Number of Edges in Random Intersection Graph on Surface of a Sphere.

by

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Abstract

In this article, we consider ‘N’ spherical caps of area $4\pi p$ were uniformly distributed over the surface of a unit sphere. We study the random intersection graph $G_N$ constructed by these caps. We prove that for $p = \frac{c}{N^\alpha}$, $c > 0$ and $\alpha > 2$, the number of edges in graph $G_N$ follow the Poisson distribution. Also we derive the strong law results for the number of isolated vertices in $G_N$: for $p = \frac{c}{N^\alpha}$, $c > 0$ for $\alpha < 1$, there is no isolated vertex in $G_N$ almost surely i.e., there are at least $N/2$ edges in $G_N$ and for $\alpha > 3$, every vertex in $G_N$ is isolated i.e., there is no edge in edge set $E_N$.

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1 Introduction.

Random intersection graphs are introduced in [8], and defined as:

Let us consider a set $V$ with $n$ vertices and another set of objects $W$ with $m$ objects. Define a bipartite graph $G^*(n, m, p)$ with independent vertex sets $V$ and $W$. Edges between $v \in V$ and $w \in W$ exist independently with probability $p$. The random intersection graph $G(n, m, p)$ derived from $G^*(n, m, p)$ is defined on the vertex set $V$ with vertices $v_1, v_2 \in V$ are adjacent if and only if there exists some $w \in W$ such that both $v_1$ and $v_2$ are adjacent to $w$ in $G^*(n, m, p)$. Also define $W_v$ be a random subset of $W$ such that each element of $W_v$ is adjacent to $v \in V$. Any two vertices $v_1, v_2 \in V$ are adjacent if and only if $W_{v_1} \cap W_{v_2} \neq \phi$, and edge set $E(G)$ is defined as

$$E(G) = \{\{v_i, v_j\} : v_i, v_j \in V, W_{v_i} \cap W_{v_j} \neq \phi\}.$$ 

Dudley, [5], derive the distribution of the degree of a vertex of random intersection graph. Also show that if $n$ be the number of vertices and $\lceil n^\alpha \rceil$ be the number of objects, the vertex degree changes sharply between $\alpha < 1$, $\alpha = 1$ and $\alpha > 1$. Bhupendra Gupta [3] derive the strong threshold for the connectivity between any two arbitrary vertices of vertex set $V$, and determine the almost sure probability bounds for the vertex degree of a typical vertex of random intersection graph.

Our Model. In this paper we considered the random intersection graph generated by the spherical caps on the surface of a 3-dimensional unit sphere.

Let $C_1, C_2, \ldots, C_N$ be the spherical caps and $X_1, X_2, \ldots, X_N$ are their respective centers on the surface of a unit sphere. Let $X_1, X_2, \ldots, X_N$ are Uniformly distributed over the surface of unit sphere. Now define a random intersection graph $G_N$ on the surface of unit sphere, with vertex set $\mathcal{X}_N = \{X_1, X_2, \ldots, X_N\}$ and edge set $\mathcal{E}_N = \{X_i X_j : C_i \cap C_j \neq \phi, i \neq j\}$.

The aim of this paper is to investigate the evolution of edges in the graph $G_N$ with vertex set $\mathcal{X}_N = \{X_1, X_2, \ldots, X_N\}$, $N = 1, 2, \ldots$, where the vertices are independently
and uniformly distributed on the surface of a unit sphere. H. Maehara, [6] gives the asymptotic results for the various properties of random intersection graph of random spheriacal caps on surface of unit sphere. Also Bhupendra Gupta, [2] gives the strong threshold function $p_0(N) = o\left(\frac{\log N}{N}\right)$ for the coverage of the surface of a unit sphere by the spherical caps. Bhupendra Gupta shown that for large $N$, if $\frac{Np}{\log N} > 1/2$ the surface of sphere is completely covered by the $N$ caps almost surely, and if $\frac{Np}{\log N} \leq 1/2$ a partition of the surface of sphere is remains uncovered by the $N$ caps almost surely.

2 Supporting Results.

Let $C_1, C_2, \ldots, C_N$ be the spherical caps on the surface of a unit sphere with their centers $X_1, X_2, \ldots, X_N$ and Uniformly distributed over the surface of unit sphere. We defined a random intersection graph $G_N$ on the surface of unite sphere, with vertex set $\mathcal{X}_N = \{X_1, X_2, \ldots, X_N\}$ and edge set $\mathcal{E}_N = \{X_iX_j : C_i \cap C_j \neq \phi, i \neq j\}$.

Let $p := p(a)$ be the probability that a point ‘$x$’ on the surface of unit sphere is covered by a specified spherical cap of angular radius ‘$a$’. Then the area of the spherical cap of angular radius ‘$a$’ is equal to $4\pi p$.

Poisson Approximation.

Let $|\mathcal{E}|$ denote the cardinality of the edge set i.e., the number of edges in the graph $G_N$.

Define a indicator function

$$\xi_i = \begin{cases} 1, & C_i \cap C_j \neq \phi, i \neq j; \\ 0, & \text{otherwise.} \end{cases} \quad (2.1)$$

that is if $X_i$ is an end point of an edge, then $\xi_i$ is equal to 1, and hence $|\mathcal{E}| = \sum_{i \in I} \xi_i$, where $I := \{i : X_iX_j \in \mathcal{E}, i \neq j\}$ is the index set.
\[
E \mid \mathcal{E} = E\left[\sum_{i=1}^{n} \xi_i \right]
\]
\[
= \sum_{i=1}^{n} E[\xi_i] = \left(\frac{N}{2}\right)4p(1-p)
\]
\[
= 2N(N-1)p(1-p) \leq 2N^2p(1-p).
\] (2.2)

**Theorem 2.1** (Arratia 1989, [1]) Suppose \( \xi_i, i \in I \) is a finite collection of Bernoulli random variables. Set \( p_i := E[\xi_i] = P[\xi_i = 1] \), and \( p_{ij} := E[\xi_i \xi_j] \). Let \( \lambda := \sum_{i \in I} p_i \), and suppose \( \lambda \) is finite. Let \( |\mathcal{E}| := \sum_{i \in I} \xi_i \). Then

\[
d_{TV}(|\mathcal{E}|, Po(\lambda)) \leq \min(3, \lambda^{-1}) \left( \sum_{i \in I} \sum_{j \in \mathcal{N}_i \setminus \{i\}} p_{ij} + \sum_{i \in I} \sum_{j \in \mathcal{N}_i} p_i p_j \right).
\] (2.3)

where, \( \mathcal{N}_i \) be the adjacency neighborhood of \( i \), i.e., the set \( \{i\} \cup \{j \in I : X_i X_j \in \mathcal{E}\} \).

## 3 Weak Law Results.

**Theorem 3.1** For \( p := p(a) = \frac{c}{N^\alpha} \), where \( c > 0 \) and \( \alpha > 2 \). Then sufficiently large \( N \),

\[
d_{TV}(|\mathcal{E}|, Po(\lambda)) \rightarrow 0,
\] (3.4)

i.e., the number of edges in the graph \( G_N \) is a Poisson random variable with parameter \( \lambda = \sum_{i \in I} p_i < \infty \).

**Proof.** First we consider,

\[
p_i = E[\xi_i] = P[\xi_i = 1].
\] (3.5)

We know there exists an edge between \( X_i \) and \( X_j \) iff \( C_i \cap C_j \neq \emptyset \), i.e. the distance between \( X_i \) and \( X_j \) is less than \( 2a \). Now consider another spherical cap \( D_i \) centered at \( X_i \) and of radius \( 2a \).

\[
P[\xi_i = 1] = P[C_i \cap C_j \neq \emptyset]
\]
\[
= P[X_j \in D_i] = p(2a).
\] (3.6)
Now, from equation (2.1), of Bhupendra [2], we have
\[ p := p(a) = \sin^2(a/2). \]  
(3.7)

Using (3.7) in (3.6), we get
\[ P[\xi_i = 1] = \sin^2(a) = \frac{1}{2}(1 - \cos(2a)) = 4p(1 - p). \]  
(3.8)

Using (3.8) in (3.5), we get
\[ p_i = E[\xi_i] = 4p(1 - p). \]  
(3.9)

Now consider
\[ p_{ij} = E[\xi_i \xi_j] = 1.P[\xi_i = 1, \xi_j = 1] = \sum_{l=1, l \neq i}^{n} \sum_{k=1, k \neq j}^{n} P[(C_i \cap C_l) \neq \phi, (C_k \cap C_j) \neq \phi] - P[(C_i \cap C_j) \neq \phi] \]
\[ = \sum_{l=1, l \neq i}^{n} P[(C_i \cap C_l) \neq \phi] \sum_{k=1, k \neq j}^{n} P[(C_k \cap C_j) \neq \phi] - P[(C_i \cap C_j) \neq \phi] \]
\[ = (4(N - 1)p(1 - p))^2 - 4p(1 - p) \]
\[ = 16((N - 1)p(1 - p))^2 \left(1 - \frac{1}{4(N - 1)^2p(1 - p)}\right) \]
\[ \leq 16((N - 1)p(1 - p))^2. \]  
(3.10)

Now by Theorem 2.1, we have
\[ d_{TV}(|\mathcal{E}|, Po(\lambda)) \leq \min(3, \lambda^{-1}) \left(\sum_{i \in I} \sum_{j \in N \setminus \{i\}} p_{ij} + \sum_{i \in I} \sum_{j \in N_i} p_ip_j\right). \]

Using (3.9) and (3.10), we get
\[ d_{TV}(|\mathcal{E}|, Po(\lambda)) \leq \min(3, \lambda^{-1}) \left(\sum_{i \in I} \sum_{j \in N \setminus \{i\}} (4(N - 1)p(1 - p))^2 + \sum_{i \in I} \sum_{j \in N_i} 4p(1 - p)4p(1 - p)\right) \]
\[ \leq \min(3, \lambda^{-1}) \left(\frac{N(N - 1)^3}{2} (4p(1 - p))^2 + \frac{N(N - 1)}{2} (4p(1 - p))^2\right). \]

Taking \( p = \frac{c}{N^\alpha} \) and \( \alpha > 2 \) in above, we get
\[ d_{TV}(|\mathcal{E}|, Po(\lambda)) \to 0, \quad N \to \infty. \]
4 Strong Law Results.

Proposition 4.1 Let $G_N$ be a random intersection graph. Let $p = \frac{c}{N^\alpha}$, then

i. For $0 < \alpha < 1$, there is no isolated vertex in $G_N$ almost surely.

ii. For $\alpha < 2$ at least one isolated vertex in $G_N$ almost surely.

iii. For $\alpha > 3$, every vertex in $G_N$ is an isolated vertex.

Proof. Let $\mathcal{X}[B]$ denote that number of vertices of the finite set point $\mathcal{X}$ that lies in the set $B$. Let $D_i$ spherical cap centered at $X_i$ and of radius $2a$.

\[
P[\text{at least one isolated vertex in } G_N] = P[\bigcup_{i=1}^{N-1} (\mathcal{X}[D_i] < 1)] \\ 
\leq \sum_{i=1}^{N-1} P[\mathcal{X}[D_i] < 1] \\ 
= \sum_{i=1}^{N-1} (1 - p(2a))^{N-1} = N(1 - p(2a))^{N-1} \\ 
\leq (N - 1) \exp(-(N - 1)p(2a)) \\ 
= (N - 1) \exp(-4(N - 1)p(1 - p)),
\]

since $p(2a) = 4p(1 - p)$. Now taking $p = \frac{c}{N^\alpha}$, we get

\[
P[\text{at least one isolated vertex in } G_N] \leq (N - 1) \exp\left(-\frac{4(N - 1)}{N^\alpha} \left(1 - \frac{1}{N^\alpha}\right)\right). \quad (4.11)
\]

The above probability is summable for $0 < \alpha < 1$, i.e.,

\[
\sum_{N=1}^{\infty} P[\text{at least one isolated vertex in } G_N] < \infty.
\]

Then by the Borel-Cantelli’s Lemma, we have

\[
P[\text{no isolated vertex in } G_N, \quad i.o.] = 1.
\]

This implies that for $\alpha < 1$ there is no isolated vertex in $G_N$ almost surely.
For the second part of proposition, we consider

\[ P\{\text{every vertex is an isolated vertex in } G_N\} = \prod_{i=1}^{N-1} P\{X[D_i] < 1\} \]
\[ \leq (1 - p(2a))^{N-1} \left( 1 - p(2a) \right)^{N-1} \]
\[ \leq \exp\left( -(N-1)p(2a) \right)^{N-1} \]
\[ \leq \exp\left( -4(N-1)^2p(1-p) \right), \]

since \( p(2a) = 4p(1-p) \). Now taking \( p = \frac{c}{N^\alpha} \), we get

\[ P\{\text{every vertex is an isolated vertex in } G_N\} \leq \exp\left( -4(N-1)^2p(1-p) \right). \] (4.12)

The above probability is summable for \( \alpha < 2 \), i.e.,

\[ \sum_{N=1}^{\infty} P\{\text{every vertex is an isolated vertex in } G_N\} < \infty. \]

Then by the Borel-Cantelli’s Lemma, we have

\[ P\{\text{at least one isolated vertex in } G_N, \ i.o.\} = 1. \]

This implies that for \( \alpha < 2 \) there is at least one isolated vertex in \( G_N \) almost surely.

For the third part of proposition, we consider

\[ P[\mathcal{E} \neq \emptyset] \leq P[|\mathcal{E}| \geq \epsilon]. \] (4.13)

By the Chebyshev’s inequality, we have

\[ P[|\mathcal{E}| \geq \epsilon] \leq \frac{E[|\mathcal{E}|]}{\epsilon} \]
\[ \leq \frac{2}{\epsilon} N^2 p(1-p). \] (4.14)
Taking $p = \frac{c}{N^\alpha}$, we get
\[ P(||E|| \geq \epsilon) \leq \frac{2N^2}{\epsilon} \frac{c}{N^\alpha}. \]
Hence from (4.13), we have
\[ P[E \neq \phi] \leq \frac{2N^2}{\epsilon} \frac{c}{N^\alpha}. \quad (4.15) \]
The above probability is summable for $\alpha > 3$, i.e.,
\[ \sum_{N=1}^{\infty} P[E \neq \phi] < \infty. \]
Then by the Borel-Cantelli’s Lemma, we have
\[ P[E = \phi, \text{i.o.}] = 1. \]
This implies that
\[ |E| = 0, \quad \text{almost surely}, \]
i.e., if $p = \frac{c}{N^\alpha}$; $\alpha > 3$, then there is no edge in the intersection graph almost surely, and hence every vertex is an isolated vertex almost surely.

**Theorem 4.2** Let $G_N$ be a random intersection graph. Let $p = \frac{c}{N^\alpha}$, then for $\alpha < 1$, there are at least $N/2$ edges in $G_N$ almost surely. For $\alpha > 3$, there is no edge in edge set $E_n$.

**Proof.** From the Proposition 4.1, we have for $\alpha < 1$, there is no isolated vertex in $G_N$ almost surely, i.e., every vertex is connected with at least one other vertex. This implies that at least $N/2$ edges in $G_n$ almost surely.

For $\alpha > 3$, every vertex in $G_N$ is isolated almost surely, implies that there is no edge in $G_N$ almost surely.
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