Generating new supergravity solutions using Ehlers-Harrison-type transformations

Henrik Gustafsson and Parviz Haggi-Mani

ITP
University of Stockholm
Box 6730, Vanadisvägen 9
S-113 85 Stockholm
SWEDEN

Abstract: The technique of generating new solutions to 4D gravity/matter systems by dimensional reduction to a $\sigma$-model is extended to supersymmetric configurations of supergravity. The conditions required for the preservation of supersymmetry under isometry transformations in the $\sigma$-model target space are found. Some examples illustrating the technique are given.

1e-mail: henrik@physto.se
2e-mail: parviz@physto.se
1 Introduction

Methods of generating new solutions from known old ones in gravity/matter systems have been discussed extensively and the techniques employed go back to Ehlers [1], Harrison [2], Neugebauer, Kramer [3], Kinnersley [4], and Geroch [5]. The cases of axion/dilaton and dilaton systems with electromagnetic fields present were discussed rather recently by [6] and [7], respectively. In this approach one considers, e.g., 4D space-times with a non-null Killing vector field. The solutions are described by 3D gravity coupled to a nonlinear $\sigma$-model. The target space of the $\sigma$-model admits isometries which are then used to generate new solutions.

In this paper we discuss in detail supergravity systems and investigate the possibility of generating new supersymmetric solutions. In this case, the equations of motion can be found from a 3D supersymmetric $\sigma$-model coupled to supergravity. This is obtained by dimensional reduction of 4D supergravity. The reduction here is a simple or naïve one, based on the assumption that all the fields are independent of one coordinate which can be chosen to be along a space- or a time-like direction. For a supersymmetric $\sigma$-model coupled to gravity to exist in 3D the target space is restricted to be either (locally) Kähler or quaternionic depending on the number of supersymmetries. The isometries of the $\sigma$-model are then used to generate new supersymmetric solutions from a given supersymmetric “seed” solution. In order to guarantee the supersymmetry of the new solutions, the Killing spinor(s) of the seed solution must be independent of the isometry direction used to reduce the original Lagrangian. In the context of superstring theory the same condition has been found necessary for T-duality to preserve supersymmetry [8, 9]. The dimensionally reduced Killing spinors are then invariant under the target space isometry transformations of the $\sigma$-model.

This paper is organised as follows: In section 2 we describe the problem in general. In section 3 we explicitly reduce the $N = 1, D = 4$ supergravity action to $N = 2, D = 3$ supergravity coupled to a supersymmetric nonlinear $\sigma$-model, find the isometries, and give examples. In section 4 we reduce the bosonic sector of $N = 2, D = 4$ supergravity which results in a $\sigma$-model with a quaternionic target space structure. It can therefore be extended to a supersymmetric $\sigma$-model. We also generate some new solutions to illustrate the method. Our conclusions are summarized in section 5.
Generalities

In this section we describe the technique of generating solutions via dimensional reduction to a $\sigma$-model [1]-[7], and extend it to the supersymmetric case.

The action

$$S = \frac{1}{4\kappa^2} \int d^4x \sqrt{-G} \left( R + L_M \right)$$

represents 4D gravity coupled to matter. Here $G_{MN}, M, N = 0, 1, 2, 3,$ is the 4D metric and $L_M$ is the matter Lagrangian. With an appropriate ansatz (see below), the solutions to the field equations are then equivalent to those resulting from an action representing 3D gravity coupled to a nonlinear $\sigma$-model

$$S = \frac{1}{4\kappa^2} \int d^3x \sqrt{|g|} \left( R - \mathcal{G}_{\alpha\beta}(\phi^\alpha) \partial_\mu \phi^\alpha \partial^\mu \phi^\beta \right),$$

where the scalar fields $\phi^\alpha$ represent both matter and 4D gravitational fields and $\mathcal{G}_{\alpha\beta}(\phi^\alpha)$ is the target space metric. In particular this equivalence holds for pure 4D gravity if we consider space-times admitting a non-null Killing vector $K$. Choosing adapted coordinates such that the corresponding isometry is just a translation, i.e. $K = K^M \partial_M = \partial/\partial y$, the metric may be written in the form

$$ds^2 = G_{MN} dx^M dx^N = f^{-2} g_{\mu\nu} dx^\mu dx^\nu + (-1)^s f^2 (dy - \omega_\mu dx^\mu)^2, \quad f > 0,$$

where $s = 0(1)$ for a space-like(time-like) $K$ and the components of the 4D metric $f, \omega_\mu$ and $g_{\mu\nu}$ are independent of the “fourth coordinate” $y$. Here $g_{\mu\nu}$ is a metric on the 3D hypersurface coordinatized by $\{x^\mu\}$. This metric will have a Lorentzian(Euclidean) signature for reduction along a space-like(time-like) direction. The vector field $\omega_\mu$ is defined up to a gauge transformation $y' = y + \lambda(x^\mu)$ under which $\omega'_\mu = \omega_\mu - \partial_\mu \lambda$. We can also include matter couplings consisting of massless scalar and abelian vector fields if they are independent of $y$ [10].

The action (2) may be obtained from (1) via naïve dimensional reduction followed by dualisation of the twist vector $\omega_\mu$ and the vector fields in $L_M$. This action is invariant under isometries $\phi'^\alpha = \phi'^\alpha(\phi^\beta)$ of the $\sigma$-model target space: Infinitesimally

$$\phi'^\alpha = \phi^\alpha + \xi^\alpha(\phi),$$

Throughout this paper we use a 4D metric with signature $(-, +, +, +)$. Curved indices are denoted by $M, N, \ldots$ and tangent space indices by $A, B, \ldots$. In 3D these are denoted $\mu, \nu, \ldots$ and $a, b, \ldots$, respectively. Underlining denotes a tangent space index, i.e. $A = (0, 1, 2, 3)$ whereas $M = (0, 1, 2, 3)$. 

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where $\xi_\alpha$ is a Killing vector. The full set of Killing vectors generate the isometries. They are found by solving the Killing equation

$$\xi(\alpha;\beta) = 0$$

where $: \,$ denotes a target space covariant derivative. The invariance implies that under an isometry transformation a set of solutions $(\phi^\alpha, g_{\mu\nu})$ transforms into a new set of solutions $(\phi'^\alpha, g'_{\mu\nu})$. Starting from one ("seed") solution an isometry thus generates a new solution, which is in general physically different from the old one (e.g. a stationary solution can be obtained from a static one).

In the supersymmetric case we expect the above results to extend to say that $4D$ supergravity/matter systems for a certain set of solutions are equivalent a $3D$ supersymmetric nonlinear $\sigma$-model coupled to supergravity. We will explicitly demonstrate this below for $N = 1$, $4D$ supergravity by dimensional reduction. For the supersymmetric $3D$ action to be invariant under the above isometries the fermionic partners $\chi^\alpha$ of $\phi^\alpha$ must also transform under the isometry transformations: Infinitesimally,

$$\delta \chi^\alpha = \partial_\beta \xi^\alpha \chi^\beta.$$  

In particular, a purely bosonic configuration cannot transform into a configuration with fermions under an isometry.

In order for a bosonic configuration of supergravity/matter to be supersymmetric, i.e. preserve some supersymmetry, there must exist a spinor $\epsilon$ satisfying the Killing spinor equations,

$$\delta_\epsilon \Psi_M = 0, \quad \delta_\epsilon \Pi = 0,$$

where $\Psi_M$ is the Rarita-Schwinger fermion and $\Pi$ represents fermionic matter fields. If the Killing spinor of the $4D$ configuration depends on $y$, the configuration will not be supersymmetric from the $3D$ point of view. We therefore make the extra assumption that the $4D$ Killing spinors are independent of $y$. It is important to note that even if the bosonic fields are taken to be independent of the isometry direction $y$, a Killing spinor does not necessarily have this property (examples will be given below). The corresponding $3D$ equations obtained by the dimensional reduction to a supersymmetric $\sigma$-model are of the form (see e.g. [11])

$$\delta_\epsilon \psi_\mu = \hat{D}_\mu \epsilon = 0,$$

$$\delta_\epsilon \chi^\alpha = \phi^\alpha \epsilon = 0,$$
where supersymmetry indices have been suppressed, $\psi_\mu$ refers to the 3D gravitino and $\hat{D}_\mu$ is a derivative containing 3D spin- and target space connections. The first equation is invariant under a target space isometry and the second one transforms as a vector; thus $\epsilon$ will satisfy $\delta_\epsilon \psi'_\mu = \delta_\epsilon \chi'_\mu = 0$. This proves that $\epsilon$ is a Killing spinor of the new configuration if it is so for the original configuration. Hence, given a supersymmetric set of solutions $(\phi^\alpha, g_{\mu\nu}, \epsilon)$ we can generate a new supersymmetric set of solutions $(\phi'^\alpha, g_{\mu\nu}, \epsilon)$ using isometry transformations $\phi'^\alpha = \phi^\alpha(\phi^\beta)$.

### 3 N=1 Supergravity

Here, following along the lines of Scherk and Schwarz [12, 13], we will derive the 3D supersymmetric action from 4D by dimensional reduction.

We start with the $N = 1, D = 4$ supergravity Lagrangian [14, 15],

$$
\mathcal{L} = \frac{1}{4\kappa^2 E R(\Omega)} - i \frac{1}{2} \varepsilon^{MNPQ} \Psi_M \Gamma_5 \Gamma_P \Psi_Q,
$$

where $\Psi_M$ is the gravitino (with Dirac conjugate $\Psi^M \equiv \Psi_M^{\dagger} i \Gamma^5$), $E$ is the determinant of the vierbein field $E^A_M$, and $R(\Omega) = E^{MA} E^{NB} R_{MNAB}(\Omega)$ is the Ricci scalar ($E^A_M$ is the inverse vierbein). The Riemann tensor and spin connection are given by

$$
R^{AB}_{MN}(\Omega) = \partial_M \Omega^A_B + \Omega^D_A \Omega^B_M - \frac{\kappa^2}{2} C^{AB}_{MN},
$$

and

$$
\Omega^{0}_{MAB} = \Omega_{MAB} - \frac{\kappa^2}{2} C_{MAB},
$$

respectively, where $\Omega^{0}_{MAB}$ is the torsionless spin connection of ordinary gravity defined by

$$
\Omega^{0}_{MAB} = \frac{1}{2} E_A^N \partial_M E_N^B - \frac{1}{2} E_A^R E_B^S E_M^D \partial_R E_{SD} - (A \leftrightarrow B)
$$

and

$$
C_{NAB} = \nabla_N \Gamma_B \Psi_A - \nabla_N \Gamma_A \Psi_B - \nabla_A \Gamma_N \Psi_B.
$$

$D_P$ is the covariant derivative which acts on the spinors as

$$
D_M \Psi_N = (\partial_M + \frac{1}{4} \Omega_M^{AB} \Gamma_{AB}) \Psi_N.
$$

---

4The 4D Gamma matrices, denoted by $\Gamma^A$, obey the Clifford algebra $\{\Gamma^A, \Gamma^B\} = 2\eta^{AB}$. We also define $\Gamma_5 = i\Gamma_1 \Gamma_2 \Gamma_3 \Gamma_0$ and $\Gamma^{AB} = \frac{1}{2} [\Gamma^A, \Gamma^B]$. 

This theory is invariant under local supersymmetry ($\epsilon$), local Lorentz ($\lambda$), and general coordinate ($\xi$) transformations:

$$\delta E_M^A = \kappa \Gamma^A \Psi_M + E_M^B \lambda_{BA} + \xi^N \partial_N E_M^A + \partial_M \xi^N E_N^A,$$

$$\delta \Psi_M = \frac{1}{\kappa} D_M(\Omega) \epsilon + \frac{1}{4} \lambda^{AB} \Gamma_{AB} \Psi_M + \xi^N \partial_N \Psi_M + \partial_M \xi^N \Psi_N.$$  

(16)  

(17)

There are no auxiliary fields included in this formulation. Therefore the supersymmetry algebra closes only modulo the fermionic field equations.

### 3.1 Dimensional Reduction

We choose to consider solutions with a space-like Killing vector ($s = 0$ in (3)). Splitting the 4-indices $A = (a, 3), M = (\mu, 3)$ and writing the vierbein as

$$E_M^A = \left(\begin{array}{cc} f^{-1} e_\mu^a & -f \omega_\mu \\ 0 & f \end{array} \right),$$

(18)

the bosonic part of the action is reduced to

$$S_{bos} = \frac{1}{4 \kappa^2} \int d^3 x \epsilon \left[ R - 2 f^{-2} \partial_\mu f \partial^\mu f - \frac{1}{4} f^4 B_{\mu \nu} B^{\mu \nu} \right],$$

(19)

where we have defined the twist field $B_{\mu \nu} = \partial_{[\mu} \omega_{\nu]}$.

To reduce the remaining part of the action it is appropriate to decompose the four-component Majorana spinors $\Psi$ above into two-component spinors which will become Majorana spinors $\psi$ in three dimensions by choosing the following representation for the Clifford algebra (see the appendix for our choice of 3D gamma matrices)

$$\Gamma^a = \gamma^a \otimes \tau_3, \quad \Gamma^3 = 1 \otimes \tau_2, \quad \Gamma^5 = -1 \otimes \tau_1,$$

(20)

where $\tau_{1,2,3}$ are the Pauli matrices. In this basis $\Psi_M$ takes the form

$$\Psi_M = \left( \begin{array}{c} \psi_1 \\ i \psi_2 \end{array} \right)_M.$$

(21)

After some simplification we are left with

$$S_f = \int d^3 x \left[ \frac{f^{-1}}{2} \varepsilon^{\mu \nu \rho} \varepsilon_{ij} \left( \bar{\psi}_i \gamma_\mu D_\nu \chi^j + \bar{\chi}^i \gamma_\mu D_\nu \psi^j \right) \\
+ \frac{f}{2} \varepsilon^{\mu \nu \rho} \left( -\omega_\mu \bar{\psi}_i \nabla_\nu \psi^j + \omega_\mu \bar{\psi}^i \nabla_\nu \psi^j - \bar{\psi}_i \nabla_\nu \psi^j \right) + e f^{-2} \varepsilon_{ij} \bar{\psi}_i \chi^j \partial^\mu f \\
+ \frac{f^3}{4} \varepsilon_{ij} \bar{\psi}_i \chi^j \omega_\mu B^{\mu \nu} + \frac{f}{4} e \bar{\psi}_i \gamma_\mu \chi^j B^{\mu \nu} - \frac{f^3}{8} e \varepsilon_{ij} \bar{\psi}_i \psi^j B^{\mu \nu} \right] + S_{\psi^4},$$

(22)
where \( \chi_i \equiv (\psi_i)_3 \) and \( S_{\psi^4} \) represents terms quartic in the fermi fields.

The local transformations (16) can in general transform the vierbein fields out of the
gauge \( E_a^3 = 0 \). To prevent this, we require \( \delta E_a^3 = 0 \). This implies
\[
\lambda_a^3 = -\kappa f^{-1} \gamma^a \chi^i. \tag{23}
\]
Furthermore, \( \delta e_a^\mu \) can be brought to canonical form,
\[
\delta e_a^\mu = \kappa \bar{e}^i \gamma^a \psi^i_\mu, \tag{24}
\]
by the following redefinitions,
\[
\tilde{\psi}^i_\mu = f^{1/2} (\psi^i_\mu + \omega_\mu \chi^i + f^{-2} \gamma^i \epsilon_{ij} \chi^j), \tag{25}
\]
\[
\tilde{\chi}^i = f^{-3/2} \chi^i, \tag{26}
\]
\[
\tilde{e}^i = f^{1/2} \dot{e}^i, \tag{27}
\]
provided this is accompanied by a compensating Lorentz transformation with parameter
\[
\lambda^{ba} = \kappa f^{-1} \gamma_{ba} \epsilon_{ij} \chi^j. \tag{28}
\]
The reduced action then reads (dropping the tildes)
\[
S = \int d^3 x \left\{ \frac{1}{4 \kappa} \bar{e}^i \left[ R - 2 f^{-2} \partial_\mu f \partial^\mu f - \frac{1}{4} f^4 B_{\mu \nu} B^{\mu \nu} \right] - \frac{1}{2} \varepsilon^{\mu \nu \rho \sigma} \bar{\psi}_i_\mu \gamma^i \partial_\rho f \partial^\sigma f \right. \\
- \frac{1}{8} f^2 \varepsilon^{\mu \nu \rho \sigma} \bar{\psi}_i_\mu \gamma^i \partial_\rho f \partial^\sigma f - e \bar{\chi}^i \partial_\mu \chi^i + \frac{1}{4} f^2 \varepsilon^{\mu \nu \rho \sigma} \gamma^i \chi^j \varepsilon_{ij} B_{\mu \nu} \right. \\
- \left. e f^{-1} \chi^i \gamma^i \partial_\mu \psi^j_\mu \varepsilon_{ij} \mu \nu \rho \sigma \psi^j_\mu \chi^i B_{\rho \sigma} - \frac{1}{2} e f^2 \chi^i \gamma^i \chi^j \varepsilon_{ij} B_{\mu \nu} \right) + S_{\psi^4}. \tag{29}
\]
In terms of the redefined fields the supersymmetry transformations are
\[
\delta e_a^\mu = \kappa \bar{e}^i \gamma^a \psi^i_\mu, \tag{30}
\]
\[
\delta f = \kappa f \dot{\psi}^i \gamma^i \varepsilon_{ij}, \tag{31}
\]
\[
\delta \omega_\mu = -f^{-2} \dot{\psi}^i_\mu \varepsilon_{ij} - 2 f^{-2} \dot{\psi}^i_\mu \gamma^i \chi^j \tag{32}
\]
\[
\delta \psi^i_\mu = \kappa^{-1} \partial_\mu \psi^i + \frac{1}{8 \kappa} f^2 \varepsilon_{\mu \rho \sigma} B_{\rho \sigma} \dot{\psi}^i \varepsilon_{ij} + O(\psi^2 \varepsilon), \tag{33}
\]
\[
\delta \chi^i = \frac{1}{8 \kappa} f^2 B_{\mu \nu} \gamma^i_{\mu \nu} \dot{\psi}^i - \frac{1}{2 \kappa} f^{-1} \partial_\mu \psi^j_\mu \varepsilon_{ij} + O(\psi^2 \varepsilon). \tag{34}
\]
As a final step, we will dualise the twist vector \( \omega_\mu \) into a scalar (the twist potential).
The field strength \( B_{\mu \nu} \) can be treated as an independent field by adding a Lagrange
multiplier field \( \sigma \) to the action (29):
\[
S' = S + \frac{1}{2} \int d^3 x \; \sigma \varepsilon^{\mu \nu \rho} \partial_\mu B_{\nu \rho}. \tag{35}
\]
This guarantees that $B_{\mu\nu}$ satisfies the Bianchi identity. The resulting field equation for $B_{\mu\nu}$ is

$$B_{\mu\nu} = -f^{-4}e^{-1}\varepsilon^{\mu\nu}\partial_{\rho}\sigma + 8\kappa^2 f^{-4}\Pi_{\mu\nu} + O(\psi^4).$$

(36)

where $\Pi_{\mu\nu}$ is a shorthand notation for the fermionic bilinears which are contracted with $B_{\mu\nu}$ in the action (29). Inserting this into $S'$ and also letting $\phi = f^2$ we finally obtain the action for a $N = 2$ (three dimensional) supersymmetric nonlinear $\sigma$-model coupled to supergravity,

$$S' = \int d^3x \left\{ \frac{1}{4\kappa^2} e \left[ R - \frac{1}{2\varphi^2} (\partial_{\mu}\phi \partial^{\mu}\phi + \partial_{\mu}\sigma \partial^{\mu}\sigma) \right] - \frac{1}{2} \varepsilon^{\mu\nu\rho}\phi \psi_{\mu} (\partial_{\nu}\psi_{\rho} - \frac{1}{4\varphi} \psi_{i} \varepsilon^{ij} \partial_{\rho}\sigma) \right. - e\chi^{i} (D_{\mu}\psi^{i} - \frac{1}{2\varphi} \partial_{\sigma}\psi^{i} \varepsilon^{ij}) - \frac{1}{2} \varphi^{-1} \chi^{i} \gamma^{\mu} \partial_{\rho}\sigma \psi_{\mu}^{i} \left. + \frac{1}{2} \varphi^{-1} e\chi^{i} \gamma^{\mu} \partial_{\sigma}\psi_{\mu}^{i} \right\} + S_{\psi^4}.$$

(37)

The local supersymmetry transformations which leave (37) invariant are

$$\delta e_{\mu} = \kappa \tau^{a} e_{\mu}$$

(38)

$$\delta \phi = 2\kappa \tau^{a} \chi^{j} \varepsilon^{ij}$$

(39)

$$\delta \sigma = 2\kappa \tau^{a} \chi^{i}$$

(40)

$$\delta \psi_{\mu} = \kappa^{-1} D_{\mu} e^{i} + \frac{1}{4\kappa} \varphi^{-1} \partial_{\mu}\sigma \varepsilon^{ij} + O(\psi^2 \epsilon)$$

(41)

$$\delta \chi^{i} = \frac{1}{4\kappa} \varphi^{-1} \partial_{\sigma} e^{i} - \frac{1}{4\kappa} \varphi^{-1} \partial_{\sigma} \varphi \varepsilon^{ij} + O(\psi^2 \epsilon).$$

(42)

The commutator of two such transformations gives local supersymmetry, general coordinate, and Lorentz transformations. Closure of the algebra requires the use of the Fermi-field equations of motion since we have no auxiliary fields.

From the action (37) we can read off the target space metric. It is given by the line element

$$ds^2 = G_{\alpha\beta} d\phi^\alpha d\phi^\beta = \frac{1}{2\varphi^2} (d\varphi^2 + d\sigma^2).$$

(43)

Changing variables $z = \sigma + i\varphi$ and $\bar{z} = \sigma - i\varphi$, it is easily seen that this transforms into

$$G_{z\bar{z}} dz d\bar{z} = -\frac{2}{(z - \bar{z})^2}dz d\bar{z} \equiv \partial_z \partial_{\bar{z}} K(z, \bar{z}).$$

(44)

Hence the target space is a Kähler manifold with Kähler potential $K(z, \bar{z}) = -2 \ln(z - \bar{z})$. This manifold is recognized as $SL(2, R)/SO(2)$. 

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We conclude this subsection by some remarks on the time-like reduction \((s = 1\) in \((3)\)). Starting from the Lagrangian \((10)\) reduction of the bosonic part will again give \((19)\) but with an opposite sign in front of the \(B^2\)-term. However, dualising the twist vector \(\omega_\mu\) gives an additional factor \((-1)^s\) from the contraction of two \(\varepsilon\)-tensors. The resulting \(\sigma\)-model will therefore have the same target space metric \((13)\) as above. Also, the \(3D\) fermions will now satisfy a Euclidean Clifford algebra. The main difference is that there are no Majorana spinors for this signature of the metric. It is, however, possible to impose a generalized Majorana condition on the spinors (see \(e.g.\) [16]). This is indeed the condition that follows from the \(4D\) Majorana condition by using a decomposition analogous to \((20)\).

### 3.2 Isometries

Given the metric \((43)\), there are three linearly independent solutions to the Killing equation \((3)\),

\[
\begin{align*}
\tilde{\xi}_1 &= 2\varphi \partial_\varphi + 2\sigma \partial_\sigma, \\
\tilde{\xi}_2 &= \partial_\sigma, \\
\tilde{\xi}_3 &= -4\varphi \sigma \partial_\varphi + 2(\varphi^2 - \sigma^2) \partial_\sigma.
\end{align*}
\]

These Killing vectors generate a 3-parameter group of target space isometries and satisfy the \(SL(2, R)\) algebra:

| \([\tilde{\xi}_i, \tilde{\xi}_j]\) | \(\tilde{\xi}_1\) | \(\tilde{\xi}_2\) | \(\tilde{\xi}_3\) |
|------------------|----------|----------|----------|
| \(\tilde{\xi}_1\) | 0        | \(-2\tilde{\xi}_2\) | \(2\tilde{\xi}_3\) |
| \(\tilde{\xi}_2\) | \(2\tilde{\xi}_2\) | 0        | \(-2\tilde{\xi}_1\) |
| \(\tilde{\xi}_3\) | \(-2\tilde{\xi}_3\) | \(2\tilde{\xi}_1\) | 0        |

Exponentiating the generators one can find how finite transformations act on the target space fields;

\[
\phi^\alpha = e^{\lambda_r \xi_r} \phi^\alpha,
\]

(no summation over \(r\)) where \(\lambda_r\) is the group parameter corresponding to the generator \(\xi_r\). The finite transformations are,

\[
\begin{align*}
\xi_1 & : \quad \varphi' = e^{2\lambda_1} \varphi, & \sigma' &= e^{2\lambda_1} \sigma \\
\xi_2 & : \quad \varphi' = \varphi, & \sigma' &= \sigma + \lambda_2 \\
\xi_3 & : \quad \varphi' = \frac{\varphi}{\gamma(\varphi^2 + \sigma^2) + 2\gamma\sigma + 1}, & \sigma' &= \frac{\gamma(\varphi^2 + \sigma^2) + \sigma}{\gamma(\varphi^2 + \sigma^2) + 2\gamma\sigma + 1}.
\end{align*}
\]
The first two correspond to a global rescaling of the 4D metric and a trivial gauge transformation of the twist potential, respectively. The third one (Ehlers transform) is somewhat more complicated. The transformations \((47)\) are most easily found by noting that the complex upper half-plane \(z = \sigma + i\varphi, \varphi > 0\) with the metric \((43)\) represents a 2D surface with constant negative curvature. As is well-known, the \(SL(2, R)\)-transformations act on the coordinates as

\[
z' = \frac{az + \beta}{\gamma z + \delta}, \quad \alpha\delta - \gamma\beta = 1, \tag{48}
\]

where \(\alpha, \beta, \gamma\) and \(\delta\) are real parameters. Choosing different combinations of the parameters we find the above transformations of \(\varphi\) and \(\sigma\). The nontrivial Ehlers transform is given by \(\alpha = 1, \beta = 0, \delta = 1\).

For the full supersymmetric \(\sigma\)-model action to be invariant under finite isometry transformations of \(\sigma\) and \(\varphi\), their fermionic partners must transform as

\[
\chi'^i = A\chi^i + B\epsilon^{ij}\chi^j, \tag{49}
\]

where

\[
A = \frac{\gamma^2(-\varphi^2 + \sigma^2) + 2\gamma\sigma + 1}{\gamma^2(\varphi^2 + \sigma^2) + 2\gamma\sigma + 1}, \quad B = \frac{2\gamma^2(\varphi\sigma) + 2\gamma\varphi}{\gamma^2(\varphi^2 + \sigma^2) + 2\gamma\sigma + 1}. \tag{50}
\]

These transformations are found by demanding that the form of the supersymmetry transformations \((38)\) are unchanged by the isometries \((47)\).

### 3.3 Examples

i) Based on the results above we exemplify the solution-generating technique on plane polarized (pp) gravitational waves. These space-times are known to be supersymmetric \([17, 18]\). We have to restrict ourselves to solutions that admit at least one space-like Killing vector. With an appropriate choice of the twist potential and the three dimensional metric elements our initial metric \((3)\) can be cast in the pp-wave solution form:

\[
ds^2 = |d\zeta|^2 + 2dUdV + 2H(\zeta, \bar{\zeta})dU^2 \tag{51}
\]

where,

\[
\zeta = x^1 + ix^2, \quad U = \frac{1}{\sqrt{2}}(x^3 - x^0), \quad V = \frac{1}{\sqrt{2}}(x^3 + x^0), \tag{52}
\]

and \(H(\zeta)\) is a harmonic function, \(\partial_\zeta \partial_{\bar{\zeta}} H = 0\). From the 4D Killing spinor equation, \(\mathcal{D}_M \epsilon = 0\), it can be shown that the Killing spinor is independent of \(x^3\) (in fact the Killing
spinor is constant in this case). Comparing the metrics (3) and (51) and using $\varphi = f^2$ we find the relations

$$H = \frac{\varphi - 1}{2}, \quad \omega_0 = 1 - \varphi^{-1}. \quad (53)$$

All functions depend on $\zeta$ and $\bar{\zeta}$ only. An isometry transformation (see (47)) results in a new pp-wave solution characterized by the transformed harmonic function $H' = \frac{1}{2}(\varphi' - 1)$. Note also that differentiating the second equation and using (36) on $e$ finds the Cauchy-Riemann equations; i.e. $z = \sigma + i\varphi$ is an analytic function of $\zeta$. This requirement will solve the 3D Killing spinor equation $\delta_\epsilon \chi^i = 0$ (also, the integrability condition for $\delta_\epsilon \psi^i = 0$ is satisfied for any static 3D metric $h_{\mu \nu}$).

ii) A simple example which illustrates the importance of the assumption that the Killing spinors are independent of the isometry direction is flat space in polar coordinates,

$$ds^2 = -dt^2 + d\rho^2 + \rho^2 d\theta^2 + dx^3 dx^3, \quad \rho^2 = (x^1)^2 + (x^2)^2, \tan \theta = x^2/x^1. \quad (54)$$

With the choice $\varphi = f^2 = \rho^2$, $\omega_\mu = 0$ and $g_{\mu \nu} = \rho^2 \eta_{\mu \nu}$, this is of the form (3) with $\theta$ as the isometry direction. The Killing spinor equation is solved by $\epsilon = e^{\frac{1}{2}F_{\perp z}^{*} e_0}$ where $e_0$ is a constant spinor. Performing a nontrivial target space isometry transformation, using the third equation in (17), we generate $\varphi' = \rho^2/((\gamma^2 \rho^4 + 1)$. The Ricci scalar for this solution is nonvanishing. Therefore it can neither be a flat space-time nor a pp-wave because these have $R = 0$. Since these are the only possible $N = 1$ supersymmetric space-times we conclude that the generated solution is not supersymmetric.

## 4 N=2 Supergravity

In this section $N = 2$ supergravity subject to the isometry assumptions given in section 2 is considered. We anticipate that $N = 2$, $D = 4$ supergravity, when reduced to 3D, will consist of an $N = 4$ locally supersymmetric nonlinear $\sigma$-model. The signature of the 3D submanifold will be Lorentzian or Euclidean depending on whether the reduction has been performed along a time- or space-like direction. To ensure $N = 4$ supersymmetry the $\sigma$-model target space is required to have quaternionic structure. Our course of action will consist of dimensional reduction of the bosonic part of the Lagrangian only. Using the isometries of the target space manifold we can then generate new supersymmetric solutions.
The $N = 2, D = 4$ supergravity Lagrangian is \[20\]
\[
\mathcal{L} = \frac{E}{4k^2} R - \frac{E}{2} \nabla^i \nabla_i \Gamma^{MPN} D_N \Psi^i - \frac{E}{4} F_{MN}^2
\]
\[
- \frac{\kappa}{4\sqrt{2}} \nabla^i_M [(F^{MN} + \hat{F}^{MN}) + \frac{\Gamma_5}{2} (\tilde{F}^{MN} + \tilde{\hat{F}}^{MN})] \Psi^i_N \epsilon^{ij},
\]
where $F_{MN}, \hat{F}_{MN}$ and $\tilde{F}_{MN}$ denote the electromagnetic, dual, and the supercovariantized field strength, respectively. Its bosonic part is the Einstein-Maxwell Lagrangian.

4.1 Reduction

The dimensional reduction of the bosonic part of the Lagrangian (55) is done along the lines of section 3. Since the most interesting solutions contain a time-like Killing vector, we will focus on this case. This amounts to having a reduced theory with a Euclidean signature of the $3D$ metric. As remarked in the previous section, the pure gravitational part will give (19) with a different sign in front of the $B^2$-term. The assumption that $F_{MN}$ is time-independent enables us to introduce electric and magnetic potentials $v$ and $u$ through

\[
F_{\mu 0} = \frac{1}{\sqrt{2}} \partial_\mu v,
\]
\[
F_{\mu \nu} = \frac{\varphi}{\sqrt{2}} \varepsilon_{\mu \nu \rho} \partial^\rho u.
\]

The twist potential $\sigma$ is now defined by the relation

\[
\partial_\mu \sigma + v \partial_\mu u - u \partial_\mu v = -\varphi^2 \varepsilon_{\mu \nu \rho} \partial^\nu \omega^\rho,
\]
to ensure consistency with the Einstein equations \[21\].

The reduction of the bosonic part of (55) can then be cast into the form (3) where the $\sigma$-model target space metric is given by

\[
ds^2 = \frac{1}{2\varphi^2} \left\{ d\varphi^2 + (d\sigma + v du - u dv)^2 \right\} - \frac{1}{\varphi} \left\{ dv^2 + du^2 \right\}.
\]

This manifold is an Einstein space with constant negative curvature; the Ricci tensor satisfies:

\[
\mathcal{R}_{\alpha \beta} = -\frac{1}{4} (8 + d) \mathcal{G}_{\alpha \beta},
\]
where $d = 4k$ is the number of scalars (four in our case). In fact, this noncompact manifold can be identified as $SU(2,1)/S(U(1,1) \times U(1)) \ [10]$. Its holonomy group is contained in $Sp(k) \times Sp(1) \ [22]$. Manifolds that have these properties are called quaternionic. This is precisely the structure needed on the target space in order to have an $N = 4, D = 3$ supersymmetric nonlinear $\sigma$-model coupled to supergravity \[11\].
4.2 Isometries

We now turn to the isometries of the $\sigma$-model. Solving the Killing vector equation (5) we find that the target space (59) possesses 8 Killing vectors:

\[
\tilde{\xi}_1 = 2\varphi \partial_{\varphi} + 2\sigma \partial_{\sigma} + u \partial_u + v \partial_v, \tag{61}
\]

\[
\tilde{\xi}_2 = v \partial_{\sigma} + \partial_u, \tag{62}
\]

\[
\tilde{\xi}_3 = u \partial_u - \partial_v, \tag{63}
\]

\[
\tilde{\xi}_4 = \partial_{\sigma}, \tag{64}
\]

\[
\tilde{\xi}_5 = v \partial_u - u \partial_v, \tag{65}
\]

\[
\tilde{\xi}_6 = 2\varphi u \partial_{\varphi} + [u\sigma + \frac{v}{2}(u^2 + v^2) - v\varphi] \partial_{\sigma} + \frac{1}{2}(u^2 - 3v^2 + 2\varphi) \partial_u + (2uv - \sigma) \partial_v, \tag{66}
\]

\[
\tilde{\xi}_7 = 2\varphi v \partial_{\varphi} + [v\sigma - \frac{u}{2}(u^2 + v^2) + u\varphi] \partial_{\sigma} + (2uv + \sigma) \partial_u + \frac{1}{2}(v^2 - 3u^2 + 2\varphi) \partial_v, \tag{67}
\]

\[
\tilde{\xi}_8 = -4\varphi \sigma \partial_{\varphi} + [-2\varphi(u^2 + v^2) + \frac{1}{2}(u^2 + v^2)^2 + 2(\varphi^2 - \sigma^2)] \partial_{\sigma}
- (2u\sigma + u^2v + v^3 - 2\varphi v) \partial_u + (-2v\sigma + uv^2 + u^3 - 2\varphi u) \partial_v, \tag{68}
\]

which satisfy the $SU(2,1)$ isometry algebra:

\[
\begin{array}{cccccccc}
| \xi_i, \xi_j | & \tilde{\xi}_1 & \tilde{\xi}_2 & \tilde{\xi}_3 & \tilde{\xi}_4 & \tilde{\xi}_5 & \tilde{\xi}_6 & \tilde{\xi}_7 & \tilde{\xi}_8 \\
\tilde{\xi}_1 & 0 & -\tilde{\xi}_2 & -\tilde{\xi}_3 & -2\tilde{\xi}_4 & 0 & \tilde{\xi}_6 & \tilde{\xi}_7 & 2\tilde{\xi}_8 \\
\tilde{\xi}_2 & 0 & 2\tilde{\xi}_4 & 0 & \tilde{\xi}_3 & \tilde{\xi}_1 & 3\tilde{\xi}_5 & -2\tilde{\xi}_7 & \tilde{\xi}_6 \\
\tilde{\xi}_3 & 0 & 0 & 0 & -\tilde{\xi}_2 & 3\tilde{\xi}_5 & -\tilde{\xi}_1 & -2\tilde{\xi}_6 & \tilde{\xi}_4 \\
\tilde{\xi}_4 & 0 & 0 & 0 & \tilde{\xi}_3 & \tilde{\xi}_2 & -2\tilde{\xi}_1 & 0 & \tilde{\xi}_5 \\
\tilde{\xi}_5 & 0 & \tilde{\xi}_7 & -\tilde{\xi}_6 & 0 & 0 & 0 & 0 & \tilde{\xi}_6 \\
\tilde{\xi}_6 & 0 & 0 & 0 & \tilde{\xi}_8 & 0 & 0 & 0 & 0 \\
\tilde{\xi}_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\tilde{\xi}_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

It is convenient to introduce Ernst potentials $\mathcal{E} = \Xi + i\sigma$ and $\Phi = \frac{1}{\sqrt{2}}(u + iv)$ where $\Xi = \varphi - \frac{1}{2}(u^2 + v^2)$. Then the finite transformations are given by [3, 4]:

\[
I. \quad \mathcal{E}' = \alpha \bar{\alpha} \mathcal{E}, \quad \Phi' = \alpha \Phi \\
II. \quad \mathcal{E}' = \mathcal{E} + ib, \quad \Phi' = \Phi \\
III. \quad \mathcal{E}' = \mathcal{E}/(1 + ic\mathcal{E}), \quad \Phi' = \Phi/(1 + ic\mathcal{E}) \tag{69}
\]

\[
IV. \quad \mathcal{E}' = \mathcal{E} - 2\bar{\beta}\Phi - \beta\bar{\beta}, \quad \Phi' = \Phi + \beta \\
V. \quad \mathcal{E}' = \mathcal{E}/(1 - 2\bar{\gamma}\Phi - \gamma\bar{\gamma}\mathcal{E}), \quad \Phi' = (\Phi + \gamma\mathcal{E})/(1 - 2\bar{\gamma}\Phi - \gamma\bar{\gamma}\mathcal{E})
\]
The parameters $b$, $c$ are real and $\alpha$, $\beta$, $\gamma$ are complex.

The first transformation is an electromagnetic duality rotation for $|\alpha| = 1$. If the modulus of $\alpha$ is different from unity, the effect of this transformation may be shown to be a scaling, $ds^2 \rightarrow |\alpha|^2 ds^2$, of the 4D metric. The second and fourth are gauge transformations of the potentials. Only the third and fifth transformations are nontrivial.

4.3 Examples

i) In our first example we use static supersymmetric space-times as “seed” solutions. It is well known that these must be of the Papapetrou-Majumdar type [17, 23],

$$ds^2 = -f^2 dt^2 + f^{-2} \delta_{\mu\nu} dx^\mu dx^\nu, \quad f = \frac{v}{\sqrt{2}}.$$  \hspace{1cm} (70)

In accordance with equation (58) we put $u = \sigma = 0$. The $N = 2$, 4D Killing spinor equation is

$$\partial_M \epsilon + \frac{1}{4} \Omega_M^{0\, AB} \Gamma_{AB} \epsilon + \frac{1}{4} F_{NP} \Gamma^{NP} \Gamma_M \epsilon = 0,$$  \hspace{1cm} (71)

where $\epsilon$ is a Dirac spinor. This equation has a solution $\epsilon$, subject to the constraint $\Gamma^0 \epsilon = \epsilon$, such that $\partial_t \epsilon = 0$. Supersymmetry will then be preserved under the transformations I-V.

The finite transformations III leave the solution invariant. The transformations V give:

$$u' = -\frac{\sqrt{2} \gamma_1 v^2}{(1 - \sqrt{2} \gamma_2 v)^2 + 2(\gamma_1 v)^2}, \quad v' = \frac{v(1 - \sqrt{2} \gamma_2 v)}{(1 - \sqrt{2} \gamma_2 v)^2 + 2(\gamma_1 v)^2},$$

$$\psi' = \frac{2 \gamma_1 v^4 + v^2 (1 - \sqrt{2} \gamma_2 v)^2}{[(1 - \sqrt{2} \gamma_2 v)^2 + 2(\gamma_1 v)^2]^2}, \quad \gamma = \gamma_1 + i \gamma_2.$$  \hspace{1cm} (72)

For $\gamma_1 = 0$ we generate a solution of the type (70). These transformation belongs to the “static” subgroup $SL(2, R)$ of the full isometry group $SU(2, 1)$. On the other hand starting from an electrically charged solution, a general $\gamma$ generates a nonzero magnetic potential. From equation (58), we see that the new solution is not necessarily static. Rather it belongs to the more general class of stationary supersymmetric solutions known as Israel-Wilson-Perjes (IWP) metrics [21]. Note that if the original solution is asymptotically flat so will the new one be.

ii) Next we start from the Robinson-Bertotti solution [24]. The Robinson-Bertotti solution is the only metric belonging to the IWP class which preserve all supersymmetries. The metric can be written in the form

$$ds^2 = (1 - \lambda y^2) dx^2 + (1 - \lambda y^2)^{-1} dy^2 + (1 - \lambda z^2)^{-1} dz^2 - (1 + \lambda z^2) dt^2.$$  \hspace{1cm} (73)
The Maxwell field strength is constant and by an electric-magnetic duality transformation can be taken to be purely electric,

\[ F_{30} = \sqrt{2\lambda}, \quad \lambda = \text{constant}. \]  

(74)

A crucial property of this class of solutions is that the Weyl tensor vanishes.

Choosing \( \varphi = 1 + \lambda z^2 \) and \( g_{\mu\nu} = \text{diag}(\varphi(1 - \lambda y^2), \varphi(1 - \lambda y^2)^{-1}, 1) \), we get the Robinson-Bertotti metric in the form (3). Applying the transformation III the fields transform into

\[
\begin{align*}
    u' &= \frac{2c\sqrt{\lambda}z\Xi}{1 + (c\Xi)^2}, \quad v' = \frac{2\sqrt{\lambda}z}{1 + (c\Xi)^2}, \\
    \varphi' &= \frac{\varphi}{1 + (c\Xi)^2}, \quad \sigma' = \frac{-c\Xi^2}{1 + (c\Xi)^2}, \quad \Xi = 1 - \lambda z^2.
\end{align*}
\]

(75)

As can be seen, we have generated a magnetic field as well as a rotation. Since this solution has a nonvanishing Weyl tensor, it does not belong to the Robinson-Bertotti class of solutions. We thus conclude that some supersymmetry is broken. However, examining the Killing spinor equation (69) one finds that there are no solutions satisfying \( \partial_t \epsilon = 0 \); which is a crucial argument required to guarantee the supersymmetry of the generated solution. This is in agreement with our general discussion in section 2.

iii) Another possibility is to start with a pp-wave as “seed” solution. The supersymmetry of these solutions has been discussed by Hull [18]. With a metric of the form (51) this solution is supersymmetric provided that \( F_{MN} \) vanishes. Furthermore the Killing spinor is time-independent (in fact constant). The transformations I-III are exactly those considered in the first example in section 3.3 and apart from the trivial transformation IV the only new feature comes from the last transformation, V. It generates a nonvanishing field strength, implying that the new solution is not a pp-wave.

5 Discussion

In this article we have shown how to generate supersymmetric solutions to \( N = 1 \) and \( N = 2 \), 4D supergravities using known techniques employed in ordinary general relativity. After a naïve dimensional reduction of \( N = 1 \), 4D supergravity we have explicitly obtained \( N = 2 \), 3D supergravity plus a nonlinear supersymmetric \( \sigma \)-model. For \( N = 2 \) we have reduced the bosonic part only. Since the target space metric of the resulting \( \sigma \)-model has a quaternionic structure it can be extended to an \( N = 4 \) nonlinear supersymmetric \( \sigma \)-model. Examining the Killing spinor equation we have shown that the supersymmetries
of the solutions are preserved under the target space isometry transformations, provided that the Killing spinor is independent of the coordinate along which the reduction is done.

Supersymmetric solutions of supergravity theories are usually discussed in the context of consistently truncated Lagrangians. It means that some bosonic fields are set to zero in the full supersymmetric Lagrangian such that the solutions of the truncated systems are also solutions, albeit specific, of the untruncated theory [25]. The $\sigma$-model resulting from the reduction of the truncated supergravity Lagrangian is a consistent truncation of the full supersymmetric nonlinear $\sigma$-model obtained from the reduction of the untruncated 4D supergravity. Since the target space $T$ of a consistently truncated $\sigma$-model is a submanifold of the target space $T'$ of the untruncated supersymmetric $\sigma$-model, the isometries of $T$ will also be isometries of $T'$ [10]. Our method would therefore be applicable for all consistently truncated 4D supergravities.

The solution-generating method used in [1]-[7] applies also to higher dimensional systems consisting of an antisymmetric tensor field coupled to $D$-dimensional dilaton-gravity. Various solutions such as black $p$-branes have been generated in this way [26]. Our investigations suggest that the solution-generating technique in such systems can also be extended along the lines described in this paper to include supersymmetry. This we will discuss elsewhere [27].

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**Appendix: 3D Gamma Matrices**

The $2 + 1$ dimensional gamma matrices $\gamma^a$ satisfy

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}, \quad \eta^{ab} = \text{diag}(-1, 1, 1). \quad (A.1)$$

In $2 + 1$ dimensions there are two inequivalent irreducible representations of the Clifford algebra (A.1) and we choose the one for which $\gamma^a \gamma^b \gamma^c = \varepsilon^{abc}$. A specific (real) representation in terms of the Pauli matrices $\tau_{1,2,3}$ is $\gamma^0 = i\tau_2, \gamma^1 = \tau_1, \gamma^2 = \tau_3$. We also define

$$\gamma^{ab} = \frac{1}{2}[\gamma^a, \gamma^b] = \varepsilon^{abc}\gamma^c. \quad (A.2)$$
The Dirac conjugate of a spinor $\psi$ is $\bar{\psi} \equiv \psi^\dagger i\gamma^0$. A Majorana spinor $\psi$ obeys the reality condition $\bar{\psi} = \psi^c$, where $\psi^c \equiv \psi^T C$ is the Majorana conjugate and $C$ the charge conjugation matrix defined through

$$
C = -C^{-1}, \quad C\gamma^a C^{-1} = -\left(\gamma^a\right)^T.
$$

(A.3)

From these properties, the “flip” identities

$$
\bar{\psi}\gamma^{a_1} \ldots \gamma^{a_k}\chi = (-1)^k \bar{\chi}\gamma^{a_k} \ldots \gamma^{a_1}\psi,
$$

(A.4)

follow for any two Majorana spinors $\psi$ and $\chi$. It is convenient to choose $C = \tau_2$. Bilinears of the type $\bar{\psi}\chi$ and $\bar{\psi}\gamma^a\chi$ will then be real.
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