An Alexander type invariant for doodles

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Abstract

We construct an Alexander type invariant for oriented doodles from a deformation of the Tits representation of the twin group. Similar to the Alexander polynomial, our invariant vanishes on unlinked doodles with more than one component. We also include values of our invariant on several doodles.

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1 Introduction

Doodles were introduced by Fenn and Taylor [9] as a finite collection of embedded circles on the 2-sphere with no triple intersections. Khovanov [19] pointed out that it was more natural to consider doodles as immersed circles on the 2-sphere than simple curves. Thus, some concepts of knot theory were successfully transferred to doodle theory. In particular, he obtained an associated group to a doodle, called the doodle group, in analogue to the fundamental group of the complement of a link. In contrast with knot theory, equivalence of doodles just considers a planar version of the first and the second Reidemeister moves (see Figure 1); the lack of the third Reidemeister move leads to significant differences in comparison with classical knot theory. For instance, every class of a doodle has a unique diagram with minimal number of crossings [19]; the calculation of Vassiliev invariants is more complicated than classical knots, but Vassiliev invariants classify doodles, problem that remains open for knots [23]. Recently, Bartholomew-Fenn-Kamada-Kamada [4] extended the study of doodles to immersed circles on closed oriented surfaces of any genus, which can be considered as virtual links analogue for doodles; in [5] they give a complete invariant for virtual doodles1.

The algebraic counterpart of doodles are the so-called twin groups, terminology due to Khovanov. The role of these groups in doodle theory is similar to that of Artin’s braid groups in knot theory. These group have appeared under different names and contexts in the literature: cartographical Gorthendiek group [25, 27], quantum symmetric group [10], twin group [2, 8, 18, 14, 15], flat braid group2 [21], traid group [13], planar braid group [11, 24]. Further, the twin group on \( n \) strands, denoted by \( T_n \), is a right angled Coxeter group generated by \( n - 1 \) involutions and its elements can be depicted as planar braids on \( n \) strands. The permutation induced by a planar braid defines a natural epimorphism from \( T_n \) to the symmetric group on \( n \) symbols, whose kernel is called the pure twin group. So far, the twin group and pure twin group have been studied in the last years, though there still exist many open questions about these groups. For details on recent developments, refer to [2, 8, 11, 13, 14, 15, 24].

Khovanov [19] proved the Alexander theorem for doodles in which any doodle in the 2-sphere is the closure of a twin, and recently Gotin [12] has established the Markov theorem for doodles the 2-sphere in which he defines the corresponding Markov moves for twins. These two theorems invite to the construction of doodle invariants in a similar way it has been done in classical knot theory. For instance, the construction of an Alexander and a Jones polynomial for doodles.

The purpose of this paper is the construction of an Alexander type invariant for dood-

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1In Bartholomew’s webpage [3] is shown a table of virtual doodles up to 10 crossings.
2This term was later used in [17] to an entirely different object
dles. We follow Burau’s work to compute the Alexander polynomial for links via certain representation of the braid group [7], applied in the context of twins and doodles.

More precisely, we define a representation \( \psi_n : T_n \to GL_{n-1}(\mathbb{Z}(x)) \) which is a deformation of the Tits representation of \( T_n \). Then, using \( \psi_n \) we construct a function \( f_n : T_n \to \mathbb{Z}(x) \) which defines an invariant of doodles up to a factor \( x^{2k} \), i.e., if two twins \( \alpha, \beta \) have equivalent closures \( \hat{\alpha} \sim \hat{\beta} \) as doodles, then their images \( f_n(\alpha), f_n(\beta) \) differ by a multiple of \( x^{2k} \) for some \( k \in \mathbb{Z} \) (see Lemma 4.9). Thus, we obtain an invariant \( Q(D) \) for a doodle \( D \) as the smallest degree polynomial over all images under functions \( f_n \) of twins with closure \( D \) (Theorem 4.14). Furthermore, we verify that these functions satisfy the skein relation

\[
f_n \left( \begin{array}{c} \lambda \\ \\end{array} \right) - f_n \left( \begin{array}{c} \lambda' \\ \\end{array} \right) = (x^2 - 1) \left( f_n \left( \begin{array}{c} \lambda \\ \\end{array} \right) - f_n \left( \begin{array}{c} \lambda' \\ \\end{array} \right) \right).
\]

It is worth noting that Chebyshev polynomials of second kind appeared unexpectedly in the construction of the functions \( f_n \). Observe that it is not the first time where Chebyshev polynomials of second kind emerge in low dimensional topology. For instance, they play an important role defining Jones-Wenzl projectors [28].

The paper is organized as follows. In Section 2, we give some preliminaries on doodles and twins; we recall the Tits representation of a Coxeter group and list necessary properties of Chebyshev polynomials. In Section 3, we study a deformation of the Tits representation of \( T_n \) as a Coxeter group and some of its properties. In Section 4, we investigate the behavior of the functions \( f_n \) under Markov moves, obtaining the definition of our invariant \( Q(D) \) and some properties. Finally, we give some examples of computations in Section 5.

## 2 Preliminaries

In the present section, we give the necessary background on doodles, twins and Chebyshev polynomials that will be used along the paper. The Alexander and Markov theorems for doodles are stated in Subsection 2.2. In Subsection 2.3, we recall the definition of Tits representation of a Coxeter group and a deformation of this one will be used later. Finally, in Subsection 2.4 we recall briefly the definition of the Chebyshev polynomials which will be used in Section 3.
2.1 Doodles and twin group

**Definition 2.1** (Cf. Khovanov, [19]). A *doodle*\(^3\) is an immersion \(D : \bigsqcup_n S^1 \to S^2\) of a disjoint union of \(n\) circles into the two-sphere\(^4\) with only transversal double points; \(n\) is the number of components. An *oriented doodle* is a doodle in which each component is oriented. We denote the set of doodles by \(\mathcal{D}\).

Two doodles \(D\) and \(D'\) are *equivalent*, denoted by \(D \sim D'\), if they can be transformed into each other through isotopies of \(S^2\) and a finite sequence of moves \(R1\) and \(R2\), see Figure 1. These moves will be called the first and the second Reidemeister moves.

\[
\begin{align*}
\begin{array}{c}
\includegraphics[height=1cm]{reidemeister_1} \\
(a) R1
\end{array} & \leftrightarrow & 
\begin{array}{c}
\includegraphics[height=1cm]{reidemeister_2} \\
(b) R2
\end{array}
\end{align*}
\]

Figure 1: Reidemeister moves.

The doodle in Figure 2a is called the trivial doodle. The first non-trivial doodle is the Borromean doodle, shown in Figure 2b, with six crossings and three components. Following [4], the doodle in Figure 2c is called the 4-poppy doodle and it has eight crossings; it is the first non-trivial 1-component doodle [4, Theorem 4.2].

\[
\begin{align*}
\begin{array}{c}
\includegraphics[height=1.5cm]{trivial} \\
(a) trivial
\end{array} & \quad \begin{array}{c}
\includegraphics[height=1.5cm]{borromean} \\
(b) Borromean
\end{array} & \quad \begin{array}{c}
\includegraphics[height=1.5cm]{4-poppy} \\
(c) 4-poppy
\end{array}
\end{align*}
\]

Figure 2: First doodles.

The algebraic counterpart of doodles are twin groups. More precisely, these groups are to doodle theory as braid groups are to knot theory.

---

\(^3\)For other definitions of doodles see [4, 23].

\(^4\)In [4], doodles in \(S^2\) are called *planar doodles*. 

Definition 2.2. The twin group $T_n$ is the group presented by generators $t_1, \ldots, t_{n-1}$ and the relations:

$$t_i^2 = 1 \quad \text{for all } 1 \leq i \leq n-1,$$

(2.1)

$$t_i t_j = t_j t_i \quad \text{for all } 1 \leq i, j \leq n-1 \text{ with } |i-j| > 1.$$  

(2.2)

In [18], Khovanov gives a geometrical interpretation of $T_n$. To be precise, the elements of $T_n$ can be regarded as planar braids, called twins. A $n$-twin is a collection of $n$ descending arcs in $\mathbb{R} \times [0, 1]$, with only transversal double points, see [18]. In particular, the generator $t_i$ is represented by the following “elementary” twin:

$$
\begin{array}{c}
\vrule \vrule \vrule \vrule \vrule \\
\vrule \vrule \vrule \vrule \vrule \\
\vrule \vrule \vrule \vrule \vrule \\
i \\
i+1 \\
\end{array}
$$

(2.3)

Relations (2.1) and (2.2) are represented, respectively, as follows:

(a) Relation (2.1)

(b) Relation (2.2)

For $n \geq 1$, let $\iota^n_R : T_n \hookrightarrow T_{n+1}$ be the natural inclusion given by $\iota^n_R(t_i) = t_i$, and let $\iota^n_L : T_n \hookrightarrow T_{n+1}$ be the inclusion given by $\iota^n_L(t_i) = t_{i+1}$, for $1 \leq i \leq n-1$ in both cases. Thus, geometrically $\iota^n_R$ (resp. $\iota^n_L$) adds a vertical strand to the right (resp. left) of the twin, see Figure 4. We denote $\iota^R = \{\iota^n_R\}_{n \geq 1}$ and $\iota^L = \{\iota^n_L\}_{n \geq 1}$ to the systems of inclusions, and we will simply write $\iota^R(\beta)$ for $\iota^n_R(\beta)$ (resp. $\iota^L(\beta)$ for $\iota^n_L(\beta)$) if $\beta \in T_n$ for some $n \geq 1$. Let $T_\infty$ denote the inductive limit associated to $\iota^R$.

Figure 4: Inclusions $\iota^n_R$ and $\iota^n_L$. 

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2.2 Alexander and Markov theorem for doodles

The closure $\hat{\beta}$ of the twin $\beta$ is the oriented doodle obtained by connecting its upper and lower ends respectively as shown in Figure 5. Khovanov proved the analogue to the classical Alexander theorem for doodles.

**Theorem 2.3** (Khovanov, [19, Theorem 2.1]). Any (oriented) doodle on the two-sphere is the closure of a twin.

![Figure 5: The Borromean doodle as the closure of the twin $(t_1t_2)^3$.](image)

Theorem 2.3 says the map $\beta \mapsto \hat{\beta}$ from $T_\infty$ onto $D$ is surjective. However, this map is not injective, for instance the twins $t_1$ and $t_1t_2$ have the same closure. Recently in [12], Gotin proves the analogue to the Markov theorem for doodles. In order to state this theorem, we set

\begin{align*}
t_{n,i} & := t_nt_{n-1} \cdots t_{n-i+1}t_{n-i}t_{n-i+1} \cdots t_{n-1}t_n, \\
t_{1,i} & := t_1t_2 \cdots t_it_{i+1}t_i \cdots t_1t_2
\end{align*}

for $0 \leq i \leq n - 1$.

**Theorem 2.4** (Gotin, [12, Theorem 4.1]). Let $\gamma$ and $\gamma'$ be two twins with $D$ and $D'$ their respective closures. Then, $D$ and $D'$ are equivalent doodles if and only if $\gamma$ can be transformed into $\gamma'$ by a finite sequence of the following moves:

- $M_0 : t^R(\beta) \sim t^L(\beta)$ for any $\beta \in T_n$,
- $M_1 : \alpha\beta \sim \beta\alpha$ for any $\alpha, \beta \in T_n$,
- $M_2 : \beta \sim t^R(\beta)t_{n,i}$ for any $\beta \in T_n$ and $0 \leq i \leq n - 1$,  
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Figure 6: $M_0$ move.

Figure 7: $M_2$ and $M_3$ moves.

$M_3 : \beta \sim t^L(\beta)t_{1,i}$ for any $\beta \in T_n$ and $0 \leq i \leq n - 1$.

We will call these moves, the Gotin-Markov moves.

Remark 2.5. When $i = 0$ in $M_2$ we have $t_{n,0} = t_n$, then the $M_2$ move becomes the classical second Markov move.

The Gotin-Markov moves define an equivalence relation on $T_\infty$, denoted by $\sim_G$. Thus, Theorem 2.3 together with Theorem 2.4 implies the bijection:

$$D/\sim \leftrightarrow T_\infty/\sim_G.$$ 

This bijection says that constructing an invariant for doodles it reduces to find a family $f = \{f_n : T_n \to S\}_{n \geq 1}$, where $S$ is a well understood set and the functions $f_n$ are invariant under the Gotin-Markov moves. In Section 4, we will construct a such family by taking $S = \mathbb{Z}(x)$ and certain $f_n$ defined in terms of a deformation of the Tits representation of the twin group.

2.3 Tits representation of Coxeter groups

We describe briefly some properties of Coxeter groups and their relation with twin groups. Let $S$ be a finite set; a Coxeter matrix on $S$ is a symmetric matrix $M = (m_{s,t})_{s,t \in S}$ such
that \( m_{s,t} \in \{2, 3, \ldots \} \cup \{\infty\} \) and \( m_{s,s} = 1 \). The Coxeter group associated to \( M \) is the group defined by the following presentation:

\[
W(M) := \langle S | (st)^{m_{s,t}} \text{ for all } s, t \text{ with } m_{s,t} \neq \infty \rangle.
\]

Tits constructed a faithful linear representation for any Coxeter group in the following way. Let \( S = \{s_1, s_2, \ldots, s_n\} \) be a finite set, \( M \) be a Coxeter matrix on \( S \) and \( V \) be a real vector space with basis \( \{e_1, \ldots, e_n\} \). Now define the symmetric bilinear form \( B \) on \( V \) by

\[
B(e_i, e_j) = \begin{cases} 
-\cos\left(\frac{\pi}{m_{i,j}}\right) & \text{if } m_{s_i, s_j} \neq \infty \\
-1 & \text{if } m_{s_i, s_j} = \infty
\end{cases}
\]

The linear map \( \sigma_i : V \to V \) given by \( \sigma_i(v) = v - 2B(e_i, v)e_i \), defines an automorphism of \( V \).

**Theorem 2.6** (Tits, see [6, p. 96]). The map \( \rho : W(M) \to GL(V) \) defined through \( \rho(s_i) = \sigma_i \) is a faithful representation of \( W(M) \).

The representation \( \rho \) above is called the Tits (or geometrical) representation of \( W(M) \). In particular for \( T_n \), the Tits representation \( \rho \) is given by \( t_i \mapsto \sigma_i \), where:

\[
\sigma_1 = \begin{pmatrix}
-1 & 0 & 0 \\
2 & 1 & 0 \\
0 & 0 & I_{n-3}
\end{pmatrix}, \quad \sigma_{n-1} = \begin{pmatrix}
I_{n-3} & 0 & 0 \\
0 & 1 & 2 \\
0 & 0 & -1
\end{pmatrix}
\]

and for \( 1 < i < n - 1 \),

\[
\sigma_i = \begin{pmatrix}
I_{i-2} & 0 & 0 & 0 \\
0 & 1 & 2 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 2 & 1 \\
0 & 0 & 0 & I_{n-i-2}
\end{pmatrix}
\]

where \( I_m \) denote the identity matrix of size \( m \).

### 2.4 Chebyshev Polynomials

There are two type of Chebyshev polynomials: the first and the second kind. For the purpose of this paper, we only need to recall the definition of the Chebyshev polynomials of second kind. For a more detailed treatment on Chebyshev polynomials see [20].
**Definition 2.7.** The *Chebyshev polynomial of second kind* \(U_n(z)\) is the polynomial of degree \(n\) in one variable \(z\) defined by the recurrence relation

\[
U_n(z) = 2zU_{n-1}(z) - U_{n-2}(z),
\]

and the initial conditions

\[
U_0(z) = 1 \quad \text{and} \quad U_1(z) = 2z.
\]

In the rest of the paper we simply refer to the polynomials \(U_i\)'s as the Chebyshev polynomials. In Section 4, we prove that certain polynomials associated to a family of twins can be written in terms of the Chebyshev polynomials (Corollary 4.4).

### 3 A Deformed Tits Representation for the Twin Group

The main purpose of this section is to define a deformation of the Tits representation and give some of its properties.

For \(n \geq 3\), we consider the following \((n-1) \times (n-1)\) matrices defined over the ring \(\Lambda := \mathbb{Z}(x,y)\):

\[
V_1 = \begin{pmatrix}
-1 & 0 & 0 \\
x & 1 & 0 \\
0 & 0 & I_{n-3}
\end{pmatrix}, \quad V_{n-1} = \begin{pmatrix}
I_{n-3} & 0 & 0 \\
0 & 1 & y \\
0 & 0 & -1
\end{pmatrix}
\]

and for \(1 < i < n - 1\),

\[
V_i = \begin{pmatrix}
I_{i-2} & 0 & 0 & 0 \\
0 & 1 & y & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & x & 1 \\
0 & 0 & 0 & I_{n-i-2}
\end{pmatrix}.
\]

For \(n = 2\), we set \(V_1 := -1\).

**Lemma 3.1.** For \(n \geq 3\) and \(1 \leq i \leq n - 2\), we have

\[
V_i V_{i+1} - V_{i+1} V_i V_{i+1} = (xy - 1)(V_i - V_{i+1}).
\]

**Proof.** It suffices to verify the case \(i = 2\) and \(n = 4\), which follows from a direct computation. \(\square\)
Matrices $V_i$’s are an analogue to the matrices used in the reduced Burau representation of the braid group [7]. In fact, we have the following proposition.

**Proposition 3.2.** For $n \geq 2$, the function $\psi_n : T_n \rightarrow GL_{n-1}(\Lambda)$ defined through $\psi_n(t_i) = V_i$ is a representation.

**Proof.** The proof follows by checking the defining relations of the twin group are satisfied by the matrices $V_i$’s. For $n = 3$ the checking results from a direct computation. In the general case, it is enough to check for $n = 4$, which follows again by direct computation:

\[(\psi_4(t_2))^2 = \begin{pmatrix} 1 & y & 0 \\ 0 & -1 & 0 \\ 0 & x & 1 \end{pmatrix} \begin{pmatrix} 1 & y & 0 \\ 0 & -1 & 0 \\ 0 & x & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\]

and

\[
\psi_4(t_1)\psi_4(t_3) = \begin{pmatrix} -1 & 0 & 0 \\ x & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & y \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ x & 1 & y \\ 0 & 0 & -1 \end{pmatrix}
\]

\[
= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & y \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ x & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \psi_4(t_3)\psi_4(t_1).
\]

\[\square\]

**Remark 3.3.** Specializing $x = y = 2$, the matrices $V_i$’s become the matrices $\sigma_i$’s seen in Subsection 2.3. Thus, $\psi_n$ can be regarded as a two parameters deformation of the Tits representation.

Observe that,

\[
\psi_{n+1}(t_i^R) = \begin{pmatrix} \psi_n(t_i) \\ v_i^R(t_i) \end{pmatrix} = \begin{pmatrix} V_i \\ v_i^R(t_i) \end{pmatrix} 1 \leq i \leq n - 1, \quad (3.1)
\]

where $v_i^R(t_i)$ is the row of length $n - 1$ equal to $0$ if $i < n - 1$ and $(0, \ldots, 0, x)$ if $i = n - 1$. This implies that for $n \geq 2$ and $\beta \in T_n$,

\[
\psi_{n+1}(t_i^R(\beta)) = \begin{pmatrix} \psi_n(\beta) \\ v_i^R(\beta) \end{pmatrix}, \quad (3.2)
\]

where $v_i^R(\beta)$ is a row of length $n - 1$ over $\Lambda$ depending on $\beta$. In Corollary 3.7 we will see how to compute $v_i^R(\beta)$. 

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Lemma 3.5. For also define the \( n \) \( \times \) \( n \) matrices \( A_n = (a_{i,j}) \) and \( Z_n = (\zeta_{i,j}) \), where \( \psi = \psi(\beta) \).

Finally, we set \( Z_n := Z_n + I_n \). For instance, for \( n = 3 \) we have:

\[
A_3 = \begin{pmatrix}
-y^{-1} & 1, 1 \leq j \leq n \\
0 & i < j - 1 \\
x & 1 \leq j \leq n - 1, i = j + 1 \\
y^{j-i}(y-1) & 1 < j \leq i.
\end{pmatrix}, \quad Z_3 = \begin{pmatrix}
xy^j - 2y^{j-1} & i = 1 \\
x^jy^j - x^jy^{j-1} & 1 < i < n \\
-x^{n-1}y^{j-1} & i = n.
\end{pmatrix}
\]

In the lemma below we use the natural map \( \theta : M_{k,i}(\Lambda) \to M_{k,i}(\Lambda) \) induced from the \( \mathbb{Z} \)-automorphism of \( \Lambda \) that interchanges \( x \) with \( y \).

Lemma 3.5. For \( n \geq 2 \) and \( 1 \leq i \leq n - 1 \), we have:

(a) \( \psi_{n+1}(t_1 \cdots t_i) = A_n \),

(b) \( \psi_{n+1}(t_{n,i}) = B_{n,i} \),

(c) \( \psi_{n+1}(t_{i,n}) = C_{n,i} \),

where \( B_{n,i} \) and \( C_{n,i} \) are defined as follows:

\[
B_{n,i} = \begin{pmatrix}
-1 & -y & -y^2 \\
x & x^2y & x^2y - y \\
0 & x & x^2y - 1
\end{pmatrix}, \quad C_{n,i} = \begin{pmatrix}
y^i & y^i & \cdots & y^i \\
y & y^2 & \cdots & y^i \\
\vdots & \vdots & \ddots & \vdots \\
y^n & y^{n-1} & \cdots & y^i
\end{pmatrix}
\]

and \( Y_i \) is the \( (n - i - 1) \times i \) matrix given by:

\[
Y_i = \begin{pmatrix}
y & y^2 & \cdots & y^i \\
y & y^2 & \cdots & y^i \\
\vdots & \vdots & \ddots & \vdots \\
y & y^2 & \cdots & y^i
\end{pmatrix}
\]

where vertical dots mean a block of 0's for \( 1 \leq i < n - 2 \).
Proof. (a) It follows by induction on $n$ and Equation (3.2).

(b) For $1 \leq i \leq n - 2$, we proceed by induction over $i$. For $i = 1$ the result is straightforward. We want to prove $\psi_{n+1}(t_{n,i+1}) = B_{n,i+1}$. Using Lemma 3.1, we obtain

$$
\psi_{n+1}(t_{n,i+1}) = \psi_{n+1}(t_{n,i}) + \psi_{n+1}(t_{n,i-1}) - (xy - 1)(\psi_{n+1}(t_{n,i-1}) - \psi_{n+1}(t_{n,i})).
$$

(3.4)

Since $\psi_{n+1}(t_{n,i}) = B_{n,i}$ by induction hypothesis, we have

$$
\psi_{n+1}(t_{n,i}) = \begin{pmatrix}
1 & 0 & 0 & r_1 \\
0 & 1 & y & r_2 \\
0 & 0 & xy - 1 & c_1 \\
\end{pmatrix}
$$

where $r_1$ and $r_2$ are row vectors of length $i$, $c_1$ is a column vector of length $i$ and $A$ is an $i \times i$ matrix. Hence

$$
\psi_{n+1}(t_{n,i-1})\psi_{n+1}(t_{n,i}) = \begin{pmatrix}
1 & xy^2 & y^2 & yr_1 \\
0 & -xy + 1 & -y & -r_1 \\
0 & 2xy - 2x & 2xy - 1 & x_1 + r_2 \\
\end{pmatrix}
$$

and

$$
\psi_{n+1}(t_{n,i-1}) - \psi_{n+1}(t_{n,i}) = \begin{pmatrix}
0 & y & 0 & 0 \\
0 & -2 & -y & -r_1 \\
0 & x & 2 - xy & -r_2 \\
0 & & c_1 & I_i - A \\
\end{pmatrix}
$$

Replacing the expressions above in Equation (3.4), we obtain

$$
\psi_{n+1}(t_{n,i+1}) = \begin{pmatrix}
1 & y & y^2 & yr_1 \\
0 & xy - 1 & y(xy - 2) & (xy - 2)r_1 \\
0 & x^2y - x & y(x^2y - x) + 1 & x_1 + xy_2 \\
\end{pmatrix}
$$

Finally, using that $xy_2 = (x^2y - 2x)r_1$, we obtain that $\psi_{n+1}(t_{n,i+1}) = B_{n,i+1}$. For the case $i = n - 1$, the result follows by using Lemma 3.1 and the formula for $B_{n,n-2}$. 

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(c) This case follows by Lemma (3.1) and a similar calculation as in (b).

From now on, we take \( x = y \). Thus, the representation \( \psi_n \) is defined over \( \Lambda = \mathbb{Z}(x) \). For \( n \geq 2 \) define the vectors \( F^R \) and \( F^L \) of \( \Lambda^n \) as follows:

\[
F^R := (U_0 \left( \frac{1}{x} \right), U_1 \left( \frac{1}{x} \right), \ldots, U_{n-2} \left( \frac{1}{x} \right), U_{n-1} \left( \frac{1}{x} \right)) \in \Lambda^n
\]

and

\[
F^L := (U_{n-1} \left( \frac{1}{x} \right), U_{n-2} \left( \frac{1}{x} \right), \ldots, U_1 \left( \frac{1}{x} \right), U_0 \left( \frac{1}{x} \right)) \in \Lambda^n,
\]

where \( U_i(z) \) is the \( i \)-th Chebyshev polynomial.

**Proposition 3.6.** For any \( \beta \in T_n \), the vectors \( F^R \) and \( F^L \) satisfy the following equations:

\[
\begin{align*}
(a) & \quad F^R \psi_{n+1}(i^R(\beta)) = F^R, \\
(b) & \quad F^L \psi_{n+1}(i^L(\beta)) = F^L.
\end{align*}
\]

**Proof.** (a) It suffices to prove it for matrices of the form (3.1). Expanding the equations:

\[
F^R \begin{pmatrix} V_i & 0 \\ v^R(t_i) & 1 \end{pmatrix} = F^R \quad \text{for } 0 \leq i \leq n - 1,
\]

turns out to be equivalent to the following system of equations

\[
\begin{align}
-2U_0 \left( \frac{1}{x} \right) + xU_1 \left( \frac{1}{x} \right) &= 0, \quad (3.5) \\
xU_{i-1} \left( \frac{1}{x} \right) - 2U_i \left( \frac{1}{x} \right) + xU_{i+1} \left( \frac{1}{x} \right) &= 0 \quad (2 \leq i \leq n - 2). \quad (3.6)
\end{align}
\]

Now, Equation (3.5) is satisfied by the initial conditions (2.7) of the Chebyshev polynomials, and Equation (3.6) is exactly their recurrence relation (2.6). So, the proof is concluded.

(b) This case is treated similarly.

**Corollary 3.7.** Let \( \beta \in T_n \) and let \((a_{i,1} a_{i,2} \cdots a_{i,n-1})\) be the \( i \)-th row of the matrix \( \psi_n(\beta) - I_{n-1} \) for \( 1 \leq i \leq n - 1 \). Then, the entries of \( v^R(\beta) = (b_1 b_2 \cdots b_{n-1}) \) in matrix (3.2) and \( v^L(\beta) = (c_1 c_2 \cdots c_{n-1}) \) in matrix (3.3) satisfy:

\[
\begin{align*}
(a) & \quad -U_{n-1} \left( \frac{1}{x} \right) b_k = \sum_{i=1}^{n-1} U_{i-1} \left( \frac{1}{x} \right) a_{i,k}, \\
(b) & \quad -U_{n-1} \left( \frac{1}{x} \right) c_k = \sum_{i=1}^{n-1} U_{n-2-i} \left( \frac{1}{x} \right) a_{i,k}.
\end{align*}
\]

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Proof. We only prove (a), a similar argument works for (b). By Proposition 3.6,

\[ F^R \left( \begin{array}{cc} \psi_n(\beta) & 0 \\ v^R(\beta) & 1 \end{array} \right) = F^R. \]

Because \( F^R I_n = F^R \), we obtain

\[ F^R \left( \begin{array}{cc} \psi_n(\beta) - I_{n-1} & 0 \\ v^R(\beta) & 0 \end{array} \right) = 0. \] (3.7)

Writing

\[ \left( \begin{array}{cc} \psi_n(\beta) - I_{n-1} \\ v^R(\beta) \end{array} \right) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,n-1} \\ a_{21} & a_{22} & \cdots & a_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} \\ b_1 & b_2 & \cdots & b_{n-1} \end{pmatrix}, \]

and expanding the product of Equation (3.7), we obtain the equations

\[ a_{1,k} + U_1 \left( \frac{1}{x} \right) a_{2,k} + U_2 \left( \frac{1}{x} \right) a_{3,k} + \cdots + U_{n-2} \left( \frac{1}{x} \right) a_{n-1,k} + U_{n-1} \left( \frac{1}{x} \right) b_k = 0, \]

where \( 1 \leq k \leq n - 1 \). These equations imply the claim (a).

\[ \square \]

4 An Invariant of Doodles

This is the main section of the paper and consists in two subsections. The first one introduces a family of functions \( f_n \)'s and we study its behavior under the Gotin-Markov moves. Also, we give a close formula to a skein-like relation that is satisfied by \( f_n \)'s. The second one proposes a polynomial invariant for doodles.

4.1 The function \( f_n \)

**Definition 4.1.** For all \( n \geq 1 \), we define the polynomial \( P_n(x) \in \Lambda \) by the formula

\[ P_n(x) = \det(\psi_{n+1}(t_1 \ldots t_n) - I_n). \]

We set \( P_0(x) = 1 \).

**Proposition 4.2.** For \( n \geq 2 \), the polynomials \( P_n \) satisfy the following recurrence relation:

\[ P_n(x) = -2P_{n-1}(x) - x^2 P_{n-2}(x). \]
Proof. By Lemma 3.5, \( P_n(x) = \det(A_n - I_n) \), where

\[
A_n - I_n = \begin{pmatrix}
-2 & -x & -x^2 & \cdots & -x^{n-2} & -x^{n-1} \\
x & x^2 - 2 & x(x^2 - 1) & \cdots & x^{n-3}(x^2 - 1) & x^{n-2}(x^2 - 1) \\
0 & x & x^2 - 2 & \cdots & x^{n-4}(x^2 - 1) & x^{n-3}(x^2 - 1) \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & x & x^2 - 2 & x(x^2 - 1) \\
0 & 0 & \cdots & 0 & x & x^2 - 2
\end{pmatrix}.
\]

Then, expanding the determinant with respect the last row, we get

\[
P_n(x) = (x^2 - 2)P_{n-1}(x) - x \det(M),
\]

where \( M \) is the following \((n - 1) \times (n - 1)\) matrix

\[
M = \begin{pmatrix}
-2 & -x & -x^2 & \cdots & -x^{n-3} & -x^{n-1} \\
x & x^2 - 2 & x(x^2 - 1) & \cdots & x^{n-4}(x^2 - 1) & x^{n-2}(x^2 - 1) \\
0 & x & x^2 - 2 & \cdots & x^{n-5}(x^2 - 1) & x^{n-3}(x^2 - 1) \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & x & x^2 - 2 & x(x^2 - 1) \\
0 & 0 & \cdots & 0 & x & x^2 - 1
\end{pmatrix}.
\]

We have now \( \det(M) = x \det(M') \), where

\[
M' = \begin{pmatrix}
-2 & -x & -x^2 & \cdots & -x^{n-3} & -x^{n-2} \\
x & x^2 - 2 & x(x^2 - 1) & \cdots & x^{n-4}(x^2 - 1) & x^{n-3}(x^2 - 1) \\
0 & x & x^2 - 2 & \cdots & x^{n-5}(x^2 - 1) & x^{n-4}(x^2 - 1) \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & x & x^2 - 2 & x(x^2 - 1) \\
0 & 0 & \cdots & 0 & x & x^2 - 1
\end{pmatrix}.
\]

Finally, it is not difficult to prove that \( \det(M') = P_{n-1}(x) + P_{n-2}(x) \). Then, the result follows by replacing it in Equation (4.1). \( \square \)

Remark 4.3. Observe that by Equation (4.2), \( P_n(x) \in \mathbb{Z}[x^2] \) for all \( n \).

Corollary 4.4. For \( n \geq 1 \), we have

\[
P_n(x) = (-x)^n U_n \left( \frac{1}{x} \right).
\]
Proof. Firstly, we have that $P_0(x) = 1 = (-x)^0 U_0 \left( \frac{1}{x} \right)$ and $P_1(x) = -2 = -x U_1 \left( \frac{1}{x} \right)$. Secondly, by the recurrence equation of the Chebyshev polynomials (2.6), we have

$$U_n \left( \frac{1}{x} \right) = \frac{2}{x} U_{n-1} \left( \frac{1}{x} \right) - U_{n-2} \left( \frac{1}{x} \right),$$

which implies $(-x)^n U_n \left( \frac{1}{x} \right) = -2(-x)^{n-1} U_{n-1} \left( \frac{1}{x} \right) - x^2(-x)^{n-2} U_{n-2} \left( \frac{1}{x} \right)$. That is, the polynomial $(-x)^n U_n \left( \frac{1}{x} \right)$ satisfies the same defining rules of $P_n(x)$, hence they are all equal. \qed

Definition 4.5. Let $f_n : T_n \to \Lambda$ be the map defined by

$$f_n(\beta) := \begin{cases} 1 & \text{if } n = 1, \\ \frac{\det(\psi_n(\beta) - I_{n-1})}{(-x)^n U_{n-1} \left( \frac{1}{x} \right)} & \text{if } n \geq 2. \end{cases}$$

In particular $f_2(1) = 0$ and $f_2(t_1) = 1$. Furthermore $f_n(t_1 t_2 \ldots t_{n-1}) = 1$ for any $n \geq 2$.

Lemma 4.6. For $n \geq 2$ and $\alpha, \beta \in T_n$, we have:

(a) $f_{n+1}(\iota^R(\beta)) = f_{n+1}(\iota^L(\beta)) = 0$,

(b) $f_n(\alpha^{-1} \beta \alpha) = f_n(\beta)$.

Proof. The proof of (a) follows by Equation (3.2) and Equation (3.3). The proof of (b) follows from conjugacy properties of determinants. \qed

Notation 4.7. In subsequent proofs, $C_{i,j}(k)$ (resp. $R_{i,j}(k)$) denotes the elementary operation which replaces the $i$-th column $C_i$ (resp. row $R_i$) by the sum $C_i + kC_j$ (resp. $R_i + kR_j$). For $k \neq 0$, $C_i(k)$ (resp. $R_i(k)$) denotes the elementary operation which multiplies by $k$ the $i$-th column (resp. row), and $C_{i,j}$ (resp. $R_{i,j}$) denotes the elementary operation that exchanges $i$-th and $j$-th columns (resp. rows).

Lemma 4.8. For $n \geq 2$ and $\alpha, \beta \in T_n$, we have:

(a) $f_{n+1}(\iota^R(\beta)t_n) = f_n(\beta)$,

(b) $f_{n+1}(\iota^L(\beta)t_1) = f_n(\beta)$.
Proof. (a) The equation $f_{n+1}(t^R(\beta) t_n) = f_n(\beta)$ is equivalent to

$$U_n \left( \frac{1}{x} \right) \det(\psi_{n+1}(t^R(\beta) t_n) - I_n) = -x U_n \left( \frac{1}{x} \right) \det(\psi_n(\beta) - I_{n-1}). \quad (4.2)$$

Let

$$\begin{pmatrix} \psi_n(\beta) & 0 \\ v_R(\beta) & 1 \end{pmatrix} = \begin{pmatrix} A & B & 0 \\ C & D & 0 \\ E & F & 1 \end{pmatrix}$$

(cf. [16, Lemma 3.12]), \hspace{1cm} (4.3)

where $A$ is a square matrix of size $(n-2)$, $B$ is a column of height $n-2$, $C$ and $E$ are rows of length $n-2$, and $D, F \in \Lambda$. By Equation (3.2),

$$\psi_{n+1}(t^R(\beta) t_n) = \begin{pmatrix} \psi_n(\beta) & 0 \\ v^R(\beta) & 1 \end{pmatrix} \begin{pmatrix} I_{n-2} & 0 \\ 0 & 1 \\ x & -1 \end{pmatrix} \begin{pmatrix} I_{n-2} & 0 \\ 0 & 1 \\ x & -1 \end{pmatrix} = \begin{pmatrix} A & B & xB \\ C & D & xD \\ E & F & xF - 1 \end{pmatrix}. \quad (4.4)$$

Subtracting the identity to (4.4), we have

$$\psi_{n+1}(t^R(\beta) t_n) - I_n = \begin{pmatrix} A - I_{n-2} & B & xB \\ C & D - 1 & xD \\ E & F & xF - 2 \end{pmatrix} \xrightarrow{C_{n,n-1}(-x)} \ldots$$

$$\ldots \rightarrow \begin{pmatrix} A - I_{n-2} & B & 0 \\ C & D - 1 & x \\ E & F & -2 \end{pmatrix} = M. \quad (4.5)$$

Hence, \(\det(\psi_{n+1}(t^R(\beta) t_n) - I_n) = \det(M).\) Observe that

$$\begin{pmatrix} \psi_n(\beta) - I_{n-1} \\ v^R(\beta) \end{pmatrix} = \begin{pmatrix} A - I_{n-2} & B \\ C & D - 1 \\ E & F \end{pmatrix}.$$
Then,
\[
U_{n-1} \left( \frac{1}{x} \right) \det(\psi_{n+1}(iR(\beta)t_n) - I_n) = U_{n-1} \left( \frac{1}{x} \right) \det(M) = \det \begin{pmatrix}
A - I_{n-2} & B & 0 \\
C & D - 1 & x \\
U_{n-1} \left( \frac{1}{x} \right) E & U_{n-1} \left( \frac{1}{x} \right) F & -2U_{n-1} \left( \frac{1}{x} \right)
\end{pmatrix} = \det(N).
\]

For \(1 \leq i \leq n - 2\), if we add \(U_{i-1} \left( \frac{1}{x} \right)\) times the \(i\)-th row of \(N\) to its \(n\)-th row, by Lemma 3.7 and the recurrence relation of Chebyshev polynomials (2.6), we have
\[
\det(N) = \det \begin{pmatrix}
A - I_{n-2} & B & 0 \\
C & D - 1 & x \\
0 & 0 & -2U_{n-1} \left( \frac{1}{x} \right) + xU_{n-2} \left( \frac{1}{x} \right)
\end{pmatrix}
= (-2U_{n-1} \left( \frac{1}{x} \right) + xU_{n-2} \left( \frac{1}{x} \right)) \det(\psi_{n}(\beta) - I_{n-1})
= -xU_{n} \left( \frac{1}{x} \right) \det(\psi_{n}(\beta) - I_{n-1}).
\]

Obtaining the Equation (4.2) as desired.

(b) This case is treated similarly by Remark 3.4.

\[\Box\]

**Lemma 4.9.** For \(n \geq 3\), \(\beta \in T_n\) and \(1 \leq i \leq n - 1\), we have:

(a) \(f_{n+1}(iR(\beta)t_{n,i}) = x^{2i} f_n(\beta)\),

(b) \(f_{n+1}(iL(\beta)t_{1,i}) = x^{2i} f_n(\beta)\).

**Proof.** (a) The equation \(f_{n+1}(iR(\beta)t_{n,i}) = x^{2i} f_n(\beta)\) is equivalent to
\[
U_{n-1} \left( \frac{1}{x} \right) \det(\psi_{n+1}(iR(\beta)t_{n,i} - I_n)) = -x^{2i+1} U_n \left( \frac{1}{x} \right) \det(\psi_n(\beta) - I_{n-1}). \tag{4.6}
\]

As in Equation (4.3) of the previous lemma, we express \(\psi_{n+1}(iR(\beta))\) in blocks as
\[
\begin{pmatrix}
\psi_{n}(\beta) \\
v^R(\beta)
\end{pmatrix} =
\begin{pmatrix}
A & B & 0 \\
C & D & 0 \\
E & F & 1
\end{pmatrix}, \tag{4.7}
\]

where \(A\) is a square matrix of size \((n - i - 2)\), \(C\) is a \((i + 1) \times (n - i - 2)\) matrix, \(D\) is a square matrix of size \((i + 1)\), \(B\) and \(F\) are rows of length \(i + 1\) and \(E\) is a row of length
For the case $i = n - 2$, we omit the blocks $A$, $C$ and $E$. For every block, we write its entries by the same letter, for instance $B = (B_l)_{1 \leq l \leq i+1}$. Using Lemma 3.5, we know the description of $\psi_{n+1}(t_{n,i})$, obtaining the product
\[
\psi_{n+1}(t^R(\beta) t_{n,i}) = \begin{pmatrix}
A & B' \\
C & D' \\
E & F'
\end{pmatrix},
\]
where $A$, $C$, $E$ are as in (4.7), $B'$ and $F'$ are rows of length $i+2$, and $D'$ is a $(i+1) \times (i+2)$ matrix. By direct computation, we obtain that the entries of the blocks are the following:

\[
B_l = \begin{cases}
B_1 & l = 1, \\
xB_1 + (x^2 - 1)B_2 + \sum_{j=1}^{i-1} x^j(x^2 - 1)B_{j+2} & l = 2, \\
x^{l-1}B_1 + x^{l-2}(x^2 - 2)B_2 + \sum_{j=1}^{i-1} [x^{j+l-2}(x^2 - 1) + \delta_j^{l-2}]B_{j+2} & 3 \leq l \leq i + 1, \\
x^{l+1}B_1 + x^l(x^2 - 2)B_2 + \sum_{j=1}^{i-1} x^{j+l}(x^2 - 1)B_{j+2} & l = i + 2,
\end{cases}
\]

\[
D_{k,l} = \begin{cases}
D_{k,1} & 1 \leq k \leq i, l = 1, \\
xD_{k,1} + (x^2 - 1)D_{k,2} + \sum_{j=1}^{i-1} x^j(x^2 - 1)D_{k,j+2} & 1 \leq k \leq i, l = 2, \\
x^{l-1}D_{k,1} + x^{l-2}(x^2 - 2)D_{k,2} + \sum_{j=1}^{i-1} [x^{j+l-2}(x^2 - 1) + \delta_j^{l-2}]D_{k,j+2} & 1 \leq k \leq i, 3 \leq l \leq i + 1, \\
x^{l+1}D_{k,1} + x^l(x^2 - 2)D_{k,2} + \sum_{j=1}^{i-1} x^{j+l+i}(x^2 - 1)D_{k,j+2} & 1 \leq k \leq i, l = i + 2,
\end{cases}
\]

\[
F_l = \begin{cases}
F_1 & l = 1, \\
xF_1 + (x^2 - 1)F_2 + \sum_{j=1}^{i-1} x^j(x^2 - 1)F_{j+2} - x^i & l = 2, \\
x^{l-1}F_1 + x^{l-2}(x^2 - 2)F_2 + \sum_{j=1}^{i-1} [x^{j+l-2}(x^2 - 1) + \delta_j^{l-2}]F_{j+2} - x^{i+l-2} & 3 \leq l \leq i + 1, \\
x^{l+1}F_1 + x^l(x^2 - 2)F_2 + \sum_{j=1}^{i-1} x^{j+l+i}(x^2 - 1)F_{j+2} - x^{2i} + 1 & l = i + 2.
\end{cases}
\]

Subtracting the identity, we obtain the matrix $\psi_{n+1}(t^R(\beta) t_{n,i}) - I_n$. Then, applying a sequence of elementary operations to this matrix, we obtain
\[
J = \begin{pmatrix}
\psi_n(\beta) - I_{n-1} & 0 \\
x & x^2 - 2 \\
x(x^2 - 1) & \vdots \\
x^{i-1}(x^2 - 1) & -x^i
\end{pmatrix}.
\]
More precisely, we have
\[
(\psi_{n+1}(t^R(\beta)I_{n,i}) - I_n) \xrightarrow{C_{n,n-i}(x^j)} \bullet \xrightarrow{R_n(x^j)} \bullet \xrightarrow{C_{n-1,2i+1,n-i}(-x^{j-1})} \bullet \xrightarrow{C_{n-i-2,t,n-i}(-x^{j-2})} \bullet \xrightarrow{C_{n-i-2,t,n-i}(-x^{j-2})} \ldots
\]
\[
\ldots \xrightarrow{\bullet} \xrightarrow{C_{n-i,n-i-1}(-x^j)} \bullet \xrightarrow{C_{n-i,n-i}(-x^j)} \bullet \xrightarrow{C_{n-i,n-i+1}(-x^j)} \bullet \xrightarrow{C_{n,n-i}} J.
\]
By Corollary 3.7, we know the description of \(v^R(\beta)\) in terms of the rows of \(\psi_{n}(\beta) - I_{n-1}\) and the Chebyshev polynomials. Thus we obtain the matrix
\[
K = \begin{pmatrix}
\psi_n(\beta) - I_{n-1} & 0 \\
x & x^2 - 2 \\
x(x^2 - 1) & \vdots \\
\vdots & x^{i-1}(x^2 - 1) & x^i U_{n-i-2}(\frac{1}{x}) + (x^2 - 2)U_{n-i-1}(\frac{1}{x}) + \sum_{k=1}^{i-1} x^{i-k}(x^2 - 1)U_{n-1-k}(\frac{1}{x}) - x^i U_{n-1}(\frac{1}{x})
\end{pmatrix},
\]
by applying the following sequence of elementary operations to \(J\),
\[
J \xrightarrow{R_n(U_{n-1}(\frac{1}{x}))} \bullet \xrightarrow{R_{n,j}(U_{j-1}(\frac{1}{x}))} K.
\]
Using iteratively the recurrence relation of the Chebyshev polynomials (2.6), we obtain
\[
-x^{i+1}U_n(\frac{1}{x}) = xU_{n-i-2}(\frac{1}{x}) + (x^2 - 2)U_{n-i-1}(\frac{1}{x}) + \sum_{k=1}^{i-1} x^{i-k}(x^2 - 1)U_{n-1-k}(\frac{1}{x}) - x^i U_{n-1}(\frac{1}{x})
\]
Therefore,
\[
K = \begin{pmatrix}
\psi_n(\beta) - I_{n-1} & 0 \\
x & x^2 - 2 \\
x(x^2 - 1) & \vdots \\
\vdots & x^{i-1}(x^2 - 1) & -x^{i+1}U_n(\frac{1}{x})
\end{pmatrix},
\]
Finally, following the sequence of elementary operations that transform \(\psi_{n+1}(t^R(\beta)I_{n,i} - I_n)\) into \(K\), we conclude that:
\[
\det(\psi_{n+1}(t^R(\beta)I_{n,n-i}) - I_n) = \frac{x^i}{U_{n-1}(\frac{1}{x})} \det(K)
\]
\[
= -x^{2i+1} \frac{U_n(\frac{1}{x})}{U_{n-1}(\frac{1}{x})} \det(\psi_n(\beta) - I_{n-1}),
\]
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and Equation (4.6) follows. For $i = n - 1$, the proof is analogue with the corresponding changes for the matrix $\psi_{n+1}(t_{n,n-1})$ (see Lemma 3.5).

(b) This case is treated similarly to (a) by Remark 3.4.

Example 4.10. The twins $\alpha = (t_1t_2)^3 \in T_3$ and $\beta = (t_1t_2)^3t_3t_3t_3 \in T_4$ are Gotin-Markov equivalent. Let us see what happens when we compute their images under $f_n$. From the definition,

$$\psi_3(t_1) = \begin{pmatrix} -1 & 0 \\ x & 0 \end{pmatrix} \quad \text{and} \quad \psi_3(t_2) = \begin{pmatrix} 1 & x \\ 0 & -1 \end{pmatrix}. $$

It follows that

$$\psi_3(t_1t_2)^3 = \begin{pmatrix} -x^4 + 3x^2 - 1 & -x^5 + 4x^3 - 3x \\ x^5 - 4x^3 + 3x & x^6 - 5x^4 + 6x^2 - 1 \end{pmatrix}. $$

Therefore

$$\det(\psi_3(\alpha) - I_2) = \det \begin{pmatrix} -x^4 + 3x^2 - 2 & -x^5 + 4x^3 - 3x \\ x^5 - 4x^3 + 3x & x^6 - 5x^4 + 6x^2 - 2 \end{pmatrix} = -x^6 + 6x^4 - 9x^2 + 4 = (-x^2 + 4)(x^4 - 2x^2 + 1).$$

Since $U_2(z) = -1 + 4z^2$, then

$$f_3(\alpha) = \frac{\det(\psi_3(\alpha) - I_2)}{(-x)^2U_2(\frac{1}{x})} = \frac{(-x^2 + 4)(x^4 - 2x^2 + 1)}{-x^2 + 4} = x^4 - 2x^2 + 1.$$

On the other hand,

$$\psi_4(t_1) = \begin{pmatrix} -1 & 0 & 0 \\ x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \psi_4(t_2) = \begin{pmatrix} 1 & x & 0 \\ 0 & -1 & 0 \\ 0 & 0 & x \end{pmatrix}, \quad \psi_4(t_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & x \\ 0 & 0 & -1 \end{pmatrix}. $$

It follows that

$$\psi_4((t_1t_2)^3t_3t_3t_3) = \begin{pmatrix} -x^4 + 3x^2 - 1 & -x^7 + 4x^3 - 4x^3 + 2x & -x^8 + 5x^6 - 8x^4 + 5x^2 \\ x^5 - 4x^3 + 3x & x^8 - 5x^6 + 7x^4 - 4x^2 + 1 & x^9 - 6x^7 + 12x^5 - 10x^3 + 2x \\ x^4 - x^2 & x^7 - 2x^5 + 2x^3 - 2x & x^8 - 3x^6 + 4x^4 - 3x^2 + 1 \end{pmatrix}. $$

Therefore

$$\det(\psi_4(\beta) - I_3) = 4x^8 - 16x^6 + 20x^4 - 8x^2 = 4x^2(x^2 - 2)(x^2 - 1)^2.$$
Since $U_3(z) = -4z + 8z^3$, then
$$f_4(\beta) = \frac{\det(\psi_4(\beta) - I_3)}{(-x)^3 U_3 \left( \frac{1}{x} \right)} = \frac{4x^2(x^2 - 2)(x^2 - 1)^2}{4x^2 - 8} = x^2(x^2 - 1)^2 = x^6 - 2x^4 + x^2.$$

Hence, $f_4(\beta) = x^2 f_3(\alpha)$.

**Remark 4.11.** Note that with Lemmas 4.6, 4.8 and 4.9, we have actually proved that if $\alpha \in T_n$ and $\beta \in T_m$ are Gotin-Markov equivalent, then $f_n(\alpha)$ and $f_m(\beta)$ are equal up to some factor $x^{2k}$ for some $k \in \mathbb{Z}$.

**Proposition 4.12.** The function $f_n$ satisfies the skein relation:
$$f_n \left( \begin{array}{c} \mathcal{X} \end{array} \right) - f_n \left( \begin{array}{c} \mathcal{Y} \end{array} \right) = (x^2 - 1) \left( f_n \left( \begin{array}{c} \mathcal{X} \end{array} \right) - f_n \left( \begin{array}{c} \mathcal{X} \end{array} \right) \right). \tag{4.8}$$

**Proof.** By (b) of Lemma 4.6, the skein relation (4.8) is equivalent to the equation
$$f_n(\beta t_{i+1} t_i) - f_n(\beta t_i t_{i+1}) = (x^2 - 1)(f_n(\beta t_i) - f_n(\beta t_{i+1})), \tag{4.9}$$
for some $\beta \in T_n$. Therefore, (4.8) is equivalent to prove the formula
$$\det(\psi_n(\beta t_{i+1} t_i) - I_{n-1}) - \det(\psi_n(\beta t_i t_{i+1}) - I_{n-1}) = (x^2 - 1)[\det(\psi_n(\beta t_i) - I_{n-1}) - \det(\psi_n(\beta t_{i+1}) - I_{n-1})].$$

The corresponding matrix for $\psi_n(t_{i+1} t_i)$ is
$$\psi_n(t_{i+1} t_i) = \begin{pmatrix} I_{i-2} & \mathcal{Y} \\ \mathcal{X} & \mathcal{X} & \mathcal{X} \\ \mathcal{X} & \mathcal{X} & \mathcal{X} & \mathcal{X} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then, we may write $\psi_n(\beta)$ in blocks as follows:
$$\psi_n(\beta) = \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & J \end{pmatrix} = \begin{pmatrix} A & B_1 & B_2 & B_3 & B_4 \\ D & E_{k,1} & E_{k,2} & E_{k,3} & E_{k,4} \\ G & H_1 & H_2 & H_3 & H_4 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}, \quad (1 \leq k \leq 4)$$
where $A$ is a square matrix of size $(i-2)$, $B$ is a $(i-2) \times (4)$ matrix, $C$ is a $(i-2) \times (n-i-3)$ matrix, $D$ is a $4 \times (i-2)$ matrix, $E$ is a square matrix of size 4, $F$ is a $4 \times (n-i-3)$ matrix, $G$ is a $(n-i-3) \times (i-2)$ matrix, $H$ is a $(n-i-3) \times 4$ matrix and $J$ is a square matrix if size $(n-i-3) \times (n-i-3)$. For $1 \leq l \leq 4$, we write $B_l$ and $H_l$ for the columns of $B$ and $H$, respectively. The resulting product of $\psi_n(\beta)$ and $\psi_n(t_it_{i+1})$ is:

$$
\begin{pmatrix}
A & B_1 & E_{k,1} & H_1 \\
\vdots & \vdots & \vdots & \vdots \\
D & E_{k,4} & H_4 & J
\end{pmatrix}
\begin{pmatrix}
x^3B_1 - (x^2-1)B_2 + x(x^2-2)B_3 + x^2B_4 \\
x^2B_1 - xB_2 + (x^2-1)B_3 + xB_4 \\
x^2E_{k,1} - xE_{k,2} + (x^2-1)E_{k,3} + xE_{k,4} \\
x^2H_1 - xH_2 + (x^2-1)H_3 + xH_4
\end{pmatrix}
\equiv
\begin{pmatrix}
B_4 \\
C
\end{pmatrix}.
$$

Subtracting the identity, we obtain $\psi_n(\beta t_it_{i+1}) - I_{n-1}$. Then, we apply the following sequence of elementary operations

$$
\psi_n(\beta t_it_{i+1}) - I_{n-1} \xrightarrow{C_{i+1,-1}(-x)} \xrightarrow{C_{i+1,-1}(-x^2)} \xrightarrow{C_{i+1,-1}(x)} \xrightarrow{C_{i+1,-1}(x)} \cdots
$$

where $M$ coincides with $\psi_n(\beta) - I_{n-1}$ except for the $i$-th and $(i+1)$-th columns. More precisely, if we write $\psi_n(\beta) - I_{n-1} = (a_1, a_2, \ldots, a_{n-1})$ and $M = (m_1, m_2, \ldots, m_{n-1})$ in vector columns $a_i, m_i \in \Lambda^{n-1}$ for $1 \leq i \leq n-1$, then

$$
m_j = \begin{cases} 
a_i + xu & \text{for } j = i, \\
a_{i+1} + u & \text{for } j = i + 1, \\
a_j & \text{otherwise},
\end{cases}
$$

where $u = \begin{pmatrix} 0 \\ -x^2 \\ x \\ -x + 2 \\ -x \\ 0 \end{pmatrix}$.

By the sequence of elementary operations applied to $\psi_n(\beta t_it_{i+1}) - I_{n-1}$, it follows that $\det(\psi_n(\beta t_it_{i+1}) - I_{n-1}) = -\det(M)$.

On the other hand, for the term $\psi_n(t_{i+1}t_it_{i+1})$ given by

$$
\psi_n(t_{i+1}t_it_{i+1}) = \begin{pmatrix}
I_{i-2} & 0 \\
0 & I_{n-i-3}
\end{pmatrix},
$$

\[ \psi_n(t_{i+1}t_it_{i+1}) = \begin{pmatrix}
I_{i-2} & 0 \\
0 & I_{n-i-3}
\end{pmatrix},
\]

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the resulting product of $\psi_n(\beta)$ and $\psi_n(t_{i+1}t_i t_{i+1})$ is

$$
\begin{pmatrix}
A & B_1 & xB_1 + (x^2 - 1)B_2 - xB_1 + x^2B_1 \\
\vdots & \vdots & \vdots \\
D & E_{k,1} & xE_{k,1} + (x^2 - 1)E_{k,2} - xE_{k,1} + x^2E_{k,1} \\
\vdots & \vdots & \vdots \\
G & H_1 & xH_1 + (x^2 - 1)H_2 - xH_1 + x^2H_1 \\
& & \vdots \\
& & \vdots \\
& & J
\end{pmatrix}.
$$

In the same way as before, we subtract the identity and apply the following sequence of elementary operations

$$
\psi_n(\beta t_{i+1}t_i t_{i+1}) - I_{n-1} \xrightarrow{C_{i,1}(x)} \xrightarrow{C_{i,2}(x)} \xrightarrow{C_{i,3}(x)} \cdots \xrightarrow{C_{i,1}(x)} \xrightarrow{C_{i,2}(x)} M',
$$

where $M'$ coincides with $\psi_n(\beta) - I_{n-1}$ except for the $i$-th and $(i+1)$-th columns. Namely, $M' = (m'_1, \ldots, m'_{n-1})$ with

$$
m'_j = \begin{cases} 
    a_i + v & \text{for } j = i, \\
    a_{i+1} + xv & \text{for } j = i + 1, \\
    a_j & \text{otherwise,}
\end{cases}
$$

where $v = \begin{pmatrix} 0 \\ -x \\ -x^2 + 2 \\ x \\ -x^2 \\ 0 \end{pmatrix}$.

The determinant is computed as $\det(\psi_n(\beta t_{i+1}t_i t_{i+1}) - I_{n-1}) = -\det(M')$. Consequently, on the left-hand side of Equation (4.9) we have

$$
\det(\psi_n(\beta t_{i+1}t_i t_{i+1}) - I_{n-1}) - \det(\psi_n(\beta t_{i+1}t_i t_{i+1}) - I_{n-1}) = -\det(M) + \det(M').
$$

Recalling that $\psi_n(\beta) - I_{n-1} = (a_1, \ldots, a_{n-1})$, in what follows we write dots for entries $a_j$ with $j \neq i, i + 1$. We thus get

$$
-\det(M) + \det(M') = -\det(\ldots, a_i + xv, a_{i+1} + u, \ldots) + \det(\ldots, a_i + v, a_{i+1} + xv, \ldots) \\
= \det(\ldots, v - xu, a_{i+1}, \ldots) + \det(\ldots, a_i, xv - u, \ldots) \\
= (x^2 - 1)[\det(\ldots, w, a_{i+1}, \ldots) - \det(\ldots, a_i, w', \ldots)],
$$

where $w, w'$ are column vectors given by

$$
w = \begin{pmatrix} 0 \\ x \\ -2 \\ x \\ 0 \\ 0 \end{pmatrix}^{i\text{-th}} \quad \text{and} \quad w' = \begin{pmatrix} 0 \\ 0 \\ x \\ -2 \\ x \\ 0 \end{pmatrix}^{i\text{-th}}.
$$
Now we work the right-hand side of Equation (4.9). The resulting product of $\psi_n(\beta)$ and $\psi_n(t_i)$ is:

\[
\psi_n(\beta t_i) = \begin{pmatrix} A & B_1 & xB_1 - B_2 + xB_3 & B_3 & B_4 & C \\ D & E_{k,1} & \vdots & \vdots & \vdots & F \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ G & H_1 & xH_1 - H_2 + xH_3 & H_3 & H_4 & J \end{pmatrix}.
\]

In a similar fashion, we subtract the identity and apply the following sequence of elementary operations

\[
\psi_n(\beta t_i) - I_{n-1} \xrightarrow{C_{i,i-1}(-x)} \xrightarrow{C_{i,i}(-x)} \xrightarrow{C_{i,1}} P,
\]

where $P$ coincides with $\psi_n(\beta) - I_{n-1}$ except for the $i$-th column. Namely, $P = (p_1, \ldots, p_{n-1})$ with

\[
p_j = \begin{cases} a_i - w & \text{for } j = i, \\ a_j & \text{otherwise}, \end{cases} \quad (1 \leq j \leq n - 1)
\]

where $w$ is as in Equation (4.12). By the sequence of elementary operations, it follows that $\det(\psi_n(\beta t_i) - I_{n-1}) = -\det(P)$.

For the other term in the right-hand side of (4.9), the resulting product of $\psi_n(\beta)$ and $\psi_n(t_{i+1})$ is:

\[
\psi_n(\beta t_{i+1}) = \begin{pmatrix} A & B_1 & B_2 & xB_2 - B_3 + xB_1 & B_4 & C \\ D & E_{k,1} & E_{k,2} & \vdots & \vdots & F \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ G & H_1 & H_2 & xH_2 - H_3 + xH_4 & H_4 & J \end{pmatrix}.
\]

Following the similar procedure of subtract the identity and applying a sequence of elementary operations, we have

\[
\psi_n(\beta t_i) - I_{n-1} \xrightarrow{C_{i,i+1}(-x)} \xrightarrow{C_{i,i+2}(-x)} \xrightarrow{C_{i,i+1}} P',
\]

where $P' = (p'_1, \ldots, p'_{n-1})$ with

\[
p'_j = \begin{cases} a_{i+1} - w' & \text{for } j = i + 1, \\ a_j & \text{otherwise}, \end{cases} \quad (1 \leq j \leq n - 1)
\]

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and \( w' \) is as in Equation (4.12). By the sequence of elementary operations applied to 
\( \psi_n(\beta t_{i+1}) - I_{n-1} \), its determinant is computed as 
\( \det(\psi_n(\beta t_{i+1}) - I_{n-1}) = -\det(P') \).
Calculating the right-hand side of Equation (4.9), we obtain
\[
det(\psi_n(\beta t_i) - I_{n-2}) - det(\psi_n(\beta t_{i+1}) - I_{n-1}) = -\det(P) + \det(P').
\] (4.13)

Simplifying,
\[
-\det(P) + \det(P') = -\det(\ldots, a_i - w, a_{i+1}, \ldots) + \det(\ldots, a_i, a_{i+1} - w', \ldots) \\
= \det(\ldots, w, a_{i+1}, \ldots) - \det(\ldots, a_i, w', \ldots) \\
= \frac{-\det(M) + \det(M')}{(x^2 - 1)},
\]
where the last equality is given by (4.11). Therefore, by Equations (4.10), (4.13) we obtain Equation (4.9), which is equivalent to the skein relation (4.8).

4.2 The invariant \( Q \)

In subsection 4.1 we studied how the function \( f_n : T_n \to \mathbb{Z}(x) \) behaves under Gotin-Markov moves. Now we use the theory of twin groups to define an invariant of doodles.

Let \( D \sim D' \) be two equivalent doodles, and let \( \alpha, \beta \) be two twins with closures \( D \) and \( D' \) respectively. Thus, \( \alpha \sim_G \beta \) are Gotin-Markov equivalent. By Remark 4.11, we know \( f_n(\alpha) \) and \( f_m(\beta) \) are the same polynomial up to a factor \( x^{2k} \) for some integer \( k \). Furthermore, by the skein relation (4.8), all the polynomials of a doodle \( D \) are polynomials in \( \mathbb{Z}[x^2] \). In this way, the lowest degree polynomial \( f_n(\alpha) \) over all twins \( \alpha \) with closure \( D \) will be an invariant for \( D \).

**Definition 4.13.** We define \( Q : \mathcal{D} \to \mathbb{Z}[x^2] \) by
\[
Q(D) := \gcd\{f_n(\alpha) \in \mathbb{Z}[x^2] \mid \hat{\alpha} = D, \ \alpha \in T_n, \ n \geq 1\}.
\] (4.14)
where \( D \in \mathcal{D} \) and \( \gcd \) is the greatest common divisor.

As a direct consequence of the previous discussion, we have the following result.

**Theorem 4.14.** The function \( Q \) is an invariant of oriented doodles.

**Example 4.15.** According to [4, Theorem 4.2], the first non-trivial doodle has 6 crossings, and is the Borromean doodle (see Figure 2b). Let us compute its polynomial under \( Q \). In Example 4.10, we compute two associated polynomials to the Borromean doodle through two Gotin-Markov equivalent twins. Furthermore, for any twin \( \beta \) with closure the Borromean doodle, we have that \( \deg(f_n((t_1t_2)^3)) \leq \deg(f_n(\beta)) \). Thus,
\[
Q(\bigcirc \bigcirc) = x^4 - 2x^2 + 1.
\]
Remark 4.16. For a doodle $D$, if exist a twin $\alpha_0$ with closure the minimal representative $D_0$ of $D$, then $Q(D)$ is obtained with the corresponding twin $\alpha_0$, i.e., $Q(D) = f_n(\alpha_0)$. In general, not every minimal representative of a doodle is realizable immediately by the closure of a twin, might be necessary to produce new double points in the diagram.

Analogously to the classical Alexander polynomial of links, the invariant $Q$ vanishes on multi-component doodles with ‘unlinked’ components.

Proposition 4.17. Let $D = D_1 \sqcup D_2$ be a doodle with two disjoint components, then $Q(D) = 0$.

Proof. If one of the components is the trivial doodle, the result follows easily from the skein relation (4.8), or by definition of $f_n$ and Lemma 4.6. Let $\alpha \in T_n$ and $\beta \in T_m$ be two twins such that $\hat{\alpha}^{-1} = D_1$ and $\hat{\beta} = D_2$, then adding vertical strands on the left or right, we have $i^R(\alpha), i^L(\beta) \in T_{n+m}$ such that $i^R(\alpha)i^L(\beta) = i^L(\beta)i^R(\alpha) = D$. By Equation (3.2) and Remark 3.4 it follows

$$
\psi_{n+m}(i^R(\alpha)i^L(\beta)) = \begin{pmatrix}
\psi_n(\alpha) & 0 & 0 \\
0 & \psi_R(\alpha) & \psi_L(\beta) \\
0 & 0 & \psi_m(\beta)
\end{pmatrix}.
$$

Thus, the $n$-th column of $\psi_{n+m}(i^R(\alpha)i^L(\beta)) - I_{n+m-1}$ is zero and $f_{n+m}(i^R(\alpha)i^L(\beta))$ vanishes. Since every polynomial associated to the doodle is the same, up to a factor $x^{2k}$ for some integer $k$, it follows $Q(D) = 0$.

Despite the function $f_n$ satisfies the skein relation (4.8), the invariant $Q$ does not. A possible alternative to use diagrams to compute $Q$ is to define a non reduced invariant $R$, i.e., invariant up to factors $x^{2k}$.

In what follows, diagrams inside dotted circles mean a local picture of doodles which are identical outside.

Conjecture. Let $R : \mathcal{D} \to \mathbb{Z}[x^2]$ be an invariant up to a factor $x^{2k}$ such that:

1. $R(\emptyset) = 1$,

2. $R\left(\begin{array}{c}
\begin{array}{c}
\hat{x}
\end{array}
\end{array}\right) - R\left(\begin{array}{c}
\begin{array}{c}
\hat{y}
\end{array}
\end{array}\right) = (x^2 - 1) \left(R\left(\begin{array}{c}
\begin{array}{c}
\hat{y}
\end{array}
\end{array}\right) - R\left(\begin{array}{c}
\begin{array}{c}
\hat{x}
\end{array}
\end{array}\right)\right)$,

3. $R\left(\begin{array}{c}
\begin{array}{c}
\hat{x}
\end{array}
\end{array}\right) = x^{2k}R\left(\begin{array}{c}
\begin{array}{c}
\hat{y}
\end{array}
\end{array}\right)$.

Then, $Q(D) = \frac{R(D)}{x^{2k_0}}$ for some $k_0 \in \mathbb{Z}$. Furthermore, $R$ is uniquely defined by these skein relations.
5 Some computations

In this section we compute the invariant $Q$ for many doodles found in the literature [4, 9, 19, 22, 23, 26]. The results are summarized in a table in Subsection 5.2.

5.1 Families of doodles

In [4], Bartholomew-Fenn-Kamada-Kamada give a geometric construction of infinite families of doodles from polygons. Furthermore, there is a note about the bijection between minimal representatives of doodles and the 1-skeleta of 3-dimensional polyhedra whose vertices have valency four. We only recall one of those examples to illustrate the geometric construction. For the rest, we describe them in a naive way or in terms of twins. For original constructions, we refer to [4].

Example 5.1. Define the doodle $B_n$ as follows. Start with two concentric $n$-gons with vertices $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$, respectively, and add the missing edges to construct the triangles $X_i Y_i X_{i+1}$ for $1 \leq i \leq n$ cyclically modulo $n$ (see Figure 8). All vertices of the resulting diagram have valency four, and it is natural how to smooth the edges to obtain the doodle $B_n$. By the symmetry of the diagram, it is easy to see that $\hat{(t_1 t_2)^n}$ is $B_n$ for any $n$. In particular, when $n$ is divisible by 3, the twin $(t_1 t_2)^n$ induces the trivial permutation, i.e., is an element of the pure twin group $PT_3$. Thus, the number of components of $B_n$ is three if $n$ is divisible by 3, called the $n$-generalized Borromean doodle; otherwise, $B_n$ has one component, called the $n$-poppy doodle.

Computations prompt the invariant $Q$ never vanishes on this family and its restriction to the family is very strong such that distinguishes, i.e., if $Q(B_n) = Q(B_m)$, then $n = m$. 

![6-Borromean](image1.png) ![7-poppy](image2.png)

Figure 8: Generalized Borromean and $n$-poppy doodles.

Example 5.2. The doodle $C_n^1$ is constructed from the previous example $D_n$, adding a circle as in Figure 9. In terms of twins, $C_n^1$ is the closure of the twin $(t_1 t_2 t_3 t_2)^n$. The doodle $C_n^1$ has four components if $n$ is divisible by 3, otherwise $C_n^1$ has two components.

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The doodle $C^1_n$ can be generalized to the doodle $D_n$ with $r$ circles added. Written in twins, define the doodle $C^r_n$ as the closure of the twin $$(t_1 (t_2 \cdots t_{r+1}) t_{r+2} (t_{r+1} \cdots t_2))^n.$$ In this case, the doodle $C^r_n$ has $r + 3$ components if $n$ is divisible by 3, otherwise $C^r_n$ has $r + 1$ components.

![Doodles](Figure 9: Doodles $C^r_n$)

Testing the invariant under different families of doodles, unfortunately we found $Q$ vanishes for an infinite family of doodles.

**Proposition 5.3.** For integers $r \geq 1$ and $n \geq 3$, $Q(C^r_n) = 0$.

**Proof.** The doodle $C^r_n$ is the closure of the twin $\alpha^n_r = (t_1 (t_2 \cdots t_{r+1}) t_{r+2} (t_{r+1} \cdots t_2))^n$. For $n \geq 3$, applying induction over $r$ and using iteratively Lemma (3.1), the matrix $\psi_{r+3}(\alpha_r)^n - I_{r+2}$ has $r + 1$ columns equal up to a power of $x$. More precisely,

$$\psi_{r+3}(\alpha_r)^n - I_{r+2} = (c_1, x^r c_{r+2}, x^{r-1} c_{r+2}, \ldots, x c_{r+2}, c_{r+2}),$$

written in columns. Hence $\det(\psi_{r+3}(\alpha_r)^n - I_{r+2}) = 0$ and the result follows. 

**Example 5.4.** Another infinite family related to $C^r_n$ (by the geometric construction given in [4]) is the family $D^1_n$. It consists of a circle as the skeleton and an array of $n$ trivial doodles $O_1, \ldots, O_n$ overlapping as in Figure 10. Doodle $D^r_n$ are a natural generalization of $D^1_n$ by adding $r$ concentric circles in the skeleton. We do not know a general formula for the twin representation of $D^r_n$, but we have some computations for small values. The invariant $Q$ vanishes for these small cases.

### 5.2 Table

The following table summarizes values of the invariant $Q$ for many doodles. We follow a format very similar to the table of knots given in [1]. The first line following the picture of the doodle, denotes the number of crossings and components of the minimal representative. The second line is the twin representation of the doodle, where the number $k$
denotes the generator $t_k$. The last line codified the polynomial invariant $Q(D) \in \mathbb{Z}[x^2]$, the first number in curly brackets denotes the half of the maximum degree of the polynomial and the next sequence in parenthesis denotes the coefficients in even powers, from higher to lower degree. For instance, $\{7\}(1, 2, -1, -2, 1)$ denotes the polynomial $x^{14} + 2x^{12} - x^{10} - 2x^8 + x^6$.

**One component doodles**

![Doodle 1](image1)

: $8^1$

: $(12)^4$

: $\{2\}(1, -4, 4)$

![Doodle 2](image2)

: $11^1$

: $(23)^2(21)^2321$

: $\{4\}(1, -2, 3, -2, 1)$

![Doodle 3](image3)

: $9^1$

: $(123)^3$

: $\{2\}(4, -4, 1)$

![Doodle 4](image4)

: $14^1$

: $(12)^7$

: $\{6\}(1, -10, 37, -62, 46, -12, 1)$

![Doodle 5](image5)

: $10^1$

: $(12)^5$

: $\{4\}(1, -6, 11, -6, 1)$

![Doodle 6](image6)

: $15^1$

: $43215(432)^245432$

: $1(32)^2$

: $\{7\}(1, 2, -1, -2, 1)$
\[
\begin{align*}
\text{15^1} & : (21)^42321323 \\
\{6\}(1, -6, 13, -14, 10, -4, 1) & \\
\text{15^1} & : (123)^5 \\
\{4\}(16, -48, 44, -12, 1) & \\
\text{16^1} & : (2143)^242342123 \\
\{5\}(1, 4, 0, -8, 4) & \\
\text{16^1} & : (12)^8 \\
\{7\}(1, -12, 56, -128, 148, -80, 16) & \\
\text{17^1} & : (43)^254321(43)^2 \\
(23)^2543212 & \\
\{9\}(1, -2, 1, 2, -2, 1) & \\
\text{17^1} & : (21)^4232132321 \\
\{7\}(1, -8, 24, -36, 32, -16, 4) & \\
\text{17^1} & : 543214345465432 \\
(43)^2567654321243 & \\
456547 & \\
\{14\}(1, -4, 4) & \\
\text{20^1} & : (12)^{10} \\
\{9\}(1, -16, 106, -376, 771, -920, 610, -200, 25) & \\
\text{21^1} & : 76543212(43)^2567 \\
654321(43)^2565432 & \\
(543)^2 & \\
\{14\}(1, 2, -2, 1, -2, 1) & \end{align*}
\]
: 21\(^1\) : 2(12\(^2\))321323

: \{9\}(1, -12, 58, -148, 223, -212, 130, -48, 9)

\[ a = 211 \]

: 22\(^1\) : (12\(^{11}\))

: \{10\}(1, -18, 137, -574, 1444, -2232, 2083, -1106, 295, -30, 1)

\[ b = 211 \]

: 21\(^1\) : 54321(32\(^2\))454321

\[ c = 211 \]

\[ (34)^4(32)^2 \]

: \{11\}(1, -6, 11, -4, -7, 8, -1, -2, 1)

\[ d = 211 \]

: \{10\}(1, -24, 218, -960, 2251, -2880, 1962, -648, 81)

\[ e = 211 \]

: 24\(^1\) : (1234\(^6\))

\[ f = 211 \]

: \{20\}(4, -4, 1)

\[ g = 211 \]

: \{12\}(1, -22, 211, -1158, 4013, -9142, 13820, -13672, 8518, -3108, 581, -42, 1)

\[ h = 211 \]

: \{6\}(64, -320, 592, -496, 184, -24, 1)
Multi-component doodles
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