Rawnsley’s ε-function on some Hartogs type domains over bounded symmetric domains and its applications

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Abstract The purpose of this paper is twofold. Firstly, we will compute the explicit expression of the Rawnsley’s ε-function $\varepsilon_{(\alpha, g(\mu; \nu))}$ of $\left(\prod_{j=1}^{k} \Omega_j^{\mathbb{B}_d(\mu)}(\mu, g(\mu; \nu))\right)$, where $g(\mu; \nu)$ is a Kähler metric associated with the Kähler potential $-\sum_{j=1}^{k} t_j \ln N_{\Omega_j}(z_j, \overline{z}_j)^{\rho_j} - \ln(\prod_{j=1}^{k} N_{\Omega_j}(z_j, \overline{z}_j)^{\rho_j} - \|w\|^2)$ on the generalized Cartan-Hartogs domain $\left(\prod_{j=1}^{k} \Omega_j^{\mathbb{B}_d(\mu)}(\mu)\right)$ and obtain necessary and sufficient conditions for $\varepsilon_{(\alpha, g(\mu; \nu))}$ to become a polynomial in $1 - \|\tilde{w}\|^2$ (see the definition (1.6) for $\tilde{w}$). Secondly, we study the Berezin quantization on $\left(\prod_{j=1}^{k} \Omega_j^{\mathbb{B}_d(\mu)}(\mu)\right)$ with the metric $g(\mu; \nu)$.

Key words: Bergman kernels · Bounded symmetric domains · Cartan-Hartogs domains · Kähler metrics · Berezin quantization

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1 Introduction

Recently, Berezin quantization has received a lot of attention (e.g., see Cahen-Gutt-Rawnsley \textsuperscript{4}, Englisch \textsuperscript{7}, Loi-Mossa \textsuperscript{18} and Zedda \textsuperscript{28}). Roughly, a quantization is a construction of a quantum system from the classical mechanics of a system. In 1927, Weyl made an attempt at a quantization known as Weyl quantization. His original ideal is associating a self-adjoint operators on a separable Hilbert space with functions on a symplectic manifold and some certain commutations are fulfilled. Later on, Berezin \textsuperscript{2} raised a new quantization procedure, i.e., Berezin quantization. A Berezin quantization on a Kähler manifold $(\Omega, \omega)$ is given by a family of associative algebra $\mathcal{A}_h$ where the parameter $h$ runs through a set $E$ of the positive reals with 0 in its closure and moreover there exist a subalgebra $\mathcal{A}$ of $\bigoplus \{\mathcal{A}_h; h \in E\}$ such that some properties are satisfied (see Berezin \textsuperscript{2} for details). More precisely, we call an associative algebra with involution $\mathcal{A}$ a quantization of $(\Omega, \omega)$ if the following properties are satisfied.

1. There exist a family of associative algebras $\mathcal{A}_h$ of functions on $\Omega$ where the parameter $h$ runs through a set $E$ of the positive reals with 0 in its closure. Moreover $\mathcal{A}$ is a subalgebra of $\bigoplus \{\mathcal{A}_h; h \in E\}$.
2. For each $f \in \mathcal{A}$ which will be written $f(h, x)$ ($h \in E$, $x \in \Omega$) such that $f(h, \cdot) \in \mathcal{A}_h$, the limit

$$\lim_{h \to 0^+} f(h, x) = \varphi(f)(x)$$

exists.

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(3) \( \varphi(f \ast g) = \varphi(f) \cdot \varphi(g) \), \( \varphi(h^{-1}(f \ast g - g \ast f)) = \frac{1}{2}\{\varphi(f), \varphi(g)\} \) for \( f, g \in \mathcal{A} \). Here \( \ast \) and \( \{,\} \) denote the product of \( \mathcal{A} \) and the Poisson bracket.

(4) For any two points \( x_1, x_2 \in \Omega \), there exists \( f \in \mathcal{A} \) such that \( \varphi(f)(x_1) \neq \varphi(f)(x_2) \).

Suppose \( D \) is a bounded domain in \( \mathbb{C}^n \) and \( \varphi \) is a strictly plurisubharmonic function on \( D \). Let \( g \) be a Kähler metric on \( D \) associated with the Kähler form \( \omega = \sqrt{-1} \partial \bar{\partial} \varphi \). For \( \alpha > 0 \), let \( \mathcal{H}_\alpha \) be the weighted Hilbert space of square integrable holomorphic functions on \( (D, g) \) with the weight \( \exp\{-\alpha \varphi\} \), that is,

\[
\mathcal{H}_\alpha := \left\{ f \in \text{Hol}(D) : \int_D |f|^2 \exp(-\alpha \varphi) \frac{\omega^n}{n!} < +\infty \right\},
\]

where \( \text{Hol}(D) \) denotes the space of holomorphic functions on \( D \). Let \( K_\alpha(z, \bar{\tau}) \) be the Bergman kernel (namely, the reproducing kernel) of the Hilbert space \( \mathcal{H}_\alpha \) if \( \mathcal{H}_\alpha \neq \{0\} \). The Rawnsley’s \( \varepsilon \)-function (see [4]) on \( D \) associated with the metric \( g \) is defined by

\[
\varepsilon_{(\alpha, g)}(z) := \exp\{-\alpha \varphi(z)\} K_\alpha(z, \bar{\tau}), \quad z \in D. \tag{1.1}
\]

Note the Rawnsley’s \( \varepsilon \)-function depends only on the metric \( g \) and not on the choice of the Kähler potential \( \varphi \). The asymptotics of the Rawnsley’s \( \varepsilon \)-function \( \varepsilon_\alpha \) was expressed in terms of the parameter \( \alpha \) for compact manifolds by Catlin [6] and Zelditch [29] (for \( \alpha \in \mathbb{N} \)) and for non-compact manifolds by Ma-Marinescu [21, 22, 23]. In some particular case it was also proved by Engliš [8, 9]. Especially, when the function \( \varepsilon_{(\alpha, g)}(z) \) is a positive constant on \( D \), the metric \( \alpha g \) is called balanced.

In order to establish a quantization procedure on a noncompact manifold \( (D, g) \), we firstly give the following two conditions (refer to [7] and [28]):

(I). The function \( \exp\{-D_g(z, u)\} \) is globally defined on \( D \times D \), \( \exp\{-D_g(z, u)\} \leq 1 \) and \( \exp\{-D_g(z, u)\} = 1 \) if and only if \( z = u \), here \( D_g(z, u) \) denotes the Calabi’s diastasis function (see Calabi [5]) which is defined by

\[
D_g(z, u) := \varphi(z, \bar{\tau}) + \varphi(u, \bar{\tau}) - \varphi(z, \bar{\tau}) - \varphi(u, \bar{\tau}).
\]

(II). The function \( a(z, \bar{\tau}) \) admits a sesquianalytic extension on \( D \times D \), that is,

\[
a(z, \bar{\tau}) = \exp\{-\varphi(z, \bar{\tau})\}
\]

and moreover there exists a infinite set \( E \) of integers such that for all \( \alpha \in E \), \( z, u \in D \),

\[
\varepsilon_{(\alpha, g)}(z, \bar{\tau}) = \exp\{-\alpha \varphi(z)\} K_\alpha(z, \bar{\tau}) = \alpha^n + B(z, \bar{\tau}) \alpha^{n-1} + C(\alpha, z, \bar{\tau}) \alpha^{n-2},
\]

where \( B(z, \bar{\tau}) \) and \( C(\alpha, z, \bar{\tau}) \) are sesquianalytic functions in \( z, \bar{\tau} \) which satisfy

\[
\sup_{z, u \in D} |B(z, \bar{\tau})| < +\infty, \quad \sup_{z, u \in D, \alpha \in E} |C(\alpha, z, \bar{\tau})| < +\infty.
\]

If \( \alpha g \) are balanced metrics on \( D \) for \( \alpha \in E \), by definition \( (D, g) \) automatically satisfies the condition (II). When the condition (I) is satisfied and \( \alpha g \) are balanced metrics for \( \alpha \in E \) on \( D \), Berezin [2] was able to establish a quantization procedure on \( (\Omega, g) \). In 1996, Engliš [7] extended the Berezin quantization to the case when the above conditions (I) and (II) are satisfied. In 2012, Loi-Mossa [18] proved that the above conditions (I) and (II) are satisfied by any homogeneous bounded domain \( \Omega \) equipped with a homogeneous Kähler metric \( g \) and thus the homogeneous bounded domain \( (D, g) \) must admit a Berezin quantization also.

Later on, Loi-Mossa (see [19] Theorem 1.2) also gave necessary and sufficient condition for a homogeneous Kähler manifold to admit a Berezin quantization. They also prove that a contractible homogeneous Kähler manifold (i.e., all the products \( (\Omega, g) \times (\mathbb{C}^m; g_0) \) where \( (\Omega, g) \) is an homogeneous bounded domain and \( g_0 \) is the standard flat metric) admits a Berezin quantization. However, except for the above cases, the known instances when the above conditions (I) and (II) are satisfied are very
few (see Engliš [7] and Loi-Mossa [18]). So it is interesting to find more complete noncompact Kähler manifolds which a quantization can be carried out.

Let \( \Omega_i \subseteq \mathbb{C}^{d_i} \) be an irreducible bounded symmetric domain \((1 \leq i \leq k)\). For given positive integer \( d_0 \), positive real numbers \( \mu_i \) \((1 \leq i \leq k)\), the generalized Cartan-Hartogs domain \( \left( \prod_{j=1}^{k} \Omega_j \right)^{\mathbb{B}^{d_0}}(\mu) \) is defined by

\[
\left( \prod_{j=1}^{k} \Omega_j \right)^{\mathbb{B}^{d_0}}(\mu) := \left\{ (z, w) \in \prod_{j=1}^{k} \Omega_j \times \mathbb{B}^{d_0} : ||w||^2 < \sum_{j=1}^{k} N_{\Omega_j}(z_j, \overline{z_j})^{\mu_j} \right\},
\]

where \( \mu = (\mu_1, \ldots, \mu_k) \in (\mathbb{R}_+)^k \), \( z = (z_1, \ldots, z_k) \in \mathbb{C}^{d_1} \times \cdots \times \mathbb{C}^{d_k} \), \( || \cdot || \) is the standard Hermitian norm in \( \mathbb{C}^{d_0} \), \( N_{\Omega_j}(z_j, \overline{z_j}) \) is the generic norm of \( \Omega_j \) \((1 \leq i \leq k)\) and \( \mathbb{B}^{d_0} := \{ w \in \mathbb{C}^{d_0} : ||w||^2 < 1 \} \).

For the reference of the generalized Cartan-Hartogs domains, see Feng-Tu [15], Tu-Wang [24] and Wang-Hao [25].

Let \( \nu = (\nu_1, \nu_2, \ldots, \nu_k) \) with \( \nu_j > -1 \) for \( 1 \leq j \leq k \). For the generalized Cartan-Hartogs domain \( \left( \prod_{j=1}^{k} \Omega_j \right)^{\mathbb{B}^{d_0}}(\mu) \), define

\[
\Phi(z, w) := -\sum_{j=1}^{k} \nu_j \ln N_{\Omega_j}(z_j, \overline{z_j})^{\mu_j} - \ln \left( \prod_{j=1}^{k} N_{\Omega_j}(z_j, \overline{z_j})^{\mu_j} - ||w||^2 \right).
\]

The Kähler form \( \omega(\mu; \nu) \) on \( \left( \prod_{j=1}^{k} \Omega_j \right)^{\mathbb{B}^{d_0}}(\mu) \) is defined by

\[
\omega(\mu; \nu) := \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \Phi.
\]

The metric \( g(\mu; \nu) \) on \( \left( \prod_{j=1}^{k} \Omega_j \right)^{\mathbb{B}^{d_0}}(\mu) \) associated with \( \omega(\mu; \nu) \) is given by

\[
ds^2 = \sum_{i,j=1}^{n} \frac{\partial^2 \Phi}{\partial Z_i \partial \overline{Z}_j} dZ_i \otimes d\overline{Z}_j,
\]

where

\[n = \sum_{j=0}^{k} d_j, \quad Z = (Z_1, \ldots, Z_n) := (z, w).\]

If \( \nu = 0 \), then the metric \( g(\mu; \nu) \) becomes the standard canonical metric (e.g., see Bi-Tu [3], Feng-Tu [14, 15], Loi-Zedda [20] and Zedda [27, 28]). In this paper, we will focus our attention on the metric \( g(\mu; \nu) \) on \( \left( \prod_{j=1}^{k} \Omega_j \right)^{\mathbb{B}^{d_0}}(\mu) \). In the following, we also present some new notations which will not be explicated in this section. Please refer to the Section 2 for details.

For the generalized Cartan-Hartogs domain \( \left( \prod_{j=1}^{k} \Omega_j \right)^{\mathbb{B}^{d_0}}(\mu), g(\mu; \nu) \) in the case of \( \nu = 0 \), Feng-Tu [15] proved that the Rawnsley’s \( \varepsilon \)-function admits the following expansion:

**Theorem 1.1** (Feng-Tu [15]). Let \( \Omega_i \) be an irreducible bounded symmetric domain in \( \mathbb{C}^{d_i} \), and denote the generic norm \( N_{\Omega_i}(z_i, \overline{z_i}) \), the dimension \( d_i \) and the genus \( p_i \) for \( \Omega_i \) \((1 \leq i \leq k)\). Set \( n = \sum_{j=0}^{k} d_j \), \( d = \sum_{j=1}^{k} d_j \) and \( \alpha > \max \left\{ \frac{p_1-1}{\mu_1(1+\nu_1)}, \ldots, \frac{p_k-1}{\mu_k(1+\nu_k)} \right\} \). Then the Rawnsley’s \( \varepsilon \)-function associated with \( \left( \prod_{j=1}^{k} \Omega_j \right)^{\mathbb{B}^{d_0}}(\mu), g(\mu; 0) \) can be written as

\[
\varepsilon_{(\alpha, g(\mu; 0))}(z, w) = \frac{1}{\prod_{i=1}^{k} \mu_i d_i} \sum_{j=0}^{d} D^j \chi(d) \left( 1 - ||w||^2 \right)^{d-j(\alpha - n) d_0},
\]

where

\[D^j \chi(d) := \sum_{j=0}^{d} (\chi(0) - \chi(d)) j^j.\]
where
\[ \tilde{w} := \frac{w}{\prod_{j=1}^{k} N_{\Omega_j}(z_j, \overline{z}_j)^{\frac{\nu_j}{2}}} \] (1.6)

So, firstly, we will compute the expression of the Rawnsley’s \( \varepsilon \)-function for any \( \nu > -1 \) as follows.

**Theorem 1.2.** Let \( \Omega_i \) be an irreducible bounded symmetric domain in \( \mathbb{C}^{d_i} \) in its Harish-Chandra realization, and denote the generic norm \( N_{\Omega_i}(z_i, \overline{z}_i) \) the dimension \( d_i \) and the genus \( p_i \) for \( \Omega_i \) (1 \( \leq i \leq k \)). For \( \alpha > \max\{n, \frac{p_1-1}{\mu_1(1+\nu_1)}, \ldots, \frac{p_k-1}{\mu_k(1+\nu_k)}\} \). Then the Rawnsley’s \( \varepsilon \)-function associated with \( ((\prod_{j=1}^{k} \Omega_{j})^{\mathbb{B}_{\nu}}(\mu), g(\mu; \nu)) \) can be written as
\[
\varepsilon_{(\alpha, g(\mu; \nu))}(z_1, \ldots, z_k, w) = (\alpha - n)_{\alpha}(1 - \|\tilde{w}\|)^2 \sum_{t=0}^{\infty} \psi(\alpha, t) \frac{\alpha_t}{t!} \|\tilde{w}\|^{2t},
\] (1.7)
where
\[
\tilde{w} = \frac{w}{\prod_{j=1}^{k} N_{\Omega_j}(z_j, \overline{z}_j)^{\frac{\nu_j}{2}}}, \quad \psi(x, y) = \prod_{i=1}^{k} \chi_{i}(\mu_i((1 + \nu_i)x + y) - p_i) / \prod_{i=1}^{k} \mu_i^d \sum_{t=0}^{\infty} \sigma(t)(x - n)_{d-1}(x + y - t)t,
\]
and
\[
\sigma(t) = \sum_{\sum_{i=1}^{k} t_i = t} \prod_{i=1}^{k} \left( \frac{d_i}{t_i} \right). \] (1.8)

Obviously, when \( \nu = 0 \), the Rawnsley’s \( \varepsilon \)-function \( \varepsilon_{(\alpha, g(\mu; \nu))} \) of \( ((\prod_{j=1}^{k} \Omega_{j})^{\mathbb{B}_{\nu}}(\mu), g(\mu; \nu)) \) becomes a polynomial in \( 1 - \|\tilde{w}\|^2 \) (see (1.5)). However, by Theorem 1.2, we know that the Rawnsley’s \( \varepsilon \)-function \( \varepsilon_{(\alpha, g(\mu; \nu))} \) may not be a polynomial in \( 1 - \|\tilde{w}\|^2 \) for a general \( \nu \). So we are interested in finding some \( \nu_0 \) such that the Rawnsley’s \( \varepsilon \)-function is a polynomial in \( 1 - \|\tilde{w}\|^2 \).

In this paper, with the expression of the Rawnsley’s \( \varepsilon \)-function \( \varepsilon_{(\alpha, g(\mu; \nu))} \) of \( ((\prod_{j=1}^{k} \Omega_{j})^{\mathbb{B}_{\nu}}(\mu), g(\mu; \nu)) \) (see Theorem 1.2), we obtain necessary and sufficient conditions for \( \varepsilon_{(\alpha, g(\mu; \nu))} \) being a polynomial in \( 1 - \|\tilde{w}\|^2 \).

**Corollary 1.3.** Let \( \Omega_i \subseteq \mathbb{C}^{d_i} \) be an irreducible bounded symmetric domain, and denote the generic norm \( N_{\Omega_i}(z_i, \overline{z}_i) \), the dimension \( d_i \) and the genus \( p_i \) for \( \Omega_i \) (1 \( \leq i \leq k \)). Set \( n = \sum_{j=0}^{k} d_j \) and \( d = \sum_{j=1}^{k} d_j \). Then the Rawnsley’s \( \varepsilon \)-function associated with \( ((\prod_{j=1}^{k} \Omega_{j})^{\mathbb{B}_{\nu}}(\mu), g(\mu; \nu)) \) can be written as
\[
\varepsilon_{(\alpha, g(\mu; \nu))}(z_1, \ldots, z_k, w) = \sum_{j=0}^{d} c_j(\alpha) (1 - \|\tilde{w}\|^2)^{d-j}
\] (1.9)
if only and if
\[
\alpha > \max\left\{ n, \frac{p_1-1}{\mu_1(1+\nu_1)}, \ldots, \frac{p_k-1}{\mu_k(1+\nu_k)} \right\}
\]
and
\[
\phi(x) = (x - d)_d \psi(\alpha, x - \alpha) = \frac{(x - d)_d \prod_{i=1}^{k} \chi_{i}(\mu_i(\nu_i\alpha + x) - p_i)}{\prod_{i=1}^{k} \mu_i^d \sum_{t=0}^{\infty} \sigma(t)(\alpha - n)_{d-1}(x - t)t}
\] (1.10)
is a polynomial in \( x \).

Furthermore, (1.9) can be re-written as
\[
\varepsilon_{(\alpha, g(\mu; \nu))}(z_1, \ldots, z_k, w) = \sum_{j=0}^{d} \frac{D^j \phi(d)}{j!}(\alpha - n)_{d_0+j} (1 - \|\tilde{w}\|^2)^{d-j},
\] (1.11)
Let \( T \) be an irreducible bounded symmetric domain, and denote the rank \( n \), the characteristic multiplicities \( a_i, b_i \), the dimension \( d_i \) and the genus \( g \) for \( T \). Let \( g(\mu; \nu) \) be the metric on the generalized Cartan-Hartogs domain \( (\prod_{j=1}^{k} \Omega_j)^{B_{\mu}}(\mu) \). Assume that \( \mu_i \in W(\Omega_i) \backslash \{0\} \), \( 1 \leq i \leq k \), where \( W(\Omega_i) \) are the Wallach sets defined by

\[
W(\Omega_i) := \left\{ 0, \frac{a_i}{2}, \frac{a_i}{2}, \ldots, (r_i - 1)\frac{a_i}{2} \right\} \cup \left\{ (r_i - 1)\frac{a_i}{2}, +\infty \right\}.
\]

If for sufficiently large \( \alpha \), \( \phi(x) \) (see (1.10)) is a polynomial in \( x \), then the generalized Cartan-Hartogs domain \( (\prod_{j=1}^{k} \Omega_j)^{B_{\mu}}(\mu), g(\mu; \nu) \) admits a Berezin quantization.

When \( \varepsilon(\alpha, g(\mu; \nu)) \) is a positive constant, we have the following result.

**Theorem 1.5.** Let \( (\prod_{j=1}^{k} \Omega_j)^{B_{\mu}}(\mu), g(\mu; \nu) \) be the generalized Cartan-Hartogs domain with canonical metric \( g(\mu; \nu) \). Then for \( \alpha > \max\{n, \frac{p_{1} - 1}{\mu_1(1 + \nu_1)}, \ldots, \frac{p_{k} - 1}{\mu_k(1 + \nu_k)}\} \), the metric \( g(\mu; \nu) \) is balanced if and only if for all \( x, y \in \mathbb{R} \),

\[
\prod_{i=1}^{k} \chi_i(\mu_i((1 + \nu_i)x + y) - p_i) = \left( \prod_{i=1}^{k} \mu_i^{d_i} \right) \sum_{t=0}^{\infty} \left( \sum_{\sum_{i=1}^{k} t_i = t} \prod_{i=1}^{k} \frac{d_i}{t_i} \nu_i^{t_i} \right) (x - n)t(x + y - d + t)d_{i-1}. \tag{1.13}
\]

Moreover, under this situation, \( (\prod_{j=1}^{k} \Omega_j)^{B_{\mu}}(\mu), g(\mu; \nu) \) must admit a Berezin quantization for \( \mu_i \in W(\Omega_i) \backslash \{0\} \) \( (1 \leq i \leq k) \).

**Remark 1.1.** When \( \nu_i = 0 \) \( (1 \leq i \leq k) \), the formula (1.13) can be re-written as

\[
\prod_{i=1}^{k} \chi_i(\mu_i(x + y) - p_i) = \prod_{i=1}^{k} \mu_i^{d_i}(x + y - d)d.
\]

Especially, if \( y = 0 \), by (2.3) and (2.4), we have

\[
\prod_{i=1}^{k} \prod_{j=1}^{r_i} (\mu_i x - p_i + 1 + (j - 1)\frac{a_i}{2} + b_i + (r_i - j)a_i) = \prod_{i=1}^{k} \mu_i^{d_i} \prod_{j=1}^{d} (x - j).
\]

This is the exactly formula (1.7) of Theorem 1.4 in Feng-Tu [15].
Example 1.1. Let \( k = d = d_0 = 1 \). Then the generalized Cartan-Hartogs domain becomes the Thullen domain

\[
\mathbb{B}^\mathbb{B}(\mu) = \left\{ (z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 < 1 \right\} \quad (\mu > 0).
\]

Obviously, in this special case, we have \( r = 1, a = 2, b = 0 \) and \( p = 2 \) (refer to [11]). Then (1.10) implies that

\[
\phi(x) = \frac{(x-1)\chi(\mu(x) - 2)}{\mu \sum_{t=0}^{1} \sigma(t)(\alpha - 2)_{t-1}(x-t)}.
\]

By the definition of \( \sigma(t) \) (see (1.8)), we know that \( \sigma(0) = \nu \) and \( \sigma(1) = 1 \). Therefore, the denominator of \( \phi(x) \) becomes \( \mu[\nu(x - 2) + (x-1)] \). Moreover, by (2.3), it follows

\[
\chi(\mu(x) - 2) = \mu(\nu(x) - 1).
\]

It is easy to see that the denominator of \( \phi(x) \) equals to \( \chi(\mu(x) - 2) \) when \( \nu = \frac{1 - \mu}{2\mu} \). This means \( \phi(x) = x - 1 \). Then by the Theorem 1.4, we conclude that \( (\mathbb{B}^\mathbb{B}(\mu), g(\mu; \frac{1 - \mu}{2\mu})) \) admits a Berezin quantization.

Example 1.2. For \( \nu_i = 0 \) \((1 \leq i \leq k)\), it is not hard to see that

\[
\sigma(0) = \sigma(1) = \ldots = \sigma(d-1) = 0, \quad \sigma(d) = 1
\]

by (1.8). The formula (1.10) yields that

\[
\phi(x) = \frac{(x-d)_{d} \prod_{i=1}^{k} \chi_i(\mu_i x - p_i)}{\prod_{i=1}^{k} \mu_i^{d_i} \sigma(d)(x-d)^{d}} = \prod_{i=1}^{k} \mu_i^{-d_i} \chi_i(\mu_i x - p_i).
\]

By (2.3), we know that \( \phi(x) \) is a polynomial of \( x \). Therefore, \( \left( (\prod_{i=1}^{k} \mathcal{B}_{\Omega_i})^{B_{\mu_0}}(\mu), g(\mu; 0, \nu_2) \right) \) admits a Berezin quantization for \( \mu_i \in \mathcal{W}(\Omega_i) \backslash \{0\}, 1 \leq i \leq k \). In particular, when \( k = 1 \), this happens to be the result of Zedda [28].

Example 1.3. Let \( \mu = (\mu_1, \mu_2), \nu_2 = \frac{1 - \mu_2(d_1 + 1)}{(d_0 + d_1 + 1)\mu_2} \). For \( \left( (\Omega_1 \times \mathbb{B})^{\mathbb{B}_{\nu_0}}(\mu), g(\mu; 0, \nu_2) \right) \), we have

\[
d_2 = 1, \quad r_2 = 1, \quad a_2 = 2, \quad b_2 = 0, \quad p_2 = 2.
\]

Combined with (2.3), we get \( \chi_2(\mu_2(\nu_2 \alpha + x) - p_2) = \mu_2(\nu_2 \alpha + x) - 1 \). Since \( \nu_2 = 0 \), hence we can obtain that

\[
\sigma(0) = \sigma(1) = \ldots = \sigma(d-2) = 0, \quad \sigma(d-1) = \nu_2, \quad \sigma(d) = 1
\]

by (1.8). Consequently, the denominator of \( \phi(x) \) can be expressed by

\[
\mu_1^{d_1} \mu_2[(x-d)_{d} + \nu_2(\alpha - n)(x-d + 1)_{d-1}] = \mu_1^{d_1} \mu_2(x-d + 1)_{d-1}[(x-d) + \nu_2(\alpha - n)].
\]

Therefore, \( \phi(x) \) can be rewritten as

\[
\phi(x) = \frac{(x-d)_{d} \chi_1(\mu_1 x - p_1)[\mu_2(\nu_2 \alpha + x) - 1]}{\mu_1^{d_1} \mu_2(x-d + 1)_{d-1}[(x-d) + \nu_2(\alpha - n)]}.
\]

Moreover, it is not hard to see \( \mu_2(\nu_2 \alpha + x) - 1 = \mu_2[(x-d) + \nu_2(\alpha - n)] \) for \( \nu_2 = \frac{1 - \mu_2(d_1 + 1)}{(d_0 + d_1 + 1)\mu_2} \). Hence we can see

\[
\phi(x) = \mu_1^{-d_1}(x-d)\chi_1(\mu_1 x - p_1)
\]

by (2.3), which means that \( \phi(x) \) is a polynomial in \( x \). So \( \left( (\Omega_1 \times \mathbb{B})^{\mathbb{B}_{\nu_0}}(\mu), g(\mu; 0, \nu_2) \right) \) admits a Berezin quantization for \( \mu_1 \in \mathcal{W}(\Omega_1) \backslash \{0\}, \mu_2 > 0 \).
The paper is organized as follows. In Section 2, we will calculate the explicit expression of the Rawnsley’s $\varepsilon$-function. Meanwhile, using the expression of the Rawnsley’s $\varepsilon$-function expansion, we obtain the necessary and sufficient conditions for $\varepsilon(\alpha, g(\mu ; \nu))$ to become a polynomial in $\|\omega\|^2$. Lastly, with the expression (1.11) of the Rawnsley’s $\varepsilon$-function expansion of $\left(\prod_{j=1}^{k} \Omega_{j}\right)^{B_{d_{0}}(\mu), g(\mu ; \nu)}$, we will prove that $\left(\prod_{j=1}^{k} \Omega_{j}\right)^{B_{d_{0}}(\mu), g(\mu ; \nu)}$ admits a Berezin quantization under some conditions.

2 The Rawnsley’s $\varepsilon$-function for $\left(\prod_{j=1}^{k} \Omega_{j}\right)^{B_{d_{0}}(\mu)}$ with the metric $g(\mu ; \nu)$

Firstly, Let us briefly recall some basic facts on irreducible bounded symmetric domains.

Let $\Omega \subseteq \mathbb{C}^{d}$ be an irreducible bounded symmetric domain and let $r, a, b, d, p, N_{\Omega}(z, \overline{w})$ be the rank, the characteristic multiplicities, the dimension, the genus and the generic norm of $\Omega$. Hence we have

$$d = \frac{r(r - 1)}{2} + a + rb + r, \quad p = (r - 1)a + b + 2. \quad (2.1)$$

For any $s > -1$, the value of the Hua integral $\int_{\Omega} N_{\Omega}(z, \overline{z})^{s}dm(z)$ is given by

$$\int_{\Omega} N_{\Omega}(z, \overline{z})^{s}dm(z) = \frac{\pi^{d}}{C_{\Omega}(s)}, \quad (2.2)$$

where $C_{\Omega} = \det(-\frac{\partial^{2}N_{\Omega}}{\partial z \partial \overline{z}})(0)$, $dm(z)$ denotes the Euclidean measure on $\mathbb{C}^{d}$, $\chi$ is the Hua polynomial

$$\chi(s) := \prod_{j=1}^{r} \left( s + 1 + (j - 1)\frac{a}{2} \right)_{1+b+(r-j)a}, \quad (2.3)$$

in which, for a non-negative integer $m$, $(s)_m$ denotes the raising factorial

$$(s)_m := \frac{\Gamma(s + m)}{\Gamma(s)} = s(s + 1) \cdots (s + m - 1). \quad (2.4)$$

Let $G$ stand for the identity connected component of the group of biholomorphic self-maps of $\Omega$, and $K$ for the stabilizer of the origin in $G$. Under the action $f \mapsto f \circ k$ ($k \in K$) of $K$, the space $\mathcal{P}$ of holomorphic polynomials on $\mathbb{C}^{d}$ admits the Peter-Weyl decomposition

$$\mathcal{P} = \bigoplus_{\lambda} \mathcal{P}_{\lambda},$$

where the summation is taken over all partitions $\lambda$, i.e., $r$-tuples $(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{r})$ of nonnegative integers such that $\lambda_{1} \geq \lambda_{2} \geq \cdots \geq \lambda_{r} \geq 0$, and the spaces $\mathcal{P}_{\lambda}$ are $K$-invariant and irreducible. For each $\lambda$, $\mathcal{P}_{\lambda} \subset \mathcal{P}_{|\lambda|}$, where $|\lambda|$ denotes the weight of partition $\lambda$, i.e., $|\lambda| := \sum_{j=1}^{r} \lambda_{j}$, and $\mathcal{P}_{|\lambda|}$ is the space of homogeneous holomorphic polynomials of degree $|\lambda|$.

Let

$$\langle f, g \rangle_{\mathcal{F}} := \int_{\mathbb{C}^{d}} f(z)\overline{g(z)}d\rho_{\mathcal{F}}(z) \quad (2.5)$$

be the Fock-Fischer inner product on the space $\mathcal{P}$ of holomorphic polynomials on $\mathbb{C}^{d}$, where

$$d\rho_{\mathcal{F}}(z) := \exp\left\{-m(z, \overline{z}) \right\} \frac{\sqrt{-1}}{2\pi} \partial \overline{\partial}m(z, \overline{z})^{d} dl$$

and $m(z, \overline{z}) := -\frac{\partial \ln N_{\Omega}(z, \overline{z})}{\partial t} \bigg|_{t=0} = -\frac{\partial N_{\Omega}(z, \overline{z})}{\partial t} \bigg|_{t=0}$.
For every partition \( \lambda \), let \( K_\lambda(z_1, \bar{z}_2) \) be the reproducing kernel of \( P_\lambda \) with respect to (2.5). The weighted Bergman kernel of the weighted Hilbert space \( A^2(C^d, \rho F) \) of square-integrable holomorphic functions on \( C^d \) with the measure \( d\rho F \) is

\[
\exp\{-m(z_1, \bar{z}_2)\} = \sum_\lambda K_\lambda(z_1, \bar{z}_2).
\]

(2.7)

The kernels \( K_\lambda(z_1, \bar{z}_2) \) are related to the generic norm \( N_\Omega(z_1, \bar{z}_2) \) by the Hua-Faraut-Korányi formula

\[
N_\Omega(z_1, \bar{z}_2)^{-s} = \sum_\lambda (s)_\lambda K_\lambda(z_1, \bar{z}_2),
\]

(2.8)

where the series converges uniformly on compact subsets of \( \Omega \times \Omega \), \( s \in C \), in which \( (s)_\lambda \) denote the generalized Pochhammer symbol

\[
(s)_\lambda := \prod_{j=1}^r \left(s - \frac{j-1}{2}a\right)^{\lambda_j}.
\]

(2.9)

For the proofs of above facts and additional details, we refer, e.g., to [10], [11] and [26].

**Lemma 2.1.** Let \( \Omega_i \subseteq C^d \) be an irreducible bounded symmetric domain, and denote the generic norm \( N_\Omega \) and the genus \( p_i \) of \( \Omega_i \) (1 \( \leq \) \( i \) \( \leq \) \( k \)). For \( z_1^i \in \Omega_i \), let \( \phi_i \) be an automorphism of \( \Omega_i \) such that \( \phi_i(z_1^i) = 0 \), 1 \( \leq \) \( i \) \( \leq \) \( k \). By [25], the function

\[
\psi(z_1, \ldots, z_k) := \prod_{i=1}^k \frac{N_{\Omega_i}(z_1^i, \bar{z}_1^i)^{\mu_i}}{N_{\Omega_i}(z_i, \bar{z}_i)^{\mu_i}} \]

satisfies

\[
|\psi(z_1, \ldots, z_k)|^2 = \prod_{i=1}^k \left( \frac{N_{\Omega_i}(\phi_i(z_i), \bar{\phi}_i(z_i))}{N_{\Omega_i}(z_i, \bar{z}_i)^{\mu_i}} \right)^{\mu_i}.
\]

(2.10)

Define the mapping

\[
F: \left( \prod_{j=1}^k \Omega_j \right)^{B^{40}}(\mu) \rightarrow \left( \prod_{j=1}^k \Omega_j \right)^{B^{40}}(\mu),
\]

\[
(z_1, \ldots, z_k, w) \mapsto (\phi_1(z_1), \ldots, \phi_k(z_k), \psi(z_1, \ldots, z_k)w).
\]

(2.12)

Then \( F \) is an isometric automorphism of \( \left( \prod_{j=1}^k \Omega_j \right)^{B^{40}}(\mu) \), namely

\[
\partial\overline{\partial}(\Phi(F(z_1, \ldots, z_k, w))) = \partial\overline{\partial}(\Phi(z_1, \ldots, z_k, w)),
\]

(2.13)

where \( \Phi(z, w) := -\sum_{j=1}^k \nu_j \ln N_{\Omega_j}(z_j, \bar{z}_j)^{\mu_j} - \ln \left( \prod_{j=1}^k N_{\Omega_j}(z_j, \bar{z}_j)^{\mu_j} - \|w\|^2 \right) \) (see (1.3)).

**Proof.** It is easy to see that \( F \) is an automorphism of \( \left( \prod_{j=1}^k \Omega_j \right)^{B^{40}}(\mu) \), and

\[
N_{\Omega_i}(\phi_i(z_i), \overline{\phi_i(z_i)})^{p_i} = J\phi_i(z_i) N_{\Omega_i}(z_i, \bar{z}_i)^{p_i} J\phi_i(z_i),
\]

(2.14)

where \( J\phi_i(z_i) \) is the holomorphic Jacobian of the automorphism \( \phi_i \) of \( \Omega_i \), 1 \( \leq \) \( i \) \( \leq \) \( k \).

By (2.11) and (2.14), we have

\[
\Phi(F) = -\sum_{i=1}^k \nu_i \ln N_{\Omega_i}(\phi_i(z_i), \overline{\phi_i(z_i)})^{\mu_i} - \ln \left( \prod_{i=1}^k N_{\Omega_i}(\phi_i(z_i), \overline{\phi_i(z_i)})^{\mu_i} - \|\psi(z_1, \ldots, z_k)w\|^2 \right)
\]

\[
= -\sum_{i=1}^k (\nu_i + 1) \ln N_{\Omega_i}(\phi_i(z_i), \overline{\phi_i(z_i)}) - \ln \left( 1 - \frac{\|w\|^2}{\prod_{i=1}^k N_{\Omega_i}(z_i, \bar{z}_i)^{\mu_i}} \right)
\]
Note the formula (2.14) implies that
\[
- \sum_{i=1}^{k} \mu_{i}(1 + \nu_{i}) \ln N_{\Omega_{i}}(\phi_{i}(z_{i}), \bar{\phi}_{i}(\bar{z}_{i})) - \ln \left( 1 - \frac{\|w\|^2}{\prod_{i=1}^{k} N_{\Omega_{i}}(z_{i}, \bar{z}_{i})^{\mu_{i}}} \right) = - \sum_{i=1}^{k} \mu_{i}(1 + \nu_{i}) \ln |J\phi_{i}(z_{i})|^{2} - \sum_{j=1}^{k} \nu_{j} \ln N_{\Omega_{j}}(z_{j}, \bar{z}_{j})^{\nu_{j}} - \ln \left( \prod_{j=1}^{k} N_{\Omega_{j}}(z_{j}, \bar{z}_{j})^{\nu_{j}} - \|w\|^2 \right).
\]

Therefore, by the definition (1.3), we obtain
\[
\Phi(F) = - \sum_{i=1}^{k} \mu_{i}(1 + \nu_{i}) \ln |J\phi_{i}(z_{i})|^{2} + \Phi,
\]
which implies (2.13) as \(J\phi_{i}(z_{i})\) \((1 \leq i \leq k)\) is a holomorphic. We complete the proof. \(\square\)

**Lemma 2.2.** Let \(\Omega_{i} \subseteq \mathbb{C}^{d_{i}}\) be an irreducible bounded symmetric domain, and denote the generic norm \(N_{\Omega_{i}}(z_{i}, \bar{z}_{i})\), the dimension \(d_{i}\) and the genus \(p_{i}\) for \(\Omega_{i}\) \((1 \leq i \leq k)\). Then we have
\[
(\partial \Phi)^{n} = \frac{\prod_{i=1}^{k} (\mu_{i} d_{i} C_{d_{i}})}{(1 - \|\tilde{w}\|^{2})^{d_{0}+1}} \prod_{i=1}^{k} \frac{\mu_{i} + \frac{1}{1 - \|\tilde{w}\|^{2}}}{N_{\Omega_{i}}(z_{i}, \bar{z}_{i})^{p_{i} + \mu_{i} d_{0}}} \left( \sum_{j=1}^{n} dZ_{j} \wedge d\bar{Z}_{j} \right)^{n}, \tag{2.15}
\]
where
\[
\Phi(z, w) := - \sum_{j=1}^{k} \nu_{j} \ln N_{\Omega_{j}}(z_{j}, \bar{z}_{j})^{\nu_{j}} - \ln \left( \prod_{j=1}^{k} N_{\Omega_{j}}(z_{j}, \bar{z}_{j})^{\nu_{j}} - \|w\|^{2} \right),
\]
\[C_{d_{i}} = \det \left( - \frac{\partial^{2} \ln N_{\Omega_{i}}(z_{i}, \bar{z}_{i})}{\partial z_{i}^{j} \partial z_{i}^{j}} \right) \bigg|_{z_{i}=0}, \quad \tilde{w} = \frac{w}{\prod_{j=1}^{k} N_{\Omega_{j}}(z_{j}, \bar{z}_{j})^{\nu_{j} \widetilde{Z}_{j}}},
\]
\[n = \sum_{j=0}^{k} d_{j}, \quad Z = (Z_{1}, \ldots, Z_{n}) = (z_{1}, \ldots, z_{k}, w).
\]

**Proof.** It is generally known that
\[
\frac{(\sqrt{2\pi})^{n}}{n!} \Phi^{n} = \det \left( \frac{\partial^{2} \Phi}{\partial Z^{i} \partial \bar{Z}^{j}} \right) \frac{\omega_{0}^{n}}{n!}, \tag{2.16}
\]
where \(\omega_{0} = \sqrt{2\pi} \sum_{j=1}^{n} dZ_{j} \wedge d\bar{Z}_{j}, \quad \frac{\partial}{\partial z_{i}} = (\frac{\partial}{\partial z_{1}}, \frac{\partial}{\partial z_{2}}, \ldots, \frac{\partial}{\partial z_{n}})^{t}, \quad \frac{\partial}{\partial \bar{z}_{i}} = (\frac{\partial}{\partial \bar{z}_{1}}, \frac{\partial}{\partial \bar{z}_{2}}, \ldots, \frac{\partial}{\partial \bar{z}_{n}})\) and \(\frac{\partial^{2}}{\partial z^{i} \partial \bar{z}^{j}} = \frac{\partial}{\partial z^{i}} \frac{\partial}{\partial \bar{z}^{j}}\).

Note the formula (2.13) implies that
\[
\det \left( \frac{\partial^{2} \Phi(F)}{\partial Z^{i} \partial \bar{Z}^{j}} \right) = \det \left( \frac{\partial^{2} \Phi}{\partial Z^{i} \partial \bar{Z}^{j}} \right). \tag{2.17}
\]
By the identity
\[
\frac{\partial^{2} \Phi(F)}{\partial Z^{i} \partial \bar{Z}^{j}} = \frac{\partial F}{\partial Z^{i}} \frac{\partial^{2} \Phi}{\partial Z^{j} \partial \bar{Z}} (F(Z)) \frac{\partial F}{\partial \bar{Z}^{j}} \tag{2.18}
\]
and (2.17), we obtain
\[
\det \left( \frac{\partial^{2} \Phi}{\partial Z^{i} \partial \bar{Z}^{j}} \right) (Z) = |JF(Z)|^{2} \det \left( \frac{\partial^{2} \Phi}{\partial Z^{i} \partial \bar{Z}^{j}} \right) (F(Z)), \tag{2.19}
\]
where
\[ F := (F_1, F_2, \ldots, F_n), \quad \frac{\partial F}{\partial Z} := \left( \frac{\partial F_1}{\partial Z}, \frac{\partial F_2}{\partial Z}, \ldots, \frac{\partial F_n}{\partial Z} \right) \]
and
\[ JF(Z) := \det \left( \frac{\partial F}{\partial Z} \right)(Z). \]

Let \( Z^0 = (z^0_1, \ldots, z^0_k, w^0) \in \left( \prod_{i=1}^k \mathbb{B}^{\delta_0} \right)^\mu \), \( \tilde{Z}^0 := (\tilde{z}^0_1, \ldots, \tilde{z}^0_k, \tilde{w}^0) = F(Z^0) \). By (2.12), we have
\[ \tilde{Z}^0 = \begin{pmatrix} 0, \ldots, 0, \frac{w^0}{\prod_{i=1}^k N_{\Omega_i}(z^0_i, \bar{z}^0_i)} \end{pmatrix} \]
and
\[ |JF(Z^0)|^2 = \prod_{i=1}^k |J\Phi_i(z^0_i)|^2 \cdot |\psi(z^0_1, \ldots, z^0_k)|^{2\delta_0}. \] (2.20)

Using \( N_{\Omega_i}(0, z_i) = 1 \) and (2.14), we can see that
\[ |J\Phi_i(z^0_i)|^2 = N_{\Omega_i}(z^0_i, \bar{z}^0_i)^{-p_i} \]
From (2.11), (2.19) and (2.20), we have
\[ |JF(Z^0)|^2 = \prod_{i=1}^k \frac{1}{N_{\Omega_i}(z^0_i, \bar{z}^0_i)^{p_i + \mu_i \delta_0}}, \] (2.21)
and
\[ \det \left( \frac{\partial^2 \Phi}{\partial Z^t \partial Z} \right)(Z^0) = \prod_{i=1}^k \frac{1}{N_{\Omega_i}(z^0_i, \bar{z}^0_i)^{p_i + \mu_i \delta_0}} \det \left( \frac{\partial^2 \Phi}{\partial Z^t \partial Z} \right)(\tilde{Z}^0). \] (2.22)

A direct calculation gives
\[
\frac{\partial^2 \Phi}{\partial Z^t \partial Z}(0, \ldots, 0, w) = \begin{pmatrix}
\mu_1 \left( \nu_1 + \frac{1}{1 - \|w\|^2} \right) & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & \mu_k \left( \nu_k + \frac{1}{1 - \|w\|^2} \right) & C_{d_k} \\
0 & \cdots & 0 & \frac{1}{1 - \|w\|^2} I_{d_0} + \frac{1}{(1 - \|w\|^2)^2} \bar{w}^t w
\end{pmatrix}
\] (2.23)
where \( I_{d_0} \) denotes the \( d_0 \times d_0 \) identity matrix, \( \bar{w}^t \) is the complex conjugate transpose of the row vector \( w = (w_1, w_2, \cdots, w_{d_0}) \), and \( C_{d_i} = -\frac{\partial^2 \ln N_{\Omega_i}}{\partial z_i^t \partial z_i} |_{z_i=0} \).

From (2.23), we have
\[ \det \left( \frac{\partial^2 \Phi}{\partial Z^t \partial Z} \right)(0, \ldots, 0, w) = \prod_{i=1}^k \left( \mu_i^{d_i} \det C_{d_i} \left( \nu_i + \frac{1}{1 - \|w\|^2} \right)^{d_i} \right) \left( 1 - \|w\|^2 \right)^{d_0 + 1}. \] (2.24)

Finally, combining (2.22) and (2.24), we have (2.15). The proof is finished. \( \square \)
Theorem 2.3. Let \( \Omega_i \subseteq \mathbb{C}^{d_i} \) be an irreducible bounded symmetric domain, and denote the generic norm \( N_{\Omega_i} \), the genus \( p_i \), the dimension \( d_i \) and the Hua polynomial \( \chi_i \) (see (2.3)) for \( \Omega_i \) \((1 \leq i \leq k)\). Endow the generalized Cartan-Hartogs domain \( \left( \prod_{j=1}^k \Omega_j \right)^{\mathbb{B}^{d_0}} (\mu) \) with the canonical metric \( g(\mu; \nu) \). For \( \alpha > \max \left\{ n, \frac{p_i - 1}{\mu_i + 1}, \ldots, \frac{p_k - 1}{\mu_k + 1} \right\} \), then the reproducing kernel \( K_{\alpha}(Z; \mathbb{Z}) \) of the weighted Hilbert space

\[
\mathcal{H}_{\alpha} = \left\{ f \in \text{Hol}\left( \left( \prod_{j=1}^k \Omega_j \right)^{\mathbb{B}^{d_0}} (\mu) \right) : \int_{\left( \prod_{j=1}^k \Omega_j \right)^{\mathbb{B}^{d_0}} (\mu)} |f|^2 \exp\{-\alpha \Phi\} \frac{\omega(\mu; \nu)^n}{n!} < +\infty \right\}
\]

can be expressed as

\[
K_{\alpha}(Z; \mathbb{Z}) = \frac{(\alpha - n)n}{\prod_{i=1}^k N_{\Omega_i}(z_i, z_i^{\mu_i(1+\nu)})} \sum_{t=0}^{+\infty} \psi(\alpha, t) \frac{(\alpha)_t}{t!} \|w\|^{2t},
\]

(2.25)

Where

\[
Z = (z_1, \ldots, z_k, w), \quad d = \sum_{j=1}^{k} d_j, \quad n = d + d_0, \quad \tilde{w} = \frac{w}{\prod_{j=1}^{k} N_{\Omega_j}(z_j, z_j^{\mu_j})},
\]

(2.26)

and

\[
\psi(x, y) = \frac{\prod_{i=1}^{k} N_{\Omega_i}(z_i, z_i^{\mu_i})}{\prod_{i=1}^{k} N_{\Omega_i}(z_i, z_i^{\mu_i})} \sum_{t=0}^{d} \sigma(t) (x - n)d_t(x + y - t),
\]

(2.27)

Proof. Firstly, by definition of \( C_{d_i} \) and \( C_{\Omega_i} \), we know that \( C_{d_i} = C_{\Omega_i} \). Therefore, according to the formula (2.15), we can express the inner product on \( \mathcal{H}_{\alpha} \) as follows

\[
(f, g) = \frac{\prod_{i=1}^{k} (\mu_i^d C_{\Omega_i})}{\pi^{n}} \int_{\left( \prod_{j=1}^k \Omega_j \right)^{\mathbb{B}^{d_0}} (\mu)} f(Z) g(Z) \prod_{i=1}^{k} N_{\Omega_i}(z_i, z_i^{\mu_i}) d_t(x - n)d_t(x + y - t),
\]

\[
\times \left( 1 - \|\tilde{w}\|^2 \right)^{\alpha - d_0 - 1} \prod_{i=1}^{k} \left( \nu_i + 1 - \|\tilde{w}\|^2 \right)^{d_i} dm(Z),
\]

(2.15)

where \( dm \) denotes the standard Euclidean measure.

Secondly, for the sake of convenience, we define \( \Omega_0 := \mathbb{B}^{d_0}, \ z_0 := w \). Let \( r_i, a_i, b_i, d_i, p_i, \chi_i, \delta_i \), \( N_{\Omega_i} \) and \( V(\Omega_i) \) be rank, characteristic multiplicities, the dimension, genus, Hua polynomial, generalized Pochhammer symbol, generic norm and the Euclidean volume of the irreducible bounded symmetric domain \( \Omega_i \) \((0 \leq i \leq k)\).

Let \( \mathcal{G}_i \) be the identity connected components of groups of biholomorphic self-maps of \( \Omega_i \subseteq \mathbb{C}^{d_i} \), and \( K_i \) be the stabilizers of the origin in \( \mathcal{G}_i \), respectively. For any \( u = (u_0, \ldots, u_k) \in \mathcal{K} := K_0 \times \cdots \times K_k \), we define the action

\[
\pi(u) f(z_1, \ldots, z_k, w) \equiv f \circ u(z_1, \ldots, z_k, w) := f(u_1 \circ z_1, \ldots, u_k \circ z_k, u_0 \circ w)
\]

of \( \mathcal{K} \), then the space \( \mathcal{P} \) of holomorphic polynomials on \( \prod_{j=0}^{k} \mathbb{C}^{d_j} \) admits the Peter-Weyl decomposition

\[
\mathcal{P} = \bigoplus_{\ell(\lambda) \leq r_i, 0 \leq i \leq k} \mathcal{P}_{\lambda_0}^{(0)} \otimes \cdots \otimes \mathcal{P}_{\lambda_k}^{(k)},
\]
where space $\mathcal{P}^{(i)}_{\lambda_i}$ is $K_i$-invariant and irreducible subspace of the space of holomorphic polynomials on $\mathbb{C}^d$, and $l(\lambda_i)$ denotes the length of partition $\lambda_i$ ($0 \leq i \leq k$).

Since $\mathcal{H}_\alpha$ is invariant under the action of $K$, that is, $\forall u \in K$, $(\pi(u)f, \pi(u)g) = (f, g)$, $\mathcal{H}_\alpha$ admits an irreducible decomposition (see [12])

$$\mathcal{H}_\alpha = \bigoplus_{l(\lambda_i) \leq r_i} \mathcal{P}^{(0)}_{\lambda_0} \otimes \ldots \otimes \mathcal{P}^{(k)}_{\lambda_k},$$

where $\bigoplus$ denotes the orthogonal direct sum.

For given partition $\lambda_i$ of length $\leq r_i$, let $K^{(i)}_{\lambda_i}(z_i; \overline{z}_i)$ be the reproducing kernel of $\mathcal{P}^{(i)}_{\lambda_i}$ with respect to (2.5). By Schur’s lemma, there exists a positive constant $c_{\lambda_0\ldots\lambda_k}$ such that $c_{\lambda_0\ldots\lambda_k} \prod_{j=0}^k K^{(j)}_{\lambda_j}(z_j; \overline{z}_j)$ is the reproducing kernel of $\mathcal{P}^{(0)}_{\lambda_0} \otimes \ldots \otimes \mathcal{P}^{(k)}_{\lambda_k}$ with respect to the above inner product $(\cdot, \cdot)$.

It is well known that the reproducing kernel can be expressed as a sum of square of the modules of the orthogonal basis, therefore we obtain

$$\prod_{i=1}^k (\mu_i^d C_{\mu_i}) \frac{1}{\pi^n} \int_{(\prod_{j=1}^k \Omega_j)^{\otimes d_0}} (\mu) c_{\lambda_0\ldots\lambda_k} \prod_{j=0}^k K^{(j)}_{\lambda_j}(z_j; \overline{z}_j) \prod_{i=1}^k N_{\Omega_i}(z_i; \overline{z}_i)^{\mu_i(1+\nu_i)\alpha-d_0-p_i}$$

$$\times \left(1 - ||\overline{w}||^2\right)^{\alpha-d_0-1} \prod_{i=1}^k \left(\nu_i + \frac{1}{1-||\overline{w}||^2}\right)^{d_i} \prod_{j=0}^k dm(z_j)$$

$$= \prod_{i=0}^k \dim \mathcal{P}^{(i)}_{\lambda_i}.$$

It follows

$$K_\alpha(Z; \overline{Z}) = \sum_{l(\lambda_i) \leq r_i} c_{\lambda_0\ldots\lambda_k} \prod_{j=0}^k K^{(j)}_{\lambda_j}(z_j; \overline{z}_j) = \sum_{l(\lambda_i) \leq r_i} \frac{\prod_{i=0}^k \dim \mathcal{P}^{(i)}_{\lambda_i}}{\prod_{j=0}^k K^{(j)}_{\lambda_j}(z_j; \overline{z}_j)} \prod_{j=0}^k K^{(j)}_{\lambda_j}(z_j; \overline{z}_j),$$

(2.28)

where $< f >$ denotes integral

$$< f > = \prod_{i=1}^k (\mu_i^d C_{\mu_i}) \frac{1}{\pi^n} \int_{(\prod_{j=1}^k \Omega_j)^{\otimes d_0}} (\mu) f(Z) \times \prod_{i=1}^k N_{\Omega_i}(z_i; \overline{z}_i)^{\mu_i(1+\nu_i)\alpha-d_0-p_i}$$

$$\times \left(1 - ||\overline{w}||^2\right)^{\alpha-d_0-1} \prod_{i=1}^k \left(\nu_i + \frac{1}{1-||\overline{w}||^2}\right)^{d_i} \prod_{j=0}^k dm(z_j).$$

In order to compute the $K_\alpha(Z; \overline{Z})$, we just need to calculate the value of $< \prod_{j=0}^k K^{(j)}_{\lambda_j}(z_j; \overline{z}_j) >$.

Since $\alpha > \max\{n, \frac{p_1-1}{\mu_1(1+\nu_1)}, \ldots, \frac{p_k-1}{\mu_k(1+\nu_k)}\}$, then $\mu_i(1+\nu_i)\alpha - p_i > -1$ and $\alpha - n - 1 > -1$. Hence we can use (see [13])

$$\int_\Omega K_\lambda(z; \overline{z}) N_\Theta(z, \overline{z})^s dm(z) = \frac{\dim \mathcal{P}_\lambda}{(p+s)\lambda} \int_\Omega N_\Theta(z, \overline{z})^s dm(z) \quad (s > -1)$$

(2.29)
to obtain
\[
\langle \prod_{j=0}^{k} K^{(j)}_{\lambda_j}(z_j; \overline{z}_j) \rangle
= \frac{\prod_{i=1}^{k} (\mu_i d_i C_{\lambda_i})}{\pi^n} \prod_{i=1}^{k} \Omega_{\lambda_i}(z_i, \overline{z}_i) N_{\Omega_{\lambda_i}}(z_i, \overline{z}_i)^{\mu_i((1+\nu_i)\alpha + \lambda_0) - p_i} \int_{\mathbb{B}_{d_0}} K^{(0)}_{\lambda_0}(w; \overline{w})(1 - \|w\|^2)^{\alpha - d_0 - 1} \prod_{i=1}^{k} \left( \nu_i + \frac{1}{1 - \|w\|^2} \right) d_i \ dm(w)
\]

By (2.2), we can simplify the above equation as follows
\[
\langle \prod_{j=0}^{k} K^{(j)}_{\lambda_j}(z_j; \overline{z}_j) \rangle
= \frac{\prod_{i=1}^{k} (\mu_i d_i C_{\lambda_i})}{\pi^n} \prod_{i=1}^{k} \Omega_{\lambda_i}(z_i, \overline{z}_i) N_{\Omega_{\lambda_i}}(z_i, \overline{z}_i)^{\mu_i((1+\nu_i)\alpha + \lambda_0) - p_i} \int_{\mathbb{B}_{d_0}} K^{(0)}_{\lambda_0}(w; \overline{w})(1 - \|w\|^2)^{\alpha - d_0 - 1} \prod_{i=1}^{k} \left( \nu_i + \frac{1}{1 - \|w\|^2} \right) d_i \ dm(w).
\tag{2.30}
\]

Since \(\Omega_0 = \mathbb{B}_{d_0}\), hence we have (refer to [16])
\[
N_{\mathbb{B}_{d_0}}(w, \overline{w}) = 1 - \|w\|^2, \ p_0 = d_0 + 1, \ \chi_0(x) = (x + 1)_{d_0}, \ C_{\mathbb{B}_{d_0}} = 1.
\]

It follows
\[
\tag{1}
= \sum_{t_0=0}^{d_1} \cdots \sum_{t_k=0}^{d_k} \left( \prod_{i=1}^{k} \left( \frac{d_i}{t_i} \right) \nu_i^{t_i} \right) \int_{\mathbb{B}_{d_0}} K^{(0)}_{\lambda_0}(w; \overline{w}) N_{\mathbb{B}_{d_0}}(w, \overline{w})^{\alpha - n - 1 + \sum_{j=1}^{k} t_j} \ dm(w)
\]
\[
= \sum_{t=0}^{d} \left( \sum_{\sum_{t_i=0}^{t} \sum_{t_i=0}^{t_i} \prod_{i=1}^{k} \left( \frac{d_i}{t_i} \right) \nu_i^{t_i} \right) \frac{\dim P^{(0)}_{\lambda_0}}{(\alpha - d + t)_{\lambda_0}} \int_{\mathbb{B}_{d_0}} N_{\mathbb{B}_{d_0}}(w, \overline{w})^{\alpha - n - 1 + t} \ dm(w)
\]
by (2.29). Applying the formula (2.2) again, we know that
\[
\tag{1}
= \sum_{t=0}^{d} \left( \sum_{\sum_{t_i=0}^{t} \sum_{t_i=0}^{t_i} \prod_{i=1}^{k} \left( \frac{d_i}{t_i} \right) \nu_i^{t_i} \right) \frac{\dim P^{(0)}_{\lambda_0}}{(\alpha - d + t)_{\lambda_0}} \chi_0(\alpha - n - 1 + t).
\]

\[
= \sum_{t=0}^{d} \left( \sum_{\sum_{t_i=0}^{t} \sum_{t_i=0}^{t_i} \prod_{i=1}^{k} \left( \frac{d_i}{t_i} \right) \nu_i^{t_i} \right) \frac{\dim P^{(0)}_{\lambda_0}}{(\alpha - d + t)_{\lambda_0}} \chi_0(\alpha - n + t).
\]
By the formula (2.4), we have

\[ 1 = \dim \mathcal{P}^{(0)}_{\lambda_0} \pi_{d_0} \sum_{t=0}^{d} \left( \sum_{\sum_{i=1}^{k} t_i = t \atop t_i \geq 0, 1 \leq i \leq k} k \prod_{i=1}^{k} \left( \frac{d_i}{t_i^\alpha} \right) \right) \frac{\Gamma(\alpha - d + t) \Gamma(\alpha - n + t)}{\Gamma(\alpha - d + t + \lambda_0) \Gamma(\alpha - n + t + d_0)} \]

Since \( n = d + d_0 \), it follows

\[ 1 = \dim \mathcal{P}^{(0)}_{\lambda_0} \pi_{d_0} \sum_{t=0}^{d} \left( \sum_{\sum_{i=1}^{k} t_i = t \atop t_i \geq 0, 1 \leq i \leq k} k \prod_{i=1}^{k} \left( \frac{d_i}{t_i^\alpha} \right) \right) \frac{\Gamma(\alpha + \lambda_0) \Gamma(\alpha - n + t) \Gamma(\alpha - n)}{\Gamma(\alpha - d + t + \lambda_0) \Gamma(\alpha - n) \Gamma(\alpha + \lambda_0)} \]

\[ = \dim \mathcal{P}^{(0)}_{\lambda_0} \pi_{d_0} \sum_{t=0}^{d} \left( \sum_{\sum_{i=1}^{k} t_i = t \atop t_i \geq 0, 1 \leq i \leq k} k \prod_{i=1}^{k} \left( \frac{d_i}{t_i^\alpha} \right) \right) \frac{\Gamma(\alpha + \lambda_0 - (d - t))_{d-t} (\alpha - n)_i}{(\alpha - n)_{n+\lambda_0}} \]

\[ = \frac{\pi_{d_0} \dim \mathcal{P}^{(0)}_{\lambda_0}}{(\alpha - n)_{n+\lambda_0}} \sum_{t=0}^{d} n \sigma (t) (\alpha + \lambda_0 - t) \ell (\alpha - n)_{d-t}. \]

Therefore, the formulas (2.30), (2.27) and a direct computation imply

\[ < \prod_{j=0}^{k} K_j^{(j)} (z_j; \overline{z_j}) > \]

\[ = \prod_{i=1}^{k} \mu_i^d \prod_{i=0}^{k} \dim \mathcal{P}_{\lambda_i} \sum_{t=0}^{d} \sigma (t) (\alpha + \lambda_0 - t) \ell (\alpha - n)_{d-t} \]

\[ = \prod_{i=0}^{k} \dim \mathcal{P}_{\lambda_i} \frac{\psi (\alpha, \lambda_0) (\alpha - n)_{n+\lambda_0} \prod_{i=1}^{k} (\mu_i (1 + \nu_i)) \lambda_i^{(i)}}{(\alpha - n)_{n+\lambda_0} \prod_{i=1}^{k} (\mu_i (1 + \nu_i))^{(i)}} \]

Hence, combining (2.28) and (2.8), we have

\[ K_{\lambda} (Z; \overline{Z}) \]

\[ = \sum_{\ell (\lambda_0) \leq r_0} \psi (\alpha, \lambda_0) (\alpha - n)_{n+\lambda_0} K_{\alpha}^{(0)} (w; \overline{w}) \prod_{i=1}^{k} (\mu_i (1 + \nu_i)) \lambda_i^{(i)} \frac{1}{K^{(j)} (z_j; \overline{z_j})} \]

\[ = \sum_{\ell (\lambda_0) \leq r_0} \prod_{i=1}^{k} \frac{1}{\mathcal{N} (z_i, \overline{z_i})} \psi (\alpha, \lambda_0) (\alpha - n)_{n+\lambda_0} K_{\alpha}^{(0)} (w; \overline{w}) \]

Since \( r_0 = 1 \), \( \lambda_0 \in \mathbb{N} \) and \( K_{\alpha}^{(0)} (w; \overline{w}) \) be the reproducing kernel of \( \mathcal{P}_{\alpha}^{(0)} \) with respect to (2.5) where \( \mathcal{P}_{\alpha}^{(0)} \) is the space of homogeneous holomorphic polynomials of degree \( \alpha_0 \). Hence we have

\[ K_{\alpha}^{(0)} (w; \overline{w}) = \sum_{|\alpha| = \lambda_0} \frac{|w|^{2\alpha}}{\prod_{i=1}^{d_0} \Gamma (\alpha_i + 1)} = \frac{|w|^{2\lambda_0}}{\lambda_0 !}. \]
Consequently, we obtain

\[
K_\alpha(Z; \overline{Z}) = \sum_{\lambda_0=0}^{+\infty} \prod_{i=1}^k \frac{1}{N_{\Omega_i}(z_i, \overline{z_i})^{\mu_i(1+\nu_i)\alpha + \lambda_0}} \psi(\alpha, \lambda_0)(\alpha - n + \lambda_0)\frac{||w||^{2\lambda_0}}{\lambda_0!}
\]

\[
= \prod_{i=1}^k N_{\Omega_i}(z_i, \overline{z_i})^{\mu_i(1+\nu_i)\alpha} \sum_{t=0}^{+\infty} \psi(\alpha, t) \frac{(\alpha - n)_{n+t}}{t!} ||\tilde{w}||^{2t}
\]

by the following fact (see (2.4))

\[(\alpha - n)_{n+t} = (\alpha - n)_n(\alpha)_t.\]

The proof is finished.

Now, combining Theorem 2.3, we can get the expression of the Rawnsley’s \(\varepsilon\)-function for the generalized Cartan-Hartogs domain \((\prod_{j=1}^k \Omega_j)^{B_{\Delta_0}}(\mu)\) with the metric \(g(\mu; \nu)\).

**Proof of Theorem 1.2.** Using the definition (1.1), we have

\[
\varepsilon_{(\alpha, g(\mu; \nu))}(z_1, \ldots, z_k, w) = e^{-\alpha \Phi(z_1, \ldots, z_k, w)} K_\alpha(z_1, \ldots, z_k, w; \overline{z_1}, \ldots, \overline{z_k}, \overline{w}),
\]

and by the definition (1.3), we get

\[
e^{-\alpha \Phi(z_1, \ldots, z_k, w)} = \prod_{i=1}^k N_{\Omega_i}(z_i, \overline{z_i})^{\mu_i(1+\nu_i)\alpha} (1 - ||\tilde{w}||^2)^\alpha.
\]

Therefore, by (2.25), we obtain (1.7). The proof is finished.

As a consequence of Theorem 1.2, we give the proof of Corollary 1.3.

**Proof of Corollary 1.3.** From (1.7) and (1.9), we obtain

\[
\sum_{j=0}^d c_j(\alpha)(1 - ||w||^2)^{-(\alpha-(d-j))} = \sum_{t=0}^{+\infty} \psi(\alpha, t) \frac{(\alpha - n)_{n+t}}{t!} ||w||^{2t}.
\]

Using

\[
(1 - ||w||^2)^{-\alpha} = \sum_{t=0}^{+\infty} \frac{(\alpha)_t}{t!} ||w||^{2t},
\]

we get

\[
\sum_{j=0}^d \frac{c_j(\alpha)}{(\alpha - n)_{d_0+j}} (\alpha + t - d)_j = \psi(\alpha, t)(\alpha + t - d)_d.
\]

This indicates that \(\psi(\alpha, t)(\alpha + t - d)_d\) is a polynomial of \(\alpha + t\). It follow \(\phi(\alpha + t) = \psi(\alpha, t)(\alpha + t - d)_d\) is a polynomial of \(\alpha + t\).

It is known that if \(\phi(x)\) is a polynomial of \(x\), then we have (refer to [13])

\[
\phi(x) = \sum_{j=0}^d \frac{D^j \phi(d)}{j!} (x - d)_j.
\]
Hence the formula (2.33) can be written as
\[ \sum_{j=0}^{d} \frac{c_j(\alpha)}{(\alpha - n)_{d_0+j}}(\alpha + t - d)_j = \sum_{j=0}^{d} \frac{D_j^\phi(d)}{j!}(\alpha + t - d)_j. \]
Therefore, we obtain
\[ c_j(\alpha) = \frac{D_j^\phi(d)}{j!}(\alpha - n)_{d_0+j}. \] (2.34)
Substituting (2.34) into (1.9), we obtain (1.11). \(\square\)

3 The proof of Theorems 1.4 and 1.5

In order to prove that a quantization procedure can be established on the generalized Cartan-Hartogs domain \(\left(\prod_{j=1}^{k} \Omega_j\right)^{\mathbb{B}^{d_0}}(\mu, g(\mu; \nu))\), we just need to prove that the Calabi’s diastasis function \(D_{g(\mu; \nu)}\) and the Rawnsley’s \(\varepsilon\)-function associated to the metric \(g(\mu; \nu)\) satisfies the condition (I) and (II), respectively. Before we do the proof, we will give the following results.

Lemma 3.1. Let \(\left(\prod_{j=1}^{k} \Omega_j\right)^{\mathbb{B}^{d_0}}(\mu)\) be the generalized Cartan-Hartogs domain, \(\mu = (\mu_1, \ldots, \mu_k)\) and \(\mu_i \in W(\Omega_i)\{0\}\), here \(W(\Omega_i)\) are the Wallach sets of \(\Omega_i\), \(1 \leq i \leq k\). Then we have
\[ \sup_{(z,w), (\xi, \eta) \in \left(\prod_{j=1}^{k} \Omega_j\right)^{\mathbb{B}^{d_0}}(\mu)} \left| 1 - w^{\xi_\Omega} \prod_{i=1}^{k} N_{\Omega_i}(z_i, \xi_i)^{-\mu_i} \right| < +\infty, \] (3.1)

where \(z = (z_1, \ldots, z_k)\) and \(\xi = (\xi_1, \ldots, \xi_k)\).

Proof. Since \(\mu_i \in W(\Omega_i)\{0\}\), we know that \(\prod_{i=1}^{k} N_{\Omega_i}(z_i, \xi_i)^{-\mu_i}\) is the reproducing kernel of some Hilbert space (see [10]). Hence there exists an orthonormal basis \(\{g_j\}\) such that
\[ \prod_{i=1}^{k} N_{\Omega_i}(z_i, \xi_i)^{-\mu_i} = \sum_{j} g_j(z)g_j(\xi), \]
where \(z = (z_1, \ldots, z_k)\) and \(\xi = (\xi_1, \ldots, \xi_k)\). Then Cauchy-Schwarz inequality yields
\[ \prod_{i=1}^{k} \left| N_{\Omega_i}(z_i, \xi_i)^{-\mu_i} \right| \leq \prod_{i=1}^{k} N_{\Omega_i}(z_i, \xi_i)^{-\frac{\mu_i}{2}} \prod_{i=1}^{k} N_{\Omega_i}(\xi_i, \xi_i)^{-\frac{\mu_i}{2}}. \]
Therefore, by \((z, w), (\xi, \eta) \in \left(\prod_{j=1}^{k} \Omega_j\right)^{\mathbb{B}^{d_0}}(\mu)\), we get the following inequality:
\[ |w^{\xi_\Omega}| \prod_{i=1}^{k} \left| N_{\Omega_i}(z_i, \xi_i)^{-\mu_i} \right| \leq \|w\|\|\eta\| \prod_{i=1}^{k} N_{\Omega_i}(z_i, \xi_i)^{-\frac{\mu_i}{2}} \prod_{i=1}^{k} N_{\Omega_i}(\xi_i, \xi_i)^{-\frac{\mu_i}{2}} < 1. \] (3.2)
The proof is finished. \(\square\)

Lemma 3.2. The noncompact Kähler manifold \(\left(\prod_{j=1}^{k} \Omega_j\right)^{\mathbb{B}^{d_0}}(\mu, g(\mu; \nu))\) satisfies condition (I).

Proof. To prove that the noncompact Kähler manifold \(\left(\prod_{j=1}^{k} \Omega_j\right)^{\mathbb{B}^{d_0}}(\mu, g(\mu; \nu))\) satisfies condition (I), we only need to prove that the noncompact Kähler manifold \(\left(\prod_{j=1}^{k} \Omega_j\right)^{\mathbb{B}^{d_0}}(\mu, \beta g(\mu; \nu))\) satisfies condition (I) for given \(\beta > \max \left\{ \frac{(r_1-1)a_1}{2\mu_1(1+\nu_1)}, \ldots, \frac{(r_k-1)a_k}{2\mu_k(1+\nu_k)} \right\} \).
By definition of the Calabi’s diastasis function of \((\prod_{j=1}^{k} \Omega_j)^{\beta g_0}(\mu, \beta g(\mu; \nu))\), we have

\[
\exp\{-D_{\beta g(\mu; \nu)}((z, w), (\xi, \eta))\} = \left| \prod_{i=1}^{k} N_{\Omega_i}(z_i, \overline{\xi}_i)^{-\beta \mu_i(1+\nu_i)} \left(1 - \frac{w^\jmath}{\prod_{i=1}^{k} N_{\Omega_i}(z_i, \overline{\xi}_i)^{\mu_i}}\right)^{-\beta} \right|^2 \\
\times \prod_{i=1}^{k} N_{\Omega_i}(z_i, \overline{\xi}_i)^{\beta \mu_i(1+\nu_i)} \prod_{i=1}^{k} N_{\Omega_i}(\xi_i, \overline{\xi}_i)^{\beta \nu_i(1+\nu_i)} \\
\times \left(1 - \frac{\|w\|^2}{\prod_{i=1}^{k} N_{\Omega_i}(z_i, \overline{\xi}_i)^{\mu_i}}\right)^{-\beta} \left(1 - \frac{\|\eta\|^2}{\prod_{i=1}^{k} N_{\Omega_i}(\xi_i, \overline{\xi}_i)^{\mu_i}}\right)^{-\beta}, \tag{3.3}
\]

By (2.8), we have

\[
\prod_{i=1}^{k} N_{\Omega_i}(z_i, \overline{\xi}_i)^{-\beta \mu_i(1+\nu_i)} \left(1 - \frac{w^\jmath}{\prod_{i=1}^{k} N_{\Omega_i}(z_i, \overline{\xi}_i)^{\mu_i}}\right)^{-\beta} = \sum_{j=0}^{+\infty} \frac{(\Gamma(\beta + j)(w^\jmath))^{j}}{\prod_{i=1}^{k} N_{\Omega_i}(z_i, \overline{\xi}_i)^{\mu_i(1+\nu_i)+j}} \\
= \sum_{j=0}^{+\infty} \sum_{\ell(\lambda^{(i)}) \leq r_i, 1 \leq \lambda^{(i)} \leq k} \frac{(\Gamma(\beta + j)(\beta_{\mu_i}(1+\nu_i) + \mu_i j)^{(i)}_{\lambda^{(i)}})}{(\jmath)!} \left(\frac{(w^\jmath)^{j}}{\prod_{i=1}^{k} N_{\Omega_i}(z_i, \overline{\xi}_i)^{\mu_i(1+\nu_i)+j}}\right) K^{(i)}_{\lambda^{(i)}}(z_i, \overline{\xi}_i), \tag{3.4}
\]

where

\[
\frac{1}{N_{\Omega_i}(z_i, \overline{\xi}_i)^{s}} = \sum_{\ell(\lambda^{(i)}) \leq r_i} (s)^{(i)}_{\lambda^{(i)}} K^{(i)}_{\lambda^{(i)}}(z_i, \overline{\xi}_i)
\]

and

\[
(s)^{(i)}_{\lambda^{(i)}} = \prod_{j=1}^{r_i} \left(s - \frac{j-1}{2} a_i\right)_{\lambda^{(i)}}.
\]

When \(\beta_{\mu_i}(1+\nu_i) > \frac{r_i - 1}{2} a_i, 1 \leq i \leq k\), namely \(\beta > \max\left\{\frac{r_1 - 1}{2a_1(1+\nu_1)}, \ldots, \frac{r_k - 1}{2a_k(1+\nu_k)}\right\}\), we have

\[
(\beta_{\mu_i}(1+\nu_i) + \mu_i j)^{(i)}_{\lambda^{(i)}} > 0
\]

for \(\forall j \geq 0\) and \(\forall \lambda^{(i)}\) with \(\ell(\lambda^{(i)}) \leq r_i\).

Since \(K^{(i)}_{\lambda^{(i)}}(z_i, \overline{\xi}_i)\) and \(\frac{(w^\jmath)^{j}}{\prod_{i=1}^{k} N_{\Omega_i}(z_i, \overline{\xi}_i)^{\mu_i(1+\nu_i)+j}}\) are reproducing kernels, then

\[
\left(\frac{\Gamma(\beta + j)}{\Gamma(\beta)} \prod_{i=1}^{k} (\beta_{\mu_i}(1+\nu_i) + \mu_i j)^{(i)}_{\lambda^{(i)}}\right) \left(\frac{(w^\jmath)^{j}}{\prod_{i=1}^{k} N_{\Omega_i}(z_i, \overline{\xi}_i)^{\mu_i(1+\nu_i)+j}}\right)
\]
are also reproducing kernels. Observe that \( K_{\lambda_i}^{(i)}(z_i, \xi_i) \) is the reproducing kernels of \( \mathcal{P}_{\lambda_i}^{(i)} \) where \( \mathcal{P}_{\lambda_i}^{(i)} \) is the subspace of homogeneous holomorphic polynomials of degree \( |\lambda_i| \), therefore, there are linearly independent holomorphic homogeneous polynomials \( f_l \) \( (1 \leq l < +\infty) \) such that

\[
\prod_{i=1}^{k} N_{\Omega_i}(z_i, \xi_i)^{-\beta \mu_i (1 + \nu_i)} \left( 1 - \frac{w \eta^\beta}{\prod_{i=1}^{k} N_{\Omega_i}(z_i, \xi_i)^{\mu_i}} \right) = \sum_{l=1}^{+\infty} f_l(z, w) f_l(\xi, \eta). \tag{3.5}
\]

Applying the Cauchy-Schwarz inequality to (3.3), we have

\[
\exp\{-D_{\beta g(\mu, \nu)}((z, w), (\xi, \eta))\} \leq 1.
\]

By \( f(z, w) = (f_1(z, w), f_2(z, w), \ldots) \neq 0 \), we get \( \exp\{-D_{\beta g(\mu, \nu)}((z, w), (\xi, \eta))\} = 1 \) if and only if there exists a constant \( c \) such that \( f_l(z, w) = c f_l(\xi, \eta), \quad \forall l \in \mathbb{N}^+ \). \tag{3.6}

Taking \( f_l = 1 \) in (3.6), we obtain \( c = 1 \).

Taking linearly independent holomorphic homogeneous polynomials \( f_l \) of degree 1 in (3.6), it follows \((z, w) = (\xi, \eta)\).

Conversely, if \((z, w) = (\xi, \eta)\), it is not hard to see that

\[
\exp\{-D_{\beta g(\mu, \nu)}((z, w), (\xi, \eta))\} = 1.
\]

So far, we complete the proof. \( \square \)

Now we can give the proof of the Theorem 1.4.

The Proof of Theorem 1.4. By Lemma 3.2, we know that condition (I) is satisfied.

In the following, we prove the condition (II) can be also fulfilled. Firstly, by the explicit expression of \( \Phi \), the function \( a((z, w), (\xi_1, \eta_1), \ldots) \) admits a sesquianalytic extension on \( (\prod_{j=1}^{k} \Omega_j)^{\beta d_0} (\mu) \times (\prod_{j=1}^{k} \Omega_j)^{\beta d_0} (\mu). \)

We define an infinite set \( E \) as follows

\[
E := \left\{ \alpha \in \mathbb{N} : \alpha > \max \left\{ n, \frac{p_1 - 1}{\mu_1 (1 + \nu_1)}, \ldots, \frac{p_k - 1}{\mu_k (1 + \nu_k)} \right\} \right\}.
\]

Let \( \alpha \in E \), by (1.11), we know that

\[
\varepsilon_{(\alpha, g(\mu, \nu))}(z, w; \xi, \eta) = \sum_{j=0}^{d} \frac{D^j \phi(d)}{j!} (\alpha - n)^{d_0+j} X(z, w; \xi, \eta)^{d-j},
\]

where

\[
X(z, w; \xi, \eta) = 1 - \frac{w \eta^\beta}{\prod_{i=1}^{k} N_{\Omega_i}(z_i, \xi_i)^{\mu_i}}.
\]

Since \( \frac{D^j \phi(d)}{d^j} = 1 \), we have

\[
\varepsilon_{(\alpha, g(\mu, \nu))}(z, w; \xi, \eta) = \alpha^n + \sum_{j=1}^{n} a_j(z, w; \xi, \eta) \alpha^{n-j} = \alpha^n + B(z, w; \xi, \eta) \alpha^{n-1} + C(\alpha, z, w; \xi, \eta) \alpha^{n-2},
\]

Therefore \( \varepsilon_{(\alpha, g(\mu, \nu))}(z, w; \xi, \eta) \) satisfies (II).
where

\[
B(z, w; \xi, \eta) = -\frac{(n+1)n}{2} + \frac{D^{d-1}\phi(d)}{(d-1)!} X(z, w; \xi, \eta), \tag{3.7}
\]

\[
C(\alpha, z, w; \xi, \eta) = \sum_{j=2}^{n} a_j(z, w; \xi, \eta)\alpha^{2-j}, \tag{3.8}
\]

here \(a_j(z, w; \xi, \eta)\) are polynomials in \(X(z, w; \xi, \eta)\).

By Lemma 3.1, it easy to see that

\[
\sup_{(z, w), (\xi, \eta) \in \left( \prod_{j=1}^{k} \Omega_j \right)^{2l_0} (\mu)} |a_j(z, w; \xi, \eta)| < +\infty, \ 2 \leq j \leq n.
\]

So, from (3.7) and (3.8), we get

\[
\sup_{(z, w), (\xi, \eta) \in \left( \prod_{j=1}^{k} \Omega_j \right)^{2l_0} (\mu)} |B(z, w; \xi, \eta)| < +\infty
\]

and

\[
\sup_{(z, w), (\xi, \eta) \in \left( \prod_{j=1}^{k} \Omega_j \right)^{2l_0} (\mu), \ \alpha \in E} |C(\alpha, z, w; \xi, \eta)| < +\infty.
\]

Hence the condition (II) is verified. So far we complete the proof. \(\square\)

Lastly, we give the proof of Theorem 1.5.

**The Proof of Theorem 1.5.** By definition, the metric \(\alpha g(\mu, \nu)\) is balanced if and only if \(\varepsilon(\alpha g(\mu, \nu))(z, w)\) is dependent of \((z, w)\). The formula (1.7) implies that \(\alpha g(\mu, \nu)\) is balanced iff there exist a constant \(\lambda(\alpha)\) with respect to \((z, w)\) such that

\[
(1 - \|\vec{\alpha}\|^2)^{-\alpha} = \lambda(\alpha) \sum_{t=0}^{+\infty} \psi(\alpha, t) \frac{(\alpha t)\|\vec{\alpha}\|^2t}. \]

Thus, by (2.32), we conclude that \(\lambda(\alpha)\psi(\alpha, t) = 1\), which means that \(\psi(\alpha, t)\) is a constant with respect to \(t\). By the expression of \(\psi(\alpha, t)\), that is,

\[
\psi(\alpha, t) = \frac{\prod_{i=1}^{k} \chi_i((1 + \nu_i)(1 + \alpha + t) - p_i)}{\prod_{i=1}^{k} \mu_i^{d_i} \sum_{j=0}^{d} \sigma(j)(\alpha - n)d - j(\alpha + t - j)j}, \]

we have \(\psi(\alpha, t)\) tends to 1 as \(t \to \infty\). Hence the metrics \(\alpha g(\mu, \nu)\) is balanced if and only if

\[
\prod_{i=1}^{k} \chi_i((1 + \nu_i)(1 + \alpha + t) - p_i) = \left( \prod_{i=1}^{k} \mu_i^{d_i} \right) \sum_{t=0}^{d} \left( \sum_{\sum_{i=1}^{k} \nu_i = t} \prod_{i=1}^{k} \mu_i^{d_i} \right) \chi_{x-n}(x + y - d + t)_{d-t}
\]

by (1.8).

Now we turn into the rest proof of Theorem 1.5. On one hand, by Lemma 3.2, we know that Condition (I) is satisfied. On the other hand, the formula (1.13) implies

\[
\varepsilon(\alpha g(\mu, \nu))(z, w; \xi, \eta) = (\alpha - n)_{n} = (\alpha - 1)(\alpha - 2) \cdots (\alpha - n).
\]

Therefore, by Berezin [2], we have \(((\prod_{j=1}^{k} \Omega_j)^{2l_0} (\mu), g(\mu; \nu))\) admits a Berezin quantization for \(\mu_{i} \in W(\Omega_{i}) \setminus \{0\}\) \((1 \leq i \leq k)\). The proof is completed. \(\square\)
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