FURTHER STABILITY ANALYSIS OF NEUTRAL-TYPE COHEN-GROSSBERG NEURAL NETWORKS WITH MULTIPLE DELAYS

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ABSTRACT. The key contribution of this paper is to study the stability analysis of neutral-type Cohen-Grossberg neural networks possessing multiple time delays in the states of the neurons and multiple neutral delays in time derivative of states of the neurons. By making the use of a proper Lyapunov functional, we propose a novel sufficient time-independent stability criterion for this model of neutral-type neural networks. The proposed stability criterion in this paper can be absolutely expressed in terms of the parameters of the neural network model considered as this newly proposed criterion only relies on the relationships established among the network parameters. A numerical example is also given to indicate the advantages of the obtained stability criterion over the previously published stability results for the same class of Cohen-Grossberg neural networks. Since obtaining stability conditions for neutral-type Cohen-Grossberg neural networks with multiple delays is a difficult task to achieve, there are only few papers in the literature dealing with this problem. Therefore, the results given in the current paper makes an important contribution to the stability problem for this class of neutral-type neural networks.

1. Introduction. In recent decades, various classes of neural networks have been employed to solve many different engineering problems arising in the real world applications such as moving image processing, control and optimization applications, parallelly computing systems, associative memory design, (The readers can refer to the references [1]-[7] for real world application of neural systems). In the design of neural networks for solving practical engineering problems, it is crucial to address the stability and equilibrium characterization of these designed neural systems. In particular, in the electronic implementation of neural systems using VLSI technology, due to the finite switching speed of amplifiers and the communication times among the neurons bring about some unavoidable time delays, which can change the aimed dynamical properties of neural systems. Because of such dynamical problems caused by time delays, it is a critical task to investigate the stability criteria for neural networks whose dynamical model involve delay parameters. In the recent literature, many various stability results have been proposed, which establish the global asymptotic stability of different classes of neural systems in the presence of

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time delays [8]-[25]. On the other hand, it is known that including the delay parameters in the time derivative of states of the neurons enables us to determine a complete characterization of dynamical properties of delayed neural systems. Such a modification in neural networks leads us to establish the delayed neutral-type neural network models. Such neutral-type systems have useful applications in population ecology, distributed networks with loss less transmission lines, propagation and diffusion models [26]-[28].

This paper will consider the neutral-type Cohen-Grossberg neural networks possessing multiple time delays in the states of the neurons and multiple neutral delays in time derivatives of the states of the neurons described by the following sets of differential equations:

\[
\dot{x}_i(t) = d_i(x_i(t)) \left( -c_i(x_i(t)) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t - \tau_{ij})) + u_i \right) + \sum_{j=1}^{n} e_{ij} \dot{x}_j(t - \zeta_{ij}), \quad i = 1, \cdots, n.
\]

(1)

in which \(x_i(t)\) denotes the state of the \(i\)th neuron, \(c_i(x_i)\) are some behaved functions and \(d_i(x_i(t))\) represent the amplification functions. The constant parameters \(a_{ij}\) and \(b_{ij}\) represent the values of the interconnections among the neurons. \(\tau_{ij} (1 \leq i, j \leq n)\) represent the time delay parameters and \(\zeta_{ij} (1 \leq i, j \leq n)\) represent the neutral delay parameters. The parameters \(e_{ij}\) denote the coefficients of the time derivative of the states involving delays. The nonlinear activation functions are denoted by \(f_j(\cdot)\) and \(u_i\) are constant inputs. In (1), if assume that \(\tau = max\{\tau_{ij}\}, \zeta = max\{\zeta_{ij}\}, 1 \leq i, j \leq n, \) and \(\xi = max\{\tau, \zeta\}.\) In this case, neural system (1) has the initial conditions given by: \(x_i(t) = \varphi_i(t)\) and \(\dot{x}_i(t) = \psi_i(t) \in C([-\xi,0],R)\) with \(C([-\xi,0],R)\) being the set of all continuous functions from \([-\xi,0]\) to \(R.\)

Before proceeding with the stability analysis of neutral system (1), we need to give the properties of \(d_i(x_i(t)), c_i(x_i(t))\) and \(f_i(x_i(t)).\) These functions are assumed to possess following main conditions:

**A_1:** For the functions \(d_i(x_i(t))\), there exist positive real numbers \(\mu_i\) and \(\rho_i\) such that the following conditions hold:

\[0 < \mu_i \leq d_i(x_i(t)) \leq \rho_i, \quad i = 1, 2, \ldots, n, \quad \forall x_i(t) \in R.\]

**A_2:** For the functions \(c_i(x_i(t))\), there exist positive real numbers \(\gamma_i\) and \(\psi_i\) such that the following conditions hold:

\[0 < \gamma_i \leq \frac{c_i(x_i(t)) - c_i(y_i(t))}{x_i(t) - y_i(t)} = \frac{|c_i(x_i(t)) - c_i(y_i(t))|}{|x_i(t) - y_i(t)|} \leq \psi_i, \quad \forall x_i(t), y_i(t) \in R, x_i(t) \neq y_i(t), \quad i = 1, 2, \ldots, n.\]

**A_3:** For the functions \(f_i(x_i(t))\) there exist positive real numbers \(\ell_i\) such that the following conditions hold:

\[|f_i(x_i(t)) - f_i(y_i(t))| \leq \ell_i |x_i(t) - y_i(t)|, \quad \forall x_i(t), y_i(t) \in R, x_i(t) \neq y_i(t), \quad i = 1, 2, \ldots, n.\]

Since, neutral-type neural system (1) cannot be stated in the matrix-vector form due to involving multiple delays, studying the stability of system (1) has been a very difficult problem to overcome. Therefore, in the past literature, many researchers have focused on the stability analysis of a special model of system (1), which is
described by the following sets of equations:

\[ \dot{x}_i(t) = d_i(x_i(t)) \left( -c_i(x_i(t)) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t - \tau_j)) + u_i \right) 
+ \sum_{j=1}^{n} e_{ij} \dot{x}_j(t - \zeta_j), \quad i = 1, \ldots, n. \]  

(2)

Note that neutral system (2) is a specialized model of neutral system (1) with the assumptions that \( \tau_{ij} = \tau_j \) and \( \zeta_{ij} = \zeta_j, \forall i, j. \) Note that system (2) can be mathematically expressed in the vector-matrix form. This makes it possible to develop and employ some suitable Lyapunov functionals to study the stability problem for system (2). There exist many results ensuring the stability of system (2) in the past literature [29]-[42] where various forms of neutral system (2) have been considered. In these papers, the proposed results have been derived by employing various and modified Lyapunov functionals and these stability results have been expressed in the various forms of linear-matrix inequalities (LMIs). On the other hand, some recent papers have presented some new algebraic stability criteria which can be considered as alternative results to those that are in the LMI forms [43]-[47].

To the best of the knowledge of the author of this paper, only a recent paper has presented some results on the stability of neutral system (1) [48]. In our current paper, by developing a novel Lyapunov functional, we will study neutral system (1) and derive some new and alternative global asymptotic stability conditions for this system.

2. Stability analysis. This section will deal with determining a new criterion that guarantees the global stability of the equilibrium point of delayed Cohen-Grossberg neural system of neutral-type given by (1). In order to achieve this task, the equilibrium points \( x^* = (x_1^*, x_2^*, \ldots, x_n^*)^T \) of delayed neutral system (1) is to be shifted to the origin. This is usually done by using the transformation \( z_i(t) = x_i(t) - x_i^* \), which can deduce the neutral system of the form

\[ \dot{z}_i(t) = \alpha_i(z_i(t)) \left( -\beta_i(z_i(t)) + \sum_{j=1}^{n} a_{ij} g_j(z_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(z_j(t - \tau_{ij})) \right) 
+ \sum_{j=1}^{n} e_{ij} \dot{z}_j(t - \zeta_{ij}), \quad i = 1, \ldots, n. \]  

(3)

where \( \alpha_i(z_i(t)) = d_i(z_i(t) + x_i^*), \beta_i(z_i(t)) = c_i(z_i(t) + x_i^*) - c_i(x_i^*), \) and \( g_i(z_i(t)) = f_i(z_i(t) + x_i^*) - f_i(x_i^*), \forall i. \) It should be pointed out that neutral-type neural system defined (3) possesses the properties of the assumptions \( A_1, A_2 \) and \( A_3. \) We can restate these assumptions for system (3) as follows:

- \( A_1: \mu_i \leq \alpha_i(z_i(t)) \leq \rho_i, \forall i, \)
- \( A_2: \gamma_i z_i^2(t) \leq \dot{z}_i(t) \beta_i(z_i(t)) \leq \psi_i z_i^2(t), \forall i, \)
- \( A_3: |g_i(z_i(t))| \leq \ell_i |z_i(t)|, \forall i. \)

We are now in the position to derive the main condition for the global asymptotic stability of system (1) which is given in the following theorem:

**Theorem 2.1.** Let the neutral-type neural system described by (3) satisfy the assumptions \( A_1 - A_3. \) Then, the origin of neural system (3) is globally asymptotically
stable if there exist positive constants \( \delta \) and \( \kappa \) such that the following conditions hold:

\[
\nu_i = \gamma_i^2 - (2 + \kappa) \sum_{j=1}^{n} a^2_{ij} \ell_i^2 - (2 + \delta) \sum_{j=1}^{n} b^2_{ij} \ell_j^2 > 0, \quad i = 1, \cdots, n
\]

\[
\nu_{ij} = \frac{1}{n} - (1 + \frac{1}{\kappa} + \frac{1}{\delta}) \sum_{j=1}^{n} \frac{1}{\mu_i^2} \varepsilon^2_{ij} > 0, \quad i, j = 1, \cdots, n.
\]

**Proof**: This theorem will be proved by utilizing the following positive definite Lyapunov functional:

\[
V(t) = \sum_{i=1}^{n} 2 \int_{0}^{z_i(t)} \frac{\beta_i(s)}{\alpha_i(z_i(t))} ds + \sum_{i=1}^{n} \sum_{j=1}^{n} \int_{t-\zeta_{ij}}^{t} \frac{1}{\alpha_j^2(z_j(s))} \hat{z}_j^2(s) ds
\]

\[
+ n(2 + \delta) \sum_{i=1}^{n} \sum_{j=1}^{n} b^2_{ij} g_j^2(z_j(t)) + \varepsilon \sum_{i=1}^{n} \sum_{j=1}^{n} \int_{t-\tau_{ij}}^{t} z_j^2(s) ds
\]

where \( \varepsilon \) is a positive constant whose numerical value will be determined later. The time derivative of \( V(t) \) along the trajectories of the neutral-type neural system described by (3) is derived to be in the following form:

\[
\dot{V}(t) = \sum_{i=1}^{n} 2 \frac{\beta_i(z_i(t))}{\alpha_i(z_i(t))} \dot{z}_i(t) + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{\alpha_j^2(z_j(t))} \hat{z}_j^2(t)
\]

\[
- \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{\alpha_j^2(z_j(t - \zeta_{ij}))} \hat{z}_j^2(t - \zeta_{ij})
\]

\[
+ n(2 + \delta) \sum_{i=1}^{n} \sum_{j=1}^{n} b^2_{ij} g_j^2(z_j(t))
\]

\[
- n(2 + \delta) \sum_{i=1}^{n} \sum_{j=1}^{n} b^2_{ij} g_j^2(z_j(t - \tau_{ij})))
\]

\[
+ \varepsilon \sum_{i=1}^{n} \sum_{j=1}^{n} z_j^2(t) - \varepsilon \sum_{i=1}^{n} \sum_{j=1}^{n} z_j^2(t - \tau_{ij})
\]

We can write the following equality:

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{\alpha_j^2(z_j(t))} \hat{z}_j^2(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{\alpha_i^2(z_i(t))} \hat{z}_i^2(t)
\]

\[
= \sum_{i=1}^{n} \frac{1}{\alpha_i^2(z_i(t))} \hat{z}_i^2(t)
\]

Using (5) in (4) results in

\[
\dot{V}(t) = \sum_{i=1}^{n} 2 \frac{\beta_i(z_i(t))}{\alpha_i(z_i(t))} \dot{z}_i(t) + \sum_{i=1}^{n} \frac{1}{\alpha_i^2(z_i(t))} \hat{z}_i^2(t)
\]

\[
- \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{\alpha_j^2(z_j(t - \zeta_{ij}))} \hat{z}_j^2(t - \zeta_{ij})
\]

\[
+ n(2 + \delta) \sum_{i=1}^{n} \sum_{j=1}^{n} b^2_{ij} g_j^2(z_j(t))
\]
Adding the term $2\beta$ from (3), we can write the equality:

$$-n(2+\delta)\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}^2 g_j^2(z_j(t-\tau_{ij}))$$

$$+\varepsilon\sum_{i=1}^{n}\sum_{j=1}^{n} z_j^2(t) - \varepsilon\sum_{i=1}^{n}\sum_{j=1}^{n} z_j^2(t-\tau_{ij})$$

$$=\sum_{i=1}^{n}\left(2\beta_i(z_i(t)) + \frac{\dot{z}_i(t)}{\alpha_i(z_i(t))}\right)\frac{\dot{z}_i(t)}{\alpha_i(z_i(t))}$$

$$-\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_j(z_j(t-\zeta_{ij}))\dot{z}_j(t-\zeta_{ij})$$

$$+n(2+\delta)\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}^2 g_j^2(z_j(t))$$

$$-n(2+\delta)\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}^2 g_j^2(z_j(t-\tau_{ij}))$$

$$+\varepsilon\sum_{i=1}^{n}\sum_{j=1}^{n} z_j^2(t) - \varepsilon\sum_{i=1}^{n}\sum_{j=1}^{n} z_j^2(t-\tau_{ij})$$

(6)

From (3), we can write the equality:

$$\frac{\dot{z}_i(t)}{\alpha_i(z_i(t))} = -\beta_i(z_i(t)) + \sum_{j=1}^{n} a_{ij}g_j(z_j(t)) + \sum_{j=1}^{n} b_{ij}g_j(z_j(t-\tau_{ij}))$$

$$+\frac{1}{\alpha_i(z_i(t))}\sum_{j=1}^{n} e_{ij}\dot{z}_j(t-\zeta_{ij}), \ i = 1, \ldots, n.$$  

(7)

Adding the term $2\beta_i(z_i(t))$ to the both sides of (7) leads to

$$2\beta_i(z_i(t)) + \frac{\dot{z}_i(t)}{\alpha_i(z_i(t))} = \beta_i(z_i(t)) + \sum_{j=1}^{n} a_{ij}g_j(z_j(t)) + \sum_{j=1}^{n} b_{ij}g_j(z_j(t-\tau_{ij}))$$

$$+\frac{1}{\alpha_i(z_i(t))}\sum_{j=1}^{n} e_{ij}\dot{z}_j(t-\zeta_{ij}), \ i = 1, \ldots, n.$$  

(8)

Multiplying (7) by (8) results in

$$\left(2\beta_i(z_i(t)) + \frac{\dot{z}_i(t)}{\alpha_i(z_i(t))}\right)\frac{\dot{z}_i(t)}{\alpha_i(z_i(t))}$$

$$= \left(-\beta_i(z_i(t)) + \sum_{j=1}^{n} a_{ij}g_j(z_j(t)) + \sum_{j=1}^{n} b_{ij}g_j(z_j(t-\tau_{ij}))\right)$$

$$+\frac{1}{\alpha_i(z_i(t))}\sum_{j=1}^{n} e_{ij}\dot{z}_j(t-\zeta_{ij})\right) \times \left(\beta_i(z_i(t)) + \sum_{j=1}^{n} a_{ij}g_j(z_j(t))$$

$$+\sum_{j=1}^{n} b_{ij}g_j(z_j(t-\tau_{ij})) + \frac{1}{\alpha_i(z_i(t))}\sum_{j=1}^{n} e_{ij}\dot{z}_j(t-\zeta_{ij})\right)$$
\[-\beta_i^2(z_i(t)) + \left( \sum_{j=1}^{n} a_{ij}g_j(z_j(t)) \right)^2 + \left( \sum_{j=1}^{n} b_{ij}g_j(z_j(t - \tau_{ij})) \right)^2 \\
+ \left( \frac{1}{\alpha_i(z_i(t))} \sum_{j=1}^{n} e_{ij} \dot{z}_j(t - \zeta_{ij}) \right)^2 \\
+ 2 \left( \sum_{j=1}^{n} a_{ij}g_j(z_j(t)) \right) \left( \sum_{j=1}^{n} b_{ij}g_j(z_j(t - \tau_{ij})) \right) \\
+ 2 \left( \sum_{j=1}^{n} a_{ij}g_j(z_j(t)) \right) \left( \frac{1}{\alpha_i(z_i(t))} \sum_{j=1}^{n} e_{ij} \dot{z}_j(t - \zeta_{ij}) \right) \\
+ 2 \left( \sum_{j=1}^{n} b_{ij}g_j(z(t - \tau_{ij})) \right) \left( \frac{1}{\alpha_i(z_i(t))} \sum_{j=1}^{n} e_{ij} \dot{z}_j(t - \zeta_{ij}) \right) \tag{9}\]

We now note the following inequalities:

\[2 \left( \sum_{j=1}^{n} a_{ij}g_j(z_j(t)) \right) \left( \sum_{j=1}^{n} b_{ij}g_j(z_j(t - \tau_{ij})) \right) \leq \left( \sum_{j=1}^{n} a_{ij}g_j(z_j(t)) \right)^2 + \left( \sum_{j=1}^{n} b_{ij}g_j(z_j(t - \tau_{ij})) \right)^2 \tag{10}\]

\[2 \left( \sum_{j=1}^{n} a_{ij}g_j(z_j(t)) \right) \left( \frac{1}{\alpha_i(z_i(t))} \sum_{j=1}^{n} e_{ij} \dot{z}_j(t - \zeta_{ij}) \right) \leq \kappa \left( \sum_{j=1}^{n} a_{ij}g_j(z_j(t)) \right)^2 + \frac{1}{\kappa} \left( \frac{1}{\alpha_i(z_i(t))} \sum_{j=1}^{n} e_{ij} \dot{z}_j(t - \zeta_{ij}) \right)^2 \tag{11}\]

and

\[2 \left( \sum_{j=1}^{n} b_{ij}g_j(z(t - \tau_{ij})) \right) \left( \frac{1}{\alpha_i(z_i(t))} \sum_{j=1}^{n} e_{ij} \dot{z}_j(t - \zeta_{ij}) \right) \leq \delta \left( \sum_{j=1}^{n} b_{ij}g_j(z(t - \tau_{ij})) \right)^2 + \frac{1}{\delta} \left( \frac{1}{\alpha_i(z_i(t))} \sum_{j=1}^{n} e_{ij} \dot{z}_j(t - \zeta_{ij}) \right)^2 \tag{12}\]

where \(\kappa\) and \(\delta\) are some positive constants. Using (10)-(12) in (9) yields:

\[(2\beta_i(z_i(t)) + \frac{\dot{z}_i(t)}{\alpha_i(z_i(t))}) \dot{z}_i(t) \]

\[\leq -\beta_i^2(z_i(t)) + 2 \left( \sum_{j=1}^{n} a_{ij}g_j(z_j(t)) \right)^2 + 2 \left( \sum_{j=1}^{n} b_{ij}g_j(z_j(t - \tau_{ij})) \right)^2 \\
+ \kappa \left( \sum_{j=1}^{n} a_{ij}g_j(z_j(t)) \right)^2 + \delta \left( \sum_{j=1}^{n} b_{ij}g_j(z_j(t - \tau_{ij})) \right)^2 \tag{13}\]

\[+ \frac{1}{\kappa} \left( \frac{1}{\alpha_i(z_i(t))} \sum_{j=1}^{n} e_{ij} \dot{z}_j(t - \zeta_{ij}) \right)^2 + \frac{1}{\delta} \left( \frac{1}{\alpha_i(z_i(t))} \sum_{j=1}^{n} e_{ij} \dot{z}_j(t - \zeta_{ij}) \right)^2 \\
+ \left( \frac{1}{\alpha_i(z_i(t))} \sum_{j=1}^{n} e_{ij} \dot{z}_j(t - \zeta_{ij}) \right)^2\]
Using (18) in (6) will give the following:

$$
= -\beta_i^2(z_i(t)) + (2 + \kappa) \left( \sum_{j=1}^{n} a_{ij} g_j(z_j(t)) \right)^2 \\
+ (2 + \delta) \left( \sum_{j=1}^{n} b_{ij} g_j(z_j(t - \tau_{ij})) \right)^2 \\
+ (1 + \frac{1}{\kappa} + \frac{1}{\delta}) \left( \sum_{j=1}^{n} c_{ij} \dot{z}_j(t - \zeta_{ij}) \right)^2
$$

Note the inequalities:

$$
\left( \sum_{j=1}^{n} a_{ij} g_j(z_j(t)) \right)^2 \leq n \sum_{j=1}^{n} a_{ij}^2 g_j^2(z_j(t)) \tag{14}
$$

$$
\left( \sum_{j=1}^{n} b_{ij} g_j(z_j(t - \tau_{ij})) \right)^2 \leq n \sum_{j=1}^{n} b_{ij}^2 g_j^2(z_j(t - \tau_{ij})) \tag{15}
$$

$$
\left( \frac{1}{\alpha_i(z_i(t))} \sum_{j=1}^{n} c_{ij} \dot{z}_j(t - \zeta_{ij}) \right)^2 \leq n \sum_{j=1}^{n} \frac{1}{\alpha_i^2(z_i(t))} c_{ij}^2 \dot{z}_j^2(t - \zeta_{ij}) \tag{16}
$$

Using (14)-(16) in (13) leads to:

$$
(2\beta_i(z_i(t)) + \frac{\dot{z}_i(t)}{\alpha_i(z_i(t))}) \dot{z}_i(t) = -\beta_i^2(z_i(t)) + (2 + \kappa) n \sum_{j=1}^{n} a_{ij}^2 g_j^2(z_j(t)) \\
+ (2 + \delta) n \sum_{j=1}^{n} b_{ij}^2 g_j^2(z_j(t - \tau_{ij})) \\
+ (1 + \frac{1}{\kappa} + \frac{1}{\delta}) n \sum_{j=1}^{n} \frac{1}{\alpha_i^2(z_i(t))} c_{ij}^2 \dot{z}_j^2(t - \zeta_{ij}) \tag{17}
$$

Thus, from (17), we can write

$$
\sum_{i=1}^{n} \left(2\beta_i(z_i(t)) + \frac{\dot{z}_i(t)}{\alpha_i(z_i(t))} \right) \dot{z}_i(t) \leq -\sum_{i=1}^{n} \beta_i^2(z_i(t)) + (2 + \kappa) n \sum_{j=1}^{n} a_{ij}^2 g_j^2(z_j(t)) \\
+ (2 + \delta) n \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij}^2 g_j^2(z_j(t - \tau_{ij})) \\
+ (1 + \frac{1}{\kappa} + \frac{1}{\delta}) n \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{\alpha_i^2(z_i(t))} c_{ij}^2 \dot{z}_j^2(t - \zeta_{ij}) \tag{18}
$$

Using (18) in (6) will give the following:

$$
\dot{V}(t) \leq -\sum_{i=1}^{n} \beta_i^2(z_i(t)) + (2 + \kappa) n \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 g_j^2(z_j(t)) \\
+ (2 + \delta) n \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij}^2 g_j^2(z_j(t)) \\
+ (2 + \delta) n \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij}^2 g_j^2(z_j(t))
$$
\[
+ (1 + \frac{1}{\kappa} + \frac{1}{\delta}) n \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{\alpha_{ij}(z_{ij}(t))} \dot{c}_{ij} z_{ij}^2(t - \zeta_{ij}) \\
- \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{\alpha_{ij}^2(z_{ij}(t - \zeta_{ij}))} \dot{z}_{ij}^2(t - \zeta_{ij}) \\
+ \varepsilon \sum_{i=1}^{n} \sum_{j=1}^{n} \dot{z}_{ij}^2(t) - \varepsilon \sum_{i=1}^{n} \sum_{j=1}^{n} \dot{z}_{ij}^2(t - \tau_{ij}) \\
= - \sum_{i=1}^{n} \beta_i^2(z_i(t)) + (2 + \kappa) n \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 g_i^2(z_i(t)) \\
+ (2 + \delta) n \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij}^2 g_i^2(z_i(t))
\]

Under the assumptions $\hat{A}_1 - \hat{A}_N$, we have $\alpha_{ij}^2(z_{ij}(t - \tau_{ij})) \leq \rho_{ij}^2$, $\alpha_{ij}^2(z_{ij}(t)) \geq \mu_{ij}^2$, \[\beta_i^2(z_i(t)) \geq \gamma_i^2 z_i^2(t)\] and $g_i^2(z_i(t)) \leq \ell_i^2 z_i^2(t)$, $i, j = 1, 2, \ldots, n$. Thus, (19) can be written as follows:

\[
\dot{V}(t) \leq - \sum_{i=1}^{n} \gamma_i^2 z_i^2(t) + (2 + \kappa) n \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 \ell_i^2 z_{ij}^2(t)
\]

\[
+ (2 + \delta) n \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij}^2 \ell_i^2 z_{ij}^2(t)
\]

\[
+ (1 + \frac{1}{\kappa} + \frac{1}{\delta}) n \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{\rho_{ij}^2} \dot{z}_{ij}^2(t - \zeta_{ij}) \\
- \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{\rho_{ij}^2} \dot{z}_{ij}^2(t - \zeta_{ij}) + \varepsilon \sum_{i=1}^{n} \sum_{j=1}^{n} \dot{z}_{ij}^2(t) - \varepsilon \sum_{i=1}^{n} \sum_{j=1}^{n} \dot{z}_{ij}^2(t - \tau_{ij}) \\
= - \sum_{i=1}^{n} \gamma_i^2 z_i^2(t) + (2 + \kappa) n \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 \ell_i^2 z_i^2(t)
\]

\[
+ (2 + \delta) n \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij}^2 \ell_i^2 z_i^2(t)
\]

\[
+ (1 + \frac{1}{\kappa} + \frac{1}{\delta}) n \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{\mu_{ij}^2} \dot{z}_{ij}^2(t - \zeta_{ij}) - \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{\mu_{ij}^2} \dot{z}_{ij}^2(t - \zeta_{ij}) \\
+ \varepsilon \sum_{i=1}^{n} \sum_{j=1}^{n} \dot{z}_{ij}^2(t) - \varepsilon \sum_{i=1}^{n} \sum_{j=1}^{n} \dot{z}_{ij}^2(t - \tau_{ij})
\]

Thus, (19) can be written as follows:
\[
\dot{V}(t) \leq - \sum_{i=1}^{n} \sum_{j=1}^{n} \nu_{ij} \dot{z}_{ij}^2(t) + \varepsilon \sum_{i=1}^{n} \sum_{j=1}^{n} \dot{z}_{ij}^2(t)
\]
\[
= - \sum_{i=1}^{n} \nu_{i} \dot{z}_{i}^2(t) + n\varepsilon \sum_{i=1}^{n} \dot{z}_{i}^2(t)
\]
\[
\leq -\nu_{m} ||z(t)||^{2} + n\varepsilon ||z(t)||^{2}
\]
\[
= -(\nu_{m} - n\varepsilon) ||z(t)||^{2}
\]
where \(\nu_{m} = \min\{\nu_{i}\}\) and \(z(t) = (z_{1}(t), z_{2}(t), \ldots, z_{n}(t))^{T}\). In (21), the choice \(\varepsilon < \frac{\nu_{m}}{n}\) will guarantee that \(\dot{V}(t) < 0\) for all \(z(t) \neq 0\). Now, consider the case where \(z(t) = 0\). (Note that \(z_{i}(t) = 0\) implies that \(g_{i}(z_{i}(t)) = 0\). In this case, from (20), we obtain
\[
\dot{V}(t) \leq - \sum_{i=1}^{n} \sum_{j=1}^{n} \nu_{ij} \dot{z}_{ij}^2(t - \zeta_{ij}) - \varepsilon \sum_{i=1}^{n} \sum_{j=1}^{n} \dot{z}_{ij}^2(t - \tau_{ij})
\]
\[
\leq -\varepsilon \sum_{i=1}^{n} \sum_{j=1}^{n} \dot{z}_{ij}^2(t - \tau_{ij})
\]
It follows from (22) that if \(z_{j}(t - \tau_{ij}) \neq 0\) for any randomly selected pairs of \(i\) and \(j\), then, \(\dot{V}(t)\) will be strictly negative definite. Now, consider the case where \(z(t) = 0\) and \(z_{j}(t - \tau_{ij}) = 0, i, j = 1, 2, \ldots, n\). (Note that \(z_{j}(t - \tau_{ij}) = 0\) implies that \(g_{j}(z_{j}(t - \tau_{ij})) = 0\)). In this case, from (20), we obtain
\[
\dot{V}(t) \leq - \sum_{i=1}^{n} \sum_{j=1}^{n} \nu_{ij} \dot{z}_{ij}^2(t - \zeta_{ij})
\]
Since \(\nu_{ij} > 0, \forall i, j\), it follows from (23) that if \(\dot{z}_{j}(t - \zeta_{ij}) \neq 0\) for any randomly selected pairs of \(i\) and \(j\), then, \(\dot{V}(t)\) will be strictly negative definite. Now, consider the case where \(z_{i}(t) = 0, g_{i}(z_{i}(t)) = 0, z_{j}(t - \tau_{ij}) = 0, g_{j}(z_{j}(t - \tau_{ij})) = 0\) and \(\dot{z}_{j}(t - \zeta_{ij}) = 0, i, j = 1, 2, \cdots, n\). In this case, we have
\[
\dot{V}(t) = \sum_{i=1}^{n} \frac{1}{\alpha_{i}^{2}(z_{i}(t))} \dot{z}_{i}^2(t)
\]
Note that if \( z_i(t) = 0, g_i(z_i(t)) = 0, z_j(t - \tau_{ij}) = 0, g_j(z_j(t - \tau_{ij})) = 0 \) and 
\( \dot{z}_j(t - \zeta_{ij}) = 0, i, j = 1, 2, \ldots, n, \) then, from (3), we have \( \dot{z}_i(t) = 0, \forall i. \) Therefore, in this case, \( \dot{V}(t) = 0. \) Thus, one can directly see that the condition \( \dot{V}(t) = 0 \) holds iff when \( z_i(t) = 0, g_i(z_i(t)) = 0, z_j(t - \tau_{ij}) = 0, g_j(z_j(t - \tau_{ij})) = 0 \) and 
\( \dot{z}_j(t - \zeta_{ij}) = 0, i, j = 1, 2, \ldots, n, \) and \( \dot{V}(t) \) is negative definite for all the other cases. 
Based on the above analysis of the time derivative of the Lyapunov functional used in 
the stability analysis, it can be stated that the origin of neutral-type neural model (3) is 
asymptotically stable. It is also worth noting that the employed Lyapunov functional is 
radially unbounded, that is to say, \( V(z(t)) \to \infty \) as \( \|z(t)\| \to \infty. \) This property 
of the Lyapunov functional ensures that the origin of neutral-type neural model (3) is 
globally asymptotically stable. Thus, one can directly conclude that 
the equilibrium point of neutral-type neural model (1) is globally asymptotically 
stable.

3. Comparisons and an example. The following theorem has been given in [48]:

**Theorem 3.1.** Let the neutral-type neural system described by (3) satisfy the 
assumptions \( A_1 - A_3. \) Then, the origin of neural system (3) is globally asymptotically 
stable if the following conditions hold:

\[
\varepsilon_i = 2\mu_i\gamma_i - \sum_{j=1}^{n} (\rho_i\ell_j|a_{ij}| + \rho_j\ell_i|a_{ji}|) - \sum_{j=1}^{n} (\rho_i\ell_j|b_{ij}| + \rho_j\ell_i|b_{ji}|) \\
- \sum_{j=1}^{n} (\rho_i\psi_j|e_{ij}| + \rho_j\psi_i|e_{ji}|) - \sum_{j=1}^{n} \sum_{k=1}^{n} (\rho_i\ell_k|a_{ki}||e_{kj}| + \rho_k\ell_i|b_{ki}||e_{kj}|) \\
- \sum_{j=1}^{n} \sum_{k=1}^{n} (\rho_j\ell_k|a_{kj}||e_{ji}| + \rho_j\ell_k|b_{kj}||e_{ji}|) > 0, \forall i
\]

and

\[
\varepsilon_i = 1 - \sum_{j=1}^{n} |e_{ji}| > 0, \forall i.
\]

Then, the origin of neutral-type system (1) is globally asymptotically stable.

We now study the following example to exploit the effectiveness and advantages 
of the criterion proposed in Theorem 2.1.

**Example.** Consider the neutral-type neural network model given by (1) with the 
following system parameters:

\[
a_{ij} = \frac{1}{16}, \quad b_{ij} = \frac{1}{16}, \quad \text{and} \quad e_{ij} = e, \quad i, j = 1, 2, 3, 4.
\]

\[
\rho_1 = \rho_2 = \rho_3 = \rho_4 = 1, \quad \ell_1 = \ell_2 = \ell_3 = \ell_4 = 1
\]

\[
\mu_1 = \mu_2 = \mu_3 = \mu_4 = 1, \quad \gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 1, 1,
\]

\[
\psi_1 = \psi_2 = \psi_3 = \psi_4 = \psi
\]

Let us first apply the results of Theorem 3.1 to this example to derive the stability 
conditions. The conditions of Theorem 3.1 are obtained as follows:

\[
\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 2, 2 - \sum_{j=1}^{4}(|a_{ij}| + |a_{ji}|) - \sum_{j=1}^{4}(|b_{ij}| + |b_{ji}|)
\]
\[-\sum_{j=1}^{4} \psi(|e_{ij}| + |e_{ji}|)\]
\[-\sum_{j=1}^{4} \sum_{k=1}^{4} (|a_{ki}| |e_{kj}| + |b_{ki}| |e_{kj}|)\]
\[-\sum_{j=1}^{4} \sum_{k=1}^{4} (|a_{jk}| |e_{ji}| + |b_{jk}| |e_{ji}|)\]
\[= 1,2 - 8\psi e - 4e > 0\]

from which the stability condition of Theorem 3.1 is derived as follows:
\[e < \frac{1,2}{8\psi + 4}\]

According to assumption $\tilde{A}_2$, for this example, the minimum value of $\psi$ is $\psi = 1,1$. For this value of $\psi$, $e$ must satisfy the condition $e < 0,09375$.

We now apply the results of Theorem 2.1 to this example to derive the stability conditions. For $\kappa = 6$ and $\delta = 6$, the conditions of Theorem 2.1 are obtained as follows:

\[\nu_i = 1,21 - 8n \sum_{j=1}^{n} a_{ji}^2 - 8n \sum_{j=1}^{n} b_{ji}^2\]
\[= 0,21 > 0, \quad i = 1,2,3,4\]

\[\nu_{ij} = \frac{1}{n} - \frac{4}{3} ne^2 > 0, \quad i, j = 1,2,3,4.\]

from which the stability condition of Theorem 2.1 is derived as follows:
\[e < \frac{\sqrt{3}}{8}\]

implying that $e < 0,2165$ is a sufficient condition for the stability of the system given in this example. When we compare stability conditions of this example imposed by Theorems 2.1 and 3.1, we can see that Theorem 2.1 imposes a less restrictive stability condition than Theorem 3.1 imposes.

4. Conclusions. This paper has made a useful contribution to the problem of the stability analysis of neutral-type Cohen-Grossberg neural networks possessing multiple time delays in the states of the neurons and multiple neutral delays in time derivative of states of the neurons. By making the use of a suitable Lyapunov functional, this paper has proposed a new sufficient time-independent stability condition for delayed neutral-type Cohen-Grossberg neural networks. The obtained stability criterion can be completely stated in terms of the parameters of the neural network model considered as the proposed criterion only relies on the relationships established among the network parameters. A instructive numerical example has also been given to show the advantages of the derived stability criterion over the previously published stability results for the same class of Cohen-Grossberg neural networks. As pointed out before, obtaining stability conditions for neutral-type Cohen-Grossberg neural networks with multiple delays is a dificult task to achieve.
Therefore, the stability condition given in the current paper makes an important contribution to the stability problem for this class of neutral-type neural networks.

REFERENCES

[1] L. O. Chua and L. Yang, Cellular neural networks: Applications, IEEE Trans. Circuits and Systems, 35 (1988), 1273–1290.
[2] M. A. Cohen and S. Grossberg, Absolute stability of global pattern formation and parallel memory storage by competitive neural networks, IEEE Trans. Systems Man Cybernet., 13 (1983), 815–826.
[3] J. Hopfield, Neural networks and physical systems with emergent collective computational abilities, Proc. Nat. Acad. Sci. U.S.A., 79 (1982), 2554–2558.
[4] A. Guez, V. Protopopescu and J. Barhen, On the stability, and design of nonlinear continuous neural networks, IEEE Trans. Syst. Man Cybernetics, 18 (1998), 80–87.
[5] J. Wang, Y. Cai and J. Yin, Multi-start stochastic competitive Hopfield neural network for frequency assignment problem in satellite communications, Expert Syst. Appl., 38 (2011), 131–145.
[6] S. C. Tong, Y. M. Li and H. G. Zhang, Adaptive neural network decentralized backstepping output-feedback control for nonlinear large-scale systems with time delays, IEEE Trans. Neural Networks, 22 (2011), 1073–1086.
[7] M. Galicki, H. Witte, J. Dorschel, M. Eiselt and G. Griessbach, Common optimization of adaptive preprocessing units and a neural network during the learning period. Application in EEG pattern recognition, Neural Networks, 10 (1997), 1153–1163.
[8] Q. Zhu, J. Cao and R. Rakkiyappan, Exponential input-to-state stability of stochastic Cohen-Grossberg neural networks with mixed delays, Nonlinear Dynam., 79 (2015), 1085–1098.
[9] Q. Zhu and J. Cao, Robust exponential stability of Markovian jump impulsive stochastic Cohen-Grossberg neural networks with mixed time delays, IEEE Trans. Neural Netw., 21 (2010), 1314–1325.
[10] R. Manivannan, R. Samidurai, J. Cao, A. Alsaedi and F. E. Alsaadi, Stability analysis of interval time-varying delayed neural networks including neutral time-delay and leakage delay, Chaos Solitons Fractals, 114 (2018), 433–445.
[11] Q. Zhu and J. Cao, Stability analysis of Markovian jump stochastic BAM neural networks with impulse control and mixed time delays, IEEE Trans. Neural Netw. Learn. Syst., 23 (2012), 467–479.
[12] Q. Song, Q. Yu, Z. Zhao, Y. Liu and F. E. Alsaadi, Boundedness and global robust stability analysis of delayed complex-valued neural networks with interval parameter uncertainties, Neural Networks, 103 (2018), 55–62.
[13] Q. Zhu and X. Li, Exponential and almost sure exponential stability of stochastic fuzzy delayed Cohen-Grossberg neural networks, Fuzzy Sets and Systems, 203 (2012), 74–94.
[14] Q. Zhu, Stability analysis of stochastic delay differential equations with Lévy noise, Systems Control Lett., 118 (2018), 62–68.
[15] W. Xie and Q. Zhu, Mean square exponential stability of stochastic fuzzy delayed Cohen-Grossberg neural networks with expectations in the coefficients, Neurocomputing, 166 (2015), 133–139.
[16] X. Tan, J. Cao and X. Li, Leader-following mean square consensus of stochastic multi-agent systems with input delay via event-triggered control, IET Control Theory Appl., 12 (2018), 299–309.
[17] Q. Zhu and J. Cao, Exponential stability analysis of stochastic reaction-diffusion Cohen-Grossberg neural networks with mixed delays, Neurocomputing, 74 (2011), 3084–3091.
[18] Q. Zhu and J. Cao, pth moment exponential synchronization for stochastic delayed Cohen-Grossberg neural networks with Markovian switching, Nonlinear Dynam., 67 (2012), 829–845.
[19] C. Ge, C. Hua and X. Guan, New delay-dependent stability criteria for neural networks with time-varying delay using delay-decomposition approach, IEEE Trans. Neural Netw. Learn. Syst., 25 (2014), 1378–1383.
[20] Z. Wang, L. Liu, Q. H. Shan and H. Zhang, Stability criteria for recurrent neural networks with time-varying delay based on secondary delay partitioning method, IEEE Trans. Neural Netw. Learn. Syst., 26 (2015), 2589–2595.
[21] X. Zhang, X. Li, J. Cao and F. Miaadi, Design of memory controllers for finite-time stabilization of delayed neural networks with uncertainty, J. Franklin Inst., 355 (2018), 5394–5413.
[22] I. Stamova, T. Stamov and X. Li, Global exponential stability of a class of impulsive cellular neural networks with suprema, *Internat. J. Adapt. Control Signal Process.*, 28 (2014), 1227–1239.

[23] Q. Zhu and R. Rakkiyappan and A. Chandrasekar, Stochastic stability of Markovian jump BAM neural networks with leakage delays and impulse control, *Neurocomputing*, 136 (2014), 136–151.

[24] H. Chen, P. Shi, C. C. Lim and P. Hu, Exponential stability for neutral stochastic Markov systems with time-varying delay and its applications, *IEEE Trans. Cybernetics*, 46 (2016), 1350–1362.

[25] L. Cheng, Z. G. Hou and M. Tan, A neutral-type delayed projection neural network for solving nonlinear variational inequalities, *IEEE Trans. Circuits Syst. II: Express Briefs*, 55 (2008), 806–810.

[26] S. I. Niculescu, *Delay Effects on Stability: A Robust Control Approach*, Lecture Notes in Control and Information Sciences, 269, Springer-Verlag, London, 2001.

[27] V. B. Kolmanovskii and V. R. Nosov, *Stability of Functional-Differential Equations*, Mathematics in Science and Engineering, 180, Academic Press, Inc., London, 1986.

[28] Y. Kuang, *Delay Differential Equations with Applications in Population Dynamics*, Mathematics in Science and Engineering, 191, Academic Press, Inc., Boston, MA, 1993.

[29] K. Shi, H. Zhu, S. Zhong, Y. Zeng and Y. Zhang, New stability analysis for neutral type neural networks with discrete and distributed delays using a multiple integral approach, *J. Franklin Inst.*, 352 (2015), 155–176.

[30] S. Muralisankar, A. Manivannan and P. Balasubramaniam, Mean square delay dependent-probability-distribution stability analysis of neutral type stochastic neural networks, *ISA Trans.*, 58 (2015), 11–19.

[31] H. Chen, Y. Zhang and P. Hu, Novel delay-dependent robust stability criteria for neutral stochastic delayed neural networks, *Neurocomputing*, 73 (2010), 2554–2561.

[32] S. Lakshmanan, J. H. Park, H. Y. Jung, O. M. Kwon and R. Rakkiyappan, A delay partitioning approach to delay-dependent stability analysis for neutral type neural networks with discrete and distributed delays, *Neurocomputing*, 111 (2013), 81–89.

[33] W. Hu, Q. Zhu and H. R. Karimi, Some improved Razumikhin stability criteria for impulsive stochastic delay differential systems, *IEEE Trans. Automat. Control*, 64 (2019), 5207–5213.

[34] S. Dharani, R. Rakkiyappan and J. Cao, New delay-dependent stability criteria for switched Hopfield neural networks of neutral type with additive time-varying delay components, *Neurocomputing*, 151 (2015), 827–834.

[35] K. Shi, S. Zhong, H. Zhu, X. Liu and Y. Zeng, New delay-dependent stability criteria for neutral-type neural networks with mixed random time-varying delays, *Neurocomputing*, 168 (2015), 896–907.

[36] G. Zhang, T. Wang, T. Li and S. Fei, Multiple integral Lyapunov approach to mixed-delay-dependent stability of neutral neural networks, *Neurocomputing*, 275 (2018), 1782–1792.

[37] H. Huang, Q. Du and X. Kang, Global exponential stability of neutral high-order stochastic Hopfield neural networks with Markovian jump parameters and mixed time delays, *ISA Trans.*, 52 (2013), 759–767.

[38] X. Liao, Y. Liu, H. Wang and T. Huang, Exponential estimates and exponential stability for neutral-type neural networks with multiple delays, *Neurocomputing*, 149 (2015), 868–883.

[39] Q. Zhu, Stabilization of stochastic nonlinear delay systems with exogenous disturbances and the event-triggered feedback control, *IEEE Trans. Automat. Control*, 64 (2019), 3764–3771.

[40] S. Arik, An analysis of stability of neutral-type neural systems with constant time delays, *J. Franklin Inst.*, 351 (2014), 4949–4959.

[41] C. H. Lien, K. W. Yu, Y. F. Lin, Y. J. Chung and L. Y. Chung, Global exponential stability for uncertain delayed neutral neural networks of neutral type with mixed time delays, *IEEE Trans. Syst. Man Cybernetics-PART B: Cybernetics*, 38 (2008), 709–720.

[42] Y. Yang, T. Liang and X. Xu, Almost sure exponential stability of stochastic Cohen-Grossberg neural networks with continuous distributed delays of neutral type, *Optik*, 126 (2015), 4628–4635.

[43] R. Samli and S. Arik, New results for global stability of a class of neutral-type neural networks with time delays, *Appl. Math. Comput.*, 210 (2009), 564–570.

[44] Z. Orman, New sufficient conditions for global stability of neutral-type neural networks with time delays, *Neurocomputing*, 97 (2012), 141–148.
[45] C. J. Cheng, T. L. Liao, J. J. Yan and C. C. Hwang, Globally asymptotic stability of a class of neutral-type neural networks with delays, *IEEE Trans. Syst. Man Cybernetics-PART B: Cybernetics*, 36 (2008), 1191–1195.

[46] H. Akca, V. Covachev and Z. Covacheva, Global asymptotic stability of Cohen-Grossberg neural networks of neutral type, *J. Math. Sci. (N.Y.)*, 205 (2015), 719–732.

[47] N. Ozcan, New conditions for global stability of neutral-type delayed Cohen-Grossberg neural networks, *Neural Networks*, 106 (2018), 1–7.

[48] N. Ozcan, Stability analysis of Cohen-Grossberg neural networks of neutral-type: Multiple delays case, *Neural Networks*, 113 (2019), 20–27.

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