LARGE DEVIATIONS, ASYMPTOTIC BOUNDS ON THE NUMBER OF POSITIVE INDIVIDUALS IN A BERNOULLI SAMPLE VIA THE NUMBER OF POSITIVE POOL SAMPLES DRAWN ON THE BERNOULLI SAMPLE

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Abstract. In this paper we define for a Bernoulli samples the empirical infection measure, which counts the number of positives (infections) in the Bernoulli sample and for the pool samples we define the empirical pool infection measure, which counts the number of positive (infected) pool samples. For this empirical measures we prove a joint large deviation principle for Bernoulli samples. We also found an asymptotic relationship between the proportion of infected individuals with respect to the samples size, $n$ and the proportion of infected pool samples with respect to the number of pool samples, $k(n)$. All rate functions are expressed in terms of relative entropies.

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1. Introduction

Sample pooling is a testing procedure mainly used in medical research to test several individuals at a time. In pool testing, samples from individuals are pooled together and tested for the presence of infectious diseases (specificity). For instance, Noguchi Memorial Institute for Medical Research (NMIMR) at the early stages of the COVID-19 pool up samples at a time and test. This testing procedure allows the detection of positive samples with sufficient Positive Predictive Value (PPV) and detects negative samples with Sufficient Negative Predictive Value (NPV). If NMIMR pools individuals and test at once and the test results is negative, all the individuals who comprise of the pool sample is declared negative of COVID-19 resulting in huge cost saving because of the inadequacy of enough testing instrument or equipment in Ghana.

In particular, fifteen samples may be tested together, using only the resources required for a single test. If a pool sample is negative, it can be inferred that all individuals were negative. If a pool sample comes back positive, then each sample needs to be tested individually to find out which individuals were positive. As samples are pooled together, ultimately fewer tests are run overall, meaning fewer testing supplies are needed, and results can be given to patients more quickly in most
cases. NMIMR, in the wake of COVID-19 in Ghana, has adopted this asymptomatic diagnostic testing approach in order to come with large number of testing requirement needs of the Ghana as we have crossed the 35,000 confirm cases required since the detection of the first positive case in March 12, 2020.

Indeed, sampling pooling or the asymptomatic Diagnostic Testing of taking up to fifteen (15) samples per pool by the NMIMR, has significantly increased the testing capacity of Ghana given the limited resources such as equipment and test kits availability in the country.

This method works well when there is a low prevalence of cases, meaning more negative results are expected. However, a major problem will arise when the specificity is very high in which case many or all the pool samples will test positive. In this case knowing how many individuals in the sample are asymptotically infected via the infected pool samples may be key to the management of the pandemic situation.

In this paper we find an asymptotic function relationship between the number of positive individual cases and positive pooled samples as the number individual to be tested become very large. To be specific, we define the empirical proportion measure which counts the number of infested individual in the sample with respect to the sample size and the empirical pooled proportion measure which counts the number of infected pooled samples with respect to the number of pool samples, $k(n)$.

In this sequel we define the two main objects for the study: the empirical infection measure and the empirical pool infection measure. And for these empirical measures we prove a joint large deviation principle with speed $n$. From this large deviation principle we an asymptotic functional relation between the number of infected individual is in the sample and the number infected pools samples as the number of individuals increase. Further, we contract from our main large deviation principle an LDP for the number of positive cases in the sample as the sample size $n$ becomes very large.

The main techniques deplore to prove our LDPs are similarly to the ones used in the paper [3] or the Ph.D Thesis [2]: These Gartner-Ellis Theorem, see [4], Combinatorics arguments via the Method of types, and method of Mixtures, see [1].

Note, the main different between our main LDP for the empirical measure we studied in this paper and the LDP for empirical measures of samples, see [4], is that while the earlier result is pooled from random sample and the later result come from a deterministic space. Also large deviation principle for sequences of maxima and minima has been found in [5].

The remaining part of the article is organized in the following manner: Section 2 contains the main background of the article and the statement of main result. In Section 3 we give the proofs of the main result. The derivation of Corollary 2.2 and Conclusion are given in Section 5.
2. Statement of Main Results

2.1 Empirical Measures of the Bernoulli Samples. Let \( X = (X_1, X_2, ..., X_n)^T \) be a random vector of independent and identically distributed Bernoulli random variables each with success probability \( \mu_n \). Let \((N_1, N_2, N_3, ..., N_{k(n)})\) be a random sample drawn uniformly from the set of integer partitions of \( n \) of length \( k(n) \). Suppose the components of \( X \) are grouped into \( Y_1 = (X_1^{(1)}, ..., X_{N_1}^{(1)}), ..., Y_{k(n)}(X_1^{(k)}, ..., X_{N_{k(n)}}^{(k)}) \) in such away that \( X_i^{(r)} \) and \( X_j^{(m)} \) are independent for all \( r \neq m \) and \( i \neq j \), where \( i, j = 1, 2, 3, ... \). We shall call \((X_1^{(1)}, ..., X_{N_1}^{(1)}), ..., (X_1^{(k)}, ..., X_{N_{k(n)}}^{(k)})\) pool samples taken from the component of \( X \). We shall study the pool samples under the the law of the pool samples:

\[
P_n\{Y_1, ..., Y_{k(n)}\} := P\{Y_1, ..., Y_{k(n)} \mid N_1 \neq 0, ..., N_{k(n)} \neq 0, N = n\},
\]
where

\[N_1 + N_2 + N_3 + \cdots + N_{k(n)} = n.\]

For any sample \( X \) we define two measures, the empirical infection measure, \( P_1^X \in \mathcal{M}(\{0, 1\}) \), by

\[
P_1^X(a) := \frac{1}{n} \sum_{i \in [n]} \delta_{X_i}(a)
\]
and the empirical pooled infection measure \( P_2^X \in \mathcal{M}(\mathbb{N} \times \mathcal{N}(\{0, 1\})) \), by

\[
P_2^{X,N}(m, \ell) := \frac{1}{n \beta_n} \sum_{j \in [k(n)]} \delta_{(N_j, L_j)}(m, \ell),
\]
where \( L_j = \left( \frac{1}{N_j} \sum_{i=1}^{N_j} \delta_{X_i^{(j)}}(x), x \in \{0, 1\} \right) \) and \( \beta_n = k(n)/n \). Note that the total mass of the empirical infection measure is \( \mathbb{1} \) and the empirical pool infection measure is also \( \mathbb{1} \). Observe also that

\[
P_1^X(x) = \beta_n \sum_{(m, \ell) \in \mathbb{N} \times \mathcal{N}(\{0, 1\})} m \ell(x) P_2^X(m, \ell).
\]

By \( \mathcal{M}(\{0, 1\}) \) we denote the space of probability vectors on the polish space \( \{0, 1\} \) equipped with the weak topology. By \( \mathcal{N}(\{0, 1\}) \) we denote space probability measures on \( \{0, 1\} \) and we assume by convention for any pair \((m, \ell)\) we have \( \ell = (0, 0) \) if \( m = 0 \). The first theorem in this Section, Theorem 2.1, is the joint LDP for the empirical infection measure and the empirical pool infection measure.

2.2 Main Results. We assume henceforth, \( k(n) \to \infty \), \( \beta_n := k(n)/n \to \beta \in (0, 1] \) as \( n \to \infty \) and \( n \mu_n/k(n) \to q \). Observe,by the fore-mentioned assumptions,we have

\[
\beta\left(q(0) + q(1)\right) = 1.
\]

We define the rate function \( J_\beta : \mathcal{M}(\{0, 1\}) \times \mathcal{M}(\mathbb{N} \times \mathcal{N}(\{0, 1\})) \to (0, \infty] \) by

\[
J_\beta(\omega, \pi) = \begin{cases} H(\omega/\beta \mid q) + H(\pi \mid \Phi^{\omega}_{\beta}), & \langle \pi \rangle = \omega/\beta, \\
\infty, & \text{otherwise}, \end{cases}
\]
(2.1)
where

\[
\langle \pi \rangle(x) = \sum_{(m, \ell) \in \mathbb{N} \times \mathcal{N}(\{0, 1\})} m \ell(x) \pi(m, \ell).
\]
and the trivariate probability distribution

\[
\Phi^\beta_j(m, \ell) = \frac{1}{(1 - e^{1/\beta})} \left[ \frac{[\omega(0)/\beta]^{ml(0)} e^{-[\omega(0)/\beta]}}{ml(0)!} \right] \left[ \frac{[\omega(1)/\beta]^{ml(1)} e^{-[\omega(1)/\beta]}}{ml(1)!} \right],
\]

(2.2)

where \( m \in \mathbb{N} \) and \( \ell \in \mathcal{N} \{0, 1\} \). The first theorem, Theorem 2.1 is our main results the joint LDP for the empirical infection measure and the empirical pool infection measure.

**Theorem 2.1.** Suppose \( X = (X_1, X_2, ..., X_n)^T \) is a random vector of independent and identically distributed Bernoulli random variables each with success probability \( n\mu_n(x)/k(n) \to q(x) \), for \( x \in \{0, 1\} \). Let \( (X^{(1)}_1, ..., X^{(1)}_{N_1}), ..., (X^{(k)}_1, ..., X^{(k)}_{N_k}) \) be pool samples drawn from components of \( X \). Then, as \( n \to \infty \), the pair \( (P^X_1, P^X_2, ..., P^X_n) \) satisfies a large deviation principle in the space \( \mathcal{M}(\mathcal{N} \times \mathcal{N}^{(0, 1)}) \) with good rate function \( \beta J^\beta(\omega, \pi) \).

From Theorem 2.1 we obtain the Corollary 2.2 which gives us the asymptotic relationship between the number of infected individual in the sample and the number of infected pool samples.

**Corollary 2.2.** Suppose \( X = (X_1, X_2, ..., X_n)^T \) is a random vector of independent and identically distributed Bernoulli random variables each with success probability \( n\mu_n(x)/k(n) \to q(x) \), for \( x \in \{0, 1\} \). Then, the proportion of positive individuals, \( I \), satisfies

\[
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}_n \left\{ I = t \mid S = \sigma \right\} = -\left[ (1 - t) \log \left( \frac{(1 - t)}{(1 - \beta q(1))} \right) + t \log \frac{t}{\beta q(1)} \right]
\]

(2.3)

where

\[
t = -\beta \log \left[ 1 - (1 - e^{-1/\beta}) \sigma \right].
\]

The next LDP, Theorem 2.3 will play a vital role in the proof of Theorem 2.1.

**Theorem 2.3.** Suppose \( X = (X_1, X_2, ..., X_n)^T \) be a random vector of independent and identically distributed Bernoulli random variables each with success probability \( n\mu_n(x)/k(n) \to q(x) \). Let \( (X^{(1)}_1, ..., X^{(1)}_{N_1}), ..., (X^{(k)}_1, ..., X^{(k)}_{N_k}) \) be pool samples drawn from components of \( X \) conditional on the event \( \{P^X_1 = \omega\} \). Then, as \( n \to \infty \), \( P^X_2, ..., P^X_n \) satisfies a large deviation principle in the space \( \mathcal{M}(\mathcal{N} \times \mathcal{N}^{(0, 1)}) \) with good rate function \( \beta J^\beta(\pi) \), where

\[
J^\beta(\pi) = \left\{ \begin{array}{ll} H(\pi \parallel \Phi^\beta), & \text{if } \langle \pi \rangle = \omega/\beta, \\ \infty, & \text{otherwise}. \end{array} \right.
\]

(2.4)

The next LDP, Theorem 2.4 will also play a vital role in the proof of Theorem 2.1.

**Theorem 2.4.** Suppose \( X = (X_1, X_2, ..., X_n)^T \) be a random vector of independent and identically distributed Bernoulli random variables each with success probability \( n\mu_n(x)/k(n) \to q(x) \). Let \( (X^{(1)}_1, ..., X^{(1)}_{N_1}), ..., (X^{(k)}_1, ..., X^{(k)}_{N_k}) \) be pool samples drawn from components of \( X \). Then, as \( n \to \infty \), \( P^X_1 \) satisfies a large deviation principle in the space \( \mathcal{M}(\{0, 1\}) \) with good rate function \( \beta I^\beta(\omega) \), while

\[
I^\beta(\omega) = H(\omega/\parallel q),
\]

(2.5)
3. Proof of Theorem 2.3 and Theorem 2.4

3.1 Proof of Theorem 2.3 by Method of Types

We begin the proof of Theorem 2.3 by first defining two main sets:

$$\mathcal{M}_n(\{0,1\}) := \{ \omega_n \in \mathcal{M}(\{0,1\}) : n\omega_n(0), n\omega_n(1) \in \mathbb{N} \}$$

and

$$\mathcal{M}_n(\mathbb{N} \times \mathcal{N}(\{0,1\})) := \{ \pi_n \in \mathcal{M}(\mathbb{N} \times \mathcal{N}(\{0,1\})) : n\pi_n(m, \ell) \in \mathbb{N}, \text{ for all } (m, \ell) \in \mathbb{N} \times \mathcal{N}(\{0,1\}) \}.$$  

The next Lemma 3.1 is a major step in the proof of Theorem 2.3

**Lemma 3.1.** Suppose $X = (X_1, X_2, \ldots, X_n)^T$ be a random vector of independent and identically distributed Bernoulli random variables each with success probability $\mu_n$. Let $(X_1^{(1)}, \ldots, X_{N_1}^{(1)}), \ldots, (X_1^{(k)}, \ldots, X_{N_k}^{(k)})$ be pool samples drawn from components of $X$. Suppose $n\omega_n, n\pi_n \rightarrow (\omega, \pi) \in \mathcal{M}(\{0,1\}) \times \mathcal{M}(\mathbb{N} \times \mathcal{N}(\{0,1\}))$. Then, we have

$$e^{-n\beta_n \min \left\{ \mathbb{H}(\frac{\pi_n}{\beta_n}), \mathbb{H}(\frac{n\omega_n}{\beta_n}) \right\} + n\eta_1(n)} \leq \mathbb{P}_n \left\{ P_{2X,N} = \pi_n \mid P_1X = \omega_n \right\} \leq e^{-n\beta_n \min \left\{ \mathbb{H}(\frac{\pi_n}{\beta_n}), \mathbb{H}(\frac{n\omega_n}{\beta_n}) \right\} + n\eta_2(n)},$$

$$\lim_{n \to \infty} \eta_1(n) = \lim_{n \to \infty} \eta_2(n) = 0.$$  

**Proof.** We begin by observing that conditional on the event $\{P_1X = \omega_n\}$ the law of $P_{2X,N} = \pi_n$ is giving by

$$\mathbb{P}_n \left\{ P_{2X,N} = \pi_n \mid P_1X = \omega_n \right\} = \left( n\beta_n \pi_n(m, \ell), (m, \ell) \in \mathbb{N} \times \mathcal{N}(\{0,1\}) \right) \times \prod_{x \in \{0,1\}} \left( m_1\ell_1(x), m_2\ell_2(x), \ldots, m_k(n)\ell_k(n)(x) \right) \left( \frac{1}{n} \right)^{n\omega_n(x)}$$

$$\times \prod_{(m, \ell) \in \mathbb{N} \times \mathcal{N}(\{0,1\})} \left[ 1 - \prod_{x \in \{0,1\}} \left( 1 - \frac{1}{k(n)} \right)^{n\omega(x)} \right]^{-n\beta_n \pi_n(m, \ell)}$$

We recall for any $n \in \mathbb{N}$, the refined Stirling’s Formula as

$$n^n e^{-n} \leq n! \leq (2\pi n)^{-1/2} n^n e^{-n+1/(12n)}.$$
Now using the refined Stirling’s Formula we have

\[-n \sum_{(m, \ell) \in \mathbb{N} \times \mathcal{N}(\{0, 1\})} \beta_n \pi_n(m, \ell) \log \beta_n \pi_n(m, \ell) - \sum_{(m, \ell) \in \mathbb{N} \times \mathcal{N}(\{0, 1\})} \frac{1}{12n \beta_n \pi_n(m, \ell)} \]

\[-n \sum_{(m, \ell) \in \mathbb{N} \times \mathcal{N}(\{0, 1\})} \beta_n \pi_n(m, \ell) \log \left(1 - e^{-1/\beta_n}\right) - o(1) \]

\[\leq \log \left( n \beta_n \pi_n(m, \ell), (m, \ell) \in \mathbb{N} \times \mathcal{N}(\{0, 1\}) \right) \prod_{(m, \ell) \in \mathbb{N} \times \mathcal{N}(\{0, 1\})} \left[ 1 - \prod_{x \in \{0, 1\}} \left( 1 - \frac{1}{k(n)} \right)^{n \omega(x)} \right] - n \beta_n \pi_n(m, \ell) \]

\[\leq -n \sum_{(m, \ell) \in \mathbb{N} \times \mathcal{N}(\{0, 1\})} \beta_n \pi_n(m, \ell) \log \beta_n \pi_n(m, \ell) + \frac{1}{12n} + \frac{1}{2} \sum_{(m, \ell) \in \mathbb{N} \times \mathcal{N}(\{0, 1\})} \log 2 \pi n \beta_n \pi_n(m, \ell) \]

\[\sum_{(m, \ell) \in \mathbb{N}(\{0, 1\})} \frac{1}{12n \beta_n \pi_n(m, \ell)} - n \sum_{(m, \ell) \in \mathbb{N}(\{0, 1\})} \beta_n \pi_n(m, \ell) \log \left(1 - e^{-1/\beta_n}\right) + o(1) \]

(3.1)

Also we have

\[\sum_{x \in \{0, 1\}} n \omega_n(x) \log \omega_n(x) - n \sum_{x \in \{0, 1\}} \omega_n(x) - n \beta_n \sum_{m \in \mathbb{N} \times \mathcal{N}(\{0, 1\})} \log \ell(x)! \pi_n(m, \ell) \]

\[\leq \log \prod_{x \in \{0, 1\}} \left( m_1 \ell_1(x), m_2 \ell_2(x), ..., m_k(n) \ell_k(n) \right) \left( \frac{1}{n} \right)^{n \omega_n(x)} \]

\[\leq \sum_{x \in \{0, 1\}} n \omega_n(x) \log \omega_n(x) - n \sum_{x \in \{0, 1\}} \omega_n(x) - n \beta_n \sum_{m \in \mathbb{N} \times \mathcal{N}(\{0, 1\})} \log \ell(x)! \pi_n(m, \ell) + n \beta_n \sum_{x \in \{0, 1\}} \frac{1}{n \omega(x)}. \]

(3.2)

Combining (3.1) and (3.2) and taking

\[\eta_1(n) = - \sum_{(m, \ell) \in \mathbb{N} \times \mathcal{N}(\{0, 1\})} \frac{1}{12n \beta_n \pi_n(m, \ell)} + o(1) \quad \text{and} \quad \eta_2(n) = \sum_{x \in \{0, 1\}} \frac{1}{n \omega(x)} + o(1) \]

we have

\[-n \beta_n H\left( \pi_n \mid \Phi_{\beta_n}^\omega \right) + n \eta_1(n) \leq \log \mathbb{P}_n \left\{ P_{2}^{X, \mathcal{N}} = \pi_n \mid P_{1}^{X} = \omega_n \right\} \leq -n \beta_n H\left( \pi_n \mid \Phi_{\beta_n}^\omega \right) + n \eta_2(n), \]

where

\[\Phi_{\beta}^\omega(m, \ell) = \frac{1}{1 - e^{-1/\beta}} \left[ \frac{[\omega(0)/\beta][m\ell(0)]e^{-[\omega(0)/\beta]}}{[m\ell(0)]!} \right] \left[ \frac{[\omega(1)/\beta][m\ell(1)]e^{-[\omega(1)/\beta]}}{[m\ell(1)]!} \right], \]

(3.3)

where \( m \in \mathbb{N} \) and \( \ell \in \mathcal{N}(\{0, 1\}) \) which ends prove of the Lemma 3.1

**Lemma 3.2.** Suppose \((\omega_n, \pi_n)\) converges to \((\omega, \pi)\) in the space \(\mathcal{M}(\{0, 1\}) \times \mathcal{M}(\mathbb{N} \times \mathcal{N}(\{0, 1\}))\). Then,

\[\lim_{n \to \infty} \left| H\left( \pi_n \mid \Phi_{\beta_n}^\omega \right) - H\left( \pi \mid \Phi_{\beta}^\omega \right) \right| = 0.\]

**Proof.** By triangle inequality we have

\[\left| H\left( \pi_n \mid \Phi_{\beta_n}^\omega \right) - H\left( \pi \mid \Phi_{\beta}^\omega \right) \right| \leq \left| H\left( \pi_n \mid \Phi_{\beta_n}^\omega \right) - H\left( \pi_n \mid \Phi_{\beta_n}^{\omega_n} \right) \right| + \left| H\left( \pi_n \mid \Phi_{\beta_n}^{\omega_n} \right) - H\left( \pi \mid \Phi_{\beta}^\omega \right) \right| \]
Now, \( |H(\pi_n \mid \Phi^{\omega}_{\beta_n}) - H(\pi \mid \Phi^{\omega}_\beta)| = 0 \) by the continuity relative entropy and
\[
H(\pi_n \mid \Phi^{\omega}_{\beta_n}) - H(\pi_n \mid \Phi^{\omega}_\beta) = \sum_{x \in \{0,1\}} \omega_n(x) \log \omega_n(x) - \sum_{x \in \{0,1\}} \omega(x) \log \omega(x) - \sum_{x \in \{0,1\}} \omega_n(x) + \sum_{x \in \{0,1\}} \omega(x) + \beta_n \sum_{(m, \ell) \in \mathbb{N} \times \mathbb{N}'(\{0,1\})} \log m \ell(x)! [\pi_n(m, \ell) - \pi(m, \ell)] + \log \left(1 - e^{-1/\beta_n}\right) - \log \left(1 - e^{-1/\beta}\right)
\]
Taking the limit as \( n \to \infty \) we have \( |H(\pi_n \mid \Phi^{\omega}_{\beta_n}) - H(\pi \mid \Phi^{\omega}_\beta)| \to 0 \), which completes the proof of Lemma 3.3.

Now, using similar Integer partitions arguments as in [3] applied to the method of types, see [4], Proof of Theorem 1.1.10 and Lemma 3.2 above, we have the LDP for the empirical pool infection measure conditional on the empirical infection measure with rate function \( \beta H(\pi \mid \Phi^{\omega}_\beta) \) which ends the proof in Theorem 2.3.

3.2 Proof of Theorem 2.4 by Gartner-Ellis Theorem The next Lemma 3.3 will be vital for using The Garner-Ellis Theorem in the Proof of Theorem 2.4.

**Lemma 3.3.** Suppose \( X = (X_1, X_2, ..., X_n)^T \) is a random vector of independent and identically distributed Bernoulli random variables each with success probability \( \nu_n(x)/k(n) \to q(x) \). Let \((X^{(1)}_1, ..., X^{(1)}_{N_1}), ..., (X^{(k)}_1, ..., X^{(k)}_{N_k})\) be pool samples drawn from components of \( X \). Then,

\[
\lim_{\lambda \to \infty} \frac{1}{n} \log \mathbb{E}\left\{e^{n(g, P^{X})}\right\} = -\sum_{x=0}^{1} \left[1 - e^{g(x)}\right] \beta q(x) = -\langle 1 - e^{g}, \beta q \rangle.
\]

**Proof.**
\[
\mathbb{E}\left\{e^{n(g, P^{X})}\right\} = \mathbb{E}\left\{\prod_{i=1}^{n} e^{g(X_i)}\right\}
\]
\[
= \prod_{i=1}^{n} \mathbb{E}(e^{g(X_i)})
\]
\[
= \prod_{x \in \{0,1\}} \left(1 - \mu_n(x) + e^{g(x)} \mu_n(x)\right)^n
\]
\[
= \prod_{x \in \{0,1\}} e^{-(1 - e^{g(x)})q(x)k(n) + o(n)}
\]
Taking limit of the normalized logarithm we have
\[
\lim_{\lambda \to \infty} \frac{1}{n} \log \mathbb{E}\left\{ e^{n(g, P_1^X)} \right\} = - \sum_{x=0}^{1} \left( \left[ 1 - e^{g(x)} \right] \beta q(x) \right).
\]

which ends the proof of the Lemma.

Now, using Gartner-Ellis Theorem, the probability measure \( P_1^X \) obeys an LDP with speed \( n \) and rate function
\[
I(\omega) = \sup_g \left\{ \left\langle g, \omega \right\rangle + \left\langle (1 - e^g), q\beta \right\rangle \right\}.
\]

By solving the variational problem we have the relative entropy
\[
I(\omega) = \beta H\left( \frac{\omega}{\beta} \middle\| \frac{\beta}{q} \right)
\]

which proves Theorem 2.4.

4. Proof of Theorem 2.1 by Method of Mixtures

For each \( n \in \mathbb{N} \) we define
\[
\mathcal{M}_n(\{0,1\}) := \left\{ \omega_n \in \mathcal{M}(\{0,1\}) : n\omega_n(x) \in \mathbb{N} \text{ for all } x \in \{0,1\} \right\},
\]
\[
\mathcal{M}_n(\mathbb{N} \times \mathcal{N}(\{0,1\})) := \left\{ n\beta_n \pi_n \in \mathcal{M}(\mathbb{N} \times \mathcal{N}(\{0,1\})) : n\beta_n \pi_n(x,y) \in \mathbb{N}, \text{ for all } x,y \in \{0,1\} \right\}.
\]

We denote by \( \Theta_n := \mathcal{M}_n(\{0,1\}) \) and \( \Theta := \mathcal{M}(\{0,1\}) \). With
\[
P_n(\omega_n) := \mathbb{P}\left\{ P_1^X = \omega_n \right\},
\]
\[
P_{\omega_n}(\pi_n) := \mathbb{P}\left\{ P_2^X = \pi_n \middle| P_1^X = \omega_n \right\}
\]

the joint distribution of \( P_1^X \) and \( P_2^X \) is the mixture of \( P_n \) with \( P_{\omega_n} \), as follows:
\[
d\tilde{P}_n(\omega_n, \pi_n) := dP_{\omega_n}(\pi_n) dP_n(\omega_n).
\]

Biggins [1, Theorem 5(b)] gives criteria for the validity of large deviation principles for the mixtures and for the goodness of the rate function if individual large deviation principles are known. The following three lemmas ensure validity of these conditions.

Observe that the family of measures \( (P_n : n \in \mathbb{N}) \) is exponentially tight on \( \Theta \).

**Lemma 4.1.** The family measures \( (\tilde{P}_n : n \in \mathbb{N}) \) is exponentially tight on \( \Theta \times \mathcal{M}(\{0,1\} \times \mathcal{N}(\{0,1\})) \).

**Proof.**

Note, \( P_{\omega_n} \) is a probability distribution on space of positive finite measures and so using similar argument as in the proof of [6, Lemma 4.3], we can conclude \( P_{\omega_n} \) is exponentially tight. Moreover, \( P_n \) is a probability distribution on the space of probability vectors on \( \{0,1\} \) so by the Chebychev’s inequality and the Prokhov’s Theorem, we can conclude \( P_n \) is exponentially tight. Hence, as \( \tilde{P}_n \) is mixture of two exponentially tight probability distributions \( (P_{\omega_n} \text{ and } P_n) \), we can conclude that the sequence of measures \( (\tilde{P}_n : n \in \mathbb{N}) \) is exponentially tight on \( \Theta \times \mathcal{M}(\{0,1\} \times \mathcal{N}(\{0,1\})) \). See, example [3].

Define the function \( I : \Theta \times \mathcal{M}(\{0,1\} \times \mathcal{N}(\{0,1\})) \to [0, \infty] \), by
\[ J_{\omega}^{\beta}(\pi) = \begin{cases} H\left(\pi \| \Phi_{\omega}^{\beta}\right), & \text{if } \langle \pi \rangle = \omega/\beta, \\ \infty & \text{otherwise}. \end{cases} \] (4.2)

and recall from Theorem 2.4 that
\[ I_{\beta}(\pi) = H\left(\omega/\beta \| q\right). \] (4.3)

**Lemma 4.2.**
(i) \( J_{\omega}^{\beta} \) is lower semi-continuous.
(ii) \( J_{\beta} \) is lower semi-continuous.

**Proof.** Observe that \( H\left(\omega \| \beta q\right) \) is relative entropy functions by definition. \( J_{\beta} \) is the sum of two relative entropy functions since \( J_{\omega}^{\beta} \) is a function of relative entropy. We conclude that both \( J_{\omega}^{\beta} \) and \( J_{\beta} \) are lower semi-continuous. \( \square \)

By [1, Theorem 5(b)], the two previous lemmas and the large deviation principles we have proved, Theorem 2.3 and Theorem 2.4 ensure that under \( \tilde{P}_{n} \) and \( P_{n} \) the random variables \((\omega_{n}, \pi_{n})\) satisfy a large deviation principle on \( \mathcal{M} \{\{0,1\}\} \times \mathcal{M}(\mathbb{N} \times \mathcal{N}(\{0,1\})) \) with good rate function \( \tilde{J} \) respectively, which ends the proof of Theorem 2.1.

5. **Proof of Corollary 2.2 and Conclusion**

5.1 **Proof of Corollary 2.2.** The proof of the Corollary is obtain from theorem 2.1 by applying the contraction Principle, see [4], to the linear mapping \((\omega, \pi) \rightarrow \omega(1)\). By theorem 2.1 \((P_{1}^{X}, P_{2}^{X,N})\) obeys a large deviation principle with speed \( n \) and rate function \( J(\omega, \pi)\). Applying the contraction principle to the linear mapping above we have that \( I = P_{1}^{X}(1) \) obeys an LDP with speed \( n \) and rate function
\[ \phi_{\sigma}(t) = \inf \left\{ J(\omega, \pi) : \omega(1) = t, 1 - \pi_{2}((0,1)) = \sigma \right\}, \] (5.1)

where
\[ \pi_{2}((0,1)) = \sum_{m=1}^{\infty} \Phi_{\omega}^{\beta}(m, (0,1)) = \frac{1}{(1 - e^{-1/\beta})} e^{-\omega(1)/\beta} (1 - e^{-(1-\omega(1))/\beta}). \]

Solving the optimization problem in 5.1 we have the rate function 2.3 which completes the proof of Corollary 2.1.

5.2 **Conclusion.** In this article we have found an asymptotic bounds for the number of infected(positive) individuals in a Bernoulli sample via the number of infected pool samples. From the asymptotic relationship in Corollary 2.2 we can infer the following: Note \( 0 \leq \sigma \leq 1 \) and

- If \( \sigma = 0 \) we shall say there are no infected pool samples and hence we no infected individuals in the sample \( X \), with a positive probability.
- If \( \sigma = 1 \) we shall say nearly every pool sample formed from \( X \) is infected and hence we have that every individual in the sample \( X \) is infected with a positive probability.
- If \( 0 < \sigma < 1 \) we shall say that some of the pool samples form from \( X \) are infected and hence some individuals in the sample \( X \) are infected with a positive probability.

In fact we shall say that \( nI \) individuals of the sample are infected, with a positive probability. Particularly, we have indirectly established that \( nI \geq n\beta\sigma \), with probability 1.
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