Tensor Fields on Self-Dual Warped AdS$_3$ Background

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Abstract Considering rank $s$ fields obey first order equation of motion, we study the dynamics of such fields in a 3-dimensional self-dual space-like warped AdS$_3$ black hole background. We obtain a Klein–Gordon-like equation for tensor fields. By using gauge constraint and traceless condition, we will find the exact solutions of the equations of motion. Then, we will compute the quasi-normal modes by imposing appropriate boundary conditions at horizon and infinity.

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1 Introduction

Gravity in three dimensions is a recently attracted subject of theoretical physics. Although, it seems that the Einstein gravity is trivial in three dimensions\cite{1} but, by adding higher corrections to the usual Einstein action with cosmological constant, one obtains theories which have propagating degrees of freedom.\cite{2}

One of such theories is Topologically Massive Gravity (TMG) which its higher derivative term is gravitational Chern–Simons term\cite{3-4}

\begin{equation}
I_{\text{TMG}} = \frac{1}{16\pi G} \left( I_{\text{EH}} + \frac{1}{\mu} I_{\text{CS}} \right),
\end{equation}

with $\mu$ is coupling constant and $I_{\text{EH}}$ includes the cosmological term $-2\Lambda = 2/l^2$.

TMG admits maximally symmetric solutions AdS$_3$ with SL$(2,\mathbb{R}) \times$ SL$(2,\mathbb{R})$ symmetry\cite{5} and BTZ black hole\cite{6} and also solutions with fewer symmetries known as warped AdS$_3$ and its corresponding black holes.\cite{7} Due to hidden conformal symmetries of the propagating modes on the warped background, it was conjectured that such gravitational warped solution has a dual conformal field theory with the corresponding central charges.\cite{5} Motivated by these observations, many works have been done in this line for better understanding the duality and other related topics of quantum gravity and black holes in three dimensions.\cite{8}

Self-dual warped AdS$_3$ solution is also solution of TMG.\cite{9} It may be defined as real line fibrations over AdS$_2$ which preserves a single SL$(2,\mathbb{R})$ isometry and a non-compact U$(1)$ isometry generated by translation along the fibre coordinate.

This background has very interesting properties and several aspects of it such as geometrical properties, thermodynamics, its CFT dual and etc., have been studied. Especially, by using an algebraic way, the author of Ref. [10] found the quasi-normal modes of scalar, vector and tensor perturbations for AdS$_3$, BTZ and warped AdS$_3$ background. This calculation has been also done using other methods such as finding the exact solution.\cite{11}

For example, in Refs. [12–13], the authors were able to find the exact solutions in BTZ black hole background for general integer or half-integer rank $s$ field and compute the quasi-normal modes.

In fact, till recently, because of some no-go theorems,\cite{14} efforts typically were focused on the dynamics of the fields with spin lower than 2 on gravitational backgrounds. But, due to the works of Ref. [15] the higher spin fields were entered into the game. Various aspects of theories with higher spin fields such as finding a Lagrangian formalism, studying the conserved current, and so on have been studied, see Ref. [16] for example.

Our aim in this paper is to continue this research line by obtaining the equations of motion of a rank $s$ field in self-dual warped AdS$_3$ background and finding the exact solutions and also the quasi-normal modes. Quasi-normal modes can be found by solving the Klein–Gordon equation for higher spin fields directly. There are also other approaches to find quasi-normal modes such as algebraic method which has been used in Ref. [10]. In this paper our starting point is a first order linear equation.

For self-dual warped AdS$_3$, we will suppose that the main equations in which higher spin fields should obey is

\begin{equation}
\epsilon^{\alpha_1\alpha_2\ldots\alpha_s} \nabla_{\alpha_1} \Phi_{\alpha_2\alpha_3\ldots\alpha_s} = -m \Phi_{\alpha_2\alpha_3\ldots\alpha_s}.
\end{equation}

Due to lower symmetry in this background, this background is not locally AdS and it was shown in Ref. [17] that such a simple first order equation can not be carried out for a metric perturbation but, one can consider Eq. (2) holds for massive spin $s$ fields in the self-dual warped AdS$_3$ background.\cite{10}

In fact, such linear equation is a natural generalization of the equation of motion of spin zero, one and two
fields at linear level in curved background. We will examine the validity of this assumption by calculating the quasi-normal modes of higher spin fields in warped AdS$_3$ backgrounds and comparing our results with the known results of Ref. [10] which has been obtained with completely different methods.

Nevertheless, in spite of BTZ background, [13] these linear equations do not imply the following simple form of second order Klein–Gordon equation

$$\nabla^2 \Phi_{\mu_1 \mu_2 \cdots \mu_s} = 0, \tag{3}$$

$$\nabla^2 \Phi_{\mu_1 \mu_2 \cdots \mu_s} = 0. \tag{4}$$

However, we will show that by imposing suitable conditions, one can write the equations for some components in the Klein–Gordon-like form and can solve them exactly. Then using the linear equation (2) and some contiguous relations between the hyper-geometric functions, we will obtain the solutions for the other components. We will also compute the quasi normal modes.

The paper is organized as follows. In Sec. 2, we shall briefly introduce the three-dimensional self dual Warped AdS$_3$ background. In the next section, we will discuss about the equations of motion and obtain the Klein–Gordon-like equations for tensor fields. In Sec. 4, we find the exact solutions for spin 2 field for two components $h_{00}$ and $h_{0t}$ and then by using the first order equation (2), we find the solutions for $h_{tt}$ and $h_{rt}$. After that, by imposing Dirichlet boundary condition, we find the quasi-normal modes for massive spin 2 fields. In Sec. 5 we will generalize our computations for higher spin fields and find the quasi normal modes.

## 2 Self-Dual Warped AdS$_3$ Background

In this section, we briefly introduce self-dual warped AdS$_3$ solution. Self dual warped AdS$_3$ is a solution of equation of motion of Topological Massive Gravity (TMG) in three dimensions. [9] The metric is given by

$$ds^2 = \frac{\ell^2}{\nu^2 + 3} \left( -(x-x_+)(x-x_-)d\tau^2 + \frac{1}{(x-x_+)(x-x_-)} dx^2 + \frac{4\nu^2}{\nu^2 + 3} \left( a d\theta + \frac{1}{2} (2x-x_+-x_-) d\tau \right)^2 \right), \tag{5}$$

where $x_+$ and $x_-$ are the outer and inner horizons radius respectively.

In fact, the warped vacua of TMG classified into three space-like, time-like and null warped types. Classification depends on the whether the norm of the Killing vector generating the U(1) isometry is positive, negative or zero. First two types can also be classified as stretched ($\mu l > 3$) or squashed ($\mu l < 3$) depending on the magnitude of the warp factor. These background space-times are not locally AdS$_3$.

Quotients of warped AdS$_3$ along various Killing directions may give rise to black holes. [5] Black hole solutions free of closed time-like curves (CTCs) can only be found in space-like stretched and null warped AdS$_3$. Self-dual solutions in AdS$_3$ are quotients of space-like warped AdS$_3$ along the U(1) isometry. Such geometries have Killing horizons and no CTCs. [20]

It was shown in Ref. [9] that under the consistent boundary conditions, the U(1) isometry enhances to a Virasoro algebra with non-vanishing left central charge $c_L = 4\nu^2/(\nu^2 + 3)$ while the SL(2, $\mathbb{R}$) isometry becomes trivial with the vanishing right central charge $c_R = 0$. Then, it is conjectured that the self-dual warped AdS$_3$ black hole is dual to a two dimensional chiral CFT. The left and right temperatures of CFT are defined as

$$T_L = \frac{\alpha}{2\pi l}, \quad T_R = \frac{x_+ - x_-}{4\pi l}. \tag{6}$$

The angular velocity of the event horizon $\Omega_H$ and the Bekenstein–Hawking entropy $S_{BH}$ of this solution are given by

$$\Omega_H = -\frac{x_+ - x_-}{2\alpha}, \quad S_{BH} = \frac{2\pi \alpha \nu}{3G(\nu^2 + 3)}. \tag{7}$$

At the end, let us mention that since the self-dual warped AdS$_3$ is not locally AdS its curvature tensor can not be written in the usual form in terms of the metric. [1]

In fact, one can show that for background (5)

$$R_{\mu \nu \lambda \theta} = R^0_{\mu \nu \lambda \theta} + r_{\mu \nu \lambda \theta}, \tag{6}$$

where

$$R^0_{\mu \nu \lambda \theta} = \frac{\lambda^2}{4} (g_{\mu \theta} g_{\nu \lambda} - g_{\nu \lambda} g_{\mu \theta}), \tag{7}$$

$$r_{\mu \tau \tau \tau} = 1 - \frac{\alpha^2}{\nu^2}, \quad r_{\mu \tau \tau \theta} = 0, \tag{7}$$

where we have defined $\lambda^2 = 4\nu^2/(\nu^2 + 3)$. Moreover, there are some relations between the metric components and Christoffel symbols as

$$g^{x \nu} \Gamma^\mu_{x \nu} = -g^\mu_\beta \Gamma^\beta_{\nu \nu} \tag{8}.$$

These relations are very useful and important for finding the final equations. In fact, our discussion about the equations and quasi normal modes can be generalized to any other metric background in 3 dimensions in which $g_{x \mu} = 0, \mu \neq x$ and $g_{x \mu} - s$ satisfy (8).

Without lose of generality, we will set $\ell^2/(\nu^2 + 3) = 1$ and $\alpha = 1$. We also define $x_0 = (x_+ + x_-)/2$ and $\bar{x}_0 = (x_+ - x_-)/2$ for later use.

## 3 Fields on Self-Dual Warped AdS$_3$

Massive integer spin $s$ fields in AdS$_3$ spaces are realized by totally symmetric tensors of rank $s$ satisfying the following equation of motion, gauge condition and traceless condition as [18–19]

$$\nabla^2 - m^2 \Phi_{\mu_1 \mu_2 \cdots \mu_s} = 0, \tag{9}$$

where $\epsilon^{x \theta} = +1/\sqrt{-g}$.
\[ \nabla^\nu \Phi_{\mu_1 \mu_2 \cdots \mu_s} = 0, \]  
(10)  
\[ g^\mu\nu \Phi_{\mu_1 \mu_2 \cdots \mu_s} = 0. \]  
(11)

Here, \( \Phi_{\mu_1 \mu_2 \cdots \mu_s} \) is a totally symmetric rank \( s \) tensor and for AdS\( 3 \) the mass of the field is given by \( m^2_s = (s - 3) + M^2 \). The first term is the mass that exists due to the curvature of AdS\( 3 \).

The above set of equations in a maximally symmetric space are equivalent with the following first order equation\(^{[13]}\)

\[ \epsilon_{\mu}^3 \nabla_{\alpha} \Phi_{\beta_1 \beta_2 \cdots \beta_3} = -m \Phi_{\mu_1 \mu_2 \cdots \mu_3}, \]  
(12)  
where \( m^2 = M^2 + (s - 1)^2 \).

However, it was shown\(^{[17]}\) in self-dual warped AdS\( 3 \) space, where the background is not locally AdS, one can not rewrite the field equations for metric perturbations in a simple form as Eq. (2). But, one can yet consider any massive tensor field \( \Phi_{\mu_1 \mu_2 \cdots \mu_3} \) obeys the above first order equation\(^{[10]}\). Adopting Eq. (2) for a tensor field \( \Phi_{\mu_1 \mu_2 \cdots \mu_3} \), one may search a set of equations similar to Eq. (3). But, it seems that satisfying the gauge constraint and tracelessness condition are problematic for a general symmetric rank \( s \) tensor field \( \Phi_{\mu_1 \mu_2 \cdots \mu_3} \) in the background (4) (see appendix A).

However, we will consider in the rest of the paper the field \( h_{\mu\nu} \) is traceless in the sense that

\[ g^{\mu\nu} h_{\mu\nu} = 0. \]  
(13)  
We also consider \( h_{\mu\theta} \) satisfies the following gauge constraint

\[ \nabla^\nu h_{\mu\theta} = 0. \]  
(14)

These two constraints are just transverse traceless conditions and help us to write down the equations of motion for \( h_{\theta\theta} \) and \( h_{\theta\theta} \) in a simple form.

The above conditions can be generalized for a higher spin field as follows

\[ g^{\alpha\beta} \Phi_{\alpha \beta \cdots \nu_s} = 0, \]  
(15)  
\[ g^{\mu\nu} \Phi_{\mu \nu \cdots \nu_s} = 0. \]  
(16)

Especially, for \( \Phi_{\theta\theta \cdots \theta} \) and \( \Phi_{\theta\theta \cdots \theta} \) we obtain

\[ (\nabla^2 - m^2) \Phi_{\theta\theta \cdots \theta} = -\frac{\lambda^2}{4} (s + 1) \Phi_{\theta\theta \cdots \theta}, \]  
(17)

As a consistency check, when we set \( \lambda = \nu = 1, l = 2 \) we recover the known Klein–Gordon equation for spin \( s \) tensor field in BTZ background\(^{[13]}\). In fact, this is due to the fact that for \( \lambda = 1 \) the geometry of self-dual warped AdS\( 3 \) becomes the usual BTZ geometry.

4 Rank 2 Field in Self-Dual Warped AdS\( 3 \)

In this section, we consider a rank 2 tensor field \( h_{\mu\nu} \) which should satisfy (2). Here, for later use, we use the calculations of Ref. [10] where it was rewritten (2) as follows

\[ \partial_t h_{\theta\theta} = \partial_t h_{\phi\phi} + \Gamma(\phi\phi) + m(h_{\theta\theta}), \]  
(18)  
\[ \partial_r h_{\theta\theta} = \partial_r h_{\phi\phi} + \Gamma(\phi\phi) + m(h_{\theta\theta}), \]  
(19)  
\[ \partial_r h_{\theta\theta} = \partial_r h_{\phi\phi} + \Gamma(\phi\phi) + m(h_{\theta\theta}), \]  
(20)  
\[ \partial_r h_{\theta\theta} = \partial_r h_{\phi\phi} + \Gamma(\phi\phi) + m(h_{\theta\theta}), \]  
(21)  
\[ \partial_r h_{\theta\theta} = \partial_r h_{\phi\phi} + \Gamma(\phi\phi) + m(h_{\theta\theta}). \]  
(22)

where

\[ \Gamma(\phi\phi) = \Gamma(\phi\phi) + \Gamma(\phi\phi), \]  
(23)  
\[ \Gamma(\phi\phi) = \Gamma(\phi\phi) + \Gamma(\phi\phi), \]  
(24)  
\[ \Gamma(\phi\phi) = \Gamma(\phi\phi) + \Gamma(\phi\phi), \]  
(25)  
\[ \Gamma(\phi\phi) = \Gamma(\phi\phi) + \Gamma(\phi\phi). \]  
(26)

Next, we focus on the equations of \( h_{\theta\theta} \) and \( h_{\theta\theta} \) components, which decouple from the other components and can be written as Eq. (17). Although, for finding the quasinormal modes we should also find the exact solutions for \( h_{\theta\theta} \) but, after solving Eq. (17) for \( h_{\theta\theta} \) and \( h_{\theta\theta} \) components

\[ \text{No. 4} \]  
Communications in Theoretical Physics 449
then we can use Eqs. (2) and (18) to find the exact solutions for \( h_{\theta t}, h_{tt}, h_{\varphi \varphi}, h_{x x} \) and \( h_{x x} \), and accordingly find the quasi-normal modes.

Recalling the fact that \( \Gamma_{\varphi \theta} = 0 \) and using the constraint (15), one can find the exact solution as follows. First, we solve Eq. (17) to find \( h_{\theta t} \) and \( h_{t \varphi} \). Then, from the first two equations of Eq. (18) we can find \( h_{\theta \varphi} \) and \( h_{x t} \). After that, from the last three equations of Eq. (18) one finds \( h_{x x}, h_{x \theta} \) and especially \( h_{r t} \) in terms of \( h_{\theta \theta}, h_{t \theta}, h_{\theta t} \) and \( h_{tt} \). Finally, inserting the \( h_{rt} \) in the third equation of Eq. (18), we obtain \( h_{tt} \). In the following, we present the results of the computations.

### 4.2 Solution for \( h_{\theta \theta} \) and \( h_{t \theta} \)

The equations of motion for \( h_{\theta \theta} \) and \( h_{t \theta} \) read as

\[
(\nabla^2 - m^2)h_{\theta \theta} = -\frac{\lambda^2}{4} h_{\theta \theta} ,
\]

\[
(\nabla^2 - m^2)\Phi_{t \theta} = \left( \frac{\lambda^2}{4} - 1 \right) h_{t \theta} + (1 - \lambda^2) \left( x - \frac{x_0 + x_+}{2} \right) h_{\theta \theta} .
\]

So, first of all, we should evaluate \( \nabla^2 \). As Ref. [13], one has

\[
\nabla^2 h_{\mu \nu} = \Delta h_{\mu \nu} - \frac{1}{\sqrt{-g}} \partial_{\alpha} (\sqrt{-g} g^{\alpha \beta} \Gamma_{\beta \gamma}^\sigma) h_{\sigma \nu} - \frac{1}{\sqrt{-g}} \partial_{\alpha} (\sqrt{-g} g^{\alpha \beta} \Gamma_{\beta \gamma}^\sigma) h_{\sigma \nu} - 2 \Gamma_{\alpha \mu}^\rho g^{\beta \gamma} \nabla_{\beta} h_{\rho \nu} - 2 \Gamma_{\alpha \nu}^\rho g^{\beta \gamma} \nabla_{\beta} h_{\rho \mu} - g^{\alpha \beta} \Gamma_{\beta \mu}^\sigma \Gamma_{\alpha \sigma}^\rho h_{\rho \nu} - g^{\alpha \beta} \Gamma_{\beta \nu}^\sigma \Gamma_{\alpha \sigma}^\rho h_{\rho \mu} - g^{\alpha \beta} \Gamma_{\beta \mu}^\sigma \Gamma_{\alpha \sigma}^\rho h_{\rho \nu} - g^{\alpha \beta} \Gamma_{\beta \nu}^\sigma \Gamma_{\alpha \sigma}^\rho h_{\rho \mu} ,
\]

where \( \Delta \) is the usual scalar Laplacian

\[
\Delta h_{\mu \nu} = \frac{1}{\sqrt{-g}} \partial_{\alpha} (\sqrt{-g} g^{\alpha \beta} \partial_{\beta} h_{\mu \nu}) .
\]

Calculations of the first and second line of Eq. (23) are straightforward. Using the first order equation (2) and the constrains (15), one can rewrite the third line in terms of \( h_{\theta \theta}, h_{t \theta}, h_{tt} \). For example

\[
\nabla_{\varphi} h_{x t} = m \frac{\sqrt{-g}}{-g} \left( -g_{\theta \varphi} h_{tt} + g_{t \theta} h_{\theta t} \right) + \nabla_{\varphi} h_{tt} ,
\]

where \( \hat{g} = g_{\theta \theta} g_{t t} - (g_{\theta t})^2 \).

Moreover, by using Eq. (25), one can show that the first and second terms in the forth line of Eq. (24) are equal to zero. Finally, in the last line, we have \( h_{xx} \) term which can be replaced with \( h_{\theta \theta} \) or \( h_{t \theta} \) by using the traceless condition.

Using the above equations and inserting the following ansatz

\[
h_{\mu \nu}(x, \theta \phi) = e^{-i(\omega t - k \theta)} R_{\mu \nu}(x) ,
\]

one can obtain the following set of differential equations for \( h_{\theta \theta} \) and \( h_{t \theta} \)

\[
(\Delta 1_{2 \times 2} + M_{2 \times 2}) \left( \begin{array}{c} H_{\theta \theta} \\ H_{t \theta} \end{array} \right) = 0 ,
\]

where the mass matrix \( M \) is given by

\[
M_{2 \times 2} = \begin{pmatrix}
-m^2 + \frac{\lambda^2}{4} & \left( 1 - \frac{2m}{\lambda} \right) (1 - \lambda^2) (x - x_0) \\
0 & -(m - \lambda^2) + \frac{\lambda^2}{4}
\end{pmatrix} ,
\]

and the operator \( \Delta \) is equal to

\[
\Delta = (x - x_+) (x - x_-) \frac{\partial^2}{\partial x^2} + 2 (x - x_0) \frac{\partial}{\partial x} + Q(x) ,
\]

where

\[
Q(x) = \frac{(\omega + k (x - x_0))^2}{(x - x_+) (x - x_-) - \frac{k^2}{\lambda^2}} .
\]

As it is clear, Eq. (27) is a set of nonlinear coupled equations. For solving such coupled equations, one may try to diagonalize \( M \) and decouple the equations but, the components of \( M \) are \( x \)-dependent and we should be careful for doing such procedure. Overcoming to this problem is easy by defining the following new fields

\[
H_{\theta \theta} = \frac{R_{\theta \theta}}{(x - x_0)} ,
\]

\[
H_{t \theta} = R_{t \theta} .
\]

Then, one can obtain the new set of equations as

\[
(\delta_{2 \times 2} + M_{2 \times 2}) \left( \begin{array}{c} H_{t \theta} \\ H_{t \theta} \end{array} \right) = 0 ,
\]

where the new matrix \( M \) is a constant matrix and is given by

\[
M_{ij} = \frac{M_{ij}}{(x - x_0)^{-1}} .
\]

Also, the new diagonal differential operator \( \delta \) is as

\[
\delta_{11} = \Delta + 2 \frac{(x - x_+)(x - x_-)}{(x - x_0)} \frac{\partial}{\partial x} + 2 ,
\]

\[
\delta_{22} = \Delta ,
\]

and all other components of \( \delta \) are zero. Now, we can diagonalize \( M \) and find new decoupled equations. Noting that the eigenvalues of \( M \) are \( M_{11} \) and \( M_{22} \), one can find the eigenvector and matrix transformation \( U \) such that \( U M U^{-1} \) becomes diagonal. The new fields in which have decoupled differential equations are \( \mathcal{H} = U H \) where

\[
H_{\theta \theta} = \frac{1}{\lambda^2} H_{\theta \theta} ,
\]

\[
H_{t \theta} = H_{\theta t} .
\]

Doing the above prescription and using the following common change of variable

\[
z = \frac{x - x_+}{x - x_-} ,
\]

one finally obtains
\[
\begin{align*}
    z(1 - z) \frac{\partial^2}{\partial z^2} H_{i\theta} + \left(1 - z + \frac{4z}{1 + z} \right) \frac{\partial}{\partial z} H_{i\theta} + \left(Q(z) + \frac{2}{1 - z} + \frac{\mathcal{M}_{22}}{1 - z}\right) H_{i\theta} &= 0, \\
    z(1 - z) \frac{\partial^2}{\partial z^2} H_{\theta\theta} + (1 - z) \frac{\partial}{\partial z} H_{\theta\theta} + \left(Q(z) + \frac{\mathcal{M}_{33}}{1 - z}\right) H_{\theta\theta} &= 0,
\end{align*}
\]

where
\[
Q(z) = -\frac{(\omega - [(x_+ - x_-)/2]k^2)}{(x_+ - x_-)^2} + \frac{(\omega + [(x_+ - x_-)/2]k^2)}{(x_+ - x_-)^2} + \frac{k^2(1 - 1/\lambda^2)}{1 - z}.
\]

The solutions of the above equations can be written in terms of the hypergeometric functions as follows
\[
\begin{align*}
    H_{\theta\theta}(z) &= z^{\alpha}(1 - z)^{1/2\gamma + 1}(C_{i\theta} F(a_{i\theta} + 1, b_{i\theta} + 1; c; z)) + D_{i\theta} F(a_{i\theta} + 1, b_{i\theta} + 1; c^*; z), \\
    H_{\theta\theta}(z) &= z^{\alpha}(1 - z)^{1/2\gamma + 1}(C_{i\theta} F(a_{i\theta} + 1, b_{i\theta} + 1; c; z) + D_{i\theta} F(a_{i\theta} + 1, b_{i\theta} + 1; c^*; z)),
\end{align*}
\]

where \(C_{ij}, D_{ij}\) are arbitrary constants and other parameters are given by
\[
\begin{align*}
    \alpha &= -\frac{1}{2} \left(k + \frac{2\omega}{x_+ - x_-}\right), \quad c = 1 + 2\alpha, \\
    a_{ij} &= \beta_{ij} - ik, \quad b_{ij} = \beta_{ij} - \frac{2i\omega}{x_+ - x_-}, \\
    \beta_{i\theta}^+ &= -\frac{1}{2} \pm \frac{1}{2} \sqrt{1 - 4k^2 \left(1 - \frac{1}{\lambda^2}\right) - 4\mathcal{M}_{11}}, \\
    \beta_{i\theta}^- &= -\frac{1}{2} \pm \frac{1}{2} \sqrt{1 - 4k^2 \left(1 - \frac{1}{\lambda^2}\right) - 4\mathcal{M}_{22}}.
\end{align*}
\]

We should mention that in the next section when we will impose the Dirichlet boundary condition, for finding a suitable solution, we will choose the plus sign for \(h_{\theta\theta}\) and \(h_{\theta\theta}\). At the end, one can use Eq. (2) to find the solutions for the remaining components \(h_{i\theta}\) and \(h_{\theta\theta}\). The results of computations have been presented in Appendix B.

### 4.3 Quasi Normal Modes

Quasi normal modes can be found by imposing Dirichlet boundary condition on the solutions (41) and (67). Before that, we would like to mention some points. First, since our aim is to find the quasi-normal modes, we consider in-going waves into the horizon and so, we choose the constants \(D_{\theta\theta} = 0\) and \(D_{i\theta} = 0\). Second, we also choose the plus sign (41). The calculations for minus sign is similar to the case of plus sign. The calculations for minus sign are similar to the case of plus sign. Third, one can see that when \(\lambda = 1\) then the solution (41) reduces to the solution that has been found in Ref. [13]. So, we can use the results of the computations of Ref. [13] for finding ratio of the coefficients \(C_{i\theta}\) and \(C_{i\theta}\). So, all coefficients of the solutions of all fields can be written in terms of \(C_{i\theta}\).

Therefore, we choose the \(C_{i\theta}\) as
\[
C_{i\theta} = C_0(a_{i\theta})(a_{i\theta} - 1),
\]
where \(C_0\) is an arbitrary constant independent of \(a_{i\theta}\) and \(b_{i\theta}\).

After all, using the following transformation relations between hypergeometric functions
\[
F(a, b, c; z) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}
\]

one can find the asymptotic behavior of the solutions when \(z \to 1\). Choosing the plus sign in Eq. (41) which means that the leading order of solutions comes from the second line of Eq. (45). Thus, one can easily find the asymptotic behavior of solutions. Let us focus, for example, on \(h_{i\theta}\) solution. Using the change of variable (38), it is not hard see that \(h_{i\theta}\) can be expanded as
\[
\lim_{z \to 1} \frac{1}{A} \frac{d}{dx} \left(\frac{\omega}{A} \frac{d}{dx} h_{i\theta}\right) = C(z)z^2 \frac{d^2}{dz^2} h_{i\theta} + \cdots
\]

\[
\sim C(z)(a_{i\theta})(a_{i\theta} - 1)z^{\alpha+2}(1 - z)^{-\beta_{i\theta}^+} \times \frac{\Gamma(c)\Gamma(a_{i\theta} + b_{i\theta} + 2 - c)}{\Gamma(a_{i\theta} + 1)\Gamma(b_{i\theta} + 1)} + \cdots,
\]

where \(C(z)\) is a function of \(z\) and \(x\) to a finite constant when \(z \to 1\).

Now, imposing Dirichlet condition on \(h_{i\theta}\) implies that
\[
\alpha + 1 = -n_1, \quad \text{or} \quad b + 1 = -n_2,
\]

where \(n_1\) and \(n_2\) are non-negative integers. One can see that these conditions are necessary and sufficient for satisfying Dirichlet condition on all fields.

Finally, one can obtain the left and right quasi normal modes as
\[
k = -i2\pi T_L l(n_1 + h_L), \quad \omega = -i2\pi T_R l(n_2 + h_R),
\]

where
\[
h_R = -\frac{1}{2} + \frac{1}{2} \sqrt{1 - 4k^2 \left(1 - \frac{1}{\lambda^2}\right) - 4\mathcal{M}_{22}},
\]

\[
h_L = -\frac{3}{2} + \frac{1}{2} \sqrt{1 - 4k^2 \left(1 - \frac{1}{\lambda^2}\right) - 4\mathcal{M}_{22}}.
\]

Notice that from Eq. (49) one has \(h_R - h_L = +2\). Note also that if one chooses \(C_{i\theta} = C_0(b_{i\theta})(b_{i\theta} - 1)\) then we will find \(h_R - h_L = -2\).

The above results for conformal weight are in precise agreement with the calculations in Ref. [10] where the con-
formal weight that have been obtained using an algebraic way from highest-weight mode.

5 Higher Spin on Self-Dual Warped AdS$_3$

In this section, we discuss how one can find the conformal weight and quasi-normal modes of a field with arbitrary spin in self-dual warped AdS$_3$ background. The key point is that, as for the spin 2 case, for finding the conformal weight and quasi-normal modes of higher spin fields, it is sufficient to find the solution of $h_{\theta\theta\cdots\theta}$ equation of motion. So, let us firstly find this solution in the next section.

5.1 Solution for $\Phi_{\theta\theta\cdots\theta}$

The equation of motion for $\Phi_{\theta\theta\cdots\theta}$ is given by Eq. (17). So, we should evaluate $\nabla^2 \Phi_{\theta\theta\cdots\theta}$ as follows$^3$

$$\nabla^2 \Phi_{\theta\theta\cdots\theta} = \Delta \Phi_{\theta\theta\cdots\theta} - \frac{1}{\sqrt{-g}} (\partial_\alpha \sqrt{-g} g^{\alpha\beta} \phi^\sigma_{\gamma\delta}) \Phi_{\theta\theta\cdots\theta} \sigma\gamma\delta$$

$$- 2 g^{\alpha\beta} \phi^\sigma_{\gamma\delta} \nabla_\beta \Phi_{\theta\theta\cdots\theta} \gamma\delta - g^{\alpha\beta} \phi^\sigma_{\theta\phi} \phi^\tau_{\sigma\tau} \Phi_{\theta\theta\cdots\theta} \phi \sigma\tau,$$

(50)

where the $\Delta$ is the scalar Laplacian (24). Again, using the first order equation (2) and the constraint (15), we are able to rewrite the $\nabla_i \Phi_{\theta\theta \cdots \theta}$ and $\Phi_{\theta\theta \cdots \theta}$ in terms of $\Phi_{\theta\theta \cdots \theta}$ and $\Phi_{\theta\theta \cdots \theta}$, without any $x$ indices.

So, using the ansatz

$$\Phi_{\theta\theta\cdots\theta} = e^{-i(\omega t-kx)} R_{\theta\theta\cdots\theta},$$

the final equation reads as

$$z(1-z) \frac{\partial^2}{\partial z^2} R_{\theta\theta\cdots\theta} + (1-z) \frac{\partial}{\partial z} R_{\theta\theta\cdots\theta}$$

$$+ \left( Q(z) + \frac{\tilde{M}_{22}}{1-z} \right) R_{\theta\theta\cdots\theta} = 0,$$

(51)

where we have used Eq. (38) and

$$\tilde{M}_{22} = -\left(m - s \frac{\lambda}{2} \right)^2 + \frac{\lambda^2}{4}.$$  

(52)

The solution of the above equation with in-going condition on horizon is

$$R_{\theta\theta\cdots\theta} = \tilde{C}_{\theta\theta\cdots\theta} (1-z)^{\tilde{\beta}_{\theta\theta\cdots\theta} + 1}$$

$$\times F_1 \left( \tilde{\beta}_{\theta\theta} - i k + 1, \tilde{\beta}_{\theta\theta} - \frac{2 i \omega}{x_+ - x_-} + 1, c, z \right),$$

(53)

where

$$\tilde{\beta}_{\theta\theta} = -\frac{1}{2} + \frac{1}{2} \sqrt{1 - 4 k^2 \left(1 - \frac{1}{\lambda^2} \right) - 4 \tilde{M}_{22}},$$

(54)

and $\alpha, c$ are given in Eq. (43).

5.2 Quasi-Normal Modes

For imposing Dirichlet boundary conditions on all fields, as for spin 2 case, we focus on $\Phi_{\theta \theta \cdots \theta}$. Using the first order equation (18), the equation of $\Phi_{\theta \theta \cdots \theta}$ is

$$\partial_x \Phi_{\theta \theta \cdots \theta} = \partial_y \Phi_{x \theta \cdots \theta} + \Gamma_{(\theta \theta \cdots \theta)} + m(\theta \theta \cdots \theta),$$

(55)

where

$$\Gamma_{(\theta \theta \cdots \theta)} = \Gamma_{\theta \theta \cdots \theta} \Phi_{\theta \theta \cdots \theta} - \Gamma_{\theta \phi} \Phi_{\theta \phi \cdots \theta},$$

$$m(\theta \theta \cdots \theta) = -m g_{x x} e^{\omega t} \left( g_{y \theta} \Phi_{x \theta \cdots \theta} - g_{\phi \theta} \Phi_{x \phi \cdots \theta} \right).$$

(56)

In the above expression, one can find the mostly-t field $\Phi_{\theta \theta \cdots \theta}$ in terms of fields with less $t$ indices. So, using Eq. (56), among the many terms, one finds a term with maximum power of $\omega$ and derivative of $\Phi_{\theta \theta \cdots \theta}$ as

$$\Phi_{\theta \theta \cdots \theta} \sim z^s \frac{d^s}{dz^s} \Phi_{\theta \theta \cdots \theta} + \cdots.$$  

(57)

Now, choosing

$$C_{\theta \theta \cdots \theta} = C_0 (\tilde{a}_{\theta \theta} \cdots (\tilde{a}_{\theta \theta} - s + 1),$$

and using the asymptotic behavior of hypergeometric functions and imposing Dirichlet condition on $\Phi_{\theta \theta \cdots \theta}$, one finds the necessary and sufficient conditions in which all field become zero at infinity are

$$\tilde{a}_{\theta \theta} + s - 1 = -n, \quad \text{or} \quad \tilde{b}_{\theta \theta} + 1 = -n,$$

(58)

where $n$ is a non-negative integer. Finally, we obtain

$$k = -i 2 \pi T L (n_1 + h_L), \quad \omega = -i 2 \pi T R (n_2 + h_R),$$

(59)

where

$$h_L = \frac{2 s - 1}{2} + \frac{1}{2} \sqrt{1 - 4 k^2 \left(1 - \frac{1}{\lambda^2} \right) - 4 \tilde{M}_{22}},$$

$$h_R = \frac{1}{2} + \frac{1}{2} \sqrt{1 - 4 k^2 \left(1 - \frac{1}{\lambda^2} \right) - 4 \tilde{M}_{22}}.$$

(60)

6 Conclusion

In this paper, we have studied the dynamics of a rank $s$ field in a 3-dimensional self-dual warped AdS$_3$ background. We have considered rank $s$ field should satisfy the equations of motion (2). We also supposed that all modes satisfy weaker constraint (15). These constrains greatly help us to proceed in calculations. In fact, we were able to obtain Klein–Gordon-like equations for $h_{\theta \phi}$ and $h_{\theta \theta \phi}$ modes.

In particular, for a rank 2 tensor field the equations of $h_{\theta \phi}$ and $h_{\theta \theta}$ are coupled nonlinear differential equations and are decoupled from the other modes. By a suitable redefinition of the fields, these equations also decoupled from each other and so we found the solutions which are in terms of the hypergeometric functions. Having found the solutions, we obtained the quasi-normal modes which are in-going waves at horizon and satisfy Dirichlet boundary condition at infinity.

At the end, we extended our computations for a general rank $s$ field and found the quasi-normal modes.

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$^3$In this section, by expressions similar to $\Gamma_{\mu \nu} \Phi_{\cdots}$ we mean that $\Gamma_{\mu \nu} \Phi_{x_2 \cdots x_n} + \Gamma_{\mu \nu} \Phi_{x_1 \cdots x_n} + \cdots + \Gamma_{\mu \nu} \Phi_{x_{n_1} \cdots x_\sigma}$. 

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**Communications in Theoretical Physics**

Vol. 68
Due to less symmetry of the warped AdS$_3$ geometry, there are many unknown aspects of such geometries and many open problems have not yet been solved.

Appendix A Gauge and Tracelessness Conditions on Self-Dual Warped AdS$_3$

Let us for simplicity consider a rank 2 field $h_{\mu\nu}$ and first suppose that it is totally symmetric and obeys Eqs. (2). It means that for a symmetric one should have
\begin{equation}
0 = \varepsilon^{\mu\nu\rho} g_{\alpha\beta} \nabla_{\alpha} h_{\beta\rho} = -g^\rho\beta \nabla^\alpha h_{\beta\alpha} + g^{\alpha\beta} \nabla_{\alpha} g^\rho\beta h_{\beta\nu}.
\tag{A1}
\end{equation}
In the other hand, using Eq. (2) for $h_{\tau\tau}$ one obtains
\begin{equation}
-mh_{\tau\tau} = g_{\rho\tau} \varepsilon^{\rho\tau\omega} (\nabla_{\omega} h_{\tau\tau} - \nabla_{\tau} h_{\tau\omega})
= g_{\rho\tau} \varepsilon^{\rho\tau\omega} (\nabla_{\omega} h_{\tau\tau} - \nabla_{\tau} h_{\tau\omega})
= g_{\rho\tau} \varepsilon^{\rho\tau\omega} (\nabla_{\omega} h_{\tau\tau} - \nabla_{\tau} h_{\tau\omega})
- m g_{\rho\tau} \varepsilon^{\rho\tau\omega} (g_{tt} h_{\tau\omega} - g_{t\omega} h_{tt} - g_{\omega\tau} h_{tt} - g_{\omega\tau} h_{tt})
= \frac{m}{g_{tt}} (0 + g^\rho\theta h_{\rho\theta} + 2 g^\theta\theta h_{\theta\theta} + \tilde{g}^\mu h_{\mu\mu}).
\tag{A2}
\end{equation}
In the third line of Eq. (A2) we have used Eq. (2)\textsuperscript{5}. The first part of the last line is true for the symmetric tensor and in the second line we used the metric (5). So, as far as the field is symmetric and obeys (2) then it automatically satisfies the tracelessness condition. Coming back to Eq. (A1), the second line is equal to zero provided that the gauge constraint be also equal to zero. But, for gauge conditions,
\begin{equation}
-m \nabla^\mu \Phi_{\mu\nu_2\cdots\nu_s} = \varepsilon^{\mu\alpha\beta} \nabla_{\mu} \Phi_{\alpha\beta\nu_2\cdots\nu_s}
= \frac{1}{2} \varepsilon^{\mu\rho\beta} \nabla_{\mu} \Phi_{\rho\beta\nu_2\cdots\nu_s}
= \varepsilon^{\mu\rho\beta} g^\rho\alpha R^\alpha_{\beta\rho\mu} \Phi_{\beta\nu_2\cdots\nu_s}
+ \varepsilon^{\mu\rho\beta} g^\rho\alpha R^\alpha_{\beta\rho\mu} \Phi_{\beta\nu_2\cdots\nu_s}
+ \varepsilon^{\mu\rho\beta} R^\alpha_{\beta\rho\mu} \Phi_{\beta\nu_2\cdots\nu_s}
+ \varepsilon^{\mu\rho\beta} R^\alpha_{\beta\rho\mu} \Phi_{\beta\nu_2\cdots\nu_s}
+ \cdots
\tag{A3}
\end{equation}
So, because of nonzero $r_{\beta\mu\alpha}$ term, the usual harmonic gauge constraint does not satisfy for any components of $\Phi_{\mu\nu_2\cdots\nu_s}$.

Appendix B Solutions for $h_{tt}$ and $h_{\tau\tau}$

In this appendix, we present the solutions of $h_{tt}$ and $h_{\tau\tau}$. The results are as follows
\begin{equation}
h_{tt} = \frac{k}{A} \frac{d}{dx} h_{tt} + \frac{\omega}{A} \frac{d}{dx} h_{\theta\theta},
- \frac{\omega B + kC}{A} h_{\theta\theta} - \frac{\omega D + kE}{A} h_{\theta\theta},
h_{\tau\tau} = \frac{1}{A} \frac{d}{dx} h_{\tau\tau} + \frac{F}{A} \frac{d}{dx} h_{\theta\theta} - \frac{G}{A} h_{\tau\tau},
- \frac{H}{A} h_{\theta\theta} - \frac{I}{A} h_{\theta\theta},
\tag{A5}
\end{equation}
where
\begin{align*}
A &= \omega \Gamma^x_{x\theta} + k(\Gamma^x_{\theta\theta} - m \epsilon^{\theta\mu\nu} g_{\mu\nu}), \\
B &= g_{xx} \epsilon^{x\theta} (\omega k + \Gamma^x_{\theta\theta} - m^2 g_{\theta\theta}), \\
C &= g_{xx} \epsilon^{x\theta} (\omega k + \Gamma^x_{\theta\theta} - m^2 g_{\theta\theta}), \\
D &= g_{xx} \epsilon^{x\theta} (\omega k + \Gamma^x_{\theta\theta} - m^2 g_{\theta\theta}), \\
E &= g_{xx} \epsilon^{x\theta} (\omega k + \Gamma^x_{\theta\theta} - m^2 g_{\theta\theta}), \\
F &= - \frac{\omega}{k} \Gamma^x_{x\theta}, \\
G &= \Gamma^x_{x\theta} - F \Gamma^x_{x\theta} - \frac{g_{xx}}{m} \epsilon^{x\theta} (\omega k + \Gamma^x_{\theta\theta} - m^2 g_{\theta\theta}), \\
H &= \frac{g_{xx}}{m} \epsilon^{x\theta} (F(k^2 - m^2 \sqrt{-g} g_{\theta\theta} + \Gamma^x_{\theta\theta} - \Gamma^x_{\theta\theta}) + \Gamma^x_{\theta\theta} - \Gamma^x_{\theta\theta} - \Gamma^x_{\theta\theta}), \\
I &= \frac{g_{xx}}{m} \epsilon^{x\theta} (F(\omega k - \Gamma^x_{\theta\theta} - m^2 \sqrt{-g} g_{\theta\theta} + \Gamma^x_{\theta\theta} - \Gamma^x_{\theta\theta} - \Gamma^x_{\theta\theta}) + \Gamma^x_{\theta\theta} - \Gamma^x_{\theta\theta} - \Gamma^x_{\theta\theta}).
\end{align*}

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\textsuperscript{5}Notice that Eq. (2) can be written as $\nabla_\lambda h_{\mu\nu} - \nabla_\nu h_{\mu\lambda} = m \epsilon_{\lambda\mu\nu} h_{\mu\nu}$. 

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