Rotating waves in a spatially nonlocal delayed feedback optical system with diffraction

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Abstract. We study self-organization in a nonlinear optical system with a feedback loop, where light is delayed and undergoes a spatial transformation within the thin annulus aperture. The dynamics of the system are modelled by a quasilinear functional differential diffusion equation on a circle that tracks the effects of diffraction by means of a coupled linear Schrodinger equation. We prove the existence of rotating wave solutions and study their stability by constructing the Faria normal form of a Hopf bifurcation.

1. Introduction
A typical nonlinear optical system with a feedback loop consists of a thin layer of a nonlinear Kerr medium and a number of mirrors that constitute the feedback loop. Do not let this simple scheme delude you: in fact, one can observe a multitude of various dynamic regimes of light field self-organization in such systems [1]. Indeed, there are natural effects of diffraction, interference between input and feedback light, diffusion of molecular excitation in the Kerr layer [2, 3]. Moreover, additional devices can be installed in the feedback loop to add nonlocality to the system: this could be a temporal delay [4] and/or a spatial transformation in the transverse direction (rotation, reflection) [3]. All in all, there are many control parameters in the system that can be tuned to produce the desired regime.

In this paper we look at a model of a nonlinear optical system with a thin annulus aperture that has delay and rotation devices in the feedback loop. It is also assumed that the feedback loop is constructed in such a way that there is no interference of light. We choose a one-dimensional model so that the equation to study is a quasilinear functional differential diffusion equation on a circle, where the nonlinear term describes the effects of diffraction.

Similar systems have been studied previously. For a system with delay, rotation, and interference (diffraction was neglected in the model), rotating-wave solutions were described as arising from a Hopf bifurcation and their stability was studied based on the analysis of the normal form on a center manifold [5]. A more complex situation was considered in [6, 7], where rotation device was removed together with interference, but diffraction was taken into account. In spite of having less control parameters, the system showed richer dynamics as it allowed both clockwise and counterclockwise rotating waves together with their sum, a standing wave.

The system we study here differs from [6, 7] in that it has an additional rotation in the feedback loop, and we are willing to describe its rotating-wave solutions. In Section 3 we will write down the equation and provide some preliminary results. In Section 4 we will prove that
rotating waves exist by passing to a rotating coordinate frame, studying the linearized problem, and applying the implicit operator theorem. Finally, in Section 5 we will construct the Faria normal form of a Hopf bifurcation on a center manifold and deduce the stability of rotating waves based on the normal form coefficients (they will be computed in closed form, hence we will have stability criteria in terms of the parameters of the system).

2. Notation

By $H^2(\mathbb{C}) = H^2(0, 2\pi; \mathbb{C})$ we denote the Sobolev space of complex-valued functions on $(0, 2\pi)$ that are Lebesgue square-integrable with their second derivative. By $H^2_{2\pi}(\mathbb{C})$ we denote the closed subspace of $H^2(\mathbb{C})$ of $2\pi$-periodic functions, which can be made into a Hilbert by a suitable choice of the inner product $\langle \cdot, \cdot \rangle_X$ and the induced norm $\| \cdot \|_X$ are taken from a unitary space $X$; lack of subscript means that $X = H(\mathbb{C}) = L^2(0, 2\pi; \mathbb{C})$. Given a function space $X(\mathbb{C})$ of complex-valued functions, we denote its real-valued counterpart by $X$.

For a Banach space $Y$ with the norm $\| \cdot \|_Y$ we write $C^k([a, b]; Y)$ for the Banach space of $k$ times continuously differentiable $Y$-valued functions with the norm

$$
\|u\|_{C^k([a, b]; Y)} = \sum_{j=0}^k \sup_{t \in [a, b]} \|u^{(j)}(t)\|_Y.
$$

Given a linear operator $A : V \rightarrow W$ on a pair of real vector spaces $V$ and $W$, we denote its complexification by $A^C$:

$$
A^C : (V + iV) \rightarrow (W + iW), \quad A^C(f + ig) = Af + iAg, \quad f, g \in V.
$$

3. Main equation

We consider a periodic boundary value problem (BVP) for a quasilinear functional differential equation (FDE) on a circle:

$$
\begin{align*}
&u_t + u = Du_{xx} + K \left| Be^{i\theta(x, t)} \right|^2, \quad x \in (0, 2\pi), \quad t > 0, \\
&u|_{x=0} = u|_{x=2\pi}, \quad u_x|_{x=0} = u_x|_{x=2\pi}.
\end{align*}
$$

Here, $B$ is a linear operator

$$
B : H(\mathbb{C}) \ni A_0(x) \mapsto A(x, z; A_0)|_{z=z_0} \in H(\mathbb{C}), \quad D(B) = H^2_{2\pi}(\mathbb{C}).
$$

that maps its input $A_0(x)$ to the solution at $z = z_0$ of a periodic initial-boundary value problem for a linear Schrödinger equation

$$
A_x + iA_{xx} = 0
$$

with $A_0(x)$ as the initial condition. We note that is has a complete orthogonal system of eigenfunctions in $H(\mathbb{C})$ such that

$$
Be^{inx} = e^{inz_0}e^{inx}.
$$

BVP (3.1) admits spatially homogeneous equilibria $u(x, t) \equiv K$. Fixing a value $\hat{K}$ of the nonlinearity parameter and considering its perturbations $K(\mu) = \hat{K} + \mu$, we get a branch of constant solutions $K(\mu)$. In their vicinity, BVP (3.1) can be written down as

$$
\begin{align*}
v_t + v &= Dv_{xx} + K(\mu) \left( \left| Be^{i\theta(x, t)} \right|^2 - 1 \right) \tag{3.3} \\
v|_{x=0} = v|_{x=2\pi}, \quad v_x|_{x=0} = v_x|_{x=2\pi},
\end{align*}
$$

for a linear Schrödinger equation

$$
A \equiv iA + K(u) \|A\|^2 A = 0
$$

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with $A(\mu) \equiv \hat{K}$ as the initial condition. We have a complete orthogonal system of eigenfunctions in $H(\mathbb{C})$ such that

$$
Be^{inx} = e^{inz_0}e^{inx}.
$$
where \( u(x, t) = K(\mu) + v(x, t) \). Extracting linear terms, we can further rewrite (3.3) as

\[
v_t + v = Dv_{xx} + L(\mu)v(x + \theta, t - T) + F(v(x + \theta, t - T), \mu),
\]

where

\[
L(\mu) \equiv L_0 + \mu L_1, \quad L_0 = -2i\dot{K}\text{Im}B, \quad L_1 = -2\text{Im}B,
\]

\[
F(w, \mu) = K(\mu) \left\{ |B(e^{iw} - 1)|^2 + 2\text{Re}B(e^{iw} - 1 - iw) \right\}.
\]

As has been shown in [6], the nonlinear operator \( F(w, \mu) : H_{2\pi}^2 \times \mathbb{R} \to H_{2\pi}^2 \) is analytic at the origin. Moreover, its expansion contains terms that are at least quadratic in \( w \) and at most linear in \( \mu \). We will make use of the following nonzero terms

\[
F_{ww}(0, 0)w^2 = 2\dot{K} \left\{ |Bw|^2 - \text{Re}Bw^2 \right\},
\]

\[
F_{www}(0, 0)w^3 = 2\dot{K} \left\{ 3\text{Im} \left[ Bw \overline{Bw}^2 \right] + \text{Im}Bw^3 \right\}, \quad F_{ww\mu}(0, 0)w^2\mu = 2\mu \left\{ |Bw|^2 - \text{Re}Bw^2 \right\}.
\]

For instance, once could represent \( F(w, \mu) \) as

\[
F(w, \mu) = \frac{1}{2} F_{ww}(0, 0)w^2 + F_3(w, \mu),
\]

where \( F_3(w, \mu) \) is analytic and contains terms with \( w^m\mu^n \), \( n \geq 2 \), \( m \leq 1 \), \( n + m > 2 \).

4. Existence of rotating waves

4.1. Rotating coordinate system

To seek rotating-wave solutions we shall pass to a rotating coordinate system with the use of rotation operators

\[
R_\alpha f(x) = f((x + \alpha) \mod 2\pi), \quad \alpha \in \mathbb{R}, \quad f \in H(\mathbb{C}).
\]

Then the solution \( v(x, t) \) is looked for in the form of

\[
v(x, t) = R_{-\Omega t}v(x), \quad \Omega \in \mathbb{R}, \quad v(x) \in H_{2\pi}^2.
\]

Plugging ansatz (4.1) into BVP (3.3), we get an ordinary functional differential equation with a shifted argument:

\[
Dv'' + \Omega v' - v + L(\mu)R_{\theta + \Omega T}v + F(R_{\theta + \Omega T}v, \mu) = 0, \quad v \in H_{2\pi}^2.
\]

4.2. Linearized problem

Let us look into the linear part of equation (4.2) at \( \mu = 0 \):

\[
A_\Omega : H \to H, \quad A_\Omega v \equiv Dv'' + \Omega v' - v + L_0R_{\theta + \Omega T}v, \quad D(A_\Omega) = H_{2\pi}^2.
\]

The adjoint operator of \( A_\Omega \) is given by

\[
A_\Omega^* : H \to H, \quad A_\Omega^* u \equiv Du'' - \Omega u' - u + R_{-\theta - \Omega T}L_0u, \quad D(A_\Omega^*) = H_{2\pi}^2.
\]

Along with \( A_\Omega \) we shall consider its complexification:

\[
A_\Omega^C u = A_\Omega f + iA_\Omega g, \quad (A_\Omega^C)^* = (A_\Omega^*)^C, \quad u = f + ig \in H_{2\pi}^2(\mathbb{C}).
\]
Lemma 4.1. The linear operators $A^0_\Omega$ and $A^\ast_\Omega$ possess a complete orthogonal system of eigenfunctions $\{e^{inx}: n \in \mathbb{Z}\}$ in $H(\mathbb{C})$. The corresponding eigenvalues are
\[
\lambda_n(A^0_\Omega) = -Dn^2 - 1 + i\Omega - 2\bar{K}e^{i(n\theta + \Omega T)}\sin(n\omega_0), \quad \lambda_n(A^\ast_\Omega) = \lambda_n(A^0_\Omega). \quad (4.5)
\]

Proof. Follows from the properties of linear operators $B$ and $R$. 

Condition 4.2. Let $\Omega = \Omega_*$ be such that
\[
N(A^C_\Omega) = N(A^\ast_\Omega) = \text{span}\{e^{inx}, e^{-inx}\}.
\]

Lemma 4.3 [(6)]. Let Condition 4.2 hold. Then
(i) $R(A_{\Omega_*}) = N(A^\ast_\Omega) = N(A_\Omega) \perp$
(ii) $A_{\Omega_*} : \mathcal{P}H^2_{2\pi} \rightarrow R(A_{\Omega_*})$ has a bounded inverse, where $\mathcal{P} : H^2_{2\pi} \rightarrow R(A_{\Omega_*})$ is an orthogonal projection.

4.3. Existence theorem
By a solution to equation (4.2) we understand a triplet $S = (v, \Omega, \mu) \in H^2_{2\pi} \times \mathbb{R} \times \mathbb{R}$. We are interested in small perturbations of a trivial solution $S_0 = (0, \Omega_*, 0)$.

Theorem 4.4. Let Condition 4.2 hold. Then there exists an $\varepsilon_0 > 0$ such that a twice continuously differentiable with respect to $\varepsilon$ branch of nontrivial solutions
\[
S_\varepsilon = (v(\cdot; \varepsilon), \Omega_\varepsilon + \omega(\varepsilon), \mu(\varepsilon)) \in H^2_{2\pi} \times \mathbb{R} \times \mathbb{R}
\]
is defined for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, and it passes through $S_0$ at $\varepsilon = 0$.

Proof. Consider a triplet $S = (v(\varepsilon), \Omega, \mu)$ defined as
\[
v(\varepsilon) = \varepsilon(\varphi + \xi), \quad \varphi = \cos(n, x) \in N(A_{\Omega_*}), \quad \xi \in D(A_{\Omega_*}) \cap N(A_{\Omega_*}) \perp, \quad \Omega = \Omega_* + \omega. \quad (4.6)
\]
We plug $S$ into (4.2) and divide by $\varepsilon \neq 0$:
\[
A_{\Omega_*}\xi + \omega\varphi + L(\mu)R_{\theta + \Omega T}\xi - L_0R_{\theta + \Omega T}\xi + \frac{1}{\varepsilon}F(R_{\theta + \Omega T}v(\varepsilon), \mu, \xi) = 0. \quad (4.7)
\]
Subspaces $N(A_{\Omega_*}) = \text{span}\{\varphi, \varphi^\prime\}$ and $R(A_{\Omega_*}) = N(A_{\Omega_*}) \perp$ are invariant under the action of the linear operators involved. Thus we can project equation (4.7) onto them:
\[
A_{\Omega_*}\xi + \omega\varphi + L(\mu)R_{\theta + \Omega T}\xi - L_0R_{\theta + \Omega T}\xi + \frac{1}{\varepsilon}F(R_{\theta + \Omega T}v(\varepsilon), \mu, \xi) = 0,
\]
\[
\omega \langle \varphi, \varphi^\prime \rangle + (L(\mu)R_{\theta + \Omega T}v(\varepsilon), \varphi) - (L_0R_{\theta + \Omega T}v(\varepsilon), \varphi) + \frac{1}{\varepsilon} \langle F(R_{\theta + \Omega T}v(\varepsilon), \mu, \varphi), \varphi^\prime \rangle = 0,
\]
\[
\omega \langle \varphi, \varphi^\prime \rangle + (L(\mu)R_{\theta + \Omega T}v(\varepsilon), \varphi^\prime) - (L_0R_{\theta + \Omega T}v(\varepsilon), \varphi^\prime) + \frac{1}{\varepsilon} \langle F(R_{\theta + \Omega T}v(\varepsilon), \mu, \varphi^\prime), \varphi \rangle = 0. \quad (4.8)
\]
Equations (4.8) are equivalent to equation (4.7) and constitute a nonlinear operator equation
\[
F(\xi, \omega, \mu, \varepsilon) = 0, \quad (4.9)
\]
where the operator
\[
F : \mathcal{P}H^2_{2\pi} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow R(A_{\Omega_*}) \times \mathbb{R} \times \mathbb{R}
\]
is analytic in the vicinity of the origin. Its $(\xi, \omega, \mu)$-Fréchet derivative there is
\[
\nabla F = \begin{pmatrix} A_{\Omega_*} & 0 \\ \nabla F_0 & \nabla F_0 \end{pmatrix}, \quad \nabla F_0 = \begin{pmatrix} \langle \varphi, \varphi \rangle + T(\xi, \xi, \varphi, \varphi) & \langle \varphi, \varphi \rangle + T(\xi, \xi, \varphi, \varphi) & \langle \varphi, \varphi \rangle + T(\xi, \xi, \varphi, \varphi) \\ \langle \varphi, \varphi \rangle + T(\xi, \xi, \varphi, \varphi) & \langle \varphi, \varphi \rangle + T(\xi, \xi, \varphi, \varphi) & \langle \varphi, \varphi \rangle + T(\xi, \xi, \varphi, \varphi) \\ \langle \varphi, \varphi \rangle + T(\xi, \xi, \varphi, \varphi) & \langle \varphi, \varphi \rangle + T(\xi, \xi, \varphi, \varphi) & \langle \varphi, \varphi \rangle + T(\xi, \xi, \varphi, \varphi) \end{pmatrix}.
\]
By Lemma 4.3, the linear operator $A_{\Omega_1}: \mathcal{P}H_{2\pi}^2 \rightarrow R(A_{\Omega_1})$ has a bounded inverse. Hence, the linear operator

$$\nabla F: \mathcal{P}H_{2\pi}^2 \times \mathbb{R} \times \mathbb{R} \rightarrow R(A_{\Omega_1}) \times \mathbb{R} \times \mathbb{R}$$

has a bounded inverse whenever the matrix $\nabla F_0$ is invertible. We now prove that $\det \nabla F_0 \neq 0$. By construction,

$$\det \nabla F_0 = -2\pi n_s^2 \sin(n_s^2 z_0) \left[2\hat{K}T \sin(n_s^2 z_0) - \cos(n_s(\theta + \Omega_s T))\right],$$

but from Condition 4.2 it follows that $n_s^2 \sin(n_s^2 z_0) \neq 0$, $\cos(n_s(\theta + \Omega_s T)) \neq 0$, and

$$2\hat{K}T \sin(n_s^2 z_0) - \cos(n_s(\theta + \Omega_s T)) = -\frac{T(1 + Dn_s^2) + \cos^2(n_s(\theta + \Omega_s T))}{\cos(n_s(\theta + \Omega_s T))} \neq 0.$$

It remains to apply the implicit operator theorem [9].

### 4.4. Expansion coefficients

The triplet $(\xi, \omega, \mu)(\varepsilon)$ can be expanded in powers of $\varepsilon$:

$$\begin{align*}
\xi(\varepsilon) &= \varepsilon \xi_1 + \varepsilon^2 \xi_2 + \mathcal{O}(\varepsilon^2) \in \mathcal{P}H_{2\pi}^2, \\
\omega(\varepsilon) &= \varepsilon \omega_1 + \varepsilon^2 \omega_2 + \mathcal{O}(\varepsilon^2) \in \mathbb{R}, \\
\mu(\varepsilon) &= \varepsilon \mu_1 + \varepsilon^2 \mu_2 + \mathcal{O}(\varepsilon^2) \in \mathbb{R}.
\end{align*}$$

(4.10)

Below we will show how to compute the first order coefficients. We plug expansion (4.10) into equation (4.7) to obtain:

$$\begin{align*}
A_{\Omega_1} \xi(\varepsilon) + \omega(\varepsilon) \frac{d}{dx}(\varphi + \xi(\varepsilon)) + L_0 \left[R_{\theta + \Omega_s T + \omega(\varepsilon)T}(\varphi + \xi(\varepsilon)) - R_{\theta + \Omega_s T}(\varphi + \xi(\varepsilon))\right] + \\
\mu(\varepsilon) L_1 R_{\theta + \Omega_s T + \omega(\varepsilon)T}(\varphi + \xi(\varepsilon)) + \frac{1}{\varepsilon} F(R_{\theta + \Omega_s T + \omega(\varepsilon)T}\varepsilon(\varphi + \xi(\varepsilon)), \mu(\varepsilon)) &\equiv 0,
\end{align*}$$

(4.11)

where $\varphi = \cos(n_s x) \in N(A_{\Omega_1})$. We note that the operator $R_{\Omega T}$ also admits an expansion in powers of $\varepsilon$:

$$R_{\theta + \Omega_s T + \omega(\varepsilon)T} y = R_{\theta + \Omega_s T} y + \varepsilon \omega_1 T R_{\theta + \Omega_s T} y' + \mathcal{O}(\varepsilon^2),$$

(4.12)

and use it together with expansion (3.7) of $F(w, \mu)$ to expand the nonlinear term in (4.11):

$$F(R_{\theta + \Omega_s T + \omega(\varepsilon)T}\varepsilon(\varphi + \xi(\varepsilon)), \mu(\varepsilon)) = \varepsilon^2 F_2 + \varepsilon^3 F_3(\varepsilon),$$

(4.13)

where

$$F_2 = \frac{1}{2} F_{w w}(0, 0) R_{\theta + \Omega_s T}[\varphi^2], \quad \|F_3(\varepsilon)\| H \leq C, \forall \varepsilon \in [\varepsilon_1, \varepsilon_2] \subset (-\varepsilon_0, \varepsilon_0).$$

We now compute the leading coefficients $\xi_1, \omega_1$, and $\mu_1$. Grouping linear in $\varepsilon$ terms in (4.11), we get:

$$A_{\Omega_1} \xi_1 = -\left[\omega_1 (\varphi' + T L_0 R_{\theta + \Omega_s T} \varphi') + \mu_1 L_1 R_{\theta + \Omega_s T} \varphi + F_2\right].$$

(4.14)

From Lemma 4.3 it follows that equation (4.14) has a unique solution whenever the right hand side is orthogonal to the kernel $N(A_{\Omega_1})$.

By its definition, $F_2$ lies in span $\{\cos(2n_s x), \sin(2n_s x)\}$ and is orthogonal to $N(A_{\Omega_1})$. Thus the solvability condition is $\nabla F_0 \cdot (\omega_1, \mu_1)^T = 0$, whence we conclude that equation (4.14) has a solution whenever

$$\omega_1 = 0, \quad \mu_1 = 0.$$
Function $\xi_1 \in \mathcal{P}H^2_\pi$ can be sought as

$$\xi_1 = c_c \cos(2n_x x) + c_s \sin(2n_x x).$$

This gives the following expression:

$$\xi_1 = -\text{Re} \left[ \frac{\hat{K} \left(1 - \cos(4n_x^2z_0)\right)}{2\lambda_{2n_x}(A_{\Omega_\varepsilon})} R_{\theta + \Omega_\varepsilon} T e^{2in_x x} \right], \quad \lambda_{2n_x}(A_{\Omega_\varepsilon}) \neq 0.$$

5. Stability of rotating waves

5.1. Abstract formulation

To rewrite BVP (3.3) in the usual FDE terms [10], we use a function space $\mathcal{C} = C([-T, 0]; X)$, $X = H^2_\pi$, and a function $v_t \in \mathcal{C}$ that acts according to $v_t(\tau) = v(t + \tau)$ where $v \in X$; we also extend $L(\mu)$ onto $\mathcal{C}$ by $\hat{L}(\mu) v = L(\mu) R_{\theta}(\varepsilon) (-T)$, hiding spatial shift inside, so that $\hat{L}(\mu)$ is linear and bounded in $\mathcal{C}$. We are ready to write (3.3) in its abstract form

$$\frac{d}{dt} v(t) = Av(t) + \tilde{L}_0 v_t + \tilde{F}(v_t, \mu), \quad v \in D(A).$$

(5.1)

Here $Aw = D^2 xw - w, D(A) = \{w \in X : Aw \in X\}$,

$$\tilde{F}(v_t, \mu) = F(R_{\theta}v_t(-T), \mu) + \mu \tilde{L}_1 v_t = \sum_{n=2}^{\infty} \frac{1}{n!} \tilde{F}_n(v_t, \mu),$$

where $\tilde{F}_n$ are the $n$-th order terms in the expansion of $\tilde{F}$.

Consider the linearization of (5.1) at $v = 0$ and $\mu = 0$:

$$\frac{d}{dt} v(t) = Av(t) + \tilde{L}_0 v_t.$$

(5.2)

The corresponding characteristic equation is

$$Ay + e^{-\lambda T} L_0 R_{\theta} y - \lambda y = 0, \quad \lambda \in \mathbb{C}, \quad y \in D(A).$$

(5.3)

We restrict our attention to $y \in \{1, \sin(nx), \cos(nx)\} \subset D(A)$; characteristic equation (5.3) is thus reduced to a countable family of equations

$$\Delta_n(\lambda) \equiv -1 - Dn^2 - 2\hat{K} \sin(n^2z_0) e^{-\lambda T + in\theta} - \lambda = 0, \quad \lambda \in \mathbb{C}, \quad n \in \mathbb{Z}.$$  

(5.4)

Hopf bifurcation occurs if the solutions $\lambda \in \mathbb{C}$ of (5.4) fit the following conditions:

1. For all solutions $\lambda$ their real parts $\text{Re} \lambda \leq 0$.
2. They are $\text{Re} \lambda = 0$ if and only if $\lambda = \pm in$, $n = \pm n_s$. (Hopf)

Consider the generator $A_0 : \mathcal{C} \to \mathcal{C}$ of the flow of equation (5.2):

$$A_0 \varphi = \dot{\varphi}, \quad D(A_0) = \left\{\varphi \in \mathcal{C}^1 : \varphi(0) \in D(A), \varphi(0) = A\varphi(0) + \tilde{L}_0 \varphi\right\}.$$

According to [10], the roots $\lambda$ of characteristic equation (5.3) are the eigenvalues of $A_0$. A two-dimensional eigenspace $P \subset \mathcal{C}$ is associated with $\lambda = \pm in$, and is spanned by

$$\Phi = \left(\varphi_1 = e^{i(n_s x + n_s \tau)}, \ \varphi_2 = \overline{\varphi_1}\right) \subset \mathcal{C}(\mathbb{C})$$
with coefficients in
\[ E^2 = \{(\zeta_1, \zeta_2) \in \mathbb{C}^2 : \zeta_2 = \overline{\zeta_1}\}. \]

Note that
\[ \frac{d}{dt} \Phi = \Phi J, \quad J = \text{diag}(iv_\alpha, -iv_\alpha). \]

It is essential here that the rotation angle \( \theta \) be nonzero. Otherwise the system would gain an additional reflection symmetry thereby making all characteristic values double and forcing the bifurcation to be degenerate \([7]\). Incidentally, Theorem 4.4 stays valid even when \( \theta = 0 \) since we are working with a rotating coordinate system there.

We wish to decompose \( \mathcal{C} \) into a direct sum of \( A_0 \)-invariant subspaces. To do so we need a function space \( \mathcal{C}^* \equiv \mathcal{C}(I; X) \) and a bilinear form \( \langle \cdot, \cdot \rangle : \mathcal{C}^* \times \mathcal{C} \to \mathbb{R} \)
\[ \langle \psi, \varphi \rangle = \langle \varphi(0), \psi(0) \rangle_X + \int_{-T}^{0} \langle \varphi(\tau), L_0 R_{-\theta} \psi(\tau + T) \rangle_X d\tau, \]
which extends to \( \langle \cdot, \cdot \rangle : \mathcal{C}(I)^* \times \mathcal{C}(I) \to \mathbb{C} \) that is antilinear in the first argument and linear in the second one:
\[ \langle \psi, \varphi \rangle = \langle \varphi(0), \psi(0) \rangle_X + \int_{-T}^{0} \langle \varphi(\tau), L_0 R_{-\theta} \psi(\tau + T) \rangle_X d\tau. \]

A formal adjoint with respect to \( \langle \cdot, \cdot \rangle \) operator \( A_0^* \) is defined as
\[ A_0^* \psi = -\hat{\psi}, \quad D(A_0^*) = \left\{ \psi \in \mathcal{C}^* : \psi(0) \in D(A), \quad -\hat{\psi}(0) = A\psi(0) + L_0\psi(T) \right\}, \]
and has the same imaginary eigenvalues. In the corresponding eigenspace we choose a basis \( \Psi \) that is biorthogonal to \( \Phi \). To this end we take
\[ \tilde{\Phi} = (\tilde{\varphi}_1 = e^{i(n_x-x_0)\tau}, \quad \tilde{\varphi}_2 = \overline{\varphi}_1)^T \subset \mathcal{C}(I)^*, \]
evaluate the form
\[ \langle \tilde{\Phi}, \Phi \rangle = \| e^{i n_x x} \|^2_{\mathcal{C}(I)} \text{diag}(\kappa^{-1}, \kappa^{-1}), \quad \kappa^{-1} = 1 - 2\kappa \sin(n_x z_0) \sin(\nu_\alpha T) e^{i m_\alpha \theta}, \]
and set
\[ \Psi = \| e^{i n_x x} \|^2_{\mathcal{C}(I)} (\kappa \tilde{\varphi}_1, \kappa \tilde{\varphi}_2)^T, \]
which is biorthogonal to \( \Phi \), i.e. \( \langle \Phi, \Psi \rangle = I \). As a result, \( Q = \{ \varphi \in \mathcal{C} : \quad \langle \Phi, \varphi \rangle = (0, 0)^T \} \)
is invariant under the action of \( A_0 \) and \( \mathcal{C} = P \oplus Q \).

Next, we define an enlarged phase space \( BC \) \([10]\) that is composed of functions of the form \( \psi = \varphi + X_0 \alpha, \quad \varphi \in \mathcal{C}, \quad \alpha \in X \), with a norm \( \| \psi \|_{BC} = \| \varphi \|_{BC} + \| \alpha \|_{X} \), where \( X_0(\tau) = 0, \quad -T \leq \tau < 0, \quad X_0(0) = I \). In other words, \( BC \) comprises functions \([-T, 0] \to X \) that are uniformly continuous on \([-T, 0) \). The operator \( A_0 \) is extended onto \( BC \) as \( \tilde{A}_0 : BC \to BC \) according to
\[ \tilde{A}_0 \psi = \hat{\psi} + X_0[A\psi(0) + \dot{L}_0 \psi - \dot{\psi}(0)], \quad D(\tilde{A}_0) = \{ \psi \in \mathcal{C}^1 : \psi(0) \in D(A) \} \equiv C^1_0. \]
This allows us to formulate equation (5.1) as an abstract ODE in \( BC \):
\[ \frac{d}{dt} v = \tilde{A}_0 v + X_0[\tilde{F}(v, \mu)], \quad v(t) = v_t \in C^1_0. \]
It is shown in [10] that \( \pi(\varphi + X_0\alpha) = \Phi(\varphi) + \langle \alpha, \Psi(0) \rangle_{\mathcal{X}} \) is a continuous projection onto \( P \), which commutes with \( A_0 \) on \( C_{0}^1 \); hence \( BC \) is decomposed into a topological direct sum \( BC = P \oplus N(\pi) \). Going back to (5.5), we express \( v(t) \in C_{0}^1 \) as a sum \( v(t) = \Phi z(t) + y(t) \), where 
\[
\zeta(t) = \langle \Upsilon, \psi \rangle \in \mathbb{E}^2, \quad y(t) = (I - \pi)v(t) \in N(\pi) \cap C_{0}^1 = Q \cap C_{0}^1 \equiv Q_0^1.
\]
This leads to an equivalent system of differential equations in \( \mathbb{E}^2 \times N(\pi) \), which we write down in a way that is suitable for the computation of the normal form:
\[
\begin{align*}
\frac{d}{dt} \zeta &= J \zeta + \sum_{j \geq 2} \frac{1}{j!} f_j^1(\zeta, y, \mu), \\
\frac{d}{dt} y &= A_1 y + \sum_{j \geq 2} \frac{1}{j!} f_j^2(\zeta, y, \mu),
\end{align*}
\]
where \( A_1 : N(\pi) \to N(\pi), D(A_1) = Q_{0}^1 \), is the restriction of \( \tilde{A}_0 \) and 
\[
f_j^1(\zeta, y, \mu) = \langle \tilde{F}_j(\Phi \zeta + y, \mu), \Psi(0) \rangle_{\mathcal{X}}, \quad f_j^2(\zeta, y, \mu) = (I - \pi)X_0 \tilde{F}_j(\Phi \zeta + y, \mu).
\]

5.2. Preliminaries for the normal form construction

Normal form construction is all about eliminating, step by step, non-resonant terms in the power series expansion by consecutive changes of variables. To formalize, we define a space \( V_j^p(N) \) of homogeneous polynomials of degree \( j \in \mathbb{N} \) in \( p \in \mathbb{N} \) variables with coefficients from a Banach space \( Y \): 
\[
V_j^p(N) = \left\{ \sum_{|q| = j} c_q w^q : q \in \mathbb{Z}_+^p, c_q \in Y \right\}.
\]
We seek changes of the form \( (\zeta, y) = (\tilde{\zeta}, \tilde{y}) + \frac{1}{j!} (U_j^1(\tilde{\zeta}, \mu), U_j^2(\tilde{\zeta}, \mu)) \), where \( \zeta, \tilde{\zeta} \in \mathbb{E}^2, y, \tilde{y} \in Q_{0}^1, U_j^1 \in V_j^3(\mathbb{E}^2) \), and \( U_j^2 \in V_j^3(Q_{0}^1) \). Assume we are on the \( k \)-th step. By a change of variables we go from \( \tilde{f}_j = (f_{j1}^1, f_{j2}^1) \) to \( g_j = (g_{j1}, g_{j2}) \), those being the \( j \)-th order terms. Then equations (5.6) take the form
\[
\begin{align*}
\frac{d}{dt} \tilde{\zeta} &= J \tilde{\zeta} + \sum_{j \geq 2} \frac{1}{j!} g_{j1}(\tilde{\zeta}, \tilde{y}, \mu), \\
\frac{d}{dt} \tilde{y} &= A_1 \tilde{y} + \sum_{j \geq 2} \frac{1}{j!} g_{j2}(\tilde{\zeta}, \tilde{y}, \mu),
\end{align*}
\]
Here \( g_{j1}(\tilde{\zeta}, \tilde{y}, \mu) = \tilde{f}_{j1}(\tilde{\zeta}, \tilde{y}, \mu), 2 \leq j \leq k - 1 \), and 
\[
g_k(\tilde{\zeta}, \tilde{y}, \mu) = \tilde{f}_k^1(\tilde{\zeta}, \tilde{y}, \mu) - (M_k^1 U_{k1}^1)(\tilde{\zeta}, \mu), \quad g_k(\tilde{\zeta}, \tilde{y}, \mu) = \tilde{f}_k^2(\tilde{\zeta}, \tilde{y}, \mu) - (M_k^2 U_{k2}^1)(\tilde{\zeta}, \mu),
\]
where the operators \( M_k^1 \) and \( M_k^2 \) are defined as 
\[
(M_k^1 h_1)(\zeta, \mu) = \nabla \zeta h_1(\zeta, \mu)J \zeta - J [h_1(\zeta, \mu)], \quad M_k^1 : V_k^3(\mathbb{E}^2) \to V_k^3(\mathbb{E}^2),
\]
\[
(M_k^2 h_2)(\zeta, \mu) = \nabla \zeta h_2(\zeta, \mu)J \zeta - A_1 [h_2(\zeta, \mu)], \quad M_k^2 : V_k^3(Q_{0}^1) \subset V_k^3(N(\pi)) \to V_k^3(N(\pi)).
\]
The flow on the center manifold \( \tilde{y} = 0 \) [11] is given by an ODE in \( \mathbb{E}^2 \)
\[
\frac{d}{dt} \tilde{\zeta} = J \tilde{\zeta} + \sum_{j \geq 2} \frac{1}{j!} g_{j1}(\tilde{\zeta}, 0, \mu).
\]
In the following we will use the following
Lemma 5.1 ([12]).

(i) The kernel $N(M_2^1)$ has the following form
\[ N(M_2^1) = \text{span}_\mathbb{R} \{ \zeta_1 \mu e_1 + \zeta_2 \mu e_2, \, i \zeta_1 \mu e_1 - i \zeta_2 \mu e_2 \}. \]

(ii) The kernel $N(M_3^1)$ has the following form
\[ N(M_3^1) = \text{span}_\mathbb{R} \{ \zeta_1^2 \zeta_2 e_1 + \zeta_1 \zeta_2^2 e_2, \, i \zeta_1^2 \zeta_2 e_1 - i \zeta_1 \zeta_2^2 e_2, \, \zeta_1 \mu^2 e_1 + \zeta_2 \mu^2 e_2, \, i \zeta_1 \mu^2 e_1 - i \zeta_2 \mu^2 e_2 \}. \]

(iii) Every $V_k^3(\mathbb{E}^2)$ can be decomposed as a direct sum $V_k^3(\mathbb{E}^2) = R(M_1^k) \oplus N(M_2^k)$.

5.3. Computation of the normal form coefficients

To construct the normal form up to the cubic terms, according to Lemma 5.1, we need to compute

\[ g_2^1(\zeta, 0, \mu) = P_{N(M_2^1)} \tilde{f}_2^1(\zeta, 0, \mu), \quad g_3^1(\zeta, 0, \mu) = P_{N(M_3^1)} \tilde{f}_3^1(\zeta, 0, \mu), \]

where $P_V$ is the projection onto $V$ and $\tilde{f}_2 = f_2$.

Using (5.7) we can evaluate $\tilde{f}_2^1(\zeta, 0, \mu)$:

\[ f_2^1(\zeta, 0, \mu) = -4 \mu \sin(n_2^2 z_0) \left[ \kappa e^{i(n_2 \theta - \nu_2 T)\zeta_1 e_1} + \kappa e^{-i(n_2 \theta - \nu_2 T)\zeta_2 e_2} \right]. \quad (5.8) \]

Thus

\[ \frac{1}{2!} g_2^1(\zeta, 0, \mu) = A_1 \zeta_1 \mu e_1 + \overline{A_1} \zeta_2 \mu e_2, \quad A_1 = -2 \kappa \sin(n_2^2 z_0) e^{i(n_2 \theta - \nu_2 T)}, \]

and we will assume that $\text{Re} A_1 \neq 0$.

After this step, $\tilde{f}_3^1$ transforms into

\[ f_3^1(\zeta, 0, \mu) = f_3^1(\zeta, 0, \mu) + \frac{3}{2} \nabla f_2^1(\zeta, 0, \mu) U_2^2(\zeta, \mu) + \frac{3}{2} \nabla \nabla f_2^1(\zeta, 0, \mu) U_2^2(\zeta, \mu) - \frac{3}{2} \nabla U_2^1(\zeta, \mu) g_2^1(\zeta, 0, \mu). \]

Hence it remains to project $f_3^1(\zeta, 0, \mu)$, $U_2^1(\zeta, \mu)$, $U_2^2(\zeta, \mu)$, and $g_2^1(\zeta, 0, \mu)$ onto $N(M_2^3)$.

Since $\text{Re} A_1 \neq 0$, we need not compute the more-than-linear in $\mu$ terms as they have no effect on the qualitative behavior of the trajectories. So we set $\mu = 0$ to calculate the cubic terms. This leads to $g_2^1(\zeta, 0, 0) = (0, 0, 0, 0)^T$. From (5.7) we get

\[ P_{N(M_2^1)} f_3^1(\zeta, 0, 0) = B_2 \zeta_1^2 \zeta_2 e_1 + \overline{B_2} \zeta_1 \zeta_2^2 e_2, \quad B_2 = 6 \kappa \tilde{K} (3 \sin(n_2^2 z_0) - \sin(3n_2^2 z_0)) e^{i(n_2 \theta - \nu_2 T)}. \]

From (5.8) we can derive that $U_2^1(\zeta, 0) = (M_2^3)^{-1} P_{R(M_2^3)} f_2^1(\zeta, 0, 0) = (0, 0, 0, 0)^T$. So we need to solve

\[ (M_2^3 U_2^3)(\zeta, 0) = f_2^3(\zeta, 0) \]

to find the polynomial $U_2^3(\zeta, 0)$. Set $h(\zeta) \equiv U_2^3(\zeta, 0)$. Based on (5.7) and the definition of $M_2^3$ we rewrite equation (5.9)

\[ [\nabla h(\zeta)](\tau), \mathcal{J} \zeta - \frac{d}{d\tau} [h(\zeta)](\tau) = -\Phi(\tau) f_2^1(\zeta, 0, 0), -T \leq \tau < 0, \]

\[ [\nabla h(\zeta)](0), \mathcal{J} \zeta - A [h(\zeta)(0)] - \tilde{L}_0 [h(\zeta)] = \tilde{F}_2(\Phi, 0) - \Phi(0) f_2^1(\zeta, 0, 0). \]

(5.10)
Continuity of $\Phi$ and $h$ lets us pass to a limit in the first equation of (5.10) as $\tau \to -0$ and subtract the limit from the second equation. As $\tilde{F}_2(\Phi, 0, 0) = 0$, equation (5.10) becomes

$$
\begin{align*}
\frac{d}{d\tau} \left[ h(\zeta) \right](\tau) &= [\nabla \zeta h(\zeta)](\tau) J \zeta, \quad -T \leq \tau < 0, \\
\frac{d}{d\tau} \left[ h(\zeta) \right](0) - A [h(\zeta)(0)] - \tilde{L}_0 [h(\zeta)] &= \tilde{F}_2(\Phi, 0).
\end{align*}
$$

(5.11)

The right hand side of the second equation of (5.11) equals to

$$
\tilde{F}_2(\Phi, 0, 0) = 2\tilde{K} \left[ 1 - \cos(4n^2z_0) \right] R_\theta [\phi_1^2(-T) + \phi_2^2(-T)].
$$

We express $h \in V^2_2(Q_0)$ as a linear combination of monomials

$$
h(\zeta) = h_{20} \zeta_1^2 + h_{02} \zeta_2^2 + 2h_{11} \zeta_1 \zeta_2, \quad h_i \in Q_0^1(\mathbb{C}).
$$

Then $(\nabla \zeta h)(\zeta) J \zeta = 2i\nu_s \left[ h_{20} \zeta_1^2 - h_{02} \zeta_2^2 \right]$, and we deduce that $h_{20} = \overline{h_{02}}$, and $h_{11} \in Q_0^1$. On grouping the monomials, we obtain two problems to solve:

$$
\begin{align*}
\frac{d}{d\tau} h_k(\tau) &= \gamma_k h_k(\tau), \quad -T \leq \tau < 0, \\
\frac{d}{d\tau} h_k(0) - A[h_k(0)] - \tilde{L}_0^C[h_k] &= G_k,
\end{align*}
$$

where

$$
G_{20} = 2\tilde{K} \left[ 1 - \cos(4n^2z_0) \right] R_\theta \phi_1^2(-T), \quad G_{11} = 0, \quad \gamma_{20} = 2i\nu_s, \quad \gamma_{11} = 0.
$$

Each problem has a unique solution:

$$
h_{20} = C_{2000} \phi_1^2, \quad C_{20} = -2\tilde{K} \left[ 1 - \cos(4n^2z_0) \right] (\Delta_{2n_s} (2i\nu_s))^{-1} \exp(2i(n_s\theta - \nu_s T)),
$$

$$
h_{11} = 0.
$$

Having found $U_2^2(\zeta, 0) = h(\zeta)$, we can calculate the remaining part of $g_3^1(\zeta, 0, 0)$:

$$
P_{N(M)} \nabla_y f_1^1(\zeta, 0, 0)[h(\zeta)] = C_2 \zeta_1^2 \zeta_2 e_1 + \overline{C_2} \zeta_1 \zeta_2^2 e_2,
$$

$$
C_2 = 4\nu \tilde{K} C_{20} [\cos(3n^2z_0) - \cos(n^2z_0)] e^{i(n_s\theta - \nu_s T)}.
$$

Collecting all the cubic terms, we find

$$
\frac{1}{3!} g_3^1(\zeta, 0, 0) = A_2 \zeta_1^2 \zeta_2 e_1 + \overline{A_2} \zeta_1 \zeta_2^2 e_2, \quad A_2 = (B_2 + C_2)/6,
$$

and the normal form

$$
\frac{d}{dt} \zeta = J \zeta + \frac{1}{2!} g_2^1(\zeta, 0, \mu) + \frac{1}{3!} g_3^1(\zeta, 0, 0) + O(|\zeta|^2 + |(\zeta, \mu)|^2).
$$

(5.12)

Passing to polar coordinates $\zeta_1 = \rho \exp(i\omega)$ in (5.12), we get our final statement.

**Theorem 5.2.** Let (Hopf) hold, $\theta \neq 0$, and $n^2z_0 \neq \pi k$, $k \in \mathbb{Z}$. Then the flow of (3.3) on a center manifold is governed by the following normal form

$$
\begin{align*}
\frac{d}{dt} \rho &= \rho (K_1 \mu + K_2 \rho^3) + O(\rho \mu^2 + |(\rho, \mu)|^4), \\
\frac{d}{dt} \omega &= \nu_s + O(|(\rho, \mu)|),
\end{align*}
$$

where $K_1 = \text{Re} A_1 \neq 0$, $K_2 = \text{Re} A_2$. 

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Coefficients of the normal form are known to govern the stability of the limit cycle of a Hopf bifurcation [13]. In our case, that is the stability of rotating-wave solutions. Indeed, they lie on the center manifold, which is itself asymptotically stable by (Hopf).

**Theorem 5.3.** Let conditions of Theorem 5.2 hold. If $K_1 > 0$ and $K_2 < 0$ then rotating waves from Theorem 4.4 correspond to an asymptotically stable limit cycle of a supercritical Hopf bifurcation.

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