Computing Directed Steiner Path Covers*

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Abstract

In this article we consider the Directed Steiner Path Cover problem on directed co-graphs. Given a directed graph $G = (V, E)$ and a set $T \subseteq V$ of so-called terminal vertices, the problem is to find a minimum number of vertex-disjoint simple directed paths, which contain all terminal vertices and a minimum number of non-terminal vertices (Steiner vertices). The primary minimization criteria is the number of paths. We show how to compute in linear time a minimum Steiner path cover for directed co-graphs. This leads to a linear time computation of an optimal directed Steiner path on directed co-graphs, if it exists. Since the Steiner path problem generalizes the Hamiltonian path problem, our results imply the first linear time algorithm for the directed Hamiltonian path problem on directed co-graphs. We also give binary integer programs for the (directed) Hamiltonian path problem, for the (directed) Steiner path problem, and for the (directed) Steiner path cover problem. These integer programs can be used to minimize change-over times in pick-and-place machines used by companies in electronic industry.

Keywords: binary integer program; combinatorial optimization; directed co-graphs; directed Steiner path cover problem; directed Steiner path problem; directed Hamiltonian path problem; pick-and-place machines

1 Introduction

For the well known Steiner tree problem there are efficient algorithms on special graph classes like series-parallel graphs [WC83], outerplanar graphs [WC82] and graphs of bounded tree-width [BCKN15, CMZ12]. The class Steiner tree problem (CSP) is a generalization of the Steiner tree problem in which the vertices are partitioned into classes of terminals [RW90]. The unit-weight version of CSP can be solved in linear time on co-graphs [WY95].

The Steiner path problem is a restriction of the Steiner tree problem such that the required terminal vertices lie on a path of minimum cost. It is also a generalization of the Hamiltonian path problem: If we choose each vertex as a terminal vertex, the Steiner path problem becomes the Hamiltonian path problem. The Euclidean bottleneck Steiner path problem was considered in [AACKS14] and a linear time solution for the Steiner path problem on trees was given in [MJV13].

While a Steiner tree always exists within connected graphs, it is not always possible to find a Steiner path, which motivates us to consider Steiner path cover problems. The Steiner path cover problem on interval graphs was considered in [CL18].

In this article we consider the directed Steiner path cover problem defined as follows. Let $G$ be a directed graph on vertex set $V(G)$ and edge set $E(G)$ and let $T \subseteq V(G)$ be a set of terminal vertices. Let $c : E(G) \rightarrow \mathbb{R}_{\geq 0}$ be a function that assigns a weight to each edge. A directed Steiner path cover for $G$ is a set $P$ of vertex-disjoint simple directed paths in $G$ that contain all terminal

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vertices of $T$ and possibly also some of the non-terminal (Steiner) vertices of $V(G) - T$. The size of a directed Steiner path cover is the number of its paths, i.e. the size is $|P|$, the cost is defined as the sum of weights of those edges used in the paths in a directed Steiner path cover of minimum size.

**Name** Directed Steiner Path Cover

**Instance** A directed graph $G$, a set of terminal vertices $T \subseteq V(G)$, and edge weights $c : E(G) \to \mathbb{R}^\geq 0$.

**Task** Find a directed Steiner path cover $P$ of minimum size for $G$ that minimizes $\sum_{p \in P} \sum_{e \in p} c(e)$.

To minimize only the sum of the edge weights makes no sense, because this sum becomes minimal, if each terminal vertex is chosen as path of length 0. So we primary have to demand that the number of paths should be minimal.

The directed Steiner path cover problem is NP-hard since it generalizes the directed Hamiltonian path problem. This motivates us to restrict the problem to special inputs. We consider a very natural class of inputs, which is defined as follows. Directed co-graphs (short for complement reducible graphs) can be generated from the single vertex graph by applying disjoint union, order composition and series composition $[\text{BdGR97}]$. They also can be characterized by excluding eight forbidden induced sub-digraphs, see $[\text{CP06}, \text{Figure 2}]$.

Directed co-graphs are exactly the digraphs of directed NLC-width 1 and a proper subset of the digraphs of directed clique-width at most 2 $[\text{BJG15}, \text{GWY16}]$. Directed co-graphs are also interesting from an algorithmic point of view since several hard graph problems can be solved in polynomial time by dynamic programming along the tree structure of the input graph, see $[\text{BJM14}, \text{GR18}, \text{GKR19a}, \text{GKR19b}, \text{GKR20}, \text{GKR21a}, \text{GKR21b}, \text{GKR21c}]$. Moreover, directed co-graphs are very useful for the reconstruction of the evolutionary history of genes or species using genomic sequence data $[\text{HSW17}, \text{NEMM}^{*}18]$.

For graphs where all edges have the same weight, the above definition of the directed Steiner path cover problem results in a minimum number of Steiner vertices. Graphs without edge weights can be considered as a special case of graphs with unit-edge weights. Since edge weights do not occur in co-graphs, we use the following problem definition.

**Name** Unit-Edge-Weight Directed Steiner Path Cover

**Instance** A directed graph $G$ and a set of terminal vertices $T \subseteq V(G)$.

**Task** Find a directed Steiner path cover of minimum size for $G$ such that the number of Steiner vertices is minimal.

In this paper we show how the value of a directed Steiner path cover of minimum size and cost for the disjoint union, order composition and series composition of two digraphs can be computed in linear time from the corresponding values of the involved digraphs. Therefore, we define a useful normal form for directed Steiner path covers in digraphs which are defined by the order composition or series composition of two digraphs. Further we give an algorithm which constructs a directed Steiner path cover of minimum size and cost for a directed co-graph in linear time.

The (unit-weight) directed Steiner tree problem is MSO$_1$-definable by $[\text{GHK}^{+}14]$, Proposition 4.14. But this does not hold for the directed Steiner path (cover) problem, since it is a generalization of the directed Hamiltonian path problem which is not MSO$_1$-definable. Thus it is not possible to obtain our solutions for the directed Steiner path (cover) problem using the fact that directed co-graphs have bounded directed clique-width $[\text{GWY16}]$ and the result from $[\text{GHK}^{+}14]$ stating that all MSO$_1$-definable digraph problems are fixed parameter tractable for the parameter directed clique-width. Since the Hamiltonian cycle problem is W[1]-hard parameterized by clique-width $[\text{FGIS10}]$ it even follows that Hamiltonian path problem and thus the (directed) Steiner path (cover) problem are W[1]-hard when parameterized by (directed) clique-width.

We also give binary integer programs for the directed Hamiltonian path problem and for the directed Steiner path (cover) problem. These integer programs can be used to minimize change-over times in pick-and-place machines used in electronic industry. The problem of minimizing change-over times is introduced in section $[\text{W}4]$.

**Name** Directed Steiner Path

**Instance** A directed graph $G$, a set $T \subseteq V(G)$ of terminal vertices, and a function $c : E(G) \to \mathbb{R}^\geq 0$ that assigns each edge some weight.

**Task** Find a directed Steiner path $p$ in graph $G$ that minimizes $\sum_{e \in p} c(e)$.
2 Preliminaries

We consider the directed Steiner path cover problem on directed co-graphs, so we first recall the definition of directed co-graphs and we will consider a normal form of directed Steiner path covers.

2.1 Directed Co-Graphs

Directed co-graphs have been introduced by Bechet et al. in [BdGR97].

Definition 2.1. The class of directed co-graphs is recursively defined as follows.

(i) Every digraph on a single vertex \(\{v\}, \emptyset\), denoted by \(\bullet_v\), is a directed co-graph.

(ii) If \(A, B\) are vertex-disjoint directed co-graphs, then

(a) the disjoint union \(A \oplus B\), which is defined as the digraph with vertex set \(V(A) \cup V(B)\) and edge set \(E(A) \cup E(B)\),

(b) the order composition \(A \bowtie B\), defined by their disjoint union plus all possible edges only directed from \(V(A)\) to \(V(B)\), and

(c) the series composition \(A \otimes B\), defined by their disjoint union plus all possible edges between \(V(A)\) and \(V(B)\) in both directions,

are directed co-graphs.

Every expression using these operations is called a directed co-expression. The recursive generation of a directed co-graph can be described by a tree structure, called directed co-tree. The leaves of the directed co-tree represent the vertices of the digraph and the inner vertices of the directed co-tree correspond to the operations applied on the subgraphs of \(G\) defined by the subtrees. For every directed co-graph one can construct a directed co-tree in linear time, see [CP06].

Directed co-graphs can also be characterized by excluding eight forbidden induced sub-digraphs, see [CP06, Figure 2] or Figure 1.

![Forbidden Sub-Digraphs](image)

Table 1: The eight forbidden induced sub-digraphs for directed co-graphs (see [CP06]).

Next we define a normal form for directed Steiner path covers in digraphs which are defined by the order composition or series composition of two digraphs.

2.2 Normal Form for Directed Steiner Path Covers

Let \(G\) be a directed co-graph, let \(T \subseteq V(G)\) be a set of terminal vertices, and let \(C\) be a directed Steiner path cover for \(G\) with respect to \(T\). Then \(s(C)\) denotes the number of Steiner vertices in the paths of \(C\).

Lemma 2.2. Let \(C\) be a directed Steiner path cover for some directed co-graph \(G = A \bowtie B\) or \(G = A \otimes B\) with respect to a set \(T \subseteq V(G)\) of terminal vertices. Then there is a directed Steiner path cover \(C'\) with respect to \(T\) which does not contain paths \(p\) and \(p'\) satisfying one of the structures (1)-(4), such that \(|C'| \geq |C'|\) and \(s(C) \geq s(C')\) applies. Let \(q_1, \ldots, q_4\) denote sub-paths which may be empty.
1. \( p = (x, q_1) \) or \( p = (q_1, x) \) where \( x \notin T \). Comment: No path starts or ends with a Steiner vertex.

2. \( p = (q_1, u, x, v, q_2) \) where \( u \in V(A) \), \( v \in V(B) \), and \( x \notin T \). Comment: On a path, the neighbors \( u, v \) of a Steiner vertex \( x \) are both contained in the same digraph.

3. \( p = (q_1, x), p' = (u, q_2) \), where \( x \in V(A), u \in V(B), p \neq p' \). Comment: No path \( p \) ends in \( A \), if there is a path \( p' \neq p \) that starts in \( B \).

4. \( p = (\ldots, x, u, v, y, \ldots) \) where \( u, v \notin T \). Comment: The paths contain no edge between two Steiner vertices.

If \( G = A \otimes B \) then cover \( C' \) also does not contain paths satisfying structures (2)-(8).

5. \( p = (x, q_1), p' = (u, q_2) \), where \( x \in V(A), u \in V(B), p \neq p' \). Comment: All paths start in the same digraph.

6. \( p = (q_1, x, y, q_2), p' = (q_3, u, v, q_4) \) where \( x, y \in V(A), u, v \in V(B) \). Comment: The cover \( C' \) contains edges of only one of the digraphs.

7. \( p = (x, q_1), p' = (q_2, u, y, v, q_3), \) where \( x, y \in V(A), u, v \in V(B) \), and \( y \notin T \). Comment: If a path starts in \( A \) then there is no Steiner vertex in \( A \) with two neighbors on the path in \( B \).

8. \( p = (x, q_1), p' = (q_2, u, v, q_3), \) where \( x \in V(A) \) and \( u, v \in V(B) \). Comment: If a path starts in \( A \), then no edge of \( B \) is contained in the cover.

**Proof**

1. If \( x \) is removed from \( p \) we get a cover with one Steiner vertex less than \( C \).

2. If \( x \) is removed from \( p \), we get a cover with one Steiner vertex less than \( C \).

3. We combine the paths to only one path \( (q_1, x, u, q_2) \) and we get a cover with one path less than \( C \).

4. Since \( G \) is a directed co-graph, the underlying undirected graph is a co-graph such that the path cannot include a \( P_4 \), i.e. a simple path of 4 vertices, as induced subgraph. Thus, there must be at least one additional arc. If such an additional arc would shorten the Steiner path by skipping \( u, v \) or both then we remove \( u \) or \( v \) or both and take the shortcut for getting a cover \( C' \). Additional arcs that do not shorten the path would create a forbidden induced subgraph from Figure 1 which is not possible. For details see Table 2.

5. The new paths are \( q_1 \) and \( (x, u, q_2) \). The cover \( C' \) is as good as \( C \).

6. If \( p \neq p' \), then \( (q_1, x, v, q_4) \) and \( (q_3, u, y, q_2) \) are the paths in cover \( C' \).

If \( p = p' \), then we have to distinguish whether \((u, v) \in q_1\), \((u, v) \in q_2\), \((x, y) \in q_3\), or \((x, y) \in q_4\). We show how to handle the first case, the other three cases are similar. Let \( p = (q_1, u, v, q_5, b, a, q_6, x, y, q_2), \) where \( b \in V(B) \) and \( a \in V(A) \). Then the new path in cover \( C' \) is \( (q_1, u, a, q_6, x, v, q_5, b, y, q_2) \). Such vertices \( a \) and \( b \) must exist because \( v \in V(B) \) and \( x \in V(A) \), possibly it holds \( a = x \) or \( b = v \). In any case cover \( C' \) is as good as \( C \).
7. If \( p \neq p' \), then \( q_1 \) and \( (q_2, u, x, v, q_3) \) are the new paths in cover \( C' \). If \( p = p' \), i.e. \( q_1 = (q'_2, u, v, q_3) \), where \( q'_2 \) is obtained from \( q_2 \) by removing \( x \), then \( (q'_2, u, x, v, q_3) \) is the new path in cover \( C' \). The cover \( C' \) is as good as \( C \). If \( p \neq p' \), then the edge \((a, b)\) is missing in the following figure.

![Diagram](image)

Operations 1, 2, 4 and 7 reduce the number of Steiner vertices by one, the remaining operations 3, 5 and 6 do not change the number of Steiner vertices. Therefore, operations 1, 2, 4 and 7 can only be executed at most \(|V - (T_A \cup T_B)|\) times.

Operation 6 reduces the number of paths by one, the remaining operations do not increase the number of paths. Therefore operation 6 can be executed at most \(\max\{|T_A|, |T_B|\}\) times.

Let us now consider those edges on a path that connect vertices of \( A \) and vertices of \( B \). The maximum number of those edges is \(|V(A)| + |V(B)| - 1\). Operation 7 can remove two such edges, operations 3 and 5 can add two such edges. Since the other operations 1, 2, 4 and 6 do not reduce the number of edges, operations 3 and 5 can be used at most \(|V(A)| + |V(B)| - 1|/2 + |V - (T_A \cup T_B)|\) times.

Since the hypothesis of Lemma 2.2 is symmetric in \( A \) and \( B \), the statement of Lemma 2.2 is also valid for co-graphs \( G = A \odot B \) if \( A \) and \( B \) are switched.

**Definition 2.3** A directed Steiner path cover \( C \) for some directed co-graph \( G = A \odot B \) or \( G = A \odot B \) is said to be in normal form if none of the operations described in the proof of Lemma 2.2 is applicable.

In the following we assume that a directed Steiner path cover for some directed co-graph \( G = A \odot B \) or \( G = A \odot B \) is always in normal form, since the operations of the proof of Lemma 2.2 do not increase the number of paths or Steiner vertices of a cover. Lemma 2.2 implies the following theorem.

**Theorem 2.4** For each directed co-graph \( G = A \odot B \) and set of terminal vertices \( T \subseteq V(G) \) any directed Steiner path cover \( C \) in normal form with respect to \( T \) does not contain an edge of digraph \( A \), and no path in \( C \) starts or ends in digraph \( A \) if \(|T_A| < |T_B|\).

**Proof** [by contradiction] Assume, the Steiner path cover \( C \) contains an edge of digraph \( A \). Then by Lemma 2.2, all paths starts in digraph \( A \). By Lemma 2.2, it holds that no Steiner vertex \( v \) of \( V(A) \) is contained in \( C \), where the neighbors of \( v \) are both of digraph \( B \). By Lemma 2.2, it holds that all vertices of \( V(B) \) from \( C \) are connected with a terminal vertex of \( V(A) \), thus \(|T_A| > |T_B|\).

Second, we have to show that no path in \( C \) starts or ends in digraph \( A \). Assume on the contrary, that there is one path that starts in \( A \). By Lemma 2.2, it holds that all paths start in \( A \). Continuing as in the first case this leads to a contradiction.

**Remark 2.5** For each directed co-graph \( G = A \odot B \) and set of terminal vertices \( T \subseteq V(G) \) any directed Steiner path cover \( C \) in normal form with respect to \( T \) it holds that each path that starts in \( A \) either remains in \( A \) or it crosses over to \( B \) and remains in \( B \). Each path that reaches a vertex of \( B \) has to stay in \( B \) since no edge from a vertex in \( B \) to a vertex in \( A \) exists.
| Graph with underlying $P_4$ and additional edges that do not shorten the path | none | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| | $a, b$ | $a, c$ | $b, c$ | $a, b, c$ |
| ![Graph](image1.png) | $D_5$ | $D_1$ | $D_4$ | $D_3$ |
| | {1,2,3} | {1,2,3} | {1,2,3} | {1,3,4} |
| ![Graph](image2.png) | $D_5$ | $D_2$ | $D_2$ | $D_4$ |
| | {2,3,4} | {1,2,4} | {1,2,3} | {2,3,4} |
| ![Graph](image3.png) | $D_6$ | $D_4$ | $D_4$ | $D_2$ |
| | {2,3,4} | {1,2,3} | {1,2,3} | {1,2,4} |
| ![Graph](image4.png) | $D_5$ | $D_4$ | $D_4$ | $D_4$ |
| | {1,2,3} | {1,2,3} | {1,2,3} | {2,3,4} |
| ![Graph](image5.png) | $D_6$ | $D_4$ | $D_4$ | $D_6$ |
| | {2,3,4} | {1,2,3} | {1,2,3} | {1,2,4} |
| ![Graph](image6.png) | $D_5$ | $D_3$ | $D_2$ | $D_1$ |
| | {1,2,3} | {1,3,4} | {1,2,3} | {1,3,4} |
| ![Graph](image7.png) | $D_5$ | $D_1$ | $D_1$ | $D_1$ |
| | {2,3,4} | {1,2,4} | {1,2,4} | {1,2,4} |
| ![Graph](image8.png) | $D_3$ | $D_2$ | $D_2$ | $D_3$ |
| | {1,2,3} | {1,2,4} | {1,2,3} | {1,3,4} |

Table 2: The leftmost column shows a graph with underlying undirected $P_4$ and at least one additional arc that do not shorten the path. The other columns show the forbidden subgraphs that are contained in the leftmost graph depending on the edges of the $P_4$. 
3 Algorithms for the Directed Steiner Path Cover Problem

3.1 Computing the optimal number of paths

Let $G$ be a directed co-graph and $T \subseteq V(G)$ be a set of terminal vertices. We define $p(G, T)$ as the minimum number of paths within a Steiner path cover for $G$ with respect to $T$. Further let $s(G, T)$ be the minimum number of Steiner vertices in a directed Steiner path cover of size $p(G, T)$ with respect to $T$. We do not specify set $T$ if it is clear from the context which set is meant.

**Lemma 3.1** Let $A$ and $B$ be two vertex-disjoint digraphs and let $T_A \subseteq V(A)$ and $T_B \subseteq V(B)$ be two sets of terminal vertices. Then the following equations hold true:

1. $p(\emptyset, \emptyset) = 0$ and $p(\emptyset, \{v\}) = 1$
2. $p(A \oplus B, T_A \cup T_B) = p(A, T_A) + p(B, T_B)$
3. $p(A \otimes B, \emptyset) = 0$
4. $p(A \otimes B, T_A \cup T_B) = \max\{1, p(B, T_B) - |V(A)|\}$ if $1 \leq |T_B|$ and $|T_A| \leq |T_B|$
5. $p(A \otimes B, T_A \cup T_B) = \max\{1, p(A, T_A) - |V(B)|\}$ if $1 \leq |T_A|$ and $|T_A| > |T_B|$
6. $p(A \otimes B, T_A \cup T_B) = p(A, T_A)$ if $p(A) \geq p(B)$
7. $p(A \otimes B, T_A \cup T_B) = p(B, T_B)$ if $p(A) < p(B)$

**Proof**

1. Obvious.

4. We show that $p(A \otimes B) \geq \max\{1, p(B) - |V(A)|\}$ applies by an indirect proof. Assume a directed Steiner path cover $C$ for $A \otimes B$ has less than $\max\{1, p(B) - |V(A)|\}$ paths. The removal of all vertices of $A$ from all paths in $C$ gives a directed Steiner path cover of size $|C| + |V(A)| < p(B)$ for $B$.  

To see that $p(A \otimes B) \leq \max\{1, p(B) - |V(A)|\}$ applies, consider that we can use any vertex of $A$ to combine two paths of the cover of $B$ to one path, since the series composition of $A$ and $B$ creates all directed edges between $A$ and $B$. If there are more terminal vertices in $T_A$ than there are paths in the cover of $B$, i.e. $p(B) < |T_A|$, then we have to split paths of $B$ and reconnect them by terminal vertices of $T_A$. This can always be done since $|T_A| \leq |T_B|$.

5. Similar to [4]

6. To see that $p(A \otimes B) \leq p(A)$ applies, consider that we can append any path of $A$ by any path of $B$, see Lemma [223]. Since no edge between $B$ and $A$ is created, no path of $B$ can be extended by a path of $A$.

We show that $p(A \otimes B) \geq p(A)$ applies by an indirect proof. Assume a directed Steiner path cover $C$ for $A \otimes B$ contains less than $p(A)$ paths. The removal of all vertices of $B$ from all paths in $C$ gives a Steiner path cover of size $|C| < p(A)$.  

7. Similar to [6]

This shows the statements of the lemma.

3.2 Computing the optimal number of Steiner vertices

**Remark 3.2** For two vertex-disjoint directed co-graphs $A$, $B$ and two sets of terminal vertices $T_A \subseteq V(A)$, $T_B \subseteq V(B)$ it holds that $s(A \oplus B, T_A \cup T_B) = s(A, T_A) + s(B, T_B)$, since the disjoint union does not create any new edges.

**Remark 3.3** Let $G = A \otimes B$ be a directed co-graph, and let $C$ be a directed Steiner path cover of $G$ such that $p = (q_1, u_1, x, q_2, v_1)$ is a path in $A$, $p_1 = (u_2, q_3)$ and $p_2 = (v_2, q_4)$ are paths in $B$, all paths are vertex-disjoint paths in $C$, where $x \notin T$, $u_1, u_2, v_1, v_2 \notin T$, and $q_1, \ldots, q_4$ are sub-paths. Then we can split $p$ at vertex $x$ into two paths, combine them with $p_1$ and $p_2$ to get $(q_1, u_1, u_2, q_3)$ and $(q_2, v_1, v_2, q_4)$ as new paths and we get a Steiner path cover without increasing the cost. If $A$ and $B$ are switched we get $(u_2, q_3, q_1, u_1)$ and $(v_2, q_4, q_2, v_1)$ as new paths and the statement also holds.
What follows is the central lemma of our work, the proof is by induction on the structure of the co-graph.

**Lemma 3.4** For every directed co-graph $G$ and every directed Steiner path cover $C$ for $G$ with respect to a set $T \subseteq V(G)$ of terminal vertices it holds that $p(G) + s(G) \leq |C| + s(C)$.

**Proof** [by induction] The statement is obviously valid for all directed co-graphs which consist of only one vertex. Let us assume that the statement is valid for directed co-graphs of $n$ vertices. Let $A$ and $B$ be vertex-disjoint directed co-graphs of at most $n$ vertices each.

**Disjoint union:** Let $G = A \oplus B$ be a directed co-graph that consists of more than $n$ vertices. By Lemma 5.1 and Remark 5.2 it holds that $p(A \oplus B) + s(A \oplus B) = p(A) + p(B) + s(A) + s(B)$. By the induction hypothesis, it holds that $p(A) + s(A) \leq |C_A| + s(C_A)$ and $p(B) + s(B) \leq |C_B| + s(C_B)$, where $C_A$ denotes the cover $C$ restricted to digraph $A$, i.e. the cover that results from $C$ when all vertices of $B$ are removed. Then the statement of the lemma follows.

$$p(A \oplus B) + s(A \oplus B) \leq |C_A| + s(C_A) + |C_B| + s(C_B) = |C| + s(C)$$

**Series composition:** Let $G = A \otimes B$ be a directed co-graph that consists of more than $n$ vertices. Without loss of generality, let $|T_A| \leq |T_B|$.

1. Let $X(A)$ denote the vertices of $A$ used in cover $C$, and let $D$ denote the cover for $B$ that we obtain by removing the vertices of $X(A)$ from cover $C$. By the induction hypothesis, it holds that $p(B) + s(B) \leq |D| + s(D)$.

2. Let $nt(X(A))$ denote the number of non-terminal vertices of $X(A)$. By Theorem 2.4 it holds that $s(C) = s(D) + nt(X(A))$ and $|C| = |D| - |T_A| - nt(X(A))$. Thus, we get $|C| + s(C) = |D| + s(D) - |T_A|$.

We put these two results together and obtain:

$$p(B) + s(B) - |T_A| \leq |D| + s(D) - |T_A| = |C| + s(C)$$

To show the statement of the lemma, we first consider the case $p(B) - 1 \leq |V(A)|$. Then it holds that $p(A \otimes B) = 1$. If $|T_A| \geq p(B) - 1$, then $d := |T_A| - (p(B) - 1)$ many Steiner vertices from $B$, if available, can be replaced by terminal vertices from $A$. Otherwise if $|T_A| < p(B) - 1$, then $d = (p(B) - 1) - |T_A|$ many Steiner vertices from $A$ are used to combine the paths. Thus, it holds that $s(A \otimes B) \leq \max\{0, s(B) - d\}$ since the number of Steiner vertices in an optimal cover is at most the number of Steiner vertices in a certain cover. Thus, since $p(A \otimes B) = 1$ we get for $s(B) \geq d$:

$$p(A \otimes B) + s(A \otimes B) \leq 1 + s(B) - d = 1 + s(B) - (|T_A| - (p(B) - 1)) = I + s(B) - |T_A| + p(B) - I \leq |C| + s(C)$$

If $s(B) < d$ then all Steiner vertices of $B$ can be replaced by terminal vertices of $A$ and since $|T_A| \leq |T_B|$ holds, some of the paths of $B$ can be reconnected by the remaining terminal vertices of $A$. Thus, $p(A \otimes B) + s(A \otimes B) = 1 \leq |C| + s(C)$ applies.

Consider now the case where $p(B) - 1 > |V(A)|$ holds, i.e. not all paths in an optimal cover for $B$ can be combined by vertices of $A$. By Lemma 5.1 it holds that $p(A \otimes B) = \max\{1, p(B) - |V(A)|\}$. Thus, for $p(A \otimes B) > 1$ we get:

$$p(A \otimes B) + s(A \otimes B) \leq p(B) - |V(A)| + s(B) + nt(A) = p(B) + s(B) - |T_A| \leq |C| + s(C)$$

The non-terminal vertices of $A$ must be used to combine paths of the cover, thus the non-terminal vertices of $A$ become Steiner vertices.

**Order composition:** Let $G = A \odot B$ be a directed co-graph that consists of more than $n$ vertices. By the induction hypothesis, it holds that $p(A) + s(A) \leq |C_A| + s(C_A)$ and $p(B) + s(B) \leq |C_B| + s(C_B)$.

Let us first consider the case $p(A) > p(B)$. By Lemma 5.1 it holds $p(A \odot B) = p(A)$. We can append any path of $A$ by any path of $B$, and by Remark 3.3 it holds that for every path that
there is more in $A$ than in $B$, a Steiner vertex of $B$ can be removed. And since an optimal cover has at most as many Steiner vertices as a concrete cover, it holds $s(A \otimes B) \leq s(C_A) + s(C_B) - \min\{s(C_B), |C_A| - |C_B|\}$. If we sum up both equations we get

$$p(A \otimes B) + s(A \otimes B) \leq p(A) + s(A) - \min\{s(C_B), |C_A| - |C_B|\}$$

If $s(C_B) \geq \min\{s(C_B), |C_A| - |C_B|\}$ applies, and since $s(C) = s(C_A) + s(C_B)$ applies, we get

$$p(A \otimes B) + s(A \otimes B) \leq p(A) + s(A) - \min\{s(C_B), |C_A| - |C_B|\}$$

The statement would be shown if $p(A) - |C_A| + |C_B| \leq |C|$ would apply. It holds $p(A) \leq |C_A|$, since an optimal cover has at most as many paths as a concrete cover, and it holds $|C_B| \leq |C|$, since $|C| = \max\{|C_A|, |C_B|\}$ by Remark 2.5. We sum up these equations and we get $p(A) - |C_A| + |C_B| \leq |C|$, which is equivalent to $p(A) - |C_A| + |C_B| \leq |C|$, thus $p(A \otimes B) + s(A \otimes B) \leq |C| + s(C)$ has been shown.

If $s(C_B) < |C_A| - |C_B|$, then it holds $p(A \otimes B) + s(A \otimes B) \leq p(A) + s(C_A)$, and we have to show that $p(A) + s(C_A) \leq |C| + s(C)$ applies. It holds $p(A) \leq |C_A|$, since an optimal cover has at most as many paths as a concrete cover, and it holds $|C_A| \leq |C|$, since $|C| = \max\{|C_A|, |C_B|\}$ by Remark 2.5. Furthermore, it holds $s(C_A) \leq s(C)$, since a part is only as big as the whole.

The other case $p(A) \leq p(B)$ can be shown in a similar way. $\square$

To see why Lemma 3.4 is crucial for the rest of this work, consider the directed graph $B$ of Figure 1 that is not a directed co-graph. Terminal vertices $T_A = \{f, g\}$ and $T_B = \{a, c, e, u, w, x, y\}$ are shown as squares. In the left part of the figure a Steiner path cover $C_\ell = \{(a, b, c, d, e), (u, v, w, x, y)\}$ for graph $B$ is shown with $|C_\ell| = 2$ and $s(C_\ell) = 4$ which is optimal. In the right part of the figure a Steiner path cover $C_r = \{(a, b, c, w, x, y), (e, (u))\}$ for $B$ is shown with $|C_r| = 3$ and $s(C_r) = 2$. The right cover can be extended to an optimal cover for $A \otimes B$ if the vertices of $A$ are used to combine the path: $\{(a, f, a, b, c, w, x, y, g, e)\}$ is an optimal cover for $A \otimes B$ with only one path and 2 Steiner vertices.

Figure 1: Small example that shows the contrapositive of the statement of Lemma 3.4 in a graph $B$ that is not directed co-graph.

In the proof of Lemma 3.6 we use the statement of Lemma 3.4 to show that optimal solutions for directed co-graphs $A$ and $B$ can be combined to an optimal solution for $A \otimes B$ and $A \otimes B$.

Remark 3.5 Let $G$ be a directed co-graph and let $C$ be a directed Steiner path cover for $G$ with respect to some set of terminal vertices $T \subseteq V(G)$. Then $s(C) \geq s(G)$ holds only if $|C| = p(G)$. If $|C| > p(G)$ then $s(C)$ might be smaller than $s(G)$.

Lemma 3.6 Let $A$ and $B$ be two vertex-disjoint digraphs, and let $T_A \subseteq V(A)$, $T_B \subseteq V(A)$ be sets of terminal vertices. Then the following equations apply:

1. $s(\{a\}, \emptyset) = 0$ and $s(\{a\}, \{v\}) = 0$

2. $s(A \otimes B, T_A \cup T_B) = s(A, T_A) + s(B, T_B)$
3. \( s(A \otimes B) = \max\{0, s(B) + p(B) - p(A \otimes B) - |T_A|\} \) if \( |T_A| \leq |T_B| \)

4. \( s(A \otimes B) = \max\{0, s(A) + p(A) - p(A \otimes B) - |T_B|\} \) if \( |T_A| > |T_B| \)

5. \( s(A \otimes B) = s(A) + s(B) \) if \( p(A) = p(B) \)

6. \( s(A \otimes B) = s(A) + s(B) - \min\{s(A), p(B) - p(A)\} \) if \( p(A) < p(B) \)

7. \( s(A \otimes B) = s(A) + s(B) - \min\{s(B), p(A) - p(B)\} \) if \( p(A) > p(B) \)

**Proof**

1. Obvious.

2. See Remark 3.2

3. First, we show \( s(A \otimes B) \leq \max\{0, s(B) + p(B) - p(A \otimes B) - |T_A|\} \).

   By Lemma 3.4, we know that \( s(A \otimes B) + p(A \otimes B) \leq s(C) + |C| \) holds true for any cover \( C \)
   for co-graph \( A \otimes B \) and any set of terminal vertices \( T \). Consider cover \( C \) for \( A \otimes B \) obtained
   by an optimal cover \( D \) for \( B \) in the following way: Use the terminal vertices of \( A \) to either
   combine paths of \( D \) or to remove a Steiner vertex of \( D \) by replacing \( v \) by some terminal
   vertex of \( A \) in a path like \((\ldots, u, v, w, \ldots) \in D \), where \( u, w \in T \). If \( |T_A| \geq s(B) + p(B) \) then
   all paths of \( D \) can be combined and all Steiner vertices can be removed by terminal vertices
   of \( A \) and since \( |T_A| \leq |T_B| \) applies, some of the paths can be split and reconnected by
   the remaining terminal vertices of \( A \). Thus, \( s(C) + |C| = 1 \) and \( s(A \otimes B) = 0 \).

   Otherwise, if \( |T_A| < s(B) + p(B) \), then we get \( s(C) + |C| = s(B) + p(B) - |T_A| \), and by Lemma
   3.4 we get the statement.

   \[
   s(A \otimes B) + p(A \otimes B) \leq s(B) + p(B) - |T_A| = s(C) + |C|
   \]

   We prove now that \( s(A \otimes B) \geq \max\{0, s(B) + p(B) - p(A \otimes B) - |T_A|\} \).

   Let \( X(A) \) be the vertices of \( V(A) \) that are contained in the paths of an optimal cover \( C \)
   for \( A \otimes B \). Let \( D \) be the cover for \( B \) obtained by removing the vertices of \( X(A) \) from \( C \).

   Then by Theorem 2.3 the following applies:

   \[
   s(A \otimes B) \geq nt(X(A)) \implies s(A \otimes B) = s(D) \geq |D| - |T_A| - s(B) + p(B) - p(A \otimes B)
   \]

   Thus, we get:

   \[
   s(A \otimes B) - nt(X(A)) = s(D) = s(A \otimes B) - |D| + p(A \otimes B) + |T_A|
   \]

   \[
   \iff s(A \otimes B) = s(D) + |D| - p(A \otimes B) + |T_A|
   \]

   The implication follows since by Lemma 3.4 it holds \( s(D) + |D| \geq s(B) + p(B) \).

4. Can be shown similar to the previous item.

5. To see that \( s(A \otimes B) \leq s(A) + s(B) \) applies, consider optimal covers \( C \) and \( D \) for \( A \) and \( B \).

   We construct a cover \( E \) for \( A \otimes B \) in such a way that any path of \( C \) is appended by a path
   of \( D \), see Lemma 2.2(b). Since \( |E| = p(A \otimes B) \) holds, we get \( s(A \otimes B) \leq s(E) = s(C) + s(D) = s(A) + s(B) \),
   because an optimal cover has at most as many Steiner vertices as a concrete cover.

   To see that \( s(A \otimes B) \geq s(A) + s(B) \) applies consider an optimal cover \( C \) for \( A \otimes B \). Then it holds
   \( s(A \otimes B) = s(C_{A}) + s(C_{B}) \geq s(A) + s(B) \), since \( |C_{A}| = p(A) = p(A \otimes B) = p(B) = |C_{B}| \).

6. We have to distinguish two cases. First, let \( s(A) > p(B) - p(A) \).

   To see that \( s(A \otimes B) \leq s(A) + s(B) - (p(B) - p(A)) \) applies, consider optimal covers \( C \) and \( D \)
   for \( A \) and \( B \). We construct a cover \( E \) for \( A \otimes B \) in such a way that we first split \( p(B) - p(A) \)
   many paths of \( C \) at Steiner vertices as described in Remark 3.3. Afterwards we put together
each of the resulting paths by a path of \( D \). Thus it holds \(|E| = p(A \odot B) = p(B)\) and therefore
\[ s(A \odot B) \leq s(C) + s(D) - (p(B) - p(A)) = s(A) + s(B) - (p(B) - p(A)). \]

Please note, a Steiner path cover \( C \) for \( A \odot B \) with \( s(C_A) > 0 \) is not optimal if \( |C_A| < |C| = p(A \odot B) \) holds. By Remark 3.8, a path of \( C_A \) could be split at a Steiner vertex and the number of Steiner vertices could be reduced.

To see that \( s(A \odot B) \geq s(A) + s(B) - (p(B) - p(A)) \) applies, consider an optimal cover \( C \) for \( A \odot B \). Then it holds \( s(A \odot B) = s(C) = s(C_A) + s(C_B) \), and by the previous note it holds \(|C| = p(A \odot B) = p(B) = |C_A|\). By Lemma 3.5, we get \( s(C_A) + |C_A| \geq s(A) + p(A) \). If we sum up these equations we get \( s(A \odot B) + p(A \odot B) = s(C_A) + |C_A| + s(C_B) \). Finally we get:
\[
\begin{align*}
s(A \odot B) &= s(C_A) + |C_A| - (p(A \odot B) + s(C_B)) \\
&\geq s(A) + p(A) - p(B) + s(C_B) \\
&\geq s(A) + p(A) - p(B) + s(B)
\end{align*}
\]

The last step holds since \( p(B) = |C_B| \) and by Remark 3.6.

Consider now the case that \( s(A) \leq p(B) - p(A) \). To see that \( s(A \odot B) \leq s(B) \) applies, consider optimal covers \( C \) and \( D \) for \( A \) and \( B \). We construct a cover \( E \) for \( A \odot B \) in such a way that we first split as many paths of \( C \) at Steiner vertices as possible in a way described in Remark 3.6. Afterwards all Steiner vertices of \( C \) have been removed and we put together each of the resulting paths by a path of \( D \). Thus it holds \(|E| = p(A \odot B) = p(B)\) and therefore \( s(A \odot B) \leq s(E) = s(B) \).

To see that \( s(A \odot B) \geq s(B) \) applies, consider an optimal cover \( C \) for \( A \odot B \). By the above note it holds \( s(C_A) = 0 \), since \( C \) would not be optimal otherwise. Thus, we get \( s(A \odot B) = s(C_B) \geq s(B) \), since \( |C_B| = p(B) \) holds and by Remark 3.6.

7. Can be shown similar to the previous item.

This shows the statements of the lemma. \( \square \)

By Lemma 3.1 and 3.6 and since a directed co-tree can be computed in linear time from the input directed co-graph [CP06], we have shown the following result.

**Theorem 3.7** The value of a directed Steiner path cover of minimum cost for a directed co-graph can be computed in linear time with respect to the size of the directed co-expression.

Lemma 3.7 allows us to minimize the following additional cost function.

**Corollary 3.8** The value of a directed Steiner path cover \( C \) for a directed co-graph \( G \) such that \(|C| + s(C)\) is minimal can be computed in linear time with respect to the size of the directed co-expression.

### 3.3 Computing an Optimal Directed Steiner Path Cover

Now we want to give an algorithm to compute an optimal directed Steiner path cover for some given directed co-graph. The function SERIESCOMP, see Algorithm 1, returns an optimal Steiner path cover for some co-graph \( G = A \odot B \), if some optimal covers \( C_A \) for \( A \) and \( C_B \) for \( B \) are given as parameter. Let \( T_A \subseteq V(A) \) and \( T_B \subseteq V(B) \) be the sets of terminal vertices, and let \(|T_A| \leq |T_B| \).

The function APPEND\((p, u)\) used in Algorithm 1 in lines 3–5, and 7 appends path \( p \) by vertex \( u \). The function COMBINE\((p, p', u)\) used in lines 14 and 20 combines path \( p \) and \( p' \) by vertex \( u \). The function REPLACE\((p, v, u)\) used in line 26 removes the vertex \( v \) from path \( p \) and replaces it by vertex \( u \). The function INSERT\((p, v, u)\) used in line 20 inserts vertex \( u \) between vertex \( v \) and its successor in path \( p \).

Similar to function SERIESCOMP we introduce a function ORDERCOMP, see algorithm 2, which returns an optimal directed Steiner path cover for some directed co-graph \( G = A \odot B \), if some optimal covers for \( A \) and \( B \) are given as parameter. The function CONCAT\((p, p')\) used in line 16 extends path \( p \) by adding path \( p' \) at its end.

Let \( G \) be a directed co-graph represented by its binary directed co-tree \( T(G) \). The function DIRECTEDSTEINERPATHCOVER, see algorithm 3, recursively computes a directed Steiner path cover
Algorithm 1 SeriesComp(set $A$, Cover $C_A$, set $B$, Cover $C_B$)

1: if $|T_A| = |T_B|$ or $|T_A| = |T_B| - 1$ then \( \triangleright \) results in only one path
2: let $T'_A := T_A$; let $T'_B := T_B$; let $p := ()$;
3: \textbf{while} $T'_A \neq \emptyset$ \textbf{do}
4: \hspace{1em} let $b \in T'_B$; $T'_B := T'_B - \{b\}$; \textbf{Append}(p, b);
5: \hspace{1em} let $a \in T'_A$; $T'_A := T'_A - \{a\}$; \textbf{Append}(p, a);
6: \hspace{1em} if $T'_B \neq \emptyset$ then
7: \hspace{2em} let $b \in T'_B$; $T'_B := T'_B - \{b\}$; \textbf{Append}(p, b)
8: \hspace{2em} \textbf{return} (p)
9: \hspace{1em} let $T'_A := T_A$; let $C'_A := C_B$;
10: \hspace{1em} let $p \in C'_B$; $C'_B := T'_B - \{p\}$;
11: \hspace{1em} \textbf{while} $T'_A \neq \emptyset$ and $C'_B \neq \emptyset$ \textbf{do}
12: \hspace{2em} let $a \in T'_A$; $T'_A := T'_A - \{a\}$;
13: \hspace{2em} let $p' \in C'_B$; $C'_B := C'_B - \{p'\}$;
14: \hspace{2em} \textbf{Combine}(p, p', a);
15: \hspace{1em} \textbf{return} (p) \cup C'_B
16: \hspace{1em} if $T'_A = \emptyset$ then
17: \hspace{2em} let $U := V(A) - T'_A$;
18: \hspace{2em} \textbf{while} $U \neq \emptyset$ and $C'_B \neq \emptyset$ \textbf{do}
19: \hspace{3em} let $u \in U$; $U := U - \{u\}$;
20: \hspace{3em} let $p' \in C'_B$; $C'_B := C'_B - \{p'\}$;
21: \hspace{3em} \textbf{Combine}(p, p', u)
22: \hspace{1em} \textbf{return} (p) \cup C'_B
23: \hspace{1em} \textbf{if} $C'_B = \emptyset$ then
24: \hspace{2em} \textbf{while} $T'_A \neq \emptyset$ \textbf{do}
25: \hspace{3em} let $a \in T'_A$; $T'_A := T'_A - \{a\}$;
26: \hspace{3em} \textbf{if} $\exists$ Steiner vertex $v$ in path $p$ \textbf{then}
27: \hspace{4em} \textbf{Replace}(p, v, a) \( \triangleright \) remove a Steiner vertex from $B$
28: \hspace{3em} \textbf{else}
29: \hspace{4em} let $(u, v)$ be an edge between to terminal vertices of $B$
30: \hspace{4em} \textbf{Insert}(p, v, a) \( \triangleright \) add terminal vertices of $A$ to the path
31: \hspace{1em} \textbf{return} (p)

of minimum cost of the subgraph of $G$ induced by the vertices of $T(G)$. For the series composition we assume that the left subtree $T_x$ of any vertex $x$ of $T(G)$ contains no more terminal vertices than its right subtree $T_x$. Otherwise we only had to swap the children of the vertex $x$.

By algorithm DirectedSteinerPathCover we obtain the following result.

Theorem 3.9 A directed Steiner path cover of minimum cost for a directed co-graph can be computed in linear time with respect to the size of the directed co-expression.

Proof The correctness follows by Lemma 3.3 and 3.6. The running time can be achieved by storing the paths using double-linked, linear lists, where the $C_B$ paths that contain Steiner vertices are stored in one set and the paths that contain no Steiner vertices are stored in another set. The lists each have a pointer to the first and last element, which are terminal vertices, and they have a pointer to the first and last Steiner vertices. Additionally, we store the number of terminal and Steiner vertices for each list. Each of the operations in Algorithm DirectedSteinerPathCover can be done in constant time.

\[ \square \]

4 Hamiltonian Path Problem

Our motivation to study the Hamiltonian Path problem comes from the problem of minimizing change-over times in pick-and-place machines in electronic industry. Before we study this problem, we recall the definition of Hamiltonian Path problem.

Name Shortest Directed Hamiltonian Path

Instance A directed graph $G$ and edge weights $c : E(G) \rightarrow \mathbb{R}^{\geq 0}$.

Task Find a directed Hamiltonian path $P$ in $G$ that minimizes $\sum_{e \in P} c(e)$.

Pick-and-place machines have been studied for many years, see [CvdKS02]. Often, the focus of planning problems in printed circuit board assembly is on optimizing the throughput of these pick-
Algorithm 2 ORDERCOMP(set A, Cover CA, set B, Cover CB)

1: let \( \mathcal{C}'_A := \mathcal{C}_A \); let \( \mathcal{C}'_B := \mathcal{C}_B \);
2: if \( |\mathcal{C}_A| < |\mathcal{C}_B| \) then
3: loop \( \min(|\mathcal{C}_A|, |\mathcal{C}_B| - |\mathcal{C}_A|) \)-times
4: let \( r \in \mathcal{C}_A \) that contains a Steiner vertex
5: \( \mathcal{C}'_A := \mathcal{C}'_A - \{r\} \);
6: \( \mathcal{C}'_A := \mathcal{C}'_A \cup \{r_1, r_2\} \) where \( r_1, r_2 \) result from \( r \) by splitting \( r \) at a Steiner vertex
7: if \( |\mathcal{C}_A| > |\mathcal{C}_B| \) then
8: loop \( \min(|\mathcal{C}_B|, |\mathcal{C}_A| - |\mathcal{C}_B|) \)-times
9: let \( r \in \mathcal{C}_B \) that contains a Steiner vertex
10: \( \mathcal{C}'_B := \mathcal{C}'_B - \{r\} \);
11: \( \mathcal{C}'_B := \mathcal{C}'_B \cup \{r_1, r_2\} \) where \( r_1, r_2 \) result from \( r \) by splitting \( r \) at a Steiner vertex
12: return \( C := \emptyset \);
13: loop \( \min(|\mathcal{C}_A|, |\mathcal{C}_B|) \)-times
14: let \( a \in \mathcal{C}_A \); \( \mathcal{C}'_A := \mathcal{C}'_A - \{a\} \);
15: return \( \mathcal{C}'_A := \mathcal{C}_A - \{a\}; \mathcal{C} := \mathcal{C} \cup \{a\} \);
16: return \( C := \mathcal{C} \cup \{a\} \);
17: While \( C \neq \emptyset \) do
18: let \( a \in \mathcal{C}_A \); \( \mathcal{C}_A := \mathcal{C}_A - \{a\}; \mathcal{C} := \mathcal{C} \cup \{a\} \);
19: while \( \mathcal{C}'_B \neq \emptyset \) do
20: let \( b \in \mathcal{C}_B \); \( \mathcal{C}_B := \mathcal{C}_B - \{b\}; \mathcal{C} := \mathcal{C} \cup \{b\} \);
21: return \( C \)

Algorithm 3 DIRECTEDSTEINERPATHCOVER(Co-Tree \( T \), Vertex \( x \))

If \( x \) is the only vertex of \( T \) then
1: If \( x \) is a terminal vertex of \( G \) then
2: return \( \{x\} \)
3: return \( \emptyset \)
4: else
5: \( \mathcal{C}_A := \) DIRECTEDSTEINERPATHCOVER\((T_{x_\ell}, x_\ell)\) \( \triangleright x_\ell \) is the left successor of \( x \)
6: \( \mathcal{C}_B := \) DIRECTEDSTEINERPATHCOVER\((T_{x_r}, x_r)\) \( \triangleright x_r \) is the right successor of \( x \)
7: if \( x \) corresponds to a disjoint union of \( T \) then
8: return \( \mathcal{C}_A \cup \mathcal{C}_B \)
9: if \( x \) corresponds to a series composition of \( T \) then
10: return \( \text{SeriesComp}(V(T_{x_\ell}), \mathcal{C}_A, V(T_{x_r}), \mathcal{C}_B) \)
11: if \( x \) corresponds to an order composition of \( T \) then
12: return \( \text{OrderComp}(V(T_{x_\ell}), \mathcal{C}_A, V(T_{x_r}), \mathcal{C}_B) \)

and-place machines to produce many printed circuit boards (short PCB) a day. We are interested in minimizing change-over times for refilling the machines.

Mid-sized companies in the electronic industry often have to produce different PCBs on one day. Each of these PCBs has to be produced in small quantities. The SMD components like capacitors, resistors, or integrated circuits are usually positioned on the boards by pick-and-place machines and are soldered in a re-flow oven. Different types of components are fed in to the machine by feeders or trays. After one type of boards has been assembled, the pick-and-place machine has to be reassembled with other types of SMD components. The time for this refilling of components is often limited by the time the last PCBs of the previous series stay in the re-flow oven. The change-over must not take too long to avoid an unnecessary downtime of the machine. Therefore, PCBs must be processed in an order that only a few component groups have to be replaced. We show how a graph model can be used to describe the problem, and we give an integer program to solve the problem.

Assume that a company produces different PCBs, and the available pick-and-place machine can be equipped with at most \( k \) different types of SMD components, since no more trays are available. At one day, \( n \) different PCBs have to be produced, that together contain \( m \) different types of SMD components \( t_1, \ldots, t_m \). Each board \( b_i \) has to be equipped with a set \( T_i \subset \{t_1, \ldots, t_m\} \) of component types. We represent each board by an \( m \)-tuple such that the \( i \)-th entry is one if and only if type \( t_i \) is needed to produce the board. The number of ones in such a tuple is at most \( k \). For example, \( b_i = (0, 1, 0, 1, 0) \) means that components of type 2 and 4 must be used, types 1, 3, and 5 are not used for producing this board.

In our graph model, each board is represented by a vertex, and two vertices are connected by an edge. To change over from a board \( u \) to a board \( v \) those types have to be removed from the
machine, that are used by \( u \) but not used by \( v \). Those types, that are not used by \( u \) but are used by \( v \) have to be inserted in to the machine. Thus, the cost of change over is described by the hamming distance of \( u \) and \( v \).

Example 4.1 Given \( n = 4 \) jobs \( b_1 = (1,0,0,0) \), \( b_2 = (0,1,0,1) \), \( b_3 = (1,0,1,0) \), and \( b_4 = (1,1,1,0) \). The resulting graph contains the artificial vertex \( b_0 = (0,0,0,0) \) to represent the initial state of the machine, where all trays are empty. It is shown in Figure 2 together with the edge costs. To minimize the overall change-over times is to produce the boards in the order \( b_1 = (1,0,0,0) \), \( b_3 = (1,0,1,0) \), \( b_4 = (1,1,1,0) \), and \( b_2 = (0,1,0,1) \) which sum up to 6.

![Figure 2: Graph modeling 4 boards. The dashed lines show a Hamiltonian path that defines an order in which the change-over times sum up to 9.](image)

A shortest Hamiltonian path that starts at the artificial vertex representing the initial state of the machine minimizes the overall change-over time. In the next section we give a mixed integer program to solve the problem.

4.1 Integer Program

In the following we assume that a directed graph \( G = (V,E,c) \) is given, where \( V \) denotes the set of vertices, \( E \) denotes the set of edges, and \( c \) denotes a function that assigns the cost \( c(e) \) to edge \( e \). We represent an undirected graph \( G' \) by a directed graph \( G \) in such a way that each edge \( e \) of \( G' \) is represented by two anti-parallel edges in \( G \), and the weight of the new edges are equal to the weight of edge \( e \).

Our integer program is based on the following idea. Choose a vertex as the first vertex on the path. This is typically an artificial vertex that represents the empty machine at the beginning of a day. The order of vertices on the path implies for two vertices \( u \) and \( v \) following immediately on the path, that the cost of edge \( (u,v) \) represents the change-over time.

Our integer program uses binary variables \( x[i,j] \), \( i,j \in [n] \), such that \( x[i,j] \) is equal to one, if and only if edge \( (i,j) \) is on the Hamiltonian path. We use integer valued variables \( p[i] \), \( i \in [n] \), such that \( p[i] = \ell \), if and only if vertex \( i \) is the \( \ell \)-th vertex on the Hamiltonian path.

The first constraints ensure that the paths are all vertex-disjoint. The path leaves each vertex \( i \) at most once, and the path enters each vertex \( j \) at most once, which can be stated as follows.

\[
\sum_{(i,j) \in E} x[i,j] \leq 1 \quad \forall i \in V \quad \text{and} \quad \sum_{(i,j) \in E} x[i,j] \leq 1 \quad \forall j \in V
\]

The following constraints prevent loops over anti-parallel edges. If there are anti-parallel edges then choose at most one of them. This is ensured by the constraints:

\[
x[i,j] \leq 1 - x[j,i] \quad \forall (i,j) \in E : (j,i) \in E
\]

Each Hamiltonian path has length \( n - 1 \):

\[
\sum_{(i,j) \in E} x[i,j] = n - 1
\]
With just this constraint on the number of edges, we may accept solutions that are not connected. It is still possible to have cycles and a path instead of one path only. The next constraints are used to avoid cycles. If edge \((i, j)\) belongs to the path then \(p[j] - p[i] = 1\) must hold true:

\[-n + (n + 1) \cdot x[i, j] \leq p[j] - p[i] \quad \forall (i, j) \in E\]

\[p[j] - p[i] \leq n - (n - 1) \cdot x[i, j] \quad \forall (i, j) \in E\]

The objective is to minimize the sum of the change-over times, i.e. to make the path length as small as possible, so we have to minimize the objective function \(\sum_{(i, j) \in E} c[i, j] \cdot x[i, j]\). Alternatively, we can use \(\max_{(i,j) \in E} c[i, j] \cdot x[i, j]\) as objective function to minimize the largest change-over time.

### 4.2 Experimental results

We ran the mixed integer program on CPLEX 12.8.0 using default settings on some standard PC with Intel i7, 2.80 GHz CPU and 16 GB RAM running Ubuntu 18.10.

Running times to solve the Hamiltonian Path problem on randomly generated tournaments with randomly generated edge weights are shown in Table 3. The running times are average values over 10 runs. Bold numbers represent times when the time limit of 1800 seconds, which is equal to 30 minutes, was reached at least once and at most twice. It is well-known that any tournament on a finite number of vertices contains a Hamiltonian path.

| \(n\) | 100 | 200 | 300 | 400 | 500 |
|------|-----|-----|-----|-----|-----|
| time | 2.1 | 22.1| 69.3| 519 | 783 |
| stdev| 0.7 | 19.1| 81.6| 741 | 885 |

Table 3: Running times of MIP to calculate Hamiltonian Path on randomly generated tournaments.

Running times to solve the Hamiltonian Path problem on complete bipartite graphs \(K_{n,m}\) for different values of \(n\) and \(m\) are shown in Table 4. The complete bipartite graph \(K_{n,m}\) contains a Hamiltonian path if and only if \(|n - m| \leq 1\). The absence of such a path can be detected very quickly by the MIP as can be seen in the columns \(m = n + 2\). All complete bipartite graphs are co-graphs so these results show that our algorithm with running time linear in the input size would be very much faster than the integer program. Running times of more than 30 minutes are indicated by a bar −.

| \(n = 50\) | \(n = 100\) | \(n = 150\) | \(n = 200\) | \(n = 250\) |
|-----|-----|-----|-----|-----|
| \(m\) | 50 | 51 | 52 | 100 | 101 | 102 | 150 | 151 | 152 | 200 | 201 | 202 | 250 | 251 | 252 |
| Time | 1.7 | 3.2 | 0.1 | 16.7 | 10.9 | 0.1 | 131 | 366 | 0.2 | − | 370 | 0.3 | − | 1774 | 0.5 |

Table 4: Running times of MIP to calculate Hamiltonian Path on complete bipartite graphs \(K_{n,m}\) for different values of \(n\) and \(m\).

### 5 Steiner Path Problem

#### 5.1 Integer Program

Here again, we assume that a directed graph \(G = (V, E, c)\) is given, where \(V\) denotes the set of vertices, \(E\) denotes the set of edges, and \(c\) denotes a function that assigns the cost \(c(e)\) to edge \(e\). We represent an undirected graph \(G'\) by a directed graph \(G\) in such a way that each edge \(e\) of \(G'\) is represented by two anti-parallel edges in \(G\), and the weight of the new edges are equal to the weight of edge \(e\).

We use additional binary variables \(y[i], i \in [n]\), such that \(y[i]\) is equal to one, if and only if vertex \(i\) is contained in the path. These variables are only used to simplify the formulation of the
following constraints [1] and [2]. As in the last chapter, our integer program uses binary variables $x[i,j]$, $i,j \in [n]$, such that $x[i,j]$ is equal to one, if and only if edge $\{i,j\}$ is on the Steiner path. We use integer valued variables $p[i]$, $i \in [n]$, such that $p[i] = \ell$, if and only if vertex $i$ is the $\ell$-th vertex on the Steiner path. Each terminal vertex $i$ has to be contained in the path.

$$p[i] \geq 1 \quad \forall i \in T$$

If $x[i,j] = 1$ then $i$ and $j$ lie on the path, i.e. if $x[i,j] = 1$ then $p[i] \neq 0$ and $p[j] \neq 0$ which can be stated by the following constraints.

$$x[i,j] \leq p[i] \quad \forall (i,j) \in E \quad \text{and} \quad x[i,j] \leq p[j] \quad \forall (i,j) \in E$$

If there are anti-parallel edges then choose at most one of them. This is ensured by the constraint:

$$x[i,j] \leq 1 - x[j,i] \quad \forall (i,j) \in E : (j,i) \in E$$

It holds $y[i] = 1$ iff $p[i] > 0$, i.e. $y[i] = 1$ iff vertex $i$ is on the path.

$$y[i] \leq p[i] \quad \forall i \in V \quad \text{and} \quad p[i] \leq n \cdot y[i] \quad \forall i \in V$$

The path has to contain one edge less than vertices.

$$\sum_{(i,j) \in E} x[i,j] = \sum_{i \in V} y[i] - 1$$  \hfill (1)

If $p[i] \neq 0$ then there has to be a $x[i,j] = 1$ or $x[j,i] = 1$.

$$p[i] \leq n \cdot \left( \sum_{(i,j) \in E} x[i,j] + \sum_{(j,i) \in E} x[j,i] \right) \quad \forall i \in V$$

For each vertex $i \in V - T$ contained in the path we have one in-going and one outgoing edge.

$$\sum_{(i,j) \in E} x[i,j] = y[i] \quad \forall i \in V - T \quad \text{and} \quad \sum_{(i,j) \in E} x[i,j] = y[i] \quad \forall j \in V - T$$  \hfill (2)

The path leaves each vertex $i \in T$ at most once and it enters each vertex $j \in T$ at most once.

$$\sum_{(i,j) \in E} x[i,j] \leq 1 \quad \forall i \in T \quad \text{and} \quad \sum_{(i,j) \in E} x[i,j] \leq 1 \quad \forall j \in T$$

If a vertex is in the path then its position must be at least one.

$$\sum_{(i,j) \in E} x[i,j] \leq p[i] \quad \forall i \in V \quad \text{and} \quad \sum_{(i,j) \in E} x[i,j] \leq p[j] \quad \forall j \in V$$

Finally, to avoid cycles, if edge $(i,j)$ belongs to the path then $p[j] - p[i] = 1$ must hold true:

$$-n + (n + 1) \cdot x[i,j] \leq p[j] - p[i] \quad \forall (i,j) \in E$$

$$p[j] - p[i] \leq n - (n - 1) \cdot x[i,j] \quad \forall (i,j) \in E$$

The objective is to minimize the sum of edge-weights $\sum_{(i,j) \in E} c[i,j] \cdot x[i,j]$. For unit-distance graphs this objective function results in a Steiner path with least number of Steiner vertices.

### 5.2 Experimental results

Running times to solve Steiner Path problems on complete bipartite graphs $K_{n,3n}$ for different values of $n$ and $t$ many randomly chosen terminal vertices are shown in Table 5. As long as at most $n$ vertices from the second set are selected as terminal vertices, a Steiner path exists. The absence of such a path can be detected very quickly by the MIP as can be seen in the columns $t = 2n$.

The running times are average values over 10 runs, bold numbers represent times when the time limit of 1800 seconds, which is equal to 30 minutes, was reached at least once and at most twice. Running times of more than 30 minutes are indicated by a bar –.
After inserting artificial vertices, the path leaves each terminal vertex exactly once, and the path enters each non-terminal vertex exactly once, which can be stated as follows.

$$\sum_{(i,j) \in E'} x[i,j] = 1 \ \forall i \in T$$

Since non-terminal vertices need not be contained in the path, the path leaves each non-terminal vertex at most once, and the path enters each non-terminal vertex at most once, which can be stated as follows.

$$\sum_{(i,j) \in E} x[i,j] \leq 1 \ \forall i \in V - T$$
The previous conditions also ensure that paths are vertex-disjoint. If a path contains a non-terminal vertex \( j \) then there has to be an edge that enters \( j \) and an edge that leaves \( j \), which can be stated as:

\[
\sum_{(i,j) \in E} x[i,j] = \sum_{(j,k) \in E} x[j,k] \quad \forall j \in V - T
\]

If there are anti-parallel edges then choose at most one of them. This is ensured by the constraint:

\[
x[i,j] \leq 1 - x[j,i] \quad \forall (i,j) \in E : (j,i) \in E
\]

Each terminal vertex \( i \) has to be contained in the path.

\[
p[i] \geq 1 \quad \forall i \in T
\]

If \( x[i,j] = 1 \) then \( i \) and \( j \) lie on the path, i.e. if \( x[i,j] = 1 \) then \( p[i] \neq 0 \) and \( p[j] \neq 0 \) which can be stated by the following constraints.

\[
x[i,j] \leq p[i] \quad \forall (i,j) \in E \quad \text{and} \quad x[i,j] \leq p[j] \quad \forall (i,j) \in E
\]

If \( p[i] \neq 0 \) then there has to be a \( x[i,j] = 1 \) or \( x[j,i] = 1 \).

\[
p[i] \leq n \cdot \left( \sum_{(i,j) \in E} x[i,j] + \sum_{(j,i) \in E} x[j,i] \right) \quad \forall i \in V
\]

Finally, to avoid cycles, if edge \((i,j)\) belongs to the path then \( p[j] - p[i] = 1 \) must hold true:

\[
-n + (n + 1) \cdot x[i,j] \leq p[j] - p[i] \quad \forall (i,j) \in E
\]

and

\[
p[j] - p[i] \leq n - (n - 1) \cdot x[i,j] \quad \forall (i,j) \in E
\]

Minimizing an objective function like \( \sum_{s \in T} x[s,v] \) would only minimize the number of paths. If there are two different path covers of minimum size, we have to select that one with the least number of Steiner vertices. By our definition of the costs of the additional edges from \( s \) to terminal vertices we can choose the same objective function as above, when we also take care of the additional edges. The objective is to minimize the sum of edge-weights \( \sum_{(i,j) \in E} c[i,j] \cdot x[i,j] \). For unit-distance graphs this objective function results in a Steiner path cover \( \hat{P} \) of minimum size, with least number of Steiner vertices \( \sum_{p \in P} \sum_{e \in p} c(e) \).

### 6.2 Experimental results

Running times to solve Steiner Path Cover problems on complete bipartite graphs \( K_{n,3n} \) for different values of \( n \) and \( t \) many randomly chosen terminal vertices are shown in Table 6.

| \( n = 25 \) | \( n = 50 \) | \( n = 75 \) | \( n = 100 \) | \( n = 125 \) |
|-------|-------|-------|-------|-------|
| \( t \) | 12 | 25 | 50 | 25 | 50 | 100 | 37 | 75 | 150 | 50 | 100 | 200 | 62 | 125 | 250 |
| time | 0.9 | 1.4 | 1.1 | 7.4 | 9.1 | 7.4 | 25.7 | 47.7 | 20.9 | 112 | 207 | 55.8 | 307 | — | 269 |

Table 6: Running times to solve Steiner Path Cover problems on complete bipartite graphs \( K_{n,3n} \) for different values of \( n \) and \( t \).

The running times are average values over 10 runs. Running times of more than 30 minutes are indicated by a bar —, bold numbers represent times when the time limit was reached at least once and at most twice. Table 7 shows running times to solve Steiner Path Cover problems on randomly generated co-graphs with \( n \) vertices and \( t \) many randomly chosen terminal vertices.

The co-graphs will be generated recursively as shown in Algorithm 4. The probability \( p \) is set to be \( 1/3 \) in our experiments.
Table 7: Running times to solve Steiner Path Cover problems on randomly generated co-graphs with \( n \) vertices and \( t \) many randomly chosen terminal vertices.

| Algorithm 4 RANDOM-CO-GRAPH(int \( t \), int \( n \)) |
|-------------------------------------------|
| choose \( t \) distinct vertices at random |
| call OTIMES(1, \( n \)) |
| function OTIMES(int \( l \), int \( r \)) |
| if \( r - 1 = 1 \) then |
| create edge \((l, r)\) and return |
| \( m := \text{rand}(l, r) \) \> choose random number between \( l \) and \( r \) |
| call OPLUS(l, \( m \)) with probability \( p \), otherwise call OTIMES(l, \( m \)) |
| call OPLUS(m+1, \( r \)) with probability \( p \), otherwise call OTIMES(m+1, \( r \)) |
| create edges \((i, j)\) for all vertices \( l \leq i \leq m \), and \( m + 1 \leq j \leq r \) |
| function OPLUS(int \( l \), int \( r \)) |
| if \( r - 1 = 1 \) then |
| return |
| \( m := \text{rand}(l, r) \) |
| call OPLUS(l, \( m \)) with probability \( p \), otherwise call OTIMES(l, \( m \)) |
| call OPLUS(m+1, \( r \)) with probability \( p \), otherwise call OTIMES(m+1, \( r \)) |

7 Conclusions and outlook

In this paper we considered the directed Steiner path cover problem for directed co-graphs. We could show a linear time solution for computing the minimum number of paths within a directed Steiner path cover and the minimum number of Steiner vertices in such a directed Steiner path cover in directed co-graphs. The results allowed us to give an algorithm which constructs a directed Steiner path cover of minimum cost for a directed co-graph in linear time. This leads to a linear time computation of an optimal directed Steiner path, if it exists, for directed co-graphs.

Undirected co-graphs are precisely those graphs which can be generated from the single vertex graph by disjoint union and join operations, see [CLSB81]. Given some undirected co-graph \( G \), we can solve the Steiner path cover problem in linear time by replacing every edge \( \{u, v\} \) of \( G \) by two directed edges \((u, v)\) and \((v, u)\) and applying our solution for directed co-graphs. This reproves our result of [GHK+20b].

The directed Hamiltonian path problem can be solved by an XP-algorithm w.r.t. the parameter directed clique-width [GHO13]. Since directed co-graphs have directed clique-width at most two [GWY16], a polynomial time solution for the directed Hamiltonian path problem follows. Such an algorithm is also given in [Gur17]. Since a directed Hamiltonian path exists if and only if we have \( T = V(G) \) and \( p(G) = 1 \), our results imply the first linear time algorithm for the directed Hamiltonian path problem on directed co-graphs. This generalizes the known results for undirected co-graphs of Lin et al. [LOP95].

In our future work we want to find out whether results can be transferred to other graph classes such as chordal graphs, interval graphs, or distance-hereditary graphs.

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