Multi-dimensional Network Security Game: How do attacker and defender battle on parallel targets?

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Abstract—In this paper, we consider a new network security game wherein an attacker and a defender are battling over "multiple" targets. This type of game is appropriate to model many current network security conflicts such as Internet phishing, mobile malware or network intrusions. In such attacks, the attacker and the defender need to decide how to allocate resources on each target so as to maximize his utility within his resource limit. We model such a multi-dimensional network security game as a constrained non-zero sum game. Two security breaching models, the product-form and the proportion-form, are considered. For each breaching model, we prove the existence of a unique Nash equilibrium (NE) based on Rosen’s theorem and propose efficient algorithms to find the NE when the games are strictly concave. Furthermore, we show the existence of multiple NEs in the product-form breaching model when the strict concavity does not hold. Our study sheds light on the strategic behaviors of the attacker and the defender, in particular, on how they allocate resources to the targets which have different weights, and how their utilities as well as strategies are influenced by the resource constraints.

1. Introduction

The economics of network security has become a thriving concern in fixed line and mobile Internet. Due to the popularity of e-commerce and other online services, malicious attacks have evolved into profit driven online crimes in the forms of Internet phishing, network intrusion, mobile malware etc. Although security defence is essential, the networking community is still witnessing an increased number of global attacks. Part of reasons are the economic benefits on performing attacks by hackers as well as the inadequate protection against the persistent attacks. Therefore, economic studies beyond the technological solutions are vitally important to reveal the behaviors of the defenders and the malicious attackers, and game theory serves as a well suited mathematical tool to bring about this fundamental understanding. A prominent application of game theory in security is intrusion detection where an attacker exploits system vulnerabilities and a defender monitors the events occurring in a network strategically [4][6]. Recent advances of network security games have two features. One is called uncertainty that incorporates incomplete information of players [6] and stochastic properties of players or environments [7]. The other is called interdependency in which the actions of players may affect other players. This type of interactions are sometimes regarded as network effects with positive or negative externality [9][10][15].

In this work, we explore a new type of network security game which is characterized by multi-dimensional attacks. We are motivated by three facts. Firstly, the effectiveness of attack or defence depends on the amount of resources that are used. The resource is an abstract representation of manpower, machines, technologies, etc. For instance, many resources are needed to create malicious websites in phishing attacks, or to camouflage malicious apps in mobiles, or to recruit zombie machines in DDoS attacks, or to probe server vulnerabilities in intrusion attacks. However, one needs to note that resource is not free for the attacker and the defender. Secondly, the attacker and the defender usually possess limited resources. For instance, the number of active bots that a botmaster can manipulate is usually constrained to a few thousands. Thirdly, the attacker can assaults multiple targets for better economic returns. These targets may represent different banks in the Internet phishing attack [11], or different Android apps in mobile malware, or different servers in network intrusion attacks. These targets vary in values or importances. Attacking (resp. protecting) more targets requires a larger amount of resources, which may exceed the resource budget of the attacker (resp. defender). As a consequence, the conflicts on multiple targets are conjointed whenever the attacker or the defender has limited amount of resources. This transforms the decision making in network security issues into myopic constrained optimization problems.

We propose a non-zero sum game-theoretic framework to characterize the constrained resource allocation between an attacker and a defender. The utility of the attacker is modeled as the profit, which is equivalent to the loss of victims minus the costs of attack resources. The utility of the defender is modeled as the loss of victims plus the costs of defence resources. Both players aim to optimize their individual utilities. We express the loss of victims on a target as a product of its weight and the security breaching probability. Two breaching models are considered; one is the product-form of attack and defence efficiencies, the other is the proportion-form of attack and defence efficiencies. In our work, we focus on the following questions: 1) How does a player select targets to attack/defend and how
A. Motivation

We are motivated by new features of network attacks and defences that are not well captured by existing works (e.g., [6] and reference therein). Most of state-of-the-art researches focus on the one dimensional strategies (i.e., monitoring probability of intrusion, channel access probability or insurance adoption of a node). Such game models are insufficient to characterize the modern days security attacks such as phishing and mobile malware, etc. Here, we present some salient features of network attacks that lead to our game formulation.

First, the attackers and the defenders are resource constrained. Resources are defined in a variety of forms. For instance, in the fast-flux phishing attack, the hijacked IP address is one type of resources of the attackers. In a mobile malware attack, the attacker’s resources can be the technology and the manpowers used to spoof the security check mechanism of the third-party apps markets. In DDoS attacks, a botmaster is usually able to control only a few thousands active bots [22]. Similarly, the defender needs to allocate resources such as technologies and manpowers to detect and remove these attacks. In general, both the attacker and the defender only possess limited resources.

Second, the efficiencies of attacks and defences depend on how many resources are allocated. While existing works (e.g., references in [6]) assume that the payoffs of the attacker and the defender are determined by whether the target is attacked or defended. We take phishing attack as an example. By creating more malicious websites, the phishing attacker is able to seduce more users and to perform more persistent attacks. If the defender allocates more resources to perform proactive detection, more malicious sites will be ferreted out in zero-day, and the attack time window will be reduced. Similarly, if more efforts are spent to create malicious Android apps, the attacker can carry out more effective camouflage, thus gaining more profits through stealing private information or sending premium SMS imperceptibly. As a countermeasure, the defender will install these apps on his cloud and examine their suspicious events for a certain amount of time.

Last but not least, the attacker and the defender battle not on a single, but rather, multiple targets. Attacker are profit-driven. They are inclined to attack many targets in parallel. The targets are specified as different E-banks in phishing, different apps in mobile malware attacks and different servers in network intrusions. Note that the targets vary in their valuations, so the attacker and the defender may allocate different amount of resources to them. To attack (resp. protect) multiple targets, more resources are required. How to perform parallel attacks becomes a challenging problem when players have certain resource limits. All these motivate our study on the strategic allocation of limited resources by the players on multiple targets simultaneously.
B. Models

Let us start with the basic security game which consists of two players, an attacker \( A \) and a defender \( D \). The attacker launches attacks on \( N \) targets (or “battlefields” interchangeably) which we denote as \( B = \{B_1, \ldots, B_N\} \). The target \( B_i \) is associated with a weight \( w_i \ (i = 1, \ldots, N) \). When \( w_i > w_j, B_i \) is more valuable than \( B_j \). Without loss of generality, we rank all targets from 1 to \( N \) in the descending order of their weights (i.e. \( w_i > w_j \) if \( i < j \)).

Attacking a target may consume some resources such as manpower to design malware, social engineering techniques to camouflage them, or dedicate many compromised machines for attacks. Defending a target needs manpower, investment in technology, and computing facilities etc. Here, we monetarize different types of resources. Let \( c \) be the price of per-unit of \( A \)'s resources, and let \( \hat{c} \) be that of \( D \)'s resources. We next define two important terms that form the utilities of the attacker and the defender.

- **Attack efficiency.** Let \( x_i \) be the amount of resources spent by \( A \) on \( B_i \), and let \( f(x_i) \) be the corresponding attack efficiency on target \( B_i \). Here, \( f(\cdot) \) reflects the ability of the attacker to intrude a system, or to camouflage the malware, etc. We assume that \( f(x_i) \) is a differentiable, strictly increasing and concave function with respect to (w.r.t) \( x_i \). The concavity means that the increment of attack efficiency decreases when \( A \) further increases \( x_i \). Without loss of generality, we let \( f(0) = 0 \) and \( 0 \leq f(x_i) \leq 1 \).

- **Defence efficiency.** Denote \( y_i \) as the resources that \( D \) uses to detect and remove the attacks on target \( B_i \). Let \( g(y_i) \) be the defence efficiency when \( D \) allocates \( y_i \) to \( B_i \). We assume that \( g(y_i) \) is a differentiable, strictly increasing and concave function of \( y_i \) with \( g(0) = 0 \) and \( 0 \leq g(y_i) \leq 1 \). For the sake of convenience, we define a complementary function \( \tilde{g}(y_i) \), the defence inefficiency, which has \( \tilde{g}(y_i) = 1 - g(y_i) \). Then, \( g(\cdot) \) is a decreasing and convex function.

It is very difficult to capture the loss of victims (also the revenue of the attacker) due to the obscure interaction between the attack efficiency of \( A \) and the defence efficiency of \( D \). Here, we formulate two simplified breaching models, one is named a “product-form” model and the other is named a “proportion-form” model. Denote by \( p_i \) the breaching probability of target \( B_i \). Then, there exist

- **Product-form model:** \( p_i = f(x_i) \tilde{g}(y_i) \);

- **Proportion-form model:** \( p_i = \frac{f(x_i)}{f(x_i)+g(y_i)} \).

In the product-form model, the change of attack (resp. defence) efficiency causes a linear change of breaching probability. For mobile phishing attacks, the defence efficiency can be regarded as the probability of detecting malware, and the attack efficiency represents the ratio of victims defrauded by the attacker. Then, the breaching probability can be taken as a product of attack efficiency and defence inefficiency. A classic example of the product-form model is the matrix-form intrusion detection game where \( f(x_i) \) and \( g(y_i) \) are linear functions [6]. The attack efficiency denotes the probability of performing an attack and the defence efficiency denotes the probability of performing a detection action. In reality, the resources of the attacker and the defender have a coupled effect on the security of a target. The increase of attack efficiency might not yield a linearly augmented breaching probability. However, it is very difficult to quantify their coupling. Here, we present a proportion-form breaching model that generalizes the cyber-security competition in [21] and the DDoS attacks on a single target in [12]. The breaching probability increases with the attack efficiency, while at a shrinking speed.

In practice, both \( A \) and \( D \) have limited resource budgets which we denote by \( X_A \) and \( Y_D \) respectively, with \( 0 < X_A, Y_D < \infty \). Our focus is to unravel the allocation strategies of the players on multiple targets with the consideration of resource limits. To achieve this goal, we make the following assumption on the attack and defence efficiencies.

**Assumption:** \( \lim_{x_i \to \infty} f(x_i) = 1 \) and \( \lim_{y_i \to \infty} g(y_i) = 1 \) in the product-form model if not mentioned explicitly.

Late on, we consider the linear \( f(x_i) \) and \( g(y_i) \) that generalize intrusion detection game to multiple targets. As a consequence of attacking \( B_i \), \( A \) receives an expected revenue of \( w_i p_i \). Let \( U_A \) be the aggregate profit of \( A \) on all the \( N \) targets. We have \( U_A = \sum_{i=1}^{N} w_i p_i - c \sum_{i=1}^{N} x_i \). The attacker \( A \) is usually profit driven and is assumed to be risk-neutral. His purpose is to maximize \( U_A \) under the resource cap \( X_A \). Then, the constrained resource allocation problem is expressed as

\[
\max_{\{x_i\}_{i=1}^{N}} U_A \\
\text{subject to } \sum_{i=1}^{N} x_i \leq X_A. \tag{1}
\]

The defender \( D \)'s objective is to minimize the revenue of the attacker \( A \) with the consideration of his resource budget. Let \( U_D \) be the disutility of \( D \) given by \( U_D = \sum_{i=1}^{N} w_i p_i - c \sum_{i=1}^{N} y_i \). When \( c \) (resp. \( c \)) is 0, \( D \) (resp. \( A \)) has a use-it-or-lose-it cost structure such that he will utilize all his resources. The resource allocation problem of \( D \) can be formulated as:

\[
\max_{\{y_i\}_{i=1}^{N}} U_D \\
\text{subject to } \sum_{i=1}^{N} y_i \leq Y_D. \tag{2}
\]

Noticing that \( A \) and \( D \) have conflicting objectives, we model the resource allocation problem as a two-player non-cooperative game and we denote it as \( G \). Let \( H \) be a convex hull expressed as \( \{ (x_i, y_i) | x_i \geq 0, y_i \geq 0, \sum_{i=1}^{N} x_i \leq X_A, \sum_{i=1}^{N} y_i \leq Y_D \} \). In what follows, we define a set of concepts for the game.

**Definition 1:** Nash Equilibrium: Let \( x = (x_1, \ldots, x_N) \) and \( y = (y_1, \ldots, y_N) \) be the feasible resource allocations by \( A \) and \( D \) in the convex hull \( H \) respectively. An allocation profile \( S = \{x^*, y^*\} \) is a Nash equilibrium (NE) if
$U_A(x^*, y^*) \geq U_A(x, y^*)$ and $U_D(x^*, y^*) \geq U_D(x^*, y)$ for any $x \neq x^*$ and $y \neq y^*$.

**Definition 2:** An *concave game* is a game in which each player $i$ has a utility function $u_i(x, i)$ such that $u_i(x, i)$ is concave in $x_i$.

**Theorem 1:** An existence and uniqueness of the Nash equilibrium and the influence of resource allocation.

In the previous section, we have shown the existence of a unique NE in the multi-dimensional security game $G$. However, we have not stated how to derive the NE, which is nontrivial in fact. Define $(x^*, y^*)$ as the NE of $G$. We show that $(x^*, y^*)$ has the following properties.

**Theorem 3:** There exist non-negative variables $\lambda$ and $\rho$ such that

$$-w_i f(x_i) \hat{g}(y_i) - \hat{c} \begin{cases} = \rho & \text{if } y_i > 0 \\ \leq \rho & \text{if } y_i = 0 \end{cases},$$

$$w_i f(x_i) \hat{g}(y_i) - c \begin{cases} = \lambda & \text{if } x_i > 0 \\ \leq \lambda & \text{if } x_i = 0 \end{cases},$$

where

$$\begin{cases} \lambda \geq 0 & \text{if } \sum_{i=1}^N x_i = X_A \\ \lambda = 0 & \text{if } \sum_{i=1}^N x_i < X_A \end{cases}$$

$$\begin{cases} \rho \geq 0 & \text{if } \sum_{i=1}^N y_i = Y_D \\ \rho = 0 & \text{if } \sum_{i=1}^N y_i < Y_D \end{cases}.$$

Herein, $\lambda$ and $\rho$ are viewed as shadow prices of violating the resource limits. From Theorem 3, one can see that $x_i^*$ and $y_i^*$ may take on 0, which occurs when $A$ or $D$ decides not to attack or defend target $B_i$. Our main question here is that given $X_A$ and $Y_D$, how $\lambda$ and $\rho$ are solved at the NE? Before answering this question, we state the sets of targets with positive resources of $A$ and $D$ at the NE.

**Lemma 1:** Let $K_A$ be the number of targets with positive resources of $A$, and $K_D$ be that with positive resources of $D$ at the NE. We have i) the set of targets being attacked is $\{B_1, \ldots, B_{K_A}\}$ and the set of targets being defended is $\{B_1, \ldots, B_{K_D}\}$; ii) $K_A \geq K_D$.

**Remark:** The utility of Lemma 1 is that it greatly reduces the space of searching $K_D$ and $K_A$, which is essential for us to compute the values of $\lambda$, $\rho$, $x_i^*$ and $y_i^*$ at the NE. In fact, we only need to test at most $(N+1)(N+2)/2$ possible sets of targets. Define two inverse functions $h_D(\lambda) := \{\hat{g}^{-1}(\cdot, \lambda)\}$ and $h_A(\rho) := \{\hat{f}^{-1}(\cdot, \rho)\}$. At the NE, the resources used by $A$ and $D$ on a target are given by

$$x_i^* = \begin{cases} h_A(\hat{f}(x_i^*, \lambda)) & \text{if } \lambda \leq K_D \\ h_A(\hat{f}(x_i^*, \lambda)) & \text{if } \lambda = 0 \end{cases},$$

$$y_i^* = \begin{cases} h_D(\hat{g}(y_i^*, \rho)) & \text{if } \rho \leq K_A \\ h_D(\hat{g}(y_i^*, \rho)) & \text{if } \rho = 0 \end{cases}.$$

In what follows, we define a set of notations w.r.t. the total resources (denoted as Tot_RES) used by both players at the NE in Table 1. The pair $(X_A^{\text{su}}, Y_D^{\text{su}})$ denote the sufficient amount of resources needed by $A$ and $D$ when $\lambda$ and $\rho$ are both 0. If both $X_A > X_A^{\text{su}}$ and $Y_D > Y_D^{\text{su}}$ hold, $A$ and $D$ have some unused resources at the NE. Then, the strategies of $A$ and $D$ on one target are independent of the other targets. We can partition the plane of $(X_A, Y_D)$ into four domains: $D_1$ $X_A > X_A^{\text{su}}$ and $Y_D > Y_D^{\text{su}}$; $D_2$ $X_A < X_A^{\text{su}}$ and $Y_D > Y_D^{\text{su}}$; $D_3$ $X_A < X_A^{\text{su}}$ and $Y_D = Y_D^{\text{su}}$; $D_4$ $X_A = X_A^{\text{su}}$ and $Y_D < Y_D^{\text{su}}$. None of the above. If $(X_A, Y_D) \in D_1$, the consumed resources of $A$ and $D$ at the NE are $X_A^{\text{su}}$ and $Y_D^{\text{su}}$ respectively. Then, $A$ uses $X_A$ resources and $D$ uses $Y_D^{\text{su}}$ at the NE. If $(X_A, Y_D) \in D_2$, the resources of $A$ and $D$ are insufficient. Then, $A$ uses $X_A^{\text{su}}$ resources and $D$ uses $Y_D$ at the NE. If $(X_A, Y_D) \in D_3$, the resources of $D$ are insufficient. Then, $A$ uses $X_A^{\text{su}}$ resources and $D$ uses $Y_D$ at the NE. The partition of $(X_A, Y_D)$ enables us to understand when the attacker (resp. the defender) possesses sufficient amount of resources for the attack (resp. defence).

**Table 1:** Notations of total amount of resources.

| $X_A$ | $Y_D$ | $X_A^{\text{su}}$ | $Y_D^{\text{su}}$ |
|------|------|----------------|----------------|
| Tot_RES used by $A$ at the NE | Tot_RES used by $D$ at the NE | Tot_RES needed by $A$ at the NE | Tot_RES needed by $D$ at the NE |
| $\lambda = 0$ | $\rho = 0$ | $\lambda = 0$ | $\rho > 0$ |
| $\lambda = 0$ | $\rho > 0$ | $\lambda > 0$ | $\lambda = 0$ |

The remaining challenge on deriving the NE is how $\lambda$ and $\rho$ are found for the given $K_A$ and $K_D$. Intuitively, we can solve $\lambda$ and $\rho$ based on Eqs. 5-8. However, there does not exist an explicit expression in general. We propose a bisection algorithm in Fig. 1 to search $\lambda$ and $\rho$. The basic idea is to express $\rho$ as two functions of $\lambda$, $\rho_1(\lambda)$ obtained from Eqs. 5-7, and $\rho_2(\lambda)$ obtained from Eqs. 6-8, and then compute their intersection. To guarantee that the bisection algorithm can find feasible $\lambda$ and $\rho$ if they exist, we show the monotonicity of $\rho_1(\lambda)$ and $\rho_2(\lambda)$ in the
Lemma 2: Suppose that feasible $\lambda$ and $\rho$ (i.e. $\lambda, \rho \geq 0$) exist for the fixed $K_A$ and $K_D$ at the NE. The following properties hold i) if $\lambda$ is 0, there has a unique $\rho \geq 0$; ii) if $\rho$ is 0, there has a unique $\lambda$; iii) $\rho_1(\lambda)$ is a strictly increasing function and $\rho_2(\lambda)$ is a strictly decreasing function.

The monotonicity property enables us to use bisection algorithm to check the existence of the pair $(\lambda, \rho)$ and solve them if they exist. When $X_A$ and $Y_D$ are sufficient, the NE can be directly computed via eqs. (7) and (8). When the resources of either $A$ or $D$ are insufficient, the NE is found by the lines 5~17 in Fig[1]. When both players have insufficient resources, the NE is obtained by the lines 18~26. The complexity order of finding the sets with positive resource allocation is merely $O(N^2)$.

Input: $N$, $X_A$, $Y_D$, $w$, $\epsilon$, $\delta$, $f(\cdot)$ and $g(\cdot)$; Output: $K_A$, $K_D$, $\lambda$, $\rho$, $x_i^*$ and $y_i^*$

1: Initialize $K_A = K_D = N$
2: Let $\lambda = \rho = 0$, compute $y_i^*$, $x_i^*$ using eqs. (7), (8) for all $i$;
3: Compute $X_A^{suf} = \sum_{i=1}^{N} x_i^*$ and $Y_D^{suf} = \sum_{i=1}^{N} y_i^*$;
4: If both $X_A \geq X_A^{suf}$ and $Y_D \geq Y_D^{suf}$, exit;
5: For $K_A \geq 1$
6: $K_D = K_A$
7: For $K_D \geq 1$
8: If $X_A \leq X_A^{suf}$
9: Find $\lambda$ by letting $\rho = 0$ and $X_A = X_A$ via (7), (8);
10: Elseif $Y_D \leq Y_D^{suf}$
11: Find $\rho$ by letting $\lambda = 0$ and $Y_D = Y_D$ via (7), (8);
12: End;
13: If $x_i^* \geq 0$, $y_i^* \geq 0$, exit;
14: $K_D = K_D - 1$
15: End
16: $K_A = K_A - 1$
17: End
18: For $K_A \geq 1$
19: $K_D = K_A = N$
20: For $K_D \geq 1$
21: Compute the fixed point $(\rho, \lambda)$ which solves (7) and (8) by setting $Y_D^D = Y_D$ and $X_A^A = X_A$; Given new pair $(\lambda, \rho)$, compute $y_i^*$ and $x_i^*$ via (7), (8);  
22: If $x_i^* \geq 0$, $y_i^* \geq 0$, exit;
23: $K_D = K_D - 1$
24: End
25: $K_A = K_A - 1$
26: End

Fig. 1. Algorithm to find $K_A$, $K_D$, $\lambda$, $\rho$, $x_i^*$ and $y_i^*$ at the NE

B. Properties of NE

Given the resource limits $X_A$, $Y_D$ and other system parameters, we now know the way that the unique NE is computed. Our subsequent question is how a player disposes resources on heterogeneous targets at the NE.

Lemma 3: The NE $(x^*, y^*)$ satisfies the following properties:

1. $y_i^* \geq y_i^*$ for $1 \leq i < j \leq K_D$;
2. $x_i^* \geq x_i^*$ for $K_D < i < j \leq K_A$;
3. $x_i^* > x_j^*$ if $\frac{g(y_i)}{g(y_j)}$ is strictly increasing w.r.t. $y$.
4. $x_i^* = x_j^*$ if $\frac{g(y_i)}{g(y_j)}$ is a constant, and
5. $x_i^* < x_j^*$ if $\frac{g(y_i)}{g(y_j)}$ is strictly decreasing w.r.t. $y$ for all $1 \leq i < j \leq K_D$.

The first property manifests that $D$ is inclined to allocate more resources to the targets with higher weights at the NE. The second property means that if two targets are not protected by $D$ at the NE, $A$ allocates more resources to the one of higher value. However, it is uncertain whether $A$ allocates more (or less) resources to a high (or lower) value target among the top $K_D$ targets with positive resources of $D$. We next use three examples to highlight that all the possibilities can happen. These examples differ in the choice of (complementary) defence efficiency functions.

We define a new term, “relative ineffectiveness of defence (RID)”, as the expression $\frac{\hat{g}(y)}{g(y)}$. Note that the first-order derivative $\frac{\hat{g}(y)}{g(y)}$ reflects how (i.e. the slope) $\hat{g}(y)$ decreases with the increase of $y$. RID reflects the relative slope that the increase of $y$ reduces $\hat{g}(y)$. If $\frac{\hat{g}(y)}{g(y)}$ decreases in $y$, further increasing $y$ makes $\hat{g}(y)$ decreases faster and faster. On the contrary, if $\frac{\hat{g}(y)}{g(y)}$ is increasing in $y$, further increasing $y$ only results in a smaller and smaller relative reduction of $\hat{g}(y)$ (considering the sign of $\frac{\hat{g}(y)}{g(y)}$).

We suppose that $A$ and $D$ allocate positive resources to $B_1$ and $B_2$.

Example 1 (InvG): $f(x) = 1 - (1 + x)^{-a}$ and $\hat{g}(y) = \frac{1}{1 + \alpha y}$. The following defence inefficiency equality holds, $\frac{\hat{g}(y)}{g(y)} = \frac{\alpha}{1 + \alpha y}$. Then, we obtain $\frac{w_i}{w_j} = \left(1 + \frac{x_i}{1 + x_j}\right)^{2(1+a)}(\frac{1}{1 + (1 + x_j)^{-a}})$. Due to $w_i > w_j$, it is easy to show $x_i > x_j$ by contradiction.

Example 2 (ExpG): $f(x) = 1 - (1 + x)^{-a}$ and $g(y) = \exp(-\theta y)$. The expression $\frac{\hat{g}(y)}{g(y)}$ is equal to $-\theta$. According to the KKT conditions in Theorem 3 there has $(\frac{1}{1 + x_j})^a(\frac{1}{1 + x_j}) = 1$. The above equation holds only upon $x_i = x_j$.

Example 3 (QuadG): $f(x) = 1 - (1 + x)^{-a}$ and $\hat{g}(y) = (1 - \theta y)^2$. Then, there exists $\frac{\hat{g}(y)}{g(y)} = -2 \frac{\theta}{1 - \theta y}$. Theorem 3 yields $\frac{w_i}{w_j} = \left(1 + \frac{x_i}{1 + x_j}\right)^a(1 - \theta y)^2$. Then, there has $x_i^* < x_j^*$.

Remark 2: For InvG-like $\hat{g}(y)$, RID is strictly increasing. The attacker’s best strategy is to allocate more resources to more important targets. In a word, the attacker and the defender have a “head-on confrontation”. For ExpG-like $\hat{g}(y)$, RID is a constant. The attacker sees a number of equally profitable targets. For QuadG-like $\hat{g}(y)$, RID is a decreasing function. The attacker tries to avoid the targets that are effectively protected by the defender.

Intuitively, when a player does not possess sufficient resources, he will gain a higher utility if his resource limit increases. This is true in a variety of cases. Suppose that not all the targets are attacked by $A$. When $X_A$ increases, $A$ can at least gain more profits by allocating the extra resources to the targets that are not under attack. We
next present a counter-intuitive example. Suppose that \( A \) and \( D \) allocate positive amount of resources to all the targets at the NE. The resources of \( A \) are insufficient while those of \( Y_D \) are sufficient, that is, \( \lambda > 0 \) and \( \rho = 0 \). When \( X_A \) increases, it is easy to show by contradiction that \( \lambda \) decreases and \( x_i \) increases. Due to the equality \(-w_i f(x_i)\tilde{g}(y_i) = \tilde{c}\) in the KKT conditions, \( y_i \) also becomes larger. The utility of the attacker on target \( B_i \) at the NE is given by \( w_i f(x_i)\tilde{g}(y_i) - cx_i = -\tilde{c} \frac{\tilde{g}'(w_i)}{\tilde{g}(w_i)} - cx_i \).

If RID of the defender, \( \tilde{g}'(w_i) \), is a constant or a decreasing function of \( y_i \), the expression \(-\tilde{c} \frac{\tilde{g}'(w_i)}{\tilde{g}(w_i)} \) is a constant or decreases as \( y_i \) increases. Hence, the utility of the attacker on target \( B_i \) decreases when \( X_A \) increases.

**Remark 3:** When the defender’s resources are insufficient, the attacker gains more profits by acquiring more resources and allocating them to more important targets. When the defender’s resources are sufficient, the attacker may explore new targets to attack, other than using all the resources to battle with the resource sufficient defender at the NE.

**C. Visualizing Whether a Target Is Attacked or Protected**

From Theorem 3, one can see that \( x^*_i \) and \( y^*_i \) may take on 0, which occurs when \( A \) or \( D \) decides not to attack or defend target \( B_i \). We next show the regions of \( \lambda \) and \( \rho \) upon which \( x^*_i \) or \( y^*_i \) hits 0. There are four possibilities, i) \( x^*_i = 0 \) and \( y^*_i = 0 \); ii) \( x^*_i > 0 \) and \( y^*_i = 0 \); iii) \( x^*_i = 0 \) and \( y^*_i > 0 \); and iv) \( x^*_i > 0 \) and \( y^*_i > 0 \). We denote \( R_{++} = \{ \lambda \geq 0; \rho \geq 0 \} \).

**Case (i):** Eqs. 3 and 4 yield the region \( R_1(\lambda, \rho) = \{ \lambda \geq \max(w_i f'(0)\tilde{g}(0) - c, 0); \rho \geq \max(-w_i f(0)\tilde{g}'(0) - \tilde{c}, 0) \} \).

**Case (ii):** Given \( x^*_i = h_A(\frac{\lambda + \rho}{w_i f(0)\tilde{g}(0)}) \), we obtain the region \( R_2(\lambda, \rho) \) wherein \( A \) attacks but \( D \) gives up target \( B_i \): \( R_2(\lambda, \rho) = \{ 0 \leq \lambda, w_i f'(0)\tilde{g}(0) - c; \rho > \max(-w_i f(h_A(\frac{\lambda + \rho}{w_i f(0)\tilde{g}(0)})\tilde{g}(0) - \tilde{c}, 0)) \} \).

**Case (iii):** Substituting \( y^*_i \) by \( h_D(-w_i f(0)\tilde{g}(0)) \) in Eq. 8, we obtain the region \( R_3(\lambda, \rho) \) wherein \( A \) gives up while \( D \) defends target \( B_i \): \( R_3(\lambda, \rho) = \{ 0 \leq \rho < -w_i f(0)\tilde{g}(0) - \tilde{c}; \lambda > \max(w_i f'(0)\tilde{g}(h_D(-\frac{\rho + \tilde{c}}{w_i f(0)}), 0)) \} \).

Due to \( f(0) = 0 \), \( \rho \) does not possess a valid value, so the region \( R_3 \) is empty.

**Case (iv):** We have the region \( R_4 = R_{++} \setminus \{ R_1 \cup R_2 \cup R_3 \} \).

For any \( i = 1, \ldots, N, x^*_i \) and \( y^*_i \) contain two unknown variables \( \lambda \) and \( \rho \). Hence, in case iv), we can rewrite \( x^*_i \) and \( y^*_i \) by \( x^*_i(\lambda, \rho) \) and \( y^*_i(\lambda, \rho) \).

**Remark 4:** The physical meanings of \( R_1 \) to \( R_4 \) are as follows: i) if \( (\lambda, \rho) \in R_1 \), both \( A \) and \( D \) do not allocate resources to this target; ii) if \( (\lambda, \rho) \in R_2 \), \( A \) attacks this target while \( D \) decides not to defend it; iii) if \( (\lambda, \rho) \in R_3 \), \( A \) does not attack this target while \( D \) defends it; iv) if \( (\lambda, \rho) \in R_4 \), \( A \) attacks this target and \( D \) defends it.

The purpose of defining \( R_1 \) to \( R_4 \) is that we can gain some insights into the system parameters (e.g., \( X_A, Y_D, w_i, c, \tilde{c} \)) on the NE without directly solving the NE. Here, for any pair \((\lambda, \rho)\), the increase of \( \lambda \) means the decrease of \( X_A \), and the increase of \( \rho \) means the decrease of \( Y_D \). This property is derived in the proof of Lemma 2.

Let us illustrate \( R_1 \sim R_4 \) by using a simple example.

**Example 4:** \( f(x) = 1 - \exp(-x); \ \tilde{g}(y) = \exp(-y); w_i = 1 \). It is easy to obtain \( h_A(x) = -\log(x) \) and \( h_D(y) = -\log(-y) \).

Substituting these expressions to \( R_1 \sim R_4 \), we derive the regions of \((\lambda, \rho)\) by \( R_1(\lambda, \rho) = \{ \lambda \geq \max(1 - c, 0), \rho \geq 0 \}; R_2 = \emptyset \) and \( R_2(\lambda, \rho) = \{ 0 \leq \lambda < 1 - c, 0 \leq \rho < 1 - c - \tilde{c} - \lambda \}; R_4 = R_{++} \setminus \{ R_1 \cup R_2 \} \). Fig. 2 shows these regions with parameters \( c \) and \( \tilde{c} \).

## IV. A Linear Intrusion Detection Game for Product-form Model

In this section, we investigate the existence and uniqueness of NE of an intrusion detection game where the attack and defence efficiencies are linear functions.

### A. A Matrix-form Game

We study a matrix-form multi-dimensional intrusion detection game. The payoff matrix on target \( B_i \) is shown in Fig. 3 where \( A \) (resp. \( N.A \)) denotes “attack” (resp. “not attack”) strategy, and \( D \) (resp. \( N.D \)) denotes “defend” (resp. “not defend”) strategy. Here, \( w_i \) denotes the loss of victims for the pair-wise strategies \((A, N.D)\) and \( \gamma w_i \) denotes that for \((A, D)\) with \( \gamma \in (0, 1) \). Let \( c \) and \( \tilde{c} \) be the costs of the “attack” and the “defend” strategies. Note that \( \tilde{c} \) refers to not only the cost of resources, but also the cost of performance such as QoS or false alarm of benign events. We consider the mixed strategies of \( A \) and \( D \) in which \( A \) attacks target \( B_i \) with probability \( x_i \) and \( D \) detects this target with probability \( y_i \). Each player only has one action on all the targets, which yields the resource constraints: \( \sum_{i=1}^{N} x_i \leq X_A \leq 1, \sum_{i=1}^{N} y_i \leq Y_D \leq 1 \) and \( 0 \leq x_i, y_i \leq 1 \).
Lemma 4: Given the attack probabilities \( y_i \) and the detection probabilities \( y_i \), the utilities of \( A \) and \( D \) can be derived easily,

\[
\begin{align*}
U_A &= w_i x_i - (1 - \gamma) w_i x_i y_i - c x_i, \\
U_D &= -w_i x_i + (1 - \gamma) w_i x_i y_i - \tilde{c} y_i.
\end{align*}
\]

The above utility functions fall in the category of our product-form game with \( D \) and \( K \), and the detection probabilities \( y_i \), the utilities of \( A \) and \( D \) can be derived easily,

\[
\begin{align*}
U_A &= w_a x_a - (1 - \gamma) w_a x_a y_a - c x_a, \\
U_D &= -w_d x_d + (1 - \gamma) w_d x_d y_d - \tilde{c} y_d.
\end{align*}
\]

The existence of a NE is guaranteed by the concavity of the game. Before diving into the solution of the NE, we present a property of the sets of targets that are attacked or defended at the NE.

**Lemma 4**: The sets of targets with positive resources at the NE are given by i) \( \{B_1, \ldots, B_{K_A}\} \) for the attacker and \( \{B_1, \ldots, B_{K_D}\} \) for the defender; ii) either \( K_A = K_D \) or \( K_A = K_D + 1 \).

Lemma 4 is the sufficient condition of the existence of NE. Similar to Lemma 1, \( A \) and \( D \) allocate resources to the subsets of more important targets. The difference lies in that \( A \) may allocate resources to more targets than \( D \) when \( f(\cdot) \) and \( g(\cdot) \) are nonlinear functions, but to at most one more target than \( D \) when \( f(\cdot) \) and \( g(\cdot) \) are linear functions. We proceed to find the NE by considering different regions of \( X_A \) and \( Y_D \) in the following theorem.

**Theorem 4**: The multi-dimensional intrusion detection game admits a NE as below

\[
\begin{align*}
P_D(k) &< Y_D < P_D(k + 1) \text{ and } X_A > P_A(k) \text{ for } 0 < k \leq N - 1. \quad \text{The NE is uniquely determined by} \\
x^*_i &= \begin{cases} \\
X_A - \frac{\sum_{j=1}^{k} \frac{w_i}{w_j}}{\sum_{j=1}^{k} \frac{w_i}{w_j}}, & \forall i \leq k \\
0, & \forall i > k + 1
\end{cases} \\
y^*_i &= \begin{cases} \\
1 - \frac{\sum_{j=1}^{k} \frac{w_i}{w_j}}{\sum_{j=1}^{k} \frac{w_i}{w_j}}, & \forall i \leq k \\
0, & \forall i > k
\end{cases}
\end{align*}
\]

Here, the sum over an empty set is 0 conventionally.
Theorem 5: Lemma for strategy unilaterally. Besides, the total consumed resources for \( D_3 \) can be mapped to an arbitrary point in this domain, in which both players have insufficient resources.

![Diagram](image)

**Remark 5:** We summarize the salient properties of the NEs for linear attacking efficiency and linear uptime as below. 1) The targets with \( x^*_i > 0 \) are equally profitable to \( A \) such that \( A \) has no incentive to change his strategy. 2) \( D \) prefers to allocate more resources to the more valuable targets. As a countermeasure, \( A \) allocates more resources to the targets that are not effectively protected by \( D \). 3) The NE is not unique with some special choices for \( X_A \) and \( Y_D \). If multiple NEs exist for a given pair \((X_A, Y_D)\), they yield the same utility for one player, but different utilities for the other player.

V. Nash Equilibrium for Proportion-form Model

In this section, we analyze the NE strategy of the players on different targets for the proportion-form breaching model.

**Nash Equilibrium and its Properties:**

We define \((x^*, y^*)\) as the NE of the game for the proportion-form model. Here, we relax the constraints to be \( f(x), g(y) \geq 0 \) (unlike \( 0 \leq f(x), g(y) \leq 1 \) in the product-form model). The attacking efficiency in the proportion-form model cannot exceed 1. Based on the KKT conditions, \((x^*, y^*)\) is given by the following theorem.

**Theorem 5:** There exist non-negative variables \( \lambda \) and \( \rho \) such that

\[
\frac{x_i \cdot f(x_i^*) g'(y_i^*)}{(x_i^*) + g(y_i^*))^2} - \hat{c} < 0 \quad \text{if} \quad y_i^* > 0, \quad \text{and} \quad (18)
\]

\[
\frac{f'(x_i^*) g(y_i^*)}{(x_i^*) + g(y_i^*))} - \hat{c} < 0 \quad \text{if} \quad x_i^* > 0, \quad \text{and} \quad (19)
\]

with the slackness conditions in Eq.5 and 6.

As the first step to find the NE, we need to investigate how many targets will be attacked by \( A \) and defended by \( D \). The following lemma shows that both \( A \) and \( D \) allocate resources to all the targets in \( E \).

**Lemma 5:** At the NE, there have \( x_i^* > 0 \) and \( y_i^* > 0 \) for all \( i = 1, \cdots, K \) if \( f(\cdot) \) and \( g(\cdot) \) are concave and strictly increasing with \( f(0) = 0 \) and \( g(0) = 0 \).

Lemma 5 simplifies the complexity to obtain the NE strategy because we do not need to test whether a target will be attacked or defended. Then, the equalities in Eqs.(18) and (19) hold. Similarly, we partition \((X_A, Y_D)\) into four domains to find the NE: \( D_1 \): \( X_A \geq X_A^{su} \) and \( Y_D \geq Y_D^{su} \); \( D_2 \): \( X_A < X_A^{su} \) and \( Y_D \geq Y_D^{su} \); \( D_3 \): \( X_A \geq X_A^{su} \) and \( Y_D < Y_D^{su} \); \( D_4 \) none of the above. The method to find the NE contains the similar steps as those of the algorithm in Fig.I. We need to check whether \((X_A, Y_D)\) is located in a domain from \( D_1 \) to \( D_4 \) one by one.

We next study how \( A \) and \( D \) allocate resources to different targets, given the resource limits \( X_A \) and \( Y_D \). The NE strategy satisfies the following properties.

**Remark 6:** In comparison to the product-form breaching model, the players in the proportion-form breaching model always allocate more resources to the more valuable targets. For the generalized proportion-form breaching model, it is usually difficult to analyze how the NE and the utilities at the NE are influenced by the resource limits. Therefore, we consider two specific examples with explicit functions \( f(\cdot) \) and \( g(\cdot) \).

**Example 5:** Let \( f(x) = x^a \) and \( g(y) = y^a \) in the breaching probability model with \( 0 < a \leq 1 \). Then, for the four cases w.r.t. the sufficiency of \( X_A \) and \( Y_D \), there have:

\( D_1 \): \( X_A \geq X_A^{su} = \frac{\lambda \sum_i w_i a(x_i^*)}{1 + (\lambda a(x_i^*))} \) and \( Y_D \geq Y_D^{su} = \frac{\lambda \sum_i w_i a(y_i^*)}{1 + (\lambda a(y_i^*))} \).

The increase of \( X_A \) or \( Y_D \) does not influence the NE and the utilities of \( A \) and \( D \). \( D_2 \): \( X_A < X_A^{su} \) and \( Y_D \geq Y_D^{su} = \frac{\lambda \sum_i w_i a(y_i^*)}{1 + (\lambda a(y_i^*))} \).

Due to \( 0 < a \leq 1 \), \( \lambda \) is a strictly decreasing function of \( X_A \). As \( X_A \) grows, \( x_i^* \) increases accordingly. Then, the utilities of \( D \) and \( A \) are given by

\[
U_D = -\sum_{i=1}^{N} \left( \frac{w_i c^a}{(c+\lambda)^{a}+c^a} - \hat{c} y_i^* \right)
\]

\[
U_A = \sum_{i=1}^{N} \left( \frac{w_i c^a}{(c+\lambda)^{a}+c^a} - \frac{w_i a c (c+\lambda) (a-1) c^a}{(c+\lambda)^{a}+c^a} \right)
\]

It is obvious to see that \( U_D \) is a decreasing function of the attacker’s resource \( X_A \). We take the first-order derivative of \( U_A \) over \( \lambda \). However, \( U_A \) does not necessarily increase when \( X_A \) grows. Let us take a look at a special situation with \( a=1 \). We then take the first order derivative of \( U_A \) over \( \lambda \) and obtain

\[
\frac{dU_A}{d\lambda} = \sum_{i=1}^{N} \frac{w_i c(\lambda - \hat{c})}{(c+\lambda + \hat{c})^3}
\]

When \( c < \lambda + \hat{c} \), \( U_A \) is a decreasing function of \( \lambda \), and hence an increasing function of \( X_A \). Otherwise, \( U_A \) decreases as \( X_A \) increases. This implies that \( A \) always benefits from obtaining more resources if his cost is smaller than that of \( D \). When \( A \)’s cost is larger than \( D \)’s, more resources may lead to a reduced utility of \( A \). \( D_3 \): \( X_A > X_A^{su} \) and \( Y_D \geq Y_D^{su} \). This case is symmetric to that of \( D_2 \), which
is not analyzed here. \(D_4\): both \(X_A\) and \(Y_D\) are insufficient. In this domain, all the resources of \(A\) and \(D\) are utilized. Then, there have \(x^*_i = \frac{w_i}{\sum_{j=1}^{N} w_j} X_A\) and \(y^*_i = \frac{w_i}{\sum_{j=1}^{N} w_j} Y_D\). The utilities of \(A\) and \(D\) are given by

\[
U_D = - \sum_{i=1}^{N} \frac{w_i (X_A)^a}{(X_A)^a + (Y_D)^a} - cY_D;
\]

\[
U_A = \sum_{i=1}^{N} \frac{w_i (X_A)^a}{(X_A)^a + (Y_D)^a} - cX_A
\]

(21)

When \(X_A\) increases, \(U_D\) decreases accordingly. However, increasing \(X_A\) does not necessarily bring a higher utility to \(A\). Similarly, increasing \(Y_D\) yields a worse utility to \(A\), but not necessarily resulting a higher utility to \(D\).

VI. Related Work

Today’s network attacks have evolved into online crimes such as phishing and mobile malware attacks. The attackers are profit-driven by stealing private information or even the money of victims. Authors in [3] measured the uptime of malicious websites in phishing attacks to quantify the loss of victims. Sheng et al. provided the interviews of experts in [18] to combat the phishing. A number of studies proposed improved algorithms to filter the spams containing links to malicious websites in [19], [20]. In mobile platforms, users usually publish root exploits that can be leveraged by malicious attackers. Authors in [11] proposed a new cloud-based mobile botnets to exploit push notification services as a means of command dissemination. They developed a stress test system to evaluate the effectiveness of the defense mechanisms for Android platform in [2]. Felt et al. surveyed the behavior of current mobile malware and evaluated the effectiveness of existing defense mechanism in [3].

Game theoretic studies of network security provide the fundamental understandings of the decision making of attackers and defenders. Authors in [17] used stochastic game to study the intrusion detection of networks. More related works on the network security game with incomplete information and stochastic environment can be found in [6], [13]. Another string of works studied the security investment of nodes whose security level depended on the his security adoption and that of other nodes connected to him. Some models did not consider the network topology [9] and some others studied either fixed graph topologies [14] or the Poisson random graph [10], [15].

Among the studies of network security game, [16], [4], [17] are closely related to our work. In [16], authors used the standard Colonel Blotto game to study the resource allocation for phishing attacks. An attacker wins a malicious website if he allocates more resources than the defender, and loses otherwise. This may oversimplify the competition between an attacker and a defender. Our work differs in that the attackers perform attacks on multiple non-identical banks or e-commerce companies, and the competition is modeled as a non-zero sum game that yields a pure strategy.

In [4], the authors formulated a linearized model for deciding the attack and monitoring probabilities on multiple servers in network intrusion attacks. Altman et al. in [17] studied a different type of multi-battlefield competition in wireless jamming attack that provides important insights of power allocation on OFDM channels.

VII. Conclusion

In this work, we formulate a generalized game framework to capture the conflict on multiple targets between a defender and an attacker that are resource constrained. A product-form and a proportion-form security breaching models are considered. We prove the existence of a unique NE and propose efficient algorithms to search this NE when the game is strictly concave. Our analysis provides important insights in the practice of network attack and defence. For the product-form breaching model, i) the defender always allocates more resources to the more important target, while the attacker may not follow this rule; ii) when the defender has sufficient amount of resources, more resources of the attacker might not bring a better utility to him; iii) when the game is not strictly concave, there may exist multiple NEs that yield different utilities of the players. For the proportion-form breaching model, iv) both the attacker and the defender allocate more resources to more important targets; v) a resource insufficient player causes a reduction of his opponent’s utility, while not necessarily gaining a better utility by himself when his resource limit increases.

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**Supplement: Proofs of Lemmas and Theorems**

**Yuedong Xu, John C.S. Lui**

**Proof of Theorem 2**

**Proof:** We prove the existence and uniqueness of the NE for the product-form and the proportion-form breaching models separately.

**Product-form:** The second-order derivatives of \( U_A \) over \( x \) can be expressed as

\[
\frac{\partial^2 U_A}{\partial x_i^2} = w_i f''(x_i) \hat{g}(y_i) < 0, \quad \text{and} \quad \frac{\partial^2 U_A}{\partial x_i \partial x_j} = 0, \quad \forall i, j.
\]

The second-order derivatives of \( U_D \) over \( y \) are given by

\[
\frac{\partial^2 U_D}{\partial y_i^2} = -w_i f(x_i) \hat{g}''(y_i) < 0, \quad \text{and} \quad \frac{\partial^2 U_D}{\partial y_i \partial y_j} = 0, \quad \forall i, j.
\]

Since \( f(x_i) \) is strictly concave w.r.t. \( x_i, U_A \) is a concave function of the strategy profile \( \{x_i, i = 1, \ldots, N\} \). Based on Rosen’s theorem, there always exists a NE in the game \( G \).

The matrix \( M \) on a target is given by

\[
M = w_i \begin{bmatrix} \varphi_1 f''(x) \hat{g}(y) & \varphi_1 f'(x) \hat{g}'(y) \\ -\varphi_2 f'(x) \hat{g}'(y) & -\varphi_2 f(x) \hat{g}''(y) \end{bmatrix}. \tag{22}
\]

Then, there has \( M+M^T = 2w_i \begin{bmatrix} 2\varphi_1 f''(x) \hat{g}(y) & (\varphi_1-\varphi_2) f'(x) \hat{g}'(y) \\ (\varphi_1-\varphi_2) f'(x) \hat{g}'(y) & -2\varphi_2 f(x) \hat{g}''(y) \end{bmatrix} \). \tag{23}

Suppose \( \varphi_1 = \varphi_2 > 0 \). Because \( f''(x) > 0 \) and \( \hat{g}''(y) > 0 \), then matrix \( -(M+M^T) \) is positive definite. Hence, \( M+M^T \) is negative definite, resulting in the unique NE in the game \( G \).

**Proportion-form:** The second-order derivatives of \( U_A \) over \( x \) can be expressed as

\[
\frac{\partial^2 U_A}{\partial x_i^2} = w_i \hat{g}(y_i) f''(x_i) (f(x_i) + g(y_i)) - 2(f'(x_i))^2 < 0
\]

and \( \frac{\partial^2 U_A}{\partial x_i \partial x_j} = 0, \quad \forall i, j \), due to \( f''(x_i) < 0 \). The second-order derivatives of \( U_D \) over \( y \) can be expressed as

\[
\frac{\partial^2 U_D}{\partial y_i^2} = w_i f(x_i) g''(y_i) (f(x_i) + g(y_i)) - 2(g'(y_i))^2 < 0
\]

and \( \frac{\partial^2 U_D}{\partial y_i \partial y_j} = 0, \quad \forall i, j \), due to \( g''(y_i) < 0 \). Hence, \( G \) is a concave game that admits a NE.

The matrix \( M \) on a target is given by

\[
M = \begin{bmatrix} \varphi_1 \frac{\partial^2 U_A}{\partial x_i \partial x_j} & \varphi_1 \frac{\partial^2 U_A}{\partial x_i \partial y_j} \\ \varphi_2 \frac{\partial^2 U_A}{\partial x_j \partial y_i} & \varphi_2 \frac{\partial^2 U_A}{\partial y_j^2} \end{bmatrix}. \tag{24}
\]

Because of \( \frac{\partial^2 U_A}{\partial x_i \partial y_j} = -\frac{\partial^2 U_A}{\partial x_j \partial y_i} \), if we let \( \varphi_1 = \varphi_2 > 0 \), the expression \( M+M^T \) is obtained by

\[
M+M^T = \begin{bmatrix} 2\varphi_1 \frac{\partial^2 U_A}{\partial x_i^2} & 0 \\ 0 & 2\varphi_2 \frac{\partial^2 U_A}{\partial y_i^2} \end{bmatrix}.
\tag{25}
\]

It is obvious to see that \( M+M^T \) is negative definite. Hence, in the proportion-form breaching model, there exists a unique NE.

**Proof of Theorem 3**

**Proof:** Recall that \( U_A \) is concave in \( x \) and \( U_D \) is concave in \( y \). Then, the best responses of \( A \) and \( D \) are the solutions to two convex optimization problems. Let \( \lambda \) and \( \rho \) be Lagrange multipliers of \( A \) and \( D \) respectively. Let \( L_D(y, \rho) \) be the Lagrangian function of the defender \( D \). We have

\[
L_D(y, \rho) = -\sum_{i=1}^{N} w_i f(x_i) \hat{g}(y_i) - \hat{c} \sum_{i=1}^{N} y_i + \rho (Y_D - \sum_{i=1}^{N} y_i). \tag{26}
\]

Our first step is to find the optimal \( y_j \) as a function of \( \rho \). Taking the derivative over \( y_j \), we obtain

\[
dL_D(y_j, \rho) = -w_i f(x_i) \hat{g}'(y_i) - (\hat{c} + \rho), \quad \forall i=1, \ldots, N. \tag{27}
\]

The optimal resource allocated to target \( i \), or \( y_i \), satisfies the following condition

\[
\rho = -w_i f(x_i) \hat{g}'(y_i) - \hat{c} \tag{28}
\]

when \( y_i \) is greater than zero. If \( y_i = 0 \) and \( \frac{dL_D(y, \rho)}{dy_i} > 0 \), we have:

\[
\rho > -w_i f(x_i) \hat{g}'(y_i) - \hat{c}. \tag{29}
\]

When \( \hat{c} < 0 \), the left hand of Equation 28 is positive for any \( x_i > 0 \). Hence, \( \rho \) is always positive if there is at least one target with \( x_i > 0 \) at the NE. This means that \( D \) will consume all the resources \( Y_D \). When \( \hat{c} > 0 \), the Karush-Kuhn-Tucker (KKT) conditions give rise to

\[
\begin{cases}
\rho \geq 0 & \text{if } \sum_{i=1}^{N} y_i = Y_D, \\
\rho = 0 & \text{if } \sum_{i=1}^{N} y_i < Y_D.
\end{cases} \tag{30}
\]

Following the same approach, we define the Lagrangian
function of $A$ as
\[ L_A(x, \lambda) = \sum_{i=1}^{N} w_i f(x_i) \tilde{g}(y_i) - c \sum_{i=1}^{N} x_i + \lambda(X_A - \sum_{i=1}^{N} x_i). \] (31)

The first-order derivatives are given by
\[ \frac{dL_A(x, \nu)}{dx_i} = w_i f'(x_i) \tilde{g}(y_i) - (c + \lambda), \quad \forall i = 1, \ldots, N. \] (32)
If $x_i^*$ is non-zero, the above derivative equals to 0. Otherwise, $L_A(x, \lambda)$ is a strictly decreasing function of $x_i$ such that $x_i^* := 0$. The Lagrange multiplier $\lambda$ also satisfies the slackness condition.

**Proof of Lemma 7**

**Proof:** Consider two targets $i, j$ with $w_i > w_j$. We assume that $x_i^* = 0$ and $x_j^* > 0$ at the NE. The utility received by $A$ is better if it shifts some $x_j^*$ to the $i$th target. This contradicts the assumption that the game is at the NE. Hence, $A$ only attacks $K_A$ targets with the descending order of their weights.

We next assume $y_i^* = 0$ and $y_j^* > 0$. There exists an inequality $-\tilde{g}(y_j^*) < -\tilde{g}(0)$. According to Theorem 3 we have $-w_j f(x_j^*) \tilde{g}(y_j^*) = -w_j f(x_j^*) \tilde{g}(y_j^*)$. Then, we can conclude $x_j^* > x_i^*$ such that $f'(x_j^*) < f'(x_i^*)$. The KKT condition in Equation (4) shows $w_i f(x_i^*) \tilde{g}(y_i^*) > w_i f'(x_i^*) \tilde{g}(y_i^*)$. Because $f'(x_j^*) < f'(x_i^*)$, $w_i > w_j$ and $\tilde{g}(y_i^*) < \tilde{g}(0)$, the inequality does not hold. Hence, $D$ attacks $K_D$ with the descending order of the weights.

For the claim $K_A \geq K_D$, this can be inferred from our preceding analysis since $D$ will not allocate resources to an target without being attacked by $A$ when $f(0) = 0$.

**Proof of Lemma 2**

**Proof:** To search $(\lambda, \rho)$, we need to consider three different cases step by step: 1) $\lambda > 0$ and $\rho = 0$, 2) $\lambda = 0$ and $\lambda > 0$, and 3) $\lambda > 0$ and $\rho > 0$. Here, the change of $\lambda$ and $\rho$ does not alter $K_A$ and $K_D$ at the NE. Recall that $h_A(\cdot)$ is a decreasing function and $h_D(\cdot)$ is an increasing function. For simplicity, we let $\uparrow$ denote "increase" and let $\downarrow$ denote "decrease". The symbol $\Rightarrow$ denotes "give rise to".

**Step 1:** $\lambda > 0$ and $\rho = 0$. When $\lambda \uparrow$, $x_i^* \downarrow$ for $K_D < i \leq K_A$. For $1 \leq i \leq K_D$, there are two possibilities, $x_i^* \uparrow$ or $x_i^* \downarrow$. In what follows, we will show that $x_i^*$ is strictly decreasing.

We assume that $x_i^* \uparrow$ as $\lambda \uparrow$. According to Eqs. (7) and (8), we have the following relationships for all $1 \leq i \leq K_D$:

\[ \begin{align*}
\lambda \uparrow &\Rightarrow x_i^* \uparrow \Rightarrow f(x_i^*) \uparrow \Rightarrow \frac{-c}{w_i f(x_i^*)} \uparrow \Rightarrow h_D\left(\frac{-c}{w_i f(x_i^*)}\right) \uparrow \\
&\Rightarrow y_i^* \uparrow \Rightarrow \tilde{g}(y_i^*) \downarrow \Rightarrow \frac{c + \lambda}{w_i \tilde{g}(y_i^*)} \uparrow \Rightarrow h_A\left(\frac{c + \lambda}{w_i \tilde{g}(y_i^*)}\right) \downarrow \Rightarrow x_i^* \downarrow
\end{align*} \]

which causes a self contradiction. Therefore, as $\lambda$ increases, $x_i^*$ cannot increases. It is easy to validate that $x_i^*$ cannot remain the same. Thus, $x_i^*$ is a strictly decreasing function of $\lambda$. According to the slackness condition, there has $\sum_{i=1}^{K_A} x_i^* = X_A$. If there exists a feasible $\lambda$ to satisfy this equality, $\lambda$ should be unique. A bisection algorithm can find the solution.

**Step 2:** $\lambda = 0$ and $\rho > 0$. We assume that $y_i^* \uparrow$ when $\rho \uparrow$. Then, the following relationship holds:

\[ \begin{align*}
\rho \uparrow &\Rightarrow y_i^* \uparrow \Rightarrow \tilde{g}(y_i^*) \downarrow \Rightarrow \frac{c}{w_i \tilde{g}(y_i^*)} \uparrow \Rightarrow h_A\left(\frac{c}{w_i \tilde{g}(y_i^*)}\right) \downarrow \\
&\Rightarrow x_i^* \downarrow \Rightarrow f(x_i^*) \downarrow \Rightarrow \frac{-(c + \rho)}{w_i f(x_i^*)} \downarrow \Rightarrow h_D\left(\frac{-(c + \rho)}{w_i f(x_i^*)}\right) \downarrow \Rightarrow y_i^* \downarrow
\end{align*} \]

which contradicts to the assumption. Similarly, we can show that $y_i^*$ cannot remain unchanged. Therefore, when $\rho$ increases, $y_i^*$ is strictly decreasing for all $1 \leq i \leq K_D$. The slackness condition gives rise to $\sum_{i=1}^{K_D} y_i^* = Y_D$. Then, we can use the bisection algorithm to find $\rho$ if it exists.

**Step 3:** $\lambda > 0$ and $\rho > 0$. We consider two cases: $K_A = K_D$ and $K_A > K_D$.

Recall that the implicit function $\rho_1(\lambda)$ is obtained from Eqs. (5)(7)(8) and the implicit function $\rho_2(\lambda)$ is obtained from Eqs. (6)(7)(8).

**Step 3.1** $K_A = K_D$. When $\lambda$ increases, there are two cases due to the constraint $\sum_{i=1}^{K_A} x_i^* = X_A$. One is that $x_i^*$ does not change for all $1 \leq i \leq K_D$. The other is that there exist two targets $B_i$ and $B_j$ ($1 \leq i, j \leq K_D$) in which $x_i^*$ increases and $x_j^*$ decreases.

If $x_i^*$ does not change for all $1 \leq i \leq K_D$, the following relationships hold

\[ \lambda \uparrow \Rightarrow \tilde{g}(y_i^*) \uparrow \Rightarrow y_i^* \downarrow, \quad \forall 1 \leq i \leq K_D. \]

Because of $\rho > 0$, there must have $\sum_{i=1}^{K_D} y_i^* = Y_D$, which contradicts to the conclusion $y_i^*$ decreases for all $1 \leq i \leq K_D$. Therefore, the case that $x_i^*$ ($1 \leq i \leq K_D$) does not change is not true.

We next turn to the second case that $x_i^*$ increases and $x_j^*$ decreases when $\lambda$ increases. The following relationships hold

\[ \lambda \uparrow \Rightarrow x_i^* \uparrow \Rightarrow f(x_i^*) \uparrow \Rightarrow -\frac{1}{w_i f(x_i^*)} \uparrow. \]

If $\rho$ increases or remains the same, we continue the induction by

\[ \begin{align*}
\frac{-1}{w_i f(x_i^*)} \uparrow &\Rightarrow \frac{-(c + \rho)}{w_i f(x_i^*)} \uparrow \Rightarrow h_D\left(\frac{-(c + \rho)}{w_i f(x_i^*)}\right) \uparrow \Rightarrow y_i^* \uparrow \\
&\Rightarrow \tilde{g}(y_i^*) \uparrow \Rightarrow \frac{c + \lambda}{w_i \tilde{g}(y_i^*)} \uparrow \Rightarrow h_A\left(\frac{c + \lambda}{w_i \tilde{g}(y_i^*)}\right) \downarrow \Rightarrow x_i^* \downarrow.
\end{align*} \]

The condition $x_i^* \uparrow$ contradicts to the conclusion $x_i^* \downarrow$. Therefore, $\rho$ must decreases when $\lambda$ increases. In a word, $\rho_1(\lambda)$ is a strictly decreasing function.

According to the slackness condition in Eq. (6), there has $\sum_{i=1}^{K_A} y_i^* = Y_D$. When $\lambda$ increases, there are also two cases w.r.t. $y_i^*$. One is that $y_i^*$ does not change for $1 \leq i \leq K_D$. The other is that there exist two targets $B_i$ and $B_j$ ($1 \leq i, j \leq K_D$) in which $y_i^*$ increases and $y_j^*$ decreases.

If $y_i^*$ does not change for $1 \leq i \leq K_D$, the following...
relationships hold
\[ \lambda \mapsto \frac{\lambda + c}{w_i g(y_i')} \mapsto h_A(\frac{\lambda + c}{w_i g(y_i')}) \mapsto x_i' \mapsto -\frac{1}{w_i f(x_i')} \downarrow. \]
Because \( y_i' \) does not change, \( \rho \) must increase.
For the second case, when \( y_i' \) increases, we obtain the following relationships
\[ \lambda \mapsto \hat{g}(y_i') \downarrow \mapsto h_D(\frac{\lambda + c}{w_i g(y_i')}) \downarrow \mapsto x_i' \downarrow \mapsto f(x_i') \mapsto -\frac{1}{w_i f(x_i')} \downarrow. \]
If \( \rho \) decreases or remains the same, there must have
\[ \frac{-1}{w_i f(x_i')} \downarrow \mapsto \frac{-(\hat{c} + \rho)}{w_i f(x_i')} \downarrow \mapsto h_D(\frac{-(\hat{c} + \rho)}{w_i f(x_i')}) \downarrow \mapsto y_i' \downarrow, \]
which contradicts to the condition \( y_i' \uparrow \). Hence, \( \rho \) must increase in this case. As a consequence, the implicit function \( \rho_2(\lambda) \) is a strictly increasing function.

Step 3.2. \( K_A > K_D \). The slackness condition in Eq. (5) is expressed as
\[ X_A = \sum_{i=1}^{K_D} h_A\left(\frac{c + \lambda}{w_i g(y_i'(\lambda, \rho))}\right) + \sum_{i=K_D+1}^{K_A} h_A\left(\frac{c + \lambda}{w_i g(0)}\right). \tag{33} \]
When \( \lambda \) increases, the expression \( h_A\left(\frac{c + \lambda}{w_i g(y_i'(\lambda, \rho))}\right) \) is strictly decreasing for \( K_D + 1 \leq i \leq K_A \). This implies that \( x_i' \) decreases for \( K_D + 1 \leq i \leq K_A \). Due to the constraint \( \sum_{i=1}^{K_A} x_i' = X_A \), \( x_i' \) increases in at least one target \( B_i \) for \( 1 \leq i \leq K_D \). In other word, the case that \( x_i' \) does not change with the increase of \( \lambda \) does not happen. Then, following the analysis in the Step 3.1, we can see that \( \rho_1(\lambda) \) is a strictly decreasing function and \( \rho_2(\lambda) \) is a strictly increasing function.

This concludes the proof.

Proof of Lemma 2

Proof: We prove this lemma by contradiction. When the both players allocate resource to targets \( B_i \) and \( B_j \) at the NE, there exists
\[ \frac{w_i f(x_i')}{w_j f(x_j')} \hat{g}(y_i') = \frac{w_i f(x_i') \hat{g}(y_i')}{w_j f(x_j') \hat{g}(y_j')} = 1. \tag{34} \]
If \( y_i' < y_j' \), the following inequality holds
\[ \hat{g}(y_i') > \hat{g}(y_j') < 0 \]
because \( g(\cdot) \) is strictly convex. The above inequality yields
\[ \frac{\hat{g}(y_i')}{\hat{g}(y_j')} > 1. \]
Combined with Eq. (34), we obtain \( \frac{f(x_i')}{f(x_j')} < 1. \)
Since \( f(\cdot) \) is strictly increasing and strictly concave, there have \( x_i' < x_j' \) and \( 0 < f'(x_i') < f'(x_j') \). Then, we can conclude
\[ \frac{w_i f'(x_i')}{w_j f'(x_j')} \hat{g}(y_i') > 1, \tag{35} \]
which contradicts to Eq. (34). Therefore, if \( D \) allocates resource to targets \( B_i \) and \( B_j \) (\( i < j \)), at the NE, there must have \( y_i' > y_j' \).

Eq. (34) can be rewritten as
\[ \frac{f(x_i') \hat{g}(y_i')}{f'(x_i') \hat{g}(y_i')} = \frac{f(x_j') \hat{g}(y_j')}{f'(x_j') \hat{g}(y_j')} \tag{36} \]
When \( \frac{\hat{g}(y_i')}{\hat{g}(y_j')} \) is a constant, there exists \( f(x_i') = f(x_j') \).
If \( x_i' > x_j' \), there have \( f(x_i') > f(x_j') \) and \( 0 \leq f'(x_i') < f'(x_j') \). This gives rise to the inequality \( \frac{f(x_i')}{f(x_j')} > \frac{f'(x_i')}{f'(x_j')} \), which contradicts to the above equality. It is also easy to show that the relationship \( x_i' < x_j' \) also contradicts to the above equality. Hence, we obtain \( x_i' = x_j' \). We next suppose that \( \frac{\hat{g}(y_i')}{\hat{g}(y_j')} \) is a strictly increasing function of \( y \). Given \( y_i' > y_j' \) for \( 1 \leq i < j \leq K_D \), we obtain \( \frac{\hat{g}(x_i')}{\hat{g}(x_j')} < \frac{\hat{g}(x_i')}{\hat{g}(x_j')} \). Then, eq. (36) yields \( \frac{f(x_i')}{f(x_j')} > \frac{f(x_i')}{f(x_j')} \), or equivalently \( x_i' > x_j' \). Similarly, when \( \frac{\hat{g}(y_i')}{\hat{g}(y_j')} \) is a strictly decreasing function of \( y \), there must have \( x_i' < x_j' \).

Proof of Lemma 2

Proof: This lemma is proved by contradiction. We consider even more general functions: \( f(x) = b_1 x \) and \( g(y) = b_2 - b_3 y \). In the intrusion detection game, we let \( b_1 = b_2 = 1 \) and \( b_3 > 0 \).

i). We assume \( x_i' = 0 \) and \( x_j' > 0 \) at the NE for two targets \( B_i \) and \( B_j \) with \( w_i > w_j \). The best response of \( D \) must satisfy \( y_i' = 0 \). Then, the following inequality holds
\[ \frac{dU_A}{dx_i}|_{x_i=0}= w_i b_1 b_2 - c > w_j b_1 b_2 - w_j b_2 b_3 y_j' - c = \frac{dU_A}{dx_j}|_{x_j=x_j'}. \]
\( A \) obtains a higher profit if he transfers the resource on \( B_j \) to \( B_i \). Thus, it is not a NE.

We further assume \( y_i' = 0 \) and \( y_j > 0 \) at the NE for two targets \( B_i \) and \( B_j \) with \( w_i > w_j \). The marginal profits on \( B_i \) and \( B_j \) satisfy
\[ w_i b_1 b_3 x_i' < w_j b_1 b_3 x_j'. \]
The above inequality gives rise to \( x_i' < x_j' \) because of \( w_i > w_j \). When \( y_i' = 0 \) and \( y_j > 0 \), the marginal profits of \( A \) on \( B_i \) and \( B_j \) satisfy
\[ \frac{dU_A}{dx_i}|_{x_i=x_i'} > \frac{dU_A}{dx_j}|_{x_j=x_j'}. \]
Then, \( A \) has a larger utility if he moves the resource on \( B_j \) to \( B_i \). This contradicts to the claim \( x_i' > x_j' \). Thus, it is not a NE.

To sum up, \( A \) allocates resources to the top \( K_A \) targets and \( D \) allocates resources to the top \( K_D \) targets. It is also very intuitive to validate \( K_D < K_A \).

ii). We assume \( K_A > K_D + 1 \) at the NE. Let \( B_i \) and \( B_j \) be two targets for \( K_D < i, j < N \). The marginal profits of \( A \) on \( B_i \) and \( B_j \) satisfy
\[ \frac{dU_A}{dx_i} = w_i b_1 b_2 - c, \quad \frac{dU_A}{dx_j} = w_j b_1 b_2 - c. \]
Because of \( w_i \neq w_j \), \( A \) can obtain a larger utility by aggregating the resources to the more profitable target.
Thus, it is not a NE. To sum up, \( K_A \) and \( K_D \) must satisfy
\[
K_D \leq K_A \leq K_D + 1.
\]
This concludes the proof.

**Proof of Theorem 2**

**Proof**: The proof utilizes the conclusions of lemma 4. According to the properties of the NE, there have
\[
w_i b_1 b_2 - w_i b_3 y_i - c = \lambda \geq 0, \quad \forall \ 1 \leq i \leq K_A; \tag{37}
\]
\[
w_i K_D b_1 b_2 - w_i K_D b_3 y_{K_D} \geq w_i K_D + b_1 b_2 \quad \text{if} \quad K_D < N; \tag{38}
\]
\[
w_i b_3 x_i - \hat{\rho} \geq 0, \quad \forall \ 1 \leq i \leq K_D; \tag{39}
\]
\[
w_i b_3 x_{K_D + 1} - \hat{\rho} \leq \rho, \quad \text{if} \quad K_D < N. \tag{40}
\]
Here, Eq. (37) means that the marginal utilities of \( A \) are non-negative and are the same on the top \( K_A \) targets. Eq. (38) means that the marginal utility of \( A \) on any top \( K_D \) target is larger than that on the top \( K_D \). This guarantees the condition \( K_D \leq K_A \leq K_D + 1 \). Eq. (39) ensures that \( D \) allocates positive resources to the top \( K_D \) targets. Eq. (40) means that \( D \) does not allocate resources to \( B_{K_D + 1} \). The above conditions give rise to the solution to the NE,
\[
x_i^* = \begin{cases} 
\hat{\rho}, & \forall i \leq K_D \\
\frac{\hat{\rho} - \frac{c + \lambda}{w_i b_3}}{1 - \frac{w_i}{b_3}}, & \forall i > K_D
\end{cases} \tag{41}
\]
\[
y_i^* = \begin{cases} 
\frac{b_3}{b_1} - \frac{c + \lambda}{w_i b_3}, & \forall i \leq K_D \\
\frac{b_3}{b_1} - \frac{c + \lambda}{w_i b_3}, & \forall i > K_D
\end{cases} \tag{42}
\]
Before commencing the analysis, we recall the following notations: \( P_A(0) = 0 \), \( P_A(k) := \sum_{i=1}^{k} \frac{\hat{\rho}}{w_i b_3} \), \( \forall \ 1 \leq k \leq N \); \( P_D(1) = 0 \), \( P_D(k) := \sum_{i=1}^{k-1} \frac{b_3}{b_1} (1 - \frac{w_i}{w_k}) \), and \( P_D(N+1) := \frac{b_3}{N} - \sum_{i=1}^{N} \frac{b_3}{b_1}. \)

**Step 1.1** \( K_D \) cannot be less than \( k \)

We assume \( K_D < k \). If \( \lambda = 0 \), \( A \) allocates all of its resources on the top \( K_A \) targets, that is,
\[
X_A = \sum_{i=1}^{K_D} \frac{\hat{\rho} + \lambda}{w_i b_1 b_3} + x_{K_D + 1} \leq \sum_{i=1}^{K_D + 1} \frac{\hat{\rho} + \lambda}{w_i b_1 b_3}. \tag{45}
\]
Because of \( \sum_{i=1}^{K_D} \frac{\hat{\rho} + \lambda}{w_i b_1 b_3} < X_A < \sum_{i=1}^{K_D + 1} \frac{\hat{\rho} + \lambda}{w_i b_1 b_3} \), there has
\[
\sum_{i=1}^{k} \frac{\hat{\rho}}{w_i b_3} < \sum_{i=1}^{K_D + 1} \frac{\hat{\rho} + \lambda}{w_i b_1 b_3}. \tag{46}
\]
Due to the condition \( K_D < k \), the above inequality gives rise to \( \rho > 0 \), which means that the marginal utility of \( D \) is positive. Thus, \( D \) allocates all the resources to the top \( K_D \) targets. According to the expression of NE, the total resources allocated by \( D \) on \( K_D \) targets satisfy
\[
Y_D \leq \sum_{i=1}^{K_D} (1 - \frac{w_i K_D}{w_i}) b_3. \tag{47}
\]
This contradicts to the condition \( Y_D > \sum_{i=1}^{k} (1 - \frac{w_i}{w_i}) b_3 \) when \( K_D < k \).

If \( \lambda = 0 \), the marginal utility on any target \( B_i \) that has no resource of \( D \) is given by \( w_i b_1 b_2 - c > 0 \). \( A \) obtains a larger utility by shifting resources to any unprotected target, which is a feasible NE. Therefore, \( K_D \) cannot be less than \( k \).

**Step 1.2** \( K_D \) cannot be larger than \( k \)

We assume \( K_D > k \). The total amount of resources used by \( A \) at the NE is given by
\[
\sum_{i=1}^{K_D} x_i^* = \sum_{i=1}^{K_D} \frac{\hat{\rho} + \lambda}{w_i b_1 b_3} + x_{K_D + 1} \geq \sum_{i=1}^{K_D} \frac{\hat{\rho}}{w_i b_1 b_3}. \tag{48}
\]
Due to the conditions \( K_D > k \) and \( X_A < \sum_{i=1}^{k} \frac{\hat{\rho}}{w_i b_1 b_3} \), we obtain \( \sum_{i=1}^{K_D + 1} x_i^* > X_A \), which is not true. Hence, \( K_D \) cannot be larger than \( k \).

**Step 1.3** \( K_D \) is equal to \( k \)

In the above analysis, we observe that \( \lambda \) must satisfy
\[
\lambda \geq w_{k+1} b_1 b_2 - c, \tag{49}
\]
given the condition \( k < N \). Otherwise, \( A \) can perform better by moving the resources to the \((k+1)\)th target. Since \( \lambda > 0 \), \( A \) fully utilizes his resources.

We then consider the value of \( \rho \). When \( \rho > 0 \), \( D \) allocates all the resources to the top \( k \) targets. This yields
\[
Y_D = \frac{b_2}{b_3} - \sum_{i=1}^{k} \frac{c + \lambda}{w_i b_1 b_3}. \tag{50}
\]
Submitting (49) to (50), we obtain the condition \( Y_D \leq \sum_{i=1}^{k} (1 - \frac{w_i}{w_k}) b_3 \). This contradicts to the initial condition \( Y_D > \sum_{i=1}^{k} (1 - \frac{w_i}{w_k}) b_3 \). Hence, \( \rho \) cannot be greater than 0. When \( \rho = 0 \), the NE strategies of \( A \) and \( D \) can be easily solved by (44) and (45).

**ii) Next we prove the second claim.**

We assume \( P_D(k) < Y_D < P_D(k+1) \) and \( X_A > P_A(k) \) for \( 1 \leq k \leq N \). The NE is uniquely determined by
\[
x_i^* = \begin{cases} 
\sum_{j=1}^{i} \frac{w_j}{w_k} X_A, & \forall i \leq k \\
0, & \forall i > k
\end{cases} \tag{51}
\]
\[
y_i^* = \begin{cases} 
\sum_{j=1}^{i} \frac{w_j}{w_k} (Y_D - b_3 k) + b_3, & \forall i \leq k \\
0, & \forall i > k
\end{cases} \tag{52}
\]

**Step 2.1** \( K_D \) cannot be less than \( k \)

We assume \( K_D < k \). If \( \lambda > 0 \), we obtain the condition \( \rho > 0 \) following the expression in (43). This means that \( D \) allocates \( Y_D \) resources to the top \( K_D \) targets. Then, there has the following inequality at the NE
\[
Y_D \leq \sum_{i=1}^{K_D} (1 - \frac{w_i K_D}{w_i}) b_3. \tag{52}
\]
Note that the feasible region of $Y_D$ is $Y_D > \sum_{j=1}^{k-1}(1 - \frac{w_j}{w_j})\frac{b_2}{b_3}$. Because of $K_D < k$, there has

$$Y_D > \sum_{j=1}^{k-1}(1 - \frac{w_k}{w_j})\frac{b_2}{b_3} - \sum_{j=1}^{k-1}(1 - \frac{w_j}{w_j})\frac{b_2}{b_3} = \sum_{j=1}^{k-1}(1 - \frac{w_k}{w_j})\frac{b_2}{b_3}, \quad (53)$$

The inequality (52) contradicts to (53), which means that $\lambda$ cannot be greater than 0.

If $\lambda=0$, all the resources of $A$ will be moved to target $B_{K_D+1}$. Then, this is not a NE. Therefore, $K_D$ cannot be less than $k$.

Step 2.2 $K_D$ cannot be larger than $k$

We assume $K_D > k$ with condition on $k < N$. The total amount of resources used by $D$ at the NE satisfy

$$\sum_{i=1}^{K_D} y_i^* = \sum_{i=1}^{K_D} \left( \frac{b_2}{b_3} - \frac{c + \lambda}{w_i b_1 b_3} \right), \quad (54)$$

There must have $\lambda > w_{K_D+1} b_1 b_2 - c$ if $A$ allocates positive resources to target $B_{K_D}$. Considering the additional condition $K_D > k$, Eq. (52) yields

$$\sum_{i=1}^{K_D} y_i^* \geq \sum_{i=1}^{k} \left( \frac{b_2}{b_3} - \frac{c + \lambda}{w_i b_1 b_3} \right), \quad (55)$$

The resource limit of $D$ should satisfy $Y_D \geq \sum_{i=1}^{k} y_i^*$. However, the inequality (55) contradicts to the condition $Y_D < \sum_{j=1}^{k-1}(1 - \frac{w_k}{w_j})\frac{b_2}{b_3}$. Therefore, $K_D$ cannot be larger than $k$.

Step 2.3 $K_D$ is equal to $k$

We consider two scenarios separately, $k < N$ and $k = N$.

If $k < N$, there must have $\lambda \geq w_{k+1} b_1 b_2 - c$ according to Eq. (53). If the equality $\lambda = w_{k+1} b_1 b_2 - c$ holds, the total amount of resources used by $D$ at the NE is given by $\sum_{i=1}^{k}(1 - \frac{w_{k+1}}{w_i})\frac{b_2}{b_3}$. This contradicts to the range of $Y_D$. Hence, there only has $\lambda > w_{k+1} b_1 b_2 - c$, which means that both $A$ and $D$ allocate positive resources to $k$ targets. Since $\lambda > w_{k+1} b_1 b_2 - c$, there exists $X_A = \sum_{i=1}^{k} x_i^* = \sum_{i=1}^{k} \frac{b_2}{b_3} - \frac{c + \lambda}{w_i b_1 b_3}$. Because $X_A > 0$, $\rho$ must be positive. Hence, by letting $X_A = \sum_{i=1}^{k} x_i^*$ and $Y_D = \sum_{i=1}^{k} y_i^*$, we can directly solve the NE as

$$x_i^* = \left( \sum_{j=1}^{k} \frac{w_j}{w_j} \right)^{-1} X_A, \quad \forall \ i \leq k \quad (56)$$

$$y_i^* = \left( \sum_{j=1}^{k-1} \frac{w_j}{w_j} \right)^{-1} (Y_D - \frac{b_2}{b_3} - \frac{c + \lambda}{w_i b_1 b_3}), \quad \forall i \leq k \quad (57)$$

If $k = N$, there has $\lambda \geq 0$. Here, when $\lambda = 0$, the total amount of resources utilized by $D$ at the NE is given by $\sum_{i=1}^{N} y_i^* = \frac{b_2}{b_3} N - \sum_{i=1}^{k} \frac{c}{w_i b_1 b_3}$. Because $Y_D < \frac{b_2}{b_3} N - \sum_{i=1}^{N} \frac{c}{w_i b_1 b_3}$, there has $\sum_{i=1}^{N} y_i^* < Y_D$, which is not true. Hence, $\lambda$ is always greater than 0. It is easy to conclude $\rho > 0$ since $X_A > \sum_{i=1}^{k} \frac{c}{w_i b_1 b_3}$ are fully utilized at the NE. Now we are clear that both $X_A$ and $Y_D$ are disposed on all $N$ targets. The NE can be computed in the same way as that in Eqs. (56) and (57).

iii) We then prove the third claim.

iv) We continue to prove the fourth claim.

v) We finally prove the fifth claim.

vi) We prove the sixth claim.

• $X_A > P_A(N)$ and $Y_D > P_D(N+1)$, the NE is given by

$$x_i^* = \left( \frac{c + \lambda}{w_i b_1 b_3} \right)^{-1} - \frac{b_2}{b_3} \quad (58)$$

To prove this claim, we only need to show that $\lambda$ and $\rho$ are both 0 at the NE. We still prove it by contradiction. If $\lambda > 0$, all the resources of $A$ are allocated to these $N$ targets. Because $X_A$ is larger than $\sum_{i=1}^{N} \frac{c}{w_i b_1 b_3}$, $\rho$ must be positive in the marginal utility functions. As a countermeasure, $D$ allocates all the resources to defend these targets. However, after $D$ allocates all of his resources, the marginal utilities of $A$ become negative due to $Y_D > \frac{b_2}{b_3} N - \sum_{i=1}^{N} \frac{c}{w_i b_1 b_3}$. The best strategy of $A$ is to give up all the targets. Hence, either $\lambda$ or $\rho$ cannot be 0 at the NE. The only possible NE must satisfy $\lambda = \rho = 0$, which leads to the expression of the NE in Eq. (58).
of A on targets from B_1 to B_k are all 0. Then, A cannot improve his utility by individually changing his strategy. At the NE, the marginal utilities of D on targets from B_1 to B_k should be non-negative and identical. Let \( X_A \) be the amount of resources used by A at the NE. There exist \( w_i, x_i^* > 0 \) for all \( i, j \leq k \) and \( \sum_{i=1}^{k} x_i^* = X_A \). Hence, the NE strategy of A is given by \( x_i^* = (\sum_{j=1}^{k} w_j)^{-1} X_A \) for \( i \leq k \) and \( x_i^* = 0 \) for \( i > k \).

This concludes the proof.

**Proof of Theorem 3**

The proof follows that of Theorem 3. Let \( \lambda \) and \( \rho \) be the Lagrange multipliers of A and D respectively. Let \( L_D(y, \rho) \) be the Lagrange function of the defender D that has

\[
L_D(y, \rho) = -\sum_{i=1}^{N} \frac{w_i f(x_i)}{f(x_i) + g(y_i)} - \hat{c} \sum_{i=1}^{N} y_i + \rho (Y_D - \sum_{i=1}^{N} y_i). \tag{65}
\]

We take the derivative of \( L_D(y, \rho) \) over \( y_i \) and obtain

\[
\frac{dL_D(y, \rho)}{dy_i} = \frac{w_i f(x_i) g(y_i)}{(f(x_i) + g(y_i))^2} - (\hat{c} + \rho), \quad \forall i. \tag{66}
\]

Here, \( L_D(y, \rho) \) is optimized in two ways. If the above derivative is 0, there exists a non-zero resource allocation strategy, i.e. \( y_i^* > 0 \). If the above derivative is less than 0, then \( y_i^* = 0 \). Similarly, we can find the conditions for the attacker to maximize his utility. For the sake of redundancy, we omit the detailed proof.

**Proof of Lemma 5**

**Proof:** According to Theorem 3, these exists a unique NE with the proportion-form breaching model. We next show by contradiction that \( x_i \) cannot be 0 on any target \( B_i \) at the NE. Suppose \( x_i^* = 0 \) on target \( B_i \). Then, there has \( f(x_i^*) = 0 \) such that \( y_i^* = 0 \). When target \( B_i \) is not protected by D, the best response of A is to allocate an arbitrarily small amount of resources to this target. Hence, \((0,0)\) is not an equilibrium strategy for A and D. Therefore, A and D allocate positive resources to all the targets at the NE.

**Proof of Lemma 6**

**Proof:** Consider two targets \( B_i \) and \( B_j \) with \( w_i > w_j \). The following equations hold at the NE.

\[
\frac{w_i f'(x_i) g(y_i)}{(f(x_i) + g(y_i))^2} = \frac{f'(x_i) g(y_i)}{f(x_i) + g(y_j)^2} = c + \lambda; \tag{67}
\]

\[
\frac{w_i f(x_i) g(y_i)}{(f(x_i) + g(y_i))^2} = \frac{w_j f(x_j) g'(y_j)}{(f(x_j) + g(y_j))^2} = \hat{c} + \rho. \tag{68}
\]

The above equations yield the following relationship

\[
\frac{f'(x_i) f(x_j)}{f'(x_j) f(x_i)} = \frac{g'(y_i) g(y_j)}{g'(y_j) g(y_i)}. \tag{69}
\]

We prove this lemma by contradiction. Let us assume that there has \( y_i < y_j \). Because \( g(\cdot) \) is a concave and strictly increasing function, we have \( g(y_i) < g(y_j) \) and \( g'(y_i) > g'(y_j) \). The right hand of Eq. (69) is greater than 1. Then, there must have \( x_i < x_j \) in the left hand of Eq. (69).

We define two functions, \( f_1(x, y) \) and \( f_2(x, y) \), where

\[
f_1(x, y) = \frac{f(x) f'(y)}{(f(x) + g(y))^2} \quad \text{and} \quad f_2(x, y) = \frac{f(x) g'(y)}{(f(x) + g(y))^2}. \tag{70}
\]

We take the derivatives of \( f_1(x, y) \) and \( f_2(x, y) \) over \( x \) and \( y \) respectively.

\[
\frac{\partial f_1}{\partial y} = \frac{f(x) f'(y)}{(f(x) + g(y))^2} - \frac{2 f'(x) f(x) g(y)}{(f(x) + g(y))^3}; \tag{71}
\]

\[
\frac{\partial f_2}{\partial y} = \frac{f(x) g'(y)}{(f(x) + g(y))^2} - \frac{2 f'(x) f(x) g'(y)}{(f(x) + g(y))^3}. \tag{72}
\]

The signs of \( \frac{\partial f_1}{\partial y} \) and \( \frac{\partial f_2}{\partial y} \) depend on whether \( f(x) \) is greater than \( g(y) \) or not. Meanwhile, \( f_1(x, y) \) is a decreasing function of \( x \) and \( f_2(x, y) \) is an increasing function of \( y \).

To prove this lemma, we consider two cases, \( f(x_i) > g(y_i) \) and \( f(x_i) < g(y_i) \).

**Case 1:** \( f(x_i) > g(y_i) \). Because there has \( f(x_i) > f(x_j) > g(y_i) \), we obtain

\[
\frac{f(x_i) g'(y_i)}{(f(x_i) + g(y_i))^2} > \frac{f(x_j) g'(y_i)}{(f(x_j) + g(y_i))^2}. \tag{75}
\]

Since \( f_2(x, y) \) is strictly decreasing w.r.t \( y \), there yields

\[
\frac{f(x_i) g'(y_i)}{(f(x_i) + g(y_i))^2} > \frac{f(x_j) g'(y_i)}{(f(x_j) + g(y_j))^2}. \tag{76}
\]

Submitting (76) to (75), we have

\[
\frac{f(x_i) g'(y_i)}{(f(x_i) + g(y_i))^2} > \frac{f(x_j) g'(y_i)}{(f(x_j) + g(y_j))^2} \tag{77}
\]

Given \( w_i > w_j \), the inequality (77) contradicts to Eq. (68).

**Case 2:** \( f(x_i) < g(y_i) \). Because of \( g(y_j) > g(y_i) > f(x_i) \), there has

\[
\frac{f(x_j) f(y_i)}{(f(x_i) + g(y_i))^2} > \frac{f(x_j) g(y_j)}{(f(x_j) + g(y_j))^2} > \frac{f(x_j) g(y_j)}{(f(x_j) + g(y_j))^2}, \tag{78}
\]

which contradicts to Eq. (67).

Therefore, for any two targets \( B_i \) and \( B_j \) with \( w_i > w_j \), there must exist \( x_i > x_j \) and \( y_i > y_j \). This concludes the proof.