On thermodynamics second law in the modified Gauss Bonnet gravity

H. Mohseni Sadjadi

1 Department of Physics, University of Tehran, P.O.B. 14395-547, Tehran, Iran.

January 4, 2013

Abstract

The second law and the generalized second law of thermodynamics in cosmology in the framework of the modified Gauss-Bonnet theory of gravity are investigated. The conditions upon which these laws hold are derived and discussed.

1 Introduction

Recently, extended theories of gravity have attracted more attentions, because besides their possible ability to describe the inflation in the early universe [1], they can be proposed as candidates to explain the present acceleration of the universe [2] without requirement to introduce exotic matter with negative pressure, dubbed as dark energy. A method to generalize the Einstein theory of gravity is to consider a Lagrangian which, in addition to the Ricci scalar, includes a general function of the Gauss-Bonnet invariant [3]. This generalization has root in low energy effective action in string theory [4].

However these modifications of Einstein theory of gravity must be in agreement with astrophysical data. Severe instabilities may be appeared in the modified Gauss Bonnet gravity. Quintessence model coupled to Gauss Bonnet invariant and its stability were studied in [5]. Also, the constraints that the stability of perturbation growths puts on modified gravity were considered in [6]. The investigation of such cosmological perturbations in the presence of a barotropic fluid can also be found in [7].

In analogy with black hole thermodynamics, investigating thermodynamics laws for cosmological horizons, which are present in many of cosmological models, has also been the subject of many studies [8]. One of
these laws is the generalized second law (GSL) which asserts that the sum of the horizon entropy and the entropy of the matter field, i.e., the total entropy, is non-decreasing [9]. Our aim is to study the constraints that this law puts on the functions introduced in modified Gauss-Bonnet theory of gravity which may be similar to what was obtained in papers like [7] with a different method.

The scheme of the paper is as follows: We consider a spatially flat Friedman-Lemaitre-Robertson-Walker (FLRW) space-time whose future event horizon is in thermal equilibrium with the matter in its environment. In the framework of the modified Gauss-Bonnet gravity, by using modified Friedman equations, and by considering the entropy evolution of dynamical future event horizon (obtained via the Noether charge method) we study the necessary conditions for validity of GSL. Via an example, we will show that this law may not hold in general. The same also occurs in some stages in phantom dominated universe in the Einstein theory of gravity as was reported in [10].

We use units with $\hbar = c = G_N = k_B = 1$ and the metric signature is $(-, +, +, +)$.

2 Thermodynamics second law and GSL in the modified Gauss-Bonnet gravity

The action of the modified Gauss-Bonnet gravity is given by

$$S = \int d^4x \sqrt{-g} \left( \frac{R + f(G)}{16\pi} + \mathcal{L}_m \right),$$

where $f(G)$ is a function of the Gauss-Bonnet invariant, $G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, and $R$ is the Ricci scalar curvature. $\mathcal{L}_m$ is the matter Lagrangian density. By variation of the action with respect to the metric components $g_{\mu\nu}$, we obtain the field equations

$$8\pi T^{\mu\nu} = -\frac{1}{2}g^{\mu\nu}(R + f(G)) + R^{\mu\nu} + 2g^{\mu\nu}R\nabla^2 f_G - 4R^{\mu\nu}\nabla^2 f_G - 2R(\nabla^\mu \nabla^\nu)f_G + 4R^{\mu\alpha\nu} \nabla_\alpha \nabla^\nu f_G + 4R^{\mu\alpha\nu} \nabla_\alpha \nabla^\nu f_G - 4R^{\mu\nu} R^{\rho\delta} f_G + 2R^{\mu\nu} R^{\rho\delta} f_G + 4R^{\rho\delta} R^{\mu\nu} f_G - 4R^{\rho\delta} R^{\mu\nu} f_G + 2R^{\rho\delta} R^{\mu\nu} f_G + 4R^{\rho\delta} R^{\mu\nu} f_G,$$

where $f_G = \frac{df}{dG}$ and $T^{\mu\nu}$ are the matter energy-momentum tensor components.

We consider a spatially flat FLRW space time, endowed with the metric (in the Cartesian comoving coordinates):

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2).$$
In terms of the Hubble parameter, $H$, the Ricci scalar and Gauss-Bonnet invariant are given by
\[ R = 6(\dot{H} + 2H^2), \quad G = 24H^2(\dot{H} + H^2). \] (4)

The over dot indicates derivative with respect to $t$. Eqs. (2) give the following modified Friedman equations:
\[ \dot{H} = -4\pi(P_m + \rho_m) + 2H^3f G - 4H\dot{H} f G - 2H^2\dot{f} G \]
\[ H^2 = \frac{8\pi}{3}\rho + \frac{1}{6}(-f + Gf G - 24H^3\dot{f} G). \] (5)

$\rho_m$ and $P_m$ are energy density and pressure of the matter component which is assumed to behave as a perfect fluid at large scale. In this paper we assume that the matter is ordinary with positive pressure: $P_m > 0$.

By setting $f = 0$, the above equations reduce to the Friedman equations in the Einstein theory of gravity. The matter satisfies the continuity equation:
\[ \dot{\rho}_m + 3H(P_m + \rho_m) = 0. \] (6)

We assume that the universe possesses a future event horizon whose radius is given by
\[ R_h(t) := a(t) \int_t^\infty \frac{dt'}{a(t')} \in \mathbb{R}. \] (7)

If there is a big rip at $t_s$, we must replace $\infty$ by $t_s$. The evolution of the horizon is given by
\[ \dot{R}_h = HR_h - 1. \] (8)

The entropy of the dynamical horizon, $S_h$, can be determined by the Noether charge method\[11\] which results in
\[ S_h = -\frac{1}{8} \int_{\text{Horizon}} \left( \frac{\partial R}{\partial R_{\alpha\beta\gamma\rho}} + f G \frac{\partial G}{\partial R_{\alpha\beta\gamma\rho}} \right) \varepsilon_{\alpha\beta} \varepsilon_{\gamma\rho} dA_h, \] (9)

where $\varepsilon_{\mu\nu}$ are binormal vectors to the horizon surface and $dA_h$ is the differential surface element. By applying this result for the future event horizon we obtain:
\[ S_h = \frac{A_h}{4} \left( 1 + \frac{4}{R_h f G} \right), \] (10)

where $A_h = 4\pi R_h^2$ is the area of the future event horizon. For a de Sitter space-time, $R_h^2 = \frac{12}{\Lambda}$ and\[10\] becomes
\[ S_h = \frac{A_h}{4} \left( 1 + \frac{R}{3} f G \right). \] (11)

Although linear terms of Gauss-Bonnet invariant in $f$ change the horizon entropy, but have no influence on $\dot{S}_h$:
\[ \dot{S}_h = 2\pi R_h (HR_h - 1) + 4\pi \dot{f} G. \] (12)
For a de Sitter space time $R_h = \frac{1}{H}$ and $\dot{S}_h = 0$.

In a super-accelerated universe (i.e. $\dot{H} > 0$, $\ddot{R}_h < 0$) [12], which in Einstein theory of gravity implies $\dot{S}_h = 2\pi R_h \ddot{R}_h < 0$. But depending on the form of $f$, this may not be the case in the modified Gauss-Bonnet theory of gravity. To see this, let us consider a quasi de Sitter space time specified by the Hubble parameter

$$H = H_0 + H_0^2 \epsilon(t) t + O(\epsilon^2)$$

where $\epsilon := \frac{H}{H_0}$, $|\epsilon| \ll 1$, and $\dot{\epsilon} \sim O(\epsilon^2)$. We can expand $R_h$ and $G$ around de Sitter point ($H_0$), up to order $\epsilon^2$, as

$$G = 24H_0^4(1 + \epsilon) + 96H_0^5t \epsilon, \quad \dot{G} = 96H_0^5\epsilon,$$

$$R_h \approx \frac{1}{H}(1 - \epsilon), \quad \dot{R}_h \approx -\epsilon.$$

We have also

$$f_G = (f_{GG}(H_0) + f_{GGG}(H_0)(G - G_0)) \dot{G}(H_0),$$

where $G_0 = 24H_0^4$. Note that for a given model by a specific $f(G)$, $H_0$ is the solution of the equation

$$G_0f_G(G_0) - f(G_0) = 6H_0^2 - 16\pi \rho.$$

Collecting all together the second law of thermodynamics for the horizon ($\dot{S}_h \geq 0$) requires:

$$(-1 + 192H_0^6f_{GG})\epsilon + O(\epsilon^2) \geq 0.$$

If the system is in quintessence (phantom) phase, i.e. $\dot{H} < (>)0$, we have $\epsilon < (>)0$ and (17) leads to $(-1 + 192H_0^6f_{GG}) \leq (>)0$.

The stability of modified Gauss-Bonnet gravity around de Sitter point, $H_0$, and in the vacuum (absence of matter) was investigated in [13]. There was obtained that for

$$0 < H_0^6f_{GG}(H_0) < \frac{1}{192},$$

the theory is stable. So for vacuum solutions and for stable models which are perturbed around de Sitter point, the thermodynamics second law for the future event horizon holds whenever $\epsilon < 0$. For $\epsilon > 0$ which leads to $\dot{H} > 0$ (super accelerated universe), we obtain $\dot{S}_h < 0$.

To investigate the thermodynamics second law for the universe, besides the horizon entropy, we must also take into account the entropy of the matter. So let us study time evolution of entropy of the matter inside the future event horizon (denoted by $S_{in}$). From the first law of thermodynamics:

$$dE = TdS_{in} - PdV,$$
we obtain $\dot{S}_{in} = \frac{P_m + \rho_m}{T} \left( -3HV + \dot{V} \right)$. Then by using (8) we obtain

$$\dot{S}_{in} = -\frac{A_h}{T} (P_m + \rho_m). \tag{20}$$

For the ordinary matter with positive pressure, we have $\dot{S}_{in} < 0$. The matter is in thermal equilibrium with the horizon, to which in analogy with the black hole thermodynamics and independent of the gravity theory resulting the considered geometry [14], the temperature $T = \frac{1}{2\pi R_h}$ is attributed [15].

By using (5) we arrive at

$$\dot{S}_{in} = \pi R^3 h (2\dot{H} - 4H^3 \dot{f}_G + 8H \dot{H} \dot{f}_G + 4H^2 \ddot{f}_G). \tag{21}$$

The generalized second law states that the total entropy of the universe is a non-decreasing function of time. So validity of this law requires

$$\dot{S}_{tot} = \dot{S}_{in} + \dot{S}_{h} \geq 0. \tag{22}$$

Note that as $\dot{S}_{in} < 0$, this law requires $\dot{S}_{h} > 0$.

If the matter ingredient is a barotropic matter $P_m = w_m \rho_m$, (21) may be written in a more simple form. Using the continuity equation

$$\rho = \tilde{\rho}_0 a^{-3\gamma}, \tag{23}$$

we find out that the generalized second law is respected whenever

$$2\pi R_h (HR_h - 1) + 4\pi \dot{f}_G - 8\pi^2 R^2 h \gamma \tilde{\rho}_0 a^{-3\gamma} \geq 0. \tag{24}$$

We have defined $\gamma = w_m + 1 \geq 1$.

In the Einstein theory of gravity, as we have expressed before, for a (super) accelerated expansion $\dot{S}_{h}(<) > 0$ holds, but we have also $\gamma(<) > 0$ (indicating that the universe is dominated by (phantom) quintessence dark energy), leading to $S_{in}(>) < 0$. Therefore the generalized second law can be still valid. In our model both the horizon entropy and Friedman equations are modified so we may have an accelerated universe even with ordinary matter with decreasing entropy. Besides, the horizon entropy is not necessarily decreasing in phantom ($\dot{H} > 0$) phase or increasing in quintessence phase ($\dot{H} < 0$) of acceleration. In a super accelerated universe with an ordinary matter as $\dot{R}_H < 0$ and $\dot{S}_{in} < 0$, a necessary conditions required for GSL to hold is $\dot{f}_G = f_{GG} \dot{G} > 0$. Note that without the correction term in the horizon entropy, GSL would be violated.

As an illustration of our results, in the following, we will try to examine the validity of the GSL through an example. Consider a super accelerated universe with a big rip singularity at $t = t_s$:

$$a = \frac{a_0}{(t_s - t)^n}, \quad 0 < n(\neq 1), \tag{25}$$
and dominated by a barotropic perfect fluid $P_m = w_m \rho_m$. The Hubble parameter is given by $H = \frac{n}{(t_s - t)^2}$. The continuity equation yields

$$\rho = \rho_0 (t_s - t)^{3n\gamma},$$

(26)

where $\rho_0 = \tilde{\rho}_0 a_0^{-3\gamma}$. From $G = \frac{24n^3(n+1)}{(t_s - t)^2}$ we can write

$$(t_s - t)^{-1} = \left( \frac{G}{24n^3(n+1)} \right)^{\frac{1}{4}}.$$

(27)

The above equation may be used to write the second equation in (5) as a differential equation for $f$ in terms of $G$:

$$-4 \frac{n + 1}{n + 1} G^2 f_{GG} + G f_G - f - 6 \left( \frac{n}{24(n+1)} \right)^{\frac{1}{4}} G G^\frac{1}{2}$$

$$+ D \left( 24n^3(n+1) \right)^{\frac{3n\gamma}{4}} G^{-\frac{3n\gamma}{4}} = 0.$$

(28)

We have defined $D = 16\pi \rho_0$. The solution of this differential equation is

$$f(G) = C'G + CG^{\frac{n+1}{4}} + \frac{(6(n(n+1))^\frac{1}{4}}{1 - n} G^\frac{1}{2}$$

$$+ \frac{4D(n+1)(24n^3(n+1))^{\frac{n\gamma}{4}}}{(3n\gamma + 4)(3n\gamma + n + 1)} G^{-\frac{3n\gamma}{4}},$$

(29)

where $C'$ and $C$ are two constants. The first term which is linear in $G$, although appears in the horizon entropy, but has influence neither on $\dot{S}$ nor on the equation [5], so we put $C' = 0$. For $f(G) = 0$ the model is the Einstein theory of gravity, where $H^2 = \frac{8\pi}{3} \rho_m$ and the continuity equation necessitates

$$D = 6n^2, \quad n = -\frac{2}{3\gamma}.$$

(30)

By choosing $C = 0$, we get a solution which by applying (30), reduces to the Einstein theory of gravity, i.e. $f = 0$. This is similar to the solution chosen in [4].

The future event horizon is given by

$$R_h = \frac{t_s - t}{1 + n}.$$

(31)

As stated before, $\dot{R}_h < 0$. The evolution of the horizon entropy is described by

$$\dot{S}_h = -2\pi \frac{(24n^3(n+1))^{1/4}}{(n+1)^2} G^{-1/4} + \frac{16\pi}{(24n^3(n+1))^{1/4}} G^{5/4} f_{GG}.$$

(32)
Without the second term we have always $\dot{S}_h < 0$. By putting \( C = C' = 0 \) in (32) we get

\[
\dot{S}_h = \frac{12\pi D\gamma n(n + 1)(24n^3(n + 1))^{\frac{1}{2}(3n\gamma - 1)}}{3n\gamma + n + 1} G^{-\frac{3}{4}(n\gamma + 1)} + 2\pi \left( \frac{24}{n(1 + n)} \right)^{\frac{1}{2}} \frac{3n + 1}{n - 1} G^{-\frac{3}{4}}. \quad (33)
\]

For $n > 1$, thermodynamics second law for the horizon is always respected. For $n < 1$, we may have $\dot{S}_h < 0$ when $t$ is enough far from $t_s$. In this case the main contribution in $\dot{S}_h$ comes from the term including $G^{-\frac{3}{4}}$ (note that $\gamma \geq 1$).

The matter entropy, is obtained as

\[
\dot{S}_{\text{in}} = -8\pi^2 \gamma \rho_0 \frac{(24n^3(n + 1))^{\frac{1}{2}(1 + n\gamma)}}{(1 + n)^3} G^{-\frac{3}{4}(1 + n\gamma)}. \quad (34)
\]

Note that as we have stated before, for ordinary matter ($\gamma > 1$), $\dot{S}_{\text{in}} < 0$. So by considering (32), we conclude that a necessary condition for GSL to be respected is $f_{GG} > 0$. The appearance of the explicit form of $f$ in $\dot{S}_{\text{tot}}$, is due to the entropy modification in modified Gauss-Bonnet gravity (see (10)). By using (33) and (34), $\dot{S}_{\text{tot}}$ may be obtained in terms of $G$. In this way $\dot{S}_{\text{tot}} \geq 0$ implies that only for times satisfying

\[
BG^{-\frac{3}{4}n\gamma - \frac{3}{4}} \geq -A, \quad (35)
\]

where

\[
A = \left( \frac{24}{n(1 + n)^7} \right)^{\frac{1}{4}} \frac{3n + 1}{n - 1},
\]

\[
B = 6Dn\gamma \left( \frac{-3\gamma n^3 + 2n^2 + 3n + 1}{(1 + n)^2(3n\gamma + n + 1)} \right) (24n^3(n + 1))^{\frac{3}{2}n\gamma - \frac{1}{4}}, \quad (36)
\]

GSL is valid.

Note that $A \neq 0$, therefore an adiabatic expansion is forbidden in this model. We have $A > 0$ for $n > 1$, and as in our model $\gamma \geq 1$, for $B > 0$ to be hold it is necessary to have $n < 1.488$. It is clear that for the case: $A > 0$, $B > 0$, GSL is respected in all times. In other cases GSL may not be valid, e.g. if $n < 1$, and in the limit $t \to t_s$ (near the big rip), GSL is not respected, this may due to quantum effects near the big rip ignored in our classical computation. Also for $t_s \gg t$ such that $DG^{-\frac{3}{4}n\gamma - \frac{3}{4}} \ll 1$ (or $G_N DG^{-\frac{3}{4}n\gamma - \frac{1}{4}} \ll 1$ in units with $\hbar = c = 1, G_N \neq 1$), GSL is clearly violated for $n > 1.488$ and $\gamma \geq 1$. 
3 Summary

In this paper we considered spatially flat FLRW universe in the framework of the modified Gauss Bonnet gravity (see (1)). We assumed that the universe has a future event horizon (see (7)). Using the modified Friedman equations (see (5)), and the Wald entropy for dynamical horizons (see (9)), thermodynamics second law for the horizon, as well as for the matter component, and finally GSL for the universe were investigated. The conditions required for validity of these laws were obtained (see (12) and (24)), and through an example by obtaining an exact solution for modified Gauss Bonnet cosmology (see 29) in a super accelerated universe, we clarified our results.

References

[1] A. A. Starobinsky, Phys. Lett. B 91, 99 (1980); M. B. Mijic, M. S. Morris and W. Suen, Phys. Rev. D 34, 2934 (1986); P. A. Anderson and W. Suen, Phys. Rev. D 35, 2940 (1987); T. P. Sotiriou, and V. Faraoni, Rev. Mod. Phys. 82, 451 (2010).

[2] A. G. Riess et al., Astron. J. 116, 1009 (1998); S. Perlmutter et al., Nature (London) 391, 51 (1998); P. M. Garnavich et al., Ap. J. 509, 74 (1998); S. Perlmutter et al., Astrophys. J. 517, 565 (1999).

[3] S. Nojiri and S. D. Odintsov, Phys. Lett. B 631, 1 (2005); S. Nojiri, S. D. Odintsov and O. G. Gorbunova, J. Phys. A 39, 6627 (2006); G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov, and S. Zerbini, Phys. Rev. D 73, 084007 (2006); A. D. Felice, S. Tsujikawa, Living Rev. Rel. 13, 3 (2010); G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov, and S. Zerbini, Phys. Rev. D 75, 086002 (2007); K. Bamba, S. D. Odintsov, L. Sebastiani, and S. Zerbini, [arXiv:0911.4390v2 [hep-th]]; M. Alimohammadi, and A. Ghalee, Phys. Rev. D 79, 063006 (2009); C. Charmousis, and A. Padilla, JHEP 0812, 038 (2008); Y. S. Myung, Y. W. Kim, and Y. J. Park Eur. Phys. J. C 58, 337 (2008); R. Chingangbam, M. Sami, P. V. Tretyakov, and A.V. Toporensky, [arXiv:0711.2122v2 [hep-th]]; A. Sheykhi and B. Wang, Phys. Lett. B 678, 434 (2009); S. Nojiri, S.D. Odintsov, [hep-th/0601213v5]; K. Bamba, C. Geng, S. Nojiri and S. D. Odintsov, [arXiv:0909.4397].

[4] M. Gasperini and G. Veneziano, Astropart. Phys. 1, 317 (1993).

[5] T. Koivisto, and David F. Mota, Phys. Rev. D 75, 023518 (2007); T. Koivisto, and David F. Mota, Phys. Lett. B 644, 104 (2007).

[6] B. Li, J. D. Barrow, and David F. Mota, Phys. Rev. D 76, 044027 (2007).
[7] A. De Felice, David F. Mota, and S. Tsujikawa, Phys. Rev. D 81, 023532 (2010).

[8] R. Bousso, Phys. Rev. D 71, 064024 (2005); T. Jacobson, Phys. Rev. Lett. 75, 1260 (1995); S. Nojiri and S. D. Odintsov, Phys. Rev. D 70, 103522 (2004); M. Akbar and R. Cai, Phys. Lett. B 635, 7 (2006); A. Sheykhi, B. Wang, and R. Cai, Nucl. Phys. B 779, 1 (2007); Y. Gong, B. Wang and A. Wang, Phs. Rev. D 75, 123516 (2007); K. Karami and S. Ghaffari, Phys. Lett. B 685, 115 (2010); H. Mohseni Sadjadi and M. Honardoost, Phys. Lett. B 647, 231 (2007); S. Chattopadhyay, and U. Debnath arXiv:1008.1722v3 [gr-qc]; K. Karami, M. Jamil, and N. Sahraei, arXiv:1003.1890v2 [gr-qc]; B. Gublerina, R. Horvat and H. Nikolic, Phys. Lett. B 636, 80 (2006); M. Jamil, E. N. Saridakis, and M. R. Setare, Phys. Rev. D 81, 023007 (2010).

[9] P. C. W. Davies, Class. Quant. Grav. 5, 1349 (1988); ibid. 4 L255 (1987); M. D. Pollock and T. P. Singh, Class. Quant. Grav. 6, 901 (1989); D. Pavon, Class. Quant. Grav. 7, 487 (1990); R. Brustein, Phys. Rev. Lett. 84, 2072 (2000); G. Izquierdo, and D. Pavon, Phys. Lett. B. 639, 420 (2006); H. Mohseni Sadjadi, Phys. Lett. B 645, 108 (2007); H. Mohseni Sadjadi and M. Jamil, arXiv:1002.3588v1 [gr-qc]; M. Akbar, Int. J. Theor. Phys. 48, 2665 (2009); S. F. Wu, B. Wang, G. H. Yang, and P. M. Zhang, Class. Quant. Grav. 25, 235018 (2008); N. Mazumder, and S. Chakraborty, arXiv:1005.5215v1 [gr-qc].

[10] G. Izquierdo, and D. Pavon, Phys. Lett. B. 639, 420 (2006).

[11] R. M. Wald Phys. Rev. D 48, (1993) 3427; V. Iyer and R. M. Wald, Phys. Rev. D 50, 846 (1994); T. Jacobson. G. Kang and R. C. Myers, Phys. Rev. D 49, 6587 (1994); I. Brevik, S. Nojiri, S. D. Odintsov and L. Vanzo, Phys. Rev. D 70, 043520 (2004); H. Maeda, Phys. Rev. D 81, 124007 (2010);

[12] H. Mohseni Sadjadi, Phys.Rev. D 73, (2006) 063525.

[13] A. D. Felice, and S. Tsujikawa, Phys. Lett. B 675, 1 (2009).

[14] P. Wang, Phys. Rev. D 72, 024030 (2005).

[15] P. C. Davies, Ann. Inst. Henri Poincare 49, 297 (1988).

[16] N. Goheer, R. Goswami, P. K. S. Dunsby and K. Ananda, Phys. Rev. D 79, 121301 (2009).