Hydrogen Atom and Time Variation of Fine-Structure Constant

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Abstract

In this paper, we have solved the de Sitter special relativistic (SRcR) Dirac equation of hydrogen in the earth-QSO (quasar) framework reference by means of the adiabatic approach. The aspects of geometry effects of de Sitter space-time described by Beltrami metric are explored and taken into account. It is found that the SRcR-Dirac equation of hydrogen is a time dependent quantum Hamiltonian system. We provide an explicit calculation to justify the adiabatic approach in dealing with this time-dependent system. Since the radius of de Sitter sphere $R$ is cosmologically large, the evolution of the system is very slow so that the adiabatic approximation legitimately works with high accuracy. We conclude that the electromagnetic fine-structure constant, the electron mass and the Planck constant are time variations. This prediction of fine-structure constant is consistent with the presently available observation data. For confirming it further, experiments/observations are required.

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Key words: Hydrogen atom, Time variation of fine structure constant, de-Sitter invariance, Special Relativity.

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I. INTRODUCTION

The life time of a stable atom, e.g., the hydrogen atom, is almost infinitely long. We can practically compare the spectra of atoms at nowadays laboratories to ones emitted from the atoms of a distant galaxy. The time interval could be on the cosmic scales. Such observation of spectra of distant astrophysical objects may encode some cosmologic information in the atomic energy levels at the position and time of emission. During last decade, several interesting experiments based on this idea in principle were reported in literature, and the fine-structure constant $\alpha$-variation in the absorption spectra of quasi-stellar objects (QSOs) were observed (see incomplete list of reference [1, 2, 3, 4, 5, 6, 7], and also the review articles [8], [9], [10] and the references within). In order to reveal the probable physics behind these experimental discoveries, we strongly suggest to reexamine the dynamic theories of atoms, typically of the hydrogen, with both fine-structure effects and cosmological effects taken into account. As is well known, in the ordinary relativistic quantum mechanics based on Einstein’s Special Relativity (denoted as $\mathcal{SR}_c$ hereafter), the hydrogen’s fine-structure spectra are independent of any cosmologic effects. For instance, the atomic spectra in this theory do not change in time due to the fact that the Hamiltonian $H_c$ is time independent.

Therefore, investigations of quantum theory of atoms at cosmic scale should be based on some extension of Einstein’s special relativity $\mathcal{SR}_c$ in which the flat Minkowski spacetime is replaced by de Sitter spacetime, and of course it should be a challenge.

A natural extension of $\mathcal{SR}_c$ is the de Sitter invariant Special Relativity (or the Special Relativity in space-time of a positive constant curvature $1/R$). By requiring the law of inertia for free particles to be true in the de Sitter special relativity, authors of [11], [12] found out that the space-time geometry is described by Beltrami metric (instead of usual Lorentz metric), and the space-time coordinate transformations to preserve Beltrami metric exists (see [14] for an English version). Thus the de Sitter special relativity was formulated in [11], [12]. In the recent years, there have been further studies on this theory in [13], [14], [15]. There is one universal parameter $c$ (speed of light) in the Einstein’s Special Relativity $\mathcal{SR}_c$. By contrast, there are two universal parameters in the de Sitter Special Relativity: $c$ and $R$ (the radius of de Sitter sphere and to character the cosmic radius). So, we will denote latter shortly as $\mathcal{SR}_{c,R}$ hereafter. In [14], the Hamiltonian formalism of de Sitter special relativity was developed. In [15], by requiring that the results of time-variation of fine
structure constant in the absorption spectra of QSOs in [1, 2, 3, 4, 5, 6, 7] are consistent with ones measured in Oklo nature fission reactor [16], the $R$ was estimated to be $10^{11} \text{ly}$ to $10^{12} \text{ly}$ approximately.

Similar to classical $\mathcal{SR}_c$ mechanics, the Lagrangian-Hamiltonian formulation of $\mathcal{SR}_{cR}$ mechanics are self-consistent, and have been appropriately established in [14]. The most significant difference between these two theories are that the free particle’s Hamiltonian for $\mathcal{SR}_c$ is spacetime independent $H_c = H_c(p_i)$, while for $\mathcal{SR}_{cR}$ the Hamiltonian is

$$H_{cR} = H_{cR}(\frac{c^2 t^2}{R^2}, \frac{(x^i)^2}{R^2}, \pi_i) \quad (1)$$

which depends on time explicitly. Of course, when $R \to \infty$, $H_{cR}(\frac{c^2 t^2}{R^2}, \frac{(x^i)^2}{R^2}, \pi_i) \to H_c(p_i)$. In $\mathcal{SR}_{cR}$ the particle’s conserved energy $E$ and momenta $p^i$ are different from its canonical energy (or Hamiltonian $H_{cR}$) and canonical momenta $\pi_i$. $E$ and $p^i$ appear as the Noether charges of the de Sitter symmetry for space-time of $\mathcal{SR}_{cR}$ mechanics [14]. The quantization of such a system is obviously nontrivial. Different from the quantization of both Newtonian mechanics and of $\mathcal{SR}_c$ mechanics, the operator ordering of “$x$” and “momentum” must be taken into account for free particle motions in $\mathcal{SR}_{cR}$-quantum mechanics [14]. It has been shown that the Weyl ordering is necessary for protecting the isometry symmetries $SO(1, 4)$ of de Sitter spacetime, and the wave-equation of spinless particle was shown to be the Klein-Gordon (KG) equation in de Sitter space-time with Beltrami metric [14]. In the present paper, we will base on such KG equation to construct the $\mathcal{SR}_{cR}$-Dirac equation for spin $1/2$ particles. Namely, the tetrad and the spin connection corresponding to the Beltrami metric will be derived. Nextly, by treating the Coulomb electric interaction between nucleus and electron as $U(1)$-EM gauge potential and basing on gauge covariant principle, we finally obtain the $\mathcal{SR}_{cR}$-Dirac equation for electron in hydrogen atom.

The main purpose of this paper is to study the spectra of hydrogen atom located on distant astrophysical objects, e.g., QSO. Because people gets astrophysics information by observation, all observable optic objects must be on the light cone of the earth. Thus, we should solve the $\mathcal{SR}_{cR}$-Dirac equation for electron in hydrogen atom in the earth-QSO reference frame, whose origin is at earth and QSO locates on the light cone. The geometry is determined in Beltrami metric of de Sitter space-time. It is expected that the solutions will show both effects of fine-structure and effects of cosmology in the spectra of such hydrogen atom. To do so, we have to solve time-dependent Hamiltonian problem due to $\mathbb{1}$ in
quantum mechanics. Our explicit calculations show that since \( R \) is cosmologically large and \( R \gg ct \), factor \((c^2t^2/R^2)\) makes the time-evolution of the system be so slow that the adiabatic approximation will legitimately works.

Generally, to a \( H(x,t) \), we may express it as \( H(x,t) = H_0(x) + H'(x,t) \). Suppose two eigenstates \(|s\rangle\) and \(|m\rangle\) of \( H_0(x) \) do not generate, i.e., \( \Delta E \equiv \hbar(\omega_m - \omega_s) = \hbar \omega_{ms} \neq 0 \). The validness of for adiabatic approximation relies on the fact that the variation of the potential \( H'(x,t) \) in the the Bohr time-period \( (\Delta T_{ms}^{(Bohr)}) \) \( \dot{H}'(x,t)_{ms} = (2\pi/\omega_{ms}) \dot{H}'(x,t)_{ms} \) is much less than \( \hbar \omega_{ms} \). That makes the quantum transition from state \(|s\rangle\) to state \(|m\rangle\) almost impossible. Thus, the non-adiabatic effect corrections are small enough (or tiny), and the adiabatic approximations are proper.

For adiabatic quantum system, the states are quasi-stationary in all instants, and hence the time variable becomes a parameter in Hamiltonian. In this approximation approach, the time-dependent Hamiltonian system was reduced to a system with time-parameter (rather than a time-dynamic variable), and then the problem becomes handleable and solvable approximatively.

By means of adiabatic approximation approach, we solve the stationary \( \mathcal{SR}_{cR} \)-Dirac equation for hydrogen atom, and the spectra of the corresponding Hamiltonian with time-parameter are obtained. As a result, we find out that the fine structure constant and the mass of electron vary as cosmic time going by. This is a interesting consequence of the theory. We will compare the prediction of our theory with the observation data of in the end of the paper. It will be pointed out that the prediction is in agreement with the observation.

\( \mathcal{SR}_{cR} \)-quantum mechanics for atom could be thought of as a cosmological atom physics theory. Since the works in this field would be helpful to reveal information about atomic energy levels of emission from cosmological distant object, the results and predictions could be interesting. In addition, the studies on \( \mathcal{SR}_{cR} \) belong to examining the base of the relativity theory from its beginning. It would be also meaningful to search what physics effects could distinguish the predictions of \( \mathcal{SR}_{cR} \) from ones of \( \mathcal{SR}_c \). The results of this paper may indicate that the special relativity for cosmologic large space-time scale may be beyond the Einstein’s special relativity \( \mathcal{SR}_c \).

The contents of the paper are organized as follows: In section II, we recall the classical mechanics of de Sitter special relativity; Section III is devoted to discuss the quantum
mechanics of de Sitter special relativistic. The quantum wave equations both of spinless particle and of spin-1/2 one are constructed; Section IV is the major part of the paper. In this section, we solve the $\mathcal{SR}_{cR}$-Dirac equation for hydrogen atom step by step: to derive $\mathcal{SR}_{cR}$-Dirac equation for hydrogen atom on QSO; to discuss Solution of usual $\mathcal{SR}_{c}$-Dirac equation for hydrogen atom at QSO, and the Beltrami-geometry effects in $\mathcal{SR}_{cR}$-Dirac equation; and then $\mathcal{SR}_{cR}$-Dirac equation for spectra of hydrogen is obtained and further solved by means of adiabatic approximation. In the end of this section we exhibit that the fine structure constant is time-variation and compare it with the observations; Finally, we briefly summarize and discuss the results of the paper. In Appendix A, we derive the electric Coulomb Law in QSO-Light-Cone Space; In Appendix B, we show the calculations of adiabatic approximative wave functions in $\mathcal{SR}_{cR}$-Dirac equation of hydrogen in detail.

II. REVIEW OF THE CLASSICAL MECHANICS FOR FREE PARTICLE IN DE SITTER SPECIAL RELATIVITY

A. The Lagrangian-Hamiltonian formalism

We begin with a brief review of the classical mechanics for a free particle in de Sitter special relativity. The Lagrangian is

$$L_{cR} = -m_0 c \frac{ds}{dt} = -m_0 c \sqrt{B_{\mu\nu}(x)dx^\mu dx^\nu} = -m_0 c \sqrt{B_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu},$$

where $\dot{x}^\mu = \frac{dx^\mu}{dt}$, $B_{\mu\nu}(x)$ is Beltrami metric:[11, 12, 13, 14, 15]:

$$B_{\mu\nu}(x) = \frac{\eta_{\mu\nu}}{\sigma(x)} + \frac{1}{R^2 \sigma(x)^2} \eta_{\rho\lambda} x^\lambda x^\rho,$$

with $\sigma(x) \equiv 1 - \frac{1}{R^2} \eta_{\mu\nu} x^\mu x^\nu$, and $R$ which is assumed to be a fundamental constant in $\mathcal{SR}_{cR}$ stands for the radius of the pseudo-sphere in $dS$-space. Setting up the time $t = x^0/c$, $B_{\mu\nu}(x)$ becomes

$$ds^2 = B_{\mu\nu}(x)dx^\mu dx^\nu = \tilde{g}_{00}d(ct)^2 + \tilde{g}_{ij} [(dx^i + N^i d(ct))(dx^j + N^j d(ct))]
= c^2 (dt)^2 \left[ \tilde{g}_{00} + \tilde{g}_{ij} \left( \frac{1}{c} \dot{x}^i + N^i \right) \left( \frac{1}{c} \dot{x}^j + N^j \right) \right],$$

(4)
where

\[ \tilde{g}_{00} = \frac{R^2}{\sigma(x)(R^2 - c^2 t^2)} \],

\[ \tilde{g}_{ij} = \frac{\eta_{ij}}{\sigma(x)} + \frac{1}{R^2 \sigma(x)^2} \eta_{ij} \eta_{jm} \dot{x}^l \dot{x}^m, \]

\[ N^i = \frac{c t \dot{x}^i}{R^2 - c^2 t^2} \].

By substituting eqs. (3)–(7) into (2), we recast the Lagrangian as

\[ L_{cR} = -m_0 c^2 \sqrt{\tilde{g}_{00} + \tilde{g}_{ij} \left( \frac{1}{c} \dot{x}^i + N^i \right) \left( \frac{1}{c} \dot{x}^j + N^j \right)}. \]

from which the following identity results in,

\[ \frac{\partial L_{cR}}{\partial x^i} = \frac{\partial^2 L_{cR}}{\partial t \partial \dot{x}^i} + \frac{\partial^2 L_{cR}}{\partial x^i \partial \dot{x}^j} \dot{x}^j. \]

Considering the Euler-Lagrangian equation

\[ \frac{\partial L_{cR}}{\partial x^i} = \frac{d}{dt} \frac{\partial L_{cR}}{\partial \dot{x}^i}, \]

we obtain the solution of equation of motion for free particle :

\[ \ddot{x}^j = 0, \quad \dot{x}^j = \text{constant}. \]

Next step is to derive the canonic momenta and the canonic energy (i.e., Hamiltonian). By the eq.(8), they reads

\[ \pi_i = \frac{\partial L_{cR}}{\partial \dot{x}^i} = -m_0 \sigma(x) \Gamma B_{0\mu} \dot{x}^\mu \]

\[ H_{cR} = \sum_{i=1}^{3} \frac{\partial L_{cR}}{\partial \dot{x}^i} \dot{x}^i - L_{cR} = m_0 c \sigma(x) \Gamma B_{0\mu} \dot{x}^\mu. \]

where

\[ \Gamma^{-1} \sigma(x) \frac{ds}{c dt} = \frac{1}{R} \sqrt{(R^2 - \eta_{ij} \dot{x}^i \dot{x}^j)(1 + \frac{\eta_{ij} \dot{x}^i \dot{x}^j}{c^2}) + 2 t \eta_{ij} \dot{x}^i \dot{x}^j - \eta_{ij} \dot{x}^i \dot{x}^j t^2 + \frac{(\eta_{ij} \dot{x}^i \dot{x}^j)^2}{c^2}}. \]

Under the equation of motion Eq.(11), we have the following relation

\[ \dot{\Gamma} \big|_{\dot{x}^i = 0} = 0, \]

whose corresponding one in \( S \mathcal{R}_c \) is

\[ \dot{\gamma} \big|_{\dot{x}^i = 0} \equiv \left. \frac{d}{dt} \left( \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \right) \right|_{v = \text{constant}} = 0. \]
It is easy to check that
\[
\lim_{R \to \infty} \Gamma = \lim_{x^i \to 0} \Gamma = \gamma \equiv \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.
\] (17)

And, in the \( R \to \infty \) limit, \( \pi_i \) and \( H_{eR} \) go back to the standard Einstein Special Relativity’s expressions:
\[
\pi_i|_{R \to \infty} = \frac{m_0 v_i}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad H_{cR}|_{R \to \infty} = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}}.
\] (18)

where \( v_i = -\eta_{ij} \dot{x}^j \). In the Table I, we listed some results of Lagrange formulism both in the ordinary special relativity \( \mathcal{SR}_c \) and in the de Sitter invariant special relativity \( \mathcal{SR}_{cR} \). Comparing the results in \( \mathcal{SR}_{cR} \) with ones in well known \( \mathcal{SR}_c \), we learned that as an extending theory of \( \mathcal{SR}_c \), \( \mathcal{SR}_{cR} \) can simply be formulated by a variable alternating in \( \mathcal{SR}_c \): 1) \( \eta_{\mu\nu} \Rightarrow B_{\mu\nu} \); 2) \( \gamma \Rightarrow \sigma \Gamma \). This is a natural and nice feature for the Lagrangian formulism of \( \mathcal{SR}_{cR} \).

| TABLE I: Metric, Lagrangian, equation of motions, canonical momenta, and Hamiltonian in the special relativity, \( \mathcal{SR}_c \), and in the de Sitter special relativity, \( \mathcal{SR}_{cR} \). The quantities \( \gamma^{-1} = \sqrt{1 + \frac{\eta_{ij} \dot{x}^i \dot{x}^j}{c^2}} \) and \( \Gamma^{-1} = \frac{1}{\sqrt{(R^2 - \eta_{ij} \dot{x}^i \dot{x}^j)(1 + \frac{\eta_{ij} \dot{x}^i \dot{x}^j}{c^2}) + 2t \eta_{ij} \dot{x}^i \dot{x}^j - \eta_{ij} \dot{x}^i \dot{x}^j t^2 + \frac{(\eta_{ij} \dot{x}^i \dot{x}^j)^2}{c^2}} \) (see eq. (14)). |
|---------------------------------|-----------------|-----------------|
| space-time metric               | \( \eta_{\mu\nu} \) | \( B_{\mu\nu}(x) \), (Eq. (3)) |
| Lagrangian                      | \( L_c = -m_0 c^2 \gamma^{-1} \) | \( L_{eR} = -m_0 c^2 \sigma^{-1} \Gamma^{-1} \) |
| equation of motion              | \( v^i = \dot{x}^i = \text{constant} \), ( or \( \dot{\gamma} = 0 \)) | \( v^i = \dot{x}^i = \text{constant} \), ( or \( \dot{\Gamma} = 0 \)) |
| canonical momenta               | \( \pi_i = -m_0 \gamma \eta_{\mu\nu} \dot{x}^\mu \) | \( \pi_i = -m_0 \sigma \Gamma B_{\mu\nu} \dot{x}^\mu \) |
| Hamiltonian                     | \( H_c = m_0 c \gamma \eta_{0\mu} \dot{x}^\mu \) | \( H_{eR} = m_0 c \sigma \Gamma B_{0\mu} \dot{x}^\mu \) |

Combining Eq. (12) with Eq. (13), the covariant 4-momentum in \( \mathcal{B} \) is:
\[
\pi_\mu \equiv (\pi_0, \pi_i) = -\frac{H_{eR}}{c} \equiv -m_0 \sigma \Gamma B_{\mu\nu} \dot{x}^\nu = -m_0 c B_{\mu\nu} \frac{dx^\nu}{ds},
\] (19)

and
\[
B^{\mu\nu} \pi_\mu \pi_\nu = m_0^2 c^2.
\] (20)

From eqs. (8) (12) (13) (20), we have the standard form of \( H_{eR}(t, x^i, \pi_i) \) as follows
\[
H_{eR} = \sqrt{g_{00}} \sqrt{m_0^2 c^4 - c^2 \bar{g}^{ij} \pi_i \pi_j - c \bar{\pi}_i N^i},
\] (21)
where \(g_{00}, N^i\) have been shown in eqs.\((5)\) \((7)\), and \(g^{ij} = \sigma(x)(\eta^{ij} - \frac{x^{i}x^{j}}{R^2 - c^2t^2})\) from eq.\((6)\). It is straightforward to get the following canonical equations

\[
\begin{align*}
\dot{x}^i &= \frac{\partial H_{cR}}{\partial \pi_i} = \{H_{cR}, x^i\}_PB, \\
\dot{\pi}_i &= -\frac{\partial H_{cR}}{\partial x^i} = \{H_{cR}, \pi_i\}_PB.
\end{align*}
\]

(22)

where the Poisson bracket

\[
\{x^i, \pi_j\}_PB = \delta^i_j, \quad \{x^i, x^j\}_PB = 0, \quad \{\pi_i, \pi_j\}_PB = 0
\]

(23)

are as usual. It is also straightforward to check \(\dot{x}^i = \text{constant}\) by eq.\((22)\).

Finally, we like to address that the canonical momenta \(\pi_i\) and the Hamiltonian \(H_{cR}\) are not the physically conserved momentum and the energy of the particle respectively, but they will play important role in the quantization of \(SR_{cR}\)-mechanics.

B. Space-time symmetry of de Sitter special relativity and the Neother charges

The space time transformations preserving the Beltrami metric were discovered about 30 years ago by Lu, Zou and Guo (LZG)\[11][12]\(\text{(see also Appendix of [14])}\). When we transform from one initial Beltrami frame \(x^\mu\) to another Beltrami frame \(\tilde{x}^\mu\), and when the origin of the new frame is \(a^\mu\) in the original frame, the transformations between them with 10 parameters is as follows

\[
\begin{align*}
x^\mu \xrightarrow{LZG} \tilde{x}^\mu &= \pm \sigma(a)^{1/2} \sigma(a, x)^{-1}(x^\nu - a^\nu)D^\mu_\nu, \\
D^\mu_\nu &= L^\mu_\nu + R^{-2} \eta_{\nu\rho} a^\rho a^\lambda (\sigma(a) + \sigma^{1/2}(a))^{-1} L^\mu_\lambda, \\
L : &= (L^\mu_\nu) \in SO(1,3), \\
\sigma(x) &= 1 - \frac{1}{R^2} \eta_{\mu\nu} x^\mu x^\nu, \\
\sigma(a, x) &= 1 - \frac{1}{R^2} \eta_{\mu\nu} a^\mu a^\nu.
\end{align*}
\]

(24)

It will be called as LZG-transformation hereafter. Under LZG-transformation, the \(B_{\mu\nu}(x)\) and the action of \(SR_{cR}\) transfer respectively as follows

\[
\begin{align*}
B_{\mu\nu}(x) \xrightarrow{LZG} \tilde{B}_{\mu\nu}(\tilde{x}) &= \frac{\partial x^\lambda}{\partial \tilde{x}^\mu} \frac{\partial x^\rho}{\partial \tilde{x}^\nu} B_{\lambda\rho}(x) = B_{\mu\nu}(\tilde{x}), \\
S_{cR} &\equiv \int dt L_{cR}(t) = -m_0c \int dt \sqrt{B_{\mu\nu}(x)dx^\mu dx^\nu} dt \xrightarrow{-LZG} \tilde{S}_{cR} = S_{cR}.
\end{align*}
\]

(25) 

(26)
By the mechanics principal, this action invariance indicates that there are 10 conserved Noether charges in $S\mathcal{R}_{cR}$ like the $S\mathcal{R}_c$ case. For $S\mathcal{R}_c$ the Noether charges are (e.g., see pp581-586 and Part 9 in ref. [19]):

Noether charges for Lorentz boost:  
\[ K_i^c = m_0 \gamma c (x_i - t \dot{x}^i) \]

Charges for space – transitions (momenta):  
\[ P_i^c = m_0 \gamma \dot{x}^i, \]

Charge for time – transition (energy):  
\[ E_c^c = m_0 c^2 \gamma \]

Charges for rotations in space (angularmomenta):  
\[ L_i^c = \epsilon_{ijk} \dot{x}^j P_k^c. \]

Here \( \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \). Note the Noether charges here are the same as the corresponding canonical quantities, because the Lagrangian for $S\mathcal{R}_c$ is time-independent and all the coordinates are cyclic. While in $S\mathcal{R}_{cR}$ there is no cyclic ignorable coordinates and the Lagrangian is spacetime dependent.

When space rotations were neglected temporarily for simplify, the LZG-transformation both due to a Lorentz-like boost and a space-transition in the $x^1$ direction with parameters $\beta = \frac{\dot{x}^1}{c}$ and $a^1$ respectively and due to a time transition with parameter $a^0$ can be explicitly written as follows:

\[ t \rightarrow \tilde{t} = \sqrt{\frac{\sigma(a)}{\epsilon(a,x)}} \gamma \left[ ct - \beta x^1 - a^0 + \beta a^1 + \frac{a^0 - \beta a^1 - \beta^2 x^1}{R^2} \frac{\sigma(a)}{\epsilon(a,x)} \sigma(a) \right] \]

\[ x^1 \rightarrow \tilde{x}^1 = \sqrt{\frac{\sigma(a)}{\epsilon(a,x)}} \gamma \left[ x^1 - \beta ct + \beta a^0 - a^1 + \frac{a^1 - \beta a^0}{R^2} \frac{\sigma(a)}{\epsilon(a,x)} \sigma(a) \right] \]

\[ x^2 \rightarrow \tilde{x}^2 = \sqrt{\frac{\sigma(a)}{\epsilon(a,x)}} x^2 \]

\[ x^3 \rightarrow \tilde{x}^3 = \sqrt{\frac{\sigma(a)}{\epsilon(a,x)}} x^3 \]  \hspace{1cm} (28)

It is easy to check when $R \rightarrow \infty$ the above transformation goes back to Poincaré transformation. Notice that in the LZG-transformation there are 3 boost parameters $\beta^i = \frac{\dot{x}^i}{c} = \frac{v^i}{c}$, 4 spacetime transition parameters $(a^0, a^1, a^2, a^3)$ (and 3 rotation parameters $\theta^i$). Here $(a^0, a^1, a^2, a^3)$ is the origin of the resulting Beltrami initial frame in the original Beltrami frame. By the standard manner and eq. (28), we have got all $S\mathcal{R}_{cR}$-Noether charges in [14], which correspond to the $S\mathcal{R}_c$-Noether charges eq. (27). Those $S\mathcal{R}_{cR}$-Noether charges are follows [14]:

Noether charges for Lorentz boost:  
\[ K_i^{cR} = m_0 \Gamma c (x^i - t \dot{x}^i) \]

Charges for space – transitions (momenta):  
\[ P_i^{cR} = m_0 \Gamma \dot{x}^i, \]

Charge for time – transition (energy):  
\[ E_c^{cR} = m_0 c^2 \Gamma \]

Charges for rotations in space (angularmomenta):  
\[ L^i_{cR} = \epsilon_{ijk} x^j P^k_{cR}. \]

\hspace{1cm} (29)
where Γ were given in eq(14). Compactly, by the above, we have the 4-momentum in \( \mathcal{SR}_{cR} \) as follows

\[
p_{cR}^\mu \equiv \{ p_0^{cR}, p_i^{cR} \} = m_0 \Gamma \frac{d\chi^\mu}{\sigma(x)} ds = -\frac{1}{\sigma(x)} B^{\mu\nu} \pi_\nu.
\]

In terms of eq.(20), the Einstein’s famous mass-energy-momentum formula \( E_c^2 = m_0^2 c^4 + c^2 p_c^2 \) now becomes

\[
E_{cR}^2 = m_0^2 c^4 + c^2 p_{cR}^2 + \frac{c^2}{R^2}(L_{cR}^2 - K_{cR}^2),
\]

where \( E_{cR}, p_{cR}, L_{cR}, K_{cR} \) are conserved physical energy, momentum, angular-momentum and boost charges in eq.(29) respectively.

### III. QUANTUM MECHANICS IN DE SITTER SPECIAL RELATIVITY

Lagrangian-Hamiltonian formulation of mechanics is the foundation of quantization. When the classical Poisson brackets in canonical equations for canonical coordinates and canonical momentum become operator’s commutators, i.e., \( \{ x, \pi \}_{PB} \Rightarrow \frac{1}{\hbar}[x, \pi] \), the classical mechanics will be quantized. In this way, for instance, the ordinary relativistic (i.e., \( \mathcal{SR}_c \)) one-particle quantum equations have been derived. To the particle with spin-0, that is just the well known Klein-Gordon equation.

Following this first principle clue, we have derived the one-particle quantum mechanics for \( \mathcal{SR}_{cR} \) in [14]. In the canonic quantization formulism, the canonic variable operators are \( x^i, \pi_i \) with \( i = 1, 2, 3 \). And due to eq.(23) the basic commutators for the free particle quantization theory of \( \mathcal{SR}_{cR} \) are the same as usual, i.e.,

\[
[x^i, \pi_j] = i\hbar \delta^i_j, \quad [\pi_i, \pi_j] = 0, \quad [x_i, x_j] = 0,
\]

hereafter the hat notations for operators are removed. Considering Wyle ordering of \( (\pi x) \) and solving (32), we have [14]

\[
\pi_\mu = -i\hbar B^{-\frac{1}{2}} \partial_\mu B^{\frac{1}{2}} = -i\hbar \partial_\mu - i\hbar B^{\frac{1}{2}} (\partial_\mu B^{\frac{1}{2}}),
\]

where \( B = \det(B_{\mu\nu}) \). The classical dispersion relation (20) can be rewritten as symmetric version \( B^{-\frac{1}{2}} \pi_\mu B^{\frac{1}{2}} B^{\mu\nu} B^{\frac{1}{2}} \pi_\nu B^{-\frac{1}{2}} = m_0^2 c^2 \), and then the \( \mathcal{SR}_{cR} \)-one particle wave equation reads

\[
B^{-\frac{1}{2}} \pi_\mu B^{\frac{1}{2}} B^{\mu\nu} B^{\frac{1}{2}} \pi_\nu B^{-\frac{1}{2}} \phi(x,t) = m_0^2 c^2 \phi(x,t),
\]
where $\phi(x,t)$ is the particle’s wave function. Substituting (33) into (34), we have
\[ \frac{1}{\sqrt{B}} \partial_\mu (B^{\mu\nu} \sqrt{B} \partial_\nu) \phi + \frac{m_c^2 c^2}{\hbar^2} \phi = 0, \] (35)
which is just the Klein-Gordon equation in curved space-time with Beltrami metric $B_{\mu\nu}$, and its explicit form is
\[ (\eta^{\mu\nu} - \frac{x^{\mu} x^{\nu}}{R^2}) \partial_\mu \partial_\nu \phi - 2 \frac{x^{\mu}}{R^2} \partial_\mu \phi + \frac{m_c^2 c^2}{\hbar^2 \sigma(x)} \phi = 0, \] (36)
which is the desired $\mathcal{SR}_{cR}$-quantum mechanics equation for free particle with spin 0. Substituting (33) into (30), we obtain the physical momentum and energy operators (noting the subscripts $cR$ for $p_\mu^{cR}$, $L^{\mu\nu}_{cR}$ in (30) will be moved hereafter):
\[ p_\mu = i\hbar [(\eta^{\mu\nu} - \frac{x^{\mu} x^{\nu}}{R^2}) \partial_\nu + \frac{5x^{\mu}}{2R^2}]. \] (37)
Operator $p_\mu$ together with operator $L^{\mu\nu} = (x^{\mu} p^{\nu} - x^{\nu} p^{\mu})/(i\hbar)$ form a algebra as follows
\[ [p_\mu, p_\nu] = \frac{\hbar^2}{R^2} L^{\mu\nu} \] (38)
\[ [L^{\mu\nu}, p_\rho] = \eta^{\rho\nu} p_\mu - \eta^{\rho\mu} p_\nu \]
\[ [L^{\mu\nu}, L^{\rho\sigma}] = \eta^{\rho\nu} L^{\mu\sigma} - \eta^{\rho\sigma} L^{\mu\nu} + \eta^{\mu\sigma} L^{\rho\nu} - \eta^{\mu\nu} L^{\rho\sigma} \]
which is just the de-Sitter algebra SO(1,4). This fact means that the quantization scheme presented in this paper preserves the external space-time symmetry of $\mathcal{SR}_{cR}$.

By the Klein-Gordon equation in curved space-time with Beltrami metric $B_{\mu\nu}$, eq.(35), we have the corresponding Dirac equation which describes the particle with spin 1/2 [20, 21]:
\[ (i e^a_\mu \gamma^a D_\mu - \frac{m_0 c}{\hbar}) \psi = 0, \] (39)
where $e^a_\mu$ is the tetrad and $D_\mu$ is the covariant derivative with Lorentz spin connection $\omega^{ab}_\mu$.

Their definitions and relations are follows (e.g., see [21])
\[ D_\mu = \partial_\mu - i \frac{\omega^{ab}_\mu}{4} \sigma_{ab}, \]
\[ \{ \gamma^a, \gamma^b \} = 2\eta^{ab}, \quad \sigma_{ab} = \frac{i}{2} [\gamma_a, \gamma_b], \quad \frac{i}{2} [\sigma_{ab}, \sigma_{cd}] = \eta_{ac} \sigma_{bd} - \eta_{ad} \sigma_{bc} + \eta_{bd} \sigma_{ac} - \eta_{bc} \sigma_{ad}, \]
\[ e^{a}_{\mu} \epsilon^{b}_{\nu} \eta_{ab} = B_{\mu\nu}, \quad e^{a}_{\mu} e^{b}_{\nu} B^{\mu\nu} = \eta^{ab}, \quad \epsilon^{\mu}_{a,\nu} = \partial_{\nu} e^{\mu}_{a} + \omega^{b}_{a,\nu} e^{\mu}_{b} + \Gamma^{\mu}_{\lambda \nu} e^{\lambda}_{a} = 0, \]
\[ \omega^{\mu}_{ab} = \frac{1}{2} (e^{a}_{\mu} \partial_{\rho} e^{\rho}_{b} - e^{b}_{\mu} \partial_{\rho} e^{\rho}_{a}) - \frac{1}{2} \Gamma^{\mu}_{a,\nu} (e^{\lambda}_{\rho} e^{\rho}_{b} - e^{b}_{\rho} e^{\rho}_{a}), \]
\[ \Gamma^{\mu}_{a,\nu} = \frac{1}{2} B^{\rho\nu} (\partial_{\lambda} B_{\nu\mu} + \partial_{\mu} B_{\nu\lambda} - \partial_{\nu} B_{\lambda\mu}). \] (40)
It is straightforward to check that the components $\psi_{\alpha}$ $(\alpha = 1, \ldots, 4)$ of the spinor satisfy the Klein-Gordon equation (33).
IV. HYDROGEN ATOM IN EARTH-QSO REFERENCE FRAME AND VARIATION OF ELECTROMAGNETIC FINE-STRUCTURE CONSTANT

Now we are going to solve $\mathcal{SR}_{cR}$-Dirac equation for hydrogen atom on QSO. In this cosmologic quantum system, there are two cosmologic length scales: cosmic radius $R \sim 10^{12} ly$ and the distance between QSO and earth $ct > R > c t > 10^8 ly$, and two microcosmic length scales: the Compton wave length of electron $a_c = \hbar/(m_e c) \approx 0.3 \times 10^{-12} m$, and Bohr radius $a = \hbar^2/(m_e e^2) \approx 0.5 \times 10^{-10} m$. The calculations for our purpose will be accurate up to $O(c^2 t^2/R^2)$. The terms proportional to $O(c^4 t^4/R^4)$, $O(cta_c/R^2)$, $O(cta/R^2)$ etc will be omitted.

A. $\mathcal{SR}_{cR}$-Dirac equation for hydrogen atom on QSO

The phenomenology of atomic physics at the cosmologic space-time scale should be discussed in terms of $\mathcal{SR}_{cR}$-quantum mechanics rather than $\mathcal{SR}_{c}$’s. Now, we show the $\mathcal{SR}_{cR}$-Dirac equation of hydrogen atom on a QSO in the earth-QSO reference frame. As illustrated in Fig.1, the earth locates at the origin of frame, the proton (nucleus of hydrogen atom) locates at $Q = \{Q^0 = c t, Q^1 = c t, Q^2 = 0, Q^3 = 0\}$, which is on QSO-light-cone $B_{\mu\nu}(Q)Q^\mu Q^\nu = \eta_{\mu\nu}Q^\mu Q^\nu = 0$. The metric of the space-time near $Q$ is

$$B_{\mu\nu}(Q) = \eta_{\mu\nu} + \frac{1}{R^2}\eta_{\mu\lambda}Q^\lambda\eta_{\nu\rho}Q^\rho,$$

and hence $B_{ij}(Q) = \eta_{ij} + \frac{c^2 t^2}{R^2}\delta_{i1}\delta_{j1}$. (41)

The electron’s coordinates are $L = \{L^0 = c t, L^1, L^2, L^3\}$, and the relative space coordinates between proton and electron are $x^i = L^i - Q^i$. The magnitude of $r \equiv \sqrt{-\eta_{ij}x^i x^j} \sim a$ (where $a \approx 0.5 \times 10^{-10} m$ is Bohr radius), and $|x^i| \sim a$.

According to gauge principle, the electrodynamic interaction between the nucleus and the electron can be taken into account by replacing the operator $D_\mu$ in eq.(39) with the $U(1)$-gauge covariant derivative $\mathcal{D}_\mu^L \equiv D_\mu^L - \delta_{\mu0}ie/(ch)\phi(x)$. Hence, the $\mathcal{SR}_{cR}$-Dirac equation for electron in hydrogen at QSO reads

$$(ie^\mu_{\alpha}a^\alpha\mathcal{D}_\mu^L - \frac{\mu c}{\hbar})\psi = 0,$$

(42)

where $\mu = m_e/(1 + m_e/m_p)$ is the reduced mass of electron, $\mathcal{D}_\mu = \frac{\partial}{\partial L^\mu} - \frac{i}{4}\omega^a_{\mu\sigma}a^\sigma - \delta_{\mu0}ie/(ch)\phi(x)$, $e^\mu_{\alpha}$ and $\omega^a_{\mu\sigma}$ have been given in eqs.(3) (40). For our purpose, we approximate $e^\mu_{\alpha}$ and $\omega^a_{\mu\sigma}$ up
FIG. 1: Sketch of the earth-QSO reference frame. The earth locates at the origin. The position vector for nucleus of atom on QSO is \( Q \), and for electron is \( L \). The distance between nucleus and electron is \( r \).

To \( \mathcal{O}(1/R^2) \),

\[
e^\mu_a = \left(1 - \frac{\eta_{cd} L^c L^d}{2R^2}\right) \eta^\mu_a - \frac{\eta_{ab} L^b L^\mu}{2R^2} + \mathcal{O}(1/R^4),
\]  \hspace{1cm} (43)

\[
\omega^{ab}_\mu = \frac{1}{2R^2} (\eta^a_\mu L^b - \eta^b_\mu L^a) + \mathcal{O}(1/R^4).
\]  \hspace{1cm} (44)

B. Solution of usual \( \mathcal{S}\mathcal{R}_c \)-Dirac equation for hydrogen atom at QSO

At first, we show the solution of usual \( \mathcal{S}\mathcal{R}_c \)-Dirac equation in the earth-QSO reference frame of Fig.1, which serves as leading order of solution for the \( \mathcal{S}\mathcal{R}_cR \)-Dirac equation with \( R \to \infty \) in that reference frame. For the hydrogen, \( \partial_{\mu} \to \mathcal{D}^L_{\mu} = \partial^L_{\mu} - \delta_{\mu0} ie/(ch)\phi_M(x) \) (noting \( \omega^{ab}_\mu|_{R \to \infty} = 0 \)), where \( \phi_M(x) \) is nucleus electric potential at \( x^i \) in Minkowski space defined by following equation

\[
- \eta^{ij} \partial_i \partial_j \phi_M(x) = \nabla^2 \phi_M(x) = -4\pi \rho(x) = -4\pi e \delta^{(3)}(x).
\]  \hspace{1cm} (45)

The solution is \( \phi_M(x) = e/r \), and hence \( \partial_0 \to \mathcal{D}^L_0 = \partial_0 - i e^2/(chr) \). Then, the \( \mathcal{S}\mathcal{R}_c \)-Dirac equation reads

\[
i\hbar \partial_t \psi = \left(-i\hbar c \vec{\alpha} \cdot \nabla_L + \mu c^2 \beta - \frac{e^2}{r}\right) \psi,
\]  \hspace{1cm} (46)
where $\beta = \gamma^0$, $\alpha^i = \beta \gamma^i$. Noting the nucleus position $Q$ = constant, we have

$$\nabla L = \frac{\partial}{\partial L} = \frac{\partial}{\partial (Q+r)} = \frac{\partial}{\partial r} \equiv \nabla,$$

and eq. (46) becomes the standard Dirac equation for electron in hydrogen at its nucleus reference frame. Energy $E$ for $\mathcal{S}\mathcal{R}_c$-mechanics is conserved, and the hydrogen is the stationary states of $\mathcal{S}\mathcal{R}_c$-Dirac equation. The stationary state condition is

$$i\hbar \partial_t \psi = E \psi.$$

As is well known, combining eqs. (46), (47) with (48), we have

$$E \psi = \left( -i \hbar c \alpha \cdot \nabla + \mu c^2 \beta - \frac{e^2}{r} \right) \psi,$$

which is the stationary $\mathcal{S}\mathcal{R}_c$-Dirac equation for hydrogen. The problem has been solved in terms of standard way, and the results are follows (see, e.g., [22][18][23])

$$E = E_{n,K} = \mu c^2 \left[ 1 + \frac{\alpha^2}{(\sqrt{K^2 - \alpha^2 + n_r})^2} \right]^{-1/2}, \quad \alpha \equiv \frac{e^2}{\hbar c}, \quad n_r = 0, 1, 2, \cdots$$

$$|K| = (j + 1/2) = 1, 2, 3, \cdots.$$

And its expansion equation in $\alpha$ is

$$E = \mu c^2 - \mu c^2 \frac{\alpha^2}{2n^2} \left[ 1 + \frac{\alpha^2}{n^2} \left( \frac{n}{j + 1/2} - \frac{3}{4} \right) + \cdots \right], \quad n = n_r + |K| = 1, 2, 3, \cdots.$$

The corresponding hydrogen’s wave functions $\psi$ have also already been finely derived (see e.g., [22][18][23]). The complete set of commutative observables is $\{H, K, j^2, j_z\}$, so that $\psi = \psi_{n,K,j,j_z}(r, \hbar, \mu, \alpha)$, where $j = 1 + \frac{\hbar}{2} \Sigma$, $\hbar K = \beta (\Sigma \cdot \mathbf{l} + \hbar)$, and $\alpha = e^2/(hc)$.

C. Beltrami-geometry effects in $\mathcal{S}\mathcal{R}_{cR}$-Dirac equation

By eqs. (42), (43) and $\partial \mu \rightarrow D^L_{\mu} = \frac{\partial}{\partial L^\mu} - \frac{i}{4} \omega^{ab}_{\mu} \gamma_{ab} - \delta_{\mu 0} ie/(ch)\phi_B(x)$, (where $\phi_B(x)$ is the electric potential in Beltrami space), we have the $\mathcal{S}\mathcal{R}_{cR}$-Dirac equation for the electron in hydrogen at the earth-QSO reference frame as follows

$$\hbar c \beta \left[ \left( 1 - \frac{\eta_{ab} L^a L^b}{2R^2} \right) \gamma^\mu D^L_{\mu} - \frac{i}{2R^2} \eta_{ab} \gamma^a L^b D^L_{\mu} - \frac{\mu c}{\hbar} \right] \psi = 0,$$

where factor $\hbar c \beta$ in the front of the equation is only for convenience. We expand each terms of (52) in order as follows:

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1. Since observed QSO must locates at the light cone, then \( \eta_{ab} L^a L^b \simeq 0 \), and the first term of (52) reads
\[
h c \beta i \gamma^\mu D^L_\mu \psi = \left( i \hbar \partial_t + i \hbar c \vec{\alpha} \cdot \nabla + \frac{\hbar c \beta}{4} \omega^{ab}_\mu \gamma^\mu \sigma_{ab} + e \phi_B(x) \right) \psi, \tag{53}
\]
where, being similar to (45), \( \phi_B(x) \) defined by following equation
\[
- B^{ij}(Q) \partial_i \partial_j \phi_B(x) = \left( \nabla^2 + \frac{c^2 t^2}{R^2} \frac{\partial^2}{\partial (x^1)^2} \right) \phi_B(x) = -4 \pi \rho_B(x) = -\frac{4 \pi e}{\sqrt{-\det(B_{ij}(Q))}} \delta^{(3)}(x). \tag{54}
\]
The solution is (see Appendix A)
\[
\phi_B = \frac{e}{r_B} \simeq \frac{e}{r} \left( 1 + \frac{c^2 t^2 (x^1)^2}{2R^2 r^2} \right), \tag{55}
\]
where \( r_B = \sqrt{(\tilde{x}^1)^2 + (x^2)^2 + (x^3)^2} \) with \( \tilde{x}^1 = (1 - c^2 t^2/(2R^2)) x^1 \). The correction factor due to \( B^{ij} \) shows a little bit of non-isotropy in \( \tilde{x}^1 \)-direction. In order to deal with this non-isotropy effect, we will use \( \{ \tilde{x}^1, x^2, x^3 \} \) (instead of usual \( \{ x^1, x^2, x^3 \} \)) to be the space coordinate variables of Dirac equation [32]. Following notations are introduced hereafter:
\[
r_B = i \tilde{x}^1 + j x^2 + k x^3, \quad |r_B| = r_B, \tag{56}
\]
\[
\nabla_B = i \frac{\partial}{\partial \tilde{x}^1} + j \frac{\partial}{\partial x^2} + k \frac{\partial}{\partial x^3}, \quad \tilde{x}^i \in \{ \tilde{x}^1, x^2, x^3 \}. \tag{57}
\]
Then the eq.(53) becomes
\[
h c \beta i \gamma^\mu D^L_\mu \psi = \left( i \hbar \partial_t + i \hbar c \vec{\alpha} \cdot \nabla_B - i \hbar c \frac{c^2 t^2}{2R^2} \alpha^1 \frac{\partial}{\partial \tilde{x}^1} + \frac{\hbar c \beta}{4} \omega^{ab}_\mu \gamma^\mu \sigma_{ab} + \frac{e^2}{r_B} \right) \psi. \tag{58}
\]
2. Estimating the contributions of the fourth term in RSH of (58) (the spin-connection contributions): By [44], the ratio of the fourth term to the first term of (58) is:
\[
\left| \frac{\hbar c \beta}{4} \omega^{ab}_\mu \gamma^\mu \sigma_{ab} \right| \sim \frac{\hbar c \beta}{4} \frac{ct}{2R^2 m_e c^2} \frac{1}{8R^2 m_e c} = \frac{ct}{8R^2} \frac{\hbar c}{8R^2} \sim 0, \tag{59}
\]
where \( a_c = \hbar/(m_e c) \simeq 0.3 \times 10^{-12} m \) is the Compton wave length of electron. \( \mathcal{O}(cta_c/R^2) \)-term is neglectable. Therefore the 3-rd term in RSH of (53) has no contribution to our approximation calculations.
3. Substituting (59) into (58) and noting \( \eta_{ab} L^a L^b \simeq 0 \), we get the first term in LHS of (52)
\[
h c \beta i \left( 1 - \frac{\eta_{ab} L^a L^b}{2R^2} \right) \gamma^\mu D^L_\mu \psi = \left( i \hbar \partial_t + i \hbar c \vec{\alpha} \cdot \nabla_B - i \hbar c \frac{c^2 t^2}{2R^2} \alpha^1 \frac{\partial}{\partial \tilde{x}^1} + \frac{e^2}{r_B} \right) \psi. \tag{60}
\]
4. The second term of (52) is

\[
- \hbar c \beta \frac{i}{2R^2} \eta_{ab} \gamma^b \gamma^a L^\mu D_\mu \psi = -i \hbar c \beta \left( \gamma^0 L^0 - \vec{\gamma} \cdot \vec{L} \right) L^\mu \left[ \partial_\mu - \delta_\mu \delta_{\mu 0} \frac{i e^2}{c \hbar r_B} \right] \psi + \mathcal{O}\left( \frac{1}{R^4} \right)
\]

\[
= -i \hbar c \beta \left( L^0 - \vec{L} \cdot \vec{\alpha} \right) \left( L^0 \partial_0 - L^0 \frac{ie^2}{c \hbar r_B} + L^i \partial_i \right) \psi
\]

\[
\approx -i \hbar c \beta \left( L^0 - L^1 \alpha^1 \right) \left( L^0 \partial_0 - L^0 \frac{ie^2}{c \hbar r_B} + L^i \partial_i \right) \psi
\]

\[
\approx -i \hbar c \beta \left( 1 - \frac{L^1}{L^0} \alpha^1 \right) \left( L^0 \right)^2 \left( \partial_0 - \frac{ie^2}{c \hbar r_B} \right)
\]

\[
+ L^0 L^i \partial_i \alpha^1 - L^2 \alpha^1 \partial_i \psi \]

(61)

where following estimations are used

\[
\frac{L^2}{R} \sim \frac{L^3}{R} \sim \frac{a}{R} \sim 0.
\]

(62)

In order to simplify (61) further, we note that \( c\vec{\alpha} = \vec{v} \) in Dirac equation theory, and hence

\[
\frac{1}{R^2} \frac{L^1}{L^0} \alpha^1 \psi \approx \frac{1}{R^2} \frac{v^1}{c} \ll \frac{1}{R^2}
\]

\[
\frac{1}{R^2} L^i \partial_i^L \psi = \frac{1}{R^2} \left( Q^i + \bar{x}^i \right) \frac{\partial}{\partial x^i} \psi + \mathcal{O}(1/R^4)
\]

\[
\Rightarrow \frac{i}{\hbar R^2} \left[ (Q^i p_i) + (\bar{x}^i p_i) \right] = \frac{i}{\hbar R^2} \left[ (\bar{Q} \cdot \vec{p}) + (\bar{r}_B \cdot \vec{p}) \right]
\]

\[
= 0.
\]

(64)

where \( Q^i \frac{\partial}{\partial x^i} \psi \Rightarrow (\bar{Q} \cdot \vec{p}) \) means that \( (\bar{Q} \cdot \vec{p}) \) serves as mean-value of operator \( \bar{Q} \cdot \vec{p} \) and can be the leading order of the operator’s approximate expansion. Since the electron does circular motion around the nucleus, and is always inside atom, we have \( (\bar{Q} \cdot \vec{p}) = (\bar{r}_B \cdot \vec{p}) = 0 \), and hence (64) holds.

Inserting (63) (64) into (61), we have

\[
- \hbar c \beta \frac{i}{2R^2} \eta_{ab} \gamma^b \gamma^a L^\mu D_\mu \psi = -i \hbar c \beta \left( \gamma^0 L^0 - \vec{\gamma} \cdot \vec{L} \right) L^\mu \left[ \partial_\mu - \delta_\mu \delta_{\mu 0} \frac{i e^2}{c \hbar r_B} \right] \psi + \mathcal{O}\left( \frac{1}{R^4} \right)
\]

\[
\approx -i \hbar c \beta \left( L^0 \right)^2 \left( \partial_0 - \frac{ie^2}{c \hbar r_B} \right) - L^2 \alpha^1 \partial_i \psi
\]

\[
= -i \hbar c \beta \left( 1 - \frac{e^2 t^2}{2R^2} \right) \psi + i \hbar c \beta \left( 1 - \frac{e^2 t^2}{2R^2} \right) \psi.
\]

(65)

5. Therefore, substituting (60) (65) into by (52), we have

\[
\hbar \left( 1 - \frac{e^2 t^2}{2R^2} \right) \partial_i \psi = \left[ -i \hbar c \vec{\alpha} \cdot \nabla_B + \mu c^2 \beta - \left( 1 - \frac{e^2 t^2}{2R^2} \right) \frac{e^2}{r_B} \right] \psi.
\]

(66)
This is a time-dependent wave equation. It is somehow difficult to deal with the time-dependent problems in quantum mechanics. Generally, there are two approximative approaches to discuss two extreme cases respectively: (i) The modification in states obtained by the wave equation depends critically on the time \( T \) during which the modification of the system’s ”Hamiltonian” take place. For this case, one would use the sudden approach; And, (ii), for case that of a very slow modification of Hamiltonian, the adiabatic approach works \[18\]. To wave equation of (66), like the discussions in Introduction of this paper, since \( R \) is cosmologically large and \( R >> ct \), factor \( (c^2t^2/R^2) \) makes the time-evolution of the system is so slow that the adiabatic approximation \[17\] may legitimately works. In the below (the subsection E), we will provide a calculations to confirm this point.

D. \( S\mathcal{R}_{cR} \)-Dirac equation for spectra of hydrogen

In order to discuss the spectra of hydrogen by \( S\mathcal{R}_{cR} \)-Dirac equation, we need to find out its solutions with certain physics energy \( E \). By eq.(37), and being similar to (48), the \( S\mathcal{R}_{cR} \)-energy eigen-state condition for (66) can be derived by means of the operator expression of momentum in \( S\mathcal{R}_{cR} \) (37):

\[
\begin{align*}
p^0 &= \frac{E}{c} = \frac{1}{\hbar} \left[ \frac{1}{c} \partial_t - \frac{ct}{R^2} x^\nu \partial_\nu \right] + \frac{5ct}{2R^2} \\
E &= \frac{\hbar}{\left( \frac{1}{c} \partial_t - \frac{ct}{R^2} \partial_t + \frac{5ct}{2R^2} \right)} \\
E\psi &\simeq \frac{\hbar}{\left( 1 - \frac{c^2t^2}{R^2} \right)} \partial_t \psi,
\end{align*}
\]

(67)

where a estimation for the ratio of the 3-rd term to the 2-nd of \( E\psi \) were used:

\[
\frac{|i\hbar \frac{c^2t^2}{2R^2} \psi|}{\frac{-c^2t^2}{R^2} \frac{\hbar \partial_t \psi}} \sim \frac{|i\hbar \frac{c^2t^2}{2R^2}|}{-2c^2t^2 \hbar E} \sim \frac{5\hbar}{2tm_e c^2} \equiv \frac{5 a_c}{2ct}
\]

where \( a_c \simeq 0.3 \times 10^{-12} \text{m} \) is the Compton wave length of electron and \( ct \) is about the distance between earth and QSO. In our approximative calculations \( a_c/(ct) \) is neglectable. For instance, to a QSO with \( ct \sim 10^9 \text{ly}, a_c/(ct) \sim 10^{-38} << (ct)^2/R^2 \sim 10^{-5} \). Hence the 3-rd term of \( E\psi \) were ignored.

Inserting (67) into (66), we have

\[
\left( 1 + \frac{c^2t^2}{2R^2} \right) E\psi = \left[ -i\hbar \mathbf{\alpha} \cdot \nabla B + mc^2 \beta - \left( 1 - \frac{c^2t^2}{2R^2} \right) \frac{e^2}{r_B} \right] \psi,
\]

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or
\[
E\psi = \left[ -\hbar c \left( 1 - \frac{c^2 t^2}{2R^2} \right) \vec{\alpha} \cdot \nabla_B + \left( 1 - \frac{c^2 t^2}{2R^2} \right) \mu c^2 \beta - \left( 1 - \frac{c^2 t^2}{R^2} \right) \frac{e^2}{r_B} \right] \psi. \tag{68}
\]
Eq. (68) is the $SR_{cR}$-Dirac equation for hydrogen spectra up to $O\left(\frac{c^2 t^2}{R^2}\right)$ (say again, $O(1/R^4)$, $O(\alpha t / R^2)$, $O(\alpha t \sqrt{e}/ R^2)$ terms have been neglected). Eq. (68) can further be written as follows
\[
E\psi = \left( -\hbar c \vec{\alpha} \cdot \nabla_B + \mu c^2 \beta - \frac{e^2}{r_B} \right) \psi, \tag{69}
\]
where
\[
h_t = \left( 1 - \frac{c^2 t^2}{2R^2} \right) \hbar, \tag{70}
\]
\[
\mu_t = \left( 1 - \frac{c^2 t^2}{2R^2} \right) \mu, \tag{71}
\]
\[
e_t = \left( 1 - \frac{c^2 t^2}{2R^2} \right) e. \tag{72}
\]
Eq. (69) is same as (49) except $\hbar$, $\mu$, $e$ be replaced by $h_t$, $\mu_t$, $e_t$. However, since the time $t$ is dynamic variable in the time-dependent Hamiltonian system, we do not know up to now whether $t$ can be approximately treated as a parameter in the system. Hence, at this stage we still cannot conclude $\hbar$, $\mu$, $e$ are time variations by (70), (71), (72). In the following, we pursue this subject.

E. Adiabatic approximation solution to $SR_{cR}$-Dirac spectra equation

Comparing (68) with (49), we can see that there are three correction terms in (68), which are proportional to $(c^2 t^2 / R^2)$. Those corrections service of the effects of $SR_{cR}$. In order to examine adiabatic approach, we rewrite spectra equation (68) into version of wave equation like eq. (46) via $E \Rightarrow i\hbar \partial_t$:
\[
\hbar \partial_t \psi = H(t)\psi = [H_0(r, e) + H'(t)]\psi, \tag{73}
\]
where
\[
H_0(r, e) = -\hbar c \vec{\alpha} \cdot \nabla_B + \mu c^2 \beta - \frac{e^2}{r_B} \text{ (see eq. (49))} \tag{74}
\]
\[
H'(t) = -\left( \frac{c^2 t^2}{2R^2} \right) H_0(r, \sqrt{2e}). \tag{75}
\]
Suppose initial state of the atom is $\psi(t = 0) = \psi_s(r_B, \hbar, \mu, \alpha)$ where $s = \{n_s, K_s, j^2_s, j_{sz}\}$, by eqs. (73) (74) (75), and catching the time-evolution effects, we have (see Chapter XVII
of Vol II of \[18\], and Appendix B)

\[
\psi(t) \simeq \psi_s(r_B, \hbar t, \mu_t, \alpha_t)e^{-i\frac{E_s}{\hbar}t} + \sum_{m \neq s} \frac{\hat{H}'(t)_{ms}}{i\hbar\omega^2_{ms}} (e^{i\omega_{ms}t} - 1) \psi_m(r_B, \hbar t, \mu_t, \alpha_t)e^{-i\int_0^t \frac{E_m(\theta)}{\hbar}d\theta},
\]

(76)

where \(h_t, \mu_t\) are given in (70) and (71), and

\[
\alpha_t \equiv \frac{e^2}{h_t c} = \left(1 - \frac{c^2 t^2}{2R^2}\right) \alpha, \ \text{with} \ \alpha = \frac{e^2}{hc}
\]

(77)

\[
\hat{H}'(t)_{ms}\big|_{(m \neq s)} = \langle m|\hat{H}'(t)|s\rangle\big|_{(m \neq s)} = -\frac{e^2 t}{R^2}\langle m|H_0(t, r, \sqrt{\frac{5}{3}} e)|s\rangle\big|_{(m \neq s)}
\]

\[
= -\frac{e^2 t}{R^2} \langle n_m, K_s, j^2_s, j_{sz}|(-2 + 1)\left(\frac{e^2}{r}\right)|n_s, K_s, j^2_s, j_{sz}\rangle e^{-i(\omega_s - \omega_m)t}
\]

\[
= \frac{e^2 t}{R^2} \langle n_m, e^2|n_s\rangle e^{-i(\omega_s - \omega_m)t},
\]

(78)

\[
\omega_{ms} = \omega_m - \omega_s, \ \omega_m = \frac{E_m}{\hbar}.
\]

(79)

Note, formula \(\langle m|H_0(t, r, e)|s\rangle\big|_{m \neq s} = 0\) has been used in the calculations of (78). The second term of Right-Hand-Side (RHS) of eq. (76) represents the quantum transition amplitudes from \(\psi_s\)-state to \(\psi_m\), which belong to non-adiabatic effect corrections (or the perturbation corrections from the adiabatic approximation). Now for showing the order of magnitude of such corrections, we estimate \(|\hat{H}'(t)_{ms}/\hbar\omega^2_{ms}|\) for \(s = 1, m = 2\) and \(t \sim 10^9\text{Yr}, R \sim 10^{12}\text{ly}\). To the leading order of \(\alpha\), the radial wave functions for hydrogen with \(n = 1, 2\) are

\[
R_{10} = \frac{2}{a^{3/2}} \exp[-r/a], \quad R_{20} = \frac{1}{\sqrt{2}a^{3/2}} \left(1 - \frac{r}{2a}\right) \exp[-r/2a],
\]

where Bohr radius \(a \simeq 0.5 \times 10^{-10}m\). Therefore, we have

\[
\left|\hat{H}'(t)_{21}/\hbar\omega^2_{21}\right| \simeq \frac{256\sqrt{2}}{243} \frac{ct \alpha}{R^2 \alpha} \simeq 1.4 \times 10^{-40} \ll \left(\frac{e^2 t^2}{R^2}\right) \sim \mathcal{O}(10^{-5}) \ll 1.
\]

(80)

Generally, the \(|\hat{H}'(t)_{ms}/\hbar\omega^2_{ms}|_{m \neq s} \sim |\hat{H}'(t)_{21}/\hbar\omega^2_{21}|\) are also tiny. The basic reason is that the variation of the potential \(H'(t)\) in the the Bohr time-period \((\Delta T_{ms}^{(\text{Bohr})})\hat{H}'(t)_{ms} = (2\pi/\omega_{ms})\hat{H}'(t)_{ms}\) are much much less than \(\hbar\omega_{ms}\). That makes the quantum transition from lower state \(|s\rangle\) to higher state \(|m\rangle\) almost impossible. Thus we conclude that the non-adiabatic effect corrections are tiny, and by eq. (76) the adiabatic wave function of leading order is legitimate and accurate enough as the solution of eq. (73):

\[
\psi(t) \simeq \psi_s(r_B, \hbar t, \mu_t, \alpha_t)e^{-i\frac{E_s}{\hbar}t}.
\]

(81)
Therefore the solution of (68) is \( \psi = \psi_s(r_B, \hbar_t, \mu_t, \alpha_t) \) with \( s = \{ n, K, j_z, j_z \} \) and \( \hbar_t, \mu_t, \alpha_t \) defined by (70) (71) (77). For the solution of \( SR_{cR}-\text{Dirac} \) equation of hydrogen at earth-QSO reference framework, very interesting result is that the electromagnetic fine-structure constant and the mass of electron are of variation with time as follows (i.e., eqs. (77), (71))

\[
\frac{\Delta \alpha}{\alpha} = \frac{\alpha_t - \alpha}{\alpha} = \frac{c^2 t^2}{2 R^2},
\]

\[
\frac{\Delta m_e}{m_e} = \frac{(m_e)_t - m_e}{m_e} = -\frac{c^2 t^2}{2 R^2}.
\]

Because \( ct \) represents the distance between earth and QSO, above equations indicate that \( \Delta \alpha/\alpha \) and \( \Delta m_e/m_e \) can also be thought of variation with distances. The observation quantity in experiments \([1, 2, 3, 4, 5, 6, 7]\) is the frequency of spectra \( \omega_t \) that is as follows

\[
\omega_t = \frac{E_t}{\hbar_t} = \frac{\mu_t c^2}{\hbar_t} \left[ 1 + \frac{\alpha_t^2}{(\sqrt{K^2 - \alpha_t^2 + n_r})^2} \right]^{-1/2}
\]

\[
= \frac{\mu c^2}{\hbar} \left[ 1 + \frac{\alpha_t^2}{(\sqrt{K^2 - \alpha_t^2 + n_r})^2} \right]^{-1/2},
\]

where fact of \( \mu_t/\hbar_t = \mu/\hbar \) due to (70) (71) has been used. Consequently, the \( t \)-dependence of \( \omega_t \) is caused by \( t \)-dependence of \( \alpha_t \) totally, and hence the time variation of \( \alpha \) could be observed by analyzing the spectra emitted from atoms on distant galaxy.

Bing equivalent with (83), (82) and (70), and noting Compton wave length of electron \( a_c = \hbar/(m_e c) \) and Bohr radius \( a = \hbar^2/(m_e e^2) = a_c/\alpha \), we can also express the variations as follows

\[
\frac{\Delta a_c}{a_c} = \frac{(a_c)_t - a_c}{a_c} = 0,
\]

\[
\frac{\Delta a}{a} = \frac{a_t - a}{a} = \frac{c^2 t^2}{2 R^2}.
\]

F. Comparing theory predictions to observations

The observations of absorption spectra of distant interstellar clouds were reported in \([1, 2, 3, 4, 5, 6, 7]\). They belong to directly exploring cosmic atom physics experimentally. Murphy and collaborators \([2]\) studied the spectra of 143 quasar absorption systems over the redshift range \( 0.2 < z_{abs} < 4.2 \). Their most robust estimate is a weighted mean

\[
\frac{\Delta \alpha}{\alpha} = (-0.57 \pm 0.11) \times 10^{-5}.
\]
Comparing with the prediction (82), we conclude that \( R^2 > 0 \). This means that the space-time symmetry for \( \mathcal{SR}_{cR} \) is de Sitter-\( SO(4,1) \) instead of anti-de Sitter-\( SO(3,2) \).

The 134 data points are assigned three epochs in ref. [24] (see table II), and the redshift \( z \)-dependence of \( \Delta \alpha/\alpha \) is shown roughly in [24]. In following, We try to further test the prediction of (82) in terms of these \( z \)-dependent data of \( \Delta \alpha/\alpha \). In order to transfer the \( t \)-dependence of \( \Delta \alpha/\alpha \) in (82) to a \( z \)-dependence prediction, a relation of \( t - z \) is needed. For this aim, an appropriate cosmological model is necessary since the description of cosmical evolution is over the \( \mathcal{SR}_{cR} \) framework. The model considerations are follows: 1) The distance between the earth and QSO in the Beltrami reference frame with origin of the earth (which is an inertial reference frame in \( \mathcal{SR}_{cR} \)) \( Q^1 \) is caused by comoving motion due to the expansion of the Universe. And \( Q^0/c = t \) is the comoving time; 2) The comoving time \( t \) is determined by \( \Lambda \)CDM model [25, 26]. In this model, we have \( t - z \) relation as follows

\[
t = \int_0^z \frac{dz'}{H(z')(1 + z')},
\]

(88)

where

\[
H(z') = H_0 \sqrt{\Omega_{m0}(1 + z')^3 + 1 - \Omega_{m0}},
\]

\[
H_0 = 100 \, h \simeq 100 \times 0.705 \, km \cdot s^{-1}/Mpc,
\]

\[
\Omega_{m0} \simeq 0.274.
\]

The \( t - z \) relation is shown in Fig.(2). Substituting this relation into (82), we obtain desireous \( z \)-dependence prediction of \( \frac{\Delta \alpha}{\alpha} (z) \), where \( R \) is free parameter. By using observation data \( \frac{\Delta \alpha}{\alpha} (z = 1.47) = -0.58 \times 10^{-5} \), we get \( R \simeq 2.73 \times 10^{12} \) (which is consistent with the estimation in [15]). Then the theory predictions are \( \frac{\Delta \alpha}{\alpha} (z = 0.65) = -0.24 \times 10^{-5} \) and \( \frac{\Delta \alpha}{\alpha} (z = 2.84) = -0.87 \times 10^{-5} \), which are in agreement with the corresponding data in [2] and [24]. The results are listed in table II, and the curve of \( \frac{\Delta \alpha}{\alpha} (z) \) is shown in Fig.(3).

The comparison concludes that the theory predictions of (82) agree with the observation data.

The observation data of time variation of \( m_p/m_e \) have also be reported [27]. However, there are no yet data of time variation of \( m_e \) up to now. Further experimental check to these predictions is expected.
TABLE II: Time variations of $\Delta \alpha/\alpha$: The first two columns are quoted from [24]. Eq. (82) with $R \approx 2.73 \times 10^{12} \text{ly}$, and the ΛCDM model’s $t - z$ relation [88] are used.

| average of redshift $\langle z \rangle$ | $(\Delta \alpha/\alpha)_{\text{expt}}$ | epoch $t$ | theory prediction of (82) |
|---------------------------------------|-------------------------------------|----------|--------------------------|
| 0.65                                  | $(-0.29 \pm 0.31) \times 10^{-5}$  | 6.04Gyr  | $-0.24 \times 10^{-5}$   |
| 1.47                                  | $(-0.58 \pm 0.13) \times 10^{-5}$  | 9.29Gyr  | $-0.58 \times 10^{-5}$   |
| 2.84                                  | $(-0.87 \pm 0.37) \times 10^{-5}$  | 11.39Gyr | $-0.87 \times 10^{-5}$   |

V. SUMMERY AND DISCUSSIONS

In this paper, we have solved the de Sitter special relativistic ($SR_{cR}$) Dirac equation of hydrogen in the earth-QSO framework reference by means of the adiabatic approach. The effects of de Sitter space-time geometry described by Beltrami metric are taken into account. The $SR_{cR}$-Dirac equation of hydrogen turns out to be a time dependent quantum Hamiltonian system. We have provided an explicit calculation to examine whether the adiabatic approach to deal with this time-dependent system is eligible. Since the radius of de Sitter sphere $R$ is cosmologically large, it makes the time-evolution of the system is so slow that the adiabatic approximation legitimately works with high accuracy. Finally, we revealed that all those facts yield important conclusions that the electromagnetic fine-structure constant, the mass of electron and the Planck constant are of variation with time. Eqs. (82), (83) and (70) describe the variations.

As is well known that the solutions of quantum mechanics equations for atom of hydrogen

FIG. 2: The $t - z$ relation in ΛCDM model (eq.(88)).
played important roles for promoting the development of quantum physics in the past century and achieved several very great successes, such as to reveal dynamic bases fundamentally for the Bohr level of hydrogen, the periodic law of elements, the fine structure of atomic spectra and so on. The studies on the fine structure of hydrogen specifically reveal the effects due to combination of the quantum mechanics and the special relativity \( \mathcal{SR}_c \). The predictions were verified by experiments. What were further promoted in this paper is that the cosmology effects are involved via the solution of the \( \mathcal{SR}_c R \)-quantum mechanics equation for hydrogen. The time-variations of the fine structure constant and the mass of electron are of cosmologic effects in atomic spectra. Obviously, the studies presented in this paper is different from other theoretic considerations for this matter from other insights, e.g., Kaluza-Klein theories \cite{28, 29}, superstring \cite{30}, accelerating Universe and dark energy \cite{31}, etc.

There are several methods to study the time-variation of fine structure constant experimentally. Among them, the observations of absorption spectra of distant interstellar clouds \cite{1, 2, 3, 4, 5, 6, 7} are the most direct verification to the predictions of this paper. Our result of the time variation of fine structure constant are consistent with the observations. This fact indicates that the effects of de Sitter special relativity become visible at the cosmic space-time scale (i.e., the distance \( \geq 10^9 \)ly). At that scale de Sitter special relativity is more reliable than Einsteinian special relativity, and the latter is the former’s approximation for the distance \( << R \). Finally we address that further experimental tests are expected.
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APPENDIX A: ELECTRIC COULOMB LAW IN QSO-LIGHT-CONE SPACE

Let’s derive (55). We start with (54), i.e.,

\[-B^{ij}(Q)\partial_i\partial_j\phi_B(x) = \left(\nabla^2 + \frac{c^2t^2}{R^2} \frac{\partial^2}{\partial(x^1)^2}\right)\phi_B(x) = \frac{-4\pi e}{\sqrt{-\det(B_{ij}(Q))}}\delta^{(3)}(x), \tag{A1}\]

where \(-B^{ij}(Q) = \eta^{ij} - \frac{c^2t^2}{R^2}\delta_{i1}\delta_{j1} + \mathcal{O}(R^{-4})\) has been used, and \(B_{ij}\) were given in (41).

Expanding (A1), we have

\[
\left[ \frac{\partial^2}{\partial(x^1/[1 + \frac{c^2t^2}{2R^2}])^2} + \frac{\partial^2}{\partial(x^2)^2} + \frac{\partial^2}{\partial(x^3)^2} \right] \phi_B(x) = -4\pi \left(1 + \frac{c^2t^2}{2R^2}\right) e\delta(x^1)\delta(x^2)\delta(x^3),
\]

and further

\[
\left[ \frac{\partial^2}{\partial(\tilde{x}^1)^2} + \frac{\partial^2}{\partial(x^2)^2} + \frac{\partial^2}{\partial(x^3)^2} \right] \phi_B(x) = -4\pi e\delta(\tilde{x}^1)\delta(x^2)\delta(x^3).
\]

Setting \(\tilde{x}^1 \equiv x^1/(1 + \frac{c^2t^2}{2R^2})\), the above equation becomes

\[
\left[ \frac{\partial^2}{\partial(\tilde{x}^1)^2} + \frac{\partial^2}{\partial(x^2)^2} + \frac{\partial^2}{\partial(x^3)^2} \right] \phi_B(x) = -4\pi e\delta(\tilde{x}^1)\delta(x^2)\delta(x^3). \tag{A2}\]

Then the solution is \(\phi_B(x) = e/r_B\) with

\[
r_B = \sqrt{(\tilde{x}^1)^2 + (x^2)^2 + (x^3)^2} = \left(1 - \frac{c^2t^2}{2R^2}\right)^{1/2} (x^1)^2 + (x^2)^2 + (x^3)^2 \right)^{1/2}
\]

\[
\simeq r \left(1 - \frac{c^2t^2(x^1)^2}{2R^2r^2}\right). \tag{A3}\]
Therefore, we have
\[ \phi_B = \frac{e}{r_B} \simeq \frac{e}{r} \left( 1 + \frac{c^2t^2(x^1)^2}{2R^2r^2} \right), \]  
(A4)
which is eq.(55) in the text.

**APPENDIX B: ADIABATIC APPROXIMATIVE WAVE FUNCTIONS IN $S\mathcal{R}_c R$-DIRAC EQUATION OF HYDROGEN**

Now we derive the wave function of (76) in the text. We start with eq.(73), i.e.
\[
i\hbar \partial_t \psi = H(t) \psi = [H_0(r_B, e) + H'(t)] \psi, \]  
(B1)
where
\[
H(t) = H_0(r_B, e) + H'(t), \]  
(B2)
\[
H_0(r, e) = -i\hbar \mathbf{\alpha} \cdot \nabla_B + \mu c^2 \beta - \frac{e^2}{r_B}, \]  
(B3)
\[
H'(t) = - \left( \frac{c^2t^2}{2R^2} \right) H_0(r_B, \sqrt{2}e). \]  
(B4)

Suppose the modification of $H(t)$ along with the time change is sufficiently slow, the system could be quasi-stationary in any instant $\theta$. Then, in the Shrödinger picture, the quasi-stationary equation of $H(\theta)$
\[
H(\theta) U_n(x, \theta) = E_n(\theta) U_n(x, \theta) \]  
(B5)
can be solved. By (B2) (B3) (B4) and $t \to \theta$, the solutions are as follows (similar to eq.(50) in text)
\[
E_n(\theta) \equiv E_{n_r, K}(\theta) = \mu_\theta c^2 \left[ 1 + \frac{\alpha_\theta^2}{(\sqrt{K^2 - \alpha_\theta^2 + n_r})^2} \right]^{-1/2} \]  
(B6)
\[
n_r = 1, 2, \ldots, \quad |K| = (j + 1/2) = 1, 2, 3, \ldots, \]
where
\[
\mu_\theta = \left( 1 - \frac{c^2\theta^2}{2R^2} \right) \mu \]  
(B7)
\[
\alpha_\theta = \left( 1 - \frac{c^2\theta^2}{3R^2} \right) \frac{e^2}{\hbar c} \]  
(B8)
\[
(\text{and } \hbar_\theta = \left( 1 - \frac{c^2\theta^2}{2R^2} \right) \hbar) \]  
(B9)
The complete set of commutative observable is \( \{ H, K, j^2, j_z \} \), so that we have

\[
U_n(x, \theta) = \psi_{n, K, j, j_z}(r_B, h_\theta, \mu_\theta, \alpha_\theta), \tag{B10}
\]

where \( j = 1 + \frac{h}{2} \Sigma, \ hK = \beta(\Sigma \cdot 1 + h) \). \( [U_n(x, \theta)] \) is complete set and satisfies

\[
\int d^3 x U_n(x, \theta) U_m^*(x, \theta) = \delta_{mn}, \quad n = \{ n_r, K, j, j_z \}. \tag{B11}
\]

Thus, the solution of time-dependent Shrödinger equation (or Dirac equation) \( \text{(B1)} \) can expanded as follows

\[
\psi(x, t) = \sum_n C_n(t) U_n(x, t) \exp \left[ -i \int_0^t \omega_n(\theta) d\theta \right], \quad \omega_n(\theta) = \frac{E_n(\theta)}{\hbar}. \tag{B12}
\]

Substituting \( \text{(B12)} \) into \( \text{(B1)} \), we have

\[
i\hbar \sum_n (\dot{C}_n U_n + C_n \dot{U}_n) \exp \left[ -i \int_0^t \omega_n(\theta) d\theta \right] = 0. \tag{B13}
\]

By multiplying \( U_m^* \exp \left[ i \int_0^t \omega_m(\theta) d\theta \right] \) to both sides of eq. \( \text{(B13)} \), and doing integral to \( x \) by using \( \text{(B11)} \), we have

\[
\dot{C}_m + C_m \int d^3 x U_m^* \dot{U}_m + \sum_n' C_n \int d^3 x U_m^* \dot{U}_n \exp \left[ -i \int_0^t (\omega_n - \omega_m) d\theta \right] = 0, \tag{B14}
\]

where \( \sum_n' \) means that \( n \neq m \) in the summation over \( n \). Noting \( \text{(B11)} \), we have

\[
\int \dot{U}_m^* U_m d^3 x + \int U_m^* \dot{U}_m d^3 x = 0, \tag{B15}
\]

and hence

\[
\int U_m^* \dot{U}_m d^3 x = i\beta \tag{B16}
\]

is purely imaginary number. Denoting

\[
\alpha_{mn} = \int U_m^* \dot{U}_n d^3 x, \quad \text{and} \quad \omega_{nm} = \omega_n - \omega_m, \tag{B17}
\]

then eq. \( \text{(B14)} \) becomes

\[
\dot{C}_m + i\beta C_m + \sum_n' C_n \alpha_{mn} \exp \left[ -i \int_0^t \omega_{nm} d\theta \right] = 0. \quad m = 1, 2, 3, \ldots \tag{B18}
\]
To further simplify it, we set
\[ V_n(x, t) = U_n(x, t) \exp \left[ -i \int_0^t \beta_n(\theta) d\theta \right], \quad (B19) \]
then
\[ \psi(x, t) = \sum_n C'_n(t) V_n(x, t) \exp \left[ -i \int_0^t \omega_n(\theta) d\theta \right], \quad (B20) \]
where \( C'_n(t) = C_n(t) \exp \left[ i \int_0^t \beta_n(\theta) d\theta \right], \) and
\[ \dot{C}'_m(t) = [\dot{C}_m + i \beta_m C_m(t)] \exp \left( i \int_0^t \beta_n(\theta) d\theta \right) \quad (B21) \]
Substituting (B21) into (B18), we finally get
\[ \dot{C}'_m + \sum_n' C'_n \alpha_{mn} \exp \left[ -i \int_0^t \omega'_{nm} d\theta \right] = 0. \quad m = 1, 2, 3, \ldots \quad (B22) \]
where
\[ \omega'_{mn} = \omega'_n - \omega'_m, \quad \omega'_n = \frac{1}{\hbar} E_n + \beta_n. \quad (B23) \]
Now let’s solve (B22). Firstly, we derive \( \alpha_{mn}. \) By (B5), we have
\[ \frac{\partial H}{\partial t} U_n + H \dot{U}_n = \dot{E}_n U_n + E_n \dot{U}_n. \quad (B24) \]
By multiplying \( U^*_m \) and doing integral over \( x, \) we have
\[ \int U^*_m \dot{H} U_n d^3 x + \int U^*_m H \dot{U}_n d^3 x = E_n \int U^*_m \dot{U}_n d^3 x \]
i.e., \[ \dot{H}_{mn} + E_m \alpha_{mn} = E_n \alpha_{mn}. \quad (B25) \]
so that
\[ \alpha_{mn} = \int U^*_m \dot{U}_n d^3 x = \frac{1}{E_n - E_m} \dot{H}_{mn}, \quad m \neq n. \quad (B26) \]
Therefore eq. (B22) becomes
\[ \dot{C}'_m + \sum_n' C'_n \frac{\dot{H}_{mn} \exp \left( -i \int_0^t \omega'_{nm} d\theta \right)}{\hbar \omega_{nm}} = 0. \quad m = 1, 2, 3, \ldots \quad (B27) \]
Suppose in the initial time the system is in \( s \)-state, i.e., \( C_n(0) = C'_n(0) = \delta_{ns}. \) To adiabatic process, \( \dot{H}(t) \to 0, \) then the 0-order approximative solution of eq. (B27) is
\[ [C'_m(t)]_0 = \delta_{ms}. \quad (B28) \]
Substituting (B28) into (B27), we get the first order correction to the approximation

\[ \dot{C}_m' = \frac{-\dot{H}_{ms}}{\hbar \omega_{ms}} \exp \left( -i \int_0^t \omega_m' dt \right) = 0, \quad m \neq s. \] (B29)

Since the dependent on time \( t \) of \( U_n(t) \) is weak for adiabatic process, eq. (B16) indicates \( \beta_n \) is small, and by (B23), we have \( \omega_m' \approx \omega_{ms} \). Then, from (B29), the first order correction to the solution is

\[ [C_m']_1 = \frac{\dot{H}_{ms}}{i \hbar \omega_{ms}} (e^{i \omega_{ms} t} - 1), \quad m \neq s. \] (B30)

Substituting (B29) (B30) into (B20) and neglecting \( \beta_n \), we get the wave function as follows

\[ \psi(x, t) \simeq U_s(x, t) e^{-i \frac{E_s t}{\hbar}} + \sum_{m \neq s} \frac{\dot{H}_{ms}}{i \hbar \omega_{ms}} (e^{i \omega_{ms} t} - 1) U_m(x, t) e^{-i \int_0^t \frac{E_m(\theta)}{\hbar} d\theta}, \] (B31)

By using eqs. (B10), (B8), (B7), (B9), we finally obtain the desired results

\[ \psi(t) \simeq \psi_s(r_B, \hbar t, \mu_t, \alpha_t) e^{-i \frac{E_s t}{\hbar}} + \sum_{m \neq s} \frac{\dot{H}'(t)_{ms}}{i \hbar \omega_{ms}^2} (e^{i \omega_{ms} t} - 1) \psi_m(r_B, \hbar t, \mu_t, \alpha_t) e^{-i \int_0^t \frac{E_m(\theta)}{\hbar} d\theta}, \] (B32)

where

\[ \alpha_t = \left( 1 - \frac{c^2 t^2}{2 R^2} \right) \alpha, \quad \text{with} \quad \alpha = \frac{e^2}{\hbar c}, \] (B33)

\[ \mu_t = \left( 1 - \frac{c^2 t^2}{2 R^2} \right) \mu, \] (B34)

\[ \hbar t = \left( 1 - \frac{c^2 t^2}{2 R^2} \right) \hbar. \] (B35)

They are just the equations (76), (77), (71) and (70) in the text.

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