A FEYNMAN-KAC APPROACH FOR THE SPATIAL DERIVATIVE OF THE SOLUTION TO THE WICK STOCHASTIC HEAT EQUATION DRIVEN BY TIME HOMOGENEOUS WHITE NOISE

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Abstract. We consider the (unique) mild solution \( u(t, x) \) of a 1-dimensional stochastic heat equation on \([0, T] \times \mathbb{R}\) driven by time-homogeneous white noise in the Wick-Skorokhod sense. The main result of this paper is the computation of the spatial derivative of \( u(t, x) \), denoted by \( \partial_x u(t, x) \), and its representation as a Feynman-Kac type closed form. The chaos expansion of \( \partial_x u(t, x) \) makes it possible to find its (optimal) Hölder regularity especially in space.

1. Introduction

As an extension of the paper [26], we will further investigate the (unique) mild solution of the 1-dimensional stochastic heat equation (SHE):

\[
\begin{aligned}
\partial_t u(t, x) &= \frac{1}{2} \partial_{xx}^2 u(t, x) + u(t, x) \diamond \dot{W}(x), \quad t \in (0, T], \ x \in \mathbb{R}, \\
u(0, x) &= u_0(x), \ x \in \mathbb{R},
\end{aligned}
\]

where \( T > 0 \), \( u_0 \) is a function satisfying certain conditions, \( \dot{W}(x) \) is a space-only Gaussian white noise on a complete probability space \((\Omega, \mathcal{F}, \mathbb{P}^W)\), and \( \diamond \) stands for the Wick product. In other words, the corresponding stochastic integration in (1) is interpreted in the Wick-Skorokhod sense.

We postpone all technical definitions to the following section.

Definition 1. We say \( u : [0, T] \times \mathbb{R} \times \Omega \to \mathbb{R} \) is said to be a mild solution of (1) if for any fixed \( (t, x) \in [0, T] \times \mathbb{R} \), \( u(t, x) \in L^2(\mathbb{P}^W) \) and it satisfies

\[
u(t, x) = \int_\mathbb{R} p(t, x - y)u_0(y)dy + \int_0^t \int_\mathbb{R} p(t - s, x - y)u(s, y) \diamond \dot{W}(y)dyds, \ \mathbb{P}^W\text{-almost surely},
\]

where \( p(t, x) := \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} \) is the Gaussian heat kernel.

Let \( \{u(t, x)\}_{(t, x) \in [0, T] \times \mathbb{R}} \) be a mild solution of (1). Then for any fixed \( (t, x) \), the random variable \( u(t, x) \) admits the following multiple Wiener chaos expansion (e.g. [9], [11] or [28]):

\[
u(t, x) = \sum_{n=0}^{\infty} I_n(F_n^W(t, x)),
\]

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where $I_n$ is the $n$-th multiple Wiener integral with respect to $W$,
\[ F_{0}^{MW}(t, x) = \int_{\mathbb{R}} p(t, x - y)u_0(y) \, dy; \]
\[ F_{n}^{MW}(t, x; y_1, \ldots, y_n) = \frac{1}{n!} \int_{[0, t]^n} p(t - r_{\rho(n)}, x - y_{\rho(n)}) \times \cdots \times p(r_{\rho(2)} - r_{\rho(1)}, y_{\rho(2)} - y_{\rho(1)}) F_{0}^{MW}(r_{\rho(1)}, x_{\rho(1)}) \, dr, \quad n \geq 1, \]
and $\rho$ denotes the permutation of $\{1, \ldots, n\}$ such that $0 < r_{\rho(1)} < \cdots < r_{\rho(n)} < t$. For simplicity, we have denoted $dr := dr_1 dr_2 \cdots dr_n$.

To distinguish among different representations of the mild solution, let us call (3) the **multiple Wiener solution** $u^{MW}(t, x)$ of (1). There are a few papers considering this representation:

(i) The paper [28, Theorem 3.1] shows that $u^{MW}$ is the unique mild solution in $C([0, T]; L^2(\mathbb{R}; L^2(\Omega)))$ if $u_0 \in L^2(\mathbb{R})$ by showing,
\[ \sup_{t \in [0, T]} \sum_{n=0}^{\infty} n! \int_{\mathbb{R}} \| F_{n}^{MW}(t, x; \bullet) \|_{L^2(\mathbb{R}^n)}^2 \, dx \leq C \| u_0 \|_{L^2(\mathbb{R})}^2 < \infty. \]

Note that $u_0 \not\in L^2(\mathbb{R})$ does not cover $u_0 \equiv 1$.

(ii) When $u_0 \in L^\infty(\mathbb{R})$, [9, Section 4] shows that
\[ \sup_{(t, x) \in [0, T] \times \mathbb{R}} \sum_{n=0}^{\infty} n! \| F_{n}^{MW}(t, x; \bullet) \|_{L^2(\mathbb{R}^n)}^2 \leq C \| u_0 \|_{L^\infty(\mathbb{R})} < \infty. \]

Hence, we can say that $u^{MW}$ is the unique mild solution in $C([0, T] \times \mathbb{R}; L^2(\Omega))$ if $u_0 \in L^\infty(\mathbb{R})$.

There is an alternative chaos expansion of the mild solution $u$ (e.g. [17, Theorem 3.11]):
\[ u(t, x) = \sum_{\alpha \in \mathcal{J}} u^{CS}_{\alpha}(t, x) \xi_{\alpha}, \quad (4) \]
where $\xi_\alpha$ and $\mathcal{J}$ are defined in Section 2.1.2. Letting $T^{n}_{[0, t]} := \{0 \leq s_1 \leq \cdots \leq s_n \leq t\}$, we can write
\[ u^{CS}_{(0)}(t, x) = \int_{\mathbb{R}} p(t, x - y)u_0(y) \, dy, \]
and for $|\alpha| = n \geq 1$,
\[ u^{CS}_{\alpha}(t, x) = \sqrt{n!} \int_{T^{n}_{[0, t]}} \int_{\mathbb{R}^n} p(t - s_n, x - y_n) \cdots p(s_2 - s_1, y_2 - y_1) u^{CS}_{(0)}(s_1, y_1) \, \varepsilon_{\alpha}(y_1, \ldots, y_n) \, ds \, dy, \]
for $\alpha \in \mathcal{J}_{n} := \{ \alpha \in \mathcal{J} : |\alpha| = n \}$, and $\{\varepsilon_{\alpha, \alpha \in \mathcal{J}_{n}\}$ forms an orthonormal basis of the symmetric part of $L^2(\mathbb{R}^n)$. We will call (4) the **chaos solution** $u^{CS}(t, x)$ of (1). The existence and uniqueness of this representation can be proved by showing the following:

(i) We can prove that $u^{CS}$ is the unique mild solution in $C([0, T]; L^2(\mathbb{R}; L^2(\Omega)))$ when $u_0 \in L^2(\mathbb{R})$ by showing (c.f. [14, Theorem 4.1])
\[ \sup_{t \in [0, T]} \sum_{n=0}^{\infty} \sum_{\alpha \in \mathcal{J}_{n}} \| u^{CS}_{\alpha}(t, \bullet) \|_{L^2(\mathbb{R})}^2 \leq C \| u_0 \|_{L^2(\mathbb{R})}^2 < \infty. \]

(ii) We can also show that $u^{CS}$ is the unique mild solution in $C([0, T] \times \mathbb{R}; L^2(\Omega))$ if $u_0 \in L^\infty(\mathbb{R})$ (c.f. [14, Theorem 4.3]) by achieving
\[ \sup_{(t, x) \in [0, T] \times \mathbb{R}} \sum_{n=0}^{\infty} \sum_{\alpha \in \mathcal{J}_{n}} | u^{CS}_{\alpha}(t, x) |^2 \leq C \| u_0 \|_{L^\infty(\mathbb{R})}^2 < \infty. \]
Indeed, (5) and (6) can be easily obtained as follows:

(5) To use the same argument as [14, Theorem 4.1], it is enough to show

\[ U_0 := \int_{\mathbb{R}} \left( \int_{\mathbb{R}} p(s, y - z_1) u_0(z_1) dz_1 \right) \cdot \left( \int_{\mathbb{R}} p(s, y - z_2) u_0(z_2) dz_2 \right) dy \leq \|u_0\|^2_{L^2(\mathbb{R})}, \]

and it is clear by semigroup property and Hölder inequality,

\[ U_0 = \int_{\mathbb{R}} \int_{\mathbb{R}} p(s + r, z_1 - z_2) u_0(z_1) u_0(z_2) dz_1 dz_2 \]

\[ = \int_{\mathbb{R}} p(s + r, z_1) \int_{\mathbb{R}} u_0(z_1 + z_2) u_0(z_2) dz_2 dz_1 \leq \|u_0\|^2_{L^2(\mathbb{R})}. \]

(6) To use the same argument as [14, Theorem 4.3], it is enough to show \(|\int_{\mathbb{R}} p(t, x - y) u_0(y) dy| \leq \|u_0\|_{L^\infty(\mathbb{R})}\), and it automatically follows from the fact \(\int_{\mathbb{R}} p(t, x) dx = 1\).

Then, it is not surprising that \(u^{\text{MW}} = u^{\text{CS}}\) if \(u_0 \in L^\infty(\mathbb{R})\) since the mild solution is unique in \(C([0, T] \times \mathbb{R}; L^2(\Omega))\).

We now discuss one more possible representation of the mild solution. In fact, the condition (2) is equivalent to the following because (7) is equivalent to the following because

\[ \mathbb{E}[F \cdot u(t, x)] = \mathbb{E}[F] \cdot \int_{\mathbb{R}} p(t, x - y) u_0(y) dy + \mathbb{E} \left[ F \cdot \int_0^t \int_{\mathbb{R}} p(t - s, x - y) u(s, y) \diamond \dot{W}(y) dy ds \right]. \]

(7)

Here, Malliavin derivative \(D\) and the Sobolev-Malliavin space \(\mathbb{D}^{1,2}\) are defined in Section 2.2.

Moreover, it is known that (7) is equivalent to

\[ \mathbb{E}[F \cdot u(t, x)] = \mathbb{E}[F] \cdot \int_{\mathbb{R}} p(t, x - y) u_0(y) dy + \mathbb{E} \left[ \left\langle \int_0^t p(t - s, x - \bullet) u(s, \bullet) ds, D(\bullet)F \right\rangle_{L^2(\mathbb{R})} \right], \]

(8)

if \(\mathbb{E} \left[ F \cdot \left( \int_{\mathbb{R}} h(y) \diamond \dot{W}(y) dy \right) \right] = \mathbb{E} \left[ \left\langle DF, h \right\rangle_{L^2(\mathbb{R})} \right] \) for \(h \in L^2(\mathbb{P}^W; L^2(\mathbb{R}))\) and \(\int_{\mathbb{R}} h(y) \diamond \dot{W}(y) dy \in L^2(\mathbb{P}^W)\) (e.g. [20, Section 2.5]).

Using (8) and a Wong-Zakai-type approximation, the paper [26] gives a Feynman-Kac representation of the unique mild solution of (1) when \(u_0 \in L^\infty(\mathbb{R})\). This is given by

\[ u(t, x) = \mathbb{E}^{\mathbb{R}} [u_0(B^x_t) \exp\{\Psi_{t,x}\}], \]

where \(\{B^x_t\}_{t \geq 0}\) is a one-dimensional Brownian motion starting at \(x\), and for fixed \((t, x) \in [0, T] \times \mathbb{R}\), the random variable \(\Psi_{t,x}\) is given by

\[ \Psi_{t,x} := \int_{\mathbb{R}} L^x_y(t) dW(y) - \frac{1}{2} \int_{\mathbb{R}} |L^x_y(t)|^2 dy. \]

Here \(L^x_a(t)\) denotes the local time of \(\{B^x_s\}_{s \geq 0}\) at level \(a\) and time \(t\). Let us call the Feynman-Kac representation the \textit{Feynman-Kac solution} \(u^{\text{FK}}(t, x)\) of (1).

Combining all, as long as \(u_0 \in L^\infty(\mathbb{R})\), we can say

\[ u := u^{\text{FK}} = u^{\text{MW}} = u^{\text{CS}} \in C \left( [0, T] \times \mathbb{R}; L^2(\Omega) \right). \]

(9)

In this paper, we will provide an alternative proof for the equivalence (9) using a more direct approach.
The main motivation for the current article is as follows: As we stated above, the equation (1) may have three possible representations for the unique mild solution \( u \), namely (I) Feynman-Kac solution \( u^{FK} \), (II) multiple Wiener-Itô integral solution \( u^{MW} \), and (III) chaos solution \( u^{CS} \). Unfortunately, there is no enough discussion on Hölder regularity of the mild solution. In particular,

(I) There is no Hölder regularity result for \( u^{FK} \) in the existing literature.

(II) For \( u^{MW} \), [28, Theorem 4.1] proves that \( u^{MW} \in C^{1/2-\varepsilon,1/2-\varepsilon}([0,T],\mathbb{R}) \) for any small \( \varepsilon > 0 \) if \( u_0 \in C^1_b(\mathbb{R}) \cap L^2(\mathbb{R}) \). Here, \( C^1_b(\mathbb{R}) \) denotes the space of all bounded differentiable functions on \( \mathbb{R} \) with bounded continuous derivatives.

(III) On the one hand, no one discusses the regularity of \( u^{CS} \) on the whole line. On the other hand, the paper [14] discuss the same equation as (1), but on a bounded domain, say \([0,\pi]\); the authors show that there exists a unique mild solution (using chaos expansion) \( u^{CS}_b \in C([0,T];L^2([0,\pi]);L^2(\Omega)) \) if \( u_0 \in L^2([0,\pi]) \), and moreover \( u^{CS}_b \in C^{3/4-\varepsilon,3/2-\varepsilon}([0,T] \times [0,\pi]) \) for any small \( \varepsilon > 0 \) if \( u_0 \in C^{3/2}([0,\pi]) \).

Since Hölder continuity is a local property, it is natural to expect that \( u^{CS} \in C^{3/4-\varepsilon,3/2-\varepsilon}([0,T] \times \mathbb{R}) \) for any small \( \varepsilon > 0 \) (under a suitable initial condition on \( u_0 \)) like the bounded case \( u^{CS}_b \). Furthermore, it is impossible that the other representations \( u^{FK} \) and \( u^{MW} \) have a different regularity from the one of \( u^{CS} \) (by uniqueness). In this sense, we would say that the existing Hölder regularity results of the mild solution on \( \mathbb{R} \) should be improved, and in this paper, we will suggest an idea of how to get the desired result. We emphasize that the regularity almost \( 3/4 \) in time and almost \( 3/2 \) in space is optimal in the classical PDE sense, since \( W \) is understood to have regularity \(-1/2-\varepsilon\) for any small \( \varepsilon > 0 \) (c.f. [6, Lemma 1.1]).

The main feature of this paper is to find the optimal spatial regularity of the unique mild solution \( u \). The first task is to find \( \partial_x u \) and check if it is well-defined. One can compute \( \partial_x u \) from \( u^{CS} \) using the same argument as [14], but we will focus on the Feynman-Kac representation and compute the spatial derivative of \( u \) using \( u^{FK} \) as a main part of this paper. This approach allows us to obtain a Feynman-Kac-type closed formula for \( \partial_x u \). We remark that we can also derive the chaos decomposition of \( \partial_x u \) using \( u^{FK} \), and it is exactly the same as the one after differentiating \( u^{CS} \) with respect to \( x \) directly. With this in hand, we can achieve the optimal Hölder regularity of \( \partial_x u \) that is almost \( 1/4 \) in time and almost \( 1/2 \) in space.

2. Preliminaries

2.1. Elements of white noise analysis

In this section, we will give a brief outline of the white noise tools which will be used in this article. The interested reader for more details is referred to [7] and [15].

For \( n \in \mathbb{N} \), let \( \mathcal{S}(\mathbb{R}^n) \) be the Schwartz space of rapidly decreasing functions on \( \mathbb{R}^n \), and \( \mathcal{S}'(\mathbb{R}^n) \) the dual of \( \mathcal{S}(\mathbb{R}^n) \), which is called the space of tempered distributions. We further define the white noise probability space by \((\mathcal{S}'(\mathbb{R}), \mathcal{B}, \mu)\), where \( \mathcal{B} \) is the Borel \( \sigma \)-algebra on \( \mathcal{S}'(\mathbb{R}) \), i.e. the \( \sigma \)-algebra generated by the cylindrical sets, and \( \mu \) is the standard Gaussian measure in \( \mathcal{S}'(\mathbb{R}) \) (see [15, Section 3.1] for more details). Specifically, the measure \( \mu \) satisfies

\[
\int_{\mathcal{S}'(\mathbb{R})} e^{i\langle \omega, \varphi \rangle} d\mu(\omega) = e^{-\frac{1}{2}||\varphi||_0^2}, \quad \varphi \in \mathcal{S}(\mathbb{R}),
\]

where \( ||\cdot||_0 \) denotes the norm in \( L^2(\mathbb{R}) \).

Let \( W \) denote the canonical coordinate process (or Wiener integral) on \( \mathcal{S}'(\mathbb{R}) \) given by

\[
W_\phi = \langle \omega, \phi \rangle, \quad \phi \in \mathcal{S}(\mathbb{R}), \quad \omega \in \mathcal{S}'(\mathbb{R}).
\]
Here, \((\cdot, \cdot)\) denotes the dual pairing between \(\mathcal{S}(\mathbb{R})\) and \(\mathcal{S}'(\mathbb{R})\).

Note that we can extend \(W\) continuously in \(L^2(\mu)\) to \(L^2(\mathbb{R})\) as
\[
\langle \omega, \phi \rangle := L^2(\mu) - \lim_{k \to \infty} \langle \omega, \phi_k \rangle, \quad \phi \in L^2(\mathbb{R}),
\]
where \(\{\phi_k\}_{k \in \mathbb{N}}\) is any sequence in \(\mathcal{S}(\mathbb{R})\) such that \(\phi_k \to \phi\) in \(L^2(\mathbb{R})\). In particular, if we define
\[
W(x)(\omega) := \begin{cases} 
\langle \omega, \chi_{[0,x]} \rangle, & \text{if } x \geq 0, \; \omega \in \mathcal{S}'(\mathbb{R}); \\
-\langle \omega, \chi_{[x,0]} \rangle, & \text{if } x < 0, \; \omega \in \mathcal{S}'(\mathbb{R}),
\end{cases}
\]
(10)
it is easy to check that \(\{W(x)\}_{x \in \mathbb{R}}\) is a Brownian motion.

**2.1.1. Chaos expansion in terms of multiple Wiener integrals**

It is well-known that any random variable \(F \in L^2(\mu)\) can be written as
\[
F = \sum_{n=0}^{\infty} I_n(f_n),
\]
where \(I_n\) is the multiple Wiener integral of order \(n\) with respect to the Brownian motion in (10), and \(f_n\) is a symmetric element of \(L^2(\mathbb{R}^n)\). This is the Wiener-Itô-Segal isomorphism between square integrable Brownian functionals and the symmetric Fock space \([13]\), i.e.
\[
L^2(\mu) \cong \bigoplus_{n=0}^{\infty} \text{Sym} L^2(\mathbb{R})^\otimes n.
\]

It turns out that the following isometry property holds
\[
\|F\|_2^2 = \sum_{n=0}^{\infty} n!|f_n|_{0,n}^2,
\]
where \(\| \cdot \|_2\) denotes the norm in \(L^2(\mu)\) and \(| \cdot |_{0,n}\) denotes the norm in \(L^2(\mathbb{R}^n)\), \(n \in \mathbb{N}\), whenever \(n = 1\) we shall omit the second sub-index.

**2.1.2. Chaos expansion in terms of Hermite polynomials**

Let \(\{e_j\}_{j \in \mathbb{N}} \subset \mathcal{S}(\mathbb{R})\) be the family of Hermite functions defined by
\[
e_j(x) := (-1)^{j-1} \left(\sqrt{\pi}2^{j-1}(j-1)!\right)^{-1/2} e^{x^2/2} \frac{d^{(j-1)}}{dx^{(j-1)}} e^{-x^2}, \; j \in \mathbb{N},
\]
where \(\frac{d^0}{dx^0}\) is the identity operator. It is known that \(\{e_j\}_{j \in \mathbb{N}}\) forms a complete orthonormal basis (CONB) of \(L^2(\mathbb{R})\). We next let \(\mathcal{J} := (\mathbb{N}_0^\mathbb{N})_c\) be the collection of multi-indices \(\alpha = (\alpha_1, \alpha_2, \ldots)\) such that every \(\alpha_j\) is a non-negative integer and there are only finitely many non-zero components. In this case, we define a few notations:

- \((0)\) is the multi-index with all zeroes;
- \(\alpha_{(j)} := (\alpha_1, \alpha_2, \ldots, \max(\alpha_j - 1, 0), \alpha_{j+1}, \ldots) \in \mathcal{J};\)
- \(\alpha^+_j := (\alpha_1, \alpha_2, \ldots, \alpha_j + 1, \alpha_{j+1}, \ldots) \in \mathcal{J};\)
- \(|\alpha| := \sum_{j=1}^{\infty} \alpha_j;\)
- \(\alpha! := \prod_{j=1}^{\infty} \alpha_j!;\)
We also define the collection of random variables $\Xi := \{ \xi_\alpha, \alpha \in J \}$ by

$$\xi_\alpha := \prod_{j=1}^{\infty} \left( \frac{H_{\alpha_j}(W_j)}{\sqrt{\alpha_j!}} \right),$$

where

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}$$

is the Hermite polynomial of order $n$.

One of the famous Cameron and Martin’s theorems [1] states that the family $\Xi$ forms an orthonormal basis in $L^2(\mu)$, and thus for $F \in L^2(\mu)$, we have the following chaos expansion

$$F = \sum_{\alpha \in J} F_\alpha \xi_\alpha$$

and

$$\|F\|_2^2 = \sum_{\alpha \in J} F_\alpha^2,$$

where $F_\alpha := \mathbb{E}[F \xi_\alpha]$.

In fact, both approaches to the chaos expansion for $F \in L^2(\mu)$ in Sections 2.1.1 and 2.1.2 are equivalent, and we can go from one to the other without particular difficulties. For example, see Section 3.2 below.

### 2.1.3. $S$-transform, Wick product, and generalized random variables

Now we introduce one of the main tools we will employ in our proofs, namely the $S$-transform. Let $F \in L^2(\mu)$ and $\phi \in L^2(\mathbb{R})$. Then the $S$-transform of $F$ is given by

$$S(F)(\phi) := \mathbb{E} \left[ F \times \mathcal{E}(\phi) \right], \quad \text{where} \quad \mathcal{E}(\phi) := \exp \left\{ W_\phi - \frac{1}{2} |\phi|^2_0 \right\}.$$

Note that $\mathcal{E}(\phi)$ is called stochastic exponential or Wick exponential (e.g. [8]).

In certain applications, one may be interested in spaces that are larger than $L^2(\mu)$, and one is naturally led to consider spaces of generalized random variables (see for instance [7], [10], [15]). In this article, we will solely consider the space of Hida distributions, and will briefly introduce them in the following.

Let $A$ be the operator given by $A = -\frac{d^2}{dx^2} + x^2 + 1$, and for $F \in L^2(\mu)$ with $F = \sum_{n=0}^{\infty} I_n(f_n)$ satisfying

$$\sum_{n=0}^{\infty} n! |A^{\otimes n} f_n|_{0,n}^2 < \infty,$$

we define $\Gamma(A)F \in L^2(\mu)$ by

$$\Gamma(A)F := \sum_{n=0}^{\infty} I_n(A^{\otimes n} f_n).$$

Sometimes $\Gamma(A)$ is referred to as the second quantization of the operator $A$.

**Remark 2.** It is known (e.g. [25, Theorem 2]) that if we define $\| \cdot \|_p := |A^p \cdot |_0$, the family of $p$-seminorms $\{ | \cdot |_p, \ p \geq 0 \}$ are equivalent to the usual seminorms in the Schwartz space $\mathcal{S}(\mathbb{R})$, i.e. they generate the same topology.

**Definition 3.** Let $(S_p)^*$ be the completion of $L^2(\mu)$ with respect to the norm

$$\| \cdot \|_{-p} := \| \Gamma(A^{-p}) \cdot \|_2, \ p > 0.$$
Then the space of Hida distributions is given by
\[(S)^* := \bigcup_{p \geq 0} (S_p)^* .\]

Any element \( \Phi \in (S)^* \) can be represented as the formal series:
\[
\Phi = \sum_{n=0}^{\infty} I_n(F_n), \quad F_n \in \mathcal{S}'_{\text{sym}}(\mathbb{R}^n) := \text{Sym}\mathcal{S}'(\mathbb{R})^\otimes n
\]
such that
\[
\sum_{n=0}^{\infty} n! |(A^{-p})^\otimes n F_n|_0^2 < \infty .
\]

In this setting, we can give a proper meaning to the white noise \( \partial_x W(x) \) or \( \dot{W}(x) \) as a Hida distribution and in a slight abuse of notation, we will denote
\[
\dot{W}(x; \omega) := \langle \omega, \delta_x \rangle
\]
Here, \( \delta_x \) stands for the Dirac-delta function. We note that [15, Section 3.4] that \( \dot{W} \in (S_p)^* \) for any \( p > 5/12 \).

Remark 4. The S-transform can be extended naturally to the space of Hida distributions by a duality argument, i.e. for \( \Phi \in (S)^* \), we define the S-transform of \( \Phi \) as
\[
S(\Phi)(\phi) := \langle \Phi, \mathcal{E}(\phi) \rangle, \quad \phi \in \mathcal{S}(\mathbb{R}),
\]
where \( \langle \bullet, \bullet \rangle \) stands for the bilinear dual paring between the space of Hida distributions \( (S)^* \) and its dual space, denoted by \( (S) \). Note that \( (S) \) is called the space of Hida test functions (see [15] for further details).

Definition 5. A function \( F : \mathcal{S}(\mathbb{R}) \to \mathbb{C} \) is called a U-functional if

1. For every \( \phi, \varphi \in \mathcal{S}(\mathbb{R}) \) the mapping \( \mathbb{R} \ni \lambda \mapsto F(\lambda \phi + \varphi) \in \mathbb{C} \) has an entire extension to \( z \in \mathbb{C} \).

2. There are constants \( 0 < K_1, K_2, p < \infty \) such that
\[
|F(\varphi)| \leq K_1 \exp(K_2|\varphi|^p), \quad \varphi \in \mathcal{S}(\mathbb{R}).
\]

We are now ready to introduce a characterization result for the space \( (S)^* \).

Theorem 6. [22, theorem 1.2] The S-transform defines a bijection between the space \( (S)^* \) and the space of U-functionals.

It is well-known that we cannot in general define the product between generalized functions, and this impossibility obviously extends also to generalized random variables. However, we are still able to define a particular type of renormalized product called Wick product and denoted by \( \diamond \) in the following way.

Definition 7. [15, Definition 8.11] Let \( \Phi, \Psi \in (S)^* \) be two Hida distributions. We define the Wick product between the two elements, denoted by \( \Phi \diamond \Psi \), to be the unique Hida distribution satisfying
\[
S(\Phi \diamond \Psi)(\phi) = S(\Phi)(\phi) \cdot S(\Psi)(\phi)
\]
for every \( \phi \in \mathcal{S}(\mathbb{R}) \).
Remark 8. The Hida space $(S)^*$ is an algebra with respect to the Wick product.

Remark 9. We can also give an alternative characterization of the Wick product between generalized random variables in terms of chaos decompositions (e.g. [7, Corollary 4.22]), namely if $\Phi, \Psi \in (S)^*$ are given by the formal series

$$
\Phi = \sum_{n=0}^{\infty} I_n(F_n), \quad \Psi = \sum_{n=0}^{\infty} I_n(G_n),
$$

then the Wick product between $\Phi, \Psi \in (S)^*$ is defined by the following chaos decomposition:

$$
\Phi \circ \Psi = \sum_{n=0}^{\infty} I_n(H_n),
$$

where $H_n = \sum_{j=0}^{n} F_{n-j} \hat{\otimes} G_j$, and $\hat{\otimes}$ denotes the symmetric tensor product.

One of the most striking properties of the Wick product is its relation with stochastic integration of Skorokhod-Itô type. In particular if $\{Y(x)\}_{x \in \mathbb{R}}$ is a Skorokhod integrable process, then we have that

$$
\int_{\mathbb{R}} Y(x) \delta W(x) = \int_{\mathbb{R}} Y(x) \circ \dot{W}(x) dx
$$

where the right hand side must be understood as a Pettis integral in $(S)^*$ (see for instance [8] or [15, Section 13.3]) and the left hand side is a Skorokhod integral (see [19]). This is the reason why in (1) we introduce the Wick product $\circ$ and say that the corresponding stochastic integral should be interpreted in the Skorokhod-Itô sense.

2.2. Elements of Malliavin calculus
For the purpose of this article, we will need a few definitions regarding Malliavin calculus. The interested reader is referred to [19] and [24] for a compressive exposition of Malliavin calculus and to [3] for the particular case in which the underlying probability space is the white noise probability space.

Let $S$ denote the class of smooth random variables $F$ having the form

$$
F = f(W_{h_1}, \ldots, W_{h_n}),
$$

where $h_1, \ldots, h_n \in L^2(\mathbb{R})$, and $f$ belong to $C^\infty_p(\mathbb{R}^n)$ which stands for the set of all infinitely continuously differentiable functions such that each function together with all its derivatives has a polynomial growth. We will refer to $S$ as the family of smooth Brownian functionals.

Definition 10. The Malliavin derivative of a smooth Brownian functional $F$ is the $L^2(\mathbb{R})$-valued random variable given by

$$
DF = \sum_{i=1}^{n} \partial_i f(W_{h_1}, \ldots, W_{h_n})h_i.
$$

In the same way, we can define the $k$-th derivative of $F$ for any $k \in \mathbb{N}$, which will be a $L^2(\mathbb{R})^{\otimes k}$-valued random variable.
**Definition 11.** Let $S$ be the space of smooth Brownian functionals, and define the following seminorm on $S$ for $k \geq 1$ and $p \geq 1$,

$$|F|_{k,p} := \left[ \mathbb{E}(|F|^p) + \sum_{j=1}^{k} \mathbb{E}\left(|D^j F|_{0,n}^p\right) \right]^{1/p}.$$ 

We will denote by $D^{k,p}$ the completion of the family $S$ with respect to the seminorm $| \cdot |_{k,p}$ and for any $F \in D^{k,p}$, we will let

$$D^k F = \lim_{n \to \infty} D^k F_n\quad \text{in } L^p(\mu; L^2(\mathbb{R})^{\otimes k}),$$

where $(F_n)_{n \in \mathbb{N}} \subset S$ is any sequence converging to $F$ in $L^p(\mu)$.

**Remark 12.** There exists an extension of the Malliavin derivative to an element of the Hida distribution space $(S)^*$ called the *Hida-Malliavin derivative* (see for instance [3]).

Furthermore, we will use the following notation

$$D^{\infty,2} := \bigcap_{k \geq 1} D^{k,2},$$

and the following lemma (e.g. [27]).

**Lemma 13.** Let $F \in D^{\infty,2}$ have the following chaos decomposition

$$F = \sum_{n=0}^{\infty} I_n(f_n).$$

Then it holds that $f_n(\bullet) = \frac{1}{n!} \mathbb{E}[D^n F]$.

Finally we introduce the following space of test functions that was introduced for the first time in [23].

**Definition 14.** For any $\lambda \in \mathbb{R}$, let $G_\lambda$ be the closure of $L^2(\mu)$ with respect to the norm $\|\Gamma(e^\lambda I) \bullet\|_2$, where $I$ stands for the identity operator. More explicitly,

$$G_\lambda := \left\{ F = \sum_{n=0}^{\infty} I_n(f_n) \in L^2(\mu) : \sum_{n=0}^{\infty} n! e^{2\lambda n} |f_n|_{0,n}^2 < \infty \right\},$$

and now we set

$$G := \bigcap_{\lambda \in \mathbb{R}} G_\lambda.$$

In particular, it is not hard to see the following inclusions:

$$(S) \subset G \subset L^2(\mu).$$

We note that if $F \in L^2(\mu)$ can be written as $\sum_{\alpha \in J} F_\alpha \xi_\alpha$ (the chaos expansion in terms of Hermite polynomials), then one can show that $F \in G_\lambda$ if

$$\sum_{n=0}^{\infty} e^{2\lambda n} \sum_{\alpha \in J_n} |F_\alpha|^2 < \infty.$$
2.3. Hölder spaces and classical Hölder regularity results

In this subsection, we first give a definition of Hölder spaces on $G \subseteq \mathbb{R}$. For $0 < \gamma < 1$, we let

$$[f]_{\gamma} := \sup_{z_1 \neq z_2 \in G} \frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|^\gamma}.$$  

We say that $f$ is Hölder continuous with Hölder exponent $\gamma$ (or Hölder $\gamma$ continuous) on $G$ if

$$\sup_{z \in G} |f(z)| + [f]_{\gamma} < \infty.$$  

The collection of Hölder $\gamma$ continuous functions on $G$ is denoted by $C^\gamma(G)$ with the norm

$$[[f]]_{\gamma} := \sup_{z \in G} |f(z)| + [f]_{\gamma}.$$  

For $k \in \mathbb{N}$, we say that $f$ is a $k$ times continuously differentiable function on $G$ if the $m$-th derivative of $f$, denoted by $\partial^m f$, exists and is continuous for all $m \leq k$. The collection of $k$ times continuously differentiable functions on $G$ such that $\partial^k f \in C^\gamma(G)$ with $0 < \gamma < 1$, is denoted by $C^{k+\gamma}(G)$ with the norm

$$[[f]]_{k+\gamma} := \sum_{1 \leq m \leq k} \sup_{z \in G} |\partial^m f(z)| + [\partial^k f]_{\gamma} < \infty.$$  

In a similar manner, we can define the Hölder spaces on $[0, T] \times \mathbb{R}$ for $T > 0$ as follows. For $0 < \gamma_1, \gamma_2 < 1$, we define

$$[f]_{\gamma_1, \gamma_2} := \sup_{(t, x) \neq (s, x) \in [0, T] \times \mathbb{R}} \frac{|f(t, x) - f(s, x)|}{|t - s|^{\gamma_1}} + \sup_{(t, x) \neq (t, y) \in [0, T] \times \mathbb{R}} \frac{|f(t, x) - f(t, y)|}{|x - y|^{\gamma_2}}.$$  

Then, $f$ is said to be Hölder $(\gamma_1, \gamma_2)$ continuous on $[0, T] \times \mathbb{R}$ if $\sup_{(t, x) \in [0, T] \times \mathbb{R}} |f(t, x)| + [f]_{\gamma_1, \gamma_2} < \infty$, and the collection of Hölder $(\gamma_1, \gamma_2)$ continuous functions on $[0, T] \times \mathbb{R}$ is denoted by $C^{\gamma_1, \gamma_2}([0, T] \times \mathbb{R})$ with the norm

$$[[f]]_{\gamma_1, \gamma_2} := \sup_{(t, x) \in [0, T] \times \mathbb{R}} |f(t, x)| + [f]_{\gamma_1, \gamma_2}.$$  

Let $k_1, k_2 \in \mathbb{N}$ and $0 < \gamma_1, \gamma_2 < 1$. The Hölder space, denoted by $C^{k_1+\gamma_1, k_2+\gamma_2}([0, T] \times \mathbb{R})$, is defined by the collection of all functions on $([0, T] \times \mathbb{R})$ such that $f$ is $k_1$ times continuously differentiable in $t$ and $k_2$ times continuously differentiable in $x$ and the norm

$$[[f]]_{k_1+\gamma_1, k_2+\gamma_2} := \sum_{0 \leq i \leq k_1, \ 0 \leq j \leq k_2} \sup_{(t, x) \in [0, T] \times \mathbb{R}} |\partial_t^i \partial_x^j f(t, x)| + [\partial_t^{k_1} \partial_x^{k_2} f]_{\gamma_1, \gamma_2} < \infty.$$  

Here, $\partial_t := \frac{\partial}{\partial t}$, $\partial_x := \frac{\partial}{\partial x}$ represents the differentiation operator with respect to $t$ (resp. $x$).

Next, we state useful regularity results for the classical solutions of standard homogeneous and inhomogeneous heat equations on $[0, T] \times \mathbb{R}$.

Recall the Gaussian heat kernel $p(t, x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$. Let us define

$$(P f)(t, x) := \int_{\mathbb{R}} p(t, x - y) f(y) dy, \quad (P * f)(t, x) := \int_0^t \int_{\mathbb{R}} p(t - s, x - y) f(s, y) dy ds.$$  

Lemma 15. [16, Chapter IV, Section 2] Let $T > 0$, $0 < \gamma \notin \mathbb{N}$, and $n, m \in \mathbb{N}_0$. Then,
3. Direct comparisons among $u^{FK}$, $u^{MW}$ and $u^{CS}$

Let $(\Omega, \mathcal{F}, \mathbb{P}^W) = (\mathcal{S}^\prime(\mathbb{R}), \mathcal{B}, \mu)$ be our main probability space and introduce an auxiliary one $(\tilde{\Omega}, \tilde{\mathcal{F}}, \mathbb{P}^B)$ carrying a one-dimensional Brownian motion $\{B_t\}_{t \geq 0}$. Furthermore, let $\mathbb{E}^W(\mathbb{E}^B)$ denote the expectation with respect to $\mathbb{P}^W$ (resp. $\mathbb{P}^B$).

Moreover, we suppose that $u_0 \in L^\infty(\mathbb{R})$ so that we have by uniqueness

$$u^{FK} = u^{MW} = u^{CS} \in C([0, T] \times \mathbb{R}; L^2(\Omega)).$$

The aim of this section is to give an alternative and more direct proof to show $u^{FK} = u^{MW} = u^{CS}$. For the sake of simplicity and consistency, we will denote, for $(t, x) \in [0, T] \times \mathbb{R},$

$$u(0)(t, x) := \int_{\mathbb{R}} p(t, x - y)u_0(y) \, dy.$$

### 3.1. $u^{FK}$ and $u^{MW}$

We recall the multiple Wiener solution of (1):

$$u^{MW}(t, x) = \sum_{n=0}^{\infty} I_n \left(F^{MW}_n(t, x)\right),$$

where

$$F^{MW}_0(t, x) = u(0)(t, x);$$

$$F^{MW}_n(t, x; y_1, \ldots, y_n) = \frac{1}{n!} \int_{[0, t]^n} \prod_{i=1}^{n} p(t - r_{\rho(i)}, x - y_{\rho(i)}) \, dr, \quad n \geq 1,$$

and $\rho$ denotes the permutation of $\{1, \ldots, n\}$ such that $0 < r_{\rho(1)} < \cdots < r_{\rho(n)} < t.$

Moreover, the Feynman-Kac solution is given by

$$u^{FK}(t, x) = \mathbb{E}^B \left[u_0(B^x_t) \exp\{\Psi_{t, x}\}\right],$$

where $\{B^x_t\}_{t \geq 0} := \{B_t + x\}_{t \geq 0}$, for fixed $(t, x) \in [0, T] \times \mathbb{R}$, and

$$\Psi_{t, x}(\omega; \tilde{\omega}) = \int_{\mathbb{R}} L^x_y(t; \tilde{\omega})dW(y; \omega) - \frac{1}{2} \int_{\mathbb{R}} |L^x_y(t; \tilde{\omega})|^2dy, \quad \mathbb{P}^W \otimes \mathbb{P}^B\text{-almost surely}, \quad (11)$$

where $L^x_a(t)$ denotes the local time of $\{B^x_s\}_{s \geq 0}$ at level $a$ and time $t$. Note that the stochastic integral in (11) is well-defined since the function $y \mapsto L^x_y(t; \tilde{\omega})$ is square integrable for each fixed $(t, x, \tilde{\omega})$.

From now on, we will omit the explicit dependence on $(\omega, \tilde{\omega})$ unless there is a risk of confusion. Furthermore, we notice that by definition $\exp\{\Psi_{t, x}\} = \mathcal{E}(L^x(t))$, i.e. the stochastic exponential (e.g. [8]) of the Brownian local time.
Using Lemma 13 and the fact that $D\mathcal{E}(L^x(t)) = \mathcal{E}(L^x(t))L^x(t)$ and $\mathbb{E}^W[\mathcal{E}(L^x(t))] = 1$, we can rewrite $u^F_k$ as

$$u^F_k(t, x) = \sum_{n=0}^{\infty} I_n \left( F^F_n(t, x) \right),$$

where

$$F^F_0(t, x) = u(0)(t, x);$$

$$F^F_n(t, x; y_1, \ldots, y_n) = \frac{1}{n!} \mathbb{E}^B \left[ (L^x(t))^{\otimes n}(y_1, \ldots, y_n)u_0(B^x_t) \right], \quad n \geq 1.$$

We will directly prove $u^F_k = u^M_k$ by showing $F^F_n = F^M_n$ for all $n \geq 0$.

For each $t \in [0, T]$ and $x \in \mathbb{R}$, it is clear that $F^F_0(t, x) = F^M_0(t, x)$. We now let $n \geq 1$. Then, by Fubini lemma,

$$F^F_n(t, x; y_1, \ldots, y_n) = \frac{1}{n!} \int_{[0, t]^n} \mathbb{E}^B \left[ \delta_0(B^x_{s_1} - y_1) \cdots \delta_0(B^x_{s_n} - y_n)u_0(B^x_t) \right] \, ds,$$

where $\delta_\varepsilon(x, y) := \lim_{\varepsilon \to 0} (\pi e)^{-1/2} e^{-|x-y|^2/\varepsilon}$, $x, y \in \mathbb{R}$ as the Dirac-delta function in the sense of distribution. Let $\sigma$ be the permutation of $\{1, \ldots, n\}$ such that $0 < s_{\sigma(1)} < \cdots < s_{\sigma(n)} < t$. Then, we have

$$F^F_n(t, x; y_1, \ldots, y_n) = \frac{1}{n!} \int_{[0,t]^n} \mathbb{E}^B \left[ \delta_0(B^x_{s_{\sigma(1)}} - y_{\sigma(1)}) \cdots \delta_0(B^x_{s_{\sigma(n)}} - y_{\sigma(n)})u_0(B^x_t) \right] \, ds,$$

$$= \frac{1}{n!} \int_{[0,t]^n} \mathbb{E}^B \left[ \delta_0(B^x_{s_{\sigma(1)}} - y_{\sigma(1)}) \cdots \delta_0(B^x_{s_{\sigma(n)}} - y_{\sigma(n)}) \mathbb{E}^B \left[ u_0(B^x_t)|F_{s_{\sigma(n)}} \right] \right] \, ds,$$

$$= \frac{1}{n!} \int_{[0,t]^n} \mathbb{E}^B \left[ \delta_0(B^x_{s_{\sigma(1)}} - y_{\sigma(1)}) \cdots \delta_0(B^x_{s_{\sigma(n)}} - y_{\sigma(n)})u_0(t - s_{\sigma(n)}, B^x_t) \right] \, ds.$$

We know that for any $t \geq s$ and $f \in L^\infty(\mathbb{R})$,

$$\mathbb{E}\left[ \delta_0(B^x_t - y)f(B^x_t)|F_s \right] = \int_{\mathbb{R}} p(t - s, B^s_x - z)f(z)\delta_0(z - y)dz = p(t - s, B^s_x - y)f(y).$$

If we use the identity iteratively, we can obtain

$$F^F_n(t, x; y_1, \ldots, y_n) = \frac{1}{n!} \int_{[0,t]^n} p(s_{\sigma(1)}, y_{\sigma(1)} - x) \times \cdots$$

$$\times p(s_{\sigma(n)} - s_{\sigma(n-1)}, y_{\sigma(n)} - y_{\sigma(n-1)}) u_0(t - s_{\sigma(n)}, x_{\sigma(n)}) \, ds. \quad (12)$$

Let $r_i = t - s_i$ for $i = 1, \ldots, n$ and $\rho(1) = \sigma(n)$, $\rho(2) = \sigma(n-1)$, $\ldots$, $\rho(n) = \sigma(1)$. Then, it is clear that $0 < r_{\rho(1)} < \cdots < r_{\rho(n)} < t$, and we can rewrite (12) as

$$F^F_n(t, x; y_1, \ldots, y_n) = \frac{1}{n!} \int_{[0,t]^n} p(t - r_{\rho(n)}, x - y_{\rho(n)}) \times \cdots$$

$$\times p(r_{\rho(2)} - r_{\rho(1)}, y_{\rho(2)} - y_{\rho(1)}) u_0(r_{\rho(1)}, x_{\rho(1)}) \, dr = F^M_n(t, x; y_1, \ldots, y_n),$$

and thus $u^F_k(t, x) = u^M_k(t, x)$.

3.2. $u^F_k$ and $u^C_k$

Let us now show $u^F_k = u^C_k$ directly. Recall

$$u^C_k(t, x) = \sum_{\alpha \in J} u^C_k(t, x) \xi_{\alpha},$$
where $u^\text{CS}_{(0)}(t, x) = u_{(0)}(t, x)$, and for $|\alpha| = n \geq 1$,

$$
 u^\text{CS}_{\alpha}(t, x) = \sqrt{n!} \int_{\mathbb{R}^n} F_n^\text{CS}(t, x; y_1, \ldots, y_n) \epsilon_\alpha(y_1, \ldots, y_n) \, dy, \quad \alpha \in \mathcal{J}_n = \{\alpha \in \mathcal{J} : |\alpha| = n\},
$$

$$
 F_n^\text{CS}(t, x; y_1, \ldots, y_n) = \int_{\mathbb{T}^n_{[0, t]}} p(t - s_n, x - y_n) \cdots p(s_2 - s_1, y_2 - y_1) u_{(0)}(s_1, y_1) \, ds,
$$

$\mathbb{T}^n_{[0, t]} = \{0 \leq s_1 \leq \cdots \leq s_n \leq t\}$, and $\{\epsilon_\alpha, \alpha \in \mathcal{J}_n\}$ is an orthonormal basis of $L^2_{\text{sym}}(\mathbb{R}^n) := \text{the symmetric part of } L^2(\mathbb{R}^n)$. Specifically,

$$
 \epsilon_\alpha = \frac{1}{\sqrt{n!}} \sum_{\sigma \in \mathcal{P}_n} e_{k_\sigma(1)}(y_1) \cdots e_{k_\sigma(n)}(y_n),
$$

(13)

where $k_\alpha = (k_1, \ldots, k_n)$ be its characteristic vector for any $\alpha \in \mathcal{J}_n$ (e.g. [14, Section 2]).

We have $u^\text{CS}_{(0)}(t, x) = u^\text{F}\text{K}_{(0)}(t, x)$ for $(t, x) \in [0, T] \times \mathbb{R}$. For $n \geq 1$, we have

$$
 \sum_{\alpha \in \mathcal{J}_n} u^\text{CS}_{\alpha}(t, x)\epsilon_\alpha = \sqrt{n!} \sum_{\alpha \in \mathcal{J}_n} \langle F_n^\text{CS}(t, x), \epsilon_\alpha \rangle_{L^2(\mathbb{R}^n)} \epsilon_\alpha.
$$

In fact, it is equal to the orthogonal projection of $F_n^\text{CS}$ on $L^2_{\text{sym}}(\mathbb{R}^n)$, and thus

$$
 \sum_{\alpha \in \mathcal{J}_n} u^\text{CS}_{\alpha}(t, x)\frac{\epsilon_\alpha}{\sqrt{n!}} = \text{Sym} \left( F_n^\text{CS}(t, x) \right) =: F_n^\text{CS}(t, x).
$$

Note that

$$
 F_n^\text{CS}(t, x) = \frac{1}{n!} \sum_{\sigma \in \mathcal{P}_n} \int_{\mathbb{T}^n_{[0, t]}} p(t - s_n, x - y_\sigma(n)) \cdots p(s_2 - s_1, y_\sigma(2) - y_\sigma(1)) u_{(0)}(s_1, y_\sigma(1)) \, ds.
$$

Now we take the $n$-fold Wiener integral $I_n(\bullet)$ on both sides to get

$$
 \sum_{\alpha \in \mathcal{J}_n} u^\text{CS}_{\alpha}(t, x)I_n \left( \frac{\epsilon_\alpha}{\sqrt{n!}} \right) = I_n \left( F_n^\text{CS}(t, x) \right).
$$

(14)

There’s a result due to Itô (see for instance [8, equation 2.2.29]) stating that

$$
 I_n \left( \text{Sym} \bigotimes_{j=1}^\infty E_j^{\otimes \alpha_j} \right) = \prod_{j=1}^\infty H_{\alpha_j} \left( I_1(e_j) \right) \quad \text{and thus} \quad I_n \left( \frac{\epsilon_\alpha}{\sqrt{n!}} \right) = \xi_\alpha.
$$

(15)

Plugging this into (14), we obtain

$$
 \sum_{\alpha \in \mathcal{J}_n} u^\text{CS}_{\alpha}(t, x)\xi_\alpha = I_n \left( F_n^\text{CS}(t, x) \right),
$$

and thus

$$
 u^\text{CS}(t, x) = u^\text{CS}_{(0)}(t, x) + \sum_{n=1}^\infty \sum_{\alpha \in \mathcal{J}_n} u^\text{CS}_{\alpha}(t, x)\xi_\alpha = u^\text{CS}_{(0)}(t, x) + \sum_{n=1}^\infty I_n \left( F_n^\text{CS}(t, x) \right).
$$
To have that \( u^{CS} = u^{FK} \), it only remains to show that \( \widehat{F^CS}_n(t, x) = F^FK_n(t, x) \) for \( n \geq 1 \). Indeed,

\[
F^FK_n(t, x) = \frac{1}{n!} \int_{[0, t]^n} \mathbb{E}^B \left[ \delta(B_{s_1}^x - y_1) \cdots \delta(B_{s_n}^x - y_n)u_0(B_1^x) \right] \, ds
\]

\[
= \frac{1}{n!} \sum_{\sigma \in \mathcal{P}_n} \int_{[0, t]^n} \mathbb{E}^B \left[ \delta(B_{s_1}^x - y_{\sigma(1)}) \cdots \delta(B_{s_n}^x - y_{\sigma(n)})u_0(B_1^x) \right] \, ds
\]

\[
= \frac{1}{n!} \sum_{\sigma \in \mathcal{P}_n} \int_{[0, t] \times \mathbb{R}^n} \left[ \delta(y_{\sigma(1)}) \otimes \cdots \otimes \delta(y_{\sigma(n)}) \right] (y_1, \ldots, y_n)p(s_n - s_{n-1}, y_n - y_{n-1}) \times \cdots
\]

\[
P(s_2 - s_1, y_2 - y_1)p(s_1, y_1 - x)u_0(t - s_n, y_n) \, dy \, ds
\]

\[
= \frac{1}{n!} \sum_{\sigma \in \mathcal{P}_n} \int_{[0, t]} p(s_n - s_{n-1}, y_{\sigma(n)} - y_{\sigma(n-1)}) \cdots p(s_1, y_{\sigma(1)} - x)u_0(t - s_n, y_{\sigma(n)}) \, ds
\]

\[
= F^CS_n(t, x) \quad \text{after rearranging the variables.}
\]

4. Basic regularity of \( u \)

We again assume that \( u_0 \in L^\infty(\mathbb{R}) \) so that \( u = u^{FK} = u^{MW} = u^{CS} \) and we will denote \( \| \cdot \|_\infty := \| \cdot \|_{L^\infty(\mathbb{R})} \). In this section, we will provide a few basic regularity of \( u \) using the Feynman-Kac representation.

**Theorem 16.** For every \((t, x) \in [0, T] \times \mathbb{R}\),

\[
u(t, x) \in \mathcal{G}.
\]

**Proof.** We have for \( \phi \in \mathcal{S}(\mathbb{R}) \),

\[
S(u(t, x))(\phi) = \mathbb{E}^W [u(t, x)\mathcal{E}(\phi)] = \mathbb{E}^W \mathbb{E}^B [u_0(B_t^x)\mathcal{E}(L^x(t))\mathcal{E}(\phi)]
\]

\[
= \mathbb{E}^B \left[ u_0(B_t^x) \exp \left( \int_{\mathbb{R}} L_y^x(t) \phi(y) dy \right) \right],
\]

where the last equality comes from Fubini Lemma and [15, Theorem 5.13].

Let \( P_m : \mathcal{S}'(\mathbb{R}) \to \mathcal{S}'(\mathbb{R}) \) be the orthogonal projection of \( \mathcal{S}'(\mathbb{R}) \) on \( \text{span}\{e_1, \ldots, e_m\}, m \geq 1 \). Then for \( \eta \in \mathcal{S}'_c(\mathbb{R}) := \mathcal{S}'(\mathbb{R}) \oplus i\mathcal{S}'(\mathbb{R}) \) and \( \lambda \in \mathbb{R} \), we have

\[
S(u(t, x))(\lambda P_m \eta) = \mathbb{E}^B \left[ u_0(B_t^x) \exp \left\{ \int_{\mathbb{R}} \lambda P_m \eta(y) L_y^x(t) dy \right\} \right].
\]

Using the fact that for \( \phi, \eta \in \mathcal{S}'_c(\mathbb{R}) \) it holds that \( \langle \eta, P_m \phi \rangle = \langle \phi, P_m \eta \rangle \), we can write

\[
S(u(t, x))(\lambda P_m \eta) = \mathbb{E}^B \left[ u_0(B_t^x) \exp \left\{ \lambda \int_{\mathbb{R}} \eta(y)(P_m L^x(t))(y) dy \right\} \right]
\]

and

\[
|S(u(t, x))(\lambda P_m \eta)|^2 = \left| \mathbb{E}^B \left[ u_0(B_t^x) \exp \left\{ \lambda \int_{\mathbb{R}} (\eta_1(y) + i\eta_2(y))(P_m L^x(t))(y) dy \right\} \right] \right|^2.
\]

By Jensen’s inequality, we have

\[
|S(u(t, x))(\lambda P_m \eta)|^2 \leq \| u_0 \|_{\infty}^2 \mathbb{E}^B \left[ \exp \left\{ \lambda \int_{\mathbb{R}} (\eta_1(y) + i\eta_2(y))(P_m L^x(t))(y) dy \right\} \right]^2.
\]
Since $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$ for any $z_1, z_2 \in \mathbb{C}$,

$$|S(u(t, x))(\lambda P_m \eta)|^2 \leq \|u_0\|_\infty^2 \mathbb{E}^B \left[ \left| \exp \left\{ \lambda \int_\mathbb{R} \eta_1(y)(P_m L^2(t))(y)dy \right\} \right|^2 \right]$$

$$= \|u_0\|_\infty^2 \mathbb{E}^B \left[ \left| \exp \left\{ 2\lambda \eta_1, P_m L^2(t) \right\} \right| \right].$$

Thus,

$$\int_{\mathcal{G}_t(\mathbb{R})} |S(u(t, x))(\lambda P_m \eta)|^2 \nu(d\eta) \leq \|u_0\|_\infty^2 \mathbb{E}^B \left[ \left| \exp \left\{ 2\lambda \eta_1, P_m L^2(t) \right\} \right| \nu(d\eta) \right],$$

where the measure $\nu$ is given by the product measure $\mu_2 \otimes \mu_2$, where $\mu_2$ is the measure on $(\Omega, \mathcal{B})$ with the characteristic function given by:

$$\int_{\mathcal{G}_t(\mathbb{R})} e^{i\langle \omega, \phi \rangle} d\mu(\omega) = e^{-\frac{1}{4}\|\phi\|_0^2}, \quad \varphi \in \mathcal{G}(\mathbb{R}).$$

It is clear that for any $\varphi \in \mathcal{G}(\mathbb{R})$, $\mu_2 \circ \langle \cdot, \varphi \rangle^{-1}$ is a centered Gaussian measure with variance 1/2 as in [8, Lemma 2.1.2]. Therefore,

$$\int_{\mathcal{G}_t(\mathbb{R})} |S(u(t, x))(\lambda P_m \eta)|^2 \nu(d\eta) \leq \|u_0\|_\infty^2 \mathbb{E}^B \left[ \frac{1}{\sqrt{\pi}} \int_\mathbb{R} e^{2\lambda y}|P_m L^2(t)|_0 e^{-y^2} dy \right]$$

$$= \|u_0\|_\infty^2 \mathbb{E}^B \left[ e^{\lambda^2 |P_m L^2(t)|_0^2} \right].$$

Finally, we obtain (by [4, page 178])

$$\lim_{m \to \infty} \int_{\mathcal{G}_t(\mathbb{R})} |S(u(t, x))(\lambda P_m \eta)|^2 \nu(d\eta) \leq \|u_0\|_\infty^2 \mathbb{E} \left[ e^{\lambda^2 |L^2(t)|^2} \right] < \infty, \quad \forall \lambda \in \mathbb{R},$$

which implies by [5, Corollary 5.1], $u(t, x)$ belongs to $\mathcal{G}$. \hfill \Box

Next, we state the basic Hölder regularity of $u$ both in time and space.

**Theorem 17.** Let $0 < \varepsilon < 1/2$ be arbitrary and $C$ be a constant.

(i) Assume that $u_0 \equiv C$. Then,

$$u \in C^{3/4-\varepsilon, 1/2-\varepsilon}([0, T] \times \mathbb{R}).$$

(ii) Assume that $u_0 \not\equiv C$ and $u_0 \in L^\infty(\mathbb{R})$ is (globally) Lipschitz continuous on $\mathbb{R}$. Then,

$$u \in C^{1/2-\varepsilon, 1/2-\varepsilon}([0, T] \times \mathbb{R}).$$

**Proof.** Let $\| \cdot \|_p := (\mathbb{E}^W \mathbb{E}^B | \cdot |^p)^{1/p}$ be the norm on the Banach space $L^p(\mathbb{P}^W \otimes \mathbb{P}^B)$ for $p \geq 1$. From [26], we have

$$\sup_{(t, x) \in [0, T] \times \mathbb{R}} \| \exp \{ \Psi_{t,x} \} \|_p < \infty. \quad (17)$$

Also, (11) conditional on $B$, becomes

$$\Psi_{t,x} \sim N \left( -\frac{1}{2} \int_\mathbb{R} |L_a(t)|^2 da, \int_\mathbb{R} |L_a(t)|^2 da \right), \quad (18)$$

where $N(\mu, \sigma^2)$ denotes the Gaussian random variable with mean $\mu$ and variance $\sigma^2$. 
(i) Let $u_0$ be a constant. Then, using the fact $|e^x - e^y| \leq (e^x + e^y)|x - y|$ for all $x, y \in \mathbb{R}$, Cauchy-Schwarz inequality, Minkowski inequality, (17), and (18), we can obtain for $p \geq 2$.

$$
\mathbb{E}^W [ |u(t, x) - u(s, y)|^p ] \leq c_p \left\{ \mathbb{E}^B \mathbb{E}^W [ |\Psi_{t,x} - \Psi_{s,y}|^2 ]^{1/2} \right\}^p = c_p \| \Psi_{t,x} - \Psi_{s,y} \|^p_2,
$$

for some $c_p > 0$. Also, the triangular inequality implies

$$
\mathbb{E}^W [ |u(t, x) - u(s, y)|^p ] \leq c_p \left\{ \| \Psi_{t,x} - \Psi_{t,y} \|_2 + \| \Psi_{t,y} - \Psi_{s,y} \|_2 \right\}^p = c_p \left( A_1^{1/2} + A_2^{1/2} \right)^p.
$$

Let us now work with

$$
A_1 = \| \Psi_{t,x} - \Psi_{t,y} \|_2^2 = \mathbb{E}^B \mathbb{E}^W [ \Psi_{t,x}^2 - 2\Psi_{t,x} \Psi_{t,y} + \Psi_{t,y}^2 ].
$$

By (18), we have

$$
A_1 = \mathbb{E}^B \left[ 2 \int_\mathbb{R} |La(t)|^2 da + \frac{1}{2} \left( \int_\mathbb{R} |La(t)|^2 da \right)^2 - 2\mathbb{E}^W [ \Psi_{t,x} \Psi_{t,y} ] \right].
$$

Recall the Dirac-delta function $\delta_{x,y}(y) = \lim_{\varepsilon \to 0} (\pi \varepsilon)^{-1/2} e^{-|x-y|^2/\varepsilon}$, $x, y \in \mathbb{R}$. Since

$$
\mathbb{E}^B \mathbb{E}^W [ \Psi_{t,x} \Psi_{t,y} ] = \mathbb{E}^B \left[ \int_0^t \int_0^t \delta_0(Bu - Br - (x - y)) dudr + \frac{1}{4} \left( \int_\mathbb{R} |La(t)|^2 da \right)^2 \right],
$$

we get

$$
A_1 = \mathbb{E}^B \left[ 2 \int_\mathbb{R} |La(t)|^2 da - 2 \int_0^t \int_0^t \delta_0(Bu - Br - (x - y)) dudr \right].
$$

The next step is done rigorously (See [10] for instance) by the translation invariant property of the Lebesgue measure:

$$
A_1 = \mathbb{E}^B \left[ \int_\mathbb{R} |La_{-x}(t)|^2 da - 2 \int_0^t \int_0^t \delta_0(Bu - Br - (x - y)) dudr + \int_\mathbb{R} |La_{-y}(t)|^2 da \right],
$$

and this yields, by [10, Proposition 9.2],

$$
A_1 = \mathbb{E}^B \left[ \int_\mathbb{R} |La_{-x}(t) - La_{-y}(t)|^2 da \right] = 4t|x - y| + O(|x - y|^2),
$$

which implies $u(t, \bullet)$ is almost Hölder $1/2$ continuous uniformly for all $t \in [0, T]$.

On the other hand, let us compute, for $0 \leq s \leq t \leq T$,

$$
A_2 = \mathbb{E}^B \mathbb{E}^W [ \Psi_{t,y}^2 - 2\Psi_{t,y} \Psi_{s,y} + \Psi_{s,y}^2 ]
$$

$$
= \mathbb{E}^B \left[ \int_\mathbb{R} |La(t)|^2 da + \frac{1}{4} \left( \int_\mathbb{R} |La(t)|^2 da \right)^2 + \int_\mathbb{R} |La(s)|^2 da + \frac{1}{4} \left( \int_\mathbb{R} |La(s)|^2 da \right)^2 \right]
$$

$$
- 2\mathbb{E}^W [ \Psi_{t,y} \Psi_{s,y} ] .
$$

Since

$$
\mathbb{E}^B \mathbb{E}^W [ \Psi_{t,y} \Psi_{s,y} ] = \mathbb{E}^B \left[ \int_0^t \int_0^s \delta_0(Bu - Br) dudr + \frac{1}{4} \left( \int_\mathbb{R} |La(t)|^2 da \right) \left( \int_\mathbb{R} |La(s)|^2 da \right) \right],
$$
we have

\[A_2 = \mathbb{E}^B \left[ \int_{\mathbb{R}} |L_a(t)|^2 da - 2 \int_0^t \int_0^s \mathbb{E}^B \left( \int_{\mathbb{R}} |L_a(t)|^2 da \right) \mathbb{E}^B \left( \int_{\mathbb{R}} |L_a(s)|^2 da \right) + \int_0^t \int_0^s \delta_0(B_r - B_z) dr dz \right]
\]

\[= \mathbb{E}^B \left[ \int_{\mathbb{R}} |L_a(t)|^2 da - 2 \int_0^t \int_0^s \delta_0(B_r - B_z) dr dz + \int_{\mathbb{R}} |L_a(s)|^2 da \right]
\]

\[+ \frac{1}{4} \mathbb{E}^B \left[ \left( \int_{\mathbb{R}} |L_a(t)|^2 da \right)^2 - 2 \left( \int_{\mathbb{R}} |L_a(t)|^2 da \right) \left( \int_{\mathbb{R}} |L_a(s)|^2 da \right) + \left( \int_{\mathbb{R}} |L_a(s)|^2 da \right)^2 \right]
\]

We note that

\[\mathbb{E}^B \left[ \int_{\mathbb{R}} |L_a(t)|^2 da \right] = \mathbb{E}^B \left[ \int_0^t \int_0^t \delta_0(B_r - B_z) dr dz \right],
\]

which implies

\[A_2 = \mathbb{E}^B \left[ \int_0^t \int_0^t \delta_0(B_r - B_z) dr dz - 2 \int_0^t \int_0^s \delta_0(B_r - B_z) dr dz + \int_0^s \int_0^s \delta_0(B_r - B_z) dr dz \right]
\]

\[+ \frac{1}{4} \mathbb{E}^B \left( \int_0^t \int_0^t \delta_0(B_r - B_z) dr dz - \int_0^s \int_0^s \delta_0(B_r - B_z) dr dz \right)^2
\]

\[= \mathbb{E}^B \left[ \int_s^t \int_s^t \delta_0(B_r - B_z) dr dz \right] + \frac{1}{4} \mathbb{E}^B \left( \int_0^t \int_0^t \delta_0(B_r - B_z) dr dz - \int_0^s \int_0^s \delta_0(B_r - B_z) dr dz \right)^2
\]

\[=: A_3 + A_4.
\]

We can easily compute \(A_3:\)

\[A_3 = \int_s^t \int_s^t (2\pi |r - z|)^{-1/2} dr dz = C(t - s)^{3/2} \text{ for some } C > 0 \text{ independent of } x.
\]

For \(A_4\), we have

\[4A_4 = \int_s^t \int_s^t \int_s^t \int_s^t \mathbb{E}^B (\delta_0(B_z - B_r) \delta_0(B_q - B_p)) dpdqdrdz
\]

\[\text{symmetry} \quad = 4! \int_s^t \int_s^t \int_s^t \int_s^t \int_s^t \mathbb{E}^B [\delta_0(B_z - B_r)] \mathbb{E}^B [\delta_0(B_q - B_p)] dpdqdrdz
\]

\[= 4! \int_s^t \int_s^t \int_s^t \int_s^t \frac{1}{\sqrt{2\pi(z - r)}} \frac{1}{\sqrt{2\pi(q - p)}} dpdqdrdz
\]

\[\leq 4! \int_s^t \int_s^t \frac{1}{\sqrt{2\pi(z - r)}} \left( \int_s^t \int_s^t \frac{1}{\sqrt{2\pi(q - p)}} \right) dpdqdrdz
\]

\[= C(t - s)^{3} \text{ for some } C > 0 \text{ independent of } x.
\]

Combining all together, we obtain

\[A_2 \leq C(t - s)^{3/2},
\]

which implies \(u(\bullet, x)\) is almost Hölder 3/4 continuous uniformly for all \(x \in \mathbb{R}\).
(ii) If $u_0$ is not a constant function on $\mathbb{R}$, then we have
\[
\mathbb{E}^W[|u(t,x) - u(s,y)|^p] = \mathbb{E}^W \left| \mathbb{E}^B \left( u_0(x + B_t \exp (\Psi_{t,x})) - u_0(y + B_s \exp (\Psi_{s,y})) \right) \right|^p \\
= \mathbb{E}^W \left( \mathbb{E}^B \left( (u_0(x + B_t) - u_0(y + B_s)) \exp (\Psi_{t,x}) \right) \right) \\
+ \mathbb{E}^B \left( u_0(y + B_s) (\exp (\Psi_{t,x}) - \exp (\Psi_{s,y})) \right) \right|^p.
\]
Since $|f + g|^p \leq 2^{p-1} (|f|^p + |g|^p)$ for $p \geq 1$,
\[
\mathbb{E}^W[|u(t,x) - u(s,y)|^p] \leq 2^{p-1} \left( \mathbb{E}^W \left| \mathbb{E}^B \left( (u_0(x + B_t) - u_0(y + B_s)) \exp (\Psi_{t,x}) \right) \right|^p \\
+ \mathbb{E}^W \left| \mathbb{E}^B \left( u_0(y + B_s) (\exp (\Psi_{t,x}) - \exp (\Psi_{s,y})) \right) \right|^p \right) \\
=: 2^{p-1} \left( \bar{A}_1 + \bar{A}_2 \right).
\]
For $\bar{A}_1$, by Cauchy-Schwarz inequality,
\[
\bar{A}_1 \leq \left( \mathbb{E}^B (u_0(x + B_t) - u_0(y + B_s))^2 \right)^{p/2} \mathbb{E}^W \left( \mathbb{E}^B (\exp (2\Psi_{t,x})) \right)^{p/2}.
\]
Since $u_0$ is Lipschitz continuous on $\mathbb{R}$, we have
\[
\bar{A}_1 \leq (|t - s| + (x - y)^2)^{p/2} \mathbb{E}^W \left( \mathbb{E}^B (\exp (2\Psi_{t,x})) \right)^{p/2} \\
\leq \left( |t - s|^{1/2} + |x - y| \right)^p \mathbb{E}^W \left( \mathbb{E}^B (\exp (2\Psi_{t,x})) \right)^{p/2}.
\]
By Minkowski inequality, Hölder inequality for $p \geq 2$, and (17), we also have
\[
\bar{A}_1 \leq \left( |t - s|^{1/2} + |x - y| \right)^p \mathbb{E}^W \mathbb{E}^B (\exp (p\Psi_{t,x})) < \infty.
\]
For $\bar{A}_2$, since $u_0 \in L^\infty(\mathbb{R})$, we can apply the same argument in (i). As a result, we can say that $u$ is Hölder continuous almost $1/2$ both in time and space. \hfill \Box

As we argued in the introduction, one expects that we can still improve the spatial regularity of $u$. We will derive our desired result in Section 5.

5. The spatial derivative of $u$

As we anticipated in the introduction, we expect that $u(t, \bullet) \in C^{3/2-\varepsilon}(\mathbb{R})$ for any small $\varepsilon > 0$. To verify this assertion, we first compute the spatial derivative of $u$ using the Feynman-Kac representation and then find its chaos expansion to see if it is well-defined in $\mathcal{G}$ and to get the optimal spatial regularity of $u$.

Let us start with a useful Lemma. The following result will serve as a key idea for finding $\partial_x u(t,x)$.

**Lemma 18.** For fixed $(t,x)$ the map $\tilde{\omega} \supset \tilde{\Omega} \mapsto \tilde{\Phi}_{t,x}(\tilde{\omega}) \in (S)^*$ given by
\[
\tilde{\Phi}_{t,x}(\tilde{\omega}) = \mathcal{E}(L^x(t; \tilde{\omega})) \circ \left[ u_0(B^x_t(\tilde{\omega})) + u_0(B^x_t) I_1 (\partial_x L^x(t; \tilde{\omega})) \right],
\]
is Bochner integrable in $(S)^*$. Here, $\partial_x L^x(t) \in \mathcal{S}'(\mathbb{R})$ denotes the pathwise distributional derivative of the local time of $\{B^x_t\}_{t \geq 0}$.

**Proof.** This immediately follows from [7, Theorem 4.51] and the facts
can conclude that Φ

We also need to show ∂x convergence theorem (DCT), we obtain is entire for any φ, φ

In order to prove that u(t, x), we start by computing the S-transform of u. From (16), for φ ∈ 𝔖(R), we have

\[ S(u(t,x))(φ) = \mathbb{E}^B \left[ u_0(B_t^x) \exp \left( \int_{\mathbb{R}} L_y^x(t)φ(y)dy \right) \right] = \mathbb{E}^B \left[ u_0(B_t^x) \exp \left( \int_0^t φ(B_s^x)ds \right) \right], \tag{19} \]

where the last equality follows by the occupation time formula.

It is clear that x ∈ R → S(u(t,x))(φ) is continuous for all φ ∈ 𝔖(R), and |S(u(t,x))(φ)| ≤ K_1 e^{K_2|φ|_p^2} for some K_1, K_2, p > 0. Then, by [18, Lemma A.1.2], we can see that u(t, x) = S^{-1}(S(u(t,x))) is weakly continuously differentiable in (S)* (with respect to the x variable).

In order to prove that u(t, x) is weakly continuously differentiable in (S)* we must first take the spatial derivative on both sides of (19). Using the fact φ, φ’ ∈ 𝔖(R) ⊂ L^∞(R), by dominated convergence theorem (DCT), we obtain

\[ \partial_x S(u(t,x))(φ) = \mathbb{E}^B \left[ u_0'(B_t^x) \exp \left( \int_0^t φ(B_s^x)ds \right) + u_0(B_t^x) \exp \left( \int_0^t φ(B_s^x)ds \right) \times \int_0^t φ'(B_s^x)ds \right], \]

and it is clear that the map x → ∂x S(u(t,x))(φ) is continuous for all φ ∈ 𝔖(R).

We also need to show ∂x S(u(t,x)) is a U-functional (see Definition 5). By direct computation, we can verify that, as in the proof of Lemma 18,

\[ |∂x S(u(t,x))(φ)| \leq K_1 e^{K_2|φ|_p^2}, \]

where K_1, K_2, p are positive real constants. Also, it is clear that the map z → ∂x S(u)(zφ + η) is entire for any φ, η ∈ 𝔖(R) and z ∈ C. Hence, ∂x S(u(t,x)) is indeed a U-functional, and thus there exists a unique Φ ∈ (S)* such that ∂x S(u(t,x)) = S(Φ); then from [18, Lemma A.3], we can conclude that u(t, x) is weakly continuously differentiable in the Hida distribution space (S)*.
Following the aforementioned reference, the weak spatial derivative \(\partial_x u\) of \(u\) is defined as the unique element in \((S)^*\) such that

\[
S(\partial_x u(t, x))(\phi) = \partial_x S(u(t, x))(\phi).
\]

Using Lemma 18, we can see that \(\partial_x S(u(t, x)) = \mathbb{E}^B \left( S(\tilde{\Phi}_{t,x}) \right)\), and furthermore we have that \(\mathbb{E}^B \left( S(\tilde{\Phi}_{t,x}) \right) = S(\mathbb{E}^B (\tilde{\Phi}_{t,x}))\) since the Bochner integral \(\mathbb{E}^B\) and the S-transform can be interchanged (e.g. [7, Theorem 4.51]). Finally, by Theorem 6, we can conclude

\[
\partial_x u(t, x) = \mathbb{E}^B \left[ \mathcal{E}(L^x(t)) \circ \left\{ I_0(u_0'(B^x_t)) + u_0(B^x_t)I_1(\partial_x L^x(t)) \right\} \right],
\]

where \(\mathbb{E}^B\) must be understood as a Bochner integral in \((S)^*\). \(\Box\)

From this result, we can only say that \(\partial_x u(t, x) \in (S)^*\) for each \((t, x) \in [0, T] \times \mathbb{R}\). But, in the following subsection, we will show that \(\partial_x u(t, x) \in \mathcal{G}\) using its chaos decomposition, and furthermore, we will investigate its Hölder regularity.

**5.1. Chaos decomposition for \(\partial_x u\)**

Let us find the chaos expansion of

\[
\partial_x u(t, x) = \mathbb{E}^B \left[ \mathcal{E}(L^x(t)) \circ \left\{ I_0(u_0'(B^x_t)) + u_0(B^x_t)I_1(\partial_x L^x(t)) \right\} \right],
\]

and notice that in this context, the Wiener integral of order 0 equals the identity operator, but nonetheless we explicitly write \(I_0\) for notational convenience.

By Lemma 13, we have

\[
\mathcal{E}(L^x(t)) = \sum_{n=0}^{\infty} \frac{1}{n!} I_n \left( L^x(t)^{\otimes n} \right), \quad \text{convergent in } L^2(\mathbb{P}^W),
\]

and by the definition of Wick product, we see that

\[
\mathcal{E}(L^x(t)) \circ \left\{ I_0(u_0'(B^x_t)) + u_0(B^x_t)I_1(\partial_x L^x(t)) \right\} = \sum_{n=0}^{\infty} I_n(h_n(t, x)), \quad \text{convergent in } (S)^*,
\]

where \(h_0(t, x) = u_0'(B^x_t)\), and

\[
\mathcal{S}'(\mathbb{R}^n) \ni h_n(t, x; \bullet) = u_0'(B^x_t) \Sym \left[ \left( \frac{L^x(t)^{\otimes(n-1)}}{(n-1)!} \right) \otimes \partial_x L^x(t) \right] (\bullet) + u_0'(B^x_t) \left( \frac{L^x(t)^{\otimes n}}{n!} \right) (\bullet), \quad n \geq 1.
\]

It is known that (e.g. [15, Chapter 13.3]) if \(\Psi(u) = \sum_{n=0}^{\infty} I_n(F_n(u))\) is Bochner integrable on \((M, \sigma(M), m)\), then \(F_n\) is Bochner integrable on \((M, \sigma(M), m)\), and it holds that

\[
\int_M \Psi(u)m(du) = \sum_{n=0}^{\infty} I_n \left( \int_M F_n(u)m(du) \right).
\]

In our case, letting \((M, \sigma(M), m) = (\tilde{\Omega}, \tilde{\mathcal{F}}, \mathbb{P}^B)\), this would imply that

\[
\partial_x u(t, x) = \sum_{n=0}^{\infty} I_n \left( \mathbb{E}^B [h_n(t, x)] \right),
\]

where \(\mathbb{E}^B\) should be understood as a Bochner integral in \(\mathcal{S}'(\mathbb{R}^n)\).
We can easily check that the first term of $\partial_x u(t, x)$ is $\mathbb{E}^B [u_0(B^x_t)] = \partial_x u(0)(t, x)$. Let’s check the general $n$-th term of $\partial_x u(t, x)$ for $n \geq 1$. Since $h_n(t, x; \bullet)$ is a symmetric element of $\mathcal{S}'(\mathbb{R}^n)$, we can expand it with respect to $\{\epsilon_\alpha : \alpha \in J_n\}$ as

$$h_n(t, x; \bullet) = \sum_{\alpha \in J_n} \langle h_n(t, x), \epsilon_\alpha \rangle_n \epsilon_\alpha(\bullet), \quad \text{convergence in } \mathcal{S}'(\mathbb{R}^n),$$

where $\langle \bullet, \bullet \rangle_n$ is the bilinear product between $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$, and

$$\epsilon_\alpha = \frac{1}{\sqrt{n!\alpha!}} \sum_{\sigma \in P_n} e_{k_0(1)}(y_1) \cdots e_{k_\alpha(n)}(y_n)$$

as defined in (13).

It is clear by direct calculations that $\langle \text{Sym} f, \text{Sym} g \rangle_n = \langle f, \text{Sym}^2 g \rangle_n = \langle f, \text{Sym} g \rangle_n$. Then, we have

$$\langle h_n(t, x), \epsilon_\alpha \rangle_n = \frac{u_0(B^x_t)}{\sqrt{n!\alpha!}(n-1)!} \left( \left[ L^x(t) \otimes (n-1) \otimes \partial_x L^x(t) \right] \sum_{\sigma \in P_n} \left[ e_{k_0(1)} \otimes \cdots \otimes e_{k_\alpha(n)} \right] \right)_n$$

$$+ \frac{u'_0(B^x_t)}{\sqrt{n!\alpha!} n!} \int_{\mathbb{R}_n} L^x(t) \otimes (y_1, \ldots, y_n) \times \sum_{\sigma \in P_n} \left[ e_{k_0(1)} \otimes \cdots \otimes e_{k_\alpha(n)} \right] (y_1, \ldots, y_n) dy$$

$$= \frac{u_0(B^x_t)}{\sqrt{n!\alpha!}(n-1)!} \sum_{\sigma \in P_n} \int_{[0,t]_n} e_{k_0(1)} \otimes \cdots \otimes e_{k_\alpha(n)}(B^x_{s_1}, \ldots, B^x_{s_n}) ds$$

$$+ \frac{u'_0(B^x_t)}{\sqrt{n!\alpha!} n!} \sum_{\sigma \in P_n} \int_{[0,t]_n} e_{k_0(1)} \otimes \cdots \otimes e_{k_\alpha(n)}(B^x_{s_1}, \ldots, B^x_{s_n}) ds, \quad \text{(21)}$$

where in the last expression we used the occupation time formula and the fact that $-\partial_x L^x$ equals distributional derivative of the Brownian local time.

Also, taking $\mathbb{E}^B$ on both sides of (20), we have

$$\mathbb{E}^B [h_n(t, x; \bullet)] = \mathbb{E}^B \left[ \sum_{\alpha \in J_n} \langle h_n(t, x), \epsilon_\alpha \rangle_n \epsilon_\alpha(\bullet) \right]. \quad \text{(22)}$$

At this point, we need the following Lemma to compute (22).

**Lemma 20.** [2, Lemma 11.45] Let $f : M \to X$ be Bochner integrable on $(M, \sigma(M), m)$ in $X$ and let $Y$ be a Banach space. If $T : X \to Y$ is a bounded operator, then $Tf : M \to Y$ is Bochner integrable on $(M, \sigma(M), m)$ in $Y$ and it holds that

$$\int_M Tf dm = T \left( \int_M f dm \right).$$

In our case, we set $\mathcal{S}'_{\text{SYM}}(\mathbb{R}^n) :=$ the symmetric part of $\mathcal{S}'(\mathbb{R}^n)$ and $T : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'_{\text{SYM}}(\mathbb{R}^n)$ equals the orthogonal projection on $\mathcal{S}'_{\text{SYM}}(\mathbb{R}^n)$, which is clearly a bounded linear operator. Even though if it is well-known that the space of tempered distributions is not a Banach space, we can think of $\mathcal{S}'(\mathbb{R})$ as the inductive limit of a family of Hilbert spaces (e.g [15, Section 3.2] or [21]), and an analogous reasoning extends to the multi-dimensional case; thus the lemma above holds true by letting $Y$ be some of those Hilbert spaces.
Then, (22) becomes
\[
\mathbb{E}^B[h_n(t, x; \bullet)] = \mathbb{E}^B \left[ \sum_{\alpha \in \mathcal{J}_n} \langle h_n(t, x), \epsilon_\alpha \rangle_n \epsilon_\alpha(\bullet) \right] = \sum_{\alpha \in \mathcal{J}_n} \langle \mathbb{E}^B[h_n(t, x)], \epsilon_\alpha \rangle_n \epsilon_\alpha(\bullet).
\]

It is known that if a function is Bochner integrable, then its Pettis and Bochner integrals coincide (see for instance the discussion on page 80 of [12]). Therefore, by definition of the Pettis integral, we have
\[
\langle \mathbb{E}^B[h_n(t, x)], \epsilon_\alpha \rangle_n = \mathbb{E}^B[\langle h_n(t, x), \epsilon_\alpha \rangle_n].
\]

Hence, we have
\[
\mathbb{E}^B[h_n(t, x; \bullet)] = \sum_{\alpha \in \mathcal{J}_n} \mathbb{E}^B[\langle h_n(t, x), \epsilon_\alpha \rangle_n \epsilon_\alpha(\bullet)].
\]

Next we compute \( \mathbb{E}^B[\langle h_n(t, x), \epsilon_\alpha \rangle_n] \), and so far from (21), we have
\[
\mathbb{E}^B[\langle h_n(t, x), \epsilon_\alpha \rangle_n] = \mathbb{E}^B \left[ \frac{u_0(B^x_t)}{\sqrt{n! \alpha!(n - 1)!}} \int_{[0, t]^n} e_{k_{\alpha}(1)} \otimes \cdots \otimes e'_{k_{\alpha}(n)}(B^x_{s_1}, \ldots, B^x_{s_n}) \right. \]
\[
\cdot \left. + \frac{u'_0(B^x_t)}{\sqrt{n! \alpha!}} \int_{[0, t]^n} e_{k_{\alpha}(1)} \otimes \cdots \otimes e_{k_{\alpha}(n)}(B^x_{s_1}, \ldots, B^x_{s_n}) \right] \cdot \] \quad (23)

To simplify the expression in (23), we first observe
\[
\partial_x [e_{k_1}(B^x_{s_1}) \cdots e_{k_n}(B^x_{s_n})] = \left[ e'_{k_1}(B^x_{s_1}) \cdots e_{k_n}(B^x_{s_n}) \right] + \cdots + \left[ e_{k_1}(B^x_{s_1}) \cdots e'_{k_n}(B^x_{s_n}) \right].
\]

Since the Lebesgue measure is invariant under rotations, we see that for any \( f : [0, t]^n \rightarrow \mathbb{R} \), it holds that
\[
\int_{[0, t]^n} f(s_1, \ldots, s_n) \, ds = \int_{[0, t]^n} \text{Sym} f(s_1, \ldots, s_n) \, ds.
\]

Thus, for any permutation \( \sigma \) of \( \{1, \ldots, n\} \), we have
\[
\int_{[0, t]^n} \frac{u_0(B^x_t)}{(n - 1)!} \sum_{\sigma \in \mathcal{P}_n} e_{k_{\sigma}(1)} \otimes \cdots \otimes e'_{k_{\sigma}(n)}(B^x_{s_1}, \ldots, B^x_{s_n}) \, ds = \int_{[0, t]^n} u_0(B^x_t) \partial_x [e_{k_1}(B^x_{s_1}) \cdots e_{k_n}(B^x_{s_n})] \, ds,
\]

and
\[
\int_{[0, t]^n} \frac{u'_0(B^x_t)}{n!} \sum_{\sigma \in \mathcal{P}_n} e_{k_{\sigma}(1)} \otimes \cdots \otimes e_{k_{\sigma}(n)}(B^x_{s_1}, \ldots, B^x_{s_n}) \, ds = \int_{[0, t]^n} u'_0(B^x_t) e_{k_1}(B^x_{s_1}) \cdots e_{k_n}(B^x_{s_n}) \, ds,
\]

which implies
\[
\mathbb{E}^B[\langle h_n(t, x), \epsilon_\alpha \rangle_n] = \frac{1}{\sqrt{n! \alpha!}} \mathbb{E}^B \left[ \int_{[0, t]^n} \partial_x [u_0(B^x_t) e_{k_1}(B^x_{s_1}) \cdots e_{k_n}(B^x_{s_n})] \, ds \right],
\]

where again \( \mathbb{T}^n_{[0, t]} = \{0 \leq s_1 \leq \cdots \leq s_n \leq t\} \).
Furthermore, we notice that

$$\int_{[0,t]^{n}} \left[ e_{k_1}(B_{s_1}^x) \cdots e_{k_n}(B_{s_n}^x) \right] \, ds = \int_{[0,t]^{n}} \text{Sym} \left[ e_{k_1}(B_{s_1}^x) \cdots e_{k_n}(B_{s_n}^x) \right] \, ds$$

$$= \int_{[0,t]^{n}} \frac{1}{n!} \sum_{\sigma \in \mathcal{P}_n} \left[ e_{k_1}(B_{s_{\sigma(1)}}^x) \cdots e_{k_n}(B_{s_{\sigma(n)}}^x) \right] \, ds = \sqrt{\alpha!} \int_{[0,t]^{n}} \partial_x e_{\alpha}(B_{s_1}^x, \ldots, B_{s_n}^x) \, ds$$

and similarly,

$$\int_{[0,t]^{n}} \partial_x \left[ e_{k_1}(B_{s_1}^x) \cdots e_{k_n}(B_{s_n}^x) \right] \, ds = \sqrt{\alpha!} \int_{[0,t]^{n}} \partial_x e_{\alpha}(B_{s_1}^x, \ldots, B_{s_n}^x) \, ds.$$

Therefore, (24) becomes

$$\mathbb{E}^B \left[ \langle h_n(t,x), e_{\alpha} \rangle_n \right] = \mathbb{E}^B \left[ \int_{[0,t]^{n}} \partial_x e_{\alpha}(B_{s_1}^x, \ldots, B_{s_n}^x) \, ds \right]$$

$$= \int_{[0,t]^{n}} \mathbb{E}^B \left[ \partial_x \left[ u_0(B_t^x) e_{\alpha}(B_{s_1}^x, \ldots, B_{s_n}^x) \right] \right] \, ds,$$

where the second equality holds true by Fubini lemma.

By conditioning iteratively on the filtration of $B$ at the sites $s_n, s_{n-1}, \ldots, s_1$, we can rewrite (25) as

$$\mathbb{E}^B \left[ \langle h_n(t,x), e_{\alpha} \rangle_n \right] = \int_{[0,t]^{n}} \int_{\mathbb{R}^n} \partial_x e_{\alpha}(y_1 + x, y_2, \ldots, y_n) p(s_n - s_{n-1}, y_n - y_{n-1}) \times$$

$$\cdots \times p(s_1, y_1) u_0(t - s_n, y_n) dy \, ds$$

$$= \int_{[0,t]^{n}} \int_{\mathbb{R}^n} \partial_y e_{\alpha}(y_1 + x, y_2, \ldots, y_n) p(s_n - s_{n-1}, y_n - y_{n-1}) \times$$

$$\cdots \times p(s_1, y_1) u_0(t - s_n, y_n) dy \, ds$$

$$= - \int_{[0,t]^{n}} \int_{\mathbb{R}^n} e_{\alpha}(y_1, y_2, \ldots, y_n) p(s_n - s_{n-1}, y_n - y_{n-1}) \times$$

$$\cdots \times \partial_y p(s_1, y_1 - x) u_0(t - s_n, y_n) dy \, ds.$$

Noticing that $-\partial_y p(s, y - x) = \partial_x p(s, x - y)$ and letting $r_i = t - s_{n+1-i}$ for $i \in \{1, \ldots, n\}$, the equation (26) becomes

$$\int_{[0,t]^{n}} \int_{\mathbb{R}^n} e_{\alpha}(y_1, y_2, \ldots, y_n) p(r_2 - r_1, y_n - y_{n-1}) \times \cdots \times \partial_x p(t - r_n, x - y_1) u_0(r_1, y_n) dy \, dr,$$

which yields, after relabeling the $y$'s,

$$\mathbb{E}^B \left[ \langle h_n(t,x), e_{\alpha} \rangle_n \right] = \int_{[0,t]^{n}} \int_{\mathbb{R}^n} \partial_x p(t - r_n, x - y_n) p(r_n - r_{n-1}, y_n - y_{n-1}) \times \cdots \times p(r_2 - r_1, y_2 - y_1)$$

$$\cdots \times e_{\alpha}(y_1, y_2, \ldots, y_n) u_0(r_1, y_1) dy \, dr.$$
Using the definition of $\varepsilon_\alpha$, it equals
\[
\frac{1}{\sqrt{\alpha!}} \frac{1}{\sqrt{n!}} \sum_{\sigma \in \mathcal{P}_n} \int_{T^p_n} \int_{\mathbb{R}^n} \partial_x p(t-r_n, x-y_n)p(r_n-r_{n-1}, y_n-y_{n-1}) \times \cdots \times p(r_2-r_1, y_2-y_1) \\
\cdots \times e_{k_{\sigma}(1)}(y_1) \times \cdots \times e_{k_{\sigma(n)}}(y_n) u(0)(r_1, y_1) dy \, dr =: \frac{1}{\sqrt{n!}} R_\alpha(t, x), \quad |\alpha| = n \geq 1.
\]

Putting all together, we obtain
\[
\mathbb{E}^B [h_n(t, x; \bullet)] = \sum_{\alpha \in \mathcal{J}_n} \frac{1}{\sqrt{n!}} R_\alpha(t, x) \varepsilon_\alpha(\bullet)
\]
and
\[
I_n (\mathbb{E}^B [h_n(t, x)]) = \sum_{\alpha \in \mathcal{J}_n} R_\alpha(t, x) I_n \left( \frac{\varepsilon_\alpha}{\sqrt{n!}} \right) = \sum_{\alpha \in \mathcal{J}_n} R_\alpha(t, x) \xi_\alpha \quad \text{by (15)}.
\]

Finally, we have the chaos expansion of $\partial_x u(t, x)$ as follows:
\[
\partial_x u(t, x) = \sum_{n=0}^{\infty} I_n (\mathbb{E}^B [h_n(t, x)]) = \sum_{n=0}^{\infty} \sum_{\alpha \in \mathcal{J}_n} R_\alpha(t, x) \xi_\alpha = \sum_{\alpha \in \mathcal{J}} R_\alpha(t, x) \xi_\alpha,
\]
where $R_{\{0\}}(t, x) = \partial_x u(0)(t, x)$, and for $|\alpha| = n \geq 1$,
\[
R_\alpha(t, x) = \frac{1}{\sqrt{\alpha!}} \sum_{\sigma \in \mathcal{P}_n} \int_{T^p_n} \int_{\mathbb{R}^n} \partial_x p(t-r_n, x-y_n)p(r_n-r_{n-1}, y_n-y_{n-1}) \times \cdots \times p(r_2-r_1, y_2-y_1) \\
\cdots \times e_{k_{\sigma(1)}}(y_1) \times \cdots \times e_{k_{\sigma(n)}}(y_n) u(0)(r_1, y_1) dy \, dr.
\]

Remark 21. Using the Feynman-Kac representation of $u(t, x)$, it was possible to compute the weak derivative of $u(t, x)$ with respect to $x$ in $(S)^c$ as follows:
\[
\partial_x u(t, x) = \sum_{\alpha \in \mathcal{J}} R_\alpha(t, x) \xi_\alpha,
\]
where $R_{\{0\}}(t, x) = \partial_x u(0)(t, x)$, and for $|\alpha| = n \geq 1$,
\[
R_\alpha(t, x) = \frac{1}{\sqrt{\alpha!}} \sum_{\sigma \in \mathcal{P}_n} \int_{T^p_n} \int_{\mathbb{R}^n} \partial_x p(t-r_n, x-y_n)p(r_n-r_{n-1}, y_n-y_{n-1}) \times \cdots \times p(r_2-r_1, y_2-y_1) \\
\cdots \times e_{k_{\sigma(1)}}(y_1) \times \cdots \times e_{k_{\sigma(n)}}(y_n) u(0)(r_1, y_1) dy \, dr.
\]

In fact, the weak spatial derivative is the usual spatial derivative in $L^2(\mathbb{P}^W)$ sense (and hence $\mathbb{P}^W$ almost sure sense). One can verify this assertion by computing $\partial_x u(t, x)$ using the chaos expansion of $u(t, x)$ directly and by the uniqueness of mild solution.

Before we show $\partial_x u(t, x) \in \mathcal{G}$ for each $t > 0$ and $x \in \mathbb{R}$, we first need the following lemma:

**Lemma 22.** Assume that $u_0 \in C^1_c(\mathbb{R})$. Then, for each $\alpha \in \mathcal{J}$, $t > 0$ and $x \in \mathbb{R}$, $R_\alpha(t, x)$ is well-defined, and moreover, for $|\alpha| \geq 1$,
\[
\lim_{\varepsilon \to 0^+} R^\varepsilon_\alpha(t, x) = R_\alpha(t, x),
\]
where
\[
R^\varepsilon_\alpha(t, x) := \frac{1}{\sqrt{\alpha!}} \sum_{\sigma \in \mathcal{P}_n} \int_{T^p_n} \int_{\mathbb{R}^n} \partial_x p(t-r_n, x-y_n)p(r_n-r_{n-1}, y_n-y_{n-1}) \times \cdots \times p(r_2-r_1, y_2-y_1) \\
\cdots \times e_{k_{\sigma(1)}}(y_1) \times \cdots \times e_{k_{\sigma(n)}}(y_n) u(0)(r_1, y_1) dy \, dr \quad \text{for } \epsilon > 0,
\]

and
\[
R_\alpha(t, x) := \frac{1}{\sqrt{\alpha!}} \sum_{\sigma \in \mathcal{P}_n} \int_{T^p_n} \int_{\mathbb{R}^n} \partial_x p(t-r_n, x-y_n)p(r_n-r_{n-1}, y_n-y_{n-1}) \times \cdots \times p(r_2-r_1, y_2-y_1) \\
\cdots \times e_{k_{\sigma(1)}}(y_1) \times \cdots \times e_{k_{\sigma(n)}}(y_n) u(0)(r_1, y_1) dy \, dr \quad \text{for } \epsilon = 0.
\]
with $T_{[0,t-\epsilon]}^n := \{0 \leq s_1 \leq \cdots \leq s_n \leq t - \epsilon\}$.

**Proof.** We will decompose $\mathfrak{R}_\alpha(t, x)$ as a finite sum of well-defined terms. Without loss of generality, we let $|\alpha| = n \geq 1$. Notice that

\[
\mathfrak{R}_\alpha(t, x) = \frac{1}{\sqrt{\alpha!}} \sum_{\sigma \in \mathcal{P}_n} \int_{T_{[0,t]}^n} \int_{\mathbb{R}^n} \frac{\partial x}{\partial \mathcal{P}_n} \frac{\partial}{\partial \mathcal{P}_n} \frac{\partial}{\partial \mathcal{P}_n} \cdots \frac{\partial}{\partial \mathcal{P}_n} \left( t - r_n, x - y_n \right) \left( r_n - r_{n-1}, y_n - y_{n-1} \right) \cdots \left( r_2 - r_1, y_2 - y_1 \right) \times e_{\mathcal{P}_n(1)}(y_1) \cdots e_{\mathcal{P}_n(n)}(y_n) u_0(r_1, y_1) dy dr
\]

\[
= \frac{1}{\sqrt{\alpha!}} \sum_{\sigma \in \mathcal{P}_n} \int_{T_{[0,t]}^n} \int_{\mathbb{R}^n} p(t - r_n, x - y_n) \partial_{y_n} \left( p(r_n - r_{n-1}, y_n - y_{n-1}) e_{\mathcal{P}_n(n)}(y_n) \right) \times \cdots \times e_{\mathcal{P}_n(1)}(y_1) \times e_{\mathcal{P}_n(n)}(y_n) u_0(r_1, y_1) dy dr
\]

\[
= \frac{1}{\sqrt{\alpha!}} \sum_{\sigma \in \mathcal{P}_n} \int_{T_{[0,t]}^n} \int_{\mathbb{R}^n} p(t - r_n, x - y_n) \partial_{y_n} p(r_n - r_{n-1}, y_n - y_{n-1}) e_{\mathcal{P}_n(n)}(y_n) \times \cdots \times e_{\mathcal{P}_n(1)}(y_1) \times e_{\mathcal{P}_n(n)}(y_n) u_0(r_1, y_1) dy dr
\]

\[
+ \frac{1}{\sqrt{\alpha!}} \sum_{\sigma \in \mathcal{P}_n} \int_{T_{[0,t]}^n} \int_{\mathbb{R}^n} p(t - r_n, x - y_n) p(r_n - r_{n-1}, y_n - y_{n-1}) e_{\mathcal{P}_n(n)}(y_n) \times \cdots \times e_{\mathcal{P}_n(1)}(y_1) \times e_{\mathcal{P}_n(n)}(y_n) u_0(r_1, y_1) dy dr
\]

\[
=: a_1 + b_1.
\]

We note that $b_1$ is well-defined since $u_0 \in L^\infty(\mathbb{R})$, and for $a_1$, we do a similar step as above:

\[
a_1 = -\frac{1}{\sqrt{\alpha!}} \sum_{\sigma \in \mathcal{P}_n} \int_{T_{[0,t]}^n} \int_{\mathbb{R}^n} p(t - r_n, x - y_n) \partial_{y_n} \left( p(r_n - r_{n-1}, y_n - y_{n-1}) e_{\mathcal{P}_n(n)}(y_n) \right) \times \cdots \times e_{\mathcal{P}_n(1)}(y_1) \times e_{\mathcal{P}_n(n)}(y_n) u_0(r_1, y_1) dy dr
\]

\[
= \frac{1}{\sqrt{\alpha!}} \sum_{\sigma \in \mathcal{P}_n} \int_{T_{[0,t]}^n} \int_{\mathbb{R}^n} p(t - r_n, x - y_n) p(r_n - r_{n-1}, y_n - y_{n-1}) e_{\mathcal{P}_n(n)}(y_n) \times \cdots \times e_{\mathcal{P}_n(1)}(y_1) \times e_{\mathcal{P}_n(n)}(y_n) u_0(r_1, y_1) dy dr
\]

\[
+ \frac{1}{\sqrt{\alpha!}} \sum_{\sigma \in \mathcal{P}_n} \int_{T_{[0,t]}^n} \int_{\mathbb{R}^n} p(t - r_n, x - y_n) p(r_n - r_{n-1}, y_n - y_{n-1}) e_{\mathcal{P}_n(n)}(y_n) \times \cdots \times e_{\mathcal{P}_n(1)}(y_1) \times e_{\mathcal{P}_n(n)}(y_n) u_0(r_1, y_1) dy dr
\]

\[
=: a_2 + b_2.
\]
Again, $b_2$ is well-defined. By iterating this process, we can get

$$
\mathfrak{R}_\alpha(t, x) = a_{n-1} + \sum_{i=1}^{n-1} b_i,
$$

where $\sum_{i=1}^{n-1} b_i$ is well-defined and

$$
a_{n-1} = \frac{1}{\sqrt{\alpha!}} \sum_{\sigma \in \mathcal{P}_n} \int_{\mathbb{T}_{[0, t]}^n} \int_{\mathbb{R}^n} p(t - r_n, x - y_n)p(r_n - r_{n-1}, y_n - y_{n-1}) \times \cdots \times p(r_2 - r_1, y_2 - y_1)
\times e_{k^{(1)}_\sigma}(y_1) \times \cdots \times e_{k^{(n)}_\sigma}(y_n) \partial y_1 u_0(r_1, y_1) dy \ dr.
$$

Since $u_0(r_1, y_1) dy = \int_{\mathbb{R}} p(r_1, y_1 - y_0) u_0(y_0) dy_0$, $a_{n-1}$ becomes

$$
a_{n-1} = \frac{1}{\sqrt{\alpha!}} \sum_{\sigma \in \mathcal{P}_n} \int_{\mathbb{T}_{[0, t]}^n} \int_{\mathbb{R}^{n+1}} p(t - r_n, x - y_n)p(r_n - r_{n-1}, y_n - y_{n-1}) \times \cdots \times p(r_2 - r_1, y_2 - y_1)
\times e_{k^{(1)}_\sigma}(y_1) \times \cdots \times e_{k^{(n)}_\sigma}(y_n) p(r_1, y_1 - y_0) u_0'(y_0) dy_0 dy \ dr,
$$

which is clearly well-defined since $u'_0 \in L^\infty(\mathbb{R})$.

Moreover, one can show that $\lim_{\epsilon \to 0^+} \mathfrak{R}_\alpha^\epsilon(t, x) = \mathfrak{R}_\alpha(t, x)$ easily by considering the same argument as $\mathfrak{R}_\alpha(t, x)$ for $\mathfrak{R}_\alpha^\epsilon(t, x)$.

**Theorem 23.** If $u_0 \in C^1_b(\mathbb{R})$, for each $t > 0$ and $x \in \mathbb{R}$,

$$
\partial_x u(t, x) \in \mathcal{G}.
$$

**Proof.** From (27), we have

$$
\mathbb{E} |\partial_x u(t, x)|^2 = |\mathfrak{R}_0(t, x)|^2 + \sum_{n=1}^{\infty} \sum_{\alpha \in \mathcal{J}_n} |\mathfrak{R}_\alpha(t, x)|^2.
$$

We note that $\mathfrak{R}_0(t, x) = \partial_x u_0(t, x) < \infty$ for each $(t, x) \in [0, T] \times \mathbb{R}$. For $|\alpha| = n \geq 1$, we use $\mathfrak{R}_\alpha^\epsilon$:

For $\epsilon > 0$, we notice that (by Fubini lemma)

$$
\mathfrak{R}_\alpha^\epsilon(t, x) = \sqrt{n!} \int_{\mathbb{T}_{[0, t]}^n} \int_{\mathbb{R}^n} \partial_x p(t - r_n, x - y_n)p(r_n - r_{n-1}, y_n - y_{n-1}) \times \cdots \times p(r_2 - r_1, y_2 - y_1)
\times e_\alpha(y_1, \ldots, y_n) u_0(r_1, y_1) dy \ dr dy.
$$

Using Bessel’s inequality, we have that

$$
\sum_{\alpha \in \mathcal{J}_n} |\mathfrak{R}_\alpha^\epsilon(t, x)|^2 \leq n! \int_{\mathbb{T}_{[0, t]}^n} \int_{\mathbb{R}^n} |\partial_x p(t - r_n, x - y_n)p(r_n - r_{n-1}, y_n - y_{n-1}) \times \cdots \times p(r_2 - r_1, y_2 - y_1) u_0(r_1, y_1) dy|^2 dy
\leq n! \int_{\mathbb{T}_{[0, t]}^n} \int_{\mathbb{R}^n} |\partial_x p(t - r_n, x - y_n)p(r_n - r_{n-1}, y_n - y_{n-1}) \times \cdots \times p(r_2 - r_1, y_2 - y_1) u_0(r_1, y_1) dy| dy
\times \int_{\mathbb{T}_{[0, t]}^n} |\partial_x p(t - r_n, x - y_n)p(r_n - r_{n-1}, y_n - y_{n-1}) \times \cdots \times p(r_2 - r_1, y_2 - y_1) u_0(r_1, y_1) dy| dy.
$$
Again using Fubini lemma and the semigroup property of \( p \), we can write the last expression as

\[ n!\|u_0\|^2_\infty (2\pi)^{-(n-1)/2} \int_{[0,t]} \prod_{k=1}^{n-1} (s_{k+1} + r_{k+1} - s_k - r_k)^{-1/2} \]

\[ \cdots \int_\mathbb{R} \partial_x p(t - r_n, x - y_n) \partial_x p(t - s_n, x - y_n) dy_n dr_s. \]

Since

\[ \int_\mathbb{R} \partial_x p(t - r_n, x - y_n) \partial_x p(t - s_n, x - y_n) dy_n = (2\pi)^{-1/2} (2t - s_n - r_n)^{-3/2}, \]

and \((a + b)^{-1/2} \leq 2^{-1/2} a^{-1/4} b^{-1/4}\) for \( a, b > 0 \), we can finally see that

\[ \sum_{\alpha \in \mathcal{J}_n} |\mathfrak{R}_\alpha(t, x)|^2 \leq n!\|u_0\|^2_\infty (2\pi)^{-n/2} \left( \int_{[0,t]} (t - s_n)^{-3/4} \prod_{k=1}^{n-1} (s_{k+1} - s_k)^{-1/4} ds \right)^2 \]

\[ \leq n!\|u_0\|^2_\infty (2\pi)^{-n/2} \left( \int_{[0,t]} (t - s_n)^{-3/4} \prod_{k=1}^{n-1} (s_{k+1} - s_k)^{-1/4} ds \right)^2 \]

\[ \leq \|u_0\|^2_\infty C^n t^{3n-4} n^{-n/2} \text{ for some } C > 0, \]

where the last inequality follows from [14, equation (4.10)].

For each \( t > 0 \) and \( x \in \mathbb{R} \), the convergence is uniform in \( \epsilon \) and \( \mathfrak{R}_\alpha^\epsilon(t, x) \rightarrow \mathfrak{R}_\alpha(t, x) \) as \( \epsilon \rightarrow 0 \) by Lemma 22,

\[ \sum_{\alpha \in \mathcal{J}_n} |\mathfrak{R}_\alpha(t, x)|^2 \leq \|u_0\|^2_\infty C^n t^{3n-4} n^{-n/2} \text{ for some constant } C > 0. \]

Since \( C^n t^{3n-4} n^{-n/2} e^{2\lambda n} \) is summable in \( n \) for any \( \lambda \in \mathbb{R} \), the conclusion follows from Definition 14. \( \square \)

Remark 24. We have the following Feynman-Kac type formula for the spatial derivative of \( u \):

\[ \partial_x u(t, x) = \mathbb{E}^B \left[ L^x(t) \circ \left\{ u_0(B^x_\cdot) + u_0(B^x_{\cdot 1}) (\partial_x L^x(t)) \right\} \right], \]

where \( \mathbb{E}^B \) must be understood as a Bochner integral in \((S)^*\). Notice that the integrand of \( \mathbb{E}^B \) on the right-hand side is in fact a Hida distribution. However, after taking the expectation \( \mathbb{E}^B \), which is interpreted as a Bochner integral in \((S)^*\), we end up with a regular random variable in \( \mathcal{G} \). This means that the white noise integral (or the Bochner integral in \((S)^*\)) has a regularizing effect.

Theorem 25. Let \( 0 < \epsilon < 1/2 \) be arbitrary and assume that \( u_0 \in C^{3/2}(\mathbb{R}) \). Then, for each \( t > 0 \),

\[ \partial_x u(t, \bullet) \in C^{1/2-\epsilon}(\mathbb{R}). \]

Proof. Let \( p > 1 \), \( t > 0 \) and \( x \in \mathbb{R} \). By [14, Proposition 2.1], we have

\[ (\mathbb{E} |\partial_x u(t, x + h) - \partial_x u(t, x)|^p)^{1/p} = \sum_{n=0}^\infty (p - 1)^{n/2} \left( \sum_{\alpha \in \mathcal{J}_n} |\mathfrak{R}_\alpha(t, x + h) - \mathfrak{R}_\alpha(t, x)|^2 \right)^{1/2}. \]

For \( |\alpha| = 0 \), by Lemma 15, we have

\[ \mathfrak{R}_{\{0\}}(t, \bullet) = \partial_x u_{\{0\}}(t, \bullet) \in C^{1/2}(\mathbb{R}). \]
For $|\alpha| = n \geq 1$, similarly to the proof of Theorem 23, we can get

$$\sum_{\alpha \in J_n} |R^e_\alpha (t, x + h) - R^e_\alpha (t, x)|^2$$

$$\leq n! \int_\mathbb{R}^n \left( \int_{\mathbb{T}^n_{[0, t-\epsilon]}} (\partial_x p(t - r_n, x + h - y_n) - \partial_x p(t - r_n, x - y_n)) p(r_n - r_{n-1}, y_n - y_{n-1}) \times \right.$$

$$\left. \cdots \times p(r_2 - r_1, y_2 - y_1) u(0)(r_1, y_1) dr \right)^2 dy$$

$$\leq n! \|u_0\|_\infty^2 (2\pi)^{-(n-1)/2} \int_{\mathbb{T}^n_{[0, t-\epsilon]}} \prod_{k=1}^{n-1} (s_{k+1} + r_{k+1} - s_k - r_k)^{-1/2}$$

$$\times \int_{\mathbb{R}} (\partial_x p(t - r_n, x + h - y_n) - \partial_x p(t - r_n, x - y_n))$$

$$\cdots \times (\partial_x p(t - s_n, x + h - y_n) - \partial_x p(t - s_n, x - y_n)) dy_n dr ds.$$

We next compute

$$\int_{\mathbb{R}} \partial_x p(t_1, x_1 - z) \partial_x p(t_2, x_1 - z) dz = \frac{1}{2\pi t_1^{3/2} t_2^{3/2}} \int_{\mathbb{R}} (x_1 - z)(x_2 - z)e^{-\frac{(x_1-z)^2}{2t_1} - \frac{(x_2-z)^2}{2t_2}} dz$$

$$= \frac{1}{2\pi t_1^{3/2} t_2^{3/2}} \int_{\mathbb{R}} z(z - (x_1 - x_2)) e^{-\frac{z^2}{2t_1} - \frac{(x_1-x_2)^2}{2t_2}} dz$$

$$= \frac{1}{2\pi t_1^{3/2} t_2^{3/2}} \int_{\mathbb{R}} (z^2 - (x_1 - x_2)z) e^{-\frac{z^2}{2t_1} - \frac{(x_1-x_2)^2}{2t_2}} dz$$

$$= \frac{1}{2\pi t_1^{3/2} t_2^{3/2}} \left(1 - \frac{(x_1-x_2)^2}{(t_1 + t_2)^2} \right).$$

The last equality can be verified using the mean and variance of a normal distribution $N \left( \frac{t_1}{t_1 + t_2}, \frac{t_1 t_2}{t_1 + t_2} \right)$. Then, we can easily check, using the fact $1 - e^{-z^2} \leq z^2$ for any $z \geq 0$ and $0 < \gamma \leq 1$,

$$\int_{\mathbb{R}} (\partial_x p(t - r_n, x + h - y_n) - \partial_x p(t - r_n, x - y_n))$$

$$\times (\partial_x p(t - s_n, x + h - y_n) - \partial_x p(t - s_n, x - y_n)) dy_n$$

$$= \sqrt{\frac{2}{\pi}} (2t - s - r)^{-3/2} \left(1 - e^{-\frac{r^2}{2(t_1 + t_2)}} + \frac{h^2 e^{-\frac{r^2}{2(t_1 + t_2)}}}{2t - s - r} \right) \leq C \hbar^{2\gamma} (2t - s - r)^{-3/2 - \gamma}. \quad (29)$$

At this point, we restrict $0 < \gamma < 1/2$ so that $(29)$ is integrable both in $s$ and $r$ variables near $t$. This leads to

$$\sum_{\alpha \in J_n} |R^e_\alpha (t, x + h) - R^e_\alpha (t, x)|^2 \leq h^{2\gamma} \|u_0\|_\infty^2 C^n(\gamma)t^{3n/4 - 1 - \frac{3}{2}n - n/2},$$

with $0 < \gamma < 1/2$ for some constant $C(\gamma) > 0$ depending only on $\gamma$.

After taking $\epsilon \to 0$, the desired result follows from $(28)$ and the Kolmogorov continuity theorem. □

**Remark 26.** Under the same initial condition in Theorem 25, we can achieve the optimal temporal regularity of $\partial_x u$ in a similar manner, i.e., $\partial_x u(\cdot, x) \in C^{1/4 - \epsilon} ([\epsilon_0, T])$ for every $x \in \mathbb{R}$, $0 < \epsilon_0 < T$, and $0 < \epsilon < 1/4$. 
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