On some special solutions to periodic Benjamin-Ono equation with discrete Laplacian

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Abstract
We investigate a periodic version of the Benjamin-Ono (BO) equation associated with a discrete Laplacian. We find some special solutions to this equation, and calculate the values of the first two integrals of motion $I_1$ and $I_2$ corresponding to these solutions. It is found that there exists a strong resemblance between them and the spectra for the Macdonald $q$-difference operators. To better understand the connection between these classical and quantum integrable systems, we consider the special degenerate case corresponding to $q = 0$ in more detail. Namely, we give general solutions to this degenerate periodic BO, obtain explicit formulas representing all the integrals of motions $I_n$ ($n = 1, 2, \cdots$), and successfully identify it with the eigenvalues of Macdonald operators in the limit $q \to 0$, i.e. the limit where Macdonald polynomials tend to the Hall-Littlewood polynomials.

Key words: Benjamin-Ono equation, nonlocal integrable system, Macdonald polynomial, integrals of motion

1. Introduction

In [7], the authors studied a doubly periodic version of the intermediate long wave (ILW) equation associated with a discrete Laplacian. Then this ILW-type equation was identified with the first member of an integrable hierarchy which is given as a certain reduction of the discrete KP theory in the framework of the Sato theory.

In this paper, we study a spacial case having a single periodicity, instead of dealing with the most general doubly periodic case which requires much more
technique based on the algebro-geometric argument. Namely we consider the periodic Benjamin-Ono equation \[2, 5\] with discrete Laplacian
\[
\frac{\partial}{\partial t} \eta(x, t) = \eta(x, t) \int_{-1/2}^{1/2} \left[ \cot \left\{ \pi (y - x) \right\} \right. \\
- 2 \cot \left\{ \pi (y - x) \right\} + \cot \left\{ \pi (y - x + \gamma) \right\} \left. \right] \eta(y, t) \frac{i dy}{2}, \tag{1}
\]
where the symbol \(\int\) denotes the Cauchy principal value integral and \(\gamma\) is a complex constant with nonzero imaginary part. Our goal in this paper is to study some special solutions to (1) and the properties of the associated integrals of motion in some detail. We also investigate the special case \(\gamma \to i \infty\) in great detail since it allows us to have very much explicit results about solutions and integrals.

We claim that (1) can be regarded as a classical integrable system associated with the Macdonald theory of symmetric functions with two parameters \(q\) and \(t\) [4]. Recall that Macdonald introduced a set of commuting \(q\)-difference operators \(D_1, D_2, \cdots\) depending on \(q\) and \(t\), which are acting on the space of symmetric polynomials, say in \(x_i\)'s. Then he proved the fundamental existence theorem for the simultaneous eigenfunctions \(P_{\lambda}(x; q, t)\) of the \(D_i\)'s, together with a certain normalization condition. Here the index \(\lambda = (\lambda_1, \lambda_2, \cdots)\) denotes the partition.

For our purpose to compare (1) with the Macdonald theory, we better consider the infinitely many variable case and use the Heisenberg representation of the Macdonald operators developed in [AMOS], [S] and [FHHSY]. Introduce the Heisenberg algebra given by the generators \(\{a_n\}_{n \in \mathbb{Z}_{\neq 0}}\) and the commutation relations
\[
[a_m, a_n] = m \frac{1 - q^{|m|}}{1 - t^{|m|}} \delta_{m+n, 0}. \tag{2}
\]
Let \(\varepsilon\) be a constant. Set
\[
\eta(z) = \sum_{n \in \mathbb{Z}} \eta_n z^{-n} = \varepsilon \exp \left( \sum_{n>0} \frac{1 - t^{-n}}{n} a_{-n} z^n \right) \exp \left( - \sum_{n>0} \frac{1 - t^n}{n} a_n z^{-n} \right). \tag{3}
\]
It was shown in [S] and [FHHSY] that we have an infinite commutative family of operators acting on the Fock space containing \(\eta_0\) as the first nontrivial member.
One of the canonical ways to write down the generators of this commutative algebra is as follows. Set
\[
I_n = \oint \cdots \oint_C \frac{dz_1}{2\pi iz_1} \cdots \frac{dz_n}{2\pi iz_n} \prod_{1 \leq j < k \leq n} \frac{z_j - z_k}{z_j - qz_k} \cdot \eta(z_1) \cdots \eta(z_n),
\]
where the integration contour \(C\) is the torus \(|z_j| = 1, j = 1, \cdots, n\), and the symbol \(\cdot \cdots \cdot\) denotes the ordinary normal ordering of the Heisenberg generators, more explicitly we have
\[
: \eta(z_1) \cdots \eta(z_n) := \prod_{1 \leq j < k \leq n} \frac{(z_j - qz_k)(z_j - t^{-1}z_k)}{(z_j - z_k)(z_j - qt^{-1}z_k)} \cdot \eta(z_1) \cdots \eta(z_n).
\]
Then we have \([I_m, I_n] = 0\) for all \(m, n \geq 1\). Moreover we can write down explicitly the spectrum of \(I_n\) as
\[
q - n(1-t^{n-1})/2 \prod_{k=1}^{n} (1-q^k) \cdot e_n(\varepsilon t^{-\lambda_1}, \varepsilon qt^{-\lambda_2}, \varepsilon q^2t^{-\lambda_3}, \cdots)
\]
where \(e_n(x_1, x_2, \cdots)\) denotes the \(n\)-th elementary symmetric function, \(\lambda = (\lambda_1, \lambda_2, \cdots)\) is a partition.

For simplicity, we set \(\alpha_n = -(1-t^n)a_n/n\). With this notation, we have \(\eta(z) = \varepsilon : \exp \left( \sum_{n \neq 0} \alpha_n z^{-n} \right) :\). Now we proceed to considering a classical limit. Set \(t = e^\hbar\). While fixing \(q\), we consider the limit \(\hbar \to 0\), namely \(t \to 1\). In this limit we have \([\alpha_m, \alpha_n] = 0\) and \([\eta_m, \eta_n] = 0\). Hence we regard the algebra generated by \(\alpha_n\)'s or \(\eta_n\)'s being our algebra of classical observables.

Induce the poisson bracket by \(\{u, v\} := \lim_{\hbar \to 0} [u, v]/\hbar\) as usual. Then we have the set of canonical commutation relations \(\{\alpha_m, \alpha_n\} = \text{sgn}(m)(1-q^{-|m|})\delta_{m+n,0}\), which gives us \(\{\eta_m, \eta_n\} = \sum_{l \neq 0} \text{sgn}(l)(1-q^{-|l|})\eta_{m-l}\eta_{n+l}\).

Let \(\eta_0\) be our Hamiltonian. For example, the time evolution of \(\eta(z)\) is given by
\[
\frac{\partial}{\partial t} \eta(z) = \{\eta_0, \eta(z)\} = \eta(z) \sum_{n \neq 0} \text{sgn}(n)(1-q^{-|n|})\eta_{-n}z^n,
\]
which is just identical to \(\Pi\) under the identification of the variables \(z = \exp(2\pi i x)\) and \(q = \exp(2\pi i \gamma)\).

Set
\[
\tau_+(z) = \exp \left( -\sum_{n>0} \frac{1}{1-q^n} \alpha_{-n} z^n \right), \tau_-(z) = \exp \left( -\sum_{n>0} \frac{1}{1-q^n} \alpha_n z^{-n} \right).
\]

We have
\[ \eta(z) = \varepsilon \frac{\tau_-(q^{-1}z) \tau_+(qz)}{\tau_-(z) \tau_+(\dot{z})}, \quad (8) \]
\[ D_t \tau_-(z) \cdot \tau_+(z) = \varepsilon \tau_-(q^{-1}z) \tau_+(qz) = -\eta_0 \tau_-(z) \tau_+(z). \quad (9) \]

Here comes a question: how one can find some special solutions to (1) or (9), calculate integrals of motion, and find some connection with the theory of Macdonald symmetric polynomials? In this paper, we show some explicit connection between the (classical) integrals and the (quantum) eigenvalues, indicating that some deep structure is hidden in our problem.

This paper is organized as follows. In §2, we give some special solutions to (1). The first few \( I_n \)'s of them are calculated explicitly. It is shown that they have precisely the same formulas as the ones for the eigenvalues of the Macdonald operators. In §3, the special case \( q = 0 \) is treated in a complete manner. The general formula for \( I_n \)'s is given.

A remark is in order here. In the work by Avanov, Bettelheim and Wiegmann [1], a classical integrable system associated with the Calogero-Sutherland model is studied. It is an interesting future problem to connect their bilinear equation and solutions with ours.

2. Some special solutions to the periodic BO equation (1)

After a little algebra, one can construct some Laurent polynomial solutions to the bilinear equation (9). Recall that \( \varepsilon \) is a parameter in (9), and we will consider it as an arbitrary parameter. We have a class of solutions parametrized by one more parameter \( a \). One may easily find \( \tau_+ \) and \( \tau_- \) given by
\[
\begin{align*}
\tau_+ &= 1 + ze^{(1-q)a}, \\
\tau_- &= 1 + \frac{\varepsilon - a}{\varepsilon - qa} z^{-1} e^{-(1-q)a},
\end{align*}
\]
(10)
satisfy (9) under the condition that \( \varepsilon q + (1 - q)a = \eta_0 = I_1 \).

We can compute higher integrals of motion (11) corresponding to this special solution. For example, we have
\[
I_1 = \varepsilon q + (1 - q)a = (1 - q)e_1(a, \varepsilon q, \varepsilon q^2, \cdots),
\]
\[
I_2 = \varepsilon^2 q^2 + (1 - q^2)a = \frac{(1 - q)(1 - q^2)}{q} e_2(a, \varepsilon q, \varepsilon q^2, \cdots),
\]
(11)
and so on.

Next, we show another class of special solutions to (9) which has two parameters \(a_1\) and \(a_2\). Set

\[
\begin{align*}
\tau_+ &= 1 + \frac{z e^{(1-q)a_1 t} + ze^{(1-q)a_2 t}}{(a_1 - a_2)^2} + \frac{z^2 e^{(1-q)(a_1 + a_2) t}}{(a_1 - qa_2)(a_1 - q^{-1}a_2)}, \\
\tau_- &= 1 + c_1 e^{-(1-q)a_1 t} + c_2 e^{-(1-q)a_2 t} + \frac{(a_1 - a_2)^2}{(a_1 - qa_2)(a_1 - q^{-1}a_2)} c_1 c_2 e^{-(1-q)(a_1 + a_2) t}, \\
c_j &= \frac{\varepsilon - qa_j}{\varepsilon - q^2 a_j} \frac{(a_1 - qa_2)(a_1 - q^{-1}a_2)}{(a_1 - a_2)^2}.
\end{align*}
\]

Then (9) holds under the condition that \(\varepsilon q^2 + (1 - q)a_1 + (1 - q)a_2 = \eta_0 = I_1\). One can calculate the higher integrals of motion (14). The first two read

\[
\begin{align*}
I_1 &= \varepsilon q^2 + (1 - q)a_1 + (1 - q)a_2 = (1 - q)e_1(a_1, a_2, \varepsilon q^2, \varepsilon q^3, \cdots), \\
I_2 &= \frac{(1 - q)(1 - q^2)}{q} e_2(a_1, a_2, \varepsilon q^2, \varepsilon q^3, \cdots),
\end{align*}
\]

and so on.

It is clearly seen from these examples that there is a strong resemblance between such formulas for the integrals of motion and the eigenvalues of the Macdonald operators \(q^{-n(n-1)/2} \prod_{k=1}^n (1 - q^k) \cdot e_n(\varepsilon t^{-\lambda_1}, \varepsilon q t^{-\lambda_2}, \varepsilon q^2 t^{-\lambda_3}, \cdots)\). Namely, if we take the limit \(t \to 1, \lambda_i \to \infty\) in such a way that we have the finite limits \(\varepsilon q^{i-1} t^{-\lambda_i} = a_i\), then we recover the above formulas for the integrals of motion. At present the reason of this beautiful correspondence has not been investigated.

3. the spacial case \(q = 0\)

At present, unfortunately, our study on the special solutions to (9) and whose integrals of motion still remains primitive and heuristic. In the special limit \(q = 0\), however, we can easily complete our program within the technique of linear algebra as we will show below.
Before we embark on the study of the bilinear equation, let us try to solve naively (6) in the limit \( q = 0 \), namely

\[
\frac{\partial}{\partial t} \eta(z) = \eta(z) \sum_{n \neq 0} \text{sgn}(n) \eta_{-n} z^n. \tag{15}
\]

Set \( \eta_{\pm}(z) := \sum_{n>0} \eta_{\pm n} z^{\mp n} \). Then we have the split equations

\[
\frac{d}{dt} \eta_{+}(z) = \eta_{+}(z)(\eta_{+}(z) + \eta_0), \quad \frac{d}{dt} \eta_{-}(z) = -\eta_{-}(z)(\eta_{-}(z) + \eta_0). \tag{16}
\]

One immediately finds the general solution,

\[
\eta_{+}(z) = \frac{-\eta_0 c_{-}(z) e^{\eta_0 t}}{d_{+}(z) + c_{+}(z) e^{\eta_0 t},} \quad \eta_{-}(z) = \frac{-\eta_0 c_{-}(z) e^{-\eta_0 t}}{d_{-}(z) + c_{-}(z) e^{-\eta_0 t}} \tag{17}
\]

where \( d_{\pm}(z), c_{\pm}(z) \) are arbitrary holomorphic functions in some neighborhoods of \( z^{\pm 1} = 0 \).

One may ask if \( d_{\pm}(z), c_{\pm}(z) \) can be restricted to Laurent polynomials in \( z^{\pm 1} \), and the denominators \( d_{\pm}(z) + c_{\pm}(z) \exp(\pm \eta_0 t) \) play the role of the tau functions \( \tau_{\pm}(z) \). The answer indeed is yes.

In the case \( q = 0 \), we have

\[
\eta(z) = \varepsilon \left( \frac{1}{\tau_{-}(z)\tau_{+}(z)} \right), \tag{18}
\]

\[
D_t \tau_{-}(z) \cdot \tau_{+}(z) = \varepsilon - \eta_0 \tau_{-}(z) \tau_{+}(z). \tag{19}
\]

Setting \( \tau_{\pm}(z) = d_{\pm}(z) + c_{\pm}(z) \exp(\pm \eta_0 t) \), one finds that (19) reduces to the functional equation

\[
d_{+}(z)d_{-}(z) - c_{+}(z)c_{-}(z) = \varepsilon/\eta_0. \tag{20}
\]

**Proposition.** Let \( n \) be a nonnegative integer, and let \( \epsilon_{\pm 1}, \ldots, \epsilon_{\pm n} \) be \( 2n \) parameters. Let \( e_m(z, t) = e_m z^m \exp\{\text{sgn}(m) \eta_0 t\} \) for \( m \in \mathbb{Z}_{\neq 0} \), and define \( \tau_{\pm}(z, t; \epsilon_{\pm 1}, \ldots, \epsilon_{\pm n}) = \tau_{\pm}(z) \) by

\[
\tau_{+}(z) = 1 + \sum_{1 \leq i \leq n} e_i + \sum_{1 \leq i_1 < i_2 \leq n} e_{i_2} e_{-i_1} + \sum_{1 \leq i_1 < i_2 < i_3 \leq n} e_{i_3} e_{-i_2} e_{i_1} + \cdots + e_n e_{-n+1} e_{-n+2} \cdots e_{-(1)^n}, \tag{21}
\]

\[
\tau_{-}(z) = 1 + \sum_{1 \leq i \leq n} e_{-i} + \sum_{1 \leq i_1 < i_2 \leq n} e_{-i_2} e_{i_1} + \sum_{1 \leq i_1 < i_2 < i_3 \leq n} e_{-i_3} e_{i_2} e_{-i_1} + \cdots + e_{-n} e_{-n-1} e_{-n-2} \cdots e_{-(1)^n}. \tag{22}
\]
Then these satisfy (19) under the condition that \( \prod_{i=1}^{n}(1 - \epsilon_i \epsilon_{-i}) = \varepsilon/\eta_0 \).

**Proof.** We have

\[
d_+ (z) = 1 + \sum_{1 \leq i_1 < i_2 \leq n} \epsilon_{i_2} \epsilon_{-i_1} z^{i_2 - i_1} + \cdots, \tag{23}
\]

\[
c_+ (z) = \sum_{1 \leq i \leq n} \epsilon_i z^i + \sum_{1 \leq i_1 < i_2 < i_3 \leq n} \epsilon_{i_3} \epsilon_{-i_2} \epsilon_{i_1} z^{i_3 - i_2 + i_1} + \cdots, \tag{24}
\]

\[
d_- (z) = 1 + \sum_{1 \leq i_1 < i_2 \leq n} \epsilon_{-i_2} \epsilon_{i_1} z^{-i_2 + i_1} + \cdots, \tag{25}
\]

\[
c_- (z) = \sum_{1 \leq i \leq n} \epsilon_{-i} z^{-i} + \sum_{1 \leq i_1 < i_2 < i_3 \leq n} \epsilon_{-i_3} \epsilon_{-i_2} \epsilon_{i_1} z^{-i_3 + i_2 - i_1} + \cdots, \tag{26}
\]

and need to show that

\[
d_+ (z) d_- (z) - c_+ (z) c_- (z) = \prod_{i=1}^{n} (1 - \epsilon_i \epsilon_{-i}). \tag{27}
\]

A way to write \( d_\pm \) and \( c_\pm \) is

\[
\begin{pmatrix}
d_+ (z) \\
c_- (z)
\end{pmatrix}
\begin{pmatrix}
c_+ (z) \\
d_- (z)
\end{pmatrix} =
\begin{pmatrix}
1 & \epsilon_n z^n \\
\epsilon_{-n} z^{-n} & 1
\end{pmatrix} \cdots
\begin{pmatrix}
1 & \epsilon_{1} z \\
\epsilon_{-1} z^{-1} & 1
\end{pmatrix} . \tag{28}
\]

The determinant of the above gives the relation (27). \( \square \)

**Remark.** For \( n \geq 0 \), and \( \epsilon_{\pm 1}, \ldots, \epsilon_{\pm n} \), we have constructed a solution to (19) and have denoted them by \( \tau_{\pm} (z, t; \epsilon_{\pm 1}, \ldots, \epsilon_{\pm n}) \). It is desirable, however, for our later purpose to have more flexible notation in which we can treat \( \tau_{\pm} \) with different numbers of parameters within a unified notation. It is clear that from infinite sequences \( (\epsilon_{\pm 1}, \epsilon_{\pm 2}, \cdots) \) with only finitely many nonzero parts, say \( \epsilon_{\pm i} = 0 \) for \( i > n \), we can construct the tau functions as above. We will denote them by \( \tau_{\pm} (z, t; \epsilon_{\pm 1}, \epsilon_{\pm 2}, \cdots) \) without the need of referring to the ‘\( n \)’.

When \( q = 0 \), the integral of motion (4) becomes the Toeplitz determinant

\[
I_n = \begin{vmatrix}
\eta_0 & \eta_{-1} & \cdots & \eta_{-n+1} \\
\eta_1 & \eta_0 & \cdots & \eta_{-n+2} \\
\vdots & \vdots & \ddots & \vdots \\
\eta_{n-1} & \eta_{n-2} & \cdots & \eta_0
\end{vmatrix}. \tag{29}
\]
**Theorem.** Let $\eta_0$ be satisfying the condition $\prod_{i=1}^{\infty} (1 - \epsilon_1 \epsilon_{-i}) = \epsilon / \eta_0$, and let $\tau_+ (z)$ be defined by (21) and (22). Hence (19) is satisfied. Then define $\eta_n$ by (18) or (17). Then we have

$$I_{k+1} = \eta_0^{k+1} (1 - \epsilon_1 \epsilon_{-1})^k (1 - \epsilon_2 \epsilon_{-2})^{k-1} \cdots (1 - \epsilon_k \epsilon_{-k}),$$

for $k \geq 1$.

**Proof.** We know that $I_n$'s are $t$ independent. Thus we may set $t = 0$ in what follows in order to make our computation simple. Set

$$\eta_{\pm}(z, 0) = \frac{-\eta_0 c_{\pm}(z)}{d_{\pm}(z) + c_{\pm}(z)} =: -\eta_0 \sum_{n>0} \nu_{\pm} z^{\pm n},$$

for notational simplicity. Then, our claim in Theorem is

$$|-1 \nu_{-1} \cdots \nu_{-k} \nu_1 -1 \cdots \nu_{-k+1} \vdots \vdots \vdots v_k \nu_{k-1} \cdots -1 | = (-1)^{k+1} \prod_{l=1}^{k} (1 - \epsilon_l \epsilon_{-l})^{k-l+1}. \tag{32}$$

This can be proved by Lemma stated below as

$$I_{k+1} = \left( \begin{array}{cccc} -1 & \nu_{-1} & \cdots & \nu_{-k} \\ \nu_1 & -1 & \cdots & \nu_{-k+1} \\ \vdots & \vdots & \ddots & \vdots \\ \nu_k & \nu_{k-1} & \cdots & -1 \end{array} \right) \left( \begin{array}{cccc} 1 & 0 & \cdots & 0 \\ \tau_{1}^{(1)} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \tau_{k}^{(1)} & 0 & \cdots & 1 \end{array} \right) \left( \begin{array}{cccc} 1 & 0 & \cdots & 0 \\ \tau_{1}^{(k)} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \tau_{k}^{(k)} & 0 & \cdots & 1 \end{array} \right)$$

$$= \left( \begin{array}{cccc} -1 & \epsilon_1 \epsilon_{-1} & \cdots & \epsilon_k \epsilon_{-k} \\ 0 & -1 & \cdots & \nu_{-k+1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \nu_{k-1} & \cdots & -1 \end{array} \right)$$

$$= -(1 - \epsilon_1 \epsilon_{-1}) \cdots (1 - \epsilon_k \epsilon_{-k}) I_k.$$

**Lemma.** For $N \geq 1$, we have

$$\left( \begin{array}{cccc} -1 & \nu_{-1} & \cdots & \nu_{-N} \\ \nu_1 & -1 & \cdots & \nu_{-(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \nu_N & \nu_{N-1} & \cdots & -1 \end{array} \right) \left( \begin{array}{c} 1 \\ \tau_{1}^{(N)} \\ \vdots \\ \tau_{N}^{(N)} \end{array} \right) = \left( \begin{array}{c} -(1 - \epsilon_1 \epsilon_{-1}) \cdots (1 - \epsilon_N \epsilon_{-N}) \\ 0 \\ \vdots \\ 0 \end{array} \right).$$

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Here, \( \tau^{(m)}_{\pm j} \) is the coefficient of \( z^{\pm j} \) in \( \tau_{\pm}(z, 0; \epsilon_{\pm 1}, \epsilon_{\pm 2}, \cdots, \epsilon_{\pm m}, 0, 0, \cdots) \), namely
\[
\tau_{\pm}(z, 0; \epsilon_{\pm 1}, \epsilon_{\pm 2}, \cdots, \epsilon_{\pm m}, 0, 0, \cdots) = \sum_{j=1}^{m} \tau^{(m)}_{\pm j} z^{\pm j}. \tag{33}
\]

**Sketch of the proof of Lemma.** We have the following three properties, which can be proved by induction on \( N \).

1. \( \nu_N \) is a polynomial depending only on \( \epsilon_{\pm 1}, \epsilon_{\pm 2}, \cdots, \epsilon_{\pm (N-1)} \) and \( \epsilon_N \), and \( \nu_{-N} \) is a polynomial depending only on \( \epsilon_{\pm 1}, \epsilon_{\pm 2}, \cdots, \epsilon_{\pm (N-1)} \) and \( \epsilon_{-N} \).

2. \[
\frac{\partial \nu_{\pm N}}{\partial \epsilon_{\pm N}} = (1 - \epsilon_1 \epsilon_{-1})(1 - \epsilon_2 \epsilon_{-2}) \cdots (1 - \epsilon_{N-1} \epsilon_{-N+1}).
\]

3. We have
\[
\begin{pmatrix}
\tau^{(N)}_{-N} & 0 & \cdots & 0 \\
\tau^{(N)}_{-N+1} & \tau^{(N)}_{-N} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\tau^{(N)}_{-1} & \tau^{(N)}_{-2} & \cdots & \tau^{(N)}_{-N}
\end{pmatrix}
\begin{pmatrix}
\nu_1 \\
\nu_2 \\
\vdots \\
\nu_N
\end{pmatrix}
= \begin{pmatrix}
d^{(N)}_{-N+1} \\
d^{(N)}_{-N+2} \\
\vdots \\
d^{(N)}_0
\end{pmatrix} - \begin{pmatrix}
0 \\
0 \\
\vdots \\
(1 - \epsilon_1 \epsilon_{-1}) \cdots (1 - \epsilon_N \epsilon_{-N})
\end{pmatrix},
\]

where, \( d^{(m)}_{\pm j} \) is defined by
\[
d_{\pm}(z, 0; \epsilon_{\pm 1}, \cdots, \epsilon_{\pm m}, 0, 0, \cdots) =: \sum_{j=1}^{m} d^{(m)}_{\pm j} z^{\pm j}.
\]

\(\square\)

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References

[1] A. G. Abanov, E. Bettelheim, P. Wiegmann Integrable hydrodynamics of Calogero-Sutherland model: Bidirectional Benjamin-Ono equation J.Phys.A 42 (2009) 135201.

[2] T.B. Benjamin, Internal waves of permanent form in fluids of great depth, J. Fluid. Mech. 29 (1967), 559-592.

[3] B. Feigin, K. Hashizume, A. Hoshino, J. Shiraishi, S. Yanagida, A commutative algebra on degenerate $\mathbb{CP}^1$ and Macdonald polynomials, J. Math. Phys. 50 (2009) 095215.

[4] I.G. Macdonald, Symmetric Functions and Hall Polynomials, second ed., Oxford University Press, New York, 1995.

[5] H. Ono, Algebraic solitary waves in stratified fluids, J. Phys. Soc. Japan, 30 (1975), 1082-1091.

[6] J. Shiraishi, A Family of Integral Transformations and Basic Hypergeometric Series, Commun. Math. Phys. 263 (2006) 439-460.

[7] J. Shiraishi, Y.Tutiya Periodic ILW equation with discrete Laplacian, J. Phys. A. 42 No.40 (2009) 404018.