A new method for probing the late-time dynamics in the Lorentzian type IIB matrix model

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Received June 2, 2017; Accepted July 5, 2017; Published August 23, 2017

The type IIB matrix model has been investigated as a possible nonperturbative formulation of superstring theory. In particular, it was found by Monte Carlo simulation of the Lorentzian version that the 9D rotational symmetry of the spatial matrices is broken spontaneously to the 3D one after some “critical time”. In this paper we develop a new simulation method based on the effective theory for the submatrices corresponding to the late time. Using this method, one can obtain the results for \( N \times N \) matrices by simulating matrices typically of the size \( O(\sqrt{N}) \). We confirm the validity of this method and demonstrate its usefulness in simplified models.

Subject Index B25

1. Introduction

The type IIB matrix model was proposed as a possible nonperturbative formulation of superstring theory in 1996 [1]. As such, it has the potential to explain the beginning of our Universe and the emergence of the real world as we observe it today. Among many works in this direction, Ref. [2] investigated the Lorentzian version of the model by Monte Carlo simulation, and demonstrated that a (3+1)D expanding universe naturally appears from the model. This implies, in particular, that the 9D rotational symmetry of the model is broken spontaneously to the 3D one at a certain time. (See Refs. [3–20] for related works.) It was also suggested from Monte Carlo simulation of simplified models that the expansion is exponential at the beginning and turns into a power law at late times [5,6]. In order to investigate such late-time behaviors, one needs to perform Monte Carlo simulation with large matrices, which is prohibitively time-consuming for the original model mainly due to the existence of fermionic matrices.

In this paper we develop a new simulation method for probing the late-time dynamics in the Lorentzian type IIB matrix model with relatively small matrices. This is made possible by simulating the effective theory for the submatrices corresponding to the late time. Typically, the size of the submatrices corresponding to the time after the spontaneous symmetry breaking (SSB) of the SO(9) symmetry is \( O(\sqrt{N}) \), where \( N \) is the size of the whole matrices. Since the computational cost of the original model is roughly \( O(N^3) \), the new method, if applied to these submatrices, enables us to extract essentially the same information with a cost of \( O(N^{5/2}) \), which is slightly less than the \( O(N^3) \) cost required for simulating the bosonic counterparts.
Our method is based on the idea that the effective theory for the submatrices has a simple action dictated by the SU($N$) symmetry and the SO(9) rotational symmetry [5]. An important new ingredient here is that we can actually make full use of the SO(9,1) Lorentz symmetry instead of just the SO(9) rotational symmetry. This is important, in particular, in applying the method to models with fermionic matrices, since otherwise we have to fine-tune one parameter in the fermionic action, which controls the ratio of the temporal and spatial parts. In fact, the SO(9,1) Lorentz symmetry is softly broken by the infrared cutoffs, which are inevitably introduced to make the model well defined, and it is not obvious that we can impose the SO(9,1) Lorentz symmetry on the effective theory. Our results indicate that this is indeed the case.

The rest of this paper is organized as follows. In Sect. 2 we briefly review the type IIB matrix model. In Sect. 3 we describe the new method, which is based on the effective theory for the submatrices corresponding to the late time. In Sect. 4 we demonstrate the usefulness of the method by applying it to a bosonic model. In Sect. 5 we discuss how one can tune the parameters in the effective theory to optimize the efficiency of the method. Section 6 is devoted to a summary and discussions. In Appendix A we give the detailed information used to make the plots in this paper. In Appendix B we present the results of the analysis in Sect. 3 for a supersymmetric model including fermionic matrices.

2. Brief review of the type IIB matrix model

The type IIB matrix model [1] is composed of 10 bosonic $N \times N$ matrices $A_\mu$ ($\mu = 0, \ldots, 9$) and 16 fermionic $N \times N$ matrices $\Psi_\alpha$ ($\alpha = 1, \ldots, 16$), which are both traceless Hermitian. Its action is given by

$$S = S_b + S_f,$$

$$S_b = \frac{1}{4g^2} \text{Tr} \left( [A_\mu, A_\nu] [A^\mu, A^\nu] \right),$$

$$S_f = -\frac{1}{2g^2} \text{Tr} \left( \Psi_\alpha (C \Gamma^\mu)_{\alpha\beta} [A_\mu, \Psi_\beta] \right),$$

where the indices $\mu, \nu$ are contracted using the Lorentzian metric $\eta_{\mu\nu} = \text{diag} (-1, 1, \ldots, 1)$. We have also introduced the 10D gamma matrices $\Gamma^\mu$ after the Weyl projection and the charge conjugation matrix $C$. The parameter $g$ is merely a scale parameter in this model instead of being a coupling constant since it can be absorbed by rescaling $A_\mu$ and $\Psi$ appropriately. The Euclidean version (studied, e.g., in Refs. [8,9]) can be obtained by making the “Wick rotation” $A_0 = iA_{10}$, where $A_{10}$ is supposed to be Hermitian.

The partition function for the Lorentzian version is proposed in Ref. [2] as

$$Z = \int dA d\Psi \ e^{iS}$$

with the action (2.1). The “$i$” in front of the action is motivated from the fact that the string worldsheet metric should also have a Lorentzian signature. By integrating out the fermionic matrices, we obtain the Pfaffian

$$\int d\Psi \ e^{iS_f} = \text{Pf} \mathcal{M} (A),$$

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which is real, unlike in the Euclidean case [9]. Note also that the bosonic action (2.2) can be written as

\[ S_b = \frac{1}{4g^2} \text{Tr} (F_{\mu\nu} F^{\mu\nu}) = \frac{1}{4g^2} \left\{ -2\text{Tr} (F_0)^2 + \text{Tr} (F_{ij})^2 \right\}, \tag{2.6} \]

where we have introduced the Hermitian matrices \( F_{\mu\nu} = i [A_\mu, A_\nu] \). Since the two terms in the last expression have opposite signs, \( S_b \) is not positive semidefinite, and it is not bounded from below.

In order to make the partition function (2.4) finite, one needs to introduce infrared cutoffs in both the temporal and spatial directions, for instance, as \[ 1 \leq \frac{1}{N} \text{Tr} (A_0)^2 \leq \kappa \frac{1}{N} \text{Tr} (A_i)^2, \tag{2.7} \]

\[ 1 \leq \frac{1}{N} \text{Tr} (A_i)^2 \leq \Lambda^2. \tag{2.8} \]

Recently we have found it important to generalize this form of the infrared cutoffs slightly in order to achieve universality in the large-\( N \) limit [7]. However, in the present methodological paper, we use the above form for simplicity.

After some manipulation and rescaling of \( A_\mu \), we can rewrite the partition function (2.4) as [2] (see Appendix A of Ref. [5] for a refined argument)

\[ Z = \int dA \text{Pf} \mathcal{M}(A) \delta \left( \frac{1}{N} \text{Tr} (F_{\mu\nu} F^{\mu\nu}) \right) \delta \left( \frac{1}{N} \text{Tr} (A_i)^2 - L^2 \right) \theta \left( \kappa L^2 - \frac{1}{N} \text{Tr} (A_0)^2 \right), \tag{2.9} \]

where \( \theta (x) \) is the Heaviside step function. The scale parameter \( L \) can be set to unity without loss of generality. This form allows us to perform Monte Carlo simulation of the Lorentzian model without the sign problem, unlike the Euclidean model\(^1\). See Appendix B of Ref. [5] for details of the Monte Carlo simulation.

It turns out [2] that one can extract a time evolution from configurations generated by simulating Eq. (2.9). For that purpose, we first use the SU(\( N \)) symmetry of the model to bring the temporal matrix \( A_0 \) into the diagonal form

\[ A_0 = \text{diag} (\alpha_1, \ldots, \alpha_N), \quad \text{where} \quad \alpha_1 < \cdots < \alpha_N. \tag{2.10} \]

In this basis, the spatial matrices \( A_i \) are found to have a band-diagonal structure. More precisely, there exists some integer \( n \) such that the elements of spatial matrices \( (A_i)_{ab} \) for \( |a - b| > n \) are much smaller than those for \( |a - b| \leq n \). Based on this observation, we may naturally consider \( n \times n \) matrices

\[ (\tilde{A}_i)_{IJ}(t) \equiv (A_i)_{K+I,K+J} \tag{2.11} \]

as representing the state of the universe at some time \( t \), where \( I, J = 1, \ldots, n \) and \( K = 0, 1, \ldots, N-n \). The time \( t \) is defined by

\[ t = \frac{1}{n} \sum_{I=1}^{n} \alpha_{K+I}, \tag{2.12} \]

\(^1\) Strictly speaking, the Pfaffian Pf in Eq. (2.9) can change its sign, but it is found that configurations with positive Pfaffian dominate at large \( N \).
corresponding to the $n \times n$ matrices $\bar{A}_i$. For example, we can define the extent of space at time $t$ as

$$R^2(t) = \left\langle \frac{1}{n} \text{tr} \sum_i (\bar{A}_i(t))^2 \right\rangle,$$

(2.13)

where the symbol $\text{tr}$ represents a trace over the $n \times n$ block. We also define the “moment of inertia tensor”

$$T_{ij}(t) = \frac{1}{n} \text{tr} (\bar{A}_i(t) \bar{A}_j(t)),$$

(2.14)

which is a $9 \times 9$ real symmetric matrix. The eigenvalues of $T_{ij}(t)$, which we denote by $\lambda_i(t)$ with the ordering

$$\lambda_1(t) > \lambda_2(t) > \cdots > \lambda_9(t),$$

(2.15)

represent the spatial extent in each of the nine directions at time $t$. Note that the expectation values $\langle \lambda_i(t) \rangle$ tend to be equal in the large-$N$ limit if the SO(9) symmetry is not spontaneously broken. This is the case at early times of the time evolution. After a critical time $t_c$, however, it was found [2] that the three largest eigenvalues $\langle \lambda_i(t) \rangle (i = 1, 2, 3)$ become significantly larger than the others, which implies that the SO(9) symmetry is spontaneously broken down to SO(3).

The block size $n$ used in calculating quantities such as Eqs. (2.13) and (2.14) by Monte Carlo simulation is determined as described in Sect. 5 of Ref. [6]. In Appendix A we present the block size used in making the plots in Figs. 1, 4, 5, and 6. There, we also present the values obtained for the critical time $t_c$ and the corresponding extent of space $R(t_c)$, which are also needed in making these plots.

In Fig. 1 we show the results [6] for the bosonic model, which is obtained by omitting the Pfaffian in Eq. (2.9). As we find in Ref. [6], the bosonic model is well defined without the infrared cutoff (2.7) for the temporal matrix $A_0$. Therefore, the constraint $\theta \left( \kappa - \frac{1}{N} \text{Tr} (A_0)^2 \right)$ for the temporal matrix $A_0$ in Eq. (2.9) can be omitted. On the left, we plot the expectation values $\langle \lambda_i(t) \rangle$ against $t$ for $N = 512$. We observe the SSB from SO(9) to SO(3) at a critical time $t_c$ similarly to the original Lorentzian type

\[ N=256, \ N=384, \ N=512 \]

\[ 2.0 \leq x \leq 2.5 \]

\[ c = 34.3(6), \ d = -55(1) \]
IIB matrix model. On the right, the extent of space $R^2(t)/R^2(t_c)$ is plotted against $(t - t_c)/R(t_c)$ for various $N \leq 512$. We observe a clear scaling behavior. It is found that the behavior of $R^2(t)$ at $t > t_c$ can be fitted to an exponential function only for a finite range, and it seems to slow down at later times, in contrast to the exponential behavior observed in Ref. [5] in a simplified model for early times. We can actually fit our $N = 512$ data within $1.85 \leq (t - t_c)/R(t_c) \leq 2.5$ to a straight line, which corresponds to the power-law expansion

$$R(t) \propto t^{1/2}$$

(2.16)

reminiscent of the expanding behavior of the Friedmann–Robertson–Walker universe in the radiation-dominated era. This is interesting given that the bosonic model used to obtain these results is expected to capture the late-time dynamics of the original model [6].

In Monte Carlo simulation, we find that the $\mathbb{Z}_2$ symmetry $A_0 \mapsto -A_0$ is not broken spontaneously, and hence the time evolution that we obtain from the generated matrices respects the time-reversal symmetry. This by no means implies that the Universe is doomed to end with a Big Crunch because one has to take the large-$N$ limit. In that limit, it could be that the turning point $t = 0$ is infinitely distant, in physical units, from the “critical time” $t_c$, at which the SSB occurs; namely $|t_c|/R(t_c) \to \infty$ as $N \to \infty$. In other words, the results for finite $N$ allow us to probe only the early-time dynamics of the model, and the appearance of the turning point at $t = 0$ can be merely a finite-$N$ artifact. In order to probe the late-time dynamics, we need to increase $N$, which is hard in the original model because the computational cost\(^2\) increases as $O(N^5)$.

It should be noted at this point that the number of data points in the SSB region in Fig. 1 (right) is 61, 73, 83 for $N = 256, 384, 512$, respectively, which roughly grows as $4\sqrt{N}$. This hints at a possible method for simulating the SSB region more efficiently.

3. The effective theory for the submatrices

In this section we consider the effective theory for the submatrices corresponding to the late times. In view of the discussion given in the previous section, we choose the SU($N$) basis in which the temporal matrix $A_0$ takes the diagonal form (2.10), and cut the $\tilde{N} \times \tilde{N}$ submatrices $\tilde{A}_\mu$ out of the whole matrices $A_\mu$, as depicted in Fig. 2. More explicitly, we define the submatrices $\tilde{A}_\mu$ as

$$(\tilde{A}_\mu)_{ab} = (A_\mu)_{s+a,s+b}, \quad s = \frac{N - \tilde{N}}{2},$$

(3.1)

where $a, b = 1, \ldots, \tilde{N}$. Using these submatrices, we define the expectation values

$$\tilde{C} \equiv \left( \frac{1}{N} \text{Tr}(\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}) \right), \quad \tilde{L}^2 \equiv \left( \frac{1}{N} \text{Tr}(\tilde{A}_0)^2 \right), \quad \tilde{k}\tilde{L}^2 \equiv \left( \frac{1}{N} \text{Tr}(\tilde{A}_0)^2 \right)$$

(3.2)

with the matrices $A_\mu$ obtained by simulating the original matrix model, where we have defined $\tilde{F}_{\mu\nu} = i[\tilde{A}_\mu, \tilde{A}_\nu]$.

Let us then consider the effective theory for the $\tilde{N} \times \tilde{N}$ submatrices $\tilde{A}_\mu$. Since the expectation values (3.2) should be reproduced from the effective theory, a natural candidate for the effective theory is $V = \tilde{L}^2 - \alpha \tilde{C}$.\(^2\) In order to make one trajectory in the hybrid Monte Carlo algorithm, the original model requires $O(N^5)$ arithmetic operations. The reason for this is that the number of iterations required for the convergence of the conjugate gradient method used to implement the effects of fermions grows as $O(N^2)$. Since matrix multiplication requires $O(N^3)$ arithmetic operations, we get $O(N^5)$.
Fig. 2. This figure shows how we cut out the $\tilde{N} \times \tilde{N}$ submatrices from the $N \times N$ matrices that appear in the Lorentzian type IIB matrix model.

The theory is

$$Z_{\text{eff}}[\tilde{N}; N, \kappa] = \int d\tilde{A} \text{Pf} \mathcal{M}(\tilde{A}) \delta \left( \frac{1}{N} \text{Tr}(\tilde{F}_{\mu\nu}^2) - \tilde{C} \right) \delta \left( \frac{1}{N} \text{Tr}(\tilde{A}_i)^2 - \tilde{L}^2 \right) \theta \left( \kappa \tilde{L}^2 - \frac{1}{N} \text{Tr}(\tilde{A}_0)^2 \right). \quad (3.3)$$

The only difference from the original Lorentzian type IIB matrix model (2.9) is the appearance of the parameter $\tilde{C}$. In order to verify this ansatz, we calculate the quantities such as Eqs. (2.13) and (2.14) using the effective theory and see whether the results agree with the ones obtained from the original model.

For simplicity, here we consider the 6D version of the bosonic model\(^3\), which can be obtained by restricting ourselves to $A_\mu \ (\mu = 0, \ldots, 5)$. The qualitative behavior of this simplified model is analogous to the original model [5,21], and, in particular, SSB from SO(5) to SO(3) occurs. We perform simulation of this model with the matrix size $N = 64$, and plot the expectation values (3.2) for the $\tilde{N} \times \tilde{N}$ submatrices against $\tilde{N}$ in Fig. 3 (left). Then, using these expectation values, we perform simulation of the effective theory (3.3) for the $\tilde{N} \times \tilde{N}$ submatrices omitting the Pfaffian $\text{Pf} \mathcal{M}$. In Fig. 3 (right) we plot the extent of space $\tilde{R}^2(\tilde{t})$ against $\tilde{t}$ for the effective theory with $\tilde{N} = 16, 32$ and compare it with the results for the original model with $N = 64$. We find that the effective theory indeed reproduces the late-time behaviors of the original model correctly, except in the region around $\tilde{t} = 0$, which is subject to finite-$N$ effects anyway. Note, in particular, that all the data points for $\tilde{N} = 16$ lie in the region where the SSB of SO(5) occurs. In Appendix B we present the results of the same analysis for the 6D version of the supersymmetric model, which includes fermionic matrices.

So far, we have determined the values of $\tilde{C}, \tilde{L}$, and $\kappa$ to be used in the effective theory (3.3) for the $\tilde{N} \times \tilde{N}$ submatrices by simulating the original model with the matrix size $N$, which is larger than $\tilde{N}$. However, the real virtue of this method lies in the fact that it can be used without simulating the

\(^3\) Let us recall that, in the bosonic models, there is no need to introduce the infrared cutoff (2.7) for the temporal matrix $A_0$. However, in the corresponding effective theory like Eq. (3.3), we need to introduce $\theta \left( \kappa \tilde{L}^2 - \frac{1}{N} \text{Tr}(A_0)^2 \right)$, which represents the infrared cutoff for the temporal matrix $A_0$. Also, when the bosonic models are considered in the argument for justifying the use of the effective theory (3.4) without simulating the original model, we assume that the parameter $\kappa$ is introduced in the original model too, where $\kappa$ should be smaller than the value of $\frac{1}{N} \text{Tr}(A_0)^2$ that is otherwise obtained.
Fig. 3. (Left) The expectation values $\bar{L}^2, \bar{\kappa} \bar{L}^2$, and $\bar{C}$ defined in Eq. (3.2) are plotted against the size $\bar{N}$ of the submatrices for the 6D version of the bosonic model with $N = 64$. (Right) The extent of space $R^2(t)$ is plotted against $t$ for the 6D version of the bosonic model. The dotted line, which is drawn to guide the eye, connects the data points (plus sign) for the original model with $N = 64$. The circles and crosses represent the results of $\bar{R}^2(\bar{t})$ for the effective theory with $\bar{N} = 16$ and $32$, respectively. The block size used in making this plot is $n = 8$ for all cases.

original model. In order to see that, let us first consider a theory

$$\hat{Z} = \int d\hat{A} \text{Pf} \mathcal{M}(\hat{A}) \delta \left( \frac{1}{N} \text{Tr}(\hat{F}_{\mu\nu} \hat{F}^{\mu\nu}) - \hat{C} \right) \delta \left( \frac{1}{N} \text{Tr}(\hat{A}_i)^2 - 1 \right) \theta \left( \hat{\kappa} - \frac{1}{N} \text{Tr}(\hat{A}_0)^2 \right), \quad (3.4)$$

which generalizes\(^4\) the original theory (2.9) by introducing nonzero $\hat{C}$. We have put hats on all the variables and the parameters of this theory to distinguish them from those in the original theory. In order for this theory to be equivalent to the effective theory (3.3), we need $\hat{N} = \bar{N}$, $\hat{A}_\mu = \bar{A}_\mu / \bar{L}$, and

$$\hat{\kappa} = \bar{\kappa}(\bar{N}; N, \kappa), \quad \hat{C} = \bar{C}(\bar{N}; N, \kappa). \quad (3.5)$$

The point here is that, whatever values we choose for $\hat{\kappa}$ and $\hat{C}$, we have two parameters $N$ and $\kappa$ at our disposal, which make it possible to satisfy Eq. (3.5) generically from counting the degrees of freedom. (Strictly speaking, this statement holds only at large $N$ since $N$ cannot be changed continuously.) Thus, the generalized theory (3.4) can always be interpreted as the effective theory for the submatrices of the original theory up to some rescaling of $A_\mu$.

4. Demonstrating the usefulness of the method

In this section we demonstrate the usefulness of our method by applying it to the 10D bosonic model. As we reviewed in Sect. 2, this model was studied in Ref. [6] and the 3D space was found to expand with the power law at late times. We will show that this result can be reproduced by using the effective theory with much smaller $N$ than that used for the original model.

Here we perform simulation of the effective theory with $N = 256$. (From now on, we omit the hats in the effective theory (3.4)). In Fig. 4 we plot the extent of space $R^2(t)/R^2(t_c)$ against $(t - t_c)/R(t_c)$.

\(^4\) It is an interesting historical remark that the same type of generalization has been proposed in the context of the space-time uncertainty principle in string theory [22]. Note, however, that the parameter $\bar{C}$ that appears in that context should be negative, unlike in our case.
Fig. 4. The extent of space $R^2(t)/R^2(t_c)$ is plotted against $x = (t - t_c)/R(t_c)$ for the effective theory of the 10D bosonic model with $N = 256$ and various values of $\kappa$ and $C$. The dashed line represents a fit of the data points for $(\kappa, C) = (1.0, 100)$ to $R^2(t)/R^2(t_c) = cx + d$ with $1.85 \leq x \leq 2.5$, which gives $c = 32.4(6)$ and $d = -52(1)$.

for the effective theory\(^5\) with various $\kappa$ and $C$. The results should be compared with the results in Fig. 1 for the original 10D bosonic model with $N \leq 512$. We observe a clear scaling behavior similar to Fig. 1 in spite of the fixed matrix size. We also find that the result for $\kappa = 1.0$ and $C = 100$ coincides with the result for the original model with $N = 512$ except around the peak. In particular, we are able to reproduce the power-law expansion $R(t)/R(t_c) \sim t^{1/2}$ at late times with the matrix size much smaller than in Fig. 1. This demonstrates the usefulness of the new method.

5. How to tune the parameters in the effective theory

We can actually optimize the efficiency of the method by tuning the parameters in the effective theory (3.4). For that purpose, let us define the “volume” $\Delta$ in the temporal direction as

$$\Delta \equiv \frac{t_{\text{peak}} - t_c}{R(t_c)}, \quad (5.1)$$

where $t_{\text{peak}}$ represents the position of the peak\(^6\) in $R^2(t)$. We also define the “lattice spacing” $\epsilon$ in the temporal direction as

$$\epsilon \equiv \frac{\Delta}{\nu}, \quad (5.2)$$

where $\nu$ is the number of data points of $R^2(t)$ contained within $t_c < t \leq t_{\text{peak}}$. This definition represents the average horizontal spacing between the adjacent data points of $R^2(t)$. Note that both $\Delta$ and $\epsilon$ can be changed by tuning $\kappa$ and $C$ in the effective theory with the matrix size fixed. In Appendix A we give the values of $\nu$, $\Delta$, and $\epsilon$ obtained from the plots in Figs. 1 (right), 4, and 5. The values plotted in Fig. 6 are also listed.

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\(^5\) In the original 10D bosonic model, there is no need to introduce the infrared cutoff (2.7) for the temporal matrix $A_0$, and we obtain $\langle \frac{1}{N} \text{Tr}(A_0)^2 \rangle = 4.376\,7(1)$ for $N = 256$. Note that the values of $\kappa$ chosen in the effective theory are considerably smaller than this value.

\(^6\) In fact, $t_{\text{peak}} \simeq 0$ due to the time-reversal symmetry $A_0 \mapsto -A_0$. 

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In Fig. 5 we plot the extent of space $R^2(t) / R^2(t_c)$ against $x(t) \equiv (t - t_c) / R(t_c)$ for various $C$ with fixed $\kappa$ (left) and for various $\kappa$ with fixed $C$ (right) in the effective theory of the 6D bosonic model with $N = 32$. Here we plot only the data points for $t \leq t_{\text{peak}}$ to visualize the achieved $\Delta$ for each case. We find that increasing $\kappa$ and increasing $C$ have similar effects; they make both $\Delta$ and $\epsilon$ larger. This is also confirmed in Fig. 6, where we plot the volume $\Delta$ (top left) and the lattice spacing $\epsilon$ (top right) against $C$ for various $\kappa$. Note, however, that $\nu = \Delta / \epsilon$ behaves differently. We find from Fig. 6 (bottom) that $\nu$ is almost independent of $C$, while it increases for smaller $\kappa$. Therefore, the...
basic strategy for tuning $\kappa$ and $C$ is to decrease $\kappa$ until $\nu$ reaches its maximum value\(^7\) $(N - n + 1)/2$ and to increase $C$ until an appropriate value of $\Delta$ or $\epsilon$ is achieved. If we decrease $\kappa$ further, all the data points will lie in the region where the SSB of rotational symmetry occurs, as in the situation with $\tilde{N} = 16$ in Fig. 3 (right). This is inconvenient since we cannot determine the critical time $t_c$ directly.

As practical applications of our method, we can use it to investigate either the infinite volume limit ($\Delta \to \infty$ with fixed $\epsilon$) or the continuum limit ($\epsilon \to 0$ with fixed $\Delta$). Figures 6 (top left) and (top right) are useful for this purpose since one can determine the value of $C$, which gives the desired $\Delta$ and $\epsilon$ for a given $\kappa$. As we mentioned earlier, $\nu$ can be as large as $(N - n + 1)/2$ in the effective theory by tuning $\kappa$, while it is $O(\sqrt{N})$ in the original model. This implies that the computational cost for the supersymmetric models can be reduced from $O(N^5)$ to $O(N^{5/2})$ by the use of the effective theory with the optimal $\kappa$.

6. Summary and discussions

In this paper we have proposed a new method for probing the dynamics of the Lorentzian type IIB matrix model after the critical time, at which the spatial rotational symmetry is spontaneously broken. The basic idea is to consider the effective theory for the submatrices corresponding to the time region after the critical time. In particular, we make full use of the Lorentz invariance to fix the form of the action for the effective theory. This ansatz is verified by explicit calculations in the simplified models despite the fact that the Lorentz invariance is softly broken by the infrared cutoffs, which are inevitably introduced to make the models well defined. We have also demonstrated the usefulness of the method by reproducing the power-law behavior of the spatial extent observed in the bosonic model. By tuning the parameters in the effective theory, one can obtain the information of the original supersymmetric model with the matrix size $N$ with the cost of $O(N^{5/2})$, which is less than the cost for the bosonic models. We therefore consider that this is a major achievement in the simulation method for the Lorentzian type IIB matrix model. It is expected, for instance, that the analysis performed in Ref. [6] for the bosonic model has become feasible for the original supersymmetric model. We hope to report on the results in future publications.

Acknowledgements

We thank S. Iso and Y. Kitazawa for valuable comments and discussions. This research was supported by MEXT as a “Priority Issue on Post-K Computer” (Elucidation of the Fundamental Laws and Evolution of the Universe) and the Joint Institute for Computational Fundamental Science (JICFuS). Computations were carried out using the computational resources of the K computer provided by the RIKEN Advanced Institute for Computational Science through the HPCI System Research Project (Project ID: hp160210). The supercomputer FX10 at the University of Tokyo was used in developing our code for parallel computing, and computational resources such as KEKCC, NTUA het clusters, and FX10 at Kyushu University were used for calculations in Appendix B. T.A. and A.T. were supported in part by Grants-in-Aid for Scientific Research (Nos. 17K05425 and 15K05046, respectively) from the Japan Society for the Promotion of Science.

Funding

Open Access funding: SCOAP\(^3\).

\(^7\) Note that the number of data points of $R^2(t)$ is $(N - n + 1)$ since the index $K$ in Eq. (2.12) is restricted as $0 \leq K \leq N - n$, where $n$ is the block size used in defining Eq. (2.11). Therefore, $\nu$ cannot be larger than $(N - n + 1)/2$.\(^{10/13}\)
Table A1. The information related to Fig. 1 is given for each N.

| N  | n  | t_c | R(t_c) | ν | Δ     | ϵ    |
|----|----|-----|--------|---|--------|------|
| 256| 24 | -0.82166(6) | 0.30045(3) | 33 | 2.7380(7) | 0.08297(2) |
| 384| 28 | -0.76930(7) | 0.26580(3) | 37 | 2.8943(6) | 0.07823(2) |
| 512| 32 | -0.76559(7) | 0.24578(3) | 42 | 3.1150(7) | 0.07417(2) |

Table A2. The information related to Fig. 4 is given for each N, κ, and C.

| N  | κ  | C  | n  | t_c | R(t_c) | ν | Δ     | ϵ    |
|----|----|----|----|-----|--------|---|--------|------|
| 256| 1.0| 50 | 32 | -0.76795(4) | 0.29690(9) | 60 | 2.5867(2) | 0.043111(3) |
| 256| 2.0| 50 | 26 | -0.75087(5) | 0.26948(7) | 44 | 2.8490(3) | 0.064750(6) |
| 256| 1.0| 100| 32 | -0.71405(5) | 0.25310(13) | 55 | 2.7730(3) | 0.050418(5) |

Table A3. The information related to Figs. 5 and 6 is given for each N, κ, and C.

| N  | κ  | C  | n  | t_c | R(t_c) | ν | Δ     | ϵ    |
|----|----|----|----|-----|--------|---|--------|------|
| 32 | 1.0| 0  | 12 | -0.8044(4)  | 0.6559(8)  | 8  | 1.2251(1) | 0.1531(2) |
| 32 | 1.0| 5  | 12 | -1.0577(3)  | 0.5350(7)  | 10 | 1.9771(1) | 0.1976(1) |
| 32 | 1.0| 10 | 10 | -0.9194(4)  | 0.4491(9)  | 9  | 2.0492(2) | 0.2277(2) |
| 32 | 1.0| 20 | 8  | -0.7922(7)  | 0.3268(10) | 8  | 2.4243(3) | 0.3030(4) |
| 32 | 1.0| 30 | 8  | -0.7875(7)  | 0.2238(6)  | 8  | 3.5155(5) | 0.4393(6) |
| 32 | 2.0| 0  | 10 | -1.1028(9)  | 0.5618(7)  | 8  | 1.9632(1) | 0.2454(3) |
| 32 | 2.0| 5  | 10 | -1.1114(8)  | 0.4896(9)  | 8  | 2.2673(3) | 0.2833(4) |
| 32 | 2.0| 10 | 8  | -0.9438(11) | 0.4085(13) | 7  | 2.3114(1) | 0.3302(5) |
| 32 | 2.0| 20 | 8  | -0.9443(11) | 0.3173(12) | 7  | 2.9835(3) | 0.4261(7) |
| 32 | 2.0| 30 | 8  | -0.9398(10) | 0.2197(7)  | 7  | 4.2727(7) | 0.6103(10) |
| 32 | 3.0| 0  | 8  | -0.9408(13) | 0.5296(12) | 6  | 1.7743(3) | 0.2957(6) |
| 32 | 3.0| 5  | 8  | -1.1365(11) | 0.4327(10) | 7  | 2.6284(4) | 0.3754(6) |
| 32 | 3.0| 10 | 8  | -1.1416(13) | 0.3860(9)  | 7  | 2.9535(5) | 0.4218(8) |
| 32 | 3.0| 20 | 8  | -1.1449(13) | 0.2981(5)  | 7  | 3.8367(7) | 0.5480(9) |
| 32 | 3.0| 30 | 6  | -0.9462(18) | 0.1857(5)  | 6  | 5.0821(14)| 0.8470(23)|

Appendix A. Detailed information used in making the plots

In this appendix we present the detailed information used in making the plots in Figs. 1, 4, 5, and 6. The information includes: the block size n used in defining the extent of space $R^2(t)$ in Eq. (2.13) and the “moment of inertia tensor” (2.14); the values obtained for the critical time $t_c$ and the corresponding extent of space $R(t_c)$; the number $\nu$ of data points of $R^2(t)$ contained within $t_c < t \leq t_{peak}$; and the volume $\Delta$ and the lattice spacing $\epsilon$ in the temporal direction defined by Eqs. (5.1) and (5.2), respectively. Tables A1 and A2 show the information for Figs. 1 and 4, respectively, whereas Table A3 shows the information for Figs. 5 and 6.

Appendix B. Results for the 6D supersymmetric model

In this appendix we show the results of the analysis in Sect. 3 for the 6D version of the supersymmetric model. This confirms, in particular, that our method also works in the case where fermionic matrices are included.

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8 See Sect. 3 of Ref. [5] for its precise definition. In Eqs. (2.9) and (3.3), the Pfaffian Pf$M$ should be replaced by the determinant det$M$, which is real but not positive semidefinite. In simulating Eqs. (2.9) and (3.3), we take the absolute value of det$M$, assuming that configurations with negative det$M$ are negligible. See footnote 1.
We perform simulation of this model with the matrix size $N = 24$ and plot the expectation values (3.2) for the $\tilde{N} \times \tilde{N}$ submatrices against $\tilde{N}$ in Fig. B1 (left). Then, using these expectation values, we perform simulation of the effective theory for the $\tilde{N} \times \tilde{N}$ submatrices. In Fig. B1 (right), we plot the extent of space $\tilde{R}^2(\tilde{t})$ against $\tilde{t}$ for the effective theory with $\tilde{N} = 16$ and compare it with the results for the original model with $N = 24$. We find that the effective theory reproduces the late-time behaviors of the original model correctly.

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