Electrodynamics in geometric algebra

Sylvain D. Brechet

Institute of Physics, Station 3, Ecole Polytechnique Fédérale de Lausanne - EPFL, CH-1015
Lausanne, Switzerland

Abstract

We consider the electrodynamics of electric charges and currents in vacuum and then generalise our results to the description of a dielectric and magnetic material medium: first in spatial algebra (SA) and then in space-time algebra (STA). Introducing a polarisation multivector \( \tilde{P} = \tilde{p} - \frac{1}{c} \tilde{M} \) and an auxiliary electromagnetic field multivector \( G = \varepsilon_0 F + \tilde{P} \), we express the Maxwell equation in the material medium in SA. Introducing a bound current vector \( \tilde{J} = J - c \nabla \cdot \tilde{P} \) in space-time, the Maxwell equation is then expressed in STA. The wave equation in the material medium is obtained by taking the gradient of the Maxwell equation. For a uniform electromagnetic medium consisting of induced electric and magnetic dipoles, the stress-energy momentum vector is written as \( \dot{T} \left( \nabla \right) = \frac{1}{c} J \cdot F = f \) where \( f \) is the electromagnetic force density vector in space-time. Finally, the Maxwell equation in the material medium can be written in STA as a wave equation for the potential vector \( A \).

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Email address: sylvain.brechet@epfl.ch (Sylvain D. Brechet)
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1. Introduction

In his seminal paper of 1865 entitled “A dynamical theory of the electromagnetic field” [1], James Clerk Maxwell obtained a fully self consistent field description of electromagnetic phenomena. The electromagnetic fields he used were introduced by Michael Faraday in order to accurately present the results of his experiments. In his earlier attempts to establish a theory of electromagnetic phenomena, Maxwell first used mechanical analogies to explain these phenomena in mechanical terms [2]. With great insight, he understood that since the mechanical scaffolding of his theory was neither relevant nor needed, he could simply get rid of it. By doing so, Maxwell revealed a beautiful field theory that served as a template for numerous other field theories. It was a paramount paradigm shift in the history of physics that paved the way for the discovery of special relativity presented by Albert Einstein as “the electrodynamics of moving bodies” [3]. The prediction of electromagnetic waves played historically a key role for the foundations of quantum mechanics that resulted from the synthesis of the wave mechanics of Erwin Schrödinger [4] and the matrix mechanics of Werner Heisenberg, Max Born and Pascual Jordan [5, 6]. The importance of Maxwell’s work was clearly stated by Einstein on the centenary of Maxwell’s birth: “We may say that, before Maxwell, physical reality, in so far as it was to represent the process of nature, was thought of as consisting in material particles, whose variations consist only in movements governed by partial differential equations. Since Maxwell’s time, physical reality has been thought of as represented by continuous fields, governed by partial differential equations, and not capable of any mechanical interpretation. This change in the conception of
Reality is the most profound and the most fruitful that physics has experienced since the time of Newton.” [7] Maxwell understood the importance of his theory of electromagnetism and shared his enthusiasm with his cousin Charles Cay: “I have also a paper afloat, containing an electromagnetic theory of light, which, till I am convinced to the contrary, I hold to be great guns”. [8]

Maxwell’s equations are usually presented in textbooks as a set of four vectorial equations [9, 10]. This was not the direct result of Maxwell’s work. It was in fact Oliver Heaviside [11] who gathered Maxwell’s equations into a set of four vectorial equations in 1884 using the vector calculus he coformulated with Josiah Willard Gibbs [12]. In his initial paper published in 1865, Maxwell wrote 20 equations in components. Subsequently, in his “Treatise on Electricity and Magnetism” [13], Maxwell recast his set of 20 equations in terms of quaternions discovered by William Rowan Hamilton in 1843. Since, the vector space of Heaviside and Gibbs is ideally suited to describe translations, it was adopted historically as the main geometric framework of physics. However, the quaternion algebra \( \mathbb{H} \) is far more suited for the description of rotations than the vector space \( \mathbb{R}^3 \) where pseudovectors need to be introduced in order to treat rotations. This naturally prompts the question: “Do we really need to choose or is it possible to take advantage of both frameworks at the same time?” In fact, the good news is that both frameworks belong to a broader mathematical framework called either geometric algebra (GA) \( \mathbb{G}^3 \) or Clifford algebra \( \text{Cl}_{\ell_3}(\mathbb{R}) \) that was discovered by William Kingdom Clifford in 1878 in an article entitled “Applications of Grassmann’s extensive algebra” [14].

In geometric algebra (GA) \( \mathbb{G}^3 \), there are four types of natural geometric entities. First, there are geometric entities of dimension 0 called scalars, which are oriented points defined by their magnitude and orientation (e.g. positive or negative). Second, there are geometric entities of dimension 1 called vectors, which are oriented lines defined by their magnitude and orientation. Third, there are geometric entities of dimension 2 called bivectors, which are oriented surfaces defined by their magnitude and orientation. Fourth, there are geometric entities of dimension 3 called trivectors or pseudoscalars, which are oriented
volume defined by their magnitude and orientation (e.g. positive or negative).
The geometric algebra (GA) $\mathbb{G}^3$ is a vector space consisting of multivectors, which are linear combinations of scalars, vectors, bivectors and pseudoscalars. The quaternion algebra $\mathbb{H}$, which consists of linear combination of scalars and bivectors is the even subalgebra of the geometric algebra (GA) $\mathbb{G}^3$. The vector space $\mathbb{R}^3$, which consists of linear combination of vectors, is a subspace of the geometric algebra (GA) $\mathbb{G}^3$. The geometric algebra (GA) $\mathbb{G}^3$ is a vector space endowed with two composition laws: the inner and outer product of two multivectors. The algebraic sum of the inner and outer product of two vectors is called the geometric product, as explained in Appendix A.

The geometric algebra (GA) $\mathbb{G}^3$ is also called the spatial algebra (SA). Since special relativity was based on electromagnetism, it is relevant to generalise the spatial algebra (SA) to include an additional temporal dimension. The generalisation of the spatial algebra (SA) $\mathbb{G}^3$ to space-time is the geometric algebra (GA) $\mathbb{G}^{1,3}$ or the Clifford algebra $\mathbb{C}\ell_{1,3} (\mathbb{R})$ called the space-time algebra (STA) developed by Hestenes [15]. The spatial algebra (SA) $\mathbb{G}^3$ is isomorphic, or equivalent in structure, to the even subalgebra of the space-time algebra (STA) $\mathbb{G}^{1,3}$. Formally, bivectors consisting of the outer product of a space vector and a time vector in $\mathbb{G}^{1,3}$ are isomorphic to spatial vectors in $\mathbb{G}^3$, the outer product of these bivectors in $\mathbb{G}^{1,3}$ are isomorphic to spatial bivectors in $\mathbb{G}^3$ and the pseudoscalar $I$ in $\mathbb{G}^3$ is isomorphic to the pseudoscalar in $\mathbb{G}^{1,3}$.

Geometric algebra in its various forms, namely spatial algebra (SA) $\mathbb{G}^3$ or space-time algebra (STA) $\mathbb{G}^{1,3}$, appears to be the most appropriate language to describe physical phenomena since its structure is based on the geometry underlying the natural world. In his memoir entitled “Reapings and Sowings”, Alexander Grothendieck, one of the greatest mathematician of the 20th century states [16]: “If there is one thing in mathematics that fascinates me more than anything else (and doubtless always has), it is neither ‘number’ nor ‘size,’ but always form. Among the thousand-and-one faces whereby form chooses to reveal itself to us, the one that fascinates me more than any other and continues to fascinate me, is the structure hidden in mathematical things.”
This article is structured as follows. In the first part, we begin by stating Maxwell’s equations in vector space (VS) in Sec. 2. Then, we recast them in spatial algebra (SA) in Sec. 3. In order to do so, we introduce a magnetic induction field bivector $B$ and an auxiliary magnetic field bivector $H$ by duality with the corresponding vectors $b$ and $h$. The duality operation in spatial algebra (SA) is defined in Appendix B.

In the second part, presented in Sec. 4-7, we examine the foundations of electromagnetism in spatial algebra (SA) in the absence of a material medium which we refer to as “vacuum”. The term “vacuum” does not mean here that there are no electric charges or no electric current but simply that the medium is vacuum. By introducing an electromagnetic multivector $F = e + cB$ in spatial algebra (SA) that is algebraically isomorphic to the complex Riemann-Silberstein vector $f = e + icb$ in complex vector space $\mathbb{C}^3$, we then show in Sec 4. that the four Maxwell equations are written as a single equation in spatial algebra (SA). In Sec. 5, we show that electromagnetic waves are described quite naturally in spatial algebra (SA). In Sec 6, we obtain an expression of the energy density and momentum density in terms of the electromagnetic field multivector $F = e + cB$, its reverse $F^\dagger = e - cB$, the auxiliary electromagnetic field multivector $G = d + \frac{1}{c}H$ and its reverse $G^\dagger = d - \frac{1}{c}H$, which allows us to establish Poynting’s theorem in spatial algebra (SA) in Sec (7). In Sec. 8, the electromagnetic field multivector $F$ is expressed in terms of the electric scalar potential $\phi$ and the magnetic vector potential $a$.

In the third part, shown in Sec. 9-12, we examine the foundations of electromagnetism in spatial algebra (SA) for a material medium with electric and magnetic dipoles, which we refer to as “matter”, where we perform a similar analysis as in “vacuum”. In the fourth part, presented in Sec. 13-19, we recast the results obtained thus far within the space-time algebra (STA). In Sec. 13 and 14, Maxwell’s equation in vacuum and matter is written very elegantly in space-time algebra (STA). In Sec. 17, the electromagnetic field multivectors $F$ and $G$ and the electric polarisation multivector $\hat{P}$ are expressed as bivectors in space-time algebra.
In Sec. 18 and 19, the stress energy momentum vector is established in vacuum and matter in space-time algebra (STA). In Sec 19, we show that the electromagnetic field bivector in space-time $F$ is the curl of the potential vector $A$ in space-time.

Appendix A sets the foundations of spatial algebra (SA) and provides identities for the products of vectors, bivectors and the pseudoscalar. The duality between geometric entities in spatial algebra is presented in Appendix B. The duality of the gradient, the divergence or the curl is estabish in Appendix C. Dual identities involving geometric entities in spatial algebra is derived in Appendix D. Dual identities involving the gradient, the divergence or the curl are shown in Appendix E. Finally, Appendix F sets the foundations of space-time algebra (STA) and provides identities for the products of vectors, bivectors and multivectors.

The notation convention adopted in this paper is the following. To distinguish typographically different types of entities in the spatial algebra (SA), we will use in general lower case letters for scalars like the charge density $q$, lower case bold letters for vectors like the electric field $e$, uppercase bold letters for bivectors like the magnetic induction field $B$ and uppercase letters for multivectors like the electromagnetic field $F$. In the space-time algebra (STA), we will in general use uppercase letters for vectors like the electric current density $J$ and for bivectors like the auxiliary electromagnetic field $G$.

2. Maxwell equations in vector space (VS)

In vector space (VS), electromagnetism is described by a set of four vector fields: the electric field vector $e$, the magnetic field pseudovector $b$, the electric displacement field vector $d$, the auxiliary magnetic field pseudovector $h$. The Maxwell equations relate these fields to the electric charge distribution described by the scalar density field $q$ and the electric current distribution described by the vector density field $j$, [9, 10]

$$\nabla \cdot d = q$$  \hspace{1cm} (1)
\[ \nabla \times h = \partial_t d + j \]  \hspace{1cm} (2) \\
\[ \nabla \times e = - \partial_t b \]  \hspace{1cm} (3) \\
\[ \nabla \cdot b = 0 \]  \hspace{1cm} (4)

The electric Gauss equation (1) is a scalar equation representing the local electric flux and the magnetic Gauss equation (4) is a pseudoscalar equation representing the local magnetic flux. The Maxwell-Ampère equation (2) is a vectorial equation representing the local electric circulation and the Faraday equation (3) is pseudovectorial equation representing the local magnetic circulation.

The divergence of the Maxwell-Ampère relation (2) is written as,

\[ \nabla \cdot (\nabla \times h) = \partial_t (\nabla \cdot d) + \nabla \cdot j = 0 \]  \hspace{1cm} (5)

Substituting the electric Gauss equation (1) into relation (5), we obtain the electric continuity equation,

\[ \partial_t q + \nabla \cdot j = 0 \]  \hspace{1cm} (6)

The Lorentz force density vector is given by, [9, 10]

\[ f = q (e + v \times b) \]  \hspace{1cm} (7)

3. Maxwell equations in spatial algebra (SA)

In spatial algebra (SA), pseudovectors are replaced by bivectors [17]. Pseudovectors are the dual of bivectors. Bivectors represent oriented plane elements that are orthogonal to the dual pseudovectors. The area or “magnitude” of the bivector corresponds to the length or “norm” of the dual pseudovector. The orientation of this duality is given by the right hand rule. If the right hand rotates along the oriented plane element defined by the bivector then the dual pseudovector is oriented along the direction of the thumb of the right hand. The converse duality is given by the left hand rule. If the thumb of the left hand is oriented along the pseudovector then the left hand rotates along the oriented plane element defined the dual bivector. Note that this orientation is
The opposite of the one that would be obtained with the right hand rule. Thus, the dual of the dual of a bivector yields a bivector with the opposite orientation, which is the opposite of the initial bivector. Similarly, the dual of the dual of a vector yields a vector with the opposite orientation, which is the opposite of the initial vector. [18] In view of this duality, we will now recast the Maxwell equations (1)-(4) in terms of vectors and bivectors.

The auxiliary magnetic field is generated by an electric current circulating in a loop, which is an oriented plane element. The auxiliary magnetic field pseudovector in vector space $\mathbf{h}$ is orthogonal to the plane of the loop. It is the dual (B.26) of the auxiliary magnetic field bivector $\mathbf{H}$ oriented along the circulation of the current density $\mathbf{j}$ [18],

$$\mathbf{H}^* = \mathbf{h} \quad \text{where} \quad |\mathbf{H}| = |\mathbf{h}|$$

where the duality is denoted with a $\ast$. The dual of this duality (B.19) is,

$$\mathbf{h}^* = (\mathbf{H}^*)^* = -\mathbf{H}$$

According to the vectorial duality (C.4), the curl of the auxiliary magnetic field pseudovector $\mathbf{h}$ is the opposite of the divergence of the auxiliary magnetic field bivector $\mathbf{H}$,

$$\nabla \times \mathbf{h} = -\nabla \cdot \mathbf{H}$$

Substituting relation (10) into the Maxwell-Ampère equation (2), it becomes,

$$\nabla \cdot \mathbf{H} = -\partial_t d - \mathbf{j}$$

In vacuum, the magnetic induction field is generated by an electric current circulating in a loop, which is an oriented plane element. The magnetic induction field pseudovector in vector space $\mathbf{b}$ is orthogonal to the plane of the loop. It is the dual (B.26) of the magnetic induction field bivector $\mathbf{B}$ oriented along the circulation of the current density $\mathbf{j}$,

$$\mathbf{B}^* = \mathbf{b} \quad \text{where} \quad |\mathbf{B}| = |\mathbf{b}|$$
The dual of this duality (B.19) is,

\[ b^* = (B^*)^* = - B \]  \hspace{1cm} (13)

According to the vectorial duality (C.4), the curl pseudovector of the electric field vector \( e \) is the dual of the curl bivector of the electric field vector \( e \),

\[ \nabla \times e = (\nabla \wedge e)^* \]  \hspace{1cm} (14)

In view of relations (12) and (14), the Faraday equation (3) is recast as,

\[ (\nabla \wedge e)^* = - \partial_t B^* \]  \hspace{1cm} (15)

The opposite of the dual of the Faraday equation (15) is given by,

\[ \nabla \wedge e = - \partial_t B \]  \hspace{1cm} (16)

Using the vectorial duality (C.5) and (C.6), the dual of the divergence of the magnetic induction field pseudovector \( b \) is the opposite of the curl of the magnetic induction field bivector \( B \),

\[ (\nabla \cdot b)^* = \nabla \wedge b^* = - \nabla \wedge B \]  \hspace{1cm} (17)

In view of relation (17), the dual the magnetic Gauss equation (4) is given by,

\[ \nabla \wedge B = 0 \]  \hspace{1cm} (18)

Thus, in spatial algebra (SA), the Maxwell equations are written as,

\[ \nabla \cdot d = q \]  \hspace{1cm} (19)

\[ \nabla \cdot H = - \partial_t d - j \]  \hspace{1cm} (20)

\[ \nabla \wedge e = - \partial_t B \]  \hspace{1cm} (21)

\[ \nabla \wedge B = 0 \]  \hspace{1cm} (22)

The electric Gauss equation (19) is a scalar equation, the Maxwell-Ampère equation (20) is a vectorial equation, the Faraday equation (21) is a bivectorial equation or pseudovectorial equation, and the magnetic Gauss equation (22) is a trivectorial equation, or pseudoscalar equation.
The divergence of the Maxwell-Ampère relation (20) is written as,

$$\nabla \cdot (\nabla \cdot H) = - \partial_t (\nabla \cdot d) - \nabla \cdot j = 0 \quad (23)$$

Substituting the electric Gauss equation (19) into relation (23), we obtain the electric continuity equation,

$$\partial_t q + \nabla \cdot j = 0 \quad (24)$$

According to the vectorial duality (B.37), (B.38) and (13), and to the antisymmetry of the inner product with a bivector (A.19), the cross product of the velocity vector $v$ and the magnetic induction field pseudovector $b$ is the inner product of the magnetic induction field bivector $B$ and the velocity vector $v$,

$$v \times b = (v \wedge b)^* = v \cdot b^* = -v \cdot B = B \cdot v \quad (25)$$

In view of the vectorial duality (25), the Lorentz force density vector (7) is recast as,

$$f = q(e + B \cdot v) \quad (26)$$

4. Maxwell equation in vacuum (SA)

We now show that in vacuum, the Maxwell equations (19)-(22) reduce to a single equation. The term “vacuum” does not mean here that there are no electric charges or no electric current but simply that the medium is vacuum. In vacuum, the vectorial linear electric constitutive relation is given by, [9, 10]

$$d = \varepsilon_0 e \quad (27)$$

and the pseudovectorial linear magnetic constitutive relation is given by; [9, 10]

$$b = \mu_0 h \quad (28)$$

The dual of the constitutive relation (28) is written as,

$$b^* = \mu_0 h^* \quad (29)$$
Using the duality relations (9) and (13), relation (29) yields the bivectorial linear magnetic constitutive relation,

\[ \mathbf{B} = \mu_0 \mathbf{H} \quad (30) \]

where the vacuum electric permittivity \( \varepsilon_0 \) and the vacuum magnetic permeability \( \mu_0 \) are related to the speed of light \( c \) by, \([9, 10]\)

\[ \varepsilon_0 \mu_0 = \frac{1}{c^2} \quad (31) \]

Substituting the linear constitutive relations (27) and (30) into the electric Gauss equation (19) and the Maxwell-Ampère equation (20) and taking into account the definition (31) of the speed of light, we obtain,

\[ \nabla \cdot \mathbf{e} = \frac{q}{\varepsilon_0} \quad (32) \]
\[ \nabla \cdot \mathbf{B} = -\frac{1}{c^2} \partial_t \mathbf{e} - \mu_0 \mathbf{j} \quad (33) \]

In vector space, the Riemann-Silberstein vector \( \mathbf{f} \) is defined as, \([19]\)

\[ \mathbf{f} = \mathbf{e} + i c \mathbf{b} \quad (34) \]

where \( i \) is the unit imaginary number. The Riemann-Silberstein vector \( \mathbf{f} \in \mathbb{C}^3 \) is a complex vector field. In spatial algebra (SA), the unit imaginary number \( i \) is replaced by the unit pseudoscalar \( \mathbf{I} \) (A.6) that has the same essential property, namely \( i^2 = \mathbf{I}^2 = -1 \). Thus, the Riemann-Silberstein vector \( \mathbf{f} \) becomes a multivector called the electromagnetic multivector field,

\[ \mathbf{F} = \mathbf{e} + c \mathbf{I} \mathbf{b} \quad (35) \]

In three dimensions, the pseudoscalar \( \mathbf{I} \) commutes with the vector field \( \mathbf{b} \). Thus, according to the duality (B.23) and (13),

\[ \mathbf{I} \mathbf{b} = \mathbf{b} \mathbf{I} = -\mathbf{b} \mathbf{I}^{-1} = -\mathbf{b}^* = \mathbf{B} \quad (36) \]

This duality operation is defined in terms of the pseudoscalar in Appendix (B). Substituting identity (36) into relation (35), the electromagnetic multivector field (35) becomes,

\[ \mathbf{F} = \mathbf{e} + c \mathbf{B} \quad (37) \]
The very elegant expression (37) of the electromagnetic multivector $F$ was used only by few authors like Arthur [20] and Macdonald [18] whereas most authors like Doran and Lasenby [21] or Hestenes [15] write the electromagnetic field multivector (35) in terms of the magnetic induction field pseudovector $b$. The choice made here seem much more natural in a sense since it reflects the underlying geometry of space. In view of the Faraday equation (21), the Maxwell-Ampère relation (20) and the speed of light (31), the partial time derivative of the electromagnetic multivector field (37) yields,

$$\partial_t F = \partial_t e + c \partial_t B$$

(38)

The gradient of the electromagnetic multivector field (37) is given by,

$$\nabla F = \nabla \cdot F + \nabla \wedge F = \nabla \cdot e + \nabla \wedge e + c \nabla \cdot B + c \nabla \wedge B$$

(39)

In view of the electric Gauss equation (32), the Faraday equation (21), the Maxwell-Ampère relation (33), the magnetic Gauss equation (22), the linear electric constitutive relation (27), the linear magnetic constitutive relation (30) and the speed of light (31), the gradient of the electromagnetic multivector field (39) becomes,

$$\nabla F = \frac{q}{\varepsilon_0} - \partial_t B - \frac{1}{c} \partial_t e - \frac{1}{\varepsilon_0 c} j$$

(40)

Taking into account the partial time derivative (38) and the gradient (40) of the electromagnetic multivector field and the definition (31) of the propagation velocity, we obtain, [21]

$$\left(\frac{1}{c} \partial_t + \nabla \right) \varepsilon_0 F = q - \frac{1}{c} j$$

(41)

which is the Maxwell equation in spatial algebra. In vacuum, the auxiliary electromagnetic multivector field is defined as,

$$G = \varepsilon_0 F$$

(42)

which is the linear electromagnetic constitutive relation. Using the definition (42) of the auxiliary electromagnetic multivector field $G$ in vacuum, the
Maxwell equation (41) is recast as,

\[
\left( \frac{1}{c} \partial_t + \nabla \right) G = q - \frac{1}{c} j
\]  

(43)

Using the constitutive relations (27) and (30) and the speed of light (31) the auxiliary electromagnetic multivector field \( G \) is recast as, [21]

\[
G = d + \frac{1}{c} H
\]  

(44)

The electric Gauss equation (19) and the Maxwell-Ampère equation (20) describing the electrodynamic phenomena driven by electric charges and currents are expressed as the divergence of the auxiliary electromagnetic multivector field \( G \) in vacuum,

\[
\nabla \cdot G = \nabla \cdot d + \frac{1}{c} \nabla \cdot H = q - \frac{1}{c} (j + \partial_t d)
\]  

(45)

Similarly, the Faraday equation (21) and the Magnetic Gauss equation (22) describing the electrodynamic phenomena that do not involve directly electric charges and currents are expressed as the curl of the electromagnetic multivector field \( F \),

\[
\nabla \wedge F = \nabla \wedge e + c \nabla \wedge B = - \partial_t B
\]  

(46)

5. Electromagnetic waves in vacuum (SA)

In the spatial algebra (SA), electromagnetic waves in vacuum are a direct and straightforward consequence of the Maxwell equation (43), as expected, but it is nonetheless quite beautiful. Indeed, multiplying the Maxwell equation (43) by \( c^{-1} \partial_t - \nabla \) and using the relations,

\[
\left( \frac{1}{c} \partial_t - \nabla \right) \left( \frac{1}{c} \partial_t + \nabla \right) G = \left( \frac{1}{c^2} \partial_t^2 - \nabla^2 \right) G
\]  

(47)

and

\[
\left( \frac{1}{c} \partial_t - \nabla \right) \left( q - \frac{1}{c} j \right) = \frac{1}{c} \partial_t q - \nabla q - \frac{1}{c^2} \partial_t j + \frac{1}{c} \nabla \cdot j + \frac{1}{c} \nabla \wedge j
\]  

(48)

together with the electric continuity equation (24), we obtain,

\[
\left( \frac{1}{c^2} \partial_t^2 - \nabla^2 \right) G = - \nabla q - \frac{1}{c^2} \partial_t j + \frac{1}{c} \nabla \wedge j
\]  

(49)
which is the wave equation for the auxiliary electromagnetic multivector field \( G \) in vacuum. Substituting the auxiliary electromagnetic multivector field (44) in vacuum into the wave equation (49) yields,

\[
\left( \frac{1}{c^2} \partial_t^2 - \nabla^2 \right) \left( d + \frac{1}{c} H \right) = -\nabla q - \frac{1}{c^2} \partial_t j + \frac{1}{c} \nabla \wedge j \quad (50)
\]

Identifying the vectorial terms in relation (50), we obtain the wave equation for the electric displacement field \( d \),

\[
\left( \frac{1}{c^2} \partial_t^2 - \nabla^2 \right) d = -\nabla q - \frac{1}{c^2} \partial_t j \quad (51)
\]

Identifying the bivectorial terms in relation (50), we obtain the wave equation for the auxiliary magnetic field \( H \),

\[
\left( \frac{1}{c^2} \partial_t^2 - \nabla^2 \right) H = \nabla \wedge j \quad (52)
\]

Dividing the wave equation (51) for the electric displacement field \( d \) by \( \varepsilon_0 \) yields the wave equation for the electric field \( e \),

\[
\left( \frac{1}{c^2} \partial_t^2 - \nabla^2 \right) e = -\frac{1}{\varepsilon_0} \nabla q - \frac{1}{\varepsilon_0 c^2} \partial_t j \quad (53)
\]

Multiplying the wave equation (52) for the auxiliary magnetic field \( H \) by \( \mu_0 \) yields the wave equation for the magnetic field \( B \),

\[
\left( \frac{1}{c^2} \partial_t^2 - \nabla^2 \right) B = \mu_0 \nabla \wedge j \quad (54)
\]

6. Electromagnetic energy and momentum in vacuum (SA)

The electromagnetic field multivector \( F \) is a linear combination of the electric field vector \( e \) and the magnetic induction field bivector \( B \) according to relation (37). Similarly, the auxiliary electromagnetic field multivector \( G \) is a linear combination of the electric displacement field vector \( d \) and the auxiliary magnetic field bivector \( H \) according to relation (44). Since the electromagnetic energy density \( e \) and the electromagnetic momentum density \( p \) in vacuum are determined by these fields, they can be recast in terms of the multivectors \( F \)
and $G$. The electromagnetic energy density in vacuum is written in vector space (VS) as, [9, 10]

\[ e = \frac{1}{2} \mathbf{d} \cdot \mathbf{e} + \frac{1}{2} \mathbf{h} \cdot \mathbf{b} \]  

(55)

Using the linear electric constitutive relation (27) and the linear magnetic constitutive relation (28), relation (55) is recast as,

\[ e = \frac{1}{2} \varepsilon_0 \mathbf{e}^2 + \frac{1}{2 \mu_0} \mathbf{b}^2 \]  

(56)

According to relations (12), (B.6), (B.7), the square of the magnetic induction field bivector $\mathbf{B}$ is the opposite of the square of the magnetic induction field pseudovector $\mathbf{b}$,

\[ \mathbf{b}^2 = |\mathbf{b}|^2 = |\mathbf{B}|^2 = \mathbf{B} \mathbf{B}^\dagger = -\mathbf{B}^2 \]  

(57)

where $\mathbf{B}^\dagger = -\mathbf{B}$ is the reverse (B.3) of the magnetic field bivector $\mathbf{B}$. In view of relation (57), the electromagnetic energy density (58) is recast in spatial algebra (SA) as,

\[ e = \frac{1}{2} \varepsilon_0 \mathbf{e}^2 - \frac{1}{2 \mu_0} \mathbf{B}^2 \]  

(58)

Taking into account the linear electric constitutive relation (27) and the linear magnetic constitutive relation (28), the electromagnetic energy density (58) is recast as,

\[ e = \frac{1}{2} \mathbf{d} \cdot \mathbf{e} - \frac{1}{2} \mathbf{H} \cdot \mathbf{B} \]  

(59)

The electromagnetic momentum density in vacuum is written in vector space (VS) as, [9, 10]

\[ \mathbf{p} = \frac{1}{c^2} \mathbf{e} \times \mathbf{h} \]  

(60)

where $\mathbf{e} \times \mathbf{h}$ is the Poynting vector. By duality using the identities (B.32), (B.35), (9) and (A.19), the electromagnetic momentum density (60) in recast in spatial algebra (SA) as,

\[ \mathbf{p} = \frac{1}{c^2} (\mathbf{e} \wedge \mathbf{h})^* = \frac{1}{c^2} \mathbf{e} \cdot \mathbf{h}^* = -\frac{1}{c^2} \mathbf{e} \cdot \mathbf{H} = \frac{1}{c^2} \mathbf{H} \cdot \mathbf{e} \]  

(61)

Thus, in spatial algebra, in view of the magnetic constitutive relation (30), the electromagnetic momentum density in vacuum (61) is written as,

\[ \mathbf{p} = \frac{1}{c^2} \mathbf{H} \cdot \mathbf{e} = \frac{1}{c^2 \mu_0} \mathbf{H} \cdot \mathbf{e} \]  

(62)
where $\mathbf{H} \cdot \mathbf{e}$ is the Poynting vector. According to the definition (37), the reverse of the electromagnetic multivector field $\mathbf{F}$ is written as,

$$F^\dagger = e^\dagger + c B^\dagger = e - c B$$

(63)

The reversion operation, defined in Appendix (B), reverses the order of the basis vectors in the different geometric entities of a multivector in spatial algebra. In view of the multivectors (37) and (63), the electric vector field is recast as,

$$e = \frac{1}{2} (F + F^\dagger)$$

(64)

and the magnetic induction bivector field is recast as,

$$B = \frac{1}{2c} (F - F^\dagger)$$

(65)

According to the definition (44), the reverse of the auxiliary electromagnetic multivector field $\mathbf{G}$ is written as,

$$G^\dagger = d^\dagger + \frac{1}{c} H^\dagger = d - \frac{1}{c} H$$

(66)

The reverse of the linear electromagnetic constitutive relation (42) is written as,

$$G^\dagger = \varepsilon_0 F^\dagger$$

(67)

In view of the multivectors (37) and (63), the electric displacement vector field is recast as,

$$d = \frac{1}{2} (G + G^\dagger)$$

(68)

and the auxiliary magnetic bivector field is recast as,

$$H = \frac{c}{2} (G - G^\dagger)$$

(69)

Using the symmetry of the inner product of two vectors (A.8) and two bivectors (A.27), the electromagnetic energy density (59) is recast as,

$$e = \frac{1}{2} (\mathbf{d} \cdot \mathbf{e}) + \frac{1}{2} (\mathbf{H} \cdot \mathbf{B})$$

(70)
Using relations (64), (64), (68) and (69), the electromagnetic energy density (70) is expressed in terms of the geometric product of the electromagnetic multivectors as,

\[ e = \frac{1}{16} \left( (G + G^\dagger) (F + F^\dagger) + (F + F^\dagger) (G + G^\dagger) \right) - \frac{1}{16} \left( (G - G^\dagger) (F - F^\dagger) + (F - F^\dagger) (G - G^\dagger) \right) \]  

which reduces to,

\[ e = \frac{1}{8} (G F^\dagger + F G^\dagger + G^\dagger F + F^\dagger G) \]  

Using the antisymmetry of the inner product of a vector and a bivector (A.19) the electromagnetic momentum density (60) is recast as,

\[ p = \frac{1}{2c^2} (H e - e H) \]  

Using relations (64), (64), (68) and (69), the electromagnetic momentum density (73) is expressed in terms of the geometric product of the electromagnetic multivectors as,

\[ p = \frac{1}{8c} \left( (G - G^\dagger) (F + F^\dagger) - (F + F^\dagger) (G - G^\dagger) \right) \]  

which is expressed as,

\[ p = \frac{1}{8c} (G F^\dagger + F G^\dagger - G^\dagger F - F^\dagger G) + \frac{1}{8c} (G F + F^\dagger G - FG - G^\dagger F^\dagger) \]  

In vacuum, in view of the linear electromagnetic constitutive relations (42), (67), the electromagnetic energy density (72) reduces,

\[ e = \frac{1}{8} (G F^\dagger + F G^\dagger + G^\dagger F + F^\dagger G) = \frac{1}{4} \varepsilon_0 (F F^\dagger + F^\dagger F) \]  

and the electromagnetic momentum density (62) reduces to, [20]

\[ p = \frac{1}{8c} (G F^\dagger + F G^\dagger - G^\dagger F - F^\dagger G) = \frac{1}{4} \varepsilon_0 c (F F^\dagger - F^\dagger F) \]  

Thus, according to relations (76) and (77) in vacuum, [20]

\[ e + p c = \frac{1}{4} (G F^\dagger + F G^\dagger) = \frac{1}{2} \varepsilon_0 F F^\dagger \]  

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7. Poynting theorem in vacuum (SA)

The continuity equation for the electromagnetic energy, called the Poynting theorem, is written in vacuum in terms of the time derivative of the energy density and the energy flux which is a multiple of the electromagnetic momentum density. The time derivative of the electromagnetic energy density (58) in vacuum is given by,

$$\partial_t e = \varepsilon_0 e \cdot \partial_t e - \frac{1}{\mu_0} B \cdot \partial_t B$$  \hspace{1cm} (79)

Using the electric constitutive equation (27) and the magnetic constitutive equation (30), relation (79) is recast as,

$$\partial_t e = e \cdot \partial_t d - H \cdot \partial_t B$$  \hspace{1cm} (80)

Using the Mawell-Ampère equation (20) and the Faraday equation (21), the time derivative of the electromagnetic energy density (80) is recast as,

$$\partial_t e = - e \cdot (\nabla \cdot H + j) + H \cdot (\nabla \wedge e)$$  \hspace{1cm} (81)

Thus,

$$\partial_t e + (\nabla \cdot H) \cdot e - H \cdot (\nabla \wedge e) = - j \cdot e$$  \hspace{1cm} (82)

Using the algebraic identity (E.23),

$$\nabla \cdot (H \cdot e) = (\nabla \cdot H) \cdot e - H \cdot (\nabla \wedge e)$$  \hspace{1cm} (83)

relation yields (82) the Poynting’s theorem,

$$\partial_t e + \nabla \cdot (H \cdot e) = - j \cdot e$$  \hspace{1cm} (84)

where $H \cdot e$ is the Poynting vector, which is the electromagnetic current density.

In view of the electromagnetic momentum density (62), Poynting’s theorem (84) in vacuum is recast as, [9, 10]

$$\frac{1}{c} \partial_t e + \nabla \cdot (p c) = - \frac{1}{c} j \cdot e$$  \hspace{1cm} (85)
8. Electric and magnetic potentials (SA)

In vector space (VS), the electric vector field \( e \) and the magnetic induction pseudovector field \( b \) can be expressed entirely in terms of the electric scalar potential \( \phi \) and magnetic vector potential \( a \). This implies that the electromagnetic multivector field \( F \) can be entirely expressed in terms of these potentials. The electric vector field \( e \) is expressed in terms of the electric scalar potential \( \phi \) and the magnetic vector potential \( a \) as, [9, 10]

\[
e = - \nabla \phi - \partial_t a \tag{86}
\]

and the magnetic induction vector field \( b \) is expressed in terms of the vector potential \( a \) as, [9, 10]

\[
b = \nabla \times a \tag{87}
\]

According to relation (13) and identity (C.1), the magnetic induction bivector field \( B \) is the dual of the magnetic induction vector field \( b \),

\[
B = - b^* = - (\nabla \times a)^* = \nabla \wedge a \tag{88}
\]

In view of relations (37) and (88), the electromagnetic multivector field \( F \) is written in terms of the potentials as,

\[
F = e + cB = - \nabla \phi - \partial_t a + c \nabla \wedge a \tag{89}
\]

The Lorentz gauge in vector space (VS) is defined as, [9, 10]

\[
\frac{1}{c^2} \partial_t \phi + \nabla \cdot a = 0 \tag{90}
\]

9. Maxwell equation in matter (SA)

We now show that in matter, the Maxwell equations (19)-(22) reduce to a single equation. The term “matter” refers here to a material medium consisting of continuum of electric and magnetic dipoles. In a dielectric and magnetic medium, the vectorial linear electric constitutive relation is given by, [9, 10]

\[
d = \varepsilon_0 e + \bar{\rho} \tag{91}
\]
where $\tilde{p}$ is the matter electric polarisation vector field, and the pseudovectorial linear magnetic constitutive relation is given by, [9, 10]

$$b = \mu_0 (h + \tilde{m})$$

(92)

where $\tilde{m}$ is the matter magnetisation pseudovector field. The dual of the constitutive relation (92) is written as,

$$b^* = \mu_0 (h^* + \tilde{m}^*)$$

(93)

The magnetisation field pseudovector $\tilde{m}$ in vector space is the dual of the auxiliary magnetisation field bivector $\tilde{M}$ in spatial algebra (B.23),

$$\tilde{M}^* = \tilde{m} \quad \text{where} \quad |\tilde{M}| = |\tilde{m}|$$

(94)

The dual of this duality is,

$$\tilde{m}^* = (\tilde{M}^*)^* = -\tilde{M}$$

(95)

Using the dualities (13),(9) and (95), relation (93) becomes the bivectorial linear magnetic constitutive relation,

$$B = \mu_0 \left( H + \tilde{M} \right)$$

(96)

Substituting the constitutive relations (91) and (96) into the auxiliary electromagnetic multivector field (44) in vacuum and taking into account the speed of light (31), we obtain,

$$G = (\varepsilon_0 e + \tilde{p}) + \frac{1}{c} \left( \frac{B}{\mu_0} - \tilde{M} \right) = \varepsilon_0 (e + cB) + \left( \tilde{p} - \frac{1}{c} \tilde{M} \right)$$

(97)

Defining the electromagnetic polarisation multivector as,

$$\tilde{P} = \tilde{p} - \frac{1}{c} \tilde{M}$$

(98)

and substituting the multivectors (42) and (98) into relation (97) we obtain the multivectorial linear electromagnetic constitutive relation,

$$G = \varepsilon_0 F + \tilde{P}$$

(99)
Substituting the multivectorial linear electromagnetic constitutive relation (99) into the Maxwell equation (43), we obtain the Maxwell equation for a dielectric and magnetic medium,

$$\left( \frac{1}{c} \partial_t + \nabla \right) \left( \varepsilon_0 F + \hat{P} \right) = q - \frac{1}{c} \hat{j} \tag{100}$$

which is in agreement with the result derived by Arthur [20]. The dynamics of the electromagnetic polarisation of the medium is described by the equation,

$$\left( \frac{1}{c} \partial_t + \nabla \right) \hat{P} = \frac{1}{c} \partial_t \hat{P} + \nabla \cdot \hat{P} + \nabla \wedge \hat{P} \tag{101}$$

Using the definition (98) of the electromagnetic polarisation multivector, we obtain the following relation,

$$\frac{1}{c} \partial_t \hat{P} + \nabla \cdot \hat{P} = \frac{1}{c} \partial_t \hat{p} + \nabla \cdot \hat{p} - \frac{1}{c^2} \partial_t \hat{M} - \frac{1}{c} \nabla \cdot \hat{M} \tag{102}$$

Using relations (98), (101) and (102), the Maxwell equation (100) is recast as,

$$\left( \frac{1}{c} \partial_t + \nabla \right) \varepsilon_0 F = \hat{q} - \frac{1}{c} \hat{j} - \nabla \wedge \hat{p} + \frac{1}{c} \nabla \wedge \hat{M} \tag{103}$$

The total electric charge density $\hat{q}$ in matter is the sum of the free electric charge density $q$ and the bound electric charge density $-\nabla \cdot \hat{p}$,

$$\hat{q} = q - \nabla \cdot \hat{p} \tag{104}$$

and total electric current density $\hat{j}$ in matter is the sum of the free electric currant density $\hat{j}$ and the bound electric current density $\partial_t \hat{p} - \nabla \cdot \hat{M}$,

$$\hat{j} = \hat{j} + \partial_t \hat{p} - \nabla \cdot \hat{M} \tag{105}$$

Using the definitions (98), (104) and (105), the Maxwell equation (103) reduces to,

$$\left( \frac{1}{c} \partial_t + \nabla \right) \varepsilon_0 F = \hat{q} - \frac{1}{c} \hat{j} - \nabla \wedge \hat{p} + \frac{1}{c} \nabla \wedge \hat{M} \tag{106}$$

Taking into account the definition (37) of the electromagnetic multivector field $F$, the trivectorial part of the Maxwell equation (106) reduces to,

$$c \varepsilon_0 \nabla \wedge B = \frac{1}{c} \nabla \wedge \hat{M} \tag{107}$$
In view of the magnetic Gauss equation (22), relation (107) yields the condition,

$$\nabla \wedge \tilde{M} = 0$$  \hspace{1cm} (108)

Taking into account the definition (37) of the electromagnetic multivector field $F$, the bivectorial part of the Maxwell equation (106) reduces to,

$$\varepsilon_0 (\partial_t B + \nabla \wedge e) = - \nabla \wedge \tilde{p}$$  \hspace{1cm} (109)

Using the Faraday equation (21), relation (109) yields the condition,

$$\nabla \wedge \tilde{p} = 0$$  \hspace{1cm} (110)

In view of the conditions (108) and (110), the Maxwell equation (106) reduces to,

$$\left( \frac{1}{c} \partial_t + \nabla \right) \varepsilon_0 F = \tilde{q} - \frac{1}{c} \tilde{j}$$  \hspace{1cm} (111)

which differs from the result derived by Arthur [20] since the conditions (108) and (110) were not properly identified. The Maxwell equation (111) in a dielectric and magnetic medium can be obtained by replacing the electric charge density $q$ by the total electric charge density $\tilde{q}$ and the electric current density $j$ by the total electric current density $\tilde{j}$ in the Maxwell equation (41) in vacuum. The difference between the Maxwell equations (100) and (111) yields the relation,

$$\left( \frac{1}{c} \partial_t + \nabla \right) \tilde{P} = (q - \tilde{q}) - \frac{1}{c} (j - \tilde{j})$$  \hspace{1cm} (112)

In view of relation of the electric charge density in matter (104) and the electric current density in matter (105), relation (112) is recast as,

$$\left( \frac{1}{c} \partial_t + \nabla \right) \tilde{P} = \nabla \cdot \tilde{p} + \frac{1}{c} \left( \partial_t \tilde{p} - \nabla \cdot \tilde{M} \right)$$  \hspace{1cm} (113)

In view of relations (108) and (110), the curl of the electromagnetic polarisation multivector (98) vanishes,

$$\nabla \wedge \tilde{P} = 0$$  \hspace{1cm} (114)

According to relation (114), the geometric product on the left-hand side of relation (113) becomes an inner product,

$$\left( \frac{1}{c} \partial_t + \nabla \right) \cdot \tilde{P} = \nabla \cdot \tilde{p} + \frac{1}{c} \left( \partial_t \tilde{p} - \nabla \cdot \tilde{M} \right)$$  \hspace{1cm} (115)
10. Electromagnetic waves in matter (SA)

In the spatial algebra (SA), electromagnetic waves in vacuum are a direct and straightforward consequence of the Maxwell equation (43), as expected, but it is nonetheless quite beautiful. Indeed, multiplying the Maxwell equation (111) by $c^{-1}\partial_t - \nabla$ yields,

$$\left(\frac{1}{c}\partial_t - \nabla\right)\left(\frac{1}{c}\partial_t + \nabla\right)\varepsilon_0 F = \left(\frac{1}{c}\partial_t - \nabla\right)\left(\hat{q} - \frac{1}{c}\hat{j}\right)$$  \hspace{1cm} (116)

which is written explicitly as,

$$\left(\frac{1}{c^2}\partial_t^2 - \nabla^2\right)\varepsilon_0 F = \frac{1}{c}\partial_t\hat{q} - \nabla\hat{q} - \frac{1}{c^2}\partial_t\hat{j} + \frac{1}{c}\nabla\cdot\hat{j} + \frac{1}{c}\nabla\wedge\hat{j}$$  \hspace{1cm} (117)

In view of the definitions of the total electric charge density (104) and the total electric current density (105), we obtain,

$$\partial_t\hat{q} + \nabla\cdot\hat{j} = \partial_t q - \partial_t (\nabla\cdot\vec{p}) + \nabla\cdot\vec{j} + \partial_t (\nabla\cdot\vec{p}) - \nabla\cdot\left(\nabla\cdot\vec{M}\right)$$  \hspace{1cm} (118)

Taking into account the electric continuity equation (24) for the free electric charge density $q$, relation (118) yields the electric continuity equation for the total electric charge density $\hat{q}$,

$$\partial_t\hat{q} + \nabla\cdot\hat{j} = 0$$  \hspace{1cm} (119)

Using the continuity equation (119), relation (117) yields the electromagnetic wave equation for the electromagnetic multivector field $F$ in a dielectric and magnetic medium,

$$\left(\frac{1}{c^2}\partial_t^2 - \nabla^2\right)\varepsilon_0 F = -\nabla\hat{q} - \frac{1}{\varepsilon_0 c^2}\partial_t\hat{j} + \frac{1}{c}\nabla\wedge\hat{j}$$  \hspace{1cm} (120)

Taking into account the definition (37) of the electromagnetic multivector field $F$, the vectorial part of the wave equation (120) yields the wave equation for the electric field $e$,

$$\left(\frac{1}{c^2}\partial_t^2 - \nabla^2\right)e = -\frac{1}{\varepsilon_0}\nabla\hat{q} - \frac{1}{\varepsilon_0 c^2}\partial_t\hat{j}$$  \hspace{1cm} (121)
Taking into account the definition (37) of the electromagnetic multivector field $F$ and using the speed of light (31), the bivectorial part of the wave equation (120) yields the wave equation for the magnetic field $B$,

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2\right) B = \mu_0 \nabla \wedge \vec{J} \quad (122)$$

11. Electromagnetic energy and momentum in matter (SA)

The electromagnetic field multivector $F$ is a linear combination of the electric field vector $e$ and the magnetic induction field bivector $B$ according to relation (37). Similarly, the auxiliary electromagnetic field multivector $G$ is a linear combination of the electric displacement field vector $d$ and the auxiliary magnetic field bivector $H$ according to relation (44). Moreover, the electric polarisation multivector $\tilde{P}$ is a linear combination of the electric polarisation vector $\tilde{p}$ and the magnetisation bivector $\tilde{M}$ according to relation (98). Since the electromagnetic energy density $e$ and the electromagnetic momentum density $p$ in matter are determined by these fields, they can be recast in terms of the multivectors $F$, $G$ and $\tilde{P}$. In view of the linear electric constitutive relation in matter (91) and the linear magnetic constitutive relation (96), relation (59) yields the electromagnetic energy density in a dielectric and magnetic medium, [9, 10]

$$e = \frac{1}{2} \mathbf{d} \cdot \mathbf{e} - \frac{1}{2} \mathbf{H} \cdot \mathbf{B} = \frac{1}{2} \varepsilon_0 e^2 - \frac{1}{2 \mu_0} B^2 + \frac{1}{2} \tilde{p} \cdot \mathbf{e} + \frac{1}{2} \tilde{M} \cdot \mathbf{B} \quad (123)$$

In view of the linear magnetic constitutive relation in matter (96), relation (62) yields the electromagnetic momentum density in matter,

$$p = \frac{1}{c^2} \mathbf{H} \cdot \mathbf{e} = \frac{1}{c^2} \left( \frac{\mathbf{B}}{\mu_0} - \tilde{M} \right) \cdot \mathbf{e} \quad (124)$$

The reverse of the linear electromagnetic constitutive relation in matter (99) is written as,

$$G^\dagger = \varepsilon_0 F^\dagger + \tilde{P}^\dagger \quad (125)$$

According the definition (98), the reverse of the electromagnetic polarisation tensor is given by,

$$\tilde{P}^\dagger = \tilde{p}^\dagger - \frac{1}{c} \tilde{M}^\dagger = \tilde{p} + \frac{1}{c} \tilde{M} \quad (126)$$
In view of the multivectors (98) and (126), the electric polarisation vector field is recast as,

\[ \vec{p} = \frac{1}{2} (\vec{P} + \vec{P}^\dagger) \]  

and the magnetisation bivector field is recast as,

\[ \vec{M} = -\frac{c}{2} (\vec{P} - \vec{P}^\dagger) \]  

Comparing the expressions (59) and (72) for the electromagnetic energy density in vacuum to the expression (123) of the electromagnetic energy density in matter, we conclude that the electromagnetic energy density in matter (123) is recast as,

\[ e = \frac{1}{8} (G F^\dagger + F G^\dagger + G^\dagger F + F^\dagger G) \]  

In view of the electromagnetic constitutive equations (99) and (125), the electromagnetic energy density in matter (129) is recast as,

\[ e = \frac{1}{4} \varepsilon_0 (F F^\dagger + F^\dagger F) + \frac{1}{8} (P F^\dagger + F P^\dagger + P^\dagger F + F^\dagger P) \]  

Comparing the expressions (62) and (75) for the electromagnetic momentum density in vacuum to the expression (124) of the electromagnetic momentum density in matter, we conclude that the electromagnetic momentum density in matter (124) is recast as,

\[ \vec{p} = \frac{1}{8} c (G F^\dagger + F G^\dagger - G^\dagger F - F^\dagger G) \]  

In view of the electromagnetic constitutive equations (99) and (125), the electromagnetic momentum density in matter (129) is recast as,

\[ \vec{p} = \frac{1}{4} c \varepsilon_0 (F F^\dagger - F^\dagger F) + \frac{1}{8} c (P F - F P + F^\dagger P^\dagger - P^\dagger F^\dagger) \]  

According to relations (129) and (131),

\[ e + \vec{p} c = \frac{1}{4} (G F^\dagger + F G^\dagger) + \frac{1}{8} (G F - F G + F^\dagger G^\dagger - G^\dagger F^\dagger) \]  

In view of the electromagnetic constitutive equations (99) and (125), relation (133) in matter is recast as,

\[ e + \vec{p} c = \frac{1}{2} \varepsilon_0 F F^\dagger + \frac{1}{8} (P F - F P + F^\dagger P^\dagger - P^\dagger F^\dagger) \]
12. Poynting theorem in matter (SA)

The continuity equation for the electromagnetic energy, called the Poynting theorem, is written in matter in terms of the time derivative of the energy density and the energy flux which is a multiple of the electromagnetic momentum density. In linear electromagnetism, the matter electric polarisation vector field $\tilde{p}(e)$ induced by the electric field $e$ is a linear map of the electric vector field $e$,

$$\tilde{p}(e) = \varepsilon_0 \chi_e(e)$$ (135)

where $\chi_e(e)$ is the electric susceptibility vector field, and the matter magnetisation bivector field $\tilde{M}(B)$ induced by the magnetic induction field $B$ is a linear map of the magnetic induction bivector field $B$,

$$\tilde{M}(B) = \frac{1}{\mu_0} \chi_B(B)$$ (136)

where $\chi_B(B)$ is the electric susceptibility bivector field. In view of the linear maps (135) and (136) the electric displacement vector field (91) is recast as,

$$d(e) = \varepsilon_0 \left( e + \chi_e(e) \right)$$ (137)

and the magnetic auxiliary bivector field (96) is recast as,

$$H(B) = \frac{1}{\mu_0} \left( B - \chi_B(B) \right)$$ (138)

In view of linear relations (135) and (136), the electromagnetic energy density (123) is recast as,

$$e(e, B) = \frac{1}{2} \varepsilon_0 \left( e + \chi_e(e) \right) \cdot e - \frac{1}{2 \mu_0} \left( B - \chi_B(B) \right) \cdot B$$ (139)

The time derivative of the linear electromagnetic energy density in matter (139) reads,

$$\partial_t e(e, B) = \varepsilon_0 e \cdot \partial_t e + \frac{1}{2} \varepsilon_0 \chi_e(e) \cdot \partial_t e + \frac{1}{2} \varepsilon_0 \partial_t \left( \chi_e(e) \right) \cdot e$$

$$- \frac{1}{\mu_0} B \cdot \partial_t B + \frac{1}{2 \mu_0} \chi_B(B) \cdot \partial_t B + \frac{1}{2 \mu_0} \partial_t \left( \chi_B(B) \right) \cdot B$$ (140)
The electric susceptibility vector $\chi_e(e)$ is a linear map of the electric field vector $e$ and the magnetic susceptibility bivector $\chi_B(B)$ is a linear map of the magnetic induction field bivector $B$. Furthermore, we assume that the electric and magnetic properties of the material medium are constant, which implies that,

$$
\partial_t \left( \chi_e(e) \right) \cdot e = \chi_e(e) \cdot \partial_t e \\
\chi_B(B) \cdot \partial_t B = \partial_t \left( \chi_B(B) \right) \cdot B
$$

(141)

In view of the conditions (141), relation (140) reduces to,

$$
\partial_t \left( e, B \right) = \varepsilon_0 e \cdot \partial_t \left( e + \chi_e(e) \right) - \frac{1}{\mu_0} \left( B - \chi_B(B) \right) \cdot \partial_t B
$$

(142)

Using the electric constitutive equation (91) and the magnetic constitutive equation (96), relation (142) is recast as,

$$
\partial_t e = e \cdot \partial_t d - H \cdot \partial_t B
$$

(143)

Using the Maxwell-Ampère equation (20) and the Faraday equation (21), the time derivative of the electromagnetic energy density (143) is recast as,

$$
\partial_t e = -e \cdot (\nabla \cdot H + j) + H \cdot (\nabla \wedge e)
$$

(144)

Thus,

$$
\partial_t e + (\nabla \cdot H) \cdot e - H \cdot (\nabla \wedge e) = -j \cdot e
$$

(145)

Using the algebraic identity (E.23),

$$
\nabla \cdot (H \cdot e) = (\nabla \cdot H) \cdot e - H \cdot (\nabla \wedge e)
$$

(146)

relation yields (145) the Poynting’s theorem,

$$
\partial_t e + \nabla \cdot (H \cdot e) = -j \cdot e
$$

(147)

where $H \cdot e$ is the Poynting vector, which is the electromagnetic current density. In view of the electromagnetic momentum density (124), Poynting’s theorem (147) in vacuum is recast as,

$$
\frac{1}{c} \partial_t e + \nabla \cdot (p c) = -\frac{1}{c} j \cdot e
$$

(148)
Poynting’s theorem (148) in matter is valid under the assumption that the electric and magnetic properties of the material medium are constant. In linear electromagnetism, according to relations (137) and (138), the auxiliary electromagnetic field multivector (44) is written as,

\[
G = d(e) + \frac{1}{c} H(B) = \varepsilon_0 \left( e + \chi_e(e) \right) + \frac{1}{\mu_0 c} \left( B - \chi_B(B) \right) \tag{149}
\]

In view of identity (31), the auxiliary electromagnetic field multivector (149) is recast as,

\[
G = \varepsilon_0 \left( e + c B \right) + \varepsilon_0 \left( \chi_e(e) - c \chi_B(B) \right) \tag{150}
\]

The susceptibilities are linear maps of the electromagnetic fields \( e \) and \( B \) that are expressed in relations (64) and (65) as linear combinations of the electromagnetic multivector \( F \) and its reverse \( F^\dagger \). Thus, we define an electromagnetic susceptibility multivector in space-time as a linear map of the electromagnetic field bivector \( F \),

\[
\chi_F(F) = \chi_e(e) - c \chi_B(B) \tag{151}
\]

In view of relations (37) and (151), the auxiliary electromagnetic field multivector (150) is written as a linear map of the electromagnetic field multivector \( F \),

\[
G(F) = \varepsilon_0 \left( F + \chi_F(F) \right) \tag{152}
\]

In view of expressions (99) and (152) for the electromagnetic field multivector \( G(F) \), the electromagnetic polarisation multivector \( \hat{P}(F) \) is a linear map of the electromagnetic field multivector \( F \),

\[
\hat{P}(F) = \varepsilon_0 \chi_F(F) \tag{153}
\]

13. Maxwell equation in vacuum (STA)

Special relativity, that is rooted in electromagnetism, is described in space-time. To describe relativistic theories like electromagnetism in vector space (VS), an additional time dimension is added such that the generalisation of the spatial vector space \( \mathbb{R}^3 \) becomes the space-time vector space \( \mathbb{R}^{1,3} \). Similarly,
the spatial algebra (SA) \( G^3 \) is generalised by including an additional time dimension and called the space-time algebra (STA) \( G^{1,3} \). As we will show, the fundamental equations of electromagnetism reach the highest degree of simplicity and beauty because it turns out that the natural language of relativistic physical phenomena is precisely space-time algebra (STA). We consider an orthonormal vector frame \( \{ e_0, e_1, e_2, e_3 \} \) in space-time in order to recast Maxwell’s equation in vacuum. The gradient operator \( \nabla \) is a covariant vector in space-time is written in coordinates as,

\[
\nabla = e^\mu \partial_\mu = e^0 \partial_0 + e^i \partial_i = e^0 \frac{1}{c} \partial_t + e^i \partial_i
\]

(154)

In view of relation (F.28), the inner product of the gradient \( \nabla \) with the time vector \( e_0 \) yields,

\[
\nabla \cdot e_0 = e_0 \cdot \nabla = \partial_0 = \frac{1}{c} \partial_t
\]

(155)

According to relation (F.29), the outer product of the gradient \( \nabla \) with the time vector \( e_0 \) yields,

\[
\nabla \wedge e_0 = -e_0 \wedge \nabla = -e_i \partial^i = -\nabla
\]

(156)

In view of the inner product (F.28) and the outer product (F.29), the geometric product of the gradient \( \nabla \) and the time vector \( e_0 \) yields,

\[
\nabla e_0 = \nabla \cdot e_0 + \nabla \wedge e_0 = \frac{1}{c} \partial_t - \nabla
\]

(157)

Taking into account the properties (F.2) and (F.4), the geometric product in reverse order is given by,

\[
e_0 \nabla = e_0 \cdot \nabla + e_0 \wedge \nabla = \nabla \cdot e_0 - \nabla \wedge e_0 = \frac{1}{c} \partial_t + \nabla
\]

(158)

The electric current density \( J \) is a contravariant vector in space-time defined as,

\[
J = J^\mu e_\mu = J^0 e_0 + J^i e_i = q c e_0 + j^i e_i
\]

(159)

In view of relation (F.17), the inner product of the electric current density \( J \) with the time vector \( e_0 \) yields,

\[
J \cdot e_0 = e_0 \cdot J = J^0 = q c
\]

(160)
According to relation (F.18), the outer product of the electric current density $J$ with the time vector $e_0$ yields,

$$J \wedge e_0 = -e_0 \wedge J = J^i (e_i \wedge e_0) = j^i e_i = j$$  \hspace{1cm} (161)$$

In view of the inner product (F.17) and the outer product (F.18), the geometric product of the electric current density $J$ and the time vector $e_0$ yields,

$$Je_0 = J \cdot e_0 + J \wedge e_0 = qc + j$$  \hspace{1cm} (162)$$

Taking into account the properties (F.2) and (F.4), the geometric product in reverse order is given by,

$$e_0 J = e_0 \cdot J + e_0 \wedge J = J \cdot e_0 - J \wedge e_0 = qc - j$$  \hspace{1cm} (163)$$

The Maxwell equation (43) in vacuum is written in spatial algebra as,

$$\left(\frac{1}{c} \partial_t + \nabla\right) G = \frac{1}{c} (qc - j)$$  \hspace{1cm} (164)$$

In view of the geometric product (158) between the time vector $e_0$ and the gradient operator $\nabla$ and the geometric product (163) of the time vector $e_0$ and the electric current density $J$, the Maxwell equation (164) in vacuum is recast in space-time algebra as,

$$e_0 \nabla G = \frac{1}{c} e_0 J$$  \hspace{1cm} (165)$$

The time vector $e_0$ defines a relative spatial frame $\{e_1, e_2, e_3\}$ that is orthogonal to it. Taking the geometric product of the time vector $e_0$ and the Maxwell equation (165) in spatial algebra, the latter is projected onto an orthonormal frame $\{e_0, e_1, e_2, e_3\}$ with a specific time vector $e_0$. Thus, in space-time algebra the frame independent Maxwell equation in vacuum reads,

$$\nabla G = \frac{1}{c} J$$  \hspace{1cm} (166)$$

The gradient of the auxiliary electromagnetic field on the left-hand side of the Maxwell equation (166) in vacuum can be written as the sum of the divergence and the curl,

$$\nabla \cdot G + \nabla \wedge G = \frac{1}{c} J$$  \hspace{1cm} (167)$$
Since the electric current density $J$ is a vector in space-time and the the auxiliary electromagnetic multivector field $G$ is the sum of a linear combination (44) of the vector field $d$ and the auxiliary magnetic bivector field $H$, the curl of the auxiliary electromagnetic multivector field, which is a bivector, has to vanish,

$$\nabla \wedge G = 0$$

which is the homogeneous Maxwell equation in vacuum that is independent of the electric current density $J$. In view of the homogeneous Maxwell equation (168) in vacuum, the Maxwell equation (167) in vacuum yields the inhomogeneous Maxwell equation in vacuum,

$$\nabla \cdot G = \frac{1}{c} J$$

In view of the electromagnetic constitutive relation in vacuum (42), the homogeneous Maxwell equation (168) in vacuum is recast as,

$$\nabla \wedge F = 0$$

and the inhomogeneous Maxwell equation (169) in vacuum is recast as,

$$\nabla \cdot F = \frac{1}{\varepsilon_0 c} J$$

The Maxwell equation in vacuum is the sum of the homogeneous and inhomogeneous Maxwell equations (168) and (169) in vacuum,

$$\nabla F = \frac{1}{\varepsilon_0 c} J$$

14. Maxwell equation in matter (STA)

In order to recast Maxwell’s equation in matter, we follow a similar approach as in vacuum except that the electromagnetic constitutive equation (99) is different due to the electric polarisation multivector $\tilde{P}$. In view of the gradient (158) and the electric current density (163), the Maxwell equation (100) is recast as,

$$\varepsilon_0 \nabla \left( \varepsilon_0 F + \tilde{P} \right) = \frac{1}{c} \varepsilon_0 J$$

32
Since $e_0^2 = 1$, multiplying relation (173) by $e_0$ yields,

$$\nabla \left( \varepsilon_0 F + \tilde{P} \right) = \frac{1}{c} J$$

(174)

In view of the electromagnetic constitutive equation (99), the Maxwell equation in matter (174) reduces to,

$$\nabla G = \frac{1}{c} J$$

(175)

We now recast the electromagnetic constitutive equation in terms of the electric current density in matter $\tilde{J}$. The electric current density in a dielectric and magnetic medium $\tilde{J}$ is a contravariant vector in space-time defined as,

$$\tilde{J} = \tilde{J}^\mu e_\mu = \tilde{J}^0 e_0 + \tilde{J}^i e_i = \tilde{q} c e_0 + \tilde{j}^i e_i$$

(176)

In view of relation (F.17), the inner product of the electric current density in matter $\tilde{J}$ with the time vector $e_0$ yields,

$$\tilde{J} \cdot e_0 = e_0 \cdot \tilde{J} = \tilde{J}^0 = \tilde{q} c$$

(177)

According to relation (F.18), the outer product of the electric current density in matter $\tilde{J}$ with the time vector $e_0$ yields,

$$\tilde{J} \wedge e_0 = - e_0 \wedge \tilde{J} = \tilde{J}^i (e_i \wedge e_0) = \tilde{j}^i e_i = \tilde{\j}$$

(178)

In view of the inner product (F.17) and the outer product (F.18), the geometric product of the electric current density in matter $\tilde{J}$ and the time vector $e_0$ yields,

$$\tilde{J} e_0 = \tilde{J} \cdot e_0 + \tilde{J} \wedge e_0 = \tilde{q} c + \tilde{\j}$$

(179)

Taking into account the properties (F.2) and (F.4), the geometric product in reverse order is given by,

$$e_0 \tilde{J} = e_0 \cdot \tilde{J} + e_0 \wedge \tilde{J} = \tilde{J} \cdot e_0 - \tilde{J} \wedge e_0 = \tilde{q} c - \tilde{\j}$$

(180)

In view of relations (104) and (105), relation (179) is recast as,

$$e_0 \tilde{J} = q c - \tilde{j} - e \nabla \cdot \tilde{p} - \left( \partial_t \tilde{p} - \nabla \cdot \tilde{M} \right)$$

(181)
Using the gradient (158), relation (115) is recast as,

\[ e_0 \nabla \cdot \hat{P} = \nabla \cdot \bar{p} + \frac{1}{c} \left( \partial_t \bar{p} - \nabla \cdot \bar{M} \right) \]  \hspace{1cm} (182)

In view of the electric current density (163) and the divergence of the electromagnetic polarisation multivector (182), the electric current density in matter (181) reduces to,

\[ e_0 \tilde{J} = e_0 J - e_0 c \nabla \cdot \hat{P} \]  \hspace{1cm} (183)

Since \( e_0^2 = 1 \), taking the geometric of the vector \( e_0 \) and equation (183) yields,

\[ \tilde{J} = J - c \nabla \cdot \hat{P} \]  \hspace{1cm} (184)

which is quite a beautiful result. The Maxwell equation (185) in a dielectric and magnetic medium is written in spatial algebra \( G^3 \) as,

\[ \left( \frac{1}{c} \partial_t + \nabla \right) \varepsilon_0 F = \tilde{q} - \frac{1}{c} \tilde{J} \]  \hspace{1cm} (185)

In view of the geometric product (158) between the time vector \( e_0 \) and the gradient operator \( \nabla \) and the geometric product (180) of the time vector \( e_0 \) and the electric current density in matter \( \tilde{J} \), the Maxwell equation (164) in matter is recast in space-time algebra \( G^3 \) as,

\[ e_0 \nabla F = \frac{1}{\varepsilon_0 c} e_0 \tilde{J} \]  \hspace{1cm} (186)

Taking the geometric product of the time vector \( e_0 \) and the Maxwell equation (186) in spatial algebra \( G^3 \), the latter is projected onto an orthonormal frame \( \{ e_0, e_1, e_2, e_3 \} \) with a specific time vector \( e_0 \). Thus, in space-time algebra \( G^3 \), the frame independent Maxwell equation in matter reads,

\[ \nabla F = \frac{1}{\varepsilon_0 c} \tilde{J} \]  \hspace{1cm} (187)

The gradient of the auxiliary electromagnetic field on the left-hand side of the Maxwell equation (187) in matter can be written as the sum of the divergence and the curl,

\[ \nabla \cdot F + \nabla \wedge F = \frac{1}{\varepsilon_0 c} \tilde{J} \]  \hspace{1cm} (188)
Since the electric current density $J$ is a vector in space-time and the electromagnetic multivector field $F$ is the sum of a linear combination (37) of the vector field $e$ and the magnetic bivector field $B$, the curl of the electromagnetic multivector field, which is a bivector, has to vanish,

$$\nabla \wedge F = 0$$  \hspace{1cm} (189)

which is the homogeneous Maxwell equation in matter that is independent of the electric current density in matter $\tilde{J}$. In view of the homogeneous Maxwell equation (189) in matter, the Maxwell equation (188) in matter yields the inhomogeneous Maxwell equation in matter,

$$\nabla \cdot F = \frac{1}{\varepsilon_0 c} \tilde{J}$$  \hspace{1cm} (190)

15. Electromagnetic waves in vacuum (STA)

In the space-time algebra (STA), electromagnetic waves in vacuum are a direct and straightforward consequence of the Maxwell equation (166) as in space-algebra but it is a way that is even simpler. The gradient of the Maxwell equation (166) in space-time is in fact the electromagnetic wave equation. The gradient of the Maxwell equation (166) in vacuum reads,

$$\nabla^2 G = \frac{1}{c} \nabla J$$  \hspace{1cm} (191)

The gradient of the electric current density on the right-hand side of relation (191) can be written as the sum of the divergence and the curl,

$$\nabla^2 G = \frac{1}{c} \nabla \cdot J + \frac{1}{c} \nabla \wedge J$$  \hspace{1cm} (192)

Since the Laplacian $\nabla^2$ is a scalar operator and the auxiliary electromagnetic multivector field $G$ is the sum of a linear combination (44) of the electric displacement vector field $d$ and the auxiliary magnetic bivector field $H$, the divergence of the current density $\nabla \cdot J$ on the right-hand side of relation (192), which is a scalar, has to vanish,

$$\nabla \cdot J = 0$$  \hspace{1cm} (193)
The electric charge conservation law (193) is recast in spatial algebra as the electric continuity equation (6) using the identity (F.33) for the scalar product of two vectors in space-time and the definition of the current density vector (165),

\[ \nabla \cdot J = \partial_t q + \nabla \cdot j = 0 \]  

(194)

In view of the electric charge conservation law (193), relation (192) reduces to,

\[ \nabla^2 G = \frac{1}{c} \nabla \wedge J \]  

(195)

which is the electromagnetic wave equation in vacuum in space-time algebra.

To show this, we recast now this equation in spatial algebra. In view of the geometric products (157), (158) and of the identity \( e_0 e_0 = 1 \), the Laplacian operator called the d'Alembertian operator is written in a specific space-time frame defined by a time vector \( e_0 \) as,

\[ \nabla^2 = \nabla \nabla = (\nabla e_0) (e_0 \nabla) = \left( \frac{1}{c} \partial_t - \nabla \right) \left( \frac{1}{c} \partial_t + \nabla \right) = \frac{1}{c^2} \partial_t^2 - \nabla^2 \]  

(196)

In view of identity (196), the left-hand side of relation (195) is recast in spatial algebra as,

\[ \nabla^2 G \]  

(197)

In view of identity (F.36), the right-hand side of relation (195) is recast in spatial algebra as,

\[ \frac{1}{c} \nabla \wedge J = - \frac{1}{c^2} \partial_t j - \nabla q + \frac{1}{c} \nabla \wedge j \]  

(198)

According to relations (197) and (198), the gradient of the Maxwell equation (195) in vacuum is recast in spatial algebra as the electromagnetic wave equation (49),

\[ \left( \frac{1}{c^2} \partial_t^2 - \nabla^2 \right) G = - \nabla q - \frac{1}{c^2} \partial_t j + \frac{1}{c} \nabla \wedge j \]  

(199)

16. Electromagnetic waves in matter (STA)

In the space-time algebra (STA), electromagnetic waves in matter are a direct and straightforward consequence of the Maxwell equation (175) or (188) as
in space-algebra but it is a way that is even simpler. The gradient of the Maxwell equation (175) or (188) in space-time is in fact the electromagnetic wave equation. The gradient of the Maxwell equation (175) in matter reads,

\[ \nabla^2 G = \frac{1}{c} \nabla J \]  

(200)

In view of equation (193), the electromagnetic wave equation in matter (200) becomes,

\[ \nabla^2 G = \frac{1}{c} \nabla \wedge J \]  

(201)

The gradient of the inhomogeneous Maxwell equation (188) in matter is given by,

\[ \nabla^2 F = \frac{1}{\varepsilon_0 c} \nabla \tilde{J} \]  

(202)

Since the Laplacian \( \nabla^2 \) is a scalar operator and the electromagnetic multivector field \( F \) is the sum of a linear combination (37) of the vector field \( e \) and the magnetic bivector field \( B \), the divergence of the current density in matter \( \nabla \cdot \tilde{J} \) on the right-hand side of relation (202), which is a scalar, has to vanish,

\[ \nabla \cdot \tilde{J} = 0 \]  

(203)

In view of the electric charge conservation law (203), relation (202) reduces to,

\[ \nabla^2 F = \frac{1}{\varepsilon_0 c} \nabla \wedge \tilde{J} \]  

(204)

which is the electromagnetic wave equation in matter in space-time algebra.

To show this, we recast now this equation in spatial algebra. In view of identity (196), the left-hand side of relation (204) is recast in spatial algebra as,

\[ \nabla^2 F = \left( \frac{1}{c^2} \partial_t^2 - \nabla^2 \right) F \]  

(205)

In view of identity (F.36), the right-hand side of relation (204) is recast in spatial algebra as,

\[ \frac{1}{\varepsilon_0 c} \nabla \wedge \tilde{J} = - \frac{1}{\varepsilon_0 c^2} \partial_t \tilde{J} - \frac{1}{\varepsilon_0} \nabla \tilde{q} + \frac{1}{\varepsilon_0 c} \nabla \wedge \tilde{J} \]  

(206)

According to relations (205) and (206), the gradient of the Maxwell equation (204) in matter is recast in spatial algebra as the electromagnetic wave
equation (49),

$$
\left( \frac{1}{c^2} \partial_i^2 - \nabla^2 \right) F = -\frac{1}{\varepsilon_0} \nabla \tilde{q} - \frac{1}{\varepsilon_0 c^2} \partial_i \tilde{j} + \frac{1}{\varepsilon_0 c} \nabla \wedge \tilde{j}
$$

(207)

17. Electromagnetic fields (STA)

The electromagnetic multivectors \( F \) and \( G \) and the electromagnetic polarisation multivector \( \tilde{P} \) in spatial algebra (SA) are in fact bivectors in space-time algebra (STA), as we will show in this section. We will also determine the components of these bivectors with respect to an orthonormal space-time frame. The electromagnetic multivector (37) is written in terms of the relative spatial vectors as,

$$
F = e + c B = e^i e_i + \frac{1}{2} c B^{ij} e_i \wedge e_j
$$

(208)

Using the definition (F.6) of the relative spatial vectors, the electromagnetic multivector (208) is recast as,

$$
F = e^i (e_i e_0) + \frac{1}{2} c B^{ij} (e_i e_0) \wedge (e_j e_0)
$$

(209)

The electric field vector \( e \) and magnetic induction field bivector \( B \) are frame dependent since they are defined in a specific spatial algebra \( \mathbb{G}^3 \). In contrast, the electromagnetic field multivector \( F \) is frame independent since it is defined in the space-time algebra \( \mathbb{G}^{1,3} \). Using identities (F.3) and (F.4), we obtain,

$$
e_i e_0 = e_i \wedge e_0
$$

$$
(e_i e_0) \wedge (e_j e_0) = e_i e_0 e_j e_0 = -e_i e_j = -e_i \wedge e_j
$$

(210)

In view of identities (210), the electromagnetic field multivector (209) is recast as,

$$
F = e^i e_i \wedge e_0 - \frac{1}{2} c B^{ij} e_i \wedge e_j
$$

(211)

In the space-time algebra, the electromagnetic field multivector (211) is in fact a bivector written as,

$$
F = \frac{1}{2} F_{\mu\nu} e_\mu \wedge e_\nu
$$

(212)
According to relations (212), (F.69) and (F.70), the components of the electromagnetic field bivector $F$ are given by,

$$F^\mu\nu = (e^\nu \wedge e^\mu) \cdot F = F \cdot (e^\nu \wedge e^\mu) \quad (213)$$

According to relations (F.23) and (F.53), the components of the electromagnetic field bivector $F$ are written as,

$$F^\mu\nu = e^\nu \cdot (e^\mu \cdot F) = (F \cdot e^\nu) \cdot e^\mu \quad (214)$$

According to identity (F.4), the components of the electromagnetic field bivector $F$ are antisymmetric,

$$F^\mu\nu = -F^{\nu\mu} \quad (215)$$

In view of identities (F.4) and (215), the electromagnetic field bivector (212) is split as,

$$F = \frac{1}{2} \left( F^{i0} e_i \wedge e_0 + F^{0i} e_0 \wedge e_i + F^{ij} e_i \wedge e_j \right) = F^{i0} e_i \wedge e_0 + \frac{1}{2} F^{ij} e_i \wedge e_j \quad (216)$$

In view of relations (211), (216) and (215), the components of the electromagnetic field bivector $F$ are given by,

$$F^{\mu\mu} = -F^{\mu\mu} = 0$$
$$F^{i0} = -F^{0i} = e^i$$
$$F^{ij} = -F^{ji} = -c B^{ij} \quad (217)$$

Thus, the components of the electromagnetic bivector field $F$ are written as an antisymmetric matrix,

$$F^\mu\nu = \begin{pmatrix} 0 & -e^1 & -e^2 & -e^3 \\ e^1 & 0 & -c B^{12} & -c B^{13} \\ e^2 & c B^{12} & 0 & -c B^{23} \\ e^3 & c B^{13} & c B^{23} & 0 \end{pmatrix} \quad (218)$$

The components of the electromagnetic field bivector $F^{\mu\nu}$ can be recast in terms of the components of the dual magnetic induction field vector $b$ where $b = B^*$,

$$b^1 e_1 + b^2 e_2 + b^3 e_3 = B^{12} (e_1 \wedge e_2)^* + B^{23} (e_2 \wedge e_3)^* + B^{31} (e_3 \wedge e_1)^* \quad (219)$$
According to the duality (B.35),
\[(e_1 \wedge e_2)^* = e_1 \times e_2 = e_3\]
\[(e_2 \wedge e_3)^* = e_2 \times e_3 = e_1\]
\[(e_3 \wedge e_1)^* = e_3 \times e_1 = e_2\] (220)

which implies that the identity (219) is recast as,
\[b^1 e_1 + b^2 e_2 + b^3 e_3 = B^{23} e_1 + B^{31} e_2 + B^{12} e_3\] (221)

Thus,
\[B^{12} = b_3 \quad \text{and} \quad B^{13} = -B^{31} = -b_2 \quad \text{and} \quad B^{23} = b_1\] (222)

Using relations (222), the components of the electromagnetic field bivector (218) are recast in vector space as,
\[
F^{\mu \nu} = \begin{pmatrix}
0 & -e^1 & -e^2 & -e^3 \\
e^1 & 0 & -c b^3 & c b^2 \\
e^2 & c b^3 & 0 & -c b^1 \\
e^3 & -c b^2 & c b^1 & 0
\end{pmatrix}
\] (223)

The frame dependence of the electric field vector \(e\) and magnetic induction field bivector \(B\) is made explicit using a geometric product with the time vector \(e_0\).

In view of relation (211), the double geometric product of the electromagnetic multivector \(F\) and the time vector \(e_0\) yields,
\[e_0 \, F \, e_0 = e^i e_0 e_i e_0 + \frac{1}{2} c B^{ij} e_0 (e_i \wedge e_j) e_0 = -e^i e_i + \frac{1}{2} c B^{ij} (e_i \wedge e_j)\] (224)

In view of relations (208), (209) and (224), we obtain,
\[e_0 \, F \, e_0 = -e + c B = -F^\dagger\] (225)

Using relations (225), (64) and (65), the electric field vector \(e\) is written entirely in terms of the electromagnetic field multivector \(F\) and the time vector \(e_0\) as,
\[e = \frac{1}{2} (F - e_0 \, F \, e_0)\] (226)
and the magnetic field bivector $B$ is written as,

$$B = \frac{1}{2c} \left( F + e_0 F e_0 \right) \quad (227)$$

The auxiliary electromagnetic multivector (44) is written in terms of the relative spatial vectors as,

$$G = d + \frac{1}{c} H = d^i e_i + \frac{1}{2c} H^{ij} e_i \wedge e_j \quad (228)$$

Using the definition (F.6) of the relative spatial vectors, the auxiliary electromagnetic multivector (228) is recast as,

$$G = d^i (e_i e_0) + \frac{1}{2c} H^{ij} (e_i e_0) \wedge (e_j e_0) \quad (229)$$

The electric displacement field vector $d$ and auxiliary magnetic field bivector $H$ are frame dependent since they are defined in a specific spatial algebra $\mathbb{G}^3$. In contrast, the electromagnetic field multivector $G$ is frame independent since it is defined in the space-time algebra $\mathbb{G}^{1,3}$. In view of identities (210), the auxiliary electromagnetic field multivector (229) is recast as,

$$G = d^i e_i \wedge e_0 - \frac{1}{2c} H^{ij} e_i \wedge e_j \quad (230)$$

In the space-time algebra, the electromagnetic field multivector (230) is in fact a bivector written as,

$$G = \frac{1}{2} G^{\mu\nu} e_\mu \wedge e_\nu \quad (231)$$

According to relations (231), (F.69) and (F.70), the components of the electromagnetic field bivector $G$ are given by,

$$G^{\mu\nu} = (e^\nu \wedge e^\mu) \cdot G = G \cdot (e^\nu \wedge e^\mu) \quad (232)$$

According to relations (F.23) and (F.53), the components of the electromagnetic field bivector $G$ are written as,

$$G^{\mu\nu} = e^\nu \cdot (e^\mu \cdot G) = (G \cdot e^\nu) \cdot e^\mu \quad (233)$$

According to identity (F.4), the components of the electromagnetic field bivector $G$ are antisymmetric,

$$G^{\mu\nu} = - G^{\nu\mu} \quad (234)$$
In view of identities (F.4) and (F.4), the auxiliary electromagnetic field bivector (231) is split as,

\[
G = \frac{1}{2} \left( G^{00} e_i \wedge e_0 + G^{0i} e_0 \wedge e_i + G^{ij} e_i \wedge e_j \right) = G^{00} e_i \wedge e_0 + \frac{1}{2} G^{ij} e_i \wedge e_j \quad (235)
\]

In view of relations (230), (231) and (234), the components of the electromagnetic field bivector \(G\) are given by,

\[
G^\mu\nu = - G^{\mu\nu} = 0
\]

\[
G^{00} = - G^{0i} = d^i
\]

\[
G^{ij} = - G^{ji} = - c^{-1} H^{ij}
\]

Thus, the components of the electromagnetic bivector field \(G\) are written as an antisymmetric matrix,

\[
G^{\mu\nu} = \begin{pmatrix}
0 & -d^1 & -d^2 & -d^3 \\
d^1 & 0 & -c^{-1} H^{12} & -c^{-1} H^{13} \\
d^2 & c^{-1} H^{12} & 0 & -c^{-1} H^{23} \\
d^3 & c^{-1} H^{13} & c^{-1} H^{23} & 0
\end{pmatrix}
\]

(237)

The components of the electromagnetic field bivector \(G^{\mu\nu}\) can be recast in terms of the components of the dual auxiliary magnetic field vector \(h\) where \(h = H^*\),

\[
h^1 e_1 + h^2 e_2 + h^3 e_3 = H^{12} (e_1 \wedge e_2)^* + H^{23} (e_2 \wedge e_3)^* + H^{31} (e_3 \wedge e_1)^* \quad (238)
\]

According to the duality (220), the identity (238) is recast as,

\[
h^1 e_1 + h^2 e_2 + h^3 e_3 = H^{23} e_1 + H^{31} e_2 + H^{12} e_3 \quad (239)
\]

Thus,

\[
H^{12} = h_3 \quad \text{and} \quad H^{13} = - H^{31} = - h_2 \quad \text{and} \quad H^{23} = h_1 \quad (240)
\]

Using relations (240), the components of the electromagnetic field bivector (237) are recast in vector space as,

\[
G^{\mu\nu} = \begin{pmatrix}
0 & -d^1 & -d^2 & -d^3 \\
d^1 & 0 & -c^{-1} h^3 & c^{-1} h^2 \\
d^2 & c^{-1} h^3 & 0 & -c^{-1} h^1 \\
d^3 & -c^{-1} h^2 & c^{-1} h^1 & 0
\end{pmatrix}
\]

(241)
The frame dependence of the electric displacement field vector $d$ and auxiliary magnetic field bivector $H$ is made explicit using a geometric product with the time vector $e_0$. In view of relation (230), the double geometric product of the auxiliary electromagnetic multivector $G$ and the time vector $e_0$ yields,

$$
e_0 G e_0 = d^i e_0 e_i e_0 + \frac{1}{2c} H^{ij} e_0 (e_i \wedge e_j) e_0$$

$$= - e^i e_i + \frac{1}{2c} H^{ij} (e_i \wedge e_j)$$

(242)

In view of relations (228), (229) and (242), we obtain,

$$e_0 G e_0 = - e + \frac{1}{c} H = - G^\dagger$$

(243)

Using relations (243), (68) and (69), the electric displacement field vector $d$ is written entirely in terms of the auxiliary electromagnetic field multivector $G$ and the time vector $e_0$ as,

$$d = \frac{1}{2} (G - e_0 G e_0)$$

(244)

and the auxiliary magnetic field bivector $H$ is written as,

$$H = \frac{c}{2} (G + e_0 G e_0)$$

(245)

The electromagnetic polarisation multivector (98) is written in terms of the relative spatial vectors as,

$$\tilde{P} = \tilde{p} - \frac{1}{c} \tilde{M} = p^i e_i - \frac{1}{2c} M^{ij} e_i \wedge e_j$$

(246)

Using the definition (F.6) of the relative spatial vectors, the electromagnetic polarisation multivector (246) is recast as,

$$\tilde{P} = p^i (e_i e_0) - \frac{1}{2c} M^{ij} (e_i e_0) \wedge (e_j e_0)$$

(247)

The electric polarisation vector $\tilde{p}$ and magnetisation bivector $\tilde{M}$ are frame dependent since they are defined in a specific spatial algebra $\mathbb{C}^3$. In contrast, the electromagnetic polarisation multivector $\tilde{P}$ is frame independent since it is defined in the space-time algebra $\mathbb{C}^{1,3}$. In view of identities (210), the electric polarisation multivector (229) is recast as,

$$\tilde{P} = \tilde{p}^i e_i \wedge e_0 + \frac{1}{2c} \tilde{M}^{ij} e_i \wedge e_j$$

(248)
In the space-time algebra, the electromagnetic polarisation multivector (248) is in fact a bivector written as,

\[ \tilde{P} = \frac{1}{2} \tilde{P}^{\mu \nu} e_\mu \wedge e_\nu \]  

(249)

According to relations (249), (F.69) and (F.70), the components of the electromagnetic polarisation bivector \( \tilde{P} \) are given by,

\[ \tilde{P}^{\mu \nu} = (e^\nu \wedge e^\mu) \cdot \tilde{P} = \tilde{P} \cdot (e^\nu \wedge e^\mu) \]  

(250)

According to relations (F.23) and (F.53), the components of the electromagnetic polarisation bivector \( \tilde{P} \) are written as,

\[ \tilde{P}^{\mu \nu} = e^\nu \cdot (e^\mu \cdot \tilde{P}) = (\tilde{P} \cdot e^\nu) \cdot e^\mu \]  

(251)

According to identity (F.4), the components of the electromagnetic polarisation bivector \( \tilde{P} \) are antisymmetric,

\[ \tilde{P}^{\mu \nu} = - \tilde{P}^{\nu \mu} \]  

(252)

In view of relations (248), (249) and (252), the components of the electromagnetic polarisation bivector \( \tilde{P} \) are given by,

\[ \tilde{P}^{\mu \mu} = - \tilde{P}^{\mu \mu} = 0 \]

\[ \tilde{P}^{i0} = - \tilde{P}^{0i} = \tilde{p}^i \]  

(254)

\[ \tilde{P}^{ij} = - \tilde{P}^{ji} = c^{-1} \tilde{M}^{ij} \]

Thus, the components of the electromagnetic polarisation bivector \( \tilde{P} \) are written as an antisymmetric matrix,

\[
\tilde{P}^{\mu \nu} =
\begin{pmatrix}
0 & -\tilde{p}^1 & -\tilde{p}^2 & -\tilde{p}^3 \\
\tilde{p}^1 & 0 & c^{-1} \tilde{M}^{12} & c^{-1} \tilde{M}^{13} \\
\tilde{p}^2 & -c^{-1} \tilde{M}^{12} & 0 & c^{-1} \tilde{M}^{23} \\
\tilde{p}^3 & -c^{-1} \tilde{M}^{13} & -c^{-1} \tilde{M}^{23} & 0
\end{pmatrix}
\]

(255)
The components of the electromagnetic polarisation bivector $\tilde{P}_{\mu\nu}$ can be recast in terms of the components of the dual auxiliary magnetisation vector $\tilde{m}$ where $\tilde{m} = \tilde{M}^*$,

$$\tilde{m}^1 e_1 + \tilde{m}^2 e_2 + \tilde{m}^3 e_3 = \tilde{M}^{12} (e_1 \wedge e_2)^* + \tilde{M}^{23} (e_2 \wedge e_3)^* + \tilde{M}^{31} (e_3 \wedge e_1)^*$$

(256)

According to the duality (220), the identity (256) is recast as,

$$\tilde{m}^1 e_1 + \tilde{m}^2 e_2 + \tilde{m}^3 e_3 = \tilde{M}^{23} e_1 + \tilde{M}^{31} e_2 + \tilde{M}^{12} e_3$$

(257)

Thus,

$$\tilde{M}^{12} = \tilde{m}_3 \quad \text{and} \quad \tilde{M}^{13} = \tilde{M}^{31} = -\tilde{m}_2 \quad \text{and} \quad \tilde{M}^{23} = \tilde{m}_1$$

(258)

Using relations (258), the components of the electromagnetic polarisation bivector (255) are recast in vector space as,

$$\tilde{P}_{\mu\nu} = \begin{pmatrix}
0 & -\tilde{p}^3 & -\tilde{p}^2 & -\tilde{p}^3 \\
\tilde{p}^3 & 0 & c^{-1} \tilde{m}_3 & -c^{-1} \tilde{m}_2 \\
\tilde{p}^2 & -c^{-1} \tilde{m}_3 & 0 & c^{-1} \tilde{m}_1 \\
\tilde{p}^3 & c^{-1} \tilde{m}_2 & -c^{-1} \tilde{m}_3 & 0
\end{pmatrix}$$

(259)

The frame dependence of the electric polarisation vector $\tilde{m}$ and magnetisation bivector $\tilde{M}$ is made explicit using a geometric product with the time vector $e_0$.

In view of relation (230), the double geometric product of the electromagnetic polarisation multivector $P$ and the time vector $e_0$ yields,

$$e_0 \tilde{P} e_0 = \tilde{p}^i e_0 e_i e_0 - \frac{1}{2c} \tilde{M}^{ij} e_0 (e_i \wedge e_j) e_0$$

$$= -\tilde{p}^i e_i - \frac{1}{2c} \tilde{M}^{ij} (e_i \wedge e_j)$$

(260)

In view of relations (246), (247) and (242), we obtain,

$$e_0 \tilde{P} e_0 = -\tilde{p} - \frac{1}{c} \tilde{M} = -\tilde{P}^\dagger$$

(261)

Using relations (261), (127) and (128), the electric polarisation field vector $\tilde{p}$ is written entirely in terms of the electromagnetic polarisation multivector $P$ and the time vector $e_0$ as,

$$\tilde{p} = \frac{1}{2} \left( \tilde{P} - e_0 \tilde{P} e_0 \right)$$

(262)
and the magnetisation field bivector $\tilde{M}$ is written as,

$$\tilde{M} = -\frac{c}{2} (\tilde{P} + e_0 \tilde{P} e_0) \quad (263)$$

### 18. Stress energy momentum in vacuum (STA)

In the space-time algebra (STA) as opposed to to tensor calculus, the stress-energy momentum is not a tensor but a vector. In fact it is a self-adjoint linear application of a space-time vector $V$ in vacuum. If this vector $V$ is the space-time gradient $\nabla$, the stress-energy momentum vector $\dot{T}(\nabla)$ reduces to the electromagnetic force density vector $f$ in space-time as we show below. In order to show this, we begin by writing the electromagnetic momentum density $P$ in space-time. The electromagnetic momentum density $P$ is a contravariant vector in space-time defined as,

$$P = P^\mu e_\mu = P^0 e_0 + P^i e_i = \frac{e}{c} e_0 + \tilde{p}^i e_i \quad (264)$$

According to the gradient (154) and the identity (F.33) for the scalar product of two vectors in space-time, the divergence of the electromagnetic momentum density (264) is given by,

$$\nabla \cdot P = \frac{1}{c^2} \partial_i e + \nabla \cdot p \quad (265)$$

In view of relations (37), (162), and $e_0^2 = 1$, the inner product of the electric current density vector $J$ and the electromagnetic multivector $F$ is written as,

$$J \cdot F = (Je_0) e_0 \cdot F = (qc + j) e_0 \cdot (e + cB) \quad (266)$$

Since the time vector $e_0$ anticommutes with the vector $e$ and commutes with the bivector $B$, using the identity (A.19), relation (266) is recast as,

$$J \cdot F = - (j \cdot e) e_0 - c (q e + B \cdot j) e_0 \quad (267)$$

where $j \cdot e$ is the electromagnetic power density and $q e + B \cdot j$ is the Lorentz force density in spatial algebra $G^3$. This means that the vector $\frac{1}{c} J \cdot F$ in the space-time algebra $G^{1,3}$ represents the electromagnetic force density,

$$f = \frac{1}{c} J \cdot F \quad (268)$$
In view of relations (159) and (212) and identities (215) and (F.54), the electromagnetic force density (268) in space-time is given by,

\[ f_\mu e^\mu = \frac{1}{2c} J^\rho F_{\rho\mu} e_\mu \cdot (e^\nu \wedge e^\mu) = \frac{1}{2c} (J^\nu F_{\nu\mu} - J^\nu F_{\mu\nu}) e^\mu = \frac{1}{c} J^\nu F_{\nu\mu} e^\mu \]  
(269)

Using relations (208) and (211), the electromagnetic force density (268) in space-time is written in the orthonormal basis \{ e_0, e_1, e_2, e_3 \} as,

\[ f = \frac{1}{c} J \cdot F = \frac{1}{c} (J^i F_{i0}) e_0 + \frac{1}{c} (J^i F_{ij}) e_j \]  
(270)

In view of relations (217), (267) and (270) and identity (F.3),

\[ f \cdot e_0 = \frac{1}{c} (J \cdot F) \cdot e_0 = \frac{1}{c} (j^i e_i) = -\frac{1}{c} j \cdot e \]  
(271)

In view of relations (265) and (271), Poynting’s theorem in vacuum (85) is recast as,

\[ \nabla \cdot P = \frac{1}{c^2} (J \cdot F) \cdot e_0 = \frac{1}{c} f \cdot e_0 \]  
(272)

The geometric product of the electromagnetic momentum density (264) and the time vector \( e_0 \) yields,

\[ P e_0 = \frac{c}{c} (e_0 e_0) + \tilde{p}^i (e_i e_0) = \frac{1}{c} (e + p c) \]  
(273)

Since \( e_0^2 = 1 \), multiplying relation (273) by \( e_0 \) we obtain,

\[ P = \frac{1}{c} (e + p c) e_0 \]  
(274)

In view of relations (78) and (225),

\[ e + p c = -\frac{1}{4} (G e_0 F e_0 + F e_0 G e_0) = -\frac{1}{2} \varepsilon_0 F e_0 F e_0 \]  
(275)

and using the identity \( e_0^2 = 1 \), the momentum vector (274) is recast as,

\[ P = -\frac{1}{4c} (G e_0 F + F e_0 G) = -\frac{1}{2c} \varepsilon_0 F e_0 F \]  
(276)

The stress energy momentum vector \( T(e_0) \) is defined as a linear mapping of the time vector \( e_0 \),

\[ T(e_0) = P c = -\frac{1}{4} (G e_0 F + F e_0 G) = -\frac{1}{2} \varepsilon_0 F e_0 F \]  
(277)
Since the stress energy momentum vector \( T(e_0) \) depends on the time vector \( e_0 \), it is frame dependent. In view of relation (277), Poynting’s theorem (272) is recast as,

\[
\nabla \cdot T(e_0) = \frac{1}{c} (J \cdot F) \cdot e_0 = f \cdot e_0
\]

(278)

To generalise the specific stress energy momentum vector (277), a stress energy momentum vector \( T(V) \) can be defined as a linear mapping of a vector \( V \),

\[
T(V) = -\frac{1}{4} (GVF + FVG) = -\frac{1}{2} e_0 F V F
\]

(279)

The inner product of a vector \( U \) and the stress energy momentum vector \( T(V) \) yields a scalar,

\[
U \cdot T(V) = -\frac{1}{4} U \cdot (GVF + FVG) = -\frac{1}{4} \left( \langle UGVF \rangle + \langle UFVG \rangle \right)
\]

(280)

where the angle brackets denote the scalar part of the multivector. Similarly, the inner product of the vector \( V \) and the stress energy momentum vector \( T(U) \) also yields a scalar,

\[
V \cdot T(U) = -\frac{1}{4} V \cdot (GU + UF) = -\frac{1}{4} \left( \langle VGF \rangle + \langle VFU \rangle \right)
\]

(281)

Using the invariance under cyclic permutation of the scalar part of the geometric product of four multivectors (F.51), we obtain,

\[
\langle UGVF \rangle = \langle VFU \rangle \quad \text{and} \quad \langle UFVG \rangle = \langle VGF \rangle
\]

(282)

Thus, according to relations (280), (281) and (282) in vacuum, the stress energy momentum vector satisfies the symmetry, [21]

\[
U \cdot T(V) = V \cdot T(U)
\]

(283)

which means that \( T(V) \) is a self-adjoint linear application of \( V \). In particular, for \( U = \nabla \) and \( V = e_0 \), relation (283) becomes,

\[
\nabla \cdot T(e_0) = e_0 \cdot \hat{T}(\hat{\nabla}) = \hat{T}(\hat{\nabla}) \cdot e_0
\]

(284)

In view of relation (284), Poynting’s theorem (278) is recast as,

\[
\hat{T}(\hat{\nabla}) \cdot e_0 = \frac{1}{c} (J \cdot F) \cdot e_0 = f \cdot e_0
\]

(285)
which means that Poynting’s theorem is an explicit expression of the stress
energy momentum vector,

\[ \dot{T}(\nabla) = \frac{1}{c} J \cdot F = f \tag{286} \]

According to relation (279), the stress energy momentum of the gradient is given
by,

\[ \dot{T}(\nabla) = -\frac{1}{4} G \nabla F - \frac{1}{4} G \nabla \dot{F} - \frac{1}{4} \dot{F} \nabla G - \frac{1}{4} F \nabla \dot{G} \tag{287} \]

where the overdot denotes on which bivector the gradient operates to the left
or the right. The stress energy momentum vector (287) is the electromagnetic
force density vector in space-time. It is explicitly written as,

\[ \dot{T}(\nabla) = \frac{1}{4} (\nabla G) F - \frac{1}{4} G (\nabla F) + \frac{1}{4} (\nabla F) G - \frac{1}{4} F (\nabla G) \tag{288} \]

where the positive signs are due to the anticommutation between the vector
\(\nabla\) and the bivectors \(G\) and \(F\). Equivalently, in vacuum, using the electromag-
netic constitutive equation (42), relations (287) and (288) for the stress energy
momentum of the gradient are recast as, [21]

\[ \dot{T}(\nabla) = -\frac{1}{2} \varepsilon_0 F \nabla F - \frac{1}{2} \varepsilon_0 F \nabla \dot{F} = \frac{1}{2} \varepsilon_0 (\nabla F) F - \frac{1}{2} \varepsilon_0 F (\nabla F) \tag{289} \]

To show the consistency of our approach, we use the Maxwell equation (166)
and (171), the electromagnetic constitutive equation (42) and the identity (F.63)
in order to recast the stress energy momentum vector (288) as,

\[
\begin{align*}
\dot{T}(\nabla) &= \frac{1}{4} \left( \frac{1}{c} J \right) F - \frac{1}{4} \varepsilon_0 F \left( \frac{1}{c \varepsilon_0} J \right) + \frac{1}{4} \left( \frac{1}{c \varepsilon_0} J \right) \varepsilon_0 F - \frac{1}{4} F \left( \frac{1}{c} J \right) \\
&= \frac{1}{2c} (J F - F J) = \frac{1}{c} J \cdot F = f 
\end{align*}
\tag{290}
\]

We now show that we recover the expression of the components of the stress
energy momentum tensor in vacuum usually obtained in the framework of tensor
calculus. In view of identity (283), the symmetric components of the stress
energy momentum tensor are written as,

\[ T^{\mu \nu} = e^\mu \cdot T (e^\nu) = \langle e^\mu T (e^\nu) \rangle = \langle e^\nu T (e^\mu) \rangle = e^\nu \cdot T (e^\mu) = T^{\nu \mu} \tag{291} \]
In view of relation (279), relation (291) becomes,

\[ T_{\mu\nu} = -\frac{1}{4} \left( \langle e^\mu G e^\nu F \rangle + \langle e^\mu F e^\nu G \rangle \right) \]

\[ = -\frac{1}{4} \left( \langle e^\nu G e^\mu F \rangle + \langle e^\nu F e^\mu G \rangle \right) = T^{\nu\mu} \] (292)

Using the invariance under cyclic permutation of the scalar part of the geometric product of four multivectors (F.51), we obtain,

\[ \langle e^\mu G e^\nu F \rangle = \langle e^\nu F e^\mu G \rangle \quad \text{and} \quad \langle e^\mu F e^\nu G \rangle = \langle e^\nu G e^\mu F \rangle \] (293)

In view of identity (293), relation (292) is recast as,

\[ T^{\mu\nu} = -\frac{1}{4} \left( \langle e^\mu G e^\nu F \rangle + \langle e^\nu G e^\mu F \rangle \right) \] (294)

According to the symmetry (293), the scalar components (292) of the symmetric stress energy momentum tensor are the result of the inner product of vectors and the double inner product of bivectors,

\[ T^{\mu\nu} = -\frac{1}{2} (e^\mu \cdot G) \cdot (e^\nu \cdot F) -\frac{1}{2} (e^\nu \cdot G) \cdot (e^\mu \cdot F) + \frac{1}{2} (e^\mu \cdot e^\nu) G : F \] (295)

where the positive sign in front of the last term is due to the anticommutation of the vectors \( e^\mu \) and \( e^\nu \) with the bivector \( G \). In view of relations (215), (214), (234) and (233),

\[ e^\nu \cdot (e^\mu \cdot F) = F^{\mu\nu} = -F^{\nu\mu} = - (F \cdot e^\mu) \cdot e^\nu \]

\[ e^\nu \cdot (e^\mu \cdot G) = G^{\mu\nu} = -G^{\nu\mu} = - (G \cdot e^\mu) \cdot e^\nu \] (296)

which implies that,

\[ e^\mu \cdot F = -F \cdot e^\mu \quad \text{and} \quad e^\mu \cdot G = -G \cdot e^\mu \] (297)

In view of the symmetry (297) and identity (F.2), relation (295) reduces to,

\[ T^{\mu\nu} = \frac{1}{2} (e^\mu \cdot G) \cdot (F \cdot e^\nu) + \frac{1}{2} (e^\nu \cdot G) \cdot (F \cdot e^\mu) + \frac{1}{2} (e^\mu \cdot e^\nu) G : F \] (298)

50
According to relations (212), (215), (231), (234), (F.23), (F.53) and (F.56),

\[
e^\mu \cdot G = \frac{1}{2} G^{\rho \sigma} (e^\mu \cdot (e_\rho \wedge e_\sigma)) = \frac{1}{2} G^{\rho \sigma} \left( (e^\mu \cdot e_\rho) e_\sigma - (e^\mu \cdot e_\sigma) e_\rho \right).
\]

\[
e^\mu \cdot G = \frac{1}{2} G^{\mu \sigma} e_\sigma - \frac{1}{2} G^{\rho \mu} e_\rho = G^{\mu \sigma} e_\sigma
\]

\[
F \cdot e^\nu = \frac{1}{2} F^{\rho \sigma} (e_\rho \wedge e_\sigma) \cdot e^\nu = \frac{1}{2} F^{\rho \sigma} \left( (e^\nu \cdot e_\sigma) e_\rho - (e^\nu \cdot e_\rho) e_\sigma \right)
\]

\[
F \cdot e^\nu = \frac{1}{2} F^{\rho \sigma} e_\rho - \frac{1}{2} F^{\nu \sigma} e_\sigma = F^{\rho \nu} e_\rho
\]

In view of relations (215), (297), (299) and (F.2),

\[
(e^\mu \cdot F) \cdot (G \cdot e^\nu) = (G^{\mu \sigma} e_\sigma) \cdot (F^{\rho \nu} e_\rho) = G^{\mu \sigma} F^{\rho \nu} (e_\sigma \cdot e_\rho)
\]

\[
= G^{\mu \sigma} F^{\rho \nu} \eta_{\sigma \rho} = G^{\mu \rho} F^{\nu \nu}
\]

According to relations (212), (215), (231), (F.23) and (F.70),

\[
G : F = \frac{1}{4} G_{\rho \sigma} F^{\rho \mu} (e_\rho \wedge e_\sigma) \cdot (e^\mu \wedge e^\nu)
\]

\[
= \frac{1}{4} G_{\rho \sigma} F^{\rho \mu} \left( (e_\sigma \cdot e^\mu) (e_\rho \cdot e^\nu) - (e_\sigma \cdot e^\nu) (e_\rho \cdot e^\mu) \right)
\]

\[
= \frac{1}{4} \left( G^{\rho \sigma} F_{\rho \sigma} - G_{\rho \sigma} F^{\rho \sigma} \right) = - \frac{1}{2} G_{\rho \sigma} F^{\rho \sigma}
\]

In view of relations (300) and (301), the components of the stress energy momentum tensor (298) are explicitly given by,

\[
T^{\mu \nu} = G^{\mu \rho} F_{\rho \nu} - \frac{1}{4} \eta^{\mu \nu} G_{\rho \sigma} F^{\rho \sigma}
\]

or equivalently by lowering the second index of the components \(T^{\mu \lambda}\) with the Minkowski metric \(\eta_{\lambda \nu}\),

\[
T^\mu_\nu = G^{\mu \rho} F_{\rho \nu} - \frac{1}{4} \delta^\mu_\nu G^{\rho \sigma} F_{\rho \sigma}
\]

In vacuum, the components of the electromagnetic constitutive relation (42) are given by,

\[
e_\nu \cdot (e^\mu \cdot G) = \varepsilon_0 e_\nu \cdot (e^\mu \cdot F)
\]

In view of identities (214) and (233), it is recast as,

\[
G^{\mu \nu} = \varepsilon_0 F^\mu_\nu
\]
In vacuum, using the constitutive relation (305), the components of the stress energy momentum tensor (303) are recast as,

\[ T^\mu_\nu = \varepsilon_0 \left( F^{\mu\rho} F_{\rho\nu} - \frac{1}{4} \delta^\mu_\nu F^{\rho\sigma} F_{\rho\sigma} \right) \]  

(306)

19. Stress energy momentum in matter (STA)

In matter the stress-energy momentum is also a vector. As we show in this section, it is a self-adjoint linear application of a space-time vector \( V \) in matter as in vacuum. If this vector \( V \) is the space-time gradient \( \nabla \), the stress-energy momentum vector \( \dot{T}(\dot{\nabla}) \) reduces to the electromagnetic force density vector \( f \) in space-time as we show below. In spatial algebra (SA), the Poynting theorem in matter (148) has the same algebraic structure as the Poynting theorem in vacuum (85). Thus, in space-time algebra, the Poynting theorem in matter has the same structure the Poynting theorem in vacuum (278),

\[ \nabla \cdot T(e_0) = \frac{1}{c} (J \cdot F) \cdot e_0 = f \cdot e_0 \]  

(307)

where according to equations (274) and (277), the stress energy momentum tensor in matter \( T(e_0) \) is given by,

\[ T(e_0) = Pc = (e + pc) e_0 \]  

(308)

In view of the identities (133) and (225), we obtain,

\[ e + pc = -\frac{1}{4} (Ge_0 F e_0 F + F e_0 G e_0 F) - \frac{1}{8} (FG - GF + e_0 F G e_0 - e_0 G F e_0) \]  

(309)

Using relation (309), and the identity \( e_0^2 = 1 \), the stress energy momentum tensor in matter (308) is recast as,

\[ T(e_0) = -\frac{1}{4} (Ge_0 F + F e_0 G) - \frac{1}{8} (FG - GF) e_0 + e_0 (FG - GF) \]  

(310)

In view of the commutator (F.73) of the bivectors \( F \) and \( G \), the stress energy momentum vector (310) is recast as,

\[ T(e_0) = -\frac{1}{4} (Ge_0 F + F e_0 G) - \frac{1}{4} (F \times G) \cdot e_0 + e_0 \cdot (F \times G) \]  

(311)
Since $F \times G$ is a bivector and $e_0$ is a vector, according to identity (F.62),

$$(F \times G) \cdot e_0 = -e_0 \cdot (F \times G)$$  \hspace{1cm} (312)

which implies that the stress energy momentum vector (311) reduces to,

$$T(e_0) = -\frac{1}{4} (G e_0 F + F e_0 G)$$  \hspace{1cm} (313)

To generalise the specific stress energy momentum vector (310), a stress energy momentum vector $T(V)$ can be defined as a linear mapping of a vector $V$,

$$T(V) = -\frac{1}{4} (G V F + F V G)$$  \hspace{1cm} (314)

According to relations (280), (281) and (282) in matter, the stress energy momentum vector satisfies the symmetry,

$$U \cdot T(V) = V \cdot T(U)$$  \hspace{1cm} (315)

which means that $T(V)$ is a self-adjoint linear application of $V$. In particular, for $U = \nabla$ and $V = e_0$, relation (315) becomes,

$$\nabla \cdot T(e_0) = e_0 \cdot \hat{T}(\nabla) = \hat{T}(\nabla) \cdot e_0$$  \hspace{1cm} (316)

In view of relation (316), Poynting’s theorem (307) is recast as,

$$\hat{T}(\nabla) \cdot e_0 = \frac{1}{c} (J \cdot F) \cdot e_0 = f \cdot e_0$$  \hspace{1cm} (317)

which means that Poynting’s theorem is an explicit expression of the stress energy momentum vector,

$$\hat{T}(\nabla) = \frac{1}{c} J \cdot F = f$$  \hspace{1cm} (318)

According to relation (279), the stress energy momentum of the gradient is given by,

$$\hat{T}(\nabla) = -\frac{1}{4} \hat{G} \nabla F - \frac{1}{4} \hat{G} \nabla \hat{F} - \frac{1}{4} \hat{F} \nabla G - \frac{1}{4} \hat{F} \nabla \hat{G}$$  \hspace{1cm} (319)

where the overdot denotes on which bivector the gradient operates to the left or the right. The stress energy momentum vector (319) is the electromagnetic force density vector in space-time. It is explicitly written as,

$$\hat{T}(\nabla) = \frac{1}{4} (\nabla G) F - \frac{1}{4} G (\nabla F) + \frac{1}{4} (\nabla F) G - \frac{1}{4} F (\nabla G)$$  \hspace{1cm} (320)
where the positive signs are due to the anticommutation between the vector $\nabla$ and the bivectors $G$ and $F$. The electromagnetic susceptibility bivector in space-time $\chi_F (F)$ is a linear map of the electromagnetic field bivector $F$. Furthermore, we assume that the electromagnetic properties of the material medium are constant and uniform, which implies that,

$$F (\nabla \chi_F (F)) = \chi_F (F) (\nabla F)$$  \hspace{1cm} (321)

According to relations (152) and (321),

$$F \left( \nabla G (F) \right) = F \left( \nabla (F + \chi_F (F)) \right) = F (\nabla F) + F (\nabla \chi_F (F))$$  \hspace{1cm} (322)

$$G (F) (\nabla F) = \left( F + \chi_F (F) \right) (\nabla F) = F (\nabla F) + \chi_F (F) (\nabla F)$$

In view of relations (321) and (322), we deduce the following identity,

$$G (F) (\nabla F) = F \left( \nabla G (F) \right)$$  \hspace{1cm} (323)

Thus, the stress energy momentum vector (319) in matter reduces to,

$$\dot{T} (\dot{\nabla}) = \frac{1}{2} (\nabla G) F - \frac{1}{2} F (\nabla G)$$  \hspace{1cm} (324)

To show the consistency of our approach, we use the Maxwell equation (175), relation (318) and identity (F.63) to recast the stress energy momentum vector (324) as,

$$\dot{T} (\dot{\nabla}) = \frac{1}{2} \left( \frac{1}{c} J \right) F - \frac{1}{2} F \left( \frac{1}{c} J \right) = \frac{1}{2c} (J F - F J) = \frac{1}{c} (J \cdot F) = f$$

For an non-uniform material medium, an additional ponderomotive force density vector in space-time has to be added to the electromagnetic force density vector $f$ in space-time. Since the stress energy momentum vector in matter (320) has the same structure as the stress energy momentum vector in vacuum (288), the components of the stress energy momentum tensor in matter have the same structure as the components of the stress energy momentum vector in vacuum,

$$T^{\mu}{}_{\nu} = G^{\mu\rho} F_{\rho\nu} - \frac{1}{4} \delta^\mu_{\nu} G^{\rho\sigma} F_{\rho\sigma}$$  \hspace{1cm} (325)
In matter, the components of the electromagnetic constitutive relation (99) are given by,
\[ e_\nu \cdot (e^\mu \cdot G) = \varepsilon_0 e_\nu \cdot (e^\mu \cdot F) + \varepsilon_0 e_\nu \cdot \left( e^\mu \cdot \tilde{P} \right) \] (326)
In view of identities (214), (233) and (251), it is recast as,
\[ G^\mu_\nu = \varepsilon_0 F^\mu_\nu + \tilde{P}^\mu_\nu \] (327)
In matter, using the constitutive relation (327), the components of the stress energy momentum tensor (325) are recast as,
\[ T^\mu_\nu = \varepsilon_0 \left( F^{\mu\rho} F_{\rho\nu} - \frac{1}{4} \delta^\mu_\nu F^{\rho\sigma} F_{\rho\sigma} \right) \]
\[ + \tilde{P}^{\mu\rho} F_{\rho\nu} - \frac{1}{4} \delta^\mu_\nu \tilde{P}^{\rho\sigma} F_{\rho\sigma} \] (328)

20. Electromagnetic potential (STA)

In vector space (VS), the electric scalar potential \( \phi \) and the magnetic vector potential \( a \) are the temporal and spatial parts of a quadrivector \( A \). Similarly, in the space-time algebra (STA), we define an electromagnetic vector potential \( A \) in space-time and express the electromagnetic field bivector \( F \) in terms of \( A \).

The electromagnetic vector potential \( A \) is a contravariant vector in space-time defined as,
\[ A = A^\mu e_\mu = A^0 e_0 + A^i e_i = \frac{\phi}{c} e_0 + a^i e_i \] (329)

In view of relation (F.17), the inner product of the electromagnetic vector potential \( A \) with the time vector \( e_0 \) yields,
\[ A \cdot e_0 = e_0 \cdot A = A^0 = \frac{\phi}{c} \] (330)

According to relation (F.18), the outer product of the electromagnetic vector potential \( A \) with the time vector \( e_0 \) yields,
\[ A \wedge e_0 = -e_0 \wedge A = A^i (e_i \wedge e_0) = a^i e_i = a \] (331)

In view of the inner product (F.17) and the outer product (F.18), the geometric product of the electromagnetic vector potential \( A \) and the time vector \( e_0 \) yields,
\[ A e_0 = A \cdot e_0 + A \wedge e_0 = \frac{\phi}{c} + a \] (332)
Taking into account the properties (F.2) and (F.4), the geometric product in reverse order is given by,

\[ e_0 A = e_0 \cdot A + e_0 \wedge A = A \cdot e_0 - e_0 \wedge e_0 = \frac{\phi}{c} - a \quad (333) \]

In view of identity (F.36), the curl of the electromagnetic vector potential \( A \) is written as,

\[ \nabla \wedge A = -\frac{1}{c} \partial_t a - \nabla \left( \frac{\phi}{c} \right) + \nabla \cdot a \quad (334) \]

According to relation (334), the electromagnetic field bivector (89) is the curl of the electromagnetic potential up to a factor \( c \),

\[ F = c \nabla \wedge A \quad (335) \]

In view of identity (F.33), the divergence of the vector potential \( A \) is written as,

\[ \nabla \cdot A = \frac{1}{c} \partial_t \left( \frac{\phi}{c} \right) + \nabla \cdot a \quad (336) \]

In view of relations (336) and (90), the Lorentz gauge in space-time algebra is written as,

\[ \nabla \cdot A = 0 \quad (337) \]

In the Lorentz gauge (337), the electromagnetic field bivector (335) is recast as,

\[ F = c \nabla \wedge A = F = c \nabla A - c \nabla \cdot A = c \nabla A \quad (338) \]

Thus, in the Lorentz gauge (337), the gradient of the electromagnetic field bivector (338) is given by,

\[ \nabla F = c \nabla^2 A \quad (339) \]

In view of relations (339) and (31), the Maxwell equation in vacuum (172) is recast in terms of the electromagnetic potential vector \( A \) as,

\[ \nabla^2 A = \mu_0 J \quad (340) \]

In view of relations (339) and (31), the Maxwell equation in matter (187) is recast in terms of the electromagnetic potential vector \( A \) as,

\[ \nabla^2 A = \mu_0 \tilde{J} \quad (341) \]
21. Conclusion

Geometric algebra (GA) has been the subject of intense study for the past half century [22, 15, 21, 18, 23]. Electrodynamics has been examined in spatial algebra (SA) and space-time algebra (STA) by numerous authors [15, 21, 17, 24, 20, 25]. This prompts the following question: “Is there any need to discuss the matter further?” The answer is “yes” for two main reasons. First, the magnetic induction field $\mathbf{b}$ and the auxiliary magnetic field $\mathbf{h}$ are treated in general as pseudovectorial fields by the vast majority of authors with the notable exception of MacDonald [18] who introduced magnetic induction field bivector $\mathbf{B}$ and the auxiliary magnetic field bivector $\mathbf{H}$ but he did not investigate the foundations of electromagnetism in spatial algebra (SA) or space-time (STA) further. Since the magnetic fields clearly geometrically as bivectors, it is necessary to treat them accordingly and this has been done in this publication. Second, the electric polarisation and magnetisation in matter was not studied in depth so far in spatial algebra (SA) and space-time (STA). Although, there was an attempt by Arthur [20] to describe electromagnetism in matter in spatial algebra (SA) but these results remain inconclusive.

In this publication, we considered an electric polarisation vector $\tilde{\mathbf{p}}$ induced linearly by the electric field vector $\mathbf{e}$ and a magnetisation bivector $\tilde{\mathbf{M}}$ induced linearly by the magnetic induction field bivector $\mathbf{B}$.

First, in the spatial algebra (SA), we introduced a polarisation multivector $\tilde{\mathbf{P}} = \tilde{\mathbf{p}} - \frac{1}{c} \tilde{\mathbf{M}}$ and an auxiliary electromagnetic field multivector $\tilde{\mathbf{G}} = \varepsilon_0 \mathbf{F} + \tilde{\mathbf{P}}$ we were able to express the Maxwell equation in matter (175) very elegantly. Using the bound electric charge density $\tilde{q}$ and the bound electric current density $\tilde{\mathbf{j}}$, the Maxwell equation (187) could then be written entirely in terms of the electromagnetic field multivector $\mathbf{F}$. This allowed us to derive the electromagnetic wave equation in matter (120) straightforwardly in the spatial algebra (SA). Writing the electromagnetic energy density $\mathbf{e}$ and momentum density $\mathbf{p}$ in terms of the electromagnetic multivectors $\mathbf{F}, \mathbf{G}$ and $\tilde{\mathbf{P}}$, we show that the electromagnetic polarisation multivector $\tilde{\mathbf{P}}$ is a linear map of the electromagnetic...
multivector $F$ written as $\tilde{P}(F) = \varepsilon_0 \chi(F)$ where $\chi(F)$ is the electromagnetic susceptibility multivector.

Second, in the space-time algebra (STA), the electromagnetic multivectors $F$, $G$ and $\tilde{P}$ defined in the spatial algebra (SA) could be interpreted as bivectors in the space-time algebra (STA). Writing the Maxwell equation (175) in terms of the auxiliary electromagnetic field multivector $G$, the stunningly elegant equation $\nabla G = \frac{1}{c} J$ had the same analytical expression as in vacuum (166). By introducing a bound current vector $\tilde{J} = J - c \nabla \cdot \tilde{P}$ in space-time, the Maxwell equation in matter (187) could be written also in terms of the electromagnetic field bivector $F$ as $\nabla F = \frac{1}{\varepsilon_0 c} \tilde{J}$. In the space-time algebra (STA), the wave equation in matter (201) is obtained by simply taking the gradient of the Maxwell equation in matter (175), namely $\nabla^2 G = \frac{1}{c} \nabla \wedge J$ since $\nabla \cdot J = 0$, or equivalently $\nabla^2 F = \frac{1}{\varepsilon_0 c} \nabla \wedge \tilde{J}$ since $\nabla \cdot \tilde{J} = 0$. The stress energy momentum vector $T(V)$ in space-time (STA) describes the energy flux across the hypersurface orthogonal to $V$. For a uniform electromagnetic medium consisting of induced electric and magnetic dipoles, the stress-energy momentum vector is a linear mapping (314) of the vector $V$ given by $T(V) = - \frac{1}{4} (GVF + FVG)$. It turns out that the stress energy momentum vector has the same analytical structure in matter as in vacuum (279). Moreover, for the gradient $V = \nabla$, the stress energy momentum vector (318) is written as $\dot{T}(\dot{V}) = \frac{1}{c} J \cdot F = f$ where $f$ is the electromagnetic force density vector in space-time. Finally, in the space-time algebra (STA), the Maxwell equation in matter (341) can be written as a wave equation for the potential vector $A$, namely $\nabla^2 A = \mu_0 \tilde{J}$ since $F = \nabla \wedge A$.

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A. Spatial algebra (SA)

We consider an orthonormal vector spatial frame \{e_1, e_2, e_3\}. The geometric product of two basis vectors reads, [21]

\[ e_i e_j = e_i \cdot e_j + e_i \wedge e_j \quad \text{with} \quad i, j = 1, 2, 3 \] (A.1)

where,

\[ e_1^2 = e_2^2 = e_3^2 = 1 \quad \text{and} \quad e_i \wedge e_i = 0 \] (A.2)

The inner product of two basis vectors is symmetric and defined as,

\[ e_i \cdot e_j = e_j \cdot e_i = \frac{1}{2} (e_i e_j + e_j e_i) \] (A.3)

and the outer product is antisymmetric,

\[ e_i \wedge e_j = -e_j \wedge e_i = \frac{1}{2} (e_i e_j - e_j e_i) \] (A.4)

The 8 basis elements of the geometric algebra (GA) \( \mathbb{G}^3 \) called the spatial algebra are :

- one scalar : 1
- three vectors : \( e_1, e_2, e_3 \)
- three bivectors : \( e_1 \wedge e_2, e_2 \wedge e_3, e_3 \wedge e_1 \)
- one pseudoscalar : \( e_1 \wedge e_2 \wedge e_3 \)

where the number of elements of each type 1, 3, 3, 1 are the binomial coefficients in three dimensions. In \( \mathbb{G}^3 \), there are \( 2^3 \) basis elements. The geometric interpretation of the basis elements is clear. A scalar \( s \) is oriented point, a vector is an oriented line element, a bivector is an oriented plane element and a pseudoscalar is an oriented volume element. The notation used is the following : scalars \( s \) are written in lower case letters, vectors \( v \) are written in lower case bold letters, bivectors \( B \) are written in upper case bold letters and pseudoscalars are written as \( s'I \). A multivector \( M \), which is linear combination of elements of \( \mathbb{G}^3 \), is written in upper case letters,

\[ M = s + v + B + s'I \] (A.5)
Taking into account the definition (A.1), the bivectors and the pseudoscalar are also written as geometric products of the basis unit vectors $e_1, e_2, e_3$:

- bivectors: $e_1 e_2, e_2 e_3, e_3 e_1$
- pseudoscalar: $e_1 e_2 e_3$

The pseudoscalar $I = e_1 e_2 e_3$ behaves as an imaginary number since,

$$I^2 = (e_1 e_2 e_3) (e_1 e_2 e_3) = (e_1 e_2 e_1 e_2) (e_3 e_3) = - (e_1 e_1) (e_2 e_2) = -1$$  \hspace{1cm} (A.6)

The geometric product of two basis vectors $u$ and $v$ reads,

$$uv = u \cdot v + u \wedge v$$  \hspace{1cm} (A.7)

The inner product between the two basis vectors is symmetric,

$$u \cdot v = v \cdot u = \frac{1}{2} (uv + vu)$$  \hspace{1cm} (A.8)

and the outer product is antisymmetric,

$$u \wedge v = - v \wedge u = \frac{1}{2} (uv - vu)$$  \hspace{1cm} (A.9)

The geometric product of a vector $v$ and a bivector $B$ is the sum of the inner product of the outer product,

$$vB = v \cdot B + v \wedge B$$  \hspace{1cm} (A.10)

and the geometric product of a bivector $B$ and a vector $v$ is the sum of the inner product of the outer product,

$$BV = B \cdot v + B \wedge v$$  \hspace{1cm} (A.11)

To determine the symmetry of these products, the spatial frame is oriented such that the bivector reads $B = B_{12} e_1 \wedge e_2 = B_{12} e_1 e_2$ and the orientation of the vector $v = v_1 e_1 + v_2 e_2 + v_3 e_3$ is arbitrary. The geometric product of the vector $v$ and the bivector $B$ reads,

$$vB = (v_1 e_1 + v_2 e_2 + v_3 e_3) (B_{12} e_1 e_2)$$
$$= v_1 B_{12} e_1 e_1 e_2 + v_2 B_{12} e_2 e_1 e_2 + v_3 B_{12} e_3 e_1 e_2$$
$$= B_{12} v_1 e_2 - B_{12} v_2 e_1 + B_{12} v_3 e_1 e_2$$  \hspace{1cm} (A.12)
The geometric product of the bivector $B$ and the vector $v$ reads,

\[
B v = (B_{12} e_1 e_2) (v_1 e_1 + v_2 e_2 + v_3 e_3)
\]
\[
= B_{12} v_1 e_1 e_2 e_1 + B_{12} v_2 e_1 e_2 e_2 + B_{12} v_3 e_1 e_2 e_3
\]
\[
= - B_{12} v_1 e_2 + B_{12} v_2 e_1 + B_{12} v_3 e_1 e_2 e_3
\]  
(A.13)

The outer product of the vector $v$ and the bivector $B$ reads,

\[
v \wedge B = (v_1 e_1 + v_2 e_2 + v_3 e_3) \wedge (B_{12} e_1 \wedge e_2)
\]
\[
= v_1 B_{12} e_1 \wedge e_1 \wedge e_2 + v_2 B_{12} e_2 \wedge e_1 \wedge e_2 + v_3 B_{12} e_3 \wedge e_1 \wedge e_2
\]
\[
= v_3 B_{12} e_1 \wedge e_2 \wedge e_3
\]  
(A.14)

The outer product of the bivector $B$ and the vector $v$ reads,

\[
B \wedge v = (B_{12} e_1 \wedge e_2) \wedge (v_1 e_1 + v_2 e_2 + v_3 e_3)
\]
\[
= B_{12} v_1 e_1 \wedge e_2 \wedge e_1 + B_{12} v_2 e_1 \wedge e_2 \wedge e_2 + B_{12} v_3 e_1 \wedge e_2 \wedge e_3
\]
\[
= B_{12} v_3 e_1 \wedge e_2 \wedge e_3
\]  
(A.15)

From relations (A.14) and (A.15) we conclude that the outer product of a vector $v$ and a bivector $B$ is symmetric,

\[
v \wedge B = B \wedge v
\]  
(A.16)

In view of relations (A.10), (A.12) and (A.14),

\[
v \cdot B = v B - v \wedge B = - B_{12} v_1 e_1 + B_{12} v_2 e_2
\]  
(A.17)

In view of relations (A.11), (A.13) and (A.15),

\[
B \cdot v = B v - B \wedge v = B_{12} v_2 e_1 - B_{12} v_1 e_2
\]  
(A.18)

From relations (A.17) and (A.18) we conclude that the inner product of a vector $v$ and a bivector $B$ is antisymmetric,

\[
v \cdot B = - B \cdot v
\]  
(A.19)

In view of relations (A.12) and (A.13), (A.16) and (A.19),

\[
v \cdot B = \frac{1}{2} (v B - B v)
\]  
(A.20)
According to relations (A.23) and (A.25), the symmetric inner product of two bivectors \( A \) and \( B \), we write them as
\[
A = A_{12} e_1 e_2 + A_{23} e_2 e_3 + A_{31} e_3 e_1 \quad \text{and} \quad B = B_{12} e_1 e_2 + B_{23} e_2 e_3 + B_{31} e_3 e_1.
\]
The geometric product of the bivectors \( A \) and \( B \) reads,
\[
AB = (A_{12} e_1 e_2 + A_{23} e_2 e_3 + A_{31} e_3 e_1) (B_{12} e_1 e_2 + B_{23} e_2 e_3 + B_{31} e_3 e_1)
\]
which reduces to,
\[
AB = -(A_{12} B_{12} + A_{23} B_{23} + A_{31} B_{31}) + (A_{31} B_{23} - A_{23} B_{31}) e_1 e_2 + (A_{12} B_{31} - A_{31} B_{12}) e_2 e_3 + (A_{23} B_{12} - A_{12} B_{23}) e_3 e_1 \quad \text{(A.23)}
\]
The geometric product of the bivectors \( B \) and \( A \) reads,
\[
BA = (B_{12} e_1 e_2 + B_{23} e_2 e_3 + B_{31} e_3 e_1) (A_{12} e_1 e_2 + A_{23} e_2 e_3 + A_{31} e_3 e_1)
\]
which reduces to,
\[
BA = -(A_{12} B_{12} + A_{23} B_{23} + A_{31} B_{31}) + (A_{31} B_{23} - A_{23} B_{31}) e_1 e_2 + (A_{12} B_{31} - A_{31} B_{12}) e_2 e_3 + (A_{23} B_{12} - A_{12} B_{23}) e_3 e_1 \quad \text{(A.25)}
\]
The geometric product (A.23) of two bivectors \( A \) and \( B \) is the sum an inner product and a cross product,
\[
AB = A \cdot B + A \times B \quad \text{(A.26)}
\]
According to relations (A.23) and (A.25), the symmetric inner product of two bivectors \( A \) and \( B \) yields a scalar,
\[
A \cdot B = B \cdot A = \frac{1}{2} (AB + BA) = -(A_{12} B_{12} + A_{23} B_{23} + A_{31} B_{31}) \quad \text{(A.27)}
\]
According to relations (A.23) and (A.25), the antisymmetric cross product of two bivectors $A$ and $B$ yields a bivector,

$$A \times B = -B \times A = \frac{1}{2} (A B - B A)$$

$$= (A_{31} B_{23} - A_{23} B_{31}) e_1 e_2 + (A_{12} B_{31} - A_{31} B_{12}) e_2 e_3$$

$$+ (A_{23} B_{12} - A_{12} B_{23}) e_3 e_1$$

(A.28)

The fact that an inner product of two bivectors yields a scalar and the cross product of two bivectors yields a bivector is specific to the spatial algebra $G^3$. To determine the geometric product of a vector $v$ and the pseudoscalar $I$, we write them as $B = B_{12} e_1 e_2 + B_{23} e_2 e_3 + B_{31} e_3 e_1$ and $I = e_1 e_2 e_3$. The geometric product of the vector $v$ and the pseudoscalar $I$ reads,

$$v I = (v_1 e_1 + v_2 e_2 + v_3 e_3) (e_1 e_2 e_3) = v_1 e_2 e_3 + v_2 e_3 e_1 + v_3 e_1 e_2$$

(A.29)

$$I v = (e_1 e_2 e_3) (v_1 e_1 + v_2 e_2 + v_3 e_3) = v_1 e_2 e_3 + v_2 e_3 e_1 + v_3 e_1 e_2$$

(A.30)

According to relation (A.29) and (A.30), the vector $v$ commutes with the pseudoscalar $I$,

$$v I = I v$$

(A.31)

To determine the geometric product of a vector $v$ and the pseudoscalar $I$, we write them as $v = v_1 e_1 + v_2 e_2 + v_3 e_3$ and $I = e_1 e_2 e_3$. The geometric product of the bivector $B$ and the pseudoscalar $I$ reads,

$$B I = (B_{12} e_1 e_2 + B_{23} e_2 e_3 + B_{31} e_3 e_1) (e_1 e_2 e_3)$$

$$= -(B_{23} e_1 + B_{31} e_2 + B_{12} e_3)$$

(A.32)

$$I B = (e_1 e_2 e_3) (B_{12} e_1 e_2 + B_{23} e_2 e_3 + B_{31} e_3 e_1) = -B_{12} e_3$$

$$= -(B_{23} e_1 + B_{31} e_2 + B_{12} e_3)$$

(A.33)

According to relation (A.32) and (A.33), the bivector $B$ commutes with the pseudoscalar $I$,

$$B I = I B$$

(A.34)
B. Duality in spatial algebra (SA)

The reverse of the scalar \( s \), the vector \( v = v_3 e_3 \), of the bivector \( B = B_{12} e_1 e_2 \) and of the pseudoscalar \( I \) is obtained by reversing the order of the basis vectors, \(^{[18]}\)

\[
\begin{align*}
  s^\dagger &= s \quad \text{(B.1)} \\
  v^\dagger &= v_3 e_3 = v \quad \text{(B.2)} \\
  B^\dagger &= B_{12} e_2 e_1 = - B_{12} e_1 e_2 = - B \quad \text{(B.3)} \\
  I^\dagger &= e_3 e_2 e_1 = - e_1 e_2 e_3 = - I \quad \text{(B.4)}
\end{align*}
\]

Thus, the reverse of the multivector (A.5) is given by,

\[
M^\dagger = s^\dagger + v^\dagger + B^\dagger + s' I^\dagger = s + v - B - s' I \quad \text{(B.5)}
\]

The square of the modulus of a vector \( v \), a bivector \( B \) and a pseudovector \( I \) are defined as,

\[
\begin{align*}
  |v|^2 &= v^\dagger \cdot v = v \cdot v = v^2 \quad \text{(B.6)} \\
  |B|^2 &= B^\dagger \cdot B = -B \cdot B = -B^2 \quad \text{(B.7)} \\
  |I|^2 &= I^\dagger \cdot I = -I \cdot I = -I^2 = 1 \quad \text{(B.8)}
\end{align*}
\]

The geometric interpretation of the modulus is clear: the modulus of a vector \(|v|\) is the length of a line element, the modulus of a bivector \(|B|\) is the surface of a plane element and the modulus of a pseudoscalar \(|s'I|\) is the volume of a space element. Thus, the modulus of the bivector obtained by taking the outer product of a vector \( u \) and a vector \( v \) is the surface of the parallelogram spanned by these vectors,

\[
|u \wedge v| = |u| \ |v| \sin \theta \quad \text{(B.9)}
\]

where \( \theta \) is the acute angle between these vectors. This modulus is the same as the modulus of the cross product of these vectors,

\[
|u \times v| = |u| \ |v| \sin \theta \quad \text{(B.10)}
\]
Thus,

\[ |u \wedge v| = |u \times v| \]  

(B.11)

The inverse of a vector \( v \), a bivector \( B \) and a pseudovector \( I \) are defined as,

\[ v^{-1} = \frac{v}{v^2} = \frac{v^\dagger}{|v|^2} \]  

(B.12)

\[ B^{-1} = \frac{B}{B^2} = \frac{B^\dagger}{|B|^2} \]  

(B.13)

\[ I^{-1} = \frac{I}{I^2} = \frac{I^\dagger}{|I|^2} = -I \]  

(B.14)

The dual of a vector \( v \), a bivector \( B \) and a pseudovector \( I \) are defined as, [18]

\[ v^* = \frac{v}{I} = v I^{-1} = -v I \]  

(B.15)

\[ B^* = \frac{B}{I} = B I^{-1} = -B I \]  

(B.16)

\[ I^* = \frac{I}{I} = I I^{-1} = 1 \]  

(B.17)

This duality is the transformation as the Hodge duality in differential forms.

The dual of the dual of a vector \( v \), a bivector \( B \) and a pseudovector \( I \) are their opposite,

\[ (v^*)^* = -v^* I = v I^2 = -v \]  

(B.18)

\[ (B^*)^* = -B^* I = B I^2 = -B \]  

(B.19)

\[ (I^*)^* = -I^* I = I I^2 = -I \]  

(B.20)

The dual of the vector \( v = v_3 e_4 \) is,

\[ v^* = -v I = -(v_3 e_3) (e_1 e_2 e_3) = -v_3 e_1 e_2 \]  

(B.21)

Defining the bivector \( V \) as,

\[ V = v_{12} e_1 e_2 \]  

where \(|V| = |v|\) and thus \( v_{12} = v_3 \)  

(B.22)

it satisfies the duality,

\[ v^* = -V = (V^*)^* \]  

and thus \( V^* = v \)  

(B.23)
The dual of the bivector \( B = B_{12} e_1 e_2 \) is,

\[
B^* = -B I = -(B_{12} e_1 e_2)(e_1 e_2 e_3) = B_{12} e_3
\]  

(B.24)

Defining the vector \( b \) as,

\[
b = b_3 e_3 \quad \text{where} \quad |b| = |B| \quad \text{and thus} \quad b_3 = B_{12}
\]  

(B.25)

it satisfies the duality,

\[
B^* = b = -(b^*)_* \quad \text{and thus} \quad b^* = -B
\]  

(B.26)

There is a duality between a vector and a bivector of same modulus and there is a duality between a scalar and a pseudoscalar of same modulus. In view of relations (B.15) and (B.26),

\[
B = -b^* = b I
\]  

(B.27)

which implies that by duality the multivector (A.5) is recast as, [15]

\[
M = (s + s'I) + (v + bI)
\]  

(B.28)

and the reverse of the multivector (B.5) is recast as,

\[
M^\dagger = (s - s'I) + (v - bI)
\]  

(B.29)

which means that the reverse of a multivector is like a complex conjugate where the pseudoscalar \( I \) is like the imaginary number \( i \). The bivector \( W \) and the vector \( w \) are defined as the wedge product and the cross product of two vectors \( u \) and \( v \) respectively,

\[
W = u \wedge v \quad \text{and} \quad w = u \times v
\]  

(B.30)

According to relation (B.11) the modulus of the bivector \( W \) and vector \( w \) are equal, which means that the vector \( w \) is the dual of the bivector \( W \),

\[
|W| = |w| \quad \text{and thus} \quad W^* = w \quad \text{and} \quad w^* = -W
\]  

(B.31)
Thus, the cross product of the vectors $u$ and $v$ is the dual of the wedge product of these vectors,

$$(u \wedge v)^* = u \times v \quad \text{and} \quad (u \times v)^* = -u \wedge v \quad (B.32)$$

To establish the duality between the inner and outer product of two vectors, we choose a vector $u = u_1 e_1 + u_2 e_2$ and a vector $v = v_1 e_1 + v_2 e_2$ in the same plane. The dual of the outer product of the vectors $u$ and $v$ yields,

$$(u \wedge v)^* = -\left( (u_1 e_1 + u_2 e_2) \wedge (v_1 e_1 + v_2 e_2) \right) (e_1 e_2 e_3)$$

$$= - (u_1 v_2 e_1 e_2 + u_2 v_1 e_2 e_1) (e_1 e_2 e_3)$$

$$= (u_1 v_2 - u_2 v_1) e_3 \quad (B.33)$$

The inner product of the vector $u$ and $v^*$ yields,

$$u \cdot v^* = - (u_1 e_1 + u_2 e_2) \cdot \left( (v_1 e_1 + v_2 e_2) (e_1 e_2 e_3) \right)$$

$$= - (u_1 e_1 + u_2 e_2) \cdot (v_1 e_2 e_3 - v_2 e_1 e_3)$$

$$= (u_1 v_2 - u_2 v_1) e_3 \quad (B.34)$$

The identification of relations (B.33) and (B.34) yields the vectorial duality,

$$(u \wedge v)^* = u \cdot v^* \quad (B.35)$$

which is recast in terms of the dual bivector $V = -v^*$ as,

$$(u \wedge v)^* = -u \cdot V = V \cdot u \quad (B.36)$$

In view of relations (B.32) and (B.35), we obtain,

$$u \times v = u \cdot v^* \quad (B.37)$$

which is recast in terms of the dual bivector $V = -v^*$ as,

$$u \times v = -u \cdot V = V \cdot u \quad (B.38)$$

The dual of the inner product of the vectors $u$ and $v$ yields,

$$(u \cdot v)^* = -\left( (u_1 e_1 + u_2 e_2) \cdot (v_1 e_1 + v_2 e_2) \right) (e_1 e_2 e_3)$$

$$= - (u_1 v_1 + u_2 v_2) e_1 e_2 e_3 \quad (B.39)$$
The outer product of the vectors $u$ and $v^*$ yields,

$$u \wedge v^* = - (u_1 e_1 + u_2 e_2) \wedge \left( (v_1 e_1 + v_2 e_2) (e_1 e_2 e_3) \right)$$

$$= - (u_1 e_1 + u_2 e_2) \wedge (v_1 e_2 e_3 - v_2 e_1 e_3)$$

$$= (u_1 v_1 + u_2 v_2) e_1 e_2 e_3$$

(B.40)

The identification of relations (B.39) and (B.40) yields the pseudoscalar duality,

$$(u \cdot v)^* = u \wedge v^*$$

(B.41)

which is recast in terms of the dual bivector $V = -v^*$ as,

$$(u \cdot v)^* = -u \wedge V$$

(B.42)

To establish the duality between the inner and outer product of a bivector and a vector, the spatial frame is oriented such that the bivector is given by $B = B_{12} e_1 e_2$ and the vector is written as $v = v_1 e_1 + v_2 e_2 + v_3 e_3$. The dual of the outer product of the vector $u$ and the bivector $B$ yields,

$$(u \wedge B)^* = - \left( (u_1 e_1 + u_2 e_2 + u_3 e_3) \wedge (B_{12} e_1 e_2) \right) (e_1 e_2 e_3)$$

$$= - (u_3 B_{12} e_1 e_2 e_3) (e_1 e_2 e_3)$$

$$= u_3 B_{12}$$

(B.43)

The inner product of the vector $u$ and the bivector $B^*$ yields,

$$u \cdot B^* = - (u_1 e_1 + u_2 e_2 + u_3 e_3) \cdot \left( (B_{12} e_1 e_2) (e_1 e_2 e_3) \right)$$

$$= (u_1 e_1 + u_2 e_2 + u_3 e_3) \cdot (B_{12} e_3)$$

$$= u_3 B_{12}$$

(B.44)

The identification of relations (B.43) and (B.44) yields the scalar duality,

$$(u \wedge B)^* = u \cdot B^*$$

(B.45)

which is recast in terms of the dual vector $b = B^*$ as,

$$(u \wedge B)^* = u \cdot b$$

(B.46)
and is the dual of the pseudoscalar duality (B.42) for \( b = v \) and \( B = V \). The dual of the inner product of the vector \( u \) and the bivector \( B \) yields,

\[
(u \cdot B)^* = -\left( (u_1 e_1 + u_2 e_2 + u_3 e_3) \cdot (B_{12} e_1 e_2) \right) (e_1 e_2 e_3)
\]

\[
= - (u_1 B_{12} e_2 - u_2 B_{12} e_1) (e_1 e_2 e_3)
\]

\[
= u_1 B_{12} e_1 e_3 + u_2 B_{12} e_2 e_3
\]

(B.47)

The outer product of the vector \( u \) and the dual of the bivector \( B^* \) yields,

\[
u \wedge B^* = -(u_1 e_1 + u_2 e_2 + u_3 e_3) \wedge \left( (B_{12} e_1 e_2) (e_1 e_2 e_3) \right)
\]

\[
= (u_1 e_1 + u_2 e_2 + u_3 e_3) \wedge (B_{12} e_3)
\]

\[
= u_1 B_{12} e_1 e_3 + u_2 B_{12} e_2 e_3
\]

(B.48)

The identification of relations (B.47) and (B.48) yields the bivectorial duality,

\[
(u \cdot B)^* = u \wedge B^*
\]

(B.49)

which is recast in terms of the dual vector \( b = B^* \) as,

\[
(u \cdot B)^* = u \wedge b
\]

(B.50)

and is the dual of the vectorial duality (B.36) for \( b = v \) and \( B = V \). To establish the duality between the inner and outer product of bivectors in the same plane, the spatial frame is oriented such that the bivectors \( A \) and \( B \) are written as \( A = A_{12} e_1 e_2 \) and \( B = B_{12} e_2 e_3 \). The dual of the inner product of the bivectors \( A \) and \( B \) yields,

\[
(A \cdot B)^* = -\left( (A_{12} e_1 e_2) \cdot (B_{12} e_1 e_2) \right) (e_1 e_2 e_3)
\]

\[
= A_{12} B_{12} e_1 e_2 e_3
\]

(B.51)

The outer product of the bivectors \( A \) and \( B^* \) yields,

\[
A \wedge B^* = -(A_{12} e_1 e_2) \wedge \left( (B_{12} e_1 e_2) (e_1 e_2 e_3) \right)
\]

\[
= (A_{12} e_1 e_2) \wedge (B_{12} e_3)
\]

\[
= A_{12} B_{12} e_1 e_2 e_3
\]

(B.52)
The identification of relations (B.51) and (B.52) yields the pseudoscalar duality,

\[(A \cdot B)^* = A \wedge B^* \] (B.53)

The pseudoscalar duality (B.53) is expressed in terms of the dual vector \( b = B^* \),

\[(A \cdot B)^* = A \wedge b \] (B.54)

In view of the identities (A.16), (B.49), the inner product of two bivectors \( A \) and \( B \) is expressed in terms of the dual vectors \( a = A^* \) and \( b = B^* \) as,

\[ (A \cdot B)^* = A \wedge b = b \wedge A = -b \wedge a^* \] (B.55)

The dual of identity (B.55) yields,

\[ A \cdot B = -a \cdot b \] (B.56)

C. Differential duality in spatial algebra (SA)

For the gradient operator \( u = \nabla \), the duality (B.32) for the curl of a vector \( v \) is expressed as,

\[ (\nabla \wedge v)^* = \nabla \times v \quad \text{and} \quad (\nabla \times v)^* = -\nabla \wedge v \] (C.1)

and the vectorial duality (B.35) yields,

\[ (\nabla \wedge v)^* = \nabla \cdot v^* \] (C.2)

In view of relations (C.1) and (C.2), we obtain,

\[ \nabla \times v = \nabla \cdot v^* \] (C.3)

which is recast in terms of the dual bivector \( V = -v^* \) as,

\[ \nabla \times v = -\nabla \cdot V \] (C.4)

For the gradient operator \( u = \nabla \), the pseudoscalar duality (B.36) is expressed as,

\[ (\nabla \cdot v)^* = \nabla \wedge v^* \] (C.5)
which is recast in terms of the dual bivector $V = -v^*$ as,

$$(\nabla \cdot v)^* = -\nabla \wedge V \tag{C.6}$$

For the gradient operator $u = \nabla$, the scalar duality (B.45) is expressed as,

$$(\nabla \wedge B)^* = \nabla \cdot B^* \tag{C.7}$$

which is recast in terms of the dual vector $b = B^*$ as,

$$(\nabla \wedge B)^* = \nabla \cdot b \tag{C.8}$$

and is the dual of the pseudoscalar duality (C.6) for $b = v$ and $B = V$. For the gradient operator $u = \nabla$, the bivectorial duality (B.49) is expressed as,

$$(\nabla \cdot B)^* = \nabla \wedge B^* \tag{C.9}$$

which is recast in terms of the dual vector $b = B^*$ as,

$$(\nabla \cdot B)^* = \nabla \wedge b \tag{C.10}$$

and is the dual of the vectorial duality (C.4) for $b = v$ and $B = V$.

D. Algebraic identities in spatial algebra (SA)

The double cross product of the three vectors $u$, $v$ and $w$ is written as,

$$u \times (v \times w) = (u \cdot w) v - (u \cdot v) w \tag{D.1}$$

In view of the duality (B.32) between the cross and wedge products, the vectorial duality (B.37), the double duality (B.19) and the antisymmetry of the outer product (A.9), the left-hand side of relation (D.1) is recast as,

$$u \times (v \times w) = u \times (v \wedge w)^* = u \cdot \left( (v \wedge w)^* \right)^* = -u \cdot (v \wedge w) = u \cdot (w \wedge v) \tag{D.2}$$

According to the identity (D.2), the double cross product (D.1) yields the triple product,

$$u \cdot (v \wedge w) = (u \cdot v) w - (u \cdot w) v \tag{D.3}$$
This triple product (D.3) represents the projection of the vector \( \mathbf{u} \) on the bivector \( \mathbf{v} \wedge \mathbf{w} \). The vectors \( \mathbf{u}, \mathbf{v} \) and \( \mathbf{w} \) satisfy the identity,

\[
\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) \tag{D.4}
\]

In view of the duality (B.32) between cross product and wedge product and the scalar duality (B.45), the three terms of identity (D.4) are recast as,

\[
\begin{align*}
\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) &= \mathbf{u} \cdot (\mathbf{v} \wedge \mathbf{w})^* = (\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w})^* \\
\mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) &= \mathbf{v} \cdot (\mathbf{w} \wedge \mathbf{u})^* = (\mathbf{v} \wedge \mathbf{w} \wedge \mathbf{u})^* \\
\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) &= \mathbf{w} \cdot (\mathbf{u} \wedge \mathbf{v})^* = (\mathbf{w} \wedge \mathbf{u} \wedge \mathbf{v})^*
\end{align*}
\tag{D.5}
\]

According to the relations (D.5), the scalar identity (D.4) becomes,

\[
(\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w})^* = (\mathbf{v} \wedge \mathbf{w} \wedge \mathbf{u})^* = (\mathbf{w} \wedge \mathbf{u} \wedge \mathbf{v})^* \tag{D.6}
\]

The dual of the scalar identity (D.6) yields the pseudoscalar identity,

\[
\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w} = \mathbf{v} \wedge \mathbf{w} \wedge \mathbf{u} = \mathbf{w} \wedge \mathbf{u} \wedge \mathbf{v} \tag{D.7}
\]

This identity represents the fact the oriented volume of the parallelepiped spanned by the vectors \( \mathbf{u}, \mathbf{v}, \mathbf{w} \) is the same as that of the parallelepiped spanned by the vectors \( \mathbf{v}, \mathbf{w}, \mathbf{u} \) and also the same as that of the parallelepiped spanned by the vectors \( \mathbf{w}, \mathbf{u}, \mathbf{v} \) because these three parallelepipeds have the same orientation. According to relations (A.20) and (A.21) for \( \mathbf{B} = \mathbf{v} \wedge \mathbf{w} \), the inner and outer products of the vector \( \mathbf{u} \) and the bivector \( \mathbf{v} \wedge \mathbf{w} \) are given by,

\[
\begin{align*}
\mathbf{u} \cdot (\mathbf{v} \wedge \mathbf{w}) &= \frac{1}{2} \left( \mathbf{u} (\mathbf{v} \wedge \mathbf{w}) - (\mathbf{v} \wedge \mathbf{w}) \mathbf{u} \right) \tag{D.8} \\
\mathbf{u} \wedge (\mathbf{v} \wedge \mathbf{w}) &= \frac{1}{2} \left( \mathbf{u} (\mathbf{v} \wedge \mathbf{w}) + (\mathbf{v} \wedge \mathbf{w}) \mathbf{u} \right) \tag{D.9}
\end{align*}
\]

To determine the mixed product of two vectors \( \mathbf{u}, \mathbf{v} \) and a bivector \( \mathbf{B} \), we use the dual vector defined as \( \mathbf{b} = -\mathbf{B} I \). Using the identity (A.19) and the triple product of vectors (D.4), we obtain,

\[
(\mathbf{u} \wedge \mathbf{v}) \cdot \mathbf{B} = (\mathbf{u} \wedge \mathbf{v}) \cdot (\mathbf{b} I) = -\mathbf{b} \cdot (\mathbf{u} \wedge \mathbf{v}) \mathbf{I} = -\mathbf{I} (\mathbf{b} \cdot \mathbf{u}) \mathbf{v} + \mathbf{I} (\mathbf{b} \cdot \mathbf{v}) \mathbf{u} \tag{D.10}
\]
which reduces to,

\[(u \wedge v) \cdot B = (B \cdot u) \cdot v - (B \cdot v) \cdot u = u \cdot (v \cdot B) - v \cdot (u \cdot B)\]  \hspace{1cm} (D.11)

To determine the first kind of mixed product of two bivectors \(A, B\) and a vector \(v\), we use the dual vectors defined as \(a = -A I\) and \(b = -B I\). Using the identities (A.19) and \(I^2 = -1\), we obtain,

\[(A \wedge v) \cdot B = (a I \wedge v) \cdot b I = -(a \wedge v) \cdot b = b \cdot (a \wedge v)\]  \hspace{1cm} (D.12)

Using the triple product of vectors (D.4), the mixed product is recast as,

\[(A \wedge v) \cdot B = b \cdot (a \wedge v) = (a \cdot b) v - (b \cdot v) a\]

\[= -(A \cdot B) v + (B \cdot v) A\]  \hspace{1cm} (D.13)

To determine the first kind of mixed product of two bivectors \(A, B\) and a vector \(v\), we use the dual vector defined as \(a = A^* = -A I\) and the duality \((v \wedge a)^* = -(v \wedge a) I\). Using the identities (A.19), (B.35) and \(I^2 = -1\), we obtain,

\[(A \cdot v) \wedge B = -(v \cdot A) \wedge B = (v \cdot a^*) \wedge B = (v \wedge a)^* \wedge B\]

\[= -(v \wedge a) I \wedge B = v \wedge (A \wedge B)\]  \hspace{1cm} (D.14)

E. Differential algebraic identities in spatial algebra (SA)

For the gradient operator \(u = \nabla\), the geometric product (A.7) implies that the gradient of the vector \(v\) is the sum of its divergence and its curl,

\[\nabla v = \nabla \cdot v + \nabla \wedge v\]  \hspace{1cm} (E.1)

For the gradient operator \(v = \nabla\), the geometric product (A.10) implies that the gradient of the bivector \(B\) is the sum of its divergence and its curl,

\[\nabla B = \nabla \cdot B + \nabla \wedge B\]  \hspace{1cm} (E.2)

The gradient of the product of two scalars \(r\) and \(s\) is written as,

\[\nabla (rs) = r \nabla s + s \nabla r\]  \hspace{1cm} (E.3)
The divergence of the product of a scalar \(s\) and a vector \(v\) reads,

\[
\nabla (s v) = s \nabla \cdot v + v \cdot \nabla s \quad (E.4)
\]

The curl of the product of a scalar \(s\) and a vector \(v\) reads,

\[
\nabla \times (s v) = s \nabla \times v - v \times \nabla s \quad (E.5)
\]

In view of the duality (B.32) between the cross and wedge products, relation (E.5) is recast as,

\[
\left( \nabla \wedge (s v) \right)^* = (s \nabla \wedge v)^* - (v \wedge \nabla s)^* \quad (E.6)
\]

The dual of relation (E.6) is given by,

\[
\nabla \wedge (s v) = s \nabla \wedge v - v \wedge \nabla s \quad (E.7)
\]

The gradient of the inner product of two vectors \(u\) and \(v\) is written as,

\[
\nabla (u \cdot v) = u \times (\nabla \times v) + v \times (\nabla \times u) + (u \cdot \nabla) v + (v \cdot \nabla) u \quad (E.8)
\]

In view of the duality (B.32) between the cross and wedge products, the vectorial duality (B.37), the double duality (B.19) and the antisymmetry of the inner product (A.19), the first two terms on the right-hand side of relation (E.8) are recast as,

\[
\begin{align*}
    u \times (\nabla \times v) &= u \times (\nabla \wedge v)^* = u \cdot \left( (\nabla \wedge v)^* \right)^* = -u \cdot (\nabla \wedge v) = (\nabla \wedge v) \cdot u \\
    v \times (\nabla \times u) &= v \times (\nabla \wedge u)^* = v \cdot \left( (\nabla \wedge u)^* \right)^* = -v \cdot (\nabla \wedge u) = (\nabla \wedge u) \cdot v
\end{align*}
\]  

(E.9)

According to relations (E.9), the gradient (E.8) of the inner product of vectors \(u\) and \(v\) is recast as,

\[
\nabla (u \cdot v) = (\nabla \wedge u) \cdot v + (\nabla \wedge v) \cdot u + (u \cdot \nabla) v + (v \cdot \nabla) u \quad (E.10)
\]

The curl of the cross product of two vectors \(u\) and \(v\) is written as,

\[
\nabla \times (u \times v) = - (\nabla \cdot u) v + (\nabla \cdot v) u - (u \cdot \nabla) v + (v \cdot \nabla) u \quad (E.11)
\]
In view of the duality (B.32) between the cross and wedge products, the vectorial duality (B.37) and the double duality (B.19), the left-hand side of relation (E.11) is recast as,

$$\nabla \times (u \times v) = \nabla \times (u \wedge v)^* = \nabla \cdot \left( (u \wedge v)^* \right)^* = - \nabla \cdot (u \wedge v) \quad \text{(E.12)}$$

According to relation (E.12), the curl (E.11) of the cross product of two vectors $u$ and $v$ is recast as the divergence of the outer product of these vectors,

$$\nabla \cdot (u \wedge v) = (\nabla \cdot u) v - (\nabla \cdot v) u + (u \cdot \nabla) v - (v \cdot \nabla) u \quad \text{(E.13)}$$

The divergence of the cross product of two vectors $u$ and $v$ is written as,

$$\nabla \cdot (u \times v) = - u \cdot (\nabla \times v) + v \cdot (\nabla \times u) \quad \text{(E.14)}$$

In view of the duality (B.32) between the cross and wedge products, the vectorial duality (B.37) and the double duality (B.19), the three terms of relation (E.14) are recast as,

$$\nabla \cdot (u \times v) = \nabla \cdot (u \wedge v)^* = \left( \nabla \wedge (u \wedge v) \right)^*$$

$$u \cdot (\nabla \times v) = u \cdot (\nabla \wedge v)^* = \left( u \wedge (\nabla \wedge v) \right)^* \quad \text{(E.15)}$$

$$v \cdot (\nabla \times u) = v \cdot (\nabla \wedge u)^* = \left( v \wedge (\nabla \wedge u) \right)^*$$

According to relations (E.15), the divergence (E.14) of the cross product of two vectors $u$ and $v$ is recast as the dual of the curl of the wedge product of these vectors,

$$\left( \nabla \wedge (u \wedge v) \right)^* = - \left( u \wedge (\nabla \wedge v) \right)^* + \left( v \wedge (\nabla \wedge u) \right)^* \quad \text{(E.16)}$$

The dual of relation (E.16) yields,

$$\nabla \wedge (u \wedge v) = - u (\wedge \nabla \wedge v) + v \wedge (\nabla \wedge u) \quad \text{(E.17)}$$

In view of the identities (B.35), (C.2) and (C.7), the divergence of the inner product of a vector $v$ and a bivector $B$ is recast in terms of the dual vector $b = B^*$ as,

$$\nabla \cdot (v \cdot B) = - \nabla \cdot (v \cdot b^*) = - \nabla \cdot (v \wedge b)^* = - \left( \nabla \wedge (v \wedge b) \right)^* \quad \text{(E.18)}$$
Taking into account relation (E.16), relation (E.18) becomes,

\[ \nabla \cdot (v \cdot B) = \left( v \wedge (\nabla \wedge b) \right)^* - \left( b \wedge (\nabla \wedge v) \right)^* \quad (E.19) \]

According to relations (B.35) and (C.2), the first term on the right-hand side of relation (E.19) is recast as,

\[ \left( v \wedge (\nabla \wedge b) \right)^* = v \cdot (\nabla \wedge b)^* = v \cdot (\nabla \cdot b) = -v \cdot (\nabla \cdot B) \quad (E.20) \]

According to relations (A.16) and (B.35) the second term on the right-hand side of relation (E.19) is recast as,

\[ \left( b \wedge (\nabla \wedge v) \right)^* = \left( (\nabla \wedge v) \wedge b \right)^* = (\nabla \wedge v) \cdot b^* = - (\nabla \wedge v) \cdot B \quad (E.21) \]

In view of relations (E.20) and (E.21), the divergence (E.19) of the inner product of a vector \( v \) and a bivector \( B \) becomes,

\[ \nabla \cdot (v \cdot B) = (\nabla \wedge v) \cdot B - v \cdot (\nabla \cdot B) \quad (E.22) \]

According to the antisymmetry (A.19) of the inner product of a vector and a bivector and the symmetry (A.27) of the inner product of two bivectors, the identity (E.22) is recast as,

\[ \nabla \cdot (B \cdot v) = (\nabla \cdot B) \cdot v - B \cdot (\nabla \wedge v) \quad (E.23) \]

In view of identities (B.35), (C.2) and (C.9), the curl of the inner product of a vector \( v \) and a bivector \( B \) is recast in terms of the dual vector \( b = B^* \) as,

\[ \nabla \wedge (v \cdot B) = - \nabla \wedge (v \cdot b^*) = - \nabla \wedge (v \wedge b)^* = - \left( \nabla \cdot (v \wedge b) \right)^* \quad (E.24) \]

Taking into account relation (E.13), relation (E.24) becomes,

\[ \nabla \wedge (v \cdot B) = - (\nabla \cdot v) b^* + (\nabla \cdot b) v^* - (v \cdot \nabla) b^* + (b \cdot \nabla) v^* \quad (E.25) \]

which is recast as,

\[ \nabla \wedge (v \cdot B) = (\nabla \cdot v) B + (\nabla \cdot B^*) v^* + (v \cdot \nabla) B + (B^* \cdot \nabla) v^* \quad (E.26) \]
In view of the dualities (C.7), (B.15), (B.16) and (A.6),

\[
\begin{align*}
(\nabla \cdot B^*) v^* &= (\nabla \wedge B)^* v^* = (\nabla \wedge B) I I v = - (\nabla \wedge B) \cdot v \\
(B^* \cdot \nabla) v^* &= (B \wedge \nabla)^* v^* = (B \wedge \nabla) I I v = - (B \wedge \nabla) \cdot v
\end{align*}
\] (E.27) (E.28)

Using the identities (E.27) and (E.28), the curl of the inner product of a vector \(v\) and a bivector \(B\) (E.26) is recast as,

\[
\nabla \wedge (v \cdot B) = (\nabla \wedge B) \cdot v + (v \cdot \nabla) B - (\nabla \wedge B) \cdot v - (B \wedge \nabla) \cdot v
\] (E.29)

According to the antisymmetry (A.19) of the inner product of a vector and a bivector, the identity (E.29) is recast as,

\[
\nabla \wedge (B \cdot v) = (\nabla \wedge B) \cdot v + (B \wedge \nabla) \cdot v - (\nabla \cdot v) B - (v \cdot \nabla) B
\] (E.30)

According to the duality (B.56) and the differential identity (E.10), the divergence of a bivector \(A\) with a bivector \(B\) is expressed in terms of the dual vectors \(a = A^*\) and \(b = A^*\) as,

\[
\nabla (A \cdot B) = - (\nabla \wedge a) \cdot b - (\nabla \wedge b) \cdot a - (a \cdot \nabla) b - (b \cdot \nabla) a
\] (E.31)

In view of the identities (C.9), (A.19), (B.35), (A.9), (B.49), we obtain the relations,

\[
\begin{align*}
(\nabla \wedge a) \cdot b &= (\nabla \wedge A^*) \cdot b = (\nabla \cdot A)^* \cdot b = - b \cdot (\nabla \cdot A)^* = - \left(b \wedge (\nabla \cdot A)\right)^* \\
&= \left((\nabla \cdot A) \wedge b\right)^* = \left((\nabla \cdot A) \wedge B^\ast\right)^* = - (\nabla \cdot A) \cdot B = B \cdot (\nabla \cdot A)
\end{align*}
\]

and

\[
\begin{align*}
(\nabla \wedge b) \cdot a &= (\nabla \wedge B^*) \cdot a = (\nabla \cdot B)^* \cdot a = - a \cdot (\nabla \cdot B)^* = - \left(a \wedge (\nabla \cdot B)\right)^* \\
&= \left((\nabla \cdot B) \wedge a\right)^* = \left((\nabla \cdot B) \wedge A^\ast\right)^* = - (\nabla \cdot B) \cdot A = A \cdot (\nabla \cdot B)
\end{align*}
\]

which implies that the divergence (E.31) of a bivector \(A\) with a bivector \(B\) is recast as,

\[
\nabla (A \cdot B) = - A \cdot (\nabla \cdot B) - B \cdot (\nabla \cdot A) - (A^* \cdot \nabla) B^* - (B^* \cdot \nabla) A^*
\] (E.32)
According to the dualities (C.7), (B.16) and (A.6),

\[(A^* \cdot \nabla) B^* = (A \wedge \nabla)^* B^* = (A \wedge \nabla) I I B = - (A \wedge \nabla) \cdot B \quad \text{(E.33)}\]
\[(B^* \cdot \nabla) A^* = (B \wedge \nabla)^* A^* = (B \wedge \nabla) I I A = - (B \wedge \nabla) \cdot A \quad \text{(E.34)}\]

In view of the identities (E.33) and (E.34), the divergence (E.32) of a bivector \(A\) with a bivector \(B\) is recast as,

\[\nabla \cdot (A \cdot B) = - A \cdot (\nabla \cdot B) - B \cdot (\nabla \cdot A) + (A \wedge \nabla) \cdot B + (B \wedge \nabla) \cdot A \quad \text{(E.35)}\]

The curl of the curl of a vector \(v\) vanishes,

\[\nabla \wedge (\nabla \wedge v) = e^i \partial_i \wedge (e^j \partial_j \wedge v) = (e^i \wedge e^j) \partial_i \partial_j v = 0 \quad \text{(E.36)}\]

In view of relations, (C.2), (C.7) and (E.36), the divergence of the divergence of a bivector \(B\) with a dual vector \(b = B^*\) vanishes,

\[\nabla \cdot (\nabla \cdot B) = - \nabla \cdot (\nabla \cdot b^*) = - \nabla \cdot (\nabla \wedge b)^* = - \left(\nabla \wedge (\nabla \wedge b)\right)^* = 0 \quad \text{(E.37)}\]

F. Space-time algebra (STA)

We consider an orthonormal vector frame \(\{e_0, e_1, e_2, e_3\}\) in space-time. The geometric product of two basis vectors reads, [15]

\[e_\mu e_\nu = e_\mu \cdot e_\nu + e_\mu \wedge e_\nu \quad \text{where} \quad \mu, \nu = 0, 1, 2, 3 \quad \text{(F.1)}\]

The inner product between two basis vectors is symmetric and defined by the Minkowski metric with the signature convention \((+,-,-,-)\),

\[e_\mu \cdot e_\nu = \eta_{\mu\nu} = 2 \delta_{\mu 0} \delta_{\nu 0} - \delta_{\mu\nu} \quad \text{(F.2)}\]

which implies that,

\[e_0^2 = 1, \quad e_i^2 = -1, \quad e_0 \cdot e_i = 0, \quad e_i \cdot e_j = - \delta_{ij} \quad \text{(F.3)}\]

and the outer product is antisymmetric,

\[e_\mu \wedge e_\nu = - e_\nu \wedge e_\mu \quad \text{(F.4)}\]
The definitions of the geometric product (F.1), the inner product (F.2) and the outer product (F.4) yield the anti-commutation relation of the space-time algebra $G^{1,3}$,

$$\frac{1}{2} (e_\mu e_\nu + e_\nu e_\mu) = \eta_{\mu\nu} \tag{F.5}$$

which is the Dirac algebra. A relative spatial orthonormal vector frame \(\{e_1, e_2, e_3\}\) attached to an observer consists of space-time bivectors defined as, [15]

$$e_i = e_i e_0 = e_i \wedge e_0 \quad \text{where} \quad i = 1, 2, 3 \tag{F.6}$$

The geometric product of two basis spatial vectors reads,

$$e_i e_j = e_i \cdot e_j + e_i \wedge e_j \tag{F.7}$$

In view of relations (F.3), (F.4) and (F.6), the symmetric inner product of the spatial orthonormal basis vectors yields,

$$e_i \cdot e_j = \frac{1}{2} (e_i \cdot e_j + e_j \cdot e_i) = \frac{1}{2} \left( (e_i e_0) (e_j e_0) + (e_j e_0) (e_i e_0) \right) \tag{F.8}$$

$$= -\frac{1}{2} \left( (e_i e_j) e_0 e_0 + (e_j e_i) e_0 e_0 \right) = -\frac{1}{2} (e_i \cdot e_j + e_j \cdot e_i) = \delta_{ij}$$

and the pseudoscalar is given by,

$$I = e_1 e_2 e_3 = (e_1 e_0) (e_2 e_0) (e_3 e_0) = (e_0 e_1 e_2 e_3) (e_0 e_0) = e_0 e_1 e_2 e_3 \tag{F.9}$$

In view of relations (F.3), (F.4) and (F.6), the geometric product of two different spatial basis vectors yields,

$$e_1 e_2 = (e_1 e_2 e_3) e_3 = I e_3 = -e_2 e_1$$

$$e_2 e_3 = e_1 (e_1 e_2 e_3) = e_1 I = I e_1 = -e_3 e_2$$

$$e_3 e_1 = - (e_1 e_3) (e_2 e_2) = (e_1 e_2 e_3) e_2 = I e_2 = -e_1 e_3 \tag{F.10}$$

which can be written as,

$$e_i e_j = \varepsilon_{ijk} I e_k \quad \text{where} \quad i \neq j \neq k \neq i \tag{F.11}$$
The inner product (F.1) and the geometric product (F.11) yield the anti-commutation and commutation relations of the spatial algebra $\mathbb{G}^3$,

\begin{align*}
\frac{1}{2} (e_i e_j + e_j e_i) &= \delta_{ij} \\
\frac{1}{2} (e_i e_j - e_j e_i) &= \epsilon_{ijk} I e_k
\end{align*}

(F.12)

which is the Pauli algebra. The spatial algebra $\mathbb{G}^3$ (Pauli algebra) is the even subalgebra of the space-time algebra $\mathbb{G}^{1,3}$ (Dirac algebra). In view of relations (F.3) and (F.4), the pseudoscalar anticommutes with the basis vectors in space-time,

\begin{align*}
I e_0 &= (e_0 e_1 e_2 e_3) e_0 = - e_0 (e_0 e_1 e_2 e_3) = - e_0 I \\
I e_1 &= (e_0 e_1 e_2 e_3) e_1 = - e_1 (e_0 e_1 e_2 e_3) = - e_1 I \\
I e_2 &= (e_0 e_1 e_2 e_3) e_2 = - e_2 (e_0 e_1 e_2 e_3) = - e_2 I \\
I e_3 &= (e_0 e_1 e_2 e_3) e_3 = - e_3 (e_0 e_1 e_2 e_3) = - e_3 I
\end{align*}

(F.13)

In view of relations (F.3) and (F.4), the pseudoscalar commutes with the basis bivector in space-time,

\begin{align*}
I e_0 e_1 &= (e_0 e_1 e_2 e_3) (e_0 e_1) = (e_0 e_1) (e_0 e_1 e_2 e_3) = e_0 e_1 I \\
I e_0 e_2 &= (e_0 e_1 e_2 e_3) (e_0 e_2) = (e_0 e_2) (e_0 e_1 e_2 e_3) = e_0 e_2 I \\
I e_0 e_3 &= (e_0 e_1 e_2 e_3) (e_0 e_3) = (e_0 e_3) (e_0 e_1 e_2 e_3) = e_0 e_3 I \\
I e_1 e_2 &= (e_0 e_1 e_2 e_3) (e_1 e_2) = (e_1 e_2) (e_0 e_1 e_2 e_3) = e_1 e_2 I \\
I e_2 e_3 &= (e_0 e_1 e_2 e_3) (e_2 e_3) = (e_2 e_3) (e_0 e_1 e_2 e_3) = e_2 e_3 I \\
I e_3 e_1 &= (e_0 e_1 e_2 e_3) (e_3 e_1) = (e_3 e_1) (e_0 e_1 e_2 e_3) = e_3 e_1 I
\end{align*}

(F.14)

In view of relations (F.3) and (F.4) and (F.13), the pseudoscalar anticommutes with the basis trivector in space-time,

\begin{align*}
I e_0 e_1 e_2 &= I^2 e_3 = - I e_3 I = - e_0 e_1 e_2 I \\
I e_1 e_2 e_3 &= I e_0 I = - e_0 I^2 = - e_1 e_2 e_3 I \\
I e_2 e_3 e_0 &= (e_0 e_1 e_2 e_3) (e_2 e_3 e_0) = - (e_2 e_3 e_0) (e_0 e_1 e_2 e_3) = - e_2 e_3 e_0 I
\end{align*}

(F.15)

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Ie_3e_0e_1 = (e_0e_1e_2e_3)(e_3e_0e_1) = −(e_3e_0e_1)(e_0e_1e_2e_3) = −e_3e_0e_1 I

A contravariant vector \( V \) in space-time is written in coordinates with respect to the orthonormal vector frame \( \{ e_0, e_1, e_2, e_3 \} \) as,

\[
V = V^\mu e_\mu = V^0 e_0 + V^i e_i \quad \text{where} \quad i = 1, 2, 3
\]  

(F.16)

where we used the Einstein summation convention. In view of relation (F.3), the inner product of the vector \( V \) with the time vector \( e_0 \) yields,

\[
V \cdot e_0 = V^0 (e_0 \cdot e_0) + V^i (e_i \cdot e_0) = V^0
\]  

(F.17)

In view of relations (F.3) and (F.6), the outer product of the vector \( V \) with the time vector \( e_0 \) is given by,

\[
V \wedge e_0 = V^0 (e_0 \wedge e_0) + V^i (e_i \wedge e_0) = V^i e_i = \nu
\]  

(F.18)

In view of the inner product (F.17) and the outer product (F.18), the geometric product of the contravariant vector \( V \) and the time vector \( e_0 \) yields,

\[
Ve_0 = V \cdot e_0 + V \wedge e_0 = V^0 + \nu
\]  

(F.19)

Taking into account the properties (F.2) and (F.4), the geometric product in reverse order is given by,

\[
e_0 V = e_0 \cdot V + e_0 \wedge V = V \cdot e_0 - V \wedge e_0 = V^0 - \nu
\]  

(F.20)

According to relations (F.13), the contravariant vector \( V = V^\mu e_\mu \) anticommutes with the pseudoscalar \( I \),

\[
VI = −IV
\]  

(F.21)

The orthonormal comoving vector frame \( \{ e^0, e^1, e^2, e^3 \} \) in space-time is related to the orthonormal vector frame \( \{ e_0, e_1, e_2, e_3 \} \) through the Minkowski metric,

\[
e_\mu = \eta_{\mu \nu} e^\nu \quad \text{and} \quad e^\mu = \eta^{\mu \nu} e_\nu
\]  

(F.22)

Thus, in view of the definition (F.2), the inner product of two different basis vectors is written as,

\[
e^\mu \cdot e_\nu = \eta^{\mu \rho} e_\rho \cdot e_\nu = \eta^{\mu \nu} \eta_{\rho \nu} = \delta^\mu_\nu
\]  

(F.23)
and the outer product of two different basis vectors is written as,

\[ e^\mu \wedge e^\nu = \eta^{\mu\rho} e^\rho \wedge e^\nu \]  \hspace{1cm} (F.24)

A relative spatial comoving orthonormal vector frame \( \{ e^1, e^2, e^3 \} \) attached to an observer consists of space-time bivectors defined as,

\[ e^i = e^i e^0 = e^i \wedge e^0 \]  \hspace{1cm} (F.25)

In view of the identities (F.22) and (F.24) and the definitions (F.6) and (F.25) of the basis vectors, a comoving basis vector is the opposite of the corresponding basis vector,

\[ e^i = e^i \wedge e^0 = \eta^{ij} \eta^{00} e^j \wedge e_0 = - \delta^{ij} e_j = - e_i \]  \hspace{1cm} (F.26)

A covariant vector \( U \) in space-time is written in coordinates with respect to the comoving orthonormal vector frame \( \{ e^0, e^1, e^2, e^3 \} \) as,

\[ U = U_\mu e^\mu = U_0 e^0 + U_i e^i \]  \hspace{1cm} (F.27)

where we used the Einstein summation convention. In view of relation (F.23), the inner product of the vector \( U \) with the time vector \( e_0 \) yields,

\[ U \cdot e_0 = U_0 (e^0 \cdot e_0) + U_i (e^i \cdot e_0) = U_0 \]  \hspace{1cm} (F.28)

In view of relations (F.3) and (F.24), the outer product of the vector \( U \) with the time vector \( e_0 \) is given by,

\[ U \wedge e_0 = U_0 (e^0 \wedge e_0) + U_i (e^i \wedge e_0) \]
\[ = U_i \eta_{00} (e^i \wedge e^0) = U_i e^i = - U^i e_i = - u \]  \hspace{1cm} (F.29)

In view of the inner product (F.28) and the outer product (F.29), the geometric product of the covariant vector \( U \) and the time vector \( e_0 \) yields,

\[ U e_0 = U \cdot e_0 + U \wedge e_0 = U_0 - u \]  \hspace{1cm} (F.30)

Taking into account the properties (F.2) and (F.4), the geometric product in reverse order is given by,

\[ e_0 U = e_0 \cdot U + e_0 \wedge U = U \cdot e_0 - U \wedge e_0 = U_0 + u \]  \hspace{1cm} (F.31)
In view of relations (F.16), (F.27) and (F.23), the inner product of the covariant vector \( U \) with the contravariant vector \( V \) yields,

\[
U \cdot V = \left( U_0 e^0 + U_i e^i \right) \cdot \left( V^0 e_0 + V^j e_j \right) \\
= U_0 V^0 (e^0 \cdot e_0) + U_i V^j (e^i \cdot e_j) = U_0 V^0 + U_i V^i
\]  

(F.32)

According to relations (F.8), (F.18) and (F.29), the inner product (F.32) is recast as,

\[
U \cdot V = U_0 V^0 + U_i V^j (e^i \cdot e_j) = U_0 V^0 + u \cdot v 
\]  

(F.33)

In view of relations (F.4), (F.24), (F.16) and (F.27), the outer product of the covariant vector \( U \) with the contravariant vector \( V \) yields,

\[
U \wedge V = \left( U_0 e^0 + U_i e^i \right) \wedge \left( V^0 e_0 + V^j e_j \right) \\
= U_0 V^j (e^0 \wedge e_j) + U_i V^0 (e^i \wedge e_0) + U_i V^j (e^i \wedge e_j) \\
= -U^0 V^j (e_j \wedge e_0) - V^0 U^i (e_i \wedge e_0) - U^i V^j (e_i \wedge e_j)
\]  

(F.34)

According to the identities (F.3), (F.4) and (F.6), the outer product of two relative spatial vectors,

\[
e_i \wedge e_j = (e_i \wedge e_0) \wedge (e_j \wedge e_0) = (e_i e_0) (e_j e_0) \\
= - (e_i e_j) (e_0 e_0) = -e_i e_j = -e_i \wedge e_j
\]  

(F.35)

According to relations (F.6), (F.18), (F.29) and (F.35) the outer product (F.34) is recast as,

\[
U \wedge V = -U^0 (V^j e_j) - (U^i e_i) V^0 + (U^i e_i) \wedge (V^j e_j) \\
= -U^0 v - u V^0 + u \wedge v
\]  

(F.36)

In view of relations (F.16), (F.27) and (F.23), the scalar part of the geometric product of two vectors \( U \) and \( V \), denoted by angle brackets, is written as,

\[
\langle U V \rangle = \langle U_\mu \epsilon^\mu V^\nu \epsilon_\nu \rangle = U_\mu V^\nu \langle \epsilon^\mu \epsilon_\nu \rangle = U_\mu V^\nu \epsilon^\mu \cdot \epsilon_\nu = U_\mu V^\mu \\
\langle V U \rangle = \langle V^\mu \epsilon_\mu U_\nu \epsilon^\nu \rangle = V^\mu U_\nu \langle \epsilon_\mu \epsilon^\nu \rangle = V^\mu U_\nu \epsilon_\mu \cdot \epsilon^\nu = V^\mu U_\mu
\]  

(F.37)

Thus, the scalar part of this geometric product is symmetric,

\[
\langle U V \rangle = \langle V U \rangle
\]  

(F.38)
We now consider a contravariant multivector in the space-time algebra $\mathbb{G}^{1,3}$ written as,

$$M = m + M^\mu e_\mu + \frac{1}{2} M^{\mu\nu} e_\mu \wedge e_\nu + \frac{1}{6} M^{\mu\nu\rho} e_\mu \wedge e_\nu \wedge e_\rho + m' I \quad \text{(F.39)}$$

and a covariant multivector given by,

$$N = n + N_\mu e^\mu + \frac{1}{2} N_{\mu\nu} e^{\mu} \wedge e^{\nu} + \frac{1}{6} N_{\mu\nu\rho} e^{\mu} \wedge e^{\nu} \wedge e^{\rho} + n' I \quad \text{(F.40)}$$

Using the algebraic relation (F.11), the pseudoscalar (F.9) and the identities (F.22) and (F.23) for different indices $i$, $j$ and $k$, we obtain,

$$e_i \wedge e_0 = - e_0 \wedge e_i = e_i$$
$$e^i \wedge e^0 = - e^0 \wedge e^i = \eta^{i0} \eta^{ij} e_j \wedge e_0 = - e_i$$
$$e_i \wedge e_j = - (e_i \wedge e_0) \wedge (e_j \wedge e_0) = - e_i \wedge e_j = - (e_i \wedge e_j \wedge e_k) e_k = - \varepsilon_{ijk} I e_k$$
$$e^i \wedge e^j = \eta^{ik} \eta^{j\ell} e_k \wedge e_\ell = e_i \wedge e_j = - \varepsilon_{ijk} I e_k \quad \text{(F.41)}$$

which implies that the bivector part of the multivectors (F.39) and (F.40) can be recast as,

$$\frac{1}{2} M^{\mu\nu} e_\mu \wedge e_\nu = (M^i + M'^i I) e_i \quad \text{(F.42)}$$
$$\frac{1}{2} N_{\mu\nu} e^{\mu} \wedge e^{\nu} = (N_i + N'_i I) e_i$$

Using the algebraic relation (F.2), the pseudoscalar (F.9) and the identities (F.22) and (F.23) for different indices $\mu$, $\nu$, $\rho$ and $\alpha$, we obtain,

$$e_\mu \wedge e_\nu \wedge e_\rho = (e_\mu \wedge e_\nu \wedge e_\rho \wedge e_\alpha) e_\beta \eta_{\alpha\beta} = I e_\beta \eta_{\alpha\beta}$$
$$e^\mu \wedge e^{\nu} \wedge e^{\rho} = (e^\mu \wedge e^{\nu} \wedge e^{\rho} \wedge e^{\alpha}) e_\alpha = I e^\beta \eta^{\alpha\beta} \quad \text{(F.43)}$$

which implies that the trivector part of the multivectors (F.39) and (F.40) can be recast as,

$$\frac{1}{6} M^{\mu\nu\rho} e_\mu \wedge e_\nu \wedge e_\rho = M^\mu I e_\mu \quad \text{(F.44)}$$
$$\frac{1}{6} N_{\mu\nu\rho} e^{\mu} \wedge e^{\nu} \wedge e^{\rho} = N_\mu I e^\mu$$

In view of relations (F.44) and (F.42), the contravariant multivector (F.39) is recast as,

$$M = (m + m' I) + (M^i + M'^i I) e_i + (M^\mu + M'^\mu I) e_\mu \quad \text{(F.45)}$$
and the covariant multivector (F.40) is recast as,

\[ N = (n + n') I + (N_i + N'_i I) e_i + (N_\mu + N'_\mu I) e^\mu \]  

(F.46)

In view of identities (F.8) and (F.23), the scalar part of the geometric products of two multivectors \( M \) and \( N \) is written as,

\[
\langle NM \rangle = (n + n') (m + m') I + (N_i + N'_i I) (M^i + M'^i I) \\
+ (N_\mu + N'_\mu I) (M^\mu + M'^\mu I)
\]

(F.47)

\[
\langle MN \rangle = (m + m') (n + n') I + (M^i + M'^i I) (N_i + N'_i I) \\
+ (M^\mu + M'^\mu I) (N_\mu + N'_\mu I)
\]

(F.48)

Thus, the scalar part of the geometric product of two multivectors \( M \) and \( N \) is symmetric,

\[
\langle MN \rangle = \langle NM \rangle
\]

(F.49)

Writing the multivector \( N \) as the geometric product of two multivectors \( R \) and \( S \) in the space-time algebra \( G^{1,3} \), relation (F.49) yields the cyclic permutation identity,

\[
\langle MRS \rangle = \langle RSM \rangle = \langle SMR \rangle
\]

(F.50)

Writing the multivector \( R \) as the geometric product of two multivectors \( N \) and \( Q \) in the space-time algebra \( G^{1,3} \), relation (F.50) yields the cyclic permutation identity,

\[
\langle MNQS \rangle = \langle NQSM \rangle = \langle QSMN \rangle = \langle SMNQ \rangle
\]

(F.51)

In the spatial algebra \( G^3 \), the triple product (D.3) for the orthonormal basis vectors \( u = e^i \), \( v = e_j \) and \( w = e_k \) is written as,

\[
e^i \cdot (e_j \wedge e_k) = (e^i \cdot e_j) e_k - (e^i \cdot e_k) e_j
\]

(F.52)

In the space-time algebra \( G^{1,3} \), the triple product for the orthonormal basis vectors is given by,

\[
e^{\mu} \cdot (e_\nu \wedge e_\rho) = (e^{\mu} \cdot e_\nu) e_\rho - (e^{\mu} \cdot e_\rho) e_\nu
\]

(F.53)
or equivalently by,
\[ e_\mu \cdot (e^\nu \wedge e^\rho) = (e_\mu \cdot e^\nu) e^\rho - (e_\mu \cdot e^\rho) e^\nu \] (F.54)

In the spatial algebra \( \mathbb{G}^3 \), using the antisymmetry (A.19), the triple product (F.52) is recast as,
\[ (e_j \wedge e_k) \cdot e^i = (e^i \cdot e_k) e_j - (e^i \cdot e_j) e_k \] (F.55)

In the space-time algebra \( \mathbb{G}^{1,3} \), the triple product for the orthonormal basis vectors is given by,
\[ (e_\nu \wedge e_\rho) \cdot e_\mu = (e_\mu \cdot e_\rho) e_\nu - (e_\mu \cdot e_\nu) e_\rho \] (F.56)

or equivalently by,
\[ (e_\nu \wedge e_\rho) \cdot e_\mu = (e_\mu \cdot e_\rho) e_\nu - (e_\mu \cdot e_\nu) e_\rho \] (F.57)

In view of identities (F.53) and (F.56),
\[ e_\mu \cdot (e_\nu \wedge e_\rho) = - (e_\nu \wedge e_\rho) \cdot e_\mu \] (F.58)

and in view of identities (F.54) and (F.57),
\[ e_\mu \cdot (e_\nu \wedge e_\rho) = - (e_\nu \wedge e_\rho) \cdot e_\mu \] (F.59)

In view of relation (F.58), the inner product of a covariant vector \( U = U_\mu e_\mu \) and a contravariant bivector \( A = \frac{1}{2} A^{\nu \rho} e_\mu \wedge e_\rho \) is antisymmetric,
\[ U \cdot A = \frac{1}{2} U_\mu A^{\nu \rho} \left( e_\mu \cdot (e_\nu \wedge e_\rho) \right) = - \frac{1}{2} A^{\nu \rho} U_\mu \left( (e_\nu \wedge e_\rho) \cdot e_\mu \right) = - A \cdot U \] (F.60)

which implies that,
\[ U \cdot A = \frac{1}{2} (U A - A U) \] (F.61)

In view of relation (F.59), the inner product of a contravariant vector \( V = V^\mu e_\mu \) and a covariant bivector \( B = \frac{1}{2} B_{\nu \rho} e_\mu \wedge e_\rho \) is antisymmetric,
\[ V \cdot B = \frac{1}{2} V^\mu B_{\nu \rho} \left( e_\mu \cdot (e_\nu \wedge e_\rho) \right) = - \frac{1}{2} B_{\nu \rho} V^\mu \left( (e_\nu \wedge e_\rho) \cdot e_\mu \right) = - B \cdot V \] (F.62)

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which implies that,
\[ V \cdot B = \frac{1}{2} (V \cdot B - B \cdot V) \] (F.63)

In view of relation (F.4), the outer product of a contravariant vector \( V = V^\mu e_\mu \) and a contravariant bivector \( A = \frac{1}{2} A^{\nu\rho} e_\mu \wedge e_\rho \) is symmetric,
\[ V \wedge A = \frac{1}{2} V^\mu A^{\nu\rho} \left( e_\mu \wedge (e_\nu \wedge e_\rho) \right) = \frac{1}{2} A^{\nu\rho} V^\mu \left( (e_\nu \wedge e_\rho) \wedge e_\mu \right) = A \wedge V \] (F.64)

which implies that,
\[ V \wedge A = \frac{1}{2} (V \wedge A + A \wedge V) \] (F.65)

In view of relations (F.4) and (F.24), the outer product of a covariant vector \( U = U_\mu e^\mu \) and a covariant bivector \( B = \frac{1}{2} B_{\nu\rho} e^\mu \wedge e^\rho \) is symmetric,
\[ U \wedge B = \frac{1}{2} U_\mu B_{\nu\rho} \left( e^\mu \wedge (e^\nu \wedge e^\rho) \right) = \frac{1}{2} B^{\nu\rho} U^\mu \left( (e^\nu \wedge e^\rho) \wedge e^\mu \right) = B \wedge U \] (F.66)

which implies that,
\[ U \wedge B = \frac{1}{2} (U \wedge B + B \wedge U) \] (F.67)

In the space-time algebra \( \mathbb{G}^{1,3} \), the inner product of the bivectors \( A \) and \( B \) is defined as the symmetric scalar part of the geometric product of the bivectors,
\[ A \cdot B = \langle AB \rangle = \frac{1}{2} \langle AB + BA \rangle \] (F.68)

The inner product of the orthonormal basis bivectors is given by,
\[ (e^\mu \wedge e^\nu) \cdot (e_\rho \wedge e_\sigma) = (e^\nu \cdot e_\rho) (e^\mu \cdot e_\sigma) - (e^\mu \cdot e_\rho) (e^\nu \cdot e_\sigma) \] (F.69)

or equivalently by,
\[ (e_\rho \wedge e_\sigma) \cdot (e^\mu \wedge e^\nu) = (e_\sigma \cdot e_\rho) (e^\mu \cdot e^\nu) - (e^\mu \cdot e_\rho) (e_\sigma \cdot e^\nu) \] (F.70)

which yields the symmetry,
\[ (e^\mu \wedge e^\nu) \cdot (e_\rho \wedge e_\sigma) = (e_\rho \wedge e_\sigma) \cdot (e^\mu \wedge e^\nu) \] (F.71)

In view of relations (F.70), (F.71), the inner product of a covariant bivector \( A = \frac{1}{2} A_{\mu\nu} e^\mu \wedge e^\nu \) and a contravariant bivector \( B = \frac{1}{2} B_{\mu\sigma} e_\mu \wedge e_\sigma \) is a scalar,
\[ A \cdot B = \frac{1}{4} A_{\mu\nu} B^{\mu\sigma} \left( \langle e^\mu \wedge e^\nu \rangle \cdot (e_\rho \wedge e_\sigma) \right) \]
\[ = \frac{1}{4} B^{\mu\sigma} A_{\mu\nu} \left( (e_\rho \wedge e_\sigma) \cdot (e^\mu \wedge e^\nu) \right) = B \cdot A \] (F.72)

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In the space-time algebra $\mathbb{C}^{1,3}$, the commutator of the bivectors $A$ and $B$ is defined as the antisymmetric part of the geometric product of the bivectors,

$$A \times B = \frac{1}{2} (AB - BA) \quad \text{(F.73)}$$

The commutator of the orthonormal basis bivectors is given by,

$$(e^\mu \wedge e^\nu) \times (e^\rho \wedge e^\sigma) = (e^\nu \cdot e^\rho) (e^\mu \wedge e^\sigma) - (e^\mu \cdot e^\rho) (e^\nu \wedge e^\sigma) - (e^\rho \cdot e^\sigma) (e^\nu \wedge e^\mu) + (e^\rho \cdot e^\mu) (e^\nu \wedge e^\sigma)$$

which yields the antisymmetry,

$$(e^\mu \wedge e^\nu) \times (e^\rho \wedge e^\sigma) = - (e^\rho \wedge e^\sigma) \times (e^\mu \wedge e^\nu) \quad \text{(F.76)}$$

In the space-time algebra $\mathbb{C}^{1,3}$, the outer product of the bivectors $A$ and $B$ is the symmetric pseudoscalar part of the geometric product of the bivectors,

$$A \wedge B = \frac{1}{4} (AB + BA) - \frac{1}{2} \left( AB + BA \right) \quad \text{(F.77)}$$

In view of relations (F.73), (F.76), the commutator of a covariant bivector $A = \frac{1}{2} A_{\mu\nu} e^\mu \wedge e^\nu$ and a contravariant bivector $B = \frac{1}{2} B^{\rho\sigma} e^\rho \wedge e^\sigma$ is a antisymmetric bivector,

$$A \times B = \frac{1}{4} A_{\mu\nu} B^{\rho\sigma} \left( (e^\mu \wedge e^\nu) \times (e^\rho \wedge e^\sigma) \right)$$

$$= - \frac{1}{4} B^{\rho\sigma} A_{\mu\nu} \left( (e^\rho \wedge e^\sigma) \times (e^\mu \wedge e^\nu) \right) = - B \times A \quad \text{(F.78)}$$

In view of relation (F.4), the outer product of a covariant bivector $A = \frac{1}{2} A_{\mu\nu} e^\mu \wedge e^\nu$ and a contravariant bivector $B = \frac{1}{2} B^{\rho\sigma} e^\rho \wedge e^\sigma$ is a symmetric pseudoscalar,

$$A \wedge B = \frac{1}{4} A_{\mu\nu} B^{\rho\sigma} \left( (e^\mu \wedge e^\nu) \wedge (e^\rho \wedge e^\sigma) \right)$$

$$= \frac{1}{4} B^{\rho\sigma} A_{\mu\nu} \left( (e^\rho \wedge e^\sigma) \wedge (e^\mu \wedge e^\nu) \right) = B \wedge A \quad \text{(F.79)}$$
In view of relations (F.68) and (F.79), the sum of the inner product and the outer product of two bivectors is written as,

\[ A \cdot B + A \wedge B = \frac{1}{2} (AB + BA) \]  

(F.80)

Thus, according to relations (F.73) and (F.80), the geometric product of two bivectors in the space-time algebra \( G^{1,3} \) is written as,

\[ AB = A \cdot B + A \times B + A \wedge B \]  

(F.81)

According to relations (F.14), the contravariant bivector \( A = A^{\mu \nu} e_\mu \wedge e_\nu \) commutes with the pseudoscalar \( I \),

\[ AI = IA \]  

(F.82)

According to relations (F.15), the contravariant trivector \( T = T^{\mu \nu \rho} e_\mu \wedge e_\nu \wedge e_\rho \) anticommutes with the pseudoscalar \( I \),

\[ TI = -IT \]  

(F.83)

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