On the discrete analog of gamma-Lomax distribution: properties and applications

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Abstract

A two parameter discrete gamma-Lomax distribution is derived as a discrete analogous to the continuous three parameters gamma-Lomax distribution (see Alzaatreh et al. (2013, 2014)) using the general approach for discretization of continuous probability distributions. Some useful structural properties of the proposed distribution are examined. Possible areas of application are also discussed.

1 Introduction

The well known discrete distributions which exists in literature (Negative binomial, Geometric, Poisson, etc.) have limited applicability in terms of modeling reliability, failure times, counts, etc to name a few. This motivates us to consider development of some discrete distributions based on popular continuous models for reliability, failure times, etc. Let \( F(x) \) be the cumulative distribution function (c.d.f.) of any random variable \( X \) and \( f(t) \) be the probability density function (p.d.f.) of a random variable \( T \) defined on \([0, 1)\). The c.d.f. of the gamma-X family of distributions defined by Alzaatreh, et al. (2013, 2014) is given by

\[
f_X(x) = \frac{1}{\Gamma(\alpha) \beta^\alpha} f_R(x) \left(- \log [1 - F_R(x)]\right)^{\alpha-1} \left(1 - F_R(x)\right)^{1/\beta-1}
\]  

(1)

If \( R \) follows the Lomax distribution with the density function \( f_R(x) = k \theta^{-1} (1 + x\theta)^{-(k+1)}, \quad x > 0 \), then (1) reduces to the gamma-Lomax distribution as

\[
f_X(x) = \frac{1}{(x + \theta) \Gamma(\alpha) c^\alpha} \left(1 + \frac{x}{\theta}\right)^{-1/c} \left(\log \left[1 + \frac{x}{\theta}\right]\right)^{\alpha-1},
\]  

(2)

\( x > 0, \quad (c, \alpha, \theta) > 0 \).

Note that if \( X \) is replaced by \( X + \theta \), then (2) reduces to the gamma-Pareto distribution which was proposed and studied by Alzaatreh et al. (2012a).

From (2), the c.d.f. of the gamma-Lomax distribution is given by

\[
F_X(x) = \frac{1}{\Gamma(\alpha)} \gamma\left\{\alpha, c^{-1}\log\left[1 + \frac{x}{\theta}\right]\right\},
\]  

(3)

where \( \gamma(\alpha, t) = \int_0^t u^{\alpha-1} e^{-u} du \) is the incomplete gamma function.

There are a variety of works available in literature that extends the Lomax distribution under the continuous paradigm. We mention them chronologically as follows. Note that (1) has been separately studied by Cordeiro et al. (2015) as a particular case of Zografos- Balakrishnan (G) family of distributions, where \( G \) is any baseline continuous distribution (from the perspective of a gamma generated model). Lemonte & Coredeiro. (2013) studied a five-parameter continuous
distribution, the so-called McDonald Lomax distribution, that extends the Lomax distribution. Ghitany et al. (2007) studied properties of a new parametric distribution generated by Marshall and Olkin extended family of distributions based on the Lomax model. However, not much work has been considered towards discrete Lomax mixture type models. The most recent reference that the authors can mention here is the work by Prieto et al. (2014), in which the authors considered the discrete generalized Pareto model (mixing with zero-inflated Poisson distribution) in modeling road accident blackspots data. It is observed that their discrete model is unimodal and the p.m.f is always a decreasing function. So the model is somehow restricted in nature. Now, our discrete Gamma Lomax distribution (henceforth, in short, DGLD) is not always decreasing. Consequently, it has greater flexibility.

The discrete gamma-Lomax distribution can be defined in the following way

\[
g(x) = P(x \leq dX < x + 1) = S(x) - S(x + 1) = \frac{1}{\Gamma(\alpha)} \left[ \gamma \left\{ \alpha, c^{-1} \log \left( 1 + \frac{x + 1}{\theta} \right) \right\} - \gamma \left\{ \alpha, c^{-1} \log \left( 1 + \frac{x}{\theta} \right) \right\} \right], \quad x \in \mathbb{N}^*, \quad (c, \alpha, \theta) > 0, \text{ where } \mathbb{N}^* = \mathbb{N} \cup \{0\}. \]

From (3), the c.d.f. of the DGLD is

\[
G(x) = \frac{1}{\Gamma(\alpha)} \gamma \left\{ \alpha, c^{-1} \log \left( 1 + \frac{\lfloor x + 1 \rfloor}{\theta} \right) \right\}, \quad x \geq 0,
\]

where \(\lfloor x \rfloor\) is the floor function, the largest integer contained in \(x\).

The paper is organized as follows. In Section 2, we discuss some structural properties of the proposed DGLD distribution, including entropy. In Section 3, we provide some real life scenarios where DGLD can be applied as an appropriate probability model. Finally, some concluding remarks are provided in Section 4.

### 2 Structural properties

In this section, we discuss some important structural properties of the DGLD. At first we have the following lemma.

**Lemma 1.** If a random variable \(Y\) follows the gamma-Lomax distribution with parameters \(c, \alpha, \theta\), then \(X = \lfloor Y \rfloor\) follows the DGLD\((c, \alpha, \theta)\).

**Proof.** Follows immediately from the definition (3).

The hazard function associated with DGLD is

\[
h(x) = \frac{\left[ \gamma \left\{ \alpha, c^{-1} \log \left( 1 + \frac{\lfloor x + 1 \rfloor}{\theta} \right) \right\} - \gamma \left\{ \alpha, c^{-1} \log \left( 1 + \frac{x}{\theta} \right) \right\} \right]}{\Gamma(\alpha) - \gamma \left\{ \alpha, c^{-1} \log \left( 1 + \frac{x}{\theta} \right) \right\}}, x \in \mathbb{N}^*.
\]

**Theorem 1.** The DGLD has a DFR property whenever \(\alpha \leq 1\).
Proof. Based on Theorem 3 in Alzaatreh et al. (2012b), it is sufficient to show that GLD has a DFR property whenever $\alpha \leq 1$. From (2), it is not difficult to show that the gamma-Lomax distribution possess a DFR property whenever $\alpha \leq 1$. Hence the result.

**Theorem 2.** The DGLD is unimodal and the mode is at $x = m$, where $m \in \{[x_0] - 1, [x_0], [x_0] + 1\}$ and $x_0 = \theta \{\exp (c(\alpha - 1)/(c + 1)) - 1\}$. Furthermore, if $[x_0] = 0$, then the mode is at $m = 0$ or 1.

Proof. From Theorem 2 in Alzaatreh et al. (2012a), the gamma-Pareto distribution is unimodal the mode is at $x_0 = 0$ or $x_0 = \theta \exp (c(\alpha - 1)/(c + 1))$. Therefore, the mode of GLD is at $x_0 = 0$ or $x_0 = \theta \{\exp (c(\alpha - 1)/(c + 1)) - 1\}$. The rest of the proof follows from Theorem 2 in Alzaatreh et al. (2012b).

### 2.1 Stochastic orderings

Stochastic ordering is an integral tool to judge comparative behaviors of random variables. Many stochastic orders exist and have various applications. Theorem 3 and Corollary 1 give some results on stochastic orderings of the DGLD. The orders considered here are the stochastic order $\leq_{st}$, and the expectation order $\leq_E$.

**Theorem 3.** The DGLD($c, \alpha, \theta$) has the following properties.

- Suppose $X_1 \sim DGLD(c, \alpha, \theta_1)$ and $X_2 \sim DGLD(c, \alpha, \theta_2)$. If $\theta_1 > \theta_2$, then $X_1 \leq_{st} X_2$.
- Suppose $X_1 \sim DGLD(c_1, \alpha, \theta)$ and $X_2 \sim DGLD(c_2, \alpha, \theta)$. If $c_1 > c_2$, then $X_1 \leq_{st} X_2$.
- Suppose $X_1 \sim DGLD(c, \alpha_1, \theta)$ and $X_2 \sim DGLD(c, \alpha_2, \theta)$. If $\alpha_1 > \alpha_2$, then $X_1 \leq_{st} X_2$.

Proof. Follows immediately from the cdf of the DGLD.

Next, we describe the expectation ordering in next corollary. Corollary 1 follows from Theorem 3.

**Corollary 1.**

- Suppose $X_1 \sim DGLD(c, \alpha, \theta_1)$ and $X_2 \sim DGLD(c, \alpha, \theta_2)$. If $\theta_1 > \theta_2$, then $X_1 \leq_E X_2$.
- Suppose $X_1 \sim DGLD(c_1, \alpha, \theta)$ and $X_2 \sim DGLD(c_2, \alpha, \theta)$. If $c_1 > c_2$, then $X_1 \leq_E X_2$.
- Suppose $X_1 \sim DGLD(c, \alpha_1, \theta)$ and $X_2 \sim DGLD(c, \alpha_2, \theta)$. If $\alpha_1 > \alpha_2$, then $X_1 \leq_E X_2$.

Before discussing other properties of the DGLD($c, \alpha, \theta$), we consider the following series expression:

1. 

$$
\gamma \{\alpha, x\} = \sum_{m=0}^{\infty} (-1)^m \frac{x^{m+\alpha}}{m!(m+\alpha)},
$$

from Nadarajah & Pal (2008).
2. For any \( c \in \mathbb{R} \)

\[
\left\{ \log \left[ 1 + \frac{x}{\theta} \right] \right\}^c = c \sum_{k=0}^{\infty} \binom{k-c}{k} \sum_{j=0}^{\infty} \frac{(-1)^j}{(c-j)} \binom{k}{j} P_{k,j} \left( \frac{x}{\theta} \right)^{k+c} = \phi_{(c,k,j)} \left( \frac{x}{\theta} \right)^{k+c},
\]

where \( \phi_{(c,k,j)} = c \sum_{k=0}^{\infty} \binom{k-c}{k} \sum_{j=0}^{\infty} \frac{(-1)^j}{(c-j)} \binom{k}{j} \), \( c_k = \frac{(-1)^{k+1}}{k+1} \), \( P_{j,0} = 1 \) and \( P_{j,k} = \frac{1}{k} \sum_{m=1}^{k} (jm - k + m) c_m P_{j,-m} \), \( k = 1, 2, \ldots \).

### 2.2 Moments

The \( r \)-th moment of DGLD is given by

\[
E(X^r) = \frac{1}{\Gamma(\alpha)} \sum_{x=0}^{\infty} x^r \left[ \gamma \{ \alpha, c^{-1} \log \left[ 1 + \frac{x+1}{\theta} \right] \} - \gamma \{ \alpha, c^{-1} \log \left[ 1 + \frac{x}{\theta} \right] \} \right] = \frac{1}{\Gamma(\alpha)} \sum_{x=0}^{\infty} x^r \sum_{m=0}^{\infty} (-1)^m \frac{(c^{-1})^{m+\alpha}}{m!(m+\alpha)} \left\{ \phi_{(m+\alpha,k,j)} \left[ \left( \frac{x+1}{\theta} \right)^{k+m+\alpha} - \left( \frac{x}{\theta} \right)^{k+m+\alpha} \right] \right\}.
\]

on applying successively (7) and (8) respectively.

**Theorem 4.** If \( c < 1/r \), then the \( r \)-th moment of the DGLD(\( c, \alpha, \theta \)) exists.

**Proof.** Assume that \( X \) follows GLD. Now, Alzaatreh et al. (2012a) showed that if \( c < 1/r \), then \( E(X^r) \) exists for all \( r \). The rest of the proof follows from the fact that \( 0 \leq |X| \leq \lfloor X \rfloor \).

### 2.3 Generating functions

Let \( X \) be a DGLD(\( c, \alpha, \theta \)). Then, the probability generating function of \( X \) can be expressed as

\[
G_X(s) = E(s^X) = \sum_{x=0}^{\infty} s^x \sum_{m=0}^{\infty} (-1)^m \frac{(c^{-1})^{m+\alpha}}{m!(m+\alpha)} \left\{ \phi_{(m+\alpha,k,j)} \left[ \left( \frac{x+1}{\theta} \right)^{k+m+\alpha} - \left( \frac{x}{\theta} \right)^{k+m+\alpha} \right] \right\}.
\]

The corresponding moment generating function is

\[
M_X(t) = E(\exp(tX)) = \sum_{x=0}^{\infty} \exp(tx) \sum_{m=0}^{\infty} (-1)^m \frac{(c^{-1})^{m+\alpha}}{m!(m+\alpha)} \left\{ \phi_{(m+\alpha,k,j)} \left[ \left( \frac{x+1}{\theta} \right)^{k+m+\alpha} - \left( \frac{x}{\theta} \right)^{k+m+\alpha} \right] \right\}.
\]

**Theorem 5.** Let \( \mu_{[r]} = E[X(X-1) \cdots (X-r+1)] \) denote the descending \( r \)-th order factorial moment. Then,
\[
\mu_n = \sum_{m=0}^{\infty} (-1)^m \frac{(c-1)^{m+\alpha}}{m!(m+\alpha)} \left\{ \phi_{(m+\alpha,k,j)} \left( r(r-1) \cdots 1 \right) \left[ \left( \frac{r+1}{\theta} \right)^{k+m+\alpha} - \left( \frac{r}{\theta} \right)^{k+m+\alpha} \right] \right\} + \mu_{r+1}.
\]

Proof. Follows immediately by successive differentiation of (10) and then substituting \( s = 1 \).

2.4 Order statistics

Suppose \( X_1, X_2, \cdots, X_n \) be a random sample drawn from (4). Then, the probability mass function and the cumulative distribution function of the \( i \)-th order statistic, \( X_{i:n} \) are given by

\[
P(X_{i:n} = x) = \frac{n!}{(i-1)! (n-i)!} \int_{F(x-1)}^{F(x)} u^{i-1} (1-u)^{n-i} du
\]

\[
= \frac{n!}{(i-1)! (n-i)!} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \int_{F(x-1)}^{F(x)} u^{i+j-1} du
\]

\[
= \frac{n!}{(i-1)! (n-i)!} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \frac{1}{(i+j) \Gamma(\alpha)^{i+j}} \times \{ A_1(x) - A_2(x) \},
\]

where

\[
A_1(x) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots \sum_{k_{i+j}=0}^{\infty} \frac{(-1)^{s_{i+j}} (c-1)^{s_{i+j}+(i+j)\alpha}}{p_{i+j}} \left\{ \phi_{(\delta_1,\delta_2,s_{i+j}+(i+j)\alpha)} \left( \frac{x}{\theta} \right)^{s_{i+j}+(i+j)\alpha} \right\},
\]

and

\[
A_2(x) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots \sum_{k_{i+j}=0}^{\infty} \frac{(-1)^{s_{i+j}} (c-1)^{s_{i+j}+(i+j)\alpha}}{p_{i+j}} \left\{ \phi_{(\delta_1,\delta_2,s_{i+j}+(i+j)\alpha)} \left( \frac{x-1}{\theta} \right)^{s_{i+j}+(i+j)\alpha} \right\},
\]

where \( s_{i+j} = \sum_{\ell=1}^{i+j} k_{\ell}, p_{i+j} = \prod_{\ell=1}^{i+j} k_{\ell}!(k_{\ell} + \alpha) \). Now, one can use (9) to get a general \( r \)-th order moment of \( X_{i:n} \).

2.5 Distribution of maximum and minimums

Maximums and minimums of random variables also arise in reliability. Let \( X_i, i = 1,2, \cdots, n \) be independent DGLD with parameters \( c_i, \alpha_i, \delta_i \). Define \( U = \min(X_1,X_2,\cdots,X_n) \) and \( W = \max(X_1,X_2,\cdots,X_n) \). Next, consider the following: The c.d.f. of \( U \) will be
\[ P(U \leq u) = 1 - \prod_{i=1}^{n} \left( 1 - \frac{1}{\Gamma(\alpha_i)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k + \alpha_i)} \left[ e^{-1} \log \left( \frac{x}{\theta} \right) \right]^{k + \alpha_i} \right) \]

\[ = 1 - \prod_{i=1}^{n} \left( 1 - \frac{1}{\Gamma(\alpha_i)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k + \alpha_i)} \left\{ \phi(k + \alpha_i, m, \alpha) \left( \frac{u}{\theta} \right)^{m + k + \alpha_i} \right\} \right) , \]

On using (8). Hence, the p.m.f. of \( U \) will be

\[ P(U = u) = P(U \leq u) - P(U \leq u - 1) \]

\[ = \prod_{i=1}^{n} \left\{ 1 - \frac{1}{\Gamma(\alpha_i)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k + \alpha_i)} \left\{ \phi(k + \alpha_i, m, \alpha) \left[ \left( \frac{u}{\theta} \right)^{m + k + \alpha_i} - \left( \frac{u - 1}{\theta} \right)^{m + k + \alpha_i} \right] \right\} \right\} . \]  

Next, by similar approach, the p.m.f. of \( W \) will be

\[ P(W = w) = P(W \leq w) - P(W \leq w - 1) \]

\[ = \prod_{i=1}^{n} \left\{ \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k + \alpha)} \left\{ \phi(k + \alpha, m, \alpha) \left[ \left( \frac{w}{\theta} \right)^{m + k + \alpha} - \left( \frac{w - 1}{\theta} \right)^{m + k + \alpha} \right] \right\} \right\} . \]

**Distribution of range**

From Kabe et al. (1969), we may write \( P(X_{1:n} = X_{n:n} = y) = \{F(y) - F(y - 1)\}^n \), for \( y = 0, 1, 2, \ldots \). If \( R = X_{n:n} - X_{1:n} \) denotes the range of the order statistics, then

\[ P(R = 0) = \left\{ \frac{1}{\Gamma(\alpha)} \right\}^n \left[ \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k + \alpha)} \phi(k + \alpha, m, \alpha) \left[ \left( \frac{y}{\theta} \right)^{m + k + \alpha} - \left( \frac{y - 1}{\theta} \right)^{m + k + \alpha} \right] \right] . \]

### 2.6 Entropy

In this section, we consider the cumulative residual entropy proposed by Rao et al. (2004). According to them, it is more general than the Shannon Entropy in that its definition is valid in both the continuous and discrete domains, secondly it possesses more general mathematical properties than the Shannon entropy and thirdly, it can be easily computed from sample data and these computations asymptotically converge to the true values. It is given by \( \eta_{CRE} = -\sum_{x=0}^{\infty} P(X > x) \log P(X > x) \). In our case it is given by

\[ \eta_{CRE} = \sum_{j=0}^{\infty} \sum_{x=0}^{\infty} \frac{1}{\Gamma(\alpha)^{j+1}} \times \left[ 1 - \left( \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k + \alpha)} \right) \left\{ \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots \sum_{k_j=0}^{\infty} \frac{(-1)^s_j}{p_j} \phi(s_j + j\alpha, \alpha) \left( \frac{x}{\theta} \right)^{s_j + j\alpha + m} \right\} \right] . \]
3 Possible applications

The proposed DGLD model can be applied to model the following real life scenarios listed below:

- Modeling road accident blackspots data.
- Modeling financial risk and other actuarial applications.
- Modeling certain types of healthcare data (which are count data in nature).

4 Conclusion

In this paper we have proposed a new discrete analog of the continuous gamma-Lomax distribution and derived its distributional properties. Several real life scenarios have also been mentioned as a possible application of the proposed DGLD. Estimation of the model parameters and associated inference with applications will be the subject matter of a future article. We will report our findings in a separate article.

References

[1] Alzaatreh, A., Famoye, F. & Lee, C. (2014). The gamma-normal distribution: Properties and applications. *Computational Statistics and Data Analysis*, 69, 67-80.

[2] Alzaatreh, A., Lee, C. & Famoye, F. (2013). A new method for generating families of continuous distributions. *Metron*, 71, 63-79.

[3] Alzaatreh, A., Famoye, F. & Lee, C. (2012a). Gamma-Pareto distribution and its applications. *Journal of Modern Applied Statistical Methods*, 11, 78-94.

[4] Alzaatreh, A., Lee, C. & Famoye, F. (2012b). On the discrete analogues of continuous distributions. *Statistical Methodology*, 9, 589-603.

[5] Cordeiro, G.M., Ortega, M.M.E., & Popovic, V. B. (2015). The gamma-Lomax distribution. *Journal of Statistical Computation and Simulation*, 85, 305-319.

[6] Ghitany, M.E., Al-Awadhi, F.A and Alkhalfan, L.A. (2007). Marshall-Olkin extended Lomax distribution and its applications to censored data. *Communications in statistics-Theory and Methods*, 36, 1855-1866.

[7] Kabe, D. (1969). Some distribution problems of order statistics from discrete populations. *Annals of the Institute of Statistics and Mathematics*, 21, 551-556.

[8] Lemonte, A.J. and Cordeiro, G.M. (2013). An extended Lomax distribution. *Statistics: A Journal of Theoretical and Applied Statistics*, 47, 800-816.

[9] Prieto, F., Gmez-Dniz, E., & Sarabia, J.M. (2014).Modeling Road Accident Blackspots Data with the Discrete Generalized Pareto distribution. *Accident Analysis & Prevention*, 71, 38-49.

[10] Rao, M., Chen, Y., Vemuri, B.C. and Wang, F. (2004).Cumulative residual entropy: a new measure of information, *IEEE Transactions on Information Theory*, 50, 1220-1228.