Shintani correspondence for Maass forms of level \(N\) and prehomogeneous zeta functions

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Abstract

A Shintani-Katok-Sarnak type correspondence for Maass cusp forms of level \(N\) is shown to be derived from analytic properties of prehomogeneous zeta functions whose coefficients involve periods of Maass forms.

Introduction

In [10], Shimura constructed a lifting from holomorphic cusp forms of half-integral weight to cusp forms of integral weight. Shimura’s original proof depends on the Rankin-Selberg method and Weil’s converse theorem [13]. In [11], Shintani constructed a lifting from holomorphic cusp forms of integral weight to cusp forms of half-integral weight by using theta functions. In the case of non-holomorphic modular forms, a prototype of the lifting had already appeared in the work of Maass [6]. Katok and Sarnak [5] developed the method of [6] to prove the Shintani correspondence for Maass cusp forms of weight 0 for \(SL_2(\mathbb{Z})\). The Katok-Sarnak formula reveals a relation between the periods of Maass forms of weight 0 and the Fourier coefficients of the corresponding form of weight \(\frac{1}{2}\), and now plays an important role in number theory. The Katok-Sarnak formula has been extended in many directions; we refer to Baruch-Mao [1], Biro [2], Duke-Imamoglu-Toth [3], Imamoglu-Lägeler-Toth [4]. On the other hand, F. Sato [9] constructed a theory of prehomogeneous zeta functions whose coefficients involve periods of automorphic forms. In this note, we show that a Shintani-Katok-Sarnak type correspondence is derived from analytic properties of a certain zeta functions investigated in [9]. The proof relies on a Weil type converse theorem for Maass forms [7].

This is an announcement whose details will appear elsewhere.

1 Statement of the result

The group \(G = SL_2(\mathbb{R})\) acts on the Poincaré upper half plane \(\mathcal{H} = \{ z = x + iy \in \mathbb{C} \mid y > 0 \}\) via the linear fractional transformation. Let \(N\) be a positive integer and take a congruence subgroup \(\Gamma_0(N)\) of level \(N\) defined by

\[
\Gamma_0(N) = \{ \gamma \in SL_2(\mathbb{Z}) \mid \gamma_2 \equiv 0 \pmod{N} \},
\]

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where $\gamma_{21}$ the $(2,1)$-entry of $\gamma$. Let $\chi$ be a Dirichlet character of mod $N$ satisfying $\chi(-1)=1$. We use the same symbol $\chi$ to denote the induced character of $\Gamma_0(N)$ defined by $\chi(\gamma) = \chi(\gamma_{22})$ for $\gamma = (\gamma_{ij}) \in \Gamma_0(N)$. A $C^\infty$-function $\Phi : \mathcal{H} \to \mathbb{C}$ is called a Maass cusp form of weight 0 for $\Gamma_0(N)$ with character $\chi$ if

1. $\Delta_0 \Phi = \lambda(1-\lambda)\Phi$ for a $\lambda \in \mathbb{C}$, where

$$\Delta_0 = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

is the hyperbolic Laplacian on $\mathcal{H}$,

2. $\Phi(\gamma z) = \chi(\gamma)\Phi(z)$ for $\gamma \in \Gamma_0(N)$, and

3. $F$ has exponential decay at all cusps of $\Gamma_0(N)$.

Let $\mathcal{S}_0(N,\lambda,\chi)$ be the space of all such functions. For $g \in G$, we put $\phi(g) = \Phi(g^{-1} \cdot \sqrt{-1})$. Let $V = \text{Sym}_2(\mathbb{R})$ be the space of real symmetric matrices of degree 2. Then $G = \mathbb{R}^x \times G$ acts on $V$ by $v \mapsto t \cdot gv^t g$ for $v \in V$ and $(t, g) \in \tilde{G}$. Let $V_+ = \{v \in V \mid \det v > 0\}$ and $V_- = \{v \in V \mid \det v < 0\}$. We have $V_+ = G \cdot I_2$ and $V_- = \tilde{G} \cdot J_2$, where

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Further, we put $H_+ = SO(I_2)$ and $H_- = SO(J_2)$ so that

$$H_+ = SO(2) = \left\{ k_{\theta} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \mid 0 \leq \theta < 2\pi \right\},$$

$$H_- = SO(1,1) = \left\{ a_y = \begin{pmatrix} y^{-1/2} & 0 \\ 0 & y^{1/2} \end{pmatrix} \mid y \in \mathbb{R}^x \right\}.$$

We normalize the Haar measures $d\mu_\pm$ on $H_\pm$ by

$$d\mu_+(k_{\theta}) = \frac{d\theta}{2}, \quad d\mu_-(a_y) = \frac{dy}{4|y|}.$$

Let $V_+^p$ (resp. $V_+^n$) be the set of positive (resp. negative) definite symmetric matrices in $V_+$. For $v \in V$ with $\det v \neq 0$, we take $t_v > 0$ and $g_v \in G$ such that

$$v = \begin{cases} t_v(g_v I_2 v) & \text{if } v \in V_+^p, \\ -t_v(g_v I_2 v) & \text{if } v \in V_+^n, \\ t_v(g_v J_2 v) & \text{if } v \in V_- \end{cases}.$$

For $v \in V \cap V_\pm$, we define the period $\mathcal{M}\phi(v)$ of $\phi$ by

$$\mathcal{M}\phi(v) = \int_{H_+/g_v^{-1}\Gamma_0,vg_v} \phi(h g_v^{-1}) d\mu_+(h),$$

2
where $\Gamma_0, \nu = \{ \gamma \in \Gamma_0(N) \mid \gamma v \gamma^t = v \}$. Then $\mathcal{M}(v)$ is absolutely convergent and does not depend on the choice of $g_v$. By \[9, \text{Lemma 6.3}\], for $v \in V_+$, we have

$$\mathcal{M}(v) = \pi \varepsilon(v) \cdot \Phi(z_v),$$

where $\varepsilon(v) = \#(\Gamma_0, v)$ and $z_v = g_v \cdot \sqrt{-1}$. Note that $z_v$ coincides with the so-called Heegner point associated with $v$. If $v \in V_-$, then $\{g_v a_y \cdot \sqrt{-1} \mid y > 0\}$ is the Heegner cycle associated with $v$, and thus $\mathcal{M}(v)$ coincides (up to constant) with a certain cycle integral of $\Phi$. Following the formulation of Shintani, we take a lattice $L_N$ defined by

$$L_N = \left\{ v = \left( \begin{array}{cc} v_1 & Nv_2 \\ Nv_2 & Nv_3 \end{array} \right) \mid v_1, v_2, v_3 \in \mathbb{Z} \right\}.$$ (see \[11, \text{p. 109}\]). Further, for $v \in L_N$, we put

$$d_N(v) := N(v_2)^2 - v_1 v_3 \quad (= -\frac{1}{N} \det v).$$

Let $V_Z$ be the set of half-integral symmetric matrices of degree 2:

$$V_Z = \left\{ w^* = \left( \begin{array}{ccc} w_1^* & w_2^* & w_3^* \\ w_2^* & w_3^* & 2 \\ w_3^* & 2 & w_1^* \end{array} \right) \mid w_1^*, w_2^*, w_3^* \in \mathbb{Z} \right\},$$

and for $w^* \in V_Z$, we put

$$\text{disc}(w^*) := (w_2^*)^2 - 4w_1^* w_3^* \quad (=-4 \det w^*).$$

We take an automorphic factor $J(\gamma, z)$ of weight $\frac{1}{2}$ defined by

$$J(\gamma, z) = \frac{\theta(\gamma z)}{\theta(z)}, \quad \text{with} \quad \theta(z) = \sum_{n=-\infty}^{\infty} \exp(2\pi in^2 z).$$

Let

$$\Delta_2 = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + i y \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$$

be the hyperbolic Laplacian of weight $\frac{1}{2}$ on $\mathcal{H}$, and $\psi$ a Dirichlet character of mod $4N$. A $C^\infty$-function $F : \mathcal{H} \to \mathbb{C}$ is called a Maass cusp form of weight $\frac{1}{2}$ for $\Gamma_0(4N)$ with character $\psi$ if

1. $\Delta_2 F = \mu(1 - \mu)F$ for a $\mu \in \mathbb{C}$,
2. $F(\gamma z) = \psi(\gamma) J(\gamma, z) F(z)$ for $\gamma \in \Gamma_0(4N)$, and
3. $F(z)$ has exponential decay at all cusps of $\Gamma_0(4N)$.

We denote by $\mathcal{S}_1(4N, \mu, \psi)$ the space of all such functions. Any $F \in \mathcal{S}_1(4N, \mu, \psi)$ has a Fourier expansion of the form

$$F(z) = \sum_{n=-\infty}^{\infty} c(n) \cdot W_{1, \mu}(n, y)e[nx], \quad (1.1)$$
where \( e[x] = \exp(2\pi \sqrt{-1}x) \) and for \( \ell \in \mathbb{Z} \),
\[
W_{\ell,\mu}(n, y) = y^{\frac{\ell}{2}} W_{\frac{\mu+\ell}{4}, \mu-\frac{1}{2}} (4\pi |n| y).
\] (1.2)

Here \( W_{\kappa,\nu}(z) \) denotes the Whittaker function. For a Dirichlet character \( \chi \) of mod\( N \), let
\[
\tau_\chi(n) = \sum_{m \mod N, (m,N)=1} \chi(m) e\left[ \frac{mn}{N} \right]
\]
be the Gauss sum associated with \( \chi \). Now we state our main theorem.

**Theorem 1.** Let \( \lambda \neq \frac{1}{2} \) and assume that \( \Phi(z) \in \mathcal{S}_0(N, \lambda, \chi^2) \). We put
\[
\mu = \frac{2\lambda+1}{4}, \quad \chi_N(r) = \chi(r) \left( \frac{N}{r} \right).
\]
Then there exists an \( F(z) \in \mathcal{S}_{1/2}(4N, \mu, \chi_N) \) such that the Fourier coefficients \( c(n) \) in (1.1) are given by
\[
c(n) = 2\pi^{-\frac{1}{2}} \cdot n^{-\frac{3}{4}} \sum_{v \in \Gamma_0(N) \setminus \mathcal{L}_N \atop d_N(v) = n} \chi(v_1) \mathcal{M}\phi(v),
\]
\[
c(-n) = n^{-\frac{3}{4}} \sum_{v \in \Gamma_0(N) \setminus \mathcal{L}_N \atop d_N(v) = -n} \chi(v_1) \Phi(z_v) e(v),
\]
for \( n = 1, 2, 3, \ldots \). Furthermore, if we put
\[
c^*(n) = 2\pi^{-\frac{1}{2}} \cdot n^{-\frac{3}{4}} \sum_{w^* \in \Gamma_0(N) \setminus \mathbb{V}_Z \atop \text{disc } w^* = n} \tau_\chi(w^*_v) \mathcal{M}\phi(w^*),
\]
\[
c^*(-n) = 2\pi^{-1} \cdot n^{-\frac{3}{4}} \sum_{w^* \in \Gamma_0(N) \setminus \mathbb{V}_Z \atop \text{disc } w^* = -n} \tau_\chi(w^*_v) \Phi(z_{w^*}) e(w^*),
\]
for \( n = 1, 2, 3, \ldots \), and define a function \( G(z) \) on \( \mathcal{H} \) by
\[
G(z) = N^{-\frac{3}{4}} \sum_{n=-\infty \atop n \neq 0}^{\infty} c^*(n) \cdot W_{1,\mu}(n, y)e[nx],
\]
then we have \( G(z) \in \mathcal{S}_{1/2}(4N, \mu, \chi_N) \) and
\[
F \left( -\frac{1}{4Nz} \right) (Nz)^{-\frac{1}{2}} = e \left[ -\frac{1}{8} \right] \cdot G(z).
\]
2 A Weil type converse theorem for Maass cusp forms

Our proof for Theorem 1 relies on a converse theorem given in [7]. Here let us recall briefly the result, with some modifications. For the convenience of readers, we give the statement for general weights. Fix an integer \( \ell \) and a positive integer \( N \). We assume that \( N \) is a multiple of 4 when \( \ell \) is odd. Let \( \alpha = \{ \alpha(n) \}_{n \in \mathbb{Z} \setminus \{0\}} \) and \( \beta = \{ \beta(n) \}_{n \in \mathbb{Z} \setminus \{0\}} \) be complex sequences of polynomial growth. For \( \alpha, \beta \), we can define the \( L \)-functions \( \xi_\pm(\alpha; s), \xi_\pm(\beta; s) \) by

\[
\xi_\pm(\alpha; s) = \sum_{n=1}^{\infty} \frac{\alpha(\pm n)}{n^s}, \quad \xi_\pm(\beta; s) = \sum_{n=1}^{\infty} \frac{\beta(\pm n)}{n^s},
\]

and the completed \( L \)-functions \( \Xi_\pm(\alpha; s) \) and \( \Xi_\pm(\beta; s) \) by \( \Xi_\pm(\alpha; s) = (2\pi)^{-s}\Gamma(s)\xi_\pm(\alpha; s) \) and \( \Xi_\pm(\beta; s) = (2\pi)^{-s}\Gamma(s)\xi_\pm(\beta; s) \).

Now we assume the following two conditions:

[C1] The \( L \)-functions \( \xi_\pm(\alpha; s), \xi_\pm(\beta; s) \) have analytic continuations to \textit{entire} functions of \( s \), and are of finite order in any vertical strip.

[C2] The following functional equation holds:

\[
\gamma(s) \begin{pmatrix} \Xi_+(\alpha; s) \\ \Xi_-(\alpha; s) \end{pmatrix} = N^{2-2\mu-s} \cdot \Sigma(\ell) \cdot \gamma(2-2\mu-s) \begin{pmatrix} \Xi_+(\beta; 2-2\mu-s) \\ \Xi_-(\beta; 2-2\mu-s) \end{pmatrix}, \tag{2.1}
\]

where \( \gamma(s) \) and \( \Sigma(\ell) \) are given by

\[
\gamma(s) = \begin{pmatrix} e^{\pi si/2} & e^{-\pi si/2} \\ e^{-\pi si/2} & e^{\pi si/2} \end{pmatrix}, \quad \Sigma(\ell) = \begin{pmatrix} 0 & i^\ell \\ 1 & 0 \end{pmatrix}.
\]

For an odd prime number \( r \) with \( (N, r) = 1 \) and a Dirichlet character \( \psi \mod r \), the twisted \( L \)-functions \( \xi_\pm(\alpha, \tau_\psi; s), \xi_\pm(\beta, \tau_\psi; s) \) are defined by

\[
\xi_\pm(\alpha, \tau_\psi; s) = \sum_{n=1}^{\infty} \frac{\alpha(\pm n)\tau_\psi(\pm n)}{n^s},
\]

\[
\xi_\pm(\beta, \tau_\psi; s) = \sum_{n=1}^{\infty} \frac{\beta(\pm n)\tau_\psi(\pm n)}{n^s},
\]

where \( \tau_\psi(n) \) is the Gauss sum associated with \( \psi \). The complete \( L \)-functions \( \Xi_\pm(\alpha, \tau_\psi; s) \) and \( \Xi_\pm(\beta, \tau_\psi; s) \) are defined by \( \Xi_\pm(\alpha, \tau_\psi; s) = (2\pi)^{-s}\Gamma(s)\xi_\pm(\alpha, \tau_\psi; s) \) and \( \Xi_\pm(\beta, \tau_\psi; s) = (2\pi)^{-s}\Gamma(s)\xi_\pm(\beta, \tau_\psi; s) \), respectively.

Let \( \mathbb{P}_N \) be a set of odd prime numbers not dividing \( N \) such that, for any positive integers \( a, b \) coprime to each other, \( \mathbb{P}_N \) contains a prime number \( r \) of the form \( r = am + b \) for some \( m \in \mathbb{Z}_{>0} \). For an \( r \in \mathbb{P}_N \), denote by \( X_r \) the set of all Dirichlet characters mod \( r \) (including the principal character). For \( \psi \in X_r \), we define the Dirichlet character \( \psi^* \) by

\[
\psi^*(k) = \overline{\psi(k)} \left( \frac{k}{r} \right)^\ell. \tag{2.2}
\]
For an odd integer \(d\), we put \(\varepsilon_d = 1\) or \(\sqrt{-1}\) according as \(d \equiv 1\) or \(3\) (mod 4). Let
\[
C_{\ell,r} = \begin{cases} 
1 & (\ell \text{ is even}), \\
\varepsilon_r^\ell & (\ell \text{ is odd}). 
\end{cases}
\]

In the following, we fix a Dirichlet character \(\chi \mod N\) that satisfies \(\chi(-1) = (\sqrt{-1})^\ell\) (resp. \(\chi(-1) = 1\)) when \(\ell\) is even (resp. odd).

For an \(r \in \mathbb{P}_N\) and a \(\psi \in X_r\), we consider the following conditions \([C1]_{r,\psi} - [C2]_{r,\psi}\) on \(\xi_\pm(\alpha, \tau_\psi; s)\) and \(\xi_\pm(\beta, \tau_\psi^*; s)\).

\([C1]_{r,\psi}\) \(\xi_\pm(\alpha, \tau_\psi; s), \xi_\pm(\beta, \tau_\psi^*; s)\) have analytic continuations to entire functions of \(s\), and are of finite order in any vertical strip.

\([C2]_{r,\psi}\) \(\Xi_\pm(\alpha, \tau_\psi; s)\) and \(\Xi_\pm(\beta, \tau_\psi^*; s)\) satisfy the following functional equation:
\[
\gamma(s) \left( \frac{\Xi_+(\alpha, \tau_\psi; s)}{\Xi_-(\alpha, \tau_\psi; s)} \right) = \chi(r) \cdot C_{\ell,r} \cdot \psi^*(-N) \cdot r^{2\mu - 2} \cdot (N_r^2)^{2 - 2\mu - s} \cdot \Sigma(\ell) \\
\cdot \gamma(2 - 2\mu - s) \left( \frac{\Xi_+(\beta, \tau_\psi^*; 2 - 2\mu - s)}{\Xi_-(\beta, \tau_\psi^*; 2 - 2\mu - s)} \right) \quad (2.3)
\]

**Lemma 1.** Let \(\mu \not\in \frac{1}{2}\mathbb{Z}\). We assume that \(\xi_\pm(\alpha; s)\) and \(\xi_\pm(\beta; s)\) satisfy the conditions \([C1]\) and \([C2]\). We assume furthermore that, for any \(r \in \mathbb{P}_N\) and \(\psi \in X_r\), \(\xi_\pm(\alpha, \psi; s)\) and \(\xi_\pm(\beta, \psi^*; s)\) satisfy the conditions \([C1]_{r,\psi}\) and \([C2]_{r,\psi}\). We define the function \(\tilde{W}_{\ell,\mu}(n,y)\) by
\[
\tilde{W}_{\ell,\mu}(n,y) = \frac{|n|^{\mu - 1}}{\Gamma \left( \mu + \frac{\text{sgn}(n)\ell}{4} \right)} \cdot W_{\ell,\mu}(n,y),
\]
where \(W_{\ell,\mu}(n,y)\) is given as (1.2), and the functions \(F_\alpha(z)\) and \(G_\beta(z)\) on \(\mathcal{H}\) by
\[
F_\alpha(z) = \sum_{n=-\infty}^{\infty} \sum_{n \neq 0} \alpha(n) \cdot \tilde{W}_{\ell,\mu}(n,y) e[nx],
\]
\[
G_\beta(z) = N^{1-\mu} \sum_{n=-\infty}^{\infty} \sum_{n \neq 0} \beta(n) \cdot \tilde{W}_{\ell,\mu}(n,y) e[nx].
\]

Then \(F_\alpha(z)\) (resp. \(G_\beta(z)\)) gives a Maass cusp form for \(\Gamma_0(N)\) of weight \(\frac{\ell}{2}\) with character \(\chi\) (resp. \(\chi_N,\ell\)), and eigenvalue \((\mu - \ell/4)(1 - \mu - \ell/4)\), where
\[
\chi_N,\ell(d) = \chi(d) \left( \frac{N}{d} \right)^{\ell}.
\]

Moreover, we have
\[
F_\alpha \left( -\frac{1}{Nz} \right) (\sqrt{Nz})^{-\ell/2} = G_\beta(z).
\]

**Remark 1.** Here we have assumed a stronger condition \(\mu \not\in \frac{1}{2}\mathbb{Z}\) than that given in the previous paper [7]. This enables us to remove conditions on zeros of \(L\)-functions (cf. [7] p. 33]). We also note that the cuspidality of \(F_\alpha(z), G_\beta(z)\) follows from the entireness of \(L\)-functions.
3 Prehomogeneous zeta functions

As an example of the theory of [9], Sato investigated the zeta functions associated to the vector space of symmetric matrices of degree 2 whose coefficients involve the periods \( M_\phi(v) \) of Maass cusp forms \( \Phi \). In this section, we introduce twisted versions of these zeta functions and give their analytic properties such as analytic continuations and functional equations.

Keep the notation as in the previous sections. We define zeta functions \( \zeta_{\pm}(\phi, \chi; s) \) and \( \zeta_{\pm}^*(\phi, \tau \chi; s) \) by

\[
\zeta_{\pm}(\phi, \chi; s) = \sum_{v \in \Gamma_0(N) \setminus \mathcal{L}_N} \frac{\chi(v_1)M_\phi(v)}{|d_N(v)|^s},
\]

\[
\zeta_{\pm}^*(\phi, \tau \chi; s) = \sum_{w^* \in \Gamma_0(N) \setminus \mathcal{L}_N} \frac{\tau_\chi(w^*_N)M_\phi(w^*)}{|\text{disc } w^*|^s}.
\]

Then we have the following lemma, whose proof is similar to that of [9, Theorem 6.7].

**Lemma 2.** The zeta functions \( \zeta_{\pm}(\phi, \chi; s) \) and \( \zeta_{\pm}^*(\phi, \tau \chi; s) \) have analytic continuations to entire functions of \( s \) and satisfy the following functional equation:

\[
\left( \begin{array}{c}
\zeta_+(\phi, \chi; 3 - s) \\
\zeta_-(\phi, \chi; 3 - s)
\end{array} \right) = \pi^{-s} N^{s-\frac{3}{2}} \Gamma \left( s + \frac{\lambda - 1}{2} \right) \Gamma \left( s - \frac{\lambda}{2} \right) \Psi_\lambda(s) \left( \begin{array}{c}
\zeta_+^*(\phi, \tau \chi; s) \\
\zeta_-^*(\phi, \tau \chi; s)
\end{array} \right),
\]

where \( \Psi_\lambda(s) \) is a \( 2 \times 2 \) matrix given by

\[
\Psi_\lambda(s) = \left( \begin{array}{cc}
\sin \pi s & \frac{2^\lambda - 1}{2} \cdot \pi \Gamma(1 - \lambda) \\
\Gamma \left( 1 - \frac{1 - \lambda}{2} \right)^2 \sin \frac{\pi \lambda}{2} & \cos \pi s
\end{array} \right).
\]

Let \( r \) be an odd prime number with \( (N, r) = 1 \), and \( \psi \) a Dirichlet character of \( \text{mod } r \). We denote by \( \psi^* \) the Dirichlet character defined as \([2,2]\) with \( \ell = 1 \). We define \( \zeta_{\pm}(\phi, \chi; \tau \psi; s) \) and \( \zeta_{\pm}^*(\phi, \tau \chi; \tau \psi^*; s) \) by

\[
\zeta_{\pm}(\phi, \chi; \tau \psi; s) = \sum_{v \in \Gamma_0(N) \setminus \mathcal{L}_N} \frac{\chi(v_1)M_\phi(v)\tau_\psi(d_N(v))}{|d_N(v)|^s},
\]

\[
\zeta_{\pm}^*(\phi, \tau \chi; \tau \psi^*; s) = \sum_{w^* \in \Gamma_0(N) \setminus \mathcal{L}_N} \frac{\tau_\chi(w^*_N)M_\phi(w^*)\tau_{\psi^*}(\text{disc } w^*)}{|\text{disc } w^*|^s}.
\]

Then we have the following lemma.
Lemma 3. The zeta functions \( \zeta_\pm(\phi, \chi; r, \psi; s) \) and \( \zeta_+^*(\phi, \tau, \tau^*; s) \) have analytic continuations to entire functions of \( s \) and satisfy the following functional equation:

\[
\begin{pmatrix}
\zeta_+(\phi, \chi; r, \psi; s) \\
\zeta_-(\phi, \chi; r, \psi; s)
\end{pmatrix} = \varepsilon_r \chi_N(r) \psi^s(-4N) \cdot r^{2s-\frac{3}{2}} \pi^{-\frac{3}{2} - 2s} \cdot \Gamma \left( s + \frac{\lambda - 1}{2} \right) \cdot \Psi(s) \left( \zeta_+^*(\phi, \tau, \tau^*; s) \\
\zeta_-(\phi, \tau, \tau^*; s) \right).
\]

The proof of Lemma 3 goes along the same line as Sato [8], Ueno [12]. In this case, however, it is necessary to calculate a kind of Gauss sums that have not appeared in the previous works. The author has learned such calculation from unpublished notes of Sato. We quote his result, which is a key ingredient and of independent interest. Let \( f_{\psi, \chi}(v) \) be a function on \( V \) defined by

\[
f_{\psi, \chi}(v) = \begin{cases} 
\tau_{\chi}(d_N(v)) \cdot \chi(v) & (v \in L_N) \\
0 & (v \notin L_N).
\end{cases}
\]

Let \( \langle v, v^* \rangle \) be the inner product on \( V \) defined by \( \langle v, v^* \rangle = \text{tr}(v^* w v w^{-1}) \) with \( w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \).

For \( v^* \in V \), we define the Fourier transform \( \hat{f}_{\psi, \chi}(v^*) \) by

\[
\hat{f}_{\psi, \chi}(v^*) = \frac{1}{[V : L]} \sum_{v \in V_0 / L} f_{\psi, \chi}(v) e(\langle v, v^* \rangle),
\]

where \( L \) is a sufficiently small lattice so that \( L \subset V \) and the value \( f_{\psi, \chi}(v) e(\langle v, v^* \rangle) \) depends only on the residue class \( v + L \).

Lemma 4 (F. Sato). If \( v^* \notin \frac{1}{N^r} V \), then we have \( \hat{f}_{\psi, \chi}(v^*) = 0 \). If \( v^* = \frac{1}{N^r} w^* \in \frac{1}{N^r} V \), we have

\[
\hat{f}_{\psi, \chi}(v^*) = \frac{\varepsilon_r}{2r^{\frac{3}{2}} N^3} \chi_N(r) \psi^s(-4N) \tau_{\chi}(w^*) \tau_{\psi^*}(\text{disc}(w^*)�).
\]

4 An outline of the proof of Theorem 1

We construct \( L \)-functions satisfying two conditions [C1] and [C2]. In the functional equation \( (2.1) \), we let \( \ell = 1 \) and \( \mu = \frac{2\lambda + 1}{4} \), and replace \( N \) by \( 4N \). Then it follows from an elementary calculation that \( (2.1) \) is transformed as

\[
\begin{pmatrix}
\xi_+(\lambda; s) \\
\xi_-(\lambda; s)
\end{pmatrix} = (4N)^{\frac{3}{2} - \lambda - s} \cdot 2^{2s + \lambda - \frac{3}{2}} \pi^{2s + \lambda - \frac{3}{2}} \cdot e \left( \frac{1}{8} \right) \Gamma(1 - s) \Gamma \left( \frac{3}{2} - \lambda - s \right) \cdot \begin{pmatrix}
-\cos \pi(s + \frac{1}{2}) & \sin \frac{\pi \lambda}{4} \\
\cos \frac{\pi \lambda}{4} & -\sin \pi(s + \frac{1}{2})
\end{pmatrix} \cdot \begin{pmatrix}
\xi_+(\beta; \frac{3}{2} - \lambda - s) \\
\xi_-(\beta; \frac{3}{2} - \lambda - s)
\end{pmatrix}.
\]

(4.1)
We put
\[
\widetilde{\zeta}_+(\phi, \chi; s) := 2^{2-\lambda} \cdot \frac{\Gamma(\lambda)}{\Gamma(\frac{3}{2})^2} \cdot \zeta_+\left(\phi, \chi; s + \frac{\lambda}{2}\right),
\]
\[
\widetilde{\zeta}_-(\phi, \chi; s) := \zeta_-\left(\phi, \chi; s + \frac{\lambda}{2}\right),
\]
\[
\widetilde{\zeta}_+^* (\phi, \tau; s) := 2^\frac{s}{2} \cdot N^{-\frac{s}{2}} \cdot \frac{\Gamma(\lambda)}{\Gamma(\frac{3}{2})^2} \cdot \zeta_+^*\left(\phi, \tau; s + \frac{\lambda}{2}\right),
\]
\[
\widetilde{\zeta}_-^* (\phi, \tau; s) := 2^{\lambda-\frac{3}{2}} \cdot N^{-\frac{3}{2}} \cdot \zeta_-^*\left(\phi, \tau; s + \frac{\lambda}{2}\right).
\]

Then (3.1) can be rewritten as
\[
\left(\frac{\widetilde{\zeta}_+(\phi, \chi; s)}{\zeta_+(\phi, \chi; s)}\right) = (4N)^{\frac{3}{2}-\lambda-s} \cdot 2^{2s+\lambda-\frac{3}{2}} \cdot N^{2s+\lambda-\frac{3}{2}} \cdot e\left[\frac{1}{8}\right] \Gamma(1-s) \Gamma\left(\frac{3}{2} - \lambda - s\right)
\]
\[
\cdot \left(- \cos \pi(s + \frac{\lambda}{2}) \quad \sin \pi\frac{\lambda}{2}\right) \cdot \left(\frac{\widetilde{\zeta}_+(\phi, \tau; \frac{3}{2}-\lambda-s)}{\widetilde{\zeta}_+(\phi, \tau; \frac{3}{2}-\lambda-s)}\right),
\]
which agrees with (4.1). Similarly, the functional equation (3.2) of the twisted zeta functions can be compared to (2.3) in the condition [C2],ψ. Now the converse theorem (Lemma 1) applies, and Theorem 1 is obtained. Further details will be discussed elsewhere.

**Remark 2.** In a paper [4] that appeared very recently, the Katok-Sarnak formula is generalized for Maass forms of even weight and odd level with trivial characters. It is an interesting problem to combine their technique, such as use of differential operators, with our method.

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