Real irradiance computed by means of the phase
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Introduction
Solving the amplitude simultaneously with the phase is a consistent issue in physics. For example, the same mathematical computations can be applied to solve the Schrödinger or the Helzmoňtz equation¹. In this way, the eikonal and amplitude equations (supplementary note 1) can estimate bright effects in microstructures.

Rewording equations for scalar electric field
The eikonal and transport equation are rewritten to the differential system (1)

\[ \begin{align*}
\left| \nabla (\delta_0) \right| &= n \\
\nabla \cdot (E_0^2 \nabla (\delta_0)) &= 0
\end{align*} \]

(1)

The solution is given in Cartesian coordinates \((x, y, z)\) whose \(z\) acts as the propagation time. Therefore, the first equation (1) successively becomes the resolving equation (2).

\[ \left( \frac{\partial \delta_0}{\partial x} \right)^2 + \left( \frac{\partial \delta_0}{\partial y} \right)^2 + \left( \frac{\partial \delta_0}{\partial z} \right)^2 = n^2 \] then \( \frac{\partial \delta_0}{\partial z} - \sqrt{n^2 - \left( \frac{\partial \delta_0}{\partial x} \right)^2 - \left( \frac{\partial \delta_0}{\partial y} \right)^2} = 0 \]

(2)

The second equation (1) evolves to times: the gradient part (3)

\[ E_0^2 \nabla_x (\delta_0) = \left[ E_0^2 \frac{\partial \delta_0}{\partial x}, E_0^2 \frac{\partial \delta_0}{\partial y}, E_0^2 \frac{\partial \delta_0}{\partial z} \right] = [a\mu, b\mu, \mu] \]

(3)

where \( a \) is given by \( a = \left( \frac{\partial \delta_0}{\partial x} \right)^{-1} \), \( b \) is given by \( b = \left( \frac{\partial \delta_0}{\partial y} \right)^{-1} \), \( \mu \) is given by \( \mu = E_0^2 \frac{\partial \delta_0}{\partial z} \) and then the divergent part (4).
\[
\frac{\partial a_u}{\partial x} + \frac{\partial b_u}{\partial y} + \frac{\partial u}{\partial z} = 0 \quad (4)
\]

It should be noted that \( a, b \) and \( I_0 = E_0^2 \) depend on the eikonal \( \delta_p \), and a unique solution exists for this differential system of the two equations (2) and (4) for very general initial boundaries\(^1\).

**Numerical schema of Lax-Friedrichs type**

The differential system obtained from the two equations (2) and (4) can be computed with equations of differences. Regular sampled grids are drawn in a three-dimensional space: the \( x \) and \( y \) coordinates are sampled with the same step \( \Delta xy \) giving sampled plans \( \mathbb{R}^2 = \bigcup_{i \in \mathbb{Z}} [x_i, x_{i+1}] \times \bigcup_{j \in \mathbb{Z}} [y_j, y_{j+1}] \), and the \( z \) coordinates are also sampled with the step \( \Delta z \) giving the sampled axis \( \mathbb{R}^+ = \bigcup_{k \in \mathbb{N}} [z^k, z^{k+1}] \). Therefore, each point in three-dimensional space is shown by \((x_i, y_j, z^k) = (i \Delta xy, j \Delta xy, k \Delta z)\) and belongs to a three-dimensional matrix.

Numerical schema for the eikonal equation (2) resolves from the equation of the differences (5).

\[
\frac{u_{i,j}^{k+1} - u_{i,j}^k}{\Delta z} = n^2(x_i, y_j, z^k) - \frac{\left| u_{i+1,j}^k - u_{i,j}^k \right|^2}{\Delta xy} - \frac{\left| u_{i,j+1}^k - u_{i,j}^k \right|^2}{\Delta xy} = 0 \quad (5)
\]

where the sample \( u_{i,j}^k \) is \( u_{i,j}^k = \delta_p(x_i, y_j, z^k) \) which is modified to the new relationship (6)

\[
u_{i,j}^{k+1} = u_{i,j}^k + \frac{\Delta z}{\Delta xy} \sqrt{(n(x_i, y_j, z^k) \Delta xy)^2 - (u_{i+1,j}^k - u_{i,j}^k)^2 - (u_{i,j+1}^k - u_{i,j}^k)^2} \]

The numerical Lax-Friedrichs\(^2\) type schema substitutes each sample by centering with its nearest neighbors (7).

\[
u_{i+1,j}^k = \frac{u_{i+1,j+1}^k + u_{i+1,j-1}^k}{2}, \quad u_{i,j+1}^k = \frac{u_{i+1,j+1}^k + u_{i-1,j+1}^k}{2}, \quad u_{i,j}^k = \frac{u_{i+1,j+1}^k + u_{i+1,j-1}^k + u_{i-1,j+1}^k + u_{i-1,j-1}^k}{4} \quad (7)
\]

It should be noted the ratio \( \frac{\Delta z}{\Delta xy} \) keeps a fixed value even if the differences becomes very small.

Numerical schema for the transport equation (4) resolve from the equation of the differences (8).

\[
u_{i,j}^{k+1} = \nu_{i,j}^k - \frac{\Delta z}{\Delta xy} [\mu(v_{i,j}^k) - \mu(v_{i,j+1}^k)] + (g(v_{i+1,j}^k) - g(v_{i,j}^k)) \quad (8)
\]

where the sample \( \nu_{i,j}^k \) is \( \nu_{i,j}^k = \mu(x_i, y_j, z^k) \) and the functions \( f(\mu) \) and \( g(\mu) \) are \( f(\mu) = a_{i,j} \) and \( g(\mu) = b_{i,j} \), respectively. The numerical Lax-Friedrichs type schema allow for computing the relationship (8). Herein, these computations
require coupling for each propagation time $z^k$ with the computation of the eikonal equation. The coefficients $a$ and $b$ and the irradiance $I_0$ use the partial differential equations $\frac{\partial \delta_0}{\partial x}$, $\frac{\partial \delta_0}{\partial y}$ and $\frac{\partial \delta_0}{\partial z}$ which are associated respectively to $\frac{u_{i+1,j}^k - u_{i,j}^k}{\Delta xy}$, $\frac{u_{i,j+1}^k - u_{i,j}^k}{\Delta xy}$ and $\frac{u_{i,j}^{k+1} - u_{i,j}^k}{\Delta z}$ with $\Delta z$ deduced from the ratio $\frac{\Delta z}{\Delta xy}$. The numerical Lax-Friedrichs type schema are renewed for these computations.

The numerical Lax-Friedrichs type schema are used in order to warrant better numerical convergence. This approach is feasible and efficient in several practical situations.

**Smooth wedge simulations**

Smooth wedges are used to illustrate results of the numerical schema (Fig. S1).

*Fig. S1: Reflective index of a smooth wedge in a 2D plane.*

The 2D numerical schema computes the phase and the irradiance (Fig. S2).

*Fig. S2: at the left, the phase image and at the right, the irradiance image.*
The 3D numerical schema allows to build images in a $xy$ plane to study the irradiance (Fig. S3).

![Figure S3](image)

*Fig. S3: at the left, a piece of the irradiance image and at the right, its associated profile.*

The profile in the radiation diagram given by the diffraction of a perfectly conducting half-plane (*supplementary note 4, at the right top, Fig. S3*) is similar to the irradiance profile (*at the left, Fig. S3*).

**References**

1. Gosse (L.), James (F.), Convergence results for an inhomogeneous system arising in various high frequency approximations, Numer. Math. 90: 721–753 (2002).

2. Breuss (M.), The correct use of the Lax-Friedrichs method, ESAIM: M2AN, 38(3):519-540 (2004).