Gluonic evanescent operators: classification and one-loop renormalization

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Abstract: Evanescent operators are a special class of operators that vanish classically in four-dimensional spacetime, while in general dimensions they are non-zero and are expected to have non-trivial physical effects at the quantum loop level in dimensional regularization. In this paper we initiate the study of evanescent operators in pure Yang-Mills theory. We develop a systematic method for classifying and constructing the $d$-dimensional Lorentz invariant evanescent operators, which start to appear at mass dimension ten. We also compute one-loop form factors for the dimension-ten operators via the $d$-dimensional unitarity method and obtain their one-loop anomalous dimensions. These operators are necessary ingredients in the study of high dimensional operators in effective field theories involving a Yang-Mills sector.

Keywords: Effective Field Theories, Renormalization and Regularization, Scattering Amplitudes

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1 Introduction

Gauge invariant local operators are important quantities in quantum field theories. For example, composite local operators give the interaction vertices in effective Lagrangians and thus are central ingredients in the study of effective field theory (EFT), see e.g. [1]. In QCD, hadron states correspond to color-singlet operators composed of gluon and quark fields. Similar gauge-invariant operators and their anomalous dimensions have been studied extensively in $\mathcal{N} = 4$ SYM and played an important role in understanding the AdS/CFT correspondence and integrability [2]. When studying the renormalization of operators, the dimensional regularization scheme is often one of the most convenient choices to regularize the divergences [3]. In this case, the spacetime dimension is generalized to $d = 4 - 2\epsilon$ dimensions, and a special subtlety appears that it is possible to construct a series of operators that vanish in four dimensions but not in $d$ dimensions, which are known as evanescent operators.

The study of evanescent operators has been considered long time ago in the context of four fermion interactions [4–7]. In general $d$-dimensional spacetime, there are infinitely many operators involving anti-symmetric tensor of Dirac matrices $\gamma^\mu$ with ranks higher than 4:

$$O^{(n)}_{4\text{-ferm}} = \bar{\psi}\gamma^{[\mu_1}\ldots\gamma_{\mu_n]}\psi\psi|_{\mu_1}\ldots|_{\mu_n}\psi, \quad n \geq 5. \quad (1.1)$$

Such operators vanish in strictly four-dimensional spacetime but not in $d$ dimensions. Similar evanescent operators also appear in the two-dimensional four-fermion models, see e.g. [8–10]. It was shown in those studies that these evanescent operators can not be ignored, as they affect the anomalous dimensions of physical operators. In the scalar $\phi^4$ theory, the high-dimensional evanescent operators were considered in [11] and it was found that the evanescent operators can give rise to negative-norm states implying that the theory is not unitary in non-integer spacetime dimensions. Counting scalar evanescent operators via Hilbert series was also considered in [12].

In this paper, we consider a new class of evanescent operators in the pure Yang-Mills theory, which are composed of field strength $F_{\mu\nu}$ and covariant derivatives $D_\mu$. A simple example of such operators can be given as

$$O_e = \frac{1}{16}\delta^{\mu_1\mu_2\mu_3\mu_4\mu_5}_{\nu_1\nu_2\nu_3\nu_4\nu_5}\text{tr}(D_{\nu_5}F_{\mu_1\mu_2}F_{\mu_3\mu_4}D_{\mu_5}F_{\nu_1\nu_2}F_{\nu_3\nu_4}). \quad (1.2)$$
where $\delta^{\mu_1 \cdots \mu_n}_{\nu_1 \cdots \nu_n} = \det(\delta^\mu_\nu)$ is the generalized Kronecker symbol (see section 2 for detail). This operator is zero in four dimensions but has non-trivial matrix elements such as form factors in general $d$ dimensions. For example, its (color-ordered) minimal tree-level form factor can be given as

$$F^{(0)}_{\mathcal{O}_e}(1, 2, 3, 4) = 2\delta^{e_1 e_2 p_1 p_2 p_3} + 2\delta^{e_1 e_4 p_1 p_4 p_2}, \quad (1.3)$$

which is a non-trivial function of Lorentz product of momenta and polarization vectors in $d$ dimensions. The main goal of this paper is to study the classification of such operators and their one-loop renormalization.

Unlike the four fermion operators in (1.1), due to the insertion of covariant derivatives and different ways of Lorentz contractions, the gluonic evanescent operators like (1.2) exhibit richer structures. Moreover, at a given mass dimension, the number of all the possible Lorentz contraction structures is finite, which means that the gluonic evanescent operators are also finite, calling for a systematic way to construct their independent basis. To classify these operators, it will be convenient to apply the correspondence between local operators and form factors [13, 14]. The main advantage is that form factors are on-shell matrix elements, thus the constraints from the equation of motion and Bianchi identities can be taken into account automatically, see e.g. [15]. Here, due to the special nature of evanescent operators, the usual spinor helicity formalism will be insufficient. Instead, one needs to consider form factors consisting of $d$-dimensional Lorentz vectors (i.e. external momenta and polarization vectors) such as in (1.3). Since the Yang-Mills operators contain non-trivial color factors, the form factor expressions provide also a useful framework to organize the color structures. One can first classify function basis at the form factor level and then map back to basis operators. We will apply a strategy to construct the basis evanescent operators along this line.

To study the quantum effect of evanescent operators, we perform one-loop computation of their form factors. The calculation is based on the unitarity method [16, 17] in $d$ dimensions. Using the form factor results, we can study their renormalization and operator-mixing behaviors. We provide explicit results of the one-loop renormalization matrices and the anomalous dimensions for the dimension-ten basis operators. These one-loop results will be necessary ingredients for the two-loop renormalization of physical operators.

This paper is organized as follows. In section 2, we first give the definition of evanescent operators and then describe the systematic construction of the operator basis. In section 3 we first explain the one-loop computation of full-color form factors using the unitarity method, then we discuss the renormalization and obtain the anomalous dimensions of the complete set of evanescent operators with dimension 10. A summary and discussion are given in section 4 followed by a series of appendices. Several technique details in the operator construction are given in appendix A–D. The basis of dimension-12 length-4 evanescent operators is given in appendix E. Efficient rules for calculating compact tree-level form factors are given in appendix F. The color-decomposition and one-loop infrared structure are discussed in appendix G–H. Finally, the full basis of dimension-10 physical operators as well as their one-loop renormalization are given in appendix I.
2 Gluonic evanescent operators

In this section, we explain the classification and the basis construction for evanescent operators. We first set up the conventions in section 2.1 and then give the definition of the evanescent operators in section 2.2. Then we discuss the systematic construction of evanescent basis operators in section 2.3. To be concrete and for simplicity, our discussion will focus on the length-4 case, and the generalization to high length cases will be given in section 2.4. The full set of dim-10 evanescent operators including length-4 and length-5 ones are summarized in section 2.5.

2.1 Setup

In this paper we consider the gauge invariant Lorentz scalar operators in pure Yang-Mills theory, which are composed of field strength $F_{\mu \nu}$ and covariant derivatives $D_{\mu}$. The field strength carries a color index as $F_{\mu \nu} = F_{a_{\mu \nu}} T^a$, where $T^a$ are the generators of gauge group satisfying $[T^a, T^b] = i f^{abc} T^c$, and the covariant derivative acts as

$$D_{\mu} \circ = \partial_{\mu} \circ + ig[A_{\mu}, \circ], \quad [D_{\mu}, D_{\nu}] \circ = -ig[F_{\mu \nu}, \circ].$$ (2.1)

A gauge invariant scalar operator can be given as

$$O(x) \sim c(a_1, \ldots, a_n) W_{m_1}^{a_1} W_{m_2}^{a_2} \cdots W_{m_L}^{a_L}, \quad \text{with} \quad W_{m_i}^{a_i} = (D_{\mu_1} \cdots D_{\mu_{m_i}} F_{\nu_1 \rho_1})^{a_i},$$ (2.2)

where $c(a_1, \ldots, a_n)$ is the color factor, and it can be written as products of traces $\text{Tr}(\ldots T^{a_1} \cdots T^{a_i} \cdots)$. Lorentz indices $\{\mu_i, \nu_i, \rho_i\}$ are contracted among different $W_{m_i}^{a_i}$.

In this paper we focus on $d$-dim Lorentz covariant operators in gauge theory, so Lorentz indices in (2.2) can only be contracted through metric. To be concrete, we will consider the YM theory with $\text{SU}(N_c)$ gauge group in this work, and the discussion can in principle be generalized to other gauge groups like $\text{SO}(N_c)$ and $\text{Sp}(N_c)$, and we will briefly comment this in section 2.3.2.

For the convenience of classifying operators, we define the length of an operator $O$ as the number of field strength $F$ (or equivalently, $W$) in it. For example, the length of the operator in (2.2) is $L$. We will classify operators according to a length hierarchy by setting two operators to be equivalent if their difference can be written in terms of high length operators. In other words, if two operators of length $l$ differ by an operator of higher length $L > l$:

$$O_l - O'_l = O_{L > l},$$ (2.3)

we say $O_l$ and $O'_l$ are equivalent at the level of length $l$, and only one of them is kept in length-$l$ operator basis. Following this length hierarchy, we first construct operator basis with lower length, then the higher length. Note the commutator of two covariant

1For simplicity, in this paper we will not distinguish upper and lower Lorentz indices. For example, $\eta^{\mu \nu}$ and $\delta^{\mu \nu}$ are regarded as equivalent. This will not cause any problem in flat spacetime.

2Since Levi-Civita tensor $\varepsilon_{\mu \nu \rho \sigma}$ breaks $d$-dimensional covariance, we do not consider it in this paper and leave it in the future work. Such tensor corresponds to parity-odd operators.
derivatives produces a field strength $F$ as in (2.1), therefore if we exchange the orders of two covariant derivatives $D$ lying in the same $\mathcal{W}$, the newly obtained operator is equivalent to the original one up to a higher length operator:

$$\text{tr}(D_{\mu_1}D_{\mu_2}\ldots D_{\mu_n}F_{\rho\sigma}\ldots) = \text{tr}(D_{\mu_2}D_{\mu_1}\ldots D_{\mu_n}F_{\rho\sigma}\ldots) + \text{higher length operator}.$$  \hfill (2.4)

We introduce an important quantity, the kinematic operator, obtained by stripping off the color factor in the full operator, which is a noncommutative product of $\mathcal{W}_i$. We denote the kinematic operator of length-$L$ by

$$[\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3, \ldots \mathcal{W}_L].$$  \hfill (2.5)

From a kinematic operator, one can create a full operator by dressing a trace color factor to it, denoted by $\mathcal{T} \circ [\ldots]$, for example,

$$\text{tr}(T^{a_1}T^{a_3}T^{a_2}T^{a_4}) \circ [\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3, \mathcal{W}_4] = \text{tr}(\mathcal{W}_1\mathcal{W}_3\mathcal{W}_2\mathcal{W}_4),$$

$$\text{tr}(T^{a_1}T^{a_4})(T^{a_2}T^{a_3}) \circ [\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3, \mathcal{W}_4] = \text{tr}(\mathcal{W}_1\mathcal{W}_4)\text{tr}(\mathcal{W}_2\mathcal{W}_3).$$  \hfill (2.6)

We will often use the short notation $\text{tr}(ij \ldots k) := \text{tr}(T^{a_i}T^{a_j} \ldots T^{a_k})$ for the trace color factor.

For high dimensional composite operators, there usually exist a number of operators with the same canonical dimensions and quantum numbers. Operators at a given dimension are in general not independent with each other, for they can be related with each other through equations of motion (EoM) or Bianchi identities (BI):

$$\text{EoM : } D_{\mu}F^{\mu\nu} = 0,$$  \hfill (2.7)

$$\text{BI : } D_{\mu}F_{\nu\rho} + D_{\nu}F_{\rho\mu} + D_{\rho}F_{\mu\nu} = 0.$$  \hfill (2.8)

In the study of EFT, the operators that are total derivatives of lower-dimensional ones are usually considered to be redundant, since these operators have zero contribution to the EFT amplitudes. Here we point out that such total-derivative operators are contained in our definition of the basis. As we will see, for the form factors (defined in (2.14)), these total-derivative operators have non-zero results and they also contribute to the renormalization matrix, thus it is natural to include them in the basis. It is also straightforward to pick them out in our basis using the form factor representation, which is described in appendix D.1.

The redundancy caused by these operator relations makes the operator counting complicated. To cure this problem, it is convenient to map operators to form factors, which are on-shell matrix elements. For the kinematic operators, one can introduce an order-preserved mapping between a kinematic operator and a polynomial composed of polarization vectors $\{e_i\}$ and external momenta $\{p_i\}$:

$$[\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_L] \to \mathcal{F}([\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_L]).$$  \hfill (2.9)

The mapping images $\mathcal{F}([\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_L])$ will be named as kinematic form factors.

The rule of the map in general $d$-dimensional kinematics is

$$\mathcal{F}_{d\text{-dim}}(\mathcal{W}_i) : \ D^\mu \to \text{i}p_i^\mu, \ F^{\mu\nu} \to \text{i}(p_i^\mu e_i^\nu - p_i^\nu e_i^\mu),$$  \hfill (2.10)
and the kinematic form factor is given by the product $\prod_{i=1}^{L} \mathfrak{F}_{d\text{-dim}}(\mathcal{W}_i)$. It is easy to see the operator relations like EoM and BI automatically vanish after the mapping (2.10). As a comparison, if we restrict to four-dimensional spacetime, it is usually convenient to decompose the field strength into self-dual and anti-self-dual components:

$$F^{\mu\nu} \rightarrow F^{\alpha\dot{\alpha}\beta\dot{\beta}} = \epsilon^{\alpha\beta} \tilde{F}^{\dot{\alpha}\dot{\beta}} + \epsilon^{\dot{\alpha}\dot{\beta}} F^{\alpha\beta},$$

and in this case one can use the spinor helicity formalism to map $D$ and $F$ as [13, 14, 18]:

$$\mathfrak{F}_{d\text{-dim}}(\mathcal{W}_i) : \quad D^{\alpha\alpha} \rightarrow \lambda_1^\alpha \lambda_1^\beta, \quad F^{\alpha\beta} \rightarrow \chi_i^\alpha \chi_i^\beta \quad \tilde{F}^{\dot{\alpha}\dot{\beta}} \rightarrow \tilde{\lambda}_i^{\dot{\alpha}} \tilde{\lambda}_i^{\dot{\beta}}. \quad (2.12)$$

Similarly, EoM and BI (equivalent to Schouten identities) are also encoded in the map (2.12). As a concrete example for these rules, for the kinematic operator is $[F_{\mu\nu}, F^{\mu\nu}]$, one can obtain its kinematic form factors in $D$-dim and 4-dim respectively as

$$\mathfrak{F}_{d\text{-dim}} = (p_1 \epsilon_1 \nu - p_1 \nu \epsilon_1 \mu)(p_2 \epsilon_2 \nu - p_2 \nu \epsilon_2 \mu), \quad \mathfrak{F}_{4\text{-dim}} = \left\{ \begin{array}{c} (\alpha_2)^2, (-, -) \\ (\alpha_2), (+, +) \end{array} \right., \quad (2.13)$$

where $(\pm, \pm)$ label the helicities of external gluons in the four-dimension case. To study the evanescent operators we will mostly use the $d$-dimensional rules.

The kinematic form factor is not yet a physical form factor for a full operator. A physical form factor is defined as a matrix element between an operator $\mathcal{O}(x)$ and $n$ on-shell states (see e.g. [19] for an introduction):

$$F_{\mathcal{O}, n} = \int d^4x \, e^{-iq \cdot x} \langle 1, \ldots, n | \mathcal{O}(x) | 0 \rangle. \quad (2.14)$$

The simplest type of physical form factors are the so-called minimal tree-level form factors, for which the number of external gluons is equal to the length of the operator. One can establish a useful map between a length-$L$ operator and its tree-level minimal form factor as

$$\mathcal{O}_L \Leftrightarrow F_{\mathcal{O}, L}(1, \ldots, L). \quad (2.15)$$

To map a full operator to its minimal form factor, one first strips off the color factor and reads the kinematic form factor from its kinematic operator, and then takes cyclic symmetrization which is caused by trace factors. As an example, consider the length-4 operator $\mathcal{O}_4 = \text{tr}(\mathcal{W}_1 \mathcal{W}_2 \mathcal{W}_3 \mathcal{W}_4)$. One can first strip off the trace factor to obtain the kinematic operator $[\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3, \mathcal{W}_4]$, then apply the above rules to obtain its kinematic form factor $\mathfrak{F}(\{\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3, \mathcal{W}_4\})$. The color-ordered minimal form factor (with color factor tr(1234)) is the cyclic symmetrization of kinematic form factor:

$$F_{\mathcal{O}_{4}, 4}(1, 2, 3, 4) = \mathfrak{F}(\{\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3, \mathcal{W}_4\}) + \text{cyclic-perm.}(1, 2, 3, 4), \quad (2.16)$$

and the full-color minimal form factor can be obtained as

$$F_{\mathcal{O}_{4}, 4}^{(0)} = \sum_{\sigma \in S_4} \text{tr}(1\sigma(2)\sigma(3)\sigma(4))F_{\mathcal{O}_{4}, 4}^{(0)}(1, \sigma(2), \sigma(3), \sigma(4)). \quad (2.17)$$

We stress that for the gluonic operators considered in this paper, the full-color minimal form factor are invariant under full $S_L$ transformation.

Besides the minimal form factor, there are higher point form factors with the number $E$ of external gluons larger than the length $L$ of the operator. The next-to-minimal and next-next-to-minimal form factors correspond to $E = L + 1$ and $E = L + 2$ respectively.
2.2 Definition of evanescent operators

Given the above preparation, we now introduce evanescent operators. An operator is called an evanescent operator, if the tree-level matrix elements of this operator have non-trivial results in general $d$ dimensions but all vanish in four dimensions. In terms of form factors, we can give a more practical definition: for an evanescent operator $O_{e}^{L}$ of length-$L$, its tree-level form factors with arbitrary numbers of external on-shell states, are all zero in four dimensions, but it has a non-trivial minimal form factor in general $d$ dimensions, namely

$$F^{(0)}_{O_{e}^{L}, n \geq L \mid 4-dim} = 0, \quad F^{(0)}_{O_{e}^{L}, L \mid d-dim} \neq 0.$$  \hfill (2.18)

Here we would like to emphasize that the vanishing of minimal form factors in four dimensions is not enough to fully characterize the property of an evanescent operator, and its higher-point non-minimal form factors are also required to vanish in four dimensions. If an operator is not an evanescent operator, i.e. its form factors do not vanish in four dimensions, we call it a physical operator.

Let us review the example of evanescent operator mentioned in the introduction (1.2). Using the map (2.9), one can obtain its color-ordered minimal form factor as (1.3), which we reproduce here

$$F^{(0)}_{O_{e}(1, 2, 3, 4)} = 2\delta^{e_{1}e_{2}p_{1}p_{2}}p_{3}p_{4} + 2\delta^{e_{1}e_{4}p_{1}p_{4}}p_{2}p_{3},$$  \hfill (2.19)

where the $\delta$ functions are Gram determinants defined as follows. We define the generalized Kronecker symbol as

$$\delta^{\mu_{1}...\mu_{n}}_{\nu_{1}...\nu_{n}} = \det(\delta^{\mu}_{\nu}).$$  \hfill (2.20)

Given two lists of Lorentz vectors $\{k_{i}\}, \{q_{i}\}, i = 1, \ldots, n$, the generalized $\delta$ function is defined as follows:

$$\delta^{k_{1}...k_{n}}_{q_{1}...q_{n}} = \det(k_{i} \cdot q_{j}).$$  \hfill (2.21)

It is easy to see that

1. If there is a pair of $\{e_{i}, p_{i}\}$ contained in $\{k_{i}\}$ or $\{q_{i}\}$, (2.21) is invariant under the gauge transformation $e_{i} \rightarrow e_{i} + \alpha p_{i}$. Thus (1.3) is manifestly gauge invariant.

2. The rank of the matrix $k_{i} \cdot q_{j}$ is determined by the smaller one of the number $d$ and $n$. If $d = 4$, then (2.21) vanishes for $n > 4$. Thus (1.3) is manifestly zero for $d = 4$.

Therefore, the minimal form factor (1.3) is nonzero in general $d$ dimensions but vanish in four dimensions. Alternatively, one may also compute the minimal form factor using the four-dimensional rule (2.12) and find it vanishing. Furthermore, one can show that the non-minimal form factors are also zero in four dimensions (which will be discussed later). Thus we can conclude that $O_{e}$ is an evanescent operator.
It is also instructive to express the gauge-invariant basis, such as used in [20], in terms of generalized $\delta$ functions. For the 4-gluon case, the gauge invariant basis can be chosen as products of building blocks $A_{ijk}$ and $B_{ij}$:

$$ A_{ijk} := \delta^{e_i}_{p_j p_k}, \quad B_{ij} := \frac{\delta^{e_{p_1 p_2 p_3 p_4}}_{e_{p_1 p_2 p_3 p_4}}}{\text{gr}_4}, $$

where $\text{gr}_n := \delta^{p_1 \cdots p_n}_{p_1 \cdots p_n}$ is a normalization factor. Note that $B_{ij}$ involves a $\delta$ function with rank larger than 4 in the numerator and thus vanishes in four dimensions. The gauge invariant basis for four-gluon form factors are in the form of $AAAA$, $AAB$ and $BB$, for example,

$$ A_{1;23}A_{2;13}A_{3;12}A_{4;23}, \quad A_{1;23}A_{2;14}B_{34}, \quad B_{12}B_{34}. $$

The numbers of independent basis for the above three types are 16, 24, 3 respectively.

In the next section, we will explain how to construct the evanescent operators in a systematic way. Before moving on, let us mention that there are no evanescent operators in the length-two and length-three cases. (The classification of length-2 and length-3 operators were considered in detail in [15].) This is consistent with the fact that: for 2-gluon or 3-gluon form factors, the set $\{e_i\}$ and $\{p_i\}$ cannot form a generalized delta function as (2.21) with rank larger than four. Thus the shortest evanescent operators are of length four, and in the next subsection we consider their basis construction.

### 2.3 Construction of length-4 evanescent operators

In this section we explain the construction of evanescent basis operators. As introduced in the previous subsection, one can first consider (color-stripped) kinematic operators, and then dress color factors to obtain full operators. Following the same logic, we will first construct the basis of evanescent kinematic operators in section 2.3.1, and then we dress proper color factors to obtain the full basis of evanescent operators in section 2.3.2.

#### 2.3.1 Evanescent kinematic operators

In this subsection, we first consider the kinematic operators $\{W_1, W_2, W_3, W_4\}$ without color factors. These operators are mapped to kinematic form factors $\tilde{F}(\{W_1, W_2, W_3, W_4\})$ according to (2.9).

To obtain the general high dimensional operator basis, one can introduce a finite set of primitive kinematic operators. Here the word “primitive” is in the sense that higher dimensional operators can be constructed by inserting pairs of covariant derivatives $\{D^\mu, D_\mu\}$ into the primitive ones; for instance, adding a $DD$ pair to the second and fourth sites of $\{W_1, W_2, W_3, W_4\}$ one has $\{W_1, D^\mu W_2, W_3, D_\mu W_4\}$.

Denoting the set of primitive operators as $\{\mathcal{K}_i\}$, the finiteness for the number of primitive kinematic operators are based on the fact that any kinematic operator of dim $\Delta \geq 12$ can always be written as a sum like

$$ \sum_i \sum_{\vec{n}_i} c_{\vec{n}_i} \prod_{1\leq a<b\leq 4} s_{ab}^{\vec{n}_i_{ab}} K_{\vec{n}_i}, \quad \dim(\mathcal{K}_i) \leq 12. \tag{2.24} $$

The $B_{ij}$ in (2.22) is denoted by $D_{ij}$ in [20].
Here \( \vec{n}_i = \{n_{i,12}, n_{i,13}, n_{i,23}, n_{i,14}, n_{i,24}, n_{i,34}\} \) and \( c_{\vec{n}_i} \) are rational numbers and \( s_{ab}^{n_{i,ab}}K_i \) refers to inserting \( n_{i,ab} \) pairs of \( \{D_\mu, D^\mu\} \) into the \( a \)-th and \( b \)-th sites of the \( i \)-th kinematic operator \( K_i \). For example:

\[
s_{13}s_{23}[W_1, W_2, W_3, W_4] = [D_\nu W_1, D_\mu W_2, D_\nu^\mu W_3, W_4] .
\]  \( (2.25) \)

To find out the minimal set of \( K_i \), one can first enumerate all possible length-4 kinematic operators within dimension 12 and then find out the subset of kinematic operators that are linearly independent, in the sense that any linear combination of their kinematic form factors \( \mathfrak{F}(K_i) \), with possible polynomial \( s_{ab} \) coefficients and total mass dimension no higher than 12, is nonzero. These independent operators are the wanted primitive kinematic operators of length four, and their number is 54. More details about the proof of (2.24) is given in appendix A.1.

The primitive kinematic operators \( K_i \) can be organized to 27 evanescent ones and 27 physical ones, denoted by \( \mathcal{E}_i \) and \( \mathcal{P}_i \) respectively. In the limit of \( d \to 4 \), \( \mathfrak{F}(\mathcal{E}_i) = 0 \) and \( \mathfrak{F}(\mathcal{P}_i) \neq 0 \). Generic high-dimensional evanescent kinematic operators can be expanded in terms of \( \{\mathcal{E}_i\} \) as

\[
\sum_{i=1}^{27} \sum_{\vec{n}_i} c_{\vec{n}_i} \prod_{1 \leq a < b \leq 4} s_{ab}^{n_{i,ab}} \mathcal{E}_i .
\]  \( (2.26) \)

We will denote the linear space of dim-\( \Delta \) evanescent basis kinematic operators by \( \mathfrak{E}_\Delta \). Formula (2.26) tells us that \( \mathfrak{E}_\Delta \) can be spanned by all the possible configurations of \( s_{12}^{n_{i,12}} \ldots s_{34}^{n_{i,34}} \mathcal{E}_i \) that satisfy:

\[
2(n_{i,12} + n_{i,13} + n_{i,14} + n_{i,23} + n_{i,24} + n_{i,34}) = \Delta - \Delta \mathcal{E}_i .
\]  \( (2.27) \)

Here we point out that kinematic operators \( \{s_{12}^{n_{i,12}} \ldots s_{34}^{n_{i,34}} \mathcal{E}_i\} \) satisfying (2.27) are linearly independent with constant coefficients, which is explained in appendix A.4. So they provide a basis of \( \mathfrak{E}_\Delta \).

Below we discuss the 27 evanescent kinematic operators \( \mathcal{E}_i \) in more detail. They can be constructed by contracting generalized \( \delta \) functions with tensor operators. We classify them as the following four classes according to their dimensions and symmetry structures.

**Class 1: \( \mathcal{E}_{1,i} \).** The first class contains twelve linearly independent kinematic operators \( \mathcal{E}_{1,i} \) of mass dimension ten. They can be chosen as

\[
\mathcal{E}_{1,1} = \frac{1}{16} \delta_{\nu_1 \nu_2 \nu_3 \nu_4} \sigma [D_\sigma F_{\mu_1 \nu_2}, F_{\mu_3 \nu_4}, D_\rho F_{\nu_1 \nu_2}, F_{\nu_3 \nu_4}] ,
\]  \( (2.28) \)

together with its \( S_4 \) permutations. Here the permutation means changing the positions of \( W_i \), i.e.

\[
\sigma \cdot [W_1, W_2, W_3, W_4] = [W_{\sigma(1)}, W_{\sigma(2)}, W_{\sigma(3)}, W_{\sigma(4)}] , \quad \sigma \in S_4 .
\]  \( (2.29) \)

\( ^4 \) Changing the ordering of \( s_{12}^{n_{i,12}} \ldots s_{34}^{n_{i,34}} \) will not affect the minimal form factor of the operator, but in general will affect the higher-point non-minimal form factors. We will discuss this more in appendix B.

\( ^5 \) One may compare numbers of \( \mathcal{E}_i \) and \( \mathcal{P}_i \) with the numbers of gauge invariant basis mentioned in section 2.2. The number of \( \mathcal{E}_i \) is equal to the number of \( AAB \) and \( BB \) basis, while the number of \( \mathcal{P}_i \) is not equal to that of \( AAAAA \) basis; this will be discussed further in appendix A.4.
The full set of operators are explicitly given in \((A.6)\). The kinematic form factor of \(E_{1,i}\) are also written in terms of \(\delta\) functions, e.g.

\[
\tilde{s}(E_{1,1}) = \delta^{e_1 e_2 p_1 p_2 p_3}.
\]  

\textbf{Class 2: } \(E_{2,i}\). The second class contains two linearly independent kinematic operators of dimension 12:

\[
E_{2,1} = \frac{1}{16} \delta^{\nu_1 \nu_2 \nu_3 \nu_4 \sigma_1 \rho_2} \{ D_{\sigma_1} F_{\mu_1 \mu_2}, D_{\sigma_2} F_{\rho_3 \mu_4}, D_{\rho_1} F_{\nu_1 \nu_2}, D_{\rho_2} F_{\nu_3 \nu_4} \},
\]

\[
E_{2,2} = E_{2,1} \begin{vmatrix}
| W_1, W_2, W_3, W_4 | \rightarrow | W_1, W_2, W_3, W_4 |
\end{vmatrix}.
\]  

Their kinematic form factors are

\[
\tilde{s}(E_{2,1}) = \delta^{e_1 e_2 p_1 p_2 p_3 p_4}, \quad \tilde{s}(E_{2,2}) = \delta^{e_1 e_2 p_1 p_2 p_3 p_4}.
\]  

\textbf{Class 3: } \(E_{3,i}\). The third class contains twelve linearly independent kinematic operators \(E_{3,i}\) of dimension 12. They are given by

\[
E_{3,1} = \frac{1}{4} \delta^{\nu_1 \nu_2 \nu_3 \nu_4 \rho_1 \rho_2 \rho_3 \rho_4} \{ D_{\nu_3} F_{\mu_1 \mu_2}, F_{\mu_3 \rho_1}, D_{\mu_3} F_{\nu_1 \nu_2}, D_{\mu_4} F_{\nu_3 \nu_4} \},
\]  

(2.33)


together with its permutations, see \((A.7)\). The kinematic form factor of \(E_{3,i}\) are also written in terms of \(\delta\) functions, e.g.

\[
\tilde{s}(E_{3,1}) = \delta^{e_2 e_3 e_4 p_1 p_2 p_3 p_4}. \quad \tilde{s}(E_{3,2}) = \delta^{e_2 e_3 e_4 p_1 p_2 p_3 p_4}.
\]  

\textbf{Class 4: } \(E_4\). The last class contains a single operator of dimension 12 which is invariant under \(S_4\) permutations:

\[
E_4 = \frac{1}{4} \delta^{\nu_1 \nu_2 \nu_3 \nu_4 \rho_1 \rho_2 \rho_3 \rho_4} \{ D_{\nu_3} F_{\mu_1 \mu_2}, D_{\mu_3} F_{\nu_1 \nu_2}, D_{\mu_4} F_{\nu_3 \nu_4}, D_{\mu_5} F_{\nu_5 \nu_6} \} + (S_4\text{-permutations}),
\]  

(2.35)

whose kinematic form factor is

\[
\tilde{s}(E_4) = \delta^{e_3 e_4 p_1 p_2 p_3 p_4} + (S_4\text{-permutations}).
\]  

(2.36)

In summary, evanescent primitive kinematic operators are given by \(\{E_{1,i}, E_{2,i}, E_{3,i}, E_4\}\), and the total number is \(12 + 2 + 12 + 1 = 27\). In the following context, if it is not necessary to give the concrete class or the explicit operator expression, we will use \(E_i\) to represent these 27 elements for simplicity. An important advantage of the above basis is that they manifest symmetry properties such that primitive kinematic operators of each class are closed under \(S_4\) action. Consequently, different ways of inserting \(DD\) pairs into the primitive ones of each class also create a set of kinematic operators \(\\{s_{12}^{i}, \ldots, s_{34}^{i}\}\) that are closed under \(S_4\) action. It is worth mentioning that the choice of basis is not unique, and we will discuss another choice of \(E_i\) in appendix \(D.2\) by including as many total derivative operators as possible.
independent color-dressed gauge-invariant operators. The main problem here is how to obtain a set of color ordering does not change the color factor in this case.

And three double traces. The number of single traces reduces to three because reversing SO(\(N\) in appendix I.5. In principle the discussion can be generalized to other gauge groups like \(SO(\(N\)) of \(N\)

\[M(\Delta-10; s_{ij}) \otimes \mathcal{R}_4\]

\[\mathcal{R}^\Delta_{\mathcal{E}} = \begin{bmatrix} M(\Delta-10; s_{ij}) \otimes \mathcal{R}_1 \oplus M(\Delta-12; s_{ij}) \otimes \mathcal{R}_2 \oplus M(\Delta-10; s_{ij}) \otimes \mathcal{R}_3 \oplus M(\Delta-12; s_{ij}) \otimes \mathcal{R}_4 \end{bmatrix}, \quad (2.37)\]

where \(M(N; s_{ij})\) refers to the linear space spanned by all the homogenous monomials \(s_{i1}^{n_{12}} \ldots s_{i4}^{n_{34}}\) with total power \(n_{12} + \ldots + n_{34} = N\). They represent all the possible ways to insert \(N\) pairs of identical \(D_s\) into the primitive kinematic operators. In appendix A.4 we show that different kinematic operators \(s_{i1}^{n_{12}} \ldots s_{i4}^{n_{34}}\) with the same mass dimension are independent with constant coefficients, so they can be chosen as the basis of \((\mathcal{R}^\Delta_{\mathcal{E}})\). As mentioned before, they are closed under \(S_4\) permutations. The counting of basis operators for \((\mathcal{R}^\Delta_{\mathcal{E}})\) with \(\Delta = 10, \ldots, 24\) are given in table 1.

### 2.3.2 Dressing color factors

From kinematic operators, one can obtain a real gauge invariant operator by dressing a color factor according to (2.6). In this subsection, based on the evanescent kinematic operators obtained in the previous subsection, we will construct the basis of evanescent operators.

For the length-four operators, there are two types of color factors, which are of single-trace and double-trace respectively:

\[\text{tr}(T^{a_i}T^{a_j}T^{a_k}T^{a_l}) = : \text{tr}(ijkl), \quad \text{tr}(T^{a_i}T^{a_j})\text{tr}(T^{a_k}T^{a_l}) = : \text{tr}(ij)\text{tr}(kl). \quad (2.38)\]

As mentioned before, our discussion will focus on the SU\((N_c)\) gauge group. For general \(N_c\), the length-4 trace basis includes six single traces and three double traces. For special values of \(N_c\) there are extra linear relations between these nine color factors, which is discussed in appendix I.5. In principle the discussion can be generalized to other gauge groups like SO\((N_c)\) and Sp\((N_c)\). For example, for SO\((N_c)\) with \(N_c > 3\), (2.38) gives three single traces and three double traces. The number of single traces reduces to three because reversing color ordering does not change the color factor in this case.

By dressing color factors to the kinematic operators, one will obtain single-trace and double-trace operators correspondingly. The main problem here is how to obtain a set of independent color-dressed gauge-invariant operators.
We first point out that the independence of the kinematic operators does not mean that their color-dressed operators are independent. For example, \(\text{tr}(1234) \circ \mathcal{E}_{1,1} \) and \(\text{tr}(1243) \circ \mathcal{E}_{1,3} \) may look different but actually they are the same operators. This is related to the fact that, any \(S_4\) permutation simultaneously acting on the color factor and the kinematic operator does not change the operator, namely,

\[
\mathcal{T}^a \circ [W_1, W_2, W_3, W_4] = (\sigma \cdot \mathcal{T}^a) \circ (\sigma^{-1} \cdot [W_1, W_2, W_3, W_4]), \quad \sigma \in S_4, \tag{2.39}
\]

where \(\mathcal{T}^a\) represents a color factor in (2.38), and for the above example, one has

\[
\text{tr}(1243) \circ \mathcal{E}_{1,3} = (\sigma \cdot \text{tr}(1234)) \circ (\sigma^{-1} \circ \mathcal{E}_{1,1}), \quad \sigma = (3 \leftrightarrow 4). \tag{2.40}
\]

Thus, one needs to avoid the over-counting and pick out the independent operators. As mentioned in section 2.3.1, the basis set of kinematic operator space \((\mathcal{R}^e_\Delta)\) can be chosen as \(\{s^{12}_{132}, \ldots, s^{34}_{341}, \mathcal{E}_i\}\) which is closed under \(S_4\) permutations. So the color dressed operator set \(\{\mathcal{T}^a \circ s^{12}_{132}, \ldots, \mathcal{T}^a \circ s^{34}_{341}, \mathcal{E}_i\}\) is also closed under \(S_4\) action defined by (2.39). Therefore we can identify different elements as the same operator if they produce the same orbit under \(S_4\) action, and we can keep only one of them as the independent operator.

An alternative more systematic method to construct independent basis operators is inspired by one-to-one correspondence between operators and minimal form factors. The problem of finding linearly independent full-color form factors is transformed to a problem of obtaining linearly independent full-color form factors. It is convenient to consider form factors since the full-color minimal form factor of a length-4 operator has the linearly independent full-color form factors. It is convenient to consider form factors since the full-color minimal form factor of a length-4 operator has the \(S_4\) symmetry by definition, as mentioned in (2.17). We can use the representation analysis of \(S_4\) to find independent full-color form factors.

The space spanned by color factors \(\{\mathcal{T}^a\}\) is denoted by \(\text{Span}\{\mathcal{T}^a\}\) and the space spanned by kinematic form factors \(\{\mathcal{F}([W_1, W_2, W_3, W_4])\}\) is denoted by \(\text{Span}\{\mathcal{F}([W_1, W_2, W_3, W_4])\}\). Color factors and kinematic form factors can be formally multiplied together to form a tensor product space:

\[
\text{Span}\{\mathcal{T}^a \mathcal{F}([W_1, W_2, W_3, W_4])\} = \text{Span}\{\mathcal{T}^a\} \otimes \text{Span}\{\mathcal{F}([W_1, W_2, W_3, W_4])\}. \tag{2.41}
\]

Since full-color form factors are invariant under \(S_4\) permutation, they must belong to the trivial representation in the space of (2.41). This means, linearly independent form factors correspond to different trivial representations of \(\text{Span}\{\mathcal{T}^a \mathcal{F}([W_1, W_2, W_3, W_4])\}\). Having this picture in mind, one can now apply some techniques of group theory, which we now explain.

First, one can expand the color and kinematic form factor spaces into irreducible representations with the representation decomposition as follows:

\[
\text{Span}\{\mathcal{T}^a\} \sim \oplus_i x_i R_i, \quad \text{Span}\{\mathcal{F}([W_1, W_2, W_3, W_4])\} \sim \oplus_j y_j R_j. \tag{2.42}
\]

\(^6\text{Here we demand }\mathcal{T}^a\text{ and }\mathcal{F}([W_1, W_2, W_3, W_4])\text{ transform individually under }S_4\text{ action, so their formal product satisfies the transformation law of a tensor product.}\)
| operator dimension | 10  | 12  | 14  | 16  | 18  | 20  | 22  | 24  |
|--------------------|-----|-----|-----|-----|-----|-----|-----|-----|
| single trace C-odd | 1   | 9   | 38  | 114 | 278 | 589 | 1128| 2001|
| single trace C-even| 3   | 16  | 54  | 145 | 330 | 671 | 1248| 2171|
| double trace       | 3   | 16  | 54  | 145 | 330 | 671 | 1248| 2171|

Table 2. The number of independent length-4 minimally-evanescent operators with dimension 10, \ldots, 24.

Here $R_i$ refer to irreducible inequivalent $S_4$ representations, which can be represented by Young diagrams as

$$R_{[4]} = \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}, \quad R_{[3,1]} = \begin{array}{c}
\bullet \\
\bullet \\
\end{array}, \quad R_{[2,2]} = \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}, \quad R_{[2,1,1,1]} = \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}.$$  \hfill (2.43)

Integers $x_i$ and $y_i$ refer to how many $R_i$ appear in $\text{Span}\{T^a\}$ and $\text{Span}\{\mathcal{F}(\lfloor W_1, W_2, W_3, W_4 \rfloor)\}$.

The representation decomposition of the tensor product (2.41) is

$$\text{Span}\{T^a\} \mathcal{F}(\lfloor W_1, W_2, W_3, W_4 \rfloor) \sim \bigoplus_k \left( \sum_{i,j} C^k_{ij} x_i y_j \right) R_k,$$

where coefficients $C^k_{ij}$ are known from representation theory [21, 22]. Especially for trivial representation $R_{[4]}$ that we are interested, $C^k_{ij} = \delta_{ij}$, so the coefficient of $R_{[4]}$ in (2.44) is

$$\sum_{i,j} C^k_{ij} x_i y_j = \sum_i x_i y_i,$$

which counts the dimension of trivial representation subspace and thus gives the number of independent form factors, or equivalent, independent operators.

In table 2, we summarize the number of basis evanescent operators based on the above method, where we have used the values of $x_i$ and $y_i$ given in (C.3) (and table 5) in appendix C. We have also introduced $C$-even and $C$-odd spaces for the single-trace color factors, which have $+1$ or $-1$ sign change under reflection of the trace:

$$C\text{-even } (T^a_{s+}) : \text{tr}(ijkl) + \text{tr}(lkji), \quad C\text{-odd } (T^a_{s-}) : \text{tr}(ijkl) - \text{tr}(lkji).$$  \hfill (2.46)

The corresponding operators are also called $C$-even and $C$-odd single-trace operators. They do not mix with each other under renormalization.

To write down explicitly the basis operators, we note that the basis of irreducible subspaces of a tensor product representation $U \otimes V$ can be written as products of the basis of irreducible subspaces of $U$ and $V$. One can thus first consider the representation decomposition of $\text{Span}\{T^a\}$ and $\text{Span}\{\mathcal{F}(\lfloor W_1, W_2, W_3, W_4 \rfloor)\}$ separately, and then write down the basis of the trivial representation subspace of the tensor product, using the method in the representation theory [21, 22].

For example, the space of $C$-even single-trace factors $\text{Span}\{T^a_{s+}\}$ is three dimensional and has representation decomposition

$$\text{Span}\{T^a_{s+}\} \sim \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array} \oplus \begin{array}{c}
\bullet \\
\bullet \\
\end{array}.$$  \hfill (2.47)
The basis belonging to the $R_{[r]}$-type sub-representation of \textbf{Span}$\{T_{s+}^{[r]}\}$ are denoted by $T_{s+}^{[r]}$. The space $(\mathcal{R}^c)_{10}$ spanned by the dim-10 evanescent kinematic operators is 12 dimensional and has representation decomposition
\begin{equation}
(\mathcal{R}^c)_{10} \sim \mathcal{R}_1^c \oplus \mathcal{R}_5^c \oplus 2 \mathcal{R}_7^c \oplus \mathcal{R}_{10}^c \oplus \mathcal{R}_{10}^c.
\end{equation}

The basis belonging to the $R_{[r]}$-type sub-representation of $(\mathcal{R}^c)_{10}$ are denoted by $E_1^{[r]}$. Since the $S_4$ actions over $[\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3, \mathcal{W}_4]$ and $\hat{\mathcal{F}}([\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3, \mathcal{W}_4])$ are related as
\begin{equation}
\hat{\mathcal{F}}([\mathcal{W}_{\sigma(1)}, \mathcal{W}_{\sigma(2)}, \mathcal{W}_{\sigma(3)}, \mathcal{W}_{\sigma(4)}]) = \sigma^{-1} \cdot \hat{\mathcal{F}}([\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3, \mathcal{W}_4]),
\end{equation}
the kinematic form factors $\hat{\mathcal{F}}(E_1^{[r]})$ also belong to the $R_{[r]}$-type sub-representation of the kinematic form factor space denoted by $\hat{\mathcal{F}}((\mathcal{R}^c)_{10})$.

The tensor product of two irreducible subspaces of the same type $R_{[r]}$ contains an element belonging to the trivial representation, and the expression of this element is
\begin{equation}
T_{s+}^{[r]} \cdot M : \hat{\mathcal{F}}(E_1^{[r]}).
\end{equation}
where $T_{s+}^{[r]}$, $\hat{\mathcal{F}}(E_1^{[r]})$ are given in (C.4), (C.8), and matrix $M$ is given in (C.15). They can be calculated according to the representation theory [21, 22]. Take $R_{[4]}$ as an example where $T_{s+}^{[4]}$ and $\hat{\mathcal{F}}(E_1^{[4]})$ are of dimension one, (2.50) becomes
\begin{equation}
T_{s+}^{[4]} \cdot \hat{\mathcal{F}}(E_1^{[4]}).
\end{equation}

Plugging in the concrete expressions of $T_{s+}^{[4]}$ and $\hat{\mathcal{F}}(E_1^{[4]})$ which are
\begin{align*}
T_{s+}^{[4]} &= \frac{1}{6} \text{tr}(1234) + \frac{1}{6} \text{tr}(1243) + \frac{1}{6} \text{tr}(1324) + \frac{1}{6} \text{tr}(1342) + \frac{1}{6} \text{tr}(1423) + \frac{1}{6} \text{tr}(1432), \\
\hat{\mathcal{F}}(E_1^{[4]}) &= \frac{1}{12} \sum_{i=1}^{12} \hat{\mathcal{F}}(E_{1,i}),
\end{align*}
and only keeping the terms proportional to $\text{tr}(1234)$,\footnote{This is enough, since the coefficients of other $\sigma \cdot \text{tr}(1234)$ give the same operator. Denote $\tau = \sigma^{-1}$:}
we obtain a sum
\begin{equation}
\frac{1}{72} \text{tr}(1234) \sum_{i=1}^{12} \hat{\mathcal{F}}(E_{1,i}).
\end{equation}
This can be understood as the form factor of the following single-trace operator
\begin{equation}
\frac{1}{72} \sum_{i=1}^{12} \text{tr}(1234) \circ E_{1,i} = \frac{1}{18} \text{tr}(1234) \circ (\frac{1}{2} E_{1,1} + \frac{1}{2} E_{1,2} + E_{1,6}).
\end{equation}
This is one of the three basis operators of the $C$-even single-trace dim-10 evanescent sector. In a similar way we can list all the basis operators of single-trace $C$-even, single-trace $C$-odd and double-trace sectors. Their expressions are summarized in (2.64) and (2.65).

A useful comment is that numbers of $C$-even single trace operators and double trace operators are always equal, as shown in table 2. This originates from the fact that in the case of length-4, the $C$-even single-trace color factors and the double-trace color factors form equivalent $S_4$ representations and they have the same invariant subgroup. Consider a pair of length-4 operators

$$O_s = \text{tr}(W_1 W_2 W_3 W_4) + \text{tr}(W_2 W_3 W_1 W_4), \quad O_d = \frac{1}{2} \text{tr}(W_1 W_3)\text{tr}(W_2 W_4). \quad (2.55)$$

The color-ordered tree form factor of $O_s$ with color factor $\text{tr}(1234)$ and the color-ordered tree form factor of $O_d$ with color factor $\text{tr}(13)\text{tr}(24)$ are equal:

$$F^{(0)}_{O_s}(p_1, p_2, p_3, p_4) = F^{(0)}_{O_d}(p_1, p_3 | p_2, p_4). \quad (2.56)$$

In this way we can establish a one-to-one correspondence between $C$-even single-trace operators and double-trace operators. We demand the $i$-th double-trace operator is always related to the $i$-th single-trace $C$-even operator through (2.55), so once the bases of single-trace operators are fixed, the bases of double-trace ones are also fixed.

In the above discussion, we consider the gauge group with general $N_c$, and the six single-trace and three double-trace color factors can be taken as linearly independent.

When $N_c$ takes a special value such as $N_c = 2, 3$, there exist extra linear relations among them, which means basis operators reduce to a smaller set. This fact provides a consistency check for loop calculation, see appendix I.5 for details.

**Remark on non-minimal form factor.** As shown above, all the evanescent operators are in the form of generalized $\delta$ functions of rank higher than 4 contracting with tensor operators, i.e.

$$O_e = \delta^{\mu_1 \cdots \mu_n}_{\nu_1 \cdots \nu_n} T_{\nu_1 \cdots \nu_n}^{\mu_1 \cdots \mu_n}, \quad n \geq 5, \quad (2.57)$$

where $T_{\nu_1 \cdots \nu_n}^{\mu_1 \cdots \mu_n}$ stands for a full tensor operator composed of $D_{\mu}$’s and $F_{\mu\nu}$’s. The form factor of $O_e$ for arbitrary $k$ also has the similar form:

$$F_{k;O_e} = \delta^{\mu_1 \cdots \mu_n}_{\nu_1 \cdots \nu_n} (F_{k;T})^{\nu_1 \cdots \nu_n}_{\mu_1 \cdots \mu_n}. \quad (2.58)$$

The $\delta$ tensor guarantees that $F_{k;O_e}$ vanishes in 4 dimensions, namely, $O_e$ is an exactly evanescent operator.

Besides, the tree form factors of $O_e$ do not explicitly depend on spacetime dimension $d$. The reason is as following. Given a rank-$n$ $\delta$ function, $d$ can only come out from the contraction

$$\delta^{\mu_1 \cdots \mu_n}_{\nu_1 \cdots \nu_n} \delta^{\nu_n}_{\mu_n} = (d - n + 1)\delta^{\mu_1 \cdots \mu_{n-1}}_{\nu_1 \cdots \nu_{n-1}}. \quad (2.59)$$

While $T_{\nu_1 \cdots \nu_n}^{\mu_1 \cdots \mu_n}$ is composed of $D_{\mu}$ and $F_{\mu\nu}$, there is no room for a $\delta^\mu$ in $F_{k;T}$ according to the Feynman rules.
One should be careful there exist operators whose minimal form factors vanish in four dimensions but higher-point form factors do not. In that case one can change them to exactly evanescent operators by adding proper higher length operators, and the detailed discussion is given in appendix B.

2.4 Higher-length evanescent operators

Our method can be generalized to construct higher-length operators. At length five, the Gram determinants of higher ranks such as $\delta^5_{p_1 p_2 \cdots p_5}$ appear, which increase the number of possible building blocks of evanescent kinematic operators. The analysis of color factor dressing also becomes more involving and one needs to consider the representations of $S_5$. Alternatively, at a fixed canonical dimension that is not too high, one can always find out the basis evanescent operators by enumerating all the possible configurations of site operators $W_i$ and picking out the linearly independent ones that vanish in four dimensions.\footnote{Enumeration in a brute-force way is also applicable to construct length-4 evanescent basis operators, so it is an alternative strategy apart from the method introduced in previous section.}

Below we briefly consider the length-5 operators.

For the case of dim-10, evanescent operators can only be of length-4 or length-5. The length-4 ones have been constructed in previous subsections. For the length-5 operators, we can first enumerate the kinematic operators. There are six linearly independent length-5 evanescent kinematic operators $U_{10;1}$. The first one is

$$U_{10;1} = \frac{1}{8} \delta^5_{\nu_1 \nu_2 \mu_3 \nu_4 \nu_5} \left[ F_{\mu_1 \mu_2}, F_{\nu_1 \nu_2}, F_{\mu_3 \mu_4}, F_{\nu_3 \nu_4}, F_{\mu_5 \nu_5} \right].$$  \hfill (2.60)

Other five ones are obtained via permutations of last three $W_i$’s of $U_{10;1}$:

$$U_{10;1} \left[ W_i, W_j, W_k, W_l, W_m \right] \rightarrow \left[ W_i, W_j, \sigma(W_k, W_l, W_m) \right], \quad \sigma \in S_3.  \hfill (2.61)$$

See details in appendix A.3.

Having the kinematic operator basis $U_{10;i}$, one can dress color factors to obtain the full operators. The color factors of length-5 operators are also classified into $C$-even ones and $C$-odd ones. Similar to (2.46), they are defined as\footnote{The definition of $C$-even and $C$-odd are based on the $C$-parity of Yang-Mills fields, where “$C$” stands for charge conjugation, see e.g. [23].}

$$C\text{-even } (T_{c+}) : \text{tr}(ijklm) - \text{tr}(mlkji), \quad \text{C\text{-odd } } (T_{c-}) : \text{tr}(ijklm) + \text{tr}(mlkji),$$

$$C\text{-even } (T_{d+}) : \text{tr}(ij)\text{tr}(klm) - \text{tr}(ij)\text{tr}(mlk), \quad \text{C\text{-odd } } (T_{d-}) : \text{tr}(ij)\text{tr}(klm) + \text{tr}(ij)\text{tr}(mlk).  \hfill (2.62)$$

For dim-10 case, all the single-trace length-5 operators are $C$-even, for example:

$$\text{tr}(12345) \circ U_{10;1} = \frac{1}{2} (\text{tr}(12345) - \text{tr}(54321)) \circ U_{10;1}.  \hfill (2.63)$$

It turns out there are two linearly independent single-trace operators and one double-trace operator. They are the final evanescent length-5 operators at dimension 10 and their explicit expressions are given in (2.66) and (2.67) in the next subsection.
2.5 Complete set of dim-10 evanescent basis operators

The complete set of dim-10 evanescent operators includes: the four length-4 single-trace ones; the three length-4 double-trace ones; two length-5 single-trace ones; one length-5 double-trace one.

Among four single-trace length-4 operators, three are \( C \)-even and one is \( C \)-odd. We label them as \( \tilde{O}^{c}_{10;5;+;1}, \tilde{O}^{c}_{10;5;+;2}, \tilde{O}^{c}_{10;5;+;3}, \tilde{O}^{c}_{10;5;+;1} \). Details of \( \tilde{O}^{c}_{10;5;+;1} \) has been given in (2.51)–(2.54). Similar construction applies to other operators. Here we just list the explicit expressions of the operators:

\[
\begin{align*}
\tilde{O}^{c}_{10;5;+;1} &= (-ig)^3 8\text{tr}(1234) \circ (\varepsilon_{1,1} + \varepsilon_{2,1} + \varepsilon_{1,2} + \varepsilon_{1,6}), \\
\tilde{O}^{c}_{10;5;+;2} &= (-ig)^3 8\text{tr}(1234) \circ (\varepsilon_{1,2} + \varepsilon_{1,6} + \varepsilon_{1,4}), \\
\tilde{O}^{c}_{10;5;+;3} &= (-ig)^3 8\text{tr}(1234) \circ (\varepsilon_{1,6} + \varepsilon_{1,4}), \\
\tilde{O}^{c}_{10;5;+;1} &= (-ig)^3 4\text{tr}(1234) \circ (\varepsilon_{1,1} - \varepsilon_{1,4}).
\end{align*}
\]

(2.64)

The three double-trace length-4 operators are denoted by \( \tilde{O}^{c}_{10;4;+;1}, \tilde{O}^{c}_{10;4;+;2}, \tilde{O}^{c}_{10;4;+;3} \), which are explicitly given as

\[
\begin{align*}
\tilde{O}^{c}_{10;4;+;1} &= (-ig)^2 4\text{tr}(12)\text{tr}(34) \circ (\varepsilon_{1,5} + \varepsilon_{1,7} + \varepsilon_{1,1}), \\
\tilde{O}^{c}_{10;4;+;2} &= (-ig)^2 4\text{tr}(12)\text{tr}(34) \circ (\varepsilon_{1,6} - 2\varepsilon_{1,1} + \varepsilon_{1,5}), \\
\tilde{O}^{c}_{10;4;+;3} &= (-ig)^2 4\text{tr}(12)\text{tr}(34) \circ (\varepsilon_{1,1} - \varepsilon_{1,5}).
\end{align*}
\]

(2.65)

The two single-trace length-5 operators and one double-trace length-5 one are all \( C \)-even, and they are labeled as \( \tilde{O}^{c}_{10;5;+;1}, \tilde{O}^{c}_{10;5;+;2}, \tilde{O}^{c}_{10;5;+;1} \), explicitly defined as

\[
\begin{align*}
\tilde{\Xi}^{c}_{10;5;+;1} &= (-ig)^3 \text{tr}(12345) \circ (U_{10;1} + U_{10;4}), \\
\tilde{\Xi}^{c}_{10;5;+;2} &= (-ig)^3 \text{tr}(12345) \circ (-2U_{10;1} + U_{10;4}), \\
\tilde{\Xi}^{c}_{10;5;+;1} &= (-ig)^3 \text{tr}(12)\text{tr}(345) \circ U_{10;1}.
\end{align*}
\]

(2.66)

(2.67)

where \( U_{10;\ell} \) are defined in (2.61).

We point out that in our definition, the operators are multiplied by a proper power of gauge coupling, as shown in (2.64)–(2.67): each length-\( L \) operator carries a factor \( (-ig)^{L-2} \). Such a convention is in accordance with (2.1) that a length-\( L \) operator containing \( [D_i, D_j] \) can be rewritten as a length-(\( L+1 \)) operator with an increasing \( -ig \) factor. In such choice, the \( n \)-point form factors of all the operators are of the same order \( \mathcal{O}(g^{n-2}) \). Besides, such an operator has canonical dimension \( n-2\epsilon \) where \( n \) is an integer, so in an EFT its Wilson coefficient times a certain integer power of mass are dimensionless. Unlike in a conformal field theory such as \( \mathcal{N} = 4 \) SYM, in QCD changing the definition of an operator by a factor of coupling will in general change its anomalous dimension due to the contribution of the beta function.

As mentioned before, our basis contains total derivative operators. Using the form factor formalism, these operators can be picked out systematically using the method described.
in appendix D.1. The final basis for dim-10 evanescent operators are linear recombinations of operators (2.64)–(2.67):

\[
\begin{pmatrix}
O_{10;s+1}^e \\
O_{10;s+2}^e \\
O_{10;s+3}^e \\
O_{10;s-1}^e \\
O_{10;d+1}^e \\
O_{10;d+2}^e \\
\Xi_{10;s+1}^e \\
\Xi_{10;s+2}^e \\
\Xi_{10;d+1}^e
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\tilde{O}_{10;s+1}^e \\
\tilde{O}_{10;s+2}^e \\
\tilde{O}_{10;s+3}^e \\
\tilde{O}_{10;s-1}^e \\
\tilde{O}_{10;d+1}^e \\
\tilde{O}_{10;d+2}^e \\
\tilde{\Xi}_{10;s+1}^e \\
\tilde{\Xi}_{10;s+2}^e \\
\tilde{\Xi}_{10;d+1}^e
\end{pmatrix}.
\tag{2.68}
\]

There are five operators in (2.68) that are total derivatives of lower dimensional operators, i.e. \( O_{10;s+1}^e \), \( O_{10;s+2}^e \), \( O_{10;s-1}^e \), \( O_{10;d+1}^e \), and \( O_{10;d+2}^e \). Their corresponding lower-dimensional operators are given in (D.1), (D.3) and (D.4).

As mentioned in section 2.3.1, the primitive evanescent kinematic operators are grouped into four classes and for dimension 10 all the evanescent operators are constructed by the first class kinematic operators. The other three classes begin to appear for operator dimension 12. In appendix E, we also summarize dim-12 length-4 basis evanescent operators, which are arranged according to the order of primitive classes.

3 One-loop renormalization of evanescent operators

In this section, we compute the one-loop form factors of evanescent operators and obtain the renormalization matrix. As an outline, we first explain the one-loop form factor calculation through the unitarity method in section 3.1, then we consider the IR and UV divergences in section 3.2, and finally we discuss the renormalization matrix and anomalous dimensions in section 3.3.

3.1 One-loop full-color form factor

To be explicit, we will mostly focus on the dim-10 evanescent basis operators given in (2.68) as concrete examples in this subsection, while it is straightforward to generalize the discussion to higher-dimensional cases as well as for physical operators. For these operators we need to calculate the following three types of form factors:

one-loop 4-gluon form factors of length-4 operators, denoted by \( F_4^{(1)} O_{4,s}^e \), \( F_4^{(1)} O_{4,d}^e \);

one-loop 5-gluon form factors of length-5 operators, denoted by \( F_5^{(1)} O_{5,s}^e \), \( F_5^{(1)} O_{5,d}^e \);

one-loop 5-gluon form factors of length-4 operators, denoted by \( F_5^{(1)} O_{4,s}^e \), \( F_5^{(1)} O_{4,d}^e \),

where the first two lines are minimal form factors and the last line are next-to-minimal form factors. The subscript ‘s’ or ‘d’ stands for single- or double-trace operators.

\[ -17 - \]
These one-loop form factors can be expanded in a set of basis integrals \( \{I_i\} \) like

\[ \mathcal{F}^{(1)} = \sum_i c_i I_i, \]

and we list the basis integrals in figure 1. The truly physical information is contained in the coefficients \( c_i \), and it is convenient to apply the unitarity method \([16, 17]\) to compute them. The central idea of unitary method is that by putting internal propagators on-shell (i.e. by performing unitarity cuts), the loop form factors can be factorized as products of tree building blocks. In this way, one can use simpler tree-level form factors and amplitudes as input to reconstruct the loop form factor coefficients \( c_i \). The final form factor is guaranteed to be the correct physical result as long as it is consistent with all possible unitarity cuts.

For the one-loop problem at hand, the complete set of cuts are shown in figure 2. Concretely, the basis integrals for minimal form factors are (a) and (b), which can be probed by cut (1) or (3) in figure 2. The integrals for next-to-minimal form factors are (a)-(e), among which (a) and (b) are probed by cut (3) and (e) is probed by cut (2). Integrals (c) and (d) can be probed by both cut (2) and (3), and their coefficients derived from these two different cuts must be the same, which provides a consistency check of the computation.

Since we consider evanescent operators, it is crucial that we perform the computation in \( d \) dimensions. We use conventional dimensional regularization scheme which provides a Lorentz covariant representation for general \( d \) dimensions.\textsuperscript{10} As mentioned before, in this

\textsuperscript{10}One may consider the dimensional reduction scheme where the 4-dimensional gauge fields may be decomposed in terms of \( D \)-dimensional ones plus the \( \epsilon \)-scalars \([24–27]\). One such example is the Konishi operator in \( \mathcal{N} = 4 \) SYM, where to get the correct two-loop anomalous dimension, it is necessary to consider the scalars with \( 6 + 2\epsilon \) components \([28]\). One could also consider integer spacetime dimensions large than

---

Figure 1. Up to particle permutations, the complete set of basis integrals contained in (3.1).

Figure 2. Up to different choices of cut channels and color orders of tree blocks, there are three classes of cuts which can probe all the topologies of one-loop 4-gluon and 5-gluon form factors of length-4 and length-5 operators.
paper we only consider operators with even $P$-parity, while for $P$-odd operators such as the Weinberg-type operators [32], the Levi-Civita tensor $\epsilon_{\mu\nu\rho\sigma}$ enters and breaks $d$-dimensional covariance, requiring other regularization schemes.\footnote{This is also related to the regularization of physical quantities involving $\gamma_5$ [3]. For the 2-loop renormalization dealing with evanescent operators and $\gamma_5$ see also [4, 33].}

We briefly outline the computational strategy as follows:

$$\mathcal{F}^{(1)}_{\text{cut}} = \sum_{\text{helicity}} \prod (\text{tree blocks}) = \text{cut integrand} \xrightarrow{\text{PV reduction}} \sum_{\text{cut permitted}} c_i I_i.$$ 

First, the input tree blocks in a cut channel can be obtained in terms of Lorentz products of momenta and polarization vectors via Feynman rules, and thus they are valid in $d$ dimensions. We summarize efficient rules for computing tree-level form factors in appendix F. Next, one needs to perform the helicity sum for the cut legs, and we use the formula:

$$\sum_{\text{helicity}} e^\mu(l_i) e^\nu(l_i) = \eta^{\mu\nu} - \frac{q_i^\mu q_i^\nu}{q_i \cdot l_i}, \quad i = 1, 2,$$

where $q_i^\mu$ are arbitrary light-like reference momenta. Furthermore, for each cut integrand, we use the Passarino-Veltman reduction method to perform integral reduction [34]. As long as the cut is allowed by the topology of $I_i$, its coefficients $c_i$ can be obtained from the given cut channel. Running through the complete set of cuts can probe all basis integrals, and in this way, one gets all the $\{c_i\}$ and obtains the full form factor results.

Since we would like to obtain the full-color one-loop form factors, there is some technical complication with respect to the color factors, which we explain in some details below. The one-loop form factor can be decomposed into single- and double-trace color basis, similar to one-loop amplitudes [35]. For example, the one-loop minimal form factor of a length-4 single-trace operator has the following color decomposition:

$$\mathbf{F}_{4;\mathcal{O}_s}^{(1)} = \sum_{\tau \in S_3} N_s \text{tr}(1\tau(2)\tau(3)\tau(4)) \mathcal{F}_{4;\mathcal{O}_s}^{(1),s}(p_1, p_{\tau(2)}, p_{\tau(3)}, p_{\tau(4)}) + \sum_{\tilde{\tau} \in Z_3} \text{tr}(1\tilde{\tau}(2)\tilde{\tau}(3)\tilde{\tau}(4)) \mathcal{F}_{4;\mathcal{O}_s}^{(1),d}(p_1, p_{\tilde{\tau}(2)}, p_{\tilde{\tau}(3)}, p_{\tilde{\tau}(4)}),$$

where $\mathcal{F}_{4;\mathcal{O}_s}^{(1),s}(p_1, p_{\tau(2)}, p_{\tau(3)}, p_{\tau(4)})$ and $\mathcal{F}_{4;\mathcal{O}_s}^{(1),d}(p_1, p_{\tilde{\tau}(2)}, p_{\tilde{\tau}(3)}, p_{\tilde{\tau}(4)})$ are the leading single-trace and sub-leading double-trace color-ordered form factors respectively. The double-trace form factors in the second line contain the mixing information with double-trace operators.

To apply unitarity method, let us consider the $s_{12}$-cut (cut (1) in figure 2) for the form factor $\mathbf{F}_{4;\mathcal{O}_s}^{(1)}$ in (3.4) as a concrete example. The corresponding cut integrand is given by the product of a 4-gluon form factor $\mathbf{F}_{4;\mathcal{O}_s}^{(0)}$ and a 4-gluon amplitude $\mathbf{A}_{4;\mathcal{O}_s}^{(0)}$,

$$\mathbf{F}_{4;\mathcal{O}_s}^{(1)} \big|_{s_{12}\text{-cut}} = \int d\text{PS}_{l_1, l_2} \sum_{\text{helicity}} \mathbf{F}_{4;\mathcal{O}_s}^{(0)}(-l_1, -l_2, p_3, p_4) \times \mathbf{A}_{4;\mathcal{O}_s}^{(0)}(p_1, p_2, l_2, l_1).$$

four, for example, six-dimensional spinor helicity formalism has been used to compute form factors in pure YM theory in [29], and operator renormalization has also been considered for gauge theories of six and eight dimensions [30, 31].
These two full-color tree blocks can be expanded in color bases as \[36\]

\[
F_{4\mathcal{O}_s}^{(0)} = \sum_{\bar{s} \in S_3} \text{tr}(l_1 \bar{\sigma}(l_2) \bar{\sigma}(3) \bar{\sigma}(4)) F_{4\mathcal{O}_s}^{(0)} (l_1, p_{\bar{\sigma}(l_2)}, p_{\bar{\sigma}(3)}, p_{\bar{\sigma}(4)}) ,
\]

\[
A_{4}^{(0)} = \sum_{\sigma \in S_3} \text{tr}(l_2 \bar{\sigma}(l_1) \bar{\sigma}(1) \bar{\sigma}(2)) A_{4}^{(0)} (l_2, p_{\sigma(l_1)}, p_{\sigma(1)}, p_{\sigma(2)})
= f^{2l_2 b} f^{l_1 1b} A_{4}^{(0)} (l_2, l_1, p_1, p_2) + f^{2l_2 b} f^{l_2 1b} A_{4}^{(0)} (l_1, l_2, p_1, p_2).
\] (3.6)

By comparing (3.4) and (3.5) and extracting the terms with wanted color factors, one can obtain the cut parts of one-loop color-ordered form factors in terms of sums of products of tree-level color-ordered blocks. For example, the color-ordered tree product

\[
\int dPS_{l_1, l_2} \sum_{\text{helicity}} F_{4\mathcal{O}_s}^{(0)} (-l_1, -l_2, p_3, p_4) A_{4}^{(0)} (l_2, l_1, p_1, p_2)
\] (3.7)

has the corresponding color factor product from (3.6)\(^{12}\)

\[
\text{tr}(l_1 l_2 34) f^{2l_2 b} f^{l_1 1b} = \frac{1}{4} \left( N_c \text{tr}(1234) + \text{tr}(12)\text{tr}(34) \right).
\] (3.8)

Comparing with the one-loop color structure in (3.4), one can see that (3.7) contributes to the \(s_{12}\)-cut of both \(F_{4\mathcal{O}_s}^{(1),s} (p_1, p_2, p_3, p_4)\) and \(F_{4\mathcal{O}_s}^{(1),d} (p_1, p_2 | p_3, p_4)\). To be complete, these two color-stripped form factors also receive contributions from other color orderings which have nonzero components of \(\text{tr}(1234)\) and \(\text{tr}(12)\text{tr}(34)\). Further details of color decomposition are provided in appendix G.

In the remaining part of this subsection, we briefly discuss some features of the evanescent form factor results. Consider the single-trace operator \(\mathcal{O}_{10,\bar{s};+;+}\) defined in (2.68). Its color-ordered 4-gluon form factors (associated with \(\text{tr}(1234)\) and \(\text{tr}(12)\text{tr}(34)\) respectively) are

\[
F_{4\mathcal{O}_{10,\bar{s};+;+};s}^{(1)} (p_1, p_2, p_3, p_4) = N_c \left[ c_{b;1} (s_{12}) I_b (s_{12}) + c_{t;1} (s_{12}) I_t (s_{12}) \right] + \text{cyclic of } (1, 2, 3, 4),
\] (3.9)

\[
F_{4\mathcal{O}_{10,\bar{s};+;+};d}^{(1)} (p_1, p_2 | p_3, p_4)
= \left[ (1 + \text{sgn} (\mathcal{O}_s)) \left( c_{b;1} (s_{12}) I_b (s_{12}) + c_{t;1} (s_{12}) I_t (s_{12}) + (3 \leftrightarrow 4) \right) + (1 \leftrightarrow 4, 2 \leftrightarrow 3) \right]
+ \left[ (1 + \text{sgn} (\mathcal{O}_s)) \left( c_{b;2} (s_{23}) I_b (s_{23}) + c_{t;2} (s_{23}) I_t (s_{23}) + (1 \leftrightarrow 3, 2 \leftrightarrow 4) \right) + (1 \leftrightarrow 2) \right].
\] (3.10)

Here \(I_b (s_{ij})\) and \(I_t (s_{ij})\) refer to the \(s_{ij}\) bubble integral and the \(s_{ij}\) one-mass triangle integral as in figure 1, and \(\text{sgn}(\mathcal{O}_s)\) is the sign change of the operator \(\mathcal{O}_s\) under reflection. Coefficients \(c_{t;1}\) and \(c_{b;1}\) can be computed using the unitarity cut (3.7), which contribute to both \(F_{4\mathcal{O}_{10,\bar{s};+;+};s}^{(1)} (p_1, p_2, p_3, p_4)\) and \(F_{4\mathcal{O}_{10,\bar{s};+;+};d}^{(1)} (p_1, p_2 | p_3, p_4)\). Coefficients \(c_{t;2}\) and \(c_{b;2}\) come

\(^{12}\)We make use of the completeness relation of \(\text{SU}(N_c)\) Lie algebra

\[
\sum_{a} T_{ai} T_{ak} = \frac{1}{2} \delta_{il} \delta_{jk} - \frac{1}{2N_c} \delta_{ij} \delta_{kl}.
\]
from a product of tree blocks with another choice of color ordering in (3.6), see details in (G.3).

The coefficients $c_i$ are functions depending on the Lorentz product of polarization vector $\{e_j\}$ and external momenta $\{p_j\}$, as well as the dimensional regularization parameter $\epsilon$. The triangle integrals capture the universal IR divergences, and their coefficients are proportional to tree-level evanescent form factors:

$$c_{t,1}(s_{12}) = -s_{12} F^{(0)}_{4;\mathcal{O}^b_{10,s+1}} (1234) = -8s_{12} \sum_{i=1}^{12} \tilde{F} (E_{1,i}) ,$$

$$c_{t,2}(s_{23}) = \frac{s_{23}}{2} \left( F^{(0)}_{4;\mathcal{O}^b_{10,s+1}} (1243) + F^{(0)}_{4;\mathcal{O}^b_{10,s+1}} (1342) \right) = 8s_{23} \sum_{i=1}^{12} \tilde{F} (E_{1,i}) ,$$

where $\tilde{F} (E_{1,i})$ are given in (2.30). The leading $\mathcal{O}(\epsilon^0)$ order of the coefficients $c_{b,1}$ and $c_{b,2}$ can also be written as linear combinations of $\tilde{F} (E_{1,i})$ and therefore also vanish in four dimensions:

$$c_{b,1}(s_{12}) = 8 \left( \sum_{i=1}^{4} \tilde{F} (E_{1,i}) - 3 \sum_{j=5}^{12} \tilde{F} (E_{1,j}) \right) + \mathcal{O}(\epsilon) ,$$

$$c_{b,2}(s_{23}) = 8 \left( - \sum_{i=9}^{12} \tilde{F} (E_{1,i}) + 3 \sum_{j=1}^{8} \tilde{F} (E_{1,j}) \right) + \mathcal{O}(\epsilon).$$

The $\mathcal{O}(\epsilon)$ order terms of bubble coefficients which are not shown above also have physical meaning, for they capture the finite mixing from evanescent operators to the physical ones and are expected to be important for two-loop calculation. They give rise to a finite part of the one-loop form factors of the evanescent operators. In the limit of dimension four, such finite contributions are equal to linear combinations of the tree-level form factors of physical operators.

### 3.2 IR subtraction and UV renormalization

In this subsection we discuss the one-loop renormalization of the gluonic operators, including both evanescent ones and physical ones. The bare form factors contain both IR and UV divergences. The renormalization $Z$-matrix can be obtained from the UV divergences of form factors. It is convenient to obtain the UV divergence by subtracting the universal IR divergences from the total divergence of a form factor. Below we first give some details about the structure of the IR divergences.

The IR divergence of a one-loop form factor can be given as a one-loop correction function acting on its tree-level form factor [37]:

$$F^{(1)}_{\mathcal{O},\text{IR}} = I^{(1)}_{\text{IR}} (\epsilon) F^{(0)}_{\mathcal{O}} ,$$

$$I^{(1)}_{\text{IR}} (\epsilon) = 1 + \frac{\beta_0}{2C_A \epsilon} \sum_{i<j} (-s_{ij})^{-\epsilon} T_i \cdot T_j ,$$

---

13The results of (3.11) and (3.12) only differ by an overall $s_{ij}$ factor, which is not a universal property of general operators. In this example, we choose the operator whose tree-level planar amplitude is invariant under $S_4$ permutation, so $c_{t,1}(s_{12})$ and $c_{t,2}(s_{23})$ happen to be proportional.
where $\beta_0 = 11C_A/3$ is the one-loop beta function, and $\mathbf{T}_i \cdot \mathbf{T}_j$ acts over the color factor through taking Lie bracket of the adjoint vector $T^a$ of the $i$-th and $j$-th gluon, for instance

$$
\mathbf{T}_i \cdot \mathbf{T}_j \text{tr}(XT^aYT^bZ) = \sum_b \text{tr}(X[T^b,T^a]Y[T^b,T^a])Z.
$$

(3.16)

Here $X$, $Y$, $Z$ represent strings of adjoint vectors which do not involve the $i$-th or $j$-th gluon. Take the minimal form factor of a single-trace length-$4$ operator $\mathcal{O}_s$ as an example. The color decomposition of its tree level form factors is given in (3.6), so together with (3.15) one has

$$
\mathbf{I}_{IR}^{(1)}(\epsilon)\mathbf{F}_4^{(0)}_{\mathcal{O}_s} = \frac{\epsilon^2 \epsilon}{\Gamma(1-\epsilon)} \left( \frac{1}{\epsilon^2} + \frac{\beta_0}{2C_A \epsilon} \right) \mathbf{C}(1,2,3,4)\mathbf{F}_4^{(0)}_{\mathcal{O}_s}(p_1,p_2,p_3,p_4) + \text{Perm.}\{2,3,4\}.
$$

(3.17)

Here $\mathbf{C}(1,2,3,4)$ is a sum of trace bases with coefficients dependent on $s_{ij}$ and $\epsilon$:

$$
\mathbf{C}(1,2,3,4) = -N_c \sum_{i=1}^{4} (-s_{ii+1})^{-\epsilon} \text{tr}(1234)
$$

$$
+ \left( -(-s_{12})^{-\epsilon} - (-s_{34})^{-\epsilon} + (-s_{13})^{-\epsilon} + (-s_{24})^{-\epsilon} \right) \text{tr}(12)\text{tr}(34)
$$

$$
+ \left( -(-s_{14})^{-\epsilon} - (-s_{23})^{-\epsilon} + (-s_{13})^{-\epsilon} + (-s_{24})^{-\epsilon} \right) \text{tr}(14)\text{tr}(23)
$$

(3.18)

Take series expansion of $\mathbf{C}(1,2,3,4)$ in $\epsilon$. The leading order of the coefficient in the first line is of $\mathcal{O}(\epsilon^0)$ and the leading orders of the coefficients in the second and third line are of $\mathcal{O}(\epsilon)$. This means that the IR divergence of $N_c$ leading form factors like $\mathcal{F}_{4,\mathcal{O}_s}^{(1)\epsilon}(p_1,p_2,p_3,p_4)$ are of $\mathcal{O}(\epsilon^{-2})$ while the IR divergences of $N_c$ sub-leading form factors like $\mathcal{F}_{4,\mathcal{O}_s}^{(1)\epsilon}(p_1,p_2|p_3,p_4)$ are of $\mathcal{O}(\epsilon^{-1})$. Similar analysis for double-trace operators and next-to-minimal form factors are given in appendix \[H\], and the structure about the $\epsilon$-expansion of the IR divergences of $N_c$ leading and sub-leading form factors are the same. We have checked that our results are consistent with these properties.

After subtracting the IR divergences, the remaining UV divergences require renormalization of the operator and the coupling constant. The renormalized form factor (with $n$ external gluons and for a length-$L_i$ operator multiplied with the bare $g$ to the power $m_i$) can be given in the following form as (see e.g. [15] for detailed discussions)

$$
\mathbf{F}_{\mathcal{O}_i,R}^{(1)} = \mathbf{F}_{\mathcal{O}_i,B}^{(1)} + \sum_j (Z^{(1)})_i \mathbf{F}_{\mathcal{O}_j,B}^{(0)} - \frac{n - L_i + m_i \beta_0}{2} \epsilon \mathbf{F}_{\mathcal{O}_i,B}^{(0)}.
$$

(3.19)

Subtracting the IR divergence from the bare form factor, (3.19) reads

$$
0 = (\mathbf{F}_{\mathcal{O}_i,B}^{(1)} - \mathbf{F}_{\mathcal{O}_i,B}^{(1)\text{IR}})_{\text{div}} + \sum_j (Z^{(1)})_i \mathbf{F}_{\mathcal{O}_j,B}^{(0)} - \frac{n - L_i + m_i \beta_0}{2} \epsilon \mathbf{F}_{\mathcal{O}_i,B}^{(0)}.
$$

(3.20)

By expanding one-loop and tree-level form factors in trace color bases, (3.19) can be decomposed to different color ordered components.
Table 3. Matrix elements of \( Z^{(1)} \) read from each type of form factors.

| Form factor | \( F^{(1)}_{4:s_{4},4:s_{4}} \) | \( F^{(1)}_{4:s_{4},d} \) | \( F^{(1)}_{5:O_{4},4:s_{4}} \) | \( F^{(1)}_{5:O_{4},d} \) | \( F^{(1)}_{5:O_{5},4:s_{4}} \) | \( F^{(1)}_{5:O_{5},d} \) |
|-------------|-------------------------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( Z^{(1)}_{4,s \rightarrow 4,s} \) | \( Z^{(1)}_{4,s \rightarrow 4,d} \) | \( Z^{(1)}_{4,d \rightarrow 4,s} \) | \( Z^{(1)}_{4,d \rightarrow 4,d} \) | \( Z^{(1)}_{5,s \rightarrow 5,s} \) | \( Z^{(1)}_{5,s \rightarrow 5,d} \) | \( Z^{(1)}_{5,d \rightarrow 5,s} \) | \( Z^{(1)}_{5,d \rightarrow 5,d} \) |
| Direct mixing | \( \sqrt{\quad} \) | \( \times \) | \( \sqrt{\quad} \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) |
| \( Z^{(1)}_{4,s \rightarrow 4,s} \) | \( Z^{(1)}_{4,s \rightarrow 4,d} \) | \( Z^{(1)}_{4,d \rightarrow 4,s} \) | \( Z^{(1)}_{4,d \rightarrow 4,d} \) | \( Z^{(1)}_{5,s \rightarrow 5,s} \) | \( Z^{(1)}_{5,s \rightarrow 5,d} \) | \( Z^{(1)}_{5,d \rightarrow 5,s} \) | \( Z^{(1)}_{5,d \rightarrow 5,d} \) |
| \( \times \) | \( \times \) | \( \times \) | \( \sqrt{\quad} \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) |
| \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \sqrt{\quad} \) | \( \times \) | \( \times \) | \( \times \) |

In the \( \overline{\text{MS}} \) scheme [38], there is no mixing from evanescent operators to the physical ones, and \( Z^{(1)} \) has the general form

\[
Z^{(1)} = \begin{pmatrix}
Z_{p \rightarrow s}^{(1)} \\
0 \\
Z_{e \rightarrow s}^{(1)}
\end{pmatrix},
\]

where the sub-matrices are all of order \( \mathcal{O}(\epsilon^{-1}) \), and subscripts p and e refer to physical and evanescent operators respectively. Note that although at one-loop level evanescent operators do not mix to physical ones at the order of \( \mathcal{O}(\epsilon^{-1}) \), physical operators in general do mix to evanescent ones.

To be concrete, consider the form factors \( F^{(1)}_{4:s_{4},i} \) of single-trace length-4 operators as (3.4), the renormalization formulæ of leading and sub-leading color-ordered components can be given as

\[
\begin{align*}
F_{4:s_{4},i,R}(p_1, p_2, p_3, p_4) &= F_{4:s_{4},i,B}(p_1, p_2, p_3, p_4) + \sum_j (Z_{4,s \rightarrow 4,s}^{(1)})_j F_{4:s_{4},i,j:B}(p_1, p_2, p_3, p_4), \\
F_{4:s_{4},d,R}(p_1, p_2 | p_3, p_4) &= F_{4:s_{4},d,B}(p_1, p_2 | p_3, p_4) + \sum_k (Z_{4,s \rightarrow 4,d}^{(1)})_k F_{4:s_{4},d,k:B}(p_1, p_2 | p_3, p_4),
\end{align*}
\]

where \( Z_{4,s \rightarrow 4,s}^{(1)} \) and \( Z_{4,s \rightarrow 4,d}^{(1)} \) represent the mixing from length-4 single-trace operators to length-4 single-trace and double-trace operators respectively. In table 3 we show that the correspondence between renormalization matrix elements and form factors (for example, for the dimension-10 evanescent operators). In particular, matrix elements of \( Z_{4,s \rightarrow 4,s}^{(1)}, Z_{4,s \rightarrow 4,d}, Z_{1,d \rightarrow 4,s}^{(1)} \) and \( Z_{4,d \rightarrow 4,d}^{(1)} \) can be obtained from two different form factors, and this provides a consistency check of our calculation.

Using the renormalization matrix, it is straightforward to obtain the dilation operator, defined as

\[
\mathbb{D} := -\frac{d}{d \log \mu} \log Z = \sum_{l=1}^{\infty} \left( \frac{\alpha_s}{4\pi} \right)^l \mathbb{D}^{(l)},
\]
and the anomalous dimensions are given as eigenvalues of the dilatation operator. The one-loop correction $\mathbb{D}^{(1)}$ of the set of dim-10 evanescent operators is

$$\mathbb{D}^{(1)} = 2\epsilon Z^{(1)},$$

(3.24)

where $Z^{(1)}$ includes $Z_{4\rightarrow 4}^{(1)}, Z_{4\rightarrow 5}^{(1)}, Z_{5\rightarrow 5}^{(1)}$.\footnote{We remark that the $\alpha_s$ expansion of $\mathbb{D}$ depends on the coupling-power in the definition of operators. In our definition, we multiply $L-2$ powers of bare gauge coupling $g$ to each length-$L$ operator as shown in (2.64)-(2.67). In this way, matrix elements $Z_{4\rightarrow 4}^{(1)}, Z_{4\rightarrow 5}^{(1)}$ and $Z_{5\rightarrow 5}^{(1)}$ contained in $Z^{(1)}$ are all of order $O(\alpha_s)$.}

The above renormalization matrix $Z^{(1)}$ is defined in $\overline{\text{MS}}$ scheme by considering only the UV divergence of the one-loop form factors. Alternatively one can choose another scheme to absorb the mixing from evanescent operators to the physical ones which takes place at the finite order of one-loop form factors and arises from the contribution of $O(\epsilon)$ terms in bubble coefficients as discussed in previous subsection. This corresponds to the finite renormalization scheme which was used for the renormalization of four-fermion evanescent operators, see e.g. [4–6, 8, 39, 40]. Such a scheme choice will be useful in the discussion of two-loop calculation, but it does not affect the one-loop order anomalous dimensions. The scheme change is realized by appending to $Z^{(1)}$ (which is of order $O(\epsilon^{-1})$) a finite term:

$$Z^{(1)} \rightarrow Z^{(1)} + \begin{pmatrix} 0 & 0 \\ Z_{e\rightarrow p}^{(1)\text{fin}} & 0 \end{pmatrix}.$$  

(3.25)

The sub-matrix $Z_{e\rightarrow p}^{(1)\text{fin}}$ is of order $O(\epsilon^0)$ and extracted from

$$0 = (\mathbf{F}^{(1),B}_{\epsilon_i} - \mathbf{F}^{(1),\text{IR}}_{\epsilon_i})|_{\text{rational,}4d} + \sum_j (Z_{e\rightarrow p}^{(1)\text{fin}})_{ij} \mathbf{F}^{(0)}_{\epsilon_i, B} |_{4d},$$

(3.26)

where $\mathcal{O}^e_i$ and $\mathcal{O}^p_j$ refer to evanescent and physical operators.

### 3.3 Renormalization matrices and anomalous dimensions

In this subsection we discuss in detail the results of the one-loop renormalization matrix of dim-10 evanescent basis operators, i.e. the sub-matrix $Z^{(1)}_{e\rightarrow e}$ in (3.21). The complete results including all dim-10 physical basis operators are given in appendix I.

The basis operators have been classified according to their $C$-even or $C$-odd properties, and since $C$-even and $C$-odd sectors do not mix to each other, their $Z$-matrices can be written separately. For $C$-even sector we arrange operators as \{\(\mathcal{O}^e_{10,s;+;i}, \mathcal{O}^e_{10,d++;i}, \Xi^e_{10,s;++i}, \Xi^e_{10,d;+;i}\)\}. For $C$-odd sector there is only one operator $\mathcal{O}^e_{10,s;--;1}$.\footnotemark
The full $Z$-matrices of dim-10 evanescent operators are

$$Z^{(1)}_{10;e^+\rightarrow e^+} = \frac{N_c}{\epsilon} \times \begin{pmatrix}
3 & -\frac{8}{3} & 0 & 0 & 16 & 0 & 0 & 0 & 0 & 0 \\
-\frac{10}{3} & \frac{14}{3} & 0 & -\frac{20}{N_c} & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & \frac{10}{3} & \frac{12}{N_c} & 0 & 0 & -\frac{112}{9} & -\frac{20}{9} & -\frac{224}{3N_c} & 0 \\
0 & -\frac{8}{N_c} & 0 & 3 & \frac{16}{3} & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{3}{N_c} & 0 & \frac{20}{3} & \frac{5}{3} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{3}{N_c} & 2 & \frac{2}{N_c} & -4 & 4 & 25 & 0 & -\frac{56}{3N_c} & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{55}{9} & \frac{52}{9} & \frac{110}{3N_c} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{0}{9} & \frac{26}{3N_c} & \frac{28}{3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{55}{9} & \frac{52}{9} & \frac{110}{3N_c} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{0}{9} & \frac{26}{3N_c} & \frac{28}{3} & 0
\end{pmatrix},$$

(3.27)

$$Z^{(1)}_{10;e^-\rightarrow e^-} = \frac{5N_c}{\epsilon}.$$

(3.28)

In the matrix (3.27), the length-4 and length-5 operators are separated by solid lines, and within each length the single-trace and double-trace operators are separated by dashed lines. There is no mixing from $O_{e_{10;x+i}}$ with $i<3$ to $O_{e_{10;\pm;3}}$ ($x = s, d$) or length-5 operators. This is because that $O_{e_{10;x+i}}$ with $i<3$ are total derivatives of lower dimensional tensor operators, see appendix D.1 for further discussion on this point.

One-loop dilatation operator $D^{(1)}$ for $C$-even and $C$-odd sectors can be obtained by plugging (3.27) and (3.28) into (3.24). We denote the eigenvalues, i.e. one-loop anomalous dimensions by $\hat{\gamma}^{(1)}_{+,e}$ for the $C$-even operators and $\hat{\gamma}^{(1)}_{-,e}$ for the $C$-odd operator. For the single $C$-odd operator, the anomalous dimension is

$$\hat{\gamma}^{(1)}_{-,e} = 10N_c.$$

(3.29)

Below we focus on the more non-trivial $C$-even sector. The 4 to 5 mixing matrix elements $Z^{(1)}_{4\rightarrow5}$ does not effect the eigenvalues of length-4 and length-5 sectors because at 1-loop level $Z$ matrix is upper-triangular.

For the length-4 operators, the six eigenvalues are determined by the following equations

$$0 = \left(\omega_1 - \frac{50}{3}\right)\left(\omega_1 - \frac{38}{3}\right) \times \left[\omega_1^4 - \frac{74\omega_1^3}{3} + \omega_1^2 \left(\frac{124}{3} - \frac{640}{N_c^2}\right) + \omega_1 \left(\frac{45448}{27} + \frac{6400}{3N_c^2}\right) - \left(-\frac{202400}{81} + \frac{317440}{9N_c^2}\right)\right],$$

(3.30)

where $\omega_1 := \hat{\gamma}^{(1)}_{+,e}/N_c$. Taking the large $N_c$ limit and expanding the anomalous dimensions up to $O(1/N_c)$, one has

$$\hat{\gamma}^{(1)}_{+,e, L=4} = N_c \left\{ -\frac{22}{3} - \frac{400}{21N_c^2} \cdot \frac{38}{3} \cdot \frac{50}{3} \cdot \frac{50}{3} + \frac{520}{3N_c^2}, \quad \frac{23 + \sqrt{345}}{3} - \frac{4(3105 \pm 211\sqrt{345})}{161N_c^2} \right\}.$$

(3.31)
We note that in the limit of large $N_c$ there is a double degeneracy of the eigenvalue $50N_c/3$, which is broken by sub-leading $N_c$ correction.

Similarly, the three eigenvalues of length-5 operators are given as solutions of

$$0 = \omega_1^3 - \frac{112\omega_1^2}{3} + \omega_1\left(360 - \frac{11440}{9N_c^2}\right) - \left(\frac{5824}{27} - \frac{45760}{27N_c^2}\right), \quad (3.32)$$

and the anomalous dimensions expanded up to $O(1/N_c)$ are

$$\hat{\gamma}^{(1),L=5}_{+,e} = N_c\left\{\frac{56}{3} + \frac{5720}{3N_c^2}, \frac{2(14 \pm \sqrt{170})}{3} - \frac{286(170 \pm 13\sqrt{170})}{51N_c^2}\right\}. \quad (3.33)$$

We have also computed the one-loop form factors and performed the one-loop renormalization for all the dim-10 physical basis operators in appendix I. There are 20 (15) single-trace (double-trace) physical operators with length four, and 4 (3) single-trace (double-trace) physical operators with length five. Their explicit definitions are given in I.1 and I.2. The operator mixing sub-matrices $Z_{10;p\rightarrow p}^{(1)}$ and $Z_{10;p\rightarrow e}^{(1)}$ in (3.21) and the one-loop anomalous dimensions of physical operators are given in appendix I.3. As discussed in section 3.2, one can choose a finite renormalization scheme as (3.25) to absorb the finite mixing from evanescent operators to the physical ones. The sub-matrix $Z_{e\rightarrow p}^{(1),\text{fin}}$ defined in (3.25) is given in appendix I.4. These results will be needed for the two-loop renormalization.

4 Summary and discussion

In this paper we initiate the study of the evanescent operators in pure Yang-Mills theory. Such operators vanish when the number of spacetime dimensions is four but have non-trivial results in general $d$ dimensions. We provide a systematic construction of the gluonic evanescent operators based on the study of their form factors in general $d$ dimensions. The gluonic evanescent operators start to appear at canonical dimension $\Delta_0 = 10$, and they are expected to take an important part in the study of high dimensional operators of $\Delta_0 \geq 10$ in any effective field theory that contains a Yang-Mills sector. We also compute the one-loop form factors of gluonic evanescent operators via unitarity method in $d$-dimensions. The one-loop operator-mixing renormalization matrices are given explicitly for the complete dimension-10 basis operators (including both evanescent and physical operators).

A concrete further study is to explore the physical effect of evanescent operators by studying the two-loop renormalization, in which it is important to include the evanescent operators to obtain the correct two-loop physical anomalous dimensions. Another interesting problem is to consider the gauge theory at the Wilson-Fisher fixed point [41], where the spacetime is in general non-integer dimensions and the evanescent operators are also physical operators; in such case it is interesting to see if the evanescent operators can render the gauge theory non-unitary, as observed for the scalar theory in [11]. We leave these studies to another work [42]. It would be also interesting to generalize the study of this work to evanescent operators in gravity theories; some discussion on the evanescent effect in gravity has been considered in [43, 44].
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A Primitive kinematic operators

In this appendix we give some details on the primitive kinematic operators defined in section 2.3.1. Appendix A.1 provides a proof of (2.24), which guarantees the completeness of primitive kinematic operators. The expressions of the primitive evanescent kinematic operators with length four are given in appendix A.2. Appendix A.4 explains why the kinematic operators $\{s_{i,12}^{n_1} \ldots s_{i,34}^{n_4} \xi_i\}$ with the same mass dimension are linearly independent, which results in the counting shown in table 1.

A.1 Proof of (2.24)

It is stated in (2.24) that a generic length-4 kinematic operator $[W_1, W_2, W_3, W_4]$ of dim $\Delta \geq 12$ can always be written as a sum like

$$[W_1, W_2, W_3, W_4] = \sum_i \sum_{\vec{n}_i} c_{\vec{n}_i} \prod_{1 \leq a < b \leq 4} s_{i,ab}^{n_{i,ab}} K_i, \quad \text{dim}(K_i) \leq 12. \quad (A.1)$$

Here $c_{\vec{n}_i}$ are rational numbers for each choice of $\vec{n}_i = \{n_{i,12}, n_{i,13}, n_{i,23}, n_{i,14}, n_{i,24}, n_{i,34}\}$, and $s_{i,ab}^{n_{i,ab}} K_i$ refers to inserting $n_{i,ab}$ pairs of $\{D_\mu, D^\mu\}$ into the $a$-th and $b$-th sites of the $i$-th kinematic operator $K_i$, as in (2.25). This can be actually generalized to generic length $L$: a length-$L$ kinematic operator $[W_1, \ldots, W_L]$ of dim $\Delta \geq 4L - 4$ can always be written as a sum like

$$[W_1, W_2, W_3, W_4] = \sum_i \sum_{\vec{n}_i} c_{\vec{n}_i} \prod_{1 \leq a < b \leq L} s_{i,ab}^{n_{i,ab}} K_i, \quad \text{dim}(K_i) \leq 4L - 4, \quad (A.2)$$

where $\vec{n} = \{n_{i,12}, n_{i,13}, \ldots, n_{i,(L-1)L}\}$. The statement can be proved as follows:

1. It suffices to only consider the kinematic operators free of $\{D_\mu, D^\mu\}$ pairs. For the case of length-$L$, the maximal mass dimension of such operator is $4L$, where every Lorentz index in $F_{\mu\nu}$ is contracted with a $D_\rho$.

2. Consider such a dim-$4L$ kinematic operator $\mathcal{D} = [W_1, \ldots, W_L]$. It contains $2L$ D and all of them contract with $F$. At least one of $L$ sites, say $W_a$, contains $D$, and therefore can be written as $W_a = (\ldots) D F_{jk}$, where $(\ldots)$ represents other $Ds$ or 1 in front of $D_i$. Applying Bianchi identity, one has

$$\mathcal{D} = [W_1, \ldots, W_{a-1}, (\ldots) D F_{jk}, W_{a+1}, \ldots, W_L]$$

$$= -[W_1, \ldots, W_{a-1}, (\ldots) D F_{ki}, W_{a+1}, \ldots, W_L]$$

$$- [W_1, \ldots, W_{a-1}, (\ldots) D F_{ij}, W_{a+1}, \ldots, W_L].$$
Since the original $F_{jk}$ must contract with $D_j, D_k$ from other sites, the first term on the r.h.s. of (A.3) contains two $D_j$ and the second term contains two $D_k$. After stripping of these identical $D$s, $\mathcal{O}$ becomes a sum of two kinematic operators with dimension $4L - 2$.

3. Consider a dim-$(4L - 2)$ kinematic operator $\mathcal{O}' = [W_1, \ldots, W_L]$ which is free of $D_i$ pairs. There exist two sites, say $W_a$ and $W_b$, whose $F$ share a pair of Lorentz indices, and therefore can be written as $(\ldots)F_{ij}$ and $(\ldots)F_{jk}$. They contract with $D_i, D_k$ from other sites.

3.a. If one of the other $L - 2$ sites, say $W_c$, can be written as $(\ldots)D_{\mu}F_{mn}$, then following the same argument as (A.3), the operator $\mathcal{O}'$ can be reduced to dimension-$(4L - 4)$ operators with a $DD$-pair insertion.

3.b. If none of the other $L - 2$ sites contains $D_\mu$, then all the $D$s must be placed in site $a$ or site $b$. Especially, $D_4$ must be contained by $W_b$ and $D_k$ must be contained by $W_a$. Applying Bianchi identity, one has

$$\mathcal{O}' = [W_1, \ldots, W_{a-1}, (\ldots)D_kF_{ij}, W_{a+1}, \ldots, W_{b-1}, (\ldots)D_lF_{jk}, W_{b+1}, \ldots, W_L]$$

$$= -[W_1, \ldots, W_{a-1}, (\ldots)D_lF_{jk}, W_{a+1}, \ldots, W_{b-1}, (\ldots)D_lF_{jk}, W_{b+1}, \ldots, W_L]$$

$$- [W_1, \ldots, W_{a-1}, (\ldots)D_jF_{ki}, W_{a+1}, \ldots, W_{b-1}, (\ldots)D_jF_{ki}, W_{b+1}, \ldots, W_L]$$

$$= -\frac{1}{2}[W_1, \ldots, W_{a-1}, (\ldots)D_jF_{jk}, W_{a+1}, \ldots, W_{b-1}, (\ldots)D_jF_{jk}, W_{b+1}, \ldots, W_L] .$$

(A.4)

The r.h.s. of (A.4) contains a pair of identical $D$s, so it can be also reduced to dimension $4L - 4$.

Thus we prove that the primitive operators in (A.1) has at most dimension 12.

To find the complete set of primitive operators $\mathcal{K}_i$, one can first enumerate all possible length-4 kinematic operators within dimension 12 and then find out the subset which is linearly independent within dimension 12, i.e.

$$\sum_i f_i(s_{ab})\tilde{\delta}(\mathcal{K}_i) \neq 0, \quad \text{with} \ dim(f_i\tilde{\delta}(\mathcal{K}_i) \leq 12, $$

(A.5)

where $f_i(s_{ab})$ are polynomials of Mandelstam variables $s_{ab}$.

This independent subset is found in following way. Start with dimension 8, the lowest operator dimension for length-4 operators, and find the subset that is linearly independent with constant coefficients, which contains six elements. Then consider dimension 10, and find the subset that is linearly independent of the dim-10 operators generated by inserting $DD$ pairs into six dim-8 ones, which contains 42 elements. Then consider dimension 12, and find the subset that is linearly independent of the dim-12 operators generated by inserting $DD$ pairs into six dim-8 ones and 42 dim-10 ones, which contains six elements. So finally one finds that there are in total 54 $\mathcal{K}_i$, which can be grouped into 27 evanescent ones and 27 physical ones, denoted by $E_i$ and $P_i$ respectively.
A.2 Primitive evanescent length-4 kinematic operators

Below we provide explicit expressions of the evanescent primitive kinematic operators $\mathcal{E}_{1,i}$ and $\mathcal{E}_{3,i}$, following the discussion in section 2.3.1.

The 12 kinematic operators in the first class are

$$\mathcal{E}_{1,1} = \frac{1}{16} \delta_{\mu_1 \nu_1} \delta_{\mu_2 \nu_2} \delta_{\mu_3 \nu_3} \delta_{\rho_1 \sigma_1} \left[ D_\sigma F_{\mu_1 \nu_2}, F_{\mu_2 \nu_3}, D_\rho F_{\nu_1 \nu_2}, F_{\nu_3 \rho_1} \right],$$

$$\mathcal{E}_{1,2} = \mathcal{E}_{1,1} \left[ W_1, W_2, W_3, W_4 \rightarrow [W_2, W_3, W_4, W_1] \right],$$

$$\mathcal{E}_{1,4} = \mathcal{E}_{1,1} \left[ W_1, W_2, W_3, W_4 \rightarrow [W_2, W_3, W_4, W_1] \right],$$

$$\mathcal{E}_{1,6} = \mathcal{E}_{1,1} \left[ W_1, W_2, W_3, W_4 \rightarrow [W_2, W_3, W_4, W_1] \right],$$

$$\mathcal{E}_{1,8} = \mathcal{E}_{1,1} \left[ W_1, W_2, W_3, W_4 \rightarrow [W_2, W_3, W_4, W_1] \right],$$

$$\mathcal{E}_{1,10} = \mathcal{E}_{1,1} \left[ W_1, W_2, W_3, W_4 \rightarrow [W_2, W_3, W_4, W_1] \right],$$

$$\mathcal{E}_{1,12} = \mathcal{E}_{1,1} \left[ W_1, W_2, W_3, W_4 \rightarrow [W_2, W_3, W_4, W_1] \right].$$

(A.6)

The 12 kinematic operators in the third class are

$$\mathcal{E}_{3,1} = \frac{1}{4} \delta_{\mu_1 \nu_1} \delta_{\mu_2 \nu_2} \delta_{\mu_3 \nu_3} \delta_{\mu_4 \nu_4} \left[ D_\nu F_{\mu_1 \mu_2}, F_{\mu_3 \mu_4}, D_\nu F_{\nu_1 \nu_2}, D_\nu F_{\nu_3 \nu_4} \right],$$

$$\mathcal{E}_{3,2} = \mathcal{E}_{3,1} \left[ W_1, W_2, W_3, W_4 \rightarrow [W_2, W_3, W_4, W_1] \right],$$

$$\mathcal{E}_{3,4} = \mathcal{E}_{3,1} \left[ W_1, W_2, W_3, W_4 \rightarrow [W_2, W_3, W_4, W_1] \right],$$

$$\mathcal{E}_{3,6} = \mathcal{E}_{3,1} \left[ W_1, W_2, W_3, W_4 \rightarrow [W_2, W_3, W_4, W_1] \right],$$

$$\mathcal{E}_{3,8} = \mathcal{E}_{3,1} \left[ W_1, W_2, W_3, W_4 \rightarrow [W_2, W_3, W_4, W_1] \right],$$

$$\mathcal{E}_{3,10} = \mathcal{E}_{3,1} \left[ W_1, W_2, W_3, W_4 \rightarrow [W_2, W_3, W_4, W_1] \right],$$

$$\mathcal{E}_{3,12} = \mathcal{E}_{3,1} \left[ W_1, W_2, W_3, W_4 \rightarrow [W_2, W_3, W_4, W_1] \right].$$

(A.7)

A.3 Dimension 10 length-5 kinematic operators

In section 2.4 we show the six length-5 kinematic operators of dimension 10, which are related with each other by $S_3$ permutations. Here we clarify the permutation for each $U_{10;i}$.

$$U_{10;2} = U_{10;1} \left[ W_1, W_j, W_k, W_l, W_m \rightarrow [W_j, W_k, W_l, W_m, W_1] \right],$$

$$U_{10;3} = U_{10;1} \left[ W_1, W_j, W_k, W_l, W_m \rightarrow [W_j, W_k, W_l, W_m, W_1] \right],$$

$$U_{10;4} = U_{10;1} \left[ W_1, W_j, W_k, W_l, W_m \rightarrow [W_j, W_k, W_l, W_m, W_1] \right],$$

$$U_{10;5} = U_{10;1} \left[ W_1, W_j, W_k, W_l, W_m \rightarrow [W_j, W_k, W_l, W_m, W_1] \right],$$

$$U_{10;6} = U_{10;1} \left[ W_1, W_j, W_k, W_l, W_m \rightarrow [W_j, W_k, W_l, W_m, W_1] \right].$$

(A.8)

A.4 Linearity independence of kinematic operators

We show in this subsection that kinematic operators $\left\{ s_{12}^{\mu_1 \gamma_1}, \ldots, s_{34}^{\mu_1 \gamma_1} \mathcal{E}_i \right\}$ with the same mass dimension are linearly independent. To prove this, it is enough to show that the evanescent
kinematic form factors $\mathcal{F}(E_i)$ are linearly independent as
\[
\sum_{i=1}^{27} \mathcal{R}_i(s_{ab})\mathcal{F}(E_i) \neq 0 , \tag{A.9}
\]
where $\mathcal{R}_i(s_{ab})$ are arbitrary nonzero rational functions of $s_{ab}$. If (A.9) is true, \{\mathcal{F}(E_i)\} are linearly independent with polynomial $s_{ab}$ coefficients of arbitrarily high mass dimensions, and thus the kinematic operators \{$s_{12}^{n_1,12} \ldots s_{34}^{n_3,34} E_i$\} are linearly independent and provide a basis of $\mathcal{R}^e_\Delta$.

The proof of (A.9) proceeds as follows. We consider $\sum_i \mathcal{R}_i \mathcal{F}(E_i) = 0$ as an equation of to-be-determined variables \{\mathcal{R}_i\}. Kinematic form factors $\mathcal{F}(E_i)$ are known functions of $s_{ab}$ together with monomials of Lorentz products containing external polarization vectors $e_1^\mu, e_2^\mu, e_3^\mu, e_4^\mu$. There are 138 such monomials such as
\[
(e_1 \cdot e_2)(e_3 \cdot e_4), \quad (e_1 \cdot p_2)(e_2 \cdot p_4)(e_3 \cdot e_4), \quad (e_1 \cdot p_2)(e_2 \cdot p_4)(e_3 \cdot p_1)(e_4 \cdot p_2), \quad \cdots . \tag{A.10}
\]
Requiring the coefficients of all these monomials vanish, one obtains 138 equations about $27 \mathcal{R}_i$, with polynomial coefficients of $s_{ab}$:
\[
\sum_{i=1}^{27} f_{n,i}(s_{ab})\mathcal{R}_i = 0, \quad n = 1, \ldots, 138 , \tag{A.11}
\]
where $f_{n,i}(s_{ab})$ are polynomials of $s_{ab}$ determined by the expressions of $\mathcal{F}(E_i)$. One finds that there is no nonzero solution of \{\mathcal{R}_i\} satisfying these 138 equations, which means no rational functions \{\mathcal{R}_i\} would turn (A.9) to an equality.

It is worth mentioning that linear independence with $s_{ab}$ rational function coefficients as shown in (A.9) is also the requirement of gauge invariant basis given in (2.23). Since $\mathcal{F}(E_i)$ already satisfy such condition, they must be linearly equivalent to the gauge invariant basis composed of $AAB$ and $BB$, which explains why their total number are both 27.

By construction, the 54 general kinematic form factors $\mathcal{F}(K_i)$ (including both physical and evanescent ones) only satisfy (A.5), and there exist 11 linear relations among them in the form of
\[
\sum_i f_i(s_{ab})\mathcal{F}(K_i) = 0, \quad \text{with } \dim(f_i\mathcal{F}(K_i)) > 12 , \tag{A.12}
\]
that explains why the number of $K_i$ is 11 larger than the number of general gauge invariant basis.

### B Comment on minimally evanescent operator

The vanishing of the minimal form factor of an operator does not mean that its higher-point form factors are also zero. An example is the following operator
\[
\Omega_1 = \frac{1}{2} \text{tr}(D_3 F_{12} F_{36} D_5 F_{12} D_{34} F_{56}) - \frac{1}{2} \text{tr}(D_3 F_{12} F_{36} D_4 F_{12} D_{34} F_{56}) - \text{tr}(D_2 F_{13} F_{46} D_5 F_{14} D_{23} F_{56}) + \text{tr}(D_4 F_{13} F_{46} D_5 F_{12} D_{23} F_{56}) - \text{tr}(D_2 F_{13} F_{46} D_2 F_{13} D_{34} F_{56}) + \text{tr}(D_2 F_{15} F_{46} D_2 F_{13} D_{34} F_{56}) - \text{tr}(D_4 F_{15} F_{46} D_3 F_{12} D_{23} F_{56}) + \text{tr}(D_1 F_{35} F_{46} D_1 F_{24} D_{23} F_{56}) . \tag{B.1}
\]
For the convenience of notation, we use integer numbers to represent Lorentz indices in $\Omega_1$ and abbreviate $D_iD_j \ldots$ as $D_{ij \ldots}$. The minimal form factor of $\Omega_1$ vanishes in four dimensions, but its next-to-minimal form factor does not, so $\Omega_1$ is minimally evanescent but not exactly evanescent.

A general minimally evanescent operator can always be modified to be an exactly evanescent one. The main idea is that by adding higher-length operators properly, one can construct evanescent operators that all their higher-point form factors vanish in four dimensions.

Let us start with a minimally-evanescent operator $O$ of length-$L$ and of canonical dimension $\Delta > 2L$. We first show that its next-to-minimal form factor $F_{O,L+1}$ in the four-dimensional limit does not have any physical pole, using proof by contradiction based on the unitarity-cut picture. Let us assume that $F_{O,L+1}$ has a physical pole in $s_{ij}$, then in the limit of $s_{ij} \to 0$ the residue of the form factor should factorize as a product of a minimal form factor and a 3-gluon amplitude, as shown in figure 3. Since the minimal form factor vanishes in four dimensions, so does the residue. Thus the 4-dim part of $F_{O,L+1}$ is a gauge invariant quantity without any poles, and therefore it can be taken as the minimal form factor and a 3-gluon amplitude, as shown in figure 3. Since the minimal form factor vanishes in four dimensions, so does the residue. Thus the 4-dim part of $F_{O,L+1}$ is a gauge invariant quantity without any poles, and therefore it can be taken as the minimal form factor of a length-$(L + 1)$ operator, denoted by $\Xi_{L+1}$. This means one should be able to construct a new operator $\tilde{O} = O - \Xi_{L+1}$ whose $L$- and $(L + 1)$-gluon form factors both vanish in four dimensions.

Similar analysis can be carried out iteratively to the higher point form factors and after subtracting finitely many operators one gets an operator $\tilde{O}$ whose form factors of $L, L + 1, \ldots, \Delta/2$ external gluons all vanish in four dimensions:

$$\tilde{O} = O - \Xi_{n+1} - \ldots - \Xi_{\Delta/2} \implies F_{\tilde{O},n}\big|_{4\text{dim}} = 0, \quad n = L, L + 1, \ldots, \Delta/2, \quad (B.2)$$

where $\Delta$ is the canonical dimension of the operator. A nice and important point is that such an operator is already an exactly evanescent operator, and there is no need to worry about the higher-point form factors. Let us consider its $(\Delta/2 + 1)$-gluon form factor $F_{\tilde{O},\Delta/2+1}$. Using the similar factorization analysis one can show that it does not have poles in four dimensions, so if it is nonzero, it should be equal to the minimal form factor of a length-$\Delta/2+1$ operator. However, there is no such operator because the highest length for mass dimension $\Delta$ is $\Delta/2$, so we conclude that the form factor of $\tilde{O}$ with external gluons more than $\Delta/2$ must vanish in four dimensions.
As an illustration, we can look into the former example (B.1). In four dimensions, the next-to-minimal form factor of $\Omega_1$ can be canceled by the minimal form factor of length-5 operator

$$
\Omega_2 = \frac{1}{2} \text{tr}(D_3 F_{12} F_{46} D_3 F_{12} [F_{54}, F_{56}]) - \frac{1}{2} \text{tr}(D_4 F_{12} F_{46} D_3 F_{12} [F_{53}, F_{56}]) - \text{tr}(D_2 F_{13} F_{46} D_2 F_{14} [F_{53}, F_{56}]).
$$

(B.3)

One can check that $\Omega_1 + \Omega_2$ is already an exactly evanescent operator, for there is no dim-10 operator with length higher than 5.

C Representation decomposition of color and kinematic factors

In this appendix we give the irreducible decomposition of the representations for both the color factors ($\text{Span}\{T^a\}$) and kinematic form factors ($\text{Span}\{\mathcal{S}[W_1, W_2, W_3, W_4]\}$) following the discussion in section 2.3.2. Below we consider $\text{SU}(N_c)$ gauge group for general $N_c$, where the color basis are given in (2.38). The color basis for special values of $N_c$ is discussed in appendix I.5.

C.1 Color factors

We first organize basis color factors so that all of them have sign change $+1$ or $-1$ under reflection

$$
T^{\sigma(1)} T^{\sigma(2)} T^{\sigma(3)} T^{\sigma(4)} \rightarrow T^{\sigma(4)} T^{\sigma(3)} T^{\sigma(2)} T^{\sigma(1)},
$$

(C.1)

which we call $C$-even and $C$-odd sectors. As defined in (2.46), for the length-4 case, the basis color factors can be classified into three types: single-trace $C$-even, single-trace $C$-odd, double-trace $C$-even, which are denoted by

$$
\{T^{s+}_a\} = \{\text{tr}(1 \sigma(2) \sigma(3) \sigma(4)) + \text{tr}(1 \sigma(4) \sigma(3) \sigma(2))\},
\{T^{s-}_a\} = \{\text{tr}(1 \sigma(2) \sigma(3) \sigma(4)) - \text{tr}(1 \sigma(4) \sigma(3) \sigma(2))\},
\{T^{d+}_a\} = \{\text{tr}(1 \sigma(2)) \text{tr}(\sigma(3) \sigma(4))\},
$$

(C.2)

where $\sigma \in Z_3$. Their representation decompositions are:

$$
\text{Span}\{T^{s+}_a\} \sim \bigoplus, \quad \text{Span}\{T^{s-}_a\} \sim \bigoplus, \quad \text{Span}\{T^{d+}_a\} \sim \bigoplus.
$$

(C.3)

The bases which realize above irreducible representations are:

$$
T^{[4]}_{s^+} = \left\{ \frac{1}{6} \text{tr}(1234) + \frac{1}{6} \text{tr}(1243) + \frac{1}{6} \text{tr}(1324) + \frac{1}{6} \text{tr}(1342) + \frac{1}{6} \text{tr}(1423) + \frac{1}{6} \text{tr}(1432) \right\},
$$

$$
T^{[2,2]}_{s^+} = \left\{ -\frac{1}{6} \text{tr}(1234) - \frac{1}{6} \text{tr}(1243) + \frac{1}{3} \text{tr}(1324) - \frac{1}{6} \text{tr}(1342) - \frac{1}{6} \text{tr}(1423) + \frac{1}{3} \text{tr}(1432) \right\},
$$

$$
T^{[2,2]}_{s^-} = \left\{ \frac{1}{3} \text{tr}(1234) - \frac{1}{6} \text{tr}(1243) - \frac{1}{6} \text{tr}(1324) - \frac{1}{6} \text{tr}(1342) - \frac{1}{6} \text{tr}(1423) + \frac{1}{3} \text{tr}(1432) \right\},
$$

$$
T^{[4]}_{d^+} = \left\{ \frac{1}{6} \text{tr}(1234) + \frac{1}{6} \text{tr}(1243) + \frac{1}{6} \text{tr}(1324) + \frac{1}{6} \text{tr}(1342) + \frac{1}{6} \text{tr}(1423) + \frac{1}{6} \text{tr}(1432) \right\}.
$$
form factors are directly obtained by transforming following context we write kinematic operators for simplicity and the results for kinematic $K$ diagonalize Therefore we can obtain the representation decomposition of $\mathbf{k}$ action over an kinematic operator $|W_1, W_2, W_3, W_4\rangle$ and its kinematic form factor $\mathbf{f}(|W_1, W_2, W_3, W_4\rangle)$ are related as

$$\mathbf{f}(|W_{\sigma(1)}, W_{\sigma(2)}, W_{\sigma(3)}, W_{\sigma(4)}\rangle) = \sigma^{-1} \cdot \mathbf{f}(|W_1, W_2, W_3, W_4\rangle).$$

(C.6)

Therefore we can obtain the representation decomposition of $\mathbf{f}(\mathbf{k}_1)$ through the representation decomposition of $\mathbf{f}(|N; s_{ij}\rangle)$ and $\mathbf{f}(|\mathbf{r}_i\rangle)$, $i = 1, 2, 3, 4$.

The $S_4$ action over an kinematic operator $|W_1, W_2, W_3, W_4\rangle$ and its kinematic form factor $\mathbf{f}(|W_1, W_2, W_3, W_4\rangle)$ are related as

$$\mathbf{f}(|W_{\sigma(1)}, W_{\sigma(2)}, W_{\sigma(3)}, W_{\sigma(4)}\rangle) = \sigma^{-1} \cdot \mathbf{f}(|W_1, W_2, W_3, W_4\rangle).$$

(C.6)

Therefore $\mathbf{r}_i$ and $\mathbf{f}(|\mathbf{r}_i\rangle)$ have the same decomposition. The basis operators that block-diagonalize $\mathbf{r}_i$ also block-diagonalize $\mathbf{f}(|\mathbf{r}_i\rangle)$ with their kinematic form factors. So in the following context we write kinematic operators for simplicity and the results for kinematic form factors are directly obtained by transforming $\mathbf{E}_{ij}$ to $\mathbf{f}(\mathbf{E}_{ij})$.

The representation decomposition of $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, $\mathbf{r}_4$:

$$\mathbf{r}_1 \sim \begin{array}{ccc} \hline & & \hline \end{array} + \begin{array}{ccc} \hline & & \hline \end{array} + 2 \begin{array}{ccc} \hline & & \hline \end{array} + \begin{array}{ccc} \hline & & \hline \end{array} + \begin{array}{ccc} \hline & & \hline \end{array}, \quad \mathbf{r}_2 \sim \begin{array}{ccc} \hline & & \hline \end{array} + \begin{array}{ccc} \hline & & \hline \end{array},$$

$$\mathbf{r}_3 \sim \begin{array}{ccc} \hline & & \hline \end{array} + 2 \begin{array}{ccc} \hline & & \hline \end{array} + \begin{array}{ccc} \hline & & \hline \end{array} + \begin{array}{ccc} \hline & & \hline \end{array}, \quad \mathbf{r}_4 \sim \begin{array}{ccc} \hline & & \hline \end{array} + \begin{array}{ccc} \hline & & \hline \end{array} + \begin{array}{ccc} \hline & & \hline \end{array} + \begin{array}{ccc} \hline & & \hline \end{array}.$$  

(C.7)

The bases which realize above irreducible representations are summarized below.

For $\mathbf{r}_1$:

$$\mathbf{E}_{1}^{[4]} = \left\{ \frac{1}{12} (\mathbf{E}_{1,1} + \mathbf{E}_{1,2} + \mathbf{E}_{1,3} + \mathbf{E}_{1,4} + \mathbf{E}_{1,5} + \mathbf{E}_{1,6} + \mathbf{E}_{1,7} + \mathbf{E}_{1,8} + \mathbf{E}_{1,9} + \mathbf{E}_{1,10} + \mathbf{E}_{1,11} + \mathbf{E}_{1,12}) \right\},$$

$$\mathbf{E}_{1}^{[3,1]} = \left\{ \frac{1}{8} (\mathbf{E}_{1,1} + \mathbf{E}_{1,2} - \mathbf{E}_{1,3} + \mathbf{E}_{1,4} + \mathbf{E}_{1,5} - \mathbf{E}_{1,6} - \mathbf{E}_{1,7} + \mathbf{E}_{1,8} + \mathbf{E}_{1,9} - \mathbf{E}_{1,10} + \mathbf{E}_{1,11} - \mathbf{E}_{1,12}),
\frac{1}{8} (-\mathbf{E}_{1,1} + \mathbf{E}_{1,2} + \mathbf{E}_{1,3} + \mathbf{E}_{1,4} + \mathbf{E}_{1,5} - \mathbf{E}_{1,6} + \mathbf{E}_{1,7} - \mathbf{E}_{1,8} + \mathbf{E}_{1,9} + \mathbf{E}_{1,10} - \mathbf{E}_{1,11} - \mathbf{E}_{1,12}), \right\}.$$
\( \mathcal{E}_{1}^{[2,1,1]} = \left\{ \frac{1}{8} \left( \mathcal{E}_{1,1} - \mathcal{E}_{1,2} + \mathcal{E}_{1,3} - \mathcal{E}_{1,4} - \mathcal{E}_{1,5} - \mathcal{E}_{1,6} + \mathcal{E}_{1,7} + \mathcal{E}_{1,8} - \mathcal{E}_{1,9} - \mathcal{E}_{1,10} + \mathcal{E}_{1,11} + \mathcal{E}_{1,12} \right) \right\} ,
\)

\( \mathcal{E}_{1}^{[1,1,1,1]} = \left\{ \frac{1}{12} \left( \mathcal{E}_{1,1} - \mathcal{E}_{1,2} - \mathcal{E}_{1,3} + \mathcal{E}_{1,4} + \mathcal{E}_{1,5} + \mathcal{E}_{1,6} + \mathcal{E}_{1,7} - \mathcal{E}_{1,8} + \mathcal{E}_{1,9} - \mathcal{E}_{1,10} + \mathcal{E}_{1,11} + \mathcal{E}_{1,12} \right) \right\} ,
\)

\( \mathcal{E}_{1}^{[2,2],a} = \left\{ x_{11} \mathcal{E}_{1}^{[2,2],1} + x_{12} \mathcal{E}_{1}^{[2,2],3} , \ x_{11} \mathcal{E}_{1}^{[2,2],2} + x_{12} \mathcal{E}_{1}^{[2,2],4} \right\} ,
\)

\( \mathcal{E}_{1}^{[2,2],b} = \left\{ x_{21} \mathcal{E}_{1}^{[2,2],1} + x_{22} \mathcal{E}_{1}^{[2,2],3} , \ x_{21} \mathcal{E}_{1}^{[2,2],2} + x_{22} \mathcal{E}_{1}^{[2,2],4} \right\} ,
\)

(C.8)

where

\( \mathcal{E}_{1}^{[2,2],1} = \frac{1}{6} \left( -\mathcal{E}_{1,5} + \mathcal{E}_{1,6} + \mathcal{E}_{1,7} - \mathcal{E}_{1,8} - \mathcal{E}_{1,9} + \mathcal{E}_{1,10} + \mathcal{E}_{1,11} - \mathcal{E}_{1,12} \right) ,
\)

\( \mathcal{E}_{1}^{[2,2],2} = \frac{1}{6} \left( -\mathcal{E}_{1,1} + \mathcal{E}_{1,2} + \mathcal{E}_{1,3} - \mathcal{E}_{1,4} + \mathcal{E}_{1,9} - \mathcal{E}_{1,10} - \mathcal{E}_{1,11} + \mathcal{E}_{1,12} \right) ,
\)

\( \mathcal{E}_{1}^{[2,2],3} = \frac{1}{6} \left( \mathcal{E}_{1,1} + \mathcal{E}_{1,2} + \mathcal{E}_{1,3} + \mathcal{E}_{1,4} - \mathcal{E}_{1,5} - \mathcal{E}_{1,8} - \mathcal{E}_{1,9} - \mathcal{E}_{1,12} \right) ,
\)

\( \mathcal{E}_{1}^{[2,2],4} = \frac{1}{6} \left( -\mathcal{E}_{1,1} - \mathcal{E}_{1,4} + \mathcal{E}_{1,5} + \mathcal{E}_{1,6} + \mathcal{E}_{1,7} + \mathcal{E}_{1,8} - \mathcal{E}_{1,10} - \mathcal{E}_{1,11} \right) ,
\)

(C.9)

and \( x_{11}, x_{12}, x_{21}, x_{22} \) can be valued in any rational numbers as long as \( x_{11} x_{22} - x_{12} x_{21} \neq 0 \). Other requirement not related to \( S_{4} \) representation helps to fix \( x_{11} = 1, x_{12} = -2 \), which we will explain in appendix D.1. There is no further constrain on \( x_{21}, x_{22} \), and in this paper we choose \( x_{21} = 0, x_{22} = 1 \).

For \( \mathcal{R}_{2} \):

\( \mathcal{E}_{2}^{[2,2]} = \left\{ \frac{1}{3} (2 \mathcal{E}_{2,2} - \mathcal{E}_{2,1}) , \ \frac{1}{3} (2 \mathcal{E}_{2,1} - \mathcal{E}_{2,2}) \right\} .
\)

(C.10)

For \( \mathcal{R}_{3} \):

\( \mathcal{E}_{3}^{[4]} = \left\{ \frac{1}{12} \left( \mathcal{E}_{3,1} + \mathcal{E}_{3,2} + \mathcal{E}_{3,3} + \mathcal{E}_{3,4} + \mathcal{E}_{3,5} + \mathcal{E}_{3,6} + \mathcal{E}_{3,7} + \mathcal{E}_{3,8} + \mathcal{E}_{3,9} + \mathcal{E}_{3,10} + \mathcal{E}_{3,11} + \mathcal{E}_{3,12} \right) \right\} ,
\)

\( \mathcal{E}_{3}^{[2,2]} = \left\{ \frac{1}{12} \left( -\mathcal{E}_{3,1} - \mathcal{E}_{3,2} - \mathcal{E}_{3,3} - \mathcal{E}_{3,4} + 2 \mathcal{E}_{3,5} + 2 \mathcal{E}_{3,6} - \mathcal{E}_{3,7} + 2 \mathcal{E}_{3,8} - \mathcal{E}_{3,9} - \mathcal{E}_{3,10} + 2 \mathcal{E}_{3,11} - \mathcal{E}_{3,12} \right) \right\} ,
\)

\( \mathcal{E}_{3}^{[2,1,1]} = \left\{ \frac{1}{8} \left( -\mathcal{E}_{3,1} - \mathcal{E}_{3,2} + \mathcal{E}_{3,3} + \mathcal{E}_{3,4} + 2 \mathcal{E}_{3,5} - 2 \mathcal{E}_{3,6} - \mathcal{E}_{3,7} + \mathcal{E}_{3,8} - \mathcal{E}_{3,9} + \mathcal{E}_{3,10} + \mathcal{E}_{3,12} \right) \right\} ,
\)

\( \mathcal{E}_{3}^{[2,1,1,a]} = \left\{ \frac{1}{8} \left( 2 \mathcal{E}_{3,1} - \mathcal{E}_{3,2} - \mathcal{E}_{3,3} - \mathcal{E}_{3,4} + \mathcal{E}_{3,5} + \mathcal{E}_{3,6} + \mathcal{E}_{3,7} + \mathcal{E}_{3,8} - 2 \mathcal{E}_{3,9} - \mathcal{E}_{3,11} + \mathcal{E}_{3,12} \right) \right\} ,
\)

\( \mathcal{E}_{3}^{[2,1,1,b]} = \left\{ \frac{1}{8} \left( -\mathcal{E}_{3,1} + 2 \mathcal{E}_{3,3} - \mathcal{E}_{3,4} + \mathcal{E}_{3,5} - 2 \mathcal{E}_{3,6} + \mathcal{E}_{3,7} + \mathcal{E}_{3,8} + \mathcal{E}_{3,9} + \mathcal{E}_{3,10} - \mathcal{E}_{3,11} \right) \right\} ,
\)

\( \mathcal{E}_{3}^{[3,1,a]} = \left\{ x_{11} \mathcal{E}_{3}^{[3,1],1} + x_{12} \mathcal{E}_{3}^{[3,1],4} , \ x_{11} \mathcal{E}_{3}^{[3,1],2} + x_{12} \mathcal{E}_{3}^{[3,1],5} , \ x_{11} \mathcal{E}_{3}^{[3,1],3} + x_{12} \mathcal{E}_{3}^{[3,1],6} \right\} ,
\)

\( \mathcal{E}_{3}^{[3,1,b]} = \left\{ x_{21} \mathcal{E}_{3}^{[3,1],1} + x_{22} \mathcal{E}_{3}^{[3,1],4} , \ x_{21} \mathcal{E}_{3}^{[3,1],2} + x_{22} \mathcal{E}_{3}^{[3,1],5} , \ x_{21} \mathcal{E}_{3}^{[3,1],3} + x_{22} \mathcal{E}_{3}^{[3,1],6} \right\} ,
\)

(C.11)
Table 4. Representation decomposition of the linear space $\mathcal{M}(N; s_{ij})$ spanned by homogenous monomials $s_{12}^{n_1} \cdots s_{34}^{n_4}$ with total power $N$, which contribute to decomposition (2.37).

| $N$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-----|--|--|--|--|--|--|--|
|     | 1 | 3 | 6 | 11 | 18 | 32 | 48 |
|     | 1 | 3 | 8 | 17 | 34 | 61 | 104 |
|     | 1 | 3 | 6 | 14 | 26 | 45 | 76 |
|     | 0 | 1 | 4 | 11 | 24 | 47 | 84 |
|     | 0 | 0 | 2 | 3 | 8 | 16 | 28 |
| total | 6 | 21 | 56 | 126 | 252 | 462 | 792 |

where

\[
\mathcal{E}_{3}^{[3,1],1} = \frac{1}{8}(2\mathcal{E}_{3,1} + 2\mathcal{E}_{3,2} - \mathcal{E}_{3,3} - \mathcal{E}_{3,4} + 2\mathcal{E}_{3,5} - \mathcal{E}_{3,7} - \mathcal{E}_{3,8} - \mathcal{E}_{3,10} - \mathcal{E}_{3,11}) , \\
\mathcal{E}_{3}^{[3,1],2} = \frac{1}{8}(-\mathcal{E}_{3,1} - \mathcal{E}_{3,2} + 2\mathcal{E}_{3,3} + 2\mathcal{E}_{3,4} + 2\mathcal{E}_{3,6} - \mathcal{E}_{3,8} - \mathcal{E}_{3,9} - \mathcal{E}_{3,11} - \mathcal{E}_{3,12}) , \\
\mathcal{E}_{3}^{[3,1],3} = \frac{1}{8}(-\mathcal{E}_{3,2} - \mathcal{E}_{3,4} - \mathcal{E}_{3,5} - \mathcal{E}_{3,6} + 2\mathcal{E}_{3,7} + 2\mathcal{E}_{3,8} + 2\mathcal{E}_{3,9} - \mathcal{E}_{3,10} - \mathcal{E}_{3,12}) , \\
\mathcal{E}_{3}^{[3,1],4} = \frac{1}{8}(-\mathcal{E}_{3,3} - \mathcal{E}_{3,4} + 2\mathcal{E}_{3,6} - \mathcal{E}_{3,7} - \mathcal{E}_{3,8} + 2\mathcal{E}_{3,9} - \mathcal{E}_{3,10} - \mathcal{E}_{3,11} + 2\mathcal{E}_{3,12}) , \\
\mathcal{E}_{3}^{[3,1],5} = \frac{1}{8}(-\mathcal{E}_{3,1} - \mathcal{E}_{3,2} + 2\mathcal{E}_{3,5} + 2\mathcal{E}_{3,7} - \mathcal{E}_{3,8} - \mathcal{E}_{3,9} + 2\mathcal{E}_{3,10} - \mathcal{E}_{3,11} - \mathcal{E}_{3,12}) , \\
\mathcal{E}_{3}^{[3,1],6} = \frac{1}{8}(2\mathcal{E}_{3,1} - \mathcal{E}_{3,2} + 2\mathcal{E}_{3,3} - \mathcal{E}_{3,4} - \mathcal{E}_{3,5} - \mathcal{E}_{3,6} - \mathcal{E}_{3,10} + 2\mathcal{E}_{3,11} - \mathcal{E}_{3,12}) , 
\]

and $x_{11}, x_{12}, x_{21}, x_{22}$ can be valued in any rational numbers as long as $x_{11}x_{22} - x_{12}x_{21} \neq 0$.

In this paper we choose $x_{11} = 1, x_{12} = 0, x_{21} = 0, x_{22} = 1$.

For $\mathcal{R}_4$:

\[
\mathcal{E}_4^{[4]} = \mathcal{E}_4 .
\]

The linear space $\mathcal{M}(N; s_{ij})$ is spanned by all the homogenous monomials $s_{12}^{n_1} \cdots s_{34}^{n_4}$ with total power $N$. We list the representation decomposition of $\mathcal{M}(N; s_{ij})$ in table 4 for $N = 1, \ldots, 7$. As shown in (2.37), the representation information of $(\mathcal{R}^e)_{\Delta}$, the space of kinematic operators of dimension $\Delta$, can be read from the representation decomposition of $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$, $\mathcal{R}_4$ and $\mathcal{M}(\Delta^{10}; s_{ij})$, $\mathcal{M}(\Delta^{12}; s_{ij})$.

We summarize the representation decomposition of $(\mathcal{R}^e)_{\Delta}$, the space of dim-$\Delta$ evanescent kinematic operators, with $\Delta$ valued from 10 to dim 24 in table 5. This is also the representation decomposition of $\mathcal{F}(\Delta)(\mathcal{R}^e)_{\Delta}$, the space of dim-$\Delta$ kinematic form factors. When constructing $(\mathcal{R}^e)_{\Delta}$ by means of inserting DD pairs into primitive kinematic operators, we have counted the dimension of $(\mathcal{R}^e)_{\Delta}$ in table 1. Such dimension can also be read from
Table 5. Representation decomposition of $\mathcal{R}_\Delta$, the spaces of dim-$\Delta$ evanescent kinematic operators, $\Delta = 10, \ldots, 24$. The integers give the counting of each irreducible representation corresponding to $y_i$ in (2.42).

| $\mathcal{R}_\Delta$ | $(\mathcal{R}_\Delta)_{10}$ | $(\mathcal{R}_\Delta)_{12}$ | $(\mathcal{R}_\Delta)_{14}$ | $(\mathcal{R}_\Delta)_{16}$ | $(\mathcal{R}_\Delta)_{18}$ | $(\mathcal{R}_\Delta)_{20}$ | $(\mathcal{R}_\Delta)_{22}$ | $(\mathcal{R}_\Delta)_{24}$ |
|-----------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
| $\mathcal{R}_\Delta$  | 1                           | 6                           | 19                          | 51                          | 114                         | 231                         | 426                         | 739                         |
| $\mathcal{R}_\Delta$  | 1                           | 10                          | 41                          | 121                         | 290                         | 609                         | 1158                        | 2045                        |
| $\mathcal{R}_\Delta$  | 2                           | 10                          | 35                          | 94                          | 216                         | 440                         | 822                         | 1432                        |
| $\mathcal{R}_\Delta$  | 1                           | 9                           | 38                          | 114                         | 278                         | 589                         | 1128                        | 2001                        |
| $\mathcal{R}_\Delta$  | 1                           | 4                           | 16                          | 43                          | 102                         | 209                         | 396                         | 693                         |

Table 5. For example, the dimension of $(\mathcal{R}_\Delta)_{10}$ is known from the first column of table 5:

$$1 \times \dim(\mathcal{R}_\Delta) + 1 \times \dim(\mathcal{R}_\Delta) + 2 \times \dim(\mathcal{R}_\Delta) + 1 \times \dim(\mathcal{R}_\Delta) + 1 \times \dim(\mathcal{R}_\Delta) = 12,$$

in accord with the number given in table 1.

As mentioned in section 2.3.2, the tensor product of two irreducible subspaces of the same type $R_{[\varphi]}$ contains an element belonging to the trivial representation. This trivial element is given by

$$T_{[\varphi]} \cdot M_{[\varphi]} \cdot \mathcal{S}(L^a_{[\varphi]}),$$

where $T_{[\varphi]}$ and $\mathcal{S}(L^a_{[\varphi]})$ refer to the basis of $R_{[\varphi]}$-type sub-representations of color space and kinematic form factor space respectively. The matrix $M_{[\varphi]}$ are known from the knowledge of representation, and here we give the result:

$$M_{[4]} = M_{[1,1,1,1]} = 1, \quad M_{[3,1]} = \frac{1}{4} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix},$$

$$M_{[2,2]} = \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad M_{[2,1,1]} = -\frac{1}{12} \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix}.$$ (C.15)

D Basis operators as total derivatives

As mentioned in section 2.1, we include total derivative operators into the basis. It is often convenient to write the operator basis in a form such that the total-derivative operators are given explicitly. In this appendix we discuss how to do this in the formalism of form factors. In section D.1, we explain how to select out the total derivative basis operators and take the dim-10 length-4 operators as concrete examples. In section D.2 we give another choice of primitive evanescent kinematic operators different from what is given in section 2.3.1, which includes more total derivatives.
D.1 Dim-10 length-4 evanescent operators

As defined in (2.14), form factors of total derivative operators are non-zero but have overall factors of the momentum \( q = \sum_{i=1}^{n} p_i \) carried by the operator, for example, \( F_{\partial T_{O,n}} = q^2 F_{O,n} \). Therefore, the basic idea to pick out the total-derivative operators is to find certain linear combinations of operators such that their form factors contain overall \( q \) factors. We can replace one of external momenta, e.g. \( p_n \), with \( q - p_1 - p_2 \ldots - p_{n-1} \) and expand the form factor in powers of \( q \). If the operator is the \( r \)-th order derivative of a rank-\( r \) tensor operator like \( D_{\mu_1} \ldots D_{\mu_r} T^{\mu_1 \ldots \mu_r} \), then the lowest degree of \( q \) in its \( n \)-point form factor is \( r \).\(^{15}\)

Based on this observation, one can apply following steps. 1) Pick out all the independent combinations of old basis whose minimal form factors and next-to-minimal form factors of the momentum \( \sigma \) have to be chosen artificially. It is permitted to take a shift \( x \rightarrow x + \delta x \) for arbitrary rational \( \delta x \). Besides, \( \text{in (C.8) introduced in (D.2)} \) belong to \( T_{s+}^{[2]} \otimes \delta(\mathcal{E}_{1}^{[2]}) \). Besides, \( \mathcal{E}_{1}^{[2]} \) is the highest possible order of derivatives. 2) Continue this step for the rest old basis operators and the derivative order \( r - 1 \), and so on.

An operator with dim-10 and length-4 is at most the second derivative of a rank-2 tensor operator with dimension 8, so \( r = 2 \). Consider the length-4 dim-10 evanescent operators (2.64)–(2.67), we find two second order derivatives and one first order derivative:

\[
\begin{align*}
\mathcal{O}_{10;\pm;1}^{e} & = D_{\mu \nu} \text{tr}(34) \circ (4P_{\mu \nu} + 2P_{\mu \nu}^{c2}) , \quad \mathcal{O}_{10;\pm;1}^{\epsilon} = D_{\mu \nu} \text{tr}(12) \text{tr}(34) \circ (P_{\mu \nu}^{c1} + 2P_{\mu \nu}^{c2}) , \\
& \quad \text{where} \\
& \quad P_{\mu \nu}^{c1} = \frac{1}{16} \delta_{\rho_1 \rho_2 \rho_3 \rho_4} \rho_5 \rho_6 F_{\sigma_3 \sigma_4} F_{\sigma_1 \sigma_2} F_{\rho_1 \rho_2} F_{\rho_3 \rho_4}, \\
& \quad P_{\mu \nu}^{c2} = |P_{\mu \nu}^{c1}| W_{\sigma_{1}} W_{\sigma_{2}} W_{\omega_{1}} W_{\omega_{2}} W_{\omega_{3}} W_{\omega_{4}} \rightarrow |W_{\sigma_{1}} W_{\sigma_{2}} W_{\omega_{1}} W_{\omega_{2}} W_{\omega_{3}} W_{\omega_{4}}| .
\end{align*}
\]

2. The minimal form factors of \( \mathcal{O}_{10;\pm;2}^{e} \) and \( \mathcal{O}_{10;\pm;3}^{\epsilon} \) (\( x = s, d \)) belong to \( T_{s+}^{[2]} \otimes \delta(\mathcal{E}_{1}^{[2]}) \). Besides, \( \mathcal{O}_{10;\pm;2}^{e} \) and \( \mathcal{O}_{10;\pm;3}^{\epsilon} \) (\( x = s, d \)) are second order total derivatives:

\[
\begin{align*}
\mathcal{O}_{10;\pm;2}^{e} & = D_{\mu \nu} \text{tr}(1234) \circ (4P_{\mu \nu} - 4P_{\mu \nu}^{c2}) , \quad \mathcal{O}_{10;\pm;3}^{\epsilon} = D_{\mu \nu} \text{tr}(12) \text{tr}(34) \circ (-2P_{\mu \nu} + 2P_{\mu \nu}^{c2}) , \\
& \quad \text{The corresponding choice of} \ x_{11} \text{and} \ x_{12} \text{introduced in (C.8) is} \ x_{11} = 1, x_{12} = -2 . \\
\end{align*}
\]

There is no remaining degree of freedom of total derivative operators, and \( \mathcal{O}_{10;\pm;3}^{\epsilon} \) have to be chosen artificially. It is permitted to take a shift \( \mathcal{O}_{10;\pm;3}^{e} \rightarrow \mathcal{O}_{10;\pm;3}^{e} + y \mathcal{O}_{10;\pm;2}^{e} \) for arbitrary rational \( y \).

\(^{15}\)The Laplacians \( \partial^2 = \partial_{\mu} \partial^{\mu} \) acting over Lorentz scalar operators are also included in this rank counting. For example, \( \partial^2 \mathcal{O} \) can be rewritten as \( \partial_\mu \partial_\nu g^{\mu \nu} \mathcal{O} \), where \( \mathcal{O} \) is a Lorentz scalar operator, so \( g^{\mu \nu} \) is the rank-2 tensor operator \( T^{\mu \nu} \) under discussion.
3. $O_{10s-1}^{e}$ is the only dim-10 C-odd operator. Its minimal form factor belongs to $T_{s}^{[2,1,1]} \otimes \mathcal{N}_{1}^{[2,1,1]}$. Besides, it can be written as a first order total derivative:

$$O_{10s-1}^{e} = D^{\mu} \text{tr}(1234) \circ (8Q_{\mu}^{e1} - 4D^{\nu}P_{\mu}^{e1}),$$

(D.4)

where

$$Q_{\mu}^{e1} = \frac{1}{16} \delta^{\rho_{1}\rho_{2}\rho_{3}\rho_{4}\rho_{5}}_{\sigma_{1}\sigma_{2}\sigma_{3}\sigma_{4}\sigma_{5}} [D_{\rho_{5}}F_{\rho_{1}\rho_{2}}, F_{\rho_{3}\rho_{4}}, F_{\sigma_{1}\sigma_{2}}, F_{\sigma_{3}\sigma_{4}}].$$

(D.5)

D.2 Evanescent primitive kinematic operators

For the choice of primitive kinematic operators $\{\mathcal{E}_{i}\}$ introduced in section 2.3.1, one can also consider total derivative operators. Another choice of primitive kinematic operators is given here, which also maintains $S_{4}$ permutation symmetry and in the form of $\delta$ functions contracting tensor operators. The difference between this new choice and the $\{\mathcal{E}_{i}\}$ given in 2.3.1 is that, it includes as many total derivative kinematic operators as basis.

Class 1. The same as $\mathcal{E}_{1,i}$ in (2.28).

Class 2. The same as $\mathcal{E}_{2,i}$ in (2.31), but we rewrite them in another form:

$$\mathcal{E}_{2,1} = \frac{1}{4} D_{\mu}D_{\nu} \delta^{\rho_{1}\rho_{2}\rho_{3}\rho_{4}\rho_{5}}_{\sigma_{1}\sigma_{2}\sigma_{3}\sigma_{4}\sigma_{5}} [D_{\rho_{5}}F_{\rho_{1}\rho_{2}}, F_{\rho_{3}\rho_{4}}, D_{\sigma_{5}}F_{\sigma_{1}\sigma_{2}}, F_{\sigma_{3}\sigma_{4}}],$$

$$\mathcal{E}_{2,2} = \mathcal{E}_{2,1} |_{W_{1},W_{2},W_{3},W_{4}} \rightarrow [W_{1},W_{2},W_{3},W_{4}].$$

(D.6)

Class 3: $\tilde{\mathcal{E}}_{3,i}$. The new class 3 kinematic operators are given by

$$\tilde{\mathcal{E}}_{3,1} = \frac{1}{4} D_{\mu} \delta^{\rho_{1}\rho_{2}\rho_{3}\rho_{4}\rho_{5}}_{\sigma_{1}\sigma_{2}\sigma_{3}\sigma_{4}\sigma_{5}} [D_{\sigma_{4}}F_{\rho_{1}\rho_{2}}, F_{\sigma_{3}\lambda}, D_{\rho_{5}}F_{\sigma_{1}\sigma_{2}}, D_{\rho_{4}}F_{\rho_{3}\rho_{5}}],$$

(D.7)

together with its permutations. They differ from $\mathcal{E}_{3,i}$ in (2.33) by sums of $DD$ insertions of $\mathcal{E}_{1,i}$, and the permutation relations between $\tilde{\mathcal{E}}_{3,i}$ and $\tilde{\mathcal{E}}_{3,1}$ are the same as (A.7).

Class 4. The same as $\mathcal{E}_{4}$ in (2.35), but we rewrite them in another form:

$$\mathcal{E}_{4} = \frac{1}{4} D_{\mu} \delta^{\rho_{1}\rho_{2}\rho_{3}\rho_{4}\rho_{5}}_{\sigma_{1}\sigma_{2}\sigma_{3}\sigma_{4}\sigma_{5}} [D_{\sigma_{4}}F_{\rho_{1}\rho_{2}}, F_{\rho_{3}\lambda}, D_{\rho_{5}}F_{\sigma_{1}\sigma_{2}}, D_{\rho_{4}}F_{\sigma_{3}\sigma_{4}}] -(\rho_{3} \leftrightarrow \sigma_{3})+(S_{4}\text{-permutations}).$$

(D.8)

E Dim-12 length-4 evanescent operators

In this section we give the complete basis evanescent operators with mass dimension 12 and length four. As counted in table 2, there are 25 single-trace ones and 16 double-trace ones. The 25 single-trace and 16 double-trace minimal evanescent operators can be classified into four classes according to the primitive kinematic operators from which they are generated.

Here we clarify the notation of operators: $O_{12s;i}^{e}$ stands for the $i$-th dim-12 length-4 evanescent single-trace operator, and $O_{12d;i}^{e}$ stands for the $i$-th dim-12 length-4 evanescent double-trace operator. Same as declared in section 2.1, we abbreviate products of covariant derivatives like $D_{i}D_{j}D_{k} \ldots$ to $D_{ijk} \ldots$ for simplicity. We also omit the $(-ig)^{2}$ in front of each length-4 operator for short.
E.1 25 single-trace operators

Twenty single-trace operators are generated by $\mathcal{E}_{1,i}$, $i = 1, \ldots, 12$. As declared in (2.24), the symbol $s_{ab}$ refers to inserting one pair of identical $D$s into the $a$-th and $b$-th sites. The inserted $D$s must be located in front of the $D$s of the primitive operators, otherwise the operators will become not exactly evanescent but only minimally evanescent.

\[
\begin{align*}
O_{12s;1}^c &= \text{tr}(1234) \circ s_{24} \mathcal{E}_{1,1}, \\
O_{12s;3}^c &= \text{tr}(1234) \circ s_{24} \mathcal{E}_{1,4}, \\
O_{12s;5}^c &= \text{tr}(1234) \circ s_{14} \mathcal{E}_{1,1}, \\
O_{12s;7}^c &= \text{tr}(1234) \circ s_{23} \mathcal{E}_{1,5}, \\
O_{12s;9}^c &= \text{tr}(1234) \circ s_{34} \mathcal{E}_{1,2}, \\
O_{12s;11}^c &= \text{tr}(1234) \circ s_{12} \mathcal{E}_{1,1}, \\
O_{12s;13}^c &= \text{tr}(1234) \circ s_{13} \mathcal{E}_{1,2}, \\
O_{12s;15}^c &= \text{tr}(1234) \circ s_{13} \mathcal{E}_{1,5}, \\
O_{12s;17}^c &= \text{tr}(1234) \circ s_{23} \mathcal{E}_{1,2}, \\
O_{12s;18}^c &= \text{tr}(1234) \circ s_{12} \mathcal{E}_{1,5}, \\
O_{12s;19}^c &= \text{tr}(1234) \circ s_{14} \mathcal{E}_{1,2}, \\
O_{12s;20}^c &= \text{tr}(1234) \circ s_{34} \mathcal{E}_{1,5}. \\
\end{align*}
\] (E.1)

One single-trace operator is generated by $\mathcal{E}_{2,i}$, $i = 1, 2$.

\[
O_{12s;21}^c = \text{tr}(1234) \circ \mathcal{E}_{2,1}. \tag{E.2}
\]

One single-trace operator is generated by kinematic operator $\mathcal{E}_4$.

\[
O_{12s;22}^c = \text{tr}(1234) \circ \mathcal{E}_4. \tag{E.3}
\]

Three single-trace operators are generated by kinematic operators $\mathcal{E}_{3,i}$, $i = 1, \ldots, 12$.

\[
\begin{align*}
O_{12s;23}^c &= \text{tr}(1234) \circ \mathcal{E}_{3,2}, \\
O_{12s;24}^c &= \text{tr}(1234) \circ \mathcal{E}_{3,3}, \\
O_{12s;25}^c &= \text{tr}(1234) \circ (\mathcal{E}_{3,1} + \mathcal{E}_{3,4}). \tag{E.4}
\end{align*}
\]

The 25 dim-12 single-trace length-4 basis evanescent operators can be recombined to 16 $C$-even ones and 9 $C$-odd ones:

\[
\begin{align*}
even : & \quad O_{12s;1}^c + O_{12s;2}^c, \quad O_{12s;3}^c + O_{12s;4}^c, \quad O_{12s;5}^c + O_{12s;6}^c, \quad O_{12s;7}^c + O_{12s;8}^c, \\
       & \quad O_{12s;9}^c + O_{12s;10}^c, \quad O_{12s;11}^c + O_{12s;12}^c, \quad O_{12s;13}^c + O_{12s;14}^c, \quad O_{12s;15}^c + O_{12s;16}^c, \\
       & \quad O_{12s;17}^c + O_{12s;18}^c, \quad O_{12s;19}^c + O_{12s;20}^c, \quad O_{12s;21}^c + O_{12s;22}^c, \quad O_{12s;23}^c + O_{12s;24}^c, \quad O_{12s;25}^c; \tag{E.5} \\
odd : & \quad O_{12s;1}^c - O_{12s;2}^c, \quad O_{12s;3}^c - O_{12s;4}^c, \quad O_{12s;5}^c - O_{12s;6}^c, \quad O_{12s;7}^c - O_{12s;8}^c, \\
       & \quad O_{12s;9}^c - O_{12s;10}^c, \quad O_{12s;11}^c - O_{12s;12}^c, \quad O_{12s;13}^c - O_{12s;14}^c, \quad O_{12s;15}^c - O_{12s;16}^c, \\
       & \quad O_{12s;23}^c - O_{12s;24}^c. \tag{E.6}
\end{align*}
\]
E.2 16 double-trace operators

Twelve double-trace operators are generated by $E_{1,i}$, $i = 1, \ldots, 12$. Similar to the single-trace case, the inserted $D$s must be located in front of the $D$s of the primitive operators, otherwise the operators will become not exactly evanescent but only minimally evanescent.

\[
\begin{align*}
\mathcal{O}^{e}_{12;d;1} &= \text{tr}(12)\text{tr}(34) \circ s_{34}\mathcal{E}_{1,5}, \\
\mathcal{O}^{e}_{12;d;2} &= \text{tr}(12)\text{tr}(34) \circ s_{12}\mathcal{E}_{1,5}, \\
\mathcal{O}^{e}_{12;d;3} &= \text{tr}(12)\text{tr}(34) \circ s_{14}\mathcal{E}_{1,5}, \\
\mathcal{O}^{e}_{12;d;4} &= \text{tr}(12)\text{tr}(34) \circ s_{14}\mathcal{E}_{1,1}, \\
\mathcal{O}^{e}_{12;d;5} &= \text{tr}(12)\text{tr}(34) \circ s_{13}\mathcal{E}_{1,5}, \\
\mathcal{O}^{e}_{12;d;6} &= \text{tr}(12)\text{tr}(34) \circ s_{13}\mathcal{E}_{1,1}, \\
\mathcal{O}^{e}_{12;d;7} &= \text{tr}(12)\text{tr}(34) \circ s_{12}\mathcal{E}_{1,6}, \\
\mathcal{O}^{e}_{12;d;8} &= \text{tr}(12)\text{tr}(34) \circ s_{12}\mathcal{E}_{1,1}, \\
\mathcal{O}^{e}_{12;d;9} &= \frac{1}{2}\text{tr}(12)\text{tr}(34) \circ s_{23}\mathcal{E}_{1,6}, \\
\mathcal{O}^{e}_{12;d;10} &= \frac{1}{2}\text{tr}(12)\text{tr}(34) \circ s_{13}\mathcal{E}_{1,1}, \\
\mathcal{O}^{e}_{12;d;11} &= \frac{1}{2}\text{tr}(12)\text{tr}(34) \circ s_{14}\mathcal{E}_{1,6}, \\
\mathcal{O}^{e}_{12;d;12} &= \frac{1}{2}\text{tr}(12)\text{tr}(34) \circ s_{24}\mathcal{E}_{1,1}. \\
\end{align*}
\]

One double-trace operator is generated by $E_{2,i}$, $i = 1, 2$.

\[\mathcal{O}^{e}_{12;d;13} = \frac{1}{2}\text{tr}(12)\text{tr}(34) \circ E_{2,2}.\]

One double-trace operator is generated by $E_{4}$.

\[\mathcal{O}^{e}_{12;d;14} = \frac{1}{2}\text{tr}(12)\text{tr}(34) \circ E_{4}.\]

Two double-trace operators are generated by $E_{3,i}$, $i = 1, \ldots, 12$.

\[
\begin{align*}
\mathcal{O}^{e}_{12;d;15} &= \text{tr}(12)\text{tr}(34) \circ E_{3,7}, \\
\mathcal{O}^{e}_{12;d;16} &= \text{tr}(12)\text{tr}(34) \circ E_{3,5}. \\
\end{align*}
\]

The double-trace operators have one-to-one correspondence to $C$-even single-trace operators. The single-double pair like (2.55) are

\[
\begin{align*}
(\mathcal{O}^{c}_{12;s;1} + \mathcal{O}^{c}_{12;s;2}, \mathcal{O}^{c}_{12;d;1}), 
(\mathcal{O}^{c}_{12;s;3} + \mathcal{O}^{c}_{12;s;4}, \mathcal{O}^{c}_{12;d;2}), 
(\mathcal{O}^{c}_{12;s;5} + \mathcal{O}^{c}_{12;s;6}, \mathcal{O}^{c}_{12;d;3}), 
(\mathcal{O}^{c}_{12;s;7} + \mathcal{O}^{c}_{12;s;8}, \mathcal{O}^{c}_{12;d;4}), 
(\mathcal{O}^{c}_{12;s;9} + \mathcal{O}^{c}_{12;s;10}, \mathcal{O}^{c}_{12;d;5}), 
(\mathcal{O}^{c}_{12;s;11} + \mathcal{O}^{c}_{12;s;12}, \mathcal{O}^{c}_{12;d;6}), 
(\mathcal{O}^{c}_{12;s;13} + \mathcal{O}^{c}_{12;s;14}, \mathcal{O}^{c}_{12;d;7}), 
(\mathcal{O}^{c}_{12;s;15} + \mathcal{O}^{c}_{12;s;16}, \mathcal{O}^{c}_{12;d;8}), 
(\mathcal{O}^{c}_{12;s;17} + \mathcal{O}^{c}_{12;d;9}), 
(\mathcal{O}^{c}_{12;s;18}, \mathcal{O}^{c}_{12;d;10}), 
(\mathcal{O}^{c}_{12;s;19}, \mathcal{O}^{c}_{12;d;11}), 
(\mathcal{O}^{c}_{12;s;20}, \mathcal{O}^{c}_{12;d;12}), 
(\mathcal{O}^{c}_{12;s;21}, \mathcal{O}^{c}_{12;d;13}), 
(\mathcal{O}^{c}_{12;s;22}, \mathcal{O}^{c}_{12;d;14}), 
(\mathcal{O}^{c}_{12;s;23} + \mathcal{O}_{12;e;24}, \mathcal{O}^{c}_{12;d;15}), 
(\mathcal{O}^{c}_{12;s;25}, \mathcal{O}^{c}_{12;d;16}) \\
\end{align*}
\]

F  Tree-level form factors

In this appendix, we give the compact expressions of tree-level minimal, next-to-minimal, and next-to-next-to-minimal form factors which are valid in general $d$ dimensions. The operators we consider are in the form of (2.2):

\[c(a_1, \ldots, a_n)(D_{\mu_1} \ldots D_{\mu_{m_1}} F_{\nu_1 \rho_1})^{a_1} \ldots (D_{\mu_n} \ldots D_{\mu_{m_n}} F_{\nu_n \rho_n})^{a_n},\]
with all $\mu, \nu, \rho$ contracted. In each site, there is a site operator like $W \sim D \ldots DF$. We will develop some useful rules for these site operators, such that the tree-level form factors can be efficiently read from them.

In following context, we write the sum $p_i + p_j + \ldots + p_k$ as $p_{ij..k}$ for short. Our results are for color-stripped form factors, where the color decomposition is based on the convention $f^{abc} = -2i\text{tr}([t^a, t^b]t^c)$, and the normalization of trace is $\text{tr}(t^a t^b) = \frac{1}{2} \delta^{ab}$.

**Minimal form factors.** How to read the minimal form factor from an operator has been illustrated in (2.10). For convenience, let us denote the polynomial read from the site operator $W_j$ which emits gluon $i$ by $\mathcal{F}_i(W_j)$.

\[
\mathcal{F}_i(D^{a_1} D^{a_2} \ldots D^{a_n} F^{\mu\nu}) = (ip^1_\mu)(ip^2_\mu) \ldots (ip^n_\mu) \varphi_i^{\mu\nu}
\]  
(F.1)

where $\varphi_i^{\mu\nu}$ is defined as

\[
\varphi_i^{\mu\nu} = \mathcal{F}_i(F^{\mu\nu}) = i(p^1_\mu e^\nu_i - p^\nu_\mu e^\mu_i).
\]  
(F.2)

In such notation, (2.10) corresponds to the special case where $i = j$. The minimal form factor is obtained by summing over all the site-gluon distributions that are permitted by the color factor. For example, the color-ordered minimal form factors of single-trace and double-trace length-4 operators are

\[
\mathcal{F}_{4;C}^{(0)}(p_1, p_2, p_3, p_4) = \sum_{\sigma \in S_4} \prod_{i=1}^4 \mathcal{F}_{i\sigma}(W_i), \quad \mathcal{F}_{4;\check{C}}^{(0)}(p_1, p_2 | p_3, p_4) = \sum_{\sigma \in H_1} \prod_{i=1}^4 \mathcal{F}_{i\sigma}(W_i).
\]  
(F.3)

Here $H_1$ is the order-8 subgroup of $S_4$ generated by $(1 \leftrightarrow 2), (3 \leftrightarrow 4)$ and $(1 \leftrightarrow 3, 2 \leftrightarrow 4)$.

**Next-to-minimal form factors.** For the case of next-to-minimal form factor, one site emits two gluons and each of the other $L - 1$ sites emits one gluon. Denote the polynomial read from the site operator $W_k$ which emits two gluons $\{i, j\}$ by $\mathcal{F}_{i,j}(W_k).$ A site operator can emit two gluons through two ways, so $\mathcal{F}_{i,j}(W_k)$ can be decomposed into two parts:

\[
\mathcal{F}_{i,j}(W_k) \equiv \mathcal{F}_{i,j}^1(W_k) + \mathcal{F}_{i,j}^2(W_k).
\]  
(F.4)

1. Polynomial $\mathcal{F}_{i,j}^1(W_k)$ is read from the contribution where two gluons $i, j$ are both emitted by the field strength $F$ in $W_k$:

\[
\mathcal{F}_{i,j}^1(D^{a_1} D^{a_2} \ldots D^{a_n} F^{\mu\nu}) = (ip^1_\mu)(ip^2_\mu) \ldots (ip^n_\mu) \varphi_{(ij)}^{\mu\nu},
\]  
(F.5)

where $\varphi_{(ij)}^{\mu\nu}$ is defined as

\[
\varphi_{(ij)}^{\mu\nu} := (-i g) \left(-2 p^\mu_{ij} e^\nu_{(ij)} + 2 p^\nu_{ij} e^\mu_{(ij)} + (e^\mu_{(i)} e^\nu_{(j)} - e^\nu_{(i)} e^\mu_{(j)})\right),
\]  
(F.6)

\[
e^\mu_{(ij)} := \frac{1}{p^2_{ij}} \left[(e_i \cdot p_j)e^\mu_j - (e_j \cdot p_i)e^\mu_i + \frac{1}{2} e_i \cdot e_j (p_i - p_j)^\mu\right].
\]

So $\varphi_{(ij)}^{\mu\nu}$ is antisymmetric under $\mu \leftrightarrow \nu$ and $i \leftrightarrow j$. 

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---
2. Polynomial \( \mathfrak{F}_{i,j}^2(W_k) \) is read from the contribution where one of the two gluons \( i, j \) is emitted by the \( F \) and the other is emitted by one \( D \) in \( W_k \):

\[
\mathfrak{F}_{i,j}^2(D^{\mu_1} D^{\mu_2} \ldots D^{\mu_n} F^{\mu\nu}) = (-i g) \sum_{m=1}^{n} \left( (i p_{ij}^{\mu_1}) \ldots (i p_{ij}^{\mu_{m-1}}) e_i^{\mu_m} (i p_{ij}^{\mu_{m+1}}) \ldots (i p_{ij}^{\mu_n}) \varphi_{ij}^{\mu\nu} - (i \leftrightarrow j) \right).
\]

The next-to-minimal form factor is obtained by distributing \( L + 1 \) gluons to \( L \) sites, and summing over all the distributions permitted by color factors. For example, the color-ordered next-to-minimal form factors of single-trace and double-trace length-4 operators are

\[
\mathcal{F}_{5,\sigma_1}^{(0)}(p_1, p_2, p_3, p_4, p_5) = \sum_{\sigma \in Z_2} \sum_{i=1}^{4} \mathfrak{F}_{\sigma(i), \sigma(i+1)}(W_i) \left[ \prod_{j=1}^{i-1} \mathfrak{F}_{\sigma(j)}(W_j) \right] \left[ \prod_{j=i+1}^{4} \mathfrak{F}_{\sigma(j+1)}(W_j) \right],
\]

\[
\mathcal{F}_{5,\sigma_2}^{(0)}(p_1, p_2 | p_3, p_4, p_5) = \sum_{\tau \in Z_2} \sum_{\sigma \in Z_3} \left( \mathfrak{F}_{\tau(1)}(W_3) \mathfrak{F}_{\tau(2)}(W_4) \left[ \mathfrak{F}_{\sigma(3), \sigma(4)}(W_1) \mathfrak{F}_{\sigma(5)}(W_2) + \mathfrak{F}_{\sigma(3), \sigma(4)}(W_2) \mathfrak{F}_{\sigma(5)}(W_1) \right] + \mathfrak{F}_{\tau(1)}(W_1) \mathfrak{F}_{\tau(2)}(W_2) \left[ \mathfrak{F}_{\sigma(3), \sigma(4)}(W_3) \mathfrak{F}_{\sigma(5)}(W_4) + \mathfrak{F}_{\sigma(3), \sigma(4)}(W_4) \mathfrak{F}_{\sigma(5)}(W_3) \right] \right).
\]

**Next-to-next-to-minimal form factors.** For the case of next-to-next-to-minimal form factor, the possible site-gluon distributions can be classified into two types:

1. One site emits three gluons and each of the other \( L - 1 \) sites emits one gluon.
2. Two sites emit two gluons and each of the other \( L - 2 \) sites emits one gluon.

Denote the polynomial read from the site operator \( W_m \) which emits three gluons \( \{i, j, k\} \) by \( \mathfrak{F}_{i,j,k}(W_m) \). A site operator can emit three gluons through four ways, so \( \mathfrak{F}_{i,j,k}(W_n) \) can be decomposed into four parts:

\[
\mathfrak{F}_{i,j,k}(W_m) = \mathfrak{F}_{i,j,k}^1(W_m) + \mathfrak{F}_{i,j,k}^2(W_m) + \mathfrak{F}_{i,j,k}^3(W_m) + \mathfrak{F}_{i,j,k}(W_m).
\]

1. Polynomial \( \mathfrak{F}_{i,j,k}^1(W_m) \) is read from the contribution where all three gluon \( i, j, k \) are emitted by the field strength \( F \) in \( W_m \):

\[
\mathfrak{F}_{i,j,k}^1(D^{\mu_1} D^{\mu_2} \ldots D^{\mu_n} F^{\mu\nu}) = (i p_{i,j,k}^{\mu_1})(i p_{i,j,k}^{\mu_2}) \ldots (i p_{i,j,k}^{\mu_n}) \varphi_{i,j,k}^{\mu\nu},
\]

where \( \varphi_{i,j,k}^{\mu\nu} \) is defined as

\[
\varphi_{i,j,k}^{\mu\nu} := -i g^2 \left[ p_{i,j,k}^{\mu_{i,j,k}} - p_{i,j,k}^{\nu_{i,j,k}} + 2 \left( e_i^{\mu} e_j^{\nu} e_k^{\mu} - e_i^{\mu} e_k^{\nu} e_j^{\mu} - e_j^{\mu} e_k^{\nu} e_i^{\mu} + e_j^{\mu} e_i^{\nu} e_k^{\mu} \right) \right],
\]

\[
e_{i,j,k}^{\mu} := \frac{1}{p_{i,j,k}^{\mu}} \left[ (e_i \cdot e_j) e_k^{\mu} + (e_j \cdot e_k) e_i^{\mu} - 2(e_i \cdot e_k) e_j^{\mu} - e_{i,j,k}^{\mu} \right],
\]

\[
e_{i,j,k}^{\mu} := \frac{1}{p_{i,j,k}^{\mu}} \left[ 4(e_i \cdot p_k) e_k^{\mu} - 4(e_i \cdot p_j) e_j^{\mu} + 2(e_i \cdot p_j) e_j^{\mu} - 2(e_i \cdot p_k) e_k^{\mu} - 2(e_i \cdot p_k) e_k^{\mu} + 2(e_i \cdot p_j) e_j^{\mu} - 2(e_i \cdot p_k) e_k^{\mu} - 2(e_i \cdot p_j) e_j^{\mu} - 2(e_i \cdot p_k) e_k^{\mu} \right].
\]

So \( \varphi_{i,j,k}^{\mu\nu} \) is antisymmetric under \( \mu \leftrightarrow \nu \) and \( i \leftrightarrow k \).
2. Polynomial $\tilde{h}^2_{i,j,k}(W_m)$ is read from the contribution where two of gluon $i, j, k$ are emitted by the $F$ and the other one is emitted by one $D$ in $W_m$:

$$\tilde{h}^2_{i,j,k}(D^{\rho_1} D^{\rho_2} \ldots D^{\rho_n} F^{\mu \nu}) = -i g \sum_{m=1}^{n} \left( p_{ij}^{\rho_1} \cdots p_{ijk}^{\rho_{m-1}} e_i^\rho^m p_{ik}^{\rho_{m+1}} \cdots p_{jk}^{\rho_n} \phi_{ijk}^{\mu \nu} - (k \rightarrow j \rightarrow i \rightarrow k) \right).$$

(F.12)

3. Polynomial $\tilde{h}^3_{i,j,k}(W_m)$ is read from the contribution where one of gluon $i, j, k$ is emitted by the $F$ and the other two are emitted by one $D$ in $W_m$:

$$\tilde{h}^3_{i,j,k}(D^{\rho_1} D^{\rho_2} \ldots D^{\rho_n} F^{\mu \nu}) = -i g^2 \sum_{r<m} \left( p_{ij}^{\rho_1} \cdots p_{ijk}^{\rho_{r-1}} e_i^\rho^r p_k^{\rho_{r+1}} \cdots p_{jk}^{\rho_n} \phi_{ijk}^{\mu \nu} - (i \leftrightarrow j) - (i \rightarrow j \rightarrow k \rightarrow i) + (i \leftrightarrow k) \right).$$

(F.13)

4. Polynomial $\tilde{h}^4_{i,j,k}(W_m)$ is read from the contribution where one of gluon $i, j, k$ is emitted by the $F$ and the other two are emitted by two $D$s in $W_m$:

$$\tilde{h}^4_{i,j,k}(D^{\rho_1} D^{\rho_2} \ldots D^{\rho_n} F^{\mu \nu}) = -g^2 \sum_{r<m} \left( p_{ij}^{\rho_1} \cdots p_{ijk}^{\rho_{r-1}} e_i^\rho^r p_k^{\rho_{r+1}} \cdots p_{jk}^{\rho_n} \phi_{ijk}^{\mu \nu} - (j \rightarrow k) - (k \rightarrow j \rightarrow k \rightarrow i) + (i \leftrightarrow k) \right).$$

(F.14)

The next-next-to-minimal form factor is obtained by distributing $L + 2$ gluons to $L$ sites, and summing over all the distributions permitted by color factors. For example, the color-ordered next-next-to-minimal form factor of a single-trace length-4 operator is

$$F^{(0)}_{6,4,2} (p_1, p_2, p_3, p_4, p_5, p_6) = \sum_{\sigma \in Z_2} \sum_{i=1}^{4} \tilde{h}_{\sigma(i),\sigma(i+1),\sigma(i+2)}(W_i) \left[ \prod_{j=1}^{i-1} \tilde{h}_{\sigma(j)}(W_j) \right] \left[ \prod_{j=i+1}^{4} \tilde{h}_{\sigma(j+2)}(W_j) \right]$$

$$+ \sum_{\sigma \in Z_2} \sum_{j=2}^{4} \sum_{i=1}^{j-1} \tilde{h}_{\sigma(i),\sigma(i+1),\sigma(j+1),\sigma(j+2)}(W_i \sigma(j+1), \sigma(j+2)) \left[ \prod_{k=1}^{i-1} \tilde{h}_{\sigma(k)}(W_k) \right] \left[ \prod_{m=i+1}^{j-1} \tilde{h}_{\sigma(m+1)}(W_m) \right]$$

$$\times \left[ \prod_{n=j+1}^{4} \tilde{h}_{\sigma(n+2)}(W_n) \right].$$

(F.15)

For a double-trace length-4 operator, there are two types of color factors appearing in next-next-to-minimal form factors. One is like $\text{tr}(12)\text{tr}(3456)$ and the other is like $\text{tr}(123)\text{tr}(456)$. The corresponding color ordered form factors are

$$F^{(0)}_{6,4,2} (p_1, p_2, p_3, p_4, p_5, p_6)$$

$$= \sum_{\sigma \in Z_4} \sum_{\tau \in Z_2} \left[ \tilde{h}_{\sigma(3),\sigma(4),\sigma(5)}(W_1) \tilde{h}_{\sigma(6)}(W_2) + \tilde{h}_{\sigma(3),\sigma(4),\sigma(5)}(W_2) \tilde{h}_{\sigma(6)}(W_1) \right] \tilde{h}_{\tau(1)}(W_3) \tilde{h}_{\tau(2)}(W_4)$$

$$+ \left[ \tilde{h}_{\sigma(3),\sigma(4)}(W_1) \tilde{h}_{\sigma(6)}(W_2) + \tilde{h}_{\sigma(3),\sigma(4)}(W_2) \tilde{h}_{\sigma(6)}(W_1) \right] \tilde{h}_{\tau(1)}(W_3) \tilde{h}_{\tau(2)}(W_4)$$

$$+ \sum_{\sigma \in Z_2} \sum_{\tau \in Z_2} \left( \tilde{h}_{\sigma(3),\sigma(4)}(W_1) \tilde{h}_{\sigma(5),\sigma(6)}(W_2) \tilde{h}_{\tau(1)}(W_3) \tilde{h}_{\tau(2)}(W_4)$$

$$+ \tilde{h}_{\sigma(3),\sigma(4)}(W_3) \tilde{h}_{\sigma(5),\sigma(6)}(W_4) \tilde{h}_{\tau(1)}(W_1) \tilde{h}_{\tau(2)}(W_2) \right).$$
and

$$F_{6;O_4}^{(0)}(p_1, p_2, p_3 | p_4, p_5, p_6) = \sum_{\sigma \in Z_3} \sum_{\tau \in Z_3} \left( \tilde{F}_{\sigma(1),\sigma(2)}(W_1)\tilde{F}_{\sigma(3)}(W_2) + \tilde{F}_{\sigma(1),\sigma(2)}(W_2)\tilde{F}_{\sigma(3)}(W_1) \right)$$

$$\times \left[ \tilde{F}_{\sigma(4),\sigma(5)}(W_3)\tilde{F}_{\sigma(6)}(W_4) + \tilde{F}_{\sigma(4),\sigma(5)}(W_4)\tilde{F}_{\sigma(6)}(W_3) \right] + (1 \leftrightarrow 4, 2 \leftrightarrow 5, 3 \leftrightarrow 6).$$

\section{Color decomposition of one-loop form factors}

In this appendix we provide the details of the color decomposition associated with the unitarity cuts.

The basic idea has been explained around (3.4)–(3.5) in section 3.1. Under each cut channel, we can analyze the color structures of tree products, which are obtained by sewing up the color factors of tree blocks through completeness relation of Lie algebra

$$\sum_a T^a_{ij} T^a_{kl} = \frac{1}{2} \delta_{il} \delta_{jk} - \frac{1}{2N_c} \delta_{ij} \delta_{kl}.$$  \hfill (G.1)

Such an explicit example is given in (3.7). By comparing with the one-loop full result (3.4), one can express the cut of the full-color form factor in terms of color-ordered tree blocks.

A word about the notation: in this section we denote single-trace operators by $O_s$ and double-trace operators by $O_d$.

One can further simplify the cut expressions by applying the relations among color-ordered tree blocks, such as Kleiss-Kuijf relations \[45\] and reflection symmetry, so that only a small set of independent color-ordered tree blocks are left in the expressions of cut integrands. To be concrete, the Kleiss-Kuijf relations reduce the independent color-ordered components of full-color tree level four gluon and five gluon amplitudes to two and six respectively:

$$A_4^{(0)} = f^{12a} f^{34a} A_4^{(0)}(p_1, p_2, p_3, p_4) + f^{13a} f^{24a} A_4^{(0)}(p_1, p_3, p_2, p_4),$$

$$A_5^{(0)} = f^{12a} f^{34a} f^{1ab} A_5^{(0)}(p_1, p_2, p_3, p_4, p_5) + (S_3\text{-perms. of } \{3, 4, 5\}).$$

Reflection symmetry identifies tree-level color ordered form factors in the following way:

\begin{align*}
\text{C-even (odd) } O_s: & \quad \mathcal{F}_{4;O_s}^{(0)}(p_i, p_j, p_k, p_l) = \pm \mathcal{F}_{4;O_s}^{(0)}(p_i, p_k, p_j, p_l), \\
& \quad \mathcal{F}_{5;O_s}^{(0)}(p_i, p_j, p_k, p_l, p_m) = \mp \mathcal{F}_{5;O_s}^{(0)}(p_m, p_i, p_k, p_j, p_l), \\
\text{C-even (odd) } O_d: & \quad \mathcal{F}_{4;O_d}^{(0)}(p_i, p_j | p_k, p_l) = \pm \mathcal{F}_{4;O_d}^{(0)}(p_j, p_i | p_k, p_l), \\
& \quad \mathcal{F}_{5;O_d}^{(0)}(p_i, p_j | p_k, p_l, p_m) = \mp \mathcal{F}_{5;O_d}^{(0)}(p_j, p_i | p_m, p_l, p_k). \hfill (G.2)
\end{align*}

Moreover, if two cut integrands are related by renaming loop momenta as $l_1 \leftrightarrow l_2$, e.g. the products $\mathcal{F}_{4;O_s}^{(0)}(p_1, p_2, l_1, l_2)A_4^{(0)}(p_3, p_4, l_2, l_1)$ and $\mathcal{F}_{4;O_s}^{(0)}(p_1, p_2, l_2, l_1)A_4^{(0)}(p_3, p_4, l_1, l_2)$, one only needs to compute one of the two. After the above steps, one finds that there is only a small set of color-ordered cut integrands which are needed to obtained the full-color form factors. See also [46] for similar applications.
In the following context, we summarize the cuts of the full-color form factors in terms of these independent color-ordered cut integrands.

We focus on the form factors listed in (3.1), and the complete set of cuts are shown in figure 2. For both $x = s$ and $x = d$, $F^{(1)}_{4:O_x}$ are fully probed by cut (1) in figure 2, $F^{(1)}_{5:O_x}$ are fully probed by cut (2) and cut (3), and $F^{(1)}_{5:O_x}$ are fully probed by cut (3).

Consider first the cut (1) in the $s_{12}$-channel. The minimal form factor of a single-trace length-4 operator satisfies the relation:

$$F^{(1)}_{4:O_x} \big|_{s_{12}} = \left( C_{11} \times \begin{array}{c} \includegraphics[scale=0.5]{fig1.png} \end{array} \right) + \left( 3 \leftrightarrow 4 \right) \left( C_{12} \times \begin{array}{c} \includegraphics[scale=0.5]{fig2.png} \end{array} \right), \quad \text{(G.3)}$$

where

$$C_{11} = (1 + \text{sgn}_{O_x})\text{tr}(12)\text{tr}(34) + N_c(\text{tr}(1234) + \text{sgn}_{O_x}\text{tr}(1432)), \quad C_{12} = -\left(1 + \text{sgn}_{O_x}\right)(\text{tr}(13)\text{tr}(24) + \text{tr}(14)\text{tr}(23)), \quad \text{(G.4)}$$

and $\text{sgn}_{O_x}$ is equal to $(-1)^{\text{length}}$ times the sign change of the operator $O_x$ ($x = s, d$) under reflection of the trace, similar to (3.10). Each grey blob in the figures in (G.3) represents a color-ordered tree form factor or amplitudes, for example, the last figure in (G.3) is

$$\begin{array}{c} \includegraphics[scale=0.5]{fig3.png} \end{array} = \int dPS_{l_1, l_2} \sum_{\text{helicty}} \mathcal{F}^{(0)}_{4:O_x} (-l_1, p_3, -l_2, p_4) A_4^{(0)} (l_2, l_1, p_1, p_2). \quad \text{(G.5)}$$

By comparing (3.4) and the permutation sum of (G.3), one obtains each color-ordered one-loop form factors. As an example, we explain how the expression of $F^{(1)}_{4:O_x}$ given in (3.10) is obtained. From (G.4) one finds that the terms contributing to $\text{tr}(12)\text{tr}(34)$ are: the first graph in (G.3), the first graph after $(1 \leftrightarrow 4, 2 \leftrightarrow 3)$, the second graph after $(2 \leftrightarrow 3)$ or $(1 \leftrightarrow 4)$, the second graph after $(1 \leftrightarrow 3)$ or $(2 \leftrightarrow 4)$. Summing above contributions together gives (3.10).

A similar relation for the minimal form factor of a double-trace length-4 operator is:

$$F^{(1)}_{4:O_x} \big|_{s_{12}} = \left( C_{13} \times \begin{array}{c} \includegraphics[scale=0.5]{fig4.png} \end{array} \right) + \left( C_{14} \times \begin{array}{c} \includegraphics[scale=0.5]{fig5.png} \end{array} \right) + \left( 3 \leftrightarrow 4 \right), \quad \text{(G.6)}$$

where

$$C_{13} = 2N_c\text{tr}(12)\text{tr}(34), \quad C_{14} = \text{tr}(1234) + \text{tr}(1432) - \text{tr}(1324) - \text{tr}(1423). \quad \text{(G.7)}$$
In the form factor blobs, the dotted lines indicate that the tree form factor is associated to double-trace color factors, such that: a dotted line connecting gluons $i$ and $j$ means that one of the double traces is ranged from $i$ to $j$ clockwise. For example, the form factor tree blocks of the first and the second diagrams in (G.6) represent the color-stripped form factors $F_{4;0}(l_1, l_2 | p_3, p_4)$ and $F_{4;0}(p_4, l_1 | p_3, l_2)$, associated with $\text{tr}(l_1 l_2)\text{tr}(34)$ and $\text{tr}(4 l_1)\text{tr}(3 l_2)$, respectively.

Next we consider for the cut (2) with channel $s_{123}$. The next-to-minimal form factor of a single-trace length-4 operator satisfies the relation:

$$F^{(1)}_{5;O_s} |_{s_{123}} = \left( C_{21} \times \begin{array}{c} 5 \\ 4 \\ 1 \\ 2 \\ 3 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \right) + S_3 \text{ perms of } \{1, 2, 3\} + \text{perms of } \{1, 2, 3\} + (1 \leftrightarrow 2), \quad (G.8)$$

where

$$C_{21} = N_c \left( \text{tr}(12345) + \text{tr}(12)\text{tr}(345) + \text{tr}(13)\text{tr}(245) + \text{tr}(23)\text{tr}(145) - \text{tr}(45)\text{tr}(132) - \text{sgn}_O \times \text{reverse} \right),$$

$$C_{22} = \text{tr}(15)\text{tr}(243) + \text{tr}(25)\text{tr}(143) + \text{tr}(35)\text{tr}(142) - \text{tr}(14)\text{tr}(235) - \text{tr}(24)\text{tr}(135) - \text{tr}(34)\text{tr}(125) - \text{sgn}_O \times \text{reverse}. \quad (G.9)$$

For a double-trace length-4 operator, its next-to-minimal form factor satisfies the relation:

$$F^{(1)}_{5;O_d} |_{s_{123}} = \left( C_{23} \times \begin{array}{c} 5 \\ 4 \\ 1 \\ 2 \\ 3 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \right) + \text{perms of } \{1, 2, 3\} + (2 \leftrightarrow 3) + (1 \leftrightarrow 2), \quad (G.10)$$

where

$$C_{23} = N_c \left( \text{tr}(45)\text{tr}(123) - \text{tr}(45)\text{tr}(132) \right),$$

$$C_{24} = \text{tr}(12345) - \text{tr}(12435) - \text{tr}(13425) + \text{tr}(14325) - \text{reverse}. \quad (G.11)$$
Finally, we consider the cut (3) with channel $s_{23}$. The next-to-minimal (minimal) form factor of a single-trace length-4 (length-5) operator satisfies the relation:

$$F_{5;O_s}^{(1)}|_{s_{23}} = \left[ (C_{31} \times \begin{array}{c}
1 \\
5 \\
4 \\
3 \\
2 \\
\end{array} ) + (1 \leftrightarrow 5) + (4 \leftrightarrow 5) + (2 \leftrightarrow 3) \right]$$

$$+ \left[ (C_{32} \times \begin{array}{c}
1 \\
5 \\
4 \\
3 \\
2 \\
\end{array} ) + Z_3 \text{ perms of } \{1, 4, 5\} + (2 \leftrightarrow 3) \right] \right], \quad \text{(G.12)}$$

where

$$C_{31} = N_c tr(12345) + tr(23)tr(451) - sgn_{O_s} \times \text{ reverse},$$

$$C_{32} = tr(45)tr(132) - tr(13)tr(245) - tr(12)tr(345) - sgn_{O_s} \times \text{ reverse}. \quad \text{(G.13)}$$

For a double-trace length-4 (length-5) operator, the next-to-minimal (minimal) form factor satisfies the relation:

$$F_{5;O_d}^{(1)}|_{s_{23}} = \left[ (C_{33} \times \begin{array}{c}
1 \\
5 \\
4 \\
3 \\
2 \\
\end{array} ) \right] + \left[ (C_{34} \times \begin{array}{c}
1 \\
5 \\
4 \\
3 \\
2 \\
\end{array} ) \right] + (1 \leftrightarrow 4) + (1 \leftrightarrow 5)$$

$$+ \left[ (C_{35} \times \begin{array}{c}
1 \\
5 \\
4 \\
3 \\
2 \\
\end{array} ) \right] + (1 \leftrightarrow 5) + (1 \leftrightarrow 4 \leftrightarrow 5 \leftrightarrow 1) + (2 \leftrightarrow 3) \right], \quad \text{(G.14)}$$

where

$$C_{33} = 2N_c tr(23)tr(451) - sgn_{O_d} tr(23)tr(541),$$

$$C_{34} = tr(45)tr(123) - sgn_{O_d} tr(45)tr(132),$$

$$C_{35} = tr(12345) - tr(12453) - tr(13452) + tr(14532) - sgn_{O_d} \times \text{ reverse}. \quad \text{(G.15)}$$

H One-loop IR divergences of form factors

In this appendix, we provide some details of the infrared divergence used in section 3.2. In following context, the operator is considered to be either $C$-even or $C$-odd, i.e. it has sign change $sgn_{O} = (-1)^{\text{length}}$ or $-(1)^{\text{length}}$ under reflection of the trace.
The IR formula of one-loop 4-gluon form factors is (see e.g. [37]):

\[ I_{\text{IR}}^{(1)}(\epsilon)F^{(0)}_{4;\text{O}} = \left( \text{tr}(1234) + sgn_{\text{O}}\text{tr}(1432) \right) F_{\text{4;IR}}^{(1),s}(p_1, p_2, p_3, p_4) + \text{cyclic perm. of } \{2,3,4\} + \text{tr}(12)\text{tr}(34) F_{\text{4;IR}}^{(1),d}(p_1, p_2 | p_3, p_4) + \text{cyclic perm. of } \{2,3,4\}. \tag{H.1} \]

If \( \text{O} \) is a single-trace operator:

\[ F_{\text{4;IR}}^{(1),s}(p_1, p_2, p_3, p_4) = -N_c \varphi(\epsilon) \sum_{i=1}^{4} (-s_{ii+1})^{-\epsilon} F^{(0)}_{\text{4;O}}(p_1, p_2, p_3, p_4), \tag{H.2} \]

\[ F_{\text{4;IR}}^{(1),d}(p_1, p_2, p_3, p_4) = \varphi(\epsilon)(1 + sgn_{\text{O}}) \times \left[ (-s_{13})^{-\epsilon} + (-s_{24})^{-\epsilon} - (-s_{12})^{-\epsilon} - (-s_{34})^{-\epsilon} \right] F^{(0)}_{\text{4;O}}(p_1, p_2, p_3, p_4) + (3 \leftrightarrow 4), \]

where

\[ \varphi(\epsilon) = e^{\gamma_E \epsilon} \frac{1}{\Gamma(1-\epsilon)} \left( \frac{1}{\epsilon^2} + \frac{\beta_0}{2N_c\epsilon} \right). \tag{H.3} \]

If \( \text{O} \) is a double-trace operator, \( sgn_{\text{O}} \) must be equal to 1, and then

\[ F_{\text{4;IR}}^{(1),s}(p_1, p_2, p_3, p_4) = \varphi(\epsilon) \left( (-s_{13})^{-\epsilon} + (-s_{24})^{-\epsilon} - (-s_{12})^{-\epsilon} - (-s_{34})^{-\epsilon} \right) F^{(0)}_{\text{4;O}}(p_1, p_2 | p_3, p_4) + (2 \leftrightarrow 4), \]

\[ F_{\text{4;IR}}^{(1),d}(p_1, p_2 | p_3, p_4) = -2N_c \varphi(\epsilon) \left( (-s_{12})^{-\epsilon} + (-s_{34})^{-\epsilon} \right) F^{(0)}_{\text{4;O}}(p_1, p_2 | p_3, p_4). \tag{H.4} \]

The IR formula of one-loop 5-gluon form factors are given as:

\[ I_{\text{IR}}^{(1)}(\epsilon)F^{(0)}_{5;\text{O}} = \left[ \left( \text{tr}(12345) - sgn_{\text{O}}\text{tr}(15432) \right) F_{\text{5;IR}}^{(1),s}(p_1, p_2, p_3, p_4, p_5) + (2 \leftrightarrow 3) \right] + S_3\text{-perm. of } \{3,4,5\} \]

\[ + \left[ \left( \text{tr}(12)\text{tr}(345) - sgn_{\text{O}}\text{tr}(12)\text{tr}(354) \right) F_{\text{5;IR}}^{(1),d}(p_1, p_2 | p_3, p_4, p_5) + (2 \leftrightarrow 3) + (1 \rightarrow 2 \rightarrow 3 \rightarrow 1) + (1 \leftrightarrow 3, 2 \leftrightarrow 4) \right] + \text{cyclic perms. of } \{3,4,5\}. \tag{H.5} \]

If \( \text{O} \) is a single-trace operator:

\[ F_{\text{5;IR}}^{(1),s}(p_1, p_2, p_3, p_4, p_5) = -N_c \varphi(\epsilon) \sum_{i=1}^{5} (-s_{ii+1})^{-\epsilon} F^{(0)}_{\text{5;O}}(p_1, p_2, p_3, p_4, p_5), \]

\[ F_{\text{5;IR}}^{(1),d}(p_1, p_2 | p_3, p_4, p_5) = \varphi(\epsilon) \left[ (-s_{13})^{-\epsilon} + (-s_{25})^{-\epsilon} - (s_{12})^{-\epsilon} - (-s_{35})^{-\epsilon} \right] F^{(0)}_{\text{5;O}}(p_1, p_2 | p_3, p_4, p_5) + \text{cyclic perm. of } \{3,4,5\} + (1 \leftrightarrow 2). \tag{H.6} \]

If \( \text{O} \) is a double-trace operator:

\[ F_{\text{5;IR}}^{(1),s}(p_1, p_2, p_3, p_4, p_5) = \varphi(\epsilon) \left[ (-s_{13})^{-\epsilon} + (-s_{25})^{-\epsilon} - (s_{12})^{-\epsilon} - (-s_{35})^{-\epsilon} \right] F^{(0)}_{\text{5;O}}(p_1, p_2 | p_3, p_4, p_5) + \text{cyclic perm. of } \{1,2,3,4,5\}, \]
I Renormalization of dim-10 physical operators

In this appendix we give the renormalization of dim-10 length-4 and length-5 physical operators, which contains nonzero mixing to the evanescent ones. As a review, the renormalization of the dimension-four operator is well known, see e.g. [47], and one-loop renormalization for dimension 6 and 8 operators were considered in [48–52]. The two-loop renormalization for dimension-6 operators (also with quark operators) were given in [53, 54]. The two-loop renormalization for length-3 operators with dimension 8-16 are given in our recent work [15].

In appendix I.1 and I.2, we give the expressions of dim-10 basis physical operators of length four and length five. In appendix I.3, we give the renormalization matrix of physical operators as well as the anomalous dimensions. In appendix I.4, we consider the finite renormalization scheme which includes finite terms into the renormalization matrix to absorb the finite mixing from evanescent operators to the physical ones. In appendix I.5, we discuss the change in basis operators and renormalization matrix when $N_c$ is reduced to 2.

For the convenience of notation, we use integer numbers to represent Lorentz indices and abbreviate $D_i D_j \ldots$ as $D_{ij\ldots}$. The same integers mean the indices are contracted, such as $F_{12} F_{12} = F_{\mu\nu} F^{\mu\nu}$.

I.1 Dim-10 length-4 physical operators

For mass dimension 10, there are 20 single-trace and 15 double-trace physical basis operators with length four. We can reorganize them so that the form factor of each new basis operator is non-vanishing only under one type of helicities, either $(−)^4$ or $(−)^3(+)$ or $(−)^2(+)^2$. We call these three helicity types $\alpha, \beta, \gamma$ sectors respectively.

In table 6, we give the counting of the dim-10 length-4 operators, where the evanescent ones have been summarized in section 2.5. The concept of $C$-even (odd) has been introduced in section 2.3.2, and it stands for the operator has a sign change $+1$ $−1$ under the reflection of traces.

We list single-trace and double-trace operators separately. The operators are arranged according to descending orders of total derivatives, i.e. we first list the second order derivatives (denoted by $D^\mu T_{\mu\nu}$), then the first order ones (denoted by $D^\mu V_\mu$), then the zero order ones (denoted by $S$). We omit the $−i g^2$ in front of each length-4 operator for simplicity.

| $\alpha$ | $\beta$ | $\gamma$ | evanescent |
|----------|---------|---------|------------|
| single, C-even | 5 | 4 | 6 | 3 |
| single, C-odd | 1 | 2 | 2 | 1 |
| double, C-even | 5 | 4 | 6 | 3 |

Table 6. Counting of dim-10 length-4 operators.

\[
\mathcal{F}_{5;\mathcal{O}_d;\text{IR}}^{(1),d}(p_1, p_2 | p_3, p_4, p_5) = -N_c \varphi(\epsilon)(2(−s_{12})^\epsilon + (−s_{34})^\epsilon + (−s_{35})^\epsilon + (s_{45})^\epsilon) \mathcal{F}_{5;\mathcal{O}_d}^{(0)}(p_1, p_2 | p_3, p_4, p_5). \tag{H.7}
\]
\[ \begin{array}{|c|c|c|c|c|} \hline \alpha & \beta & \gamma & \text{evanescent} \\
\hline D^{\mu
u}T_{\mu\nu} & 2 & 2 & 3 & 2 \\
D^\mu V^\mu & 1 & 1 & 0 & 0 \\
S & 2 & 1 & 3 & 1 \\
\text{total} & 5 & 4 & 6 & 3 \\
\hline \end{array} \]

Table 7. Counting of dim-10 length-4 single-trace C-even operators.

**Single-trace, C-even.** The $\alpha$-sector:

\[
\mathcal{O}_{10,\alpha^{+}+1} = D^{\mu} \left[ \frac{1}{8} \text{tr}(F_{12}F_{13}F_{34}F_{43}) - \frac{1}{8} \text{tr}(F_{12}F_{34}F_{13}F_{42}) + \frac{1}{16} \text{tr}(F_{12}F_{34}F_{12}F_{34}) \right] \\
+ \frac{3}{8} \text{tr}(F_{12}F_{34}F_{23}F_{14}) ,
\]

\[
\mathcal{O}_{10,\alpha^{+}+2} = D^{\mu} \left[ - \frac{1}{2} \text{tr}(F_{12}F_{13}F_{34}F_{42}) - \text{tr}(F_{12}F_{34}F_{13}F_{42}) - \frac{1}{4} \text{tr}(F_{12}F_{34}F_{12}F_{34}) \right] ,
\]

\[
\mathcal{O}_{10,\alpha^{+}+3} = D^{\mu} \text{tr}(1234) \circ \left( \mathcal{Q}^{\alpha 1}_\mu + \frac{1}{4} D^{\nu} \mathcal{P}^{\nu 2}_{\mu\nu} \right) ,
\]

\[
\mathcal{O}_{10,\alpha^{+}+4} = \frac{3}{4} \text{tr}(F_{12}F_{13}D_5F_{34}F_{53}) + \frac{1}{4} \text{tr}(F_{12}D_5F_{13}D_5F_{34}) + 2 \text{tr}(F_{12}D_3F_{51}D_5F_{34}) \\
+ \text{tr}(F_{12}D_3F_{51}D_5F_{34}) + \frac{3}{4} \text{tr}(F_{12}D_3F_{51}D_5F_{34}) + 2 \text{tr}(F_{12}D_1F_{34}D_5F_{23}F_{45}) \\
- \text{tr}(F_{13}D_2F_{34}D_5F_{14}F_{53}) ,
\]

\[
\mathcal{O}_{10,\alpha^{+}+5} = - \frac{1}{2} \text{tr}(F_{12}D_3F_{51}D_5F_{34}) - \frac{1}{2} \text{tr}(F_{12}D_3F_{51}D_5F_{34}) - 3 \text{tr}(F_{12}D_3F_{13}F_{34}D_4F_{34}) \\
+ \text{tr}(F_{12}D_3F_{51}D_5F_{34}) - \frac{1}{2} \text{tr}(F_{12}D_3F_{51}D_5F_{34}) ,
\]

(1.1)

where $\mathcal{P}^{\nu 2}_{\mu\nu}$ is given in (D.2) and

\[
\mathcal{Q}^{\alpha 1}_\mu = - \frac{1}{2} \left[ F_{12}, F_{13}, F_{34}, F_{43} \right] - \left[ F_{13}, F_{24}, F_{34} \right] - \left[ F_{12}, F_{24}, F_{34} \right] - \left[ F_{13}, F_{24}, F_{34} \right] - \left[ F_{12}, F_{24}, F_{34} \right] - \left[ F_{13}, F_{24}, F_{34} \right] - \left[ F_{12}, F_{24}, F_{34} \right] \\
- \frac{1}{2} \left[ F_{34}, F_{12}, F_{13}, F_{24} \right] + \left[ F_{34}, F_{12}, F_{13}, F_{24} \right] + \left[ F_{34}, F_{12}, F_{13}, F_{24} \right] + \left[ F_{34}, F_{12}, F_{13}, F_{24} \right] .
\]

(1.2)

The $\beta$-sector:

\[
\mathcal{O}_{10,\beta^{+}+1} = D^{\mu} \text{tr}(1234) \circ \mathcal{P}^{\beta 1}_{\mu\nu} ,
\]

\[
\mathcal{O}_{10,\beta^{+}+2} = D^{\mu} \text{tr}(1234) \circ \left( - \frac{2}{3} \mathcal{P}^{\beta 1}_{\mu\nu} + \mathcal{P}^{\beta 2}_{\mu\nu} - \mathcal{P}^{\nu 2}_{\mu\nu} \right) ,
\]

\[
\mathcal{O}_{10,\beta^{+}+3} = D^{\mu} \text{tr}(1234) \circ \left( \mathcal{Q}^{\beta 1}_\mu + D^{\nu} \left( - \frac{1}{3} \mathcal{P}^{\beta 1}_{\mu\nu} + \frac{1}{2} \mathcal{P}^{\beta 4}_{\mu\nu} \right) \right) ,
\]

\[
\mathcal{O}_{10,\beta^{+}+4} = - \frac{1}{3} \text{tr}(F_{12}F_{51}F_{34}F_{23}F_{45}) + \frac{2}{3} \text{tr}(F_{13}D_1F_{54}F_{23}F_{45}) - \frac{2}{3} \text{tr}(F_{13}D_1F_{54}F_{23}F_{45}) \\
+ \frac{2}{3} \text{tr}(F_{13}D_2F_{45}F_{23}D_1F_{45}) ,
\]

(1.3)

where

\[
\mathcal{P}^{\beta 1}_{\mu\nu} = \frac{1}{2} \left[ F_{12}, F_{34}, F_{12}, F_{34} \right] + \left[ F_{13}, F_{24}, F_{34} \right] - \left[ F_{34}, F_{12}, F_{13}, F_{24} \right] = \frac{1}{2} \left[ F_{34}, F_{12}, F_{34}, F_{12} \right] \\
- \frac{1}{4} g_{\mu\nu} \left[ F_{34}, F_{12}, F_{34}, F_{12} \right] ,
\]
Table 8. Counting of dim-10 length-4 single-trace C-odd operators.

|   | \(\alpha\) | \(\beta\) | \(\gamma\) | evanescent |
|---|---|---|---|---|
| \(D^{\mu
\nu}T_{\mu\nu}\) | 0 | 1 | 0 | 0 |
| \(D^{\mu}V_{\mu}\) | 1 | 1 | 2 | 1 |
| \(S\) | 0 | 0 | 0 | 0 |
| total | 1 | 2 | 2 | 1 |

\[ p^{\mu \nu} = [F_{2\mu}, F_{12}, F_{13}, F_{3\nu}] + [F_{2\mu}, F_{3\nu}, F_{13}, F_{12}] + [F_{3\nu}, F_{13}, F_{12}, F_{2\mu}] + [F_{3\nu}, F_{2\mu}, F_{12}, F_{13}] + g_{\mu\nu}[F_{14}, F_{12}, F_{23}, F_{34}], \]

\[ p^{\mu \nu} = -\frac{1}{2} [F_{12}, F_{13}, F_{3\mu}, F_{3\nu}] + [F_{13}, F_{3\mu}, F_{12}, F_{2\mu}] + \frac{1}{2} [F_{3\nu}, F_{3\mu}, F_{12}, F_{13}], \]

\[ Q^{\mu 1}_\nu = [D_4 F_{12}, F_{34}, F_{13}, F_{2\mu}] + [D_4 F_{12}, F_{3\mu}, F_{13}, F_{24}] - [D_4 F_{2\mu}, F_{13}, F_{34}, F_{12}], \]  

\( (1.4) \)

The \(\gamma\)-sector:

\[ O_{10,7;3;+;1} = D^2 \left( \frac{1}{8} \text{tr}(F_{12} F_{34} F_{12} F_{34}) + \frac{1}{2} \text{tr}(F_{12} F_{34} F_{23} F_{14}) \right), \]

\[ O_{10,7;3;+;2} = D^2 \left( - \frac{1}{4} \text{tr}(F_{12} F_{12} F_{34} F_{34}) - \frac{1}{2} \text{tr}(F_{12} F_{23} F_{34} F_{14}) + \frac{1}{8} \text{tr}(F_{12} F_{34} F_{12} F_{34}) \right), \]

\[ O_{10,7;3;+;3} = D^{\mu\nu} \text{tr}(1234) \circ (\mathcal{P}^{\mu 1}_\nu + \mathcal{P}^{\nu 1}_\mu - \frac{1}{2} \mathcal{P}^{\mu \nu}), \]

\[ O_{10,7;3;+;4} = \frac{1}{4} \text{tr}(F_{12} F_{12} D_5 F_{34} D_5 F_{34}) + \frac{1}{4} \text{tr}(F_{12} D_5 F_{12} D_5 F_{34} F_{34}) + \text{tr}(F_{12} F_{23} D_5 F_{34} D_5 F_{14}) + \frac{1}{8} \text{tr}(F_{12} F_{34} D_5 F_{12} D_5 F_{34}), \]

\[ O_{10,7;3;+;5} = \text{tr}(F_{12} F_{23} D_5 F_{14} D_5 F_{34}) + \text{tr}(F_{12} D_5 F_{23} D_5 F_{14} F_{34}) + \frac{1}{2} \text{tr}(F_{12} F_{34} D_5 F_{12} D_5 F_{34}), \]

\[ O_{10,7;3;+;6} = \frac{7}{4} \text{tr}(F_{12} D_5 F_{12} D_5 F_{34} D_5 F_{34}) + \frac{5}{4} \text{tr}(F_{12} D_5 F_{12} D_5 F_{34} F_{34}) - 4 \text{tr}(F_{12} F_{23} D_5 F_{14} D_5 F_{34}) + \frac{5}{4} \text{tr}(F_{12} F_{34} D_5 F_{12} D_5 F_{34}) - \frac{3}{4} \text{tr}(F_{12} F_{34} D_5 F_{12} D_5 F_{34}) - 2 \text{tr}(F_{12} D_5 F_{13} D_5 F_{23} D_5 F_{24} F_{25}) + \text{tr}(F_{13} F_{23} D_5 F_{24} F_{25} D_5 F_{34}). \]  

\( (1.5) \)

\[ where \; \mathcal{P}^{\mu 1}_\nu \; is \; given \; in \; (D.2) \; and \]

\[ \mathcal{P}^{\mu 1}_\nu = -\frac{1}{4} [F_{12}, F_{12}, F_{3\mu}, F_{3\nu}] - \frac{1}{4} [F_{12}, F_{12}, F_{3\nu}, F_{3\mu}] + [F_{12}, F_{2\mu}, F_{13}, F_{3\nu}] - \frac{1}{4} [F_{12}, F_{3\mu}, F_{12}, F_{3\nu}] + \frac{1}{4} [F_{12}, F_{3\nu}, F_{12}, F_{3\mu}] + \frac{1}{4} [F_{12}, F_{3\mu}, F_{12}, F_{3\nu}] - \frac{1}{4} [F_{12}, F_{3\nu}, F_{12}, F_{3\mu}] - \frac{1}{4} [F_{12}, F_{3\nu}, F_{12}, F_{3\mu}] - \frac{1}{4} [F_{3\mu}, F_{12}, F_{2\mu}, F_{3\nu}] + \frac{1}{4} [F_{3\nu}, F_{12}, F_{2\mu}, F_{3\mu}] + \frac{1}{4} [F_{3\mu}, F_{12}, F_{2\mu}, F_{3\nu}] + \frac{1}{4} [F_{3\nu}, F_{12}, F_{2\mu}, F_{3\mu}] - \frac{3}{2} [F_{3\nu}, F_{3\mu}, F_{12}, F_{13}] + g_{\mu\nu} \left( \frac{1}{8} [F_{34}, F_{12}, F_{12}, F_{34}] - \frac{1}{8} [F_{34}, F_{12}, F_{34}, F_{12}] + \frac{1}{8} [F_{34}, F_{34}, F_{12}, F_{12}] \right). \]  

\( (1.6) \)

\textbf{Single-trace, C-odd.} \; The \(\alpha\)-sector:

\[ O_{10,\alpha;8;+;1} = D^{\mu} \text{tr}(1234) \circ \left( 2 Q^{\mu 1}_\mu + 2 Q^{\mu 2}_\mu + 2 Q^{\mu 1}_\mu - D^{\nu} \left( \frac{1}{2} \mathcal{P}^{\alpha 1}_{\mu \nu} + \frac{3}{2} \mathcal{P}^{\nu 1}_{\mu \nu} \right) \right), \]  

\( (1.7) \)
where $Q_\mu^{c1}$ is given in (D.5) and
\begin{align}
\mathcal{P}_{\mu\nu}^{\mu1} &= -\frac{1}{2} [F_{12}, F_{12}, F_{3\mu}, F_{3\nu}] + \frac{1}{2} [F_{12}, F_{13}, F_{2\mu}, F_{3\nu}] + \frac{1}{2} [F_{12}, F_{3\mu}, F_{3\nu}, F_{12}] - \frac{1}{2} [F_{12}, F_{3\mu}, F_{12}, F_{3\nu}] \\
&- \frac{1}{2} [F_{12}, F_{3\nu}, F_{3\mu}, F_{12}] + [F_{12}, F_{3\nu}, F_{2\mu}, F_{12}] + [F_{2\mu}, F_{3\nu}, F_{12}, F_{12}] - \frac{1}{2} [F_{3\mu}, F_{3\nu}, F_{12}, F_{2\mu}] \\
&- \frac{1}{2} [F_{3\nu}, F_{2\mu}, F_{12}] + [F_{3\nu}, F_{3\mu}, F_{12}, F_{2\mu}] - \frac{1}{4} g_{\alpha\beta} [F_{34}, F_{12}, F_{34}], \\
Q_\mu^{\alpha2} &= [D_4 F_{12}, F_{13}, F_{24}, F_{3\mu}] + [D_4 F_{12}, F_{13}, F_{3\mu}, F_{24}] - [D_4 F_{12}, F_{24}, F_{13}, F_{3\mu}] + [D_4 F_{12}, F_{2\mu}, F_{13}, F_{34}] \\
&- [D_4 F_{12}, F_{3\mu}, F_{13}, F_{24}] - [D_4 F_{12}, F_{2\mu}, F_{3\mu}, F_{24}, F_{13}] - [D_4 F_{2\mu}, F_{3\mu}, F_{13}, F_{13}] \\
&+ \frac{1}{2} [D_4 F_{3\mu}, F_{34}, F_{12}, F_{12}], \tag{1.8}
\end{align}

The $\beta$-sector:
\begin{align}
\mathcal{O}_{10;\beta\gamma_{-};1} &= D^{\mu\nu} \text{tr}(1234) \circ \mathcal{P}_{\mu\nu}^{\beta4}, \\
\mathcal{O}_{10;\beta\gamma_{-};2} &= D^{\mu\nu} \text{tr}(1234) \circ \left( -2 Q_\mu^{\beta1} - 2 Q_\mu^{\beta2} + D^\nu \left( -\mathcal{P}_{\mu\nu}^{\beta4} + \frac{1}{2} \mathcal{P}_{\mu\nu}^{\beta2} + \mathcal{P}_{\mu\nu}^{\beta1} - \frac{1}{2} \mathcal{P}_{\mu\nu}^{\beta2} \right) \right), \tag{1.9}
\end{align}

where
\begin{align}
Q_\mu^{\beta2} &= [D_4 F_{12}, F_{13}, F_{2\mu}, F_{34}] - [D_4 F_{12}, F_{3\mu}, F_{24}, F_{13}] - [D_4 F_{2\mu}, F_{3\mu}, F_{24}, F_{13}] \\
&+ \frac{1}{2} [D_4 F_{3\mu}, F_{12}, F_{34}, F_{12}], \tag{1.10}
\end{align}

The $\gamma$-sector:
\begin{align}
\mathcal{O}_{10;\gamma\beta_{-};1} &= D^{\mu\nu} \text{tr}(1234) \circ \left( 2 Q_\mu^{\gamma1} + \frac{1}{2} D^\nu \left( \mathcal{P}_{\mu\nu}^{\gamma1} - \mathcal{P}_{\mu\nu}^{\gamma2} - \mathcal{P}_{\mu\nu}^{\epsilon1} \right) \right), \\
\mathcal{O}_{10;\gamma\beta_{-};2} &= D^{\mu\nu} \text{tr}(1234) \circ \left( 2 Q_\mu^{\gamma2} + 2 Q_\mu^{\epsilon1} + \frac{1}{2} D^\nu \left( -\mathcal{P}_{\mu\nu}^{\gamma1} + \mathcal{P}_{\mu\nu}^{\gamma2} - 3 \mathcal{P}_{\mu\nu}^{\epsilon1} \right) \right), \tag{1.11}
\end{align}

where
\begin{align}
\mathcal{P}_{\mu\nu}^{\gamma2} &= -\frac{1}{4} [F_{12}, F_{12}, F_{3\mu}, F_{3\nu}] + \frac{1}{2} [F_{12}, F_{13}, F_{2\mu}, F_{3\nu}] + \frac{1}{2} [F_{12}, F_{3\mu}, F_{3\nu}, F_{12}] - \frac{1}{2} [F_{12}, F_{3\mu}, F_{12}, F_{3\nu}] \\
&- \frac{1}{4} [F_{12}, F_{3\nu}, F_{3\mu}, F_{12}] + \frac{1}{4} [F_{12}, F_{3\nu}, F_{2\mu}, F_{12}] + \frac{1}{4} [F_{12}, F_{3\nu}, F_{3\mu}, F_{12}] + \frac{1}{4} [F_{2\mu}, F_{3\nu}, F_{12}, F_{12}] \\
&+ \frac{1}{4} [F_{3\mu}, F_{12}, F_{2\mu}, F_{12}] - \frac{1}{4} [F_{3\mu}, F_{12}, F_{3\nu}, F_{12}] - \frac{1}{4} [F_{3\nu}, F_{2\mu}, F_{12}, F_{12}] - \frac{1}{4} [F_{3\nu}, F_{3\mu}, F_{12}, F_{12}] \\
&- \frac{1}{4} [F_{3\mu}, F_{3\nu}, F_{12}, F_{12}] + \frac{1}{4} [F_{3\nu}, F_{3\mu}, F_{12}, F_{12}] \\
&+ g_{\mu\nu} \left( \frac{1}{8} [F_{34}, F_{12}, F_{12}, F_{34}] + \frac{1}{8} [F_{34}, F_{12}, F_{34}, F_{12}] - \frac{1}{8} [F_{34}, F_{34}, F_{12}, F_{12}] \right), \\
Q_\mu^{\gamma1} &= -[D_4 F_{12}, F_{13}, F_{3\mu}, F_{24}] - [D_4 F_{13}, F_{24}, F_{3\mu}, F_{13}] + [D_4 F_{12}, F_{24}, F_{3\mu}, F_{13}] - [D_4 F_{2\mu}, F_{3\mu}, F_{24}, F_{13}] \\
&+ [D_4 F_{12}, F_{3\mu}, F_{13}, F_{24}] + [D_4 F_{12}, F_{3\mu}, F_{13}, F_{24}] - [D_4 F_{2\mu}, F_{3\mu}, F_{13}, F_{13}] \\
&- [D_4 F_{2\mu}, F_{3\mu}, F_{13}, F_{13}] + \frac{1}{2} [D_4 F_{3\mu}, F_{12}, F_{34}, F_{12}] \\
Q_\mu^{\gamma2} &= -[D_4 F_{12}, F_{24}, F_{3\mu}, F_{13}] + [D_4 F_{12}, F_{2\mu}, F_{3\mu}, F_{24}, F_{13}] - [D_4 F_{2\mu}, F_{3\mu}, F_{13}, F_{13}] \\
&+ \frac{1}{2} [D_4 F_{3\mu}, F_{34}, F_{12}, F_{12}]. \tag{1.12}
\end{align}
\[ O_{10,\alpha,d+1} = D^\mu \left( \frac{1}{16} \text{tr}(F_{12} F_{34})^2 - \frac{1}{16} \text{tr}(F_{12} F_{34}) \text{tr}(F_{23} F_{34}) + \frac{3}{16} \text{tr}(F_{12} F_{23}) \text{tr}(F_{34} F_{14}) + \frac{1}{32} \text{tr}(F_{12} F_{12}) \text{tr}(F_{34} F_{34}) \right), \]

\[ O_{10,\alpha,d+2} = D^\mu \left( -\frac{1}{4} \text{tr}(F_{12} F_{34})^2 - \frac{1}{2} \text{tr}(F_{12} F_{34}) \text{tr}(F_{23} F_{14}) - \frac{1}{8} \text{tr}(F_{12} F_{12}) \text{tr}(F_{34} F_{34}) \right), \]

\[ O_{10,\alpha,d+3} = D^\mu \text{tr}(12) \text{tr}(34) \circ \left( \frac{1}{2} Q^\mu_{\mu} + \frac{1}{8} D^\nu (P^\alpha_{\mu} + P^\nu_{\mu} - P^\rho_{\mu}) \right), \]

\[ O_{10,\alpha,d+4} = \frac{3}{8} \text{tr}(F_{12} D_{12} F_{34})^2 + \frac{1}{8} \text{tr}(F_{12} D_{12} F_{34}) \text{tr}(D_{5} D_{12} F_{34}) + \frac{1}{2} \text{tr}(F_{12} D_{12} F_{34}) \text{tr}(F_{23} D_{5} F_{14}) + \frac{3}{8} \text{tr}(F_{12} D_{5} F_{5}) \text{tr}(F_{34} D_{5} F_{34}) + \frac{1}{2} \text{tr}(F_{12} D_{5} F_{5}) \text{tr}(F_{34} D_{5} F_{34}) + \frac{1}{4} \text{tr}(F_{12} D_{5} F_{5}) \text{tr}(F_{34} D_{5} F_{34}). \tag{1.13} \]

The 13-sector:

\[ O_{10,\beta,d+1} = D^\mu \text{tr}(12) \text{tr}(34) \circ P^\beta_{\mu}, \]

\[ O_{10,\beta,d+2} = D^\mu \text{tr}(12) \text{tr}(34) \circ \left( - P^\beta_{\mu} - \frac{1}{3} P^\beta_{\mu} \right), \]

\[ O_{10,\beta,d+3} = D^\mu \text{tr}(12) \text{tr}(34) \circ \left( - \frac{1}{2} Q^\beta_{\mu} + \frac{1}{4} D^\nu (- P^\beta_{\nu} + \frac{1}{2} P^\beta_{\nu} \right), \]

\[ O_{10,\beta,d+4} = - \frac{1}{6} \text{tr}(F_{12} F_{12}) \text{tr}(D_{5} F_{34} D_{5} F_{34}) + \frac{2}{3} \text{tr}(F_{12} F_{23}) \text{tr}(D_{1} F_{15} F_{5} F_{5}) \]

\[ - \frac{1}{3} \text{tr}(F_{12} F_{23}) \text{tr}(D_{1} F_{5} F_{5} F_{5}). \tag{1.14} \]

where

\[ P^\beta_{\mu} = [F_{12}, F_{34}, F_{13}, F_{24}, F_{12} - \frac{1}{2} F_{12}, F_{34}, F_{12}, F_{24}, F_{12}] - [F_{24}, F_{13}, F_{34}, F_{12}] - \frac{1}{2} [F_{34}, F_{12}, F_{12}, F_{34}], \]

\[ P^\beta_{\mu} = P^\beta_{\mu} |\omega_{i}, \omega_{2}, \omega_{3}, \omega_{4}, \omega_{5}, \omega_{6}, \omega_{7}, \omega_{8}, \omega_{9}| \].
\[ Q_{\mu_3}^{\alpha_3} = [D_4 F_{12}, F_{2\mu}, F_{34}, F_{13}] - [D_4 F_{12}, F_{3\mu}, F_{13}, F_{24}] + [D_4 F_{12}, F_{3\mu}, F_{24}, F_{13}] \\
- \frac{1}{2} [D_4 F_{2\mu}, F_{12}, F_{13}, F_{34}] - \frac{1}{2} [D_4 F_{2\mu}, F_{12}, F_{34}, F_{13}] - \frac{1}{2} [D_4 F_{2\mu}, F_{13}, F_{12}, F_{34}] \\
+ \frac{1}{2} [D_4 F_{2\mu}, F_{13}, F_{34}, F_{12}] + \frac{1}{2} [D_4 F_{2\mu}, F_{34}, F_{12}, F_{13}] - \frac{1}{2} [D_4 F_{2\mu}, F_{34}, F_{13}, F_{12}] \\
+ \frac{1}{4} [D_4 F_{3\mu}, F_{12}, F_{13}, F_{24}] - \frac{1}{4} [D_4 F_{3\mu}, F_{12}, F_{34}, F_{12}] + \frac{1}{4} [D_4 F_{3\mu}, F_{34}, F_{12}, F_{12}] \ . \] (I.15)

The \( \gamma \)-sector:

\[ O_{10,\gamma,d+;1} = D^2 \left( \frac{1}{4} \text{tr}(F_{12} F_{23}) \text{tr}(F_{34} F_{14}) + \frac{1}{16} \text{tr}(F_{12} F_{12}) \text{tr}(F_{34} F_{34}) \right) , \]

\[ O_{10,\gamma,d+;2} = D^2 \left( -\frac{1}{8} \text{tr}(F_{12} F_{34})^2 - \frac{1}{4} \text{tr}(F_{12} F_{34}) \text{tr}(F_{23} F_{14}) + \frac{1}{16} \text{tr}(F_{12} F_{12}) \text{tr}(F_{34} F_{34}) \right) , \]

\[ O_{10,\gamma,d+;3} = D^{\mu\nu} \text{tr}(12) \text{tr}(34) \circ \left( \frac{1}{2} \gamma^\mu - \frac{1}{4} p_1^{\mu\nu} + \frac{1}{2} p_2^{\mu\nu} \right) , \]

\[ O_{10,\gamma,d+;4} = \frac{1}{8} \text{tr}(F_{12} D_5 F_{34})^2 + \frac{1}{8} \text{tr}(F_{12} D_5 F_{34}) \text{tr}(D_5 F_{12} F_{34}) + \frac{1}{2} \text{tr}(F_{12} D_5 F_{34}) \text{tr}(F_{23} D_5 F_{14}) \\
- \frac{1}{8} \text{tr}(F_{12} D_5 F_{12}) \text{tr}(F_{34} D_5 F_{34}) , \]

\[ O_{10,\gamma,d+;5} = \frac{1}{2} \text{tr}(F_{12} D_5 F_{14}) \text{tr}(F_{23} D_5 F_{34}) + \frac{1}{2} \text{tr}(F_{12} D_5 F_{23}) \text{tr}(F_{34} D_5 F_{14}) \\
+ \frac{1}{4} \text{tr}(F_{12} D_5 F_{12}) \text{tr}(F_{34} D_5 F_{34}) , \]

\[ O_{10,\gamma,d+;6} = \frac{1}{8} \text{tr}(F_{12} D_5 F_{34})^2 + \frac{5}{8} \text{tr}(F_{12} D_5 F_{34}) \text{tr}(D_5 F_{12} F_{34}) - \frac{1}{2} \text{tr}(F_{12} D_5 F_{34}) \text{tr}(F_{23} D_5 F_{14}) \\
- 2 \text{tr}(F_{12} D_5 F_{14}) \text{tr}(F_{23} D_5 F_{34}) + \frac{1}{2} \text{tr}(F_{13} D_5 F_{45}) \text{tr}(F_{23} D_5 F_{45}) + \text{tr}(F_{12} D_5 F_{23}) \text{tr}(F_{34} D_5 F_{14}) \\
- \frac{3}{8} \text{tr}(F_{12} D_5 F_{12}) \text{tr}(F_{34} D_5 F_{34}) - \text{tr}(F_{12} D_5 F_{23}) \text{tr}(F_{13} D_5 F_{35}) . \] (I.16)

### I.2 Dim-10 length-5 physical operators

For mass dimension 10, there are 4 single-trace and 3 double-trace physical operators with length five. All of them are C-even operators. Similar to the length-4 operators, they can be chosen to belong to certain helicity sector. For length-5 operators, we still use subscript \( \alpha, \beta, \gamma \) to label the helicity sector, but in length-5 case they correspond to \((-)^5, (-)^4(+), (-)^3(+)^2\), not as in length-4 case where they refer to \((-)^4, (-)^3(+), (-)^2(+)^2\). We omit the \((-ig)^3\) in front of each length-5 operator for simplicity.

\[ \Xi_{10,\alpha d+;1} = 5 \text{tr}(F_{12} F_{13} F_{24} F_{35} F_{45}) - 5 \text{tr}(F_{12} F_{13} F_{24} F_{35} F_{45}) + \text{tr}(F_{12} F_{13} F_{34} F_{45} F_{25}) , \]

\[ \Xi_{10,\alpha d+;2} = \frac{5}{2} \text{tr}(F_{12} F_{13} F_{24} F_{35} F_{45}) - 3 \text{tr}(F_{12} F_{13} F_{24} F_{35} F_{45}) + \text{tr}(F_{12} F_{13} F_{24} F_{45} F_{35}) , \]

\[ \Xi_{10,\gamma d+;1} = \frac{3}{2} \text{tr}(F_{12} F_{13} F_{24} F_{35} F_{45}) - 2 \text{tr}(F_{12} F_{13} F_{24} F_{35} F_{45}) - \text{tr}(F_{12} F_{13} F_{24} F_{45} F_{35}) \\
+ \text{tr}(F_{12} F_{13} F_{34} F_{45} F_{25}) , \]

\[ \Xi_{10,\gamma d+;2} = \text{tr}(F_{12} F_{13} F_{24} F_{35} F_{45}) - 2 \text{tr}(F_{12} F_{13} F_{24} F_{35} F_{45}) - 2 \text{tr}(F_{12} F_{13} F_{24} F_{45} F_{35}) . \] (I.17)

\[ \Xi_{10,\alpha d+;1} = 2 \text{tr}(F_{12} F_{34}) \text{tr}(F_{12} F_{35} F_{45}) - 4 \text{tr}(F_{12} F_{13}) \text{tr}(F_{24} F_{35} F_{45}) \\
+ \text{tr}(F_{12} F_{12}) \text{tr}(F_{34} F_{35} F_{45}) , \]
\[
\Xi_{10;\alpha d+;2} = -4\text{tr}(F_{12}F_{13})\text{tr}(F_{24}F_{35}F_{45}) + \text{tr}(F_{12}F_{12})\text{tr}(F_{34}F_{35}F_{45}),
\]
\[
\Xi_{10;\alpha d+;3} = \frac{1}{2}\text{tr}(F_{12}F_{34})\text{tr}(F_{12}F_{35}F_{45}) + \text{tr}(F_{12}F_{13})\text{tr}(F_{24}F_{35}F_{45})
\]
\[
- \frac{1}{12}\text{tr}(F_{12}F_{12})\text{tr}(F_{34}F_{35}F_{45}).
\]

(I.18)

I.3 One-loop renormalization of complete dim-10 operators

In this appendix, we provide the renormalization matrices and anomalous dimensions for physical operators of mass dimension 10.

The complete basis operators of mass dimension 10 can be separated into $C$-even and $C$-odd sectors. Since $C$-even and $C$-odd sectors do not mix to each other, the general form of one-loop renormalization matrix (3.21) can be written for these two sectors separately:

\[
\begin{bmatrix}
Z^{(1)}_{10;+} & 0 \\
0 & Z^{(1)}_{10;-}
\end{bmatrix} = \begin{bmatrix}
Z^{(1)}_{10;p\rightarrow p+} & Z^{(1)}_{10;p\rightarrow e+} \\
0 & Z^{(1)}_{10;p\rightarrow e-}
\end{bmatrix}.
\]

(I.19)

The values of $Z^{(1)}_{10;+\rightarrow e+}$ and $Z^{(1)}_{10;e\rightarrow e-}$, characterizing the renormalization of evanescent operators, have been given in (3.27) and (3.28). In this subsection we show the values of other sub-matrices $Z^{(1)}_{10;p\rightarrow p+}$, $Z^{(1)}_{10;p\rightarrow e+}$, $Z^{(1)}_{10;p\rightarrow p-}$, $Z^{(1)}_{10;p\rightarrow e-}$, which are obtained from the renormalization of physical operators.

Similar to the notation of section 3.1, the operators are understood as carrying a coupling factor, for example, $O^{e}_{10;\gamma s+;1}$ refers to $(-ig)^2O^{e}_{10;\gamma s+;1}$, and $\Xi^{e}_{5\alpha s+;1}$ refers to $(-ig)^3\Xi^{e}_{5\alpha s+;1}$.

First, let us see the renormalization of $C$-odd sector, which is relatively simple. We arrange $C$-odd physical operators in following order:

\[
\{O_{10;\alpha s-;1}, O_{10;\beta s-;1}, O_{10;\beta s-;2}, O_{10;\gamma s-;1}, O_{10;\gamma s-;2}\}. 
\]

(I.20)

As mentioned in section 3.3, there is only one $C$-odd evanescent operator $O^{e}_{10;\alpha s-;1}$. The renormalization matrix of the $C$-odd sector can be given as

\[
\begin{pmatrix}
\frac{16}{3} & 0 & 0 & 0 & \frac{1}{17} \\
0 & \frac{17}{4} & 0 & 0 & 0 \\
0 & -1 & \frac{25}{4} & 0 & 0 & -\frac{1}{17} \\
0 & 0 & 0 & \frac{37}{19} & -\frac{5}{19} & \frac{19}{19} \\
0 & 0 & -\frac{3}{19} & \frac{82}{19} & \frac{13}{19} & \frac{13}{19} \\
0 & 0 & 0 & 0 & 0 & 5
\end{pmatrix}
\]

(I.21)

Second, we consider the renormalization of $C$-even sector. We divide the $C$-even physical operators into four groups: length-4 $\alpha$-sector, length-4 $\beta$-sector, length-4 $\gamma$-sector, and length-5. They are arranged in following order:

\[
4\alpha + : \{O_{10;\alpha s+;1}, O_{10;\alpha d+;1}\},
\]
\[
4\beta + : \{O_{10;\alpha s+;1}, O_{10;\alpha d+;1}\},
\]

\[
5: \{O_{10;\alpha s+;1}, O_{10;\alpha d+;1}\},
\]

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\[ 4\gamma^+ : \{ \mathcal{O}_{10;\alpha;\beta;\gamma;\delta}^+, \mathcal{O}_{10;\alpha;\beta;\gamma;\delta}^+ \}, \]
\[ 5\rho^+ : \{ \Xi_{10;\alpha;\beta;\gamma;\delta}, \Xi_{10;\alpha;\beta;\gamma;\delta}, \Xi_{10;\alpha;\beta;\gamma;\delta}, \Xi_{10;\alpha;\beta;\gamma;\delta}, \Xi_{10;\alpha;\beta;\gamma;\delta} \}. \] (I.22)

The symbol \( 5\rho \) refers to length-5 physical operators, distinguished from the length-5 evanescent operators listed in (2.66) and (2.67). The arrangement of \( C \)-even evanescent operators is the same as given in section 3.3:

\[ \{ \mathcal{O}_{10;\alpha;\beta;\gamma;\delta}^e, \mathcal{O}_{10;\alpha;\beta;\gamma;\delta}^e, \Xi_{10;\alpha;\beta;\gamma;\delta}^e, \Xi_{10;\alpha;\beta;\gamma;\delta}^e \}. \] (I.23)

The full \( C \)-even renormalization matrix \( Z^{(1)}_{10;\rho^+} \) in (I.19) can be divided into sub-blocks:

\[
Z^{(1)}_{10;\rho^+} = \begin{pmatrix}
Z^{(1)}_{4\alpha^+ \to 4\alpha^+} & 0 & 0 & Z^{(1)}_{4\alpha^+ \to 5\rho^+} \\
0 & Z^{(1)}_{4\beta^+ \to 4\beta^+} & 0 & Z^{(1)}_{4\beta^+ \to 5\rho^+} \\
0 & 0 & Z^{(1)}_{4\gamma^+ \to 4\gamma^+} & Z^{(1)}_{4\gamma^+ \to 5\rho^+} \\
0 & 0 & 0 & Z^{(1)}_{5\rho^+ \to 5\rho^+}
\end{pmatrix}, \quad Z^{(1)}_{10;\rho^+ \to \rho^+} = \begin{pmatrix}
Z^{(1)}_{4\alpha^+ \to \rho^+} \\
Z^{(1)}_{4\beta^+ \to \rho^+} \\
Z^{(1)}_{4\gamma^+ \to \rho^+} \\
Z^{(1)}_{5\rho^+ \to \rho^+}
\end{pmatrix}. \] (I.24)

The blocks of \( Z^{(1)}_{10;\rho^+ \to \rho^+} \) are

\[
Z^{(1)}_{4\alpha^+ \to 4\alpha^+} = \frac{N_c}{\epsilon} \begin{pmatrix}
0 & \frac{5}{12} & 0 & 0 & 0 & 0 & \frac{5}{N_c} & 0 & 0 & 0 \\
0 & \frac{16}{3} & \frac{17}{3} & 0 & 0 & 0 & \frac{64}{3N_c} & 0 & 0 & 0 \\
0 & \frac{16}{3} & -\frac{5}{12} & \frac{16}{3} & 0 & 0 & 0 & -\frac{5}{N_c} & 0 & 0 \\
0 & \frac{1}{6} & \frac{2}{3} & 8 & -\frac{2}{3} & 8 & \frac{8}{N_c} & 0 & \frac{6}{N_c} & 0 & -\frac{6}{N_c} \\
0 & \frac{8}{3} & -\frac{5}{12} & \frac{2}{3} & -\frac{10}{3} & \frac{14}{3} & 0 & 0 & 0 & \frac{30}{N_c} & 0 \\
-\frac{5}{N_c} & -\frac{5}{4N_c} & 0 & 0 & 0 & \frac{25}{3} & 0 & 0 & 0 & 0 \\
\frac{16}{N_c} & \frac{4}{N_c} & 0 & 0 & 0 & 0 & -\frac{11}{3} & 0 & 0 & 0 \\
\frac{8}{N_c} & \frac{5}{N_c} & \frac{3}{N_c} & 0 & 0 & 0 & 0 & \frac{25}{3} & 0 & 0 \\
-\frac{2}{3N_c} & -\frac{7}{12N_c} & \frac{5}{6N_c} & -\frac{25}{6N_c} & -\frac{5}{6N_c} & 0 & 0 & 0 & -\frac{11}{3} & 0 \\
\frac{22}{3N_c} & -\frac{5}{6N_c} & \frac{41}{6N_c} & -\frac{115}{6N_c} & -\frac{23}{6N_c} & 0 & 0 & 0 & 0 & \frac{25}{3}
\end{pmatrix}, \] (I.25)

\[
Z^{(1)}_{4\alpha^+ \to 5\rho^+} = \frac{N_c}{\epsilon} \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{17}{3} & 8 & 0 & 0 & -\frac{8}{N_c} & 0 & 0 & 0 & 0 & 0 \\
-\frac{26}{3} & 10 & 0 & 0 & -\frac{15}{N_c} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{65}{6N_c} & \frac{20}{N_c} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{83}{6N_c} & \frac{20}{N_c} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}. \] (I.26)
\[
Z^{(1)}_{4\beta^+\rightarrow4\beta^+} = \frac{N_c}{\epsilon} \begin{pmatrix}
\frac{5}{3N_c} & -\frac{3}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{5}{3} & \frac{9}{4} & 0 & 0 & 0 & -\frac{40}{3N_c} & 0 & 0 \\
-\frac{5}{21} & -\frac{3}{8} & \frac{21}{4} & 0 & 0 & -\frac{5}{3N_c} & 0 & 0 \\
\frac{1}{6} & -\frac{1}{2} & -\frac{1}{6} & 0 & 0 & 0 & 0 & 0 \\
\frac{5}{3N_c} & -\frac{3}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{35}{9N_c} & -\frac{35}{12N_c} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{35}{72N_c} & -\frac{11}{24N_c} & -\frac{3}{4N_c} & 0 & 0 & 0 & 0 & 0 \\
\frac{7}{18N_c} & -\frac{1}{6N_c} & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
\] 
(I.27)

\[
Z^{(1)}_{4\beta^+\rightarrow5\pi^+} = \frac{N_c}{\epsilon} \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
\] 
(I.28)

\[
Z^{(1)}_{4\gamma^+\rightarrow4\gamma^+} = \frac{N_c}{\epsilon} \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & -\frac{1}{2} & 9 & 2 & 0 & 0 & 0 & 0 \\
-\frac{1}{2} & 2 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
-\frac{2}{7} & 3 & -\frac{2}{7} & -\frac{2}{7} & 3 & -\frac{2}{7} & \frac{2}{7} & -\frac{2}{7} \\
\frac{64}{15} & \frac{2}{5} & \frac{14}{15} & \frac{14}{15} & \frac{14}{15} & \frac{14}{15} & \frac{14}{15} & \frac{14}{15} \\
\frac{16}{N_c} & -\frac{22}{5N_c} & \frac{4}{N_c} & -\frac{4}{N_c} & \frac{4}{N_c} & -\frac{4}{N_c} & \frac{4}{N_c} & -\frac{4}{N_c} \\
\end{pmatrix},
\] 
(I.29)
\[
Z_{4\gamma \rightarrow 5p^+}^{(1)} = \frac{N_c}{\epsilon} \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{5} & -\frac{1}{15} & 0 & -\frac{31}{5N_c} & 0 & 0 & 0 \\
0 & 0 & -\frac{5}{36} & -\frac{7}{36} & 0 & -\frac{3}{2N_c} & -\frac{8}{N_c} & 0 & 0 \\
0 & 0 & -\frac{197}{60} & -\frac{169}{60} & 0 & -\frac{5}{6N_c} & -\frac{110}{5N_c} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{111}{20N_c} & -\frac{29}{20N_c} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{31}{20N_c} & \frac{859}{360N_c} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{701}{20N_c} & -\frac{671}{360N_c} & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}, \quad (I.30)
\]

\[
Z_{5p^+ \rightarrow 5p^+}^{(1)} = \frac{N_c}{\epsilon} \begin{pmatrix}
-\frac{11}{3} & 10 & 0 & 0 & -\frac{35}{N_c} & 0 & 0 \\
-9 & \frac{49}{3} & 0 & 0 & -\frac{35}{N_c} & 0 & 0 \\
0 & 0 & 9 & \frac{1}{3} & 0 & \frac{1}{2N_c} & \frac{10}{N_c} \\
0 & 0 & 1 & \frac{37}{7} & 0 & -\frac{13}{3N_c} \frac{1}{3N_c} \\
\frac{12}{N_c} & -\frac{24}{N_c} & 0 & 0 & \frac{7}{3} & 0 & 0 \\
0 & 0 & -\frac{20}{3N_c} & -\frac{10}{3N_c} & 0 & \frac{20}{3} \frac{7}{3} \\
0 & 0 & -\frac{1}{2N_c} & -\frac{7}{4N_c} & 0 & 0 \frac{7}{3} \\
\end{pmatrix}, \quad (I.31)
\]

The blocks of \(Z_{10p^+ \rightarrow e^+}^{(1)}\) are

\[
Z_{10p^+ \rightarrow e^+}^{(1)} = \frac{N_c}{\epsilon} \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{5}{6N_c} & \frac{4}{3N_c} & 0 & 0 & 0 & 0 \\
-\frac{41}{72} & -\frac{23}{36} & -\frac{7}{12} & -\frac{11}{12N_c} & \frac{5}{12N_c} & -\frac{3}{2N_c} & 0 & 0 & 0 \\
\frac{19}{72} & \frac{13}{36} & \frac{5}{12} & -\frac{37}{12N_c} & \frac{65}{12N_c} & -\frac{15}{2N_c} & -\frac{11}{12} & -\frac{23}{3N_c} & \frac{41}{N_c} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{4N_c} & -\frac{11}{12N_c} & 0 & -\frac{1}{8} & \frac{1}{6} & 0 & 0 & 0 & 0 \\
\frac{5}{18N_c} & \frac{1}{18N_c} & \frac{1}{3N_c} & \frac{4}{3} & \frac{7}{3} & 10 & 0 & 0 & 0 \\
\frac{37}{36N_c} & \frac{20}{36N_c} & \frac{10}{3N_c} & \frac{1}{3} & -\frac{1}{3} & 0 & -\frac{46}{9N_c} & -\frac{49}{36N_c} & 0 \\
\end{pmatrix}, \quad (I.32)
\]
the mixing from length-4 operators to length-5 operators do not effect the eigenvalues. The anomalous dimensions are given as the eigenvalues of one-loop dilation operators defined in (3.24). The existence of non-zero $Z_{p+\rightarrow e+}^{(1)}$ and $Z_{p-\rightarrow e-}^{(1)}$ do not effect the eigenvalues because at the order of $O(\epsilon^{-1})$ there is no mixing in the opposite direction. Similarly, the mixing from length-4 operators to length-5 operators do not effect the eigenvalues. The eigenvalues determined by sub-matrices $Z_{e+\rightarrow e+}^{(1)}$ and $Z_{e-\rightarrow e-}^{(1)}$ are given in (3.29)-(3.33).
Here we give the eigenvalues determined by sub-matrices $Z^{{(1)}_{p^+\to p^+}}$ and $Z^{{(1)}_{p^-\to p^-}}$. Furthermore, at one-loop order, different helicity sectors do not mix to each other, so the eigenvalues can be obtained from the sub-matrices of each helicity sector separately.

Denote the eigenvalues of $C$-odd length-4 $\alpha, \beta, \gamma$-sector by $\gamma^{(1)}_{-;\alpha}, \gamma^{(1)}_{-;\beta}, \gamma^{(1)}_{-\gamma}$. They are determined by $Z^{(1)}_{10;\gamma^+\to \gamma^-}$ given in (I.21):

$$\gamma^{(1)}_{-;\alpha} = \frac{32N_c}{3}, \quad \gamma^{(1)}_{-;\beta} = \left(\frac{17N_c}{2} - \frac{25N_c}{2}\right), \quad \gamma^{(1)}_{-\gamma} = \left(\frac{11N_c - 22N_c}{3}\right).$$  \hfill (I.36)

Denote the eigenvalues of $C$-even length-4 $\alpha, \beta, \gamma$-sector by $\gamma^{(1)}_{+;\alpha}, \gamma^{(1)}_{+;\beta}, \gamma^{(1)}_{+\gamma}$. Respectively they are determined by sub-matrices $Z^{{(1)}_{10;\alpha^+\to \alpha^-}}, Z^{{(1)}_{10;\beta^+\to \beta^-}}, Z^{{(1)}_{10;\gamma^+\to \gamma^-}}$ given in (I.25), (I.27), (I.29). Denote the eigenvalues of $C$-even length-5 sector by $\gamma^{(1)}_{+;\gamma}$. They are determined by $Z^{(1)}_{10;5\gamma^+\to 5\gamma^-}$ given in (I.31). For convenience we introduce $\omega_h = \gamma^{(1)}_{+;h}/N_c$ where $h = \alpha, \beta, \gamma, 5$.

For $C$-even length-4 $\alpha$-sector, $\omega_\alpha$ are roots of equation

$$0 = \left(\omega_\alpha - \frac{50}{3}\right)\left(\omega_\alpha - \frac{32}{3}\right) \times \left(\omega_\alpha^4 - \frac{104\omega_\alpha^3}{3} + \omega_\alpha^2 \left(\frac{764}{3} - \frac{360}{N_c^2}\right) + \omega_\alpha \left(\frac{48208}{27} - \frac{2400}{N_c^2}\right) - \frac{1390400}{81} + \frac{73760}{N_c^2}\right) \times \left(\omega_\alpha^4 - \frac{62\omega_\alpha^3}{3} + \omega_\alpha^2 \left(\frac{76}{3} + \frac{640}{N_c^2}\right) + \omega_\alpha \left(\frac{39640}{27} + \frac{14080}{3N_c^2}\right) + \left(\frac{88000}{81} - \frac{486400}{9N_c^2}\right)\right).$$  \hfill (I.37)

The order four factorized polynomial in the second line is inherited from the $\alpha$-sector with mass dimension 8. Expand eigenvalues up to $O(N_c^{-1})$:

$$\gamma^{(1)}_{+\alpha} = N_c\left\{\left(\frac{17}{3} \pm \sqrt{\frac{1921}{3}}\right) - \frac{20(41 \pm 19\sqrt{41})}{41N_c^2}, \frac{38}{3} \pm 2\sqrt{5}, \frac{3}{3} \pm \frac{18(360 \pm 173\sqrt{5})}{19N_c^2}, \frac{32}{3}\right\}$$

$$+ \left[\frac{50}{3} - \frac{690}{N_c^2}, \frac{50}{3} + \frac{80}{N_c^2}, \frac{22}{3} - \frac{40}{N_c^2}, \frac{22}{3} + \frac{150}{N_c^2}\right].$$  \hfill (I.38)

The sub-leading $N_c$ correction breaks the triple degeneracy of $50N_c/3$ and double degeneracy of $-22N_c/3$.

For $C$-even length-4 $\beta$-sector, $\omega_\beta$ are roots of equation

$$0 = (\omega_\beta - 12)^3 \left(\omega_\beta - \frac{21}{2}\right) \left(\omega_\beta^4 - \frac{53\omega_\beta^3}{2} + \omega_\beta^2 \left(214 - \frac{160}{N_c^2}\right) - \omega_\beta \left(480 - \frac{1520}{N_c^2}\right) - \frac{6400}{N_c^2}\right).$$  \hfill (I.39)

Expand eigenvalues up to $O(N_c^{-1})$:

$$\gamma^{(1)}_{+\beta} = N_c\left\{\frac{1}{4}(29 \pm \sqrt{201}) - \frac{40(201 \pm 19\sqrt{201})}{201N_c^2}, \frac{21}{2}, \frac{12, 12, 12, 12 + \frac{280}{3N_c^2} - \frac{40}{3N_c^2}}{3}\right\}. \hfill (I.40)$$

The sub-leading $N_c$ correction breaks the quartet degeneracy of $12N_c$ to triple degeneracy.
For C-even length-4 $\gamma$-sector, $\omega_\gamma$ are roots of equation
\[
0 = (\omega_\gamma - 9) \left( \omega_\gamma - \frac{22}{9} \right) \times \left( \omega_\gamma^4 - \frac{31\omega_\gamma^3}{3} - \left( \frac{418}{9} + \frac{1936}{9N_c^2} \right) \omega_\gamma^2 + \left( \frac{15004}{27} - \frac{60016}{27N_c^2} \right) \omega_\gamma - \left( \frac{10648}{27} - \frac{42592}{27N_c^2} \right) \right)
\]
\[
\times \left( \omega_\gamma^6 - \frac{736\omega_\gamma^5}{15} + \left( \frac{3887}{5} - \frac{5170}{9N_c^2} \right) \omega_\gamma^4 - \left( \frac{384932}{135} - \frac{42736}{3N_c^2} \right) \omega_\gamma^3 + \left( \frac{13919744}{405} + \frac{7114772}{81N_c^2} \right) \omega_\gamma^2 - \left( \frac{119573168}{405} - \frac{1362032}{81N_c^2} \right) \omega_\gamma - \left( \frac{224969008}{405} - \frac{156611312}{81N_c^2} \right) \right). \tag{I.41}
\]
The order four factorized polynomial in the first line is inherited from the $\gamma$-sector with mass dimension 8.

Expand eigenvalues up to $O(N_c^{-1})$:
\[
\hat{\gamma}_{\gamma^3}^{(1)} = N_c \left\{ \frac{9}{6} \left( \frac{1}{6} + \sqrt{697} \right) + \frac{121(21607 + 811\sqrt{697})}{43911N_c^2}, x_i + \frac{y_i}{N_c^2}, \right. \nonumber
\]
\[
\left. \frac{22}{3} - \frac{23260}{8559N_c^2}, \frac{110058}{9N_c^2}, \frac{22}{3} + \frac{32454N_c^2}{17N_c^2}, \frac{22}{3} \right\}. \tag{I.42}
\]
There are three pairs of $\{x_i, y_i\}$, where $x_1, x_2, x_3$ and $y_1, y_2, y_3$ are respectively the roots of equations
\[
0 = x^3 - \frac{446x^2}{15} + \frac{769x}{3} - \frac{8014}{15}, \nonumber
\]
\[
0 = y^3 + \frac{35538354643y^2}{600128550} + \frac{131096322792295265y}{2216037779059699} - \frac{4091084565828125}{633153651159914}. \tag{I.43}
\]
The sub-leading $N_c$ correction breaks the triple degeneracy of $22N_c/3$ and double degeneracy of $-22N_c/3$.

For C-even length-5 sector, $\omega_5$ are roots of equation
\[
0 = \left( \omega_5^8 - 30\omega_5^7 + \omega_5 \left( \frac{716}{3} - \frac{720}{N_c^2} \right) \right) \times \left( \omega_5^4 - \frac{118\omega_5^3}{3} + \omega_5 \left( \frac{5006}{9} - \frac{220}{9N_c^2} \right) \right) - \left( \frac{15176}{27} - \frac{12000}{N_c^2} \right) \omega_5 \left( \frac{89300}{27} - \frac{680}{27N_c^2} \right) + \left( \frac{61600}{9} + \frac{42400}{27N_c^2} \right). \tag{I.44}
\]
The order three factorized polynomial in the first line belongs to the $\alpha$-sector of length-5, and order four factorized polynomial in the second line belongs to the $\gamma$-sector of length-5.

Expand eigenvalues up to $O(N_c^{-1})$:
\[
\hat{\gamma}_{\gamma^3}^{(1)} = N_c \left\{ \frac{38}{3} \pm 2\sqrt{10} + \frac{180}{N_c^2}, \frac{4}{3} - \frac{360}{N_c^2}, \right. \nonumber
\]
\[
\left. \frac{1}{3} \left( \frac{32}{3} \pm \sqrt{34} \right) - \frac{5(39304 + 6887\sqrt{34})}{4437N_c^2}, \frac{40}{3} + \frac{760}{9N_c^2}, \frac{14}{3} + \frac{120}{29N_c^2} \right\}. \tag{I.45}
\]
The sub-leading $N_c$ correction breaks the double degeneracy of $14N_c/3$. 
I.4 Finite renormalization as a scheme change

As mentioned in (3.25), one can choose another renormalization scheme by appending finite terms to $Z^{(1)}$ to absorb the mixing from the evanescent operators to the physical ones at the order of $O(\varepsilon^0)$. Discussing $C$-even and $C$-odd sectors separately, the finite terms added to $Z^{(1)}_{10e+}$ and $Z^{(1)}_{10e-}$ in (I.19) are

\[
Z^{(1),\text{fin}}_{10e+} = \begin{pmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
\end{pmatrix}, \quad Z^{(1),\text{fin}}_{10e-} = \begin{pmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
\end{pmatrix}.
\] (I.46)

In this subsection we give the values of $Z^{(1),\text{fin}}_{10e+\rightarrow p+}$ and $Z^{(1),\text{fin}}_{10e-\rightarrow p-}$.

\[
Z^{(1),\text{fin}}_{10e+\rightarrow p+} = N_c \begin{pmatrix}
\frac{16}{3} & -2 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\frac{-32}{3 N_c} & -\frac{8}{3 N_c} & 0 & 0 \\
\end{pmatrix}, \quad Z^{(1),\text{fin}}_{10e-\rightarrow p-} = N_c \begin{pmatrix}
\frac{10}{3} & -2 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\frac{-32}{3 N_c} & -\frac{8}{3 N_c} & 0 & 0 \\
\end{pmatrix}.
\] (I.47)

where

\[
Z^{(1),\text{fin}}_{4e+\rightarrow 4\alpha+} = N_c \begin{pmatrix}
\frac{16}{3} & -2 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\frac{-32}{3 N_c} & -\frac{8}{3 N_c} & 0 & 0 \\
\end{pmatrix}, \quad Z^{(1),\text{fin}}_{4e+\rightarrow 4\beta+} = N_c \begin{pmatrix}
\frac{10}{3} & -2 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\frac{-32}{3 N_c} & -\frac{8}{3 N_c} & 0 & 0 \\
\end{pmatrix}.
\] (I.48)

\[
Z^{(1),\text{fin}}_{4e+\rightarrow 4\gamma+} = N_c \begin{pmatrix}
\frac{16}{3} & -2 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\frac{-32}{3 N_c} & -\frac{8}{3 N_c} & 0 & 0 \\
\end{pmatrix}.
\] (I.49)
I.5 Discussion on $N_c = 2, 3$

So far our discussion is for general $N_c$, which is the rank of the special unitary gauge group. As mentioned in the end of section 2.3.2, for special values like $N_c = 2, 3$, there exist extra linear relations between the color factors and as a result, the number of trace basis are reduced. Here we take length-4 sector with $N_c = 2$ as an example to discuss the basis reduction in detail, and we briefly comment on the length-5 sector and also the $N_c = 3$ case in the end.

When $N_c = 2$, the length-4 single traces and double traces have linear relations

$$2 \text{tr}(jklm) = \text{tr}(jk)\text{tr}(lm) - \text{tr}(jl)\text{tr}(km) + \text{tr}(jm)\text{tr}(kl).$$  \hspace{1cm} (1.53)

So nine length-4 color factors reduce to three double traces. As a consequence, all single-trace length-4 $C$-odd operators become zero, while the single-trace length-4 $C$-even basis operators are equal to combinations of double-trace ones

$$O_{s+;i} \frac{N_c=2}{N_c} \sum_k M^i_k O_{d;k},$$  \hspace{1cm} (1.54)

where $M^i_k$ depends on the basis choice. As a result, the independent set of dim-10 length-4 evanescent operators only contains three double-trace ones which are given in (2.65).

The special property of operators basis in $N_c = 2$ provides a consistency check of the renormalization matrix obtained for general $N_c$. One can first write out the renormalization of a single-trace operator $O_{s+;i}$ and then replace the single-trace operators $O_{s+;j}$ appearing in the mixing pattern with the double-trace ones according to (1.54):

$$O_{s+;i;R} - O_{s+;i;B} \xrightarrow{N_c=2} \alpha_s \sum_j (Z^{(1)}_{s+r})^{i} j \sum_k M^i_k O_{d;k} + \alpha_s \sum_k (Z^{(1)}_{s+d})^{i} k O_{d;k} + O(\alpha_s^2)$$

On the other hand, one can first rewrite $O_{s+;i}$ in terms of double-trace operators $O_{d;k}$ according to (1.54) and then write out the renormalization of $O_{d;k}$:

$$O_{s+;i;R} - O_{s+;i;B} \xrightarrow{N_c=2} \sum_k M^i_k \left( \alpha_s \sum_j (Z^{(1)}_{d+s})^{i} j \sum_l M^l_j O_{d;l} + \alpha_s \sum_l (Z^{(1)}_{d+d})^{l} k O_{d;l} + O(\alpha_s^2) \right).$$
The two processes should yield the same result, which means

$$N_c = 2 : \quad (Z_{s\to s}^{(1)}) \cdot M + (Z_{s\to d}^{(1)}) = M \cdot (Z_{d\to s}^{(1)}) \cdot M + M \cdot (Z_{d\to d}^{(1)}). \quad (I.55)$$

This provides a consistency condition. Take the dim-10 length-4 evanescent sector as an example, under the basis choice given by (2.68), the matrix $M$ in (I.54) is given as

$$
\begin{pmatrix}
O_{10s;+;1}^c \\
O_{10s;+;2}^c \\
O_{10s;3}^c
\end{pmatrix}
\xrightarrow{N_c=2}
\begin{pmatrix}
1 & 0 & 0 \\
0 & -2 & 0 \\
0 & 0 & -2
\end{pmatrix}
\begin{pmatrix}
O_{10d;+;1}^c \\
O_{10d;+;2}^c \\
O_{10d;3}^c
\end{pmatrix}, \quad (I.56)
$$

Here we skip the $C$-odd operators because as mentioned before they vanishes when $N_c = 2$. One can see this $M$ matrix together with $Z_{10;eva:+}^{(1)}$ given in (3.27) satisfies equation (I.55).

Similar analysis can be done for other cases. When $N_c = 3$, the length-4 single traces and double traces have linear relations

$$\text{tr}(1234) + (S_3\text{-perms.}) = \text{tr}(12)\text{tr}(34) + (Z_3\text{-perms. of } (234)), \quad (I.57)$$

thus the nine color factors reduce to eight ones. For the length-5 case, the color factors have 24 single traces and 10 double traces as given in (2.62). In this case, there are 28 and 2 linear relations for $N_c = 2$ and $N_c = 3$ respectively. The operator reduction under these situations can also be discussed and thus provides more consistency checks like (I.55) and our results agree with all the consistency conditions.

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