Abstract. We show that any \( nD \) measures in \( \mathbb{R}^n \) can be bisected by an arrangement of \( D \) hyperplanes, when \( n \) is a power of two.

1. Introduction

Let \( \mathcal{H} = \{ H_1, H_2, \ldots, H_D \} \) be a finite set of hyperplanes, \( \{ A_1, A_2, \ldots, A_D \} \) affine functions such that the zero set of \( A_i \) is \( H_i \), and \( P^H = A_1 A_2 \ldots A_D \) the product of these affine functions. If \( \mu \) is a measure in \( \mathbb{R}^n \), we will say that \( \mathcal{H} \) bisects \( \mu \) if

\[
\mu \left\{ v \in \mathbb{R}^n : P^H(v) > 0 \right\} \leq \frac{\mu(\mathbb{R}^n)}{2} \quad \text{and} \quad \mu \left\{ v \in \mathbb{R}^n : P^H(v) < 0 \right\} \leq \frac{\mu(\mathbb{R}^n)}{2}.
\]

Theorem 1. Let \( n \) and \( D \) be integers such that \( D > 0 \) and \( n > 1 \) is a power of two. Given \( nD \) finite measures \( \mu_1, \mu_2, \ldots, \mu_{nD} \) in \( \mathbb{R}^n \), there exists an arrangement of at most \( D \) hyperplanes that bisect each of the measures.

Observe that a family of \( nD + 1 \) delta masses based at a set of points, no \( n + 1 \) of which lie on the same hyperplane cannot be simultaneously bisected by less than \( D + 1 \) hyperplanes. Barba and Schnider [2] conjectured that the previous theorem holds for any \( n \) and confirmed this conjecture for the case of four measures in the plane (\( n = D = 2 \)). Notice that the case \( D = 1 \) of this conjecture corresponds to the classical ham sandwich theorem (see the book [3] for many other ham sandwich type results).

2. Parametrization of arrangements

Parametrize hyperplanes in \( \mathbb{R}^n \) by elements of \( S^n \) mapping \( (a_0, a_1, \ldots, a_n) \in S^n \) to the affine function

\[
A(x) = a_0 + a_1 x_1 + \cdots + a_n x_n.
\]

Parametrize hyperplane arrangements by elements of \( (S^n)^D \). An element of \( (S^n)^D \) corresponds to \( D \) affine functions \( A_1, \ldots, A_D \) and the polynomial corresponding to \( \mathcal{H} = \{ A_1^{-1}(0), A_2^{-1}(0), \ldots, A_D^{-1}(0) \} \) of degree \( D \) is given by

\[
P^H(x) = A_1(x) \ldots A_D(x).
\]

Let \( S_D \) be the symmetric group of permutations of \( D \) elements, and \( \mathbb{Z}/2^D \) be the \( D \)-fold product of the abelian group on two elements. Let \( G = S_D \ltimes \mathbb{Z}/2^D \) be their semi-direct product. The group \( \mathbb{Z}/2^D \) acts on \( (S^n)^D \) by the antipodal map \( A \mapsto -A \) on each \( S^n \) factor, this action is free, the group \( S_D \) acts on \( (S^n)^D \) by permuting the factors. Their semi-direct product acts by permuting and applying antipodal maps on some of the factors.

The action of \( G \) on \( (S^n)^D \) is not free. Its non-free part \( \Sigma \) corresponds to \( D \)-tuples \( (A_1, \ldots, A_D) \) such that \( A_i = A_j \) or \( A_i = -A_j \) for some \( i \neq j \).

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3. Approximation of measures

We prove the theorem for a subspace of measures, which is a dense subset of \( \mathcal{P} \), the space of Borel probability measures with the weak topology; then we deduce the general case by approximation. We denote by \( \mathcal{P}^k \) its \( k \)-fold Cartesian product, whose elements are sets \( \{\mu_1, \mu_2 \ldots \mu_k\} \) of Borel probability measures in \( \mathbb{R}^n \). The material that we need from measure theory is covered in many analysis books, see for instance [5, 7].

Lemma 2. For \( k, D > 0 \), the set of ordered \( k \)-tuples of Borel measures that are not bisectable by \( D \) hyperplanes is open in \( \mathcal{P}^k \).

Proof. Assume that \( M = (\mu_1, \ldots, \mu_k) \in \mathcal{P}^k \) is a \( k \)-tuple of Borel probability measures that cannot be bisected by an arrangement of \( D \) hyperplanes \( \mathcal{H} \). For any polynomial \( P^H \), there exists a sign \( \pm \) and an \( i \in \{1 \ldots, k\} \) such that

\[
\mu_i \{ \pm P^H > 0 \} > 1/2.
\]

From continuity of the measure \( \mu_i \) we can choose an open set \( W \) whose closure is compact and is contained in \( \{ \pm P^H > 0 \} \) such that,

\[
\mu_i(W) > 1/2.
\]

By definition, the ordered \( k \)-tuples of Borel probability measures \( M' = (\mu'_1, \ldots, \mu'_k) \) such that \( \mu'_i(W) > 1/2 \) constitute a neighborhood \( \mathcal{U} \ni M \) in the weak topology. The arrangements of hyperplanes \( \mathcal{H}' \) such that \( P^{H'} \) is positive on the closure of \( W \) constitute a neighborhood \( \mathcal{V} \ni \mathcal{H} \) in the topology on the space of arrangements. Any pair of \( M' \in \mathcal{U} \) and \( \mathcal{H}' \in \mathcal{V} \) have the property that \( \mathcal{H}' \) does not bisect \( M' \).

Since the space of arrangements \( (S^n)^D \) is compact, a finite number of such \( \mathcal{V}_1, \ldots, \mathcal{V}_N \) cover the whole space of arrangements. The intersection of the respective \( \mathcal{U}_1, \ldots, \mathcal{U}_N \) produce a neighborhood of \( M \) every member of which cannot be bisected with any arrangement of hyperplanes. \( \square \)

Corollary 3. Theorem 1 for Borel measures follows from its validity on any dense subset of \( \mathcal{P}^k \).

Denote by \( \delta_v \) the Dirac delta mass at the point \( v \), i.e. for a Borel set \( X \), \( \delta_v(X) = 1 \) if \( v \in X \) and \( \delta_v(X) = 0 \) otherwise. We call measures of the form \( \frac{1}{N} \sum_{k=1}^N \delta_{v_k} \) with odd \( N \) and \( v_1, \ldots, v_N \) in general position, oddly supported measures. We say that a finite family of measures is in general position if no hyperplane intersects \( n+1 \) connected components of the union of their supports.

Lemma 4. Oddly supported measures in general position are dense in \( \mathcal{P} \). Ordered \( k \)-tuples of oddly supported measures in general position are dense in \( \mathcal{P}^k \).

Proof. Assume the contrary, then there is a Borel probability measure \( \mu \) whose weak neighborhood \( \mathcal{V} \) contains no oddly supported measure. It is sufficient to consider \( \mathcal{V} \) from the base of the weak topology given by a finite set of inequalities

\[
\mu'(U_1) > m_1, \ldots, \mu'(U_\ell) > m_\ell
\]

for open \( U_i \) and real \( m_i \). Let \( N \) be an odd number. Sample \( N \) points \( v_k \) independently, distributed according to \( \mu \) and consider the random measure

\[
\nu_N = \frac{1}{N} \sum_{k=1}^N \delta_{v_k}.
\]
The random variable $\nu_N(U_i)$ is given by,

$$\nu_N(U_i) = \frac{\#\{k = 1, \ldots, N : v_k \in U_i\}}{N}$$

This is a sum of $N$ independent Bernoulli random variables with expectation $\mu(U_i)$. By the law of large numbers $\nu_N(U_i)$ converges almost surely to $\mu(U_i)$. Hence for sufficiently large $N$ the probability of satisfying the inequalities $\nu_N(U_i) > m_i$ simultaneously is arbitrarily close to 1; and we might perturb the points $v_k$ so that none of them leave any $U_i$ it belonged to so that for the perturbed measure $\nu_N(U_i)$ is still in $\mathcal{V}$.

The second statement follows immediately, we do the same for $k$ measures and take a single sufficiently large odd $N$. After that we perturb the total $Nk$ support points so that none of them leaves any $U_i$ (from the definition of a weak neighborhood) it belonged to.

Let $\eta_v$ be an $\varepsilon$-smoothening of the delta mass at $v$. More precisely $\eta_v$ is a Borel probability measure centrally symmetric around $v$, which is supported inside a ball $B_v(\varepsilon)$ of radius $\varepsilon$ centered at $v$ and has a continuous density. Now take points in general position $v_1, \ldots, v_N$ and consider a sum of $\varepsilon$-smoothenings

$$\mu = \frac{1}{N} \sum_{k=1}^{N} \eta_{v_k}.$$ 

If $N$ is an odd number and no $n+1$ tuple of the $B_{v_k}(\varepsilon)$ are intersected by a hyperplane, then we include $\mu$ in the set $\mathcal{M}_\varepsilon$. Finally we put $\mathcal{M} := \cup_{\varepsilon>0} \mathcal{M}_\varepsilon$, this is the set of measures we will work with.

**Lemma 5.** The set $\mathcal{M}$ is dense in the space of probability measures with the weak topology, moreover the set of ordered $k$-tuples of measures in general position in $\mathcal{M}^k$ is dense in $P^k$.

**Proof.** For any oddly supported measure, we weakly approximate every delta mass $\delta_{v_k}$ by its respective $\eta_{v_k}$ supported in the respective $B_{v_k}(\varepsilon)$. If $\varepsilon$ is sufficiently small then no $n+1$ of the balls will be intersected by a single hyperplane. So $\mathcal{M}$ is dense in the space of oddly supported measures which by Lemma 4, is dense in $P$. Similarly, $\mathcal{M}^k$ is dense in $P^k$. $\square$

4. Bisecting well separated sets of measures

We say that a family of sets $X_1, X_2, \ldots, X_m$ in $\mathbb{R}^n$ is well separated if no $n$-tuple of their convex hulls $\text{conv}(X_1), \text{conv}(X_2) \ldots \text{conv}(X_m)$ is intersected by an $(n-2)$-dimensional affine space. A family of measures is well separated if their supports are well separated. The following lemma was shown in [1] for absolutely continuous measures.

**Lemma 6.** For any family of well separated measures in general position $\mu_1, \mu_2, \ldots, \mu_n \in \mathcal{M}$ there exists a unique hyperplane $H$ that bisects each of the measures.

**Proof.** The existence of this hyperplane is provided by the ham sandwich theorem, we only need to show the uniqueness. Assume we have a pair of halving hyperplanes $H$ and $H'$, since the measures are well-separated, the intersection $H \cap H'$ does not touch the convex hull of the support of some $\mu_i$. The both hyperplanes must intersect the interior of the support of $\mu_i$, since it is constructed from an odd number of equal measures. Now it is clear that one of the halves $H_- \cap \text{conv supp } \mu_i$ and $H_+ \cap \text{conv supp } \mu_i$ strictly
contains some of $H'_+ \cap \text{conv supp } \mu_i$ and $H'_- \cap \text{conv supp } \mu_i$ and therefore $H$ and $H'$ cannot equipartition $\mu_i$ at the same time. \hfill \Box

The following lemma describes the bisecting arrangements of hyperplanes in the case when the measures are well separated.

**Lemma 7.** For any family of $nD$ well separated measures in general position, an arrangement of $D$ hyperplanes $\mathcal{H}$ is bisecting, if and only if, there is bijection $\varphi$ between the elements of $\mathcal{H}$ and a partition of $[nD]$ into $n$-tuples such that the hyperplane $H_i \in \mathcal{H}$ bisects the $n$-tuple of measures with indices in $\varphi(H_i)$.

**Proof.** Given a partition $Y$ of $[nD]$ into $n$-tuples $\{Y_1, Y_2 \ldots Y_D\}$. By Lemma 6, for each $n$-tuple $Y_i$, the corresponding measures are bisected by a unique ham sandwich cut, this defines a bijection $\varphi^{-1}: Y \rightarrow \mathcal{H}$. Since the measures are well separated, any measure with index not in $\varphi(H)$ is not intersected by $H$. So the arrangement is simultaneously bisecting. Conversely, since the supports are well separated, each hyperplane of a bisecting arrangement must intersect the supports of precisely $n$ of the measures, otherwise at least one measure cannot be bisected. In this situation each hyperplane bisects $n$ measures and does not touch the convex hulls of the supports the remaining measures. By Lemma 6, such a hyperplane must be the unique ham sandwich cut of the corresponding $n$-tuple of measures. \hfill \Box

Let $N(n, D)$ be the number of unordered partitions of a set of $nD$ elements into $D$ sets of $n$ elements each. Clearly

$$N(n, D) = \frac{(nD)!}{D!(n!)^D},$$

but we will not use this formula.

**Lemma 8.** If $n$ is a power of two then $N(n, D)$ is odd.

**Proof.** Consider the action of the 2-Sylow subgroup $S \subset \mathfrak{S}_{nD}$ on these partitions. To describe this Sylow subgroup we need to make a binary tree with $2^m$ leaves, where $2^m$ is the smallest power of two not smaller than $nD$. Then we drop the leaves that have numbers strictly greater than $nD$ and drop the corresponding higher vertices of the tree. Then $S$ is the symmetry group of the remaining tree and its embedding into $\mathfrak{S}_{nD}$ is obtained by looking at the leaves of the tree and how they are permuted by $S$. It is a Sylow subgroup just because by construction its order equals

$$2\sum_{k \geq 1} [nD/2^k],$$

which is the largest power of two that divides $(nD)! = |\mathfrak{S}_{nD}|$.

The set $nD$ has a decomposition into consecutive $n$-tuples $P_1 \cup \cdots \cup P_D$. As it is easily seen, when $n$ is a power of two, each $P_i$ corresponds to a full binary subtree. Hence the group $S$ can permute transitively each of $P_i$ while fixing all elements of the other $P_j$, $j \neq i$. This guarantees that an unordered partition into $n$-tuples that is fixed by $S$ must coincide with the chosen partition $P_1 \cup \cdots \cup P_D$. Other partitions are not fixed under the $S$ action, so they come in orbits. Since $S$ is a 2-group, all such orbits are even, hence the total number of partitions into $n$-tuples is odd. \hfill \Box
5. Proof of the Theorem

By Lemmas 5 and Corollary 3 it is sufficient to prove the theorem for measures in $\mathcal{M}$ (smoothed oddly supported measures) in general position. Denote by $A_i$ the support of the measure $\mu_i$, and by $C_i$ the set of centers of balls whose union is $A_i$. We say that the points on $A_i$ are of color $i$. Denote by $M$ the family $\{\mu_1, \mu_2, \ldots, \mu_D\}$. Arguing similarly to Lemma 6 observe that for any family of measures in general position in $\mathcal{M}$ (not necessarily well-separated) a bisecting arrangement has to be the union of $D$ hyperplanes each of which intersects a heterochromatic set of $n$ connected components, otherwise some of the measures will not be bisected. We only need to count such arrangements.

We deform the measures $\mu_i$ continuously to a situation where we can easily count the number of bisecting arrangements of the family. We use measures in $\mathcal{M}$ throughout, so we might prescribe a trajectory of $C_i$ and choose $\varepsilon > 0$ later. In the following all the objects that we deal with depend on $t \in [0, 1]$ which we call time, and denote this time with a subindex $t$. For each $t \in [0, 1]$ we consider a measure $\mu_{i,t} \in \mathcal{M}$ that depends continuously on $t$ such that $\mu_{i,0} := \mu_i$ and the family $M_1 := \{\mu_{1,1}, \mu_{2,1}, \ldots, \mu_{nD,1}\}$ (at time $t = 1$) is well separated and in general position. By Lemma 7 we know that the family $M_1$ has exactly $N(n, D)$ bisecting arrangements.

Let us further describe the motion of $M_t$ in more detail. We want to describe a generic trajectory of measures in general position. Consider a point $b_i$ from a general position set $b_1, b_2, \ldots, b_{nD}$. Choose $\alpha > 0$ so that the balls $B(b_i, \alpha)$ are well separated. Then move each of the points of $C_i$ towards $b_i$ in such a way that each set $C_i$ is always in general position within itself and at the end, the support of the $\mu_{i,1}$ is contained in $B(b_i, \alpha)$. For example, the deformation could follow a homothety with center $b_i$. By perturbing the speed of the trajectories if necessary, we can assume that at no moment of time there exist two $(n+1)$-tuples of connected components of the $A_i$, each of which is intersected by a hyperplane. In particular, at no time $t$, an $(n+2)$-tuple of connected components is intersected by a single hyperplane. To put it short, in a generic trajectory the events when some $n+1$ supporting balls of the measures can be intersected by a hyperplane come one by one.

Denote by $Z_t$ the subset of points of $(\mathbb{S}^n)^D$ corresponding to bisecting arrangements of the family $M_t$. Our crucial observation is that $Z_t$ does not touch the non-free part $\Sigma \subset (\mathbb{S}^n)^D$. An assumed $G$-fixed point of $Z_t$ corresponds to a set of hyperplanes in which two of the hyperplanes coincide. From the assumption on the generic trajectory it follows that we thus have at most $D - 1$ distinct hyperplanes that intersect at least $nD$ supporting balls of the measures in the set $M_t$. But there is a unique $(n+1)$-tuple of such balls that can be intersected by a single hyperplane, in all other situations the hyperplanes intersect at most $n$ balls each. The inequality $n(D - 1) + 1 < nD$ thus gives a contradiction, so the non-free part of the space of arrangements is not touched during the motion.

Let us show that the parity of the number of bisecting arrangements stays invariant during the motion; then Lemma 8 delivers the result in the case we are interested in.

Consider the continuous $G$-equivariant map $f : (\mathbb{S}^n)^D \times [0, 1] \to \mathbb{R}^{nD}$ given by

$$(f_t(x))_i = \mu_{i,t}\{P > 0\} - \mu_{i,t}\{P < 0\},$$

where $P$ is the polynomial we associate to $x \in (\mathbb{S}^n)^D$. We have the solution set $Z_t = f_t^{-1}(0) \subset (\mathbb{S}^n)^D \setminus \Sigma$ at time $t$. We need to show that $Z_0 \neq \emptyset$ and let us assume the contrary, that $Z_0 = \emptyset$. 


If fact, the union of all such $Z_t$ for $t \in [0, 1]$ is the preimage of zero $Z = f^{-1}(0)$, a closed subset of the product $(S^n)^D \times [0, 1]$, not touching the non-free part of this product $\Sigma \times [0, 1]$. Denote the free part

$$F = ((S^n)^D \setminus \Sigma) \times [0, 1]$$

for brevity. Using the Thom transversality theorem \cite{6, 4} (on the free part $F$ we just apply the non-equivariant transversality for the sections of the vector bundle $F$ with $t = 0, 1$ since for $t = 0, 1$ the map $f$ was already transversal to zero. Now we have $Z'$ with $Z'_0 = Z_0 = \emptyset$ and such that $Z'_1$ consists of an odd number of $G$-orbits. But $Z'_1$ is the boundary of the one-dimensional compact manifold $Z'$ with free action of $G$ that cannot consist of an odd number of $G$-orbits, a contradiction.

**Remark 9.** The previous version of this paper incorrectly claimed Theorem 1 for any $n$. It was claimed that the cohomology class that was denoted there by $e_i$ vanished on the complement of the set of arrangements of $D$ hyperplanes bisecting a single measure. Actually the argument given there with the curve $\gamma_i$ provides this fact for the class $\sum_{i=1}^D e_i$, the modulo two Euler class of the one-dimensional representation of $(Z/2)^D$, on which each generator of every $Z/2$ acts antipodally. The vanishing lemma implies that if $(e_1 + \cdots + e_D)^k$ is nonzero in the cohomology ring of the product of projective spaces, then for every $k$ measures there exist an arrangement of $D$ hyperplanes bisecting the measures. This in turn amounts to finding an odd multinomial coefficient, $\binom{k}{k_1, k_2, \ldots, k_D} = \binom{k}{k_1} \binom{k-k_1}{k_2} \cdots \binom{k-k_1-\cdots-k_{D-1}}{k_D}$ with $k_1, \ldots, k_D \leq n$. For such a coefficient to be odd, when we add the numbers in the sum $k_1 + \cdots + k_D$ in binary representation then no carry should occur. Consider the largest $m$ such that, $2^m \leq n$, then we need $k_1 + \cdots + k_D \leq 2^{m+1} - 1$. There is an example of such a sum with no carry if we put for $D \leq m + 2$, $k_1 = 2^m, k_2 = 2^{m-1}, \ldots, k_{D-1} = 2^{m-D+2}, k_D = 2^{m-D+2} - 1$, and for $D \geq m + 2$, $k_1 = 2^m, k_2 = 2^{m-1}, \ldots, k_m = 2, k_{m+1} = 1, k_{m+2} = \cdots = k_D = 0$. From which we can conclude that we can bisect $2^{m+1} - 1 \leq 2n - 1$ measures with at most 2 hyperplanes, and taking more hyperplanes, does not yield anything new with this technique (not using the permutations $\mathfrak{S}_D$).

On the other hand, if $2^m \leq n$ and we have $2^mD$ measures in $\mathbb{R}^n$, we can project linearly to $\mathbb{R}^{2m}$, apply Theorem 1 to obtain a bisecting arrangement of $D$ hyperplanes in $\mathbb{R}^{2m}$ and look at their inverse image, an arrangement of $D$ hyperplanes in $\mathbb{R}^n$ that bisects the original measures. Since $2^mD > 2^{m+1} - 1$ in the nontrivial case $D \geq 2$, Theorem 1 always provides a better result then the above cohomological argument.

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