Hyperbolic kaleidoscopes and Chaos in foams and Hele-Shaw cell

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Abstract. Liquid foams have fascinating optical properties, which are caused by the large number of light refractions and reflections by liquid films and Plateau borders. Due to refraction and reflection at the interfaces, the direction of the rays leaving a Plateau border can vary greatly for the same incident angle and a small positional offset. A close look in some configurations of the Plateau borders or liquid bridges reveals the existence of some triangular patterns surrounded by a complex structure, and these patterns bear a resemblance to those observed in some systems involving chaotic scattering and multiple light reflections between spheres. Provided the optical properties of the sphere surfaces are chosen appropriately, fractals are natural consequences of multiple scattering of light rays in these cavities. The cavity acts as a hyperbolic kaleidoscope multiplying the scattering of light rays generating patterns related to Poincaré disks and Sierpinski gaskets in comparison to linear kaleidoscopes. We present some experimental results and simulations of these patterns explained by the light of the chaotic scattering.

1. Introduction

Foams have been intensively investigated for many years as natural phenomena that are prominent in everyday life. The propagation of light in foams has received attention, and a number of patterns are generated spontaneously due to the reflection and refraction of light. One of these patterns is observed by the formation of some triangular images inside liquid bridges in a single layer of bubbles between two plates, known as the Hele-Shaw cell. By inspection, these patterns bear a resemblance to those observed in some systems involving chaotic scattering and multiple light reflections between spheres [1]. These patterns can be obtained using mirrored spheres, and basically the image obtained is due to the fact that the different spheres are mirrored in each of the other spheres giving rise to multiple mirror images [2]. Similar systems presenting fractal-like patterns have been studied by some authors [3, 4, 5].

The use of fractal characteristics may have a potential to control and localize light in a different way from the conventional optical devices, and lead to new engineering applications of light spreading.
devices such as optical links [6]. For the case of liquid bridges and Plateau borders, the curvature of liquid/air interface acts as scatterer, and the direction of the rays leaving the liquid bridges can vary greatly for the same incident angle with only a small positional offset, due to refraction and reflection at the interface, and consequently the occurrence of chaotic scattering is also possible. In this paper we experimentally observed these fractal-like patterns of chaotic scattering of light in hyperbolic kaleidoscopes and foams.

The goal of this paper is to describe the existence of these triangular patterns in foams and their relation with the amazing images obtained from the chaotic scattering of light in spheres and spherical shells. We are going to look how these patterns observed in spheres are related with fractal dimension. After that, we associate some aspects of chaotic scattering of cavities with negative curvature with patterns observed in hyperbolic kaleidoscopes. Some dynamical features of the light scattering of these hyperbolic optical elements can be explored using non-Euclidean geometry and recurrence equations. In addition to this, we look at some aspects of Kolmogorov-Sinai entropy in soft billiards in order to throw some light in similar systems with light scattering in the presence of reflection and refraction.

2. Experimental Apparatus and methods

We have used different setups, involving Hele-Shaw cell, reflective spheres and hyperbolic prisms. Some images were obtained in a transparent Hele-Shaw cell, the setup I, consisting of two plain parallel Plexiglas plates separated by a small gap (20 x 20 x 0.2 cm³). The cell contains only air and an amount of commercial dishwashing liquid (V = 30 cm³). This liquid is manufactured by Bombril, and is used without dilution. The essential surfactant is Linear Alkylbenzene Sulfonate (LAS). The surface tension is γ = 25 dyne/cm, and the density of this detergent is ρ = 0.95 g/cm³, with refractive indices of liquid n_l = 1.333, and n_g = 1.0 for the air, see reference [7] for more details. For the case of reflective spheres and spherical shells, the configurations were constructed with two sets of reflective surfaces, the setup II is a set of three reflective spheres.

The setup III is a set of three reflective spherical shells in order to construct hyperbolic kaleidoscopes. A small sphere was placed between the spherical shells as the real object of a hyperbolic kaleidoscope. We also have constructed hyperbolic prisms with refractive index n_p = 1.32, in order to obtain images of multiple reflections of light in “solid” foams using a beam from a diode laser (650 nm, 5 mW), that we call setup IV. The light source used when photographing setups I, II

Figure 1. (a) The liquid bridges observed in a Hele-Shaw cell and some intriguing triangular patterns. (b) Image obtained from three Christmas ball ornaments similar to those obtained from liquid bridges in Hele-Shaw cells. Inset, simulation obtained with ray tracing technique for a cavity inside three spheres showing the triangular pattern.

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and III was a 30W fluorescent lamp, with a diffusive screen and backlighting technique. We have used some routines written in C and commercial software of ray-tracing image (Mathematica) in order to simulate the light scattering in spheres and spherical shells.

3. Triangular patterns
In figure 1, there are some examples of these patterns obtained with our experimental apparatus. In figure 1(a), the liquid bridges observed in a Hele-Shaw cell have scattered the light forming some intriguing triangular patterns. These images are compared to the image obtained from three Christmas ball ornaments in figure 1(b). The inset of figure 1(b) is the simulation obtained with ray tracing technique for a cavity inside three spheres showing the same triangular pattern. The liquid bridges shown in figure 1(a) act as curved refractive-reflective surfaces which change with time. For example, in figures 2(a)-(b), the foam gets thinner over time, leading to a variation of these patterns with time.

![Figure 2.](image)

**Figure 2.** (a)-(b) The effects of the light refraction in the liquid bridges can be observed by the distortion of the frame behind the Hele-Shaw cell. From (a) to (b) we can observe the difference provided by the liquid drainage. (c)-(d) Sketch of the rays paths in pure reflective cavities, and (e) the rays paths in liquid bridges.

We can observe the net result of the foam drainage by the optical deformation of a square grid behind the cell. Assuming that the liquid wets perfectly the plates, the liquid bridge boundary is made up of arcs of circles. For a pure reflective case of figures 2(c)-(d), the geometry of the cross-section of
spherical shells affects its scattering properties. Taking into account the refraction and reflection, the light rays with only a small difference in the incident position impacting the liquid/air interface may cause different light paths, as it is shown in figure 2(e), and consequently the occurrence of chaotic scattering is also possible. Moreover, we can see that light reflection is much stronger in the pure reflective case than in the liquid bridges and the Plateau borders, which implies that it is mainly the multiple reflections, which attribute to the generation of the triangular patterns. Quite generally, these patterns are related to some geometrical features of the set up and the fractal dimension of the chaotic repeller formed by the cavity between the spheres. We will discuss some of these features in the next section.

4. Fractal dimension and kaleidoscopes
The fractal mirror images can be explored using optical billiards, which consist of cavities between polished, reflective surfaces of spheres. Exploiting multiple reflections of light between the spheres, a variety of fractal optical effects can be observed, and some interesting properties of these optical systems were summarized in reference [8].

![Figure 3](image)

**Figure 3.** (a) The lateral reduction is related to the size of the image $h_i$ and the apparent size of the sphere $h_o$ given by the semiangle $\phi_{\text{max}}$. Example of $r/L = 0.5$. (b) Example of the lateral reduction for the case of $r/L = 0.4$. (c) The fractal dimension obtained with equation (7) compared to numerical values obtained from references [3, 9, 10]. (d) Multiple scattering of a pencil of trajectories in three sphere system for the case of $r/L = 0.49$, with the fractal dimension close to 1.
A three dimensional optical billiard can be considered as a discrete time dynamical system mapping a vector describing an incident ray at the $n$th reflection from the scatterer to the $(n+1)$th reflection, and these reflections create the mirror images. These mirror images show a fractal structure, whose fractal dimension increases with the number of spheres in a cluster as well as with the ratio of the radius of each sphere, $r$, to the distance between two spheres, $L$. The fractal dimension $D$ has been discussed and calculated numerically for the case of relatively small $r/L$ ratios by Eckhardt [9], while Korsch and Wagner [3] studied the fractal dimension $D$ of mirror images for larger values of $r/L$ ratios. We have to define the lateral reduction factor $p$, which is characterized by the ratio involving the image of the object and the object, creating the effect of lateral reduction. Assuming large angle of reflection, and taking into account the distance $L$ between two identical spheres with radius $r$, the lateral reduction is defined by the height of the image $h_i$ divided by the height of the part of the object $h_o$, figures 3(a)-(b), given by $p = h_i/h_o$, and it is approximately given by $p = [2(L/r-1)]^{-1}$ [3]. The rays reaching the surface of the object from the center of the mirrored sphere must lie within a cone of semiangle $\phi_{\text{max}}$ given by $\sin(\phi_{\text{max}}) = r/L$, which gives $h_o$ smaller than the sphere diameter $d = 2r$. For example, for the case of spheres in contact ($r = L/2$) represented in figure 3(a), the lateral reduction is $p = 0.5$. In figure 3(b), increasing the distance between the spheres so that $r/L = 0.4$, the lateral reduction is $p = 1/3$.

For the case of three spheres the number of mirror images, $m = 2$, there is an approximate formula for the fractal dimension. The fractal sets can be considered as a limiting set of $n$ reflections, for $n$ going to infinite, where each reflection is the reduction of the set to $1/m$ of the previous set (See reference [10]). The scaling factor $s^{-1}$ enables us to generalize the dimensionality of this approach. When $A_n$ is the total measure of reflection $n$, it can be covered by $N(\varepsilon_n) \sim \varepsilon^{-D}$ intervals of size $\varepsilon = A_n m^{-n}$ and we obtain:

$$A_n = \varepsilon_n N(\varepsilon_n) \sim \varepsilon^{1-D} = (A_n m^{-n})^{1-D},$$

and the $A_n$ will decrease exponentially for large values of $n$, so that

$$A_n \sim m^{-n(1/D-1)} = e^{-\gamma n},$$

with the rate $\gamma$ given by

$$\gamma = (1/D - 1) \ln m.$$  

It follows that

$$D = \frac{\ln m}{\ln m + \gamma} = \frac{\ln m}{\ln(m\delta)},$$

where

$$\delta = e^{\gamma} \approx \delta_n = \frac{A_{n-1}}{A_n}$$

is approximately the ratio between the measure of two successive reflections. For a self-similar set with scaling factor $s$ we have $A_n = m A_{n-1}/s$, and we recover the definition of self-similar set as

$$D = \frac{\ln m}{\ln p},$$

which gives in the case of spheres:

$$D = \frac{\ln m}{\ln[2(L/r-1)]}.$$
In figure 3(c) is shown the fractal dimension for the case of three spheres with different $r/L$ ratios. This fractal dimension is a measure of the chaoticity of the dynamics. In order to explain this system, we can represent the three spheres as a sequence of partitions formed by planar hard disk system. The dimension of each partition reduces to zero in the limit $r/L$ approaching to 0, and to unity for $r = L/2$, for the case of spheres in contact. For example, for the case of $r/L = 0.4$ the fractal dimension $D = 0.631$. In this way, for three spheres the fractal dimension ranges from 0 to 1 depending on the $r/L$ ratio.

The existence of chaotic scattering was obtained numerically by Poon et al. [5] in a similar system, in which a three-disk system presents chaotic saddle with Wada property, and depending on the angle of incidence, it is conjectured the existence of trajectories trapping the light rays. However, in real systems it is not possible to observe this kind of trajectories because the reflectance of the sphere surface is not perfect. In addition to this, the observed images are created by the light rays which enter from one side of the cavity and exit at the opposite side. In systems like these, there are no trapped orbits, but there are orbits in which the light ray will leave the scattering region only after many collisions. José et al. [11] have reported the existence of stable and unstable manifolds in the elastic particle scattering from two hard disks. It is important to note that, two orbits starting out close together on the same side of the stable manifold may have their last hits on different disks and hence leave the scattering region in a quite different direction. This sensitiveness to initial condition can be observed in figure 2(c). The closer to the stable manifold, the greater is the difference of the directions of the light rays. Consequently, the light rays in the three sphere system are scattered by a cavity with properties very close to those proposed by the theory of chaotic scattering.

5. Hyperbolic geometry and Sierpinski gasket

At this point it is interesting to understand the geometric structure of the images obtained from the experiment. These multiple reflections are related to some properties of kaleidoscopes. For the case of regular three-mirror kaleidoscope, it creates an image that repeats one triangular pattern continuously, as it is shown in figure 4(a), hence producing a faceted pattern that fills the entire field of view, with a uniform tiling of the plane with triangles. For the case of the cavity between three spheres, the image produced shows a three-dimensional appearance in comparison with a regular kaleidoscope. We can consider that the cavity acts as a hyperbolic kaleidoscope, multiplying and distorting the images of some object placed at the center of the cavity.

The coding of generalized straight lines on curved surfaces or geodesics on a surface of negative curvature can be found in reference [12]. In our case, this coding represents the tessellation of triangles forming the border of the sequence of reflections from the initial conditions on a spherical surface. This approach involves the Fuchsian groups, which do for non-Euclidean geometry what crystallographic groups do for Euclidean geometry. In order to give a hint of this kind of image distortion, we can explore some aspects of non-Euclidean geometry. In general, the mathematics of kaleidoscopes in $N$ dimensions is the study of finite groups of orthogonal $N \times N$ real matrices that are generated by reflection matrices. In that way, patterns observed in kaleidoscopes are related to Möbius transformations such as reflection symmetries, even in a non-Euclidean space. The three mirrored spheres could be an analogy to a stereographic projection of a regular kaleidoscope. Such stereographic projection is the Poincaré hyperbolic disk, as the image obtained experimentally by us in figure 4(b), with hyperbolic tensor metric [13]:

$$ds^2 = \frac{dx^2 + dy^2}{(1 - x^2 - y^2)}.$$ (8)
Figure 4. (a) Diagram of the reflection of a regular three-mirror kaleidoscope. (b) Poincaré disc model and lines connecting the reflected object forming the Serpinski-like gasket.

Using this metric, the lines are circular arcs that cut the boundary of the disk at right angles, and the angles in this model are the same as the Euclidean angle, the lines are badly distorted. We can observe these effects of the kaleidoscope placing an object at the center of the cavity (figure 5). Some of these transformations, called isometries, are indicated by the reflections and rotations of the letters ABC in the Poincaré disk model, as it is seen in figure 4(b). The parallel lines extending to the infinity in the kaleidoscope of plane mirrors are transformed in accumulation points at the border of the Poincaré disk. The regions ABC related to different initial conditions of the multiple reflections form a fractal-like pattern. Notice that as the lines separating each region approaches the border of the disk, the image repeats in finer and finer detail, approaching infinity.

The outstanding aspect of these images (figures 4(b) and 5(a)-(b)) is the representation of the Sierpinski gasket in the reflections [4]. However, it is not an exact Sierpinski gasket because it is distorted by the curvature of the sphere surfaces, consequently the sections are only triangular in shape instead of being true triangles, and the vertices are lost in the multiple reflections that arise where two spheres touch.

To illustrate a similar Sierpinski gasket effect in a three-sphere system, we have obtained the image of figure 5(a). Using ray trace technique, we have simulated three spherical shells touching each other. Inside the cavity between these spherical shells, there is a small sphere acting as object of this kaleidoscope. The bright and the contrast were adjusted to enhance the sequence of reflections of this small sphere. Figure 5(b) reproduces the image of a hyperbolic kaleidoscope with a small sphere acting as the object placed in the cavity. The configuration of this hyperbolic kaleidoscope correlates with ray tracing arrangements of spheres of different sizes.
6. Reflection and Refraction

The analogy between chaotic scattering in dynamical systems and light scattering in liquid bridges is based on the fact that in both cases there is a bounded area in which light rays or particles bounce back and forth for a certain time. For example, in figure 6(a), we present a configuration of liquid bridges in a Hele-Shaw cell containing three triangular zones with multiple reflections and refractions. We have compared one of these regions (figure 6(b)) with the pure reflective case of a similar configuration using reflective shells (figure 6(c)). In both images we can observe the existence of a distorted triangle surrounded by a toothed edge. We can observe some differences between these two images. In figure 6(b) there are black regions in the edge of these triangles, because most part of the light was refracted and reflected internally, while in figure 6(c) there is no black region, and the edges are well defined.
The high curvature shapes of the liquid bridges can cause incident light to be scattered in many
different directions through reflection, in addition to refraction [14]. Given an arbitrary frame of
reference, any incoming ray can be identified by its direction $\theta_i$ and by the impact parameter $b_i$. The
impact parameter is proportional to the angular momentum $I_i = k b_i$, where $k$ is the wave number. The
geometrical relationship between the incident and outgoing ray is given by Snell’s law

$$ n_i \sin \theta_i = n_t \sin \theta_t, $$

where $n_i, n_t$ are the refractive indices of two media, $\theta_i$ is the angle of incidence and $\theta_t$ is the angle of
refraction.

For the case of a gas/liquid interface the law of reflection states that $\theta_i = \theta_r$ (reflected), and when $n_i > n_t$ there exists the total internal reflection if the angle of incidence is above the critical angle, $\theta_c$, and
the reflectivity at normal incidence, $R_{rf}$, are given by

$$ \theta_c = \arcsin \left( \frac{n_g}{n_l} \right), $$

$$ R_{rf} = \left( \frac{n_l - n_g}{n_g + n_l} \right)^2. $$

The fraction of the light that is reflected is given by the Fresnel reflection coefficient, a value which
depends on the indices of refraction of liquid and gas ($n_l, n_o$) and also on the incident angle of the light. A fraction of the light entering the liquid will also be reflected inside the liquid. According to Suffern [4], the Sierpinski gaskets obtained by reflection are result of reflections at nearly normal incidence. For the case of gas/liquid interfaces, the reflectivity decreases the observation of Sierpinski gaskets, because the reflectivity of this interface at normal incidence is about 2 %, in contrast to the case of polished gold, about 50% in white light [15]. Due to the curvature and physical properties, the liquid
bridges can be seen as hyperbolic prisms.

The mapping relating the incoming coordinates between successive collisions described in
reference [16] is given by:

$$ \theta_o = \theta_i, $$

$$ a_o = \begin{cases} a_i, & \text{if } \sin^2(\theta_i) > 1 - U, \\ a_i + 2 \arccos \left( \frac{\sin \theta_i}{\sqrt{1 - U}} \right), & \text{otherwise}, \end{cases} $$

where the $o$ subscripted stands for output and we called this mapping of $a$ map.

Equations (12) represent the cases of reflection and refraction of the particle, in which $U$ is the
finite potential height of the interior of all disks with radius $r$ measured in units of kinetic energy. We
also have the light rays interacting with the scatterers in the plane centered at sites of a triangular
lattice, with distance $L = 2\sqrt{3}$, in which we have the T map with the relations between $(\theta_o, a_o, (\theta_i, a_i)$
and $(b_o, \beta_o)$ [16]:

$$ T = \begin{cases} R(\sin \beta_i - \sin \theta_o) = M \sin(a_i + \theta_o) + N \cos(a_i + \theta_o), \\ R(\cos \beta_i - \cos \theta_o) = -M \cos(a_i + \theta_o) + N \sin(a_i + \theta_o), \\ \pi + b_i = a_o + \theta_o + \beta_o. \end{cases} $$
The location of the scatterers is determined by a pair \((N, M)\). The output of this model is used in the computations of the entropy of section 8.

7. Hyperbolic Prism acting as a hyperbolic kaleidoscope

We have used three spheres, touching each other, as a mold to make hyperbolic prisms. The material has a refractive index equal to 1.32 and it scatters some of the light passing inside, so we can have the image of the light beam inside the prism. The three spheres mold corresponds to the bubbles of the foam. As in the liquid bridges, we have a material more refractive than the air in the bubbles. Similarly to the case of hyperbolic kaleidoscopes, the total internal reflection of light can be observed in hyperbolic prisms. For an incoming laser beam we can observe the quasi-self-similarity of the reflections as it is shown in figure 7(a).

![Image of a laser beam injected in a hyperbolic prism with multiple internal reflections](image)

![Plot of recurrence equation obtained from hyperbolic prism](image)

![Zooming the region inside the dashed lines of figure 7(b)](image)

![Another magnification of the middle left region inside the dashed lines of figure 7(c)](image)

**Figure 7.** (a) Image of a laser beam injected in a hyperbolic prism with multiple internal reflections (b) Plot of recurrence equation obtained from hyperbolic prism. (c) Zooming the region inside the dashed lines of figure 7(b) we can observe a similar pattern. (d) Another magnification of the middle left region inside the dashed lines of figure 7(c) contains a small copy of the previous pattern.

The projection of a straight line in a spherical surface is a circle in the plane determined by the line and the center of the spherical surface. In any other plane, the image is an ellipse. Considering the Möbius transformations due to multiple reflections discussed previously, we have obtained a recurrence equation given by fitting an elliptical curve to the image of Fig. 7(a):

\[
f_n(x) = \sqrt{P_1 x^2 + \frac{P_2}{2^{2n}} + \sqrt{P_2 \left(1 - \frac{1}{2^n}\right)}},
\]  

(14)
with \( P_1 = 0.695(2) \) and \( P_2 = 203,100(200) \), using pixel units. Using the symmetry of the pattern of Fig. 7(a), the recurrence equation \( f_n(x) \) represents the upper side of figure 7(a), while \( -f_n(x) \) represents the down side. We can define a sequence in which each ellipse is a reflection of the preceding one, as it is shown in figure 7(b). This pattern appears approximately identical at different scales. For example, figures 7(c)-(d) are magnifications of the middle left of the previous pattern. This result demonstrates the existence of patterns due Möbius transformations in a hyperbolic prism acting as a reflecting prism, similar to those observed in foams.

8. The Kolmogorov-Sinai entropy

The question at this point is: is there chaos in a system with refraction and reflection? Due to refraction and reflection at the interfaces, the direction of the rays leaving the liquid bridges can vary greatly for the same incident and only a small positional offset, and consequently there is the possibility of chaotic scattering, but how to characterize this chaotic effect? In the case of foam, it is not possible to define the exit between the interfaces in a similar way as in the case of spheres, because of the transparency of the foam. So we have to explore other ways to characterize this dynamical system.

In systems with escape, the rate of escape of particles from an open system is expressed in terms of the sum of the positive Lyapunov exponents and the Kolmogorov-Sinai entropy on the repeller, it can be found in reference [17]. Using similar maps (equations (12) and (13)) as mentioned above, some models of physical systems present chaotic scattering involving both reflection and refraction [16]. Such physical systems represent the propagation of rays passing from one medium to another, such as composite billiards [18], or soft billiard systems [16]. Baldwin has provided an empirical formula for the Kolmogorov-Sinai entropy \( h \) [16]:

\[
h = \ln \left[ 1 + \frac{U}{r_l^2} + 1.29 \left( \frac{U}{r_l^2} \right)^{0.4} \right],
\]

where, again \( U \) is a finite potential height defined by the refractive index \( n \), and \( r_l \) is the radius of each scatterer. Equation (15) is valid for the general case of a geometrical optical system, where a light ray travels through a lattice of circular scatterers with refractive index \( n = (U - 1)^{1/2} \), in a medium where \( n = 1 \). For the pure reflective case, \( U = 1 \). The signature of chaos is obtained by the computation of the Kolmogorov-Sinai entropy \( h \). The Kolmogorov-Sinai entropy \( h \) is the measure of the information loss in \( N \)-dimensional phase space [19]. The phase space is divided in cells of size \( \epsilon \), and takes the measurements at time increments \( \tau \). The existence of finite positive values of \( h \) indicates the existence of chaotic behavior [19]. Using equation (15) for the pure reflective case the value of \( h \) is around 1.98. For the liquid case, we have used \( U_l = n^2 - 1 \), due to the fact that the refractive index of the scatterer is smaller than the index of the medium, so that \( U_l = 0.777 \). In both cases, we have used \( r_l = 0.5 \), meaning that each scatterer (sphere or bubble) touch its nearest neighbors. For the case of the presence of refraction and reflection, we have obtained \( h \) around 1.81. This value of entropy demonstrates that a system with refraction and reflection presents chaotic behavior. Comparing both entropy values, we can notice the both case have the same order of magnitude, but the system with refraction and reflection has the entropy value (1.81) which is a little bit smaller than the entropy of the pure reflective case (1.98).

It is interesting to note that using an approach based on determining optical absorption lengths for foams of different values of liquid fraction and the transport theory, some authors [20, 21] determined the fraction \( f \) on the path that a ray of light spends in the liquid phase of foam. They have obtained that though \( 1/f \) decayed linearly with most values of liquid fraction, for intermediate values of liquid fraction \( 1/f \) is lower than predicted by random sampling. In comparison to the values of the Kolmogorov-Sinai entropy, these results indicate there is the possibility of \( h \) even bigger than those obtained by equation (15) for the case of a light ray traversing a 2D periodic foam in a random walk with a \( 1/f \) decay.
9. Conclusion
The most interesting feature of all the ideas mentioned in this paper is the concept of chaotic scattering of light in foams based on hyperbolic geometry. In doing so, we have presented the analogy between chaotic scattering and the effects of light rays in liquid bridges observed in Hele-Shaw cells. Some of the main aspects of image formation in three reflective spherical shells are compared to the case of hyperbolic kaleidoscopes using the Poincaré disk model. This three-mirrored sphere system can be regarded as a chaotic system where a wave or a particle is scattered by a cavity composed by surfaces with negative curvature. In that way, the incoming wave undergoes successive Möebius transformations, such as translations, rotations, inversions, and reductions. The effects of the refraction and reflection of the light rays were studied using some properties of soft billiards. In addition to this, even a hyperbolic prism behaves like a hyperbolic kaleidoscope, because the pure case of reflection can be observed in the hyperbolic prism, and consequently foams retain properties of ordinary billiards. The existence of finite positive values of Kolmogorov-Sinai entropy is an indicative that light can be channeled through the network of Plateau borders.

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