TRANSVERSE ELECTRIC CONDUCTIVITY OF QUANTUM COLLISIONAL MAXWELLIAN PLASMA

A. V. Latyshev[1], A. A. Yushkanov[2]

Faculty of Physics and Mathematics, Moscow State Regional University, 105005, Moscow, Radio str., 10–A

Formulas for calculation of transverse dielectric function and transverse electric conductivity in quantum collisional Maxwellian plasma are obtained. The Wigner – Vlasov – Boltzmann kinetic equation with collision integral in BGK (Bhatnagar, Gross and Krook) form in coordinate space is used. Various special cases are investigated. Comparison with Lindhard’s formula has been carried out.

Key words: collisional plasma, electric conductivity, dielectric permeability, Lindhard formula.

PACS numbers: 52.25.Dg Plasma kinetic equations, 52.25.-b Plasma properties

1. Introduction

In the present work formulas for calculation of electric conductivity and dielectric permeability in quantum collisional Maxwellian plasma are deduced.

During the derivation of the kinetic equation we generalize the approach, developed by Klimontovich and Silin [1].

Dielectric permeability in the collisionless quantum gaseous plasma was studied by many authors (see, for example, [1] – [10]).

In the work [6], where the one-dimensional case of the quantum plasma is investigated, the importance of derivation of dielectric permeability with
use of the quantum kinetic equation with collision integral in the form of
BGK – model (Bhatnagar, Gross, Krook) \[11\], \[12\] was noted.

The present work is devoted to the performance of this task.

A dielectric permeability is one of the most significant characteristics
of a plasma. This quantity is necessary for description of the skin effect
\[13\], for analysis of surface plasmons \[14\], for description of the process
of propagation and damping of the transverse plasma oscillations \[10\], the
mechanism of electromagnetic waves penetration in plasma \[9\], and for
analysis of other problems of plasma physics \[15\], \[16\], \[17\], \[19\] and \[20\].

In the present work the Wigner — Vlasov — Boltzmann kinetic equation
with collision integral in the form of relaxation $\tau$ – model in coordinate
space is used.

This equation is considered about the Wigner function, which is quantum
analog of classical distribution function. The Wigner function has been entered in work \[21\] and then it was investigated in works \[22\] – \[25\].

Kliwer and Fuchs were the first who have noticed \[4\], that the dielectric
function for quantum plasma deduced by Lindhard in collisional case does
not pass into dielectric function for classical plasma in the limit when
Planck’s constant $\hbar$ tends to zero. This means, that dielectric Lindhard’s
function does not take into account electron collisions correctly. Kliwer
and Fuchs have corrected Lindhard’s dielectric function ”by hands” so
that it passed into classical one under condition $\hbar \to 0$.

In the works \[14\], \[15\] the dielectric function received by them (Kliwer
and Fuchs) was applied to consideration of various questions of metal
optics.

In the work \[5\] the correct account of collisions in framework of the
relaxation model in electron momentum space for the case of longitudinal
dielectric function has been carried out. At the same time the correct
account of influence of collisions for transverse dielectric function has not
been implemented till now.
The aim of the present work is the elimination of this lacuna.

2. Transverse conductivity in maxwellian plasma

In a considered case of plasma locally equilibrium maxwellian distribution function looks like (see [26]):

\[ f_{eq} \equiv f^{(0)} = \frac{N_0}{v_T \pi^{3/2}} \exp \left\{ -\left( \frac{P - e}{c p_T} A(\mathbf{r}, t) \right)^2 - \frac{eU}{k_B T} \right\}. \quad (1.1) \]

Here \( v_T = \sqrt{\frac{2k_BT}{m}} \) is the thermal electron velocity, \( m \) is the electron mass, \( e \) is the electron charge, \( p_T = mv_T \) is the thermal electron momentum, \( P \) is the dimensionless electron momentum, \( N_0 \) is the electron number density in equilibrium state, \( A(\mathbf{r}, t) \) is the vector potential, \( U = U(\mathbf{r}, t) \) is the scalar potential (which is considered further equal to zero), \( c \) is the light velocity, \( k_B \) is the Boltzmann constant, \( T \) is the plasma temperature, \( f_0(P) \) is the absolute Maxwell distribution,

\[ f_0(P) = \frac{N_0}{v_T^3 \pi^{3/2}} e^{-p^2}. \]

The Wigner — Vlasov functional in this case equals to [26]:

\[ W[f] = (PA) \frac{i e v_T}{\hbar c} (f_0^+(P) - f_0^-(P)), \quad (1.2) \]

where

\[ f_0^\pm \equiv f_0^\pm(P) = \frac{N_0}{v_T^3 \pi^{3/2}} \exp \left\{ -\left( P \mp \frac{q}{2} \right)^2 \right\}, \]

\( q \) is the dimensionless wave vector, \( k_T \) is the wave number corresponding to thermal movement of molecules,

\[ q = \frac{k}{k_T}, \quad k_T = \frac{p_T}{\hbar} = \frac{mv_T}{\hbar}. \]

We consider the kinetic equation with the Wigner — Vlasov functional (1.2)

\[ \frac{\partial f}{\partial t} + \mathbf{v} \frac{\partial f}{\partial \mathbf{r}} = \nu (f_{eq} - f) + (PA) \frac{i e v_T}{\hbar c} (f_0^+ - f_0^-). \quad (1.3) \]
Here $\nu$ is the effective electron collision frequency. Linearization of equilibrium distribution function (1.1) gives:

$$f_{eq} = f_0(P) + f_0(P) \frac{2e}{c_p T}(PA).$$

Therefore we seek the solution of equation (1.3) in the form

$$f = f_0(P) + f_0(P) \frac{2e}{c_p T}(PA) + f_0(P) \frac{2e}{c_p T}(PA) h(P). \quad (1.4)$$

Substituting (1.4) in (1.3) we receive

$$(PA)f_0(P) \frac{2e}{c_p T}(\nu - i \omega + ikv) = \frac{2ie}{c_p T}(PA)(\omega - kv)f_0(P)$$

$$+(PA)\frac{i\nu v_T}{c h}(f_0^+ - f_0^-). \quad (1.5)$$

Let’s replace in the equation (1.5) $v$ on

$$v = \frac{p}{m} - \frac{eA}{mc} = v_T P - \frac{eA}{mc}$$

and in the received equation we will produce linearization on the vector field $A$. We receive the equation from which it is found

$$(PA)h(P)f_0(P) \frac{2ie}{c_p T}(\omega - v_T kP) + \frac{i\nu v_T}{c h}(f_0^+ - f_0^-)$$

Substituting (1.6) in (1.4) we receive

$$f = f_{eq} + \frac{2ie}{c_p T}(PA) \frac{(\omega - v_T kP)f_0(P) + \frac{\mathcal{E}_T}{\hbar}(f_0^+ - f_0^-)}{\nu - i \omega + iv_T kP}, \quad (1.7)$$

where $\mathcal{E}_T$ is the thermal electron energy,

$$\mathcal{E}_T = \frac{mv_T^2}{2}.$$

Electric current in an equilibrium condition is equal to zero $[26]$. Then we receive using (1.7)

$$j(r, t) = e \int v f(r, v, t) d^3 v$$
\[ \frac{2ie^2}{c \rho T} \int \mathbf{v}(\mathbf{P} \mathbf{A}) \left( \omega - v_T k \mathbf{P} \right) f_0(P) \frac{(\mathcal{E}_T / \hbar)(f_0^+ - f_0^-)}{\nu - i\omega + iv_T k \mathbf{P}} d^3 v. \]  \quad (1.8)

After linearization on the vector field we obtain
\[ j = \frac{2ie^2 v_T^4}{c \rho T} \int \mathbf{P}(\mathbf{P} \mathbf{A}) \left( \omega - v_T k \mathbf{P} \right) f_0(P) \frac{(\mathcal{E}_T / \hbar)(f_0^+ - f_0^-)}{\nu - i\omega + iv_T k \mathbf{P}} d^3 P. \]  \quad (1.9)

We take unit vector \( \mathbf{e}_1 = \frac{\mathbf{A}}{\mathbf{A}} \), directed alongside to the vector \( \mathbf{A} \). Then equality (1.9) may be rewritten in the form
\[ j(r, t) = \frac{2ie^2 v_T^3 A(r, t)}{cm} \int \mathbf{P}(\mathbf{P} \mathbf{e}_1) S(P^2, k_1 \mathbf{P}) d^3 P, \]  \quad (1.9’)

where \( k_1 = v_T \pi k = \|\mathbf{k}\) is the dimensionless wave vector,
\[ S(P^2, k_1 \mathbf{P}) = \frac{(\omega - k_1 \mathbf{P}) f_0(P) + (\mathcal{E}_T / \hbar)(f_0^+ - f_0^-)}{1 - i\omega \pi + ik_1 \mathbf{P}}. \]

We take other unit vector \( \mathbf{e}_2 \), which is perpendicular to the vector \( \mathbf{k}_1 \), i.e.
\[ \mathbf{e}_2 = \frac{\mathbf{A} \times \mathbf{k}_1}{|\mathbf{A} \times \mathbf{k}_1|} = \frac{\mathbf{A} \times \mathbf{k}_1}{Ak_1}, \]
where \( \mathbf{A} \times \mathbf{k}_1 \) is the vector product.

We expand the vector \( \mathbf{P} \) on three orthogonal directions \( \mathbf{e}_1, \mathbf{e}_2 \) and \( \mathbf{n} = \frac{\mathbf{k}_1}{k_1} = \frac{\mathbf{k}}{k} \):
\[ \mathbf{P} = (\mathbf{P} \mathbf{n}) \mathbf{n} + (\mathbf{P} \mathbf{e}_1) \mathbf{e}_1 + (\mathbf{P} \mathbf{e}_2) \mathbf{e}_2. \]

We receive with the help of this expansion
\[ (\mathbf{P} \mathbf{A}) \mathbf{P} = A(\mathbf{P} \mathbf{e}_1) \mathbf{P} = \]
\[ = A(\mathbf{P} \mathbf{e}_1)(\mathbf{P} \mathbf{n}) \mathbf{n} + A(\mathbf{P} \mathbf{e}_1)^2 \mathbf{e}_1 + A(\mathbf{P} \mathbf{e}_1)(\mathbf{P} \mathbf{e}_2) \mathbf{e}_2. \]

Substituting this decomposition in (1.9’), and, considering, that integrals from odd functions on a symmetric interval are equal to zero, we receive:
\[ j(r, t) = \frac{2ie^2 v_T^3 A(r, t)}{cm} \int (\mathbf{P} \mathbf{e}_1)^2 S(P^2, k_1 \mathbf{P}) d^3 P. \]  \quad (1.10)
In view of symmetry value of integral will not change, if the vector \( \mathbf{e}_1 \) to replace with any other unit vector \( \mathbf{e}_2 \), perpendicular to the vector \( \mathbf{k}_1 \). Therefore

\[
\int (\mathbf{P} \mathbf{e}_1)^2 S(P^2, \mathbf{k}_1 \mathbf{P}) d^3 P = \int (\mathbf{P} \mathbf{e}_2)^2 S(P^2, \mathbf{k}_1 \mathbf{P}) d^3 P
\]

\[
= \frac{1}{2} \int \left[ (\mathbf{P} \mathbf{e}_1)^2 + (\mathbf{P} \mathbf{e}_2)^2 \right] S(P^2, \mathbf{k}_1 \mathbf{P}) d^3 P.
\]

Hence, for current density we have:

\[
\mathbf{j}(\mathbf{r}, t) = \frac{ie^2 v_3^3 \mathbf{A}(\mathbf{r}, t)}{cm} \int \left[ (\mathbf{P} \mathbf{e}_1)^2 + (\mathbf{P} \mathbf{e}_2)^2 \right] S(P^2, \mathbf{k}_1 \mathbf{P}) d^3 P.
\]

Let’s notice, that the value \( P^2 = \mathbf{P} \mathbf{P} \) is equal to:

\[
P^2 = (\mathbf{P} \mathbf{e}_1)^2 + (\mathbf{P} \mathbf{e}_2)^2 + (\mathbf{P} \mathbf{n})^2,
\]

whence

\[
(\mathbf{P} \mathbf{e}_1)^2 + (\mathbf{P} \mathbf{e}_2)^2 = P^2 - \frac{(\mathbf{P} \mathbf{k}_1)^2}{k_1^2} = P^2 - (\mathbf{P} \mathbf{n})^2 = P_{\perp}^2,
\]

where \( P_{\perp} \) is the projection of vector \( \mathbf{P} \) on a straight line, perpendicular to plane \( (\mathbf{e}_1, \mathbf{e}_2) \).

Then for current density we receive the following expression

\[
\mathbf{j}(\mathbf{r}, t) = \frac{ie^2 v_3^3 \mathbf{A}(\mathbf{r}, t)}{cm} \int P_{\perp}^2 S(P^2, \mathbf{k}_1 \mathbf{P}) d^3 P.
\]

We consider the connection between electric field and potentials

\[
\mathbf{E}(\mathbf{r}, t) = -\frac{1}{c} \frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} - \frac{\partial U(\mathbf{r}, t)}{\partial \mathbf{r}},
\]

or

\[
\mathbf{E}(\mathbf{r}, t) = \frac{i\omega}{c} \mathbf{A}(\mathbf{r}, t).
\]

Hence, the current is connected with vector potential as:

\[
\mathbf{j}(\mathbf{r}, t) = \sigma_{tr} \mathbf{E}(\mathbf{r}, t) = \sigma_{tr} \frac{i\omega}{c} \mathbf{A}(\mathbf{r}, t).
\]
Replacing a current in the left part of this equality expression through a field, we receive
\[
\sigma_{tr} \frac{i\omega}{c} \mathbf{A} (\mathbf{r}, t) = \frac{ie^2 v_3^2 A (\mathbf{r}, t)}{cm} \int P^2 S(P^2, \mathbf{k}_1 \mathbf{P}) d^3 P.
\]

From here we receive expression for transverse electric conductivity
\[
\sigma_{tr} = \frac{e^2 v_3^2}{m\omega} \int P^2 S(P^2, \mathbf{k}_1 \mathbf{P}) d^3 P.
\]

Let’s replace in this expression the absolute Maxwell distribution by its explicit representation. We receive that:
\[
\sigma_{tr} = \frac{e^2 N_0}{m\omega \pi^{3/2}} \int \left( \omega - v_T k \mathbf{P} \right) e^{-P^2} + \mathcal{E}_T \left( \frac{e^{-P^2} - e^{-P^2}}{\hbar \omega} \right) P^2 d^3 P. \quad (1.11)
\]

Let’s transform the formula (1.11) with the use of standard expression for static electric conductivity \(\sigma_0 = (e^2 N_0)/(m\nu)\):
\[
\sigma_{tr} = \frac{\sigma_0}{\pi^{3/2}} \int \left[ \left( 1 - \frac{k \mathbf{P}}{\omega_T} \right) e^{-P^2} + \frac{\mathcal{E}_T}{\hbar \omega} \left( e^{-P^2} - e^{-P^2} \right) \right] \frac{P^2 d^3 P}{1 - i\omega_T + ik \mathbf{P}}. \quad (1.12)
\]

On the basis of formulas (1.11) or (1.12) we will write the formula for transverse dielectric permeability \(\varepsilon_{tr} = 1 + \frac{4\pi i}{\omega} \sigma_{tr}\) in Maxwellian plasma
\[
\varepsilon_{tr} = 1 + \frac{i\omega_p^2}{\omega^2 \pi^{3/2}} \int \left[ \left( \omega_T - k \mathbf{P} \right) e^{-P^2} + \frac{\mathcal{E}_T}{\hbar \nu} \left( e^{-P^2} - e^{-P^2} \right) \right] \frac{P^2 d^3 P}{1 - i\omega_T + ik \mathbf{P}}.
\]

Let’s present (1.11) in the form of the sum
\[
\sigma_{tr} = \sigma^{\text{classic}}_{tr} + \sigma^{\text{quant}}_{tr}. \quad (1.13)
\]

Here
\[
\sigma^{\text{classic}}_{tr} = \sigma_0 f_{\text{classic}}, \quad (1.14)
\]

where
\[
f_{\text{classic}} = \frac{1}{\pi^{3/2}} \int \frac{e^{-P^2} P^2 d^3 P}{1 - i\omega_T + ik \mathbf{P}}, \quad (1.15)
\]

and
\[
\sigma^{\text{quant}}_{tr} = \sigma_0 f_{\text{quant}}, \quad (1.16)
\]
where
\[
f_{\text{quant}} = \frac{1}{\pi^{3/2}} \int \left\{ -\frac{v_T}{\omega} kP e^{-P^2} + \frac{k_T v_T}{2\omega} (e^{-P^2} - e^{-P^2_{\perp}}) \right\} \frac{P^2_{\perp} d^3 P}{1 - i\omega + ilkP}. \tag{1.17}
\]
We will present the formula (1.16) in the form of the sum of two terms
\[
\sigma^{\text{quant}} = \sigma_1 + \sigma_2. \tag{1.18}
\]
In the formula (1.18) following designations are entered
\[
\sigma_1 = -\frac{l}{\omega \tau} \frac{1}{\pi^{3/2}} \int \frac{kP e^{-P^2} P^2_{\perp} d^3 P}{1 - i\omega + ilkP}, \tag{1.19}
\]
and
\[
\sigma_2 = \frac{lk_T}{2\omega \tau \pi^{3/2}} \frac{1}{\pi^{3/2}} \int \frac{(e^{-P^2_{\perp}} - e^{-P^2}) P^2_{\perp} d^3 P}{1 - i\omega + ilkP}. \tag{1.20}
\]
After obvious replacement of variables we receive
\[
\frac{e^{-(P_{\perp} + q/2)^2}}{1 - i\omega + ilkP} \rightarrow \frac{e^{-P^2}}{1 - i\omega + ilk(P \pm q/2)}.
\]
Let’s notice, that
\[
\frac{1}{1 - i\omega + ilkP + ilkq/2} - \frac{1}{1 - i\omega + ilkP - ilkq/2} = \frac{-ilkq}{(1 - i\omega + ilkP)^2 + (ilkq/2)^2}.
\]
By means of two last equalities we receive expression for \(\sigma_2\):
\[
\sigma_2 = -i\sigma_0 \frac{(lk)^2}{\omega \tau} \frac{1}{\pi^{3/2}} \int \frac{e^{-P^2} P^2_{\perp} d^3 P}{(1 - i\omega + ilkP)^2 + (ilkq/2)^2}. \tag{1.21}
\]
Let’s consider, that \(k = kn, \ \kP = knP = kP_n\), thus \(P^2 = P^2_n + P^2_{\perp}\), besides,
\[
kq = knq = kn \frac{k}{k_T} = k^2 \frac{n^2}{k_T} = k^2 \frac{k^2}{k_T} = kq.
\]
Therefore we may write down the following expression for the component of transverse conductivity $\sigma_2$:

$$\sigma_2 = i\sigma_0 \frac{\nu}{\omega} \int \frac{e^{-P^2} P^2_{\perp} d^3P}{(P_n - z/q)^2 - (q/2)^2},$$

where dimensionless parameters are entered

$$z = x + iy = \frac{\omega + \nu}{k_T v_T}, \quad x = \frac{\omega}{k_T v_T}, \quad y = \frac{\nu}{k_T v_T}.$$

Let’s notice, that the formula (1.12) can be deduced from the general formula for transverse conductivity of quantum plasma at arbitrary degree degeneration of electronic gas (see, for example, [26])

$$\frac{\sigma_{tr}}{\sigma_0} = \frac{1}{4\pi f_2(\alpha)} \int \left\{ \frac{1 - k_1 P}{\omega_T} g(P) \right\} \frac{P^2_{\perp} d^3P}{1 - i\omega_T + ik_1 P}.$$  

(1.23)

In the formula (1.23) following designations are entered

$$f_2(\alpha) = \int_0^\infty x^2 f_F(x) dx = \int_0^\infty \frac{x^2}{1 + e^{x^2 - \alpha}},$$

$$g(P) = \frac{e^{P^2 - \alpha}}{(1 + e^{P^2 - \alpha})^2},$$

where $\alpha$ is the dimensionless chemical potential, $\alpha = \frac{\mu}{k_B T}.$

Let’s notice that by $\alpha \to -\infty$ $f_2(\alpha) = e^{\alpha} \sqrt{\frac{\pi}{4}}.$ Therefore we have two following limiting transitions

$$\lim_{\alpha \to -\infty} \frac{g(P)}{4\pi f_2(\alpha)} = \frac{e^{-P^2}}{\pi^{3/2}},$$

$$\lim_{\alpha \to -\infty} \frac{f_+^F(P) - f_-^F(P)}{4\pi f_2(\alpha)} = \frac{e^{-P^2_+} - e^{-P^2_-}}{\pi^{3/2}}.$$

Taking into account two last limiting transitions it is clear, that the formula (1.23) at $\alpha \to -\infty$ passes in the formula (1.12).
3. Special cases and properties of transverse conductivity

When wave number \( k = 0 \) from the formula (1.12) we obtain the known classical formula for conductivity:

\[
\sigma_{\text{tr}}(k = 0) = \frac{\sigma_0}{\pi^{3/2}} \int \frac{e^{-P^2} P_+^2 d^3P}{1 - i\omega\tau} = \frac{\sigma_0}{1 - i\omega\tau} = \sigma_0 \frac{\nu}{\nu - i\omega},
\]

because

\[
\frac{1}{\pi^{3/2}} \int e^{-P^2}(P_y^2 + P_z^2) d^3P = 1.
\]

We will spread out the expression in braces (1.17) on degrees of \( q \)

\[
\frac{v_T}{\omega} \left\{ - Pk e^{-P^2} + \frac{k_T}{2} \left( e^{-P_+^2} - e^{-P_-^2} \right) \right\}
\]

\[
= \frac{v_T}{\omega} e^{-P^2} \frac{1}{6} k_T (P|q|)^2 - \frac{3}{2} q^2
\]

\[
= \frac{v_T k_T}{6\omega} e^{-P^2} q^2 P_n (P_n^2 - \frac{3}{2}), \quad P_n = P_n.
\]

On the basis of this decomposition at small \( q \) we receive

\[
f_{\text{quant}} = \frac{v_T k_T}{6\omega \pi^{3/2}} \int (P|q|)^2 \frac{P_+^2 d^3P}{1 - i\omega\tau + ilP_k}. \quad (2.1)
\]

By means of (2.1) from the formula (1.12) at small \( q \) we obtain

\[
\frac{\sigma_{\text{tr}}}{\sigma_0} = \frac{1}{\pi^{3/2}} \int \frac{e^{-P^2} P_+^2 d^3P}{1 - i\omega\tau + ilkP}
\]

\[
+ \frac{k_T v_T}{6\omega \pi^{3/2}} \int (P|q|)^2 \frac{P_+^2 d^3P}{1 - i\omega\tau + ilkP}. \quad (2.2)
\]

From the formula (2.2) follows, that at \( q \to 0 \) (or, when Planck’s constant tends to zero \( \hbar \to 0 \)), the formula for transverse conductivity of quantum plasma passes in the formula of transverse conductivity of classical plasma.

Vector \( k \) we will direct along an axis \( x \), \( k = k\{1, 0, 0\} \), then

\[
P_k = kP_x, \quad P_+^2 = \left( P_x + \frac{q}{2} \right)^2.
\]
Hence, the formula (1.12) can be rewritten in the form

\[
\frac{\sigma_{tr}}{\sigma_0} = \frac{1}{\pi^{3/2}} \int \left\{ \left( 1 - \frac{v_T k P_x}{\omega} \right) e^{-p_x^2} \right. \\
\left. + \frac{kTv_T}{2\omega} \left( e^{-p_y^2} - e^{-p_z^2} \right) \right\} \frac{(P_y^2 + P_z^2)d^3P}{1 - i\omega\tau + ilkP_x}. \tag{2.3}
\]

The internal double integral can be easily calculated in the polar coordinates in a plane \((P_y, P_z)\):

\[
\int e^{-P_y^2 - P_z^2}(P_y^2 + P_z^2)dP_ydP_z = \pi.
\]

Hence, according to (2.3) the transverse conductivity is expressed by one-dimensional integral:

\[
\frac{\sigma_{tr}}{\sigma_0} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left\{ \left( 1 - \frac{v_T k P_x}{\omega} \right) e^{-p_x^2} \right. \\
\left. + \frac{kTv_T}{2\omega} \left[ e^{-(P_x - q/2)^2} - e^{-(P_x + q/2)^2} \right] \right\} \frac{dP_x}{1 - i\omega\tau + ilkP_x}. \tag{2.4}
\]

The formula (2.4) can be transformed to the form

\[
\frac{\sigma_{tr}}{\sigma_0} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left\{ \left( 1 - \frac{lk}{\omega\tau} \mu \right) e^{-\mu^2} \right. \\
\left. + \frac{lkTe^{-q^2/4}}{2\omega\tau} (e^{q\mu} - e^{-q\mu}) \right\} \frac{e^{-\mu^2}d\mu}{1 - i\omega\tau + ilk\mu}. \tag{2.5}
\]

Let’s us notice that at small \(q\)

\[
e^{-q^2/4}(e^{q\mu} - e^{-q\mu}) = 2q\mu + \mu(\mu^2 - \frac{3}{2})\frac{q^3}{3} + \cdots.
\]

Hence, at small \(q = k/k_T\) we receive

\[
\sigma_{tr} = \sigma^{\text{classic}}_{tr} + \frac{\sigma_0 k^2 k^3}{6\omega m^2 v_T \sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\mu^2}(\mu^2 - 3/2)d\mu}{1 - i\omega\tau + ilk\mu}. \tag{2.6}
\]
In the formula (2.6) the first term is the transverse conductivity in classical plasma

$$\sigma_{\text{tr}}^{\text{classic}} = \frac{\sigma_0}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\mu^2} d\mu}{1 - i\omega \tau + ilk\mu}.$$  

Let’s transform the denominator of the formula (2.4)

$$\frac{1}{1 - i\omega \tau + ilk\mu} = \frac{1}{ilk\mu + (1 - i\omega \tau)/(ilk)} = \frac{1}{ilk \mu - z/q},$$

where dimensionless parameters are entered

$$x = \frac{\omega}{k_T v_T}, \quad y = \frac{\nu}{k_T v_T}, \quad z = x + iy = \frac{\omega + i\nu}{k_T v_T}.$$  

By means of these designations we receive

$$\sigma_{\text{tr}}^{\text{classic}} = -i\sigma_0 \frac{y}{q} t\left(\frac{z}{q}\right), \quad \text{(2.7)}$$

where

$$t(z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\tau^2} d\tau}{\tau - z}.$$  

Let’s rewrite the formula (2.6) in dimensionless variables

$$\frac{\sigma_{\text{tr}}}{\sigma_0} = -i\frac{y}{q} t\left(\frac{z}{q}\right) - i\frac{y}{q} q^2 \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\tau^2} \tau(\tau^2 - 3/2) d\tau}{\tau - z/q}. \quad \text{(2.8)}$$

Let’s transform the formula (2.8) in the form

$$\frac{\sigma_{\text{tr}}}{\sigma_0} = -i\frac{y}{q} t\left(\frac{z}{q}\right) - i\frac{y}{x} q^2 \left[\frac{1}{2} + \left(\frac{z^2}{q^2} - \frac{3}{2}\right) \lambda_C\left(\frac{z}{q}\right)\right]. \quad \text{(2.9)}$$

In (2.9) $\lambda_C(z)$ is the plasma dispersion function entered by Van Campen

$$\lambda_C(z) = 1 + z t(z) = 1 + \frac{z}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\tau^2} d\tau}{\tau - z} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\tau e^{-\tau^2} d\tau}{\tau - z}. $$

Let’s return to the formula (2.5) and we will present it in the dimensionless variables in the following form

$$\frac{\sigma_{\text{tr}}}{\sigma_0} = -\frac{iy}{q \sqrt{\pi}} \int_{-\infty}^{\infty} \left[1 - \frac{q}{x} \tau + \frac{e^{-q^2/4}}{2x} (e^{q\tau} - e^{-q\tau}) \right] \frac{e^{-\tau^2} d\tau}{\tau - z/q}, \quad \text{(2.10)}$$
or
\[
\frac{\sigma_{tr}}{\sigma_0} = -\frac{i}{q} y t \left( \frac{z}{q} \right) + i \frac{y}{x} \lambda_C \left( \frac{z}{q} \right) - \frac{iy e^{-q^2/4}}{2q x \sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{qt} - e^{-qt}}{t - z/q} e^{-t^2} dt.
\]

Besides, the formula (2.10) can be written down in the form
\[
\frac{\sigma_{tr}}{\sigma_0} = -\frac{i}{q} y t \left( \frac{z}{q} \right) + i \frac{y}{x} \lambda_C \left( \frac{z}{q} \right) + \frac{iy}{2x} T(z, q),
\]
where
\[
T(z, q) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-t^2} dt}{(t - z/q)^2 - q^2/4}.
\]

4. Comparison with Lindhard’s formula

Let’s consider the Lindhard’s formula (5.3.4) from [16] for transverse conductivity. After limiting transition at \( \eta \to 0 \) from (5.3.4) we receive
\[
\hat{\sigma}^{\text{Lind}}(q, \omega) = \frac{i N e^2}{\omega m} - \frac{ie^2 \hbar^2}{\Omega \omega m^2} \sum_k \left[ \frac{f^0(\mathcal{E}_k)}{\mathcal{E}_{k+q} - \mathcal{E}_k - \hbar(\omega + iv)} - \frac{f^0(\mathcal{E}_k)}{\mathcal{E}_k - \mathcal{E}_{k-q} - \hbar(\omega + iv)} \right] |\langle k + q|p|k \rangle^*|^2. \quad (3.1)
\]

Here \( f^0(\mathcal{E}_k) = \frac{N}{v_T^3 \pi^{3/2}} e^{-p^2} \) is absolute Maxwell — Boltzmann distribution, \( \mathcal{E}_k = \frac{\hbar^2 k^2}{2m} \); besides, the sum from (3.1) is understood as integral
\[
\frac{1}{\Omega} \sum_k = \int \frac{\hbar^3}{m^3} d^3 k = \int \frac{\hbar^3}{m^3} \frac{p_T^3}{\hbar^3} d^3 P = \int v_T^3 d^3 P.
\]

Let’s present the formula (3.1) in the form
\[
\hat{\sigma}^{\text{Lind}}(q, \omega) = \frac{i \sigma_0}{\omega \tau} + \hat{\sigma}_2, \quad (3.2)
\]
where
\[
\hat{\sigma}_2 = -\frac{ie^2 \hbar^2}{\Omega \omega m^2} \sum_k \left[ \frac{f^0(\mathcal{E}_k)}{\mathcal{E}_{k+q} - \mathcal{E}_k - \hbar(\omega + iv)} - \frac{f^0(\mathcal{E}_k)}{\mathcal{E}_k - \mathcal{E}_{k-q} - \hbar(\omega + iv)} \right] |\langle k + q|p|k \rangle^*|^2.
\]
\[-\frac{f^0(\mathcal{E}_k)}{\mathcal{E}_k - \mathcal{E}_{k-q} - \hbar(\omega + i\nu)} \right] |\langle k + q | p | k \rangle_*|^2. \quad (3.3)\]

Let us notice that
\[\mathcal{E}_{k+q} - \mathcal{E}_k = \frac{\hbar^2}{2m} \left[ q^2 + 2kq - \frac{2m}{\hbar}(\omega + i\nu) \right],\]
\[\mathcal{E}_k - \mathcal{E}_{k-q} = -\frac{\hbar^2}{2m} \left[ q^2 - 2kq + \frac{2m}{\hbar}(\omega + i\nu) \right].\]

Let’s present the formula (3.3) in the integrated form
\[\hat{\sigma}_2 = -\frac{2ie^2N}{m\omega\pi^{3/2}} \int \left[ k^2 - \left( \frac{kq}{q} \right)^2 \right] \times \]
\[\times \left[ \frac{1}{q^2 + 2kq - \frac{2m}{\hbar}(\omega + i/\tau)} + \frac{1}{q^2 - 2kq + \frac{2m}{\hbar}(\omega + i/\tau)} \right] e^{-P^2} d^3P. \quad (3.4)\]

Let’s transform the formula (3.4). We will direct a wave vector \( q \) lengthways \( x \)-component of momentum, i.e. we take \( q = \{k, 0, 0\} \), and instead of vector \( k \) we will enter a dimensionless vector \( P \) by the following equality
\[k = \frac{P}{\hbar} = \frac{p_F}{\hbar} P, \quad p_F = mv_F.\]

Then the first square bracket is equal to
\[k^2 - \left( \frac{kq}{q} \right)^2 = \frac{p_F^2}{\hbar^2} \left( P^2 - P_x^2 \right) = \frac{p_F^2}{\hbar^2} \left( P_y^2 + P_z^2 \right) = \frac{p_F^2}{\hbar^2} P_\perp^2.\]

The second square bracket is equal to
\[\left[ \frac{1}{q^2 + 2kq - \frac{2m}{\hbar}(\omega + i/\tau)} + \frac{1}{q^2 - 2kq + \frac{2m}{\hbar}(\omega + i/\tau)} \right] =
\frac{i\hbar}{2mv} \left[ \frac{1}{1 - i\omega + ilkP + i\hbar k^2\tau/2m} - \frac{1}{1 - i\omega + ilkP - i\hbar k^2\tau/2m} \right] =
\frac{\hbar^2 k^2\tau}{2m^2 \tau} \frac{1}{(1 - i\omega + ilkP)^2 + \left( \frac{\hbar k^2\tau}{2m} \right)^2}.\]
Here
\[ \frac{\hbar k^2 \tau}{2m} = \frac{k^2 v_T \tau}{2mv_T / \hbar} = \frac{k^2 l}{2k_T} = \frac{lq}{2}. \]

Now we represent integral Lindhard’s term in the form
\[ \hat{\sigma}_2 = -i\sigma_0 \frac{(lk)^2}{\omega \tau} \frac{1}{\pi^{3/2}} \int \frac{e^{-P^2} P_\perp^2 3P}{(1 - i\omega \tau + ilkP)^2 + (lkq/2)^2}. \] \hspace{1cm} (5.5)

The formula (3.5) for \( \hat{\sigma}_2 \) exactly coincides with the formula (1.21) for \( \sigma_2 \). It means, that deduced in the present work the formula for calculation of the electric conductivity in quantum collisional plasma, does not coincide with the corresponding formula deduced by Lindhard.

Let’s write down their difference (in dimensionless parameters)
\[ \sigma_{tr} - \sigma_{tr}^{Lind} = -i\sigma_0 \frac{y^2}{xq} \left[ 1 + \frac{x}{q} t\left( \frac{z}{q} \right) - \lambda_c \left( \frac{z}{q} \right) \right] = -\sigma_0 \frac{y^2}{xq} t\left( \frac{x + iy}{q} \right). \]

From this formula follows, that at \( q \to \infty \) the difference \( \sigma_{tr} - \sigma_{tr}^{Lind} \to 0 \).

On Figs. 1 – 10 we will give the graphic analysis of electric conductivity. On Figs. 1, 3 and 5 plots of the real part of relation \( \sigma_{tr}/\sigma_0 \) are presented, and on Figs. 2, 4 and 6 plots of the imaginary part of this relation are presented as function od dimensionless wave number \( q \).

The analysis of plots on Figs. 1 - 6 shows, that at small values of dimensionless wave number the curves corresponding to the quantum plasma (these are curves of 1), coincide with the curves corresponding to classical plasma (these are curves of 3). Intermediate values of dimensionless wave number, where curves of 1 and 3 not coincide, make an interval of \( 0.1 < q < 1 \) in a case \( x = 0.1, y = 0.01 \).

On Figs. 7 – 10 dependences for real (Figs. 7 and 9) and imaginary (Figs. 8 and 10) parts of relation \( \sigma_{tr}/\sigma_0 \) are presented as functions of dimensionless frequency of a field \( x \) at various values of dimensionless wave number \( q \).
At great values $x$ the curves corresponding to various values $q$, coincide. So, in the case $y = 0.01$ and small values of dimensionless wave number $q = 0.1, 0.2, 0.3$ the real and imaginary parts of relations $\sigma_{tr}/\sigma_0$ coincide at $x > 1$. In a case $y = 0.01$ and great values of dimensionless wave number $q = 2, 3, 4$ the real parts of relation $\sigma_{tr}/\sigma_0$, coincide at $x > 20$, and imaginary coincide at $x > 10$.

5. Conclusions

In the present work formulas for calculation of transverse conductivity and permeability in quantum collisional Maxwellian plasma are obtained. The kinetic Wigner – Vlasov – Boltzmann equation with collision integral in the BGK form in coordinate space is used.

Expand of transverse conductivity by degrees dimensionless of wave vector is derived.

When Planck’s constant $\hbar \to 0$, the deduced formula for conductivity pass in the formula of conductivity for classical plasma.

Comparison of the deduced formula for the transverse conductivity with Lindhard conductivity is carried out.

The graphic analysis of the dependences of real and imaginary parts of transverse conductivity on the dimensionless wave number, and on dimensionless frequency electric field is given.

At small values of the dimensionless wave numbers the transverse conductivity is well described by the formula for classical conductivity, and at great values by Lindhard’s formula.

REFERENCES

1. Klimontovich Y. and Silin V. P. About spectra of systems of interacting particles and collective losses at passage of the charged particles
through substance. - Uspekhi Fiz. Nauk. 1960. V. 70(2), 247–286 (in Russian); Physics-Uspekhi (Advances in Physical Sciences); J. Exp. Theor. Fiz. 23, 151 (1952); The Spectra of Systems of Interacting Particles. In Plasma Physics, Ed. J. E. Drummond (McGraw-Hill, New York). 1961. Chap. 2 pp. 35–87.

2. **Lindhard J.** On the properties of a gas of charged particles. - Kongelige Danske Videnskabernes Selskab, Matematisk–Fysiske Meddelelser. V. 28, No. 8 (1954), 1–57.

3. **Roos von, O. Boltzmann — Vlasov Equation for a Quantum Plasma.** - Phys. Rev. 119. No. 4. (1960), 1174–1179.

4. **Kliewer K. L. and Fuchs R.** Lindhard Dielectric Functions with a Finite Electron Lifetime. - Phys. Rev. 181, No. 2 (1969), 552–558.

5. **Mermin N. D.** Lindhard Dielectric Functions in the Relaxation–Time Approximation. - Phys. Rev. B. 1970. V. 1, No. 5. P. 2362–2363.

6. **Manfredi G.** How to model quantum plasmas. - ArXiv:quant-ph/0505004.

7. **Anderson D., Hall B., Lisak M. , and Marklund M.** Statistical effects in the multistream model for quantum plasmas. - Phys. Rev. E 65 (2002), 046417.

8. **de Andrés P., Monreal R. , and Flores F.** Relaxation–time effects in the transverse dielectric function and the electromagnetic properties of metallic surfaces and small particles. - Phys. Rev. B. 1986. Vol. 34, No. 10. Pp. 7365–7366.

9. **Shukla P. K. and Eliasson B.** Nonlinear aspects of quantum plasma physics. - Uspekhy Fiz. Nauk 180, 55–82 (2010) (in Russian), Volume 53, Number 1 (2010) in English.
10. *Eliasson B. and Shukla P. K.* Dispersion properties of electrostatic oscillations in quantum plasmas. - arXiv:0911.4594v1.

11. *Bhatnagar P. L., Gross E. P., and Krook M.* A model for collision processes in gases. I. Small amplitude processes in charged and neutral one-component systems. - Phys. Rev. 94 (1954), 511–525.

12. *Opher M., Morales G. J., and Leboeuf J. N.* Krook collisional models of the kinetic susceptibility of plasmas. - Phys. Rev. E. V.66, 016407, 2002.

13. *Gelder van, A. P.* Quantum Corrections in the Theory of the Anomalous Skin Effect. - Phys. Rev. 1969. Vol. 187. No. 3. P. 833–842.

14. Fuchs R., and Kliewer K. L. Surface plasmon in a semi–infinite free–electron gas. - Phys. Rev. B. 1971. V. 3. No. 7. P. 2270–2278.

15. *Fuchs R., and Kliewer K. L.* Optical properties of an electron gas: further studies of a nonlocal description. - Phys. Rev. 1969. V. 185. No. 3. P. 905–913.

16. *Dressel M., and Grüner G.* Electrodynamics of Solids: Optical Properties of Electrons in Matter. - ( Univ. Press Cambridge, New York, 2003), p. 487. 2008.

17. *Wierling A.* Interpolation between local field corrections and the Drude model by a generalized Mermin approach. - arXiv:0812.3835v1 [physics.plasm-ph] 19 Dec 2008.

18. *Faramarzi Sh., Hervieux P.–A., and Bigot J.–Y.* Temperature dependence of longitudinal and transverse dielectric functions of inhomogeneous Fermi systems in the local density approximation. - J. of Optoelectronics and Materials. V. 7, No. 6, December 2005, p. 3083 – 3092.
19. **Brodin G., Marklund M., and Manfredi G.** Quantum Plasma Effects in the Classical Regime. - Phys. Rev. Letters. **100**, (2008). P. 175001-1 – 175001-4.

20. **Manfredi G., and Haas F.** Self-consistent fluid model for a quantum electron gas. - Phys. Rev. B **64** (2001), 075316.

21. **Wigner E. P.** On the quantum correction for thermodynamic equilibrium. - Phys. Rev. **40** (1932), 749–759.

22. **Tatarskii V. I.** The Wigner representation of quantum mechanics. - Sov. Phys. Usp. **26** (1983), 311–327 [Uspekhy Fiz. Nauk. **139**, 587 (1983) (in Russian)].

23. **Hillery M., O’Connell R. F., Scully M. O., and Wigner E. P.** Distribution functions in physics: Fundamentals. - Phys. Rep. **106** (1984), 121–167.

24. **Arnold A., and Steinrück H.** The ‘electromagnetic’ Wigner equation for an electron with spin. - Z. Angew. Math. Phys. **40** (1989), 793–815.

25. **Kozlov V. V., and Smolyanov O. G.** Vigner’s function and diffusion in a collisionless medium of quantum particles. - Teoriya veroyatnostey i eye primeneniya, 2006, vol. 51, no. 1, pp. 109-125 (In Russian); Teor. Veroyatn. Primen. **51**, 109 (2006) ; translation in Theory Probab. Appl. **51**(1), 168 (2007).

26. **Latyshev A. V., and Yushkanov A. A.** Transverse electric conductivity of quantum collisional plasmas. - ArXiv: 1002. 1017v2 [math-ph] 11 Feb 2010, 44 pages.
Figure 1: The case: $x = 0.1, y = 0.01$. Dependence $\text{Re}(\sigma_{tr}/\sigma_0)$ on the dimensionless wave number $q$.

Figure 2: The case: $x = 0.1, y = 0.01$. Dependence $\text{Im}(\sigma_{tr}/\sigma_0)$ on the dimensionless wave number $q$. 
Figure 3: The case: \( x = 0.001, y = 0.01 \). Dependence \( \text{Re}(\sigma_{tr}/\sigma_0) \) on the dimensionless wave number \( q \).

Figure 4: The case: \( x = 0.001, y = 0.01 \). Dependence \( \text{Im}(\sigma_{tr}/\sigma_0) \) on the dimensionless wave number \( q \).
Figure 5: The case: $q = 0.5, y = 0.01$. Dependence $\text{Re}(\sigma_{tr}/\sigma_0)$ on the dimensionless frequency $x$.

Figure 6: The case: $q = 0.5, y = 0.01$. Dependence $\text{Im}(\sigma_{tr}/\sigma_0)$ on the dimensionless frequency $x$. 
Figure 7: The case: $y = 0.01$. Dependence $\text{Re}(\sigma_{tr}/\sigma_0)$ on the dimensionless frequency $x$. Curves 1, 2, 3 correspond to $q = 0.1, 0.2, 0.3$.

Figure 8: The case: $y = 0.01$. Dependence $\text{Im}(\sigma_{tr}/\sigma_0)$ on the dimensionless frequency $x$. Curves 1, 2, 3 correspond to $q = 0.1, 0.2, 0.3$. 
Figure 9: The case: $y = 0.01$. Dependence $\text{Re}(\sigma_{tr}/\sigma_0)$ on the dimensionless frequency $x$. Curves 1, 2, 3 correspond to $q = 2, 3, 4$.

Figure 10: The case: $y = 0.01$. Dependence $\text{Im}(\sigma_{tr}/\sigma_0)$ on the dimensionless frequency $x$. Curves 1, 2, 3 correspond to $q = 2, 3, 4$. 