A numerical characterization of polarized manifolds \((X, L)\) with \(K_X = -(n - i)L\) by the \(i\)th sectional geometric genus and the \(i\)th \(\Delta\)-genus

YOSHIKAI FUKUMA

Abstract

Let \((X, L)\) be a polarized manifold of dimension \(n\). In this paper, by using the \(i\)th sectional geometric genus and the \(i\)th \(\Delta\)-genus, we will give a numerical characterization of \((X, L)\) with \(K_X = -(n - i)L\) for the following cases (i) \(i = 2\), (ii) \(i = 3\) and \(n \geq 5\), (iii) \(\max\{2, \dim \text{Bs}|L| + 2\} \leq i \leq n - 1\).

1 Introduction

Let \(X\) be a projective variety with \(\dim X = n\) defined over the field of complex numbers, and let \(L\) be an ample line bundle on \(X\). Then \((X, L)\) is called a polarized variety. If \(X\) is smooth, then we say that \((X, L)\) is a polarized manifold. The main purpose of this paper is to give a numerical characterization of \((X, L)\) with \(K_X = -(n - i)L\). Then the following is well-known:

\begin{proposition}
Let \((X, L)\) be a polarized manifold of dimension \(n \geq 2\).

\begin{enumerate}
\item \((X, L)\) is a polarized manifold with \(K_X = -(n + 1)L\) (resp. \(K_X = -nL\)) if and only if \(2g(X, L) - 2 = -2L^n\) (resp. \(2g(X, L) - 2 = -L^n\)).
\item (See [7, (1.9) Theorem].) \((X, L)\) is a polarized manifold with \(K_X = -(n - 1)L\), which is called a Del Pezzo manifold, if and only if \(2g(X, L) - 2 = 0\) and \(\Delta(X, L) = 1\).
\end{enumerate}
\end{proposition}

(Here \(g(X, L)\) (resp. \(\Delta(X, L)\)) denotes the sectional genus (resp. the \(\Delta\)-genus) of \((X, L)\).)

As the next step, we want to give a numerical characterization of polarized manifolds with \(K_X = -(n - i)L\) and \(i \geq 2\) by using some invariants of \((X, L)\). In [15] and [17], we define the \(i\)th sectional geometric genus \(g_i(X, L)\) and the \(i\)th \(\Delta\)-genus \(\Delta_i(X, L)\) of \((X, L)\) for every integer \(i\) with \(0 \leq i \leq n\). The \(i\)th sectional geometric genus (resp. the \(i\)th \(\Delta\)-genus) is a generalization of the sectional genus (resp. \(\Delta\)-genus), that is, \(g_i(X, L) = g(X, L)\) (resp. \(\Delta_i(X, L) = \Delta(X, L)\)). So by looking at the Proposition [14] above carefully, the author thought maybe we were able to give a numerical characterization of polarized manifolds \((X, L)\) with \(K_X = -(n - i)L\) by using the \(i\)th sectional geometric genus and the \(i\)th \(\Delta\)-genus.

In this paper, as the main results, we prove the following:

\begin{theorem}
(See Theorems 4.2.1, 4.3.1 and 4.4.1 below.) Let \((X, L)\) be a polarized manifold of dimension \(n \geq 3\). Assume that one of the following types holds:

\begin{itemize}
\item[*] Key words and phrases. Polarized manifolds, Fano manifolds, sectional genus, \(\Delta\)-genus, sectional geometric genus, \(i\)th \(\Delta\)-genus.
\item[†] 2000 Mathematics Subject Classification. Primary 14C20; Secondary 14J30, 14J35, 14J40, 14J45.
\item[‡] This research was partially supported by the Grant-in-Aid for Scientific Research (C) (No.20540045), Japan Society for the Promotion of Science, Japan.
\end{itemize}
\end{theorem}
(a) \( i = 2 \).

(b) \( i = 3 \) and \( n \geq 5 \).

(c) \( \max\{2, \dim B_5 + 2\} \leq i \leq n - 1 \).

Then the following are equivalent one another.

- \( C(i, 1) \) \( K_X + (n - i)\mathcal{L} = \mathcal{O}_X \).
- \( C(i, 2) \) \( \Delta_i(X, \mathcal{L}) = 1 \) and \( 2g_1(X, \mathcal{L}) - 2 = (i - 1)\mathcal{L}^n \).
- \( C(i, 3) \) \( \Delta_i(X, \mathcal{L}) > 0 \) and \( 2g_1(X, \mathcal{L}) - 2 = (i - 1)\mathcal{L}^n \).
- \( C(i, 4) \) \( g_i(X, \mathcal{L}) = 1 \) and \( 2g_1(X, \mathcal{L}) - 2 = (i - 1)\mathcal{L}^n \).
- \( C(i, 5) \) \( g_i(X, \mathcal{L}) > 0 \) and \( 2g_1(X, \mathcal{L}) - 2 = (i - 1)\mathcal{L}^n \).

The author would like to thank Dr. Hironobu Ishihara for giving some comments about this paper.

**Notation and Conventions**

We say that \( X \) is a variety if \( X \) is an integral separated scheme of finite type. In particular \( X \) is irreducible and reduced if \( X \) is a variety. Varieties are always assumed to be defined over the field of complex numbers. In this article, we shall study mainly a smooth projective variety. The words “line bundles” and “Cartier divisors” are used interchangeably. The tensor products of line bundles are denoted additively.

\( \mathcal{O}(D) \): invertible sheaf associated with a Cartier divisor \( D \) on \( X \).

\( \mathcal{O}_X \): the structure sheaf of \( X \).

\( \chi(F) \): the Euler-Poincaré characteristic of a coherent sheaf \( F \).

\( h^i(F) := \dim H^i(X, F) \) for a coherent sheaf \( F \) on \( X \).

\( h^i(D) := h^i(\mathcal{O}(D)) \) for a Cartier divisor \( D \).

\( q(X)(= h^1(\mathcal{O}_X)) \): the irregularity of \( X \).

\( h^i(X, \mathbb{C}) := \dim H^i(X, \mathbb{C}) \).

\( b_i(X) := h^i(X, \mathbb{C}) \).

\( K_X \): the canonical divisor of \( X \).

\( \mathbb{P}^n \): the projective space of dimension \( n \).

\( \mathbb{Q}^n \): a quadric hypersurface in \( \mathbb{P}^{n+1} \).

\( \sim \) (or \( = \)): linear equivalence.

\( \det(\mathcal{E}) := \bigwedge^r \mathcal{E} \), where \( \mathcal{E} \) is a vector bundle of rank \( r \) on \( X \).

\( \mathbb{P}_X(\mathcal{E}) \): the projective space bundle associated with a vector bundle \( \mathcal{E} \) on \( X \).

\( H(\mathcal{E}) \): the tautological line bundle on \( \mathbb{P}_X(\mathcal{E}) \).

\( \mathcal{F}^\vee := \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) \).

\( c_i(\mathcal{E}) \): the \( i \)-th Chern class of a vector bundle \( \mathcal{E} \).

\( c_i(X) := c_i(T_X) \), where \( T_X \) is the tangent bundle of a smooth projective variety \( X \).

For a real number \( m \) and a non-negative integer \( n \), let

\[
[m]^n := \begin{cases} 
  m(m+1) \cdots (m+n-1) & \text{if } n \geq 1, \\
  1 & \text{if } n = 0.
\end{cases}
\]

\[
[m]_n := \begin{cases} 
  m(m-1) \cdots (m-n+1) & \text{if } n \geq 1, \\
  1 & \text{if } n = 0.
\end{cases}
\]

Then for \( n \) fixed, \([m]^n\) and \([m]_n\) are polynomials in \( m \) whose degree are \( n \).

For any non-negative integer \( n \),

\[
n! := \begin{cases} 
  [n]_n & \text{if } n \geq 1, \\
  1 & \text{if } n = 0.
\end{cases}
\]
Assume that \( m \) and \( n \) are integers with \( n \geq 0 \). Then we put
\[
\binom{m}{n} := \frac{[m]_n}{n!}
\]
We note that \( \binom{m}{n} = 0 \) if \( 0 \leq m < n \), and \( \binom{m}{0} = 1 \).

## 2 Preliminaries

Here we list up some facts which will be used later.

**Definition 2.1** (1) Let \( X \) (resp. \( Y \)) be an \( n \)-dimensional projective manifold, and let \( L \) (resp. \( A \)) be an ample line bundle on \( X \) (resp. \( Y \)). Then \((X, L)\) is called a simple blowing up of \((Y, A)\) if there exists a birational morphism \( \pi : X \to Y \) such that \( \pi \) is a blowing up at a point of \( Y \) and \( L = \pi^*(A) - E \), where \( E \) is the exceptional divisor.

(2) Let \( X \) (resp. \( M \)) be an \( n \)-dimensional projective manifold, and let \( L \) (resp. \( A \)) be an ample line bundle on \( X \) (resp. \( M \)). Then we say that \((M, A)\) is a reduction of \((X, L)\) if \((X, L)\) is obtained by a composite of simple blowing ups of \((M, A)\), and \((M, A)\) is not obtained by a simple blowing up of any polarized manifold. The morphism \( \mu : X \to M \) is called the reduction map.

**Definition 2.2** Let \((X, L)\) be a polarized manifold of dimension \( n \). We say that \((X, L)\) is a scroll (resp. quadric fibration, Del Pezzo fibration) over a normal projective variety \( Y \) with \( \dim Y = m \) if there exists a surjective morphism with connected fibers \( f : X \to Y \) such that \( K_X + (n - m + 1)L = f^*(A) \) (resp. \( K_X + (n - m)L = f^*(A) \), \( K_X + (n - m - 1)L = f^*(A) \)) for some ample line bundle \( A \) on \( Y \).

**Remark 2.1** If \((X, L)\) is a scroll over a smooth curve \( C \) (resp. a smooth projective surface \( S \)) with \( \dim X = n \geq 3 \), then by \([6]\) (3.2.1 Theorem) and \([5]\) Proposition 3.2.1 and Theorem 14.1.1] there exists an ample vector bundle \( E \) of rank \( n \) (resp. \( n - 1 \)) on \( C \) (resp. \( S \)) such that \((X, L) \cong (\mathbb{P}_C(E), H(E)) \) (resp. \((\mathbb{P}_S(E), H(E))\)).

**Theorem 2.1** Let \((X, L)\) be a polarized manifold with \( \dim X = n \geq 3 \). Then \((X, L)\) is one of the following types.

1. \((\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))\).
2. \((\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))\).
3. A scroll over a smooth projective curve.
4. \(K_X \sim -(n - 1)L\), that is, \((X, L)\) is a Del Pezzo manifold.
5. A quadric fibration over a smooth curve.
6. A scroll over a smooth projective surface.
7. Let \((M, A)\) be a reduction of \((X, L)\).
   7.1 \( n = 4 \), \((M, A) = (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))\).
   7.2 \( n = 3 \), \((M, A) = (\mathbb{Q}^4, \mathcal{O}_{\mathbb{Q}^4}(2))\).
   7.3 \( n = 3 \), \((M, A) = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))\).
   7.4 \( n = 3 \), \( M \) is a \( \mathbb{P}^2 \)-bundle over a smooth curve \( C \) and for any fiber \( F' \) of it, \((F', A|_{F'}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))\).
   7.5 \( K_M \sim -(n - 2)A\), that is, \((M, A)\) is a Mukai manifold.
(7.6) $(M, A)$ is a Del Pezzo fibration over a smooth curve.

(7.7) $(M, A)$ is a quadric fibration over a normal surface.

(7.8) $n \geq 4$ and $(M, A)$ is a scroll over a normal projective variety of dimension 3.

(7.9) $K_M + (n-2)A$ is nef and big.

**Proof.** See [5] Proposition 7.2.2, Theorem 7.2.4, Theorem 7.3.2, Theorem 7.3.4, and Theorem 7.5.3. See also [10] Chapter II, (11.2), (11.7), and (11.8). □

**Remark 2.2** Let $(X, L)$ be a polarized manifold with $\dim X = n \geq 3$.

1. $\kappa(K_X + (n-2)L) = -\infty$ if and only if $(X, L)$ is one of the types from (1) to (7.4) in Theorem 2.1.

2. $\kappa(K_X + (n-2)L) = 0$ if and only if $(X, L)$ is (7.5) in Theorem 2.1.

3. $\kappa(K_X + (n-2)L) \geq 1$ if and only if $(X, L)$ is one of the types from (7.6) to (7.9) in Theorem 2.1.

**Definition 2.3** (7.5.7 Definition-Notation]) Let $(X, L)$ be a polarized manifold of dimension $n \geq 3$, and let $(M, A)$ be a reduction of $(X, L)$. Assume that $K_M + (n-2)A$ is nef and big. Then for large $m \gg 0$ the morphism $\varphi : M \to W$ associated to $|m(K_M + (n-2)A)|$ has connected fibers and normal image $W$. Then we note that there exists an ample line bundle $K$ on $W$ such that $K_M + (n-2)A = \varphi^*(K)$. Let $D : = (\varphi^*A)^{\vee\vee}$, where $^{\vee\vee}$ denotes the double dual. Then the pair $(W, D)$ together with $\varphi$ is called the second reduction of $(X, L)$.

**Remark 2.3** (1) If $K_M + (n-2)A$ is nef and big but not ample, then $\varphi$ is equal to the nef value morphism of $A$.

(2) If $K_M + (n-2)A$ is ample, then $\varphi$ is an isomorphism.

(3) If $n \geq 4$, then $W$ has isolated terminal singularities and is 2-factorial. Moreover if $n$ is even, then $X$ is Gorenstein (see [5] Proposition 7.5.6)).

Here we consider the characterization of $(X, L)$ with $\kappa(K_X + (n-3)L) = -\infty$. We note that $\kappa(K_X + (n-1)L) = -\infty$ (resp. $\kappa(K_X + (n-2)L) = -\infty$) if and only if $(X, L)$ is one of the types from (1) to (3) (resp. from (1) to (7.4)) in Theorem 2.1. Here we consider the case where $\kappa(K_X + (n-3)L) = -\infty$. If $(X, L)$ is one of the types from (1) to (7.8) in Theorem 2.1, then $\kappa(K_X + (n-3)L) = -\infty$ holds. So we assume that $K_M + (n-2)A$ is nef and big. Then there exist a normal projective variety $W$ with only 2-factorial isolated terminal singularities, a birational morphism $\phi_2 : M \to W$ and an ample line bundle $K$ on $W$ such that $K_M + (n-2)A = (\phi_2)^*(K)$. Let $D : = (\phi_2)_*(A)^{\vee\vee}$. Then $D$ is a 2-Cartier divisor on $W$ and $K = K_W + (n-2)D$ (see [5] Lemma 7.5.8). Then the pair $(W, D)$ is the second reduction of $(X, L)$ (see Definition 2.3). Here we note that if $K_M + (n-2)A$ is ample, then $(W, K) \cong (M, K_M + (n-2)A)$.

Then the following properties hold:

1. $\kappa(K_X + (n-3)L) = \kappa(K_W + (n-3)K)$ holds [5] Corollary 7.6.2.

2. $(n-2)(K_W + (n-3)D) = K_W + (n-3)K$ and $K_M + (n-3)A = \phi_2^*(K_W + (n-3)D) + \Delta$ for an exceptional $\mathbb{Q}$-effective divisor $\Delta$ of $\phi_2$. Therefore

\[
m(n-2)(K_X + (n-3)L) = m(n-2)\phi_1^*(K_M + (n-3)A) + E_1
\]

\[
= m(n-2)\phi_1^* \circ \phi_2^*(K_W + (n-3)D) + E_1 + m(n-2)\Delta
\]

\[
= m\phi_1^* \circ \phi_2^*(K_W + (n-3)K) + E_1 + m(n-2)\Delta.
\]

(Here $\phi_1 : X \to M$ is a reduction of $(X, L)$ and $E_1$ is a $\phi_1$-exceptional effective divisor.)
Proposition 2.1

Let $\mathcal{L}$ be a polarized manifold of dimension $n \geq 5$, $(M, A)$ a reduction of $(X, \mathcal{L})$, and $(W, K)$ the second reduction of $(X, \mathcal{L})$. Then $\kappa(K_X + (n-3)\mathcal{L}) = -\infty$ if and only if $(X, \mathcal{L})$ satisfies one of the following:

1. $(X, \mathcal{L})$ is one of the types $(1), (2), (3), (4), (5), (6), (7.5), (7.6), (7.7)$ in Theorem 2.1.
2. $K_M + (n-2)A$ is nef and big, and $(W, K)$ is one of the following:
   1. $\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(1)$.
   2. $1, 2, 3$ in Theorem 7.7.5.

Proof. See [5] Theorems 7.7.2, 7.7.3, 7.7.5 and Proposition 7.7.9. $\Box$

So we consider the case of $n = 4$. In this case $M^\sharp$ and $W$ are Gorenstein (see [5] Proposition 7.5.6 and 7.5.7 Definition-Notation). Then by the proof of [11] Section 4, we see that $M^\sharp$ is one of the types in [11] (4.4.1). If $(W, K)$ or $M^\sharp$ is either (4.2), (4.4.1), (4.4.2), (4.6.0.0), (4.6.0.1.0), (4.6.0.2.1), (4.6.1), (4.7) or (4.8.0) in [11] (4.4.1), then we see that $\kappa(K_X + \mathcal{L}) = -\infty$.

Assume that $(W, K)$ is the type (4.4.4) in [11] (4.4.1). Then we note that $\tau(K) = 3$ and there exist a normal Gorenstein projective variety $W_2$, an ample line bundle $K_2$ on $W_2$ and a birational morphism $\mu : W \to W_2$ such that $\mu$ is the simultaneous contraction to distinct smooth points of divisors $E_i \cong \mathbb{P}^3$ such that $E_i \subset \text{reg}(W), E_i|_E_i \cong \mathcal{O}_{\mathbb{P}^3}(-1), K_W + 3K = \mu^*(K_{W_2} + 3K_2)$ and $K_{W_2} + 3K_2$ is ample, that is, $\tau(K_2) < 3$. Moreover we infer that $W_2$ has the same singularities as $W$ by above. Since $E_i \subset \text{reg}(W)$, we have $\psi^{-1}(E_i) \cong E_i$ by the definition of $\psi$. Hence there exist a normal Gorenstein projective variety $W_2^3$ and birational morphisms $\mu^\sharp : M^\sharp \to W_2^3$ and $\psi^\sharp : W_2^3 \to W_2$ such that $\mu \circ \psi = \psi^\sharp \circ \mu^\sharp$. We note that $\mu^\sharp : M^\sharp \to W_2^3$ is the contraction of $\psi^{-1}(E_2)$ and $W_2^3$ has the same singularities as $M^\sharp$. The pair $(W_2, K_2)$ is a reduction of $(W, K)$ and is called the $2\frac{1}{2}$ reduction of $(W, K)$ in [3] (2.2 Theorem-Definition). We also note that $h^j(O_X) = h^j(O_M) = h^j(O_W) = h^j(O_{W_2}) = h^j(O_{W_2^3})$ for this $\psi^\sharp : W_2^3 \to W_2$ and $(W_2, K_2)$, we can apply the same argument as in [11] Section 4. If $\tau(K_2) \leq 1$, then we can prove that $\kappa(K_X + \mathcal{L}) \geq 0$. So we assume that $\tau(K_2) > 1$. Then $(W_2, K_2)$ is either (4.6.0.0), (4.6.0.1.0), (4.6.0.2.1), (4.6.1), (4.6.4), (4.7) or (4.8.0) in [11] (4.4.1). If $(W_2, K_2)$ is either (4.6.0.0), (4.6.0.1.0), (4.6.0.2.1), (4.6.1), (4.7) or (4.8.0) in [11] (4.4.1), then we see that $\kappa(K_X + \mathcal{L}) = -\infty$.

If $(W_2, K_2)$ is the type (4.6.4) in [11] (4.4.1), then by the same argument as in [11] Section 4 we see that there exist a normal Gorenstein projective variety $W_3$, an ample line bundle $K_3$ on $W_3$ and a birational morphism $\mu_2 : W_2 \to W_3$ such that $W_3$ has the same singularities as $W_2, K_{W_3} + 2K_3 = \mu_2^*(K_{W_2} + 2K_3)$ and $K_{W_3} + 2K_3$ is ample, that is, $\tau(K_3) < 2$. Here we note that $\kappa(K_X + \mathcal{L}) = \kappa(K_{W_2} + K_2) = \kappa(K_{W_3} + K_3)$.

If $\tau(K_3) \leq 1$, then $\kappa(K_X + \mathcal{L}) = \kappa(K_{W_3} + K_3) \geq 0$.

If $\tau(K_3) > 1$, then $(W_3, K_3)$ is either (4.7) or (4.8.0) by the same argument as in [11] Section 4 and we have $\kappa(K_X + \mathcal{L}) = \kappa(K_{W_3} + K_3) = -\infty$.

By the above argument, we get the following:
Theorem 2.2 Let \((X, \mathcal{L})\) be a polarized manifold of dimension \(n = 4\).

1. The inequality \(\kappa(K_X + \mathcal{L}) \geq 0\) holds if and only if there exist a normal projective variety \(W_3\) with only isolated terminal singularities, an ample line bundle \(K_3\) on \(W_3\), and a birational morphism \(\Phi : X \to W_3\) such that \(\tau(K_3) \leq 1\) and \(h^0(2m(K_X + \mathcal{L})) = h^0(m(K_{W_3} + K_3))\) for every positive integer \(m\).

2. The equality \(\kappa(K_X + \mathcal{L}) = -\infty\) holds if and only if \((X, \mathcal{L})\) satisfies one of the following:

2.1 The pair \((X, \mathcal{L})\) is either (1), (2), (3), (4), (5), (6), (7.1), (7.5), (7.6), (7.7) or (7.8) in Theorem 2.2.

2.2 There exist a normal projective variety \(W_3\) with only isolated terminal singularities, an ample line bundle \(K_3\) on \(W_3\), and a birational morphism \(\Phi : X \to W_3\) such that \((W_3, K_3)\) is either (4.2), (4.4.0), (4.4.1), (4.6.0.0), (4.6.0.1.0), (4.6.0.2.1), (4.6.1), (4.7) or (4.8.0) in [11] (4.11).

Furthermore we need the following two lemmas.

Lemma 2.1 Let \((X, \mathcal{L})\) be a polarized manifold of dimension \(n \geq 4\), and let \((M, \mathcal{A})\) be a reduction of \((X, \mathcal{L})\). If \(\kappa(K_X + (n-3)\mathcal{L}) = -\infty\), then \(h^j(\mathcal{O}_X) = 0\) for any \(j\) with \(j \geq 3\) unless \((M, \mathcal{A})\) is a scroll over a normal projective variety of dimension \(3\). If \((M, \mathcal{A})\) is a scroll over a normal projective variety of dimension \(3\), then \(h^j(\mathcal{O}_X) = h^j(\mathcal{O}_M) = 0\) for every integer \(j\) with \(j \geq 4\).

Proof. (I) First we assume that \(n \geq 5\). By assumption and Proposition 2.1, \((X, \mathcal{L})\) satisfies either (1), (2.1) or (2.2) in Proposition 2.1. Here we note that since \(h^j(\mathcal{O}_X) = h^j(\mathcal{O}_M) = h^j(\mathcal{O}_W)\), we have only to prove that \(h^j(\mathcal{O}_W) = 0\). But by Proposition 2.1 it is easy to show this and left to the reader.

(II) Next we assume that \(n = 4\). By Theorem 2.2, \((X, \mathcal{L})\) satisfies either (2.1) or (2.2) in Theorem 2.2. Here we use notation above. Here we note that \(h^j(\mathcal{O}_X) = h^j(\mathcal{O}_M) = h^j(\mathcal{O}_W) = h^j(\mathcal{O}_{W_3}) = h^j(\mathcal{O}_{M_2}) = h^j(\mathcal{O}_{W_4})\).

(II.A) If \((W_3, K_3)\) is a \((\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))\)-fibration over a smooth curve \(C\), then there exists an extremal ray \(R\) such that \((K_{W_3} + 2K_3)\mathcal{R} = 0\) (see [11] (4.6.1)). Let \(\rho\) be the contraction morphism of \(R\). Then \(\dim \rho(X) = 1\), \(\rho(X) = C\) and \(\rho : X \to C\) is the \((\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))\)-fibration. Moreover there exists a line bundle \(E\) on \(C\) such that \(K_{W_3} + 2K_3 = \rho^*(E)\). Since \(K_3\) is ample, by [25] Theorem 1-2-5 we see that \(R^j\rho_* (K_{W_3} + 2K_3) = 0\) for every integer \(j\) with \(j > 0\). Hence \(0 = R^j\rho_* (\rho^*(E)) \cong R^j\rho_* (\mathcal{O}_{W_3}) \otimes \mathcal{B}\) and we have \(R^j\rho_* (\mathcal{O}_{W_3}) = 0\) for every positive integer \(j\). Therefore \(h^j(\mathcal{O}_{W_3}) = 0\) for every integer \(j\) with \(j \geq 2\).

(II.B) Assume that \((W_3, K_3)\) is the cone over \((\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))\). Then the following holds: Let \(E := \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(2), \mathcal{P} := \mathbb{P}_{\mathbb{P}^3}(E)\) and \(H(E)\) the tautological line bundle on \(P\). Let \(\pi : P \to \mathbb{P}^N\) be the morphism associated to \(H(E)\). Then \(W_3 = \pi(P)\) (see [11] 1.1.8 in Chapter I). First we note that \(h^j(\mathcal{O}_P) \geq h^j(\mathcal{O}_{W_3})\) for every nonnegative integer \(j\). On the other hand \(h^j(\mathcal{O}_P) = h^j(\mathcal{O}_{\mathbb{P}^3}) = 0\) for every integer \(j\) with \(j \geq 1\). Therefore we get \(h^j(\mathcal{O}_{W_3}) = 0\) for every integer \(j\) with \(j \geq 1\).

(II.C) For other cases it is easy and left to the reader. □

Lemma 2.2 Let \((X, \mathcal{L})\) be a polarized manifold of dimension \(n \geq 3\), and let \((M, \mathcal{A})\) be a reduction of \((X, \mathcal{L})\). Assume that \(K_M + (n-2)\mathcal{A}\) is nef and big. Let \((W, \mathcal{D})\) be the second reduction of \((X, \mathcal{L})\) and \(\phi : M \to W\) its morphism. (Here we use notation in Definition 2.3) Then \(h^j(\mathcal{A}) = h^j(\mathcal{D})\) for every integer \(j \geq 3\).
Proof. As we said in Definition 2.3 there exists an ample line bundle $K$ on $Y$ such that $K_M + (n-2)A = \varphi^*(K)$. By [25] Theorem 1-2-5 we have $R^j\varphi_* (K_M + (n-1)A) = 0$. On the other hand $R^j\varphi_* (K_M + (n-1)A) = R^j\varphi_* (\varphi^*(K) \otimes A) = \mathcal{K} \otimes R^j\varphi_* (A)$. Therefore $R^j\varphi_* (A) = 0$ and we have $h^j(A) = h^j(\varphi_* (A))$ for every positive integer $j$.

Since $A$ is a line bundle on $M$, we see that $\varphi_*(A)$ is a torsion free coherent sheaf on $W$. Then there exists an injective homomorphism $\mu : \varphi_*(A) \to (\varphi_*(A))^{\vee \vee}$. Hence we get the following exact sequence

$$0 \to \varphi_*(A) \to (\varphi_*(A))^{\vee \vee} \to \text{Coker} \mu \to 0.$$  

Here we note that $\dim \text{Supp}(\text{Coker} \mu) \leq 1$ because there exists a closed subset $Z$ on $W$ such that $\dim Z \leq 1$ and $M \setminus \varphi^{-1}(Z) \cong W \setminus Z$. Therefore $h^j(\text{Coker} \mu) = 0$ for every integer $j$ with $j \geq 2$ by [22] Theorem 2.7 in Chapter III or [23] Theorem 4.6*. Hence we have $h^j(\varphi_*(A)) = h^j((\varphi_*(A))^{\vee \vee})$ for every integer $j$ with $j \geq 3$. Since $D = (\varphi_*(A))^{\vee \vee}$, we get the assertion. □

**Definition 2.4** Let $X$ be a smooth projective variety and let $\mathcal{F}$ be a vector bundle on $X$. Then for every integer $j$ with $j \geq 0$, the $j$-th Segre class $s_j(\mathcal{F})$ of $\mathcal{F}$ is defined by the following equation: $c_i(\mathcal{F}^\vee)s_i(\mathcal{F}) = 1$, where $c_i(\mathcal{F}^\vee)$ is the Chern polynomial of $\mathcal{F}^\vee$ and $s_i(\mathcal{F}) = \sum_{j \geq 0} s_j(\mathcal{F})t^j$.

**Remark 2.4** (1) Let $X$ be a smooth projective variety and let $\mathcal{F}$ be a vector bundle on $X$. Let $s_j(\mathcal{F})$ be the Segre class which is defined in [21] Chapter 3. Then $s_j(\mathcal{F}) = \delta_j(\mathcal{F}^\vee)$. (2) For every integer $i$ with $1 \leq i$, $s_i(\mathcal{F})$ can be written by using the Chern classes $c_j(\mathcal{F})$ with $1 \leq j \leq i$. (For example, $s_1(\mathcal{F}) = c_1(\mathcal{F})$, $s_2(\mathcal{F}) = c_1(\mathcal{F})^2 - c_2(\mathcal{F})$, and so on.)

### 3 Review on the $i$th sectional geometric genus and the $i$th $\Delta$-genus of polarized varieties.

Here we are going to review the $i$th sectional geometric genus and the $i$th $\Delta$-genus of polarized varieties $(X, \mathcal{L})$ for every integer $i$ with $0 \leq i \leq \dim X$. Up to now, there are many investigations of $(X, \mathcal{L})$ via the sectional genus and the $\Delta$-genus. In order to analyze $(X, \mathcal{L})$ more deeply, the author extended these notions. In [15] Definition 2.1 we defined an invariant called the $i$th sectional geometric genus which is thought to be a generalization of the sectional genus. First we recall the definition of this invariant.

**Notation 3.1** Let $(X, \mathcal{L})$ be a polarized variety of dimension $n$, and let $\chi(t\mathcal{L})$ be the Euler-Poincaré characteristic of $t\mathcal{L}$. Then $\chi(t\mathcal{L})$ is a polynomial in $t$ of degree $n$, and we can describe this as

$$\chi(t\mathcal{L}) = \sum_{j=0}^{n} \chi_j(X, \mathcal{L}) \binom{t+j-1}{j}.$$  

**Definition 3.1** ([15] Definition 2.1) Let $(X, \mathcal{L})$ be a polarized variety of dimension $n$. Then for any integer $i$ with $0 \leq i \leq n$ the $i$th sectional geometric genus $g_i(X, \mathcal{L})$ of $(X, \mathcal{L})$ is defined by the following.

$$g_i(X, \mathcal{L}) = (-1)^i(\chi_{n-i}(X, \mathcal{L}) - \chi(\mathcal{O}_X)) + \sum_{j=0}^{n-i} (-1)^{n-i-j} h^{n-j}(\mathcal{O}_X).$$  

**Remark 3.1** (1) Since $\chi_{n-i}(X, \mathcal{L}) \in \mathbb{Z}$, the invariant $g_i(X, \mathcal{L})$ is an integer by definition.  

(2) If $i = \dim X = n$, then $g_n(X, \mathcal{L}) = h^n(\mathcal{O}_X)$ and $g_n(X, \mathcal{L})$ is independent of $\mathcal{L}$.  

(3) If $i = 0$, then $g_0(X, \mathcal{L}) = \mathcal{L}^n$.  

7
(4) If \(i = 1\), then \(g_1(X, \mathcal{L}) = g(\mathcal{L})\), where \(g(\mathcal{L})\) is the sectional genus of \((X, \mathcal{L})\). If \(X\) is smooth, then \(g_1(X, \mathcal{L}) = 1 + (1/2)(K_X + (n-1)\mathcal{L})\mathcal{L}^{n-1}\), where \(K_X\) denotes the canonical line bundle on \(X\).

(5) Let \((X, \mathcal{L})\) be a polarized manifold of dimension \(n\) and let \((M, \mathcal{A})\) be a reduction of \((X, \mathcal{L})\). Then \(g_i(X, \mathcal{L}) = g_i(M, \mathcal{A})\) for every integer \(i\) with \(1 \leq i \leq n\).

The following are main problems about the \(i\)th sectional geometric genus.

**Problem 3.1**  
(i) Does the \(i\)th sectional geometric genus have a property similar to that of the sectional genus? For example, there are the following two questions.

(i.1) Does \(g_i(X, \mathcal{L}) \geq 0\) hold? More strongly, does \(g_i(X, \mathcal{L}) \geq h^i(\mathcal{O}_X)\) hold?

(i.2) Can we get the \(i\)th sectional geometric genus version of the theory on sectional genus?

(ii) Are there any relationship between \(g_i(X, \mathcal{L})\) and \(g_{i+1}(X, \mathcal{L})\)?

(iii) Classify \((X, \mathcal{L})\) by the value of the \(i\)th sectional geometric genus.

(iv) What is the geometric meaning of the \(i\)th sectional geometric genus?

**Remark 3.2**  
(1) First we consider Problem 3.1 (i.1). At present we can prove that the non-negativity of \(g_i(X, \mathcal{L})\) holds if \((a)\) \(i = 0\), \((b)\) \(i = 1\), \((c)\) \(i = 2\) and \(n = 3\), \((d)\) \(i = n\). But in general it is unknown whether \(g_i(X, \mathcal{L})\) is non-negative or not. Next we consider the second inequality. Of course, if the second inequality holds, then non-negativity of \(g_i(X, \mathcal{L})\) also holds. If \(i = 0\) or \(n\), then this inequality holds. But it is unknown whether this inequality holds or not in general.

(2) Next we consider Problem 3.1 (ii). If \(Bs|\mathcal{L}| = \emptyset\), then \(g_i(X, \mathcal{L}) = 0\) implies \(g_{i+1}(X, \mathcal{L}) = 0\).

(3) Next we consider Problem 3.1 (iii). If \(i = 1\), then the classification of polarized manifolds \((X, \mathcal{L})\) with \(g_i(X, \mathcal{L}) \leq 2\) was obtained (see [3], [24], [3], and [9]).

If \(i = 2\), then the classification of polarized manifolds \((X, \mathcal{L})\) with the following is obtained (see [15], Corollary 3.5 and Theorem 3.6] and [20]):

(ii.1) The case where \(Bs|\mathcal{L}| = \emptyset\) and \(g_2(X, \mathcal{L}) = h^2(\mathcal{O}_X)\).

(ii.2) The case where \(\mathcal{L}\) is very ample and \(g_2(X, \mathcal{L}) = h^2(\mathcal{O}_X) + 1\).

(4) Finally we consider Problem 3.1 (iv). Namely we will explain the geometric meaning of the \(i\)th sectional geometric genus. First we will give the following definition.

**Definition 3.2** Let \((X, \mathcal{L})\) be a polarized variety of dimension \(n\). Then \(\mathcal{L}\) has a \(k\)-ladder if there exists a sequence of irreducible and reduced subvarieties \(X \supset X_1 \supset \cdots \supset X_k\) such that \(X_i \in |\mathcal{L}_{i-1}|\) for \(1 \leq i \leq k\), where \(X_0 := X\), \(\mathcal{L}_0 := \mathcal{L}\) and \(\mathcal{L}_i := \mathcal{L}|_{X_i}\). Here we note that \(\dim X_j = n - j\). Let \(r_{p,q} : H^p(X_q, \mathcal{L}_q) \to H^p(X_{q+1}, \mathcal{L}_{q+1})\) be the natural map.

**Theorem 3.1** ([16] Propositions 2.1 and 2.3, and Theorem 2.4) Let \(X\) be a projective variety of dimension \(n \geq 2\) and let \(\mathcal{L}\) be an ample line bundle on \(X\). Assume that \(h^i(-s\mathcal{L}) = 0\) for every integers \(t\) and \(s\) with \(0 \leq t \leq n - 1\) and \(1 \leq s\), and \(|\mathcal{L}|\) has an \((n - i)\)-ladder for an integer \(i\) with \(1 \leq i \leq n\). Then the \(i\)th sectional geometric genus has the following properties.

(1) \(g_i(X_j, \mathcal{L}_j) = g_i(X_{j+1}, \mathcal{L}_{j+1})\) for every integer \(j\) with \(0 \leq j \leq n - i - 1\). (Here we use the notation in Definition 3.2)
Let Proposition 3.1 from (1).

(2) By using the Serre duality and the Kawamata-Viehweg vanishing theorem, we get the assertion

Assume that $(X, L)$ is a polarized manifold with $Bs|L| = \emptyset$, then $\mathcal{L}$ has an $(n - i)$-ladder $X \supset X_1 \supset \cdots \supset X_{n-i}$ such that each $X_j$ is smooth, and from Theorem 3.1 (1) and Remark 3.1 (2) we see that $g_i(X, \mathcal{L}) = g_i(X_{n-i}, h^0(\mathcal{O}_{X_{n-i}})) = h^0(\Omega_{X_{n-i}})$, that is, the $i$th sectional geometric genus is the geometric genus of $i$-dimensional projective variety $X_{n-i}$. This is a reason why we call this invariant the $i$th sectional geometric genus.

From Theorem 3.1 we see that the $i$th sectional geometric genus is expected to have properties similar to those of the geometric genus of projective surfaces and we can propose several problems which can be considered as a generalization of theorems in the theory of surfaces. In [19], we investigated them. See [19] for further detail.

For other results concerning the $i$th sectional geometric genus, for example, see [16], [18] and [19].

The following result will be used later.

**Theorem 3.2** Let $X$ be a projective variety with $\dim X = n$ and let $\mathcal{L}$ be a nef and big line bundle on $X$.

(1) For any integer $i$ with $0 \leq i \leq n - 1$, we have

$$g_i(X, \mathcal{L}) = \sum_{j=0}^{n-i-1} (-1)^{n-j} \binom{n-i}{j} \chi(-(n-i-j)\mathcal{L}) + \sum_{k=0}^{n-i} (-1)^{n-i-k} h^{n-k}(\mathcal{O}_X).$$

(2) Assume that $X$ is smooth. Then for any integer $i$ with $0 \leq i \leq n - 1$, we have

$$g_i(X, \mathcal{L}) = \sum_{j=0}^{n-i-1} (-1)^j \binom{n-i}{j} h^0(K_X + (n-i-j)\mathcal{L}) + \sum_{k=0}^{n-i} (-1)^{n-i-k} h^{n-k}(\mathcal{O}_X).$$

**Proof.** (1) By the same argument as in the proof of [15 Theorem 2.2], we obtain

$$\chi_{n-i}(X, \mathcal{L}) = \sum_{j=0}^{n-i} (-1)^{n-i-j} \binom{n-i}{j} \chi(-(n-i-j)\mathcal{L}).$$

Hence by Definition 3.1 we get the assertion.

(2) By using the Serre duality and the Kawamata-Viehweg vanishing theorem, we get the assertion from (1). □

**Proposition 3.1** Let $(X, \mathcal{L})$ be a polarized manifold of dimension $n \geq 4$ and let $(M, \mathcal{A})$ be a reduction of $(X, \mathcal{L})$. If $\kappa(K_X + (n-3)\mathcal{L}) = -\infty$, then $g_j(X, \mathcal{L}) = g_j(M, \mathcal{A}) = 0$ for every integer $j \geq 3$ unless $(M, \mathcal{A})$ is a scroll over a normal projective variety of dimension 3.

**Proof.** Assume that $(M, \mathcal{A})$ is not a scroll over a normal projective variety of dimension 3. Then by Lemma 2.1 we have $h^j(\mathcal{O}_X) = h^j(\mathcal{O}_M) = 0$ for every integer $j \geq 3$. By assumption, we get $h^0(K_M + t\mathcal{A}) = 0$ for every integer $t$ with $1 \leq t \leq n-3$. Hence by Theorem 3.2 (2) and Remark 3.1 (5) we get $g_j(X, \mathcal{L}) = g_j(M, \mathcal{A}) = 0$ for every integer $j \geq 3$. This completes the proof. □

**Remark 3.3** If $(M, \mathcal{A})$ is a scroll over a normal projective variety of dimension 3, then by [15 Example 2.10 (8)] we have $g_j(X, \mathcal{L}) = g_j(M, \mathcal{A}) = 0$ for every integer $j \geq 4$ and $g_3(X, \mathcal{L}) = g_3(M, \mathcal{A}) = h^3(\mathcal{O}_M) = h^3(\mathcal{O}_X)$. 

9
In particular, we have $\Delta(X, \mathcal{L}) \leq 1$ in Definition 3.3. Remark 3.4 may be unknown. So it is important to investigate the following problems in order to understand the meaning and properties of the $i$th $\Delta$-genus.

Problem 3.2

(i) Does the $i$th $\Delta$-genus have properties similar to those of the $\Delta$-genus? For example, there are the following two questions.

Theorem 3.3 (See e.g. [10] §3 in Chapter I.) Let $X$ be a projective variety of dimension $n \geq 2$ and let $\mathcal{L}$ be an ample line bundle on $X$. We use the notation in Definition 3.2. If $|\mathcal{L}|$ has an $(n-1)$-ladder and $h^0(\mathcal{L}_{n-1}) > 0$, then

$$\Delta(X, \mathcal{L}) = \sum_{j=0}^{n-1} \dim \text{Coker}(r_{0,j}).$$

In particular, we have $\Delta(X, \mathcal{L}) \geq \Delta(X_1, \mathcal{L}_1) \geq \cdots \geq \Delta(X_{n-1}, \mathcal{L}_{n-1}) \geq 0$.

Here we want to give the definition of the $i$th $\Delta$-genus which satisfies a generalization of Theorem 3.3. Now we are going to give the definition of the $i$th $\Delta$-genus.

Definition 3.3 (See [17] Definition 2.1.) Let $(X, \mathcal{L})$ be a polarized variety of dimension $n$. For every integer $i$ with $0 \leq i \leq n$, the $i$th $\Delta$-genus $\Delta_i(X, \mathcal{L})$ of $(X, \mathcal{L})$ is defined by the following formula:

$$\Delta_i(X, \mathcal{L}) = \begin{cases} 0 & \text{if } i = 0, \\ g_{i-1}(X, \mathcal{L}) - \Delta_{i-1}(X, \mathcal{L}) + (n-i+1)h^{i-1}(\mathcal{O}_X) - h^{i-1}(\mathcal{L}) & \text{if } 1 \leq i \leq n. \end{cases}$$

Remark 3.4

(1) If $i = 1$, then $\Delta_1(X, \mathcal{L})$ is equal to the $\Delta$-genus of $(X, \mathcal{L})$.

(2) If $i = n$, then $\Delta_n(X, \mathcal{L}) = h^n(\mathcal{O}_X) - h^n(\mathcal{L})$ (see [17] Proposition 2.4).

(3) For every integer $i$ with $1 \leq i \leq n$, by the definition of the $i$th $\Delta$-genus, we have the following equality which will be used later.

$$\Delta_{i-1}(X, \mathcal{L}) = g_{i-1}(X, \mathcal{L}) - \Delta_i(X, \mathcal{L}) + (n-i+1)h^{i-1}(\mathcal{O}_X) - h^{i-1}(\mathcal{L}).$$

(4) Let $(X, \mathcal{L})$ be a polarized manifold of dimension $n$ and let $(M, \mathcal{A})$ be a reduction of $(X, \mathcal{L})$. Then $\Delta_i(X, \mathcal{L}) = \Delta_i(M, \mathcal{A})$ for every integer $i$ with $2 \leq i \leq n$ (see [17] Corollary 2.11).

Then, for the case of the $i$th $\Delta$-genus, we can prove the following.

Theorem 3.4 (See [17] Theorem 2.8 and Corollary 2.9] and [10] Proposition 2.1.) Let $X$ be a projective variety of dimension $n \geq 2$ and let $\mathcal{L}$ be an ample line bundle on $X$. We use the notation in Definition 3.2. Assume that $h^t(-s\mathcal{L}) = 0$ for every integers $t$ and $s$ with $0 \leq t \leq n-1$ and $1 \leq s$. If $|\mathcal{L}|$ has an $(n-i)$-ladder and $h^0(\mathcal{L}_{n-i}) > 0$ for an integer $i$ with $1 \leq i \leq n$, then

$$\Delta_i(X, \mathcal{L}) = \sum_{j=0}^{n-i} \dim \text{Coker}(r_{i-1,j}).$$

In particular, we have $\Delta_i(X, \mathcal{L}) \geq \Delta_i(X_1, \mathcal{L}_1) \geq \cdots \geq \Delta_i(X_{n-i}, \mathcal{L}_{n-i}) \geq 0$.

The definition of the $i$th $\Delta$-genus is so complicated that a lot of things about the $i$th $\Delta$-genus are unknown. So it is important to investigate the following problems in order to understand the meaning and properties of the $i$th $\Delta$-genus.

Problem 3.2

(i) Does the $i$th $\Delta$-genus have properties similar to those of the $\Delta$-genus? For example, there are the following two questions.
Remark 3.5 If $X$ is smooth and $\mathcal{L}$ is ample, then the following facts on Problem 3.2 are known.

(1) First we consider Problem 3.2 (i.1). If $i = 1$, then $\Delta_1(X, \mathcal{L}) \geq 0$ holds (see [10] (4.2 Theorem)). Moreover if $\mathcal{L}$ is base point free, then $\Delta_i(X, \mathcal{L}) \geq 0$ holds for every integer $i$ with $0 \leq i \leq n$. But unfortunately, there exists an example $(X, \mathcal{L})$ such that $\Delta_i(X, \mathcal{L}) < 0$ (see [17] Section 4).

(2) Next we consider Problem 3.2 (ii). If $i = 1$ and $\mathcal{L}$ is merely ample, then it is known that $g_1(X, \mathcal{L}) = 0$ if and only if $\Delta_1(X, \mathcal{L}) = 0$ (see [10] (12.1 Theorem)). Next we consider the case of $i \geq 2$. Then under the assumption that $Bs|\mathcal{L}| = \emptyset$ we see that $g_i(X, \mathcal{L}) = 0$ if and only if $\Delta_i(X, \mathcal{L}) = 0$ (see [14] Theorem 3.13).

(3) Next we consider Problem 3.2 (iii) under the assumption that $\mathcal{L}$ is base point free. Then, for example, we get the following: If $\Delta_i(X, \mathcal{L}) \leq i - 1$, then $\Delta_{i+1}(X, \mathcal{L}) = 0$ (see [17] Proposition 3.9). In particular, if $\Delta_i(X, \mathcal{L}) = 0$, then $\Delta_{i+1}(X, \mathcal{L}) = 0$. Maybe there will be several relationship between $\Delta_i(X, \mathcal{L})$ and $\Delta_{i+1}(X, \mathcal{L})$ other than this.

(4) For Problem 3.2 (iv), there exists the classification of $(X, \mathcal{L})$ by the value of $\Delta_2(X, \mathcal{L})$ as follows:

(4.1) The classification of polarized manifolds $(X, \mathcal{L})$ such that $Bs|\mathcal{L}| = \emptyset$ and $\Delta_2(X, \mathcal{L}) = 0$ (see [17] Theorem 3.13 and Remark 3.13.1).

(4.2) The classification of polarized manifolds $(X, \mathcal{L})$ such that $\mathcal{L}$ is very ample and $\Delta_2(X, \mathcal{L}) = 1$ (see [17] Theorem 3.17 and [20] Remark 2).

(5) At present, we do not know much about any answer to Problem 3.2 (v). This problem seems to be the most difficult problem among the above problems even in the case where $\mathcal{L}$ is base point free or very ample.

In this paper, we consider Problem 3.1 (i.2) and Problem 3.2 (i.2). In [17] (1.9 Theorem), Fujita proved that $(X, \mathcal{L})$ is a Del Pezzo manifold (namely $K_X = -(n - 1)\mathcal{L}$) if and only if $g(X, \mathcal{L}) = 1$ and $\Delta(X, \mathcal{L}) = 1$, that is, $g_1(X, \mathcal{L}) = 1$ and $\Delta_1(X, \mathcal{L}) = 1$. So in this paper, we consider an analogous characterization of $(X, \mathcal{L})$ with $K_X = -(n - i)\mathcal{L}$ by using $g_i(X, \mathcal{L})$ and $\Delta_i(X, \mathcal{L})$.

4 Main Theorems

4.1 A conjecture

First we provide the following conjecture which is the main theme of this paper.

Conjecture 4.1.1 Let $(X, \mathcal{L})$ be a polarized manifold of dimension $n \geq 3$. Then, for every integer $i$ with $2 \leq i \leq n - 1$, the following are equivalent one another.

$C(i, 1)$: $K_X = -(n - i)\mathcal{L}$

$C(i, 2)$: $\Delta_i(X, \mathcal{L}) = 1$ and $2g_i(X, \mathcal{L}) - 2 = (i - 1)\mathcal{L}^n$. 

11
Then $K - C$ of dimension $m$.

Remark 4.1.1 If $i = 1$, then $C(1, 1)$ and $C(1, 2)$ in Conjecture 4.1.1 are equivalent each other for any ample line bundle $L$. Of course $C(1, 1)$ implies $C(1, 3)$ (resp. $C(1, 4)$, $C(1, 5)$). But $C(1, 3)$ (resp. $C(1, 4)$, $C(1, 5)$) does not imply $C(1, 1)$ because $(X, L)$ is possibly a scroll over an elliptic curve.

Remark 4.1.2 As a generalization of the case where $i = 1$, it is natural to consider that $C(i, 1)$ is equivalent to the following condition:

$$C(i, 6): \Delta_i(X, L) = 1 \quad \text{and} \quad g_i(X, L) = 1.$$  

We can easily see that $C(i, 1)$ implies $C(i, 6)$. But from the following examples (Examples 4.1.1 and 4.1.2) we see that its converse is not true in general.

Example 4.1.1 Let $n$ be a natural number with $n \geq 3$, and let $Y$ be a smooth projective variety of dimension $m$ with $1 \leq m \leq n - 2$. Let $Y$ be an ample line bundle on $X$ such that $E$ is a polarized manifold $(Y, \mathcal{H})$ like this. For example, let $Y$ be a principally polarized Abelian variety with $\dim Y = m = n - 2$ and let $\mathcal{H}$ be an ample line bundle on $Y$ such that $\mathcal{H}^n = m!$. Then $K_Y + (n - m - 1)\mathcal{H} = \mathcal{H}$ and $h^0(K_Y + (n - m - 1)\mathcal{H}) = h^0(\mathcal{H}) = 1$.

Next we take a Del Pezzo manifold $(F, \mathcal{A})$ of dimension $n - m$. Then we note that $K_F = -(n - m - 1)\mathcal{A}$.

Here we set $X := Y \times F$ and $L := p_1^*(\mathcal{H}) \otimes p_2^*(\mathcal{A})$, where $p_i$ denotes the $i$th projection map. Then $K_X + (n - m - 1)L = p_1^*(K_Y + (n - m - 1)\mathcal{H})$. By [17] Lemma 1.6 we also get $h^j(\mathcal{O}_X) = 0$ and $h^2(L) = 0$ for every integer $j$ with $j \geq m + 1$. Hence $\Delta_n(X, L) = 0$ by Remark 3.3 (2), and by Theorem 4.2.1 (2) we see that $g_j(X, L) = 0$ for every integer $j$ with $j \geq m + 2$. Moreover we see that $\Delta_j(X, L) = 0$ for every integer $j \geq m + 2$ and

$$\Delta_{m+1}(X, L) = \Delta_{m+2}(X, L) = (n - m - 1)h^{m+1}(\mathcal{O}_X) - h^{m+1}(L) = 1.$$

Therefore $g_{m+1}(X, L) = \Delta_{m+1}(X, L) = 1$. But $K_X \neq -(n - m - 1)L$ and this $(X, L)$ is an example.

Example 4.1.2 Let $k$ be a natural number with $k \geq 2$ and set $n := 2k + 1$ and $i := (n - 1)/2$. Here we consider $(M, \mathcal{A}) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2))$. Then $K_M = -(k + 1)\mathcal{A} = -(n - i)\mathcal{A}$. Moreover we see that $g_i(M, \mathcal{A}) = 1$ and $\Delta_i(M, \mathcal{A}) = 1$ (see (1) in the proof of Theorem 4.1.1 below). Let $\pi : X \to \mathbb{P}^n$ be the blowing up at a general point on $\mathbb{P}^n$ and let $\mathcal{L} := \pi^*(\mathcal{A}) - E$, where $E$ is the exceptional divisor. Then by [11] Theorem 0.1, we see that $(X, \mathcal{L})$ is a polarized manifold with $K_X + (n - i)\mathcal{L} = (i - 1)E$. On the other hand, we note that $(M, \mathcal{A})$ is a reduction of $(X, \mathcal{L})$ and $2 \leq i < n - 1$. Hence by Remarks 4.1.1 (5) and 4.1.1 (4) we get $g_i(X, \mathcal{L}) = g_i(M, \mathcal{A}) = 1$ and $\Delta_i(X, \mathcal{L}) = \Delta_i(M, \mathcal{A}) = 1$.

4.2 The case where $\max\{2, \dim Bs|\mathcal{L}| + 2\} \leq i \leq n - 1$

First we consider the case where $\max\{2, \dim Bs|\mathcal{L}| + 2\} \leq i \leq n - 1$.

Theorem 4.2.1 Let $(X, \mathcal{L})$ be a polarized manifold of dimension $n \geq 3$, and let $m = \dim Bs|\mathcal{L}|$. (If $Bs|\mathcal{L}| = \emptyset$, then we set $m = -1$.) Assume that $i$ is an integer with $\max\{2, m + 2\} \leq i \leq n - 1$. Then Conjecture 4.1.1 is true.

Proof. By assumption and [16] Proposition 1.12 (2), we see that the following hold:
Claim 4.2.1. \( \mathcal{L} \) has an \((n-i)\)-ladder \( X_{n-i} \subset \cdots \subset X_1 \subset X \).

\((B_i)\) \( h^0(\mathcal{L}_{n-i}) > 0 \). 
\((C)\) \( h^j(\mathcal{L}^{n-i}) = 0 \) for any \( j \) and \( t \) with \( 0 \leq j \leq n-1 \) and \( t > 0 \).
\((D_i)\) \( X_j \) is normal for any \( j \) with \( 0 \leq j \leq n-i \).
\((E_i)\) \( X_j \) is Cohen-Macaulay for any \( j \) with \( 0 \leq j \leq n-i \).

(I) Assume that \( C(i, 1) \) holds. Then by Remark \[3.1\] (4) we see that 
\[
2g_1(X, \mathcal{L}) - 2 = (K_X + (n-i)\mathcal{L} + (i-1)\mathcal{L})\mathcal{L}^{n-1} = (i-1)\mathcal{L}^n.
\]
Here we note that \( h^j(\mathcal{O}_X) = 0 \) and \( h^j(\mathcal{L}) = 0 \) for every integer \( j \) with \( j \geq 2 \). Moreover \( h^0(K_X + (n-i)\mathcal{L}) = 1 \) and \( h^0(K_X + k\mathcal{L}) = 0 \) for every integer \( k \) with \( 1 \leq k \leq n-i - 1 \). Hence we see that \( g_i(X, \mathcal{L}) = 1 \) and \( g_k(X, \mathcal{L}) = 0 \) for every integer \( k \) with \( k \geq i+1 \) by Theorem \[3.2\] (2) (this means that \( C(i, 1) \) implies \( C(i, 4) \)) and by Remark \[3.3\] (2) and (3), we have \( \Delta_k(X, \mathcal{L}) = 0 \) for every integer \( k \) with \( k \geq i+1 \) and 
\[
\Delta_i(X, \mathcal{L}) = g_i(X, \mathcal{L}) - \Delta_{i+1}(X, \mathcal{L}) + (n-i)h^i(\mathcal{O}_X) - h^i(\mathcal{L}) = 1.
\]
Therefore we see that \( C(i, 1) \) implies \( C(i, 2) \) and \( C(i, 4) \) above.

(II) It is trivial that \( C(i, 2) \) implies \( C(i, 3) \), and \( C(i, 4) \) implies \( C(i, 5) \).

(III) Assume that \( C(i, 3) \) holds. Then we will prove that \( C(i, 5) \) holds. In order to prove \( C(i, 5) \), it suffices to show that \( g_i(X, \mathcal{L}) > 0 \). Here we note that \( g_i(X, \mathcal{L}) \geq 0 \) by \[16\] Theorem 2.4. Assume that \( g_i(X, \mathcal{L}) = 0 \). Here we prove the following.

Claim 4.2.1 If \( g_i(X, \mathcal{L}) = 0 \), then \( \Delta_i(X, \mathcal{L}) = 0 \).

**Proof.** Assume that \( g_i(X, \mathcal{L}) = 0 \). Then \( 0 = g_i(X, \mathcal{L}) = g_i(X_{n-i}, \mathcal{L}_{n-i}) = h^i(\mathcal{O}_X_{n-i}) \) by Theorem \[3.1\] and Remark \[3.1\] (2). Therefore \( h^j(\mathcal{O}_X) = h^j(\mathcal{O}_X_1) = \cdots = h^j(\mathcal{O}_X_{n-i}) \leq h^j(\mathcal{O}_X_{n-i}) = 0 \) by \[16\] Proposition 2.1 (b)]. Hence \( H^{i-1}(\mathcal{L}_j) \rightarrow H^{i-1}(\mathcal{L}_{j+1}) \) is surjective for \( 0 \leq j \leq n-i \). Namely \( \dim \text{Coker}(r_{i-1,j}) = 0 \) for \( 0 \leq j \leq n-i \). On the other hand by Theorem \[3.4\] we have 
\[
\Delta_i(X, \mathcal{L}) = \sum_{k=0}^{n-i} \dim \text{Coker}(r_{i-1,k}).
\]
Therefore we get 
\[
\Delta_i(X, \mathcal{L}) = \sum_{k=0}^{n-i} \dim \text{Coker}(r_{i-1,k}) = 0.
\]
This completes the proof of Claim 4.2.1. \( \square \)

But this contradicts the assumption. Therefore \( g_i(X, \mathcal{L}) > 0 \) and we see that \( C(i, 5) \) holds.

(IV) Assume that \( C(i, 5) \) holds. Then 
\[
1 + \frac{1}{2}(i-1)\mathcal{L}^n = g_1(X, \mathcal{L})
\]
\[ = 1 + \frac{1}{2} (K_X + (n-1)\mathcal{L}) \mathcal{L}^{n-1} \]
\[ = 1 + \frac{1}{2} (K_{X_{n-i}} + (i-1)\mathcal{L}_{n-i}) \mathcal{L}_{n-i}^{i-1} \]
\[ = 1 + \frac{1}{2} (i-1)\mathcal{L}^n + \frac{1}{2} K_{X_{n-i}} \mathcal{L}_{n-i}^{i-1} . \]

Hence \( K_{X_{n-i}} \mathcal{L}_{n-i}^{i-1} = 0 \). On the other hand, we get \( g_i(X, \mathcal{L}) = h^i(\mathcal{O}_{X_{n-i}}) \) by \((A_i)\) and \((C)\) (see also \cite{16} Propositions 2.1 and 2.3). Furthermore by \((D_i)\), \((E_i)\) and the Serre duality, we obtain \( h^0(K_{X_{n-i}}) = h^1(\mathcal{O}_{X_{n-i}}) \). Hence we have \( 0 < g_i(X, \mathcal{L}) = h^i(\mathcal{O}_{X_{n-i}}) = h^0(K_{X_{n-i}}) \). Hence we see that \( K_{X_{n-i}} = \mathcal{O}_{X_{n-i}} \).

Next we prove the following claim.

**Claim 4.2.2** A natural map \( \text{Pic}(X_j) \to \text{Pic}(X_{j+1}) \) is injective for any \( j \) with \( 0 \leq j \leq n - i - 1 \).

**Proof.** From the following exact sequence
\[ 0 \to \mathcal{O}_{X_j} \to \mathcal{O}_{X_j}^* \to 0, \]
we get the following commutative diagram.

\[
\begin{array}{cccc}
H^1(X_j, \mathbb{Z}) & \longrightarrow & H^1(\mathcal{O}_{X_j}) & \longrightarrow & H^1(\mathcal{O}_{X_j}^*) & \longrightarrow & H^2(X_j, \mathbb{Z}) \\
\varphi_1 & \downarrow & \varphi_2 & \downarrow & \varphi_3 & \downarrow & \varphi_4 \\
H^1(X_{j+1}, \mathbb{Z}) & \longrightarrow & H^1(\mathcal{O}_{X_{j+1}}) & \longrightarrow & H^1(\mathcal{O}_{X_{j+1}}^*) & \longrightarrow & H^2(X_{j+1}, \mathbb{Z})
\end{array}
\]

So in order to prove that \( \varphi_3 \) is injective, it suffices to show the following for every integer \( j \) with \( 0 \leq j \leq n - i - 1 \) because \( \text{Pic}(X_j) \cong H^1(\mathcal{O}_{X_j}^*) \) and \( \text{Pic}(X_{j+1}) \cong H^1(\mathcal{O}_{X_{j+1}}^*) \).

(a) \( h^1(\mathcal{O}_{X_j}(-X_{j+1})) = 0 \).

(b) \( H^1(X_j, \mathbb{Z}) \cong H^1(X_{j+1}, \mathbb{Z}) \).

(c) The map \( H^2(X_j, \mathbb{Z}) \to H^2(X_{j+1}, \mathbb{Z}) \) is injective.

By \((C)\) we can prove \( h^t(\mathcal{L}_j^{\otimes s}) = 0 \) for every \( j, t \) and \( s \) with \( 0 \leq j \leq n - i - 1, 0 \leq t \leq n - j - 1 \) and \( 1 \leq s \). Therefore we get (a) since \( \dim X_{n-i-1} = i + 1 \geq 3 \).

Next we consider (b) and (c). In this case we need to take an \((n-i)\)-ladder carefully. Namely, we take general members \( X_1 \in |\mathcal{L}|, X_2 \in |\mathcal{L}|_{X_1}, \ldots, X_n \in |\mathcal{L}|_{X_{n-1}} \). Then \( X \supset X_1 \supset \ldots \supset X_{n-i} \) is an \((n-i)\)-ladder such that \( X_j - X_{j+1} \) is smooth for every \( j \). Hence \( X_j - X_{j+1} \) is locally complete intersection. Here we use \cite{3} Corollary 2.3.3. Since \( X_{j+1} \) is an ample line bundle on \( X_j \), we see that if \( 2 \leq i = \dim X_{n-i-1} - 1 \), then \( H^t(X_j, \mathbb{Z}) \to H^t(X_{j+1}, \mathbb{Z}) \) is an isomorphism (resp. injective) for \( t = 1 \) (resp. \( t = 2 \)) and every \( j \) with \( 0 \leq j \leq n - i - 1 \).

Therefore we get the assertion of Claim \ref{claim:4.2.2} \( \Box \)

By this claim we have \( K_X + (n-i)\mathcal{L} = \mathcal{O}_X \). So we get \( C(i, 1) \). This completes the proof of Theorem \ref{theorem:4.2.1} \( \Box \)
4.3 The case of $i = 2$

Next we consider the case where $i = 2$ and $L$ is ample in general. Then we can prove the following:

**Theorem 4.3.1** Let $(X, L)$ be a polarized manifold of dimension $n \geq 3$. Then Conjecture 4.1.1 for $i = 2$ is true.

**Proof.** (I) First we assume that $C(2, 1)$. Then by the same argument as in the proof of Theorem 4.2.1 we see that $C(2, 2)$ and $C(2, 4)$ hold.

(II) It is trivial that $C(2, 2)$ (resp. $C(2, 4)$) implies $C(2, 3)$ (resp. $C(2, 5)$).

(III) Assume that $C(2, 3)$. Then we will prove that $C(2, 5)$ holds. In order to prove $C(2, 5)$, it suffices to show that $g_2(X, L) > 0$. By the assumption that $2g_1(X, L) - 2 = L^n$, we get $(K_X + (n - 2)L)L^{n-1} = 0$. Hence by 3 Lemma 2.5.9 we have $\kappa(K_X + (n - 2)L) \leq 0$.

(II.1) If $\kappa(K_X + (n - 2)L) = -\infty$, then $(X, L)$ is one of the types (1) to (7.4) in Theorem 2.1 by Remark 2.2.1. By [15, Example 2.10] and [17, Example 2.12] we may assume that $(X, L)$ is a scroll over a smooth surface $S$ because we assume that $\Delta_2(X, L) > 0$. In this case, by [9, 3.2.1] and [10, (11.8.6)] in the proof of (11.8) Theorem, there exists an ample vector bundle $E$ of rank $n - 1$ on such that $X = \mathbb{P}(E)$, $L = H(E)$. Let $\pi : X \rightarrow S$ be its morphism. Here we calculate $(K_X + (n - 2)L)L^{n-1}$.

$$(K_X + (n - 2)L)L^{n-1} = (H(E) + \pi^*(K_S + c_1(E))H(E))^{n-1} = K_Sc_1(E) + c_2(E).$$

If $h^2(O_S) = 0$, then $h^2(O_X) = h^2(O_S) = 0$ and $\Delta_2(X, L) = (n - 1)h^2(O_X) - h^2(L) = -h^2(L) \leq 0$ and this contradicts the assumption. Hence $h^2(O_S) \geq 1$ and by the Serre duality we have $h^0(K_S) \geq 1$. Since $E$ is ample, we see that $K_Sc_1(E) \geq 0$ and $c_2(E) > 0$. Therefore $(K_X + (n - 2)L)L^{n-1} > 0$ and this contradicts the assumption. Therefore there does not exist any $(X, L)$ with $\kappa(K_X + (n - 2)L) = -\infty$, $\Delta_2(X, L) > 0$ and $2g_1(X, L) - 2 = L^n$.

(II.2) Assume that $\kappa(K_X + (n - 2)L) = 0$. Let $(M, A)$ be a reduction of $(X, L)$. Then by Theorem 2.1 $(M, A)$ is a Mukai manifold and by 15 Proposition 2.6 and Example 2.10 (7) we see that $g_2(X, L) = g_2(M, A) > 0$. Hence by (II.1) and (II.2) we get $C(2, 5)$.

(IV) Assume that $C(2, 5)$. Then by the same argument as (III) above, we see that $\kappa(K_X + (n - 2)L) \leq 0$. If $\kappa(K_X + (n - 2)L) = -\infty$, then $(X, L)$ is a scroll over a smooth surface $S$ since $g_2(X, L) > 0$. On the other hand $h^2(O_X) > 0$ because $g_2(X, L) = h^2(O_X)$. Hence by the same argument as (III.1) above, we see that $(K_X + (n - 2)L)L^{n-1} > 0$ and this is impossible. Therefore $\kappa(K_X + (n - 2)L) = 0$ and $K_M + (n - 2)A = O_X$, where $(M, A)$ is a reduction of $(X, L)$. Here we prove that $(X, L) \cong (M, A)$. So we assume that $(X, L) \not\cong (M, A)$. Then $(K_X + (n - 2)L)L^{n-1} > (K_M + (n - 2)A)A^{n-1}$ holds. But since $(K_X + (n - 2)L)L^{n-1} = 0$ and $(K_M + (n - 2)A)A^{n-1} = 0$, this is impossible. Hence $(X, L) \cong (M, A)$ and we get $C(2, 1)$.

This completes the proof of Theorem 4.3.1.

**Corollary 4.3.1** Let $(X, L)$ be a polarized manifold of dimension $n \geq 3$. Assume that $\dim \text{Bs}L \leq 1$. Then Conjecture 4.1.1 is true.

**Proof.** Since $\dim \text{Bs}L \leq 1$, we see that Conjecture 4.1.1 is true for $i \geq 3$ by Theorem 4.2.1. On the other hand, if $i = 2$, then Conjecture 4.1.1 is also true by Theorem 4.3.1 Therefore we get the assertion.
4.4 The case where $i = 3$ and $n \geq 5$

Next we consider the case where $i = 3$ and $n \geq 5$.

**Theorem 4.4.1** Let $(X, \mathcal{L})$ be a polarized manifold of dimension $n \geq 5$. Then Conjecture 4.1.1 for $i = 3$ is true.

**Proof.** (I) By the same argument as in the proof of Theorem 4.1.1 we see that $C(3, 1)$ implies $C(3, 2)$ and $C(3, 4)$.

(II) It is trivial that $C(3, 2)$ (resp. $C(3, 4)$) implies $C(3, 3)$ (resp. $C(3, 5)$).

(III) Assume that $C(3, 3)$. Then we will prove that $C(3, 1)$ holds.

Since $2g_1(X, \mathcal{L}) - 2 = 2\mathcal{L}^n$, we have $(K_X + (n - 3)\mathcal{L})\mathcal{L}^{n-1} = 0$. Hence by an argument similar to (III) in the proof of Theorem 4.3.1 we see that $\kappa(K_X + (n - 3)\mathcal{L}) \leq 0$.

(III.1) Assume that $\kappa(K_X + (n - 3)\mathcal{L}) = 0$. Then since $\kappa(K_X + (n - 3)\mathcal{L}) = 0$ and $(K_X + (n - 3)\mathcal{L})\mathcal{L}^{n-1} = 0$, there exists a positive integer $t$ such that $t(K_X + (n - 3)\mathcal{L}) = \mathcal{O}_X$. But by Lemma 3.3.2 we have $K_X + (n - 3)\mathcal{L} = \mathcal{O}_X$.

(III.2) Assume that $\kappa(K_X + (n - 3)\mathcal{L}) = -\infty$.

**Lemma 4.4.1** There does not exist any $(X, \mathcal{L})$ with $\kappa(K_X + (n - 3)\mathcal{L}) = -\infty$, $\Delta_3(X, \mathcal{L}) > 0$ and $2g_1(X, \mathcal{L}) - 2 = 2\mathcal{L}^n$.

**Proof.** If $\kappa(K_X + (n - 3)\mathcal{L}) = -\infty$, then by Proposition 2.1 $(X, \mathcal{L})$ satisfies either (1), (2.1) or (2.2) in Proposition 2.1.

(i) If $(X, \mathcal{L})$ satisfies (1) in Proposition 2.1 then by using Example 2.12 we see that $\Delta_3(X, \mathcal{L}) = 0$ unless $(M, \mathcal{A})$ is a scroll over a normal projective variety of dimension 3.

We consider the case where a reduction $(M, \mathcal{A})$ of $(X, \mathcal{L})$ is a scroll over a normal projective variety $Y$ with $\dim Y = m \geq 2$. Then by [6 (3.2.1) Theorem] and [2] Proposition 2.5, we get the following:

**Proposition 4.4.1** Let $(X, \mathcal{L})$ be a scroll over a 3-dimensional normal projective variety $Y$. If $\dim X \geq 5$, then $Y$ is smooth and $(X, \mathcal{L})$ is a classical scroll over $Y$.

So we see that $(M, \mathcal{A})$ is a classical scroll, that is, $Y$ is smooth and $(M, \mathcal{A}) = (\mathbb{P}_Y(\mathcal{E}), H(\mathcal{E}))$, where $\mathcal{E}$ is an ample vector bundle on $Y$. Then in general the following claim holds.

**Claim 4.4.1** Let $(X, \mathcal{L})$ be a polarized manifold of dimension $n$ and let $(M, \mathcal{A})$ be a reduction of $(X, \mathcal{L})$. Assume that there exists a smooth projective variety $Y$ of dimension $m \geq 2$ and an ample vector bundle $\mathcal{E}$ on $Y$ of rank $n - m + 1$ such that $M = \mathbb{P}_Y(\mathcal{E})$ and $A = H(\mathcal{E})$.

(1) If $g_m(X, \mathcal{L}) > 0$, then $2g_1(X, \mathcal{L}) - 2 > (m - 1)\mathcal{L}^n$.

(2) If $\Delta_m(X, \mathcal{L}) > 0$, then $2g_1(X, \mathcal{L}) - 2 > (m - 1)\mathcal{L}^n$.

**Proof.** Assume that $h^m(\mathcal{O}_Y) \geq 1$. Then $\kappa(Y) \geq 0$. On the other hand,

\[
2g_1(X, \mathcal{L}) - 2 - (m - 1)\mathcal{L}^n \\
\geq 2g_1(M, \mathcal{A}) - 2 - (m - 1)\mathcal{A}^n \\
= (K_M + (n - m)\mathcal{A})\mathcal{A}^{n-1} \\
= (H(\mathcal{E}) + f^*(K_Y + \det \mathcal{E}))H(\mathcal{E})^{n-1} \\
= -s_m(\mathcal{E}) + (K_Y + \det \mathcal{E})s_{m-1}(\mathcal{E}) \\
= s_{m-1}(\mathcal{E})s_1(\mathcal{E}) - s_m(\mathcal{E}) + K_Ys_{m-1}(\mathcal{E}).
\]

(Here $f : M \to Y$ denotes the projection.) Since $\mathcal{E}$ is ample, $m \geq 2$ and $\kappa(Y) \geq 0$, we have $s_{m-1}(\mathcal{E})s_1(\mathcal{E}) - s_m(\mathcal{E}) > 0$ and $K_Ys_{m-1}(\mathcal{E}) \geq 0$ by [21] Example 12.1.7 and Lemma 14.5.1].
Therefore we see that $2g_1(X, \mathcal{L}) - 2 > (m - 1)\mathcal{L}^n$.

(1) If $g_m(X, \mathcal{L}) > 0$, then $h^m(\mathcal{O}_Y) > 0$ because $g_m(X, \mathcal{L}) = g_m(M, \mathcal{A}) = h^m(\mathcal{O}_M) = h^m(\mathcal{O}_Y)$ by [15 Example 2.10 (8)]. Hence by the above argument we get the assertion (1).

(2) Assume that $\Delta_m(X, \mathcal{L}) > 0$. Here we note that $g_j(M, \mathcal{A}) = 0$, $h^j(\mathcal{O}_M) = 0$ and $h^j(\mathcal{A}) = 0$ for every $j \geq m + 1$ and $g_m(M, \mathcal{A}) = h^m(\mathcal{O}_M)$ by [13 Example 2.10 (8)] and [17 Lemma 1.6]. Therefore we get $\Delta_j(M, \mathcal{A}) = 0$ for every integer $j$ with $j \geq m + 1$, and by using Remark 3.4 (3) we have $\Delta_m(M, \mathcal{A}) = (n - m + 1)h^m(\mathcal{O}_M) - h^m(\mathcal{A})$. If $h^m(\mathcal{O}_M) = 0$, then $\Delta_m(M, \mathcal{A}) = -h^m(\mathcal{A}) \leq 0$ and this contradicts the assumption because $\Delta_m(X, \mathcal{L}) = \Delta_m(M, \mathcal{A})$ by [17 Corollary 2.11]. Therefore $h^m(\mathcal{O}_M) > 0$. Hence by the above argument we get the assertion (2). Therefore we get the assertion of Claim 4.4.1. □

Since $\dim Y = 3$ and we assume that $2g_1(X, \mathcal{L}) - 2 = 2\mathcal{L}^n$, we have $\Delta_3(X, \mathcal{L}) \leq 0$ by Claim 4.4.1 if $(M, \mathcal{A})$ is a scroll over a normal projective variety of dimension 3.

Therefore we have $\Delta_3(X, \mathcal{L}) \leq 0$ if $(X, \mathcal{L})$ satisfies (1) in Proposition 2.1.

(ii) Next we assume that $(X, \mathcal{L})$ satisfies (2.1) or (2.2) in Proposition 2.1.

Assume that $(W, K)$ is the type 3 in [5 Theorem 7.7.5]. Then $2g_1(X, \mathcal{L}) - 2 = 2\mathcal{L}^5 - 1 \neq 2\mathcal{L}^5$. So we may assume that $(W, K)$ is not of this type.

By Proposition 3.11 and the assumption, we have $h^j(\mathcal{O}_X) = h^j(\mathcal{O}_M) = 0$ and $g_j(X, \mathcal{L}) = g_j(M, \mathcal{A}) = 0$ for every integer $j$ with $j \geq 3$. By Remark 3.4 (2) we have $\Delta_n(X, \mathcal{L}) = \Delta_n(M, \mathcal{A}) = h^n(\mathcal{O}_M) - h^n(\mathcal{A})$. Moreover by Remark 3.4 (3) and (4) we have

$$\Delta_j(X, \mathcal{L}) = \Delta_j(M, \mathcal{A}) = g_j(M, \mathcal{A}) - \Delta_{j+1}(M, \mathcal{A}) + (n - j)h^j(\mathcal{O}_M) - h^j(\mathcal{A}).$$

So in order to calculate $\Delta_3(X, \mathcal{L})$, we have to calculate $h^j(\mathcal{A})$ with $j \geq 3$. Then by Lemma 2.2 we see that $h^j(\mathcal{A}) = h^j(\mathcal{D})$ for every $j \geq 3$.

(ii.1) If $(W, K) \cong (\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(1))$, then $\mathcal{D} = \mathcal{O}_{\mathbb{P}^6}(2)$ and we see that $h^j(\mathcal{D}) = 0$ for every $j \geq 2$.

(ii.2) Assume that $(W, K)$ is the type 1 in [3 Theorem 7.7.5], that is, $(W, K) \cong (\mathbb{Q}^5, \mathcal{O}_{\mathbb{Q}^5}(1))$. Then $K_W$ is a Cartier divisor and $K = K_W + 3\mathcal{D}$. Hence $3\mathcal{D}$ is also Cartier. On the other hand $2\mathcal{D}$ is Cartier by [5 Lemma 7.5.8]. Hence $\mathcal{D} = 3\mathcal{D} - 2\mathcal{D}$ is also Cartier and $\mathcal{D} = \mathcal{O}_{\mathbb{Q}^5}(2)$. Therefore by the Kawamata-Viehweg vanishing theorem [25 Theorem 1-2-5], we have $h^j(\mathcal{D}) = h^j(\mathcal{O}_{\mathbb{Q}^5}(2)) = h^j(K_W + \mathcal{O}_{\mathbb{Q}^5}(7)) = 0$ for every $j \geq 1$.

(ii.3) Assume that $(W, K)$ is the type 2 in [5 Theorem 7.7.5]. Let $\pi : W \to C$ be the $\mathbb{P}^4$-bundle over a smooth curve $C$. Then by [5 Proposition 3.2.1], there exists an ample vector bundle $\mathcal{E}$ on $C$ such that $W \cong \mathbb{P}_C(\mathcal{E})$ and $H(\mathcal{E}) = K$. Since $W$ is smooth in this case, $\mathcal{D}$ is a Cartier divisor and $\mathcal{D} = 2H(\mathcal{E}) + \pi^*(\mathcal{B})$ for $\mathcal{B} \in \text{Pic}(C)$.

Claim 4.4.2 $h^j(2H(\mathcal{E}) + \pi^*(\mathcal{B})) = 0$ for every $j \geq 2$.

Proof. Since $R^k\pi_*(2H(\mathcal{E}) + \pi^*(\mathcal{B})) = R^k\pi_*(2H(\mathcal{E})) \otimes \mathcal{B} = 0$ for every positive integer $k$, we see that $h^j(2H(\mathcal{E}) + \pi^*(\mathcal{B})) = h^j(\pi_*(2H(\mathcal{E}) + \pi^*(\mathcal{B})))$ for every $j \geq 0$. Since $\dim C = 1$, we get $h^j(\pi_*(2H(\mathcal{E}) + \pi^*(\mathcal{B}))) = 0$ for every integer $j$ with $j \geq 2$. Hence we get the assertion of Claim 4.4.2. □

By the above argument we have $h^j(\mathcal{A}) = 0$ for every integer $j$ with $j \geq 3$. Hence by Remark 3.4 (4) we see that $\Delta_j(X, \mathcal{L}) = \Delta_j(M, \mathcal{A}) = 0$ for every $j \geq 3$. Therefore $\Delta_3(X, \mathcal{L}) = 0$ if $(X, \mathcal{L})$ satisfies (2.1) or (2.2) in Proposition 2.1.

From (i) and (ii) above, we see that $\Delta_3(X, \mathcal{L}) \leq 0$ if $\kappa(K_X + (n-3)\mathcal{L}) = -\infty$ and $2g_1(X, \mathcal{L}) - 2 = 2\mathcal{L}^n$. Therefore we get the assertion of Lemma 4.4.1. □

From (III.1) and (III.2) we see that $C(3, 3)$ implies $C(3, 1)$.
(IV) Assume that \( C(3, 5) \). Then we will prove that \( C(3, 1) \) holds. First we see that \( \kappa(K_X + (n - 3)L) \leq 0 \) by the same reason of the case (III) above.

(IV.1) Assume that \( \kappa(K_X + (n - 3)L) = 0 \). Then by the same argument as (III.1) above, we get \( K_X + (n - 3)L = O_X \) in this case.

(IV.2) Assume that \( \kappa(K_X + (n - 3)L) = -\infty \). By Proposition 4.2.1 we see that \( g_3(X, L) = 0 \) unless \( (M, A) \) is a scroll over a normal projective variety of dimension 3. Next we assume that \( (M, A) \) is a scroll over a normal projective variety \( Y \) of dimension 3. Here we note that \( Y \) is smooth and \( (M, A) \) is a classical scroll over \( Y \) by Proposition 4.4.1. Then \( g_3(X, L) = g_3(M, A) = h^3(O_M) = h^3(O_X) \) (see Remark 3.2). If \( h^3(O_X) > 0 \), then by Claim 4.4.1 we have \( 2g_1(X, L) - 2 > 2L^n \) and this contradicts the assumption \( C(3, 5) \). Hence \( h^3(O_X) = 0 \), and \( g_3(X, L) = 0 \) also holds in this case. Therefore there does not any \( (X, L) \) with \( \kappa(K_X + (n - 3)L) = -\infty, g_3(X, L) > 0 \) and \( 2g_1(X, L) - 2 = 2L^n \).

By (IV.1) and (IV.2) we get \( C(3, 1) \). Therefore these complete the proof of Theorem 4.3.1 □

By Theorems 4.2.1, 3.1.1 and 4.4.1 we get the following corollary.

**Corollary 4.1** Let \( (X, L) \) be a polarized manifold of dimension \( n \geq 5 \). Assume that \( \dim Bs|L| = 2 \). Then Conjecture 4.1.1 is true.

**Remark 4.4.1** Next we consider the case where \( n = 4 \). By the same argument as the proof of Theorem 4.4.1 we can check that the following implications hold: \( C(3, 1) \implies C(3, 2), C(3, 1) \implies C(3, 4), C(3, 2) \implies C(3, 4), \) and \( C(3, 3) \implies C(3, 5) \).

Next we consider the implication \( C(3, 3) \implies C(3, 1) \). By the same argument as in the case (III) in Theorem 4.4.1 we have \( \kappa(K_X + L) \leq 0 \).

If \( \kappa(K_X + L) = 0 \), then we can prove that \( K_X + L = O_X \). So we assume that \( \kappa(K_X + L) = -\infty \). Since \( n = 4 \), we see that \( (X, L) \) satisfies \( 2 \) in Theorem 2.2. Then \( h^4(O_M) = 0 \) and \( h^4(A) = 0 \) because \( h^4(A) = h^4(K_M - A) \) and \( \kappa(M) = -\infty \). By Proposition 3.1 and Lemma 2.1 we have \( g_3(X, L) = g_3(M, A) = 0 \) and \( h^3(O_X) = h^3(O_M) = 0 \) unless \( (M, A) \) is a scroll over a normal projective variety of dimension 3. Therefore \( \Delta_4(M, A) = 0 \) and \( \Delta_3(M, A) = -h^3(A) \leq 0 \) unless \( (M, A) \) is a scroll over a normal 3-fold. Since \( \Delta_3(X, L) = \Delta_3(M, A) \), we get \( \Delta_4(X, L) \leq 0 \). But this contradicts the assumption. Therefore if \( (M, A) \) is not a scroll over a normal 3-fold, then \( C(3, 3) \) implies \( C(3, 1) \).

If \( (M, A) \) is a classical scroll over a smooth 3-fold, then by Claim 4.4.1 we also see that \( C(3, 3) \) implies \( C(3, 1) \). By the same argument as above, we see that \( C(3, 5) \) implies \( C(3, 1) \) if \( (M, A) \) is a classical scroll over a smooth 3-fold.

So in order to prove that Conjecture 4.1.1 for \( n = 4 \) and \( i = 3 \) is true, it suffices to consider the case where a reduction of \( (X, L) \) is a scroll over a normal projective variety \( Y \) with \( \dim Y = 3 \), but not a classical scroll over a smooth 3-fold \( Y \).

### 4.5 Some remarks

Finally we would like to give a comment about Conjecture 4.1.1

(a) We can easily see that \( C(i, 1) \) implies \( C(i, 2) \) and \( C(i, 4) \), and \( C(i, 2) \) (resp. \( C(i, 4) \)) implies \( C(i, 3) \) (resp. \( C(i, 5) \)) by the same argument as the proof of Theorem 4.2.1

(b) By looking at the proof of Theorems 4.2.1, 3.1.1 or 4.4.1 we can prove the following:

**Proposition 4.5.1** If there does not exist any polarized manifold \( (X, L) \) with \( \kappa(K_X + (n - i)L) = -\infty, \Delta_4(X, L) = 0 \) (resp. \( g_i(X, L) > 0 \)) and \( 2g_1(X, L) - 2 = (i - 1)L^n \), then we see that \( C(i, 3) \) (resp. \( C(i, 5) \)) implies \( C(i, 1) \).
So it is important to know whether there exists an example of \((X, \mathcal{L})\) with \(\kappa(K_X + (n-i)\mathcal{L}) = -\infty\), \(\Delta_i(X, \mathcal{L}) > 0\) (resp. \(g_i(X, \mathcal{L}) > 0\)) and \(2g_1(X, \mathcal{L}) - 2 = (i - 1)\mathcal{L}^n\).

(c) We can regard the following result as the case where \(i = n\) in Conjecture 4.1.1.

**Proposition 4.5.2** Let \(X\) be a smooth projective variety of dimension \(n\). Then the following are equivalent one another.

(i) \(K_X \sim \mathcal{O}_X\).

(ii) \(\Delta_n(X, \mathcal{L}) = 1\) and \(2g(X, \mathcal{L}) - 2 = (n - 1)\mathcal{L}^n\) hold for any ample line bundle \(\mathcal{L}\).

(iii) \(\Delta_n(X, \mathcal{L}) > 0\) and \(2g(X, \mathcal{L}) - 2 = (n - 1)\mathcal{L}^n\) hold for any ample line bundle \(\mathcal{L}\).

(iv) \(g_n(X, \mathcal{L}) = 1\) and \(2g(X, \mathcal{L}) - 2 = (n - 1)\mathcal{L}^n\) hold for any ample line bundle \(\mathcal{L}\).

(v) \(g_n(X, \mathcal{L}) > 0\) and \(2g(X, \mathcal{L}) - 2 = (n - 1)\mathcal{L}^n\) hold for any ample line bundle \(\mathcal{L}\).

**Proof.** (i)⇒(ii): By Remark 3.3 (2) we have \(\Delta_n(X, \mathcal{L}) = h^n(\mathcal{O}_X) - h^n(\mathcal{L})\). By assumption we get \(h^n(\mathcal{O}_X) = h^0(K_X) = 1\). Next we calculate \(h^n(\mathcal{L})\). Since \(h^n(\mathcal{L}) = h^0(K_X - \mathcal{L}) = h^0(-\mathcal{L})\), we see that \(h^n(\mathcal{L}) = 0\) because \(\mathcal{L}\) is ample. Therefore \(\Delta_n(X, \mathcal{L}) = 1\). Of course \(2g(X, \mathcal{L}) - 2 = (n - 1)\mathcal{L}^n\) holds since \(K_X \sim \mathcal{O}_X\).

(ii)⇒(i): By Remark 3.3 (2) we have \(\Delta_n(X, \mathcal{L}) = h^n(\mathcal{O}_X) - h^n(\mathcal{L})\), we have \(h^n(\mathcal{O}_X) \geq 1\). By the Serre duality we see that \(h^0(K_X) \geq 1\). On the other hand, since \(2g(X, \mathcal{L}) - 2 = (n - 1)\mathcal{L}^n\), we have \(K_X \mathcal{L}^{n-1} = 0\). Therefore we get \(K_X \sim \mathcal{O}_X\).

(i)⇒(vi): Since \(K_X \sim \mathcal{O}_X\), we see that \(2g(X, \mathcal{L}) - 2 = (K_X + (n - 1)\mathcal{L})\mathcal{L}^{n-1} = (n - 1)\mathcal{L}^n\) and \(h^0(K_X) = 1\). By the Serre duality we have \(h^n(\mathcal{O}_X) = h^0(K_X) = 1\). Hence \(g_n(X, \mathcal{L}) = h^n(\mathcal{O}_X) = 1\) by Remark 3.3 (2).

(v)⇒(i): By assumption, we have \(h^n(\mathcal{O}_X) = g_n(X, \mathcal{L}) > 0\). We also note that \(K_X \mathcal{L}^{n-1} = 0\) by the assumption that \(2g(X, \mathcal{L}) - 2 = (n - 1)\mathcal{L}^n\). Hence we see that \(K_X \sim \mathcal{O}_X\) because \(h^0(K_X) = h^n(\mathcal{O}_X) > 0\). □

**References**

[1] E. Ballico, *Ample divisors on the blow up of \(\mathbb{P}^n\) at points* Proc. Amer. Math. Soc. 127 (1999), 2527–2528.

[2] E. Ballico and J. A. Wiśniewski, *On Bănică sheaves and Fano manifolds*, Compos. Math. 102 (1996), 313–335.

[3] M. C. Beltrametti, A. Lanteri, and M. Palleschi, *Algebraic surfaces containing an ample divisor of arithmetic genus two*, Ark. Mat. 25 (1987), 189–210.

[4] M. C. Beltrametti and A. J. Sommese, *Special results in adjunction theory in dimension four and five*, Ark. Mat. 31 (1993), 197–208.

[5] M. C. Beltrametti and A. J. Sommese, *The adjunction theory of complex projective varieties*, de Gruyter Expositions in Math. 16, Walter de Gruyter, Berlin, NewYork, (1995).

[6] M. C. Beltrametti, A. J. Sommese and J. A. Wiśniewski, *Results on varieties with many lines and their applications to adjunction theory (with an appendix by M. C. Beltrametti and A. J. Sommese)*, in Complex Algebraic Varieties, Bayreuth 1990, ed. by K. Hulek, T. Peternell, M. Schneider, and F.-O. Schreyer, Lecture Notes in Math., 1507 (1992), 16–38, Springer-Verlag, New York.
[7] T. Fujita, *On the structure of polarized manifolds th total deficiency one*, I, J. Math. Soc. Japan 32 (1980), 709–725.

[8] T. Fujita, *On polarized manifolds whose adjoint bundles are not semipositive*, in Algebraic Geometry Sendai 1985, pp.167–178, Adv. Stud. Pure Math. 10, Kinokuniya, 1987.

[9] T. Fujita, *Classification of polarized manifolds of sectional genus two*, the Proceedings of “Algebraic Geometry and Commutative Algebra” in Honor of Masayoshi Nagata (1987), 73–98.

[10] T. Fujita, *Classification Theories of Polarized Varieties*, London Math. Soc. Lecture Note Ser. 155, Cambridge University Press, (1990).

[11] T. Fujita, *On Kodaira energy and adjoint reduction of polarized manifolds*, Manuscripta Math. 76 (1992), 59–84.

[12] Y. Fukuma, *A lower bound for the sectional genus of quasi-polarized surfaces*, Geom. Dedicata 64 (1997), 229–251.

[13] Y. Fukuma, *A lower bound for sectional genus of quasi-polarized manifolds*, J. Math. Soc. Japan 49 (1997), 339–362.

[14] Y. Fukuma, *On sectional genus of quasi-polarized 3-folds*, Trans. Amer. Math. Soc. 351 (1999), 363–377.

[15] Y. Fukuma, *On the sectional geometric genus of quasi-polarized varieties, I*, Comm. Algebra 32 (2004), 1069–1100.

[16] Y. Fukuma, *On the sectional geometric genus of quasi-polarized varieties, II*, Manuscripta Math. 113 (2004), 211–237.

[17] Y. Fukuma, *A genearlization of the ∆-genus of polarized varieties*, J. Math. Soc. Japan 57 (2005), 1003–1044.

[18] Y. Fukuma, *A lower bound for the second sectional geometric genus of polarized manifolds*, Adv. Geom. 5 (2005), 431–454.

[19] Y. Fukuma, *On the second sectional H-arithmetic genus of polarized manifolds*, Math. Z. 250 (2005), 573–597.

[20] Y. Fukuma, *Addendum: “On the sectional geometric genus of quasi-polarized varieties, I”*, Comm. Algebra 36 (2008), 3250–3252.

[21] W. Fulton, *Intersection Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete 2 (1984), Springer-Verlag.

[22] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.

[23] S. Iitaka, *Algebraic Geometry. An introduction to birational geometry of algebraic varieties*, Graduate Texts in Mathematics, No. 76. Springer-Verlag, New York-Berlin, 1982.

[24] P. Ionescu, *Generalized adjunction and applications*, Math. Proc. Cambridge Philos. Soc. 99 (1986), 457–472.

[25] Y. Kawamata, K. Matsuda and K. Matsuki, *Introduction to the minimal model problem*, Algebraic geometry, Sendai, 1985, 283–360, Adv. Stud. Pure Math., 10, 1987.
Department of Mathematics
Faculty of Science
Kochi University
Akebono-cho, Kochi 780-8520
Japan
E-mail: fukuma@kochi-u.ac.jp