ON BOUNDEDNESS OF CHARACTERISTIC CLASS VIA QUASI-MORPHISM

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Abstract. In this paper, we characterize the second bounded characteristic classes of foliated bundles in terms of the non-descendible quasi-morphisms on the universal covering of the structure group. As its application, we study the boundedness of obstruction classes for (contact) Hamiltonian fibrations and show the non-existence of foliated structures on some Hamiltonian fibrations. Moreover, for any closed symplectic manifold, we show the non-triviality of the second bounded cohomology group of the Hamiltonian diffeomorphism group.

1. Key Theorems

Let $G$ be a connected topological group which admits the universal covering $\pi: \tilde{G} \to G$ and $G^\delta$ denote the group $G$ with the discrete topology. Cohomology classes of the classifying spaces $BG$ and $BG^\delta$ are considered as universal characteristic classes of principal $G$-bundles and foliated $G$-bundles (or flat $G$-bundles), respectively. In this paper, we concentrate our interest on the characteristic classes in degree two. The identity homomorphism $\iota: G^\delta \to G$ induces a continuous map $B\iota: BG^\delta \to BG$ and a homomorphism $B\iota^*: H^2(BG; \mathbb{R}) \to H^2(BG^\delta; \mathbb{R})$.

In this article, an element of $\text{Im}(B\iota^*)$ is simply called a characteristic class of foliated $G$-bundles. Hence in our terminology, if a characteristic class is non-zero for a foliated $G$-bundle $E$, the bundle $E$ is non-trivial not only as a foliated $G$-bundle but also as a $G$-bundle.

Let $H^2_{\text{grp}}(G; \mathbb{R})$ and $H^2_{\text{b}}(G; \mathbb{R})$ be the second group cohomology and second bounded cohomology of $G$, respectively. Then, there is a canonical map $c_G: H^2_{\text{b}}(G; \mathbb{R}) \to H^2_{\text{grp}}(G; \mathbb{R})$ called the comparison map. A group cohomology class $\alpha \in H^2_{\text{grp}}(G; \mathbb{R})$ is called bounded if it is in the image of $c_G$.

Since the cohomology group $H^2(BG^\delta; \mathbb{R})$ is canonically isomorphic to $H^2_{\text{grp}}(G; \mathbb{R})$, we can consider the intersection

$$\text{Im}(c_G) \cap \text{Im}(B\iota^*)$$

as a subspace of $H^2_{\text{grp}}(G; \mathbb{R})$. This intersection $\text{Im}(c_G) \cap \text{Im}(B\iota^*)$ is the vector space of bounded characteristic classes of foliated $G$-bundles.

Our main theorem stated below characterizes the space $\text{Im}(c_G) \cap \text{Im}(B\iota^*)$ in terms of the homogeneous quasi-morphisms on the universal covering $\tilde{G}$ of $G$. Let $Q(\tilde{G})$ and $Q(G)$ be the vector space of all homogeneous quasi-morphisms on $\tilde{G}$ and on $G$, respectively (see Subsection 3.1 for the definition). Let $\pi^*: Q(G) \to Q(\tilde{G})$ be the pullback induced from $\pi: \tilde{G} \to G$.

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Theorem 1.1 (Theorem 6.3). There exists an isomorphism

\[ Q(\tilde{G})/(H^1_{\text{grp}}(\tilde{G}; \mathbb{R}) + \pi^*Q(G)) \cong \text{Im}(c_G) \cap \text{Im}(B^*). \]

The following corollary, which is mainly used in applications, immediately follows from Theorem 1.1.

Corollary 1.2 (Corollary 6.5). Let \( G \) be a topological group whose universal covering \( \tilde{G} \) satisfies \( H^1_{\text{grp}}(\tilde{G}; \mathbb{R}) = 0 \). Then there exists an isomorphism

\[ Q(\tilde{G})/\pi^*Q(G) \cong \text{Im}(c_G) \cap \text{Im}(B^*) \left( \subset H^2_{\text{grp}}(G; \mathbb{R}) \right). \]

In particular, if \( \mu \in Q(\tilde{G}) \) does not descend to \( G \), i.e., \( \mu \notin \pi^*Q(G) \), then \( \mu \) gives rise to a non-trivial element of \( H^2_{\text{grp}}(G; \mathbb{R}) \).

In the present paper, we apply the above results to the group of (contact) Hamiltonian diffeomorphisms. As an important topic of symplectic and contact topology, many researchers have studied quasi-morphisms on these groups (for examples, see [EP03], [CG04], [FY00a], [Fy00b], [FOOO19], [BZ15] and [FPR18]). By combining these outcomes and our results (Theorem 1.1 and Corollary 1.2), we obtain some results on the ordinary group cohomology of these groups (see Corollaries 2.1 and 2.2, Example 2.4, 2.6, 2.5 and 2.7, Corollary 2.9, Proposition A.2 and A.4).

Remark 1.3. Let \( G = \text{Homeo}_+(S^1) \) be the group of orientation preserving homeomorphisms of the circle. By the theorem of Thurston [Thu74], we have \( H^2_{\text{grp}}(G; \mathbb{R}) = \text{Im}(B^*) \cong \mathbb{R} \cdot e \), where \( e \) is the Euler class of \( \text{Homeo}_+(S^1) \). It is known that the space \( Q(\tilde{G}) \) is spanned by Poincaré’s rotation number \( \text{rot}: \tilde{G} \to \mathbb{R} \), that is, \( Q(\tilde{G}) \cong \mathbb{R} \cdot \text{rot} \) (see [Ghy01]). Therefore we have \( Q(\tilde{G}) \cong H^2_{\text{grp}}(G; \mathbb{R}) \). Note that the cohomology \( H^2_{\text{grp}}(G; \mathbb{R}) \) is equal to \( \text{Im}(B^*) \cap \text{Im}(c_G) \) since the Euler class is bounded. Moreover, the space \( Q(\tilde{G}) \) is equal to \( Q(\tilde{G})/(H^1_{\text{grp}}(\tilde{G}; \mathbb{R}) + \pi^*Q(G)) \) since \( G \) is uniformly perfect and \( \tilde{G} \) is perfect. Thus, Theorem 1.1 can be seen as a generalization of this isomorphism to an arbitrary topological group.

2. Applications to symplectic and contact geometry

We apply Corollary 1.2 to symplectic and contact geometry. A symplectic manifold \( (M, \omega) \) has the natural transformation group \( \text{Ham}(M, \omega) \) called the group of Hamiltonian diffeomorphisms [Ban97] DEFINITION 4.2.4. [PR14] Subsection 1.2]. A contact manifold \( (M, \xi) \) also has the natural transformation group \( \text{Cont}_0(M, \xi) \) called the group of contact Hamiltonian diffeomorphisms [Gei08].

2.1. Boundedness of characteristic classes. It is an interesting and difficult problem to determine whether a given characteristic class is bounded. The Milnor-Wood inequality ([Mil58], [Woo71]) asserts that the Euler class of foliated SL(2, \( \mathbb{R} \))-bundles (and foliated Homeo\(_+_i(S^1)\)-bundles) is bounded. It was shown that any element of \( \text{Im}(B^*) \) is bounded for any real algebraic subgroups of \( GL(n, \mathbb{R}) \) ([Gro82]) and for any virtually connected Lie group with linear radical ([CMPSC11]).

As far as the authors know, for homeomorphism groups and diffeomorphism groups, in contrast, the boundedness of characteristic classes is known only for the following specific examples.

- The Euler class of \( \text{Homeo}_+(S^1) \) is bounded [Woo71].
- The Godbillon-Vey class integrated along the fiber on \( \text{Diff}_+(S^1) \) is unbounded [Thu72].
- Any non-zero cohomology class of \( \text{Homeo}_0(\mathbb{R}^2) \) is unbounded [Chal94].
- Any non-zero cohomology class of \( \text{Homeo}_0(T^2) \) is unbounded, where \( T^2 \) is the two-dimensional torus [MR18].
• Some cohomology classes of Homeo₀(M) are unbounded, where M is a closed Seifert-fibered 3-manifold such that the inclusion SO(2) → Homeo₀(M) induces an inclusion of π₁(SO(2)) as a direct factor in π₁(Homeo₀(M)) \cite{Man20} (see also \cite{MN21}).

Using Corollary 1.2 we show the boundedness and unboundedness of characteristic classes on (contact) Hamiltonian diffeomorphism groups. Let us consider the symplectic manifold (S² × S², \omega_λ) and the contact manifold (S³, \xi). The symplectic form \omega_λ is defined by \omega_λ = pr₁^* \omega_0 + \lambda \cdot pr₂^* \omega_0, where \omega_0 is the area form on S² and pr₂ : S² × S² → S² is the j-th projection. The contact structure \xi is the standard one on S³.

To simplify the notation, we set G_λ = Ham(S² × S², \omega_λ) and H = Cont₀(S³, ξ). For 1 < \lambda ≤ 2, we have H²(BG_λ; \mathbb{Z}) ≅ \mathbb{Z} and H²(BH; \mathbb{Z}) ≅ \mathbb{Z} (see Section 6). Let \phi_λ ∈ H²(BG_λ; \mathbb{Z}) and \phi_H ∈ H²(BH; \mathbb{Z}) be the generators (or the “primary obstruction classes with coefficients in \mathbb{Z}” of G_λ-bundles and H-bundles, respectively).

Using Corollary 1.2 we can clarify the difference between these classes in terms of the boundedness. For any c ∈ H².grp(G; \mathbb{Z}), let c_ρ ∈ H².grp(G; \mathbb{R}) denote the corresponding cohomology class with coefficients in \mathbb{R}.

**Corollary 2.1.** The following properties hold.

1. The cohomology class
   
   \[ B_1^*(\phi_\lambda)_\mathbb{R} ∈ H^2.grp(G_\lambda; \mathbb{R}) \]
   
   is bounded.

2. The cohomology class
   
   \[ B_1^*(\phi_H)_\mathbb{R} ∈ H^2.grp(H; \mathbb{R}) \]
   
   is unbounded.

We will prove Corollary 2.1 in Section 6. In order to show Corollary 2.1 we use Ostrover’s Calabi quasi-morphism, which is a Hamiltonian Floer theoretic invariant.

Moreover, we will show the Milnor-Wood type inequality in Section 7 (Theorem 7.1). Applying it to the obstruction class (\phi_\lambda)_\mathbb{R}, we obtain the following:

**Corollary 2.2.** Let \Sigma_h be a closed orientable surface of genus h ≥ 1. Then, there exist infinitely many isomorphism classes of Hamiltonian fibrations over \Sigma_h with the structure group G_λ = Ham(S² × S², \omega_λ) which do not admit foliated G_λ-bundle structures.

**Remark 2.3.** The boundedness of c and c_ρ are equivalent, that is, the integer cohomology class c is bounded if and only if the real cohomology class c_ρ is bounded (this is shown by the same arguments in \cite{CMPS11} Lemma 29). Hence, the statement same as in Corollary 2.1 holds for the integer coefficients cohomology classes B_1^*(\phi_\lambda) and B_1^*(\phi_H).

Corollaries 2.1 and 2.2 will be restated in more general form (see Corollary 6.6 and Theorem 7.2 respectively).
al.). From these homogeneous quasi-morphisms on \(\text{Ham}(M, \omega)\), we can construct non-trivial elements of \(H^2_\text{grp}(\text{Ham}(M, \omega); \mathbb{R})\) under the canonical map
\[
d: Q(\text{Ham}(M, \omega)) \to H^2_\text{grp}(\text{Ham}(M, \omega); \mathbb{R})
\]
(for the map \(d\), see Section 3). Note that the classes obtained by this map are trivial as ordinary group cohomology classes in \(H^2_\text{grp}(\text{Ham}(M, \omega); \mathbb{R})\).

On the other hand, using Corollary 2.2 and homogeneous quasi-morphisms on the universal covering groups, we can construct non-trivial second bounded cohomology class of \(\text{Ham}(M, \omega)\) and \(\text{Cont}_0(M, \xi)\), which are also non-trivial as ordinary cohomology classes. Note that the universal covering \(\text{Ham}(M, \omega)\) and \(\text{Cont}_0(M, \xi)\) are perfect for closed symplectic and contact manifolds ([Ban78], [Ryb10]). Therefore these groups satisfy the assumption in Corollary 1.2.

In the following cases, Corollary 1.2 provides non-trivial cohomology classes in \(H^2_\text{grp}(\text{Ham}(M, \omega); \mathbb{R})\) and \(H^2_\text{grp}(\text{Cont}_0(M, \xi); \mathbb{R})\).

**Example 2.4.** Ostrover [Ost06] constructed quasi-morphism \(\mu^\lambda\) on \(\widehat{\text{Ham}}(S^2 \times S^2, \omega_\lambda)\) for \(\lambda > 1\). The homogeneous quasi-morphism \(\mu^\lambda\) does not descend to \(\text{Ham}(S^2 \times S^2, \omega_\lambda)\) (Proposition 2.2). Hence, we obtain a non-trivial cohomology class of \(\text{Ham}(S^2 \times S^2, \omega_\lambda)\) from \(\mu^\lambda\).

**Example 2.5.** Ostrover and Tyomkin [OT09] constructed two homogeneous quasi-morphisms \(\mu_1, \mu_2: \widehat{\text{Ham}}(M, \omega) \to \mathbb{R}\) when \((M, \omega)\) is the 1 points blow up of \(\mathbb{C}P^2\) with some toric symplectic form. The restrictions of \(\mu_1, \mu_2\) to \(\pi_1(\text{Ham}(M, \omega))\) are linear independent. Hence, Corollary 1.2 implies the dimension of \(H^2_\text{grp}(\text{Ham}(M, \omega); \mathbb{R})\) is larger than one.

**Example 2.6.** Fukaya, Oh, Ohta and Ono [FOOO19] THEOREM 1.10 (3) constructed quasi-morphisms on \(\text{Ham}(M, \omega)\) when \((M, \omega)\) is the \(k\) points blow up of \(\mathbb{C}P^2\) with some toric symplectic form, where \(k \geq 2\). Their quasi-morphisms do not descend to \(\text{Ham}(M, \omega)\) [FOOO19] THEOREM 30.13. Hence, we can construct a non-trivial element of \(H^2_\text{grp}(\text{Ham}(M, \omega); \mathbb{R})\) from their quasi-morphisms.

**Example 2.7.** Givental [Giv90] constructed a homogeneous quasi-morphism \(\mu\) on \(\text{Cont}_0(\mathbb{R}P^{2n+1}, \xi)\) that is called the non-linear Maslov index (see also Sim07, BZ15). This quasi-morphism \(\mu\) does not descend to \(\text{Cont}_0(\mathbb{R}P^{2n+1}, \xi)\). Hence we obtain a non-trivial element of \(H^2_\text{grp}(\text{Cont}_0(\mathbb{R}P^{2n+1}, \xi); \mathbb{R})\).

In Section 6 we also show the following:

**Corollary 2.8.** Let \((M, \omega)\) be a closed symplectic manifold. Then there exists an injective homomorphism
\[
\delta_0 : Q(\widehat{\text{Ham}}(M, \omega)) \to H^2_\text{grp}(\text{Ham}(M, \omega); \mathbb{R}).
\]

In [She14], for every closed symplectic manifold \((M, \omega)\), Shelukhin constructed a non-trivial homogeneous quasi-morphism \(\mu_S: \widehat{\text{Ham}}(M, \omega) \to \mathbb{R}\). Therefore, the following corollary follows from Corollary 2.8.

**Corollary 2.9.** For every closed symplectic manifold \((M, \omega)\), the bounded cohomology group \(H^2_\text{grp}(\text{Ham}(M, \omega); \mathbb{R})\) is non-zero.

**Remark 2.10.** Quasi-morphisms in EP03, Ost06, OT09, McD10, FOOO19, Ush11, Bor13, Cas17 and Vial18 are constructed via the Hamiltonian Floer theory. As good textbooks on this topic, we refer to PR14 and FOOO19.

**Disclaimer 2.11.** Throughout the present paper, we tacitly assume that topological group \(G\) is path-connected, locally path-connected, and semi-locally simply-connected. In particular, every topological group \(G\) in the present paper admits the universal covering \(\pi: \tilde{G} \to G\).
2.3. Organization of the paper. Section 3 collects preliminary facts. Section 4 and Section 5 are devoted to show an isomorphism theorem (Theorem 5.4) for an arbitrary group extension. In Section 6 we prove Theorem 1.1 by applying the isomorphism theorem to a topological group and its universal covering. We give applications in Section 7 and Section 8. In Section 6, we prove a Milnor-Wood type inequality and show the non-existence of foliated structures on Hamiltonian fibrations. In Section 5, we consider an extension problem of homomorphisms on $\pi_1(G)$ to $\tilde{G}$. In Appendix A we give examples of non-trivial (contact) Hamiltonian fibrations.

3. Preliminaries

3.1. (Bounded) group cohomology and quasi-morphism. We briefly review the (bounded) cohomology of (discrete) group and the quasi-morphism. Let $G$ be a group and $A$ an abelian group. Let $C_{\text{grp}}^n(G; A)$ denote the set of $n$-cochains $c: G^n \to A$ and $\delta: C_{\text{grp}}^n(G; A) \to C_{\text{grp}}^{n+1}(G; A)$ the coboundary map. For $c \in C_{\text{grp}}^1(G; A)$, its coboundary $\delta c \in C_{\text{grp}}^2(G; A)$ is defined by

$$\delta c(g_1, g_2) = c(g_1) + c(g_2) - c(g_1 g_2)$$

for $g_1, g_2 \in G$ (see [Bro82] for the precise definition of $\delta$). The cohomology $H_{\text{grp}}^n(G; A)$ of the cochain complex $(C_{\text{grp}}^\bullet(G; A), \delta)$ is called the (ordinary) group cohomology of $G$.

It is known that the cohomology of group $G$ is canonically isomorphic to the cohomology of classifying space $BG^d$ of discrete group $G^d$. This isomorphism is given by an isomorphism of cochains (see, for example, [Dup78]). Under this isomorphism, we identify $H^n(BG^d; A)$ with $H^n_{\text{grp}}(G; A)$.

Let $A = \mathbb{Z}$ or $\mathbb{R}$. Let $C_{\text{grp}}^n(G; A)$ denote the set of bounded $n$-cochains, i.e., $c \in C_{\text{grp}}^n(G; A)$ such that

$$\|c\|_\infty = \sup_{g_1, \ldots, g_n \in G} |c(g_1, \ldots, g_n)| < +\infty.$$  

The cohomology $H_{\text{grp}}^n(G; A)$ of the cochain complex $(C_{\text{grp}}^\bullet(G; A), \delta)$ is called the bounded cohomology of $G$. The inclusion map from $C_{\text{grp}}^\bullet(G; A)$ to $C_{\text{grp}}^\bullet(G; A)$ induces the homomorphism $c_{\text{grp}}: H_{\text{grp}}^\bullet(G; A) \to H_{\text{grp}}^\bullet(G; A)$, which is called the comparison map.

Definition 3.1. A real-valued function $\mu$ on a group $G$ is called a quasi-morphism if

$$D(\mu) = \sup_{g, h \in G} |\mu(gh) - \mu(g) - \mu(h)|$$

is finite. The value $D(\mu)$ is called the defect of $\mu$. A quasi-morphism $\mu$ on $G$ is called homogeneous if $\mu(g^n) = n\mu(g)$ for all $g \in G$ and $n \in \mathbb{Z}$. Let $Q(G)$ denote the real vector space of homogeneous quasi-morphisms on $G$.

It is known that any homogeneous quasi-morphism is conjugation-invariant, that is, $\mu \in Q(G)$ satisfies

$$\mu(ghg^{-1}) = \mu(h)$$

for any $g, h \in G$ (see [Cal09] Section 2.2.3 for example).

By definition, the coboundary $\delta \mu$ of a homogeneous quasi-morphism $\mu \in Q(G)$ defines a bounded two-cocycle on $G$. This induces the following exact sequence

$$0 \to H_{\text{grp}}^1(G; \mathbb{R}) \to Q(G) \xrightarrow{D} H_{\text{grp}}^2(G; \mathbb{R}) \xrightarrow{\text{defect}} H_{\text{grp}}^2(G; \mathbb{R})$$

(see [Cal09] Theorem 2.50 for example).
The following property of homogeneous quasi-morphisms is important in the present paper:

\[ \mu(fg) = \mu(f) + \mu(g) = \mu(gf) \]

(see [PRT14] Proposition 3.1.4) for any \( f, g \in G \) with \( fg = gf \).

In the present paper, we often refer to Ostrover’s Calabi quasi-morphism and so we explain here. Let \( (S^2 \times S^2, \omega) \) be the symplectic manifold defined in Subsection 2.1. Entov and Polterovich [EP04] constructed a homogeneous quasi-morphism \( \mu^\lambda \) on \( \text{Ham}(S^2 \times S^2, \omega_1) \) using the Hamiltonian Floer theory. More precisely, \( \mu^1 \) is constructed as the homogenization of Oh-Schwarz’s spectral invariants, which is a Hamiltonian Floer theoretic invariant [Sch00, Ost05]. (See also Remark 2.10.) They also proved that \( \mu^1 \) descends to \( \text{Ham}(S^2 \times S^2, \omega_1) \).

After their work, Ostrover [Ost06] applied Entov-Polterovich’s idea to \( \widehat{\text{Ham}}(S^2 \times S^2, \omega_\lambda) \) for \( \lambda > 1 \) and studied a quasi-morphism \( \mu^\lambda : \widehat{\text{Ham}}(S^2 \times S^2, \omega_\lambda) \to \mathbb{R} \). In contrast to Entov-Polterovich’s quasi-morphisms, Ostrover’s Calabi quasi-morphism \( \mu^\lambda \) does not descend to \( \text{Ham}(S^2 \times S^2, \omega_\lambda) \).

**Proposition 3.2** ([Ost06]). For \( \lambda > 1 \), there exists \( \tilde{\gamma} \in \pi_1(\text{Ham}(S^2 \times S^2, \omega_\lambda)) \) such that \( \mu(\tilde{\gamma}) \neq 0 \). In particular, \( \mu^\lambda \) does not descend to \( \text{Ham}(S^2 \times S^2, \omega_\lambda) \).

### 3.2. Characteristic classes

For a fibration, the primary obstruction class is defined as an obstruction to the construction of a cross-section. We briefly recall the definition of the obstruction class via the Serre spectral sequence (see [Whi78] for details). Let \( F \to E \to B \) be a fibration. For simplicity, we suppose the following: the base space \( B \) is one-connected, the fiber \( F \) is path-connected, and the fundamental group \( \pi_1(F) \) is abelian. Let \((E^p_q, \partial^p_q)\) be the Serre spectral sequence with coefficients in \( \pi_1(F) \). Since \( B \) is one-connected, any local coefficient system on \( B \) is simple, and therefore we have

\[ E^2_{0,0} \cong H^0(B; H^0(F; \pi_1(F))). \]

Hence we obtain \( E^2_{2,0} \cong H^2(B; \pi_1(F)) \) and \( E^2_{0,1} \cong H^1(F; \pi_1(F)) \). Since the cohomology group \( H^1(F; \pi_1(F)) \) is isomorphic to \( \text{Hom}(\pi_1(F), \pi_1(F)) \), the derivation map \( \partial^2_{0,1} : E^2_{0,1} \to E^2_{2,0} \) defines a map

\[ \partial^2_{0,1} : \text{Hom}(\pi_1(F), \pi_1(F)) \to H^2(B; \pi_1(F)). \]

Here we abuse the symbol \( \partial^2_{0,1} \).

We are now ready to state the definition of the primary obstruction class of fibrations.

**Definition 3.3.** Let \( F \to E \to B \) be a fibration such that \( B \) is one-connected, \( F \) is path-connected, and \( \pi_1(F) \) is abelian. Let \((E^p_q, \partial^p_q)\) be the Serre spectral sequence of the fibration. The cohomology class \( \alpha(E) = -\partial^2_{0,1}(\text{id}_{\pi_1(F)}) \in H^2(B; \pi_1(F)) \) is called the primary obstruction class of \( E \), where \( \text{id}_{\pi_1(F)} \in \text{Hom}(\pi_1(F), \pi_1(F)) \) is the identity homomorphism.

**Remark 3.4.** It is known that the above definition is equivalent to the classical definition of the obstruction class to the construction of a cross-section (see, for example, [Whi78] (6.10) Corollary in Chapter VI and (7.9*) Theorem in Chapter XIII).

By the naturality of the spectral sequence, the primary obstruction class is a characteristic class. Its universal element \( \phi \) is given as the primary obstruction class of the principal universal bundle \( G \to EG \to BG \). Note that the classifying space \( BG \) is one-connected and \( \pi_1(G) \) is abelian.
Remark 3.5. The class $\phi$ is also obtained as follows. By taking classifying spaces of the central extension $0 \to \pi_1(G) \to \tilde{G} \to G \to 1$, we obtain the following fibration

(3.4) \[ B\pi_1(G) \to B\tilde{G} \to BG. \]

Note that the fundamental group of $B\pi_1(G)$ is isomorphic to $\pi_1(G)$ and this is abelian. Then, the primary obstruction class of fibration (3.4) is the class $\phi \in H^2(BG; \pi_1(G))$.

Let $f : \pi_1(G) \to \mathbb{R}$ be a homomorphism and

\[ f_* : H^\bullet(-; \pi_1(G)) \to H^\bullet(-; \mathbb{R}) \]

denote the change of coefficients homomorphism. Let $(E^{p,q}_r, d^{p,q}_r)$ be the Serre spectral sequence of (3.4) with coefficients in $\mathbb{R}$. Since $E^{0,1}_2 \cong H^1(B\pi_1(G); \mathbb{R}) \cong \text{Hom}(\pi_1(G); \mathbb{R})$ and $E^{2,0}_2 \cong H^2(BG; \mathbb{R})$, the derivation $d^{0,1}_2 : E^{0,1}_2 \to E^{2,0}_2$ defines a homomorphism

\[ d^{0,1}_2 : \text{Hom}(\pi_1(G), \mathbb{R}) \to H^2(BG; \mathbb{R}). \]

Proposition 3.6. Let $(E^{p,q}_r, d^{p,q}_r)$ be the Serre spectral sequence of (3.4) with coefficients in $\mathbb{R}$. For a homomorphism $f : \pi_1(G) \to \mathbb{R}$, the equality

\[ -d^{0,1}_2(f) = f_*\phi \in H^2(BG; \mathbb{R}) \]

holds.

Proof. Let $(E^{p,q}_r, d^{p,q}_r)$ be the Serre spectral sequence of (3.4) with coefficients in $\pi_1(G)$. Then the equality $-d^{0,1}_2(id_{\pi_1(G)}) = \phi$ holds. Since the derivation maps in the Serre spectral sequence is compatible with the change of coefficients homomorphisms, we have the following commutative diagram

\[
\begin{array}{ccc}
\text{Hom}(\pi_1(G), \mathbb{R}) & \cong & E^{0,1}_2 \\
\downarrow{f_*} & & \downarrow{f_*} \\
\text{Hom}(\pi_1(G), \mathbb{R}) & \cong & E^{2,0}_2 \\
\end{array}
\]

Since $f = f_* (id_{\pi_1(G)})$, we obtain

\[ -d^{0,1}_2(f) = -d^{0,1}_2(f_* (id_{\pi_1(G)})) = f_*(-d^{0,1}_2(id_{\pi_1(G)})) = f_*\phi \]

and the proposition follows. \qed

Remark 3.7. Let $(E^{p,q}_r, d^{p,q}_r)$ be the Serre spectral sequence of (3.4) with coefficients in $\mathbb{R}$. Then, the map $d^{0,1}_2$ is an isomorphism. Indeed, the $E_2$-page of the spectral sequence induces an exact sequence

\[ 0 \to H^1(BG; \mathbb{R}) \to H^1(B\tilde{G}; \mathbb{R}) \to H^1(B\pi_1(G); \mathbb{R}) \]

\[ \xrightarrow{d^{0,1}_2} H^2(BG; \mathbb{R}) \to H^2(B\tilde{G}; \mathbb{R}). \]

Since $\tilde{G}$ is one-connected, the classifying space $B\tilde{G}$ is two-connected. Hence the cohomology groups $H^1(B\tilde{G}; \mathbb{R})$ and $H^2(BG; \mathbb{R})$ are trivial, and this implies that the derivation map $d^{0,1}_2$ is an isomorphism. In particular, the class $f_*\phi = -d^{0,1}_2(f)$ is non-zero if and only if the homomorphism $f$ is non-zero.
4. Construction of group cohomology classes

Let us consider an exact sequence
\begin{equation}
1 \to K \xrightarrow{j} \Gamma \xrightarrow{\pi} G \to 1
\end{equation}
of discrete groups.

**Definition 4.1.** A subspace $\mathcal{C}(\Gamma)$ of $C^1(\Gamma; A)$ is defined by
\begin{equation}
\mathcal{C}(\Gamma) = \{ F \in C^1_{\text{grp}}(\Gamma; A) | F(k) = F(\gamma) + F(\gamma k) \text{ for any } \gamma, k \in K \}.
\end{equation}

We define a map $\mathfrak{D}: \mathcal{C}(\Gamma) \to C^2_{\text{grp}}(G; A)$ by setting
\[ \mathfrak{D}(F)(g_1, g_2) = F(\gamma_2) - F(\gamma_1, \gamma_2) + F(\gamma_1), \]
where $\gamma_j$ is an element of $\Gamma$ satisfying $\pi(\gamma_j) = g_j$.

**Lemma 4.2.** The map $\mathfrak{D}: \mathcal{C}(\Gamma) \to C^2_{\text{grp}}(G; A)$ is well-defined.

**Proof.** Let $\gamma'_j$ be another element of $\Gamma$ satisfying $\pi(\gamma'_j) = g_j$. Then there exist $k_1, k_2 \in K$ satisfying $\gamma'_1 = k_1 \gamma_1$ and $\gamma'_2 = k_2 \gamma_2$. By the definition of $\mathcal{C}(\Gamma)$, we have
\[ F(\gamma'_j) - F(\gamma'_1, \gamma'_2) + F(\gamma'_2) = F(k_2 \gamma k) - F(k, 1) + F(k_1) + F(\gamma) \]
\[ = F(k_2 \gamma k) - F(k_1, k_2) + F(k_1) + F(\gamma), \]
This implies the well-definedness of the map $\mathfrak{D}$. \hfill \Box

**Lemma 4.3.** For any $F \in \mathcal{C}(\Gamma)$, the cochain $\mathfrak{D}(F)$ is a cocycle.

**Proof.** Since $\pi^* \mathfrak{D}(F) = -\delta F$ by the definition of $\mathfrak{D}(f)$, we have
\[ \pi^*(\delta \mathfrak{D}(F)) = -\delta \pi^* F = 0. \]
By the surjectivity of $\pi: \Gamma \to G$, we have $\delta \mathfrak{D}(F) = 0$. \hfill \Box

**Definition 4.4.** A homomorphism $\mathfrak{d}: \mathcal{C}(\Gamma) \to H^2_{\text{grp}}(G; A)$ is defined by
\[ \mathfrak{d}(F) = [\mathfrak{D}(F)] \in H^2_{\text{grp}}(G; A). \]

For an element $F$ of $\mathcal{C}(\Gamma)$, the restriction $F|_K = i^* F$ to $K$ is a homomorphism. Moreover, $F|_K$ is $\Gamma$-invariant since
\[ F(\gamma^{-1} k \gamma) = F(\gamma^{-1} k \gamma) - F(\gamma) = F(k \gamma) - F(\gamma) = F(k). \]
Let $H^1_{\text{grp}}(K; A)\Gamma$ denote the space of $\Gamma$-invariant homomorphisms from $K$ to $A$. Then the restriction to $K$ defines a homomorphism $i^*: \mathcal{C}(\Gamma) \to H^1_{\text{grp}}(K; A)\Gamma$.

**Lemma 4.5.** The homomorphism $i^*: \mathcal{C}(\Gamma) \to H^1_{\text{grp}}(K; A)\Gamma$ is surjective.

**Proof.** Let $s: G \to \Gamma$ be a section of $p: \Gamma \to G$ satisfying $s(1_G) = 1_\Gamma$, where $1_G \in G$ and $1_\Gamma \in \Gamma$ are the unit elements of $G$ and $\Gamma$, respectively. Since $\gamma \cdot s(\pi(\gamma))^{-1}$ is in $\text{Ker}(\pi: \Gamma \to G)$, we regard $\gamma \cdot s(\pi(\gamma))^{-1}$ as an element of $K$ under the injection $i: K \to \Gamma$. For an element $f$ of $H^1_{\text{grp}}(K; A)\Gamma$, define $f_s: \Gamma \to A$ by
\[ f_s(\gamma) = f(\gamma \cdot s(\pi(\gamma))^{-1}). \]
Note that the restriction of $f_s$ to $K$ is equal to $f$. Moreover, the equalities
\[ f_s(k \gamma) = f(k \gamma \cdot s(\pi(k \gamma))^{-1}) = f(k \gamma \cdot s(\pi(k \gamma))^{-1}) = f_s(k) + f_s(\gamma) \]
and
\[ f_s(\gamma k) = f(\gamma k \cdot s(\pi(\gamma k))^{-1}) = f(\gamma k \cdot s(\pi(\gamma)))^{-1} \]
\[ = f(\gamma k \gamma^{-1}) + f_s(\gamma \cdot s(\pi(\gamma))^{-1}) \]
\[ = f(k) + f_s(\gamma \cdot s(\pi(\gamma))^{-1}) \]
\[ = f_s(k) + f_s(\gamma \cdot s(\pi(\gamma k))^{-1}) = f_s(k) + f_s(\gamma) \]
hold, where we use the \( \Gamma \)-invariance of \( f \) in the second equalities. Hence \( f_s \) is an element of \( \mathcal{C}(\Gamma) \) and the surjectivity follows. \( \square \)

For sequence (4.1), there is an exact sequence
\[ 0 \rightarrow H^1_{\grp}(G; \mathbb{R}) \overset{\partial}{\rightarrow} H^1_{\grp}(\Gamma; \mathbb{R}) \overset{i^*}{\rightarrow} H^1_{\grp}(K; \mathbb{R})^\Gamma \]
\[ \overset{\partial}{\rightarrow} H^2_{\grp}(G; \mathbb{R}) \overset{i^*}{\rightarrow} H^2_{\grp}(\Gamma; \mathbb{R}) \]
called the five-term exact sequence. This five-term exact sequence is obtained by the Hochschild-Serre spectral sequence \( (E_p^q) \) of (4.1), and the map \( \partial \) is the derivation \( \partial^0_{2}: E^0_{2} = H^1_{\grp}(K; \mathbb{R})^\Gamma \rightarrow E^2_{0} = H^2_{\grp}(G; \mathbb{R}) \).

**Lemma 4.6.** The diagram
\[ \begin{array}{ccc}
\mathcal{C}(\Gamma) & \xrightarrow{\delta} & H^1_{\grp}(K; A)^\Gamma \\
\downarrow i^* & & \downarrow \tau \\
H^1_{\grp}(K; A)^\Gamma & \xrightarrow{\partial} & H^2_{\grp}(G; A). \\
\end{array} \]
commutes.

**Proof.** By Definition 4.3 and Proposition 4.7 below, the commutativity follows. \( \square \)

**Proposition 4.7 ([NSW08] (1.6.6) Proposition).** For any \( \Gamma \)-invariant homomorphism \( f \in H^1_{\grp}(K; A)^\Gamma \), there exists a one-cochain \( F: \Gamma \rightarrow A \) such that \( i^*F = f \) and that \( \delta F(\gamma_1, \gamma_2) \) depends only on \( \pi(\gamma_1) \) and \( \pi(\gamma_2) \), that is, there exists a cocycle \( c \in C^2_{\grp}(G; A) \) satisfying \( c(\pi(\gamma_1), \pi(\gamma_2)) = \delta F(\gamma_1, \gamma_2) \) for any \( \gamma_1, \gamma_2 \in \Gamma \). Moreover, the class \( \tau(F) \) is equal to \([c] \in H^2_{\grp}(G; A)\).

5. A DIAGRAM VIA BOUNDED COHOMOLOGY AND QUASI-MORPHISM

From this section, we mainly consider cohomology with coefficients in \( \mathbb{R} \). In this section, we refine the commutative diagram in view of bounded cohomology and homogeneous quasi-morphism. Recall that a cohomology class \( \alpha \in H^2_{\grp}(G; \mathbb{R}) \) is called **bounded** if \( \alpha \) is in the image of the comparison map \( c_G: H^2_{\grp}(G; \mathbb{R}) \rightarrow H^2_{\grp}(G; \mathbb{R}) \).

**Proposition 5.1.** There is a commutative diagram
\[ \begin{array}{ccc}
\mathcal{C}(\Gamma) \cap Q(\Gamma) & \xrightarrow{\delta} & H^2_{\grp}(G; \mathbb{R}) \\
\downarrow i^* & & \downarrow \tau \\
H^1_{\grp}(K; \mathbb{R})^\Gamma & \xrightarrow{\partial} & H^2_{\grp}(G; \mathbb{R}). \\
\end{array} \]

**Proof.** Let \( F \) be an element of \( \mathcal{C}(\Gamma) \cap Q(\Gamma) \). Then, the cocycle \( \mathcal{D}(F) \) is bounded since \( F \) is a quasi-morphism and \( \mathcal{D}(F)(g_1, g_2) = F(\gamma_2) - F(\gamma_1 \gamma_2) + F(\gamma_1) \) for any \( g_1, g_2 \in G \) and their lifts \( \gamma_1, \gamma_2 \in \Gamma \). Hence the homomorphism \( \mathcal{D}: \mathcal{C}(\Gamma) \rightarrow C^2_{\grp}(G; \mathbb{R}) \) induces a homomorphism
\[ \delta_b: \mathcal{C}(\Gamma) \cap Q(\Gamma) \rightarrow H^2_{\grp}(G; \mathbb{R}). \]
By the definition of the comparison map $c_G$, we have $\mathcal{d} = c_G \circ \mathcal{d}_b$. 

**Remark 5.2.** For a central extension

$$0 \to A \xrightarrow{i} \Gamma \xrightarrow{\pi} G \to 1,$$

the space $Q(\Gamma)$ is contained in $C(\Gamma)$. Indeed, by the definition of central extension, we have $a\gamma = \gamma a$ for any $a \in A$ and $\gamma \in \Gamma$. Hence, by (3.3), any homogeneous quasi-morphism $\mu \in Q(\Gamma)$ satisfies

$$\mu(a\gamma) = \mu(\gamma a) = \mu(a) + \mu(\gamma).$$

This implies that $Q(\Gamma) \subset C(\Gamma)$. Moreover, any homomorphism $f : A \to \mathbb{R}$ is $\Gamma$-invariant since $\gamma^{-1}a\gamma = a\gamma^{-1}\gamma = a$ for any $\gamma \in \Gamma$ and any $a \in A$. Hence, together with Proposition 5.1, we obtain the following commutative diagram

$$\begin{array}{ccc}
Q(\Gamma) & \xrightarrow{\delta_b} & H^2_b(G; \mathbb{R}) \\
\downarrow \phi & & \downarrow c_G \\
H^1_{\text{grp}}(A; \mathbb{R}) & \xrightarrow{\iota} & H^2_{\text{grp}}(G; \mathbb{R})
\end{array}$$

for a central extension $0 \to A \to \Gamma \to G \to 1$.

**Lemma 5.3.** Let $\mu$ be a homogeneous quasi-morphism on $\Gamma$ whose restriction to $K$ is a homomorphism. Then $\mu$ is contained in $C(\Gamma)$.

**Proof.** For any $\gamma \in \Gamma$, $k \in K$, and $n \in \mathbb{N}$, the equalities

$$(k\gamma)^n = k \cdot \gamma k \gamma^{-1} \cdot \gamma^2 k \gamma^{-2} \cdot \ldots \cdot \gamma^{n-1} k \gamma^{-(n-1)} \cdot \gamma^n$$

and

$$(\gamma k)^n = \gamma^n \gamma^{-1} k \gamma^{-1} \cdot \gamma^2 k \gamma^{-2} \cdot \gamma^3 k \gamma^{-3} \cdot \ldots \cdot \gamma^{n-1} k \gamma^{-(n-1)}$$

hold. By (3.13), the restriction $\mu|_K$ is $\Gamma$-invariant. Hence we have

$$\mu(k \cdot \gamma k \gamma^{-1} \cdot \gamma^2 k \gamma^{-2} \cdot \ldots \cdot \gamma^{n-1} k \gamma^{-(n-1)}) = \mu(k^n)$$

and

$$\mu(\gamma^{-1} k \gamma^{-1} \cdot \gamma^{-2} k \gamma^{-2} \cdot \gamma^{-3} k \gamma^{-3} \cdot \ldots \cdot \gamma^{n-1} k \gamma^{-(n-1)}) = \mu(\gamma^n).$$

These equalities imply that

$$n \cdot |\mu(k\gamma) - \mu(k) - \mu(\gamma)| = |\mu((k\gamma)^n) - \mu(k^n) - \mu(\gamma^n)| < D(\mu)$$

and

$$n \cdot |\mu(\gamma k) - \mu(\gamma) - \mu(k)| = |\mu((\gamma k)^n) - \mu(\gamma^n) - \mu(k^n)| < D(\mu).$$

Hence we obtain $\mu(k\gamma) = \mu(k) + \mu(\gamma)$ and $\mu(\gamma k) = \mu(\gamma) + \mu(k)$. 

**Theorem 5.4.** The homomorphism $\mathcal{d} : C(\Gamma) \to H^2_{\text{grp}}(G; \mathbb{R})$ induces an isomorphism

$$(C(\Gamma) \cap Q(\Gamma))/(H^1_{\text{grp}}(\Gamma; \mathbb{R}) + \pi^*Q(G)) \to \text{Im}(\pi) \cap \text{Im}(c_G).$$
Proof. Let us consider the following commutative diagram whose rows and columns are exact:

\[
\begin{array}{ccccccccc}
H^1(K; \mathbb{R})^Γ & \rightarrow & Q(K)^Γ & \rightarrow & H^2(K; \mathbb{R})^Γ \\
\uparrow & & \uparrow & & \uparrow \\
H^1(Γ; \mathbb{R}) & \rightarrow & Q(Γ) & \rightarrow & H^2(Γ; \mathbb{R}) & \rightarrow & H^2(Γ; \mathbb{R})^Γ \\
\downarrow & \pi^* & \downarrow & \pi^* & \downarrow & \pi^* \\
H^1(G; \mathbb{R}) & \rightarrow & Q(G) & \rightarrow & H^2(G; \mathbb{R}) & \rightarrow & H^2(G; \mathbb{R})^Γ \\
& & 0 & \rightarrow & H^1(K; \mathbb{R})^Γ, & & \\
\end{array}
\]

where the exactness of the third column was shown in [Bou95]. By the definition of \( \partial_0 \), we have \( \pi^* \partial_0(μ) = d(μ) \) for \( μ ∈ C(Γ) \cap Q(Γ) \). Hence the map \( \pi^* : H^2_0(Γ; \mathbb{R}) \rightarrow H^2(Γ; \mathbb{R}) \) gives an isomorphism

\[ \pi^* : H^2_0(Γ; \mathbb{R}) \xrightarrow{\cong} d(C(Γ) \cap Q(Γ)). \]

Then, in this diagram, the map \( d \) is given as the composite

\[ c_G \circ (π^*)^{-1} \circ d : C(Γ) \cap Q(Γ) \rightarrow H^2_0(Γ; \mathbb{R}). \]

The equality \( \text{Ker}(d) = H^1_{\text{grp}}(Γ; \mathbb{R}) + π^* Q(G) \) is verified by a diagram chasing argument. By Lemma 5.3 and a diagram chasing argument, the surjectivity of the map \( d : C(Γ) \cap Q(Γ) \rightarrow \text{Im}(π) \cap \text{Im}(c_G) \) also follows. \( \square \)

Remark 5.5. For a central extension \( Γ \) of \( G \), the homomorphism \( d : C(Γ) \rightarrow H^2_0(G; \mathbb{R}) \) induces an isomorphism

\[ Q(Γ)/(H^1_{\text{grp}}(Γ; \mathbb{R}) + π^* Q(G)) \rightarrow \text{Im}(π) \cap \text{Im}(c_G) \]

since \( C(Γ) \cap Q(Γ) = Q(Γ) \) (see Remark 5.2).

6. ON TOPOLOGICAL GROUPS

6.1. General topological groups. Let \( G \) be a topological group and \( π : \tilde{G} \rightarrow G \) the universal covering. Since the exact sequence

\[ 0 \rightarrow π_1(G) \rightarrow \tilde{G} \xrightarrow{π} G \rightarrow 1 \]

is a central extension ([Pon86, Theorem 15]), we obtain the commutative diagram

\[
\begin{array}{cccccc}
Q(\tilde{G}) & \rightarrow & H^2_0(Γ; \mathbb{R}) \\
\downarrow & \pi^* & \downarrow & c_G \\
H^1_{\text{grp}}(π_1(G); \mathbb{R}) & \rightarrow & H^2_{\text{grp}}(G; \mathbb{R})
\end{array}
\]

by Remark 5.2. Moreover, by Remark 5.5, the homomorphism \( d : Q(\tilde{G}) \rightarrow H^2_{\text{grp}}(G; \mathbb{R}) \) induces an isomorphism

\[ Q(\tilde{G})/(H^1_{\text{grp}}(\tilde{G}; \mathbb{R}) + π^* Q(G)) \rightarrow \text{Im}(π) \cap \text{Im}(c_G) \]

In this section, we clarify the relation between the class \( d(μ) ∈ H^2_{\text{grp}}(G; \mathbb{R}) \) and the primary obstruction class \( o ∈ H^2(BG; π_1(G)) \).
By taking the classifying spaces of Corollary 6.3, we obtain a commutative diagram of fibrations

\[ \begin{array}{ccc}
B\pi_1(G) & \longrightarrow & B\tilde{G}^g \\
\downarrow \quad & & \downarrow \\
B\pi_1(G) & \longrightarrow & B\tilde{G}
\end{array} \]

In what follows, we regard the pullback \( Bt^* : H^*(BG; \mathbb{R}) \to H^*(BG^g; \mathbb{R}) \) as a homomorphism

\[ Bt^* : H^*(BG; \mathbb{R}) \to H^*_{\text{grp}}(G; \mathbb{R}) \]

under the isomorphism \( H^*(BG^g; \mathbb{R}) \cong H^*_{\text{grp}}(G; \mathbb{R}) \).

**Lemma 6.1.** Let \( (E^r_q, d^r_q) \) be the \( \mathbb{R} \)-coefficient cohomology Serre spectral sequence of the fibration \( B\pi_1(G) \to B\tilde{G} \to B\tilde{G}^g \). Then the equality

\[ Bt^* \circ d^0_{2} = \tau : H^1_{\text{grp}}(\pi_1(G); \mathbb{R}) \to H^2_{\text{grp}}(G; \mathbb{R}) \]

holds, where we identify \( E^0_{2,1} = H^1(B\pi_1(G); \mathbb{R}) \) with \( H^1_{\text{grp}}(\pi_1(G); \mathbb{R}) \).

**Proof.** Let \( (\delta E^p_q, \delta d^p_q) \) be the Hochschild-Serre spectral sequence of central extension Corollary 6.2. Note that the spectral sequence \( (\delta E^p_q, \delta d^p_q) \) is isomorphic to the Serre spectral sequence of the fibration \( B\pi_1(G) \to B\tilde{G}^g \to B\tilde{G}^g \) (see [Ben91] for example). Since the map \( \tau \) is equal to the derivation map \( \delta d^0_{1,1} \) by definition, the naturality of the Serre spectral sequence asserts that

\[ Bt^* \circ d^0_{2} = \delta d^0_{2} = \tau, \]

and the lemma follows. \( \square \)

**Corollary 6.2.** Let \( \sigma \in H^2(BG; \mathbb{R}) \) be the primary obstruction class for \( G \)-bundles. Then, for any homogeneous quasi-morphism \( \mu \in Q(\tilde{G}) \), the equality

\[ \Phi(\mu) = -Bt^*((\mu|_{\pi_1(G)}), \sigma) \]

holds.

**Proof.** Let \( (E^r_q, d^r_q) \) be the Serre spectral sequence as in Lemma 6.1. Using Proposition 3.6 we obtain

\[ Bt^* \circ d^0_{2}(\mu|_{\pi_1(G)}) = -Bt^*((\mu|_{\pi_1(G)}), \sigma). \]

On the other hand, using Lemma 6.1 and commutative diagram 6.2, we obtain

\[ Bt^* \circ d^0_{2}(\mu|_{\pi_1(G)}) = \tau(\mu|_{\pi_1(G)}) = \tau(i^*(\mu)) = \sigma(\mu). \]

Hence the equality \( \Phi(\mu) = -Bt^*((\mu|_{\pi_1(G)}), \sigma) \) holds. \( \square \)

**Corollary 6.3.** If \( H^1(\tilde{G}; \mathbb{R}) \) is trivial, then the homomorphism

\[ Bt^* : H^2(BG; \mathbb{R}) \to H^2_{\text{grp}}(G; \mathbb{R}) \]

is injective.

**Proof.** By the five-term exact sequence

\[ 0 \to H^1_{\text{grp}}(G; \mathbb{R}) \to H^1_{\text{grp}}(\tilde{G}; \mathbb{R}) \to H^1_{\text{grp}}(\pi_1(G); \mathbb{R}) \]

\[ \to H^2_{\text{grp}}(G; \mathbb{R}) \to H^2_{\text{grp}}(\tilde{G}; \mathbb{R}), \]

the triviality of \( H^1_{\text{grp}}(\tilde{G}; \mathbb{R}) \) implies the injectivity of the map \( \tau \). Hence the map \( Bt^* \) is injective by Lemma 6.1 and Remark 3.7. \( \square \)
Theorem 6.4. The homomorphism $\vartheta: Q(\tilde{G}) \to H^2_{\text{grp}}(G; \mathbb{R})$ induces an isomorphism

$$Q(\tilde{G})/(H^2_{\text{grp}}(G; \mathbb{R}) + \pi^*Q(G)) \to \text{Im}(B\tau^*) \cap \text{Im}(c_G).$$

Proof. The equality

$$\text{Im}(B\tau^*) = \text{Im}(\tau)$$

holds by Lemma 6.1 and Remark 3.7. Hence, isomorphism 6.3 implies the theorem. □

The following corollary immediately follows from Theorem 6.3.

Corollary 6.5. If the first cohomology $H^1_{\text{grp}}(\tilde{G}; \mathbb{R})$ is trivial, then the homomorphism $\vartheta$ induces an isomorphism

$$Q(\tilde{G})/\pi^*Q(G) \to \text{Im}(B\tau^*) \cap \text{Im}(c_G).$$

In particular, if $\mu \in Q(\tilde{G})$ does not descend to $G$, then the class $\vartheta(\mu) \in H^2_{\text{grp}}(G; \mathbb{R})$ is non-zero.

By using Corollary 6.2, Corollary 6.3 and Theorem 6.4, we obtain the following corollary.

Corollary 6.6. Let $G$ be a topological group and $\tilde{G}$ the universal covering of $G$.

1. Let $\mu: \tilde{G} \to \mathbb{R}$ be a homogeneous quasi-morphism which does not descend to $G$. Let $\varphi \in H^2(BG; \pi_1(G))$ denote the primary obstruction class of $G$. Then, the cohomology class

$$B\varphi^*(((\mu|_{\pi_1(G)})\varphi)_{\mathbb{R}}) \in H^2_{\text{grp}}(G; \mathbb{R})$$

is bounded. Here, $(\mu|_{\pi_1(G)})\varphi: H^2(BG; \pi_1(G)) \to H^2(BG; \mathbb{R})$ is the change of coefficients homomorphism induced from $\mu|_{\pi_1(G)}: \pi_1(G) \to \mathbb{R}$.

2. Assume that the space $Q(\tilde{G})$ is trivial. Then, for any non-zero element $c$ of $H^2(BG; \mathbb{R})$, a cohomology class

$$(B\varphi)^*(c) \in H^2_{\text{grp}}(G; \mathbb{R})$$

is unbounded.

6.2. Hamiltonian and contact Hamiltonian diffeomorphism groups. We set $G_\lambda = \text{Ham}(S^2 \times S^2, \omega_\lambda)$ and $H = \text{Cont}_0(S^3, \xi)$. For $1 < \lambda \leq 2$, it is known that $\pi_1(G_\lambda) \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (Anjos02) and $\pi_1(H) \cong \mathbb{Z}$ (Eliashberg92). By Remark 3.7, we have

$$H^2(BG_\lambda; \mathbb{Z}) \cong H^1(B\pi_1(G_\lambda); \mathbb{Z}) \cong \text{Hom}(\pi_1(G_\lambda), \mathbb{Z}) \cong \mathbb{Z}$$

and

$$H^2(BH; \mathbb{Z}) \cong H^1(B\pi_1(H); \mathbb{Z}) \cong \text{Hom}(\pi_1(H), \mathbb{Z}) \cong \mathbb{Z}.$$
Proof of Corollary 2.8. First, we prove (1). Recall that the restriction \( \mu^\lambda|_{\pi_1(G_\lambda)} \) of Ostrover’s Calabi quasi-morphism is a non-trivial homomorphism to \( \mathbb{R} \) (Proposition 6.2). Hence there exists a non-zero constant \( a \) such that
\[
\phi = a\mu^\lambda|_{\pi_1(G_\lambda)} : \pi_1(G_\lambda) \to \mathbb{R},
\]
where \( \phi \) is the homomorphism given as (6.4). Therefore we have
\[
Bt^*(o_{G_\lambda})_\mathbb{R} = a \cdot Bt^*((\mu|_{\pi_1(G_\lambda)}))_\mathbb{R}.
\]
Since Ostrover’s Calabi quasi-morphism does not descend to \( G_\lambda \), the class \( Bt^*(o_{G_\lambda})_\mathbb{R} \) is bounded by Corollary 6.6 (1).

Next, we prove (2). Because the universal covering group \( \tilde{H} = \widehat{\text{Cont}}_0(S^3, \xi) \) is uniformly perfect (see [FPK18, Corollary 3.6 and Remark 3.7]), we have \( Q(\tilde{H}) = 0 \). By Corollary 6.6 (2), the class \( Bt^*(o_{\tilde{H}})_\mathbb{R} \) is unbounded. \( \square \)

In Section 7, we provide another proof of Corollary 2.1 (2) by using a Milnor-Wood type inequality (Theorem 7.1) instead of Corollary 6.6 (2) (see Remark 7.4).

We end this section with a proof of Corollary 2.8. To do this, we prepare the following lemma.

Lemma 6.7. For any topological group \( G \) whose universal covering group \( \tilde{G} \) satisfies \( H^1_{\text{grp}}(\tilde{G}; \mathbb{R}) = 0 \), the map
\[
\vartheta_b : Q(\tilde{G}) \to H^2_b(\tilde{G}; \mathbb{R})
\]
is injective.

Proof. By exact sequence (3.2) and the assumption \( H^1_{\text{grp}}(\tilde{G}; \mathbb{R}) = 0 \), the map \( Q(\tilde{G}) \to H^2_b(\tilde{G}; \mathbb{R}) \) is injective. Hence, for any homogeneous quasi-morphism \( \mu \in Q(\tilde{G}) \), the bounded cohomology class \( [\delta\mu] \in H^2_b(\tilde{G}; \mathbb{R}) \) is non-zero. Since
\[
\pi^*(\vartheta_b(\mu)) = [\delta\mu] \in H^2_b(\tilde{G}; \mathbb{R}),
\]
where \( \pi^* : H^2_b(G; \mathbb{R}) \to H^2_b(\tilde{G}; \mathbb{R}) \) is the homomorphism induced by the universal covering \( \pi : \tilde{G} \to G \), the class \( \vartheta_b(\mu) \) is also non-zero. \( \square \)

Proof of Corollary 2.8. Because \( \widehat{\text{Ham}}(M, \omega) \) is perfect ([Ban78], Lemma 6.7 implies Corollary 2.8). \( \square \)

7. Milnor-Wood type inequality and bundles with no flat structures

In this section, we show the existence of bundles over a surface which do not admit foliated (flat) structures. To this end, first we introduce a Milnor-Wood type inequality.

Let \( e \) be a universal characteristic class of foliated principal \( G \)-bundles. Then the characteristic class \( e \) is given as an element in \( H^2(\text{BG}^\delta; \mathbb{R}) \). For a foliated principal \( G \)-bundle \( G \to E \to B \), the characteristic class \( e(E) \) of \( E \) associated to \( e \) is defined by
\[
e(E) = f^* e \in H^2(B; \mathbb{R}),
\]
where \( f : B \to \text{BG}^\delta \) is the classifying map of \( E \).

Let \( \Sigma_h \) denote a closed oriented surface of genus \( h \geq 1 \) and \( G \to E \to \Sigma_h \) be a foliated \( G \)-bundle. Let \( \rho : \pi_1(\Sigma_h) \to G \) be a holonomy homomorphism of the bundle \( E \). Then the classifying map of the bundle \( E \) is given by
\[
B\rho : \Sigma_h \simeq B\pi_1(\Sigma_h) \to \text{BG}^\delta.
\]
Theorem 7.1. Let \( c \) be an element of \( \text{Im}(c_G) \cap \text{Im}(B t^*) \) and \([\Sigma_h] \in H_2(\Sigma_h; \mathbb{Z})\) the fundamental class of \( \Sigma_h \). Then, for any foliated principal \( G \)-bundle \( G \to E \to \Sigma_h \), an inequality
\[ |\langle c(E), [\Sigma_h] \rangle| \leq D(\mu)(4h - 4) \]
holds, where \( \mu \in Q(\hat{G}) \) is a homogeneous quasi-morphism satisfying \( \mathcal{O}(\mu) = c \).

Proof. By Theorem 6.4, there exists a homogeneous quasi-morphism \( \mu \in Q(\hat{G}) \) satisfying \( \mathcal{O}(\mu) = [\mathcal{O}(\mu)] = c \). Since \( \pi^* \mathcal{O}(\mu) = \delta \mu \), we have
\[ \|\mathcal{O}(\mu)\|_{\infty} = \|\delta \mu\|_{\infty} = D(\mu). \]
In particular, we have \( \|\mathcal{O}(\mu)\|_{\infty} \leq D(\mu) \). Let \( \rho: \pi_1(\Sigma_h) \to G \) be a holonomy homomorphism associated with the foliated bundle \( G \to E \to \Sigma_h \). Since the operator norm of \( \rho^*: H^2_\rho(G; \mathbb{R}) \to H^2_\rho(\pi_1(\Sigma_h); \mathbb{R}) \) is equal or lower than 1, we have
\[ \|\rho^*(\mathcal{O}(\mu))\|_{\infty} \leq \|\mathcal{O}(\mu)\|_{\infty} \leq D(\mu). \]
Note that the bounded cohomology of a topological space \( X \) is isometrically isomorphic to the bounded cohomology of the fundamental group \( \pi_1(\Sigma) \) [Gro82]. Hence we have
\[ \|c(E), [\Sigma_h]\| \leq \|c\|_{\infty}\|\Sigma_h\| \leq D(\mu)(4h - 4). \]

Theorem 7.2. Let \( G \) be a topological group and \( \Sigma_h \) a closed surface of genus \( h \geq 1 \). Assume that there exist a homogeneous quasi-morphism \( \mu \in Q(\hat{G}) \) and \( \gamma \in \pi_1(G) \) satisfying \( \mu(\gamma) \neq 0 \). Then, there exist infinitely many isomorphism classes of principal \( G \)-bundles over \( \Sigma_h \) which do not admit foliated \( G \)-bundle structures.

Proof. We normalize the homogeneous quasi-morphism \( \mu \) as \( \mu(\gamma) = 1 \) by a non-zero constant multiple. We set \( c = \mathcal{O}(\mu) = B t^*((\mu|_{\pi_1(\Sigma)}), \varnothing) \in H^2(B G^5; \mathbb{R}) \), then \( c \) belongs to \( \text{Im}(c_G) \cap \text{Im}(B t^*) \). Assume that a principal \( G \)-bundle \( E \to \Sigma_h \) admits a foliated structure. Then, there exists a continuous map \( f_\Sigma: \Sigma_h \to B G^5 \) such that \( f = B t \circ f_\Sigma \), where \( f: \Sigma_h \to B G \) is the classifying map of \( E \). Let \( E_\delta \) be a foliated \( G \)-bundle on \( \Sigma_h \) induced by \( f_\delta \). Then,
\[ c(E_\delta) = f_\delta^* c = f_\delta^*(B t^* \mu, \varnothing) = \mu_\delta(f_\delta^* B t^* \varnothing) = \mu_\delta(f^* \varnothing) = \mu_\delta(\varnothing) = \mu(\varnothing). \]
Hence we obtain that
\[ (\langle \mu|_{\pi_1(\Sigma)}, \varnothing(E), [\Sigma_h] \rangle) = (\langle c(E), [\Sigma_h] \rangle) \leq D(\mu)(4h - 4) \tag{7.1} \]
by Theorem 7.1.

For each \( n \in \mathbb{Z} \), we now construct a principal \( G \)-bundle \( E_n \) over \( \Sigma_h \) whose characteristic number \( (\langle \mu|_{\pi_1(\Sigma)}, \varnothing(E_n), [\Sigma_h] \rangle) = n \). Let us fix a triangulation \( T \) of \( \Sigma_h \) and take a triangle \( \Delta \in T \). For \( n \in \mathbb{Z} \), we now take a loop \( \{g_t\}_{0 \leq t \leq 1} \) in \( G \) which represents \( \gamma^n \in \pi_1(G) \). Let \( E \to \Sigma_h \setminus \text{Int}(\Delta) \) and \( E' \to \Delta \) be trivial \( G \)-bundles, where \( \text{Int}(\Delta) \) is the interior of \( \Delta \). Then, we obtain a bundle \( E_n \) by gluing the bundles \( E \) and \( E' \) along \( \partial \Delta \approx S^1 \) with the transition function \( S^1 \to G; t \to g_t \).

Since the class \( \varnothing(E_n) \) is the primary obstruction to the cross-sections (see Remark 3.4), we have \( (\varnothing(E_n), [\Sigma_h]) = \gamma^n \) and therefore we obtain
\[ (\langle \mu|_{\pi_1(\Sigma)}, \varnothing(E_n), [\Sigma_h] \rangle) = \mu(\gamma^n) = n. \]
Hence, by equation (7.1), for a sufficiently large \( n \), \( E_n \) do not admit foliated \( G \)-bundle structures and we complete the proof. \( \square \)
Remark 7.3. Any non-trivial principal $G$-bundle over the 2-sphere $\Sigma_0$ does not admit foliated structures since the fundamental group of $\Sigma_0$ is trivial. Hence, if the order of $\pi_1(G)$ is infinite, there exist infinitely many isomorphism classes of principal $G$-bundles over $\Sigma_0$ which do not admit foliated structures.

Proof of Corollary 2.2. Ostrover’s Calabi quasi-morphism satisfies the assumption in Theorem 7.2. Hence Theorem 7.2 implies the corollary. □

Remark 7.4. As an application of Theorem 7.1 one can prove Corollary 2.1 (2) by constructing explicitly a foliated $H$-bundle with arbitrary large characteristic number. Indeed, for any $N \in \mathbb{Z} = \pi_1(H)$, there exist $2k$ elements $\tilde{g}_1, \ldots, \tilde{g}_{2k}$ of $\tilde{H}$ such that the equality $N = [\tilde{g}_1, \tilde{g}_2] \ldots [\tilde{g}_{2k-1}, \tilde{g}_{2k}]$ since $\tilde{H}$ is uniformly perfect. Note that the number $k$ does not depend on $N$. We set $g_j = p(\tilde{g}_j) \in H$ for any $j$, where $p: \tilde{H} \to H$ is the universal covering. Let $\Sigma_k$ be a closed surface of genus $k$ and $a_j \in \pi_1(\Sigma_k)$ the canonical generator with the relation
$$[a_1, a_2] \ldots [a_{2k-1}, a_{2k}] = 1.$$ 

Let $\varphi: \pi_1(\Sigma_k) \to \text{Cont}_0(S^3, \xi)$ be a homomorphism defined by $\varphi(a_j) = g_j$ for any $j$. Then, the characteristic number of the foliated $H$-bundle with the holonomy homomorphism $\varphi$ is equal to $N$ (this computation of the characteristic number is known as Milnor’s algorithm [Mi58]).

8. Non-extendability of homomorphisms on $\pi_1(G)$ to homogeneous quasi-morphisms on $\tilde{G}$

In Section 2.1, we use the homogeneous quasi-morphisms on the universal covering $\tilde{G}$ to show the (un)boundedness of characteristic classes. In this section, on the contrary, we use the (un)boundedness of characteristic classes to study the extension problem of homomorphism on $\pi_1(G)$ to $\tilde{G}$. The extension problem of homomorphisms and homogeneous quasi-morphisms have been studied by some researchers (for example, see [Ish14, Shi16, KK19, KKMM20, KKMM21, Mar22]).

Let $T = S^1 \times S^1$ be the two-dimensional torus and $\text{Homeo}_0(T)$ the identity component of the homeomorphism group of $T$ with respect to the compact-open topology. In [Ham65], it was shown that the fundamental group $\pi_1(\text{Homeo}_0(T))$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

Corollary 8.1. Any non-trivial homomorphism in $\text{Hom}(\pi_1(\text{Homeo}_0(T)), \mathbb{R})$ cannot be extended to $\text{Homeo}_0(T)$ as a homogeneous quasi-morphism.

Proof. It is enough to show that the equality
$$Q(\widehat{\text{Homeo}_0(T)}) = \pi^* Q(\text{Homeo}_0(T))$$
holds, where $\pi: \widehat{\text{Homeo}_0(T)} \to \text{Homeo}_0(T)$ is the universal covering. Because the universal covering $\text{Homeo}_0(T)$ is perfect [KR11], we have
$$Q(\widehat{\text{Homeo}_0(T)})/\pi^* Q(\text{Homeo}_0(T)) = \text{Im}(c_G) \cap \text{Im}(Bt^*)$$
by Corollary 1.2. Because any non-zero classes in $\text{Im}(Bt^*)$ are unbounded [MR18], we have $Q(\widehat{\text{Homeo}_0(T)})/\pi^* Q(\text{Homeo}_0(T)) = 0$, and the corollary holds. □

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Appendix A. Examples of (contact) Hamiltonian fibrations

Recall that $G_\lambda = \text{Ham}(S^2 \times S^2, \omega_\lambda)$, $H = \text{Cont}_0(S^3, \xi)$, and the cohomology classes
$$o_{G_\lambda} \in H^2(BG_\lambda; \mathbb{Z}) \quad \text{and} \quad o_H \in H^2(BH; \mathbb{Z})$$
are the primary obstruction classes. Our main concern in this paper (e.g., Corollary A.1) was the classes $B^r(o_{G_\lambda})_\mathbb{R} \in H^2(BG_\lambda^r; \mathbb{R})$ and $B^r(o_H)_\mathbb{R} \in H^2(BH^r; \mathbb{R})$. In this appendix, we rather use the classes $o_{G_\lambda}$ and $o_H$ to study (not necessarily foliated) Hamiltonian fibrations and contact Hamiltonian fibrations.

We begin with the following general proposition.

**Proposition A.1.** Let $G$ and $K$ be topological groups and $i: G \to K$ be a continuous homomorphism. Assume that the universal covering $\tilde{G}$ is perfect. If there exists a non-trivial element $\tilde{g}$ of $\pi_1(G)$ satisfying $i_*(\tilde{g}) = 0 \in \pi_1(K)$, then there exists a non-trivial principal $G$-bundle $E$ over $\Sigma_h$ such that the bundle $E$ is trivial as a principal $K$-bundle.

**Proof.** By the perfectness of $\tilde{G}$, we can take $\tilde{g}_j \in \tilde{G}$ ($j = 1, \ldots, 2h$) such that $\tilde{g} = [\tilde{g}_1, \tilde{g}_2] \cdots [\tilde{g}_{2h-1}, \tilde{g}_{2h}]$. Let us define a homomorphism $\rho: \pi_1(\Sigma_h) \to G$ by setting
$$\rho(a_j) = \pi(\tilde{g}_j)$$
for any $j$, where $\pi: \tilde{G} \to G$ is the universal covering. Then the principal $G$-bundle $G \to E_\rho \to \Sigma_h$ associated to the holonomy homomorphism $\rho$ is non-trivial (see [Mi83]). We show the principal $K$-bundle $E_{i\circ\rho}$ is trivial. By the assumption of $\tilde{g}$, we have
$$0 = i_*(\tilde{g}) = [i_*(\tilde{g}_1), i_*(\tilde{g}_2)] \cdots [i_*(\tilde{g}_{2h-1}), i_*(\tilde{g}_{2h})].$$
Let us define $i \circ \rho: \pi_1(\Sigma_h) \to \tilde{K}$ by
$$i \circ \rho(a_j) = i_*(\tilde{g}_j),$$
then this map $i \circ \rho$ is a homomorphism satisfying $\pi \circ (i \circ \rho) = i \circ \rho$, where $\pi: \tilde{K} \to K$ is the universal covering. Thus the classifying map $B(i \circ \rho): B\pi_1(\Sigma_h) \to BG \to BK$ factors into
$$B(i \circ \rho) = B\pi \circ B(i \circ \rho): \Sigma_h \simeq B\pi_1(\Sigma_h) \to B\tilde{K} \to BK.$$ Since the fundamental group and second homotopy group of the classifying space $B\tilde{K}$ are trivial, the map
$$B(i \circ \rho): \Sigma_h \to B\tilde{K}$$
is null-homotopic and so is the classifying map $B(i \circ \rho)$ of the bundle $E_{i\circ\rho}$. Thus the bundle $E_{i\circ\rho}$ is a trivial bundle. \qed

### A.1. Contact Hamiltonian fibrations.

Let $M$ be a manifold with a contact structure $\xi$. Let $\text{Cont}_0(M, \xi)$ be a contact Hamiltonian diffeomorphism group, that is, the identity component of the group
$$\text{Cont}(M, \xi) = \{g \in \text{Diff}(M) \mid g^*\xi = \xi\}$$
with the $C^\infty$-topology. A fiber bundle $M \to E \to B$ is called a contact Hamiltonian fibration if the structure group is reduced to the contact Hamiltonian diffeomorphism group.

The orientation preserving diffeomorphism group $\text{Diff}_+(S^3)$ of the 3-sphere is homotopy equivalent to $SO(4)$ ([Hat83]). Hence the fundamental group $\pi_1(\text{Diff}_+(S^3))$
is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Let $\xi$ be the standard contact structure on the 3-sphere. The fundamental group of $\text{Cont}_0(S^3, \xi)$ is isomorphic to $\mathbb{Z}$ (\cite{Eli92}, \cite{CS16}). Let $i$: $\text{Cont}_0(S^3, \xi) \rightarrow \text{Diff}_+(S^3)$ be the inclusion, then the induced map
\[ i_\ast: \pi_1(\text{Cont}_0(S^3, \xi)) \cong \mathbb{Z} \rightarrow \pi_1(\text{Diff}_+(S^3)) \cong \mathbb{Z}/2\mathbb{Z} \]
is surjective (\cite{CS16}). Let $\tilde{g} \in \pi_1(\text{Cont}_0(S^3, \xi))$ be a non-zero even number in $\mathbb{Z} \cong \pi_1(\text{Cont}_0(S^3, \xi))$, then we have $i_\ast(\tilde{g}) = 0 \in \pi_1(\text{Diff}_+(S^3)) \cong \mathbb{Z}/2\mathbb{Z}$. By the perfectness of $\text{Cont}_0(S^3, \xi)$ (\cite{Ryb10}) and Proposition A.1 there is a non-trivial principal $\text{Cont}_0(S^3, \xi)$-bundle over a closed surface that is trivial as a principal $\text{Diff}_+(S^3)$-bundle. In other words, there is a sphere bundle that is non-trivial as a contact Hamiltonian fibration but trivial as an oriented sphere bundle.

For a contact Hamiltonian fibration $S^3 \rightarrow E \rightarrow \Sigma_h$, let $\sigma(E) \in H^2(\Sigma_h; \mathbb{Z})$ be the obstruction class. Let $\chi(E) \in \mathbb{Z}$ denote the characteristic number
\[ \chi(E) = \langle \sigma(E), [\Sigma_h] \rangle. \]

\textbf{Proposition A.2.} Let $S^3 \rightarrow E \rightarrow \Sigma_h$ be a foliated contact Hamiltonian fibration. If the characteristic number $\chi(E) \in \mathbb{Z}$ is even, the bundle $E$ is trivial as an oriented sphere bundle. If $\chi(E)$ is odd, the bundle $E$ is non-trivial as an oriented sphere bundle.

\textit{Proof.} Let $p: \pi_1(\Sigma_h) \rightarrow \text{Cont}_0(S^3, \xi)$ be a holonomy homomorphism of $E$. Set $g_j = \psi(a_j)$ and take lifts $\tilde{g}_j \in \text{Cont}_0(S^3, \xi)$ of $g_j$‘s, where $a_j \in \pi_1(\Sigma_h)$ are the generators. Set $\tilde{g} = [g_1, g_2] \cdots [g_{2g-1}, g_{2g}]$. Then, by the algorithm (\cite{MIL58}) for computing the characteristic number of foliated bundles, we have
\[ \chi(E) = [g_1, g_2] \cdots [g_{2g-1}, g_{2g}] = \tilde{g} \in \mathbb{Z} \cong \pi_1(\text{Cont}_0(S^3, \xi)). \]

If $\chi(E)$ is even, we have $i_\ast(\tilde{g}) = 0 \in \pi_1(\text{Diff}_+(S^3)) \cong \mathbb{Z}/2\mathbb{Z}$. Thus, the bundle $E$ is trivial as an oriented sphere bundle by the same arguments in Proposition A.1. If $\chi(E)$ is odd, we have $i_\ast(\tilde{g}) = 1 \in \pi_1(\text{Diff}_+(S^3)) \cong \mathbb{Z}/2\mathbb{Z}$. Since the characteristic number $\chi(E)$ is non-zero, the bundle $E$ is non-trivial as an oriented sphere bundle. □

A.2. Hamiltonian fibrations. Let $M$ be a manifold with a symplectic form $\omega$. A fiber bundle $M \rightarrow E \rightarrow B$ is called a Hamiltonian fibration if the structure group is reduced to the Hamiltonian diffeomorphism group $\text{Ham}(M, \omega)$.

Let us consider the 4-manifold $S^2 \times S^2$.

By Propositions A.1 and 3.2 we obtain the following:

\textbf{Proposition A.3.} There exists a positive integer $h_0$ and a non-trivial Hamiltonian fibration $p_0: E_0 \rightarrow \Sigma_{h_0}$ over a closed surface.

We can also prove that the Hamiltonian fibration $p_0: E_0 \rightarrow \Sigma_{h_0}$ in Proposition A.3 is stably non-trivial in the following sense.

\textbf{Proposition A.4.} Let $(N, \omega_N)$ be a closed symplectic manifold and $p: \epsilon_N = \Sigma_{h_0} \times N \rightarrow \Sigma_{h_0}$ the trivial $N$-bundle. Then, the Whitney sum
\[ E_0 \oplus \epsilon_N \rightarrow \Sigma_{h_0} \]
is non-trivial as a Hamiltonian fibration.

To prove Proposition A.3 we use the following theorem essentially proved by Entov and Polterovich.

\textbf{Theorem A.5 (Theorem 5.1 of [EP09]).} Let $(N, \omega_N)$ be a closed symplectic manifold. For $\lambda \geq 1$, let $\omega_{\lambda N}$ denote the symplectic form $\text{pr}_1^\ast \omega_\lambda + \text{pr}_2^\ast \omega_N$ where $\text{pr}_1: S^2 \times S^2 \times N \rightarrow S^2 \times S^2$, $\text{pr}_2: S^2 \times S^2 \times N \rightarrow N$ are the first, second projection,
respectively. Then, there exists a function $\mu^{X,N}: \widetilde{\text{Ham}}(S^2 \times S^2 \times N, \omega_{X,N}) \rightarrow \mathbb{R}$ such that

$$\mu^{X,N}(\tilde{\phi}_N) = \mu^X(\tilde{\phi})$$

for every $\tilde{\phi} \in \widetilde{\text{Ham}}(S^2 \times S^2, \omega_{X,N})$.

Here, $\tilde{\phi}_N$ is the element of $\widetilde{\text{Ham}}(S^2 \times S^2 \times N, \omega_{X,N})$ represented by the path $\{\tilde{\phi}_N^t\}_{t \in [0,1]}$ defined by $\tilde{\phi}_N^t(x,y) = (\phi^t(x), y)$ where $\{\phi^t\}_{t \in [0,1]}$ is a path in $\text{Ham}(S^2 \times S^2 \times N, \omega_X)$ representing $\tilde{\phi}$.

**Remark A.6.** The function $\mu^{X,N}$ satisfies the conditions of “partial Calabi quasi-morphism” ([Ent14, Theorem 3.2]). However, the authors do not know whether the restriction of $\mu^{X,N}$ to the fundamental group is homomorphism or not.

**Proof of Proposition A.4.** Let $\tilde{g} = \{\tilde{g}^t\}_{t \in [0,1]}$ be a path in $\text{Ham}(S^2 \times S^2, \omega_X)$ corresponding to the bundle $E_0$. Define a loop $\tilde{g}_N = \{\tilde{g}_N^t\}_{t \in [0,1]}$ in $\text{Ham}(S^2 \times S^2 \times N, \omega_{X,N})$ by $\tilde{g}_N^t(x,y) = (g^t(x), y)$. Then, by Theorem A.3 and Proposition 3.2 we have that $\mu^{X,N}(\tilde{g}_N) = \mu^X(\tilde{g}) \neq 0$, in particular, $\tilde{g}_N$ is a non-trivial element of $\pi_1(\text{Ham}(S^2 \times S^2 \times N, \omega_{X,N}))$. By Proposition A.1, the proposition follows. \qed

**References**

[Anj02] Silvia Anjos, *Homology type of symplectomorphism groups of $S^2 \times S^2$*, Geom. Topol. **6** (2002), 195–218.

[Ban78] Augustin Banyaga, *Sur la structure du groupe des difféomorphismes qui préservent une forme symplectique*, Comment. Math. Helv. **53** (1978), no. 2, 174–217.

[Ban97] ———, *The structure of classical diffeomorphism groups*, Mathematics and its Applications, vol. 400, Kluwer Academic Publishers Group, Dordrecht, 1997.

[Ben91] D. J. Benson, *Representations and cohomology*, Cambridge Studies in Advanced Mathematics, vol. 2, Cambridge University Press, 1991.

[BG92] J. Barge and É. Ghys, *Cocycles d’Euler et de Maslov*, C. R. Acad. Sci. Paris Sér. I Math. **314** (1992), no. 2, 235–265.

[Bor13] Matthew Strom Borman, *Quasi-states, quasi-morphisms, and the moment map*, Int. Math. Res. Not. IMRN (2013), no. 11, 2497–2533.

[Bou95] Abdessalam Bouarich, *Suites exactes en cohomologie bornée réelle des groupes discrets*, C. R. Acad. Sci. Paris Sér. I Math. **320** (1995), no. 11, 1355–1359.

[Bra15] Michael Brandenbursky, *B-s-invariant metrics and quasi-morphisms on groups of Hamiltonian diffeomorphisms of surfaces*, Internat. J. Math. **26** (2015), no. 9, 1550066, 29.

[Bro82] Kenneth S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, vol. 87, Springer-Verlag, New York-Berlin, 1982.

[BZ15] Matthew Strom Borman and Frol Zapolsky, *Quasimorphisms on contactomorphism groups and contact rigidity*, Geom. Topol. **19** (2015), no. 1, 365–411.

[Cal04] Danny Calegari, *Circular groups, planar groups, and the Euler class*, Proceedings of the Casson Fest, Geom. Topol. Monogr., vol. 7, Geom. Topol. Publ., Coventry, 2004, pp. 431–491.

[Cal09] ———, * scl*, MSJ Memoirs, vol. 20, Mathematical Society of Japan, Tokyo, 2009.

[Cas17] Alexander Caviedes Castro, *Calabi quasi-morphisms for monotone coadjoint orbits*, J. Topol. Anal. **9** (2017), no. 4, 689–706.

[CMPSC11] Indira Chatterji, Guido Mislin, Christophe Pittet, and Laurent Saloff-Coste, *A geometric criterion for the boundedness of characteristic classes*, Math. Ann. **351** (2011), no. 3, 541–569.

[CS16] Roger Casals and Oldřich Spáčil, *Chern-Weil theory and the group of strict contactomorphisms*, J. Topol. Anal. **8** (2016), no. 1, 59–87.

[Dup78] Johan L. Dupont, *Curvature and characteristic classes*, Lecture Notes in Mathematics, Vol. 640, Springer-Verlag, Berlin-New York, 1978.

[Eli92] Yakov Eliashberg, *Contact 3-manifolds twenty years since J. Martinet’s work*, Ann. Inst. Fourier (Grenoble) **42** (1992), no. 1-2, 165–192.

[Ent14] Michael Entov, *Quasi-morphisms and quasi-states in symplectic topology*, Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. II, Kyung Moon Sa, Seoul, 2014, pp. 1147–1171.
References

[EP03] Michael Entov and Leonid Polterovich, Calabi quasimorphism and quantum homology, Int. Math. Res. Not. (2003), no. 30, 1635–1676.

[EP09] ———, Rigid subsets of symplectic manifolds, Compos. Math. 145 (2009), no. 3, 773–826.

[FOOO19] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono, Spectral invariants with bulk, quasi-morphisms and Lagrangian Floer theory, Mem. Amer. Math. Soc. 260 (2019), no. 1254, x+266.

[FPR18] Maia Fraser, Leonid Polterovich, and Daniel Rosen, On Sandon-type metrics for contactomorphism groups, Ann. Math. Qué. 42 (2018), no. 2, 191–214.

[Fri17] Roberto Frigerio, Bounded cohomology of discrete groups, Mathematical Surveys and Monographs, vol. 227, American Mathematical Society, Providence, RI, 2017.

[Gei08] Hansjörg Geiges, An introduction to contact topology, Cambridge Studies in Advanced Mathematics, vol. 109, Cambridge University Press, Cambridge, 2008.

[Ghy01] Étienne Ghys, Groups acting on the circle, Enseign. Math. (2) 47 (2001), no. 3-4, 329–407.

[Giv90] A. B. Givental, Nonlinear generalization of the Maslov index, Theory of singularities and its applications, Adv. Soviet Math., vol. 1, Amer. Math. Soc., Providence, RI, 1990, pp. 71–103.

[Ham65] Mary-Elizabeth Hamstrom, The space of homeomorphisms on a torus, Illinois J. Math. 9 (1965), 59–65.

[Hat83] Allen E. Hatcher, A proof of the Smale conjecture, Diff(S^3) ≃ O(4), Ann. of Math. (2) 117 (1983), no. 3, 553–607.

[Ish14] Tomohiko Ishida, Quasi-morphisms on the group of area-preserving diffeomorphisms of the 2-disk via braid groups, Proc. Amer. Math. Soc. Ser. B 1 (2014), 43–51.

[JK02] Tadeusz Januszkiewicz and Jarek Kędra, Characteristic classes of smooth fibrations, 2002.

[KK19] Morimichi Kawasaki and Mitsuaki Kimura, G-invariant quasimorphisms and symplectic geometry of surfaces, to appear in Israel J. Math., arXiv:1911.10855v2 (2019).

[KKMM20] Morimichi Kawasaki, Mitsuaki Kimura, Takahiro Matsushita, and Masato Mimura, Bavard’s duality theorem for mixed commutator length, arXiv:2007.02257v3, to appear in Enseign. Math. (2020).

[KKMM21] Morimichi Kawasaki, Mitsuaki Kimura, Takahiro Matsushita, and Masato Mimura, Commuting symplectomorphisms on a surface and the flux homomorphism, preprint, arXiv:2102.12161v1 (2021).

[KR11] Agnieszka Kowalik and Tomasz Rybicki, On the homeomorphism groups of manifolds and their universal coverings, Cent. Eur. J. Math. 9 (2011), no. 6, 1217–1231.

[Mar20] Shuhei Maruyama, The dixmier-douady class, the action homomorphism, and group cocycles on the symplectomorphism group, 2020.

[Mar22] ———, Extensions of quasi-morphisms to the symplectomorphism group of the disk, Topology Appl. 305 (2022), Paper No. 107880.

[McD04] Dusa McDuff, Lectures on groups of symplectomorphisms, Rend. Circ. Mat. Palermo (2) Suppl. (2004), no. 72, 43–78.

[McD10] ———, Monodromy in Hamiltonian Floer theory, Comment. Math. Helv. 85 (2010), no. 1, 95–133.

[Mil58] John Milnor, On the existence of a connection with curvature zero, Comment. Math. Helv. 32 (1958), 215–223.

[MN21] Nicolas Monod and Sam Nariman, On the bounded cohomology of certain homeomorphism groups, preprint, arXiv:2111.04365 (2021).

[MR18] Kathryn Mann and Christian Rosendal, Large-scale geometry of homeomorphism groups, Ergodic Theory Dynam. Systems 38 (2018), no. 7, 2748–2779.

[NSW08] Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg, Cohomology of number fields, second ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 323, Springer-Verlag, Berlin, 2008.
ON BOUNDEDNESS OF CHARACTERISTIC CLASS VIA QUASI-MORPHISM

Yong-Geun Oh, Construction of spectral invariants of Hamiltonian paths on closed symplectic manifolds, The breadth of symplectic and Poisson geometry, Progr. Math., vol. 232, Birkhäuser Boston, Boston, MA, 2005, pp. 525–570.

Yaron Ostrover, Calabi quasi-morphisms for some non-monotone symplectic manifolds, Algebr. Geom. Topol. 6 (2006), 405–434.

Yaron Ostrover and Ilya Tyomkin, On the quantum homology algebra of toric Fano manifolds, Selecta Math. (N.S.) 15 (2009), no. 1, 121–149.

L. S. Pontryagin, Selected works vol. 2, topological groups, Translated from the Russian and with a preface by A. Brown. With additional material translated by P. S. V. Naidu., Gordon and Breach Science Publishers, Inc., New York-London-Paris, 1986.

Leonid Polterovich and Daniel Rosen, Function theory on symplectic manifolds, CRM Monograph Series, vol. 34, American Mathematical Society, Providence, RI, 2014.

Pierre Py, Quasi-morphisms of Calabi and graph of Reeb on the torus, C. R. Math. Acad. Sci. Paris 343 (2006), no. 5, 323–328.

Pierre Py, Quasi-morphisms et invariant de Calabi, Ann. Sci. École Norm. Sup. (4) 39 (2006), no. 1, 177–195.

Alexander G. Reznikov, Characteristic classes in symplectic topology, Selecta Math. (N.S.) 3 (1997), no. 4, 601–642, Appendix D by Ludmil Katzarkov.

Tomasz Rybicki, Commutators of contactomorphisms, Adv. Math. 225 (2010), no. 6, 3291–3326.

Matthias Schwarz, On the action spectrum for closed symplectically aspherical manifolds, Pacific J. Math. 193 (2000), no. 2, 419–461.

Egor Shelukhin, The action homomorphism, quasimorphisms and moment maps on the space of compatible almost complex structures, Comment. Math. Helv. 89 (2014), no. 1, 69–123.

A. I. Sljern, Extension of pseudocharacters from normal subgroups, III, Proc. Jangjeon Math. Soc. 19 (2016), no. 4, 609–614.

Gabi Ben Simon, The nonlinear Maslov index and the Calabi homomorphism, Commun. Contemp. Math. 9 (2007), no. 6, 769–780. MR 2372458

Yasha Savelyev and Egor Shelukhin, K-theoretic invariants of Hamiltonian fibrations, J. Symplectic Geom. 18 (2020), no. 1, 251–289.

William Thurston, Noncobordant foliations of $S^3$, Bull. Amer. Math. Soc. 78 (1972), 511–514.

William Thurston, Foliations and groups of diffeomorphisms, Bull. Amer. Math. Soc. 80 (1974), 304–307.

Michael Usher, Deformed Hamiltonian Floer theory, capacity estimates and Calabi quasi-morphisms, Geom. Topol. 15 (2011), no. 3, 1313–1417.

Renato Vianna, Continuum families of non-displaceable Lagrangian tori in $(\mathbb{C}P^1)^{2m}$, J. Symplectic Geom. 16 (2018), no. 3, 857–883.

George W. Whitehead, Elements of homotopy theory, Graduate Texts in Mathematics, vol. 61, Springer-Verlag, New York-Berlin, 1978.

John W. Wood, Bundles with totally disconnected structure group, Comment. Math. Helv. 46 (1971), 257–273.

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