Strong correlation, electron-phonon interaction and critical fluctuations: isotope effect, pseudogap formation, and phase diagram of the cuprates

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Within the Hubbard-Holstein model with long-range Coulomb forces, we revisit the charge-ordering scenario for the superconducting cuprates and account for the presence or the absence of a formed stripe phase in different classes of cuprates. We also evaluate the mean-field and the fluctuation-corrected critical lines for charge ordering and we relate them with the various pseudogap crossover lines occurring in the cuprates and we discuss a mechanism for their peculiar isotopic dependence. Considering the dynamical nature of the charge-ordering transition, we explain the spread of \( T^* \) and of its isotopic shift, obtained with experimental probes with different characteristic time scales.

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I. INTRODUCTION

The phase diagram of the cuprates, like, e.g., \( \text{La}_{2-x}\text{Sr}_x\text{CuO}_4 \) (LSCO), is characterized by three distinct regions which are reached by varying the doping \( x \): (i) the overdoped region, where the system displays a markedly metallic character, which can be reasonably described within the standard Fermi-liquid picture; (ii) the region around optimal doping, where the superconducting critical temperature \( T_c \) is maximum, and no other energy scale besides the temperature \( T \) is seen in various experiments, so that the metallic phase above \( T_c \) strongly deviates from the Fermi-liquid behavior; (iii) the underdoped region where, on the contrary, there is an abundance of energy scales, and an even stronger violation of the Fermi-liquid behavior. These are related to a doping-dependent temperature \( T^0 \), below which a suppression of the density of states at the Fermi level is revealed in static measurements and to a doping-dependent temperature \( T^* \) below which various experiments reveal a pseudogap \( \Delta_p \). \( T^* \) strongly depends on the characteristic time scale of the probe used to measure the pseudogap properties. Other energy scales of the underdoped phase are the superconducting gap \( \Delta \) and the critical temperature \( T_c \).

The generic partition of the phase diagram into three qualitatively quite distinct regions with markedly different behaviors and the absence of any energy scale but the temperature in the intermediate (optimally doped) region, suggest the presence of a quantum critical point (QCP) near optimal doping \( x=0 \). If this is the case, the identification of the optimally doped region with the region where quantum critical fluctuations are present naturally accounts for the peculiar non-Fermi-liquid behavior. Various proposals have been made to specify the broken symmetry phase to be associated with the underdoped region, ranging from a circulating-current phase \( \sigma_0 \), to a change in symmetry of the superconducting order parameter \( \sigma_0^* \). The scenario that we have been proposing along the years \[x+1\] is related to the occurrence of an instability for charge ordering (CO), along a line \( T_{CO}(x) \) ending in the QCP near optimal doping \( x=0 \). This proposal of a (dynamical) CO instability finds support from neutron scattering experiments in Ne-doped LSCO \[12\] and \( \text{YBa}_2\text{Cu}_3\text{O}_{7-x} \) systems \[13\] where charge domain walls (stripes) have been detected.

In this paper, by extending previous results, we account for the various pseudogap-formation temperatures and give a reason why formed stripes may be present in some classes of cuprates and be absent in other classes. We also address the issue of the anomalous isotopic effects of \( T_c \) and \( T^* \) in the cuprates and provide a novel mechanism based on the relevance of CO fluctuations.

II. THE CHARGE-ORDERING SCENARIO AND THE CUPRATES PHASE DIAGRAM

The above CO scenario found one of its possible realizations within a minimal model of strongly correlated electrons interacting with phonons, i.e., the one-band Hubbard-Holstein (HH) model in the presence of long-range Coulomb forces \[14\]. In this model there is a hopping term \( t \) representing the electron kinetic energy, a (large) on-site Hubbard repulsion \( U \), representing the strong electron-electron (e-e) correlation, a coupling \( g \) between a local lattice distortion and the local electron density, a phonon frequency \( \omega_0 \), characterizing the relevant lattice distortion, and a long-range e-e Coulomb repulsion \( V_C \). The simplification with respect to models taking into account the complex structure of the cuprates was adopted for two main reasons: first of all it makes the model manageable despite the formal complications related to the treatment of strong e-e correlations via the slave-boson large-\( N \) expansion formalism. Moreover it allows to concentrate on the relevant aspects providing both a non-Fermi-liquid behavior and a strong pairing mechanism. Although the relevant features related to quantum criticality should be generic, different cuprates
should of course show different aspects related to their specific structure.

Two major results were obtained within the HH model \[14,15\]. First of all, at mean field, one obtains the \( T = 0 \) phase diagram reported in Fig. 1 with a line of the electron-phonon (e-ph) coupling \( g \) as a function of the doping \( x \). This line marks a second-order transition for the onset of CO characterized by an order parameter \( \rho_{q} \), representing the microscopic charge modulation at a wave vector \( q \). Each point on the curve \( g(x) \) is a QCP, which is the \( T = 0 \) end point of a critical line \( T_{CO}^{0}(x) \). Secondly, charge collective fluctuations mediate a doping- and temperature-dependent singular scattering among the quasiparticles

\[
\Gamma(q, \omega) \approx \tilde{U} - \frac{V}{\xi_{q}^{-2} + |q - q_c|^2 - i\gamma\omega},
\]

where \( \tilde{U} \) is a residual repulsion, \( V \) is the strength of the singular interaction, \( \gamma \sim t^{-1} \) is a characteristic time scale of the charge fluctuations, and \( \xi_{q}^{-2} \) is the inverse square correlation length, which measures the distance from criticality. This interaction diverges at \( \omega = 0 \) and at a critical wave vector \( q_c \) when \( \xi_{0}^{-2} \) vanishes. This occurs at \( T = 0 \) when the doping is reduced down to a mean-field critical value \( x_{0}^{c} \), and at finite \( T \), when approaching the mean-field critical line \( T_{CO}^{0}(x) \). As we discuss in Sec. II B, the CO fluctuations beyond mean field shift the mean-field critical line \( T_{CO}^{0}(x) \) to a lower value \( T_{CO}(x) \), ending in the fluctuation-corrected QCP at \( x_{c} < x_{0}^{c} \), and modify accordingly the expression for the coherence length \( \xi \).

The singular part of the interaction \( (1) \) (with \( \xi_{0} \to \xi \) if fluctuations are taken into account) is attractive both in the particle-particle and in the particle-hole channels. Therefore, the quasiparticles feel an increasingly strong attraction by approaching the CO critical line. In the particle-hole channel, this interaction can produce a pseudogap due to the incipient CO. At the same time in the particle-particle channel, the strong attraction can lead to pair formation even in the absence of phase coherence \[14\]. Therefore the pseudogap-formation temperature \( T^{*}(x) \) closely tracks the underlying CO transition line.

We point out that the assumption of an infinitely large Hubbard repulsion \( U \) and the large-\( N \) expansion do not allow for a correct description of the spin degrees of freedom \[14\]. However, at large but finite \( U \) the above scenario is not expected to drastically change, except that the modulation of the charge profile within the CO state should favor antiferromagnetic ordering in the charge-poor regions. This state with charge and (enslaved) spin modulation mimics the stripe phase observed in the cuprates, when approaching it from the large-doping side, where charge degrees of freedom play a major role. Enslaved nearly critical antiferromagnetic fluctuations with a characteristic wave vector \( q_{AF} \simeq (\pi, \pi) \) are expected to mediate an interaction similar to Eq. (1) in the particle-particle channel, which becomes repulsive in the Cooper channel. This additional contribution is relevant to properly reproduce the main features of the single-particle spectra \[8,9\]. The \( d \)-wave symmetry of the superconducting order parameter, which is already favored by the residual repulsive term in Eq. (1) \[20\], is further stabilized by the antiferromagnetic fluctuations.

\[\text{FIG. 1. Phase diagram e-ph coupling } g \text{ vs doping } x \text{ of the single-band infinite-}U \text{ Hubbard-Holstein model with nearest-neighbor hopping } t = 0.5 \text{ eV, next-to-nearest-neighbor hopping } t' = -(1/6)t, \text{ phonon frequency } \omega_{\text{ph}} = 40 \text{ meV, and in the presence of long-range Coulomb forces with strength } V_{C} = 0.55 \text{ eV. See Ref. } [11] \text{ for a detailed description of the model. An empty circle on the } g \text{ axis marks the minimum value } g_{\text{min}} \text{ allowing for CO.}\]

A. The zero-temperature phase diagram

The separation between the homogeneous phase and the CO phase at \( T = 0 \), reported in Fig. 1, is marked by a line having a minimum value \( g_{\text{min}} \), below which no CO (or stripes) could be formed as a static phase. Of course strong quantum fluctuations mediating the singular scattering \[14\], which leads to pairing and non-Fermi-liquid behavior in the normal phase, would be present both above and below the quantum critical line.

The crucial point in obtaining the CO instability is that strong electronic correlations reduce the effect of the homogenizing electron kinetic-energy term, thus favoring the possibility of an instability induced by a residual interaction, which in our model is provided by the e-ph interaction. The relevant role of the lattice in driving the instability is, indeed, strongly suggested by the peculiar isotopic effect of \( T^{*} \) and \( T_{c} \) \[12\] and by recent EXAFS experiments \[21\], which identified interesting features associated to the lattice structure. Specifically a microstrain \( \varepsilon \) of the Cu-O bonds was measured in terms of the deviation of the Cu-O distance with respect to a reference distance \( d_{0} \), which introduces a mismatch in the
lattice between the CuO$_2$ layers and the rock-salt layers. Actually there is a close resemblance between the experimental ($\varepsilon, x$) and the theoretical ($g, x$) phase diagrams indicating that the Cu-O microstrain and the e-ph coupling $g$ of the simplified HH model are strictly related. From the microscopic point of view it is indeed quite natural that a lattice contraction in the CuO$_2$ planes can enhance the effective coupling between the electrons and the ions. Therefore the comparison between experiments and theory allows to draw the conclusion that a one-to-one monotonic relation $g = g(\varepsilon)$ is likely to exist.

Once this general framework is settled, one can move to identify more detailed possible scenarios. In particular the order of the homogeneous-metal to charge-ordered-metal transition is of obvious relevance. The simplest possibility is that in the real materials this transition is of the second order. In this case, at $T = 0$, the different microstrains determined by the rock-salt layers directly correspond to different e-ph couplings and are reflected in the similar phase diagrams. Of course this simple theoretical picture is not necessarily realized in the quite complex real materials where i) anharmonic effects and/or ii) additional non-ordering fields can partially or entirely transform the second-order transition into a first-order one. In this last case, the above scenario maintains its validity, as long as this transition is weakly first-order.

![Phase Diagram](image)

**FIG. 2.** The phase diagram of the cuprates according to the CO-QCP scenario for LSCO (a) and Bi2212 (b). The solid line is the m-f critical line $T_{CO}^0(x)$ ending at $T = 0$ in the m-f QCP at $x_c$. The lowest dashed line in panel (a) marks the 3D critical line $T_{CO}(x)$ in the presence of fluctuations, ending in the QCP at $x_c$. The dot-dashed line in panels (a,b) indicates the “dynamical instability” condition (see text) for $\omega_{probe} = 1$ meV. The intermediate dashed line in panel (a) represents the “dynamical instability” condition for $\omega_{probe} = 1$ meV. The experimental points for $T^0$ (stars) and for $T^*$ measured with fast (crosses) and slow (squares) probes for LSCO are from Ref. [1], those for Bi2212 are from Refs. [24,25]. The critical temperatures $T_c$ are also shown (pluses).

We stressed in several papers [3] that the actual onset of a fully developed CO phase is competing with local or coherent pair formation, which modifies the fermionic spectrum stabilizing the system against the electronic CO transition. There is the complementary possibility that the charge-ordered state can never be reached because $g$ is below $g_{min}$. The corresponding observable quantity identified by the experiments in Ref. [21] is the critical microstrain $\varepsilon_c$, below which no stripe phase can be observed. Therefore the cuprates can deviate from CO criticality not only by $x - x_c$, in the (underdoped) quantum-disordered region, and by $T$ or $T - T_{CO}(x)$ in the quantum-critical or underdoped region, but also because the microstrain $\varepsilon$ is smaller or larger than $\varepsilon_c$, thereby tending to a homogeneous or an inhomogeneous phase respectively. As long as $\varepsilon - \varepsilon_c$ is too large and $T$ is finite, for $x \sim x_c$ the quantum-critical region, with strong dynamical fluctuations, can be reached in both cases. According to the proposed mechanism of pairing mediated by critical fluctuations, the stronger are these fluctuations and the larger is $T_c$. Indeed in the experiment the mercury compound Hg1212, having $\varepsilon \sim \varepsilon_c$, has the the largest $T_c$, while Hg1201 (with $\varepsilon < \varepsilon_c$) and Bi2212 or LSCO (with $\varepsilon > \varepsilon_c$) have a lower $T_c$. Moreover, since $\varepsilon < \varepsilon_c$ in Hg1201, we expect no $T_{CO}(x)$, and therefore no $T^*$, for this material.

### B. Charge-ordering transition at finite temperature and pseudogap formation

In this section we calculate the temperature for the onset of the CO instability within the mean-field approximation, $T_{CO}^0(x)$, and its corrected value, $T_{CO}(x)$, in the presence of the leading fluctuations beyond mean field. For establishing the onset of the CO phase coming from high-doping and high-temperature regime, the direct involvement of the spin degrees of freedom can be safely ignored.

For $T < T_{CO}^0$ the CO fluctuations become substantial, leading to a reduction of the quasiparticle density of states. We therefore identify our mean-field line $T_{CO}(x)$ with the weak-pseudogap crossover line $T_{CO}^0(x)$ observed in Knight-shift, transport, and static susceptibility measurements [1], as the incipient reduction of the quasiparticle density of states. The fluctuation-corrected critical line $T_{CO}(x) \approx T_{CO}^0(x)$, the correlation length $\xi$ diverges, and the scattering [1] becomes effectively singular, is instead closely tracked by the pseudogap crossover line $T^*(x)$, according to the discussion of Sec. II.

All the complicated formal structure of the quasiparticle scattering, mediated by phonons and by the Coulomb interaction within the slave-boson approach, is, at the end, simply represented, at least near criticality, by a RPA resummation $\Gamma(q, \omega_n) \approx V_{eff}(q)/[1 + V_{eff}(q)\Pi(q, \omega_n)]$ of an effective static interaction $V_{eff}(q) = \hat{U}(q) + V_C(q) - \lambda T$ [23]. Here $\Pi$ is
the fermionic polarization bubble, $\lambda \equiv 2g^2/\hbar \omega_0$ is the dimensionless e-ph coupling, and $U(q) \simeq A + B |q|^2$ is the residual short-range repulsion between quasiparticles, and $V_C(q)$ is the Fourier transform of the e-e Coulomb repulsion. For the correspondence of $V_{eff}(q)$ with the parameters of the original HH model see Ref. [23]. The mean-field CO instability condition, which occurs for reasonable values $\lambda \sim 1$, is $1 + V_{eff}(q_0)\Pi(q_0, \omega = 0) = 0$. At $T = 0$ this determines $\eta_0$ and the position of the mean-field QCP $x_0^c$. By expanding $1 + V_{eff}\Pi$ near the instability at $T = 0$, we find $\xi_0^{-2} \propto x - x_0^c$. At $T > 0$, considering the $T$ dependence of the bare polarization bubble $\Pi$, the instability condition determines the mean-field critical line $T_{CO}^0(x)$ (the solid-line curves in Fig. 2), which ends at $x_0^c$. We evaluate $T_{CO}^0(x)$ by taking standard quasiparticle (i.e., dressed by the slave bosons) band parameters ($t = 200$ meV, $\omega_0 = 70$ meV, leading to $A = 200$ meV and $B = 170$ meV in the residual repulsion $U$), $V_C = 220$ meV and $g = 210$ meV are adjusted to match with the extrapolated experimental values of $T^0(x)$ for LSCO $T^0_c(x_0 \approx 0.22)$ with the mean-field QCP $x_0^c$. Without any further adjustment of the parameters the curve $T_{CO}^0(x)$ agrees remarkably well with the experimental data for $T^0(x)$. Similar parameters are taken for Bi2212 to fit the data in Refs. [24,25], with $V_C = 220$ eV and $g = 230$ eV.

FIG. 3. The vertex and selfenergy corrections beyond RPA for the fermionic bubbles (solid line) due to critical CO fluctuations [the wavy line represents the singular part of the effective interaction, Eq. (1)].

Due to the strong anisotropy of the layered cuprates, the contribution of the fluctuations related to any instability is significant [23,24]. Specifically the CO fluctuations which mediate the critical effective interaction, Eq. (1), can be included in the polarization bubble $\Pi$ (entering the instability condition) via the diagrams of Fig. 3, leading to corrections beyond RPA. From the explicit evaluation of these diagrams we find the self-consistent correction to the inverse square correlation length

$$\xi^{-2} = \xi_0^{-2} + 12\tilde{u} T \sum_{|\omega_n| < \omega_0} \frac{V}{\xi^{-2} + |q - q_c|^2 + \gamma |\omega_n|},$$

(2)

where $\omega_n$ are the bosonic Matsubara frequencies, and the coupling constant $\tilde{u} \sim V/t^3$. Since the phonons are responsible for the instability, $\omega_0$ appears as the frequency cut-off [11]. The fluctuation-corrected instability condition is $\xi = 0$. As it stands, Eq. (2) is written for the two-dimensional Cu-O planes. At $T = 0$, Eq. (2) with $\xi^{-2} = 0$ leads to a finite shift of the two-dimensional (2D) QCP $x_0^c - x_c \propto \omega_0$ (see Fig. 2). At finite $T$, purely 2D fluctuations suppress the transition. However, by considering the more realistic anisotropic 3D character of the critical fluctuations, we obtain a finite fluctuation-corrected transition line $T_{CO}^0(x)$, which is reported as the lowest dashed line in Fig. 2(a). Thus, the inclusion of fluctuations brings the critical line from temperatures of the order of typical electronic energies ($T_{CO}^0 \sim T$) down to much lower temperatures $T_{CO}$ of the order of the observed $T^*$’s.

The spread in the measured values of $T^*$ depending on the typical frequency of the experimental probe $\omega_{probe}$ (see, e.g., Ref. [1]) suggests, however, within our identification of $T_{CO}$ with $T^*$, that the CO instability may be “dynamical”. In this case, the critical condition $\xi^{-2} = 0$ is replaced by $\xi^{-2} = \gamma \omega_{probe}$ in the self-consistency condition, Eq. (2). Two examples are reported in Figs. 2 and $\omega_{probe} = 1$ meV (dot-dashed line in Figs. 2(a,b), typical of neutron scattering experiments, and $\omega_{probe} = 1$ meV (second dashed line from bottom in Fig. 2(a), typical of static experiments (NQR, NMR). The corresponding experimental data for $T^*$ in LSCO are also reported for comparison. The coupling $V$ between the charge fluctuations and the quasiparticles, which is the only parameter for which an a priori estimate is difficult, is fixed by matching the position of the fluctuation-corrected QCP with the $T = 0$ extrapolation of the experimental values of $T^*$. We used $\gamma = 0.7$ eV$^{-1}$ and $\gamma = 0.4$ eV$^{-1}$ for LSCO and Bi2212 respectively, and $V = 0.54$ eV for both. Note that, similarly to the case of $T_{CO}^0(x)$, the agreement between the calculated $T_{CO}^0(x)$ and the experimental points for $T^*$ is obtained without adjustment of other parameters.

III. NOVEL ISOPOTE EFFECTS

In the present framework, any mechanism shifting the position of the QCP and the line $T_{CO}(x)$ is mirrored by a corresponding shift of the superconducting critical line $T_c(x)$ and of the pseudogap formation temperature $T^*(x)$. This is the case of the shift induced by isotopic substitution, and we want to connect our results with the observation of isotopic effects on $T_c$ [29] and on $T^*$ [30,31]. As shown in Sec. II A, the mean-field weak-pseudogap crossover temperature $T_{CO}^0 \sim T^0$ depends on the phonon frequency only through the dimensionless e-ph coupling $\lambda$, which is known to display no isotopic dependence. Therefore $T_{CO}^0$, and consequently $T^0$ and the
portion of $T_c(x)$ near and above $T^0(x)$, are not expected to display isotope effects.

On the other hand, the CO fluctuations, as shown above, crucially involve $\omega_0$, and new physical effects can be induced by the isotopic substitutions, on fluctuation-dependent physical quantities, such as $x_c$ and $T_{\text{CO}}(x)$. Since $\omega_0$ decreases with increasing ionic mass, the effect of fluctuations is reduced, and $x_c$ and $T_{\text{CO}}(x)$ remain closer to their mean-field values. In Fig. 4, we report the line $T_{\text{CO}}(x)$, calculated via Eq. (3), with the same parameters used in Sec. II B to fit the $T^*$ data of LSCO, together with its isotopic shift calculated for $^{16}O \rightarrow ^{18}O$ substitution (i.e., for a five percent reduction of $\omega_0$).

FIG. 4. Calculated effect of the isotopic change $^{16}O \rightarrow ^{18}O$ (i.e. $\omega_0 = 70$ meV and $\omega'_0 = 66$ meV) on $T^*$ in LSCO, both for a fast- and a slow-probe measurement. The inferred shift on $T_c$ is also reported, although hardly visible on this scale.

Contrary to standard theories based on CO pseudo-gap [33], the isotopic shift is positive (i.e. anomalous) for $T_{\text{CO}} \sim T^*$. Moreover, if the slope of $T_{\text{CO}}(x)$ is large, a small isotopic shift in $x_c$ can result in a substantial shift in $T_{\text{CO}} \sim T^*$, i.e., the steeper is $T^*$, the larger is the isotopic effect. Since the slope of $T_{\text{CO}}(x)$ increases by increasing $\omega_{\text{probe}}$, whereas the QCP is unshifted, the isotope effect on $T_{\text{CO}} \sim T^*$ is enhanced, and we have the general trend that faster probes should detect a larger isotope effect on $T^*$. Although we cannot account for the near-absent or negative isotope effect on $T^*$ within the almost static probes in YBa$_2$Cu$_3$O$_6$ [30,31], this general trend is in qualitative agreement with a much stronger effect observed in the isostructural HoBa$_2$Cu$_4$O$_8$ (HBCO-124) with fast neutron scattering [32]. Indeed, this fast-probe experiment should be represented by the curve $T_{\text{CO}}(x)$ corresponding to $\omega_{\text{probe}} = 1$ meV in Fig. 4. The huge isotope effect on $T^*$ observed in HBCO-124 suggests that the curve $T_{\text{CO}}(x)$ in this system is steeper than in LSCO.

As far as $T_c$ in the underdoped region is concerned, we can only infer some expected consequences. Assuming that the shift in $x_c$ produces a rigid shift of $T_c(x)$ along the doping axis, the isotopic effect for $T_c$ is negative (i.e. normal) upon reducing $\omega_0$, and much smaller than for $T^*$ (the $T_c(x)$ curve being much flatter), in agreement with long-standing experiments [23,24].

This large difference in the isotope effect for $T_c$ and $T^*$, \( (\Delta T_c/\Delta M)/(\Delta T^*/\Delta M) \ll 1 \), is indeed experimentally observed in HBCO-124, and reported in Ref. [22], where a "striking similarity between isotope substitution and underdoping with respect to both $T_c$ and $T^*$" is pointed out. Although we are not aware of any systematic analysis of the doping dependencies of $T_c$, $T^*$ and their isotopic shifts in HBCO-124, allowing for a strict comparison, this observation finds its natural interpretation within our theory, where the isotopic substitution increases $x_c$ and is therefore nearly equivalent to underdoping.

In the strongly underdoped materials, where substantial isotope effects in the penetration depth [34] and in the x-ray absorption [35] have been observed, our results can provide only qualitative indications, as other effects (magnetic, polaronic, lattice-pinning) not included in our HH model become relevant.

IV. CONCLUSIONS

In conclusion we have revisited the CO scenario, and explicitly evaluated the mean-field and the fluctuation-corrected critical lines, $T_{\text{CO}}^{0}(x)$ and $T_{\text{CO}}(x)$, for the onset of the charge inhomogeneous phase, which should represent the stripe phase of the cuprates, when present. The nearly singular effective interaction among the electrons mediated by the CO fluctuations near the instability, suggests the natural identification of the experimentally observed weak-pseudogap crossover line $T^0(x)$ with $T_{\text{CO}}^{0}(x)$, and of the pseudogap formation temperature $T^*$ with $T_{\text{CO}}(x)$. We obtain in this way a good fitting of the experimental data, and by assuming that the CO transition has a dynamical character, we account for the spread of the measurements of $T^*(x)$, depending on the characteristic time scale of the experimental probe. The existence of the CO transition, depending on the strength of the e-ph coupling, as produced by our zero-temperature phase diagram, can give an answer to the puzzling question why features of formed stripes are observed in some cuprates and absent in others. The CO instability condition at mean-field level depends only on the dimensionless e-ph coupling which is not changed by isotopic substitution. Therefore, in our model, $T_{\text{CO}}^{0}(x)$ should not undergo any isotopic shift. On the contrary, $T_{\text{CO}}(x)$, and therefore $T^*$, are strongly modified by the effect of CO fluctuations, which in our approach is proportional to the phonon frequency $\omega_0$. Within our model, therefore, $T^*$ shows a novel isotope effect produced by charge fluctuations, with a positive shift, which depends on the slope of $T_{\text{CO}}(x)$. It results, therefore, that the isotope effect is stronger when detected with faster probes. We also infer that $T_c(x)$ should have no isotope effect in...
the optimal and overdoped regimes, and a small negative effect in the underdoped region, which increases upon underdoping. In this new isotope effect, the phonon do not appear directly as mediators of pairing, but are indirectly involved, via the CO fluctuations, and qualitatively explain the complex behavior of $T^*$ and $T_c$ upon isotopic substitution.

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