AdaTerm: Adaptive T-Distribution Estimated Robust Moments towards Noise-Robust Stochastic Gradient Optimizer

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Abstract

With deep learning applications becoming more practical, practitioners are inevitably faced with datasets corrupted by a variety of noise such as measurement errors, mislabeling and estimated surrogate inputs/outputs, which can have negative impacts on the optimization results. As a safety net, it is natural to improve the robustness to noise of the optimization algorithm which updates the network parameters in the final process of learning. Previous works revealed that the first momentum used in Adam-like stochastic gradient descent optimizers can be modified based on the Student’s t-distribution to produce updates robust to noise. In this paper, we propose AdaTerm which derives not only the first momentum but also all of the involved statistics based on the Student’s t-distribution, providing for the first time a unified treatment of the optimization process under the t-distribution statistical model. When the computed gradients statistically appear to be aberrant, AdaTerm excludes them from the update and reinforce its robustness for subsequent updates; otherwise, it normally updates the network parameters and relaxes its robustness for the following updates. With this noise-adaptive behavior, AdaTerm’s excellent learning performance was confirmed via typical optimization problems with several cases where the noise ratio is different and/or unknown. In addition, we proved a new general trick for deriving a theoretical regret bound without AMSGrad.

1. Introduction

Deep learning ((LeCun et al., 2015)) is one of the most successful technologies in the last decade. Most of its success comes from the use of stochastic gradient descent (SGD) algorithms ((Robbins and Monro, 1951)) to optimize deep neural networks using first-order gradients over the network parameters obtained by a back-propagation technique for a given optimization problem. Therefore, in parallel with the development of various network structures like residual networks ((He et al., 2016)), various SGD-based optimizers that can optimize the networks in a more stable and efficient manner have been pursued. The most representative optimizer would be Adam ((Kingma and Ba, 2014)), and a lot of its variants with respective features have been proposed (see the survey for details in (Sun et al., 2019; Schmidt et al., 2021)). Among these, in our knowledge, RAdam ((Liu et al., 2020)) and AdaBelief ((Zhuang et al., 2020)) have illustrated the state-of-the-art (SOTA) learning performance.

One of the features of the SGD optimizers is robustness to noise in gradients, which can result for example from the use of noisy datasets with sensory errors, mislabeling ((Mirylenka et al., 2017; Suchi et al., 2019)) and from optimization problems that require the use of estimated inputs and/or outputs like long-term dynamics learning ((Chen et al., 2018; Kishida et al., 2020)), reinforcement learning (RL) ((Sutton and Barto, 2018)), and distillation from the trained teacher(s) ((Rusu et al., 2015; Gou et al., 2021)). This feature is for example essential in robot learning problems, where the available datasets can be small, making easily apparent the adverse effects of noise. Not only that, but it has been shown empirically in ((Simsekli et al., 2019)) and ((Zhou et al., 2020)) that the norm of the gradient noise in both Adam and SGD had heavy tails, even in the absence of input/output noise.

In order to deal with such noise and robustly carry out efficient gradient updates, previous studies have proposed to detect and exclude the aberrant gradients affected by noise. In particular, the work by (Ilboudo et al., 2020) has focused on the fact that the first-order momentum used in recent Adam-like optimizers is computed with exponential moving average (EMA), which can be regarded as the mean of the normal distribution, known to be sensitive to noise. By con-
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Verting such a noise-sensitive first-order momentum into the t-momentum — derived from the Student’s t-distribution — which is robust to noise, most of the Adam-like optimizers can acquire robustness. Furthermore, in a follow-up work ((Ilboudo et al., 2021)), the authors showed how the degrees of freedom of the t-momentum which determines the level of robustness, could be tuned automatically by combining the t-momentum algorithm with a heuristic estimation method. This latest version of the t-momentum algorithm, called At-momentum, was then applied to a practical behavioral cloning dataset and shown to have acquired some adaptive behaviors to different noise ratio in the optimization problem to be solved while suppressing excessive robustness.

However, these two previous works modify only the first-order momentum, and the second-order momentum, which also usually appears in the Adam-like optimizers, is updated by the basic EMA. (Ilboudo et al., 2020) motivated this choice by relying on the fact that the corresponding smoothness parameter $\beta_2$ is usually set to be large and therefore has the potential to reduce the effects of noise. Unfortunately, such design induces a modeling discrepancy between the Student’s t-distribution model applied for the first-order moment and the Gaussian distribution applied to the second moment. Similarly, although the latest version makes the degrees of freedom adjustable, it is unclear whether the heuristic adjustment violates the other assumptions and/or interferes with the other parts of the algorithm. In the end, the lack of a unified derivation of the algorithm as a noise-robust optimizer could allow potential problems to be included.

This paper therefore proposes the Adaptive T-distribution estimated robust moments algorithm, called AdaTerm, with a unified derivation of all the distribution parameters based on their gradients for online maximum likelihood estimation of the Student’s t-distribution. AdaTerm introduces adaptive step sizes specifically chosen to turn the gradient-based update of the statistics into an interpolation between the past statistics values and the update amounts. Such adaptive step sizes also allow the smoothness parameters $\beta$ to be common for all the involved statistics. In addition, since the gradient of the degrees of freedom in the multi-dimensional case has been reported to not be consistent with our expectation (briefly stated, it is too small; see details in (Ley and Neven, 2012)), it is appropriately approximated as in the one-dimensional case where our expectation holds. This approximated value can also be used as the upper bound. As expected, AdaTerm can obtain the following qualitative behaviors from the above implementations: if the given gradients are considered aberrant, it reinforces the robustness while excluding the gradients; otherwise, it relaxes the robustness to facilitate updates.

Including the proposed AdaTerm, all optimizers are required to derive their regret bounds to theoretically analyze their convergence. However, since the appearance of the AMSGrad paper ((Reddi et al., 2018)), all subsequent literature have assumed its usage in order to derive a regret bound, even when the target optimizer does not usually use AMSGrad. To avoid this theoretical and practical contradiction, we devise a new trick for deriving a theoretical regret bound without AMSGrad, which can be employed for other optimizers.

Our contributions in this paper are four folds:

1. Unified derivation of a novel SGD algorithm that is adaptively robust to noise;
2. Easing the difficulty of tuning hyper-parameters by using a common smoothness parameter for all the distribution parameters;
3. Theoretical proof of the regret bound without the necessity of AMSGrad;
4. Numerical verification of usefulness in major test functions and typical problems (i.e. classification problems with mislabeling, long-term prediction problems, reinforcement learning, and policy distillation).

In the last verification, we compared not only AdaBelief and RAdam as the state-of-the-art algorithms but also t-Adam variants developed in the related work.

2. Problem statement and related works

2.1. Optimization problem solved by SGD optimizer

Let us briefly define the optimization (minimization without loss of generality) problem that we will solve using either of the SGD optimizers. Suppose that some input data $x$ and output data $y$ are generated according to the problem-specific (stochastic) rule, $p(x, y)$. Then, the problem-specific minimization target, $L$, is given as follows:

$$L = \mathbb{E}_{x,y \sim p(x,y)}[\ell(f(x; \theta), y)]$$

(1)

where $\ell$ denotes the loss function for each data, and $f(x; \theta)$ denotes the mapping function (e.g. from $x$ to $y$) with the parameter set $\theta$ which is optimized through this minimization (e.g. network weights and biases).

The above expectation operation can be approximated by Monte Carlo method. In other words, let a dataset containing $N$ pairs of $(x, y)$, $D = \{ (x_n, y_n) \}_{n=1}^N$, be constructed according to the problem-specific rule. With $D$, the above minimization target is replaced as follows:

$$L_D = \frac{1}{|D|} \sum_{x_n, y_n \in D} \ell(f(x_n; \theta), y_n)$$

(2)
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where $|D|$ denotes the size of dataset ($N$ in this case).

We can then compute the gradient w.r.t. $\theta$, $g = \nabla \theta L_D$, and use gradient descent to obtain the (sub)optimal $\theta$ that (locally) minimizes $L_D$. However, if $D$ is large, the above gradient computation would be infeasible due to the limitation of computational resources. The SGD optimizer (Robbins and Monro, 1951) therefore extracts a subset (a.k.a. mini batch) at each update step $t$, $B_t \subset D$, and updates $\theta$ from $\theta_{t-1}$ to $\theta_t$ as follows:

$$g_t = \nabla \theta_{t-1} L_{B_t}$$

$$\theta_t = \theta_{t-1} - \alpha \eta (g_t)$$

where $\alpha > 0$ denotes the learning rate, and $\eta$ represents a function used to modify $g_t$ in order to improve the learning performance. Namely, various SGD optimizers have their own $\eta$.

For example, in the case of Adam ((Kingma and Ba, 2014)), which is the most popular optimizer in recent years, $\eta$ is given with three hyper-parameters, $\beta_1 \in (0, 1)$, $\beta_2 \in (0, 1)$, and $\epsilon \ll 1$.

$$\eta_{Adam}(g_t) = \frac{m_t (1 - \beta_2)^{-1}}{v_t (1 - \beta_2^{-1}) + \epsilon}$$

where

$$m_t = \beta_1 m_{t-1} + (1 - \beta_1) g_t$$

$$v_t = \beta_2 v_{t-1} + (1 - \beta_2) g_t^2$$

A simple interpretation of Adam is that, since $g_t$ fluctuates depending on how $B_t$ is sampled, the first momentum $m_t$ smoothes $g_t$ with the past gradients to stabilize the update, and the second moment $v_t$ scales $g_t$, which is different depending on the problem, to increase the genericity.

2.2. Inaccurate datasets in practical cases

If the above problem setting is satisfied, we can stably acquire one of the local solutions through one of the SGD optimizers, although their regret bounds may be different ((Reddi et al., 2018; Alacaoglu et al., 2020)). However, in real problems, it is difficult to make datasets that follow the problem-specific rules exactly.

For example, in RL ((Sutton and Barto, 2018)), the optimal actions for the target tasks cannot be explicitly defined, so an agent learns the policy (i.e. mapping from state inputs to action outputs) by estimating the optimal actions based on reward values given from the tasks. Likewise, when distilling knowledge and skills from trained large-scale models to a smaller one ((Gou et al., 2021)), biases resulting in training will prevent the generation of accurate supervised signals. As a third example, in the problem of learning dynamics from time-series data, the long-term prediction accuracy may be involved in the loss function with predicted values as inputs ((Chen et al., 2018)). Even in classification problems, it is not realistic to expect all the data to be correctly labelled, especially when employing (semi-)automatic annotation techniques ((Suchi et al., 2019)).

All of the above examples can be interpreted as the problem of learning with an inaccurate dataset $D = D \cup E$ with a noisy subset, $E \not\subseteq D$, caused by the estimated values, mislabeling, and aforementioned difficulties. Please note that the optimization for $D$ would be impossible if $|D| < |E|$ since the majority is switched; hence, we assume $|D| > |E|$.

2.3. Related works

The inherent noisiness of the stochastic gradient combined with the ubiquity of imperfect data in practical settings has encouraged the propositions of more robust and efficient machine learning algorithms against noisy or heavy-tailed datasets. All these methods can be divided into two main approaches, going from methods that produce robust estimates of the loss function, to methods based on the detection and attenuation of wrong gradient updates ((Gulcehre et al., 2017; Holland and Ikeda, 2019; Prasad et al., 2020; Kim and Choi, 2021)). Each approach has its own pros/cons as summarized in Table 1.

As mentioned in the introductory section, the latter approach is the one we take in this paper. In particular, we draw inspiration from the work by (Ilboudo et al., 2020) which first proposed the use of the Student’s t-distribution as a statistical model for the gradients of the optimization process.

Indeed, in that work, the popular EMA-based momentum at the heart of the SOTA most recent optimization algorithms and defined by $m_t = \beta_1 m_{t-1} + (1 - \beta_1) g_t$ was identified with the iterative arithmetic mean $m_t = \frac{n-1}{n} m_{t-1} + \frac{1}{n} g_t$ such that $\beta_1 = \frac{n-1}{n}$ and $n$ is a fixed number of recent samples independent of $t$. This classical EMA, derived from the arithmetic mean estimator which is of Gaussian origin was then replaced by a Student’s t-based mean estimator $m_t = \frac{W_{t-1} + w_t m_{t-1}}{W_{t-1} + w_t}$, where $w_t = (\nu + d) / (\nu + \sum_{j=1}^{d} (g_{i,j} - m_{t-1,j})^2)$, and to match the fixed number of recent samples, the sum $W_t = \sum_{i=1}^{t} w_i$ used in the usual estimator was replaced by a decaying sum $W_t = \frac{2\beta_t-1}{\beta_t} W_{t-1} + w_i$.

Unfortunately, the second-order momentum $v_t$ is still based on the regular EMA, i.e. $v_t = \beta_2 v_{t-1} + (1 - \beta_2) g_t^2$ where $s_t$ is a function of the squared gradient, e.g. $s_t = g_t^2$ for Adam ((Kingma and Ba, 2014)) and $s_t = (g_t - m_t)^2$ for AdaBelief ((Zhuang et al., 2020)). This stands in contrast to its usage in the computation of $w_t$ which makes the underlying assumption that $v_t$ is also derived from the maximum likelihood (ML) Student’s t-distribution scale estimator, re-
sulting in the unnatural blending of two statistical models. Furthermore, the degrees of freedom $\nu$ is treated as an hyperparameter and kept constant throughout the optimization process.

To alleviate this latter problem, the follow-up work (Ilboudo et al., 2021) proposed to adapt and apply the direct incremental degrees of freedom estimation algorithm developed by Aeschliman et al. (2010) in order to automatically update the degrees of freedom. However, this algorithm is based on an approximation of $E[\log ||g||^2]$ and $\text{Var}[\log ||g||^2]$ and therefore uses a different approach than the ML approach that gave rise to the t-momentum. In the experiment section below, we show that this difference in approach results in a degrees of freedom with insufficient adaptability, most likely due to the fact that the direct incremental degrees of freedom estimation algorithm was originally developed to work with a specific estimation method for the location and scale parameters that the ML-based t-momentum does not match.

In this paper, we tackle the lack of unified approach by requiring that all of the parameters be estimated under the ML estimation of the t-distribution. In particular, the previous work was able to derive Student’s t-based EMA by relying on the analytic solution of the maximum log-likelihood (i.e. by setting the gradient of the log-likelihood to zero and solving explicitly). However, although such approach can also be used to estimate a scale parameter $\nu$, by also requiring a fixed number of samples in the given solution, the extension to the degrees of freedom is much more difficult due its nonlinearity and the absence of a closed-form solution to the ML problem. To avoid such problem, we therefore propose to use a gradient ascent algorithm for all of the parameters, combined with adaptive step sizes. We then apply some further tricks to ensure that the algorithm still recovers the EMA in the Gaussian limit and does not violate the positivity of both the scale and the degrees of freedom parameter. Namely, we surrogate when necessary the gradients by their upper bound. The details of the derivation are given in the next section.

### Table 1. Pros/Cons of the two main approaches to robustness

| Approach                      | Pros                                      | Cons                                      |
|-------------------------------|-------------------------------------------|-------------------------------------------|
| Robust Loss Estimation        | Robustness independent of batch size      | Usually problem specific                   |
|                               |                                           | Usually require the use of all the available data |
|                               |                                           | Can be both unstable and costly in high dimensions |
| Robust Gradient Descent       | Widely applicable                         | Robustness dependent of batch size and outliers repartition |
|                               |                                           | Rely on only estimates of true gradient   |

### 3. Derivation of AdaTerm

#### 3.1. The Student’s t distribution statistical model

Since the gradient for $E$ is disturbed, this paper develops a noise-robust optimizer to achieve learning along $D$ while properly detecting and excluding such disturbed gradient as noise. Although it is possible to make the loss function $\ell$ noise-robust, e.g. as done by (Ma et al., 2020), the noise-robust optimizer proposed in this paper is highly useful in that it can be employed very widely as a safety net.

In order to exclude the aforementioned disturbed gradients, we assume that the gradients $g$ can be modeled by the Student’s t-distribution, which has a heavy tail and is robust to outliers, referring to the related work (Ilboudo et al., 2020). Note that this statistical model is further supported by the fact that it has been shown empirically (Simsekli et al., 2019) that the norm of the gradient noise in SGD had heavy tails and furthermore that in continuous-time, the gradient’s stationary distribution was found (Ziyin et al., 2021) to obey a Student’s t-like distribution.

Specifically, $g$ is assumed to be generated from a $d$-dimensional diagonal Student’s t-distribution which is characterized by three kinds of parameters: a location parameter $m \in \mathbb{R}^d$, a scale parameter $\nu \in \mathbb{R}_{>0}$, and a degrees of freedom parameter $\nu \in \mathbb{R}_{>0}$. That is,

$$
g \sim \frac{\Gamma\left(\frac{\nu + d}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\left(\nu\pi\right)^{\frac{d}{2}} \prod_d \sqrt{v_d}} \left(1 + \frac{1}{\nu} \sum_d \left(g_d - m_d\right)^2 v_d^{-1}\right)^{-\frac{\nu + d}{2}}
=: T(g \mid m, \nu, \nu)
$$

where $\Gamma$ denotes the gamma function. Here, $d$ corresponds to the dimension size of each subset of parameters, meaning that, following the related work (Ilboudo et al., 2020)) and PyTorch implementation (Paszke et al., 2017), AdaTerm is applied to a weight matrix and a bias in each layer separately.

In the paper by (Ilboudo et al., 2020), only $m$ has been estimated based on its maximum likelihood solution, and in a later paper (Ilboudo et al., 2021), $\nu$ has been adjusted by a heuristic approximation of the maximum likelihood (Aeschliman et al., 2010). The present paper simultaneously estimates $m$, $\nu$, and $\nu$ based on their surrogated.
gradients to maximize the log-likelihood. The unified derivation yields a more adaptive parameter $\nu$ (see 4), with the same computational complexity (i.e. $O(d)$) as the other major optimizers.

3.2. Gradients for maximum log-likelihood

To derive AdaTerm, let us consider the problem of maximizing the log-likelihood, $\ln T(g \mid m, v, \nu)$, in order to estimate the parameters $m$, $v$, and $\nu$ that can adequately model the recent $g$. Since this model is expected to be time-varying, we simply employ a gradient ascent at each step instead of $\psi$ the response to outliers can be intuitively analyzed by $\nabla \nu m$ defined.

In addition, as will be explained later, the pure gradients of $v$ and $\nu$ does not provide the expected behavior, hence some surrogated versions are introduced.

Nevertheless, we first show the pure gradient for each statistic. To simplify the notation, the following variables are defined.

\[
s = (g - m)^2, \quad D = \frac{1}{d} s^T v^{-1}, \quad \nu D = \nu D^{-1}, \quad w_{mv} = \frac{\nu + 1}{\nu + D}
\]

When $\nu$ is defined to be proportional to $d$, as in the literature (Ilboudo et al., 2020), $\nu D$ corresponds to the proportionality coefficient. With these, the gradients w.r.t. $m$, $v$, and $\nu$ can be derived, respectively.

\[
\nabla_m \ln T = \frac{-\nu d - \nu^{-1}(g - m)v^{-1}}{2(1 + \nu^{-1}dD)} = \frac{\nu D^{-1} + 1 - g - m}{2v} = w_{mv} \frac{g - m}{2v} \tag{9}
\]

\[
\nabla_v \ln T = \frac{-\nu d - \nu^{-1}(g - m)^2v^{-2}}{2(1 + \nu^{-1}dD)} = \frac{1}{2v^2} \left( \frac{\nu + 1}{\nu + D} - v \right)
= w_{mv} \frac{\nu}{2v^2(\nu + 1)} \left\{ (s - v) + (s - Dv)\nu^{-1} \right\} \tag{10}
\]

\[
\nabla_\nu \ln T = \frac{1}{2} \psi \left( \frac{\nu + d}{2} \right) - \frac{1}{2} \psi \left( \frac{\nu}{2} \right) - \frac{d}{2\nu} - \frac{1}{2} \ln(1 + \nu^{-1}dD) - \frac{\nu + d}{2(1 + \nu^{-1}dD)} \tag{11}
\]

where the gradients w.r.t. $m$ and $v$ are transformed so that the response to outliers can be intuitively analyzed by $w_{mv}$. $\psi$ denotes the digamma function.

3.3. Online maximum likelihood updates with adaptive step sizes

Following the success of the EMA scheme in outlier-free optimization, we specifically choose the adaptive step sizes so that in the Gaussian limit (i.e. when $\nu \to \infty$), the gradient ascent updates of the mean $m$ and scale $v$ parameters revert to simple EMA updates. Under this requirement, we now derive the update rules and the corresponding step sizes for each parameters.

3.3.1. The first moment

Since $m$ is defined in the whole real space, gradient-based updates can be applied with no restriction. The gradient ascent equation is therefore given by:

\[
m_t = m_{t-1} + \kappa_m g_m
= m_{t-1} + \kappa_m w_{mv} \frac{(g_t - m_{t-1})}{2v_{t-1}} \tag{12}
\]

where $\kappa_m$ is the update step size. Since we restrict ourselves to a EMA-like update rule, we also have:

\[
m_t = m_{t-1} + \tau_m (g_t - m_{t-1})
\]

where $\tau_m$ is the interpolation factor satisfying $\tau_m \in (0, 1)$. Therefore,

\[
\tau_m = \frac{\kappa_m w_{mv}}{2v_{t-1}} \Rightarrow 0 < \kappa_m < \frac{2v_{t-1}}{w_{mv}}
\]

\[
\Rightarrow \kappa_m = \frac{2kv_{t-1}}{w_{mv}}, \quad k \in (0, 1) \quad \Rightarrow \tau_m = k \tag{13}
\]

Furthermore, for $\nu \to \infty$, we require $\tau_m \to (1 - \beta)$ with $\beta \in (0, 1)$ the common smoothness parameter (i.e. we recover a Gaussian model in the limit), meaning that $k \to (1 - \beta)$. Since the interpolation ratio $\tau_m$ should be adaptive to outliers, namely, it should ultimately involve $w_{mv}$ and since $k \in (0, 1)$, in this paper, we set $k = (1 - \beta) w_{mv} \overline{w}_{mv}$ where $w_{mv} \leq \overline{w}_{mv} = (\nu + 1)\nu^{-1}$. With this, the adaptive step size $\kappa_m$ and the update rule for the first moment $m$ is given by:

\[
\kappa_m = 2v_{t-1} \frac{1 - \beta}{\overline{w}_{mv}} \tag{14}
\]

and the update rule becomes:

\[
m_t = m_{t-1} + \kappa_m g_m = m_{t-1} + (1 - \beta) \frac{w_{mv}}{w_{mv}} (g_t - m_{t-1})
= (1 - \tau_m)m_{t-1} + \tau_m g_t \tag{15}
\]

where $\tau_m = (1 - \beta)\overline{w}_{mv}$. 

3.3.2. The Central Second Moment

Similarly, the gradient ascent update rule for the scale parameter \( v \) is given by:

\[
v_t = v_{t-1} + \kappa_v g_v \\
= v_{t-1} + \kappa_v \frac{w_{mv}}{2v_{t-1}^2} [s_t + \Delta s - v_{t-1}] \quad (16)
\]

where \( \kappa_v \) is the update step size and \( \Delta s = (s - Dv)\hat{v}^{-1} \).

Again, we restrict ourselves to an EMA-like update rule:

\[
v_t = v_{t-1} + \tau_v [s_t + \Delta s - v_{t-1}] \]

where \( \tau_v \in (0, 1) \) as well as \( \tau_m \). In order to preserve a limiting Gaussian model, we simply require that for \( \hat{v} \to \infty \), \( \tau_v \to (1 - \beta) \), since we already have \( \Delta s \to 0 \). Therefore, we can write:

\[
\tau_v = \frac{\kappa_v w_{mv} \hat{v}}{2v_{t-1}^2 (\hat{v} + 1)} \implies 0 < \kappa_v < \frac{2v_{t-1}^2 (\hat{v} + 1)}{w_{mv} \hat{v}} \\
\implies \kappa_v = \frac{2k v_{t-1}^2 (\hat{v} + 1)}{w_{mv} \hat{v}}, \quad k \in (0, 1) \implies \tau_v = k
\]

(17)

Here again, \( k \) must satisfy the same conditions as in the first-moment derivation. This means that we can again set \( k = (1 - \beta) w_{mv}^{-1} \), which results in the following step size:

\[
\kappa_v = 2v_{t-1}^2 (1 - \beta)
\]

(18)

and the corresponding update rule:

\[
v_t = v_{t-1} + \kappa_v g_v \\
= v_{t-1} + (1 - \beta) \frac{w_{mv}}{w_{mv}} [(s_t + \Delta s) - v_{t-1}] \\
= (1 - \tau_v) v_{t-1} + \tau_v (s_t + \Delta s)
\]

(19)

where \( \tau_v = \tau_m = (1 - \beta) \frac{w_{mv}}{w_{mv}} \).

Although as stated before we have that for \( \hat{v} \to \infty \), \( \Delta s = (s - Dv)\hat{v}^{-1} \to 0 \), we notice that its value can be negative for \( \hat{v} \ll \infty \). This means that the update rule above can potentially take the scale parameter \( v \) into a negative region during the learning process, which is undesired. There are many ways to satisfy the positive constraint such as reparametrization or the mirror descent (MD) strategies. However, in order to preserve the EMA-like interpolation directly on the parameter \( v \), we employ the projected gradient method. In particular, we use a simple gradient clipping method where we put a lower bound on \( \Delta s \):

\[
\Delta s =: \max(\epsilon^2, (s - Dv)\hat{v}^{-1}) \quad (20)
\]

where \( \epsilon \ll 1 \). Such a gradient upper bound has the disadvantage that it can cause the algorithm to overestimate the scale parameter \( v \). In that case, when \( v \) is larger than the exact value, \( D \) and \( w_{mv} \) would become smaller and larger respectively, namely, the robustness to noise may be impaired. However, this drawback is attenuated by the fact that Adam-like optimizers scale the descent update amounts by \( v \), which means that the effect of noise is expected to still be insignificant. We also argue that the slow decrease of \( v \) permitted by the gradient clipping strategy is advantageous in preventing too much robustness. Indeed, it is possible to derive an update rule over \( v \) such that the positiveness is directly built in the equation from the start as detailed in H. However, we found that such strategies (which includes the fixed-number-of-samples-based ML analytic solution \( \beta v_{t-1} + (1 - \beta) w_{mv} s_t \)), although having similar performance on the simple regression task described in the experiments, proves to be less effective (and for some, unstable) on the prediction task.

As a remark, since \( \tau_m = \tau_v = (1 - \beta) w_{mv}^{-1} \), we adopt one common notation for both of them as \( \tau_{mv} = (1 - \beta) w_{mv}^{-1} \).

3.3.3. The Degrees of Freedom

Simplifying the gradient

Compared to \( m \) and \( v \), the update rule of \( \nu \) is much more delicate to handle. In particular, when the positive constraint on \( \nu \) would be violated by a simple gradient ascend update cannot be identified. This is mainly due to the digamma function \( \psi \), hence the first step is to get rid of it. In this paper, we use the upper and lower bounds of \( \psi \) to find the upper bound of the gradient w.r.t. \( \nu \). Specifically, the upper and lower bounds of \( \psi \) are given as follows:

\[
\ln x - \frac{1}{x} \leq \psi(x) \leq \ln x - \frac{1}{2x}
\]

This leads to the following upper bound for eq. (11):

\[
\nabla_{\nu} \ln \mathcal{T} \leq \frac{1}{2} \left\{ \ln(\nu + d) - \ln 2 - \frac{1}{\nu + d} - \ln \nu + \ln 2 + \frac{2}{\nu} \\
- \frac{d}{\nu} - \ln(\nu + dD) + \ln \nu - \frac{\nu + d}{\nu + dD} + 1 + \frac{d}{\nu} \right\} \\
= \frac{1}{2} \left\{ -w_{mv} + \ln w_{mv} + 1 + \frac{\nu + d}{\nu + 1} \right\} \\
= \frac{1}{2} \left\{ -w_{\nu} + 1 + \frac{\nu + 21}{\nu + 1} \right\}
\]

(21)

where \( w_{\nu} = w_{mv} - \ln w_{mv} \). Although the use of an upper bound can impair the noise robustness in theory (as in the case of \( \nu \)), in practice, as the maximum likelihood solution for \( \nu \) has been reported to be overly robust, this modification is not expected to affect the learning performance too much (see below).
Tackling the curse of dimensionality The overestimated gradient above is very simple and its behavior is easy to understand. Notably, we found that its behavior is greatly affected by \( d \), as shown in Fig. 1 with \( \nu = d \) (that is \( \nu = 1 \)). Note that it is not necessary to follow the literature ((Ilboudo et al., 2020)) for setting the degrees of freedom as \( \nu = \nu d \), but this procedure is important to provide the same level of robustness to the respective subsets of parameters regardless of their respective size. As can be seen, when \( d = 1 \), the gradient w.r.t. \( \nu \) is negative only when \( w_m \) is very small corresponding to a very large deviation \( D \) and the negative gradient effectively decreases \( \nu \) to exclude the noisy gradients of the network’s parameters \( g \); otherwise, the gradient in (21) becomes positive and \( \nu \) is increased to suppress the robustness and make it easier to update the parameters. Unfortunately, with \( d \sim 10,000 \) which is common for a neural network’s weights’ matrix, it is always negative and almost zero even when \( w_m = 1 \), which means that \( \nu \) can only decrease. Such a pessimistic behavior is consistent with the non-intuitive behavior of the multivariate Student’s t-distribution reported in the literature (Ley and Neven, 2012)). This problem has also been empirically confirmed in (Ilboudo et al., 2021)), where the excessive robustness was forcibly suppressed by correcting the obtained (approximated) estimate by multiplying it with \( d \).

To alleviate this problem, we further consider an upper bound where \( \nu \) is replaced by \( \nu^\prime \). In addition, since \( \nu \) does not appear directly in any of the other maximum likelihood gradients, it is more natural to deal with the gradient of \( \nu \).

The above modifications are finally summarized below.

\[
\nabla \rho \ln \mathcal{T} = d \nabla \rho \ln \mathcal{T} \\
\leq d \left\{ -w_\rho + 1 \right\} (\nu + 1 \nu^\prime) \\
= w_\rho \left\{ -1 \right\} (\nu + 1 \nu^\prime) \frac{1}{\nu w_\rho} =: g_\rho \tag{22}
\]

Where the second line is obtained by using \( \frac{1}{\nu^\prime} \leq \frac{1}{\rho} \). This new upper bound can be interpreted as an approximation of the multivariate distribution by a univariate distribution. Although it is possible to model the distribution as a univariate distribution from the beginning of the derivation, there remains a concern that the robustness may be degraded since it would capture the average without focusing on the amount of deviation in each axis.

**Update rule** With this upper bound, we now proceed to the derivation of a proper update rule for the robustness parameter \( \nu \). To do so, we draw similarities with the update rules obtained previously for \( m \) and \( v \) and set our goal to an EMA-like equation with an adaptive smoothness parameter \( \tau_\rho = (1 - \beta)w_\rho \overline{w}_\rho^{-1} \), where \( w_\rho \leq \overline{w}_\rho \). That is, we want the following update rule:

\[
\nabla \rho \ln \mathcal{T} = d \nabla \rho \ln \mathcal{T} \\
\leq d \left\{ -w_\rho + 1 \right\} (\nu + 1 \nu^\prime) \\
= w_\rho \left\{ -1 \right\} (\nu + 1 \nu^\prime) \frac{1}{\nu w_\rho} =: g_\rho \tag{22}
\]

\[
\nu_t = (1 - \tau_\rho)\nu_{t-1} + \tau_\rho \nu_t \tag{23}
\]

where \( \lambda_t \) is some function of \( w_{mv} \). To derive such an equation, we focus first on \( \tau_\rho \) and in particular on finding the maximum value of \( w_\rho, \overline{w}_\rho \). We start by noticing that \( w_\rho \) is a convex function over \( w_{mv} \) with a minimum value at \( w_{mv} = 1 \). The maximum value is therefore determined when \( w_{mv} \) is the largest \( w_{mv} \) or smallest \( w_{mv} \). The largest \( w_{mv} \) value has already been derived as \( \overline{w}_{mv} \), but the smallest value cannot exactly be given since \( w_{mv} \to 0 \) when \( D \to \infty \). Therefore, instead of the exact minimum value, we employ the tiny value of float \( \epsilon_{\text{float}} \), which is the closest to zero numerically (in the case of float32, \( w_{mv} = \epsilon_{\text{float}} \) \( \approx 87.3365 \)). In summary, the maximum \( w_\rho, \overline{w}_\rho \), can be defined as follows:

\[
\overline{w}_\rho = \max(\overline{w}_{mv} - \log(\overline{w}_{mv}), \epsilon_{\text{float}} - \log(\epsilon_{\text{float}})) \tag{24}
\]

If we design the step size with \( \overline{w}_\rho \), we can obtain an interpolation update rule for \( \nu \) similar to \( m \) and \( v \). However, although the minimum value of \( \nu \) is expected to be positive, an excessive robustness could inhibit the learning process. On that account, it is desirable that the user retain some level of control on how much robustness the algorithm is allowed to achieve. Therefore, our final trick is to transform \( \nu = \nu + \Delta \nu \) with a user-specified minimum value, \( \nu > 0 \), and the deviation, \( \Delta \nu > 0 \), automatically controlled by the algorithm. With this transformation and the appropriate step size \( \kappa_{\Delta \nu} \) in a gradient ascent applied on \( \Delta \nu \), whose surrogate gradient is the same as eq. (22), the update rule can be given as follows:

\[
\Delta \nu_t = \Delta \nu_{t-1} + \kappa_{\Delta \nu} \nu_t \\
\nu_t = (1 - \tau_\rho)\nu_{t-1} + \tau_\rho \nu_t \\
\nu_t = (1 - \tau_\rho)\nu_{t-1} + \tau_\rho \nu_t \tag{23}
\]

\[
\kappa_{\Delta \nu} = 2\Delta \nu_{t-1} \frac{1 - \beta}{d \overline{w}_\rho} \tag{25}
\]

\[

Figure 1. w_{mv} vs. the surrogated gradient in eq. (21) with \( \nu = d \) according to several dimension sizes
Then, by adding $\tilde{\nu} + \epsilon$ to both sides of the update rule for $\Delta \tilde{v}$, we get the update rule for $\hat{\nu}$ directly:

$$\hat{\nu}_t = (1 - \tau_\nu)\hat{\nu}_{t-1} + \tau_\nu \lambda_t$$  \hspace{1cm} (26)

$$\tau_\nu = (1 - \beta)w_\nu \bar{\nu}^{-1}$$  \hspace{1cm} (27)

$$\lambda_t = \left( \frac{\hat{\nu}_{t-1} + 2}{\hat{\nu}_{t-1} + 1 + \hat{\nu}_{t-1}} \right) \frac{\hat{\nu}_{t-1} - \frac{\hat{\nu}}{\tau_\nu} + \tilde{\nu} + \epsilon}{\hat{\nu}_{t-1}w_\nu}$$ \hspace{1cm} (28)

Note that the minimum value of $\Delta \hat{\nu} = \hat{\nu} - \tilde{\nu}$ is given by $\epsilon$ so that $\Delta \hat{\nu} > 0$ is satisfied. This process also prevents the first term of $\lambda$ from becoming 0 and stopping the update of $\hat{\nu}$.

### 3.4 Algorithm

Finally, the update amount of the optimization parameters $\theta$ for AdaTerm is given as follows:

$$\eta_{\text{AdaTerm}}(g_t) = \frac{m_t(1 - \beta^t)^{-1}}{\sqrt{v_t(1 - \beta^t)^{-1}}}$$ \hspace{1cm} (29)

Unlike Adam (Kingma and Ba, 2014)), the small amount usually added to the denominator, $\epsilon$, is removed since $\sqrt{v} \geq \epsilon$ in our implementation. Note that AdaBelief (Zhuang et al., 2020)) could also remove it. However, it is still added in the AdaBelief algorithm, even though the small amount in AdaBelief has little effect since the minimum of the scaling factor is, in comparison, absolutely larger.

The pseudo-code of AdaTerm is summarized in Alg. 1. The regret bound is also analyzed with a new approach different from the literature (Reddi et al., 2018)) in A by combining the trick proposed by (Alacaoglu et al., 2020), with a new trick based on the Lemma A.1 in order to remove the need for the AMSGrad assumption from the regret analysis.

### 3.5 Behavior analysis

#### 3.5.1 Convergence analysis

Our convergence proof adopts the approach highlighted in (Alacaoglu et al., 2020). As such, we start by enunciating the same assumptions:

**Assumption 3.1.** Necessary assumptions:

1. $\mathcal{F} \subset \mathbb{R}^d$ is a compact convex set
2. $\mathcal{L}_{B_t}: \mathcal{F} \to \mathbb{R}$ is a convex lower semicontinuous (lsc) function
3. $\mathcal{F}$ has a bounded diameter, i.e. $D = \max_{x,y \in \mathcal{F}} \|x - y\|_\infty$, and $G = \max_{t \in [T]} \|g_t\|_\infty$

Then, the following convergence result, whose proof can be found in A, can be stated:

**Theorem 3.2.** Let $\tau_t$ be the value of $\tau_{mv}$ at time step $t$ and let $T \leq \tau_t, \forall t$. Under Assumption 3.1, and with $\beta < 1, \alpha_t = \alpha/\sqrt{T}$, AdaTerm achieves a regret $R_T = \sum_{t=1}^{T} \mathcal{L}_{B_t}(\theta_t) - \mathcal{L}_{B_t}(\theta^*)$ such that:

$$R_T \leq \frac{D^2\sqrt{T}}{4\tau_T \alpha} \sum_{i=1}^{d} v_{T,i}^{1/2} + \left[ \frac{\tau T + 1 - (\beta + \epsilon)}{2T} \right] \sum_{t=1}^{T-1} \frac{D^2}{\alpha_t} \sum_{i=1}^{d} v_{t,i}^{1/2}$$

$$+ \left[ \frac{(1 - \beta)^2 \alpha}{\epsilon^2 \sqrt{T}} \right] \sum_{t=1}^{T-1} \sum_{i=1}^{d} (1 - \tau) T - k g_{k,i}^{2}$$

$$+ \left[ \frac{\tau T + 1 - (\beta + \epsilon)}{2T} \right] \left[ \frac{(1 - \beta)^2 \alpha}{\epsilon^2 \sqrt{T}} \right] \sqrt{1 + \log(T - 1)} \sum_{i=1}^{d} g_{T-1,i}^{2}$$ \hspace{1cm} (30)

**Corollary 3.3** (Non robust case regret bound). If $\tilde{\nu} \to \infty$,
then $\tau \to \text{constant} = (1 - \beta)$. Then, the regret becomes:

$$R_T \leq \frac{D^2 \sqrt{T}}{4(1 - \beta)} \sum_{t=1}^{T} \alpha t^{1/2} + \frac{1}{2} \sum_{t=1}^{T-1} \frac{D^2 \sum_{i=1}^{d} v_{t,i}^{1/2}}{\alpha t^{1/2}} + \frac{\alpha}{\epsilon v \sqrt{T}} \sum_{t=1}^{T-1} \sum_{i=1}^{d} \beta^{T-k} g_{k,i}^2$$

$$+ \frac{(1 + \beta) \alpha \sqrt{T + \log(T - 1)}}{2(1 - \beta) \epsilon} \sum_{i=1}^{d} \|g_{1:T-1,i}\|_2^2$$

(31)

Note the similarity between this regret bound and the one derived by (Reddi et al., 2018) and by (Zhuang et al., 2020) using AMSGrad. In particular, this regret can be bounded by $O(G\sqrt{T})$ and thus the regret is upper bounded by a minimum of $O(G\sqrt{T})$. This leads to a worst case dependence of the regret on $T$ to still be $O(\sqrt{T})$ despite the non-usage of AMSGrad.

We emphasize that this approach to the regret bound is not specific to AdaTerm, but can be used to bound the regret of other momentum based optimizers, including Adam and AdaBelief.

3.5.2. Qualitative Comparison with Related Work and Robustness Analysis

The difference between AdaTerm and the related work ((Ilboudo et al., 2020)) is threefold. First, for $m$, the difference lies in how the interpolation between the value before the update and the update amount is performed. In the past work, the discounted sum of $w_{mv}$ was used, but in AdaTerm, $\overline{w}_{mv}$ is used instead, which eliminates the need to store the discounted sum in memory.

Secondly, $v$ is now updated robustly depending on $w_{mv}$, ensuring that the observation of aberrant gradients $g$ would not cause a sudden increase of $v$ and inadvertently loosen the threshold for other aberrant update amounts in the next and subsequent steps. In addition, it is expected to coordinate among the axes. That is, if an anomaly is detected only on a particular axis, $\Delta s$ on that axis will become larger, making $v$ on that axis larger and mitigating the anomaly detection. Conversely, if anomalies are detected in most axes, $\Delta s \approx \epsilon^2$ will continue to exclude the anomalies as described above. Such adaptive behaviors would yield stable updates even if $\beta$ is smaller than the conventional $\beta_2 (= 0.999$ in most optimizers).

Thirdly, the robustness indicator $\tilde{\nu}$ will be increased when the deviation $D$ is small (i.e. no aberrant value), in which case the first term of $\lambda \nu$ becomes larger and $\kappa_{\Delta \phi} g_{\phi}$ becomes positive. However, the increased speed will be limited by $\overline{\nu}_v$ thanks to its inclusion in the step size $\kappa_{\Delta \phi}$. On the other hand, if an aberrant value is observed, $w_{\phi} \gg 1$ and $\lambda$ will get smaller such that $\kappa_{\Delta \phi} g_{\phi}$ becomes negative. The decay of $\lambda$ towards $\tilde{\nu}$ will then happen more quickly compared to the increase in speed. An intuitive visualization of the described behavior obtained by the AdaTerm equations can be found in B. Thus, although the robustness tuning mechanism substitutes the upper bound of its gradient, it still behaves conservatively and can be expected to retain its excellent robustness to noise.

Finally, if $\tilde{\nu} \to \infty$, the robustness is lost by design and AdaTerm essentially matches a slightly different version of AdaBelief, with performance difference due to the simplification of the hyper-parameters and the variance computation mechanism (AdaBelief estimates $E[(g_t - m_t)^2]$, while AdaTerm estimates $E[(g_t - m_{t-1})^2]$ which is the usual variance estimator). Therefore, in problems where AdaBelief would be effective, AdaTerm would perform effectively as well.

4. Experiments

4.1. Analysis with test functions

Before solving more practical benchmark problems, we analyze the behavior of AdaTerm through minimization of typical test functions. In other words, we aim to find the two-dimensional point where the respective potential fields are minimized by relying on the gradients of the potential fields. To analyze the robustness to noise, uniformly distributed
noise \((\in (-0.1, 0.1))\) at a specified ratio is added to the point coordinates to be optimized.

The results are summarized in Figs. 2 and 3 (details are in C). Fig. 2, which shows the error norm from the analytically-optimal point, shows that the convergence accuracy of McCormick function with AdaTerm was not good in lower noise ratio probably due to large learning rate. In other scenes, however, AdaTerm was able to maintain its normal update performance while mitigating the effect of noise. The performance of Adam ((Kingma and Ba, 2014)) was significantly degraded when some noise is given, indicating the sensitivity to noise. On the Michalewicz function, At-Adam ((Ilboudo et al., 2021)) judges the steep gradients near the optimal value to be due to noise and tends to exclude them, causing the optimal solution to not be obtained. This result implies that the automatic mechanism for tuning \(v\) employed by At-Adam has insufficient robustness adaptability to noise. Indeed, Fig. 3, which plots the final \(v\), shows that \(v\) with At-Adam converged to a nearly constant value independent of the noise ratio. In contrast, in AdaTerm, \(v\) was inversely proportional to the noise ratio, and succeeded in achieving an intuitively natural behavior of increasing \(v\) when the noise is rare, and decreasing \(v\) when the noise became more frequent (high noise ratio).

### 4.2. Robustness display on simple regression task

**Problem settings** Following the same process as in ((Ilboudo et al., 2020)), we consider the problem of fitting a ground truth function \(f(x) = x^2 + \ln(x + 1) + \sin(2\pi x) \cos(2\pi x)\) given some noisy observations \(y = f(x) + \zeta\) with \(\zeta\) given by:

\[
\zeta \sim T(1, 0, 0.05) \cdot \text{Bern}\left(\frac{p}{100}\right) ; \quad p = 0, 10, 20, \ldots, 100
\]

(32)

where \(T(1, 0, 0.05)\) designates a Student’s t-distribution with degrees of freedom \(\nu_{\zeta} = 1\), location \(\mu_{\zeta} = 0\) and scale \(\nu_{\zeta} = 0.05\) and \(\text{Bern}(p/100)\) is a Bernoulli distribution with the ratio \(p\) as its parameter. 50 trials with different random seeds are conducted for each noise ratio \(p\) and each optimization method, using 40000 \((x, y)\) pairs sampled as observations and split into batches of size 10.

The model used is a fully-connected neural network with 5 linear layers, each composed of 50 neurons, equipped with the ReLU activation function ((Shang et al., 2016)) for all the hidden layers. The training and the test loss functions are the Mean Squared Error (MSE) applied on \((\hat{y}, y)\) and \((\hat{y}, f(x))\) respectively, where \(\hat{y}\) is the network’s estimate given \(x\).

**Result** The test loss against the noise ratio \(p\) is plotted in Fig. 4. As can be seen, AdaTerm performed even more robustly than t-Adam and At-Adam and is able to keep its prediction loss almost constant across all noise ratio. The effect of the batch size on the learning performance is also studied and can be found in F.

### 4.3. Configurations of benchmark problems

After verifying the optimization efficiency and robustness on some test functions and a simple regression task, we now perform four practical benchmark problems to compare the performance of typical optimizers, including the proposed AdaTerm. Detailed setups (e.g. network architectures) can be found in D.

#### 4.3.1. Classification of mislabeled CIFAR-100

The first problem is an image classification problem with CIFAR-100 dataset. As artificial noise, some proportion of the labels (0 % and 10 %) on the training data is randomized to be anything other than the true labels. As a simple data augmentation, random horizontal flip is introduced along with 4 padding only when training.

The loss function is given to be the cross-entropy. In order to stabilize the learning performance, we introduce a label smoothing technique ((Szegedy et al., 2016)). The degree of smoothing is set to 20 %, referring to the literature ((Lukasik et al., 2020)).

#### 4.3.2. Long-term prediction of robot locomotion

The second problem is a motion prediction problem. The dataset used was borrowed from the literature ((Kobayashi, 2020)) and contains 150 trajectories as training data, 25 trajectories as validation data, and 25 trajectories as test data, all gathered from a hexapod locomotion task with 18 observed joint angles and 18 reference joint angles. An agent continually predicts the states within the given time interval.
(1 and 30 steps) from the initial true state and subsequent predicted states. Therefore, the predicted states used as inputs would deviate from the true ones and become noise for learning.

For the loss function, instead of the mean squared error (MSE), which is commonly used, we employ the negative log likelihood (NLL) of the predictive distribution, which allows the scale difference of each state to be considered internally. A NLL is computed at each prediction step, then their sum is used as the loss function. Because of the high cost of back-propagation through time (BPTT) over the entire trajectory, truncated BPTT (30 steps on average) ((Puskorius and Feldkamp, 1994; Tallec and Ollivier, 2017)) is employed.

4.3.3. Reinforcement Learning on Pybullet Simulator

The third problem is RL simulated by Pybullet engine with OpenAI Gym ((Brockman et al., 2016; Coumans and Bai, 2016)). The tasks to be solved are Hopper and Ant, both of which require an agent to move as straight as possible on flat terrain. As mentioned before, RL relies on estimated values due to no true signals, which can easily introduce noise in the training.

The RL algorithm implemented is an actor-critic algorithm based on the literature ((Kobayashi, 2021)). The agent only learns after each episode using experience replay. This experience replay samples 128 batches after each episode, and its buffer size is set to be small enough (i.e. 10,000) to reveal the influence of noise.

4.3.4. Policy Distillation by Behavioral Cloning

The fourth problem is policy distillation ((Rusu et al., 2015)). The three policies for Ant that were properly learned in the RL problem above were taken as experts, and 10 trajectories of state-action pairs were collected from each of them. We also collected three trajectories from the one policy that fell into a local solution as an amateur. In the dataset constructed with these trajectories, not only the amateur trajectories, but also the expert trajectories would not be always truly optimal, and thus they cause noise.

The loss function is the negative log likelihood of the policy according to behavioral cloning ((Bain and Sammut, 1995)), which is the simplest imitation learning method. In addition, a weight decay is added to prevent the small networks (i.e. the distillation target) from over-fitting particular behaviors.

4.4. Results of benchmark problems

In all the experiments, we compare the following six optimizers: Adam ((Kingma and Ba, 2014)), AdaBelief ((Zheng et al., 2020)), and RAdam ((Gulcehre et al., 2017)) as the state-of-the-art optimizers in the cases without noise; t-Adam ((Ilboudo et al., 2020)) and At-Adam ((Ilboudo et al., 2021)) as the noise-robust optimizers; and AdaTerm as our proposal. Note that the initial \( \nu \) in At-Adam is set to be the same value used in t-Adam, in order to evaluate its adaptiveness. In each condition, 24 trials are conducted with different random seeds, and the mean and standard deviation of the scores for the respective benchmarks are evaluated. All the test results after learning can be found in Table 2. An ablation test was also conducted for AdaTerm, as summarized in E.

Since we prepared benchmarks that favor noise-robust optimizers, AdaTerm obtained the best in most of the problems (except for 1-step prediction and RL tasks). Since 1-step prediction is a relatively simple supervised learning, it does not need robustness to noise. Therefore, as like the error norm of McCormick function in Fig. 2, the too high learning rate caused by \( \beta < \beta_2 \) contributed to the performance degradation. Although AdaTerm for RL tasks was certainly not the best, we can confirm its usefulness from the two facts that i) it achieved the second-best in Hopper task and ii) Ant task was only successful with noise-robust optimizers. In summary, we can say that AdaTerm can maintain its learning performance better than other optimizers in practical problems with noise.

We analyze the remarkable results of the benchmark problems, respectively. In the classification problem, AdaTerm achieved the highest classification performance in the compared optimizers even for the 0% label error. This is because CIFAR-100 contains general images, which are prone to be with noise, and the training dataset has only 500 images per class, revealing the adverse effects of noise.

In the 30-step prediction, AdaTerm was the only one that succeeded in making MSE approximately one. We can easily expect that this problem is noise-contained due to inaccurate estimated inputs. However, as learning progresses, the accuracy of the estimated inputs should be improved, and the noise robustness would gradually become unnecessary. In AdaTerm, the adjustment of the noise robustness worked properly (like Fig. 3), and was successful in accelerating learning.

In the RL problem, as mentioned before, it is clear that Ant task required the high robustness to noise. In addition, as one of the implementation tips in RL, it is often pointed out that setting a relatively large \( \epsilon \) contributes to the stability of learning. AdaTerm has larger \( \epsilon \) (i.e. 1e-5) as the default value for the enough minimal adjustment speed of \( \tilde{\nu} \), which may also contribute to the performance improvement.

Finally, in the policy distillation, the performance improvement by AdaTerm was confirmed even when amateur data, which is a source of noise, was not included. This is due
5. Discussion

The above experiments and simulations showed the robustness improvement of AdaTerm against the related works. However, we have to discuss its limitations as below.

5.1. Performance and drawbacks of AdaTerm on noise-free problems

As can be seen in Fig. 2 and in the 1 step prediction error of Table 2, there is a drawback of using AdaTerm on a noise-free optimization problem when compared to a non-robust optimizer such as Adam. Indeed these results show that in the absence of noise, employing an optimization method which assumes from the start an absence of aberrant data points can give a better result. This is not surprising and should be expected, since AdaTerm (with its default initial value $\tilde{\nu}_0 = 1$) has a non-zero adaptation time to converge to a non-robust behavior. Despite that, the overall results also show that the drawback or penalty incurred from using the AdaTerm algorithm instead of Adam or AdaBelief in the case of noise-free applications does not constitute much of a problem when taking into consideration its ability to deal with the possible presence of unknown noise ratios. In particular, given how hard it is to gather perfect datasets in practice.

In addition, when suspecting a noise-free problem, AdaTerm endows the practitioner with the ability to set a large initial value for the degrees of freedom and then let the algorithm decide if the dataset is indeed noise-free. Such freedom is only possible thanks to the adaptive capability of AdaTerm as clearly displayed in Fig. 3.

5.2. Gap between theoretical and experimental convergence analysis

Although Theorem 3.2 gives a theoretical upper bound on the regret achieved by AdaTerm as a typical approach to convergence analysis, it completely eludes the robustness factor brought in by AdaTerm. In particular, Corollary 3.3 appears to have a better bound compared to Theorem 3.2 which, as displayed in the ablation study of E, contrasts with the practical application. This implies a gap between the theoretical analysis and the experimental analysis and stems on the fact that the theoretical bound relies on $\tau$.

As a remedy to this shortcoming of the regret analysis, we therefore relied on an intuitive analysis of the robustness factor based on the behavior of the different components of our algorithm. Although this qualitative analysis has a weak theoretical convergence value, the experimental analysis against different noise settings show that AdaTerm is not only robust but also efficient as an optimization algorithm.

Nevertheless, we acknowledge the need for a stronger theoretical analysis that takes into consideration the noisiness of the gradients, while allowing for a theoretical comparison between the robustness and efficiency of different optimizers.
AdaTerm towards Noise-Robust Stochastic Gradient Optimizer

5.3. Normalization of gradient

By considering the gradients to be generated from Student’s t-distribution, AdaTerm normalizes the gradients (more precisely, its first momentum) using the estimated scale, instead of the second moment. This is similar to the normalization of AdaBelief, as mentioned before. However, as shown in the G, the normalization with the second moment can sometimes perform better.

Basically, since the second moment is larger than the scale, the normalization by the second moment makes the update conservative, while the one by the scale is expected to break through the stagnation of updates. While both characteristics are certainly important, the answer to the question “which one is desirable?” remains situation-dependent. Therefore, we need to pursue the theory concerning this point, and introduce a mechanism to adaptively switch the use of both according to the required characteristics.

6. Conclusion

We presented the AdaTerm optimizer, which is adaptively robust to noise and outliers, for deep learning. AdaTerm was derived from the assumption that the gradients are generated from multivariate diagonal Student’s t-distribution, and then, the distribution parameters are optimized through the surrogated maximum log-likelihood estimation. Optimization of test functions revealed that AdaTerm can adjust its robustness to noise in accordance with the impact of the noise. Through the four typical benchmarks, we confirmed that the robustness to noise and the learning performance of AdaTerm are as good as or better than those of conventional optimizers. In addition, we derived the new regret bound for the Adam-like optimizers without the assumption of the use of AMSGrad.

This paper focused on computing the moments’ values, but in recent years, the importance of integration with raw SGD (i.e. decay of scaling in Adam-like optimizers) has been confirmed ((Luo et al., 2019; Zhou et al., 2020)). We will therefore investigate a natural integration of AdaTerm and the raw SGD by reviewing ∆s, which may enable the normalization to be constant. In addition, as mentioned in the discussion, we argue that a new framework for analyzing optimization algorithms both in terms of robustness and efficiency and such that they can be compared, is required. One such analysis was done by (Scaman and Malherbe, 2020) on SGD, but its extention to momentum-based optimizers and its ability to allow theoretical comparison across different algorithms remain limited. We will therefore seek in the future, a similar but better approach and apply it to analyze the robustness and efficiency of different optimization algorithms, including AdaTerm.

References

Aeschliman, C., Park, J., Kak, A.C., 2010. A novel parameter estimation algorithm for the multivariate t-distribution and its application to computer vision, in: European conference on computer vision, Springer. pp. 594–607.

Alacaoglu, A., Malitsky, Y., Mertikopoulos, P., Cevher, V., 2020. A new regret analysis for adam-type algorithms, in: International Conference on Machine Learning, PMLR. pp. 202–210.

Ba, J., Kiros, J.R., Hinton, G.E., 2016. Layer normalization. arXiv preprint arXiv:1607.06450.

Bain, M., Sammut, C., 1995. A framework for behavioural cloning., in: Machine Intelligence 15, pp. 103–129.

Brockman, G., Cheung, V., Pettersson, L., Schneider, J., Schulman, J., Tang, J., Zaremba, W., 2016. Openai gym. arXiv preprint arXiv:1606.01540.

Chen, R.T., Rubanova, Y., Bettencourt, J., Duvenaud, D., 2018. Neural ordinary differential equations, in: International Conference on Neural Information Processing Systems, pp. 6572–6583.

Chung, J., Gulcehre, C., Cho, K., Bengio, Y., 2014. Empirical evaluation of gated recurrent neural networks on sequence modeling. arXiv preprint arXiv:1412.3555.

Coumans, E., Bai, Y., 2016. Pybullet, a python module for physics simulation for games, robotics and machine learning. GitHub repository.

De Ryck, T., Lanthaler, S., Mishra, S., 2021. On the approximation of functions by tanh neural networks. Neural Networks 143, 732–750.

Doğru, F.Z., Bulut, Y.M., Arslan, O., 2018. Doubly reweighted estimators for the parameters of the multivariate t-distribution. Communications in Statistics-Theory and Methods 47, 4751–4771.

Elfwing, S., Uchibe, E., Doya, K., 2018. Sigmoid-weighted linear units for neural network function approximation in reinforcement learning. Neural Networks 107, 3–11.

Gou, J., Yu, B., Maybank, S.J., Tao, D., 2021. Knowledge distillation: A survey. International Journal of Computer Vision 129, 1789–1819.

Gulcehre, C., Sotelo, J., Moczulski, M., Bengio, Y., 2017. A robust adaptive stochastic gradient method for deep learning, in: International Joint Conference on Neural Networks, IEEE. pp. 125–132.

He, K., Zhang, X., Ren, S., Sun, J., 2016. Deep residual learning for image recognition, in: IEEE conference on computer vision and pattern recognition, pp. 770–778.
AdaTerm towards Noise-Robust Stochastic Gradient Optimizer

Holland, M.J., Ikeda, K., 2019. Efficient learning with robust gradient descent. Machine Learning 108, 1523–1560.

Ilboudo, W.E.L., Kobayashi, T., Sugimoto, K., 2020. Robust stochastic gradient descent with student-t distribution based first-order momentum. IEEE Transactions on Neural Networks and Learning Systems.

Ilboudo, W.E.L., Kobayashi, T., Sugimoto, K., 2021. Adaptive t-momentum-based optimization for unknown ratio of outliers in amateur data in imitation learning, in: IEEE/RSJ International Conference on Intelligent Robots and Systems, IEEE. pp. 7851–7857.

Keskar, N.S., Mudigere, D., Nocedal, J., Smelyanskiy, M., Tang, P.T.P., 2017. On large-batch training for deep learning: Generalization gap and sharp minima, in: International Conference on Learning Representations.

Kishida, M., Ogura, M., Yoshida, Y., Wadayama, T., 2020. Deep learning-based average consensus. IEEE Access 8, 142404–142412.

Kobayashi, T., 2019. Student-t policy in reinforcement learning to acquire global optimum of robot control. Applied Intelligence 49, 4335–4347.

Kobayashi, T., 2020. q-vae for disentangled representation learning and latent dynamical systems. IEEE Robotics and Automation Letters 5, 5669–5676.

Kobayashi, T., 2021. Proximal policy optimization with relative pearson divergence, in: 2021 IEEE International Conference on Robotics and Automation (ICRA), IEEE. pp. 8416–8421.

Lange, K.L., Little, R.J., Taylor, J.M., 1989. Robust statistical modeling using the t distribution. Journal of the American Statistical Association 84, 881–896.

LeCun, Y., Bengio, Y., Hinton, G., 2015. Deep learning. nature 521, 436.

Lee, C., Cho, K., Kang, W., 2018. Directional analysis of stochastic gradient descent via von mises-fisher distributions in deep learning. arXiv preprint arXiv:1810.00150.

Ley, C., Neven, A., 2012. The value at the mode in multivariate t distributions: a curiosity or not? arXiv preprint arXiv:1211.1174.

Liu, L., Jiang, H., He, P., Chen, W., Liu, X., Gao, J., Han, J., 2020. On the variance of the adaptive learning rate and beyond, in: International Conference on Learning Representations.

Lukasik, M., Bhojanapalli, S., Menon, A., Kumar, S., 2020. Does label smoothing mitigate label noise?, in: International Conference on Machine Learning, PMLR. pp. 6448–6458.

Luo, L., Xiong, Y., Liu, Y., Sun, X., 2019. Adaptive gradient methods with dynamic bound of learning rate, in: International Conference on Learning Representations.

LeCun, Y., Bengio, Y., Hinton, G., 2015. Deep learning. nature 521, 436.
Shang, W., Sohn, K., Almeida, D., Lee, H., 2016. Understanding and improving convolutional neural networks via concatenated rectified linear units, in: international conference on machine learning, pp. 2217–2225.

Simsekli, U., Sagun, L., Gurbuzbalaban, M., 2019. A tail-index analysis of stochastic gradient noise in deep neural networks, in: International Conference on Machine Learning, PMLR. pp. 5827–5837.

Suchi, M., Patten, T., Fischinger, D., Vincze, M., 2019. Easylabel: a semi-automatic pixel-wise object annotation tool for creating robotic rgb-d datasets, in: International Conference on Robotics and Automation, IEEE. pp. 6678–6684.

Sun, S., Cao, Z., Zhu, H., Zhao, J., 2019. A survey of optimization methods from a machine learning perspective. IEEE transactions on cybernetics 50, 3668–3681.

Sutton, R.S., Barto, A.G., 2018. Reinforcement learning: An introduction. MIT press.

Szegedy, C., Vanhoucke, V., Ioffe, S., Shlens, J., Wojna, Z., 2016. Rethinking the inception architecture for computer vision, in: IEEE conference on computer vision and pattern recognition, pp. 2818–2826.

Tallec, C., Ollivier, Y., 2017. Unbiasing truncated backpropagation through time. arXiv preprint arXiv:1705.08209.

Xu, J., Sun, X., Zhang, Z., Zhao, G., Lin, J., 2019. Understanding and improving layer normalization. Advances in Neural Information Processing Systems 32, 4381–4391.

Zhou, P., Feng, J., Ma, C., Xiong, C., Hoi, S.C.H., et al., 2020. Towards theoretically understanding why sgd generalizes better than adam in deep learning. Advances in Neural Information Processing Systems 33.

Zhuang, J., Tang, T., Ding, Y., Tatikonda, S.C., Dvornek, N., Papademetris, X., Duncan, J., 2020. Adabelief optimizer: Adapting stepsizes by the belief in observed gradients. Advances in Neural Information Processing Systems 33.

Ziyin, L., Liu, K., Mori, T., Ueda, M., 2021. Strength of minibatch noise in sgd. arXiv preprint arXiv:2102.05375.
A. Proof of regret bounds

Below is the proof and the detailed analysis for the AdaTerm algorithm. Following the literature, we ignore the bias correction term. We also consider a scheduled learning rate \( \alpha_t = \alpha (\sqrt{t})^{-1} \) and the non-expansive weighted projection operator \( \Pi_{\mathcal{F}, \Omega}(\theta) = \arg \min_{\theta' \in \mathcal{F}} \| \theta' - \theta \|_{\Omega} \). Note however that the use of the projection operator is not a necessary step for deriving the upper bound and as the reader may notice, no projection is used in Algorithm 1. Despite that, it provides a more general setting and therefore, following the literature, we consider its potential use in our proof.

A.1. Preliminary

We start by stating intermediary results that we shall use later to bound the regret.

**Lemma A.1** (Peter–Paul inequality or Young’s inequality with \( \zeta \) and exponent 2). \( \forall \zeta > 0 \) and \( \forall (x, y) \in \mathbb{R}^2 \) we have that 
\[ xy \leq \frac{\zeta}{2} x^2 + \frac{1}{2\zeta} y^2. \]

From which we get the following results:

**Corollary A.2** (Partial bound of \( \langle m_t, \theta_t - \theta^* \rangle \)). Let \( (m_t)_{t \geq 0}, (v_t)_{t \geq 0} \) and \( (\theta_t)_{t \geq 0} \) be the sequence of momentum, variance and iterates produced by AdaTerm with a scheduled learning rate \( \alpha_t > 0, \forall t \). Then, we have:
\[ \langle m_t, \theta_t - \theta^* \rangle \leq \frac{D^2}{2\alpha_t} \left\| v_t^{1/4} \right\|^2 + \frac{1}{2} \alpha_t \left\| m_t \right\|_{v_t^{1/2}}^2 \] (33)

**Proof.** This follows from the application of Lemma A.1 with the identification:
\[ \zeta = v_t^{1/2}, \quad x_i = (\theta_{t,i} - \theta^*_t), \quad y_i = m_{t,i} \]

Combined with the fact that \( \forall i \in [d], (\theta_{t,i} - \theta^*_t) \leq D \). \qed

A.2. Introduction

Let \( m_t \) and \( v_t \) be the momentum and variance produced by AdaTerm, then:
\[ m_t = (1 - \tau_t)m_{t-1} + \tau_t g_t, \quad v_t = (1 - \tau_t)m_{t-1} + \tau_t \left( (g_t - m_{t-1})^2 + \Delta s \right) \] (34)

From the definition of \( m_t \), we can derive:
\[ g_t = \frac{1}{\tau_t} m_t - \frac{1 - \tau_t}{\tau_t} m_{t-1} \] (35)

which leads to:
\[ \langle g_t, \theta_t - \theta^* \rangle = \frac{1}{\tau_t} \langle m_t, \theta_t - \theta^* \rangle - \frac{1 - \tau_t}{\tau_t} \langle m_{t-1}, \theta_t - \theta^* \rangle - \frac{1 - \tau_t}{\tau_t} \langle m_{t-1}, \theta_{t-1} - \theta^* \rangle - \frac{1 - \tau_t}{\tau_t} \langle m_{t-1}, \theta_t - \theta_{t-1} \rangle \] (36)

\[ = \frac{1}{\tau_t} \left( \langle m_t, \theta_t - \theta^* \rangle - \langle m_{t-1}, \theta_{t-1} - \theta^* \rangle \right) + \langle m_{t-1}, \theta_{t-1} - \theta^* \rangle - \frac{1 - \tau_t}{\tau_t} \langle m_{t-1}, \theta_t - \theta_{t-1} \rangle \] (37)

We therefore have that:
\[ \sum_{t=1}^{T} \langle g_t, \theta_t - \theta^* \rangle = R_{1,T} + R_{2,T} + R_{3,T} \] (38)

\[ R_{1,T} = \sum_{t=1}^{T} \langle m_{t-1}, \theta_{t-1} - \theta^* \rangle = \sum_{t=1}^{T-1} \langle m_t, \theta_t - \theta^* \rangle \quad \text{(since } m_0 = 0) \] (39)

\[ R_{2,T} = - \sum_{t=1}^{T} \frac{1 - \tau_t}{\tau_t} \langle m_{t-1}, \theta_t - \theta_{t-1} \rangle = \sum_{t=1}^{T} \frac{1 - \tau_t}{\tau_t} \langle m_{t-1}, \theta_{t-1} - \theta_t \rangle \] (40)

\[ R_{3,T} = \sum_{t=1}^{T} \frac{1}{\tau_t} \left( \langle m_t, \theta_t - \theta^* \rangle - \langle m_{t-1}, \theta_{t-1} - \theta^* \rangle \right) \] (41)
A.3. Bound of $R_{1,T}$

Using Corollary A.2, we have that:

$$R_{1,T} = \sum_{t=1}^{T-1} \langle m_t, \theta_t - \theta^* \rangle \leq \sum_{t=1}^{T-1} \frac{D^2}{2\alpha_t} \| \nu_t^{1/4} \|^2 + \frac{1}{2} \sum_{t=1}^{T-1} \alpha_t \| m_t \|^2 \nu_t^{1/2}$$  \hspace{1cm} (42)

A.4. Bound of $R_{2,T}$

With $\tilde{m}_{t-1} = \frac{1-\tau_t}{\tau_{t+1}} m_{t-1}$ and by using $m_0 = 0$, we have:

$$R_{2,T} = \sum_{t=1}^{T} \langle \tilde{m}_{t-1}, \theta_{t-1} - \theta_t \rangle = \sum_{t=2}^{T} \langle \tilde{m}_{t-1}, \theta_{t-1} - \theta_t \rangle = \sum_{t=1}^{T-1} \langle \tilde{m}_t, \theta_t - \theta_{t+1} \rangle$$

Then, by applying Holder’s inequality, followed by the use of the projection operator where $\theta_t = \Pi_{F,\delta^{1/2}_t} (\theta_t)$, since $\theta_t \in F$, we obtain:

$$R_{2,T} \leq \sum_{t=1}^{T-1} \| \tilde{m}_t \| \delta_t^{-1/2} \| \theta_t - \theta_{t+1} \| \delta_t^{1/2} = \sum_{t=1}^{T-1} \| \tilde{m}_t \| \delta_t^{-1/2} \| \Pi_{F,\delta^{1/2}_t} (\theta_t) - \Pi_{F,\delta^{1/2}_t} (\theta_t - \alpha_t \tilde{v}_t^{-1/2} m_t) \| \delta_t^{1/2}$$

The non-expansive property of the operator then allows:

$$R_{2,T} \leq \sum_{t=1}^{T-1} \| \tilde{m}_t \| \delta_t^{-1/2} \| \theta_t - \theta_{t+1} \| \delta_t^{1/2} \leq \sum_{t=1}^{T-1} \| \tilde{m}_t \| \delta_t^{-1/2} \| \theta_t - \alpha_t \tilde{v}_t^{-1/2} m_t \| \delta_t^{1/2} = \sum_{t=1}^{T-1} \| \tilde{m}_t \| \delta_t^{-1/2} \| \alpha_t \tilde{v}_t^{-1/2} m_t \| \delta_t^{1/2} = \sum_{t=1}^{T-1} \alpha_t \| m_t \| \delta_t^{1/2}$$

Finally, by restoring $\tilde{m}_t = \frac{1-\tau_{t+1}}{\tau_t} m_t$ and using $\frac{1-\tau_{t+1}}{\tau_t} \leq \frac{1-\tau}{T}$, we have:

$$R_{2,T} \leq \sum_{t=1}^{T-1} \alpha_t \frac{1-\tau_{t+1}}{\tau_t} \| m_t \|^2 \delta_t^{-1/2} \leq \frac{1-\tau}{T} \sum_{t=1}^{T-1} \alpha_t \| m_t \|^2 \delta_t^{-1/2}$$  \hspace{1cm} (43)

A.5. Bound of $R_{3,T}$

$$R_{3,T} = \sum_{t=1}^{T} \frac{1}{\tau_t} \langle \langle m_t, \theta_t - \theta^* \rangle - \langle m_{t-1}, \theta_{t-1} - \theta^* \rangle \rangle = \sum_{t=1}^{T} A_t$$

$$A_t = \frac{1}{\tau_t} \langle m_t, \theta_t - \theta^* \rangle - \frac{1}{\tau_{t-1}} \langle m_{t-1}, \theta_{t-1} - \theta^* \rangle + \frac{\tau_t - \tau_{t-1}}{\tau_t \tau_{t-1}} \langle m_{t-1}, \theta_{t-1} - \theta^* \rangle$$

$$\leq \frac{1}{\tau_t} \langle m_t, \theta_t - \theta^* \rangle - \frac{1}{\tau_{t-1}} \langle m_{t-1}, \theta_{t-1} - \theta^* \rangle + \frac{|\tau_t - \tau_{t-1}|}{2\tau_t \tau_{t-1}} \left\{ \frac{D^2}{\alpha_{t-1}} \| \nu_{t-1}^{1/4} \|^2 + \alpha_{t-1} \| m_{t-1} \| \nu_{t-1}^{1/2} \right\}$$

where the last line is obtained by application of Corollary A.2 on time step $t-1$. Finally, using the telescoping sum over the first two terms in $A_t$ with the particularity of $m_0 = 0$, combined with $|\tau_t - \tau_{t-1}| \leq (1 - \beta) - \tau = [1 - (\beta + \tau)]$, we arrive at the following relation:

$$\sum_{t=1}^{T} A_t \leq \frac{1}{\tau_T} \langle m_T, \theta_T - \theta^* \rangle + \frac{1 - (\beta + \tau)}{2\tau^2} \left\{ \sum_{t=1}^{T} \frac{D^2}{\alpha_{t-1}} \| \nu_{t-1}^{1/4} \|^2 + \sum_{t=1}^{T} \alpha_{t-1} \| m_{t-1} \| \nu_{t-1}^{1/2} \right\}$$

$$R_{3,T} \leq \frac{1}{\tau_T} \langle m_T, \theta_T - \theta^* \rangle + \frac{1 - (\beta + \tau)}{2\tau^2} \left\{ \sum_{t=1}^{T-1} \frac{D^2}{\alpha_t} \| \nu_t^{1/4} \|^2 + \sum_{t=1}^{T-1} \alpha_t \| m_t \| \nu_t^{1/2} \right\}$$  \hspace{1cm} (44)
A.6. Bound of \( \frac{1}{\tau T} \langle m_T, \theta_T - \theta^* \rangle \)

Applying Lemma A.1 with \( \zeta = \frac{\sigma^2}{\alpha_T} \) and \( x_i = (\theta_{T,i} - \theta^*_i) \), \( y_i = m_{T,i} \), we get:

\[
\frac{1}{\tau T} \langle m_T, \theta_T - \theta^* \rangle \leq \frac{1}{\tau T} \left[ \frac{D^2}{4\alpha_T} \left\| v_T^{1/4} \right\|^2 + \alpha_T \left\| m_T \right\|^2_{v_T^{1/2}} \right] \leq \frac{1}{\tau T} \frac{D^2}{4\alpha_T} \left\| v_T^{1/4} \right\|^2 + \frac{(1 - \beta)^2 \epsilon_T^2}{\epsilon_T^2} \sum_{i=1}^{d} \sum_{k=1}^{T} (1 - \tau)^{T-k} g_{k,i}^2 \\
(45)
\]

A.7. Bound of \( \sum_{t=1}^{T-1} \alpha_t \left\| m_t \right\|^2_{v_t^{-1/2}} \)

With \( \alpha_t = \frac{\alpha}{\sqrt{t}} \), we have:

\[
\sum_{t=1}^{T-1} \alpha_t \left\| m_t \right\|^2_{v_t^{-1/2}} = \sum_{t=1}^{T-1} \alpha_t \left\| \tilde{v}_t^{-1/4} m_t \right\|^2_2 \leq \sum_{t=1}^{T-1} \frac{\alpha}{\epsilon \sqrt{t}} \left\| m_t \right\|^2_2 = \sum_{t=1}^{T-1} \frac{\alpha}{\epsilon \sqrt{t}} \sum_{k=1}^{t} \sum_{i=1}^{d} \tau_k g_{k,i} \left( 1 - \tau_{k+j} \right)^{t-k} \left( \sum_{i=1}^{d} \left( 1 - \tau \right)^{t-k} g_{k,i}^2 \right)^2 \leq \frac{(1 - \beta)^2 \alpha T-1}{\epsilon} \sum_{i=1}^{d} \sum_{k=1}^{T-1} (1 - \tau)^k g_{k,i}^2 \frac{1}{\sqrt{t}} \leq \frac{(1 - \beta)^2 \alpha T-1}{\epsilon \sqrt{t}} \sum_{i=1}^{d} \sum_{k=1}^{T-1} (1 - \tau)^k g_{k,i}^2 \frac{1}{\sqrt{t}} (Since \( 1 - (1 - \tau)^{T-t} < 1 \))
\]

Finally, using Cauchy-Schwarz’s inequality (without the squares) with \( u_t = g_{T,i}^2 \) and \( v_t = \frac{1}{\sqrt{t}} \), and with \( \sum_{t=1}^{T-1} \frac{1}{t} \leq 1 + \log(T - 1) \), we get:

\[
\sum_{t=1}^{T-1} \alpha_t \left\| m_t \right\|^2_{v_t^{-1/2}} \leq \frac{(1 - \beta)^2}{\epsilon \sqrt{T}} \sum_{i=1}^{d} \left\| g_{T-1,i} \right\|^2_2 \sqrt{T-1} \leq \frac{(1 - \beta)^2}{\epsilon \sqrt{T}} \sum_{i=1}^{d} \left\| g_{T-1,i} \right\|^2_2 \sqrt{1 + \log(T - 1)} \sum_{i=1}^{d} \left\| g_{T-1,i} \right\|^2_2 (46)
\]

where \( \left\| g_{T-1,i} \right\|^2_2 = \sqrt{T-1} \sum_{i=1}^{T-1} g_{i,i}^4 \leq G_{\infty} \sqrt{T-1} \) where \( G_{\infty} = \max_{t \in [T-1]} \| g_t \|_\infty \).

A.8. Bound of the regret

With

\[
R_T = \sum_{t=1}^{T} \mathcal{L}_{B_t}(\theta_t) - \mathcal{L}_{B_t}(\theta^*) \leq \sum_{t=1}^{T} (g_t, \theta_t - \theta^*) = R_{1,T} + R_{2,T} + R_{3,T}
\]
the bound of the regret is obtained by bringing everything together and a straightforward factorization:

$$R_T \leq \frac{D^2 \sqrt{T}}{4T \alpha} \sum_{i=1}^{d} v_{T,i}^{1/2} + \left[ \frac{T^2 + 1 - (\beta + \tau)}{2\tau^2} \right] \sum_{t=1}^{T-1} \alpha_t \sum_{i=1}^{d} v_{t,i}^{1/2}$$

$$+ \left[ \frac{(1-\beta)^2 \alpha}{\tau^2 \sqrt{T}} \right] \sum_{t=1}^{T-1} \sum_{i=1}^{d} (1-\tau)^{T-k} g_{k,i}^2$$

$$+ \left[ \frac{1}{2} + \frac{1-\tau}{\tau} + \frac{1-\beta}{\epsilon \tau^2} \right] \left[ \frac{(1-\beta)^2 \alpha}{\epsilon \tau^2} \right] \sqrt{1 + \log(T-1)} \sum_{i=1}^{d} \|g_{1:T-1,i}\|_2$$

(47)

B. Visualization of the key elements of AdaTerm

Figures 5 and 6 respectively show the visual description of the interpolation factor $\tau_\text{mv}$ and of the gradient ascent increment $\kappa_\text{g}_\text{mv}$ for the robustness parameter $\tilde{\nu}$. As can be seen in Fig. 5, for small deviations $D_t$, $\tau_\text{mv} \to (1-\beta)$, and conversely, when $D_t$ is large, $\tau_\text{mv} \to 0$ in order to reduce the influence of the aberrant gradient on both $m_t$ and $v_t$. Similarly, Fig. 6 shows how the degrees of freedom is incremented based on the value of $D_t$. The blue region corresponds to a region where $\lambda_t$ is larger than the current robustness value $\tilde{\nu}_{t-1}$ which leads to an increase of $\tilde{\nu}_t$. On the other hand, the orange region, which occurs for large $D_t$ prompts a decrease in the value of $\tilde{\nu}_t$ since $\kappa_\text{g}_\text{mv}$ is negative. As can be seen, the boundary $D_t$ does not change monotonically, but tends to increase when $\tilde{\nu}_{t-1}$ is small, and conversely tends to decrease when $\tilde{\nu}_{t-1}$ is large, thus maintaining balance.

![Figure 5](image-url)  
**Figure 5.** Visualization of $\tau_\text{mv}$ against $\tilde{\nu}_{t-1}$ and $D_t$ values (with $\beta = 0.9$)
A remarkable result was obtained for Michalewicz function (see Fig. 9c). Specifically, AdaTerm initially moved only in the $y$-direction, as in AdaBelief, and then paused at steep gradients in the $x$-direction. This is because the steep gradients toward

Figure 6. Visualization of $\kappa_{\Delta t}g_{\Delta t}$ against $\tilde{v}_{t-1}$ and $D_t$ values (with $\beta = 0.9$)

C. Details of test functions

The test functions tested $f(x, y)$ were Rosenbrock with a valley, McCormick with a plate, and Michalewicz with steep drops, as defined below.

\[
  f(x, y) = \begin{cases} 
    100(y - x^2)^2 + (x - 1)^2 & \text{Rosenbrock} \\
    \sin(x + y) + (x - y)^2 - 1.5x + 2.5y + 1 & \text{McCormick} \\
    -\sin(x)\sin^{20}(x^2/\pi) - \sin(y)\sin^{20}(2y^2/\pi) & \text{Michalewicz} 
  \end{cases}
\] (48)

Using the gradients of $f(x, y)$, $(x, y)$ can be optimized to minimize $f(x, y)$. The initial values of $(x, y)$ were fixed to $(-2, 2)$, $(4, -3)$, and $(1, 1)$, respectively. The noise added to $(x, y)$ for computing the gradients was sampled from the uniform distribution with range $(-0.1, 0.1)$. The probability of noise was set to six conditions: $\{0, 1, 2, 5, 10, 15\}%$.

The optimizers tested were Adam, At-Adam with adaptive $\nu$, and the proposed AdaTerm, AdaBelief ((Zhuang et al., 2020)), which is similar to AdaTerm with $\nu \to \infty$, was also tested, but was excluded from Figs. 2 and 3 due to its similarity to Adam. The learning rate was set to $\alpha = 0.01$, which is higher than that for typical network updates, but other hyper-parameters were left at their default settings. The update was performed 15000 times and the error norm between the final $(x, y)$ and the analytically-optimal point was computed. To evaluate the statistics, 100 trials were run with different random seeds.

Trajectories of the convergence process are illustrated in Figs. 7–9. Note that the color of each point indicates the elapsed time, changing from blue to red. AdaTerm was less likely to update in the wrong direction caused by noise than the others, indicating stable updating as well as the cases with almost no noise. In addition, we found the faster update speed of AdaTerm than others. This is probably because $\beta < \beta_0$ could quickly adapt to small gradients in the saddle area, although this caused the overshoot around the optimal point in Fig. 8c, resulting in the larger error norm than the others in Fig. 2.

A remarkable result was obtained for Michalewicz function (see Fig. 9c). Specifically, AdaTerm initially moved only in the $y$-direction, as in AdaBelief, and then paused at steep gradients in the $x$-direction. This is because the steep gradients toward
the optimum were considered as noise for a while. In fact, as time passed, AdaTerm judged that the steep gradients are not noise. Then, as $\nu$ became larger, the update was resumed, finally reaching the optimal point. On the other hand, At-Adam was unable to adapt $\nu$ to the steep gradients, and the point moved to avoid the optimal point (see Fig. 9b). When noise was added, however, the steep gradients started to be utilized with the help of noise, and the optimal point was finally reached.

![Figure 7. Trajectories on Rosenbrock function](image)

**D. Learning setups**

The models for learning the four benchmark problems are implemented using PyTorch ((Paszke et al., 2017)) with the respective setups, as summarized in Table 3. The parameters that are not listed in the table are basically set to be the recommended default values. Note that the learning rate for RL was smaller than ones for the other problems to avoid getting stuck in one of the local solutions before good samples are collected. In addition, the batch size was basically kept 32 to make the noise more likely to affect the gradients, but for the classification problem only, the batch size was increased to 256, which is within the general range, due to the too increased learning cost.

For the classification problem, ResNet18 ((He et al., 2016)) was employed as the network model. The official PyTorch implementation was basically employed, but to take the size difference of the input image into account, the first layer of convolution was changed to one with kernel size of 3 and stride of 1 and the subsequent MaxPool layer was excluded.

For the prediction problem, a gated recurrent unit (GRU) ((Chung et al., 2014)) was employed for constructing the network model, which consists of two serial hidden layers with each holding 128 GRU units. In the output layer, the state-dimensional means and standard deviations, which are parameters of the multivariate diagonal Gaussian distribution, are outputted. Note that, since extending the number of prediction steps delays the learning process, we set the epoch to 100 for 1-step prediction and 300 for 30-step prediction.

For the RL problem, five fully-connected layers with 100 neurons for each were implemented as the hidden layers. Each activation function was an unlearned LayerNorm ((Ba et al., 2016; Xu et al., 2019)) and Swish function ((Elfwing et al., 2018)). The output is a multivariate diagonal student’s t-distribution to improve the efficiency of the exploration ((Kobayashi, 2019)).

For the policy distillation problem, two fully-connected layers with only 32 neurons for each were implemented as the hidden layers. This structure is clearly smaller than that for the RL problem described above. Each activation function was only Tanh function, with reference to the fact that this design has been reported to have sufficient expressive capability ((De
Figure 8. Trajectories on McCormick function

Ryck et al., 2021)). In the output layer, the state-dimensional means and standard deviations, which are parameters of the multivariate diagonal Gaussian distribution, are outputted.

Table 3. Learning setups

| Parameter               | Classification | Prediction | RL   | Distillation |
|-------------------------|----------------|------------|------|--------------|
| Learning rate           | 1e-3           | 1e-3       | 1e-4 | 1e-3         |
| Batch size              | 256            | 32         | 32   | 32           |
| #Epoch                  | 100            | 100/300    | 2000 | 200          |
| Label smoothing         | 0.2            | -          | -    | -            |
| Truncation              | -              | 30         | -    | -            |
| #Batch/#Replay buffer   | -              | 128/1e+4   | -    | -            |
| Weight decay            | -              | -          | 1e-4 | -            |

E. Ablation study

For the ablation test of AdaTerm, we test the following three conditions: $\Delta \tilde{\nu} = 0$, called no adaptiveness; $\tilde{\nu} = \infty$, called no robustness; and $\tilde{\nu}_0 = 100$, called large init. All other conditions are the same as the experiments in the main text.

The test results are summarized in Table 4. As mostly expected, the no adaptiveness outperformed the no robustness for the problems with high noise effects; and vice versa. From the results of such as the 30-step prediction and the policy distillation, we can say that the optimal solution may lie somewhere in the middle, rather than at the two extremes, and the normal AdaTerm found it. On the other hand, in the case where $\tilde{\nu}_0$ is increased and the noise robustness is poor at the beginning of training, the performance was worse than the normal case where $\tilde{\nu}_0 = \tilde{\nu} + \epsilon$ and the noise robustness is maximized at the beginning, except for the classification problem. This tendency suggests that even if $\tilde{\nu}$ is adjusted to gain the appropriate noise robustness after optimization without considering the effects of noise, the performance would be prone to get stuck in one of the local solutions. For the classification problem only, the size of the network architecture was larger than for the other problems, and its redundancy allowed the classifier to escape from the local solutions. In such a case, using more gradients from the beginning resulted in the performance improvement.
F. Batch size effect on robustness analysis

As illustrated in the literature ([Lee et al., 2018]), the batch size has an impact on the stochastic gradient’s noise strength. Furthermore, intuitively, larger batch sizes would help mitigate the effect of corrupted data points by drowning them in a majority of good points inside the mini batch. However, in typical machine learning tasks, given some dataset, an increase of batch size also implies a decrease in the number of updates (and therefore also a decrease in the number of gradients observed). It has also been reported that the large batch size degrades generalization performance ([Keskar et al., 2017]). To replicate such trade-off effect and to study how it can affect the robustness, we used the regression task where the number of samples is fixed at 40000 and then split into batches of different sizes ∈ {8, 16, 32, 64}.

The results show that for all optimizers in noiseless scenarios, the performance degraded when increasing the batch size, consistent with the results of previous studies, although this is more pronounced for AdaTerm and (A)t-Adam. On the other hand, robustness to noise behaved slightly differently than expected. Specifically, AdaTerm and (A)t-Adam lost their robustness as the batch size increased, which was natural because it gets harder to detect aberrant gradients hidden in the majority of good points. However, the majority could not mitigate the effect of noise, as can be confirmed in the fact that Adam and AdaBelief remained sensitive to noise.
**G. AdaTerm variants**

Our main proposed algorithm is given in Alg. 1. However, we consider and study two alternative variants as described below.

**G.1. Uncentered second-moment-based updates**

The derived AdaTerm equations use the estimate of the centered second moment, i.e. $\mathbb{E}[(g - m)^2]$, in the update based on the same logic outlined in ((Zhuang et al., 2020)). However, the Adam algorithm employs an estimate of the uncentered second moment, i.e. $\mathbb{E}[g^2]$. We therefore also consider a gradient descent update based on the uncentered estimated moment:

$$
\eta_{\text{AdaTerm,Uncentered}}(g_t) = m_t(1 - \beta_t)^{-1} - \frac{1}{\sqrt{(v_t + m_t^2)(1 - \beta_t)^{-1}}}
$$

Here, we approximately use the relationship, $\mathbb{E}[(g - m)^2] = \mathbb{E}[g^2] - \mathbb{E}[g]^2$. Note that, in that case, $\eta_{\text{AdaTerm,Uncentered}} < 1$ is always satisfied, which would yield more conservative learning.

**G.2. Adaptive bias correction term**

The bias correction term has its origin in the Adam paper ((Kingma and Ba, 2014)). In that paper, the authors derive the bias term as:

$$
(1 - \beta) \sum_{i=1}^{t} \beta^{t-i} = 1 - \beta^t
$$

and they interpret it as the term needed to correct the bias caused by initializing the EMA with zeros.

Another interpretation (which do no require taking the expectation of the EMA), is to see this term as the normalization term in a weighted average. Indeed, given $\{g_1, \cdots, g_t\}$, the weighted sum of $g$ with respect to some weights $\{w_1, \cdots, w_t\}$ is given by:

$$
\sum_{i=1}^{t} w_i g_i
$$

If $\sum_{i=1}^{t} w_i \neq 1$, then to get the proper weighted average, a correction is required as:

$$
\text{ave}(g) = \left(\sum_{i=1}^{t} w_i\right)^{-1} \sum_{i=1}^{t} w_i g_i
$$

In the case of the EMA, the weights $w_i$ are given by:

$$
m_t = \sum_{i=1}^{t} (1 - \beta)\beta^{t-i} g_i \Leftrightarrow w_i = (1 - \beta)\beta^{t-i}
$$

$$
\Rightarrow \sum_{i=1}^{t} w_i = \sum_{i=1}^{t} (1 - \beta)\beta^{t-i} = 1 - \beta^t
$$
The bias correction term of Adam can therefore be seen as a correction needed for a proper weighted average. We can also notice that the bias correction term at time step $t$, $c_t$, corresponds to the EMA of 1:

$$c_t = \sum_{i=1}^{t} (1 - \beta) \beta^{t-i} = 1 - \beta^t = \text{EMA} (x_i = 1)$$

such that $c_t = \beta c_{t-1} + (1 - \beta)$, with $c_0 = 0$

c_0 = 0 corresponds to the fact that EMA is initialized at 0. If instead $m_0 \neq 0$, then it should be $c_0 = 1$.

In AdaTerm, the EMA is given by:

$$m_t = (1 - \tau_{mv,t}) m_{t-1} + \tau_{mv,t} g_t = \sum_{i=1}^{t} \tau_{mv,i} \prod_{j=1}^{t-i} (1 - \tau_{mv,j}) g_i$$

$$\implies w_t = \tau_{mv,t} \prod_{j=1}^{t-i} (1 - \tau_{mv,j})$$

(52)

Since $(1 - \beta)$ linearly enters in the expression of $\tau_{mv}$, the bias correction of Adam can also be used in AdaTerm as given in eq. 29. However, following the weighted average interpretation, a different correction term can be obtained as:

$$c_t = \sum_{i=1}^{t} \tau_{mv,i} \prod_{j=1}^{t-i} (1 - \tau_{mv,j})$$

$$\implies c_t = \tau_{mv} c_{t-1} + (1 - \tau_{mv})$$

(53)

This correction term is adaptive and in the Gaussian limit, it will be the same as the Adam/Adabelief one in eq. (50). Using this adaptive correction term, a slightly modified AdaTerm update rule is used:

$$\eta^\text{AdaTerm,AdaBias} (g_t) = \frac{m_t c_t^{-1}}{\sqrt{v_t c_t^{-1}}}$$

(54)

G.3. Comparison

As can be seen in the Table 5, the adaptive bias correction is a complete downgrade compared to the vanilla bias correction. However, the uncentered version of AdaTerm sometimes outperform the main algorithm and proves to have a certain merit that deserves to be further investigated, as mentioned in the conclusion section above.

H. Alternative $v$ update rule

The gradient ascent update rule for the scale parameter $v$ is given by:

$$v_t = v_{t-1} + \kappa_v g_v = v_{t-1} + \kappa_v \frac{w_{mv} \tilde{v}}{2 \nu_{t-1}^2 (\tilde{v} + 1)} [(s_t + \Delta s) - v_{t-1}] = v_{t-1} + \kappa_v \frac{w_{mv} \tilde{v}}{2 \nu_{t-1}^2 (\tilde{v} + 1)} [(s_t + (s_t - D_t v_{t-1}) \nu^{-1}) - v_{t-1}]$$

$$= v_{t-1} + \kappa_v \frac{w_{mv} \tilde{v}}{2 \nu_{t-1}^2 (\tilde{v} + 1)} [(\tilde{v} + 1) v_{t-1} - (\tilde{v} + D_t) v_{t-1}] = v_{t-1} + \frac{\kappa_v}{2 \nu_{t-1}^2} [w_{mv} s_t - v_{t-1}] = v_{t-1} + \frac{\kappa_v}{2 \nu_{t-1}^2} [s_t + \epsilon_t - v_{t-1}]$$

(55)

$$= v_{t-1} + \tau_v [s_t + \epsilon_t - v_{t-1}]$$

(56)

$$= v_{t-1} + \tau_v [s_t + \epsilon_t - v_{t-1}]$$

(57)

where $\kappa_v$ is the update step size and $\epsilon_t = (w_{mv} - 1) s_t$. In order to preserve a limiting Gaussian model, we simply require that for $\tilde{v} \to \infty$, $\tau_v \to (1 - \beta)$, since we already have $\epsilon_t \to 0$. Therefore, we have:

$$\tau_v = \frac{\kappa_v}{2 \nu_{t-1}} \implies 0 < \kappa_v < 2 \nu_{t-1} \implies \kappa_v = 2 k \nu_{t-1}^2, \ 0 < k < 1 \implies \tau_v = k$$

(58)
Table 5. Results for AdaTerm’s variants

| Method                        | Classification | Prediction | RL | Distillation |
|-------------------------------|----------------|------------|----|--------------|
|                              | Accuracy       | MSE at final prediction | The sum of rewards | The sum of rewards |
|                              | 0 %            | 10 %       | 1 step | 30 steps | Hopper | Ant | w/o amateur | w/ amateur |
| AdaTerm                       | 0.7315         | 0.6815     | 0.0335 | 1.0016    | 1550.25 | 2021.37    | 1770.17 | 1411.02 |
| (Ours)                        | (3.66e-3)      | (4.46e-3)  | (3.09e-4) | (2.31e-1) | (5.88e+2) | (3.87e+2) | (2.17e+2) | (1.92e+2) |
| AdaTerm                       | 0.7309         | 0.6815     | 0.0338 | 0.9085    | 1688.59 | 2113.81    | 1649.53 | 1372.08 |
| (Uncentered)                  | (3.66e-3)      | (4.46e-3)  | (3.09e-4) | (2.31e-1) | (5.88e+2) | (3.87e+2) | (2.17e+2) | (1.92e+2) |
| AdaTerm                       | 0.7294         | 0.6810     | 0.0336 | 1.0342    | 1595.10 | 2017.69    | 1672.71 | 1400.88 |
| (AdaBias)                     | (4.26e-3)      | (3.13e-3)  | (4.65e-4) | (2.53e-1) | (6.45e+2) | (3.21e+2) | (2.50e+2) | (2.45e+2) |
| AdaTerm                       | 0.7299         | 0.6800     | 0.0336 | 1.0342    | 1649.53 | 2113.81    | 1624.56 | 1302.55 |
| (Uncentered + AdaBias)        | (3.70e-3)      | (3.76e-3)  | (4.31e-4) | (6.43e-1) | (6.72e+2) | (3.74e+2) | (2.21e+2) | (2.47e+2) |
| t-AdaBelief                   | 0.7231         | 0.6827     | 0.0318 | 1.1517    | 1104.87 | 2185.45    | 1639.93 | 1303.19 |
| (4.23e-3)                     | (4.02e-3)      | (4.42e-4)  | (1.17e-1) | (4.83e+2) | (3.77e+2) | (2.72e+2) | (2.72e+2) |
| At-AdaBelief                  | 0.7223         | 0.6815     | 0.0318 | 1.1306    | 1214.47 | 2196.84    | 1671.65 | 1325.70 |
| (4.50e-3)                     | (3.46e-3)      | (5.02e-4)  | (1.17e-1) | (4.23e+2) | (3.77e+2) | (2.72e+2) | (2.72e+2) |

Since $s_t + \epsilon_t = w_{mv}s_t$, it is no longer necessary to have $k$ as a function of $w_{mv}$. In particular, we can directly set $k = (1 - \beta)$. However, in practice, we found that this can lead to instabilities and NaN values. Therefore, we instead use $k = (1 - \beta)(w_{mv})^{-1}$, which results in the following update rule:

\[
\begin{align*}
\kappa_v &= 2v_{t-1}^2(1 - \beta), \quad \tau_v = (1 - \beta)(w_{mv})^{-1} \\
\nu_t &= v_{t-1} + \kappa_v g_v = v_{t-1} + \frac{1 - \beta}{w_{mv}} [(s_t + \epsilon_t) - v_{t-1}] = (1 - \tau_v)v_{t-1} + \tau_v(w_{mv}s_t) \\
v_t &= \left(1 - \frac{1 - \beta}{w_{mv}}\right)v_{t-1} + \frac{1 - \beta}{w_{mv}}(w_{mv}s_t + \epsilon^2) 
\end{align*}
\]

(59)

(60)

where $\epsilon^2$ is added to avoid the zero value problem. Since $w_{mv}$ and $s_t$ are both positive, the gradient ascent rule will never violate the constraint.

This version of AdaTerm is called AdaTerm2 in the results below, where we can see that on the simple regression task, it performs similarly to our main algorithm, but performs less efficiently on the prediction task. Notably, we can see signs of its instability in the 30 steps prediction task, around epoch 100.
AdaTerm towards Noise-Robust Stochastic Gradient Optimizer

(a) Training loss (1 step)
(b) Eval. prediction error (1 step)
(c) Training loss (30 steps)
(d) Eval. prediction error (30 steps)