ALGEBRAIC CYCLES REPRESENTING COHOMOLOGY OPERATIONS

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Abstract. In this paper we will show that certain universal homology classes which are fundamental in topology are algebraic. To be specific, the products of Eilenberg-MacLane spaces $K_{2q} \cong K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 4) \times \ldots \times K(\mathbb{Z}, 2q)$ have models which are limits of complex projective varieties. Precisely, we have $K_{2q} = \lim_{d \to \infty} \mathcal{C}^q_d(P^n)$ where $\mathcal{C}^q_d(P^n)$ denotes the Chow variety of effective cycles of codimension $q$ and degree $d$ on $P^n$. It is natural to ask which elements in the homology of $K_{2q}$ are represented by algebraic cycles in these approximations. In this paper we find such representations for the even dimensional classes which are known as Steenrod squares (as well as their Pontrjagin and join products). These classes are dual to the cohomology classes which correspond to the basic cohomology operations also known as the Steenrod squares.

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§0. Introduction

In this paper we will show that certain universal homology classes which are fundamental in topology are algebraic. To be specific, the products of Eilenberg-MacLane spaces \( \mathcal{K}_{2q} \equiv K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 4) \times \cdots \times K(\mathbb{Z}, 2q) \) have models which are limits of complex projective varieties. In the simplest case we can write \( \mathcal{K}_{2q} = \lim_{d \to \infty} SP^d(\mathbb{P}^n_C) \) where \( SP^d \) denotes the \( d \)-fold symmetric product (cf. [DT1], [DT2]). Here the projections \( SP(\mathbb{P}^n_C) \to SP(\mathbb{P}^m_C) \) were given explicitly in [FL]. More generally, \( \mathcal{K}_{2q} = \lim_{d \to \infty} C^q_d(\mathbb{P}^n) \) where \( C^q_d(\mathbb{P}^n) \) denotes the Chow variety of effective cycles of codimension \( q \) and degree \( d \) on \( \mathbb{P}^n_C \) [L1], [L2]. It is natural to ask which elements in the homology of \( \mathcal{K}_{2q} \) are represented by algebraic cycles in these approximations. In this paper we find such representations for the even dimensional classes which correspond to the basic cohomology operations known as the Steenrod squares (as well as their Pontrjagin and join products).

More precisely, the Steenrod squares, \( Sq^i \), are stable cohomology operations which go from \( H^m(X; \mathbb{Z}_2) \) to \( H^{m+i}(X; \mathbb{Z}_2) \). They are stable in the sense that they commute with suspension. There are canonical dual homology operations, cf. [H-Mc], denoted \( \overline{Sq}_i \) defined by

\[
\langle \overline{Sq}_i \alpha, \beta \rangle = \langle \alpha, Sq^i \beta \rangle
\]

where \( \langle \ , \ \rangle \) is the non-degenerate Kronecker pairing.

Precomposing with reduction mod 2, denoted \( \rho \), we have \( Sq^i \equiv Sq^i \rho \) which are also called Steenrod squares and are stable cohomology operations which go from \( H^m(X; \mathbb{Z}) \) to \( H^{m+i}(X; \mathbb{Z}_2) \). Under the general isomorphism \( H^m(X; G) \cong [X, K(G, m)] \) these operations correspond to classes in \( [K(\mathbb{Z}, m), K(\mathbb{Z}_2, m + i)] \) acting by composition and therefore, in turn, correspond to stable cohomology classes in \( H^{m+i}(K(\mathbb{Z}, m); \mathbb{Z}_2) \).
They are denoted \( Sq^i \iota_m \) or \( Sq^i \rho \iota_m \) where \( \iota_m \) is the fundamental class of \( K(\mathbb{Z}, m) \).

The basic homology classes which we show to be algebraic are two-torsion classes, \( Sq_{2k,2n} \in H_{2k+2n}\left(K(\mathbb{Z}, 2n); \mathbb{Z}\right) \), which after reduction mod 2 are sent by \( \overline{Sq}_{2k} \) to the fundamental class of \( K(\mathbb{Z}, 2n) \) reduced mod 2. The classes \( Sq_{2k,2n} \) are homology counterparts to the stable cohomology classes \( Sq^{2k} \iota_{2n} = Sq^{2k} \rho \iota_{2n} \in H_{2k+2n}\left(K(\mathbb{Z}, 2n); \mathbb{Z}_2\right) \).

These classes \( Sq_{2k,2n} \in H_{2k+2n}\left(K(\mathbb{Z}, 2n); \mathbb{Z}\right) \) are exactly the classes of Cartan which correspond to the admissible sequences \((0, 2k)\) in \([C_1]\), and also to the admissible sequences \((2k)\) of \([C_2]\).

From these basic classes we can show that very much of the even homology of \( K_{2q} \) is algebraic. There are two natural pairings on the varieties \( C^q_d(\mathbb{P}^n) \) which approximate \( K_{2q} \): The addition map

\[
C^q_d(\mathbb{P}^n) \times C^q_d(\mathbb{P}^n) \rightarrow C^q_{d+d'}(\mathbb{P}^n)
\]

and the biadditive \textit{join} pairing

\[
C^q_d(\mathbb{P}^n) \times C^q_{d'}(\mathbb{P}^{n'}) \rightarrow C^q_{d+d'}(\mathbb{P}^{n+n'+1})
\]

Both are algebraic maps (morphisms of varieties). Thus, beginning with the basic classes one can show that a great deal of the even-degree homology of \( K_{2q} \) is algebraic.

We will attempt to make this paper accessible both to the algebraic topologist and to the algebraic geometer and in so doing, make it also accessible, for example, to the differential geometer who may have become intrigued by related work on the Chern map \([LM_1], [BLM], [LLM_1] \).

The results in this paper suggest the possibility of being able to explicitly construct Steenrod operations in motivic theory or in morphic
cohomology [FL]. However, such constructions do not follow immediately and will form the subject of future research.

§1. Cycle space basics

We begin by a quick recap of some basic facts about the Chow spaces $C^q(P^n)$. For further details see [LM1], [L2]. Let $\pi : C^{n+1} - \{0\} \to P^n$ be the projection onto complex projective $n$-space. Recall that an algebraic set is a subset $V$ of $P^n$ such that $\pi^{-1}(V) \cup \{0\}$ is the set of zeroes of a finite set of homogeneous polynomials. An algebraic variety is an irreducible algebraic set, i.e., if $V = V_1 \cup V_2$, with $V_1, V_2$ algebraic, then $V_1 \subset V_2$ or $V_2 \subset V_1$. An effective algebraic cycle of dim $p$ is a formal finite sum $c = \Sigma n_i V_i$ where the $V_i$’s are algebraic varieties of dimension $p$ and the $n_i$’s are positive integers. The degree of the cycle $c$ is $\deg c = \Sigma n_i \deg V_i$ where $\deg V_i$ is the homological degree of $V_i$. The space of effective algebraic cycles in $P^n$ of codimension $q$ (i.e., of dimension $n - q$) and degree $d$ is itself an algebraic variety, called a Chow variety, which we will denote $C^q_d(P^n)$. Now we can embed $C^q_d(P^n)$ into $C^q_{d+1}(P^n)$ by choosing a fixed codimension $q$ plane $L_0$ in $P^n$ and mapping a cycle $c$ in $C^q_d(P^n)$ to $c + L_0$ in $C^q_{d+1}(P^n)$. Then by passing to the direct limit we have

$$C^q(P^n) \equiv \lim_{\to} C^q_d(P^n)$$

We now consider the complex join of a variety $V$ in $P^n$ and a variety $W$ in $P^m$: we place $P^n$ and $P^m$ in general position in $P^{n+m+1}$ and define the complex join of $V$ and $W$, denoted $V \sharp W$, to be the point-set union of all complex lines from points in $V$ to points in $W$. cf. [H]. Now, for $P^{n-1} \subset P^n$ we have $P^n = P^{n-1} \sharp p_n$ for a point $p_n$ in $P^n - P^{n-1}$.
§2. The integral homology of $SP^2(S^{2n})$

We denote by $SP^k(X)$ the $k$-fold symmetric product of $X$, namely, the space of unordered $k$-tuples of points in $X$. Then by choosing a fixed base point, $x_0$, in $X$ and mapping a $k$-tuple, $\{x_1, \ldots, x_k\}$ to $\{x_1, \ldots, x_k, x_0\}$ and passing to the direct limit we have the infinite symmetric product of $X$, which we denote by $SP(X)$. Note that $C^n(P^n) = SP(P^n)$.

We know from Dold-Thom [DT$_2$] that there is a homotopy equivalence
\[ SP(S^{2n}) \cong K(\mathbb{Z}, 2n). \] (2.0)
We are, therefore, interested in the homology of $SP^2(S^{2n})$. In fact, it will be fundamental for us that the inclusion $SP^k(S^{2n}) \subset SP(S^{2n})$ induces an injection
\[ H_*(SP^k(S^{2n}); \mathbb{Z}) \rightarrow H_*(K(\mathbb{Z}, 2n); \mathbb{Z}) \]
for all $k$.

To understand the classes of interest we shall decompose $SP^2(S^{2n})$ as follows. Set
\[ \Delta = \{ \{x, x\} \in SP^2(S^{2n}) : x \in S^{2n} \} = S^{2n} \]
and
\[ \tilde{\Delta} = \{ \{x, -x\} \in SP^2(S^{2n}) : x \in S^{2n} \} = \mathbb{P}^{2n}_{\mathbb{R}} \]
and let $U$ be an $\epsilon$-tubular neighborhood of $\Delta$ in $SP^2(S^{2n})$ in the natural metric, for small $\epsilon$. Set
\[ V = SP^2(S^{2n}) - \Delta. \]

LEMMA 2.1. There are homotopy equivalences
\[ U \cong S^{2n}, \quad V \cong \mathbb{P}^{2n}_{\mathbb{R}}, \quad U \cap V \cong PTS^{2n} \]
where $PTS^{2n}$ denotes the projectivization of the tangent bundle of $S^{2n}$. 
Proof. Note that $S^{2n} = \Delta \subset U$ is a deformation retract; hence, $S^{2n} \cong U$. We claim that $P^{2n}_R = \tilde{\Delta} \subset V$ is also a deformation retract. To see this consider an element $\{x, y\} \in V$. Since $x \neq y$, we see that neither $\{x, -y\}$ nor $\{-x, y\}$ is an antipodal pair on $S^{2n}$. Hence each pair is joined by a unique shortest geodesic arc. We move $x$ toward $-y$ and $y$ toward $-x$ uniformly in time so that when $t = 1/2$ they have each gone half the distance. Specifically, we set
\[ x_t = \frac{(1-t)x - ty}{||(1-t)x - ty||}, \quad y_t = \frac{(1-t)y - tx}{||(1-t)y - tx||} \]
for $0 \leq t \leq 1/2$. Then $\rho_t : V \rightarrow V$ given by $\rho_t(\{x, y\}) = \{x_t, y_t\}$ for $0 \leq t \leq 1/2$ is a smooth homotopy with $\rho_0 = \text{Id}$ and
\[ \rho_{1/2}(\{x, y\}) = \left\{ \frac{x - y}{||x - y||}, \frac{y - x}{||x - y||} \right\} \in \tilde{\Delta} \]
for all $\{x, y\}$. This proves $P^{2n}_R = \tilde{\Delta} \cong V$.

For the last assertion we note that $U \cap V \cong \partial U = \partial \hat{U}/\mathbb{Z}_2$ where $\partial \hat{U}$ is the boundary of a tubular neighborhood, $\hat{U}$, of the diagonal in $S^{2n} \times S^{2n}$. The normal bundle, $N$, to $\Delta \subset S^{2n} \times S^{2n}$ is equivalent to the tangent bundle of $S^{2n}$, and the action induced by $\varphi(x, y) = (y, x)$ on $N$ is just the bundle map $v \mapsto -v$. Since $\hat{U}$ is equivariantly equivalent to $N$, we have $U \cap V \cong \partial U \cong P^T S^{2n}$ as claimed. \hfill \Box

Applying Mayer-Vietoris to the pair $(U, V)$ gives a long exact sequence
\[ \cdots \rightarrow H_k(\partial U) \rightarrow H_k(S^{2n}) \oplus H_k(P^{2n}_R) \rightarrow H_k(SP^2(S^{2n})) \rightarrow H_{k-1}(\partial U) \rightarrow \cdots \]
from which one concludes that
\[ \partial : H_{k+1}(SP^2(S^{2n}); \mathbb{Z}) \cong H_k(\partial U; \mathbb{Z}) \text{ for all } k > 2n. \quad (2.2) \]

We now use the Serre spectral homology sequence for the fibration
\[ P^{2n-1}_R \rightarrow \partial U \rightarrow S^{2n} \quad (2.3) \]
which has $E^2$-term $E^2 = H_\ast(S^{2n}; H_\ast(\mathbf{P}^{2n-1}_R))$ to conclude that

$$H_k(\partial U; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \text{ or } 4n - 1 \\ \mathbb{Z}_2 & \text{if } k = 1, 3, ..., 4n - 3 \text{ but } k \neq 2n - 1 \\ \mathbb{Z}_4 & \text{if } k = 2n - 1 \\ 0 & \text{otherwise} \end{cases}$$

$$H_k(SP^2(S^{2n}); \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0, 2n \text{ or } 4n \\ \mathbb{Z}_2 & \text{if } k = 2n + 2, 2n + 4, ..., 4n - 2 \\ 0 & \text{otherwise} \end{cases}$$  \hspace{1cm} (2.4)

(Note that the $E^2$-term is lacunary and there is only one differential to compute. See [C2].)

The classes which are of interest to us here are the generators of the torsion classes in $H_\ast(SP^2(S^{2n}); \mathbb{Z})$. Let

$$\text{Sq}_{2k, 2n} \in H_{2(n+k)}(SP^2(S^{2n}); \mathbb{Z})$$  \hspace{1cm} (2.5)

denote the non-zero element for $0 < k < n$.

§ 3. Cycles representing $\text{Sq}_{2k, 2n}$

In this section we shall present explicit real algebraic cycles which represent the 2-torsion classes (2.5). For this we write $S^{2n}$ as

$$S^{2n} = C^n \cup \{\infty\}$$  \hspace{1cm} (3.1)

where the embedding $C^n \subset S^{2n}$ is given by the inverse of stereographic projection. This gives embeddings

$$C^n \times C^n \subset S^{2n} \times S^{2n}$$  \hspace{1cm} (3.2)

and

$$SP^2(C^n) \subset SP^2(S^{2n})$$  \hspace{1cm} (3.3)

Let $(z, \zeta) = (z_1, ..., z_n; \zeta_1, ..., \zeta_n)$ be coordinates for $C^n \times C^n$, so that the unordered pairs $\{z, \zeta\}$ parametrize $SP^2(C^n)$. For each $k$, $0 < k < n,$
we define cycles $S_{2k,2n}$ by setting

$$
S_{2k,2n} \equiv \text{the closure in } SP^2(S^{2n}) \text{ of }
\{ \{z, \zeta\} \in SP^2(C^n) : z_{k+1} = \zeta_{k+1}, z_{k+2} = \zeta_{k+2}, \ldots, z_n = \zeta_n \}
$$

To see that this is real algebraic we write

$$
S^{2n} = \{(w, t) \in C^n \times \mathbb{R} : |w|^2 + t^2 = 1\}
$$

and include $C^n \hookrightarrow S^{2n}$ by $z \mapsto \left(\frac{2z}{1+|z|^2}, 1 - \frac{2}{1+|z|^2}\right)$ that is, by stereographic projection. Now the cycle $S_{2k,2n}$ is the quotient by $\mathbb{Z}_2$ of

$$
\{( (w, t), (w', t) ) \in S^{2n} \times S^{2n} : w_i = w'_i, i = k+1, \ldots, n \}
$$

and is clearly a real algebraic subset of $SP^2(S^{2n})$ of dimension $2(n + k)$. It is regular and canonically oriented (by the complex structure) outside the diagonal $\Delta$. Thereby $S_{2k,2n}$ becomes an integral cycle in our space (as a rectifiable current [Fed] or via triangulation of the pair $(SP^2(S^{2n}), S_{2k,2n})$).

**Theorem 3.5.** The homology class of $S_{2k,2n}$ is not zero. Hence, $S_{2k,2n}$ represents the 2-torsion class $S_{2k,2n} \in H_2(S^{n+k})(SP^2(S^{2n}); \mathbb{Z})$ defined in §2.

**Proof.** It will suffice to prove that $S_{2k,2n}$ is not homologous to zero as a cycle modulo 2. To do this we produce a mod-2 cycle $T$ of dimension $2(n - k)$ which meets $S_{2k,2n}$ transversely in exactly one regular point. The cycle $T$ is defined as follows. Note that the "normal" to $\Delta$ in $SP^2(S^{2n})$ is the cone on $P_{\mathbb{R}}^{2n-1}$ (since we are dividing by the flip along the diagonal). We shall take $T$ to be a linear subspace $P_{\mathbb{R}}^{2n-k} \subset P_{\mathbb{R}}^{2n-1}$.

Specifically, let $\epsilon > 0$ be fixed. Set

$$
SN_\epsilon = \{(z, -z) \in C^n \times C^n : ||z|| = \epsilon \} \approx S^{2n-1},
$$

and let

$$
P_N_\epsilon = SN_\epsilon / \mathbb{Z}_2
$$

$$
= \{(z, -z) \in C^n \times C^n : ||z|| = \epsilon \} / \mathbb{Z}_2 \approx P_{\mathbb{R}}^{2n-1},
$$
Note that $Sq_{2k,2n}$ meets $PN_\epsilon$ transversally and that
\[
Sq_{2k,2n} \cap PN_\epsilon = \{ (z, -z) \in PN_\epsilon : z_{k+1} = \ldots = z_n = 0 \}
\cong \mathbb{P}^{2k-1}_R \tag{3.6}
\]
We now choose $\mathbb{P}^{2(n-k)}_R = T \subset PN_\epsilon = \mathbb{P}^{2n-1}_R$ to be a projective linear subspace of dimension $2(n-k)$ which meets the projective linear subspace (3.6) in one point. (Almost any choice will do.) Then $T$ is a mod-2 cycle of dimension $2(n-k)$ which is contained in the regular locus of $SP^2(S^{2n})$ and meets $Sq_{2k,2n}$ transversely in one point as claimed.

Taking the intersection number with $T$ mod 2 defines a $\mathbb{Z}_2$-valued cocycle on $SP^2(S^{2n})$ (cf [Sc]. We have just seen that this cocycle is non-zero on $Sq_{2k,2n}$. Hence $r[Sq_{2k,2n}] \neq 0$ where $r : H_{2(n+k)}(SP^2(S^{2n}); \mathbb{Z}) \rightarrow H_{2(n+k)}(SP^2(S^{2n}); \mathbb{Z}_2)$ is reduction mod 2, and so $[Sq_{2k,2n}] \neq 0$ as claimed. □

Note 3.7 An alternative argument for Theorem 3.5 can be given along the following lines. Consider the manifold with boundary $V \equiv SP^2(S^{2n}) - \Delta_\epsilon$, where $\Delta_\epsilon$ is the $\epsilon$-tubular neighborhood of the diagonal, $\Delta$, in $SP^2(S^{2n})$. Proper intersection defines the non-degenerate Lefschetz duality pairing
\[
H_\ell(V, \partial V) \otimes H_{4n-\ell}(V) \rightarrow \mathbb{Z}_2,
\]
and so $[Sq_{2k,2n}] \neq 0$ in $H_\ell(V, \partial V; \mathbb{Z}_2)$. We “cone off” $\partial(Sq_{2k,2n} \cap V)$ down to $\Delta$ to give a cycle in $SP^2(S^{2n})$ which is not zero. This can be read directly out of the Mayer-Vietoris sequence.

We recall from [St] that the natural inclusion $j : SP^2(S^{2n}) \hookrightarrow SP(S^{2n}) \cong K(\mathbb{Z}, 2n)$ induces an injection
\[
j_* : H_*(SP^2(S^{2n}); \mathbb{Z}) \hookrightarrow H_*(K(\mathbb{Z}, 2n); \mathbb{Z}).
\]
By the computations of H. Cartan [C1], [C2] there is exactly one 2-torsion class in $H_{2(n+k)}(K(\mathbb{Z}, 2n); \mathbb{Z})$, which we shall denote by $Sq_{2k,2n}$. Theorem 3.5 therefore gives us the following result.

**THEOREM 3.8.** $j_* Sq_{2k,2n} = Sq_{2k,2n}.$

**Note 3.9** When pushed into $H_{2(n+k)}(K(\mathbb{Z}, 2n); \mathbb{Z})$, the class $Sq_{2k,2n}$ can be interpreted in terms of the homology operations described in [HMc]. Recall that if $\varphi_{2n} \in H^{2k}(K(\mathbb{Z}, 2n); \mathbb{Z})$ denotes the fundamental cohomology class and $\rho$ is reduction mod 2, then the Steenrod operation $Sq^{2k}$ corresponds to a non-trivial element $Sq^{2k}\rho \varphi_{2n} \in H^{2(n+k)}(K(\mathbb{Z}, 2n); \mathbb{Z})$.

Let $\eta_{2n} \in H_{2n}(K(\mathbb{Z}, 2n); \mathbb{Z})$ be the fundamental homology class and again let $\rho$ denote reduction mod 2. Let $\overline{Sq^{2k}}$ denote the homology operation dual to $Sq^{2k}$ as in [HMc]. Recall that for a cohomology operation $\varphi$, its dual $\overline{\varphi}$ is defined by

$$\langle \overline{\varphi} \alpha, \beta \rangle = \langle \alpha, \varphi \beta \rangle$$

for all $\alpha, \beta$.

where $\langle , \rangle$ is the Kroneker pairing. Then we claim that $\overline{Sq^{2k}}\rho Sq_{2k,2n} = \rho \eta_{2n}$. That is, we claim that $\langle \rho \eta_{2n}, \beta \rangle = \langle \overline{Sq^{2k}}\rho Sq_{2k,2n}, \beta \rangle$ all $\beta$. But the only non-trivial $\beta$ is $\varphi_{2n}$ and we have $\langle \rho Sq_{2k,2n}, Sq^{2k}\rho \varphi_{2n} \rangle \neq 0$.

**§4. The algebraic cycles $\widehat{Sq}_{2k,2n}$**

In this section we shall define (complex) algebraic cycles in $SP^2(\mathbb{P}^n_C)$ which represent 2-torsion elements in integral homology and push forward to the classes $Sq_{2k,2n}$ under the natural projection $SP^2(\mathbb{P}^n_C) \rightarrow SP^2(\mathbb{P}^n_C/\mathbb{P}^{n-1}_C) = SP^2(S^{2n})$. Let us fix homogeneous coordinates $[z] = [z_0, \ldots, z_n]$ for $\mathbb{P}^n_C$ and bihomogeneous coordinates $([z], [\zeta])$ for
\( \text{PROPOSITION 4.1.} \)  
The cycles \( Q_{A,B} = \tilde{D}_A - \tilde{D}_B \) for \((A,B) \in \text{Skew} \times \text{Sym}\) represent 2-torsion elements in the Chow group of rational equivalence classes of algebraic cycles on \( SP^2(\mathbb{P}_C^n) \).

\begin{proof}
Note that \( 2Q_{A,B} = \xi_*(D_A - D_B) \) and that \( D_A - D_B \) is rationally equivalent to zero on \( \mathbb{P}_C^n \times \mathbb{P}_C^n \).
\end{proof}
We generalize this construction to higher codimension as follows. Let $V \subset \mathbb{C}^{n+1}$ be a linear subspace of dimension $k$, set $\text{Skew}_V = \{ A \in \text{Skew} : V \subset \text{Null}(A) \}$, and consider the subvariety $Q(V) \subset \mathbb{P}_\mathbb{C}^n \times \mathbb{P}_\mathbb{C}^n$ given by

$$Q(V) \equiv \bigcap_{A \in \text{Skew}_V} D_A. \quad (4.2)$$

If $k = n - 1$, then $Q(V) = D_A$ for some $A \in \text{Skew}$ of rank 2. However, for $k < n - 1$, $Q(V)$ is not a complete intersection. As above we observe that $Q(V)$ is $\mathbb{Z}_2$-invariant and define the reduced cycle

$$\tilde{Q}(V) = \frac{1}{2} \xi_*(Q(V)) \quad \text{on } SP^2(\mathbb{P}_\mathbb{C}^n). \quad (4.3)$$

We now introduce an explicit family of cycles of type $Q(V)$ and examine systems of equations which define them. Consider the subspace $V_{k,n} \subset \mathbb{C}^{n+1}$ of dimension $k$ given by

$$V_{k,n} = \{ z \in \mathbb{C}^{n+1} : z_0 = 0 \text{ and } z_{k+1} = z_{k+2} = \cdots = z_n = 0 \}, \quad (4.4)$$

and define

$$Q_{k,n} = Q(V_{k,n}) \quad \text{and} \quad \tilde{Q}_{k,n} = \tilde{Q}(V_{k,n}). \quad (4.5)$$

Let $\mathbb{C}_0^n \times \mathbb{C}_0^n \subset \mathbb{P}_\mathbb{C}^n \times \mathbb{P}_\mathbb{C}^n$ be the affine chart defined in homogeneous coordinates by $z_0 = \zeta_0 = 1$. Then

$$Q^0_{k,n} \equiv Q_{k,n} \cap (\mathbb{C}_0^n \times \mathbb{C}_0^n) = \{ (z, \zeta) \in \mathbb{C}^n \times \mathbb{C}^n : z_j = \zeta_j \text{ for } k+1 \leq j \leq n \}$$

and

$$Q_{k,n} = \overline{Q^0_{k,n}} \quad \text{in } \mathbb{P}_\mathbb{C}^n \times \mathbb{P}_\mathbb{C}^n. \quad (4.6_t)$$

Consider now the rational family of affine varieties $Q^0_{k,n}(t) \subset \mathbb{C}_0^n \times \mathbb{C}_0^n$ defined by the equations

$$\begin{cases} (1 - t)(z_{k+1} - \zeta_{k+1}) + t = 0 \\ z_i - \zeta_i = 0 \quad \text{for } k + 2 \leq i \leq n \end{cases} \quad (4.6_t)$$

for $t \in \mathbb{C}$. For $t \neq 1$, we let $Q_{k,n}(t)$ be the closure of $Q^0_{k,n}(t)$ in $\mathbb{P}_\mathbb{C}^n \times \mathbb{P}_\mathbb{C}^n$. It is straightforward to verify that in bihomogeneous coordinates
([z_0, \ldots, z_n], [\zeta_0, \ldots, \zeta_n]) on \mathbb{P}^n_C \times \mathbb{P}^n_C, Q^0_{k,n}(t) is defined by the system of equations:

\[
\begin{align*}
(1 - t)(\zeta_0 z_{k+1} - \zeta_{k+1} z_0) + t \zeta_0 z_0 &= 0 \\
(1 - t)(\zeta_i z_{k+1} - \zeta_{k+1} z_i) + t \zeta_0 z_i &= 0 \\
(1 - t)(\zeta_i z_{k+1} - \zeta_{k+1} z_i) + t \zeta_i z_0 &= 0 \\
\zeta_0 z_i - \zeta_i z_0 &= 0 & \text{for } k + 2 \leq i \leq n \\
\zeta_j z_i - \zeta_i z_j &= 0 & \text{for } k + 2 \leq i, j \leq n,
\end{align*}
\]

(4.7t)

(Check that when \(\zeta_0 = z_0 = 1\), equations (4.6t) and (4.7t) are equivalent, and on the divisor \(\zeta_0 z_0 = 0\), there are no components of dimension \(n + k\) provided \(t \neq 1\).)

Let \(R_{k,n} \equiv Q_{k,n}(1) \subset \mathbb{P}^n_C \times \mathbb{P}^n_C\) be the subvariety defined by the equations (4.7_1), i.e., with \(t = 1\). Note that by the first equation \(\zeta_0 z_0 = 0\), we have

\[
\text{supp } (R_{k,n}) \subset (\mathbb{P}^{n-1} \times \mathbb{P}^n_C) \cup (\mathbb{P}^n_C \times \mathbb{P}^{n-1}_C).
\]

The cycle \(R_{k,n}\) is also \(\mathbb{Z}_2\)-invariant, and we can define the reduced cycle

\[
\tilde{R}_{k,n} = \frac{1}{2} \zeta_*(R_{k,n}) \quad \text{on } SP^2(\mathbb{P}^n_C).
\]

**Definition 4.10** For \(0 < k < n\), let \(\tilde{S}_{q_{2k,2n}}\) be the algebraic \((n + k)\)-cycle on \(SP^2(\mathbb{P}^n_C)\) defined by

\[
\tilde{S}_{q_{2k,2n}} = \tilde{Q}_{k,n} - \tilde{R}_{k,n}.
\]

**Proposition 4.11.** The cycle \(\tilde{S}_{q_{2k,2n}}\) represents a 2-torsion element in the Chow group of rational equivalence classes of algebraic cycles on \(SP^2(\mathbb{P}^n_C)\). In particular, it represents a 2-torsion element \(\tilde{S}_{q_{2k,2n}}\) in \(H_{2(n+k)}(SP^2(\mathbb{P}^n_C); \mathbb{Z})\).
Proof. Note that $2\widehat{S}_{2k,2n} = \xi_*(Q_{k,n} - R_{k,n})$ and that the cycle $Q_{k,n} - R_{k,n} = Q_{k,n}(0) - Q_{k,n}(1)$ is rationally equivalent to zero on $\mathbb{P}_C^n \times \mathbb{P}_C^n$ (via the family (4.7)).

**Note 4.12** Let $\omega \in H^2(\mathbb{P}_C^n; \mathbb{Z})$ be the canonical generator, and let $\pi_i : \mathbb{P}_C^n \times \mathbb{P}_C^n \to \mathbb{P}_C^n$ denote projection onto the $i$th factor for $i = 1, 2$. Set $\omega_i = \pi_i^*(\omega)$. Then a straightforward calculation shows that

$$\mathcal{P}(Q_{k,n}) = \begin{cases} (\omega_1 \omega_2)^{\ell}(\omega_1 + \omega_2) & \text{if } n - k = 2\ell + 1 \\ (\omega_1 \omega_2)^{\ell}(\omega_1^2 + \omega_2^2) & \text{if } n - k = 2\ell + 2 \end{cases}$$

where $\mathcal{P} : H_{2(n+k)}(\mathbb{P}_C^n \times \mathbb{P}_C^n; \mathbb{Z}) \xrightarrow{\approx} H^{2(n-k)}(\mathbb{P}_C^n \times \mathbb{P}_C^n; \mathbb{Z})$ is the Poincaré duality map.

§5. The main theorem

Fix homogeneous coordinates $[z_0, ..., z_n]$ for $\mathbb{P}_C^n$ and let $\mathbb{P}_C^{n-1} = \{[z] \in \mathbb{P}_C^n : z_0 = 0\}$. There is an affine chart $\Psi : \mathbb{C}^n \xrightarrow{\approx} \mathbb{C}_0^n \equiv \mathbb{P}_C^n - \mathbb{P}_C^{n-1}$ given by $(z_1, ..., z_n) \mapsto [1, z_1, ..., z_n]$. Consider the real analytic map

$$\mathbb{P}_C^n \xrightarrow{\pi} S^{2n} \quad (5.1)$$

with the defining properties that

$$\pi(\mathbb{P}_C^{n-1}) = \{\infty\} \quad (5.2)$$

and $\pi$ is represented by the identity in the distinguished $\mathbb{C}^n$-charts, i.e., the diagram

$$\begin{array}{ccc}
\mathbb{C}^n & \xrightarrow{\Psi} & \mathbb{P}_C^n \\
\downarrow \text{Id} & & \downarrow \pi \\
\mathbb{C}^n & \xrightarrow{\Psi_0} & S^{2n} \\
\end{array} \quad (5.3)$$

commutes, where $\Psi_0$ is the chart considered in §3.
Taking the cartesian product gives a map
\[ \mathbf{P}_C^n \times \mathbf{P}_C^n \stackrel{\pi \times \pi}{\longrightarrow} S^{2n} \times S^{2n} \]
with
\[ (\pi \times \pi)(\mathbf{P}_C^n \times \mathbf{P}_C^{n-1} \cup \mathbf{P}_C^{n-1} \times \mathbf{P}_C^n) = S^{2n} \times \{\infty\} \cup \{\infty\} \times S^{2n}. \] (5.4)
This descends to a map
\[ SP^2(\mathbf{P}_C^n) \stackrel{\Pi}{\longrightarrow} SP^2(S^{2n}). \] (5.5)

**Proposition 5.6.**
\[ \Pi_*(\widehat{\mathrm{Sq}}_{2k,2n}) = \mathrm{Sq}_{2k,2n} \]

**Proof.** Restricting to our affine coordinates gives an open dense subset (cf.(5.3))
\[
\begin{array}{ccc}
SP^2(C^n) & \longrightarrow & SP^2(\mathbf{P}_C^n) \\
\downarrow \mathrm{Id} & & \downarrow \Pi \\
SP^2(C^n) & \longrightarrow & SP^2(S^{2n})
\end{array}
\]
where the horizontal arrows are inclusions and where \( SP^2(C^n) = \{[[z],[\zeta]] : z_0 = \zeta_0 = 1\} \). In this open set the divisor \( \widehat{Q}_{k,n} \) is defined by the equations
\[ z_i - \zeta_i = 0 \quad \text{for} \quad k + 1 \leq i \leq n. \] (5.7)
(See (4.6) of (4.7) with \( t = 0 \).) Furthermore, in \( SP^2(\mathbf{P}_C^n) \) the cycle \( \widehat{R}_{k,n} \) is contained in the divisor \( \zeta_0 z_0 = 0 \) (cf. (4.8)). Consequently in the open subset \( SP^2(C^n) \), the cycles \( \widehat{\mathrm{Sq}}_{2k,2n} \) and \( \mathrm{Sq}_{2k,2n} \) are defined by the same equations (5.7). The other component \( \widehat{R}_{k,n} \) of \( \widehat{\mathrm{Sq}}_{2k,2n} \) is contained in the complement of \( SP^2(C^n) \), which by (5.4) and (4.8) is mapped into the “axis” \( 2n \)-sphere in \( SP^2(S^{2n}) \). Now the real dimension of \( \widehat{R}_{k,n} \) is \( 2(n+k) \), and since \( k > 0 \), this implies that \( \Pi_*(\widehat{R}_{k,n}) = 0 \) as a real \( 2(n+k) \)-cycle. Hence, \( \Pi_*(\widehat{\mathrm{Sq}}_{2k,2n}) = \Pi_*(\widehat{Q}_{k,n}) = \mathrm{Sq}_{2k,2n} \) as claimed. \( \square \)
We now consider the commutative diagram

\[
\begin{array}{c}
SP^2(P^n_C) \subset \Pi \rightarrow SP(P^n_C) \cong K(\mathbb{Z}, 2) \times \cdots \times K(\mathbb{Z}, 2n) \\
SP^2(S^{2n}) \subset SP(S^{2n}) = SP(P^n_C/P^n_{C-1}) \cong K(\mathbb{Z}, 2n) \\
\end{array}
\]

and let \( J : SP^2(P^n_C) \rightarrow K(\mathbb{Z}, 2n) \) be the map given by composition in this diagram. Then combining 3.5, 3.8, 4.11 and 5.6 gives the following main result.

**THEOREM 5.9.** The algebraic \((n+k)\)-cycles \( \hat{S}_{q,2k,2n} \) in \( SP^2(P^n_C) \) represent non-zero 2-torsion elements in integral homology which push forward under \( J \) to the universal classes \( Sq_{2k,2n} \in H_{2(n+k)}(K(\mathbb{Z}, 2n); \mathbb{Z}) \).

§ 6. Assembling the classes

Let \( C^2 \subset C^3 \subset C^4 \subset \cdots C^{n+1} \subset \cdots \) be the flag defined by embedding \( C^i \subset C^j \) as the first \( i \) coordinates. This gives filtrations

\[
P^i_C \subset P^2_C \subset P^3_C \subset \cdots \subset P^n_C.
\]

\[SP^2(P^1_C) \subset SP^2(P^2_C) \subset SP^2(P^3_C) \subset \cdots \subset SP^2(P^n_C). \tag{6.1}\]

We use these inclusions \( j_m : SP^2(P^m_C) \subset SP^2(P^n_C) \) to push our cycles forward (without changing notation):

\[
\hat{S}_{q,2k,2n} = (j_m)_*(\hat{S}_{q,2k,2m})
\]
for \( 0 < k < m \leq n \). Equations (4.7t) show directly that in \( P^k_C \times P^k_C \) one has

\[
Q_{k,m} = Q_{k,n} \cap (P^m_C \times P^m_C) \quad \text{and} \quad R_{k,m} = R_{k,n} \cap (P^m_C \times P^m_C)
\]
from which it follows that

\[
\hat{S}_{q,2k,2m} = \hat{S}_{q,2k,2n} \cap SP^2(P^m_C). \tag{6.2}
\]
In [FL] Friedlander and Lawson define algebraic “projection maps”

\[ \rho_m : SP(P^n_C) \longrightarrow SP(P^m_C) \]

for \( 0 < m \leq n \), with the following properties:

(i) \( \rho_m(P^l_C) \subseteq SP(P^l_C) \) for all \( l < m \).

(ii) The map

\[ \psi_m : P^m_C/P^{m-1}_C \longrightarrow SP(P^m_C/P^{m-1}_C) \]

induced by \( \rho_m \) on the subquotient is the fundamental, degree-one inclusion. In particular, \( \psi_m \) induces a homotopy equivalence

\[ (\psi_m)_* : SP(P^m_C/P^{m-1}_C) \approx SP(P^m_C/P^{m-1}_C). \]

Let \( \bar{\rho}_m \) denote the composition \( SP(P^n_C) \xrightarrow{\rho_m} SP(P^m_C) \longrightarrow SP(P^m_C/P^{m-1}_C) \)

(iii) The map

\[ \rho = \prod_{m=1}^n \bar{\rho}_m : SP(P^n_C) \longrightarrow \prod_{m=1}^n SP(P^m_C/P^{m-1}_C) \cong \prod_{m=1}^n K(Z, 2m) \]

induced by these projections is a homotopy equivalence.

It follows that the classes \( \widehat{Sq}_{2k,2m} = [\widehat{Sq}_{2k,2m}] \in H_{2k+2m}(SP(P^n_C); Z) \) satisfy

\[ (\bar{\rho}_l)_* (\widehat{Sq}_{2k,2m}) = \begin{cases} \text{Sq}_{2k,2m} & \text{if } l = m \\ 0 & \text{if } l \neq m \end{cases} \]

since \( \widehat{Sq}_{2k,2m} \subset SP(P^m_C) \).

§7. Proliferation The fact that the basic classes \( Sq_{2k,2n} \) are represented by algebraic cycles implies that in fact much of the homology of

\[ \lim_{d \to \infty} SP^d(P^n_C) \cong \prod_{j=1}^q K(Z, 2j) \]
is algebraic. To begin we consider the algebraic suspension map $\Sigma_{n-\nu} : SP^d(P^q_C) \rightarrow C^q_d(P^n)$ which associates to a 0-cycle $c$ its join $\Sigma_{n-\nu}(c) = c \# P^q_n - \nu - 1$ with a linear subspace complementary to $P^q_C$ (cf. [L2]). This map induces a homotopy equivalence

$$\Sigma_{n-\nu} : \lim_{d \rightarrow \infty} SP^d(P^q_C) \xrightarrow{\simeq} \lim_{d \rightarrow \infty} C^q_d(P^n),$$

and since $\Sigma_{n-\nu}$ is an algebraic map, the corresponding homology classes $Sq_{2k,2n}$ on $\lim_{d \rightarrow \infty} C^q_d(P^n)$ have algebraic representatives (already at level $d = 2$).

There are two algebraic pairings: the addition map

$$C^q_d(P^n) \times C^q_{d'}(P^n) \xrightarrow{\pm} C^q_{d+d'}(P^n) \quad \text{(7.1)}$$

and the biadditive join pairing (cf. §2)

$$C^q_d(P^n) \times C^q_{d'}(P^{n'}) \xrightarrow{\#} C^q_{d+d'}(P^{n+n'+1}). \quad \text{(7.2)}$$

The addition map (7.1) induces the standard H-space multiplication on $K(Z, 2q)$ and therefore induces the standard Pontrjagin product in homology. This map clearly takes pairs of algebraic classes to algebraic classes.

The join mapping (7.2) gives a way of composing our classes. We define

$$Sq_{2k_1,2n_1} \cdots \cdot Sq_{2k_l,2n_l} = \big[
\widehat{S}q_{2k_1,2n_1} \# \left( \widehat{S}q_{2k_2,2n_2} \# \left( \cdots \# \widehat{S}q_{2k_l,2n_l} \right) \right) \big].$$

(7.3)

All such classes are also algebraic.

Thus the integral homology classes on $\prod K(Z, 2j)$ generated by the basic classes $Sq_{2k,2n}$ under the Pontrjagin product and the join composition are all algebraic.
It is reasonable to conjecture that the set of classes $Sq_{2k_1,2n_1} \ast \cdots \ast Sq_{2k_\ell,2n_\ell}$ correspond to the set of admissible sequences $(2k_1, 2k_2, ..., 2k_\ell)$ appearing in Cartan [C$_2$]. If so, then a great deal of the 2-torsion in the canonical image

$$H_{2*}(K(\mathbb{Z}, 2k); \mathbb{Z}) \subset H_{2*}(\prod_j K(\mathbb{Z}, 2j); \mathbb{Z})$$

for each $i$, will be algebraic. We will, of course, not retrieve any elements with a "Bockstein" appearing.

In a sequel, [M], analogous results will be established for the $p$-torsion where $p$ is an odd prime.

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