THE PRINCIPLE OF OPEN INDUCTION ON \([0,1]\) AND THE APPROXIMATE-FAN THEOREM

WIM VELDMAN

Est enim per contrapositionem conversio ut si dicas
omnis homo animal est omne non animal non homo est.

Boethius, de Syll. Cat.

Abstract. In the earlier papers [30], [31] and [32] we collected statements that are, in a weak formal context, equivalent to Brouwer’s Fan Theorem. This time, we do the same for the Principle of Open Induction on \([0,1]\) and the Approximate-Fan Theorem. These principles follow from Brouwer’s Thesis on Bars and imply the Fan Theorem.

1. Introduction

1.1. L.E.J. Brouwer wanted to do mathematics differently. He restored the logical constants to their natural constructive meaning and introduced some new principles or axioms. His famous Continuity Theorem claims that every function from the unit interval \([0,1]\) to the set \(\mathcal{R}\) of the real numbers is uniformly continuous. This theorem states two things: a function from \([0,1]\) to \(\mathcal{R}\) is continuous everywhere, and a function from \([0,1]\) to \(\mathcal{R}\) that is continuous everywhere is continuous uniformly on \([0,1]\). The principle underlying the first statement now is called Brouwer’s Continuity Principle and the principle underlying the second statement is Brouwer’s Fan Theorem.

The Fan Theorem asserts that every thin bar in Cantor space \(2^{\omega}\) is finite. The theorem extends to every subset of Baire space \(\omega^\omega\), that, like \(2^{\omega}\), is a fan. Brouwer derived the Fan Theorem from his Thesis on Bars in \(\omega^\omega\) although a Thesis on Bars in \(2^{\omega}\) would have been sufficient for the purpose at hand. Brouwer’s Thesis on Bars in \(\omega^\omega\) is a much stronger statement than the Fan Theorem.

The Principle of Open Induction on \(2^{\omega}\) asserts that every open subset of \(\omega^\omega\) that is progressive in \(2^{\omega}\) under the lexicographical ordering contains \(2^{\omega}\). T. Coquand, (see [7] and [26], Section 11), saw that this principle follows from Brouwer’s Thesis on Bars in \(\omega^\omega\). The principle is a contraposition of the (classical) fact that a non-empty closed subset of \(2^{\omega}\) must have a smallest element under the lexicographical ordering.

The Approximate-Fan Theorem asserts that every thin bar in an approximate fan is almost-finite. Every fan is an approximate fan, but not conversely, and every finite subset of the set \(\omega\) of the natural numbers is almost-finite, but not conversely.

We shall show that, in a weak formal context, the Principle of Open Induction on \(2^{\omega}\) is a consequence of the Approximate-Fan Theorem and that the Approximate-Fan Theorem follows from the Thesis on Bars in \(\omega^\omega\). The Principle of Open Induction on \(2^{\omega}\) implies the Fan Theorem but not conversely. Both the Principle of Open Induction on \(2^{\omega}\) and the Approximate-Fan Theorem have many equivalents.

The important work done in classical reverse mathematics and beautifully described in [17] has been a source of inspiration. Our results belong to intuitionistic mathematics.

\[1^{\text{See [33].}}\]
reverse mathematics and are an intuitionistic counterpart to the Sections III.1, III.2, III.7 and part of Section V.1 from [17].

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1.2. The contents of the paper. Apart from this introductory Section 1, the paper contains Sections 2-13.

In Section 2, we introduce BIM, a formal system for Basic Intuitionistic Mathematics also used in Veldman [30], [31] and [32].

In Section 3, we introduce Sections 2-13. We prove:

In Section 4, we introduce BI, Brouwer’s Principle of Induction on Bars in \( \omega^\omega \) and OI([0,1]), the Principle of Open Induction on \([0,1] \).

We prove that BIM ⊢ BI → OI([0,1]) and that BIM ⊢ OI([0,1]) → HB.

In Section 5, we introduce Ded, Dedekind’s Theorem which says that every infinite bounded and nondecreasing sequence of real numbers is convergent. We show that this statement is constructively false and introduce Ded, a contraposition of Ded. We introduce Ded^−, a statement similar to Ded, intended for \( \omega^\omega \) rather than for \( \mathbb{R} \). We also introduce OI(2^ω), the Principle of Open Induction on Cantor space 2^ω. We prove that BIM ⊢ OI([0,1]) ↔ Ded ↔ Ded^− ↔ OI(2^ω).

In Section 6 we introduce EnDec?!, a positive statement with the negative consequence that no enumerable subset of \( \omega \) positively fails to be a decidable subset of \( \omega \). We prove that BIM ⊢ OI([0,1]) ↔ EnDec?!.

In Section 7, we introduce the statement WFKB: for every decidable subset B of \( \omega \) that is a bar in \( \omega^\omega \), the set of all (codes of) finite sequences below the bar B is well-founded under the Kleene-Brouwer-ordering. We prove: BIM ⊢ OI([0,1]) ↔ WFKB. As a corollary we obtain the result that, in BIM, OI([0,1]) proves the principle of transfinite induction on the ordinal number \( \varepsilon_0 \), and we conclude, using a result of Troelstra’s, that BIM + FT ⊬ OI([0,1]).

In Section 8, we introduce approximate fans and the Approximate-Fan Theorem AppFT. We prove that BIM ⊢ BI → AppFT.

In Section 9, we introduce BW, the Bolzano-Weierstrass Theorem which says that every infinite and bounded sequence of real numbers has a convergent subsequence. As this statement is constructively false, we introduce BW, a contraposition of BW, and also BW_ω and (BW_ω)\(^+\), two similar statements for \( \omega^\omega \) rather than for \( \mathbb{R} \). We prove that BIM ⊢ AppFT ↔ BW ↔ BW_ω ↔ (BW_ω)\(^+\).

We observe that BIM ⊢ BW → Ded.

In Section 10, we introduce Asc, Ascoli’s Lemma, the constructively wrong statement saying that every infinite sequence of uniformly continuous functions from \([0,1] \) to \([0,1] \), all obeying the same modulus of uniform continuity, has a convergent subsequence. We also introduce Asc, a contraposition of Asc. We prove: BIM ⊢ AppFT ↔ Asc.

In Section 11, we introduce IRT = ∀k > 0[IRT(k)], the Intuitionistic Ramsey Theorem. We prove that BIM ⊢ AppFT ↔ IRT ↔ IRT(3). We also prove that BIM ⊢ (OI([0,1]) + IRT(2)) → AppFT. We introduce PH, the Paris-Harrington Theorem. We prove that BIM ⊢ IRT → PH, and thus find a second argument proving that BIM + FT ⊬ AppFT.

In Section 12, we introduce MP_1, Markov’s Principle: ∀a[¬¬∃n[α(n) = 0] → ∃n[α(n) = 0]]. MP_1 is intuitionistically not acceptable. We prove that BIM + MP_1 ⊢ OI([0,1]) → AppFT. We introduce \( \Sigma^1_0 \)-BI, the restriction of BI to enumerable subsets of \( \omega \). We introduce \( \Sigma^1_0 \)-ND, the principle that every enumerable subset
A of \( \omega \) is nearly-decidable, i.e. \( \neg \exists \beta \forall m (m \in A \leftrightarrow \beta(m) \neq 0) \). We prove that \( \text{BIM} + \text{MP}_1 + \text{OI}(0,1) \leftrightarrow \Sigma^0_1-\text{BI} \leftrightarrow \Sigma^0_1-\text{ND} \). Section 12 builds upon earlier work by R. Solovay and J.R. Moschovakis.

In Section 13, we have collected some notations and conventions and some basic facts. The reader may consult this Section whenever he feels the need to do so.

2. Basic Intuitionistic Mathematics \( \text{BIM} \)

2.1. The axioms of \( \text{BIM} \). The formal system \( \text{BIM} \) (for Basic Intuitionistic Mathematics) that we now introduce has also been used in Veldman \[31\] and \[32\].

There are two kinds of variables, numerical variables \( m, n, p, \ldots \), whose intended range is the set \( \omega \) of the natural numbers, and function variables \( \alpha, \beta, \gamma, \ldots \), whose intended range is the set \( \omega^\omega \) of all infinite sequences of natural numbers, that is, the set of all functions from \( \omega \) to \( \omega \). There is a numerical constant \( 0 \). There are unary function constants \( \text{Id} \), a name for the identity function, \( \underline{0} \), a name for the zero function, and \( S \), a name for the successor function, and \( K, L \), names for the projection functions from \( \omega \times \omega \) to \( \omega \). There is one binary function symbol \( J \), a name for the (surjective) pairing function from \( \omega \times \omega \) to \( \omega \). From these symbols numerical terms are formed in the usual way. The basic terms are the numerical variables and the numerical constant and more generally, a term is obtained from earlier terms by the use of a function symbol. The function constants \( \underline{0}, S, K \) and \( L \) and the function variables are, in the beginning stage of the development of \( \text{BIM} \), the only function terms. As the theory develops, names for operations on infinite sequences are introduced and more complicated function terms appear.

There are two equality symbols, \( =_0 \) and \( =_1 \). The first symbol may be placed between numerical terms only and the second one between function terms only. When confusion seems improbable we simply write \( = \) and not \( =_0 \) or \( =_1 \). A basic formula is an equality between numerical terms or an equality between function terms. A basic formula in the strict sense is an equality between numerical terms. We obtain the formulas of the theory from the basic formulas by using the connectives, the numerical quantifiers and the function quantifiers.

Theorems are obtained from the axioms by the rules of intuitionistic logic.

The first axiom is the Axiom of Extensionality.

Axiom 1. \( \forall \alpha \forall \beta ( \alpha =_1 \beta \leftrightarrow \forall n (\alpha(n) =_0 \beta(n))) \).

Axiom 1 guarantees that every formula will be provably equivalent to a formula built up by means of connectives and quantifiers from basic formulas in the strict sense.

The second axiom is the axiom on the function constants \( \text{Id}, \underline{0}, S, J, K \) and \( L \).

Axiom 2. \( \forall n (\text{Id}(n) = n) \land \forall n (\underline{0}(n) = 0) \land \forall n (S(n) \neq 0) \land \forall n \forall m (S(m) = S(n) \rightarrow m = n) \land \forall m \forall n (K(J(m, n)) = m \land L(J(m, n)) = n \land n = J(K(n), L(n))) \).

Thanks to the presence of the pairing function we may treat binary, ternary and other non-unary operations on \( \omega \) as unary functions. \( \alpha(m, n, p) \) for instance will be an abbreviation of \( \alpha(J(m, n), p)) \).

We introduce the following notation: for each \( n, n' := K(n) \) and \( n'' := L(n) \), and for all \( m, n, (m, n) := J(m, n) \). The last part of Axiom 2 now reads as follows: \( \forall m \forall n (m, n') = m \land (m, n'') = n \land n = (n', n'') \). Given any \( \alpha \), we let \( \alpha' \) and \( \alpha'' \) be the elements of \( \omega^\omega \) defined by: \( \forall n (\alpha'(n) = (\alpha(n))') \land \alpha''(n) = (\alpha(n))'' \).

The third axiom asks for the closure of the set \( \omega^\omega \) under the operations composition, primitive recursion and unbounded search.\footnote{This axiom underwent an improvement with respect to its version in \[31\].}
Axiom 3. \( \forall \alpha \forall \beta \exists \gamma \forall n [\gamma (n) = \alpha (\beta (n))] \land \forall \alpha \forall \beta \forall \gamma \exists \delta \forall n [\delta (n) = \gamma (\alpha (n), \beta (n))] \land \forall \alpha \forall \beta \exists \gamma [\gamma (0) = p \land \forall n [\gamma (S(n)) = \beta (n, \gamma (n))]] \land \forall \alpha \forall \beta \forall \gamma \forall m \forall n [\gamma (m, n) = \alpha (m) \land \gamma (m, S(n)) = \beta (m, n, \gamma (m, n)) \land \forall [\forall n \exists m [\alpha (m, n) = 0] \rightarrow \exists \gamma \forall m [\alpha (m, \gamma (m)) = 0 \land \forall n < \gamma (m) [\alpha (m, n) \neq 0]]]. \)

We introduce \( \circ \), composition, as a binary operation on functions, with defining axiom: \( \forall \alpha \forall \beta [\alpha \circ \beta (n) = \alpha (\beta (n))]. \)

The fourth axiom is the Unrestricted Axiom Scheme of Induction:

Axiom 4. For every formula \( \phi = \phi (n) \) the universal closure of the following formula is an axiom:

\[ (\phi (0) \land \forall n [\phi (n) \rightarrow \phi (S(n))]) \rightarrow \forall n [\phi (n)] \]

The system consisting of the axioms mentioned up to now will be called BIM.

2.2. Extending BIM. We mention some axioms and assumptions that may be studied in the context of BIM.

2.2.1. Brouwer’s Unrestricted Continuity Principle, BCP:

For every subset \( A \) of \( \omega^\omega \times \omega \), if \( \forall \alpha \exists \exists \forall \exists \beta [\exists m = \exists m \rightarrow \beta (n)] \).

(“\( \alpha An \)” abbreviates “\( \alpha, n \in A \).”)

Note that BCP is an axiom scheme and not a single axiom. The revolutionary impact of BCP on the development of intuitionistic mathematics is treated in [23].

2.2.2. Church’s Thesis, CT:

\[ \exists \exists \exists \forall \exists \forall \exists \exists [r(e, n, z) \neq 0 \land \forall i < z [r(e, n, i) = 0] \land \psi (z) = \alpha (n)]. \]

CT contradicts BCP, see, for instance, [19] Proposition 6.7.

2.2.3. The Unrestricted First Axiom of Countable Choice, AC\(_{0,0}\):

For every subset \( A \) of \( \omega \): \( \forall n \exists m [n \in A \rightarrow \exists \gamma \forall n [\alpha (m, n) = 0]] \).

(“\( \alpha Am \)” abbreviates “\( \alpha, m \in A \).”)

Also AC\(_{0,0}\) is an axiom scheme.

The formula \( \forall [\forall m \exists n [\alpha (m, n) = 0] \rightarrow \exists \gamma \forall n [\alpha (m, \gamma (m)) = 0]] \) is called the Minimal Axiom of Countable Choice \( \text{Min-AC}_{0,0} \). Min-AC\(_{0,0}\) follows from Axiom 3.

Another special case is the \( \Pi^0_1 \)-First Axiom of Countable Choice, \( \text{AC}^0_1 \):

\[ \forall [\forall m \exists n [n \notin E_{\alpha}] \rightarrow \exists \gamma \forall n [\gamma (n) \notin E_{\alpha}]]. \]

i.e.

\[ \forall [\forall m \exists n [\alpha (m, n) \neq n + 1] \rightarrow \exists \gamma \forall n [\alpha (m, n) \neq n + 1] \].

A more cautious, almost innocent, version is the Weak \( \Pi^0_1 \)-First Axiom of Countable Choice, Weak-\( \Pi^0_1 \)-AC\(_{0,0}\):

\[ \forall [\forall m \exists n [n \geq m [\alpha (m, p) = 1] \rightarrow \exists \gamma \forall n [\gamma (n) \geq n [\alpha (m, p) = 1]]]. \]

This axiom may be used for proving that every fan is an explicit fan.

Lemma 2.1. BIM + Weak-\( \Pi^0_1 \)-AC\(_{0,0}\) ⊩ \( \forall \beta [\text{Fan} (\beta) \rightarrow \text{Fan}^+ (\beta)] \).

Proof. Let \( \beta \) be given such that \( \text{Fan} (\beta) \). Note: \( \forall s \exists n \forall p \geq n [\beta (s \cdot \langle p \rangle) = 0] \). Find \( \gamma \) such that \( \forall s \forall p \geq \gamma (s) [\beta (s \cdot \langle p \rangle) = 0] \) and note that \( \gamma \) shows that \( \beta \) is an explicit fan-law.

\footnote{For each \( \alpha, E_{\alpha} := \{ m \mid \exists p [\alpha (p) = m + 1]\} \), see Subsection 13.3.}

\footnote{For the notions ‘fan’ and explicit fan’, see Subsection 13.7.}
2.2.4. The Limited Principle of Omniscience, LPO:
\[\forall \alpha [\exists n [\alpha(n) \neq 0] \lor \forall n [\alpha(n) = 0]].\]

From a constructive point view, LPO makes no sense. The following is a well-known counterexample in Brouwer’s style. Let \(d : \omega \to \omega\) be the decimal expansion of the real number \(\pi\), i.e., \(\pi = 3 + \sum_{n=0}^{\infty} d(n) \cdot 10^{-n}\). Define \(\alpha\) such that, for all \(n\), \(\alpha(n) \neq 0\) if and only if \(\forall k < 99 \left[d(n + k) = 9\right]\). We then have no proof of \(\exists n [\alpha(n) \neq 0]\) and also no proof of \(\forall n [\alpha(n) = 0]\). This example shows that LPO is constructively unjustified.

In BIM + BCP one may prove \(\neg\text{LPO}\), i.e. LPO is even contradictory, see, for instance, [32]. If we find that some given statement implies LPO, we may conclude that this statement is unjustified, and, in the presence of BCP, even contradictory.

3. The Fan Theorem and the Heine-Borel Theorem

3.1. Brouwer’s argument for the Fan Theorem. Brouwer claims, in [4] and [5], implicitly, as he jumps at once to a larger claim, see Subsection 2.2.1, that, for every subset \(B\) of \(2^{<\omega}\), if \(Bar_{2^s}(B)\), i.e. \(\forall \alpha \in 2^\omega [\exists n [\alpha(n) \in B]]\), then there exists a canonical proof of this fact. The canonical proof is an arrangement of statements “\(Bar_{2^s}(\alpha)\)”, i.e. \(\forall \alpha \in 2^\omega [s \subseteq \alpha \to \exists n [\alpha(n) \in B]]\), where \(s \in 2^{<\omega}\). The conclusion of the proof is: \(Bar_{2^s}(\alpha)(B)\), and the proof uses only steps of one of the following kinds:

(i) ‘starting points:’
\(s \in B\), and, therefore, \(Bar_{2^s}(\alpha)(B)\),

(ii) ‘forward steps:’
\(Bar_{2^s}(\alpha)(0)(B)\) and \(Bar_{2^s}(\alpha)(1)(B)\), and, therefore, \(Bar_{2^s}(\alpha)(B)\),

(iii) ‘backward steps:’
\(Bar_{2^s}(\alpha)(B)\), and, therefore, \(Bar_{2^s}(\alpha)(0)(B)\), and:
\(Bar_{2^s}(\alpha)(B)\), and, therefore, \(Bar_{2^s}(\alpha)(1)(B)\).

One may prove that one can do without the backward steps.

**Definition 1.** Let \(B\) be a subset of \(2^{<\omega}\).

\(B\) is inductive in \(2^{<\omega}\) if and only if \(\forall s \in 2^{<\omega} [\forall i < 2 [s \ast (i) \in B ] \to s \in B ]\).

\(B\) is monotone in \(2^{<\omega}\) if and only if \(\forall s \in 2^{<\omega} [\forall i < 2 [s \ast (i) \in B ] \to s \in B ]\).

**Definition 2.** \(\text{BI}_{2^s}\), the Principle of Bar Induction in Cantor space is the following statement:

For all \(B \subseteq \omega\), if \(Bar_{2^s}(B)\) and \(B\) is monotone and inductive in \(2^{<\omega}\), then \(\langle \rangle \in B\).

\(\text{BI}_{2^s}\) may be added to BIM as an axiom scheme. We introduce two special cases of this axiom scheme:

\(\Delta_1^0 - \text{BI}_{2^s}\): \(\forall \alpha [Bar_{2^s}(D_\alpha) \land \forall s \in 2^{<\omega} [\forall i < 2 [s \ast (i) \in D_\alpha ] ] \to \langle \rangle \in D_\alpha ]\),

and

\(\Sigma_1^0 - \text{BI}_{2^s}\): \(\forall \alpha [Bar_{2^s}(E_\alpha) \land \forall s \in 2^{<\omega} [\forall i < 2 [s \ast (i) \in E_\alpha ] ] \to \langle \rangle \in E_\alpha ]\).

We also introduce the Fan Theorem (for \(2^\omega\)):

\(\text{FT} = \text{FT}_{2^\omega} : \forall \alpha [Bar_{2^\omega}(D_\alpha) \to \exists n [Bar_{2^\omega}(D_\alpha)n]]\).

Let \(B\) be a subset of \(2^{<\omega}\) such that \(Bar_{2^s}(B)\) and \(B\) is both inductive and monotone in \(2^{<\omega}\). Take the canonical proof of “\(Bar_{2^s}(B)\)” and replace in this proof every statement “\(Bar_{2^s}(\alpha)\)” by the statement: “\(s \in B\)”. We then obtain a proof of: “\(\langle \rangle \in B\)”.

This argument justifies \(\text{BI}_{2^s}\).

**Theorem 3.1.** (i) BIM \(\vdash \Delta_1^0 - \text{BI}_{2^s}\).

(ii) BIM \(\vdash \Sigma_1^0 - \text{BI}_{2^s} \iff \text{FT}_{2^\omega}\).
Proof. (i) Let \( \alpha \) be given such that \( \text{Bar}_{2^\omega}(D_\alpha) \) and \( \forall s \in 2^{<\omega}[s \in D_\alpha \iff \forall i < 2[s \ast (i) \in D_\alpha] \). Assume \( \langle \rangle \notin D_\alpha \). Define \( \gamma \) such that \( \forall n[\gamma(n) = \mu < 2[\bar{n} \ast (i) \notin D_\alpha] \).\) Note: \( \gamma \in 2^{<\omega} \) and \( \forall n[\gamma(n) \in D_\alpha] \). \( \text{Contradiction. Conclude that } \langle \rangle \in D_\alpha \).

(ii). First, assume \( \Sigma^0_1\text{-BI}_{2^\omega} \). Let \( \alpha \) be given such that \( \text{Bar}_{2^\omega}(D_\alpha) \). Define \( \beta \) such that, for each \( n \), if \( n' \in 2^{<\omega} \) and \( \text{length}(n') \leq n'' \) and \( \forall t \in 2^{<\omega} \cap \omega^m \[n' \subseteq t \rightarrow \exists s \subseteq t[s \in D_\alpha] \), then \( \beta(n) = n' + 1 \) and, if \( n \) not, then \( \beta(n) = 0 \). Note that \( E_\beta = \{ u \in 2^{<\omega} \mid \exists m \geq \text{length}(u) \forall t \in 2^{<\omega} \cap \omega^m [u \subseteq t \rightarrow \exists s \subseteq t[s \in D_\alpha] \} \). Note that \( D_\alpha \subseteq E_\beta \) and \( E_\beta \) is both monotone and inductive in \( 2^{<\omega} \). Conclude that \( \langle \rangle \in E_\beta \), and find \( n \) such that \( \beta(n) = \langle \rangle + 1 \). Note that \( n' = 0 \) and \( \text{length}(n') = 0 \) and \( \forall t \in 2^{<\omega} \cap \omega^m \exists s \subseteq t[s \in D_\alpha] \), and, therefore, \( \exists m[\text{Bar}_{2^\omega}(D_{\bar{m}n})] \). Conclude that \( \text{FT} \) holds.

Now assume \( \text{FT} \). Let \( \alpha \) be given such that \( \text{Bar}_{2^\omega}(E_\alpha) \) and \( \forall s \in 2^{<\omega}[s \in E_\alpha \leftrightarrow \forall i < 2[s \ast (i) \in E_\alpha] \). Define \( \beta \) such that, for each \( s \), \( \beta(s) \neq 0 \) if and only if \( s \in 2^{<\omega} \) and \( \exists p < s \bar{m} \leq s[\alpha(p) = \bar{m}n + 1] \). Note that \( \text{Bar}_{2^\omega}(\alpha) \). Using \( \text{FT} \), find \( p \) such that \( \text{Bar}_{2^\omega}(\alpha(\bar{p}) \). Find \( m : = \max\{\text{length}(s) \mid s \in D_{\alpha(\bar{p})} \} \). Note that \( \forall s \in 2^{<\omega} \cap \omega^m \exists q \leq m[\bar{n} \in E_\alpha] \). We now prove that \( \forall k \leq m \forall s \in 2^{<\omega} \cap \omega^k \exists q \leq k[\bar{n} \in E_\alpha] \) and do so by backwards induction, starting from the case \( k = m \). Suppose \( k + 1 \leq m \) and \( \forall s \in 2^{<\omega} \cap \omega^{k+1} \exists q \leq k + 1[\bar{n} \in E_\alpha] \). Assume that \( s \in 2^{<\omega} \cap \omega^k \). As \( \forall i < 2^{m \bar{p}} \subseteq s \ast (i) \in E_\alpha] \), one may distinguish two cases. Either \( \exists q \leq k[\bar{n} \in E_\alpha] \), or \( \forall i < 2[s \ast (i) \in E_\alpha] \), and, therefore, \( s \in E_\alpha \). In both cases \( \exists q \leq k[\bar{n} \in E_\alpha] \). We thus see that \( \langle \rangle \in E_\alpha \). Conclude that \( \Sigma^0_1\text{-BI}_{2^\omega} \) holds.

There are many equivalents of \( \text{FT} \), see [30], [31] and [32].

3.2. Extending the Fan Theorem to arbitrary fans. \( 2^\omega \) is a fan, but it is not the only one. The notions spread, fan and explicit fan are introduced in Subsection 2.2.3. In Subsection 2.2.3 we saw that, in \( \text{BIM + \text{Weak-IT}^1_\omega \text{-AC}_{0,0}} \), one may prove \( \forall \beta[\text{Fan}(\beta) \rightarrow \text{Fan}^+(\beta)] \), i.e. every fan is an explicit fan.

Definition 3. The following two statements are versions of the Extended Fan Theorem,
\[
\text{FT}_{\text{ext}} : \forall \beta \forall \alpha([\text{Fan}^+(\beta) \land \text{Bar}_{\alpha}(D_\alpha)) \rightarrow \exists n[\text{Bar}_{\alpha}(D_{\bar{n}m})]], \quad \text{and} \\
\text{FT}^*_{\text{ext}} : \forall \beta \forall \alpha([\text{Fan}(\beta) \land \text{Bar}_{\alpha}(D_\alpha)) \rightarrow \exists n[\text{Bar}_{\alpha}(D_{\bar{n}m})]].
\]

\( \text{FT}_{\text{ext}} \) and \( \text{FT}^*_{\text{ext}} \) extend \( \text{FT} \) from Cantor space \( C \) tot arbitrary (explicit) fans.

Theorem 3.2. \( \text{BIM + \text{FT} \rightarrow \text{FT}^*_{\text{ext}}} \).

Proof. The proof may be found in [18] and [31], but we find it useful to give it here.

Assume \( \text{FT} \). Let \( \beta, \alpha \) be given such that \( \text{Fan}^+(\beta) \) and \( \text{Bar}_{\alpha}(D_\alpha) \). One may assume that \( \beta(\langle \rangle) = 0 \). Find \( \gamma \) such that, for each \( s \), if \( \beta(s) = 0 \), then \( \gamma(s) \) is the greatest \( n \) such that \( \beta(s \ast (n)) = 0 \).

For each \( s \), we define \( \delta(s) \) in \( 2^{<\omega} \), as follows, by induction on \( \text{length}(s) \). \( \delta(\langle \rangle) = \langle \rangle \) and, for all \( s \), \( \delta(s \ast (n)) = \delta(s) \ast \bar{n} \ast (1) \).

We then define \( \eta \) such that, for each \( t, \eta(t) \neq 0 \) if and only if there exist \( s, i \) such that \( t = \delta(s) \ast \bar{i} \) and either: \( \beta(s) \neq 0 \) or: \( \beta(s) = 0 \) and \( i > \gamma(s) \) or: \( \alpha(s) \neq 0 \).

We now prove that \( \text{Bar}_{2^\omega}(D_\eta) \). Assume \( \varepsilon \in 2^{<\omega} \). Define \( \varepsilon^* \) as follows, by induction. For each \( n \), if there exists \( j \leq \varepsilon(\bar{n} \ast j) \) such that \( \beta(\bar{n} \ast j) = 0 \) and \( \delta(\bar{n} \ast j) \subseteq \varepsilon \), then \( \varepsilon'(n) \) is the least such \( j \), and, if no such \( j \) exists, then \( \varepsilon'(n) = \gamma(\bar{n}) \). Note: \( \varepsilon^* \in \beta \) and \( \exists \bar{m} \in D_\alpha \). Either \( \delta(\bar{n} \ast j) \subseteq \varepsilon \) or there exist \( s, i \) such \( \beta(s) = 0 \) and \( \delta(s) \ast \bar{i} \subseteq \varepsilon \) and \( i > \gamma(s) \) or \( \alpha(s) \neq 0 \). In all three cases, \( \exists m[\bar{m} \in D_\eta] \). Conclude that \( D_\eta \) is a bar in \( 2^{<\omega} \).

Using \( \text{FT} \), find \( m \) such that \( \text{Bar}_{2^\omega}(D_{\bar{m}n}) \). Find \( p \) such that, for each \( s \), if \( s \geq p \), then \( \delta(s) \geq m \). Conclude: \( \text{Bar}_{\alpha}(D_{\bar{m}p}) \).

Conclude \( \text{FT}^*_{\text{ext}} \).
3.3. Extending FT and extending WKL. The following statement, Weak König’s Lemma WKL, is studied in classical reverse mathematics and constructively false:
\[ \forall \alpha \forall n \exists s \in 2^{<\omega} [\text{length}(s) = m \land \forall n \leq m [\alpha(n) = 0] \rightarrow \exists \gamma \in \mathcal{F}_B \forall n [\alpha(n) = 0]]. \]

FT is a contraposition of WKL.

Extending WKL from subtrees of \(2^{<\omega}\) to arbitrary finitely branching trees, one obtains König’s Lemma KL:
\[ \forall \beta \forall \alpha ([\text{Fan}(\beta) \land \forall m \exists s [\text{length}(s) = m \land \beta(s) = 0 \land \forall n \leq m [\alpha(n) = 0]]) \rightarrow \exists \gamma \in \mathcal{F}_B \forall n [\alpha(n) = 0]]. \]

\(\text{FT}^{\text{ext}}\) is a contraposition of KL.

Restricting KL to explicitly finitely branching trees, one obtains Bounded König’s Lemma BKL:
\[ \forall \beta \forall \alpha ([\text{Fan}^+(\beta) \land \forall m \exists s [\text{length}(s) = m \land \beta(s) = 0 \land \forall n \leq m [\alpha(n) = 0]]) \rightarrow \exists \gamma \in \mathcal{F}_B \forall n [\alpha(n) = 0]]. \]

\(\text{FT}^{\text{ext}}\) is a contraposition of BKL.

In the classical formal context of RCA₀, KL is equivalent to ACA₀ and definitely stronger than WKL. WKL and BKL, on the other hand, are equivalent, see [17] Lemma IV.1.4, page 130 and our Theorem [17].

We must conclude that Weak-\(\text{II}^0_1\)-AC₀₀, if introduced in the classical context of RCA₀, would bring us from WKL to KL and thus, in this context, would have serious consequences.

In the intuitionistic context of BIM, however, quantifiers are read constructively, so the assumption from which Weak-\(\text{II}^0_1\)-AC₀₀ draws its conclusion is, if it would be possible to say so, stronger than in the classical context. The intuitionistic step from \(\text{FT}^{\text{ext}}\) to \(\text{FT}^{\text{ext}}\) seems to be a more innocent step than the classical step from BKL to KL.

3.4. Strong bars and thin bars.

Definition 4. For every \(\beta\), for every \(B \subseteq \omega\), \(B\) is a strong bar in \(\mathcal{F}_B\), notation: Strongbar\(\mathcal{F}_B\)(\(B\)), if and only if \(\forall \zeta \in [\omega]^\omega [\forall n [\beta(\zeta(n)) = 0] \rightarrow \exists m [\zeta(n)m \in B]]\).

Note that \(B\) is a strong bar in \(\mathcal{F}_B\) if and only if every increasing sequence of finite sequences admitted by \(\beta\) contains an element that ‘meets’ \(B\).

We now prove that, for every explicit fan \(\mathcal{F}\), for every decidable subset \(B\) of \(\omega\), \(B\) is a strong bar in \(\mathcal{F}\) if and only if \(B\) has a finite subset that is a bar in \(\mathcal{F}\).

Lemma 3.3. BIM \(\vdash \forall \beta [\text{Fan}^+(\beta) \rightarrow \forall \alpha [\text{Strongbar}_{\mathcal{F}_B}(D_\alpha) \leftrightarrow \exists n [\text{Bar}_{\mathcal{F}_B}(D_m)]]]\).

Proof. Let \(\beta\) be given such that \(\text{Fan}^+(\beta)\). Find \(\delta\) such that, for each \(n\), for each \(s\) in \(\omega^n\), if \(\beta(s) = 0\), then \(s < \delta(n)\).

(i) Let \(\alpha\) be given such that Strongbar\(\mathcal{F}_B\)(\(D_\alpha\)). Define \(\zeta\) such that, for every \(n\), \(\beta(\zeta(n)) = 0\) and, if there exists \(s < \delta(n)\) such that \(s \in \omega^n\) and \(\neg \exists m \leq n [\delta(m) \in D_\alpha]\), then \(\zeta(n)\) is the least such \(s\). Find \(\eta\) in \([\omega]^\omega\) such that \(\zeta \circ \eta \in [\omega]^\omega\). Find \(n\) such that \(\exists m [\zeta \circ \eta(n)m \in D_\alpha]\) and conclude that \(\forall s \in \omega^n [\beta(s) = 0 \rightarrow \exists m \leq n [\delta(m) \in D_\alpha]]\) and that Bar\(\mathcal{F}_B\)(\(D_m\)).

(ii) Assume \(\exists n [\text{Bar}_{\mathcal{F}_B}(D_m)]\). Find \(n\) such that Bar\(\mathcal{F}_B\)(\(D_m\)). Find \(p\) such that, for each \(s\) in \(D_m\), \(\text{length}(s) < p\). Let \(\zeta\) in \([\omega]^\omega\) be given such that \(\forall m [\beta(\zeta(m)) = 0]\). Consider \(\zeta(p)\) and note that \(\zeta(p) \geq p\). Conclude that \(\exists q \leq \text{length}(\zeta(p)) \exists q [\zeta(p)q \in D_\alpha]\). Conclude that \(\forall \zeta \in [\omega]^\omega [\forall m [\beta(\zeta(m)) = 0] \rightarrow \exists p \exists q [\zeta(p)q \in D_\alpha]]\), i.e. Strongbar\(\mathcal{F}_B\)(\(D_\alpha\)). \(\square\)
Definition 5. \( B \subseteq \omega \) is thin if and only if \( \forall p \in B \exists q \in B[p \neq q \rightarrow p \perp q] \). For every \( B \subseteq \omega \), for every \( F \subseteq \omega^\omega \), \( B \) is a thin bar in \( F \), notation: \( \text{Thin}^F(B) \), if and only if \( B \) is thin and \( \text{Bar}^F(B) \).

We now prove that \( \text{FT}_\text{ext} \) is, in \( \text{BIM} \), equivalent to the statement that, in an explicit fan, every thin bar is finite:

**Theorem 3.4.** The following statements are equivalent in \( \text{BIM} \):

(i) \( \text{FT}_\text{ext} \) : \( \forall \beta [\text{Fan}^+(\beta) \rightarrow \forall \alpha [\text{Bar}^\omega_\beta(P_m) \rightarrow \exists n [\text{Bar}^\omega_\beta(D_m^n)]]] \).

(ii) \( \forall \beta [\text{Fan}^+(\beta) \rightarrow \forall \alpha [\text{Thin}^\omega_\beta(P_m) \rightarrow \exists n \forall m > n [\beta(m) = 0 \rightarrow \alpha(m) = 0]]] \).

**Proof.**

(i) \( \Rightarrow \) (ii). Let \( \beta, \alpha \) be given such that \( \text{Fan}^+(\beta) \) and \( \text{Bar}^\omega_\beta(P_m) \). Using (ii), find \( n \) such that \( \text{Bar}^\omega_\beta(D_m^n) \). Note that \( D_m^n \) is thin and that \( \forall s [\beta(s) = 0 \rightarrow (s \in D_m^n \leftrightarrow s \in D_m)] \) and \( \forall m > n [\beta(m) = 0 \rightarrow \alpha(m) = 0] \).

(ii) \( \Rightarrow \) (i). Let \( \beta, \alpha \) be given such that \( \text{Fan}^+(\beta) \) and \( \text{Bar}^\omega_\beta(P_m) \). Define \( \gamma \) such that, for each \( s, \gamma(s) \neq 0 \) if and only if \( \alpha(s) \neq 0 \) and \( \forall m < \text{length}(s) [\alpha(\alpha m) \neq 0] \). Note: \( D_m \subseteq D_m^n \) and \( \text{Bar}^\omega_\beta(D_m) \). Using (ii), find \( n \) such that \( \forall m > n [\beta(m) = 0 \rightarrow \gamma(m) = 0] \) and conclude \( \text{Bar}^\omega_\beta(D_m^n) \) and \( \text{Bar}^\omega_\beta(D_m) \).

\( \square \)

3.5. The Heine-Borel Theorem. We will use some notions introduced in Subsections 3.3 and 3.4.

**Definition 6.** For every subset \( \mathcal{X} \) of \( \mathcal{R} \), for every subset \( \mathcal{C} \) of \( \mathcal{S} \), covers \( \mathcal{X} \), notation: \( \text{Cov}^\mathcal{X}_\mathcal{S}((\mathcal{C})) \), if and only if \( \forall o \in \mathcal{X} \exists s \in \mathcal{C}[\delta(o) \subseteq s] \).

For every \( s \in \mathcal{S} \), \( L(s) := (s', s'' \circ s') \) and \( R(s) := (s' \circ s'', s') \). \( L(s) \) is the left half of \( s \) and \( R(s) \) is the right half of \( s \).

\( B \) is a mapping from \( 2^{<\omega} \) to \( \mathcal{S} \) such that \( B(\langle \rangle) = (0_1, 1_1) \), and for each \( c \) in \( 2^{<\omega} \), \( B(c \circ \langle 0 \rangle) = L(B(c)) \) and \( B(c \circ \langle 1 \rangle) = R(B(c)) \).

For each \( k, \bar{k} \) is the element of \( \omega^\omega \) such that \( \forall n [\alpha(n) = k] \).

\( Q_2 \) is the set of (code numbers of) binary rationals \( \frac{m}{2^n} \), where \( m \in \mathbb{Z} \) and \( n \geq 0 \).

The Heine-Borel Theorem is the statement:

\( \text{HB} : \forall \alpha [\text{Cov}^\mathcal{S}_{\text{Bar}^\omega_\beta(D_m)}(D_m) \rightarrow \exists m [\text{Cov}^\mathcal{S}_{\text{Bar}^\omega_\beta(D_m)}(D_m^n)]] \).

**Theorem 3.5.** \( \text{BIM} \vdash \text{FT} \Leftrightarrow \text{HB} \).

**Proof.** Assume \( \text{FT} \). Let \( \alpha \) be given such that \( \text{Cov}^\mathcal{S}_{\text{Bar}^\omega_\beta(D_m)}(D_m) \). Define \( \beta \) such that \( \forall c [\beta(c) \neq 0 \leftrightarrow (c \in 2^{<\omega} \land \exists s \leq c [s \in D_m \land B(c) \subseteq s]] \). Note that \( \text{Bar}^\omega_\beta(D_m) \).

Find \( m \) such that \( \text{Bar}^\omega_\beta(D_m^n) \). Conclude that \( \text{Cov}^\mathcal{S}_{\text{Bar}^\omega_\beta(D_m^n)}(D_m) \).

Conclude that \( \forall \alpha [\text{Cov}^\mathcal{S}_{\text{Bar}^\omega_\beta(D_m^n)}(D_m) \rightarrow \exists m [\text{Cov}^\mathcal{S}_{\text{Bar}^\omega_\beta(D_m^n)}(D_m^n)]] \), i.e. \( \text{HB} \).

Now assume \( \text{HB} \). Let \( \alpha \) be given such that \( \text{Bar}^\omega_\beta(D_m) \). Define \( \beta \) such that, for all \( s, \beta(s) \neq 0 \) if and only if \( s \in \mathcal{S} \) and either

1. \( \exists n < s [\alpha(\alpha n) \neq 0 \land s' < q \land s'' < s' < q' \text{ or } \exists m [\text{Bar}^\omega_\beta(D_m^n)] \),
2. \( \exists m < s [\alpha(\alpha m) \neq 0 \land B'(\alpha m) < s' < q \text{ or } \exists m [\text{Bar}^\omega_\beta(D_m^n)] \),
3. \( \exists c \exists m < s [\exists l [c \in 2^{<\omega} \land \alpha(c \circ \langle 0 \rangle \circ \alpha m) \neq 0 \land (c \circ \langle 1 \rangle \circ \alpha m) \neq 0 \land B'(c \circ \langle 0 \rangle \circ \alpha m) < s' < q \text{ or } B''(c \circ \langle 1 \rangle \circ \alpha m)] \).

We first show that \( \forall q \in Q_2 \exists S_1 \exists S_2 \exists \exists D_m[S' < q < q < D_m \text{ and we do so in three steps}]

First consider \( 0_q \). Find \( n \) such that \( \alpha(\alpha n) \in D_m \). Find \( s \in \mathcal{S} \) such that \( n \leq s \) and \( s < q \land s'' < s' < q' \text{ or } \exists m [\text{Bar}^\omega_\beta(D_m^n)] \).

Then consider \( 1_q \). Find \( m \) such that \( m \leq s \) and \( B'(\alpha m) < s' < q \text{ or } \exists m [\text{Bar}^\omega_\beta(D_m^n)] \).

Finally, assume \( n > 0 \) and \( 0 < m < 2^n \) and \( m \) is odd and consider \( \frac{m}{2^n} \). Find \( c \) in \( 2^{<\omega} \) such that \( \text{length}(c) = n \) and \( c(n - 1) = 1 \) and \( \sum_{i < n} c(i) \frac{1}{2^i} = \frac{m}{2^n} \). Note that
\[ B(c) = \left( \frac{m}{2^n}, \frac{m+1}{2^n} \right). \] Find \( p, r \) such that \( \beta(n-1) \ast (0) \ast \frac{1}{p} \in D_\alpha \) and \( \frac{c}{2^n} \in D_\alpha \).

Find \( s \in S \) such \( \beta(n-1) < s \) and \( p < s \) and \( r < s \) and \( B'(\beta(n-1) \ast (0) \ast \frac{1}{p}) < Q \)

\[ s' < \frac{s}{2^n} < s'' < Q B'((c + 2^n) \cdot \frac{1}{2^n}). \]

Note that \( s \in D_\beta \) and \( s' < \frac{m}{2^n} < s'' \).

Using the above, find \( \zeta \) in \( \omega^\omega \) such that, for each \( q \) in \( Q_2 \), if \( 0_q \leq q \leq 1_q \), then \( \zeta(q) \) is the least \( s \) in \( D_\beta \) such that \( s' < Q q < s'' \).

Let \( \delta \) be an element of \([0,1]\). Let \( \text{QED} \) denote the proposition: \( \exists s \in S \exists m[s \in D_\beta \land \delta(m) \subseteq S \] s\], i.e. \( \text{Cov}_{[0,1]}(D_\beta) \). Applying \( \text{HB} \), we find \( m \) such that \( \text{Cov}_{[0,1]}(D_{\beta m}) \). Assume we find \( c \in 2^{\omega^\omega} \) such that \( c > m \) and \( \neg \exists \zeta \in D_{\beta m} \exists s \in S \) \( s < \zeta \) \( s' < Q \delta(m) < Q s'' \).

Contradiction. Conclude that \( \text{Bar}_{2^\omega}(D_{\beta m}) \).

Conclude that \( \forall \alpha[\text{Bar}_{2^\omega}(D_\alpha) \rightarrow \exists m[\text{Bar}_{2^\omega}(D_{\beta m})] \), i.e. \( \text{FT} \).

\( \Box \)

The equivalence of \( \text{FT} \) and \( \text{HB} \) is also proven in \([12]\) and \([31]\). \( \text{HB} \) is equivalent, in BIM, to the statement: \( \forall \alpha[\text{Cov}_{[0,1]}(E_\alpha) \rightarrow \exists m[\text{Cov}_{[0,1]}(E_{\beta m})] \), see \([31]\) Corollary 9.8.(xi).

4. Bar Induction and Open Induction on \([0,1]\)

4.1. Brouwer's argument for the principle of Bar Induction. Let \( B \) be a subset of \( \omega \) such that \( \text{Bar}_{\omega^\omega}(B) \), i.e. \( \forall \alpha \exists m[\zeta \in B] \). Brouwer claims, see \([3]\) and \([5]\), that there must exist a “canonical” proof of this fact. The canonical proof is an arrangement of statements of the form \( \text{Bar}_{\omega^\omega \cdot m}(B) \), i.e. \( \forall \alpha[s \subseteq \alpha \rightarrow \exists m[\zeta \in B]] \). The conclusion of the proof is: \( \text{Bar}_{\omega^\omega \cdot m}(B) \), and the proof uses only steps of one of the following kinds:

(i) ‘starting points:’
\[ s \in B, \text{ and, therefore, } \text{Bar}_{\omega^\omega \cdot m}(B). \]

(ii) ‘forward steps:’
\[ \text{Bar}_{\omega^\omega \cdot m}(0)(B), \text{Bar}_{\omega^\omega \cdot m}(1)(B), \text{Bar}_{\omega^\omega \cdot m}(2)(B), \ldots, \]
and, therefore, \( \text{Bar}_{\omega^\omega \cdot m}(B) \)

(iii) ‘backward steps:’
\[ \text{Bar}_{\omega^\omega \cdot m}(B), \text{ and, therefore, } \text{Bar}_{\omega^\omega \cdot m}(n)(B). \]

Unlike in the special case considered in Subsection \([31]\) backward steps can not be missed, see, for instance, \([28]\).

Note that the forward steps have \textit{infinitely many premises} and that the canonical proof can not be visualized as a finite tree.

\( ^5 \) quod est demonstrandum, what still has to be proven
Definition 7. $B \subseteq \omega$ is inductive if and only if $\forall s[\forall n[s * (n) \in B] \rightarrow s \in B]$ and monotone if and only if $\forall s[\forall n[s \in B \rightarrow s * ⟨n⟩ \in B]]$.

BI, the Principle of Induction on Bars in $\omega^\omega$, is the following statement:
For all $B \subseteq \omega$, if $Bar_\omega(B)$ and $B$ is monotone and inductive, then $⟨\rangle \in B$.

BI may be added to BIM as an axiom scheme. The Principle of Induction on Enumerable Bars in Baire space is the following restricted statement:

$\Sigma^0_1$-BI : $\forall α[Bar_\omega(E_α) \land \forall s[s \in E_α \leftrightarrow \forall n[s * ⟨n⟩ \in E_α])] \rightarrow 0 \in E_α$.

Let $B \subseteq \omega$ be given such that $Bar_\omega(B)$ and $B$ is inductive and monotone. If we replace, in a canonical proof of “$Bar_\omega(B)$”, every statement of the form “$Bar_\omega(\gamma(s))$” by the statement “$s \in B$”, we obtain a proof of “$⟨\rangle \in B$”. The intuitionistic mathematician therefore accepts BI.

4.2. The Principle of Open Induction on the unit interval $[0,1]$. We will make use of some definitions in Subsections 13.8 and 13.10.

Definition 8. $A \subseteq R$ is progressive in $[0,1]$ if and only if $\forall γ \in [0,1][[0,γ) \subseteq A] \rightarrow γ \in A$.

$OI([0,1])$, the Principle of Open Induction on $[0,1]$ is the following statement:
For every open subset $H$ of $R$, if $H$ is progressive in $[0,1]$, then $[0,1] \subseteq H$.

For each $s$ in $\mathbb{S}$, $\mathfrak{S} := ⟨(s')_R, (s'')_R⟩$, see Subsection 13.3.

Theorem 4.1. $\text{BIM} \vdash \Sigma^0_1$-BI $\rightarrow$ $OI([0,1])$.

Proof. Assume $\Sigma^0_1$-BI. Let $\alpha$ be given such that $\forall γ \in [0,1][[0,γ) \subseteq H_\alpha \rightarrow γ \in H_\alpha]$. Define $δ$ such that $δ(⟨⟩) = (0_Q,1_Q)$, and, for each $s$, for each $n$, if $[0,1]δ(δ(n) + q)$ $δ''(s)) \subseteq H_{\alpha n}$, then $δ(s * (n)) = R(δ(s))$, and, if not, then $δ(s * (n)) = L(δ(s))$.

We prove that for each $s$, $\exists p[0,δ(s)) \subseteq H_{\alpha n}$ and we do so by induction on $length(s)$. First note: $\delta''(⟨⟩) = (0_Q,0_Q)$ and $[0_{R},0_{R}) = 0 \in H_{\alpha n}$. Now let $s, p$ be given such that $[0,δ'(s)) \subseteq H_{\alpha n}$. Let also $n$ be given. If $δ(s * (n)) = L(δ(s))$, then $[0,δ'(s * (n))) = [0,δ'(s)) \subseteq H_{\alpha n}$, and, if $δ(s * (n)) = R(δ(s))$, then $[0,δ'(s * (n))) = [0,1]δ''(s)) \subseteq H_{\alpha n}$. Conclude that $\forall n\exists p[0,δ'(s * (n))) \subseteq H_{\alpha n}$. We thus see that $\exists \forall s[0,δ'(s)) \subseteq H_{\alpha n}$.

Define $β$ be such that, for each $n$, if $δ(n) \subseteq H_{\alpha n}$, then $β(n) = n'' + 1$, and, if not, then $β(n) = 0$. Note that $\forall s[s \in E_β \leftrightarrow \exists n[δ(n) \subseteq H_{\alpha n}]]$.

We first prove that $Bar_\omega(E_β)$. Let $ε$ be given. Define $γ$ such that $\forall n[γ(n) = \text{double}_2(δ(\bar{n})))$. Note that $ε \in [0,1]$ and $γ \in [0,γ) \exists n[γ(n) < ε \land γ(n) < ε \land δ''(n)) \subseteq H_{\alpha n}]$. Conclude that $[0,γ) \subseteq H_{\alpha n}$ and $γ \in H_{\alpha n}$. Find $n, t$ such that $α(t) \neq 0$ and $γ(n) \subseteq t$ and note that $δ(\bar{n}) \subseteq γ(n) \subseteq t$ and that $\bar{n} \in E_β$. Conclude that $\forall s[\exists n[\exists n[\exists n \in E_β]]$, i.e. $Bar_\omega(E_β)$.

Note that $\forall s[s \in E_β \rightarrow \exists n[s * (n) \in E_β]]$, i.e. $E_β$ is monotone. Now prove that $E_β$ is inductive. Let $s$ be given such that $\forall n[s * (n) \in E_β]$. Find $n'$ such that $β(n') = s * (0) + 1$ and note that $n'' = s * (0)$.

Define $q := \text{max}(n', p)$. Conclude that $δ(s * (q')) \subseteq H_{\alpha n}$ and $s \in E_β$. We thus see that $\forall s[\forall n[s * (n) \in E_β] \rightarrow s \in E_β]$, i.e. $E_β$ is inductive. Using $\Sigma^0_1$-BI, we conclude that $⟨⟩ \in E_β$, i.e. $\exists n[δ(⟨⟩) = [0,1] \subseteq H_{\alpha n} \subseteq H_\alpha]$.

The above argument is due to Th. Coquand, see [7].

Theorem 4.2. $\text{BIM} \vdash OI([0,1]) \rightarrow \text{HB}$.

Proof. Assume $OI([0,1])$. Let $β$ be given such that $D_β$ covers $[0,1]$. Define $ε$ such that $\forall s \in S[ε(⟨⟩) \neq 0 \leftrightarrow [0_Q < s'' \land D_{\bar{n}} \text{ covers } [0, s'')]].$ Note that $\forall γ \in [0,1][γ \in H_ε \leftrightarrow \exists n[D_{\bar{n}} \text{ covers } [0, γ]]].$ We prove that $\forall γ \in [0,1][[0, γ) \subseteq H_ε \rightarrow γ \in H_ε].$
Let $\gamma$ in $[0,1]$ be given such that $[0, \gamma) \subseteq \mathcal{H}_\epsilon$. Find $s, n$ such that $s \in D_\beta$ and $\gamma(n) \subseteq s$ and distinguish two cases.

**Case** (1), $s' < 0 \mathcal{Q}_2$. Note that $D_{\mathcal{P}(s+1)}$ covers $[0, \gamma'(n)]$. Find $m > n$ such that $t := \gamma(m) > s$. Note that $(\gamma(m+1) \subseteq s$ and $D_{\mathcal{P}m}$ covers $[0, t^n]$. Conclude that $t \in D_\epsilon$ and $\gamma \in \mathcal{H}_\epsilon$.

**Case** (2), $s' \geq 0 \mathcal{Q}_2$. Note that $0 \mathcal{R} (s') \mathcal{R} \gamma$, and, therefore, $(s') \mathcal{R} \mathcal{H}_\epsilon$. Find $t, n$ such that $t \in D_\epsilon$ and $(s') \mathcal{R}(n) \subseteq s$. Note that $D_{\mathcal{P}(t+1)}$ covers $[0, t^n]$. Define $q := \max(s + 1, f + 1)$ and note that $t \in D_{\mathcal{P}q}$ and $D_{\mathcal{P}q}$ covers $[0, \gamma''(n)]$. Find $m > n$ such that $u := \gamma(m) > q$. Note that $\gamma(m+1) \subseteq u$ and $D_{\mathcal{P}m}$ covers $[0, u^n]$. Conclude that $u \in D_\epsilon$ and $\gamma \in \mathcal{H}_\epsilon$.

We thus see that $\mathcal{H}_\epsilon$ is progressive in $[0,1]$. Using $\mathcal{O}([0,1])$, conclude that $1 \mathcal{R} \mathcal{H}_\epsilon$, and that there exists $n$ such that $D_{\mathcal{P}n}$ covers $[0,1]$. Conclude that, for each $\beta$, if $D_\beta$ covers $[0,1]$, then there exists $n$ such that $D_{\mathcal{P}n}$ covers $[0,1]$, i.e. HB.

**4.3. Borel’s first proof.** One finds Borel’s first proof of the Heine-Borel Theorem in his thèse from 1895, see [1]. One might paraphrase his (classical) argument as follows.

Assume $D \subseteq \mathbb{S}$ covers $[0,1]$. We define a mapping $f$ from the first uncountable ordinal $\omega_1$ to $\mathbb{Q}$ such that,

(i) for all $\alpha, \beta$ in $\omega_1$, if $\alpha < \beta$, and $f(\alpha) <_\mathbb{Q} 1$, then $f(\alpha) <_\mathbb{Q} f(\beta)$, and,

(ii) for every $\alpha$ in $\omega_1$, there is a finite subset of $D$ covering $[0, f(\alpha)]$.

Define $f(0) := 0$. Note that there exists indeed a finite subset of $D$ covering $[0, f(0)] = [0, 0] = \{0\}$. Let $\alpha$ be an element of $\omega_1$ such that, for every $\beta < \alpha$, $f(\beta)$ has been defined already. Calculate $\gamma := \sup\{f(\beta) \mid \beta < \alpha\}$. If $\gamma >_\mathbb{R} 1$, let $\beta_0$ be the least $\beta$ such that $f(\beta) >_\mathbb{Q} 1$ and define $f(\alpha) := f(\beta_0)$. If $\gamma \leq_\mathbb{R} 1$, find $s$ in $D$ and $n$ in $\omega$ such that $\gamma(n) \subseteq s$. Find $\beta < \alpha$ such that $s' < f(\beta)$. Find a finite subset $D'$ of $D$ covering $[0, f(\beta)]$. Define $f(\alpha) := (\gamma(n))''$. Note that the finite set $D' \cup \{s\}$ covers $[0, f(\alpha)]$.

Now observe there must exist $\alpha < \omega_1$ such that $f(\alpha) >_\mathbb{Q} 1$, because, if there is no such $\alpha$, $f$ is an injective map from the uncountable set $\omega_1$ into the countable set $\mathbb{Q}$. It follows that there exists a finite subset of $D$ covering $[0,1]$.

**4.4. Achilles.** In his argument for the Heine-Borel Theorem sketched in Subsection 4.3, Borel actually uses and proves the principle of open induction on $[0,1]$. His indirect argument of course is not convincing from a constructive point of view. We are confronted with the following question. If Achilles starts from 0 and makes a step of positive length in the direction of 1 from every position that he reaches, and if he also reaches the limit of every convergent sequence of reached positions, can we see that he will arrive at 1?

Achilles first will arrive at $f(1) > f(0) = 0$, then, unless $f(1) \geq 1$, at $f(2) > f(1)$, and then, unless $f(2) \geq 1$, at $f(3) \geq f(2)$, and then, taking a bold leap, at $p := \lim_{n \to \infty} f(n)$, a position beyond every $f(n)$. If $p \leq 1$, one may find a rational $q > p$ that is reached by Achilles and define $f(\omega) := q$. If $p > 1$, one defines $f(\omega) = f(n_0)$, where $n_0$ is the least $n$ such that $f(n) > 1$.

It is not so easy, alas, to calculate $p := \lim_{n \to \infty} f(n)$ and even if one finds $p$, it may be difficult to decide: $p < 1$ or $1 \leq p$. It thus is not clear how to actually find $f(\omega)$. And if one should find it, and tries to continue the process, many more problems of the same kind will arise.

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6. In [2], one finds another proof, more like the one most of us are used to.

7. Dedekind’s Theorem, i.e. the theorem that every bounded infinite and monotone sequence of real numbers converges is not true constructively, see Subsection 5.2.
How might Achilles convince himself constructively that he will reach the point 1? He would first have to convince himself that he will reach $\frac{1}{2}$. The problem of finding out if and when he reaches $\frac{1}{2}$, however, does not seem easier than the problem of finding out if and when he reaches 1.

Achilles will be very surprised and happy seeing Theorem 4.1, Brouwer’s prediction that he will indeed arrive at 1.

4.5. A stronger formulation.

**Definition 9.** The Stronger Principle of Open Induction on $[0,1]$ is the following statement.

$$OI^+([0,1]) : \forall \alpha [\forall \gamma \in [0,1][[0,\gamma) \subseteq \mathcal{H}_\alpha \rightarrow \gamma \in \mathcal{H}_\alpha] \rightarrow \exists n[[0,1) \subseteq \mathcal{H}^n]].$$

From the proof of Theorem 4.1 one may conclude that $BI \vdash \Sigma^0_1$-$BI \rightarrow OI^+([0,1])$ and not only that $BI \vdash \Sigma^0_1$-$BI \rightarrow OI([0,1])$. Theorem 4.2 makes one see that $BI \vdash OI([0,1]) \leftrightarrow OI^+([0,1])$.

5. The contraposition of Dedekind’s Theorem

5.1. Preliminaries.

**Definition 10.** $[\omega]^\omega$ is the set of all $\zeta$ such that $\forall n[\zeta(n) < \zeta(n+1)]$.

$Q^\omega$ is the set of all $\gamma$ such that $\forall n[\gamma(n) \in Q]$.

$\gamma \in Q^\omega$ converges if and only if $\forall n \exists m \forall p \geq m[[\gamma(m+p) - q \gamma(m)] \leq q \frac{1}{2^{|n|}}]$.

$\gamma \in Q^\omega$ converges explicitly if and only if $\exists \delta \forall n \forall p[[\gamma(\delta(n)+p) - \gamma \circ \delta(n)] \leq q \frac{1}{2^{|n|}}]$.

$\gamma \in Q^\omega$ has a converging subsequence if and only if

$$\exists \zeta \in [\omega]^\omega \forall n[[\gamma \circ \zeta(n+1) - q \gamma \circ \zeta(n)] < q \frac{1}{2^{|n|}}],$$

and positively fails to have a converging subsequence if and only if

$$\forall \zeta \in [\omega]^\omega \exists n[[\gamma \circ \zeta(n+1) - q \gamma \circ \zeta(n)] > q \frac{1}{2^{|n|}}].$$

$\gamma \in Q^\omega$ is non-decreasing if and only if $\forall n[\gamma(n) \leq q \gamma(n+1)]$.

If $\gamma \in Q^\omega$ is non-decreasing, then $\gamma$ positively fails to converge if and only if

$$\forall \zeta \in [\omega]^\omega \exists n[[\gamma \circ \zeta(n+1) - q \gamma \circ \zeta(n)] > q \frac{1}{2^{|n|}}].$$

$BI$ proves that $\gamma \in Q^\omega$ converges explicitly if and only if $\exists \delta \in R[\lim_{n \rightarrow \infty} \gamma(n) = \delta]$, i.e. $\exists \delta \in R \forall n \exists m \forall p \geq m[[\gamma(m+p) - q \gamma(m)] < q \frac{1}{2^{|n|}}]$. Using Weak-$\Pi^1_1$-$AC_{0,0}$ one may prove in $BI$ that, if $\gamma \in Q^\omega$ converges, then $\gamma$ converges explicitly. $BI$ also proves that, if $\gamma \in Q^\omega$ is non-decreasing, then $\gamma$ converges explicitly if and only if $\gamma$ has a converging subsequence.

**Definition 11.** Let $\varepsilon$ be an element of $Q^\omega$ such that, for each $n$, $\varepsilon(n) > q_0 \varepsilon$. We let $Sum(\varepsilon)$ be the element of $Q^\omega$ such that, for each $n$, $(Sum(\varepsilon))(n) =_Q \sum_{i=0}^{n} \varepsilon(i)$. $\varepsilon$ is called (explicitly) summable if and only if $Sum(\varepsilon)$ converges (explicitly).

**Lemma 5.1.** Let $\gamma$ be an element of $Q^\omega$ and let $\lambda$, $\varepsilon$ be explicitly summable elements of $Q^\omega$ such that, for all $n$, $\gamma(n) > q_0 \lambda(n)$ and $\lambda(n) > q_0 \varepsilon(n)$. $BI$ proves the following.

(i) If $\exists \zeta \in [\omega]^\omega \forall n[[\gamma \circ \zeta(n+1) - q \gamma \circ \zeta(n)] < q \varepsilon(n)]$, then $\exists \zeta \in [\omega]^\omega \forall n[[\gamma \circ \zeta(n+1) - q \gamma \circ \zeta(n)] < q \lambda(n)].$

(ii) If $\forall \zeta \in [\omega]^\omega \forall n[[\gamma \circ \zeta(n+1) - q \gamma \circ \zeta(n)] > q \varepsilon(n)]$, then $\forall \zeta \in [\omega]^\omega \forall n[[\gamma \circ \zeta(n+1) - q \gamma \circ \zeta(n)] > q \lambda(n)].$

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8See Subsection 2.7.23
Proof. Let \( \lambda, \varepsilon \) be explicitly summable elements of \( \mathbb{Q}^\omega \). Find \( \delta \in [\omega]^\omega \) such that
\[
\forall n \forall \gamma \forall i \left( \sum_{i=\delta(n)+p}^{\delta(n)+p} \varepsilon(i) \leq \lambda(n) \right).
\]
Note that \( \forall n[\varepsilon(n) > \lambda_0] \) and conclude that
\[
\forall n \forall \gamma \forall i \left( \sum_{i=\delta(n)+p}^{\delta(n)+p} \varepsilon(i) < \lambda(n) \right).
\]

(i) Let \( \zeta \) be an element of \( [\omega]^\omega \) such that \( \forall n[|\gamma \circ \zeta(n+1) - \gamma \circ \zeta(n)| < \lambda(n)] \). Let \( \eta \) be an element of \( [\omega]^\omega \) such that, for each \( n \), \( \zeta(\eta(n)) > \delta(n) \) and define \( \zeta^* = \zeta \circ \eta \). Note that, for each \( n \), \( |\gamma \circ \zeta^*(n+1) - \gamma \circ \zeta^*(n)| = |\gamma \circ \zeta \circ \eta(n+1) - \gamma \circ \zeta \circ \eta(n)| \leq \lambda(n+1) - \lambda(n) \leq \lambda(n) \).

(ii) Assume \( \forall \xi \in [\omega]^\omega \exists \gamma \circ \zeta(n+1) - \gamma \circ \zeta(n) > \lambda(n) \). Let \( \zeta \) be an element of \( [\omega]^\omega \). Find \( \eta \) in \( [\omega]^\omega \) such that, for each \( n \), \( \zeta(\eta(n)) > \delta(n) \) and define \( \zeta^* = \zeta \circ \eta \). Note that, for each \( n \), \( |\gamma \circ \zeta^*(n+1) - \gamma \circ \zeta^*(n)| > \lambda(n) \). Conclude that
\[
\sum_{i=\eta(n)}^{\eta(n)+1} |\gamma \circ \zeta^*(i+1) - \gamma \circ \zeta^*(i)| > \lambda(n) \]
and find \( i \) such that \( \eta(n) \leq i < \eta(n+1) \) and \( \gamma \circ \zeta^*(i+1) - \gamma \circ \zeta^*(i) > \varepsilon(i) \).

5.2. Dedekind’s Theorem fails constructively. R. Dedekind wrote his 5 in order to justify the “intuitively clear” statement: Every infinite sequence of reals that is bounded and nondecreasing converges.

Definition 12. We call the following statement Dedekind’s Theorem:

Ded: For all \( \gamma \in \mathbb{Q}^\omega \),
if \( \forall n[\gamma(n) \leq \gamma(n+1) \leq \lambda_0] \), then \( \exists \zeta \in [\omega]^\omega \forall n[\gamma(n) \leq \gamma(n+1) + \frac{1}{n}] \).

The following Theorem shows that Ded is constructively false.

Theorem 5.2. \( \text{BIM} \vdash \text{Ded} \rightarrow \text{LPO} \).

Proof. Assume Ded. Let \( \alpha \) be given. Define \( \gamma \in \mathbb{Q}^\omega \) such that for all \( n \), if \( \forall i < n[\alpha(i) = 0] \), then \( \gamma(n) = 0 \), and, if \( \exists i < n[\alpha(i) \neq 0] \), then \( \gamma(n) = 1 \). Using Ded, find \( \zeta \) in \( [\omega]^\omega \) such that \( \forall n[\gamma \circ \zeta(n+1) \leq \gamma \circ \zeta(n) + \frac{1}{n}] \). Define \( m = \zeta(1) \) and note: \( \forall n > n[m] \gamma(n) < \gamma(m) + \frac{1}{n} \). Either \( \gamma(m) = 1 \) and: \( \exists i < m[\alpha(i) \neq 0] \) or: \( \gamma(m) = 0 \) and \( \forall n[\gamma(n) < \gamma(m)] \) and \( \forall i[\alpha(i) = 0] \). We thus see that \( \forall n[\exists i[\alpha(i) \neq 0] \lor \forall i[\alpha(i) = 0]] \), i.e. LPO.

5.3. A contraposition of Dedekind’s Theorem.
We introduce a contraposition of Dedekind’s Theorem.

Definition 13. The contraposition of Dedekind’s Theorem is the following statement:

\( \text{Ded} \): For all \( \gamma \in \mathbb{Q}^\omega \), if \( \forall n[\gamma(n) \leq \gamma(n+1)] \) and \( \forall \zeta \in [\omega]^\omega \exists \gamma \circ \zeta(n) + \frac{1}{n} < \gamma \circ \zeta(n+1) \), then \( \exists \gamma[1 < \gamma(n)] \).

i.e. every non-decreasing infinite sequence of rationals that positively fails to converge grows beyond 1.

Theorem 5.3. \( \text{BIM} \vdash \text{OI}([0,1]) \leftrightarrow \text{Ded} \).

Proof. (i) Assume \( \text{OI}([0,1]) \). Let \( \gamma \in \mathbb{Q}^\omega \) be given such that \( \forall n[\gamma(n) \leq \gamma(n+1)] \) and \( \forall \zeta \in [\omega]^\omega \exists \gamma \circ \zeta(n) + \frac{1}{n} < \gamma \circ \zeta(n+1) \). Define \( \alpha \) such that \( \forall s \in S[\alpha(s) \neq 0 \leftrightarrow \exists n < s[\varepsilon(n) > \gamma(n)] \] Note that \( \forall \eta \in \mathbb{R}[\eta \in \mathcal{H}_\alpha \leftrightarrow \exists \eta[\eta \in \mathcal{R} (\gamma(n))] \] We now prove that \( \mathcal{H}_\alpha \) is progressive in \([0,1]\). Let \( \eta \) be an element of \([0,1]\) such that \( [0, \eta] \subseteq \mathcal{H}_\alpha \). Note that \( \forall n[\exists \eta[\eta \in \mathcal{H}_\alpha \land \mathcal{H}_\alpha \land \eta \in \mathcal{H}_\alpha] \). Define \( \beta \) such that \( \forall n[\beta(n) = \mu \rho n' + \rho' \eta''] < \gamma(\eta')] \). Define \( \zeta \) such that \( \zeta(0) = \beta''(0) \) and \( \forall n[\zeta(n+1) = \max(\zeta(n+1), \beta''(n+1))] \). Note that \( \zeta \in [\omega]^\omega \) and \( \forall n[\eta \in \mathcal{H}_\alpha \land \mathcal{H}_\alpha \land \eta \in \mathcal{H}_\alpha] \). Find \( n \) such that \( \gamma \circ \zeta(n) + \frac{1}{n} < \gamma \circ \zeta(n+1) \). Conclude that \( \eta \in \mathcal{H}_\alpha \land \mathcal{H}_\alpha \land \mathcal{H}_\alpha \land \eta \in \mathcal{H}_\alpha \).
We thus see that \( \forall n \in [0,1] [0,n] \subseteq \mathcal{H}_\alpha \rightarrow \eta \in \mathcal{H}_\alpha \). Using \( \text{OI}(0,1) \), we conclude that \([0,1] \subseteq \mathcal{H}_\alpha \), and \( \exists n \{1 \not< \gamma(n) \} \). We thus see that, for each \( \gamma \) in \( \mathbb{Q}^\omega \), if \( \gamma \) is nondecreasing and positively fails to converge, then \( \exists n \{1 \not< \gamma(n) \} \), i.e. \( \mathbb{D} \).

(ii) Assume \( \mathbb{D} \). Let \( \alpha \) be given such that \( \mathcal{H}_\alpha \) is progressive in \([0,1] \). Define \( \gamma \) in \( \mathbb{Q}^\omega \) such that, for each \( n \), \( \gamma(n) = \max_0 \{q \in \mathbb{Q} : (q,0,1) \subseteq \mathcal{H}_\alpha \} \). Note that \( \forall n \gamma(n) \leq \gamma(n+1) \). We now prove that \( \gamma \) positively fails to converge. Let \( \zeta \in [\omega]^\omega \) be given. Define \( \delta \) in \( \mathbb{Q}^\omega \) such that \( \delta(0) = \gamma(0) \), and, for each \( n \), \( \delta(n+1) = \min_{\mathbb{N}} (\delta(n) + \frac{1}{2^n}, \gamma \circ \zeta(n+1)) \). Note that the infinite sequence \( \delta \) converges and find \( \varepsilon := \lim_{n \to \infty} \delta(n) \in \mathbb{R} \). Note that, for each \( n \), \( [0,\delta(n)] \subseteq \mathcal{H}_\alpha \), and \( [0,\varepsilon] = \bigcup_{n=0}^\infty [0,\delta(n)] \). Conclude that \( [0,\varepsilon] \subseteq \mathcal{H}_\alpha \), and that \( \varepsilon \in \mathcal{H}_\alpha \). Find \( s \) such that \( s' < \eta \subseteq \frac{e'}{e'}(m) < \frac{e'}{e'}(m) < \frac{s'}{s'}(n) \) and \( \alpha(s) \neq 0 \). Find \( p \) such that \( [0,\eta'] \subseteq \mathcal{H}_\alpha \). Define \( q := \max(s+1, p) \) and note that \([0,\eta'] \subseteq \mathcal{H}_\alpha \). Conclude that \( q > R(\gamma(q)) \rightleftharpoons q \circ \zeta(q) \). Conclude that \( \delta(q) < \gamma \circ \zeta(q) + 1 \) and \( \exists i < q \gamma \circ \zeta(i) + \frac{1}{q} < i \). We thus see that \( \forall \zeta \in [\omega]^\omega \exists \gamma \circ \zeta(i) + \frac{1}{q} < q \circ \zeta(i+1) \). Using \( \mathbb{D} \), we find \( n \) such that \( \gamma(n) \not> 1 \) and conclude that \([0,1] \subseteq \mathcal{H}_\alpha \). We thus see that, for each \( \alpha \), if \( \mathcal{H}_\alpha \) is progressive in \([0,1] \), then \( n \in \mathcal{H}_\alpha \), i.e. \( \text{OI}(0,1) \).

5.4. The contraposition of Dedekind’s Theorem in \( \omega^\omega \).

**Definition 14.** For each \( n \), for all \( a,b \),

\[
\neg_n a \iff \exists j < n \iff j < \min(\text{length}(a), \text{length}(b)) \land a(j) \neq b(j)), \text{ and } a =_n b \iff \neg(a \neq_n b). \tag{10}
\]

The following Lemma may be compared to Lemma 5.1 (ii).

**Lemma 5.4.** \( \text{BIM} \) proves the following.

For all \( \gamma \), for all \( \eta \) in \( [\omega]^\omega \), if \( \forall \zeta \in [\omega]^\omega \exists n \gamma \circ \zeta(n+1) = \eta(n) \), then \( \forall \zeta \in [\omega]^\omega \exists n \gamma \circ \zeta(n+1) \neq \eta(n) \).

**Proof.** Let \( \gamma, \eta \) be given such that \( \eta \in [\omega]^\omega \) and \( \forall \zeta \in [\omega]^\omega \exists n \gamma \circ \zeta(n+1) = \eta(n) \). Let \( \zeta \) be an element of \([\omega]^\omega \). Note that \( \zeta \circ \eta \in [\omega]^\omega \) and find \( n \) such that \( \gamma \circ \zeta \circ \eta(n+1) \neq \eta(n) \). Find \( i \) such that \( \eta(n) \leq i \leq \eta(n+1) \) and \( \gamma \circ \zeta(i) \neq \eta(n) \). Note that \( \eta(n) \leq i \) and \( \gamma \circ \zeta(i) 
eq \gamma \circ \zeta(i+1) \). We thus see that \( \forall \zeta \in [\omega]^\omega \exists n \gamma \circ \zeta(n+1) \neq \gamma \circ \zeta(n) \).

**Definition 15.** For all \( a,b \),

\[
a \leq_{\text{lex}} b \iff \exists i < \min(\text{length}(a), \text{length}(b)) [a_i = b_i \land a(i) < b(i)].
\]

\( \gamma \in \omega^\omega \) is 1-non-decreasing if and only if \( \forall n [\gamma(n) <_{\text{lex}} \gamma(n+1) \lor \gamma(n+1) \leq \gamma(n+1)] \). If \( \gamma \in \omega^\omega \) is 1-non-decreasing, then \( \gamma \) is explicitly 1-convergent if and only if \( \exists \zeta \in [\omega]^\omega \forall \zeta \gamma \circ \zeta(n) = \gamma \circ \zeta(n+1) \). If \( \gamma \in \omega^\omega \) is 1-non-decreasing, then \( \gamma \) positively fails to be 1-convergent if and only if \( \forall \zeta \in [\omega]^\omega \exists n \gamma \circ \zeta(n) \neq \gamma \circ \zeta(n+1) \).

The contrapositive of Dedekind’s Theorem for Baire space is the statement:

**\( \mathbb{D}_{\omega^\omega} \):** For all \( \gamma \), if \( \forall n [\gamma(n) <_{\text{lex}} \gamma(n+1) \lor \gamma(n) \leq \gamma(n+1)] \) and \( \forall \zeta \in [\omega]^\omega \exists n \gamma \circ \zeta(n) \neq \gamma \circ \zeta(n+1) \), then \( \exists n [\gamma(n) \not\in 2^\omega^\omega] \).

i.e. every 1-non-decreasing element of \( \omega^\omega \) that positively fails to be 1-convergent will leave \( 2^\omega^\omega \).

**Theorem 5.5.** \( \text{BIM} \vdash \mathbb{D} \leftrightarrow \mathbb{D}_{\omega^\omega} \).

**Proof.** (i) Assume \( \mathbb{D} \).

Define \( \delta_0 \) such that, for all \( a \) in \( 2^\omega^\omega \), \( \delta_0(a) = \sum_{i < \text{length}(a)} \frac{a(i)+1}{3} \). Note that, for all \( n \), for all \( a, b \) in \( 2^\omega^\omega \), if \( a <_{\text{lex}} b \) and \( a \neq b \), then \( \delta_0(a) + \frac{1}{3} < \delta_0(b) \).
Let $\gamma$ be given such that $\forall n[\gamma(n) \leq_{lex} \gamma(n+1) \lor \gamma(n) = \gamma(n+1)]$ and $\forall \zeta \in [\omega]^\omega \exists n[\zeta(n) \neq n \lor \zeta(n+1) \neq n]$. Define $\eta$ in $\mathbb{Q}^\omega$ such that, for each $p$, either $\forall m \leq p[\gamma(m) \in 2^\omega]$ and $\eta(p) = \delta_0(\gamma(p))$, or $\exists m \leq p[\gamma(m) \notin 2^\omega]$ and $\eta(p) = (p + 1)^\omega$. Note that $\forall n[\eta(n) \leq \eta(n+1)]$. We now prove that $\eta$ positively fails to have a converging subsequence. Let $\zeta \in [\omega]^\omega$ be given. Find $p$ such that $\gamma \circ \zeta(p) \neq_p \gamma \circ \zeta(p+1)$ and distinguish two cases.

1. For all $m \leq (p+1)[\gamma(m) \in 2^\omega]$, then $\eta \circ \zeta(p) = \delta_0(\gamma \circ \zeta(p))$ and $\eta \circ \zeta(p+1) = \delta_0(\gamma \circ \zeta(p+1))$ and $\eta \circ \zeta(p) + (p+1) \leq \eta \circ \zeta(p+1)$.

2. For all $m \leq (p+1)[\gamma(m) \notin 2^\omega]$, then $\eta \circ \zeta(p+2) = \gamma(\zeta(p+2))$, and $\eta \circ \zeta(p+1) \geq \eta \circ \zeta(p+1)$.

We thus see that $\forall \gamma \in [\omega]^\omega \exists p[\eta \circ \zeta(p) + (p+1) \leq \eta \circ \zeta(p+1)]$ and $\Delta_{\omega}$. We thus see that, if $\gamma \in \omega^\omega$ is 1-nondecreasing and positively fails to be 1-convergent, then $\exists m[\gamma(m) \notin 2^\omega]$, i.e. $\Delta_{\omega}$.

Assume $\Delta_{\omega}$. Define $\delta_1$ such that, for all $a$ in $2^\omega$, $\delta_1(a) = \sum_{i<\text{length}(a)} a(i) \Delta_{\omega}$. Note that, for all $n$, for all $a, b$ in $2^\omega$, if $a \leq b$ then $|\delta_1(a) - \delta_1(b)| \leq \frac{1}{2^n}$.

Let $\gamma \in \mathbb{Q}^\omega$ be given such that $\forall n[\gamma(n) \leq \gamma(n+1)]$ and $\forall \zeta \in [\omega]^\omega \exists n[\gamma \circ \zeta(n)]$. We want to prove that $\exists n[\eta(n) \leq \gamma(n)]$. First note that $\forall n[\eta(n) > m[\gamma(n) \leq \gamma(n+1)]$. Then find $\zeta \in [\omega]^\omega$ such that $\forall n[\gamma \circ \zeta(n) \leq \gamma \circ \zeta(n+1)]$ and define $\gamma \circ \zeta$. Note that $\forall n[\gamma \circ \zeta(n) \leq \gamma \circ \zeta(n+1)]$ and $\forall x \in [\omega]^\omega \exists n[\gamma \circ \zeta(n) \geq \gamma \circ \zeta(n+1)]$. Define $\rho$ such that, for each $n$, if $\exists m \leq n + 1[\gamma \circ \zeta(n)]$, then $\rho(n) = (n + 2)$, and, if $\forall n \leq n + 1[\gamma \circ \zeta(n)]$, then $\rho(n)$ is the least $a$ in $2^\omega$ such that $\gamma \circ \zeta(n) \leq \delta_1(a) \gamma \circ \zeta(n+1)$ and $\delta_1(a) \gamma \circ \zeta(n) \leq \frac{1}{2^n}[\gamma \circ \zeta(n+1)$]. Note that $\rho$ is $1$-non-decreasing. We now prove that $\rho$ positively fails to be 1-convergent. Assume that $\eta \in [\omega]^\omega$. Find $n$ such that $\gamma \circ \eta(n) + (p+1) \leq \gamma \circ \eta(n+1)$ and distinguish two cases.

1. For all $i \leq (n+1)[\gamma(i) > \gamma(i+1)]$, then $\rho \circ \eta(n+1) \leq \rho \circ \eta(n)+1 \gamma \circ \eta(n) + (p+1) \gamma \circ \eta(n+1) \leq \rho \circ \eta(n+1)$ and $\rho \circ \eta(n+1) = (\rho \circ \eta(n) + (p+1)) \leq \rho \circ \eta(n+1)$. We thus see that $\forall \gamma \in [\omega]^\omega \exists n[\rho \circ \eta(n+1) \leq \rho \circ \eta(n)]$. Using Lemma 5.4 and $\Delta_{\omega}$, we find $p$ such that $\eta \circ \rho \notin 2^\omega$. Conclude that $\eta \circ \rho = (p + 2)$, and $\exists n \leq p[\eta \circ \rho \leq \gamma(n)]$. We thus see that, for all $\gamma$ in $\mathbb{Q}^\omega$, if $\gamma$ is non-decreasing and positively fails to converge, then $\exists n[\eta \circ \rho \leq \gamma]$.

5.5. The Principle of Open Induction on Cantor space $2^\omega$.

Definition 16. For all $A \subseteq \omega^\omega$, for all $\gamma$ in $A$, $A \leq_{\text{lex}} \gamma = \{ \delta \in A \mid \delta \leq_{\text{lex}} \gamma \}$.

$A \subseteq \omega^\omega$ is progressive in $2^\omega$ if and only if $\forall \gamma \in 2^\omega[\forall \gamma \in 2^\omega \leq_{\text{lex}} \gamma \subseteq A \rightarrow \gamma \in A]$. The Principle of Open Induction on Cantor space is the following statement:

$\text{OL}(2^\omega)$: For every open subset $G$ of $\omega^\omega$, if $G$ is progressive in $2^\omega$, then $2^\omega \subseteq G$.

Theorem 5.6. BIM $\vdash \Delta_{\omega} \leftrightarrow \text{OL}(2^\omega)$.

Proof. (i) Assume $\Delta_{\omega}$. Let $\alpha$ be given such that $\forall \gamma \in 2^\omega[\forall \gamma \in 2^\omega \leq_{\text{lex}} \gamma \subseteq G_\alpha \rightarrow \gamma \in G_\alpha]$. Define $\gamma$ such that $\gamma(0) := \gamma$ and, for each $n$, (1) if $2^\omega \not\subseteq G_{\gamma(n)}$, then $\gamma(n)$ is the least element $s$ of $2^\omega$ satisfying $\gamma(n) \leq_{\text{lex}} s \subseteq G_{\gamma(n)}$ and $2^\omega \cap s \not\subseteq G_{\gamma(n)}$, and, (2) if $2^\omega \subseteq G_{\gamma(n)}$, then $\gamma(n) = (n + 2)$. Note that $\forall \gamma \in 2^\omega[\forall \gamma \leq_{\text{lex}} \gamma(n) \subseteq G_{\gamma(n)}$ and $\forall \gamma \leq_{\text{lex}} \gamma(n) \vee \gamma(n) \not\subseteq \gamma(n+1)]$. We now prove that $\gamma$ positively fails
to be 1-convergent. Let \( \zeta \in [\omega]^\omega \) be given. Define \( \delta(0) = \gamma \circ \zeta(0) \), and, for each \( n \), if \( \forall i \leq n[\gamma \circ \zeta(i) = \gamma \circ \zeta(i + 1), \text{ then } \delta(n + 1) = \gamma \circ \zeta(n + 1) \), and, if not, then \( \delta(n + 1) = \delta(n) \ast \langle 0 \rangle \). Note that, for each \( n, \delta(n) \equiv \delta(n + 1) \). Define \( \varepsilon \) such that \( \forall n[\varepsilon = n \cdot \delta(n)] \) and note that, for all \( \beta \) in \( 2^\omega \), if \( \beta \leq \langle \varepsilon \rangle \), then \( \exists \beta \ast \delta(n) \rangle \), and \( \exists \beta \ast \langle \delta(n) \rangle \), and \( \beta \in \gamma \). Thus we see that \( 2^n \leq \gamma \) and conclude that \( \varepsilon \in \gamma \). Find \( n \) such that \( \alpha(\gamma(n)) \neq 0 \). Define \( p := \max(\alpha(\gamma(n)), \gamma(n) + 1) \) and note that \( e < \langle \gamma(n) \rangle \). Conclude that \( \exists i \leq n[\gamma \circ \zeta(i) \neq i \gamma \circ \zeta(i + 1)] \). We thus see: \( \forall \zeta \in [\omega]^\omega \exists i[\gamma \circ \zeta(i) \neq i \gamma \circ \zeta(i + 1)] \). Using \( \text{Ded}_\omega \), we find \( n \) such that \( \gamma(n) \notin 2^\omega \) and conclude: \( 2^n \subseteq \gamma \).

We thus see that, for all \( \alpha \), if \( \gamma_a \) is progressive in \( 2^\omega \), then \( 2^\omega \subseteq \gamma_a \), i.e., \( \text{OI}(2^\omega) \).

(ii) Assume \( \text{OI}(2^\omega) \). Let \( \gamma \) be given such that \( \forall n[\gamma(n) < \langle \gamma(n) \rangle < \langle \gamma(n + 1) \rangle \) and \( \forall \zeta \in [\omega]^\omega \exists n[\gamma \circ \zeta(n) \neq n \gamma \circ \zeta(n + 1)] \). Note that \( \exists n[\gamma(n) < \langle \gamma(n) \rangle + \langle \gamma(n + 1) \rangle] \) and conclude that \( \exists \alpha \exists m \exists n < \langle \gamma(n) \rangle \). Also note that \( \forall m \exists p > m \exists \gamma(p) \). We may conclude that, \( \forall n, \exists \gamma(n) \not\in 2^\omega \). We thus see that \( \gamma \) is 1-non-decreasing and positively fails to converge, then \( \exists n[\gamma(n) \notin 2^\omega \), i.e., \( \text{Ded}_\omega \).

\[ \square \]

**Corollary 5.7.** \( \text{BIM} \vdash \text{OI}([0, 1]) \leftrightarrow \text{OI}(2^\omega) \leftrightarrow \text{Ded} \leftrightarrow \text{Ded}_\omega \).

**Proof.** See Theorems 5.5 and 5.6.

6. Enumerables subsets of \( \omega \) that positively fail to be decidable

For each \( \alpha, D_\alpha := \{ \gamma : \alpha(\gamma(n)) \neq 0 \} \) and \( E_\alpha := \{ m : \exists n[\alpha(\gamma(n)) = m + 1] \} \). \( D_\alpha \) is the subset of \( \omega \) decided by \( \alpha \) and \( E_\alpha \) is the subset of \( \omega \) enumerated by \( \alpha \). \[ \square \]

**Lemma 6.1.** \( \text{BIM} \vdash \forall \alpha \exists \gamma[D_\alpha = E_\alpha] \).

**Proof.** Let \( \alpha \) be given. Define \( \gamma \) such that, for each \( n, \) if \( \alpha(n) = 0 \), then \( \gamma(n) = 0 \), and, if \( \alpha(n) \neq 0 \), then \( \gamma(n) = n + 1 \). Clearly, \( D_\alpha = E_\gamma \).

**Lemma 6.2.** \( \text{BIM} + \text{BCP} \vdash \forall \gamma \exists \alpha[D_\alpha = E_\gamma] \).

**Proof.** Assume that \( \forall \gamma \exists \alpha[E_\gamma = D_\alpha] \). Conclude that \( \forall \gamma[0 \in E_\gamma \lor 0 \notin E_\gamma] \). Using Brouwer’s Continuity Principle \( \text{BCP} \), find \( p \) such that \( \forall \gamma\forall p = \langle \gamma \rangle \to 0 \in E_\gamma \), or \( \forall \gamma\forall p = \langle \gamma \rangle \to 0 \notin E_\gamma \). Note that \( 0 \notin E_\gamma \) and \( 0 \in E_\gamma^{\text{BCP}} \), and conclude that both alternatives are false.

**Lemma 6.3.** \( \text{BIM} + \text{CT} \vdash \exists \gamma \forall \alpha[D_\alpha \subseteq E_\gamma \to \exists n \in E_\gamma[n \notin D_\alpha]] \).

\[ \square \]

\[ \text{11See Subsection } \text{1.4.5} \]

\[ \text{12For } \text{BCP, see Subsubsection } \text{2.2.1.} \]
Proof. Using Church’s Thesis \(\text{CT}\) find \(\tau\), \(\psi\) such that \(\forall \alpha \exists \forall n \exists z [\tau(e, n, z) \neq 0 \land \forall i < z [\tau(e, n, i) = 0] \land \psi(z) = \alpha(n)]\). Define, for each \(n\), \(W_n := \{m \mid \exists z [\tau(n, m, z) \neq 0]\}\). Define the so-called \((self-)halting\) problem \(K := \{n \mid n \in W_n\}\). Define \(\gamma\) such that, for each \(n\), if \(\gamma(n) = 0\) and \(\forall i < n [\tau(n, i, i) = 0]\), then \(\gamma(n) = n + 1\), and, if not, then \(\gamma(n) = 0\), and note that \(K = \gamma\). Let \(\alpha\) be given such that \(D_\alpha \subseteq \gamma\). Define \(\delta\) such that, for each \(n\), if \(\alpha(n) = 0\), then \(\delta(n) = 0\), and if \(\alpha(n) \neq 0\), then \(\delta(n) = \psi(z) + 1\) where \(z\) is the least number \(i\) such that \(\tau(n, i, i) \neq 0\). Find \(e\) such that \(\forall n \exists z [\tau(e, n, z) \neq 0 \land \forall i < z [\tau(e, n, i) = 0] \land \psi(z) = \delta(n)]\). Note that \(e \in K = \gamma\). Assume \(\alpha(e) \neq 0\). Let \(z\) be the least \(i\) such that \(\tau(e, i, i) = 1\) and conclude that \(\psi(z) = \psi(z) + 1\). Contradiction. Conclude that \(\alpha(e) = 0\) and \(e \in \gamma \setminus D_\alpha\). \(\square\)

**Definition 17.** Let \(D, E\) be subsets of \(\omega\) such that \(D \subseteq E\). \(D\) is properly contained in \(E\) or \(D\) is surpassed by \(E\) if and only if \(\exists n \in E[n \notin D]\). \(X \subseteq \omega\) positively fails to be decidable if and only if for every \(\alpha\), if \(D_\alpha \subseteq X\), then \(D_\alpha\) is properly contained in \(X\).

Lemma [6,3] shows that \(\text{CT}\) enables one to find \(\gamma\) such that \(E_\gamma\) positively fails to be decidable.

**Definition 18.** We introduce the following statement.

\(\text{EnDec}?!: \forall \gamma [\forall \alpha ([D_\alpha \subseteq E_\gamma \land \exists n [n \notin D_\alpha]) \rightarrow \exists p [p \notin D_\alpha \land p \in E_\gamma] \rightarrow E_\gamma = \omega]\).

i.e. for every \(\gamma\), if, for each \(\alpha\), if \(D_\alpha \subseteq E_\gamma\) is surpassed by \(\omega\), then \(D_\alpha\) is surpassed by \(E_\gamma\), then \(E_\gamma = \omega\).

Note that \(\text{EnDec}?!\) implies: “There is no \(\gamma\) such that \(E_\gamma\) positively fails to be decidable.” and thus contradicts \(\text{CT}\).

**Theorem 6.4.** \(\text{BIM} \vdash \text{OI}(2^\omega) \rightarrow \text{EnDec}?!\).

*Proof.* Assume \(\text{OI}(2^\omega)\).

Let \(\gamma, n\) be given such that \(\forall \alpha \in 2^\omega ([D_\alpha \subseteq E_\gamma \land n \notin D_\alpha] \rightarrow \exists p [p \notin D_\alpha \land p \in E_\gamma])\). Define \(\zeta\) such that, for all \(a\) in \(2^{<\omega}\), \(\zeta(a) \neq 0\) if and only if \(\exists \mu [\text{length}(a) \exists i < \text{length}(a))[(\mu(p) = 0 \land \gamma(i) = p + 1) \lor \gamma(i) = n + 1]\). Note that \(G_\zeta = \{\alpha \in 2^\omega \mid \exists \mu [\text{length}(a) \exists i < \text{length}(a)) [(\mu(p) = 0 \land \gamma(i) = p + 1) \lor \gamma(i) = n + 1]\}}\). We now prove that \(G_\zeta\) is progressive in \(2^{<\omega}\). Let \(\alpha\) in \(2^{<\omega}\) be such that \((2^{<\omega})_{\zeta,\alpha} \subseteq G_\zeta\). Let \(\delta\) in \(D_\alpha\) be given. Define \(\beta\) in \(2^{<\omega}\) such that \(\beta(k) = 0\) and \(\forall m \neq k [\beta(m) = \alpha(m)]\). Note that \(D_\beta = D_\alpha \setminus \{k\} \land \beta_{\leq} \alpha\) and, therefore, \(\beta \in G_\zeta\). Find \(p\) such that either \(p \in E_\gamma \setminus D_\beta\) or \(p \in E_\gamma\). Note that (1) if \(p \in E_\gamma \setminus D_\beta\) and \(p \neq k\), then \(p \in E_\gamma\) and \(\alpha \in G_\zeta\), and, (2) if \(p \in E_\gamma \setminus D_\beta\) and \(p = k\), then \(k \in E_\gamma\), and (3) if \(n \in E_\gamma\), then \(\alpha \in G_\zeta\). We thus see that \(\forall k \in D_\alpha [\alpha \in G_\zeta \land k \in E_\gamma]\), i.e. \(\forall k \in D_\alpha [\exists \mu [\exists q \in D_\zeta \lor \gamma(q) = k + 1]]\). Define \(\delta\) such that \(\forall k \in D_\alpha [\delta(k) = \mu \exists q \in D_\zeta \lor \gamma(q) = k + 1]\). Define \(\alpha^*\) such that \(\forall k [\alpha^*(k) \neq 0 \land \gamma(\delta(k)) = k + 1]\). Note that \(D_{\alpha^*} \subseteq E_\gamma\) and find \(p\) in \(E_\gamma \setminus D_{\alpha^*}\). Either \(\alpha(p) = 0\) and \(p \in E_\gamma \setminus D_\alpha\) and \(\alpha \in G_\zeta\), or \(\alpha(p) \neq 0\) and \(\gamma(\delta(p)) \neq p + 1\) and \(\exists q \in D_\zeta\) and \(\alpha \in G_\zeta\). In both cases, \(\alpha \in G_\zeta\). We thus see that \(\forall \alpha \in 2^{<\omega} [\exists \mu [\exists q \in D_\zeta \lor \gamma(q) = k + 1]\}. Using \(\text{OI}(2^\omega)\) we conclude that \(2^{<\omega} \subseteq G_\zeta\) and, in particular, \(1 \in G_\zeta\) and \(n \in E_\gamma\). Conclude that, for all \(\gamma, n\), if \(\forall \alpha \in 2^{<\omega} ([D_\alpha \subseteq E_\gamma \land n \notin D_\alpha] \rightarrow \exists p [p \notin D_\alpha \land p \in E_\gamma])\), then \(n \in E_\gamma\).

Now let \(\gamma\) be given such that \(\forall \alpha \in 2^{<\omega} ([D_\alpha \subseteq E_\gamma \land \exists n [n \notin D_\alpha]) \rightarrow \exists p [p \notin D_\alpha \land p \in E_\gamma])\). Conclude that \(\forall n \in E_\gamma\) and \(E_\gamma = \omega\).

Conclude that, for each \(\gamma\), if every decidable subset of \(\omega\) that is contained in \(E_\gamma\) and properly contained in \(\omega\) is also properly contained in \(E_\gamma\), then \(E_\gamma = \omega\), i.e. \(\text{EnDec}?!\).

\(\blacksquare\)

\(^{13}\)For \(\text{CT}\), see Subsubsection [3,2].
Theorem 6.5. $\text{BIM} \vdash \text{EnDec}?! \rightarrow \text{OI}([0, 1])$.

Proof. Assume $\text{EnDec}?!$.

Let $\alpha$ be given such that $\forall \delta \in [0, 1]$, $\forall \delta \in [0, 1]$. We shall prove that $[0, 1] \subseteq H_\alpha$. Define $\zeta$ such that $\forall n \in \zeta(n) = \mu i [i \in Q \land 0 \leq i \leq 1 \land \forall j < i]$. Note that $\{ \zeta(n) : n \in \omega \} = \{ q \in Q \mid 0 \leq q \leq 1 \}$. Define $\gamma$ such that, for each $n$, if $[0, \zeta(n') \subseteq H_{\text{EnDec}?!}$, then $\gamma(n) = n' + 1$, and, if not, then $\gamma(n) = 0$. Note that $\forall n \in E_\gamma \equiv \exists m\{ [0, \zeta(n)] \subseteq H_{\text{EnDec}?!} \}$. We shall prove that every decidable subset of $\omega$ that is surpassed by $\omega$ and contained in $E_\gamma$ is also surpassed by $E_\gamma$. Let $\beta, n$ be given such that $n \notin D_\beta$ and $D_\beta \subseteq E_\gamma$. We shall prove that $\exists p \in E_\gamma \setminus D_\beta$. Define $k_0 := \mu i [\zeta(i) = 0]$. Note that $0 \in H_\alpha$ and $k_0 \in E_\gamma$. Now assume that $k_0 \in D_\beta$. Define $\delta$ as follows, by induction. Define $\delta(0) := (k_0, n)$. Note that $\delta(0) \notin D_\beta$ and $\delta(n) \notin D_\beta$ and $\zeta \circ \delta(0) \subseteq \zeta \circ \delta(n)$. Let $m$ be given such that $\delta(n)$ has been defined and $\delta(m) \notin D_\beta$ and $\delta(n) \notin D_\beta$ and $\zeta \circ \delta(m) \subseteq \zeta \circ \delta(n)$. Find $l$ such that $\zeta(l) = \frac{1}{2} (\zeta(\delta(m)) \cup \zeta(\delta(n)))$. If $l \in D_\beta$, define $\delta(m + 1) = (l, \delta(m))$, and, if $l \notin D_\beta$, define $\delta(m + 1) := (\delta(m), l)$. Note that, for each $m$, $\delta(m) \subseteq D_\gamma$, and $\exists m\{ [0, \zeta \circ \delta(m)] \subseteq H_{\text{EnDec}?!} \}$. Note that, for each $m$, $\delta(m) \notin D_\beta$. Define $\zeta$ such that $\forall n \in \{ \text{doubles}((\zeta \circ \delta(n), \zeta \circ \delta(n))) \}$. Note that $\exists \in [0, 1]$ and $\forall \gamma \in [0, \zeta \circ \delta(n)] \in R$ and $\forall \eta \in [0, \zeta \circ \delta(n)] \in R$. Find $r$ such that $\exists \beta \subseteq H_{\text{EnDec}?!}$. We conclude that $E_\gamma = \omega$. Define $k_1 := \mu i [\zeta(k_1) = 1]$. Find $n$ such that $\gamma(n) = k_1 + 1$. Conclude that $n'' = k_1$ and $D_{\text{EnDec}?!}$ covers $[0, 1]$. If $\forall n \in [0, 1]$, then $\exists n(\zeta(n) \notin Y)$. We thus see that, for each $\alpha$, if $H_\alpha$ is progressive in $[0, 1]$, then $[0, 1] \subseteq H_\alpha$, i.e. $\text{OI}([0, 1])$.

Corollary 6.6. $\text{BIM} \vdash \text{OI}([0, 1]) \leftrightarrow \text{\textit{\textbf{Ded}}} \leftrightarrow \text{\textit{\textbf{\text{Ded}}}''} \leftrightarrow \text{\textit{\textbf{OI}}}(\omega) \leftrightarrow \text{\textit{\textbf{EnDec}?!}}$.

Proof. Use Theorems 6.3 and 6.5 and Corollary 6.7.

7. Below a bar, $\zeta_{KB}$ is a well-ordering

Definition 19. $\zeta_{KB}$ is a binary relation on $\omega$ satisfying $\forall s \forall t[\zeta_{KB} t \leftrightarrow (t \subset s \lor s \subset \zeta_{KB} t)]$.

The ordering $\zeta_{KB}$ is called the Kleene-Brouwer ordering of $\omega$.

For every decidable $X \subseteq \omega$, $\text{Below}(X) := \{ s \mid \forall t [s \notin X] \}$.

For every decidable $Y \subseteq \omega$, $\zeta_{KB}$ well-orders $Y$ if and only if, for all $\zeta$, if $\forall n([n(\zeta(n) + 1) \in \zeta_{KB} \zeta(n)]$, then $\exists n(\zeta(n) \notin Y]$.

Lemma 7.1. Let $Y$ be a decidable subset of $\omega$. The following statements are equivalent in $\text{BIM}$.

(i) $\zeta_{KB}$ well-orders $Y$.

(ii) for all $\zeta$, there exists $n$ such that either $\neg(\zeta(n + 1) \in \zeta_{KB} \zeta(n))$ or $\zeta(n) \notin Y$.

Proof. (i) $\Rightarrow$ (ii). Let $\zeta$ be given. Define $\zeta^*$ such that $\zeta^*(0) = \zeta(0)$ and, for each $n$, if $\forall i \leq i(\zeta(i + 1) \in \zeta_{KB} \zeta(i))$, then $\zeta^*(n + 1) = \zeta(n + 1)$, and, if not, then $\zeta^*(n + 1) = \zeta^*(n) + (0)$. Note that $\forall n([n(\zeta(n + 1) \in \zeta_{KB} \zeta(n))$ and find $n$ such that $\zeta^*(n) \notin Y$. Either $\zeta(n) = \zeta^*(n) \notin Y$ or $\exists i < [\neg(\zeta(i + 1) \in \zeta_{KB} \zeta(i))]$.

(ii) $\Rightarrow$ (i). Obvious.
Definition 20. For all \( s, n \), if \( n \geq \text{length}(s) \), then \( s(n) = 0 \).

We define \( \delta \) such that \( \delta(\langle \rangle) = \langle \rangle \) and, \( \forall s \forall n[\delta(s * \langle n \rangle) = \delta(s) * \langle n \rangle + (1)] \).

Note that, for all \( \gamma \in 2^{\omega} \), \( \exists \delta[\delta(s) \in \langle KB \rangle \land \forall t <_{KB} t \leftrightarrow (\delta(t) <_{\text{lex}} \delta(s) \lor \delta(t) \sqsubseteq \delta(s))] \). Note that, for all \( \gamma \in 2^{\omega} \), \( \exists \beta[\gamma = \delta[\beta] \text{ if and only if } \forall m \exists n > m[\delta(n) = 1] \).

Theorem 7.2. The following statements are equivalent in BIM:

(i) \( \overrightarrow{\text{Ded}}_{\omega} \).

(ii) For all \( \alpha \), if \( \text{Bar}_{\omega}(D_{\alpha}) \), then \( <_{KB} \text{ well-orders Below}(D_{\alpha}) \).

(iii) \( \text{Endec}! \).

Proof. (i) \( \Rightarrow \) (ii). Assume \( \overrightarrow{\text{Ded}}_{\omega} \).

Let \( \alpha, \zeta \) be such that \( \text{Bar}_{\omega}(D_{\alpha}) \) and \( \forall n[\zeta(n + 1) <_{KB} \zeta(n)] \). We shall prove that \( \exists m \exists m' \leq \text{length}(\zeta(n)) \langle \zeta(n) \rangle \in D_{\alpha} \).

Define \( \delta \) such that for \( n \), if \( \exists j \exists n \exists m \leq \text{length}(\zeta(n)) \langle \zeta(i) \rangle \in D_{\alpha} \), then \( \zeta(n) = \delta \circ \zeta(n) \text{, and, if } \exists j \exists n \exists m \leq \text{length}(\zeta(n)) \langle \zeta(i) \rangle \in D_{\alpha} \), then \( \zeta(n) = \langle n + 2 \rangle \).

Note that, for all \( n \), \( \zeta(n + 1) <_{\text{lex}} \zeta(n) \lor \zeta(n) \in \zeta(n + 1) \lor \zeta(n) <_{\text{lex}} \zeta(n + 1) \lor \zeta(n) \in \zeta(n + 1) \). We now first prove that \( \forall n \exists j[\zeta(n + j) <_{\text{lex}} \zeta(n + j + 1)] \).

Let \( n \) be defined. Define \( \beta \) such that \( \beta(0) = \zeta(n) \) and, for each \( j, \) if \( \forall j \left[ \zeta(n + i) <_{\text{lex}} \zeta(n + i + 1) \right] \) then \( \beta(j + 1) = \zeta(n + j + 1) \), and, if not, then \( \beta(j) = \beta(j) \lor (1) \). Note that, for all \( j, \) \( \beta(j) \sqsubseteq (j + 1) \) and \( \text{length} \beta(j) \geq j \).

Find \( \gamma \) in \( 2^{\omega} \) such that \( \forall j \left[ \gamma(j) \sqsubseteq \gamma \right] \). Note that, for each \( j, \) \( \text{length} \beta(j) > 0 \) and \( \left( \beta(j) \right) \left( \text{length} \beta(j) - 1 \right) = 1 \) and conclude that \( \forall j \exists j \left[ \gamma(j) = 1 \right] \). Find \( \zeta \) such that \( \delta \zeta = \gamma \). Find \( k \) such that \( \forall k \in D_{\alpha} \). Find \( q := \text{length} \left( \delta \zeta(k) \right) \). Assume that \( \beta(q) = \zeta(n + q) \). Then \( \forall k \in D_{\alpha} \) and \( \zeta(n + q) = \langle n + q + 2 \rangle \).

Contradiction. Conclude that \( \zeta(n + j) \sqsubseteq \zeta(n + j + 1) \) and \( \zeta(n + j) <_{\text{lex}} \zeta(n + j + 1) \).

Now define \( \theta \) such that \( \theta(0) = \mu j \left[ \zeta(n) <_{\text{lex}} \zeta(n + 1) \right] \) and \( \forall j \left[ \theta(i + 1) = \mu j \left[ j > \theta(i) \land \zeta(i) <_{\text{lex}} \zeta(j + 1) \right] \right] \). Define \( \zeta := \zeta \circ \theta \) and note that \( \forall j \left[ \zeta \circ \theta \right] \left( \text{length} \zeta \circ \theta - 1 \right) = 1 \).

Let \( n \) in \( 2^{\omega} \) be defined. Define \( \gamma \) in \( 2^{\omega} \) such that, for each \( j, \) if \( \forall j \leq j \left[ \zeta \circ \theta \left( \zeta(i + 1) \right) \right] \), then \( \gamma(j) = \left( \zeta \circ \theta \left( \zeta(i + 1) \right) \right) \). Let \( \gamma(n) = \gamma(i) \).

We thus see that \( \forall m \exists n > m[\delta(n) = 1] \).

Find \( \delta \) such that \( \forall n \exists \delta[\delta(n) \sqsubseteq \gamma] \). Find \( n \) such that \( \exists n \in D_{\alpha} \). Define \( q := \text{length} \left( \delta \zeta(n) \right) \). Assume that \( \forall i \left[ \zeta \circ \theta \left( \zeta(i + 1) \right) \right] \). Then \( \gamma = \delta \zeta(n) \sqsubseteq \zeta \circ \theta \circ \eta \left( \zeta(q) \right) \).

Contradiction. Conclude that \( \exists i \left[ \zeta \circ \theta \left( \zeta(i + 1) \right) \right] \). We thus see that
\( \forall \eta \in [\omega]^{\omega} \exists \exists [\zeta \circ \eta(i) \neq \zeta \circ \eta(i+1)] \). Using \( \mathbb{Ded}_\omega \), we conclude that \( \exists n[\zeta(n) \notin 2^\omega] \) and \( \exists \exists m \leq \text{length}(\zeta(n)) \{\zeta(m) \in D_\alpha\} \).

We thus see that, for all \( \alpha, \zeta \), if \( \text{Bar}_\omega(D_\alpha) \) and \( \forall n[\zeta(n+1) <_{KB} \zeta(n)] \), then \( \exists n \in [\zeta(n) \notin \text{Below}(D_\alpha)] \) i.e. \( <_{KB} \) well-orders \( \text{Below}(D_\alpha) \).

(ii) \( \Rightarrow \) (iii). Assume: for all \( \alpha \), if \( \text{Bar}_\omega(D_\alpha) \), then \( <_{KB} \) well-orders \( \text{Below}(D_\alpha) \).

Let \( \gamma, n \) be given such that \( \forall \alpha \in 2^\omega \{n \notin D_\alpha \land D_\alpha \subseteq E_\gamma \} \rightarrow \exists p[p \in E_\gamma \setminus D_\alpha] \).

We shall prove that \( n \in E_\gamma \), i.e. \( \exists p[\gamma(p) = n + 1] \).

Note that \( \forall m \exists \alpha[E_\alpha = D_\alpha] \). It follows that \( \forall m \exists \eta \eta > m [E_\eta \subseteq E_\eta \cap n \in E_\eta] \).

Define \( \eta \) such that \( \eta(0) = 0 \) and \( \forall n[\eta(m+1) = \mu > \eta(m)][E_{\eta(m+1)} \subseteq E_{\eta} \cap n \in E_{\eta}] \).

Note that \( \forall n[n \notin E_{\eta(m+1)}] \rightarrow \exists s[s(0) \in \omega^m \forall p[p \in E_{\eta(m+1)}] \Rightarrow \exists i < m] \).

Define \( \zeta \) such that \( \zeta(0) = (\gamma) \) and, for each \( m \), either:

1. \( n \notin E_{\eta(m+1)} \) and \( \zeta(m+1) = \zeta(m) \ast (n) \), or:
2. \( n \notin E_{\eta(m+1)} \) and \( \zeta(m+1) = \mu \in \omega^m + 1 \forall p[p \in E_{\eta(m+1)}] \Rightarrow \exists j < m\gamma(s(j) = p)] \).

Note that, for all \( m, \zeta(m) \in \omega^m \) and \( \forall i < m[\zeta(m)(i) \in E_\gamma] \) and \( \zeta(m+1) <_{KB} \zeta(m) \).

Define \( \lambda \) such that, for all \( m, \lambda(m) \in \omega^m \) and \( \forall i < m[\lambda(m)(i) \in E_\gamma] \) and \( \zeta(m+1) <_{KB} \lambda(m) \).

For all \( k, s \) such that \( s \in \omega^{2k} \), we define: \( s \) is fine if and only if

1. \( \forall i < k[\gamma(s(2i + 1)) = s(2i) + 1] \) and
2. \( \forall i[i + 1 < k \rightarrow (s(2i) < s(2i + 2) \lor s(2i + 2) = n)] \) and
3. \( \forall i < k[\gamma(s(2i)) = n \rightarrow s(2i + 2) = n] \).

Note that, if \( s \in \omega^{2k} \) is fine, then \( \forall i < k[s(2i) \in E_\gamma] \) and, if \( \forall i < k[s(2i) \notin n] \) then \( \forall i[i + 1 < k \rightarrow s(2i) < s(2i + 2)] \). Note that one may decide, given any \( s \) in \( \omega^{2k} \), if \( s \) is fine or not. Note that, if \( s \) is fine, then for every \( j < k \), \( s(2j) \) is fine. Note that, for every \( m, \lambda(m) \) is fine.

For all \( k, s \) such that \( s \in \omega^{2k} \), we define: \( s \) is forgetful if and only if \( s \) is fine and, for some \( j < k, \gamma(j) \) is forgetful or \( \gamma(j) \) is not fine and \( \gamma(j) \) is forgetful and \( \gamma(j) \) is not fine and \( \gamma(j) \) is forgetful.

If \( s \in \omega^{2k} \) is forgetful, then \( \exists \exists \mu < m[\gamma(i) = n + 1], \) then \( \lambda(m) \) is not forgetful.

Define \( B := \{s \in \bigcup_k \omega^{2k} \mid s \) is not fine \( \lor s \) is forgetful \}. \( B \) is a decidable subset of \( \omega \) as one may define \( \tau \) such that \( B = D_\tau \). We now prove that \( \text{Bar}_\omega(B) \).

Let \( \beta \) be given. Define \( \nu \) such that, for each \( m, \) if \( \beta(2m + 2) \) is fine, then \( \beta(2m + 2) = \beta(2m + 2) \), and, if \( \beta(2m + 2) \) is not fine, then \( \nu(2m + 1) = \mu[\gamma(q) > 0 \lor (\gamma(q) - 1 = n \land \nu(q) - 1 > \nu(2m - 2)] \) and \( \nu(2m) = \gamma(\nu(2m + 1) - 1) \).

Note that, for all \( m, \beta(2m) \) is fine and either \( \nu(2m) < \nu(2m + 2) \) or \( \nu(2m + 2) = n \).

Define \( \alpha \) such that, for each \( m, \alpha(m) \neq 0 \leftrightarrow (m \neq n \land \exists j < m \gamma[j = \nu(2j)] \).

Note that \( D_\alpha \subseteq E_\gamma \) and that \( n \notin D_\alpha \). Find \( p = \mu[\gamma(q') = q' + 1 \land q' \notin D_\alpha \) and distinguish two cases. Case (a). \( p'' = n \). Note that \( \beta(2p) \) is forgetful. Case (b). \( p'' \neq n \). Find \( i \) such that \( \nu(p'' < \nu(0) \lor \nu(2i) < p'' \lor \nu(2i + 2) \). Define \( m := \max(i + 1, p'' + 1) \). Note that \( \beta(2m) \) is forgetful. In both cases we thus find \( m \) such that \( \beta(2m) \) is forgetful. Either \( \beta(2m) \neq \beta(2m) \) and \( \beta(2m) \) is not fine, or \( \beta(2m) = \beta(2m) \) is forgetful. Conclude that \( \beta(2m) \in B \). We thus see that, for each \( \beta \), there exists \( m \) such that \( \beta(2m) \in B, \) i.e. \( \text{Bar}_\omega(B) \).

Using the fact that \( <_{KB} \) well-orders \( \text{Below}(B) \), find \( m, p \) such that \( \lambda(m)p \in B \).

As \( \lambda(m) \) is fine, \( \lambda(m)p \) is forgetful and also \( \lambda(m) \) itself is forgetful. Conclude that \( \exists j < m[\gamma(j) = n + 1] \) and \( n \in E_\gamma \).
We thus see that, for each $\eta$, for each $\gamma$, if $\forall \alpha \in 2^\omega [n \notin D_\alpha \land D_\alpha \subseteq E_\gamma] \rightarrow \exists p [p \in E_\gamma \land p \notin D_\alpha]$, then $n \in E_\gamma$. We conclude that, for each $\gamma$, if $\forall \alpha \in 2^\omega [D_\alpha \subseteq E_\gamma \land \exists n [n \notin D_\alpha]] \rightarrow \exists p [p \notin D_\alpha \land p \in E_\gamma]$, then $E_\gamma = \omega$, i.e. $\text{EnDec}!!$.

(iii) $\Rightarrow$ (i). See Corollary 6.6.

\[ \text{Theorem 7.2 is a counterpart to [17] Lemma V.1.3 and Exercise V.1.11.} \]

In Definition 3 we defined:

\begin{align*}
B \subseteq \omega & \text{ is thin if and only if } \forall p \in B \forall q \in B [p \neq q \rightarrow p \perp q]. \\
B \subseteq \omega & \text{ is a thin bar in } F \subseteq \omega^\omega; \text{ notation } \text{Thinbar}_F(B) \text{ if and only if } B \text{ is thin and } \text{Bar}_F.
\end{align*}

**Corollary 7.3. The following are equivalent in BIM**

(i) $\text{OI}(2^\omega)$.

(ii) For all $\alpha$, if $\text{Bar}_\omega(D_\alpha)$, then $<_{KB}$ well-orders $\text{Below}(D_\alpha)$.

(iii) For all $\alpha$, if $\text{Thinbar}_\omega(D_\alpha)$, then $<_{KB}$ well-orders $D_\alpha$.

**Proof.** (i)$\Leftrightarrow$(ii). See Theorem 7.2 and Corollary 6.6.

(ii)$\Rightarrow$(iii). Assume (ii). Let $\alpha$ be given such that $\text{Thinbar}(D_\alpha)$. Define $\beta$ such that $\beta(\langle \rangle) = 0$ and $\forall \forall n [\beta(s \upharpoonright n) \neq 0 \leftrightarrow \alpha(s) \neq 0]$. Note that $\text{Bar}_\omega(D_\beta)$ and $D_\alpha \subseteq \text{Below}(D_\beta)$. As $<_{KB}$ well-orders $\text{Below}(D_\beta)$, $<_{KB}$ also well-orders $D_\alpha$.

Note that, as $D_\alpha$ is thin, the relations $<_{KB}$ and $<_{lex}$ coincide on $D_\alpha$. Conclude that (iii) holds.

(iii)$\Rightarrow$(ii). Assume (iii). Let $\alpha$ be given such that $\text{Bar}_\omega(D_\alpha)$. Define $\beta$ such that $\forall s [\beta(s) \neq 0 \rightarrow (s \in D_\alpha \land \forall i \in s(t \notin D_\alpha)])$. Note that $\text{Thinbar}_\omega(D_\alpha)$. Let $\zeta$ be given. We have to prove $QED := \exists n [\neg [\zeta(n+1) <_{KB} \zeta(n)] \lor \zeta(n) \notin \text{Below}(D_\alpha)]$. We claim that $\forall i j \geq i [\zeta(j+1) <_{lex} \zeta(j) \lor QED]$. We prove this claim as follows.

Let $i$ be given. Define $\zeta^*$ such that $\zeta^*(0) = \zeta(i)$ and, for each $j$, if $\forall i j \leq i [\zeta^*(i) \subseteq \zeta(i+1)]$, then $\zeta^*(j+1) = \zeta(j+1)$, and, if not, then $\zeta^*(j+1) = \zeta^* (j) \lor 0$. Note that, for all $j$, $\zeta^*(j) \subseteq \zeta^*(j+1)$, and find $\gamma$ such that $\forall n [\zeta^*(n) \subseteq \gamma]$. Then find $p$ such that $\exists p \in D_\alpha$ and $n$ such that $\exists p \subseteq \gamma^*(n)$. Note: $\zeta^*(n) \notin \text{Below}(D_\alpha)$.

If $\zeta^*(n) = \zeta(n)$, then $QED$, and, if not, $\exists k < n [\zeta(i+k+1) <_{lex} \zeta(i+k)]$. Conclude that $\forall i j \geq i [\zeta(j+1) <_{lex} \zeta(j) \lor QED]$. Find $\eta$ in $[\omega]^{\omega}$ such that $\forall n [\zeta \circ \eta(n+1) <_{lex} \zeta \circ \eta(n) \lor QED]$. Find $s$ such that $s \notin D_\beta$. Now define $\rho$ such that, for each $n$, if $\forall i \leq n [\zeta \circ \eta(i) \in \text{Below}(D_\beta)]$, then $\zeta \circ \eta(n) \in \rho(n)$ and $\rho(n) \in D_\beta$, and, if not, then $\rho(n) = s_0$. Using (iii), find $n$ such that either $\neg (\rho(n+1) <_{lex} \rho(n))$ or $\rho(n) \notin D_\beta$. Conclude that either $\neg (\zeta \circ \eta(n+1) <_{lex} \zeta \circ \eta(n))$ and $\exists n [\neg (\zeta(n+1) <_{lex} \zeta(n))]$ or $\exists i \leq n + 1 [\zeta \circ \eta(i) \notin \text{Below}(D_\alpha)]$, so, in any case, $QED$.

One may introduce some of the usual countable ordinals as elements $\beta$ of $\omega^\omega$ with the property: $D_\beta$ is a thin bar in $\omega^\omega$. We sketch how this could be done.

**Definition 21.** Define $\varphi : 2^\omega \rightarrow 2^\omega$ such that, for every $\alpha$, $(\varphi(\alpha))(\langle \rangle) = 0$, and, for every $n$, for every $t$, $(\varphi(\alpha))(\langle n \upharpoonright t \rangle) = 0$ if and only if $\exists s [\text{length}(s) = n \land \forall i < n [\alpha(s(i)) \neq 0] ] \land t = s(0) \ast s(1) \ast \ldots \ast s(n-1)$.

\[ \omega \text{ is the element of } 2^\omega \text{ satisfying } \forall s [\omega(s) = 1 \leftrightarrow \exists n [s = \langle n \rangle]]. \]

\[ \bar{\omega} \text{ is the element of } 2^\omega \text{ such that } \bar{\omega}(\langle \rangle) = 0 \text{ and } (\bar{\omega})^0 = \omega \text{ and } \forall n [(\bar{\omega})^n = \varphi(\bar{\omega})^n]]. \]

**Theorem 7.4. BIM proves the following.**

(i) $\forall \alpha \in \text{Thinbar}_\omega(D_\alpha) \rightarrow \text{Thinbar}_\omega(D_{\varphi(\alpha)})$.

(ii) $\text{Thinbar}(D_\omega) \land \text{Thinbar}(D_{\bar{\omega}})$.

(iii) $\text{OI}(2^\omega) \rightarrow <_{lex}$ well-orders $D_{\bar{\omega}}$.

**Proof.** The proof of (i) and (ii) is left to the reader. For (iii), use Theorem 7.2.
Theorem 7.4 is a sharpening of a result obtained by W. Howard and G. Kreisel, see [9], Appendix 2, to the effect: $\text{BIM} \vdash \text{BI} \rightarrow \ltex$ well-orders $D_{\omega}$. Using the fact: $\ltex$ well-orders $D_{\omega}$, G. Gentzen was able to prove the consistency of arithmetic, that is, the consistency of BIM. A. S. Troelstra, see [12], proved, making use of techniques developed for eliminating choice sequences, that the Fan Theorem FT is conservative over Heyting Arithmetic, that is, over BIM. Using Gödel’s Second Incompleteness Theorem we obtain:

**Corollary 7.5.** $\text{BIM} \not\vdash \ltex$ well-orders $D_{\omega}$ and $\text{BIM} + \text{FT} \not\vdash (2\omega)$.

8. The Almost-Fan Theorem and the Approximate-Fan Theorem

We need a statement stronger than FT, that plays, in the intuitionistic context of BIM, a role comparable to the rôle fulfilled by KL in the classical context of RCA0. FT probably is not a good candidate, see Subsection 5.5.

8.1. Notions of finiteness.

**Definition 22.** For each $n$, $\omega^n := \{s \mid \text{length}(s) = n\}$ and $[\omega]^n := \{s \in \omega^n \mid \forall i (i + 1 < n \rightarrow s(i) < s(i + 1))\}$.

For each $\alpha$, $D_{\alpha} := \{n \in \omega \mid \alpha(n) \neq 0\}$. $D_{\alpha}$ is finite if and only if $\exists n \forall m \geq n (\alpha(m) = 0)$. For each $n$, $D_{\alpha}$ has at most $n$ members if and only if $\forall s \in [\omega]^{n+1} \exists i \leq n (s(i) \notin D_{\alpha})$. $D_{\alpha}$ is bounded-in-number if and only if there exists $n$ such that $D_{\alpha}$ has at most $n$ members. $D_{\alpha}$ is almost-finite if and only if $\forall \zeta \in [\omega]^{\omega} \forall n ([\zeta(n) \notin D_{\alpha}])$.

Obviously, for each $\alpha$, if $D_{\alpha}$ is finite, then $D_{\alpha}$ is bounded-in-number, and, if $D_{\alpha}$ is bounded-in-number, then $D_{\alpha}$ is almost-finite. The converse statements are not true, see [21]. Almost-finite subsets of $\omega$ are also studied in [22], [27] and [34].

$B \subseteq \omega$ is called a decidable subset of $\omega$ if and only if $\exists \alpha [B = D_{\alpha}]$.

**Lemma 8.1.** One may prove in BIM:

(i) For all decidable subsets $A, B$ of $\omega$, if $A \subseteq B$ and $B$ is almost-finite, then also $A$ is almost-finite.

(ii) For all decidable subsets $A, B$ of $\omega$, if $A, B$ are almost-finite, then $A \cup B$ is almost-finite.

(iii) For every $k$, for all decidable subsets $B_0, B_1, \ldots, B_k$ of $\omega$, if, for each $n \leq k$, $B_n$ is almost-finite, then $\bigcup_{n \leq k} B_n$ is almost-finite.

(iv) For every infinite sequence $A, B_0, B_1, \ldots$ of decidable subsets of $\omega$, if $A = \bigcup_{n \in \omega} B_n$ and, for each $n$, $B_n$ is almost-finite, and $\forall \zeta \in [\omega]^{\omega} \exists n [B_n = \emptyset]$, then $A$ is almost-finite.

**Proof.** (i) The proof is left to the reader.

(ii) Let $A, B$ be given decidable and almost-finite subsets of $\omega$. Assume $\zeta \in [\omega]^{\omega}$. Define $\eta$ as follows, by induction. $\eta(0) := \mu p [\zeta(p) \notin A]$ and, for each $n$, $\eta(n+1) := \mu p [p > \eta(n) \land \zeta(p) \notin A]$. Note that $\zeta \in [\omega]^{\omega}$ and $\forall n [\zeta \circ \eta(n) \notin A]$. Find $q$ such that $\zeta \circ \eta(q) \notin B$. Define $q := \eta(q)$ and note that $\zeta(q) \notin A \cup B$. We thus see that $\forall \zeta \in [\omega]^{\omega} \exists q [\zeta(q) \notin A \cup B]$, i.e. $A \cup B$ is almost-finite.

(iii) Use (ii) and induction.

(iv) Let $A, B_0, B_1, \ldots$ be an infinite sequence of decidable subsets of $\omega$ such that, for each $n$, $B_n$ is almost-finite and $A = \bigcup_{n \in \omega} B_n$ and $\forall \zeta \in [\omega]^{\omega} \exists n [B_n = \emptyset]$. Assume $\zeta \in [\omega]^{\omega}$. Using (ii), define $\eta$ as follows, by induction. $\eta(0) := \mu p [p \notin B_0]$, and, for each $n$, $\eta(n+1) := \mu p [p > \eta(n) \land \forall i \leq n+1 [\zeta(p) \notin B_i]]$. Note that $\zeta \circ \eta \in [\omega]^{\omega}$ and $\forall n \forall i \leq n \forall j \ni [\zeta \circ \eta(j) \notin B_i]$. Define $\varepsilon$ as follows. For each $n$, if $\zeta \circ \eta(n) \notin A$,  16See Subsection 13.3.
\(\varepsilon(n) = n + 1\), and, if \(\zeta \circ \eta(n) \in A\), then \(\varepsilon(n) := \mu p[\zeta \circ \eta(n) \in B_p]\). Note that \(\forall n[\varepsilon(n) > n \land (\zeta \circ \eta(n) \in A \rightarrow \zeta \circ \eta(n) \in B_{\varepsilon(n)})]\). Define \(\lambda\) as follows. \(\lambda(0) = \varepsilon(0)\) and, for each \(n\), \(\lambda(n + 1) := \mu p[\varepsilon(p) > \varepsilon \circ \lambda(n)]\). Note that \(\varepsilon \circ \lambda \in [\omega]^{\omega}\). Find \(n\) such that \(B_{\varepsilon \circ \lambda(n)} = \emptyset\). Define \(q := \eta \circ \lambda(n)\) and note that \(\zeta(q) \notin A\). We thus see that \(\forall \alpha \in [\omega]^{\omega} q[\zeta(q) \notin A]\), i.e. \(A\) is almost-finite.

\[\square\]

8.2. Fans, approximate fans and almost-fans.

Definition 23. \(\beta \in \omega^\omega\) is an approximate-fan-law, notation: \(\text{Appfan}(\beta)\), if and only if \(\text{Spr}(\beta)\) and, for each \(n\), the set \(\{t \in \omega^n \mid \beta(t) = 0\}\) is bounded-in-number.

\(\beta\) is an explicit approximate-fan-law, notation: \(\text{Appfan}^+(\beta)\), if and only if \(\text{Spr}(\beta)\) and there exists \(\gamma\) such that for each \(n\), the set \(\{t \in \omega^n \mid \beta(t) = 0\}\) has at most \(\gamma(n)\) members.

\(\beta\) is an almost-fan-law, notation: \(\text{Almfan}(\beta)\), if and only if \(\text{Spr}(\beta)\) and, for each \(s\), the set \(\{n \mid \beta(s \cdot \langle n\rangle) = 0\}\) is almost-finite.

\(\mathcal{F} \subseteq \omega^\omega\) is an (explicit) approximate fan/almost-fan if and only if there exists an (explicit) approximate-fan-law/almost-fan-law \(\beta\) such that \(\mathcal{F} = \mathcal{F}_\beta\).

Weak-\(\Pi^1_1\)-\(\text{AC}_{0,0}\), see Subsection 2.2.3, does not seem to be sufficient for proving that every approximate-fan-law is an explicit approximate-fan-law, but \(\Pi^1_1\)-\(\text{AC}_{0,0}\) clearly is sufficient.

Note that every fan is an approximate fan, but that, conversely, it is not true that every approximate fan is a fan.

Theorem 8.2. A spread-law \(\beta\) is an almost-fan-law if and only if, for each \(n\), the set \(\{t \in \omega^n \mid \beta(t) = 0\}\) is almost-finite.

Proof. First, let \(\beta\) be an almost-fan-law. We prove by induction that, for each \(n\), the set \(\{t \in \omega^n \mid \beta(t) = 0\}\) is almost-finite. Obviously, the set \(\{t \in \omega^n \mid \beta(t) = 0\}\) either is empty or coincides with \(\{\langle\rangle\}\) and so is almost-finite. Now let \(n\) be given such that the set \(\{t \in \omega^n \mid \beta(t) = 0\}\) is almost-finite. Note that, for each \(t\) in \(\omega^n\) such that \(\beta(t) = 0\), the set \(\{s \in \omega^{n+1} \mid \beta(s) = 0 \land t \sqsubseteq s\}\) is almost-finite, like the set \(\{n \mid \beta(t \cdot \langle n\rangle) = 0\}\). Note that the set \(\{s \in \omega^{n+1} \mid \beta(s) = 0\}\) is an (explicit) approximate fan/almost-fan if and only if there exists an (explicit) approximate-fan-law/almost-fan-law \(\beta\) such that \(\mathcal{F} = \mathcal{F}_\beta\).

As to the converse, let \(\beta\) be a spread-law such that, for each \(n\), the set \(\{t \in \omega^n \mid \beta(t) = 0\}\) is almost-finite. Let \(t, n\) be given such that \(t \in \omega^n\) and \(\beta(t) = 0\). Note that the set \(\{s \in \omega^{n+1} \mid \beta(s) = 0 \land t \sqsubseteq s\}\) is a subset of the set \(\{s \in \omega^{n+1} \mid \beta(s) = 0\}\) and thus, by Lemma 5.7(iv), almost-finite. Conclude that the set \(\{k \mid \beta(t \cdot k) = 0\}\) is also almost-finite.

One may ask if every spread-law \(\beta\) satisfying the condition that, for each \(s\), the set \(\{n \mid \beta(s \cdot \langle n\rangle) = 0\}\) is bounded-in-number, is an approximate-fan-law. The answer is no: assuming this statement we prove LPO.

Let \(\alpha\) be given. Define \(\beta\) be such that, for each \(s\), \(\beta(s) = 0\) if and only if

(i) if \(\text{length}(s) > 0\), then \(s(0) = 0\) or \(s(0) = \mu n[\alpha(n) \neq 0]\), and

(ii) if \(\text{length}(s) > 1\) then \(s(0) \leq s(1) \leq 2 \cdot s(0)\), and

(iii) if \(\text{length}(s) > 2\), then \(\forall j > 1[j < \text{length}(s) \rightarrow s(j) = 0]\).

Note that \(\beta\) is a spread-law, and that, for each \(s\), if \(\beta(s) = 0\), then either \(s = \langle\rangle\) and the set \(\{n \mid \beta(s \cdot \langle n\rangle) = 0\}\) has at most 2 members or \(\text{length}(s) = 1\) and the set \(\{n \mid \beta(s \cdot \langle n\rangle) = 0\}\) has \(s(0) + 1\) members or \(\text{length}(s) > 1\) and the set \(\{n \mid \beta(s \cdot \langle n\rangle) = 0\}\) has 1 member. By our assumption, \(\beta\) is an approximate-fan-law. Find \(k\) such that the set \(\{s \in \omega^2 \mid \beta(s) = 0\}\) has at most \(k\) members. Conclude that, for each \(m\), if \(\beta(\langle m\rangle) = 0\), then \(m < k\). Conclude that, for all
If \( m = \mu [\alpha(n) \neq 0] \), then \( m < k \). Conclude that \( \exists n [\alpha(n) \neq 0] \) if and only if \( \exists n < k [\alpha(n) \neq 0] \). Conclude that \( \exists n [\alpha(n) \neq 0] \lor \forall n [\alpha(n) = 0] \).

We thus see that, for each \( \alpha \), \( \exists n [\alpha(n) \neq 0] \lor \forall n [\alpha(n) = 0] \), i.e. LPO.

Note that every approximate fan is an almost-fan, but that, conversely, it is not true that every almost-fan is an approximate fan.

**Definition 24.** The following statement is called the Almost-fan Theorem:

**AlmFT**: \( \forall \beta \forall \alpha [(\text{Almfan}(\beta) \land \text{Thinbar}_{F_\beta}(D_\alpha) \land \forall s [s \in D_\alpha \rightarrow \beta(s) = 0]) \rightarrow \forall \zeta \in [\omega]^{\omega} \exists n [\zeta(n) \notin D_\alpha]] \),

i.e. in an almost-fan, every thin bar is almost-finite.

We also introduce the Approximate-fan Theorem:

**AppFT**: \( \forall \beta \forall \alpha [(\text{Appfan}^+(\beta) \land \text{Thinbar}_{F_\beta}(D_\alpha) \land \forall s [s \in D_\alpha \rightarrow \beta(s) = 0]) \rightarrow \forall \zeta \in [\omega]^{\omega} \exists n [\zeta(n) \notin D_\alpha]] \),

i.e. in an explicit approximate fan, every thin bar is almost-finite.

Note that \( \text{BIM} \vdash \text{AlmFT} \rightarrow \text{AppFT} \). One may also see that \( \text{AppFT} \) extends \( \text{FT} \), see Theorem 3.3 (ii).

**AlmFT** has been introduced in [24] and [25]. It was shown in these papers that \( \text{AlmFT} \) is a consequence of Brouwer’s Thesis on Bars in \( \omega^\omega \) and that \( \text{AlmFT} \) implies \( \text{OI}(0, 1) \). As we shall see, also \( \text{AppFT} \) implies \( \text{OI}(0, 1) \). \( \text{AppFT} \) seems to be a little bit weaker than \( \text{AlmFT} \).

### 8.3. Proving AlmFT.

**Definition 25.** For each \( \beta \), \( \text{CA}_\beta := \{s \mid \forall \gamma \exists n [\beta(s, \gamma n) \neq 0]\} \). \( X \subseteq \omega \) is co-analytic if and only if \( \exists \beta [X = \text{CA}_\beta] \).

The Principle of Induction on co-analytic bars in Baire space is the following statement:

**\( \Pi_1^1 \text{-BI} \)**: \( \forall \beta[(\text{Bar}_{\omega^\omega}(\text{CA}_\beta) \land \forall s [s \in \text{CA}_\beta \leftrightarrow \forall i [s \ast (i) \in \text{CA}_\beta]]) \rightarrow 0 \in \text{CA}_\beta] \).

**Theorem 8.3.** \( \text{BIM} \vdash \Pi_1^1 \text{-BI} \rightarrow \Sigma^0_1 \text{-BI} \).

**Proof.** Note that every enumerable subset of \( \omega \) is co-analytic as \( \forall \beta \exists \delta [E_\delta = \text{CA}_\delta] \).

One proves the latter fact as follows. Given \( \beta \), define \( \delta \) such that, for each \( a \), for each \( s \), \( \delta(s, a) \neq 0 \) if and only if \( \gamma(\text{length}(a)) = s + 1 \).

**Theorem 8.4.** \( \text{BIM} \vdash \Pi_1^1 \text{-BI} \rightarrow \text{AlmFT} \).

**Proof.** Let \( \beta, \alpha \) be given such that \( \text{Almfan}(\beta) \) and \( \text{Thinbar}_{F_\beta}(D_\alpha) \) and \( \forall s [s \in D_\alpha \rightarrow \beta(s) = 0] \).

Define \( B := \{s \mid \forall \zeta \in [\omega]^{\omega} \exists n [s \subseteq \zeta(n) \rightarrow \zeta(n) \notin D_\alpha]\} \), i.e. \( B \) is the set of all \( s \) such that the set \( \{t \mid s \subseteq t \land t \in D_\alpha\} \) is almost-finite. Note that \( B \) is a co-analytic subset of \( \omega^\omega \).

We claim that \( B \) is a bar in \( \omega^\omega \). For let \( \gamma \) be given. Define \( \gamma^* \) such that, for each \( n \), if \( \beta(\gamma^* n \ast \langle \gamma(n) \rangle) = 0 \), then \( \gamma^*(n) = \gamma(n) \), and, if not, then \( \gamma^*(n) = m \beta(\gamma^* n \ast \langle p \rangle) = 0 \). Note that \( \gamma^* \in F_\beta \). Find \( m \) such that \( \gamma^m \in D_\alpha \).

Either \( \gamma^m \in \text{D}_\alpha \) and, as \( D_\alpha \) is thin, \( \{t \mid \gamma^m \subseteq t \land t \in D_\alpha\} = \{\gamma^m\} \) is almost-finite, or \( \gamma^m \neq \gamma^m \) and \( t \mid \gamma^m \subseteq t \land t \in D_\alpha\} = \emptyset \) is almost-finite. In any case, \( \gamma^m \in B \).

We now prove that \( B \) is inductive. Let \( s \) be given such that, for each \( n \), \( s(n) \in B \).

Define \( A := \{t \mid s \subseteq t \cup t \in D_\alpha\} \), and, for each \( n \), \( B_n := \{t \mid s(n) \subseteq t \land t \in D_\alpha\} \).

First assume that \( \exists t \subseteq s \in D_\alpha \). Then \( A \) has at most one element. Next, assume that \( \neg \exists t \subseteq s \in D_\alpha \). Then \( A = \bigcup_{n \in \omega} B_n \), and \( \forall \zeta \in [\omega]^{\omega} \exists n [\beta(s \ast \langle \zeta(n) \rangle) \neq 0] \).

Note that \( \forall n [\beta(s \ast (n)) \neq 0 \rightarrow B_n \neq 0] \), and, for each \( n \), \( B_n \) is almost-finite. One may conclude, using Lemma 3.1 (iv), that \( A \) is almost-finite. In any case, \( A \) is almost-finite, i.e. \( s \in B \).
Note that $B$ is also monotone. Applying $\Pi^1_1$-BI, conclude that $\langle \rangle \in B$ i.e. $D_\alpha$ is almost-finite.

9. The Semi-approximate-fan Theorem and the contraposition of the Bolzano-Weierstrass Theorem

9.1. The Semi-approximate-fan Theorem.

Definition 26. Let $\gamma, n$ be given. The set $E_n = \{m \mid \exists p[\gamma(p) = m + 1]\}$ has at most $n$ members if and only if $\forall s \in [\omega]^{n+1}\exists i \leq n\exists j \leq n[\gamma \circ s(i) = 0 \vee (i < j \wedge \gamma \circ s(i) = \gamma \circ s(j))]$. The set $E_n$ is bounded in number if and only if, for some $n$, the set $E_n$ has at most $n$ members.

Definition 27. For every $\delta$, $SF_{\delta} := \{\gamma \mid \forall n[\exists n \in E_\delta]\} = \{\gamma \mid \forall n\exists p[\delta(p) = \gamma + n + 1]\}$. $\delta \in \omega^\omega$ is a semi-spreadlaw, notation: $\text{Semi}appfan(\delta)$, if and only if $\forall s[\exists \delta(s) \in E_\delta \leftrightarrow \exists s \langle \gamma \rangle \in E_\delta]$.

$\delta \in \omega^\omega$ is a semi-approximate-fan-law, notation: $\text{Semi}appfan^+(\delta)$, if and only if $\exists n[\exists \gamma \langle \gamma \rangle \in E_\delta]$ has at most $n$ members. $\delta \subseteq \omega^\omega$ is an (explicit) semi-approximate fan if and only if there exists an (explicit) semi-approximate-fan-law $\delta$ such that $F = SF_{\delta}$.

Semi-spreads are called closed-and-semilocated subsets of $\omega^\omega$ in [23]. It is shown in [21] that an inhabited subset $F$ of $\omega^\omega$ is closed-and-semilocated if and only if $F$ is closed-and-separable, i.e. $F$ is closed in $\omega^\omega$ and $\exists \gamma[\gamma F \leftrightarrow \forall n \exists m[\exists n = \alpha^m n]]$.

Using $\Pi^1_1$-AC$_0$ [18] one may prove that every semi-approximate-fan-law is an explicit semi-approximate-fan-law.

Definition 28. The following statement is the semi-approximate-fan theorem:

$\text{SemiappFT} : \forall \delta \forall \alpha[\langle \text{Semi}appfan^+ (\delta) \rangle \wedge \text{Thinbar}_{SF_{\delta}}(D_\alpha) \wedge D_\alpha \subseteq E_\delta] \rightarrow \forall \zeta \in [\omega]^\omega \exists \eta[\zeta(n) \notin D_\alpha]$, i.e. in an explicit semi-approximate-fan, every decidable thin bar is almost-finite.

Theorem 9.1. $\text{BIM} \vdash \text{AppFT} \rightarrow \text{SemiappFT}$.

Proof. Let $\delta, \alpha$ be given such that $\text{Semi}appfan(\delta)$ and $\text{Thinbar}_{SF_{\delta}}(D_\alpha) \subseteq E_\delta$.

17 $X \subseteq \omega$ is enumerable if and only if $\exists \gamma[X = E_\gamma]$.

$\alpha$ subset $F$ of $\omega^\omega$ is closed if and only if, for some $\beta$, $F = \{a \mid \forall n[\beta(\gamma(n)) = 0]\}$, see Subsection 9.3.

19 See Subsubsection 9.3.
each $n$, the set $\{t \in \omega^n \mid \beta(s) = 0\}$ has at most $\varepsilon(n)$ elements. Conclude that $\beta$ is an explicit approximate-fan-law. Let $\theta$ be an element of $\mathcal{F}_\beta$. Define $\nu$ such that, for each $n$, $\mathfrak{n}_i = \eta(\mathfrak{n}_i)$. Note that $\nu$ belongs to $\mathcal{S}\mathcal{F}_\beta$ and find $n$ such that $\mathfrak{n}_n \in D_\alpha$. Conclude that $\mathfrak{n}_n \in D_{\alpha_0q}$. We thus see that $\text{Bar}_{\mathcal{F}_\beta}(D_{\alpha_0q})$. Note that $D_{\alpha_0q}$ is thin, and, using $\text{AppFT}$, conclude that $D_{\alpha_0q}$ is almost-finite. We now prove that also $D_\alpha$ itself is almost-finite. Let $\zeta$ be an element of $[\omega]^\omega$. Define $\kappa$ such that, for each $n$, either $\zeta(n) \in D_\alpha$ and $\kappa(n)$ is the least $s$ such that $\beta(s) = 0$ and $\eta(s) = \zeta(n)$, (and therefore: $\eta \circ \kappa(n) = \zeta(n)$), or $\zeta(n) \notin D_\alpha$ and $\kappa(n)$ is the least $s$ such that $\beta(s) \neq 0$ and $\forall i < n[\kappa(i) < s]$. Note that $\kappa$ is well-defined, as there are infinitely many numbers $s$ such that $\beta(s) \neq 0$, and note that $\kappa$ is one-to-one, as $\eta$ is a one-to-one function from the set $\{t \mid \beta(t) = 0\}$ to $E_\delta$. Define $\rho$ such that $\rho(0) = \kappa(0)$ and, for each $n$, $\rho(n+1)$ is the least $j > \rho(n)$ such that $\kappa(\rho(n)) < \kappa(j)$. Note that $\kappa \circ \rho \in [\omega]^\omega$ and find $n$ such that $\kappa \circ \rho(n) \notin D_{\alpha_0q}$. Conclude that $\eta \circ \kappa \circ \rho(n) \notin D_\alpha$. If $\zeta(\rho(n)) \in D_\alpha$, then $\eta \circ \kappa \circ \rho(n) = \zeta(\rho(n))$ and $\zeta(\rho(n)) \notin D_\alpha$. Conclude that $\zeta(\rho(n)) \notin D_\alpha$. Conclude that $\forall \zeta \in [\omega]^\omega \exists m[\zeta(m) \notin D_\alpha]$, i.e. $D_\alpha$ is almost-finite. □

The next Definition extends Definition 4

**Definition 29.** For every semi-spread-law $\delta$, for every $B \subseteq \omega$, $B$ is $s$ strong bar in $\mathcal{S}\mathcal{F}_\delta$, notation: Strongbar$_{\mathcal{S}\mathcal{F}_\delta}(B)$, if and only if $\forall \zeta \in [\omega]^\omega, \forall n[\zeta(n) \in E_\delta] \rightarrow \exists m[\zeta(m) \notin B]$. $\text{Lemma 9.2}$ and Corollary 9.3 extend Lemma 9.3

**Lemma 9.2.** BIM proves that, for every semi-spread-law $\delta$, for every $\alpha$, if $\text{Bar}_{\mathcal{S}\mathcal{F}_\delta}(D_\alpha)$ and $D_\alpha$ is almost-finite, then Strongbar$_{\mathcal{S}\mathcal{F}_\delta}(D_\alpha)$.

**Proof.** Let $\delta, \alpha$ be given such that $\text{Semispr}(\delta)$ and $\text{Bar}_{\mathcal{S}\mathcal{F}_\delta}(D_\alpha)$ and $D_\alpha$ is almost-finite. Assume that $\zeta \in [\omega]^\omega$ and $\forall n[\zeta(n) \in E_\delta]$. For each $n$, for each $s$ in $\omega^n \cap E_\delta$, define $s^+ \in \omega^{n+1} \cap E_\delta$, as follows. Given $n$, $s$, find $k_0 := \mu k[\delta(k) > 0 \land s \subset \delta(k) - 1]$ and define $s^+ := \delta(k_0) - 1(n+1)$. Define $\gamma$ such that, for each $n$, $\zeta(n) \subseteq \gamma^n$ and, for each $p \geq \text{length}(\zeta(n))$, $\gamma(n) = (\gamma(n))^\omega(p+1)$. Note that, for each $n$, $\zeta(n) \subseteq \gamma^n$ and $\gamma^n \in \mathcal{S}\mathcal{F}_\delta$. Define $\eta$ such that, for each $n$, $\eta(n) = \gamma^n_I$ where $I = \mu q[\gamma^n_I \in D_\alpha]$. Note that, for all $m, n$ either $\eta(n) = \gamma(n)$ or $\eta(m) \notin \eta(n)$. Note that, for each $n$, either $\zeta(n) \subseteq \eta(n)$ or $\eta(n) \subseteq \zeta(n)$. Define $QED := \exists m[\exists n[\zeta(n) \notin D_\alpha]]$ and note that $QED \leftrightarrow \forall n[\zeta(n) \subseteq \zeta(n)]$. Note that, for each $n$, the set $\{k \mid \zeta(k) \subseteq \eta(n)\}$ has at most $p := \text{length}(\eta(n))$ elements. We define $\rho$ in $[\omega]^\omega$. We define $\rho(0) = \eta(0)$ and, for each $n > 0$, we distinguish two possibilities.

(i) There exists $s$ in $[\omega]^n$ such that $\mathfrak{m} = \eta \circ s$ and $\forall i < n[\zeta \circ s(i) \subseteq \eta \circ s(i)]$. We calculate $N := \sum_{i < n} \text{length}(\rho(i))$ and $p := s(n-1)$ and determine $k < N$ such that, for all $i < n$, $\eta(i) \subseteq \eta(p+k+1)$, or $\eta(p+k+1) \subseteq \zeta(p+k+1)$ and define $\rho(n) = \eta(p+k+1)$.

(ii) If (i) does not apply, $n > 0$ and we define $\rho(n) = \rho(n-1) * \langle n \rangle$, where $k = \mu \rho(i) < n[\rho(i) < n - 1 \times \langle n \rangle]$.

Note that, for each $n$ either there exists $s$ in $[\omega]^n$ such that $\mathfrak{m} = \eta \circ s$ or $\exists i < n^3[\zeta(j) \subseteq \eta(j) = \rho(i)]$ and $QED$. Note that $\forall m \forall n[m < n \rightarrow \rho(n) \neq \rho(n)]$ and find $\theta$ in $[\omega]^\omega$ such that $\rho \circ \theta \in [\omega]^\omega$. Using the fact that $D_\alpha$ is almost-finite, find $n$ such that $\rho \circ \theta(n) \notin D_\alpha$. Find $p$ such that $\rho \circ \theta(n) \subseteq \zeta(p)$ and conclude that $\zeta(p) \subseteq \eta(p) \notin D_\alpha$. Conclude $QED$.

We thus see that $\forall \zeta \in [\omega]^\omega, \forall n[\zeta(n) \in E_\delta] \rightarrow \exists m[\exists n[\zeta(n) \notin D_\alpha]]$, i.e. Strongbar$_{\mathcal{S}\mathcal{F}_\delta}(D_\alpha)$. □

20QED: ‘quod est demonstrandum’, ‘what still has to be proven’.
Corollary 9.3. BIM proves that the following two statements are equivalent.

(i) **AppFT.**

(ii) For all \( \beta, \alpha \), if \( \text{Appfan}^+ (\beta) \) and \( \forall s [s \in D_\alpha \rightarrow \beta(s) = 0] \) and \( \text{Bar}_{F_\alpha} (D_\alpha) \), then \( \text{StrongBar}_{F_\alpha} (D_\alpha) \).

Proof. (i) \( \Rightarrow \) (ii) Let \( \beta, \alpha \) be given such that \( \text{Appfan}^+ (\beta) \) and \( \forall s [s \in D_\alpha \rightarrow \beta(s) = 0] \) and \( \text{Bar}_{F_\alpha} (D_\alpha) \). Using **AppFT**, conclude that \( D_\alpha \) is almost-finite. Note that there exists \( \delta \) such that \( F_\beta = F_\delta \). Using Lemma [22] conclude that \( \text{StrongBar}_{F_\delta} (D_\alpha) \).

(ii) \( \Rightarrow \) (i). Let \( \beta, \alpha \) be given such that \( \text{Appfan}^+ (\beta) \) and \( \forall s [s \in D_\alpha \rightarrow \beta(s) = 0] \) and \( \text{ThinBar}_{F_\alpha} (D_\alpha) \). Define \( \gamma \) such that for all \( t, \gamma (t) \neq 0 \) if and only if \( \exists s \exists n [t = s * (n) \land s \in D_\gamma \land \beta (s * (n)) = 0] \). Note that \( \text{Bar}_{F_\alpha} (D_\gamma) \) and \( \forall s [s \in D_\gamma \rightarrow \beta(s) = 0] \). Conclude that \( \text{StrongBar}_{F_\gamma} (D_\gamma) \).

Let \( \lambda \in [\omega]^\omega \) be given. Find \( n, m \) such that \( \lambda (n) n \in D_\alpha \) and conclude that \( \lambda (n) \notin D_\alpha \). Conclude that \( \forall \lambda \in [\omega]^\omega \exists n [\lambda (n) \notin D_\alpha] \), i.e. \( D_\alpha \) is almost-finite.

9.2. The contraposition of the Bolzano-Weierstrass Theorem in \( \omega^\omega \).

**Definition 30.** The contraposition of the Bolzano-Weierstrass Theorem in \( \omega^\omega \) is the following statement:

\( \text{AppBar}_{\omega^\omega} \) : For all \( \gamma \), if \( \forall \lambda \in [\omega]^\omega \exists n [\lambda \circ \gamma (n) \neq \lambda \circ \gamma (n + 1)] \), then \( \exists n [\gamma (n) \notin 2^{<\omega}] \).

**Definition 31.** Define \( \tau \) such that \( \tau (\langle \rangle) := \langle \rangle \) and, for all \( s \), for all \( n \), \( \tau (s * \langle n \rangle) := \tau (s) * \bar{n} (n + 1) * \langle 1 \rangle \).

**Theorem 9.4.** BIM \( \vdash \text{SemiApp} \rightarrow \text{AppBar}_{\omega^\omega} \).

Proof. Let \( \tau \) be as in Definition [31]. Observe the following.

(i) \( \forall s [\tau (s) \in 2^{<\omega}] \)

(ii) \( \forall s \exists i [i + 1 < \text{length} (s) \land (\tau (s))(i) = 1 \rightarrow (\tau (s))(i + 1) = 0] \)

(iii) \( \forall s \in 2^{<\omega} [i + 2 < \text{length} (\tau (s)) \land (\tau (s))(i) = (\tau (s))(i + 1) = 0 \rightarrow (\tau (s))(i + 2) = 1] \)

(iv) \( \forall s \in 2^{<\omega} [\text{length} (\tau (s)) \leq 3 \cdot \text{length} (s)] \)

Let \( \lambda \) be given such that \( \forall \lambda \in [\omega]^\omega \exists n [\lambda \circ \gamma (n) \neq \lambda \circ \gamma (n + 1)] \). Define \( \delta \) such that, for each \( n \), if \( n' \leq n \), then \( \delta (n) \rightarrow n' + 1 \), and, if \( n' = n \), then \( \delta (n) = 0 \). Note that \( \forall s \in E_\delta \leftrightarrow \exists n [s \subseteq n] \land \exists \gamma (n) \subseteq 1 \) and conclude that \( \delta \) is a semi-spread.

Note that \( \forall s [s \in E_\delta \leftrightarrow \exists n [s \subseteq n \land s \in E_\delta] \) and that, for each \( n \), the set \( \{ s \in \omega^n \mid s \in E_\delta \} \) has at most \( 2^n \) members, and conclude that \( \delta \) is an explicit semi-approximate-fan-law. Define \( B := \{ s \in 2^{<\omega} \mid \exists i [i + 2 < \text{length} (s) \land s (i) = s (i + 1) = s (i + 2) = 1] \lor \exists i [\text{length} (s) [\gamma (i) \notin 2^{<\omega}]] \} \). Now prove that \( B \) is a bar in \( SF_\delta \).

Let \( \beta \in SF_\delta \) be given. Define \( \zeta \) as follows. For each \( p \), having to define \( \zeta (p) \), we distinguish two cases.

**Case (1).** \( \neg \exists i [i + 2 < p \land \beta (i) = \beta (i) = \beta (i + 2) = 1] \). Find \( k := \min \{ \delta (n) = \bar{p} + 1 \} \) and define \( \zeta (p) = k' \). Note that \( k' = \bar{p} \subseteq \tau (\gamma \circ \zeta (p)) \).

**Case (2).** \( \exists i [i + 2 < p \land \beta (i) = \beta (i) = \beta (i + 2) = 1] \). Note that \( p > 0 \) and define \( \zeta (p) := \bar{p} (p + 1) + 1 \).

This completes the definition of \( \zeta \). Note that \( \forall p \exists q [\zeta (q) > p] \). Define \( \lambda \) such that \( \lambda (0) = 0 \) and \( \forall n [\lambda (n + 1) = n \mu q > \lambda (n)] [\zeta (q) > \zeta (\lambda (n))] \). Consider \( \eta := \zeta \circ \lambda \).

21Note that \( \tau \) slightly differs from the function \( \delta \) given in Definition [20].
\[ \tau(\gamma \circ \eta(3n)) = \tau(\gamma \circ \eta(3n + 3)) \] As \( \gamma \circ \eta(3n) \in 2^{<\omega} \) and \( \gamma \circ \eta(3n + 3) \in 2^{<\omega} \), also \( \gamma \circ \eta(3n) = \tau(\gamma \circ \eta(3n + 3)) \). Contradiction. Conclude that either \( \overline{\beta}(\lambda(3n + 3)) \in B \) or \( \gamma \circ \eta(3n) \in 2^{<\omega} \) or \( \gamma \circ \eta(3n + 3) \in 2^{<\omega} \). In the latter two cases, \( \overline{\beta}(\lambda(3n + 3) + 1) \in B \).

We thus see that \( \forall \beta \in SF \exists n[\beta(n) \in B] \), i.e. \( \text{Bar}_{SF}(B) \). Note that \( \forall n \exists m \gamma(n) > m \gamma(n) \). Define \( \eta' \) in \( [\omega]^n \) such that \( \forall n(\gamma \circ \eta'(n + 1) = \gamma \circ \eta'(n)) \) and define: \( \gamma' := \gamma \circ \eta' \). Using SenniappFF and Lemma \ref{lem:9.2} find \( m, n \) such that: \( \overline{\tau} \circ \gamma' \in SF \). Note that \( \exists n(\gamma(n) \in 2^{<\omega}) \).

We thus see that, for all \( \gamma \), if \( \forall \zeta \in [\omega]^* \exists n[\gamma \circ \zeta(n) \neq \gamma \circ \zeta(n + 1)] \), then \( \exists n(\gamma(n) \notin 2^{<\omega}) \), i.e. \( \gamma \circ \eta(3n) \neq \gamma \circ \eta(3n + 3) \).

9.3. The contrapositive of the Bolzano-Weierstrass Theorem in \( \mathcal{R} \).

**Definition 32.** The following statement is a version of the classical Bolzano-Weierstrass Theorem:

**BW:** For all \( \gamma \) in \( Q^\omega \), if \( \forall n(\gamma(n) \leq Q 1q) \), then \( \exists \gamma [\omega]^* \gamma(n) \leq Q 1q] \).

i.e. every infinite sequence \( \gamma \) of rationals in \( [-1, 1] \) has a convergent subsequence:

The following statement is the contrapositive of the Bolzano-Weierstrass Theorem:

**\( \overline{BW} \):** For all \( \gamma \) in \( Q^\omega \), if \( \forall \gamma \in [\omega]^* [\gamma(n) \neq \gamma(n + 1) - Q 1q] \), then \( \exists n(\gamma(n) < Q 1q] \).

i.e. every infinite sequence of rationals that positively fails to have a convergent subsequence leaves \( [-1, 1] \).

**BW** extends Dedekind’s Theorem Ded, see Subsection \ref{sec:5.2} and thus is constructively false.

**Theorem 9.5.** \( \text{BiM} \vdash \overline{BW}_{\omega^\omega} \leftrightarrow \overline{BW} \).

**Proof.** (i) Assume \( \overline{BW}_{\omega^\omega} \).

Define \( \delta_1(a) \) such that, for all \( a \in 2^{<\omega} \), \( \delta_1(a) = \sum_{i<\text{length}(a)} a(i) \). Note that, for all \( a, b \), for all \( a \in 2^{<\omega} \), if \( a \neq b \) then \( |\delta_1(a) - \delta_1(b)| < \frac{1}{2^n} \).

Let \( \gamma \) in \( Q^\omega \) be given such that \( \forall \gamma \in [\omega]^* [\gamma(n) \neq \gamma(n + 1) - Q 1q] \). We want to prove: \( \exists n(\gamma(n) < Q 1q) \). Define \( \eta \) such that, for each \( n \), if \( \exists n \leq n + 1(\gamma(n) < Q 1q) \), then \( \eta(n) = (n + 2) \), and, if \( \forall n \leq n + 1(\gamma(n) \neq Q 1q) \), then \( \eta(n) = n \) is the least \( a \) in \( 2^{<\omega} \) such that \( |\delta_1(a) - \delta_1(n)| < \frac{1}{2^n} \). Assume that \( \exists n[\gamma(n) < Q 1q] \). Find \( n \) such that \( \gamma(n + 1) - \gamma(n) > Q 1q \). and distinguish two cases.

(1) \( \forall i \leq \gamma(n + 1) + 1(\gamma(i) \neq Q 1q) \). Observe that \( |\delta_1(\gamma(n + 1) - \delta_1(\gamma(n))| > Q 1q \neq Q 1q, \gamma(n + 1) \neq Q 1q \).

Note that \( \eta(n + 1) \neq Q 1q \).

Conclude that \( \gamma(n + 1) \neq Q 1q \).

(2) \( \exists i \leq \gamma(n + 1) + 1(\gamma(i) \neq Q 1q) \).

Note that \( \eta(n + 1) = Q 1q \).

We thus see that \( \forall \gamma \in [\omega]^* \exists n[\gamma(n) \neq Q 1q] \). Using Lemma \ref{lem:9.4} and \( \overline{BW}_{\omega^\omega} \), we find \( p \) such that \( \eta(p) \neq 2^{<\omega} \). Conclude that \( \eta(p) = \langle p + 2 \rangle \) and \( \exists n(\gamma(n) < Q 1q] \).

We thus see that, for each \( \gamma \) in \( Q^\omega \), if \( \gamma \) positively fails to have a convergent subsequence, then \( \exists n(\gamma(n) < Q 1q] \), i.e. \( \overline{BW} \).

(ii) Assume \( \overline{BW} \).

Define \( \delta_0(a) \) such that, for all \( a \) in \( 2^{<\omega} \), \( \delta_0(a) = \sum_{i<\text{length}(a)} a(i) \). Note that, for all \( a, b \), for all \( a \in 2^{<\omega} \), if \( a \neq b \), then \( |\delta_0(a) - \delta_0(b)| < \frac{1}{2^n} \).

Let \( \gamma \) be given such that \( \forall \gamma \in [\omega]^* \exists n[\gamma(n) \neq \gamma(n + 1) \neq \gamma(n + 1)] \). Define \( \eta \) in \( Q^\omega \) such that, for each \( p, \) either \( \forall n \leq p[\gamma(n) \in 2^{<\omega}] \) and \( \eta(n) = \delta_0(\gamma(n)) \), or...
We thus see that $\forall \eta \in [\omega]^\omega$ such that $\exists \gamma$ such that, for each $\gamma$, if $\gamma$ positively fails to have a 1-convergent subsequence, then $\exists m[\gamma(m) \notin 2^{<\omega}]$, i.e. $\text{BW}_{\omega^\omega}$.

9.4. Extending $\text{BW}_{\omega^\omega}$.

**Definition 33.** The Extended contrapositive of the Bolzano-Weierstrass Theorem is the following statement:

$$(\text{BW}_{\omega^\omega})^+: \text{For all } \beta, \gamma, \text{ if } \text{Appfan}^+(\beta) \text{ and } \forall \zeta \in [\omega]^\omega \exists n[\gamma \circ \zeta(n) \neq n \gamma \circ \zeta(n+1)], \text{ then } \exists n[\beta(\gamma(n)) \neq 0].$$

$(\text{BW}_{\omega^\omega})^+$ generalizes $\text{BW}_{\omega^\omega}$ from Cantor space $2^\omega$ to any explicit approximate fan $F_\beta$.

**Theorem 9.6.** $\text{BIM} \vdash (\text{BW}_{\omega^\omega})^+ \rightarrow (\text{BW}_{\omega^\omega})^+$.

**Proof.** Let $\beta$ be given such that $\text{Appfan}^+(\beta)$. Find $\delta$ such that for all $n$, the set $\{ s \in \omega^\omega : \beta(s) = 0 \}$ has at most $\delta(n)$ elements. Define $\eta$ such that, for each $n$, $\eta(n) = \sum_{i \leq n} \delta(i) + 1$. Define $\psi$ such that

(i) $\psi(\langle \rangle) = \langle \rangle$, and,

(ii) for every $t, j$, if $\beta(t * j) = 0$, and there are exactly $k$ numbers $l$ such that $l < j$ and $\beta(t * \langle l \rangle) = 0$, then $\psi(t * \langle j \rangle) = \psi(t) * (j * 1)$.

Note that, for all $n$, for all $s, t, j$, if $\beta(s) = \beta(t) = 0$ and $s \neq t$ then $\psi(s) \neq (n_0)$ $\psi(t)$. Let $\gamma$ be given such that $\forall \zeta \in [\omega]^\omega \exists n[\gamma \circ \zeta(n) \neq n \gamma \circ \zeta(n+1)]$. Define $\nu$ such that, for each $p$, either $\forall n \leq p[\beta(\gamma(p)) = 0]$ and $\nu(p) = \psi(\gamma(p))$, or $\exists n \leq p[\beta(\gamma(n)) = 0]$ and $\nu(p) = (p + 2)$. Let $\zeta$ be an element of $[\omega]^\omega$. Find $p$ such that $\gamma \circ \zeta(p) \neq \gamma \circ \zeta(p + 1)$. If $\forall n \leq p[\beta(\gamma(n)) = 0]$, then $\nu(p) = \psi(\gamma \circ \zeta(p))$. If $\exists n \leq p[\beta(\gamma(n)) \neq 0]$, then $\nu(p + 1) = \nu(\zeta(p+1))$, and $\nu(\zeta(p+2)) = \nu(\zeta(p+1) + 1)$.

We thus see that $\forall \zeta \in [\omega]^\omega \exists n[\nu(p) \neq \nu(\zeta(p))]$. Applying $\text{BW}_{\omega^\omega}$, we find $p$ such that $\nu(p) \neq 2^{<\omega}$. Conclude that $\nu(p) = (p + 2)$ and $\exists m \leq p[\beta(\gamma(m)) = 0]$, and $\exists m[\beta(\gamma(n)) \neq n \gamma \circ \zeta(n+1)]$, i.e. $(\text{BW}_{\omega^\omega})^+$.

**Theorem 9.7.** $\text{BIM} \vdash (\text{BW}_{\omega^\omega})^+ \rightarrow \text{AppFT}$.

**Proof.** Assume $(\text{BW}_{\omega^\omega})^+$. Let $\beta, \alpha$ be given such that $\text{Appfan}^+(\beta)$ and $\forall s \in D_\alpha \rightarrow \beta(s) = 0$ and $\text{Bar}_{\text{fan}}(D_\alpha)$. Find $\delta$ such that, for each $n$, the set $\{ s \in \omega^\omega : \beta(s) = 0 \}$ has at most $\delta(n)$ elements. Let $\zeta$ in $[\omega]^\omega$ be given such that $\zeta(0) \notin 2^{<\omega}$ and, for each $n$, if $\forall i \leq n + 1[\zeta(i) \in D_\alpha]$, then $\zeta(n + 1) = \zeta(n) = \zeta(n+1) = \zeta(n+1)$, and, if not, then $\zeta(n+1) = \zeta(n) + (p)$ where $p = \mu q[\beta(\zeta(n) * \langle q \rangle) = 0]$, and $\zeta(n+1) = (p)$ where $p = \mu q[\zeta(n) > \zeta(n) + \beta(q)] \neq 0]$. Note that both $\zeta^*, \zeta^+$ are in $[\omega]^\omega$ and,
for each $n$, $\beta(\zeta^*(n)) = 0$. Also note that, for each $n$, there exists $m \leq \sum_{i \leq n} \delta(i)$ such that $\text{length}(\zeta^*(m)) > n$. Define $\varepsilon$ in $[\omega^\omega]$ such that $\varepsilon(0) = 0$ and, for each $n$, $\varepsilon(n + 1) = \varepsilon(n) + \text{length}(\zeta^*(\varepsilon(n)))$ and define $\zeta^* = \zeta^* \circ \varepsilon$. Note that $\forall n \{\text{length}(\zeta^*(n)) \geq n\}$. Let $\eta$ in $[\omega^\omega]$ be given. Define $\theta$ such that, for each $n$, if $\forall i \leq n[\zeta^* \circ \eta(i) = i \circ \zeta^* \circ \eta(i + 1)]$, then $\theta(n) = (\zeta^* \circ \eta(n + 1))(n)$, and, if not, then $\theta(n)$ is the least $i$ such that $\beta([\theta n \star (i)]) = 0$. Note that $\theta \in F_\beta$, and find $n$ such that $\theta n \in D_\alpha$. Note that $\zeta^* \circ \eta(n + 1) = i \circ \zeta^* \circ \eta(n + 1)$ and distinguish two cases.

(a) $\exists i \leq \varepsilon \circ \eta(n + 1)[\zeta(i) \notin D_\alpha]$, and note that $\zeta^+ \circ \varepsilon \circ \eta(n + 1) \neq \varepsilon \circ \eta(n + 1)$. Find $p := \mu_i[\zeta(i) \notin D_\alpha]$ and note that $\zeta^+ \circ \varepsilon \circ \eta(p + 1) \neq \varepsilon \circ \eta(p + 2)$.

(b) $\forall i \leq \varepsilon \circ \eta(n + 1)[\zeta(i) \in D_\alpha]$. Then $\forall i \leq \varepsilon \circ \eta(n + 1)[\zeta(i) = \zeta^+(i) = \zeta^+(i)]$. Assume that $\forall i \leq n[\zeta^* \circ \eta(i) = i \circ \zeta^* \circ \eta(i + 1)]$. Then $\delta n = \zeta^* \circ \eta(n + 1) \in \zeta^* \circ \eta(n + 1)$. Note that $D_\alpha$ is thin and conclude that $\zeta^* \circ \eta(n + 1) \notin D_\alpha$. Contradiction. Conclude that $\exists i \leq n[\zeta^* \circ \eta(i) \neq i \circ \zeta^* \circ \eta(i + 1)]$.

We thus see that $\forall n \in [\omega^\omega] \exists \alpha \in [\omega^\omega] \exists \alpha(\zeta(n) \notin D_\alpha)$. Using $(\text{BW}_\omega)^+$, we find $n$ such that $\beta(\zeta(n)) \neq 0$. Conclude that there exists $i \leq \varepsilon(n)$ such that $\zeta(i) \notin D_\alpha$. We thus see that $\forall \zeta \in [\omega^\omega] \exists \alpha \in [\omega^\omega] \exists \alpha(\zeta(n) \notin D_\alpha)$. Hence $D_\alpha$ is almost-finite. We thus see that, even in explicit approximate fan, every thin bar is almost-finite, i.e.

**AppFT.**

**Corollary 9.8.**

$\text{BIM} \vdash \text{AppFT} \iff \text{SemiappFT} \iff \text{BW}_\omega \iff \text{BW} \iff (\text{BW}_\omega)^+ \iff \text{BW}$. 

**Proof.** Use Theorems [9.1] [9.2] [9.3] [9.6] and [9.7].

**Corollary 9.9.** $\text{BIM} \vdash \text{AppFT} \iff \text{BW}$ and: $\text{BIM} \vdash \text{OI}([0, 1]) \iff \text{BIM} \vdash \text{BW} \iff \text{BW}$.

**10. Ascoli’s Lemma**

10.1. **Again the Bolzano-Weierstrass Theorem.** One might say that $\text{BW}$ states the sequential compactness of $[-1, 1]$. We want to get rid of the reference to points not lying in $[-1, 1]$.

**Theorem 10.1.** The following statements are equivalent in $\text{BIM}$:

(i) $\text{BW}^+$: For all $\gamma$ in $[\omega]^\omega$,

if $\forall \zeta \in [\omega]^\omega \exists \eta[\gamma \circ \zeta(n + 1) - \eta \circ \zeta(n)] > Q[n]$, then $\exists \eta \in [\omega^{\omega}]$.

(ii) For all $\gamma$ in $(\text{Q} \cap [-1, 1])^\omega$, for all $\alpha$,

if $\forall \zeta \in [\omega]^\omega \exists \eta[\gamma \circ \zeta(n + 1) - \eta \circ \zeta(n)] > Q[n]$, then $\exists \eta \in [\omega^{\omega}]$. 

Taking $\alpha = 0$, we see that (ii) implies the constructively weaker statement: $\forall \gamma \in (\text{Q} \cap [-1, 1])^\omega \forall \zeta \in [\omega]^\omega \exists \eta[\gamma \circ \zeta(n + 1) - \eta \circ \zeta(n)] > Q[n]$, that is: the assumption that, for some $\gamma$ in $(\text{Q} \cap [-1, 1])^\omega$, every subsequence positively fails to converge, leads to a contradiction.

**Proof.** (i) $\Rightarrow$ (ii). Let $\gamma$ in $(\text{Q} \cap [-1, 1])^\omega$ and $\alpha$ be a given such that $\forall \zeta \in [\omega]^\omega \exists \eta[\gamma \circ \zeta(n + 1) - \eta \circ \zeta(n)] > Q[n]$. Define $\delta$ in $[\omega]^\omega$ such that, for each $n$, if $\forall i \leq n[\alpha(i) = 0]$, then $\delta(n) = \gamma(n)$, and, if $\forall i \leq n[\alpha(i) \neq 0]$, then $\delta(n) = Q[n + 1]$. We prove that every subsequence of $\delta$ positively fails to converge. Assume $\zeta \in [\omega]^\omega$. Find $n$ such that $\gamma \circ \zeta(n + 1) - \eta \circ \zeta(n)] > Q[n]$, and $\delta \circ \zeta(n) = \gamma \circ \zeta(n)$ and $\delta \circ \zeta(n + 1) = \gamma \circ \zeta(n + 1)$. 


\(\zeta(n + 1) + |\delta \circ \zeta(n + 1) - q| < q < \frac{1}{2^q}, \text{ or } \exists i \leq \zeta(n + 1)[\alpha(i) \neq 0]\) and 
\(|\delta \circ \zeta(n + 2) - q \delta \circ \zeta(n + 1)| = q \zeta(n + 2) - q \zeta(n + 1) > R > \frac{1}{2^{(n + 2)}}. \text{ We thus see that } \forall \gamma \in [\omega]^{\infty} \exists n [\delta \circ \zeta(n + 1) - q \delta \circ \zeta(n)] > Q > \frac{1}{2^q}. \text{ Now use (i) and find } n \text{ such that } [\delta(n)] > Q > 1Q. \text{ Conclude that } \delta(n) \neq \gamma(n) \text{ and } \exists i \leq n[\alpha(i) \neq 0].

(ii) \implies (i). \text{ Let } \gamma \in Q^\omega \text{ be given such that } \forall \gamma \in [\omega]^{\infty} [\gamma \circ \zeta(n + 1) - q \gamma \circ \zeta(n) | > Q > \frac{1}{2^q}]. \text{ Define } \alpha \text{ such that, for each } n, \alpha(n) \neq 0 \text{ if and only if } \exists i \leq n[\gamma(i) > Q > 1Q]. \text{ Define } \delta \text{ in } (Q \cap [0, 1])^\omega \text{ such that, for each } n, \text{ if } |\gamma(n)| > Q > 1Q, \text{ then } \delta(n) = \gamma(n), \text{ and if not, then } \delta(n) = 0Q. \text{ Note that, for each } n, \text{ if } \gamma(n) \neq \delta(n), \text{ then } |\gamma(n)| > Q > 1Q \text{ and } \exists i \leq n[\alpha(i) \neq 0]. \text{ Let } \zeta \text{ be an element of } [\omega]^{\infty}. \text{ Find } n \text{ such that } |\gamma \circ \zeta(n + 1) - q \gamma \circ \zeta(n) | > Q > \frac{1}{2^q}. \text{ Either } |\delta \circ \zeta(n + 1) - q \delta \circ \zeta(n)| > R > \frac{1}{2^q} \text{ or } \exists i \leq (n + 1)[\alpha(i) \neq 0]. \text{ We thus see that } \forall \gamma \in [\omega]^{\infty} \exists n [\delta \circ \zeta(n + 1) - q \delta \circ \zeta(n)] > Q > \frac{1}{2^q} \supseteq \alpha(n) \neq 0]. \text{ Using (ii), find } n \text{ such that } \alpha(n) \neq 0, \text{ and conclude that } \exists i \leq n[\gamma(i) > Q > 1Q]. \square

We now give a similar reformulation of \((BW_\omega)^+\).

**Theorem 10.2.** The following statements are equivalent in \(BIM\).

(i) \((BW_\omega)^+\): For all \(\beta, \gamma, \delta, \text{ if } Appfan^+ (\beta) \text{ and } \forall \gamma \in [\omega]^\infty \exists n [\gamma \circ \zeta(n) \neq \gamma \circ \zeta(n + 1)], \text{ then } \exists n[\beta \circ \gamma(n) \neq 0].

(ii) For all \(\beta, \gamma, \alpha, \text{ if } Appfan^+ (\beta) \text{ and } \forall n[\beta \circ \gamma(n) = 0] \text{ and } \forall \gamma \in [\omega]^\infty \exists n [\gamma \circ \zeta(n + 1) \neq \gamma \circ \zeta(n) \supseteq \alpha(n) \neq 0], \text{ then } \exists n[\alpha(n) \neq 0].

(iii) For all \(\beta, \gamma, \alpha, \text{ if } Appfan^+ (\beta) \text{ and } \forall n[\gamma \circ \zeta(n) \neq \gamma \circ \zeta(n + 1)], \text{ then } \exists n[\alpha(n) \neq 0].

**Proof.** (i) \implies (ii). \text{ Let } \beta, \alpha, \gamma \text{ be given such that } Appfan^+ (\beta) \text{ and } \forall n[\beta \circ \gamma(n) = 0] \text{ and } \forall \gamma \in [\omega]^\infty \exists n [\gamma \circ \zeta(n) \neq \gamma \circ \zeta(n + 1)]. \text{ We define } \delta \text{ such that, for each } n, \text{ if } \exists i \leq n[\alpha(i) \neq 0], \text{ then } \delta(n) = (n + 2), \text{ and, if not, then } \delta(n) = \gamma(n). \text{ Let } \zeta \text{ in } [\omega]^\omega \text{ be given. Find } n \text{ such that } \gamma \circ \zeta(n + 1) \neq \gamma \circ \zeta(n) \supseteq \alpha(n) \neq 0. \text{ Either } \delta \circ \zeta(n + 1) \neq \delta \circ \zeta(n) \text{ or } \exists i \leq (n + 1)[\alpha(i) \neq 0] \text{ and } \delta \circ \zeta(n + 1) \neq \delta \circ \zeta(n + 2). \text{ We thus see that } \forall \gamma \in [\omega]^\infty \exists n [\delta \circ \zeta(n + 1) \neq \delta \circ \zeta(n)] \supseteq \alpha(n) \neq 0]. \text{ Now use (ii) and find } n \text{ such that } \beta \circ \delta(n) \neq 0. \text{ Conclude that } \delta(n) = (n + 2) \text{ and } \exists i \leq n[\alpha(n) \neq 0].

(ii) \implies (i). \text{ Let } \beta \text{ be given such that } Appfan^+ (\beta) \text{ and let } \gamma \text{ be given such that } \forall \gamma \in [\omega]^\infty \exists n [\gamma \circ \zeta(n + 1) \neq \gamma \circ \zeta(n)]. \text{ Define } \alpha \text{ such that, for each } n, \text{ if } \alpha(n) \neq 0 \text{ if and only if } \exists i \leq n[\beta \circ \gamma(i) \neq 0]. \text{ Define } \delta \text{ such that, for each } n, \text{ if } \forall i \leq n[\beta \circ \gamma(i) = 0], \text{ then } \delta(n) = \gamma(n), \text{ and, if not, then } \delta(n) = \gamma(n). \text{ Let } \zeta \text{ be an element of } [\omega]^\omega. \text{ Find } n \text{ such that } \gamma \circ \zeta(n + 1) \neq \gamma \circ \zeta(n) \supseteq \alpha(n) \neq 0. \text{ Either } \delta \circ \zeta(n + 1) \neq \delta \circ \zeta(n) \text{ or } \exists i \leq (n + 1)[\alpha(i) \neq 0]. \text{ We thus see that } \forall \gamma \in [\omega]^\infty \exists n [\delta \circ \zeta(n + 1) \neq \delta \circ \zeta(n)] \supseteq \alpha(n) \neq 0]. \text{ Now use (ii) and find } n \text{ such that } \beta \circ \delta(n) \neq 0. \text{ Conclude that } \delta(n) = (n + 2) \text{ and } \exists i \leq n[\alpha(n) \neq 0].

(iii) \implies (ii). \text{ Let } \beta \text{ be given such that } Appfan^+ (\beta) \text{ and let } \gamma, \alpha \text{ be given such that } \forall n[\gamma \in F_\beta] \text{ and } \forall \gamma \in [\omega]^n [\gamma \circ \zeta(n + 1) \neq \gamma \circ \zeta(n)] \supseteq \alpha(n) \neq 0]. \text{ Define } \delta \text{ such that, for each } n, \text{ if } \gamma(n) \neq \delta(n), \text{ then } \gamma(n) = \delta(n), \text{ and } \delta(n) = \delta(n). \text{ Let } \exists i \leq n[\beta \circ \gamma(i) \neq 0]. \text{ One may do so by requiring that, for each } n, \text{ for each } i \geq length(\gamma(n)), \delta(i) = \mu\in[\beta(\delta^n(i) \ast (p(i))) = 0]. \text{ Note that } \forall \gamma \in [\omega]^\infty \exists n [\delta \circ \zeta(n + 1) \neq \delta \circ \zeta(n)] \supseteq \alpha(n) \neq 0]. \text{ Using (iii), conclude that } \exists n[\alpha(n) \neq 0]. \square

10.2. Introducing \(C([-1, 1])\).

**Definition 34.** \(s \in \omega^n \) is a block if and only if

(i) for each \(i < n, \ (s(i))' \in S \) and \((s(i))'' \in S, \) and


\[31\]
(ii) \((s(0))'' = -1\) and, for all \(i\), if \(i + 1 < n\), then \(((s(i))'')'' = ((s(i + 1))'')'\), and \(((s(n))'')'' = 1\), and,

(iii) for all \(i\), if \(i + 1 < n\), then \((s(i))'' \approx_{\mathbb{S}} (s(i + 1))''\).

Block := \(\{s \mid s \text{ is a block}\}\).

For each \(n\), for each \(s\) in Block \(\cap \omega^n\), height\((s) := \max_{\mathbb{Q}} \{((s(i))'')'' - ((s(i))'')' \mid i < n\}\), and mesh\((s) := \min \{((s(i))'')'' - ((s(i))'')' \mid i < n\}\).

For all \(s, t\) in Block, \(t \subseteq_{\text{Block}} s\), the block \(t\) is a refinement of the block \(s\), if and only if \(\forall i < \text{length}(t) \exists j < \text{length}(s) [[(t(i))']' \subseteq_{\mathbb{S}} (s(j))'' \land (t(i))'' \subseteq_{\mathbb{S}} (s(j))'''\]; and \(s \#_{\text{Block}} t\), \(s\) deviates from \(t\), if and only if \(\exists i < \text{length}(s) \exists j < \text{length}(t) [[(s(i))'' \approx_{\mathbb{S}} (t(j))''] \land (s(i))'' \#_{\mathbb{S}} (t(j))'''\], and \(s \approx_{\text{Block}} t\), \(s\) does not deviate from \(t\), if and only if \(\neg((s \#_{\text{Block}} t) \land \varphi)\) in Block holds if and only if \(\forall n[\varphi(n + 1) \subseteq_{\text{Block}} \varphi(n)]\) and \(\varphi\) (twists if and only if \(\forall n \exists n[\text{height}(\varphi(n)) < \frac{1}{n^2}]\). C([−1, 1]) is the set of all shrinking and dwindling elements of Block. For all \(\varphi, \psi\) in C([−1, 1]), \(\varphi \#_{\text{Block}} \psi\), \(\varphi\) deviates from \(\psi\), if and only if \(\forall n[\varphi(n) \#_{\text{Block}} \psi(n)]\) and \(\varphi =_{\text{Block}} \psi\), \(\varphi\) does not deviate from \(\psi\), if and only if \(\forall n[\varphi(n) \approx_{\text{Block}} \psi(n)]\). For all \(\varphi\) in C([−1, 1]), for all \(\alpha \in [-1, 1]\), for all \(\beta \in \mathbb{R}\), we define \(\varphi : \alpha \mapsto \beta\), \(\varphi\) maps \(\alpha\) onto \(\beta\), if and only if, for each \(n\), there exist \(n, i\) such that \(i < \text{length}(\varphi(n))\) and \(\alpha(n) \approx_{\mathbb{S}} ((\varphi(n))'(i))''\) and \(((\varphi(n))(i))'' \approx_{\mathbb{S}} (\beta(m))''\).

**Lemma 10.3.** One may prove in BIM, that, for all \(\varphi\) in C([−1, 1]),

(i) for all \(\alpha, \gamma\) in [−1, 1], for all \(\beta, \delta\) in \(\mathbb{R}\), if \(\varphi : \alpha \mapsto \beta\) and \(\varphi : \gamma \mapsto \delta\), then,

(a) for each \(n\), if \(|\alpha - \gamma| < \frac{1}{2}\text{mesh}(\varphi(n))\), then \(|\beta - \delta| < 2 \cdot \text{heigth}(\varphi(n))\), and,

(b) if \(\alpha =_{\mathbb{R}} \gamma\), then \(\beta =_{\mathbb{R}} \delta\), and,

(ii) for all \(\alpha \in [0, 1]\), there exists \(\beta \in \mathbb{R}\) such that \(\varphi : \alpha \mapsto \beta\).

**Proof.** The proof is left to the reader. \(\square\)

### 10.3. Introducing moduli of uniform continuity.

**Definition 35** (Canonical elements of C([−1, 1])). \(S_0 := \{s \in S \mid s''_s = 0\}_{\mathbb{Q}} \land 2 \cdot s' \in \mathbb{Z}\), and, for each \(n\), \(S_{n+1} := \{s \in S \mid (2 \cdot q \cdot s', 2 \cdot s''_s) \in S_{n+1}\}\). For all \(m, n,\) CBlock\(_{m,n} := \{s \in \text{Block} \mid \forall i < \text{length}(s) [[(s(i))'') \subseteq_{\mathbb{S}} (s(i))'' \subseteq_{\mathbb{S}} S_m \land (s(i))'' \subseteq_{\mathbb{S}} S_n\}\). For each \(\delta\) in \([\omega]\), \(m, n,\) CBlock\(_{m,n} := \{s \in \text{Block} \mid \forall i < \text{length}(s) [[(s(i))'') \subseteq_{\mathbb{S}} (s(i))'' \subseteq_{\mathbb{S}} S_m \land (s(i))'' \subseteq_{\mathbb{S}} S_n\}\).

**Lemma 10.4.** One may prove in BIM that, for all \(\delta\) in \([\omega]\), for all \(\varphi\) in C\(_{\delta}([−1, 1])\), for all \(n > 0\), for all \(\alpha, \beta\) in C\(_{\delta}([−1, 1])\), for all \(\gamma, \varepsilon\) in \(\mathbb{R}\), if \(\varphi : \alpha \mapsto \beta\) and \(\varphi : \gamma \mapsto \varepsilon\) and \(|\alpha - \beta| < \frac{1}{2} \cdot \text{length}(\varphi(n)) < \frac{1}{2\varepsilon}\), then \(|\varphi(n) - \varepsilon| < \frac{1}{2} \cdot \text{length}(\varphi(n))\). Proof. The proof is left to the reader.

So \(\delta\) may be seen as a modulus of uniform continuity that is valid for every member of C\(_{\delta}([−1, 1])\). Note that, for each \(\varphi\) in C\(_{\delta}([−1, 1])\), for each \(n\), \(\text{length}(\varphi(n)) = 2^{\delta(n+1)}\).

**Definition 36** (Bounded elements of C([−1, 1])). For each \(\delta\) in \([\omega]\), C\(_{\delta}([−1, 1], [−1, 1]) := \{\varphi \in C([−1, 1]) \mid \forall n < 2^{\delta(n+1)} \exists n((\varphi(n))(i))'' \subseteq_{\mathbb{S}} (−1, 1)\}\).

**Definition 37.** We call the following statement Ascoli’s Lemma,

\[
\text{Asc: For all } \delta \text{ in } [\omega], \text{ for all } \varphi \text{ in } (C\(_{\delta}([−1, 1], [−1, 1]))'' \exists \varphi \text{ in } C\(_{\delta}([−1, 1], [−1, 1]))\text{.}
\]

The following statement is a contrapositive of Ascoli’s Lemma,

\[
\text{Asc: For all } \delta \text{ in } [\omega], \text{ for all } \varphi \text{ in } (C\(_{\delta}([−1, 1], [−1, 1]))'' \text{, for all } \alpha, \text{ if } \forall \xi \in [\omega] \exists n[\varphi^\xi(n+1)(n) \#_{\text{Block}} \varphi^\xi(n) \lor \alpha(n) \neq 0], \text{ then } \exists n[\alpha(n) \neq 0].
\]

\footnote{Brouwer already made use of these canonical rational segments, see \S 5.}
The following argument show that \textbf{Asc} proves \textbf{BW}.

Assume \textbf{Asc}. Let $\gamma$ in $Q^\omega$ be given such that $\forall n[[\gamma(n)] \leq Q 1_q]$. Define $\delta$ such that $\forall n[\delta(n) = n]$. Define $\varphi$ in $(C_5([-1,1],[-1,1]))^\omega$ such that, for each $n$, for all $\alpha$ in $[-1,1]$ for $\varphi^n$ such that, for all $n$ such that, for each $n$, $\varphi^n : \alpha \mapsto (\gamma(n))_{\mathcal{R}}$. Applying \textbf{Asc}, find $\zeta$ in $[\omega]^\omega$ such that $\forall n[\varphi_\zeta(n) = n \varphi_\zeta(n)]$. Conclude that $\forall n[[\gamma \circ \zeta(n) - Q \gamma \circ (n+1)] \leq Q \frac{1}{2}]$. Conclude $\textbf{BW}$.

As $\textbf{BW}$ is constructively false, so is $\textbf{Asc}$.

We now observe that, for every $\delta$ in $[\omega]^\omega$, $C_5([-1,1],[[-1,1]]$ is an explicit fan. Given $\delta$, define $\delta^*$ such that, for each $t$, $\delta^*(t) = 0$ if and only if $\forall i < length(t)[t(i) \in CBlock_{(i,1)}]$ and $\forall i < length(t) - 1[t(i+1) \subseteq Block_{(i)}]$. Note that $\delta^*$ is a spread-law and that $F_{\text{det}}$ coincides with $C_5([0,1],[0,1])$. Note that, for each $n$, for each $t$ in $\omega^n$ such that $\delta^*(t) = 0$, the set $\{m \mid \delta^*(t \cdot m) = 0\}$ is a subset of $CBlock_3(n,n)$ and has at most $2^n + n^2$ elements. Conclude that $\delta^*$ is an explicit fan-law.

**Theorem 10.5.** The following statements are equivalent in BIM.

(i) $\left(\textbf{BW}_{\omega^\omega}\right)^+$.

(ii) $\textbf{Asc}$.

(iii) $\textbf{AppFT}$. \\

**Proof.** (i) $\Rightarrow$ (ii). This follows from the fact that, for every $\delta$, $\delta^*$ is an explicit fan-law, see Theorem 10.2(iii).

Use the observation that, for each $\varphi$ in $C_5([-1,1],[[-1,1]]^\omega$, if $\forall \zeta \in [\omega]^\omega[3n[\varphi(n+1)] \# Block \varphi(n) \land \alpha(n) \neq 0]$, then $\forall \zeta \in [\omega]^\omega[3n[\varphi(n+1) \neq \varphi(n)] \lor \alpha(n) \neq 0]$. (ii) $\Rightarrow$ (i). Assume (ii). According to Corollary 9.8, it suffices to prove $\tilde{\textbf{BW}}$. We also use Theorem 10.1. Let $\gamma$, $\alpha$ be given such that $\gamma \in Q \cap [-1,1]^\omega$ and $\forall \zeta \in [\omega]^\omega[3n[\gamma \circ (n+1) - Q \gamma \circ (n)] > Q \frac{1}{2} + \alpha(n) \neq 0]$. Slightly adapting the proof of Lemma 5.3(ii), one may conclude that $\forall \zeta \in [\omega]^\omega[3n[\gamma \circ (n+1) - Q \gamma \circ (n)] > Q \frac{1}{2} + \alpha(n) \neq 0]$. Let $\delta$ in $[\omega]^\omega$ be given. Define $\varphi$ such that, for each $n$, $\varphi^n \in C_5([-1,1])$, and, for all $\alpha$ in $[-1,1]$ $\varphi : \alpha \mapsto (\gamma(n))_{\mathcal{R}}$. Note that, for all $p,q,n$, if $\varphi^q = n+1$, $\varphi^t$, then $|\gamma(p) - Q \gamma(q)| < Q \frac{1}{2}$. Conclude that $\forall \zeta \in [\omega]^\omega[3n[\varphi(n+1) \# Block \varphi(n)] \lor \alpha(n) \neq 0]$. Using (ii), conclude that $\exists n[\alpha(n) \neq 0]$. We thus see that, for all $\alpha$, for all $\gamma$ in $(Q \cap [-1,1])^\omega$, if $\forall \zeta \in [\omega]^\omega[3n[\gamma \circ (n+1) - Q \gamma \circ (n)] > Q \frac{1}{2} + \alpha(n) \neq 0]$, then $\exists n[\alpha(n) \neq 0]$, i.e. $\tilde{\textbf{BW}}$, see Theorem 10.1.

(i) $\Leftrightarrow$ (iii). According to Corollary 5.3(i) is equivalent, in BIM, to $\textbf{AppFT}$. \hfill \square

11. RAMSEY’S THEOREM

11.1. The intuitionistic (infinite) Ramsey Theorem.

**Definition 38.** For all $k > 0$, for every $X \subseteq \omega$, we define: $X$ is $k$-almost-full, Almost-full$_k(X)$, if and only if $\forall \zeta \in [\omega]^\omega \exists s \in [\omega]^k[\zeta \circ s \in X]$.

For each $k > 0$, the following statement is called the $k$-dimensional Intuitionistic Ramsey Theorem, IRT$_k$(k):

$\forall \alpha \forall \beta \forall \zeta \exists D_{\alpha} \forall \exists s \in [\omega]^k[\zeta \circ s \in D_{\alpha}]$.

The statement IRT$_k$(1) is called the Intuitionistic Pigeonhole Principle.

The statement IRT := $\forall k > 0[IRT_k(k)]$ is called the Intuitionistic Ramsey Theorem.

For each $k \geq 1$, the following statement is called the $k$-dimensional Classical Ramsey Theorem, CRT$_k$(k):

$\forall \alpha \exists \zeta \exists s \in [\omega]^k[\zeta \circ s \in D_{\alpha}] \lor \exists \zeta \in [\omega]^\omega \forall s \in [\omega]^k[\zeta \circ s \notin D_{\alpha}]$. 33
The statement \( \text{CRT} := \forall k > 0[\text{CRT}(k)] \) is called the Classical Ramsey Theorem.

**Theorem 11.1.** (i) \( \text{BIM} + \text{CRT}(1) \rightarrow \text{LPO} \).
(ii) \( \text{BIM} + X \lor \neg X \lor \forall k > 0[\text{IRT}(k)] \leftrightarrow \text{CRT}(k) \).

**Proof.** (i) Assume \( \text{CRT}(1) \). Let \( \alpha \) be given. Define \( \beta \) such that \( \forall n[\beta((n)) \neq 0 \leftrightarrow \exists i \leq n[\alpha(i) \neq 0]] \). Note that \( \exists \xi \in [\omega]^\omega \forall n[[\xi(n)] \in D_\beta] \leftrightarrow \forall n[\alpha(n) \neq 0] \) and that \( \exists s \in [\omega]^\omega \forall n[[\xi(n)] \notin D_\beta] \leftrightarrow \forall n[\alpha(n) = 0] \). Using \( \text{CRT}(1) \), conclude that \( \forall n[\alpha(n) \neq 0] \lor \forall n[\alpha(n) = 0] \). Conclude that \( \forall \alpha[\exists n[\alpha(n) \neq 0] \lor \forall n[\alpha(n) = 0]] \), i.e. \( \text{LPO} \).

(ii) Let \( k > 0 \) be given and assume \( \text{IRT}(k) \). Let \( \alpha \) be given and note that \( D_\alpha \) and \( \omega \setminus D_\alpha \) can not both be \( k \)-almost-full. If \( D_\alpha \) is not \( k \)-almost-full, then \( \exists \xi \in [\omega]^\omega \forall s \in [\omega]^k[\alpha \circ s \notin D_\alpha] \) and, if \( \omega \setminus D_\alpha \) is not \( k \)-almost-full, then \( \exists s \in [\omega]^\omega \forall s \in [\omega]^k[\alpha \circ s \notin D_\alpha] \). We may conclude that \( \text{CRT}(k) \).

Conversely, let \( k > 0 \) be given and assume \( \text{CRT}(k) \). Let \( \alpha, \beta \) be given such that \( D_\alpha, D_\beta \) are both \( k \)-almost-full. Let \( \zeta \) be an element of \( [\omega]^\omega \). Define \( \gamma = \alpha \circ \zeta \) and note that \( D_\gamma \) is also \( k \)-almost-full. Conclude \( \neg \exists n \in [\omega]^\omega \forall s \in [\omega]^k[\eta \circ s \notin D_\gamma] \). Using \( \text{CRT}(k) \), find \( \eta \) in \( [\omega]^\omega \) such that \( \forall s \in [\omega]^k[\eta \circ s \in D_\gamma] \), i.e. \( \forall s \in [\omega]^k[\zeta \circ \eta \circ s \in D_\alpha] \).

Find \( s \in [\omega]^k \) such that \( \zeta \circ \eta \circ s \in D_\beta \) and define \( t := \eta \circ s \). Note that \( \zeta \circ t \in D_\alpha \cap D_\beta \). We thus see that \( \exists s \in [\omega]^\omega \forall t \in [\omega]^k[\zeta \circ t \in D_\alpha \cap D_\beta] \), i.e. \( D_\alpha \cap D_\beta \) is \( k \)-almost-full. We may conclude that \( \text{IRT}(k) \).

Ramsey’s Theorem is proven in [16] and the intuitionistic version is treated in [20], [26] and [29]. In the latter three papers, the full Intuitionistic Ramsey Theorem is considered. This theorem states that for all \( k \geq 1 \), for all \( R, T \subseteq \omega \), if both \( R, T \) are \( k \)-almost-full, then \( R \cap T \) is \( k \)-almost-full.

In the statement \( \forall k > 0[\text{IRT}(k)] \) one restricts oneself to decidable subsets of \( \omega \).

**Theorem 11.2.** The following statements are equivalent in \( \text{BIM} \):

(i) \( \text{AppFT} \).
(ii) \( \forall k > 0[\text{IRT}(k)] \).
(iii) \( \text{IRT}(3) \).
(iv) \( \text{EnDec} \land \text{IRT}(2) \).

**Proof.** (i) \( \Rightarrow \) (ii). We use induction and first prove \( \text{IRT}(1) \).

Let \( \alpha, \beta \) be given such that both \( D_\alpha \) and \( D_\beta \) are \( 1 \)-almost-full. Let \( \zeta \in [\omega]^\omega \) be given. Note that \( \forall n[\exists m > n[[\zeta(m)] \in D_\alpha] \). Define \( \eta \) such that \( \eta(0) = \mu m[[\zeta(m)] \in D_\alpha \) and \( \forall n[[\zeta \circ \eta(n + 1) = \mu m > \eta(n)][[\zeta(m)] \in D_\alpha] \). Note that \( \zeta \circ \eta \in [\omega]^\omega \) and \( \forall n[[\zeta \circ \eta(n)] \in D_\alpha \). Find \( m \) such that \( [\zeta \circ \eta(m)] \in D_\beta \), define \( q := \eta(m) \) and note that \( [\zeta(q)] \in D_\alpha \cap D_\beta \). We thus see that \( \forall \zeta \in [\omega]^\omega \exists q[[\zeta(q)] \in D_\alpha \cap D_\beta] \), i.e. \( D_\alpha \cap D_\beta \) is \( 1 \)-almost-full.

Now assume \( k \geq 1 \) and \( \text{IRT}(k) \). We are going to prove \( \text{IRT}(k + 1) \).

Let \( \alpha, \beta \) be given such that \( D_\alpha, D_\beta \) both are \( (k + 1) \)-almost-full. We have to prove that also \( D_\alpha \cap D_\beta \) is \( (k + 1) \)-almost-full. We first set ourselves the modest goal of proving that the set \( D_\alpha \cap D_\beta \) is inhabited, i.e. \( \exists u \in [\omega]^{k+1}[u \in D_\alpha \cap D_\beta] \).

For each \( s \in [\omega]^{<\omega} \) and each \( X \subseteq \omega \), define

\[
s \text{ is } (k + 1)\text{-prehomogeneous for } X
\]

if and only if \( \forall t \in [\omega]^{k+1}[\forall j[t(k + 1) < i < j < \text{length}(s) \rightarrow (s \circ (t * \langle i \rangle) \in X \leftrightarrow s \circ (t * \langle j \rangle) \in X)]\).

Define \( \varphi \) such that \( \varphi(0) = (\) and, for each \( n \), \( \varphi(n + 1) = \varphi(j) * \langle n \rangle \) where \( j \) is the largest \( i \leq n \) such that \( \varphi(i) * \langle n \rangle \) is \( (k + 1) \)-prehomogeneous for \( D_\alpha \) and for \( D_\beta \). Note that \( \varphi(n + 1) \) is well-defined, as \( \varphi(0) * \langle n \rangle \) is \( (k + 1) \)-prehomogeneous for \( D_\alpha \) and for \( D_\beta \).
Define $\delta$ such that $\delta((\cdot)) \neq 0$ and $\forall \forall s [\delta(t \ast (\cdot)) \neq 0 \leftrightarrow \varphi(n) = t \ast (\cdot)]$. Note that $\forall s[\delta(s) \neq 0 \leftrightarrow \exists n[\varphi(n) = s]]$. Also note that $D_5 \subseteq [\omega]^\omega$ and that $D_5$ is a
tree, i.e. $\forall \forall s \ast (n) \in D_5 \rightarrow s \in D_5$. $D_5$ is called the $(k + 1)$-Erdős-Rado tree belonging
to $\alpha$ and $\beta$. $D_5$ is infinite and, for each $n$, for each $s$ in $D_5 \cap [\omega]^n$, there are at most $d(\delta)$ numbers $i$ such that $\delta(s \ast (i)) \neq 0$.
This is because, for each $n$, for each $s$ in $D_5 \cap [\omega]^n$, there are $\binom{n}{\delta}$ elements $t$ of $[\omega]^k$ such that $t(k - 1) < n$, and, for each such $t$, for each $i$, $(s \circ t) \ast (i)$ belongs to one of the four sets $D_0 \cap D_\beta, D_\alpha \setminus D_\beta, D_\beta \setminus D_\alpha$ and $\omega \setminus (D_\alpha \cup D_\beta)$. Note that, for all $i, j$, if $s \ast (i)$ belongs to $D_\beta$ and $i < j$, and, for all elements $t$ of $[\omega]^k$ such that $t(k - 1) < n$, $(s \circ t) \ast (i)$ and $(s \circ t) \ast (j)$ belong to the same of these four sets, then $s \ast (i, j)$ is $(k + 1)$-prehomogeneous for both $D_\alpha$ and $D_\beta$ and $s \ast (j)$ does not belong to $D_\beta$.
Define $\gamma$ such that $\gamma(0) = 1$, and, for each $n$, $\gamma(n + 1) = \gamma(n) \cdot 4(\delta)$. Note that, for each $n$, there are at most $\gamma(n)$ elements $s$ of $D_\beta$ such that $\text{length}(s) = n$. Define $\varepsilon$ such that $\forall s[\varepsilon(s) = 0 \leftrightarrow \exists n[t \in D_\beta \land s = t \ast \varepsilon(s)]$. Note that $\forall s[\varepsilon(s) = 0 \leftrightarrow \exists n[\varepsilon(s(n)) = 0]]$. Define $\eta$ such that $\eta(0) = 1$, and $\forall n[\eta(n + 1) = \eta(n) \cdot (4(\delta) + 1)]$. Note that, for each $n$, there are at most $\eta(n)$ numbers $s$ in $\omega^n$ such that $\varepsilon(s) = 0$. We thus see that the set $\mathcal{F}_\varepsilon$ is an explicit approximate fan.
Now define $B := \{s \mid \varepsilon(s) = 0 \land \exists t \in [\omega]^{k+1}[t(k) < \text{length}(s) \land s \circ t \in D_\alpha \cap D_\beta] \land s \circ t \in D_\alpha \cap D_\beta \lor \delta(s) = 0\}$. We prove that $B$ is a bar in $\mathcal{F}_\varepsilon$. Let $\gamma \in \mathcal{F}_\varepsilon$ be given. Define $\gamma^*$ such that $\gamma^*(0) = \gamma(0)$, and, for each $n$, if $\gamma(n + 1) \neq 0$ then $\gamma^*(n + 1) = \gamma(n + 1)$, and, if $\gamma(n + 1) = 0$, then $\gamma^*(n + 1) = \gamma^*(n) + 1$. Note that $\gamma^* \in [\omega]^{\omega}$ and, if $\gamma \in [\omega]^{\omega}$, then $\gamma^* = \gamma$, and $\forall n[\gamma^*(n) \neq \gamma(n) \rightarrow \delta(\gamma^*(n)) = 0]$. Define $\alpha^*$ and $\beta^*$ such that, for each $t$ in $[\omega]^k$, $\alpha^*(t) = \alpha(\gamma^* \circ t \ast (j))$ and $\beta^*(t) = \beta(\gamma^* \circ t \ast (j))$, where $j := t(k - 1) + 1$.
Let $\zeta$ in $[\omega]^{\omega}$ be given. Find $s$ in $[\omega]^{k+1}$ such that $\gamma^* \circ \zeta \ast s \in D_\alpha$. Define $n := ((\zeta \circ s)(k) + 1) + 1$ and $t := s(k)$, and $\exists n[\zeta \circ s \in D_\alpha]$. Conclude that $\forall \gamma \in [\omega]^{\omega} \land \gamma^* \circ \zeta \ast s \ast \gamma \in D_\alpha$, and, for all elements $t$ of $[\omega]^k$ such that $t(k - 1) + 1$, and, therefore, $\gamma^* \circ t \in D_\alpha$, or $\gamma^* \circ t = (k + 1)$-prehomogeneous for $D_\alpha$, and, therefore, $\delta(\gamma^* \circ t) = 0$. One may prove by a similar argument that $\forall \gamma \in [\omega]^{\omega} \land \gamma^* \circ \zeta \ast s \ast \gamma \in D_\alpha$. Use $\text{IRT}(k)$ and conclude that $\forall \zeta \in [\omega]^{\omega} \land \gamma^* \circ \zeta \ast s \ast \gamma \in D_\alpha$. Use $\text{IRT}(k)$ and conclude that $\forall \zeta \in [\omega]^{\omega} \land \gamma^* \circ \zeta \ast s \ast \gamma \in D_\alpha$.
Find $t$ in $[\omega]^k$ such that $t \in D_\alpha \cap D_\beta$. Either $t \in D_\alpha \cap D_\beta$ or $\exists n[\delta(\gamma^*(n)) = 0]$, i.e. $\gamma^*(t \ast (j)) = (\gamma^* \circ t \ast (j)) \in D_\alpha \cap D_\beta$, where $j := t(k - 1) + 1$, or $\exists n[\delta(\gamma^*(n)) = 0]$. In both cases, we find $n$ such that $\gamma^n \in B$. Note that $\forall \gamma \in [\omega]^{\omega} \land \gamma^* \circ \zeta \ast s \ast \gamma \in D_\alpha$. We thus see that $\forall \gamma \in [\omega]^{\omega} \land \gamma^* \circ \zeta \ast s \ast \gamma \in D_\alpha$, i.e. $\text{Bar}_{\delta^*}(B)$.
Find $\zeta \in [\omega]^{\omega}$ such that $\zeta(0) = \mu \langle m \rangle \in D_\beta$, and, for each $n$, $\zeta(n + 1) = \mu \langle m \rangle \in D_\beta$. Note that $\forall m[\mu \rightarrow \varepsilon(m) = 0]$. Using $\text{AppFT}$ and $\text{Corollary 3.3}$, conclude that $\text{StrongBar}_{\delta^*}(B)$ and find $n, m$ such that $\zeta(n) \in B$ and $\zeta(n) \in B$. Conclude that $\exists n \in [\omega]^{k+1}[s \circ t \in D_\alpha \cap D_\beta, \land s \circ t \in D_\alpha \cap D_\beta]$. We thus see that, for all $\alpha, \beta$, if $D_\alpha, D_\beta$ are $(k + 1)$-almost-full, then $\exists u \in [\omega]^{k+1}[u \in D_\alpha \cap D_\beta]$.

Again, let $\alpha, \beta$ be given such that $D_\alpha, D_\beta$ both are $(k + 1)$-almost-full and let $\gamma$ in $[\omega]^{\omega}$ be given. Define $\alpha', \beta'$ such that $\forall \gamma \in [\omega]^{\omega} \land \alpha'(t) = \alpha(\gamma \circ t) \land \beta'(t) = \beta(\gamma \circ t)$. Note that $D_\alpha'$ is $(k + 1)$-almost-full, as $\forall \delta \in [\omega]^{\omega} \exists s \in [\omega]^{k+1}[\lambda \circ \delta \circ s \in D_\alpha]$ and $\forall \delta \in [\omega]^{\omega} \exists s \in [\omega]^{k+1}[\delta \circ \delta \circ s \in D_\alpha]$. Also $D_\beta'$ is $(k + 1)$-almost-full. Conclude that
∃u ∈ [ω]^{k+1}[u ∈ D_α ∩ D_β], and that \( \forall u ∈ [ω]^{k+1}[γ ⊢ D_α ∩ D_β] \). We thus see that \( \forall γ ∈ [ω]^{k+1}[γ ⊢ D_α ∩ D_β] \), i.e. \( D_α ∩ D_β \) is \((k+1)\)-almost-full.

Conclude that \( ∀k > 0[IRT(k) → IRT(k+1)] \) and \( ∀k > 0[IRT(k)] \).

(iii) \( \Rightarrow \) (iv). Note that BIM \( \vdash IRT(3) → IRT(2) \). It thus suffices to show that BIM \( \vdash IRT(3) → EnDec \).!

Assume IRT(3). Let \( γ, n \) be given such that \( ∀n[(n /∈ D_α ∧ D_α ⊆ E_γ) → ∃p[p /∈ D_α ∧ p ∈ E_γ]] \). We will prove that \( n ∈ E_γ \), i.e. \( ∃j[γ(j) = n + 1] \).

Define \( α \) such that for all \( s \in [ω]^3 \), \( α(s) \neq 0 \) if and only if \( ∃i < s(0)[i ∈ E_{n(2)} \setminus E_{n(3)}] \). We now prove that \( D_α \) is 3-almost-full. Assume \( δ ∈ [ω]^ω \). Define \( ε \) such that, for all \( p, ε(p) \neq 0 \) if and only if \( p /∈ n ∧ ∀k[ε ⊢ p < δ(k)] \). Define \( p /∈ n ∧ D_α \). Find \( l \) such that \( j < δ(l) \) and note that \( δ(k), δ(k + 1), δ(l) \in D_α \).

We thus see that \( ∀i,j \in [ω]^{k+1}[∃s ∈ [ω]^3[δ ⊢ s] ∈ D_α, i.e. D_α \) is 3-almost-full.

Define \( β \) such that \( ∀s ∈ [ω]^3[δ(s) ≠ 0] → ∀i < s(0)[i ∈ E_{n(2)} \setminus E_{n(3)}[i ∈ E_{n(1)}]] \).

We now prove that \( D_β \) is 3-almost-full. Assume \( δ ∈ [ω]^ω \). Also assume that \( ∀k < δ[δ(0), δ(k), δ(k + 1)] /∈ D_β \). Then \( ∀k < δ(0)[∃i < δ(0)[i ∈ E_{n(k+1)} \setminus E_{n(k)}] \).

Clearly, this is impossible. Conclude that \( ∀k < δ(0)[[δ(0), δ(k), δ(k + 1)] ∈ D_β \).

We thus see that \( ∀i,j ∈ [ω]^{k+1}[∃s ∈ [ω]^3[δ ⊢ s] ∈ D_β \), i.e. \( D_β \) is 3-almost-full.

Using IRT(3), conclude that \( D_α ∩ D_β \) is 3-almost-full and, in particular, \( ∃s ∈ [ω]^3[s ∈ D_α ∩ D_β \).

Find such \( s \) and conclude that \( n ∈ E_{n(2)} ⊆ E_γ \).

We thus see that, for all \( γ, n \), if \( ∀n[(n /∈ D_α ∧ D_α ⊆ E_γ) → ∃p[p /∈ D_α ∧ p ∈ E_γ]] \), then \( n ∈ E_γ \). One easily concludes EnDec ?!

(iv) \( \Rightarrow \) (i). As BIM proves EnDec !\( \leftrightarrow \) Ded, see Corollary 5.6 and also AppFT \( \leftrightarrow \) BW. see Corollary 5.7. It suffices to show that BIM proves (\( \text{Ded} \) + IRT(2)) \( \rightarrow \) BW !\( \text{23} \).

Assume Ded and IRT(2). Let \( γ \in Q^ω \) be given such that \( ∀ζ ∈ [ω]^ω[∃n[|γ ⊢ ζ(n + 1) > γ ⊢ ζ(n)] ≥ 1̄] \). We shall prove that \( ∃n[|γ(n)| ≥ 1̄] \).

Let \( γ \in [ω]^ω \) be given. Define \( δ \in [ω]^ω \) such that \( δ(0) := γ(0) \), and, for each \( n, if \( γ ⊢ ζ(n + 1) ≥ γ ⊢ ζ(n) \), then \( δ(n + 1) = γ(0)(n + 1) \), and, if not, then \( δ(n + 1) = δ(n + 1) + q \).

Note that \( ∀n[δ(n + 1) ≥ 1̄] \) and \( ∀n[|η ⊢ γ(n)| ≥ 1̄] \).

Using Ded, find \( n \) such that \( δ(n) > 1̄ \). Note that \( either δ(n) = γ(0) \) or \( γ ⊢ ζ(n) \) or \( ζ(n) > 1̄ \). \( \text{or} \) \( δ(n) ≠ γ(0) \) and \( ζ(n) ≤ 1̄ \) \( \text{or} \) \( γ ⊢ ζ(i) > γ ⊢ ζ(i + 1) \). We thus see that \( ∀ζ ∈ [ω]^ω[∃n[|γ ⊢ ζ(n)| ≥ 1̄] \) \( ∀γ ⊢ ζ(n) > γ ⊢ ζ(0) ∧ γ(0) ⊢ ζ(n + 1) \). One may also prove that \( ∀ζ ∈ [ω]^ω[∃n[|γ ⊢ ζ(n)| ≥ 1̄] \) \( ∀γ ⊢ ζ(n) < γ ⊢ ζ(0) ∧ ζ(n + 1) > γ ⊢ ζ(n) \). Use IRT(2) and conclude that \( ∃n[|γ(n)| ≥ 1̄] \).

We thus see that, for each \( γ \in Q^ω \), if \( ∀ζ ∈ [ω]^ω[∃n[|γ ⊢ ζ(n + 1) > γ ⊢ ζ(n)] ≥ 1̄] \), then \( ∃n[|γ(n)| ≥ 1̄] \), i.e. BW. \( \square \)

11.2. The Paris-Harrington Theorem. We first show that the Intuitionistic Ramsey Theorem extends from \textit{decidable} to \textit{enumerable} subsets of \( ω \).

Corollary 11.3. One may prove, in BIM + \( ∀k > 0[IRT(k)] \):

\[ ∀k > 0[∀α/\bar{0}((\text{Almost full}_k(E_α) ∧ \text{Almost full}_k(E_β)) → \text{Almost full}_k(E_α ∩ E_β))]. \]

\( \text{23} \) The argument that follows is also used for 35. Theorem 35.
Proof. Let $k, \alpha, \beta$ be given such that both $E_\alpha$ and $E_\beta$ are $k$-almost-full. Define $\delta, \varepsilon$, such that $\forall s[\delta(s) \neq 0 \leftrightarrow (s \in [\omega]^{k+1} \land \exists k \in E_{\alpha})]$ and $\forall s[\varepsilon(s) \neq 0 \leftrightarrow (s \in [\omega]^{k+1} \land \exists k \in E_{\beta})]$. Note that $D_\delta$ and $D_\varepsilon$ are $k+1$-almost-full. Using IRT$(k+1)$, conclude that $D_\delta \cap D_\varepsilon$ is $k+1$-almost-full. Note that $\forall s[s \in D_\delta \cap D_\varepsilon \rightarrow \exists k \in E_\alpha \cap E_\beta]$. Conclude that $E_{\alpha} \cap E_{\beta}$ is $k$-almost-full.

\[\Box\]

Definition 40. For all positive integers $m, k$, $[m]^k := \{s \in [\omega]^k \mid s(k-1) < m\}$. For all positive integers $m, k$, for all $c, r$, we define $c : [m]^k \rightarrow r$ if and only if $\forall s \in [m]^k[s < \text{length}(c) \land c(s) < r]$. For all $r, k, n, M$ such that $n \geq k$, we define $M \rightarrow^* (n)^k$ if and only if, for all $c : [M]^k \rightarrow r$, there exist $l$ such that $s \in [M]^l$ and $l \geq n$ and $l \geq s(0) \land \forall u \in [l]^k \forall v \in [l]^k[c(s \circ u) = c(s \circ v)]$.

The Paris-Harrington Theorem $\text{PH}$ is the following statement:

$$\forall r \forall k \forall n \geq k \exists M[M \rightarrow^* (n)^r]$$

In [20], it is explained how $\text{PH}$ may be derived from IRT. It seems useful to consider this argument again in the formal context of BIM.

Definition 40. $[\omega]^\omega := \bigcup_{k \in \omega} [\omega]^k$.

$X \subseteq [\omega]^\omega = \omega$ is called $\omega$-almost-full, $\text{Almostfull}_\omega(X)$, if and only if $\forall \zeta \in [\omega]^\omega \exists s \in [\omega]^\omega[\zeta \circ s \in X]$.

Let $r$ be a positive integer and let $\chi$ be given such that $\forall n[\chi(n) < r]$. One calls $\chi$ an $r$-colouring of $\omega$.

For each $X \subseteq [\omega]^\omega$, for each $k > 0$, we let $X^{\chi,k}$ be the set of all $s$ in $X$ that are $\chi, k$-monochromatic, i.e., for all $u, v$ in $[\omega]^k$, if both $u(k-1) < \text{length}(s)$ and $v(k-1) < \text{length}(s)$, then $\chi(u \circ v) = \chi(s \circ v)$.

Note that, if $X$ is an enumerable subset of $\omega$, then also $X^{\chi,k}$ is an enumerable subset of $\omega$.

Corollary 11.4. One may prove, in BIM + $\forall k > 0[\text{IRT}(k)]$:

$$\forall k > 0 \forall r > 0 \forall \chi : \omega \rightarrow r \forall \alpha \exists (E_\alpha)^{\chi \circ \alpha} \subseteq \text{Almostfull}_\omega((E_\alpha)^{\chi \circ \alpha})$$

Proof. Let $k > 0$ be given. We use induction on $r$. Note that the case $r = 1$ is trivial. Now assume $r \geq 1$ is given such that the case $r$ of the statement has been established.

Let $\chi$ be given such that $\forall n[\chi(n) < r + 1]$. Define $\chi_0$ such that, for all $n$, if $\chi(n) < r$, then $\chi_0(n) = \chi(n)$, and, if $\chi(n) = r$, then $\chi_0(n) = r - 1$. Define $\chi_1$ such that, for all $n$, if $\chi(n) > 0$, then $\chi_1(n) = \chi(n) - 1$, and, if $\chi(n) = 0$, then $\chi_1(n) = 0$. Note that $\forall n[\chi(n) < r]$.

Let $X \subseteq [\omega]^\omega$ be given. Note that, for each $s$, if $\zeta \circ s \in (E_\alpha)^{\chi \circ \alpha}$, then either $\zeta \circ s \in (E_\alpha)^{\chi \circ \alpha}$, or, for some $t \in [\omega]^k$, $t(k-1) < \text{length}(s)$ and $\chi(\zeta \circ s \circ t) = r$, and, if $\zeta \circ s \in (E_\alpha)^{\chi \circ \alpha}$, then either $\zeta \circ s \in (E_\alpha)^{\chi \circ \alpha}$, or, for some $t \in [\omega]^k$, $t(k-1) < \text{length}(s)$ and $\chi(\zeta \circ s \circ t) = r$.

Again, let $\zeta$ in $[\omega]^\omega$ be given. Let $\text{QED}$ be the statement: $\exists s \in [\omega]^\omega[\zeta \circ s \in X^{\chi,k}]$. Note that, for each $\eta$ in $[\omega]^\omega$, there exists $s$ such that $\zeta \circ \eta \circ s \in X^{\chi,k}$ and, therefore, either $\text{QED}$ or $\exists u \in [\omega]^k[\chi(\zeta \circ \eta \circ u) = r]$. Define $Y_0 := \{t \in [\omega]^k \mid \chi(\zeta \circ t) = r \lor \text{QED}\}$. Note that $Y_0$ is $k$-almost-full and an enumerable subset of $\omega$. Also define $Y_1 := \{t \in [\omega]^k \mid \chi(\zeta \circ t) = 0 \lor \text{QED}\}$. Note that also $Y_1$ is $k$-almost-full and an enumerable subset of $\omega$. Using Corollary 11.3 conclude that $Y_0 \cap Y_1$ is $k$-almost-full. Find $t$ such that $\zeta \circ t \in Y_0 \cap Y_1$. Then $\chi(\zeta \circ t) = r$ or $\text{QED}$, and $\chi(\zeta \circ t) = 0$ or $\text{QED}$. Note that $r > 0$ and $\text{QED}$.

We thus see that $\exists \zeta \in [\omega]^\omega \exists s[\zeta \circ s \in (E_\alpha)^{\chi \circ \alpha}]$, i.e., $(E_\alpha)^{\chi \circ \alpha}$ is $\omega$-almost-full.\[\Box\]
Definition 41. Let \( r, k \) be positive integers.

Let \( c \) be given such that \( \forall n < \text{length}(c)[c(n) < r] \).

For all \( X \subseteq [\omega]^\omega \), \( X^{c,k} \) is the set of all \( s \) in \( X \) that are \( c,k \)-monochromatic, i.e. for all \( u, v \) in \([\omega]^k\), if \( u(k - 1) < \text{length}(s) \) and \( v(k - 1) < \text{length}(s) \), then \( s \circ u < \text{length}(c) \) and \( s \circ v < \text{length}(c) \) and \( c(s \circ u) = c(s \circ v) \).

Recall that \( \text{PH} \) is the statement: \( \forall r > 0 \forall k > 0 \forall n \exists M[M \to_s (n)^k] \).

Theorem 11.5. \( \text{BIM} + \forall k > 0[\text{IRT}(k)] \vdash \text{PH} \).

Proof. Let \( r, k, n \) be positive integers. Note that \( F := \{ \chi \mid \forall m[\chi(m) < r] \} \) is a fan. Define
\[
X := \{ s \in [\omega]^\omega \mid \text{length}(s) \geq n \land \text{length}(s) \geq s(0) \}.
\]
Note that \( X \) is \( \omega \)-almost-full and a decidable subset of \( \omega \). According to Corollary 11.4, for each \( \chi \in F \), the set \( X^{\chi,k} \) is \( \omega \)-almost-full, and, in particular, \( \exists s[s \in X^{\chi,k}] \).

Define
\[
B := \{ c \mid \forall m[\text{length}(c)[c(m) < r] \land \exists s[s \in X^{\chi,k}] \}.
\]
Note that \( B \) is a decidable subset of \( \omega \) and a bar in \( F \).

Note that, in \( \text{BIM} + \forall k > 0[\text{IRT}(k)] \) one may prove the Fan Theorem \( \text{FT} \), see Theorem 11.2 Corollaries 9.8 and 9.9 and Theorems 11.2 and 5.3.

Using \( \text{FT} \), find \( M \) such that \( \forall \chi \in F \exists m \leq M[\exists \eta \in B] \), and, therefore, \( \forall \chi \in F[\exists M \in B] \). Assume \( c : [M]^k \to r \). Note that \( M \leq \text{length}(c) \). Find \( \chi \) in \( F \) such that \( \exists M[\exists M \in B] \). Find \( s \) in \( X^{\exists M,k} \) and note that \( s \in X^{\chi,k} \). We thus see that \( \forall c : [M]^k \to r \exists s[\text{length}(s) \geq n \land \text{length}(s) \geq s(0) \land s \in c, k \text{-monochromatic}] \). i.e. \( M \to_s (n)^k \).

It is a famous fact that the Paris-Harrington Theorem can not be proven in classical or intuitionistic arithmetic, see [15], that is: \( \text{BIM} \not\vdash \text{PH} \). Using Troelstra’s result that \( \text{FT} \) is conservative over intuitionistic arithmetic, see [18], we conclude:

Corollary 11.6. \( \text{BIM} \not\vdash \text{PH} \) and \( \text{BIM} \not\vdash \text{FT} \to \text{AppFT} \).

Note that the second part of Corollary 11.6 is an easy consequence of the second part of Corollary 7.6.

Note that, in \( \text{BIM} \), the axiom scheme of induction is not restricted to arithmetical formulas. The classical system \( \text{ACA}_0 \) (implicitly) has such a restriction and \( \text{ACA}_0 \) is conservative over classical arithmetic, see [17] page 367, Remark IX. In \( \text{ACA}_0 \), one may prove (proper versions of) \( \text{CRT}(1) \) and \( \forall k[\text{CRT}(k + 1) \to \text{CRT}(k + 2)] \) and, therefore, \( \text{RT}(3) \), but not the Paris-Harrington Theorem and not \( \forall k > 0[\text{CRT}(k)] \), see [17] Section III.7.

12. Markov’s Principle

We consider two semi-classical axioms: Markov’s Principle and Kuroda’s Principle. There is no good argument why either one of these principles should be taken as an axiom for constructive arithmetic or analysis.

Definition 42. Markov’s Principle (MP1) is the statement
\[
\forall \alpha[\neg\exists n[\alpha(n) \neq 0] \to \exists n[\alpha(n) \neq 0]].
\]

Kuroda’s Principle of Double Negation Shift (DNS0) is the statement for every subset \( R \) of \( \omega \), \( \forall n[\neg\neg R(n)] \to \neg\neg\neg R(n) \).

or, equivalently,

for every subset \( R \) of \( \omega \), \( \neg\neg\neg R(n) \lor \neg R(n) \).

\( X \subseteq \omega \) is \( X \) nearly-decidable, or classically decidable\(^{24}\), if and only if 
\( \neg\neg\exists \beta[D_\beta = X] \), that is, \( \neg\neg\exists \beta \forall n[\exists x \leftrightarrow \beta(n) \neq 0] \).

\(^{24}\)This expression is used in [13].
\(\Sigma^0_1\text{-ND}\) is the statement

Every enumerable subset of \(\omega\) is nearly-decidable, \(\forall\gamma \neg\exists\beta[D_\beta = D_\gamma]\).

**Theorem 12.1.** In BIM + MP\(_1\), the following statements are equivalent:

(i) \(\text{EnDec}?!\).

(ii) \(\Sigma^0_1\text{-ND}\).

(iii) \(\Sigma^0_1\text{-BI}\).

**Proof.** (i) \(\Rightarrow\) (ii). Assume \(\text{EnDec}?!\). Let \(\gamma\) be given and assume that \(\neg\exists\beta[D_\beta = D_\gamma]\). Let \(\beta\) be given such that \(D_\beta \subseteq E_\gamma\). Conclude that \(\neg(E_\gamma \subseteq D_\beta)\), and \(\forall n[\mathcal{E}(\gamma) \rightarrow \beta(n) \neq 0]\), and \(\forall n[\mathcal{E}(\gamma) \rightarrow \beta(n) \neq 0]\), and \(\forall \exists \beta[p|\beta(p') = p'' + 1 \land \beta(p'') = 0]\). Use MP\(_1\) and conclude that \(\exists p|\beta(p') = p'' + 1 \land \beta(p'') = 0\) and \(\exists q[q \in E_\gamma \setminus D_\beta]\). We thus see that \(\forall \beta \in E_\gamma \rightarrow \exists q[q \in E_\gamma \setminus D_\beta]\).

Using \(\text{EnDec}?!\), we conclude that \(E_\gamma = \omega\), i.e. \(E_\gamma = D_\gamma\). Contradiction. We thus see that \(\forall \gamma \neg\exists\beta[E_\gamma = D_\beta]\), i.e. \(\Sigma^0_1\text{-ND}\).

(ii) \(\Rightarrow\) (iii). Assume \(\Sigma^0_1\text{-ND}\). Let \(\gamma\) be given such that \(\text{Bar}(E_\gamma)\) and \(\forall s[s \in E_\gamma \leftrightarrow \forall n[s \ast \langle n\rangle \in E_\gamma]]\). Assume we find \(\beta\) such that \(E_\gamma = D_\beta\). Note that \(\forall n[\beta(s) = 0 \rightarrow \neg \exists n[\beta(s \ast \langle n\rangle) = 0]]\) and: \(\forall n[\beta(s) = 0 \rightarrow \neg \exists n[\beta(s \ast \langle n\rangle) = 0]]\).

Using MP\(_1\), conclude that \(\forall n[\beta(s) = 0 \rightarrow \exists n[\beta(s \ast \langle n\rangle) = 0]]\).

Assume \(\beta((\langle \rangle)) = 0\). Find \(\delta\) such that, for each \(n\), \(\delta(n)\) is the least \(p\) such that \(\beta(\delta n \ast (p)) = 0\). Note that \(\forall n[\beta(\delta n) = 0]\), and: \(\forall n[\beta(s) = 0 \rightarrow \neg \exists n[\beta(s \ast \langle n\rangle) = 0]]\).

We thus see that, if \(\exists \beta \in E_\gamma = D_\beta\), then \(\exists \beta \in E_\gamma\). Using \(\Sigma^0_1\text{-ND}\), note that \(\neg \exists \beta \in E_\gamma\), and conclude that \(\neg \exists \beta \in E_\gamma\), i.e. \(\neg \exists \beta \in E_\gamma\). Using MP\(_1\) once more, conclude that \(\exists p|\beta(p) = 1\) and \(\exists p|\beta(p') = 1\).

Conclude that, for all \(\gamma\), if \(\text{Bar}(E_\gamma)\) and \(\forall s[s \in E_\gamma \leftrightarrow \forall n[s \ast \langle n\rangle \in E_\gamma]\), then \(\exists p|\beta(p) = 1\).

(iii) \(\Rightarrow\) (i). See Theorem 11.1 and Corollary 5.6. \(\square\)

The surprising observation that, in a context like BIM + MP\(_1\), \(\Sigma^0_1\text{-BI}\) implies \(\Sigma^0_1\text{-ND}\) is due to R. Solovay, see Lemma 5.3 in [13]. J.R. Moschovakis made me see that BIM + MP\(_1\) + \(\Sigma^0_1\text{-ND}\) \(\Rightarrow\) \(\text{EnDec}?!\).

**Definition 43.** We extend the language of BIM by introducing an infinite sequence of binary predicate symbols \(S^i_0, P^i_0, P^i_1, P^i_2, \ldots\) with the following defining axioms:

(i) \(\forall m[S^i_0(m, \gamma) \leftrightarrow m \in E_\gamma]\),

(ii) \(\forall m[P^i_0(m, \gamma) \leftrightarrow \neg S^i_0(m, \gamma)]\), and

(iii) for each \(n > 0\), \(\forall m[P^i_n(m, \gamma) \leftrightarrow \exists p[P^0_n((m, p), \gamma)]\), and

(iv) for each \(n > 0\), \(\forall m[P^i_n(m, \gamma) \leftrightarrow \exists \gamma[P^0_n((m, p), \gamma)]\).

A subset \(X\) of \(\omega\) is (positively) arithmetical if and only if there exist \(n > 0\), \(\gamma\) such that \(\forall m[m \in X \leftrightarrow S^0_n(m, \gamma)]\) or \(\forall m[m \in X \leftrightarrow P^0_n(m, \gamma)]\).

For each \(n\), we let \(\Sigma^0_n\text{-ND}\) be the statement

Every \(\Sigma^0_n\)-subset of \(\omega\) is nearly-decidable, \(\forall \gamma \forall \exists \beta \forall m[S^0_n(m, \gamma) \leftrightarrow \beta(m) \neq 0]\), and we let \(\Pi^0_n\text{-ND}\) be the statement

Every \(\Pi^0_n\)-subset of \(\omega\) is nearly-decidable, \(\forall \gamma \forall \exists \beta \forall m[P^0_n(m, \gamma) \leftrightarrow \beta(m) \neq 0]\).

**Theorem 12.2.** For each \(n\), BIM + \(\Sigma^0_1\text{-ND}\) \(\Leftrightarrow\) \(\Sigma^0_n\text{-ND} \land \Pi^0_n\text{-ND}\).

**Proof.** We use induction. Assume \(\Sigma^0_1\text{-ND}\).

Let \(\gamma, \beta\) be given such that \(\forall m[S^0_1(m, \gamma) \leftrightarrow \beta(m) \neq 0]\). Define \(\delta\) such that \(\forall m[\delta(m) = 0 \leftrightarrow \beta(m) \neq 0]\). Note that \(\forall m[P^0_1(m, \gamma) \leftrightarrow \delta(m) \neq 0]\). Conclude that, if \(\exists \gamma \forall m[S^0_1(m, \gamma) \leftrightarrow \beta(m) \neq 0]\), then \(\exists \beta \forall m[P^0_1(m, \gamma) \leftrightarrow \beta(m) \neq 0]\). Note that

\[\text{in the metalanguage}\]
\[ \neg \exists \beta \forall m [S^0_\beta(m, \gamma) \leftrightarrow \beta(m) \neq 0] \] and conclude that \[ \neg \exists \beta \forall m [P^0_\beta(m, \gamma) \leftrightarrow \beta(m) \neq 0] \] Conclude \( \Pi^0_1 \)-ND.

We thus see that \( \text{BIM} + \Sigma^0_1 \text{-ND} \vdash \Pi^0_1 \text{-ND} \).

Assume \( n > 0 \) and \( \Pi^0_1 \text{-ND} \). Let \( \gamma, \beta \) be given such that \( \forall m [P^0_\beta(m, \gamma) \leftrightarrow \beta(m) \neq 0] \). Note that \( \forall m [S^0_\beta(m, \gamma) \leftrightarrow \exists \beta \forall \delta (m, \gamma) \neq 0] \). Define \( \delta \) such that, for all \( m, p \), if \( \beta(m, p) \neq 0 \), then \( \delta(m, p) = m + 1 \), and, if not, then \( \delta(m, p) = 0 \). Note that \( \forall m [m \in E_\delta \leftrightarrow S^0_{m+1}(m, \gamma)] \), i.e. \( \forall m [S^0_1(m, \delta) \leftrightarrow S^0_{m+1}(m, \gamma)] \). Use \( \Sigma^0_1 \text{-ND} \) and conclude that \( \neg \exists \beta \forall m [S^0_{n+1}(m, \gamma) \leftrightarrow \beta(m) \neq 0] \). Conclude that, if \( \exists \beta \forall m [P^0_\beta(m, \gamma) \leftrightarrow \beta(m) \neq 0] \), then \( \neg \exists \beta \forall m [S^0_{n+1}(m, \gamma) \leftrightarrow \beta(m) \neq 0] \). Note that \( \neg \exists \beta \forall m [P^0_\beta(m, \gamma) \leftrightarrow \beta(m) \neq 0] \) and conclude that \( \neg \exists \beta \forall m [S^0_{n+1}(m, \gamma) \leftrightarrow \beta(m) \neq 0] \). Conclude \( \Sigma^0_{n+1} \text{-ND} \).

We thus see that \( \text{BIM} + \Sigma^0_1 \text{-ND} \vdash \Pi^0_1 \text{-ND} \rightarrow \Sigma^0_{n+1} \text{-ND} \).

Thus, Theorem 12.2 is also due to R. Solovay, see Lemma 5.5 and Theorem 5.6 in [13], see also [14]. J.R. Moschovakis showed that \( \text{BIM} + \Sigma^0_1 \text{-ND} \) proves the theorem that the constructive arithmetical hierarchy is proper and also an intuitionistic version of \( \Delta^1_1 \)-comprehension, see Corollaries 5.8 and 5.9 in [13].

**Theorem 12.3.** \( \text{BIM} + M^1 \vdash \Sigma^0_1 \text{-ND} \rightarrow \text{BW} \).

**Proof.** Assume that \( \gamma \in Q^\omega \) and \( \forall \zeta \in [\omega]^\omega \exists \eta \exists \delta \forall (\gamma \circ (n+1) - \gamma \circ \zeta(n)) > Q \frac{1}{z} \). Define \( \delta \in Q^\omega \) such that, for each \( n \), if \( \forall i \leq n [\eta(i) \in [-1, 1]] \), then \( \delta(n) = \gamma(n) \), and, if not, then \( \delta(n) = 0 \). Note that \( \forall \eta(\delta(n)) \in [-1, 1] \).

Define \( C := \{ s \in S | s \subseteq (1, 0, 1) \} \) and \( \exists \nu \forall m > m \delta(n) < s' \lor s'' < \delta(n) \}. \)

Recall that, for each \( s \in S \), \( L(s) = (s', s'' \circ s) \) and \( R(s) = (s'' \circ s, s') \). Note that, for each \( s \) such that \( s \subseteq (1, 0, 1) \), if both \( L(s) \in C \) and \( R(s) \in C \), then \( s \in C \).

Also note that \( C = \Sigma^0_1 \)-subset of \( \omega \) and find \( \varepsilon \) such that \( \forall s \in C \leftrightarrow S_0(s, \varepsilon) \).

Assume we find \( \eta \) such that \( C = D_\eta \), i.e. \( \forall s \in C \leftrightarrow \eta(s) \neq 0 \). Note that, for each \( s \) such that \( s \subseteq (1, 0, 1) \) and \( \eta(L(s)) \neq 0 \) or \( \eta(R(s)) \neq 0 \), then \( \eta(s) \neq 0 \), and, therefore, if \( \eta(s) = 0 \) then either \( \eta(L(s)) = 0 \) or \( \eta(R(s)) = 0 \). Note that \( \eta((0, 1, 1)) = 0 \). Define \( \lambda \) such that \( \lambda(0) = (1, 0, 1) \), and, for each \( n \), if \( \eta(L(\lambda(n))) = 0 \), then \( \lambda(n+1) = L(\lambda(n)) \), and, if not, then \( \lambda(n+1) = R(\lambda(n)) \).

Note that, for each \( n \), \( \eta(\lambda(n)) = 0 \).

Let \( s \) in \( S \) be given such that \( s \subseteq (1, 0, 1) \) and \( \eta(s) = 0 \). Then \( \neg \exists \nu \forall m > m \delta(n) < s' \lor s'' < \delta(n) \) and, therefore, \( \forall m \neg \exists \nu > m |s' \leq \delta(n) \leq s''| \). Use \( M^1 \) and conclude that \( \forall m \forall n > m |s' \leq \delta(n) \leq s''| \).

Conclude that \( \forall \nu \forall m > m \delta(n) \leq Q \delta(n) \leq Q (\lambda(p))' \). Define \( \zeta \) such that \( \zeta(0) = 0 \) and, for each \( n \), \( \zeta(n+1) \) is the least \( p \) such that \( p > \zeta(n) \) and \( (\lambda(p))' \leq Q \delta(p) \leq Q (\lambda(n))' \). Note that, for each \( n \), \( |\delta \circ \zeta(n+2) - \delta \circ \zeta(n+1)| \leq \frac{1}{s} \). Find \( n \) such that \( |\gamma \circ \zeta(n+2) - \gamma \circ \zeta(n+1)| \leq \frac{1}{s} \) and conclude that \( \exists \iota \leq \zeta(n+2)(\gamma(i)) > Q \frac{1}{s} \).
We thus see that, if $\exists n \forall s[S_0^2(s, e) \leftrightarrow \eta(s) \neq 0]$, then $\exists n[\gamma(n) >_Q 1_Q]$. Using $\Sigma^0_1$-ND and Theorem 12.2 we conclude that $\neg\neg\exists n \forall s[S_0^2(s, e) \leftrightarrow \eta(s) \neq 0]$. We thus find that $\neg\neg\exists n[\gamma(n) >_Q 1_Q]$, and, using $\text{MP}_1$ once more, that $\exists n[\gamma(n) >_Q 1_Q]$.

We may conclude $\text{BW}$. $\square$

The first item of the next theorem occurs already in [27], Section 3.20.

Theorem 12.4.

(i) $\text{BIM} \vdash \forall n[\forall \zeta \in [\omega^2] \exists n[\zeta(n) \notin D_\alpha] \rightarrow \neg\neg\exists n \forall m > n[m \notin D_\alpha]]$.

(ii) $\text{BIM} \vdash \text{MP}_1 \vdash \forall n[\neg\neg\exists n \forall m > n[m \notin D_\alpha] \rightarrow \forall \zeta \in [\omega^2] \exists n[\zeta(n) \notin D_\alpha]]$.

(iii) $\text{BIM} + \text{MP}_1 + \Sigma^0_1$-ND $\vdash \forall \beta [\text{Almfan}(\beta) \rightarrow \neg\neg \text{Fan}(\beta)]$.

(iv) $\text{BIM} + \text{MP}_1 + \text{AppFT} \vdash \text{AlmfT}$.

Proof. (i) Let $\alpha$ be given such that $D_\alpha$ is almost-finite, i.e. $\forall \zeta \in [\omega^2] \exists n[\zeta(n) \notin D_\alpha]$. Assume that $\neg\neg\exists n \forall m > n[m \notin D_\alpha]$. Then $\forall n \neg\neg\exists n \forall m > n[m \notin D_\alpha]$, and, by $\text{MP}_1$, $\forall n \exists m > n[m \notin D_\alpha]$. Define $\zeta$ such that $\zeta(0) = \mu \forall p [p \in D_\alpha]$ and, for each $n$, $\zeta(n + 1) = \mu \forall p [\zeta(n) \land p \in D_\alpha]$. Note that $\forall n[\zeta(n) \in D_\alpha]$. Contradiction. Conclude that $\neg\neg\exists n \forall m > n[m \notin D_\alpha]$, i.e. $D_\alpha$ is not-not-finite.

(ii) Assume $\text{MP}_1$. Let $\alpha$ be given such that $D_\alpha$ is not-not-finite, i.e. $\neg\neg\exists n \forall m > n[m \notin D_\alpha]$. Assume $\zeta \in [\omega^2]$. Note that, if $\exists n \forall m > n[m \notin D_\alpha]$, then $\exists n[\zeta(n) \notin D_\alpha]$. Conclude that $\neg\neg\exists n [\zeta(n) \notin D_\alpha]$ and, using $\text{MP}_1$, that $\exists n[\zeta(n) \notin D_\alpha]$. We thus see that $\forall \zeta \in [\omega^2] \exists n[\zeta(n) \notin D_\alpha]$, i.e. $D_\alpha$ is almost-finite.

Now assume that $\forall \alpha[\neg\neg\exists n \forall m > n[m \notin D_\alpha] \rightarrow \forall \zeta \in [\omega^2] \exists n[\zeta(n) \notin D_\alpha]]$, i.e. every decidable subset of $\omega$ that is not-not-finite is also almost-finite. Let $\alpha$ be given such that $\neg\neg\exists n[\alpha(n) \neq 0]$. Define $\beta$ such that $\forall \alpha[\beta(n) \neq 0 \leftrightarrow (\alpha(n) \neq 0 \land \forall i < n[\alpha(i) = 0])]$. Note that $\neg\neg\exists n \forall m > n[m \notin D_\beta]$, i.e. $D_\beta$ is not-not-finite. Conclude that $D_\beta$ is almost-finite and find $n$ such that $n \notin D_\alpha$ and, therefore, $\alpha(n) \neq 0$. We thus see that $\exists n[\alpha(n) \neq 0]$. Conclude that $\forall \alpha[\neg\neg\exists n[\alpha(n) \neq 0] \rightarrow \exists n[\alpha(n) \neq 0]]$, i.e. $\text{MP}_1$.

(iii) Assume $\text{MP}_1$. Let $\beta$ be given such that Almostfan$\beta$, i.e. $\text{Spr}(\beta) \land \forall s[\beta(s) = 0 \rightarrow \forall \zeta \in [\omega^2] \exists n[\beta(s * \zeta(n)) \neq 0]]$. Using (i), conclude that $\forall s[\beta(s) = 0 \rightarrow \neg\neg\exists n \forall m > n[\beta(s * \zeta(n)) = 0]]$. Assume we find $\delta$ such that $\forall s \forall \zeta \delta(s * \zeta(n)) \neq 0 \leftrightarrow \forall m > n[\beta(s * \zeta(n)) \neq 0]$. Note that $\forall \alpha \forall \zeta \forall s \forall n \delta(s * \zeta(n)) \neq 0$. Using $\text{MP}_1$, conclude that $\forall s \forall \zeta \forall n \delta(s * \zeta(n)) \neq 0$. Conclude that Fan$^+(\beta)$, i.e. $\beta$ is an explicit fan-law. Conclude that, if $\exists \delta \forall \zeta \forall s \forall n \delta(s * \zeta(n)) \neq 0 \leftrightarrow \forall m > n[\beta(s * \zeta(n)) = 0]$, then Fan$^+(\beta)$. Using $\Sigma^0_1$-ND, and its consequence $\Pi^0_2$-ND, see Theorem 12.2, conclude that $\neg\neg\exists \delta \forall \zeta \forall s \forall n \delta(s * \zeta(n)) \neq 0 \leftrightarrow \forall m > n[\beta(s * \zeta(n)) = 0]$. Conclude $\neg\neg\text{Fan}^+(\beta)$.

(iv) Assume $\text{MP}_1$ and $\text{AppFT}$. Note that, in $\text{BIM}$, $\text{AppFT}$ implies EnDec$\exists \forall$?, see Corollary 9.9 and, together with $\text{MP}_1$, also $\Sigma^0_1$-ND, see Corollary 9.6 and Theorem 12.4. We will prove $\text{AlmfT}$. Let $\beta$ be given such that Almfan$\beta$. Let $\alpha$ be given such that Thinbar$\beta(D_\alpha)$. Note that, if $\text{Fan}^+(\beta)$, then $\exists n \forall m \notin D_\alpha$, by Theorem 12.4. Conclude that $\neg\neg\exists n \forall m > n[m \notin D_\alpha]$. Now use (i) and conclude that $\forall \zeta \in [\omega^2] \exists n[\zeta(n) \notin D_\alpha]$, i.e. $D_\alpha$ is almost-finite. We thus see that $\forall \beta \forall \alpha[\text{Almfan}(\beta) \land \text{Thinbar}_\beta(D_\alpha)] \rightarrow \forall \zeta \in [\omega^2] \exists n[\zeta(n) \notin D_\alpha]$, i.e. $\text{AlmfT}$. $\square$

Corollary 12.5. $\text{BIM} + \text{MP}_1 \vdash \text{OII}([0, 1]) \leftrightarrow \text{AppFT} \leftrightarrow \Sigma^0_1$-ND $\leftrightarrow \text{AlmfT}$.

Proof. Use Corollaries 6.6 and 9.9 and Theorems 12.2, 12.3 and 12.4. $\square$
13. Some notations and conventions and basic facts

13.1. Finite sequences and infinite sequences of natural numbers. We assume that the language of BIM contains constants for the primitive recursive functions and relations and that their defining equations have been added to the axioms of BIM. In particular, there is a constant \( p \) denoting the function enumerating the prime numbers, \( p(0) = 2, p(1) = 3, \ldots \).

For each \( k \), there is a \( k \)-ary function symbol \( (\ )_k \) for the function coding sequences of natural numbers of length \( k \) by natural numbers:

\[
\langle m_0, \ldots, m_{k-1} \rangle = (p(0))^{m_0} \cdots (p(k-2))^{m_{k-2}} \cdot (p(k-1))^{m_{k-1}+1} - 1
\]

In practice, we omit the subscript \( k \) and write \( \langle \rangle \) for \( (\ )_k \).

\( \text{length}(0) := 0 \) and, for each \( a > 0 \), \( \text{length}(a) \) is the greatest number \( k \leq a \) such that \( p(k - 1) \) divides \( a + 1 \).

For each \( a \), for each \( i \), if \( i + 1 \leq \text{length}(a) \), then \( a(i) \) is the greatest number \( q \) such that \( (p(i))^q \) divides \( a + 1 \), and, if \( i + 1 = \text{length}(a) \), \( a(i) \) is the greatest number \( q \) such that \( (p(i))^{q+1} \) divides \( a + 1 \), and if \( i \geq \text{length}(a) \), then \( a(i) = 0 \). Observe that \( a = (a(0), a(1), \ldots, a(i-1)) \), where \( i = \text{length}(a) \).

For each \( m \), \( \omega^m := \{ a \mid \text{length}(a) = m \} \).

For each \( m \), \( [\omega]^m := \{ a \in \omega^m \mid \forall i + 1 < m \rightarrow a(i) < a(i + 1) \} \).

For each \( a, b \), \( a+b \) is the number \( s \) satisfying \( \text{length}(s) = \text{length}(a) + \text{length}(b) \) and, for each \( n \), if \( n < \text{length}(a) \), then \( s(n) = a(n) \) and, if \( \text{length}(a) \leq n < \text{length}(s) \), then \( s(n) = b(n - \text{length}(a)) \).

For each \( a \), for each \( \alpha \), \( a + \alpha \) is the element \( \beta \) of \( \omega^\omega \) satisfying, for each \( n < \text{length}(\alpha) \), then \( \beta(n) = a(n) \) and, if \( \text{length}(\alpha) \leq n \), then \( \beta(n) = \alpha(n - \text{length}(\alpha)) \).

For each \( a \), for each \( n < \text{length}(\alpha) \) we define: \( \pi(n) = (a(0), \ldots, a(n-1)) \).

If confusion seems unlikely, we sometimes write: \( \pi(n) \) and not: \( \pi(n) \).

For all \( a, b, a \subseteq b \leftrightarrow \exists n < \text{length}(b) [a = b(n)] \), and \( a \subseteq b \leftrightarrow (a \subseteq b \land a \neq b) \).

For each \( n \), for all \( a, b, a \neq b \leftrightarrow \exists j < \text{length}(a) \land (a(j) = b(j)) \land a = b \).

For each \( a \), for each \( n, \pi(n) := (\alpha(0), \ldots, \alpha(n - 1)) \).

If confusion seems unlikely, we sometimes write: \( \pi(n) \) and not: \( \pi(n) \).

For each \( n \), for each \( s \), \( s \subseteq \alpha \leftrightarrow \exists n [\pi(n) = s] \).

For each \( s, \omega^\omega \cap s := \{ \alpha \mid s \subseteq \alpha \} \).

\( 2^{<\omega} \) is the set of all natural numbers \( s \) coding a finite binary sequence, that is, such that, for all \( n < \text{length}(s) \), \( s(n) = 0 \lor s(n) = 1 \). Cantor space \( 2^\omega \) is the set of all \( \gamma \) such that \( \forall n [\pi(n) \in 2^{<\omega}] \).

For each \( s \) in \( 2^{<\omega} \), \( 2^\omega \cap s := \{ \alpha \in 2^\omega \mid s \subseteq \alpha \} \).

For each \( a, \alpha \circ s \) is the element \( t \) of \( \omega \) such that \( \text{length}(t) = \text{length}(s) \), and for all \( i < \text{length}(t) \), then \( t(i) = a(s(i)) \).

For each \( a, \alpha \circ s \) is the element \( \gamma \) of \( \omega^\omega \) satisfying \( \forall n [\gamma(n) = \alpha(n)] \).

For each \( X \subseteq \omega, X^\omega := \{ \alpha \mid \forall n [\alpha(n) \in X] \} \).

For each \( a, \alpha \circ s \) is the element \( \beta \) of \( \omega^\omega \) satisfying \( \forall m [\beta(m) = \alpha(n, m)] \).

For each \( X \subseteq \omega, X^\omega := \{ \alpha \mid \forall n [\alpha(n) \in X] \} \).

13.2. Lexicographical ordering and Kleene-Brouwer-ordering.

For all \( s, t, s <_{\text{lex}} t \leftrightarrow \exists i [i < \text{length}(s) \land i < \text{length}(t) \land \pi(i) = \pi(i) \land s(i) < t(i)] \).

For all \( a, \alpha <_{\text{lex}} \beta \leftrightarrow \exists n [\pi(n) <_{\text{lex}} \beta(n)] \).

For all \( a, \alpha <_{\text{lex}} \beta \leftrightarrow \exists n [\pi(n) <_{\text{lex}} \beta(n)] \).

For all \( a, \alpha <_{\text{lex}} \beta \leftrightarrow \exists n [\pi(n) <_{\text{lex}} \beta(n)] \).

For all \( \gamma, \mathcal{X} <_{\text{lex}} \gamma := \{ \alpha \mid \alpha \in \mathcal{X} \} \).

For all \( \mathcal{A}, \mathcal{X} \subseteq \omega^\omega, \mathcal{A} \text{ is progressive in } \mathcal{X} \text{ if and only if } \forall \gamma \in \mathcal{Y} [\mathcal{X} <_{\text{lex}} \gamma \subseteq \mathcal{A} \rightarrow \gamma \in \mathcal{X}] \).

For all \( s, t, s <_{KB} t \leftrightarrow (s \sqsubset t \lor s <_{\text{lex}} t) \). The ordering \( <_{KB} \) is called the Kleene-Brouwer ordering of \( \omega \), and sometimes the Lusin-Sierpiński ordering.
of $\omega$. The Kleene-Brouwer ordering $<_{KB}$ is a decidable and linear ordering of $\omega$:

\[
BIM \vdash \forall s,t : [s <_{KB} t \lor s = t \lor t <_{KB} s].
\]

13.3. Increasing sequences. We define, for each $n$, $[\omega]^n := \{s | \text{length}(s) = n \land \forall i[i + 1 < n \rightarrow s(i) < s(i + 1)]\}$. We also define: $[\omega]^<_\omega := \bigcup_{n \in \omega} [\omega]^n$ and $[\omega]^\omega := \{\alpha | \alpha(n) < \alpha(n + 1)\}$.

13.4. Bars and thin bars. For each subset $X$ of $\omega$, for each subset $B$ of $\omega$, $B$ is a bar in $X$, notation: $\text{Bar}_X(B)$, if and only $\forall n \exists \alpha(n) \in B$, and $B$ is a thin bar in $X$, notation: $\text{Thinbar}_X(B)$ if and only if $\forall n (s <_{KB} t \rightarrow s = t)$.

13.5. Decidable and enumerable subsets of $\omega$. For each $\alpha$, $D_\alpha := \{i | \alpha(i) \neq 0\}$. The set $D_\alpha$ is the subset of $\omega$ decided by $\alpha$. $A \subseteq \omega$ is a decidable subset of $\omega$ if and only if $\exists \alpha(A = D_\alpha)$. For each $\alpha$, $D_\alpha := \{i | \text{length}(a(a(i) \neq 0)\}$. $X \subseteq \omega$ is a finite subset of $\omega$ if and only if $\exists \alpha(X = D_\alpha)$. Note that, for each $\alpha$, $D_\alpha = \bigcup_{n \in \omega} D_\alpha^n$. Note that, for each $\alpha$, $D_\alpha$ is a finite subset of $\omega$ if and only if $\exists m \forall n \geq m (\alpha(n) = 0)$.

For each $\alpha$, $E_\alpha := \{n | 3p < (a(p) = n + 1)\]$. The set $E_\alpha$ is the subset of $\omega$ enumerated by $\alpha$. $A \subseteq \omega$ is an enumerable subset of $\omega$, or $A$ belongs to the class $\Sigma_0^1$, if and only if $\exists \alpha (A = E_\alpha)$. For each $\alpha$, $E_\alpha := \{n | 3p < (a(p) = n + 1)\]$. Note that, for each $\alpha$, $E_\alpha$ is a finite subset of $\omega$ and, for each $\alpha$, $E_\alpha = \bigcup_{n \in \omega} E_\alpha^n$.

13.6. Open subsets of $\omega$. For every $\alpha$, $G_\alpha := \{\gamma | \exists n (\gamma \in D_\alpha^n)\}$. $G \subseteq \omega$ is an open subset of $\omega$ if and only if $\exists \alpha [G = G_\alpha]$. For each $t$ in $\alpha$, $G_t := \{\gamma | \exists n (\gamma \in D_\alpha^n)\}$. Note that, for every $\alpha$, $G_\alpha = \bigcup_{n \in \omega} G_\alpha^n$. Also note that, for every $X \subseteq \omega$, $\forall \alpha [X \subseteq G_\alpha \iff \exists X \subseteq G_t \iff \exists X \subseteq G_t(D_t)]$.

13.7. Spreads, fans and explicit fans. For each $\beta$, $F_\beta := \{\alpha | \forall n (\beta(n) = 0)\}$. $F \subseteq \omega$ is a closed subset of $\omega$ if and only if $\exists \beta[F = F_\beta]$. $\beta$ is a spread-law, notation $\text{Spr}(\beta)$, if and only if $\forall s[\beta(s) = 0 \rightarrow \exists n (\beta(s * n) = 0)]$. $F \subseteq \omega$ is a spread or a located-and-closed subset of $\omega$ if and only if $\exists \beta[F = F_\beta]$. $X \subseteq \omega$ is inhabited if and only if $\exists \alpha (\alpha \in X)$. Note that, for every $\beta$, if $\text{Spr}(\beta)$, then $F_\beta$ is inhabited if and only if $\exists (\beta(\beta) = 0)$ and $F_\beta = \emptyset$ if and only if $\beta(\beta) = 0$.

$\beta$ is a finitary spread-law, or a fan-law, notation: $\text{Fan}(\beta)$, if and only if $\exists \beta (\text{Spr}(\beta)$ and $\forall s (s(n) = 0 \rightarrow m \leq n)$, $\beta$ is an explicit fan-law, notation $\text{Fan}^+(\beta)$, if and only if $\exists \beta (\text{Spr}(\beta)$ and $\exists s (s(n) = 0 \rightarrow m \leq \gamma(s))$. One may prove in BIM that, for every $\beta$, if $\text{Fan}(\beta)$, then $\text{Fan}^+(\beta)$ if and only if $\exists \beta \forall s \in \omega^n (\beta(s) = 0 \rightarrow s < \delta(n))$. $F \subseteq \omega$ is an (explicit) fan if and only if there exists an (explicit) fan-law $\beta$ such that $F = F_\beta$.

13.8. Real numbers. The development of real analysis in BIM has been described in [31]. Recall that in Subsection 2.1 we introduced the following notation: for each $n, n' := K(n)$ and $n'' := L(n)$, and for all $m, n, (m, n) := J(m, n)$. The last part of Axiom 2 then reads as follows: $\forall m \forall n[m, n' = m \land (m, n)n' = n \land n = (n', n'')]$.

For all $m, n$ in $\omega, m =_2 n \leftrightarrow m' + n'' = m' + n'' \land m <_z n \leftrightarrow m' + n' < m' + n'$ and $m \leq z n \leftrightarrow m' + n'' \leq m' + n'$ and $m + z n := (m' + n', m'' + n'')$ and $m - z n := (m' - n', m'' - n')$ and $m \leq z n := (m' \cdot n', m'' \cdot n')$ and $m \leq z n := (m' \cdot n' + m'' \cdot n', m' \cdot n'' + m'' \cdot n')$ and $0_2 := (0, 0)$ and $1_2 := (1, 0)$. $\mathbb{Q}$ is the set of all $m$ such that $m'' > 0_2$. For all $p, q \in \mathbb{Q}, p =_Q q \leftrightarrow p' = q' \land p'' = q''$ and $p <_Q q \leftrightarrow p' < q' \land p'' < q''$ and $p <_Q q \leftrightarrow p' < q' \land p'' < q''$. $\mathbb{Q}$ is the set of the code numbers of the rational segments, i.e., $\forall s[s \in \mathbb{Q} \leftrightarrow \exists \alpha(s' \in \mathbb{Q} \land s'' \in \mathbb{Q} \land s <_Q s'')$. For every $s$ in $\mathbb{S}$, $\text{length}_2(s) := s'' - q s'$. For all $s, t$ in $\mathbb{S}, s \leq t \leftrightarrow t' \leq Q s' <_Q s''$. For every $s$ in $\mathbb{S}, s + \mathbb{Q} t$.
13.9. Coverings. Let $X$ be a subset of $\mathcal{R}$ and let $C$ be a subset of $\mathcal{S}$. We define: $C$ covers $X$, notation: $\text{Cov}_C(X)$, if and only if $\forall \alpha \in X \exists \exists \in C[\alpha(n) \sqsubseteq \mathcal{S}]$.

Let $C$ be a finite subset of $\mathcal{S}$, and let $s$ be an element of $\mathcal{S}$. Note that $C$ covers $X$ if and only if $\exists \exists \in \mathcal{S}[\text{length}(u) = n \land \forall i < n[u(i) \in C] \land (u(0))' < Q s' < Q (u(0))'' \land \forall i < n-1[u(i) \equiv Q (u(i+1))] \land (u(n-1))' < Q s'' < Q (u(n-1))'']$. We thus may decide, if $C$ covers $X$ or not. Also note that, if $C$ covers $X$, then $\exists \forall \gamma \in \mathcal{S}[\gamma \equiv \mathcal{S}] \land (\gamma(0)) = \mathcal{S}$. For each $s < n$, $p_C(n) = (p - q \frac{1}{n}, p + q \frac{1}{n})$. For each $s$ in $\mathcal{S}$, we define $\overline{\mathcal{S}} := [(s')_\mathcal{R}, (s'')_\mathcal{R}]$.

13.10. Open subsets of $\mathcal{R}$. For every $\alpha$ we let $H_\alpha$ be the set of all $\gamma$ in $\mathcal{R}$ such that $\exists s \in \mathcal{S}[s \in D_\alpha \land \gamma(n) \sqsubseteq \mathcal{S}]$. A subset $H$ of $\mathcal{R}$ is open if and only if, for some $\alpha$, $H = H_\alpha$. For each $t$ in $\omega$ we let $H_t$ be the set of all $\gamma$ in $\mathcal{R}$ such that, for some $\exists s \in \mathcal{S}[s \in D_t \land \gamma(n) \sqsubseteq \mathcal{S}]$. Note that, for every $\alpha$, $H_\alpha = \bigcup H_{\alpha_t}$. Also note that, for every $X \subseteq \mathcal{R}$, $\forall t[X \subseteq H_\alpha \leftrightarrow \text{Cov}_C(D_\alpha)]$ and $\forall t[X \subseteq H_t \leftrightarrow \text{Cov}_C(D_t)]$.

For all $\forall A \subseteq \mathcal{R}$, for all $\alpha, \beta$ in $\mathcal{R}$ such that $\alpha \leq \beta$, $A$ is progressive in $[\alpha, \beta]$ if and only if $\forall \gamma \in [\alpha, \beta][0 \leq \gamma \subseteq A]) \rightarrow \gamma \in A]$.

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Institute for Mathematics, Astrophysics and Particle Physics, Faculty of Science, Radboud University Nijmegen, Postbus 9010, 6500 GL Nijmegen, the Netherlands

Email address: W.Veldman@science.ru.nl