Sufficient conditions for the filtration of a stationary processes to be standard

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Abstract

Let $X$ be a stationary process with values in some $\sigma$-finite measured state space $(E, \mathcal{E}, \pi)$, indexed by $\mathbb{Z}$. Call $\mathcal{F}^X$ its natural filtration. In [2], sufficient conditions were given for $\mathcal{F}^X$ to be standard when $E$ is finite. The proof of this result used a coupling of all probabilities on the finite set $E$.

In this paper, we construct a coupling of all laws having a density with regard to $\pi$, which is much more involved. Then, we provide sufficient conditions for $\mathcal{F}^X$ to be standard, generalizing those in [2].

1 Introduction

1.1 Setting

In this paper we study stationary processes $X = (X_n)_{n \in \mathbb{Z}}$ indexed by the integer line $\mathbb{Z}$ and with values in a measured space $(E, \mathcal{E}, \pi)$. We will see that such processes can be defined recursively as follows: for every $n \in \mathbb{Z}$, $X_n$ is a function of the “past” $X_{n-1} = (X_k)_{k \leq n-1}$ of $X$ and of a “fresh” random variable $U_n$, which brings in some “new” randomness. In particular the process $U = (U_n)_{n \in \mathbb{Z}}$ is independent. To be more specific, we introduce some notations and definitions about $\sigma$-fields.

All $\sigma-$fields are assumed to be complete. For every process $\xi = (\xi_n)_{n \in \mathbb{Z}}$ and every $n \in \mathbb{Z}$, let $\xi^\preceq_n = (\xi_k)_{k \leq n}$ and $\mathcal{F}^\xi_n = \sigma(\xi^\preceq_n)$. The natural filtration of $\xi$ is the nondecreasing sequence $\mathcal{F}^\xi = (\mathcal{F}^\xi_n)_{n \in \mathbb{Z}}$. Furthermore, $\mathcal{F}^\xi_\infty = \sigma(\xi_k; k \in \mathbb{Z})$ and $\mathcal{F}^\xi_{-\infty}$ is the tail $\sigma$-field $\mathcal{F}^\xi_{-\infty} = \bigcap_{k \in \mathbb{Z}} \mathcal{F}^\xi_k$.

We say that a process $U$ is a governing process for $X$, or that $U$ governs $X$ if, for every $n \in \mathbb{Z}$, (i) $U_{n+1}$ is independent of $\mathcal{F}^X_n$, and (ii) $X_{n+1}$ is measurable with
respect to $\sigma(U_{n+1}) \lor F_n^X$. In particular any governing process is independent. If moreover the $U_n$ are uniform on $[0, 1]$, the process $(U, X)$ is – according to Schachermayer’s definition [8] and up to a time reversal – a parametrization of the process $X$. For any process $X = (X_n)_{n \in \mathbb{Z}}$ with values in a separable space, the natural filtration $F^X$ possesses a parametrization (see a proof in [3], section 5.0.5).

Likewise, we say that a governing process $U$ is a generating process for $X$, or that $U$ generates $X$ if, for every $n \in \mathbb{Z}$, $X_n$ is measurable with respect to $F_n^U$. This is equivalent to the condition that $F_n^X \subset F_n^U$ for every $n \in \mathbb{Z}$, a property which, from now on, we write as $F^X \subset F^U$.

Governing and generating processes are related to immersions of filtrations. Recall that the filtration $F^X$ is immersed in the filtration $F^U$ if $F^X \subset F^U$ and if, for every $n \in \mathbb{Z}$, $F^X_{n+1}$ and $F^U_n$ are independent conditionally on $F^X_n$. Roughly speaking, this means that $F^U_n$ gives no further information on $X_{n+1}$ than $F^X_n$ does. Equivalently, $F^X$ is immersed in $F^U$ if every $F^X$-martingale is an $F^U$-martingale.

The following easy fact holds (see a proof in [3]).

**Lemma 1.1** If $U$ is a generating process for $X$, then $F^X$ is immersed in $F^U$.

Another notable property of filtrations is standardness. Recall that $F^X$ is standard if, modulo an enlargement of the probability space, one can immerse $F^X$ in a filtration generated by an i.i.d. process with values in a separable space. Vershik introduced standardness in the context of ergodic theory. Examples of non-standard filtrations include the filtrations of $[T, T^{-1}]$ transformations, introduced in [6]. Split-word processes, inspired by Vershik’s $(r_n)$-adic sequences of decreasing partitions [10] and studied in [9] and [7], for instance, also provide non-standard filtrations.

Obviously, lemma 1.1 above implies that if $X$ has a generating and governing process, then $F^X$ is standard. Whether the converse holds is not known.

Necessary and sufficient conditions for standardness include Vershik’s self-joining criterion and Tsirelson’s notion of $I$-cosiness. Both notions are discussed in [5] and are based on conditions which are subtle and not easy to use nor to check in specific cases.

Our goal in this paper is to provide sufficient conditions for the existence of a generating sequence for $F^X$ (and therefore of standardness) that are easier to use than the ones mentioned above. Each of our conditions involves a quantification of the influence of the distant past of the process on its present. We introduce them in the next section.
1.2 Measuring the influence of the distant past

We now introduce some quantities measuring the influence of the past of a process on its present. We need to introduce some notations.

Recall that $X$ is a stationary process indexed by the integer line $\mathbb{Z}$ with values in some measurable space $(E, \mathcal{E})$ and with natural filtration $\mathcal{F}^X$. From now on, we fix a reference measure $\pi$ on $E$ which is $\sigma$-finite.

**Notation 1** (1) Slabs: For any sequence $(\xi_n)_{n \in \mathbb{Z}}$ in $E^\mathbb{Z}$, deterministic or random, and any integers $i \leq j$, $\xi_{i:j}$ is the $(j - i + 1)$-uple $(\xi_n)_{i \leq n \leq j}$ in $E^{j-i+1}$.

(2) Shifts: If $k - i = \ell - j$, $\xi_{i:k} = \xi_{j:\ell}$ means that $\xi_{i+n} = \xi_{j+n}$ for every integer $n$ such that $0 \leq n \leq k - i$.

Infinite case: Let $E^\infty$ denote the space of sequences $(\xi_n)_{n \leq -1}$. For every $i$ in $\mathbb{Z}$, a sequence $(\xi_n)_{n \leq i}$ is also considered as an element of $E^\infty$ since, similarly to the finite case, one identifies $\xi_i^\infty = (\xi_n)_{n \leq i}$ and $\xi_j^\infty = (\xi_n)_{n \leq j}$ if $\xi_{i+n} = \xi_{j+n}$ for every integer $n \leq 0$.

(3) Concatenation: for all $i \geq 0$, $j \geq 0$, $x = (x_n)_{1 \leq n \leq i}$ in $E^i$ and $y = (y_n)_{1 \leq n \leq j}$ in $E^j$, $xy$ denotes the concatenation of $x$ and $y$, defined as

$$xy = (x_1, \ldots, x_i, y_1, \ldots, y_j), \quad xy \in E^{i+j}.$$

Infinite case: for all $j \geq 0$, $y = (y_n)_{1 \leq n \leq j}$ in $E^j$ and $x = (x_n)_{n \leq -1}$ in $E^\infty$, $xy$ denotes the concatenation of $x$ and $y$, defined as

$$xy = (\ldots, x_{-2}, x_{-1}, y_1, \ldots, y_j), \quad xy \in E^\infty.$$

**Assumption 1** From now on, assume that one can choose a regular version of the conditional law of $X_0$ given $X_{-1}^\infty$ in such a way that for every $x \in E^\infty$, the law $\mathcal{L}(X_0 | X_{-1}^\infty = x)$ has a density $f(\cdot | x)$ with respect to the reference $\sigma$-finite measure $\pi$ on $E$. Then, for every $n \geq 0$ and for every $x_{1:n} \in E^n$, the law $\mathcal{L}(X_0 | X_{-n-1} = x_{1:n})$ has a density $f(\cdot | x_{1:n})$ with respect to $\pi$.

When $E$ is countable and $\pi$ is the counting measure on $E$, the densities $f(\cdot | x)$ coincide with the functions $p(\cdot | x)$ used in [2] and defined by

$$p(a|x) = \mathbb{P}(X_0 = a \mid X_{-n-1} = x), \quad \text{for } x \in E^n \text{ and } a \in E;$$

and

$$p(a|x) = \mathbb{P}(X_0 = a \mid X_{-1}^\infty = x), \quad \text{for } x \in E^\infty \text{ and } a \in E.$$

We now introduce three quantities $\gamma_n$, $\alpha_n$ and $\delta_n$ measuring the pointwise influence at distance $n$. 

3
Definition 1 For every $n \geq 0$, let

$$
\gamma_n = 1 - \inf \left\{ \frac{f(a|xz)}{f(a|yz)} ; a \in E, \ x \in E^<, \ y \in E^<, \ z \in E^n, \ f(a|yz) > 0 \right\},
$$

$$
\alpha_n = 1 - \inf_{z \in E^n} \int_{E} \inf \{ f(a|yz) ; y \in E^< \} \, d\pi(a),
$$

$$
\delta_n = \sup \left\{ \| f(\cdot|xz) - f(\cdot|yz) \| ; \ x \in E^<, \ y \in E^<, \ z \in E^n \right\},
$$

where, for every densities $f$ and $g$ on $E$,

$$
\| f - g \| = \frac{1}{2} \int_{E} |f(x) - g(x)| \, d\pi(x) = \int_{E} [f(x) - g(x)]_+ \, d\pi(x).
$$

Note that the values $\gamma_n$, $\alpha_n$ and $\delta_n$ depend on the choice of the regular version of the conditional law of $X_0$ given $X^{-1}$. One wants small influences to apply the theorems below and one needs a “good” version of the conditional law to get small influences. Yet, replacing $\pi$ by an equivalent measure do not modify the quantities $\gamma_n$, $\alpha_n$ and $\delta_n$.

The sequences $(\gamma_n)_{n \geq 0}$, $(\alpha_n)_{n \geq 0}$ and $(\delta_n)_{n \geq 0}$ are non-increasing, $[0,1]$-valued, $\delta_n \leq \gamma_n$ and $\delta_n \leq \alpha_n$ for every $n \geq 0$ (see the proof in [2], section 5).

For every $[0,1]$-valued sequence $(\varepsilon_n)_{n \geq 0}$, we consider the condition

$$
\sum_{k=0}^{+\infty} \prod_{n=0}^{k} (1 - \varepsilon_n) = +\infty. \quad (\mathcal{H}(\varepsilon))
$$

For instance, $\mathcal{H}(\gamma)$ and $\mathcal{H}(2\delta/(1+\delta))$ are respectively

$$
\sum_{k=0}^{+\infty} \prod_{n=0}^{k} (1 - \gamma_n) = +\infty, \quad \text{and} \quad \sum_{k=0}^{+\infty} \prod_{n=0}^{k} \frac{1 - \delta_n}{1 + \delta_n} = +\infty.
$$

Observe that if two $[0,1]$-valued sequences $(\varepsilon_n)_{n \geq 0}$ and $(\zeta_n)_{n \geq 0}$ are such that $\varepsilon_n \leq \zeta_n$ for every $n \geq 0$, then $\mathcal{H}(\zeta)$ implies $\mathcal{H}(\varepsilon)$. Hence condition $\mathcal{H}(\varepsilon)$ asserts that $(\varepsilon_n)_{n \geq 0}$ is small in a certain way.

1.3 Statement of the main results

The definition of $(\gamma_n)_{n \geq 0}$ and the assumption $\mathcal{H}(\gamma)$ are both in [1]. The main result of [1] is that when $E$ if of size 2, then $\mathcal{H}(\gamma)$ implies that $\mathcal{F}^X$ admits a governing generating process. This result is restricted by the following three conditions. First, the size of $E$ must be 2. Second, one must control the ratios...
of probabilities which define $\gamma_n$. Third, $\mathcal{H}(\gamma)$ implies that $\gamma_0 < 1$, therefore one can show that $\mathcal{H}(\gamma)$ implies the existence of $c > 0$ such that $f(a|x) \geq c$ for every $x$ in $E^\subset$ and $a$ in $E$.

Theorem 2 in [2] improves on this and gets rid of the first two restrictions. And Theorem 1 below improves on Theorem 2 in [2].

**Theorem 1**

1. Assume that $E$ is finite, and that $\mathcal{H}(2\delta/(1 + \delta))$ holds. Then $\mathcal{F}^X$ admits a governing generating process.

2. If the size of $E$ is 2, $\mathcal{H}(\delta)$ alone implies that $\mathcal{F}^X$ admits a governing generating process.

The improvements that theorem 1 brings to the first result in [2] are to replace the assumption $\mathcal{H}(2\delta)$ is replaced by $\mathcal{H}(2\delta/(1 + \delta))$ and to get rid of the extra hypothesis $\delta_0 < 1/2$.

Another measure of influence, based on the quantities $\alpha_n$, is introduced and used in [4] (actually the notation there is $a_n = 1 - \alpha_n$). The authors show that if $\mathcal{H}(\alpha)$ holds, there exists a perfect sampling algorithm for the process $X$, a result which implies that $\mathcal{F}^X$ admits a governing generating process.

Our Theorem 1, the result in [1] and the exact sampling algorithm of [4] all require an upper bound of some pointwise influence sequence. The second theorem in [2] uses a less restrictive hypothesis based on some average influences $\eta_n$, defined below.

**Definition 2** For every $n \geq 0$, let $\eta_n$ denote the average influence at distance $n$, defined as

$$\eta_n = \mathbb{E}[\| f(\cdot|Y_{n-1}) - f(\cdot|X_\subset^{\leq n-1}Y_{n-1})\|],$$

where $Y$ is an independent copy of $X$, and call $\mathcal{H}'(\eta)$ the condition

$$\sum_{k=0}^{+\infty} \eta_k < +\infty. \quad (\mathcal{H}'(\eta))$$

As before, the definition of $\eta_n$ depends on the choice of the regular version of the conditional law of $X_0$ given $X_\subset^{\leq 1}$, but it does not depend on the choice of the reference measure $\pi$. We recall the result from [2].

**Theorem 2** (Theorem 3 of [2]) Assume that $E$ is finite and that for every $a$ in $E$, $p(a|X_\subset^{\leq 1}) > 0$ almost surely (priming condition). Then, $\mathcal{H}'(\eta)$ implies that $\mathcal{F}^X$ admits a governing generating process.
Note that the sequence \((\eta_n)_{n\geq 0}\) is \([0,1]\)-valued. If \(\eta_n < 1\) for every \(n \leq 0\), then \(\mathcal{H}'(\eta)\) implies \(\mathcal{H}(\eta)\). Yet, since \(\eta_n \leq \delta_n\) for every \(n \geq 0\), the condition \(\mathcal{H}'(\eta)\) cannot be compared to the conditions \(\mathcal{H}(\delta)\) and \(\mathcal{H}(2\delta/(1+\delta))\).

The main result of the present paper is the theorem below which extends theorem 3 of [2] to any \(\sigma\)-finite measured space \((E, \mathcal{E}, \pi)\).

**Theorem 3** Let \((E, \mathcal{E}, \pi)\) be any \(\sigma\)-finite measured space. Assume that one has chosen a regular version of the conditional law of \(X_0\) given \(X_{-1}\) in such a way that for every \(x \in E^\circ\), the law \(\mathcal{L}(X_0|X_{-1} = x)\) has a density \(f(\cdot|x)\) with respect to \(\pi\) and that \(f(a|X_{-1}^\circ) > 0\) for \(\pi\)-almost every \(a\) in \(E\), almost surely (priming condition). Then, \(\mathcal{H}'(\eta)\) implies that \(\mathcal{F}^X\) admits a governing generating process, hence is standard.

In this paper, we call **priming condition** the condition that \(f(a|X_{-1}^\circ) > 0\) for almost every \(a\) in \(E\), almost surely. This generalises the priming condition of [2]. Indeed, when \(E\) is finite and \(\pi\) is the counting measure on \(E\), the priming condition is equivalent to the assumption that for every \(a\) in \(E\), \(p(a|X_{-1}^\circ) > 0\) almost surely, that is, the condition involved in theorem 2. An equivalent form, which is actually the one used in the proof of theorem 2, is

\[
\inf_{a \in E} p(a|X_{-1}^\circ) > 0 \text{ almost surely.}
\]

But this condition cannot be satisfied anymore when \(E\) is infinite. Therefore, the proof of theorem 3 requires new ideas to use the priming condition, see our priming lemma in section 3 (lemma 3.1).

### 1.4 Global couplings

A key tool for the construction of our governing process is a new global coupling. This global coupling, which applies when \(E\) is infinite, is much more involved than the coupling introduced in [2] when \(E\) is finite.

Recall that if \(p\) and \(q\) are two fixed probabilities on a countable space \(E\), then for every random variables \(Z_p\) and \(Z_q\) with laws \(p\) and \(q\) defined on the same probability space,

\[
\mathbb{P}[Z_p \neq Z_q] \geq \|p - q\|,
\]

where \(\|p - q\|\) is the distance in total variation between \(p\) and \(q\), defined as

\[
\|p - q\| = \frac{1}{2} \sum_{a \in E} |p(a) - q(a)| = \sum_{a \in E} [p(a) - q(a)]_+.
\]

Conversely, a standard construction in coupling theory provides some random variables \(Z_p\) and \(Z_q\) with laws \(p\) and \(q\) such that \(\mathbb{P}[Z_p \neq Z_q] = \|p - q\|\).

Couplings can be generalized to any set of laws, on a measurable space \((E, \mathcal{E})\).
Definition 3 Note \( \mathcal{P}(E) \) the set of probabilities on the measurable space \((E, \mathcal{E})\). Let \( \mathcal{A} \) be a subset of \( \mathcal{P}(E) \). A global coupling for \( \mathcal{A} \) is a random variable \( U \) with values in some measurable space \((F, \mathcal{F})\) and a function \( g \) from \( \mathcal{A} \times F \) to \( E \) such that for every \( p \in \mathcal{A} \), \( g(U, p) \) is a random variable of law \( p \) on \( E \).

Good global couplings are such that for every \( p \) and \( q \) in \( \mathcal{A} \), the probability \( \mathbb{P}[g(U, p) \neq g(U, q)] \) is small when \( \|p - q\| \) is small.

When \( E = \{0, 1\} \), a classical coupling used in [1] is given by \( g(U, p) = \mathbf{1}_{\{U \leq p(1)\}} \), where \( U \) is a uniform random variable on \([0, 1]\). This coupling satisfies the equality \( \mathbb{P}[g(U, p) \neq g(U, q)] = \|p - q\| \), for every \( p \) and \( q \) in \( \mathcal{P}(E) \).

This construction can still be extended to any countable set, as follows.

**Proposition 1.2** Assume that \( E \) is countable. Let \( \varepsilon = (\varepsilon_a)_{a \in E} \) be an i.i.d. family of exponential random variables with parameter 1. Then, for any probability \( p \) on \( E \), there exists almost surely one and only one \( a \in E \) such that

\[
\frac{\varepsilon_a}{p(a)} = \inf_{b \in E} \frac{\varepsilon_b}{p(b)} \quad (\star),
\]

with the convention that \( \varepsilon_b/p(b) = +\infty \) if \( p(b) = 0 \).

- Define \( g(\varepsilon, p) \) (almost surely) as the only index \( a \) such that \( (\star) \) holds. Then the law of the random variable \( g(\varepsilon, p) \) is \( p \).
- For every \( p, q \) in \( \mathcal{P}(E) \),

\[
\mathbb{P}[g(\varepsilon, p) \neq g(\varepsilon, q)] \leq 2 \frac{\|p - q\|}{1 + \|p - q\|} \leq 2 \|p - q\|.
\]

The bound \( 2\|p - q\|/(1 + \|p - q\|) \) improves on the bound \( 2\|p - q\| \) given in proposition 2.6 of [2] and explains that the assumption \( \mathcal{H}(2\delta/(1 + \delta)) \) of theorem 1 is weaker than the assumptions \( \mathcal{H}(2\delta) \) and \( \delta_0 < 1/2 \) required in theorem 2 of [2], although the same proof works. The proof of proposition 1.2 will be given in section 5.

We now briefly introduce a new global coupling that works on any measured \( \sigma \)-finite space \((E, \mathcal{E}, \pi)\). The construction and the properties of the coupling \((g, U)\) will be detailed in section 2.

Let \( \mathcal{A} \) be the subset of \( \mathcal{P}(E) \) of all probability laws \( p \) on \( E \) admitting a density \( f_p \) with respect to \( \pi \). Let \( \lambda \) be the Lebesgue measure on \( \mathbb{R}^+ \). Let \( U \) be a Poisson point process on \( E \times \mathbb{R}^+ \times \mathbb{R}^+ \) of intensity \( \pi \otimes \lambda \otimes \lambda \).

In the following, any element \((x, y, t)\) of \( U \) is seen as a point \((x, y)\) in \( E \times \mathbb{R}^+ \) appearing at the time \( t \). Since \( \lambda \) is diffuse, the third components of the elements
of $U$ are almost surely all distinct. For any density function $f$ on $(E, \mathcal{E}, \pi)$, call $t_f$ the instant of appearance of a point under the graph of $f$. Then, almost surely, $t_f$ is positive and finite and the set $U \cap (A \times [0, t_f])$ is almost surely reduced to a single point $(x_f, y_f, t_f)$. The random variable $x_f$ has density $f$ with respect to $\pi$.

**Proposition 1.3** With the notations above, the formula $g(U, p) = x_f$ defines a global coupling for $A$ such that for every $p$ and $q$ in $A$,

$$
\mathbb{P}[g(U, p) \neq g(U, q)] \leq \frac{2\|p - q\|}{1 + \|p - q\|} \leq 2\|p - q\|.
$$

The paper is organised as follows: In section 2, we detail the construction of the global coupling just mentionned, we prove proposition 1.3 and we construct a governing sequence for the process $(X_n)_{n \in \mathbb{Z}}$. Section 3 is devoted to the proof of a key intermediate result which we call priming lemma, which uses the priming condition. In section 4, we complete the proof of theorem 3, showing that the governing sequence $(U_n)_{n \in \mathbb{Z}}$ constructed in section 2 generates the filtration $\mathcal{F}^X$.

# 2 Global couplings and construction of a governing sequences

## 2.1 Construction of a global coupling

Let $(E, \mathcal{E}, \pi)$ be a $\sigma$-finite measured space. Let $\lambda$ be the Lebesgue measure on $\mathbb{R}^+$. Set $\mu = \pi \otimes \lambda$ on $E \times \mathbb{R}^+$. Denote by $\mathcal{D}$ the set of countable subsets of $E \times \mathbb{R}^+ \times \mathbb{R}^+$.

Let $U$ be a Poisson point process on $E \times \mathbb{R}^+ \times \mathbb{R}^+$ of intensity $\pi \otimes \lambda \otimes \lambda$. By definition, $U$ is a $\mathcal{D}$-valued random variable such that for any pairwise disjoint sets $B_k$ in $\mathcal{E}$, of finite measure for $\mu \otimes \lambda = \pi \otimes \lambda \otimes \lambda$, the cardinalities $|U \cap B_k|$ are independent Poisson random variables with respective parameters $(\mu \otimes \lambda)(B_k)$.

In the following, any element $(x, y, t)$ of $U$ is seen as a point $(x, y)$ in $E \times \mathbb{R}^+$ appearing at time $t$. Since $\lambda$ is diffuse, the third components of the elements of $U$ are almost surely all distinct.

**Notation 2** For any measurable subset $A$ of $E \times \mathbb{R}^+$, the time of appearance of a point in $A$ is

$$
t_A(U) = \inf \{t \geq 0 : U \cap (A \times [0, t]) \neq \emptyset\}.
$$

Let us recall some classical and useful properties of Poisson point processes. If the measure $\mu(A)$ is positive and finite, the random variable $t_A(U)$ has exponential
law of parameter $\mu(A)$. Moreover the set $U \cap (A \times [0, t_A(U)])$ is almost surely reduced to a single point $(x_A(U), y_A(U), t_A(U))$. The couple $(x_A(U), y_A(U))$ thus defined is a random variable of law $\mu(\cdot|A) = \mu(\cdot \cap A)/\mu(A)$ which is independent from $t_A(U)$.

Moreover if the sets $B_k$ are pairwise disjoint measurable subsets of $E \times \mathbb{R}^+$, the random variables $(x_{B_k}(U), y_{B_k}(U), t_{B_k}(U))$ are independent.

The coordinates $x_A(U), y_A(U), t_A(U)$ vary as a function of the variable $U$. This notation will be used later on when we will consider several Poisson point processes, but, by abuse of notation it will be abbreviated to $x_A, y_A, t_A$ when there is no ambiguity on the Poisson point process.

**Lemma 2.1** If the $n$ sets $A_k$ are measurable subsets of $E \times \mathbb{R}^+$, of finite positive measures for $\mu$, then almost surely

$$t_{A_1} = \ldots = t_{A_n} \iff t_{A_1 \cap \ldots \cap A_n} = t_{A_1 \cup \ldots \cup A_n}.$$  

**Proof** The reverse implication follows from the inequalities

$$t_{A_1 \cup \ldots \cup A_n} = \min\{t_{A_1}, \ldots, t_{A_n}\} \leq \max\{t_{A_1}, \ldots, t_{A_n}\} \leq t_{A_1 \cap \ldots \cap A_n}.$$  

The direct implication follows from the fact that, almost surely, the third components of the elements of $U$ are all distinct. Therefore if $t_{A_1} = \ldots = t_{A_n}$, then the point $(x_{A_k}, y_{A_k}, t_{A_k})$ does not depend on $k$, therefore $(x_{A_k}, y_{A_k})$ belongs to $A_1 \cap \ldots \cap A_n$. This ends the proof. \qed

We now study the dependence of $x_A$ with respect to $A$.

**Proposition 2.2** Let $A$ and $B$ be two measurable subsets of $E \times \mathbb{R}^+$ of measures finite and positive, one has that

$$\{x_A = x_B\} \supset \{t_A = t_B\} \text{ almost surely}$$

and

$$P[t_A = t_B] = \frac{\mu(A \cap B)}{\mu(A \cup B)}.$$  

**Proof** Since the third components of the elements of $U$ are almost surely all distinct, one has

$$t_A = t_B \Rightarrow x_A = x_B \text{ almost surely.}$$

By lemma 2.1, and the equality $t_{A \cup B} = \min\{t_{A \cap B}, t_{A \triangle B}\}$,

$$t_A = t_B \iff t_{A \cap B} = t_{A \cup B} \iff t_{A \cap B} \leq t_{A \triangle B}.$$
Since $t_{A \cap B}$ and $t_{A \Delta B}$ are independent exponential random variables with parameters $\mu(A \cap B)$ and $\mu(A \Delta B)$, one gets
\[ P[t_A = t_B] = \frac{\mu(A \cap B)}{\mu(A \cup B)}, \]
which completes the proof.

\[ \square \]

**Remark 1** Note that if $\pi$ is diffuse, then $\{x_A = x_B\} = \{t_A = t_B\}$ almost surely, thus $P[x_A = x_B] = \frac{\mu(A \cap B)}{\mu(A \cup B)}$.

Let us introduce some subsets to which we will apply proposition 2.2.

**Notation 3** For any measurable map $f : E \to \mathbb{R}^+$, denote by $D_f$ the part of $E \times \mathbb{R}^+$ located below the graph of $f$:
\[ D_f = \{(x, y) \in E \times \mathbb{R}^+ : y \leq f(x)\}. \]

Then,
\[ \mu(D_f) = \int_E f(x) \, d\pi(x). \]

If this measure is positive and finite, then we denote in abbreviated form :
\[ (x_f, y_f, t_f) = (x_{D_f}, y_{D_f}, t_{D_f}). \]
Figure 1: First point of $U$ below the graph of $f$. 
Proposition 1.3 is a direct consequence of the next lemma.

**Lemma 2.3** Let \( \alpha > 0 \) and \( f \) be a probability density function on \((E, \pi)\). Then

- the random variable \( x_{\alpha f} \) has density \( f \) with respect to \( \pi \). Thus \((U, x)\) is a global coupling for the set of all laws on \( E \) with a density with respect to \( \pi \)
- for all probabilities \( f \) and \( g \) on \( E \),

\[
P[x_f = x_g] \geq P[t_f = t_g] = \frac{1 - \|f - g\|}{1 + \|f - g\|}.
\]

**Proof** The first result follows from the fact that the law of \((x_{\alpha f}, y_{\alpha f})\) is \( \mu(\cdot | D_f) \). The second result follows from proposition 2.2. \( \square \)
2.2 Construction of a governing random variable

Let $X$ be a random variable with values in $E$ with a density $f$ with respect to $\pi$. Let $W$ be a Poisson point process on $E \times \mathbb{R}^+ \times \mathbb{R}^+$ of intensity $\pi \otimes \lambda \otimes \lambda$, and $V$ be a random variable uniform on $[0, 1]$, such that $X, V, W$ are independent. Let us modify $W$ in such a way that the first two components of the first point which appears in $D_f$ are $\langle X, V f(X) \rangle$. We set

$$U = \left[ W \setminus \{(x_f(W), y_f(W), t_f(W))\} \right] \cup \{(X, V f(X), t_f(W))\}.$$
Proposition 2.4 The process $U$ thus defined is a Poisson point process on $E \times \mathbb{R}^+ \times \mathbb{R}^+$ of intensity $\pi \otimes \lambda \otimes \lambda$, such that $x_f(U) = X$.

Proof One has directly $x_f(U) = X$. Let us show that $U$ is a Poisson point process. Denote by $\overline{D_f}$ the complementary of $D_f$ in $E \times \mathbb{R}^+$. Let us split $W$ into two independent Poisson point processes $W_1$ and $W_2$ by setting:

$$W_1 = W \cap (D_f \times \mathbb{R}^+) \text{ and } W_2 = W \cap (\overline{D_f} \times \mathbb{R}^+).$$

In a similar way, set

$$U_1 = U \cap (D_f \times \mathbb{R}^+) \text{ and } U_2 = U \cap (\overline{D_f} \times \mathbb{R}^+).$$

It suffices to show that $\mathcal{L}(U_1, U_2) = \mathcal{L}(W_1, W_2)$. Note that $U_2 = W_2$ and $U_1$ is a function of $X, V, W_1$, therefore $U_1$ is independent from $U_2$ and $\mathcal{L}(U_2) = \mathcal{L}(W_2)$. Thus what remains to be proved is that $\mathcal{L}(U_1) = \mathcal{L}(W_1)$.

Let $(w_k, t_k)_{k \geq 1}$ be the sequence of the points of $W_1$ in $D_f \times \mathbb{R}^+$ ordered in such a way that $(t_k)_{k \geq 1}$ is an increasing sequence. Then

$$U_1 = \bigcup_{k \geq 1} \{(u_k, t_k)\}$$

with $u_1 = (X, V f(X))$ and $u_k = w_k$ for $k \geq 2$. The random variables $(w_k)_{k \geq 1}$ form an i.i.d. sequence of law $\mu(\cdot|D_f)$, independent of $((t_k)_{k \geq 1}, X, V)$.

A direct computation shows that the random variable $u_1 = (X, V f(X))$ has law $\mu(\cdot|D_f)$. Moreover, $u_1$ is independent of $((w_k)_{k \geq 2}, (t_k)_{k \geq 1})$ by independence of $X, V$ and $W$. Therefore $\mathcal{L}(U_1) = \mathcal{L}(W_1)$ which completes the proof. \hfill $\square$

2.3 Construction of a governing sequence

Assume that $(X_n)_{n \in \mathbb{Z}}$ is a stationary process with values in $E$ such that given $X_{\leq 1}$, $X_0$ admits conditional densities $(f(\cdot|x))_{x \in E^0}$ with respect to $\pi$. One wants to apply the previous construction at each time $n \in \mathbb{Z}$.

Recall that for every $n \in \mathbb{Z}^+$ and $x \in E_{\geq n}$ the function $f(\cdot|x)$ is the density of the law $\mathcal{L}(X_0|X_{<n} = x)$ with respect to $\pi$.

Let $(W_n)_{n \in \mathbb{Z}}$ be a sequence of independent Poisson point processes on $E \times \mathbb{R}^+ \times \mathbb{R}^+$ of intensity $\pi \otimes \lambda \otimes \lambda$, and let $(V_n)_{n \in \mathbb{Z}}$ be a sequence of independent and uniform random variables on $[0, 1]$, such that the sequences $(X_n)_{n \in \mathbb{Z}}, (V_n)_{n \in \mathbb{Z}}, (W_n)_{n \in \mathbb{Z}}$ are independent. Set

$$U_n = W_n \cup \{(X_n, V_nf_{n-1}(X_n), t_{f_{n-1}}(W_n))\} \setminus \{(x_{f_{n-1}}(W_n), y_{f_{n-1}}(W_n), t_{f_{n-1}}(W_n))\},$$

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where $f_{n-1} = f(\cdot | X_{n-1}^o)$. Since $X$ is stationary, $f_{n-1}$ is the law of $X_n$ given the past $X_{n-1}^o$. Proposition 2.4 implies that for every $n \in \mathbb{Z}$, $U_n$ is a Poisson point process independent from $\mathcal{F}^{X,Y,W}_{n-1}$. Moreover, $X_n = x f_{n-1}(U_n)$ and $U_n$ is a function of $W_n, V_n$ and $X_{n-1}^o$. Therefore we have the following result.

**Proposition 2.5** The process $(U_n)_{n \in \mathbb{Z}}$ thus defined is an i.i.d. sequence of Poisson point processes on $E \times \mathbb{R}^+ \times \mathbb{R}^+$ of intensity $\pi \otimes \lambda \otimes \lambda$, which governs $\mathcal{F}^X$ with the recursion formula $X_n = x f_{n-1}(U_n)$.

### 3 Priming lemma

Fix an integer $\ell \geq 0$. The following lemma provides a random variable $Z_{1,\ell}$ and an event $H_{\ell}$ which both depend only on $U_{1,\ell} = (U_1, \ldots, U_\ell)$ such that when $H_{\ell}$ occurs, $X_{1,\ell} = Z_{1,\ell}$ with probability close to 1.

**Lemma 3.1 (Priming lemma)** Assume that $f(a|X_{1,\ell}) > 0$ almost surely for $\pi$-almost every $a$ in $E$. For every $\varepsilon > 0$ and $\ell \geq 0$, there exists a random variable $Z_{1,\ell} = (Z_1, \ldots, Z_\ell)$ with values in $E^\ell$ and an event $H_{\ell}$, which are both functions of $U_{1,\ell} = (U_1, \ldots, U_\ell)$ only, such that

- $\mathbb{P}[H_{\ell}] > 0$,
- $\mathcal{L}(Z_{1,\ell}) = \mathcal{L}(X_{1,\ell})$,
- $H_{\ell}$ is independent of $Z_{1,\ell}$,
- $\mathbb{P}[X_{1,\ell} \neq Z_{1,\ell} | H_{\ell}] \leq \varepsilon$.

Note that the hypothesis and the conclusions of lemma 3.1 are not modified if one replace $\pi$ by an equivalent measure. Thus in the proof, one may assume without loss of generality that $\pi$ is a probability measure.

The proof proceeds by induction. For $\ell = 0$ the result is trivial with $H_0 = \Omega$ and $Z_0$ equal to the empty word. The inductive step from $\ell$ to $\ell + 1$ uses the next two lemmas. These lemmas show that with a probability close to 1, most of the graph of $f(\cdot | X_{1,\ell}^o)$ is between $m^{-1} f(\cdot | Z_{1,\ell})$ and $n$ for suitable constants $m$ and $n$.

**Lemma 3.2** Let $\ell \geq 0$ be an integer, $H$ an event of positive probability, and $Z$ be a random variable with values in $E^\ell$. For every $\varepsilon > 0$, there exists a real number $m \geq 1$ such that

$$
\mathbb{E}
\left[
\int_E \left[ f(a|Z_{1,\ell}) - m f(a|X_{1,\ell}^o) \right]_+ \, d\pi(a) \, \bigg| \, H
\right] \leq \varepsilon.
$$
Proof. By the priming condition and the stationarity of $X$, almost surely, $f(a|X^\omega_{\ell'}) > 0$ for $\pi$-almost every $a \in E$, one has
\[ [f(a|Z_{1,\ell}) - m f(a|X^\omega_{\ell'})]_+ \to 0 \text{ as } m \to +\infty. \]
Hence, by dominated convergence
\[ \mathbb{E} \left[ \int_E [f(a|Z_{1,\ell}) - m f(a|X^\omega_{\ell'})]_+ d\pi(a) \right] \to 0 \text{ as } m \to +\infty. \]
The same result holds with $\mathbb{E}[\cdot|H]$ instead of $\mathbb{E}$ since $\mathbb{P}[\cdot|H] \leq \mathbb{P}[H]^{-1} \mathbb{P}[\cdot]$, which concludes the proof. \hfill \square

Lemma 3.3 Let $\ell \geq 0$ be an integer, $H$ an event of positive probability and $Z$ be a random variable with values in $E^\ell$. For every $\varepsilon > 0$, there exists a real number $n \geq 1$ such that
\[ \mathbb{E} \left[ \int_E [f(a|X^\omega_{\ell}) - n]_+ d\pi(a) \bigg| H \right] \leq \varepsilon. \]
For such a real number $n$, there exists a random variable $M$, with values in $[n, n+1]$, which is a function of $Z$ only, such that
\[ \int_E \sup(m^{-1} f(x|Z), M) d\pi(x) = n + 1 \]
and
\[ \mathbb{E} \left[ \int_E [f(a|X^\omega_{\ell}) - M]_+ d\pi(a) \bigg| H \right] \leq \varepsilon. \]

Proof. The same method as in the proof of lemma 3.2 provides a real number $n \geq 1$ such that
\[ \mathbb{E} \left[ \int_E [f(a|X^\omega_{\ell}) - n]_+ d\pi(a) \bigg| H \right] \leq \varepsilon. \]
Define a random application $\phi$ from $\mathbb{R}^+$ to $\mathbb{R}^+$ by
\[ \phi(s) = \int_E \sup(m^{-1} f(x|X^\omega_{\ell}), s) d\pi(x). \]
Since for every $s \in \mathbb{R}^+$, $s \leq \sup(m^{-1} f(x|X^\omega_{\ell}), s) \leq m^{-1} f(x|X^\omega_{\ell}) + s$, one has $s \leq \phi(s) \leq m^{-1} + s \leq 1 + s$ (recall that $\pi$ is assumed to be a probability). Hence $\phi(n) \leq n + 1 \leq \phi(n + 1)$. Then, since $\phi$ is continuous, the random variable
\[ M = \inf \{ s \in \mathbb{R}^+ : \phi(s) = n + 1 \} \]
is well defined and satisfies the conclusion of the lemma. \hfill \square

We can now prove the induction step of the proof of lemma 3.1.
Proof. Let $\varepsilon > 0$ and $\ell \in \mathbb{N}$. Assume that one has constructed $H_\ell$ and $Z_{1:\ell}$ such that they are both function of $U_{1:\ell}$ only and such that

- $\mathbb{P}[H_\ell] > 0$,
- $\mathcal{L}(Z_{1:\ell}) = \mathcal{L}(X_{1:\ell})$,
- $H_\ell$ is independent of $Z_{1:\ell}$,
- $\mathbb{P}[Z_{1:\ell} \neq X_{1:\ell} \mid H_\ell] \leq \varepsilon/3$.

Lemmas 3.2 and 3.3, applied to $\ell$, $H = H_\ell$, $Z = Z_{1:\ell}$ and $\varepsilon/3$ provide two real numbers $m, n$ and a random variable $M$. Set $f_\ell = f(\cdot | X_{1:\ell}^{2})$, $f'_\ell = f(\cdot | Z_{1:\ell})$ and

$$H' = \left\{ t_m f'_\ell(U_{\ell+1}) \leq t_M(U_{\ell+1}) \right\}$$

$$= \left\{ t_{\sup (m^{-1} f'_\ell, M)}(U_{\ell+1}) = t_m f'_\ell(U_{\ell+1}) \right\}.$$

$$Z_{\ell+1} = x_{m^{-1} f'_\ell(U_{\ell+1})},$$

$$H_{\ell+1} = H_\ell \cap H'.$$

Set

$$A = D_{m^{-1} f'_\ell} = \left\{ (x, y) \in E \times \mathbb{R}^+ : y \leq m^{-1} f(x | Z_{1:\ell}) \right\}$$

$$A_z = D_{m^{-1} f(\cdot | z)}$$

$$B = D_M$$

$$C = D_{f'_\ell}.$$

Then $Z_{\ell+1} = x_A(U_{\ell+1})$, $H' = \{ t_A(U_{\ell+1}) \leq t_B(U_{\ell+1}) \}$ and $X_{\ell+1} = x_C(U_{\ell+1})$ (see proposition 2.5).
By the first point of lemma 2.3 and by the independence of \( U_{\ell+1} \) and \( f'_\ell \), the random variable \( Z_{\ell+1} \) has the density \( f'_\ell \) conditionally on \( Z_\ell \). Hence for \( z \in E^\ell \) and \( B \in \mathcal{E} \),

\[
P[Z_{\ell+1} \in B \mid Z_1:\ell = z] = \int_B f(x|z) \, d\pi(x) = P[X_{\ell+1} \in B \mid X_1:\ell = z].
\]

But \( \mathcal{L}(Z_1:\ell) = \mathcal{L}(X_1:\ell) \), hence the random variable \( Z_{1:\ell+1} \) has the same law as \( X_{1:\ell+1} \).

The event \( H' \) holds when the first point of \( U_{\ell+1} \) which falls below the graph of \( \max(m^{-1}f'_\ell, M) \) is below the graph of \( m^{-1}f'_\ell \). Since \( U_{\ell+1} \) is independent of \( (Z_1:\ell, H_\ell) \), proposition 2.2 and lemma 3.3 provide the equalities

\[
P[H' \mid \sigma(Z_1:\ell, H_\ell)] = \frac{\mu(A)}{\mu(A \cup B)} = \frac{\int_E m^{-1}f(a|Z_1:\ell) \, d\pi(a)}{\int_E \sup(m^{-1}f(a|Z_1:\ell), M) \, d\pi(a)} = \frac{1}{m(n+1)}.
\]

Hence \( H' \) is independent of \( (Z_1:\ell, H_\ell) \) and since \( H_{\ell+1} = H_\ell \cap H' \), one has

\[
P[H_{\ell+1}] = P[H']P[H_\ell] = \frac{P[H_\ell]}{m(n+1)} > 0.
\]
Moreover, for almost every \( z \in E^d \),
\[
\mathbb{P}[H'] = \mathbb{P}[H'|Z_{1:t} = z] = \mathbb{P}[t_{m-1,t+1}(U_{t+1}) \leq t_M(U_{t+1})] = \mathbb{P}[t_{A_z}(U_{t+1}) \leq t_B(U_{t+1})],
\]
by independence of \( Z_{1:t} \) and \( U_{t+1} \).

Let us show that \( Z_{1:t} \) and \( H_{t+1} \) are independent. Let \( F \in \mathcal{E} \), \( G \in \mathcal{E} \) and \( p = \mathbb{P}[Z_{1:t} \in F \times G; H_{t+1}] = \mathbb{P}[Z_{1:t} \in F; Z_{t+1} \in G; H_{t}; H'] \). Then
\[
p = \int_F \mathbb{P}[x_{A_z}(U_{t+1}) \in G; H_{t}; t_{A_z}(U_{t+1}) \leq t_B(U_{t+1}) | Z_{1:t} = z] \mathbb{P}[Z_{1:t} \in dz].
\]
By independence of \( H_t, Z_{1:t} \) and \( U_{t+1} \), one has
\[
p = \int_F \mathbb{P}[H_t] \mathbb{P}[x_{A_z}(U_{t+1}) \in G; t_{A_z}(U_{t+1}) \leq t_B(U_{t+1})] \mathbb{P}[Z_{1:t} \in dz].
\]
But \( \{t_{A_z}(U_{t+1}) \leq t_B(U_{t+1})\} = \{t_{A_z}(U_{t+1}) \leq t_{B \setminus A_z}(U_{t+1})\} \). By independence of \( x_{A_z}(U_{t+1}), t_{A_z}(U_{t+1}) \) and \( t_{B \setminus A_z}(U_{t+1}) \) and by equality (2), one gets
\[
p = \int_F \mathbb{P}[H_{t+1}] \mathbb{P}[x_A(U_{t+1}) \in G | Z_{1:t} = z] \mathbb{P}[Z_{1:t} \in dz]
= \mathbb{P}[H_{t+1}] \int_F \mathbb{P}[x_A(U_{t+1}) \in G | Z_{1:t} = z] \mathbb{P}[Z_{1:t} \in dz]
= \mathbb{P}[H_{t+1}] \mathbb{P}[Z_{1:t+1} \in F \times G],
\]
which shows that \( Z_{1:t+1} \) and \( H_{t+1} \) are independent.

What remains to be proved is the upper bound of \( \mathbb{P}[Z_{t+1} \neq X_{t+1}|H_{t+1} = 1] \).

By proposition 2.2 and lemma 2.1,
\[
\{X_{t+1} = Z_{t+1}\} \cap H' \supset \{t_C(U_{t+1}) = t_A(U_{t+1}) = t_{A \cup B}(U_{t+1})\}
= \{t_{A \cap C \cap (A \cup B)}(U_{t+1}) = t_{A \cup C \cup (A \cup B)}(U_{t+1})\}
= \{t_{A \cap C}(U_{t+1}) = t_{A \cup C \cup B}(U_{t+1})\}.
\]
Since \( A, B \) and \( C \) are measurable for \( \mathcal{F}_{t}^{X,U} \) and since \( U_{t+1} \) is independent from \( \mathcal{F}_{t}^{X,U} \), one has
\[
\mathbb{P}[X_{t+1} = Z_{t+1}; H' \mid \mathcal{F}_{t}^{X,U}] \geq \frac{\mu(A \cap C)}{\mu(A \cup C \cup B)}.
\]
Moreover
\[
\mu(A \cap C) = \frac{1}{m} \int_E \min(f(a|Z_{1:t}), m_f(a|X_{1:t}^z)) d\pi(a)
= \frac{1}{m} \left( \int_E f(a|Z_{1:t}) d\pi(a) - \int_E [f(a|Z_{1:t}) - m_f(a|X_{1:t}^z)]_+ d\pi(a) \right)
\]
\[
= \frac{1}{m} \left( 1 - \int_E [f(a|Z_{1:t}) - m_f(a|X_{1:t}^z)]_+ d\pi(a) \right)
\]
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and
\[
\mu(A \cup C \cup B) = \int_E \sup (m^{-1} f(a|Z_1, \ell), f(a|X_1^\omega), M) \, d\pi(a)
\]
\[
= \int_E \sup(m^{-1} f(a|Z_1, \ell), M) \, d\pi(a)
\]
\[
+ \int_E [f(a|X_1^\omega) - \sup(m^{-1} f(a|Z_1, \ell), M)]_+ \, d\pi(a)
\]
\[
\leq n + 1 + \int_E [f(a|X_1^\omega) - M]_+ \, d\pi(a)
\]
\[
\leq (n + 1) \left(1 + \int_E [f(a|X_1^\omega) - M]_+ \, d\pi(a)\right).
\]
Thus
\[
\mu(A \cup C \cup B)^{-1} \geq \frac{1}{n + 1} \left(1 - \int_E [f(a|X_1^\omega) - M]_+ \, d\pi(a)\right).
\]

and
\[
\frac{\mu(A \cap C)}{\mu(A \cup C \cup B)} \geq \frac{1}{m(n + 1)} \left(1 - \int_E [f(a|Z_1, \ell) - mf(a|X_1^\omega)]_+ \, d\pi(a)
\]
\[
- \int_E [f(a|X_1^\omega) - M]_+ \, d\pi(a)\right).
\]

Since \(H_\ell \in \mathcal{F}_\ell^{X,U}\) one has
\[
\mathbb{P}[X_{\ell + 1} = Z_{\ell + 1} \mid H_\ell] \geq \mathbb{E}\left[\frac{\mu(A \cap C)}{\mu(A \cup C \cup B)} \mid H_\ell\right]
\]
\[
\geq \frac{1}{m(n + 1)} \left(1 - \mathbb{E}\left[\int_E [f(a|Z_1, \ell) - mf(a|X_1^\omega)]_+ \, d\pi(a) \mid H_\ell\right]
\]
\[
- \mathbb{E}\left[\int_E [f(a|X_1^\omega) - M]_+ \, d\pi(a) \mid H_\ell\right]\right)
\]
\[
\geq \frac{1 - 2\varepsilon/3}{m(n + 1)},
\]
where the last inequality stands from lemmas 3.2 and 3.3.

But
\[
\mathbb{P}[X_{\ell + 1} = Z_{\ell + 1}, H' \mid H_\ell] = \mathbb{P}[H' \mid H_\ell] \mathbb{P}[X_{\ell + 1} = Z_{\ell + 1} \mid H' \cap H_\ell]
\]
\[
= \frac{1}{m(n + 1)} \mathbb{P}[X_{\ell + 1} = Z_{\ell + 1} \mid H_{\ell + 1}].
\]

Hence,
\[
\mathbb{P}[Z_{\ell + 1} = X_{\ell + 1} \mid H_{\ell + 1}] \geq 1 - 2\varepsilon/3,
\]
and
\[
P[Z_{1:t+1} \neq X_{1:t+1}|H_{t+1}] \leq P[Z_t \neq X_{1:t}|H_{t+1}] + P[Z_{t+1} \neq X_{t+1}|H_{t+1}] \leq \varepsilon,
\]
which ends the proof.

\[\square\]

4 End of the proof of theorem 3

In this section, we show that the governing sequence \((U_n)_{n \in \mathbb{Z}}\) given by proposition 2.5 is a generating sequence. By stationarity, it suffices to show that \(X_0\) is a function of \((U_n)_{n \in \mathbb{Z}}\) only. We proceed by successive approximations.

4.1 Approximation until a given time

Choose \(\varepsilon > 0\) and \(\ell \geq 1\) such that \(\sum_{n \geq \ell} \eta_n \leq \varepsilon\), let \(J = [s, t]\) be an interval of integers such that \(t - s + 1 = \ell\). Note \(X_J = X_{s:t}\).

By stationarity of \(X\), lemma 3.1 provides an event \(H_J\) and a random variable \(Z_J\), functions of \(U_J\) only, such that

- \(\mathbb{P}[H_J] > 0\),
- \(\mathcal{L}(Z_J) = \mathcal{L}(X_J) = \mathcal{L}(X_{1:t})\),
- \(H_J\) is independent of \(Z_J\),
- \(\mathbb{P}[X_J \neq Z_J \mid H_J] \leq \varepsilon\).

Using \(Z_J\) and the governing sequence \((U_n)_{n \geq t+1}\), we consider the random variables \((X'_n)_{n \geq s}\) defined by \(X'_J = Z_J\) and for every \(n \geq t + 1\),

\[X'_n = x_{f'_{n-1}}(U_n)\text{ where } f'_{n-1} = f(\cdot|X'_{s:n-1}).\]

Let us establish some properties of the process \(X'\) thus defined.

Lemma 4.1 For every \(n \geq s\), the law of \(X'_{s:n}\) is the law of \(X_{s:n}\).

Proof. Choose \(n \geq t + 1\), \(z \in \mathbb{E}^{n-s}\) and \(B \in \mathcal{E}\). By lemma 2.3 and by independence of \(U_n\) and \(f'_{n-1}\), the random variable \(X'_n\) admits the density \(f'_{n-1} = f(\cdot|X'_{s:n-1})\) conditionally on \(X'_{s:n-1}\). Hence

\[
\mathbb{P}[X'_n \in B \mid X'_{s:n-1} = z] = \int_B f(x|z) \, d\pi(x) = \mathbb{P}[X_n \in B \mid X_{s:n-1} = z].
\]

Since \(X'_J = Z_J\) as the same law as \(X_J\), the result follows by induction. \[\square\]
Lemma 4.2 One has \( \mathbb{P}[X' \neq X \mid s, +\infty[ \mid H_J] \leq 3\varepsilon \).

Proof. Let \( n \geq t + 1 \). Since \( X_n = x_{f_{n-1}}(U_n) \) and \( X'_n = x_{f'_{n-1}}(U_n) \), proposition 2.3 and the independence of \( U_n \) and \( \mathcal{F}_{n-1}^{X,U} \) yield,

\[
\mathbb{P}[X'_n \neq X_n \mid \mathcal{F}_{n-1}^{X,U}] \leq 2\|f'_{n-1} - f_{n-1}\|.
\]

Let

\[
p_n = \mathbb{P}[X'_n \neq X_n ; X'_{s:n-1} = X_{s:n-1} ; H_J].
\]

Since \( \{X'_{s:n-1} = X_{s:n-1} ; H_J\} \in \mathcal{F}_{n-1}^{X,U} \), one gets

\[
p_n \leq \mathbb{E}\left[2\|f_{n-1} - f'_{n-1}\| 1_{\{X'_{s:n-1} = X_{s:n-1}\}H_J}\right] = 2 \mathbb{E}\left[\|f(\cdot|X'_{s:n-1}X_{s:n-1}) - f(\cdot|X'_{s:n-1})\| 1_{\{X'_{s:n-1} = X_{s:n-1}\}H_J}\right] \leq 2 \mathbb{E}\left[\|f(\cdot|X'_{s:n-1}X'_{s:n-1}) - f(\cdot|X'_{s:n-1})\| H_J\].
\]

But \( X_{s-1}^\alpha, Z_J, H_J \) and \( U_{t+1:n-1} \) are independent hence \( X_{s-1}^\alpha, X'_{s:n-1} \) and \( H_J \) are independent since \( X'_{s:n-1} \) is a function of \( Z_J \) and \( U_{t+1:n-1} \) only. Thus,

\[
p_n \leq 2 \mathbb{E}\left[\|f(\cdot|X_{s-1}^\alphaX'_{s:n-1}) - f(\cdot|X_{s:n-1})\|\right] \mathbb{P}[H_J] = 2\eta_{n-s} \mathbb{P}[H_J].
\]

Hence,

\[
\mathbb{P}[X'_n \neq X_n ; X'_{s:n-1} = X_{s:n-1} | H_J] \leq 2\eta_{n-s},
\]

therefore,

\[
\mathbb{P}[X'_{s:n} \neq X_{s:n} | H_J] \leq \mathbb{P}[X'_{s:n-1} \neq X_{s:n-1} | H_J] + 2\eta_{n-s}.
\]

By induction, one gets

\[
\mathbb{P}[X'_{s:n} \neq X_{s:n} | H_J] \leq \mathbb{P}[X'_J \neq X_J | H_J] + 2 \sum_{m=t}^{n-s} \eta_m.
\]

Since \( X'_J = Z_J \) and \( \mathbb{P}[X_J \neq Z_J | H_J] \leq \varepsilon \), this yields

\[
\mathbb{P}[X' \neq X | s, +\infty[ H_J] \leq \varepsilon + 2 \sum_{m=t}^{\infty} \eta_m \leq 3\varepsilon,
\]

which ends the proof. \( \square \)
4.2 Successive approximations

Our next step in the proof of theorem 3 is to approach the random variable \( X_0 \) by measurable functions of the governing sequence \((U_n)_{n \in \mathbb{Z}}\) of proposition 2.5. To this aim, we group the innovations by intervals of times. For every \( m \geq 1 \) one chooses an integer \( L_m \) such that

\[
\sum_{n \geq L_m} \eta_n \leq 1/m.
\]

Lemma 3.1 (the priming lemma) applied to \( \ell = L_m \) and \( \varepsilon = 1/m \) provides an event \( H_{L_m} \) of positive probability \( \alpha_m \) and a random variable \( Z_{L_m} \) such that

\[
P[X_{1:L_m} = Z_{L_m} \mid H_{L_m}] \geq 1 - 1/m.
\]

Choose an integer \( M_m \geq 1/\alpha_m \). Split \( Z^* \) into \( M_1 \) intervals of length \( L_1 \), \( M_2 \) intervals of length \( L_2 \), and so on. More precisely set, for every \( n \geq 1 \), \( \ell_n = L_{m(n)} \), \( \varepsilon_n = 1/m(n) \) and \( \alpha_n = \alpha_{m(n)} \) where \( m(n) \) is the only integer such that

\[
M_1 + \cdots + M_{m(n)} - 1 < n \leq M_1 + \cdots + M_{m(n)}.
\]

Therefore, for every \( k \geq 0 \), one gets

\[
\sum_{n \geq t_k} \eta_n \leq \varepsilon_k.
\]

Finally, for every \( k \geq 0 \), set

\[
t_k = - \sum_{1 \leq n \leq k} \ell_n
\]

that is to say \( t_0 = 0 \) and \( t_k = t_{k-1} - \ell_k \) for \( k \geq 1 \). Define, for \( k \geq 0 \), the interval of integers

\[
J_k = [t_k, t_k + \ell_k - 1] = [t_k, t_{k-1} - 1] \text{ and } X_{J_k} = X_{t_k:t_{k-1} - 1}.
\]
Lemma 3.1 applied to \((\varepsilon_k)_{k \geq 1}\) and \((U_j)_{k \geq 1}\) provides events \((H_{J_k})_{k \geq 1}\) and random variables \((Z_{J_k})_{k \geq 1}\). For every \(k \geq 0\), let us use the construction of section 4.1: set \(X^k_{J_k} = Z_{J_k}\), then for every \(n \geq t_k + \ell_k = t_{k-1}\)

\[
X^k_n = x_{f_{n-1}^k}(U_n) \text{ where } f_{n-1}^k = f(\cdot | X^k_{t_k:n-1}).
\]

Therefore lemma 4.2 yields the inequality

\[
P[X_0 \neq X^k_0] \leq P[X_{t_k:0} \neq X^k_{t_k:0} | H_{J_k}] \leq 3\varepsilon_k,
\]

which shows that

\[
P[X^k_0 \neq X_0 | H_{J_k}] \to 0 \text{ when } k \to +\infty.
\]

Moreover the events \(H_{J_k}\) are independent as functions of random variables \(U_j\) for disjoint sets of indices \(j\) and

\[
\sum_{k \geq 1} P[H_{J_k}] = \sum_{n \geq 1} \alpha_{\ell_m(n)} = \sum_{m=1}^{+\infty} M_m \alpha_m = +\infty
\]

since \(M_m \alpha_m \geq 1\) by choice of \(M_m\).

Lemma 4.3, stated below, provides a deterministic increasing function \(\theta\) such that

\[
\sum_{k \geq 1} P[X^k_0 \neq X_0 \mid H_{J_{\theta(k)}}] < +\infty
\]

and

\[
\sum_{k \geq 1} P[H_{J_{\theta(k)}}] = +\infty.
\]

Using Borel-Cantelli’s lemma, one deduces that

- \(\{X^k_0 \neq X_0\} \cap H_{J_{\theta(k)}}\) is realized for a finite number of \(k\) only, a.s.
• $H_{J_0(k)}$ is realized for an infinite number of $k$ a.s.

Thus, for every $B \in \mathcal{E}$,

$$\{X_0 \in B\} = \limsup_{k \to \infty} H_{J_0(k)} \cap \{X_0^{\theta(k)} \in B\},$$

hence $\{X_0 \in B\} \in \mathcal{F}_U^X$. By stationarity of the process $(X,U)$, one gets the inclusion of the filtration $\mathcal{F}^X$ into the filtration $\mathcal{F}^U$, which ends the proof. □

**Lemma 4.3** Let $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$ denote two bounded sequences of nonnegative real numbers such that the series $\sum b_n$ diverges and such that $a_n \ll b_n$.

Then there exists an increasing function $\theta : \mathbb{N} \to \mathbb{N}$ such that the series $\sum_n a_{\theta(n)}$ converges and the series $\sum_n b_{\theta(n)}$ diverges.

A proof of this lemma can be found in citeceilliersplit.

### 5 Proof of proposition 1.2

Assume that $E$ is countable. Let $\varepsilon = (\varepsilon_a)_{a \in E}$ be an i.i.d. family of exponential random variables with parameter 1.

Let us show that, for any probability $p \in E$, there exists almost surely one and only one $a \in E$ such that

$$\frac{\varepsilon_a}{p(a)} = \inf_{b \in E} \frac{\varepsilon_b}{p(b)} \quad (\star),$$

with the convention that $\varepsilon_b/p(b) = +\infty$ if $p(b) = 0$.

For every positive real number $r$,

$$\sum_{b \in E} \mathbb{P}[\varepsilon_b/p(b) \leq r] = \sum_{b \in E} (1 - e^{-p(b)r}) \leq \sum_{b \in E} p(b)r = r < +\infty,$$

hence Borel-Cantelli’s lemma ensures that the event $\{\varepsilon_b/p(b) \leq r\}$ occurs only for a finite number of $b \in E$. Thus the infimum in $(\star)$ is achieved at some $a \in E$. The uniqueness follows from the equalities $\mathbb{P}[\varepsilon_a/p(a) = \varepsilon_b/p(b) < +\infty] = 0$ for every $a \neq b$, since $\varepsilon_a$ and $\varepsilon_b$ are independent random variables with diffuse laws.

Define $g(\varepsilon,p)$ (almost surely) as the only index $a$ verifying $(\star)$. Let us show that, for every $p,q \in \mathcal{P}(E)$,

$$\mathbb{P}[g(\varepsilon,p) \neq g(\varepsilon,q)] \leq 2 \frac{\|p - q\|}{1 + \|p - q\|}.$$
For every $a, b \in E$, set $C_a = \{ g(\varepsilon, p) = g(\varepsilon, q) = a \}$ and

$$\lambda_{b/a} = \max \left( \frac{p(b)}{p(a)}, \frac{q(b)}{q(a)} \right).$$

Fix $a \in E$. Then up to negligible events,

$$\{ g(\varepsilon, p) = a \} = \bigcap_{b \neq a} \left\{ \frac{\varepsilon(a)}{p(a)} \leq \frac{\varepsilon(b)}{p(b)} \right\} = \left\{ \frac{\varepsilon_a}{p(a)} = \min_b \frac{\varepsilon_b}{p(b)} \right\}.$$  

Thus,

$$C_a = \left\{ \frac{\varepsilon_a}{p(a)} = \min_b \frac{\varepsilon_b}{p(b)} ; \frac{\varepsilon_a}{q(a)} = \min_b \frac{\varepsilon_b}{q(b)} \right\} = \left\{ \varepsilon_a \leq \min_b \left( \frac{p(a)\varepsilon_b}{p(b)} , \frac{q(a)\varepsilon_b}{q(b)} \right) \right\} = \bigcap_{b \neq a} \left\{ \varepsilon_b \geq \lambda_{b/a}\varepsilon_a \right\},$$

Conditioning on $\varepsilon_a$ and using the fact that the random variables $(\varepsilon_b)_{b \neq a}$ are i.i.d., exponentially distributed and independent of $\varepsilon_a$, one gets

$$\mathbb{P}[C_a | \varepsilon_a] = \mathbb{P}\left[ \bigcap_{b \neq a} \left\{ \varepsilon_b \geq \lambda_{b/a}\varepsilon_a \right\} | \varepsilon_a \right] = \prod_{b \neq a} \exp \left( -\lambda_{b/a}\varepsilon_a \right),$$

hence

$$\mathbb{P}[C_a] = \mathbb{E} \left[ \exp \left( -\left( \sum_{b \neq a} \lambda_{b/a} \right)\varepsilon_a \right) \right] = \left( 1 + \sum_{b \neq a} \lambda_{b/a} \right)^{-1} = \left( \sum_b \lambda_{b/a} \right)^{-1}.$$  

(3)  

(4)  

(5)

But $\lambda_{b/a} \leq \max(p(b), q(b))/\min(p(a), q(a))$, hence

$$\mathbb{P}[C_a] \geq \frac{\min(p(a), q(a))}{\sum_b \max(p(b), q(b))} \frac{\min(p(a), q(a))}{1 + \|p - q\|}.$$  

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Therefore

\[ \mathbb{P}[g(\varepsilon, p) = g(\varepsilon, q)] = \sum_{a \in E} \mathbb{P}[C_a] \geq \sum_{a \in E} \frac{\min(p(a), q(a))}{1 + \|p - q\|} = \frac{1 - \|p - q\|}{1 + \|p - q\|}. \]

Last note that if \( p = q \), then for each \( a \in E \), equation (5) becomes

\[ \mathbb{P}[g(\varepsilon, p) = a] = \mathbb{P}[C_a] = \left( \sum_b \frac{p(b)}{p(a)} \right)^{-1} = p(a), \]

which shows that the law of \( g(\varepsilon, p) \) is \( p \). The proof is complete. \( \square \)

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