An Approach Towards the Proof of the Strong Goldbach’s Conjecture for Sufficiently Large Even Integers

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**Summary.** We approach a new proof of the strong Goldbach’s conjecture for sufficiently large even integers by applying the Dirichlet’s series. Using the Perron formula and the Residue Theorem in complex variable integration, one could show that any large even integer is demonstrated as a sum of two primes. In this paper, the Riemann Hypothesis is assumed to be true throughout the paper. A novel function is defined on the natural numbers set. This function is a typical sieve function. Then based on this function, several new functions are represented and using the Prime Number Theorem, Sabihi’s theorem, and the second Sabihi’s conjecture, the strong Goldbach’s conjecture is proved for sufficiently large even integers.

**Keywords:** Strong Goldbach’s conjecture, Dirichlet’s series, Prime Number Theorem, Perron formula, Sabihi’s theorem

**Mathematics Subject Classification:** 11A41, 11P32, 11M06, 11N05, 11N36

1 **INTRODUCTION**

The strong Goldbach’s conjecture refers to Christian Goldbach’s theory which was represented by him with a letter to the great Swiss mathematician Leonard Euler in 1742. He asked him for solving this problem. He spent much time trying to prove it, but never succeeded. In this theory, he claims that “every even integer greater than two could be a sum of two prime numbers”. The word “strong” is opposition to word “weak” where refers to “every odd number greater than five could be a sum
of three prime numbers”. Since two hundreds years ago, several relevant proofs on this subject have been conducted. Using the sieve method, a lot of mathematicians as Brun [1] in 1920 have verified various results who expressed every even integer is a sum of product at most 9 primes and product at most another 9 primes”. Wang [2] expressed in 1962, ”every even integer is a sum of 1 prime and product at most 4 primes”. Pan [3] obtained in 1962 that every even integer is a sum of 1 prime and product at most 5 primes. Richert [4] obtained same result in 1969, but product at most 3 primes instead at most 5 primes. Chen [5] in 1973 proved same for 1 prime and product at most 2 primes. Rényi [6] in 1947 represented that every even integer is a sum of 1 prime and product at most c primes. Montgomery and Vaughan found an exceptional set in Goldbach’s problem by the circle method [7]. We presented [8-9] by the two papers in 2009 and 2010 stating 5 new Functions,11 new Lemmas, a novel explicit formula for computing \( \pi(x) \), several methods, a new conjecture and some new experimental computations to prove Goldbach’s conjecture experimentaly. C.D. Pan and C.B. Pan in [10] made the proof of some Theorems in analytic number Theory so that we make use of some of them here. Most of these Theorems are about the proof of the Prime Number Theorem.

2 ELEMENTARY DEFINITIONS

The following new function is defined as \( n \) be a natural number and \( N \) an even positive integer,then

\[
\eta(n) = \begin{cases} 
0 & \text{if } N \equiv n \mod(p) \text{ and } p \leq \sqrt{N} \\
1 & \text{otherwise}
\end{cases} \quad (1-2)
\]

where \( p \) denotes a prime number and \( \gcd(p, N) = 1 \). From the function \( \eta(n) \) is recognized properties of sieve function. Just, based on the above function and known functions as \( \zeta(s) \), \( \theta(x) \), \( \pi(x) \) and \( \psi(x) \) one is able to define the several new following functions (2-2) to (6-2).

\[
\zeta(s, \eta) = \sum_{n=1}^{\infty} \frac{\eta(n)}{n^s} \qquad (2-2)
\]

\[
\theta(x, \eta) = \sum_{p \leq x} \eta(p) \log(p) \qquad (3-2)
\]

\[
\pi(x, \eta) = \sum_{p \leq x} \eta(p) \qquad (4-2)
\]

\[
\psi(x, \eta) = \sum_{n \leq x} \eta(n) \Lambda(n) \qquad (5-2)
\]

\[
\Psi(s, \eta) = \sum_{n=1}^{\infty} \frac{\eta(n) \Lambda(n)}{n^s} \quad (6-2)
\]

where

\[
\Lambda(n) = \begin{cases} 
\log(p) & \text{if } \log(n) = m\log(p) \\
0 & \text{otherwise}
\end{cases} \quad (7-2)
\]

\[
\theta(x) = \sum_{p \leq x} \log(p) \quad (8-2)
\]
The main scope of this paper is to show that $\pi(N, \eta) \geq 1$ where $N$ denotes a sufficient large even integer. This proves the strong Goldbach’s conjecture for sufficiently even integers.

**Lemma 2.1**

If $\pi(N, \eta) \geq 1$, then the strong Goldbach’s conjecture holds

**Proof:**

Let

$$\pi(N, \eta) = \sum_{p \leq N} \eta(q) \left\{ \begin{array}{ll} 0 & \text{if } N \equiv q \mod(p) \text{ and } p \leq \sqrt{N} \\ 1 & \text{otherwise} \end{array} \right. $$

(11-2)

Consider a contradiction as: $\pi(N, \eta) = 0$

If $\pi(N, \eta) = 0$ then consider a typical prime as $2 < q < N$, therefore based on the above relation, one should have for any $q$: $N - q < \sqrt{N}$ or $N - q > \sqrt{N}$. If $N - q < \sqrt{N}$ then $N - q = kp$ and based on the equation (1-2), one concludes that $\eta(q) = 0$. If $N - q > \sqrt{N}$, one has two cases: the first $N - q$ is prime and based upon the equation (1-2), $\eta(q) = 1$. In the second case, $N - q$ is not a prime but at least one of its factors will be under $\sqrt{N}$ (every integer as $m$ has at least a prime factor less than $\sqrt{m}$, because otherwise product of its prime factors will be greater than itself i.e. $m$). Hence, based on the equation (1-2), $\eta(q) = 0$. From this argument, one could conclude if $\pi(N, \eta) \geq 1$ then the first case will be happened. This means that $N - q$ is a prime number and the strong Goldbach’s conjecture holds. Using the Prime Number Theorem [10] could be written:

$$\theta(x, \eta) = \pi(x, \eta) \log(x) - \int_2^x \frac{\pi(u, \eta)}{u} du$$

(12-2)

$$\pi(x, \eta) = \theta(x, \eta) \log(x) + \int_2^x \frac{\theta(u, \eta)}{u \log(u)^2} du$$

(13-2)

$$\psi(x, \eta) = \theta(x, \eta) + O(x^{1/2})$$

(14-2)

### 3 LEMMAS AND THEOREMS TO APPROACH TO THE PROOF OF $\pi(N, \eta) \geq 1$

In this section, we prove four Lemmas using both Perron formula of Dirichlets series and the Residue Theorem. These Lemmas are the base of a basic Theorem which will be given in the next Section. Here, we give a theorem and some lemmas along with their proofs.

**Lemma 3.1**
Let $A(s) = \sum_{n=1}^{\infty} a(n)n^{-s}, \sigma_a < +\infty$ and $H(u)$ and $B(u)$ be increasing functions so that $|a(n)| \leq H(n), n = 1, 2, ...$ and $\sum_{n=1}^{\infty} a(n)n^{-\sigma} \leq B(\sigma), \sigma > \sigma_a$ for any $s_0 = \sigma_0 + it_0$ and $b_0 \geq \sigma_0 + b > \sigma_a, T \geq 1$. Also, $x \geq 1$, then we have Perron formula as follows [10]:

$$
\sum_{n \leq x} a(n)n^{-s_0} + \frac{1}{2}a(x)x^{-s_0} = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} A(s_0 + s) \frac{x^s}{s} ds + O\left(\frac{x^b B(b + \sigma_0)}{T}\right) + O(x^{1-\sigma_a}H(2x)\min\left(1, \frac{\log(x)}{T}\right)) (1-3)
$$

where "O" denotes a constant depending on $\sigma_a$ and $b_0$. $\sigma_a$ denotes absolute convergence abscissa of Dirichlet’s series. Refer to [10] in order to see the proof.

If $s_0 = 0$, then the relation (1-3) will be:

$$
\sum_{n \leq x} a(n) + \frac{1}{2}a(x) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} A(s) \frac{x^s}{s} ds + O\left(\frac{x^b B(b)}{T}\right) + O(xH(2x)\min\left(1, \frac{\log(x)}{T}\right)) (2-3)
$$

Lemma 3.2

Let $c_1$ be a positive constant, then $\zeta(s, \eta)$ has no zero point for $\sigma \geq 1 - \frac{c_1}{\log(|t|+2)}$ where $s = \sigma + it$.

**Proof:**

Let’s define a new function as:

$$
\zeta(s, \hat{\eta}) = \sum_{n=1}^{\infty} \frac{(1 - \eta(n))}{n^s} (3-3)
$$

If $\sigma \geq 1 - \frac{c_1}{\log(|t|+2)}$ then can be written:

$$
|\zeta(s, \eta)| \leq |\zeta(s, \hat{\eta})| (4-3)
$$

Also,

$$
|\zeta(s, \eta) + \zeta(s, \hat{\eta})| \leq |\zeta(s)| (5-3)
$$

Using Rouche’s Theorem, the number of zeros of the function $\zeta(s, \eta)$ is equal to the zeros of the function $\zeta(s)$. Since $\zeta(s)$ has no zero in the set $\sigma \geq 1 - \frac{c_1}{\log(|t|+2)}$, therefore $\zeta(s, \eta)$ has also no zero in the same set. This proves the Lemma.

Lemma 3.3

Let Dirichlet’s series $\Psi(s, \eta) = \sum_{n=1}^{\infty} \frac{n\Lambda(n)}{\phi(n)}\eta(n), \sigma_c = \sigma_\alpha = 1$ where $\sigma_c$ and $\sigma_\alpha$ denote the convergence and absolute convergence abscissa series respectively.

**Proof:**

Using the Prime Number Theorem for Arithmetic Progressions and $\gcd(k, N) = 1$ one can write by referring to [11]:

$$
\sum_{n \leq x, n \equiv N \pmod{k}} A(n) = \frac{x}{\phi(k)} + O\left(\frac{x}{\log^2(x)}\right) (6-3)
$$
\[
\lim_{x \to \infty} \sum_{n \leq x, \ n \equiv N \mod(k)} A(n) = \frac{x}{\phi(k)} \quad (7-3)
\]

For sufficiently large number \( N \):

\[
\eta(n) = \prod_{p > \sqrt{N}, \gcd(p, N) = 1} (1 - \frac{1}{\phi(p)}) \quad (8-3)
\]

Generally, one can write:

\[
\sum_{n \leq x} \eta(n)A(n) = \sum_{n \leq x} \prod_{p > \sqrt{N}, \gcd(p, N) = 1} (1 - \frac{1}{\phi(p)}) \sum_{n \leq x} A(n) \quad (9-3)
\]

and for sufficiently large number \( N \) is:

\[
\sum_{n \leq x} \eta(n)A(n) = \sum_{n \leq x} \prod_{p > \sqrt{N}, \gcd(p, N) = 1} (1 - \frac{1}{\phi(p)}) \sum_{n = 1}^{\infty} A(n) = \\
x \lim_{x \to \infty} \sum_{n \leq x} \prod_{p > \sqrt{N}, \gcd(p, N) = 1} (1 - \frac{1}{p - 1}) \quad (10-3)
\]

Just, we give a conjecture namely the second Sabihi’s conjecture on the Goldbach’s conjecture as below (to see our first conjecture refer to [8]):

**Second Sabihi’s Conjecture (SSC)**

Let \( \gamma \) be the Euler constant, \( p \) a prime number, \( N \) a sufficiently large even number, and Riemann Hypothesis holds, then:

\[
\log(N) = \frac{4e^{-\gamma} \prod_{p \geq 2} (1 - \frac{1}{(p-1)^2}) \prod_{p > 2, p \nmid N} \frac{p-1}{p-2} (1 + O(\frac{1}{\log(N)}))}{\sum_{n \leq x} \prod_{p > \sqrt{N}, \gcd(p, N) = 1} (1 - \frac{1}{p-1})} \quad (11-3)
\]

Having the conjecture the relation (10-3) gives:

\[
\prod_{p > 2} (1 - \frac{1}{(p-1)^2}) \prod_{p > 2, p \nmid N} \frac{p-1}{p-2} (1 + O(\frac{1}{\log(N)})) \frac{x}{\log(N)} = 4e^{-\gamma} \times \\
\sum_{n \leq x} \prod_{p > 2} (1 - \frac{1}{(p-1)^2}) \prod_{p > 2, p \nmid N} \frac{p-1}{p-2} (1 + O(\frac{1}{\log(N)})) \frac{x}{\log(N)} \quad (12-3)
\]

Assume we hold \( N \) to a constant value and \( x \to \infty \) then could be written:

\[
\sigma_c = \lim_{x \to \infty} \frac{\log(\sum_{n \leq x} \eta(n)A(n))}{\log(x)} = 1 \quad (13-3)
\]

Therefore, \( \sigma_c = \sigma_a = 1 \) and the Lemma is proven.

**Lemma 3.4**

Let Dirichlet’s series in Lemma 3.3 have a pole at \( s = 1 \), then it has a residue of the following form at same pole:

\[
4e^{-\gamma} \prod_{p > 2} (1 - \frac{1}{(p-1)^2}) \prod_{p > 2, p \nmid N} \frac{p-1}{p-2} (1 + O(\frac{1}{\log(N)})) \frac{1}{\log(N)} \quad (14-3)
\]

**Proof:**

It is well-known
\[ \lim_{s \to 1} (1 - s) \sum_{n=1}^{\infty} \frac{A(n)}{n^s} = 1 \]  \hfill (15-3)

On the other hand
\[ \lim_{x \to \infty} \sum_{n \leq x, \ n \equiv N \mod (k)} \frac{A(n)}{n^s} = \frac{1}{\varphi(k)} \sum_{n=1}^{\infty} \frac{A(n)}{n^s} \]  \hfill (16-3)

Consequently
\[ \lim_{x \to \infty} \sum_{n \leq x, \ n \equiv N \mod (k)} \frac{\eta(n)A(n)}{n^s} = \sum_{n=1}^{\infty} \prod_{p > \sqrt{N}, \ \gcd(p,N)=1} \left(1 - \frac{1}{p-1} \right) \sum_{n=1}^{\infty} \frac{A(n)}{n^s} \]  \hfill (17-3)

and this is equal to
\[ 4e^{-\gamma} \prod_{p>2} (1 - \frac{1}{(p-1)^2}) \prod_{p>2, \ p \mid N} \frac{p-1}{p-2} \left(1 + O\left(\frac{1}{\log(N)}\right)\right) \frac{1}{\log(N)} \sum_{n=1}^{\infty} \frac{A(n)}{n^s} \]  \hfill (18-3)

Just, one could obtain the residue of the Dirichlet’s series as below:
\[ \lim_{s \to 1} (1 - s) \sum_{n=1}^{\infty} \frac{\eta(n)A(n)}{n^s} = 4e^{-\gamma} \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p>2, \ p \mid N} \frac{p-1}{p-2} \left(1 + O\left(\frac{1}{\log(N)}\right)\right) \times \]  
\[ \frac{1}{\log(N)} \left(\lim_{s \to 1} (1 - s)\right) \sum_{n=1}^{\infty} \frac{A(n)}{n^s} = 4e^{-\gamma} \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) \times \]  
\[ \prod_{p>2, \ p \mid N} \frac{p-1}{p-2} \left(1 + O\left(\frac{1}{\log(N)}\right)\right) \frac{1}{\log(N)} \]  \hfill (19-3)

\section*{4 PROOF OF INEQUALITY \( \pi(N, \eta) \geq 1 \) FOR SUFFICIENTLY LARGE EVEN INTEGER \( N \)}

In this section, we prove one main theorem. On the basis of this theorem, the inequality \( \pi(N, \eta) \geq 1 \) under two conditions will be proved. The first condition is to be assumed the trueness of Riemann Hypothesis (RH) and the second is to be assumed to hold true second Sabihi’s conjecture (SSC).

\textbf{Sabihi’s Theorem}

Let \( N \) be a sufficiently large even integer. Let both second Sabihi’s Conjecture (SSC) and Riemann Hypothesis (RH) hold then:
\[ \psi(N, \eta) = 4e^{-\gamma} \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p>2, \ p \mid N} \frac{p-1}{p-2} \left(1 + O\left(\frac{1}{\log(N)}\right)\right) \frac{N}{\log(N)} + \]  
\[ O(N \{H(2N)\min\left(1, \frac{\log(N)}{T}\right) - e^{-c\sqrt{\log(N)}}\}) \]  \hfill (20-4)
\[ \pi(N, \eta) = 4e^{-\gamma} \prod_{p_2} (1 - \frac{1}{(p-1)^2}) \prod_{p_2 . | N} \frac{p - 1}{p - 2} (1 + O(\frac{1}{\log(N)})) \frac{N}{\log^2 N} + \]
\[
\int_{2}^{N} \frac{du}{\log^2 u} + O(\frac{XH(2N)\min(1, \frac{\log(N)}{p}) - NE^{-c\sqrt{\log(N)}}}{\log(N)}) + O(\int_{2}^{N} (e^{-c\sqrt{\log(u)}} - u^{-p})du) \]
\[
(21-4) \]

**Proof:**

Let \( a = 1 - c_1 \frac{1}{\log(T+2)} \), \( b = 1 + \frac{1}{\log(z)} \), \( \log T = (\log(x))^{1/\alpha} \) for \( 0 < \alpha < 1 \), and \( H(u) \leq \log u, B(u) \leq c_2 \log(x) \) where \( c_2 \) is a positive constant. By applying the Lemma 3.1, the relation (2-3) and assuming \( A(s) = \sum_{n=1}^{\infty} a(n)n^{-s} \) from Lemma 3.1:

\[
\sum_{n \leq x} a(n) = \sum_{n \leq x} \eta(n)A(n) = \psi(x, \eta) \]
\[
(22-4) \]

Let \( C_n \) be a rectangle contour with vertices \( a \pm iT \) and \( b \pm iT \) then if \( T \) tends to infinity

\[
I = \frac{1}{2\pi i} \int_{C_n} \Psi(s, \eta) \frac{x^s}{s} ds = J + K \]
\[
(23-4) \]

where \( J \) denotes integral along the line joining \( b - iT \) to \( b + iT \) and \( K \) denotes the integral along the other three sides of rectangle. Applying the relation (2-3) to the relation (23-4) gives

\[
I = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} \Psi(s, \eta) \frac{x^s}{s} ds + O(f(x)) + O(g(x)) = \]
\[
(24-4) \]

\[
J = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} \Psi(s, \eta) \frac{x^s}{s} ds \]
\[
(25-4) \]

and

\[
K = \frac{1}{2\pi i} \{ \int_{b-iT}^{a-iT} + \int_{a+iT}^{a-iT} + \int_{a+iT}^{b+iT} \} \Psi(s, \eta) \frac{x^s}{s} ds \]
\[
(26-4) \]

\[
O(f(x)) = \frac{x \log(x)}{T} \lim_{T \to \infty} = 0 \]
\[
(27-4) \]

\[
O(g(x)) = \left( \int_{a-i\infty}^{b+i\infty} + \int_{b-i\infty}^{a+i\infty} \right) \Psi(s, \eta) \frac{x^s}{s} ds \leq \]
\[
\left( \int_{b-i\infty}^{a-i\infty} + \int_{a+i\infty}^{b+i\infty} \right) \sum_{n=1}^{N} A(n)n^{-s} \frac{x^s}{s} ds = \]
\[
\left( \int_{b-i\infty}^{a-i\infty} + \int_{a+i\infty}^{b+i\infty} \right) (Z(s)) \frac{x^s}{s} ds \leq \]
\[
\ll \left( \int_{b-i\infty}^{a-i\infty} + \int_{a+i\infty}^{b+i\infty} \right) \log^2 |t| \frac{x^s}{s} ds \]
\[
(28-4) \]

Referring to [10], one can conclude that \( z(s) \ll \log^2 |t| \) since \( \sigma > 1 - c_1 \log^2 |t| \). By manipulating the last right hand term of inequality (28-4) we obtain as below:

\[
O(g(x)) = xe^{-c\sqrt{\log(x)}} \]
\[
(29-4) \]
Just, residue theorem expresses that

\[ I = 4e^{-\gamma} \prod_{p > 2} \left(1 - \frac{1}{(p - 1)^2}\right) \prod_{p > 2, p \mid N} \frac{p - 1}{p - 2} \left(1 + O\left(\frac{1}{\log(N)}\right)\right) \frac{x}{\log(N)} \]  

(30-4)

The relation (2-3) gives:

\[ \sum_{n \leq x} a(n) = \psi(x, \eta) = I - O(f(x)) - O(g(x)) + O\left(\frac{x^h B(b)}{T}\right) + O\left(x H(2x) \min(1, \frac{\log(x)}{T})\right) \]  

(31-4)

Consequently

\[ \psi(x, \eta) = 4e^{-\gamma} \prod_{p > 2} \left(1 - \frac{1}{(p - 1)^2}\right) \prod_{p > 2, p \mid N} \frac{p - 1}{p - 2} \left(1 + O\left(\frac{1}{\log(N)}\right)\right) \frac{x}{\log(N)} + \]  

\[ O\left(x \left\{ H(2x) \min(1, \frac{\log(x)}{T}) - e^{-c \sqrt{\log(x)}} \right\}\right) \]  

(32-4)

If we apply \( x = N \) for a sufficiently large even integer, then the first formula of the theorem is proven. By applying and combining the relations (12-2) to (14-2) and (32-4) could be written:

\[ \pi(x, \eta) = \frac{\psi(x, \eta) - O(x^{1/2})}{\log(x)} + \int_{2}^{x} \frac{\psi(u, \eta) - O(u^{1/2})}{u \log^2(u)} \, du = 4e^{-\gamma} \prod_{p > 2} \left(1 - \frac{1}{(p - 1)^2}\right) \times \]  

\[ \prod_{p > 2, p \mid N} \frac{p - 1}{p - 2} \left(1 + O\left(\frac{1}{\log(N)}\right)\right) \frac{x}{\log(N)} + \]  

\[ O\left(\frac{x H(2x) \min(1, \frac{\log(x)}{T})}{\log(x)}\right) - O\left(\frac{x e^{-c \sqrt{\log(x)}}}{\log(x)}\right) + \int_{2}^{x} \frac{du}{u \log^2 u} + O\left(\int_{2}^{x} \frac{e^{-c \sqrt{\log(u)}} \, du}{\log^2 u}\right) - \]  

\[ O\left(\frac{x^{1/2}}{\log(x)}\right) - O\left(\int_{2}^{x} \frac{du}{u^{1/2} \log^2 u}\right) \]  

(33-4)

Again, applying \( x = N \) to the relation (33-4) for a sufficiently large even integer, the second formula of the theorem is also proven. In the above relation, the following relation and inequality have been applied. Applying the Prime Number Theorem:

\[ \psi(x) = x + O(x e^{-c \sqrt{\log(x)}}) \]  

(34-4)

and

\[ \int_{2}^{x} \frac{\psi(u, \eta) \, du}{u \log^2 u} \leq \int_{2}^{x} \frac{\psi(u) \, du}{u \log^2 u} \]  

(35-4)

From the relation (33-4) and replacing \( x \) by \( N \) and tending \( N \) to infinity, we easily see that \( \pi(N, \eta) > 1 \) or \( \pi(N, \eta) \neq 0 \) and the theorem is proven. This proves the strong Goldbach’s conjecture for sufficiently large even integers.

References

1. V. Brun "Descrile d’Eratosthene et Theorhme de Goldbach" Videnskabsselskabets:Kristiania Skrifter I,Mate-Naturvidenskapelige Klasse,3,1920,1-36
2. Y. Wang "On the Representation of a Large Integer as the Sum of a Prime and an Almost Prime" Sci.Sin.11,1962,1033-1054.
3. C.T.Pan "On the Representation of an Even Number as the Sum of a Prime and an Almost Prime" Acta Math.Sin.12,1962,95-106.
4. H.E. Richert "Selberg's Sieve with Weights" Mathematica 16, 1969, 1-22.
5. J.R. Chen "On the Representation of a Large Even Integer as the Sum of a Prime and the Product of at most Two Primes" Sci.Sin. 17, 1973, 157-176.
6. A. Rényi "On the Representation of an Even Integer as the Sum of a single Prime and a Single Almost Prime Number" Dokl. Akad. Nauk SSSR 56, 1947, 455-458.
7. H.L. Montgomery and R.C. Vaughan "The Exceptional Set in Goldbach's Problem" Acta. Arith. 27, 1975, 353-370.
8. Ahmad Sabihi "The Novel Researches Toward the Proof of the Goldbach's Conjecture by the Novel Functions, the Novel Conjecture, the Riemann Zeta Function, and the Novel Experimental Computations" Bull. Allahabad Math. Soc. 25, 2010, 77-123.
9. Ahmad Sabihi "A Novel Explicit Formula for Computing \( \pi(x) \) the Number of Primes \( \leq x \)" Int. J. Math. Analysis 3, 2009, 1893-1903.
10. C.D. Pan and C.B. Pan "Basic Analytic Number Theory" Science Press, Beijing, 1999.
11. Serge Lang "Algebraic Number Theory" Springer-Verlag, New York 1986.