DIFFERENTIAL COCYCLES AND DIXMIER-DOUADY BUNDLES

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Abstract. This paper exhibits equivalences of 2-stacks between certain models of $S^1$-gerbes and differential 3-cocycles. We focus primarily on the model of Dixmier-Douady bundles, and provide an equivalence between the 2-stack of Dixmier-Douady bundles and the 2-stack of differential 3-cocycles of height 1, where the ‘height’ is related to the presence of connective structure. Differential 3-cocycles of height 2 (resp. height 3) are shown to be equivalent to $S^1$-bundle gerbes with connection (resp. with connection and curving). These equivalences extend to the equivariant setting of $S^1$-gerbes over Lie groupoids.

1. Introduction and Preliminaries

Originally due to Giraud [9], $S^1$-gerbes are ‘geometric models’ representing cohomology classes in $H^3(M;\mathbb{Z})$ of a smooth manifold $M$; these geometric models are analogous to principal $S^1$-bundles over $M$, which by Weil’s Theorem [22] represent cohomology classes in $H^2(M;\mathbb{Z})$. There are several concrete constructions for $S^1$-gerbes in the literature, and this paper focuses primarily on the model of Dixmier-Douady bundles (DD-bundles), which are locally trivial fibre bundles of $C^*$-algebras, with typical fibre $\mathbb{K}(\mathcal{H})$, the compact operators on a separable complex Hilbert space $\mathcal{H}$. Other concrete constructions for $S^1$-gerbes appearing in the literature are $S^1$-bundle gerbes (see [6, 10, 15]), $S^1$-central extensions of Lie groupoids (see [3]), and principal Lie 2-group bundles (see [2, 18, 23]).

Over a fixed manifold $M$, these constructions naturally result in bicategories. In the case of DD-bundles, 1-arrows or Morita isomorphisms $\mathcal{E}: \mathcal{A}_1 \rightarrow \mathcal{A}_2$ are Banach space bundles $\mathcal{E} \rightarrow M$ of fibrewise $(\mathcal{A}_2, \mathcal{A}_1)$-bimodules, and 2-arrows $\tau: \mathcal{E}_1 \Rightarrow \mathcal{E}_2$ are Banach space bundle isomorphisms (see Section 3 for details). Analogous to the (first) Chern class for principal $S^1$-bundles, we can associate to a DD-bundle $\mathcal{A} \rightarrow M$ its Dixmier-Douady class (DD-class) $\text{DD}(\mathcal{A}) \in H^3(M;\mathbb{Z})$, and by a theorem of Dixmier and Douady [8], Morita isomorphism classes of DD-bundles are classified by their DD-class. The need to relax the notion of isomorphism of DD-bundles, from usual ‘structure-preserving’ fibre bundle isomorphisms...
to Morita isomorphisms, is related to the fact that two non-isomorphic (in the sense of fibre bundles) DD-bundles can have the same Dixmier-Douady class. An alternate fix is to restrict to the case \( \dim \mathcal{H} = \infty \) (see [10, Theorem 4.85]); however, there are naturally occurring examples of interest with finite dimensional fibres (e.g. the Clifford algebra bundle of an even rank Euclidean vector bundle). (For bundle gerbes, the situation is similar — there are non-isomorphic bundle gerbes with the same DD-class, which is what motivated the definition of stable isomorphism (see [16, 21]). In that setting, one also has 2-arrows, namely transformations of stable isomorphisms to obtain a bicategory of bundle gerbes over a space.) Additionally, since DD-bundles can be pulled back along smooth maps, we obtain a presheaf \( B^2S^1 \) of bicategories over the category of smooth manifolds \( \text{Mfld}, M \mapsto B^2S^1(M) \) (see Proposition 3.4).

The local nature of the bicategory \( B^2S^1(M) \) of DD-bundles over \( M \) is competently described in the language of stacks. Roughly speaking, since DD-bundles are locally trivial fibre bundles, the bicategory \( B^2S^1(M) \) can be reconstructed from the bicategories \( B^2S^1(U_\alpha) \), where \( \{U_\alpha\} \) is any open cover of \( M \). Such a reconstruction, however, should accommodate the more general notion of Morita isomorphism, which should ultimately be used to ‘glue’ together DD-bundles \( \mathcal{A}_\alpha \to U_\alpha \) with (possibly non-isomorphic) fibres \( \mathbb{K}(\mathcal{H}_\alpha) \). This notion of gluing is made more precise in Theorem 3.3 which states that the presheaf \( B^2S^1 \) of bicategories is a 2-stack. In more detail, following [17], introduce the descent bicategory \( B^2S^1(U_\bullet) \) associated to the cover \( \{U_\alpha\} \) of \( M \) (see Definition 1.4), which naturally comes with a functor \( B^2S^1(M) \to B^2S^1(U_\bullet) \) induced by restriction to the open sets in the cover. That \( B^2S^1 \) is a 2-stack means that this restriction functor is an equivalence of bicategories for every \( M \) and every cover of \( M \).

This paper relates DD-bundles to differential 3-cocycles of height 1, following ideas in [14] that considered the case of principal \( S^1 \)-bundles and differential 2-cocycles. In [12, Section 3.2], Hopkins and Singer introduce a category of differential \( k \)-cocycles \( DC^k_s \) (see Section 2.1), where \( s > 0 \) is an integer, which we shall refer to as the height. The cochain complexes \( DC^s \) provide a kind of refinement of singular cohomology. For example, the cohomology group \( H^2(\text{DC}^1_s(M)) \) classifies principal \( S^1 \)-bundles over \( M \) when \( s = 1 \), and when \( s = 2 \) it classifies principal \( S^1 \)-bundles with connection (up to connection preserving isomorphism). An important feature of this complex is that \( H^k(\text{DC}^s_s(M)) \) is isomorphic to the group of differential characters, due to Cheeger and Simons [7]. The perspective from [14] adopted here views the cocycles in \( DC^3_s(M) \) as objects of a 2-category \( \text{DC}^3_s(M) \), where by construction cohomology classes correspond precisely to isomorphism classes of objects (see Section 2.1). In fact, since cochains can be pulled back along smooth maps, we have a presheaf of 2-categories \( \text{DC}^3_s \), and Theorem 2.7 verifies that \( \text{DC}^3_s \) is a 2-stack, which we refer to as the 2-stack of differential 3-cocycles of height \( s \). When \( s = 3 \), we will follow [14] and call \( \text{DC}^3_3 \)
the 2-stack of differential characters of degree 3. Our first main result, Theorem 4.3, shows that there is an equivalence of 2-stacks $B^2S^1 \cong DC^3_1$.

Theorem 4.3 has some immediate consequences. First, for any manifold $M$ the equivalence $B^2S^1(M) \cong DC^3_1(M)$ results in a ‘strictification’ of the bicategory of DD-bundles to the strict 2-category of differential cocycles, which can be useful in practice. (For example, they were used in [13] to verify the compatibility among certain definitions of prequantization in the context of Hamiltonian actions of quasi-symplectic/twisted presymplectic groupoids.)

Second, the equivalence as 2-stacks provides an equivalence of equivariant objects as well. That is, for any Lie groupoid $\Gamma_1 \rightrightarrows \Gamma_0$, we may consider the bicategories of $\Gamma_\bullet$-equivariant DD-bundles $B^2S^1(\Gamma_\bullet)$ (for an action groupoid $G \times M \rightrightarrows M$, this is a weakening of the usual notion of $G$-equivariant DD-bundles) and $\Gamma_\bullet$-equivariant differential cocycles $DC^3_1(\Gamma_\bullet)$ (see Section 1.3 and Definition 3.5). The equivalence of 2-stacks automatically gives the corresponding equivalence of equivariant objects $B^2S^1(\Gamma_\bullet) \cong DC^3_1(\Gamma_\bullet)$.

Third, since the isomorphism classes of objects in the 2-categories $DC^3_1(M)$ and $DC^3_1(\Gamma_\bullet)$ are easily computed (by taking cohomology of the corresponding cochain complexes), we obtain the Dixmier-Douady classification of Morita isomorphism classes of DD-bundles over $M$ by $H^3(M;\mathbb{Z})$ (see Corollary 4.4) and its equivariant counterpart, classifying Morita isomorphism classes of $\Gamma_\bullet$-equivariant DD-bundles by $H^3(\Gamma_\bullet;\mathbb{Z})$ for proper Lie groupoids $\Gamma_1 \rightrightarrows \Gamma_0$ (see Corollary 4.5). For compact Lie groups $G$, it is well known that $G$-equivariant DD-bundles are classified by $H^3_G(M;\mathbb{Z})$. An interesting consequence of Corollary 4.5 is that every $G \times M \rightrightarrows M$ equivariant DD-bundle (a weaker notion that the usual notion of $G$-equivariance) is Morita equivalent to a genuine $G$-equivariant DD-bundle over $M$.

We also obtain refinements of Theorem 4.3 in the setting of $S^1$-bundle gerbes with connective structures. By making use of the technology in [17] showing that $S^1$-bundle gerbes with connection and curving form a 2-stack $Grb^\nabla,B$, our second main theorem (Theorem 4.13) establishes a corresponding equivalence $Grb^\nabla,B \cong DC^3_3$ with the 2-stack of differential characters of degree 3. (Theorem 4.9 verifies the expected equivalences to differential 3-cocycles for bundle gerbes without connections, $Grb \cong DC^3_1$, and with connections but no specified curving $Grb^\nabla \cong DC^3_2$.)

Similar to the corollaries listed above, we immediately obtain the classification of stable isomorphism classes of $S^1$-bundle gerbes over $M$ with (or without) connection by $H^3(M;\mathbb{Z})$ (see Corollary 4.10), and the classification of stable isomorphism classes of $S^1$-bundle gerbes with connection and curving by differential characters of degree 3, $H^3(\text{DC}^*_3(M))$ (see Corollary 4.14). The equivariant versions state that (Corollary 4.11) stable isomorphism classes of $\Gamma_\bullet$-equivariant $S^1$-bundle gerbes are classified by $H^3(\Gamma_\bullet;\mathbb{Z})$ (for proper Lie groupoids $\Gamma_1 \rightrightarrows \Gamma_0$); (Corollary 4.12) stable isomorphism classes of $\Gamma_\bullet$-equivariant bundle gerbes with
connection are classified by $H^3(\text{DC}_3^*(\Gamma_{\bullet}))$; and (Corollary 4.15) $\Gamma_{\bullet}$-equivariant bundle gerbes with connection and curving are classified by differential characters $H^3(\text{DC}_3^*(\Gamma_{\bullet}))$.

The paper is organized as follows. In the remainder of this section, we collect some preliminaries on the simplicial manifold $\Gamma_{\bullet}$ associated to a Lie groupoid $\Gamma_1 \rightrightarrows \Gamma_0$, and recall some terminology related to presheaves of bicategories and 2-stacks.

Section 2 recalls constructions and notation regarding differential cocycles and verifies in Theorem 2.7 that differential 3-cocycles form a 2-stack.

In Section 3, we review some definitions surrounding the bicategory of DD-bundles over a manifold, as well as their equivariant counterparts on a Lie groupoid. We show in Theorem 3.9 that DD-bundles form a 2-stack.

Section 4 contains the main theorems of the paper. Namely, this section contains Theorem 4.3 exhibiting the equivalence between DD-bundles and differential 3-cocycles of height 1, together with the Corollaries mentioned above. At the end of this section, we establish the refinements of this result to $S^1$-bundle gerbes with connective structures mentioned above (Theorems 4.9 and 4.13).

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1.1. Lie Groupoids. We briefly recall some aspects related to the simplicial manifold $\Gamma_{\bullet}$ associated to a Lie groupoid $\Gamma_1 \rightrightarrows \Gamma_0$, as well as the resulting double complex arising from a presheaf of chain complexes. Denote the source and target maps by $s, t : \Gamma_1 \to \Gamma_0$, respectively, and write multiplication $\Gamma_1 \times_{\Gamma_0} \Gamma_1 \to \Gamma_1$ as $(g_1, g_2) \mapsto g_1 g_2$.

For $k \geq 2$, write

$$\Gamma_k = \underbrace{\Gamma_1 \times_{\Gamma_0} \Gamma_1 \times_{\Gamma_0} \cdots \times_{\Gamma_0} \Gamma_1}_{\text{$k$ factors}}$$

whose elements are $k$-tuples $(g_1, \ldots, g_k)$ of composable arrows (with $s(g_i) = t(g_{i+1})$). For $0 \leq i \leq k$, let $\partial_i : \Gamma_k \to \Gamma_{k-1}$ be the face maps given by

$$\partial_i(g_1, \ldots, g_k) = \begin{cases} (g_2, \ldots, g_k) & \text{if } i = 0 \\ (g_1, \ldots, g_i g_{i+1}, \ldots, g_k) & \text{if } 0 < i < k \\ (g_1, \ldots, g_{k-1}) & \text{if } i = k. \end{cases}$$

For convenience, we set $\partial_0 = s$ and $\partial_k = t$ on $\Gamma_1$. It is easily verified that the face maps satisfy the simplicial identities $\partial_i \partial_j = \partial_{j-1} \partial_i$ for $i < j$. (We will not require degeneracy maps in this paper.)
Let \((A^*, d)\) denote a presheaf of cochain complexes, and consider the double complex \(A^*(\Gamma_* )\), depicted below.

\[
\begin{array}{c c c c c}
\vdots & \vdots & \vdots \\
\downarrow d & \downarrow -d & \downarrow d \\
A^2(\Gamma_0) & \overset{\partial}{\longrightarrow} & A^2(\Gamma_1) & \overset{\partial}{\longrightarrow} & A^2(\Gamma_2) & \overset{\partial}{\longrightarrow} & \cdots \\
\downarrow d & \downarrow -d & \downarrow d \\
A^1(\Gamma_0) & \overset{\partial}{\longrightarrow} & A^1(\Gamma_1) & \overset{\partial}{\longrightarrow} & A^1(\Gamma_2) & \overset{\partial}{\longrightarrow} & \cdots \\
\downarrow d & \downarrow -d & \downarrow d \\
A^0(\Gamma_0) & \overset{\partial}{\longrightarrow} & A^0(\Gamma_1) & \overset{\partial}{\longrightarrow} & A^0(\Gamma_2) & \overset{\partial}{\longrightarrow} & \cdots \\
\end{array}
\]

The horizontal differential is the alternating sum of pullbacks of face maps, \(\partial = \sum (-1)^i \partial_i\). Denote the total complex by \((A^*(\Gamma_* ))_{\text{tot}}\) with \((A^*(\Gamma_* ))_{\text{tot}}^n = \bigoplus_{p+q=n} A^p(\Gamma_q)\), and total differential \(\delta = (-1)^q d \oplus \partial\).

In this paper, we will use the de Rham complex \(\Omega^*\), smooth singular cochains \(C^*(-; \mathbb{Z})\) and \(C^*(-; \mathbb{R})\), and a cochain complex of Hopkins and Singer [12], denoted \(\text{DC}_s^*\) following the notation in [14] (reviewed in Section 2.2). Note that we will abuse notation and use integration of forms to view \(\Omega^*(M) \subset C^*(M; \mathbb{R})\) and also view \(C^*(M; \mathbb{Z}) \subset C^*(M; \mathbb{R})\).

For an open cover \(\{U_\alpha\}\) of a manifold \(M\), write \(U = \bigsqcup_\alpha U_\alpha\) and let \(\pi: U \to M\) be the natural map induced by inclusions of open sets. We denote the Čech groupoid \(U \times_M U \rightrightarrows U\) corresponding to the cover \(\{U_\alpha\}\) by \(U_*\).

1.2. Presheaves of Bicategories. We recall some details regarding a presheaf in bicategories. (See [4] for background on higher categories.) Let \(\text{BiCat}\) denote the 3-category of bicategories, whose objects are weak 2-categories; 1-arrows are pseudo-functors; 2-arrows are pseudo-natural transformations; and 3-arrows are modifications. (We shall often omit the prefix pseudo in the text; unless stated otherwise, functors and natural transformations are of the pseudo variety.)

**Definition 1.1 (Presheaf of bicategories).** A presheaf of bicategories (over manifolds) is a lax functor \(\mathfrak{X}: \text{Mfd}^{\text{op}} \to \text{BiCat}\). It consists of the following data:

1. for every manifold \(T\), a bicategory \(\mathfrak{X}(T)\);
2. for every map \(f: S \to T\), a functor \(f^*: \mathfrak{X}(T) \to \mathfrak{X}(S)\);
3. for every pair of composable maps \(R \xrightarrow{f} S \xrightarrow{g} T\), a natural isomorphism \(\phi_{f,g}: f^* \circ g^* \Rightarrow (gf)^*\);
(4) for every triple of composable maps $Q \xrightarrow{f} R \xrightarrow{g} S \xrightarrow{h} T$, a modification $\theta = \theta_{f,g,h}$ between the composite natural transformations

$$f^*g^*h^* \Rightarrow (gf)^*h^* \Rightarrow (hgf)^*$$

and

$$f^*g^*h^* \Rightarrow f^*(hg)^* \Rightarrow (hgf)^*.$$ 

The modifications $\theta$ are required to satisfy the following coherence condition. For each quadruple of composable maps $P \xrightarrow{f} Q \xrightarrow{g} R \xrightarrow{h} S \xrightarrow{k} T$, the two induced modifications between the composite natural transformations,

$$f^*g^*h^*k^* \Rightarrow f^*g^*(kh)^* \Rightarrow f^*(khg)^* \Rightarrow (khgf)^*,$$

and

$$f^*g^*h^*k^* \Rightarrow (gf)^*h^*k^* \Rightarrow (hgf)^*k^* \Rightarrow (khgf)^*$$

must coincide.

To elaborate further, the natural transformation in (3) of Definition 1.1 above consists of a 1-isomorphism $\phi^A: f^*g^*A \Rightarrow (gf)^*A$ in $\mathfrak{X}(R)$ for each object $A$ in $\mathfrak{X}(T)$ as well as 2-isomorphisms $\sigma_F: \phi^{A'} \circ f^*g^*F \Rightarrow (gf)^*F \circ \phi^A$ for each 1-arrow $F: A \to A'$ in $\mathfrak{X}(T)$. The modifications in (4) are given as follows. For each object $A$ in $\mathfrak{X}(T)$ let $\alpha^A = \phi^A_{gf,h} \circ \phi^{*A}_{g,h}$ and $\beta^A = \phi^A_{f,hg} \circ f^*\phi^A_{g,h}$ be the 1-isomorphisms in $\mathfrak{X}(Q)$ given by the composite natural transformations in (4), and let

$$\mu_E: \alpha^{A'} \circ f^*g^*h^*E \Rightarrow (hg)^*E \circ \alpha^A, \quad \nu_E: \beta^{A'} \circ f^*g^*h^*E \Rightarrow (hg)^*E \circ \beta^A$$

be the natural isomorphisms corresponding to a 1-arrow $E: A \to A'$. The modification $\theta$ consists of 2-arrows $\theta(A): \alpha^A \Rightarrow \beta^A$. These are required to satisfy the following property. For any 2-arrow $\rho: E \Rightarrow E'$ between 1-arrows $E, E': A \to A'$, we have the equality

$$\nu_{E'} \circ (\theta(A') \star f^*g^*h^*\rho) = ((hg)^*\rho \star \theta(A)) \circ \mu_E,$$  \tag{1.3}

where $\star$ denotes ‘horizontal’ composition of 2-arrows.

The coherence condition on the modifications can be stated as follows. For any object $A$ in $\mathfrak{X}(T)$, the modifications $\theta$ result in the following two 2-cells from (1.2) to (1.1): namely, the composition

\[
\begin{array}{ccc}
(gf)^*h^*k^*A & \xrightarrow{\theta} & (hg)^*k^*A \\
\downarrow \sigma & & \downarrow \theta \\
 f^*g^*h^*k^*A & \xrightarrow{\rho} & (gf)^*(kh)^*A \\
\downarrow \theta & & \downarrow \theta \\
 f^*g^*(kh)^*A & \xrightarrow{\rho} & f^*(khg)^*A \\
\end{array}
\]
(where $\sigma$ denotes the 2-isomorphism $\sigma_{\phi A}$) and the composition

$$
\begin{align*}
(gf)^*h^*k^*A &\xrightarrow{(g)\star}(hf)^*k^*A \\
\downarrow_{\psi} &\downarrow_{\psi} \\
(fg)^*h^*k^*A &\xrightarrow{(f)\star}(hgf)^*A \\
\downarrow_{f^*\psi} &\downarrow_{f^*\psi} \\
f^*(hk)^*A &\xrightarrow{(k)\star}(khg)^*A
\end{align*}
$$

(where he has omitted the subscripts on each modification $\psi$). These 2-cells must agree for every $A$.

1.3. **Equivariant Objects in a Presheaf.** Recall the construction from [17] of equivariant objects in a presheaf of bicategories.

**Definition 1.2.** Let $\mathfrak{X}$ be a presheaf of bicategories over manifolds, and let $\Gamma_1 \supseteq \Gamma_0$ be a Lie groupoid. The bicategory $\mathfrak{X}(\Gamma_\bullet)$ of $\Gamma_\bullet$-equivariant objects of $\mathfrak{X}$ is given by the following:

1. objects consist of triples $(A, E, \tau)$ where $A$ is an object in $\mathfrak{X}(\Gamma_0)$; $E: \partial_0^* A \to \partial_1^* A$ is a 1-isomorphism in $\mathfrak{X}(\Gamma_1)$; and $\tau: \partial_2^* E \circ \partial_0^* E \Rightarrow \partial_1^* E$ is a 2-isomorphism in $\mathfrak{X}(\Gamma_2)$ satisfying the coherence condition $\partial_2^* \tau \circ (\text{id} \star \partial_0^* \tau) = \partial_1^* \tau \circ (\partial_2^* \tau \star \text{id})$ in $\mathfrak{X}(\Gamma_3)$;
2. 1-arrows $(F, \alpha): (A, E, \tau) \to (A', E', \tau')$ consist of a 1-arrow $F: A \to A'$ in $\mathfrak{X}(\Gamma_0)$; and a 2-arrow $\alpha: E' \circ \partial_0^* F \Rightarrow \partial_1^* F \circ E$ in $\mathfrak{X}(\Gamma_1)$ satisfying
   $$(\text{id} \star \tau) \circ (\partial_2^* \alpha \star \text{id}) \circ (\text{id} \star \partial_0^* \alpha) = \partial_1^* \alpha \circ (\tau' \star \text{id})$$
   in $\mathfrak{X}(\Gamma_2)$;
3. 2-arrows $(F, \alpha) \Rightarrow (F', \alpha')$ consist of a 2-arrow $\beta: F \Rightarrow F'$ in $\mathfrak{X}(\Gamma_0)$ satisfying $\alpha' \circ (\text{id} \star \partial_0^* \beta) = (\partial_1^* \beta \star \text{id}) \circ \alpha$ in $\mathfrak{X}(\Gamma_1)$.

**Remark 1.3.** Definition [12] implicitly makes use of the simplicial identities on the simplicial manifold $\Gamma_\bullet$ associated to the Lie groupoid $\Gamma_1 \supseteq \Gamma_0$. For example, since $\partial_i \circ \partial_j = \partial_{j-1} \circ \partial_i \ (i < j)$, Definition [11] [3] gives natural isomorphisms $\chi_{ij}: \partial_j^* \circ \partial_i^* \Rightarrow \partial_i^* \circ \partial_{i-1}^*$. A priori, the 2-isomorphism $\tau$ in Definition [12] is not well-defined and should instead be written more precisely as a 2-isomorphism $\tau: \chi_{12}^A \circ \partial_2^* E \circ (\chi_{02}^A)^{-1} \circ \partial_0^* E \Rightarrow \partial_1^* E \circ (\chi_{01}^A)^{-1}$. Throughout this paper, we will freely make use of simplicial identities and suppress the resulting 1-isomorphisms $\chi_{ij}$, as in the above definition.

1.4. **2-Stacks.** We briefly recall some notions related to 2-stacks. For further details, the reader may wish to consult [3][17].

**Definition 1.4.** [17] Def. 2.12] Let $\mathfrak{X}$ be a presheaf of bicategories over manifolds.
(1) Let \( M \) be a manifold. Given an open cover \( \{ U_\alpha \} \) of \( M \), the **descent bicategory of \( M \) corresponding to the cover** \( \{ U_\alpha \} \) is the bicategory \( \mathcal{X}(U_\ast) \) of \( U_\ast \)-equivariant objects of \( \mathcal{X} \).

(2) We say \( \mathcal{X} \) is a **2-prestack** (or simply, prestack) if for every manifold \( M \) and every open cover \( \{ U_\alpha \} \) of \( M \), the natural restriction functor \( \pi^* : \mathcal{X}(M) \to \mathcal{X}(U_\ast) \) is fully faithful.

(3) We say \( \mathcal{X} \) is a **2-stack** (or simply, stack) if for every manifold \( M \) and every open cover \( \{ U_\alpha \} \) of \( M \), the natural restriction functor \( \pi^* : \mathcal{X}(M) \to \mathcal{X}(U_\ast) \) is an equivalence of bicategories.

Given a prestack \( \mathcal{X}_0 \), one can associate to it a **stackification** (see [5, Section 1.10]), which is a stack \( \mathcal{X} \) together with a morphism (a pseudo-natural transformation) \( F : \mathcal{X}_0 \to \mathcal{X} \) such that (i) for any \( M \), the functor \( F_M : \mathcal{X}_0(M) \to \mathcal{X}(M) \) is fully faithful (i.e. an equivalence on Hom categories), and (ii) every object in \( \mathcal{X}(M) \) is locally isomorphic to one in the image of \( \mathcal{X}_0(M) \) (i.e. for every object \( A \) in \( \mathcal{X}(M) \), there exists a cover \( \pi : U \to M \) and an object \( A_0 \) in \( \mathcal{X}_0(U) \) together with an isomorphism \( F_M(A_0) \to \pi^*A \) in \( \mathcal{X}(U) \)). In [17, Section 3], the authors provide a concrete construction for a stackification, called the **plus construction** of \( \mathcal{X}_0 \).

### 2. Differential Cocycles

This section recalls the construction of the (2-)category of **differential cocycles** from [12], and establishes some properties analogous to those in [14], adapted to 2-stacks and differential cocycles of degree 3. Specifically, differential cocycles of degree 3 are constructed as a presheaf of bicategories (strict 2-groupoids, in fact) associated to a certain presheaf of cochain complexes. We briefly review the more general construction of **cocycle 2-categories** associated to any presheaf of cochain complexes in Section 2.1, turning our attention to the case of differential cocycles in Section 2.2, where we show in Theorem 2.7 that differential cocycles form a 2-stack.

#### 2.1. Cocycle 2-Categories

Following [14], but adapting to the setting of 2-categories, we review the construction of a 2-category from a cochain complex \((A^\ast, d)\). In this paper, we will assume all cochain complexes are concentrated in non-negative degrees (i.e. \( A^n = 0 \) for all \( n < 0 \)).

**Definition 2.1.** Let \((A^\ast, d)\) be a cochain complex of abelian groups. Fix an integer \( k \geq 0 \). Define the **cocycle 2-category** \( \mathcal{H}^k(A^\ast) \) as follows:

- (0) objects are \( k \)-cocycles: \( c \in A^k \) such that \( dc = 0 \),
- (1) a 1-arrow \( c_1 \to c_2 \) is a \((k - 1)\)-cochain \( b \) such that \( c_1 - c_2 = db \),
(2) a 2-arrow \( b_1 \Rightarrow b_2 \) between two 1-arrows \( b_1, b_2 : c_1 \to c_2 \) is an equivalence class \([a]\) of \((k-2)\)-cochains such that \( b_2 - b_1 = da \), where \( a_1 \) is equivalent to \( a_2 \) if there is a \((k-3)\)-cochain \( z \) such that \( a_2 - a_1 = dz \).

Composition is given by addition of cochains, and the identity is the 0-cochain. It follows that \( \mathcal{H}^k(A^*) \) is a strict 2-groupoid.

**Remark 2.2.** Analogous to some of the properties of cochain categories listed in [13] Section 3.1, we note that isomorphism classes of objects in \( \mathcal{H}^k(A^*) \) are in one-to-one correspondence with \( H^k(A^*) \), and that the automorphism category of any object in \( \mathcal{H}^k(A^*) \) is the cochain category \( \mathcal{H}^{k-1}(A^*) \).

**Remark 2.3.** Note that the cocycle 2-category \( \mathcal{H}^k(\tau_{k-2}A^*) \) obtained by replacing \( A^* \) above with its good truncation

\[
\tau_{k-2}A^n = \begin{cases} 
A^n & \text{if } n > k-2 \\
A^{k-2}/\text{im } d & \text{if } n = k-2 \\
0 & \text{if } n < k-2.
\end{cases}
\]

is identical to \( \mathcal{H}^k(A^*) \).

This construction behaves well with respect to morphisms of cochain complexes. In particular, a morphism of cochain complexes \( f : (A^*, d_A) \to (B^*, d_B) \) naturally induces a 2-functor \( \mathcal{H}^k(f) : \mathcal{H}^k(A^*) \to \mathcal{H}^k(B^*) \), and a cochain homotopy \( s : A^* \to B^{*-1} \) between cochain maps \( f \) and \( g \) induces a pseudo-natural transformation \( \mathcal{H}^k(s) : \mathcal{H}^k(f) \Rightarrow \mathcal{H}^k(g) \).

**Lemma 2.4.** Let \((A^*(-), d)\) be a presheaf of complexes of abelian groups over the category \( \text{Mfld} \). Then the assignment \( M \mapsto \mathcal{H}^k(A^*(M)) \) is a presheaf of strict 2-groupoids.

**Proof.** Fix a presheaf of cochain complexes \((A^*(-), d)\) over \( \text{Mfld} \). Then we already have that \( \mathcal{H}^k(A^*(-)) \) is a strict 2-groupoid. For a smooth map \( f : M \to N \) of manifolds, we obtain the pullback map \( f^* : \mathcal{H}^k(A^*(N)) \to \mathcal{H}^k(A^*(M)) \). And for a pair of composable maps \( M \xrightarrow{f} N \xrightarrow{g} P \), we have \( f^* g^* = (g \circ f)^* : A^*(P) \to A^*(M) \) on cochains; therefore, we have trivial natural transformations and, in turn, trivial modifications. \(\square\)

Let \( k \geq 0 \). Given a presheaf of cochain complexes \((A^*, d)\) and a Lie groupoid \( \Gamma_1 \rightrightarrows \Gamma_0 \), consider the bicategory \( \mathcal{H}^k(A^*(\Gamma_\bullet)) \) of \( \Gamma_\bullet \)-equivariant objects of \( \mathcal{H}^k(A^*(-)) \). It is straightforward to verify that there is an isomorphism of bicategories \( \mathcal{H}^k(A^*(\Gamma_\bullet)) \cong \mathcal{H}^k((\tau_{k-2}A^*(\Gamma_\bullet))_{\text{tot}}) \), where \( \tau_{k-2}A^* \) is the good truncation of \( A^* \) at \( k-2 \) (cf. Remark 2.3). In subsequent sections of this paper, we will be particularly interested in the case \( k = 3 \). If the 0th cohomology of the presheaf of complexes vanishes identically, a straightforward argument shows the equivalence of bicategories \( \mathcal{H}^3(A^*(\Gamma_\bullet)) \cong \mathcal{H}^3((\tau_1A^*(\Gamma_\bullet))_{\text{tot}}) \) can be improved.
Proposition 2.5. Let $(\mathcal{A}^s(-), d)$ be a presheaf of cochain complexes on $\text{Mfld}$. Suppose that $H^0(\mathcal{A}^s(M)) = 0$ for all manifolds $M$. Then for any Lie groupoid $\Gamma_1 \rightrightarrows \Gamma_0$, the bicategory $\mathcal{H}^3(\mathcal{A}^s(\Gamma_*))$ of $\Gamma_*$-equivariant objects of $\mathcal{H}^3(\mathcal{A}^s(-))$ is equivalent to the cocycle 2-category $\mathcal{H}^3((\mathcal{A}^s(\Gamma_*))_{\text{tot}})$.

2.2. Differential Cocycles as a 2-Stack. In this section, we review the construction of the complex of differential cocycles on a manifold and the resulting 2-category of differential cocycles (see [12][14]), paying special attention to differential cocycles of degree 3. In Theorem 2.7, we show that the corresponding presheaf of cocycle 2-categories forms a 2-stack, extending the treatment in [14] of the degree 2 case.

Definition 2.6. Fix an integer $s > 0$. Let $M$ be a manifold. The complex of differential cochains of $M$ of height $s$, denoted $\mathcal{D}C^s_{\omega}(M)$, is defined as

$$\mathcal{D}C^s_{\omega}(M) := \{(c, h, \omega) \in C^k(M; \mathbb{Z}) \times C^{k-1}(M; \mathbb{R}) \times \Omega^k(M) \mid \omega = 0 \text{ if } k < s\},$$

with differential $d: \mathcal{D}C^k_{\omega}(M) \to \mathcal{D}C^{k+1}_{\omega}(M)$ given by

$$d(c, h, \omega) := (dc, \omega - c - dh, d\omega).$$

Then $\mathcal{D}C^s_{\omega}$ defines a presheaf of cochain complexes on $\text{Mfld}$, and we let $\mathcal{D}C^k_{\omega}$ denote the presheaf of bicategories, $\mathcal{D}C^k_{\omega}(M) := \mathcal{H}^k(\mathcal{D}C^s_{\omega}(M))$, of the resulting cocycle 2-category. Using the Theorem below, we call $\mathcal{D}C^3_{\omega}$ the 2-stack of differential cocycles of degree 3 and height $s$.

Theorem 2.7. Let $s > 0$. The presheaf of differential 3-cocycles $\mathcal{D}C^3_{\omega}$ is a 2-stack over $\text{Mfld}$.

Proof. The proof is very similar to the proof of [14] Prop. 3.4, which shows that the presheaf $\mathcal{D}C^2_{\omega}$ is a 1-stack. We summarize the main points here. Let $M$ be a manifold and $\pi: U \to M$ a covering. We need to show that the restriction functor $\pi^*: \mathcal{D}C^3_{\omega}(M) \to \mathcal{D}C^3_{\omega}(U)$ is an equivalence. By Remark 2.3 and Proposition 2.5, it suffices to verify that the restriction $\mathcal{H}^3(\tau_1\mathcal{D}C^3_{\omega}(M)) \to \mathcal{H}^3(\tau_1\mathcal{D}C^3_{\omega}(U))$ is an equivalence. Such an equivalence follows directly from the triviality of the cohomology of the double complex

\[
\begin{array}{cccc}
\vdots & \vdots & \vdots \\
\downarrow d & \downarrow d & \downarrow -d \\
\mathcal{D}C^3_{\omega}(M) & \mathcal{D}C^3_{\omega}(U_0) & \mathcal{D}C^3_{\omega}(U_1) & \cdots \\
\downarrow d & \downarrow d & \downarrow -d \\
\mathcal{D}C^2_{\omega}(M) & \mathcal{D}C^2_{\omega}(U_0) & \mathcal{D}C^2_{\omega}(U_1) & \cdots \\
\downarrow d & \downarrow d & \downarrow -d \\
\mathcal{D}C^1_{\omega}(M) & \mathcal{D}C^1_{\omega}(U_0) & \mathcal{D}C^1_{\omega}(U_1) & \cdots \\
\end{array}
\]
where $\tilde{DC}_s^1(-) = DC_s^1(-)/\text{im} \ d$. By the acyclic assembly lemma of homological algebra, it suffices to verify that the above rows are exact, which is shown directly in [14]. □

We will be mainly interested in the case of differential 3-cocycles with heights $s = 1, 2, 3$. We record the following Proposition for later use, whose proof is completely analogous to the one appearing in [14, Sections 4.2 and 4.3] for differential cocycles of degree 2.

**Proposition 2.8.**

1. Let $s \in \{1, 2\}$. For any manifold $M$, the natural projection $\text{pr}: DC_s^3(M) \to C^3(M; \mathbb{Z})$ induces an isomorphism on cohomology: $H^3(DC_s^*(M)) \cong H^3(M; \mathbb{Z})$.

2. For any proper Lie groupoid $\Gamma_1 \rightrightarrows \Gamma_0$, the natural projection $\text{pr}: \bigoplus_{p + q = 3} DC_1^p(\Gamma_q) \to \bigoplus_{p + q = 3} C^p(\Gamma_q; \mathbb{Z})$

induces an isomorphism on cohomology: $H^3(DC_1^*(\Gamma_{\bullet})_{\text{tot}}) \cong H^3(\Gamma_{\bullet}; \mathbb{Z})$.

3. **Dixmier-Douady Bundles**

We begin by recalling some definitions surrounding Dixmier-Douady bundles. For further background, we refer to [1] and [19].

**Definition 3.1 (Dixmier-Douady Bundles).** Fix a manifold $M$. A Dixmier-Douady bundle (DD-bundle) $A \to M$ is a locally trivial bundle of $C^*$-algebras with typical fibre $K(H)$, the $C^*$-algebra of compact operators on a separable complex Hilbert space $H$, and with structure group $\text{Aut}(K(H)) = PU(H)$. Here, we use the strong operator topology.

A Morita isomorphism of DD-bundles $E: (A_1 \to M) \dashv (A_2 \to M)$ is a locally trivial Banach space bundle $E \to M$ with typical fibre $K(H_1, H_2)$, the compact operators from $H_1$ to $H_2$ (where the typical fibre of $A_i$ is $K(H_i)$, $i = 1, 2$). The bundle $E$ comes equipped with a natural fibrewise $(A_2, A_1)$-bimodule structure

$$A_2 \odot E \odot A_1,$$

locally modelled on the natural $(K(H_2), K(H_1))$-bimodule structure on $K(H_1, H_2)$ given by post- and pre-composition of operators. The composition of two Morita isomorphisms $E_1: A_1 \to A_2$ and $E_2: A_2 \to A_3$ is given by $E_2 \odot E_1 = E_2 \otimes_{A_2} E_1$, the fibrewise completion of the (algebraic) tensor product over $A_2$.

Given two Morita isomorphisms $E_1, E_2: A_1 \to A_2$, a 2-isomorphism $\tau: E_1 \Rightarrow E_2$ is a continuous bundle isomorphism $\tau: E_1 \to E_2$ that intertwines the norms and the $(A_2, A_1)$-bimodule structures.
Remark 3.2. One can also define a Morita morphism \((\Phi, \mathcal{E}): (\mathcal{A}_1 \to M_1) \to (\mathcal{A}_2 \to M_2)\) of two DD-bundles \(\mathcal{A}_1 \to M_1\) and \(\mathcal{A}_2 \to M_2\) as a pair: a continuous map \(\Phi: M_1 \to M_2\), and a Morita isomorphism \(\mathcal{E}: \mathcal{A}_1 \to \Phi^* \mathcal{A}_2\). If one were to view a 2-stack as a bicategory fibred in 2-groupoids, then this definition would be required. However, since we are taking the (equivalent) sheaf perspective of a 2-stack, we will not need to work with these.

Lemma 3.3. Let \(M\) be a smooth manifold, and let \(\mathcal{A}_1, \mathcal{A}_2 \to M\) be DD-bundles over \(M\). Suppose that \(\mathcal{E}, \mathcal{E}': \mathcal{A}_1 \to \mathcal{A}_2\) are Morita isomorphisms, and set \(L = \text{Hom}_{\mathcal{A}_2 - \mathcal{A}_1}(\mathcal{E}, \mathcal{E}')\) denote the bundle of bimodule homomorphisms \(\mathcal{E} \to \mathcal{E}'\). Then

1. \(L\) is a Hermitian line bundle over \(M\); and
2. the fibrewise ‘evaluation map’ \(\mathcal{E} \otimes L \to \mathcal{E}'\) is an isomorphism of \((\mathcal{A}_2, \mathcal{A}_1)\)-bimodule bundles.

Proof. That \(L\) is a 1-dimensional complex line bundle follows from the fact that any bimodule homomorphism \(\mathbb{K}(\mathcal{H}_1, \mathcal{H}_2) \to \mathbb{K}(\mathcal{H}_1, \mathcal{H}_2)\) is a scalar. To see this, choose (possibly finite) bases \(\{e_1, e_2, \ldots\}\) for \(\mathcal{H}_1\) and \(\{f_1, f_2, \ldots\}\) for \(\mathcal{H}_2\). Let \(\phi: \mathbb{K}(\mathcal{H}_1, \mathcal{H}_2) \to \mathbb{K}(\mathcal{H}_1, \mathcal{H}_2)\) be a bimodule homomorphism. Then for any \(a \in \mathbb{K}(\mathcal{H}_1, \mathcal{H}_2), x \in \mathbb{K}(\mathcal{H}_2),\) and \(y \in \mathbb{K}(\mathcal{H}_1)\), we have \(\phi(xay) = x\phi(a)y\). Recall that compact operators are the norm closure of the subspace generated by rank one operators. For \(u \in \mathcal{H}_2\) and \(v \in \mathcal{H}_1\), let \(u \otimes v\) be the rank-one operator \((u \otimes v)(w) = (w, v)u\), where \((-,-)\) is the inner product on \(\mathcal{H}_1\). For \(k, l = 1, 2, \ldots\), write \(\phi(f_k \otimes \overline{e_l}) = \sum_{ij} \phi^{(k,l)} f_{ij} \otimes e_j\). Recall that \((f_i \otimes \overline{f_j})(f_k \otimes \overline{e_l}) = f_i \otimes \overline{e_l}\) if \(j = k\) and 0 otherwise, and similarly \((f_k \otimes \overline{e_l})(e_m \otimes \overline{e_n}) = f_k \otimes \overline{e_n}\) if \(m = n\) and 0 otherwise. Then multiplying on the left and right by appropriate elements to isolate coefficients, it is easy to see that the coefficients \(\phi^{(k,l)}\) vanish unless \(k = i\) and \(l = j\). That is \(\phi\) is a diagonal operator. A similar argument shows that \(\phi\) is a scalar. \(\square\)

Any \(*\)-bundle isomorphism \(\phi: \mathcal{A}_1 \to \mathcal{A}_2\) gives rise to a Morita isomorphism: namely, \(\mathcal{A}_2: \mathcal{A}_1 \to \mathcal{A}_2\) with the natural left \(\mathcal{A}_2\)-module structure and right \(\mathcal{A}_1\)-module structure induced by \(\phi\).

Recall that given a Morita isomorphism \(\mathcal{E}: \mathcal{A}_1 \to \mathcal{A}_2\), the \textbf{opposite} Morita isomorphism \(\mathcal{E}^*: \mathcal{A}_2 \to \mathcal{A}_1\) is given by \(\mathcal{E}^* = \mathcal{E}\) as real vector bundles, with opposite (conjugate) scalar multiplication. There are natural 2-isomorphisms \(\mathcal{E}^* \otimes_{\mathcal{A}_2} \mathcal{E} \cong \mathcal{A}_1\) and \(\mathcal{E} \otimes_{\mathcal{A}_1} \mathcal{E}^* \cong \mathcal{A}_2\).

It is straightforward to verify that for any manifold \(M\), the collection of DD-bundles over \(M\) form the objects of a bigroupoid (a weak 2-category in which 1-arrows are coherently invertible and 2-arrows are invertible), with Morita isomorphisms as 1-arrows and 2-isomorphisms as 2-arrows. Denote this bigroupoid by \(\mathcal{B}^2 \mathcal{S}^1(M)\).
Given a map \( f: M_1 \to M_2 \) of manifolds, pullbacks of DD-bundles, as well as 1- and 2-arrows are defined in the usual way, resulting in a pseudofunctor \( f^*: \mathcal{B}^2 \mathcal{S}^1(M_2) \to \mathcal{B}^2 \mathcal{S}^1(M_1) \). In fact, we obtain a (lax) functor \( \mathcal{B}^2 \mathcal{S}^1 \) from \( \text{Mfld}^{\text{op}} \) to \( \text{BiCat} \).

**Proposition 3.4.** The assignment \( M \mapsto \mathcal{B}^2 \mathcal{S}^1(M) \) defines a presheaf of bicategories.

**Proof.** Given a pair of composable maps

\[
R \xrightarrow{f} S \xrightarrow{g} T
\]

and a DD-bundle \( \mathcal{A} \to T \), there is a canonical bundle \(*\)-isomorphism

\[
\phi_{f,g}: f^*(g^* \mathcal{A})) \to (gf)^* \mathcal{A}
\]

and hence a corresponding canonical Morita isomorphism, the \(( (gf)^* \mathcal{A}, f^*(g^* \mathcal{A}))\)-bimodule \( \mathcal{E}_{f,g}^A = \mathcal{E}^A = (gf)^* \mathcal{A} \). Moreover, if \( \mathcal{F}: \mathcal{A} \dashv \to \mathcal{A}' \) is a Morita isomorphism, there is a canonical 2-isomorphism

\[
\sigma_{\mathcal{F}}: \mathcal{E}^{A'} \otimes_{f^* \mathcal{A}'} f^* \mathcal{F} \to (gf)^* \mathcal{F} \otimes_{(gf)^* \mathcal{A}} \mathcal{E}^A
\]

induced from the \(( \mathcal{A}', \mathcal{A})\)-bimodule action on \( \mathcal{F} \). In other words, we have a natural isomorphism of functors \( f^* \circ g^* \cong (gf)^* \).

Given composable maps

\[
Q \xrightarrow{f} R \xrightarrow{g} S \xrightarrow{h} T
\]

we show next that there exists a modification between the composite natural isomorphism

\[
f^* \circ g^* \circ h^* \cong (gf)^* \circ h^* \cong (hgf)^*
\]

(3.1)

and the composite natural isomorphism

\[
f^* \circ g^* \circ h^* \cong f^* \circ (hg)^* \cong (hgf)^*.
\]

(3.2)

Indeed, given a DD-bundle \( \mathcal{A} \to T \), the first composition is given by the Morita isomorphism

\[
\mathcal{E}^A_{gf,h} \otimes_{(gf)^* \mathcal{A}} \mathcal{E}^{h* \mathcal{A}}_{f,g} = (hgf)^* \mathcal{A} \otimes_{(gf)^* \mathcal{A}} (gf)^* \mathcal{A}
\]

while the second is given by

\[
\mathcal{E}^A_{f,hg} \otimes_{f^* (hg)^* \mathcal{A}} f^* \mathcal{E}^A_{g,h} = (hgf)^* \mathcal{A} \otimes_{f^* (hg)^* \mathcal{A}} f^* (hg)^* \mathcal{A}.
\]

Each of these is 2-isomorphic (via the respective right-action maps) to \( (hgf)^* \mathcal{A} \). To verify that this results in a family \( \theta(\mathcal{A}) \) of modifications from the composition (3.1) to (3.2) (as in Definition 1.1 (4)), let \( \rho: \mathcal{F} \to \mathcal{G} \) be a 2-isomorphism between Morita isomorphisms.
\( \mathcal{F}, \mathcal{G} : \mathcal{A} \to \mathcal{A}' \). The required equality follows from the commutativity of the diagram below (where we have omitted the subscripts under the \( \otimes \) symbols, for simplicity).

\[
\begin{array}{c}
(hgf)^* \mathcal{A} \otimes (gf)^* h^* \mathcal{A}' \otimes f^* g^* h^* \mathcal{F} \\
\xrightarrow{\text{id} \otimes \sigma_{h^*} \otimes \mathcal{F}} \\
(hgf)^* \mathcal{A} \otimes f^* (hg)^* \mathcal{A}' \otimes f^* g^* h^* \mathcal{G}
\end{array}
\]

The commutativity follows from the fact that \( \rho \) respects the bimodule actions on (pullbacks of) \( \mathcal{F} \) and \( \mathcal{G} \). The coherence condition is similarly verified; it follows from the axioms of a bimodule action. \( \square \)

Applying Definition to the presheaf \( B^2 S^1 \) of DD-bundles over manifolds, we make the following definition.

**Definition 3.5 (Equivariant DD-bundles).** Let \( \Gamma_1 \rightrightarrows \Gamma_0 \) be a Lie groupoid.

1. A **\( \Gamma \)-equivariant DD-bundle** is a triple \( (\mathcal{A}, \mathcal{E}, \tau) \) consisting of a DD-bundle \( \mathcal{A} \to \Gamma_0 \), a Morita isomorphism \( \mathcal{E} : \partial^*_0 \mathcal{A} \rightrightarrows \partial^*_1 \mathcal{A} \) and a 2-isomorphism \( \tau : \partial^*_2 \mathcal{E} \otimes \partial^*_0 \partial^*_1 \mathcal{A} \partial^*_0 \mathcal{E} \Rightarrow \partial^*_1 \mathcal{E} \) satisfying the coherence condition \( \partial^*_2 \tau \circ (\text{id} \star \partial^*_0 \tau) = \partial^*_1 \tau \circ (\partial^*_0 \tau \star \text{id}) \).

2. A **\( \Gamma \)-equivariant Morita isomorphism** \( (\mathcal{E}, \mathcal{A}, \tau) \to (\mathcal{E}', \mathcal{A}', \tau') \) is a pair \( (\mathcal{G}, \phi) \) consisting of a Morita isomorphism \( \mathcal{G} : \mathcal{A} \rightrightarrows \mathcal{A}' \) and a 2-isomorphism \( \phi : \mathcal{E}' \otimes \partial^*_0 \mathcal{G} \Rightarrow \partial^*_1 \mathcal{G} \otimes \mathcal{E} \) satisfying a coherence condition \( (\text{id} \star \tau') \circ (\partial^*_2 \phi \star \text{id}) \circ (\text{id} \star \partial^*_0 \phi) = \partial^*_1 \phi \circ (\tau' \star \text{id}) \).

3. A **\( \Gamma \)-equivariant 2-isomorphism** \( (\mathcal{G}, \phi) \Rightarrow (\mathcal{G}', \phi') \) is a 2-isomorphism \( \beta : \mathcal{G} \Rightarrow \mathcal{G}' \) satisfying \( \phi' \circ (\text{id} \star \partial^*_0 \beta) = (\partial^*_1 \beta \star \text{id}) \circ \phi \).

**Remark 3.6.** Along the lines of Remark, Definition freely uses simplicial identities—for example, by viewing \( \partial^*_1 \mathcal{E} \) as a \( (\partial^*_1 \partial^*_1 \mathcal{A}, \partial^*_0 \partial^*_0 \mathcal{A}) \)-bimodule and \( \partial^*_2 \mathcal{E} \) as a \( (\partial^*_1 \partial^*_1 \mathcal{A}, \partial^*_0 \partial^*_0 \mathcal{A}) \)-bimodule.

**Remark 3.7.** The usual notion of a \( G \)-equivariant DD-bundle over \( M \) is more restrictive than that of a \( (G \times M \rightrightarrows M) \)-equivariant DD-bundle. A DD-bundle \( \mathcal{A} \to M \) equipped with a \( G \)-action that lifts the \( G \)-action on \( M \) gives an equivariant DD-bundle in the sense of Definition with \( \mathcal{E} = \partial^*_0 \mathcal{A} \) coming from the \( \ast \)-isomorphisms given by the \( G \)-action on \( \mathcal{A} \), and with trivial 2-isomorphism component. However, as noted in Remark for compact Lie groups \( G \), every \( (G \times M \rightrightarrows M) \)-equivariant DD-bundle is Morita isomorphic to a genuine \( G \)-equivariant DD-bundle.

We will need the following Lemma for Theorem below.
Lemma 3.8. Suppose \( g \in \mathrm{U}(\mathcal{H}) \) implements an automorphism \( \text{Ad}_g : \mathbb{K}(\mathcal{H}) \to \mathbb{K}(\mathcal{H}) \). View \( \mathbb{K}(\mathcal{H}) \) as a \( \mathbb{K}(\mathcal{H}) \)-bimodule, with right action via \( \text{Ad}_g \). Let \( e_1 \in \mathcal{H} \) be a unit vector. The map \( \tilde{g} : \mathcal{H}^{\text{op}} \to \mathcal{H}^{\text{op}} \otimes \mathbb{K}(\mathcal{H}) \) given by \( \tilde{g}(v) = e_1 \otimes (e_1 \otimes \overline{v})g^* \) is an isomorphism of \( (\mathbb{C}, \mathbb{K}(\mathcal{H})) \)-bimodules. In other words, \( \tilde{g} \) fills in the 2-cell:

\[
\begin{array}{c}
\mathcal{C} \\
\downarrow \quad \downarrow \tilde{g} \\
\mathcal{H}^{\text{op}} \\
\downarrow \quad \quad \downarrow \\
\mathbb{K}(\mathcal{H}) \\
\downarrow \quad \quad \downarrow \\
\mathbb{K}(\mathcal{H})
\end{array}
\]

(Here, \( u \otimes \overline{v} \) denotes the rank 1 operator \( w \mapsto (w, v)u \), where \( (\cdot, \cdot) \) is the inner product on \( \mathcal{H} \).)

Proof. Let \( e_1 \) be a unit vector in \( \mathcal{H}^{\text{op}} \), and let \( \tilde{g} \) be as in the statement of the lemma. Since \( \tilde{g} \) is \( \mathbb{C} \)-linear, it remains to check that \( \tilde{g} \) is a map of right \( \mathbb{K}(\mathcal{H}) \)-modules. Let \( x \in \mathbb{K}(\mathcal{H}) \), and observe that for \( v \in \mathcal{H}^{\text{op}} \)

\[
\tilde{g}(v \cdot x) = \tilde{g}(x^*v)
\]

\[
= e_1 \otimes (e_1 \otimes \overline{x}v)g^*
\]

\[
= e_1 \otimes (e_1 \otimes \overline{v})xg^*
\]

\[
= \tilde{g}(v) \cdot x
\]

A direct calculation shows that \( \tilde{g} \) is independent of the choice of unit vector. \( \square \)

Theorem 3.9. The presheaf of DD-bundles \( B^2S^1 \) is a 2-stack over \( \text{Mfld} \).

Proof. To show that \( B^2S^1 \) is a 2-stack, we verify that for any \( M \) and any cover \( \pi : U \to M \) by open sets \( \{U_\alpha\} \) in \( M \), the restriction \( \pi^* : B^2S^1(M) \to B^2S^1(U_\bullet) \) induces an equivalence of bicategories.

We begin by verifying that \( \pi^* \) is fully faithful on Hom categories (i.e. bijections on the corresponding 2-morphisms 2-Hom). Let \( \mathcal{A} \) and \( \mathcal{B} \) be DD-bundles over \( M \). Denote the restriction by \( (-)|_U \). Let \( \mathcal{E}, \mathcal{F} \in \text{Hom}(\mathcal{A}, \mathcal{B}) \), and consider the restriction \( 2\text{-Hom}(\mathcal{E}, \mathcal{F}) \to 2\text{-Hom}(\mathcal{E}|_U, \mathcal{F}|_U) \). This is a bijection because a continuous bundle map is uniquely determined by its restrictions to open sets in a cover that agree on overlaps.

Next, we show that the restriction functor is essentially surjective on Hom categories, which shows that the restriction functor (on bicategories) is fully faithful (and hence \( B^2S^1 \) is a 2-prestack). Recall a 1-morphism in \( \mathcal{A}|_U \to \mathcal{B}|_U \) in \( B^2S^1(U_\bullet) \) consists of a collection \( (\mathcal{E}_\alpha, \phi_{\alpha\beta}) \) of bundles \( \mathcal{E}_\alpha \to U_\alpha \) of bimodules together with 2-morphisms \( \phi_{\alpha\beta} : \mathcal{E}_\alpha|_{U_\alpha\beta} \to \mathcal{E}_\beta|_{U_\alpha\beta} \) satisfying

\[
\phi_{\beta\gamma}|_{U_{\alpha\beta\gamma}} \circ \phi_{\alpha\beta}|_{U_{\alpha\beta\gamma}} = \phi_{\alpha\gamma}|_{U_{\alpha\beta\gamma}}.
\]
Hence the bundles $\mathcal{E}_a$ glue together to give a 1-morphism $\mathcal{E} \in \text{Hom}(\mathcal{A}, \mathcal{B})$; therefore, restriction is essentially surjective on Hom categories, as desired.

Finally, we show that $\pi^*$ is an equivalence of bicategories, by showing it is essentially surjective on objects. Let $(\mathcal{A}_*, \mathcal{E}_a, \tau_{\alpha\beta\gamma})$ be an object in $\mathcal{B}^1\mathcal{S}^1(U_*)$, where $\mathcal{A}_a \to U_\alpha$ are DD-bundles, equipped with $\left(\mathcal{A}_a\bigg|_{U_{\alpha\beta}}, \mathcal{A}_\beta\bigg|_{U_{\alpha\beta}}\right)$-bimodules $\mathcal{E}_{\alpha\beta} \to U_{\alpha\beta}$ and 2-isomorphisms $\tau_{\alpha\beta\gamma} : \mathcal{E}_{\alpha\beta}\bigg|_{U_{\alpha\beta\gamma}} \otimes \mathcal{E}_{\beta\gamma}\bigg|_{U_{\alpha\beta\gamma}} \to \mathcal{E}_{\alpha\gamma}\bigg|_{U_{\alpha\beta\gamma}}$ satisfying the coherence condition $\partial \tau = 0$—i.e., such that the following diagram commutes (over $U_{\alpha\beta\gamma}$):

$$\begin{array}{ccc}
\mathcal{E}_{\alpha\beta} \otimes \mathcal{E}_{\beta\gamma} \otimes \mathcal{E}_{\gamma\delta} & \xrightarrow{id \otimes \tau_{\beta\gamma\delta}} & \mathcal{E}_{\alpha\beta} \otimes \mathcal{E}_{\beta\delta} \\
\tau_{\alpha\beta\gamma} \otimes id & & \tau_{\alpha\beta\delta} \\
\mathcal{E}_{\alpha\gamma} \otimes \mathcal{E}_{\gamma\delta} & \xrightarrow{\tau_{\alpha\gamma\delta}} & \mathcal{E}_{\alpha\delta}
\end{array}$$

(3.3)

We wish to find a DD-bundle $\mathcal{B} \to \mathcal{M}$ and a 1-morphism $(\mathcal{G}_a, \phi_{\alpha\beta}) : \mathcal{B}\big|_U \to (\mathcal{A}_a, \mathcal{E}_{\alpha\beta}, \tau_{\alpha\beta\gamma})$.

Note that it suffices to assume that $\{U_\alpha\}$ is a good cover. Indeed, suppose $V$ is a refinement of $U$. Then by [17] Lemma 4.3, $V_* \to U_*$ is a weak equivalence of groupoids, and hence, by [17] Theorem 2.1.6, the restriction functor $r : \mathcal{B}^1\mathcal{S}^1(U_*) \to \mathcal{B}^1\mathcal{S}^1(V_*)$ is fully faithful. Suppose we have $\mathcal{B}$ and a morphism $\mathcal{B}\big|_V \cong (\mathcal{B}\big|_U)|_V \to (\mathcal{A}_a, \mathcal{E}_{\alpha\beta}, \tau_{\alpha\beta\gamma})|_V$. Since $r$ is fully faithful, we get the desired morphism $\mathcal{B}\big|_U \to (\mathcal{A}_a, \mathcal{E}_{\alpha\beta}, \tau_{\alpha\beta\gamma})$.

The DD-bundle $\mathcal{B}$ will result from an $\mathcal{S}^1$-valued 2-cocycle, defined by gluing trivial $\mathbb{K}(\mathcal{H})$-bundles (with $\dim \mathcal{H} = \infty$) over $U_\alpha$ with transition maps $g_{\alpha\beta} : U_{\alpha\beta} \to \text{PU}(\mathcal{H})$. In this case, the restriction $\mathcal{B}\big|_U$ in $\mathcal{B}^1\mathcal{S}^1(U_*)$ may be described as follows. Let $\mathcal{B}_a = \mathcal{B}\big|_{U_\alpha} = U_\alpha \times \mathbb{K}(\mathcal{H})$. The transition maps $g_{\alpha\beta}$ give bundle isomorphisms $\mathcal{B}_\beta \to \mathcal{B}_\alpha$, and with corresponding Morita isomorphism $\mathcal{B}_{\alpha\beta} = \mathcal{B}_a$ with the right $\mathcal{B}_\beta$-action obtained via $g_{\alpha\beta}$. The cocycle condition $g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma}$ guarantees that the action map

$$\lambda_{\alpha\beta\gamma} : \mathcal{B}_{\alpha\beta} \otimes \mathcal{B}_\beta \to \mathcal{B}_\alpha$$

is a map of $(\mathcal{B}_a, \mathcal{B}_\gamma)$-bimodules. Hence, $\mathcal{B}\big|_U = (\mathcal{B}_a, \mathcal{B}_{\alpha\beta}, \lambda_{\alpha\beta\gamma})$.

To get the 2-cocycle defining $\mathcal{B}$, begin by choosing Morita trivializations $\mathcal{F}_a : \mathcal{A}_a \to \mathbb{C}$ (using the contractibility of the open sets $U_\alpha$). Then over each $U_{\alpha\beta}$, we get the pair of Morita isomorphisms,

$$\mathcal{E}_{\alpha\beta}, \mathcal{F}_a^* \otimes \mathcal{F}_\beta : \mathcal{A}_\beta \to \mathcal{A}_a.$$

Since we are assuming a good cover, the line bundles $L_{\alpha\beta} = \text{Hom}(\mathcal{E}_{\alpha\beta}, \mathcal{F}_a^* \otimes \mathcal{F}_\beta)$ are trivializable. Therefore, we may choose 2-isomorphisms (i.e. sections of $L_{\alpha\beta}$) $\sigma_{\alpha\beta} : \mathcal{E}_{\alpha\beta} \to \mathcal{F}_a^* \otimes \mathcal{F}_\beta$. That is, we have the following 2-cell:
Over triple intersections, we get the pair of Morita isomorphisms,
\[ \mathcal{E}_{\alpha\beta} \otimes \mathcal{E}_{\beta\gamma} \xrightarrow{\tau_{\alpha\beta\gamma}} \mathcal{E}_{\alpha\gamma}, \] and
\[ \mathcal{E}_{\alpha\beta} \otimes \mathcal{E}_{\beta\gamma} \xrightarrow{\sigma_{\alpha\beta\gamma} \otimes \sigma_{\beta\gamma}} \mathcal{F}_{\alpha}^* \otimes \mathcal{F}_{\beta}^* \otimes \mathcal{F}_{\gamma}^* \rightarrow \mathcal{F}_{\alpha}^* \otimes \mathcal{F}_{\gamma} \rightarrow \mathcal{E}_{\alpha\gamma}, \] which correspond to the two ways of filling in the 2-cell shown below.

That is, we have two sections of the line bundle \( L'_{\alpha\beta\gamma} = \text{Hom}(\mathcal{E}_{\alpha\beta} \otimes \mathcal{E}_{\beta\gamma}, \mathcal{E}_{\alpha\gamma}) \), which must therefore differ by an \( S^1 \)-valued function \( s_{\alpha\beta\gamma} : U_{\alpha\beta\gamma} \rightarrow S^1 \) defined by:
\[ \tau_{\alpha\beta\gamma} = s_{\alpha\beta\gamma} \sigma_{\alpha\gamma}^{-1} \circ (\sigma_{\alpha\beta} \otimes \sigma_{\beta\gamma}). \]

We claim that \( s_{\alpha\beta\gamma} \) defines a 2-cocycle. To see this, consider \( \rho_{\alpha\beta\gamma} = \sigma_{\alpha\gamma}^{-1} \circ (\sigma_{\alpha\beta} \otimes \sigma_{\beta\gamma}) \). A direct calculation shows that \( \rho \) also satisfies the coherence condition “\( \partial \rho = 0 \)” (similar to (3.3)). Hence taking \( \partial \) of both sides of the equation above gives the desired cocycle condition
\[ s_{\alpha\beta\gamma} s_{\alpha\gamma\delta} = s_{\alpha\beta\delta} s_{\beta\gamma\delta}. \]

Let \( B \rightarrow M \) be a DD-bundle defined by this 2-cocycle.

For later use, we note that since \( \{U_\alpha\} \) is a good cover, we can find lifts \( \hat{g}_{\alpha\beta} : U_{\alpha\beta} \rightarrow U(H) \) (with \( \text{Ad}\hat{g}_{\alpha\beta} = g_{\alpha\beta} \)) so that
\[ (\hat{g}_{\alpha\beta} \hat{g}_{\beta\gamma})^* = s_{\alpha\beta\gamma} \hat{g}_{\alpha\gamma}^*. \] (3.4)
(A priori, (3.4) may only be true up to a coboundary; however, one can choose lifts that give equality on the nose. See the proof of [19, Proposition 4.83].)
Next we construct an isomorphism \((\mathcal{G}, \phi): (\mathcal{B}_\alpha, \mathcal{E}_{\alpha\beta}, \lambda_{\alpha\beta\gamma}) \to (\mathcal{A}_\alpha, \mathcal{E}_{\alpha\beta}, \tau_{\alpha\beta\gamma})\). Since each \(\mathcal{B}_\alpha\) is a trivial \(\mathbb{K}(\mathcal{H})\)-bundle, we have canonical Morita isomorphisms \(\mathcal{H}^{op}: \mathcal{B}_\alpha \to \mathbb{C}\) (recall \(\mathcal{H}^{op}\) is the trivial \(\mathcal{H}\)-bundle over \(U_\alpha\) with conjugate scalar multiplication, viewed as a \((\mathbb{C}, \mathbb{K}(\mathcal{H}))\)-bimodule.) Let

\[
\mathcal{G}_\alpha = \mathcal{F}_\alpha^* \otimes \mathcal{H}^{op}: \mathcal{B}_\alpha \to \mathcal{A}_\alpha.
\]

To construct \(\phi\), we use Lemma 3.8 to get the \((\mathbb{C}, \mathcal{B}_\beta)\)-bimodule map

\[
\tilde{g}_{\alpha\beta}: \mathcal{H}^{op} \to \mathcal{H}^{op} \otimes B_{\alpha\beta}.
\]

Let \(\phi_{\alpha\beta}: \mathcal{E}_{\alpha\beta} \otimes \mathcal{G}_\beta \to \mathcal{G}_\alpha \otimes B_{\alpha\beta}\) denote the 2-morphism given by the interior of the 2-cell below.

We claim that the coherence condition on \(\phi\) (see Definition 1.2 (3)) is satisfied, and hence \((\mathcal{G}, \phi)\) defines an isomorphism. Indeed, the coherence condition amounts to the commutativity of the following diagram.

\[
\begin{array}{ccc}
\mathcal{E}_{\alpha\beta} \otimes \mathcal{E}_{\beta\gamma} \otimes \mathcal{F}_\gamma^* \otimes \mathcal{H}^{op} & \xrightarrow{\mathcal{E}_{\alpha\beta} \otimes \mathcal{F}_\beta^* \otimes \mathcal{H}^{op} \otimes B_{\beta\gamma}} & \mathcal{F}_\alpha^* \otimes \mathcal{H}^{op} \otimes B_{\alpha\beta} \otimes B_{\beta\gamma} \\
\tau_{\alpha\beta\gamma} \otimes \mathcal{E}_{\beta\gamma} \otimes \mathcal{F}_\gamma^* \otimes \mathcal{H}^{op} & \xrightarrow{\phi_{\alpha\beta} \otimes \mathcal{F}_\beta^* \otimes \mathcal{F}_\gamma^* \otimes \mathcal{H}^{op} \otimes B_{\beta\gamma}} & \mathcal{F}_\alpha^* \otimes \mathcal{H}^{op} \otimes B_{\alpha\gamma} \\
\end{array}
\]

To see that the diagram commutes, observe that the composition along the top can be written as

\[
\begin{array}{ccc}
\mathcal{E}_{\alpha\beta} \otimes \mathcal{E}_{\beta\gamma} \otimes \mathcal{F}_\gamma^* \otimes \mathcal{H}^{op} & \xrightarrow{\sigma_{\alpha\beta} \otimes \mathcal{F}_\beta^* \otimes \mathcal{E}_{\beta\gamma} \otimes \mathcal{H}^{op} \otimes B_{\beta\gamma}} & \mathcal{F}_\alpha^* \otimes \mathcal{F}_\beta^* \otimes \mathcal{F}_\gamma^* \otimes \mathcal{F}_\gamma^* \otimes \mathcal{H}^{op} \otimes B_{\beta\gamma} \\
\mathcal{F}_\alpha^* \otimes \mathcal{H}^{op} \otimes B_{\beta\gamma} & \xrightarrow{id \otimes \tilde{g}_{\alpha\beta} \otimes \mathcal{F}_\gamma^*} & \mathcal{F}_\alpha^* \otimes \mathcal{H}^{op} \otimes B_{\alpha\beta} \otimes B_{\beta\gamma} \\
\mathcal{F}_\alpha^* \otimes \mathcal{H}^{op} \otimes B_{\beta\gamma} & \xrightarrow{\mathcal{F}_\alpha^* \otimes \mathcal{H}^{op} \otimes B_{\alpha\beta} \otimes B_{\beta\gamma}} & \mathcal{F}_\alpha^* \otimes \mathcal{H}^{op} \otimes B_{\alpha\gamma} \\
\end{array}
\]
(where the curved arrow pair uses the natural pairing given by the fibrewise inner product),
while the composition along the bottom can be written as
\[
\mathcal{E}_{\alpha\beta} \otimes \mathcal{E}_{\beta\gamma} \otimes F^*_\gamma \otimes \mathcal{H}^{op} \xrightarrow{\tau_{\alpha\beta} \otimes \text{id} \otimes \tilde{g}_{\alpha\gamma}} \mathcal{E}_{\alpha\gamma} \otimes F^*_\gamma \otimes \mathcal{H}^{op} \otimes B_{\alpha\gamma}
\]
\[
\sigma_{\alpha\gamma} \otimes \text{id} \otimes \text{id}
\]
\[
F^*_\alpha \otimes F_\gamma \otimes F^*_\gamma \otimes \mathcal{H}^{op} \otimes B_{\alpha\gamma} \xrightarrow{\text{pair}} F^*_\alpha \otimes \mathcal{H}^{op} \otimes B_{\alpha\gamma}
\]
Comparing the above compositions, we see that they agree because equation (3.4) holds.

It follows that the presheaf \( B^2S^1 \) of DD-bundles is a 2-stack. \qed

4. THE DIXMIER-DOUADY 2-FUNCTOR

In this section we prove the main result of this paper, Theorem 4.3, which states that the 2-stack of Dixmier-Douady bundles is equivalent to the 2-stack of differential 3-cocycles. We also relate differential 3-cocycles with \( S^1 \)-bundle gerbes, and establish analogous results for \( S^1 \)-bundle gerbes with connection and curving.

**Definition 4.1.** Let \( B^2S^1_{\text{triv}} \) be the presheaf of bicategories with a single object in each \( B^2S^1_{\text{triv}}(M) \), the trivial DD-bundle \( \mathcal{A}_0 := M \times \mathbb{K}(\mathcal{H}) \) (with \( \dim \mathcal{H} = \infty \)), with 1-arrows all Morita isomorphisms from \( \mathcal{A}_0 \) to itself, along with all associated 2-arrows. That is, the morphism category over \( M \) is \( \text{Hom}_{B^2S^1(M)}(\mathcal{A}_0, \mathcal{A}_0) \).

**Lemma 4.2.** \( B^2S^1_{\text{triv}} \) is a prestack with stackification \( B^2S^1 \).

**Proof.** This follows directly from Theorem 3.9. \qed

**Theorem 4.3.** The 2-stack \( B^2S^1 \) of DD-bundles is equivalent to the 2-stack \( DC^3_1 \) of differential 3-cocycles of height 1.

**Proof.** Fix an infinite dimensional separable Hilbert space \( \mathcal{H} \). Let \( M \) be a manifold, and let \( \mathcal{A}_0 \) be the trivial DD-bundle as in Definition 4.1. By Lemma 3.3, we have an equivalence \( \text{Hom}_{B^2S^1(M)}(\mathcal{A}_0, \mathcal{A}_0) \cong BS^1(M) \), where \( BS^1(M) \) denotes the category of principal \( S^1 \)-bundles over \( M \). Define a morphism of 2-prestacks \( DD_{\text{triv}}: B^2S^1_{\text{triv}} \to DC^3_1 \) by setting \( DD_{\text{triv}}(M)(\mathcal{A}_0) = 0 = (0, 0, 0) \), while on morphisms, \( DD_{\text{triv}}(M) = \text{Ch}(M): BS^1(M) \to \text{Hom}_{DC^3_1(M)}(0, 0) = DC^3_1(M) \) (see Remark 2.2), where \( \text{Ch} \) is the equivalence of stacks in \( [14] \). Since \( DD_{\text{triv}} \) is fibrewise fully faithful, and every object in \( DC^3_1 \) is locally isomorphic to one in the image of \( B^2S^1_{\text{triv}} \) (since isomorphism classes of objects in \( DC^3_1(M) \) are classified by \( H^3(DC^3_1(M)) \cong H^3(M; \mathbb{Z}) \), by Proposition 2.8, and \( M \) can be covered by contractible open sets), then \( DC^3_1 \) is a 2-stackification of \( B^2S^1_{\text{triv}} \) (see \( [5] \) Section 1.10). Since \( B^2S^1 \) is a 2-stackification of \( B^2S^1_{\text{triv}} \), the morphism \( DD_{\text{triv}} \) extends to an equivalence \( DD: B^2S^1 \to DC^3_1 \) of 2-stacks. \qed
Theorem 4.3 has some immediate consequences. For any manifold $M$, the DD-functor gives an equivalence of bicategories $B^2 S^1(M) \cong DC^3_1(M)$. Hence,

**Corollary 4.4.** Let $M$ be a manifold. There is a one-to-one correspondence between Morita isomorphism classes of DD-bundles over $M$ and $H^3(M; \mathbb{Z})$.

*Proof.* Isomorphism classes of objects in $DC^3_1(M)$ (see Remark 2.2) are in one-to-one correspondence with $H^3(DC^*_1(M)) \cong H^3(M; \mathbb{Z})$, by Proposition 2.8.

More generally, for a Lie groupoid $\Gamma_1 \rightrightarrows \Gamma_0$, the DD-functor gives an equivalence of bicategories $B^2 S^1(\Gamma) \cong DC^3_1(\Gamma)$. Hence,

**Corollary 4.5.** Let $\Gamma_1 \rightrightarrows \Gamma_0$ be a proper Lie groupoid. There is a one-to-one correspondence between Morita isomorphism classes of $\Gamma$-equivariant DD-bundles and $H^3(\Gamma; \mathbb{Z})$.

*Proof.* Isomorphism classes of objects in $DC^3_1(\Gamma) = H^3(DC^*_1(\Gamma))$ (see Remark 2.2) are in one-to-one correspondence with $H^3((DC^*_1(\Gamma))_{tot}) \cong H^3(\Gamma; \mathbb{Z})$, by Proposition 2.5 and Proposition 2.8.

**Remark 4.6.** Composing the DD-functor with $pr$ from Proposition 2.8 for any DD-bundle we obtain the classical DD-class in $H^3(M; \mathbb{Z})$ (or $H^3(\Gamma; \mathbb{Z})$ in the equivariant case).

In the case that $\Gamma_1 \rightrightarrows \Gamma_0$ is an action groupoid corresponding to an action of a Lie group on a manifold, we obtain:

**Corollary 4.7.** Let $G$ be a Lie group acting smoothly and properly on a manifold $M$. There is a one-to-one correspondence between Morita isomorphism classes of $(G \times M \rightrightarrows M)$-equivariant DD-bundles and (the simplicial model for) $G$-equivariant cohomology $H^3_G(M; \mathbb{Z})$.

**Remark 4.8.** For compact Lie groups $G$, it is well known that (ordinary) $G$-equivariant DD-bundles over $M$ are classified by $H^3_G(M; \mathbb{Z})$. Corollary 4.7 implies that every $(G \times M \rightrightarrows M)$-equivariant bundle is Morita isomorphic to a genuine $G$-equivariant DD-bundle.

**Differential Characters, Bundle Gerbes, and Connections.** In this section, we briefly comment on the equivalence of Dixmier-Douady bundles and $S^1$-bundle gerbes, and provide refinements of Theorem 4.3 for bundle gerbes with connective structures. Bundle gerbes were defined by Murray [16] (see also [6, 10, 11] and [20] for details). In [17], the authors show that bundle gerbes form a 2-stack, namely the 2-stackification of the pre-stack $Grb_{triv}$, which by definition is the presheaf of bicategories with a single object in $Grb_{triv}(M)$ and morphisms the category $BS^1(M)$ of principal $S^1$-bundles over $M$. In particular, since the prestacks $Grb_{triv}$ and $B^2 S^1_{triv}$ are equivalent, we immediately obtain the equivalence between $S^1$-bundle gerbes $Grb$ and DD-bundles $B^2 S^1$. This gives part (1) of the Theorem below:
Theorem 4.9.

(1) The 2-stack \( \text{Grb} \) of \( S^1 \)-bundle gerbes is equivalent to the 2-stack \( \mathcal{DC}_1^3 \) of differential 3-cocycles of height 1.

(2) The 2-stack \( \text{Grb}^\nabla \) of \( S^1 \)-bundle gerbes with connection (but no specified curving) is equivalent to the 2-stack \( \mathcal{DC}_2^3 \) of differential 3-cocycles of height 2.

Proof of (2). The 2-stack \( \text{Grb}^\nabla \) is the 2-stackification of the presheaf of bicategories \( \text{Grb}_{\text{triv}}^\nabla \), which by definition is the presheaf of bicategories with a single object in \( \text{Grb}_{\text{triv}}^\nabla (M) \) and morphisms the category \( \mathcal{DBS}^1(M) \) of principal \( S^1 \)-bundles over \( M \) with connection. The rest of the argument is the same as in the proof of Theorem 4.3, using instead the equivalence \( \mathcal{DBS}^1 \cong \mathcal{DC}_2^3 \) from \([14]\). \( \square \)

Similar to the corollaries following Theorem 4.3, we obtain the following:

Corollary 4.10. Let \( M \) be a manifold. There is a one-to-one correspondence between (stable) isomorphism classes of \( S^1 \)-bundle gerbes over \( M \) (with or without connection) and \( H^3(M; \mathbb{Z}) \).

Corollary 4.11. Let \( \Gamma_1 \to \Gamma_0 \) be a proper Lie groupoid. There is a one-to-one correspondence between (stable) isomorphism classes of \( \Gamma_\bullet \)-equivariant \( S^1 \)-bundle gerbes without connection and \( H^3(\Gamma_\bullet; \mathbb{Z}) \).

Corollary 4.12. Let \( \Gamma_1 \to \Gamma_0 \) be a Lie groupoid. There is a one-to-one correspondence between (stable) isomorphism classes of \( \Gamma_\bullet \)-equivariant \( S^1 \)-bundle gerbes with connection and \( H^3(\mathcal{DC}_2^\ast(\Gamma_\bullet)_{\text{tot}}) \).

In fact, the argument used to prove the equivalence of Theorem 4.3 (and hence also Theorem 4.9) can be used to show an equivalence between differential characters of degree 3 and bundle gerbes with connection and curving.

Theorem 4.13. The 2-stack of \( S^1 \)-bundle gerbes with connection and curving \( \text{Grb}^\nabla, B \) is equivalent to the 2-stack \( \mathcal{DC}_3^3 \) of differential characters of degree 3.

Proof. Recall from \([17]\), that the 2-stack of bundle gerbes with connection and curving, denoted \( \text{Grb}^{\nabla, B} \), is a stackification of a 2-prestack \( \text{Grb}_{\text{triv}}^{\nabla, B} \), defined as follows. Objects in \( \text{Grb}_{\text{triv}}^{\nabla, B}(M) \) are 2-forms \( \beta \in \Omega^2(M) \); 1-arrows are line bundles over \( M \) with connection \( (L, \nabla) : \beta_1 \to \beta_2 \) where \( \text{curv}(\nabla) = \beta_2 - \beta_1 \); 2-arrows are connection-preserving bundle maps \( (L, \nabla) \to (L', \nabla') \).

Similar to the proof of Theorem 4.3, we define a morphism of 2-prestacks \( F_{\text{triv}} : \text{Grb}_{\text{triv}}^{\nabla, B} \to \mathcal{DC}_3^3 \) as follows. Let \( M \) be a manifold. For an object \( \beta \in \Omega^2(M) \) of \( \text{Grb}_{\text{triv}}^{\nabla, B}(M) \), set \( F_{\text{triv}}(\beta) = (0, \beta, d\beta) \). On morphism categories, we use the equivalence \( \text{DCh} : \mathcal{DBS}^1 \to \mathcal{DC}_2^3 \) from \([14]\, Section 4.4\), between \( S^1 \)-bundles with connection and differential characters of degree 2. In
particular, send \((L, \nabla): \beta_1 \to \beta_2\) to \((c, h, 0) \in \text{DC}^3_2(M)\), where \(\text{DCh}(L, \nabla) = (c, h, \beta_2 - \beta_1)\), and send \(\phi: (L_1, \nabla_1) \to (L_2, \nabla_2)\) to \(\text{DCh}(\phi) \in \text{DC}^3_2(M) = \text{DC}^3_3(M)\).

That \(F_{\text{triv}}\) is an equivalence on morphism categories follows from the equivalence \(\text{DCh}\). Indeed, given a morphism \((c, h, 0): (0, \beta_1, d\beta_1) \to (0, \beta_2, d\beta_2)\) in \(\text{DC}^3_3(M)\), \((c, h, \beta_2 - \beta_1)\) is a cocycle in \(\text{DC}^3_2(M)\) (i.e. an object in \(\text{DC}^3_2(M)\)); therefore, there exists a line bundle with connection \((L, \nabla)\) so that \(\text{DCh}(L, \nabla) = (c', h', \beta_2 - \beta_1)\) and a morphism \((b, g, 0): (c', h', \beta_2 - \beta_1) \to (c, h, \beta_2 - \beta_1)\) in \(\text{DC}^3_2(M)\), which may be viewed as a 2-arrow in \(\text{DC}^3_3(M)\), \((b, g, 0): (c, h, 0) \Rightarrow (c', h', 0)\), and thus \(F_{\text{triv}}\) is essentially surjective on morphism categories. That it is fully faithful on morphism categories is clear, since this is true for \(\text{DCh}\).

Since \(F_{\text{triv}}\) is fibrewise fully faithful, and every object in \(\text{DC}^3_3\) is locally isomorphic to one in the image of \(F_{\text{triv}}\) (since isomorphism classes of objects in \(\text{DC}^3_3(M)\) are classified by \(H^3(\text{DC}_3^*\text{triv}(M))\), and \(M\) can be covered by contractible open sets \(U\) on which \(H^3(\text{DC}_3^*\text{triv}(U)) = 0\)), then \(\text{DC}^3_3\) is a 2-stackification of \(\text{Grb}^\nabla_{\text{triv}}\) (see [3 Section 1.10]). Since \(\text{Grb}^\nabla_{B, \text{triv}}\) is a 2-stackification of \(\text{Grb}^\nabla_{B, \text{triv}}\), the morphism \(F_{\text{triv}}\) extends to an equivalence \(F: \text{Grb}^\nabla_{B, \text{triv}} \to \text{DC}^3_3\) of 2-stacks.

Similar to above, we obtain the following corollaries to Theorem 4.13.

**Corollary 4.14.** Let \(M\) be a manifold. There is a one-to-one correspondence between (stable) isomorphism classes of \(\mathbb{S}^1\)-bundles gerbes with connection and curving and \(H^3(\text{DC}_3^3(M))\) differential characters of degree 3.

**Corollary 4.15.** Let \(\Gamma_1 \Rightarrow \Gamma_0\) be a Lie groupoid. There is a one-to-one correspondence between (stable) isomorphism classes of \(\Gamma_\bullet\)-equivariant \(\mathbb{S}^1\)-bundle gerbes with connection and curving and \(H^3(\text{DC}_3^3(\Gamma_\bullet))\) equivariant differential characters of degree 3.

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