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TOWARDS CANONICAL REPRESENTATIONS OF FINITE
HEISENBERG GROUPS

S. LYSENKO

Abstract. We consider a finite abelian group $M$ of odd exponent $n$ with a symplectic form $\omega : M \times M \rightarrow \mu_n$ and the Heisenberg extension $1 \rightarrow \mu_n \rightarrow H \rightarrow M \rightarrow 1$ with the commutator $\omega$. According to the Stone - von Neumann theorem, $H$ admits an irreducible representation with the tautological central character (defined up to a non-unique isomorphism). We construct such irreducible representation of $H$ defined up to a unique isomorphism, so canonical in this sense.

1. Introduction

1.0.1. Consider a finite abelian group $M$ of odd exponent $n$ with a symplectic form $\omega : M \times M \rightarrow \mu_n$. It admits a unique symmetric Heisenberg extension $1 \rightarrow \mu_n \rightarrow H \rightarrow M \rightarrow 1$ with the commutator $\omega$. According to the Stone - von Neumann theorem, $H$ admits an irreducible representation with the tautological central character (defined up to a non-unique isomorphism). We construct such irreducible representation of $H$ defined up to a unique isomorphism, so canonical in this sense, over a suitable finite extension of $\mathbb{Q}$.

1.0.2. We are motivated by the following question of Dennis Gaitsgory about [6]. Let $X$ be a smooth projective connected curve over an algebraically closed field $k$. Let $T$ be a torus over $k$ with a geometric metaplectic data $\mathcal{G}$ as in [3]. To fix ideas, consider the sheaf-theoretic context of $\ell$-adic sheaves on finite type schemes over $k$. Let $(H, \mathcal{G}_{\mathcal{Z}_H}, \epsilon)$ be the metaplectic Langlands dual datum associated to $(T, \mathcal{G})$ in ([3], Section 6), so $H$ is a torus over $\mathbb{Q}_\ell$ isogenous to the Langlands dual of $T$. Let $\sigma$ be a twisted local system on $X$ for $(H, \mathcal{G}_{\mathcal{Z}_H})$ in the sense of ([3], Section 8.4). To this data in ([3], Section 9.5.3) we attached the DG-category of Hecke eigensheaves. The question is whether this category identifies canonically with the DG-category $\text{Vect}$ of $\mathbb{Q}_\ell$-vector spaces. In [6] we constructed such an irreducible Hecke eigensheaf for $\sigma$ out of a given irreducible representation of certain finite Heisenberg group (denoted by $\Gamma$ given by formula (33) in [6], Section 5.2.4) with the tautological central character.

This is why we are interested in constructing a canonical irreducible representation of finite Heisenberg groups as in Section 1.0.1. We do this only assuming the order of $M$ odd, the case of even order remains open.

2. Main result

2.0.1. Let $e$ be an algebraically closed field of characteristic zero. Let $M$ be a finite abelian group, $\omega : M \times M \rightarrow e^*$ a bilinear form, which is alternating, that is, $\omega(m, m) =$
0 for any \( m \in M \). Assume the induced map \( M \to \text{Hom}(M, e^* ) \) is an isomorphism, that is, the form is nondegenerate.

If \( L \subset M \) is a subgroup, \( L^\perp = \{ m \in M \mid \omega(m, l) = 0 \text{ for all } l \in L \} \) is its orthogonal complement, this is a subgroup. The group \( L \) is isotropic if \( L \subset L^\perp \). The subgroup \( L \) is lagrangian if \( L^\perp = L \).

For a lagrangian subgroup we get an exact sequence
\[
0 \to L \to M \to L^* \to 0,
\]
where \( L^* = \text{Hom}(L, e^* ) \). Namely, we send \( m \in M \) to the character \( l \mapsto \omega(m, l) \) of \( L \).

This exact sequence always admits a splitting \( L^* \to M \), which is a homomorphism, see for example ([7], 4.1). For such a splitting after the obtained identification \( M \cong L \times L^* \) the form \( \omega \) becomes
\[
\omega((l_1, \chi_1), (l_2, \chi_2)) = \frac{\chi_1(l_2)}{\chi_2(l_1)}
\]
for \( l_i \in L, \chi_i \in L^* \).

By ([7], Theorem 2), up to an isomorphism, there is a unique central extension
\[
1 \to e^* \to H_{e^*} \to M \to 1
\]
with the commutator \( \omega \). We are interested in understanding the category of representations of \( H_{e^*} \) with the tautological central character.

2.0.2. For a finite abelian group \( L \) its exponent is the least common multiple of the orders of the elements of \( L \). Let \( n \) be the exponent of \( M \), this is a divisor of \( \sqrt{|M|} \in \mathbb{N} \).

Let \( \mu_n = \mu_n(e) \). Let us be given a central extension
\[
1 \to \mu_n \to H \to M \to 1
\]
together with a symmetric structure \( \sigma \) in the sense of ([1], Section 1.1) and commutator \( \omega \). That is, \( \sigma \) is an automorphism of \( H \) such that \( \sigma^2 = \text{id} \), \( \sigma|_{\mu} = \text{id} \), and \( \sigma \mod \mu_b \) is the involution \( m \mapsto -m \) of \( M \).

From now on assume \( n \) odd. Then by ([7], Section 1), there is a unique symmetric central extension (4) up to a unique isomorphism. Besides, (3) is isomorphic to the push-out of (4) under the tautological character \( \iota : \mu_n \hookrightarrow e^* \).

The extension \( H \) is constructed as follows. Let \( \beta : M \times M \to \mu_n \) be the unique alternating bilinear form such that \( \beta^2 = \omega \). We take \( H = M \times \mu_n \) with the product
\[
(m_1, a_1)(m_2, a_2) = (m_1 + m_2, a_1a_2\beta(m_1, m_2))
\]
for \( m_i \in M, a_i \in \mu_n \). Then \( \sigma(m, a) = (-m, a) \) for \( m \in M, a \in \mu_n \).

Let \( G = \text{Sp}(M) \), the group of automorphisms of \( M \) preserving \( \omega \). Let \( g \in G \) act on \( H \) sending \( (m, a) \) to \( (gm, a) \). This gives the semi-direct product \( H \rtimes G \).

2.0.3. The following version of the Stone - von Neumann theorem holds for \( H \), the proof is left to a reader.

**Proposition 2.0.4.** Up to an isomorphism, there is a unique irreducible representation of \( H \) over \( e \) with the tautological central character \( \iota : \mu_d \hookrightarrow e^* \).
2.0.5. Write \( \mathcal{L}(M) \) for the set of lagragian subgroups in \( M \). For \( L \in \mathcal{L}(M) \) let \( \bar{L} \) be the preimage of \( L \) in \( H \), this is a subgroup. If \( \chi_L : \bar{L} \to e^* \) is a character extending the tautological character \( \iota : \mu_b \hookrightarrow e^* \), set
\[
\mathcal{H}_L = \{ f : H \to e \mid f(\bar{lh}) = \chi_L(\bar{l})f(h), \text{ for } \bar{l} \in \bar{L}, h \in H \}
\]
This is a representation of \( H \) by right translations. It is irreducible with central character \( \iota \).

2.0.6. We study the following.

Problem: Describe the category \( \text{Rep}_\iota(H) \) of representations of \( H \) over \( e \) with central character \( \iota : \mu_b \hookrightarrow e^* \). Is there an object of \( \text{Rep}_\iota(H) \), which is irreducible and defined up to a unique isomorphism? (If yes, it would provide an equivalence between \( \text{Rep}_\iota(H) \) and the category of \( e \)-vector spaces).

2.0.7. Let \( I \) be the set of primes appearing in the decomposition of \( n \), write \( n = \prod_{p \in I} p^{r(p)} \) with \( r(p) > 0 \). Let \( K \subset e \) be the subfield generated over \( \mathbb{Q} \) by \( \{ \sqrt{p} \mid p \in I \} \) and \( \mu_n \).

Theorem 2.0.8. There is an irreducible representation \( \pi \) of \( H \) over \( K \) with central character \( \iota : \mu_n \hookrightarrow K^* \) defined up to a unique isomorphism. The \( H \)-action on \( \pi \) extends naturally to an action of \( H \rtimes G \).

Remark 2.0.9. Let \( K' \subset e \) be the subfield generated over \( \mathbb{Q} \) by \( \mu_n \). The field of definition of the character of \( \pi \) is \( K' \). However, we do not expect that for any \( H \) with \( n \) odd Theorem 2.0.8 holds already with \( K \) replaced by \( K' \), but we have not checked that. Our choices of \( \sqrt{p} \) for \( p \in I \) are made to use the results of \([5]\), and the formulas from \([5]\) do not work without these choices. Note also the following. If \( L \) is an odd abelian group, and \( b : L \times L \to e^* \) is a nondegenerate symmetric bilinear form then the Gauss sum of \( b \) is defined as
\[
G(L, b) = \sum_{l \in L} b(l, l)
\]
Using the classification of such symmetric bilinear forms given in \([4]\), one checks that \( G(L, b)^4 = |L|^2 \). Since the construction of \( \pi \) in Theorem 2.0.8 is related to representing the corresponding 2-cocyle (given essentially by certain Gauss sums) as a coboundary (after some minimal additional choices), we expect that our choices of \( \sqrt{p} \) for \( p \in I \) are necessary.

Remark 2.0.10. For \( L \in \mathcal{L}(M) \), the \( H \)-representation \( \mathcal{H}_L \) from Section 2.0.5 is defined over \( K \). We sometimes view it as a representation over \( K \), the precise meaning is hopefully clear from the context.

3. Proof of Theorem 2.0.8

3.0.1. Reduction. For \( p \in I \) let
\[
H_p = \{ h \in H \mid h^{(p^s)} = 1 \text{ for } s \text{ large enough} \}
\]
and similarly for \( M_p \). Then \( H_p \subset H \) is a subgroup that fits into an exact sequence \( 1 \to \mu_{p^{r(p)}} \to H_p \to M_p \to 1 \), and \( H = \prod_{p \in I} H_p \), product of groups. Indeed, \( \omega(H_p, H_q) = 1 \)
for \( p, q \in I, p \neq q \). Besides, \( \sigma \) preserves \( H_p \) for each \( p \in I \), so \((H_p, \sigma|_{H_p})\) is a symmetric Heisenberg extension of \((M_p, \omega_p)\) by \( \mu_{p^d} \). Here \( \omega_p : M_p \times M_p \to \mu_{p^d} \) is the restriction of \( \omega \). So, Problem 2.0.6 reduces to the case of a prime \( n \). If \( M_p = 0 \) then take \( \pi_p \) be the 1-dimensional representation given by the tautological character \( \mu_{p^d} \to e^x \).

For \( p \in I \) odd let \( K_p \subset e \) be the subfield generated over \( \mathbb{Q} \) by \( \mu_{p^d} \) and \( \sqrt{p} \). We prove Theorem 2.0.8 in the case of an odd prime \( n \) getting for \( p \in I \) a representation \( \pi_p \) of \( H_p \) over \( K_p \), hence over \( K \) also. Then for any odd \( n \), \( \pi = \otimes_{p \in I} \pi_p \) is the desired representation.

3.0.2. From now on we assume \( n = p^r \) for an odd prime \( p \).

3.1. **Case** \( r = 1 \).

3.1.1. In this section we assume \( M \) is a \( \mathbb{F}_p \)-vector space of dimension \( 2d \). To apply the results of [5], pick an isomorphism \( \psi : \mathbb{F}_p \to \mu_p \). It allows to identify \( H \) with \( M \times \mathbb{F}_p \). We then view \( \mathcal{L}(M), H \) as algebraic varieties over \( \mathbb{F}_p \). We allow the case \( d = 0 \) also.

3.1.2. Recall the following construction from ([5], Theorem 1).

Pick a prime \( \ell \neq p \), and an algebraic closure \( \mathbb{Q}_\ell \) of \( \mathbb{Q}_\ell \). We assume \( \mathbb{Q}_\ell \) is chosen in such a way that \( K \subset \mathbb{Q}_\ell \) is a subfield. In particular, we get \( \sqrt{p} \in K \). It gives rise to the \( \mathbb{Q}_\ell \)-sheaf \( \mathbb{Q}_\ell(\tfrac{1}{2}) \) over \( \text{Spec} \mathbb{F}_p \).

Pick a 1-dimensional \( \mathbb{F}_p \)-vector space \( J \) of parity \( d \mod 2 \) as \( \mathbb{Z}/2\mathbb{Z} \)-graded. Let \( A \) be the line bundle (of parity zero as \( \mathbb{Z}/2\mathbb{Z} \)-graded) on \( \mathcal{L}(M) \) with fibre \( J \otimes \det L \) at \( L \in \mathcal{L}(M) \). Write \( \mathcal{L}(M) \) for the gerbe of square roots of \( A \).

In loc.cit we have constructed an irreducible perverse sheaf \( F \) on \( \mathcal{L}(M) \times \mathcal{L}(M) \times H \). Though in loc.cit. we mostly worked over an algebraic closure \( \overline{\mathbb{F}}_p \), \( F \) is defined over \( \mathbb{F}_p \).

**Lemma 3.1.3.** For any \( i : \text{Spec} \mathbb{F}_p \to \mathcal{L}(M) \times \mathcal{L}(M) \times H \), \( \text{tr}(\text{Fr}, i^* F) \in K \). Here \( \text{Fr} \) is the geometric Frobenius endomorphism.

**Proof.** This follows from formula (10) in ([5], Section 3.3). Namely, after a surjective smooth localization (a choice of an additional lagrangian in \( M \)), there is an explicit formula for \( F \) as the convolution along \( H \) of two explicit rank one local systems. Their traces of Frobenius lie in \( K \), as their definition involves only the Artin-Schreier sheaf and Tate twists. So, the same holds after the convolution along the finite group \( H(\mathbb{F}_p) \).

3.1.4. For an algebraic stack \( S \to \text{Spec} \mathbb{F}_p \) we write \( S(\mathbb{F}_p) \) for the set of isomorphism classes of its \( \mathbb{F}_p \)-points. In view of the isomorphism \( \psi : \mathbb{F}_p \cong \mu_p \) fixed above, for \( L \in \mathcal{L}(M)(\mathbb{F}_p) \) we identify \( \bar{L} = L \times \mu_p \) with \( L \times \mathbb{F}_p \). Let

\[
F^{cd} : \mathcal{L}(M)(\mathbb{F}_p) \times \mathcal{L}(M)(\mathbb{F}_p) \times H(\mathbb{F}_p) \to K
\]

be the function trace of Frobenius of \( F \).

For \( L \in \mathcal{L}(M)(\mathbb{F}_p) \) its preimage in \( \mathcal{L}(M)(\mathbb{F}_p) \) consists of two elements. We let \( \mu_2 \) act on \( \mathcal{L}(M)(\mathbb{F}_p) \) over \( \mathcal{L}(M)(\mathbb{F}_p) \) permuting the elements in the preimage of each \( L \in \mathcal{L}(M)(\mathbb{F}_p) \). We call a function \( h : \mathcal{L}(M)(\mathbb{F}_p) \to K \) genuine if it changes by the

\footnote{For this construction we adopt the conventions of loc.cit about \( \mathbb{Z}/2\mathbb{Z} \)-gradings and étale \( \mathbb{Q}_\ell \)-sheaves on schemes over \( \mathbb{F}_p \).}
nontrivial character of $\mu_2$ under this $\mu_2$-action. Recall that $F^{cl}$ is genuine with respect the first and the second variable.

Let us write $L^0$ for a point of $\mathcal{L}(M)(\mathbb{F}_p)$ over $L \in \mathcal{L}(M)(\mathbb{F}_p)$. As in ([5], Section 2), for $L^0, N^0 \in \mathcal{L}(M)(\mathbb{F}_p)$ viewing $\mathcal{H}_L, \mathcal{H}_N$ as $H$-representations over $K$, we define the canonical intertwining operator

$$F_{N^0, L^0} : \mathcal{H}_L \to \mathcal{H}_N$$

by

$$(F_{N^0, L^0} f)(h_1) = \int_{h_2 \in H} F_{N^0, L^0}(h_1 h_2^{-1}) f(h_2) dh_2,$$

where our measure $dh_2$ is normalized by requiring that the volume of a point is one.

Let $G = \text{Sp}(M)$ viewed as an algebraic group over $\mathbb{F}_p$. It acts naturally on $\mathcal{L}(M), H,$ and $\mathcal{L}(M)$. By definition, for $g \in G, (m, a) \in H$, $g(m, a) = (gm, a)$ for $m \in M, a \in \mathbb{F}_p$, and this action preserves the symmetric structure $\sigma$ on $H$. If $g \in G$, $f : H \to K$ then $gf : H \to K$ is given by $(gf)(h) = f(g^{-1}h)$. Then $g \in G(\mathbb{F}_p)$ yields an isomorphism $\mathcal{H}_L \cong \mathcal{H}_{gL}$. We let $G$ act diagonally on $\mathcal{L}(M) \times \mathcal{L}(M) \times H$.

The above intertwining operators satisfy the following properties

- $F_{L^0, L^0} = \text{id}$;
- $F_{R^0, N^0} \circ F_{N^0, L^0} = F_{R^0, L^0}$ for any $R^0, N^0, L^0 \in \mathcal{L}(M)(\mathbb{F}_p)$;
- for any $g \in G(\mathbb{F}_p)$ we have $g \circ F_{N^0, L^0} \circ g^{-1} = F_{gN^0, gL^0}$.

**Definition 3.1.5.** Let $\pi$ be the $K$-vector space of collections $f_{L^0} \in \mathcal{H}_L$ for $L^0 \in \mathcal{L}(M)(\mathbb{F}_p)$ satisfying the property: for $N^0, L^0 \in \mathcal{L}(M)(\mathbb{F}_p)$ one has

$$F_{N^0, L^0}(f_{L^0}) = f_{N^0}$$

This is our canonical $H$-representation over $K$.

We let $G(\mathbb{F}_p)$ act on $\mathcal{L}(M)(\mathbb{F}_p) \times H(\mathbb{F}_p)$ diagonally. This yields a $G(\mathbb{F}_p)$-action on $\pi$ sending $\{f_{L^0}\} \in \pi$ to the collection $L^0 \mapsto g(f_{g^{-1}L^0})$.

### 3.2. Case $r \geq 1$.

3.2.1. Let $L$ be a finite abelian group, $p$ be any prime number. For $k \geq 0$ let $L[p^k] = \{l \in L \mid p^k l = 0\}$ and

$$\rho_k(L) = L[p^k]/(L[p^{k-1}] + pL[p^{k+1}]).$$

Each $\rho_k(L)$ is a vector space over $\mathbb{F}_p$. Note that

$$\rho_k(\mathbb{Z}/p^m \mathbb{Z}) \cong \begin{cases} \mathbb{Z}/p \mathbb{Z}, & m = k \\ 0, & \text{otherwise} \end{cases}$$

For finite abelian groups $L, L'$ one has canonically $\rho_k(L \times L') \cong \rho_k(L) \times \rho_k(L')$. 
3.2.2. Canonical isotropic subgroup. Let $p$ be any prime, $M$ is a finite abelian $p$-group of exponent $n = p^r$ with an alternating nondegenerate bilinear form $\omega : M \times M \to \mu_n$. We first construct by induction a canonical isotropic subgroup $S \subset M$ such that $\text{Aut}(M)$ fixes $S$ and $S^\perp/S$ is a $\mathbb{F}_p$-vector space.

Write the set \( \{ r > 0 \mid p_r(M) \neq 0 \} \) as \( \{ r_1, \ldots, r_s \} \) with \( 0 < r_1 < r_2 < \ldots < r_s \). There is an orthogonal direct sum \( (M, \omega) \cong \bigoplus_{i=1}^s (M_i, \omega_i) \), where $\omega_i : M_i \times M_i \to \mu_n$ is an alternating nondegenerate bilinear form, and $M_i$ is a free $\mathbb{Z}/p^{r_i}$-module of finite rank.

Let
\[
\rho = \begin{cases} \frac{r_s}{2}, & \text{if } r_s \text{ is even} \\ \frac{r_s+1}{2}, & \text{if } r_s \text{ is odd} \end{cases}
\]

Set $S_1 = p^\rho M$. Since $\omega$ takes values in $\mu_{p^{r_s}}$, $S_1$ is isotropic and fixed by $\text{Aut}(M)$. By induction hypothesis, we have a canonical isotropic subgroup $S' \subset M_1 := S_1^\perp/S_1$ such that $S'^\perp/S'$ is a $\mathbb{F}_p$-vector space, where $S'^\perp$ denotes the orthogonal complement of $S'$ in $M_1$. Let $S$ be the preimage of $S'$ under $S_1^\perp \to M_1$. This is our canonical isotropic subgroup in $M_1$.

Set $M_c = S_1^\perp/S_1$, it is equipped with the induced alternating nondegenerate bilinear form $\omega_c : M_c \times M_c \to \mu_p$, the subscript $c$ stands for ‘canonical’.

3.2.3. We keep the assumptions of Theorem 2.0.8, so $p$ is odd. View $S$ as a subgroup of $H$ via $s \mapsto (s,0) \in H$ for $s \in S$. Let $H^S = S^\perp \times \mu_n$, this is a subgroup of $H$. Since $S$ lies in the kernel of $\beta : S^\perp \times S^\perp \to \mu_n$, we get the alternating nondegenerate bilinear form $\beta_c : M_c \times M_c \to \mu_p$ given by $\beta_c(m_1, m_2) = \beta(m_1, m_2)$ for $m_i \in S^\perp$ over $m_i$.

Set $H_c = M_c \times \mu_p$ with the product
\[
(m_1, a_1)(m_2, a_2) = (m_1 + m_2, a_1 a_2 \beta_c(m_1, m_2))
\]
This is a central extension $1 \to \mu_p \to H_c \to M_c \to 1$ with the commutator $\omega_c$ and the symmetric structure $\sigma_c(m, a) = (-m, a)$ for $(m, a) \in H_c$.

Let $\alpha_S : H^S \to H_c$ be the homomorphism sending $(m, a)$ to $(m \bmod S, a)$ for $m \in S^\perp$, its kernel is $S$.

As in Section 3.1, we get the algebraic stack $\widetilde{\mathcal{L}}(M_c), \mathcal{L}(M_c), H_c$ over $\mathbb{F}_p$. Let $G = \text{Sp}(M, \omega)$ be the group of automorphisms of $M$ preserving $\omega$, this is a finite group. We let $g \in G$ act on $H$ sending $(m, a)$ to $(gm, a)$. Let $g \in G$ act on functions $f : H \to K$ by $(gf)(h) = f(g^{-1}h)$ for $h \in H$. For $L \in \mathcal{L}(M)$ this yields an isomorphism $g : \mathcal{H}_L \cong \mathcal{H}_{gL}$ of $K$-vector spaces.

Since $G$ preserves $S^\perp$, we have the homomorphism $G \to G_c := \text{Sp}(M_c)(\mathbb{F}_p)$. Via this map, $G$ acts on $\mathcal{L}(M_c)(\mathbb{F}_p), \widetilde{\mathcal{L}}(M_c)(\mathbb{F}_p), H_c$.

3.2.4. We denote elements of $\mathcal{L}(M_c)$ by a capital letter with a subscript $c$. For $L_c \in \mathcal{L}(M_c)$ let $L \in \mathcal{L}(M)$ denote the preimage of $L_c$ under $S^\perp \to M_c$.

For $L_c \in \mathcal{L}(M_c)$ we have the representation $\mathcal{H}_{L_c}$ of $H_c$ over $K$ defined in Section 3.1.4, and the $H$-representation $\mathcal{H}_L$ over $K$ defined in Section 2.0.5.

For $L_c \in \mathcal{L}(M_c)$ any $f$ in the space of invariants $\mathcal{H}^S_L$ is the extension by zero under $H^S \hookrightarrow H$. The space $\mathcal{H}^S_L$ is naturally a $H_c$-module. We get an isomorphism of $H_c$-modules $\tau_{L_c} : \mathcal{H}_{L_c} \cong \mathcal{H}^S_L$ sending $f$ to the composition $H^S \cong H_c \to K$ extended by zero to $H$. 

For $g \in G$, $L_c \in \mathcal{L}(M_c)$ the isomorphism $g : \mathcal{H}_L \to \mathcal{H}_{gL}$ yields an isomorphism $g : \mathcal{H}^S_L \to \mathcal{H}^S_{gL}$ of $S$-invariants.

3.2.5. Given $L^0_c, N^0_c \in \tilde{\mathcal{L}}(M_c)(\mathbb{F}_p)$, we define a canonical intertwining operator
\begin{equation}
\mathcal{F}_{N^0_c, L^0_c} : \mathcal{H}_L \to \mathcal{H}_N
\end{equation}
as the unique isomorphism of $H$-modules such that the diagram commutes
\begin{align*}
\mathcal{H}_L^S & \xleftarrow{\mathcal{F}_{N^0_c, L^0_c}^S} \mathcal{H}_N^S \\
\uparrow \tau_{L_c} & \quad \uparrow \tau_{N_c} \\
\mathcal{H}_{L_c} & \xrightarrow{F_{N^0_c, L^0_c}} \mathcal{H}_{N_c}
\end{align*}
Here $F_{N^0_c, L^0_c}$ are the canonical intertwining operators from Section 3.1.4. The properties of the canonical intertwining operators of Section 3.1.4 imply the following properties of (5):

- $\mathcal{F}_{L^0_c, L^0_c} = \text{id}$ for $L^0_c \in \tilde{\mathcal{L}}(M_c)(\mathbb{F}_p)$;
- for $R^0_c, N^0_c, L^0_c \in \tilde{\mathcal{L}}(M_c)(\mathbb{F}_p)$ one has
  \[ \mathcal{F}_{R^0_c, N^0_c} \circ F_{N^0_c, L^0_c} = F_{R^0_c, L^0_c} \]
- for $g \in G, N^0_c, L^0_c \in \tilde{\mathcal{L}}(M_c)(\mathbb{F}_p)$ we have $g \circ F_{N^0_c, L^0_c} \circ g^{-1} = F_{g N^0_c, g L^0_c}$.

**Definition 3.2.6.** Let $\pi$ be the $K$-vector space of collections $f_{L^0_c} \in \mathcal{H}_L$ for $L^0_c \in \tilde{\mathcal{L}}(M_c)(\mathbb{F}_p)$ satisfying the property: for $N^0_c, L^0_c \in \tilde{\mathcal{L}}(M_c)(\mathbb{F}_p)$ one has
\[ \mathcal{F}_{N^0_c, L^0_c}(f_{L^0_c}) = f_{N^0_c} \]
The element $h \in H$ sends $\{f_{L^0_c}\} \in \pi$ to the collection $\{h(f_{L^0_c})\} \in \pi$. This is our canonical $H$-representation over $K$.

The group $G$ acts on $\pi$ sending $\{f_{L^0_c}\} \in \pi$ to the collection $L^0_c \mapsto g(f_{g^{-1}L^0_c})$. This is a version of the Weil representation of $G$. (In the case when the field of coefficients is $\mathbb{C}$, this $G$-representation was also obtained in [2], however a canonical representation of $H$ was not constructed in [2]).

The above actions of $H$ and $G$ on $\pi$ combine to an action of the semi-direct product $H \rtimes G$ on $\pi$. Theorem 2.0.8 is proved.

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