ON THE ABSOLUTELY CONTINUOUS SPECTRUM
OF THE LAPLACE-BELTRAMI OPERATOR ACTING
ON \(p\)-FORMS FOR A CLASS OF WARPED PRODUCT
METRICS

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Abstract. We explicitly compute the absolutely continuous
spectrum of the Laplace-Beltrami operator for \(p\)-forms for the class
of warped product metrics \(ds^2 = y^{2a} dy^2 + y^{2b} d\theta_{\partial M}^2\), where \(y\) is a
boundary defining function on the unit ball \(B(0,1)\) in \(\mathbb{R}^N\).

1. Introduction

In the present paper we continue the investigation of the spectrum of
the Laplace-Beltrami operator acting on \(p\)-forms for a class of warped
product metrics started in [1]. The Riemannian manifolds consid-
ered in that paper were constructed as follows: let \(\overline{M}\) be a compact
\(N\)-dimensional manifold with boundary, and let \(y\) be any boundary-
defining function. We endowed the interior \(M\) of \(\overline{M}\) with a Riemannian
metric \(ds^2\) such that in a small tubular neighbourhood of \(\partial M\) in \(M\)
\(ds^2\) takes the form

\[ ds^2 = e^{-2(a+1)t} dt^2 + e^{-2b} d\theta_{\partial M}^2, \]

where \(t = -\log y \in (c, +\infty)\) and \(d\theta_{\partial M}^2\) is a Riemannian metric on \(\partial M\).
For \(a \leq -1\), the manifold \(M\) is complete, hence the Laplace-Beltrami
operator \(\Delta^p_M\) is essentially selfadjoint on the smooth compactly sup-
ported \(p\)-forms. Moreover, under the assumption of rotational symmetry,
that is, assuming that \(\partial M = S^{N-1}\), we were able to check the belonging
of \(0\) to the essential spectrum of \(\Delta^p_M\), and hence to achieve a complete
description of the essential spectrum.

In the present paper, instead, we are concerned with the absolutely
continuous (and, partly, with the singularly continuous) spectrum.

In [3], Eichhorn showed that the essential spectrum of \(\Delta^p_M\) coincides
with the essential spectrum of the Friedrichs extension \((\Delta^p_M)^F\) of

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the restriction of $\Delta^p_M$ to the smooth $p$-forms with compact support in $(c, +\infty) \times \partial M$. Hence, in order to achieve the results in \cite{1}, it sufficed to know the behaviour of the Riemannian metric only in a tubular neighbourhood of the boundary.

As for the absolutely continuous (and the singularly continuous) spectrum of $\Delta^p_M$, instead, to our knowledge no result of the sort of \cite{6} is available. As a consequence, in order to compute these parts of the spectrum, we need global information on the Riemannian manifold.

In the present paper we restrict our attention to the case in which $M$ is the unitary open ball $B(0, 1)$ in $\mathbb{R}^N$ endowed with a Riemannian metric $ds^2$ given by

\begin{equation}
\tag{1.1}
ds^2 := f(t)dt^2 + g(t)d\theta^2_{S^{N-1}},
\end{equation}

where $t = 2\text{sech}(|x|)$, $f(t) > 0$, $g(t) > 0$ for every $t \in (0, +\infty)$, and $d\theta^2_{S^{N-1}}$ is the standard Riemannian metric on $S^{N-1}$. Moreover, we assume that $f(t) = 1$ and $g(t) = t^2$ for $0 < t < \epsilon$, whilst $f(t) = e^{-2(a+1)t}$ and $g(t) = e^{-2bt}$ for $t > c > \epsilon$.

On one hand, these assumptions give us a complete knowledge of the essential spectrum (see \cite{1}); on the other hand, they let us employ the radial decomposition techniques developed by Dodziuk (\cite{3}), Donnelly (\cite{5}) and Eichhorn (\cite{6}, \cite{7}). The decomposition consists of two steps: first, thanks to the Hodge decomposition on $S^{N-1}$, we write any $p$-form $\omega$ on $M$ as

\begin{equation}
\tag{1.2}
\omega = \omega_{1\delta} \oplus \omega_{2d} \land dt \oplus (\omega_{1d} \oplus \omega_{2\delta} \land dt),
\end{equation}

where $\omega_{1\delta}$ (resp. $\omega_{1d}$) is a coclosed (resp. closed) $p$-form on $S^{N-1}$ parametrized by $t$ and $\omega_{2\delta}$ (resp. $\omega_{2d}$) is a coclosed (resp. closed) $(p-1)$-form on $S^{N-1}$ parametrized by $t$. This gives the orthogonal decomposition

$L^2_p(M) = \mathcal{L}_{p,1}(M) \oplus \mathcal{L}_{p,2}(M) \oplus \mathcal{L}_{p,3}(M),$

and, since $\Delta^p_M$ is invariant, the corresponding decomposition

$$\Delta^p_M = \Delta^p_{M1} \oplus \Delta^p_{M2} \oplus \Delta^p_{M3}.$$ 

Since

$$\sigma_{ac}(\Delta^p_M) = \bigcup_{i=1}^{3} \sigma_{ac}(\Delta^p_{Mi}),$$

$$\sigma_{sc}(\Delta^p_M) = \bigcup_{i=1}^{3} \sigma_{sc}(\Delta^p_{Mi}),$$

we can reduce ourselves to the study of the absolutely continuous (and of the singularly continuous) spectrum of $\Delta^p_{Mi}$ for $i = 1, 2, 3$. 

The second step consists of the decomposition of \( \omega_{1\delta} \) (resp. of \( \omega_{2\delta} \), \( \omega_{3\delta} \)) according to an orthonormal basis of coclosed \( p \)-eigenforms (resp. closed and coclosed \( (p-1) \)-eigenforms) of \( \Delta^p_{S^{N-1}} \) (resp. of \( \Delta^{p-1}_{S^{N-1}} \)) on \( S^{N-1} \). In this way, up to a unitary equivalence, the spectral analysis of \( \Delta^p_{M_i} \), for \( i = 1, 2, 3 \), can be reduced to the investigation of the spectra of a countable number of selfadjoint Sturm-Liouville operators \( D_{i\lambda_k} \) on the half-line \((0, +\infty)\), parametrized by the eigenvalues \( \lambda^p_k \), \( k \in \mathbb{N} \) of \( \Delta^p_{S^{N-1}} \) on \( S^{N-1} \) if \( i = 1 \), and by the eigenvalues \( \lambda^{p-1}_k \) of \( \Delta^{p-1}_{S^{N-1}} \) if \( i = 2, 3 \). In particular, we have that for \( i = 1, 2, 3 \)

\[
\sigma_{ac}(\Delta^p_{M_i}) = \bigcup_{k \in \mathbb{N}} \sigma_{ac}(D_{i\lambda_k}),
\]

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\]

Actually, since the Hodge \( * \) operator maps isometrically \( p \)-forms of \( \mathcal{L}_{p,1}(M) \) into \( (N-p) \)-forms of \( \mathcal{L}_{N-p,2}(M) \) and vice versa, it suffices to consider the cases \( i = 1, 3 \). Moreover, it turns out that, since the absolutely continuous spectrum of \( \Delta^p_M \) is contained in the essential spectrum of \( \Delta^p_M \), which we know from \( \square \), in order to compute the absolutely continuous spectrum of \( \Delta^p_M \) it suffices to study the absolutely continuous spectrum of \( D_{1\lambda_k}^p \) for any \( k \in \mathbb{N} \): indeed, for any \( a \leq -1 \), \( b \in \mathbb{R}, p \in [0, N] \), we find that \( \bigcup_{k \in \mathbb{N}} \sigma_{ac}(D_{1\lambda_k}^p) = \sigma_{ess}(\Delta^p_{M_{\lambda_k}}) \).

The absolutely continuous spectrum (and the singularly continuous spectrum) of the operators \( D_{1\lambda_k}^p \) is computed through perturbation theory; on one hand, this required a subtle investigation of their domains. On the other hand, since the operators \( D_{1\lambda_k}^p \) act on the one-dimensional half-line \((0, +\infty)\) and have strongly divergent potential terms at zero, in order to study their spectra we had to prove modified versions of the classical Agmon-Kato-Kuroda Theorem (\( \square \)) and Lavine Theorem (\( \square \)). In particular, we had to choose properly the unperturbed operators employed in the perturbation techniques.

Let us briefly discuss our results. If \( a = -1, b < 0 \), the situation is similar to the hyperbolic case; we find that for \( 0 \leq p \leq N \)

\[
\sigma_{ac}(\Delta^p_M) = \left[ \min \left\{ \left( \frac{N-2p-1}{2} \right)^2 b^2, \left( \frac{N-2p+1}{2} \right)^2 b^2 \right\}, +\infty \right).
\]

If \( a \leq -1 \) and \( b = 0 \), for \( 0 \leq p \leq N \)

\[
\sigma_{ac}(\Delta^p_M) = [\overline{\lambda}_p, +\infty),
\]

where \( \overline{\lambda}_p \) is the minimum between the lowest eigenvalue \( \lambda^p_0 \) of \( \Delta^p_{S^{N-1}} \) and the lowest eigenvalue \( \lambda^{p-1}_0 \) of \( \Delta^{p-1}_{S^{N-1}} \).
For $a = -1$ and $b > 0$, if $1 < p < N - 1$ $\sigma_{ac}(\Delta^p_M) = \emptyset$, whilst if $p \in \{0, 1, N - 1, N\}$ $\sigma_{ac}(\Delta^p_M) = \left(\frac{N-1}{2}b^2, +\infty\right)$.

If $a < -1$ and $b < 0$, for $0 \leq p \leq N$ $\sigma_{ac}(\Delta^p_M) = [0, +\infty)$; finally, if $a < -1$ and $b > 0$, for $1 < p < N - 1$ $\sigma_{ac}(\Delta^p_M) = \emptyset$, whilst for $p \in \{0, 1, N - 1, N\}$ $\sigma_{ac}(\Delta^p_M) = [0, +\infty)$.

As for the singularly continuous spectrum, in any case we found that $\sigma_{sc}(\Delta^p_M) = \sigma_{sc}((\Delta^p_M)^3)$, whilst $\sigma_{sc}(\Delta^p_M) = \sigma_{sc}((\Delta^p_M)^2) = \emptyset$.

It would be interesting to complete the analysis of the spectrum of $\Delta^p_M$, computing its singularly continuous spectrum. This problem can be reduced to the determination of the singularly continuous spectrum of $D_{3\lambda^p_k - 1}$ for any $k \in \mathbb{N}$; this turns out to be a hard task because $D_{3\lambda^p_k - 1}$ is a coupled system of Sturm-Liouville operators on the half-line $(0, +\infty)$ with strongly divergent potentials at zero, for which the application of perturbation techniques is difficult.

The paper is organized as follows: in section 2 we introduce some preliminary facts and basic notations. In section 3 we describe in some detail the decomposition techniques. The calculus of the absolutely continuous spectrum (and, partly, of the singularly continuous spectrum) of $\Delta^p_M$ is performed in section 4 for $a = -1$ and in section 5 for $a < -1$.

2. Preliminary facts and notations

For $N \geq 2$, let $\overline{B}(0, 1)$ denote the closed unit ball

$$\overline{B}(0, 1) = \{x = (x_1, ..., x_N) \in \mathbb{R}^N \mid x_1^2 + ... + x_N^2 \leq 1\},$$

and let $\mathbb{S}^{N-1}$ denote the sphere

$$\mathbb{S}^{N-1} = \{(x_1, ..., x_N) \in \mathbb{R}^N \mid x_1^2 + ... + x_N^2 = 1\},$$

endowed with a coordinate system $(U_i, \Theta_i)$, $i = 2, ..., k + 1$, $\Theta_i : U_i \to \mathbb{R}^{N-1}$.

Let us consider the interior of $\overline{B}(0, 1)$,

$$B(0, 1) = \{(x_1, ..., x_N) \in \mathbb{R}^N \mid x_1^2 + ... + x_N^2 < 1\},$$

with the coordinate system $(V_i, \Phi_i)$, for $i = 1, ..., k + 1$, defined in the following way: in a neighbourhood of 0, for some $\delta > 0$,

$$V_1 = \{(x_1, ..., x_N) \in \mathbb{R}^N \mid x_1^2 + ... + x_N^2 < \delta\}$$

and

$$\Phi_1(x_1, ..., x_N) = (x_1, ..., x_N),$$
whilst for $i > 1$, $x \neq 0$,
\[ V_i = \left\{ x \in \mathbb{R}^N \mid \frac{x}{\|x\|} \in U_i \right\}, \]
\[ \Phi_i : V_i \longrightarrow (0, +\infty) \times \Theta_i(U_i), \]
\[ \Phi_i(x_1, \ldots, x_N) = \left( 2 \tanh(\|x\|), \Theta_i\left( \frac{x}{\|x\|} \right) \right) =: (t, \theta_i). \]

We denote by $M$ the manifold $B(0, 1)$, endowed with a Riemannian metric $ds^2$ such that on $\Phi_i(V_i)$, for $i > 1$,
\[ ds^2 := f(t) dt^2 + g(t) d\theta^2_{S^{N-1}}, \tag{2.1} \]
where $f(t) > 0$, $g(t) > 0$ for every $t \in (0, +\infty)$ and $d\theta^2_{S^{N-1}}$ is the standard metric on $S^{N-1}$. $ds^2$ is well-defined on $B(0, 1) \setminus \{0\}$.

We suppose that for $t > c > 0$, $a \in \mathbb{R}$, $b \in \mathbb{R}$
\[ f(t) = e^{-2(a+1)t}, \quad g(t) = e^{-2bt}. \tag{2.2} \]
As for the behaviour as $t \to 0$, we suppose that for $t \in (0, \epsilon)$ ($\epsilon = 2 \tanh(\delta)$)
\[ f(t) \equiv 1, \quad g(t) = t^2. \tag{2.3} \]
This assures that $ds^2$ can be extended to a smooth Riemannian metric on the whole manifold $M$; indeed, for $t \in (0, \epsilon)$, $ds^2$ is the expression, in polar coordinates, of the Euclidean metric on $\mathbb{R}^N$.

It is well-known (see [10]) that a Riemannian metric of this kind is complete if and only if $a \leq -1$. Hence throughout the paper we will suppose that $a \leq -1$.

For $p = 0, \ldots, N$, we will denote by $C^\infty(\Lambda^p(M))$ the space of all smooth $p$-forms on $M$, and by $C^\infty_c(\Lambda^p(M))$ the set of all smooth, compactly supported $p$-forms on $M$. For any $\omega \in C^\infty(\Lambda^p(M))$, we will denote by $|\omega(t, \theta)|_M$ the norm induced by the Riemannian metric on the fiber over $(t, \theta)$, given in local coordinates by
\[ |\omega(t, \theta)|^2_M = g^{j_1j_2}(t, \theta) \cdots g^{j_pj_p}(t, \theta) \omega_{i_1 \cdots i_p}(t, \theta) \omega_{j_1 \cdots j_p}(t, \theta), \]
where $g^{ij}$ is the expression of the Riemannian metric in local coordinates. We will denote by $d^p_M$, $\ast_M$, $\delta^p_M$, respectively, the differential, the Hodge $\ast$ operator and the codifferential on $M$, defined as in [2]. $\Delta^p_M$ will stand for the Laplace-Beltrami operator acting on $p$-forms
\[ \Delta^p_M = d^p_M \delta^p_M + \delta^p_M d^p_M, \]
which is expressed in local coordinates by the Weitzenböck formula
\[ ((\Delta^p_M) \omega)_{i_1 \cdots i_p} = -g^{ij} \nabla_i \omega_{j i_1 \cdots i_p} + \sum_j R^\alpha_{j i_1 \cdots i_p} \omega_\alpha + \sum_{j, l \neq j} R^\alpha_{ijij} \omega_\alpha, \]
where $\nabla_i \omega$ is the covariant derivative of $\omega$ with respect to the Riemannian metric, and $R^i_j$, $R^i_{jk}$ denote respectively the local components of the Ricci tensor and the Riemann tensor induced by the Riemannian metric. As usual, $L_p^2(M)$ will denote the completion of $C_\infty^p(\Lambda^p(M))$ with respect to the norm $\|\omega\|_{L_p^2(M)}$ induced by the scalar product $\langle \omega, \tilde{\omega} \rangle_{L_p^2(M)} := \int_M \omega \wedge \ast_M \tilde{\omega}$; 

$$\|\omega\|_{L_p^2(M)}^2 := \int_M |\omega(t, \theta)|_M^2 dV_M,$$

where $dV_M$ is the volume element of $(M, ds^2)$.

It is well-known that, since the Riemannian metric on $M$ is complete, the Laplace-Beltrami operator is essentially selfadjoint on $C_\infty^p(\Lambda^p(M))$, for $p = 0, ..., N$. We will denote by $\Delta_M^p$ also its closure.

Let us end this section with some notations and preliminary facts in spectral analysis. If $H$ is any selfadjoint operator acting in a Hilbert space $\mathcal{H}$, $H : D(H) \subseteq \mathcal{H} \rightarrow \mathcal{H}$, we will denote by $\sigma_{\text{ess}}(H)$ the essential spectrum of $H$, that is, the spectrum of $H$ minus the isolated eigenvalues of finite multiplicity. Following [8], $E_H(\mu)$ (\(\mu \in \mathbb{R}\)) will stand for the spectral family associated to the operator $H$; moreover, $P_H(\mu)$ will denote the projection $E_H(\mu) \ominus E_H(\mu - 0)$ (where $E_H(\mu - 0) = s - \lim_{\epsilon \to 0} E_H(\mu)$), whilst $E_H(S)$ will stand for the spectral measure of any Borel set $S \subseteq \mathbb{R}$. As usual, we will denote by $\mathcal{H}_p(H)$ the closed subset of $\mathcal{H}$ spanned by all the eigenfunctions of $H$, and by $\mathcal{H}_c(H)$ its orthogonal complement in $\mathcal{H}$; correspondingly, we will denote by $\sigma_p(H)$ the set of all the eigenvalues of $H$ and by $\sigma_c(H)$ the spectrum of the restriction of $H$ to $\mathcal{H}_c(H)$. Following [11], we will denote by $\mathcal{H}_{ac}(H)$ the subset of absolute continuity of $H$, defined as the set of all $u \in \mathcal{H}$ such that $\langle E_H(S)u, u \rangle_\mathcal{H} = 0$ for any Borel set $S$ whose Lebesgue measure $|S|$ is equal to zero. $\mathcal{H}_{ac}(H)$ will stand for the set $\mathcal{H}_c(H) \ominus \mathcal{H}_{ac}(H)$. Accordingly, we will denote by $\sigma_{ac}(H)$ (resp. $\sigma_{sc}(H)$) the absolutely (resp. singularly) continuous spectrum of $H$, defined as the spectrum of the restriction of $H$ to the subspace $\mathcal{H}_{ac}(H)$ (resp. $\mathcal{H}_{sc}(H)$).

Finally, let us recall the following basic facts, whose proof is elementary and is therefore omitted:

**Lemma 2.1.** Let $H$ be a selfadjoint operator acting on a Hilbert space $\mathcal{H}$, $H : D(H) \subseteq \mathcal{H} \rightarrow \mathcal{H}$. Then
(1) if \( \mu \in \mathbb{R} \) is an isolated eigenvalue of \( H \), then \( \mu \notin \sigma_{ac}(H) \) (resp. \( \mu \notin \sigma_{sc}(H) \)); as a consequence, \( \sigma_{ac}(H) \subseteq \sigma_{ess}(H) \) (resp. \( \sigma_{sc}(H) \subseteq \sigma_{ess}(H) \));

(2) if \( \mathcal{H} = \bigoplus_{k \in \mathbb{N}} \mathcal{H}_k \), where \( \mathcal{H}_k \), for every \( k \in \mathbb{N} \), is a closed subspace of \( \mathcal{H} \) (possibly empty), and if \( H \) splits accordingly as \( H = \bigoplus_{k \in \mathbb{N}} H_k \), where for every \( k \in \mathbb{N} \) \( H_k = \mathcal{H}_k \), then \( \sigma_{ac}(H) = \bigcup_{k \in \mathbb{N}} \sigma_{ac}(H_k) \) and \( \sigma_{sc}(H) = \bigcup_{k \in \mathbb{N}} \sigma_{sc}(H_k) \);

(3) for any constant \( K \in \mathbb{R} \), \( \mathcal{H}_{ac}(H + K) = \mathcal{H}_{ac}(H) \) (resp. \( \mathcal{H}_{sc}(H + K) = \mathcal{H}_{sc}(H) \)); as a consequence, \( \sigma_{ac}(H + K) = \sigma_{ac}(H) + K \) (resp. \( \sigma_{sc}(H + K) = \sigma_{sc}(H) + K \)).

3. Hodge decomposition

In the present section let us suppose that the Riemannian metric \( ds^2 \) in \( (0, +\infty) \times S^{N-1} \) takes the form

\[
(3.1) \quad ds^2 = f(t) \, dt^2 + g(t) \, d\theta_{S^{N-1}}^2,
\]

where \( f(t) > 0 \) and \( g(t) > 0 \) for any \( t \in (0, +\infty) \).

Given \( \omega \in C^\infty(\Lambda^p(M)) \), let us write, for \( (t, \theta) \in (0, +\infty) \times S^{N-1} \)

\[
(3.2) \quad \omega(t, \theta) = \omega_1(\theta) + \omega_2(\theta) \wedge dt,
\]

where \( \omega_1 \) and \( \omega_2 \) are respectively a \( p \)-form and a \((p-1)\)-form on \( S^{N-1} \) depending on \( t \). An easy computation shows that \( *_M \omega \) can be expressed in terms of decomposition \( (3.2) \) as

\[
(3.3) \quad *_M \omega = (-1)^{N-p} g^{\frac{N-2p+1}{2}}(t) f^{-\frac{1}{2}}(t) *_S^{N-1} \omega_2
\]

\[
+ g^{\frac{N-2p-1}{2}}(t) f^{\frac{1}{2}}(t) {*}_S^{N-1} \omega_1 \wedge dt,
\]

where \( *_S^{N-1} \) denotes the Hodge \( * \) operator on \( S^{N-1} \). Moreover, \( d^p_M \) and \( \delta^p_M \) split respectively as

\[
(3.4) \quad d^p_M \omega = d^p_{S^{N-1}} \omega_1 + \left\{ (-1)^p \frac{\partial \omega_1}{\partial t} + d^p_{S^{N-1}} \omega_2 \right\} \wedge dt,
\]

\[
(3.5) \quad \delta^p_M \omega = g^{-1}(t) \delta^p_{S^{N-1}} \omega_1 + (-1)^p f^{-\frac{1}{2}} g^{\frac{N-1+2p}{2}} \frac{\partial}{\partial t} \left( f^{-\frac{1}{2}} g^{\frac{N+1-2p}{2}} \omega_2 \right)
\]

\[
+ g^{-1} \delta^p_{S^{N-1}} \omega_2 \wedge dt,
\]

where \( p \) is the degree of \( \omega \), \( d^p_{S^{N-1}} \) is the differential on \( S^{N-1} \) and \( \delta^p_{S^{N-1}} \) is the codifferential on \( S^{N-1} \).
Moreover, the $L^2$-norm of $\omega \in C^\infty(\Lambda^p(M)) \cap L^2_p(M)$ can be written as

\[
\|\omega\|^2_{L^2_p(M)} = \int_0^{+\infty} g^{N-2p-1} \frac{1}{2} f^{\frac{1}{2}}(s) \|\omega_1(s)\|^2_{L^2_p(S^{N-1})} ds
\]

\[
+ \int_0^{+\infty} g^{N+1-2p} \frac{1}{2} f^{-\frac{1}{2}}(s) \|\omega_2(s)\|^2_{L^2_p(S^{N-1})} ds,
\]

where $\|\cdot\|_{L^2_p(S^{N-1})}$ is the $L^2$-norm for $p$-forms on $S^{N-1}$.

From (3.4) and (3.5), a lengthy but straightforward computation gives

\[
\Delta^p_M \omega = (\Delta^p_M \omega)_1 + (\Delta^p_M \omega)_2 \wedge dt,
\]

where

\[
(\Delta^p_M \omega)_1 = g^{-1}(t) \Delta^p_{S^{N-1}} \omega_1 + (-1)^p f^{-1}(t) g^{-1}(t) \frac{\partial g}{\partial t} \omega_1 \wedge \omega_2
\]

\[
- f^{-\frac{1}{2}}(t) g^{-N+1-2p} \frac{1}{2} (t) \frac{\partial}{\partial t} \left( f^{-\frac{1}{2}}(t) g^{-N-1-2p} \frac{1}{2} (t) \frac{\partial \omega_1}{\partial t} \right)
\]

and

\[
(\Delta^p_M \omega)_2 = g^{-1}(t) \Delta^p_{S^{N-1}} \omega_2 + (-1)^p g^{-2}(t) \frac{\partial g}{\partial t} \omega_1
\]

\[
- \frac{\partial}{\partial t} \left\{ f^{-\frac{1}{2}}(t) g^{-N+1-2p} \frac{1}{2} (t) \frac{\partial}{\partial t} \left( f^{-\frac{1}{2}}(t) g^{-N-1-2p} \frac{1}{2} (t) \omega_2 \right) \right\}.
\]

Here we denote by $\Delta^p_{S^{N-1}}$ the Laplace-Beltrami operator on $S^{N-1}$.

Since for every smooth $\omega \in L^2_p(M)$ we have that $\omega_1 \in L^2_p(M)$, $\omega_2 \wedge dt \in L^2_p(M)$ and

\[
\langle \omega_1, \omega_2 \wedge dt \rangle_{L^2_p(M)} = 0,
\]

(3.2) gives rise to an orthogonal decomposition of $L^2_p(M)$ into two closed subspaces. However, (3.7) and (3.8) show that $\Delta^p_M$ is not invariant under this decomposition, and further decompositions are required.

It is well-known that, for $0 \leq p \leq N - 1$,

\[
C^\infty(\Lambda^p(S^{N-1})) = dC^\infty(\Lambda^{p-1}(S^{N-1})) \oplus \delta C^\infty(\Lambda^{p+1}(S^{N-1})) \oplus H^p(S^{N-1}),
\]

where $H^p(S^{N-1})$ is the space of harmonic $p$-forms on $S^{N-1}$, and the decomposition is orthogonal in $L^2_p(S^{N-1})$. Hence, for $1 \leq p \leq N - 1$, every $\omega \in L^2_p(M) \cap C^\infty(\Lambda^p(M))$ can be written as

\[
\omega = \omega_{1\delta} \oplus \omega_{2d} \wedge dt \oplus (\omega_{1d} \oplus \omega_{2s} \wedge dt),
\]

where $\omega_{1\delta}$ (resp. $\omega_{1d}$) is a coclosed (resp. closed) $p$-form on $S^{N-1}$ parametrized by $t$, and $\omega_{2s}$ (resp. $\omega_{2d}$) is a coclosed (resp. closed) $p-$
1)-form on $\mathbb{S}^{N-1}$ parametrized by $t$. By closure, we get the orthogonal decomposition

$$L^2_p(M) = \mathcal{L}_{p,1}(M) \oplus \mathcal{L}_{p,2}(M) \oplus \mathcal{L}_{p,3}(M),$$

where, for every $\omega \in L^2_p(M) \cap C^\infty(\Lambda^p(M))$,

$$\omega_{1\delta} \in \mathcal{L}_{p,1}(M),$$

$$\omega_{2d} \wedge dt \in \mathcal{L}_{p,2}(M)$$

and

$$(\omega_{1d} \oplus \omega_{2\delta} \wedge dt) \in \mathcal{L}_{p,3}(M).$$

Since

$$d_{S^{N-1}}^p \Delta_{S^{N-1}}^p = \Delta_{S^{N-1}}^{p+1} d_{S^{N-1}}^p, \quad \delta_{S^{N-1}}^p \Delta_{S^{N-1}}^p = \Delta_{S^{N-1}}^{p-1} \delta_{S^{N-1}}^p,$$

$$\frac{\partial}{\partial t} d_{S^{N-1}}^p = d_{S^{N-1}}^p \frac{\partial}{\partial t}, \quad \frac{\partial}{\partial t} \delta_{S^{N-1}}^p = \delta_{S^{N-1}}^p \frac{\partial}{\partial t},$$

the Laplace-Beltrami operator is invariant under this decomposition, and

$$\Delta^p_M = \Delta^p_{M_1} \oplus \Delta^p_{M_2} \oplus \Delta^p_{M_3},$$

where, for $i = 1, 2, 3$, $\Delta^p_{M_i}$ is the restriction of $\Delta^p_M$ to $\mathcal{L}_{p,i}(M)$. We remark that, for $i = 1, 2, 3$, $\Delta^p_{M_i}$ is essentially selfadjoint on $C^\infty_c(\Lambda^p(M)) \cap \mathcal{L}_{p,i}(M)$. In view of Lemma 2.1 for $1 \leq p \leq N - 1$,

$$\sigma_{ac}(\Delta^p_M) = \bigcup_{i=1}^3 \sigma_{ac}(\Delta^p_{M_i}),$$

$$\sigma_{sc}(\Delta^p_M) = \bigcup_{i=1}^3 \sigma_{sc}(\Delta^p_{M_i}).$$

For $p = 0$ (resp. $p = N$), any $\omega \in L^2_p(M)$ can be written as $\omega = \omega_{1\delta}$ (resp. $\omega = \omega_{2d} \wedge dt$), where $\omega_{1\delta}$ (resp. $\omega_{2d}$) is a coclosed (resp. closed) 0-form (resp. $(N - 1)$-form) parametrized by $t$ on $\mathbb{S}^{N-1}$. Hence $L^2_0(M) = \mathcal{L}_{0,1}(M)$ (resp. $L^2_N(M) = \mathcal{L}_{N,2}(M)$), and $\Delta^0_M = \Delta^0_{M_1}$ (resp. $\Delta^N_M = \Delta^N_{M_2}$). Again, $\Delta^0_{M_1}$ (resp. $\Delta^N_{M_2}$) is essentially selfadjoint on $C^\infty_c(\Lambda^0(M)) \cap \mathcal{L}_{0,1}(M)$ (resp. on $C^\infty_c(\Lambda^N(M)) \cap \mathcal{L}_{N,2}(M)$).

Hence, for every $p \in [0, N]$, in order to determine the spectral properties of $\Delta^p_M$ it suffices to study the corresponding properties of $\Delta^p_{M_i}$, $i = 1, 2, 3$.

To this purpose, let us introduce further decompositions. First of all, for any $\omega \in L^2_p(M) \cap C^\infty(\Lambda^p(M))$ we decompose $\omega_{1\delta}$ according to an orthonormal basis $\{\tau_{1k}\}_{k \in \mathbb{N}}$ of coclosed eigenforms of $\Delta^p_{S^{N-1}}$; this yields

$$(3.9) \quad \omega_{1\delta} = \oplus_k h_k(t) \tau_{1k},$$
where \( h_k(t) \tau_{1k} \in L^2_p(M) \) for every \( k \in \mathbb{N} \), and the sum is orthogonal in \( L^2_p(M) \), thanks to (2.1). By closure, we get the decomposition
\[
L_{p,1}(M) = \oplus_{k \in \mathbb{N}} L_{p,1,k}(M),
\]
where for \( \omega \in L^2_p(M) \cap C^\infty(\Lambda^p(M)) \)
\[
h_k(t) \tau_{1k} \in L_{p,1,k}(M)
\]
for every \( k \in \mathbb{N} \). We will call \( p \)-form of type I any \( p \)-form \( \omega \in L^2_p(M) \) such that \( \omega \in L_{p,1,k}(M) \) for some \( k \in \mathbb{N} \).

For every \( k \in \mathbb{N} \), let us denote by \( \lambda^p_k \) the eigenvalue associated to \( \tau_{1k} \).

Since for every \( k \in \mathbb{N} \)
\[
(3.10) \quad \Delta^p_{M1}(h(t) \tau_{1k}) = \left\{ \frac{\lambda^p_k}{g(t)} f(t)^{-\frac{1}{2}} g(t)^{-\frac{N+1+2p}{2}} \frac{\partial}{\partial t} \left( f(t)^{-\frac{1}{2}} g(t)^{\frac{N-1-2p}{2}} \frac{\partial h}{\partial t} \right) \right\} \tau_{1k},
\]
\( \Delta^p_{M1} \) is invariant under decomposition (3.9). Moreover, if \( \omega = h(t) \tau_{1k} \)
\[
\|\omega\|_{L^2_p(M)}^2 = \int_0^{+\infty} g(s)^{\frac{N-2p-1}{2}} f(s)^{\frac{1}{2}} h(s)^2 ds.
\]
Thus, \( \Delta^p_{M1} \) is unitarily equivalent to the direct sum over \( k \in \mathbb{N} \) of certain selfadjoint operators \( \Delta^p_{1\lambda^p_k} \) in \( L^2((0, +\infty), g^{-\frac{N-2p-1}{2}} f^{\frac{1}{2}}) \) such that
\[
C^\infty_c(0, +\infty) \subseteq \mathcal{D}(\Delta^p_{1\lambda^p_k})
\]
and for every \( h \in C^\infty_c(0, +\infty) \)
\[
(3.11) \quad \Delta^p_{1\lambda^p_k} h = \frac{\lambda^p_k}{g(t)} h(t) - f(t)^{-\frac{1}{2}} g(t)^{-\frac{N+1+2p}{2}} \frac{\partial}{\partial t} \left( f(t)^{-\frac{1}{2}} g(t)^{\frac{N-1-2p}{2}} \frac{\partial h}{\partial t} \right).
\]
If we set
\[
(3.12) \quad w(t) = h(t)f(t)^{\frac{1}{2}} g(t)^{\frac{N-2p-1}{4}},
\]
a direct (but lengthy) computation shows that \( \Delta^p_{M1} \) is unitarily equivalent to the direct sum, over \( k \in \mathbb{N} \), of some selfadjoint operators \( D^p_{1\lambda^p_k} \) in \( L^2(0, +\infty) \) such that
\[
C^\infty_c(0, +\infty) \subseteq \mathcal{D}(D^p_{1\lambda^p_k})
\]
and for every \( w \in C_c^\infty(0, +\infty) \)

\[
D_{1\lambda_k} w = -\frac{\partial}{\partial t} \left( \frac{1}{f} \frac{\partial w}{\partial t} \right) + \left\{ -7 \frac{1}{16} \frac{1}{f^3} \left( \frac{\partial f}{\partial t} \right)^2 + \frac{11}{4} \frac{\partial^2 f}{\partial t^2} - \frac{11}{2} \frac{\partial f}{\partial t} \frac{(N-1-2p)}{4} \frac{1}{g \partial t} + \frac{1}{f} \frac{(N-2p-1)}{4} \frac{(N-2p-5)}{4} \frac{1}{g^2} \left( \frac{\partial g}{\partial t} \right)^2 \\
+ \frac{1}{f} \frac{(N-2p-1)}{4} \frac{1}{g \partial t^2} + \frac{\lambda_k^p}{g} \right\} w.
\]

Analogously, for every \( \omega \in L_p^2(M) \cap C^\infty(\Lambda^p(M)) \) we decompose \( \omega_{2d} \)

\[
\omega_{2d} \wedge dt = \oplus_k h_k(t) \tau_{2k} \wedge dt.
\]

Correspondingly, by closure we get the orthogonal decomposition

\[
\mathcal{L}_{p,2}(M) = \oplus_{k \in \mathbb{N}} \mathcal{L}_{p,2,k}(M);
\]

we will call \( p \)-form of type II a \( p \)-form \( \omega \in L_p^2(M) \) such that \( \omega \in L_{p,2,k}(M) \) for some \( k \in \mathbb{N} \).

For every \( k \in \mathbb{N} \)

\[
\Delta_{M^2}^p(h(t)\tau_{2k} \wedge dt) = (\Delta_{2\lambda_k^{-1}}^p h)\tau_{2k} \wedge dt,
\]

where

\[
\Delta_{2\lambda_k^{-1}}^p h = \frac{\lambda_k^{p-1}}{g(t)} h(t)
\]

\[
- \frac{\partial}{\partial t} \left\{ f(t)^{-\frac{1}{2}} g(t)^{-\frac{N-1-2p}{2}} \frac{\partial}{\partial t} \left( f(t)^{-\frac{1}{2}} g(t)^{\frac{N+1-2p}{2}} h(t) \right) \right\}.
\]

Here, again, for every \( k \in \mathbb{N} \) we denote by \( \lambda_k^{p-1} \) the eigenvalue of \( \Delta_{\Lambda^{N-1}}^p \)

corresponding to the eigenform \( \tau_{2k} \).

If \( \omega = h(t)\tau_{2k} \wedge dt \), then

\[
\|\omega\|^2_{\mathcal{L}_{p}^2(M)} = \int_0^{+\infty} g(s)^{-\frac{N-2p+1}{2}} f(s)^{-\frac{1}{2}} h(s)^2 ds.
\]

Thus, if we set

\[
w(t) = h(t) f(t)^{-\frac{1}{2}} g(t)^{\frac{N+1-2p}{4}},
\]

we find that \( \Delta_{M^2}^p \) is unitarily equivalent to the direct sum, over \( k \in \mathbb{N} \),

of certain selfadjoint operators \( D_{1\lambda_k^{-1}} \) in \( L^2(0, +\infty) \) such that

\[
C_c^\infty(0, +\infty) \subseteq \mathcal{D}(D_{1\lambda_k^{-1}})
\]
and for every $w \in C_c^\infty(0, +\infty)$

\[
(3.17) \quad D_{2\lambda_k^{-1}} w = -\frac{\partial}{\partial t} \left( \frac{1}{f} \frac{\partial w}{\partial t} \right) + \left\{ -\frac{7}{16} f \frac{\partial f}{\partial t} + \frac{1}{4} \frac{\partial^2 f}{\partial t^2} - \frac{1}{2} \frac{\partial f}{\partial t} \left( N - 1 + 2p \right) \right. \\
\left. \quad + \frac{1}{4} \frac{\partial g}{\partial t} + \frac{(N - 2p + 1)(N - 2p + 5)}{4} f^4 \left( \frac{\partial g}{\partial t} \right)^2 \right. \\
\left. \quad + \frac{1}{4} \frac{(-N + 2p - 1)}{g \partial t^2} + \frac{\lambda_k^{-1}}{g} \right\} w.
\]

Finally, for every $\omega \in L^2_p(M) \cap C^\infty(A^p(M))$ we decompose $\omega_{2\delta}$ with respect to an orthonormal basis of coclosed eigenforms $\{\tau_{3k}\}_{k \in \mathbb{N}}$ of $\Delta_{\mathbb{S}^{N-1}}^{p-1}$. For every $k \in \mathbb{N}$ we denote by $\lambda_k^{-1}$ the eigenvalue corresponding to the eigenform $\tau_{3k}$; then $\left\{ \frac{1}{\lambda_k^{-1}} \tau_{3k} \right\}$ is an orthonormal basis of closed eigenforms of $\Delta_{\mathbb{S}^{N-1}}^p$. Hence, we get the following decomposition for any $\omega_{1d} \oplus \omega_{2\delta} \wedge dt$

\[
(3.18) \quad \omega_{1d} \oplus \omega_{2\delta} \wedge dt = \bigoplus_k \left( \frac{1}{\sqrt{\lambda_k^{-1}}} h_{1k} d_{\mathbb{S}^{N-1}}^{p-1} \tau_{3k} \oplus (-1)^p h_{2k} \tau_{3k} \wedge dt \right),
\]

whence, by closure

$$
\mathcal{L}_{p,3}(M) = \bigoplus_{k \in \mathbb{N}} \mathcal{L}_{p,3,k}(M).
$$

We call $p$-form of type III any $p$-form $\omega \in L^2_p(M)$ such that $\omega \in \mathcal{L}_{p,3,k}(M)$ for some $k \in \mathbb{N}$.

A direct computation shows that, for every $k \in \mathbb{N}$,

\[
(3.19) \quad \Delta_{M^3} \left( \frac{1}{\sqrt{\lambda_k^{-1}}} h_{1k}(t) d_{\mathbb{S}^{N-1}}^{p-1} \tau_{3k} \oplus_M (-1)^p h_{2k}(t) \tau_{3k} \wedge dt \right)
\]

$$
= \left( \Delta_1 \lambda_k^{-1} h_{1k} + \frac{1}{f(t)} \frac{1}{g(t)} \frac{\partial g}{\partial t} \sqrt{\lambda_k^{-1}} h_{2k} \right) \left( \frac{1}{\sqrt{\lambda_k^{-1}}} d_{\mathbb{S}^{N-1}}^{p-1} \tau_{3k} \right)
\]

$$
\oplus \left( \Delta_2 \lambda_k^{-1} h_{2k} + \frac{1}{g^2(t)} \frac{\partial g}{\partial t} \sqrt{\lambda_k^{-1}} h_{1k} \right) ((-1)^p \tau_{3k} \wedge dt);
$$

moreover, if $\omega = \frac{1}{\sqrt{\lambda}} h_1(t) d_{\mathbb{S}^{N-1}}^{p-1} \tau_3 \oplus_M (-1)^p h_2(t) \tau_3 \wedge dt$, then

$$
\|\omega\|_{L^2_p(M)}^2 = \int_c^{+\infty} g(s) \frac{N-2p-1}{2} f(s) \frac{1}{2} h_1(s)^2 ds
$$
\[ + \int_c^{+\infty} g(s) \frac{N+1-2p}{2} f(s)^{-\frac{1}{2}} h_2(s)^2 \, ds. \]

Hence, if we set
\[
(3.20) \quad w_1(t) = g^{\frac{N-2p-1}{4}(t)} f^{\frac{1}{4}}(t) h_1(t), \\
      w_2(t) = g^{\frac{N-2p-1}{4}(t)} f^{-\frac{1}{4}}(t) h_2(t),
\]
we find that $\Delta_{M3}^p$ is unitarily equivalent to the direct sum, over $k \in \mathbb{N}$, of certain selfadjoint operators $D_{3\lambda_k^{p-1}}$ in $L^2(0, +\infty) \oplus L^2(0, +\infty)$ such that
\[ C_c^\infty(0, +\infty) \oplus C_c^\infty(0, +\infty) \subseteq D(D_{3\lambda_k^{p-1}}) \]
and for every $w_1 \oplus w_2 \in C_c^\infty(0, +\infty) \oplus C_c^\infty(0, +\infty)$
\[
(3.21) \quad D_{3\lambda_k^{p-1}}(w_1 \oplus w_2) = \left( D_{1\lambda_k^{p-1}} w_1 + g(t)^{-\frac{1}{2}} f(t)^{-\frac{1}{2}} \frac{\partial g}{\partial t} \sqrt{\lambda_k^{p-1}} w_2 \right) \oplus \left( D_{2\lambda_k^{p-1}} w_2 + g(t)^{-\frac{1}{2}} f(t)^{-\frac{1}{2}} \frac{\partial g}{\partial t} \sqrt{\lambda_k^{p-1}} w_1 \right).
\]

For $i = 1, 2, 3$, for any $k \in \mathbb{N}$, we will denote by $T_{p,i,k}(M)$ the unitary equivalence between $L_{p,i,k}(M)$ and $L^2(0, +\infty)$ (resp. $L^2(0, +\infty) \oplus L^2(0, +\infty)$ if $i = 3$) given by $(3.12)$ (resp. $(3.16)$, $(3.20)$).

We remark that even if the orthogonal decompositions depend on the Riemannian metric (since we have to take closures in the $L^2$-norm), the eigenvalues $\lambda_k^p$ and the eigenforms $\tau_{1k}$ depend only on $\mathbb{S}^{N-1}$ and hence are the same for any choice of the functions $f(t)$ and $g(t)$.

As a consequence, we have:

**Lemma 3.1.** Let $M$ be the $N$-dimensional unitary ball $B(0, 1)$ endowed with any complete Riemannian metric of type
\[
(3.22) \quad ds^2 = f(t) \, dt^2 + g(t) \, d\theta_{\mathbb{S}^{N-1}}^2,
\]
where $t = \text{sn}(\|x\|)$, $d\theta_{\mathbb{S}^{N-1}}^2$ is the standard Riemannian metric on $\mathbb{S}^{N-1}$, $f(t) > 0$ and $g(t) > 0$ for any $t \in (0, +\infty)$. Moreover, let us suppose that $f$, $g$ fulfill condition $(2.3)$. Then for every $p \in [0, N]$, for any $i = 1, 2, 3$, for every $k \in \mathbb{N}$, the set
\[
(3.23) \quad X_{p,i,k} := (T_{p,i,k}(M))(C_c^\infty(\mathcal{N}^p(M)) \cap L_{p,i,k}(M))
\]
does not depend on the choice of the functions $f$, $g$, provided they fulfill condition $(2.3)$.

**Proof.** Let $(f_1, g_1)$, $(f_2, g_2)$ be two couples of smooth positive functions on $(0, +\infty)$ such that
(1) the corresponding Riemannian metrics
\[ f_j(t)dt^2 + g_j(t)d\theta_{S^{N-1}}, \quad j = 1, 2 \]
are complete on \( B(0, 1) \);

(2) for \( j = 1, 2 \), \( f_j(t) = 1 \) and \( g_j(t) = t^2 \) for \( t \in (0, \epsilon) \).

For sake of simplicity, let us consider the case \( i = 1 \) (the proofs of the other cases are analogous). Let \( \omega = h(t)\tau_{1k} \in C_c^\infty(\Lambda^p(M)) \cap L_{p,1,k}(M, g_1) \); then if we consider the differential form \( \tilde{\omega} \) on \( M \) defined as
\[ \tilde{\omega}(t, \theta) = f_2(t)^{-\frac{1}{4}}g_2(t)^{-\frac{N-2p-1}{4}}f_1(t)^{\frac{1}{4}}g_1(t)^{\frac{N-2p-1}{4}}\omega(t, \theta), \]
then \( \tilde{\omega} \in C_c^\infty(\Lambda^p(M)) \). Moreover, it is immediate to see that
\[ \mathcal{T}_{p,i,k}(M, g_1)\omega = \mathcal{T}_{p,i,k}(M, g_2)\tilde{\omega}. \]

The set \( X_{p,i,k} \) defined above is a natural core for the operator \( D_{i\lambda_k} \). Namely, we have the following characterization of \( \mathcal{D}(D_{i\lambda_k}) \) for any \( i = 1, 2, 3 \) and for every \( k \in \mathbb{N} \):

**Lemma 3.2.** Let \( M \) be as in Lemma 3.1. Then, for every \( p \in [0, N] \), for every \( i = 1, 2, 3 \), for every \( k \in \mathbb{N} \), the operator \( D_{i\lambda_k} \) is essentially selfadjoint on the set \( X_{p,i,k} \) defined by (3.23).

**Proof.** Since \( \Delta_p^M \) is essentially selfadjoint on \( C_c^\infty(\Lambda^p(M)) \), then, for \( i = 1, 2, 3 \), \( \Delta_{Mi}^p \) is essentially selfadjoint on \( C_c^\infty(\Lambda^p(M)) \cap L_{p,i}(M) \). Analogously, for any \( i = 1, 2, 3 \) and for any \( k \in \mathbb{N} \) the restriction of \( \Delta_{Mi}^p \) to the subspace \( L_{p,i,k}(M) \) is essentially selfadjoint on \( C_c^\infty(\Lambda^p(M)) \cap L_{p,i,k}(M) \). Hence, for every \( k \in \mathbb{N} \) the operator \( D_{i\lambda_k} \) is essentially selfadjoint on the set \( X_{p,i,k} \). \( \square \)

Applying the decomposition techniques described above to the Friedrichs extension \( (\Delta^p_M)^F \) of the restriction of \( \Delta^p_M \) to \( C_c^\infty(\Lambda^p(M) \setminus B(0, c)) \) for some arbitrarily chosen \( c > 0 \), in \( \mathbb{I} \) we computed explicitly the essential spectrum of \( \Delta^p_M \) (it was shown by Eichhorn (\( \mathbb{H} \)) that \( \sigma_{ess}(\Delta^p_M) = \sigma_{ess}((\Delta^p_M)^F) \)). Namely, we obtained the following result:

**Theorem 3.3.** Let \( M \) be the unitary ball \( B(0, 1) \) in \( \mathbb{R}^N \) endowed with a Riemannian metric \( ds^2 \) which, in a tubular neighbourhood of the boundary \( S^{N-1} \), is given by
\[ ds^2 = e^{-2(a+1)t} dt^2 + e^{-2mt} d\theta_{S^{N-1}}^2, \]
where \( a \leq -1 \), \( t = \text{sech} (||x||) \) and \( d\theta_{S^{N-1}}^2 \) is the standard Riemannian metric on \( S^{N-1} \). Then
(1) if \( a = -1 \) and \( b < 0 \), if \( p \neq N/2 \)

\[
\sigma_{\text{ess}}(\Delta_M^p) = \min \left\{ \left( \frac{N - 2p - 1}{2} \right)^2 b^2, \left( \frac{N - 2p + 1}{2} \right)^2 b^2 \right\}, +\infty
\]

whilst if \( p = N/2 \)

\[
\sigma_{\text{ess}}(\Delta_M^p) = \{0\} \cup \left[ \frac{b^2}{4}, +\infty \right); \]

(2) if \( a = -1 \) and \( b = 0 \), for every \( p \in [0, N] \)

\[
\sigma_{\text{ess}}(\Delta_M^p) = [\lambda_p, +\infty),
\]

where \( \lambda_p \) is the minimum between the smallest eigenvalue of \( \Delta_{SN-1}^p \) and the smallest eigenvalue of \( \Delta_{SN-1}^{p-1} \);

(3) if \( a = -1 \) and \( b > 0 \), if \( 1 < p < N - 1 \)

\[
\sigma_{\text{ess}}(\Delta_M^p) = \emptyset,
\]

whilst if \( p \in \{0, 1, N - 1, N\} \)

\[
\sigma_{\text{ess}}(\Delta_M^p) = \left[ \left( \frac{N - 1}{2} \right)^2 b^2, +\infty \right);
\]

(4) if \( a < -1 \) and \( b < 0 \), for every \( p \in [0, N] \)

\[
\sigma_{\text{ess}}(\Delta_M^p) = [0, +\infty);
\]

(5) if \( a < -1 \) and \( b = 0 \), for every \( p \in [0, N] \)

\[
\sigma_{\text{ess}}(\Delta_M^p) = [\lambda_p, +\infty),
\]

where \( \lambda \) is the minimum between the smallest eigenvalue of \( \Delta_{SN-1}^p \) and the smallest eigenvalue of \( \Delta_{SN-1}^{p-1} \);

(6) if \( a < -1 \) and \( b > 0 \), if \( 1 < p < N - 1 \)

\[
\sigma_{\text{ess}}(\Delta_M^p) = \emptyset,
\]

whilst if \( p \in \{0, 1, N - 1, N\} \)

\[
\sigma_{\text{ess}}(\Delta_M^p) = [0, +\infty).
\]

As for the absolutely continuous spectrum and the singularly continuous spectrum, in view of Lemma 2.1 for \( i = 1, 2, 3 \) we have that

\[
\sigma_{\text{sc}}(\Delta_{Mi}^p) = \bigcup_{k \in \mathbb{N}} \sigma_{\text{sc}}(D_{i\lambda_k})
\]

and

\[
\sigma_{\text{sc}}(\Delta_{Mi}^p) = \bigcup_{k \in \mathbb{N}} \sigma_{\text{sc}}(D_{i\lambda_k});
\]
thus, we can reduce ourselves to the analysis of the absolutely continuous and of the singularly continuous spectra of the selfadjoint operators
\[ D_1 \lambda_k^p, \ D_2 \lambda_k^{p-1} \text{ and } D_3 \lambda_k^{p-1}. \]
Since the Hodge \( * \) operator maps \( p \)-forms of type I isometrically onto \((N-p)\)-forms of type II, it suffices to consider the cases \( i = 1 \) and \( i = 3 \).

Finally, let us observe that the decomposition techniques described above work also in the case of the Euclidean space (this will be essential in the construction of the unperturbed operators employed in the computation of the absolutely continuous spectrum).

Namely, let us consider the Euclidean space \((\mathbb{R}^N, e)\), that is, \(\mathbb{R}^N\) endowed with the Euclidean metric. From now on, we will denote by \(\Delta^p_e\) the Laplace-Beltrami operator acting on \(p\)-forms on \((\mathbb{R}^N, e)\). In polar coordinates the Euclidean metric has the expression
\[
ds^2 = dr^2 + r^2 d\theta_{S^{N-1}}^2,
\]
where \(d\theta_{S^{N-1}}^2\) is the standard Riemannian metric on \(S^{N-1}\). Then it is possible to introduce the decompositions
\[
L^2_p(\mathbb{R}^N, e) = L_{p,1}(\mathbb{R}^N, e) \oplus L_{p,2}(\mathbb{R}^N, e) \oplus L_{p,3}(\mathbb{R}^N, e)
\]
and, for \(i = 1, 2, 3\),
\[
L_{p,i}(\mathbb{R}^N, e) = \oplus_{k \in \mathbb{N}} L_{p,i,k}(\mathbb{R}^N, e).
\]
For any \(k \in \mathbb{N}\), we will denote by \(T_{p,1,k}(\mathbb{R}^N, e)\) the unitary equivalence between \(L_{p,1,k}(\mathbb{R}^N, e)\) and \(L^2(0, +\infty)\).

4. The Case \(a = -1\)

Let us introduce the change of coordinates
\[
r : (0, +\infty) \longrightarrow (0, +\infty),
\]
\[
r(t) := \int_0^{+\infty} \sqrt{f(s)} \, ds;
\]
the Riemannian metric in the new coordinate system \((r, \theta)\) on \((0, +\infty) \times S^{N-1}\) is given by
\[(4.1) \quad d\sigma^2 = dr^2 + \tilde{g}(r) d\theta_{S^{N-1}}^2,
\]
where
\[
\tilde{g}(r) = r^2 \quad \text{for} \quad r \in (0, +\epsilon)
\]
and
\[
\tilde{g}(r) = e^{-2b_{\text{r}}} \quad \text{for} \quad r > \overline{c} = K + \epsilon,
\]
where \( K = \int_\epsilon^c \sqrt{f(s)} \, ds \). Applying the orthogonal decomposition of Section 4 in the new coordinate system we find the following expressions for the operators \( D_{1\lambda_k^p} \): for any \( w \in C^\infty(0, +\infty) \cap \mathcal{D}(D_{1\lambda_k^p}) \)

\[
D_{1\lambda_k^p} w = -\frac{\partial^2 w}{\partial r^2} + V_1(r) w,
\]

where

\[
V_1(r) = \begin{cases} 
\frac{(N-2p+1)}{2}\frac{(N-2p+3)}{2} \frac{1}{r^2} + \frac{\lambda_{k}^p}{r^2} & \text{for } r \in (0, \epsilon) \\
\text{a smooth function} & \text{for } r \in [\epsilon, \bar{c}] \\
\frac{(N-2p+1)}{2} b^2 + \lambda_{k}^{p-1} e^{2b r} & \text{for } r > \bar{c}.
\end{cases}
\]

Analogously, for any \( w \in C^\infty(0, +\infty) \cap \mathcal{D}(D_{2\lambda_k^{p-1}}) \)

\[
D_{2\lambda_k^{p-1}} w = -\frac{\partial^2 w}{\partial r^2} + V_2(r) w,
\]

where

\[
V_2(r) = \begin{cases} 
\frac{(N-2p+1)}{2}\frac{(N-2p+3)}{2} \frac{1}{r^2} + \frac{\lambda_{k}^{p-1}}{r^2} & \text{for } r \in (0, \epsilon) \\
\text{a smooth function} & \text{for } r \in [\epsilon, \bar{c}] \\
\frac{(N-2p+1)}{2} b^2 + \lambda_{k}^{p-1} e^{2b r} & \text{for } r > \bar{c}.
\end{cases}
\]

Finally, for every \((w_1 \oplus w_2) \in (C^\infty(0, +\infty) \oplus C^\infty(0, +\infty)) \cap \mathcal{D}(D_{3\lambda_k^{p-1}})\),

\[
D_{3\lambda_k^{p-1}}(w_1 \oplus w_2) = \left( D_{1\lambda_k^{p-1}} w_1 + V_3(r) \sqrt{\lambda_{k}^{p-1}} w_2 \right) \oplus \left( D_{2\lambda_k^{p-1}} w_2 + V_3(r) \sqrt{\lambda_{k}^{p-1}} w_1, \right).
\]

where

\[
V_3(r) = \begin{cases} 
\frac{2}{r^2} & \text{for } r \in (0, \epsilon) \\
\text{a smooth function} & \text{for } r \in [\epsilon, \bar{c}] \\
-2b e^{br} & \text{for } r > \bar{c}.
\end{cases}
\]

The behaviour of the potential at infinity depends strongly on the sign of \( b \in \mathbb{R} \). Hence, we will consider separately the cases \( b < 0 \), \( b = 0 \) and \( b > 0 \).

4.1. The case \( b < 0 \). We begin with the study of the absolutely continuous (and of the singularly continuous) spectrum of the operators \( D_{1\lambda_k^p} \). To this purpose, we need some preliminary Lemmas. The first is a classical statement in functional analysis:
Lemma 4.1. \textbf{([12])} Let $A$, $C$ be symmetric operators. Suppose that $\mathcal{D}$ is a linear subspace satisfying $\mathcal{D} \subseteq \mathcal{D}(A)$, $\mathcal{D} \subseteq \mathcal{D}(C)$, and that
\[\|(A - C)\varphi\| \leq a(\|A\varphi\| + \|C\varphi\|) + b\|\varphi\|\]
for all $\varphi \in \mathcal{D}$, where $0 \leq a < 1$, $b \geq 0$. Then
(1) $A$ is essentially selfadjoint on $\mathcal{D}$ if and only if $C$ is essentially selfadjoint on $\mathcal{D}$;
(2) $\mathcal{D}(A_{|D}) = \mathcal{D}(C_{|D})$.

Proof. For a proof see \textbf{[12]}.

The second Lemma is an easy generalization to the case of differential forms of the Agmon-Kato-Kuroda Theorem (see \textbf{[13]}).

We recall that a potential $V(x)$ on $\mathbb{R}^N$ is called an Agmon potential if for some $\epsilon > 0$ the potential $W(x) := (1 + |x|^2)^{\frac{1}{2} + \epsilon}V(x)$ is a relatively compact perturbation of the scalar Laplacian $-\Delta$. Moreover, it is well-known that if for some $\epsilon > 0$ $(1 + |x|^2)^{\frac{1}{2} + \epsilon}V(x) \in L^\infty(\mathbb{R}^N)$ then $V(x)$ is an Agmon potential (see \textbf{[13]}).

Lemma 4.2. Let $V$ be an Agmon potential on $\mathbb{R}^N$. If $H = \Delta^p + V$, then:
(1) the set $\mathcal{E}_+$ of positive eigenvalues of $H$ is a discrete subset of $(0, +\infty)$, and each eigenvalue has finite multiplicity;
(2) $\sigma_{sc}(H) = \emptyset$;
(3) the wave operators $W^\pm(H, \Delta^p)$ exist and are complete.

Proof. For the scalar case (i.e. $p = 0$) see \textbf{[13]}. For $p > 0$ the conclusion follows applying to each component the result in the scalar case.

We are now in position to prove our first result:

Lemma 4.3. For $a = -1$, $b < 0$, for $0 \leq p \leq N - 1$, for every $k \in \mathbb{N}$
\[\sigma_{ac}(D_{1\lambda_k^p}) = \left(\frac{N - 2p - 1}{2}\right)^2 b^2, +\infty\]
and $\sigma_{sc}(D_{1\lambda_k^p}) = \emptyset$.

Proof. We will compute the absolutely continuous and the singularly continuous spectrum of $D_{1\lambda_k^p}$ through perturbation techniques. Since for $b < 0$ we have that $e^{2br} \to 0$ as $r \to +\infty$, it might seem natural to apply directly the Agmon-Kato-Kuroda Theorem for functions (see \textbf{[13]}) to
the couple of operators \((D_{1\lambda_k}, H)\) on the half-line \((0, +\infty)\), where

\[
H := -\frac{\partial^2}{\partial r^2} + \left(\frac{N - 2p - 1}{2}\right)^2 b^2.
\]

However, on one hand, the Agmon-Kato-Kuroda Theorem holds for operators acting on the whole \(\mathbb{R}^N\), whilst the operators \(H\) and \(D_{1\lambda_k}\) act on the half-line. On the other hand, the potential part of the operator \(D_{1\lambda_k}\) has a singularity at zero. Hence, we developed a different argument. The idea is to “move” the problem to the \(N\)-dimensional Euclidean space \((\mathbb{R}^N, e)\), where the singularity disappears.

Let us consider, on \((\mathbb{R}^N, e)\), the operators

\[
\tilde{H}_0 = \Delta_e^p + \left(\frac{N - 2p - 1}{2}\right)^2 b^2,
\]

\[
\tilde{H}_1 = \tilde{H}_0 + \tilde{V}(|x|),
\]

where

\[
\tilde{V}(|x|) = \begin{cases}
- \left(\frac{N-2p-1}{2}\right)^2 b^2 & \text{for } |x| \in (0, \epsilon) \\
V_1(|x|) - \left(\left(\frac{N-2p-1}{2}\right)^2 \left(\frac{N-2p-3}{2}\right) + \lambda_k^p\right) \frac{1}{|x|^2} & \text{for } |x| \in [\epsilon, \bar{c}] \\
- \left(\frac{N-2p-1}{2}\right)^2 b^2 & \text{for } |x| \in [\epsilon, \bar{c}] \\
\end{cases}
\]

\[
- \left(\frac{N-2p-1}{2}\right)^2 \left(\frac{N-2p-3}{2}\right) + \lambda_k^p \frac{1}{|x|^2} + \lambda_k^p e^{2b|x|} & \text{for } |x| > \bar{c}.
\]

Since \(\Delta_e^p\) is essentially selfadjoint on \(C_c^\infty(\Lambda^p(\mathbb{R}^N, e))\), in view of Lemma 4.1 both \(\tilde{H}_0\) and \(\tilde{H}_1\) are essentially selfadjoint on \(C_c^\infty(\Lambda^p(\mathbb{R}^N, e))\). We denote again by \(\tilde{H}_0\) and \(\tilde{H}_1\) their closures. Since an easy computation shows that for \(0 < \epsilon < \frac{1}{2}\)

\[
(1 + |x|^2)^{\frac{1}{p} + \epsilon} \tilde{V}(|x|) \in L^\infty(\mathbb{R}^N, e),
\]

\(\tilde{V}(|x|)\) is an Agmon potential on \(\mathbb{R}^N\). As a consequence, Lemma 4.2 implies that

1. the set \(\tilde{\mathcal{E}}\) of the eigenvalues of \(\tilde{H}_1\) greater than \((\frac{N-2p-1}{2})^2 b^2\) is a discrete subset of \(\left((\frac{N-2p-1}{2})^2 b^2, +\infty\right)\), and each eigenvalue has finite multiplicity;

2. \(\sigma_{sc}(\tilde{H}_1) = \emptyset\);

3. the wave operators \(W^\pm(\tilde{H}_1, \tilde{H}_0)\) exist and are complete.

Now, let us consider the restrictions \(P_{L_{p,1,k}(\mathbb{R}^N, e)} \tilde{H}_1, P_{L_{p,1,k}(\mathbb{R}^N, e)} \tilde{H}_0\) of \(\tilde{H}_1\) and \(\tilde{H}_0\) to \(L_{p,1,k}(\mathbb{R}^N, e)\), and let us apply the unitary transformation...
$\mathcal{T}_{p,1,k}(\mathbb{R}^N, e)$. In this way we find two operators

\begin{align*}
H_0 & := \mathcal{T}_{p,1,k}(\mathbb{R}^N, e) \circ \tilde{H}_0 \circ (\mathcal{T}_{p,1,k}(\mathbb{R}^N, e))^{-1}, \\
H_1 & := \mathcal{T}_{p,1,k}(\mathbb{R}^N, e) \circ \tilde{H}_1 \circ (\mathcal{T}_{p,1,k}(\mathbb{R}^N, e))^{-1},
\end{align*}

both essentially selfadjoint on the set $X_{p,1,k}$ defined by (3.23).

Since a simple computation shows that for any $w \in X_{p,1,k}$

\begin{equation*}
H_1 w = D_{1} \lambda_k w,
\end{equation*}

in view of Lemma 3.2 we find that $H_1 = D_{1} \lambda_k$.

Recalling Lemma 2.1, we find immediately that $\sigma_{sc}(D_{1} \lambda_k) = \sigma_{sc}(H_1) \subseteq \sigma_{sc}(\tilde{H}_1) = \emptyset$. Moreover, since the projection $P_{\zeta_{p,1,k}(\mathbb{R}^N, e)}$ commutes with both $\tilde{H}_0$ and $\tilde{H}_1$, we find that the existence and completeness of the wave operators $W^\pm(\tilde{H}_1, \tilde{H}_0)$ implies the existence and completeness of the wave operators $W^\pm(D_{1} \lambda_k, H_0)$. As a consequence, we have that

\begin{equation*}
\sigma_{ac}(D_{1} \lambda_k) = \sigma_{ac}(H_0).
\end{equation*}

Since the spectrum of $\Delta^p_M$ is purely absolutely continuous, equal to $[0, +\infty)$ and of constant multiplicity, $\sigma_{ac}(H_0) = \left[\left(\frac{N-2p-1}{2}\right)^2 b^2, +\infty\right)$. This completes the proof. \hfill $\Box$

Hence:

**Proposition 4.4.** For $a = -1$, $b < 0$, for $0 \leq p \leq N - 1$,

\begin{equation*}
\sigma_{ac}(\Delta^p_M) = \left[\left(\frac{N-2p}{2}\right)^2 b^2, +\infty\right), \quad \text{and} \quad \sigma_{ac}(\Delta^p_{M1}) = \emptyset.
\end{equation*}

By duality:

**Proposition 4.5.** For $a = -1$, $b < 0$, for $1 \leq p \leq N$,

\begin{equation*}
\sigma_{ac}(\Delta^p_M) = \left[\left(\frac{N-2p+1}{2}\right)^2 b^2, +\infty\right), \quad \text{and} \quad \sigma_{ac}(\Delta^p_{M2}) = \emptyset.
\end{equation*}

As a consequence, since we already know from Theorem 3.3 that for $a = -1$, $b < 0$, for every $p \in [0, N]$ the essential spectrum of $\Delta^p_M$ is equal to $\left[\min \left\{\left(\frac{N-2p-1}{2}\right)^2 b^2, \left(\frac{N-2p+1}{2}\right)^2 b^2\right\}, +\infty\right)\)$, we can state the following:

**Theorem 4.6.** For $a = -1$, $b < 0$, for $0 \leq p \leq N$,

\begin{equation*}
\sigma_{ac}(\Delta^p_M) = \min \left\{\left(\frac{N-2p-1}{2}\right)^2 b^2, \left(\frac{N-2p+1}{2}\right)^2 b^2\right\}, +\infty\right)\), \quad \sigma_{ac}(\Delta^p_{M2}) = \sigma_{ac}(\Delta^p_{M3}).
\end{equation*}
4.2. The case $b = 0$. As in the previous case, we begin with the study of $D_{1\lambda_k^p}$ for any $k \in \mathbb{N}$. If $b = 0$, the potential $V_1(r)$ in (4.2) is simply given by

$$V_1(r) = \begin{cases} \frac{(N-2p-1)}{2} \frac{(N-2p-3)}{2} \frac{1}{r^2} + \lambda_k^p \frac{1}{r^2} & \text{for } r \in (0, \varepsilon) \\ \text{a smooth function} & \text{for } r \in [\varepsilon, \bar{r}] \\ \lambda_k^p & \text{for } r > \bar{r} \end{cases}$$

Lemma 4.7. For $a = -1$, $b = 0$, for $0 \leq p \leq N - 1$, for every $k \in \mathbb{N}$

$$\sigma_{ac}(D_{1\lambda_k^p}) = [\lambda_k^p, +\infty) \quad \text{and} \quad \sigma_{sc}(D_{1\lambda_k^p}) = \emptyset.$$ 

Proof. Let us consider, on $(\mathbb{R}^N, e)$, the operators

$$\tilde{H}_0 = \Delta_e + \lambda_k^p,$$

$$\tilde{H}_1 = \Delta_e + \lambda_k^p + \tilde{V}(|x|),$$

where

$$\tilde{V}(|x|) = \begin{cases} -\lambda_k^p & \text{for } |x| \in (0, \varepsilon) \\ \text{a smooth function} & \text{for } |x| \in [\varepsilon, \bar{r}] \\ -\left(\frac{N-2p-1}{2} \frac{(N-2p-3)}{2}\right) \frac{1}{|x|^2} - \lambda_k^p \frac{1}{|x|^2} & \text{for } |x| > \bar{r}. \end{cases}$$

Again, in view of Lemma 4.4, both $\tilde{H}_1$ and $\tilde{H}_0$ are essentially selfadjoint on $C_c^\infty(\Lambda^p(\mathbb{R}^N, e))$. Hence, the operators

$$H_0 := T_{p,1,k}(\mathbb{R}^N, e) \circ \tilde{H}_0 \circ (T_{p,1,k}(\mathbb{R}^N, e))^{-1},$$

$$H_1 := T_{p,1,k}(\mathbb{R}^N, e) \circ \tilde{H}_1 \circ (T_{p,1,k}(\mathbb{R}^N, e))^{-1},$$

are both essentially selfadjoint on the set $X_{p,1,k}$. In particular, as in the proof of Lemma 4.3, we have that $H_1 = D_{1\lambda_k^p}$. Since an easy computation shows that $\tilde{V}(|x|)$ is an Agmon potential on $\mathbb{R}^N$ (indeed, for $0 < \varepsilon < \frac{1}{2}$, $(1+|x|^2)^{\frac{4}{2}+\varepsilon} \in L^\infty(\mathbb{R}^N))$, reasoning as in the proof of Lemma 4.3 we find that $\sigma_{sc}(D_{1\lambda_k^p}) = \emptyset$ and $\sigma_{ac}(D_{1\lambda_k^p}) = \sigma_{ac}(\tilde{H}_0) = [\lambda_k^p, +\infty)$. □

As a consequence, by Lemma 2.1 we have:

Proposition 4.8. For $a = -1$, $b = 0$, for $0 \leq p \leq N - 1$, 

$$\sigma_{ac}(\Delta_{M_1}^p) = [\lambda_0^p, +\infty),$$

where $\lambda_0^p$ is the lowest eigenvalue of $\Delta_{S_{N-1}}^p$, and 

$$\sigma_{sc}(\Delta_{M_1}^p) = \emptyset.$$ 

By duality:
Proposition 4.9. For $a = -1$, $b = 0$, for $1 \leq p \leq N$, 
\[ \sigma_{ac}(\Delta_{M2}) = \left[ \lambda_0^{p-1}, +\infty \right), \]
where $\lambda_0^{p-1}$ is the lowest eigenvalue of $\Delta_{S_{N-1}}^{p-1}$, and
\[ \sigma_{sc}(\Delta_{M2}) = \emptyset. \]

Since we already know from Theorem 3.3 that for $a = -1, b = 0$ the essential spectrum of $\Delta_{M}^{p}$ is equal to $[\lambda_p, +\infty)$ for every $p \in [0, N]$, where $\lambda_p = \min \{ \lambda_p^0, \lambda_0^{p-1} \}$, we obtain the following result:

Theorem 4.10. For $a = -1$, $b = 0$, for $0 \leq p \leq N$,
\[ \sigma_{ac}(\Delta_{M}^{p}) = \left[ \lambda_p, +\infty \right), \]
where $\lambda_p = \min \{ \lambda_p^0, \lambda_0^{p-1} \}$, and
\[ \sigma_{sc}(\Delta_{M}^{p}) = \sigma_{sc}(\Delta_{M3}^{p}). \]

4.3. The case $b > 0$. As in the previous cases, in order to compute the absolutely continuous spectrum of $\Delta_{M}^{p}$ it suffices to study the absolutely continuous spectrum of $D_{1}^{a} \lambda_k^p$ for any $k \in \mathbb{N}$:

Lemma 4.11. For $a = -1$, $b > 0$, for every $k \in \mathbb{N}$ if $\lambda_k^p > 0$
\[ \sigma_{ac}(D_{1}\lambda_k^p) = \emptyset \quad \text{and} \quad \sigma_{sc}(D_{1}\lambda_k^p) = \emptyset, \]
whilst if $\lambda_k^p = 0$
\[ \sigma_{ac}(D_{1}\lambda_k^p) = \left[ \left( \frac{N-1}{2} \right)^2 b^2, +\infty \right) \quad \text{and} \quad \sigma_{sc}(D_{1}\lambda_k^p) = \emptyset. \]

Proof. It was proved in [1] that for $a = -1, b > 0$, if $\lambda_k^p > 0$ then $\sigma_{ess}(D_{1}\lambda_k^p) = \emptyset$; as a consequence, in this case $\sigma_{ac}(D_{1}\lambda_k^p) = \sigma_{sc}(D_{1}\lambda_k^p) = \emptyset$.

If, on the contrary, $\lambda_k^p = 0$, we have that $V_1(r)$ is simply
\[ V_1(r) = \begin{cases} 
\left( \frac{N-2p-1}{2} \right) \left( \frac{N-2p-3}{2} \right) \frac{1}{r^p} & \text{for } r \in (0, \epsilon) \\
\text{a smooth function} & \text{for } r \in [\epsilon, \overline{c}] \\
\left( \frac{N-2p-1}{2} \right)^2 b^2 & \text{for } r > \overline{c}.
\end{cases} \]

Let us consider, on $(\mathbb{R}^N, e)$, the operators
\[ \tilde{H}_0 = \Delta_e + \left( \frac{N-2p-1}{2} \right)^2 b^2, \]
\[ \tilde{H}_1 = \Delta_e + \left( \frac{N-2p-1}{2} \right)^2 b^2 + \tilde{V}(|x|), \]
where
\[
\tilde{V}(|x|) = \begin{cases} 
- \left( \frac{N-2p-1}{2} \right)^2 b^2 & \text{for } |x| \in (0, \epsilon) \\
\text{a smooth function} & \text{for } |x| \in [\epsilon, \overline{c}] \\
- \left( \frac{N-2p-1}{2} \right) \left( \frac{N-2p-3}{2} \right) \frac{1}{|x|^2} & \text{for } |x| > \overline{c}.
\end{cases}
\]

Again, in view of Lemma 4.1, both \( \tilde{H}_1 \) and \( \tilde{H}_0 \) are essentially selfadjoint on \( C^\infty_c(\Lambda^p(\mathbb{R}^N, e)) \). Hence, the operators
\[
H_0 := \mathcal{T}_{p,1,k}(\mathbb{R}^N, e) \circ \tilde{H}_0 \circ (\mathcal{T}_{p,1,k}(\mathbb{R}^N, e))^{-1},
\]
\[
H_1 := \mathcal{T}_{p,1,k}(\mathbb{R}^N, e) \circ \tilde{H}_1 \circ (\mathcal{T}_{p,1,k}(\mathbb{R}^N, e))^{-1},
\]
are both essentially selfadjoint on the set \( X_{p,1,k} \). This fact, jointly with a simple computation, shows that \( H_1 = D_{1\lambda^p_k} \).

Since \( \tilde{V}(|x|) \) is an Agmon potential on \( \mathbb{R}^N \), reasoning as in the proof of Lemma 4.3 we find that \( \sigma_{sc}(D_{1\lambda^p_k}) = \emptyset \) and \( \sigma_{ac}(D_{1\lambda^p_k}) = \sigma_{ac}(\tilde{H}_0) = \left( \left( \frac{N-2p-1}{2} \right)^2 b^2, +\infty \right) \). \( \square \)

Now, it is well-known that on \( S^{N-1} \) we can have \( \lambda^p_k = 0 \) (that is, there exist harmonic \( p \)-forms) only for \( p = 0 \) or for \( p = N - 1 \).

Hence:

**Proposition 4.12.** For \( a = -1, b > 0 \), if \( 0 < p < N - 1 \)
\[
\sigma_{ac}(\Delta^p_{M_1}) = \emptyset \quad \text{and} \quad \sigma_{sc}(\Delta^p_{M_1}) = \emptyset,
\]
whilst if \( p \in \{0, N - 1\} \)
\[
\sigma_{ac}(\Delta^p_{M_1}) = \left[ \left( \frac{N-1}{2} \right)^2 b^2, +\infty \right] \quad \text{and} \quad \sigma_{sc}(\Delta^p_{M_1}) = \emptyset.
\]

By duality:

**Proposition 4.13.** For \( a = -1, b > 0 \), if \( 1 < p < N \)
\[
\sigma_{ac}(\Delta^p_{M_2}) = \emptyset \quad \text{and} \quad \sigma_{sc}(\Delta^p_{M_2}) = \emptyset,
\]
whilst if \( p \in \{1, N\} \)
\[
\sigma_{ac}(\Delta^p_{M_1}) = \left[ \left( \frac{N-1}{2} \right)^2 b^2, +\infty \right] \quad \text{and} \quad \sigma_{sc}(\Delta^p_{M_1}) = \emptyset.
\]

As a consequence, in view of Theorem 3.3 we have the following result:
**Theorem 4.14.** For \( a = -1, \) \( b > 0, \) if \( 1 < p < N - 1 \)
\[
\sigma_{ac}(\Delta_M^p) = \emptyset \quad \text{and} \quad \sigma_{sc}(\Delta_M^p) = \sigma_{sc}(\Delta_{M^3}^p),
\]
whilst if \( p \in \{0, 1, N - 1, N\} \)
\[
\sigma_{ac}(\Delta_M^p) = \left[ \left( \frac{N - 1}{2} \right)^2 b^2, +\infty \right) \quad \text{and} \quad \sigma_{sc}(\Delta_M^p) = \sigma_{sc}(\Delta_{M^3}^p).
\]

5. **The case** \( a < -1 \)

As in the previous section, we introduce the change of coordinates
\[
r : (0, +\infty) \rightarrow (0, +\infty),
\]
\[
r(t) := \int_0^t \sqrt{f(s)} \, ds;
\]
the Riemannian metric in the new coordinate system \((r, \theta)\) on \((0, +\infty) \times S^{N-1}\) is given by
\[
d\sigma^2 = dr^2 + \tilde{g}(r) \, d\theta_{S^{N-1}}^2,
\]
where
\[
\tilde{g}(r) = r^2 \quad \text{for} \quad r \in (0, +\epsilon)
\]
and
\[
\tilde{g}(r) = |a + 1|^{-\frac{2b}{|a+1|}} (r - c_1)^{-\frac{2b}{|a+1|}} \quad \text{for} \quad r > \bar{c} = K + \epsilon,
\]
where \( K = \int_\epsilon^c \sqrt{f(s)} \, ds \) and \( c_1 = K + \epsilon - \frac{e|a+1|}{|a+1|} > 0. \) Applying the orthogonal decomposition of Section 4 in the new coordinate system we find the following expression for the operators \( D_{1\lambda_k} : \) for any \( w \in C^\infty(0, +\infty) \cap D(D_{1\lambda_k}^p) \)
\[
D_{1\lambda_k} w = -\frac{\partial^2 w}{\partial r^2} + V_1(r) w,
\]
where
\[
V_1(r) = \begin{cases} \left( \frac{N-2p-1}{2} \right) \left( \frac{N-2p-3}{2} \right) \frac{1}{r^2} + \frac{\lambda_k^p}{r^2} & \text{for} \; r \in (0, \epsilon) \\ \text{a smooth function} & \text{for} \; r \in [\epsilon, \bar{c}] \\ \tilde{K}_1 (r - c_1)^{-2} + \lambda_k^p |a + 1|^{-\frac{2b}{|a+1|}} (r - c_1)^{-\frac{2b}{|a+1|}} & \text{for} \; r > \bar{c}, \end{cases}
\]
where
\[
\tilde{K}_1 = \left( \frac{N - 2p - 1}{2} \right)^2 \frac{b^2}{|a+1|^2} + \frac{N - 2p - 1}{2} \frac{b}{|a+1|}.
\]
Analogously, for any \( w \in C^\infty(0, +\infty) \cap D(D_{2\lambda_k}^{p-1}) \)
\[
D_{2\lambda_k}^{p-1} w = -\frac{\partial^2 w}{\partial r^2} + V_2(r) w,
\]
where

\[ V_2(r) = \begin{cases} \left( \frac{N-2p+1}{2} \right)^2 \left( \frac{N-2p+3}{2} \right) \frac{b^2}{r^2} + \frac{\lambda_{k-1}^{p-1}}{r^2} & \text{for } r \in (0, \epsilon) \\
\text{a smooth function} & \text{for } r \in [\epsilon, \bar{r}] \\
\bar{K}_2(r-c_1)^{-2} + \lambda_{k}^{p-1} |a+1|^{\frac{2p}{m+1}} (r-c_1)^{\frac{2p}{m+1}} & \text{for } r > \bar{r}, \end{cases} \]

where

\[ \bar{K}_2 = \left( \frac{N-2p+1}{2} \right)^2 \frac{b^2}{|a+1|^2} + \frac{N-2p+1}{2} \frac{b}{|a+1|}. \]

Finally, for every \((w_1 \oplus w_2) \in (C^\infty(0, +\infty) \oplus C^\infty(0, +\infty)) \cap \mathcal{D}(D_{3\lambda_k^{p-1}})\),

\[ (5.2) \quad D_{3\lambda_k^{p-1}}(w_1 \oplus w_2) = \left( D_{1\lambda_k^{p-1}}w_1 + V_3(r) \sqrt{\lambda_k^{p-1}w_2} \right) \]
\[ \oplus \left( D_{2\lambda_k^{p-1}}w_2 + V_3(r) \sqrt{\lambda_k^{p-1}w_1} \right), \]

where

\[ V_3(r) = \begin{cases} \frac{2}{r^2} & \text{for } r \in (0, \epsilon) \\
\text{a smooth function} & \text{for } r \in [\epsilon, \bar{r}] \\
|a+1|^{\frac{b}{m+1}} (r-c_1)^{\frac{b}{m+1}-1} & \text{for } r > \bar{r}. \end{cases} \]

As in the previous section, the behaviour of the potential at \(+\infty\) depends strongly on the sign of \(b \in \mathbb{R}\), thus we will consider separately the cases \(b < 0\), \(b = 0\) and \(b > 0\).

5.1. The case \(b < 0\). Let us begin with the study of the spectrum of \(D_{1\lambda_k^{p}}\) for any \(k \in \mathbb{N}\). To this purpose, let us introduce the following Theorem, which is an easy generalization to the case of \(p\)-forms of a result due to Lavine (see [9]):

**Theorem 5.1.** Let \(\bar{V}\) be a multiplication operator acting on \(L_p^2(\mathbb{R}^N, e)\), where

\[ \bar{V}(x) = V_\alpha(x) + V_\beta(x), \]

with

1. \(V_\alpha \in C^1(\mathbb{R}^N)\),
2. \(\lim_{|x| \to +\infty} V_\alpha(x) = 0\)
3. \(\frac{|\partial V_\alpha|}{|e|} \leq c(1 + r)^{-\gamma} \text{ for some } \gamma > 1 \text{ (here } r = |x|)\),
4. \(V_\beta(x) = (1 + |x|)^{-\gamma} (f_p + f_\infty) \text{ for some } \gamma > 1, f_\infty \in L^\infty(\mathbb{R}^N), f_p \in L^p(\mathbb{R}^N) \text{ for } p > \max(\frac{N}{2}, 1). \)
Then there exists a unique selfadjoint operator $H$ with $D(H) \subseteq \mathcal{D}(\Delta^p_e)^{1/2}$ such that for every $\omega \in D(H)$

$$ (H\omega, \omega)_{L^2_p(\mathbb{R}^N, e)} = \sum_{i,j=1}^N \int_{\mathbb{R}^N} \left( \frac{\partial \omega_i}{\partial x_j} \right)^2 \, dx + \int_{\mathbb{R}^N} \tilde{V}(x)|\omega(x)|^2 \, dx. $$

The positive eigenvalues of $H$ have finite multiplicity and can accumulate only at 0. Moreover,

$$ \mathcal{H}_{ac}(H) = (\mathcal{H}_p(H))^\perp. $$

**Proof.** For the scalar case (i.e. $p = 0$) see [9]. For $p > 0$ the assert follows applying to each component the result in the scalar case. \qed

**Remark 5.2.** Under the assumptions of Theorem 5.1 we do not get the existence and completeness of the wave operators $W^\pm(H, \Delta^p_e)$ (indeed, for certain potentials they might not exist, as shown in [4]).

We are now in position to prove the following

**Lemma 5.3.** For $a = -1$, $b < 0$, for $0 \leq p \leq N - 1$, for every $k \in \mathbb{N}$

$$ \sigma_{ac}(D_1\lambda^p_k) = [0, +\infty) \quad \text{and} \quad \sigma_{sc}(D_1\lambda^p_k) = \emptyset. $$

**Proof.** Let us consider, on the Euclidean space $(\mathbb{R}^N, e)$, the operators

$$ \tilde{H}_0 := \Delta^p_e, \quad \tilde{H}_1 := \Delta^p_e + \tilde{V}(|x|), $$

where

$$ \tilde{V}(|x|) = \begin{cases} 0 & \text{for } |x| \in (0, \epsilon) \\ \text{a smooth function} & \text{for } |x| \in [\epsilon, \bar{c}] \\ \tilde{K}_1(|x| - c_1)^{-2} + \lambda^p_k|a + 1|^{-\frac{2|b|}{|a|+1}}(r - c_1)^{-\frac{2|b|}{|a|+1}} \\ - \left( \frac{N-2p-1}{2} \right) \left( \frac{N-2p-3}{2} \right) \frac{1}{|x|^2} - \lambda^p_k \frac{1}{|x|^2} & \text{for } |x| > \bar{c}. \end{cases} $$

Since $\tilde{V}(|x|)$ is bounded, we have that $\tilde{H}_1$ is essentially selfadjoint on $C_c^\infty(\mathbb{R}^N, e)$. Hence, the operators

$$ H_0 := \mathcal{T}_{p,1,k}(\mathbb{R}^N, e) \circ \tilde{H}_0 \circ (\mathcal{T}_{p,1,k}(\mathbb{R}^N, e))^{-1}, \quad H_1 := \mathcal{T}_{p,1,k}(\mathbb{R}^N, e) \circ \tilde{H}_1 \circ (\mathcal{T}_{p,1,k}(\mathbb{R}^N, e))^{-1}, $$

are both essentially selfadjoint on the set $X_{p,1,k}$. Since $D_1\lambda^p_k$ is essentially selfadjoint on $X_{p,1,k}$ and $D_1\lambda^p_k w = H_1 w$ for every $w \in X_{p,1,k}$, we have that $H_1 = D_1\lambda^p_k$.

Now, $\tilde{V}(|x|)$ is not an Agmon potential for any possible value of $a < -1$, $b < 0$. 


If $|b| > \frac{|a+1|}{2}$, then for $0 < \varepsilon < \min \left\{ \frac{1}{2}, \frac{1}{2} \left( \frac{2|b|}{|a+1|} - 1 \right) \right\}$ we have that

$$(1 + |x|^2)^{\frac{1}{2} + \varepsilon} \tilde{V}(|x|) \in L^\infty(\mathbb{R}^N),$$

hence $\tilde{V}(|x|)$ is an Agmon potential on $\mathbb{R}^N$. As a consequence, following the argument of Lemma 4.3 we find that for $|b| > \frac{|a+1|}{2}$, $\sigma_{ac}(D_{1\lambda^p_k}) = [0, +\infty)$ and $\sigma_{sc}(D_{1\lambda^p_k}) = \emptyset$.

If, on the contrary, $|b| \leq \frac{|a+1|}{2}$, $\tilde{V}(|x|)$ is no more an Agmon potential; however, $\tilde{V}(|x|)$ fulfills the assumptions of Theorem 5.1. Indeed, $\tilde{V}(|x|)$ can be written as

$$\tilde{V}(|x|) = V_\alpha(|x|) + V_\beta(|x|),$$

where

$$V_\alpha(|x|) = V_\beta(|x|) = 0,$$

for $|x| \in (0, \epsilon)$, whilst for $|x| > \tau$

$$V_\alpha(|x|) = \lambda_p^k |a + 1|^{-\frac{2|b|}{|a+1|}}(|x| - c_1)^{-\frac{2|b|}{|a+1|}},$$

and

$$V_\beta(|x|) = K_1(|x| - c_1)^{-2} - \left( \frac{N - 2p - 1}{2} \frac{N - 2p - 3}{2} + \lambda_p^k \right) \frac{1}{|x|^2}.$$ 

It is immediate to see that $V_\alpha \in C^1(\mathbb{R}^N)$, $V_\alpha(|x|) \to 0$ as $|x| \to +\infty$ and

$$|\partial V_\alpha/\partial r| \leq C(1 + r)^{-\left(\frac{2|b|}{|a+1|} + 1\right)}$$

for some positive constant $C$.

Moreover, for $\varepsilon < 1$

$$(1 + |x|)^{1+\varepsilon} V_\alpha(|x|) \in L^\infty(\mathbb{R}^N).$$

As a consequence, by Theorem 5.1

$$\mathcal{H}_{ac}(\tilde{H}_1) = \left( \mathcal{H}_p(\tilde{H}_1) \right)^\perp;$$

moreover, the positive eigenvalues of $\tilde{H}_1$ have finite multiplicity and can accumulate only at 0. These facts hold also for the restriction of $\tilde{H}_1$ to the subspace $L_{p, 1, k}(\mathbb{R}^N, e)$, Hence, we find that, for every $k \in \mathbb{N},$

$$\mathcal{H}_{ac}(D_{1\lambda^p_k}) = \left( \mathcal{H}_p(D_{1\lambda^p_k}) \right)^\perp;$$

moreover, for every $k \in \mathbb{N}$ the positive eigenvalues of $D_{1\lambda^p_k}$ have finite multiplicity and can accumulate only at 0.
From (5.3) we immediately get $H_{sc}(D_{1\lambda^p_k}) = \emptyset$, whence $\sigma_{sc}(D_{1\lambda^p_k}) = \emptyset$.

As for the absolutely continuous spectrum, since $\sigma(D_{1\lambda^p_k}) = \sigma_{ac}(D_{1\lambda^p_k}) \cup \sigma_p(D_{1\lambda^p_k})$ and, in view of Theorem 5.1 in [1], $\sigma(D_{1\lambda^p_k}) = [0, +\infty)$, we find $[0, +\infty) \setminus \sigma_p(D_{1\lambda^p_k}) \subseteq \sigma_{ac}(D_{1\lambda^p_k})$, whence $\sigma_{ac}(D_{1\lambda^p_k}) = [0, +\infty)$.

Hence, by Lemma 2.1,

**Proposition 5.4.** For $a < -1$, $b < 0$, for $0 \leq p \leq N - 1$, $\sigma_{ac}(\Delta^p_{M1}) = [0, +\infty)$ and $\sigma_{sc}(\Delta^p_{M1}) = \emptyset$.

By duality:

**Proposition 5.5.** For $a < -1$, $b < 0$, for $1 \leq p \leq N$, $\sigma_{ac}(\Delta^p_{M2}) = [0, +\infty)$ and $\sigma_{sc}(\Delta^p_{M2}) = \emptyset$.

As a consequence, since from Theorem 3.3 we already know that for $a < -1$, $b < 0$ the essential spectrum of $\Delta^p_M$ is equal to $[0, +\infty)$ for every $p \in [0, N]$, we can state the following

**Theorem 5.6.** For $a < -1$, $b < 0$, for $0 \leq p \leq N$,

$\sigma_{ac}(\Delta^p_M) = [0, +\infty)$ and $\sigma_{ac}(\Delta^p_{M}) = \sigma_{sc}(\Delta^p_{M3})$.

5.2. **The case $b = 0$.** First of all, we study the spectral properties of $D_{1\lambda^p_k}$ for every $k \in \mathbb{N}$.

**Lemma 5.7.** For $a < -1$, $b = 0$, for $0 \leq p \leq N - 1$, for every $k \in \mathbb{N}$ $\sigma_{ac}(D_{1\lambda^p_k}) = [\lambda^p_k, +\infty)$ and $\sigma_{sc}(D_{1\lambda^p_k}) = \emptyset$.

**Proof.** For $b = 0$, the potential $V_1(r)$ is simply given by

$$V_1(r) = \begin{cases} 
\left(\frac{N-2p-1}{2}\right) \left(\frac{N-2p-3}{2}\right) - \frac{1}{r^2} + \lambda^p_k \frac{1}{r^2} & \text{for } r \in (0, \varepsilon) \\
\text{a smooth function} & \text{for } r \in [\varepsilon, \overline{c}] \\
\overline{K}_1(r-c_1)^{-2} + \lambda^p_k & \text{for } r > \overline{c}.
\end{cases}$$

Let us consider, on the Euclidean space $(\mathbb{R}^N, e)$, the operators $\tilde{H}_0 := \Delta^p_e + \lambda^p_k$. 
\[ \tilde{H}_1 := \Delta^p_e + \lambda_k^p + \tilde{V}(|x|), \]

where

\[ \tilde{V}(|x|) = \begin{cases} 
-\lambda_k^p & \text{for } |x| \in (0, \epsilon) \\
\text{a smooth function} & \text{for } |x| \in [\epsilon, \bar{c}] \\
\tilde{K}_1(|x| - c_1)^{-2} - \left( \frac{N-2p-3}{2} \frac{N-2p-1}{2} + \lambda_k^p \right) \frac{1}{|x|^2} & \text{for } |x| > \bar{c}.
\end{cases} \]

Since \( \tilde{V}(|x|) \) is an Agmon potential on \( \mathbb{R}^N \), following the argument of Lemma 4.3 we find that \( \sigma_{sc}(D_{1\lambda_k^p}) = \emptyset \) and \( \sigma_{ac}(D_{1\lambda_k^p}) = [\lambda_k^p, +\infty) \) for every \( k \in \mathbb{N} \).

Hence, by Lemma 2.1:

**Proposition 5.8.** For \( a < -1 \), \( b = 0 \), for \( 0 \leq p \leq N - 1 \),

\[ \sigma_{ac}(\Delta^p_{M1}) = [\lambda_0^p, +\infty) , \]

where \( \lambda_0^p \) is the lowest eigenvalue of \( \Delta^p_{S_{N-1}} \) on \( p \)-forms, and

\[ \sigma_{sc}(\Delta^p_{M1}) = \emptyset. \]

By duality:

**Proposition 5.9.** For \( a < -1 \), \( b = 0 \), for \( 1 \leq p \leq N \),

\[ \sigma_{ac}(\Delta^p_{M2}) = [\lambda_0^{p-1}, +\infty) , \]

where \( \lambda_0^{p-1} \) is the lowest eigenvalue of \( \Delta^p_{S_{N-1}} \) on \( (p-1) \)-forms, and

\[ \sigma_{sc}(\Delta^p_{M2}) = \emptyset. \]

As a consequence, since we know from Theorem 3.3 that for \( a < -1 \), \( b = 0 \), for every \( p \in [0, N] \) the essential spectrum of \( \Delta^p_{M} \) is equal to \( [\bar{\lambda}_p, +\infty) \), where \( \bar{\lambda}_p = \min \{ \lambda_0^p, \lambda_0^{p-1} \} \), we find the following result:

**Theorem 5.10.** For \( a < -1 \), \( b = 0 \), for \( 0 \leq p \leq N \),

\[ \sigma_{ac}(\Delta^p_{M}) = [\bar{\lambda}, +\infty) , \]

where \( \bar{\lambda} = \min \{ \lambda_0^p, \lambda_0^{p-1} \} \), and

\[ \sigma_{sc}(\Delta^p_{M}) = \sigma_{sc}(\Delta^p_{M2}) . \]

5.3. **The case** \( b > 0 \). As in the previous cases, in order to compute the absolutely continuous spectrum of \( \Delta^p_{M} \) it suffices to study the absolutely continuous spectrum of \( D_{1\lambda_k^p} \) for every \( k \in \mathbb{N} \):

**Lemma 5.11.** For \( a < -1 \), \( b > 0 \), for every \( k \in \mathbb{N} \) if \( \lambda_k^p > 0 \)

\[ \sigma_{ac}(D_{1\lambda_k^p}) = \emptyset \quad \text{and} \quad \sigma_{sc}(D_{1\lambda_k^p}) = \emptyset , \]

whilst if \( \lambda_k^p = 0 \)

\[ \sigma_{ac}(D_{1\lambda_k^p}) = [0, +\infty) \quad \text{and} \quad \sigma_{sc}(D_{1\lambda_k^p}) = \emptyset. \]
Proof. It was proved in [1] that for \( a < -1, \ b > 0, \) if \( \lambda_k^p > 0 \) then \( \sigma_{\text{ess}}(D_{1\lambda_k^p}) = \emptyset; \) thus, if \( \lambda_k^p > 0, \) then \( \sigma_{\text{ac}}(D_{1\lambda_k^p}) = \sigma_{\text{sc}}(D_{1\lambda_k^p}) = \emptyset. \)

If, on the contrary, \( \lambda_k^p = 0, \) then \( V_1(r) \) is simply given by

\[
V_1(r) = \begin{cases} 
\left( \frac{N-2p-1}{2} \right) \left( \frac{N-2p-3}{2} \right) \frac{1}{r^s} & \text{for } r \in (0, \epsilon) \\
\text{a smooth function} & \text{for } r \in [\epsilon, \bar{c}] \\
\tilde{K}_1(r-c_1)^{-2} & \text{for } r > \bar{c}.
\end{cases}
\]

Let us consider, on \((\mathbb{R}^N, e)\), the operators

\[
\tilde{H}_0 := \Delta^p_e, \\
\tilde{H}_1 := \Delta^p_e + \tilde{V}(|x|),
\]

where

\[
\tilde{V}(|x|) = \begin{cases} 
0 & \text{for } |x| \in (0, \epsilon) \\
\text{a smooth function} & \text{for } |x| \in [\epsilon, \bar{c}] \\
- \left( \frac{N-2p-1}{2} \right) \left( \frac{N-2p-3}{2} \right) \frac{1}{|x|^s} + \tilde{K}_1(|x| - c_1)^{-2} & \text{for } |x| > \bar{c}.
\end{cases}
\]

Since \( \tilde{V}(|x|) \) is an Agmon potential on \( \mathbb{R}^N, \) following the argument of Lemma 4.3 we find that if \( \lambda_k = 0 \) then \( \sigma_{\text{sc}}(\Delta_{1\lambda_k^p}) = \emptyset \) and \( \sigma_{\text{ac}}(\Delta_{1\lambda_k^p}) = [0, +\infty). \)

This completes the proof. \( \square \)

Hence, by Lemma 2.1:

**Proposition 5.12.** For \( a < -1, \ b > 0, \) if \( 0 < p < N-1 \)

\[
\sigma_{\text{ac}}(\Delta^p_{M_1}) = \emptyset \quad \text{and} \quad \sigma_{\text{sc}}(\Delta^p_{M_1}) = \emptyset,
\]

whilst if \( p \in \{0, N-1\} \)

\[
\sigma_{\text{ac}}(\Delta^p_{M_1}) = [0, +\infty) \quad \text{and} \quad \sigma_{\text{sc}}(\Delta^p_{M_1}) = \emptyset.
\]

By duality:

**Proposition 5.13.** For \( a < -1, \ b > 0, \) if \( 1 < p < N \)

\[
\sigma_{\text{ac}}(\Delta^p_{M_2}) = \emptyset \quad \text{and} \quad \sigma_{\text{sc}}(\Delta^p_{M_2}) = \emptyset,
\]

whilst if \( p \in \{1, N\} \)

\[
\sigma_{\text{ac}}(\Delta^p_{M_2}) = [0, +\infty) \quad \text{and} \quad \sigma_{\text{sc}}(\Delta^p_{M_2}) = \emptyset.
\]

As a consequence, in view of Theorem 3.3 we get the following result:

**Theorem 5.14.** For \( a < -1, \ b > 0, \) if \( 1 < p < N-1 \)

\[
\sigma_{\text{ac}}(\Delta^p_M) = \emptyset \quad \text{and} \quad \sigma_{\text{sc}}(\Delta^p_M) = \sigma_{\text{sc}}(\Delta^p_{M_3}),
\]

whilst if \( p \in \{0, 1, N-1, N\} \)

\[
\sigma_{\text{ac}}(\Delta^p_M) = [0, +\infty) \quad \text{and} \quad \sigma_{\text{sc}}(\Delta^p_M) = \sigma_{\text{sc}}(\Delta^p_{M_3}).
\]


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