RANDOM ATTRACTORS FOR STOCHASTIC NAVIER-STOKES EQUATION ON A 2D ROTATING SPHERE WITH STABLE LÉVY NOISE

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Abstract. In this paper we prove that the stochastic Navier-Stokes equations with stable Lévy noise generate a random dynamical systems. Then we prove the existence of random attractor for the Navier-Stokes equations on 2D spheres under stable Lévy noise (finite dimensional). We also deduce the existence of a Feller Markov Invariant Measure.

1. Introduction and motivation. The asymptotic behaviour of dynamical systems is one of fundamentals in mathematical physics. One of the central notions is that of an attractor, which conveys crucial geometric information about the asymptotic regime of a dynamical system as $t \to \infty$. It is well known that the 2D Navier-Stokes equations is dissipative and so have a global attractor, see for instance [29, 34].

More precisely, there exists compact subset $K$ of the original phase space in which all asymptotic dynamics lie. Much of the theory of infinite dynamical systems is devoted to study the properties of this set $K$, which is called the global attractor (see for instance [34, 29]). For instance, one can show that, under certain mild conditions on $K$, one can meaningfully define a group of solution operator $S(t)$ sensibly for all $t \in \mathbb{R}$, this defines a standard dynamical system,

$$(K, \{S(t)\}_{t \in \mathbb{R}})$$

A random (pullback) attractor is the pullback attractor for which time-dependent forcing become random. Readers are referred to [11] for a comparison of the three frameworks for the study of attractors, namely attractors, pullback attractors and random attractors.

As in the deterministic case, the theory of random attractor plays an important role in the study of the asymptotic behaviour of dissipative random dynamical system. The authors in [14, 13] developed a theory for the existence of random attractors for stochastic systems that closely comparable to the deterministic theory. Roughly speaking, a random attractor is a random invariant compact set which attracts every trajectory as time goes to infinity.

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The strategy to prove the existence of random attractor is analogous to the method of proving global attractor in deterministic case, which involves two main methods. The first of which requires one to find a bounded absorbing set and prove asymptotic compactness of $S(t)$; the second being to find compact absorbing set - the method we employ herein in proving existence of random attractor. The existence theorems and continuous dependence of initial datum results in [16] allows us to define a flow $\varphi: \mathbb{R} \times \Omega \times H \to H$ in the following sense:

$$\varphi_t(\omega)v_0 = v(t, \omega; v_0), \quad t \in T, \omega \in \Omega, \ v_0 \in H.$$ 

In the paper [7], the authors proved the existence of random attractor for the Navier-Stokes equations on 2D spheres under Gaussian-type forcing. To our knowledge, there is only one result on these systems when they are perturbed instead by non-Gaussian noise [22]. Gaussian-type models has been used widely to model phenomenons in engineering, science and finance.

The trajectories of a particle driven by Brownian motion are continuous in time almost surely, the displacement increases linearly in time (in a mean square sense). Moreover, the probability density function decays exponentially in space. These models do not capture fluctuation with peculiar properties such as anomalous diffusion (See for instance [3] and [36]). A good candidate can be used to model complex phenomena involve irregular fluctuation with peculiar properties is Lévy motion. Particularly non-Gaussian Lévy motion, has been widely applied to Biology, Image processing, Climate forecast and certainly in Finance and Physics [20, 32, 37, 26, 18].

A stable noise distribution permits a variety of simultaneous variation including noise with a large number of small random impulses and occasional large random disturbances with infinite moment. From a fluid modeling point of view, although continuous models are good enough in a macroscopic scale, at an atomic scale, the model breaks down, and the use of a Lévy process is compelling as fluid is not continuous at a microscopic scale [25].

As a special non-Gaussian stochastic process, the stable-type process has increasingly attracted mathematical interests due to the properties which the Gaussian process does not possess. Also, the tail of Gaussian random variable decays exponentially, which does not fit well for modeling processes with high variability or for extreme events such as earthquakes or stock market crashes. In contrast, the stable Lévy motion has a ‘heavy tail’ that decays polynomially and can be useful for these applications. For instance, when heavier tails (relative to a Gaussian distribution) of asset returns are more pronounced, the asymmetric $\alpha$-stable distribution becomes an appropriate alternative for modeling [27].

The goal of this work is to investigate the dynamical behavior of the SNSE on 2D rotating spheres with additive stable Lévy noise with $\beta \in (1, 2)$. That is,

$$du(t) = [-Au(t) - B(u(t), u(t)) + Cu(t) + f]dt + \sum_{i=1}^{m} \sigma_i dL_i(t)e_i, \quad u(0) = u_0. \quad (1)$$

The Stokes operator $A$ on the sphere is defined as

$$A : D(A) \subset H \to H, \quad A = -P(\Delta + 2\text{Ric}), \quad D(A) = H^2(\mathbb{S}^2) \cap V, \quad (2)$$

where Ric is the Ricci tensor on the 2D sphere $\mathbb{S}^2$. More precisely, since

$$\text{Ric} = \begin{pmatrix} E & F \\ F & C \end{pmatrix},$$
where the coefficients $E, F, G$ of the first fundamental form are given by
\[
E = x_\theta \cdot x_\theta = 1; \\
F = x_\theta \cdot x_\phi = x_\phi \cdot x_\theta = 0; \\
C = x_\phi \cdot x_\phi = \sin^2 \theta,
\]
we find that
\[
\text{Ric} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \theta \end{pmatrix}.
\] (3)

The bilinear form $B : V \times V \to V'$ is defined by
\[
(B(u, v), w) = b(u, v, w) = \sum_{i,j=1}^{3} \int_{S^2} u_i \frac{\partial (v_k)}{\partial x_i} u_j dx, \quad w \in V.
\] (4)

With a slight abuse of notation, we denote $B(u) = B(u, u)$ and $B(u) = \pi(u, \nabla u)$.

The Coriolis operator $C_1 : L^2(S^2) \to L^2(S^2)$ is bounded linear on $L^2(S^2)$ and is defined by the formula
\[
(C_1 v)(x) = 2\Omega(x \times v(x)) \cos \theta, \quad x \in S^2.
\] (5)

Moreover, we need a well defined bounded operator $C = PC_1$ in $H$. One can relate $C$ and $C_1$ as follows,
\[
(Cu, u) = (C_1 u, Pu) = \int_{S^2} 2\Omega \cos \theta ((x \times u) \cdot u(x)) dS(x) = 0, \quad \text{for} \ u \in H
\] (6)

Furthermore, $f \in H$ and $e_1, \cdots, e_m \in H$ are the eigenfunctions of the Stokes operator $A$, $\{\sigma_l\}$ is a sequence of real numbers, $L_i(t)$, $(1 \leq l \leq m)$ are mutually independent two-sided $\beta$-stable Lévy processes, $u = u(t, x, \omega)$ is now a random velocity of the fluid.

The goal in this work is in threefold:

- Prove (1) generates a Random Dynamical System $\varphi$;
- Prove the existence of random attractors for (1);
- Prove the existence of a Feller Markov Invariant Measure supported by the random attractor.

To this end, we study the stationary ergodic solution of an Ornstein-Uhlenbeck, make a transformation to obtain some estimates of the solution respectively in space $H$ and $V$, then using the compact embedding of Sobolev space, we obtain the existence of a compact random set which absorbs any bounded non-random subset of space $H$.

In section 2, we recall the concepts of a stable process, RDS, random dynamical system, random attractors and Markov-Invariant Measures. In section 3, we prove that the solution to equation (1) defines a random dynamical system associated with a stochastic flow $\varphi$. To this end, we first identify a suitable canonical probability space for the linear stationary stochastic Stokes equation. Then, using a priori estimates for strong solutions of (1) obtained in our work [16], we identify a compact absorbing set. Then using the assumption of a finite-dimensional noise, we deduce the existence of a random attractor. Finally, some standard properties of random attractors yield the existence of a random invariant measure which is supported by the random attractor of equation (1).

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1The angular velocity vector of earth is denoted as $\Omega$ for consistency with geophysical fluid dynamics Literature. It shall not be confused with the notation for probability space $\Omega$. 
2. Preliminary. In this section, we first review some necessary notions and preliminaries. The notion of a Lévy process follows closely the book [31]. The presentation here follows closely [4, 14, 2] with some slight modification to jump case based on the various papers on RDS under Lévy noise [19, 21]. The notion of a Random Dynamical System is simply a generalisation of a deterministic dynamical system. In brief, an RDS has two features: one is the measurable dynamical system $\varphi$, which is used to model the random perturbations, and the other is the cocycle mapping $\vartheta$ defined over the dynamical system (see [2] for more detail).

2.1. Basic definitions.

**Definition 2.1 (Lévy process).** An $H$-valued Lévy process is a stochastic process $X = \{X(t), t \in [0, \infty)\}$ such that

- $X(0) = 0$ a.s. and $X$ is stochastically continuous: $\forall \varepsilon > 0$, 
  \[
  \lim_{t \downarrow 0} \mathbb{P}(|X(t)| > \varepsilon) = 0.
  \]
- $X$ has independent increments, that is, $\forall 0 \leq t_0 < t_1 < \cdots < t_n$, the random vectors $X(t_0), X(t_1) - X(t_0), X(t_2) - X(t_1), \cdots, X(t_n) - X(t_{n-1})$ are independent,
- $X$ has stationary increments: 
  \[
  X_{t+s} - X_t \overset{d}{=} X_s \quad \forall s, t \geq 0,
  \]
- $t \mapsto X(t)$ is càdlàg a.s. .

Note, that without the assumption of stationary increments, we have an additive process. We will need also a two-sided Lévy process defined as follows. Let $X_1$ and $X_2$ be two independent Lévy processes defined on the same probability space and with the same distribution. Then we define the two-sided Lévy process

\[
X(t) = \begin{cases} 
X_1(t) & \text{if } t \geq 0 \\
X_2(t) & \text{if } t < 0 
\end{cases} \quad t \in \mathbb{R}.
\]

For a two-sided Lévy process we will consider the filtration $\mathcal{F}_t = \sigma(X(s) : s \leq t)$ for all $t \in \mathbb{R}$.

The cylindrical Lévy Noise used in this work is obtained by subordinating a cylindrical Wiener process with any $\beta/2$-stable process. Let us recall some basic facts from [30]. First, recall that a real random variable $X$ is said to be $\beta$-stable with the, scale parameter $\sigma$, skewness parameter $\delta$, and shift parameter $\nu$, in short, $X \sim S_\beta(\sigma, \delta, \nu)$ if

\[
\mathbb{E}e^{i\theta X} = e^{i\theta \nu - |\sigma \theta|^\beta (1 - i\delta \text{sgn}(\theta))},
\]

where

\[
c = \begin{cases} 
((\sigma \theta)^{1-\beta} - 1) \tan \frac{\pi \beta}{2} & \text{if } \beta \neq 1 \\
-\frac{2}{\pi} \log |\sigma \theta| & \text{if } \beta = 1
\end{cases}
\]

Note that, in particular, $S_2(\sigma, 0, \nu) = \mathcal{N}(\nu, 2\sigma^2)$ is Gaussian.

We have also the following definition:

**Definition 2.2.** A real valued random variable $X$ is said to be symmetric $\beta$-stable, $0 < \beta \leq 2$, if $X \sim S_2(\sigma, 0, \nu)$ or, explicitly

\[
\mathbb{E}e^{i\theta X} = e^{-\sigma^2 |\theta|^\beta / 2}, \quad \theta \in \mathbb{R}.
\]
The name “$\beta$-stable” means that if $X_1, \ldots, X_m$ are independent and $\beta$-stable, then $\sum_{j \leq m} \alpha_j X_j$ is $\beta$-stable, and

$$\sigma\left( \sum_{j \leq m} \alpha_j X_j \right) = \left( \sum_{j \leq m} |\alpha_j|^\beta \sigma(X_j)^\beta \right)^{1/\beta},$$

which is obvious from (7).

**Definition 2.3.** A random vector $X = (X_1, \ldots, X_N)$ with values in $\mathbb{R}^N$ is $\beta$-stable if each linear combination $\sum_{i=1}^N \alpha_i X_i$ is a real $\beta$-stable variable.

A random process $X = (X_t, t \in I)$ indexed by $I$ is called $\beta$-stable if for every $t_1, \ldots, t_N$ in $I$, $(X_{t_1}, \ldots, X_{t_N})$ is a $\beta$-stable random vector. (p.131 in [24], p.233 in [33])

A natural generalisation of the $\mathbb{R}^n$ definition of stable Lévy motion (see for instance p.113 [30]) to the Hilbert space is the following

**Lemma 2.4.** A Lévy process $\{X(t), t \geq 0\}$ on a Hilbert space is a $\beta$-stable Lévy motion if and only if $X(t) - X(s) \sim S_\beta((t-s)^{1/\beta}, \delta, 0)$ for some $0 < \beta \leq 2$, $-1 \leq \delta \leq 1$.

Now let us recall the definition of Random Dynamical System (RDS) and cocycle property.

**Definition 2.5.** A triple $\mathfrak{X} = (\Omega, \mathcal{F}, \vartheta)$ is said to be a measurable dynamical system (DS) if $(\Omega, \mathcal{F})$ is a measurable space and $\vartheta : \mathbb{R} \times \Omega \ni (t, \omega) \mapsto \vartheta_t \omega \in \Omega$ is a $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}$-measurable map such that for all $t, s \in \mathbb{R}$, $\vartheta_{t+s} = \vartheta_t \circ \vartheta_s$. A quadruple $\mathfrak{X}(\Omega, \mathcal{F}, \mathcal{P}, \vartheta)$ is called a metric dynamical system (RDS) iff. $(\Omega, \mathcal{F}, \mathcal{P})$ is a probability space and $\mathfrak{X}' := (\Omega, \mathcal{F}, \vartheta)$ is a measurable DS such that for each $t \in \mathbb{R}$, the map $\vartheta_t : \Omega \to \Omega$ is $\mathcal{P}$-preserving.

**Definition 2.6.** Given a metric DS $\mathfrak{X}$ and a Polish space $(X, d)$, a map $\varphi : \mathbb{R} \times \Omega \times X \ni (t, \omega, x) \mapsto \varphi(t, \omega)x \in X$ is called a measurable random dynamical system (on $X$ over $\vartheta$), iff

- $\varphi$ is $(\mathcal{B}(\mathbb{R}) \times \mathcal{F} \times \mathcal{B}, \mathcal{B})$-measurable.
- The trajectories $\varphi(\cdot, \omega)x : \mathbb{R} \to X$ are càdlàg $\forall (\omega, x) \in \Omega \times \mathbb{R}$;
- $\varphi$ is $\vartheta$-cocycle:

$$\varphi(t+s, \omega) = \varphi(t, \vartheta_s \omega) \circ \varphi(s, \omega) \quad \forall \ s, t \in \mathbb{R} \setminus \{0\}, \varphi(0, \omega) = \text{id} \setminus \{0\}, \varphi(\omega, \omega) = \vartheta_\omega \in \Omega$$

(8)

It follows from the cocycle property that $\varphi(t, \omega)$- is automatically invertible. $(\forall t \in T$ and $\forall \mathcal{P}$ a.e. $\omega$) In fact, $\varphi(t, \omega)^{-1} = \varphi(-t, \vartheta_t \omega)$ for $t \in T$. Instead of assuming (8) for all $\omega \in \Omega$, it suffices to assume (8) for all $\omega$ from a measurable ($\vartheta_t$)-invariant subset of $\Omega$ of full measure.

2.2. Stochastic calculus for two-sided time. While we will assume our metric dynamical system has two sided time $T = \mathbb{R}$, in this subsection we briefly discuss the extension of stochastic calculus to two sided time. The material follows closely with section 2.3.2 [2].

Let $(\Omega, \mathcal{F}, \mathcal{P})$ from now denotes a complete probability space.

**Definition 2.7** (Two-Parameter Filtration, p.71 [2]). Assume $\mathcal{F}_s^t$, $s, t \in \mathbb{R}$, $s \leq t$, is a two parameter family of sub $\sigma$-algebras of $\mathcal{F}$ with the following properties

- $\mathcal{F}_s^t \subset \mathcal{F}_u^v$ for $u \leq s \leq t \leq v$
Let us assume that $(\Omega, \mathcal{F}, \mathbb{P})$ be a metric DS, let $\mathcal{F}$ be the $\mathbb{P}$-completion of $\mathcal{F}^0$, and let $\{\mathcal{F}_t\}_{s \leq t}$, be a filtration in $(\Omega, \mathcal{F}, \mathbb{P})$. We call $(\Omega, \mathcal{F}, \mathbb{P}, \{\vartheta_t\}_{t \in \mathbb{R}}, \{\mathcal{F}_t\}_{s \leq t})$ a filtered DS, if for all $s, t, u \in \mathbb{R}$, $s \leq t$, we have

$$\vartheta_u^{-1}\mathcal{F}_s = \mathcal{F}_{s+u}.$$

**Definition 2.8** (Filtered DS, p.72 [2]). Let $(\Omega, \mathcal{F}^0, \mathbb{P}, \{\vartheta_t\}_{t \in \mathbb{R}})$ be a metric DS, let $\mathcal{F}$ be the $\mathbb{P}$-completion of $\mathcal{F}^0$, and let $\{\mathcal{F}_t\}_{s \leq t}$, be a filtration in $(\Omega, \mathcal{F}, \mathbb{P})$. We call $(\Omega, \mathcal{F}, \mathbb{P}, \{\vartheta_t\}_{t \in \mathbb{R}}, \{\mathcal{F}_t\}_{s \leq t})$ a filtered DS, if for all $s, t, u \in \mathbb{R}$, $s \leq t$, we have

$$\vartheta_u^{-1}\mathcal{F}_s = \mathcal{F}_{s+u}.$$

2.3. **Attraction and absorption.** For two random sets $A, B \subset X$, we put

$$d(A, B) = \sup_{x \in A} d(x, B) \quad \text{and} \quad \rho(A, B) = \max\{d(A, B), d(B, A)\}.$$

In fact, $\rho$ restricted to the family $\mathcal{C}$ of all nonempty closed subsets on $X$ is a metric (see [10]), and it is the so-called Hausdorff metric. From now on, let $\mathcal{X}$ be the Borel $\sigma$-field on $\mathcal{C}$ generated by open sets w.r.t. the metric $\rho$ [5, 10, 12].

**Definition 2.9.** Let us assume that $(\Omega, \mathcal{F})$ is a measurable space and let $(X, d)$ be a Polish space. A set-valued map $C : \Omega \to \mathcal{C}(X)$ is said to be measurable iff. $C$ is $(\mathcal{F}, \mathcal{X})$-measurable. Such a map is often called a closed and bounded random set. A closed and bounded random set $C$ will be called a compact random set on $X$ if for every $\omega \in \Omega$, $C(\omega)$ is a compact subset of $X$.

**Example 1.** A closed set valued map $K : \Omega \to 2^X$ is a random closed set.

**Remark.** Let $f : X \to \mathbb{R}$, be a continuous function on the Polish space $X$ and $\Omega \to \mathbb{R}$ an $\mathcal{F}$-measurable random variable. If the set $C_{f,R}(\omega) := \{x \in X : f(x) \leq R(\omega)\}$ is nonempty for each $\omega \in \Omega$, then $C_{f,R}$ is a closed and bounded random set.

**Definition 2.10.** Let $\varphi : \mathbb{R} \times \Omega \times X \ni (t, \omega, x) \mapsto \varphi(t, \omega)x \in X$ be measurable RDS on a Polish space $(X, d)$ over a metric DS $\mathfrak{T}$. A closed random set $B$ is said to be $\varphi$ forward invariant if and only if for all $\omega \in \Omega$,

$$\varphi(t,\omega)B(\omega) \subseteq B(\vartheta_t \omega) \quad \forall \ t > 0.$$

A closed random set $B$ is said to be strictly $\varphi$ invariant iff. $\forall \omega \in \Omega$,

$$\varphi(t,\omega)B(\omega) = B(\vartheta_t \omega) \quad \forall \ t > 0.$$

**Remark.** By substituting $\vartheta_{-t} \omega$ for $\omega$, we have the equivalent version of the above definition:

$$\varphi(t,\vartheta_{-t} \omega)B(\vartheta_{-t} \omega) \subseteq B(\omega), \quad \forall \ t > 0,$$

$$\varphi(t,\vartheta_{-t} \omega)B(\vartheta_{-t} \omega) = B(\omega), \quad \forall \ t > 0.$$

**Definition 2.11.** For a given closed random set $B$, the $\Omega$-limit set of $B$ is defined to be the set

$$\Omega(B, \omega) = \Omega_B(\omega) = \bigcap_{T \geq 0} \bigcup_{t \geq T} \varphi(t, \vartheta_{-t} \omega)B(\vartheta_{-t} \omega).$$

**Remark.**

(i) A priori $\Omega(B, \omega)$ can be an empty set.

(ii) One has the following equivalent version of Definition 2.11:

$$\Omega_B(\omega) = \{y : \exists n \to \infty, \{x_n\} \subset B(\vartheta_{-t_n} \omega), \lim_{n \to \infty} \varphi(t_n, \vartheta_{-t_n} \omega)x_n = y\}.$$

- $\mathcal{F}_s^+ := \cap_{u \geq t} \mathcal{F}_u^s = \mathcal{F}_s^t$, $\mathcal{F}_s^- := \cap_{u \leq s} \mathcal{F}_u^s = \mathcal{F}_s^t$ for $t \leq s$.
- $\mathcal{F}_s^r$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$ for every $s \leq t$.
(iii) Since \( \bigcup_{t \geq T} \varphi(t, \vartheta_t\omega) \) is closed, \( \Omega_B(\omega) \) is closed as well.

Given a probability space, a random attractor is a compact random set, invariant for the associated RDS and attracting every bounded random set in its basis of attraction. More precisely,

**Definition 2.12.** A random set \( A : \Omega \to \mathcal{C}(X) \) is a random attractor iff

- \( A \) is a compact random set;
- \( A \) is \( \varphi \)-invariant, i.e. \( \mathbb{P}\text{-a.s.} \)

\[
\varphi(t,\omega)A(\omega) = A(\vartheta_t\omega),
\]

(9)

- \( A \) is attracting, in the sense that, for all \( B \in X \) it holds

\[
\lim_{t \to \infty} \rho(\varphi(t, \vartheta_{-t}\omega)B(\varphi_{-t}\omega), A(\omega)) = 0.
\]

The random attractor \( A \) in the present work shall not be confused with the Stokes operator \( A \) introduced in [16].

Let us now state a result on the existence of a random attractor, which is an extension of the Gaussian noise case in the pioneering work in [14] to the general Lévy noise case.

**Theorem 2.13.** Let \( \varphi \) be a continuous in space, but càdlàg in time RDS on \( X \). Assume the existence of a compact random set \( K \) absorbing every deterministic bounded set \( B \subseteq H \). Then there exists a random attractor \( A \) given by

\[
A(\omega) = \bigcup_{B \subseteq X, B \text{ bounded}} \Omega_B(\omega), \ \omega \in \Omega.
\]

**Proof.** The proof is analogous to the proof of Theorem 3.11 in [14]. \( \square \)

### 2.4. Invariant measures on random sets.

In the final section, we prove the existence of invariant measure for the RDS \( \varphi \) (Put in another way, the existence of random invariant measure). Let us now discuss the notion of random invariant measure.

**Definition 2.14.** Let \( \varphi \) be a RDS on \( X \). The map

\[
\Theta : T \times X \times \Omega \to X \times \Omega,
\]

\[
(t, x, \omega) \mapsto (\varphi(t, \omega)x, \vartheta_t\omega)
\]
is said to be the *skew product* induced by \( \varphi \).

**Definition 2.15.** Let \( \varphi \) be a given RDS over a metric DS \( T \). A probability measure \( \mu \) on \( (\Omega \times X, \mathcal{F} \times B) \) is said to be an invariant measure for \( \varphi \) iff

- \( \Theta_t \) preserves \( \mu : \Theta_t(\mu) = \mu \) for all \( t > 0 \);
- The first marginal of \( \mu \) is \( \mathbb{P} \), i.e. \( \pi_\Omega(\mu) = \mathbb{P} \) where

\[
\pi_\Omega : \Omega \times X \ni (\omega, x) \mapsto \omega \in \Omega.
\]

The following corollary gives the existence of Invariant Measures for a RDS \( \varphi \). The proof follows from the Markov-Kuratowski fixed point theorem (see p.87 [12] for more detail.).

**Corollary 1** (p.374,[14]). Let \( \varphi \) be an RDS, and suppose \( \omega \mapsto A(\omega) \) is a compact measurable forward invariant set for \( \varphi \). Then there exist invariant measure for \( \varphi \) which are supported by \( A \).

Alternatively, one can construct a random invariant measure more explicitly via a Krylov-Bogoliubov type argument; we refer readers to p.87 [12].
2.4.1. Markov invariant measures. Based on the conditions in Theorem 2.13, it is clear the attractor is measurable with respect to the past $\mathcal{F}^-$, since $\Omega_B$ is measurable for any nonrandom $B$.

Define two $\sigma$-algebra corresponding to the future and the past, respectively by

$$\mathcal{F}^+ = \sigma\{\omega \mapsto \varphi(\tau, \vartheta_t \omega) : \tau, t \geq 0\},$$

and

$$\mathcal{F}^- = \sigma\{\omega \mapsto \varphi(\tau, \vartheta_t \omega) : \tau, 0 \leq \tau \leq t\}.$$ 

Then $\vartheta_t^{-1}\mathcal{F}^+ \subset \mathcal{F}^+$ for all $t \geq 0$ and $\vartheta_t^{-1}\mathcal{F}^- \subset \mathcal{F}^-$ for all $t \leq 0$.

**Proposition 2.16.** Suppose $\omega \mapsto A(\omega)$ is an $\varphi$-invariant compact set which is measurable with respect to the past $\mathcal{F}^-$ for an RDS $\varphi$. Then there exist an invariant measure $\mu$ supported by $A$ such that also $\omega \mapsto \mu_\omega$ is measurable with respect to $\mathcal{F}^-$. 

**Corollary 2** (p.374[14]). Under the conditions of Proposition 2.16, suppose in addition that $\varphi$ is an RDS whose individual trajectories form a Markov family, and such that $\mathcal{F}^+$ and $\mathcal{F}^-$ are independent. Then there exists an invariant measure $\rho$ for the associated Markov semigroup. Furthermore, the limit

$$\mu_\omega = \lim_{t \to \infty} \varphi(t, \vartheta_{-t}\omega)\rho$$

exists $P$ a.s., $\rho = \int \mu_\omega dP(\omega) = E(\mu_\omega)$, and $\mu$ is a Markov measure.

2.4.2. Feller Markov invariant measures. By Corollary 1 for a given RDS $\varphi$ on a Polish space $X$, one can find an invariant probability measure if an invariant compact random set $K(\omega), \omega \in \Omega$ can be identified. Hence Corollary 1 is generalised as the following.

**Corollary 3.** A continuous in space, càdlàg in time RDS which has an invariant compact random set $K(\omega), \omega \in \Omega$ has at least one invariant probability measure $\mu$ in the sense of definition 2.15.

One of the desirable properties of solutions of stochastic PDE is the Feller property. Let us now define a Feller Invariant Measure for a Markov RDS $\varphi$. If $f : X \to \mathbb{R}$ is bounded Borel measurable function, then put

$$(P_t f)(x) = Ef(\varphi(t, x)), \quad t \geq 0, x \in X. \quad (10)$$

It is clear that $P_t f$ is also a bounded and Borel measurable function. Moreover, one has the following result.

**Proposition 2.17.** Assume that that RDS $\varphi$ is a.s. continuous in space for every $t \geq 0$. Then the family $(P_t, t \geq 0)$ is Feller, i.e. $P_t f \in C_b(X)$ if $f \in C_b(X)$. Moreover, if the RDS $\varphi$ is càdlàg in time, then for any $f \in C_b(X)$, $(P_t f)(x) \to f(x)$ as $t \downarrow 0$.

**Proof.** For the first part, let us fix $t > 0$. If $x_n \to x$ in $X$, then it follows from the space continuity of $\varphi(t, \cdot)$ that $(P_t f)(x_n) \to (P_t f)(x)$ using the Lebesgue dominated convergence theorem.

For the second part, note that for a given $x \in X$ from the càdlàg property of $\varphi(\cdot, x, \omega) : [0, \infty) \to X$ for a.e. $\omega$ it follows that one has $(P_t f)(x) \to f(x)$ as $t \to 0$ if $x \in X$.

A RDS $\varphi$ is called Markov iff the family $(P_t, t \geq 0)$ is a semigroup on $C_b(X)$, that is, $P_{t+s} = P_t \circ P_s$ for all $t, s \geq 0$. 

Definition 2.18. A Borel probability measure $\mu$ in $H$ is said to be invariant w.r.t. $P_t$ if
\[ P_t^* \mu := \int_X P_t(x, \Gamma) \mu(dx) = \mu(\Gamma), \quad \forall \Gamma \in \mathcal{B}(X), \quad \forall t, \]
where $(P_t^* \mu) (\Gamma) = \int_H P_t(x, \Gamma) \mu(dx)$ for $\Gamma \in \mathcal{B}(H)$ and $P_t(x, \cdot)$ is the transition probability, $P_t(x, \Gamma) = P_t(1_{\Gamma})(x)$

Finally, a Feller invariant probability measure for a Markov RDS $\varphi$ on $H$ is, by definition, an invariant probability measure for the semigroup $(P_t^*, t \geq 0)$ defined by (10).

In view of Corollary 2, if a Markov RDS $\varphi$ on a Polish space $H$ has an invariant compact random set $K(\omega), \omega \in \Omega$, then there exists a Feller invariant probability measure $\mu$ for $\varphi$. More precisely we have the following corollary.

Corollary 4. If a càdlàg time and space continuous RDS $\varphi$ contains an invariant compact random set, then there exists a Feller invariant probability measure $\mu$ for $\varphi$.

3. Random dynamical systems generated by the SNSE on a rotating unit sphere. Having established the well-posedness in the earlier work [16], we are in a position to define an RDS $\varphi$ corresponding to the problem
\[ du(t) + Au(t) + B(u(t), u(t)) + Cu = f dt + GdL(t), \quad u(0) = u_0, \quad (11) \]
where $L$ is an $H$-valued stable Lévy process and $G : H \to H$ is a bounded operator.

But first, we need to determine a sample (canonical) probability space for which the dynamics of the driving noise remains stationary.

3.1. Some analytic facts. Recall that $X = L^4(S^2) \cap H$ denote the Banach space endowed with the norm
\[ |x|_X = |x|_H + |x|_{L^4(S^2)}. \]

Recall Assumption 1, page 18 in [16], namely, the space $K \subset H \cap L^4$ is a Hilbert space such that for any $\delta \in (0, 1/2)$,
\[ A^{-\delta} : K \to H \cap L^4(S^2) \quad \text{is } \gamma\text{-radonifying.} \quad (12) \]
This assumption is satisfied if $K = D(A^s)$ for some $s > 0$.

Remark. Under the above assumption the space $K$ can be taken as the RKHS of the cylindrical Wiener process $W(t)$ on $H \cap L^4$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. We identify $\Omega$ with $D(\mathbb{R}, E)$, which denotes the space of càdlàg functions defined on $\mathbb{R}$ which takes value in $E$. The Skorohod metric for $D$ is defined as
\[ d(l_1, l_2) = \sum_{i=1}^{\infty} (1 \land d_i ^2(l_1, l_2)) \quad \forall l_1, l_2 \in D. \]

We define $l_1^i(t) := g_i(t)l_1(t)$ and $l_2^i(t) := g_i(t)l_2(t)$ with
\[ g_i(t) := \begin{cases} 1, & \text{if } 0 \leq |t| \leq i - 1 \\ i - t, & \text{if } 0 \leq i - 1 \leq |t| \leq i \\ 0, & \text{if } 0 \leq c|t| \leq i \end{cases} \]
and
\[
d^*_{i}(l_1, l_2) = \inf_{\lambda \in \Lambda} \left( \sup_{-i \leq s \leq t} \frac{\lambda(t) - \lambda(s)}{t-s} \vee \sup_{-i \leq t \leq i} |l_1(t) - l_2(\lambda(t))| \right).
\]

The notation $\Lambda$ denotes the set of strictly increasing, continuous function $\lambda(t)$ from $\mathbb{R}$ to itself with $\lambda(0) = 0$. This Skorohod space is a complete separable metric space which is taken as the canonical sample space. Let $\mathcal{F}$ be the Borel $\sigma$-algebra of the Polish space $(D(\mathbb{R}, X), d)$. For every $t \in \mathbb{R}$ we have the evaluation map $L_t : D(\mathbb{R}, X) \to \mathbb{R}$ denote by $L_t(\omega) = \omega(t)$. Then we have $\mathcal{F} = \sigma(L_t, t \in \mathbb{R})$, that is, $\mathcal{F}$ is the smallest $\sigma$-algebra generated by the family of maps $\{L_t : t \in \mathbb{R}\}$. Let $\mathbb{P}$ be the unique probability measure which makes the canonical process a two-sided Lévy process with paths in $D(\mathbb{R}; E)$, that is, $\omega(t) = L_t(\omega)$.

Define the shift $(\vartheta_t \omega)(\cdot) = \omega(t + \cdot) - \omega(t)$ $t \in \mathbb{R}, \omega \in \Omega$. Then the map $(t, \omega) \to \vartheta_t(\omega)$ is continuous and measurable [2] and the (Lévy) probability measure $\mathbb{P}$ is $\vartheta$ invariant, that is, $\mathbb{P}(\vartheta^{-1}_t(T)) = \mathbb{P}(T)$ for all $T \in \mathcal{F}$. This flow is an ergodic dynamical system with respect to $\mathbb{P}$. Thus $(\Omega, \mathcal{F}, \mathbb{P}, (\vartheta_t)_{t \in \mathbb{R}})$ is a metric DS.

3.2. Ornstein-Uhlenbeck process. In the following subsections we are concerned with the linear evolutionary Stokes equations.

Recall, the equation
\[
\begin{cases}
\dot{u}(t) + (A + \alpha)u(t) = f(t), & t \in [0, T], \\
u(0) = u_0.
\end{cases}
\]
Suppose $A$ generates a strongly continuous and analytic semigroup in a Banach space $E$ and $f : [0, T] \to E$ is such a function that
\[
\int_0^T |f(t)|_E dt < \infty,
\]
then (13) has a unique continuous solution and the solution is given by
\[
u(t) = e^{-tA}u_0 + \int_0^t e^{-\alpha(t-s)A} f(s)ds.
\]
In particular, we have

**Definition 3.1.** Let $L$ be a $H$-value Lévy process given by the formula
\[
L(t) = \sum_{i=1}^{\infty} \sigma_i L^i(t)
\]
where $L^i$ are independent and identically distributed one-dimensional Lévy processes. We assume that
\[
\int_0^T |L(t)|_H dt < \infty \quad \text{a.s.}
\]
Then the solution of the differential equation
\[
\dot{z}(t) + (\alpha + A)z(t) = L(t), \quad z(0) = 0, \quad \alpha > 0
\]
is given by
\[
z(t) = \int_0^t e^{-(t-s)(\alpha + A)}(L(s))ds.
\]
3.2.1. **Stochastic convolution and integrating by parts.** Here we quote a useful integration by part formula from [35] which allows us to attain the desired regularity for which the RDS \( \varphi \) exist.

Consider the following Ornstein-Uhlenbeck process generated by the Stokes operator on \( S^2 \),

\[
z_t = \int_0^t e^{-A(t-s)}dL(s) = \sum_{l=1}^\infty z_l(t)e_l,
\]

where \( \{e_l : l = 1, \ldots \} \) is the complete orthonormal system of eigenfunctions of \( A \) in \( H \) and

\[
z(t) = \int_0^t e^{-\lambda_l(t-s)}\sigma_l dL^l(s),
\]

where \( L^l(t) = \langle L(t), e_l \rangle \) and \( \lambda_l \) are the eigenvalues of the Stokes operator \( A \). By the Itô product formula, see Theorem 4.4.13 of [1] for any \( l \geq 1 \), one has that

\[
\sigma_l L^l(t) = \sigma_l \int_0^t \lambda_l e^{-\lambda_l(t-s)}L^l(s)ds + \int_0^t e^{-\lambda_l(t-s)}\sigma_l dL^l(s) + \sigma_l \int_0^t \lambda_l e^{-\lambda_l(t-s)}\Delta L^l(s)ds,
\]

where \( \Delta L^l(s) = L^l(s) - L^l(s-). \) Since \( L^l(t) \) is a \( \beta \) stable process, \( \Delta L^l(s) = 0 \) a.e. for \( s \in [0, t] \) and so we have

\[
\int_0^t \lambda_l e^{-\lambda_l(t-s)}\Delta L^l(s)ds = 0.
\]

Therefore,

\[
z(t) = \sigma_l L^l(t) - \int_0^t \lambda_l e^{-\lambda_l(t-s)}\sigma_l L^l(s)ds.
\]

Hence, if we assume that \( \sigma_l = 0 \) for \( l > m \) for a certain finite \( m > 1 \) then

\[
z(t) = L(t) - Y(t),
\]

where

\[
Y(t) = \int_0^t Ae^{-A(t-s)}L(s)ds.
\]

In this case we clearly have

\[
Y(t) \in H \quad \text{a.s.} \quad (t \geq 0).
\]

3.2.2. **Regularity of shifting flow.** To prove that our stochastic Navier-Stokes system generates a RDS, we will transform it into a random PDE in \( X \) with the aid of the integration by parts technique introduced earlier. We need to give a meaning to the Ornstein-Uhlenbeck process in the metric dynamical system \( (\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}}) \) given by

\[
z(\theta_t \omega) := \tilde{z}(t) = \int_{-\infty}^t \hat{A}^{1+\delta} e^{-\lambda_l(r-t)\hat{A}}(\bar{w}(t) - \bar{w}(r))dr, \quad t \in \mathbb{R}.
\]

Our goal now is to show \( \tilde{z}(t) \) is a well defined element in \( X := L^4(S^2) \cap H \) for a.e. \( \omega \). But first, we need to give the following two results, which can be viewed as a generalisation of Theorem 4.1 and Theorem 4.4 in [9] to the case where the Ornstein-Uhlenbeck generator is \( \hat{A} = \nu A + C, \quad D(\hat{A}) = D(A), \quad A = -P(\Delta + 2\text{Ric}). \)
Recall that
\[ |A^{1+\delta} e^{-(A+\alpha t)}|_{L^2(X,X)} \leq C t^{-1-\delta} e^{-(\mu+\alpha)t}, \quad t > 0. \]  

**Proposition 3.2.** Assume \( \beta \in (1, 2), \ p \in (0, \beta) \) and
\[ \sum_{l \geq 1} |\sigma_l|^\beta \lambda_l^{\beta/2} < \infty. \]

Then
\[ E \int_{-\infty}^t |\hat{A} e^{-(t-r)A}(\tilde{\omega}(t) - \tilde{\omega}(r))|_X^p \ dr < \infty. \]  

Moreover, for \( P \) almost every \( \tilde{\omega} \in D(\mathbb{R}, X) \), \( t \in \mathbb{R} \) the function
\[ \hat{\xi}(t) = \hat{\xi}(\tilde{\omega})(t) = \int_{-\infty}^t \hat{A} e^{-(t-r)A}(\tilde{\omega}(t) - \tilde{\omega}(r)) dr, \quad t \in \mathbb{R} \]

is well defined and càdlàg in \( X \). Furthermore, for any \( \kappa > 0 \) such that \( \kappa p > 1 \) there exists a random variable \( C \) depending on \( \beta, \ p, \ \sigma, \ \delta \) such that
\[ |\hat{\xi}(\tilde{\omega})(t)|_X \leq C(\beta, p, \sigma, \delta, \tilde{\omega})(1 + |t|^{\kappa}). \]  

**Proof. Part I.** We will show first that the Lévy process \( L \) is càdlàg in \( X \) and
\[ E \sup_{t \leq T} |L(t)|_X^p < \infty. \]  

Follow Proposition 3.4 in [16], taking \( Ge_l = \sigma_l e_l \), the process \( L \) is càdlàg in \( V \) by Lemma 3.5 in [16], hence in \( L^2(S^2) \) and finally in \( X \). It remains to show (22).

Recall Lemma 3.1, that is,
\[ E \sup_{t \leq T} |A^{\theta} L(t)|^p \leq C(\beta, p, \sigma, \delta, \tilde{\omega})(1 + |t|^{\kappa}). \]  

Putting \( \delta = 0 \), we have,
\[ E \sup_{t \leq T} |L(t)|^p \leq C(\beta, p, \sigma, \delta, \tilde{\omega})(1 + |t|^{\kappa}). \]  

and putting \( \delta = \frac{1}{2} \) we have,
\[ E \sup_{t \leq T} |A^{\frac{1}{2}} L(t)|^p \leq C(\beta, p, \sigma, \delta, \tilde{\omega})(1 + |t|^{\kappa}). \]  

So
\[ E \sup_{s \leq T} |L(s)|_X^p \]
\[ \leq c E \sup_{s \leq T} |L(s)|^p + c E \sup_{s \leq T} |L(s)|^p_{L^1(S^2)} \]
\[ \leq c E \sup_{s \leq T} |L(s)|^p + c E \sup_{s \leq T} \left( |L(s)|^\frac{p}{2} |L(s)|^{\frac{p}{2}} \right) \quad \text{via Ladyzhenskaya inequality} \]
\[ \leq c E \sup_{s \leq T} |L(s)|^p + c E \sup_{s \leq T} |L(s)|^p_{V} \quad \text{via Poincé inequality in [16]} \]
\[= C(\beta, p) \left( \sum_{|\ell| \geq 1} |\sigma|^{p\beta}\right) T^\frac{p}{\beta} + C(\beta, p) \left( \sum_{|\ell| \geq 1} |\sigma|^{\beta p/\ell} \right) T^\frac{p}{\beta}\]

\[\leq C(\beta, p) \left( \sum_{|\ell| \geq 1} |\sigma|^{\beta p/\ell} \right) T^\frac{p}{\beta}.

Part II. In what follows we use the fact that \(\tilde{\omega}(t) = L(t)\) \(\mathbb{P}\) a.s. Using the change of variables \(s = t - r\), we obtain

\[
\mathbb{E} \int_{-\infty}^{t} |\hat{A}e^{-(t-r)\hat{A}}(\tilde{\omega}(t) - \tilde{\omega}(r))|^p \, dr = \int_{0}^{t} \mathbb{E} |\hat{A}e^{-s\hat{A}}(\tilde{\omega}(t) - \tilde{\omega}(t-s))|^p \, ds
\]

Using (18) with \(\gamma = \alpha + \mu\) we have

\[
\int_{0}^{t} \mathbb{E} |\hat{A}e^{-s\hat{A}} \tilde{\omega}(s)|^p \, ds \leq C \int_{0}^{t} e^{-\frac{\gamma s}{p}}|A^{\hat{A}} \tilde{\omega}(s)|^p \, ds
\]

\[
\leq C \int_{0}^{t} e^{-\frac{\gamma s}{p}} C(\beta, p) s^{\beta} \sum_{|\ell| \geq 1} |\sigma|^{\beta p/\ell} \tilde{\omega}(s) \, ds < \infty,
\]

since \(p - \frac{p}{\beta} < 1\) and we infer that \(\hat{z}(t)\) is well defined in \(X\) \(\mathbb{P}\) a.s. using the same arguments as in the proof of (22) above.

We will prove (21). Applying Lemma 3.6 in [16], with the Banach space \(B = X\) and \(\kappa\) such that \(\kappa p > 1\) we obtain

\[
|\hat{z}(\tilde{\omega})|_{X} \leq C(\beta, p, \sigma, \delta, \kappa, \tilde{\omega})(1 + |t|^\kappa),
\]

and (21) follows.

Part III. One has to check \(\hat{z}\) is right continuous with left limit in \(X\). To this end note first that

\[
\hat{z}(t) = \int_{-\infty}^{t} \hat{A}e^{-(t-s)\hat{A}}(\omega(t) - \omega(s)) \, ds
\]

\[
= \left( \hat{A} \int_{0}^{t} e^{-s\hat{A}} \, ds \right) \omega(t) - \int_{-\infty}^{t} \hat{A}e^{-(t-s)\hat{A}} \omega(s) \, ds
\]

\[
= \omega(t) - \int_{-\infty}^{t} \hat{A}e^{-(t-s)\hat{A}} \omega(s) \, ds,
\]

since \(\hat{A}\) is invertible. The function \(\omega\) is càdlàg in \(X\) by assumption. We will show that the function

\[
F(t, \omega) = \int_{-\infty}^{t} \hat{A}e^{-(t-s)\hat{A}} \omega(s) \, ds
\]

is continuous in \(X\) for \(\mathbb{P}\) a.e. \(\omega\). Indeed, for \(s, t \in \mathbb{R}\) such \(r < t\) we have

\[
\int_{-\infty}^{t} \hat{A}e^{-(t-s)\hat{A}} \omega(s) \, ds = \int_{-\infty}^{r} \hat{A}e^{-(t-s)\hat{A}} \omega(s) \, ds + \int_{r}^{t} \hat{A}e^{-(t-s)\hat{A}} \omega(s) \, ds
\]

\[
= \hat{A}e^{-(t-r)\hat{A}} \int_{-\infty}^{r} e^{-(r-s)\hat{A}} \omega(s) \, ds + \int_{r}^{t} \hat{A}e^{-(t-s)\hat{A}} \omega(s) \, ds
\]

\[
= \hat{A}e^{-(t-r)\hat{A}} \hat{h} + I(t),
\]

where

\[
\hat{h} = \int_{-\infty}^{t} \hat{A}e^{-(t-s)\hat{A}} \omega(s) \, ds
\]

and

\[
I(t) = \int_{r}^{t} \hat{A}e^{-(t-s)\hat{A}} \omega(s) \, ds.
\]
Since the semigroup \((e^{-\delta \hat{A}})_{\delta \geq 0}\) is analytic, we find that the function \(t \to \hat{A}e^{-(t-r)\hat{A}}h\) is continuous for \(t > r\). Let us consider \(I(t)\). By Sobolev embeddings we have a continuous embedding \(H^{1,2} \subset W^{1,4}\). Therefore for \(\delta\) small enough the function \(t \to A^{\delta}\omega(t)\) is locally bounded in \(L^4\) for a.e. \(\omega\). Then
\[
I(t) = \int_{t-r}^{t} \hat{A}^{\frac{1}{2}}e^{-(t-s)\hat{A}}A^{\delta}\omega(s)\, ds
\]
is continuous for \(t > r\), again by standard properties of analytic semigroups. \(\square\)

**Theorem 3.3.** Under the assumption of Proposition 3.2, for \(\mathbb{P}\text{-a.e. } \omega \in D(\mathbb{R}, X)\),
\[
\hat{\omega}(\partial_s \omega)(t) = \hat{\omega}(\omega(t + s), \ t, s \in \mathbb{R}.
\]

**Proof.** The proofs of the first three parts follows from closely from Theorem 4.8 and Proposition 8.4 in [9], see also Theorem 9 in [23]. For the last part, since \((\partial_s \omega)(r) = \omega(r + s) - \omega(s), \ r \in \mathbb{R}\), we have
\[
\hat{\omega}(\partial_s \omega)(t) = \int_{-\infty}^{t} Ae^{-(t-r)A} [\partial_s \omega(t) - \partial_s \omega(r)] dr
\]
\[
= \int_{-\infty}^{t} \hat{A}e^{-(t-r)A} [\omega(t + s) - \omega(r + s)] dr
\]
\[
= \int_{-\infty}^{t+s} \hat{A}e^{-(t+s-r')A} [\omega(t + s) - \omega(r')] dr' = \hat{z}(\omega)(t + s).
\]
\(\square\)

Now, put \((\tau_s \zeta)(t) = \zeta(t + s), \ t, s \in \mathbb{R}\). Therefore \(\tau_s\) is a linear, bounded map from \(D(\mathbb{R}, X)\) into \(D(\mathbb{R}, X)\). Moreover, the family \((\tau_s)_{s \in \mathbb{R}}\) is a \(C_0\) group on \(D(\mathbb{R}, X)\). Hence the shifting property could be expressed as

**Corollary 5.** For \(\mathbb{P}\text{-a.e. } \omega \in D(\mathbb{R}, X)\) For \(s \in \mathbb{R}, \ \tau_s \circ \hat{\omega} = \hat{\omega} \circ \partial_s\), that is
\[
\tau_s(\hat{\omega}(\omega)) = \hat{\omega}(\partial_s(\omega)), \ \omega \in D(\mathbb{R}; X).
\]

**Proposition 3.4.** The process
\[
z_\alpha(t) = \int_{-\infty}^{t} e^{-(t-s)(\hat{A}+\alpha I)} dL(s),
\]
where the integral is interpreted in the sense of [9] is well defined and is identified as a solution to the equation
\[
dz_\alpha(t) + (\hat{A} + \alpha I)z_\alpha dt = dL(t), \ t \in \mathbb{R}.
\]
The process \(z_\alpha, \ t \in \mathbb{R}\) is a stationary OU process.

We define
\[
z_\alpha(\omega) := \hat{z}(\hat{A} + \alpha I; \omega) \in D(\mathbb{R}, X),
\]
i.e. for any \(t \geq 0\),
\[
z_\alpha(\omega)(t) := \int_{-\infty}^{t} \hat{A}e^{-(t-s)(\hat{A}+\alpha I)}(\omega(t) - \omega(s))\, ds
\]
By Proposition 3.1,
\[
\frac{d^+}{dt} z_\alpha(\omega)(t) + (\hat{A} + \alpha I) \int_{-\infty}^{t} \hat{A}e^{-(t-s)(\hat{A}+\alpha I)}(\omega(t) - \omega(s))\, ds = L(t).
\]
Therefore $z_\alpha(t)$ satisfies
\[
\frac{d^+}{dt}z_\alpha(t) = (\hat{A} + \alpha I)z_\alpha(t) = \dot{\omega}(t), \quad t \in \mathbb{R}.
\] (27)

It follows from Theorem 3.3 that
\[ z_\alpha(\vartheta_s(\omega); t) = z_\alpha(\omega)(t + s), \quad \omega \in D(\mathbb{R}, X), \quad t, s \in \mathbb{R}. \]

Similar to our definition of Lévy process $L_t$, i.e. $L_t(\omega) := \omega(t)$, we can view the ODE as a definition of $z_\alpha(t)$ on $(\Omega, \mathcal{F}, \mathbb{P})$; equation (27) suggests that this process is an Ornstein-Uhlenbeck process.

Now we have enough tools to prove the cocycle property of RDS, and this allows us to prove that $(\varphi, \theta)$ is an RDS. The proof follows same lines as Theorem 6.15 in [4].

3.3. Random dynamical system generated by the SNSE on a sphere with Lévy noise. Let us fix $\alpha \geq 0$, put $\Omega = \Omega(E)$. Let $z(t)$ be the Ornstein-Uhlenbeck process generated by the Stokes operator on $\mathbb{S}^2$ defined in section 3.2.1.

We define a map $\varphi = \varphi_\alpha : \mathbb{R} \times \Omega \times H \to H$ by
\[
(t, \omega, x) \mapsto v(t, \hat{z}_\alpha(\omega))(x - \hat{z}_\alpha(\omega)(0)) + \hat{z}_\alpha(\omega)(t),
\]
where $v(t) = u(t) - z_\alpha(t)$. In what follows, we write $z = z_\alpha$ for simplicity.

Put in another way,
\[
\varphi = \varphi_\alpha(t, \omega)x := v(t, z_\alpha(\omega))(x - z_\alpha(\omega)(0)) + z_\alpha(\omega)(t)
\]
\[= u(t, \omega; x) \quad \forall t \in T, \omega \in \Omega, \ x \in H,
\]
where $u(\cdot; \omega, u_0)$ is the solution of the integral equation corresponding to given $\omega \in \Omega$, $u_0 \in H$ and $\varphi$ satisfies the definition of RDS.

Since $\varphi(t) = \varphi(t, \vartheta_t(\omega))v_0$ and $v(0) = v_0$, then $\varphi(0, \omega) = I$. It is clear that $\varphi(0, \omega) = I$. Because $z(\omega) \in D(\mathbb{R}; X)$, $z(\omega)(0)$ is a well-defined element of $H$ and hence $\varphi$ is well defined. Furthermore, we have the first main result of this work.

**Theorem 3.5.** $(\varphi, \theta)$ is a Random Dynamical System.

To prove the claim, one can simply check the definition of a random dynamical system (see subsection 2.6).

**Proof.** First, we check **Measurability.** Suppose $u_0 \in V$ and $t \in T$ is fixed, then the map $\omega \mapsto \varphi(t, \omega)u_0 \in H$ is measurable because the solution $u(t, \omega; u_0)$ is constructed as the (pointwise in $\omega$) limit of successive approximations of the contraction, which is measurable, being explicitly defined in terms of measurable objects. Finally, if $u_0 \in H$, then $u(t, \omega; u_0)$ is the limit of $u(t, \omega; u_0^n)$ with $u_0^n \in V$. The required measurability is assured.

Next, we check **Continuous dependence on initial data.**

The proof follows a similar line of the proof of uniqueness for the strong solution in [16].

Thirdly, we check the Càdlàg property of $\varphi(t, \omega)$. The càdlàg property of $\varphi, u_0$ follows from equation (19) since the $\omega$ is càdlàg. See also chapter 3, p.116 of the book [17].

Lastly, we will check the cocycle property of $\varphi$, namely, for any $x \in H$, one has to check,
\[
\varphi(t + s, \omega)x = \varphi(t, \vartheta_s(\omega))\varphi(s, \omega), \quad t, s \in \mathbb{R}. \] (28)
From the definition of \( \varphi \), noting from the cocycle property of \( z \), \( \dot{z}(\vartheta_s \omega)(t) = \dot{z}(\omega)(t + s) \), \( z(\omega)(s) = z(\vartheta_s \omega)(0) \) for all \( s \in \mathbb{R} \), we have for all \( t, s \in \mathbb{R} \),
\[
\varphi(t + s, \omega)x = v(t + s, z(\omega)(t + s))(x - z(\omega)(0)) + z(\omega)(t + s),
\]
\[
\varphi(t, \vartheta_s \omega)\varphi(s, \omega)x = v(t, z(\vartheta_s \omega)(t))(x - z(\omega)(0)) + z(\omega)(s) - z(\vartheta_s \omega)(0) + z(\vartheta_s \omega)(t)
\]
\[
= v(t, z(\vartheta_s \omega)(t))(v(s, z(\omega)(s))(x - z(\omega)(0))) + z(\vartheta_s \omega)(t).
\]
In view of (3.3), to prove (28), we need to prove
\[
v(t + s, z(\omega)(t + s))(x - z(\omega)(0)) = v(t, z(\vartheta_s \omega)(t))(v(s, z(\omega)(0))).
\]
Following the idea from [4], let us fix \( s \in \mathbb{R} \), define \( v_1, v_2 \) by
\[
v_1(t) = v(t + s, z(\omega)(t + s))(x - z(\omega)(0)) \quad t \in \mathbb{R},
\]
\[
v_2(t) = v(t, z(\vartheta_s \omega)(t))(v(s, z(\omega)(s))(x - z(\omega)(0))), \quad t \in \mathbb{R}.
\]
Because \( v(0, z(\vartheta_s \omega)(0))(x - z(\vartheta_s \omega)(0)) = x - z(\vartheta_s \omega)(0) \), one infer that
\[
v_1(0) = v(s, z(\omega)(s))(x - z(\omega)(0))
\]
\[
= v(0, z(\vartheta_s \omega)(0))(v(s, z(\omega)(s))(x - z(\omega)(0))) = v_2(0).
\]
Since \( \mathbb{R} \ni t \mapsto v(t, z(\omega)) \) is a solution to
\[
\begin{cases}
  \frac{dv}{dt} = -\nu Av - B(v) - B(v, z) - B(z, v) - \alpha z + f, \\
  v(0) = v_0.
\end{cases} \tag{29}
\]
and
\[
v'_1(t) = \frac{dv(\cdot, z(\omega))}{dt} \tag{30}
\]
On the other hand, in view of our earlier existence uniqueness results, the fact that \( v \) takes a value in \( D(A) \) implies that \( v(t) \) is differentiable for almost every \( t \).
We have
\[
v'(t) = -\nu Av_1(t + s, z(\omega)(t + s)) - B(v_1(t + s, z(\omega)(t + s))
\]
\[
+ z(\omega)(t + s) + \alpha z(\omega)(t + s) + f
\]
\[
= -\nu Av_1(t, z(\omega)) - B(v_1(t, z(\omega)) + z(\omega)(t + s)) + \alpha z(\omega)(t + s) + f.
\]
On the other hand for \( v_2 \),
\[
\frac{dv_2(t, z(\vartheta_s \omega)(t))}{dt} = -\nu Av_2(t, z(\vartheta_s \omega)(t)) - B(v_2(t, z(\vartheta_s \omega)(t))
\]
\[
+ z(\vartheta_s \omega)(t) + \alpha z(\vartheta_s \omega)(t) + f.
\]
Therefore, \( v_1, v_2 \) solve respectively
\[
\begin{cases}
  v'_1(t) = -\nu Av'_1 - B(v'_1(t) + z(\omega)(t + s)) + \alpha z(\omega)(t + s) + f, \\
  v_1(0) = v(z(\omega))(s)(x - z(\omega)(0)),
\end{cases}
\]
\[
\begin{cases}
  v'_2(t) = -\nu Av'_2 - B(v'_2(t) + z(\vartheta_s \omega)(t)) + \alpha z(\vartheta_s \omega)(t) + f, \\
  v_1(0) = v(z(\omega))(s)(x - z(\omega)(0)).
\end{cases}
\]
By cocycle property of \( z \), \( z(\vartheta_s \omega)(t) = z(\omega)(t + s) \) for \( t \in \mathbb{R} \).

Therefore, \( v_1, v_2 \) are solutions to (29) with the same initial data \( v(s, z(\omega)(s))(x - z(\omega)(0)) \) at \( t = 0 \). Then it follows from the uniqueness of solution to (29) that \( v_1 = v_2 \), \( t \in \mathbb{R} \).
3.4. Existence of random attractors. This subsection establishes the existence of random attractors. The main arguments follow from classical lines of proving global attractors by finding compact absorbing sets. However, as pointed out in the paper [14], the analysis of Navier-Stokes equations with additive noise in our case requires some non-trivial consideration. In particular, a critical question arises when analyzing the estimate \( \frac{d}{dt} |v(t)|^2 \), the usual estimates for the nonlinear term \( b(v(t), z(t), v(t)) \) yield a term \( |v(t)|^2 |z(t)|^4 \), so we were not able to deduce any bound in \( H \) for \( |v(t)|^2 \) under the classical lines (see for instance section 6 in [6]). Nevertheless, in light of the method developed in [14], via the usual change of variable and by writing the noise and the Ornstein-Uhlenbeck process as an infinite sequence of 1D processes, we are able to show there exist random attractors to our system 1 as well. In what follow, we will detail our proof. First we need a few Lemmas from our two companion papers.

Lemma 3.6. (Lemma 3.2, [16]) Suppose that there exists some \( \delta > 0 \) such that

\[
\sum_{l \geq 1} |\sigma_l| \lambda_l^{\beta \delta} < \infty.
\]

Then for all \( p \in (0, \beta) \),

\[
E|A^\delta L(t)|^p \leq C(\beta, p) \left( \sum_{l \geq 1} |\sigma_l| \lambda_l^{\beta \delta} \right)^{\frac{p}{\beta}} t^{\frac{\beta}{p}} < \infty.
\]

(31)

Lemma 3.7. (Lemma 1.7, [15]) We have

\[
\sup_{-1 \leq t \leq 0} |Az(t)|^2 < \infty.
\]

Now, using Lemma 3.7 and Lemma 3.6 applied with \( \delta = \frac{1}{2} \) we find that the process \( z \) is \( \text{c` ad` ag} \) in \( V \) and

\[
\sup_{-1 \leq t \leq 0} (|z(t)|^2 + |z(t)|^2 + |Az(t)|^2) < \infty \quad \mathbb{P} \text{ a.s.}.
\]

(32)

Using equation (4.12) in [28], one can now choose \( \alpha > 0 \) such that

\[
4\eta m E|z_1(0)| \leq \frac{\lambda_1}{4},
\]

(33)

where \( \lambda_1 \) is the first eigenvalue of \( A \), since \( E|z_1(0)|^p \to 0 \) as \( \alpha \to \infty \).

From (33) and the Ergodic Theorem we obtain

\[
\lim_{t_0 \to -\infty} \frac{1}{1 - t_0} \int_{t_0}^{-1} 4\eta \sum_{l=1}^{m} |z_l(s)| ds = 4\eta m E|z_1(0)| < \frac{\lambda_1}{4}.
\]

Put \( \gamma(t) = -\frac{\lambda_1}{4} + 4\eta \sum_{l=1}^{m} |z_l(t)| \), we get

\[
\lim_{t_0 \to -\infty} \frac{1}{1 - t_0} \int_{t_0}^{-1} \gamma(s) ds < -\frac{\lambda_1}{4}.
\]

(34)

From this fact and by stationarity of \( z_1 \) we finally obtain

\[
\lim_{t_0 \to -\infty} e^{\int_{t_0}^{-1} \gamma(s) ds} = 0 \quad \mathbb{P} - \text{a.s.},
\]

(35)

\[
\sup_{t_0 < -1} e^{\int_{t_0}^{-1} \gamma(s) ds} |z(t_0)|^2 < \infty, \quad \mathbb{P} - \text{a.s.}
\]

(36)
for all $1 \leq j, l \leq m$. Indeed, note for instance that for $t < 0$,
\[
\frac{z_l(t)}{t} = \frac{z_l(0)}{t} - \frac{1}{t} (\alpha + A_l) \int_t^0 z_l(s) ds + \frac{L_l(t)}{t},
\]
whence \(\lim_{t \to -\infty} \frac{z_l(t)}{t} = 0\) \(\mathbb{P}\)-a.s., which implies (35) and (36). Consider the abstract SNSE
\[
du + [Au + B(u) + Cu] dt = f dt + GdL(t)
\]
and the Ornstein-Uhlenbeck equation
\[
dz + (\dot{A} + \alpha I) z dt = GdL(t),
\]
where \(L(t) = \sum_{l=1}^m c_l L_l(t)\). We now use the change of variable \(v(t) = u(t) - z(t)\). Then, by subtracting the Ornstein-Uhlenbeck equation from the abstract SNSE, we find that \(v\) satisfies the equation
\[
\frac{dv(t)}{dt} = -\nu Av(t) - C v(t) - B(u, u) + f + \alpha z(t).
\]

Recall the Poincare inequalities
\[
|u|_V^2 \geq \lambda_1 |u|^2, \quad \forall \ u \in V, \quad (39) \\
|Au|^2 \geq \lambda_1 |u|^2, \quad \forall \ u \in D(A). \quad (40)
\]

**Lemma 3.8.** Suppose that \(v\) is a solution to problem (29) on the time interval \([t_0, \infty)\) with \(z \in L^1_{\text{loc}}(\mathbb{R}, L^4(S^2)) \cap L^2_{\text{loc}}(\mathbb{R}, V')\) and \(t_0 \geq 0\). Then, for any \(t \geq \tau \geq t_0\), one has
\[
|v(t)|^2 \leq |v(\tau)|^2 e^{\int_{\tau}^t \gamma(s) ds} + \int_{\tau}^t e^{\int_{s}^t \gamma(s') ds'} 2p(s) ds,
\]
where
\[
p(t) = c |f|^2 + c\alpha |z|^2 + \delta |z|^2 \sum_{l=1}^m |z_l(t)|, \quad (42)
\]
\[
\gamma(t) = -\frac{\lambda_1}{2} + 4\delta \sum_{l=1}^m |z_l(t)| \quad (43)
\]
for all \(t_0 \leq \tau \leq t\) and \(c\) depends only on \(\lambda_1\).

**Proof.** The proof will be provided shortly. \(\square\)

Let \(H, A : D(A) \subset H \to H, V = D(A^{1/2}) = D(\dot{A}^{1/2})\) and \(B(u, v) : V \times V \to V', Cu\) be spaces and operators introduced in the previous section. Suppose that there exists a constant \(c_B > 0\) such that
\[
\langle B(u, v), w \rangle = |b(u, v, w)| \leq c_B |u|^{1/2} |u|^1 |v|^{1/2} |v|^1 |w|, \quad \forall \ u, v, z \in V, \quad (44)
\]
\[
\langle B(u, v), v \rangle \leq c_B |u|^{1/2} |Au|^{1/2} |v|^{1/2} |v| |z|
\]
for all $u \in D(A)$, $v \in V$ and $z \in H$. Moreover, let $f \in H$, $e_1, \ldots, e_m \in H$ be given, \{\sigma_l\} is a sequence of real numbers. Consider 1 again,

$$du(t) = [-Au(t) - B(u(t), u(t)) + Cu(t) + f]dt + \sum_{l=1}^{m} \sigma_l dL(t)e_l, \quad u(0) = u_0.$$ 

As in [16], assume that $e_l$ are the eigenfunctions of the Stoke operator $A$, $1 \leq l \leq m$, there exists $\delta > 0$ such that

$$\langle B(u, e_l), u \rangle \leq \delta |u|^2, \quad u \in V, l = 1, \ldots, m. \quad (45)$$

**Remark.** In bounded domain or in $\mathbb{S}^2$, one has

$$\langle B(u, e_l), u \rangle = \sum_{i,j=1}^{3} \int_{\mathbb{S}^2} u_i \frac{\partial(e_l)_{,j}}{\partial x_i} u_j dx. \quad (46)$$

In this case assumption (45) is satisfied when $e_l$ are Lipschitz continuous in $\mathbb{S}^2$. Put $L(t) = \sum_{l=1}^{m} e_l L_l(t)$.

### 3.4.1. Stochastic flow

Consider the abstract SNSE

$$du + [Au + B(u) + Cu]dt = f dt + GdL(t),$$

and the Ornstein-Uhlenbeck equation

$$dz + (\hat{A} + \alpha I)z dt = GdL(t).$$

From the discussion from the earlier sub-subsection, it is clear that $z(t)$ is a stationary ergodic solution with continuous trajectories taking values in $D(A)$. So we can transform the SNSE to a random PDE. The main advantage is that we can solve the equation $\omega$-wise due to the absence of the stochastic integral.

We now use the change of variable $v(t) = u(t) - z(t)$. Then, by subtracting the Ornstein-Uhlenbeck equation from the abstract SNSE, we find that $v$ satisfies the following equation

$$\frac{dv(t)}{dt} = -\nu Av(t) - Cv(t) - B(u, u) + f + \alpha z. \quad (47)$$

Now recall the following theorem from [16], namely,

**Theorem 3.9.** Assume that $\varphi_l \in D(A)$, $1 \leq l \leq m$, there exists $\eta > 0$ such that

$$\langle B(u, \varphi_l), u \rangle \leq \eta |u|^2, \quad u \in V, l = 1, \ldots, m \quad (48)$$

is satisfied. Then for $\mathbb{P}$-a.s. $\omega \in \Omega$, there hold

- For all $t_0 \in \mathbb{R}$ and all $v_0 \in H$, there exists a unique solution $v \in C([t_0, +\infty); H) \cap L^2_{\text{loc}}([t_0, +\infty); V)$ of equation (47) with initial value $v_0$.
- If $v_0 \in V$, then the solution belongs to $C([t_0, +\infty); V) \cap L^2_{\text{loc}}([t_0, +\infty); D(A))$.
- Hence, for every $\varepsilon > 0, v(t) \in C([t_0 + \varepsilon, +\infty); V) \cap L^2_{\text{loc}}([t_0 + \varepsilon, +\infty); D(A))$.
- Denoting the solution by $v(t, t_0; \omega, v_0)$, then the map $v_0 \mapsto v(t, t_0; \omega, v_0)$ is continuous for all $t \geq t_0$, $v_0 \in H$.

Now let us define the transition semigroup for the flow $\varphi$ as

$$P_tf(x) = \mathbb{E}f(\varphi(t, x)).$$

**Corollary 6.** It follow from Theorem 3.9 the transition semigroup for the Markov RDS $\varphi$ has Feller property in $H$. That is, $P_t : C_0(H) \to C_0(H)$ holds for $t \geq 0^2$. 

---

2 The case of $t = 0$ is trivial since $P_0f = f$.
Having the map \( v_0 \rightarrow v(t, t_0; \omega, v_0) \), where \( v(t, t_0; \omega, v_0) \) is the solution to (47) with \( v(t_0) = v_0 \), we can now define a stochastic flow \( \varphi(t, \omega) \) in \( H \) by setting
\[
\varphi(t, \omega)u_0 = v(t, 0; \omega, u_0 - z_\alpha(\omega)(0)) + z_\alpha(\omega)(t).
\]

3.4.2. Absorbing in \( H \) at time \( t = -1 \). In what follows, assume \( \omega \in \Omega \) is fixed; the results will hold \( \mathbb{P} \) a.s.. Suppose \( t_0 \leq -1 \) and \( u_0 \in H \) be given, and let \( v \) be the solution of equation (47) for \( t \geq t_0 \), with \( v(t_0) = u_0 - z(t_0, \omega) \) (which was denoted above by \( v(t, 0; \omega, u_0 - z(0, \omega)) \)). Using the well known identity \( \frac{1}{2} \partial_t |v(t)|^2 = (v(t), v(t)) \), and the assumption \( \langle B(u, v), v \rangle = 0 \) and the antisymmetric term \( \langle Cv, v \rangle = 0 \) we have
\[
\frac{1}{2} \frac{d}{dt} |v|^2 = -\nu(Av, v) - \langle B(u, z), u \rangle + (\alpha z, v) + \langle f, v \rangle \quad \text{(49)}
\]
\[
\leq -\nu|v|_V^2 - \langle B(u, z), u \rangle + \alpha|z||v| + |f||v|. \quad \text{(50)}
\]

By the definition of \( z \) and assumptions (48),
\[
|\langle B(u, z), u \rangle| = \left| \sum_{l=1}^{m} (B(u, e_l), u)e_l \right| \leq \delta |u|^2 \sum_{l=1}^{m} |z_l| \leq 2\delta |u|^2 \sum_{l=1}^{m} |z_l| + 2\delta |z|^2 \sum_{l=1}^{m} |z_l|.
\]

and the inequalities
\[
(\alpha z, v) = c\alpha|z|^2 + c'|v|^2;
\]
\[
\langle f, v \rangle \leq c|f|^2 + c'|v|^2.
\]

For simplicity we take \( \nu = 1 \). Then via Young’s inequality, one can show that there exists \( c, c' > 0 \) depending only on \( \lambda_1 \) such that
\[
\frac{1}{2} \frac{d}{dt} |v|^2 + \frac{1}{2} |v|_V^2 \leq \left( -\frac{\lambda_1}{4} + 2\delta \sum_{l=1}^{m} |z_l| + 2c' \right) |v|^2 + c|f|^2 + c\alpha|z|^2 + 2c|z|^2 \sum_{l=1}^{m} |z_l| + 2\delta |z|^2 \sum_{l=1}^{m} |z_l|.
\]

Let \( \gamma(t) \), and \( p(t) \) be defined as in 3.8. Namely,
\[
p(t) = c|f|^2 + c\alpha|z|^2 + \delta|z|^2 \sum_{k=1}^{m} |z_k(t)|,
\]
\[
\gamma(t) = -\frac{\lambda_1}{2} + 4\delta \sum_{l=1}^{m} |z_l(s)|,
\]
we have
\[
\frac{1}{2} \frac{d}{dt} |v|^2 + \frac{1}{2} |v|_V^2 \leq \frac{1}{2} \gamma(t)|v|^2 + p(t), \quad \text{(51)}
\]
\[
\frac{d}{dt} |v(t)|^2 \leq \gamma(t)|v(t)|^2 + 2p(t).
\]

Invoking Gronwall Lemma over the interval \([a, \infty)\), we have (41).
Lemma 3.10. There exists a random radius $r_1(\omega) > 0$ such that for all $\rho > 0$ there exists (a deterministic) $\bar{t} \leq -1$ such that the following holds $\mathbb{P}$-a.s. For all $t_0 \leq \bar{t}$ and for all $u_0 \in H$ with $|u_0| \leq \rho$, the solution $v(t, t_0; \omega, u_0 - z(s))$ of equation (29) over $[t_0, \infty]$ with $v(t_0) = u_0 - z_\alpha(t_0)$ satisfies the inequality

$$|v(-1, t_0; \omega, u_0 - z(t_0, \omega))|^2 \leq r_1^2(\omega).$$

Proof. Apply Lemma 3.8 with $t_0$, $\omega$. The claim then follows from (41). We remark that $t_0$ depends on $\omega$. 

Put

$$\nu_1^2(\omega) = 2 + 2 \sup_{t_0 \leq -1} e^{\int_{0}^{-t_0} \gamma(\xi) d\xi} |z(t_0)|^2 + \int_{-\infty}^{-1} e^{\int_{s}^{-t_0} \gamma(\xi) d\xi} 2p(s) ds$$

which is finite $\mathbb{P}$ a.s. due to the stationarity of $z_t$ (namely, equations (1.20) and (1.21) in [15]).

So, given $\rho > 0$, choose $\bar{t}$ such that

$$e^{\int_{0}^{-\bar{t}} \gamma(\xi) d\xi} \rho^2 \leq 1$$

for all $t_0 \leq \bar{t}$. The claim then follows from (41). We remark that $t_0$ depends on $\omega$. 

Taking $t \in [-1, 0]$ and $\tau = -1$ in (41) we have

$$|v(t)|^2 \leq |v(-1)|^2 e^{\int_{-1}^{t} \gamma(\xi) d\xi} + \int_{-1}^{t} e^{\int_{s}^{-1} \gamma(\xi) d\xi} 2p(s) ds.$$ 

Let us now come back to (51):

$$\frac{d}{dt}|v|^2 + |v|^2 \leq \gamma(t)|v|^2 + 2p(t).$$

Integrate over $[-1, 0]$,

$$|v(0)|^2 - |v(-1)|^2 + \int_{-1}^{0} |v(s)|^2 \gamma ds \leq \left( \int_{-1}^{0} \gamma(\xi) d\xi \right) \left( \sup_{-1 \leq t \leq 0} |v(t)|^2 \right) + \int_{-1}^{0} 2p(s) ds.$$ 

Therefore,

$$\int_{-1}^{0} |v(s)|^2 \gamma ds \leq |v(-1)|^2 + \left( \int_{-1}^{0} \gamma(\xi) d\xi \right) \left( \sup_{-1 \leq t \leq 0} |v(t)|^2 \right) + \int_{-1}^{0} 2p(s) ds.$$ 

Therefore, from the above lemma we deduce

Lemma 3.11. There exist two random variables $c_1(\omega)$ and $c_2(\omega)$ depending on $\lambda_1, e_1, \ldots, e_m$ and $|f|$ such that for all $\rho > 0$ there exists $\bar{t}(\omega) \leq -1$ such that the following holds $\mathbb{P}$-a.s. $\forall t_0 \leq \bar{t}$ and for all $u_0 \in H$ with $|u_0| \leq \rho$, the solution $v(t, \omega; t_0, u_0 - z(t_0, \omega))$ of equation (47) over $[t_0, \infty]$ with $v(t_0) = u_0 - z(t_0)$ satisfies

$$|v(t, \omega; t_0, u_0 - z(t_0, \omega))|^2 \leq c_1(\omega) \quad \forall \ t \in [-1, 0],$$

$$\int_{-1}^{0} |v(s, \omega; t_0, u_0 - z(t_0, \omega))|^2 ds \leq c_2(\omega).$$
Proof. Put
\[ c_1(\omega) = e^{\int_{t}^{t+1} \gamma(\xi) d\xi} t_1(\omega) + \int_{t-1}^{t} e^{\int_{s}^{t+1} \gamma(\xi) d\xi} p(s) ds, \]
\[ c_2(\omega) = r_2(\omega) \left( 1 + \int_{t-1}^{t} \gamma(\xi) d\xi \right) + \int_{t-1}^{t} 2p(s) ds, \]
with \( r_1(\omega) \) as in the previous lemma. Then, given \( \rho > 0 \), it suffices to choose \( t(\omega) \) as in the proof of that previous lemma. \( \square \)

3.4.3. Absorption in \( V \) at \( t = 0 \). From (29) we have (by multiplying \( Av \) left and right and noting \( (v_t, Av) = \frac{1}{2} \int_{\mathbb{R}} |v|^2 \), using inequality (44), and use the Young’s inequality with \( ab = \sqrt{a^2 + b^2} \), \( p = 2 \)

\[ ab \leq \frac{a^2}{2} + \frac{b^2}{2}, \]

With the choice of \( e = \frac{\nu}{2} \), one has that
\[ \langle f, Av \rangle \leq \frac{2}{\nu} |f|^2 + \frac{\nu}{8} |Av|^2, \]
\[ \langle \alpha z, Av \rangle \leq \frac{2}{\nu} |\alpha z|^2 + \frac{\nu}{8} |Av|^2, \]
\[ c_B |u|^{1/2} |Au|^{1/2} |u| |Av| \leq 2
\[ \int_{t}^{t+1} \gamma(\xi) d\xi \int_{t-1}^{t} 2p(s) ds, \]

\[ \frac{1}{2} \frac{d}{dt} |v|^2_V = -\nu |Av|^2 - \langle B(u, u), Av \rangle + \langle f, Av \rangle + \langle \alpha z, Av \rangle \]
\[ \leq -\nu |Av|^2 + c_B |u|^{1/2} |Au|^{1/2} |u| |Av| + |f||Av| + |\alpha z||Av| \]
\[ \leq -\frac{5\nu}{8} |Av|^2 + 2c_B^2 |u||Av| |u|^2_V + \frac{2}{\nu} (|f|^2 + |\alpha z|^2) \]

With \( q(t) = \frac{2}{\nu} (|f|^2 + |\alpha z|^2) \) and noticing \( |Av| \leq |Av| + |Az| \),
\[ \leq -\frac{5\nu}{8} |Av|^2 + 2c_B^2 |u||Av||u|^2_V + 2c_B^2 |u||Az||u|^2_V + q(t) \]

Apply Young inequality with \( e = \nu/2 \) for the nonlinear term,
\[ \leq -\frac{5\nu}{8} |Av|^2 + \frac{\nu}{8} |Av|^2 + 4\nu c_B^4 |u|^2 |u|^2_V |u|^2_V + 2c_B^2 |u||Az||u|^2_V + q(t) \]
\[ \leq -\frac{\nu}{8} |Av|^2 + 4\nu c_B^4 |u|^2 |u|^2_V |u|^2_V + 2c_B^2 |u||Az||u|^2_V + q(t) \]
\[ \leq -\nu |Av|^2 + 8\nu c_B^4 |u|^2 |u|^2_V |u|^2_V + 8\nu c_B^4 |u|^2 |u|^2_V |z|^2_V + 2c_B^2 |u||Az||u|^2_V + q(t). \]

Temporarily disregarding the \( |Av|^2 \) term, we have
\[ \frac{d}{dt} |v|^2_V \leq 16\nu c_B^4 |u|^2 |u|^2_V |v|^2_V + 16\nu c_B^4 |u|^2 |u|^2_V |z|^2_V + 4\nu c_B^4 |u||Az||u|^2_V + 2q(t). \]

Invoking Gronwall Lemma, we get for \( s \in [-1, 0], \)
\[ |v(0)|^2_V \leq e^{\int_{-1}^{0} 16\nu c_B^4 |u|^2 |u|^2_V |v|^2_V + 16\nu c_B^4 |u|^2 |u|^2_V |z|^2_V + 4\nu c_B^4 |u||Az||u|^2_V + 2q(t) \} ds \]
\[ \times \left( |v(s)|^2 + \int_{s}^{0} (16\nu c_B^4 |u|^2 |u|^2_V |z|^2_V + 4\nu c_B^4 |u||Az||u|^2_V + 2q(t)) d\sigma \right). \]
Lemma 3.12. Let $B_\rho$ be a ball with radius $\rho$. For every $\varepsilon > 0$, there exists a random radius $r_2(\omega, \varepsilon) > 0$, depending only on $\lambda_1, e_1, \cdots, e_m$ and $f$ such that for every $\rho$ there exists a deterministic $\bar{t} < -1$ such that

$$\mathbb{P} \left\{ \omega : \sup_{t \in [-\bar{t}, \bar{t} + N]} |\varphi(t, \theta_{-t}\omega)B_\rho|^2 > r_2^2 \right\} < \varepsilon, \quad \forall N \geq 1.$$ 

For all $t_0 \leq \bar{t}$ and for all $u_0 \in H$ with $|u_0| \leq \rho$, let $v = v(t, t_0, \omega; u_0 - z(t_0, \omega))$ be the solution of equation (47) over $[t_0, \infty]$ with $v(t_0) = u_0 - z_\alpha(t_0)$. Put $u(t, t_0, \omega; u_0) = z(t, \omega) + v(t, t_0, \omega; u_0 - z(t_0, \omega))$. Then

$$|u(0, \omega; t_0, u_0 - z_\alpha(t_0, \omega))|^2 \leq r_2^2(\omega, \varepsilon).$$

Proof. In view of (54), we need to estimate the term.

$$\int_{-1}^{0} |u(s)|^2 |u(s)|^2 dv.$$ 

Now using the fact $u = v + z$ and $|u|^2 \leq 2|v|^2 + 2|z|^2 = 2(|v|^2 + |z|^2)$, then the two terms can be estimated as follows.

$$\int_{-1}^{0} |u(s)|^2 |u(s)|^2 ds$$

$$\leq \sup_{-1 \leq t \leq 0} |u(t)|^2 \int_{-1}^{0} |v^2| ds$$

$$\leq \sup_{-1 \leq t \leq 0} 2(|v(t)|^2 + |z(t)|^2) \left( \int_{-1}^{0} 2|v|^2 + 2|z|^2 \right)$$

$$\leq 2(c_1(\omega) + \sup_{-1 \leq t \leq 0} |z(t)|^2)2(c_2(\omega) + \int_{-1}^{0} |z(s)|^2 ds)$$

$$= 2c_3(\omega)2c_4(\omega),$$

$$\int_{-1}^{0} |u(s)||u(s)|^2 ds$$

$$\leq \sup_{-1 \leq t \leq 0} |u(t)| \int_{-1}^{0} |u(s)|^2 ds$$

$$\leq (c_1(\omega) + \sup_{-1 \leq t \leq 0} |z(t)|)2c_4(\omega).$$

Putting

$$c_3(\omega) = c_1(\omega) + \sup_{-1 \leq t \leq 0} |z(t)|^2,$$

$$c_4(\omega) = c_2(\omega) + \int_{-1}^{0} |z(t)|^2 ds,$$

$$c_5(\omega) = c_1(\omega)^{1/2} + \sup_{-1 \leq t \leq 0} |z(t)|.$$
Then, (54) becomes
\[
|u(0)|_V^2 
\leq 2|z(0)|_V^2 + 2|v(0)|_V^2 
\leq 2|z(0)|_V^2 + 2 \left[ c_2(\omega) + 64c_4c_3(\omega)c_4(\omega) \sup_{-1 \leq t \leq 0} |z(t)| + 8c_2c_3(\omega)c_4(\omega) \sup_{-1 \leq t \leq 0} |Az(t)| + \int_{-1}^{0} 2q(s)ds \right]
\times e^{64c_4c_3(\omega)c_4(\omega)} =: r_2^2(\omega).
\]

Hence there exists a random ball in $V$ which absorbs the bounded sets of $H$. Since $V$ is compactly embedded in $H$, there exists a compact set $K \subset H$ such that, for all bounded set $B \subset H$ there exists $\bar{t} \leq -1$ such that $\varphi B \subset K$ $\mathbb{P}$ almost surely.

3.5. **Existence of Feller Markov invariant measures.** In this subsection, we prove that the existence of random attractors implies the existence of Feller Markov invariant measures.

For a random set $A$, we define the $\Omega$-limit set to be
\[
\Omega_A(\omega) = \Omega(A, \omega) = \bigcap_{T \geq 0} \bigcup_{t \geq T} \varphi(t, \vartheta - t\omega)A(\vartheta - t\omega)
\]

Recall Corollary 2, that is, **Corollary 2.** Under the conditions of Proposition 2.16, suppose in addition that $\varphi$ is an RDS whose individual trajectories form a Markov family, and such that $\mathcal{F}^+$ and $\mathcal{F}^-$ are independent. Then there exists an invariant measure $\rho$ for the associated Markov semigroup. Furthermore, the limit
\[
\mu_\omega = \lim_{t \to \infty} \varphi(t, \vartheta - t\omega)\rho
\]
exists $\mathbb{P}$ a.s., $\rho = \int \mu_\omega d\mathbb{P}(\omega) = \mathbb{E}(\mu, \cdot)$, and $\mu$ is a Markov measure.

**Theorem 3.13.** Let $\varphi$ be a continuous in space, but càdlàg in time RDS on $X$. Assume the existence of a compact random set $K$ absorbing every deterministic bounded set $B \subset H$. The stochastic flow associated with the SNSE with additive Lévy noise (1) has a compact random attractor given by
\[
A(\omega) = \bigcup_{B \subset H \text{ bounded}} \Omega_B(\omega), \quad \omega \in \Omega.
\]

Moreover, the Markov semigroup induced by the flow of $H$ has an invariant measure $\rho$ by Corollary 2. The associated flow-invariant Markov measure $\mu$ on $H \times \Omega$ has the property that its disintegration $\omega \mapsto \mu_\omega$ is supported by the attractor.

**Proof.** Recall that, in the language of the stochastic flow associated with our SNSE (1),
\[
u(0, \omega; t_0, u_0) = \varphi(t_n, \vartheta - t_0\omega)u_0 = v(0, \omega; t_0, u_0 - z(s)) + z(t).
\]
Then by Lemma 3.12, there exists a random ball in $V$ which absorbs the bounded sets of $H$. Since $V$ is compactly embedded in $H$, there exists a compact set $K \subset H$ such that, for any bounded set $B \subset H$ there exists $t \leq -1$ such that $\varphi B \subset K$ $\mathbb{P}$ a.s.. Defining $K(\omega) := \{u \in H : |u| \leq r_2(\omega)\}$, we have proved the existence of
a compact absorbing set. Then by Theorem 2.13, there exists a random attractor to (1). The existence of an invariant Markov measure is a direct consequence of Corollary 4, provided we can show that the one-point motions associated with the flow \( \varphi(t, \omega) \) define a family of Markov processes. The proof of this is analogous to the proof of the Markov property of solutions to the (1) in [16]. For the convenience of the reader, we repeat the proof here. Let \( \varphi \) be the proof of the Markov property of solutions to the (1) in [16]. For the convenience of the reader, we repeat the proof here. Let \( \varphi \) and 1

\[
\varphi(t + s, \omega)x = \varphi(s, \omega)\varphi(t, \omega)x
\]

over \([t, \infty]\) with \( \mathcal{F}_t \)-measurable initial condition \( \varphi(t, \delta) = \delta \). It suffices to prove

\[
\mathbb{E}[f(\varphi(t + s)x) | \mathcal{F}_t] = P_t(f)(\varphi(t)x),
\]

for all \( 0 \leq s \leq t \) and all bounded continuous functions \( f \) over \( H \), which implies that \( \varphi(t + s)x \) is a Markov process with transition semigroup \( P_t \). By uniqueness, the following holds

\[
\varphi(t + s, \omega)x = \varphi(s, \omega)\varphi(t, \omega)x
\]

over \([t, \infty]\) with \( \mathcal{F}_t \)-measurable initial condition \( \varphi(t, \delta) = \delta \). It suffices to prove

\[
\mathbb{E}[f(\varphi(t + s)x) | \mathcal{F}_t] = P_t(f)(\delta)
\]

for every \( H \) integrable, \( \mathcal{F}_t \)-measurable random variable \( \delta \).

Note, (55) not only holds for every \( f \in C_b(H) \), but also holds for \( \varphi = 1_{\Gamma} \), where \( \Gamma \) is an arbitrary Borel set of \( H \) and consequently for all \( \varphi \in B_b(H) \). Without loss of generality, we assume \( \varphi \in C_b(H) \). We know that, if \( \delta = \delta_i \), \( \mathbb{P} \) a.s., then the random variable, \( \varphi(t, t + s)\delta_i \) is independent of \( \mathcal{F}_t \), since \( \varphi(t, t + s)\delta_i \) is \( \mathcal{F}_{t, t+s} \) measurable. Hence,

\[
\mathbb{E}(f(\varphi(t, t + s)\delta_i) | \mathcal{F}_t) = \mathbb{E}(f(\varphi(t, t + s)\delta_i)) = P_t\{f(\delta_i)\} = P_t(f(\delta)), \quad \mathbb{P} \text{ a.s.}
\]

Since the coefficient of the equation for \( \varphi(t, t + s) \) are independent, one can see that the \( H \) random variable \( \varphi_{t, t+s} \) and \( \varphi_{x,x} \) have the same law. If \( \delta \) has the form

\[
\delta = \sum_{i=1}^{N} \delta_i 1_{\Gamma_i},
\]

where \( \delta_i \in H \) and \( \Gamma^{(i)} \subset \mathcal{F}_t \) is a partition of \( \Omega \), \( 1_{\Gamma_i} \) are elements of \( H \). Then

\[
\varphi(t, t + s)\delta_i = \sum_{i=1}^{N} \varphi(t, t + s, \delta_i) 1_{\Gamma_i}, \quad \mathbb{P}, \text{ a.s.}
\]

Hence,

\[
\mathbb{E}(f(\varphi(t, t + s)\delta) | \mathcal{F}_t) = \sum_{i=1}^{N} \mathbb{E}(f(\varphi(t, t + s)\delta_i) 1_{\Gamma_i}) | \mathcal{F}_t) \quad \mathbb{P} \text{ a.s.}
\]

Taking into account the random variable \( u(t, t + s)\delta_i \) independent to \( \mathcal{F}_t \) and \( 1_{\Gamma_i} \) are \( \mathcal{F}_t \) measurable, \( i = 1, \cdots, l \), one deduces that

\[
\mathbb{E}[f(\varphi(t, t + s, \delta)) | \mathcal{F}_t] = \sum_{i=1}^{N} P_t f(\delta_i) 1_{\Gamma_i} = P_t f(\delta), \quad \mathbb{P} \text{ a.s.}
\]
and so (55) is proved. For a general \( \delta \) there exists a sequence of \( \delta_n \) for which (55) converges to \( \delta \) in \( L^2(\Omega; H) \) a.s., that is,
\[
E|\delta - \delta_n|^2 \to 0.
\]

By continuity of \( f \) one can pass in the identity (55), with \( \delta \) replaced with \( \delta_n \), to the limit and (55) holds if \( E|\delta|^2 < \infty \). So \( \varphi(t, \omega) \) defines a family of Markov processes.

The proof of existence of a Markov measure is completed.

**Theorem.** Let \( \varphi \) be a continuous in space, but càdlàg in time RDS on \( X \). Assume the existence of a compact random set \( K \) absorbing every deterministic bounded set \( B \subseteq H \). Then there exists a random attractor \( A \) given by
\[
A(\omega) = \bigcup_{B \subseteq X, B \text{ bounded}} \Omega_B(\omega), \quad \omega \in \Omega.
\]

**Remark.** Although the same results hold in the \( \beta \)-stable Lévy case as in the Gaussian case (see [14]), there is some difference between dealing with Brownian motion and Lévy motions. First, we need to consider càdlàg function in the Skorohod metric, which are different from the continuous case in the metric under the compact-open topology. Second, one has to consider solutions in the sense of Carathéodory and the right-hand derivatives.

Let \( u(t, x) \) be the unique solution to problem (1). Let us recall from [16] that such a unique solution exists for each \( x \in H \). Let us define the transition operator \( P_t \) by a standard formula. For \( f \in C_b(H) \), put
\[
(P_tf)(x) = E f(\varphi(t, x)), \quad t \geq 0, x \in X.
\]

In view of Proposition 2.17, \( (P_t, t \geq 0) \) is a family of Feller operators, i.e. \( P_t : C_b(H) \to C_b(H) \) and, for any \( f \in C_b(H) \) and \( x \in H \), \( P_tf(x) \to f(x) \) as \( t \downarrow 0 \). Moreover, following the identical lines of the proof of Theorem 3.13 in the last subsection, one can prove that \( \varphi \) is a Markov RDS. Invoking corollary 4, we deduce the existence of Feller invariant measure for our stochastic Navier-Stokes equations (1).

**Corollary 7.** There exists a Feller Markov Invariant Measure for the SNSE (1)

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