The field of the Reals and the Random Graph are not Finite-Word Ordinal-Automatic

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Abstract. Recently, Schlicht and Stephan lifted the notion of automatic-structures to the notion of (finite-word) ordinal-automatic structures. These are structures whose domain and relations can be represented by automata reading finite words whose shape is some fixed ordinal $\alpha$. We lift Delhommé’s relative-growth-technique from the automatic and tree-automatic setting to the ordinal-automatic setting. This result implies that the random graph is not ordinal-automatic and infinite integral domains are not ordinal-automatic with respect to ordinals below $\omega_1 + \omega^\omega$ where $\omega_1$ is the first uncountable ordinal.

1. Introduction

Finite automata play a crucial role in many areas of computer science. In particular, finite automata have been used to represent certain infinite structures. The basic notion of this branch of research is the class of automatic structures (cf. [10]). A structure is automatic if its domain as well as its relations are recognised by (synchronous multi-tape) finite automata processing finite words. This class has the remarkable property that the first-order theory of any automatic structure is decidable. One goal in the theory of automatic structures is a classification of those structures that are automatic (cf. [3, 12, 11, 9, 13]). Besides finite automata reading finite or infinite (i.e., $\omega$-shaped) words there are also finite automata reading finite or infinite trees. Using such automata as representation of structures leads to the notion of tree-automatic structures [1]. The classification of tree-automatic structures is less advanced but some results have been obtained in the last years (cf. [3, 5, 7]). Schlicht and Stephan [14] and Finkel and Todorčević [4] have started research on a new branch of automatic structures based on automata processing $\alpha$-words where $\alpha$ is some ordinal. An $\alpha$-word is a map $w \in \Sigma^\alpha$ for some finite alphabet $\Sigma$. We call $w$ a finite $\alpha$-word if there is one symbol $\diamondsuit$ such that $w(\beta) = \diamondsuit$ for all but finitely many ordinals $\beta < \alpha$. We call the structures represented by finite-word $\alpha$-automatic structures $(\alpha)$-automatic. Many of the

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fundamental results on automatic structures have analogues in the setting of \((\alpha)\)-automatic structures.

- The first-order theory of every \((\alpha)\)-automatic structure is decidable and the class of \((\alpha)\)-automatic structures is closed under expansions by first-order definable relations for all \(\alpha < \omega_1 + \omega^\omega\) [6].
- Lifting a result of Blumensath [1] from the word- and tree-automatic setting, there is an \((\alpha)\)-automatic structure which is complete for the class of \((\alpha)\)-automatic structures under first-order interpretations [6].
- The sum-of-box-augmentation technique of Delhommé [3] for tree-automatic structures has an analogue for ordinal-automatic structures which allows to classify all \((\alpha)\)-automatic ordinals [14] and give sharp bounds on the ranks of \((\alpha)\)-automatic scattered linear orderings [14] and well-founded order trees [8].
- The word-automatic Boolean algebras [11] and the \((\omega^n)\)-automatic Boolean algebras [6] have been classified. In contrast, a classification of the tree-automatic Boolean algebras is still open.

In summary one can say that all known techniques which allow to prove that a structure is not tree-automatic have known counterparts for ordinal-automaticity. The only exception to this rule has been Delhommé’s growth-rate-technique [3]. We close this gap by showing that the maximal growth rates of ordinal automatic structures also has a polynomial bound. This allows to show that the Rado graph is not \((\alpha)\)-automatic. In fact, we show that the bound on the maximal growth-rate of \((\alpha)\)-automatic structure that we provide is strictly smaller than the bound for tree-automatic and strictly greater than the bound for word-automatic structures. Exhibiting this fact, we provide a new example of a structure that is \((\omega^2)\)-automatic but not word-automatic. This example also shows that our growth-rate bound for \((\alpha)\)-automatic structure is essentially optimal.

One of the long-standing open problems in the field of automatic structures is the question whether the field of the reals \(\mathcal{R} = (\mathbb{R}, +, \cdot, 0, 1)\) has a presentation based on finite automata. Due to cardinality reasons it is clear that this structure is not word- or tree-automatic. Recently, Zaid et al. [15] have shown that \(\mathcal{R}\) (as well as every infinite integral domain) is not infinite-word-automatic. This leaves infinite-tree-automata as the last classical candidate that might allow to represent \(\mathcal{R}\). Note that the cardinality argument also shows that \(\mathcal{R}\) is not \((\alpha)\)-automatic for all countable \(\alpha\) (because the set of finite \(\alpha\)-words is countable). Nevertheless the set of finite \(\omega_1\)-words is uncountable whence \(\mathcal{R}\) may be a priori \((\alpha)\)-automatic for some uncountable ordinal \(\alpha\). Using the growth rate argument we can show that no infinite integral domain is \((\alpha)\)-automatic for any ordinal \(\alpha < \omega_1 + \omega^\omega\). Let us mention that it also remains open whether \(\mathcal{R}\) is automatic with respect to automata that also accept infinite \(\alpha\)-words for some \(\alpha \geq \omega^2\).

1.1. Outline of the Paper. In the next section we recall the necessary definitions on \((\alpha)\)-automatic structures and the fundamental notions concerning growth rates. In Section 3 we recall basic results on \((\alpha)\)-automatic structures which are needed to obtain the growth rate bound in Section 4. Finally, Section 5 contains applications of the growth rate argument to the random graph, integral domains and concludes with the construction of a new example of an \((\omega^2)\)-automatic structure which is not word-automatic because its growth-rate exceeds the known bound for word-automatic structures.
2. Definitions

2.1. Ordinals. We identify an ordinal \( \alpha \) with the set of smaller ordinals \( \{ \beta \mid \beta < \alpha \} \). We say \( \alpha \) has countable cofinality if \( \alpha = 0 \) or there is a sequence \( \langle \alpha_i \rangle_{i \in \omega} \) of ordinals such that \( \alpha = \sup \{ \alpha_i + 1 \mid i \in \omega \} \). Otherwise we say \( \alpha \) has uncountable cofinality. We denote the first uncountable ordinal by \( \omega_1 \). Note that it is the first ordinal with uncountable cofinality.

For every ordinal \( \alpha \) and every \( n \in \mathbb{N} \), let \( \alpha \sim n \) be the ordinal of the form \( \alpha \sim n = \omega^{n+1} \beta \) for some ordinal \( \beta \) such that
\[
\alpha = \alpha \sim n + \omega^n m_n + \omega^{n-1} m_{n-1} + \cdots + m_0
\]
for some natural numbers \( m_0, \ldots, m_n \).

2.2. Ordinal-Shaped Words. First of all, we agree on the following convention: In this article, every alphabet \( \Sigma \) contains a distinguished blank symbol which is denoted by \( \diamond \) or, if the alphabet is clear from the context, just by \( \circ \). Moreover, for alphabets \( \Sigma_1, \ldots, \Sigma_r \), the distinguished symbol of the alphabet \( \Sigma_1 \times \cdots \times \Sigma_r \) will always be \( \diamond \Sigma_1 \times \cdots \times \Sigma_r = (\diamond \Sigma_1, \ldots, \diamond \Sigma_r) \).

For some limit ordinal \( \beta \leq \alpha \) and a map \( w : \alpha + 1 \to A \) we introduce the following notation for the set of images cofinal in \( \beta \):
\[
\lim_{\beta} w := \{ a \in A \mid \forall \beta' \beta' < \beta \exists \beta'' < \beta' w(\beta'') = a \}.
\]

Definition 1. An \((\alpha)\)-word (over \( \Sigma \)) (called a finite \( \alpha \) word over \( \Sigma \)) is a map \( w : \alpha \to \Sigma \) whose support, i.e., the set
\[
\text{supp}(w) = \{ \beta \in \alpha \mid w(\beta) \neq \circ \},
\]
is finite. The set of all \((\alpha)\)-words over \( \Sigma \) is denoted by \( \Sigma^{(\alpha)} \). We write \( \circ^\alpha \) for the constantly \( \circ \) valued word \( w : \alpha \to \Sigma \), \( w(\beta) = \circ \) for all \( \beta < \alpha \).

Definition 2. If \( \gamma \leq \delta \leq \alpha \) are ordinals and \( w : \alpha \to \Sigma \) some \((\alpha)\)-word, we denote by \( w|_{[\gamma, \delta)} \) the restriction of \( w \) to the subword between position \( \gamma \) (included) and \( \delta \) (excluded).

2.3. Automata and Automatic Structures. Büchi [2] has already introduced automata that process \((\alpha)\)-words. These behave like usual finite automata at successor ordinals while at limit ordinals a limit transition that resembles the acceptance condition of a Muller-automaton is used.

Definition 3. An ordinal automaton is a tuple \( (Q, \Sigma, I, F, \delta) \) where \( Q \) is a finite set of states, \( \Sigma \) a finite alphabet, \( I \subseteq Q \) the initial states, \( F \subseteq Q \) the final states and
\[
\delta \subseteq (Q \times \Sigma \times Q) \cup (2^Q \times Q)
\]
is the transition relation.

Definition 4. A run of \( \mathcal{A} \) on the \((\alpha)\)-word \( w \in \Sigma^{(\alpha)} \) is a map \( r : \alpha + 1 \to Q \) such that
- \( (r(\beta), w(\beta), r(\beta + 1)) \in \Delta \) for all \( \beta < \alpha \)
- \( (\lim_{\beta} r, r(\beta)) \in \Delta \) for all limit ordinals \( \beta \leq \alpha \).

The run \( r \) is accepting if \( r(0) \in I \) and \( r(\alpha) \in F \). For \( q, q' \in Q \), we write \( q \xrightarrow{w}^{A} q' \) if there is a run \( r \) of \( \mathcal{A} \) on \( w \) with \( r(0) = q \) and \( r(\alpha) = q' \).
In the following, we always fix an ordinal $\alpha$ and then concentrate on the set of $(\alpha)$-words that a given ordinal automaton accepts. In order to stress this fact, we will call the ordinal-automaton an $(\alpha)$-automaton.

**Definition 5.** Let $\alpha$ be some ordinal and $A$ be an $(\alpha)$-automaton. The $(\alpha)$-language of $A$, denoted by $L_{\Sigma^{(\alpha)}}(A)$, consists of all $(\alpha)$-words $w \in \Sigma^{(\alpha)}$ which admit an accepting run of $A$ on $w$. Whenever $\alpha$ is clear from the context, we may omit the subscript $\Sigma^{(\alpha)}$ and just write $L(A)$ instead of $L_{\Sigma^{(\alpha)}}(A)$.

Automata on words (or infinite words or (infinite) trees) have been applied fruitfully for representing structures. This can be lifted to the setting of $(\alpha)$-words and leads to the notion of $(\alpha)$-automatic structures. In order to use $(\alpha)$-automata to recognise relations of $(\alpha)$-words, we need to encode tuples of $(\alpha)$-words by one $(\alpha)$-word:

**Definition 6.** Let $\Sigma$ be an alphabet and $\tau \in \mathbb{N}$.

1. We regard any tuple $\bar{w} = (w_1, \ldots, w_\tau) \in (\Sigma^{(\alpha)})^\tau$ of $(\alpha)$-words over some alphabet $\Sigma$ as an $(\alpha)$-word $\bar{w} \in (\Sigma^\tau)^{(\alpha)}$ over the alphabet $\Sigma^\tau$ by defining
   \[ \bar{w}(\beta) = (w_1(\beta), \ldots, w_\tau(\beta)) \]
   for each $\beta < \alpha$.

2. An $r$-dimensional $(\alpha)$-automaton over $\Sigma$ is an $(\alpha)$-automaton $A$ over $\Sigma^r$. The $r$-ary relation on $\Sigma^{(\alpha)}$ recognised by $A$ is denoted
   \[ R(A) = \left\{ \bar{w} \in (\Sigma^{(\alpha)})^r \mid \bar{w} \in L(A) \right\}. \]

Usually, this interpretation of $\bar{w}$ as an $(\alpha)$-word is called convolution of $\bar{w}$ and denoted $\otimes \bar{w}$. For the sake of convenience, we just omit the symbol $\otimes$.

**Definition 7.** Let $\tau = \{R_1, R_2, \ldots, R_\tau\}$ be a finite relational signature and let relation symbol $R_i$ be of arity $r_i$. A structure $\mathfrak{A} = (A, R_1^A, R_2^A, \ldots, R_\mu^A)$ is $(\alpha)$-automatic if there are an alphabet $\Sigma$ and $(\alpha)$-automata $A, A_\approx, A_1, \ldots, A_m$ such that
   - $A$ is an $(\alpha)$-automaton over $\Sigma$,
   - for each $R_i \in \tau$, $A_i$ is an $r_i$-dimensional $(\alpha)$-automaton over $\Sigma$ recognising an $r_i$-ary relation $R(A_i)$ on $L(A)$,
   - $A_\approx$ is a 2-dimensional $(\alpha)$-automaton over $\Sigma$ recognising a congruence relation $R(A_\approx)$ on the structure $\mathfrak{A}' = (L(A), L(A_1), \ldots, L(A_m))$, and
   - the quotient structure $\mathfrak{A}'/R(A_\approx)$ is isomorphic to $\mathfrak{A}$, i.e., $\mathfrak{A}'/R(A_\approx) \cong \mathfrak{A}$.

In this situation, we call the tuple $(A, A_\approx, A_1, \ldots, A_m)$ an $(\alpha)$-automatic presentation of $\mathfrak{A}$. This presentation is said to be injective if $L(A_\approx)$ is the identity relation on $L(A)$. In this case, we usually omit $A_\approx$ from the tuple of automata forming the presentation.

### 2.4. Definitions Concerning Growth Rates.

The basic idea behind the growth rate technique is the question how many elements of a structure can be distinguished using a fixed finite set of relations and a set of parameters which has $n$ elements. We call two elements $a$ and $b$ distinguishable by a $(1+p)$-ary relation $R$ with parameters from $E$ if there are $e_1, e_2, \ldots, e_p \in E$ such that $(a,e_1,e_2,\ldots,e_p) \in R$ while $(b,e_1,e_2,\ldots,e_p) \notin R$. If $|E| = n$ and $R$ is some relation, it is clear that there are at most $2^n$ many elements that are pairwise distinguishable by $E$ with parameters from $E$. Delhommé [3] has shown that for every tree-automatic relation $R$ there are always sets $E$ with $n$ elements such that there
are at most \( n^c \) pairwise distinguishable elements where \( c \) is a constant only depending on \( R \) (and not on \( n \) or \( E \)). For word-automatic structures this bound even drops to \( n \cdot c \). We now provide basic definitions that allow to derive a similar bound for \((\alpha)\)-automatic structures.

**Definition 8.** Let \( \mathfrak{A} \) be an \((\alpha)\)-automatic structure with domain \( A \) and \( \Phi \) be a finite set of \((\alpha)\)-automata such that each \( A \in \Phi \) recognises a \( 1 + p \)-ary relation \( R_A \subseteq A^{1+p} \). Let \( E \subseteq A \) be a finite set and let \( F \) be an infinite family of subsets of \( A \) with \( \emptyset \in F \).

1. For all \( a, a' \in A \) we write \( a \sim_\Phi E a' \) if \((a, e_1, \ldots, e_p) \in R_A \iff (a', e_1, \ldots, e_p) \in R_A \) for all \( e_1, \ldots, e_p \in E \) and all \( A \in \Phi \), i.e., \( a \) and \( a' \) are indistinguishable with the automata from \( \Phi \) and parameters in \( E \).
2. We say \( S \subseteq A \) is \( E \)-\( \Phi \)-free if \( a \not\sim_\Phi E a' \) for all \( a, a' \in S \).
3. We say some set \( G \subseteq E \) is maximal \( E \)-\( \Phi \)-free if \( G \) is \( E \)-\( \Phi \)-free and there is no \( E \)-\( \Phi \)-free strict superset of \( G \).
4. For all \( S \subseteq A \) we write \( |S| \) for \( \max \{ |F| : F \in F \text{ with } F \subseteq S \} \).

Set \( \nu_\Phi^F(E) = \min \{ |G| : G \text{ maximal } E-\Phi \text{-free} \} \) and for \( n \in \mathbb{N} \), set \( \nu_\Phi^F(n) = \inf \{ \nu_\Phi^F(E) : E \in F, |E| = n \} \in \mathbb{N} \cup \{ \infty \} \) (where \( \inf \emptyset = \infty \)).

\( \nu_\Phi^F \) measures the minimal growth rate of sets definable from \( \Phi \) with a finite set of parameters with respect to some infinite family \( F \). In most applications \( F \) can be defined to be the set of all subsets. In this case \( \nu_\Phi^F \) just measures the growth rate of sets definable from \( \Phi \) with a finite set of parameters. Let us comment on how such a function \( \nu_\Phi^F \) is usually used. Typical results on growth rate are of the form “there are infinitely many \( n \in \mathbb{N} \) such that \( \nu_\Phi^F(n) \leq p(n) \)” for a certain polynomial \( p \). If \( F \) is the set of all subsets of the domain of the given structures, this says that for infinitely many values of \( n \) there is a subset \( E \) of size \( n \) such that every maximal \( E-\Phi \) free set \( G \) has size at most \( p(n) \).

3. Basic Results

In this Section we cite some results from [6] that turn out to be useful in the following sections.

**Proposition 9** (Proposition 3.6 of [6]). Let \( \alpha \geq 1 \) be an ordinal of countable cofinality and let \( \mathcal{A} = (S, \Sigma, I, F, \Delta) \) be an automaton with \( |S| \leq m \). For all \( s_0, s_1 \in S \) and \( \sigma \in \Sigma \),

\[
s_0 \xrightarrow{\sigma^m} A s_1 \iff s_0 \xrightarrow{\sigma^m} A s_1.
\]

**Proposition 10** (cf. Proposition 3.7 of [6]). Let \( \alpha \geq 1 \) be an ordinal of uncountable cofinality and let \( \mathcal{A} = (S, \Sigma, I, F, \Delta) \) be an automaton with \( |S| \leq m \). For all \( s_0, s_1 \in S \) and \( \sigma \in \Sigma \),

\[
s_0 \xrightarrow{\sigma^\alpha} A s_1 \iff s_0 \xrightarrow{\sigma^\alpha} A s_1.
\]
Lemma 11 (Lemma 3.19 of [6]). For every finite alphabet $\Sigma$, there is an $\alpha$-automaton that recognises a well-order $\sqsubseteq$ on the set $\Sigma^{(\alpha)}$. Moreover the relation $\subseteq_{\text{supp}}$ given by $w \subseteq_{\text{supp}} v$ if and only if $\text{supp}(w) \subseteq \text{supp}(v)$ is $(\alpha)$-automatic.

4. The Growth Rate Technique

Delhommé proved the following bounds on the growth rates of maximal $\mathcal{F}$-$E$-$\Phi$-free sets in the word- and tree-automatic setting.

Proposition 12 ([3]). For each set $\Phi$ of word-automatic relations, there is a constant $k$ such that $\nu^\Phi_F(n) \leq k \cdot n$ for infinitely many $n \in \mathbb{N}$.

For each set $\Phi$ of tree-automatic relations, there is a constant $k$ such that $\nu^\Phi_F(n) \leq n^k$ for infinitely many $n \in \mathbb{N}$.

The basic proof idea is to show that any $E$-$\Phi$-free set $G$ can be transformed into an $E$-$\Phi$-free set $G'$ such that $|G| = |G'|$ whose elements are all words (or trees) that have a domain that is similar to the union of the domains of all parameters from $E$. In order to prove a similar result we first provide a notion of having similar domains for $(\alpha)$-words.

Definition 13. Let $m \in \mathbb{N}$, $X$ a finite set of ordinals and $\beta$ and ordinal of the form

$$
\beta = \beta_{\sim_m} + \omega^m n_m + \omega^{m-1} n_{m-1} + \cdots + n_0.
$$

- Let $U_m(\beta)$ denote the set of ordinals $\gamma = \gamma_{\sim_m} + \omega^m l_m + \omega^{m-1} l_{m-1} + \cdots + l_0$ such that one of the following holds:
  - $\gamma = \beta,$
  - $\gamma_{\sim_m} = \beta_{\sim m}$ and for $k$ maximal with $l_k \neq n_k$, we have $l_k \leq n_k + m$ and $l_i \leq m$ for all $i < k$, or
  - $\gamma_{\sim_m} = \beta_{\sim m} + \omega_1 c$ for some $1 \leq c \leq m$ and $l_i \leq m$ for all $0 \leq i \leq m$.
- Let $U_m(X, \delta) = (\bigcup_{\gamma \in X \cup \{0\}} U_m(\gamma)) \cap \delta$.
- Let $U_m^i(X, \delta) = U_m(X, \delta)$ and $U_m^{i+1}(X, \delta) = U_m(U_m^i(X, \delta), \delta)$ for $i \in \mathbb{N}$.

A crucial observation is that, roughly speaking, there are few $(\alpha)$-words with support in $U_m^i(X, \alpha)$. Using the following abbreviations, we make this idea precise in the following lemma:

1. $c_m(\beta) = \max_{i \leq m} n_i$,
2. $c_m(X) = \max_{\gamma \in X} c_m(\gamma)$, and
3. $d_m(X) = |\{\gamma_{\sim m} | \gamma \in X \cup \{0\}\}|$.

Lemma 14. Suppose that $X$ is a finite set of ordinals and $i \geq 1$. Then

$$
|U_m^i(X, \alpha)| \leq (c_m(X \cup \{\alpha\}) + im)^{m+1} \cdot (i \cdot m + 1) \cdot d_m(X \cup \{\alpha\}).
$$

Proof. A simple induction shows that all $\gamma \in U_m^i(\beta)$ satisfy $\gamma_{\sim m} = \beta_{\sim m} + \omega_1 \cdot k$ for some $0 \leq k \leq (i \cdot m)$.

One also proves inductively that the coefficient of $\omega^j$ of an element of $U_m^i(X, \alpha)$ is bounded by $(c_m(w) + im)$. □
In this section, we fix an ordinal $\alpha$, and a finite set of $(\alpha)$-automata
\[ \Phi = (A_1, A_2, \ldots, A_n). \]
Without loss of generality, each automaton has the same state set $Q$. We fix the constant
\[ K = |2^Q \times Q|^n + 1. \]
The following proposition contains the main technical result that allows to use the relative growth technique for ordinal-automatic structures. This proposition implies that for all $E$ and $\Phi$ there is an $E, \Phi$-free set of maximal size with support in $U_K(\text{supp}(E))$. Since $U_K(\text{supp}(E))$ is small this provides an upper bound on the minimal size of maximal $E, \Phi$-free sets.

**Proposition 15.** Let $E \subseteq \Sigma^{(\alpha)}$ and $v \in \Sigma^{(\alpha)}$. There is a word $w \in U_K(\text{supp}(E), \alpha)$ such that $v \sim_E^\Phi w$.

For better readability we first provide a simple tool for the proof

**Lemma 16.** Let $E \subseteq \Sigma^{(\alpha)}$ and $v \in \Sigma^{(\alpha)}$. Let $n \in \mathbb{N}$ and $\beta$ some ordinal such that $\beta + \omega^{n+1} \leq \alpha$. If there is an ordinal $\gamma$ such that $\gamma < \gamma + \omega^n(K - 1) \leq \beta < \gamma + \omega^{n+1}$ and
\[ \text{supp}(E) \cap [\gamma, \gamma + \omega^{n+1}) = \emptyset \] (4.1)
then there are natural numbers $n_1 < n_2 \leq K$ such that for
\[ w := v \upharpoonright [0, \gamma + \omega^n n_1) + v \upharpoonright [\gamma + \omega^n n_2, \alpha) \]
then $w \sim_E^\Phi v$, i.e., for all $\bar{e} \in E^k$ and $\bar{A} \in \Phi$
\[ \bar{A} \text{ accepts } v \otimes \bar{e} \text{ iff } \bar{A} \text{ accepts } w \otimes \bar{e} \]

**Proof.** Set $I = \{1, 2, \ldots, n\}$. We define the function
\[ f : \{0, 1, \ldots, K\} \to 2^Q \times Q \times I \]
such that $f(j)$ contains $(q, p, i)$ if and only if there is a run of $A_i$ from state $q$ to state $p$ on $v \upharpoonright [\gamma, \gamma + \omega^j)$ by $A_i$. By choice of $K$ there are $n_1 < n_2$ with $f(n_1) = f(n_2)$. Thus,
\[ q \xrightarrow{v \otimes \bar{e}}_{A_i} p \]
\[ \iff \exists r, s \left( q \xrightarrow{(v \otimes \bar{e})[0, \gamma]}_{A_i} r \land r \xrightarrow{v[\gamma, \gamma + \omega^n n_2) \otimes \omega^n n_1}_{A_i} s \land s \xrightarrow{(v \otimes \bar{e})[\gamma + \omega^n n_2, \alpha]}_{A_i} p \right) \]
\[ \iff \exists r, s \left( q \xrightarrow{(v \otimes \bar{e})[0, \gamma]}_{A_i} r \land r \xrightarrow{v[\gamma, \gamma + \omega^n n_1) \otimes \omega^n n_2}_{A_i} s \land s \xrightarrow{(v \otimes \bar{e})[\gamma + \omega^n n_2, \alpha]}_{A_i} p \right) \]
\[ \iff q \xrightarrow{w \otimes \bar{e}}_{A_i} p \]

from which we immediately conclude that $v \sim_E^\Phi w$. \qed

**Proof of Proposition 15.** The proof is by outer induction on $|\text{supp}(v) \setminus U_K(\text{supp}(E), \alpha)|$ and by inner (transfinite) induction on
\[ \beta = \min(\text{supp}(v) \setminus U_K(\text{supp}(E), \alpha)). \]
Fix the presentation
\[ \beta = \beta_{K} + \omega^K b_K + \omega^{K-1} b_{K-1} + \cdots + b_0 \]
with $b_0, \ldots, b_K \in \mathbb{N}$ and proceed as follows.
• If there is some \( n \leq K \) such that \( b_n + 1 - K \geq 0 \) and 
\[
(\text{supp}(E) \cup \{\alpha\}) \cap [\epsilon_1, \epsilon_2) = \emptyset
\]
where 
\[
\epsilon_1 = \beta_{\sim K} + \omega^K b_K + \omega^{K-1} b_{K-1} + \cdots + \omega^n (b_n + 1 - K)
\]
and 
\[
\epsilon_2 = \beta_{\sim K} + \omega^K b_K + \omega^{K-1} b_{K-1} + \cdots + \omega^{n+1} (b_{n+1} + 1),
\]
then we can apply the previous lemma and obtain a word \( v' \) such that \( v \sim_E^E v' \) and 
\[
|\text{supp}(v') \setminus U_K(\text{supp}(E), \alpha)| < |\text{supp}(v) \setminus U_K(\text{supp}(E), \alpha)|
\]
or
\[
\beta' = \min(\text{supp}(v') \setminus U_K(\text{supp}(E), \alpha)).
\]
has a presentation
\[
\beta' = \beta_{\sim K} + \omega^K b_K + \omega^{K-1} b_{K-1} + \cdots + \omega^{n+1} b_{n+1} + \omega^n b' + \omega^{n-1} b_{n-1} + \cdots + b_0
\]
with \( b' < b_n \).

• Assume that the conditions for the previous case are not satisfied. Either \( b_i \leq K - 1 \) for \( 0 \leq i \leq K \) or there is a minimal \( i \leq K \) such that \( b_i \geq K \). We first show that the latter case cannot occur. Assuming \( b_i \geq K \) we have
\[
(\text{supp}(E) \cup \{\alpha\}) \cap [\epsilon_1, \epsilon_2) \neq \emptyset
\]
where 
\[
\epsilon_1 = \beta_{\sim K} + \omega^K b_K + \omega^{K-1} b_{K-1} + \cdots + \omega^i (b_i + 1 - K)
\]
and 
\[
\epsilon_2 = \beta_{\sim K} + \omega^K b_K + \omega^{K-1} b_{K-1} + \cdots + \omega^{i+1} (b_{i+1} + 1).
\]

Thus, there is some \( \gamma \in \text{supp}(E) \cup \{\alpha\} \) with
\[
\gamma = \beta_{\sim K} + \omega^K b_K + \omega^{K-1} b_{K-1} + \cdots + \omega^{i+1} b_{i+1} + \omega^i c_i + \omega^{i-1} c_{i-1} + \cdots + c_0
\]
such that \( c_i + K - 1 > b_i \) whence \( \beta \in U_K(\gamma) \subseteq U_K(\text{supp}(E), \alpha) \) contradicting the definition of \( \beta \).

Thus, we can assume that \( b_i \leq K - 1 \) for all \( 0 \leq i \leq K \). By definition of \( \beta \) we conclude that \( \beta_{\sim K} \neq \gamma_{\sim K} + \omega^c c \) for \( c \in \{0, 1, \ldots, K\} \) for all \( \gamma \in \text{supp}(E) \cup \{0, \alpha\} \).

We proceed with one of the following cases depending on the cofinality of \( \beta_{\sim K} \).

1. If \( \beta_{\sim K} \) has countable cofinality, let
\[
\gamma = \max((\text{supp}(E) \cup \{v\}) \cap \beta_{\sim K}) + 1
\]
\[
\delta = \min((\text{supp}(E) \cap [\beta, \alpha]) \cup \{\alpha\})
\]
and
\[
\delta' = \max(\text{supp}(v) \cap \delta_{\sim K}) + 1.
\]

From the definition of \( \beta \) it follows that \( \gamma_{\sim K} < \beta_{\sim K} < \delta_{\sim K} \) and that \( \text{sup}(E) \cap [\gamma, \delta) = \emptyset \). Note that \( [\gamma, \beta_{\sim K}) \) is of shape \( \omega^{K+1} \eta_1 \) for some ordinal \( \eta_1 \geq 1 \) of countable cofinality. By definition of \( \delta' \), \( [\delta', \delta_{\sim K}) \) is of shape \( \omega^{K+1} \eta_2 \) for some ordinal \( \eta_2 \geq 1 \). Choose an ordinal \( \eta \) such that \( [\beta_{\sim K}, \delta') + \eta \) is isomorphic to \( [\gamma, \delta_{\sim K}) \) and define
\[
w := v \upharpoonright [0, \gamma) + \omega^\omega + v \upharpoonright [\beta_{\sim K}, \delta') + \omega^\eta + v \upharpoonright [\delta_{\sim K}, \alpha).
\]
For all $\bar{e} \in E^k$ and $A \in A$ we conclude that 
\[ q \xrightarrow{\ast} A \ x \rightarrow p \]

\[ \Leftrightarrow \exists r, s, t, u \in Q \begin{cases} q \xrightarrow{\ast} A r \land r \xrightarrow{\ast} s \land s \xrightarrow{\ast} t \xrightarrow{\ast} A p \end{cases} \]

whence $w \sim_{E^j} v$. Moreover

\[ |\text{supp}(w) \cup U_K(\text{supp}(E), \alpha)| < |\text{supp}(v) \cup U_K(\text{supp}(E), \alpha)| \]

because the letter at position $\beta$ in $v$ has been shifted to position $\gamma$. Since all $b_i \leq K - 1$ we conclude that this position belongs to $U_K(\text{supp}(E), \alpha)$.

(2) If $\beta_{\sim K}$ has uncountable cofinality, Let

\[ \gamma = \max(\text{supp}(E) \cap [\beta_{\sim K}]) + 1 \]
\[ \delta = \min((\text{supp}(E) \cap [\beta, \alpha)) \cup \{\alpha\}) \text{ and} \]
\[ \delta' = \max(\text{supp}(v) \cap [\delta_{\sim K}]) + 1. \]

From the definition of $\beta$ and the uncountable cofinality of $\beta_{\sim K}$ it follows that $\gamma_{\sim K} + \omega_1 K < \beta_{\sim K} < \delta_{\sim K}$ and that $\text{supp}(E) \cap [\gamma, \delta) = \emptyset$. Let

\[ \gamma' = \max(\text{supp}(E \cup \{v\}) \cap [\beta_{\sim K}]) + 1 \]

and note that $\gamma' \leq \gamma_{\sim K} + \omega_1 (K + 1)$ because $\gamma' \in U_K(E, \alpha)$ since $\beta$ has been chosen minimal. If $\gamma' \geq \gamma_{\sim K} + \omega_1 K$, then we do the following preparatory step that locally changes $v$ to some $v'$ such that $v \sim_{E^j} v'$ and shrinking the corresponding value of $\gamma'$. For this purpose, note that $e \upharpoonright [\gamma, \beta] = \omega([\gamma, \beta])$ for all $e \in E$. By choice of $K$ there are numbers $i < j \leq K$ such that for all $A \in \Phi$, all $q, p \in Q$ and all $\bar{e} \in E^k$ we have

\[ q \xrightarrow{(e \otimes [\gamma, \gamma + \omega_{i+1}])} A \ x \rightarrow p \Leftrightarrow q \xrightarrow{(e \otimes [\gamma, \gamma + \omega_{j+1}])} A \ x \rightarrow p \]

Choose an ordinal $\eta$ such that $\omega_1 \cdot (j - i) + [\gamma + \omega_1 i, \gamma'] = \omega_1 j + \gamma' + \eta$ and set

\[ v' = v \upharpoonright [0, \gamma_{\sim K} + \omega_1 i] + v \upharpoonright [\gamma_{\sim K} + \omega_1 j, \gamma'] + \omega \upharpoonright [\gamma', \alpha). \]
Now \( v \sim_E v' \) because for all \( \bar{e} \in E^k \)
\[
q \xrightarrow{\bar{e} \otimes} p
\]
\[
\iff \exists r, s, t \in Q \quad \left( \frac{q}{r} \xrightarrow{\bar{e}} \exists \frac{r}{s} \xrightarrow{\bar{e}} t \xrightarrow{s} \frac{r}{p} \right)
\]
\[
\iff \exists r, s, t \in Q \quad \left( \frac{q}{r} \xrightarrow{\bar{e}} \exists \frac{r}{s} \xrightarrow{\bar{e}} t \xrightarrow{s} \frac{r}{p} \right)
\]
\[
\iff \exists r, s, t \in Q \quad \left( \frac{q}{r} \xrightarrow{\bar{e}} \exists \frac{r}{s} \xrightarrow{\bar{e}} t \xrightarrow{s} \frac{r}{p} \right)
\]
\[
\iff \exists \frac{q}{r} \xrightarrow{\bar{e}} p
\]

Note that the definitions of \( \beta, \gamma, \delta \) and \( \delta' \) with respect to \( v' \) agree with those for \( v \). Thus, from now on we replace \( v \) by \( v' \) whence we can assume that \( \gamma' < \gamma + \omega_1 K \).

Note that \([\gamma', \beta_{\sim} K] \) is of shape \( \omega^{K+1} \eta_1 \) for some ordinal \( \eta_1 \geq 1 \) of uncountable cofinality. By definition of \( \delta', [\delta', \delta_{\sim} K] \) is of shape \( \omega^{K+1} \eta_2 \) for some ordinal \( \eta_2 \geq 1 \). Choose an ordinal \( \eta \) such that \( [\beta_{\sim} K, \delta'] + \eta \) is isomorphic to \( [\gamma' + \omega_1, \delta_{\sim} K] \) and define
\[
w := v \restriction [0, \gamma') + \omega^\omega + v \restriction [\beta_{\sim} K, \delta') + \omega^\eta + v \restriction [\delta_{\sim} K, \alpha).
\]

For all \( \bar{e} \in E^k \) and \( A \in A \) we conclude that
\[
q \xrightarrow{\bar{e}} p
\]
\[
\iff \exists r, s, t, u \in Q \quad \left( \frac{q}{r} \xrightarrow{\bar{e}} \exists \frac{r}{s} \xrightarrow{\bar{e}} t \xrightarrow{s} \frac{r}{p} \right)
\]
\[
\iff \exists r, s, t, u \in Q \quad \left( \frac{q}{r} \xrightarrow{\bar{e}} \exists \frac{r}{s} \xrightarrow{\bar{e}} t \xrightarrow{s} \frac{r}{p} \right)
\]
\[
\iff \exists \frac{q}{r} \xrightarrow{\bar{e}} p
\]

whence \( w \sim_E^p v \). Moreover
\[
|\text{supp}(w) \setminus U_K(\text{supp}(E), \alpha)| < |\text{supp}(v) \setminus U_K(\text{supp}(E), \alpha)|
\]
because the letter at position \( \beta \) in \( v \) has been shifted to position
\[
\gamma_{\sim} K + \omega_1 + \omega^K b_K + \omega^{K-1} b_{k-1} + \omega^{K-2} b_{k-2} + \cdots + b_0
\]
and since all \( b_i < K - 1 \) we conclude that this is position is contained in \( U_K(\supp(E), \alpha) \) because \( \gamma \sim K = \eta \sim K + \omega_1 c \) for some \( \eta \in \supp(E) \cup \{0\} \) and some \( c \in \{0, 1, \ldots, K - 1\} \).

The previous result allows us to directly deduce the following bound on the growth rates of ordinal-automatic relations.

**Theorem 17.** Fix an infinite family \( \mathcal{F} \) of sets of \((\alpha)\)-words with \( \emptyset \in \mathcal{F} \). For every \( c > 1 \),

\[
\nu^\Phi(n) \leq n^c
\]

for infinitely many \( n \in \mathbb{N} \).

**Proof.** Heading for a contradiction, assume that there is a natural number \( N \) such that

\[
\forall E \in \mathcal{F} \text{ with } |E| > N \forall G \text{ maximal } E-\Phi\text{-free } \exists F \in \mathcal{F} (F \subseteq G \text{ and } |F| \geq |E|^c)
\]

where \( c > 1 \).

Take a finite set \( F_0 \) of parameters with \( |F_0| > N \). Having defined a finite set \( F_{i-1} \) such that \( \supp(F_{i-1}) \in U_K^{i-1}(\supp(F_0)) \) we can use the previous lemma to choose some maximal \( F_{i-1}\text{-}\Phi\text{-free set } G_i \) with \( \supp(G_i) \in U_K^i(\supp(F_0)) \). By assumption, there is some \( F_i \in \mathcal{F} \) with \( F_i \subseteq G_i \) and \( |F_i| \geq |F_{i-1}|^c \). By induction we obtain \( |F_i| \geq |F_0|^{c^i} \).

On the other hand, all elements of \( F_i \) have support in \( U_K^i(\supp(F_0)) \) which by Lemma 14 is at most of size

\[
(|\Sigma| + 1)^{c_0(i+1)(c_1 + Ki)^K}
\]

for some constants \( c_0 \) and \( c_1 \). Since \( c^i \) grows faster than any polynomial in \( i \) we have \( c^i > c_0(i+1)(c_1 + Ki)^K \) for some large \( i \) which leads to a contradiction.

## 5. Applications

### 5.1. Random Graph

The random graph (or Rado graph) \((V,E)\) is the unique countable graph that has the property that for any choice of finite subsets \( V_0, V_1 \subseteq V \) there is a node \( v' \) which is adjacent to every element of \( V_0 \) but not adjacent to any element of \( V_1 \).

**Theorem 18.** Given an ordinal \( \alpha \), the random graph is not \((\alpha)\)-automatic.

**Proof.** Heading for a contradiction assume that the random graph was \((\alpha)\)-automatic. As shown by Delhommé [3], the random graph satisfies \( \nu^\Phi_F(n) = 2^n \) for all \( n \in \mathbb{N} \) where \( \Phi \) consists of only one automaton recognising the edge relation of the random graph and \( \mathcal{F} \) contains all subsets of the domain of the random graph. This contradicts Proposition 15.

**Remark 19.** Similarly, taking \( \mathcal{F} \) to be the family of all antichains, one proves that the random partial order is not \((\alpha)\)-automatic (cf. [11] for an analogous result for automatic structures).
5.2. Integral Domains. In this part we show that there is no infinite \((\alpha)-\)automatic integral domain for any \(\alpha < \omega_1 + \omega^\omega\). We cannot use the growth rate theorem directly but use a variant of its proof. The difference is that we do not use a fixed set of relations \(\Phi\) when defining the sequence \((F_i)_{i \in \mathbb{N}}\) but in each step we take a different relation but ensure that we can still apply Proposition 15 with a fixed constant \(K\) in each step. This is ensured by using relations defined by a fixed automaton \(A\) which has an additional parameter which is chosen very carefully. In fact, we follow the proof of Khoussainov et al. [11] from the automatic case. Let us recall the basic definitions and some observations from their proof. An integral domain is a commutative ring with identity \((D, +, 0, 1)\) such that \(d \cdot e = 0 \Rightarrow d = 0\) or \(e = 0\) for all \(d, e \in D\).

**Lemma 20** (cf. Proof of Theorem 3.10 from [11]). Let \((D, +, 0, 1)\) be an integral domain and \(E \subseteq D\) a finite subset. There is some \(d \in D\) such that for all \(a_1, a_2, b_1, b_2 \in E\), if \(a_1 d + b_1 = a_2 d + b_2\), then \(a_1 = a_2\) and \(b_1 = b_2\), i.e., the function \(f_d : E^2 \rightarrow D\), \((e_1, e_2) \mapsto e_1 d + e_2\) is injective.

**Proposition 21.** Let \(\mathfrak{A} = (D, +, 0, 1)\) be an \((\alpha)-\)automatic integral domain for some \(\alpha < \omega_1 + \omega^\omega\). \(^1\)

There is a constant \(m\) such that for every finite set \(X \subseteq \alpha\) of ordinals we have

\[
|\{ d \in D \mid \text{supp}(d) \subseteq U_m(X, \alpha) \}| \geq |\{ d \in D \mid \text{supp}(d) \subseteq X \}|^2.
\]

**Proof.** As an abbreviation, we use the expression \(x_1, \ldots, x_n \subseteq \text{supp} y\) for

\[
x_1 \subseteq \text{supp} y \land \cdots \land x_n \subseteq \text{supp} y.
\]

Let \(\psi(x, p)\) denote the formula

\[
x \in D \land \forall a_1, a_2, b_1, b_2 \subseteq \text{supp} p \ ((a_1 x + b_1 = a_2 x + b_2) \Rightarrow (a_1 = a_2 \land b_1 = b_2))
\]

and \(\psi_{\text{min}}(x, p)\) the formula

\[
\psi(x, p) \land \forall y (\psi(y, p) \Rightarrow x \subseteq y)
\]

where \(\subseteq\) denotes the \((\alpha)-\)automatic well-order from Lemma 11. Due to the previous lemma for every \(p \in \Sigma^{(\alpha)}\) there is a unique \(x\) satisfying \(\psi_{\text{min}}(x, p)\). Moreover, the map \(f : (a, b) \mapsto ax + b\) is injective when the domain is restricted to words with support contained in \(\text{supp}(p)\).

Since \(\subseteq\) is \((\alpha)-\)automatic and \((\alpha)-\)automatic structures are close under first-order definitions, there is an automaton \(A_{\varphi}\) corresponding to the following formula

\[
\varphi(p, a, b, c) = a \subseteq \text{supp} p \land b \subseteq \text{supp} p \land \exists x (\psi_{\text{min}}(x, P) \land c = ax + b)
\]

For each finite set \(X\) of ordinals, choose an \((\alpha)-\)word \(p_X\) such that \(\text{supp}(p) = X\). Set

\[D_X = \{ d \in D \mid c \subseteq \text{supp} p \} \quad \text{and} \quad F_X = \{ c \in D \mid \exists a, b \in \Sigma^{(\alpha)} A_{\varphi} \text{ accepts } (p_X, a, b, c) \}.
\]

Since we are dealing with an injective presentation, Proposition 15 implies that for every \(a, b \in D_X\) there is some \(c_{a,b} \in F_X\) such that \(A_{\varphi}\) accepts \((p_X, a, b, c_{a,b})\) and \(\text{supp}(c_{a,b}) \subseteq \text{supp}(p_X)\).

\(^1\)Without loss of generality, we can assume that \(\mathfrak{A}\) has a injective representation by the automata \((A_D, A_I, A_0, A_1)\) such that \(L(A_D) = D\), i.e., \(D\) is a set of \((\alpha)-\)words (cf. [6]).
\[ U_K(\supp(p_X) \cup \supp(a) \cup \supp(c), \alpha) = U_K(X, \alpha) \]. Moreover, \( c_{a,b} = c_{a',b'} \) implies \( a = a' \) and \( b = b' \) whence we conclude that
\[ |\{ d \in D \mid \supp(d) \subseteq U_K(X, \alpha) \}| \geq |F_X| \geq |D_X|^2. \]

\[ \square \]

**Remark 22.** We crucially rely on \( \alpha < \omega_1 + \omega^\omega \) because otherwise we cannot be sure that there is an automaton corresponding to \( \psi(x, p) \).

**Corollary 23.** Let \( \alpha < \omega_1 + \omega^\omega \). There is no \( (\alpha) \)-automatic infinite integral domain. In particular, there is no \( (\alpha) \)-automatic infinite field.

**Proof.** Assume \( D \) is the domain of an \( (\alpha) \)-automatic infinite integral domain. Choose two elements \( d_1 \neq d_2 \) from \( D \) and let \( X = \supp(d_1) \cup \supp(d_2) \). Set
\[ F_0 = \{ d \in D \mid \supp(d) \subseteq \supp(d_1) \cup \supp(d_2) \}. \]

Iterated application of the previous lemma yields that
\[ F_i := \{ d \in D \mid \supp(d) \subseteq U_K(\supp(F_0), \alpha) \} \]
satisfies \( |F_{i+1}| \geq |F_i|^2 \). Since \( |F_0| \geq 2 \) we conclude that \( |F_n| \geq 2^{2^n} \) and \( \supp(F_n) \subseteq U^n_K(X, \alpha) \). From Lemma 14 we conclude that there are only \( 2^{p(n)} \) many elements in \( D \) with support in \( U^n_K(X, \alpha) \) for some polynomial \( p(n) \) which results in a contradiction for large \( n \).

\[ \square \]

6. **Optimality of the Bound on the Growth-Rate**

Recall that word-automatic structures satisfy that \( \nu_\Phi^F(n) < n \cdot k \) for some constant \( k \) where \( F \) is a family as before and \( \Phi \) is a finite set of word-automatic relations. In contrast, our bound for \( (\alpha) \)-automatic structures is only \( n^k \) for every constant \( k > 1 \). In this section, we give an example that, if \( \alpha \geq \omega^2 \), then there are \( (\alpha) \)-automatic structures violating any bound of the form \( n \cdot k \) for every constant \( k \).

**Definition 24.** For every \( n \in \mathbb{N} \), let
\[ D_n = \{ \omega n_1 + n_2 \mid n_1 + n_2 \leq n \} \]
and let \( T_n \) be the set \( (\omega^2) \)-words over \( \Sigma = \{ a, b, \diamond \} \) such that \( w \in T_n \) if and only if \( \supp(w) = D_n \). For \( i \in \{ a, b \} \), we also define functions
\[ f_i : \Sigma^{(\omega^2)} \times \Sigma^{(\omega^2)} \to \Sigma^{(\omega^2)} \] by
\[ f_i(w, v)(\alpha) = \begin{cases} i & \text{if } \alpha = 0, \\ w(\omega n_1 + n_2) & \text{if } \alpha = \omega n_1 + n_2 + 1, \\ v(\omega n_1) & \text{if } \alpha = \omega(n_1 + 1). \end{cases} \]

\[ \text{It is easily shown that for } \alpha < \omega^2 \text{ every } (\alpha) \text{-automatic structure is word-automatic and vice versa whence the stronger bound from the word-automatic case applies.} \]
It is not difficult to see that the graphs of $f_a$ and $f_b$ are $(\omega^2)$-automatic relations. Let $\Phi$ consist of two automata, one corresponding to the graph of $f_a$ and one corresponding to the graph of $f_b$. It is straightforward to verify that $T_{n+1} = f_a(T_n \times T_n) \cup f_b(T_n \times T_n)$. Moreover, since $f_a$ and $f_b$ are functions, it follows that any maximal $T_n$-\Phi-free set is of the form $T_{n+1} \cup \{\omega\}$ for some $(\omega^2)$-word $\omega \not\in T_{n+1}$. A simple calculation shows that $|D_n| = \frac{(n+1)(n+2)}{2}$ whence $|T_n| = 2^{\frac{(n+1)(n+2)}{2}}$.

**Corollary 25.** Setting $F$ to be the family of the sets $T_n$ for every $n \in \mathbb{N}$, we obtain that $\nu_F^\Phi(n) = \begin{cases} m \cdot 2^{n+2} & \text{if } m = 2^{\frac{(n+1)(n+2)}{2}}, \\ \infty & \text{otherwise}. \end{cases}$

**Proof.** Just note that $|T_{n+1}| = 2^{\frac{(n+3)(n+2)}{2}} = 2^{\frac{(n+1)(n+2)}{2} + 3} = |T_n| \cdot 2^{n+2}$. \hfill \Box

Since there for every constant $k$ there is some value $n_0 \in \mathbb{N}$ such that $2^{n+2} \geq k$ this shows that $\nu_F^\Phi(m) \leq m \cdot k$ only holds for finitely many $m \in \mathbb{N}$. This shows that there is no word-automatic presentation of the $(\omega^2)$-automatic structure $(\Sigma^{(\omega^2)}, f_a, f_b)$ and that the bound in Theorem 17 cannot be replaced by $n \cdot k$.

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