Order $\rho^2$ Corrections to Randall-Sundrum I Cosmology

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Abstract: We derive the corrections to the Friedmann equation of order $\rho^2$ in the Randall-Sundrum (RS) model, where two 3-branes bound a slice of five-dimensional Anti-deSitter space. The effects of radion stabilization by the Goldberger-Wise mechanism are taken into account. Surprisingly, we find that an inflaton on either brane will experience no order $\rho^2$ corrections in the Hubble rate $H$ due to its own energy density, although an observer on the opposite brane does see such a correction. Thus there is no enhancement of the slow-roll condition unless inflation is simultaneously driven by inflatons on both branes. Similarly, during radiation domination, the $\rho^2$ correction to $H$ on a given brane vanish unless there is nonvanishing energy density on the opposite brane. During the electroweak phase transition the correction can be large, but it has the wrong sign for causing sphalerons to go out of thermal equilibrium, so it cannot help electroweak baryogenesis. We discuss the differences between our results and exact solutions in RS-II cosmology.
1. Introduction

The idea of our world being a 3-brane embedded in a higher dimensional bulk has provided
an attractive explanation for the weakness of gravity compared to the other fundamental
forces [1, 2, 3]. In this framework, 4-D gravity is weak at distances much greater than the
fundamental Planck length, usually assumed to be the TeV$^{-1}$ scale, but it quickly becomes
strong at shorter distances. This idea has rich consequences for LHC physics, and it also
implies modifications to cosmology at temperatures approaching 1 TeV. Although it may
not be feasible to test deviations from standard cosmology which happened so early, this
scale is sufficiently close to the electroweak phase transition to give some hope of observable
consequences, for example to electroweak baryogenesis. Moreover the nature of inflation
can be significantly altered by extra-dimensional effects [1, 2, 3, 18].

A seminal work on brane cosmology [3] observed that the Friedmann equation on
a 3-brane embedded in a flat 5-D spacetime takes the form $H^2 = \frac{\rho^2}{36M_5^2}$ instead of the
standard form $H^2 = \frac{8\pi G \rho}{3}$. Here $M_5$ is the 5-D Planck mass and $\rho$ is the 4-D energy
density on the brane. This strange result arises from the fact that the brane is assumed to
be infinitesimally thin compared to the size of the extra dimension ($y$), which introduces
a discontinuity in the derivatives of metric elements with respect to $y$ at the position of
the brane. The \( \rho^2 \) dependence is strongly contradicted by big bang nucleosynthesis. It was subsequently observed [8, 9] that by adding a negative cosmological constant \( \Lambda_5 \) to the 5-D bulk, and a tension \( T \) to the brane, the Friedmann equation generalizes to \( H^2 = \frac{\Lambda_5}{6M_5^3} + \frac{(T+\rho)^2}{36M_5^3} \), which takes the form \( H^2 = \frac{8\pi G}{3}(1 + O(\frac{\rho}{M_5^2})) \) if one chooses \( T^2 = -6\Lambda_5M_5^3 \) and \( \frac{T}{18M_5^5} = \frac{8\pi G}{9} \). In other words, we recover the standard Friedmann equation, modified by \( \rho^2 \) corrections which are small as long as \( \rho \ll T \sim \text{TeV} \). In addition, the two relations between \( T \) and \( M_5 \) are equivalent to those needed for the Randall-Sundrum solutions, where the bulk is a region of 5-D Anti-DeSitter space.

Although the problem of how to recover the correct Friedmann equation was partially resolved, refs. [8, 9] pointed out that in the two-brane model of [2] (RS-I), \( H^2 \) could be real only if \( \rho \) was assumed to be negative on the second brane, whose tension is negative. In addition one had to assume a fine-tuned relation between the energy densities on the two branes. This untenable result was especially troubling because the negative tension, or TeV brane, is the one where we are assumed to be living in order to perceive 4-D gravity as being weak. Ref. [10] pointed out that these problems could be avoided if the \( T_{55} \) component of the bulk stress-energy tensor had the right properties. It was subsequently shown [11, 12] that stabilization of the size of the extra dimension (the radion mode) was necessary and sufficient for recovering acceptable cosmology on the TeV brane. The unphysical requirement of a fine-tuned negative value for \( \rho \) was merely a consequence of assuming the extra dimension was static when generically, in the absence of any stabilizing force, it would tend to expand or contract. The positive effects of stabilization were subsequently confirmed in greater detail by references [13, 14] and [15]. The problem of losing stabilization at temperatures above the TeV scale and tunneling back to the stabilized ground state was explored in ref. [16].

The issue of radion stabilization exists only in the RS-I and ADD [1] solutions; in RS-II [3] the extra dimension is infinite in size, there is only a single brane, and there is no radion—its wave function diverges away from the brane, and so it is not a normalizable mode if there is no second brane to compactify the space. Because of this simplicity, much of the early work on brane cosmology has focused on the RS-II solution. In this case the above-quoted form of the modified Friedmann equation becomes exact,

\[
H^2 = \frac{8\pi G}{3}\rho \left(1 + \frac{\rho}{2T}\right) \tag{1.1}
\]
as long as higher-derivative corrections to the gravitational action are neglected\(^1\) and the bulk is empty. Perhaps the most important application of the \( \rho/T \) correction is to increase the expansion rate over its normal value during inflation [5, 6]. This allows for the possibility of fulfilling the slow-roll condition with steeper inflationary potentials than would normally be admissible, with the possible observable consequence of a spectral index of density perturbations which is smaller than in conventional models of inflation. It should be kept in mind that the effects of the \( \rho/T \) correction should not be trusted quantitatively whenever

\(^1\)Since the discontinuity in the derivative of the warp factor at the brane is proportional to \( \rho \) as a consequence of the 5-D Einstein equations, one may expect that higher derivative terms will bring higher powers of \( \rho \).
they become large, precisely because of the neglect of higher derivative operators which are only suppressed by the energy scale $\sim T^{1/4}$. Beyond this scale one really needs the full quantum theory of gravity, presumably string theory, to make quantitative predictions. Thus the $\rho/T$ corrections should be regarded as indicative of the kinds of new qualitative effects that could be expected at energy densities above the quantum gravity scale.

More recently, the cosmology of the RS-I model has attracted renewed attention. The AdS/CFT correspondence has been used as a way of better quantifying the quantum gravity effects: Kaluza-Klein excitations of the bulk graviton modes are supposed to be equivalent to bound states of a strongly coupled, nearly conformal field theory residing on the TeV brane [17]. At temperatures above the TeV scale, the TeV brane is supposed to be hidden behind a horizon [18] associated with a black hole which formed in the bulk [19]. It has been noted that the emergence of the TeV brane from the horizon may occur around the same time as the electroweak phase transition [20]. Ref. [16] provided an alternative picture of a first order phase transition which leads to the appearance of the TeV brane at this epoch.

There have been numerous investigations of the effects of a bulk scalar field on brane-world cosmology [21]. In the present work we shall be concerned with the $O(\rho^2)$ corrections to the Friedmann equation which arise from the 5-D nature of the underlying geometry in the RS-I model, using the Goldberger-Wise (GW) [22] mechanism to stabilize the size of the extra dimension. In general one no longer expects eq. (1.1) to hold in the RS-I model because of the additional effect of the dynamical radion. The radius is displaced by the cosmological expansion, which changes the strength of gravity and thus back-reacts on the expansion rate. Although one should not trust these corrections quantitatively when they start to become large (for the reasons discussed above), they accurately predict the deviations from standard cosmology as one starts to approach the quantum gravity scale around 1 TeV.

The plan of the paper is as follows. In section 2 we expand the 5-D Einstein equations to second order in a perturbation series in $\rho$ and $\rho_\ast$, the energy densities on the TeV and Planck branes. In section 3 the equations are solved to find the ingredients from which one can infer the effective 4-D Friedmann equations. In section 4 we present the results for the Hubble rate and the acceleration as measured on either brane, for arbitrary equations of state, and we discuss the implications for cosmology. We also compare the results to where there is only a single brane or where there is no bulk scalar. The reader who is not interested in the details of solving the Einstein equations can go directly to this section. Section 5 gives a summary and conclusions. Some technical details can be found in the appendices.

2. $O(\rho^2)$ Einstein Equations

Following closely the formalism of ref. [15], we will look for cosmological solutions to 5-D gravity coupled to the Goldberger-Wise scalar field $\Phi$, which is responsible for stabilizing the radion, and to two branes: the Planck brane, located at $y_0 \equiv 0$, and the TeV brane at
We will make a perturbative expansion in the energy densities \( \rho, \rho_\star \) of the branes around the static solution, where \( \rho = \rho_\star = 0 \):

\[
N(t, y) = A_0(y) + \delta N_1(t, y) + \delta N_2(t, y); \quad A(t, y) = A_0(y) + \delta A_1(t, y) + \delta A_2(t, y)
\]

\[
b(t, y) = b_0 + \delta b_1(t, y) + \delta b_2(t, y); \quad \Phi(t, y) = \Phi_0(y) + \delta \Phi_1(t, y) + \delta \Phi_2(t, y).
\]

The subscripts on the perturbations indicate their order in powers of \( \rho \) or \( \rho_\star \), both of which are taken to be formally of the same order for the purposes of developing the perturbation expansion, although in actual order of magnitude we may consider them to have very different values. This ansatz is to be substituted into the scalar field equation,

\[
\partial_t \left( \frac{1}{n} b a^3 \dot{\Phi} \right) - \partial_y \left( \frac{1}{b} a^3 n \dot{\Phi}' \right) + a^3 n \left[ b V' + V_0' \delta(y) + V_1' \delta(y - 1) \right] = 0,
\]

and into the Einstein equations, \( G_{mn} = \kappa^2 T_{mn} \). Here and in the following, primes on functions of \( y \) denote \( \frac{\partial}{\partial y} \), while primes on potentials of \( \Phi \) will mean \( \frac{\partial}{\partial \Phi} \). The components of the Einstein tensor which do not vanish identically are

\[
G_{00} = 3 \left[ \left( \frac{\dot{a}}{a} \right)^2 + \frac{\dot{b}}{b} - \frac{n}{a} \left( \frac{a''}{a} + \left( \frac{a'}{a} \right)^2 - \frac{a'b'}{ab} \right) \right]
\]

\[
G_{ii} = \frac{a^2}{b^2} \left[ \left( \frac{a'}{a} \right)^2 + 2 \frac{a' n'}{a n} b' \frac{n'}{b} + 2 \frac{b'}{b} b + 2 \frac{a''}{a} + \frac{n''}{n} \right]
\]

\[
+ \frac{a^2}{n^2} \left[ - \left( \frac{\dot{a}}{a} \right)^2 + 2 \frac{\dot{a} n}{a n} - 2 \frac{\dot{a}}{a} + \frac{\dot{b}}{b} \left( -\frac{2}{a} + \frac{\dot{a}}{a} \right) - \frac{\ddot{b}}{b} \right]
\]
and the stress energy tensor is \( T_{mn} = g_{mn}(V(\Phi) + \Lambda) + \partial_m \Phi \partial_n \Phi - \frac{1}{2} \partial^i \Phi \partial_i \Phi g_{mn} \) in the bulk. On the branes, \( T_{m}^{n} \) is given by
\[
T_{m}^{n} = \frac{\delta(y)}{b(t,0)} \text{diag}(V_{0} + \rho_{*}, V_{0} - p_{*}, V_{0} - p_{*}, V_{0} - p_{*}, 0) + \frac{\delta(y - 1)}{b(t,1)} \text{diag}(V_{1} + \rho, V_{1} - p, V_{1} - p, V_{1} - p, 0) \tag{2.7}
\]

(Later we will assume that the potentials \( V_{0} \) and \( V_{1} \) are very stiff and are vanishing at their minima, so they can be neglected.)

The perturbative solution has been given to first order in \( \rho, \rho_{*} \) in [15]; here we want to extend the analysis to second order. Following [13], it is useful to work with the following linear combinations of the metric tensor perturbations:
\[
\Psi_{2} = \delta A_{2} - A_{0} \frac{\delta b_{2}}{b_{0}} - \frac{\kappa^{2}}{3} \Phi'_{0} \delta \Phi_{2} - \frac{\kappa^{2}}{6} (\delta \Phi'_{1} + \Phi''_{0} \frac{\delta b_{1}}{b_{0}}) \delta \Phi_{1}; \quad \Upsilon_{2} = \delta N'_{2} - \delta A'_{2}. \tag{2.8}
\]

These are convenient variables because they appear in a natural way in the boundary conditions at the two branes. By integrating the (00) and (ii) Einstein equations in the vicinity of either brane, we obtain the perturbed Israel junction conditions,
\[
\Psi_{2}(t, 0) = + \frac{\kappa^{2}}{6} b_{0} \rho_{*} \frac{\delta b_{1}}{b_{0}} \bigg|_{t; y = 0}; \quad \Psi_{2}(t, 1) = - \frac{\kappa^{2}}{6} b_{0} \rho_{*} \frac{\delta b_{1}}{b_{0}} \bigg|_{t; y = 1} \tag{2.9}
\]
\[
\Upsilon_{2}(t, 0) = - \frac{\kappa^{2}}{2} b_{0} (\rho_{*} + p_{*}) \frac{\delta b_{1}}{b_{0}} \bigg|_{t; y = 0}; \quad \Upsilon_{2}(t, 1) = + \frac{\kappa^{2}}{2} b_{0} (\rho + p) \frac{\delta b_{1}}{b_{0}} \bigg|_{t; y = 1} \tag{2.10}
\]

The analogous quantities at first order were found to be
\[
\Psi_{1} = \delta A' - A'_{0} \frac{\delta b_{1}}{b_{0}} - \frac{\kappa^{2}}{3} \Phi'_{0} \delta \Phi_{1}; \quad \Upsilon_{1} = \delta N'_{1} - \delta A'_{1} \tag{2.11}
\]
in [15], and they satisfy the same boundary conditions as in (2.9,2.10), but with the replacement \( \frac{\delta b_{1}}{b_{0}} \rightarrow 1 \).

The Einstein equations take a form resembling ordinary second order differential equations in \( y \) for the variables \( \Psi_{2} \) and \( \Upsilon_{2} \), where all the time dependence of the latter is implicit, through \( \rho(t), \rho_{*}(t), p(t) \) and \( p_{*}(t) \). The quantities \( \frac{\dot{A}}{A} \) and \( \frac{\dot{b}}{b} \) function like constants of integration in this context, whose values are fixed by imposing the boundary conditions (2.9,2.10). This procedure results in o.d.e.'s in time for \( a_{0}(t) \), which are the Friedmann equations we are seeking. Accordingly, we need to expand \( \frac{\dot{A}}{A} \) and \( \frac{\dot{b}}{b} \) in powers of \( \rho \). Recall that in ordinary cosmology, \( (\frac{\dot{a}}{a})^{2} \sim \frac{\dot{a}}{a} \sim \rho \). Therefore each time derivative of \( a_{0} \) counts as

\[
G_{05} = 3 \left[ \frac{n' \dot{a}}{n a} + \frac{a' \dot{b}}{a b} - \frac{\dot{a}}{a} \right] \tag{2.6}
\]
\[
G_{55} = 3 \left[ \frac{a' \left( \frac{n'}{a} + \frac{n'}{n} \right)}{a b} - \frac{\dot{b}}{n^{2}} \left( \frac{\dot{a}}{a} - \frac{n}{n} \right) + \frac{\dot{a}}{a} \right] \]

\[
\Psi_{2}(t, 0) = + \frac{\kappa^{2}}{6} b_{0} \rho_{*} \frac{\delta b_{1}}{b_{0}} \bigg|_{t; y = 0}; \quad \Psi_{2}(t, 1) = - \frac{\kappa^{2}}{6} b_{0} \rho_{*} \frac{\delta b_{1}}{b_{0}} \bigg|_{t; y = 1} \tag{2.9}
\]
\[
\Upsilon_{2}(t, 0) = - \frac{\kappa^{2}}{2} b_{0} (\rho_{*} + p_{*}) \frac{\delta b_{1}}{b_{0}} \bigg|_{t; y = 0}; \quad \Upsilon_{2}(t, 1) = + \frac{\kappa^{2}}{2} b_{0} (\rho + p) \frac{\delta b_{1}}{b_{0}} \bigg|_{t; y = 1} \tag{2.10}
\]
half a power of \( \rho \). In the brane cosmology we have \( (\frac{\dot{a}}{a})^2 \sim \frac{\ddot{a}}{a} \sim \rho, \rho_* + O(\rho_*^2, \rho_* \rho, \rho^2) \), so we expect that

\[
\frac{\dot{a}_0}{a_0} = \sqrt{\frac{8\pi G}{3}} (\rho_* + \Omega^4 \rho) \left[ 1 + O(\rho_* , \rho) \right] \equiv \left( \frac{\dot{a}_0}{a_0} \right)_{(1)} + \left( \frac{\dot{a}_0}{a_0} \right)_{(2)} + \ldots
\]

\[
\left( \frac{\dot{a}_0}{a_0} \right)_{(2)}^2 = \frac{8\pi G}{3} (\rho_* + \Omega^4 \rho) \left[ 1 + O(\rho_* , \rho) \right] \equiv \left( \frac{\dot{a}_0}{a_0} \right)_{(1)}^2 + \left( \frac{\dot{a}_0}{a_0} \right)_{(2)}^2 + \ldots
\]

where we have used subscripts to denote the power of \( \rho \) or \( \rho_* \) contained in the associated term. Here \( \Omega \equiv e^{-A_0(1)} \) is the value of the warp factor at the TeV brane, which solves the hierarchy problem in the RS scenario, and the linear combination \( \rho_* + \Omega^4 \rho \) is the result that has been proven \([11, 15]\) to appear in the lowest order Friedmann equation when the radion is stabilized. We therefore expand \( \frac{\dot{a}}{a} \) and \( \frac{\ddot{a}}{a} \) in the following manner:

\[
\left( \frac{\dot{a}}{a} \right) = \left( \frac{\dot{a}_0}{a_0} - \dot{\bar{A}} \right)^2 = \left( \frac{\dot{a}_0}{a_0} \right)_{(1)}^2 + \left( \frac{\dot{a}_0}{a_0} \right)_{(2)}^2 + O(\rho^3)
\]

\[
= \left( \frac{\dot{a}_0}{a_0} \right)_{(1)}^2 + \left[ 2 \left( \frac{\dot{a}_0}{a_0} \right)_{(2)}^2 - 2 \left( \frac{\dot{a}_0}{a_0} \right)_{(2)} \right] \delta \dot{A}_1 + O(\rho^3);
\]

\[
\frac{\ddot{a}}{a} = \frac{\ddot{a}_0}{a_0} - 2 \frac{\dot{a}_0}{a_0} \dot{\bar{A}} - \ddot{\bar{A}} = \left( \frac{\ddot{a}_0}{a_0} \right)_{(1)} + \left( \frac{\ddot{a}_0}{a_0} \right)_{(2)} + O(\rho^3)
\]

\[
= \left( \frac{\ddot{a}_0}{a_0} \right)_{(1)} + \left[ \left( \frac{\ddot{a}_0}{a_0} \right)_{(2)} - 2 \left( \frac{\ddot{a}_0}{a_0} \right)_{(2)} \right] \delta \dot{A}_1 - \delta \ddot{A}_1 + O(\rho^3).
\]

In terms of these variables, we can write the second order Einstein equations (specifically, the linear combinations (00), (00) + \( \frac{n^2}{n_0^2} (ii) \), (05) and (55), in that order) as

\[
\left( \frac{\dot{a}_0}{a_0} \right)_{(2)}^2 b_0^2 e^{2A_0} = 4A'_0 \Psi_2 - \Psi'_1 + F_\Psi
\]

\[
2 \left( \left( \frac{\dot{a}_0}{a_0} \right)_{(2)}^2 - \left( \frac{\ddot{a}_0}{a_0} \right)_{(2)} \right) b_0^2 e^{2A_0} = -4A'_0 \Upsilon_2 + \Upsilon'_1 + F_\Upsilon
\]

\[
0 = - \left( \frac{\dot{a}_0}{a_0} \right)_{(1)} \Upsilon_2 + \Psi_2 + F_{05}
\]

\[
\left( \left( \frac{\dot{a}_0}{a_0} \right)_{(2)}^2 + \left( \frac{\ddot{a}_0}{a_0} \right)_{(2)} \right) b_0^2 e^{2A_0} = A'_0 (4\Psi_2 + \Upsilon_2) + \frac{k^2}{3} \left( \Phi'_0 \delta \Phi_2 - \Phi_0 \delta \Phi'_2 + \Phi'_0 \delta \frac{b_0}{b_0} \right) + F_{55}
\]

and the scalar field equation as

\[
\delta \Phi''_2 = (4\Psi_2 + \Upsilon_2) \Phi'_0 + \left( \frac{4k^2}{3} \Phi'^2_0 + b_0^2 \Phi''(\Phi_0) \right) \delta \Phi_2 + (2b_0^2 V''(\Phi_0) + 4A'_0 \Phi'_0) \delta \frac{b_0}{b_0}
\]
\[ + \Phi'_0 \frac{\delta b'_2}{b_0} + \mathcal{F}_\Phi \]  

(2.19)

where all the dependence on first order quantities squared is contained in the functions \( \mathcal{F}_\Psi, \mathcal{F}_\Upsilon, \mathcal{F}_{05}, \mathcal{F}_{55} \) and \( \mathcal{F}_\Phi \), which can be found in appendix A.

We have used the symbolic manipulation features of Matlab and Maple to arrive at this and the other results in this paper, giving us confidence that there are no algebraic errors, despite the complexity of the formulas. We emphasize that all the terms with subscripts “1” (or \( \frac{1}{2} \)) are already explicitly known from the solution given by ref. [15]. The unknown quantities, with subscripts “2” (or \( \frac{3}{2} \)) thus constitute a relatively manageable part of these complicated looking equations. Having thus derived the second order equations, we are now ready to find their solutions.

3. Solutions

We wish to solve the perturbed Einstein equations (2.13-2.18) for the unknown quantities \((\dot{a}_0/a_0)^2\) and \((\dot{a}_0/a_0)\), which are needed to find the \( \rho^2 \) corrections to the Friedmann equations on the TeV brane. To simplify this task, we will henceforth work in the limit of stiff brane potentials, \( \lambda_i \to \infty \), which ensures that \( \delta \Phi_i = 0 \) at each brane. This allows us to choose a gauge in which \( \delta \Phi_1 \) vanishes everywhere in the bulk [17], and to solve explicitly for \( b_1 \):\(^2\)

\[ \frac{\delta b_1}{b_0} = \frac{b_0}{2\Phi'_0} \left[ \Omega^4 (\rho - 3p) + (\rho_\ast - 3p_\ast) (G - A'_0 e^{4A_0}) \right] \]  

(3.1)

where

\[ G(y) = \left[ \frac{1}{2} e^{2A_0(y)} + A'_0 e^{4A_0(y)} \int_0^y e^{-2A_0} dy \right] / \int_0^1 e^{-2A_0} dy \]  

(3.2)

The other terms which are first order in \( \rho \) were also found in [17] to be:

\[ \Psi_1 = \frac{\kappa^2 b_0}{6(1 - \Omega^2)} e^{4A_0(y)} \left( F(y)(\Omega^4 \rho + \rho_\ast) - (\Omega^4 \rho + \Omega^2 \rho_\ast) \right) \]  

(3.3)

\[ \Upsilon_1 = \frac{\kappa^2 b_0}{2(1 - \Omega^2)} e^{4A_0(y)} \left( -F(y)(\Omega^4 \rho + p + \rho_\ast + p_\ast) + (\Omega^4 (\rho + p) + \Omega^2 (\rho_\ast + p_\ast)) \right) \]  

(3.4)

where

\[ F(y) = 1 - (1 - \Omega^2) \frac{\int_0^y e^{-2A_0} dy}{\int_0^1 e^{-2A_0} dy} \]  

(3.5)

and

\[ \Omega = e^{-A_0(1)} \]  

(3.6)

is the warp factor evaluated at the TeV brane.

\(^2\)In fact, the procedure of using a small coordinate transformation to set \( \delta \Phi_1 = 0 \) can be reiterated to set \( \delta \Phi_i(t, y) = 0 \) to all orders, so that all effects of \( \rho \) and \( \rho_\ast \) on the bulk stress-energy component \( T_{55} \) are contained in \( b_i(t, y) \).
where we have normalized $A_0(0) = 0$, and introduced

$$\epsilon = \sqrt{4 + \frac{m^2}{k^2} - 2} \cong \frac{m^2}{4k^2}. \quad (3.8)$$

Here

$$k = \sqrt{-\kappa^2 \Lambda/6} \quad (3.9)$$

is the inverse AdS curvature scale, which should be somewhat below the Planck scale so that higher derivative operators like $R^2$ do not invalidate the solution.

The above solution is an approximate one in the limit of small $v_0$ for the model considered by GW [22–23], where the bulk scalar $\Phi$ has only a mass term. It was shown by ref. [23] that this solution is actually exact when $\Phi$ has a quartic interaction which is related to the mass in a certain way. We are therefore justified in considering (3.7) to be exact, for the appropriate choice of $V(\Phi)$, in all that will follow. If on the other hand we decided to use the pure $m^2 \Phi^2$ potential for $V(\Phi)$, the solutions (3.7) receive corrections which are a power series in $\kappa^2 v_0^3 e^{-2k_0 y} v$:

$$A_0(y) = k_0 y^2 + \frac{\kappa^2 \phi_0^2}{12} (e^{-2k_0 y} - 1) + \frac{\kappa^4 \phi_0^4}{96(1 + \epsilon)} (e^{-2k_0 y} - 1) + \ldots$$

$$\phi_0(y) = \phi_0 e^{-k_0 y} + \frac{\kappa^2 \phi_0^3 \epsilon}{12(1 + \epsilon)} e^{-3k_0 y} + \ldots \quad (3.10)$$

Here $\phi_0$ differs from $v_0$ by terms of order $\kappa^2 v_0^3$, to satisfy the boundary condition $\phi_0(0) = v_0$. We shall find that none of our main results depend on which bulk scalar potential is used.

As was previously shown, the first two Einstein equations yield differential equations for $\Psi_2$ and $\Upsilon_2$. They are solved by:

$$\Psi_2 = e^{4A_0(y)} \left[ C_\Psi + \int_0^y e^{-4A_0(y)} \left( F_\Psi(y, t) - \left( \frac{\dot{a}_0}{a_0} \right)_2^2 b_0^2 e^{2A_0} \right) dy \right] \quad (3.11)$$

$$\Upsilon_2 = e^{4A_0(y)} \left[ C_\Upsilon + \int_0^y e^{-4A_0(y)} \left( 2 \left( \frac{\dot{a}_0}{a_0} \right)_2^2 - \left( \frac{\ddot{a}_0}{a_0} \right)_2 \right) b_0^2 e^{2A_0(y)} - F_\Upsilon(y, t) \right) dy \right]. \quad (3.12)$$

We do not require the $\Phi$ equation of motion, since this is derivable from the Einstein equations. The boundary conditions (2.9, 2.10) allow us to eliminate the constants of integration $C_\Psi$ and $C_\Upsilon$ to find $(\ddot{a}_0/a_0)_{(2)}$ and $(\dot{a}_0/a_0)_{(2)}$:

$$\left( \frac{\dot{a}_0}{a_0} \right)_2^2 = \frac{1}{b_0^2 \int_0^1 e^{-2A_0} dy} \left[ \Psi_2(0) - \Omega^4 \Psi_2(1) + \int_0^1 e^{-4A_0(y)} F_\Psi dy \right] \quad (3.13)$$

$$\left( \frac{\dot{a}_0}{a_0} \right)_2 - \left( \frac{\ddot{a}_0}{a_0} \right)_2 = \frac{1}{2b_0^2 \int_0^1 e^{-2A_0} dy} \left[ \Omega^4 \Upsilon_2(1) - \Upsilon_2(0) + \int_0^1 e^{-4A_0(y)} F_\Upsilon dy \right]. \quad (3.14)$$
Notice that the terms $\Psi_2(0)$, $\Psi_2(1)$, $\Upsilon_2(0)$, $\Upsilon_2(1)$, are given in terms of $\rho$, $\rho_*$, $p$ and $p_*$ by the jump conditions (2.9-2.10). We will denote the equations of state for matter on the branes by

$$p = \omega \rho, \quad p_* = \omega_* \rho_*.$$  \hfill (3.15)

To evaluate the integrals of $\mathcal{F}_\Psi$ and $\mathcal{F}_\Upsilon$ in (3.13-3.14), we need the first order solutions $\Psi_1$, $\Upsilon_1$, given by (3.3-3.4), as well as $\delta A_1$ and $\delta N_1$. These can be obtained from the definitions (2.11) and the choice of gauge $\delta \Phi = 0$, as $\delta A_1(y) = \int_0^y (A'_0 + \Psi_1(y)) dy$ and $\delta N_1(y) = \delta A_1 + \int_0^y \Upsilon_1(y) dy$. Moreover, we need $(\dot{a}_0/a_0)(1/2)$, given by (2.12), and, time derivatives of the first order perturbations, $\delta \dot{A}_1$, $\delta \dot{N}_1$, $\delta b_1$. To reexpress the latter in terms of $\rho$ and $\rho_*$, we use the following relations, which can be derived from the (05) Einstein equation

$$\dot{\rho} = -3 \frac{\dot{a}_0}{a_0} (1 + \omega) \rho + O(\rho^{5/2}) \cong -\sqrt{24\pi G (\rho + \Omega^4 \rho)(1 + \omega)} \rho$$

$$\dot{\rho}_* = -3 \frac{\dot{a}_0}{a_0} (1 + \omega_*) \rho_* + O(\rho_*^{5/2}) \cong -\sqrt{24\pi G (\rho_* + \Omega^4 \rho)(1 + \omega_*)} \rho_*$$

$$\ddot{\rho} \cong 12\pi G (1 + \omega) \rho \left[3(1 + \omega)\Omega^4 \rho + (3 + \omega_*) + 2\omega \right]$$

$$\ddot{\rho}_* \cong 12\pi G (1 + \omega_*) \rho_* \left[3(1 + \omega_*)\rho_* + (3 + \omega_*) + 2\omega \right]$$  \hfill (3.16)

where we have assumed that $\omega$ and $\omega_*$ are constant in time. (This assumption would only be important during a period of transition such as going from radiation to matter domination.) In these expressions, the 4-D effective Newton’s constant is defined by integrating the 5-D gravitational action over the extra dimension to obtain

$$8\pi G = \kappa^2 \left(2b_0 \int_0^1 e^{-A_0} dy \right)^{-1}. \hfill (3.17)$$

We now have general expressions allowing us to obtain the second order corrections to the Friedmann equations. However the integrals appearing in these expressions (of the form $\int e^{\text{const.} \times A_0} dy$) cannot be performed analytically for $A_0$ given by (3.7). To overcome this difficulty, we will need to make a further approximation, namely that the radion mass is small. In the small-$v_0$ limit, the radion mass was found to be

$$m_r^2 \cong \frac{4}{3} \kappa^2 \varepsilon^2 \Omega^2 + 2\varepsilon$$  \hfill (3.18)

in ref. [14], which also agrees up to factors of order unity with the result found by ref. [24]. By expanding in the small parameter $\eta \equiv \kappa^2 v_0^2/12$, we can express the warp factor as $e^{cA_0} = e^{(k b_0 y + \eta e^{-2 k b_0 y} - 1)} = e^{k b_0 y} \sum_n \frac{1}{n!} (c \eta e^{-2 k b_0 y} - 1)^n$. The integrals can then be performed exactly, order by order in powers of $v_0^2$. In fact, the appearance of $1/\Phi^2_0$ in (3.1) means that the leading order term in this expansion will actually be $O(v_0^2)$.

At this point we are ready to combine the terms like $(\dot{a}_0/a_0)(2)$ and $(\dot{a}_0/a_0)(3)$ with those coming from $(\dot{a}_0/a_0)(1/2)\delta A_1$ and $\delta \dot{A}_1$, to obtain the complete results for $\dot{a}/a$ and $\dot{a}/a$, as dictated by eqs. (2.13-2.14). However, these are not yet the physical Hubble rate nor acceleration when evaluated on the TeV brane, due to a further correction from $\delta N_1$ that must be applied, and which will be explained in the next section. Because $\delta A_1$ and $\delta N_1$ depend on $y$, the rates $\dot{a}/a$ as measured by observers on the two branes will differ.
4. Physical Friedmann Equations on the Branes

Although the main results found above, (3.13)-(3.14), will resemble the Friedmann equations when expressed as functions of the energy densities on the branes (after being combined with the appropriate corrections from $\delta \dot{A}_1$ and $\delta \ddot{A}_1$), they are not yet written in terms of the standard Friedmann-Robertson-Walker (FRW) time variable for TeV brane observers, which we shall denote by $\tau$. On the TeV brane, the lapse function $n^2(t,1)$ has also received corrections of the kind we are interested in. Therefore we should transform to $\tau$ using

$$d\tau = \frac{n(t,1)}{n_0(t,1)} dt = e^{-\delta N_1(t,1) - \delta N_2(t,1) - \cdots} dt$$

(4.1)

where $n_0(t,1) = a_0(t,1) = \Omega$ is the warp factor without any perturbations due to matter.

At first it may not be obvious why we should use only the perturbations, $\delta N_i$, to relate $\tau$ to $t$ instead of using the full warp factor. Of course, one is always free to rescale the time coordinate by a constant factor; this cannot change any physical observables. There are several ways of seeing why the above choice is the most straightforward one. First, it gives the same time coordinate as that in which the resolution of the gauge hierarchy problem was couched in the original RS paper [2]. Specifically, using the time coordinate as defined above, in the absence of the cosmological perturbation, the action for a free scalar field on the TeV brane has the form

$$S = \frac{1}{2} \int d^4x \Omega^4 \left( \dot{\phi}^2 - (\nabla \phi)^2 - m_0^2 \phi^2 \right)$$

$$= \frac{1}{2} \int d^4x \left( (\dot{\phi}^2) - (\nabla \phi)^2 - \Omega^2 m_0^2 \phi^2 \right)$$

(4.2)

where $\phi = \Omega \phi$. This was the argument used to show that a bare mass of $m_0 \sim M_p$ translates into a physical mass of $m = \Omega m_0$. If on the other hand we had defined $d\tau = \Omega dt$ as we might have been tempted to do in (4.1), we would get

$$S \to \frac{1}{2} \int d\tau d^3x \Omega^3 \left( (\dot{\phi}^2) - \Omega^{-2}(\nabla \phi)^2 - m_0^2 \phi^2 \right)$$

$$= \frac{1}{2} \int d\tau d^3x \left( (\dot{\phi}^2) - \Omega^{-2}(\nabla \phi)^2 - m_0^2 \phi^2 \right)$$

(4.3)

where now $\phi = \Omega^{3/2} \phi$. In these variables it is no longer obvious that physical masses are suppressed relative to the Planck scale. The resolution is that one must also reconsider the 4-D effective gravitational action in the new time coordinate. The result is that the 4-D Planck scale is exponentially larger than the underlying 5-D gravity scale, $(8\pi G)^{-1} = \Omega^{-2} \kappa^{-2}/k$. In this picture, the fundamental scales are all taken to be of order TeV, while the 4-D Planck scale is enhanced by $\Omega^{-1}$. Another, simpler, way of deriving the same result is to do everything with the warp factor defined to be unity on the TeV brane, and large on the Planck brane. In either method, the ratio of physical particle masses to the 4-D Planck mass is the same.

The consequence of all this is simple: to find the physical Hubble rate at some position in the bulk $y$ in terms of the complete expression for $\dot{a}/a$, one merely multiplies by $dt/d\tau =$
\[ e^{\delta N_1 + \delta N_2} |_{y}\quad: \quad H \cong \left(1 + \delta N_1\right) \frac{\dot{a}}{a} \quad (4.4) \]

Since the leading contribution to \( H \) is already \( O(\rho^{1/2}) \), we can ignore the \( \delta N_2 \) corrections; these would be \( O(\rho^{5/2}) \). Similarly, the second Friedmann equation, which goes like \( \dot{H} \), becomes

\[
\frac{dH}{d\tau} = (1 + \delta N_1) \frac{d}{dt} H \\
\cong (1 + 2\delta N_1) \left( \frac{\ddot{a}}{a} - \left( \frac{\dot{a}}{a} \right)^2 \right) + \delta \dot{N}_1 \frac{\dot{a}_0}{a_0} \quad (4.5)
\]
to the order we are calculating. All dots are still time derivatives with respect to \( t \), not \( \tau \).

The general expressions for the order \( \rho^2 \) corrections to \( H^2 \) and \( dH/d\tau \) are rather cumbersome if expressed at an arbitrary position in the bulk. Here we give the results for \( H \) evaluated on either the TeV or the Planck brane, to leading order in the radion mass squared, \( m^2_r \), eq. (3.18). We have defined \( \bar{\rho} = \Omega^4 \rho \) since this is the physically observable energy density on the TeV brane, whereas \( \rho \) is the bare value. For each term \( \rho^2, \rho^* \rho, \bar{\rho}^2 \), we give only the leading dependence on the warp factor \( \Omega \), which is usually assumed to be \( \ll 1 \) to solve the hierarchy problem. Thus we have found the leading corrections to the Hubble rate and acceleration in a simultaneous expansion in \( \rho, \rho^*, 1/m_r \) and \( \Omega \):³

\[
H^2 |_{y=1} = \frac{8\pi G}{3} \left( \bar{\rho} + \rho_* + \frac{2\pi G}{3m^2_r\Omega^2} \left( 9(1 - 3\omega)(1 + \omega)\bar{\rho}^2 + 4(1 - 3\omega)(4 + 3\omega)\rho_*^2 \Omega^2 + 4(1 - 3\omega)(4 + 3\omega)\rho_*^2 \right) \right) \quad (4.6)
\]

\[
H^2 |_{y=0} = \frac{8\pi G}{3} \left( \bar{\rho} + \rho_* + \frac{2\pi G}{3m^2_r\Omega^2} \left( 9(1 - 3\omega)(1 + \omega)\rho_*^2 \Omega^4 - (1 - 3\omega)(7 + 3\omega)\bar{\rho}^2 + 2\Omega^2 \rho_*^2 [2(1 - 3\omega)(2 + 3\omega) - 3(1 - 3\omega)(1 + \omega)] \right) \right) \quad (4.7)
\]

\[
\left. \frac{dH}{d\tau} \right|_{y=1} = -4\pi G \left( \bar{\rho}(1 + \omega) + \rho_* (1 + \omega_*) \right) + \frac{4\pi G}{3m^2_r\Omega^2} \left( \Omega^2 (1 + \omega_*)(1 - 3\omega_*)(13 + 9\omega_*)\rho_*^2 + 9(1 - 3\omega)(1 + \omega)^2 \rho_*^2 \right) \left( 1 - 3\omega)(2(1 + \omega) + 6(1 + \omega)^2 + 3(1 + \omega)(1 + \omega_*)\bar{\rho} \rho_* \right) \right) \quad (4.8)
\]

\[
\left. \frac{dH}{d\tau} \right|_{y=0} = -4\pi G \left( \bar{\rho}(1 + \omega) + \rho_* (1 + \omega_*) \right) + \frac{4\pi G}{3m^2_r\Omega^2} \left( -4(1 + \omega)(1 - 3\omega)\rho_*^2 + 9(1 + \omega_*)^2 (1 - 3\omega_*)^2 \right) \right) \]

³We have factored these expressions to make it more clear that they vanish when \( \omega = \omega_* = 1/3 \). The asymmetrical appearance of terms like \( 2(1 - 3\omega)(2 + 3\omega) - 3(1 - 3\omega)(1 + \omega) \) is not due to a typographical error.
\[ \Omega^2 \left[ 6(1 - 3\omega)(1 + \omega)^2 + 2(1 - 3\omega)(1 + \omega) - 2(1 - 3\omega)(1 + \omega_s) \\
- 4(1 - 3\omega_s)(1 + \omega) + 3(1 + \omega_s)(1 - 3\omega)(1 + \omega) \right] \rho_s \rho \] (4.9)

These, finally, can be considered to be the main results of this paper, except in the case of \( \omega = \omega_s = 1/3 \). As can be seen, in that case the quadratic corrections vanish at order \( 1/m_r^2 \), so we have to go to higher order in \( m_r^2 \) to get the leading result for radiation, as will be described below. Two comments are in order. (1) These results are independent of the choice of bulk potential \( V(\Phi) \) for the scalar. Any dependence on \( V \) comes in starting at the next order in \( v_0^2 \). (2) It may seem strange that on the Planck brane (\( y = 0 \)) powers of \( \Omega^2 \) track powers of \( \rho \), but not so on the TeV brane. The difference is due to the corrections to \( \frac{\dot{\rho}}{\rho_0} \) from \( \delta A_1 \sim H \delta A_1 \), eq. (2.13), and \( \delta N_1 \), eq. (4.4). They give contributions to \( H^2 \) of order \( H^2 \times [\delta A_1(1), \delta N_1(1)] \). Since \( H^2 \sim \rho_s + \bar{\rho} \) at lowest order, the correlation of powers of \( \Omega^2 \) and powers of \( \rho \) is no longer respected.

Equipped with these results, we can now specialize to several situations of cosmological interest.

### 4.1 Inflation

The earliest epoch of interest is an inflationary one, where the equation of state is \( \omega = -1 \) or \( \omega_s = -1 \), depending on whether we put the inflaton on the TeV or the Planck brane. Let us first suppose the inflaton is on the the TeV brane, so that \( \rho_s = 0 \). Then eq. (4.6) tells us that there is no quadratic correction to the Hubble rate at \( y = 1 \), in contrast to the situation in pure RS-II (single brane) cosmology. Not only is this true for the leading term in the expansion in \( m_r^2 \), but in fact we have checked that it is true order by order for the terms \( 1/m_r^2 \) and \( m_r^0 \). The fact that many terms must seemingly miraculously cancel to achieve this suggests that it is true to all orders.

The analogous statements hold true for an inflaton on the Planck brane: if \( \bar{\rho} = 0 \) and only \( \rho_s \) is nonzero, there is no quadratic correction to \( H^2 \) at \( y = 0 \). Again, we have shown this to be exactly true for the first few orders in an expansion in \( m_r^2 \). There is such a correction to \( H^2 \) at the TeV brane, but unfortunately this has no impact on the rolling of a scalar field on the Planck brane, since its own Hubble rate is unaffected. Thus we cannot take advantage of the modification to the Friedmann equation to admit steeper inflationary potentials than are normally allowed by the slow-roll conditions, as was explored in refs. [3, 6]. Only if both \( \rho_s \) and \( \bar{\rho} \) are nonzero do we get this kind of effect (through the \( \rho_s \bar{\rho} \) term in eq. (4.7)). But even there, its sign tends to be the wrong one for the desired purpose. For example, suppose that some of the inflaton’s energy density has been converted energy on the TeV brane, with an arbitrary value of \( \omega \) while \( \omega_s = -1 \). Then (4.7) predicts that \( H^2 \) is decreased by the quadratic corrections, regardless of the value of \( \omega \).

Even though there might be no presently observable effect, it is still of theoretical interest to consider the modification to the TeV brane’s Hubble rate due to a Planck brane inflaton, to contrast with simpler models of brane cosmology. With \( \rho = 0 \) but \( \rho_s \neq 0 \) and
\( H^2 \rvert_{y=1} \) is corrected by the factor

\[
H^2 \rvert_{y=1} = \frac{8\pi G}{3} \rho_\ast \left( 1 + \frac{32\pi G}{3m_r^2} \rho_\ast \right) \tag{4.10}
\]

The correction becomes important at temperatures of order the intermediate scale \( T \sim \sqrt{m_r M_p} \sim 10^{10} \text{ GeV} \). We can compare this to the situation where there is no stabilizing scalar field, but a fine-tuning \( \rho = -\rho_\ast /\Omega^2 \) is required between the energy densities on the two branes, in order to keep the bulk from expanding. In this case, the relation (1.1) applies to the given order at both branes, since for \( \omega_\ast = -1 \) the difference \( H^2(0) - H^2(1) = O(\rho_\ast^4) \) \[11\]. But this correction only becomes important when \( T \sim M_p \) in the unstabilized case, since the natural scale for the Planck brane tension is \( M_p^4 \).

The effect in the present case is easy to understand in terms of the usual slow roll condition for scalar fields during inflation. If the Hubble rate exceeds the mass of the radion, it will start rolling, hence the size of the extra dimension is destabilized, which in turn changes the strength of gravity from the 4-D point of view, due to the dependence of \( G \) on \( b_0 \) in (3.17). The criterion that \( H < m_r \) translates to \( T < \sqrt{m_r M_p} \), which is precisely the condition we found for the \( \rho_\ast^2 \) correction to \( H^2 \) to be small. On the other hand, the reader should not be misled into thinking that the \( \rho_\ast^2 \) term can be deduced from this effect (Hubble expansion giving an \( m_r^2 \)-dependent shift in \( b \), leading to a shift in \( G \) alone). The coefficient of \( \rho_\ast^2 \) is numerically different from that which is obtained solely from the effect of shifting \( G \).

We noted that an inflaton on the TeV brane gives no \( \tilde{\rho}^2 \) correction to \( H^2 \) measured on the TeV brane. A similar story applies for an observer on the Planck brane: he sees no \( \rho_\ast^2 \) correction to \( H^2 \) on his brane. Again, we find this to be true order by order in an expansion about \( m_r^2 = 0 \), so it is presumably an exact statement. Moreover we have found this to be true for either of the choices of the bulk potential \( V(\Phi) \) that we have considered. At first sight this may be a surprising result, because it holds even in the limit that the infrared brane is removed to \( y \to \infty \). In this limit we might expect physics to coincide with that of the RS-II model, where the \( \rho_\ast^2 \) correction to \( H^2 \rvert_{y=0} \) (eq. (1.1)) is well known to occur, at least in the case of no bulk scalar field. We will explain the apparent discrepancy in a subsection below.

### 4.2 Radiation era

Our results (4.7, 4.8) predict that there is no effect of order \( \rho^2 /m_r^2 \) when the equation of state on the two branes is that of radiation, \( \omega = \omega_\ast = 1/3 \). The origin of the \( 1/m_r^2 \) dependence is the large shift in \( \delta b_1 \) (1.1) for generic equations of state. In the special case of radiation, \( \delta b_1 = 0 \) since the radion couples to the trace of the stress energy tensor, which vanishes in this case. We have therefore done a separate treatment when \( \omega = \omega_\ast = 1/3 \) to find the leading effect. We obtain

\[
H^2 \rvert_{y=1} = \frac{8\pi G}{3} \left( \rho_\ast + \tilde{\rho} + \frac{2\pi G}{3k^2 \Omega^2} (\Omega^2 \rho_\ast - \tilde{\rho}) \rho_\ast \right) \tag{4.11}
\]

\[
H^2 \rvert_{y=0} = \frac{8\pi G}{3} \left( \rho_\ast + \tilde{\rho} - \frac{2\pi G}{3k^2 \Omega^2} (\Omega^2 \rho_\ast - \tilde{\rho}) \tilde{\rho} \right) \tag{4.12}
\]
and the equations for \( dH/d\tau \) have vanishing corrections at this order in \( \rho \) and \( \rho_* \). It is curious that, just like in the previously discussed case of inflation, the Hubble rate on a given brane receives an order \( \rho^2 \) correction only if the energy density on the other brane is nonvanishing.

We can make the preceding results much stronger in the case where the energy density vanishes on one of the two branes. If \( \rho_* = 0 \), then eq. (3.13) implies that the second order correction to the Hubble rate on the TeV brane vanishes identically, with no approximations, provided that

\[
\int_0^1 \left[ e^{4A_0(y_1)} \left( \int_0^{y_1} e^{-2A_0(y_2)} dy_2 \right) \left( \int_0^{y_1} e^{-2A_0(y_3)} dy_3 \right) - e^{4A_0(y_1)} \left( \int_0^{y_1} e^{-2A_0(y_2)} dy_2 \right)^2 \right. \\
- \left. e^{-2A_0(y_1)} \left( \int_0^{y_1} e^{4A_0(y_2)} \int_0^{y_2} e^{-2A_0(y_3)} dy_3 \right) dy_2 \right] dy_1 = 0 
\]

(4.13)

In the inverse case where \( \rho = 0 \) and \( \omega_* = 1/3 \), the second order correction to \( H^2 \) evaluated on the Planck brane vanishes if

\[
\int_0^1 \left[ e^{4A_0(y_1)} \left( \int_0^{y_1} e^{-2A_0(y_2)} dy_2 - \int_0^{y_1} e^{-2A_0(y_2)} dy_2 \right)^2 \\
+ e^{-2A_0(y_1)} \left( \int_0^{y_1} e^{4A_0(y_2)} \left( \int_0^{y_2} e^{-2A_0(y_3)} dy_3 \right) dy_2 \right) \left( \int_0^{y_2} e^{-2A_0(y_3)} dy_3 \right) dy_2 \right] dy_1 = 0
\]

(4.14)

Writing

\[
u_1 = \int_0^{y_1} \left( e^{4A_0(y_2)} \int_0^{y_2} e^{-2A_0(y_3)} dy_3 \right) dy_2 \\
u_2 = \int_0^{y_1} e^{4A_0(y_2)} \left( \int_0^{y_2} e^{-2A_0(y_3)} dy_3 \right) \left( \int_0^{y_2} e^{-2A_0(y_3)} dy_3 \right) dy_2 \\
dv = e^{-2A_0(y_1)} dy_1
\]

(4.15)

where \( u_1 \) and \( u_2 \) are to be used in the first and second case respectively, we can use integration by parts (\( \int u_i dv = u_i v - \int v du_i \)) to show that these integrals do indeed vanish identically.

Let us now focus on physics on the TeV brane \( (y = 1) \). By the time of nucleosynthesis no more than 10% of the radiation energy density can be on the Planck brane, so one expects that \( \rho_* \lesssim \bar{\rho} \) at TeV temperatures as well. Therefore we can ignore the \( \Omega^2 \rho_* \) term, and the fractional correction to \( H^2 \) is of order

\[
\frac{\delta H^2}{H^2} - 1 \sim - \left( \frac{\rho_*}{\text{TeV}^4} \right)
\]

(4.16)

It is interesting that the order of magnitude is nearly correct for this to be relevant at the time of the electroweak phase transition. (We could change TeV \( \rightarrow 100 \) GeV by making \( k \) a few orders of magnitude smaller than \( M_p \).) However the minus sign shows that the Hubble rate is suppressed relative to standard cosmology. This is disappointing for electroweak baryogenesis, since one would have liked to increase \( H^2 \) make sphalerons go
out of equilibrium at the electroweak phase transition, a possibility that has been recently analyzed in ref. [26]. Of course, it may happen that the sign turns out to be the right one in some other model of brane cosmology.

4.3 Relation to RS-I model without stabilization and to RS-II

As a check on our results, one would like to verify that they are consistent with related cosmological solutions. For example, when there is no scalar field and the equation of state is inflationary, Kaloper [27] found the exact solution

\[ ds^2 = a^2(w) \left(-dt^2 + e^{2Ht}d\vec{x}^2\right) + dw^2; \]
\[ a(w) = e^{-kw} - \hat{\rho}_* \sinh kw \]

(4.17)

where \( \hat{\rho}_* = \rho_*/T \), the ratio of the excess energy density to the tension on the Planck brane, and the Hubble rate is given by

\[ H^2 = 2k^2((1 + \hat{\rho}_*)^2 - 1) \]

(4.18)

As in the static case, one can decide whether or not to compactify the extra dimension by inserting a second brane at some position \( w_1 \). Without compactification, the extra dimension is still effectively cut off by the presence of a horizon at \( w = \frac{1}{2k} \ln(1 + \frac{2}{\rho_*}) \). If we compactify explicitly by inserting a second brane, its tension \((-T)\) and excess energy density \( \rho \) are completely fixed by the value of \( a'(w_1) \), leading to the constraint

\[ \hat{\rho} \equiv \rho \frac{T}{\rho_*} = -1 + \frac{e^{-kw_1} - \hat{\rho}_* \cosh kw_1}{e^{-kw_1} - \hat{\rho}_* \sinh kw_1} \]

(4.19)

Expanding in \( \rho_* \), to leading order this gives the well-known constraint [11] \( \rho = -\rho_*/\Omega^2 \) needed for a static bulk in the absence of stabilization (recall that the \( \Omega = e^{-kw_1} \) is the warp factor). However, the exact tuning is a power series in \( \rho_* \), and our second order formalism correctly reproduces the \( O(\rho_*^2) \) term, as we describe in more detail in appendix C. Of course, our perturbative treatment also correctly reproduces the solution \( a(w) \) and \( a_0(t) \) expanded in powers of \( \rho_* \).

The previous check was specifically for the equation of state \( \omega_* = -1 \). The same can be done for an arbitrary equation of state, for which the (27) solution generalizes to

\[ ds^2 = -n^2(t,w)dt^2 + a^2(t,w)d\vec{x}^2 + dw^2; \]
\[ a(t,w) = a_0(t)\left(e^{-kw} - \hat{\rho}_*(t) \sinh kw\right); \]
\[ n(t,w) = e^{-kw} + \hat{\rho}_*(t)(2 + 3\omega_*) \sinh kw, \]

(4.20)

where \( a_0(t) \) is determined by the modified Friedmann equation [18] with \( H = \frac{\dot{a}a}{a_0} \). Again, our perturbative formalism agrees with the expansion of the exact solution at order \( \rho_*^2 \). It is interesting to note in passing that for \( \omega_* > -1 \), the metric function \( a(t,y) \) vanishes at a

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4This is related to the coordinate system we have been using by \( w = by \), where now \( b \) is taken to be constant by fiat.
position between the brane at $y = 0$ and the horizon where $n(t, y) = 0$. Nevertheless, all components of the 5D Riemann tensor are well behaved throughout the bulk, and neither the vanishing of $a(t, y)$ nor $n(t, y)$ leads to any curvature singularities.

Now we come to a slightly more subtle issue: what happens if we try to smoothly remove the TeV brane from the problem, to recover the known single-brane results? We have noted that our results for $H^2$ do not reduce to those of the RS-II model, even in the case where the TeV brane is taken to infinity. The presence of the bulk scalar is crucial for understanding the difference between the two situations, because it modifies the bulk stress-energy component $T_{55}$ in response to matter on the branes. Even though we are working in a gauge where $\delta \Phi_i = 0$, $T_{55} = g_{55}(V(\Phi) + \Lambda) - \frac{1}{2} \Psi^2$ is nevertheless modified by the shift in $g_{55} = b(t, y)^2$. It is the $G_{55} = \kappa^2 T_{55}$ Einstein equation which implies the exotic form of the Friedmann equation $H^2 \sim (\rho_s + T)^2$ when there is no matter in the bulk. This argument no longer holds when $T_{55}$ is present. Therefore it is not so surprising that the $\rho^2_s$ corrections no longer coincide with those of the RS-II model, when we have just a single brane but also a scalar field.

Mathematically, we cannot recover the RS-II model simply by taking the limit $v_0 \to 0$, which corresponds to removing the scalar field from the problem. If we attempt to do this, our perturbation expansion breaks down because $\delta b_1$, eq. (3.1), diverges—unless the matter on the branes is tuned so that the term in brackets in (3.1) vanishes. This tuning is the condition $\rho = -\rho_s/\Omega^2$ discussed above. A special case is that of radiation, where no such breakdown of perturbation theory occurs.

### 4.4 Dark Radiation

In the present work we have neglected the presence of dark radiation [28, 30, 31, 32], which appears in the Friedmann equation in the normal way,

$$H^2 = 2k^2 \left( (\dot{\rho}_s + 1)^2 - 1 + \frac{c}{a_0(t)^4} \right);$$ \hspace{1cm} (4.21)

in particular, there is no quadratic correction $c^2/a_0(t)^8$ even in the unstabilized case. The AdS/CFT interpretation is that the dark radiation is simply the thermalized degrees of freedom of the conformal field theory [17], but from the 5D viewpoint, the dark radiation is associated with the presence of a black-hole-like singularity that appears in the bulk at a finite distance from the brane. This can be most clearly seen in the coordinate system of ref. [31] (see also [33]), where the brane is moving through a static bulk whose geometry is that of the 5D AdS-Schwarzschild metric.

Again using the results of [28], the exact solution takes the form (4.20), but now

$$a^2(t, w) = a_0^2(t) \left( e^{-kw} - \dot{\rho}_s(t) \sinh kw \right)^2 + \frac{c}{a_0(t)^2} \sinh^2 kw;$$

$$n(t, w) = \frac{\dot{a}(t, w)}{a_0(t)} = \frac{a_0(t)}{a(t, y)} \left( \left( e^{-kw} - \dot{\rho}_s(t) \sinh kw \right)^2 + 3\rho_s(t)(1 + \omega_s) \left( e^{-kw} - \dot{\rho}_s(t) \sinh kw \right) \sinh kw - \frac{2c}{a_0^2(t)} \sinh^2 kw \right).$$ \hspace{1cm} (4.22)
From this exact solution we see that the metric functions do not factorize in the way which we assumed in our original ansatz (2.3). A more general form would therefore be needed to study the interactions, if any, between $\rho^2$ corrections and dark radiation in the stabilized model.

5. Summary and conclusions

In this paper we have focused on the corrections to the Friedmann equation which arise in the possibly more realistic 5D brane cosmologies where the bulk stress energy is not trivial, but has some inhomogeneity in the extra dimension due to a scalar field. The presence of the scalar was motivated by the need to stabilize the size of the extra dimension when the extra dimension is compactified by the presence of a second (TeV) brane, but it could also be present on more general grounds.

In a perturbative expansion in powers of energy densities on the branes, the Hubble parameter $H$ generally should receive corrections to all orders in $\rho$ and $\rho_*$ rather than terminating at order $\rho_*^2$ as happens in the unstabilized solutions. We found the rather surprising result that the $\rho^2$ correction vanishes in the special cases where the energy density is confined to the same brane as that where $H$ is measured and the equation of state is either that of inflation or radiation. If either of these conditions are not fulfilled, the $\rho^2$ correction does not vanish, but its sign and magnitude can be different from the case of a single brane with no bulk scalar field. In these cases, the correction has a coefficient of order $(\text{TeV})^{-2} m_r^{-2}$, where $m_r$ is the mass of the radion, which is expected to be somewhat lighter than the TeV scale.

We noted two implications of this result. First, it is impossible to obtain order $\rho^2$ corrections to $H^2$ during inflation on a given brane unless the inflaton is on the opposite brane. This means the corrections have no effect on the dynamics of the inflaton, which is disappointing from the point of view of potentially observing the effects of branes on cosmology. Second, it is equally true during radiation domination that $\rho^2$ effects can appear for a given brane only if the energy density on the other brane is nonzero. If some fraction of the radiation of the universe is on the Planck brane, there could be significant deviations from standard cosmology during the electroweak epoch. Unfortunately, the sign of the corrections is such as to decrease the Hubble rate. It would have been more interesting for the purposes of electroweak baryogenesis to obtain the other sign.

On a more optimistic note, we should reiterate that the exact cosmological behavior cannot be predicted from the $\rho^2$ terms when they become as important as important as the linear in $\rho$ terms, since the perturbative expansion is breaking down. It would therefore be interesting to find exact cosmological solutions in the presence of the stabilizing field.

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A. 1st order functions appearing in 2nd order Einstein equations

These are the functions appearing in eqs. (2.15–2.19), which give the solutions for the second order perturbations.

\[
\mathcal{F}_\psi = 2 \left( \delta N_1 + \frac{\delta b_1}{b_0} \right) \Psi_1 + 2 \Psi_1^2 - \frac{\kappa^2}{6} \Phi_0' \Phi_1' \Phi_1 \left( \frac{\delta b_1}{b_0} + 2 \delta N_1 \right) + \Psi_1 \left( \frac{\delta b_1}{b_0} - 4 A_0' \left( 2 \delta N_1 + \frac{\delta b_1}{b_0} \right) \right) + \frac{\kappa^2}{6} \Phi_0' \left( \frac{\delta b_1}{b_0} \right)^2 + \frac{\kappa^2}{6} \left( \Phi_0' (4 \Psi_1 - \Upsilon_1) + \Phi_0'' \frac{\delta b_1}{b_0} \right) \Phi_1' \delta \Phi_1 + \left( \frac{\dot{a}_0}{a_0} \right) e^{2 A_0 b_0^2} \left( 2 \delta \dot{A}_1 - \frac{\delta b_1}{b_0} \right) \tag{A.1}
\]

\[
\mathcal{F}_\Upsilon = -4 \Upsilon_1 \Psi_1 - \Upsilon_1^2 - 2 \left( \delta N_1 + \frac{\delta b_1}{b_0} \right) \Psi_1' + \left( 4 A_0' \left( 2 \delta N_1 + \frac{\delta b_1}{b_0} \right) - \frac{\delta b_1''}{b_0} \right) \Upsilon_1 \tag{A.2}
\]

\[
\mathcal{F}_{055} = \left( \delta \dot{A}_1 - \frac{\dot{a}_0}{a_0} \right) \Upsilon_1 - \frac{\delta b_1}{b_0} \Psi_1 + \frac{\kappa^2}{6} \left( \Phi_1' - \Phi_0' \frac{\delta b_1}{b_0} \right) \Phi_1 + \left( \Phi_0' \frac{\delta b_1}{b_0} - \delta \Phi_1' \right) \delta \Phi_1 \tag{A.3}
\]

\[
\mathcal{F}_{55} = -2 \left( \delta N_1 + \frac{\delta b_1}{b_0} \right) A_0' \Upsilon_1 + 2 \Psi_1^2 + \left( \Upsilon_1 - 4 A_0' \left( 2 \delta N_1 + \frac{\delta b_1}{b_0} \right) \right) \Psi_1 + \frac{\kappa^2}{3} \left( -2 \Phi_0' \delta N_1 \frac{\delta b_1}{b_0} - \frac{3}{2} \Phi_0' \left( \frac{\delta b_1}{b_0} \right)^2 \right) + \left( \frac{\dot{a}_0}{a_0} \right) e^{2 A_0 b_0^2} \left( \delta \dot{N}_1 - 4 \delta \dot{A}_1 \right) + e^{2 A_0 b_0^2} \frac{\delta \dot{A}_1}{b_0} - \frac{\kappa^2}{6} \delta \Phi_1'^2 + 2 \frac{\kappa^2}{3} \left( \Phi_0' \frac{\delta b_1}{b_0} + \delta \Phi_1' \right) \delta \Phi_1' \tag{A.4}
\]

\[
\mathcal{F}_\Phi = \frac{\delta b_1^2}{b_0^3} \left( -3 \Phi_0' + 4 A_0' \Phi_0' \right) - \Phi_0' \frac{\delta b_1}{b_0} \left( 8 \Psi_1 + 3 \frac{\delta b_1'}{b_0} + 2 \Upsilon_1 \right) + \frac{V''(\Phi_0)}{2} b_0^3 \delta \Phi_1^2 + 2 \kappa^2 \Phi_0' \delta \Phi_1' \left( \frac{\delta b_1}{b_0} \Phi_0' + \delta \Phi_1' \right) + \delta \Phi_1' \left( -4 \frac{\delta b_1}{b_0} A_0' + \Upsilon_1 + 4 \Psi_1 + \frac{\delta b_1'}{b_0} \right) + 2 \Phi_1'' \frac{\delta b_1}{b_0} \tag{A.5}
\]

B. Complete second order corrections

In the previous sections, we presented simplified expression for $H^2$ and $\dot{H}$. Here we present
the complete expressions for the second order in $\rho$ corrections to $O(v_0^{-2})$, evaluated on the branes.

\[ H_2^2(y = 1) = \frac{8\pi^2 G^2(\Omega^{-2\epsilon} - \Omega^4)}{3k^2\kappa^2 v_0^2 e^2(2 + \epsilon)(1 - \Omega^2)\Omega^2} \left[ 9\Omega^6(1 - 3\omega)(1 + \omega)\rho^2 
\right.
\]
\[ + (1 - 3\omega_s)(4(4 + 3\omega_s) - \Omega^2(7 + 3\omega_s))\rho_s^2 
\]
\[ + 2\Omega^2 \left\{ \Omega^2((1 - 3\omega)(1 - 3\omega_s) + 6(1 + \omega)(1 - 3\omega_s)) - 2(1 - 3\omega_s) - 4(1 - 3\omega) \right\} 
\]
\[ + 2(1 - 3\omega) + 6(1 + \omega)(1 - 3\omega) \right\} \rho \rho_s \right] \] (B.1)

\[ H_2^2(y = 0) = \frac{8\pi^2 G^2(\Omega^{-2\epsilon} - \Omega^4)}{3k^2\kappa^2 v_0^2 e^2(2 + \epsilon)(1 - \Omega^2)\Omega^2} \left[ 9(1 - 3\omega_s)(1 + \omega_s)\rho_s^2 
\right.
\]
\[ + \Omega^4(1 - 3\omega)(4\Omega^2(4 + 3\omega) - (7 + 3\omega))\rho^2 
\]
\[ + 2\Omega^2 \left\{ (1 - 3\omega)(1 - 3\omega_s) + 6(1 + \omega)(1 - 3\omega) - 2(1 - 3\omega) - 4(1 - 3\omega_s) \right\} 
\]
\[ + \Omega^2(2(1 - 3\omega_s) + 6(1 + \omega_s)(1 - 3\omega_s)) \right\} \rho \rho_s \] (B.2)

\[ \dot{H}_2(y = 1) = \frac{-8\pi^2 G^2(\Omega^{-2\epsilon} - \Omega^4)}{k^2\kappa^2 v_0^2 e^2(2 + \epsilon)(1 - \Omega^2)\Omega^2} \left[ 9\Omega^6(1 - 3\omega)(1 + \omega)^2\rho^2 
\right.
\]
\[ + (1 - 3\omega_s)(1 + \omega_s)(4(1 - \Omega^2) + 9(1 + \omega_s))\rho_s^2 
\]
\[ + \Omega^2 \left\{ (1 - 3\omega)(2(1 + \omega_s) + 6(1 + \omega)^2 + 2(1 + \omega) + 3(1 + \omega)(1 + \omega_s) \right\} 
\]
\[ + \Omega^2(-2(1 + \omega)(1 - 3\omega_s) + 2(1 + \omega_s)(1 - 3\omega_s) + 3(1 + \omega_s)(1 - 3\omega_s)(1 + \omega) 
\]
\[ - 4(1 - 3\omega)(1 + \omega_s) + 6(1 + \omega_s)^2(1 - 3\omega_s) \right\} \rho \rho_s \] (B.3)

\[ \dot{H}_2(y = 0) = \frac{-8\pi^2 G^2(\Omega^{-2\epsilon} - \Omega^4)}{k^2\kappa^2 v_0^2 e^2(2 + \epsilon)(1 - \Omega^2)\Omega^2} \left[ 9(1 - 3\omega_s)(1 + \omega_s)^2\rho_s^2 
\right.
\]
\[ - \Omega^4(1 - 3\omega)(1 + \omega)(4(1 - \Omega^2) + 9\Omega^2(1 + \omega))^2 
\]
\[ + \Omega^2 \left\{ (1 - 3\omega)(2(1 + \omega_s) + 6(1 + \omega_s)^2 + 2(1 + \omega) + 3(1 + \omega)(1 + \omega_s) \right\} 
\]
\[ - 2(1 + \omega_s)(1 - 3\omega) + 2(1 + \omega)(1 - 3\omega) + 3(1 + \omega)(1 - 3\omega)(1 + \omega_s) 
\]
\[ - 4(1 - 3\omega_s)(1 + \omega) + 6(1 + \omega)^2(1 - 3\omega) \right\} \rho \rho_s \] (B.4)

C. Absence of a scalar field and infinite extra dimension

In order to trust the perturbative approach which was used to derive the results presented in this paper, we must convince ourselves that the same approach allows us to reproduce the well known results of RS-I and RS-II cosmology in the absence of a scalar field. In this section, we will show that this is indeed the case.

In the absence of a stabilization mechanism, the configuration with two parallel branes lying at fixed positions along the extra dimension is generally unstable. In order to look for static solutions, we must impose $b = \text{constant}$ from the start, which means that any derivatives of $b$ are absent from the Einstein equations (2.4).

Working to zeroth order in $\rho$, it is straightforward to show that:

\[ V_0 = -V_1 = \frac{6k}{\kappa^2} \]
\[ A_0 = kb_0 y, \] (C.1)
so that we recover the expected fine-tuning between the branes’ tensions.

The (00) and (00-ii) combinations of the Einstein equations, linearized in \( \rho \), can be solved in the same manner as in [15]. However, the absence of a scalar field means that the (55) equation no longer allows one to solve for \( \delta b \). Rather, plugging the solutions for \( \Psi_1, \Upsilon_1, \left( \frac{\mu}{a_0} \right)_1 \) and \( \left( \frac{\mu}{a_0} \right)_1 \) into the (55) equation leads to the following constraint:

\[
\Omega^4(\rho - 3p) + \Omega^2(\rho_\ast - 3p_\ast) = 0. \tag{C.2}
\]

The only way to satisfy this constraint without running into inconsistencies at higher order in \( \rho \) is to set

\[
\rho = -\Omega^{-2}\rho_\ast; \quad \omega_\ast = \omega = -1, \tag{C.3}
\]

where the second equation is found by demanding that \( \frac{d}{dt} \left( \frac{\rho}{\rho_\ast} \right) \) be constant in time, as dictated by the first equation.

In short, setting \( \Phi \) and \( \delta b \) to zero has made our system of equations over-determined: the (00), (00-ii) and (55) Einstein equations constitute a system of three equation for two unknowns. The way out of this problem consists of turning one of our parameters (\( \rho \) and \( \rho_\ast \)) into a variable to be expanded in powers of the other. We will therefore choose to write \( \rho = \rho_1 + \rho_2 + \ldots \), where the subscripts indicate the order in powers of \( \rho_\ast \), and \( \rho_1 \) is given by (C.3).

We can then repeat the same steps (solving the first two Einstein equations and plugging the results in the (55) equation) for the equations at second order in \( \rho_\ast \) to find:

\[
\rho_2 = -\frac{\kappa^2(1 - \Omega^2)}{12k\Omega^4}\rho_\ast^2. \tag{C.4}
\]

These results for the relation between \( \rho \) and \( \rho_\ast \) agree with the exact results presented in [27], when the latter are expanded as a powers series in \( \rho_\ast \).

Imposing these constraints we find the Hubble rate on the Planck brane to be

\[
H^2|_{y=0} = \frac{8\pi G}{3}(1 - \Omega^2)\rho_\ast \left[ 1 + \frac{\kappa^2}{12k}\rho_\ast \right] + O(\rho_\ast^2) \tag{C.5}
\]

with \( 8\pi G \) given by (3.17). Using (C.4) and the fact that \( \Omega \to 0 \) as \( b_0 \to \infty \), we see that we do indeed recover the expected RS-II behaviour (1.1) when the second brane is taken to infinity. We emphasize however that this is only true insofar as \( \rho_1 \) and \( \rho_2 \) are written as in (C.3) and (C.4). If we had instead decided to take the TeV brane out of the picture by setting \( \rho = 0 \) at the start, we would not have gotten this result. Specifically, there would be a missing factor of 1/2 in the second order correction, and more importantly, the constraint equations (C.3, C.4) would demand that \( \rho_\ast = 0 \).

Suppose now that we start out with a single brane with an extra dimension of infinite size. The difference between this approach and the one described above lies in the fact that setting \( \rho = 0 \) also sets the boundary conditions for \( \Psi \) and \( \Upsilon \) at \( y = 1 \) to be \( \Psi(y = 1) = \Upsilon(y = 1) = 0 \). Here however, since the second brane is explicitly absent from the setup, we only have one set of boundary conditions on \( \Psi \) and \( \Upsilon \), i.e. these two variables are only fixed on the single brane, located at \( y = 0 \).
We start by redefining the extra dimension’s coordinate \( y \) as
\[
\hat{y} = b_0 y,
\]
so that \( \hat{y} \) goes from 0 to \( b_0 \) as \( y \) goes from 0 to 1 (and \( b_0 \) can be chosen to be infinite). We also choose a gauge where the fluctuations in \( b \) vanish. To order \( \rho^0 \), the equations tell us that
\[
A_0 = k\hat{y},
V_0 = \frac{6k}{\kappa^2}
\]
The order \( \rho^1 \) Einstein equations will again look like those of [15]. Even though we only have one set of boundary conditions for \( \Psi_1 \) and \( \Upsilon_1 \), we can still solve the first two Einstein equations, leaving \( \Psi_1(\hat{y} = b_0) \) and \( \Upsilon_1(\hat{y} = b_0) \) unspecified. If we then plug the results in the (55) equation, we find the following constraint:
\[
(4\Psi_1 + \Upsilon_1) |_{\hat{y}=b_0} = \Omega^{-2} (4\Psi_1 + \Upsilon_1) |_{\hat{y}=0}
\]
Taking this into account, we can write the Friedmann equations as:
\[
H_1^2 = \frac{2k}{(1 - \Omega^2)} \left( \frac{\kappa^2}{6} \rho_* - \Omega^4 \Psi_1 |_{\hat{y}=b_0} \right)
\]
\[
\dot{H}_1 = \frac{k}{6(1 - \Omega^2)} \left( \kappa^2 (-\Omega^2 + 3\Omega^2 \omega_* - 3 - 3\omega_*) \rho_* + 24\Omega^4 \Psi_1 |_{\hat{y}=b_0} \right)
\]
It would appear that we have complete freedom in choosing a value for \( \Psi_1(\hat{y} = b_0) \), but there are a couple of points to consider.

Firstly, we expect our results to reproduce the standard Friedmann equations at \( O(\rho_*^1) \). One can easily see that with the following choice:
\[
\Psi_1(\hat{y} = b_0) = \frac{\kappa^2}{6} \rho_*,
\]
this requirement is satisfied. Similarly, recovering the correct \( O(\rho_*^2) \) term would require setting \( \Psi_2 \) to the appropriate value at \( \hat{y} = b_0 \).

Secondly, we know from [27] that if the tension on the brane is larger than the static solution’s value, there will be a horizon in the bulk at the (finite) value of \( \hat{y} \) for which \( a(\hat{y}) = 0 \). Given the form we have chosen for \( a \), it is not immediately obvious how we can reproduce this behaviour. However, if we expand \( e^{-\delta A_1} \) as \( 1 - \delta A_1 \), then in the limit where we neglect any higher order in \( \rho \) contributions, \( a \) will vanish when \( \delta A_1 = 1 \). (Notice that this is also the point at which our perturbative approach breaks down completely). Choosing \( \Psi_1(\hat{y} = b_0) \) in the manner mentioned above, and recalling that
\[
\delta A_1(\hat{y}) = \int_0^{\hat{y}} \Psi_1(\hat{y}) \, d\hat{y}
\]
we find that \( \delta A_1(\hat{y}) = 1 \) at the position
\[
\hat{y}_{a=0} = \frac{1}{2k} \ln \left( 1 + \frac{12k}{\kappa^2 \rho_*} \right)
\]
which corresponds precisely to what we would find using the exact solutions of [27].

One more subtlety remains. If we solve for $\delta N_1$, we can see that there will be a point in the bulk where $n$ vanishes. (In the same manner as $a$ vanishes, i.e., where $\delta N_1(\hat{y}) = 1$.) In general, this point and the one where $a$ vanishes do not coincide. Indeed, it can be shown that $\delta N_1 = 1$ at the position

$$\hat{y}_{n=0} = \frac{1}{2k} \ln \left( 1 + \frac{12k}{\kappa^2 \rho_*} + 3(1 + \omega_*)(e^{2k} - 1) \right).$$  \hfill (C.14)

Intuitively, one might expect that $a = 0$ corresponds to a geometrical singularity while $n = 0$ corresponds to a horizon. We would then demand that $\hat{y}_{n=0} \leq \hat{y}_{a=0}$ in order to avoid the appearance of a naked singularity in the bulk. One can readily see that this would impose $\omega_* \leq -1$. However, as we stated in the main text, examining the behaviour of the Riemann tensor at these special values of $\hat{y}$ shows no evidence of curvature singularities in the bulk.

To summarize, we have shown in this section that our perturbative approach allows us to:

a) reproduce the expected RS-II result as a limit of unstabilized RS-I, as long as $\rho$ is treated as a dependent rather than a free parameter;

b) reproduce the expected RS-II results starting from a single brane setup, as long as we fix the values of $\Psi$ and $\Upsilon$ at $\hat{y} = b_0$ appropriately;

c) calculate the points in the bulk at which $a$ and $n$ vanish and find a result which matches the exact one.

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