Additional Light Waves in Hydrodynamics and Holography

Antonio Amariti\textsuperscript{a}, Davide Forcella\textsuperscript{b}, Alberto Mariotti\textsuperscript{c}

\textsuperscript{a}Department of Physics, University of California, San Diego
9500 Gilman Drive, La Jolla, CA 92093-0319, USA

\textsuperscript{b}Laboratoire de Physique Théorique de l’École Normale Supérieure
and CNRS UMR 8549
24 Rue Lhomond, Paris 75005, France

\textsuperscript{c}Theoretische Natuurkunde, Vrije Universiteit Brussel
and The International Solvay Institutes
Pleinlaan 2, B-1050 Brussels, Belgium

Abstract

We study the phenomenon of additional light waves (ALWs), observed in crystal optics: two or more electromagnetic waves with the same polarization, but different refractive index, propagate simultaneously in an isotropic medium. We show that ALWs are common in relativistic hydrodynamics, and in particular in strongly coupled systems that admit a dual gravitational description, where the ALWs are dual to quasi normal modes in the AdS gravity. We study both the transverse and the longitudinal light wave propagation. In the longitudinal channel we find a transition between regimes with different number of excitonic resonances which resembles the transition to standard optics observed in crystals.

October 8, 2010

\textsuperscript{1}amariti@ucsd.edu
\textsuperscript{2}forcella@lpt.ens.fr
\textsuperscript{3}alberto.mariotti@vub.ac.be
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Introduction

In the recent years an intense research on light propagation in some artificial media, called metamaterials, has been developed [1]. Some very uncommon properties, theoretically predicted, have become viable.

One of the most attractive is the negative refraction of light, originally predicted in [2]. It is the phenomenon for which the phase velocity and energy flux of a electromagnetic wave in a material have opposite directions of propagation. One may ask if there exist any generic class of media, for which the EM response functions are exactly computable, that have a negative refractive index. With this motivation, in [3], we studied negative refraction in relativistic hydrodynamics and in particular in strongly coupled relativistic media. We established a new connection between optics and geometry through the gauge/gravity correspondence [4], and we observed that negative refraction is quite ubiquitous for this class of media. The negative refraction of light appears because the dispersion relations are dissipative and they violate the locality, i.e. there is spatial dispersion.

Spatial dispersion is an interesting property of media that leads to new amazing non-standard optics. In media with spatial dispersion there is a linear non local relation between the electric field and the electric induction. This translates in the dependence of the permittivity on the wave vector.
A non-standard optic phenomenon associated to spatial dispersion is the propagation of additional light waves (ALWs) in crystal optics. The existence of ALWs has been predicted in 1957 by Pekar [5]. He claimed that spatial dispersion effects are relevant in presence of resonances given by electron-hole bound states, called excitons. At the resonance frequency the interaction among the excitons and the photons is strong, and the light wave that propagates in the material is a mixture of these states, called polaritons. The energy necessary to create the resonance is proportional to the momentum of the incoming photon. As a consequence the linear relation between the electric field and the electric induction violates locality, and the permittivity has a pole that depends on the wave vector.

In absence of strong spatial dispersion the dispersion equation has a single solution for $n^2$, the square of the refractive index. If there are strong spatial dispersion effects new solutions to the dispersion relation arise. At the resonance frequency these new solutions are associated to the propagation of ALWs. This exotic picture, in which many different propagating light waves are associated to the same incoming photon, has been subsequently observed in experiments (see [6] and references therein).

The lifetime of an exciton depends on the order parameters of the theory. If its lifetime becomes too short, the ALW associated to this resonance does not propagate anymore. Pekar’s theory predicts that by varying the order parameters the number of waves propagating in the medium varies too. This transition among different optical regimes has been observed in crystals with one exciton. In that case for low temperature the lifetime of the exciton is large enough for both waves to propagate. As the temperature increases the exciton lifetime reduces and above a critical value only the standard single wave propagates.

In this paper we investigate the existence of ALWs in relativistic hydrodynamics and in particular in strongly coupled media with a gravity dual description. In such media the propagation of light, in the hydrodynamical regime, typically shows spatial dispersion. We revisit the model studied in [3], where the refraction of transverse light waves was studied. It was found that at finite temperature and chemical potential the refractive index was negative in the low frequency and small wave-vector regime. The refractive index was computed by considering only the lowest order dependence of the permittivity from the wave-vector. Here we observe that if we take into account the whole spatial dispersion effects there is a second solution of the dispersion equation, as in the case of crystals. We study this additional light wave and we show that it propagates for every value of the order parameters.

Then we analyze the propagation of longitudinal electromagnetic waves in the same system. In absence of spatial dispersion longitudinal light waves do not propagate in a medium. On the contrary here the permittivity is strongly dependent on the wave-vector, and there is a propagating wave at finite temperature and zero charge density. If a chemical potential is turned on, there is another solution to the dispersion equation, and hence an ALW. The analysis of the propagation of these two waves depends on the values of the order parameters. Indeed, by varying them, there is an optical transition, between the regimes with one and two propagating waves. This is similar to the transition...
to standard optics predicted by Pekar’s theory.

The paper is organized as follows. In section 1 we review the linear response theory of media with spatial dispersion. In section 2 we discuss the propagation of the light waves in media with spatial dispersion and we review the basic properties of Pekar’s theory. In section 3 we observe that if the medium is described by relativistic hydrodynamics, then ALWs are generated. In section 4 we study the ALWs in a holographic theory in the hydrodynamical regime. In section 5 we study some properties of these waves and then we conclude. In the Appendix A we discuss the relation between the current correlators and the linear response function of a medium with spatial dispersion. In the Appendix B we discuss the Poynting vector and its relation with the sign of the refractive index for the ALWs.

1 Linear response and spatial dispersion

In this section we review the linear response electromagnetic functions for media with spatial dispersion \[ \text{[7–10]} \] i.e. we take into account the momentum dependence. In this case it is common to use the \( E, B, D \) approach to macroscopic electromagnetism for which the Maxwell equations in the Fourier space are

\[
\begin{align*}
k \cdot B &= 0, \\
k \cdot D &= 0, \\
k \times E &= \omega B, \\
k \times B &= -\omega D
\end{align*}
\]

(1)

and \( D_i = \epsilon_{ij}(\omega, k)E_j \). The linear response of the medium is encoded in the permittivity tensor, \( \epsilon_{ij}(\omega, k) \). For isotropic media this tensor naturally decomposes in transverse \( \epsilon_T(\omega, k) \) and longitudinal part \( \epsilon_L(\omega, k) \), and

\[
D_i = \epsilon_{ij}(\omega, k)E_j = \epsilon_T(\omega, k)E^T_i + \epsilon_L(\omega, k)E^L_i
\]

(2)

where \( E^T_i \) and \( E^L_i \) are the transverse and longitudinal components of the electric field, and they are defined in terms of the projectors \( P^T_{ij} \) and \( P^L_{ij} \) as

\[
E^T_i = P^T_{ij}E_j = \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right) E_j, \\
E^L_i = P^L_{ij}E_j = \frac{k_i k_j}{k^2} E_j
\]

(3)

The functions \( \epsilon_T(\omega, k) \) and \( \epsilon_L(\omega, k) \) are obtained in linear response theory by applying an external electromagnetic field potential \( A_j \) and measuring the electromagnetic current \( J_i = q^2 G_{ij}A_j \), where the \( G_{ij} \) is the retarded current correlator in the medium and \( q \) is the four-dimensional EM coupling. As we show in the Appendix A the transverse and longitudinal permittivity are related to the transverse and longitudinal retarded current correlator by

\[
\begin{align*}
\epsilon_T(\omega, k) &= 1 - \frac{4\pi}{\omega^2} q^2 G_T(\omega, k), \\
\epsilon_L(\omega, k) &= 1 - \frac{4\pi}{\omega^2} q^2 G_L(\omega, k)
\end{align*}
\]

(4)

Thermodynamical stability requires the positivity of the outgoing heat flow. This is associated to the quantities

\[
Q_{\text{heat}} = \frac{\omega}{8\pi} \text{Im}(\epsilon_T(\omega, k))|E_T|^2, \\
Q_{\text{heat}} = \frac{\omega}{8\pi} \text{Im}(\epsilon_L(\omega, k))|E_L|^2
\]

(5)
for transverse and longitudinal waves respectively. They are positive because the function \( \epsilon_T \) and \( \epsilon_L \) are computed from the retarded Green functions as in (1) and both \( \text{Im}(G_T(\omega, k)) < 0 \) and \( \text{Im}(G_L(\omega, k)) < 0 \), which implies \( \text{Im}(\epsilon_T(\omega, k)) > 0 \) and \( \text{Im}(\epsilon_L(\omega, k)) > 0 \).

Using Maxwell equation (1), the dispersion relations for the transverse and longitudinal waves read

\[
\epsilon_T(\omega, k) = \frac{k^2}{\omega^2}, \quad \epsilon_L(\omega, k) = 0
\]

Due to spatial dispersion \( \epsilon_T(\omega, k) \) and \( \epsilon_L(\omega, k) \) are functions of both \( \omega \) and \( k \). Differently from [3], here we do not expand at \( k^2 \) order. In such a way we can capture the effects of large spatial dispersion, i.e. we allow for large momentum dependence in the permittivity. By using the dispersion relation (6) one can find the on-shell ratio \( k^2/\omega^2 \) and from this ratio the refractive index \( n^2(\omega) \). This is in general a complex quantity that takes into account the dissipation of the medium. Note that, since the permittivity depends on the wave vector, the dispersion relation can give in general more than one solution for \( n^2(\omega) \).

For a propagating electromagnetic wave, the sign of the refraction is given by the reciprocal sign of the phase velocity (Re[\( n \)]) and of the flux of energy (the Poynting vector). In presence of strong spatial dispersion and strong dissipation, the definition of the Poynting vector and the determination of the flux of energy are not well established [7]. Hence in this case the sign of the refractive index cannot be study without the knowledge of a microscopic description of the system. We discuss about this issue in the Appendix [B].

2 Pekar’s theory and ALW

In 1957 Pekar [5] discussed the existence of an additional light wave in the optical spectra of crystals in presence of strong spatial dispersion. This phenomenon is associated to the presence of a resonance in the crystal, namely an exciton. See [11] for review. For example in a semiconductor crystal, an exciton is associated to the transition of an electron, from the valence to the conduction band. When such a transition occurs a positive charge carrier, a hole, remains in the valence band. This hole and the electron interact throughout a Coulomb potential. The bound state generated by this interaction is the exciton. This exciton couples to light and its coupling can be described at macroscopic level through the dielectric tensor \( \epsilon(\omega, k) \).

The exciton has the same momentum of the absorbed photon, i.e. the energy necessary to create the exciton depends on the wave-vector \( k \). This translates in the \( k \) dependence of the dielectric tensor. In the simplest case, a single exciton, with effective mass \( M \) and inverse lifetime \( \Gamma \), modifies the transverse conductivity as

\[
\epsilon_T(\omega, k) = \epsilon_0 + \frac{A}{\omega_0 + \frac{k^2}{2M} - \omega - i\Gamma}
\]

where \( \omega_0 \) is the energy required to create an exciton without momentum.
On shell the dispersion equation is \( \epsilon_T(\omega, k) = \frac{k^2}{\omega^2} \). Solving this equation and using the definition \( n^2 = \frac{k^2}{\omega^2} \) for the refractive index one finds

\[
n_{1,2}^2 = \frac{w^2 \epsilon_0 + 2M(w + i\Gamma - \omega_0) \mp \sqrt{8AMw^2 + (w^2 \epsilon_0 - 2M(w + i\Gamma - \omega_0))^2}}{2w^2}
\]

In [5] Pekar concluded that the two solutions of the dispersion relation in presence of an exciton are associated to two light waves, identically polarized. If there are multi-excitons resonances (other poles depending on the wave-vector in the dispersion relation) there is an ALW associated to each pole. These waves are distinguished by their different velocity and damping. This phenomenon is similar to the birefringence for anisotropic crystals. The main difference is that in the case of birefringence the two waves with different refractive index have also different polarization. Moreover birefringence is strongly related to anisotropy, while ALWs appear in isotropic systems too.

ALWs can be studied also in longitudinal waves. In that case, in absence of spatial dispersion, there is no light propagating in the crystal. Otherwise if spatial dispersion is taken into account the propagation is possible, and every pole involving \( k \) in \( \epsilon_L(\omega, k) \) is associated to a light wave.

**Propagation of ALW**

The propagation of the ALWs in the crystal does not directly follow from having multiple solutions to the dispersion relation. Indeed at frequency far from the excitonic resonance the photon does not mix with the exciton, and a single light wave propagates. The problem of propagation of the ALW is related to the choice of the boundary conditions. Indeed in the case of strong spatial dispersion the usual Maxwell boundary conditions are not enough to determine the amplitudes of the reflected and transmitted waves. Pekar faced the problem of propagation of the ALWs by introducing an additional boundary condition (ABC). There are many possible phenomenological choices of these boundary conditions, and they are related to the currents that flow at the crystal surface. These conditions define an effective refractive index which is used to study the propagation of the ALWs in the material. This index selects which one of the light waves propagates at every frequency.

Here we discuss the effective index obtained from the Pekar’s boundary condition. This condition is \( P_{ex} = 0 \), and it represents the continuity of the excitonic polarization at the crystal surface. If there are two light waves the effective refractive index is defined by the relation

\[
n_{eff} = \frac{n_1}{1 - Q} + \frac{n_2}{1 + 1/Q}
\]

where \( Q = -E_1/E_2 \), and \( E_1 \) and \( E_2 \) are the complex amplitudes of the two waves. From Pekar’s ABC it follows that \( Q = (\epsilon_0 - n_1^2)/(\epsilon_0 - n_2^2) \) and

\[
n_{eff} = \frac{\epsilon_0 + n_1n_2}{n_1 + n_2}
\]
By changing the ABC the behavior of the ratio $Q$ and of the effective refractive index changes. Indeed a different choice of the ABC leads to a different choice of $E_1$ and $E_2$, and it affects the propagation. There are many controversial issues related to the consistency of the ABCs. In the rest of the paper we assume that in our case it is possible to define $n_{eff}$ as in (10). The effective index $n_{eff}$ determines which of the two waves is propagating or if both of them propagate as follows.

If the effective index $n_{eff}$ corresponds to one light wave at lower frequencies and to the other at higher frequencies, there is a frequency regime in which $n_{eff}$ switches from one index to the other. This means that both the waves are propagating in that frequency range. In this case the KK relations are not satisfied separately on the two solutions of (8). The standard optics cannot be applied and the correct description is furnished by Pekar’s theory. The effective index, indeed, satisfies the KK relations and solves the apparent violation of causality.

Standard optics is recovered if the lifetime of the exciton is too short. Indeed in this case the exciton does not live enough to modify the linear response of the system and a single wave propagates. This possibility is theoretically predicted in Pekar’s theory because the lifetime of the exciton resonance varies with the temperature. If the temperature increases up to a critical value $T_c$, also the inverse lifetime increases, and it reaches a critical value $\Gamma_c$. Below $T_c$ the Pekar’s theory must be applied, and there is a range of frequencies in which both the waves are propagating. Above $T_c$ the standard optics holds, and the effective index $n_{eff}$ coincides with only one of the two solutions.

3 ALW from hydrodynamics

In this section we observe that the propagation of an ALW is possible in media described by relativistic hydrodynamics if the transverse current, that couples to the external EM field, has a retarded correlator dominated by a diffusive pole.

Let us consider a system with a $U(1)$ conserved current $J$ which has a diffusive behavior in its transverse part $J_T$: $(\partial_t - D\nabla^2)J_T = 0$, where $D$ is the diffusion coefficient. The retarded Green function of $J_T$ has the pole

$$G_T(\omega, k) = \frac{iB\omega}{i\omega - Dk^2} \quad (11)$$

with $B$ real, up to higher order in $\omega$, $k^2$. The analysis of the hydrodynamic equations for a relativistic system at finite temperature and chemical potential shows that the transverse current indeed satisfies a diffusion equation; the constants $B$ and $D$ are related to the values of the transport coefficients and thermodynamical quantities by

$$B = \frac{\rho^2}{\varepsilon + P}, \quad D = \frac{\eta}{\varepsilon + P} \quad (12)$$

$\varepsilon$ is the energy density, $\rho$ the charge density, $P$ the pressure and $\eta$ the shear viscosity. If the retarded correlator of the current is dominated by the pole (11), the transverse
permittivity is

$$\varepsilon_T(\omega, k) = 1 - i \frac{4\pi q^2 B}{\omega(i\omega - Dk^2)}$$  \hspace{1cm} (13)

where \(q\) is the EM coupling. By solving the dispersion equation for (13) we find two different solutions for \(n^2\)

$$n_{1,2}^2 = \frac{D + i\omega \pm \sqrt{16\pi q^2 DB + (D - i\omega)^2 \omega}}{2D}$$  \hspace{1cm} (14)

There are two different refractive indexes associated to two different light waves. The appearance of multiple solutions to the dispersion relation may give origin to the propagation of ALWs, like in Pekar’s theory. The computation of the transport coefficients is crucial for the study of the existence and of the propagation of the ALWs in a system described by relativistic hydrodynamics.

### 4 ALW in holographic optics

As we discussed above the analysis of ALWs in hydrodynamics requires the knowledge of the transport coefficients. There is a class of strongly coupled media in which they are exactly calculable, because the Green function follows from the application of the holographic principle in string theory. Indeed the holographic correspondence in string theory implies that a strongly coupled quantum field theory in \(d\)-dimensional Minkowski space-time is equivalent to classical gravity in AdS\(_{d+1}\). In term of this gauge/gravity duality the order parameter of the \(d\)-dimensional field theory are related to the order parameter of a black hole solution in AdS\(_{d+1}\). In this paper we focus on the case \(d = 4\). In the hydrodynamical regime we can analytically compute the EM response functions of this strongly coupled system, at finite temperature and charge density. If these response functions have one or more poles in the \((k, \omega)\) plane we can observe the generation of one or more ALWs.

In crystal optics the ALWs are the consequence of the strong spatial dispersion, which is associated to the excitonic resonances. The interaction of the excitons with the photon is described at macroscopic level by the presence of particular poles in the electromagnetic response function \(\varepsilon(\omega, k)\). In the holographic correspondence we know that the poles of the response functions are associated to quasi normal frequencies in AdS gravity [26].

Quasi normal modes are solutions to the linearized equations of the classical fluctuations of the black hole with ingoing boundary conditions. Ingoing boundary conditions are chosen because classically the horizon does not emit radiation. The classical fluctuations are interpreted as small deviations from thermodynamic equilibrium in the dual field theory, and the incoming boundary conditions correspond to dissipation. The dispersion relations of this dissipative process are encoded in the poles of the retarded Green function, which are located at the (complex) eigenfrequencies of the gravitational quasi normal modes.
We conclude that the quasi normal frequencies in AdS gravity are the holographic duals of the excitonic resonances in the strongly coupled field theory.

In this section we study the propagation of light waves both for transverse and for longitudinal waves in media described by a dual gravitational background, in the hydrodynamical regime\(^4\). In the simplest case, at zero charge density but at finite temperature, the transverse propagator does not posses any pole, and there is only one propagating light wave in the medium. At finite charge density a diffusive pole is generated, and indeed this corresponds to a new branch for the refractive index \(n^2\), which is associated to the propagation of an ALW.

In the longitudinal case, at zero charge density but finite temperature, the correlator has a single diffusive pole, and there is a single light wave. Giving a non zero value to the chemical potential produces another pole: the second sound. The emergence of this new quasi-normal mode generates the second solution of \(n^2\) which is associated to an ALW.

4.1 Transverse waves without chemical potential: single wave

![Figure 1: Re(n) for transverse waves with \(T = 1, \mu = 0\) and \(q = 0.1\).](image1)

![Figure 2: Im(n) for transverse waves with \(T = 1, \mu = 0\) and \(q = 0.1\).](image2)

We concentrate on the properties of a medium described by its 3+1 dimensional energy momentum tensor \(T^{\mu\nu}\) and its 3+1 dimensional \(U(1)\) conserved current \(J^\mu\). This subsector of the 3+1 dimensional medium is encoded in a 5 dimensional metric \(g_{mn}\) and a five dimensional gauge field \(A_m\), with the action:

\[
S = \frac{1}{2e^2l^2} \int d^5x \sqrt{-g} \left( R - \frac{6}{l^2} \right) - \frac{1}{4e^2} \int d^5x \sqrt{-g} F_{mn}F^{mn}
\]

(15)

where \(e\) is the five dimensional EM coupling constant and \(3/l^2\) is the cosmological constant. We consider the 3+1 dimensional plasma in an homogeneous and isotropic state that is in thermal equilibrium, with zero charge density. This ground state of the system is

\(^4\)Observe that the calculation is valid if \(|k| = |n|\omega\) is still hydrodynamical
described by a particular solution of (15), an uncharged AdS black hole:
\[
\begin{align*}
    ds^2 &= \frac{l^2}{4b^2u} \left( dx^2 + dy^2 + dz^2 - f(u)dt^2 \right) + \frac{l^2}{4u^2f(u)}du^2
\end{align*}
\] (16)

where \( b = \frac{1}{2\pi T} \) and \( f(u) = 1 - u^2 \). The black hole horizon is at \( u = 1 \) and the AdS boundary is at \( u = 0 \). To compute the linear response of the system at an external electric perturbation we need to linearize the equation of motion for the component of the gauge field \( A_m(u)e^{-i\omega t + ikz} \) parallel to the wave vector \( k \). We need then to solve these equations with the infalling boundary condition at \( u = 1 \) and arbitrary Dirichlet boundary conditions at \( u = 0 \).

The retarded correlator for the transverse current has been computed in [18] at small frequencies and wave vector:
\[
    G_T(\omega, k) = -\frac{i\omega}{4b}
\] (17)

with \( b = \frac{1}{2\pi T} \). In this case transverse permittivity \( \epsilon_T(\omega, k) \) is
\[
    \epsilon_T(\omega, k) = \epsilon_T(\omega) = 1 + \frac{i\pi q^2}{b\omega}
\] (18)

The holographic calculation leaves an ambiguity in the definition of the correlator: a term proportional to \( (\omega^2 - k^2) \). The value of this term has to be fixed by the holographic renormalization procedure [19]. The physical requirement we use is that at large \( \omega \) the permittivity goes to its vacuum value: \( \epsilon(\omega \to \infty) = \epsilon_T(\omega \to \infty, 0) \to 1 \). We computed numerically the large \( \omega \) behavior of the correlators and we fixed in this way the value of the renormalization constant. Anyway, we observe that all our results are qualitatively independent of this renormalization constant.

Spatial dispersion in (18) does not give any effects, and a single wave is expected. Indeed the squared refractive index is given by
\[
    n^2 = 1 + \frac{i\pi q^2}{b\omega}
\] (19)

and the refractive index, which real and imaginary part are plotted in Figures 1 and 2, is
\[
    n = \sqrt{1 + \frac{\pi^2 q^4}{2b^2\omega^2} + i\frac{\pi q^2}{\sqrt{2}b\omega}}
\] (20)

As explained in the Appendix B in this case the sign of the refractive index can be studied by comparing \( \text{Re}[n(\omega)] \) with the Poynting vector. As claimed in [3] the light wave is positively refracted.

4.2 Transverse waves with chemical potential: ALW

In this section we study a 3+1 dimensional medium at finite charge density. This ground state of the system is described by a different solution of (15), a charged AdS-RN black
The retarded correlator for the transverse current has been computed \[20\] at small frequencies and wave vector:

\[
G_T(\omega, k) = \frac{3a}{4(1+a)b^2} \left( \frac{i\omega}{Dk^2 + i\omega} \right) - \frac{(2-a)i\omega}{8(1+a)^2b} \tag{24}
\]

where \(D = \frac{2-a}{4\pi T}\)

This case corresponds to the one studied in \[3\]. In that case we expanded for small \(k\) and we ignored strong effects coming from spatial dispersion. In that approximation the region of very small frequency cannot be explored, and we found only one wave (with negative refraction) propagating in the system. In order to study the effects of strong spatial dispersion, we should not expand in the momentum. The transverse permittivity is then given by

\[
\epsilon_T(\omega, k) = 1 + \frac{4i\pi q^2}{\omega} \left( \frac{3a}{4(1+a)b^2} \left( \frac{1}{Dk^2 - i\omega} \right) + \frac{(2-a)}{8(1+a)^2b} \right) \tag{25}
\]
whose pole structure is analogous to the one studied in section 3. Indeed, by solving the dispersion relation $\epsilon_T = n^2$, we obtain two different solutions for $n^2$. One of the solution corresponds to the one studied in [3]. The other instead is the new solution (the ALW) associated to the strong spatial dispersion effects. We plot in blue and in red the refractive index of the two waves: in Figure 3 the real part while in Figure 4 its imaginary part.

To conclude this section we comment on the sign of the refractive index for the light waves. Both light waves are strongly dissipative, as can be seen in Figure 4. The wave corresponding to the red line in the Figure 3 is the one studied in [3]. There we studied a frequency window where the spatial dispersion effects are small. The Poynting vector can then be easily determined and we were able to prove that there is negative refraction. The other wave (blue line in Figure 3) is everywhere in frequencies characterized by strong spatial dispersion. Hence the flux energy direction cannot be easily determined and as a consequence we need additional informations on the microscopic model to study the sign of the refraction (see appendix B).

4.3 Longitudinal waves without chemical potential: single wave

![Graph](image)

Figure 5: $\text{Re}(n)$ for longitudinal waves with $T = 1$, $\mu = 0$ and $q = 0.1$.

![Graph](image)

Figure 6: $\text{Im}(n)$ for longitudinal waves with $T = 1$, $\mu = 0$ and $q = 0.1$.

At finite temperature light waves propagate also in the longitudinal channel. Usually longitudinal waves do not propagate in the vacuum because the dispersion relation, $\epsilon_L(\omega) = 0$, does not generate any refractive index. Nevertheless at finite temperature the background studied above gives origin to a diffusive pole in the current correlator, even in absence of finite charge density. This pole in the $(k, \omega)$ plane implies that a solution to the dispersion relation exists and there is a propagating light wave in the longitudinal channel.

The retarded correlator for the longitudinal current has been computed in [18] at small frequencies and wave vector:

$$G_L(\omega, k) = \frac{l}{2e^2b} \frac{\omega^2}{i\omega - k^2b}$$  \hspace{1cm} (26)
In this case the longitudinal permittivity $\epsilon_L(\omega, k)$ is

$$\epsilon_L(\omega, k) = 1 - \frac{2\pi q^2 l}{e^2 b(i\omega - k^2 b)}$$

(27)

The holographic calculation leaves an ambiguity in the definition of the correlator: a term proportional to $\omega^2$. The value of this term has to be fixed by the holographic renormalization procedure [19]. We use the same requirement of the previous section: $\epsilon(\omega \to \infty) = \epsilon_L(\omega \to \infty, 0) \to 1$. The large $\omega$ behavior of the correlators has been computed numerically, and this fixes the value of the renormalization constant. Once again we observe that our results do not depend qualitatively on this renormalization constant. The refractive index is computed by solving the dispersion relation $\epsilon_L(\omega, k) = 0$. There is only a single wave propagating in the medium. In Figure 5 and Figure 6 we plotted the real and imaginary part of the refractive index.

4.4 Longitudinal waves with chemical potential: ALW

Figure 7: $Re(n)$ and $Im(n)$ for longitudinal waves, with $\mu/T = 1$ (above) and $\mu/T = 3$ (below) and $q = 0.1$.

Here we proceed by analyzing the optics of the system when we add a chemical potential. At finite charge density there is another pole, corresponding to a new collective mode. This mode, called the second sound, is associated to another excitonic resonance. Both resonances are peaked around the same frequency $\omega \simeq 0$, and in this case there are
two different light waves, two mixing states of the photon with the excitons, propagating in the medium.

The retarded correlator for the longitudinal current has been computed \[21\] at small frequencies and wave vector:

\[
G_{zz}(\omega, k) = \frac{-w^2}{4(1 + a)b^2} \left( \frac{9a}{k^2 - 3\omega^2} + \frac{2(2 - a)^2b}{2(2 + a)bk^2 - 4i(1 + a)\omega - (2 - a)^2bD_m\omega^2} \right)
\]

with

\[
D_m = \frac{2 \left(4(1 + a)^3(1 + 4a)^2 \log(2 - a) - 27(2 - a)a^2(1 + 4a) + 2(1 + a)^2\sqrt{1 + 4a(1 + a - 4)} \log \left( \frac{3 + \sqrt{1 + 4a}}{3 - \sqrt{1 + 4a}} \right) \right)}{(2 - a)^4(1 + 4a)^2}
\]

The sound pole \((k^2 - 3\omega^2)\) leads to a second solution for the dispersion relation and hence to a second light wave. We fix \(T = 1\) and vary the ratio \(\mu/T\), from low to larger values. We plot the real and the imaginary part of the refractive index in Figure 7, for \(\mu/T = 1\) and for \(\mu/T = 3\). From the figure we observe that the two solutions \(n_1\) and \(n_2\) intersect at small frequency for small chemical potential. They do not intersect anymore for larger values of \(\mu\). This effects is associated to the transition between optical regimes, where the number of propagating waves changes. We give a more detailed analysis in next section.

5 Propagation of the ALW in our media

In this section we discuss the propagation of the ALWs in our media. At finite temperature and charge density we have found more solutions to the dispersion relation, leading to ALWs. The propagation of ALWs is related to the boundary condition, as explained in section 2.

An important difference between the situation that we studied here and the case of crystals, for which the Pekar’s’ theory has been developed, is that, here, we deal with an infinite medium. In crystal optics the ABCs are imposed at the surface of the medium. Here we assume that similar conditions can be imposed at infinity such that it is possible to define \(n_{\text{eff}}\) as in (10), with \(\epsilon_0 = 1\).

We plot the effective index \(n_{\text{eff}}\) for the transverse waves in Figure 8. From the figure we observe that \(n_{\text{eff}}\) switches from \(n_1\) to \(n_2\) at small frequencies, near the resonance. By borrowing the crystal optics physical interpretation, the behavior of \(n_{\text{eff}}\) suggests that both the light waves propagate. We can imagine an alternative check of this result by looking at the propagation of each light wave with respect to the dissipation. This property is encoded in the ratio \(\text{Re}(n)/\text{Im}(n)\). We plotted this ratio in Figure 10. At low frequency both the waves are strongly dissipative, and the ratio \(\text{Re}(n)/\text{Im}(n)\) is comparable for them. As the frequency increases the wave selected by \(n_{\text{eff}}\) is preferred because the absorption is lower for that one.

An analogous phenomenon is found in the longitudinal waves. We plot in Figure 9 the effective index and in Figure 11 the ratio \(\text{Re}(n)/\text{Im}(n)\).
We conclude that both in the transverse and in the longitudinal channel we have propagating ALWs.

5.1 Transitions between different optical regimes

As explained in section 2 in Pekar’s theory for crystals, a transition from propagating ALWs to standard optics takes places at some critical value of the order parameter, where the lifetime of the exciton become short.

Here, in the transverse case, at finite temperature and charge density, this transition does not occur. At every value of the order parameters there is a range of frequency where both waves propagate. Indeed $n_{\text{eff}}$ always switches from $n_1$ to $n_2$ for every temperature and chemical potential.

In the longitudinal case the transition to standard optics cannot exist, because, without taking into account spatial dispersion, there are no light waves propagating in the longitudinal channel. Nevertheless we observe here a transition between different optical regimes, as can be seen in Figures 12. For low values of the chemical potential, $n_{\text{eff}}$ coincides with a single light wave in the whole hydrodynamical regime. In this case this is the only light wave propagating in the medium. If the chemical potential increases it
reaches a critical value $\mu_{\text{crit}}$. For $\mu > \mu_{\text{crit}}$ the effective index $n_{\text{eff}}$ switches from $n_1$ to $n_2$. In the frequency range where the transition of $n_{\text{eff}}$ takes place there are two propagating light waves.

The physical explanation of the transition is that the chemical potential is an order parameter for the mixing of the two excitons with the photon. For small values of $\mu$ there is only one light propagating in the medium, and there is no large mixing between the two excitons. The single propagating wave is associated to the sound pole in [28], which is not dissipative and it is the dominant one at low $\mu$. For larger $\mu$ there is larger mixing between the two quasi-normal modes, and both can propagate, depending on the frequency.

![Figure 12](image)

Figure 12: Optical transition for $T = 1$ and $q = 0.1$, at $\mu = \mu_{\text{crit}}$ for longitudinal waves: for $\mu < \mu_c$ only one light wave is propagating, while for $\mu > \mu_c$ both waves propagate at low frequency.

**Conclusions**

In this paper we observed that theories described by relativistic hydrodynamics with a diffusive behavior for the currents can give origin to the phenomenon of additional light waves. This property has been theoretically predicted by Pekar in crystal optics and then experimentally verified. As in the case of crystals, here the ALWs are connected with the strong effects of spatial dispersion, i.e. the $k$ dependence of the poles of the Green functions.

We verified our conjecture on the propagation of ALWs in media described by relativistic hydrodynamics by studying a strongly coupled field theory that can be holographically described. In this case the Green functions and the transport coefficients are exactly calculable thanks to the gauge/gravity duality.

Note that with our analysis we can only capture the hydrodynamical regime of the theory. Indeed at larger frequency and momentum there are other quasi-normal modes. They were first computed in [22] for black holes in AdS. These modes exist only for some discrete complex frequencies associated to new poles of the Green functions [23–25]. For the transverse and longitudinal R-current correlators the quasi-normal modes have been
In the dual field theory these poles correspond to the presence of excitons that can lead to other ALWs, propagating at the various resonance frequencies. It would be interesting to perform a detailed study of the existence and the propagation of the ALWs out of the hydrodynamical regime. For this computation the use of the gauge/gravity correspondence will be fundamental.

Another issue is the study of the transition to standard optics. In crystal optics it is driven by the $i\Gamma$ term appearing in the pole of the dielectric tensor. In our case this term is set to zero in an ideal system, but it arises if impurities are taken into account. There is a general strategy to compute the contribution of the impurities to the Green functions in the gauge/gravity duality \cite{30}. Once this contribution is added we expect that there is some critical value of the order parameters at which the exciton lifetime is too short for the ALWs to propagate. It would be nice to check this prediction in explicit models.

This paper is a second step in the understanding of optical properties of strongly coupled media via holography. Indeed as in \cite{3} the gauge/gravity duality is an useful laboratory to verify some general statements common to many systems.\footnote{See also \cite{29}, in which the authors studied the optical properties of an holographic superconductor.} With this motivation many other optical properties of strongly coupled systems can be studied.

Acknowledgments

It is a great pleasure to thank David Mateos for comments on the draft. We also thank V. M. Agranovich, Y. Gartstein, V. Ginis and A. Paredes for nice discussions. A.A. is supported by UCSD grant DOE-FG03-97ER40546; A. M. is a Postdoctoral researcher of FWO-Vlaanderen. A. M. is also supported in part by the Belgian Federal Science Policy Office through the Interuniversity Attraction Pole IAP VI/11 and by FWO-Vlaanderen through project G.0114.10N; D. F. is supported by CNRS and ENS Paris.

A $\epsilon_T$ and $\epsilon_L$

In this Appendix we derive the expression for $\epsilon_T$ and $\epsilon_L$ in term of the transverse and longitudinal correlators. The relevant equation is

$$\partial_t E + 4\pi J = \partial_t D$$

where $D_i = \epsilon_{ij}(\omega, k)E_j$.

The vector potential is $A_i = P_{ij}^T A_i + P_{ij}^L A_i \equiv A_i^T + A_i^L$. The field $E_i$ is expressed in term of the vector potential $A$ as $E_i = F_{0i} = -i\omega A_i - ikA_0$. From the last equality in \cite{30} we have

$$4\pi J_i = \partial_t D_i - \partial_t E_i = -i\omega(D_i - E_i) = -i\omega(\epsilon_{ij} - \delta_{ij})E_j = -\omega(\epsilon_{ij} - \delta_{ij})(k_jA_0 + \omega A_j)$$

\footnote{See also \cite{29} for the computation of the residues.}
where \( k_i = (0, 0, k) \). In the linear response theory the Green function is related to the current through the relation

\[
J_i = q^2 \left( G_{ij}(\omega, k)A^j + G_{i0}(\omega, k)A^0 \right) = q^2 \left( G_{ij}(\omega, k)A_j - G_{i0}(\omega, k)A_0 \right)
\]  (32)

We now make use of the identities \( G_{z0} = -\frac{k}{\omega} G_{zz} \) and \( G_{x0} = G_{y0} = G_{xx} = G_{zy} = 0 \). From these relations and from (31) and (32) we obtain

\[
\epsilon_{ij}(\omega, k) = \delta_{ij} - \frac{4\pi}{\omega^2} q^2 G_{ij}(\omega, k)
\]  (33)

By projecting this relation on the transverse and longitudinal channels we obtain

\[
\epsilon_T(\omega, k) = 1 - \frac{4\pi}{\omega^2} q^2 G_T(\omega, k), \quad \epsilon_L(\omega, k) = 1 - \frac{4\pi}{\omega^2} q^2 G_L(\omega, k)
\]  (34)

### B. Poynting vector

In the paper we showed different physical systems exhibiting the phenomenon of ALW. It is interesting to determine if a propagating wave shows negative refraction too. There is negative refraction when the phase vector is opposed to the energy flux vector. The sign of the phase velocity is given by \( \text{Re}[n(\omega)] \) while the energy flux is given by the Poynting vector. However, the Poynting vector is not always well defined [8].

In the case of small dissipation, even if spatial dispersion is present, the Poynting vector is oriented as the group velocity \( v_g \), and has the simple expression

\[
S = \text{Re} \left( E^* \wedge B - \frac{\omega}{2} \frac{\partial \epsilon_{ij}}{\partial k} E^*_i E_j \right)
\]  (35)

Negative refraction in this context has been studied in [7].

When dissipation effects are not negligible, the Poynting vector is not in general directed as the group velocity. If spatial dispersion effects are small, we can compute it using the classical \( \epsilon, \mu \) approach [7] (see [32, 33]).

\[
S = \text{Re} \left( \frac{n}{\mu} \right) |E|^2
\]  (36)

In [3] we showed that this expression is equivalent to (35) for transverse waves, by expanding the permittivity at second order in the wave vector. In that case we can studied negative refraction by comparing the sign of the Poynting vector with the sign of \( \text{Re}[n(\omega)] \).

In the general case of strong spatial dispersion and strong dissipation we cannot identify precisely the direction of the flux of energy [8]. Hence we cannot determine if there is negative refraction. Except for the wave also analyzed in [3], all the other waves we studied in this paper fall in this category.

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7Where \( \mu \) here refers to the magnetic permeability.
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