The metric of extended Einstein equation and Schwarzschild solution in six dimensions

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Abstract. The metric of a warped space-time with two extra dimensions is established by means of the Einstein equations in six dimensions and the compactification of two extra dimensions on a square. It is shown that at every among two fixed points our manifold reduces to the five-dimensional Randall - Sundrum space-time and the hierarchy problem could be solved. The Schwarzschild solutions have been extended in six dimensions. We also proposed an equation for the mass operator generated from extra dimensional coordinates.

1. Introduction

Besides the extended models of Standard Model (SM), the extra dimensions have been considered as the ways to investigate both well-studied and new problems of particle physics and Cosmology, in the new physics beyond the SM to unify gravity with gauge interactions. Extra dimensional theories can lead to really significant physical consequences.

We started with the following important observations

(i) Solving the mass hierarchy problem by means of the wrap factor attaching to the four-dimensional metric is very successful [1,2];

(ii) The proton stability can be explained within the framework of a theory with two extra dimensions [3,4] that ensures a lifetime of proton longer than the current experimental bounds [4], even in the presence of baryon violation at TeV scale, and forces the neutrino masses to be of the Dirac type [5]. It also explains the origin of electroweak symmetry breaking [6], the number of fermion generation [7] and the breaking of grand unified gauge groups [8-10].

(iii) The existence of chiral fermions [3] in the four-dimensional effective theory has good prospect for setting up the six-dimensional standard model within a warped space-time.

We assume that the geometry of our theory is a 6D Riemann manifold, whose metric is expressed as

\[ ds^2 = G_{AB} dx^A dx^B = e^{-2f(y_1, y_2)} g_{\mu\nu} dx^\mu dx^\nu - dy_1^2 - dy_2^2, \]

where \( g_{\mu\nu} = \text{diag}(1, -1, -1, -1) \) is the 4-dimensional tensor metric. The coordinates are \( x^A = x^\mu, y_1, y_2 \) with \( \mu, \nu = 0, 1, 2, 3 \) labelling the brane directions, \( 0 \leq y_1, y_2 \leq L \).
The general 6-dimensional action takes the form

\[ S = S_{\text{grav}} + S_{\text{hid}} + S_{\text{vis}} \]

\[ = \int d^6x \int dy_1 dy_2 \sqrt{-G} \left[ (-\Lambda + 2M^4)R \right. \]

\[ \left. - \sqrt{g_{\text{vis}}} V_{\text{vis}} \delta(y_1 - L) \delta(y_2 - L) - \sqrt{g_{\text{hid}}} V_{\text{hid}} \delta(y_1) \delta(y_2) \right]. \]  \hfill (2)

The 6-dimensional Einstein’s Equation reads

\[ \sqrt{-G} \left( R_{AB} - \frac{1}{2} G_{AB} R \right) = \frac{1}{4M^4} \left[ \Lambda \sqrt{-G} G_{AB} \right. \]

\[ + V_{\text{vis}} \sqrt{-g_{\text{vis}}} g_{\mu \nu}^{\text{vis}} \delta^{\rho \sigma} \delta(y_1 - r_c) \delta(y_2) \]

\[ \left. + V_{\text{hid}} \sqrt{-g_{\text{hid}}} g_{\mu \nu}^{\text{hid}} \delta^{\rho \sigma} \delta(y_1) \delta(y_2) \right], \]  \hfill (3)

where the \((3 + 1)\)-dimensional space-time is connected with the \((4 + 2)\)-dimensional by

\[ g_{\mu \nu}^{\text{vis}}(x) = G_{AB}(x, y_1 = L, y_2 = L), \]

\[ g_{\mu \nu}^{\text{hid}}(x) = G_{AB}(x, y_1 = 0, y_2 = 0). \]  \hfill (4)

Our aim was to determine the analytical expression for \(f(y_1, y_2)\) so that above metric is a solution of Einstein’s equation extended in six dimensions. We also obtained the physical consequences when the 6D metric was considered by the 4D effective theory. Similarly in the case of RS space-time, the hierarchy problem was solved due to the Planck mass. We also consider the metric in the central symmetric system of 6D coordinates. In addition, the equation for mass operator in extra dimensions was proposed.

This paper is organized as follows. In Section 2, the Einstein equations extended in 6D space-time are solved in the cylindrical coordinate system. Section 3 is devoted to investigate the hierarchy problems. The Schwarzschild solutions in 6D are extended in Section 4. In Section 5, we present an equation for mass operator inclusive extra dimensional coordinates. The discussions and conclusions are given in Section 6.

2. The six-dimensional solution of Einstein’s equation in the cylindrical symmetry

We impose two external variables in the cylindrical coordinates system

\[ y_1 = \rho \cos \theta; y_2 = \rho \sin \theta, \]  \hfill (5)

where \(0 \leq \rho \leq r_c, -\pi \leq \theta \leq \pi, f(y_1, y_2) = f(\rho, \theta).\)

This leads to the following metric

\[ ds^2 = e^{-2f(\rho, \theta)} g_{\mu \nu} dx^\mu dx^\nu - d\rho^2 - \rho^2 d\theta^2. \]  \hfill (6)

We determine the bulk metric which is the solution of 6-dim Einstein’s equation

\[ \sqrt{-G} \left( R_{AB} - \frac{1}{2} G_{AB} R \right) = \frac{1}{4M^4} \left[ \Lambda \sqrt{-G} G_{AB} \right. \]

\[ + V_{\text{vis}} \sqrt{-g_{\text{vis}}} g_{\mu \nu}^{\text{vis}} \delta^{\rho \sigma} \delta(\rho - r_c) \delta(\theta - \pi) \]

\[ \left. + V_{\text{hid}} \sqrt{-g_{\text{hid}}} g_{\mu \nu}^{\text{hid}} \delta^{\rho \sigma} \delta(\rho) \delta(\theta) \right], \]  \hfill (7)

where

\[ g_{\mu \nu}^{\text{vis}}(x^\mu) = G_{AB}(x^\mu, \rho = r_c, \theta = \pi), \]

\[ g_{\mu \nu}^{\text{hid}}(x^\mu) = G_{AB}(x^\mu, \rho = 0, \theta = 0). \]  \hfill (8)
and
\[
G = \text{det}G_{AB}; \sqrt{-G} = r_c e^{-4f(\rho,\theta)},
\]
\[
R = 4\left[2 f^\rho_{\rho}(\rho, \theta) + \frac{2}{r_c^2} f^\theta_{\theta}(\rho, \theta) - 5 f^2_\rho(\rho, \theta) - \frac{5}{r_c^2} f^2_\theta(\rho, \theta)\right].
\]

By separating the indices \(\mu\nu, \theta\theta, \) and \(\rho\rho,\) we derive the system of equations from the Einstein equation (7)
\[
4 f^\rho_{\rho}(\rho, \theta) + 10 f^2_\rho(\rho, \theta) + \frac{6}{r_c^2} f^2_\theta(\rho, \theta) = -\frac{\Lambda}{4M^4}, \quad (9)
\]
\[
- \frac{4}{r_c^2} f^\theta_{\theta}(\rho, \theta) + \frac{10}{r_c^2} f^2_\theta(\rho, \theta) + 6 f^2_\rho(\rho, \theta) = -\frac{\Lambda}{4M^4}, \quad (10)
\]
\[
3 f^\rho_{\rho}(\rho, \theta) + \frac{3}{r_c^2} f^\theta_{\theta}(\rho, \theta) - 6 f^2_\rho(\rho, \theta) - \frac{6}{r_c^2} f^2_\theta(\rho, \theta),
\]
\[
= -\frac{\Lambda}{4M^4} + \frac{V_{\text{vis}}}{4M^4 r_c} \delta(\rho - r_c)\delta(\theta - \pi) + \frac{V_{\text{hid}}}{4M^4 r_c} \delta(\rho)\delta(\theta). \quad (11)
\]

It is clear that when \(\rho\) coordinate is absent, i.e. the sixth dimension disappears, Eq. (9) - (11) reduce to the Einstein’s Eq. in the RS space - time.

Here the single scale \(k\) is related to cosmological constant \(\Lambda\) by
\[
\Lambda = -24k^2 M^4, \quad (12)
\]
and
\[
V_{\text{vis}} = -24M^4 k; \quad V_{\text{hid}} = 24M^4 k. \quad (13)
\]

The solution for the bulk metric reads
\[
ds^2 = e^{-2k^2 \mu_\rho d\mu dx^\rho} - d\rho^2 - \rho^2 d\theta^2. \quad (14)
\]

Generally, the solution - metric of Einstein’s Equation in 6 dimensional space - time takes the form
\[
ds^2 = e^{-2k|z|} g_{\mu\nu} dx^\mu dx^\nu - dy_1^2 - dy_2^2, \quad (15)
\]
where \(z = y_1 + iy_2.\)

It is clear that (15) will reduce to the RS metric in the following cases:

(i) Compactification on a 2 - dimensional sphere \((S^2/Z_2)\)
\[
y_1^2 + y_2^2 \leq r_c^2.
\]
At \(y_2 = 0\) we have \(-r_c \leq y_1 \leq r_c,\) then the metric (15) will takes the form
\[
ds^2 = e^{-2k|z|} g_{\mu\nu} dx^\mu dx^\nu - r_c^2 d\phi^2, \quad (16)
\]
in which \(y_1 = r_c \phi, -1 \leq \phi \leq 1.\)

Metric (16) is exactly the RS metric if we identify (16) at \(\phi = 1\) with that at \(\phi = -1.\) Hence, the manifold (15) contains two RS space - time (16) as its submanifolds at \(y_1 = 0\) and \(y_2 = 0.\)

(ii) Compactification on a square \((T^2/Z_2)\)
\[
0 \leq y_1, y_2 \leq L.
\]

In this case (15) is also to be reduced to (16) at \(y_1 = 0, L\) and \(y_2 = 0, L.\) As a result, the manifold (15) contains two RS manifolds at the square summits if we identify the point \(y_1 = 0\) with \(y_1 = L\) and the point at \(y_2 = 0\) with \(y_2 = L.\)
3. The Planck mass and hierarchy problem

We consider scalar field $T(x)$ so that its VEV is the compact radius $r_c = \text{const}$. The local 4-dimensional metric is given by

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad (17)$$

where $h_{\mu\nu}$ is symmetric tensor and very small. It is identified to physical gravitational field $h_{\mu\nu}$ of 4-dimensional effective theory, $|h_{\mu\nu}| \ll 1$.

The 6-dim metric takes the form

$$ds^2 = e^{-2k|T(x)|} [\eta_{\mu\nu} + h_{\mu\nu}] dx^\mu dx^\nu - dT^2(x) - T^2(x) d\theta^2. \quad (18)$$

The four dimensional effective action is given by

$$S_{\text{eff}} = \int d^4x \int dT(x) \int r_c d\theta \sqrt{-\bar{g}} e^{-4k|T(x)|} \left[ -\Lambda + 2M^4 e^{2k|T(x)|} \bar{R} \right], \quad (19)$$

where $\bar{g} = \det g_{\mu\nu}, \bar{R} = R e^{-2k|T(x)|}$.

We focus on the mass term in the 4-dim gravitational action

$$S_{\text{grav}} = -\int d^4x \sqrt{-\bar{g}} 2M_{Pl}^2 \bar{R}, \quad (20)$$

where $M_{Pl}$ is 4-dimensional Planck mass.

Compare (19) to (20), we derive

$$M_{Pl}^2 = -M^4 \int_{-\pi}^{\pi} r_c d\theta \int e^{-2k|T(x)|} dT(x) = \frac{M^4 r_c}{2k} \int_{-\pi}^{\pi} d\theta e^{-2k|T(x)|}. \quad (21)$$

With $|T(x)| = r_c|\theta|$, it becomes

$$M_{Pl}^2 = \frac{M^4}{2k^2} \left( 1 - e^{-2kr_c\pi} \right). \quad (22)$$

Similarly in the case of 5 dimensions, the result (22) shows that $M_{Pl}$ depends weakly on $r_c$ in the large $kr_c$ limit. By means of an adequate value of $k$, the 6-dimensional mass $M^4$ has the same power with 4-dimensional Planck mass. That means the hierarchy problem could be solved.

4. The Schwarzchild solution extptended in the 6D space-time

In this section we find the tensor metric that satisfies the central symmetry in 6D extended Einstein equation in the vacuum

$$R^M_N - \frac{1}{2} \delta^M_N R = -\Lambda \delta^M_N, \quad (23)$$

where we denote the Latin labels $M, N = 0, 1, 2, 3, 5, 6$ with $x^0 = t, x^1 = r, x^2 = \theta, x^3 = \varphi, x^5 = \rho, x^6 = \phi$. The metric in the spherical coordinates system corresponding to (23) reads

$$ds^2 = dt^2 - [dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)] - d\rho^2 - \rho^2 d\phi^2. \quad (24)$$

Based on the Schwarzchild solution in 4D, we impose the metric in the form

$$G_{MN} = \text{diag} \left( e^{2\alpha(r)}, -e^{2\beta(r)}, -r^2, -r^2 \sin^2 \theta, -1, -\rho^2 e^{2\sigma(r)} \right). \quad (25)$$
Tensor Ricci $R^M_M$ and $R$ in the Eq (23) are calculated from the non vanishing Christoffel symbols $\Gamma^P_{MN}$. In two external dimensions it is determined by

$$\Gamma^5_{66} = - \{ \rho^2 \sigma' (\rho) + \rho \} e^{2\sigma(\rho)}; \quad \Gamma^6_{56} = \Gamma^6_{65} = \sigma' (\rho) + \frac{1}{\rho};$$  \tag{26}

$$R_{55} = \sigma'' + \left( \sigma' \right)^2 + \frac{2\sigma'}{\rho}; \quad R_{66} = \rho^2 \left( \sigma'' + \left( \sigma' \right)^2 + \frac{2\sigma'}{\rho} \right) e^{2\sigma(\rho)}. \tag{27}$$

The functions on the exponents of (25) are given by

$$\alpha(r) = - \beta(r) = \ln \left( 1 - \frac{C_1}{r} \right)\frac{1}{2}; \tag{28}$$

and

$$\sigma(\rho) = \ln \left( 1 - \frac{C_2}{\rho} \right). \tag{29}$$

The constants $C_1, C_2$ are derived from the condition of the weak (Newton) gravitational field at the limit $r \to \infty$ and $\rho \to \infty$ (in the linear approximation)

$$C_1 = 2GM, C_2 = -GM. \tag{30}$$

Here Newton constant $G$ is given by volumetric scaling of truly fundamental gravity scale $G_\star$, which has dimension $[G_\star] = [\text{Energy}]^{-(n+2)}$ \cite{11}.

$$G = \frac{G_\star}{V_n}. \tag{31}$$

The extended Schwarzschild solution in 6D space-time takes the form

$$ds^2 = \left( 1 - \frac{2GM}{r} \right) dt^2 - \left( 1 - \frac{2GM}{r} \right)^{-1} dr^2 - r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) - d\rho^2 - \left( 1 - \frac{2GM}{\rho} \right)^{-1} \rho^2 d\phi^2. \tag{31}$$

Note that the metric (31) has singularities at $r = \rho = 2GM = r_0$, within which the spheres are trapped surfaces \cite{12}. When $M \to 0$ the metric (31) is restored to 6D normal space-time (24). When $r \to \infty$ and $\rho \to 0$ this metric approaches the flat asymptotic.

5. The mass operator in extra dimensions

We have already determined the bulk metric in 6 dimensions. Let us now discuss the equation like Klein-Gordon equation, from which the mass of particles in extra dimensions is presented.

We start from the Klein-Gordon in the (1, 3) dimensions

$$\left( \Box - m^2 \right) \varphi(x_\mu) = 0. \tag{32}$$

Note that the spinor field which is described by the Dirac equation

$$i\gamma^\mu \frac{d\psi}{dx^\mu} + m\psi = 0,$$

also satisfies the Klein-Gordon equation. Hence, equation (32) could also determine the mass of fermions.
In the \((1, 3 + n)\) dimensions, this equation takes the general form

\[
\left( \Box + \sum_{a=1}^{n} \frac{\partial^2}{\partial y_a^2} - M^2 \right) \Phi(x, y_a) = 0.
\] (33)

By separating the variables

\[
\Phi(x, y_a) = \varphi(x) \psi(y_a),
\]

we have

\[
\left( \Box + \sum_{a=1}^{n} \frac{\partial^2}{\partial y_a^2} - M^2 \right) \varphi(x) \psi(y_a) = 0,
\]

\[
\Box \varphi(x) = -\sum_{a=1}^{n} \left( \frac{\partial^2}{\partial y_a^2} - M^2 \right) \psi(y_a) = m^2.
\] (34)

Hence we derive two equations

\[
( \Box - m^2 ) \varphi(x) = 0,
\] (35)

\[
M^2 - \sum_{a=1}^{n} \frac{\partial^2}{\partial y_a^2} \psi(y_a) = m^2 \psi(y_a).
\] (36)

Equation (35) expresses that \(m\) is the mass of particles in physical space-time and equation (36) shows that \(m\) is the eigenvalue of spectrum mass operator corresponding to eigenfunction \(\psi(y_a)\) in extra dimensions.

\[
\tilde{m}^2 = M^2 - \sum_{a=1}^{n} \frac{\partial^2}{\partial y_a^2}.
\] (37)

By transferring to the momentum space

\[
\psi(y_a) = \int \frac{d^n y}{(2\pi)^{n/2}} u(q_a) \exp(ip^a y_a),
\] (38)

we obtain

\[
m^2 = M^2 - \sum_{a=1}^{n} p_a^2 \quad \text{or} \quad \sum_{a=1}^{n} p_a^2 = m^2 - M^2.
\] (39)

It shows that the extra momenta are compactified on a sphere if \(m^2 - M^2 > 0\). If \(m^2 - M^2 < 0\) the extra momenta are compactified on a pseudo-sphere when one of the \(n\) extra dimensions is time-like. We are led to the conclusion that even in the models according to the universal extra-dimensions theory, where the extra dimensional space is not compactified, but the extra-momentum space must be compactified.

6. Discussion and conclusion

In the above sections, we have built a formalism for the warped six-dimensional space-time. It shows that this metric could be reduced to the RS space-time in the case of compactifications on two dimensional sphere \((S^2/Z_2)\) or on a square \((T^2/Z_2)\). Similarly in the case of RS space-time, the hierarchy problems is solved due to the Planck mass. In addition, the Schwarzschild solution in six dimensions is extended. We also proposed an equation for the eigenvalues of mass operator inclusive extra dimensions and showed that the extra momentum space must be compactified in any theory on extra dimensions.

It is hoped that this formalism could pave the way for better understanding of various physical phenomena of high energy physics.
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