Singularities of Slant Focal Surfaces along Lightlike Locus on Mixed Type Surfaces

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Abstract: There are generally the mixed type surfaces with lightlike locus in the Lorentz-Minkowski 3-space. To investigate the geometry of lightlike locus, we define slant focal surfaces and slant evolutes associated to the original mixed type surface by using a moving frame field along the lightlike locus defined by Honda et al. We obtain that singularities of slant focal surfaces and slant evolutes depend on the differential geometric properties of the lightlike locus. Furthermore, we investigate the relationship between slant focal surfaces and slant evolutes. We also consider the relationship between slant evolutes and the lightlike locus on the lightcone.

Keywords: mixed type surfaces; lightlike locus; slant focal surfaces; slant evolutes; singularities

1. Introduction

The research aim of this paper is to investigate the differential geometric properties of the mixed type surfaces in Lorentz-Minkowski 3-space. Let $f : U \to \mathbb{R}^3$ be a frontal, where $\mathbb{R}^3$ is a 3-dimensional Lorentz-Minkowski space. If frontal $f$ is an immersion, spacelike, timelike and lightlike points can be defined on it in terms of the induced metric. The lightlike points are independent from singular points of frontal $f$. It can be a singular point of the induced metric. When $p$ is the lightlike point of the first kind [1,2], the lightlike locus $f|_{L(f)}$ may be a spacelike regular curve in [2]. Then we have tangent vector $e$ of $f|_{L(f)}$ and two lightlike vectors $L, N$ along $L(f)$. Thus they construct a moving frame along $L(f)$.

The focal surface and evolute of regular space curves are investigated as classical objects in differential geometry (cf. [3–7]). The focal surface is the envelope of family of normal planes. The evolute is not only the locus of the centre of osculating spheres but also set of singular values of the focal surfaces. Since lightlike locus is a spacelike regular curve, we can give the definitions of focal surface and evolute of lightlike locus. It is the envelope of family of normal planes which are spanned by two lightlike vectors $L$ and $N$ satisfying a symmetric (or, dual) condition along $L(f)$. Moreover, the osculating lightlike surface of lightlike locus is considered in [1]. It is the envelope of family of osculating planes which are also limiting tangent plane spanned by $e$ and $L$ of $f$ along $f|_{L(f)}$.

On the other hand, the evolute of plane curve is the envelope of family of its normal lines and the envelope of family of tangent lines is the original curve. It is natural to ask what lies between normal lines and tangent lines. In [8], P. J. Giblin and J. P. Warder give the definition of straight lines $L$ which is obtained by rotating tangent lines counterclockwise through a angle $\alpha$ along plane curve. Moreover, envelope of lines $L$ is called by $\tau_\alpha$. Then $\tau_0$ is original curve and $\tau_\alpha$ is the evolute of plane curve. Inspired by this thought, since osculating planes of $f$ along $f|_{L(f)}$ are similar to tangent lines of plane curve and normal planes of $f$ along $f|_{L(f)}$ are similar to normal lines of plane curves, we can define $\theta$-planes which move between osculating planes and normal planes along lightlike locus and study new symmetric properties about these planes. The $\theta$ is given by angle of between $\theta$-planes and osculating planes. It follows that the slant focal surface of lightlike locus is defined.
by the envelope of family of \( \theta \)-planes of \( f \) along \( f|_L(f) \). If \( \theta = 0 \), the slant focal surfaces is the osculating lightlike surface. If \( \theta = \frac{\pi}{2} \), the slant focal surface is the focal surface of lightlike locus, and the \textit{slant evolute} is given by singular set of the slant focal surface.

In this paper, we give some basic notions including frame \( \{e, L, N\} \) along lightlike locus in Section 2. In Section 3, by using this frame, we give the definitions of the slant focal surface and the slant evolute of lightlike locus \( L(f) \). Then geometry and singularities of them can be investigated by the moving frame. On the other hand, wave front is firstly given in [9]. Furthermore, many articles on the frontal or front have been published during the two decades [10–16]. Thus, by the criterions of singularities of frontal or front, we obtain that singularities of slant focal surfaces have not only cuspidal edge, swallowtail but also cuspidal beaks under this frame. But cuspidal cross cap and cuspidal lips are never appeared on the slant focal surface. Moreover, we investigate relationship between slant focal surfaces and slant evolutes from viewpoint of singularity theory. We obtain that the image of slant evolute is precisely the set of non-degenerate singular values of slant focal surface. Since the geometry of moving frame is related to the properties of \( f|_L(f) \), thus the geometry and singularities of slant focal surfaces and slant evolutes are deeply dependent on geometric properties of the frontal \( f \). In Section 4, slant focal surfaces and slant evolutes are given by the discriminant set and the secondary discriminant sets of \( \theta \)-function. Moreover, if the slant evolute is a constant point under a certain condition, then the lightlike locus is on a lightcone whose vertex is the slant evolute, meanwhile this lightcone is the osculating lightlike surface. Finally, if the evolute of lightlike locus is a constant point, then the lightlike locus is on a pseudo sphere. In recent years, some of the latest connected studies can be seen in [17–26]. In the future work, we are going to proceed to study some applications combine with singularity theory and submanifold theory, etc. to obtain new results and theorems about symmetry.

All manifolds and mappings are \( C^\infty \) unless otherwise stated.

2. Preliminaries

In this section, we prepare some basic notions about frontal and lightlike locus in \( \mathbb{R}^3 \).

Firstly, we give the definition of the frontal in \( \mathbb{R}^3 \). About lightlike locus on mixed surfaces, please see [2] for details.

2.1. Frontals in \( \mathbb{R}^3 \)

Let \( \mathbb{R}^3 = \{(x_0, x_1, x_2) | x_i \in \mathbb{R} (i = 0, 1, 2)\} \) be a 3-dimensional vector space. For any \( x = (x_0, x_1, x_2), y = (y_0, y_1, y_2) \in \mathbb{R}^3 \), the \textit{pseudo scalar product} of \( x \) and \( y \) is defined by

\[
\langle x, y \rangle = -x_0 y_0 + x_1 y_1 + x_2 y_2,
\]

we call \((\mathbb{R}^3, \langle , \rangle)\) the \textit{Minkowski space} and write \( \mathbb{R}^3 \) instead of \((\mathbb{R}^3, \langle , \rangle)\). A non-zero vector \( x \in \mathbb{R}^3 \) is \textit{spacelike} (respectively, \textit{timelike}, \textit{lightlike}) if \( \langle x, x \rangle \) is positive (respectively, negative, zero). The \textit{norm} of a non-zero vector \( x \) is defined by \( |x| = \sqrt{\langle x, x \rangle} \). A plane \( P(v, c) = \{x \in \mathbb{R}^3 | \langle x, v \rangle = c\} \) is called \textit{spacelike}, \textit{timelike} or \textit{lightlike} when \( v \) is timelike, spacelike or lightlike respectively. We can define the vector product

\[
x \times y = \det \begin{pmatrix}
-e_0 & e_1 & e_2 \\
x_0 & x_1 & x_2 \\
y_0 & y_1 & y_2
\end{pmatrix},
\]

where \( x = (x_0, x_1, x_2) \) and \( y = (y_0, y_1, y_2) \in \mathbb{R}^3 \).

We can consider a subbundle \( \mathbb{R}^3 \times Gr(2, 3) \subseteq \mathbb{R}^3 \times \mathbb{R}^3 = T\mathbb{R}^3 \), where \( Gr(2, 3) \) is the Grassmannian of 2-planes in \( \mathbb{R}^3 \). We can identify it with the projective tangent bundle

\[
PT\mathbb{R}^3 = \mathbb{R}^3 \times \mathbb{P} \mathbb{R}^3
\]

via \( PT\mathbb{R}^3 \ni V_q \rightarrow (V_q)^{-1} \) by the scalar product \( \langle , \rangle \), where \( x^+ = \{y \in \mathbb{R}^3 | \langle x, y \rangle = 0\} \).

As we know that \( PT\mathbb{R}^3 = \mathbb{R}^3 \times \mathbb{P} \mathbb{R}^3 \) is a contact manifold. A map \( f : U \rightarrow \mathbb{R}^3 \) is a \textit{frontal} if there exists a map \( F = (f, [v]) : U \rightarrow \mathbb{R}^3 \times \mathbb{P} \mathbb{R}^3 \) such that \( F \) is an isotropic map, that is, \( \langle df(X), v \rangle = 0 \) for any \( X \in T_pU \) and \( p \in U \). We can call \( F \) is an \textit{isotropic lift} of \( f \). Since we
can identify $PT\mathbb{R}^3_1$ with $\mathbb{R}^3_1 \times Gr(2,3)$, $F = (f, [\nu])$ can be identified with $(f, v^\perp)$, where $v^\perp: U \to Gr(2,3)$ be a 2-dimensional subspace-valued map. We can call $v^\perp(p)$ the limiting tangent plane at $p \in U$. A frontal $f$ is a front if the isotropic lift $F$ is an immersion. If $f$ is an immersion and $[\nu] = [f_u \times f_\nu]$, then $f$ is a front. We can call $[\nu]$ a lightcone Gauss map of $f$ [27, 28].

It is known that a frontal (front) may have singularities. The generic singularities of frontals are cuspidal edges, swallowtails and cuspidal cross cap. Cuspidal edges and swallowtails can appear on the fronts, but cuspidal cross cap is a frontal which is not a front. A singular point of a frontal germ $f$ at $p$ is a cuspidal edge (briefly, CE) if $f$ is $A$-equivalent to $(u, v) \mapsto (u, v^2, v^3)$ at the origin. Here we recall two map germs $f, g: (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ are $A$-equivalent if there exist diffeomorphism $\Phi: (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ such that $\Phi \circ f \circ \Phi^{-1} = g$. A singular point of a frontal germ $f$ at $p$ is a swallowtail (briefly, SW) if it is $A$-equivalent to $(u, v) \mapsto (u, 4v^3 + 2uv, 4v^4 + uv^2)$ at the origin. A singular point of a frontal germ $f$ at $p$ is a cuspidal cross cap (briefly, CBK) if $f$ is $A$-equivalent to $(u, v) \mapsto (u, v^3, uv^2)$ at the origin. In this decade, differential geometric properties of frontals or fronts in $\mathbb{R}^3$ are non-empty. We can find many studies on differential geometric property of spacelike surfaces or surfaces whose lightlike point set $L(f)$ is non-empty in $\mathbb{R}^3_1$ (cf. [27, 28, 30–40]).

2.2. Frame Field along Lightlike Locus

Let $f: U \to \mathbb{R}^3_1$ be a frontal with non-empty lightlike point set $L(f)$. Then $F = (f, [\nu])$ is an isotropic lift of $f$. We can assume that $L(f)$ can be parameterized by a regular curve $\gamma: (-e, e) \to U$ near $p = \gamma(0) \in L(f)$. Under the assumption we call $f(L(f))$ the lightlike locus and set $\dot{\gamma} = f \circ \gamma$. Furthermore, we also assume that $\dot{\gamma}'(u) = f_{\dot{u}}$ is non-zero and spacelike. This indicates that the lightlike locus $\dot{\gamma}$ is a spacelike regular curve in $\mathbb{R}^3_1$. Then we call frontal $f$ which satisfies the above assumptions an admissible frontal.

We consider a frame along $L(f)$ as a regular curve on the frontal $f$ (cf. [2]). We take a parameter $u$ which satisfies $|e(u)| = 1$, where $e$ is a unit tangent vector field of $\dot{\gamma}$. Then we have a frame $\{e, L, N\}$ along $\dot{\gamma}(u)$ satisfying

$$\langle e, e \rangle = 1, \langle e, L \rangle = 0, \langle e, N \rangle = 0, \langle L, L \rangle = 0, \langle L, N \rangle = 1, \langle N, N \rangle = 0,$$

(1)

and the following Frenet-Serret type formula

$$\begin{pmatrix} e'(u) \\ L'(u) \\ N'(u) \end{pmatrix} = \begin{pmatrix} 0 & \alpha_N(u) & \alpha_L(u) \\ -\alpha_L(u) & -\alpha_G(u) & 0 \\ -\alpha_N(u) & 0 & \alpha_G(u) \end{pmatrix} \begin{pmatrix} e(u) \\ L(u) \\ N(u) \end{pmatrix},$$

(2)

where

$$\alpha_L = \langle e', L \rangle, \quad \alpha_N = \langle e', N \rangle, \quad \alpha_G = \langle N', L \rangle.$$

(3)

Moreover, we can set $\dot{L}(u) = \psi(u)L(u)$, where $\psi: I \to \mathbb{R}$ is a non-vanishing function. Thus, we assume $\dot{N}(u) = N(u)/\psi(u)$ in order to make the frame $\{e, \dot{L}, \dot{N}\}$, satisfy (1) along $L(f)$. By (3), we can also define three functions $\pi_L, \pi_N, \pi_G$ relating to the frame $\{e, L, N\}$ and have the following relationships:

$$\pi_L = \psi \alpha_L, \quad \pi_N = \alpha_N/\psi, \quad \pi_G = \alpha_G + \psi(1/\psi)' \cdot$$

(4)

Then $\alpha_L, \alpha_N, \alpha_G$ can be regarded as invariants of $f$ as follows:

We consider a vector field $\eta$ on $U$ satisfying that $df(\eta(q)) = L(q)$. We set function $\beta = \eta(\dot{f}(\eta), df(\eta))|_{L(f)}$ which does not vanish along $L(f)$. Let $\kappa_L, \kappa_N$ and $\kappa_G$ the lightlike
Symmetry 2022, 14, 1203

singular curvature, the lightlike normal curvature and the lightlike geodesic torsion of\( f \) along\( L(f) \), respectively ([2], [Definition 3.2]). Then by [2], [Proposition 3.5], we obtain that
\[
\alpha_L = \beta^{1/3} \kappa_L, \quad \alpha_N = \beta^{-1/3} \kappa_N, \quad \alpha_G = \kappa_G + \beta^{1/3}(\beta^{-1/3})'.
\] (5)

If we take vector field \( \eta \) satisfying that \( \beta = 1 \), then \( \alpha_L = \kappa_L, \alpha_N = \kappa_N \) and \( \alpha_G = \kappa_G \) hold.

To simplify the expressions in Section 3, we define four smooth functions as follows:
\[
\sigma^\theta : I \to \mathbb{R}, \quad \theta \in [0, \pi/2], \\
h^\theta : I \to \mathbb{R}, \quad \theta \in [0, \pi/2],
\]
\[
\sigma^\theta(u) = \cos^2 \theta \alpha_L(u) + \sin^2 \theta \alpha_N(u) + \sin \theta \cos \theta \alpha_G(u).
\] (6)
\[
h^\theta(u) = \sin^3 \theta(-\alpha'_L + \alpha_N \alpha_G)(u) + \sin^2 \theta \cos \theta(-\alpha'_G + \alpha'_N + 2\alpha_L \alpha_N)(u) + \sin \theta \cos^2 \theta(3\alpha'_N \alpha_G)(u) + \cos^3 \theta \alpha'_L(u).
\] (7)
\[
h^1_1 : I \to \mathbb{R}, \quad \theta \in [0, \pi/2],
\]
\[
h^1_1(u) = \sin^3 \theta(-\alpha'_L + \alpha_N \alpha_G)(u) + \sin^2 \theta \cos \theta(-\alpha'_G + \alpha'_N + 2\alpha_L \alpha_N)(u) + \sin \theta \cos^2 \theta(-\alpha'_L + 2\alpha_N \alpha_G)(u) + \cos^3 \theta \alpha'_L(u).
\] (8)
\[
h^2_2 : I \to \mathbb{R}, \quad \theta \in [0, \pi/2],
\]
\[
h^2_2(u) = \sin^2 \theta(\alpha'_L + \alpha_N \alpha_G)(u).
\] (9)

3. Singularities of Slant Focal Surfaces of the Lightlike Locus

In this section, we investigate singularities of slant focal surfaces of the lightlike locus and give the relationship between the slant focal surface and the slant evolute from the viewpoint of singularity theory.

Let \( f : U \to \mathbb{R}^3 \) be an admissible frontal. Under the notations in Section 2.2, we can define slant focal surfaces of the lightlike locus \( \hat{\gamma} \) as follows:

**Definition 1.** For a fixed \( \theta \in [0, \pi/2] \), the slant focal surface \( \mathcal{FS}^\theta_{\hat{\gamma}} : I \times I \to \mathbb{R}^3 \) of lightlike locus \( \hat{\gamma} \) is given by
\[
\mathcal{FS}^\theta_{\hat{\gamma}}(u, m, n) = \hat{\gamma}(u) - mL(u) - n(\cos \theta e(u) + \sin \theta N(u)),
\]
where \( I = \{(m, n) \in \mathbb{R}^2 | m \sin \theta \alpha_L(u) + n \sigma^\theta(u) + \sin \theta = 0 \text{ for all } u \in I \} \).

At least locally, we can easily see that \( (\sigma^\theta, \sin \theta \alpha_L)(u) \neq (0, 0) \). If \( \sigma^\theta(u) \neq 0 \), the slant focal surface \( \mathcal{FS}^\theta_{\hat{\gamma}} : I \times \mathbb{R} \to \mathbb{R}^3 \) is given by
\[
\mathcal{FS}^\theta_{\hat{\gamma}}(u, m) = \hat{\gamma}(u) - mL(u) + \frac{\sin \theta(\alpha_L'(u) + 1)}{\sigma^\theta(u)}(\cos \theta e(u) + \sin \theta N(u)).
\] (10)

If \( \sin \theta \alpha_L(u) \neq 0 \), the slant focal surface \( \mathcal{FS}^\theta_{\hat{\gamma}} : I \times \mathbb{R} \to \mathbb{R}^3 \) is given by
\[
\mathcal{FS}^\theta_{\hat{\gamma}}(u, m) = \hat{\gamma}(u) + \frac{n \sigma^\theta(u) + \sin \theta \alpha_L(u)}{\sin \theta \alpha_L(u)}L(u) - n(\cos \theta e(u) + \sin \theta N(u)).
\] (11)
In the case when $\theta = 0$, we consider the slant focal surface under the assumption $\sigma^0 \neq 0$. It is given by

$$FS^0_\theta(u, m) = \hat{\gamma} - mL(u), \quad (12)$$

then we have the followings.

**Remark 1.** If $\theta = 0$, $FS^0_\theta$ coincides with the osculating lightlike surface $f_L$ of $f$ along $L(f)$ (cf. [1]). Moreover, if $\theta = \frac{\pi}{2}$, we call $FS^{\pi/2}_\theta$ the focal surfaces of the lightlike locus $\hat{\gamma}$.

Assume that $\sigma^\theta(u) \neq 0$ and $U = I \times \mathbb{R}$, by Equations (2) and (10), we have

$$
(FS^\theta_\theta)_{u(m, m)} = \left(\frac{ma_L(u) + 1}{\sigma^\theta(u)}\right)' \sin \theta \cos \theta
+ \frac{(ma_L(u) + 1)(\cos^2 \theta a_L(u) + \sin \theta \cos \theta a_G(u))}{\sigma^\theta(u)}e(u)
+ \frac{(ma_L(u) + 1)\sin \theta \cos \theta a_N(u)}{\sigma^\theta(u)}L(u)
+ \left(\frac{\sin \theta (ma_L(u) + 1)}{\sigma^\theta(u)}\right)' \sin \theta
+ \frac{(ma_L(u) + 1)(\sin \theta \cos \theta a_L(u) + \sin^2 \theta a_G(u))}{\sigma^\theta(u)}N(u). \quad (13)
$$

To simplify (13), we define three functions $a^\theta_1, a^\theta_2, a^\theta_3: U \to \mathbb{R}$ satisfying

$$(FS^\theta_\theta)_{u(m, m)} = a^\theta_1(u, m)e(u) + a^\theta_2(u, m)L(u) + a^\theta_3(u, m)N(u).$$

Furthermore, we define a mapping $\nu^\theta: I \to \mathbb{R}^3$ for a fixed $\theta \in [0, \pi/2]$ by,

$$\nu^\theta(u) = \sin \theta e(u) - \cos \theta L(u).$$

It follows that

$$\langle (FS^\theta_\theta)_{u(m, m)}, \nu^\theta(u) \rangle = 0, \quad \langle (FS^\theta_\theta)_{m(m, m)}, \nu^\theta(u) \rangle = 0.$$ 

Then $(FS^\theta_\theta, [\nu^\theta])$ is an isotropic lift of $FS^\theta_\theta$. Thus, $FS^\theta_\theta$ is a frontal for a fixed $\theta \in [0, \pi/2]$.

To investigate singularities of $FS^\theta_\theta$, we define a mapping $T^\theta: I \to \mathbb{R}^3$ for a fixed $\theta \in [0, \pi/2]$ which is transverse to $(\nu^\theta)^\perp$ as follows (cf. [1]):

$$T^\theta(u) = \delta^\theta (\sin \theta e(u) - \cos \theta L(u)) + \epsilon^\theta N(u),$$

where

$$\delta^\theta = \begin{cases} 1 & \text{if } \theta \neq 0, \\ 0 & \text{if } \theta = 0, \end{cases} \quad \epsilon^\theta = \begin{cases} 0 & \text{if } \theta \neq 0, \\ 1 & \text{if } \theta = 0. \end{cases} \quad (15)$$
We can define a smooth function \( \lambda^\theta : U \to \mathbb{R} \) (cf. [1]) for a fixed \( \theta \in [0, \pi/2] \) by

\[
\lambda^\theta(u, m) = \det((\mathcal{FS}_r^\theta)_u(u, m), (\mathcal{FS}_r^\theta)_m(u, m), T^\theta(u))
\]

\[
= \delta^\theta(\sin^3 \theta \delta^\theta_2(u, m) \frac{\alpha_L(u)}{\alpha^3(u)} + \sin \theta \delta^\theta_3(u, m))
\]

\[
+ \epsilon^\theta(\sin \theta \cos \theta \delta^\theta_2(u, m) \frac{\alpha_L(u)}{\alpha^3(u)} + \delta^\theta_1(u, m)),
\]

then \( p \) is a singular point if and only if \( \lambda^\theta(p) = 0 \). Moreover, \( p \) is non-degenerate if and only if \( d\lambda^\theta(p) \neq 0 \). By implicit function theorem, the singular set \( S(\mathcal{FS}_r^\theta) = \{(u, m)|\lambda^\theta(u, m) = 0\} \) is parameterized by a regular curve \( \xi^\theta : I \to U \) in a neighborhood of \( p \). For the singular set \( S(\mathcal{FS}_r^\theta) \), there exists a non-zero vector field \( \eta^\theta \) near \( p \) satisfying \( \eta^\theta = \ker d\mathcal{FS}_r^\theta \) at \( p \in S(\mathcal{FS}_r^\theta) \). We call \( \xi^\theta \) null vector field. Furthermore, we call \( (\xi^\theta)_u(u) \) singular direction and \( \eta^\theta(u) \) null direction if \( \xi^\theta \) is parameterized by \( m \). When \( \xi^\theta \) is parameterized by \( m \), we also call \( (\xi^\theta)_m(m) \) singular direction and \( \eta^\theta(m) \) null direction.

**Theorem 1.** Under the assumption \( \sigma^\theta \neq 0 \), we assume that \( p_0 = (u_0, m_0) \) is a singular point of \( \mathcal{FS}_r^\theta \) and have the followings.

(a) If \( h^\theta(u_0) \neq 0 \) and \( \theta \neq 0 \), then

1. \( \mathcal{FS}_r^\theta \) at \( p_0 \) is \( A \)-equivalent to the cuspidal edge if and only if \( m'(u_0) - a^\theta_2(u_0, m_0) \neq 0 \).
2. \( \mathcal{FS}_r^\theta \) at \( p_0 \) is \( A \)-equivalent to the swallowtail if and only if \( m'(u_0) - a^\theta_2(u_0, m_0) = 0, m''(u_0) - \frac{d}{du}(a^\theta_2(u_0, m(u_0))) \neq 0 \).
3. \( \mathcal{FS}_r^\theta \) at \( p_0 \) is never \( A \)-equivalent to the cuspidal cross cap.

(b) If \( h^\theta(u_0) = 0 \), then

1. \( \mathcal{FS}_r^\theta \) at \( p_0 \) is \( A \)-equivalent to the cuspidal edge if and only if \( (m_0 h^\theta + h^\theta)_u(u_0) \neq 0 \).
2. \( \mathcal{FS}_r^\theta \) at \( p_0 \) is \( A \)-equivalent to the cuspidal beaks if and only if

\[
(m_0 h^\theta + h^\theta)_u(u_0) = 0, (h^\theta)'(u_0) \neq 0, (m_0 h^\theta + h^\theta)_{uu}(u_0) + 2h'(u_0)a^\theta_2(u_0, m_0) \neq 0.
\]
3. \( \mathcal{FS}_r^\theta \) at \( p_0 \) is never \( A \)-equivalent to the swallowtail, cuspidal cross cap and cuspidal lips.

(c) If \( \theta = 0 \), then

1. \( \mathcal{FS}_r^\theta \) at \( p_0 \) is \( A \)-equivalent to the cuspidal edge if and only if \( m'(u_0) - a^\theta_2(u_0, m_0) \neq 0 \).
2. \( \mathcal{FS}_r^\theta \) at \( p_0 \) is \( A \)-equivalent to the swallowtail if and only if

\[
m'(u_0) - a^\theta_2(u_0, m_0) = 0, m''(u_0) - \frac{d}{du}(a^\theta_2(u_0, m(u_0))) \neq 0.
\]
3. \( \mathcal{FS}_r^\theta \) at \( p_0 \) is never cuspidal cross cap, cuspidal beaks and cuspidal lips.

Here, a singular point \( p \) of \( f \) is a cuspidal beaks if \( f \) is \( A \)-equivalent to \((u, v) \mapsto (u, 2v^3 - u^2v, 3u^4 - u^2v^2) \) at 0. A singular point \( p \) of \( f \) is a cuspidal lips (briefly, CL) if \( f \) is \( A \)-equivalent to \((u, v) \mapsto (u, 2v^3 + u^2v, 3u^4 + u^2v^2) \) at 0. A singular point \( p \) of \( f \) is a cuspidal cross cap (briefly, CCR) if \( f \) is \( A \)-equivalent to \((u, v) \mapsto (u, v^2, uv^3) \) at 0. We can draw the pictures of these singularities by software “MATHEMATICA” in Figures 1 and 2. About the criteria for CE, SW, please see [13]. Criteria for CCR, see [10]. Criteria for CBK, CL, see [11].
Then \( p \) is a singular point if and only if there exists a frontal which is not front, then the assertion (3) of (a) holds.

If \( \sigma^\theta \neq 0 \), then \( \eta^\theta \nu^\theta \neq 0 \). Thus, \( \mathcal{F}S^\theta_{FS} \) is a front.

If \( \theta \neq 0 \), by (16),

\[
\lambda^\theta(u, m) = \sin^2 \theta \frac{m h^\theta(u) + h^\theta_1(u)}{\sigma^\theta(u)}.
\]

Then we have

\[
\lambda^\theta_m(u, m) = \sin^2 \theta \frac{h^\theta_2(u)}{\sigma^\theta(u)} \quad \lambda^\theta_u(u, m) = \sin^2 \theta \frac{m h^\theta(u) + h^\theta_1(u)}{\sigma^\theta(u)} u.
\]

If \( h^\theta(u_0) \neq 0 \), then \( d\lambda^\theta(u_0, m_0) \neq 0 \), so that \( p \) is non-degenerate singular point. By implicit theorem, we have \( \xi^\theta(u) = (u, m(u)) \). Then \( \zeta^\theta(u) = \partial/\partial u + a^\theta_2(u, m(u)) \partial/\partial m \).

We consider

\[
\det(\eta^\theta, (\zeta^\theta)')(u) = \begin{vmatrix}
1 & a^\theta_2(u, m(u)) \\
1 & m'(u)
\end{vmatrix}, \quad \frac{d}{du} \det(\eta^\theta, (\zeta^\theta)')(u) = m''(u) - (a^\theta_2)'(u, m(u)).
\]

Then \( \det(\eta^\theta, (\zeta^\theta)')(u_0) \neq 0 \) proves the assertion (1) of (a), \( \det(\eta^\theta, (\zeta^\theta)')(u_0) = 0 \) and \( \frac{d}{du} \det(\eta^\theta, (\zeta^\theta)')(u_0) \neq 0 \) prove the assertion (2) of (a). Since \( \mathcal{F}S^\theta_{FS} \) is a front and cuspidal cross cap is a frontal which is not front, then the assertion (3) of (a) holds.

When \( h^\theta(u_0) = 0 \), we have \( \xi^\theta(u_0, m_0) = \lambda^\theta_m(u_0, m_0) = 0 \). Point \( p \) is non-degenerate singular point if and only if \( \lambda^\theta_u(u_0, m_0) \neq 0 \). If \( p \) is non-degenerate, we have \( \zeta^\theta(u) = (u_0, m) \).

Then \( (\xi^\theta)' = \partial/\partial m \) and \( \eta^\theta(m) = \partial/\partial u + a^\theta_2(u(m), m) \partial/\partial m \). We consider

\[
\det(\eta^\theta, \xi^\theta_m)(m) = \begin{vmatrix}
1 & a^\theta_2(u(m), m) \\
0 & 1
\end{vmatrix} = 1,
\]

then \( p_0 \) is a cuspidal edge and there is no swallowtail, thus the assertion (1) of (b) holds.

**Figure 1.** there exist above singular points on the slant focal surface.

**Figure 2.** there never exist singular points on the slant focal surface.
If \( p_0 \) is degenerate singular point, we can consider
\[
\text{det } \text{Hess} \lambda^\theta(p_0) = \det \left( \begin{array}{cc} \lambda_{mm}^\theta & \lambda_{mu}^\theta \\ \lambda_{um}^\theta & \lambda_{uu}^\theta \end{array} \right)(p_0) = -\left( \lambda_{mm}^\theta \right)^2(p_0) = -\sin^4 \theta \left( \frac{h^\theta(u)}{\sigma^\theta(u)} \right)' \leq 0
\]  
(21)

and
\[
\eta^\theta \eta^\theta \lambda^\theta(p_0) = \sin^2 \theta \frac{m h^\theta(u_0) + h^\theta_1(u_0) u + 2(h^\theta)'(u_0) a^\theta_2(u_0, m_0)}{\sigma^\theta(u_0)}.
\]  
(22)

Thus, \( \text{det } \text{Hess} \lambda^\theta(p_0) \neq 0 \) and \( \eta^\theta \eta^\theta \lambda^\theta(p_0) \neq 0 \) prove the assertion (2) of (b).

Under the assumption \( \sin \theta \neq 0 \), and have the followings.

If \( \theta = 0 \),
\[
\lambda^0(u, m) = m a_L(u) + 1,
\]  
(23)

Then \( \lambda_m^0(u, m) = (h^0)^{1/3}(u) = a_L(u) \). Since \( \sigma^0(u_0) = a_L(u_0) \neq 0 \), then \( d\lambda^0(u_0) \neq 0 \). Thus, \( p \) on \( F S^0_L \) is non-degenerate singular point. By implicit theorem, we have \( \xi^0(u) = (u, m(u)) \). Then \( (\xi^0)'(u) = d/\sigma + m'(u) \sigma/\sigma_m \) and \( \eta^0(u) = d/\sigma + a^2_2(u, m(u)) \).

We consider
\[
\text{det}(\eta^0, (\xi^0)')(u) = \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \frac{\partial^2(u, m(u))}{m'(u)}, \quad \frac{d}{du} \text{det}(\eta^0, (\xi^0)')(u) = m''(u) - (a^0_2)'(u, m(u)).
\]  
(24)

Then \( \text{det}(\eta^0, (\xi^0)')(u) \neq 0 \) gives a condition for cuspidal edge on \( F S^0_L \). \( \text{det}(\eta^0, (\xi^0)')(u) = 0 \) and \( \frac{d}{du} \text{det}(\eta^0, (\xi^0)')(u) \neq 0 \) give a condition for swallowtail on \( F S^0_L \). Since \( F S^0_L \) is a front and cuspidal cross cap is a frontal which is not front, there is no cuspidal cross cap. Since \( p \) is only non-degenerate singular point, then there is no cuspidal beaks and cuspidal lips. \( \square \)

Assume that \( \sin \theta a_L(u) \neq 0 \) at least locally and \( U = I \times \mathbb{R} \), by Equations (2) and (11), we have
\[
(F S^\theta_L)(u, n) = (- \frac{n \sigma^\theta(u)}{\sin \theta} + n \sin \theta a_N(u)) e(u)
\]  
(25)

Under the assumption \( \sin \theta a_L \neq 0 \), we investigate singularities of \( F S^\theta_L \). Then we define a function \( \overline{\lambda}^\theta: U \to \mathbb{R} \) by:
\[
\overline{\lambda}^\theta(u, n) = \text{det}(F S^\theta_L)(u, n), (F S^\theta_L)(u, n), T^\theta(u))
\]  
(26)

then we obtain that singular set \( S(F S^\theta_L) = \{(u, n)|\overline{\lambda}^\theta(u, n) = 0\} \). We give the condition for singular points of \( F S^\theta_L \) under the assumption \( a_L \neq 0 \) and \( \theta \neq 0 \) as follows.

**Theorem 2.** Under the assumption \( a_L \neq 0 \) and \( \theta \neq 0 \), we assume that \( p_0 = (u_0, n_0) \) is a singular point of \( F S^\theta_L \) and have the followings.
(a) If \( h^\theta \neq 0 \), then

1. \( \mathcal{FS}_l^\theta \) at \( p_0 \) is \( A \)-equivalent to the cuspidal edge if and only if \( \sin \theta n'(u_0) - b^\theta_3(u_0, n_0) \neq 0 \).
2. \( \mathcal{FS}_l^\theta \) at \( p_0 \) is \( A \)-equivalent to the swallowtail if and only if

\[
\sin \theta n'(u_0) - b^\theta_3(u_0, n_0) = 0, \quad \sin \theta n''(u_0) - \frac{\partial}{\partial u} b^\theta_3(u_0, n_0) \neq 0.
\]

(3) \( \mathcal{FS}_l^\theta \) at \( p_0 \) is never \( A \)-equivalent to the cuspidal cross cap

(b) If \( h^\theta = 0 \), then

1. \( \mathcal{FS}_l^\theta \) at \( p_0 \) is \( A \)-equivalent to the cuspidal edge if and only if

\[
\frac{\partial}{\partial u}(nh^\theta(u_0) + h^\theta_2(u_0)) \neq 0.
\]

(2) \( \mathcal{FS}_l^\theta \) at \( p_0 \) is \( A \)-equivalent to the cuspidal beaks if and only if

\[
\frac{\partial}{\partial u}(nh^\theta(u_0) + h^\theta_2(u_0)) = 0, \quad (h^\theta)'(u_0) \neq 0,
\]

\[
\sin \theta(nh^\theta(u_0) + h^\theta_2(u_0)) + 2b^\theta_3(u_0, n_0)(h^\theta)'(u_0) \neq 0
\]

(3) \( \mathcal{FS}_l^\theta \) at \( p_0 \) is never \( A \)-equivalent to swallowtail, cuspidal cross cap and cuspidal lips.

**Remark 2.** Since the proof of Theorem 2 is similar to the one of Theorem 1 under the assumption \( \theta \neq 0 \). Then we omit it here.

On the other hand, we give the definition of slant evolutes of lightlike locus. Then we give that the image of the set of non-degenerate singular points of the slant focal surfaces coincide with the image of the slant evolute. Moreover, we give relationships between singularities of the slant evolutes and singularities of slant focal surfaces.

**Definition 2.** For a fixed \( \theta \in [0, 2\pi] \), the slant evolute \( \mathcal{E}_l^\theta : I \to \mathbb{R}^3 \) of lightlike locus \( \hat{\gamma} \) with \( h^\theta(u) \neq 0 \) for \( u \in I \) is given by

\[
\mathcal{E}_l^\theta(u) = \hat{\gamma}(u) + \frac{h_2^\theta(u)}{h^\theta(u)} L(u) + \frac{h_3^\theta(u)}{h^\theta(u)} (\cos \theta e(u) + \sin \theta N(u)),
\]

**Theorem 3.** Let \( \hat{\gamma} \) be a lightlike locus with \( h^\theta(u) \neq 0 \). Point \( p_0 = (u_0, m_0, n_0) \) is a singular point of \( \mathcal{FS}_l^\theta \). Then we have the following:

1. the image of set of non-degenerate singular point of \( \mathcal{FS}_l^\theta \) coincide with the image of \( \mathcal{E}_l^\theta \).
2. \( \mathcal{FS}_l^\theta \) at \( (u_0, m_0, n_0) \) is \( A \)-equivalent to the cuspidal edge if and only if \( \mathcal{E}_l^\theta \) at \( u_0 \) is a regular point.
3. \( \mathcal{FS}_l^\theta \) at \( (u_0, m_0, n_0) \) is \( A \)-equivalent to the swallowtail of and only if \( \mathcal{E}_l^\theta \) at \( u_0 \) is locally diffeomorphic to (2,3,4)-cusp.

**Proof.** For slant focal surface, we can assume that \( h^\theta(u) \neq 0 \) under \((\sigma^\theta, \sin \theta \nu_l) \neq (0, 0)\). If \( \sigma^\theta \neq 0 \), we have the Equation (10). Then we give a proof as follows:

Since \( h^\theta \neq 0 \), by Equations (17) and (18), we have non-degenerate singular set \( \hat{\gamma}(u) = (u, m(u)) \), where

\[
m(u) = -\frac{h_3^\theta(u)}{h^\theta(u)}.
\]

(27)
By a direct calculation, non-degenerate singular locus \( FS^0_\gamma(u, m(u)) = E^\theta_\gamma(u) \) under \( \sigma^\theta \neq 0 \). Similarly, we easily have that the \( FS^0_\gamma(u, n(u)) = E^\theta_\gamma(u) \) under \( \alpha_L \neq 0 \). Thus, the assertion (1) holds.

By Equations (10) and (27), we have

\[
E^\theta_\gamma(u) = \gamma(u) - m(u)L(u) + \frac{\sin \theta(m(u)\alpha_L(u) + 1)}{\sigma^\theta(u)}(\cos \theta e(u) + \sin \theta N(u)). \tag{28}
\]

To simplify the follow equations, we define three functions \( \epsilon_1^\theta, \epsilon_2^\theta, \epsilon_3^\theta : I \to \mathbb{R} \) as follows:

\[
\epsilon_1^\theta(u) = \frac{\sin \theta \cos \theta \alpha_L(u)(m'(u) - a_2^\theta(u, m(u)))}{\sigma^\theta(u)},
\]

\[
\epsilon_2^\theta(u) = a_2^\theta(u, m(u)) - m'(u),
\]

\[
\epsilon_3^\theta(u) = \frac{\sin^2 \theta \alpha_L(u)(m'(u) - a_2^\theta(u, m(u)))}{\sigma^\theta(u)}. \tag{29}
\]

By (2), (13), (14) and (28), we have

\[
(E^\theta_\gamma)'(u) = (a_1^\theta(u, m(u)) + m'(u) \sin \theta \cos \theta \alpha_L(u))e(u)
+ (a_2^\theta(u, m(u))) - m'(u))L(u)
+ (a_3^\theta(u, m(u))) + m'(u) \sin^2 \theta \alpha_L(u))N(u)
\]

\[
= \epsilon_1^\theta(u)e(u) + \epsilon_2^\theta(u)L(u) + \epsilon_3^\theta(u)N(u), \tag{30}
\]

\[
(E^\theta_\gamma)''(u) = (\epsilon_1^\theta)'(u)e(u) + (\epsilon_2^\theta)'(u)L(u) + (\epsilon_3^\theta)'(u)N(u)
+ \epsilon_1^\theta(u)e'(u) + \epsilon_2^\theta(u)L'(u) + \epsilon_3^\theta(u)N'(u), \tag{31}
\]

\[
(E^\theta_\gamma)'''(u) = (\epsilon_1^\theta)''(u)e(u) + (\epsilon_2^\theta)''(u)L(u) + (\epsilon_3^\theta)''(u)N(u)
+ 2(\epsilon_1^\theta)'(u)e'(u) + 2(\epsilon_2^\theta)'(u)L'(u) + 2(\epsilon_3^\theta)'(u)N'(u)
+ \epsilon_1^\theta(u)e''(u) + \epsilon_2^\theta(u)L''(u) + \epsilon_3^\theta(u)N''(u). \tag{32}
\]

By Equations (29) and (30), we have \( \epsilon_2^\theta(u) = m'(u) - a_2^\theta(u, m(u))) \neq 0 \) if and only if \( (E^\theta_\gamma)'(u) \neq 0 \). By the assertion (1) of (a) of Theorem 1, the assertion (2) holds under the assumption \( \sigma^\theta \neq 0 \).

If \( p_0 \) is a (2,3,4)-cusp of \( E^\theta_\gamma \), then \( (E^\theta_\gamma)'(u_0) = 0, \) \( \text{rank}((E^\theta_\gamma)''(u_0)), (E^\theta_\gamma)'''(u_0)) = 2 \). Hence, \( (E^\theta_\gamma)'(u_0) = 0 \) if and only if \( E^\theta_\gamma(p_0) = 0 \). Using (29), (31) and (32), a long but straightforward computation gives that

\[
((\epsilon^\theta_2)^2(u_0)N^\theta(u_0) \neq 0
\]

if and only if \( \text{rank}((E^\theta_\gamma)''(u_0)), (E^\theta_\gamma)'''(u_0)) = 2 \). Since \( h^\theta(u_0) \neq 0 \) and \( \sigma^\theta(u_0) \neq 0 \), thus, by the assertion (2) of (a) of Theorem 1, the assertion (3) holds under the assumption \( \sigma^\theta \neq 0 \).

If \( \sin \theta \alpha_L \neq 0 \), we have the similar conclusions to those under the assumption \( \sigma^\theta \neq 0 \). Since this proof is similar to the above proof, so we omit it. \( \square \)

4. Properties of Non-Degenerate Singular Set of Slant Focal Surfaces \( FS^0_\gamma \)

In this section, we consider the properties of non-degenerate singular set of \( FS^0_\gamma \) under \( h^\theta(u) \neq 0 \).
4.1. \( \theta \)-Functions

Let \( \gamma : I \rightarrow \mathbb{R}^3 \) be the lightlike locus of admissible frontal \( f \). For a fixed \( \theta \in [0, \pi/2] \), we define a function \( F_\theta : I \times \mathbb{R}^3 \rightarrow \mathbb{R} \) by

\[
F_\theta(u, v) = \langle \dot{\gamma}(u) - v_\theta, -\sin \theta e(u) + \cos \theta L(u) \rangle,
\]

then \( F_\theta(u, v) = 0 \) represents a family of \( \theta \)-planes. If \( \theta = 0 \), the 0-planes are osculating lightlike planes. If \( \theta = \pi/2 \), the \( \pi/2 \)-planes are normal planes. We denote \( F_\theta(v_0)(u) = F_\theta(u, v_0) \), for any \( v_0 \in \mathbb{R}^3 \). Then we have the following proposition.

Proposition 1. Under the above notations, then we have the followings:

1. \( F_\theta(v_0)(u_0) = 0 \) if and if there exist \( \lambda_1, \lambda_2 \in \mathbb{R} \) such that

\[
\dot{\gamma}(u_0) - v_0 = \lambda_1(\cos \theta e(u_0) + \sin \theta N(u_0)) + \lambda_2 L(u_0).
\]

2. \( F_\theta(v_0)(u_0) = (F_\theta)_u(v_0)(u_0) = 0 \) if and only if there exists \( \lambda_1, \lambda_2 \in \mathbb{R} \) such that

\[
\dot{\gamma}(u_0) - v_0 = \lambda_1(\cos \theta e(u_0) + \sin \theta N(u_0)) + \lambda_2 L(u_0),
\]

where

\[
\lambda_1 = \frac{h^2(u_0)}{h^\theta(u_0)}, \lambda_2 = -\frac{h_1^\theta(u_0)}{h^\theta(u_0)}.
\]

Proof. Since \( F_\theta(v_0)(u) = \langle \dot{\gamma}(u) - v_\theta, -\sin \theta e(u) + \cos \theta L(u) \rangle \), we have the following calculations:

\[
\begin{align*}
(a) \quad & F_\theta(v_0)(u) = \langle e(u), -\sin \theta e(u) + \cos \theta L(u) \rangle + \langle \dot{\gamma}(u) - v_\theta, -\sin \theta e(u) + \cos \theta L(u) \rangle, \\
(b) \quad & F_\theta(v_0)(u) = \langle e(u), -\sin \theta e(u) + \cos \theta L(u) \rangle + \langle \dot{\gamma} - v_\theta, -\sin \theta e(u) + \cos \theta L(u) \rangle \\
& + \langle \dot{\gamma} - v_\theta, -\sin \theta e(u) + \cos \theta L(u) \rangle.
\end{align*}
\]

By the Equation (2), \( F_\theta(v_0)(u_0) = 0 \) if and only if there exists \( \lambda_1, \lambda_2 \in \mathbb{R} \) such that

\[
\dot{\gamma}(u_0) - v_0 = \lambda_1(\cos \theta e(u_0) + \sin \theta N(u_0)) + \lambda_2 L(u_0).
\]

Moreover, if \( F_\theta(v_0)(u_0) = (F_\theta)_u(v_0)(u_0) = 0 \), by the formula (a) and Equation (2), we can also have that

\[
\lambda_1 = \frac{h^2(u_0)}{h^\theta(u_0)}, \lambda_2 = -\frac{h_1^\theta(u_0)}{h^\theta(u_0)},
\]

then the assertion (1) and (2) holds.

Under \( h^\theta \neq 0 \), if \( F_\theta(v_0)(u_0) = (F_\theta)_u(v_0)(u_0) = (F_\theta)_{uu}(v_0)(u_0) = 0 \), by the formula (a), formula (b) and Equation (2), a long but straightforward calculation gives that

\[
\lambda_1 = \frac{h_2^\theta(u_0)}{h^\theta(u_0)}, \lambda_2 = -\frac{h_1^\theta(u_0)}{h^\theta(u_0)},
\]

then the assertion (3) holds.

For \( \theta \)-functions \( F_\theta \) under a fixed \( \theta \in [0, \pi/2] \), its discriminant set is defined as follows

\[
D_{F_\theta} = \{ v \in \mathbb{R}^3 | \text{there exists } u \in I \text{ such that } F_\theta = \frac{\partial F_\theta}{\partial u} = 0 \text{ at } (u, v) \}.
\]
and its second discriminant set is

$$\mathcal{D}_E^2 = \{ v \in \mathbb{R}_1^3 \mid \text{there exists } u \in I \text{ such that } F_0 = \frac{\partial F_0}{\partial u} = \frac{\partial^2 F_0}{\partial u^2} = 0 \text{ at } (u, v) \}. $$

We can easily see that the slant focal surface $\mathcal{FS}_\gamma^b$ coincides with the discriminant set $\mathcal{D}_E$, and the slant evolute $\mathcal{E}_\gamma^b$ coincides with the second discriminant set $\mathcal{D}_E^2$. Furthermore, the second discriminant set is also the set of non-degenerate singular values of the discriminant set by Theorem 3.

4.2. Slant Evolutes of Lightlike Locus and Pseudo Spheres

In this subsection, we consider relationship between slant evolutes of lightlike locus and pseudo spheres.

**Lemma 1.** For a fixed $\theta \in [0, \pi/2]$, $(d/du)\mathcal{E}_\gamma^b(u) = 0$ for all $u \in I$ if and only if there exist a constant vector $v^\rho \in \mathbb{R}_1^3$ such that $\langle \mathring{\gamma}(u) - v^\rho, -\sin \theta e(u) + \cos \theta L(u) \rangle = 0$ for all $u \in I$.

**Proof.** If $(d/du)\mathcal{E}_\gamma^b(u) = 0$ for all $u \in I$ and a fixed $\theta \in [0, \pi/2]$, we set $v^\rho = \mathcal{E}_\gamma^b(u)$, then we have $\langle \mathring{\gamma} - v^\rho, \sin \theta e(u) + \cos \theta L(u) \rangle = 0$. Conversely, if there exist a constant vector $v^\rho \in \mathbb{R}_1^3$ such that $\langle \mathring{\gamma}(u) - v^\rho, -\sin \theta e(u) + \cos \theta L(u) \rangle = 0$, then there exist functions $x^\rho, y^\rho : I \to \mathbb{R}$ such that $\mathring{\gamma}(u) - v^\rho = x^\rho(u)L(u) + y^\rho(u)(\cos \theta e(u) + \sin \theta N(u))$ for $u \in I$ and $\theta \in [0, \pi/2]$. By taking the derivative of both the sides as follows:

$$e(u) = x^\rho(u)(-a_L e(u) + a_G(u)L(u)) + (x^\rho)'(u)L(u) + (y^\rho)'(u)(\cos \theta e(u) + \sin \theta N(u)) + y^\rho(u)(\cos \theta a_N(u)L(u) + a_L(u)N(u)) + \sin \theta(-a_N e(u) + a_G(u)N(u)).$$

(33)

Since (1), then we have

$$\begin{align*}
(x^\rho)'(u) - x^\rho(u)a_G(u) + \cos \theta y^\rho(u)a_N(u) &= 0, \quad (34) \\
(y^\rho)'(u)\sin \theta + y^\rho(u)a_L(u) \cos \theta + y^\rho(u)a_G(u) \sin \theta &= 0, \quad (35) \\
1 + x^\rho(u)a_L(u) - (y^\rho)'(u) \cos \theta + y^\rho(u)a_N(u) \sin \theta &= 0. \quad (36)
\end{align*}$$

If $\theta \neq 0$, by (35) and (36), we have

$$\sin \theta + x^\rho(u)a_L(u) \sin \theta + y^\rho(u)e^\rho(u) = 0. \quad (37)$$

By differentiating (37), then

$$0 = x^\rho(u)(a_L(u) + a_L(u)a_G(u)) + (x^\rho)'(u)(\sin^2 \theta + y^\rho(u)(\sin^2 \theta a_L(u) - a_N(u)a_G(u))) + \sin^2 \theta \cos \theta(-a_G^2 + a_G^2(u) - 2a_L(u)a_N(u)) + \sin \cos^2 \theta(-2a_L(u)a_G(u) + a_L^2(u)) + \cos^2 \theta(-a_G^2).$$

(38)

By (37) and (38), we have

$$x^\rho(u) = -\frac{h_L^0(u)}{h^0(u)} a_L(u), \quad y^\rho(u) = -\frac{h_N^0(u)}{h^0(u)} a_G(u)$$

for $\theta \neq 0$.

If $\theta = 0$, since $h^0(u) = a_L^2(u) \neq 0$, by (34)–(36), we have $x^0(u) = -\frac{1}{a_L(u)}$ and $y^0(u) = 0$. Thus, constant vector $v^\rho = \mathcal{E}_\gamma^b(u)$ for any $u \in I$ and a fixed $\theta \in [0, \pi/2]$, then $(d/du)\mathcal{E}_\gamma^b(u) \equiv 0$ for a fixed $\theta \in [0, \pi/2]$. □
Let $M^2(r) = \{ v \in \mathbb{R}^3_1 \mid \langle v, v \rangle = \delta r^2, \text{ a constant } r \in \mathbb{R} \}$, where $|v| = r$. If $\delta = 1, M^2(r) = S^2_1(r)$. If $\delta = -1, M^2(r) = H^2(r)$. If $\delta = 0, M^2(r) = L^2$. Moreover, we denote $M^2(x, r) = \{ v \in \mathbb{R}^3_1 \mid \langle v - x, v - x \rangle = \delta r^2, \text{ a constant } r \in \mathbb{R} \}$ whose vertex is $x$.

**Proposition 2.** When $\theta = 0$, $(d/du)\mathcal{E}_\gamma^0 (u) = 0$ for all $u \in I$ if and only if $\tilde{\gamma}$ lies in lightcone $LC_{v^0}$ whose vertex is $v^0 = \tilde{\gamma}(u) + \frac{1}{\alpha(x)}L(u)$. When $\theta = \pi/2$, $(d/du)\mathcal{E}_\gamma^\pi/2 (u) = 0$ for all $u \in I$ if and only if there exists a constant vector $v^0 \in \mathbb{R}^3_1$ and a non-negative real number $r \in \mathbb{R}$ such that $\tilde{\gamma} \in M^2(v^0, r)$.

**Proof.**
We fix $\alpha(x) = \frac{1}{\alpha(x)}$. For a fixed $u \in I$, $\theta \in [0, \pi/2]$, then $\tilde{\gamma}(u) \in \mathcal{E}_\gamma^0 \equiv \mathcal{E}_\gamma^\pi/2 \equiv \mathcal{E}_\gamma^\theta$. By Lemma 1, we set $\mathcal{E}_\gamma^0 \equiv \mathcal{E}_\gamma^\pi/2 \equiv \mathcal{E}_\gamma^\theta$.

**Proposition 3.** For a fixed $\theta \in [0, \pi/2]$, if we assume that $\alpha(x) + \alpha_\mathcal{L}_G \equiv 0$ then $(d/du)\mathcal{E}_\gamma^\theta (u) \equiv 0$ for all $u \in I$ if and only if there exists a constant vector $v = \tilde{\gamma}(u) + \frac{1}{\alpha(x)}L(u) \in \mathbb{R}^3_1$ such that $\tilde{\gamma}$ is on the lightcone $LC_v$. Moreover, $LC_v = FS_\gamma^0 (u)$ for all $u \in I$.

**Proof.**
Under the assumption $\alpha(x) + \alpha_\mathcal{L}_G \equiv 0$, it follows that $\mathcal{E}_\gamma^\theta (u) = \hat{\mathcal{E}}(u) + \frac{\hat{h}(u)}{\hat{h}(u)}L(u) = \tilde{\gamma}(u) + \frac{1}{\alpha(x)}L(u)$. ByLemma 1, we set $\mathcal{E}_\gamma^0 \equiv \mathcal{E}_\gamma^\pi/2 \equiv \mathcal{E}_\gamma^\theta \equiv \tilde{\gamma}(u) + \frac{1}{\alpha(x)}L(u)$, then $\hat{\mathcal{F}}_{\gamma} = FS_\gamma^0 (u)$ for all $u \in I$. We can easily see that $\hat{\mathcal{F}}_{\gamma}^0$ is a lightcone whose vertex is $\mathcal{E}^\theta_\gamma$. Thus, $LC_v = FS_\gamma^0 (u)$.

**4.3. Examples**

We give an example in order to understand the slant focal surfaces and slant evolutes of lightlike locus from intuitional viewpoint.

**Example 1.** Let $f : U \rightarrow \mathbb{R}^3$ is a mixed surfaces, $f(u, v) = (\sin \sqrt{2}u \cos v, \cos \sqrt{2}u \cos v, \sin v)$. Then lightlike set $\gamma(u) = (u, \pi/4)$ and lightlike locus

$$\tilde{\gamma} = f \circ \gamma(u) = (\sqrt{2}/2 \sin \sqrt{2}u, \sqrt{2}/2 \cos \sqrt{2}u, \sqrt{2}/2).$$

We can see the mixed surfaces (yellow surfaces) and lightlike locus (red curves) in Figure 3.
Figure 3. Mixed surfaces and lightlike locus.

By a direct calculation, $\sigma^\theta(u) = 1 \neq 0$, slant focal surfaces $\mathcal{FS}_\gamma^\theta$ are given by

$$
\mathcal{FS}_\gamma^\theta(u, m) = ((1 + m)(\sqrt{2}/2 \cos^2 \theta \sin \sqrt{2}u + 1/2 \sin 2\theta \cos \sqrt{2}u),
(1 + m)(\sqrt{2}/2 \cos^2 \theta \sin \sqrt{2}u - 1/2 \sin 2\theta \cos \sqrt{2}u),
\sqrt{2}/2 \cos^2 \theta - \sqrt{2}/2(1 + \sin^2 \theta)m)\). \tag{39}
$$

If $\theta = 0$, then

$$
\mathcal{FS}_\gamma^0 = ((1 + m)(\sqrt{2}/2 \sin \sqrt{2}u), (1 + m)(\sqrt{2}/2 \cos \sqrt{2}u), \sqrt{2}/2 - \sqrt{2}/2m).
$$

We can see that $\mathcal{FS}_\gamma^0$ is a tangent flat approximation of $f$ along $L(f)$ (cf. [1]). Then we have the $\mathcal{FS}_\gamma^0$ (green surface) in the following picture. Tangent plane (blue plane) of $\mathcal{FS}_\gamma^0(u, m)$ at $\hat{\gamma}(0)$ coincides with that of $f(u, v)$ at $\hat{\gamma}(0)$ in Figure 4.

Figure 4. $\mathcal{FS}_\gamma^0$ and tangent plane at $\hat{\gamma}$.

If $\theta = \pi/2$,

$$
\mathcal{FS}_\gamma^{\pi/2} = ((1 + m)(-1/2 \cos \sqrt{2}u), (1 + m)(1/2 \sin \sqrt{2}u), -\sqrt{2}m),
$$

then focal surface $\mathcal{FS}_\gamma^{\pi/2}$ is in Figure 5.
Since \( h^\theta(u) = \cos \theta(1 + \sin^2 \theta) \), when \( \theta \neq \pi/2 \), we have that slant evolutes are given by
\[
\mathcal{E}^\theta_\gamma(u) = (0, 0, \sqrt{2}).
\] (40)

Since \( (d/du)\mathcal{E}^\theta_\gamma(u) = 0 \) and \( a'_L(u) + a_L(u)a_G(u) = 0 \) for \( u \in I \), by Proposition 3, \( \tilde{\gamma} \) is on the lightcone \( \mathcal{F}_\gamma \mathcal{S}^0_\gamma \) whose vertex is \( \mathcal{E}^\theta_\gamma(u) = (0, 0, \sqrt{2}) \). The \( \mathcal{F}_\gamma \mathcal{S}^0_\gamma \) is the green surface in Figure 6.

Figure 5. focal surfaces \( \mathcal{F}_\gamma \mathcal{S}^\pi/2_\gamma \).

Figure 6. lightcone \( \mathcal{F}_\gamma \mathcal{S}^0_\gamma \) and vertex \( \mathcal{E}^\theta_\gamma \).

5. Conclusions

To investigate the geometry of lightlike locus on the mixed type surfaces in the Lorentz-Minkowski 3-space, we define slant focal surfaces and slant evolutes by using a moving frame field along the lightlike locus. By the criterions of singularities of front, the classification theorem of singularities of the slant focal surface is given. This theorem deeply depends on geometric properties of lightlike locus. Furthermore, we obtain the relationship between slant focal surfaces and slant evolutes from the viewpoint of singularity theory. When the slant evolute is a constant point under a certain condition, the lightlike locus is on the lightcone whose vertex is the slant evolute (cf. Example 1). Meanwhile, this lightcone is
the osculating lightlike surfaces. Finally, if the evolute of lightlike locus is a constant point, then the lightlike locus is on a pseudo sphere.

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