A CLASS OF INVERSE CURVATURE FLOWS FOR STAR-SHAPED
HYPERSURFACES EVOLVING IN A CONE

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Abstract. Given a smooth convex cone in the Euclidean $(n+1)$-space ($n \geq 2$), we consider strictly mean convex hypersurfaces with boundary which are star-shaped with respect to the center of the cone and which meet the cone perpendicularly. If those hypersurfaces inside the cone evolve by a class of inverse curvature flows, then, by using the convexity of the cone in the derivation of the gradient and H"older estimates, we can prove that this evolution exists for all the time and the evolving hypersurfaces converge smoothly to a piece of a round sphere as time tends to infinity.

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1. Introduction

Recently, Chen, Mao, Tu and Wu\textsuperscript{3} considered the evolution of a one-parameter family of closed, star-shaped and strictly mean convex hypersurfaces $M_t^n$, given by $X(\cdot, t) : S^n \times [0, T) \to \mathbb{R}^{n+1}$ with some $T < \infty$, under the flow

$$\frac{\partial}{\partial t} X = \frac{1}{|X|^\alpha H(X)^\nu},$$

where $\nu$ is the unit outward normal vector of $M_t^n$, $H$ is the mean curvature of $M_t^n$, and $|X|$ is the distance from the point $X(x, t)$ to the origin of $\mathbb{R}^{n+1}$. For $\alpha \geq 0$, they showed the long-time existence and the asymptotical behavior of the flow (1.1). Clearly, when $\alpha = 0$, the flow (1.1) degenerates into the classical inverse mean curvature flow (IMCF for short), and therefore Gerhardt’s or Urbas’s classical result for the IMCF in $\mathbb{R}^{n+1}$ (see [7, 17]) is covered by the main conclusion of [3] as a special case.

As we know, the classical IMCF is scale invariant. However, generally, the flow (1.1) is non-scale-invariant except the case $\alpha = 0$. The meaning of studying the non-scale-invariant version of inverse curvature flows (ICFs for short) have been revealed clearly by Gerhardt\textsuperscript{9}, where he investigated the non-scale-invariant version of the classical ICF considered by himself in [7]. This improvement from the scale invariant case to the non-scale-invariant case permits that Gerhardt’s main conclusion in [9] covers some interesting conclusions in [12, 15] for the inverse Gauss curvature flow (IGCF for short) and the power of the IGCF. Based on this reason, it also should be interesting to investigate properties of the non-scale-invariant flow (1.1) in different settings – for this purpose, please see the series work [3, 4] of Prof. Jing Mao and his collaborators. The flow (1.1) is an initial value problem of second-order parabolic PDEs. Could we consider the case of boundary value problems? This motivation forces us to consider the evolution of hypersurfaces with boundary under the ICFs considered in [3].

Marquardt\textsuperscript{14} considered the classical IMCF with a Neumann boundary condition (NBC for short), where the embedded flowing hypersurfaces were supposed to be perpendicular to a...
smooth convex cone in \( \mathbb{R}^{n+1} \). He proved that the flow exists for all the time and after rescaling, the evolving hypersurfaces converge smoothly to a piece of a round sphere. Later, Lamber and Scheuer [11] extended this interesting conclusion to the situation that the hypersurfaces are perpendicular to the prescribed sphere. In 2017, Chen, Mao, Xiang and Xu [2] improved Marquardt’s main conclusion above to the case that the ambient space is the warped product \( I \times \lambda(r) N^n \), where \( I \subset \mathbb{R} \) is an unbounded interval of \( \mathbb{R} \), \( N^n \) is an \( n \)-dimensional Riemannian manifold with nonnegative Ricci curvature, and the warping function \( \lambda(r) \) satisfies some growth assumptions. Inspired by these works, it should be interesting to consider the flow (1.1) with a prescribed NBC. In fact, we can prove the following:

**Theorem 1.1.** Let \( \alpha > 0 \), \( M^n \subset \mathbb{S}^n \subset \mathbb{R}^{n+1} \), and \( \Sigma^n := \{ r \in \mathbb{R}^{n+1} \mid r > 0, x \in \partial M^n \} \) be the boundary of a smooth, convex cone that is centered at the origin and has outward unit normal \( \mu \). Let \( X_0 : M^n \to \mathbb{R}^{n+1} \) be such that \( M^n_0 := X_0(M^n) \) is a compact, strictly mean convex \( C^{2,\gamma} \)-hypersurface \((0 < \gamma < 1)\) which is star-shaped with respect to the center of the cone. Assume that

\[
M^n_0 = \text{graph}_M^mu_0
\]

is a graph over \( M^n \) for a positive map \( u_0 : M^n \to \mathbb{R} \) and

\[
\partial M^n_0 \subset \Sigma^n, \quad (\mu \circ X_0, \nu_0 \circ X_0)|_{\partial M^n} = 0,
\]

where \( \nu_0 \) is the unit normal to \( M^n_0 \). Then

(i) there exists a family of strictly mean convex hypersurfaces \( M^n_t \) given by the unique embedding

\[
X \in C^{2+\gamma,1+\frac{\gamma}{2}}(M^n \times [0, \infty), \mathbb{R}^{n+1}) \cap C^\infty(M^n \times (0, \infty), \mathbb{R}^{n+1})
\]

with \( X(\partial M^n, t) \subset \Sigma^n \) for \( t \geq 0 \), satisfying the following system

\[
\begin{aligned}
\frac{\partial}{\partial t} X &= \frac{1}{|X|^\alpha} H \nu \quad \text{in } M^n \times (0, \infty) \\
\langle \mu \circ X, \nu \circ X \rangle &= 0 \quad \text{on } \partial M^n \times (0, \infty) \\
X(\cdot, 0) &= M^n_0 \quad \text{in } M^n
\end{aligned}
\]

where \( H \) is the mean curvature of \( M^n_t := X(X^n, t) = X_t(M^n) \), \( \nu \) is the unit outward normal vector of \( M^n_t \) and \( |X| \) is the distance from the point \( X(x, t) \) to the origin. Moreover, the Hölder norm on the parabolic space \( M^n \times (0, \infty) \) is defined in the usual way (see, e.g., [3] Note 2.5.4).

(ii) the leaves \( M^n_t \) are graphs over \( M^n \), i.e.,

\[
M^n_t = \text{graph}_M^nu(\cdot, t).
\]

(iii) Moreover, the evolving hypersurfaces converge smoothly after rescaling to a piece of a round sphere of radius \( r_\infty \), where \( r_\infty \) satisfies

\[
\frac{1}{\sup_{M^n} u_0} \left( \mathcal{H}^n(M^n_0) \right)^{\frac{n}{n}} \leq r_\infty \leq \frac{1}{\inf_{M^n} u_0} \left( \mathcal{H}^n(M^n_0) \right)^{\frac{n}{n}},
\]

where \( \mathcal{H}^n(\cdot) \) stands for the \( n \)-dimensional Hausdorff measure of a prescribed \( n \)-manifold.

**Remark 1.1.** In order to avoid any potential confusion with the mean curvature \( H \), we use \( C^{m+2,\gamma, \frac{m+2+\gamma}{2}} \), not \( H^{m+2,\gamma, \frac{m+2+\gamma}{2}} \), used in [3] to represent the parabolic Hölder norm. It is easy to check that all the arguments in the sequel are still valid for the case \( \alpha = 0 \) except some minor changes should be made. For instance, if \( \alpha = 0 \), then (3.1) becomes \( \phi(x, t) = \frac{1}{n} t + c \). However, in this setting, one can also get the \( C^0 \) estimate as well. Clearly, when \( \alpha = 0 \), the
flow degenerates into the parabolic system with the vanishing NBC in \[14, \text{Theorem 1}], and correspondingly, our main conclusion here covers \[14, \text{Theorem 1}] as a special case.

2. The corresponding scalar equation

2.1. The geometry of graphic hypersurfaces.

For an \(n\)-dimensional Riemannian manifold \((M^n, g)\), the Riemann curvature \((3,1)\)-tensor \(Rm\) is defined by

\[ Rm(X,Y)Z = -\nabla_X\nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z. \]

Pick a local coordinate chart \(\{x^i\}_{i=1}^n\) of \(M\). The component of the \((3,1)\)-tensor \(Rm\) is defined by

\[ Rm\left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) \frac{\partial}{\partial x^k} = R^l_{ijk} \frac{\partial}{\partial x^l} \]

and \(R_{ijkl} := g_{lm} R^m_{ijk}\). Then, we have the standard commutation formulas (the Ricci identities):

\[ (\nabla_i \nabla_j - \nabla_j \nabla_i) \alpha_{k_1 \cdots k_r} = \sum_{l=1}^r R^m_{ijkl} \alpha_{k_1 \cdots k_{l-1} m k_{l+1} \cdots k_r}. \]

If furthermore \((M^n, g)\) is an immersed hypersurface in \(\mathbb{R}^{n+1}\) with \(R_{ijkl}\) the Riemannian curvature of \(M^n\). Let \(\nu\) be a given unit outward normal vector and \(h_{ij}\) be the second fundamental form of the hypersurface \(M\) with respect to \(\nu\), that is,

\[ h_{ij} = -\left\langle \frac{\partial^2 X}{\partial x^i \partial x^j}, \nu \right\rangle_{\mathbb{R}^{n+1}}. \]

Set \(X_{ij} = \partial_i \partial_j X - \Gamma^k_{ij} X_k\), where \(\Gamma^k_{ij}\) is the Christoffel symbol of the metric on \(M^n\). We need the following identities

(2.1) \[ X_{ij} = -h_{ij} \nu, \quad \text{Gauss formula} \]

(2.2) \[ \nu_i = h_{ij} X^j, \quad \text{Weingarten formula} \]

(2.3) \[ R_{ijkl} = h_{ik} h_{jl} - h_{il} h_{jk}, \quad \text{Gauss equation} \]

(2.4) \[ \nabla_k h_{ij} = \nabla_j h_{ik}, \quad \text{Codazzi equation}. \]

Then, using the Codazzi equation we get

\[ \nabla_i (\nabla_j h_{kl}) = \nabla_i (\nabla_k h_{lj}) = \nabla_i \nabla_k h_{lj}. \]

Then, using the Ricci identities we have

\[ \nabla_i \nabla_j h_{kl} = \nabla_k \nabla_i h_{lj} + R_{iklm} h^m_j + R_{ikjm} h^m_l. \]

Using the Codazzi equation again, it follows that

\[ \nabla_i \nabla_j h_{kl} = \nabla_k (\nabla_i h_{lj}) + R_{iklm} h^m_j + R_{ikjm} h^m_l \]

\[ = \nabla_k \nabla_i h_{lj} + R_{iklm} h^m_j + R_{ikjm} h^m_l. \]

Using the Gauss equation, we have

(2.5) \[ \nabla_i \nabla_j h_{kl} = \nabla_k \nabla_i h_{lj} + h^m_j (h_{il} h_{km} - h_{im} h_{kl}) + h^m_l (h_{ij} h_{km} - h_{im} h_{kj}). \]
2.2. The corresponding scalar equation.

In coordinates on the sphere $\mathbb{S}^n$, we equivalently formulate the problem by the corresponding scalar equation. Since the initial $C^{2,\gamma}$-hypersurface is star-sharped, there exists a scalar function $u_0 \in C^{2,\gamma}(M^n)$ such that $X_0 : M^n \to \mathbb{R}^{n+1}$ has the form $x \mapsto (x, u_0(x))$. The hypersurface $M^n_t$ given by the embedding

$$X(\cdot, t) : M^n \to \mathbb{R}^{n+1}$$

at time $t$ may be represented as a graph over $M^n \subset \mathbb{S}^n$, and then we can make ansatz

$$X(x, t) = (x, u(x, t))$$

for some function $u : M^n \times (0, T) \to \mathbb{R}$.

**Lemma 2.1.** Define $p := X(x, t)$ and assume that a point on $\mathbb{S}^n$ is described by local coordinates $\xi^1, \ldots, \xi^n$, that is, $x = x(\xi^1, \ldots, \xi^n)$. Let $\partial_i$ be the corresponding coordinate fields on $\mathbb{S}^n$ and $\sigma_{ij} = g_{S^n}(\partial_i, \partial_j)$ be the metric on $\mathbb{S}^n$. Let $u_i = D_i u$, $u_{ij} = D_j D_i u$, and $u_{ijk} = D_k D_j D_i u$ denote the covariant derivatives of $u$ with respect to the round metric $g_{S^n}$ and let $\nabla$ be the Levi-Civita connection of $M^n_t$ with respect to the metric $g$ induced from the standard metric of $\mathbb{R}^{n+1}$. Then, the following formulas hold:

(i) The tangential vector on $M_t$ is

$$X_i = \partial_i + u_i \partial_r$$

and the corresponding outward unit normal vector is given by

$$\nu = \frac{1}{v} \left( \partial_r - \frac{1}{u^2} u^i \partial_i \right),$$

where $u^i = \sigma^{ij} u_j$, and $v := \sqrt{1 + u^{-2} |Du|^2}$ with $Du$ the gradient of $u$.

(ii) The induced metric $g$ on $M_t$ has the form

$$g_{ij} = u^2 \sigma_{ij} + u_i u_j$$

and its inverse is given by

$$g^{ij} = \frac{1}{u^2} \left( \sigma^{ij} - \frac{u^i u^j}{u^2 v^2} \right).$$

(iii) The second fundamental form of $M_t$ is given by

$$h_{ij} = \frac{1}{v} \left( -u_{ij} + u \sigma_{ij} + \frac{2}{u} u_i u_j \right).$$

and

$$h^i_j = g^{ik} h_{kj} = \frac{1}{uv} \delta^i_j - \frac{1}{uv} \bar{\sigma}^{ik} \varphi_{kj}, \quad \bar{\sigma}^{ij} = \sigma^{ij} - \frac{\varphi^i \varphi^j}{v^2}.$$

Naturally, the mean curvature is given by

$$H = \sum_{i=1}^n h^i_i = \frac{1}{uv} \left( n - (\sigma^{ij} - \frac{\varphi^i \varphi^j}{v^2}) \varphi_{ij} \right),$$

where $\varphi = \log u$.

(iv) Let $p \in \Sigma^n$, $\hat{\mu}(p)$ be the normal to $\Sigma^n$ at $p$ and $\mu = \mu^i(x) \partial_i$ be the normal to $\partial M^n$ at $x$. Then

$$\langle \hat{\mu}(p), \nu(p) \rangle = 0 \iff \mu^i(x) u_i(x, t) = 0.$$
Proof. Since
\[ h_{ij} = -\langle \nabla_{ij} X, \nu \rangle = -\langle \nabla_{\partial_i \partial_j} + u_i \nabla_{\partial_j} \partial_r + u_j \nabla_{\partial_i} \partial_r + u_i u_j \nabla_{\partial_r} \partial_r, \nu \rangle, \]
these formulas can be verified by direct calculation. The details can also be found in [1]. \qed

Using techniques as in Ecker [5], (see also [7, 8, 14]), the problem (1.2) is degenerated into solving the following scalar equation with the corresponding initial data
\[
\begin{align*}
\frac{\partial u}{\partial t} &= \frac{v}{u^\alpha} H \quad \text{in } M^n \times (0, \infty) \\
\nabla_\mu u &= 0 \quad \text{on } \partial M^n \times (0, \infty) \\
u u(\cdot, 0) &= u_0 \quad \text{in } M^n.
\end{align*}
\]
By Lemma 2.1, define a new function \( \varphi(x,t) = \log u(x,t) \) and then the mean curvature can be rewritten as
\[
H = \sum_{i=1}^{n} h^i_i = \frac{e^{-\varphi}}{v} \left( n - \frac{\sigma^{ij} \varphi_j}{v^2} \right) \varphi_{ij}.
\]
Hence, the evolution equation in (2.6) can be rewritten as
\[
\frac{\partial \varphi}{\partial t} = e^{-\alpha \varphi} (1 + |D\varphi|^2) \frac{1}{n - \left( \sigma_{0}^{ij} - \frac{\varphi_{0}^{i} \varphi_{0}^{j}}{v^2} \right) \varphi_{0,ij}} := Q(\varphi, D\varphi, D^2 \varphi).
\]
In particular,
\[
n - \left( \sigma_{0}^{ij} - \frac{\varphi_{0}^{i} \varphi_{0}^{j}}{v^2} \right) \varphi_{0,ij}
\]
is positive on \( M^n \), since \( M_0 \) is strictly mean convex. Thus, the problem (1.2) is again reduced to solve the following scalar equation with the NBC
\[
\begin{align*}
\frac{\partial \varphi}{\partial t} &= Q(\varphi, D\varphi, D^2 \varphi) \quad \text{in } M^n \times (0, T) \\
\nabla_\mu \varphi &= 0 \quad \text{on } \partial M^n \times (0, T) \\
\varphi(\cdot, 0) &= \varphi_0 \quad \text{in } M^n,
\end{align*}
\]
where
\[
n - \left( \sigma_{0}^{ij} - \frac{\varphi_{0}^{i} \varphi_{0}^{j}}{v^2} \right) \varphi_{0,ij}
\]
is positive on \( M^n \). Clearly, for the initial surface \( M_0 \),
\[
\frac{\partial Q}{\partial \varphi_{ij}} \bigg|_{\varphi_0} = \frac{1}{u^{2+\alpha} H^2} (\sigma_{0}^{ij} - \frac{\varphi_{0}^{i} \varphi_{0}^{j}}{v^2})
\]
is positive on \( M^n \). Based on the above facts, as in [1, 8, 14], we can get the following short-time existence and uniqueness for the parabolic system (1.2).

**Lemma 2.2.** Let \( X_0(M^n) = M_0^n \) be as in Theorem 1.1. Then there exist some \( T > 0 \), a unique solution \( u \in C^{2+\gamma,1+\frac{\gamma}{2}}(M^n \times [0,T]) \cap C^\infty(M^n \times (0,T]) \), where \( \varphi(x,t) = \log u(x,t) \), to the parabolic system (2.7) with the matrix
\[
n - (\sigma^{ij} - \frac{\varphi^{i} \varphi^{j}}{v^2}) \varphi_{ij}
\]
positive on $M^n$. Thus there exists a unique map $\psi : M^n \times [0,T] \to M^n$ such that $\psi(\partial M^n, t) = \partial M^n$ and the map $\hat{X}$ defined by

$$\hat{X} : M^n \times [0, T) \to \mathbb{R}^{n+1} : (x, t) \mapsto X(\psi(x,t), t)$$

has the same regularity as stated in Theorem 1.1 and is the unique solution to the parabolic system (1.2).

Let $T^*$ be the maximal time such that there exists some $u \in C^{2+\gamma,1+\gamma}_\gamma(M^n \times [0,T^*)) \cap C_\infty(M^n \times (0, T^*))$ which solves (2.7). In the sequel, we shall prove a priori estimates for those admissible solutions on $[0,T]$ where $T < T^*$.

### 3. $C^0$, $\dot{\varphi}$ and Gradient Estimates

**Lemma 3.1 (C$^0$ estimate).** Let $\varphi$ be a solution of (2.7). Then for $\alpha > 0$, we have

$$c_1 \leq u(x, t) \Theta^{-1}(t, c) \leq c_2, \quad \forall x \in M^n, \ t \in [0,T],$$

for some positive constants $c_1, c_2$, where $\Theta(t, c) := \left\{ \frac{\alpha t}{n} + e^{\alpha c} \right\}^{\frac{1}{\alpha}}$ with

$$\inf_{M^n} \varphi(\cdot, 0) \leq c \leq \sup_{M^n} \varphi(\cdot, 0).$$

**Proof.** Let $\varphi(x, t) = \varphi(t)$ (independent of $x$) be the solution of (2.7) with $\varphi(0) = c$. In this case, the first equation in (2.7) reduces to an ODE

$$\frac{d}{dt} \varphi = e^{-\alpha \varphi} \frac{1}{n}.$$

Therefore,

$$(3.1) \quad \varphi(t) = \frac{1}{\alpha} \ln \left( \frac{\alpha t}{n} + e^{\alpha c} \right), \quad \text{for } \alpha > 0.$$

Using the maximum principle, we can obtain that

$$(3.2) \quad \frac{1}{\alpha} \ln \left( \frac{\alpha t}{n} + e^{\alpha \varphi_1} \right) \leq \varphi(x, t) \leq \frac{1}{\alpha} \ln \left( \frac{\alpha t}{n} + e^{\alpha \varphi_2} \right),$$

where $\varphi_1 := \inf_{M^n} \varphi(\cdot, 0)$ and $\varphi_2 := \sup_{M^n} \varphi(\cdot, 0)$. The estimate is obtained since $\varphi = \log u$. \qed

**Lemma 3.2 ($\dot{\varphi}$ estimate).** Let $\varphi$ be a solution of (2.7) and $\Sigma^n$ be a smooth, convex cone, then for $\alpha > 0$,

$$\min \left\{ \inf_{M^n} \dot{\varphi}(\cdot, 0) \cdot \Theta(0)^{\alpha}, \frac{1}{n} \right\} \leq \dot{\varphi}(x, t) \Theta(t)^{\alpha} \leq \max \left\{ \sup_{M^n} \dot{\varphi}(\cdot, 0) \cdot \Theta(0)^{\alpha}, \frac{1}{n} \right\}.$$

**Proof.** Set

$$\mathcal{M}(x, t) = \dot{\varphi}(x, t) \Theta(t)^{\alpha}.$$

Differentiating both sides of the first evolution equation of (2.7), it is easy to get that

$$(3.3) \quad \begin{cases}
\frac{\partial \mathcal{M}}{\partial t} = Q^{ij} D_{ij} \mathcal{M} + Q^k D_k \mathcal{M} + \alpha \Theta^{-\alpha} \left( \frac{1}{n} - \mathcal{M} \right) \mathcal{M} & \text{in } M^n \times (0, T) \\
\nabla_{\mu} \mathcal{M} = 0 & \text{on } \partial M^n \times (0, T) \\
\mathcal{M}(\cdot, 0) = \dot{\varphi}_0 \cdot \Theta(0)^{\alpha} & \text{on } M^n,
\end{cases}$$
where \( Q^{ij} := \frac{\partial Q}{\partial \varphi_{ij}} \) and \( Q^k := \frac{\partial Q}{\partial \varphi_k} \). Then the result follows from the maximum principle.

\[ \square \]

**Lemma 3.3** (Gradient estimate). Let \( \varphi \) be a solution of (2.7) and \( \Sigma^n \) be a smooth, convex cone described as in Theorem 1.1. Then we have for \( \alpha > 0 \),

\[ |D\varphi| \leq \sup_{M^n} |D\varphi(\cdot, 0)|, \quad \forall \ x \in M^n, \ t \in [0, T]. \]  

Proof. Set \( \psi = \frac{|D\varphi|^2}{2} \). By differentiating \( \psi \), we have

\[ \frac{\partial \psi}{\partial t} = \frac{\partial}{\partial t} \varphi_m \varphi^m = \dot{\varphi}_m \varphi^m = Q_{mm}. \]

Then using the evolution equation of \( \varphi \) in (2.7) yields

\[ \frac{\partial \psi}{\partial t} = Q_{ij} \varphi_{ij} + Q_{k} \varphi_k - \alpha Q |D\varphi|^2. \]

Interchanging the covariant derivatives, we have

\[ \psi_{ij} = D_j(\varphi_m \varphi^m) = \varphi_{mj} \varphi^m + \varphi_m \varphi^m_j = (\varphi_{mj} - R_{jmi}^l) \varphi^m + \varphi_m \varphi^m_j. \]

Therefore, we can express \( \varphi_{mj} \varphi^m \) as

\[ \varphi_{mj} \varphi^m = \psi_{ij} + R_{jmi}^l \varphi^m - \varphi_m \varphi^m_j. \]

Then, in view of the fact \( R_{jmi}^l = \sigma_{ji} \sigma_{ml} - \sigma_{ij} \sigma_{lm} \) on \( S^n \), we have

\[ \frac{\partial \psi}{\partial t} = Q_{ij} \psi_{ij} + Q_k \psi_k - \alpha Q |D\varphi|^2. \]

Since the matrix \( Q_{ij} \) is positive definite, the third and the fourth terms in the RHS of (3.5) are non-positive. Noticing that the last term in the RHS of (3.5) is also non-positive if \( \alpha > 0 \). Since \( \Sigma^n \) is convex, using a similar argument to the proof of [14, Lemma 5] (see pp. 1308) implies that

\[ \nabla_{\mu} \psi = - \sum_{i,j=1}^{n-1} h_{ij}^{\partial M^n} \nabla_{e_i} \varphi \nabla_{e_j} \varphi \leq 0 \quad \text{on} \quad \partial M^n \times (0, T), \]

where an orthonormal frame at \( x \in \partial M^n \), with \( e_1, \ldots, e_{n-1} \in T_x \partial M^n \) and \( e_n := \mu \), has been chosen for convenience in the calculation, and \( h_{ij}^{\partial M^n} \) is the second fundamental form of the boundary \( \partial M^n \subset \Sigma^n \). So, we can get

\[ \begin{cases}
\frac{\partial \psi}{\partial t} \leq Q_{ij} \psi_{ij} + Q_k \psi_k & \text{in} \ M^n \times (0, T) \\
\nabla_{\mu} \psi \leq 0 & \text{on} \ \partial M^n \times (0, T) \\
\psi(\cdot, 0) = \frac{|D\varphi(\cdot, 0)|^2}{2} & \text{in} \ M^n.
\end{cases} \]

Using the maximum principle, we get the gradient estimate of \( \varphi \) in Lemma 3.3.

\[ \square \]
Remark 3.1. It is worth pointing out that the evolving surface $M_t^a$ is always star-shaped under the assumption of Theorem 1.1, since, by Lemma 3.3, we have

$$\left\langle \frac{X}{|X|}, \nu \right\rangle = \frac{1}{v}$$

is bounded from below by a positive constant.

Combing the gradient estimate with $\dot{\varphi}$ estimate, we can obtain

Corollary 3.4. If $\varphi$ satisfies (2.7), then we have

$$0 < c_3 \leq H \varphi \leq c_4 < +\infty,$$

where $c_3$ and $c_4$ are positive constants independent of $\varphi$.

4. Hölder Estimates and Convergence

Set $\Phi = \frac{1}{|X|^n}$, $w = \langle X, \nu \rangle$ and $\Psi = \frac{\Phi}{w}$. We can get the following evolution equations.

Lemma 4.1. Under the assumptions of Theorem 1.1, we have

$$\frac{\partial}{\partial t} g_{ij} = 2\Phi h_{ij},$$

$$\frac{\partial}{\partial t} g_{ij}^{\dot{}} = -2\Phi h_{ij}^{\dot{}},$$

$$\frac{\partial}{\partial t} \nu = -\nabla \Phi,$$

$$\partial_t h_{ij} - \Phi H^{-1} \Delta h_{ij} = \Phi H^{-1} |A|^2 h_{ij} + \frac{2\Phi}{H^2} H_i H_j - 2\Phi h_{ik} h^{kj}$$

$$- \alpha \Phi (\nabla_i \log u \nabla_j \log H + \nabla_j \log u \nabla_i \log H)$$

$$+ \alpha \Phi u^{-1} u_{ij} - \alpha (\alpha + 1) \Phi \nabla_i \log u \nabla_j \log u,$$

and

$$\frac{\partial \Psi}{\partial t} = \text{div}_y (u^{-\alpha} H^{-2} \nabla \Psi) - 2H^{-2} u^{-\alpha} \Psi^{-1} |\nabla \Psi|^2$$

$$- \alpha \Psi^2 - \alpha \Psi^2 u^{-1} \nabla_i u \langle X, X_i \rangle - \alpha u^{-\alpha - 1} H^{-2} \nabla_i u \nabla_i \Psi.$$

Proof. The first three evolution equations are easy to obtain and so are omitted. Using the Gauss formula, we have

$$\partial_t h_{ij} = \partial_i (\partial_j X, -\nu)$$

$$= \langle \partial_i \partial_j (\Phi \nu), -\nu \rangle - \langle \Gamma_{ij}^k \partial_k X - h_{ij} \nu, \partial_i \nu \rangle$$

$$= -\partial_i \partial_j \Phi - \Phi \langle \partial_i \partial_j \nu, \nu \rangle + \Gamma_{ij}^k \Phi_k$$

$$= -\nabla^2_{ij} \Phi - \Phi \langle \partial_i (h_{ij} \partial_k X), \nu \rangle$$

$$= -\nabla^2_{ij} \Phi + \Phi h_{ik} h_{kj}.$$

Direct calculation results in

$$\nabla_{ij}^2 \Phi = \Phi (\frac{1}{H} H_{ij} + 2 \frac{H_i H_j}{H^2})$$

$$+ \alpha \Phi (\nabla_i \log u \nabla_j \log H + \nabla_j \log u \nabla_i \log H)$$

$$- \alpha \Phi u^{-1} u_{ij} + \alpha (\alpha + 1) \Phi \nabla_i \log u \nabla_j \log u.$$
Hence
\[ \Delta h_{ij} = H_{ij} + H h_{ik} h_j^k - h_{ij} |A|^2, \]
so
\[ \nabla_i^2 \Phi = -\Phi H^{-1} \Delta h_{ij} + \Phi h_{ik} h_j^k - \Phi H^{-1} |A|^2 h_{ij} + \frac{2H_i H_j}{H^2} \]
\[ + \alpha \Phi(\nabla_i \log u \nabla_j \log H + \nabla_j \log u \nabla_i \log H) \]
\[ - \alpha \Phi u^{-1} u_{ij} + \alpha (\alpha + 1) \Phi \nabla_i \log u \nabla_j \log u. \]
Thus,
\[ \partial_t h_{ij} - \Phi H^{-1} \Delta h_{ij} = \Phi H^{-1} |A|^2 h_{ij} - \frac{2\Phi}{H^2} H_i H_j \]
\[ - \alpha \Phi(\nabla_i \log u \nabla_j \log H + \nabla_j \log u \nabla_i \log H) \]
\[ + \alpha \Phi u^{-1} u_{ij} - \alpha (\alpha + 1) \Phi \nabla_i \log u \nabla_j \log u. \]
Then
\[ \partial_t H = \partial_t g^{ij} h_{ij} + g^{ij} \partial_t h_{ij} \]
\[ = -2\Phi g^{ij} h_{ij} + g^{ij} \left( \Phi H^{-1} \Delta h_{ij} + \Phi H^{-1} |A|^2 h_{ij} - \frac{2\Phi}{H^2} \nabla_i H \nabla_j H \right) \]
\[ + \alpha \Phi g^{ij} (-\nabla_i \log u \nabla_j \log H - \nabla_j \log u \nabla_i \log H + u^{-1} u_{ij} - (\alpha + 1) \nabla_i \log u \nabla_j \log u) \]
\[ = \Phi H^{-1} \Delta H - \frac{2\Phi}{H^2} |\nabla H|^2 - \Phi |A|^2 + \alpha \Phi g^{ij} (-2\nabla_i \log u \nabla_j \log H + u^{-1} u_{ij} - (\alpha + 1) \nabla_i \log u \nabla_j \log u) \]
\[ = u^{-\alpha} H^{-2} \Delta H - 2u^{-\alpha} H^{-3} |\nabla H|^2 - u^{-\alpha} H^{-1} |A|^2 - 2\alpha u^{-\alpha - 1} H^{-2} \nabla_i u \nabla^i H + \alpha u^{-\alpha - 1} H^{-1} \Delta u \]
\[ - \alpha (\alpha + 1) u^{-\alpha - 2} H^{-1} |\nabla u|^2. \]
Clearly,
\[ \partial_t w = \Phi + \alpha \Phi u^{-1} \nabla^i u \langle X, X_i \rangle + \Phi H^{-1} \nabla^i H \langle X, X_i \rangle, \]
using the Weingarten equation, we have
\[ w_i = h_i^k \langle X, X_k \rangle, \]
\[ w_{ij} = h_{ik}^j \langle X, X_k \rangle + h_{ij} - h_i^k h_{kj} \langle X, \nu \rangle = h_{ij,k} \langle X, X^k \rangle + h_{ij} - h_i^k h_{kj} \langle X, \nu \rangle. \]
Thus,
\[ \Delta w = H + \nabla^i H \langle X, X_i \rangle - |A|^2 \langle X, \nu \rangle \]
and
\[ \partial_t w = u^{-\alpha} H^{-2} \Delta w + u^{-\alpha} H^{-2} w |A|^2 + \alpha u^{-\alpha - 1} H^{-1} \nabla^i u \langle X, X_i \rangle. \]
Hence
\[ \frac{\partial \Psi}{\partial t} = -\alpha \frac{1}{u^{1+\alpha}} \frac{1}{H w} \dot{u} - \frac{1}{u^\alpha H^2} w \partial_t H - \frac{1}{u^\alpha H} \frac{1}{w^2} \partial_t w \]
\[ = -\alpha \frac{1}{u^{1+\alpha}} \frac{1}{H w} \frac{1}{w^{\alpha - 1}} H w - \frac{1}{w^\alpha H^2} w \partial_t H - \frac{1}{u^\alpha H} \frac{1}{w^2} \partial_t w \]
\[ = -\alpha u^{-2\alpha} H^{-2} w^{-2} + (\alpha + 1) u^{-2\alpha - 2} H^{-3} w^{-1} |\nabla u|^2 + 2u^{-2\alpha} H^{-5} w^{-1} |\nabla H|^2 \]
\[ + 2\alpha u^{-2\alpha - 1} H^{-4} w^{-1} \nabla_i u \nabla^i H - \alpha u^{-2\alpha - 1} H^{-3} w^{-1} \Delta u - u^{-2\alpha} H^{-4} w^{-1} \Delta H - u^{-2\alpha} H^{-3} w^{-2} \Delta w \]
\[ - \alpha u^{-2\alpha - 1} H^{-2} w^{-2} \nabla^i u \langle X, X_i \rangle. \]
In order to prove (4.1), we calculate
\[ \nabla_i \Psi = -\alpha u^{-\alpha-1} H^{-1} w^{-1} \nabla_i u - u^{-\alpha} H^{-2} w^{-1} \nabla_i H - u^{-\alpha} H^{-1} w^{-2} \nabla_i w \]
and
\[ \nabla_i^2 \Psi = \alpha (\alpha + 1) u^{-\alpha-2} H^{-1} w^{-1} \nabla_i u \nabla_j u + \alpha u^{-\alpha-1} H^{-2} w^{-1} \nabla_i u \nabla_j H + \alpha u^{-\alpha-1} H^{-1} w^{-2} \nabla_i u \nabla_j w \\
- \alpha u^{-\alpha-1} H^{-1} w^{-1} \nabla_i^2 u + \alpha u^{-\alpha-1} H^{-2} w^{-1} \nabla_i H \nabla_j u + 2u^{-\alpha} H^{-3} w^{-1} \nabla_i H \nabla_j H \\
+ u^{-\alpha} H^{-2} w^{-1} \nabla_i H \nabla_j w - u^{-\alpha} H^{-2} w^{-1} \nabla_i^2 H + \alpha u^{-\alpha-1} H^{-1} w^{-2} \nabla_i w \nabla_j u \\
+ u^{-\alpha} H^{-2} w^{-2} \nabla_i w \nabla_j H + 2u^{-\alpha} H^{-1} w^{-3} \nabla_i w \nabla_j w - u^{-\alpha} H^{-1} w^{-2} \nabla_i^2 w. \]
Thus
\[ u^{-\alpha} H^{-2} \Delta \Psi = \alpha (\alpha + 1) u^{-2\alpha-2} H^{-3} w^{-1} |\nabla u|^2 + 2u^{-2\alpha} H^{-5} w^{-1} |\nabla H|^2 + 2u^{-2\alpha} H^{-3} w^{-3} |\nabla w|^2 \\
+ 2\alpha u^{-2\alpha-1} H^{-4} w^{-1} \nabla_i u \nabla_j H + 2\alpha u^{-2\alpha-1} H^{-3} w^{-2} \nabla_i u \nabla_i w + 2u^{-2\alpha} H^{-4} w^{-2} \nabla_i H \nabla^i w \\
- \alpha u^{-2\alpha-1} H^{-3} w^{-1} \Delta u - u^{-2\alpha} H^{-4} w^{-1} \Delta H - u^{-2\alpha} H^{-3} w^{-2} \Delta w. \]
So we have
\[ \text{div}(u^{-\alpha} H^{-2} \nabla \Psi) = -\alpha u^{-\alpha-1} H^{-2} \nabla_i \Psi \nabla^i u - 2u^{-\alpha} H^{-3} \nabla_i \Psi \nabla^i H + u^{-\alpha} H^{-2} \Delta \Psi \]
\[ = (2\alpha + \alpha) u^{-2\alpha-2} H^{-3} w^{-1} |\nabla u|^2 + 5\alpha u^{-2\alpha-1} H^{-4} w^{-1} \nabla_i u \nabla^i H + 3\alpha u^{-2\alpha-1} H^{-3} w^{-2} \nabla_i u \nabla^i w \\
+ 4u^{-2\alpha} H^{-5} w^{-1} |\nabla H|^2 + 4u^{-2\alpha} H^{-4} w^{-2} \nabla_i w \nabla^i H + 2u^{-2\alpha} H^{-3} w^{-3} |\nabla w|^2 \\
- \alpha u^{-2\alpha-1} H^{-3} w^{-1} \Delta u - u^{-2\alpha} H^{-4} w^{-1} \Delta H - u^{-2\alpha} H^{-3} w^{-2} \Delta w. \]
and
\[ 2H^{-1} w |\nabla \Psi|^2 = 2\alpha u^{-2\alpha-2} H^{-3} w^{-1} |\nabla u|^2 + 2u^{-2\alpha} H^{-5} w^{-1} |\nabla H|^2 + 2u^{-2\alpha} H^{-3} w^{-3} |\nabla w|^2 \\
+ 4\alpha u^{-2\alpha-1} H^{-4} w^{-1} \nabla_i u \nabla^i H + 4\alpha u^{-2\alpha-1} H^{-3} w^{-2} \nabla_i u \nabla^i w + 4u^{-2\alpha} H^{-4} w^{-2} \nabla_i H \nabla^i w. \]
As above, we have
\[ \frac{\partial \Psi}{\partial t} - \text{div}(u^{-\alpha} H^{-2} \nabla \Psi) + 2H^{-1} w |\nabla \Psi|^2 \\
= -\alpha u^{-2\alpha} H^{-2} w^{-2} - \alpha u^{-2\alpha-1} H^{-2} w^{-2} \nabla^i u \langle X, X_i \rangle + \alpha^2 u^{-2\alpha-2} H^{-3} w^{-1} |\nabla u|^2 \\
+ \alpha u^{-2\alpha-1} H^{-4} w^{-1} \nabla_i u \nabla^i H + \alpha u^{-2\alpha-1} H^{-3} w^{-2} \nabla_i u \nabla^i w \\
= -\alpha \Psi^2 - \alpha \Psi^2 u^{-1} \nabla^i u \langle X, X_i \rangle - \alpha u^{-\alpha-1} H^{-2} \nabla_i u \nabla^i \Psi. \]
The proof is finished. \( \square \)

Now, we define the rescaled flow by
\[ \tilde{X} = X \Theta^{-1}. \]
Thus,
\[ \tilde{u} = u \Theta^{-1}, \]
\[ \tilde{\varphi} = \varphi - \log \Theta, \]
and the rescaled Gauss curvature is given by
\[ \tilde{H} = H \Theta. \]
Then, the rescaled scalar curvature equation takes the form
\[ \frac{\partial}{\partial t} \tilde{u} = \frac{v}{w^\alpha H} \Theta^{-\alpha} - \frac{1}{n} \tilde{u}^\alpha. \]
Defining $t = t(s)$ by the relation

$$\frac{dt}{ds} = \Theta^\alpha$$

such that $t(0) = 0$ and $t(S) = T$. Then $\tilde{u}$ satisfies

$$\begin{cases}
\frac{\partial}{\partial s} \tilde{u} = \frac{v}{\tilde{u}^\alpha} - \frac{\tilde{u}}{n} & \text{in } M^n \times (0, S) \\
\nabla \cdot \tilde{u} = 0 & \text{on } \partial M^n \times (0, S) \\
\tilde{u}((x, 0) = \tilde{u}_0 & \text{in } M^n.
\end{cases}$$

(4.2)

**Lemma 4.2.** Let $X$ be a solution of (1.2) and $\tilde{X} = X \Theta^{-1}$ be the rescaled solution. Then

$$D\tilde{u} = Du \Theta^{-1}, \ D\tilde{\varphi} = D\varphi, \ \frac{\partial \tilde{u}}{\partial s} = \frac{\partial u}{\partial t} \Theta^{-1} - \frac{1}{n} u \Theta^{-1},$$

$$\tilde{g}_{ij} = \Theta^{-2} g_{ij}, \ \tilde{g}^{ij} = \Theta^2 g^{ij}, \ \tilde{h}_{ij} = h_{ij} \Theta^{-1}.$$

*Proof.* These relations can be computed directly. \qed

**Lemma 4.3.** Let $u$ be a solution to the parabolic system (2.7), where $\varphi(x, t) = \log u(x, t)$, and $\Sigma^n$ be a smooth, convex cone described as in Theorem 1.1. Then there exist some $\beta > 0$ and some $C > 0$ such that the rescaled function $\tilde{u}(x, s) := u(x, t(s)) \Theta^{-1}(t(s))$ satisfies

$$[D\tilde{u}]_{x, \beta} + \left[\frac{\partial \tilde{u}}{\partial s}\right]_{\beta} + [\tilde{H}]_{\beta} \leq C(\|u_0\|_{C^{2+\gamma, 1+\gamma}(M^n); n, \beta, M^n}),$$

where $[f]_{x, \beta} := [f]_{x, \beta} + [f]_{s, \frac{\beta}{\gamma}}$ is the sum of the Hölder coefficients of $f$ in $M^n \times [0, S]$ with respect to $x$ and $s$.

*Proof.* We divide our proof in three steps.$^1$

**Step 1:** We need to prove that

$$[D\tilde{u}]_{x, \beta} + [D\tilde{u}]_{s, \frac{\beta}{\gamma}} \leq C(\|u_0\|_{C^{2+\gamma, 1+\gamma}(M^n); n, \beta, M^n}).$$

According to Lemmas 3.1, 3.2 and 3.3 it follows that

$$[D\tilde{u}]_{x, \beta} + \left[\frac{\partial \tilde{u}}{\partial s}\right]_{\beta} \leq C(\|u_0\|_{C^{2+\gamma, 1+\gamma}(M^n); n, \beta, M^n}).$$

Then we can easily obtain the bound of $[\tilde{u}]_{x, \beta}$ for any $0 < \beta < 1$. Lemma 3.1 in [6, Chap. 2] implies that the bound for $[D\tilde{u}]_{x, \beta}$ follows from a bound for $[\tilde{u}]_{s, \frac{\beta}{\gamma}}$ and $[D\tilde{u}]_{x, \beta}$. Hence it remains to bound $[D\varphi]_{x, \beta}$ since $D\tilde{u} = \tilde{u} D\tilde{\varphi}$. For this, fix $s$ and the equation (2.7) can be rewritten as an elliptic Neumann problem

$$\text{div}_\sigma \left( \frac{D\tilde{\varphi}}{\sqrt{1 + |D\tilde{\varphi}|^2}} \right) = \frac{n}{\sqrt{1 + |D\tilde{\varphi}|^2}} - e^{-n_\beta^2} \sqrt{1 + |D\tilde{\varphi}|^2} \tilde{\varphi} - \frac{n}{\tilde{\varphi} + \frac{1}{n}}.$$

In fact, the equation (4.4) is of the form $D_i(a^i(p)) + a(x) = 0$, where the bound of $a$, the smallest and largest eigenvalues of $a(p) := \frac{\partial a}{\partial \Theta}$ are controlled due to the estimate for $|D\varphi|$ and $|\tilde{u}|$. The estimate of $[D\tilde{\varphi}]_{x, \beta}$ for some $\beta$ follows from a Morrey estimate by calculations similar to the arguments in [8, Chap. 4, § 6; Chap. 10, § 2] (interior estimate and boundary estimate). For a rigorous proof of this estimate the reader is referred to [13].

$^1$In the proof of Lemma 4.3 the constant $C$ may differ from each other. However, we abuse the symbol $C$ for the purpose of convenience.
Step 2: The next thing to do is to show that

\[
\frac{\partial \tilde{u}}{\partial s} + \frac{\partial \tilde{u}}{\partial s} \leq C(\| u_0 \|_{C^{2+\gamma, 1+\gamma}(\tilde{M}^n), n, \beta, M^n}),
\]

As \( \frac{\partial}{\partial s} \tilde{u} = \tilde{u} \left( \frac{v}{\tilde{u}^{1+\alpha} H} - \frac{1}{n} \right) \), it is enough to bound \( \frac{\partial}{\partial s} \left( \frac{v}{\tilde{u}^{1+\alpha} H} \right) \). Set \( w(s) := \frac{v}{\tilde{u}^{1+\alpha} H} = \Theta^\alpha \Psi \), and then we have

\[
\frac{\partial w}{\partial s} = \frac{\partial}{\partial t}(\Theta^\alpha \Psi) \frac{\partial t}{\partial s} = \frac{\alpha}{n} + \Theta^\alpha \frac{\partial \Psi}{\partial t}.
\]

Let \( \tilde{\nabla} \) be the Levi-Civita connection of \( \tilde{M}_s := \tilde{X}(\tilde{M}^n, s) \) with respect to the metric \( \tilde{g} \). Combine with (4.1) and Lemma 4.2 we get

\[
\frac{\partial w}{\partial s} = \text{div}_\tilde{g}(\tilde{u}^{-\alpha} \tilde{H}^{-2} \tilde{\nabla} w) - 2 \tilde{H}^{-2} \tilde{u}^{-\alpha} w^{-1} |\tilde{\nabla} w|_\tilde{g}^2 + \frac{\alpha}{n} w - \alpha w^2 - \alpha u^2 P - \alpha \tilde{u}^{-\alpha} H^{-2} \tilde{v}_i \tilde{u} \tilde{\nabla}^i w,
\]

where \( P := u^{-1} \nabla^i u(X, X_i) \). Applying Lemma 3.3 we have

\[
|P| \leq |\nabla u|_g = \frac{|D \varphi|}{v} \leq C.
\]

The weak formulation of (4.5) is

\[
\int_{s_0}^{s_1} \int_{\tilde{M}_s} \frac{\partial w}{\partial s} \eta d\mu_s ds = \int_{s_0}^{s_1} \int_{\tilde{M}_s} \text{div}_\tilde{g}(\tilde{u}^{-\alpha} \tilde{H}^{-2} \tilde{\nabla} w) \eta - 2 \tilde{H}^{-2} \tilde{u}^{-\alpha} w^{-1} |\tilde{\nabla} w|_\tilde{g}^2 \eta d\mu_s ds
\]

(4.6)

+ \int_{s_0}^{s_1} \int_{\tilde{M}_s} \left( \frac{\alpha}{n} w - \alpha w^2 - \alpha u^2 P - \alpha \tilde{u}^{-\alpha} \tilde{H}^{-2} \tilde{v}_i \tilde{u} \tilde{\nabla}^i w \right) \eta d\mu_s ds.

Since \( \nabla \varphi = 0 \), the interior and boundary estimates are basically the same. We define the test function \( \eta := \xi^2 w \), where \( \xi \) is a smooth function with values in \([0, 1] \) and is supported in a small parabolic neighborhood. Then

\[
\int_{s_0}^{s_1} \int_{\tilde{M}_s} \frac{\partial w}{\partial s} \xi^2 w d\mu_s ds = \frac{1}{2} \| w \xi \|_{L^2, \tilde{M}_s}^2 - \int_{s_0}^{s_1} \int_{\tilde{M}_s} \xi \psi w^2 d\mu_s ds.
\]

Using integration by parts and Young’s inequality, we can obtain

\[
\int_{s_0}^{s_1} \int_{\tilde{M}_s} \text{div}_\tilde{g}(\tilde{u}^{-\alpha} \tilde{H}^{-2} \tilde{\nabla} w) \xi^2 w d\mu_s ds
\]

(4.7)

\[
= - \int_{s_0}^{s_1} \int_{\tilde{M}_s} \tilde{u}^{-\alpha} \tilde{H}^{-2} \tilde{v}_i \tilde{u} \tilde{\nabla}^i w d\mu_s ds - 2 \int_{s_0}^{s_1} \int_{\tilde{M}_s} \tilde{u}^{-\alpha} \tilde{H}^{-2} \xi \tilde{v}_i \tilde{u} \tilde{\nabla}^i \xi d\mu_s ds
\]

\[
\leq \int_{s_0}^{s_1} \int_{\tilde{M}_s} \frac{\alpha}{n} w - \alpha w^2 - \alpha u^2 P - \alpha \tilde{u}^{-\alpha} \tilde{H}^{-2} \tilde{v}_i \tilde{u} \tilde{\nabla}^i w \xi d\mu_s ds
\]

\[
\leq C \alpha \int_{s_0}^{s_1} \int_{\tilde{M}_s} \xi \psi(w^2 + |w|^3) d\mu_s ds + \int_{s_0}^{s_1} \int_{\tilde{M}_s} \alpha \tilde{u}^{-\alpha} \tilde{H}^{-2} \tilde{\nabla} \tilde{u} |\tilde{\nabla} w| \xi d\mu_s ds
\]

(4.8)

\[
\leq C \alpha \int_{s_0}^{s_1} \int_{\tilde{M}_s} \xi^2 (w^2 + |w|^3) d\mu_s ds + \frac{\alpha}{2} \int_{s_0}^{s_1} \int_{\tilde{M}_s} \tilde{u}^{-\alpha} \tilde{H}^{-2} |\tilde{\nabla} w|^2 \xi^2 d\mu_s ds
\]
Combing (4.7), (1.8) and (1.9), we have
\[
\frac{1}{2} \| u \xi \|_{2, \widetilde{M}_s}^{s_1} + (2 - \frac{\alpha}{2}) \int_{s_0}^{s_1} \int_{\widetilde{M}_s} \tilde{a}^{-\alpha} \tilde{H}^{-1} |\nabla \tilde{w}|^2 \xi^2 d\mu_s ds \\
\leq \int_{s_0}^{s_1} \int_{\widetilde{M}_s} \xi |\xi| w^2 d\mu_s ds + \int_{s_0}^{s_1} \int_{\widetilde{M}_s} \tilde{a}^{-\alpha} \tilde{H}^{-1} |\nabla \xi|^2 w^2 d\mu_s ds \\
+ C \alpha \int_{s_0}^{s_1} \int_{\widetilde{M}_s} \xi^2 (w^2 + |w|^3) d\mu_s ds + \frac{\alpha}{2} \int_{s_0}^{s_1} \int_{\widetilde{M}_s} \tilde{a}^{-\alpha} \tilde{H}^{-1} |\nabla \tilde{u}|^2 \xi^2 w^2 d\mu_s ds,
\]
which implies that
\[
\frac{1}{2} \| w \xi \|_{2, \widetilde{M}_s}^{s_1} + (2 - \frac{\alpha}{2}) \int_{s_0}^{s_1} \int_{\widetilde{M}_s} \tilde{a}^{-\alpha} \tilde{H}^{-1} |\nabla \tilde{w}|^2 \xi^2 d\mu_s ds \\
\leq (1 + \frac{1}{\text{min}(\tilde{a}^{-\alpha} \tilde{H}^2)}) \int_{s_0}^{s_1} \int_{\widetilde{M}_s} w^2 (|\xi| + |\nabla \xi|^2) d\mu_s ds \\
+ \alpha \left( C + \frac{\max(|\nabla \tilde{u}|^2)}{2 \text{min}(\tilde{a}^{-2\alpha} \tilde{H}^2)} \right) \int_{s_0}^{s_1} \int_{\widetilde{M}_s} \xi^2 w^2 + \xi^2 |w|^3 d\mu_s ds.
\]
(4.10)

This means that \( w \) belong to the De Giorgi class of functions in \( M^n \times [0, S) \). Similar to the arguments in [3] Chap. 5, §1 and §7, there exist constants \( \beta \) and \( C \) such that
\[
[w]_{\beta} \leq C \| w \|_{L^\infty(M^n \times [0, S])} \leq C(\| u_0 \|_{C^{2+\gamma, 1+\frac{\gamma}{2}}(M^n)}, n, \beta, M^n).
\]

Step 3: Finally, we have to show that
\[
[\tilde{H}]_{x, \beta} + [\tilde{H}]_{s, \frac{\beta}{2}} \leq C(\| u_0 \|_{C^{2+\gamma, 1+\frac{\gamma}{2}}(M^n)}, n, \beta, M^n).
\]

This follows from the fact that
\[
\tilde{H} = \sqrt{1 + |D\varphi|^2 \tilde{a}^{-1}}
\]

then we can obtain the following higher-order estimates

**Lemma 4.4.** Let \( u \) be a solution to the parabolic system (2.1), where \( \varphi(x, t) = \log u(x, t) \), and \( \Sigma^n \) be a smooth, convex cone described as in Theorem 1.1. Then for any \( s_0 \in (0, S) \) there exist some \( \beta > 0 \) and some \( C > 0 \) such that
\[
\| \varphi \|_{C^{2k+\beta, k+\frac{\beta}{2}}(M^n)} \leq C(\| u_0 \|_{C^{2+\gamma, 1+\frac{\gamma}{2}}(M^n)}, n, \beta, M^n)
\]
and for all \( k \in \mathbb{N} \),
\[
\| \varphi \|_{C^{2k+\beta, k+\frac{\beta}{2}}(M^n \times [s_0, S])} \leq C(\| u_0 \|_{C^{2k+\beta, k+\frac{\beta}{2}}(M^n)}, n, \beta, M^n).
\]

**Proof.** By Lemma 2.1 we have
\[
uvH = n - \left( \sigma^{ij} - \frac{\varphi^i \varphi^j}{u^2} \right) \varphi_{ij} = n - u^2 \Delta_g \varphi.
\]

Since
\[
u^2 \Delta_g \varphi = \tilde{u}^2 \Delta_{g\tilde{u}} = -|\nabla u|^2 + \tilde{u} \Delta_{g\tilde{u}},
\]

...
then
\[
\frac{\partial \tilde{u}}{\partial s} = \frac{\partial u}{\partial t} \Theta^{\alpha - 1} \frac{1}{n} \tilde{u} - \frac{uv H}{u^{1+\alpha} H^2} \Theta^{\alpha - 1} \frac{1}{n} \tilde{u} = \frac{\Delta \tilde{u}}{u^{\alpha} H^2} + \frac{2v}{u^{\alpha} H} \frac{1}{n} \tilde{u} - \frac{n + |\tilde{\nabla} \tilde{u}|^2}{u^{1+\alpha} H^2},
\]
which is a uniformly parabolic equation with Hölder continuous coefficients. Therefore, the linear theory (see [10, Chap. 4]) yield the inequality (4.11).

Set \( \tilde{\varphi} = \log \tilde{u} \), and then the rescaled version of the evolution equation in (4.2) takes the form
\[
\frac{\partial \tilde{\varphi}}{\partial s} = e^{-\alpha \tilde{\varphi}} \left[ n - \left( \sigma^{ij} - \frac{\tilde{\varphi} \tilde{\varphi}_i \tilde{\varphi}_j}{v^2} \right) \tilde{\varphi}_{ij} \right] \frac{v^2}{n} - \frac{1}{n},
\]
where \( v = \sqrt{1 + |D\tilde{\varphi}|^2} \). According to the \( C^{2+\beta,1+\beta} \) estimate of \( \tilde{u} \) (see Lemma 4.3), we can treat the equations for \( \frac{\partial \tilde{\varphi}}{\partial s} \) and \( D_i \tilde{\varphi} \) as second-order linear uniformly parabolic PDEs on \( M^n \times [s_0, S] \). At the initial time \( s_0 \), all compatibility conditions are satisfied and the initial function \( u(\cdot, t_0) \) is smooth. We can obtain a \( C^{3+\beta,3+\beta} \) estimate for \( D_i \tilde{\varphi} \) and a \( C^{2+\beta,2+\beta} \) estimate for \( \frac{\partial \tilde{\varphi}}{\partial s} \) (the estimates are independent of \( T \)) by Theorem 4.3 and Exercise 4.5 in [10, Chap. 4]. Higher regularity can be proven by induction over \( k \).

**Theorem 4.5.** Under the hypothesis of Theorem 1.1, we conclude \( T^* = +\infty \).

**Proof.** The proof of this result is quite similar to the corresponding argument in [14, Lemma 8] and so is omitted. \( \Box \)

### 5. Convergence of the Rescaled Flow

We know that after the long-time existence of the flow has been obtained (see Theorem 4.5), the rescaled version of the system (2.7) satisfies
\[
\begin{cases}
\frac{\partial}{\partial s} \tilde{\varphi} = \tilde{Q}(\tilde{\varphi}, D\tilde{\varphi}, D^2\tilde{\varphi}) & \text{in } M^n \times (0, \infty) \\
\nabla \mu \tilde{\varphi} = 0 & \text{on } \partial M^n \times (0, \infty) \\
\tilde{\varphi}(\cdot, 0) = \tilde{\varphi}_0 & \text{in } M^n,
\end{cases}
\]
where
\[
\tilde{Q}(\tilde{\varphi}, D\tilde{\varphi}, D^2\tilde{\varphi}) := e^{-\alpha \tilde{\varphi}} \left[ n - \left( \sigma^{ij} - \frac{\tilde{\varphi} \tilde{\varphi}_i \tilde{\varphi}_j}{v^2} \right) \tilde{\varphi}_{ij} \right] \frac{v^2}{n} - \frac{1}{n}
\]
and \( \tilde{\varphi} = \log \tilde{u} \). Similar to what has been done in the \( C^1 \) estimate (see Lemma 3.3), we can deduce a decay estimate of \( \tilde{u}(\cdot, s) \) as follows.

**Lemma 5.1.** Let \( u \) be a solution of (2.6), then for \( \alpha > 0 \), we have
\[
|D\tilde{u}(x, s)| \leq \sup_{M^n} |D\tilde{u}(\cdot, 0)| e^{-\lambda s},
\]
where \( \lambda \) is a positive constant.
Proof. Set \( \psi = \frac{|D\varphi|^2}{2} \). Similar to that in Lemma 3.3, we can obtain
\[
\frac{\partial \psi}{\partial s} = \bar{Q}^{ij} \psi_{ij} + \bar{Q}^k \psi_k - \bar{Q}^{ij} (\sigma_{ij} |D\varphi|^2 - \varphi_i \varphi_j)
- \bar{Q}^{ij} \varphi_m \bar{\varphi}^m - \alpha \tilde{Q} |D\varphi|^2,
\]
(5.3)
with the boundary condition
\[D_\mu \psi \leq 0.\]
By the \( C^2 \) estimate, we can find a positive constant \( \lambda \) such that
\[
\begin{cases}
\frac{\partial \psi}{\partial s} \leq \bar{Q}^{ij} \psi_{ij} + \bar{Q}^k \psi_k - \lambda \psi & \text{in } M^n \times (0, \infty) \\
D_\mu \psi \leq 0 & \text{on } \partial M^n \times (0, \infty) \\
\psi(\cdot, 0) = \frac{|D\varphi(\cdot, 0)|^2}{2} & \text{in } M^n.
\end{cases}
\]
Using the maximum principle and Hopf’s lemma, we can get the gradient estimates of \( \bar{\varphi} \), and then the inequality (5.2) holds from the estimate for \( D\tilde{u} \).

Lemma 5.2. Let \( u \) be a solution of the flow (2.6). Then,
\[
\tilde{u}(\cdot, s)
\]
converges to a real number as \( s \to +\infty \).

Proof. Set \( f(t) := \mathcal{H}^n(M^n_t) \), which, as before, represents the \( n \)-dimensional Hausdorff measure of \( M^n_t \) and is actually the area of \( M^n_t \). According to the first variation of a submanifold, see e.g. [16], and the fact \( \text{div}_{M^n_t} \nu = H \), we have
\[
f'(t) = \int_{M^n_t} \text{div}_{M^n_t} \left( \frac{\nu}{|X|^{n}H} \right) d\mathcal{H}^n
= \int_{M^n_t} \sum_{i=1}^n \langle \nabla e_i \left( \frac{\nu}{|X|^{n}H} \right), e_i \rangle d\mathcal{H}^n
= \int_{M^n_t} |u|^{-\alpha} d\mathcal{H}^n,
\]
(5.4)
where \( \{e_i\}_{1 \leq i \leq n} \) is some orthonormal basis of the tangent bundle \( TM^n_t \). We know that (3.2) implies
\[
\left( \frac{\alpha}{n} t + e^{\alpha \varphi_2} \right)^{-1} \leq u^{-\alpha} \leq \left( \frac{\alpha}{n} t + e^{\alpha \varphi_1} \right)^{-1},
\]
where \( \varphi_1 = \inf_{M^n} \varphi(\cdot, 0) \) and \( \varphi_2 = \sup_{M^n} \varphi(\cdot, 0) \). Hence
\[
\left( \frac{\alpha}{n} t + e^{\alpha \varphi_2} \right)^{-1} f(t) \leq f'(t) \leq \left( \frac{\alpha}{n} t + e^{\alpha \varphi_1} \right)^{-1} f(t).
\]
Combining this fact with (5.4) yields
\[
\frac{\left( \frac{\alpha}{n} t + e^{\alpha \varphi_2} \right) \frac{\mathcal{H}^n(M^n_t)}{e^{n \varphi_2}}}{\left( \frac{\alpha}{n} t + e^{\alpha \varphi_1} \right) \frac{\mathcal{H}^n(M^n_t)}{e^{n \varphi_1}}} \leq f(t) \leq \frac{\left( \frac{\alpha}{n} t + e^{\alpha \varphi_1} \right) \frac{\mathcal{H}^n(M^n_t)}{e^{n \varphi_1}}}{\left( \frac{\alpha}{n} t + e^{\alpha \varphi_2} \right) \frac{\mathcal{H}^n(M^n_t)}{e^{n \varphi_2}}}.
\]
Therefore, the rescaled hypersurface \( \tilde{M}_s = M^n_t \Theta^{-1} \) satisfies the following inequality
\[
\frac{\left( \frac{\alpha}{n} t + e^{\alpha \varphi_2} \right) \frac{\mathcal{H}^n(M^n_t)}{e^{n \varphi_2}}}{\left( \frac{\alpha}{n} t + e^{\alpha \varphi_1} \right) \frac{\mathcal{H}^n(M^n_t)}{e^{n \varphi_1}}} \leq \mathcal{H}^n(\tilde{M}_s) \leq \frac{\left( \frac{\alpha}{n} t + e^{\alpha \varphi_1} \right) \frac{\mathcal{H}^n(M^n_t)}{e^{n \varphi_1}}}{\left( \frac{\alpha}{n} t + e^{\alpha \varphi_2} \right) \frac{\mathcal{H}^n(M^n_t)}{e^{n \varphi_2}}},
\]
which implies that the area of $\tilde{M}_s$ is bounded and the bounds are independent of $s$. Together with Lemma 5.11, Lemma 5.1 and the Arzelà-Ascoli theorem, we conclude that $\tilde{u}(\cdot, s)$ must converge in $C^\infty(M^n)$ to a constant function $r_\infty$ with
\[
\frac{1}{e^{\phi^2}} \left( \frac{\mathcal{H}^n(M_0^n)}{\mathcal{H}^n(M^n)} \right)^{\frac{1}{n}} \leq r_\infty \leq \frac{1}{e^{\phi^1}} \left( \frac{\mathcal{H}^n(M_0^n)}{\mathcal{H}^n(M^n)} \right)^{\frac{1}{n}},
\]
i.e.,
\[
(5.5) \quad \frac{1}{\sup_{M^n} u_0} \left( \frac{\mathcal{H}^n(M_0^n)}{\mathcal{H}^n(M^n)} \right)^{\frac{1}{n}} \leq r_\infty \leq \frac{1}{\inf_{M^n} u_0} \left( \frac{\mathcal{H}^n(M_0^n)}{\mathcal{H}^n(M^n)} \right)^{\frac{1}{n}}.
\]
This completes the proof. □

So, we have

**Theorem 5.3.** The rescaled flow
\[
\frac{d\tilde{X}}{ds} = \frac{1}{|\tilde{X}|^\alpha \nu} \tilde{X} - \frac{\tilde{X}}{n}
\]
exists for all time and the leaves converge in $C^\infty$ to a piece of round sphere of radius $r_\infty$, where $r_\infty$ satisfies (5.5).

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