A note on choosability with defect 1 of graphs on surfaces

Vida Dujmović*  Djedjiga Outioua

Abstract

This note proves that every graph of Euler genus $\mu$ is $\lceil 2 + \sqrt{3\mu + 3} \rceil$-choosable with defect 1 (that is, clustering 2). Thus, allowing defect as small as 1 reduces the choice number of surface embeddable graphs below the chromatic number of the surface. For example, the chromatic number of the family of toroidal graphs is known to be 7. The bound above implies that toroidal graphs are 5-choosable with defect 1. This strengthens the result of Cowen, Goddard and Jesurum (1997) who showed that toroidal graphs are 5-colourable with defect 1.

1 Introduction

In a vertex coloured graph, a monochromatic component is a connected component of $G$ where each vertex has the same colour. A vertex colouring is proper if every monochromatic component has 1 vertex. The following notion, introduced formally by Cowen, Cowen and Woodall [6] and studied as early as 1966 by Lovász [11], generalizes proper graph colourings. A graph $G$ is $k$-colourable with defect $d$, that is, $(k, d)^*$-colourable, if the vertices of $G$ can be coloured with $k$ colours such that the subgraph induced by the vertices of each monochromatic component has maximum degree $d$. Colourings with defect zero are proper, otherwise they are called defective. In this note we are interested in defective colourings that are as close as possible to being proper, that is, in colourings with defect at most 1. In particular, we study such colourings for graphs on surfaces. Typically, the goal of this line of research is to reduce the number of colours below the number required by proper colourings. For example, Voigt proved [13] that there are planar graphs that are not 4-choosable, but Cushing and Kierstead [7] proved that planar graphs are 4-choosable if defect 1 is allowed, thereby answering an open problem posed by several authors [17, 1, 14, 8]. Defective choosability will be formally defined below. For some classes of graphs, however, allowing even arbitrarily big defect does not reduce the number of colours below the chromatic number of the class. For example, $k$-trees have chromatic number $k + 1$, and yet for every $d$ there is $k$-tree that is not $(k, d)^*$-colourable (see the standard example in [15]). This implies that, for any $d$, there is a planar graph (in fact a series-parallel graph), that is not $(2, d)$-colourable.

A similar notion to that of defective colouring is clustered colouring. A graph $G$ is $k$-colourable with clustering $c$ if the vertices of $G$ can be coloured with $k$ colours such that each monochromatic component has at most $c$ vertices. Note that a colouring of a graph has defect at most 1 if and only if it has clustering at most 2. This is not the case for higher defect/clustering values. Thus, our results also

*School of Electrical Engineering and Computer Science, University of Ottawa, emails: vida.dujmovic@uottawa.ca, douti102@uottawa.ca. Research supported by NSERC and Ministry of Research, Innovation and Science of Ontario. The first author would like to thank Frédéric Havet for introducing her to defective list colourings more than a decade ago, during her visit to INRIA Sophia Antipolis, funded by French Consulat Général de France à Montréal.
give colourings of graphs on surfaces with clustering at most 2. There is a plethora of recent work on the subject of clustered and defective graph colourings/choosability. For an extensive coverage of the topic see the recent survey by Wood [15] and Sections 3, 4 and 5 in the survey by Woodall [16].

There is a rich history of various colouring problems on graphs on surfaces. For the background, we only focus on the previous work on colourings and choosability with defect 1. The goal of this note is to show that for every surface, graphs embeddable in that surface have choice number with defect 1 less than the chromatic number of that surface.

A surface \( \Sigma \) is a compact connected 2-manifold without boundary. Surfaces are classified into two classes. Each orientable (non-orientable) surface is homeomorphic to a sphere with \( g \geq 0 \) handles \( (g \geq 1 \) crosscaps) attached. For a surface \( \Sigma \), its Euler genus, \( \text{eg}(\Sigma) \), is 0 if \( \Sigma \) is orientable and \( h \) if \( \Sigma \) is non-orientable. Given a graph \( G \), its Euler genus \( \text{eg}(G) \) is the minimum Euler genus of a surface \( \Sigma \) where \( G \) can be embedded in.

In their 1997 paper, Cowen, Goddard, and Jesurum [4, 5] studied defective colouring of graphs on surfaces. In their conclusions, they suggest a study of defective choosability, that is, a list version of the problem. This is what we study in this note, namely rather than studying the colouring problem of graphs on surfaces while allowing defect 1, we study a list version of the problem which is its strengthening. For planar graphs, the defective list version was first studied by Eaton and Hull [8] and Škrekovski [14].

A list assignment for a graph \( G \) is a function \( L \) that assigns a set \( L(v) \), also called a list \( L(v) \), of colours to each vertex \( v \in V(G) \). A list \( L(v) \) with \( |L(v)| \geq k \) is a \( k \)-list. A list assignment \( L \) is a \( k \)-list assignment if each vertex is assigned a \( k \)-list. For a list assignment \( L \), \( G \) is \( L \)-colourable with defect \( d \) if \( G \) can be coloured such that each vertex \( v \) gets a colour from its list \( L(v) \) and such that the defect is at most \( d \). \( G \) is \( (k, d)^*-\)choosable if for every \( k \)-list assignment \( L \), \( G \) is \( L \)-colourable with defect \( d \).

In an \( L \)-colouring of a graph, a vertex \( v \) is proper if none of its neighbours have the same colour as \( v \). The choice-\( i \) number of a graph \( G \) is the minimum \( k \) such that \( G \) is \( (k, i)^* \)-choosable. The chromatic-\( i \) number of a graph \( G \) is the minimum \( k \) such that \( G \) is \( (k, i)^* \)-colourable. For proper colourings, \( i = 0 \) is omitted in this notation.

The following is our main result.

\textbf{Theorem 1.} Every graph \( G \) is \( \left( 2 + \sqrt{3 \text{eg}(G)} + 3, 1 \right)^* \)-choosable.

Cowen, Goddard and Jesurum [5, 4] proved that toroidal graphs are \( (5, 1)^* \)-colourable. Theorem 1 implies that they are, in fact, \( (5, 1)^* \)-choosable thus the theorem constitutes a strengthening of that result. In addition, the proof of Theorem 1 does not use the four-colour theorem. For planar graphs, the theorem states that they are \( (4, 1)^* \)-choosable. This has been proved by Cushing and Kierstead [7]. Our proof works for \( (t, 1)^* \)-choosability where \( t \geq 5 \), thus it does not imply Cushing and Kierstead’s result.

It is well-known (by Heawood’s conjecture [10] proved by Ringel and Young [12]) that for every surface \( \Sigma \), except the Klein bottle, there is a graph (in fact a complete graph) that embeds in that surface whose chromatic number is exactly \( \frac{2 + \sqrt{2 \text{eg}(\Sigma) + 1}}{2} = \left[ 3.5 + \sqrt{6 \text{eg}(\Sigma)} + 0.25 \right] \). Since this lower bound is achieved for complete graphs \( G \), it follows that the choice number of such graphs is at least \( \left[ 3.5 + \sqrt{6 \text{eg}(G)} + 0.25 \right] \). Thus, Theorem 1 shows that for every surface, the choice-1 number of graphs embeddable in the surfaces can be reduced below what the chromatic number of the surface allows.

The bound in Theorem 1 is surely not tight. The only lower bound available however, follows from the fact that having choice-1 number at most \( p \) implies having a chromatic number at most \( 2p \).
Thus, the above $[3.5 + \sqrt{6\operatorname{eg}(\Sigma)} + 0.25]$ bound on the chromatic number implies that for every surface $\Sigma$, except the Klein bottle, there is a graph that embeds in that surface whose choice-1 number is at least $[1.75 + \sqrt{1.5 \operatorname{eg}(G)} + 1/16]$. For example, this lower bound and Theorem 1 imply that the correct bound on choice-1 number of toroidal graphs is either 4 or 5, leading to an open question of whether toroidal graphs are $(4, 1)^*–$choosable. Unfortunately it is still not even known if toroidal graphs are $(4, 1)^*–$colourable, which is an open question from 1997 by Cowen, Goddard, and Jesurum [4, 5].

Even the question on whether all planar graphs are $(4, 1)^*–$choosable has been open until recently. The question, which received considerably attention, was asked in several articles over the years [17, 1, 14, 8] and was finally settled in the positive by Cushing and Kierstead [7]. A well-known result by Voigt states that there are planar graphs that are not 4-choosable [13]. Cowen, Cowen and Woodall [6] proved that there are planar graphs that are not $(3, 1)^*–$colourable and thus they are not $(3, 1)^*–$choosable. Cowen, Goddard, and Jesurum [5] show that in fact testing if a planar graph is $(2, 1)^*–$colourable or $(3, 1)^*–$colourable is NP-complete. The hardness of $(2, 1)^*–$colourability remains unchanged even for planar graphs with maximum degree 4, as proved by Corrêa, Havet and Sereni [3]. For more on the complexity of defective colouring problems, see the recent results by Belmonte, Lampis and Mitsou [2].

Note that the above negative result on $(3, 1)^*–$colourability of planar graphs implies that the aforementioned lower bound, $[1.75 + \sqrt{1.5 \operatorname{eg}(G)} + 1/16]$, on chromatic-1 and choice-1 number is not tight, in fact it is off by 2 for planar graphs. Thus, the lower bound is possibly quite weak for choice-1 number for graphs on surfaces. We conclude this section by asking for improvements on the lower and upper bound on choice-1 number of such graphs, starting with the question of whether toroidal graphs are $(4, 1)^*–$choosable or $(4, 1)^*–$colourable. Colouring and choosability problems often benefit from proving a non-trivial maximum average degree results for a colouring problem. Unfortunately, while such results are known for defective-\( d \) choosability with \( d \geq 2 \), by the results of Havet and Sereni [9], no such results are known for defective-1 choosability. Any non-trivial bound on maximum average degree for defective-1 choosability, could likely be used to improve the result in Theorem 1.

2 Useful Lemmas

We consider simple and undirected graphs $G = (V, E)$ with vertex set $V(G)$ and edge set $E(G)$. The number of vertices is denoted by $n = |V(G)|$ and the number of edges by $m = |E(G)|$. For a set $S \subseteq V(G)$, $G[S]$ denotes the subgraph of $G$ induced by the vertices of $S$ and $G - S$ denotes the subgraph $G[V(G) \setminus S]$. If $S$ is comprised of one vertex, $v$, then $G - v$ denotes $G - \{v\}$. For every $v \in V$, let $N(v)$ denote the set of neighbours of $v$. The degree of $v$ is $\deg_G(v) = |N(v)|$. A vertex of degree $d$ is called degree-$d$ vertex. The minimum and maximum degree in $G$ are denoted respectively by $\delta(G)$, and $\Delta(G)$. We omit “$G$” from this notation whenever the graph is clear from the context.

We start with a useful observation. Lovász’ [11] proof for defective colouring of graphs of maximum degree $\Delta$ extends easily to choosability. We include the proof for completeness.

Lemma 1. [11] For every integer $k > 0$, every graph $G$ with maximum degree $\Delta$ is $(k, \lceil \frac{\Delta}{k} \rceil)^*–$choosable. In fact, $G$ is $k$-choosable such that each vertex $v \in V(G)$ has at most $\lceil \frac{\deg(v)}{k} \rceil$ neighbours with the same colour as $v$.

Proof. For a $k$-list assignment $L$ of $G$ consider an $L$-colouring of $G$ that minimizes the number of monochromatic edges. Assume for the sake of contradiction that this is not a desired list colouring. Let $V_1, V_2, \ldots, V_{|L(G)|}$ denote the resulting colour classes (some possibly empty), where $L(G) = \cup \{L(v) \mid v \in V(G)\}$. Assume that there is a vertex $v \in V(G)$ coloured $c \in L(v)$ such that there are at least
Let \(\lfloor \frac{\deg(v)}{k} \rfloor + 1\) neighbours of \(v\) in the colour class \(V_c\). In that case, there is a colour class \(V_p, p \in L(v)\) such that the number of neighbours of \(v\) in \(L_p\) is at most \(\lfloor \frac{\deg(v)}{k} \rfloor\). Changing the colour of \(v\) from \(c\) to \(p\) reduces the number of monochromatic edges, thus the contradiction. \(\square\)

The proof of Theorem 1 uses the discharging technique. The following lemma provides some key observations that will be used for discharging rules.

**Lemma 2.** Let \(G\) a vertex minimal graph of Euler genus at most \(\mu\) such that \(G\) is not \((t, 1)^\ast\)–choosable, \(3 \geq t \in \mathbb{N}\). Then \(G\) has the following properties.

1. (a) The minimum degree, \(\delta(G)\), of \(G\) is at least \(t\)
   (b) The maximum degree, \(\Delta(G)\), of \(G\) is at least \(2t\)
   (c) The number of vertices in \(G\) is at least \(2t + 1\).
2. (a) The set of degree-\(t\) vertices of \(G\) forms an independent set in \(G\).
   (b) Each vertex \(v\) of \(G\) has at most \(\left\lceil \frac{\deg(v)}{2} \right\rceil\) degree-\(t\) neighbours, whenever \(G\) is edge maximal.
3. (a) There is no 3-cycle \(v,w,u\) in \(G\) such that \(\deg(v) = t, \deg(w) = t + 1\) and \(\deg(u) = t + 1\).
   (b) Each degree-\(t\) vertex \(v\) of \(G\), has at least \(\left\lceil \frac{t}{2} \right\rceil\) degree-\(d\), \(d \geq t + 2\), neighbours whenever \(G\) is edge-maximal.
4. Whenever \(G\) is edge-maximal, if \(t\) is even and a degree-\(t\) vertex \(v\) has exactly \(\frac{t}{2}\) degree-\(d\), \(d \geq t + 2\), neighbours, then at least one of them has degree at least \(t + 3\).

**Proof.** By the assumptions of the lemma there is a \(t\)-list assignment \(L\) such that \(G\) is not \(L\)-colourable.

1a. Assume on the contrary that \(G\) has a vertex \(v\) of degree at most \(t - 1\). Then \(G - v\) is a nonempty graph of Euler genus at most \(\mu\) and thus by the vertex minimality of \(G\) it is \(L\)-colourable with defect 1. Since \(|N(v)| \leq t - 1\), there is a colour \(c_1\) in the \(t\)-list \(L(v)\) that is not used by any vertex in \(|N(v)|\), thus the \(L\)-colouring of \(G - v\) can be extended to \(L\)-clouring of \(G\), giving the contradiction.

1b. If \(\Delta(G) < 2t\), then \(\left\lceil \frac{\Delta(G)}{t} \right\rceil = 1\) and thus \(G\) is \((t, 1)^\ast\)–choosable by Lemma 1, thus contradicting the assumptions of the lemma.

1c. Follows from 1b.

2a. Assume on the contrary that \(G\) has two degree-\(t\) vertices, \(v\) and \(w\), that are adjacent. Then \(G - \{v, w\}\) is a nonempty graph of Euler genus at most \(\mu\) and thus by the vertex minimality of \(G\) it is \(L\)-colourable with defect 1. Since \(|\{N(v) - \{v\}\}| \leq t - 1\) and \(|\{N(w) - \{v\}\}| \leq t - 1\), there is a colour \(c_1\) in the \(t\)-list \(L(v)\) and \(c_2\) in the \(t\)-list \(L(w)\) such that no vertex in \(\{N(v) - \{v\}\}\) is coloured \(c_1\) and no vertex in \(\{N(w) - \{v\}\}\) is coloured \(c_2\) in the \(L\)-colouring of \(G - \{v, w\}\). Thus, assigning colour \(c_1\) to \(v\) and \(c_2\) to \(w\) gives \(L\)-colouring of \(G\) with defect 1 (even if \(c_1 = c_2\)).

2b. We now can assume \(G\) is edge maximal graph with a 2-cell embedding in a surface of Euler genus at most \(\mu\). The cyclic ordering of edges around \(v\) in the embedding. By edge maximality of \(G\), for any pair of consecutive edges \(vw\) and \(vy\) around \(v\), we have that \(x\) and \(y\) are adjacent. Thus, by 1a and 2a, \(x\) or \(y\) must have degree at least \(t + 1\), implying the claim.

3a. \(G - \{v, w, u\}\) is a nonempty graph of Euler genus at most \(\mu\) and thus by the vertex maximality, \(G\) it is \(L\)-colourable with defect 1. Since \(|\{N(w) - \{v, u\}\}| \leq t - 1\) and \(|\{N(u) - \{v, w\}\}| \leq t - 1\), there is a colour \(c_1\) in the \(t\)-list \(L(w)\) and \(c_2\) in the \(t\)-list \(L(u)\) such that no vertex in \(\{N(w) - \{v, u\}\}\) is coloured \(c_1\) and no vertex in \(\{N(u) - \{v, w\}\}\) is coloured \(c_2\) in the \(L\)-colouring of \(G - \{v, w, u\}\). Assign colour \(c_1\) to \(w\) and \(c_2\) to \(u\). That gives \(L\)-colouring of \(G - v\) with defect 1. If there is a colour \(c\) in the \(t\)-list \(L(v)\) such that no vertex in \(N(v)\) is coloured \(c\) in that \(L\)-colouring of \(G - v\), then assigning \(v\) colour \(c\) gives \(L\)-colouring of \(G\) with defect 1. Otherwise, \(c_1 \in L(v)\) and \(c_2 \in L(v)\) and \(c_1 \neq c_2\). In that case, both \(w\) and \(u\) are proper in the \(L\)-colouring of \(G - v\) and thus assigning \(v\) colour \(c_1\) gives \(L\)-colouring of \(G\) with defect 1.
3b. We now can assume $G$ is edge maximal graph with a 2-cell embedding in a surface of Euler genus at most $\mu$. The edge maximality of $G$ implies that every pair of consecutive edges $vx$ and $vw$ around $v$, defines a 3-cycle $v, x, y$ in $G$. Then 3a and the fact that $\deg(v) = t$ imply the claim.

4. We now can assume $G$ is edge maximal graph with a 2-cell embedding in a surface of Euler genus at most $\mu$. Assume for the sake of contradiction that all the neighbours of $v$ have degree at most $t + 2$. Let $H = G[N(v)]$ and $H' = G[v \cup N(v)]$. Consider vertices in $N(v)$ in the cyclic order, $v_1, v_2, \ldots, v_t$, around $v$ as determined by the embedding and starting with a degree-$(t + 2)$ vertex $v_1$. By the edge maximality of $G$, $C = v_1, \ldots, v_t, v_1$ is a cycle (non necessarily induced) in $G$. Then the degrees in $G$ of vertices in $N(v)$ are as follows $\deg(v_i) = t + 2$ for $i \equiv (1 \mod 2)$ and $\deg(v_i) = t + 1$ for $i \equiv (0 \mod 2)$. (The fact that the degrees alternate, between $t + 1$ and $t + 2$, along $C$ is the consequence of Property 2a, Property 3a and the assumption that $v$ has exactly $\frac{t}{2}$ degree-$(t + 2)$ neighbours). Thus, since $t$ is even, for each degree-$(t + 2)$ vertex in $C$, its two neighbours along $C$ are degree-$(t + 1)$ vertices. $G' = G - \{v \cup N(v)\}$ has genus at most $\mu$ and it is not an empty graph by Property 1c and the fact that $|v \cup N(v)| = t + 1$, and $t > 0$. Thus, $G'$ is $L$-clourable with defect 1. This list colouring of $G'$ can be extended to a $L$-colouring of $G$ with defect 1 as follows.

Define the list assignment $L'$ of $H'$ and $(H)$ as follows. For every $w \in V(H')$ the list $L'(w)$ is equal to the list $L(w)$ minus the colours used by the neighbours of $w$ in $L$-colouring of $G'$. Clearly any $L'$-list colouring of $H'$ with defect 1, extends the colouring of $G'$ to $L$-colouring of $G$ with defect 1. By considering degrees in $G$ of vertices in $H'$ we get, $|L'(v)| = |L(v)|, |L'(v_i)| \geq \deg_{H'}(v_i) - 2$ for $i \equiv (1 \mod 2)$ and $|L'(v_i)| \geq \deg_{H'}(v_i) - 1$ for $i \equiv (0 \mod 2)$. Moreover, in $H$, $|L'(v_i)| \geq \deg_H(v_i) - 1$ for $i \equiv (0 \mod 2)$. Let $S = \{v_i | i \equiv (0 \mod 2)\}$. By the above observation, each vertex $v_i, i \equiv (1 \mod 2)$, is adjacent to at least 2 vertices of $S$ (its neighbours along $C$), thus $\deg_H(v_i) \geq \deg_{H-S}(v_i) + 2$. Thus, in $H - S$, $|L'(v_i)| \geq \deg_{H-S}(v_i) + 1$ for all $i \equiv (1 \mod 2)$. Therefore, each vertex $x$ in $H - S$ has smaller degree in $H - S$ than the number of colours in its list as determined by $L'$ list assignment, that is, $\deg_{H-S}(x) < |L'(x)|$. The greedy colouring then implies that $H - S$ has $L'$-colouring where every vertex in $H - S$ is proper.

By property 3a, $S$ forms an independent set in $H$. Thus, all of $\deg_{H}(v_i)$ neighbours of vertex $v_i \in S$ in $H$ are in $V(H) - S$. Let $A$ denote the vertex set comprised of the vertices $v_i \in S$ whose neighbours in $H - S$ use at most $|L'(v_i)| - 1 \geq \deg_{H}(v_i) - 1$ colours. Let $B = S - A$. Since $|L'(v_i)| \geq \deg_{H}(v_i)$ for each vertex $v_i \in S$, each vertex in $v_i \in A$ can choose a colour from its list $L'(v_i)$ such that $v_i$ is proper in $L'$-colouring of $H - B$ (and in $H$ as will be seen later). Then $H - B$ is $L'$-coloured such that every vertex of $H - B$ is proper. For each vertex $v_i \in B$, each of its colours in $L'(v_i)$ is used by exactly one of its neighbours in $H$, and each of its neighbours in $H$ uses actually one colour in $L'(v_i)$. Thus, giving each such vertex $v_i$ the colour equal to the colour of its first counterclockwise neighbour along $C$ defines $L'$-colouring of $H$. It remains to show that this $L'$-colouring of $H$ has defect at most 1. Clearly, the vertices of $A$ are proper in the colouring of $H$. Vertices of $H - S$ are proper in the colouring of $H - B$, thus all the monochromatic edges have one endpoint in $B$ and the other in $V(H) - S$. Assume a vertex $w \in B$ has two neighbours in $V(H) - S$ coloured with the same colour as $w$. That implies that the number of colours used by the neighbours of $w \in V(H) - S$ is at most $\deg_B(w) - 1$ thus $w \in A$, contradiction. Finally, assume a vertex $w \in V(H) - S$ has two neighbours $x$ and $y$ in $B$ coloured with the same colour as $w$. That implies that $w$ is the first counter clockwise neighbour on $C$ of both $x$ and $y$, which is impossible since no pair of vertices of $S$ are adjacent. Thus, we have an $L'$-colouring of $H$ with defect 1. Recall that $|L'(v)| = t$. In $H'$, if the vertices in $H$ do not use all the colours in $L'(v)$, then $v$ can be coloured such that it is proper in $L'$-colouring of $H'$, thus resulting in the $L'$-colouring of $H'$ with defect at most one. Otherwise, each colour in $L'(v)$ is used by exactly one vertex in $H$ and each vertex in $H$ uses exactly one colour in $L'(v)$. Therefore, the $L'$-colouring of $H$ is proper and no two
neighbours of $v$ use the same colour. Thus, $v$ can use any colour in $L'(v)$ and the resulting $L'$-colouring of $H'$ has defect at most one.

By the choice of $L'$ the resulting $L'$-colouring of $H'$ with defect 1 extends the $L$-colouring of $G'$ to give $L$-colouring of $G$ with defect 1, which is a desired contradiction. $\square$

3 Proof of Theorem 1

Let $\mu := \text{eg}(G)$ and $t := \left\lfloor 2 + \sqrt{3\mu + 3} \right\rfloor$. If $G$ is planar, the statement of the theorem is true by the result of Cushing and Kierstead[7]. Thus, we may assume that $G$ is not planar, that is $\mu > 0$. Thus, $t \geq 5$. Assume for the sake of contradiction that the statement of the theorem is false. We may assume that $G$ is a vertex minimal, and subject to that edge maximal, connected graph that is a counter example to the theorem. Specifically, assume $G$ is a connected graph with a 2-cell embedding on a surface of Euler genus at most $\mu$, and no edge can be added to $G$ without introducing edge crossings or making $G$ non-simple, and $G$ is not $(t,1)$-choosable, but for every $v \in V(G)$, $G-v$ is $(t,1)$-choosable. Thus, $G$ satisfies the properties listed in Lemma 2.

Let the number of faces of the embedding of $G$ be denoted by $f$. The Euler formula gives: $f = m - n + 2 - \mu$. Each face in an embedding of the edge maximal graph has size at most 3, and each edge is in at most 2 faces, thus $f \leq \frac{2m}{3}$.

For each $v \in V(G)$ let its charge $w(v) := \deg(v)$. Since $2m = \sum_{v \in V(G)} \deg(v)$, the Euler formula and the above inequality give

$$\sum_{v \in V(G)} (w(v) - 6) \leq 6\mu - 12. \quad (1)$$

We will move these charges from one vertex to another such that the overall sum $\sum_{v \in V(G)} (w(v) - 6)$ remains unchanged. We move the vertex charges according to the following discharging rules:

|(★1)| Each degree–$(t+2)$ vertex $v$ sends the charge of $\frac{1}{\left\lfloor \frac{t}{2} \right\rfloor + 1}$ to each degree–$t$ vertex in $N(v)$.
|(★2)| If $t$ is odd, each degree–$d$ vertex $v$, $d \geq t + 3$, sends the charge of $\frac{1}{\left\lfloor \frac{t}{2} \right\rfloor + 1}$ to each degree–$t$ vertex in $N(v)$.
|(★3)| If $t$ is even, each degree–$d$ vertex $v$, $d \geq t + 3$, sends the charge of $\frac{2}{\left\lfloor \frac{t}{2} \right\rfloor + 1}$ to each degree–$t$ vertex in $N(v)$.

After above discharging rules are applied to all the vertices in $G$, their new charges are as follows. The weights of degree–$(t+1)$ vertices remain unchanged.

Consider now the new weight of a degree–$t$ vertex $v$. By property 3b in Lemma 2, $v$ has at least $\left\lfloor \frac{t}{2} \right\rfloor$ degree–$d$, $d \geq t + 2$ neighbours. Thus, if $t$ is odd, each degree–$t$ vertex $v$ has new weight $w(v) \geq t + \left\lfloor \frac{t}{2} \right\rfloor \frac{1}{\left\lfloor \frac{t}{2} \right\rfloor + 1} = t + \frac{t+1}{2} \frac{1}{\left\lfloor \frac{t}{2} \right\rfloor + 1} = t + 1$. If $t$ is even, we have two cases to consider. First consider the case that $v$ has at least $\frac{t}{2} + 1$ degree–$d$, $d \geq t + 2$ neighbours. Then the new weight of $v$ is $w(v) \geq t + \left(\frac{t}{2} + 1\right) \frac{1}{\left\lfloor \frac{t}{2} \right\rfloor + 1} = t + 1$.

The second case to consider is that $v$ has exactly $\frac{t}{2}$ degree–$d$, $d \geq t + 2$ neighbours, in which case by property 4 in Lemma 2, one of the neighbours if $v$ has degree at least $t + 3$. Then by the rule (★3), the new weight of $v$ is $w(v) \geq t + (\frac{t}{2} - 1) \frac{2}{\left\lfloor \frac{t}{2} \right\rfloor + 1} + \frac{2}{\left\lfloor \frac{t}{2} \right\rfloor + 1} = t + 1$.

Since the minimum degree in $G$ is $t$, by Property 1a of Lemma 2, it remains to consider the weights of the degree–$d$, $d \geq t + 2$ vertices. By property 2b in Lemma 2, each degree–$(t+2)$ vertex has at
most \( \lceil \frac{t+2}{2} \rceil \) degree-\( t \) neighbours. The rule (\( \ast \)) applies to each such vertex \( v \) and thus its new weight is \( w(v) \geq t + 2 - \lceil \frac{t+2}{2} \rceil = t + 1 \) (both when \( t \) is odd and even).

Consider now a degree-\( d \) vertex \( v, d \geq t + 3 \). If \( t \) is odd, then the rule (\( \ast \)) applies and by property 2b in Lemma 2, the new weight of \( v \) is \( w(v) \geq d - \lfloor \frac{d}{\lceil \frac{d}{t+1} \rceil} \rfloor > t + 1 \). (That is because, \( d - \lfloor \frac{d}{\lceil \frac{d}{t+1} \rceil} \rfloor \geq d - \frac{d}{t+1} = d - \frac{d}{t+1} \). Now \( d - \frac{d}{t+1} > t + 1 \) whenever \( d > t + 2 + \frac{1}{t} \). Since \( t > 1 \), that is our case.)

If \( t \) is even, then the rule (\( \ast \)) applies and by property 2b in Lemma 2, the new weight of \( v \) is \( w(v) \geq d - \lfloor \frac{d}{\lceil \frac{d}{t+1} \rceil} \rfloor = d - \frac{d}{t+1} = d - \frac{d}{t+1} \). Since \( d \geq t + 4 \), and \( w(v) \geq d - \lfloor \frac{d}{\lceil \frac{d}{t+1} \rceil} \rfloor > t + 1 \). (That is because, \( d - \lfloor \frac{d}{\lceil \frac{d}{t+1} \rceil} \rfloor \geq d - \frac{2d}{t+2} \). Now \( d - \frac{2d}{t+2} > t + 1 \) whenever \( d > t + 3 + \frac{2}{t} \). Since \( t > 2 \), that is our case.)

Therefore, after discharging every vertex has charge at least \( t + 1 \). Finally, we show that there is at least one vertex with the charge greater than \( t + 1 \). By the above arguments each degree \( d, d \geq t + 4 \), vertex \( v \) has weight \( w(v) \geq d - \frac{d}{t+1} \) when \( t \) is odd, and weight \( w(v) \geq d - \frac{2d}{t+2} \) when \( t \) is even. Thus, in either case, \( w(v) \geq \frac{dt}{t+2} \). By Lemma 1, \( G \) has a vertex \( v \) of degree at least \( 2t \). Since \( t \geq 4, 2t \geq t + 4 \). Therefore, \( G \) has a vertex \( v \) with weight \( w(v) \geq \frac{dt}{t+2} \geq \frac{2t^2}{t+2} \).

The new weights and the inequality 1 give

\[
\sum_{i=1}^{n} (w(v_i) - 6) \geq (n - 1)(t + 1 - 6) + \frac{2t^2}{t+2} - 6
\]

\[
= (n+1)(t-5) + \frac{8}{t+2}
\]

\[
> (n+1)(t-5).
\]

By Property 1c of Lemma 2, \( n \geq 2t + 1 \), and with \( t \geq 5 \) and inequality 1 we get

\[
(2t + 2)(t - 5) < \sum_{i=1}^{n} (w(v_i) - 6) \leq 6\mu - 12
\]

The inequality \((2t + 2)(t - 5) < 6\mu - 12\) is only true for \( t < 2 + \sqrt{3\mu + 3} \) thus for \( t = \left[ 2 + \sqrt{3\mu + 3} \right] \) we get a contradiction thereby completing the proof.

\( \square \)

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