Finsler geometry as a model for relativistic gravity
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Abstract
We give an overview on the status and on the perspectives of Finsler gravity, beginning with a discussion of various motivations for considering a Finslerian modification of General Relativity. The subjects covered include Finslerian versions of Maxwell’s equations, of the Klein-Gordon equation and of the Dirac equation, and several experimental tests of Finsler gravity.

1 Introduction

All gravitational phenomena are well described within the theory of General Relativity (GR). In particular, the Universality of Free Fall, the Local Lorentz Invariance and the Local Position Invariance which are at the basis of GR have been experimentally verified with a high precision. Together they tell us that the orbit of a pointlike test particle in a gravitational field is uniquely determined by the initial conditions and that all non-gravitational experiments give the same result in any local inertial frame irrespective of where and when they are carried out [1]. This leads to the result that gravity can be described by means of a pseudo-Riemannian metric of Lorentzian signature. The question of how this metric is related to the energy content of the spacetime is answered by Einstein’s field equation. Einstein arrived at this field equation in 1915 after a long and arduous process of trial and error. Much later Lovelock [2, 3] proved that Einstein’s field equation (including a cosmological constant) is uniquely determined by the requirements that it contains derivatives of at most second order of the metric and that it implies a local conservation law of energy. So if one sticks with metrical theories of gravity, then there is not much freedom of considering field equations other than Einstein’s. For tests of Einstein’s theory in the Solar system and with binary pulsars one usually resorts to the parametrized post-Newtonian formalism; until now, all predicted effects have been confirmed by observation with high precision [4, 5]. Einstein’s theory is also in agreement with observations at the scale of galaxies and clusters of galaxies (if one accepts the existence of dark matter) and at cosmological scales (if one accepts the existence of dark energy).

This, however, does not mean that there is no reason for considering possible modifications of GR. One motivation for such modifications comes from the fact that classical GR and Quantum Theory are incompatible. It is usually expected that these two theories should merge into a new theory called Quantum Gravity which is still to be found. It is furthermore expected that there should be a regime interpolating between GR and Quantum Gravity where gravity may be well approximated by a classical (i.e., non-quantum, effective) theory with small deviations from GR. This interpolating effective gravity theory is generally expected to lead to tiny violations of the above-mentioned foundations of GR which may have two consequences. Firstly, the new theory may have additional gravitational fields beyond the pseudo-Riemannian metric as, e.g., scalar fields or a space-time torsion. Secondly, the pseudo-Riemannian metric itself may be modified. In particular, it has been suggested that it may be necessary to replace the pseudo-Riemannian with a Finslerian metric [6]. Such a replacement has the consequence that differential operators such as the d’Alembert operator have to be replaced with pseudo-differential operators, thereby introducing non-local features into the theory. This is in agreement with the general expectation that the implementation of quantum gravity effects leads to a non-local theory.

In this paper we want to review, albeit biased by our personal preferences, the present status and the perspectives of Finslerian spacetime theories. One motivation for our interest in Finsler spacetimes comes from Quantum Gravity, as indicated above, but this is not the only one. Finslerian geometry
can also be used for an elegant description of the symmetry given by Very Special Relativity \[7\] which, although different from Special Relativity, is still compatible with all current experimental limits on violations of Lorentz invariance and spatial isotropy. Moreover, a quite different and particularly strong motivation for considering Finsler spacetimes comes from the fact that they naturally arise from a slight modification of the axiomatic approach to spacetime theory by Ehlers, Pirani and Schild \[8\] as will be outlined in Section 3 below.

2 Mathematical foundation of Finsler geometry

2.1 Positive definite case

It is worthwhile to recall that Riemannian geometry was originally introduced as a theory of positive definite metrics, before it was generalised to allow for indefinite (‘pseudo-Riemannian’ or ‘semi-Riemannian’) metrics. Similarly, Finsler geometry was originally restricted to the positive definite case. In this version the theory was brought forward by Paul Finsler in his PhD Thesis of 1919 \[9\], following a brief remark in Riemann’s Habilitation Thesis \[10\], and in this version it is treated in mathematical standard text-books such as the one by Rund \[11\] or by Bao, Chern and Shen \[12\]. We briefly summarise the main features of positive definite Finsler metrics before turning to the indefinite case which is of more interest in view of the applications we have in mind.

A positive definite Finsler structure is usually given in terms of a Finsler function \( F(x, \dot{x}) \) which is defined on the tangent bundle \( TM \) of a manifold \( M \) with the zero section removed, \( (x, \dot{x}) \in TM \setminus \{0\} \). In addition to being at least three times continuously differentiable, it should be strictly positive, \( F(x, \dot{x}) > 0 \), positively homogeneous of degree one, \( F(x, \lambda \dot{x}) = \lambda F(x, \dot{x}) \) for all \( \lambda > 0 \), and such that the Finsler metric

\[
 g_{\mu\nu}(x, \dot{x}) = \frac{\partial^2 F(x, \dot{x})^2}{\partial \dot{x}^\mu \partial \dot{x}^\nu}
\]

is non-degenerate. Here and in the following we write, by abuse of notation as usual, \( x = (x^\mu) \) for a point in the base manifold represented by its coordinates and \( \dot{x} = (\dot{x}^\mu) \) for a point in the tangent space at \( x \) represented by the induced coordinates. The homogeneity of \( F \) implies that

\[
 F(x, \dot{x})^2 = \frac{1}{2} g_{\mu\nu}(x, \dot{x}) \dot{x}^\mu \dot{x}^\nu,
\]

so the requirement of \( F \) being strictly positive implies that the Finsler metric is positive definite. Moreover, the homogeneity also implies that the length functional

\[
 \ell(x) = \int_a^b F(x(s), \dot{x}(s)) ds
\]

is invariant under reparametrisation for each curve \( x : [a, b] \to M \) in \( M \). The extremals of the length functional are, by definition, the (unparametrised) geodesics of the Finsler structure. The Euler-Lagrange equations of the Lagrangian \( L(x, \dot{x}) = F(x, \dot{x})^2 \) give the affinely parametrised geodesics.

The Finsler metric is independent of the \( \dot{x} \) if and only if \( F(x, \dot{x})^2 \) is a quadratic form. In this case the Finsler structure reduces to a Riemannian structure. So roughly speaking Finsler geometry introduces a way of measuring lengths that is more general than the one known from Riemannian geometry. In applications of positive definite Finsler metrics to physics the underlying manifold is to be interpreted as (three-dimensional) space or as a submanifold thereof. We will now turn to indefinite Finsler metrics where the underlying manifold may be interpreted as (four-dimensional) spacetime.
2.2 Indefinite case

From (4) we read that a generalisation to indefinite Finsler metrics may be achieved in one of two ways: Either one replaces \( F(x, \dot{x})^2 \) by a function that is allowed to take negative values, or one restricts the domain of definition of \( F(x, \dot{x}) \) to a subset of tangent vectors which are then to be interpreted as timelike with respect to the Finsler metric. Both possibilities have been worked out in the literature; the first one was pioneered by Beem [13], the second by Asanov [14].

Beem’s definition of a (possibly indefinite) Finsler structure on a manifold \( M \) is given in terms of a real-valued function \( L(x, \dot{x}) \) which generalises the square \( F(x, \dot{x})^2 \) of the Finsler function. \( L(x, \dot{x}) \) is required to be defined and at least three times continuously differentiable on the tangent bundle with the zero section removed. It should be positively homogeneous of degree two,

\[
L(x, \lambda \dot{x}) = \lambda^2 L(x, \dot{x}) \quad \text{for all } \lambda > 0,
\]

and such that the Finsler metric

\[
g_{\mu\nu}(x, \dot{x}) = \frac{\partial^2 L(x, \dot{x})}{\partial \dot{x}^\mu \partial \dot{x}^\nu}
\]

is non-degenerate. In analogy to (4), these assumptions imply that

\[
L(x, \dot{x}) = \frac{1}{2} g_{\mu\nu}(x, \dot{x}) \dot{x}^\mu \dot{x}^\nu.
\]

As \( L \) may take positive or negative values, the Finsler metric may be indefinite. It must have a specific signature which cannot change from one point to another because the assumptions guarantee that the determinant of the Finsler metric is continuous and nowhere zero. In view of applications to physics, we are particularly interested in the case that \( M \) is four-dimensional and that the signature is Lorentzian, \((+, -, -, -)\). Mathematically, however, the definition makes sense for any dimension and any signature. At each point \( x \), we can classify vectors \( \dot{x} \) as timelike, lightlike or spacelike according to whether \( L(x, \dot{x}) \) is positive, zero or negative. In the case of Lorentzian signature the homogeneity assumption guarantees that, at each point \( x \), the lightlike vectors form a cone which, however, may have more than two connected components. Criteria for having exactly two connected components (to be interpreted as a future and a past light cone) have been worked out by Minguzzi [15].

From a mathematical point of view Beem’s definition is quite satisfactory. In view of physics, however, it is a bit too restrictive because it excludes several cases which are of interest. Here are two examples. Firstly, light propagation in a biaxial crystal can be described in terms of two Finsler metrics (see Perlick [16]) which, however, violate Beem’s differentiability assumption on a set of measure zero in \( TM \setminus \{0\} \). Secondly, in some static Finsler spacetimes, see Section 5.2 below, the Lagrangian fails to be well-defined on a set of measure zero in \( TM \setminus \{0\} \). These observations motivated us to relax Beem’s definition in [20] by requiring the Lagrangian to be defined and at least three times continuously differentiable only almost everywhere on \( TM \setminus \{0\} \). A stronger modification of Beem’s definition had been brought forward, already earlier, by Pfeifer and Wohlfarth [17]. In addition to relaxing the regularity conditions in a certain way they also allowed for positive homogeneity of any degree.

The other approach to Finsler metrics, which is detailed in the book by Asanov [14], is restricted to the case of Lorentzian signature. Here one sticks with a positive-valued Finsler function \( F(x, \dot{x}) \) that is positively homogeneous of degree one, but one restricts its domain of definition, at each point \( x \), to an open conic subset of the tangent space. Asanov calls the vectors in this domain, which are to be interpreted as timelike, the ‘admissible vectors’. In its original version this approach suffered from the disadvantage that practically nothing was known about the boundary of the domain of admissible vectors, so one did not have any control of the vectors one would like to interpret as lightlike. However, this disadvantage was overcome in more recent work by Javaloyes and Sánchez [18] who introduced a refined definition of Lorentzian Finsler structures in terms of cones and worked out several interesting examples.
In view of applications to physics, it seems fair to say that the most appropriate definition of indefinite Finsler structures is still a matter of debate. We emphasise that here the mathematical details are important. As a physicist, one usually does not pay much attention to domains of definition and conditions of differentiability, and in most cases one gets away with that. In the case at hand, however, such mathematical subtleties may have a big impact.

3 A constructive axiomatic approach to Finsler spacetimes

Inspired by earlier work of Reichenbach, Carathéodory, Weyl and others, Ehlers, Pirani and Schild [8] described a constructive axiomatic approach to spacetime theory by which the pseudo-Riemannian structure of spacetime and gravity is justified and can be tested with a finite number of different experiments. This axiomatic approach uses light rays and freely falling particles as the primitive objects. In the following we briefly review the axioms and we indicate where a slight modification leads to a Finslerian spacetime. The observation that a modification of the Ehlers-Pirani-Schild axiomatics leads to a Finsler structure was made already in 1985 by Tavakol and Van Den Bergh [19].

The first group of axioms makes spacetime into a differentiable manifold and the worldlines of light rays and freely falling particles into one-dimensional submanifolds. As this part is of no relevance in view of the Finsler modification, we do not discuss it here. In the next step the conformal structure of spacetime is established. To that end one considers a particle worldline $P$ parametrised with a time $t$. This may be just any parametrisation; note that at this stage it does not make sense to ask if $t$ is proper time because the latter notion is not yet defined. An axiom requires that Einstein’s synchronisation procedure can be carried through: For every event $e$ on $P$ there is a neighbourhood $U$ and a bigger neighbourhood $V$ such that every event $p$ in $U\setminus P$ can be connected to $P$ by exactly two light rays that are contained in $V$. If the events where these light rays meet $P$ are denoted $e_1$ and $e_2$, this construction can be extended to points $p \in U \cap P$ by setting $e_1 = e_2 = p$ in this case, thereby defining a function $g_e: p \mapsto (t(e) - t(e_1)) (t(e_2) - t(e))$ on all of $U$. The crucial step in this part of the axiomatics is that they required $g_e$ to be two times continuously differentiable on all of $U$. In combination with the axioms on light propagation this implies that the quantity

$$g_{\mu\nu}(e) = \lim_{p \to e} \frac{\partial^2 g_e(p)}{\partial p^\mu \partial p^\nu}$$

(9)

is a well-defined symmetric second rank tensor which is non-degenerate and of Lorentzian signature. Changing the parametrisation on $P$ has the only effect of multiplying $g_{\mu\nu}(e)$ with a non-zero factor, so this construction defines a conformal equivalence class of Lorentzian metrics on the spacetime.

From the axioms on light propagation it could be shown that the light rays are lightlike geodesics of these Lorentzian metrics. Since in view of physics differentiability is to be understood as an idealisation that can never be verified by a finite number of measurements, a postulate of differentiability has to be taken with care in a constructive axiomatics. To a certain extent, such postulates are necessary and they had been used in the Ehlers-Pirani-Schild axiomatics already in the first step where the differentiable structure was established. However, such postulates are questionable if relaxing them leads to a different type of mathematical structure. This is exactly what happens in the case at hand. If we require $g_e$ to be two times continuously differentiable only on $U \setminus P$, the limit on the right-hand side of (9) will in general depend on the direction from which $p$ approaches the event $e$. As a consequence, one gets a conformal equivalence class of Finsler metrics. The rest of the Ehlers-Pirani-Schild axiomatics is about the projective geometry established with the help of freely falling particles and about the compatibility of the conformal and the projective structure. The details of how to establish a Finsler spacetime precisely in the sense of Beem’s definition (or some modification thereof) have still to be worked out, but it is clear that dropping the above-mentioned differentiability postulate in the Ehlers-Pirani-Schild axiomatics leads to some kind of Finsler geometry.
4 Finsler gravity

As motivated in the preceding section, we will now discuss the perspectives of a Finslerian theory of gravity. We assume that spacetime is a four-dimensional manifold $M$ and that gravity is coded in an indefinite Finsler structure of Lorentzian signature. Unless explicitly referring to another definition, we assume that the Finsler structure is defined in terms of a Lagrangian function $L(x, \dot{x})$ in the sense of Beem, see Sec. 2.2, but for the reasons outlined there we require $L(x, \dot{x})$ to be defined and at least three times differentiable only almost everywhere on $TM \setminus \{0\}$. We use Einstein’s summation convention for greek indices taking value 0,1,2,3 and for latin indices taking values 1,2,3. Our choice of signature is $(+−−−)$. We use units making $\hbar = 1$, but we keep the vacuum speed of light $c$.

If it is possible to find coordinates (on an open neighbourhood $U$ in $M$) such that the Lagrangian is independent of $x$, we call the Finsler structure flat (on $U$). In this case we have no gravitational field but a spacetime that generalises special relativity in a way that violates Lorentz invariance.

We will now discuss equations of motion for particles and fields on a Finslerian spacetime. In the last part of this section we will then briefly review the various attempts of finding a Finsler generalisation of Einstein’s field equation.

4.1 The geodesic equation

With the help of the Lagrangian $L(x, \dot{x})$ we define affinely parametrised geodesics as the solutions to the Euler Lagrange equations

$$\frac{d}{ds} \left( \frac{\partial L(x, \dot{x})}{\partial \dot{x}^\mu} \right) - \frac{\partial L(x, \dot{x})}{\partial x^\mu} = 0. $$

(10)

By homogeneity, the Lagrangian $L(x, \dot{x})$ is a constant of motion,

$$\frac{d}{ds} L(x, \dot{x}) = 0 $$

(11)

along every solution of (10). As a consequence we may classify geodesics as timelike, lightlike or spacelike according to whether $L(x, \dot{x})$ is positive, zero or negative. Moreover, the fact that $L(x, \dot{x})$ is homogeneous of degree two implies that a geodesic remains a geodesic under an affine reparametrisation $s \mapsto as + b$ where $a \neq 0$ and $b$ are real numbers. This is the reason why $s$ is called an ‘affine parameter’. Along timelike geodesics we may choose the constant $a$ such that in the new parametrisation $L(x, \dot{x}) = 1$. In this case the affine parameter is called (Finsler) proper time.

As an alternative to the Lagrangian formulation, the Finsler geodesics may also be written in terms of a Hamiltonian. To that end one has to introduce the canonical momenta

$$p_\mu = \frac{\partial L(x, \dot{x})}{\partial \dot{x}^\mu} = g_{\mu\nu}(x, \dot{x})\dot{x}^\nu $$

(12)

and the Hamiltonian

$$H(x, p) = p_\mu \dot{x}^\mu - L(x, \dot{x}) = \frac{1}{2} g^{\mu\nu}(x, p)p_\mu p_\nu $$

(13)

where $g^{\mu\nu}(x, p)$ is the contravariant metric,

$$g^{\mu\nu}(x, p)g_{\nu\sigma}(x, \dot{x}) = \delta^\mu_\sigma. $$

(14)

In (13) and (14) one has to express $\dot{x}^\mu$ as a function of $x$ and $p$ with the help of (12). The timelike, lightlike and spacelike Finsler geodesics are then the solutions to Hamilton’s equations with $H(x, p)$ positive, zero and negative, respectively. As $L(x, \dot{x})$ is homogeneous of degree two with respect to the $\dot{x}^\mu$, the Hamiltonian $H(x, p)$ is homogeneous of degree two with respect to the $p_\mu$, hence

$$p_\mu H^\mu(x, p) = 2 H(x, p) $$

(15)
where
\[
H^\mu(x, p) = \frac{\partial H(x, p)}{\partial p_\mu} .
\] (16)

For the observable features of a Finslerian spacetime structure it is of crucial importance that timelike geodesics are to be interpreted as freely falling (massive, structureless) particles and that lightlike geodesics are to be interpreted as (freely propagating) light rays. This interpretation is a direct consequence of the axiomatic approach outlined in Sec. 3. The interpretation of lightlike geodesics as light rays can also be justified by considering the high-frequency limit of appropriately Finsler-modified Maxwell equations, see next section.

4.2 Maxwell equations

In the standard formalism Maxwell’s equations, and other field equations, are partial differential equations for tensor fields on the spacetime manifold \( M \). When generalising to a Finsler setting some features of this familiar situation have to be given up. Either one has to allow the fields to live on the tangent bundle \( TM \) rather than on \( M \), or one has to allow the equations to become pseudo-differential equations rather than differential equations. The first possibility was advertised by Pfeifer and Wohlfarth \[17\]. In their approach, the components \( F_{\mu\nu} \) of the electromagnetic field strength depend not only on \( x \) but also on \( \dot{x} \). This requires a rather radical change of the interpretation of an electromagnetic field which is no longer given in terms of an invariant geometric object on the spacetime manifold. The second possibility is more conservative. It was brought forward in the appendix of Lämmerzahl et al. \[20\] and by Itin et al. \[21\] and will be briefly discussed in the following.

We begin by considering a flat Finsler spacetime. We can then find coordinates such that the Lagrangian and, hence, the Hamiltonian (13) is independent of \( x \). As a consequence, (16) simplifies to
\[
H^\mu(p) = \frac{\partial H(p)}{\partial p_\mu} = g^{\mu\nu}(p)p_\nu .
\] (17)

If the space-time metric is the standard Minkowski metric, \( g^{\mu\nu} = \eta^{\mu\nu} \) where \((\eta^{\mu\nu}) = \text{diag}(1, -1, -1, -1)\), Maxwell’s equations read
\[
\partial_\mu F_{\nu\sigma} + \partial_\nu F_{\sigma\mu} + \partial_\sigma F_{\mu\nu} = 0 .
\] (18)
\[
\eta^{\rho\sigma} \partial_\sigma F_{\rho\nu} = -\mu_0 J_\nu
\] (19)

where \( F_{\rho\nu} \) is the electromagnetic field strength, \( J_\nu \) is the current density and \( \mu_0 \) is the permeability of the vacuum.

If we replace the Minkowski metric \( \eta^{\rho\sigma} \) with our flat Finsler metric \( g^{\rho\sigma}(p) \), it is natural to replace
\[
\eta^{\rho\sigma} \partial_\sigma \mapsto g^{\rho\sigma}(-i\partial)\partial_\sigma
\] (20)

where \( i \) is the imaginary unit. Then (19) becomes a pseudo-differential equation,
\[
g^{\rho\sigma}(-i\partial)\partial_\rho F_{\sigma\nu} = -\mu_0 J_\nu
\] (21)

whereas (18) remains unchanged. Eq. (21) can also be written as
\[
iH^\rho(-i\partial)F_{\rho\nu} = -\mu_0 J_\nu .
\] (22)

If the current is given, (18) and (22) define a perfectly reasonable system of first-order equations for the electromagnetic field which is a second-rank antisymmetric tensor field on the spacetime manifold, just as in the standard theory. If the Hamiltonian is specified, it is an interesting problem to determine the resulting modification of the Coulomb potential. This problem was solved in the above-mentioned paper by Itin et al. \[21\] for the case that the metric differs from the Minkowski metric by a term of
fourth order with respect to the spatial momentum components. Such a modification of the Coulomb
potential implies, of course, a modification of the hydrogen spectrum, see below.

It is not difficult to perform the passage to ray optics from the equations (18) and (22). If one
applies the operator \( \partial_\tau \) to (22) for the case that \( J_\nu = 0 \) and uses (18), one finds after a bit of algebra
that the electromagnetic field strength satisfies a generalised wave equation,

\[
H(-i\partial)F_{\mu\nu} = 0 .
\]  
(23)

This equation is solved by a plane-wave ansatz

\[
F_{\mu\nu}(x) = \text{Re}\left\{ f_{\mu\nu} \exp(ik_\sigma x^\sigma) \right\} ,
\]  
(24)

only if the wave covector \( k_\sigma \) satisfies the equation

\[
H(k) = 0 ,
\]  
(25)

which demonstrates that on our flat Finsler spacetime electromagnetic waves propagate along lightlike
straight lines.

On a curved Finsler spacetime the partial derivatives in (22) and, thus, in (23) have to be replaced
by some kind of covariant derivatives to give a coordinate independent meaning to these equations.
By a generalisation of the above argument one can then show that on a curved Finsler spacetime high-
frequency electromagnetic waves propagate along lightlike geodesics, see the Appendix of Lämmerzahl
et al. [20].

### 4.3 Klein-Gordon equation

In analogy to Maxwell’s equations, the Klein-Gordon equation can be generalised into a Finsler setting
in two quite different ways: In the first approach, which was advertised in particular by Asanov [14],
one allows the field to depend not only on the \( x^\mu \) but also on the \( \dot{x}^\mu \); the Klein-Gordon equation
is then a differential equation involving a generalised wave operator which also differentiates with
respect to the \( \dot{x}^\mu \). In the second approach one leaves the field, as in the standard theory, to depend
on the \( x^\mu \) only and allows the Klein-Gordon equation to become a pseudo-differential equation. We
will here follow the second approach.

As in the case of Maxwell’s equations, we first consider a flat Finsler spacetime given, in appropri-
ately chosen coordinates, by a Hamiltonian that is independent of \( x \),

\[
H(p) = \frac{1}{2} g^{\mu\nu}(p)p_\mu p_\nu .
\]  
(26)

If the metric is the usual Minkowski metric, \( g^{\mu\nu} = \eta^{\mu\nu} \), the Klein-Gordon equation for a complex-
valued scalar field \( \Phi \) with mass parameter \( m \) reads

\[
\eta^{\rho\sigma} \partial_\rho \partial_\sigma \Phi + m^2 \Phi = 0 .
\]  
(27)

If we replace the Minkowski metric \( \eta^{\rho\sigma} \) with our flat Finsler metric \( g^{\rho\sigma}(p) \), we replace the wave
operator in (27) according to the same rule as in (20),

\[
\eta^{\rho\sigma} \partial_\rho \partial_\sigma \mapsto g^{\rho\sigma}(-i\partial_\rho \partial_\sigma) .
\]  
(28)

This gives us the Finslerian version of the Klein-Gordon equation,

\[
g^{\rho\sigma}(-i\partial_\rho \partial_\sigma) \Phi + m^2 \Phi = 0
\]  
(29)

which can also be written as

\[
2 H(-i\partial)^2 \Phi + m^2 \Phi = 0 .
\]  
(30)
Clearly, (30) is a pseudo-differential equation for the scalar field \( \Phi \).

The non-relativistic limit of (30) gives a Finsler-modified free Schrödinger equation. This has been worked out by Itin et al. \[21\] for the case of a flat Finsler metric that differs from the Minkowski metric by terms of fourth order in the spatial momentum coordinates. In the same paper, the free Schrödinger equation was then replaced by the Schrödinger equation with a Finsler-modified Coulomb potential which allowed calculating Finsler perturbations of the hydrogen atom. We will come back to this work in Section 5.1 below.

On a curved Finsler spacetime, (29) has to be modified in two ways. Firstly, the Hamiltonian is then a function not only on the \( p^\mu \) but necessarily also of the \( x^\mu \). Secondly, one needs to make the differential operator coordinate-independent by adding terms involving Finslerian Christoffel symbols. Therefore, the resulting equation is of the form

\[
2H(x, -i\partial)\Phi + m^2\Phi + \ldots = 0 \tag{31}
\]

where the ellipses indicate a term involving a pseudo-differential operator that is homogeneous of degree one acting on \( \Phi \).

4.4 Dirac equation

A Finsler generalisation of the Dirac equation is not straightforward since there are conceptually different approaches. This is related to how the transition from the Klein-Gordon equation to the Dirac equation is performed. In the following we discuss in some detail two different routes from the Finslerian Klein-Gordon equation to a Finslerian Dirac equation. We restrict to the case of a flat Finsler spacetime before commenting, at the end of this section, on the case of a curved spacetime.

4.4.1 Reducing the Finslerian Klein-Gordon equation to first order

We first take the route which starts from the Klein-Gordon equation (30) on a flat Finsler spacetime and ask for a related first-order differential equation of the form

\[
0 = \gamma(-i\partial)\psi + m\psi. \tag{32}
\]

Here we consider a field variable \( \psi(x) \in \mathbb{C}^r \), for some \( r \in \mathbb{N} \), and \( \gamma(p) \) is an \( r \times r \) matrix that is positively homogeneous of degree one, \( \gamma(\lambda p) = \lambda\gamma(p) \) for \( \lambda > 0 \). In the following we write

\[
\gamma^\mu(p) = \frac{\partial\gamma(p)}{\partial p^\mu}, \quad \gamma^{\mu\nu}(p) = \frac{\partial^2\gamma(p)}{\partial p^\mu \partial p^\nu}. \tag{33}
\]

Owing to the homogeneity (32) can be rewritten as

\[
0 = -i\gamma^\mu(-i\partial)\partial_\mu\psi + m\psi, \tag{34}
\]

which looks like the ordinary Dirac equation but with generalised Dirac matrices \( \gamma^\mu(p) \) that are homogeneous of degree zero with respect to the \( p_\mu \). The compatibility with (30) then requires

\[
\gamma^2(p) = H(p)\mathbf{1}, \tag{35}
\]

where \( \mathbf{1} \) is the \( r \times r \) unit matrix. This equation can be rewritten as \( \gamma^{\mu\nu}(p)\gamma^{\nu\nu}(p)p_\mu p_\nu = 2g^{\mu\nu}(p)p_\mu p_\nu\mathbf{1} \). Because of the \( p \)-dependence of the \( \gamma^\mu \) it is clear that the usual Clifford algebra has to be modified,

\[
\gamma^{\mu}(p)\gamma^{\mu}(p) + \gamma^{\nu}(p)\gamma^{\nu}(p) + \gamma^{\mu\nu}(p)\gamma(p) + \gamma(p)\gamma^{\mu\nu}(p) = 2g^{\mu\nu}(p)\mathbf{1}. \tag{36}
\]

Only if \( \gamma \) depends linearly on \( p \) do we recover the usual Clifford algebra.
For a given Finsler Hamiltonian $H(p)$ it is possible to calculate the $\gamma$ matrices in terms of a series expansion with respect to the Finslerian deviation from the Minkowski space Hamiltonian $H_0(p) = \frac{1}{2} \eta^{\mu\nu} p_\mu p_\nu$. We work this out for the special case that $H(p)$ is of the form

$$2H(p) = p_0^2 - \left( (\delta^{i_1i_2} \ldots \delta^{i_{2n-1}i_{2n}} + \psi^{i_1 \ldots i_{2n}}) p_{i_1} \cdots p_{i_{2n}} \right)^{1/n}$$

(37)

with some integer $n$. The $\psi^{i_1 \ldots i_{2n}}$ are the components of a purely spatial symmetric tensor of rank $2n$. Upon writing the Hamiltonian as

$$2H(p) = p_0^2 - |\vec{p}|^2 \left( 1 + \frac{\psi(\vec{p})}{|\vec{p}|^{2n}} \right)^{1/n}$$

(38)

where

$$|\vec{p}|^2 = \delta^{ij} p_i p_j, \quad \psi(\vec{p}) = \psi^{i_1 \ldots i_{2n}} p_{i_1} \cdots p_{i_{2n}}.$$  

(39)

Taylor expansion yields

$$2H(p) = p_0^2 - |\vec{p}|^2 \left( 1 + \frac{1}{n} \frac{\psi(\vec{p})}{|\vec{p}|^{2n}} + \frac{1 - n}{2n^2} \left( \frac{\psi(\vec{p})}{|\vec{p}|^{2n}} \right)^2 + \frac{(1 - n)(1 - 2n)}{3n^3} \left( \frac{\psi(\vec{p})}{|\vec{p}|^{2n}} \right)^3 \ldots \right)$$

(40)

We want to solve (35) with the ansatz

$$\gamma(p) = \gamma^{(\mu)}(p) p_\mu = \gamma^{(0)}_\mu p_\mu + \gamma^{(1)}_\mu(p) p_\mu + \gamma^{(2)}_\mu(p) p_\mu + \ldots$$

(41)

where the $\gamma^{(\mu)}_\mu(p)$ are of the $\mu$th order in $\psi$. Inserting (41) and (40) into (35) and comparing terms of equal order in $\psi$ gives a hierarchy of equations that can be solved successively. A first step in this direction was taken in [22].

Though being quite natural and straightforward this is a rather cumbersome approach. In principle, it works for every Finsler Hamiltonian that admits a Taylor expansion about the Minkowski Hamiltonian. The resulting Dirac equation is, in general, a pseudo-differential equation for an $r$-component (spinor) field whose components are functions only of the $x^\mu$. We will now briefly outline another approach which gives us a differential equation rather than a pseudo-differential equation, but it works only for very special Finslerian structures.

### 4.4.2 Reducing a higher-order scalar differential equation to first order

For the second route we start out from a Klein-Gordon-like equation of the form

$$\eta^{\mu_1 \ldots \mu_{2n}} \partial_{\mu_1} \cdots \partial_{\mu_{2n}} \Phi = \left( \frac{m^2}{2} \right)^n \Phi$$

(42)

where $n$ is a positive integer and $\eta^{\mu_1 \ldots \mu_{2n}}$ is a symmetric tensor of rank $2n$. We may think of this equation as coming from a Finsler structure in the sense of Pfeifer and Wohlfarth [17] who allow for homogeneity of any degree. We ask the question of whether this higher-order equation can be reduced to a first-order equation which has the form of a generalised Dirac equation, $i \gamma^\mu \partial_\mu \psi - m \psi = 0$. Here we mean by ‘reduced’ that any solution of the first-order generalised Dirac equation is also a solution of the higher-order equation. A $2n$ fold iteration of the Dirac equation then leads to the condition

$$\gamma^{(\mu_1 \ldots \mu_{2n})} = \eta^{\mu_1 \ldots \mu_{2n}} 1$$

(43)

which clearly is a generalisation of the standard Clifford algebra for $n = 1$.

In the positive definite case, such systems have been discussed in the literature. Roby [23] used the concept of a generalised Clifford algebra for the linearisation of $n$-forms. Nono [24] discussed the linearisation of higher-order equations $g^{\mu_1 \ldots \mu_m} \partial_{\mu_1} \cdots \partial_{\mu_m} \Phi = \epsilon^m \Phi$ which is similar to our approach. He also derived the condition (43) and also a slightly more general condition. A short review on various concepts of generalised Clifford algebras has been given by Childs [25].
4.4.3 Other approaches and generalisations to curved Finsler spacetimes

In addition to the two routes sketched above, other approaches to a Finslerian Dirac equation have been suggested. For the case of flat Finsler spacetimes, to which we have restricted here, we mention that Finsler-type modifications of Special Relativity naturally arise in the Standard Model Extension, see e.g. Kostelecký [26]. In this context modified Dirac equations have been discussed by various authors, see e.g. Lehmert [27]. Moreover, several authors have considered modified Dirac equations on curved Finsler manifolds. To the best of our knowledge, the first attempt of formulating a Finsler version of the Dirac equation was brought forward by Asanov [14]. His Dirac spinors depend on two variables where the first one ranges over the spacetime manifold and the second one is related to a parametric representation of the indicatrix $L = 1$. Asanov calls this the ‘parametric representation of physical fields’. Bogoslovsky and Goenner [28] discussed a Finslerian metric which is conformally related to the Minkowski metric. Within this framework they also proposed a Dirac equation. The $\gamma$-matrices are still the usual ones; the modifications come in through the modified transformations of the vectors and spinors. Finally, we mention the work by Vacaru, see e.g. [29], on Clifford-Finsler algebroids and modified Dirac equations.

4.5 Generalised Einstein equation

Until now we have assumed that a Finslerian spacetime is given and we have discussed equations of motion for particles or fields on this spacetime. Now we have to address the question of how the Finslerian spacetime (i.e., the gravitational field) is determined by the distribution of energy, i.e., how Einstein’s field equation has to be modified to fit into a Finslerian setting. Einstein’s field equation is of the general form that the curvature of spacetime is algebraically related to its energy content. If one wants to preserve this general form one faces a major problem: In Finsler geometry there are various different curvature tensors but none of them lives on the base manifold; if written in local coordinates, the components of the curvature tensor are functions not only of the $x^\mu$ but also of the $\dot{x}^\mu$. Therefore, a straightforward way of generalising Einstein’s field equation would require also the energy content of the spacetime to be described by an object that depends on the $x^\mu$ and on the $\dot{x}^\mu$, i.e., by an energy-momentum tensor that lives on the cotangent bundle. This would mean a major change in interpretation: We are used to modelling the energy content of a spacetime in terms of tensor or spinor fields on the spacetime; letting the fields depend on the $\dot{x}^\mu$ would be a way of saying that they are not such invariant (i.e., coordinate independent and observer independent) fields on the spacetime. We emphasise that this problem occurs only at the level of the gravitational field equation: As long as we consider (Maxwell, Klein-Gordon, Dirac ... ) fields on a Finsler background spacetime, without taking the self-gravity of these fields into account, we may very well consider these fields as tensor or spinor fields on the spacetime, as demonstrated in the preceding sections.

Several different Finsler generalisations of Einstein’s field equation have been suggested, but it seems fair to say that up to now none of them is generally accepted. To the best of our knowledge, the first such generalisation was brought forward by Rund and Beare [30] in 1972. Their set-up is based on curvature and energy quantities that depend on the $x^\mu$ and on the $\dot{x}^\mu$, as indicated above. Since the authors were not able to formulate a law of local energy conservation, as it is valid in the standard formalism, they themselves doubted that their approach gives a physically reasonable generalisation of Einstein’s field equation. A few years later, Asanov [31] made a different suggestion. He reduced the curvature quantities from the tangent bundle over spacetime to the spacetime itself by what he called the notion of ‘osculation’: He chose an auxiliary timelike vector field $V^\mu(x)$ and replaced, in the argument of the curvature quantities, the $\dot{x}^\mu$ with $V^\mu(x)$. The problem with this approach is, of course, that the geometry of spacetime is not invariant but depends on a vector field that may be interpreted, at each point, as the four-velocity of an observer. Such an observer-dependent geometry is very much against the spirit of general relativity.
A completely different approach was suggested by Rutz [32]. She restricted herself to the question of how the *vacuum* Einstein equation could be generalised into a Finslerian setting. As a guiding principle, she used the idea that the vacuum field equation should express the fact that the tidal tensor be trace-free. This is true in Newtonian theory, where the tidal tensor is the Hessian of the Newtonian potential, and also in Einstein’s theory, where the tidal tensor is the Riemannian curvature tensor. Consequences of the resulting vacuum equation have been discussed in some detail. In particular a plethora of spherically symmetric solutions has been found, see Rutz [33]. The generalisation to the matter case, however, remains an open problem.

Another version of a vacuum field equation, in this case for Finsler spacetimes with a certain product structure, was investigated in several papers by Vacaru, see for instance [34]. In particular, black-hole solutions of this field equation have been worked out.

Finally, we mention the work by Pfeifer and Wohlfarth [35] who brought forward a Finsler generalisation of Einstein’s field equation, based on their definition of Finsler spacetimes which is a generalisation of Beem’s, see Section 2.2. These authors decidedly take the view that curvature and energy quantities should, indeed, depend on the $x^\mu$ and on the $\dot{x}^\mu$, and they discuss the corresponding notions of observers and measurements in some detail. Although their approach is certainly satisfactory from a mathematical point of view, we believe that there are still open questions in view of the physical interpretation. Therefore, in our view, the problem of finding a Finsler generalisation of Einstein’s field equation is still open.

5 Experimental tests

5.1 Finslerian violation of Lorentz invariance

In sufficiently small spacetime regions we may neglect gravity, i.e., we may approximate the spacetime metric by a flat metric. At this level of approximation, a hypothetical Finsler modification of spacetime theory comes up to replacing the Minkowski metric of special relativity with a flat Finsler metric. A characteristic feature of such a modification is a violation of Lorentz invariance, in particular of spatial isotropy, which can be experimentally tested in various ways.

One example is the test of anisotropies in the propagation of light with the help of Michelson interferometry, see Lämmerzahl et al. [36]. If optical resonators are used instead of traditional Michelson interferometers, the isotropy of the velocity of light has been verified with extremely high accuracy. This gives very strong limitations on Finsler perturbations based, however, on the assumption that only the light propagation but not the length of the resonator (or of the interferometer arms) is affected by the Finsler perturbation in a measurable way. In the above-mentioned paper arguments are given why this assumption is, indeed, justified.

As another possibility, Finslerian anisotropies may also be observed with the help of spectroscopy. We have seen in Section 4.2 that in a Finsler spacetime the Coulomb law will be modified. As a consequence, the energy levels of the hydrogen atom will change. We have worked this out for a flat Finsler metric given by a Hamiltonian $H$ of the form

$$2 \ H (p) = p_0^2 - \sqrt{\left( \delta^{ij} \delta^{kl} + \psi^{ijkl} \right) p_i p_j p_k p_l} ,$$  \hspace{1cm} (44)

see Itin et al. [21]. Here $\psi^{ijkl}$ is a totally symmetric spatial fourth-rank tensor that describes a Finsler perturbation of the Minkowski metric. Assuming that the $\psi^{ijkl}$ are so small that all equations can be linearised with respect to them, the Hamiltonian can be simplified by a linear coordinate transformation to the form

$$2 \ H (p) = \eta^{\mu\nu} p_\mu p_\nu - \frac{2 \phi^{ijkl} p_i p_j p_k p_l}{\delta^{mn} p_m p_n} ,$$  \hspace{1cm} (45)

where the redefined Finsler perturbation tensor $\phi^{ijkl}$ is totally symmetric and trace-free, so there are 9 independent components. After determining the Finsler modified Coulomb potential we have set up
the Finsler modified Schrödinger equation and calculated the eigenvalues for the quantum numbers \( n = 1, \, n = 2 \) and \( n = 3 \) of the Hamiltonian with the help of perturbation theory. In general, the Finsler coefficients give rise to a splitting of the Lyman-\( \alpha \) line (transition from \( n = 2 \) to \( n = 1 \)) and of the Lyman-\( \beta \) line (transition from \( n = 3 \) to \( n = 1 \)). If we observe, with a measuring accuracy \( \delta \omega \) of the frequency, that these two lines do not split, our calculated values of the shifts lead to upper bounds on the \( |\phi_{ijkl}| \) in the order of \( 10^{-17} \delta \omega/\text{Hz} \). As frequencies can be measured in the optical and in the ultraviolet with an accuracy of up to \( \delta \omega \approx 10^{-7} \text{Hz} \), we see that atom spectroscopy gives us bounds on the Finsler coefficients in the order of \( 10^{-24} \). Nuclear spectroscopy might give even smaller bounds, but this has not been worked out until now.

5.2 Solar system tests of Finsler gravity

The PPN formalism, which is routinely used as a mathematical framework for modelling possible deviations from Einstein’s theory, is restricted to theories where the gravitational field is described in terms of a pseudo-Riemannian metric. However, similar post-Newtonian expansions have also been developed for special Finsler metrics which have been suggested as hypothetical Finslerian models of the Solar system. In particular, Lämmerzahl et al. [20] used such expansions for a certain quartic Finsler metric. In another paper [38] he considered a Finsler metric that differs from a pseudo-Riemannian metric only by a (nowhere vanishing) scalar factor; such Finsler spacetimes are, of course, very special because they have the same lightlike geodesics as a pseudo-Riemannian metric, so the laws of light deflection are unaffected by this kind of Finsler modification.

For this reason, we suggested a different mathematical setting for Solar system tests of Finsler gravity, see Lämmerzahl et al. [20]. In spirit it is similar to the PPN formalism but the mathematical technicalities are different. We start out from a Finsler spacetime with a Lagrangian \( L \) of the form

\[
2L = (h_{tt} + c^2 \psi_0) \dot{t}^2 - \left( h_{ij} \dot{x}^i \dot{x}^j + \psi_{ijkl} \dot{x}^i \dot{x}^j \dot{x}^k \dot{x}^l \right)^{\frac{2}{3}}.
\]

Here

\[
h_{tt} \dot{t}^2 - h_{ij} \dot{x}^i \dot{x}^j = \left( 1 - \frac{2GM}{c^2 r} \right) c^2 \dot{t}^2 - \frac{dr^2}{1 - \frac{2GM}{c^2 r}} - r^2 \left( \sin^2 \theta \, d\varphi^2 + d\theta^2 \right)
\]

is the Schwarzschild metric, the spatial perturbation tensor field \( \psi_{ijkl} \) is spherically symmetric and independent of \( t \), and the time perturbation \( \psi_0 \) is a function of \( r \) only. The fourth-order term \( \psi_{ijkl} \dot{x}^i \dot{x}^j \dot{x}^k \dot{x}^l \) may be viewed as the leading order term in a general Finsler power–law perturbation of the spatial part of the metric. Note that (46) is an example of a Finsler Lagrangian which does not satisfy Beem’s definition because it is not defined and three times continuously differentiable on all of \( TM \setminus \{0\} \): It is not well behaved on vectors tangent to a \( t \)-line. The fact that we want to include static metrics of this form is the main motivation why we relaxed Beem’s definition by requiring \( L \) to be defined and three times continuously differentiable only almost everywhere on \( TM \setminus \{0\} \), recall Section 2.2.

We assume that the Finsler perturbation is so small that we may linearise all expressions with respect to \( \psi_{ijkl} \) and \( \psi_0 \). Moreover, because of the spherical symmetry we may restrict to the equatorial plane \( \theta = \pi/2 \) when discussing timelike and lightlike geodesics. After an appropriate transformation of the radius coordinate the Lagrangian can then be rewritten as

\[
2L = (1 + \phi_0) h_{tt} \dot{t}^2 - (1 + \phi_1) h_{rr} r^2 - r^2 \dot{\varphi}^2 - \frac{\phi_2 h_{rr} r^2 \dot{\varphi}^2 \varphi^2}{h_{rr} r^2 + r^2 \dot{\varphi}^2}
\]

with redefined Finsler perturbations \( \phi_0, \phi_1 \) and \( \phi_2 \) which are functions of \( r \) only. Note that \( \phi_0 \) just changes the time measurement while \( \phi_1 \) changes the radial length measurement, i.e., these two perturbations just lead to a modified pseudo-Riemannian metric. By contrast, \( \phi_2 \) describes a genuine Finsler perturbation. We call \( \phi_2 \) the ‘Finslerity’.
In Lämmerzahl et al. [20] we calculated the orbits of timelike and lightlike geodesics in the geometry given by (48). Considering this geometry as a model for the Solar system, this allowed us to determine the effect of the Finsler perturbation on the perihelion precession of planets and on the time delay and the deflection of light. It was our main goal to find observational bounds on the Finslery. Assuming that the Finsler perturbations have a fall-off behaviour as

$$\phi_A(r) = \phi_{A1} \frac{2GM}{c^2 r} + O\left(\left(\frac{2GM}{c^2 r}\right)^2\right), \quad A = 0, 1, 2,$$

we found from Solar system observations that

$$|\phi_{21}| \lesssim 10^{-3}.$$  

This bound is surprisingly weak, much weaker than the bounds on Finsler perturbations from atom spectroscopy, cf. Section 5.1.

We conclude this section with the remark that tests of GR with Solar system ephemerides are usually based on the PPN formalism and do not cover Finsler perturbations. It might be worthwhile to include some kind of ‘Finsler parameter’ into these considerations.

### 5.3 Redshift experiments

If light is emitted with a certain frequency $\omega_1$ by an observer, it will in general arrive with a different frequency $\omega_2$ when received by another observer. The quantity

$$z = \frac{\omega_1 - \omega_2}{\omega_2}$$

is called the redshift. Here it is understood that $\omega_1$ and $\omega_2$ are measured with respect to standard clocks by the respective observer. In general, $z$ comes about as a combination of effects from the relative motion (Doppler shift) and from the spacetime geometry (gravitational redshift). Redshift experiments are appropriate for testing spacetime theories on Earth, in the Solar system and at cosmological scales.

Obviously, redshift experiments crucially depend on the notion of standard clocks. On a Finsler spacetime, a standard clock is defined as a clock that parametrises its worldline with a parameter $\tau$, called (Finsler) proper time, according to

$$g_{\mu\nu}(\gamma(\tau), \dot{\gamma}(\tau)) \dot{\gamma}^\mu \dot{\gamma}^\nu = 1.$$  

Here $g_{\mu\nu}$ denotes the Finsler metric and $\gamma^\mu(\tau)$ is the parametrised worldline. Einstein’s synchronisation procedure can then be used in the usual way for assigning a radar time and a radar distance to events in a neighbourhood of a standard clock, cf. Pfeifer [39].

For any two worldlines parametrised with proper time, $\gamma_1(\tau_1)$ and $\gamma_2(\tau_2)$, the redshift can be written as

$$1 + z = \frac{g_{\mu\nu}(x(s_1), \dot{x}(s_1)) d^{s_1}_\tau(\gamma_1)}{g_{\rho\sigma}(x(s_2), \dot{x}(s_2)) d^{s_2}_\tau(\gamma_2)}.$$  

where $x^\mu(s)$ is a light ray connecting the emission event at parameter value $s_1$ to the reception event at parameter value $s_2$. For a derivation and a detailed discussion of this general redshift formula in Finsler spacetimes we refer to a forthcoming article by Hasse and Perlick [40]. The formula looks exactly the same as the familiar redshift formula in a general-relativistic spacetime (see, e.g., Straumann [41]), with the only modification that now the metric depends also on the tangent vector to the light ray.
The general redshift formula allows to test Finsler geometry on Earth, in the Solar system and in cosmology. Details will be given in the above-mentioned paper by Hasse and Perlick. Moreover, for cosmological redshift tests we refer to Hohmann and Pfeifer [42] who discuss the distance-redshift relation in a cosmological Finsler spacetime based on the field equation suggested by Pfeifer and Wohlfarth [35].

6 Conclusions

Finsler geometry is a very natural generalisation of pseudo-Riemannian geometry and there are good physical motivations for considering Finsler spacetime theories. We have mentioned the Ehlers-Pirani-Schild axiomatics and also the fact that a Finsler modification of GR might serve as an effective theory of gravity that captures some aspects of a (yet unknown) theory of Quantum Gravity. We have addressed the somewhat embarrassing fact that there is not yet a general consensus on fundamental Finsler equations, in particular on Finslerian generalisations of the Dirac equation and of the Einstein equation, and not even on the question of which precise mathematical definition of a Finsler spacetime is most appropriate in view of physics. We have seen that the observational bounds on Finsler deviations at the laboratory scale are quite tight. By contrast, at the moment we do not have so strong limits on Finsler deviations at astronomical or cosmological scales.

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