Entanglement in algebraic quantum mechanics: Majorana fermion systems

F Benatti\(^1,2\) and R Floreanini\(^2\)

\(^1\)Dipartimento di Fisica, Università di Trieste, I-34151 Trieste, Italy
\(^2\)Istituto Nazionale di Fisica Nucleare, Sezione di Trieste, I-34151 Trieste, Italy

E-mail: floreanini@ts.infn.it

Received 29 February 2016, revised 13 May 2016
Accepted for publication 27 May 2016
Published 24 June 2016

Abstract

Many-body entanglement is studied within the algebraic approach to quantum physics in systems made of Majorana fermions. In this framework, the notion of separability stems from partitions of the algebra of observables and properties of the associated correlation functions, rather than on particle tensor products. This allows a complete characterization of non-separable Majorana fermion states to be obtained. These results may have direct application in quantum metrology: using Majorana systems, sub-shot-noise accuracy in parameter estimations can be achieved without preliminary resource-consuming, state entanglement operations.

Keywords: entanglement, identical particles, Majorana fermions, quantum metrology

1. Introduction

Majorana fermions describe real fermion excitations that can be thought of as half of normal fermions, in the sense that a complex fermion mode can be obtained by putting together two real ones. Although originally introduced in particle physics\(^1,2\), as fermions that are their own antifermions, they have found applications in various branches of physics\(^3\).

Most notably, Majorana fermions appear as quasi-particle excitations in the so-called topological superconductors\(^2–6\). They turn out to be spatially separated and this delocalized character protects them from decoherence effects generated by any local interaction; furthermore, they exhibit non-Abelian statistics. These two characteristic properties make Majorana excitations in superconductors very attractive as building blocks for topological

\(^3\) For instance, see the reviews\(^2–8\) and references therein.
quantum computation, where logical qubits are encoded in states of non-Abelian anyons and logical operations are performed through their braiding transformations [7–10].

Typical resources needed in quantum computation and communication are non-classical correlations and entanglement. While in the case of distinguishable particles the notion of separability and entanglement is well understood, in systems made of identical constituents, like Majorana fermions, these notions are still unsettled. The key observation is that the tensor product structure of the multiparticle Hilbert space is, in general, no longer available when the particles are indistinguishable. As a consequence the standard definition of entanglement based on this structure loses its meaning when dealing with bosons and fermions, or more so in general anyons. In such cases, the emphasis should shift from the system states to the algebra of its observables [34–43], treating the system Hilbert space as an emergent concept.

This change in perspective is dictated by physical considerations: since particles are identical, they cannot be singly addressed nor their individual properties measured, with only collective global observables actually being experimentally accessible [44, 45]. In other words, when dealing with many-body systems, the presence of entanglement should be identified through the properties of the observable correlation functions and not by a priori properties of the system states [46–58].

This more general approach to separability and entanglement finds its more natural formulation in the so-called algebraic approach to quantum physics [59–68]. In this more general framework, a quantum system is defined through the algebra \( \mathcal{A} \) containing all its observables. A state \( \Omega \) for the system is just a positive linear map from \( \mathcal{A} \) to the complex numbers, so that, for any observable \( \alpha \in \mathcal{A} \) (i.e. a Hermitian element of \( \mathcal{A} \)) the real number \( \Omega(\alpha) \) gives its experimentally accessible mean value. From the couple \((\mathcal{A}, \Omega)\), one then deduces through a standard procedure (the so-called Gelfand–Naimark–Segal construction) the Hilbert space \( \mathcal{H}_\Omega \) containing the system states. In this very general framework, many-body entanglement can be naturally identified by the presence of non-classical correlations among suitable observables.

In the following, we shall apply this general approach to the study of the notions of separability and entanglement in systems made of Majorana fermions. In this case the algebra \( \mathcal{A} \) containing the observables of the systems turns out to be a Clifford algebra [69–73], whose representation theory is quite different from the usual Fock representation of the algebra of creation and annihilation operators of standard fermions. This poses new questions concerning the relation between entanglement and the reducibility of the algebra representation, making the theory of Majorana fermion entanglement much richer than that of standard fermions or bosons.

These results may have direct application in quantum technology, especially in using quantum interferometric devices [74–84] to perform metrological tasks. Indeed, as in the case for boson and fermion systems [50, 52, 57], for Majorana systems, parameter estimation accuracies going beyond the classical shot-noise limit can be obtained by exploiting some sort of quantum non-locality embedded in the measuring apparatus, without the need of preliminary state operations. In this respect, Majorana fermions might turn out to be very useful not only in performing decoherence-protected quantum computational tasks, but also in the development of the next generation of highly sensitive quantum sensors.

\[ \text{J. Phys. A: Math. Theor. 49 (2016) 305303 F Benatti and R Floreanini} \]

\[ \text{F. Benatti and R. Floreanini} \]

\[ \text{4 The literature on entanglement in many-body systems is vast, e.g. see [11–33]; however, only a limited part of the reported results are applicable to systems composed of identical constituents.} \]
2. Algebraic approach to quantum mechanics

For completeness, in this section we shall briefly summarize the main features of the algebraic formulation of quantum physics, outlining the concepts and tools that will be needed in the following discussions.

Any quantum system can be characterized by the collections of observations that can be made on it through suitable measurement processes [66]. The physical quantities that are thus accessed are the observables of the system, forming an algebra \( \mathcal{A} \) under multiplication and linear combinations: the algebra of observables.

\* \( \mathcal{C}^* \)-algebras

In general, \( \mathcal{A} \) turns out to be a non-commutative \( \mathcal{C}^* \)-algebra; this means that it is a linear associative algebra (with unity) over the field of complex numbers \( \mathbb{C} \), i.e. a vector space over \( \mathbb{C} \), with an associative product that is linear in both factors. Further, \( \mathcal{A} \) is endowed with an operation of conjugation: it possesses an antilinear involution \( \star : \mathcal{A} \to \mathcal{A} \), such that \( (\alpha \beta)' = \alpha \beta \), for any element \( \alpha \) of \( \mathcal{A} \). In addition, a norm \( \| \cdot \| \) is defined on \( \mathcal{A} \), satisfying \( \| \alpha \beta \| \leq \| \alpha \| \| \beta \| \), for any \( \alpha, \beta \in \mathcal{A} \) (thus implying that the product operation is continuous), and such that \( \| \alpha' \| = \| \alpha \| \), so that \( \| \alpha \| = \| \alpha' \| \). Moreover, \( \mathcal{A} \) is closed under this norm, meaning that \( \mathcal{A} \) is a complete space with respect to the topology induced by the norm (a property that, in turn, makes \( \mathcal{A} \) a Banach algebra).

In the case of an \( n \)-level system, \( \mathcal{A} \) can be identified with the \( \mathcal{C}^* \)-algebra \( \mathcal{M}_n(\mathbb{C}) \) of complex \( n \times n \) matrices; the \( \star \)-operation coincides now with the Hermitian conjugation, \( M' = M^\dagger \), for any element \( M \in \mathcal{M}_n(\mathbb{C}) \), while the norm \( \| M \| \) is given by the largest eigenvalue of \( M'M \). Nevertheless, the description of a physical system through its \( \mathcal{C}^* \)-algebra of observables is particularly appropriate in the presence of an infinite number of degrees of freedom, where the canonical formalism becomes problematic. Indeed, the algebra \( \mathcal{B}(\mathcal{H}) \) of all bounded operators on an infinite-dimensional Hilbert space \( \mathcal{H} \) is another canonical example of a \( \mathcal{C}^* \)-algebra when equipped with the usual operator norm and adjoint operation. Actually, any \( \mathcal{C}^* \)-algebra is isomorphic to a norm-closed self-adjoint subalgebra of \( \mathcal{B}(\mathcal{H}) \), for some Hilbert space \( \mathcal{H} \) (Gelfand–Naimark theorem). It is worth adding that the non-commutativity of the algebra of observables \( \mathcal{A} \) can be taken as the distinctive property characterizing quantum systems.

\* States on \( \mathcal{C}^* \)-algebras

Although the system observables, i.e. the Hermitian elements of \( \mathcal{A} \), can be identified with the physical quantities measured in experiments, the explicit link between the algebra \( \mathcal{A} \) and the outcome of the measurements is given by the concept of a state \( \Omega \), through which the expectation value \( \Omega(\alpha) \) of the observable \( \alpha \in \mathcal{A} \) can be defined.

In general, a state \( \Omega \) on a \( \mathcal{C}^* \)-algebra \( \mathcal{A} \) is a linear map \( \Omega : \mathcal{A} \to \mathbb{C} \), with the property of being positive, i.e. \( \Omega(\alpha'\alpha) \geq 0, \forall \alpha \in \mathcal{A} \), and normalized, \( \Omega(1_\mathcal{A}) = 1 \), with \( 1_\mathcal{A} \) being the unit of \( \mathcal{A} \). It immediately follows that the map \( \Omega \) is also continuous: \( \| \Omega(\alpha) \| \leq \| \alpha \| \) for all \( \alpha \in \mathcal{A} \).

This general definition of a state of a quantum system comprises the standard one in terms of normalized density matrices on a Hilbert space \( \mathcal{H} \); indeed, any density matrix \( \rho \) defines a state \( \Omega_\rho \) on \( \mathcal{B}(\mathcal{H}) \) through the relation

\[
\Omega_\rho(\alpha) = \text{Tr}[\rho \alpha], \quad \forall \alpha \in \mathcal{B}(\mathcal{H}),
\]

which for pure states, \( \rho = |\psi\rangle\langle\psi| \), reduces to the standard expectation: \( \Omega_\rho(\alpha) = \langle\psi|\alpha|\psi\rangle \).

Nevertheless, the definition in terms of \( \Omega \) is more general, holding even for systems with

\[\text{5 For a more detailed discussion, see the reference textbooks [59–65].}\]
infinite degrees of freedom, for which the usual approach in terms of state vectors may be unavailable.

As for density matrices on a Hilbert space \( \mathcal{H} \), a state \( \Omega \) on a C*-algebra \( \mathcal{A} \) is said to be pure if it cannot be decomposed as a convex sum of two states, i.e. if the decomposition \( \Omega = \lambda \Omega_1 + (1 - \lambda)\Omega_2 \), with \( 0 \leq \lambda \leq 1 \), holds only for \( \Omega_1 = \Omega_2 = \Omega \). If a state \( \Omega \) is not pure, it is called `mixed'. It is worth noticing that, for consistency, the assumed completeness of the relation between observables and measurements on a physical system requires that the observables separate the states, i.e. \( \Omega_1(\alpha) = \Omega(\alpha) \) for all \( \alpha \in \mathcal{A} \) implies \( \Omega_1 = \Omega_2 = \Omega \), and similarly that the states separate the observables, i.e. \( \Omega(\alpha) = \Omega(\beta) \) for all states \( \Omega \) on \( \mathcal{A} \) implies \( \alpha = \beta \).

* GNS Construction

Although the above description of a quantum system through its C*-algebra of observables (its measurable properties) and states over it (giving the observable expectations) looks rather abstract, it actually allows a Hilbert space interpretation, through the so-called Gelfand-Naimark-Segal (GNS) construction.

**Theorem 1.** Any state \( \Omega \) on the C*-algebra \( \mathcal{A} \) uniquely determines (up to isometries) a representation \( \pi_\Omega \) of the elements of \( \mathcal{A} \) as operators in a Hilbert space \( \mathcal{H}_\Omega \), containing a reference vector \( |\Omega\rangle \), whose matrix elements reproduce the observable expectations:  
\[
\Omega(\alpha) = \langle \Omega | \pi_\Omega(\alpha) | \Omega \rangle, \quad \alpha \in \mathcal{A}.
\]  

**Proof.** The algebra \( \mathcal{A} \) can be viewed as a vector space by associating to each element \( \alpha \in \mathcal{A} \) a vector \( |\psi_\alpha\rangle \), and (assuming the state \( \Omega \) to be faithful, i.e. \( \Omega(\alpha^*\alpha) > 0 \) for all non-vanishing \( \alpha \)) by introducing the positive definite inner product \( \langle \psi_\alpha | \psi_\beta \rangle = \Omega(\alpha^*\beta) \). The completion of \( \mathcal{A} \) in the corresponding norm gives a Hilbert space \( \mathcal{H}_\Omega \). The representation \( \pi_\Omega : \mathcal{A} \to B(\mathcal{H}_\Omega) \) on \( \mathcal{H}_\Omega \) can then be defined by: \( \pi_\Omega(\alpha) |\psi_\beta\rangle = |\psi_{\alpha^*\beta}\rangle \); indeed it satisfies \( \pi_\Omega(\alpha)\pi_\Omega(\beta) = \pi_\Omega(\alpha\beta) \) and \( \pi_\Omega(\alpha^*\alpha) = \pi_\Omega(\alpha^*\alpha) \). The element \( |\psi_\Omega\rangle \equiv |\Omega\rangle \) of \( \mathcal{H}_\Omega \) is cyclic with respect to \( \pi_\Omega \), as any element \( |\psi_\alpha\rangle \) in \( \mathcal{H}_\Omega \) can be written as \( |\psi_\alpha\rangle = \pi_\Omega(\alpha) |\Omega\rangle \), or in more precise terms, \( \pi_\Omega(\mathcal{A}) |\Omega\rangle \) is dense in \( \mathcal{H}_\Omega \). If the state \( \Omega \) is not faithful, the same construction holds by identifying \( \mathcal{H}_\Omega \) with the completion of \( \mathcal{A}/\mathcal{N}_\Omega \), where \( \mathcal{N}_\Omega \) is the kernel of the form \( \langle \cdot | \cdot \rangle \) defined above. \( \square \)

This result makes apparent that the notion of Hilbert space associated to a quantum system is not a primary concept, but an emergent tool, a consequence of the C*-algebra structure of the system observables. Further, the whole construction sketched above is unique up to unitary transformations. Indeed, if \( \pi'_\Omega \) is another representation of \( \mathcal{A} \) on a Hilbert space \( \mathcal{H}'_\Omega \) with cyclic vector \( |\Omega'\rangle \) such that \( \Omega(\alpha) = \langle \Omega' | \pi'_\Omega(\alpha) |\Omega'\rangle \) for all \( \alpha \in \mathcal{A} \), then \( \pi_\Omega \) and \( \pi'_\Omega \) are unitarily equivalent, i.e. there exists an isometry \( U : \mathcal{H}_\Omega \to \mathcal{H}'_\Omega \) such that \( U \pi_\Omega U^{-1} = \pi'_\Omega \).

* Reducibility and phases

A representation \( \pi \) of the algebra \( \mathcal{A} \) on a Hilbert space \( \mathcal{H} \) is irreducible if \( \mathcal{H} \) and the null space are the only closed subspaces invariant under the action of \( \pi(\mathcal{A}) \). One can prove that the GNS representation \( \pi_\Omega \) is irreducible if and only if the state \( \Omega \) is pure. When the representation \( \pi_\Omega \) is not irreducible, it can be decomposed in general into the direct sum of irreducible representations \( \pi'^{(i)} \):  
\[
\pi_\Omega = \bigoplus_i \pi'^{(i)}_\Omega, \quad (3)
\]
and similarly, the Hilbert space $\mathcal{H}_\Omega$ also decomposes into the direct sum of invariant subspaces $\mathcal{H}_\Omega^{(r)}$ carrying the irreducible representation $\pi_\Omega^{(r)}$:

$$\mathcal{H}_\Omega = \bigoplus_r \mathcal{H}_\Omega^{(r)}. \tag{4}$$

As we shall see in the following, irreducibility is an important issue in the classification of entangled states.

Any vector $|\psi\rangle \in \mathcal{H}_\Omega$ defines a new GNS representation via the state $\Omega_\psi$ defined by: $\Omega_\psi(\alpha) = \langle \psi | \pi_\Omega(\alpha) | \psi \rangle$, for all $\alpha \in \mathcal{A}$. It turns out that the new state $\Omega_\psi$ give rise to a GNS representation unitarily equivalent to the one constructed over $\Omega$; in other terms, $\mathcal{H}_\Omega$ and $\mathcal{H}_{\Omega_\psi}$ can be identified as the two representations $\pi_\Omega$ and $\pi_{\Omega_\psi}$. Similarly, a density matrix $\rho$ on $\mathcal{H}_\Omega$ also defines a state $\Omega_\rho$ on $\mathcal{A}$ through the identification $\Omega_\rho(\alpha) = \text{Tr}(\rho \pi_\Omega(\alpha))$, while the corresponding GNS representation $\pi_{\Omega_\rho}$ can be expressed in terms of representations equivalent to the representation $\pi_\Omega$. The set of all states of the form $\Omega_\rho$ forms the so-called folium of the representation $\pi_\Omega$; it contains all the states accessible by the operators $\pi_\Omega(\alpha)$, $\alpha \in \mathcal{A}$, and constitutes a quantum phase of the physical systems. Systems with infinite degrees of freedom, as in many-body physics, generally exhibit more than one inequivalent phase, i.e. they admit more than one inequivalent representation of the associated observable algebra $\mathcal{A}$.

3. Entanglement in algebraic quantum mechanics

As seen in the previous section, given any quantum system, once a state $\Omega$ on its algebra of observables $\mathcal{A}$ is chosen, i.e. a set of expectation values for the elements of $\mathcal{A}$ are fixed, one can always construct the associated Hilbert space $\mathcal{H}_\Omega$ and use it for its description. This space contains a reference vector $|\Omega\rangle$ through which one can generate the whole $\mathcal{H}_\Omega$ by applying elements of $\mathcal{A}$ to it. In more physical terms, all states of the system can be obtained from $|\Omega\rangle$ by the action of all possible physically acceptable operations.

This algebraic approach to quantum physics turns out to be the most suitable for discussing issues related to the notions of quantum non-locality and entanglement in very general terms: it does not make explicit reference to the specific structure of the system under study that can, indeed, be formed even by an infinite number of elementary constituents and thus possibly exhibit more than one physical phase. Although the definition of separability and entanglement within this approach to quantum theory was introduced long ago [34–37], only recently has it been applied to characterize non-classical correlations in systems involving identical particles, both in the case of bosons and fermions [50–58].

Given a quantum system, its observable algebra $\mathcal{A}$ and a state $\Omega$ on it, one immediately faces a problem with the standard textbook definition of separability: the associated GNS Hilbert space $\mathcal{H}_\Omega$ does not generally result a tensor product of single-particle Hilbert spaces and, therefore, the usual notion of entanglement, based on this structure, is inapplicable.

In line with the characterization of a physical system through its algebra of observables $\mathcal{A}$, one should instead focus attention on this algebra rather than on the Hilbert space $\mathcal{H}_\Omega$; in this way, the presence of entanglement can be identified by the existence of non-classical correlations among mean values of system observables, belonging to different subalgebras of $\mathcal{A}$. As a preliminary step, it is then necessary to introduce the notion of partition of the operator algebra $\mathcal{A}$. In the following, we shall consider operator algebras constructed by means of elementary mode operators, e.g. annihilation and creation operators, generating an algebra $\mathcal{A}$ either of boson or fermion character; these algebras can be infinite-dimensional. Within this general framework, we then introduce the following basic definition:
Definition 1. An algebraic bipartition of the operator algebra $\mathcal{A}$ is any pair $(\mathcal{A}_1, \mathcal{A}_2)$ of subalgebras of $\mathcal{A}$, namely $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{A}$, such that $\mathcal{A}_1 \cap \mathcal{A}_2 = \mathbb{I}_\mathcal{A}$, that is they can share only scalar multiples of the identity. Further, in the boson case the two subalgebras are assumed to commute, $[\mathcal{A}_1, \mathcal{A}_2] = 0$ for all $\mathcal{A}_i \in \mathcal{A}_i$, $i = 1, 2$, while in the fermion case, only the even part $\mathcal{A}_e^\mathcal{A}_1$ of $\mathcal{A}_1$ is required to commute with the whole $\mathcal{A}_2$, or $\mathcal{A}_e^\mathcal{A}_2$ with $\mathcal{A}_1$; in general, the even part $\mathcal{A}_e^\mathcal{A}$ of a fermion algebra $\mathcal{A}$ is defined as the norm closure of the algebra of polynomials constructed with even powers of elementary mode operators.

Remark. This definition differs from the one given in \cite{34–43}, in which elements belonging to different bipartitions are required to commute, both for bosons and fermions; as already mentioned in \cite{56–58} and further discussed in the next section, the previous, more general, definition allows for a more physically complete treatment of fermion entanglement.

In general, the linear span of products of elements of the two subalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$ need not reproduce the whole algebra $\mathcal{A}$, i.e. $\mathcal{A}_1 \cup \mathcal{A}_2 \subset \mathcal{A}$. However, in the cases of partitions defined in terms of modes, as discussed below, one has $\mathcal{A}_1 \cup \mathcal{A}_2 = \mathcal{A}$, a condition that will hereafter always be assumed.

Any algebraic bipartition encodes in a natural way the definition of the system local observables:

Definition 2. An element of $\mathcal{A}$ is said to be local with respect to a given bipartition $(\mathcal{A}_1, \mathcal{A}_2)$, or simply $(\mathcal{A}_1, \mathcal{A}_2)$-local, if it is the product $\alpha_1 \alpha_2$ of an element $\alpha_1$ of $\mathcal{A}_1$ and an element $\alpha_2$ in $\mathcal{A}_2$.

From this notion of operator locality, a natural definition of state separability and entanglement follows \cite{50, 57, 6}:

Definition 3. A state $\Omega$ on the algebra $\mathcal{A}$ will be called separable with respect to the bipartition $(\mathcal{A}_1, \mathcal{A}_2)$ if the expectation $\Omega(\alpha_1 \alpha_2)$ of any local operator $\alpha_1 \alpha_2$ can be decomposed into a linear convex combination of products of expectations:

$$\Omega(\alpha_1 \alpha_2) = \sum_k \lambda_k \Omega_k^{(1)}(\alpha_1) \Omega_k^{(2)}(\alpha_2), \quad \lambda_k \geq 0, \quad \sum_k \lambda_k = 1,$$

where $\Omega_k^{(1)}$ and $\Omega_k^{(2)}$ are given states on $\mathcal{A}$; otherwise the state $\Omega$ is said to be entangled with respect the bipartition $(\mathcal{A}_1, \mathcal{A}_2)$.

Remark. (i) This generalized definition of separability can be easily extended to the case of more than two partitions; for instance, in the case of an $n$-partition, equation (5) would extend to

$$\Omega(\alpha_1 \alpha_2 \cdots \alpha_n) = \sum_k \lambda_k \Omega_k^{(1)}(\alpha_1) \Omega_k^{(2)}(\alpha_2) \cdots \Omega_k^{(n)}(\alpha_n), \quad \lambda_k \geq 0, \quad \sum_k \lambda_k = 1.$$

(ii) When dealing with systems of $N$-distinguishable constituents, the algebra $\mathcal{A}$ usually acts on a Hilbert space $\mathcal{H}$; if the state $\Omega$ on $\mathcal{A}$ is normal, i.e. it can be represented by a density matrix $\rho_\Omega$, so that $\Omega(\alpha) = \text{Tr}[\rho_\Omega \alpha^\dagger \alpha]$, for any $\alpha \in \mathcal{A}$, definition 3 gives the standard notion of separability, namely $\rho_\Omega$ can be expressed as a convex combination of product states:

As already observed, the algebras $\mathcal{A}$ need not be finitely generated, so that the sums appearing below could, in principle, contain an infinite number of terms; in such a case, we shall assume their convergence in a proper topology.
\[ \rho_\Omega = \sum_k p_k \rho_k^{(1)} \otimes \rho_k^{(2)} \otimes \cdots \otimes \rho_k^{(N)}, \quad p_k \geq 0, \quad \sum_k p_k = 1, \]  

where \( \rho^{(i)} \) represents a state for the \( i \)-th constituent. Indeed, in this case, the partition of the system into its elementary constituents induces a natural tensor product decomposition both of the Hilbert space \( \mathcal{H} \) and of the algebra \( \mathcal{A} \) of its operators; a direct application of the condition (6) to this natural tensor product multipartition immediately yields the decomposition (7).

(iii) In systems of identical particles there is no \textit{a priori} given natural bipartition to be used for the definition of separability; therefore, issues about entanglement and non-locality are meaningful only with reference to a choice of a specific partition in the associated operator algebra [46–58]. This general observation, often overlooked, is at the origin of much confusion in the recent literature.

When the state \( \Omega \) is pure, the separability condition (5) simplifies, and the following result holds:

**Lemma 1.** Pure states \( \Omega \) on the operator algebra \( \mathcal{A} \) are separable with respect to a given bipartition \((\mathcal{A}_1, \mathcal{A}_2)\) if and only if

\[ \Omega(\alpha_1 \alpha_2) = \Omega(\alpha_1) \Omega(\alpha_2), \]  

for all local operators \( \alpha_1, \alpha_2 \).

In other words, separable pure states are just product states. We shall first give a proof of this result for boson operator algebras, leaving the analysis of fermion algebras to the next section.

**Proof.** For the \textit{if} part of the proof, observe that, according to \textit{definition 3}, states satisfying (8) are automatically \((\mathcal{A}_1, \mathcal{A}_2)\)-separable since they obey (5) with only one contribution to the convex sum. For the \textit{only if} part of the proof, recall that any element \( \alpha \in \mathcal{A} \) can be written as \( \alpha = \sum_i \alpha_i^{(1)} \alpha_i^{(2)}, \) with \( \alpha_i^{(1)} \in \mathcal{A}^{(1)} \) and \( \alpha_i^{(2)} \in \mathcal{A}^{(2)} \). Therefore, if by hypothesis a state \( \Omega \) is separable, i.e. it can be written as in (8) on all \((\mathcal{A}_1, \mathcal{A}_2)\)-local operators, then

\[ \Omega(\alpha) = \sum_i \Omega(\alpha_i^{(1)} \alpha_i^{(2)}) = \sum_i \lambda_k \Omega_k^{(1)}(\alpha_i^{(1)}) \Omega_k^{(2)}(\alpha_i^{(2)}) = \sum_k \omega_k(\alpha), \]  

where \( \omega_k \) are linear maps defined on the whole algebra \( \mathcal{A} \) by the relation

\[ \omega_k(\alpha) = \sum_i \Omega_k^{(1)}(\alpha_i^{(1)}) \Omega_k^{(2)}(\alpha_i^{(2)}). \]  

One can easily see that these maps are positive. Indeed, for any \( \alpha \in \mathcal{A} \) one has \((T \text{ signifies matrix transposition})\)

\[ \omega_k(\alpha \alpha^*) = \text{Tr}[M_k^{(1)}(M_k^{(2)^T})], \quad [M_k^{(1)}]_{ij} = \Omega_k^{(1)}(\alpha_i^{(1)} \alpha_j^{(1)}), \quad \ell = 1, 2, \]  

with the matrices \( M_k^{(1)}, M_k^{(2)} \) Hermitian and positive; since the trace of the product of two positive matrices is positive, one immediately gets: \( \omega_k(\alpha \alpha^*) \geq 0 \). In addition, the maps \( \omega_k \) are normalized, \( \Omega(1) = \Omega_k^{(1)}(1) \Omega_k^{(2)}(1) = 1 \), and therefore represent states for the algebra \( \mathcal{A} \). But, since \( \Omega \) is pure by hypothesis, only one term in the convex combination (9) must be different from zero. Dropping the superfluous label \( k \), we then find: \( \Omega(\alpha^{(1)} \alpha^{(2)}) = \Omega^{(1)}(\alpha^{(1)}) \Omega^{(2)}(\alpha^{(2)}). \) By separately taking \( \alpha^{(1)} \) and \( \alpha^{(2)} \) to coincide with the identity operator, one finally obtains the result (8). \( \square \)
Remark. As we shall see explicitly later on, given a bipartition of the algebra $\mathcal{A}$, a pure separable state $\Omega$ on it is, in general, no longer pure when restricted to a proper subalgebra $\mathcal{B} \subset \mathcal{A}$; nevertheless, since in any case it obeys the condition (8), it will remain separable.

Using the previous definitions and the result of lemma 1, one can study the entanglement with respect to a given bipartition $(\mathcal{B}_1, \mathcal{B}_2)$ of the boson algebra $\mathcal{A}$ of states in a folium (see the discussion at the end of section 2) of the representation $\pi_\beta$ corresponding to a given state $\Omega$ on $\mathcal{A}$, assuming $\Omega$ to be separable.

The specific separability condition (8) allows one to obtain the generic form of any pure separable state:

**Proposition 1.** Let $(\mathcal{A}, \Omega)$ be operator algebra and state associated to a given boson quantum system and assume $\Omega$ to be separable with respect to a given bipartition $(\mathcal{B}_1, \mathcal{B}_2)$. Then a normalized pure state $|\psi\rangle$ in the GNS Hilbert space $\mathcal{H}_\Omega$ is $(\mathcal{B}_1, \mathcal{B}_2)$-separable if and only if it can be written in the form

$$|\psi\rangle = \pi_\beta(\beta^{(1)})\pi_\beta(\beta^{(2)})|\Omega\rangle,$$

with $\beta^{(i)} \in \mathcal{A}_i$, $i = 1, 2$, while $\pi_\beta(\beta^{(i)})$ denotes the corresponding operator representation on the Hilbert space $\mathcal{H}_\Omega$.

**Proof.** For the if part of the proof, notice that the normalization condition together with the assumed $(\mathcal{B}_1, \mathcal{B}_2)$-separability of $\Omega$ yield:

$$\langle \psi|\psi\rangle = \langle \Omega|\pi_\beta(\beta^{(2)})\pi_\beta(\beta^{(1)})\pi_\beta(\beta^{(2)})|\Omega\rangle = \langle \Omega|\pi_\beta(\beta^{(1)})\pi_\beta(\beta^{(2)})|\Omega\rangle = 1.$$

Using this result and again the separability condition of $\Omega$, one then has

$$\langle \psi|\pi_\beta(\alpha^{(1)})\pi_\beta(\alpha^{(2)})|\psi\rangle = \langle \Omega|\pi_\beta(\beta^{(2)})\pi_\beta(\alpha^{(2)})\pi_\beta(\beta^{(2)})|\Omega\rangle \langle \Omega|\pi_\beta(\beta^{(1)})\pi_\beta(\alpha^{(1)})\pi_\beta(\beta^{(1)})|\Omega\rangle.$$

For the only if part of the proof, observe that due to the cyclicity of the GNS state $|\Omega\rangle$, one can surely write $|\psi\rangle = \pi_\beta(\beta)|\Omega\rangle$, for some $\beta \in \mathcal{A}$. Further, since $\mathcal{A}_1 \cup \mathcal{A}_2 = \mathcal{A}$, $\beta$ can be written as a combination of suitable local operators, $\beta = \sum_i \beta_i^{(1)} \beta_i^{(2)}$, with $\beta_i^{(1)} \in \mathcal{A}_i$ and $\beta_i^{(2)} \in \mathcal{A}_2$. Then, for any local operator $\alpha^{(1)}\alpha^{(2)}$, the separability condition (5) implies

$$\langle \psi|\pi_\beta(\alpha^{(1)})\pi_\beta(\alpha^{(2)})|\psi\rangle = \sum_{i,j} \langle \psi_j^{(2)}|\psi_j^{(1)}\rangle \langle \psi_j^{(1)}|\pi_\beta(\alpha^{(1)})|\psi_j^{(1)}\rangle \times \sum_{r,s} \langle \psi_r^{(1)}|\psi_r^{(1)}\rangle \langle \psi_r^{(1)}|\pi_\beta(\alpha^{(2)})|\psi_r^{(2)}\rangle,$$

where

$$|\psi_j^{(\ell)}\rangle = \pi_\beta(\beta^{(\ell)})|\Omega\rangle, \quad \ell = 1, 2.$$

The matrices $\langle \psi_j^{(\ell)}|\psi_j^{(\ell)}\rangle$ are Hermitian and positive semi-definite and, therefore, they can be diagonalized by suitable unitary transformations:

$$\langle \psi_j^{(\ell)}|\psi_j^{(\ell)}\rangle = \sum_r |U^{(\ell)}|_r \lambda^{(\ell)}_r |U^{(\ell)}|_r, \quad U^{(\ell)} U^{(\ell)*} = 1, \quad \lambda^{(\ell)}_r \geq 0, \quad \ell = 1, 2.$$
Then, by defining
\[ |\phi_r^{(1)}\rangle = \sum_j |U^{(2)}|_{ij} |\psi_j^{(1)}\rangle, \quad |\phi_r^{(2)}\rangle = \sum_j |U^{(1)}|_{ij} |\psi_j^{(2)}\rangle, \]
and recalling again that any element \( \alpha \in \mathcal{A} \) can be decomposed in terms of local operators, \( \alpha = \sum_i \alpha_i^{(1)} \alpha_i^{(2)} \), using (13) one can write:
\[ \langle \psi | \pi_1(\alpha) |\psi\rangle = \sum_{r,s} \lambda_r^{(1)} \lambda_s^{(2)} \omega_{rs}(\alpha), \]
with
\[ \omega_{rs}(\alpha) = \sum_i \langle \phi_r^{(1)} | \alpha_i^{(1)} |\phi_s^{(1)}\rangle \langle \phi_r^{(2)} | \alpha_i^{(2)} |\phi_s^{(2)}\rangle. \]

As in the proof of lemma 1, one easily sees that the linear maps \( \omega_{rs} \) are actually (un-normalized) states on \( \mathcal{A} \). Then, by the purity of the state \( |\psi\rangle \), the convex combination on the rhs of (16) must contain just one term. This implies that there are just two positive constants \( \lambda^{(1)} \) and \( \lambda^{(2)} \) and therefore, no sum over \( r \) in (15): \( \langle \psi^{(1)} | \psi^{(1)}\rangle = |V^{(1)}_r |^2 V^{(1)}_r, \ell = 1, 2, \) with \( V^{(1)}_r = \sqrt{\lambda^{(1)}} T^{(1)} |V^{(1)}_r, \ell \rangle, \ell \in \mathbb{C} \). These inner products are thus in factorized form and this is possible only if the vectors \( |\psi^{(1)}_r\rangle \) are all proportional: \( |\psi^{(1)}_r\rangle \equiv |V^{(1)}_r | V^{(1)}_r\). Recalling their definition from (14), this then implies \( \beta^{(1)} = V^{(1)}_r \beta^{(1)}, \) up to operators annihilating \( |\Omega\rangle \); as a consequence the factorized form (12) follows. \( \square \)

As an immediate consequence of this result, one has the following corollary, which will be very useful in characterizing the general form of entangled states in \( \mathcal{H}_\Omega \).

**Corollary 1.** Let \( (\mathcal{A}, \Omega, \pi_1) \) be the triple characterizing the operator algebra, state and GNS representation of a given boson quantum system. Given any bipartition \( (\mathcal{A}_1, \mathcal{A}_2) \) of \( \mathcal{A} \) with \( \Omega \) separable, it is always possible to choose a basis in the corresponding Hilbert space \( \mathcal{H}_\Omega \) made of separable pure states.

**Proof.** In each of the two subalgebras \( \mathcal{A}_1, \mathcal{A}_2 \) viewed as linear spaces fix a basis \( \{ e^{(1)}_k \}, \{ e^{(2)}_k \} \), so that any element \( \alpha^{(i)} \in \mathcal{A}_i, i = 1, 2 \), can be decomposed as a linear combination of the basis elements, \( \alpha^{(i)} = \sum_k e^{(i)}_k \alpha^0_k \), with \( e^{(i)}_k \) complex coefficients. Then, the following set of vectors \( |k; \ell\rangle = \pi_1(e^{(1)}_k) \pi_2(e^{(2)}_\ell) |\Omega\rangle \), obtained by applying products of the chosen basis elements to the cyclic vector \( |\Omega\rangle \), are pure and separable by construction; further, they span the whole \( \mathcal{H}_\Omega \), since otherwise the condition \( \mathcal{A}_1 \cup \mathcal{A}_2 = \mathcal{A} \) would not be satisfied. \( \square \)

In many relevant cases, one can choose the basis elements \( e^{(i)}_k \in \mathcal{A}_i, i = 1, 2 \), so that the resulting basis \( |k; \ell\rangle \) of \( \mathcal{H}_\Omega \) is orthonormal. In such cases, any (normal) mixed state \( \Omega_{\rho} \), represented by the density matrix \( \rho \) on \( \mathcal{H}_\Omega \), can be decomposed as
\[ \rho = \sum_{j,k,l,m} \rho_{j,k,l,m} |j; k\rangle \langle \ell; m|, \quad \sum_{j,k} \rho_{j,k,j,k} = 1. \]
A density matrix \( \rho_{\Omega} \) in diagonal form
\[ \rho_{\Omega} = \sum_{j,k} \rho_{j,k,j,k} |j; k\rangle \langle j; k|, \]
is clearly separable, being a convex combination of projectors on separable pure states. More explicitly, for any local operator \( \alpha^{(1)} \alpha^{(2)} \), one has:
which is precisely of the separable form (5). This observation, together with proposition 1 and the fact that generic separable mixed states belong to the convex hull of pure separable states, can be used to characterize separable mixed states:

**Corollary 2.** A mixed boson state \( \rho \) (as in (17)) is separable with respect to the given bipartition \((A_1, A_2)\) if and only if it is a convex combination of projectors onto pure \((A_1, A_2)\)-separable states; otherwise, the state \( \rho \) is \((A_1, A_2)\)-entangled.

In general, to determine whether a given density matrix \( \rho \) can be written in diagonal separable form is a hard task and one is forced to rely on suitable separability tests, although these are generally not exhaustive. As discussed in the next section, one such test is peculiar to fermion systems, and it is connected to the anticommuting character of the corresponding operator algebra \( \mathcal{A} \).

### 4. Entanglement in fermion systems

In this section, we extend the results previously obtained in the case of boson systems to many-body systems made of fermion elementary constituents. Adopting a second-quantized point of view, for fermion systems the observable algebra \( \mathcal{A} \) coincides with the complex algebra \( \mathcal{A}_f \) of canonical anticommutation relations. It is generated by elements \( a_i, a_i^* \) obeying the relations

\[
\{a_i, a_j^*\} = \delta_{ij}, \quad \{a_i, a_j\} = \{a_i^*, a_j^*\} = 0, \quad i, j = 1, 2, \ldots, M, \tag{20}
\]

where, for simplicity, we have assumed that the fermions can occupy \( M \) different modes (with \( M \) possibly infinite). This framework is quite general and can accommodate various situations arising in atomic and condensed matter physics; in particular, it can be used to describe ultracold fermions confined in optical lattices [74–84]. The norm closure of all polynomials in the creation and annihilation operators gives the full fermion operator algebra \( \mathcal{A}_f \).

This algebra has a natural gradation in terms of its even and odd part:

**Definition 4.** Introduce the automorphism \( \vartheta \) of \( \mathcal{A}_f \) defined by its action on the basic operators \( a_i \) and \( a_i^* \) as follows: \( \vartheta(a_i) = -a_i \), \( \vartheta(a_i^*) = -a_i^* \). Then, the even component \( \mathcal{A}^e_f \) of \( \mathcal{A}_f \) is the subset containing the elements \( \alpha^e \in \mathcal{A}_f \) such that \( \vartheta(\alpha^e) = \alpha^e \), while the odd component \( \mathcal{A}^o_f \) of \( \mathcal{A}_f \) consists of elements \( \alpha^o \in \mathcal{A}_f \), for which \( \vartheta(\alpha^o) = -\alpha^o \).

The even part \( \mathcal{A}^e_f \) is a subalgebra of \( \mathcal{A}_f \), the one generated by even polynomials in all creation and annihilation operators; on the other hand, \( \mathcal{A}_f^o \) is only a linear space and not an algebra, since the product of two odd elements is even. Nevertheless, using the two projectors \( \mathcal{P}^e = (1 + \vartheta) / 2 \) and \( \mathcal{P}^o = (1 - \vartheta) / 2 \), any element \( \alpha \in \mathcal{A}_f \) can be decomposed in its even \( \alpha^e \equiv \mathcal{P}^e(\alpha) \) and odd \( \alpha^o \equiv \mathcal{P}^o(\alpha) \) parts: \( \alpha = \alpha^e + \alpha^o \).

---

7 Part of the results of this section have been already discussed in [56–58].
A bipartition of the $M$-mode fermion algebra $A_f$ can be easily obtained by splitting the collection of operators $\{a_i, a_i^\dagger\}$ into two disjoint sets $\{a_i, a_i^\dagger\} i = 1, 2, \ldots, m$ and $\{a_j, a_j^\dagger\} j = m + 1, m + 2, \ldots, M$. All polynomials in the first set (together with their norm closures) form a subalgebra $A_1$, while the second set generates a subalgebra $A_2$. These two algebras have only the unit element in common and further $A_1 \cup A_2 = A_f$. Further, one defines the even $A_e^2$ and odd $A_o^2$ components of the two subalgebras $A_i, i = 1, 2$, as done above for the full algebra $A_f$, through the automorphism $\vartheta$. Only the operators of the first partition belonging to the even component $A_e^2$ commute with any operator of the second partition and, similarly, only the even operators of the second partition commute with the whole subalgebra $A_i$. Recalling now definition 1, $(A_1, A_2)$ is indeed an algebraic bipartition of $A_f$; in practice, it is determined by the choice of the integer $m$, with $0 < m < M$.

Let us now come back to the definition of separability introduced in definitions 1–3 and to the apparent difference with which it treats bosonic and fermionic systems. As already noticed, in the boson case the two subalgebras $A_1, A_2$ defining the algebraic bipartition $(A_1, A_2)$ naturally commute, i.e. each element $\alpha_1$ of the operator algebra $A_1$ commutes with any element $\alpha_2$ in $A_2$. Instead, in the case of fermion systems the two subalgebras $A_1, A_2$ do not, in general, commute. Nevertheless, in such systems only self-adjoint operators belonging to the even components $A_e^2, A_o^2$ qualify as physical observables and these do commute as required by the definition of bipartition.

At this point, two different attitudes are possible regarding the definition of separability expressed by the condition (5): (i) allow in it all operators from the two subalgebras $A_1, A_2$, or (ii) restrict all considerations to observables only. The first approach is in line with the notion of ‘microcausality’ adopted in constructive quantum field theory [67, 68], where the emphasis is on quantum fields, which are required either to commute (boson fields) or anticommute (fermion fields) if defined on (causally) disjoint regions. On the other hand, the second point of view reminds us of the notion of ‘local commutativity’ in algebraic quantum field theory [60, 61], where only observables are considered, and assumed to commute if localized in disjoint regions.

These two points of view are not equivalent, as can be appreciated by the following simple example. Let us consider a system consisting of just one fermion that can occupy two modes, $M = 2$, with the bipartition defined by the two modes themselves. In the standard Fock representation, i.e. the GNS construction built out of the vacuum state $|\Omega_0\rangle$, such that $\pi_{\Omega_0}(a_i)|\Omega_0\rangle = 0, i = 1, 2$ (see below for details), consider the following state:

$$\Omega = |\phi \rangle \langle \phi|, \quad |\phi \rangle = \frac{1}{\sqrt{2}}(|1, 0\rangle + |0, 1\rangle),$$

which is a combination of the two manifestly separable Fock states $|1, 0\rangle = \pi_{\Omega_0}(a_1^\dagger)|\Omega_0\rangle$ and $|0, 1\rangle = \pi_{\Omega_0}(a_2^\dagger)|\Omega_0\rangle$. Clearly, it appears to be entangled and indeed, as discussed in [57], in a suitable $N$-fermion generalization its quantum non-locality can be used in quantum metrology to achieve sub-shot-noise accuracy in parameter estimation.

Nevertheless, in the second approach mentioned above, it is found to satisfy the condition (5), hence to be separable. Indeed, only observables, i.e. even self-adjoint operators, can be used in this case as $\alpha_1$ and $\alpha_2$. In practice, only the two partial number operators $a_1^\dagger a_1$ and $a_2^\dagger a_2$ together with the identity are admissible, and for these observables the state (21) behaves as the separable state $(|0, 1\rangle \langle 0, 1| + |1, 0\rangle \langle 1, 0|)/2$. This is different to the situation within the first approach: in this case, all operators are admissible, for instance $a_1^\dagger$ and $a_2$, which indeed prevent the separability condition (5) to be satisfied. In view of this, as in [56–58], we here advocate and adopt the first point of view, i.e. point (i) above: it gives a more general and physically complete treatment of fermion entanglement.
In this respect, it should be added that the anticommuting character of the fermion algebra gives stringent constraints on the form of the states defined on it, specifically on the ones that can be represented as a product of other states.

As for any operator algebra, a state on \( A_f \) is a positive, linear functional \( \Omega : A_f \to \mathbb{C} \). Then, the following result holds (see [57, 85] for the rather simple proof):

**Lemma 2.** Consider a bipartition \( (A_1, A_2) \) of the fermion algebra \( A_f \) and two states \( \Omega_1, \Omega_2 \) on it. Then, the linear functional \( \Omega \) on \( A_f \) defined by \( \Omega(\alpha_1\alpha_2) = \Omega_1(\alpha_1)\Omega_2(\alpha_2) \) for all \( \alpha_1 \in A_1 \) and \( \alpha_2 \in A_2 \) is a state on \( A_f \) only if at least one \( \Omega_i \) vanishes on the odd component of \( A_i \).

This result implies that the product \( \Omega_k^{(1)}(\alpha_1)\Omega_k^{(2)}(\alpha_2) \) on the rhs of (5) in definition 3, vanishes whenever \( \alpha_1 \) and \( \alpha_2 \) are both odd. Since the even component \( A_0 \) commutes with the entire subalgebra \( A_2 \), and similarly \( A_2 \) commutes with \( A_0 \), it follows that for fermions the decomposition (5) is non-trivial only for local operators \( \alpha_1\alpha_2 \) such that \([\alpha_1, \alpha_2] = 0\), thus making the definition of separability it encodes completely analogous to the one for bosons.

**Remark.** Given a partition \( (A_1, A_2) \) of \( A_f \), consider a product state \( \Omega \) such that \( \Omega(\alpha_1\alpha_2) = \Omega_1(\alpha_1)\Omega_2(\alpha_2) \) for all \( \alpha_1 \in A_1 \) and \( \alpha_2 \in A_2 \); because of lemma 2, it must be zero on the odd elements of at least one partition. Indeed, fixing two odd elements \( \alpha_1^o \) and \( \alpha_2^o \) in the two partitions, by lemma 2 at least one of the two expectations \( \Omega(\alpha_1^o) \) or \( \Omega(\alpha_2^o) \) must be zero. Assume \( \Omega(\alpha_1^o) = 0 \); then, again using the previous lemma, one has \( \Omega(\beta_1^o\alpha_2^o) = \Omega(\beta_1^o)\Omega(\alpha_2^o) = 0 \), and thus \( \Omega(\beta_1^o) = 0 \), for any odd element \( \beta_1^o \in A_1 \).

Motivated by this last remark, in the following we shall limit our considerations to states on \( A_f \) that are left invariant by the action of the automorphism \( \vartheta \), \( \Omega \circ \vartheta = \Omega \), namely states that are vanishing on the odd component of the fermion algebra: this physical, ‘gauge-invariance’ condition is always tacitly assumed in the discussion of any fermion many-body system.

Within the framework introduced above, most of the results discussed in the previous section in the case of boson algebras remain true also for the fermion algebra \( A_f \). In particular, the characterization of pure, separable states given by lemma 1 and proposition 1 is unaltered; however, the proofs of these results need refinements in order to comply with the anticommuting character of \( A_f \).

- **Proof of lemma 1:** The if part of the proof is unaltered, while for the only if part, one notices that also in this case any element \( \alpha \in A_f \) can be written as \( \alpha = \sum \alpha_1^{(1)}\alpha_2^{(2)} \), with \( \alpha_1^{(1)} \in A^{(1)}, \alpha_2^{(2)} \in A^{(2)} \); however, as previously observed, the elements \( \alpha_1^{(l)}, l = 1, 2 \), can be decomposed as the sum of their even and odd parts, so that in the previous decomposition of \( \alpha \) in terms of local operators, one can assume all \( \alpha_1^{(l)} \) to be of given parity. The proof than proceeds as in the boson case, since the expressions in (9) and (10) are unaltered. The only troubling point is to show that the linear maps \( \omega_k(\alpha) \) in (9) are really states on \( A_f \), i.e. that \( \omega_k(\alpha \, \alpha^* ) \geq 0 \). This is done by explicit calculation showing that \( \omega_k(\alpha \, \alpha^* ) \) can be expressed as in (11) plus additional pieces that are nonetheless vanishing due to the result of lemma 2.

- **Proof of proposition 1:** In this case it is the if part of the proof that requires some care, while the only if part of the proof proceeds exactly as in the boson case, recalling that averages of odd operators on the GNS state \( \Omega \) vanish. Given the bipartition \( (A_1, A_2) \), let us

---

8 This is apparent in the standard Fock representation of the fermion algebra discussed below, since the vacuum expectation of an odd number of elements \( a_i \) and \( a_i^* \) is always vanishing.
assume that the pure, normalized state $|\psi\rangle$ can be written as in (12), i.e.

$$|\psi\rangle = n_1(\beta^{(1)}) n_2(\beta^{(2)}) |\Omega\rangle,$$

where $\beta^{(1)}, \beta^{(2)} \in A_\ell$, $\ell = 1, 2$; we have to prove that:

$$\langle \psi | n_1(\alpha^{(1)}) n_2(\alpha^{(2)}) |\psi\rangle = \langle \psi | n_1(\alpha^{(2)}) |\psi\rangle \langle \psi | n_2(\alpha^{(1)}) |\psi\rangle,$$

for any $\alpha^{(i)} \in A_\ell$. First of all, notice that if (22) is true for $\alpha^{(1)}, \alpha^{(2)}$ of definite parity, then it is also true for generic $\alpha^{(1)}, \alpha^{(2)}$, since they can always be decomposed as the sum of their even and odd parts. The proof then splits into four parts, according to all the possible combinations of parities that the elements $\alpha^{(1)}, \alpha^{(2)}$ can take. By writing also $\beta^{(1)}$ and $\beta^{(2)}$ as the sum of their even and odd parts and using the normalization condition $\langle \psi | \psi \rangle = 1$, explicit computation then shows that the result (22) is indeed true, keeping in mind that $\Omega$ is separable and vanishing on odd elements of $A_\ell$.

As a further consequence of lemma 2, the following criterion of entanglement holds:

**Corollary 3.** Given the bipartition $(A_1, A_2)$ of the fermion algebra $A_\ell$, if a state $\Omega$ is non-vanishing on a local operator $\alpha^1_1 \alpha^2_2$, with the two components $\alpha^1_i \in A^1_i$, $\alpha^2_2 \in A^2_2$ both belonging to the odd part of the two subalgebras, then $\Omega$ is entangled.

**Proof.** Indeed, if $\Omega(\alpha^1_1 \alpha^2_2) \neq 0$, then, by lemma 2, $\Omega$ cannot be written as in (5), and therefore it is entangled. \qed

The standard GNS construction for the algebra $A_\ell$ is based on the vacuum state $\Omega_0$ giving rise to the so-called Fock representation. It is characterized by the condition $\Omega_0(a_i) = 0$, for all annihilation operators $a_i$, or equivalently, $\pi_{A_\ell}(a_i)|\Omega_0\rangle = 0$; the corresponding Hilbert space $\mathcal{H}_{\Omega_0}$ is spanned by the states obtained by applying creation operators, $\pi_{A_\ell}(a_i^\dagger) \equiv [\pi_{A_\ell}(a_i)]^\dagger$ to the cyclic vector $|\Omega_0\rangle$. A basis in $\mathcal{H}_{\Omega_0}$ is then given by the set of Fock states:

$$|n_1, n_2, \ldots, n_M\rangle = [\pi_{A_\ell}(a^\dagger_i)]^{n_i} [\pi_{A_\ell}(a^\dagger_j)]^{n_j} \cdots [\pi_{A_\ell}(a^\dagger_k)]^{n_k} |\Omega_0\rangle,$$

with the integers $n_1, n_2, \ldots, n_M$ representing the occupation numbers of the different modes; due to algebraic relations (20), they can take only the two values 0 or 1. In this representation, the total number $N = \sum_i N_i$, with $N_i = \pi_{A_\ell}(a^\dagger_i) \pi_{A_\ell}(a_i)$ counting the occupation number of the $i$th mode, is a well-defined operator on $\mathcal{H}_{\Omega_0}$; As a consequence, Fock states with different occupation numbers are orthogonal.

**Remark.** Notice that the operator $\hat{N}$ commutes with all physical observables, since coherent mixtures of states with different total occupation numbers are not physical due to the conservation of the fermion parity operator: we are in presence of a so-called superselection rule [86–88].

One can easily see that the Fock representation of $A_\ell$ is irreducible, so that the Fock states in (23) are pure on $A_\ell$. Further, they are separable with respect to any fermion bipartition $(A_1, A_2)$ as in definition 1, since they are in the product form (8). Therefore, they can be used to give a convenient decomposition of any fermion state in the folium of $\Omega_0$ in particular, for any density matrix $\rho$ on $\mathcal{H}_{\Omega_0}$.

First, note that due to the above-mentioned fermion number superselection rule, a general fermion density matrix can be written as an incoherent superposition of states $\rho_N$ with a fixed number $N$ of fermions:
\[ \rho = \sum_{N} \lambda_{N} \rho_{N}, \quad \lambda_{N} \geq 0, \quad \sum_{N} \lambda_{N} = 1. \]

One can then limit the discussion to the states \( \rho_{N} \), with \( N \) fixed. Indeed, notice that two density matrices \( \rho_{N_{1}} \) and \( \rho_{N_{2}} \), with \( N_{1} = N_{2} \), have supports on orthogonal subspaces of \( \mathcal{H}_{\Omega_{0}} \); as a result, the Fock Hilbert space decomposes as \( \mathcal{H}_{\Omega_{0}} = \bigoplus_{N} \mathcal{H}_{N} \), where \( \mathcal{H}_{N} \) are Hilbert spaces spanned by Fock vectors (23) which have exactly \( N \) fermions, i.e. \( \sum_{i} n_{i} = N \).

As discussed above, a bipartition of \( \mathcal{F} \) is given by a partition of the fermion modes into two disjoint sets, one containing the first \( m \) modes, while the second contains the remaining \( M - m \) ones; we can refer to such a choice as the \((m,M-m)\)-partition. Given such a bipartition the Fock basis in \( \mathcal{H}_{\Omega_{0}} \) can be relabeled in a more convenient way as \( \langle k, \sigma; N - k, \sigma' \rangle \), where the integer \( k \) gives the number of occupied modes in the first partition, while \( \sigma \) counts the different ways in which these modes can be taken out of the available \( m \) \((k \leq m)\); similarly, \( \sigma' \) distinguishes the ways in which the remaining \( N - k \) occupied modes can be distributed in the second partition.

Then, a generic density matrix \( \rho_{N} \) on \( \mathcal{H}_{N} \) can be decomposed as
\[ \rho_{N} = \sum_{k,l=0}^{N} \sum_{\sigma,\sigma',\tau,\tau'} \rho_{k\sigma',l\tau'\tau} \langle k, \sigma; N - k, \sigma' \rangle \langle l, \tau; N - l, \tau' \rangle, \quad \sum_{k=0}^{N} \sum_{\sigma,\sigma'} \rho_{k\sigma',k\sigma'} = 1, \]
where \( N = \max\{0, N - M + m\} \) and \( N_{e} = \min\{N, m\} \) are the minimum and maximum number of fermions that the first partition can contain, due to the exclusion principle. Using this decomposition, one can obtain a full characterization of the structure of entangled fermion states (see [57] for further details):

**Proposition 2.** A generic \((m, M - m)\)-mode bipartite state (25) is entangled if and only if it cannot be cast in the following block diagonal form:
\[ \rho_{N} = \sum_{k=0}^{N} P_{k} \rho_{k}, \quad \sum_{k=0}^{N} P_{k} = 1, \quad \text{Tr}[\rho_{k}] = 1, \]
with
\[ \rho_{k} = \sum_{\sigma,\sigma',\tau,\tau'} \rho_{k\sigma',k\tau'\tau} \langle k, \sigma; N - k, \sigma' \rangle \langle k, \tau; N - k, \tau' \rangle, \quad \sum_{\sigma,\sigma'} \rho_{k\sigma',k\sigma'} = 1, \]
(i.e. at least one of its non-diagonal coefficients \( \rho_{k\sigma',k\tau'\tau}, k \neq 1 \), is non-vanishing) or, if it can, if and only if at least one of its diagonal blocks \( \rho_{k} \) is non-separable.

The extension of this result to the case of Majorana fermions requires some care since, as we shall see in the coming sections, for real fermions the GNS representations of the observable algebra \( \mathcal{A} \) are generally reducible.

## 5. Algebraic approach to Majorana fermions

For a system made of real fermions, the structure of the algebra of observables \( \mathcal{A} \) turns out to be quite different from that of \( \mathcal{A}_{f} \) characterizing complex fermions and discussed in the previous section. As in that case, we shall describe the Majorana observable algebra in terms of real mode operators \( c_{i}, \quad i = 1, 2, \ldots, N \), with \( c_{i}^{\dagger} = c_{i} \), satisfying the following anticommutation relations:9

9 Although the number \( N \) of Majorana modes can also be infinite, for simplicity, hereafter we shall limit our considerations to the more physically relevant case of finite \( N \).
The linear span of all products of these mode operators, together with the unit element \( c_0 \equiv 1 \), form the (Euclidean) complex Clifford algebra \( \mathcal{C}_N(\mathbb{C}) \) \cite{69–73}, which is the operator \( C^* \)-algebra relevant to describe Majorana fermion systems. One can easily show that the monomials
\[
(c_1)^{n_1} (c_2)^{n_2} \ldots (c_N)^{n_N}, \quad n_i = 0, 1,
\]
with \((c_1)^{0} (c_2)^{0} \ldots (c_N)^{0}\) interpreted as the identity, form a basis in \( \mathcal{C}_N \), which therefore has dimension \( 2^N \). Finite-dimensional Clifford algebras are isomorphic to matrix algebras \cite{69}; however, one has to distinguish two cases, according to whether \( N \) is even or odd. When \( N = 2n \), the algebra \( \mathcal{C}_{2n}(\mathbb{C}) \) is isomorphic to the \( 2^n \times 2^n \) matrix algebra \( \mathcal{M}_{2^n}(\mathbb{C}) \), while for \( N = 2n + 1 \) the algebra \( \mathcal{C}_{2n+1}(\mathbb{C}) \) is isomorphic to the direct sum \( \mathcal{M}_{2^n}(\mathbb{C}) \oplus \mathcal{M}_{2^n}(\mathbb{C}) \).

Explicitly, up to unitary equivalences, in the case \( \mathcal{C}_{2n} \) the isomorphism is given by
\[
c_{2k} \longleftrightarrow m_{2k} \equiv \sigma_0 \otimes \ldots \otimes \sigma_0 \otimes \sigma_1 \otimes \sigma_3 \otimes \ldots \otimes \sigma_3 ,
\]
\[
c_{2k+1} \longleftrightarrow m_{2k+1} \equiv \sigma_0 \otimes \ldots \otimes \sigma_0 \otimes \sigma_2 \otimes \sigma_3 \otimes \ldots \otimes \sigma_3 ,
\]
where \( \sigma_i, i = 1, 2, 3 \) are the Pauli matrices, with \( \sigma_0 \) the \( 2 \times 2 \) identity matrix, while for \( \mathcal{C}_{2n+1} \) one obtains
\[
c_k \longleftrightarrow m_{2k} \oplus m_{2k}, \quad k = 1, 2, \ldots, 2n
\]
\[
c_{2n+1} \longleftrightarrow (\sigma_1 \otimes \ldots \otimes \sigma_1) \oplus (-\sigma_3 \otimes \ldots \otimes -\sigma_3).
\]

Although out of \( 2n \) Clifford modes one can construct \( n \) ordinary complex fermions modes through the relations
\[
a_k = \frac{1}{2}(c_{2k-1} + ic_{2k}),
\]
\[
a_k^* = \frac{1}{2}(c_{2k-1} - ic_{2k}), \quad k = 1, 2, \ldots, n,
\]
the properties of the Clifford algebra \( \mathcal{C}_N \) are quite different from those of the fermion algebra \( \mathcal{A}_f \). First of all, there is no exclusion principle for real fermion modes, since \( (c_i)^2 = 1 \), and not zero, as for the complex fermion operators \( a_k, a_k^* \) in (32). In fact, one cannot even speak of occupancy of a Clifford mode, since there is no number operator in \( \mathcal{C}_N \)\(^{10}\). In a sense, a Clifford mode is always filled and empty at the same time. Recalling the discussion of the previous section, this implies that the Clifford algebras do not admit Fock representations; therefore, all the results regarding separability and entanglement given before for standard fermions need to be reconsidered.

As in the case of \( \mathcal{A}_f \), the automorphism \( \vartheta \), defined by its action on the mode operators as \( \vartheta(c_i) = -c_i \), allows decomposing \( \mathcal{C}_N \) in its even \( \mathcal{C}_N^e \) and odd \( \mathcal{C}_N^o \) parts. Then, following definition 1, a bipartition \((\mathcal{A}_i, \mathcal{A}_j)\) of the Clifford algebra \( \mathcal{C}_N \) is given by two Clifford sub-algebras \( \mathcal{A}_i, \mathcal{A}_j \subset \mathcal{C}_N \), having only the unit element in common, and such that \( \mathcal{A}_i \cup \mathcal{A}_j = \mathcal{C}_N \), together with \([\mathcal{A}_i^e, \mathcal{A}_j^o] = [\mathcal{A}_i^o, \mathcal{A}_j^e] = 0\).

In practice, the bipartition \((\mathcal{A}_i, \mathcal{A}_j)\) is obtained by splitting the collection of modes \( \{c_i \mid i = 1, 2, \ldots, N\} \) into two disjoint sets \( \{c_i \mid i = 1, 2, \ldots, p\} \) and \( \{c_j \mid j = p + 1, p + 2, \ldots, N\} \). The linear span of the monomials \((c_i)^{n_i} (c_j)^{n_j} \ldots (c_p)^{n_p}\), with \( n_i = 0, 1 \), gives the subalgebra \( \mathcal{A}_i \).

\(^{10}\) One can certainly form Hermitian bilinears in the Clifford modes, e.g. \( i c_{2k-1} c_{2k} \), but these are related to the occupation number operator of the corresponding complex fermion modes and not the Clifford modes.
while that of \((c_{p+1})^{p_{p+1}}(c_{p+2})^{p_{p+2}} \ldots (c_N)^{p_N}\) generates the subalgebra \(\mathcal{A}_2\). In practice, also in this case, a bipartition of \(\mathcal{C}_N\) is determined by the choice of the integer \(p\), with \(0 < p < N\).

As for any \(C^*\)-algebra, a state on the Clifford algebra \(\mathcal{C}_N\) is given by a positive linear map from \(\mathcal{C}_N\) to \(\mathcal{C}_N\). The most simple state is given by the map \(\Omega\) that sends all elements of \(\mathcal{C}_N\) to \(0\), except for the identity, which is mapped to \(1\). As for any \(\Sigma\)-algebra, a state on the Clifford algebra \(\mathcal{C}_N\) is given by a positive linear map from \(\mathcal{C}_N\) to \(\mathcal{C}_N\). The most simple state is given by the map \(\Omega\) that sends all elements of \(\mathcal{C}_N\) to \(0\), except for the identity, which is mapped to \(1\).

Through the standard GNS construction, one constructs a Hilbert space \(\mathcal{H}_{\Omega}\) and a representation \(\pi_{\Omega}\) on it; the space \(\mathcal{H}_{\Omega}\) is generated by applying elements of \(\mathcal{C}_N\) to the cyclic vector \(\Omega\) for any element \(g \in \mathcal{C}_N\) one has \(\pi_{\Omega}(g)\).

Such a state is separable with respect to the bipartition \((\mathcal{A}_1, \mathcal{A}_2)\) defined above. Indeed, as already mentioned, the elements \(\alpha^{(1)} \in \mathcal{A}_1\) and \(\alpha^{(2)} \in \mathcal{A}_2\) have the generic form:

\[
\alpha^{(1)} = \sum_{n_1,n_2,\ldots,n_p} \alpha^{(1)}_{n_1,n_2,\ldots,n_p} (c_1)^{n_1} (c_2)^{n_2} \ldots (c_N)^{n_N},
\]

\[
\alpha^{(2)} = \sum_{n_{p+1},n_{p+2},\ldots,n_N} \alpha^{(2)}_{n_{p+1},n_{p+2},\ldots,n_N} (c_{p+1})^{n_{p+1}} (c_{p+2})^{n_{p+2}} \ldots (c_N)^{n_N},
\]

with \(n_i = 0, 1\) and coefficients \(\alpha^{(1)}_{n_1,n_2,\ldots,n_p}\) \(\alpha^{(2)}_{n_{p+1},n_{p+2},\ldots,n_N} \in \mathbb{C}\). Then,

\[
\Omega(\alpha^{(1)}\alpha^{(2)}) = \alpha^{(1)}\alpha^{(2)} = \Omega(\alpha^{(1)})\Omega(\alpha^{(2)}).
\]

Remark. Due to the anticommutative character of the Clifford modes, \(c_ic_j = -c_jc_i\), for \(i \neq j\), the restrictions on the form of the states of a fermion algebra given in lemma 2 hold also for the Clifford algebra \(\mathcal{C}_N\), as the entanglement criterion given in corollary 3.

A basis in the Hilbert space \(\mathcal{H}_{\Omega}\) can be obtained by applying the basis elements in \((29)\) to the cyclic vector:

\[|\mu_1, \mu_2, \ldots, \mu_M\rangle = |\pi_{\Omega}(c_1)^{\mu_1}\pi_{\Omega}(c_2)^{\mu_2} \ldots |\pi_{\Omega}(c_N)^{\mu_N}\rangle|\Omega\rangle;\]

these vectors are clearly orthogonal among themselves thanks to \((33)\). This basis contains \(2^N\) vectors so that the GNS representation \(\pi_{\Omega}\) turns out to be highly reducible. For instance, when \(N = 2n\), any element \(\gamma \in \mathcal{C}_N\) will be represented in \(\pi_{\Omega}\) by a \(2^n \times 2^n\) matrix, i.e. by elements of \(\mathcal{M}_{2^n}\) while, as explicitly shown by \((30)\), \(\mathcal{C}_N\) is isomorphic to \(\mathcal{M}_{2^n}\).

In order to study separability and entanglement in the case of reducible GNS representations, one needs to generalize the treatment presented in section 3, which is appropriate only for irreducible GNS representations, as the Fock representation used to discuss standard fermions. We shall see that reducibility allows for richer structures in the classification scheme of entangled Majorana states.

11 For even \(N\) even, this state corresponds to a thermal state for the algebra \(\mathcal{A}_1\) generated by the complex fermion modes in \((32)\), in the limit of infinite temperature.
6. Reducible GNS representations

In order to properly treat reducible GNS representations, one needs to generalize the algebraic approach to quantum systems presented in section 2. As we have seen, for any quantum system defined by the operator algebra \( \mathcal{A} \) and a state \( \Omega \), the GNS construction allows the building of a triple \((\mathcal{H}_\Omega, \pi_\Omega, \{|\Omega\rangle\})\), so that the system can be described in terms of bounded operators \( \mathcal{B}(\mathcal{H}_\Omega) \) acting on the Hilbert space \( \mathcal{H}_\Omega \) spanned by the (completion of the) set of vectors \( \{\pi_\Omega(\mathcal{A})|\Omega\rangle\}\).

When the representation \( \pi_\Omega \) is not irreducible, as for any algebra representation, it can always be decomposed into irreducible representations \( \pi^{(\mu,r)} \):

\[
\pi_\Omega = \bigoplus_{\mu,r} \pi^{(\mu,r)}. \tag{37}
\]

Two indices \( \mu \) and \( r \) will be used to label such representations: the Greek index \( \mu \) distinguishes among different irreducible representations, while the Latin index \( r \) counts the multiplicity of a given irreducible representation. In other words, \( \pi^{(\mu,r)} \) and \( \pi^{(\nu,s)} \), with \( \mu \neq \nu \), are different irreducible representations, while \( \pi^{(\mu,r)} \) and \( \pi^{(\mu,s)} \), with \( r \neq s \), are two copies of the same irreducible representation \( \pi^{(\mu)} \). We shall call \( d_\mu \) and \( m_\mu \) the dimension and the multiplicity of \( \pi^{(\mu)} \).

Remark. Notice that, contrary to the usual convention, in the decomposition (37) unitarily equivalent representations are treated as distinct. This is necessary to discuss quantum separability issues, since unitary transformations might map a given bipartition \( (A_1, A_2) \) of \( A \) into a different one \([50, 57]\).

To the decomposition of representations, as in (37), there corresponds a similar decomposition of the Hilbert space \( \mathcal{H}_\Omega \):

\[
\mathcal{H}_\Omega = \bigoplus_{\mu,r} \mathcal{H}_\Omega^{(\mu,r)}, \tag{38}
\]

so that for any element \( \alpha \in \mathcal{A} \), the operator \( \pi^{(\mu,r)}(\alpha) \) acts non-trivially only on the irreducible subspace \( \mathcal{H}_\Omega^{(\mu,r)} \). Let \( \{|e_i^{(\mu,r)}\rangle\}_{i=1,2,\ldots,d_\mu} \) be a set of elements of \( \mathcal{H}_\Omega \) forming an orthonormal basis for the subspace \( \mathcal{H}_\Omega^{(\mu,r)} \). Since the GNS representation \( \pi^{(\mu,r)} \) is irreducible, these states are pure, as any element of this subspace, and the whole \( \mathcal{H}_\Omega^{(\mu,r)} \) can be obtained by applying the operators \( \pi_\Omega(\mathcal{A}) \) to the normalized cyclic vector:

\[
|\Omega^{(\mu,r)}\rangle = \frac{1}{\sqrt{N^{(\mu,r)}}} \sum_i (e_i^{(\mu,r)}|\Omega\rangle|e_i^{(\mu,r)}\rangle), \quad N^{(\mu,r)} = \sum_i |\langle e_i^{(\mu,r)}|\Omega\rangle|^2. \tag{39}
\]

On the other hand, a generic element in the full Hilbert space \( \mathcal{H}_\Omega \) turns out to generally be a mixed state when restricted to the operator algebra \( \pi_\Omega(\mathcal{A}) \)[12]. Indeed, any normalized state \( |\psi\rangle \in \mathcal{H}_\Omega \) can be expanded using the collection of basis elements \( \{|e_i^{(\mu,r)}\rangle\} \) introduced above as:

\[
|\psi\rangle = \sum_{\mu,r,i} \langle e_i^{(\mu,r)}|\psi\rangle |e_i^{(\mu,r)}\rangle, \quad \sum_{\mu,r,i} |\langle e_i^{(\mu,r)}|\psi\rangle|^2 = 1. \tag{40}
\]

Further, thanks to the irreducibility of the representations \( \pi^{(\mu,r)} \), one has:

\[
\langle e_1^{(\mu,r)}|\pi_\Omega(\alpha)|e_j^{(\mu,r)}\rangle = \delta_{\mu,\nu} \delta_{r,s} \langle \pi_\Omega^{(\mu)}(\alpha)|e_{ij}\rangle, \tag{41}
\]

[12] Although it is surely a pure state for the full algebra \( \mathcal{B}(\mathcal{H}_\Omega) \) of bounded operators on \( \mathcal{H}_\Omega \).
where $[\pi^{(\mu)}_{\Omega}(\alpha)]_{ij}$ is the matrix representation of the element $\alpha \in \mathcal{A}$ in the irreducible representation $\pi^{(\mu)}_{\Omega}$. In fact, recall that $\pi^{(\mu)}_{\Omega}$, with $r = 1, 2, \ldots, m_\mu$, are all copies of the same representation $\pi^{(\mu)}_{\Omega}$, thus, the matrix elements

$$[\pi^{(\mu)}_{\Omega}(\alpha)]_{ij} \equiv \langle e^{(\mu,r)}_i | \pi_{\Omega}(\alpha) | e^{(\mu,r)}_j \rangle,$$

are actually independent from the multiplicity index $r$, or equivalently, the representation matrix $[\pi^{(\mu)}_{\Omega}(\alpha)]$ is the same for all the $m_\mu$ copies $\pi^{(\mu)}_{\Omega}$, $r = 1, 2, \ldots, m_\mu$. As a consequence, the mean value of any element $\alpha \in \mathcal{A}$ with respect to the state $|\psi\rangle$ can be represented by means of a density matrix $\rho_\psi$, using the trace operation:

$$\langle \psi | \pi_{\Omega}(\alpha) | \psi \rangle = \text{Tr}[\rho_\psi \pi_{\Omega}(\alpha)],$$

where

$$\rho_\psi = \sum_{\mu} \sum_{y} \lambda^{(\mu)}_{y} | e^{(\mu)}_y \rangle \langle e^{(\mu)}_y |,$$

with $\lambda^{(\mu)}_{y} = \sum_r \langle \psi | e^{(\mu,r)}_y \rangle \langle e^{(\mu,r)}_x | \psi \rangle$, and $| e^{(\mu)}_y \rangle$ any basis in $\mathcal{H}_\Omega$ carrying the irreducible representation $\pi^{(\mu)}_{\Omega}$; in practice, a convenient choice for $| e^{(\mu)}_y \rangle$ is the basis $| e^{(\mu,r)}_y \rangle$ in $\mathcal{H}^{(\mu,r)}_{\Omega}$, with any fixed index $r$ since, as remarked upon above, each one of these spaces carries the same irreducible representation $\pi^{(\mu)}_{\Omega}$ of $\mathcal{A}$.

In particular, the cyclic GNS vectors $|\Omega\rangle$ turns out to be represented by the density matrix $\rho_{\Omega}$ of the general form (43), with $\lambda^{(\mu)}_{y} = \sum_r \langle \Omega | e^{(\mu,r)}_y \rangle \langle e^{(\mu)}_x | \Omega \rangle$, so that, for any $\alpha \in \mathcal{A}$,

$$\Omega(\alpha) = \text{Tr}[\rho_{\Omega} \pi_{\Omega}(\alpha)].$$

Similarly, any mixed state $\rho$ on $\mathcal{H}_\Omega$, which can generally be decomposed as

$$\rho = \sum_{\mu,r,s} \lambda^{(\mu,r,s)}_{y} | e^{(\mu,r)}_y \rangle \langle e^{(\mu,s)}_x |,$$

when restricted to the algebra $\pi_{\Omega}(\mathcal{A})$, also reduces to the generic form (43).

There are, however, notable exceptions to this general rule. Let us fix the irreducible representation $\pi^{(\mu)}_{\Omega}$ and consider the following linear combination in $\mathcal{H}_\Omega$:

$$| f^{(\mu,r)}_i \rangle = \sum_{s,j} U_{ij} V_{sj} | e^{(\mu,s)}_j \rangle,$$

with $U$ and $V$ unitary matrices and $| e^{(\mu,s)}_j \rangle$ the orthonormal basis in $\mathcal{H}^{(\mu,s)}_{\Omega}$ and $r = 1, 2, \ldots, m_\mu$, the $m_\mu$ Hilbert subspaces carrying the representation $\pi^{(\mu)}_{\Omega}$. When restricted to $\pi_{\Omega}(\mathcal{A})$, the vectors $| f^{(\mu,r)}_i \rangle$ behave as the linear combinations $| e^{(\mu,r)}_j \rangle = \sum_j V_{sj} | e^{(\mu,s)}_j \rangle$, since, due to the identity (41) above, the following matrix elements:

$$\langle f^{(\mu,r)}_i | \pi_{\Omega}(\alpha) | f^{(\mu,r)}_j \rangle = \sum_{k} U_{ik} U_{jk} V_{kj} V_{ji} \langle e^{(\mu,p)}_k | \pi_{\Omega}(\alpha) | e^{(\mu,q)}_i \rangle = \sum_{k} V_{kj} V_{ji} \langle \pi^{(\mu)}_{\Omega}(\alpha) | e^{(\mu,q)}_i \rangle \langle e^{(\mu,p)}_k | \pi^{(\mu)}_{\Omega}(\alpha) \rangle,$$

are actually independent from the index $r$. Being combinations of basis states, the vectors in $| e^{(\mu,s)}_j \rangle$ $i = 1, 2, \ldots, d_\mu$ are pure, forming another orthonormal basis in $\mathcal{H}^{(\mu,s)}_{\Omega}$, as a consequence, also the more general combinations $| f^{(\mu,r)}_i \rangle$ in (46) represent pure states on the subalgebra $\pi^{(\mu,r)}_{\Omega}(\mathcal{A})$. This result will be important for the discussions that will follow.

Remark. The vectors in (46) forming the set $| f^{(\mu,r)}_i \rangle$ $i = 1, 2, \ldots, d_\mu$ are clearly orthonormal and, as shown by (47), span an invariant subspace of $\mathcal{H}_\Omega$, which is, however,
different from $\mathcal{H}_{\Omega}^{(\mu,r)}$, it carries a representation of $\mathcal{A}$ unitarily equivalent to $\pi_{\Omega}^{(\mu)}$, coinciding with it only when $V = 1$. This means that the partial decomposition $\oplus_r \mathcal{H}_{\Omega}^{(\mu,r)}$ into subspaces carrying the representation $\pi_{\Omega}^{(\mu)}$ is, in general, not unique\(^{13}\).

7. Reducibility and entanglement

When a state $\Omega$ for the operator algebra $\mathcal{A}$ gives rise to a reducible GNS representation $\pi_{\Omega}$, the analysis of the notions of separability and entanglement according to definition 3 in section 3 becomes more involved. Following the previous discussion, one can decompose $\pi_{\Omega}$ into its irreducible components

$$\pi_{\Omega} = \oplus_{\mu,r} \pi_{\Omega}^{(\mu,r)},$$

(48)

where $\pi_{\Omega}^{(\mu,r)}$, for $\mu$ fixed and $r = 1, 2, \ldots, m_{\mu}$, are $m_{\mu}$ copies of the same irreducible representation $\pi_{\Omega}^{(\mu)}$. Correspondingly, one has a similar decomposition for the GNS Hilbert space, $\mathcal{H}_{\Omega} = \oplus_{\mu,r} \mathcal{H}_{\Omega}^{(\mu,r)}$, where, for $\mu$ fixed, the $m_{\mu}$ subspaces $\mathcal{H}_{\Omega}^{(\mu,r)}$ are all isomorphic, and, without loss of generality, they can be identified.

Let us now fix a bipartition $(A_1, A_2)$ of $\mathcal{A}$ and consider an orthonormal basis $\{|e_{i}^{(\mu,r)}\rangle\}$ in each Hilbert space $\mathcal{H}_{\Omega}^{(\mu,r)}$; these states are pure since they carry the irreducible representation $\pi_{\Omega}^{(\mu)}$. In addition, they can be chosen to be separable:

**Lemma 3.** Given any bipartition $(A_1, A_2)$ of the algebra $\mathcal{A}$, and a separable state $\Omega$ leading to the reducible representation $\pi_{\Omega}$ with decomposition as in (37) and (38), it is always possible to select in $\mathcal{H}_{\Omega}^{(\mu,r)}$ an orthonormal basis $\{|e_{i}^{(\mu,r)}\rangle\}$ of separable pure states.

**Proof.** The statement can be proven by explicitly constructing the basis. For simplicity, we shall consider $\mathcal{A}$ to be a boson algebra; however, using the techniques presented in section 4, the proof can be easily extended to the fermion case. The building procedure involves selecting two self-adjoint elements $\alpha_1 \in A_1$ and $\alpha_2 \in A_2$, $\alpha_i^* = \alpha_i$ in the two partitions. On the space $\mathcal{H}_{\Omega}^{(\mu,r)}$, these elements are represented by the Hermitian operators $\pi_{\Omega}^{(\mu)}(\alpha_i), i = 1, 2$, with spectral decomposition:

$$\pi_{\Omega}^{(\mu,r)}(\alpha_1) = \sum_k \alpha_1(1)^k P_k^{(\mu,r)}, \quad \pi_{\Omega}^{(\mu,r)}(\alpha_2) = \sum_{\ell} \alpha_2(2)^\ell Q_\ell^{(\mu,r)}, \quad \alpha_1(1), \alpha_2(2) \in \mathbb{R}. \quad (49)$$

Since the GNS vector $|\Omega\rangle \in \mathcal{H}_{\Omega}$ is assumed to be separable, by acting on it with the projectors $P_\ell^{(\mu,r)} \subseteq \pi_{\Omega}^{(\mu)}(A_1)$ and $Q_\ell^{(\mu,r)} \subseteq \pi_{\Omega}^{(\mu)}(A_2)$ one builds a basis of manifestly separable pure states of the form\(^{14}\)

$$|e_{i}^{(\mu,r)}\rangle = \frac{1}{\langle \Omega | P_\ell^{(\mu,r)} | \Omega \rangle \langle \Omega | Q_\ell^{(\mu,r)} | \Omega \rangle} P_\ell^{(\mu,r)} Q_\ell^{(\mu,r)} |\Omega\rangle, \quad i \equiv (k, \ell). \quad (50)$$

These states satisfy the separability condition (8) and thus, by taking in $\alpha_1 = \alpha_2 = 1$, they are orthonormal:

\(^{13}\) This might have some consequences when evaluating the von Neumann entropy of the state $\Omega$ [92, 93].

\(^{14}\) Here, we are assuming that the projectors $P_\ell^{(\mu,r)}$ and $Q_\ell^{(\mu,r)}$ correspond to elements belonging to the algebra $\mathcal{A}$ and, as we shall see, this is indeed the case when $\mathcal{A}$ is a Clifford algebra. However, in more general cases, this condition might not hold; in such instances, one simply applies all considerations to the von Neumann algebra extension of $\mathcal{A}$ [59].
\[
\langle e_1^{(\mu,r)} | e_j^{(\nu,s)} \rangle = \frac{\langle \Omega | P_k^{(\mu,r)} P_k^{(\nu,s)} \Omega \rangle \langle \Omega | Q_\ell^{(\mu,r)} Q_\ell^{(\nu,s)} \Omega \rangle}{\langle \Omega | P_k^{(\mu,r)} | \Omega \rangle \langle \Omega | Q_\ell^{(\mu,r)} | \Omega \rangle} = \delta_{\mu,1} \delta_{\nu,j} \equiv \delta_{ij}.
\]

Furthermore, the set of vectors (50) form a basis for the space \( \mathcal{H}_{\Omega}^{(\mu,r)} \). Indeed, the existence of an element \( \psi \in \mathcal{H}_{\Omega}^{(\mu,r)} \) not belonging to the span of the set \( \{ |e_i^{(\mu,r)}\rangle \} \) would be in contradiction with the assumption that \( \mathcal{A}_1 \cup \mathcal{A}_2 = \mathcal{A} \); in fact, by construction, the set \( \{ P_k^{(\mu,r)} \} \) generates \( \pi_{\Omega}^{(\mu,r)}(\mathcal{A}_1) \), while \( \{ Q_\ell^{(\mu,r)} \} \) generates \( \pi_{\Omega}^{(\mu,r)}(\mathcal{A}_2) \).

\[\Box\]

Remark. If some of the above projectors turn out to annihilate \( |\Omega\rangle \), e.g. \( P_k^{(\mu,r)} |\Omega\rangle = 0 \), for some \( k \), one considers the subalgebra generated by them and repeats the previous construction by choosing a suitable self-adjoint element in it. For a finitely generated algebra \( \mathcal{A} \), the successive application of this procedure will surely come to an end and provide the wanted separable basis \( \{ |e_i^{(\mu,r)}\rangle \} \).

Having constructed in each space \( \mathcal{H}_{\Omega}^{(\mu,r)} \) a basis \( \{ |e_i^{(\mu,r)}\rangle \} \) of vectors that are separable with respect to the chosen bipartition, one can now consider arbitrary linear combinations of these vectors. In general, such combinations will no longer be separable. For instance, even limiting our attention to a single space \( \mathcal{H}_{\Omega}^{(\mu,r)} \), with fixed indices \( \mu \) and \( r \), the following combinations of vectors \( |e_i^{(\mu,r)}\rangle = \sum_j V_j |e_j^{(\mu,r)}\rangle \), with \( V_j \) arbitrary complex coefficients, are still pure states in \( \mathcal{H}_{\Omega}^{(\mu,r)} \), as discussed in the previous section; however, they are no longer separable, since in general the expectation \( \langle \tilde{e}_i^{(\mu,r)} | \pi_{\Omega}^{(\mu,r)}(\mathcal{A}_1) | \tilde{e}_j^{(\mu,r)} \rangle \langle \tilde{e}_j^{(\mu,r)} | \pi_{\Omega}^{(\mu,r)}(\mathcal{A}_2) | \tilde{e}_i^{(\mu,r)} \rangle \) of any local operator \( \alpha_1 \alpha_2, \alpha_i \in \mathcal{A}_i, i = 1, 2 \), cannot be written in product form as in (8).

Nevertheless, there are linear combinations involving basis vectors in spaces \( \mathcal{H}_{\Omega}^{(\mu,r)} \) with a different index \( r \) that remain separable.

Lemma 4. Within the hypothesis of the previous lemma, let us consider the following linear combinations of basis states:

\[
|g_j^{(\mu,r)}\rangle = \sum_s U_{js} |e_s^{(\mu,r)}\rangle,
\]

with \( U \) a unitary matrix. These states are pure and separable.

\[\textbf{Proof.}\] As already shown, the matrix elements of any operator \( \pi_{\Omega}(\alpha) \), \( \alpha \in \mathcal{A} \), with respect to the vectors of the set \( \{ |e_i^{(\mu,r)}\rangle \} \) coincide with those of the corresponding vectors in the set \( \{ |e_i^{(\mu,r)}\rangle \} \), since both set of vectors carry the same irreducible representation \( \pi_{\Omega}^\mu \) of \( \mathcal{A} \) (see (47) with \( V = 1 \)). Then, since the separability condition (8) holds by construction for the elements of the basis \( \{ |e_i^{(\mu,r)}\rangle \} \), it is automatically also true for the vectors in \( \{ |g_j^{(\mu,r)}\rangle \} \).

\[\Box\]

More generally, a state on the algebra \( \mathcal{A} \) that belongs to the folium of \( \mathcal{H}_{\Omega} \) is mixed, and thus represented by a density matrix \( \rho \) on \( \mathcal{H}_{\Omega} \). It can be decomposed as in (43):

\[
\rho = \sum_{\mu,j} \lambda_j^{(\mu)} |e_j^{(\mu)}\rangle \langle e_j^{(\mu)}|, \quad \sum_{\mu,j} \lambda_j^{(\mu)} = 1,
\]

where the set \( \{ |e_j^{(\mu)}\rangle \} \) is a separable basis in \( \mathcal{H}_{\Omega} \) carrying the irreducible representation \( \pi_{\Omega}^{(\mu)} \); in practice, as mentioned earlier, it can be taken to coincide with any separable basis \( \{ |e_i^{(\mu,r)}\rangle \} \) in \( \mathcal{H}_{\Omega}^{(\mu,r)} \) (introduced above) with arbitrary, but fixed \( r \).
It follows that a state in diagonal form,
\[ \rho = \sum_{\mu} \lambda^{(\mu)} \left| e^{(\mu)}_i \right\rangle \left\langle e^{(\mu)}_i \right|, \]
is surely separable, being the convex combination of separable, rank-1 projectors. One can then conclude that:

**Proposition 3.** A generic mixed state \( \rho = \sum_{\mu} \rho^{(\mu)} \) as in (53) is entangled with respect to the given bipartition \( (\mathcal{A}_1, \mathcal{A}_2) \), if and only if at least one of its irreducible components \( \rho^{(\mu)} \),
\[ \rho^{(\mu)} = \frac{1}{\lambda^{(\mu)}} \sum_{ij} \lambda^{(\mu)}_{ij} \left| e^{(\mu)}_i \right\rangle \left\langle e^{(\mu)}_j \right|, \quad \lambda^{(\mu)} \equiv \sum_i \lambda^{(\mu)}_{ii}, \]
turns out to be non-separable.

As a consequence, the study of quantum correlations in the reducible representation \( \pi_\Omega \) of the algebra \( \mathcal{A} \), as given by the state \( \Omega \) through the GNS construction, reduces to the analysis of entanglement in each of its irreducible components \( \pi^{(\mu)}_\Omega \), for which the results given in section 3 apply.

All the above discussion can be made very explicit in the case of Majorana fermion systems, i.e. when the operator algebra \( \mathcal{A} \) coincides with the Clifford algebra \( \mathcal{C}_N \) and for \( \Omega \) the state introduced in (33) is chosen.

8. Structure of entangled Majorana states: \( C_2 \)

We shall start by discussing the simplest Majorana system, the one defined by the operator algebra \( \mathcal{C}_2 \), generated by the two mode elements \( c_1 \) and \( c_2 \). As discussed in section 5, the entire Clifford algebra \( \mathcal{C}_2 \) is then obtained as the linear span of the following four basis elements: \{1, \( c_1 \), \( c_2 \), \( c_1c_2 \)\}. As state \( \Omega \) on this algebra, we shall choose the one given in (33), so that: \( \Omega(c_1) = \Omega(c_2) = \Omega(c_1c_2) = 0 \), while \( \Omega(1) = 1 \).

Given the state \( \Omega \), the GNS construction provides a representation \( \pi_\Omega \) of \( \mathcal{C}_2 \) on a four-dimensional Hilbert space \( \mathcal{H}_\Omega \), which is given by the linear span of the four vectors obtained by applying the basis elements \( 1, c_1, c_2 \) and \( c_1c_2 \) to the cyclic vector \( \left| \Omega \right\rangle \)\(^{15}\). Since, as discussed in section 5, \( \mathcal{C}_2 \) is isomorphic to \( \mathcal{M}_2(\mathbb{C}) \), the algebra of \( 2 \times 2 \) complex matrices, the four-dimensional GNS representation \( \pi_\Omega \) is reducible. This means it can be decomposed as
\[ \pi_\Omega = \pi^{(1)}_\Omega \oplus \pi^{(2)}_\Omega, \]
in terms of two, equal, two-dimensional representations \( \pi^{(r)}_\Omega \), acting on two-dimensional Hilbert subspaces \( \mathcal{H}^{(r)}_\Omega \), \( r = 1, 2 \) such that \( \mathcal{H}_\Omega = \mathcal{H}^{(1)}_\Omega \oplus \mathcal{H}^{(2)}_\Omega \)\(^{16}\). As a consequence, the chosen state \( \Omega \) is not a pure state for \( \mathcal{C}_2 \).

The only non-trivial bipartition \( (\mathcal{A}_1, \mathcal{A}_2) \) of the algebra \( \mathcal{C}_2 \) is the one in which the subalgebra \( \mathcal{A}_1 \) is the linear span of \{1, \( c_1 \)\}, while the subalgebra of \( \mathcal{A}_2 \) is that of \{\( c_2 \)\}. For the construction of two orthonormal bases in \( \mathcal{H}^{(r)}_\Omega \) formed by separable pure states we follow the general scheme outlined in the proof of lemma 3 in the previous section. The procedure involves choosing two generic self-adjoint elements, \( \alpha = a_0 + a_1 c_1 \) in \( \mathcal{A}_1 \) and

\(^{15}\) For the sake of simplicity, here and in the following, we shall use the same symbol to indicate the elements \( c_i \) of the abstract Clifford algebra and its corresponding GNS representation \( \pi_\Omega(c_i) \) as operators acting on \( \mathcal{H}_\Omega \).

\(^{16}\) In this situation, all irreducible representations are equal, so that the index \( r \) takes only one value and can be suppressed; thus in the decomposition (56) only the multiplicity index \( r = 1, 2 \) appears.
\[ \beta = b_0 + b_1 c_2 \in \mathbb{A}_2, \text{ with } a_i, b_i \in \mathbb{R}. \]  
Their spectral decomposition,

\[ \alpha = (a_0 + a_1)P_+ + (a_0 - a_1)P_- \quad \text{and} \quad P_\pm = (1 \pm c_1)/2, \]

\[ \beta = (b_0 + b_1)Q_+ + (b_0 - b_1)Q_- \quad \text{and} \quad Q_\pm = (1 \pm c_2)/2, \]  
allows the construction of the following four orthonormal vectors:

\[ |e_i^{(r)}\rangle = \frac{1}{2} \hat{e}_i^{(r)}(c_1, c_2)|\Omega\rangle, \quad r = 1, 2, \quad i = 1, 2, \]  
where \( \hat{e}_i^{(r)}(c_1, c_2) \) are (suitably normalized) products of the projectors \( P_\pm \) and \( Q_\pm \):

\[ \begin{cases} 
\hat{e}_1^{(1)}(c_1, c_2) = (1 + c_1)(1 + c_2) \\
\hat{e}_1^{(2)}(c_1, c_2) = (1 + c_1)(1 - c_2) \\
\hat{e}_2^{(1)}(c_1, c_2) = (1 - c_1)(1 + c_2) \\
\hat{e}_2^{(2)}(c_1, c_2) = (1 - c_1)(1 - c_2). 
\end{cases} \]  

One can easily check that the set \( \{|e_i^{(r)}\rangle \mid i = 1, 2\} \) is a basis for the subspace \( \mathcal{H}_{12} \subset \mathcal{H}_\Omega \) carrying the irreducible representation \( \pi_{12}^{(1)} \) for which

\[ \begin{array}{ccc}
1 & \rightarrow & \sigma_0 \\
c_1 & \rightarrow & \sigma_3 \\
c_2 & \rightarrow & \sigma_1, 
\end{array} \]  
and consequently \( c_1c_2 \rightarrow i\sigma_2 \). Similarly, the set \( \{|e_i^{(2)}\rangle \mid i = 1, 2\} \) is a basis in \( \mathcal{H}_{12}^{(2)} \subset \mathcal{H}_\Omega \), carrying another copy of the same irreducible representation. In view of this, as discussed before, the four states \( |e_i^{(r)}\rangle \) are pure. Furthermore, they are manifestly separable with respect to the given bipartition; indeed, one can explicitly check that they satisfy the condition (8) for any local operator in \( \mathcal{C}_2 \).

The decomposition \( \mathcal{H}_\Omega = \mathcal{H}_{12}^{(1)} \oplus \mathcal{H}_{12}^{(2)} \) is, however, not unique due to the fact that the representation (60) has multiplicity 2. In fact, as discussed in the final remark of section 6, the linear combinations

\[ |g_i^{(r)}\rangle = \sum_{s=1}^{2} U_{gs} |e_i^{(r)}\rangle, \]  
with \( U \) unitary, define two orthonormal bases \( \{|g_i^{(1)}\rangle \mid i = 1, 2\} \) and \( \{|g_i^{(2)}\rangle \mid i = 1, 2\} \), spanning two subspaces \( \tilde{\mathcal{H}}_{12}^{(r)} \subset \mathcal{H}_\Omega \) giving a new decomposition \( \mathcal{H}_\Omega = \tilde{\mathcal{H}}_{12}^{(1)} \oplus \tilde{\mathcal{H}}_{12}^{(2)} \) of the GNS Hilbert space. However, as already noticed in the general case, the irreducible representations \( \pi_i^{(r)} \) of \( \mathcal{C}_2 \) corresponding to this new decomposition of \( \mathcal{H}_\Omega \) coincide with old ones, i.e. one has \( \pi_i^{(1)} = \pi_i^{(2)} = \pi_i^{(1)} = \pi_i^{(2)} \). In addition, by lemma 4, the new pure basis vectors \( |g_i^{(r)}\rangle \) still represent separable states.

These considerations became very explicit by taking for instance \( U = (c_1 + c_3)/\sqrt{2} \), so that the two basis states result in:

\[ \begin{cases} 
|g_1^{(1)}\rangle = \frac{1}{\sqrt{2}} (1 + c_1)\Omega \\
|g_1^{(2)}\rangle = \frac{1}{\sqrt{2}} (1 + c_1)c_2\Omega \\
|g_2^{(1)}\rangle = \frac{1}{\sqrt{2}} (1 - c_1)c_2\Omega \\
|g_2^{(2)}\rangle = \frac{1}{\sqrt{2}} (1 - c_1)\Omega. 
\end{cases} \]  

These states are manifestly separable and give rise to two copies of the same matrix representation of \( \mathcal{C}_2 \) given in (60).

On the other hand, if one instead considers, as in (46), linear unitary combinations of the vectors \( |e_i^{(r)}\rangle \) also involving the lower index.
the resulting sets of vectors \( \{ |f_{i}^{(1)}\rangle \mid i = 1, 2 \} \) and \( \{ |f_{i}^{(2)}\rangle \mid i = 1, 2 \} \) are still a basis carrying the \( C_{2} \) irreducible representations \( V_{\Omega(r)} V_{\dagger} \), \( r = 1, 2 \), unitarily equivalent to the original ones, but the pure states \( |f_{i}^{(r)}\rangle \) are, in general, no longer separable. An interesting example, which will become useful in the following, is given by the choice

\[
U = \frac{1}{2} \begin{pmatrix} 1 - i & 1 + i \\ 1 + i & 1 - i \end{pmatrix}, \quad V = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix},
\]

giving rise to the basis vectors

\[
|f_{i}^{(r)}\rangle = \frac{1}{\sqrt{2}} f_{i}^{(r)}(c_{1}, c_{2}) |\Omega\rangle, \quad r = 1, 2, \quad i = 1, 2,
\]

with

\[
\begin{align*}
&f_{1}^{(1)}(c_{1}, c_{2}) = (c_{1} + c_{2}) \\
&f_{2}^{(1)}(c_{1}, c_{2}) = (1 + i c_{1} c_{2}) \\
&f_{1}^{(2)}(c_{1}, c_{2}) = (1 - i c_{1} c_{2}) \\
&f_{2}^{(2)}(c_{1}, c_{2}) = (c_{1} - c_{2})
\end{align*}
\]

Using the entanglement criterion given in corollary 3, one immediately sees that the vectors (65) are non-separable, since \( \langle f_{i}^{(r)}|c_{1}c_{2}|f_{i}^{(r)}\rangle = 0 \).

Coming now to the GNS state \( |\Omega\rangle \), one can easily see that, although generating the whole Hilbert space \( H_{\Omega} \), it is not a pure state on the Clifford algebra \( C_{2} \). In fact, recalling (62), one can write:

\[
|\Omega\rangle = \frac{1}{\sqrt{2}} (|g_{1}^{(1)}\rangle + |g_{2}^{(2)}\rangle).
\]

Since \( \langle g_{i}^{(1)}|\alpha|g_{j}^{(2)}\rangle = 0 \), \( i, j = 1, 2 \), for any element \( \alpha \in C_{2} \), due to the irreducibility of the representations carried by \( \{ |g_{i}^{(1)}\rangle \} \) and \( \{ |g_{i}^{(2)}\rangle \} \), the mean value \( \Omega(\alpha) \) can be expressed in terms of a density matrix \( \rho_{\Omega} \) such that

\[
\langle \Omega|\alpha|\Omega\rangle = \text{Tr}[\rho_{\Omega} \alpha], \quad \forall \alpha \in C_{2},
\]

with

\[
\rho_{\Omega} = \frac{1}{2} (|g_{1}^{(1)}\rangle \langle g_{1}^{(1)}| + |g_{2}^{(2)}\rangle \langle g_{2}^{(2)}|);
\]

the state \( |\Omega\rangle \) is therefore a mixed state when restricted to \( C_{2} \). In addition, being a convex combination of projectors onto separable states, \( \rho_{\Omega} \) is itself, as already observed in section 5 (cf equation (35)). A similar conclusion also holds for the states \( c_{1}|\Omega\rangle \), \( c_{2}|\Omega\rangle \) and \( c_{1}c_{2}|\Omega\rangle \) that together with \( |\Omega\rangle \) generate the whole GNS Hilbert space \( H_{\Omega} \); one easily finds that, when restricted to the algebra \( C_{2} \), also these three states are represented by separable density matrices.

More generally, any state on \( C_{2} \) can be represented by a density matrix that, following (53), can be written in the form

\[
\rho = \sum_{i,j} \lambda_{ij} |e_{i}\rangle \langle e_{j}|, \quad \sum_{i} \lambda_{ii} = 1,
\]

where \( \{ |e_{i}\rangle \mid i = 1, 2 \} \) is any separable basis carrying a two-dimensional representation of \( C_{2} \); in particular, one can choose one of the two bases given in (58) and (59).
In order to characterize its entanglement properties, one has to distinguish the cases in which the coefficient $\lambda_{12}$ is complex or real. In the first case, one has:

**Lemma 5.** The density matrix $\rho$, as in (70), with $\lambda_{12}$ a non-vanishing complex number is never separable.

**Proof.** The density matrix in (70) can be decomposed into its diagonal part

$$\rho_D = \sum \lambda_i |e_i\rangle \langle e_i|,$$

and the off-diagonal one $\eta \equiv \rho - \rho_D$. While $\rho_D$ is clearly a separable state, $\eta$ being the difference of two density matrices is not even a state. However, for $\lambda_{12} \in \mathbb{C}$, the quantity $\text{Tr}[\eta c_{12}] \equiv \text{Tr}[\rho c_{12}] = 2i \text{Im}(\lambda_{12})$ is non-vanishing, so that by the criterion of corollary 3, $\rho$ is surely entangled. $\square$

When $\lambda_{ij}$ is a real matrix, the situation is more involved, since $\text{Tr}[\rho c_{12}]$ is always zero and the entanglement criterion in corollary 3 gives no information: one has then to resort to the fact that separable mixed states are convex combinations of pure separable states.

**Lemma 6.** The density matrix $\rho$, as in (70), with $\lambda_{12} \in \mathbb{R}$ is separable if and only if $\lambda_{11} \geq |\lambda_{12}|$ and $\lambda_{22} \geq |\lambda_{12}|$.

**Proof.** First consider a pure state $|\psi\rangle = a_1 |e_1\rangle + a_2 |e_2\rangle$, with $|a_1|^2 + |a_2|^2 = 1$; one computes

$$\langle \psi | e_1 c_2 | \psi \rangle = 2 i \text{Im}(a_1 a_2^*), \quad \langle \psi | e_1 | \psi \rangle = |a_1|^2 - |a_2|^2, \quad \langle \psi | c_2 | \psi \rangle = 2 \text{Re}(a_1 a_2^*).$$

Using the separability condition (8) of lemma 1, it follows that $|\psi\rangle$ is a separable pure state with respect to the considered bipartition if and only if, together with $\text{Im}(a_1 a_2^*) = 0$, at least one of the following two conditions is satisfied: $|a_1|^2 = |a_2|^2$ or $\text{Re}(a_1 a_2^*) = 0$. Therefore, the only separable pure states are:

$$|\psi_1\rangle = |e_1\rangle, \quad |\psi_2\rangle = |e_2\rangle, \quad |\psi_3\rangle = \frac{|e_1\rangle + |e_2\rangle}{\sqrt{2}}, \quad |\psi_4\rangle = \frac{|e_1\rangle - |e_2\rangle}{\sqrt{2}}.$$

Take now a generic mixed, separable state $\rho$ that can be expressed as in (70); it must be obtainable as a convex combination of the projectors onto the above separable pure states, and therefore must be of the form

$$\rho = \sum_{i=1}^4 \mu_i |\psi_i\rangle \langle \psi_i|, \quad \mu_i \geq 0, \quad \sum_{i=1}^4 \mu_i = 1.$$ 

(72)

As a consequence, comparing (70) and (72), one obtains:

$$\lambda_{11} - \lambda_{12} = \mu_1 + \mu_4 \geq 0, \quad \lambda_{11} + \lambda_{12} = \mu_1 + \mu_3 \geq 0, \quad \lambda_{22} - \lambda_{12} = \mu_2 + \mu_4 \geq 0, \quad \lambda_{22} + \lambda_{12} = \mu_2 + \mu_3 \geq 0,$$

which are possible only if: $|\lambda_{12}| \leq \min \{\lambda_{11}, \lambda_{22}\}$.

On the other hand, using the above relations, one can express three of the convex coefficients appearing in the decomposition (72) in terms of the remaining one and $\lambda_{11}, \lambda_{12}$; for instance:
The conditions \(0 \leq \mu_i \leq 1, \ i = 1, 2, 3,\) then yields
\[
\max \{\lambda_{11} - \lambda_{12} - 1, \lambda_{11} + \lambda_{12} - 1, 2\lambda_{11} - 1\} \leq \mu_1 \leq \min \{\lambda_{11} - \lambda_{12}, \lambda_{11} + \lambda_{12}, 2\lambda_{11}\}.
\]
Therefore, assuming \(\mu_1 \leq \min \{\lambda_{11}, \lambda_{22}\},\) one can always choose coefficients \(m_i,\) satisfying
\[
m_i = \lambda_{11} + \lambda_{11} - \mu_1,
\]
\[
\mu_1 = \lambda_{11} - \lambda_{12} - \mu_1,
\]
\[
\mu_4 = \lambda_{11} - \lambda_{12} - \mu_1.
\]

Remark. Surprisingly, as explicitly shown in the above proof, the two combinations
\[
|\psi_{3,4}\rangle = (|e_i\rangle \pm |e_2\rangle)/\sqrt{2}
\]
are separable in \(\mathcal{C}_2.\) This behavior is clearly quite different from the case of two distinguishable qubits, or two-mode boson/fermion systems, where superpositions of pure separable states give entangled states.

In conclusion, using the powerful machinery of algebraic quantum mechanics, we have been able to classify all entangled states of the Clifford algebra \(\mathcal{C}_2.\) As we shall see, one can also treat the case of the general algebra \(\mathcal{C}_N\) in a similar way.

9. Structure of entangled Majorana states: \(\mathcal{C}_N\)

Before treating the case of a general Clifford algebra, it is useful to explicitly discuss the construction of a basis of separable states in \(\mathcal{H}_2\) carrying the irreducible representations of \(\mathcal{C}_N\) with \(N = 3, 4,\) by extending the techniques previously adopted for \(\mathcal{C}_3.\)

The algebra \(\mathcal{C}_3,\) the simplest Clifford algebra \(\mathcal{C}_N\) with \(N\) odd, is the linear span of the set \(\{1, c_1, c_2, c_3\}.\) Its only non-trivial bipartition \((\mathcal{A}_1, \mathcal{A}_2)\) is the one in which \(\mathcal{A}_1\) is the linear span of \(\{1, c_1, c_2\},\) while \(\mathcal{A}_2\) is that of \(\{1, c_3\},\) since all other possible bipartitions can be reduced to this one by a suitable reordering of the mode labels.

Choosing again the state \(\Omega,\) as in (33), the GNS construction gives a representation \(\pi_{\Omega}\) of \(\mathcal{C}_3\) on the Hilbert space \(\mathcal{H}_2,\) which is now eight-dimensional. As outlined earlier for \(\mathcal{C}_2,\) a separable basis in it can be constructed following the procedure presented in the proof of lemma 3; in practice, such a basis can be obtained by augmenting the four operators \(\hat{e}^{(r,s)}(c_1, c_2) = (1 \pm c_1)(1 \pm c_2)\) introduced in (59) with the additional two projectors \((1 \pm c_3)/2,\) yielding the eight orthonormal vectors:
\[
|e_i^{(r,s)}\rangle = \frac{1}{2\sqrt{2}} \hat{e}^{(r)}(c_1, c_2)\hat{e}^{(s)}(c_3)|\Omega\rangle, \quad r = 1, 2, \quad i = 1, 2, \quad s = 1, 2, \quad (73)
\]
where
\[
\hat{e}^{(1)}(c_3) = (1 + c_3) \quad \hat{e}^{(2)}(c_3) = (1 - c_3). \quad (74)
\]

However, as discussed in section 5, the irreducible representations of \(\mathcal{C}_3\) are four-dimensional, so that \(\pi_{\Omega}\) as given by the GNS construction decomposes as \(\pi_{\Omega} = \pi_{\Omega}^{(1)} \oplus \pi_{\Omega}^{(2)}\) into two equal four-dimensional irreducible representations \(\pi_{\Omega}^{(s)},\) \(s = 1, 2\) acting on two subspaces \(\mathcal{H}_{(1)}^{(s)},\) such that \(\mathcal{H}_2 = \mathcal{H}_{(1)}^{(1)} \oplus \mathcal{H}_{(1)}^{(2)}.\) One can check that the first subspace \(\mathcal{H}_{(1)}^{(1)}\) is spanned by the four basis vectors in (73) with \(s = 1,\) i.e. \(|e_i^{(r,1)}\rangle, \ r, i = 1, 2,\) while the second can be checked with \(s = 2.\) Explicitly, one finds:
Therefore, the vectors in (73) represent pure states for \( \mathcal{C}_3 \); further, due to \textit{proposition 1}, they are separable.

In order to explicitly obtain the decomposition of the GNS representation into its irreducible components in the general case \( \mathcal{C}_N \), with \( N > 3 \), more effort is required. The discussion of the case \( N = 4 \) suffices for one to grasp the general structure.

In the case of \( \mathcal{C}_4 = \text{span} \{1, c_1, c_2, c_3, c_4 \} \), the Hilbert space \( \mathcal{H}_\Omega \) is 16-dimensional and can be spanned by the sixteen orthonormal states:

\[
|\psi^{(rs)}_{(ij)}\rangle = \frac{1}{4} \phi^{(rs)}_{(ij)} |\Omega\rangle, \quad r, s = 1, 2, \quad i, j = 1, 2, \tag{76}
\]

with

\[
\phi^{(rs)}_{(ij)} \equiv \hat{\phi}^{(rs)}_i(c_1, c_2) \hat{\phi}^{(rs)}_j(c_3, c_4), \tag{77}
\]

where \( \hat{\phi}^{(rs)}_i(c_1, c_2) \) are the four combinations in (59), while \( \hat{\phi}^{(rs)}_j(c_3, c_4) \) are of exactly the same form, but with \( c_1 \) replaced by \( c_3 \), and \( c_2 \) by \( c_4 \). For instance, one explicitly has

\[
\begin{align*}
\phi^{(1,1)}_{(1,1)} & \equiv (1 + c_1)(1 + c_2)(1 + c_3)(1 + c_4) \\
\phi^{(1,1)}_{(1,2)} & \equiv (1 - c_1)(1 + c_2)(1 + c_3)(1 + c_4) \\
\phi^{(1,1)}_{(1,2)} & \equiv (1 + c_1)(1 + c_2)(1 - c_3)(1 + c_4) \\
\phi^{(1,1)}_{(1,2)} & \equiv (1 - c_1)(1 + c_2)(1 - c_3)(1 + c_4).
\end{align*}
\tag{78}
\]

The 16 states (76) look separable for any choice of bipartition of \( \mathcal{C}_4 \), but unfortunately cannot be simply grouped into sets of four in order to form a basis for subspaces of \( \mathcal{H}_\Omega \) carrying irreducible representations of \( \mathcal{C}_4 \), as done before for \( \mathcal{C}_2 \) and \( \mathcal{C}_3 \). Nevertheless, this can be obtained through unitary transformations similar to the ones introduced in (63) and (64); this will allow one to conclude that the vectors \( |\psi^{(rs)}_{(ij)}\rangle \) are also pure states for \( \mathcal{C}_4 \). Recall that, in general, a state in \( \mathcal{H}_\Omega \) is mixed when restricted to the Clifford algebra, while in order to characterize entangled Clifford states, a basis of pure separable states is needed.

For the sake of definitiveness, let us fix the bipartition \( (A_1, A_2) \) of \( \mathcal{C}_4 \) for which \( A_1 = \text{span} \{1, c_1, c_2 \} \) and \( A_2 = \text{span} \{1, c_3, c_4 \} \)\(^{17}\). Using the unitary matrices \( U \) and \( V \) in (64), one can then write

\[
\phi^{(rs)}_{(ij)} = \sum_{p,k} U_{kp} f^p_k(c_1, c_2) \sum_{q,l} U_{ql} V_{rj} f^q_l(c_3, c_4). \tag{79}
\]

where \( f^p_k(c_1, c_2), p, k = 1, 2, \) coincide with the monomials in (66), while \( f^q_l(c_3, c_4), q, l = 1, 2, \) are of exactly the same form with the substitution \( c_1 \rightarrow c_3 \) and \( c_2 \rightarrow c_4 \). This implies that the states \( |\psi^{(rs)}_{(ij)}\rangle \) in (76) can be expressed as linear combinations of the vectors:

\[
|f^{(rs)}_{(ij)}\rangle = \frac{1}{4} f^{(r)}_i(c_1, c_2) f^{(s)}_j(c_3, c_4) |\Omega\rangle, \quad r, s = 1, 2, \quad i, j = 1, 2. \tag{80}
\]

Using the results presented in the previous section, one can easily check that the four vectors \( |f^{(rs)}_{(ij)}\rangle \) with the indices \( r \) and \( s \) fixed, span a four-dimensional subspace

\(^{17}\) The other independent bipartition, for which \( A_1 = \text{span} \{1, c_1, c_2, c_3 \} \) and \( A_2 = \text{span} \{1, c_4 \} \), can be similarly treated using the results obtained above for \( \mathcal{C}_3 \).
\( \mathcal{H}_{(x)} \subset \mathcal{H}_{\Omega} \) carrying an irreducible representation \( \pi^{(x)}_{\Omega} \) of \( C_4 \). Then, the original GNS representation \( \pi_{\Omega} \) decomposes as

\[
\pi_{\Omega} = \oplus r,s \pi^{(r,s)}_{\Omega},
\]

where \( r,s = \pm \) into four, four-dimensional irreducible representations \( \pi^{(r,s)}_{\Omega} \) which turn out to be all equal. As a result, since the basis vectors \( | f^{(r,s)}_{(i,j)} \rangle \) are pure, the original vectors \( | v^{(r,s)}_{(i,j)} \rangle \), being linear combinations of these, are also pure. In addition, they are also separable, being essentially product states.

**Remark.** Notice that the states \( | f^{(r,s)}_{(i,j)} \rangle \) are manifestly separable for the chosen bipartition \((A_1, A_2)\); however, as discussed in the previous section, these states become non-separable when restricted to the two-dimensional Clifford subalgebra \( A_1 \) or \( A_2 \).

The whole construction can now be easily generalized to the case of a generic Clifford algebra \( C_N \). Given the state \( \Omega \) in (33), when \( N \) is even one can build a basis of pure, separable states in the Hilbert space \( \mathcal{H}_{\Omega} \) by acting with products of the four elements \( e^{(r)}_i (c_a, c_b) = (1 \pm c_a) (1 \pm c_b) \) on the cyclic GNS vector \( |\Omega\rangle \), explicitly obtaining

\[
| e^{r}_i \rangle = \frac{1}{\sqrt{2^N}} e^{(r)(i)}_0 (c_1, c_2) e^{(r)(i)}_1 (c_3, c_4) ... e^{(r)(N-1)}_{N-1} (c_N) |\Omega\rangle,
\]

where \( r = (r_1, r_3, ..., r_{N-1}) \) and \( i = (i_1, i_2, ..., i_{N-1}) \); on the other hand, when \( N \) is odd, the two elements \( e^{(r)}_i (c_a) = (1 \pm c_a) \) are also needed, so that

\[
| e^{r}_i \rangle = \frac{1}{\sqrt{2^N}} e^{(r)(i)}_0 (c_1, c_2) e^{(r)(i)}_1 (c_3, c_4) ... e^{(r)(N-1)}_{N-1} (c_N) |\Omega\rangle.
\]

In the above expressions, all the ‘\( r \)’s and ‘\( i \)’s of the indices take the two values 1 and 2. These states are all manifestly separable for any bipartition of the algebra \( C_N \); further they are pure since, as in the case of \( C_4 \) discussed above, they can be unitarily related to states carrying irreducible representations of the Clifford algebra.

For the sake of definiteness, let us assume \( N \) to be even and fix a bipartition \((A_1, A_2)\) for which \( A_1 = \text{span} \{1, c_1, c_2, ..., c_{2k}\} \) and \( A_2 = \text{span} \{1, c_{2k+1}, c_{2k+2}, ..., c_N\} \), with \( 1 \leq k \leq N/2 - 2 \); in this case, the multi-indices \( r \) and \( i \) take \( 2^N/2 \) possible values that we shall henceforth take to be \( 1, 2, ..., 2^N/2 \). Then, by generalizing the transformation in (79), the states \( | e^{r}_i \rangle \) in (82) can be unitarily related to the following ones:

\[
| f^{r}_i \rangle = \frac{1}{\sqrt{2^N}} f^{(r)(i)}_0 (c_1, c_2) f^{(r)(i)}_1 (c_3, c_4) ... f^{(r)(N-1)}_{N-1} (c_N) |\Omega\rangle,
\]

with \( f^{(r)}_i (c_a, c_b) = \{ c_a \pm c_b, (1 \pm ic_a c_b) \} \), as in (66)\(^{18}\). For fixed indices \( r \), these states span a \( 2^{N/2} \)-dimensional subspace \( \mathcal{H}_{(r)}^{(r)} \) of \( \mathcal{H}_{\Omega} \) carrying an irreducible representation of \( C_N \). Indeed, the \( 2^{N/2} \) sets of basis vectors \( \{| f^{r}_i \rangle \} | i = 1, 2, ..., 2^{N/2} \rangle \), with \( r = 1, 2, ..., 2^{N/2} \), induce a decomposition \( \mathcal{H}_{\Omega} = \oplus_r \mathcal{H}_{(r)}^{(r)} \) of the GNS Hilbert space into subspaces, each carrying an irreducible representation \( \pi^{(r)}_{\Omega} \) of \( C_N \), so that the GNS representation \( \pi_{\Omega} \) has the following decomposition into irreducible components: \( \pi_{\Omega} = \oplus_r \pi^{(r)}_{\Omega} \).

Since all the representations \( \pi^{(r)}_{\Omega} \) turn out to be equal, any state of \( C_N \) in the *folium* of \( \Omega \), represented by a density matrix \( \rho \), can be decomposed as

\(^{18}\) Similarly, for the independent bipartition with \( A_1 = \text{span} \{1, c_1, c_2, ..., c_{2k+1}\} \) and \( A_2 = \text{span} \{1, c_{2k+2}, c_{2k+3}, ..., c_N\} \), the states in (82) can be unitarily related to the following (un-normalized) ones:

\[
| f^{r}_i \rangle (c_1, c_2) ... f^{(r)(i)}_{2k-1} (c_{2k-1}, c_{2k}) e^{(r)(2k)}_{2k+1} (c_{2k+1}) f^{(r)(2^{N/2}+2)}_{2k+2} (c_{2k+2}, c_{2k+3}) ... e^{(r)(N)} (c_N) |\Omega\rangle.
\]
\[ \rho = \sum_{j,k} \lambda_{jk} |f_j^{(k)}\rangle \langle f_k^{(j)}|, \quad \sum_k \lambda_{kk} = 1, \quad (85) \]

where for the vector basis \( \{|f_k^{(j)}\rangle\} \) one can choose any of the sets \( \{|f_k^{(j)}\rangle\} \) introduced above, with \( r \) fixed. Clearly, also in this general case, a diagonal state of the form

\[ \rho_D = \sum_k \lambda_{kk} |f_k^{(k)}\rangle \langle f_k^{(k)}|, \quad (86) \]

is manifestly separable. On the other hand, provided not all off-diagonal coefficients \( \lambda_{jk} \), \( j \neq k \), are real, one can always find a monomial \( c_i c_j \), with \( c_i \in A_1 \) and \( c_j \in A_2 \) for which \( \text{Tr}[\eta \ c_i c_j] = 0 \), with \( \eta = \rho - \rho_D \). Therefore, in this generic case, by the criterion of corollary 3 any state \( \rho \) becomes entangled if and only if it is not in the diagonal form (66). However, a full characterization of entangled states in the case in which the coefficients \( \lambda_{jk} \) are all real cannot be given in general, since one has to resort to the general separability condition (5).

As an interesting example of an entangled state in \( C_N \), let us fix \( N = 2n \) and consider the balanced bipartition \((A_1, A_2)\) for which \( A_1 = \text{span}\{1, c_1, c_2, \ldots, c_n\} \) and \( A_2 = \text{span}\{1, c_{n+1}, c_{n+2}, \ldots, c_{2n}\} \). The monomial

\[ \gamma = \gamma^{(1)}_p \gamma^{(2)}_p \], \quad \gamma^{(1)} = c_i c_i \ldots c_i, \quad \gamma^{(2)} = c_j c_j \ldots c_j, \quad (87) \]

with \( 1 \leq i \leq n \) and \( (n + 1) \leq j \leq 2n \), is an element of \( C_{2n} \) which is manifestly local with respect to the chosen bipartition, since \( \gamma^{(1)} \in A_1 \), while \( \gamma^{(2)} \in A_2 \). Furthermore, when the integer \( p \) is odd, \( \gamma^{(1)} \) and \( \gamma^{(2)} \) are odd elements, \( \{\gamma^{(1)}, \gamma^{(2)}\} = 0 \), such that \( \gamma^2 = -1 \). Consider then the following vector in \( \mathcal{H}_\Omega \):

\[ |\phi\rangle = \frac{1}{\sqrt{2}} |1 + i\gamma\rangle |\Omega\rangle. \quad (88) \]

When restricted to the algebra \( C_{2n} \) it becomes a mixed state, since both \( |\Omega\rangle \) and \( \gamma |\Omega\rangle \) are no longer pure; further, the expectation \( \langle \phi | \gamma | \phi \rangle \) is non-vanishing, so that, again by corollary 3, the state is entangled. This result will be useful in the following section, while discussing metrological applications of Majorana systems.

10. Application to quantum metrology

Using quantum physics in metrological applications is surely one of the most promising developments in quantum technology; it allows the determination of a physically interesting parameter \( \theta \), typically a phase, with unprecedented accuracy\(^{19}\). This result is achieved through a \( \theta \)-dependent state transformation that occurs inside a suitable measurement apparatus, generally an interferometric device. In the most common case of linear setups, this transformation can be modeled using a unitary mapping, \( \rho \rightarrow \rho_B \), sending the initial state \( \rho \) into the final parameter-dependent outcome state:

\[ \rho_B = e^{i\theta J} \rho e^{-i\theta J}, \quad (89) \]

where \( J \) is the device-dependent, \( \theta \)-independent operator generating the state transformation. The task of quantum metrology is to determine the ultimate bounds on the accuracy with which the parameter \( \theta \) can be obtained through a measurement of \( \rho_B \) and to study how these bounds scales with the available resources.

\(^{19}\) The literature on the subject is growing fast; for a partial list, see [94–118] and references therein.
General quantum estimation theory allows a precise determination of the accuracy $\delta \theta$ with which the phase $\theta$ can be obtained in a measurement involving the operator $J$ and the initial state $\rho$; one finds that $\delta \theta$ is limited by the following inequality [119–122]:

$$\delta \theta \geq \frac{1}{\sqrt{F[\rho, J]}},$$

(90)

where the quantity $F[\rho, J]$ is the so-called quantum Fisher information; it is a continuous convex function of the state $\rho$, satisfying the inequality [123, 124]

$$F[\rho, J] \leq 4 \Delta^2 J,$$

(91)

where $\Delta^2 J \equiv [\langle J^2 \rangle - \langle J \rangle^2]$ is the variance of the operator $J$ in the state $\rho$, with the equality holding only for pure states. In order to reach a better resolution in $\theta$-estimation one should obtain larger quantum Fisher information; thus for a given measuring apparatus, i.e. a given operator $J$, one can still optimize the precision with which $\theta$ is determined by choosing an initial state $\rho$ that maximizes $F[\rho, J]$.

For measuring devices made of $N$-distinguishable particles, the following bound on the quantum Fisher information holds for any separable state $\rho_{\text{sep}}$ [114]:

$$F[\rho_{\text{sep}}, J] \leq N.$$

(92)

In other words, feeding the measuring apparatus with separable initial states, the best achievable precision in the determination of the phase shift $\theta$ is bounded by the so-called shot-noise limit:

$$\delta \theta \geq \frac{1}{\sqrt{N}}.$$  

(93)

This is also the best result attainable using classical (i.e. non-quantum) devices: the accuracy in the estimation of $\theta$ scales at most with the inverse square root of the number of available resources. Instead, quantum-equipped metrology allows one to reach sub-shot-noise sensitivities by using suitable detection protocols and entangled input states.

This conclusion holds when the metrological devices used to estimate the physical parameter $\theta$ are based on systems of distinguishable particles. When dealing with identical particles, the above statement needs to be rephrased. Indeed, in the case of both boson and fermion systems it has been explicitly shown that sub-shot-noise sensitivities may also be obtained via a non-local operation acting on separable input states [50, 57]. In other words, although some sort of non-locality is needed in order to go below the shot-noise limit, this can be provided by the measuring apparatus itself and not necessarily by the input state $\rho$, which indeed can be separable. This result clearly has direct experimental relevance since the preparation of suitably entangled input states may require a large amount of resources in practice.

When dealing with fermions, the situation appears more involved due to the anti-commuting character of the associated operator algebra $\mathcal{A}$. Indeed, while in the case of bosons a two-mode apparatus (e.g. a standard two-way interferometer) filled with $N$ particles is sufficient to reach sub-shot-noise efficiencies, with fermions a multimode apparatus is needed in order to reach comparable sensitivities [125–130].

With Majorana fermions, things become even more complicated since the notion of mode occupation loses its meaning, as a number operator is no longer available. It is the number $N$ of available Majorana modes that now quantifies the amount of resources available for the process of parameter estimation and it is in reference to this number that the shot-noise limit
in (93) should be considered. In other words, in dealing with systems of Majorana fermions, it is the mode structure of the measuring apparatus that becomes relevant.

As an example, consider again a system with an even number \( N = 2n \) of Majorana modes and choose for it the balanced bipartition \( (\mathcal{A}_1, \mathcal{A}_2) \), with \( \mathcal{A}_1 = \text{span} \{ 1, c_1, c_2, \ldots, c_n \} \) and \( \mathcal{A}_2 = \text{span} \{ 1, c_{n+1}, c_{n+2}, \ldots, c_{2n} \} \). As a generator of the transformations inside the measuring apparatus, let us take the following operator:

\[
J = i \sum_{k=1}^{n} \omega_k c_k c_{n+k},
\]

where \( \omega_k \) is a given spectral function, e.g. \( \omega_k \approx k^p \), with \( p \) integer. The unitary transformation \( U_\theta = e^{i\theta J} \) implementing the state transformation inside the apparatus is clearly non-local, since it cannot be written as the product \( \rho_1 \rho_2 \), with \( \rho_1 \in \mathcal{A}_1 \) and \( \rho_2 \in \mathcal{A}_2 \). It represents a sort of generalized multimode beam-splitter, so that the whole measuring device behaves as a multimode interferometer.

Let us feed the interferometer with a pure initial state \( |\psi\rangle \langle \psi| \), coinciding with one of the basis elements \( |f_k^i \rangle \) in (84) carrying the irreducible representation \( \pi_{11}^\text{irr} \) of \( C_{2n} \) presented in the previous section. For instance, choose

\[
|\psi\rangle = \frac{1}{2^n} (1 + ic_1 c_2) \ldots (1 + ic_{n-1} c_n)(1 - ic_{n+1} c_{n+2}) \ldots (1 - ic_{2n-1} c_{2n}) |\Omega\rangle;
\]

as discussed before, this state is separable with respect to the chosen bipartition.

The quantum Fisher information can be easily computed, since in this case it is proportional to the variance of \( J \) with respect to \( |\psi\rangle \); assuming for simplicity \( n \) even, one finds

\[
F[\rho, J] = \sum_{k=1}^{N/4} (\omega_{2k-1} + \omega_{2k})^2,
\]

which turns out to be larger than \( N \). In particular, for \( \omega_k \sim k \), one finds that in the limit of large \( N, F[\rho, J] \) behaves as \( N^3/3 \). Therefore, also with Majorana fermions, a suitably devised interferometric apparatus can beat the shot-noise limit in \( \theta \) estimation even starting with a separable state as in (95). One can check that this sub-shot-noise gain in precision can also be obtained using any other state vector belonging to the separable basis (82), although it is for the state (95) that the value actually attained by the quantum Fisher information is maximal.

**Remark.** Notice that, in general, the obtained value for \( F[\rho, J] \) scales with a power of \( N \) greater than two. When dealing with systems made of \( N \)-distinguishable particles, the following general bound holds:

\[
F[\rho, J] \leq N^2,
\]

for any state \( \rho \) and generator \( J \), providing an absolute lower bound for the accuracy in \( \theta \) estimation called the Heisenberg limit: \( 8\theta \geq 1/N \). Instead, in the scenario described above, one is able to reach sub-Heisenberg sensitivities, which is another advantage of using fermion systems\(^{20}\).

Some sort of quantum non-locality is nevertheless needed in order to attain sub-shot-noise accuracies. In order to appreciate this point, let us consider the same Majorana system as before, but using a different generator \( \tilde{J} \), i.e. a different measuring apparatus, where:

\(^{20}\) The possibility of getting sensitivities beyond the Heisenberg limit has been discussed before, although using non-linear metrology [131–138].
\[ \tilde{J} = i \sum_{k=1}^{n} \tilde{\omega}_k c_{2k-1} c_{2k}, \]  
with \( \tilde{\omega}_k \) a given spectral density. The unitary transformation \( \tilde{U}_\theta = e^{i\theta J} \) implementing state transformation inside the interferometer is now local with respect to the chosen bipartition, since it is the product of \( n \) transformations depending on couples of contiguous modes:

\[ \tilde{U}_\theta = \prod_{k=1}^{n} e^{-i \tilde{\omega}_k c_{2k-1} c_{2k}}. \]

If one feeds the apparatus with any vector belonging to the separable basis in (82), one does not obtain any advantage in parameter estimation accuracy with respect to the shot-noise limit; actually, the quantum Fisher information vanishes.

However, using an entangled state as the initial state, the situation changes. Indeed, let us consider the entangled state \( |\psi\rangle \) in (88) introduced at the end of the previous section. Although not a pure state for \( \mathcal{C}_N \), the corresponding quantum Fisher information, being representation-independent, can be computed in the full Hilbert space \( \mathcal{H}_\Omega \), obtaining

\[ F[|\psi\rangle, J] = \sum_{k=1}^{n} (\tilde{\omega}_k)^2. \]

For a spectral density of generic form \( \tilde{\omega}_k \approx k^p \), one finds, for large \( N \), \( F[|\psi\rangle, J] \approx N^{2p+1} \), obtaining again a sub-shot-noise accuracy for \( \theta \) estimation; actually, for \( p \geq 1 \), the sensitivity in the determination of \( \theta \) goes even beyond the Heisenberg limit.

11. Outlook

Non-classical correlations are the basis of most of the recent advances in modern quantum physics, and in particular in quantum technology, leading to the possibility of the realization of devices outperforming those presently available. The characterization and quantification of these resources is, therefore, of utmost importance, especially in many-body systems since, thanks to the recent advances in ultracold and superconducting physics, they are becoming the preferred laboratories for studying new quantum effects.

For systems made of identical constituents, the usual notions of separability and entanglement need a revision, since the particle Hilbert space tensor structure on which these concepts are based is no longer available due to particle indistinguishability. Attention, then, should shift from the Hilbert space paradigm to a new one, focusing on the system observables and the algebra they obey; quantum non-separability can then be signaled by the behavior of observable correlation functions.

This change of perspective can be most simply formulated using the algebraic approach to quantum physics. Here, a quantum system is identified by its operator algebra \( \mathcal{A} \) containing all its observables, while the Hilbert space \( \mathcal{H}_\Omega \) of its states is an emergent concept, determined by the choice of a state \( \Omega \) on \( \mathcal{A} \) through the so-called GNS construction. The state \( \Omega \), a positive, normalized linear form on \( \mathcal{A} \), determines the expectation values of the observables, thus making the connection with measurable quantities. It also provides an explicit representation \( \pi_\Omega \) of \( \mathcal{A} \), so that the observables act as operators on \( \mathcal{H}_\Omega \). In this framework, the notion of locality is no longer given a priori, once and for all; rather, it is based on the choice of a bipartition (or, more generally, a multipartition) of the algebra \( \mathcal{A} \) into subalgebras \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \), such that \( \mathcal{A}_1 \cup \mathcal{A}_2 = \mathcal{A} \) and \( \mathcal{A}_1 \cap \mathcal{A}_2 = 1_\mathcal{A} \). An element of \( \mathcal{A} \) is local if it is the product of an element of \( \mathcal{A}_1 \) times an element of \( \mathcal{A}_2 \). A state \( \Omega \) on \( \mathcal{A} \) is then
separable if its expectation on all local operators can be written as a convex combinations of products of expectations.

This general definition of separability, previously studied in boson or fermion settings, has been applied here to the study of systems made of Majorana fermions. In view of the attention they are receiving in superconducting physics and as possible building blocks in topological quantum computations, Majorana excitations are becoming the focus of a rapidly increasing number of investigations: studying their entanglement properties is therefore of great relevance.

For Majorana systems, the operator algebra \( \mathcal{A} \) containing all observables becomes a Clifford algebra \( \mathcal{C} \). These algebras do not admit a Fock representation; this implies that for such systems the notions of number operator and of mode occupation are no longer available. Furthermore, given a state \( \Omega \), the corresponding representation of \( \mathcal{C} \) on the Hilbert space \( \mathcal{H}_\Omega \) turns out to be generally reducible: this makes the characterization of entangled states much more involved than for bosons or ordinary fermions, since now \( \Omega \) is no longer a pure state for the algebra \( \mathcal{C} \).

The relation between quantum non-separability and reducibility of the GNS representation \( \pi_\Omega \) has not been studied much in the literature. Here, instead, a general detailed treatment of entanglement theory in the presence of reducible operator algebra representations has been given, and then applied to the case of Clifford algebras for a specific physically relevant choice of the state \( \Omega \). This has allowed us to obtain a rather complete characterization of general entangled Majorana states; as an illustration, the cases of systems containing just few Majorana modes have been analyzed in great detail. The whole treatment is very general and can be easily applied to the discussion of different choices for \( \Omega \).

Among promising quantum technological applications, quantum metrology is the natural context in which the above results can be fruitfully employed. Indeed, as discussed in the last section, multimode Majorana quantum interferometers can be used to improve accuracy in the measurement of relevant physical parameters far beyond the so-called shot-noise limit (the best limit reachable by classical devices). Some sort of quantum non-locality is clearly needed in order to reach these sub-classical sensitivities; however, this need not be encoded in the initial state, and it can be provided by the interferometric apparatus itself. As a result, no preliminary resource-consuming entanglement operation on the state entering the apparatus is needed in order to get sub-shot-noise accuracies. In this respect, Majorana fermion systems may turn out to play a central role in the development of new generations of quantum sensors capable of outperforming any currently available apparatus dedicated to the detection of faint physical signals.

References

[1] Majorana E 1937 Nuovo Cimento 14 171
[2] Elliott S R and Franz M 2015 Rev. Mod. Phys. 87 137
[3] Alicea J 2012 Rep. Prog. Phys. 75 076501
[4] Leijnse M and Flensberg K 2012 Semicond. Sci. Technol. 27 124003
[5] Beenakker C W J 2013 Annu. Rev. Con. Mat. Phys. 4 113
[6] Stancu T D and Tewari S 2013 J. Phys. C 25 233201
[7] Nayak C et al 2008 Rev. Mod. Phys. 80 1083
[8] Hassler F 2014 arXiv:1404.0897
[9] Kitaev A Y 2001 Phys.-Usp. 44 131
[10] Kitaev A Y 2003 Ann. of Phys. 303 2
[11] Schliemann J, Cirac J I, Kus M, Lewenstein M and Loss D 2001 Phys. Rev. A 64 022303
[12] Paskauskas R and You L 2001 Phys. Rev. A 64 042310
[64] Strocchi F 2008 *An Introduction to the Mathematical Structure of Quantum Mechanics* 2nd edn (Singapore: World Scientific)
[65] Strocchi F 2008 *Symmetry Breaking* 2nd edn (Heidelberg: Springer)
[66] Strocchi F 2012 *Eur. Phys. J. Plus* 127 12
[67] Streater R and Wightman A 1964 *PCT, Spin and Statistics, and All That* (New York: Benjamin)
[68] Strocchi F 2004 *Found. Phys.* 34 501
[69] Gilbert J E and Murray M A M 1991 *Clifford Algebras and Dirac Operators in Harmonic Analysis* (Cambridge: Cambridge University Press)
[70] Lawson H B and Michelsohn M L 1989 *Spin Geometry* (Princeton, NJ: Princeton University Press)
[71] Porteous I R 1995 *Clifford Algebras and the Classical Groups* (Cambridge: Cambridge University Press)
[72] Lourenço P 2001 *Clifford Algebras and Spinors* (Cambridge: Cambridge University Press)
[73] Meinrenken E 2013 *Clifford Algebras and Lie Theory* (Heidelberg: Springer)
[74] Leggett A J 2001 *Rev. Mod. Phys.* 73 307
[75] Pitaevskii L and Stringari S 2003 *Bose–Einstein Condensation* (Oxford: Oxford University Press)
[76] Pethick C J and Smith H 2004 *Bose–Einstein Condensation in Dilute Gases* (Cambridge: Cambridge University Press)
[77] Gerry C C and Knight P 2005 *Introductory Quantum Optics* (Cambridge: Cambridge University Press)
[78] Haroche S and Raimond J-M 2006 *Exploring the Quantum: Atoms, Cavities and Photons* (Oxford: Oxford University Press)
[79] Leggett A J 2006 *Quantum Liquids* (Oxford: Oxford University Press)
[80] Köhl M and Esslinger T 2006 *Europhys. News.* 37 18
[81] Inguscio M, Ketterle W and Salomon C (ed) 2006 *Ultra-cold Fermi Gases* (Amsterdam: IOS Press)
[82] Giorgini S, Pitaevskii L and Stringari S 2008 *Rev. Mod. Phys.* 80 1215
[83] Cronin A D, Schmiedmayer J and Pritchard D E 2009 *Rev. Mod. Phys.* 81 1051
[84] Yukalov V I 2009 *Laser Phys.* 19 1
[85] Araki H and Moriya H 2003 *Commun. Math. Phys.* 237 105
[86] Bartlett S D, Rudolph T and Spekkens R W 2007 *Rev. Mod. Phys.* 79 555
[87] Wick G C, Wightman S and Wigner E 1952 *Phys. Rev.* 88 101
[88] Moriya H 2006 *Commun. Math. Phys.* 264 411
[89] Price G L 1987 *Can. J. Math.* 39 492
[90] Powers R T 1988 *Can. J. Math.* 40 86
[91] Alicki R and Fannes R 2001 *Quantum Dynamical Systems* (Oxford: Oxford University Press)
[92] Balachandran A P, de Queiroz A R and Vaidya S 2013 *Eur. Phys. J. Plus* 128 112
[93] Balachandran A P, Queiroz A and Vaidya S 2013 *Phys. Rev. D* 88 025001
[94] Caves C M 1981 *Phys. Rev. D* 23 1693
[95] Yurke B 1986 *Phys. Rev. Lett.* 56 1515
[96] Yurke B, McCall S L and Klauder J R 1986 *Phys. Rev. A* 33 4033
[97] Holland M J and Burnett K 1993 *Phys. Rev. Lett.* 71 1355
[98] Kitagawa M and Ueda M 1993 *Phys. Rev. A* 47 5138
[99] Wineland D J *et al.* 1994 *Phys. Rev. A* 50 67
[100] Sanders B C and Milburn G J 1995 *Phys. Rev. Lett.* 75 2944
[101] Bollinger J J *et al.* 1996 *Phys. Rev. A* 54 R4649
[102] Bouyer P and Kasevich M A 1997 *Phys. Rev. A* 56 R1083
[103] Dowling J P 1998 *Phys. Rev. A* 57 4736
[104] Sørensen A, Duan L-M, Cirac J I and Zoller P 2001 *Nature* 409 63
[105] Holland H, Kok P and Dowling J P 2002 *J. Mod. Opt.* 49 2325
[106] Dunningham J A, Buenett K and Barnett S M 2002 *Phys. Rev. Lett.* 89 150401
[107] Wang X and Sanders B C 2003 *Phys. Rev. A* 68 012101
[108] Giovannetti V, Lloyd S and Maccone L 2004 *Science* 306 1330
[109] Korbicz J K, Cirac J I and Lewenstein M 2005 *Phys. Rev. Lett.* 95 120502
[110] Higgins B L *et al.* 2007 *Nature* 450 393
[111] Uys H and Meystre P 2007 *Phys. Rev. A* 76 013804
[112] Dowling J P 2008 *Contemp. Phys.* 49 125
[113] Dorner U *et al.* 2009 *Phys. Rev. Lett.* 102 040403

34
114 Pezzè L and Smerzi A 2009 Phys. Rev. Lett. 102 100401
115 Boixo S et al 2009 Phys. Rev. A 80 032103
116 Tóth G, Knapp C, Gühne O and Briegel H J 2009 Phys. Rev. A 79 042334
117 Kacprowicz M et al 2010 Nat. Photon. 4 357
118 Giovannetti V, Lloyd S and Maccone L 2011 Nat. Photon. 5 222
119 Helstrom C W 1976 Quantum Detection and Estimation Theory (New York: Academic Press)
120 Holevo A S 1982 Probabilistic and Statistical Aspects of Quantum Theory (Amsterdam: North-Holland)
121 Petz D 2008 Quantum Information Theory and Quantum Statistics (Berlin: Springer)
122 Paris M G A 2009 Int. J. Quant. Inf. 7 125
123 Luo S 2000 Lett. Math. Phys. 53 243
124 Braunstein S L, Caves C M and Milburn G J 1996 Ann. Phys. 247 135
125 D’Ariano G M and Paris M G A 1997 Phys. Rev. A 55 2267
126 D’Ariano G M, Macchiavello C and Sacchi M F 1998 Phys. Lett. A 248 103
127 Söderholm J et al 2003 Phys. Rev. A 67 053803
128 Vourdas A and Dunningham J A 2005 Phys. Rev. A 71 013809
129 Cooper J J et al 2009 J. Phys. B 42 105301
130 Cooper J J et al 2012 Phys. Rev. Lett. 108 130402
131 Luis A 2004 Phys. Lett. A 329 8
132 Boixo S et al 2007 Phys. Rev. Lett. 98 090401
133 Rey A M, Jiang L and Lukin M D 2007 Phys. Rev. A 76 053617
134 Choi S and Sundaram B 2008 Phys. Rev. A 77 053613
135 Woolley M J, Milburn G J and Caves C M 2008 New J. Phys. 10 125018
136 Napolitano N et al 2011 Nature 471 486
137 Datta A and Shaji A 2012 Mod. Phys. Lett. B 26 1230010
138 Hall M J W and Wiseman H M 2012 Phys. Rev. X 2 041006