Quintessence Universe and cosmic acceleration in $f(Q, T)$ gravity

M. Koussour$^{1,*}$, N. Myrzakulov$^{2,3,†}$, S.H. Shekh$^{4,‡}$ and M. Bennai$^{1,5,§}$

$^1$Quantum Physics and Magnetism Team, LPMC, Faculty of Science Ben M’sik, Casablanca Hassan II University, Morocco.
$^2$L. N. Gumilyov Eurasian National University, Nur-Sultan 010008, Kazakhstan.
$^3$Ratbay Myrzakulov Eurasian International Centre for Theoretical Physics, Nur-Sultan 010009, Kazakhstan.
$^4$Department of Mathematics. S. P. M. Science and Gilani Arts Commerce College, Ghatanji, Dist. Yavatmal, Maharashtra-445301, India.
$^5$Lab of High Energy Physics, Modeling and Simulations, Faculty of Science, University Mohammed V-Agdal, Rabat, Morocco.

(Dated: September 12, 2022)

The problem of cosmic acceleration and dark energy is one of the mysteries presently posed in the scientific society that general relativity has not been able to solve. In this work, we have considered alternative models to explain this late-time acceleration in a flat Friedmann-Lemaitre-Robertson-Walker (FLRW) Universe within the framework of the $f(Q, T)$ modified gravity theory (where $Q$ is the non-metricity and $T$ is the trace of the energy-momentum tensor) recently proposed by Xu et al. (Eur. Phys. J. C 79 (2019) 708), which is an extension of $f(Q)$ gravity with the addition of the $T$ term. Here, we presume a specific form of $f(Q, T) = \alpha Q + \beta Q^2 + \gamma T$ where $\alpha$, $\beta$, and $\gamma$ are free model parameters, and obtained the exact solutions by assuming the cosmic time-redshift relation as $t(z) = \frac{mH_0}{\alpha} n z$ which produces the Hubble parameter of the form $H(z) = \frac{mH_0}{m+n} \left[ \frac{1}{g(z)} + 1 \right]$ where $m$ and $n$ are the non-negative constants, we find the best values for them using 57 data points of the Hubble parameter $H(z)$. Also, we find the behavior of different cosmological parameters as the deceleration parameter ($q$), energy density ($\rho$), pressure ($p$) and EoS parameter ($\omega$) and compare them with the observational results. To ensure the validity of the results, we studied the energy conditions along with jerk parameter. Finally, we found that our model behaves similarly to the quintessence Universe.

I. INTRODUCTION

General Relativity (GR) is based that space and time constitute a unified structure assigned on Riemannian manifolds with the metric and the Levi-Civita connection. As we know GR is established on some main assumptions like Relativity principle, Equivalence principle, General Covariance principle, Causality principle and Lorentz covariance. Moreover, it is well-known that GR has two other equivalent descriptions, that are based on different connections. When we investigate a metric compatible but flat connection (i.e. the curvature is zero), we get the Teleparallel Equivalent of General Relativity (TEGR), where gravity is formulated through torsion [12]. Finally, the last description is Symmetric Teleparallel Equivalent of GR (STEGR) which works in a flat and torsion-free connection where non-metricity is assumed as gravitational interactions [13]. On the other hand, modern observations in cosmology of SNIa (type Ia Supernova) [1, 2], LSS (Large Scale Structure) [3, 4], WMAP (Wilkinson Microwave Anisotropy Probe) data [5–7], CMB (Cosmic Microwave Background) [8, 9], and BAO (Baryonic Acoustic Oscillations) [10, 11] show that the expansion of the Universe has entered an acceleration phase. Moreover, the same observational data display that everything we see around us is only 5% of the total content of the Universe, and the remaining content, i.e. 95% is in the form of unknown species dubbed Dark Matter (DM) and Dark Energy (DE). The results of these observations contradict GR, in particular the standard Friedmann equations, which are part of the applications of GR on a homogeneous and isotropic Universe on a large scale. Consequently, GR is not the final theory of gravity, it might be a special case of a more general theory of gravity.

To account for the recent observational data associated with the acceleration of the Universe, in recent years in the literature, there is a huge effort to modify gravity in order to be able to describe the evolution of the Universe and solve the mysteries of DE and DM. Most of the works start from the curvature-based Einstein-Hilbert action formulation and extend it in the form $f(R)$ [14]. Also successfully built a gravitational modification torsion-based on TEGR namely $f(T)$ grav-
ity [15, 16]. Note that TEGR at the level of equations coincides completely with general relativity, but equations of their modifications are different because field equations of \( f(T) \) gravity are of second order while those of \( f(R) \) are of fourth-order. Very recently new modified \( f(Q) \) gravity theory was proposed as a geometric interpretation and attracted a lot of attention in which gravity is attributed to non-metricity, which geometrically describes the variation of the length of a vector in the parallel transport i.e. \( Q_{\gamma\mu\nu} = \nabla_\gamma S_{\mu\nu} \) [17, 18]. The differential geometry in this case is called Weyl geometry which is a generalization of Riemannian geometry i.e. the geometric basis of GR. These non-metricity-based \( f(Q) \) theories represent a generalization of the STEGR, like \( f(R) \) and \( f(T) \) gravity. The theory \( f(Q) \) gravity has been explored in different contexts and cosmological applications. In [19] investigated the evolution of linear perturbations in the \( f(Q) \) gravity and considered different evolutions of the effective dark energy equation of state. Non-metricity scalar \( Q \) and the equations of motion for generic static and spherically symmetric geometry with an anisotropic fluid is derived in [20]. Application of Dirac’s method for the quantization of constrained systems in the context of \( f(Q) \) gravity is presented in [21]. In [22, 23] derived the gravitational equations for \( f(Q) \) gravity in the homogeneous, anisotropic locally, rotationally, symmetric Bianchi-I Universe in the presence of a single anisotropic perfect fluid.

One of interesting extensions of symmetric teleparallel gravity newly proposed as geometric alternatives to DE based on the coupling between non-metricity \( Q \) and the trace of the energy-momentum tensor \( T \), i.e., considering an arbitrary function \( f(Q, T) \) in the gravitational action [24]. It is clear that for \( T = 0 \) i.e. the case of vacuum, this theory reduces to the \( f(Q) \) gravity, which is equivalent to GR and passes all solar system tests. The full set of field equations of this theory are obtained by varying the gravitational action with respect to both metric and connection, separately. The covariant divergence of the gravitational equations are obtained. Such coupling can lead to the non-conservation of the energy-momentum tensor. This conservation violation has substantial physical clues that predict large changes in the thermodynamics of the Universe similar to those predicted by \( f(R, T) \) gravity. Note that the resulting theory differs from well-known \( f(R, T) \) gravity [25] and \( f(T, T) \) gravity [26] in that it is a novel modified gravitational theory based on a more general geometric framework than Riemannian geometry (Weyl geometry), with no curvature-equivalent and no torsion-equivalent, and its cosmological implications are very interesting. In the literature, there are active works in the framework \( f(Q, T) \) gravity theory [27–31]. Thus, there is a motivation to examine several theoretical, observational and cosmological aspects of \( f(Q, T) \) gravity. It has newly been found out that \( f(Q, T) \) gravity dramatically alters the nature of tidal forces and the equation of motion in the Newtonian limit [32]. In [33] authors developed the cosmological linear theory of perturbations for \( f(Q, T) \) gravity. They claim that results might also enable to test with CMB and standard siren data. Energy conditions constraints on different forms of \( f(Q, T) \) gravity was investigated in [34, 35]. This analysis were carried using the actual values of the deceleration parameters, Hubble and verify the compatibility with \( \Lambda \)CDM model. Weyl form of \( f(Q, T) \) gravity model in which the scalar non-metricity is fully determined by a vector field \( w_\mu \) proposed in [36]. This gravity theory can be considered as an useful and alternative approach for the description of the early phases and late phases of cosmological evolution.

In this work, we have also investigated the cosmological model with Friedmann-Lemaître-Robertson-Walker (FLRW) Universe in \( f(Q, T) \) theory taking into account the coupling of the form \( f(Q, T) = aQ + bQ^2 + \gamma T \), where \( a, b, \) and \( \gamma \) are free model parameters. This supported by \( f(R, T) \) gravity form \( f(R, T) = aR + bR^2 + \gamma T \) in which the presence of square term of \( R \) reveals the existence of the late-time acceleration phase. Using the hybrid expansion law of the scale factor which leads to the time-redshift relation in the form of the Lambert function i.e. \( t(z) = \frac{\ln b}{2\pi i} g(z) \) that has been studied in several modified theories of gravity [37, 38], we have analyzed the various cosmological parameters like deceleration parameter, energy density, pressure, and the equation of state (EoS) parameter with the energy conditions for our cosmological model. In addition, we try to constrain the model parameters using the recent 57 Hubble datasets points by minimizing the chi-square function.

The manuscript is organizing in the following form: In Sect. II presented field equations of the theory by varying action. Gravitational field equations along with it’s solution for the FLRW line element are shown in Sect. III. Comparison with observational data constraints from \( H(z) \) datasets demonstrated in Sect. IV. In Sect. V, we studied some physical parameters including energy conditions and the Cosmographic jerk parameter for particular case of \( f(Q, T) \) theory while the conclusions is given in the last section in detail.
II. BASIC FORMALISM IN $f(Q,T)$ GRAVITY

The modified Einstein-Hilbert action for the $f(Q,T)$ extended symmetric teleparallel gravity is given by [24]

$$S = \int \left( \frac{1}{16\pi} f(Q,T) + L_m \right) \sqrt{-g} d^4x,$$

where $f(Q,T)$ being the general function of the non-metricity scalar $Q$ and the trace of the energy-momentum tensor $T$. $g$ is the determinant of the metric tensor $g_{\mu\nu}$ i.e. $g = \det \left( g_{\mu\nu} \right)$, and $L_m$ is the usual matter Lagrangian. The non-metricity scalar $Q$ is defined as

$$Q = -\delta^\mu_{(\nu} L^\alpha_{\nu\beta)} - L^\beta_{\alpha\nu} - L^\alpha_{\nu\beta},$$

where the disformal tensor $L^\beta_{\alpha\gamma}$ is given by

$$L^\beta_{\alpha\gamma} = -\frac{1}{2} \delta^\beta_\gamma \left( \nabla_\alpha g_{\beta\gamma} + \nabla_\gamma g_{\beta\alpha} - \nabla_\beta g_{\alpha\gamma} \right).$$

The non-metricity tensor is defined by the following form

$$Q_{\gamma\mu\nu} = \nabla_\gamma g_{\mu\nu},$$

and the trace of the non-metricity tensor is obtained as

$$Q_\beta = g^\mu\nu Q_{\beta\mu\nu} \quad \bar{Q}_\beta = g^\mu\nu Q_{\mu\beta\nu}.$$  

Further, we define the superpotential tensor as follows

$$P^\beta_{\mu\nu} = -\frac{1}{2} L^\beta_{\mu\nu} + \frac{1}{4} (Q^\beta - \bar{Q}^\beta) g_{\mu\nu} - \frac{1}{4} \delta^\beta_\mu Q_\nu, \quad (6)$$

and using this definition above, the non-metricity scalar is given as

$$Q = -Q_{\beta\mu\nu} P^\beta_{\mu\nu}. \quad (7)$$

Here, the definition of the energy-momentum tensor of the matter is given by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}L_m)}{\delta g_{\mu\nu}}.$$  

and

$$\Theta_{\mu\nu} = \delta^a_\beta \delta T_{a\beta}.$$  

The variation of energy-momentum tensor with respect to the metric tensor $g_{\mu\nu}$ read as

$$\frac{\delta g^{\mu\nu} T_{\mu\nu}}{\delta g^a_{\beta}} = T_{a\beta} + \Theta_{a\beta}. \quad (10)$$

In addition, the field equations of $f(Q,T)$ gravity are given by varying the action ($S$) with respect to metric tensor $g_{\mu\nu}$

$$-\frac{2}{\sqrt{-g}} \nabla_\beta (f_Q \sqrt{-g} P^\beta_{\mu\nu} - \frac{1}{2} f T_{\mu\nu} + f_T (T_{\mu\nu} + \Theta_{\mu\nu}) - f_Q (P_{\mu\beta\alpha} Q^\alpha_{\beta\nu} - 2Q^\beta_{\mu} P_{\beta\alpha\nu}) = 8\pi T_{\mu\nu},$$  

where $f_Q = \frac{df(Q,T)}{dQ}$, $f_T = \frac{df(Q,T)}{dT}$, and $\nabla_\beta$ denotes the covariant derivative. From Eq. (11) it appears that the field equations of $f(Q,T)$ extended symmetric teleparallel gravity are possible. Originally, Xu et al. [24] obtained three models using following functional forms of $f(Q,T)$: (i) $f(Q,T) = \alpha Q + \beta T$, (ii) $f(Q,T) = \alpha Q^{n+1} + \beta T$, (iii) $f(Q,T) = -\alpha Q - \beta T^2$ where $\alpha$, $\beta$ and $n$ are constants.

III. FLAT FLRW UNIVERSE IN $f(Q,T)$ COSMOLOGY

To solve field equations in $f(Q,T)$ extended symmetric teleparallel gravity it is usually necessary to make simplifying assumptions such as the choice of a metric.

Therefore, in this work, we consider the spatially homogeneous and isotropic flat FLRW Universe given by the following metric,

$$ds^2 = -N^2(t) dt^2 + a^2(t) \left( dx^2 + dy^2 + dz^2 \right),$$  

where $a(t)$ is the scale factor of the Universe, depending only on cosmic time (where the cosmic time is measure in Gyr) and $N(t)$ is the lapse function considered to be 1 in the standard case. The rates of expansion and dilation are determined as $H \equiv \frac{\dot{a}}{a}$, $T \equiv \frac{\dot{N}}{N}$ respectively. Thus, the corresponding non-metricity scalar is given by $Q = \frac{6N^2}{a^2}$. In this article, we presume that the content of the Universe as a perfect fluid whose energy-momentum tensor is given by

$$T^\mu_\nu = \text{diag} \left( -\rho, p, p, p \right),$$  

Here, \( p \) is the perfect fluid pressure and \( \rho \) is the energy density of the Universe. Thus, for tensor \( \theta^a_v \) the expression is obtained as \( \theta^a_v = \text{diag} (2\rho + p, -p, -p, -p) \). Considering the case as \( N = 1 \), the Einstein field equations using the metric (12) are given as,

\[
8\pi\rho = \frac{f}{2} - 6FH^2 - \frac{2\tilde{G}}{1 + \tilde{G}} \left( \dot{F}H + F\dot{H} \right), \quad (14)
\]

\[
8\pi p = -\frac{f}{2} + 6FH^2 + 2 \left( \dot{F}H + F\dot{H} \right). \quad (15)
\]

where, we used \( Q = 6H^2 \) and \((\cdot)\) dot represents a derivative with respect to cosmic time \((t)\). In this case,

\[
3H^2 = 8\pi\rho_{\text{eff}} = \frac{f}{4F} - \frac{4\pi}{F} \left[ (1 + \tilde{G}) \rho + \tilde{G}p \right], \quad (17)
\]

\[
2\dot{H} + 3H^2 = -8\pi p_{\text{eff}} = \frac{f}{4F} - \frac{2\dot{F}H}{F} + \frac{4\pi}{F} \left[ (1 + \tilde{G}) \rho + (2 + \tilde{G})p \right]. \quad (18)
\]

In order to obtain the exact solutions to the field equations above, we assume the Lambert function distribution for the time-redshift relation \( t(z) \) as following

\[
t(z) = \frac{nt_0}{m} g(z), \quad (19)
\]

and

\[
g(z) = \text{LambertW} \left[ k (1 + z)^{-\frac{1}{n}} \right], \quad (20)
\]

where \( k = \frac{m}{\pi} e^{\pi} \), \( n \) and \( m \) are non-negative constants and \( t_0 \) represents the present age of the Universe. Using the relation between the scale factor and redshift of the Universe \( a(t) = a_0 (1 + z)^{-1} \) where \( a_0 \) represent the present value of scale factor, we find the Hubble parameter as

\[
H = \frac{\dot{a}}{a} = \frac{m}{t_0} + \frac{n}{t}. \quad (21)
\]

The motivation behind the above choice is that the relation (21) produces the scale factor of the hybrid type, and it is known in the literature that this type depicts the transition from the early decelerating phase to the present accelerating phase, which create a time-dependent deceleration parameter. Also, the ansatz reduces to the usual power law for \( m = 0 \) and de Sitter solutions for certain values of \( m \) and \( n \). The corresponding deceleration parameter is given as

\[
q = -1 + \frac{d}{dt} \left( \frac{1}{H} \right) = -1 + nt_0^2 (nt_0 + mt)^{-2}. \quad (22)
\]

For the model constant parameters \( m \) and \( n \), we found the appropriate values in the next section using the observational Hubble data \( H(z) \) (OHD) as \( m = 0.2239 \) and \( n = 0.6886 \). Fig. 1 clearly shows the transition of the Universe from the deceleration phase \( (q > 0) \) to the acceleration phase \( (q < 0) \) with the transition redshift \( z_{tr} = 0.5234 \) for \( m = 0.2239 \). Thus, the transition redshift value for our model is in conformity with the observational data.

In addition, in Fig. 1 we examine the effect of parameter \( m \) on the model through three different values, namely \( m = 0.2239, 0.3, 0.4 \).

### IV. OBSERVATIONAL CONSTRAINTS FROM \( H(z) \) DATASETS

To get the best fit value of the model parameters \( m \) and \( n \) of our model under study and to compare our results with observation data, we need to constrain the parameters using some observational datasets. In this section, we will use Hubble datasets with 57 data points. In [39] Sharov and Vasiliev prepared a list of 57 data points
of measurements of the Hubble parameter in the cosmological redshift range $0.07 \leq z \leq 2.42$, 31 points from the differential age method (DA method) and the other 26 points were evaluated using BAO data and other methods (See Tab. I).

Using Eqs. (19)-(21), the Hubble parameter $H$ in terms of the cosmological redshift $z$ as

$$H(z) = \frac{mH_0}{m+n} \left[ \frac{1}{8(z)} + 1 \right],$$

where $H_0 = \frac{m+n}{t_0}$ is the present value of Hubble parameter. From Eq. (23) we can see that the parameters of the model which are needed to constrain are $m, n$ and $H_0$. Thus, the best fit values of model parameters $m, n$ and $H_0$ are determined by minimizing the following chi-square function

$$\chi^2_{OHD}(m, n, H_0) = \sum \frac{[H_{th}(m, n, H_0, z_i) - H_{obs}(z_i)]^2}{\sigma_i^2},$$

where $H_{th}(m, n, H_0, z_i)$ and $H_{obs}(z_i)$ are theoretical and observed values of Hubble parameter respectively, and $\sigma_i^2$ represents the standard error in the observed Hubble parameter measurements. Also, $\sigma_i^2$ errors of differential age method and BAO and other methods are represented in Tab. I. The best fit values of model parameters $m, n$ and $H_0$ is obtained as $m = 0.2239 (-0.0962, 0.544)$, $n = 0.6886 (0.4018, 0.9754)$ and $H_0 = 62.73 km.s^{-1}.Mpc^{-1} (54.3, 71.16)$ respectively. Fig. 2 shows the best fit curve of $H(z)$ versus the cosmological redshift $z$ using 57 Hubble parameter measurements.

![Fig. 1. Deceleration parameter $(q)$ versus redshift $(z)$ with $n = 0.6886$.](image1)

![Fig. 2. Best fit curve of Hubble function $H(z)$ versus redshift $z$. The blue dots represents error bars of 57 data points, the red line is the curve gained for our model.](image2)

**V. COSMOLOGICAL $f(Q, T)$ MODEL**

Many researchers have studied several models of $f(R, T)$ gravity in the form $f(R, T) = \alpha R + \beta R^2 + \gamma T$ (where $R$ is the Ricci scalar and $T$ is the trace of energy-momentum tensor) and obtained good results, with the factor $R^2$ is added to explain the late time acceleration in the expansion of the Universe. Motivated by this research and by replacing $R$ by $Q$, we obtain

$$f(Q, T) = \alpha Q + \beta Q^2 + \gamma T$$

where $\alpha$, $\beta$, and $\gamma$ are model parameters. The values of $f_Q$ and $f_T$ in the field equations (14) and (15) are derived as $F = f_Q = \alpha + 2\beta Q$ and $8\pi G = f_T = \alpha$. Using Eq. (25) in Eqs. (14) and (15) the values of the energy density $\rho$, pressure $p$ and equation of state (EoS) parameter $\omega = \frac{p}{\rho}$ are obtained as

$$\rho = \frac{\alpha \gamma \dot{H} - 3H^2(8\pi \alpha + \gamma(\alpha - 12\beta \dot{H})) - 54\beta(\gamma + 8\pi)H^4}{2(\gamma + 4\pi)(\gamma + 8\pi)},$$

$$p = \frac{(3\gamma + 16\pi)H \left( \alpha + 36\beta H^2 \right) + 3(\gamma + 8\pi)H^2 \left( \alpha + 18\beta H^2 \right)}{2(\gamma + 4\pi)(\gamma + 8\pi)},$$

$$\omega = \frac{(3\gamma + 16\pi)H \left( \alpha + 36\beta H^2 \right) + 3(\gamma + 8\pi)H^2 \left( \alpha + 18\beta H^2 \right)}{\alpha \gamma \dot{H} - 3H^2(8\pi \alpha + \gamma(\alpha - 12\beta \dot{H})) - 54\beta(\gamma + 8\pi)H^4}.$$

**A. Evaluation of $\rho, p, \omega, T$ and $f(Q, T)$:**

By using Eq. (21) in Eqs. (26)-(28) the expressions for $\rho, p, \omega, T$ and $f(Q, T)$ model are obtained as follows:
Energy density ($\rho$):

$$\rho = -\frac{1}{2(\gamma + 4\pi)(\gamma + 8\pi)} \left[ 36\beta \gamma n (mt + nt_0)^2 t^{-4} t_0^{-2} + 3\alpha (\gamma + 8\pi) \left( \frac{m}{t_0} + \frac{n}{t} \right)^2 
+ 54\beta (\gamma + 8\pi) \left( \frac{m}{t_0} + \frac{n}{t} \right)^4 + \alpha \gamma nt^{-2} \right],$$  \hspace{1cm} (29)

Pressure ($p$):

$$p = -\frac{1}{2(\gamma + 4\pi)(\gamma + 8\pi)} \left[ \beta n (108\gamma + 576\pi) (mt + nt_0)^2 t^{-4} t_0^{-2} - 3\alpha (\gamma + 8\pi) \left( \frac{m}{t_0} + \frac{n}{t} \right)^2 
- 54\beta (\gamma + 8\pi) \left( \frac{m}{t_0} + \frac{n}{t} \right)^4 + \alpha n (3\gamma + 16\pi)t^{-2} \right].$$  \hspace{1cm} (30)

Fig. 3 represents the evolution of the energy density of the Universe as a function of redshift for three values of the parameter $m = 0.2239, 0.3, 0.4$. From this figure, we can see that the energy density remains positive for all $z$ values and is an increasing function of the cosmological redshift. It starts with a positive value and approaches zero when $z \to -1$. The pressure behavior as a function of redshift is shown in Fig. 4, we observe that the pressure in the current model is a decreasing function of the cosmological redshift, and it starts from a large negative value and approaches zero at the present time. According to recent observations, the Universe is in an accelerating expansion phase due to the so-called dark energy that has negative pressure. Thus the pressure for our model is consistent with recent observations.

![Image](image1.png)

**FIG. 4.** Pressure ($p$) versus redshift ($z$) with $n = 0.6886$, $\alpha = -3$, $\beta = 0.003$ and $\gamma = -\pi$.

![Image](image2.png)

**FIG. 3.** Energy density ($\rho$) versus redshift ($z$) with $n = 0.6886$, $\alpha = -3$, $\beta = 0.003$ and $\gamma = -\pi$.

![Image](image3.png)

**FIG. 5.** EoS parameter ($\omega$) versus redshift ($z$) with $n = 0.6886$, $\alpha = -3$, $\beta = 0.003$ and $\gamma = -\pi$.
EoS parameter ($\omega$):

$$\omega = \frac{\beta n (108\gamma + 576\pi) (mt + nt_0)^2 t^{-4} t_0^{-2} - 3\alpha (\gamma + 8\pi) \left(\frac{m}{t_0} + \frac{n}{T}\right)^2 - 54\beta (\gamma + 8\pi) \left(\frac{m}{t_0} + \frac{n}{T}\right)^4 + an (3\gamma + 16\pi) t^{-2}}{36\beta \gamma m (mt + nt_0)^2 t^{-4} t_0^{-2} + 3\alpha (\gamma + 8\pi) \left(\frac{m}{t_0} + \frac{n}{T}\right)^2 + 54\beta (\gamma + 8\pi) \left(\frac{m}{t_0} + \frac{n}{T}\right)^4 + a\gamma nt^{-2}}.$$  

(31)

The EoS parameter is an essential tool for describing the epochs the Universe has gone through, and understanding the nature of dark energy. This parameter takes various values for each different model of dark energy. In the case that the nature of dark energy is the cosmological constant ($\Lambda$CDM), $\omega = -1$. While if $-1 < \omega < -0.33$, we say that the nature of dark energy is quintessence, and $\omega < -1$, indicates phantom nature of the model. From Fig. 5, we can see that $-1 < \omega < -0.2$, it indicates quintessence nature of the model in the present, and in the future the model approaches to the $\Lambda$CDM region. The current value of $\omega$ is obtained as $\omega_0 = -0.5079$ for the values of the constants constrained by the observational Hubble parameter $H(z)$ data i.e. $m = 0.2239$ and $n = 0.6886$. Thus, the current value of EoS parameter of the model is consistent with Planck’s 2018 results.

The value of the trace of energy-momentum tensor $T$ is obtained as

$$T = 3p - \rho = \frac{1}{(\gamma + 4\pi)(\gamma + 8\pi)t_0^4} \left[ 2(54\beta(\gamma + 8\pi)m^4 t^4 + 216\beta(\gamma + 8\pi) m^3 nt_0^3 t_0 
+ 3m^2 t_0^2 \left( \gamma (12\beta n(9n - 2) + n t^2) + 8\pi \left(18\beta n(6n - 1) + n t^2\right) \right) + 6mnt_0^3 \left( \gamma (12\beta n(3n - 2) + n t^2) \right) 
+ 8\pi \left(18\beta n(2n - 1) + n t^2\right) \right) + nt_0^3 \left( \gamma (18\beta(n - 4)n^2 + n(3n - 2)t^2) \right) 
+ 12\pi \left(36\beta(n - 1)n^2 + n(2n - 1)t^2\right) \right]$$

(32)

Using the definition of non-metricity $Q$ for flat FLRW Universe and Eq. (25), the function $f(Q, T)$ is obtained as

$$f(Q, T) = \frac{1}{(\gamma + 4\pi)(\gamma + 8\pi)t_0^4} \left[ 4(36\beta \left(\gamma^2 + 9\pi\gamma + 8\pi^2\right) m^4 t^4 + 144\beta \left(\gamma^2 + 9\pi\gamma + 8\pi^2\right) m^3 nt_0^3 t_0 
+ 3m^2 t_0^2 \left(16\pi^2 \left(36\beta n^2 + n t^2\right) + \gamma^2 \left(12\beta n(6n - 1) + n t^2\right) + 2\pi \gamma \left(36\beta n(9n - 1) + 5n t^2\right) \right) 
+ 6mnt_0^3 \left(16\pi^2 \left(12\beta n^2 + n t^2\right) + \gamma^2 \left(12\beta n(2n - 1) + n t^2\right) + 2\pi \gamma \left(36\beta n(3n - 1) + 5n t^2\right) \right) 
+ nt_0^3 \left(36\beta \left(\gamma^2 + 9\pi\gamma + 8\pi^2\right) n^3 - 36\beta \gamma(\gamma + 6\pi)n^2 + 3\alpha \left(\gamma^2 + 10\pi\gamma + 16\pi^2\right) n t^2 - \alpha \gamma(\gamma + 6\pi)t^2\right) \right]$$

(33)

The behavior of $f(Q, T)$ gravity model versus redshift ($z$) and $m$ is clearly shown in Fig. 6.

B. Energy conditions (ECs):

We previously studied some cosmological parameters that plays an important role in studying the evolution of the Universe, such as the deceleration parameter, EoS parameter, etc. But in order to predict the cosmic ac-
Acceleration in modern cosmology, a set of energy conditions appeared that relates the energy density of the Universe and pressure and can be derived from equation of Raychaudhuri [40]. In GR, the role of these energy conditions is to prove the theorems for the existence of space-time singularity and black holes [41]. Several authors have worked on energy conditions in various backgrounds [33–35]. In this paper, we will consider the famous energy conditions in order to check the validity of the model in the context of cosmic acceleration. There are different forms of energy conditions such as the weak energy conditions (WEC), null energy condition (NEC), dominant energy conditions (DEC), and strong energy conditions (SEC) are given for the content of the Universe in form of a perfect fluid in $f(Q, T)$ modified gravity as follows

- **WEC:** $\rho \geq 0$,
- **NEC:** $\rho + p \geq 0$,
- **DEC:** $\rho - p \geq 0$,
- **SEC:** $\rho + 3p \geq 0$.

The significance of these energy conditions above shows that when the NEC is violated, all other energy conditions are violated. This violation of the NEC represents the depletion of energy density as the Universe expands. Also, the violation of the SEC represents the acceleration of the Universe. We can see this from the standard Friedmann equations, in order to explain the late-time cosmic acceleration with $\omega \simeq -1$, it must be $\rho + 3p = \rho (1 + 3\omega) < 0$. In Fig. 7, we can see the evolution of the energy conditions WEC, NEC, DEC and SEC as functions of the cosmological redshift and $m$, respectively. From the figures, we observe that WEC, NEC and DEC are satisfied while the SEC is violated in the present and future. Hence, the violation of SEC leads to the acceleration of the Universe (see Fig. 7)

**C. Cosmographic jerk parameter**

The jerk parameter is one of the basic physical quantities to explain the dynamics of the Universe. The Jerk parameter is a dimensionless third derivative of the scale factor $a(t)$ with respect to cosmic time $t$ and is specified as [35]

$$j = \frac{\dddot{a}}{aH^3}. \quad (34)$$

Eq. (34) can be written in terms of a deceleration parameter $q$ as

$$j = q + 2q^2 - \frac{\ddot{q}}{H}. \quad (35)$$

Using Eqs. (21) and (22), the jerk parameter for our model is

$$j = \left[ m^3t^3 + 3m^2nt^2t_0 + 3m(n-1)nt^2_0 + n\left(n^2-3n+2\right)t_0^3 \right] (mt + nt_0)^{-3} \quad (36)$$

The value of the jerk parameter is $j = 1$ for $\Lambda CDM$ model. The Universe is transitioning from an early deceleration phase to the current phase of acceleration with a positive jerk parameter $j_0 > 0$ and a negative DP $q_0 < 0$ corresponding to $\Lambda CDM$ model. From Fig. 8 we can see that the jerk parameter remains positive for $m = 0.2239$ and $n = 0.6886$ and approaches 1 later. The current jerk parameter value $j_0$ is positive. Thus, our model is similar to the $\Lambda CDM$ model in the future.
In the current analysis, we studied one of interesting extension of $f(Q)$ gravity theory in form $f(Q, T)$ for geometric alternatives to dark energy in which the term $Q$ is non-metricity scalar and $T$ is the trace of the matter energy-momentum tensor using a source as perfect fluid. In this analysis, we found the best fit value of the model parameters $m$ and $n$ or constrain this parameters using some observational datasets such as 57 data points in the cosmological redshift range $0.07 \leq z \leq 2.42$, 31 points from the differential age method (DA method) and the other 26 points were evaluated using BAO data and other methods. In the derived model, the transition of the Universe from the deceleration phase ($q > 0$) to the acceleration phase ($q < 0$) with the transition redshift $z_{tr} = 0.5234$ for $m = 0.2239$ which is the evidence that the transition redshift value for our model is in conformity with the observational data.
In addition, we have studied the evolution of the energy density of the Universe as a function of redshift which remains positive for all $z$ and is an increasing function of the cosmological redshift. The pressure behavior as a function of redshift of the model was also studied and is a decreasing function of the cosmological redshift, and it starts from a large negative value and approaches zero at the present time. Thus, the model is consistent with recent observations. As for the EoS parameter behavior of our model, it is in the range $-1 < \omega < -0.2$ which indicates a quintessence nature of the model at present, and in the future the model approaches $\Lambda$CDM region whereas the current value of $\omega$ is obtained as $\omega_0 = -0.5079$ for the values of the constants $m = 0.2239$ and $n = 0.6886$. Therefore, the current value of EoS parameter of the model is consistent with Planck’s 2018 results. Finally, the energy conditions: WEC, NEC, and DEC are satisfied while the SEC is violated in the present and future. The violation of SEC leads to the acceleration of the Universe while the jerk parameter remains positive and approaches 1 later.

Thus, our model is similar to the $\Lambda$CDM model in the future.

**ACKNOWLEDGMENTS**

We are very much grateful to the honorary referee and the editor for the illuminating suggestions that have significantly improved our work in terms of research quality and presentation. This research was funded by the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan (Grant No. AP09058240).

Data availability There are no new data associated with this article

Declaration of competing interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

---

[1] A.G. Riess et al., Astron. J. 116, 1009 (1998).
[2] S. Perlmutter et al., Astrophys. J. 517, 565 (1999).
[3] T. Koivisto, D.F. Mota, Phys. Rev. D 73, 083502 (2006).
[4] S.F. Daniel, Phys. Rev. D 77, 103513 (2008).
[5] C.L. Bennett et al., Astrophys. J. Suppl. 148, 119-134 (2003).
[6] D.N. Spergel et al., [WMAP Collaboration], Astrophys. J. Suppl. 148, 175 (2003).
[7] G. Hinshaw et al., Astrophys. J. Suppl. 208, 19 (2013).
[8] R.R. Caldwell, M. Doran, Phys. Rev. D 69, 103517 (2004).
[9] Z.Y. Huang et al., JCAP 0605, 013 (2006).
[10] D.J. Eisenstein et al., Astrophys. J. 633, 560 (2005).
[11] W.J. Percival at el., Mon. Not. R. Astron. Soc. 401, 2148 (2010).
[12] K.Hayashi and T.Shirafuji, Phys. Rev. D 19, 3524 (1979).
[13] J.M. Nester and H-J Yo, Chin. J. Phys. 37, 2 (1999).
[14] H. A. Buchdahl, MNRAS 150, 1 (1970).
[15] G.Bengochea and R. Ferraro, Phys. Rev. D 79, 124019 (2009).
[16] E.V. Linder, Phys. Rev. D 81, 127301 (2010); Phys.Rev.D 82, 109902 (2010) (erratum).
[17] J.B. Jimenez et al., Phys. Rev. D 98, 4 (2018).
[18] J.B. Jimenez et al., Universe 5, 7 (2019).
[19] I.S. Albuquerque and N.Frusciante, Phys. Dark Univ. 35, 100980 (2022).
[20] W. Wang et al., Phys. Rev. D 105, 2 (2022).
[21] N. Dimakis et al., Class. Quant. Grav. 38, 22 (2021).
[22] A. De et al., Eur. Phys. J. C 82, 1 (2022).
[23] M. Koussour et al., Phys. Dark Universe 36, 101051 (2022).
[24] Y. Xu et al., Eur. Phys. J. C 79, 8 (2019).
[25] T. Harko et al., Phys. Rev. D 84, 024020 (2011).
[26] T. Harko et al., J. Cosmol. Astropart. Phys. 12, 021 (2014).
[27] O. Sokoliuk and A.Baransky, Astron.Nachr. 343, 5, e220003 (2022).
[28] A. Pradhan, A. Dixit, Int.J.Geom.Meth.Mod.Phys. 18 10, 2150159 (2021).
[29] N. Godani, G.C. Samanta Int.J.Geom.Meth.Mod.Phys. 18 9, 2150134 (2021).
[30] S. Bhattacharjee Int.J.Mod.Phys. A 37 06, 2250017 (2022).
[31] A. Najera and A. Fajardo, Phys.Dark Univ. 34,100889 (2021).
[32] J-Z. Yang et al., Eur. Phys. J. C, 81, 111 (2021).
[33] A. Najera and A. Fajardo, J. Cosmol. Astropart. Phys. 2022, 03 (2022).
[34] S. Arora and P.K. Sahoo, Phys. Scripta 95, 9 (2020).
[35] S. Arora et al., Phys. Dark Univ. 31, 100790 (2021).
[36] Y. Xu et al., Eur. Phys. J. C 80, 5 (2020).
[37] M. Koussour et al., J. High Energy Astrophys, 35, 43-51 (2022).
[38] M. Koussour and M. Bennai, Int. J. Mod. Phys. A 37, 2250027 (2022).
[39] G. S. Sharov and V. O. Vasiliev, Math. Model. Geom. 6, 1-20 (2018).
[40] A. Raychaudhuri, Phys. Rev. D 98, 1123 (1955).
[41] R. M. Wald, General Relativity, (Chicago, IL: University of Chicago Press, 1984).
[42] R. D. Blandford et al., ASP Conf. Ser. 339, 27 (2005).
| $z$ | $H(z)$ | $\sigma_H$ | $z$ | $H(z)$ | $\sigma_H$ |
|-----|--------|------------|-----|--------|------------|
| 0.070 | 69 | 19.6 | 0.4783 | 80 | 99 |
| 0.90 | 69 | 12 | 0.480 | 97 | 62 |
| 0.120 | 68.6 | 26.2 | 0.593 | 104 | 13 |
| 0.170 | 83 | 8 | 0.6797 | 92 | 8 |
| 0.1791 | 75 | 4 | 0.7812 | 105 | 12 |
| 0.1993 | 75 | 5 | 0.8754 | 125 | 17 |
| 0.200 | 72.9 | 29.6 | 0.880 | 90 | 40 |
| 0.270 | 77 | 14 | 0.900 | 117 | 23 |
| 0.280 | 88.8 | 36.6 | 1.037 | 154 | 20 |
| 0.3519 | 83 | 14 | 1.300 | 168 | 17 |
| 0.3802 | 83 | 13.5 | 1.363 | 160 | 33.6 |
| 0.400 | 95 | 17 | 1.430 | 177 | 18 |
| 0.4004 | 77 | 10.2 | 1.530 | 140 | 14 |
| 0.4247 | 87.1 | 11.2 | 1.750 | 202 | 40 |
| 0.4497 | 92.8 | 12.9 | 1.965 | 186.5 | 50.4 |
| 0.470 | 89 | 34 |  |  |  |

| $z$ | $H(z)$ | $\sigma_H$ | $z$ | $H(z)$ | $\sigma_H$ |
|-----|--------|------------|-----|--------|------------|
| 0.24 | 79.69 | 2.99 | 0.52 | 94.35 | 2.64 |
| 0.30 | 81.7 | 6.22 | 0.56 | 93.34 | 2.3 |
| 0.31 | 78.18 | 4.74 | 0.57 | 87.6 | 7.8 |
| 0.34 | 83.8 | 3.66 | 0.57 | 96.8 | 3.4 |
| 0.35 | 82.7 | 9.1 | 0.59 | 98.48 | 3.18 |
| 0.36 | 79.94 | 3.38 | 0.60 | 87.9 | 6.1 |
| 0.38 | 81.5 | 1.9 | 0.61 | 97.3 | 2.1 |
| 0.40 | 82.04 | 2.03 | 0.64 | 98.82 | 2.98 |
| 0.43 | 86.45 | 3.97 | 0.73 | 97.3 | 7.0 |
| 0.44 | 82.6 | 7.8 | 2.30 | 224 | 8.6 |
| 0.44 | 84.81 | 1.83 | 2.33 | 224 | 8 |
| 0.48 | 87.90 | 2.03 | 2.34 | 222 | 8.5 |
| 0.51 | 90.4 | 1.9 | 2.36 | 226 | 9.3 |

**TABLE I.** 57 points of $H(z)$ data: 31 (DA) and 26 (BAO+other) [39].