REMARKS ON THE HARDY–LITTLEWOOD INEQUALITY FOR 
\emph{m}-HOMOGENEOUS POLYNOMIALS AND \emph{m}-LINEAR FORMS

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Abstract. The Hardy–Littlewood inequality for \( m \)-homogeneous polynomials on \( \ell_p \) spaces is valid for \( p > m \). In this note, among other results, we present an optimal version of this inequality for the case \( p = m \). We also show that the optimal constant, when restricted to the case of 2-homogeneous polynomials on \( \ell_2(\mathbb{R}^2) \) is precisely 2. In an Appendix we justify why, curiously, the optimal exponents of the Hardy–Littlewood inequality do not behave smoothly.

1. Introduction

The Hardy–Littlewood inequality for (complex or real) bilinear forms defined on \( \ell_p \) spaces for \( p > 2 \) dates back to 1934 [12]. This inequality together with the Bochner–Hille inequality [7] and Littlewood’s 4/3 theorem [14] are the cornerstones of the birth of the fruitful theory of multiple summing operators. There are, of course, natural counterparts of the Hardy–Littlewood inequality for \( m \)-homogeneous polynomials and \( m \)-linear forms defined on \( \ell_p \) spaces for \( p > m \) (see [11] and the references therein).

For \( K = \mathbb{R} \) or \( \mathbb{C} \) and \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \), we define \( |\alpha| := \alpha_1 + \cdots + \alpha_n \). By \( x^\alpha \) we shall denote the monomial \( x_1^{\alpha_1} \cdots x_n^{\alpha_n} \) for any \( x = (x_1, \ldots, x_n) \in K^n \). The polynomial Littlewood’s 4/3 theorem asserts that there is a constant \( B_{K,2}^{\text{pol}} \geq 1 \) such that

\[
\left( \sum_{|\alpha|=2} |a_\alpha|^\frac{4}{3} \right)^\frac{3}{4} \leq B_{K,2}^{\text{pol}} \|P\|
\]

for all 2-homogeneous polynomials \( P : \ell_\infty^n \rightarrow K \) given by

\[
P(x_1, ..., x_n) = \sum_{|\alpha|=2} a_\alpha x^\alpha,
\]

and all positive integers \( n \), where \( \|P\| := \sup_{z \in B_{\ell_\infty^n}} |P(z)| \). When we replace \( \ell_\infty^n \) by \( \ell_p^n \) we obtain the polynomial Hardy–Littlewood inequality whose optimal exponents are \( \frac{4p}{3p-4} \) for \( 4 \leq p \leq \infty \) and \( \frac{p}{p-2} \) for \( 2 < p \leq 4 \). In other words, for \( 4 \leq p \leq \infty \) and \( n \geq 1 \), there is a constant \( C_{K,2,p}^{\text{pol}} \geq 1 \) (not depending on \( n \)) such that

\[
\left( \sum_{|\alpha|=2} |a_\alpha|^\frac{4p}{3p-4} \right)^\frac{3p-4}{4p} \leq C_{K,2,p}^{\text{pol}} \|P\|,
\]
for all 2-homogeneous polynomials on $\ell_n^p$ given by $P(x_1, \ldots, x_n) = \sum_{|\alpha|=m} a_{\alpha} x^\alpha$. When $2 < p \leq 4$ the optimal exponent $\frac{4p}{3p-4}$ is replaced by $\frac{p}{p-2}$, which is also sharp.

When $m < p < 2m$ the above inequality has a polynomial version due to Dimant and Sevilla-Peris [11]: given an $m$-homogeneous polynomial $P(x_1, \ldots, x_n) = \sum_{|\alpha|=m} a_{\alpha} x^\alpha$ defined on $\ell_n^p$ with $m < p < 2m$, there is a constant $C_{\text{pol}}^{m,p} \geq 1$ (not depending on $n$) such that

$$\left(\sum_{|\alpha|=m} |a_{\alpha}|^{\frac{p-m}{p-m}}\right)^{\frac{p-m}{p-m}} \leq C_{\text{pol}}^{m,p} \|P\|.$$ 

Moreover the exponent $\frac{p}{p-m}$ is sharp. For $p \geq 2m$, a similar inequality replacing the optimal exponent $\frac{p}{p-m}$ by the optimal exponent $\frac{2mp}{mp+p-2m}$ holds (this case is due to Praciano-Pereira [16]).

In this note we extend the above inequality (keeping its sharpness) to the case $p = m$ (we mention [4] for a different approach for multilinear forms; here, contrary to what happens in [4], we allow the left hand side of the inequality to be the sup norm). We also obtain the optimal constant when we are restricted to 2-homogeneous polynomials defined on $\ell_2^2$ over the real scalar field. In a final appendix we show why the optimal exponents of the bilinear Hardy–Littlewood inequality do not behave smoothly (a similar argument holds for $m$-linear forms and $m$-homogeneous polynomials).

## 2. The Hardy–Littlewood Inequality for $m$-Homogeneous Polynomials on $\ell_m$

Let us recall the $m$-linear Hardy–Littlewood inequalities:

- (Hardy–Littlewood/Praciano-Pereira [12, 16], 1934/1981) Let $m \geq 2$ be a positive integer and $p \geq 2m$. For all $m$–linear forms $T : \ell_n^p \times \cdots \times \ell_n^p \to \mathbb{K}$ and all positive integers $n$,

$$\left(\sum_{j_1, \ldots, j_m=1}^n |T(e_{j_1}, \ldots, e_{j_m})|^{2mp/(mp+p-2m)}\right)^{\frac{mp+p-2m}{2mp}} \leq \left(\sqrt{2}\right)^{m-1} \|T\|.$$ 

Moreover, the exponent $2mp/(mp+p-2m)$ is optimal.

- (Hardy–Littlewood/Dimant–Sevilla-Peris [12, 11], 1934/2014) Let $m \geq 2$ be a positive integer and $m < p < 2m$. For all $m$–linear forms $T : \ell_n^p \times \cdots \times \ell_n^p \to \mathbb{K}$ and all positive integers $n$,

$$\left(\sum_{j_1, \ldots, j_m=1}^n |T(e_{j_1}, \ldots, e_{j_m})|^{\frac{p}{p-m}}\right)^{\frac{p-m}{p-m}} \leq \left(\sqrt{2}\right)^{m-1} \|T\|.$$ 

Moreover, the exponent $p/(p-m)$ is optimal.

From now on the optimal (and unknown) constants satisfying the above inequalities are denoted by $C^{m,p}_{\mathbb{K}}$.

We begin with the following lemma which is an adaptation of [3, Proposition 2.2]. We present a proof for the sake of completeness:
Lemma 2.1. If $P$ is an $m$-homogeneous polynomial of degree $m$ on $\ell^p$, with $m < p < 2m$, given by $P(x_1, \ldots, x_n) = \sum_{|\alpha|=m} a_\alpha x^\alpha$, then

$$\left( \sum_{|\alpha|=m} |a_\alpha|^{\frac{p}{p-m}} \right)^{\frac{p-m}{p}} \leq C_{K,m,p}^{\text{pol}} \|P\|$$

with

$$C_{K,m,p}^{\text{pol}} \leq C_{K,m,p}^{\text{mult}} \frac{m^m}{(m)!^{\frac{p-m}{p}}}. $$

Proof. Let $L$ be the symmetric $m$-linear form associated to $P$. From [10] we have

$$\sum_{|\alpha|=m} |a_\alpha|^{\frac{p}{p-m}} = \sum_{|\alpha|=m} \left( \binom{m}{\alpha} |L(e_1^{\alpha_1}, \ldots, e_n^{\alpha_n})|^{\frac{p}{p-m}} \right)$$

$$\quad \quad \quad = \sum_{|\alpha|=m} \binom{m}{\alpha} \frac{p}{p-m} |L(e_1^{\alpha_1}, \ldots, e_n^{\alpha_n})|^{\frac{p}{p-m}}. $$

For all $\alpha$, the term $|L(e_1^{\alpha_1}, \ldots, e_n^{\alpha_n})|^{\frac{p}{p-m}}$ appears $\binom{m}{\alpha}$ times in $\sum_{i_1, \ldots, i_m=1}^n |L(e_{i_1}, \ldots, e_{i_m})|^{\frac{p}{p-m}}$. Hence

$$\sum_{|\alpha|=m} \binom{m}{\alpha} \frac{p}{p-m} |L(e_1^{\alpha_1}, \ldots, e_n^{\alpha_n})|^{\frac{p}{p-m}} = \sum_{i_1, \ldots, i_m=1}^n \binom{m}{\alpha} \frac{1}{\binom{m}{\alpha}} |L(e_{i_1}, \ldots, e_{i_m})|^{\frac{p}{p-m}}$$

and, since $\binom{m}{\alpha} \leq m!$ we obtain

$$\sum_{|\alpha|=m} \binom{m}{\alpha} \frac{p}{p-m} |L(e_1^{\alpha_1}, \ldots, e_n^{\alpha_n})|^{\frac{p}{p-m}} \leq (m!)^{\frac{p-m}{p}} \sum_{i_1, \ldots, i_m=1}^n |L(e_{i_1}, \ldots, e_{i_m})|^{\frac{p}{p-m}}. $$

We thus have, from the $m$-linear Hardy–Littlewood inequality,

$$\left( \sum_{|\alpha|=m} |a_\alpha|^{\frac{p}{p-m}} \right)^{\frac{p-m}{p}} \leq \left( (m!)^{\frac{p-m}{p}} \sum_{i_1, \ldots, i_m=1}^n |L(e_{i_1}, \ldots, e_{i_m})|^{\frac{p}{p-m}} \right)^{\frac{p-m}{p}}$$

$$\quad \quad \quad = (m!)^{1-p}\left( \sum_{i_1, \ldots, i_m=1}^n |L(e_{i_1}, \ldots, e_{i_m})|^{\frac{p}{p-m}} \right)^{\frac{p-m}{p}}$$

$$\quad \quad \quad \leq (m!)^{1-p} \frac{m^m}{m!} C_{K,m,p}^{\text{mult}} \|L\|.$$ 

On the other hand, it is well-known that

$$\|L\| \leq \frac{m^m}{m!} \|P\|$$

and hence

$$\left( \sum_{|\alpha|=m} |a_\alpha|^{\frac{p}{p-m}} \right)^{\frac{p-m}{p}} \leq \left( (m!)^{1-p} \frac{m^m}{m!} \right) C_{K,m,p}^{\text{mult}} \|P\| = \left( \frac{m^m}{(m!)^{\frac{p-m}{p}}} \right) C_{K,m,p}^{\text{mult}} \|P\|.$$ 

Now we are ready to state and prove our first result:
Proposition 2.2 (The Hardy–Littlewood inequality for 2-homogeneous polynomials in $\ell_2$). For all positive integers $n$ we have

$$ \max_{|\alpha|=2} |a_\alpha| \leq 4\sqrt{2} \|P\| $$

for all $P = \sum_{|\alpha|=2} a_\alpha x^\alpha$ in $\mathcal{P}(\ell^n_2)$. Moreover this result is optimal in the sense that the sup norm in the left hand side cannot be replaced by any $\ell_r$-norm without keeping the constant independent of $n$.

Proof. Let $2 < p < 4$. It is well-known, from the Hardy–Littlewood inequality (see also [11]) for bilinear forms $T : \ell^n_p \times \ell^n_p \to \mathbb{K}$, that

$$ \left( \sum_{i,j=1}^n |T(e_i, e_j)|^{\frac{p}{p-2}} \right)^{\frac{p-2}{p}} \leq \sqrt{2} \|T\|. \tag{3} $$

From the previous lemma we conclude that for all $Q = \sum_{|\alpha|=2} c_\alpha x^\alpha$ in $\mathcal{P}(\ell^n_p)$ we have

$$ \left( \sum_{|\alpha|=2} |c_\alpha|^{\frac{p}{p-2}} \right)^{\frac{p-2}{p}} \leq 2^{\frac{p}{p-2}} \sqrt{2} \|Q\|. $$

Let $P = \sum_{|\alpha|=2} a_\alpha x^\alpha$ be a polynomial in $\mathcal{P}(\ell^n_2)$. For all $p \in (2, 4)$ let us consider $P_p \in \mathcal{P}(\ell^n_p)$ given by the same rule as $P$. We have

$$ \left( \sum_{|\alpha|=2} |a_\alpha|^{\frac{p}{p-2}} \right)^{\frac{p-2}{p}} \leq 2^{\frac{p}{p-2}} \sqrt{2} \sup \{|P_p(x_1, ..., x_n)| : \sum |x_i|^p = 1\} $$

$$ = 2^{\frac{p}{p-2}} \sqrt{2} \sup \{|P(x_1, ..., x_n)| : \sum |x_i|^p = 1\}. $$

Making $p \to 2$ we obtain

$$ \max_{|\alpha|=2} |a_\alpha| \leq 4\sqrt{2} \|P\|. $$

Now we prove the optimality. Suppose that there is a $r < \infty$ and a constant $C \geq 1$ (not depending on $n$) such that

$$ \left( \sum_{|\alpha|=2} |a_\alpha|^r \right)^{\frac{1}{r}} \leq C \|P\| $$

for all $P = \sum_{|\alpha|=2} a_\alpha x^\alpha$ in $\mathcal{P}(\ell^n_2)$ and all $n$. Let $p \in (2, 4)$ be so that

$$ r < \frac{p}{p-2}. $$

Let $R = \sum_{|\alpha|=2} \beta_\alpha x^\alpha$ be a polynomial in $\mathcal{P}(\ell^n_p)$ and let $R_2$ be the same polynomial, but with domain $\ell^n_2$. We thus have

$$ \left( \sum_{|\alpha|=2} |\beta_\alpha|^r \right)^{\frac{1}{r}} \leq C \sup \{|R_2(x_1, ..., x_n)| : \sum |x_i|^2 = 1\} $$

$$ = C \sup \{|R(x_1, ..., x_n)| : \sum |x_i|^2 = 1\} $$

$$ \leq C \sup \{|R(x_1, ..., x_n)| : \sum |x_i|^p = 1\} $$

for all $n$ and this is a contradiction in view of the optimality of the exponent $\frac{p}{p-2}$ in the classical Hardy–Littlewood inequality. \qed
Remark 2.3. We recall the definition of the polarization constants for polynomials on $\ell_p$ spaces:

$$K(m, p) := \inf \{ M > 0 : \|L\| \leq M\|P\| \},$$

where the infimum is taken over all $P \in \mathcal{P}(\ell^n_p)$ and $L$ is the unique symmetric $m$-linear form associated to $P$. As we have used in Lemma 2.1, it is well known that in general

$$\|L\| \leq m^{m!} \|P\|$$

but for $\ell_p^n$ spaces the above estimate may be improved if we use $K(m, p)$. As we have used in Lemma 2.1, it is well known that in general

$$\|L\| \leq m^{m!} \|P\|$$

but for $\ell_p^n$ spaces the above estimate may be improved if we use $K(m, p)$. For instance, a result due to Harris [13] asserts that

$$C(m, p) \leq \left( \frac{m^m}{m!} \right)^{\frac{p-2}{p}} \left( \frac{2}{2^m} \right)^{\frac{p-2}{2}} \|P\|$$

for all $p \geq 1$, whenever $m$ is a power of 2 (see also [3]). In particular, if $m = 2$ and $p > 2$, we have,

$$\left( \sum_{|\alpha|=2} |a_\alpha|^{p-2} \right)^{\frac{p-2}{p}} \leq \left( 2^{1-\frac{p-2}{p}} \left( \frac{2}{2^m} \right)^{\frac{p-2}{2}} \right) C_{m,2,p} \|P\|$$

and

$$= 2 \cdot C_{m,2,p} \|P\|$$

for all $P$ on $\ell_p^n$, when working with complex scalars.

If we look for better constants we can isolate the case of complex scalars of Proposition (2.2) and obtain the following (note that a careful examination of [11] shows that we can replace $\sqrt{2}$ by $\frac{2}{\sqrt{\pi}}$ in (3) for the case of complex scalars):

**Proposition 2.4** (The 2-homogeneous Hardy–Littlewood inequality for $\ell_2$ and complex scalars). For all $n \geq 1$, we have

$$\max_{|\alpha|=2} |a_\alpha| \leq \frac{4}{\sqrt{\pi}} \|P\|$$

for all $P = \sum_{|\alpha|=2} a_\alpha x^\alpha$ in $\mathcal{P}(2\ell_2^n)$ over the complex scalar field. Moreover this result is optimal in the sense of Theorem 2.2.

A simple adaptation of the proof of Proposition 2.2 combined with the $m$-linear version of the Hardy–Littlewood inequality due to Dimant and Sevilla-Peris for $m < p < 2m$ (see [11], Proposition 4.1 (i)) gives us the following general extension for the case $p = m$:

**Theorem 2.5** (The Hardy–Littlewood inequality for $m$-homogeneous polynomials in $\ell_m$). Let $m \geq 2$ be a positive integer. Given $n \geq 1$, there is an optimal constant $C_{m,m} \geq 1$ (not depending on $n$) such that

$$\max_{|\alpha|=m} |a_\alpha| \leq C_{m,m} \|P\|$$

for all $P \in \mathcal{P}(m\ell_m^n)$, with

$$C_{R,m} \leq \left( \frac{\sqrt{2}}{\sqrt{\pi}} \right)^{m-1} m^m,$$

$$C_{C,m} \leq \left( \frac{2}{\sqrt{\pi}} \right)^{m-1} m^m.$$

Moreover this result is optimal in the sense that the sup norm in the left hand side cannot be replaced by any $\ell_r$-norm without keeping the constant independent of $n$. 
Remark 2.6. If we look for better constants we can write the above estimate depending on the polarization constants and we get

\[ C_{\mathbb{R},m} \leq \left( \sqrt{2} \right)^{m-1} (m!) \mathbb{R}(m,m) \]
\[ C_{\mathbb{C},m} \leq \left( \frac{2}{\sqrt{\pi}} \right)^{m-1} (m!) \mathbb{C}(m,m). \]

3. The optimal constant for the case \( m = 2 \) and \( \ell_2^2 \)

For all fixed \( n \geq 1 \) let us define \( C_{\mathbb{K}}(n) \) as the optimal constant satisfying

\[ \max_{|\alpha| = 2} |a_\alpha| \leq C_{\mathbb{K}}(n) \|P\| \]

for all \( P: \ell_2^2 \to \mathbb{K} \). It is simple to show that \( C_{\mathbb{R}}(2) \geq 2 \). In fact, the 2-homogeneous polynomial \( P_2: \ell_2^2 \to \mathbb{R} \) given by

\[ P_2(x) = x_1 x_2. \]

has norm 1/2. From

\[ \max_{|\alpha| = 2} |a_\alpha| \leq C_{\mathbb{R}}(2) \|P_2\| \]

we conclude that

\[ C_{\mathbb{R}}(2) \geq 2. \]

In order to show that the optimal constant \( C_{\mathbb{R}}(2) \) is precisely 2 we will use the expression of the extremal polynomials on the unit ball of \( \mathcal{P}(\ell_2^2) \). The following result is due to Choi and Kim [9]:

Theorem 3.1 (Choi–Kim). For \( p = 2 \), a 2-homogeneous norm one polynomial \( P(x,y) = ax^2 + by^2 + cxy \) is an extreme point of the unit ball of \( \mathcal{P}(\ell_2^2) \) if, and only if,

(i) \( |a| = |b| = 1, c = 0 \) or
(ii) \( a = -b, 0 < |c| \leq 2 \) and \( 4a^2 = 4 - c^2 \).

From the Krein–Milman Theorem, we already know that the optimal constants shall be searched within the extreme polynomials of the unit ball of \( \mathcal{P}(\ell_2^2) \). So we have:

Theorem 3.2. For \( \mathbb{K} = \mathbb{R} \), the optimal constant for the Hardy–Littlewood inequality for 2-homogeneous polynomials in \( \mathcal{P}(\ell_2^2) \) is 2.

Proof. Let us denote by \( C_{\mathbb{R}}(2) \) the optimal constant. For all extremal polynomials given by the previous theorem we have

\[ \max \{|a|, |b|, |c|\} \leq C_{\mathbb{R}}(2) \|P\| = C_{\mathbb{R}}(2). \]

In the case (i) we have \( C_{\mathbb{R}}(2) \geq 1 \) and in the case (ii) we have

\[ 2 = \max \{ |a|, \sqrt{4-4a^2} : 0 < a < 1 \} \leq C_{\mathbb{R}}(2) \|P\| = C_{\mathbb{R}}(2), \]

and thus the optimal constant \( C_{\mathbb{R}}(2) \) is 2. \( \square \)
Remark 3.3. It was recently proved in [8] that, when \( K = \mathbb{R} \), the optimal constants for the Hardy–Littlewood inequality for 2-homogeneous polynomials in \( P(2^2 \ell_p^2) \) and \( 2 < p < 4 \) is \( 2^{2/p} \) (the case \( p = 4 \) is proved in [5]). The above result shows that this formula is also valid for our new version of the Hardy–Littlewood inequality for \( p = 2 \), since \( 2^{2/2} = 2 \).

4. Appendix: why are the optimal exponents of the Hardy–Littlewood inequality not smooth?

The original versions of the Hardy–Littlewood inequality for bilinear forms can be stated as follows:

- **[12] Theorems 2 and 4** If \( p, q \geq 2 \) are such that
  \[
  \frac{1}{2} < \frac{1}{p} + \frac{1}{q} < 1
  \]
  then there is a constant \( C \geq 1 \) such that
  \[
  \left( \sum_{j,k=1}^{\infty} |A(e_j, e_k)|^{\frac{pq}{pq-p-q}} \right)^{\frac{pq-p-q}{pq}} \leq C \|A\|
  \]
  for all continuous bilinear forms \( A : \ell_p \times \ell_q \to \mathbb{R} \) (or \( \mathbb{C} \)). Moreover the exponent \( \frac{pq}{pq-p-q} \) is optimal.

- **[12] Theorems 1 and 4** If \( p, q \geq 2 \) are such that
  \[
  \frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}
  \]
  then there is a constant \( C \geq 1 \) such that
  \[
  \left( \sum_{j,k=1}^{\infty} |A(e_j, e_k)|^{\frac{4pq}{3pq-2p-2q}} \right)^{\frac{3pq-2p-2q}{4pq}} \leq C \|A\|
  \]
  for all continuous bilinear forms \( A : \ell_p \times \ell_q \to \mathbb{R} \) (or \( \mathbb{C} \)). Moreover the exponent \( \frac{4pq}{3pq-2p-2q} \) is optimal.

Looking at both results, the natural question is: why does \( \frac{1}{p} + \frac{1}{q} = \frac{1}{2} \) separate two different expressions for the optimal exponents in [12] and [5]? In this appendix we revisite the bilinear Hardy–Littlewood inequalities to justify this lack of smoothness. In fact we show that, in a more precise sense (that will be clear soon in Remark 4.2) the exponent \( \frac{pq}{pq-p-q} \) in (4) is not optimal. We present a “smooth” and optimal version (Theorem 4.3) of the above Hardy–Littlewood theorems which, surprisingly, is not entirely encompassed even by the ultimate very general recent extensions of the Hardy–Littlewood inequalities (as those from [2]).

We begin by recalling a general version of the Kahane–Salem–Zygmund inequality, which appears in [1] Lemma 6.2].
Lemma 4.1 (Kahane–Salem–Zygmund inequality (extended form)). Let $m, N \geq 1$, $p_1, \ldots, p_m \in [1, \infty]$ and let, for $p \geq 1$,

$$\alpha(p) = \begin{cases} \frac{1}{2} - \frac{1}{p} & \text{if } p \geq 2 \\ 0 & \text{otherwise.} \end{cases}$$

Then there exists an $m$-linear map $A : \ell_{p_1}^N \times \cdots \times \ell_{p_m}^N \to \mathbb{K}$ of the form

$$A(z^{(1)}, \ldots, z^{(m)}) = \sum_{i_1, \ldots, i_m} \pm z_1^{(i_1)} \cdots z_m^{(i_m)}$$

such that

$$\|A\| \leq C_m N^{1/2 + \alpha(p_1) + \cdots + \alpha(p_m)}$$

for some constant $C_m > 0$.

If we look at [12, Theorem 2] we can realize (see [12, page 247]) that in fact the authors prove that, for $1/2 < \frac{1}{p} + \frac{1}{q} < 1$, there is a constant $C \geq 1$ such that

$$\left( \sum_{k=1}^{\infty} \left( \sum_{j=1}^{\infty} |A(e_j, e_k)|^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \leq C \|A\|$$

with $\lambda = \frac{pq}{pq - p - q}$, for all continuous bilinear forms $A : \ell_p \times \ell_q \to \mathbb{R}$ (or $\mathbb{C}$). Since in this case we have $2 < \lambda$, the authors use a trivial estimate to conclude, from (6), that

$$\left( \sum_{j,k=1}^{\infty} |A(e_j, e_k)|^{\lambda} \right)^{\frac{1}{\lambda}} \leq C \|A\|,$$

with $\lambda = \frac{pq}{pq - p - q}$, for all continuous bilinear forms $A : \ell_p \times \ell_q \to \mathbb{R}$ (or $\mathbb{C}$). The proof that the exponent $\frac{pq}{pq - p - q}$ in (7) is sharp is quite simple (we now use an idea taken from [11]) and stress that the usual approach via Kahane–Salem–Zygmund inequality is not effective here (why? due to the “rough” estimation when passing from (6) to (7)). To prove the optimality, it suffices to consider the bilinear form $A_n : \ell_p \times \ell_q \to \mathbb{R}$ (or $\mathbb{C}$) given by $A_n(x, y) = \sum_{j=1}^{n} x_j y_j$ and use Hölder’s inequality. In fact, since $\frac{1}{p} + \frac{1}{q} + \frac{1}{\lambda} = 1$, we have

$$\|A_n\| \leq n^{1/\lambda}.$$

If (7) would hold for a certain $r$ instead of $\lambda$, combining with (8) we would obtain

$$n^{\frac{1}{r}} \leq C n^{\frac{1}{\lambda}}$$

for all $n$, and thus

$$r \geq \lambda = \frac{pq}{pq - p - q}.$$

As a matter of fact, even if we consider sums in just one index (i.e., $j = k$), the exponent $\frac{pq}{pq - p - q}$ in (7) is still optimal (observe that $A_n$ is a kind of diagonal form). However, what does it exactly mean that $\frac{pq}{pq - p - q}$ is optimal in (7) in the usual sense? It means (also in the sense of [12]) that for
both indexes \( j, k \) we can not take simultaneously exponents smaller than \( \frac{pq}{pq-p-q} \). In other words, re-writing (7) as

\[
\left( \sum_{j,k=1}^{\infty} \left( \sum_{k=1}^{\infty} |A(e_j, e_k)|^r \right)^{\frac{1}{r}} \right)^{\frac{1}{s}} \leq C \|A\|
\]

we cannot have \( r = s < \frac{pq}{pq-p-q} \). But a different question, motivated by (6), would be: is it possible to have \((r, s)\) satisfying the above inequality with \( r = 2 \) and \( s < \frac{pq}{pq-p-q} \) or with \( r < 2 \) and \( s = \frac{pq}{pq-p-q} \)?

So, a new question arises: Is (6) sharp in this sense? We stress that the trivial fact that

\[
\left( \sum_{j,k=1}^{\infty} |A(e_j, e_k)|^\lambda \right)^{\frac{1}{\lambda}} \leq C \|A\|
\]

plus the fact that the exponent \( \lambda = \frac{pq}{pq-p-q} \) is sharp in (7) in the sense of [12] does not assure that the exponents \( \lambda \) or 2 in (6) are sharp in our sense: for this task we need the Kahane–Salem–Zygmund inequality. In fact, if

\[
\left( \sum_{j=1}^{\infty} \left( \sum_{k=1}^{\infty} |A(e_j, e_k)|^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{s}} \leq C \|A\|
\]

using the bilinear form \( A \) from the Kahane–Salem–Zygmund inequality we have, for all \( N \),

\[
\left( N \cdot \left( N^{\frac{1}{2}} \right)^s \right)^{\frac{1}{s}} \leq CN^{\frac{1}{2}+\left(\frac{1}{p}-\frac{1}{2}\right)+\left(\frac{1}{q}-\frac{1}{2}\right)},
\]

and thus

\[
N^{\frac{1}{2}+\frac{1}{p}} \leq CN^{\frac{1}{2}-\frac{1}{p}+\frac{1}{q}},
\]

i.e.,

\[
s \geq \lambda.
\]

In the case in which the exponent \( \frac{pq}{pq-p-q} \) is untouched we show that the exponent 2 can not be improved using a similar argument.

**Remark 4.2.** We note that in our “more precise” sense of optimality, the exponent \( \frac{pq}{pq-p-q} \) in (7) is not optimal, because the “first” exponent \( \lambda = \frac{pq}{pq-p-q} \) can be improved to 2.

Now, if we turn our attention to [12, Theorem 1] we can also realize that from [12] we also have

\[
\left( \sum_{k=1}^{\infty} \left( \sum_{j=1}^{\infty} |A(e_j, e_k)|^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{s}} \leq C \|A\|
\]

for \( \lambda = \frac{pq}{pq-p-q} \) (this fact is also observed in [15, Theorem 1], however with no mention to its eventual optimality, and for the case of complex scalars). Again, a simple consequence of the Kahane–Salem–Zygmund inequality asserts that the exponents of (7) are sharp (in the sense that \( \lambda \) can not be improved keeping the exponent 2 as it is and vice-versa); we left the details for the reader. So, from (6) and (7) we can rewrite, in a unifying and optimal form, the results of Hardy and Littlewood as follows:
Theorem 4.3 (Hardy and Littlewood - revisited). If \( p, q \geq 2 \) and \( \frac{1}{p} + \frac{1}{q} < 1 \), then there is a constant \( C \geq 1 \) such that

\[
\left( \sum_{k=1}^{\infty} \left( \sum_{j=1}^{\infty} |A(e_j, e_k)|^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{\lambda}} \leq C \|A\|
\]

for \( \lambda = \frac{pq}{pq - p - q} \), for all continuous bilinear forms \( A : \ell_p \times \ell_q \to \mathbb{R} \) (or \( \mathbb{C} \)). We, as usual, consider \( c_0 \) instead of \( \ell_\infty \) when \( p = \infty \). The exponents are optimal in the sense that \( \lambda \) can not be improved keeping the exponent 2 nor the exponent \( \lambda \) can be improved keeping the exponent \( 2 \).

Remark 4.4. For \( \frac{1}{2} < \frac{1}{p} + \frac{1}{q} < 1 \) we have \( 2 < \lambda \) and we can not use a Minkowski’s type result as in [1, Section 3] to interchange the positions of 2 and \( \lambda \). For this reason, even making a “rough” approximation when replacing 2 by \( \lambda \) when passing from (6) to (7), the resulting estimate (7) is still sharp in the usual sense, as we mentioned before. We stress that it is in fact impossible in this case to interchange 2 and \( \lambda \) (even looking for stronger arguments than a Minkowski’s type result). The reason is quite simple: if this was possible, by “interpolating” the resulting exponents \((2, \lambda)\) and \((\lambda, 2)\) with \( \theta = 1/2 \) in the sense of [1, Section 2] we would obtain an improvement of (7) (in the usual sense, i.e., a smaller exponent would be valid for all indexes), and we know that this is not possible.

Remark 4.5. The fact that 2 and \( \lambda \) can not be interchanged when \( \frac{1}{2} < \frac{1}{p} + \frac{1}{q} < 1 \) is certainly the reason of the absence of the full content of Theorem 4.3 in the very general paper [2] (see [2, Theorem 1.1]).

For \( \frac{1}{p} + \frac{1}{q} \geq \frac{1}{2} \), we have \( \lambda < 2 \), and since there is an obvious symmetry between \( j \) and \( k \), a consequence of Minkowski’s inequality allows us (as in [1, Section 2]) to interchange the positions of 2 and \( \lambda \) and obtain

\[
\left( \sum_{k=1}^{\infty} \left( \sum_{j=1}^{\infty} |A(e_j, e_k)|^\lambda \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{2}} \leq C \|A\|.
\]

Now “interpolating the multiple exponents” \((\lambda, 2)\) and \((2, \lambda)\) with \( \theta = 1/2 \) in the sense of [1, Section 2], or using the Hölder’s inequality for mixed sums (see [6]), we obtain (5) as a corollary. In fact, varying the weight \( \theta \) from 0 to 1, we recover a family of optimal inequalities as in [1 2].

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