Higher dimensional extremal black strings

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Abstract

We investigate the five and six-dimensional black strings within Einstein-Maxwell theory. The extremal black strings are endowed with the null Killing symmetry. We study the propagation of Einstein-Maxwell modes in the extremal black string background by using this symmetry. It turns out that one graviton is a propagating mode, while both the Maxwell($\mathcal{F}$) and three-form ($\mathcal{H}$) fields are non-propagating modes. Further we discuss the stability and classical hair of the extremal black strings.
I. INTRODUCTION

Recently there has been much progress in understanding the microscopic origin of the black hole entropy. This was made possible by using a new description of solitonic states in string theory \([1-6]\). For the simplest five-dimensional extremal black hole, Strominger and Vafa \([1]\) counted the number of degeneracy corresponding to BPS-saturated states in the string theory for given charge. And they showed that for large charge, the number of states increases as \(e^{A/4}\), where \(A\) is the area of the horizon for extremal black hole. That is, the statistical interpretation for the Bekenstein-Hawking entropy was made possible in five dimensions. Further, the five-dimensional black hole is just a six-dimesional black string which winds around a compact internal circle \([7]\). The microstates of five-dimensional extremal black hole are based on the fields moving around a circle in the internal dimensions. In order to understand this situation, it is useful to take this internal direction as a space-time direction explicitly. This is a six dimensional black string solution.

On the other hand, the four-dimensional extreme dilation black hole with the coupling constant \(a = \sqrt{\frac{p}{p+2}}\) could be interpreted as a non-dilatonic, black \(p\)-brane in \((4 + p)\) dimensional Einstein-Maxwell system \([8]\). For example, for \(p = 1\) one has the \(a = 1/\sqrt{3}\) dilaton black hole. Thus the four-dimensional extreme dilaton black hole can be considered as the double-dimensional reduction of a non-singular five-dimensional black string. The double dimensional reduction implies that the spacetime dimension is reduced from five to four and simultaneously the the dimension of the extended object from one (black string) to zero (black hole). The special case of \(a = 1/\sqrt{3}\) is of particular interest because the five-dimensional black string is a solution to the pure five-dimensional supergravity. In this case one can derive a Bogomolnyi-type bound on the energy per unit length and this bound is saturated by the black string solution. We also note the relevance of this model to three-dimensional black string.

In this paper we study the higher dimensional black strings within Einstein-Maxwell theory. We remind the reader that the Einstein-Maxwell model is the simplest one to
study the higher dimensional black strings. In order to relate these to the counting of the black hole entropy one has to consider the ten-dimensional string theories \[7\]. Actually the five-dimensional black string should be investigated in the context of the type IIA string theory \[10\]. This theory contains the fields from the NS-NS sector: a metric \((G_{MN})\), a two-form \((B_{MN})\), and a dilaton \((\phi)\) \[9\]. Also the fields from the RR sector are a one-form \((A_M)\) and a three-form \((C_{MNP})\). This carries three charges: a magnetic charge with respect to the RR two-form \((G = dA)\), an electric charge with respect to the RR four-form \((F = dC)\), and a magnetic charge with respect to the NS-NS three-form \((H = dB)\). We consider the compactification from ten to four: \(M^{10} \rightarrow M^4 \times M^6\). One takes six dimensions to be compactified to form a torus and assumes translational symmetry in five of these dimensions \((M^6 \rightarrow S^1 \times \tilde{S}^1 \times \tilde{T}^4)\). The sixth direction along \(S^1\) will have a length \(L\) much longer than the others, and will be the direction in which the waves propagate. Hence this solution corresponds to a black string with traveling waves in five dimensions. Four toroidal directions \((\tilde{T}^4)\) form a torus of volume \(\tilde{V} = (2\pi)^4V\), and one along \(\tilde{S}^1\) forms a circle of length \(\tilde{L}\). The other four dimensions \((M^4)\) will be realized into a space-time.

For a complete study on the six-dimensional black string, one has to consider the ten-dimensional type IIB string theory compactified on a four torus with volume \(\mathcal{V}\). This ten-dimensional theory consists of (1) NS-NS sector : a dilaton \((\varphi)\), a metric \((g_{MN})\) and a two-form field \((B_{MN}^{(1)})\) (2) RR sector : an axion field \((\chi)\), a two-form \((B_{MN}^{(2)})\), and a self dual four-form \((\tilde{C}_{MNPQ})\). This compactification turns out to be \((M^{10} \rightarrow M^6 \times T^4)\). We assume one additional spatial direction in six is toroidally compactified to form a circle \((S^1)\) of length \(L\) and choose \(L >> \sqrt{\mathcal{V}}\) so that the solutions resemble strings in six dimensions. Here this solution carries also electric and magnetic charges with respect to the RR three-form \((H^{(2)} = dB^{(2)})\). The object which carries a charge under the NS-NS gauge field \((B_{MN}^{(1)})\) is the elementary string itself. At low energies the elementary string is described by a classical solution of the effective field theory. However there are no states which carry charges of the RR gauge field \((B_{MN}^{(2)})\). Recently, it turns out that these missing states carrying RR charges are D-branes \[11\]. We note that these solutions with charges are not precisely our case but
are very similar to the kind of problems we are interested in. As a first step to analyze the five and six-dimensional black strings generically, we consider here the Einstein-Maxwell theory.

Garfinkle used a generating technique to find out the traveling waves of the black string \[12,13\]. Starting from a known static solution, this technique produces a new solution representing waves traveling on the old black string background. However, this method has some limitations, since it always requires that the background metric \((\bar{g}_{MN})\) possess a null, orthogonal Killing vector. Further, Horowitz and Tseytlin have worked on finding traveling wave solutions of string theories in various dimensions \(14,15\). They related a limit of their \(F\)-model to the extremal black string in three dimensions. Recently Horowitz and Marolf counted the number of microstates for black string with traveling waves to obtain its entropy \(10\).

We use the standard scheme for studying the black hole as well as black string. Our method is based on that for the black hole analysis \(16\). But there are some differences. For the black holes, we choose the metric perturbation \((h_{\mu\nu})\) in such a way that the background symmetry should be restored at the perturbation level \(17\). However, it is also necessary for the black strings \(12,13\) to exploit such a background symmetry. For the extremal black strings, the background symmetry corresponds to the null Killing isometry. Furthermore in the black hole physics, the counting of degrees of freedom for physical field is crucial for initial setting of the perturbing fields. On the other hand, it is more important for the extremal black string physics to use the null Killing vector field in choosing the perturbations.

The paper is organized as follows. In the next section, we will review the mathematical formalism for the subsequent study. Especially we explain the relationship between the dilaton black holes and higher dimensional black strings. In Sec.III we study the propagation of graviton and Maxwell modes in five-dimensional black string background. The propagation of six-dimensional black string is investigated in Sec.IV. The last section offers conclusion and discussion for our results.
II. FORMALISM

Given a solution of the 

\((d+p)\)-dimensional Einstein equations with a black \(p\)-brane (a horizon and \(p\)-fold translational symmetry), one can always find a dilaton black hole solution of \(d\)-dimensional gravity by the double-dimensional reduction. The 

\((d+p)\)-dimensional action is given by

\[
S_{d+p} = \int d^{(d+p)}x \sqrt{-g} \left( R_{d+p} - \frac{2}{(d-2)!} F_{d-2}^2 \right).
\]

We perform the double dimensional reduction of \(p\) dimensions by taking the \((d+p)\)-dimensional metric and \((d-2)\)-form \(F_{d-2}\) to be

\[
\tilde{g}_{MN} = \begin{pmatrix}
  e^{2\beta(x)} g_{\mu\nu} & 0 \\
  0 & e^{2\alpha(x)} g_{mn}
\end{pmatrix}
\]

and

\[
F_{d-2} = \frac{1}{(d-2)!} F_{\mu_1 \cdots \mu_{d-2}} dx^{\mu_1} \cdots dx^{\mu_{d-2}},
\]

where \(x^\mu(y^m)\) are the coordinates for the \(d\)-dimensional spacetime (\(p\)-brane). Note that both the metric and Maxwell field are independent of \(y^m\). We do not include any \(x-y\) cross term in the metric, which give rise to additional gauge fields in \(d\)-dimensional spacetime. Using the relation for the conformal transformation by \(\tilde{g}_{MN} = \Omega^2 g_{MN}\), we show that for \(p\alpha + (d-2)\beta = 0\), (1) turns out to be the \(d\)-dimensional action. After fixing the normalization of the dilaton kinetic term as

\[
\alpha^2 = \frac{2(d-2)}{p(d+p-2)},
\]

the \(d\)-dimensional dilaton action is given by

\[
S_d = \int d^d x \sqrt{-g} \left( R_d - 2(\nabla \phi)^2 - \frac{2}{(d-2)!} e^{-2\alpha(x)} F_{d-2}^2 \right),
\]

where

\[
a = \frac{(d-3)\sqrt{2p}}{\sqrt{(d-2)(d+p-2)}}.
\]
Then the magnetically charged black hole solutions to the Euler-Lagrange equations of (5) are

\begin{align*}
    ds_d^2 &= -\left[1 - \left(\frac{r_+}{r}\right)^{d-3}\right]\left[1 - \left(\frac{r_-}{r}\right)^{d-3}\right]^{1-(d-3)\gamma} dt^2 \\
    &+ \left[1 - \left(\frac{r_+}{r}\right)^{d-3}\right]^{-1}\left[1 - \left(\frac{r_-}{r}\right)^{d-3}\right]^{\gamma-1} dr^2 + r^2\left[1 - \left(\frac{r_-}{r}\right)^{d-3}\right]^{\gamma} d\Omega_{(d-2)}^2, \\
    e^{a\phi} &= \left[1 - \left(\frac{r_-}{r}\right)^{d-3}\right]^{\gamma-1}, \\
    F_{(d-2)} &= Q\varepsilon_{d-2},
\end{align*}

where \(\varepsilon_{d-2}\) is the volume form for the unit \((d-2)\)-sphere and

\begin{equation}
    \gamma = \frac{2p}{(d-2)(p+1)}. \tag{8}
\end{equation}

Here a single charge \(Q\) is related to the inner \((r_-)\) and outer \((r_+)\) horizons by

\begin{equation}
    Q^2 = \frac{(d + p - 2)(d - 3)}{2(p + 1)}(r_+ r_-)^{d-3}. \tag{9}
\end{equation}

A black \(p\)-brane solution of the action (1) is given by

\begin{align*}
    ds_{d+p}^2 &= -\left[1 - \left(\frac{r_+}{r}\right)^{d-3}\right]\left[1 - \left(\frac{r_-}{r}\right)^{d-3}\right]^{\frac{2-p}{d-3}} dt^2 + \left[1 - \left(\frac{r_-}{r}\right)^{d-3}\right]^{\frac{p}{d-3}} dy_p^2 \\
    &+ \left[1 - \left(\frac{r_+}{r}\right)^{d-3}\right]^{-1}\left[1 - \left(\frac{r_-}{r}\right)^{d-3}\right]^{-1} dr^2 + r^2d\Omega_{(d-2)}^2. \tag{10}
\end{align*}

For the nonextremal case \((r_+ > r_-)\), both the black holes and black \(p\)-branes have event horizons at \(r = r_+\) and Cauchy horizons at \(r = r_-\). However the situation is quite interesting for the extremal case \((r_+ = r_- = \mu)\). In this case, the black hole horizon becomes singular. On the other hand, the black \(p\)-brane solution leads to

\begin{equation}
    ds_{d+p,ext}^2 = \left[1 - \left(\frac{\mu}{r}\right)^{d-3}\right]^{\frac{2}{d-3}+p} (-dt^2 + dy_p^2) + \left[1 - \left(\frac{\mu}{r}\right)^{d-3}\right]^{-2} dr^2 + r^2d\Omega_{(d-2)}^2. \tag{11}
\end{equation}

This metric is invariant under the full \((p+1)\)-Poincare group. For appropriate values of \(d\) and \(p\), (11) includes all the known extreme, non-dilatonic, extented object solutions of higher dimensional supergravities. These are membrane and fivebrane solutions of the eleven-dimensional supergravity, the self-dual three-brane of the ten-dimensional supergravity and
the self-dual string of six-dimensional supergravity. Here we are interested only in one-
dimensional extended object with horizons (black strings). The black strings are realized
only for three, five and six dimensions [8]. In [18], we have already investigated the propaga-
tion of string fields in the three-dimensional extremal black string background. Unlike the
higher dimensional cases, the exact conformal field theory is known and thus the particle
contents can be determined.

III. FIVE-DIMENSIONAL BLACK STRING

We start with the five-dimensional Einstein-Maxwell action,

\[ S_{5d} = \int d^5x \sqrt{-g} \left( R - \frac{1}{4} F_{MN}^2 \right), \]  

(12)

where \( x^M \) are the coordinates \((t, x, r, \theta, \phi)\) for five dimensions. The equations of motion are
given by

\[ R_{MN} + \frac{1}{12} F^2 g_{MN} - \frac{1}{2} F_{MP} F^P_N = 0, \]  

(13)

\[ \nabla_M F^{MN} = 0. \]  

(14)

The static black string solution to the above equations in the extremal limit is given by

\[ \bar{g}_{MN} = \begin{pmatrix}
-\left(1 - \frac{\mu}{r}\right) & 0 & 0 & 0 & 0 \\
0 & \left(1 - \frac{\mu}{r}\right) & 0 & 0 & 0 \\
0 & 0 & \left(1 - \frac{\mu}{r}\right)^{-2} & 0 & 0 \\
0 & 0 & 0 & r^2 & 0 \\
0 & 0 & 0 & 0 & r^2 \sin^2 \theta^2
\end{pmatrix}, \]  

(15)

\[ \bar{F}_{\theta\phi} = Q \sin \theta. \]  

(16)

Here the constant \( \mu \) is related the ADM mass per unit length of the string \((M)\) and magnetic
charge per unit length \((Q)\) as
\[ \mu = \frac{2|Q|}{\sqrt{3}} = \frac{4M}{3}. \]  

Eq.(15) represents a straight, static black string which is an one-dimensional extended object with horizon at \( r = \mu \). The above metric is not only translationally invariant, but also boost invariant. This allows us to introduce two null-coordinates \((v = x + t, u = x - t)\) and null Killing vector field \((\frac{\partial}{\partial v})\). Using these, the line element can be rewritten as

\[ ds_{4+1}^2 = (1 - \frac{\mu}{r})dudv + (1 - \frac{\mu}{r})^{-2}dr^2 + r^2d\Omega^2, \]  

where \( d\Omega^2 \) denotes the metric of the unit two-sphere. After the double dimensional reduction of this balck string, one can find a four-dimensional extremal dilaton black hole with \( a = 1/\sqrt{3} \),

\[ ds_4^2 = -\left((1 - \frac{\mu}{r})^\frac{2}{3}dt^2 + (1 - \frac{\mu}{r})^{-\frac{2}{3}}dr^2 + (1 - \frac{\mu}{r})^\frac{4}{3}r^2d\Omega^2 \right). \]  

For our purpose, we introduce the perturbation fields \((\mathcal{F}(x), h(x))\) around the black string background as

\[ F_{\theta\phi} = \bar{F}_{\theta\phi} + F_{\theta\phi} = \bar{F}_{\theta\phi}(1 + \mathcal{F}), \]  
\[ g_{MN} = \bar{g}_{MN} + h_{MN} = \bar{g}_{MN} + hk_Mk_N, \]

where \( k_M \) is the null, orthogonal Killing vector which satisfies

\[ k^Mk_M = 0, \quad \bar{\nabla}_Mk_N = 0, \quad k_M\bar{\nabla}_Nk_L = 0. \]

Here we choose \( k^M = (\frac{\partial}{\partial v})^M \). For this metric perturbation, the harmonic gauge condition \((\nabla_Mh^M_N = \frac{1}{2}\nabla_Nh^M_M)\) is trivially satisfied. This ansatz for the metric perturbation is valid for the extremal black strings, but not for the extremal black holes. This is because only the extremal balck string has the null Killing symmetry.

In order to obtain the equations governing the perturbations, one has to linearize (13) and (14) as

\[ \delta R_{MN}(h) + \frac{1}{6}\bar{F}_{\theta\phi}\bar{F}^{\theta\phi}(h_{MN} + 2\bar{g}_{MN}\mathcal{F}) - (\bar{F}_{M\theta}\bar{F}_{N}^{\theta} + \bar{F}_{M\phi}\bar{F}_{N}^{\phi})\mathcal{F} = 0, \]
\[ \nabla_M (\tilde{F}^{MN}F) + \delta \Gamma^M_{NP}(h) \tilde{F}^{PN} = 0, \]  

(24)

where

\[ \delta R_{MN}(h) = -\frac{1}{2}(\nabla_M \nabla_N h^P{}_P + \nabla^P \nabla_P h_{MN} - \nabla^P \nabla_N h_{MP} - \nabla^P \nabla_M h_{NP}), \]  

(25)

\[ \delta \Gamma^M_{NP}(h) = \frac{1}{2}g^{ML}(\nabla_P h_{NL} + \nabla_N h_{PL} - \nabla_L h_{NP}). \]  

(26)

From (24), taking \( N = \theta \) and \( \phi \) respectively, one can easily find

\[ \partial_\theta F = 0, \quad \partial_\phi F = 0. \]  

(27)

This implies that if \( F \neq 0 \), \( F \) depends only on \( r \), \( t \) and \( x \) as

\[ F = F(r, t, x). \]  

(28)

From (23), one finds fifteen equations. However, only nine of them are nontrivial and these are given by

\[(v, u) : \quad \partial^2_v h + \frac{2\mu^2}{r} F = 0, \]  

(29)

\[(u, u) : \quad (r - \mu)^2 \partial_r^2 h + \frac{2(r^2 - \mu^2)}{r} \partial_r h + \frac{2\mu^2}{r^2} h + \cot \theta \partial_\theta h + \frac{1}{\sin^2 \theta} \partial_\phi^2 h = 0, \]  

(30)

\[(u, r) : \quad (\partial_r - \frac{\mu}{r(r - \mu)}) \partial_r (k^2_r h) = 0, \]  

(31)

\[(u, \theta) : \quad \partial_\theta \partial_v h = 0, \]  

(32)

\[(u, \phi) : \quad \partial_\phi \partial_v h = 0, \]  

(33)

\[(r, r) : \quad F = 0, \]  

(34)

\[(\theta, \theta) : \quad F = 0, \]  

(35)

\[(\phi, \phi) : \quad F = 0, \]  

(36)

From (27) and (34)-(36), we obtain \( F = 0 \). This means that the Maxwell mode is not the propagating one in the five-dimensional black string background. Substituting this into (29) and (31)-(33) leads to

\[ \partial_v h = 0. \]  

(37)
In order to solve one remaining (30), let us define \( h \) as
\[
  h \equiv p(r) h'(r) U(u) Y(\theta, \phi). 
\]
Here the spherical harmonics \( Y(\theta, \phi) \) on \( S^2 \) satisfies the angular eigenvalue equation
\[
  \frac{\partial^2 Y}{\partial \theta^2} + \cot \theta \frac{\partial Y}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2 Y}{\partial \phi^2} = -l(l + 1)Y, 
\]
where \( l \) is a non-negative integer. Then for \( p(r) = r/(r - \mu) \), (30) is reduced to
\[
  (r - \mu)^2 \partial^2_r h' + 2(r - \mu) \partial_r h' - l(l + 1)h' = 0. 
\]
Assuming the form of solution \( h' = (r - \mu)\alpha \), the general solution to this is
\[
  h' = C_1 (r - \mu)^l + \frac{C_2}{(r - \mu)^{l+1}}. 
\]

At this stage we can compare our results with that of three-dimensional black string. The \( l = 0 \) case corresponds to the three-dimensional black string. In this case, from (39) one finds
\[
  \partial^2_r h' + \frac{2}{(r - \mu)} \partial_r h' = 0. 
\]
This is exactly the equation for three-dimensional black string and the corresponding solution is given by \( h'_3 = \frac{C_2}{r - \mu} [13][18] \). Therefore we can interpret the five-dimensional black string as the direct product of three-dimensional black string and two-sphere \( (S^2) \).

**IV. SIX-DIMENSIONAL BLACK STRING**

The six-dimensional action from (1) is given by
\[
  S_{6d} = \int d^6x \sqrt{-g} (R - \frac{1}{3} H_{MNP}^2), 
\]
where \( x^M \) are the coordinates \((t, x, r, \chi, \theta, \phi)\) for six dimensions and an \( H_{MNP} \) is the three-form \( H = dB \). The equations of motion are given by
\[
  R_{MN} + \frac{1}{6} H^2 g_{MN} - H_{MPQ} H_M^{\quad PQ} = 0, 
\]
\[
  \nabla_M H^{MPQ} = 0. 
\]
The static black string solution to the above equations in the extremal limit is given by

\[ ds_{5+1}^2 = -\left[1 - \left(\frac{\mu}{r}\right)^2\right] dt^2 + \left[1 - \left(\frac{\mu}{r}\right)^2\right] dx^2 + \left[1 - \left(\frac{\mu}{r}\right)^2\right]^{-2} dr^2 + r^2 d\Omega_3^2, \]  

(45)

\[ \bar{H}_{\chi\theta\phi} = Q \sin^2 \chi \sin \theta, \]  

(46)

where \( d\Omega_3^2 \) represents the metric for three sphere (\( S^3 \)). Here the constant \( \mu \) is related to the magnetic charge per unit length (\( Q \)) in the extremal limit as

\[ \mu^4 = \frac{Q^2}{2}. \]  

(47)

Eq.(45) represents a six-dimensional, static black string. By the same argument as in the five-dimensional case, one can introduce two null-coordinates (\( v = x + t, u = x - t \)) and null Killing vector field (\( \frac{\partial}{\partial v} \)). The line element can be rewritten as

\[ ds^2 = \left[1 - \left(\frac{\mu}{r}\right)^2\right] dv du + \left[1 - \left(\frac{\mu}{r}\right)^2\right]^{-2} dr^2 + r^2 d\Omega_3^2. \]  

(48)

After the double dimensional reduction of this black string, one can find a five-dimensional extremal dilaton black hole with \( a = \sqrt{2/3} \),

\[ ds_5^2 = \left[1 - \left(\frac{\mu}{r}\right)^2\right]^\frac{1}{2} \left[- \left(1 - \left(\frac{\mu}{r}\right)^2\right) dt^2 + \left(1 - \left(\frac{\mu}{r}\right)^2\right)^{-2} dr^2 + r^2 d\Omega_3^2\right]. \]  

(49)

Except the conformal factor, this metric is similar to the five-dimensional Reissner-Nordström black hole, which was considered in studying microscopic origin of the Bekenstein-Hawking entropy [1].

Also one can introduce the perturbation fields (\( \mathcal{H}, h \)) around the extremal black string background as [18,19]

\[ H_{\chi\theta\phi} = \bar{H}_{\chi\theta\phi} + \mathcal{H}_{\chi\theta\phi} = \bar{H}_{\chi\theta\phi}(1 + \mathcal{H}), \]  

(50)

\[ g_{MN} \equiv \bar{g}_{MN} + h_{MN} = \bar{g}_{MN} + h k_M k_N, \]  

(51)

where \( \bar{g}_{MN} \) represents (45) and \( k_M \) is the null, orthogonal Killing vector in six dimensions. After linearizing (43) and (44), we have
\[\delta R_{MN}(h) + \frac{1}{6} \bar{H}^2 (h_{MN} + 2 \bar{g}_{MN} \mathcal{H}) - 2 \bar{H}_{MPQ} \bar{H}_N^{\ PQ} \mathcal{H} = 0, \quad (52)\]

\[\nabla_M (\bar{H}^MNP \mathcal{H}) + \delta \Gamma^M_{MQ}(h) \bar{H}^QNP = 0. \quad (53)\]

From (53), taking \(N = \chi, \theta \) and \(\phi \) respectively one finds

\[\partial_\chi \mathcal{H} = 0, \quad \partial_\theta \mathcal{H} = 0, \quad \partial_\phi \mathcal{H} = 0. \quad (54)\]

This implies that if \(\mathcal{H} \neq 0\), \(\mathcal{H}\) depends only on \(r, t\) and \(x\),

\[\mathcal{H} = \mathcal{H}(r, t, x). \quad (55)\]

From (52), one finds twenty-one equations. However, ten nontrivial equations are given by

\((v, u) : \quad \partial_v^2 h + \frac{8 u^4}{r^6} \mathcal{H} = 0, \quad (56)\)

\[\quad (u, u) : \quad \frac{(r^2 + \mu^2)^2}{r^4} \partial_r^2 h + \frac{(r^2 + \mu^2)(3r^2 + 5u^2)}{r^6} \partial_r h + \frac{8u^4}{r^6} h\]

\[+ \frac{1}{r^2} (\partial^2 \chi + \frac{\partial^2_\chi}{\sin^2 \chi} + \frac{\partial^2_\phi}{\sin^2 \chi \sin^2 \theta} + 2 \cot \chi \partial_\chi + \frac{\cot \theta}{\sin^2 \chi} \partial_\theta) h = 0, \quad (57)\]

\[\quad (u, r) : \quad (\partial_r - \frac{2u}{r(r^2 + \mu^2)})(\partial_v (k^2_{\chi} h)) = 0, \quad (58)\]

\[\quad (u, \chi) : \quad \partial_\chi \partial_v h = 0, \quad (59)\]

\[\quad (u, \theta) : \quad \partial_\theta \partial_v h = 0, \quad (60)\]

\[\quad (u, \phi) : \quad \partial_\phi \partial_v h = 0, \quad (61)\]

\[\quad (r, r) : \quad \mathcal{H} = 0, \quad (62)\]

\[\quad (\chi, \chi) : \quad \mathcal{H} = 0, \quad (63)\]

\[\quad (\theta, \theta) : \quad \mathcal{H} = 0, \quad (64)\]

\[\quad (\phi, \phi) : \quad \mathcal{H} = 0. \quad (65)\]

From (54) and (62)-(65), we obtain \(\mathcal{H} = 0\). This means that the three-form \(\mathcal{H}_{\chi \theta \phi}\) is not the propagating one in the six-dimensional black string background. Putting this into (56) and (58)-(61), we have
\partial_v h = 0. \quad (66)

In order to solve (57), we define \( h \) as \( h \equiv q(r)h''(r)U'(u)Y(\chi, \theta, \phi) \), where the spherical harmonics \( Y(\chi, \theta, \phi) \) on \( S^3 \) satisfies the angular eigenvalue equation

\[
\frac{\partial^2 Y}{\partial \chi^2} + 2 \cot \chi \frac{\partial Y}{\partial \chi} + \frac{1}{\sin^2 \chi \sin^2 \theta} \frac{\partial^2 Y}{\partial \theta^2} + \frac{1}{\sin^2 \chi} \frac{\partial^2 Y}{\partial \phi^2} = -L(L + 2)Y, \quad (67)
\]

where \( L(L + 2) \) is an eigenvalue for the Laplacian operator on the three sphere. Choosing \( q(r) = \frac{r^2}{r^2 - \mu^2} \), then (57) is reduced to

\[
\frac{(r^2 - \mu^2)^2}{r^4} \frac{\partial^2 h''}{\partial r^2} + \frac{(r^2 - \mu^2)(3r^2 + \mu^2)}{r^5} \frac{\partial h''}{\partial r} - \frac{L(L + 2)}{r^2} h'' = 0. \quad (68)
\]

In order to solve the above equation, we introduce a new variable \( (z) \) as

\[
z \equiv \frac{r^2}{\mu^2} - 1. \quad (69)
\]

Then (68) can be rewritten as

\[
4\frac{\partial^2 h''(z)}{\partial z^2} + \frac{8}{z} \frac{\partial h''(z)}{\partial z} - \frac{L(L + 2)}{z^2} h''(z) = 0. \quad (70)
\]

Assuming the form of solution \( h''(z) = z^\alpha \), one finds

\[
\alpha = \frac{L}{2}, \quad -\frac{L + 2}{2}. \quad (71)
\]

The general solution to (68) is

\[
h''(r) = C_1'\left(\frac{r^2 - \mu^2}{\mu^2}\right)^L + C_2'\left(\frac{r^2 - \mu^2}{\mu^2}\right)^{-L-2} \quad (72)
\]

with two arbitrary constants \( C_1' \) and \( C_2' \).

**V. DISCUSSION**

Let us first discuss the stability of higher-dimensional black strings. Conventionally, in deciding whether a black hole is stable or not, we start with a perturbation which is regular everywhere in space at the initial time \( t \). In the cases of most black holes, the
linearized equation which governs the perturbation is the Schrödinger-type equation \[16,17\]. And we then investigate whether such a perturbation grows with time. If there exists an exponentially growing mode, the black hole is unstable. This implies that one finds a physical mode with the potential well around the black hole. The potential barrier implies the scattering state, while the potential well gives us the scattering state as well as the bound state. It is well known that the bound state solution takes the form of exponentially increasing or decreasing functions.

Here we have no additional constraint for determining \( U(x, t) \) except the chiral constraint:
\[
\partial_v U(x, t) = 0 \quad \text{within this scheme.}
\]
We assume the normal mode solution of the form
\[
U(x, t) = e^{-iEt} e^{-i\Pi x}.
\]
(73)
From \( \partial_v U(x, t) = 0 \), one finds \( \Pi = -E \). Then the form of \( U(x, t) \) is determined as
\[
U(u) = e^{i\Pi u}.
\]
(74)
This is a plane wave along the \( v=\text{constant} \) null line. And this is called the longitudinal wave, since it carries only momentum \( \Pi \) along the string direction (\( x \)-direction) \[10\]. Hence the graviton mode \( (h = p(r)\dot{h}'(r)U(u)Y_{lm}(\theta, \phi)) \) can be the propagating wave in the black string background.

On the other hand, we may choose \( \Pi = i\alpha \). Then (74) leads to an exponentially growing mode \( \tilde{U}(u) = e^{\alpha(t-x)} \) with respect to the time. Applying the argument of black hole stability to the extremal black strings, one finds that it is unstable. This is because there is no restriction on choosing \( \Pi \) as either real or imaginary within the extremal black strings. This point contradicts to that of the black holes. Namely, one obtained the Schrödinger-type equation for the \( a = \frac{1}{\sqrt{3}} \) extremal dilaton balck hole \[20\], whereas we cannot obtain the corresponding equation for the five-dimensional extremal black string. This is a result of our setting of \( h_{MN} \) in (21). Although one can easily obtain the solutions from this choosing of \( h_{MN} \), one cannot find the Schrödinger-type equation for the extremal black strings.

Next, we consider the problem of classical hair in the black string theory. The classical no-hair theorem of general relativity severely restricts the kinds of fields that can exist
outside a black hole in four-dimensional spacetime. For example, in a static geometry with a smooth event horizon the only field which is well behaved both in the asymptotically flat region and on the event horizon is monopole gravitational and electromagnetic fields. Here we define a black string with hair to be a geometry with a smooth event horizon and a field which is nonsingular at that horizon. Conventionally, one takes a static solution to the linearized equations as a kind of hair \[21\]. In this case we require that the static solution be smooth both at the horizon \((r = \mu)\) and at spatial infinity \((r = \infty)\) \[22\]. Let us consider our case. We note that (39) and (68), as they stand, are static equations for the graviton. Then the full static solutions for graviton are given by

\[
p(r)h'(r) = C_1 r(r - \mu)^{l-1} + \frac{C_2 r}{(r - \mu)^{l+2}},
\]

(75)

\[
q(r)h''(r) = C'_1 \left(\frac{r^2 - \mu^2}{\mu^2}\right)^{L-1} + C'_2 \left(\frac{r^2 - \mu^2}{\mu^2}\right)^{-\frac{L+4}{2}}.
\]

(76)

The first terms in (75) and (76) are smooth near the horizon and diverge as \(r \to \infty\), while the second terms are singular on the horizon and converge in the asymptotically flat region. Thus one cannot find a smooth solution which behaves well on both limits. This means that in the strict sense of black hole hair, there is no hair in the five and six-dimensional extremal black strings.

In conclusion, we found the solutions which propagate in the higher dimensional extremal black string backgrounds. These correspond to the graviton mode. Both the Maxwell\((\mathcal{F})\) and three-form \((\mathcal{H})\) fields are non-propagating modes. This seems to be a controversial result, compared with the black holes. We consider the conventional counting of degrees of freedom. The number of degrees of freedom for the gravitational field \((h_{\mu\nu})\) in \(D\)-dimensions is \((1/2)D(D-3)\). We have \(-1\) for \(D = 2\). This means that in two dimensions the contribution of graviton is equal and opposite to that of a spinless particle (dilaton). In the 2d dilaton black hole, two graviton-dilaton modes are thus trivial gauge artefacts \[17\]. In the case of \(D = 3\), we have no propagating gravitons \[19\]. For \(D = 4\) Schwarzschild black hole, we obtain two degrees of freedom. These correspond to Regge-Wheeler mode for odd-parity
perturbation and Zerilli mode for even-parity perturbation $[16, 23]$. In the cases of $D = 5, 6$, one finds five and nine degrees of freedom respectively. However, it is emphasized that this is a conventional counting which is suitable for the black holes. We note that our model is the extremal black string with the null Killing symmetry. In this case we have only one propagating graviton in five dimensions and also one graviton in six dimensions. Furthermore, the Maxwell and three-form field have $D - 2$ and $(D - 2)(D - 3)/2$ degrees of freedom respectively. A naive counting gives us 3 for the five-dimensional Maxwell field and 6 for the six-dimensional three-form field. However, it turns out that both the Maxwell ($\mathcal{F}$) and three-form ($\mathcal{H}$) fields are non-propagating modes in the extremal black string backgrounds. It seems that the conventional counting for degrees of freedom is not suitable for the extremal black strings.

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