Stability of ground state degeneracy to long-range interactions

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We show that some gapped quantum many-body systems have a ground state degeneracy that is stable to long-range (e.g., power-law) perturbations, in the sense that any ground state energy splitting induced by such perturbations is exponentially small in the system size. More specifically, we consider an Ising symmetry-breaking Hamiltonian with several exactly degenerate ground states and an energy gap, and we then perturb the system with Ising symmetric long-range interactions. For these models we prove (1) the stability of the gap, and (2) that the residual splitting of the low-energy states below the gap is exponentially small in the system size. Our proof relies on a convergent polymer expansion that is adapted to handle the long-range interactions in our model. We also discuss applications of our result to several models of physical interest, including the Kitaev p-wave wire model perturbed by power-law density-density interactions with an exponent greater than 1.

I. INTRODUCTION

Some gapped quantum many-body systems have the interesting property that they have multiple nearly degenerate ground states with an energy splitting that is exponentially small in the system size [1–3]. Furthermore, this nearly exact degeneracy is a robust phenomenon in the sense that it does not require fine tuning parameters in the Hamiltonian. There is an ongoing effort to realize systems of this kind in the laboratory, as many of them have been argued to be useful platforms for a reliable quantum memory or even a fault-tolerant quantum computer [4].

This kind of nearly exact ground state degeneracy is a well-established phenomenon for systems with short-range interactions (e.g., finite-range interactions). In this case, there are rigorous arguments proving the existence of an exponentially small ground state splitting in many concrete models [5–11]. On the other hand, much less is known about exponentially small ground state splitting for systems with long-range interactions (e.g., power-law decaying interactions). This is despite the fact that long-range interactions are present in many candidate systems with ground state degeneracy, such as topological superconductors and fractional quantum Hall liquids [1, 3]. The main purpose of this paper is to present a rigorous result showing that a robust, exponentially small ground state splitting also occurs in some models with long-range interactions.

The simplest place to study these issues is in the context of perturbations of exactly solvable models. Suppose $H_0$ is an exactly solvable short-range Hamiltonian, and suppose further that $H_0$ has several exactly degenerate ground states which are separated from the rest of the spectrum by a finite energy gap. Consider a generic perturbation of $H_0$ of the form

$$H = H_0 + \lambda V,$$

where $\lambda$ is a real coefficient and $V$ is an interaction with both a short-range and a long-range part. In this context, our question becomes one of stability: in particular, does the energy gap stay open when we turn on a small $\lambda$, and if so, how large is the residual splitting of the states below the gap?

For short-range perturbations, an exponential bound on this residual splitting has been obtained in particular cases in Refs. 5 and 6, and in great generality for topologically-ordered $H_0$’s in Refs. 7–11.† These papers are all part of a much larger collection of works [5–21] that focused on proving various stability results for gapped Hamiltonians subjected to short-range perturbations (Ref. 9 also considered long-range perturbations – see the next paragraph). The papers in this collection utilize a variety of different methods and apply to a variety of different kinds of unperturbed Hamiltonians, for example the case where $H_0$ is a classical Hamiltonian [12–14] and the case where $H_0$ is a quantum Hamiltonian obeying a Topological Quantum Order condition [7–11].

In contrast, when $V$ is a long-range perturbation, say containing power-law interactions, it is not even clear that one should expect an exponentially small bound on the residual splitting of the low-energy states. For example, one might guess that power-law interactions would generically lead to a power-law bound on the residual splitting (i.e., the splitting is bounded from above by a constant times the system size raised to a negative power). Indeed, the only known stability results in the power-law case were obtained in Ref. 9, and those results imply a power-law bound on the residual splitting.

To gain some intuition on the difficulty of obtaining an exponential splitting bound in the long-range case, it is useful to review two of the main ways of arguing for the exponential splitting bound in the case of a short-range perturbation. For simplicity of exposition, we review these methods in the case of the one-dimensional transverse field Ising chain of length $L$, for...
where \( H_0 = - \sum_{j=1}^{L} \sigma_j^z \sigma_{j+1}^z \) and \( V = \sum_{j=1}^{L} \sigma_j^y \). In this case \( H_0 \) has two exactly degenerate low-energy states given by \(|\pm\rangle = (|\uparrow\rangle \pm |\downarrow\rangle)/\sqrt{2} \), where \(|\uparrow\rangle\) and \(|\downarrow\rangle\) are the states of the chain with all spins up and all spins down, respectively.

The first method relies on the technique of quasiadiabatic continuation (QAC) and was already discussed in the original paper on that technique [22]. This method can be used to prove an exponential splitting bound when the perturbation parameter \( \lambda \) is small enough so that the energy gap of the system stays open. The basic idea is as follows: given our assumption that the energy gap of \( H \) stays open, \( H \) will have two low-energy states \(|\pm, \lambda\rangle\) that evolve out of the unperturbed states \(|\pm\rangle\). Quasiadiabatic continuation\(^2\) allows one to construct a unitary operator \( U_\lambda \) with the following two properties:

1. \( U_\lambda \) maps the low-energy states of \( H_0 \) to those of \( H: |\pm, \lambda\rangle = U_\lambda |\pm\rangle \).

2. Conjugation by \( U_\lambda \) takes local operators to local operators, up to a rapidly decaying tail (the decay is faster than any power in the distance).

One can then express the energy splitting \( \delta \) between the states \(|+, \lambda\rangle\) and \(|-, \lambda\rangle\) as

\[
\delta = \langle +|U_\lambda^\dagger H U_\lambda|+\rangle - \langle -|U_\lambda^\dagger H U_\lambda|-\rangle = 2\text{Re} \left\{ \langle \uparrow|U_\lambda^\dagger H U_\lambda| \downarrow \rangle \right\} .
\]

We see that the energy splitting \( \delta \) evaluates to the matrix element of the transformed Hamiltonian \( U_\lambda^\dagger H U_\lambda \) between the all-up and all-down states. This expression is quite intuitive as it is reminiscent of an instanton-type tunneling effect. Note that, for any operator \( \mathcal{O} \), \( \langle \uparrow| \mathcal{O}| \downarrow \rangle = 0 \) unless \( \mathcal{O} \) acts nontrivially on all \( L \) sites of the chain. Since \( H \) is a sum of local terms, the part of \( U_\lambda^\dagger H U_\lambda \) that acts on all \( L \) sites decays rapidly with \( L \) by the properties of the QAC operator \( U_\lambda \). Therefore we find that \( \delta \) decays faster than any power of \( L \), which is very close to the exponential splitting result that we wanted to prove.\(^3\)

While the QAC method works very well for short-range interactions, it has serious limitations for long-range interactions such as power-law interactions. The problem is that conjugation by \( U_\lambda \) takes local operators to local operators with power-law (or larger) tails. Therefore, the QAC-based method only seems capable of proving a power-law splitting bound in the presence of power-law interactions. Indeed, the results of Ref. 9 are based on this technique and, as we mentioned above, in the power-law case they imply a power-law bound on the residual splitting.

The other well-known method for arguing for an exponential splitting is an intuitive argument based on perturbation theory [2].

The basic idea is that the matrix elements \( \langle \uparrow|V^p| \downarrow \rangle \) of the \( p \)-th power of \( V \) between the all-up and all-down states vanish for all powers \( p \) until \( p = L \), the length of the chain. Hence, the energies of the states \(|+, \lambda\rangle\) and \(|-, \lambda\rangle\) will not deviate until order \( L \) in perturbation theory in \( \lambda \), which suggests that the splitting between these two states will scale as \( \lambda^L \) and will therefore be exponentially small in the system size. This argument can be made precise if one can show that the perturbative expansion for the ground state energy has a finite radius of convergence in the thermodynamic limit. One way to prove results of this nature is to use a convergent “polymer expansion”, a type of perturbation theory for the free energy of a model. This method was used in the stability proofs in Refs. 6, 12–14, and 18 for systems with short-range interactions.

Like the QAC method, the standard perturbative method encounters difficulties for systems with long-range interactions. The problem is that the combinatorial arguments that guarantee the convergence of the polymer expansion typically rely on bounding the number of polymers of a particular size that contain a given space-time point, and the presence of long-range interactions invalidates these bounds by vastly increasing the number of possible polymers.\(^4\) Fortunately, this difficulty can be overcome with more sophisticated combinatorial arguments. Indeed, in the closely related problem of classical statistical mechanical models with long range interactions, convergent polymer expansions have been established in a number of systems [26–35].

In this paper, we put these ideas together and extend the perturbative method to a class of quantum systems with long-range interactions. We focus on a simple case where \( H_0 \) is a classical symmetry-breaking Hamiltonian with several exactly degenerate low-energy states that are separated from the rest of the spectrum by a finite energy gap. Thus, our initial setting is similar to Refs. 5, 12–16. We then add to this initial Hamiltonian a (symmetric) perturbation \( \lambda V \) where \( V \) consists of both short- and long-range parts,

\[
V = V_{\text{short}} + V_{\text{long}} .
\]

In the long-range part of \( V \), we allow a large class of power-law and other long-range interactions, including not only 2-body terms but also more general \( K \)-body terms (for \( K \) of order 1). For these Hamiltonians, and for \( \lambda \) small enough, we prove (1) stability of the spectral gap, and (2) an exponentially small (in system size) bound on the residual splitting of the low-energy states below the gap. Thus, we are able to establish an exponential splitting bound for a large family of Hamiltonians with

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\(^2\) Actually, what we are describing here is known as exact QAC and was developed starting in Ref. 23. For an introduction to exact QAC, see Ref. 24.

\(^3\) It is possible to prove a true exponential splitting bound using the approximate (or Gaussian) version of QAC as in the original paper [22].

\(^4\) The standard combinatorial arguments for systems with short-range interactions are reviewed in Ch. V of Ref. 25.
long-range interactions. We also apply our results to several models of current interest including Kitaev’s p-wave wire model [3] perturbed by power-law density-density interactions, as well as a number-conserving model of a topological superconducting qubit [36, 37].

To conclude this introduction, we highlight several previous works that studied gapped and/or topological phases in the presence of long-range interactions. First, several works have presented evidence for the existence of gapped topological phases in the presence of long-range (usually power-law) interactions [38–40]. In addition, there is a large body of work, beginning with Ref. 41, that focused on the effects of long-range hopping and pairing terms in free fermionic systems (in one and higher dimensions) as well as related Ising spin chains (in the one-dimensional case). For our purposes here, Refs. 42–47 are of particular interest, as all of these works presented negative results for exponentially small ground state degeneracy splitting in systems with long-range interactions, including some systems where it was possible to check the stability of the spectral gap (i.e., stability of the phase). In contrast to these works, we have been able to establish not only the stability of the spectral gap in our models, but also the exponentially small bound on the residual splitting of the low-energy states below the gap.

This paper is organized as follows. In Sec. II we introduce an example of the kind of model that our result applies to, and then we state our main result in the context of this model. In Sec. III we prove our main result for this model and discuss the physical intuition that underlies the formal proof. In Sec. IV we explain how our main result can be generalized to a much larger family of models than we considered in Secs. II and III. Section V presents our conclusions. Finally Appendixes A, B, C and D contain important details for the proofs of our main results from Sections III and IV.

II. STABILITY OF THE TRANSVERSE FIELD ISING MODEL TO LONG-RANGE INTERACTIONS

In this section we state our main result in the context of a prototypical model that is both complicated enough to contain all of the physics we are interested in, and simple enough to allow for a straightforward demonstration of our method of proof. In Sec. IV we explain how our result can be generalized to a much larger family of models.

A. Description of the model and its symmetry

We consider a one dimensional spin-1/2 chain with \( L \) sites and periodic boundary conditions, and we assume that \( L \) is even for convenience, although our results would also hold without this assumption. The sites on the chain are labeled by \( j \in \{1, \ldots, L\} \), and on each site we have the three Pauli operators \( \sigma^x_j, \sigma^y_j, \text{ and } \sigma^z_j \). We measure distances on the chain using the periodic distance function, |\( j - k \)|\(_p\) := \( \min_{n \in \mathbb{Z}} |j - k + nL| \). The Hamiltonian for our model takes the form

\[
H = H_0 + \lambda V ,
\]

where \( H_0 \) is a classical Ising Hamiltonian,

\[
H_0 = \frac{\Delta}{2} \sum_{j=1}^{L} (1 - \sigma^z_j \sigma^z_{j+1}) ,
\]

and where \( \Delta > 0 \) is an overall energy scale and \( \sigma^{x,y,z}_{L+1} \equiv \sigma^{x,y,z}_1 \) because of the periodic boundary conditions. The perturbation term \( V \) takes the form

\[
V = \hbar \sum_{j=1}^{L} \sigma^y_j + \frac{1}{2} \sum_{j \neq k} f(|j - k|_p) \sigma^x_j \sigma^x_k ,
\]

where \( \hbar \) is a dimensionless real parameter and \( f(r) \) is a dimensionless real function that determines the long-range interaction. Note that the first term here (with coefficient \( \hbar \)) is the usual transverse field term.

As we mentioned above, the function \( f(r) \) determines the long-range part of \( V \). Throughout the paper we assume that \( f(r) \) satisfies two conditions. First, we assume that \( f(r) \) is normalized so that \( |f(r)| \leq 1 \) for all \( r \). Next, we assume that \( f(r) \) satisfies a summability condition of the form

\[
\sum_{r=1}^{L/2} |f(r)| \leq \frac{c}{2} ,
\]

with a constant \( c \) that can be chosen to be independent of \( L \).\(^5\) A typical example of a long-range interaction that satisfies this condition is a power law interaction \( f(r) = \frac{1}{r^\alpha} \), provided that the power \( \alpha \) satisfies \( \alpha > 1 \) (note the strict inequality).

\(^5\) The upper bound of \( L/2 \) on the range of this sum is due to the fact that \( |j - k|_p \leq L/2 \) for all \( j \) and \( k \).
Our assumption that $c$ can be chosen to be independent of $L$ is crucial for our main result. This assumption guarantees that the long-range term in the Hamiltonian has an operator norm that is extensive (i.e., linear in $L$):

\[
\left| \frac{1}{2} \sum_{j \neq k} f(|j - k|_p) \sigma^z_j \sigma^z_k \right| \leq \frac{cL}{2}.
\] (2.5)

Later in this section, we demonstrate the importance of the summability condition (2.4), by giving an example of an instability for a long-range interaction that violates this condition.

Just like the usual transverse field Ising model, our model has a $\mathbb{Z}_2$ Ising symmetry, generated by the operator

\[ S = \prod_{j=1}^{L} \sigma^z_j. \] (2.6)

The Hilbert space $\mathcal{H}$ of our spin chain can be divided into two sectors, which we denote by $\mathcal{H}_+$ and $\mathcal{H}_-$, such that any state $|\psi_\pm\rangle \in \mathcal{H}_\pm$ satisfies $S|\psi_\pm\rangle = \pm |\psi_\pm\rangle$. Then, since $H$ commutes with $S$, we can separately diagonalize $H$ within each sector.

**B. Main result**

To state our main result, we need to review three key properties of the unperturbed Hamiltonian $H_0$. The first property of $H_0$ is that it has a unique ground state in each sector $\mathcal{H}_\pm$. Specifically, the ground state in the sector $\mathcal{H}_\pm$ is the state $|\pm\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle \pm |\downarrow\rangle)$ where $|\uparrow\rangle$ denotes the state with all spins pointing up ($\sigma^z_j = 1 \forall j$), and $|\downarrow\rangle$ denotes the state with all spins pointing down ($\sigma^z_j = -1 \forall j$). The second property of $H_0$ is that it has a finite energy gap ($= 2\Delta$) above the ground state within each sector. The final property of $H_0$ is that the two ground states $|+\rangle$ and $|-\rangle$ are exactly degenerate.

Our main stability result says that $H$ possesses approximately these same properties, in the limit $L \to \infty$, for small but finite values of $\lambda$:

**Theorem 1.** There exists a $L$-independent constant $\lambda_0 > 0$ such that, if $|\lambda| < \lambda_0$, then (1) $H$ has a unique ground state and a finite energy gap in each sector $\mathcal{H}_\pm$ of the Hilbert space with fixed $S$ eigenvalue, and (2) the ground state energy splitting $|E_+(\lambda) - E_-(\lambda)|$ between sectors satisfies the exponential bound

\[
|E_+(\lambda) - E_-(\lambda)| \leq c_1 L e^{-c_2 L},
\] (2.7)

where $c_1$ and $c_2$ are positive constants that depend on $\Delta$, $h$, and $\lambda$, but not on $L$.

**C. Applications to other models**

We now discuss some applications of Theorem 1 to important models in condensed matter physics. The first model that we discuss is Kitaev’s p-wave wire (KpW) model $[3]$. The degrees of freedom in this model are spinless fermions hopping on the sites $j \in \{1, \ldots, L\}$ of a one-dimensional chain. The Hamiltonian with open boundary conditions takes the form

\[
H_{\text{KpW}} = - \sum_{j=1}^{L-1} \left( \frac{t}{2} c^\dagger_j c^\dagger_{j+1} + \frac{\Delta}{2} c_j c_{j+1} + \text{h.c.} \right) - \mu \sum_{j=1}^{L} \left( n_j - \frac{1}{2} \right),
\] (2.8)

where $c_j$ and $c^\dagger_j$ are the spinless fermion operators, $n_j = c^\dagger_j c_j$, and $t, \Delta, \text{and } \mu$ are the hopping energy, pairing energy, and chemical potential, respectively.

This model is in its topological phase for $|\mu| < t$ and $\Delta \neq 0$. The special point $t = \Delta$ and $\mu = 0$ (where the Majorana zero modes at the ends decouple from the bulk) is known to have the following stability properties when perturbed by a generic weak short-range interaction $V_{\text{short}}$: 6

1. $H = H_{\text{KpW}} + V_{\text{short}}$ has a unique ground state and a finite energy gap in each sector of the fermion Fock space with fixed fermion parity.

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6 [This can be proven using the general results from Ref. 7. Reference 48 contains an alternative proof that holds for the case of quadratic perturbations only.]
2. Let $E_\pm$ be the ground state energy of $H$ in the sector with fermion parity equal to $\pm 1$. Then, $|E_+ - E_-| \leq c_1 L e^{-c_2 L}$, for some $L$-independent constants $c_1$ and $c_2$.

A major open question about this model is whether these stability properties persist in the presence of long-range interactions. As we mentioned in the introduction, Refs. 42–44 have provided evidence that the exponential bound on the ground state energy splitting does not survive in the presence of long-range hopping and pairing terms. Our result does not apply to those models, but it does apply to the arguably more physical case of a long-range density-density interaction of the form

$$V_{\text{long}} = \frac{\lambda}{2} \sum_{j \neq k} f_{jk} n_j n_k.$$  

(2.9)

Here $\lambda$ is a coupling constant and $f_{jk} = f_{kj}$ is a two-body interaction whose absolute value is bounded from above by a positive function $g(|j - k|)$ of the distance between two sites, $|f_{jk}| \leq g(|j - k|)$, with $g(r)$ satisfying the summability condition $\sum r g(r) \leq c$ for some constant $c$.

To see why our results apply to this model, note that at the special point where $t = \Delta$, the perturbed KpW Hamiltonian $H = H_{\text{KpW}} + V_{\text{long}}$ can be mapped, via a Jordan-Wigner transformation, onto a spin model of the form

$$H = -\frac{\Delta}{2} \sum_{j=1}^L \sigma_j^z \sigma_{j+1}^z + \sum_{j=1}^L h_j \sigma_j^x + \frac{\lambda}{8} \sum_{j \neq k} f_{jk} \sigma_j^x \sigma_k^x + \text{constant},$$  

(2.10)

where the transverse fields $h_j$ are given in terms of the parameters $\mu$ and $\lambda$ by $h_j = \frac{\mu}{2} - \frac{\lambda}{2} \sum_{k \neq j} f_{jk}$. This model is clearly very similar to the spin model that we study in this paper, up to a breaking of translation invariance and a redefinition of the parameters.

Our stability result does indeed apply to this model, after some minor modifications to accommodate the breaking of translation invariance. Therefore, we conclude that the energy gap and exponentially small ground state splitting of the KpW model survives in the presence of sufficiently weak long-range density-density interactions $V_{\text{long}}$. We can also extend this result to the case where $t \neq \Delta$, by thinking of the deviation from the point $t = \Delta$ as an additional (short-range) perturbation. For more details on this we refer the reader to our discussion on generalizations of our result in Sec. IV.

Our second example relates to the subject of number-conserving models of topological superconductivity. We will explore this example in more detail in a separate paper [37], and so we only give a brief description here. Most studies of topological superconductivity rely on a mean-field description of superconductivity – a description that violates the physical symmetry of particle number conservation. Recently, several authors have argued that this breaking of number conservation could pose a problem for the proposed applications of topological superconductors to quantum computation. In Ref. 36 we began an investigation of this issue in the context of a number-conserving version of the KpW model that features an additional degree of freedom that represents a bulk s-wave superconductor. This additional degree of freedom can exchange Cooper pairs with the environment, and the total number of fermions in the model is conserved (see Ref. 36 for the details of the model). In Ref. 36 we proved one stability property of this model, namely the existence of a unique ground state and a finite energy gap in each sector of the Hilbert space with fixed total particle number.

One issue that was not addressed in Ref. 36 was the size of the ground state degeneracy splitting in a two-wire “qubit” setup in which the model has two low-energy states below a finite energy gap in each sector of fixed total particle number. Using our main result in this paper, we can now address this question: our results imply that the residual splitting between these two low-energy states is exponentially small in the length $L$ of the wires. The key to applying our result to that model is a mapping, which holds in each sector of fixed total particle number, that maps our number-conserving topological superconductor model to a spin model with an all-to-all long-range interaction with $f(r) = 1/L$ for all separations $r$ (this interaction originates from the charging energy term in the model from Ref. 36).

D. Instability for interactions that violate the summability condition

To close this section, we now present a simple variational argument that shows why the summability condition (2.4) is necessary for our stability result. This argument establishes an instability for models of the same form as our Hamiltonian but where $f(r)$ no longer satisfies (2.4). The instability that we discuss appears for negative values of $\lambda$ of arbitrarily small magnitude. As in a famous argument of Dyson [49], this instability is strong evidence that any perturbation theory in $\lambda$ has a vanishing radius of convergence when $f(r)$ does not satisfy the summability condition.

To demonstrate the instability, we compute the expectation value of $H$ in two different variational trial states. For the first trial state we choose the “ferromagnetic” state $| \uparrow \rangle$ that has all of the spins pointing in the positive $z$-direction: $\sigma_j^z | \uparrow \rangle = | \uparrow \rangle$ for all $j$. For the second trial state we choose the “paramagnetic” state $| \Rightarrow \rangle$ that has all of the spins pointing in the positive $x$-direction:
σ_j^± |⇒⟩ = |⇒⟩ for all j. If the variational calculation shows that |⇒⟩ is favored over |⇑⟩, even for very small values of |λ|, then we will interpret that finding as indicating an instability of $H_0$ to the interaction $V$.

Following this plan, and setting $h = 0$ for simplicity, we find
\[ \langle ⇑ | H | ⇑⟩ = 0, \quad \langle ⇒ | H | ⇒⟩ = \frac{\Delta}{2} L + \frac{\lambda}{2} \sum_{j \neq k} f(|j - k|_p). \] (2.11)

To proceed, let us consider the case where the interaction $f(r)$ is positive and does not satisfy Eq. (2.4). For example, consider the standard Coulomb interaction, $f(r) = 1/r$. With this choice we find that
\[ \sum_{j \neq k} f(|j - k|_p) \sim c'L \ln(L) \] (2.12)
for some positive constant $c'$. Taking $\lambda$ to be negative, the energy of the trial state $|⇒⟩$ is then
\[ \langle ⇒ | H | ⇒⟩ \sim \frac{\Delta}{2} L - \frac{c|\lambda|}{2} L \ln(L). \] (2.13)

Because of the additional $\ln(L)$ factor, the paramagnetic trial state $|⇒⟩$ is always (for $\lambda < 0$) a better trial state than the ferromagnetic state $|⇑⟩$ for large enough system size $L$. On the other hand, for an interaction that does satisfy the summability condition we instead find that
\[ \langle ⇒ | H | ⇒⟩ \geq \frac{\Delta}{2} L - \frac{c|\lambda|}{2} L, \] (2.14)
and so in this case the ferromagnetic state $|⇑⟩$ is always a better choice for $|\lambda| < \Delta/c$. These results clearly show that the summability condition (2.4) is necessary for a general stability result like Theorem 1.

III. PROOF OF THE MAIN RESULT

In this section we present the proof of Theorem 1. We begin with an outline of the proof, summarizing the two main steps. We then explain these steps in detail in Secs. III B - III D.

A. Outline of the proof

Our proof of Theorem 1 can be divided into two steps. In the first step, we study the partition function of our model at inverse temperature $\beta$ in each sector $H_±$ of the Hilbert space. We define partition functions $Z_±$ by
\[ Z_± = \text{Tr}_± \{ e^{-\beta H} \}, \] (3.1)
where $\text{Tr}_± (\cdots)$ denotes a trace over $H_±$. For later use we also define the unperturbed partition function $Z_0$ in each sector by
\[ Z_0 = \text{Tr}_+ \{ e^{-\beta H_0} \} = \text{Tr}_- \{ e^{-\beta H_0} \}. \] (3.2)

In the context of quantum statistical mechanics, one can often express partition functions as a sum over configurations (usually worldlines of some kind) on an appropriate spacetime region. In the first step of the proof we develop a precise representation of this kind by showing that $Z_±$ can be written in the form
\[ \frac{Z_±}{Z_0} = \sum_X W_±(X), \] (3.3)
where the sum is taken over a finite set of geometric configurations $X$ on the spacetime region $[1, L+1] \times [0, \beta]$, and $W_±(X) \in \mathbb{C}$ is a complex weight for the configuration $X$. In what follows we refer to the configurations $X$ as support sets. This part of our proof closely follows Kennedy and Tasaki (KT) [12].

The support sets $X$ are constructed from a set of basic building blocks that includes "boxes", "plaquettes", and "dashed lines." In addition, the support sets $X$ that appear in the expression for $Z_±$ all have the further property that they can be decomposed into a union of non-intersecting elementary support sets $X_1, X_2, \ldots$,
\[ X = X_1 \cup X_2 \cup \cdots \] (3.4)

We give a precise definition of the notion of intersection for these support sets later in this section.
where the weights \( W_\pm(X) \) factor as
\[
W_\pm(X) = W_\pm(\chi_1)W_\pm(\chi_2) \cdots
\]  
(3.5)

For reasons that we explain later, we refer to \( \chi_1, \chi_2, \ldots \) as weakly-connected support sets.

Note that the weights \( W_\pm(X) \) will generally depend on \( \lambda \) (as well as the other parameters in \( H \)). In what follows, we will allow this parameter \( \lambda \) to take on complex values: \( \lambda \in \mathbb{C} \). Thus, we are now studying a “complexified” version of \( Z_\pm \) whose restriction to real values of \( \lambda \) is equal to the partition function for our original model.

Next, in order to gain analytic control over the expansion (3.3), we prove an upper bound on the absolute values of the weights \( W_\pm(X) \).

Lemma 1. For any \( \mu, \delta > 0 \) there exists a \( \lambda_0 > 0 \) such that, if \( |\lambda| \leq \lambda_0 \), then the weight \( W_\pm(X) \) for any support set \( X \) (not necessarily weakly-connected) satisfies the bound
\[
|W_\pm(X)| \leq e^{-\mu |X|} \delta^{d(X)} \left[ \prod_{r=1}^{d} |f(r)|^{d_r(X)} \right],
\]  
(3.6)

where \( |X| \) is the number of boxes and plaquettes in \( X \), \( d_r(X) \) is the number of dashed lines of length \( r \) in \( X \), and \( d(X) = \sum_{r=1}^{d} d_r(X) \).

To summarize, the key results from the first step of the proof are Eqs. (3.3), (3.4), and (3.5), and Lemma 1.

In the second step we use the representation of \( Z_\pm / Z_0 \) from (3.3) to develop an expansion, known as a polymer expansion, for the logarithm, \( \ln(Z_\pm / Z_0) \). The main difficulty in this step is to prove that our expansion for \( \ln(Z_\pm / Z_0) \) is absolutely convergent (for small enough \( |\lambda| \)). This is the most important part of our proof and it is the part where the long-range nature of the interactions in our model really comes into play. As we review below, the key to proving the convergence of the polymer expansion is the following lemma, which we prove as part of the second step of our proof. To state the lemma, we first define a modified weight \( \tilde{W}(X) \) using the upper bound in Lemma 1:
\[
\tilde{W}(X) = e^{-\mu |X|} \delta^{d(X)} \left[ \prod_{r=1}^{d} |f(r)|^{d_r(X)} \right].
\]  
(3.7)

Lemma 2. There exists a \( \lambda \)- and \( \beta \)-independent constant \( K_0 > 0 \) of order one such that, if \( \mu - 3\delta c > K_0 \), then
\[
q = \sum_{\chi \ni v} \tilde{W}(\chi) e^{14|\chi|} < 1,
\]  
(3.8)

where the sum is taken over all weakly-connected support sets \( \chi \) that contain a fixed box or plaquette \( v \).

To understand the physical meaning of this lemma, it is useful to interpret \( \tilde{W}(\chi) \) as a Boltzmann weight of a classical statistical mechanics model in two dimensions. Then the sum \( q = \sum_{\chi \ni v} \tilde{W}(\chi) e^{14|\chi|} \) can be thought of as a sum over all the Boltzmann weights of configurations containing a fixed point \( v \). The condition that \( q < 1 \) is then analogous to the standard energy-entropy condition of Peierls [50] (see Refs. 51–53 for related results). Specifically, in the expression \( q = \sum_{\chi \ni v} \tilde{W}(\chi) e^{14|\chi|} \), the weight \( \tilde{W}(\chi) \) can be thought of as a Boltzmann weight or energy contribution, while the sum over weakly-connected support sets \( \chi \) is the entropy contribution.\(^8\) The above lemma effectively shows that energy dominates over entropy, which tells us that the polymers do not proliferate and hence the perturbed model should be in the same phase as the unperturbed model.

Turning back to the proof: by combining Lemma 1 and Lemma 2 with standard results about the polymer expansion, it is straightforward to show that the polymer expansion for \( \ln(Z_\pm / Z_0) \) is absolutely convergent in the region of \( \mathbb{C} \) defined by \( |\lambda| < \lambda_0 \) that is independent of \( L \) and \( \beta \). With this convergence result in hand, we can apply the same reasoning as in KT (see pg. 470 of Ref. 12) to conclude that, for real \( \lambda \) satisfying \( |\lambda| < \lambda_0 \), \( H \) has a unique ground state and a finite energy gap in each sector \( \mathcal{H}_\pm \). This completes the proof of part (1) of Theorem 1. In the rest of the second step of the proof we use the existence of the absolutely convergent polymer expansion for \( \ln(Z_\pm / Z_0) \) to deduce several pieces of information about our model. Specifically, the polymer expansion allows us to prove the following two claims.

\(^8\) The exact physical meaning of the exponential factor \( e^{14|\chi|} \) is not as clear, but one interpretation for this factor is given in Ref. 54.
Claim 1. For all $\lambda \in \mathbb{C}$ satisfying $|\lambda| < \lambda_0$, the complexified free energy $-\ln(Z_+)/\beta$ is a holomorphic function of $\lambda$. In addition, $-\ln(Z_+)/\beta$ is bounded from above as
\[
\left| -\frac{1}{\beta} \ln \left( \frac{Z_+}{Z_0} \right) \right| \leq c_3 L, \tag{3.9}
\]
where the constant $c_3$ depends on $\Delta$ but not on $\lambda, \hbar, \beta,$ and $L$.

The first part of this claim states that the free energy of our original Hermitian model (with real $\lambda$) possesses an analytic continuation to the region of $\mathbb{C}$ defined by $|\lambda| < \lambda_0$. The second part states that the complexified free energy at any temperature is at most extensive in the system size. This result is not a surprise for real values of $\lambda$, but it is a nontrivial result for complex $\lambda$.

In particular, for complex $\lambda$ it is possible for $Z_{\pm}$ to be equal to zero for some choice of parameters, and in that case the logarithm of $Z_{\pm}$ would diverge and be ill-defined at those parameter values. Claim 1 proves that this scenario cannot occur for $|\lambda| < \lambda_0$.

Claim 2. Let $f(\lambda, \beta)$ be the difference between the complexified free energies for the two sectors $\mathcal{H}_\pm$.
\[
f(\lambda, \beta) = -\frac{1}{\beta} \ln (Z_+) + \frac{1}{\beta} \ln (Z_-). \tag{3.10}
\]
Then the derivatives $\frac{d^n}{dn} f(\lambda, \beta)$ vanish at $\lambda = 0$ for all $n$ satisfying $n < L/2$.

This claim follows from the fact that one must flip all $L$ spins on the chain to go from the state with all spins up to the state with all spins down. This result is not surprising and is clear from ordinary Schrödinger perturbation theory.

Claims 1 and 2 are the key results from the second step of the proof. In the rest of this outline we explain how to finish the proof of Theorem 1 using these claims. This last part of the proof is a formalization of the ideas of Klich [6] and relies on a few important theorems from complex analysis.

To show that $E(\lambda)$ is at most extensive in the system size, this result is not a surprise for real values of $E(\lambda)$, but it is a nontrivial result for complex $\lambda$. In particular, for complex $\lambda$ it is possible for $Z_{\pm}$ to be equal to zero for some choice of parameters, and in that case the logarithm of $Z_{\pm}$ would diverge and be ill-defined at those parameter values. Claim 1 proves that this scenario cannot occur for $|\lambda| < \lambda_0$.

For real values of $\lambda$ this limit clearly exists and is equal to the ground state energy $E_+(\lambda)$ of our model. Then, by the Vitali convergence theorem (Theorem B.25 of Ref. 55), this fact (convergence for real $\lambda$), combined with Claim 1, implies that the limit exists and that $E_+(\lambda)$ is a holomorphic function of $\lambda$ for all $\lambda \in \mathbb{C}$ satisfying $|\lambda| < \lambda_0$. In addition, since $\lim_{\beta \to \infty} Z_0 = 1$, and since the constant $c_3$ from Claim 1 was independent of $\beta$, the bound from Claim 1 carries over to $E_+(\lambda)$ and we have
\[
|E_+(\lambda)| \leq c_4 L. \tag{3.12}
\]

We now show that these properties imply an exponential bound on the energy difference $E_+(\lambda) - E_-(\lambda)$. Our strategy is to show that the complexified energy difference
\[
f(\lambda) := E_+(\lambda) - E_-(\lambda) \tag{3.13}
\]
satisfies an exponential bound throughout the entire convergence region inside $\mathbb{C}$. This result then immediately implies an exponential bound on $E_+(\lambda) - E_-(\lambda)$ for real values of $\lambda$ within the convergence region.

To show that $f(\lambda)$ satisfies an exponential bound we first note that $f(\lambda)$ is a holomorphic function of $\lambda$ for $|\lambda| < \lambda_0$ since $E_\pm(\lambda)$ are holomorphic in this region. In addition, (3.12) guarantees that $|f(\lambda)| \leq 2c_3 L$, while Claim 2 implies that
\[
\left| \frac{d^n f(\lambda)}{d\lambda^n} \right|_{\lambda=0} = 0, \quad n \in \{0, 1, \ldots, \frac{L}{2} - 1\}. \tag{3.14}
\]

With these three properties in hand, we can now use a version of Schwarz’s lemma$^9$ to conclude that, for $|\lambda| < \lambda_0$, $f(\lambda)$ satisfies the bound
\[
|f(\lambda)| \leq \left( \frac{|\lambda|}{\lambda_0} \right)^{\frac{L}{2}} f_0, \quad f_0 = 2c_3 L. \tag{3.15}
\]

In particular, for real $\lambda$ the energy difference $|E_+(\lambda) - E_-(\lambda)|$ obeys a bound of the form (2.7) with $c_1 = 2c_3$ and $c_2 = \frac{1}{2} \ln(\lambda_0/|\lambda|)$.

---

$^9$ The usual version of Schwarz’s lemma states that if $f(z)$ is holomorphic for $|z| < z_0$, satisfies $f(0) = 0$, and is bounded as $|f(z)| \leq f_0$ for all $|z| < z_0$, then $|f(z)| \leq \frac{f_0}{z_0}$ for all $|z| < z_0$. To prove this one applies the maximum modulus principle to the function $\frac{f(z)}{z^{\frac{L}{2}}}$, which is also holomorphic for $|z| < z_0$ by virtue of the fact that $f(0) = 0$. To prove the version of Schwarz’s lemma that we use here, one should instead apply the maximum modulus principle to the function $\frac{f(z)}{z^{\frac{L}{2}}}$, which is holomorphic on $|z| < z_0$ if $f(z)$ and the first $L/2 - 1$ derivatives of $f(z)$ vanish at $z = 0$. 


B. Step 1: A formula for $Z_{\pm}$

In this section we present the first step of our proof. In particular, we show how to derive the expression (3.3) for $Z_{\pm}/Z_0$ and the properties of the support sets $X$ and weights $W_{\pm}(X)$ that are stated in Eqs. (3.4), (3.5), and Lemma 1. Since this construction largely follows KT [12], we only outline the main steps here. We provide all of the details for this step in Appendix A.

1. Setting up the expansion of $Z_{\pm}$

To set up the expansion of $Z_{\pm}$ we first expand $e^{-\beta H}$ in a Dyson series by iterating the Duhamel formula,

$$e^{-\beta(H_0+\lambda V)} = e^{-\beta H_0} - \lambda \int_0^\beta d\tau e^{-(\beta-\tau)H_0} V e^{-\tau(H_0+\lambda V)} .$$

(3.16)

While KT [12] used the Trotter product formula for this step, the Duhamel formula was used in this way in Refs. 13 and 14. If we iterate the Duhamel formula then we end up with a series expansion for $e^{-\beta H}$ of the form

$$e^{-\beta H} = e^{-\beta H_0} + \sum_{n=1}^\infty (-\lambda)^n \int_0^\beta d\tau_1 \cdots \int_0^{\tau_2} d\tau_1 e^{-\beta H_0} V(\tau) \cdots V(\tau_1) ,$$

(3.17)

where the integration in the $n$th term is over the region of $\mathbb{R}^n$ defined by $0 \leq \tau_1 \leq \tau_2 \leq \cdots \leq \tau_n \leq \beta$, and where

$$V(\tau) = e^{\tau H_0} V e^{-\tau H_0} .$$

(3.18)

One important fact about this expansion is that it is absolutely convergent for a lattice system with a finite lattice size $L$ and a finite-dimensional Hilbert space on each site. Indeed, we have the bound\(^10\)

$$\sum_{n \geq 0} \left||(-\lambda)^n \int_0^\beta d\tau_1 \cdots \int_0^{\tau_2} d\tau_1 e^{-\beta H_0} V(\tau) \cdots V(\tau_1)\right|| \leq e^{\beta|\lambda|\|V\|} < \infty ,$$

(3.19)

where $\|O\|$ denotes the usual operator norm of $O$ (the largest singular value of $O$). In addition, one can see from the derivation that this bound continues to hold for complex values of $\lambda$.

The next step is to rewrite $V$ as a sum over terms $V_Y$ that act on (unordered) subsets $Y \subset \{1, \ldots, L\}$ of lattice sites,

$$V = \sum_Y V_Y .$$

(3.20)

For our model there are two types of subsets that contribute non-zero terms to $V$. First, if $Y = \{j\}$ contains the single site $j$,

$$V_{\{j\}} = h \sigma_j^z .$$

(3.21)

Next, if $Y = \{j, k\}$ is a set of two distinct lattice sites $j$ and $k$, then we have

$$V_{\{j, k\}} = f(|j - k|_p) \sigma_j^z \sigma_k^z .$$

(3.22)

With this notation the $n$th term in the expansion of $e^{-\beta H}$ now takes the form

$$(-\lambda)^n \sum_{Y_1, \ldots, Y_n} \int_0^\beta d\tau_1 \cdots \int_0^{\tau_2} d\tau_1 e^{-\beta H_0} V_{Y_n}(\tau_n) \cdots V_{Y_1}(\tau_1) .$$

(3.23)

Finally, to evaluate $Z_{\pm}$ we need to trace over the sector $H_{\pm}$ of the full Hilbert space. We choose to evaluate this trace in a basis of a simultaneous eigenstates of $H_0$ and the Ising symmetry operator $S$. To define this basis, let $s = (s^{(2)}, \ldots, s^{(L)})$ be an $(L - 1)$-tuple of spin values, with $s^{(j)} \in \{\uparrow, \downarrow\}$ for $j \in \{2, \ldots, L\}$. Here, $s^{(j)}$ labels the $z$-projection of the spin on site $j$. Using these values, we first define the set of $2^{L-1}$ states $|s\rangle$ via

$$|s\rangle = |\uparrow, s^{(2)}, \ldots, s^{(L)}\rangle .$$

(3.24)

\(^{10}\) To derive this bound one should unpack the $V(\tau_k)$’s and note that $\|e^{-(\tau_{k+1} - \tau_k)H_0}\| \leq 1$ because $H_0$ has a positive spectrum.
These states have the spin at site 1 pointing up, and then the configurations of the rest of the spins are specified by \( s^{(2)}, \ldots, s^{(L)} \). Using the states \( |s\rangle \), we can then construct an orthonormal basis of states \( |s, \pm\rangle \) for \( \mathcal{H}_\pm \) via

\[
|s, \pm\rangle = \frac{1}{\sqrt{2}} (1 \pm S) |s\rangle .
\]

As we mentioned above, the states \( |s, \pm\rangle \) are also eigenstates of the unperturbed Hamiltonian \( H_0 \),

\[
H_0 |s, \pm\rangle = \epsilon_s |s, \pm\rangle ,
\]

where the energy \( \epsilon_s \) is given by

\[
\epsilon_s = \Delta \times ( \text{ number of domain walls in } |s\rangle )
\]

Note that for the unperturbed Hamiltonian \( H_0 \), the eigenvalues \( \epsilon_s \) do not depend on the Ising symmetry sector, and so all eigenstates of \( H_0 \) are (at least) doubly-degenerate. This exact degeneracy will be split once we turn on the interaction \( V \).

For any operator \( \mathcal{O} \), we can then compute \( \text{Tr}_\pm \{ \mathcal{O} \} \) using this basis in the standard way: \( \text{Tr}_\pm \{ \mathcal{O} \} = \sum_s \langle s, \pm | \mathcal{O} | s, \pm \rangle \), where we sum over all \( (L-1) \)-tuples \( s = (s^{(2)}, \ldots, s^{(L)}) \). Therefore, at this point our expansion for \( Z_\pm \) takes the form

\[
Z_\pm = Z_0 + \sum_{n=1}^{\infty} (-\lambda)^n \sum_{Y_1, \ldots, Y_n} \int_0^\beta d\tau_n \cdots \int_0^\tau_2 d\tau_1 \sum_s \langle s, \pm | e^{-\beta H_0} V_{Y_n}(\tau_n) \cdots V_{Y_1}(\tau_1) | s, \pm \rangle .
\]

The next step in obtaining Eq. (3.3) is to follow KT [12] and introduce a “blocking” in the (imaginary) time direction: we divide the time interval \([0, \beta]\) into subintervals of length \( \tau \), where \( \tau \) is a lattice spacing in the time direction, and we assume that \( \beta/\tau \) is an integer that we call \( M \), i.e. \( \frac{\beta}{\tau} = M \in \mathbb{N} \).

Using this blocking in the time direction, we can obtain Eq. (3.3) via the following steps. We first show that each term in the Duhamel expansion of \( Z_\pm \) has a representation in terms of a microscopic configuration \( C \) on the region \([1, L+1] \times [0, \beta]\). The configuration \( C \) will consist of the worldlines of domain walls in the spin chain. Next, each microscopic configuration \( C \) in turn be assigned a support set \( s(C) \), which is a geometric configuration on the blocked spacetime lattice defined by the temporal lattice spacing \( \tau \). By collecting all microscopic terms with the same support set \( s(C) = X \), we can rewrite \( Z_\pm/Z_0 \) in the form (3.3) where, roughly speaking, we have

\[
W_\pm(X) = \sum_{C: s(C)=X} W_\pm(C) ,
\]

and where \( W_\pm(C) \) is the term in the Duhamel expansion that corresponds to the microscopic configuration \( C \). Strictly speaking, this formula for \( W_\pm(X) \) is not quite correct because the Duhamel expansion for \( Z_\pm \) involves both discrete summation as well as time integration. In Appendix A, we present a precise formula for \( W_\pm(X) \) that takes care of this notational issue.

2. Definition of the support sets \( X \) and factorization of the weights

We now define the microscopic configurations \( C \) and the support sets \( X \) in our model, and then we discuss the factorization properties of the weights \( W_\pm(X) \). To define the microscopic configurations we study the integrand of a typical term at \( n \)th order in our expansion of \( Z_\pm \), for example the term

\[
\sum_s \langle s, \pm | e^{-\beta H_0} V_{Y_n}(\tau_n) \cdots V_{Y_1}(\tau_1) | s, \pm \rangle .
\]

Each matrix element

\[
\langle s, \pm | e^{-\beta H_0} V_{Y_n}(\tau_n) \cdots V_{Y_1}(\tau_1) | s, \pm \rangle
\]

in this sum is associated with a microscopic configuration \( C \). To define \( C \), we insert a complete set of states for \( \mathcal{H}_\pm \) between every perturbation term in (3.30) to obtain an expression of the form

\[
\sum_{s_1, \ldots, s_{n-1}} \langle s_1, \pm | V_{Y_n} | s_{n-1}, \pm \rangle \cdots \langle s_1, \pm | V_{Y_1} | s, \pm \rangle e^{-\beta H_0} e^{-(\tau_n-\tau_1) \epsilon_{e_1}} e^{-(\tau_n-\tau_{n-1}) \epsilon_{e_{n-1}}} \cdots e^{-(\tau_2-\tau_1) \epsilon_{e_1}} .
\]
Since each perturbation term $V_Y$ flips either one or two spins and does nothing else, the summand here is only non-zero for a single choice of the intermediate states $|s_1, \pm\rangle, \ldots, |s_{n-1}, \pm\rangle$. If we denote these particular states by $|s'_1, \pm\rangle, \ldots, |s'_{n-1}, \pm\rangle$, then we find that (3.30) now takes the form

$$
\langle s, \pm| V_{Y_{11}} |s'_{n-1}, \pm\rangle \cdots \langle s'_1, \pm| V_{Y_{12}} |s, \pm\rangle \ e^{-(\beta - \tau_n + \tau_1)\epsilon} e^{-(\tau_n - \tau_{n-1})\epsilon} s_{n-1} \cdots e^{-(\tau_2 - \tau_1)\epsilon} s'_1 .
$$

(3.31)

We can associate (3.31) with a microscopic configuration $C$ of domain wall worldlines on $[1, L + 1] \times [0, \beta)$. To do this, we first draw vertical lines in the interval $[\tau_1, \tau_2]$ at the horizontal locations of all domain walls in $|s'_1, \pm\rangle$, then draw vertical lines in the interval $[\tau_2, \tau_3]$ at the horizontal locations of all domain walls in $|s'_2, \pm\rangle$, and so on. The last time interval wraps around the time direction (the trace enforces periodic boundary conditions in time), and so in that case we draw the vertical lines for domain walls in $|s, \pm\rangle$ in the interval $[\tau_n, \beta]$ and then continue them in the interval $[0, \tau_1]$. At this point the configuration $C$ consists of vertical segments, but has no horizontal segments. It turns out that this is enough information for our derivation, and we do not need to assign any horizontal segments to $C$.

For every microscopic configuration $C$, we define a corresponding support set $X = s(C)$. Each support set is constructed from plaquettes, boxes, and dashed lines on the blocked spacetime lattice. The plaquettes and boxes were already considered in KT [12], but the dashed lines are a new ingredient that is necessary to keep track of the long-range interactions in our model. The plaquettes can be viewed as subsets of $\mathbb{R}^2$ and they take the form $[j, j + 1] \times [\tau(\ell - 1), \tau\ell]$, where $j \in \{1, \ldots, L\}$ and $\ell \in \{1, \ldots, M\}$. The boxes can also be viewed as subsets of $\mathbb{R}^2$, and they take the form $[j - \frac{1}{2}, j + \frac{1}{2}] \times [\tau(\ell - 1), \tau\ell]$ for $j \in \{1, \ldots, L\}$ and $\ell \in \{1, \ldots, M\}$. In these definitions, horizontal coordinates should always be interpreted modulo $L$.

Finally, each dashed line in a support set $X$ connects two different boxes or two different plaquettes in the same time slice of $X$. As such, dashed lines are always parallel to the spatial direction of the spacetime lattice. Finally, our rule for assigning dashed lines to a support set $X$ will be such that any two boxes within a given time slice are connected by at most one dashed line.

We now present the rules for determining which plaquettes, boxes, and dashed lines are included in the support set $s(C)$ for the configuration $C$ associated with a particular matrix element like (3.30). If $C$ has a worldline that passes all the way through a given plaquette, then $s(C)$ contains this plaquette. For this to occur, $C$ must contain an unbroken vertical segment of length at least $\tau$. Next, if (3.30) contains a perturbation $V_Y$ with support on site $j$ and acting within the time interval $[\tau(\ell - 1), \tau\ell]$, then $s(C)$ contains the box centered on $j$ within this time slice. With these rules, the worldlines in $C$ will always begin and end inside boxes that are part of $s(C)$.

We now come to the rule for assigning dashed lines to $s(C)$. If (3.30) contains a perturbation $V_{(j,k)}$ (i.e., $V_Y$ with $Y = \{j, k\}$ and $j \neq k$) acting within the time interval $[\tau(\ell - 1), \tau\ell]$, then $s(C)$ contains a single dashed line connecting the boxes centered on $j$ and $k$ within this time slice (those boxes will already be in $s(C)$ because of the rule for assigning boxes to a support set). The dashed line that we draw to connect the two boxes is always the dashed line of minimal length according to the periodic distance function $| \cdot |_p$. Therefore, the maximum length that a dashed line can have is $L/2$. In Fig. 1 we show an example of a microscopic domain wall worldline configuration (left panel) and its corresponding support set (right panel).

For a given support set $X$, we define $p(X)$ and $b(X)$ to be the number of plaquettes and boxes, respectively, in $X$. Next, we define $d_r(X)$ to be the number of dashed lines of length $r$ in $X$, where $r \in \{1, \ldots, \frac{L}{2}\}$. The total number of dashed lines in $X$ is then given by $d(X) := \sum_{r=1}^{\frac{L}{2}} d_r(X)$. Finally, we define the “size” $|X|$ of the support set $X$ to be equal to the total number of plaquettes and boxes in $X$.

$$
|X| := p(X) + b(X) .
$$

(3.32)

Note that we do not include the number of dashed lines in the definition of $|X|$.

To proceed further, we define two different notions of connected support sets which will be important in our proof. We call a support set $X$ connected if (i) $X$ has no dashed lines, and (ii) the boxes and plaquettes in $X$ form a connected subset of $\mathbb{R}^2$. Note that in this definition, two boxes that only touch at a corner form a connected subset of $\mathbb{R}^2$ since we have defined the boxes to be closed on all sides. The same goes for two plaquettes that only touch at a corner. Thus, each box or plaquette is touching 14 neighboring boxes and/or plaquettes. We use the special notation $\chi$ to denote a connected support set.

Likewise, we call a support set $X$ weakly-connected if it is possible to get from any connected component of $X$ (regarded as a subset of $\mathbb{R}^2$) to any other connected component of $X$ by traversing a finite sequence of dashed lines belonging to $X$. We use the notation $\chi$ (no tilde) to denote a weakly-connected support set.

We also need to define a notion of intersection for weakly-connected support sets. We say that two weakly-connected support sets $\chi_1$ and $\chi_2$ intersect, denoted by

$$
\chi_1 \cap \chi_2 \neq \emptyset ,
$$

11 The horizontal location of a domain wall is the midpoint of the horizontal bond that lies between the two oppositely oriented spins that form that domain wall.

12 As we will see below, support sets that contain dashed lines connecting plaquettes to plaquettes have vanishing weight and do not appear in our expression for $Z_{\pm}$. The reason we include these “illegal” configurations is to simplify the structure of the support sets by putting boxes and plaquettes on an equal footing.
FIG. 1. A microscopic domain wall worldline configuration (the red lines in the left panel) and its corresponding support set (shown in the right panel). The support set is made up of boxes, plaquettes, and dashed lines, and the plaquettes and boxes are shaded in light blue. The red crosses indicate the action of a $\sigma^x$ operator at the given position in spacetime.

if $\chi_1$ and $\chi_2$ share a box or plaquette or if there is a box or plaquette in $\chi_1$ that is touching a box or plaquette in $\chi_2$ (including touching at corners as we discussed above). The right panel of Fig. 1 features one weakly-connected support set and one connected support set, and these two do not intersect with each other.

With these definitions it is clear that any support set $X$ can be expressed as a collection of a finite number of non-intersecting weakly-connected support sets,

$$X = \chi_1 \cup \chi_2 \cup \cdots , \quad \chi_i \cap \chi_j = \emptyset \forall i, j .$$

Furthermore, it is not hard to see that the weights $W_\pm(X)$ factor into a product of weights for each weakly-connected support set in $X$,

$$W_\pm(X) = \prod_{i=1}^{k} W_\pm(\chi_i) .$$

The proof of this factorization property for our model is identical to the proof of the corresponding property in KT [12] (see their Lemma 4.5), and so we do not repeat it here. It follows from the fact that the weights $W_\pm(C)$ for the microscopic configurations $C$ have this factorization property, and this is because $W_\pm(C)$ is proportional to the matrix element (3.31).

3. The bound on the weights

The last result from step one of the proof is Lemma 1, which shows that the weights $W_\pm(X)$ are bounded by

$$|W_\pm(X)| \leq e^{-\mu|X|} g_d(X) \left[ \prod_{r=1}^{\frac{\delta}{4}} |f(r)|^{d_r(X)} \right]$$

for any $\delta, \mu > 0$, as long as $|\lambda|$ is smaller than some $\lambda_0$. Recall that in this bound, $|X|$ is the number of boxes and plaquettes in $X$, $d_r(X)$ is the number of dashed lines of length $r$ in $X$, and $d(X) = \sum_{r=1}^{\frac{\delta}{4}} d_r(X)$ is the total number of dashed lines in $X$. This bound will be important for the second step of our proof, where we use it as part of the proof that the polymer expansion for our model is absolutely convergent. We present a detailed proof of Lemma 1 in Appendix A.

C. Review of the polymer expansion

Before moving on to the second step of the proof, we first introduce the main tool that is used in that step, namely the polymer expansion (also known as the cluster expansion). The polymer expansion is a tool from rigorous statistical mechanics that, under certain conditions, allows one to obtain an absolutely convergent series expansion for the logarithm of the partition function of a given model. One can then prove many things about the model using this expansion. In this subsection we give a brief overview of the polymer expansion and the conditions that guarantee its convergence (we discuss the issue of convergence in more detail in Appendix C). For our presentation of these results we follow Ref. 54, Ch. 5 of Ref. 55, and Ch. 20 of Ref. 56.
In practice, the polymer expansion arises as follows. In many statistical mechanical models, the partition function $Z$ admits an absolutely convergent series expansion in which each term in the series has a geometric interpretation in terms of disconnected geometric objects. These geometric objects are the polymers that give this expansion its name. A good example to have in mind is the contours of domain walls that appear in the low temperature expansion of the classical Ising model in two dimensions. When this geometric expansion for $Z$ exists, one can sometimes find a region of the model’s parameter space in which the logarithm $\ln(Z)$ of the partition function also admits an absolutely convergent series expansion in terms of the same geometric objects. This expansion for $\ln(Z)$ is the polymer expansion.

Let us now be more explicit. Let $\Gamma$ be a finite set whose elements are called polymers and are denoted by $\gamma$. The polymers have a notion of intersection so that, for any two polymers $\gamma_1$ and $\gamma_2$, we can have $\gamma_1 \cap \gamma_2 = \emptyset$ or $\gamma_1 \cap \gamma_2 \neq \emptyset$. Note that the intersection of a polymer with itself is non-empty, $\gamma \cap \gamma \neq \emptyset$. To define a statistical model we also need to define the weight of a polymer. The weight $W(\gamma) \in \mathbb{C}$ of a polymer $\gamma$ is just a complex number associated with that polymer.

Let $\Gamma'$ be a subset of $\Gamma$. It is an unordered collection of a certain number of polymers. A subset $\Gamma'$ of polymers of the form $\Gamma' = \{\gamma_1, \ldots, \gamma_k\}$ is said to be disconnected if the polymers $\gamma_1, \ldots, \gamma_k$ are pairwise disjoint, i.e., $\gamma_i \cap \gamma_j = \emptyset$ for $1 \leq i < j \leq k$. If $\Gamma' = \{\gamma_1, \ldots, \gamma_k\}$ is disconnected, then we define the weight of $\Gamma'$ to be the product of the weights of all the $\gamma_i$,

$$W(\Gamma') := \prod_{i=1}^{k} W(\gamma_i).$$

(3.36)

In terms of these quantities, the polymer partition function $Z$ is defined as

$$Z := \sum_{\Gamma' \subseteq \Gamma, \text{ disconnected}} W(\Gamma'),$$

(3.37)

where the sum is taken only over disconnected subsets $\Gamma'$. In this sum we also include the empty set $\Gamma' = \emptyset$, and we assign it a weight of 1, $W(\emptyset) := 1$. Note also that, although we have called $Z$ the polymer “partition function”, we have allowed the weights $W(\gamma)$ to be complex numbers. As a result, the polymer expansion can be applied to both quantum and classical statistical mechanics.

The polymer expansion is an expansion for $\ln(Z)$ in terms of the same weights $W(\gamma)$ that appear in the definition of $Z$. To write down the series for $\ln(Z)$, we first need to introduce some more notation. If we are given a collection of polymers $\gamma_1, \ldots, \gamma_k$, then we define $G(\gamma_1, \ldots, \gamma_k)$ to be the graph with the following properties. First, the vertices of $G(\gamma_1, \ldots, \gamma_k)$ are labeled from 1 to $k$. Second, $G$ has an edge between vertices $i$ and $j$ (with $i \neq j$) if $\gamma_i \cap \gamma_j \neq \emptyset$ (for $i, j \in \{1, \ldots, k\}$), and no other edges besides these. Next, we need to define the index of a connected graph. Let $C_k$ be the set of connected graphs with $k$ vertices labeled from 1 to $k$, and let $G \in C_k$. The index of $G$, denoted by $n(G)$, is defined by

$$n(G) := \sum_{H \in C_k, H \subseteq G} (-1)^{\ell(H)},$$

(3.38)

where the sum is taken over all $H$ in $C_k$ that are also subgraphs of $G$ (including $H = G$), and where $\ell(H)$ is the number of edges (or lines) in $H$.

Using these notations, the polymer expansion for $\ln(Z)$ is given by

$$\ln(Z) = \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{H \in C_k} n(H) \sum_{G(\gamma_1, \ldots, \gamma_k) = H} \prod_{i=1}^{k} W(\gamma_i)$$

$$:= \sum_{k=1}^{\infty} I_k,$$

(3.39)

(4.0)

where we defined $I_k$ to be the $k$th term in the sum in the first line. Note that $\ln(Z)$ only receives contributions from (ordered) collections $\gamma_1, \ldots, \gamma_k$ of polymers whose associated graph $G(\gamma_1, \ldots, \gamma_k)$ is connected.

We now review the conditions that ensure the convergence of the polymer expansion for $\ln(Z)$. For this purpose we specialize to the concrete setting of polymer models defined on a finite set $\Lambda$ whose elements are called “vertices.” In this setting each polymer $\gamma$ contains a subset of the vertices in $\Lambda$. A polymer can also have additional internal structure beyond the vertices that it contains, and we will see an example of this when we apply the polymer expansion to our model. We define the size $|\gamma|$ of a polymer to be the number of vertices contained in $\gamma$ (so $|\gamma| \geq 1$), and the total number of vertices in $\Lambda$ is denoted by $|\Lambda|$. Finally, the notion of intersection is simply sharing of vertices (i.e., $\gamma_1 \cap \gamma_2 \neq \emptyset$ if $\gamma_1$ and $\gamma_2$ have at least one vertex in common).
We now present a simple convergence criterion for the polymer expansion in this setting. (See Appendix C for a derivation of this criterion starting from the general convergence criterion of Ref. 54.) To use this criterion, our polymer model needs to have the following property: the weights need to satisfy an upper bound of the form
\[
|W(\gamma)| \leq \tilde{W}(\gamma),
\]
where the set of polymers \( \Gamma \) is isotropic with respect to the weight \( \tilde{W}(\gamma) \), in the sense that the sum \( \sum_{\gamma \in \Gamma} \tilde{W}(\gamma) g(|\gamma|) \) is independent of the vertex \( v \in \Lambda \) for any function \( g(|\gamma|) \) of the polymer size \(|\gamma|\) (the sum is taken over all \( \gamma \in \Gamma \) containing \( v \)).

Let us assume that the above property holds. If we define
\[
q := \sum_{\gamma \ni v} \tilde{W}(\gamma)e^{\gamma_1},
\]
then one can establish the following convergence criterion:

**Theorem 2** (Simplified convergence criterion). If \( q < 1 \), then the polymer expansion (3.40) for \( \ln(Z) \) is absolutely convergent for fixed \( |\Lambda| \), and furthermore \( \ln(Z) \) satisfies the bound
\[
|\ln(Z)| \leq \sum_{k=1}^{\infty} |I_k| \leq -|\Lambda| \ln(1 - q).
\]

To complete our review of the polymer expansion, we need to discuss the properties of this expansion when the weights \( W(\gamma) \) depend on an additional parameter \( z \in \mathbb{C} \), so that we have \( W(\gamma) \to W_z(\gamma) \) in our original expansion for \( Z \). Suppose that \( W_z(\gamma) \) is a holomorphic function of \( z \) for \( z \in \mathbb{D} \), where \( \mathbb{D} \) is a connected open subset of \( \mathbb{C} \). In this situation it is natural to ask whether \( \ln(Z) \) is also a holomorphic function of \( z \) on \( \mathbb{D} \) when the polymer expansion converges. The answer to this question is provided by Theorem 5.8 of Ref. 55, which can be summarized as follows.\(^{13}\) Suppose that \( \sup_{z \in \mathbb{D}} |W_z(\gamma)| \leq \tilde{W}(\gamma) \), where \( \tilde{W}(\gamma) \) is a real and positive weight that is independent of \( z \), and let \( Z \) be the partition function obtained by replacing \( W_z(\gamma) \) with \( \tilde{W}(\gamma) \) in Eq. (3.37). If the polymer expansion for \( \ln(Z) \) is absolutely convergent, then the polymer expansion for \( \ln(Z) \) is absolutely convergent for any \( z \in \mathbb{D} \), and \( \ln(Z) \) is a holomorphic function of \( z \) for \( z \in \mathbb{D} \). As we discussed above, this holomorphic property of \( \ln(Z) \) is a crucial ingredient in our stability proof.

### D. Step 2: Convergent polymer expansion and its consequences

#### 1. Mapping our model to a polymer model

We start by interpreting our formula (3.3) for \( Z_\pm/Z_0 \) as a polymer partition function. To this end, recall that the support sets \( X \) can be decomposed into a union of non-intersecting weakly-connected support sets \( \chi_1, \chi_2, \ldots \), and that \( W_\pm(X) \) factors according to this decomposition (3.5). It follows that the polymers in our model correspond exactly to the weakly-connected support sets \( \chi \).

To simplify the notation, we now reformulate this polymer model so that each polymer is described by a subset of vertices in a particular lattice \( \Lambda \). This lattice \( \Lambda \) consists of spacetime points that are at the plaquette centers and vertical bond midpoints of the blocked spacetime lattice. Since the blocked spacetime lattice has sites at locations \( (j, (\ell - 1)\tau) \) for \( j \in \{1, \ldots, L\} \) and \( \ell \in \{1, \ldots, M\} \), the vertices in \( \Lambda \) are located at \( (j + 1/2, (\ell - 1/2)\tau) \) (the plaquette centers) and \( (j, (\ell - 1/2)\tau) \) (the vertical bond midpoints), for \( j \in \{1, \ldots, L\} \) and \( \ell \in \{1, \ldots, M\} \). The total number of vertices in \( \Lambda \) is then \( |\Lambda| = 2LM \) where \( M \) is the integer \( M = \beta/\tau \), defined previously.

We can map each weakly-connected support set \( \chi \) to a polymer \( \gamma_\chi \) that consists of a subset of vertices of \( \Lambda \) together with appropriate dashed lines connecting these vertices. The mapping that we use is the obvious one:

1. If \( \chi \) contains a certain box or plaquette, then \( \gamma_\chi \) contains the vertex at the center of that box or plaquette.
2. If two boxes or two plaquettes in \( \chi \) are connected by a dashed line, then the corresponding vertices in \( \gamma_\chi \) are also connected by a dashed line.

\(^{13}\) Note that Ref. 55 refers to a connected open subset of \( \mathbb{C} \) as a domain.
It is very important to note that these edges are different from the dashed lines that we previously discussed.

We also define the weight of $\gamma_\chi$ to be equal to the weight of $\chi$, that is: $W_{\pm}(\gamma_\chi) := W_{\pm}(\chi)$. Finally, we use the special notation $\tilde{\gamma}_\chi$ to denote a polymer that is obtained from a connected support set $\tilde{\chi}$.

The price we pay for this simple mapping from $\chi$ to $\gamma_\chi$ is that our notion of intersection of weakly-connected support sets does not quite map onto the simple notion of sharing of vertices of $\Lambda$ that we considered in Sec. III C. We now discuss the notion of intersection in our model in more detail, and also clarify the structure of the set $\Gamma$ of allowed polymers in our model.

To describe the set of allowed polymers and the notion of intersection, it is convenient to add additional structure to the set $\Lambda$ to turn it into a connected graph. We do this by adding edges\(^{14}\) that connect each vertex to 14 of its neighbors in the manner shown in Fig. 2. To understand these connections, recall that in our original picture on the blocked spacetime lattice, each box touches 14 neighboring boxes and/or plaquettes, and likewise for each plaquette. The 14 connections shown in Fig. 2 correspond exactly to these neighboring boxes/plaquettes.

Now that we have endowed $\Lambda$ with a graph structure, we are almost ready to describe the set of allowed polymers $\Gamma$. To do this we first need to review the definition of an induced subgraph of a graph. Consider an abstract graph $G = (V, E)$, where $V$ and $E$ are the sets of vertices and edges of $G$. If $V' \subset V$ is a subset of vertices, then the induced subgraph of $G$ determined by $V'$ takes the form $(V', E')$, where $E' \subset E$ contains all edges of $G$ that have both of their endpoints contained in $V'$. In general, an induced subgraph will not be connected and will instead have several connected components. An induced subgraph that is connected is called a connected induced subgraph.

We are now ready to describe $\Gamma$. The set $\Gamma$ contains all polymers $\gamma$ with the following structure:

1. $\gamma$ contains a subset of the vertices in $\Lambda$.
2. $\gamma$ can contain horizontal dashed lines connecting pairs of vertices with the same time coordinate. The two vertices connected by the dashed line must come from the same type of object (box or plaquette) on the blocked spacetime lattice.
3. Two vertices in $\gamma$ are connected by at most one dashed line.
4. Let $\Lambda_\gamma$ be the induced subgraph of $\Lambda$ determined by the vertices in $\gamma$. If $\Lambda_\gamma$ has more than one connected component, then the dashed lines in $\gamma$ must be such that it is possible to get from one connected component of $\Lambda_\gamma$ to any other one by traversing sufficiently many dashed lines.

Although it looks different, this definition of the allowed polymers is exactly the same as the definition of the allowed weakly-connected support sets that we presented in Sec. III B 2. We have simply translated that definition from the language of boxes and plaquettes into the language of polymers $\gamma$ containing vertices of the connected graph $\Lambda$. We also note that our definition of connected support sets maps onto special polymers $\tilde{\gamma}$ that (i) do not contain any dashed lines and (ii) whose induced subgraph $\Lambda_\tilde{\gamma}$ has only one connected component, i.e., $\Lambda_\tilde{\gamma}$ is a connected induced subgraph of $\Lambda$.

Finally, using the graph structure of $\Lambda$, we can now translate the notion of intersection of weakly-connected support sets into the polymer language. Specifically, we find that two polymers $\gamma_1$ and $\gamma_2$ intersect if (i) they have one or more vertices in common, or (ii) a vertex in $\gamma_1$ is connected to a vertex in $\gamma_2$ via an edge of $\Lambda$.

2. Lemma 2 and the convergence of the polymer expansion

At this point we have succeeded in showing that our expansion for $Z_{\pm}/Z_0$ has exactly the structure required to apply the polymer expansion to study $\ln(Z_{\pm}/Z_0)$. Now we just need to show that the resulting expansion is absolutely convergent.

\(^{14}\) It is very important to note that these edges are different from the dashed lines that we previously discussed.
Specifically, the combinatorial factors given by
\[ \ln(q) \]
prove convergence in our case. Specifically, if we re-define \( q \) as
\[ q := \sum_{\gamma \in 3^v} \tilde{W}(\gamma)e^{14|\gamma|}, \]  
(3.44)
where the only change is the multiplicative factor of 14 in the exponent, then our polymer expansion for \( \ln(Z_{\pm}/Z_0) \) will be absolutely convergent if \( q < 1 \). We explain the derivation of this modified convergence criterion in Appendix C.

As in Sec. III C, this (modified) convergence criterion only holds if our polymer model satisfies the following property: the weights need to satisfy an upper bound of the form \( |W_{\pm}(\gamma)| \leq \tilde{W}(\gamma) \) where the set of polymers \( \Gamma \) is isotropic with respect to the weight \( \tilde{W}(\gamma) \) in the sense of Sec. III C. But we have already established exactly such a bound in step one of the proof. Indeed, Lemma 1 implies that the weight \( W_{\pm}(\gamma) \) obeys the bound \( |W_{\pm}(\gamma)| \leq \tilde{W}(\gamma) \) with
\[ \tilde{W}(\gamma) = e^{-\mu|\gamma|\delta d(\gamma)} \left[ \prod_{r=1}^{\beta} \left| f(r) \right|^{d_r(\gamma)} \right], \]  
(3.45)
where \( d_r(\gamma) \) is the number of dashed lines of length \( r \) in \( \gamma \) and \( d(\gamma) \) is the total number of dashed lines in \( \gamma \). Moreover, it is clear from the form of \( \tilde{W}(\gamma) \), that our set of polymers \( \Gamma \) is isotropic with respect to this weight.

At this point, all that remains to prove the convergence of the polymer expansion is to show that \( q < 1 \). More precisely, we need to show that \( q < 1 \) at least one choice of \( \mu, \delta > 0 \). Then, as a result of Lemma 1, there exists \( \lambda_0 \) so that, for all \( \lambda \in \mathbb{C} \) satisfying \( |\lambda| < \lambda_0 \), we have \( |W_{\pm}(\gamma)| \leq \tilde{W}(\gamma) \), and hence the polymer expansion converges.

The desired bound, \( q < 1 \), follows from Lemma 2, which we prove in Appendix B using combinatorial arguments.

### 3. Claims 1 and 2 as consequences of the convergent polymer expansion

To wrap up our discussion of step two of the proof, we now explain how Claims 1 and 2 can be deduced from the existence of the convergent polymer expansion for \( \ln(Z_{\pm}/Z_0) \). To start, the first part of Claim 1 follows from the general results about the holomorphic properties of the polymer expansion that we reviewed in the last paragraph of Sec. III C. Recall from that paragraph that the key to proving holomorphicity in a connected open set \( D \) of the (complexified) parameter space is to show that the polymer expansion also converges if we replace the weights \( W_{\pm}(X) \) by a uniform upper bound \( \tilde{W}(X) \) that is independent of the parameters and holds throughout the region \( D \). But we have already established this, since we can use for \( \tilde{W}(X) \) the same upper bound \( \tilde{W}(X) \) that we obtained in Lemma 1 and used in Lemma 2. Thus, we can conclude that the complexified free energy \( -\ln(Z_{\pm})/\beta \) in our model is a holomorphic function of \( \lambda \) in the region of \( \mathbb{C} \) where the polymer expansion converges.

Next, the second part of Claim 1 is also a simple consequence of the convergence of the polymer expansion. Since the expansion for \( \ln(Z_{\pm}/Z_0) \) converges absolutely, Theorem 2 implies that \( \ln(Z_{\pm}/Z_0) \) obeys the bound
\[ \left| \ln \left( \frac{Z_{\pm}}{Z_0} \right) \right| \leq -|\lambda| \ln(1 - q). \]  
(3.46)
[This bound has the same form as in Theorem 2, but \( q \) here is our modified version with the extra factor of 14 in the exponent.] For our model the size of \( \Lambda \) is given by \( |\Lambda| = 2LM \), where \( M = \beta/\tau \). Thus, Claim 2 holds with the constant \( c_3 \) given by
\[ c_3 = -\frac{2}{\tau} \ln(1 - q). \]  
(3.47)
Since \( c_3 \) can be expressed only in terms of \( \tau \) and \( q \), we find that \( c_3 \) depends only on \( \Delta \), the constant \( c \) from the summability condition (2.4), and geometric properties of the model.\(^\dagger\) Note that \( c_3 \) does not depend on \( \lambda, h, \beta, \) or \( L \).

Finally, we explain how Claim 2 follows from the form of the polymer expansion for \( \ln(Z_{\pm}/Z_0) \). To understand this claim we first recall that for any support set \( X \) the weight \( W_{\pm}(X) \) is given by a combined sum and integral over terms of the form
\[ \frac{1}{Z_0} \text{Tr}_{\pm} \{ e^{-\beta H_0} V_{Y_n}(\tau_n) \cdots V_{Y_1}(\tau_1) \}, \]  
(3.48)
\(^\dagger\) Specifically, the combinatorial factors \( K \) and \( K_0 \) that appear in our proofs in Appendixes A and B.
where $\tau_1, \ldots, \tau_n$ and $Y_1, \ldots, Y_n$ are a sequence of times and perturbation terms that are consistent with the support set $X$. Now, for any operator $O$, the difference between the trace of $O$ over $H_+$ and the trace of $O$ over $H_-$ can be written in the form

$$\text{Tr}_+\{O\} - \text{Tr}_-\{O\} = \text{Tr}\{P_+ O\} - \text{Tr}\{P_- O\} = \text{Tr}\{SO\},$$

(3.49)

where $S$ is the Ising symmetry operator from Eq. (2.6) and $P_\pm = \frac{1}{2}(1 \pm S)$ is the projector onto $H_\pm$. Since $S$ flips all $L$ spins on the chain, we can see from Eq. (3.49) that $\text{Tr}_+\{O\} - \text{Tr}_-\{O\}$ will vanish unless $O$ acts nontrivially on all $L$ spins.

We can now apply this reasoning to a typical term

$$\frac{1}{Z_0} \text{Tr}_+\{e^{-\beta H_0} V_{Y_n}(\tau_n) \cdots V_{Y_1}(\tau_1)\} - \frac{1}{Z_0} \text{Tr}_-\{e^{-\beta H_0} V_{Y_n}(\tau_n) \cdots V_{Y_1}(\tau_1)\}$$

(3.50)

that would appear in the (summand and integrand of the) difference $W_+(X) - W_-(X)$. Since each operator $V_Y$ acts on at most two spins, we see from Eq. (3.49) that this difference must vanish if $n$ is less than $L/2$. Thus, the difference of weights $W_+(X) - W_-(X)$ only receives contributions from terms of order $\lambda^n$ with $n \geq L/2$. This is because $n$ here keeps track of the order of the Duhamel expansion, and in that expansion the order $n$ term is accompanied by a factor of $\lambda^n$. Finally, since the polymer expansion expresses $\ln(Z_+ / Z_0)$ as a sum over products of the weights $W_+(\chi)$, the same result holds for the difference of free energies $f(\lambda, \beta) = -\frac{1}{\beta} \ln(Z_+) + \frac{1}{\beta} \ln(Z_-)$. This can be proven order by order in the polymer expansion (since it converges absolutely). This completes the proof of Claim 2 and of the second step of our proof of Theorem 1.

IV. GENERALIZATION TO OTHER MODELS

Although we have focused on the specific model Hamiltonian from Sec. II A, it is possible to generalize our stability result to a much larger family of Hamiltonians. In particular, our result also holds for models (i) without translation invariance, (ii) in dimensions greater than one, and (iii) with a much wider variety of interaction terms. In particular, in regards to point (iii) we will show that our result is not limited to the case of two-body interactions. In this section we briefly discuss a more general family of models that our main result also applies to.

The family of models that we consider here all involve spin $1/2$ degrees of freedom in $D \geq 1$ spatial dimensions and on a (hyper)cubic lattice of linear size $L$. Let $\sigma_\tau^{x,y,z}$ be the spin operators (really, the Pauli matrices) acting on site $\tau$ of the lattice, and let $S = \prod_\tau \sigma_\tau^z$ be the Ising symmetry operator. Our starting point is again an unperturbed Hamiltonian $H_0$ of the Ising form,

$$H_0 = \frac{\Delta}{2} \sum_{\tau, \tau'} (1 - \sigma_\tau^z \sigma_{\tau'}^z),$$

(4.1)

where $\langle \tau, \tau' \rangle$ denotes a sum over nearest neighbors (counting each pair only once). As before, $H_0$ commutes with $S$ and has a unique ground state and a finite energy gap in each sector $H_{\pm}$ of fixed $S$ eigenvalue (we still have $\Delta > 0$). The ground state $|\pm\rangle$ in the sector $H_{\pm}$ is again given by the appropriate superposition of the all-up and all-down states, $|\pm\rangle = (|\uparrow\rangle \pm |\downarrow\rangle)/\sqrt{Z}$. Our full Hamiltonian takes the form $H = H_0 + \lambda V$, where we assume that the perturbation $V$ preserves the Ising symmetry, $[S, V] = 0$. The main difference from before is that we now allow $V$ to take the general form

$$V = \sum_\tau h_\tau V_\tau,$$

(4.2)

where the $h_\tau$ are real coefficients satisfying $|h_\tau| \leq h$ for some positive real number $h$, and the $V_\tau$ are operators that we now describe. Specifically, $V_\tau = T_\tau V_0 T_\tau^{-1}$, where $T_\tau$ is the operator that performs translations by $\tau$, and $V_0$ is of the form

$$V_0 = \sum_\gamma V_\gamma,$$

(4.3)

where the sum runs over subsets $\gamma$ of lattice sites containing the origin $\tau = 0$, and having at most $K$ sites, and where $V_\gamma$ is supported on $\gamma$. Note that in our setup we have broken translation symmetry because we allow each term $V_\tau$ to appear with a different coefficient $h_\tau$. Note also that $V_0$ (and therefore $H$) contains at most $K$-body interactions. We assume that $K$ remains fixed and finite in the thermodynamic limit $L \to \infty$.

In this general case we need to impose two additional conditions on the interaction terms $V_\gamma$ to guarantee that our results hold. The first condition is a generalization of the summability condition (2.4), namely

$$\sum_\gamma ||V_\gamma|| \leq c,$$

(4.4)
where $\| \cdot \|$ is the usual operator norm and the constant $c$ is again required to be independent of the lattice size. As before, this condition guarantees that the operator norm of the Hamiltonian is extensive in the system size.

For the second condition, we require the $V_Y$ to obey an additional symmetry relation of the form

$$[S_{Y_i}, V_Y] = 0 \quad \forall \ i,$$

(4.5)

where the $Y_i$ are the connected components of $Y$, and $S_{Y_i}$ is the Ising symmetry operator restricted to $Y_i$: $S_{Y_i} = \prod_{\sigma_{r} \in Y_i} \sigma_r^z$.

To understand why we impose the condition (4.5), consider a simple $V$ that violates this condition, namely a long range $\sigma_r^x \sigma_r^z$ interaction. It is known that (antiferromagnetic) interactions of this kind can cause a phase transition for arbitrarily small $\lambda$ even when the summability condition (4.4) holds [57–59]; therefore we cannot drop (4.5) without imposing stricter requirements on the decay of the interactions. Furthermore, at a technical level, the condition (4.5) is essential to our proof as it guarantees the factorization of the weights in our polymer model. We also note that (4.5) is a natural property of many long-range interactions – for example, those that are induced by integrating out gapless degrees of freedom (e.g., phonons or photons) that are even under the Ising symmetry.

For any $V$ of this kind, we can prove the following result:

**Theorem 3.** There exists a $L$-independent constant $\lambda_0 > 0$ such that, if $|\lambda| < \lambda_0$, then (1) $H$ has a unique ground state and a finite energy gap in each sector $\mathcal{H}_\pm$ of the Hilbert space with fixed $S$ eigenvalue, and (2) the ground state energy splitting $|E_+ (\lambda) - E_- (\lambda)|$ between sectors satisfies the exponential bound

$$|E_+ (\lambda) - E_- (\lambda)| \leq c_1 L^D e^{-c_2 L^D},$$

(4.6)

where $D \geq 1$ is the spatial dimension and $c_1, c_2$ are positive constants that depend on $\Delta, \lambda, h$, and the form of the interactions, but not on the system size $L$.

In Appendix D we summarize the changes that we need to make in our setup and in the proofs of Lemma 1 and 2 in order to prove this more general result.

## V. CONCLUSION

In this paper we investigated the stability of certain gapped Hamiltonians $H_0$ to perturbations $V$ that contain long-range interactions, such as those that decay as a power law of the distance. For the family of models that we considered we were able to prove not only the stability of the spectral gap, but also a more detailed stability property for the states below the gap. Specifically, for an $H_0$ with two exactly degenerate ground states below the gap, we were able to prove that the residual splitting of these two states in the perturbed model was exponentially small in the system size. As we mentioned in the introduction, this exponential splitting bound is of great interest for quantum computation applications.

There are many possible directions for future work. One of the most interesting directions would be to widen the class of unperturbed Hamiltonians $H_0$ that we can study. In this paper we were mostly limited to the case where $H_0$ is classical (although the Jordan-Wigner transformation allowed us to prove results about some one dimensional quantum models, such as Kitaev’s p-wave wire model [3]). It would be interesting to consider the case where $H_0$ is a truly quantum Hamiltonian in a dimension higher than one, for example the toric code model in two dimensions.

It would also be interesting to understand how to establish stability (if it does exist) in the presence of long-range interactions that violate our summability condition, for example the $1/r^3$ Coulomb interaction in one or higher dimensions. To investigate that case it may be necessary to include the contributions to the Hamiltonian from the background charges that make the entire system neutral, as is known to be important for proving the stability of matter [60].

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**Appendix A: Properties of the weights $W_{\pm}(X)$ and proof of Lemma 1**

In this appendix we present a detailed analysis of the weights $W_{\pm}(X)$ for our model. We first give a precise definition of $W_{\pm}(X)$. This definition can be thought of as a more careful form of Eq. (3.29). In the second part of this appendix we then
present the proof of Lemma 1, which establishes the bound (3.6) on $|W_\pm (X)|$. Finally, in the third part of this appendix we discuss two small technical details that arise in the derivation of the bound on $|W_\pm (X)|$.

1. Precise definition of $W_\pm (X)$

To write down a precise expression for the weights we first decompose the weight $W_\pm (X)$ for any non-empty support set $X$ into a sum over contributions from all orders $n \geq 1$ in the Duhamel expansion of $Z_\pm$, and a sum over contributions from all possible subsets $Y_1, \ldots, Y_n$ that determine the perturbation terms that appear at order $n$,

$$W_\pm (X) = \sum_{n \geq 1} \sum_{Y_1, \ldots, Y_n} W_{\pm Y_1, \ldots, Y_n}^{(n)}(X). \quad (A1)$$

With this decomposition, it is possible to write down a precise expression for the weight $W_{\pm Y_1, \ldots, Y_n}^{(n)}(X)$ with the help of two indicator functions that we now define.

In order to compute $W_{\pm Y_1, \ldots, Y_n}^{(n)}(X)$, we need to integrate over all times $0 \leq \tau_1 \leq \cdots \leq \tau_n \leq \beta$ such that the pairs $(Y_1, \tau_1), \ldots, (Y_n, \tau_n)$ are consistent with the support set $X$. For fixed $Y_1, \ldots, Y_n$, there may be several different ways of distributing the pairs $(Y_1, \tau_1), \ldots, (Y_n, \tau_n)$ among the different time intervals of the blocked spacetime lattice such that the resulting configuration is consistent with $X$. We will need to sum/integrate over all of these consistent ways of distributing the pairs $(Y_1, \tau_1), \ldots, (Y_n, \tau_n)$ in our final expression for $W_{\pm Y_1, \ldots, Y_n}^{(n)}(X)$.

Consider first the set of times where one or more of $\tau_1, \ldots, \tau_n$ is an exact integer multiple of the temporal lattice spacing $\tau$. This set of times is a subset of $\mathbb{R}^n$ of Lebesgue measure zero, and so it does not actually contribute to the weight $W_{\pm Y_1, \ldots, Y_n}^{(n)}(X)$ (which involves an integral over $\tau_1, \ldots, \tau_n$). Therefore, in studying the distribution of the pairs $(Y_1, \tau_1), \ldots, (Y_n, \tau_n)$ among the different time intervals, we can restrict our attention to the cases where each of the times $\tau_1, \ldots, \tau_n$ lies in an open interval $(\tau (\ell - 1), \tau \ell)$ for some $\ell \in \{1, \ldots, M\}$. We now proceed to consider these cases.

A particular distribution of the $n$ pairs $(Y_1, \tau_1), \ldots, (Y_n, \tau_n)$ among the open time intervals $(\tau (\ell - 1), \tau \ell)$ is described by a set of non-negative integers $p_1, \ldots, p_M$ satisfying $\sum_{\ell=1}^{M} p_\ell = n$ and such that the first $p_1$ pairs are located in the first (open) time slice $(0, \tau)$, the next $p_2$ pairs are located in the second time slice $(\tau, 2\tau)$, and so on. Let $\bar{Y} = (Y_1, \ldots, Y_n)$ denote the $n$-tuple of subsets, and let $\bar{p} = (p_1, \ldots, p_M)$ denote the $M$-tuple of integers that determine a particular distribution of the pairs $(Y_1, \tau_1), \ldots, (Y_n, \tau_n)$ among the different time intervals. In addition, let $\bar{\tau} = (\tau_1, \ldots, \tau_n)$ be the $n$-tuple containing the times where a perturbation acts in the integrand of a term at nth order in the Duhamel expansion.

To keep track of the distributions that are consistent with $X$, we define two indicator functions $I_X[\bar{Y}; \bar{p}]$ and $I_{\bar{p}}[\bar{\tau}]$ as follows. The first function $I_X[\bar{Y}; \bar{p}]$ is equal to 1 if the set $\bar{Y}$ and the distribution described by $\bar{p} = (p_1, \ldots, p_M)$ is consistent with $X$, and equal to 0 otherwise. The second function $I_{\bar{p}}[\bar{\tau}]$ is equal to 1 if the first $p_1$ times in $\bar{\tau}$ lie in $(0, \tau)$, the next $p_2$ times in $\bar{\tau}$ lie in $(\tau, 2\tau)$, etc., and equal to 0 otherwise. With these notations we have

$$W_{\pm Y_1, \ldots, Y_n}^{(n)}(X) = \frac{(-\lambda)^n}{Z_0} \sum_{\bar{p}_1, \ldots, \bar{p}_M \geq 0} I_X[\bar{Y}; \bar{p}] \int_0^\beta d\tau_n \cdots \int_0^{\tau_2} d\tau_1 I_{\bar{p}}[\bar{\tau}] \operatorname{Tr}_\pm \left\{ e^{-\beta H_0} V_{\bar{Y}_1}(\tau_1) \cdots V_{Y_1}(\tau_1) \right\}, \quad (A2)$$

where we sum over all allowed distributions $p_1, \ldots, p_M$ of the pairs $(Y_1, \tau_1), \ldots, (Y_n, \tau_n)$ into the different time intervals. This expression for $W_{\pm Y_1, \ldots, Y_n}^{(n)}(X)$, combined with Eq. (A1), gives a precise definition of the weight $W_\pm (X)$ for each non-empty support set $X$. Note also that the empty support set $X = \emptyset$ can be consistently assigned a weight of 1, $W_\pm (\emptyset) = 1$, and this particular contribution to $Z_\pm / Z_0$ comes from the zeroth order term in the Duhamel expansion of $Z_\pm$.

2. Proof of Lemma 1

We now prove the bound (3.6) on $|W_\pm (X)|$ that is stated in Lemma 1 from the main text. To prove this bound we start with the precise expression for the weights $W_{\pm Y_1, \ldots, Y_n}^{(n)}(X)$ from (A2) above. To derive an upper bound on $|W_{\pm Y_1, \ldots, Y_n}^{(n)}(X)|$, we first use the trivial bound $Z_0 \geq 1$ to find that $(Z_0)^{-1} \leq 1$ (and this bound is expected to be reasonably tight in the low temperature regime that we are interested in). Next, we note that

$$\operatorname{Tr}_\pm \left\{ e^{-\beta H_0} V_{Y_n}(\tau_n) \cdots V_{Y_1}(\tau_1) \right\} = \sum_{s_1, \ldots, s_n} \langle s_n, \pm | V_{Y_n} | s_{n-1}, \pm | s_1, \pm | V_{Y_1} | s_n, \pm \rangle e^{-(\beta - \tau_n + \tau_1)\epsilon_{s_n}} \cdots e^{-(\tau_2 - \tau_1)\epsilon_{s_1}} , \quad (A3)$$
where \( |s, \pm \rangle \) are the eigenstates of \( H_0 \) that we introduced in the main text. Using the expression (3.27) for the energies \( \epsilon_s \), we find that

\[
e^{-(\beta - \tau_n + \tau_1)} e_{s_n} \cdots e^{-(\tau_2 - \tau_1)} e_{s_1} = e^{-\Delta \times (\text{total vertical length of worldlines in } C)} \leq e^{-\Delta \tau p(X)},
\]

where \( C \) is the microscopic configuration of worldlines determined by \( (Y_1, \tau_1), \ldots, (Y_n, \tau_n) \) and \( |s_n, \pm \rangle \), and \( p(X) \) is the number of plaquettes in \( X \). In addition, because of the time-ordering and the grouping of the pairs \( (Y_1, \tau_1), \ldots, (Y_n, \tau_n) \), we have

\[
\int_0^\beta dt_1 \cdots \int_0^\tau_2 dt_1 \prod_{p \neq \epsilon} f_{p} = \prod_{p \neq \epsilon} \frac{\tau^n}{p_1! \cdots p_M!}.
\]

To proceed, let \( n_1 \) be the number of transverse field terms that appear in the set \( V_{Y_1}, \ldots, V_{Y_n} \) of perturbation terms. Similarly, let \( n_2, r \) be the number of long-range perturbation terms acting at a distance \( r \) that appear in the set \( V_{Y_1}, \ldots, V_{Y_n} \). [The subscripts 1 and 2 indicate that these keep track of one-body and two-body terms, respectively.] Since there are \( n \) perturbation terms in total, we have the identity

\[
n = n_1 + n_2, \quad n_2 = \sum_{r=1}^L n_{2,r}.
\]

If we use these definitions, and also use Eqs. (A32) to bound the matrix elements of the perturbation terms \( V_{Y} \), then we find the bound (note the plus subscript on the trace on the right-hand side)

\[
|W^{(n)}_{\pm; Y_1, \ldots, Y_n}(X)| \leq (|\lambda||h|\tau)^{n_1} (|\lambda|\tau)^{n_2} \prod_{r=1}^L |f(r)|^{n_{2,r}} e^{-\Delta \tau p(X)} \sum_{p_1, \ldots, p_M \geq 0 \atop p_1 + \cdots + p_M = n} I_X[Y_1^\pm; \tilde{p}] \frac{1}{p_1! \cdots p_M!} \text{Tr} \left\{ \tilde{V}_{Y_1} \cdots \tilde{V}_{Y_1} \right\},
\]

where we also defined the operators \( \tilde{V}_{Y} \) by

\[
\tilde{V}_{Y} = \begin{cases} \sigma_j^x & \text{if } Y = \{ j \}, \\ \sigma_j^x \sigma_k^x & \text{if } Y = \{ j, k \}. \end{cases}
\]

We now simplify the prefactors that depend on \( \lambda \) and \( h \) in our bound on \( |W^{(n)}_{\pm; Y_1, \ldots, Y_n}(X)| \). We start by writing

\[
(|\lambda|\tau)^{n_2} = (\sqrt{|\lambda|\tau})^{2n_2} (\sqrt{|\lambda|\tau})^{n_2}.
\]

We then have

\[
(|\lambda||h|\tau)^{n_1} (|\lambda|\tau)^{n_2} \prod_{r=1}^L |f(r)|^{n_{2,r}} = (|\lambda||h|\tau)^{n_1} (\sqrt{|\lambda|\tau})^{2n_2} (\sqrt{|\lambda|\tau})^{n_2} \prod_{r=1}^L |f(r)|^{n_{2,r}}.
\]

Next, we assume that there exists some real number \( \alpha_0 > 0 \) such that

\[
|\lambda||h|\tau \leq \alpha_0, \quad (\sqrt{|\lambda|\tau})^{n_2} \leq \alpha_0.
\]

We will choose a particular value for \( \alpha_0 \) later. With this assumption we find that

\[
(|\lambda||h|\tau)^{n_1} (|\lambda|\tau)^{n_2} \prod_{r=1}^L |f(r)|^{n_{2,r}} \leq (\alpha_0)^{n_1 + 2n_2} (\sqrt{|\lambda|\tau})^{n_2} \prod_{r=1}^L |f(r)|^{n_{2,r}}.
\]

If we now assume that \( \alpha_0 \leq 1 \) and that \( \sqrt{|\lambda|\tau} \leq 1 \), and use our assumption that \( |f(r)| \leq 1 \) for all \( r \), then we have

\[
(|\lambda||h|\tau)^{n_1} (|\lambda|\tau)^{n_2} \prod_{r=1}^L |f(r)|^{n_{2,r}} \leq (\alpha_0)^b(X) (\sqrt{|\lambda|\tau})^{d(X)} \prod_{r=1}^L |f(r)|^{d_r(X)},
\]
where we used the bounds

\[ b(X) \leq n_1 + 2n_2, \quad d_r(X) \leq n_2, \quad r. \]  

(A14)

[Recall that \( b(X) \) is the number of boxes in \( X \) and \( d_r(X) \) is the number of dashed lines of length \( r \) in \( X \).] At this point our bound on the weights takes the form

\[ |W^{(n)}_{Y_1, \ldots, Y_n}(X)| \leq (a_0)^{b(X)}(\sqrt{\lambda/\tau})^{d_r(X)} \left[ \prod_{r=1}^{R} |f(r)|^{d_r(X)} \right] e^{-\Delta \tau p(X)} \sum_{p_1, \ldots, p_M \geq 0 \atop p_1 + \cdots + p_M = n} I_X[\tilde{Y}; \tilde{p}] \frac{1}{p_1! \cdots p_M!} \text{Tr}_+ \left\{ \tilde{V}_{Y_n} \cdots \tilde{V}_{Y_1} \right\}. \]  

(A15)

The last step in obtaining a bound on \( |W_{\pm}(X)| \) is to sum over all choices of \( Y_1, \ldots, Y_n \), and then to sum over all \( n \geq 1 \) (we sum over \( n \geq 1 \) because we are assuming that \( X \) is non-empty). In particular, we need a bound on

\[ \sum_{n \geq 1} \sum_{Y_1, \ldots, Y_n} \sum_{p_1, \ldots, p_M \geq 0 \atop p_1 + \cdots + p_M = n} I_X[\tilde{Y}; \tilde{p}] \frac{1}{p_1! \cdots p_M!} \text{Tr}_+ \left\{ \tilde{V}_{Y_n} \cdots \tilde{V}_{Y_1} \right\}. \]  

(A16)

To obtain this bound we first define, for each time slice \((\tau(\ell - 1), \tau)\), a set \( S_\ell(X) \) of subsets \( Y \) of lattice sites as follows. First, \( S_\ell(X) \) contains \( \{j\} \) if the box centered at site \( j \) in time slice \( \ell \) is in \( X \). Next, \( S_\ell(X) \) contains \( \{j, k\} \) if \( X \) contains a dashed line between the boxes centered on sites \( j \) and \( k \) in time slice \( \ell \). Using the sets \( S_\ell(X) \), we now define the “partial perturbation terms” \( V_\ell \) as

\[ V_\ell = \sum_{Y \in S_\ell(X)} \tilde{V}_Y. \]  

(A17)

The next ingredient we need is a certain projector associated with the support set \( X \) at time zero. Let \( X_0 \) be the restriction of \( X \) to the spatial slice at time zero. By the construction of \( X \), any bond \((j, j + 1)\) at time zero that is contained in the complement of \( X_0 \) must be in its ground state (i.e., no domain wall) for all microscopic configurations \( C \) with the support set \( X \). We define \( \Pi_0 \) to be the projector that projects these bonds onto their ground state,

\[ \Pi_0 = \prod_{(j, j+1) \subset X_0^c} \left( \frac{1 + \sigma_j^z \sigma_{j+1}^z}{2} \right). \]  

(A18)

In terms of the projector \( \Pi_0 \) and the partial perturbation terms \( V_\ell \), the sum from Eq. (A16) can be bounded as

\[ \sum_{n \geq 1} \sum_{Y_1, \ldots, Y_n} \sum_{p_1, \ldots, p_M \geq 0 \atop p_1 + \cdots + p_M = n} I_X[\tilde{Y}; \tilde{p}] \frac{1}{p_1! \cdots p_M!} \text{Tr}_+ \left\{ \tilde{V}_{Y_n} \cdots \tilde{V}_{Y_1} \right\} \leq \text{Tr}_+ \left\{ \Pi_0 e^{V_1} e^{V_2} \cdots e^{V_M} \right\}, \]  

(A19)

and this bound is similar to the bound in Eq. 4.19 of KT [12]. The way to understand this bound is to note that the trace on the right-hand side contains all terms on the left-hand side, and possibly even more terms. However, any extra terms are still positive, so this is an upper bound. In addition, the factorials that appear on the left-hand side are recovered after Taylor-expanding the exponentials \( e^{V_\ell} \) on the right-hand side.

The final step is to bound \( \text{Tr}_+ \left\{ \Pi_0 e^{V_1} e^{V_2} \cdots e^{V_M} \right\} \), and we can do this using the inequality from Eq. (A33). To apply that inequality we choose \( A = \Pi_0 \) and \( B = e^{V_1} e^{V_2} \cdots e^{V_M} \). For \( B \) we find that

\[ ||B|| \leq e^{\sum_{\ell=1}^{M} ||V_\ell||} \leq e^{b(X) + d(X)}. \]  

(A20)

Next, for \( A \) we have

\[ \text{Tr}_+ \{ A \} \leq 2^# \text{ of bonds overlapping with } X_0. \]  

(A21)

To simplify this factor, let \( p_1(X) \) and \( b_1(X) \) be the number of plaquettes and boxes, respectively, in \( X \) in the first time slice. Then we have

\[ # \text{ of bonds overlapping with } X_0 \leq p_1(X) + 2b_1(X), \]  

(A22)

which follows from the fact that each box overlaps with two adjacent bonds. At this point our full bound on the weights takes the form
\[ |W_{\pm}(X)| \leq (\alpha_0)^{b(X)} (\sqrt{|\lambda| \tau})^{d(X)} \left[ \prod_{r=1}^{L} |f(r)|^{d_r(X)} \right] e^{-\Delta \tau p(X)} e^{b(X) + d(X) 2 p_1(X) + 2 b_1(X)} \]

\[ \leq (\alpha_0)^{b(X)} e^{-\Delta \tau p(X)} (e \sqrt{|\lambda| \tau})^{d(X)} \left[ \prod_{r=1}^{L} |f(r)|^{d_r(X)} \right], \quad (A23) \]

where \( K \) is a numerical constant of order 1.\(^{16}\) Next, we make our choice for \( \alpha_0 \). We follow KT [12] and choose

\[ \alpha_0 = e^{-\Delta \tau}, \quad (A24) \]

and so we find that

\[ |W_{\pm}(X)| \leq e^{-\Delta \tau |X| + K |X|} (e \sqrt{|\lambda| \tau})^{d(X)} \left[ \prod_{r=1}^{L} |f(r)|^{d_r(X)} \right]. \quad (A25) \]

Now suppose that we are given some \( \mu, \delta > 0 \), and we want to be able to satisfy the bound from (3.6). If we first choose \( \tau \) large enough to satisfy

\[ \tau \geq \frac{\mu + K}{\Delta}, \quad (A26) \]

and then choose \( \lambda \) small enough to satisfy

\[ e \sqrt{|\lambda| \tau} \leq \delta, \quad (A27) \]

then we find the desired bound

\[ |W_{\pm}(X)| \leq e^{-\mu |X|} \sqrt{\lambda}^{d(X)} \left[ \prod_{r=1}^{L} |f(r)|^{d_r(X)} \right]. \quad (A28) \]

Finally, from our assumptions that \( |\lambda||h| \tau \leq \alpha_0 \) and \( \sqrt{|\lambda| \tau} \leq \alpha_0 \), we find that our bound on \( |W_{\pm}(X)| \) holds for all \( \lambda \in \mathbb{C} \) satisfying \( |\lambda| \leq \lambda_0 \), where

\[ \lambda_0 = \min \left( \frac{1}{|h| \tau} e^{-\Delta \tau}, \frac{1}{\tau} e^{-4 \Delta \tau}, \frac{1}{\tau} \delta^2 \right), \quad (A29) \]

and the value of \( \tau \) is determined by (A26). It is crucial for our results that \( \lambda_0 \) is independent of the system size \( L \) and the inverse temperature \( \beta \). Note also that \( \alpha_0 \leq 1 \) by construction, since \( \Delta, \tau \geq 0 \). In addition, if \( |\lambda| \leq \lambda_0 \), then we also have \( \sqrt{|\lambda| \tau} \leq 1 \). This completes the proof of Lemma 1.

3. Technical details for the bound on the weights

We close this appendix by discussing two small technical details that arose in our derivation of the bound on \( W_{\pm}(X) \).

a. Signs of matrix elements in Ising symmetry sectors

The first technical detail that we address here is the signs of the matrix elements \( \langle s, \pm | V_Y | s', \pm \rangle \) of the perturbation terms \( V_Y \), where \( |s, \pm \rangle \) and \( |s', \pm \rangle \) are two of the eigenstates of \( H_0 \) in the sector \( \mathcal{H}_{\pm} \), defined in Sec. III B 1.

\(^{16}\) For example, this bound will hold if we choose \( K = 1 + 2 \ln(2) \).
Since $V_{ij}$ contains either a single operator $\sigma_j^x$ or a product $\sigma_j^x \sigma_k^x$ of two Pauli $x$ operators, we need to consider the signs of matrix elements of $\sigma_j^x$ and $\sigma_j^x \sigma_k^x$ between two states $|s, \pm\rangle$ and $|s', \pm\rangle$. Using $[S, \sigma_j^x] = 0$ and $S^2 = 1$, we find that

\begin{align}
\langle s, \pm | \sigma_j^x | s', \pm \rangle &= \langle s | \sigma_j^x | s' \rangle \pm \langle s | \sigma_j^x S | s' \rangle \quad (A30) \\
\langle s, \pm | \sigma_j^x \sigma_k^x | s', \pm \rangle &= \langle s | \sigma_j^x \sigma_k^x | s' \rangle \pm \langle s | \sigma_j^x \sigma_k^x S | s' \rangle \quad (A31)
\end{align}

The key point here is that, for given $|s\rangle$ and $|s'\rangle$, only one of the terms on the right-hand sides of these equations can be non-zero. Therefore we find that

\begin{align}
|\langle s, -| \sigma_j^x | s', -\rangle| &= \langle s, +| \sigma_j^x | s', +\rangle \\
|\langle s, -| \sigma_j^x \sigma_k^x | s', -\rangle| &= \langle s, +| \sigma_j^x \sigma_k^x | s', +\rangle 
\end{align}

(A32a, A32b)

[The matrix elements of $\sigma_j^x$ and $\sigma_j^x \sigma_k^x$ in “$+$” states are always positive.] These relations are the reason why there is only a “$+$” subscript on the trace in the bound in Eq. (A7).

b. Modified trace bound

The next issue that we address here is the problem of obtaining upper bounds on traces of the form $\text{Tr}_+ \{AB\}$, where the trace is taken over the sector $\mathcal{H}_+$ of the total Hilbert space. In our derivation of the bound on $|W_+(X)|$, we encountered a trace of this form where both operators $A$ and $B$ commuted with the Ising symmetry operator $S$, and where the operator $A$ was semipositive definite. In that case it is possible to obtain a useful bound on $\text{Tr}_+ \{AB\}$ in the following way. Let $|n, +\rangle$ be a basis of simultaneous eigenstates of $S$ and $A$ in the plus sector, with $S|n, +\rangle = |n, +\rangle$ and $A|n, +\rangle = a_n |n, +\rangle$, where $a_n \geq 0$ are the eigenvalues of $A$ within $\mathcal{H}_+$. Then we have

$$\text{Tr}_+ \{AB\} = \sum_n a_n \langle n, + | B | n, + \rangle \leq ||B|| \sum_n a_n .$$

Since $\sum_n a_n = \text{Tr}_+ \{A\}$, we find that $\text{Tr}_+ \{AB\}$ is bounded as

$$\text{Tr}_+ \{AB\} \leq ||B|| \text{Tr}_+ \{A\} .$$

(A33)

We used this inequality to bound $\text{Tr}_+ \{\prod_i e^{V_i} e^{V_2} \cdots e^{V_M}\}$ in Eqs. (A20)-(A22).

Appendix B: Proof of Lemma 2

Our goal in this appendix is to prove Lemma 2, which establishes sufficient conditions for the bound $q < 1$ to hold, where

$$q = \sum_{\gamma \in v} \tilde{W}(\gamma) e^{14|\gamma|} .$$

(B1)

The sum is taken over all polymers that contain the particular vertex $v$, and the weight $\tilde{W}(\gamma)$ is given by

$$\tilde{W}(\gamma) = e^{-\mu |\gamma| \delta(\gamma)} \left[ \prod_{r=1}^{L} |f(r)|^{d_v(\gamma)} \right] .$$

(B2)

This lemma is the key to proving the convergence of the polymer expansion for our model.

To gain some intuition for why it is still possible to prove that $q < 1$ with long-range interactions, and for why there must be some restriction on the form of $f(r)$, it is useful to look at a simple example. Suppose we have a weakly-connected polymer $\gamma$ which is made up of two connected polymers $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ that are connected to each other by a single dashed line, and where $v \in \tilde{\gamma}_1$. The contribution of this polymer to $q$ involves (among other things) a sum over the length $r$ of the single dashed line that connects $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$, with each term in the sum weighted by a factor of $|f(r)|$. If $f(r)$ satisfies the summability condition (2.4), then this contribution is bounded by a finite constant (even as $L \to \infty$), and we might expect that more complicated contributions to $q$ will also be finite. On the other hand, if $\sum_r |f(r)|$ diverges as $L \to \infty$, then this simple example already shows that we can never have $q < 1$, and so the polymer expansion will not converge and the system is likely to be unstable.

We now move on to the formal proof of Lemma 2.
a. Setting up the calculation

We start by introducing some notation. For each polymer $\gamma$, we define $n_c(\gamma)$ to be the number of connected components in $\Lambda_\gamma$ (the induced subgraph of $\Lambda$ determined by the vertices in $\gamma$). It is convenient to organize the sum in (B1) in terms of the number of connected components, $N \equiv n_c(\gamma)$, and the total size of the polymer, $\ell \equiv |\gamma|$. Defining

$$q(N, \ell) = \sum_{\gamma: n_c(\gamma) = N, |\gamma| = \ell} \tilde{W}(\gamma)e^{14|\gamma|}$$

we have

$$q = \sum_{\ell=1}^{\infty} \sum_{N=1}^{\ell} q(N, \ell) .$$

Here, the upper limit in the second sum is $\ell$ because a polymer of size $\ell$ can have at most $N \leq \ell$ connected components.

b. Bounding $q(N, \ell)$ in terms of $q_{\text{tree}}(N, \ell)$

We now bound $q(N, \ell)$ in terms of a closely related quantity $q_{\text{tree}}(N, \ell)$. To define the latter quantity, note that there is a natural way to define a graph associated with each polymer $\gamma$. The nodes of this graph correspond to the connected components of $\gamma$ after we erase the dashed lines (i.e., the connected components of $\Lambda_\gamma$), while the edges correspond to the dashed lines. We will say that a polymer $\gamma$ is tree-like if the corresponding graph is a tree: i.e. there is exactly one way to get from any connected component of $\gamma$ to any other connected component of $\gamma$ by traversing dashed lines.

We define $q_{\text{tree}}(N, \ell)$ to be a sum over tree-like $\gamma$ with $N$ connected components and total size $\ell$:

$$q_{\text{tree}}(N, \ell) = \sum_{\substack{\gamma: n_c(\gamma) = N, \gamma \text{ tree-like} \\ |\gamma| = \ell}} \tilde{W}(\gamma)e^{14|\gamma|} .$$

To derive a bound on $q(N, \ell)$ in terms of $q_{\text{tree}}(N, \ell)$, note that, given any polymer $\gamma$, we can always obtain a tree-like configuration by removing an appropriate subset of the dashed lines. Turning this around, it follows that we can construct every polymer $\gamma$ by starting with an appropriate tree-like configuration and then adding dashed lines.

The key question is then: how many ways can we add dashed lines to a tree-like $\gamma$ of size $\ell$? To answer this question, notice that dashed lines in some $\gamma$ can be parameterized by two vertices $v, v' \in \gamma$ that share the same time-coordinate: $\tau(v) = \tau(v')$. (Here, $\tau(v)$ denotes the time coordinate of $v$). If we imagine choosing $v$ first and $v'$ second, then it is clear that there are at most $\ell$ possibilities for $v$, and at most $L$ possibilities for $v'$ (since it must have the same time coordinate as $v$, and in addition the dashed lines only connect boxes to boxes or plaquettes to plaquettes, but not boxes to plaquettes). It follows that the maximum number of possible dashed lines is $\ell L$. Then, since there are at most $\ell L$ possible dashed lines that can be added to $\gamma$, the total number of ways to decorate $\gamma$ by dashed lines is at most $2^{\ell L}$ since we can choose to add or not add each possible dashed line.

The above counting argument suggests that $q(N, \ell) \leq 2^{\ell L} q_{\text{tree}}(N, \ell)$, but this bound does not use the fact that a dashed line of length $r$ comes with a factor of $\delta(f(r))$ (according the weight $\tilde{W}(\gamma)$). Let us again consider the number of ways we can add dashed lines to a tree-like $\gamma$ of size $\ell$. There are again at most $\ell$ choices for the first end of the dashed line, and at most $L - 1$ choices for the second end (not $L$ choices since the dashed line must end in a different place than it started). We again have the option to add or not add each dashed line. However, each dashed line comes with a different weight according to its length. Suppose we have chosen a fixed location for the vertex at the first end of a dashed line. Then choosing to add (with weight $\delta(f(r))$) or not add (with weight 1) each dashed line with this fixed starting vertex yields a factor of

$$\left[\frac{\delta^{-1}}{\prod_{r=1}^{L-1} (1 + \delta(f(r)))}\right]^2 \left[1 + \delta(\frac{L-1}{2})\right] ,$$

---

17 This graph is yet another graph that should not be confused with the graphs that appear in the polymer expansion of $\ln(Z)$ or the graph structure that we have added to $\Lambda$. 
where the first factor in this equation is squared because we can go either way around the spatial circle to find the second end of the dashed line (and there is only one dashed line with the maximum length of \( L/2 \)). It is convenient to now bound this factor from above as

\[
\left[ \prod_{r=1}^{\frac{L}{2} - 1} (1 + \delta |f(r)|) \right]^2 \leq \left[ \prod_{r=1}^{\frac{L}{2}} (1 + \delta |f(r)|) \right]^2.
\]  

This is the factor that we obtain for a fixed location of the vertex at the first end of the dashed line. We already mentioned that there are \( \ell \) choices for the location of this first vertex, and so in the end we find a bound with this factor raised to the \( \ell \)th power,

\[
q(N, \ell) \leq \left[ \prod_{r=1}^{\frac{L}{2}} (1 + \delta |f(r)|) \right]^{2\ell} q_{\text{tree}}(N, \ell).
\]

Finally, for later use we note that we can use the bound \( 1 + x \leq e^x \) and the summability condition (2.4) for \( f(r) \) to find that

\[
\left[ \prod_{r=1}^{\frac{L}{2}} (1 + \delta |f(r)|) \right]^{2\ell} \leq e^{2\ell \delta \sum_{r=1}^{\frac{L}{2}} |f(r)|} \leq e^{\ell \delta c}.
\]

This leads us to a final bound on \( q(N, \ell) \) of the form

\[
q(N, \ell) \leq e^{\ell \delta c} q_{\text{tree}}(N, \ell).
\]

Our task is now to bound \( q_{\text{tree}}(N, \ell) \). To this end, we now describe a way to “grow” a tree-like polymer starting with some more basic components. To explain our construction, we first need to fix an ordering on the vertices in our graph \( \Gamma \). This ordering can be arbitrary – i.e. it need not have any geometric significance.

Our “growing” procedure takes several pieces of data as input:

1. An \( N \)-tuple of connected polymers \((\tilde{\gamma}_1, \ldots, \tilde{\gamma}_N)\), each of which contain the distinguished vertex \( v \) and none of which have any dashed lines. This collection of polymers should have total size \( \ell \): i.e. \( \sum_{i=1}^{N} \ell_i = \ell \) where \( \ell_i = |\tilde{\gamma}_i| \).

2. An \((N-1)\)-tuple of vertices \((\bar{\sigma}_1, \ldots, \bar{\sigma}_{N-1})\), where \( \bar{\sigma}_k \) (for \( k \in \{1, \ldots, N-1\} \)) belongs to the connected polymer \( \tilde{\gamma}_{i_k} \) for some integer \( i_k \) satisfying \( 1 \leq i_k \leq k \). We also assume that the pairs \((\bar{\sigma}_k, i_k)\) form an increasing sequence with respect to an ordering \( \prec \) defined by \((\bar{\sigma}_k, i_k) \prec (\bar{\sigma}_{k'}, i_{k'})\) if either (1) \( i_k < i_{k'} \), or (2) \( i_k = i_{k'} \) and \( \bar{\sigma}_k \) has a larger number than \( \bar{\sigma}_{k'} \) in the original ordering of the vertices on \( \Gamma \).

3. An \((N-1)\)-tuple \((x_2, \ldots, x_N)\) of spatial (i.e., horizontal) displacements.

Given this input data, we can construct a corresponding tree-like polymer \( \gamma \) as follows. First, we define \((2N-1)\) vertices, \( \{s_1, \ldots, s_{N-1}, v_1, v_2, \ldots, v_N\} \) by setting \( v_1 = v \) and using the following recursive equations:

\[
s_k = \hat{s}_k + (v_{i_k} - v) \quad \text{and} \quad v_{k+1} = s_k + (x_{k+1}, 0).
\]  

Given \( s_k \) by translating \( \hat{s}_k \) by the vector \( v_{i_k} - v \). In the second equation, we define \( v_{k+1} \) by translating \( s_k \) in the spatial direction by the amount \( x_{k+1} \) (the notation \((x_{k+1}, 0)\) means a vector in the two-dimensional spacetime with spatial component equal to \( x_{k+1} \) and temporal component equal to 0).

To build our configuration \( \gamma \), we then take the union

\[
\bigcup_{k=1}^{N} [\tilde{\gamma}_k + (v_k - v)]
\]

where the above notation means that we translate each \( \tilde{\gamma}_k \) by the vector \( v_k - v \) before taking the union. We then add dashed lines between the pairs \((s_k, v_{k+1})\) for \( k \in \{1, \ldots, N - 1\} \). Note that \( s_k \) and \( v_{k+1} \) have the same time-coordinate by construction. Also \( v_k \in \tilde{\gamma}_k + (v_k - v) \) and \( s_k \in \tilde{\gamma}_k + (v_k - v) \).
Note that this construction can generate illegal polymers in some cases because the different connected components, \([\tilde{\gamma}_k + (v_k - v)]\), may overlap with one another. However, this is not important for our purposes: the only property that we need is the converse result that every tree-like polymer with \(N\) connected components can be generated by this procedure. This property implies that the number of tree-like polymers \(\gamma\) containing the distinguished vertex \(v\) is less than or equal to the number of possible choices of the above data.

We now use this property of our construction to obtain an upper-bound on \(q_\text{tree}(N, \ell)\). To this end, we first define

\[
q_0(\ell) = \sum_{\gamma \ni v \atop |\gamma| = \ell} e^{-\mu|\gamma|} e^{14|\gamma|}
\]

(B13)

where the sum is taken over all connected polymers \(\tilde{\gamma}\) of size \(\ell\) containing the vertex \(v\) and not having any dashed lines. With this notation, our construction implies the following upper bound on \(q_\text{tree}(N, \ell)\):

\[
q_\text{tree}(N, \ell) \leq \sum_{\ell_1, \ldots, \ell_N = \ell} \prod_{i=1}^{N} q_0(\ell_i) \sum_{(\pi_1, \ldots, \pi_{N-1}) \text{ increasing}} \sum_{x_2, \ldots, x_N} \delta_{\pi_1}^{N-1} \prod_{k=2}^{N} |f(|x_k|_p)| .
\]

(B14)

We now explain the different factors in this expression. The first part \(\sum_{\ell_1, \ldots, \ell_N = \ell} \prod_{i=1}^{N} q_0(\ell_i)\) accounts for the sum over different connected components \(\tilde{\gamma}_1, \ldots, \tilde{\gamma}_N\) of lengths \(\ell_1, \ldots, \ell_N\), and with total length \(\ell\). Next, we sum over all \((N-1)\)-tuples \((\pi_1, \ldots, \pi_{N-1})\) of vertices that satisfy the properties discussed in item 2 in the list of data required for our tree-growing procedure. Next, we sum over all the spatial displacements \((x_2, \ldots, x_N)\). Finally, the factor of \(\delta_{\pi_1}^{N-1} \prod_{k=2}^{N} |f(|x_k|_p)|\) assigns the correct weight to the \(N - 1\) dashed lines in the tree-like configurations that we are generating.

Using the summability condition (2.4) for \(f(r)\), we find that \(\sum_{x_k} |f(|x_k|_p)| \leq c\) for all \(k\), and so

\[
\sum_{\pi_1, \ldots, \pi_{N-1} \text{ increasing}} \sum_{x_2, \ldots, x_N} \delta_{\pi_1}^{N-1} \prod_{k=2}^{N} |f(|x_k|_p)| \leq \sum_{(\pi_1, \ldots, \pi_{N-1}) \text{ increasing}} (\delta c)^{N-1} \leq \left(\ell + N - 2\right) (\delta c)^{N-1} ,
\]

(B15)

where \(\binom{\ell + N - 2}{N - 1}\) is the number of increasing (but not necessarily strictly increasing) sequences of length \(N - 1\) taken from a set of size \(\ell\). Using these results, we obtain a bound on \(q_\text{tree}(N, \ell)\) of the form

\[
q_\text{tree}(N, \ell) \leq \sum_{\ell_1, \ldots, \ell_N = \ell} \prod_{i=1}^{N} q_0(\ell_i) \left(\frac{\ell + N - 2}{N - 1}\right) (\delta c)^{N-1} .
\]

(B16)

To proceed further we note that the partition function \(q_0(\ell)\) obeys a bound of the form

\[
q_0(\ell) \leq e^{(K' - \mu)\ell}
\]

(B17)

where \(K'\) is some constant of order 1. This bound follows from the fact that the number of connected polymers \(\tilde{\gamma}\) of size \(\ell\) that contain \(v\) can be bounded by \(e^{K''\ell}\) for some \(K''\). The latter bound is a standard combinatorial result about the counting of connected subsets of a graph; see e.g. Lemma V.8.5 of Ref. 25, which gives \(K'' = 2 \ln(14)\), and then \(K' = 2 \ln(14) + 14\) for our polymer model.

Substituting this bound into (B16) and simplifying, we derive

\[
q_\text{tree}(N, \ell) \leq \sum_{\ell_1, \ldots, \ell_N = \ell} \left(\frac{\ell + N - 2}{N - 1}\right) (\delta c)^{N-1} \cdot e^{(K' - \mu)\ell}
\]

(B18)

Next we note that the number of \(N\)-tuples \((\ell_1, \ldots, \ell_N)\) of positive integers with \(\ell_1 + \cdots + \ell_N = \ell\) is given by \(\binom{\ell - 1}{N - 1}\). Therefore, evaluating the sum gives

\[
q_\text{tree}(N, \ell) \leq \binom{\ell - 1}{N - 1} \cdot \left(\frac{\ell + N - 2}{N - 1}\right) (\delta c)^{N-1} \cdot e^{(K' - \mu)\ell} .
\]

(B19)
To simplify this expression we use the two inequalities
\[
\left( \frac{\ell - 1}{N - 1} \right) \leq 2^\ell, \quad \left( \frac{\ell + N - 2}{N - 1} \right) \leq \left( \frac{2\ell}{N - 1} \right)
\]
where the second inequality follows from the fact that \( N \leq \ell \). Substituting these inequalities into \((B19)\) gives
\[
q_{\text{tree}}(N, \ell) \leq \left( \frac{2\ell}{N - 1} \right) (\delta c)^{N-1} \cdot e^{(K' - \mu + \ln(2))\ell}
\]
\[(B21)\]

\text{d. Bounding } q

We are now ready to bound \( q \). First, we combine \((B10)\) and \((B21)\) to derive
\[
q(N, \ell) \leq \left( \frac{2\ell}{N - 1} \right) (\delta c)^{N-1} \cdot e^{(K' - \mu + \delta c + \ln(2))\ell}.
\]
\[(B22)\]
Next, we use the binomial expansion to deduce
\[
\sum_{N=1}^{\ell} q(N, \ell) \leq (1 + \delta c)^{2\ell} \cdot e^{(K' - \mu + \delta c + \ln(2))\ell}.
\]
\[(B23)\]
Applying the inequality \((1 + x)^n \leq e^{nx}\), we derive
\[
\sum_{N=1}^{\ell} q(N, \ell) \leq e^{(K' - \mu + 3\delta c + \ln(2))\ell}.
\]
\[(B24)\]
It follows that
\[
q \leq \sum_{\ell=1}^{\infty} e^{(K' - \mu + 3\delta c + \ln(2))\ell}
\]
\[(B25)\]
Clearly the right hand side is a geometric series so we can sum it exactly. In particular we can see that \( q < 1 \) if we choose \( e^{(K' - \mu + 3\delta c + \ln(2))} < 1/2 \). This corresponds to the requirement that
\[
\mu - 3\delta c > K_0, \quad K_0 = K' + 2 \ln(2).
\]
\[(B26)\]
This completes the proof of Lemma 2.

Appendix C: Convergence criteria for the polymer expansion

In this appendix we present more details on the different convergence criteria for the polymer expansion. We first explain how the simplified convergence criterion that we presented in Sec. III C follows from a general convergence criterion that is derived in Ref. 54. We then explain how to derive the modified convergence criterion that we used in Sec. III D 2 to handle the case where the notion of intersection of polymers is more complicated than simple sharing of vertices of \( \Lambda \).

1. Derivation of the simplified convergence criterion

As in Sec. III C, we consider the setting where each polymer contains a subset of the vertices in the set \( \Lambda \), and where two polymers are defined to intersect if they have one or more vertices in common. In this case, it is possible to derive a general criterion for the convergence of the polymer expansion [54]. To state this general criterion we first define, for any integer \( d \geq 1 \), a positive real number \( \tilde{Q}(d) \) by
\[
\tilde{Q}(d) := \sup_{v \in \Lambda} \sum_{\gamma \ni v} |W(\gamma)| |\gamma|^{d-1},
\]
\[(C1)\]
where the sum is taken over all polymers $\gamma$ that contain the vertex $v$, and then we take the supremum over all vertices $v$ in $\Lambda$. In terms of the $Q(d)$, we also define the positive real number $\tilde{q}$ by

$$
\tilde{q} := \sum_{d=1}^{\infty} \frac{1}{(d-1)!} \tilde{Q}(d).
$$

(C2)

In terms of $\tilde{q}$, one can show that for all $k \geq 1$, the $k$th term $I_k$ in the expansion (3.40) of $\ln(Z)$ satisfies the bound

$$
|I_k| \leq \frac{1}{k} \tilde{q}^k |\Lambda|.
$$

(C3)

We can see from this expression that the polymer expansion for $\ln(Z)$ will be absolutely convergent if $\tilde{q} < 1$ (again, for a fixed value of $|\Lambda|$). Therefore, in this general case we have the following result.

**Theorem** (Theorem 3.4 of Ref. 54). If $\tilde{q} < 1$, then the polymer expansion (3.40) for $\ln(Z)$ is absolutely convergent for fixed $|\Lambda|$ and furthermore $\ln(Z)$ satisfies the bound

$$
|\ln(Z)| \leq \sum_{k=1}^{\infty} |I_k| \leq -|\Lambda| \ln(1 - \tilde{q}).
$$

(C4)

Using this general criterion, we now derive the simplified convergence criterion stated in Theorem 2. Recall that the simplified criterion applies in the case where we have an upper bound on the weights of the form $|W(\gamma)| \leq \bar{W}(\gamma)$, and where the set of polymers is *isotropic* with respect to $\bar{W}(\gamma)$ in the sense that the sum $\sum_{\gamma \ni v} \bar{W}(\gamma) g(|\gamma|)$ is independent of $v$ for any function $g(|\gamma|)$ of $|\gamma|$. In this case we can see that

$$
\sum_{\gamma \ni v} |W(\gamma)| |\gamma|^{d-1} \leq \sum_{\gamma \ni v} \bar{W}(\gamma) |\gamma|^{d-1},
$$

(C5)

and the value of the sum on the right-hand side of this inequality no longer depends on the specific vertex $v$. Then for all $d$ we have

$$
\tilde{Q}(d) \leq \sum_{\gamma \ni v} \bar{W}(\gamma) |\gamma|^{d-1},
$$

(C6)

with no supremum on the right-hand side, and so $\tilde{q} \leq q$, where $q = \sum_{\gamma \ni v} \bar{W}(\gamma)e^{|\gamma|}$ was defined in Eq. (3.42) of the main text. Since $\tilde{q} \leq q$, we can prove the convergence of the polymer expansion if we can verify our simplified convergence criterion, i.e., if we can prove that $q < 1$.

### 2. Derivation of the modified convergence criterion

We now explain the origin of the modified convergence criterion that we used in Sec. III D 2. Recall that we needed to use this modified criterion because in our model the notion of intersection of polymers is more complicated than simple sharing of vertices of $\Lambda$.

In the derivation of the general convergence criterion for the polymer expansion, as discussed in Ref. 54, the specifics of the notion of intersection of polymers is only used near the end of the proof. Specifically, this information is used to bound sums of the form

$$
\sum_{\gamma' \ni \gamma \neq \emptyset} |W(\gamma')| g(|\gamma'|),
$$

(C7)

where the sum is taken over all polymers $\gamma'$ that intersect with a particular fixed polymer $\gamma$, and $g(|\gamma'|) \geq 0$ is a positive function of the polymer size $|\gamma'|$.

Consider first the simple case where the notion of intersection is equivalent to sharing vertices. In this case we can follow Ref. 54 and bound the sum (C7) as

$$
\sum_{\gamma' \ni \gamma \neq \emptyset} |W(\gamma')| g(|\gamma'|) \leq \sum_{v \in \gamma} \sum_{\gamma' \ni v} |W(\gamma')| g(|\gamma'|),
$$

(C8)
where we sum over all vertices $v$ in $\gamma$, and then we have an inner sum over all $\gamma'$ that contain $v$. Now suppose that we have a bound $|W(\gamma)| \leq \tilde{W}(\gamma)$, and that $\Gamma$ is isotropic with respect to $\tilde{W}(\gamma)$. Then we can proceed further with this bound to obtain

$$
\sum_{\gamma' \cap \gamma \neq \emptyset} |W(\gamma')|g(|\gamma'|) \leq \sum_{v \in \gamma} \sum_{\gamma' \ni v} \tilde{W}(\gamma')g(|\gamma'|) = |\gamma| \sum_{\gamma' \ni v} \tilde{W}(\gamma')g(|\gamma'|),
$$

where the last equality holds because our additional assumptions imply that the inner sum is independent of $v$. If we continue to follow the proof in Ref. 54, and use this method to bound overlap sums like (C7), then we will recover the simplified convergence criterion that we presented in Theorem 2 of Sec. III C.

Next, we discuss the modification to this procedure that we need for our model, where two polymers $\gamma_1$ and $\gamma_2$ intersect if (i) they have one or more vertices in common, or (ii) a vertex in $\gamma_1$ is connected to a vertex in $\gamma_2$ via an edge in $\Lambda$. To handle this case we define, for each polymer $\gamma$, a thickened polymer $\gamma_{\text{thick}}$, where $\gamma_{\text{thick}}$ satisfies:

1. $\gamma_{\text{thick}}$ contains the same dashed lines as $\gamma$.
2. $\gamma_{\text{thick}}$ contains all vertices in $\gamma$ and all vertices that are connected to vertices in $\gamma$ by an edge of $\Lambda$.

For later use we note that, since each vertex in $\Lambda$ is connected by edges to 14 other vertices, we have the upper bound

$$
|\gamma_{\text{thick}}| \leq 14|\gamma|.
$$

Using this definition of thickened polymers, we can bound a sum of the form (C7) for our model as

$$
\sum_{\gamma' \cap \gamma \neq \emptyset} |W(\gamma')|g(|\gamma'|) \leq \sum_{v \in \gamma_{\text{thick}}} \sum_{\gamma' \ni v} |W(\gamma')|g(|\gamma'|),
$$

where the outer sum is now taken over all $v$ in $\gamma_{\text{thick}}$. If we again assume a bound $|W(\gamma)| \leq \tilde{W}(\gamma)$, and that $\Gamma$ is isotropic with respect to $\tilde{W}(\gamma)$, then we can obtain a bound that looks similar to the one from the previous case with the simpler notion of intersection,

$$
\sum_{\gamma' \cap \gamma \neq \emptyset} |W(\gamma')|g(|\gamma'|) \leq \sum_{v \in \gamma_{\text{thick}}} \sum_{\gamma' \ni v} \tilde{W}(\gamma')g(|\gamma'|)
$$

$$
= |\gamma_{\text{thick}}| \sum_{\gamma' \ni v} \tilde{W}(\gamma')g(|\gamma'|)
$$

$$
\leq 14|\gamma| \sum_{\gamma' \ni v} \tilde{W}(\gamma')g(|\gamma'|),
$$

where we used the inequality $|\gamma_{\text{thick}}| \leq 14|\gamma|$ in the last line. If we now follow the proof in Ref. 54, and use this method to bound overlap sums like (C7), then we will recover the modified convergence criterion that we used for our model in Sec. III D 2.

Appendix D: Setup and the proofs of Lemma 1 and Lemma 2 for the general model

In this appendix we summarize the changes that need to be made in our setup and in the proofs of Lemma 1 and 2 in order to prove the more general result that we presented in Sec. IV.

1. Setup

The main change to our setup is in the form of the support sets $X$. As before, these still consist of boxes and plaquettes. However, since we are no longer restricted to the case of 2-body interactions, the dashed lines that we previously included in the support sets are not capable of capturing the structure of the interactions in our more general model. As we explain below, we replace the dashed lines with a new kind of structure, namely subsets of boxes or plaquettes. Thus, our support sets will be built from boxes, plaquettes, and subsets. Before explaining the form of these subsets, we first discuss the form of the boxes and plaquettes for models in spatial dimensions larger than one.
For ease of visualization we consider the case of $D = 2$ spatial dimensions. The spatial lattice is a square lattice and the blocked spacetime lattice is a cubic\textsuperscript{18} lattice. The boxes from the $D = 1$ case now become cubes that have a vertical bond of the blocked spacetime lattice running up their center. The rule for assigning such a cube to the support set of a microscopic configuration is the same as in the $D = 1$ case: if a perturbation with support on site $r$ acts within a certain time slice, then the box (now a cube) centered on $r$ within that time slice is included in the support set.

Moving on to the plaquettes, in the $D = 2$ case, the domain wall worldlines now live on \textit{vertical faces} of the blocked spacetime lattice. These vertical faces are the analogues, in $D = 2$, of the plaquettes from the $D = 1$ case. If a microscopic configuration has a domain wall worldline running all the way through a vertical face of the blocked spacetime lattice, then we include that vertical face in the support set of that configuration.

Next, we come to the description of the \textit{subsets}. Each subset contains between 2 and $K$ boxes in the same time slice, or between 2 and $K$ plaquettes in the same time slice. This rule comes from the fact that the interactions in our general model act on a subset of up to $K$ lattice sites. In the two-body case a dashed line connecting two boxes is equivalent to the subset containing the two boxes at its ends, but for higher-body interactions we instead work with subsets that contain all of the boxes that participate in an interaction term. While the dashed lines were labeled by their length $r$, the possible subsets are labeled by $Y$ and a spacetime vector $R$, where $Y$ is one of the subsets with $|Y| \geq 2$ that contributes to $V_0$, and $R$ is the location of a vertical bond midpoint (for a subset of boxes) or a plaquette center (for a subset of plaquettes). In what follows, it will be convenient to denote such a subset by $Y_R$ and to say that this subset has “shape” $Y$.

One small technical point about these subsets is that, if we have a subset of plaquettes, then we require all of the plaquettes in this subset to be separated from each other by a Bravais lattice vector of the (hyper)cubic lattice. We require this because, as in the $D = 1$ case, $Z_k$ does not actually receive contributions from configurations consisting of subsets of plaquettes, and so we can choose the rule for assembling plaquettes into subsets to be whatever is most convenient for our combinatorial analysis.

Finally, in this setting we define the connected support sets $\tilde{\chi}$ in exactly the same manner as before. We then define the weakly-connected support sets $\chi$ to be those support sets made up of several connected support sets $\tilde{\chi}_1, \tilde{\chi}_2, \ldots$ together with subsets such that it is possible to get from one connected support set in $\chi$ to any other by hopping between sites belonging to the same subset. For any support set $X$ we define $s_Y(X)$ to be the number of subsets of shape $Y$ in $X$, and then we define $s(X) = \sum_Y s_Y(X)$ to be the total number of subsets in $X$.

2. \textbf{Lemma 1}

We now discuss the changes to the proof of Lemma 1. In fact, this proof proceeds exactly as before, after we define some notation and explain how to use the new summability condition (4.4).

First, we perform all traces using a basis $|s, \pm\rangle$ for $H_{\pm}$ that is defined in a similar way as before (the tuple $s$ is again defined with a fixed value of one particular spin, say the spin at $r = 0$).

Next, we expand each operator $V_Y$ as a sum over Pauli strings,

$$V_Y = \sum_A v_Y^{(A)} \sigma^A,$$

(D1)

where the multi-index $A$ labels Pauli strings that act on the sites contained in $Y$, and the $v_Y^{(A)}$ are coefficients. To be more precise, $A$ is a $|Y|$-tuple of the form $A = (a_1, \ldots, a_{|Y|})$ where each component $a_i$ takes values in the set $\{0, x, y, z\}$, and then $\sigma^A$ is defined to be the product

$$\sigma^A := \sigma^{a_1}_{r_1} \cdots \sigma^{a_{|Y|}}_{r_{|Y|}},$$

(D2)

where $\sigma^0 := I$ is the $2 \times 2$ identity matrix and $r_1, \ldots, r_{|Y|}$ are the lattice sites contained in $Y$ (one of them is the origin 0). There are at most $4^K$ values of the multi-index $A$ since $|Y| \leq K$.

For each $V_Y$ we now define an operator $\text{abs}(V_Y)$ by replacing all $\sigma^z$’s in $V_Y$ by $I$’s, replacing all $\sigma^v$’s in $V_Y$ by $\sigma^z$’s, and finally replacing the coefficients $v_Y^{(A)}$ by their absolute values $|v_Y^{(A)}|$. The operator $\text{abs}(V_Y)$ has the important property that all of its matrix elements in the basis $|s, +\rangle$ of $H_+$ are positive.

Given this definition of $\text{abs}(V_Y)$, we then define numbers $f_Y$ and operators $\tilde{V}_Y$ by

$$f_Y := ||\text{abs}(V_Y)||, \quad \tilde{V}_Y := \frac{\text{abs}(V_Y)}{f_Y},$$

(D3)

\textsuperscript{18} For simplicity, in this discussion we ignore the different lattice spacing in the temporal direction.
where we assume that $f_Y \neq 0$. In what follows we also assume that the $f_Y$ satisfy the bound $|f_Y| \leq 1$ for all $Y$, as the perturbation term can always be made larger by increasing the value of $\lambda$.

If the $V_Y$ satisfy the summability condition (4.4), then the $f_Y$ satisfy the condition $\sum_Y f_Y \leq c'$, where $c' = 4^K c$, which more closely resembles the original summability condition (2.4) from our simple model. To prove this, note first that $|\text{abs}(V_Y)| \leq \sum_A |\psi_Y^{(A)}|$, and that $|\psi_Y^{(A)}| \leq ||V_Y||$. (This last bound can be proven using the formula $\psi_Y^{(A)} = \text{Tr}\{V_Y \sigma^A\}/2^{L^D}$, where the trace is taken over the full Hilbert space, which has dimension $2^{L^D}$). Then, since there are at most $4^K$ values of $A$, we find that $|\text{abs}(V_Y)| \leq 4^K ||V_Y||$ and the modified summability condition $\sum_Y f_Y \leq 4^K c$ follows.

With this notation we can prove Lemma 1 following almost the same steps as before. The main difference from our previous proof in Appendix A is that we must now consider the contributions to $W_{\pm}(X)$ from the individual Pauli strings $\sigma^A$ that contribute to each perturbation term $V_r$,

$$V_r = \sum_Y T_r V_Y T_r^{-1} = \sum_Y \sum_A \psi_Y^{(A)} T_r \sigma^A T_r^{-1}. \quad \text{(D4)}$$

In particular, when we construct a precise formula for $W_{\pm}(X)$ (i.e., the analogue of Eqs. (A1) and (A2)), we use an indicator function that is equal to one only if the translated Pauli strings $T_r \sigma^A T_r^{-1}$, which are labeled by triples $(r, Y, A)$, are consistent with the support set $X$ (of course, this perturbation term should also be acting at a time that is consistent with $X$). This allows us to extract the exponential factors associated with the plaquettes in $X$, and we can then bound the rest of the weight using the operators $V_Y$ in a similar way as before. We then find that the final bound on the weights takes the form

$$|W_{\pm}(X)| \leq e^{-\mu |X| \delta^N(X)} \prod_Y \delta_Y^{q_Y(X)}, \quad \text{(D5)}$$

where $|X| = b(X) + p(X)$ as before. Of course, the exact value of $\lambda_0$ for which Lemma 1 holds will change due to the new form of the interaction.

### 3. Lemma 2

We now discuss the changes in the proof of Lemma 2. The weights $\tilde{W}(\chi)$ that appear in this lemma are now given by the right-hand side of (D5), and we can also map each support set $\chi$ to a polymer $\gamma$ in the same manner as before. In this case our strategy is again to build up each weakly-connected polymer $\gamma$ by adding subsets to a special basic set of weakly-connected polymers. In our original proof of Lemma 2 in Appendix B these were the tree-like polymers. In this more general case the correct objects are not actually trees, and so we refer to them as minimal weakly-connected polymers. We say that a weakly-connected polymer is minimal if it becomes disconnected upon the removal of any subset that it contains.

The key to the proof is now a growing procedure that generates a minimal weakly-connected polymer from a set of more basic data. In this case the data is as follows. We again require an ordering of all the vertices in $\Lambda$. Next, we require these ingredients:

1. An $N$-tuple of connected polymers $(\tilde{\gamma}_1, ..., \tilde{\gamma}_N)$, each of which contain $v$. This collection of polymers should have total size $\ell$: i.e. $\sum_{i=1}^N \ell_i = \ell$ where $\ell_i = |\tilde{\gamma}_i|$.

2. An $m$-tuple of vertices $(\bar{\pi}_1, ..., \bar{\pi}_m)$, where $(N - 1)/(K - 1) \leq m \leq N - 1$.

3. An $m$-tuple $(Y^{(1)}, ..., Y^{(m)})$ of subsets, where the subset $Y^{(k)}$ contains $n_k \leq K$ points.

4. The vertex $\bar{\pi}_k$ belongs to $\tilde{\gamma}_{i_k}$ for some integer $i_k$ satisfying $1 \leq i_k \leq 1 + \sum_{k'=1}^{k-1} (\hat{n}_{k'} - 1)$. (Here the integer $\hat{n}_{k'}$ is defined below). We again assume that the pairs $(\bar{\pi}_k, i_k)$ form an increasing sequence with respect to an ordering “$\prec$” defined by $(\bar{\pi}_k, i_k) \prec (\bar{\pi}_{k'}, i_{k'})$ if either (1) $i_k < i_{k'}$, or (2) $i_k = i_{k'}$ and $\bar{\pi}_k$ has a larger number than $\bar{\pi}_k$ in the original ordering of the vertices of $\Lambda$.

Using these ingredients, the growing procedure now takes the following form. First, let $v_1 = v$ and $s_1 = \bar{\pi}_1$. Also, let $Y^{(1)}_{s_1}$ be the translation of the subset $Y^{(1)}$ to the vertex $s_1$, and let $\{y_1^{(1)}, ..., y_{n_1}^{(1)}\}$ be the $n_1 - 1$ vertices in $Y^{(1)}_{s_1}$ that are not equal to $s_1$ (which came from the point at the origin $(0, 0)$ of spacetime in the untranslated subset $Y^{(1)}$). We have also ordered these vertices according to our original ordering of the vertices of $\Lambda$. We now define $\hat{n}_1 - 1$ vertices $v_2, ..., v_{\hat{n}_1}$, where $\hat{n}_1 \leq n_1$, to be equal to the subset of the vertices $\{y_1^{(1)}, ..., y_{n_1}^{(1)}\}$ that are not contained in the connected polymer $\tilde{\gamma}_1$ (we also preserve the ordering when defining these vertices). We then translate $\tilde{\gamma}_2$ by $v_2 - v, \tilde{\gamma}_3$ by $v_3 - v$, and so on, ending with a translation
of $\tilde{\gamma}_n$ by $v_n - v$. At this point the minimal weakly-connected polymer that we are building consists of the union of translated polymers

$$\bigcup_{k=1}^{\tilde{n}_1} [\tilde{\gamma}_k + v_k - v]$$

(D6)

together with the subset $Y_s^{(1)}$.

Next, we define a new vertex $s_2$ by the relation $s_2 = s_2 + (v_{n_2} - v)$. In addition, let $Y_s^{(2)}$ be the translation of $Y^{(2)}$ to the vertex $s_2$, and let $\{y^{(2)}_1, \ldots, y^{(2)}_{n_2-1}\}$ be the $n_2 - 1$ vertices in $Y_s^{(2)}$ that are not equal to $s_2$ (and we again order these according to the original ordering on $A$). We then define $\tilde{n}_2 - 1$ new vertices $v_{\tilde{n}_1+1}, v_{\tilde{n}_1+2}, \ldots, v_{\tilde{n}_1+\tilde{n}_2-1}$, where $\tilde{n}_2 \leq n_2$, to be equal to the subset of the vertices $\{y^{(2)}_1, \ldots, y^{(2)}_{n_2-1}\}$ that are not contained in the union (D6) that we have already constructed. We then add the translated polymers $\bigcup_{k=2}^{\tilde{n}_2} [\tilde{\gamma}_n + v_n + k - 1] - v]$ and the subset $Y_s^{(2)}$ to the minimal weakly-connected polymer that we are growing. At this point our minimal weakly-connected polymer consists of the union

$$\bigcup_{k=1}^{\tilde{n}_1+\tilde{n}_2-1} [\tilde{\gamma}_k + v_k - v]$$

together with the subsets $Y_s^{(1)}$ and $Y_s^{(2)}$.

We now continue the construction in this manner until all of the original connected polymers $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_N$ are used up. As in our original growing procedure from Appendix B, this procedure is capable of growing all minimal weakly-connected polymers (and many others as well), and so it leads to an upper bound on the number of minimal weakly-connected polymers that contain the vertex $v$. Let $q_{\text{min}}(N, \ell)$ be the following sum over all minimal $\gamma$ with $N$ connected components and total size $\ell$:

$$q_{\text{min}}(N, \ell) = \sum_{\substack{\gamma \supseteq v \\
\gamma \text{ minimal} \\
|\gamma| = \ell}} W(\gamma) e^{14|\gamma|} .$$

(D7)

Then our growing procedure leads to an upper bound on this quantity of the form

$$q_{\text{min}}(N, \ell) \leq \sum_{\ell_1, \ldots, \ell_N = \ell} \prod_{i=1}^N q_0(\ell_i) \sum_{m=(N-1)/(K-1)}^{N-1} \prod_{(\pi_1, i_1), \ldots, (\pi_m, i_m) \text{increasing w.r.t.} \prec} Y^{(1)}(\pi_1, i_1), \ldots, Y^{(m)}(\pi_m, i_m) \sum_{k=1}^m e^m \prod_{k=1}^m f_Y(k) ,$$

(D8)

where the $q_0(\ell_i)$ are again the contributions from the $\tilde{\gamma}_i$. The main difference between this expression and Eq. (B14) is that we now have an extra sum over $m$, which counts the number of subsets that are needed to build up each minimal configuration (in the case of two-body interactions, where $K = 2$, we must have $m = N - 1$ to build a tree-like polymer out of $N$ connected components). As before, we also find that $q(N, \ell) \leq e^{6\delta c'} q_{\text{min}}(N, \ell)$ (recall that $c' = 4K c$) by the same argument of adding or not adding every possible subset (and then using the summability condition for the $f_Y$).

Finally, we proceed to bound $q = \sum_{\ell=1}^\ell \sum_{N=1}^N q(N, \ell)$ as before. The main new feature that we encounter here is the sum

$$\sum_{N=1}^\ell \sum_{m=(N-1)/(K-1)}^{N-1} \binom{2\ell}{m} (\delta c')^m ,$$

which can be bounded as follows,

$$\sum_{N=1}^\ell \sum_{m=(N-1)/(K-1)}^{N-1} \binom{2\ell}{m} (\delta c')^m \leq \sum_{N=1}^\ell \sum_{m=0}^{2\ell} \binom{2\ell}{m} (\delta c')^m \leq \ell (1 + \delta c')^{2\ell} \leq e^{(2\delta c' + 1)\ell} .$$

We then complete the proof in the same way as before.

[1] X. G. Wen and Q. Niu, Phys. Rev. B 41, 9377 (1990).
