Compressive Phase Retrieval of Structured Signal

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Abstract

Compressive phase retrieval is the problem of recovering a structured vector \( x \in \mathbb{C}^n \) from its phaseless linear measurements. A compression algorithm aims to represent structured signals with as few bits as possible. As a result of extensive research devoted to compression algorithms, in many signal classes, compression algorithms are capable of employing sophisticated structures in signals and compress them efficiently. This raises the following important question: Can a compression algorithm be used for the compressive phase retrieval problem? To address this question, COmpressive PhasE Retrieval (COPER) optimization is proposed, which is a compression-based phase retrieval method. For a family of compression codes with rate-distortion function denoted by \( r(\delta) \), in the noiseless setting, COPER is shown to require slightly more than \( \lim_{\delta \to 0} \frac{r(\delta)}{\log(1/\delta)} \) observations for an almost accurate recovery of \( x \).

I. INTRODUCTION

Consider the problem of recovering \( x_o \in \mathcal{Q} \), where \( \mathcal{Q} \) denotes a compact subset of \( \mathbb{C}^n \), from \( m \) noisy phase-less linear observations

\[ y = |Ax_o| + \epsilon, \]

where \( \epsilon \in \mathbb{R}^n \) denotes the measurement noise. Here, \(| \cdot |\) denotes the element-wise absolute value operator. Further assume that the class of signals \( \mathcal{Q} \) is “structured”, but instead of the set \( \mathcal{Q} \), or its underlying structure for recovering \( x_o \) from \( y \), we have access to a compression code that takes advantage of the structure of signals in \( \mathcal{Q} \) to compress them efficiently. A rate-\( r \) compression code is composed of an encoder mapping \( E \) and a decoder mapping \( D \), where

\[ E : \mathbb{C}^n \to \{0, 1\}^r, \quad D : \{0, 1\}^r \to \mathbb{C}^n. \]

The distortion performance of compression code defined by mappings \((E, D)\) on set \( \mathcal{Q} \) is measured as

\[ \delta \triangleq \sup_{x \in \mathcal{Q}} \| x - D(E(x)) \|. \]

Given a family of compression codes \((E_r, D_r)\) for set \( \mathcal{Q} \) indexed by their rate \( r \), let \( \delta(r) \) denote the distortion performance of the code operating at rate \( r \). Then, the distortion-rate function of this family of codes is defined as

\[ r(\delta) \triangleq \inf\{ r : \delta(r) < \delta \}. \]

It can be shown that, for a class of structured signals, \( r(\delta) \ll n \log(1/\delta) \) \[\|\]. Note that, up to a constant, \( n \log(1/\delta) \) is the rate distortion of an optimal code for an \( \ell_2 \) ball in \( \mathbb{R}^n \). In this paper, we would like to answer the following two questions:

1) Is it possible to use a given compression algorithm to recover \( x_o \) from its undersampled set of phaseless observations?
2) What is the required number of observations (in terms of the rate-distortion of the code), for almost zero-distortion recovery of \( x_o \)?

To address these two questions, we propose COMPressive PhasE Retrieval (COPER) that aims to use a compression code for the phase retrieval problem. We study the performance of this method in the noiseless setting \( \varepsilon = 0 \) and characterize the required number of phaseless measurements for an almost zero-distortion recovery. The number of measurements is connected with the level of structure of signals in \( Q \) that is captured by the employed family of compression codes. Given a family of compression codes \( F \) could successfully run the algorithm with \( m \) observations experimentally, but no theoretical guarantee is proved in this paper for convergence of the algorithm.

As shown in [1], there exist some compression codes that achieve these lower bounds in both cases. It is reasonable to think of \( \dim_\alpha(Q) \) as the true dimension of compact set \( Q \). Thus, the smaller \( \dim_\alpha(Q) - \dim_\alpha(F) \), the more structure of \( Q \) is captured by compression family \( F \). Similarly, it can be shown that, for \( k \)-sparse signals in the unit ball of \( \mathbb{C}^n \), the \( \alpha \)-dimension of any family of compression codes is above \( 2k \), and this bound is achievable. We show that \( m > \dim_\alpha(F) \) observations suffice for an accurate recovery of \( x_o \).

A. Related work

The problem of phase retrieval has been studied extensively in the literature. Most papers are concerned with standard structures such as sparsity [3]-[8]. Assuming the signal is \( k \)-sparse, i.e. all of its coordinates but \( k \) ones are 0, a variety of recovery algorithms exist in the literature. In the following, we briefly review some of such methods.

[3] assumes that the signal is sparse, or can be approximated well with few non-zero coefficients. Furthermore, the authors suppose that \( l_1 \)-norm of the signal is known, it employs an iterative algorithm to solve the phase retrieval problem and can guarantee the convergence of algorithm, having \( k^2 \ln \frac{n}{2k} \) observations. However, in practice, they could successfully run the algorithm with \( k \ln \frac{n}{2k} \) measurements. [6] uses the lifting technique to convexify the problem and take advantage of semi definite programming (SDP), to recover the signal of interest. Since the \( x \in \mathbb{C}^n \) is lifted to the space of \( \mathbb{C}^{n \times n} \) matrices, the complexity of the algorithm grows fast with \( n \). Therefore this method is practical only in very limited scenarios. [7] adds an \( l_1 \)-penalty to overcome the ambiguity in phase and to encourage the algorithm to recover a sparse signal. It proves that in the non-negative real case, where there is no phase ambiguity anymore, the correct signal \( x \) is the minimizer of the cost function it defines. They also apply their method to some image signals. In all of the simulations, the number of measurements is proportional to \( n \), the ambient dimension of \( x \), where the least sampling rate they use, is 0.2. [4] uses generalized approximate message passing to recover the signal with \( 2k \ln \left( \frac{n}{k} \right) \) observations experimentally, but no theoretical guarantee is proved in this paper for convergence of the algorithm. [8] considers the case of noisy observations for a \( k \)-sparse signal. Inspired by Wirtinger flow [2], it proposes an algorithm that requires \( m \gg k \ln n \), where \( m \) denotes the number of measurements. They choose \( m = O(k^2) \) to show their algorithm converges. [1] studies the problem of phase retrieval for general signals, where no structure is assumed. They propose a non-convex problem to recover the signal from its phase-less measurements with a careful initialization.
We measure the quality of our estimate $\hat{\delta}y$ which is our main contribution at this work, and it is followed by section IV that concludes the paper.

It is straightforward to confirm that $\dim_{\alpha}(F)$ measurements which in the case of $k$-sparse signal is $2k$.

The organization of the paper is as follows. Section II reviews the problem of compressible phase retrieval and introduces our proposed COMpressive PhasE Retrieval (COPER) optimization. Section III states and proves the theorem which is our main contribution at this work, and it is followed by section IV that concludes the paper.

II. COMPRESSIBLE PHASE RETRIEVAL

Consider a family of compression codes $F = \{ (E_r, D_r) \}_{r}$ for a set $Q \subset \mathbb{C}^n$ indexed by their rates $r$. The codebook of compression code $(E_r, D_r)$ at rate $r$ is defined as

$$C_r \triangleq D_r(E_r(Q)) = \{ D_r(E_r(x)) : x \in Q \}.$$  

It is straightforward to confirm that $|C_r| \leq 2^r$. Recall that $\|x - D(E(x))\| \leq \delta$ for all $x \in Q$ by definition of distortion $\delta$. As discussed in the introduction, we assume that

$$\dim_{\alpha}(F) = \lim_{\delta \to 0} \frac{r(\delta)}{\log 2} \ll n,$$

i.e., the set $Q$ denotes a class of structured signals. The main goal of this paper is to employ the described family of compression codes for $Q$ to recover a signal $x \in Q$ from its noiseless phaseless observations $y = |Ax|$. Given measurement matrix $A \in \mathbb{C}^{m \times n}$, define the distortion measure $\hat{d} : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{R}^+$ as follows

$$d(x, e) \triangleq \frac{1}{m} \sum_{k=1}^{m} \left( |A_k^T x|^2 - |A_k^T e|^2 \right)^2 = \frac{1}{m} \sum_{k=1}^{m} \left( A_k^T(x - ce)^*A_k^T \right)^2,$$

where $A_k^T$ denotes the $k$th row of $A$. Throughout the paper, for complex matrix $A$, $A^*$ and $\tilde{A}$ denote its transposed-conjugate, and conjugate, respectively. Based on the defined distance measure, consider the following discrete non-convex optimization problem named COMpressive PhasE Retrieval (COPER), for recovering $x$:

$$\hat{x} = \arg \min_{c \in C_r} d(x, c).$$

In other words, among all elements of the codebook, COPER finds the one for which under distortion measure defined in (3) $Ac$ is closest to measurements $y$. In the next section, we study the performance of COPER.

Note that in phase retrieval, since the measurements are phaseless, the recovery of $x$ can never be exact; if $x$ satisfies $y = |Ax|$, then so does $e^{i\theta}x$, for any $\theta \in \mathbb{R}$. Hence, following the standard procedure in the phase retrieval literature, we measure the quality of our estimate $\hat{x}$ as

$$\inf_{\theta \in [0, 2\pi]} \| e^{i\theta}x - \hat{x} \|^2.$$
In the next section, we would bound \( \inf_{\theta} \| e^{i\theta} x - \hat{x} \|^2 \) in terms of the number of measurements and the rate-distortion function of the code. We will then characterize the number of measurements COPER requires to perform an accurate recovery.

### III. MAIN RESULT

The main goal of this section is to analyze the performance of the COPER optimization defined in (3). Toward this goal, we make the following assumptions that are standard in the analysis of phase retrieval algorithms:

1) For every \( x \in \mathbb{Q} \), we have \( \| x \|_1 \leq 1 \).

2) The elements of \( A \) are iid drawn from \( \mathcal{N}(0, 1) + i\mathcal{N}(0, 1) \), where \( i \) is square root of \(-1\).

The following theorem and corollary describe our main contributions in this work.

**Theorem 1.** Let \( (\mathcal{E}_r, \mathcal{C}_r) \) be a rate-\( r \) compression code with distortion \( \delta \). Let \( x \in \mathbb{Q} \) denotes the desired signal, and define sensing matrix \( A \), as above. Let \( \hat{x} \) denotes the solution of COPER optimization. That is, \( \arg \min_{c \in \mathcal{C}_r} d(x, c) \). Then we have

\[
\inf_{\theta} \| e^{i\theta} x - \hat{x} \|^2 \leq 16\sqrt{3} \frac{1 + \tau_2}{\sqrt{\tau_1}} m \delta,
\]

with probability at least

\[
1 - 2^r e^{-\frac{\tau_1}{2} (K + \ln \tau_1 - \ln m)} - e^{-2m \left( \frac{\tau_2}{2} - \ln(1 + \tau_2) \right)},
\]

where \( K = \ln 2 \pi e \), and \( \tau_1, \tau_2 \) are arbitrary positive real numbers.

**Corollary 1.** For large enough \( r \), we have

\[
\mathbb{P} \left( \inf_{\theta} \| e^{i\theta} x - \hat{x} \|^2 \leq C \delta^e \right) \geq 1 - 2^{-c_r r} - e^{-0.6m},
\]

where \( C = 32\sqrt{3} \), and \( m = \eta \frac{r}{\log \tau} \). Given \( \eta > \frac{1}{1 - e} \), \( c_r \) is a positive number greater than \( \eta (1 - e) - 1 \).

**Proof of corollary** Let \( \epsilon > 0, \eta > 0 \), in theorem 1 let \( \tau_1 = m^2 \delta^{2-2e} \), and \( \tau_2 = 1 \). It follows that,

\[
\inf_{\theta} \| e^{i\theta} x - \hat{x} \|^2 \leq 1 - e^{-r \left( \ln 2 + \frac{\ln \tau_1}{2} (K + \ln m^2 \delta^{2-2e} - \ln m) \right)} - e^{-2m (1 - \ln 2)},
\]

Note that \( 1 - \ln 2 > 0.3 \), and

\[
1 + \frac{\eta}{2 \ln \frac{1}{\tau}} \left( K + \ln m^2 \delta^{2-2e} - \ln m \right) = 1 + \frac{\eta (K + \ln m)}{2 \ln \frac{1}{\tau}} - \eta (1 - e).
\]

Since \( K, \eta \) are constants, and \( m \to \eta \dim_{\alpha}(\mathcal{F}) \) as \( \delta \to 0 \). Therefore,

\[
\frac{\eta (K + \ln m)}{2 \ln \frac{1}{\tau}} \xrightarrow{\delta \to 0} 0.
\]

Set any positive number \( c_\eta \) such that \( 0 < c_\eta < \eta (1 - e) - 1 \), so for large enough \( r \) we have

\[
1 + \frac{\eta (K + \ln m)}{2 \ln \frac{1}{\tau}} - \eta (1 - e) < -c_\eta,
\]

Thus

\[
e^{-r \ln 2 \left( 1 + \frac{\ln \tau_1}{2} (K + \ln m^2 \delta^{2-2e} - \ln m) \right)} < 2^{-c_r r}.
\]
Proof of Theorem 1

Let
\[ \tilde{x} = D(E(x)) \in C_r. \]

Note that by definition of \( \delta(r) \), \( \|x - \tilde{x}\| \leq \delta(r) \). Moreover, by definition of \( \tilde{x} \), we have
\[ d(\|Ax\|, |A\tilde{x}|) \leq d(\|Ax\|, |A\tilde{x}|). \]

In appendix B we prove some basic properties of \( d(...) \). In particular theorem 2 characterizes concentration of \( d(...) \), which implies that
\[
\lambda_{\text{max}}^2(\tilde{x})\tau_1 < d(\|Ax\|, |A\tilde{x}|) \\
\leq d(\|Ax\|, |A\tilde{x}|) \\
< \lambda_{\text{max}}^2(\tilde{x})(4m(1 + \tau_1))^2, \tag{9}
\]
where for the fixed signal \( x \) and a complex vector \( e \), \( \lambda_1(e) \), \( \lambda_2(e) \) are non-zero eigenvalues of \( xx^* - ee^* \), and \( \lambda_{\text{max}}(e) \) is the one with the largest absolute value. Therefore,
\[
\lambda_{\text{max}}^2(\tilde{x}) < \frac{16m^2(1 + \tau_2)^2}{\tau_1}, \tag{10}
\]
with a probability larger than \( 1 - 2^e e^{2K(1+\ln \tau_1 - \ln m)} - e^{-2m(\tau_2 - \ln(1+\tau_2))} \), where \( K = \ln 2\pi e \), \( \tau_1, \tau_2 > 0 \). Note that by (11)
\[
\lambda_{\text{max}}^2(\tilde{x}) \geq \frac{1}{2} \left( \lambda_1^2(\tilde{x}) + \lambda_2^2(\tilde{x}) \right) \\
= \frac{1}{2} \left( \|x\|^2 - \|\tilde{x}\|^2 \right)^2 + \left( \|x\|^2 \|\tilde{x}\|^2 - \|x^*\tilde{x}\|^2 \right). \tag{11}
\]
Recall \( \|x - \tilde{x}\| \leq \delta \) and since \( x, \tilde{x} \in Q \) we have \( \|x\|, \|\tilde{x}\| \leq 1 \), thus
\[
(\|x\| + \|\tilde{x}\|)^2 (\|x\| - \|\tilde{x}\|)^2 \leq 4\delta^2. \tag{12}
\]
Moreover,
\[
\delta^2 \geq \|x - \tilde{x}\|^2 \\
= \|x\|^2 + \|\tilde{x}\|^2 - x^*\tilde{x} - \tilde{x}^*x \\
\geq \|x\|^2 + \|\tilde{x}\|^2 - 2|x^*\tilde{x}| \\
\geq 2 (\|x\| \|\tilde{x}\| - |x^*\tilde{x}|),
\]
so we have \( (\|x\| \|\tilde{x}\| - |x^*\tilde{x}|) \leq \frac{\delta^2}{2} \), which implies
\[
\left( \|x\|^2 \|\tilde{x}\|^2 - |x^*\tilde{x}|^2 \right) \leq \delta^2. \tag{13}
\]
So by (11) we get
\[
\lambda_{\text{max}}^2(\tilde{x}) \leq \left( \lambda_1^2(\tilde{x}) + \lambda_2^2(\tilde{x}) \right) \\
= \left( \|x\|^2 - \|\tilde{x}\|^2 \right)^2 + 2 \left( \|x\|^2 \|\tilde{x}\|^2 - |x^*\tilde{x}|^2 \right) \\
\leq 6\delta^2. \tag{14}
\]
Therefore, combining (10), (11), (14), it follows that, with probability larger than 
\(1 - 2r e^{(K + \ln \tau_1 - \ln m) - e^{-2m(\tau_2 - \ln(1 + \tau_2))}}\), we have
\[
\frac{1}{2} \left( \|x\|^2 - \|\hat{x}\|^2 \right)^2 + \left( \|x\|^2 \|\hat{x}\|^2 - |x^* \hat{x}|^2 \right) \leq \chi^2_{\max}(\hat{x}) \leq \frac{16m^2(1 + \tau_2)^2}{\tau_1} \chi^2_{\max}(\hat{x}) \leq \frac{96m^2(1 + \tau_2)^2}{\tau_1} \delta^2.
\]

Having (15) and lemma 1 we get
\[
\left( \inf_\theta \| e^{i\theta} x - \hat{x} \|^2 \right)^2 \leq \frac{768m^2(1 + \tau_2)^2}{\tau_1} \delta^2,
\]
which means
\[
P \left( \inf_\theta \| e^{i\theta} x - \hat{x} \|^2 \leq 16\sqrt{3} \frac{1 + \tau_2}{\sqrt{\tau_1}} m \delta \right) \geq 1 - 2r e^{\frac{K + \ln \tau_1 - \ln m}{\sqrt{\tau_1}}} - e^{-2m(\tau_2 - \ln(1 + \tau_2))},
\]
where \(K = \ln 2\pi e, \tau_1, \tau_2 > 0\).

**Remark 1.** Corollary 1 shows, COPER can recover the signal \(x\) with \(\eta \dim(\mathcal{Q})\) for any \(\eta > 1\) with desired small distortion. This happens with a high probability that tends to 1 as \(m, r \to \infty\). In the case of \(k\)-sparse complex signal, COPER needs \(2\eta k\) measurements for almost accurate recovery which is less than all previous results, at least by a log factor, due to the best of our knowledge.

**IV. Conclusions**

In this paper we have studied the problem of employing compression codes to solve the phase retrieval problem. Given a class of structured signals and a corresponding compression code, we have proposed COPER, which provably recovers structured signals in that class form their phaseless measurements using the compression code. Our results have shown that, in noiseless phase retrieval, asymptotically, the required sampling rate for almost zero-distortion recovery, modulo the phase, is the same as noiseless compressed sensing.

**APPENDIX A**

**Preliminaries**

**Lemma 1.**

(a) \(\inf_{\theta \in [0, 2\pi]} \| e^{i\theta} x - y \|\) achieves its minimum at a value of \(\theta\) that makes \(e^{-i\theta} x^* y\) a positive real number, and for that \(\theta\) we have
\[
\left\| e^{i\theta} x - y \right\|^2 = \|x\|^2 + \|y\|^2 - 2 |x^* y|.
\]

(b) For any two vectors \(x\) and \(\hat{x}\) in \(\mathbb{C}^n\), we have
\[
\frac{1}{8} \left( \inf_{\theta} \left\| e^{i\theta} x - \hat{x} \right\|^2 \right)^2 \leq \frac{1}{2} \left( \|x\|^2 - \|\hat{x}\|^2 \right)^2 + \left( \|x\|^2 \|\hat{x}\|^2 - |x^* \hat{x}|^2 \right) .
\]
Proof. Let $z = e^{i\theta}x$

$$
\|z - y\|^2 = (z - y)^* (z - y)
= \|z\|^2 + \|y\|^2 - 2 R(z^*y)
\geq \|z\|^2 + \|y\|^2 - 2|z^*y|
= \|x\|^2 + \|y\|^2 - 2|x^*y|
$$

To prove (b), note that by part (a) we have

$$
\left( (\|z\| - \|\hat{x}\|)^2 + 2 (\|z\|\|\hat{x}\| - |x^*\hat{x}|) \right)^2
\leq (1 + 1) \left( (\|z\| - \|\hat{x}\|)^4 + 4 (\|z\|\|\hat{x}\| - |x^*\hat{x}|)^2 \right)
\leq 2 (\|z\|^2 - \|\hat{x}\|^2)^2 + 8 (\|z\|^2\|\hat{x}\|^2 - |x^*\hat{x}|^2),
$$

and so

$$
\frac{1}{8} \left( \inf_{\theta} |e^{i\theta}x - \hat{x}|^2 \right)^2
\leq \frac{1}{4} (\|x\|^2 - \|\hat{x}\|^2)^2 + (\|x\|^2\|\hat{x}\|^2 - |x^*\hat{x}|^2)
\leq \frac{1}{2} (\|x\|^2 - \|\hat{x}\|^2)^2 + (\|x\|^2\|\hat{x}\|^2 - |x^*\hat{x}|^2).
$$

Lemma 2. For any $u > 0$,

$$
g(u) = e^{-\frac{u}{2}} \Phi \left( -\frac{\sqrt{u}}{u} \right) \leq 1.
$$

Proof. With a change of variable it is equivalent to show $h(v) = e^{-\frac{v^2}{2}} - \Phi(-v) \geq 0$ for all $v \geq 0$. To see this note that

$$
h'(v) = \left(-v + \frac{1}{\sqrt{2\pi}}\right) e^{-\frac{v^2}{2}} \Rightarrow \begin{cases} h'(v) \geq 0 & v \leq \frac{1}{\sqrt{2\pi}} \\ h'(v) < 0 & v > \frac{1}{\sqrt{2\pi}}. \end{cases}
$$

In addition, $h(0) = \frac{1}{2} > 0$ and $h(\infty) = 0$.

Lemma 3 (Chi squared concentration). For any $\tau \geq 0$ we have

$$
P \left( \chi^2(m) > m(1 + \tau) \right) \leq e^{-\frac{m}{2}(1-\ln(1+\tau))} \quad \tau > 0.
$$

Appendix B

Properties of $d(\cdot, \cdot)$

Recall from (3) that

$$
d(\|Ax\|,|Ac|) = \sum_{k=1}^{m} \left( A_k^T (xx^* - cc^*) A_k \right)^2.
$$

(16)

First, for fixed $x$ and $c$, we derive the distribution and the moment-generating function (mgf) of $d(\|Ax\|,|Ac|)$. Note that $xx^* - cc^*$ is a Hermitian matrix of rank at most two, and therefore it can be written as

$$
xx^* - cc^* = Q^T \begin{pmatrix} \lambda_1(c) \\ \lambda_2(c) \\ 0 \end{pmatrix} Q,
$$

(17)
\[\lambda_1(c) + \lambda_2(c) = \text{Tr}(xx^* - cc^*) = \|x\|^2 - \|c\|^2,\]  
(18)

and
\[\lambda_1(c)^2 + \lambda_2(c)^2 = \text{Tr}(xx^* - cc^*)^2\]
\[= \text{Tr}(xx^*xx^*) + \text{Tr}(cc^*cc^*)\text{Tr}(cc^*xx^*) - \text{Tr}(xx^*cc^*)\]
\[= \|x\|^4 + \|c\|^4 - 2|x^*c|^2\]
\[= \left(\|x\|^2 - \|c\|^2\right)^2 + 2 \left(\|x\|^2\|c\|^2 - |x^*c|^2\right).\]  
(19)

Also, we have
\[2\lambda_1(c)\lambda_2(c) = (\lambda_1(c) + \lambda_2(c))^2 - (\lambda_1(c)^2 + \lambda_2(c)^2)\]
\[= 2 \left(\|x^*c|^2 - \|x\|^2\|c\|^2\right)\]
\[\leq 0.\]

Hence, without loss of generality, we can assume that
\[\lambda_1(c) \geq 0, \text{ and } \lambda_2(c) < 0.\]

On the other hand, combining (16) and (17), one has
\[
\sum_{k=1}^{m} \left( A^*_k (xx^* - cc^*) \tilde{A}_k \right)^2 = \sum_{k=1}^{m} \left( A^*_k Q^T \begin{pmatrix} \lambda_1(c) \\ \lambda_2(c) \end{pmatrix} 0 \right) \tilde{Q} \tilde{A}_k \right)^2
\]
\[= \sum_{k=1}^{m} \begin{pmatrix} B_k^* \begin{pmatrix} \lambda_1(c) \\ \lambda_2(c) \end{pmatrix} \end{pmatrix} B_k \right)^2
\]
\[= \sum_{k=1}^{m} \left( \lambda_1(c)|B_{k,1}|^2 + \lambda_2(c)|B_{k,2}|^2 \right)^2,
\]
where \(B_k = \tilde{Q} \tilde{A}_k\).

Since \(\tilde{Q}\) is an orthonormal matrix, \(B = \tilde{Q} \tilde{A}\) has the same distribution as \(A\), and therefore the \(\chi^2\) variables in the above sum are all independent. Let \(Z_k = \left( \lambda_1(c)|B_{k,1}|^2 + \lambda_2(c)|B_{k,2}|^2 \right)^2\). Then we have
\[d(Ax|,|Ac|) = \sum_{i=1}^{m} Z_i,\]  
(20)

where \(Z_1, \ldots, Z_m\) are i.i.d. as \((\lambda_1(c)U + \lambda_2(c)V)^2\), where \(U\) and \(V\) are independent \(\chi^2(2)\) random variables.

To proceed, we derive the mgf of \(Z_1\) defined as \(\left( \lambda_1(c)|B_{1,1}|^2 + \lambda_2(c)|B_{1,2}|^2 \right)^2\). Recall that \(\lambda_1(e) > 0, \lambda_2(e) < 0\). Consider \(\alpha > 0\), then
\[f(\alpha) = E\left[e^{-\alpha Z_1}\right] = \int_{x,y \geq 0} e^{-\alpha(x + y)} e^{-\frac{x^2}{2} - \frac{y^2}{2}} dxdy.\]  
(21)

Consider changing the variable \((x, y)\) in the above integral to \((u, v)\) by defining
\[(u, v) = \left( \lambda_1(c)x + \lambda_2(c)y, \frac{x + y}{2} \right),\]
where
\[
\frac{\partial u, v}{\partial x, y} = \begin{bmatrix} \lambda_1(c) & \lambda_2(c) \\ \frac{1}{\pi} & \frac{1}{\pi} \end{bmatrix} = \frac{\lambda_1(c) - \lambda_2(c)}{2}.
\]

But
\[
v = \frac{u}{2\lambda_2(c)} = \left(\frac{1}{2} + \frac{\lambda_1(c)}{-2\lambda_2(c)}\right)x.
\]

Since \(\frac{\lambda_1(c)}{-2\lambda_2(c)} > 0\) we have
\[
x \geq 0 \iff v \geq \frac{u}{2\lambda_3(c)}.
\]

Similarly,
\[
v \geq \frac{u}{2\lambda_1(c)}.
\]

Therefore,
\[
f(\alpha) = \frac{2}{4(\lambda_1(c) - \lambda_2(c))} \int_{u \geq 2\lambda_2(c)} e^{-\alpha u^2} e^{-v} dvdu
\]
\[
= \frac{1}{2(\lambda_1(c) - \lambda_2(c))} \int_{u \geq 2\lambda_2(c)} e^{-\alpha u^2} e^{-\frac{v}{2\lambda_2(c)}} dvdu + \frac{1}{2(\lambda_1(c) - \lambda_2(c))} \int_{u < 0} e^{-\alpha u^2} e^{-\frac{v}{2\lambda_2(c)}} dvdu
\]
\[
= \frac{1}{2(\lambda_1(c) - \lambda_2(c))} \int_{u = 0} e^{-\alpha u^2 - \frac{v}{2\lambda_2(c)}} du + \frac{1}{2(\lambda_1(c) - \lambda_2(c))} \int_{u = -\infty} e^{-\alpha u^2 - \frac{v}{2\lambda_2(c)}} du
\]
\[
= \frac{e^{\frac{v}{2\lambda_2(c)}}}{2(\lambda_1(c) - \lambda_2(c))} \int_{u = 0} e^{-\alpha (u + \frac{v}{2\lambda_2(c)})} du + \frac{e^{\lambda_2(c)\alpha^2 - \frac{v}{4\lambda_2(c)}}}{2(\lambda_1(c) - \lambda_2(c))} \int_{u = -\infty} e^{-\alpha (u + \frac{v}{2\lambda_2(c)})^2} du
\]
\[
= \frac{\sqrt{\pi}}{2(\lambda_1(c) + |\lambda_2(c)|)\sqrt{\alpha}} e^{\frac{v}{4\lambda_1(c)\sqrt{\alpha}}} \Phi\left(\frac{-\sqrt{2}}{4|\lambda_1(c)|\sqrt{\alpha}}\right) + \frac{\sqrt{\pi}}{2(\lambda_1(c) + |\lambda_2(c)|)\sqrt{\alpha}} e^{\frac{\lambda_2(c)\alpha^2}{4\lambda_2(c)}} \Phi\left(\frac{-\sqrt{2}}{4|\lambda_2(c)|\sqrt{\alpha}}\right),
\]
(22)

where \(\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{1}{2}u^2} du\).

Define \(\lambda_{\text{min}}(c), \lambda_{\text{max}}(c)\) to denote \(\lambda_1(c), \lambda_2(c)\) with smaller and larger absolute value respectively, i.e.,
\[
|\lambda_{\text{min}}(c)| = \min\{ |\lambda_1(c)|, |\lambda_2(c)| \}, \quad |\lambda_{\text{max}}(c)| = \max\{ |\lambda_1(c)|, |\lambda_2(c)| \}.
\]

**Lemma 4.** For any \(\alpha > 0\), we have
\[
f(\alpha) \leq \left(\frac{\pi}{\lambda_{\text{max}}(c)^2\alpha}\right)^\frac{1}{2}.
\]

**Proof.** It is straightforward with (22) and **Lemma 3** by pointing out that
\[
e^{\lambda_{\text{max}}(c)^2\alpha} \Phi\left(\frac{-\sqrt{2}}{4|\lambda_1(c)|\sqrt{\alpha}}\right) = g\left(4|\lambda_1(c)|\sqrt{\alpha}\right) \leq 1,
\]
and
\[
\frac{1}{|\lambda_1(c)| + |\lambda_2(c)|} \leq \frac{1}{\lambda_{\text{max}}(c)^2}.
\]

**Theorem 2** (Concentration of \(d(\ldots)\)). Suppose \(C_r\) is set of code-words at rate \(r\), and \(x\) denotes the signal of interest.

For a given \(c \in \mathbb{C}^n\), let \(\lambda_{\text{min}}^2(\cdot) \leq \lambda_{\text{max}}^2(\cdot)\) be squared of the two non-zero eigenvalues of \(xx^* - cc^*\). For any positive real numbers \(\tau_1, \tau_2\),
\[
P\left(d(|Ax|, |Ac|) > \lambda_{\text{max}}^2(c)\tau_1 \forall c \in C_r\right) \geq 1 - 2^r e^{\frac{\tau_1}{(K + \ln \tau_1 - \ln m)}},
\]
(23)
where $K = \ln 2\pi e$ and

$$
P \left( d[|Ax|,|Ac|] < \lambda_{\max}^2(e) \left( 4m(1 + \tau_2) \right)^2 \right) \geq 1 - e^{-2m(\tau_2 - \ln(1 + \tau_2))}. \tag{24}
$$

**Proof.** To derive (23), note that by lemma 4, we have

$$
P \left( d[|Ax|,|Ac|] \leq t \right) = \mathbb{P} \left( e^{-\alpha \sum_{i=1}^{m} Z_i} \geq e^{-\alpha t} \right)
$$

$$
\leq e^{\alpha t} \mathbb{E} \left[ e^{-\alpha Z_i} \right]^m
\leq e^{\alpha t} f(\alpha)^m
\leq e^{\alpha t} \left( \frac{\pi}{\lambda_{\max}(e)^{2\alpha}} \right)^{\frac{m}{2}},
$$

where $\alpha > 0$ is a free parameter. Let $\alpha = \frac{m}{2\lambda_{\max}(e)\tau_1}$ and $t = \lambda_{\max}^2(e)\tau_1$, therefore

$$
P \left( d[|Ax|,|Ac|] \leq \lambda_{\max}^2(e)\tau_1 \right) \leq e^{\frac{m}{2} \left( 2\pi \tau_1 \right)^{\frac{m}{2}}}
\leq e^{\frac{m}{2}(K + \ln \tau_1 - \ln m)},
$$

so we get

$$
P \left( d[|Ax|,|Ac|] > \lambda_{\max}^2(e)\tau_1 \right) \geq 1 - e^{\frac{m}{2}(K + \ln \tau_1 - \ln m)},
$$

and with an union bound on $C_r$ we get

$$
P \left( d[|Ax|,|Ac|] > \lambda_{\max}^2(e)\tau_1 \quad \forall c \in C_r \right) \geq 1 - 2^{r} e^{\frac{m}{2}(K + \ln \tau_1 - \ln m)},
$$

where $K = \ln 2\pi e$.

To prove (24), note that for $Z_i$ defined in (20), one has $Z_i \leq \left( \lambda_{\max}(e) \chi^2(4) \right)^2$, thus

$$
\sum_{i=1}^{m} Z_i \leq \lambda_{\max}^2 \sum_{i=1}^{m} \chi^2(4)
\leq \lambda_{\max}^2 \left( \sum_{i=1}^{m} \chi^2(4) \right)^2
\sim \lambda_{\max}^2 \left( \chi^2(4m) \right)^2,
$$

therefore by lemma 3 we have

$$
P \left( d[|Ax|,|Ac|] \geq \lambda_{\max}^2(e) \left( 4m(1 + \tau_2) \right)^2 \right) = P \left( \sum_{i=1}^{m} Z_i \geq \lambda_{\max}^2 \left( 4m(1 + \tau_2) \right)^2 \right)
\leq P \left( \chi^2(4m) \geq 4m(1 + \tau_2) \right)
\leq e^{-2m(\tau_2 - \ln(1 + \tau_2))},
$$

and so

$$
P \left( d[|Ax|,|Ac|] < \lambda_{\max}^2(e) \left( 4m(1 + \tau_2) \right)^2 \right) \geq 1 - e^{-2m(\tau_2 - \ln(1 + \tau_2))},
$$

for any $\tau_2 > 0$. \qed
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