The gravitational growth index formalism provides a model independent way to look for deviations from general relativity by testing dark energy physics distinct from its effects on the cosmic expansion history. Here we extend the approach to incorporate an early time parameter \( \gamma \), in addition to the growth index in describing the growth of large scale structure. We illustrate its utility for models with modified gravity at high redshift, early acceleration, or early dark energy. Future data will have the capability to constrain the dark energy equation of state, the growth index \( \gamma \), and \( \gamma \), simultaneously, with no degradation in the equation of state determination.

I. INTRODUCTION

Exploration beyond the Standard Model of cosmology is an active area of investigation, trying to understand the nature of the physics causing the acceleration of the cosmic expansion. For the expansion itself, this can be parametrized in a model dependent fashion through the equation of state of an (effective) dark energy field. Within general relativity, and for a field described entirely by its equation of state, this completely describes the cosmic growth history of large scale structure as well. However, one should test this minimal framework.

In general, changes in the expansion history affect the growth history. Rather than treating these as unrelated, the gravitational growth index formalism \([1, 2]\) treats in a unified fashion those effects from a common physical origin, and introduces a new parameter, the gravitational growth index \( \gamma \), to keep distinct new physics such as modification of gravity that breaks the relation between expansion and growth. Model independent parametrization of expansion, through the dark energy equation of state \( w(a) = w_0 + w_a (1 - a) \), where \( a \) is the cosmic scale factor, and of growth through \( \gamma \), were merged into what was called Minimal Modified Gravity in \([3]\). Data would then be used to fit simultaneously the set \( \{ w_0, w_a, \gamma \} \). If \( \gamma \) was consistent with the general relativity value then no modification of gravity would be required.

This framework depended on a standard matter dominated regime for growth at early times (see the derivation by \([2]\)) and could not fully treat models in which the growth was enhanced, rather than suppressed as expected from an accelerating component, relative to the matter dominated limit. Here we broaden the formalism to include growth enhancement, breakdown of high redshift pure matter domination, and other effects of gravitational modifications.

In Section II we introduce the expanded formalism, Beyond the Standard Model 3 (BSM-3), and apply it in Section III to both enhanced and suppressed growth in various modified early universe scenarios. We investigate in Section IV the ability of future data to constrain simultaneously the four parameters of BSM-3 and deliver clear results on the nature of the new physics, and conversely the penalty in biasing results if the extended framework is ignored.

II. EXTENDING GROWTH

In seeking to understand the growth history of cosmic structure, one can either parametrize it directly \([4, 5, 6, 7]\) or seek to use the knowledge that the expansion history already determines a major part of it (indeed all of it within the standard, general relativity scenario). The first path conflates the effects of expansion and growth, so we adopt the second approach as more likely to reveal the physics clearly. Also see the consistency approach of \([8]\). One can also explore modifications of gravity in terms of parametrizing the metric potential functions (see, e.g., \([9, 10, 11]\) and numerous others); this follows the formal application of the field equations but it is unclear how practical it is to relate to observations due to a number of unresolved issues (see the discussion in §4.4 of \([12]\), and also \([13]\)). For a general overview of modified gravity theories see \([14]\) and for classes of acceleration physics see \([15]\).

The gravitational growth index formalism parameterizes the growth itself, but takes account of the expansion history effects. This is more directly related to the observations and can be carried out practically and straightforwardly, but may not include all possible gravitational modifications. In particular, it is not very apt at including anisotropic stress and other scale dependent effects. However, many of the issues in \([12, 13]\) for the metric potential approach also concern the handling of anisotropic stress and scale dependence. Certainly for linear growth on large scales the growth index formalism is a superb approximation, with a accuracy relative to the exact solutions in a wide range of dark energy and modified gravity models at the \(10^{-3}\) level, and a formal derivation by considering deviations from the matter dominated universe \([2]\).

A first step beyond the Standard Model of cosmology with general relativity and a cosmological constant is to use a time varying dark energy equation of state to describe the expansion and growth histories. This adds two parameters \( w_0 \) and \( w_a \) to describe the equation of state as a function of cosmic scale factor, \( w(a) = w_0 + w_a (1 - a) \), and this is generally accurate to the \(10^{-3}\) level \([16]\). We call this case with the two parameter set \( \{ w_0, w_a \} \) as Beyond the Standard Model 1, or BSM-1.
The formalism introduced in [1] to test the gravitational framework extends this to BSM-2. The new parameter $\gamma$, the gravitational growth index, was defined through

$$g(a) = e^{\int_0^a (\delta \rho / \rho)(\Omega_m(a')^{-1})}$$

where $g = (\delta \rho / \rho) / a$ is the linear growth factor of matter density perturbations normalized to the pure matter dominated case, and $\Omega_m(a)$ is the matter density fraction of the critical density as a function of scale factor $a$. Thus BSM-2 involves the parameter set $\{w_0, w_a, \gamma\}$. Within general relativity, $\gamma \approx 0.55$, with a very slight variation with $w_0, w_a$ (see [1] for fitting forms and [2] for a derivation).

In the matter dominated epoch, $\Omega_m(a) \rightarrow 1$, and the presence of (effective) dark energy impels $\Omega_m(a) < 1$ so the integrand is negative and growth is suppressed relative to the pure matter dominated case: $g < 1$. This is exactly what we would expect in the presence of an accelerating component. However, one could consider particular circumstances where growth is enhanced. This can formally be handled by allowing for a negative growth index, $\gamma < 0$. However as the matter dominates more completely going back to high redshifts, $\gamma$ is driven toward $-\infty$ if growth is for some reason enhanced. Thus, the formalism gives the alarm that something non-standard is going on, but is not that useful in characterizing exactly what. In addition, if the matter domination at high redshifts is itself non-standard, for example part of the nonrelativistic energy density is not due to matter, which clusters, but to early dark energy with an equation of state $w = 0$, which does not cluster, then this also affects the growth index formalism, despite still suppressing the growth.

Thus, although the gravitational growth index works quite well in a large variety of cases, there do exist exceptions that alter the framework in such a way that could mislead us in interpreting the parameter fits. What is required is basically a calibration of the growth behavior at early times, how it deviates from the matter dominated high redshift expectation. This is what we correct for in BSM-3 with the introduction of an early time calibration parameter $g_\ast$, making the simple yet palpable change:

$$g(a) = g_\ast e^{\int_0^a (\delta \rho / \rho)(\Omega_m(a')^{-1})}.$$

We justify and discuss this form in the Appendix. In the standard case, $g_\ast = 1$. Enhanced growth involves $g_\ast > 1$, and any deviation from $g_\ast = 1$ signals a non-standard early universe behavior.

So BSM-3 describes the cosmological observations at a deeper level with the parameter set $\{w_0, w_a, \gamma, g_\ast\}$. We show in [1V] that this enlarged set indeed delivers new insights while preserving the information from the expansion history at the constraint level of BSM-1 and most of the gravitational information at the level of BSM-2. Simultaneously, however, it avoids the bias that would occur in BSM-1 or BSM-2 if their assumptions of the restricted framework were invalid.

Note that $g_\ast$ serves as a necessary anchor, or calibration, for growth just as the parameter $\mathcal{M}$ (invoking the absolute luminosity) is required for anchoring supernova distances to low redshift or $\mathcal{S}$ (invoking the absolute sound horizon scale) is required for anchoring baryon acoustic oscillation distances to high redshift (see [17, 18, 19]). In any of the three cases one can avoid the need for calibration by considering only relative quantities, e.g. $d(z_1)/d(z_2)$ or $g(a_1)/g(a_2)$, but this comes at the price of degrading the cosmological leverage of the data; [18] showed the degradation factor is of order 2 for both the supernovae and baryon acoustic oscillation cases.

### III. APPLICATIONS

We now apply the new formalism to several specific examples to show that the parameters are well defined, extend the reach of the previous framework, and how each probes particular aspects of the physics.

#### A. Early Dark Energy

Early dark energy refers to dark energy with a non-negligible fraction $\Omega_e$ of the total energy density at high redshift, such as during recombination. Models that include dilatation symmetries, including one of the first dark energy models [20], can have energy densities scaling at high redshift as the dominant component, so possessing $w = 0$ and a constant fraction of the matter density during the matter dominated epoch. The dark energy does not cluster though and the exact solution in linear growth theory (for $\Omega_e \ll 1$) would give $\gamma = 0.6$.

However, the equation of state of early dark energy (EDE) does not remain at $w = 0$, but evolves toward $w = -1$ to give acceleration. For the EDE parametrization of [21], the expansion history over $z \approx 0 - 2$ is fit to 0.02% by a non-EDE model with the same present equation of state $w_0$ and $w_a \approx 5\Omega_e$ [18]. Thus expansion history observations would lead us to expect a growth index given by the fitting formula calibrated to $10^{-3}$ accuracy on $w_0 - w_a$ models [1] as

$$\gamma = 0.55 + 0.05 \left[ 1 + w(z = 1) \right] = 0.55 + 0.05 \left[ 1 + w_0 + (5/2)\Omega_e \right].$$

Note that the equation of state is supposed to be evaluated at $z = 1$, and so one should not take $w = 0$ for EDE. For viable models, this gives $\gamma \approx 0.55$, just like normal $w_0 - w_a$ models. However, the growth in EDE models can be very different (even from the $\gamma = 0.6$ case) because the EDE impacts the high redshift matter domination.

The early time growth calibration factor $g_\ast$ is precisely what is needed to resolve the discrepancy. Figure [1] plots $g_\ast$ as a function of scale factor for the case $w_0 = -0.95$, $\Omega_e = 0.03$. Over the range where growth observations can be made, $g_\ast$ is constant to high precision, 0.2% for
The constancy of \( g_* \) becomes even stronger for smaller \( \Omega_e \), with any deviations vanishing linearly with \( \Omega_e \) (e.g. \( \delta g_*/g_* < 0.0005 \) for \( z < 3 \) when \( \Omega_e = 0.01 \)). We have thus found that \( g_* \) is truly a single parameter and one distinct from the previous (BSM-2) parameters, capable of extending our physics knowledge through testing the high redshift growth framework.

\[ \delta Q \approx a^{(\sqrt{1+2\delta Q}-1)/4} \approx a^{-(3/5)\delta Q} , \]

where \( \delta Q = Q - 1 \). However, constant \( \delta Q \) is restrictive. Considering the source term \( GR_m(a) \), if we leave the matter density to be given by the expansion history, then the gravitational modification enters through \( G \). By definition of the present gravitational coupling as \( G_{\text{Newton}} \) (we do not consider scale dependence here, see [11]), the variation goes to zero today so we cannot have a time independent \( \delta Q \).

Within scalar-tensor theories of gravity, \( G \) can vary. The impact of this on the gravitational growth index framework was discussed in [2] for a model that preserved the high redshift matter domination. Here we consider the opposite situation of a deviation at early times, but having \( G_{\text{Newton}} \) today. As a toy model we take

\[ \delta Q = \frac{B}{1 + (a/a_i)^{3/5}} . \]

This smoothly transitions from \( G = (1 + B)G_{\text{Newton}} \) at high redshift \( (a \ll a_i) \) to \( G_{\text{Newton}} \) today. When \( B > 0 \) we expect enhanced growth and so \( g_* > 1 \). As long as \( a_* \) is not too close to the present, this modification is an early time phenomenon; solving the growth equation we indeed find a signal in the deviation \( g_* \neq 1 \) while \( \gamma \) is unaffected. Again, we see the separation of physical effects, with \( g_* \) serving as an independent early time growth calibration parameter.

For example, taking the expansion history to be the standard model of matter plus a cosmological constant, the growth within this modified gravity model preserves \( \gamma = 0.55 \), but not \( g_* = 1 \). With \( B = 0.03 \), \( q = 3 \) (e.g. the scalar-tensor theory thaws from a frozen scalar field), and \( a_i = 10^{-3} \) (evolution begins near matter-radiation equality), we find \( g_* = 1.042 \). Again, describing this by a single parameter rather than a function of scale factor is

\[ \delta \rho/\rho \sim a^{(\sqrt{1+2\delta Q}-1)/4} \approx a^{-(3/5)\delta Q} , \]

because \( g_* \) can be so well approximated as a constant over the observational epochs, this allows us to robustly define the growth index \( \gamma \) without confusion from \( g_* \). Specifically, \( g_* \) cancels out in the ratio \( g(a_2)/g(a_1) \) for \( a_1, a_2 > 0.1 \). Fitting the growth index for this interval gives \( \gamma = 0.556 \), in excellent agreement with Eq. (3) which gives \( \gamma = 0.55625 \). Indeed the deviations from the exact growth ratio as solved numerically are below 0.025% for \( z < 4 \) (0.11% for \( a > 0.1 \)). This approach makes clear that the growth index \( \gamma \) describes the relative growth behavior and the new parameter \( g_* \) serves as a calibration for the absolute growth, compared to the standard high redshift matter domination.

In the EDE case, an excellent fitting form is

\[ g_* = 1 - 4.4 \Omega_e . \]

Thus a determination of \( g_* \) from data leads to a constraint on the early dark energy density of \( \sigma(\Omega_e) \approx 0.23 \sigma(g_*). \) A 10% estimation of \( g_* \) would give a tighter bound on \( \Omega_e \) than current constraints. (We return to this in [LV])

\[ \gamma = 0 \text{ to } 3 (1.4\% \text{ out to } a = 0.1) \). This justifies the treatment of the calibration as a single parameter, rather than a function of scale factor, in the same way that the growth index \( \gamma \) is a single, constant parameter. Moreover, we find \( g_* = 0.87 \), distinct from the standard case of \( g_* = 1 \), giving a clear sign that at high redshift there is deviation from the standard matter domination scenario.
an excellent approximation, with constancy preserved to 0.03% over $a = 0.1 - 1$. If we choose $B < 0$ then we have growth suppression instead, $g_* = 0.959$. The results are quite insensitive to the exact form of the transition in $G$, i.e. the value of $q$. A general fitting form for $a > 10^{-4}$ is

$$\delta g_* \approx 1.4 B \log(a/10^{-4}). \quad (7)$$

We emphasize that $\gamma$ is the parameter seeing deviations in the form of the gravitational growth equations, and $g_*$ probes deviations in the early time growth. Gravitational modifications such as DGP braneworld theory, which look like general relativity at early times, give $g_* = 1$. The signal of deviation from general relativity for this theory appears in linear growth via the far from standard value of $\gamma = 0.68 \ [1, 20]$. Thus $g_*$ and $\gamma$ probe different aspects of the gravitational framework.

C. Early Time Acceleration

One of the puzzles of cosmic acceleration is the coincidence problem: why does acceleration happen now, within the last factor 2 in expansion out of the $\sim 10^{58}$ since inflation? While this can be solved, or at least ameliorated, by use of certain time or length scales (e.g. the transition from radiation to matter domination or the magnitude of the scalar curvature or Hubble parameter), an alternate solution is removing the coincidence by having acceleration occur several times since inflation. See, for example, models by [27, 28, 29].

Since periods of acceleration act to suppress growth, stringent limits can be placed on the length of any such epochs, with [29] constraining the length $\Delta \ln a$ to less than $\sim 5\%$ of the Hubble e-folding time based on the total growth to the present. The effect on early growth due to a high redshift epoch of acceleration should be captured precisely by $g_*$, and furthermore we can test that $\gamma$ is unaffected – i.e. again $g_*$ is probing distinct physics.

We consider a period, beginning at $a_{\ast }$ and ending at $a_{\gamma } = a_{\ast } e^{\Delta \ln a}$, when the dark energy density $\Omega_{\rm de} = 1$ and $w = -1$, i.e. a cosmological constant-like dark energy completely dominates at some early epoch. (In this toy model we take the matter density to be restored after this epoch to its previous value, $\Omega_{m}(a_{\ast }) = \Omega_{m}(a_{\gamma })$.) This indeed suppresses growth, and we find that for $z_{\gamma } \geq 30$ the value of the gravitational growth index $\gamma$ for observations at $z \lesssim 3$ is unaffected\(^1\), while $g_*$ is offset from unity in a manner accurately described by a single constant parameter. A fitting formula is

$$g_* = 1 - 1.2 \Delta \ln a. \quad (8)$$

\(^1\) Recall that for strong suppression of growth, $\gamma$ gets large, and due to the inertia of the growth rate, $\gamma$ does not fully recover until a couple of $e$-folds of expansion have passed.

D. Dark Coupling and Scale Dependence

While the BSM-3 framework is valuable for probing gravity and early time growth, it is not all encompassing. One can have theories with individual features which cannot be reduced to two gravitational parameters. These would be compared to observations on a theory by theory basis – the point of the Beyond the Standard Model framework is to obtain model independent guidance to testing the physics.

Two classes of theories also require further specialization. One is theories involving scale dependence, where even the linear growth regime is not dependent purely on scale factor but behaves differently for different waveforms. An example is $f(R)$ theories (see, e.g., [31]). Here a successful fitting formalism is the parametrized post-Friedmann approach of [32]. For mild scale dependence due to the dark energy sound speed differing from the speed of light the $\gamma$ fitting formula has been extended by [33]. Another class is theories that introduce non-gravitational coupling between (dark) matter and dark energy.

Dark coupling is problematic for several reasons: it 1) violates the Equivalence Principle, 2) introduces additional, non-gravitational forces, e.g. a non-Hubble friction term, and 3) is not well constrained regarding the form of the coupling. Nevertheless, for small values of the coupling strength it has been shown to be well approximated by the $\gamma$ formalism [1]. For working within the coupling theory class (at least for some forms of coupling), an accurate fitting form has been developed by [34, 35].

Because of remaining issues with general, rigorous treatment of scale dependence (see §4.4.2 of [12], and [13]) and the ability of arbitrary forms of coupling to generate arbitrary growth behaviors, we do not address these classes further. No finite parametrization will fit every class of model imaginable, so we focus on broadly applicable, model independent, compact yet highly accurate parametrizations. The growth parameters $\gamma$ and $g_*$ serve these theoretical purposes well. We next address whether they have observational practicality.

IV. GLOBAL PARAMETER FITS

While the parameter $g_*$ carries new physics insights with it, we have to make sure that the addition of it is
practical: it must be reasonably constrained, and including it in the fit must not substantially degrade the other parameter constraints. For example, the inclusion of $\gamma$ in going from BSM-1 to BSM-2 had little deleterious effect on the estimation of $w_0$, $w_a$.

Because $g_\ast$ calibrates growth, it is degenerate with other absolute growth parameters such as the primordial scalar perturbation amplitude $A_s$ or the present equivalent, the mass fluctuation amplitude $\sigma_8$. Thus it will require constraints from CMB data or techniques sensitive to $\sigma_8$ such as weak gravitational lensing or cluster abundances to separate out cleanly $g_\ast$. Since with present data these other parameters are already estimated to better than 5%, and should improve further with forthcoming data, this is not an insurmountable obstacle.

### A. Future Constraints

To analyze the constraints on the BSM-3 parameters \(\{w_0, w_a, \gamma, g_\ast\}\) we consider linear growth measurements over various redshift ranges, together with CMB data of Planck quality (in the form of a 0.2% prior on the reduced distance to last scattering), and supernova distance data for \(z = 0 - 1.7\) as from a SNAP-type JDEM. The default growth data has 2% precision at \(z = 0.1 - 1\) every 0.1 in redshift, and the fiducial cosmology is flat ΛCDM, so \((w_0, w_a, \gamma, g_\ast) = (-1, 0, 0.55, 1)\).

The correlation coefficients between $g_\ast$ and the other cosmological parameters are quite small, e.g. \(r(\Omega_m, g_\ast) = 0.12\), with the largest one being \(r(\gamma, g_\ast) = 0.86\). This lack of strong degeneracies is heartening and indicates that $g_\ast$ can function as a distinct parameter. The uncertainties and full confidence contour of the equation of state parameters are unaffected by the addition of $g_\ast$ in conjunction with the growth data, while the fully marginalized \(\sigma(\gamma) = 0.081\) and \(\sigma(g_\ast) = 0.018\). However, this does represent a factor 2.0 degradation in knowledge of $\gamma$, relative to fixing $g_\ast = 1$, and a factor 2.0 in the area of the main growth parameters $\Omega_m$-$\gamma$ confidence contour, as Fig. 2 illustrates. This is in line with the other “calibration” parameters, where marginalizing over the supernova distance calibration $\mathcal{M}$ changes the expansion parameters $w_0$-$w_a$ contour area by a factor 1.9, and marginalizing over the baryon acoustic oscillation sound horizon calibration $S$ changes the $w_0$-$w_a$ contour area by 2.3.

Because the dependence of growth on $g_\ast$ is flat with redshift, using higher redshift data per se would not be expected to directly help estimation of $g_\ast$. Indeed, because sensitivity to $\gamma$ peaks at low redshift (see Fig. 1 of [30]), degeneracies are broken best at $z < 1$. However, enough residual degeneracy remains that additional (not substitute) measurements at higher redshift do tighten constraints; see Fig. 3. Extending the redshift range of growth measurements over $z = 0.1 - 2$ improves the constraints more than statistically, with two times the number of measurements giving an almost factor two (not $\sqrt{2}$) improvement, to \(\sigma(g_\ast) = 0.0096\). This carries further to higher redshifts, with measurements over \(z = 0.1 - 3\) giving a factor three improvement to \(\sigma(g_\ast) = 0.0060\), and \(\sigma(\gamma) = 0.042\). (Note this value of \(\sigma(\gamma)\) is nearly the same as what would have been estimated for $z = 0.1 - 1$ data if $g_\ast$ had been (improperly) ignored.)

Growth measurements over $z = 0.1 - 1$ but with 1% (4%) precision lead to parameter estimation of \(\sigma(\gamma) = 0.042\) (0.16) and \(\sigma(g_\ast) = 0.011\) (0.034).

For realistic survey design, one would have to take into account the observational issues in achieving the measurement precision. While measuring linear growth at low redshift requires surveys covering huge areas, the tracing objects themselves may be easier to see at these redshifts. At high redshift, a broader range of scales is in the linear growth regime, but tracers may be more difficult to measure. The Lyman-$\alpha$ forest provides a promising probe for the linear growth at high redshift [37, 38], and the BOSS experiment [39] now underway will include a redshift range $z = 2.3 - 2.8$ (in the present context the amplitude of the matter power spectrum rather than the baryon acoustic oscillations is the goal). Finally, note that all the leverage of the growth measurements essentially goes into the growth parameters; increasing the growth redshift range from $z < 1$ to $z < 2$ changes the area of the marginalized $w_0$-$w_a$ contour by 1% (again showing that the new parameter is probing distinct physics).

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**FIG. 2:** The early time calibration parameter $g_\ast$ plays a crucial role not only for physics insight, but in obtaining realistic estimates of the gravitational growth index $\gamma$. The uncertainty in $\gamma$ when fixing $g_\ast$ is equivalent to assuming measurements extend over $z < 3.5$ rather than $z < 1$. 

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**FIG. 3:** Extending the redshift range of growth measurements over $z = 0.1 - 2$ improves the constraints more than statistically, with two times the number of measurements giving an almost factor two (not $\sqrt{2}$) improvement, to $\sigma(g_\ast) = 0.0096$. This carries further to higher redshifts, with measurements over $z = 0.1 - 3$ giving a factor three improvement to $\sigma(g_\ast) = 0.0060$, and $\sigma(\gamma) = 0.042$. (Note this value of $\sigma(\gamma)$ is nearly the same as what would have been estimated for $z = 0.1 - 1$ data if $g_\ast$ had been (improperly) ignored.)

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We could consider another form of growth measurement—redshift space distortions. This arises from a more nonlinear regime, where the density fluctuations induce velocities distorting the Hubble flow. This was proposed as a test of the gravitational framework in [36, 40] and first analyzed in terms of BSM-2 and constraints on both the expansion history and gravitational growth index \( \gamma \) [36]. Other papers elaborating on this or related approaches include [41, 42, 43, 44, 45].

The distortions \( \beta \) measured through galaxy redshifts depend on the logarithmic derivative of the linear growth factor, and a bias parameter \( b \) relating the galaxy density to the matter density: \( \beta = f/b \) where

\[
f = \frac{d\ln g}{d\ln a} + 1 = \Omega_m(a)^\gamma + \frac{d\ln g_*}{d\ln a}.
\]

This involves the expansion history through \( \Omega_m(a) \), the gravitational growth index \( \gamma \), and the early time calibration \( g_* \). However, because \( g_* \) is so constant, especially at late times when redshift distortion measurements would be evaluated, the second term in the last line above is negligible. For example, for the \( \Omega_e = 0.03 \) early dark energy case where \( g_* \approx 0.87 \) the first term is 100 times larger than the second.

Adding 10\% measurements of \( f \) over \( z = 0.1 - 1 \) every 0.1 in redshift, as in [30], to the supernova, CMB, and growth data does not necessarily have a significant effect on the parameter constraints. Treating the bias parameter \( b \) as a single number and marginalizing over it improves the uncertainties to \( \sigma(\gamma) = 0.068 \) and \( \sigma(g_*) = 0.016 \) (i.e., by 15\% and 10\% respectively). Measurements of \( f \) at the 5\% level lower these uncertainties to 0.051 and 0.014 respectively. These constraints weaken as a more realistic treatment of bias is included. Because of this, a redshift distortion survey would need to be carefully designed for it to have a useful impact. Note that neglecting consideration of bias or of the need to fit for \( g_* \) could lead to overoptimistic conclusions; for example, ignoring \( g_* \) alone makes the \( \gamma \) constraint look a factor 1.8 better.

### B. Parameter Bias from Ignoring \( g_* \)

Inclusion of the early time parameter \( g_* \) not only opens new windows on physics but is necessary for accurate estimation of the gravitational growth index \( \gamma \) and the expansion history parameters. As seen in Fig. 2 estimation of the uncertainty on \( \gamma \) is significantly affected by fitting for \( g_* \). However, more important is that neglect of \( g_* \) will bias the values of the other parameters.

This can calculated within the Fisher analysis bias formalism and we find as a rule of thumb that \( \delta \gamma \approx 3.1 \delta g_* \). To preserve an accuracy of 0.05, say, in \( \gamma \) to distinguish modified gravity from general relativity, requires that the offset in \( g_* \) be no more than 0.016. That is, if we neglect to fit \( g_* \) and take by fiat \( g_* = 1 \) (as in BSM-2), then the parameter estimation is substantially biased for models where the true \( g_* < 0.98 \) or \( g_* > 1.02 \).

Figures 4 and 5 illustrate this dramatically for the early dark energy case with \( \Omega_e = 0.023 \). Even this small amount of early dark energy gives \( \delta g_* = 0.1 \), as in Eq. (4).

Through the dependence of the growth on both the gravitational growth index and the expansion history, all the cosmological parameters become substantially biased if \( g_* = 1 \) is incorrectly assumed. Thus, since we do not know a priori that early time density perturbations were “standard weights”\(^2\), to protect against such bias requires simultaneously fitting for the early time calibration parameter \( g_* \).

### V. CONCLUSIONS

In exploring the nature of the physics behind the cosmic acceleration we can investigate both the expansion history of the universe and the growth history of large scale structure. Comparison of the two is one of the best

\(^2\) Issues of density calibration in physics date back to Archimedes’ test of whether the king’s crown was solid gold, and Newton’s tests for counterfeit coins, but a phrase like “standard candles” or even “standard rulers” is lacking.
methods for testing our understanding of gravitation on large scales. Some of the main desiderata for such a formalism are a compact parametrization that keeps the origins of the effects distinct, without conflating the physics. Establishing a framework for carrying this out in a model independent manner allows us to search generally for the physics, including the possibility of surprises.

Here we have extended the framework to take in important classes of models of both gravity and dark energy where the early time behavior of growth does not follow the standard matter dominated scenario. In addition to the gravitational growth index $\gamma$ that reflects deviations in the form of the growth equations, distinct from the expansion history, the new parameter $g_*$ calibrates the early growth, distinct from $\gamma$ – they probe different aspects of gravitation. This calibration is an essential element to take into account, even if no non-standard growth is expected, just as $\mathcal{M}$ is needed for robust use of supernova distances and $\mathcal{S}$ is necessary for robust use of baryon acoustic oscillation distances. We have demonstrated that the biasing effects on other parameters if one simply assumes $g_* = 1$ can be severe, and one will also misestimate the area of the growth parameters confidence contour by a factor 2.

Future data will be capable of globally fitting the four expansion plus growth “Beyond the Standard Model” parameters $\{w_0, w_a, \gamma, g_*\}$. Accurate measurements of $g_*$ can reveal exciting aspects of early dark energy, early gravity, and early acceleration, with 2% precision delivering $\sigma(\Omega_c) = 0.005$, or $\delta G(z \gg 1)/G_{\text{Newton}}$ to 1.4%, or constraining the length of an early acceleration period to 1.7% of a Hubble time.

These prospects are exciting. To achieve them will require good measurements of the degenerate parameters of the primordial or current density perturbation amplitude, $A_s$ or $\sigma_8$, and linear growth measurements, perhaps through the Lyman-$\alpha$ forest, of $\sim 2\%$. The inclusion of $g_*$ necessarily increases the uncertainty on $\gamma$ by a factor 2. To recover the precision will require a very broad redshift range $z \approx 0 - 3.5$ of linear growth measurements. Alternately, weak gravitational lensing data may help, though we have seen that redshift distortions do not give substantial tightening, and nonlinear structure may not even $[46, 47]$ (but see $[48]$).

The BSM-3 framework delivers extended physics information over a broader class of models, including those with enhanced growth or breaking standard matter domination. Adding a single early time parameter $g_*$ is a highly accurate approximation, constant over the observable epoch to the $10^{-3}$ level and agreeing with the exact growth solution at the $10^{-3}$ level. Moreover, the framework is practical, able to constrain deviations from general relativity and the standard model while keeping the physical interpretations clear and distinct.
Appendix: Early time treatment

Rather than the approach adopted in this article, one might have considered defining a parameter \( \dot{g} \), so as to represent the entire growth history – that is,

\[
g(a) = \dot{g} e^{\int_{a_*}^a \frac{da'}{a'} |\Omega_m(a')^{\gamma-1}|}, \tag{10}
\]

in contrast to the lower limit of 0 for the integral in Eq. (2). However, this conflates the expansion history with non-standard growth physics. For example, for a model with \((w_0, w_a) = (-0.8, 0.5)\) and standard gravity, one would derive \( \dot{g} \approx 0.95 \), apparently indicating a non-standard high redshift framework. In other words, using the definition of \( \dot{g} \), rather than \( g \), one has introduced a parameter that is not a distinct probe from \( \gamma \) and the expansion history \( \Omega_m(a) \), possibly leading to confusion. By keeping the full evolution of the \( \gamma \) term and defining \( g_* \), as in Eq. (2), one clearly separates the different physical effects beyond standard cosmology.

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[1] E.V. Linder, Phys. Rev. D 72, 043529 (2005) arXiv:astro-ph/0507263
[2] E.V. Linder & R.N. Cahn, Astropart. Phys. 28, 481 (2007) arXiv:astro-ph/0701317
[3] D. Huterer & E.V. Linder, Phys. Rev. D 75, 023519 (2007) arXiv:astro-ph/0608681
[4] L. Knox, Y-S. Song, J.A. Tyson, Phys. Rev. D 74, 023512 (2006) arXiv:astro-ph/0503644
[5] M. Ishak, A. Upadhye, D.N. Spergel, Phys. Rev. D 74, 043513 (2006) arXiv:astro-ph/0507184
[6] S. Wang, L. Hui, M. May, Z. Haiman, Phys. Rev. D 76, 063503 (2007) arXiv:0705.0165
[7] G.M. Bernstein, arXiv:0808.3400
[8] M.J. Mortonson, W. Hu, D. Huterer, arXiv:0810.1744
[9] R. Caldwell, A. Cooray, A. Melchiorri, Phys. Rev. D 76, 023507 (2007) arXiv:astro-ph/0703375
[10] B. Jain & P. Zhang, Phys. Rev. D 78, 063503 (2008) arXiv:0709.2375
[11] P. Zhang, R. Bean, M. Liguori, S. Dodelson, arXiv:0809.2836
[12] E.V. Linder, Rep. Prog. Phys. 71, 056901 (2008) arXiv:0801.2968
[13] S. Bashinsky, arXiv:0707.0692
[14] R. Durrer & E.V. Linder, arXiv:0811.4132
[15] J-P. Uzan, Gen. Rel. Grav. 39, 307 (2007) arXiv:astro-ph/0605313
[16] R. de Putter & E.V. Linder, JCAP 0810, 042 (2008) arXiv:0808.0189
[17] E.V. Linder, J. Phys. A 40, 6097 (2007) arXiv:astro-ph/0610173
[18] E.V. Linder & G. Robbers, JCAP 0806, 004 (2008) arXiv:0803.2877
[19] F. De Bernardis et al., arXiv:0812.3557
[20] C. Wetterich, Nucl. Phys. B 302, 668 (1988)
[21] M. Doran & G. Robbers, JCAP 0606, 026 (2006) arXiv:astro-ph/0601544
[22] S. Rappaport, J. Schwab, S. Burles, G. Steigman, Phys. Rev. D 77, 023515 (2008) arXiv:0710.5300
[23] O. Zahn & M. Zaldarriaga, Phys. Rev. D 67, 063002 (2003) arXiv:astro-ph/0212690
[24] K. Umeznu, K. Ichikl, M. Yahiro, Phys. Rev. D 72, 044010 (2005) arXiv:astro-ph/0503578
[25] K.C. Chan & M-C. Chu, Phys. Rev. D 75, 083521 (2007) arXiv:astro-ph/0611851
[26] A. Luo, R. Scoccimarro, G. Starkman, Phys. Rev. D 69, 124015 (2004) arXiv:astro-ph/0401515
[27] K. Griest, Phys. Rev. D 66, 123501 (2002) arXiv:astro-ph/0202052
[28] S. Dodelson, M. Kaplinghat, E. Stewart, Phys. Rev. Lett. 85, 5276 (2000) arXiv:astro-ph/0002360
[29] G. Barenboim & J. Lykken, Phys. Lett. B 635, 453 (2006) arXiv:astro-ph/0504090
[30] E.V. Linder, Astropart. Phys. 26, 16 (2006) arXiv:astro-ph/0603584
[31] R. Bean, D. Bernat, L. Pogosian, A. Silvestri, M. Trodden, Phys. Rev. D 75, 064020 (2007) arXiv:astro-ph/0611321
[32] W. Hu & I. Sawicki, Phys. Rev. D 76, 104043 (2007) arXiv:0708.1190
[33] G. Ballesteros & A. Riotto, Phys. Lett. B 668, 171 (2008) arXiv:0807.3343
[34] L. Amendola & C. Quercellini, Phys. Rev. Lett. 92, 181102 (2004) arXiv:astro-ph/0403019
[35] C. Di Porto & L. Amendola, Phys. Rev. D 77, 083508 (2008) arXiv:0707.2686
[36] E.V. Linder, Astropart. Phys. 29, 424 (2008) arXiv:0710.0373
[37] P. McDonald et al., ApJ 635, 761 (2005) arXiv:astro-ph/0407377
[38] U. Seljak, A. Slosar, P. McDonald, JCAP 0610, 014 (2006) arXiv:astro-ph/0604335
[39] http://cosmology.lbl.gov/boss
[40] L. Guzzo et al., Nature 451, 541 (2008) arXiv:0802.1944
[41] K. Yamamoto, T. Sato, G. Huetsi, Prog. Theory. Phys. 120, 609 (2008) arXiv:0805.4789
[42] V. Acquaviva, A. Hajian, D.N. Spergel, S. Das, Phys. Rev. D 78, 043514 (2008) arXiv:0803.2236
[43] A. Abate & O. Lahav, MNRAS 389, L47 (2008) arXiv:0805.3180
[44] S.A. Thomas, F.B. Abdalla, J. Weller, arXiv:0810.4863
[45] M. Francis, G.F. Lewis, E.V. Linder, MNRAS in press arXiv:0808.2840
[46] M. Grossi & V. Springel, arXiv:0809.3404
[47] M. Baldi, V. Pettorino, G. Robbers, V. Springel, arXiv:0812.3901