Infinite Chain of Harmonic Oscillators Under the Action of the Stationary Stochastic Force

A.A. Lykov*    M.V. Melikian*

September 7, 2022

Abstract

We consider countable system of harmonic oscillators on the real line with quadratic interaction potential with finite support and local external force (stationary stochastic process) acting only on one fixed particle. In the case of positive definite potential and initial conditions lying in $l^2(\mathbb{Z})$-space the presentation of the deviations of the particles from their equilibrium points are found. Precisely, deviation of each particle could be represented as the sum of some stationary process (it is also time limiting process in distribution for that function) and the process which converges to zero as $t \to +\infty$ with probability one. The time limit for the mean energy of the whole system is found as well.

1 Introduction

Systems of coupled harmonic oscillators and their generalizations are the classical object of study in mathematical physics, generally speaking, and in physics itself. The point is that in physics the harmonic oscillator model plays an important role, especially in the study of small oscillations of systems near stable equilibrium position. Vibrations of a load attached to the spring (in a horizontal position) could serve as an example of such fluctuations in classical mechanics, in quantum mechanics it can be vibrations of atoms in solids, molecules, etc. Existence of solutions and their ergodic properties have been studied in [25]. There has also been extensive research on the convergence to equilibrium of the countable harmonic chain in contact with the thermostat [2, 13, 26, 27].

In physics methods of study of many systems are of probabilistic nature: typically, researchers describe how the movement of particles in a system affects on the average picture of the behavior of the system as a whole.

Here one cannot fail to note the series of works [1, 2, 3, 4], in which chains of harmonic oscillators with different random initial conditions (this group of authors has more early works, however, the above are seen as basic). Interest to such models has not faded away

*Mechanics and Mathematics Faculty, Lomonosov Moscow State University, Leninskie Gory 1, Moscow, 119991, Russia
so far, see, for example, [5]. Here we would also like to note the works of Dudnikova T.V., for example, [11, 12, 13], where the author studies the behavior of solutions at large times, deduces variance estimates for them, proves the existence of wave operators or the convergence of the distributions of solutions to some limiting measure.

Randomness can be introduced into the model in other ways. For instance, in a number of works, where the heat flux in a finite disordered chain of oscillators, the masses of particles can be assumed to be random. Dyson [7] was the first to consider such a model. Later results were obtained by Matsud and Ishii [8], Leibovitz (see. [9, 10]). There are also models where external influence is stochastic (see the article of Lykov A.A. [28]).

We would like to note the series of works of physicists (performed rather in mathematical spirit) [14, 15, 16, 17], where the authors investigate the propagation of heat along an infinite chain of harmonic oscillators at the micro level to obtain a connection between micro and macro descriptions (see also links inside), as well articles [18, 19, 20].

2 Model and main results

In this paper, we consider large countable systems. The interaction between particles we consider only in the context of Newton’s classical mechanics (see [21, 35, 36, 37]).

More precisely, we consider a countable system of point particles with unit masses on the real line \( \mathbb{R} \) with coordinates \( \{ x_k \}_{k \in \mathbb{Z}} \) and velocities \( \{ v_k \}_{k \in \mathbb{Z}} \). We define the formal Hamiltonian (total energy of the system) by formula:

\[
H(x(t), v(t)) = \sum_{k \in \mathbb{Z}} \frac{v_k^2}{2} + \sum_{k \in \mathbb{Z}} \frac{a_{kk}}{2} (x_k(t) - ka)^2 + \sum_{k, j \in \mathbb{Z}} \frac{a_{kj}}{2} (x_k(t) - x_j(t) - (k - j)a)^2,
\]

where parameters \( a > 0, a_{kk} \geq 0 \), and \( (V)_{kj} = a_{kj} \) – linear operator in some linear space (conditions will be discussed below). We call Hamiltonian “formal” due to the fact that, generally speaking, the question arises here on the convergence of the series involved in its definition. In this case, the first sum corresponds to the kinetic energy of the system and the remaining – to the potential. Namely, the second sum in the Hamiltonian means that the particle with number \( k \), where \( a_{kk} > 0 \), is a harmonic oscillator (oscillation occurs near the position \( ka \)), the last sum is responsible for interaction between particles with numbers \( k \) and \( j \), where \( a_{kj} \neq 0 \), where, depending on the sign of \( a_{kj}(x_k(t) - x_j(t) - (k - j)a) \), there is attraction or repulsion between the corresponding particles. The distance \( (k - j)a \) here is the distance to which these particles “tend to”. Particle dynamics is determined by the countable ODE system:

\[
\ddot{x}_k(t) = -\frac{\partial H}{\partial x_k} = -a_{kk}(x_k(t) - ka) + \sum_{j \in \mathbb{Z}} a_{kj}(x_k(t) - x_j(t) - (k - j)a), \quad k \in \mathbb{Z},
\]

(1)
with initial conditions \( x_k(0), v_k(0) \). Equilibrium position (i.e. the particle configuration with minimum of energy) will be:

\[
x_k = ka, \quad v_k = 0, \quad k \in \mathbb{Z}.
\]

This means that if particles at the initial moment are in this configuration then the particles will not move at all, i.e. we will have \( x_k(t) = ka \), \( v_k(t) = 0 \) for all \( t \geq 0 \). In this case, it will be convenient to move on to new variables – deviations:

\[
q_k(t) = x_k - ka, \quad p_k(t) = \dot{q}_k(t) = v_k(t).
\]

It is easy to see that new variables \( q_k(t) \) satisfy the following ODE system:

\[
\ddot{q}_k = -a_{kk}q_k + \sum_{\substack{j \in \mathbb{Z} \setminus \{k\}}} a_{k,j}(q_k(t) - q_j(t)), \quad k \in \mathbb{Z}. \tag{2}
\]

Therefore, we will further describe the systems of particles on a straight line by introducing the Hamiltonian depending immediately on the deviations. Let us introduce the notation:

\[
q(t) = \{q_j(t)\}_{j \in \mathbb{Z}}, \quad p(t) = \{p_j(t)\}_{j \in \mathbb{Z}}.
\]

Now consider a countable system of harmonic oscillators on a real line with formal Hamiltonian:

\[
H(q(t), p(t)) = \frac{1}{2} \sum_{k \in \mathbb{Z}} p_k^2(t) + \frac{1}{2} \sum_{k, j \in \mathbb{Z}} a(k - j)q_k(t)q_j(t),
\]

where \( q_j(t), p_j(t) = \dot{q}_j(t) \in \mathbb{R} \) — are the deviation and the particle momentum (of particle number \( j \)) respectively, and the real-valued function \( a(k) \) satisfies three conditions:

1. symmetry: \( a(k) = a(-k) \);
2. bounded support, i.e.: there exists \( K \in \mathbb{N} \) such that for all \( |k| > K \) holds \( a(k) = 0 \);
3. for all \( \lambda \in \mathbb{R} \) holds:

\[
\omega^2(\lambda) = \sum_{k \in \mathbb{Z}} a(k)e^{ik\lambda} > 0. \tag{3}
\]

We assume also that the initial conditions \( \{q_j(0)\}_j, \{p_j(0)\}_j \) lie in Hilbert space \( L \):

\[
L = \{\psi = (q, p) : q \in l_2(\mathbb{Z}), \quad p \in l_2(\mathbb{Z})\}.
\]

Let \( V \) be the linear operator over \( \mathbb{Z} \) corresponding to \( \{a(k)\} \), (i.e. \( V_{k,j} = a(k - j) = a(j - k) \)).

For all \( \psi \in L \) we have the Fourier transform \( \hat{\psi}(\lambda) \in L^2([0; 2\pi]) \), where \( \hat{\psi}(\lambda) = (\hat{q}(\lambda), \hat{p}(\lambda)) \),

\[
\hat{q}(\lambda) = \sum_{k \in \mathbb{Z}} q_k e^{ik\lambda}. \quad \text{Then} \quad \hat{V}q(\lambda) = \omega^2(\lambda)\hat{q}(\lambda). \quad \text{Indeed,}
\]

\[
\hat{V}q(\lambda) = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}, |k-j| \leq K} a(k - j)q_k e^{ij\lambda} = \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}, |l| \leq K} a(l)e^{i(k-l)\lambda}q_k =
\]
\[ V \text{ for } \lambda \in \mathbb{R}, \quad \sum_{k \in \mathbb{Z}} \sum_{t \in \mathbb{Z}, |t| \leq K} a(-l)e^{-il\lambda}e^{ik\lambda}q_k = \sum_{k \in \mathbb{Z}} \sum_{t \in \mathbb{Z}, |t| \leq K} a(l)e^{il\lambda}e^{ik\lambda}q_k = \omega^2(\lambda)\tilde{q}(\lambda). \]

Further, the operator \( V \) is positive definite:

\[ \sum_{k,j \in \mathbb{Z}} a(k - j)q_kq_j = (q, Vq) = (\tilde{q}, \tilde{V}q) = (\tilde{q}(\lambda), \omega^2(\lambda)\tilde{q}(\lambda)) = \omega^2(\lambda)(\tilde{q}(\lambda), \tilde{q}(\lambda)) > 0, \]

for \( q \neq 0 \) due to (3). Here in the second equality is used that if \( x, y \in l_2 \) then \( (x, y) = (\hat{x}, \hat{y}) \), where \( \hat{x} \) is the Fourier transform of the element \( x \), i.e. that \( (q_1, q_2)_{l_2(\mathbb{Z})} = \frac{1}{2\pi} \int_0^{2\pi} \psi_1(\lambda)\overline{\psi_2(\lambda)}d\lambda \).

Suppose, moreover, that on particle with a fixed number \( n \in \mathbb{Z} \) external force \( f(t) \) acts. Then the motion of the system is described by the following infinite ODE system:

\[ \dot{q}_j = -\sum_k a(k - j)q_k + f(t)\delta_{j,n}, \quad j \in \mathbb{Z}, \]

where \( \delta_{j,n} \) is the Kronecker symbol. We assume that \( f(t) \) is a stochastic process satisfying the following condition:

\( \text{A1) real-valued centered second-order stationary process with continuous covariance function (see [38], p. 361).} \)

We say that sequences of stochastic processes \( \{q_k(t)\}_{k \in \mathbb{Z}} \), \( \{p_k(t)\}_{k \in \mathbb{Z}} \) solve the system of equations (4) if they are continuously differentiable in mean square and when substituted to (4), the right and left sides are equal almost surely. More precisely, for any \( j \in \mathbb{Z} \) and any \( t \geq 0 \) equalities

\[ \dot{q}_j = p_j, \]
\[ \dot{p}_j = -\sum_k a(k - j)q_k + f(t)\delta_{j,n}, \]

hold with probability one.

The following lemma on the existence and uniqueness of a solution of main system (4) holds.

**Proposition 2.1.** Let the condition A1) be satisfied. Then for all \( \psi(0) \in L \) there exists and is unique solution \( \psi(t) = (q(t), p(t)) \) of the system (4) with initial condition \( \psi(0) \) such that \( P(\psi(t) \in L) = 1 \) for all \( t \geq 0 \).

Uniqueness here means that if there is another solution \( \varphi(t) \) of the system (4) with initial condition \( \psi(t) \) such that \( P(\varphi(t) \in L) = 1 \) for all \( t \geq 0 \) then \( \psi(t) \) and \( \varphi(t) \) are stochastically equivalent, i.e. \( P(\varphi(t) = \varphi(t)) = 1 \) for all \( t \geq 0 \).

We are interested in the question of how the solution \( \psi(t) \) and the average energy of the system \( \mathbf{E}H(\psi(t)) \) as \( t \to \infty \) behave. Before formulating the main results, we introduce one more condition on the external force.

Consider the set \( E = \{\omega^2(\lambda) : \lambda \in \mathbb{R} \} \) — range of the function \( \omega^2(\lambda) \) (spectral set of our system). Since \( \omega^2(\lambda) \) is trigonometric polynomial, then \( E \) is a segment \([e_1; e_2]\) of the real line. Denote by \( \mu(dx) \) the spectral measure of the process \( f(t) \) and introduce the condition:

\( \text{A2) the support of the spectral measure } \mu \text{ is isolated from the plus or minus “of the root” of the set } E, \text{ i.e. there is an open set } U \text{ containing } \pm [\sqrt{e_1}, \sqrt{e_2}] \text{ such that } \mu(U) = 0.} \)
Theorem 2.1. Consider conditions A1) and A2) and \( \psi(0) = (q(0), p(0)) \in L \) hold. Then there is random process \( \eta(t) = (q^\infty(t), p^\infty(t)) \) such that the following conditions hold:

1. \( \eta(t) \) is a solution to the system \( \{4\} \) with some initial conditions;
2. the difference \( \psi(t) - \eta(t) \) converges to zero as \( t \to +\infty \) component-by-component with probability one, and the trajectories of the process are continuous and infinitely differentiable a.s.;
3. each component of \( \eta(t) \) is a stationary process, satisfying condition A1) and \( P(\eta(t) \in L) = 1 \) for all \( t \geq 0 \);
4. there exist positive constants \( c_1, c_2 \) and \( 0 < r < 1 \) such that

\[
Dq_k^\infty(0) \leq c_1 r^{|n-k|}, \quad Dp_k^\infty(0) \leq c_2 r^{|n-k|}.
\]

Thus, the process \( \psi(t) \) is in some sense close to the stationary process \( \eta(t) \). Besides this in addition, \( \eta(t) \) has “nice” properties. In particular, the variance of the \( \eta(0) \) components decreases exponentially with increasing distance to the point of application of the external force.

Generally speaking, this assertion does not imply the weak convergence of \( \psi(t) \) components to the corresponding \( \eta(0) \) components. However, the following assertion can be proven:

Theorem 2.2. In addition to conditions A1) and A2) assume that \( f(t) \) is a strictly stationary process. Then each component of \( \psi(t) \) converges in distribution to the corresponding component \( \eta(0) \), i.e. for all \( k \in \mathbb{Z} \) takes place the convergence

\[
q_k(t) \overset{d}{\to} q^\infty_k(0), \quad p_k(t) \overset{d}{\to} p^\infty_k(0)
\]

as \( t \to \infty \).

Let us formulate theorems on mean energy of the system.

Theorem 2.3. Let conditions A1) and A2) be satisfied then

\[
\lim_{t \to +\infty} EH(\psi(t)) = \alpha + H(\psi(0)),
\]

\[
EH(\eta(0)) = \alpha,
\]

where we introduced the following constant

\[
\alpha = \frac{1}{4\pi} \int_{\mathbb{R}} \int_0^{2\pi} \frac{\omega^2(\lambda) + x^2}{(\omega^2(\lambda) - x^2)^2} d\lambda \mu(dx).
\]

Thus, the time limit for the average energy of the system generally differs from the average energy of the limiting distribution, which does not depend on the initial energy level, however, they will coincide in the case of zero initial conditions.
2.1 Proofs

To begin with, let us note that it follows from the spectral theory that for \( f(t) \) – second-order stationary centered stochastic process:

\[
B(s) = \int_{\mathbb{R}} e^{isx} \mu(dx), \quad f(s) \overset{a.s.}{=} \int_{\mathbb{R}} e^{isx} Z(dx),
\]

(5)

where \( Z(dx) \) is an orthogonal measure, \( \mu(dx) \) is a spectral measure, and \( B(s) \) is the covariance function.

2.1.1 Proof of Proposition 2.1

Let us denote an operator in \( L \):

\[
A = \begin{pmatrix} 0 & E \\ -V & 0 \end{pmatrix}.
\]

Let’s rewrite the system (4) in Hamiltonian form:

\[
\begin{aligned}
\dot{q}_j &= p_j, \\
\dot{p}_j &= -\sum_k a(k - j)q_k + f(t)\delta_{j,n}.
\end{aligned}
\]

(6)

Let’s introduce the vector \( \psi(t) = \begin{pmatrix} q(t) \\ p(t) \end{pmatrix} \). Then the system will be rewritten in the form:

\[
\dot{\psi} = A\psi + f(t)g,
\]

(7)

g = (0, e_n)^T, \quad 0, e_n \in l_2(\mathbb{Z}), \quad e_n(j) = \delta_{j,n}.

The uniqueness of the solution follows from the linearity of the system. Indeed, let \( \psi(t) \) and \( \varphi(t) \) be solutions of the system (7) with the same initial condition. Then \( \epsilon(t) = \psi(t) - \varphi(t) \) is a solution to the homogeneous equation

\[
\dot{\epsilon} = A\epsilon,
\]

(8)

with zero initial condition \( \epsilon(0) = 0 \) and, moreover, \( \epsilon(t) \in L \) almost certainly for all \( t \geq 0 \). Thus, similarly to the arguments of the classical theory of ODEs in Banach spaces we have the required statement (see [29]).

The solution of the system (7) can be expressed with the classical formula for solving a nonhomogeneous ODE (see [29]):

\[
\psi(t) = e^{At}(\psi(0) + \int_0^t e^{-As}gf(s)ds) = \psi_0(t) + \psi_1(t),
\]

where

\[
\psi_0(t) = e^{At}\psi(0),
\]

\[
\psi_1(t) = \int_0^t e^{A(t-s)}gf(s)ds.
\]
Note that since \(V_{i,j} = a(i - j)\) and \(a(k)\) has bounded support the operator \(A\) is a bounded linear operator on \(L\), hence the operator \(e^{At}\) is well-defined bounded operator on \(L\). Hence \(\psi_0(t) \in L\) for all \(t \geq 0\).

Next, we turn to the consideration of \(\psi_1(t)\). For this we need lemma

**Lemma 2.1.** For the operator \(A\) the following is true:

\[
e^{At} = \begin{pmatrix}
\cos(\sqrt{V}t) & (\sqrt{V})^{-1}\sin(\sqrt{V}t) \\
-\sqrt{V}\sin(\sqrt{V}t) & \cos(\sqrt{V}t)
\end{pmatrix},
\]

where the sine and cosine of the operator are defined by the corresponding series.

**Proof.** Direct check or see [29]. \(\square\)

Let’s return to the proof of the proposition 2.1. Denote \(\psi_1(t) = (q^{(1)}(t), p^{(1)}(t))^T\). Then from the lemma 2.1 follows:

\[
q_k^{(1)}(t) = \int_0^t f(s)S_{k,n}(t - s)ds, \quad S(t) = (\sqrt{V})^{-1}\sin(\sqrt{V}t),
\]

\[
p_k^{(1)}(t) = \int_0^t f(s)C_{k,n}(t - s)ds, \quad C(t) = \cos(\sqrt{V}t).
\]

Note that consideration of the root of the operator is possible due to its positive definiteness. It is necessary to prove that \(\{q_k^{(1)}(t)\}_k, \{p_k^{(1)}(t)\}_k \in l_2(\mathbb{Z})\) almost certainly. The proof is based on the lemma:

**Lemma 2.2.** For all \(t \geq 0\) the following is true:

\[
|C_{k,n}| \leq \frac{\nu^{|kn|}2^\rho}{(2\rho)!}e^{\sqrt{v}t}, \quad |S_{k,n}| \leq \frac{\nu^{|kn|}2^\rho+1}{(2\rho + 1)!}e^{\sqrt{v}t},
\]

where \(S(t)\) and \(C(t)\) are defined in (10) – (11), \(v = \|V\|_{l_2(\mathbb{Z})}\), \(\rho = \lceil\frac{|kn|}{K}\rceil\) (here \(\lfloor x\rfloor\) is the smallest integer not less than \(x\)).

**Proof.** See Lemma 3.2 in [28] (p. 7). \(\square\)

Let’s continue the proof of the proposition.

\[
E|q_k^{(1)}(t)|^2 = E\left(\int_0^t f(s)S_{k,n}(t - s)ds\int_0^t f(\tau)S_{k,n}(t - \tau)d\tau\right) =
\]

\[
= \int_0^t \int_0^t E\left(\int_{\mathbb{R}} e^{ixZ(d\mathbf{x})}\int_{\mathbb{R}} e^{-iy\mu(d\mathbf{y})}S_{k,n}(t - s)S_{k,n}(t - \tau)d\mathbf{y}d\mathbf{x}\right)S_{k,n}(t - s)S_{k,n}(t - \tau)d\tau =
\]

\[
= \int_0^t \int_0^t e^{i(s-\tau)x}\mu(d\mathbf{x})S_{k,n}(t - s)S_{k,n}(t - \tau)d\mathbf{y}d\mathbf{x}d\tau =
\]
\begin{align*}
&= \int_0^t \int_0^t B(s-\tau)S_{k,n}(t-s)\overline{S_{k,n}(t-\tau)}
&\leq \sup_{s \in [0,t]} B(s) \int_0^t \int_0^t \nu^\rho(t-s)^{2\rho+1} t^{2\rho+1}
&\quad \cdot e^{\sqrt{\nu}(t-s)} e^{\sqrt{\nu}(t-\tau)} ds d\tau \\
&\leq \sup_{s \in [0,t]} B(s) \left( \frac{\nu^\rho t^{2\rho+2}}{(2\rho+2)!} e^{\sqrt{\nu}t} \right)^2.
\end{align*}

Let us show the correctness of the second equality. Possibility of permutation of integration and taking the mathematical expectation follows from the continuity of the covariance functions of the process \( f(s) \) (from the condition) and the existence of integrals

\begin{equation}
\int_0^t f(s)S_{k,n}(t-s)ds
\end{equation}

(in the mean square sense) and the Riemann integral

\begin{equation}
\int_0^t \int_0^t B(s-\tau)S_{k,n}(t-s)\overline{S_{k,n}(t-\tau)}
\end{equation}

The last integral exists due to continuity and boundedness of integrands, and for the existence of the integral \([12]\) already described conditions are enough (see [30] pp. 94–129).

From here:

\begin{align*}
\sum_k E|q_k^{(1)}(t)|^2 &\leq \sup_{s \in [0,t]} B(s) \sum_{\rho} \left( \frac{\nu^\rho t^{2\rho+2}}{(2\rho+2)!} e^{\sqrt{\nu}t} \right)^2 \\
&\leq \sup_{s \in [0,t]} B(s) e^{2\sqrt{\nu}t} t^2 \sum_{\rho} \left( \frac{(vt)^{2\rho}}{(2\rho)!} \right) = \\
&= \sup_{s \in [0,t]} B(s) e^{2\sqrt{\nu}t} t^2 ch(vt^2) < \infty,
\end{align*}

whence, by the corollary of Levy’s monotone convergence theorem (see [31] p. 306) it follows that

\begin{equation*}
\sum_k |q_k^{(1)}(t)|^2 < \infty \text{ a.s.,}
\end{equation*}

i.e. \( \{q_k^{(1)}(t)\}_k \in l_2(\mathbb{Z}) \) almost certainly. Similarly, \( \{p_k^{(1)}(t)\}_k \in l_2(\mathbb{Z}) \) almost certainly. The assertion has been completely proven.

### 2.1.2 Proof of the Theorem [2.1]

Let’s introduce \( \eta(t) \) by the formula:

\begin{equation*}
\eta(t) = -\int_R e^{itx} R_A(ix) Z(dx) g,
\end{equation*}

where \( R_A(z) = (A-zI)^{-1} \) is the resolvent of the operator \( A \) (here \( I \) is the identity operator over \( \mathbb{Z} \times \mathbb{Z} \)), and we prove that the stochastic process introduced in this way satisfies all conditions of the theorem. Moreover, the resolvent is bounded due to the conditions of
Theorem 2.1 (namely, condition A2) on page 13), which implies the convergence of the considered integral.

First, we prove that it is a solution to the system (6):

\[-\dot{\eta}(t) + A\eta(t) + f(t)g = \int_{\mathbb{R}} ix e^{itx} R_A(ix)Z(dx)g - \int_{\mathbb{R}} e^{itx} A R_A(ix)Z(dx)g + f(t)g = \]

\[= \int_{\mathbb{R}} e^{itx} (ixI - A) R_A(ix)Z(dx)g + \int_{\mathbb{R}} e^{itx} Z(dx)g = \]

\[= \int_{\mathbb{R}} e^{itx} ((ixI - A)(A - ixI)^{-1} + I)Z(dx)g = 0. \]

It is possible to introduce differentiation under the integral sign in view of the existence of integral

\[\int_{\mathbb{R}} (R_A(ix)g, R_A(ix)g) \mu(dx)\]

(See [32], p. 94).

**Proof of item 4.** Since the limit vector is stationary, consider \(\eta(0)\) and for brevity we denote it by \(\xi = \eta(0)\). Since \(E\eta(0) = 0\), then

\[cov(\eta_j(0), \eta_k(0)) = E\eta_j(0)\eta_k(0) = E\xi_j\xi_k. \]

Further, in view of the fact that the relation

\[E\left(\int f Z(dx) \overline{\int g Z(dx)}\right) = \int f \overline{\mu}(dx), \]

is true we have

\[C \equiv E\xi \xi^T = \int_{\mathbb{R}} e^{itx} R_A(ix)gg^T R_A(ix)^T e^{-itx} \mu(dx) = \int_{\mathbb{R}} R_A(ix)gg^T R_A(ix)^T \mu(dx). \quad (14)\]

Denote

\[C(x) = R_A(ix)gg^T R_A(ix)^T. \quad (15)\]

Let us find the resolvent of the operator \(A\):

\[R_A(\lambda) = (A - \lambda I)^{-1}, \]

\[
\begin{pmatrix}
-\lambda E & E \\
-V & -\lambda E
\end{pmatrix}
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
= \begin{pmatrix}
E & 0 \\
0 & E
\end{pmatrix},
\]

\[
\begin{pmatrix}
-\lambda A + C & -\lambda B + D \\
-V A - \lambda C & -V B - \lambda D
\end{pmatrix}
= \begin{pmatrix}
E & 0 \\
0 & E
\end{pmatrix},
\]

from where

\[R_A(\lambda) = \begin{pmatrix}
-\lambda R_V(-\lambda^2) & -R_V(-\lambda^2) \\
E - \lambda^2 R_V(-\lambda^2) & -\lambda R_V(-\lambda^2)
\end{pmatrix}, \]
thus

\[ R_A(ix) = \begin{pmatrix} -ixR_V(x^2) & -R_V(x^2) \\ E + x^2R_V(x^2) & -ixR_V(x^2) \end{pmatrix}. \]

\[ R_A(ix)g = \begin{pmatrix} -R_V(x^2)e_n \\ -ixR_V(x^2)e_n \end{pmatrix}, \quad (16) \]

\[ g^TR_A(ix)^T = ( -e_n^TR_V(x^2), ix e_n^TR_V(x^2)). \]

For convenience of notation we denote

\[ \rho = R_V(x^2), \Gamma = e_ne_n^T, \]

then (15) becomes:

\[ C(x) = \begin{pmatrix} \rho \Gamma \rho & -ix(\rho \Gamma \rho) \\ ix(\rho \Gamma \rho) & x^2(\rho \Gamma \rho) \end{pmatrix}. \]

In (14) \( \int_{\mathbb{R}} ix(\rho \Gamma \rho) \mu(dx) = 0 \), since \( \mu \) is a symmetric measure due to the realness of the process, \( xR_V(x^2) \) is an odd function. Let us pass to the integral \( \int_{\mathbb{R}} \rho \Gamma \rho \mu(dx) \). Denote

\[ c_{k,j} = (Ce_k, e_j) = \int_{\mathbb{R}} (\rho \Gamma \rho e_k, e_j) \mu(dx), \]

\[ c_{k,j}(x) = (\rho \Gamma \rho e_k, e_j) = (\Gamma \rho e_k, e_j), \]

Note that

\[ \widehat{\Gamma}x(\lambda) = \sum_j (\Gamma x)_j e^{ij\lambda} = x_ne^{in\lambda}, \]

\[ \widehat{\rho}e_j = \frac{\hat{e}_j}{\omega^2(\lambda) - x^2} = \frac{e^{ij\lambda}}{\omega^2(\lambda) - x^2} = e^{ij\lambda}b_k(\lambda), \]

\[ \widehat{\Gamma} \rho e_k = (\rho e_k)_n e^{in\lambda} = (\rho e_k, e_n) e^{in\lambda} = e^{in\lambda}(\widehat{\rho}e_k, \hat{e}_n) = \]

\[ = \frac{e^{in\lambda}}{2\pi} \int_0^{2\pi} e^{-in\nu} e^{ik\nu} d\nu \]

\[ = \int_0^{2\pi} \frac{e^{i(n-\nu)\lambda} d\nu}{\omega^2(\lambda) - x^2}. \]

hence, in view of the self-adjointness of the operator \( \rho \):

\[ (\rho \Gamma \rho e_k, e_j) = (\Gamma \rho e_k, \rho e_j) = (\widehat{\Gamma} \rho e_k, \hat{e}_j) = \]

\[ = \left( \frac{1}{2\pi} \right)^2 \int_0^{2\pi} \frac{e^{i(n-j)\lambda} d\lambda}{\omega^2(\lambda) - x^2} \int_0^{2\pi} \frac{e^{i(k-n)\lambda} d\lambda}{\omega^2(\lambda) - x^2}. \]

and, taking into account the parity and periodicity of the integrands, we arrive at to

\[ (\rho \Gamma \rho e_k, e_j) = \left( \frac{1}{2\pi} \right)^2 \int_0^{2\pi} \frac{\cos((n-j)\lambda) d\lambda}{\omega^2(\lambda) - x^2} \int_0^{2\pi} \frac{\cos((k-n)\lambda) d\lambda}{\omega^2(\lambda) - x^2}. \]

Consider the cases:
1. \( j = k \neq n \to +\infty \). Here
\[
\int_0^{2\pi} b_x(\lambda) \cos((k - n)\lambda) d\lambda = \frac{1}{k} (b_x(\lambda) \sin((k - n)\lambda)|_{\lambda=0}^{\lambda=2\pi} - \int_0^{2\pi} b'_x(\lambda) \sin((k - n)\lambda) d\lambda) = \frac{1}{k^2} (b''_x(\lambda) \cos((k - n)\lambda)|_{\lambda=0}^{\lambda=2\pi} - \int_0^{2\pi} b''_x(\lambda) \cos((k - n)\lambda) d\lambda) = \ldots,
\]
thus there is an exponentially fast decay (faster than any degree, i.e. \( o\left(\frac{1}{(k-n)^{\infty}}\right)\)).

2. In other cases, consider \( c_{k,j} = \int_R c_{k,j}(x) \mu(dx) \), where \( c_{k,j}(x) \) is entered in (18). Denote:
\[
h_k(x) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i(n-k)\lambda}}{\omega^2(\lambda) - x^2} d\lambda,
\]
We make a replacement \( z = e^{i\lambda} \):
\[
h_k(x) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{1}{P(z) - x^2} z^{n-k-1} dz,
\]
where \( P(z) = \omega^2(\lambda)|_{\lambda=\lambda(z)} \), i.e.
\[
P(z) = \sum_j a(j) e^{ij\lambda} = a(0) + \sum_{j=1}^{K} a(j)(z^j + z^{-j}).
\]
We are interested in the roots of the equation
\[
P(z) - x^2 \equiv a(0) + \sum_{j=1}^{K} a(j)(z^j + z^{-j}) - x^2 = 0,
\]
\[
a(K)z^{2K} + \ldots + (-x^2 + a(0))z^K + \ldots + a(K) = 0.
\]
Obviously, \( P(z) = P(1/z) \), hence, if \( z \) is a root, then \( 1/z \) is the root, so the zeros are invariant under the inversion of the unit circle \(|z| = 1\). These values are the values from the spectrum, which we ”avoid”. In total, we have \( K \) inverse pairs, which needs to be bypassed. You can choose a ring (neighborhood of the unit circle), where the inverse function is holomorphic
\[
g(z) = \frac{1}{P(z) - x^2}.
\]
To do this, we find the maximum modulo root of the equation (20), lying inside the unit circle. We denote its modulus by \( R(x) \). Then its inverse pair has modulus equal to \( \frac{1}{R(x)} \).
moreover this will be the smallest modulus of roots lying outside the unit circle. Thus, \( g(z) \) is holomorphic in the ring

\[
R(x) < |z| < \frac{1}{R(x)}, \quad 0 < R(x) < 1 < \frac{1}{R(x)}.
\]

We choose \( \epsilon \) close to zero and denote by \( \rho = \frac{1-\epsilon}{R(x)} \), then the contour \(|z| = \rho\) lies in the holomorphy ring \( g(z) \), hence it is possible in (19) to replace the integration contour with the considered one, then

\[
h_k(x) = \frac{1}{2\pi i} \int_{|z| = \rho} \frac{1}{P(z) - x^2} z^{n-k-1} dz,
\]

let’s evaluate the module:

\[
|h_k(x)| \leq \frac{1}{2\pi} \int_{|z| = \rho} |g(z)z^{n-k-1}| \cdot |dz| = \frac{1}{2\pi} \int_{|z| = \rho} |g(z)|\rho^{n-k-1} \cdot |dz|,
\]

in the last integral we make the change \( z = \rho e^{i\phi} \), then

\[
|h_k(x)| \leq \frac{\rho^{n-k-1}}{2\pi} \int_0^{2\pi} |g(\rho e^{i\phi})|\rho d\phi = \frac{\rho^{n-k}}{2\pi} \int_0^{2\pi} |g(\rho e^{i\phi})|d\phi = \frac{\rho^{n-k}}{2\pi} r(x).
\]

Note that since there exists \( q \) such that for all \( x \) from \( \mathbb{R} \setminus E \), \( \rho \geq 1/q > 1 \) is true, whence \( 1/\rho \leq q < 1 \) and

\[
|h_k(x)| \leq \frac{q^{k-n}}{2\pi} r(x),
\]

\[
c_{k,K} \leq \int_{\mathbb{R}} |h_k(x)|^2 \mu(dx) \leq \frac{q^{2(k-n)}}{4\pi^2} \int_{\mathbb{R}} r^2(x)\mu(dx) \to 0, \quad k \to +\infty,
\]

if \( \int r^2(x)\mu(dx) \) converges. Let’s prove that this is indeed the case:

\[
r(x) = \frac{1}{2\pi} \int_0^{2\pi} |g(\rho e^{i\phi})|d\phi \leq \frac{1}{2\pi} \max_{z:|z|=\rho} |g(z)| \cdot 2\pi = \max_{z:|z|=\rho} |g(z)|,
\]

where \( g(z) \) is introduced in (21). Let us notice, that

\[
|P(z) - x^2| \geq ||P(z)| - x^2| \geq \min\{|I - x^2|, |S - x^2|\} > 0,
\]

where

\[
I = I(\rho) = \inf_{z:|z|=\rho} |P(z)|, \quad S = S(\rho) = \sup_{z:|z|=\rho} |P(z)|,
\]

and the last strict inequality holds due to the choice of radius \( \rho \) of the circle.

Further, for \( \rho = 1 \) the segment \([I(\rho), S(\rho)]\) coincides with the set \( E \). Since \( I \) and \( S \) are continuous functions of \( \rho \), there exists a neighborhood of the point \( \rho = 1 \) such that for any \( \rho \) from this neighborhood segment \([I(\rho), S(\rho)]\) lies in \( U \), where \( U \) is defined in condition A2.
Then there is a positive constant $\epsilon > 0$ such that for any $x \in \mathbb{R} \setminus U$ we have the following inequality:

$$|P(z) - x^2| \geq \epsilon > 0,$$

wherefrom

$$|g(z)| \leq \frac{1}{\epsilon}.$$

So, for all $z : |z| = \rho$, and all $x \in \mathbb{R} \setminus U$ we have an estimate:

$$r(x) \leq \frac{1}{\epsilon}.$$

For $x \to \infty$ the function $r(x)$ decreases as $\frac{1}{x^2}$ due to the estimate (22), whence the convergence of the integral follows.

**Next, let’s move on to item 3.** It follows from item 4 that the initial conditions $\eta(0) \in L$. Indeed:

$$\sum_{k \in \mathbb{Z}} Dq_k^\infty(0) \leq c_1 \sum_{k \in \mathbb{Z}} r^{n-k} \leq c_1 \sum_{k \in \mathbb{Z}} p^{|k|} < \infty,$$

$$\sum_{k \in \mathbb{Z}} Dp_k^\infty(0) \leq c_2 \sum_{k \in \mathbb{Z}} r^{n-k} \leq c_2 \sum_{k \in \mathbb{Z}} p^{|k|} < \infty,$$

whence, by the corollary of Levy’s monotone convergence theorem (see [31] p. 306) it follows that

$$\sum_k |q_k^\infty(0)|^2 < \infty \ a.s.,$$

i.e. $\{q_k^\infty(0)\}_k \in l_2(\mathbb{Z})$ almost certainly. Similarly, $\{p_k^\infty(0)\}_k \in l_2(\mathbb{Z})$ is almost surely. Then the statement [27] implies what is required.

**Let’s prove the second item.** The difference $\psi(t) - \eta(t)$ at the initial time point lies in $L$ and is a solution of the homogeneous equation, hence, componentwise tends to zero almost surely. Indeed, $\epsilon(t) = \psi(t) - \eta(t)$ is a solution to the homogeneous equation

$$\dot{\epsilon} = A\epsilon,$$

with the initial condition $\epsilon(0) \in L$ almost surely for all $t \geq 0$, and has the form:

$$\epsilon(t) = e^{At}\epsilon(0) = e^{At}\begin{pmatrix} q(0) \\ p(0) \end{pmatrix},$$

which obviously implies continuity and infinite differentiability a.s. of process trajectories. Consider one of the coordinates:

$$\begin{pmatrix} \epsilon(t), e_k \\ 0 \end{pmatrix} = e^{At}\begin{pmatrix} q(0) \\ p(0) \end{pmatrix}, \begin{pmatrix} e_k \\ 0 \end{pmatrix} = e^{At}\begin{pmatrix} q(0) \\ p(0) \end{pmatrix}, \begin{pmatrix} e_k \\ 0 \end{pmatrix} \quad (3)$$
\[
\begin{pmatrix}
\cos(\omega(\lambda)t)q(0)(\lambda) + \frac{\sin(\omega(\lambda)t)}{\omega(\lambda)} p(0)(\lambda) \\
-\omega(\lambda) \sin(\omega(\lambda)t) q(0)(\lambda) + \cos(\omega(\lambda)t) p(0)(\lambda)
\end{pmatrix}
+ \left( e^{i k \lambda} \right)
\]
\[
= \frac{1}{2\pi} \int_{0}^{2\pi} e^{i k \lambda} \left( \cos(\omega(\lambda)t)q(0)(\lambda) + \frac{\sin(\omega(\lambda)t)}{\omega(\lambda)} p(0)(\lambda) \right) d\lambda
\]

The equation \((3)\) uses the formulas \((28)\) and \((29)\). And converging to zero takes place due to Corollary 2 in \([33]\) (p. 6), since:

\[
\left| \frac{1}{2\pi} \int_{0}^{2\pi} e^{i k \lambda} \left( \cos(\omega(\lambda)t)q(0)(\lambda) + \frac{\sin(\omega(\lambda)t)}{\omega(\lambda)} p(0)(\lambda) \right) d\lambda \right| \leq c \sqrt{t}, \quad (24)
\]

where the constant \(c\) does not depend on \(x\). Indeed, the corollary asserts:

**Corollary 2 (Arkhipov, Karatsuba, Chubarikov).** Let \(g(x)\) – piecewise monotone continuous function, \(p\) – number of its monotonicity segments, \(\max_{0 \leq x \leq 1} |g(x)| = H\). Let real-valued function \(f(x)\) for \(0 < x < 1\) has an \(n\)-order derivative, \(n > 1\), moreover, for some \(A > 0\), for all \(0 < x < 1\), the inequality \(|f^{(n)}(x)| \geq A\) holds. Then for \(G = \int_{0}^{1} g(x)e^{2\pi if(x)}dx\) holds:

\[
|G| \leq H \min \{1; 24pnA^{-1/n}\}. \quad (25)
\]

Let’s make a change in the integral \((24)\):

\[
\frac{1}{2\pi} \int_{0}^{2\pi} e^{i k \lambda} \left( \cos(\omega(\lambda)t)q(0)(\lambda) + \frac{\sin(\omega(\lambda)t)}{\omega(\lambda)} p(0)(\lambda) \right) d\lambda =
\]
\[
= \int_{0}^{1} e^{i k 2\pi \zeta} \left( \cos(\omega(2\pi \zeta)t)q(0)(2\pi \zeta) + \frac{\sin(\omega(2\pi \zeta)t)}{\omega(2\pi \zeta)} p(0)(2\pi \zeta) \right) d\zeta =
\]
\[
= \frac{1}{2} \int_{0}^{1} \cos(2\pi k \zeta) q(0)(2\pi \zeta) \left( e^{i\omega(2\pi \zeta)} + e^{-i\omega(2\pi \zeta)} \right) d\zeta +
\]
\[
+ \frac{i}{2} \int_{0}^{1} \sin(2\pi k \zeta) q(0)(2\pi \zeta) \left( e^{i\omega(2\pi \zeta)} + e^{-i\omega(2\pi \zeta)} \right) d\zeta +
\]
\[
+ \frac{1}{2i} \int_{0}^{1} \cos(2\pi k \zeta) \frac{p(0)(2\pi \zeta)}{\omega(2\pi \zeta)} \left( e^{i\omega(2\pi \zeta)} - e^{-i\omega(2\pi \zeta)} \right) d\zeta +
\]
\[
+ \frac{1}{2} \int_{0}^{1} \sin(2\pi k \zeta) \frac{p(0)(2\pi \zeta)}{\omega(2\pi \zeta)} \left( e^{i\omega(2\pi \zeta)} - e^{-i\omega(2\pi \zeta)} \right) d\zeta,
\]

thus it is necessary to estimate 8 integrals. Consider one of them (for the rest the estimate is obtained similarly):

\[
\int_{0}^{1} \sin(2\pi k \zeta) \frac{p(0)(2\pi \zeta)}{\omega(2\pi \zeta)} e^{i\omega(2\pi \zeta)} d\zeta,
\]
According to the notation of the corollary \( g(\zeta) = \sin (2\pi k\zeta) \frac{\hat{h}(0) (2\pi\zeta)}{\omega(2\pi\zeta)} \). Consider

\[
\omega^2(2\pi\zeta) = a(0) + 2 \sum_{k=1}^{K} a(k) \cos(2\pi k\zeta).
\]

As a linear combination of cosines it is continuous and piecewise monotonic on \([0; 1]\) function. The function \( h(t) = \sqrt{t} \) is continuous and monotonic over the entire domain of definition, whence we obtain that the composition \( \omega(2\pi\zeta) \) of these two functions is continuous and piecewise monotonic on \([0; 1]\). Further, in view of (3), as well as the monotonicity and continuity of function \( h_1(t) = \frac{1}{t} \), function \( \frac{1}{\omega(2\pi\zeta)} \) is continuous and piecewise monotonic as a composition of functions. Function \( \hat{p}(0)(2\pi\zeta) \) is also continuous as the Fourier transform of an element from \( l_2(\mathbb{Z}) \), hence \( g(\zeta) \) is continuous and piecewise monotone as a product of functions. According to the Weierstrass theorem, on \([0; 1]\) it reaches its maximum on the interval, which we denote as \( H \). Now consider \( f(\zeta) = t \frac{\omega(2\pi\zeta)}{2\pi} \).

This function is \( n \) times differentiable (we can take arbitrary \( n > 1 \)):

\[
f'(\zeta) = t \omega'(2\pi\zeta) = -t \frac{\sum_{k=1}^{K} a(k) k \sin(2\pi k\zeta)}{\omega(2\pi\zeta)},
\]

\[
f''(\zeta) = 2\pi t \omega''(2\pi\zeta) = -2\pi t \frac{\left( \sum_{k=1}^{K} a(k) k^2 \cos(2\pi k\zeta) \right) \omega(2\pi\zeta) - \omega'(2\pi\zeta) \sum_{k=1}^{K} a(k) k \sin(2\pi k\zeta)}{\omega^2(2\pi\zeta)} =
\]

\[
= -2\pi t \frac{\left( \sum_{k=1}^{K} a(k) k^2 \cos(2\pi k\zeta) \right) \omega^2(2\pi\zeta) + \left( \sum_{k=1}^{K} a(k) k \sin(2\pi k\zeta) \right)^2}{\omega^3(2\pi\zeta)}.
\]

Numerator as a finite linear combination of trigonometric functions has a finite number of zeros on the segment \([0; 1]\) (if it has zeros somewhere at all). If there are no zeros, then the second derivative is separable from zero, the required constant \( A \) exists. If zeros exist, then consider their \( \varepsilon \)-neighborhoods. Out of these neighborhoods the second derivative is separable from zero. In these surroundings you can see derivatives of order three or higher. In view of the analyticity of the function, and also that it is not a constant, there exists the number \( n \) of the derivative, under which the corresponding derivative has no zeros in the chosen neighborhood. In this case, in each neighborhood we have the estimate (25). Whence, in view of the fact that \( A = A_0 t \), by the corollary we get the estimate (24). Which is what was required.

### 2.1.3 Proof of the Theorem 2.2

Strictly stationary process with finite first two moments is a second-order stationary process, therefore, we have the right to use the results of the previous Theorem. Then \( \psi_k(t) - \eta_k(t) \xrightarrow{a.s.} 0 \).

The lemma is required:
Lemma 2.3. Let for \( \xi(t), \eta(t) \) – one-dimensional real random processes be true: \( \xi(t) \xrightarrow{a.s.} 0, t \to +\infty, \) and for any \( t > 0 \) \( \eta(t) \xrightarrow{d} \eta_0. \) Then \( \xi(t) \xrightarrow{d} \eta_0. \)

Proof. \( \xi(t) \xrightarrow{a.s.} 0 \) implies convergence \( \xi(t) \xrightarrow{P} 0. \) Further, from the condition for all \( t \) \( \eta(t) \xrightarrow{d} \eta_0, \) convergence of \( \eta(t) \xrightarrow{d} \eta_0 \) follows. From where, according to the Slutsky lemma for the function \( g(x,y) = x + y, \) we obtain the assertion of the lemma. \( \square \)

2.1.4 Proof of the Theorem 2.3

Let us represent the energy of the system in the following form:

\[
H(\psi) = \frac{1}{2} (\psi, G\psi), \quad G = \begin{pmatrix} V & 0 \\ 0 & E \end{pmatrix}.
\]

Let us represent the solution as

\[
\psi(t) = \epsilon(t) + \eta(t),
\]

\[
2H(\psi) = (\psi, G\psi) = (\psi, \psi)_H = (\epsilon, \epsilon)_H + (\epsilon, \eta)_H + (\eta, \epsilon)_H + (\eta, \eta)_H.
\]

Then the averages of the second and third terms are equal to zero due to the fact that \( E\eta = 0. \) Further,

\[
\eta^T G\eta = \int_0^t g^T e^{A(t-s_1)} f(s_1) ds_1 \int_0^t G e^{A(t-s_2)} g f(s_2) ds_2 =
\]

\[
= \int_0^t \int_0^t (e^{A(t-s_1)} g, e^{A(t-s_2)} g)_H f(s_1) f(s_2) ds_1 ds_2.
\]

Possibility of permutation of integration and taking the mathematical expectation again follows from the continuity of the covariance function of the process \( f(s) \) (from the condition of the theorem) and the existence of integrals

\[
\int_0^t f(s) G e^{A(ts)} g ds, \quad \int_0^t f(s) e^{A(ts)} g ds,
\]

(in the mean square sense) and the Riemann integral existence

\[
\int_0^t \int_0^t B(s - \tau) g^T e^{A(t-s)} G e^{A(t-\tau)} g ds d\tau.
\]

The last integral exists due to integrands continuity and boundedness, and for the existence of integrals \( 26 \) already described conditions are enough (see \[30\] p. 94 – 129). Then

\[
E(\eta, \eta)_H = \int_0^t \int_0^t (e^{A(t-s_1)} g, e^{A(t-s_2)} g)_H B(s_1 - s_2) ds_1 ds_2.
\]

We introduce the scalar product by the formula

\[
(\psi_1, \psi_2)_H = \frac{1}{2\pi} \int_0^{2\pi} \hat{p}_1(\lambda) \overline{\hat{p}_2}(\lambda) + \omega^2(\lambda) \hat{q}_1(\lambda) \overline{\hat{q}_2}(\lambda) d\lambda.
\]
Then

\[ (\psi, \psi)_H = \frac{1}{2\pi} \int_0^{2\pi} |\tilde{p}(\lambda)|^2 + \omega^2(\lambda)|\tilde{q}(\lambda)|^2 d\lambda. \]

Further, taking into account (9) we get:

\[ e^{At} \left( \begin{array}{c} \hat{q} \\ \hat{p} \end{array} \right) = \left( \begin{array}{cc} \cos(\omega(\lambda)t)\tilde{q}(\lambda) + \frac{\sin(\omega(\lambda)t)}{\omega(\lambda)}\tilde{p}(\lambda) \\ -\omega(\lambda)\sin(\omega(\lambda)t)\tilde{q}(\lambda) + \cos(\omega(\lambda)t)\tilde{p}(\lambda) \end{array} \right), \]  

where

\[ \hat{g} = \left( \begin{array}{c} \hat{0} \\ e_n \end{array} \right) = \left( \begin{array}{c} 0 \\ e^{in\lambda} \end{array} \right), \]  

so

\[ \hat{e}^{At}g = e^{in\lambda} \left( \begin{array}{c} \frac{\sin t\omega(\lambda)}{\omega(\lambda)} \\ \cos t\omega(\lambda) \end{array} \right), \]  

whence from (27) follows:

\[ E(\eta, \eta)_H = \frac{1}{2\pi} \int_0^t \int_0^t \int_0^{2\pi} \cos((t - s_1)\omega(\lambda))\cos((t - s_2)\omega(\lambda))B(s_1 - s_2)d\lambda ds_1 ds_2 + \]

\[ + \frac{1}{2\pi} \int_0^t \int_0^t \int_0^{2\pi} \sin((t - s_1)\omega(\lambda))\sin((t - s_2)\omega(\lambda))B(s_1 - s_2)d\lambda ds_1 ds_2 = \]

\[ = \frac{1}{2\pi} \int_0^t \int_0^t \int_0^{2\pi} \cos((s_1 - s_2)\omega(\lambda))B(s_1 - s_2)d\lambda ds_1 ds_2. \]

Further, in view of (5):

\[ E(\eta, \eta)_H = \frac{1}{2\pi} \int_0^t \int_0^t \int_0^{2\pi} \cos((s_1 - s_2)\omega(\lambda)) \int_{\mathbb{R}} e^{i2(s_1 - s_2)\mu(dx)}d\lambda ds_1 ds_2 = \]

\[ = \frac{1}{2\pi} \int_{\mathbb{R}} \int_0^{2\pi} f(t, \lambda, x)d\lambda \mu(dx), \]

where

\[ f(t, \lambda, x) = \int_0^t \int_0^t \cos((s_1 - s_2)\omega(\lambda))e^{i2(s_1 - s_2)}ds_1 ds_2 = \frac{1}{2}(f_+(t, \lambda, x) + f_-(t, \lambda, x)), \]

where

\[ f_{\pm}(t, \lambda, x) = \int_0^t \int_0^t e^{i(s_1 - s_2)(x \pm \omega(\lambda))}ds_1 ds_2. \]

Denote \( \gamma = x \pm \omega(\lambda), \) then

\[ f_{\pm}(t, \lambda, x) = \int_0^t \int_0^t e^{i(s_1 - s_2)\gamma}ds_1 ds_2 = \left| \int_0^t e^{is\gamma} ds \right|^2 = \left| \frac{e^{is\gamma}}{i\gamma} \right|^2 = \frac{1}{\gamma^2} |e^{i\gamma} - 1|^2 = \]

17
\[ f(t, \lambda, x) = \frac{1 - \cos t(x + \omega(\lambda))}{(x + \omega(\lambda))^2} + \frac{1 - \cos t(x - \omega(\lambda))}{(x - \omega(\lambda))^2} = I_+(t, \lambda, x) + I_-(t, \lambda, x), \]

further

\[ E(\eta, \eta)_H = \frac{1}{2\pi} \int_\mathbb{R} \int_0^{2\pi} (I_+(t, \lambda, x) + I_-(t, \lambda, x)) \, d\lambda \, \mu(dx), \]

\[ \int_0^{2\pi} I_+(t, \lambda, x) \, d\lambda = \int_0^{2\pi} \frac{1 - \cos t(x + \omega(\lambda))}{(x + \omega(\lambda))^2} \, d\lambda = \int_0^{2\pi} \frac{d\lambda}{(x + \omega(\lambda))^2} - \int_0^{2\pi} \frac{\cos t(x + \omega(\lambda))}{(x + \omega(\lambda))^2} \, d\lambda. \]

By corollary 2 in [33] (p. 6) has place estimation:

\[ r_+(t, x) = \left| \int_0^{2\pi} \frac{\cos t(x + \omega(\lambda))}{(x + \omega(\lambda))^2} \, d\lambda \right| \leq \frac{c}{\sqrt{t}}, \quad (31) \]

where the constant \( c \) does not depend on \( x \). Indeed, we make the substitution in the integral in (31):

\[ \int_0^{2\pi} \frac{\cos t(x + \omega(\lambda))}{(x + \omega(\lambda))^2} \, d\lambda = 2\pi \int_0^1 \frac{\cos t(x + \omega(2\pi \zeta))}{(x + \omega(2\pi \zeta))^2} \, d\zeta = \pi \int_0^1 \frac{e^{it(x+\omega(2\pi \zeta))} + e^{-it(x+\omega(2\pi \zeta))}}{(x + \omega(2\pi \zeta))^2} \, d\zeta. \]

According to the notation of the corollary \( g(\zeta) = \frac{1}{(x+\omega(2\pi \zeta))^2} > 0 \). From the proof of item 2 of Theorem 2.1 we have that \( \omega(2\pi \zeta) \) is continuous and piecewise monotonic on \([0; 1]\). Further, the support of the measure \( \mu \) is separated from the root of the spectral set of the operator \( V \), and therefore the function \( x + \omega(2\pi \zeta) \) is separated from zero for all \( \zeta \in [0; 1] \). Then \( (x + \omega(\lambda))^2 \) is separated from zero and, in view of the monotonicity and continuity of the function \( h_1(t) = \frac{1}{t} \), \( g(\zeta) \) is continuous and piecewise monotonic as a composition of functions. By the Weierstrass theorem it reaches its maximum on \([0; 1]\), which we denote by \( H \). Now consider \( f(\zeta) = \frac{t(x+\omega(2\pi \zeta)^2)}{2\pi} \). This function is \( n \) times differentiable (we can take arbitrary \( n > 1 \)):

\[ f'(\zeta) = t\omega'(2\pi \zeta) = -t \sum_{k=1}^{K} a(k)k \sin(2\pi k \zeta) / \omega(2\pi \zeta), \]

\[ f''(\zeta) = 2\pi t \omega''(2\pi \zeta) = -2\pi t \frac{\left( \sum_{k=1}^{K} a(k)k^2 \cos(2\pi k \zeta) \right) \omega(2\pi \zeta) - \omega'(2\pi \zeta) \sum_{k=1}^{K} a(k)k \sin(2\pi k \zeta)}{\omega^2(2\pi \zeta)} = \]

\[ = -2\pi t \frac{\left( \sum_{k=1}^{K} a(k)k^2 \cos(2\pi k \zeta) \right) \omega^2(2\pi \zeta) + \left( \sum_{k=1}^{K} a(k)k \sin(2\pi k \zeta) \right)^2}{\omega^3(2\pi \zeta)}. \]

Numerator as a finite linear combination of trigonometric functions has a finite number of zeros on the segment \([0; 1]\) (if it has zeros somewhere). If there are no zeros, then the
second derivative is separable from zero, the required constant \( A \) exists. If zeros exist, then consider their \( \varepsilon \)-neighborhoods. Out of these neighborhoods the second derivative is separable from zero. In these surroundings you can see derivatives of order three or higher. In view of the analyticity of the function, and also that it is not a constant, there is an index \( n \) of the derivative, which has no zeros in the chosen neighborhood. In that case, in each neighborhood, the estimate (28) takes place. From where in view \( A = A_0 t \) we obtain the estimate (31) by the corollary. Which is what was required.

Hereof

\[
E(\eta, \eta)_H = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{0}^{2\pi} \left( \frac{1}{(x + \omega(\lambda))^2} + \frac{1}{(x - \omega(\lambda))^2} \right) d\lambda \mu(dx) + R(t),
\]

\[
|R(t)| \leq \frac{1}{2\pi} \int_{\mathbb{R}} \frac{2c}{\sqrt{t}} \mu(dx) = \frac{c}{\pi \sqrt{t}} B(0) = \frac{c \sigma^2}{\pi \sqrt{t}}, \quad \sigma^2 = E f_2^2.
\]

Where do we get that

\[
\lim_{t \to +\infty} E(\eta, \eta)_H = \frac{1}{4\pi} \int_{\mathbb{R}} \int_{0}^{2\pi} \left( \frac{1}{(x + \omega(\lambda))^2} + \frac{1}{(x - \omega(\lambda))^2} \right) d\lambda \mu(dx) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{0}^{2\pi} \frac{\omega^2(\lambda) + x^2}{(\omega^2(\lambda) - x^2)^2} d\lambda \mu(dx).
\]

It remains to find \( E(\epsilon, \epsilon)_H \):

\[
(\epsilon, \epsilon)_H = \frac{1}{2\pi} \int_{0}^{2\pi} |\hat{p}(\lambda)|^2 + \omega^2(\lambda)|\hat{q}(\lambda)|^2 d\lambda,
\]

where

\[
q_\epsilon = \cos(\omega(\lambda)t)\tilde{q}(0)(\lambda) + \frac{\sin(\omega(\lambda)t)}{\omega(\lambda)}\tilde{p}(0)(\lambda),
\]

\[
p_\epsilon = -\omega(\lambda)\sin(\omega(\lambda)t)\tilde{q}(0)(\lambda) + \cos(\omega(\lambda)t)\tilde{p}(0)(\lambda),
\]

which follows from (28). Hence:

\[
|\hat{p}(\lambda)|^2 + \omega^2(\lambda)|\hat{q}(\lambda)|^2 = (-\omega(\lambda)\sin(\omega(\lambda)t)\tilde{q}(0)(\lambda) + 
\]

\[
+ \cos(\omega(\lambda)t)\tilde{p}(0)(\lambda))(-\omega(\lambda)\sin(\omega(\lambda)t)\tilde{q}(0)(\lambda) + \cos(\omega(\lambda)t)\tilde{p}(0)(\lambda)) + 
\]

\[
+ \omega^2(\lambda)(\cos(\omega(\lambda)t)\tilde{q}(0)(\lambda) + \frac{\sin(\omega(\lambda)t)}{\omega(\lambda)}\tilde{p}(0)(\lambda))(\cos(\omega(\lambda)t)\tilde{q}(0)(\lambda) + 
\]

\[
+ \frac{\sin(\omega(\lambda)t)}{\omega(\lambda)}\tilde{p}(0)(\lambda)) = |\tilde{p}(0)(\lambda)|^2 + \omega^2(\lambda)|\tilde{q}(0)(\lambda)|^2,
\]

which in view of (32) leads to

\[
(\epsilon, \epsilon)_H = \frac{1}{2\pi} \int_{0}^{2\pi} |p(0)(\lambda)|^2 + \omega^2(\lambda)|q(0)(\lambda)|^2 d\lambda = 2H(\psi(0)),
\]

19
whence, in view of the non-randomness of the last expression:

\[ E(\epsilon, \epsilon)_H = 2H(\psi(0)). \]

Now we find the average energy of the limiting distribution:

\[ \eta(t) = (q^\infty(t), p^\infty(t)). \]

Since this vector is a stationary solution of the system under study (and the only one with such initial conditions) and the Fourier transform is a linear operator then the Fourier transform of this vector will go into the stationary solution of the resulting equation:

\[ \ddot{Q}_\lambda(t) = -\omega^2 Q_\lambda(t) + f_i e^{in\lambda}, \quad \omega = \omega(\lambda), \quad (33) \]
\[ Q_\lambda(0) = q^\infty(0)(\lambda), \quad \dot{Q}_\lambda(0) = p^\infty(0)(\lambda). \quad (34) \]

Functions given by formulas:

\[ \widehat{q^\infty(t)}(\lambda) = \int_\mathbb{R} \frac{e^{itx} e^{in\lambda}}{\omega^2(\lambda) - x^2} Z(dx), \]
\[ \widehat{p^\infty(t)}(\lambda) = \int_\mathbb{R} \frac{ix e^{itx} e^{in\lambda}}{\omega^2(\lambda) - x^2} Z(dx), \]

give stationary solution of the equation with the corresponding initial conditions. Indeed, direct verification shows that this is the solution of equations (33). It remains to prove that the initial conditions also correspond to it. To do this, we find the Fourier transform \( \eta(0) \):

\[ \eta(0) = -\int_\mathbb{R} R_A(ix) g Z(dx), \]
\[ \widehat{\eta(0)}(\lambda) = -\int_\mathbb{R} \widehat{R_A(ix)} g Z(dx), \]

where the series summation and integration are interchanged which is possible due to the convergence of the integral \( \int_\mathbb{R} |R_A(ix)|^2 \mu(dx) \).

Let us find \( R_A(ix) g \). From (16) we have:

\[ \widehat{R_A(ix) g} = \begin{pmatrix} -R_V(x^2)e_n \\ -ix R_V(x^2)e_n \end{pmatrix}. \]

Because

\[ R_V(x^2)e_n = \frac{e^{in\lambda}}{\omega^2(\lambda) - x^2}, \]

(see [34], pp. 357–359), we arrive at

\[ R_A(ix) g = -\frac{e^{in\lambda}}{\omega^2(\lambda) - x^2} \begin{pmatrix} 1 \\ ix \end{pmatrix}, \]
i.e. to (34). Which is what was required.

Then:

\[
EH(\eta(0)) = \frac{1}{4\pi} \int_{\mathbb{R}} \int_{0}^{2\pi} \left( \frac{x^2}{(\omega^2(\lambda) - x^2)^2} + \frac{\omega^2(\lambda)}{(\omega^2(\lambda) - x^2)^2} \right) d\lambda \mu(dx) = \\
= \frac{1}{4\pi} \int_{\mathbb{R}} \int_{0}^{2\pi} \frac{\omega^2(\lambda) + x^2}{(\omega^2(\lambda) - x^2)^2} d\lambda \mu(dx) = \alpha.
\]

The theorem is proved completely.

2.2 Acknowledgment

We would like to thank professor Vadim Malyshev for stimulating discussions and numerous remarks.

References

[1] R.L. Dobrushin and J. Fritz (1977) Non-equilibrium dynamics of one-dimensional infinite particle systems with a hard-core interaction. Commun. Math. Phys. 55, 275–292.

[2] C. Boldrighini, A. Pellegrinotti and L. Triolo (1983) Convergence to stationary states for infinite harmonic systems. J. Stat. Phys. 30 (1), 123–155.

[3] C. Boldrighini, R.L. Dobrushin and Yu.M. Sukhov (1983) One-dimensional hard rod caricature of hydrodynamics. J. Stat. Phys. 31 (3), 123–155.

[4] R.L. Dobrushin, A. Pellegrinotti, Yu.M. Sukhov and L. Triolo (1986) One-dimensional harmonic lattice caricature of hydrodynamics. J. Stat. Phys. 43 3–4.

[5] C. Bernardini, F. Huveneers and S. Olla (2019) Hydrodynamic limit for a disordered harmonic chain. Commun. Math. Phys. 365:215

[6] A.A. Lykov and V.A. Malyshev (2018) Convergence to equilibrium due to collisions with external particles. Markov Processes and Related Fields 24 (2), 197–227.

[7] F.J. Dyson (1953) The dynamics of a disordered linear chain. Phys. Rev. 92 (6), 1331–1338.

[8] H. Matsuda and K. Ishii (1970) Localization of normal modes and energy transport in the disordered harmonic chain. Prog. Theor. Phys. Suppl. 45, 56–88.

[9] A.J. O’Connor and J.L. Lebowitz (1974) Heat conduction and sound transmission in isotopically disordered harmonic crystals. J. Math. Phys. 15, 692–703.

[10] A. Casher and J.L. Lebowitz (1971) Heat flow in regular and disordered harmonic chains. J. Math. Phys. 12 (8), 1701–1711.
[11] T. Dudnikova (2018) Behavior for large time of a two-component chain of harmonic oscillators. *Russian Journal of Mathematical Physics* **4** (25), 470–491.

[12] T. Dudnikova (2016) Long-time asymptotics of solutions to a hamiltonian system on a lattice. *Journal of Mathematical Sciences* **219** (1), 69–85.

[13] T. Dudnikova, A. Komech and H. Spohn (2003) On the convergence to statistical equilibrium for harmonic crystals. *J. Math. Phys.* **44** (6), 2596–2620.

[14] V.A. Kuzkin and A.M. Krivtsov (2018) Energy transfer to a harmonic chain under kinematic and force loadings: Exact and asymptotic solutions. *J. Micromech. and Mol. Phys.* **3**, 1–2.

[15] A.M. Krivtsov (2019) The ballistic heat equation for a one-dimensional harmonic crystal. In: Altenbach H., Belyaev A., Eremeev V., Krivtsov A., Porubov A. (eds) *Dynamical Processes in Generalized Continua and Structures. Advanced Structured Materials* **103**.

[16] A.M. Krivtsov, M.B. Babenkov and D.V. Tsvetkov (2020) Heat propagation in a one-dimensional harmonic crystal on an elastic foundation. *Physical Mesomechanics* **23** (2), 109–119.

[17] S.N. Gavrilov and A.M. Krivtsov (2020) Steady-state kinetic temperature distribution in a two-dimensional square harmonic scalar lattice lying in a viscous environment and subjected to a point heat source. *Continuum Mech. Thermodyn.* **32**, 41–61.

[18] J. Hemmen (1980) Dynamics and ergodicity of the infinite harmonic crystal. *Physics Reports* **65** (2), 43–149.

[19] R. Fox (1983) Long-time tails and diffusion. *Phys. Rev. A* **27** (6), 3216–3233.

[20] J. Florencio and Howard Lee (1985) Exact time evolution of a classical harmonic-oscillator chain. *Phys. Rev. A* **31** (5), 3221–3236.

[21] A. Lykov and V. Malyshev (2017) From the $N$-body problem to Euler equations. *Russian Journal of Mathematical Physics* **1** (24), 79–95.

[22] I. Prigogine and R. Herman (1971) Kinetic theory of vehicular traffic. *N.Y.: Elsevier*.

[23] D. Helbing (2001) Traffic and related self-driven many particle systems. *Rev. Mod. Phys.* **73**, 1067–1141.

[24] A. Feintuch and B. Francis (2012) Infinite chains of kinematic points. *Automatic* **48**, 901–908.

[25] O. Lanford and J. Lebowitz (1975) Time evolution and ergodic properties of harmonic systems. In: *Dynamical Systems, Theory and Applications*, J. Moser (eds). Lect. Notes Phys. **38**, 144–177. Springer, Berlin, Heidelberg.
[26] N.N. Bogolyubov (1945) *On Some Statistical Methods in Mathematical Physics*. Ac. Sci. USSR, Kiev.

[27] H. Spohn and J. Lebowitz (1977) Stationary non-equilibrium states of infinite harmonic systems. *Commun. Math. Phys.* **54**, 97–120.

[28] A.A. Lykov (2020) Energy growth of infinite harmonic chain under microscopic random influence. *Markov Processes and Related Fields* **26**, 287–304.

[29] Ju.L. Dalecki and M.G. Krein (1974) *Stability of Solutions of Differential Equations in Banach Space*. AMS.

[30] H. Cramer and M. Lidbetter (1969) *Stationary Stochastic Processes*. Mir, Moscow. (In Russian).

[31] A.N. Kolmogorov and S.V. Fomin (1999) Elements of the theory of functions and functional analysis. *Dover Publ. Inc.* (English transl.)

[32] A.D. Wentzell (1981) A Course in the Theory of Stochastic Processes. *McGraw-Hill Inc. US* (English transl.)

[33] G. I. Arkhipov, A. A. Karatsuba and V. N. Chubarikov (1979) Trigonometric integrals. *Izv. Akad. Nauk SSSR Ser. Mat.* **43**, (5) 971–1003. *Izv. Math* **15**, (2) 211–239.

[34] V.I. Bogachev and O.G. Smolyanov (2020) Real and Functional Analysis. *Springer Nature Switzerland AG*

[35] A.A. Lykov and V.A. Malyshev (2012) Harmonic chain with weak dissipation. *Markov Processes and Related Fields* **18**, 1–10.

[36] V.N. Chubarikov, A.A. Lykov and V.A. Malyshev (2016) Regular continuum systems of point particles. I: systems without interaction. *Chebyshevskii Sbornik* **17** (3), 148–165. [arXiv:1611.02417](https://arxiv.org/abs/1611.02417). (In Russian).

[37] A.A. Lykov and V.A. Malyshev (2013) Convergence to Gibbs equilibrium — Unveiling the Mystery. *Markov Processes and Related Fields* **19**, 643–666.

[38] R. Grimmett and D.R. Stirzaker (2001) Probability and random processes. *Oxford University Press, third edition.*