On the usage of lines in $GC_n$ sets

Hakop Hakopian$^1$ · Vahagn Vardanyan$^2$

Received: 3 September 2018 / Accepted: 10 May 2019 / Published online: 3 June 2019 © Springer Science+Business Media, LLC, part of Springer Nature 2019

Abstract
A planar node set $\mathcal{X}$, with $|\mathcal{X}| = \binom{n+2}{2}$, is called $GC_n$ set if each node possesses fundamental polynomial in form of a product of $n$ linear factors. We say that a node uses a line if the line is a component of the fundamental polynomial of this node. A line is called $k$-node line if it passes through exactly $k$ nodes of $\mathcal{X}$. At most, $n+1$ nodes can be collinear in any $GC_n$ set and an $(n+1)$-node line is called a maximal line. The Gasca-Maeztu conjecture (1982) states that every $GC_n$ set has a maximal line. Until now, the conjecture has been proved only for the cases $n \leq 5$. Here, we provide a correct statement and prove a conjecture proposed by V. Bayramyan and H. H. in a recent paper. Namely, by assuming that the Gasca-Maeztu conjecture is true, we prove that for any $GC_n$ set $\mathcal{X}$ and any $k$-node line $\ell$ the following statement holds: Either the line $\ell$ is not used at all, or it is used by exactly $\binom{s}{2}$ nodes of $\mathcal{X}$, where $k - \delta \leq s \leq k$, $\delta = n + 1 - k$. If in addition $k - \delta \geq 3$ and the number of maximal lines of the set $\mathcal{X}$ is greater than 3, then the first case here is excluded, i.e., $\ell$ necessarily is a used line. Finally, we provide a characterization for the usage of a $k$-node line in a $GC_n$ set $\mathcal{X}$ concerning the case $k - \delta = 2$.

Keywords Polynomial interpolation · Gasca-Maeztu conjecture · $n$-poised set · $GC_n$ set · Maximal line

Mathematics Subject Classification (2010) 41A05 · 41A63

1 Introduction

An $n$-poised set $\mathcal{X}$ in the plane is a node set for which the interpolation problem with bivariate polynomials of total degree at most $n$ is unisolvent. Node sets with
geometric characterization: $GC_n$ sets, introduced by Chung and Yao [11], form an important subclass of $n$-poised sets. In a $GC_n$ set, the fundamental polynomial of each node is a product of $n$ linear factors. We say that a node uses a line if the line is a factor of the fundamental polynomial of this node. A line is called $k$-node line if it passes through exactly $k$-nodes of $\mathcal{X}$. It is a simple fact that at most $n + 1$ nodes can be collinear in $GC_n$ sets. An $(n + 1)$-node line is called a maximal line. The conjecture of M. Gasca and J. I. Maeztu [12] states that every $GC_n$ set has a maximal line. Until now the conjecture has been proved only for the cases $n \leq 5$ (see [2] and [14]). For a maximal line $\lambda$ in a $GC_n$ set $\mathcal{X}$, the following statement is evident: the line $\lambda$ is used by all $\binom{n+1}{2}$ nodes in $\mathcal{X} \setminus \lambda$.

Recently, it was proved in [1] and [17] that an $n$-node line in a $GC_n$ set $\mathcal{X}$, where $n \geq 4$, is used either by exactly $\binom{n}{2}$ or by $\binom{n-1}{2}$ nodes. Also, a conjecture was proposed in the paper [1] by V. Bayramyan and H. H., concerning the usage of any $k$-node line in $GC_n$ sets, $2 \leq k \leq n + 1$. In this paper, we make a correction in the mentioned conjecture and then prove it. Namely, by assuming that the Gasca-Maeztu conjecture is true, we prove that for any $GC_n$ set $\mathcal{X}$ and any $k$-node line $\ell$ the following statement holds: The line $\ell$ is not used at all, or it is used by exactly $\binom{s}{2}$ nodes of $\mathcal{X}$, where $s$ satisfies the condition $k - \delta \leq s \leq k$, $\delta = n + 1 - k$. If in addition $k - \delta \geq 3$ and $\mu(\mathcal{X}) > 3$, then the first case here is excluded, i.e., the line $\ell$ is necessarily a used line. Here, $\mu(\mathcal{X})$ denotes the number of maximal lines of $\mathcal{X}$. We prove also that the subset of nodes of $\mathcal{X}$ that use the line $\ell$ forms a $GC_{s-2}$ set if it is not an empty set. Moreover, we prove that actually it is an $\ell$-proper subset of $\mathcal{X}$, meaning that it can be obtained from $\mathcal{X}$ by removing the nodes in subsequent maximal lines, which do not intersect the line $\ell$ at a node of $\mathcal{X}$ or the nodes in pairs of maximal lines intersecting $\ell$ at the same node of $\mathcal{X}$. At the last step, when the line $\ell$ becomes maximal, the nodes in $\ell$ are removed (see the forthcoming Definition 2.10).

Finally, we provide a characterization for the usage of $k$-node lines in $GC_n$ sets when $k - \delta = 2$ and $\mu(\mathcal{X}) > 3$.

Let us mention that earlier Carnicer and Gasca proved that a $k$-node line $\ell$ can be used by at most $\binom{k}{2}$ nodes of a $GC_n$ set $\mathcal{X}$ and in addition there are no $k$ collinear nodes that use $\ell$, provided that GM conjecture is true (see [5], Theorem 4.5).

### 1.1 Poison sets

Denote by $\Pi_n$ the space of bivariate polynomials of total degree at most $n$: 

$$\Pi_n = \left\{ \sum_{i+j \leq n} c_{ij} x^i y^j \right\}.$$ 

We have that 

$$N := \dim \Pi_n = \binom{n+2}{2}. \quad (1.1)$$

Let $\mathcal{X}$ be a set of $s$ distinct nodes (points): 

$$\mathcal{X} = \{(x_1, y_1), (x_2, y_2), \ldots, (x_s, y_s)\}. \quad (1.2)$$
The Lagrange bivariate interpolation problem is: for given set of values \( C := \{c_1, c_2, \ldots, c_s\} \) find a polynomial \( p \in \Pi_n \) satisfying the conditions
\[
p(x_i, y_i) = c_i, \quad i = 1, 2, \ldots, s.
\] (1.3)

**Definition 1.1** A set of nodes \( \mathcal{X} \), \( |\mathcal{X}| = s \), is called \( n \)-poised if for any set of values \( C \) there exists a unique polynomial \( p \in \Pi_n \) satisfying the conditions (1.3).

It is an elementary Linear Algebra fact that if a node set \( \mathcal{X} \) is \( n \)-poised then \( |\mathcal{X}| = N \). Thus, from now on, we will consider sets \( \mathcal{X} \), with \( |\mathcal{X}| = s = N \) in (1.2), when \( n \)-poisedness is studied. If a set \( \mathcal{X} \) is \( n \)-poised then we say that \( n \) is the degree of the set \( \mathcal{X} \).

**Proposition 1.2** The set of nodes \( \mathcal{X} \), with \( |\mathcal{X}| = N \), is \( n \)-poised if and only if the following implication holds:
\[
p \in \Pi_n, \ p|_{\mathcal{X}} = 0 \implies p = 0,
\]
where \( p|_{\mathcal{X}} \) means the restriction of \( p \) to \( \mathcal{X} \).

A polynomial \( p \in \Pi_n \) is called an \( n \)-fundamental polynomial of a node \( A \in \mathcal{X} \), if
\[
p(A) \neq 0 \quad \text{and} \quad p|_{\mathcal{X}\{A\}} = 0.
\]
We shall denote such a polynomial by \( p^*_A,\mathcal{X} \).

**Definition 1.3** Given an \( n \)-poised set \( \mathcal{X} \). We say that a node \( A \in \mathcal{X} \) uses a line \( \ell \in \Pi_1 \), if
\[
p^*_A,\mathcal{X} = \ell q, \text{ where } q \in \Pi_{n-1}.
\]

The following proposition is well-known (see, e.g., [13] Proposition 1.3):

**Proposition 1.4** Suppose that a polynomial \( p \in \Pi_n \) vanishes at \( n + 1 \) points of a line \( \ell \). Then we have that
\[
p = \ell r, \text{ where } r \in \Pi_{n-1}.
\]

Thus, at most, \( n + 1 \) nodes of an \( n \)-poised set \( \mathcal{X} \) can be collinear. A line \( \lambda \) passing through \( n + 1 \) nodes of the set \( \mathcal{X} \) is called a maximal line. Clearly, in view of Proposition 1.4, any maximal line \( \lambda \) is used by all the nodes in \( \mathcal{X}\setminus\lambda \).

Below we bring other properties of maximal lines:

**Corollary 1.5** ([3], Prop. 2.1) Let \( \mathcal{X} \) be an \( n \)-poised set. Then we have that

(i) Any two maximal lines of \( \mathcal{X} \) intersect necessarily at a node of \( \mathcal{X} \);
(ii) Any three maximal lines of \( \mathcal{X} \) cannot be concurrent;
(iii) \( \mathcal{X} \) can have at most \( n + 2 \) maximal lines.
2 GC\(_n\) sets and the Gasca-Maeztu conjecture

Now, let us consider a special type of \(n\)-poised sets satisfying a geometric characterization (GC) property introduced by K.C. Chung and T.H. Yao:

**Definition 2.1** [11] An \(n\)-poised set \(\mathcal{X}\) is called GC\(_n\) set (or GC set) if the \(n\)-fundamental polynomial of each node \(A \in \mathcal{X}\) is a product of \(n\) linear factors.

So, GC\(_n\) sets are \(n\)-poised sets such that each of its nodes uses exactly \(n\) lines.

**Corollary 2.2** ([5], Prop. 2.3) Let \(\lambda\) be a maximal line of a GC\(_n\) set \(\mathcal{X}\). Then, the set \(\mathcal{X} \setminus \lambda\) is a GC\(_{n-1}\) set. Moreover, for any node \(A \in \mathcal{X} \setminus \lambda\), we have that

\[
p_A^* = \lambda p_A^*_{\mathcal{X} \setminus \lambda}. \tag{2.1}
\]

Next, we present the Gasca-Maeztu conjecture, briefly called GM conjecture:

**Conjecture 2.3** ([12], Sect. 5) Any GC\(_n\) set possesses a maximal line.

Till now, this conjecture has been confirmed for the degrees \(n \leq 5\) (see [2, 14]). For a generalization of the Gasca-Maeztu conjecture to maximal curves, see [15].

Let us mention the following important result:

**Theorem 2.4** ([5], Thm. 4.1) If the Gasca-Maeztu conjecture is true for all \(k \leq n\), then any GC\(_n\) set possesses at least three maximal lines.

This yields, in view of Corollary 1.5 (ii) and Proposition 1.4, that each node of a GC\(_n\) set \(\mathcal{X}\) uses at least one maximal line.

Denote by \(\mu := \mu(\mathcal{X})\) the number of maximal lines of the node set \(\mathcal{X}\).

**Proposition 2.5** ([5], Crl. 3.5) Let \(\lambda\) be a maximal line of a GC\(_n\) set \(\mathcal{X}\) such that \(\mu(\mathcal{X} \setminus \lambda) \geq 3\). Then we have that

\[
\mu(\mathcal{X} \setminus \lambda) = \mu(\mathcal{X}) \quad \text{or} \quad \mu(\mathcal{X}) - 1.
\]

**Definition 2.6** [4] Given an \(n\)-poised set \(\mathcal{X}\) and a line \(\ell\). Then, \(\mathcal{X}_\ell\) is the subset of nodes of \(\mathcal{X}\) which use the line \(\ell\).

Note that a statement on maximal lines we have already mentioned can be expressed as follows

\[
\mathcal{X}_\ell = \mathcal{X} \setminus \ell, \quad \text{if } \ell \text{ is a maximal line}. \tag{2.2}
\]

Suppose that \(\lambda\) is a maximal line of \(\mathcal{X}\) and \(\ell \neq \lambda\) is any line. Then, in view of the relation (2.1), we have that

\[
\mathcal{X}_\ell \setminus \lambda = (\mathcal{X} \setminus \lambda)_\ell. \tag{2.3}
\]

In the sequel, we will use frequently the following two lemmas of Carnicer and Gasca.
Let \( \mathcal{X} \) be an \( n \)-poised set and \( \ell \) be a line with \( |\ell \cap \mathcal{X}| \leq n \). A maximal line \( \lambda \) is called \( \ell \)-disjoint if
\[
\lambda \cap \ell \cap \mathcal{X} = \emptyset.
\] (2.4)

**Lemma 2.7** ([5], Lemma 4.4) Let \( \mathcal{X} \) be an \( n \)-poised set and \( \ell \) be a line with \( |\ell \cap \mathcal{X}| \leq n \). Suppose also that a maximal line \( \lambda \) is \( \ell \)-disjoint. Then, we have that
\[
\mathcal{X}_\ell = (\mathcal{X} \setminus \lambda)_\ell.
\] (2.5)

Moreover, if \( \ell \) is an \( n \)-node line then we have that \( \mathcal{X}_\ell = \mathcal{X} \setminus (\lambda \cup \ell) \), hence \( \mathcal{X}_\ell \) is an \((n-2)\)-poised set.

Let \( \mathcal{X} \) be an \( n \)-poised set and \( \ell \) be a line with \( |\ell \cap \mathcal{X}| \leq n \). Two maximal lines \( \lambda', \lambda'' \) are called \( \ell \)-adjacent if
\[
\lambda' \cap \lambda'' \cap \ell \in \mathcal{X}.
\] (2.6)

**Lemma 2.8** ([5], proof of Thm. 4.5) Let \( \mathcal{X} \) be an \( n \)-poised set and \( \ell \) be a line with \( 3 \leq |\ell \cap \mathcal{X}| \leq n \). Suppose also that two maximal lines \( \lambda', \lambda'' \) are \( \ell \)-adjacent. Then, we have that
\[
\mathcal{X}_\ell = (\mathcal{X} \setminus (\lambda' \cup \lambda''))_\ell.
\] (2.7)

Moreover, if \( \ell \) is an \( n \)-node line then we have that \( \mathcal{X}_\ell = \mathcal{X} \setminus (\lambda' \cup \lambda'' \cup \ell) \), hence \( \mathcal{X}_\ell \) is an \((n-3)\)-poised set.

Next, by the motivation of above two lemmas, let us introduce the concept of an \( \ell \)-reduction of a \( GC_n \) set.

**Definition 2.9** Let \( \mathcal{X} \) be a \( GC_n \) set, \( \ell \) be a \( k \)-node line, \( k \geq 2 \). We say that a set \( \mathcal{Y} \subset \mathcal{X} \) is an \( \ell \)-reduction of \( \mathcal{X} \), and briefly denote this by \( \mathcal{X} \setminus_\ell \mathcal{Y} \), if
\[
\mathcal{Y} = \mathcal{X} \setminus (C_0 \cup C_1 \cup \cdots \cup C_s),
\]
where
\begin{enumerate}
\item \( C_0 \) is an \( \ell \)-disjoint maximal line of \( \mathcal{X} \), or \( C_0 \) is the union of a pair of \( \ell \)-adjacent maximal lines of \( \mathcal{X} \);
\item \( C_i \) is an \( \ell \)-disjoint maximal line of the \( GC \) set \( \mathcal{Y}_i := \mathcal{X} \setminus (C_0 \cup C_1 \cup \cdots \cup C_{i-1}) \), or \( C_i \) is the union of a pair of \( \ell \)-adjacent maximal lines of \( \mathcal{Y}_i \), \( i = 1, \ldots, s \);
\item \( \ell \) passes through at least 2 nodes of \( \mathcal{Y} \).
\end{enumerate}

We get immediately from Lemmas 2.7 and 2.8 that
\[
\mathcal{X} \setminus_\ell \mathcal{Y} \Rightarrow \mathcal{X}_\ell = \mathcal{Y}_\ell.
\] (2.8)

Notice that we cannot do any further \( \ell \)-reduction with the set \( \mathcal{Y} \) if the line \( \ell \) is a maximal line here. For this situation, we have the following:

**Definition 2.10** Let \( \mathcal{X} \) be a \( GC_n \) set, \( \ell \) be a \( k \)-node line, \( k \geq 2 \). We say that the set \( \mathcal{X}_\ell \) is an \( \ell \)-proper \( GC_m \) subset of \( \mathcal{X} \) if there is a \( GC_{m+1} \) set \( \mathcal{Y} \) such that
(i) $\mathcal{X} \backslash \ell \subseteq \mathcal{Y}$;
(ii) The line $\ell$ is a maximal line in $\mathcal{Y}$.

In view of the relations (2.8) and (2.2), we have that
$$\mathcal{X}_\ell = \mathcal{Y} \setminus \ell = \mathcal{X} \setminus (C_0 \cup C_1 \cup \cdots \cup C_s \cup \ell),$$
where the sets $C_i$ satisfy conditions listed in Definition 2.9.

Let us mention that, in view of Corollary 2.2, the above $\ell$-proper $GC_m$ subset $\mathcal{X}_\ell$ is indeed a $GC_m$ set, with $m = n - \sum_{i=0}^s \delta_i - 1$, where $\delta_i \in \{1, 2\}$ is the number of the maximal lines contained in $C_i$.

Note that if $\ell$ is an $n$-node line then the node set $\mathcal{X}_\ell$ in Lemma 2.7 or in Lemma 2.8 is an $\ell$-proper $GC_{n-2}$ or $GC_{n-3}$ subset of $\mathcal{X}$, respectively.

We immediately get from Definitions 2.9 and 2.10 the following:

**Proposition 2.11** Suppose that $\mathcal{X}$ is a $GC_n$ set. If $\mathcal{X} \backslash \ell \subseteq \mathcal{Y}$ and $\mathcal{Y}_\ell$ is an $\ell$-proper $GC_m$ subset of $\mathcal{Y}$ then $\mathcal{X}_\ell$ is an $\ell$-proper $GC_m$ subset of $\mathcal{X}$.

### 2.1 Classification of $GC_n$ sets

Here, we will consider the results of Carnicer, Gasca, and Godés, concerning the classification of $GC_n$ sets according to the number of maximal lines the sets possess. Let us start with

**Theorem 2.12** [9] Let $\mathcal{X}$ be a $GC_n$ set with $\mu(\mathcal{X})$ maximal lines. Suppose also that $GM$ conjecture is true for the degrees not exceeding $n$. Then, $\mu(\mathcal{X}) \in \{3, n - 1, n, n + 1, n + 2\}$.

1. **Lattices with $n + 2$ maximal lines - the Chung-Yao natural lattices.**

   Let a set $\mathcal{M}$ of $n + 2$ lines be in general position, i.e., no two lines are parallel and no three lines are concurrent, $n \geq 0$. Then, the Chung-Yao set is defined as the set $\mathcal{X}$ of all $\binom{n+2}{2}$ intersection points of these lines. Note that the black nodes in Fig. 1 form a Chung-Yao lattice for $n = 3$. We have that the $n + 2$ lines of $\mathcal{M}$ are maximal for $\mathcal{X}$. Each fixed node here is lying in exactly 2 lines and does not belong to the remaining $n$ lines. Observe that the product of the latter $n$ lines gives the fundamental polynomial of the fixed node. Thus, $\mathcal{X}$ is a $GC_n$ set. Let us mention that any $n$-poised set $\mathcal{X}$, with $\mu(\mathcal{X}) = n + 2$, clearly forms a Chung-Yao lattice. Recall that there are no $n$-poised sets with more maximal lines (Corollary 1.5, (iii)).

2. **Lattices with $n + 1$ maximal lines - the Carnicer-Gasca lattices.**

   Let a set $\mathcal{M}$ of $n + 1$ lines be in general position, $n \geq 2$. Then, the Carnicer-Gasca lattice $\mathcal{X}$ is defined as $\mathcal{X} := \mathcal{X}^{(2)} \cup \mathcal{X}^{(1)}$, where $\mathcal{X}^{(2)}$ is the set of all intersection nodes of these $n + 1$ lines, and $\mathcal{X}^{(1)}$ is a set of other $n + 1$ non-collinear nodes, one in each line, to make the line maximal. Note that the black nodes and the white nodes together in Fig. 1 form a Carnicer-Gasca lattice for $n = 4$. We have that $|\mathcal{X}| = \binom{n+1}{2} + (n + 1) = \binom{n+2}{2}$. It is easily seen that $\mathcal{X}$ is a $GC_n$ set and has exactly $n + 1$ maximal lines, i.e., the lines of $\mathcal{M}$. Let us mention that any $n$-poised set $\mathcal{X}$, with $\mu(\mathcal{X}) = n + 1$, clearly forms a Carnicer-Gasca lattice (see [3], Proposition 2.4).
3. Lattices with $n$ maximal lines.

Let a set $\mathcal{M}$ of $n$ lines be in general position, $n \geq 3$. Then, consider the lattice $\mathcal{X}$ defined as

$$\mathcal{X} := \mathcal{X}^{(2)} \cup \mathcal{X}^{(1)} \cup \mathcal{X}^{(0)},$$

where $\mathcal{X}^{(2)}$ is the set of all intersection nodes of these $n$ lines, $\mathcal{X}^{(1)}$ is a set of other $2n$ nodes, two in each line, to make the line maximal and $\mathcal{X}^{(0)}$ consists of a single node, denoted by $O$, which does not belong to any line from $\mathcal{M}$. Correspondingly, we have that $|\mathcal{X}| = \left(\begin{array}{c} n \\ 2 \end{array}\right) + 2n + 1 = \left(\begin{array}{c} n+2 \\ 2 \end{array}\right)$.

In the sequel, we will need the following characterization of $GC_n$ set $\mathcal{X}$, with $\mu(\mathcal{X}) = n$, due to Carnicer and Gasca (see Fig. 7):

**Proposition 2.13** ([3], Prop. 2.5) A node set $\mathcal{X}$ is a $GC_n$ set with the set of maximal lines $\mathcal{M}$, $\#\mathcal{M} = n$, if and only if the representation (2.9) holds with the following additional properties:

(i) There are 3 lines $\ell_1^o, \ell_2^o, \ell_3^o$, concurrent at the node $O : O = \ell_1^o \cap \ell_2^o \cap \ell_3^o$, such that $\mathcal{X}^{(1)} \subset \ell_1^o \cup \ell_2^o \cup \ell_3^o$;

(ii) No line $\ell_i^o$, $i = 1, 2, 3$, contains $n + 1$ nodes of $\mathcal{X}$.

4. Lattices with $n - 1$ maximal lines.

Let a set $\mathcal{M} = \{\lambda_1, \ldots, \lambda_{n-1}\}$ of $n - 1$ lines be in general position, $n \geq 4$. Then, consider the lattice $\mathcal{X}$ defined as

$$\mathcal{X} := \mathcal{X}^{(2)} \cup \mathcal{X}^{(1)} \cup \mathcal{X}^{(0)},$$

where $\mathcal{X}^{(2)}$ is the set of all intersection nodes of these $n - 1$ lines, $\mathcal{X}^{(1)}$ is a set of other $3(n - 1)$ nodes, three in each line, to make the line maximal and $\mathcal{X}^{(0)}$ consists of exactly three nodes, denoted by $O_1, O_2, O_3$, which do not belong to any line from $\mathcal{M}$.
Correspondingly, we have that $|X| = \binom{n-1}{2} + 3(n - 1) + 3 = \binom{n+2}{2}$. Note that all the nodes of $X^{(k)}$ belong to exactly $k$ maximal lines and are called $k_m$-nodes, $k = 0, 1, 2$.

Clearly, the three $0_m$-nodes $O_1, O_2, O_3$ are non-collinear. Indeed, otherwise, the set $X$ is lying in $n$ lines, which are the $n - 1$ lines of $M$ and the line passing through the three nodes. This, in view of Proposition 1.2, contradicts the $n$-poisedness of $X$.

Denote by $\ell_i^{oo}$, $1 \leq i \leq 3$, the line passing through the two $0_m$-nodes in $\{O_1, O_2, O_3\} \setminus \{O_i\}$. We call this lines $OO$ lines. Suppose that $X^{(1)} = \{A_1^1, A_2^2, A_3^3 \in \lambda_i : 1 \leq i \leq n - 1\}$.

In the sequel, we will need the following characterization of $GC_n$ set $X$, with $\mu(X) = n - 1$, due to Carnicer and Godés (see Figs. 3 and 6) (see [10], Section 5, Case d=3, for a proof detail):

**Proposition 2.14** ([8], Thm. 3.2) A set $X$ is a $GC_n$ set with exactly $n - 1$ maximal lines $\lambda_1, \ldots, \lambda_{n-1}$, where $n \geq 4$, if and only if, with some permutation of the indexes of the maximal lines and $1_m$-nodes, the representation (2.10) holds with the following additional properties:

(i) $X^{(1)} \setminus \{A_1^1, A_2^2, A_3^3\} \subset \ell_1^{oo} \cup \ell_2^{oo} \cup \ell_3^{oo}$;

(ii) Each line $\ell_i^{oo}, i = 1, 2, 3$, passes through exactly $(n - 2) 1_m$-nodes (and through two $0_m$-nodes). Moreover, $\ell_i^{oo} \cap \lambda_i \notin X$, $i = 1, 2, 3$;

(iii) The triples $\{O_1, A_2^2, A_3^3\}, \{O_2, A_1^1, A_3^3\}, \{O_3, A_1^1, A_2^2\}$ are collinear.

5. Lattices with 3 maximal lines - generalized principal lattices.
A principal lattice is defined as an affine image of the set (see Fig. 2)

\[ \mathcal{P}L_n := \left\{ (i, j) \in \mathbb{N}_0^2 : \ i + j \leq n \right\}. \]

Let us set \( I = \{0, 1, \ldots, n + 1\} \). Observe that the following 3 sets of \( n + 1 \) lines, namely \( \{x = i : i \in I\} \), \( \{y = j : j \in I\} \), and \( \{x + y = k : k \in I\} \), intersect at \( \mathcal{P}L_n \). We have that \( \mathcal{P}L_n \) is a \( \mathcal{G}C_n \) set. Moreover, the following is the fundamental polynomial of the node \((i_0, j_0) \in \mathcal{P}L_n\):

\[
p^{\star}_{i_0,j_0}(x, y) = \prod_{0 \leq i < i_0, 0 \leq j < j_0, 0 \leq k < k_0} (x - i)(y - j)(x + y - n + k), \tag{2.11}
\]

where \( k_0 = n - i_0 - j_0 \).

Next, let us bring the definition of the generalized principal lattice due to Carnicer, Gasca, and Godés (see [6, 7]):

**Definition 2.15** [7] A node set \( \mathcal{X} \) is called a generalized principal lattice, briefly \( \mathcal{G}P\mathcal{L}_n \), if there are 3 sets of lines each containing \( n + 1 \) lines

\[
\ell^j_i(\mathcal{X})_{i \in \{0,1,\ldots,n\}, \ j = 0, 1, 2,} \tag{2.12}
\]

such that the 3n + 3 lines are distinct,

\[
\ell^0_i(\mathcal{X}) \cap \ell^1_j(\mathcal{X}) \cap \ell^2_k(\mathcal{X}) \cap \mathcal{X} \neq \emptyset \iff i + j + k = n
\]

and

\[
\mathcal{X} = \left\{ x_{ijk} \mid x_{ijk} := \ell^0_i(\mathcal{X}) \cap \ell^1_j(\mathcal{X}) \cap \ell^2_k(\mathcal{X}), 0 \leq i, j, k \leq n, i + j + k = n \right\}.
\]

Observe that if \( 0 \leq i, j, k \leq n, \ i + j + k = n \) then the three lines \( \ell^0_i(\mathcal{X}), \ell^1_j(\mathcal{X}), \ell^2_k(\mathcal{X}) \) intersect at a node \( x_{ijk} \in \mathcal{X} \). This implies that each node of \( \mathcal{X} \) belongs to only one line of each of the three sets of \( n + 1 \) lines. Therefore \( |\mathcal{X}| = (n + 1)(n + 2)/2 \).

One can find readily, as in the case of \( \mathcal{P}L_n \), the fundamental polynomial of each node \( x_{ijk} \in \mathcal{X}, \ i + j + k = n \):

\[
p^{\star}_{i_0,j_0,k_0}(x, y) = \prod_{0 \leq i < i_0, 0 \leq j < j_0, 0 \leq k < k_0} \ell^0_i(\mathcal{X})\ell^1_j(\mathcal{X})\ell^2_k(\mathcal{X}). \tag{2.13}
\]

Thus, \( \mathcal{X} \) is a \( \mathcal{G}C_n \) set.

Let us bring a characterization for \( \mathcal{G}P\mathcal{L}_n \) set due to Carnicer and Godés:

**Theorem 2.16** ([7], Thm. 3.6) Assume that GM Conjecture holds for all degrees up to \( n - 3 \). Then, the following statements are equivalent:

(i) \( \mathcal{X} \) is generalized principal lattice of degree \( n \);

(ii) \( \mathcal{X} \) is a \( \mathcal{G}C_n \) set with \( \mu(\mathcal{X}) = 3 \).
3 The main result

Now, we are in a position to formulate and prove the corrected version of the conjecture proposed by V. Bayramyan and H. H. in ([1], Conj. 3.7) as:

**Theorem 3.1** Let $\mathcal{X}$ be a $GC_n$ set, and $\ell$ be a $k$-node line, $k \geq 2$. Assume that GM Conjecture holds for all degrees up to $n$. Then, we have that

$$X_\ell = \emptyset, \text{ or }$$

where $k - \delta \leq s \leq k$ and $\delta = n + 1 - k$.

Moreover, if $k - \delta \geq 3$ and $\mu(\mathcal{X}) > 3$, then $X_\ell \neq \emptyset$, i.e., (3.2) holds (with $s \geq 3$). Furthermore, in the case $X_\ell \neq \emptyset$, we have for any maximal line $\lambda$:

$$|\lambda \cap X_\ell| = 0 \text{ or } |\lambda \cap X_\ell| = s - 1.$$ 

Let us mention that (3.1) was missing in the original conjecture in [1] and the possibility that the set $X_\ell$ may be empty was associated only with the case $k - \delta \leq 1$, i.e., with the possibility of the equality $|X_\ell| = \binom{s}{2} = 0$ in (3.2). Thus, we assume that a $GC_{s-2}$ subset with $s < 2$ is empty set.

Note that we added here the statement that $X_\ell$ is an $\ell$-proper $GC_n$ set.

In the last subsection, we characterize constructions of $GC_n$ sets for which there is a non-used $k$-node line with $k - \delta = 2$ and $\mu(\mathcal{X}) > 3$.

3.1 Some known special cases of Theorem 3.1

The following theorem concerns the special case $k = n$ of Theorem 3.1 (which corresponds to $\delta = 1$). It is a corrected version of the original result in ([1], Theorem 3.3). This result was the first step toward Theorem 3.1. The corrected version appears in ([17], Theorem 3.1).

**Theorem 3.2** Let $\mathcal{X}$ be a $GC_n$ set, $n \geq 1$, $n \neq 3$, and $\ell$ be an $n$-node line. Assume that GM Conjecture holds for all degrees up to $n$. Then, we have that

$$|X_\ell| = \binom{n}{2} \text{ or } \binom{n-1}{2}.$$ 

Moreover, the following hold:

(i) $|X_\ell| = \binom{n}{2}$ if and only if there is an $\ell$-disjoint maximal line $\lambda$, i.e., $\lambda \cap \ell \cap X = \emptyset$. In this case, we have that $X_\ell = \mathcal{X} \setminus (\lambda \cup \ell)$. Hence it is an $\ell$-proper $GC_{n-2}$ set;

(ii) $|X_\ell| = \binom{n-1}{2}$ if and only if there is a pair of $\ell$-adjacent maximal lines $\lambda', \lambda''$, i.e., $\lambda' \cap \lambda'' \cap \ell \in \mathcal{X}$. In this case, we have that $X_\ell = \mathcal{X} \setminus (\lambda' \cup \lambda'' \cup \ell)$. Hence, it is an $\ell$-proper $GC_{n-3}$ set.

Next, let us bring a characterization for the case $n = 3$, which is not covered in above Theorem.
Proposition 3.3 ([17], Prop. 3.3) Let $\mathcal{X}$ be a $GC_3$-set and $\ell$ be a 3-node line. Then we have that
\[ |\mathcal{X}_\ell| = 3, \quad 1, \quad \text{or} \quad 0. \tag{3.4} \]
Moreover, the following hold:

(i) $|\mathcal{X}_\ell| = 3$ if and only if there is a maximal line $\lambda_0$ such that $\lambda_0 \cap \ell \cap \mathcal{X} = \emptyset$. In this case, we have that $\mathcal{X}_\ell = \mathcal{X} \setminus (\lambda_0 \cup \ell)$. Hence it is an $\ell$-proper $GC_1$ set.

(ii) $|\mathcal{X}_\ell| = 1$ if and only if there are two maximal lines $\lambda', \lambda''$, such that $\lambda' \cap \lambda'' \cap \ell \in \mathcal{X}$. In this case, we have that $\mathcal{X}_\ell = \mathcal{X} \setminus (\lambda' \cup \lambda'' \cup \ell)$. Hence it is an $\ell$-proper $GC_0$ set.

(iii) $|\mathcal{X}_\ell| = 0$ if and only if there are exactly three maximal lines in $\mathcal{X}$ and they intersect $\ell$ at three distinct nodes.

Note that (3.4) holds for 3-node lines in any $n$-poised set (see [16], Cor. 6.1).

Let us mention that, in view of the relation (2.2), Theorem 3.1 is true if the line $\ell$ is a maximal line ($\delta = 0$).

Theorem 3.1 is also true in the case when $GC_n$ set $\mathcal{X}$ is a Chung-Yao lattice. Indeed, in this lattice, the only used lines are the maximal lines. Next, for any $k$-node line $\ell$ with $k \leq n$, we have that $2k \leq \mu(\mathcal{X}) = n + 2$, since through any node there pass two maximal lines. Thus, for $\ell$, we have $k - \delta \leq 1$ (see [1]) and the fact $\mathcal{X}_\ell = \emptyset$ is in accordance with Theorem 3.1.

Next, proposition reveals a rich structure of the Carnicer-Gasca lattice.

Proposition 3.4 ([1], Prop. 3.8) Let $\mathcal{X}$ be a Carnicer-Gasca lattice of degree $n$ and $\ell$ be a $k$-node line, $k \geq 2$. Then, we have that
\[ \mathcal{X}_\ell \text{ is an } \ell \text{-proper } GC_{s-2} \text{ subset of } \mathcal{X}, \text{ hence } |\mathcal{X}_\ell| = \binom{s}{2}, \tag{3.5} \]
where $k - \delta \leq s \leq k$ and $\delta = n + 1 - k$.

Moreover, in the case $\mathcal{X}_\ell \neq \emptyset$, we have for any maximal line $\lambda$:
\[ |\lambda \cap \mathcal{X}_\ell| = 0 \text{ or } |\lambda \cap \mathcal{X}_\ell| = s - 1. \]

Furthermore, for each $n, k, \text{ and } s$, with $k - \delta \leq s \leq k$, there is a Carnicer-Gasca lattice $\mathcal{X}$ of degree $n$ and a $k$-node line $\ell$ such that (3.5) is satisfied.

Note that the phrase “$\ell$-proper” is not present in the formulation of Proposition in [1] but it follows readily from the proof there.

The following result is due to Carnicer and Gasca (see also [1], eq. (1.4)).

Proposition 3.5 ([5], Prop. 4.2) Let $\mathcal{X}$ be a $GC_n$ set and $\ell$ be a 2-node line, then
\[ |\mathcal{X}_\ell| = 1 \quad \text{or} \quad 0. \]

Let us complement this with the following

Lemma 3.6 Let $\mathcal{X}$ be a $GC_n$ set, $\ell$ be a 2-node line, and $|\mathcal{X}_\ell| = 1$. Assume that GM Conjecture holds for all degrees up to $n$. Then $\mathcal{X}_\ell$ is an $\ell$-proper $GC_0$ subset.
Proof Indeed, suppose that $X_\ell = \{A\}$ and $\ell$ passes through the nodes $B$, $C \in X$. The node $A$ uses a maximal $(n+1)$-node line in $X$ which we denote by $\lambda_0$. Next, $A$ uses a maximal $n$-node line in $X \setminus \lambda_0$ which we denote by $\lambda_1$. Continuing this way, we find consecutively the lines $\lambda_2, \lambda_3, \ldots, \lambda_{n-1}$, and obtain that

$$\{A\} = X \setminus (\lambda_0 \cup \lambda_1 \cup \cdots \cup \lambda_{n-1}).$$

To finish the proof, it suffices to show that $\lambda_{n-1} = \ell$ and the remaining lines $\lambda_i, i = 0, \ldots, n-2$, are $\ell$-disjoint. Indeed, the node $A$ uses $\ell$, and since it is a 2-node line, it may coincide only with the last maximal line $\lambda_{n-1}$. Now, suppose conversely that a maximal line $\lambda_k$, $0 \leq k \leq n-2$, intersects $\ell$ at a node, say $B$. Then, consider the polynomial of degree $n$

$$p = \ell_{A, C} \prod_{i \in [0, \ldots, n-2]} \lambda_i,$$

where $\ell_{A, C}$ is the line through $A$ and $C$. Clearly, $p$ passes through all the nodes of $X$ which contradicts Proposition 1.2.

Now, in view of Proposition 3.5 and Lemma 3.6, we conclude that Theorem 3.1 is true for the case of 2-node lines in any $GC_n$ sets.

### 3.2 Some preliminaries for the proof of Theorem 3.1

Now let us show that Theorem 3.1 is true for the node set $X$ if $\mu(X) = 3$.

**Proposition 3.7** Let $X$ be a $GC_n$ set with $\mu(X) = 3$ and $\ell$ be a $k$-node line, $k \geq 2$. Assume that GM Conjecture holds for all degrees up to $n-3$. Then, we have that

$$X_\ell = \emptyset,$$

or $X_\ell$ is an $\ell$–proper $GC_{k-2}$ subset of $X$, hence $|X_\ell| = \left(\begin{smallmatrix} k \\ 2 \end{smallmatrix}\right)$.

Moreover, if $k \leq n$ and $X_\ell \neq \emptyset$, then for a maximal line $\lambda_1$ of $X$, we have that $\lambda_1 \cap \ell \neq \emptyset$ and $|\lambda_1 \setminus X_\ell| = 0$.

For the remaining two maximal lines, we have that $|\lambda \cap X_\ell| = k-1$.

Furthermore, if the line $\ell$ intersects each maximal line at a node then $X_\ell = \emptyset$.

**Proof** According to Theorem 2.16, the set $X$ is a generalized principal lattice of degree $n$ with some three sets of $n+1$ lines: $\ell_j^i(X)_{i \in [0, 1, \ldots, n]}$, $j = 0, 1, 2$, given in (2.12). Then, we obtain from (2.13) that the only used lines in $X$ are the lines $\ell_j^i(X)$, where $0 \leq s < n$, $r = 0, 1, 2$. Therefore, the only used $k$-node lines are the lines $\ell_{n-k+1}^r(X)$, $r = 0, 1, 2$. Consider the line, say with $r = 0$, i.e., $\ell \equiv \ell_{n-k+1}^0(X)$. It is used by all the nodes $x_{ijl} \in X$ with $i > n-k+1$, i.e., $i = n-k+2, n-k+3, \ldots, n$.

Thus, $\ell$ is used by exactly $\left(\begin{smallmatrix} k \\ 2 \end{smallmatrix}\right) = (k-1) + (k-2) + \cdots + 1$ nodes. This implies also that $X_\ell = X \setminus (\ell_0^0 \cup \ell_1^0 \cup \cdots \cup \ell_{n-k+1}^0)$. Hence $X_\ell$ is an $\ell$-proper $GC_{k-2}$ subset of $X$.

The part “Moreover” also follows readily from here. Now it remains to notice that the part “Furthermore” is a straightforward consequence of the part “Moreover.”

$\square$
Next statement is on the presence and usage of \((n - 1)\)-node lines in \(GC_n\) sets with \(\mu(X) = n - 1\) (cf. Proposition 4.2, [17]).

**Proposition 3.8** Let \(X\) be a \(GC_n\) set with \(\mu(X) = n - 1\), and \(\ell\) be an \((n - 1)\)-node line, where \(n \geq 4\). Assume also that through each node of \(\ell\) there passes exactly one maximal line. Then, we have that either \(n = 4\) or \(n = 5\). Moreover, in both these cases we have that \(X_\ell = \emptyset\).

**Proof** Consider a \(GC_n\) set with \(\mu(X) = n - 1\). In this case, we have the representation (2.10), i.e., \(X = X^{(2)} \cup X^{(1)} \cup X^{(0)}\) satisfying the conditions of Proposition 2.14. Here, \(X^{(k)}\) is the set of all \(k_m\)-nodes, i.e., nodes belonging exactly to \(k\) maximal lines. Recall that \(X^{(0)}\) consists of three non-collinear nodes: \(X^{(0)} = \{O_1, O_2, O_3\}\) outside the maximal lines.

Let \(\ell\) be an \((n - 1)\)-node line. First notice that, according to the hypothesis of Proposition, all the nodes of the line \(\ell\) are \(1_m\)-nodes. Therefore, \(\ell\) does not coincide with any \(O O\) line, i.e., line passing through two \(0_m\)-nodes.

From Proposition 2.14, (i), we have that all the nodes of \(X^{(1)}\), except the three nodes \(A_1, A_2, A_3\), which are called here special nodes, belong to the three \(O O\) lines. We have also, in view of Proposition 2.14, (iii), that the nodes \(A_1, A_2, A_3\) are not collinear. Therefore there are three possible cases:

(i) \(\ell\) does not pass through any special node,
(ii) \(\ell\) passes through two special nodes,
(iii) \(\ell\) passes through one special node.

In the first case, \(\ell\) may pass only through nodes lying in three \(O O\) lines. Then, it may pass through at most three nodes, i.e., \(n \leq 4\). Therefore, in view of the hypothesis \(n \geq 4\), we get that \(n = 4\) and \(\mu(X) = 3\). Now, in view of Proposition 3.7, part “Furthermore,” we get that \(X_\ell = \emptyset\).

Next, consider the case when \(\ell\) passes through two special nodes. Then, according to Proposition 2.14, (iii), it passes through an \(0_m\)-node. Recall that this case is excluded since \(\ell\) passes through \(1_m\)-nodes only.

Finally, consider the third case when \(\ell\) passes through exactly one special node. Then, it may pass through at most three other \(1_m\)-nodes lying in \(O O\) lines. Therefore, \(\ell\) may pass through at most four nodes.

First suppose that \(\ell\) passes through exactly 3 nodes. Then again, we obtain that \(n = 4\), \(\mu(X) = 3\) and \(X_\ell = \emptyset\).

Next, suppose that \(\ell\) passes through exactly 4 nodes. Then, we have that \(n = 5\). Without loss of generality, we may assume that the special node \(\ell\) passes through is, say, \(A_1\). Next, let us show first that \(|X_\ell| \leq 1\). Here, we have exactly \(4 = n - 1\) maximal lines. Consider the maximal line \(\lambda_4\), for which, in view of Proposition 2.14, the intersection with each \(O O\) line is a node in \(X\) (see Fig. 3). Denote \(B := \ell \cap \lambda_4\). Assume that the node \(B\) belongs to the line \(\ell^{i^{o}}\), \(1 \leq i \leq 3\), i.e., the line passing through \(\{O_1, O_2, O_3\} \setminus \{O_i\}\) (\(i = 2\) in Fig. 3). According to the condition (ii) of Proposition 2.14, we have that \(\ell^{i^{o}} \cap \lambda_i \notin X\). Denote \(C := \lambda_i \cap \lambda_4\). Now, let us prove that \(X_\ell \subset \{C\}\) which implies \(|X_\ell| \leq 1\).
Consider the $GC_4$ set $X_i = \mathcal{X} \setminus \lambda_i$. Here, we have two maximal lines intersecting at the node $B \in \ell$, i.e., $\lambda_4$ and $\ell_{oo}$. Therefore, we conclude from Lemma 2.8 that no node from these two maximal lines uses $\ell$ in $X_i$. Thus, in view of (2.3), no node from $\lambda_4$ except possibly $C$, uses $\ell$ in $X$. Now consider the $GC_4$ set $X_4 = \mathcal{X} \setminus \lambda_4$. Observe, on the basis of the characterization of Proposition 2.14, that $X_4$ has exactly 3 maximal lines. On the other hand, here, the line $\ell$ intersects each maximal line at a node. Therefore, in view of Proposition 3.7, part “Furthermore,” we have that $(X_4)_{\ell} = \emptyset$. Hence, in view of (2.3), we conclude that $X_{\ell} \subset \{C\}$.

Now, to complete the proof, it suffices to show that the node $C$ does not use $\ell$. Let us determine the lines the node $C$ uses. Since $C = \lambda_i \cap \lambda_4$, first of all, it uses the two maximal lines in $\{\lambda_1, \lambda_2, \lambda_3\} \setminus \{\lambda_i\}$. It is easily seen that the next two lines $C$ uses are $OO$ lines: $\{\ell_{oo}^1, \ell_{oo}^2, \ell_{oo}^3\} \setminus \{\ell_{i_{oo}}\}$. Now, notice that the two nodes, except $C$, which do not belong to the four used lines are $B$ and the special node $A^\ell_i$. Hence, the fifth line used by $C$ is the line passing through the latter two nodes. Now, observe that this line coincides with $\ell$ if and only if $i = 1$. Note that Fig. 3 depicts the case $X_{\ell} = \emptyset$ with $i \neq 1 \ (i = 2)$ and is not valid for the later discussion.

In the final and most interesting part of the proof, we will show that in the case $i = 1$, i.e., when the special node $\ell$ passes is $A^\ell_1$ and $B = \ell \cap \lambda_4 \cap \ell_{oo}^1$, the node $C$ can not use the line $\ell$. More precisely, we will do the following. By assuming that (see Fig. 4)

(i) the maximal lines $\lambda_i$, $i = 1, \ldots, 4$, and the three $O_1, O_2, O_3$ are given, $D := \lambda_2 \cap \lambda_3$, hence $D \in \mathcal{X}^{(2)}$,
(ii) the two $OO$ lines $\ell_{2}^{oo}$, $\ell_{3}^{oo}$ do not pass through the nodes of $X^{(2)}$, $E := \ell \cap \lambda_{3} \cap \ell_{2}^{oo}$, $F := \ell \cap \lambda_{2} \cap \ell_{3}^{oo}$, and

(iii) the conditions in Proposition 2.14, (iii), are satisfied, i.e., the line through the two special nodes in $\{A_{1}^{1}, A_{2}^{2}, A_{3}^{3}\} \setminus \{A_{i}^{i}\}$ passes through the node $O_{i}$ for each $i = 1, 2, 3$.

we will prove that the third $OO$ line $\ell_{1}^{oo}$ passes necessarily through the node $D \in X^{(2)}$, which contradicts Proposition 2.14, (ii).

To this end, we simplify Fig. 4 by deleting from it the maximal lines $\lambda_{1}$ and $\lambda_{4}$ to obtain the following Fig. 5. Let us now apply the well-known Pappus hexagon theorem for the pair of triple collinear nodes here

$$A_{1}^{1}, E, F;$$

$$O_{1}, A_{2}^{2}, A_{3}^{3}.$$ 

Now observe that

$$\ell(A_{1}^{1}, A_{2}^{2}) \cap \ell(E, O_{1}) = O_{3}, \ell(E, A_{3}^{3}) \cap \ell(F, A_{2}^{2}) = D, \ell(A_{1}^{1}, A_{3}^{3}) \cap \ell(F, O_{1}) = O_{2},$$

where $\ell(A, B)$ denotes the line passing through the points $A$ and $B$. Thus, according to the Pappus theorem we get that the triple of nodes $D, O_{2}, O_{3}$ is collinear, leading to contradiction. 

Remark 3.9 Let us show that the case of non-used 4-node line in Fig. 3 is possible nevertheless. The problem with this is that we have to confirm that the three conditions in Proposition 2.14 are satisfied. More precisely:
Fig. 5 The set $\mathcal{X}$ of Fig. 4 without the maximal lines $\lambda_1$ and $\lambda_4$

(i) $\ell_i^{oo} \cap \lambda_i = \emptyset$ for each $i = 1, 2, 3$;
(ii) The line through the two special nodes in $\{A_1^1, A_2^2, A_3^3\} \setminus \{A_i^i\}$ passes through the outside node $O_i$ for each $i = 1, 2, 3$.

Let us outline how one can get a desired figure (see Fig. 6). Let us start the figure with the three maximal lines (non-concurrent) $\lambda_1, \lambda_2, \lambda_3$. Then, we choose two $\ell^*$

Fig. 6 A non-used 4-node line $\ell_4^*$ in a $GC_5$ set $\mathcal{X}^*$
On the usage of lines in $\text{GC}_n$ sets

Let us start the proof with a list of the major cases in which Theorem 3.1 is true.

**Step 1.** Theorem 3.1 is true in the following cases:

(i) The line $\ell$ is a maximal line.

Indeed, as we have mentioned already, in this case we have $X_\ell = X \setminus \ell$ and all the conclusions of Theorem can be readily verified.

(ii) The line $\ell$ is an $n$-node line, $n \in \mathbb{N}$.

In this case Theorem 3.1 is valid by virtue of Theorem 3.2 (for $n \in \mathbb{N} \setminus \{3\}$) and Proposition 3.3 (for $n = 3$).

(iii) The line $\ell$ is a 2-node line.

In this case Theorem 3.1 follows from Proposition 3.5 and Lemma 3.6.

Now, let us prove Theorem by complete induction on $n$ - the degree of the node set $X$. Obviously, Theorem is true in the cases $n = 1, 2$. Note that this follows also from Step 1 (i) and (ii).

Assume that Theorem is true for any node set of degree not exceeding $n - 1$. Then, let us prove that it is true for the node set $X'$ of degree $n$. Suppose that we have a $k$-node line $\ell$.

**Step 2:** Suppose additionally that there is an $\ell$-disjoint maximal line $\lambda$. Then, we get from Lemma 2.7 that

$$X_\ell = (X' \setminus \lambda)_\ell.$$  \hspace{1cm} (4.1)

Therefore by using the induction hypothesis for the $\text{GC}_{n-1}$ set $X' := X \setminus \lambda$, we get the relation (3.2), i.e., $k - \delta' \leq s \leq k$ and $\delta' = \delta(X', \ell) = (n - 1) + 1 - k = n - k = \delta - 1$. Thus, we get $k - \delta + 1 \leq s \leq k$. Next, we use Proposition 2.11 in checking that $X_\ell$ is an $\ell$-proper subset of $X$.

Now, let us verify the part “Moreover.” Suppose that $k - \delta = 2k - n - 1 \geq 3$, i.e., $2k \geq n + 4$, and $\mu(X) > 3$. For the line $\ell$ in the $\text{GC}_{n-1}$ set $X'$, we have $k - \delta' = k - \delta + 1 \geq 4$. Thus, if $\mu(X') > 3$, then by the induction hypothesis, we
have that \((\mathcal{X}')_\ell \neq \emptyset\). Therefore, we get, in view of (4.1), that \(\mathcal{X}_\ell \neq \emptyset\). It remains to consider the case \(\mu(\mathcal{X}') = 3\). In this case, in view of Proposition 2.5, we have that \(\mu(\mathcal{X}) = 4\), which, in view of Theorem 2.12, implies that \(4 \in \{n - 1, n, n + 1, n + 2\}\), i.e., \(2 \leq n \leq 5\).

The case \(n = 2\) was verified already. Now, since \(2k \geq n + 4\), we deduce that either \(k \geq 4\) if \(n = 3, 4\) or \(k \geq 5\) if \(n = 5\). These cases follow from Step 1 (i) or (ii). The part “Furthermore” follows readily from the relation (4.1).

**Step 3:** Suppose additionally that there is a pair of \(\ell\)-adjacent maximal lines \(\lambda', \lambda''\). Then, we get from Lemma 2.8 that

\[
\mathcal{X}_\ell = \left(\mathcal{X} \setminus (\lambda' \cup \lambda'')\right)_\ell.
\]

Therefore, by using the induction hypothesis for the \(GC_{n-2}\) set \(\mathcal{X}'' := \mathcal{X} \setminus (\lambda' \cup \lambda'')\), we get the relation (3.2), i.e., \(n - 2 + 1 - (k - 1) = n - k = \delta - 1\). Thus, we get \(k - \delta \leq s \leq k - 1\). Next, we use Proposition 2.11 to check that \(\mathcal{X}_\ell\) is an \(\ell\)-proper subset of \(\mathcal{X}\).

Now, let us verify the part “Moreover.” Suppose that \(k - \delta = 2k - n - 1 \geq 3\), i.e., \(2k \geq n + 4\), and \(\mu(\mathcal{X}) > 3\). The line \(\ell\) is \((k - 1)\)-node line in the \(GC_{n-2}\) set \(\mathcal{X}''\) and we have that \(k - 1 - \delta'' = k - \delta \geq 3\). Thus, if \(\mu(\mathcal{X}'') > 3\), then by the induction hypothesis, we have that \((\mathcal{X}'')_\ell \neq \emptyset\) and therefore we get, in view of (4.2), that \(\mathcal{X}_\ell \neq \emptyset\). It remains to consider the case \(\mu(\mathcal{X}'') = 3\). Then, in view of Proposition 2.5, we have that \(\mu(\mathcal{X}) = 4\) or 5, which, in view of Theorem 2.12, implies that \(4\) or \(5 \in \{n - 1, n, n + 1, n + 2\}\) i.e., \(2 \leq n \leq 6\).

The cases \(2 \leq n \leq 5\) were considered in the previous step. Thus, suppose that \(n = 6\). Then, since \(2k \geq n + 4\), we deduce that \(k \geq 5\). In view of Step 1, (ii), we may suppose that \(k = 5\).

Now the set \(\mathcal{X}''\) is a \(GC_4\) and the line \(\ell\) is a 4-node line there. Thus, in view of Step 1 (ii), we have that \((\mathcal{X}'')_\ell \neq \emptyset\). Therefore, we get, in view of (4.2), that \(\mathcal{X}_\ell \neq \emptyset\).

The part “Furthermore” follows readily from the relation (4.2).

**Step 4.** Now consider any \(k\)-node line \(\ell\) in a \(GC_n\) set \(\mathcal{X}\). In view of Step 1 (iii), we may assume that \(k \geq 3\). In view of Theorem 2.12 and Propositions 3.3 and 3.7, we may assume also that \(\mu(\mathcal{X}) \geq n - 1\) and \(n \geq 4\).

Next suppose that \(k \leq n - 2\). Since then \(\mu(\mathcal{X}) > k\), we necessarily have either the situation of Step 2 or Step 3.

Thus, we may assume that \(k \geq n - 1\). Then, in view of Step 1 (i) and (ii), it remains to consider the case \(k = n - 1\), i.e., \(\ell\) is an \((n - 1)\)-node line. Again, if \(\mu(\mathcal{X}) \geq n\), then we necessarily have either the situation of Step 2 or Step 3. Therefore, we may assume also that \(\mu(\mathcal{X}) = n - 1\). By the same argument, we may assume that each of the \(n - 1\) nodes of the line \(\ell\) is an intersection node with one of the \(n - 1\) maximal lines.

Therefore, the conditions of Proposition 3.8 are satisfied and we arrive to the two cases: \(n = 4, k = 3, k - \delta = 1\) or \(n = 5, k = 4, k - \delta = 2\). In both cases, we have that \(\mathcal{X}_\ell = \emptyset\) and \(k - \delta \leq 2\). Thus, in this case, Theorem is true. □

From the above proof, we get immediately
On the usage of lines in $GC_n$ sets

Corollary 4.1 Let $X_0$ be a $GC_n$ set, with $\mu(X_0) > 3$ and $n \geq 6$. Let also $\ell$ be a $k$-node line, $2 \leq k \leq n$. Assume that GM Conjecture holds for all degrees up to $n$. Then, there is either $\ell$-disjoint maximal line or a pair of $\ell$-adjoint maximal lines.

Indeed, assume conversely that the conclusion here is not true. Then, in the above proof for the set $X_0$ and the $k$-node line $\ell$, we arrive necessarily to Step 4. Next, in Step 4, as above, we conclude that $n = 4$, or $n = 5$, which contradict the condition $n \geq 6$.

4.1 The characterization of the case $k - \delta = 2, \mu(X) > 3$

Here, for each $n$ and $k$, with $k - \delta = 2k - n - 1 = 2$, we bring two constructions of $GC_n$ sets with a non-used $k$-node line. At the end (see forthcoming Proposition 4.2), we prove that these are the only constructions with the mentioned possession.

Let us start with a counterexample in the case $n = k = 3$ (see [17], Section 3.1). Consider a $GC_3$ set $\mathcal{Y}^*$ of 10 nodes with exactly three maximal lines: $\lambda_1, \lambda_2, \lambda_3$ (see Fig. 7). This set is of the form (2.9) and satisfies the conditions listed in Proposition 2.13. Now observe that the 3-node line $\ell_3^*$ here intersects all the three maximal lines at nodes. Therefore, in view of Proposition 3.3, (iii), the line $\ell_3^*$ is non-used, i.e., $(\mathcal{Y}^*)_{\ell_3^*} = \emptyset$. **Fig. 7** A $GC_3$ set $\mathcal{Y}^*$ with 3 maximal lines and a 3-node line $\ell_3^*$
Let us outline how one can get Fig. 7. We start the mentioned figure with the three lines $\ell^o_1, \ell^o_2, \ell^o_3$ through $O$, i.e., the $0_m$-node. Then, we choose the maximal lines $\lambda_1, \lambda_2, \lambda_3$ intersecting $\ell^o_1, \ell^o_2$ at 4 distinct points. Let $A_i := \lambda_i \cap \ell^o_i$, $i = 1, 2$. We choose the points $A_1$ and $A_2$ such that the line through them: $\ell^o_2$ intersects the line $\ell^o_3$ at a point $A_3$. Next, we choose a third maximal line $\lambda_3$ passing through $A_3$. Let us mention that we choose the maximal lines such that they are not concurrent and intersect the three lines through $O$ at nine distinct points. Finally, all the specified intersection points in Fig. 7 we declare as the nodes of $Y^*$.

In the general case of $k - \delta = 2k - n - 1 = 2$, we set $k = m + 3$ and obtain $n = 2m + 3$, where $m = 0, 1, 2, \ldots$. Let us describe how the previous $GC_3$ node set $Y^*$ together with the 3-node line $\ell^*_3$ can be modified to $GC_n$ node set $\bar{Y}^*$ with a $k$-node line $\bar{\ell}^*_3$ such that $(\bar{Y}^*)_{\bar{\ell}^*_3} = \emptyset$.

To this end, we just leave the line $\ell^*_3$ unchanged, i.e., $\bar{\ell}^*_3 \equiv \ell^*_3$ and extend the set $Y^*$ to a $GC_n$ set $\bar{Y}^*$ in the following way (see Fig. 8). We fix $m$ points: $B_i, i = 1, \ldots, m$, in $\ell^*_3$ different from $A_1, A_2, A_3$ ($m = 2$ in Fig. 8). Then, we add $m$ pairs of (maximal) lines $\lambda'_i, \lambda''_i$, $i = 1, \ldots, m$, intersecting at these $m$ points, respectively: $\lambda'_i \cap \lambda''_i = B_i$, $i = 1, \ldots, m$.

We assume that the following condition is satisfied:

(i) The $2m$ lines $\lambda'_i, \lambda''_i$, $i = 1, \ldots, m$, together with $\lambda_1, \lambda_2, \lambda_3$, are in general position, i.e., no two lines are parallel and no three lines are concurrent;

(ii) The mentioned $2m + 3$ lines intersect the lines $\ell^o_1, \ell^o_2, \ell^o_3$ at distinct $3(2m + 3)$ points.

Now, all the points of the intersections of the $2m + 3$ lines $\lambda'_i, \lambda''_i$, $i = 1, \ldots, m$, together with $\lambda_1, \lambda_2, \lambda_3$ are declared as the nodes of the set $\bar{Y}^*$. Next, for each of the lines $\lambda'_i, \lambda''_i$, $i = 1, \ldots, m$, also two from the three intersection points with the lines

Fig. 8 A non-used $k$-node line $\bar{\ell}^*_3$ in a $GC_n$ set $\bar{Y}^*$ with $k - \delta = 2, m = 2$
\( \ell_1, \ell_2, \ell_3 \) are declared as \((1_m^-)\) nodes. After this, the lines \( \lambda_1, \lambda_2, \lambda_3 \) and the lines \( \lambda'_i, \lambda''_i, i = 1, \ldots, m \) become \((2m + 4)\)-node lines, i.e., maximal lines.

Now, one can verify readily that \( Y^* \) is a \( GC_n \) set of form \((2.9)\) and satisfies the conditions in Proposition 2.13 with \( n = 2m + 3 \) maximal lines: \( \lambda'_i, \lambda''_i, i = 1, \ldots, m \), together with \( \lambda_1, \lambda_2, \lambda_3 \). Finally, in view of Lemma 2.8 and the relation \((2.7)\), applied \( m \) times with respect to the pairs \( \lambda'_i, \lambda''_i, i = 1, \ldots, m \), gives: \((Y^*)\tilde{\ell}_3 = (Y^*)\tilde{\ell}_3 = \emptyset\).

Let us call the set \( Y^* \) an \( m \)-modification of the set \( Y^* \). In the same way, we could define \( X^* \) as an \( m \)-modification of the set \( X^* \) from Fig. 3, with the 4-node non-used line \( \ell^*_4 \) (see Remark 3.9). The only differences from the previous case here are:

(i) Now \( k = m + 4 \), \( n = 2m + 5 \), \( m = 0, 1, 2, \ldots \) (again \( k − δ = 2 \));
(ii) We have \( 2m + 4 \) maximal lines: \( \lambda'_i, \lambda''_i, i = 1, \ldots, m \), and the lines \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \);
(iii) The lines \( \ell^*_{1}, \ell^*_{2}, \ell^*_{3} \) are replaced with the lines \( \ell^{oo}_{1}, \ell^{oo}_{2}, \ell^{oo}_{3} \);
(iv) For each of the lines \( \lambda'_i, \lambda''_i, i = 1, \ldots, m \), all three intersection points with the lines \( \ell^{oo}_{1}, \ell^{oo}_{2}, \ell^{oo}_{3} \), are declared as \((1_m^-)\) nodes.

Now, one can verify readily that the set \( X^* \) is a \( GC_n \) set of the form \((2.10)\) and satisfies the conditions in Proposition 2.14 with \( n − 1 = 2m + 4 \) maximal lines. In the same way as above, we get: \((X^*)\tilde{\ell}_4 = (X^*)\tilde{\ell}_4 = \emptyset\). Thus, we obtain another construction of \( GC_n \) sets, with non-used \( k \)-node lines, where \( k = m + 4 \), \( n = 2m + 5 \) and \( k − δ = 2 \).

At the end, let us prove the following

**Proposition 4.2** Let \( X \) be a \( GC_n \) set and \( \ell \) be a \( k \)-node line with \( k − δ := 2k − n − 1 = 2 \) and \( μ(X) > 3 \). Suppose that the line \( \ell \) is a non-used line. Then, we have that either \( X = X^* \), \( \ell = \tilde{\ell}^*_4 \), or \( X = Y^* \), \( \ell = \tilde{\ell}^*_3 \).

**Proof** Notice that \( n \) is an odd number and \( n ≥ 3 \). In the case \( n = 3 \), Proposition 4.2 follows from Proposition 3.3.

Thus, suppose that \( n ≥ 5 \). Since \( μ(X) > 3 \), we get, in view of Theorem 2.12, that

\[
μ(X) ≥ n − 1.
\]

(4.3)

Now, let us prove that there is no \( \ell \)-disjoint maximal line \( λ \) in \( X \).

Suppose conversely that \( λ \) is a maximal line with \( λ \cap \ell \notin X \). Denote by \( X'_λ := X \setminus λ \). Since \( X_λ = ∅ \) therefore, by virtue of the relation \((2.3)\), we obtain that \((X'_λ)\ell = ∅ \). Then, we have that \( k − δ' := k − δ(X'_λ, \ell) = k − ((n − 1) + 1 − k) = 2k − n = 3 \). By taking into account the latter two facts, i.e., \((X'_λ)\ell = ∅ \) and \( k − δ' = 3 \), we conclude from Theorem 3.1, part “Moreover,” that \( μ(X'_λ) = 3 \). Next, by using \((4.3)\) and Proposition 2.5, we obtain that \( μ(X) = 4 \). By applying again \((4.3)\), we get that \( 4 ≥ n − 1 \), i.e., \( n ≤ 5 \). Therefore, we arrive to the case: \( n = 5 \). Since \( k − δ = 2 \), we conclude that \( k = 4 \). Then, observe that the line \( \ell \) is 4-node line in the \( GC_4 \) set \( X'_λ \). By using Theorem 3.2, we get that \((X'_λ)\ell \neq ∅ \), which is a contradiction.

Next, let us prove Proposition in the case \( n = 5 \). As we mentioned above, then \( k = 4 \). We have that there is no \( \ell \)-disjoint maximal line. Suppose also that there is no pair of \( \ell \)-adjacent maximal lines. Then, in view of \((4.3)\), we readily get that \( μ(X) = 4 \), and through each of the four nodes of the line \( \ell \), there passes a maximal
line. Therefore, Proposition 2.14 yields that $\mathcal{X}$ coincides with $\mathcal{X}^*$ (or, in other words, $\mathcal{X}$ is a 0-modification of $\mathcal{X}^*$) and $\ell$ coincides with $\ell_4^*$.

Next suppose that there is a pair of $\ell$-adjacent maximal lines: $\lambda', \lambda''$. Denote by $\mathcal{X}' := \mathcal{X} \setminus (\lambda' \cup \lambda'')$. Then, we have that $\ell$ is a 3-node non-used line in $\mathcal{X}'$. Thus, we conclude readily that $\mathcal{X}$ coincides with $\mathcal{Y}^*$ and $\ell$ coincides with $\ell_3^*$ (with $m = 1$).

Now, let us continue by using induction on $n$. Assume that Proposition is valid for all degrees up to $n - 1$. Let us prove it in the case of the degree $n$. We may suppose that $n \geq 7$. It suffices to prove that there is a pair of $\ell$-adjacent maximal lines: $\lambda', \lambda''$. Indeed, in this case, we can complete the proof just as in the above case $n = 5$.

Suppose by way of contradiction that there is no pair of $\ell$-adjacent maximal lines. Also, we have that there is no $\ell$-disjoint maximal line. Therefore, we have that $\mu(\mathcal{X}) \leq k$. Now, by using (4.3), we get that $k \geq n - 1$. Therefore, $2 = k - \delta = 2k - n - 1 \geq 2n - 2 - n - 1 = n - 3$. This implies that $n - 3 \leq 2$, i.e., $n \leq 5$, which is a contradiction.

Acknowledgments We thank reviewers for thorough reviews and appreciate comments, suggestions, and corrections that have made a significant contribution to improving the quality of the paper.

References

1. Bayramyan, V., Hakopian, H.: On a new property of $n$-poised and $GC_n$ sets. Adv. Comput. Math. 43, 607–626 (2017)
2. Busch, J.R.: A note on Lagrange interpolation in $\mathbb{R}^2$. Rev. Un. Mat. Argentina 36, 33–38 (1990)
3. Carnicer, J.M., Gasca, M.: Planar Configurations with Simple Lagrange Formula. In: Lyche, T., Schumaker, L.L. (eds.) Mathematical Methods in CAGD: Oslo 2000, pp. 55–62. Vanderbilt University Press, Nashville (2001)
4. Carnicer, J.M., Gasca, M.: A conjecture on multivariate polynomial interpolation. Rev. R. Acad. Cienc. Exactas Fís. Nat. (Esp.), Ser. A Mat. 95, 145–153 (2001)
5. Carnicer, J.M., Gasca, M.: On Chung and Yao’s Geometric Characterization for Bivariate Polynomial Interpolation. In: Lyche, T., Mazure, M.-L., Schumaker, L.L. (eds.) Curve and Surface Design: Saint Malo 2002, pp. 21–30. Nashboro Press, Brentwood (2003)
6. Carnicer, J.M., Gasca, M.: Generation of lattices of points for bivariate interpolation. Numer. Algorithms 39, 69–79 (2005)
7. Carnicer, J.M., Godés, C.: Geometric characterization and generalized principal lattices. J. Approx. Theory 143, 2–14 (2006)
8. Carnicer, J.M., Godés, C.: Geometric Characterization of Configurations with Defect Three. In: Cohen, A., Merrien, J.L., Schumaker, L.L. (eds.) Curve and Surface Fitting: Avignon 2006, pp. 61–70. Nashboro Press, Brentwood (2007)
9. Carnicer, J.M., Godés, C.: Configurations of nodes with defects greater than three. J. Comput. Appl. Math. 233, 1640–1648 (2010)
10. Carnicer, J.M., Godés, C.: Extensions of planar $GC$ sets and syzygy matrices. Adv. Comput. Math. 45, 655–673 (2019)
11. Chung, K.C., Yao, T.H.: On lattices admitting unique Lagrange interpolations. SIAM J. Numer. Anal. 14, 735–743 (1977)
12. Gasca, M., Maeztu, J.I.: On Lagrange and Hermite interpolation in $\mathbb{R}^k$. Numer. Math. 39, 1–14 (1982)
13. Hakopian, H., Jetter, K., Zimmermann, G.: A new proof of the Gasca-Maeztu conjecture for $n = 4$. J. Approx. Theory 159, 224–242 (2009)
14. Hakopian, H., Jetter, K., Zimmermann, G.: The Gasca-Maeztu conjecture for $n = 5$. Numer. Math. 127, 685–713 (2014)
15. Hakopian, H., Rafayelyan, L.: On a generalization of Gasca-Maeztu conjecture. New York J. Math. 21, 351–367 (2015)
16. Hakopian, H., Toroyan, S.: On the uniqueness of algebraic curves passing through n-independent nodes. New York J. Math. 22, 441–452 (2016)
17. Hakopian, H., Vardanyan, V.: On a correction of a property of $GC_n$ sets. Adv. Comput. Math. 45, 311–325 (2019)

Publisher’s note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.