FREE NILPOTENT GROUPS ARE $C^*$-SUPERRIGID

TRON OMLAND

Abstract. The free nilpotent group $G_{m,n}$ of class $m$ and rank $n$ is the free object on $n$ generators in the category of nilpotent groups of class at most $m$. We show that $G_{m,n}$ can be recovered from its reduced group $C^*$-algebra, in the sense that if $H$ is any group such that $C^*_\lambda(H)$ is isomorphic to $C^*_\lambda(G_{m,n})$, then $H$ must be isomorphic to $G_{m,n}$.

Introduction

Group $C^*$-algebras play an important role in the theory of operator algebras. A natural question to ask, yet not much studied, is to what extent a group can be recovered from its (reduced) group $C^*$-algebra. The analog problem for group von Neumann algebras has received some attention in the last decades, and a group $G$ is called $W^*$-superrigid if it can be recovered from its group von Neumann algebra $L(G)$, that is, if $H$ is any group such that $L(H) \cong L(G)$, then $H \cong G$. The group von Neumann algebra of any nontrivial countable amenable group with infinite conjugacy classes is isomorphic to the hyperfinite II$_1$ factor, so in general, much of the structure is lost in the construction. However, some examples of $W^*$-superrigid groups are known, in particular, some classes of generalized wreath products.

Inspired by this terminology, a group $G$ is said to be $C^*$-superrigid if $C^*_\lambda(H) \cong C^*_\lambda(G)$ implies that $H \cong G$. It has been known for some time that torsion-free abelian groups are $C^*$-superrigid, and more recently, it has been shown that certain torsion-free virtually abelian groups \cite{1} and all finitely generated torsion-free 2-step nilpotent groups \cite{2} are $C^*$-superrigid.

In a somewhat different direction, specific examples of amalgamated free products are proven to be $C^*$-superrigid \cite{1}. For more background on the topic, see \cite{2} and references therein.

In this short note, we show that also the free nilpotent groups are $C^*$-superrigid.

1. Preliminaries and various results

Let $G$ be a discrete group. As usual, $C^*(G)$ denotes the full group $C^*$-algebra of $G$, and we let $g \mapsto u_g$ be the canonical inclusion of $G$ into $C^*(G)$. The left regular representation $\lambda$ of $G$ on $l^2(G)$ is given by $\lambda_g(\delta_h) = \delta_{gh}$ for all $g,h \in G$, and the reduced group $C^*$-algebra $C^*_r(G)$ of $G$ is the $C^*$-subalgebra of $B(l^2(G))$ generated by the image of $\lambda$. It follows that $\lambda$ induces a homomorphism of $C^*(G)$ onto $C^*_r(G)$, mapping $u_g$ to $\lambda_g$ for all $g \in G$. Moreover, it is well-known that if $C^*(G) \cong C^*_r(G)$, then $\lambda$ must be faithful, and in this case, $G$ is called amenable, and we use $\lambda$ to identify $C^*(G)$ with $C^*_r(G)$.

The subgroup $G'$ of $G$ generated by all the elements $ghg^{-1}h^{-1}$ for $g,h \in G$ is called the commutator subgroup of $G$. It is normal in $G$, and the quotient $G_{ab} = G/G'$ is an abelian group, called the abelianization of $G$. The group $G_{ab}$ is the largest abelian quotient of $G$, that is, whenever $N$ is normal subgroup of $G$ and $G/N$ is abelian, $G' \subseteq N$.
Let $\tilde{\pi}_{ab} : C^*(G) \to C^*(G_{ab})$ denote the homomorphism induced by the quotient map $\pi_{ab} : G \to G_{ab}$. Finally, we remark that $\pi_{ab}$ induces a map $C^*_r(G) \to C^*_r(G_{ab}) = C^*(G_{ab})$ only if $G^r$, or equivalently, $G$ is amenable.

For a $C^*$-algebra $A$, the commutator ideal $J$ of $A$ is the ideal generated by all elements $xy - yx$ for $x, y \in A$. Let $\phi : A \to A/J$ denote the quotient map. The Gelfand spectrum $\Gamma_A$ of a $C^*$-algebra $A$ is given by

$$
\Gamma_A = \left\{ \text{nonzero algebra homomorphisms } \gamma : A \to \mathbb{C} \right\}.
$$

If $\rho \in \Gamma_{A/J}$, then $\rho \circ \phi$ belongs to $\Gamma_A$, and every $\gamma \in \Gamma_A$ defines an element $\rho \in \Gamma_{A/J}$ given by $\rho(x + J) = \gamma(x)$. Together, this gives that $\Gamma_{A/J} = \Gamma_A$. Moreover, if $x \notin J$, then $0 \neq \phi(x) \in A/J$, which is commutative, so there exists $\rho \in \Gamma_{A/J}$ such that $\rho(\phi(x)) \neq 0$. We conclude that

$$
J = \bigcap_{\gamma \in \Gamma_A} \ker \gamma.
$$

**Lemma 1.1.** The commutator ideal $J$ of $C^*(G)$ coincides with the kernel of $\tilde{\pi}_{ab}$.

**Proof.** First, since $C^*(G_{ab})$ is commutative, $\ker \tilde{\pi}_{ab}$ must contain all commutators in $C^*(G)$, and thus $J \subseteq \ker \tilde{\pi}_{ab}$. Next, we note that

$$
\Gamma_{C^*(G_{ab})} = \text{Hom}(G_{ab}, \mathbb{T}) = \text{Hom}(G, \mathbb{T}) = \Gamma_{C^*(G)}.
$$

The second identification is given by $\chi' \mapsto \chi' \circ \pi_{ab}$, for $\chi' \in \text{Hom}(G_{ab}, \mathbb{T})$, and the inverse by $\chi \mapsto \chi'$ for $\chi \in \text{Hom}(G, \mathbb{T})$, where $\chi'(g + G') = \chi(g)$. The last identification is the usual integrated form, with inverse $\gamma \mapsto \chi$ for $\gamma \in \Gamma_{C^*(G)}$, where $\chi(g) = \gamma(g) - \gamma(1)$; and the first equality is similar. Combined, the first and last space is identified via $\gamma' \mapsto \gamma' \circ \pi_{ab}$ for $\gamma' \in \Gamma_{C^*(G_{ab})}$.

Thus, if $x \notin J$, then by (1) there is $\gamma \in \Gamma_{C^*(G)}$ such that $\gamma(x) \neq 0$. Since $\gamma = \gamma' \circ \tilde{\pi}_{ab}$ for some $\gamma' \in \Gamma_{C^*(G_{ab})}$, we have $\gamma'(\tilde{\pi}_{ab})(x) \neq 0$, and hence $x \notin \ker \tilde{\pi}_{ab}$. □

The following result is proven in [3] Theorem 8.58.

**Proposition 1.2.** Suppose that $G$ is torsion-free and abelian and let $H$ be any group such that $C^*(H) \cong C^*(G)$. Then $H \cong G$.

**Corollary 1.3.** If $H$ is any group such that $C^*(H) \cong C^*(G)$, then $C^*(H_{ab}) \cong C^*(G_{ab})$. In particular, if $G_{ab}$ is torsion-free, then $H_{ab} \cong G_{ab}$.

**Proof.** Any isomorphism $C^*(H) \cong C^*(G)$ takes the commutator ideal of $C^*(H)$ to the commutator ideal of $C^*(G)$, and thus, the quotients $C^*(H_{ab})$ and $C^*(G_{ab})$ must be isomorphic. □

The upper central sequence of $G$, denoted $Z_0 \subset Z_1 \subset Z_2 \subset \cdots$, is defined by $Z_0 = \{e\}$, $Z_1 = Z(G)$, and for all $i \geq 0$,

$$
Z_{i+1} = \{ g \in G : [g, h] \in Z_i \text{ for all } h \in G \}.
$$

In particular, we remark that $Z_i$ is a normal subgroup of $Z_{i+1}$ and $Z_{i+1}/Z_i = Z(G/Z_i)$ for all $i \geq 0$. If there exists an $m$ such that $G = Z_m$, then $G$ is called a nilpotent group, and the smallest such $m$ is said to be the class of $G$.

**Lemma 1.4.** Suppose that $G$ is a nilpotent group and let $S \subseteq G$ be a set such that $\pi_{ab}(S)$ generates $G_{ab}$. Then $S$ generates $G$.

**Proof.** Let $m$ be the nilpotency class of $G$, and let $\{e\} = Z_0 \subset Z_1 \subset \cdots \subset Z_{m-1} \subset Z_m = G$ be the upper central series of $G$. Denote by $H$ the subgroup of $G$ generated by $S$. For $0 \leq i \leq m$, set $H_i = HZ_i$. Then $H_i$ is a subgroup of $G$ and a normal subgroup of $H_{i+1}$ for all $i$. Indeed, for $h, h' \in H_i$, $z_i, z'_i \in Z_i$, $$(h'z_i)(h'z'_i) = h'hz_i[z_i^{-1}, h^{-1}]z'_i \in HZ_iZ_{i-1}Z_i = HZ_i = H_i.$$
since \([z_i^{-1}, h^{-1}] \in Z_{i-1}\). Moreover, for \(h, h' \in H\), \(z_i, z_i' \in Z_i\),
\[(hz_i+1)(h'z_i)(hz_i+1)^{-1} = h[z_i+1, h']h'z_i+1z_i^{-1}z_i[z_i^{-1}, z_i+1]h^{-1} \in HZ,HZ,Z_{i-1}H = H_i.\]
If \(H \neq G\), there would exist some \(0 \leq k < m\) such that \(H_k \neq G\) and \(H_{k+1} = G\). Then
\[G/H_k = H_{k+1}/H_k = HZ_{k+1}/HZ_k \cong Z_{k+1}/Z_k,\]
where the last identification is the second isomorphism theorem, and the last quotient is abelian. Thus, \(H_k\) contains the commutator subgroup \(G'\), and therefore also \(HG'\). Since \(\pi_{ab}(H) = G_{ab}\), then \(HG' \cong G\).

Hence, we conclude that \(H = G\).

2. \(C^+\)-superrigidity of free nilpotent groups

The free nilpotent group \(G_{m,n}\) of class \(m\) and rank \(n\) is the free object on \(n\) generators in the category of nilpotent groups of class at most \(m\). It is defined by a set of generators \(\{g_i\}_{i=1}^n\) subject to the relations that all commutators of length \(m+1\) involving the generators are trivial, i.e., \([\cdots[g_i, g_{i_2}, g_{i_3}, \cdots, g_{i_m}], g_{i_{m+1}}]\) is trivial for any choice of sequence of generators.

For all \(m \geq 1\), we have \(G_{m,1} \cong Z\), while \(G_{m,n}\) is an \(m\)-step nilpotent group for every \(n \geq 2\). See [5] and [6] for more about free nilpotent groups.

The group \(G_{m,n}\) satisfies the following universal property: If \(H\) is any nilpotent group of class at most \(m\) and \(h_1, \ldots, h_n\) are elements in \(H\), there exists a unique homomorphism \(G_{m,n} \to H\) mapping \(g_i\) to \(h_i\) for all \(i\).

The abelianization of \(G_{m,n}\) is isomorphic to \(\mathbb{Z}^n\) and \(\pi_{ab}\) maps \(g_i\) to the generator \(e_i\) of the \(i\)th summand of \(\mathbb{Z}^n\).

The center \(Z(G_{m,n})\) of \(G_{m,n}\) is a free abelian group (its rank can be computed, but it is not relevant here), and for \(m, n \geq 2\) we have
\[(2)\quad G_{m,n}/Z(G_{m,n}) \cong G_{m-1,n},\]
as seen by mapping generators to generators.

**Lemma 2.1.** Let \(m, n \geq 2\), and let \(H\) be a nilpotent group of class at most \(m\) that can be generated by \(n\) elements. Suppose that \(H/Z(H) \cong G_{m-1,n}\). Then \(H \cong G_{m,n}\).

**Proof.** The universal property of \(G_{m,n}\) means that there exists a surjective map \(\varphi : G_{m,n} \to H\). Clearly, \(\varphi(Z(G_{m,n})) \subseteq Z(H)\), and we set \(K = \varphi^{-1}(Z(H))\). Consider the maps
\[G_{m,n}/Z(G_{m,n}) \to G_{m,n}/K \to H/Z(H),\]
given by \(aZ(G_{m,n}) \mapsto aK\) and \(aK \mapsto \varphi(a)Z(H)\). The composition map \(\psi\) is surjective since \(\varphi\) is surjective. Since finitely generated nilpotent groups are Hopfian, \(G_{m-1,n} \cong G_{m,n}/Z(G_{m,n})\) does not have any proper quotient isomorphic to itself. Hence, the composition map \(\psi\) must be an isomorphism, and \(K = Z(G_{m,n})\). We get the following commutative diagram
\[
\begin{array}{ccc}
1 & \longrightarrow & Z(G_{m,n}) & \overset{1}{\longrightarrow} & G_{m,n} & \overset{q}{\longrightarrow} & G_{m,n}/Z(G_{m,n}) & \longrightarrow & 1 \\
\phi & \cong & \varphi & \quad \quad & \psi & \cong & \psi & \quad \quad & \psi \\
1 & \longrightarrow & Z(H) & \overset{1}{\longrightarrow} & H & \overset{q}{\longrightarrow} & H/Z(H) & \longrightarrow & 1
\end{array}
\]
By the five lemma, \(\varphi\) is an isomorphism. \(\square\)

**Theorem 2.2.** For all natural numbers \(m\) and \(n\), the free nilpotent group \(G_{m,n}\) of class \(m\) and rank \(n\) is \(C^+\)-superrigid.
Proof. The case $n = 1$ is obvious, so let $n \geq 2$. We do this by induction on $m$. Note first that $G_{1,n} \cong \mathbb{Z}^n$, which is $C^*$-superrigid (see Proposition 1.2). Let $m \geq 2$, and suppose that $G_{m-1,n}$ is $C^*$-superrigid. Let $H$ be any group and assume that $C^*(H) \cong C^*(G_{m,n})$. It follows from [2, Theorem B] that $H$ is a torsion-free nilpotent group of class $m$.

Moreover, $C^*(H/Z(H)) \cong C^*(G_{m,n}/Z(G_{m,n}))$ by [2, Proof of Lemma 4.3], and [2] implies that the latter is isomorphic to $C^*(G_{m-1,n})$. By the induction hypothesis, $G_{m-1,n}$ is $C^*$-superrigid, so $H/Z(H) \cong G_{m-1,n}$.

The abelianization of $G_{m,n}$ is isomorphic to $\mathbb{Z}^n$ and thus $H_{ab} \cong \mathbb{Z}^n$ by Corollary 1.3. For each $1 \leq i \leq n$, choose an element $s_i$ of $H$ that is mapped to the generator $e_i$ of $\mathbb{Z}^n \cong H_{ab}$. If $S = \{s_i : 1 \leq i \leq n\}$, then $\pi_{ab}(S)$ generates $H_{ab}$, so $S$ generates $H$ by Lemma 1.4, i.e., $H$ can be generated by $n$ elements.

Therefore, we may apply Lemma 2.1 to conclude that $H \cong G_{m,n}$. \qed

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Department of Mathematics, University of Oslo, P.O.Box 1053 Blindern, NO-0316 Oslo, Norway

E-mail address: trono@math.uio.no