Section [2] other than the following inequality which leads to contradiction. Theorem [1.2] then follows.

$$|E(G - S)| \geq |E(G)| - 13 > \frac{10(n - s)}{7} - 13 > \frac{10(n' - s)}{7}.$$
In this section we show our lemma on the case $D(e) \geq \langle 5, 5, 4 \rangle$. Our proof follows similar techniques as in [1], [2] and [3]. In particular, [1] analyzed the case $D(e) \geq \langle 4, 4, 4 \rangle$, [2] analyzed the case $D(e) \geq \langle 5, 5, 5 \rangle$, and [3] analyzed the case $D(e) \geq \langle 5, 5, 3 \rangle$. We use a slight variation of their methods to prove our lemma.

Without loss of generality, assume $d(y), d(z) \geq 5$ and $d(x) \geq 4$. As we must not have $D(e) \geq \langle 6, 4, 2 \rangle$, we must have $d(y) = d(z) = 5$. For $p \in \{x, y, z\}$, let $G(p)$ be the set of all vertices distinct from $x, y, z$ that lie on the same edge with $p$. We first note that we must have $G(y) = G(z)$. Suppose the contrary, and some edge $e_1 \neq e$ adjacent to $y$ contain some vertex not in $G(z)$. Then at most one edge adjacent to $z$ other than $e$ contains a vertex in $e_1$, so at least three edges $F$ adjacent to $z$ are disjoint from $e_1$. Thus, we can take an edge $e_2$ containing $x$ that is disjoint from $e_1$, then take an edge $e_3$ from $F$ that is disjoint from $e_2$. So $e, e_1, e_2, e_3$ forms a $C_{13}$, contradiction.

Similarly, we must have $G(x) \subset G(y)$. Suppose the contrary, and some edge $e_1 \neq e$ adjacent to $x$ contain some vertex not in $G(y)$. Then, we can take an edge $e_3$ containing $z$ that is disjoint from $e_1$. Among the four edges adjacent to $y$ distinct from $e$, at most two can intersect $e_3$, and at most one can intersect $e_1$. Thus, we can choose $e_2$ containing $y$ that is disjoint from $e_1$ and $e_3$. So $e, e_1, e_2, e_3$ forms a $C_{13}$, contradiction.

Thus $S \{x, y, z\} = G(y) = G(z) \supset G(x)$. We define $F$ as the set of all edges in $E(G)$ that contains one of the vertices in $S$, but is disjoint from $\{x, y, z\}$. It suffices to show that $F$ must be empty.

We denote the vertices in $G(y)$ by $a, b, c, d, r, s, p, q$, such that $\{z, a, b\}, \{z, c, d\}, \{z, r, s\}, \{z, p, q\}$ are edges in $E(G)$.

**Step I.** We construct an auxiliary bipartite graph $H = (X_H, Y_H, E_H)$, where $X_H = \{e_i | y \in e_i\}, Y_H = \{e_j | z \in e_j\}, E_H = \{e_i, e_j | e_i \cap e_j \neq \emptyset\}$. $H$ is a 2-regular bipartite graph with order $8$. Thus, $H = C_8$ or $H = C_4 \uplus C_4$.

We claim that if $G$ contains no crown, $H$ contains a $K_{2, 2}$. Arbitrarily choose $e \in G(x)$. Define $V_1 = e \cap S \subset E_H, W_1 = \{e_i \cap V_1 \neq \emptyset\} \subset X_H \uplus Y_H$, we have $|V_1| \leq 2, |W_1| \leq 4, |H - W_1| \geq 4$. To find a crown, we only need to choose $e_i \in Y_G, e_j \in X_G$ s.t. $\{e_i, e_j\} \not \in E_H - W_1$. Therefore, if there is no crown in $H$, $H - W_1$ has to be a completed bipartite graph. Since $|G - W_1| \geq 4$ and two parts have the same order, there is definitely a $K_{2, 2}$ in $H - W_1$. So $H$ contains a $K_{2, 2}$, furthermore, $H = C_4 \uplus C_4$.

By symmetry, we can assume $\{z, a, b\}, \{z, c, d\}$ are in a $C_4$ and $\{z, r, s\}, \{z, p, q\}$ are in the other one. Without loss of generality, we can further assume $\{y, b, d\}, \{y, a, c\}$ lie in $E(G)$, and $\{y, s, q\}, \{y, r, p\}$ lie in $E(G)$.

**Step II.** Now let $V_1 = \{a, b, c, d\}, V_2 = \{r, s, p, q\}$. We have symmetry between $V_1$ and $V_2$, and symmetry inside $V_i, i = 1, 2$ as well. We claim that there exists no edge containing $x$ that contains exactly one vertex in $V_1$ and another one in $V_2$. Otherwise we can let it be $\{x, a, r\}$ by symmetry. Then $\{z, a, b\}, \{y, b, d\}, \{z, p, q\}, \{x, a, r\}$ form a $C_{13}$, contradiction. Thus the edges other than $e$ containing $x$ must be a subset of $\{\{x, a, d\}, \{x, b, c\}, \{s, r, q\}, \{x, s, p\}\}$.

**Step III.** Let $f$ be any element of $F$. By symmetry we can let $a \in f$ without loss of generality. Then we can see $b, c \notin f$. Firstly, we claim that $f$ cannot contain exactly one element $a$ of $S$. Otherwise $\{z, a, b\}, \{y, b, d\}, \{z, r, s\}, f$ form a $C_{13}$, contradiction. Secondly, we claim that $d \notin f$. Otherwise $G(x) = \{\{x, b, c\}, \{s, r, q\}, \{x, s, p\}\} \mid d(x) \geq 4$. Since at most one edge of $\{z, r, s\}$ and $\{z, p, q\}$ intersect $f$, we can assume $\{z, r, s\} \cap f = \emptyset$. Then $\{z, a, b\}, \{x, b, c\}, \{z, r, s\}, f$ form a $C_{13}$, contradiction.

Therefore we can assume $r \in f$ by symmetry. Similarly we know that $q \notin f$ since $a, d$ and $r, q$ are symmetric. So $f$ has exactly two elements $a, r$ of $S$. While $\{z, a, b\}, \{x, b, c\}, \{z, p, q\}, f$ form a $C_{13}$ in this case, contradiction.
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