A HOMOTOPY INVARIANT OF IMAGE SIMPLE FOLD MAPS TO ORIENTED SURFACES

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ABSTRACT. The singular set of a generic map $f : M \to F$ of a manifold $M$ of dimension $m \geq 2$ to an oriented surface $F$ is a closed smooth curve $\Sigma(f)$. We study the parity of the number of components of $\Sigma(f)$.

The image $f(\Sigma)$ of the singular set inherits canonical local orientations via so-called chessboard functions. Such a local orientation gives rise to the cumulative winding number $\omega(f) \in \frac{1}{2}\Z$ of $\Sigma(f)$. When the dimension of the manifold $M$ is even we also define an invariant $I(f)$ which is the residue class modulo 4 of the sum of the number of components of $\Sigma(f)$, the number of cusps, and twice the number of self-intersection points of $f(\Sigma)$. Using the cumulative winding number and the invariant $I(f)$, we show that the parity of the number of connected components of $\Sigma(f)$ does not change under homotopy of $f$ provided that one of the following conditions is satisfied: (i) the dimension of $M$ is even, (ii) the singular set of the homotopy is an orientable manifold, or (iii) the image of the singular set of the homotopy does not have triple self-intersection points.

1. Introduction

Singular sets of smooth maps $f : M \to F$ of smooth $n$-manifolds into surfaces played a strong role in recent various discoveries. Studying singular sets of maps, Gay and Kirby [4] proved that any smooth closed oriented connected 4-manifold admits a trisecting map to $\mathbb{R}^2$, in analogy to the existence of Heegaard splittings for oriented connected closed 3-manifolds, see also the paper [2] by Baykur and Saeki for the existence of a simplified trisection. Kalmar and Stipsicz [8] obtained upper bounds on the complexity of the singular set of maps from 3-manifolds to the plane. These upper bounds are expressed in terms of certain properties of the link $L \subset S^3$, where the 3-manifold is obtained via integral surgery along $L$. Ryabichev [13] gave precise conditions for the existence of maps of surfaces with prescribed loci of singularities. Kitazawa [9] studied simple stable maps (of non-negative dimension) of smooth manifolds to Euclidean target spaces, ($\mathbb{R}^2$, in particular) whose singular sets are concentric spheres. Saeki [15] and [16] showed that every closed connected oriented 3-manifold admits a stable map to a sphere without definite fold points. Many

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$\mathbb{Z}_2$-invariants of stable maps of 3-manifolds into the plane were found by M. Yamamoto in [19]. In [17] Saeki constructed an integral invariant of stable maps of oriented closed 3-manifolds into $\mathbb{R}^2$.

A smooth map is image simple if its restriction to the singular set is a topological embedding. In the present paper we study under what conditions the numbers $\#|\Sigma(f)|$ and $\#|\Sigma(g)|$ of components of singular sets of two homotopic image simple fold maps $f$ and $g$ of manifolds of dimension $m \geq 2$ to a surface are congruent modulo two.

This question has been solved in the case $m = 2$, and it is partially answered in the case $m = 3$. Namely, M. Yamamoto [20] showed that if $f$ is a map of degree $d$ between oriented closed surfaces of genera $g$ and $h$ respectively, then the parity of $\#|\Sigma(f)|$ is the same as that of $d(h - 1) - (g - 1)$. On the other hand, in [17] Saeki gave an example of two image simple fold maps $f, g: S^2 \to \mathbb{R}^2$ such that the parities of the number of components of the singular sets of $f$ and $g$ are different.

Our main result is split into three cases; the first being the case when the source manifold is of even dimension and the remaining two cases consider an odd-dimensional source manifold. The following theorem is the main result in the case where the source manifold $M$ is of even dimension.

**Theorem 1.1.** Let $f$ and $g$ be two homotopic image simple fold maps from a closed oriented manifold $M$ of even dimension $m \geq 2$ to an oriented surface $F$ of finite genus. Then, the number of components of $\Sigma(f)$ is congruent modulo two to the number of components of $\Sigma(g)$.

To prove Theorem 1.1 we define the cumulative winding number $\omega(f) \in \frac{1}{2}\mathbb{Z}$ for generic maps to surfaces. In general, $\omega$ is not a homotopy invariant. However, for image simple fold maps $f, g: M^n \to \mathbb{R}^2$, the parities of $\omega(f)$ and $\omega(g)$ agree. Thus, for image simple fold maps, $\omega \in \mathbb{Z}$ is a $\mathbb{Z}_2$-homotopy invariant. We note that the cumulative winding number we introduce in the present paper is different from the rotation numbers considered by Levine [11], Chess [3], and Yonebayashi [21].

For odd-dimensional source manifolds we state and prove two theorems; the first theorem requires that $\Sigma(f)$ does not undergo any $R_3$ moves (see Fig. 2) during homotopy, while the second requires that the singular set of the homotopy is orientable.

**Theorem 1.2.** Let $f$ and $g$ be two homotopic image simple fold maps from a closed oriented manifold $M$ of odd dimension $m \geq 2$ to an oriented surface $F$. Suppose that $F$ is $\mathbb{R}^2$ or $S^2$ if $m > 3$. Suppose that no $R_3$ moves occur during the homotopy from $f$ to $g$. Then, the number of components of $\Sigma(f)$ is congruent modulo two to the number of components of $\Sigma(g)$.

We note that $R_3$-moves are closely related to triple points of the singular sets $\Sigma(h)$ of maps $h$ to $\mathbb{R}^3$. These are studied by Saeki and T. Yamamoto [18].
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The proof of Theorem 1.2 also utilizes the cumulative winding number $\omega(f)$.

**Theorem 1.3.** Let $f$ and $g$ be two homotopic image simple fold maps from a closed oriented manifold $M$ of dimension $m \geq 2$ to an oriented surface $F$ of finite genus. Suppose the surface $\Sigma(H)$ of singular points of the homotopy $H$ between $f$ and $g$ is orientable. Then, the number of components of $\Sigma(f)$ is congruent modulo two to the number of components of $\Sigma(g)$.

Let $\#|A_2(f)|$ be the number of cusps of the map $f$, $\Delta(f)$ the number of self-intersections of $f(\Sigma)$, and $\#|\Sigma(f)|$ the number of connected components of $f(\Sigma)$. To prove Theorem 1.3 we introduce a modulo 4 function

$$I(f) \equiv \#|A_2(f)| + 2\Delta(f) + 2\#|\Sigma(f)| \mod 4,$$

and show that it is invariant under generic homotopy whose singular set is orientable. In particular, the function $I(f)$ is a homotopy invariant provided that the dimension of the manifold $M$ is even. In [6], Gromov introduces and more deeply studies $I(f)$ as an integer-valued function.

The paper is structured as follows. In section 2 we review the notions of generic maps, stable maps, and generic families of maps. We note that there are several conflicting definitions of a generic family of maps in the literature and chose one which is the most convenient for the present paper. In section 3 we review singularities $A_i(f)$ of Morin maps and introduce the manifolds $A_I(f) \subset M$ related to multi-singularities of smooth maps. In section 4, using the manifolds $A_I(f)$, we list all moves of singularities which occur under a generic homotopy of maps to $\mathbb{R}^2$. For completeness, we give a proof that no other moves are possible. Section 5 serves to introduce the notion of an abstract singular set diagram. In section 6 we define chessboard functions and in section 7 we look at examples of chessboard functions. In sections 8 and 9 we define the cumulative winding number and record how homotopy affects the cumulative winding number, respectively. In section 10 we prove Theorems 1.1 and 1.2 and in section 11, we prove that $I(f)$ is indeed invariant under homotopy whose singular set is orientable and use it to provide proof of Theorem 1.3. We finish our discussion in section 12 by listing and proving a few interesting applications of our results.

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2. Stable and generic maps

In this section we recall the definition of stable maps, generic maps, generic families of maps, and $n$-functions.

Let $f$ be a smooth map of a non-negative dimension $m - n$ of a manifold $M$ of dimension $m$ to a manifold $N$ of dimension $n$. We say that a point $x \in M$ is regular
if the kernel rank of $f$ at $x$ is $m - n$. Otherwise, the point $x$ is said to be singular.

Recall that a smooth map is a Thom-Boardman map if for each $k$, its $k$-jet extension is transverse to each Thom-Boardman submanifold of the $k$-jet space. The singular set $\Sigma(f)$ of a Thom-Boardman map $f : M \to N$ is stratified by smooth submanifolds $\Sigma^I(f) \subset M$ parametrized by Thom-Boardman symbols $I$.

2.1. Generic maps. Let $f : M \to N$ be a Thom-Boardman map. Let $x_j \in \Sigma^I_j$ be distinct singular points in $M$ with $j = 1, \ldots, r$ such that

$$f(x_1) = f(x_2) = \cdots = f(x_r) = y.$$ 

We say that $f$ satisfies the normal crossing condition if for each tuple of points $x_1, \ldots, x_r$ as above the vector spaces $d_{x_1}f(T\Sigma^I_1), \ldots, d_{x_r}f(T\Sigma^I_r)$ are in general position in the vector space $T_y N$.

**Definition 2.1.** We say that a smooth map $f$ is generic if it is a Thom-Boardman map satisfying the normal crossing condition.

It is known that generic maps are residual in $C^\infty(M, N)$, e.g., see [5, p.157].

2.2. Stable maps. There are various equivalent definitions of stability of smooth maps $f : M \to N$ of a closed manifold $M$ to an arbitrary manifold $N$, e.g., see [5, Theorem 7.1]. It follows that $f$ is stable if any $k$-parametric deformation of $f$ is trivial in the sense of [5, Definition 2.1].

2.3. Generic families of maps. Let $f_t : M \to N$ be a parametric family of maps parametrized by a smooth manifold $T$. It defines a map $F : M \times T \to N \times T$ by $F(x, t) = (f_t(x), t)$, and a stratification of $M \times T$ by submanifolds $\Sigma^I(F)$, where $I$ ranges over Thom-Boardman symbols. It is common to define a generic homotopy $f_t$ by requiring that the associated map $F$ is generic, e.g., see [7]. However, we will need a more restrictive definition. Let $\pi_T$ denote the projection of $M \times T$ onto the second factor. We say that a parametric family $\{f_t\}$ is a generic parametric family if the associated map $F$ is generic, and the restrictions $\pi_T|_{\Sigma^I(F)}$ are generic for each Thom-Boardman symbol $I$. A parametric family $f_t$ is a stable parametric family if any $k$-parametric deformation $f_{t,s}$ of $f_t$ is trivial.

2.4. $n$-functions. In some cases it is helpful to study maps to manifolds of dimension $n$ by means of $(n - 1)$-parametric families of functions, or, $n$-functions. More precisely, given a manifold $X$ of dimension $m$, and a manifold $Y$ of dimension $n \leq m$, a smooth proper map $f : X \to Y$ is an $n$-function if for each $q \in Y$, there is a compact neighborhood $U$ of $q$ with a diffeomorphism $\psi : U \to [0, 1]^n$, and a diffeomorphism $\varphi : f^{-1}(U) \to [0, 1]^{n-1} \times M$ for an $(m - n + 1)$-manifold $M$, such that $\psi \circ f \circ \varphi^{-1}$:
Let \( f : X \to Y \) be a generic smooth proper map of corank 1 to a manifold of dimension \( n \). Then \( f \) is an \( n \)-function.

**Proof.** Let \( q \) be a point in \( Y \). Since \( f \) is of corank 1, there is a diffeomorphism \( \psi : U \to [0, 1]^n \) of a neighborhood of \( U \) such that the composition \( \pi_n^1 \circ \psi \circ f|_{f^{-1}(U)} \) is a submersion, where \( \pi_n^1 : [0, 1]^n \to [0, 1]^{n-1} \) is the projection \((x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1})\). We may choose \( U \) so that the resulting proper submersion to a disc is a trivial fiber bundle. Then, there is a diffeomorphism \( \varphi : f^{-1}(U) \to [0, 1]^{n-1} \times [0, 1] \) such that the map \( \psi \circ f \circ \varphi^{-1} \) is of the form \((t, p) \mapsto (t, g_t(p))\), for a parametric family \( g_t \) of functions on the fiber \( M \). \( \square \)

### 3. Singularities of maps

In this section we review the definition of generic singularities of smooth maps to surfaces and manifolds of dimension 3.

Let \( f \) be a smooth map \( f : M \to N \) of non-negative dimension \( m - n \) of a manifold \( M \) of dimension \( m \) to a manifold \( N \) of dimension \( n \). The set \( A_0(f) \) of regular points of \( f \) is an open submanifold of \( M \) of codimension 0. We now review the definition of relatively simple singularity types \( A_r \) for \( r \geq 1 \) with Thom-Boardman symbol \( I_r = (m - n + 1, 1, \ldots, 1, 0) \) of length \( r + 1 \).

We say that a point \( x \in M \) is a **fold point** if there is a neighborhood \( U \cong \mathbb{R}^{n-1} \times \mathbb{R}^{m-n+1} \) about \( x \), with coordinates \((x_1, \ldots, x_m)\) in \( M \), and a coordinate neighborhood \( V \cong \mathbb{R}^{n-1} \times \mathbb{R} \) about \( f(x) \) in \( N \) such that \( f(U) \subset V \) and \( f|_U \) is given by a product of the identity map \( \text{id}_{\mathbb{R}^{n-1}} : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1} \) and a standard Morse function \( \mathbb{R}^{m-n+1} \to \mathbb{R} \) with a unique critical point, i.e.,

\[
  f(x_1, x_2, \ldots, x_m) = (x_1, \ldots, x_{n-1}, \pm x_n^2 \pm x_{n+1}^2 \pm \ldots \pm x_m^2).
\]

The set of fold singular points of \( f \) is denoted by \( A_1(f) \). The number \( i \) of terms among \( x_n, \ldots, x_m \) with positive signs is called a **relative index** of \( f \). We may always choose coordinate neighborhoods such that \( i \leq m - 1 - i \). The number \( i \) with respect to such a coordinate system is said to be the (absolute) **index** of the fold point. If the index of the critical point is 0, then \( x \) is said to be a **definite** fold point. Otherwise, the fold point \( x \) is **indefinite**.

**Definition 3.1.** We say that the map \( f \) is a **fold map** if every singular point \( x \) is a fold point. Furthermore, a fold map \( f \) is an **indefinite fold map** if every fold point is indefinite.
It immediately follows that if $f$ is a fold map, then the set of singular points $\Sigma(f)$ of $f$ is a closed submanifold of $M$ of dimension 1, and $f|_{\Sigma(f)}$ is an immersion.

We say that a point $x \in M$ is an $A_r$-singular point for $r > 1$, if there is a neighborhood $U \subset M$ of $x$, with coordinates $(t_1, \ldots, t_{n-r}, \ell_2, \ldots, \ell_r, x_1, \ldots, x_{m-n+1})$, and a neighborhood $V \subset N$ of $f(x)$, with coordinates $(T_1, \ldots, T_{n-r}, L_2, \ldots, L_r, Z)$, such that $f(U) \subset V$ and the restriction $f|_{U}$ is given by
\[
T_i = t_i \quad \text{for} \quad i = 1, \ldots, n-r, \\
L_i = \ell_i \quad \text{for} \quad i = 2, \ldots, r, \\
Z = \pm x_1^2 \pm x_2^2 \pm \cdots \pm x_{m-n+1}^2 + \ell_2 x_{m-n} + \ell_3 x_{m-n+1}^2 + \cdots + \ell_r x_{m-n+1}^{r-1} \pm x_{m-n+1}^r.
\]
The sets $A_r(f)$ of singular points of type $A_r$ are submanifolds of $M$ of dimension $n - r$.

**Definition 3.2.** Singular points of types $A_2$ and $A_3$ are called *cusp* and *swallowtail* singular points, respectively.

**Definition 3.3.** We say that a stable map $f : M \to N$ of a smooth manifold $M$ into a surface $N$ is *simple* if $A_2(f) = \emptyset$, and for every singular value $y$, every connected component of the singular fiber $f^{-1}(y)$ contains at most one singular point. Also, a generic map $f : M \to N$ is said to be *image simple* if its restriction to the singular set $f|_{\Sigma(f)}$ is a topological embedding.

We note that the term ‘image simple map’ is introduced by Saeki in his forthcoming paper.

### 3.1. Morin Maps

We say that a smooth map $f$ is a *Morin map* if all its singular points are of type $A_r$ for $r \geq 1$. It is known that for $n \leq 3$, all generic maps $M^m \to \mathbb{R}^n$ of non-negative dimension $m - n$ are Morin. The singular set $\Sigma(f)$ of a Morin map is a closed smooth submanifold of $M$ of dimension $n - 1$. Given a Morin map $f$, for each $i$, the closure $\text{Cl}(A_i(f))$ is a smooth submanifold of $M$. Furthermore, for each $i$ and $j$ such that $i < j$, the manifold $\text{Cl}(A_i(f))$ is a submanifold of $\text{Cl}(A_j(f))$.

For a generic Morin map $f$, we denote by $A_{ij}(f)$ the set of points $x \in A_i(f)$ for which there is a distinct point $y \in A_j(f)$ such that $f(x) = f(y)$. Similarly, we denote by $A_{ijk}(f)$ the subset of points $x \in A_i$ for which there are distinct points $y \in A_j$ and $z \in A_k$ such that $f(x) = f(y) = f(z)$. We will denote the restriction of $f$ to $A_I(f)$ by $f|_{A_I}$, for short, where $I$ is either a single index $i$, or a multi-index $ij$ or $ijk$.

When the non-negative dimension $m - n$ of a map $f : M \to N$ is even, we have the following theorem.

**Theorem 3.4.** Let $f : M \to N$ be a Morin map of non-negative even dimension $m - n$ into an oriented manifold $N$. Then, the set $\Sigma(f)$ is a canonically oriented submanifold of $M$. 

We emphasize that the manifold \( M \) in Theorem 3.4 is not necessarily orientable.

**Proof.** Since \( \text{Cl}(A_3(f)) \) is a proper submanifold of \( \Sigma(f) \) of codimension 2, it suffices to introduce an orientation of \( \Sigma(f) \) in the complement to \( \text{Cl}(A_3(f)) \), i.e., only over the union of \( A_1(f) \) and \( A_2(f) \).

We note that the index of a fold singular point \( x \) depends on the choice of coorientation of the immersed submanifold \( f(A_1) \) at the point \( f(x) \). If a fold singular point \( x \) is of index \( i \) for one choice of coorientation, then \( x \) is of index \( m - i - n + 1 \) for the other choice of coorientation. Since the (non-negative) dimension \( m - n \) of the map \( f \) is even, it follows that the parity of the index is changed when the coorientation is changed. Consequently, the immersed manifold of fold singular points \( f(A_1) \) admits a unique coorientation at each point \( f(x) \), such that the index of the fold singular point \( x \) with respect to the coorientation is odd. We say that such a coorientation is canonical.

We orient the immersed manifold \( f(A_1) \) so that the orientation of \( f(A_1) \) followed by the coorientation of \( f(A_1) \) agrees with the orientation of \( N \). In turn, the orientation of \( f(A_1) \) defines an orientation of \( A_1(f) \). We claim that the so-defined orientation of \( A_1(f) \) extends to an orientation of \( A_1(f) \cup A_2(f) \). Indeed, in a coordinate neighborhood \( U \) about an \( A_2 \)-singular point \( x \) and a coordinate neighborhood about \( f(x) \), the map \( f \) is given by:

\[
T_i = t_i, \quad i = 1, \ldots, n - 2,
\]

\[
L_2 = \ell_2,
\]

\[
Z = \varphi(t_1, \ldots, t_{n-2}, \ell_2)(x_{m-n+1}) \pm x_1^2 \pm x_2^2 \pm \cdots \pm x_{m-n},
\]

where \( (t_1, \ldots, t_{n-2}, \ell_2, x_1, \ldots, x_{m-n}, x_{m-n+1}) \) are local coordinates about \( x \) in \( M \) and \((T_i, L_2, Z)\) are local coordinates about \( f(x) \) in \( N \), and for each parameter \( (t_1, \ldots, t_{n-2}, \ell_2) \),

\[
\varphi(t_1, \ldots, t_{n-2}, \ell_2)(x_{m-n+1}) = \ell_2 x_{m-n+1} + x_{m-n+1}^2
\]

is either a regular function, a Morse function with a cancelling pair of critical points, or a function with a unique birth-death singularity. For each fold critical value in \( f(\Sigma \cap U) \), the direction \( \frac{\partial}{\partial z} \) defines a coorientation of \( f(\Sigma) \) at the corresponding critical value. It follows that for each Morse function \( \varphi(t_1, \ldots, t_{n-2}, \ell_2) \), the parities of the indices of the two cancelling Morse critical points are different. Therefore, the canonical coorientation of one critical point of \( \varphi(t_1, \ldots, t_{n-2}, \ell_2) \) is given by \( \frac{\partial}{\partial z} \), while the canonical coorientation for the other critical point is \( -\frac{\partial}{\partial z} \). Thus, the coorientation of \( f(A_1) \) extends to a coorientation of an immersed smoothing of \( f(A_1 \cup A_2) \). This implies that \( \Sigma(f) \) is orientable. \( \square \)
3.2. **Singularities of generic maps to 2-manifolds.** Let \( f : M \to N \) be a generic smooth map of a manifold of dimension \( m \geq 2 \) to a manifold \( N \) of dimension 2. The map \( f \) may only have regular, fold, and cusp map germs. The set of regular map germs forms an open submanifold \( A_0(f) \) of \( M \). The complement to the submanifold \( A_0(f) \) in \( M \) is the submanifold of singular points \( \Sigma(f) \) of dimension 1. It contains a discrete set of cusp singular points \( A_2(f) \). The rest of \( \Sigma(f) \) is a disjoint union of arcs and circles of fold singular points \( A_1(f) \). The restriction of \( f \) to \( A_0(f) \) is a submersion. The restriction of \( f \) to \( A_1(f) \) is a self-transverse immersion with 0-dimensional self-crossings. In general, the images of \( f|_{A_1} \) and \( f|_{A_2} \) are disjoint.

3.3. **Singularities of generic maps to 3-manifolds.** Let \( F : M \to N \) be a generic smooth map of a manifold of dimension \( m \geq 3 \) to a manifold of dimension 3. The map \( F \) may only have regular, fold, cusp, and swallowtail map germs. Since \( F \) satisfies the normal crossing condition, the set \( A_{11}(F) \) is a submanifold which consists of open arcs and circles. We note that the image of \( A_{11}(F) \) is the self-crossing of the immersion \( F|_{A_{11}} \), while the image of \( A_{12}(F) \cong A_{21}(F) \) is the set of intersections of folds with cusps. The image of the set \( A_{111}(F) \) is the set of triple self-intersections of folds. The submanifolds \( A_{12}(F) \cong A_{21}(F) \) and \( A_{111}(F) \) are of dimension 0, while all other manifolds \( A_{ij} \) and \( A_{ijk} \) (except for the aforementioned manifold \( A_{111} \)) are empty.

4. **Generic homotopies of maps to \( \mathbb{R}^2 \)**

In this section we study how the singular set of a map to \( \mathbb{R}^2 \) is modified under generic homotopy.

Let \( F : M \times [0, 1] \to \mathbb{R}^2 \times [0, 1] \) be a homotopy between two generic maps, and let \( \pi : M \times [0, 1] \to [0, 1] \) denote the projection onto the second factor.

**Definition 4.1.** The homotopy \( F \) is a *generic homotopy* if \( F \) is a generic map and \( \pi|_{A_I(F)} \) is a Morse function for each \( I \in \{\{1\}, \{2\}, \{11\}\} \).

**Lemma 4.2.** The set of generic homotopies is open and dense in the space of all homotopies.

**Proof.** Any homotopy sufficiently close to a generic homotopy is also generic. Consequently, the set of generic homotopies is open. Let \( F : M \times [0, 1] \to \mathbb{R}^2 \times [0, 1] \) be an arbitrary homotopy. To show that there exists a generic homotopy close to \( F \), we may assume that \( F \) is a generic map. Choose a diffeomorphism \( \varphi \in C^\infty(M \times [0, 1], M \times [0, 1]) \) arbitrarily close to the identity map \( \iota_{M \times [0, 1]} \) such that \( \pi : M \times [0, 1] \to [0, 1] \) restricted to the curve \( A_2(F) \cup A_{11}(F) \) is a Morse function. Since \( \varphi \) is arbitrarily close to \( \iota_{M \times [0, 1]} \), we deduce that \( F \circ \varphi \) is also a homotopy. Suppose that \( \pi|_{A_2(F)} \) and \( \pi|_{A_{11}(F)} \) are Morse functions. Then there is a diffeomorphism \( \psi \) arbitrarily close to \( \iota_{M \times [0, 1]} \) such that \( \pi \circ \psi|_{A_{11}(F)} \) is a Morse function. If the
diffeomorphism $\psi$ is sufficiently close to $\iota_{\mathbb{M}\times[0,1]}$, then $\pi \circ \psi|_{A_2(F)}$ and $\pi \circ \psi|_{A_1(F)}$ are still Morse functions. This completes the proof of Lemma 4.2.

We note that members $f_t$ of a generic family $F = \{f_t\}$ may not be generic maps. We will next list several instances when a member $f_t$ of a generic homotopy of maps to $\mathbb{R}^2$ is not generic.

4.1. List of generic moves.

4.1.1. Reidemeister-II fold crossing. The restriction $f_t|_{A_1}$ may not be a self-transverse immersion for a discrete set of moments $t \in [0, 1]$. If $f_t$ is a generic homotopy, and $f_t|_{A_1}$ is not self-transverse at $t = t_0$, then as $t$ ranges in the interval $(t_0 - \varepsilon, t + \varepsilon)$, the map $f_t|_{A_1}$ undergoes a Reidemeister-II fold crossing, see Fig. 1.

4.1.2. Reidemeister-III fold crossing. Similarly, the map $f_t|_{A_1}$ may undergo a Reidemeister-III fold crossing, see Fig. 2.

4.1.3. Cusp-fold crossing. The cusp-fold crossing occurs when $f_t(x) = f_t(y)$, for a cusp point $x \in A_2(f_t)$ and a fold point $y \in A_1(f_t)$, see Fig. 3.
In Figures 3, 4, and 5 the numbers $i$ and $i + 1$ indicate the relative index of each fold curve. The relative index for each curve is considered in the direction of the corresponding blue arrow.

4.1.4. Wrinkle singularity. Under a generic homotopy, a new path component of singular points may appear in the form of a wrinkle, see Fig. 4.

4.1.5. Merge singularity. Under a merge singularity move, a canceling pair of cusp points disappear while the singular set changes by a surgery of index 1 along the canceling pair of cusp points, see Fig. 5.

4.1.6. Swallowtail singularity. Under a swallowtail singularity move, two cusp points and a self-intersection point of the singular set appear, see Fig. 6.
Theorem 4.3. Under a generic homotopy $F = \{f_t\}$ of maps to $\mathbb{R}^2$, the singular set $\Sigma(F)$ is modified by isotopy, as well as the above listed moves.

Proof. Let $F : M \times [0, 1] \to \mathbb{R}^2 \times [0, 1]$ be a generic homotopy, and let $\pi : M \times [0, 1] \to [0, 1]$ denote the projection to the second factor. If $\pi|_{A_1(F)}$ does not have critical points on the level $M \times \{t_0\}$, then for sufficiently small $\varepsilon > 0$, the singular set $A_1(f_t)$, parametrized by $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$, is modified by an ambient isotopy. Thus, it remains to study modifications of the singular set of $f_t$ corresponding to critical points of the Morse functions $\pi|_{A_1(F)}$. We claim that $\pi|_{A_1(F)}$ has no critical points when $I = \{1\}$.

Lemma 4.4. The map $\pi|_{A_1(F)}$ is a submersion.

Proof. Over the set $A_1(F)$ of critical points, there is a well-defined kernel bundle $K_1(F)$ of $dF$. In fact, over $A_1(F)$ there is a splitting

$$T(M \times [0, 1])|_{A_1(F)} \cong K_1(F)|_{A_1(F)} \oplus T \Lambda_1(F).$$

Assume that there is a critical point $p \in A_1(F)$ of the function $\pi|_{A_1(F)}$. Then $T_p(A_1(F))$ is in the kernel of $d_p\pi$. On the other hand, the projection $d_p\pi$ coincides with the composition

$$T_p(M \times [0, 1]) \to T_{F(p)}(\mathbb{R}^2 \times [0, 1]) \to T_{\pi(p)}([0, 1])$$

of $d_pF$ and the differential of the projection $\mathbb{R}^2 \times [0, 1] \to [0, 1]$ onto the second factor. Since $K_1(F)|_p$ is in the kernel of $d_pF$, it follows that $K_1(F)|_p$ is in the kernel of $d_p\pi$. To summarize, we have shown that $T_p(M \times [0, 1])$ is in the kernel of $d_p\pi$, which contradicts the fact that $\pi$ is a submersion. \hfill \square

Let us now consider critical points of the function $\pi|_{A_2(F)}$.

Lemma 4.5. Let $p \in A_2(F)$ be a critical point of $\pi|_{A_2(F)}$. Then $p$ is a critical point of $\pi|_{\Sigma(F)}$.

Proof. As above, over $A_2(F)$, there is a well-defined kernel bundle $K_1(F)$ and cokernel bundle $Q_1(F)$ of $dF|_{A_2(F)}$. Let $L$ denote the vector subbundle of $T(M \times [0, 1])|_{A_2(F)}$...
given by $K_1(F) \cap T(\Sigma(F))$. It follows that $\dim L = 1$, and there is a splitting
\[ T(\Sigma(F))|_{A_2(F)} \cong L \oplus T(A_2(F)). \]
The argument of Lemma 4.4 shows that $L_p$ belongs to the kernel of $d_p \pi$. In particular, it belongs to the kernel of $d_p \pi|_{A_2(F)}$. On the other hand, if $p$ is a critical point of $\pi|_{A_2(F)}$, then $T_p(A_2(F))$ is also in the kernel of $d_p \pi|_{A_2(F)}$. Thus, the point $p$ is a critical point of $\pi|_{\Sigma(F)}$. □

By Lemma 4.5, if $p$ is a critical point of $\pi|_{A_2(F)}$, then $p$ is also a critical point of the function $\pi|_{\Sigma(F)}$. If the index of the critical point $p$ is 0, then $p$ corresponds to the appearance (birth) of a wrinkle singularity in $\Sigma(f_t)$. A critical point of index 1 corresponds to the cusp merge move or its inverse, while a critical point of index 2 corresponds to the disappearance (death) of a wrinkle singularity.

The critical points of $\pi$ restricted to the submanifold of double points of $A_{11}(F)$ correspond to Reidemeister-II fold crossings. All points of $A_{12}(F), A_{111}(F)$, and $A_3(F)$ are critical in the sense that the differential of $\pi|_{A_i(F)}$ in these cases vanishes. It remains to observe that points of $A_{12}(F)$ correspond to cusp-fold crossings, $A_{111}(F)$ correspond to Reidemeister-III fold crossings, and $A_3(F)$ correspond to swallowtail singularities. □

Remark 4.6. The counterpart of Lemma 4.4 for a generic concordance $F: M \times [0, 1] \to \mathbb{R}^2 \times [0, 1]$ of smooth maps is not valid. Furthermore, there are moves of generic concordances that do not occur under a generic homotopy. Specifically, under a generic concordance, an embedded circle of fold singular points may appear or disappear, and the curves of fold singular points may be modified by embedded surgery of index 1.

5. Oriented abstract singular set diagrams

The proof of the main result relies on so-called abstract singular set diagrams, which we introduce now.

Let $S$ denote a closed (possibly not path-connected) manifold of dimension 1 together with two disjoint families $P \subset S$ and $Q \subset S$ of finitely many distinguished points. We require that the number of points in $Q$ is even, and that the points in $Q$ are paired. We denote the distinguished points in the family $P$ by $p_1, p_2, \ldots$, and the points in $Q$ by $q_1, q_1', q_2, q_2', \ldots$, where the points $q_i$ and $q_i'$ are paired. We say that a compact subset of $S$ is an arc if its interior contains no distinguished points, and its boundary is either empty or consists of the distinguished points.

Definition 5.1. An orientated abstract singular set diagram consists of the manifold $S$, the families $P$ and $Q$, and an orientation of all arcs on $S$ such that
if two arcs $\alpha$ and $\beta$ share a common point $p_i \in P$, then the orientations of $\alpha$ and $\beta$ agree

- if $q_j \in Q$ is a common point of arcs $\alpha$ and $\beta$, while $q'_j \in Q$ is a common point of arcs $\alpha'$ and $\beta'$, then the orientations on $\alpha$ and $\beta$ agree if and only if the orientations on $\alpha'$ and $\beta'$ agree.

In the stated requirements, we allow that some of the arcs $\alpha, \beta, \alpha'$ and $\beta'$ may coincide. We note that as a point $x$ traverses a path component of $S$, the orientation of $S$ at $x$, that agrees with the orientation of an arc containing $x$, may change only at a point in $Q$. Furthermore, at a point in $Q$ the orientation of $S$ may or may not change. For the sake of convenience, we will simply refer to an oriented abstract singular set diagram as a diagram.

6. Chessboard functions

In order to properly equip a singular set diagram with a so-called canonical local orientation and coorientation, we first need to introduce the concept of a chessboard function. Let $f : M \to \mathbb{R}^n$ be a generic smooth map of a closed manifold. We say that a curve $\gamma$ in $\mathbb{R}^n$ is a generic curve with respect to $f(\Sigma)$ if it intersects each Thom-Boardman stratum $f(\Sigma')$ of the singular set transversely. In particular, we have $\gamma \cap f(\Sigma) = \gamma \cap f(\Sigma^{d+1,0})$, where $d = m - n$ is the dimension of the map $f$.

If necessary, we can further perturb the generic curve $\gamma$, so that it avoids self-intersection points of the immersed fold surface $f(A_1)$.

Definition 6.1. We say that a locally constant function $c : \mathbb{R}^n \setminus f(\Sigma) \to \mathbb{Z}$ (respectively, $c : \mathbb{R}^n \setminus f(\Sigma) \to \mathbb{Z}_2$) is an integral chessboard function (respectively, a $\mathbb{Z}_2$-valued chessboard function) if the values $c(\gamma(-1))$ and $c(\gamma(1))$ differ by precisely 1 for each generic curve $\gamma : [-1, 1] \to \mathbb{R}^n$ intersecting $f(\Sigma)$ at a unique point $\gamma(0)$.

We say that a singular value $y$ of a map $f$ is a simple singular value if the fiber $f^{-1}(y)$ contains a unique critical point. We note that for a generic smooth map, the submanifold of $\mathbb{R}^n$ of simple fold values is dense in $f(\Sigma)$. A local orientation of $f(\Sigma)$ is an orientation of the submanifold of simple fold values. Similarly, a local coorientation of $f(\Sigma)$ is a coorientation in $\mathbb{R}^n$ of the submanifold of simple fold values. We say that a local orientation of $f(\Sigma)$ agrees with the local coorientation of $f(\Sigma)$ if the local orientation of $f(\Sigma)$ followed by the local coorientation of $f(\Sigma)$ agrees with the standard orientation of $\mathbb{R}^n$.

Definition 6.2. An integral (respectively, $\mathbb{Z}_2$-valued) chessboard function $c$ defines a canonical local coorientation on $f(\Sigma)$ in the direction of the region over which $c$ assumes the smaller value (respectively, the even value). The local orientation that agrees with the canonical local coorientation is said to be a canonical local orientation.
Let \( f : M \to \mathbb{R}^2 \) be a stable map of a manifold of dimension \( m \geq 2 \), and \( c \) a chessboard function. Then, the pair \((f, c)\) gives rise to a diagram \((\Sigma(f); P, Q)\), where \( \Sigma(f) \) is the singular set of the map \( f \), and the subsets \( P \) and \( Q \) of distinguished points are the sets \( A_2(f) \) and \( A_{11}(f) \) respectively. The pairs \((q_i, q'_i)\) of points in \( Q \) are the fold points with the same image in \( \mathbb{R}^2 \), i.e. self-intersection points. Finally, the orientation of the arcs of \( \Sigma(f) \) is the canonical local orientation of \( \Sigma(f) \).

Proposition 6.3. Let \( f : M \to \mathbb{R}^2 \) be a generic map of non-negative dimension, and \( c \) an integral or \( \mathbb{Z}_2 \)-valued chessboard function on \( \mathbb{R}^2 \). Then \((\Sigma(f); P, Q)\) is an oriented singular set diagram, where \( \Sigma(f) \) is equipped with the canonical local orientation.

Proof. Given a generic map \( f : M \to \mathbb{R}^2 \) of a manifold \( M \), we have defined a manifold \( \Sigma(f) \), together with two families of points \( P \) and \( Q \) that break \( f(\Sigma) \) into canonically oriented arcs. By Lemma 6.4 below, the orientations of arcs that share a common point in \( P \) agree. By Lemma 6.6 below, if \( q_j \) is a common point of arcs \( \alpha \) and \( \beta \), and \( q'_j \) is a common point of arcs \( \alpha' \) and \( \beta' \), then the orientations on the arcs \( \alpha \) and \( \beta \) agree if and only if the same is true for the arcs \( \alpha' \) and \( \beta' \). Thus, indeed, each generic map \( f \) of non-negative dimension, together with a chessboard function, defines an oriented singular set diagram. To complete the proof of Proposition 6.3, it remains to provide proof of Lemma 6.4 and Lemma 6.6.

Lemma 6.4. Let \( \alpha \) and \( \beta \) be two arcs in \( \Sigma(f) \) that share a common endpoint \( p \in P \). Then, the canonical orientations of arcs \( \alpha \) and \( \beta \) agree.

Notice that in the statement of Lemma 6.4, we do not require that \( \alpha \) and \( \beta \) are distinct.

Proof. Consider a neighborhood \( W \) of a cusp point \( p \in P \). We may assume that the curve \((\alpha \cup \beta) \cap W \) splits \( W \) into two regions. The coorientation of \( \alpha \) and \( \beta \) are in the direction of the region where the chessboard function assumes the smaller value for integer chessboard functions, and an even value for \( \mathbb{Z}_2 \)-valued chessboard functions. In particular, these coorientations agree. Thus, the orientations of \( \alpha \) and \( \beta \) agree. \( \square \)

Suppose now that \( \alpha, \alpha', \beta, \beta' \) are four arcs in \( S \), such that \( \alpha \) and \( \beta \) share a common endpoint \( q \in Q \), while \( \alpha' \) and \( \beta' \) share a common endpoint \( q' \in Q \), where \( q \) and \( q' \) are paired points, i.e. \( f(q) = f(q') \). Then, the curve \( f(\Sigma) \cap U \) breaks a neighborhood \( U \) of \( y \) in \( \mathbb{R}^2 \) into four regions. We call these regions \( L, T, R, B \) for left, top, right, and bottom, respectively. Let \( (a, b, c, d) \) be the values of the chessboard function \( c \) at four points that are pairwise in different regions \( L, T, R, B \), e.g., see Fig. 7. We say that \( (a, b, c, d) \) is a type of the double point. We note that the order of entries
Figure 7. Coorientation of arcs near double points of types \((a - 1, a, a + 1, a)\) on the left, and \((a + 1, a, a + 1, a)\) on the right.

\((a, b, c, d)\) depends on the choice of, say, the left region \(L\). However, the cyclic order of entries \((a, b, c, d)\) is an invariant of the double point.

**Lemma 6.5.** Up to a cyclic permutation, each type of double points is either of the form \((a, a + 1, a, a - 1)\) or \((a, a + 1, a, a + 1)\) for some \(a\), where \(a\) is a non-negative integer if the chessboard function is integral, and it is an element of \(\mathbb{Z}_2\) if the chessboard function is \(\mathbb{Z}_2\)-valued.

The proof of Lemma 6.5 is straightforward; we omit it. We note that if the chessboard function is \(\mathbb{Z}_2\)-valued, then the two types of double points in Lemma 6.5 are the same. The canonical orientation of arcs with respect to the \(\mathbb{Z}_2\)-valued chessboard function near the double point is the same as for the integral chessboard function near the double point of the type \((a, a + 1, a, a + 1)\).

**Lemma 6.6.** The canonical orientations of \(\alpha\) and \(\beta\) agree if and only if the canonical orientations of \(\alpha'\) and \(\beta'\) agree.

**Proof.** We will give an argument for an integral chessboard function; for a \(\mathbb{Z}_2\)-valued chessboard function the argument is similar. Without loss of generality, we may assume that the arcs are labeled \(\alpha, \beta, \alpha', \beta'\) as in Fig. 7. By Lemma 6.5, up to a cyclic permutation, the type of the double point is either of the form \((a, a + 1, a, a - 1)\) or \((a, a + 1, a, a + 1)\). If the double point is of the form \((a, a + 1, a, a - 1)\), then up to a cyclic permutation, the values of \(c\) are as shown on the left schematic of Fig. 7. Therefore, the coorientations, and hence orientations, of \(\alpha\) and \(\beta\) agree. Similarly, the orientations of \(\alpha'\) and \(\beta'\) agree. If the double point is of the form \((a, a + 1, a, a + 1)\), then the values of \(c\) are as on the right schematic of Fig. 7 and therefore, the coorientations, and hence orientations, of \(\alpha\) and \(\beta\) do not agree. Similarly, the orientations of \(\alpha'\) and \(\beta'\) disagree, as well.

This completes the proof of Proposition 6.3.

\(\square\)
7. Examples of Chessboard functions

In this section we give several examples of integral chessboard functions. We note that the reduction modulo 2 turns any integral chessboard function into a $\mathbb{Z}_2$-valued chessboard function.

7.1. The chessboard function for maps of dimension 0 counting path components of the fiber. Let $f: M \to \mathbb{R}^n$ be a proper generic map of a manifold of dimension $n$. We say that the map $f$ is of odd degree if the number of points in the inverse image of any regular value of $f$ is odd. Otherwise, we say that $f$ is of even degree. For a regular value $y \in \mathbb{R}^n$ of $f$, let $|f^{-1}(y)|$ denote the number of path-connected components in the fiber $f^{-1}(y)$. Consider the following integer-valued function:
\[
c(y) = \begin{cases} 
|f^{-1}(y)| & \text{if } f \text{ is of even degree,} \\
\frac{|f^{-1}(y)|+1}{2} & \text{if } f \text{ is of odd degree.}
\end{cases}
\]
It immediately follows that $c$ is an integral chessboard function.

7.2. The chessboard function for maps of dimension 1 counting path components of the fiber. Let $f: M \to \mathbb{R}^n$ be a generic map of a closed oriented manifold of dimension $n+1$. For a regular value $y \in \mathbb{R}^n$ of $f$, let $c(y)$ denote the number of path components in the fiber $f^{-1}(y)$, i.e.
\[
c(y) = |f^{-1}(y)|
\]
We claim that $c(y)$ is a chessboard function on $\mathbb{R}^n \setminus f(\Sigma)$. Indeed, let $z$ be a fold singular value of $f$ that is not a self-intersection point of $f(\Sigma)$. Then there is a disc neighborhood $U \ni z$ such that $U \setminus (U \cap f(\Sigma))$ consists of two open discs $U_1$ and $U_2$.

**Lemma 7.1.** Suppose that $f: M \to \mathbb{R}^n$ is a generic proper map of an oriented manifold of dimension $n+1$ to $\mathbb{R}^n$. Let $y_1 \in U_1$ and $y_2 \in U_2$ be two points. Then, the number of path components in the fiber $f^{-1}(y_1)$ differs from the number of path components in the fiber $f^{-1}(y_2)$ precisely by 1, i.e.
\[
|f^{-1}(y_2)| = |f^{-1}(y_1)| \pm 1
\]
**Proof.** Without loss of generality, we may assume that $U \cong (-1,1) \times (-1,1)$, while $f(\Sigma) \cap U$ coincides with $(-1,1) \times \{0\}$. Let $\gamma$ denote the embedded curve $\{0\} \times (-1,1)$. We may assume that $y_1$ and $y_2$ are points on $\gamma$. Now, let $\pi_1: U \to (-1,1)$ denote the projection of $U$ onto the first factor. Then the composition $\pi_1 \circ f|_{M_0}: f^{-1}(U) \to (-1,1)$ is a proper submersion of the manifold $M_0 := f^{-1}(U)$, since
\[
\text{Im } d(\pi_1 \circ f|_{M_0}) = d\pi_1(\text{Im } d(f|_{M_0})).
\]
Consequently, the map $\pi_1 \circ f|_{M_0}$ is a trivial fiber bundle with fiber diffeomorphic to $V := f^{-1}(\gamma)$, i.e. $M_0 \cong V \times [0,1]$. In view of the inherited orientation on $M_0$, we deduce that the manifold $V$ is also orientable.

Now, we examine the number of components of the preimages $f^{-1}(y_1)$, $f^{-1}(y_2)$ which are subsets of the surface $V$. Since the restriction $f|_V : V \to \gamma$ is a Morse function, the manifold $f^{-1}(y_2)$ is obtained from $f^{-1}(y_1)$ by an elementary oriented surgery. We conclude that the numbers of path components in $f^{-1}(y_1)$ and $f^{-1}(y_2)$ differ by exactly 1. □

Lemma 7.1 shows that the function $c$ counting the number of path components in the regular fibers of $f$ is an integral chessboard function. In particular, the image of the singular set $f \Sigma$ carries a canonical local coorientation.

7.3. The Euler chessboard function. Let $f : M \to \mathbb{R}^n$ be a proper generic map of a manifold of dimension $n + 2q$ for some $q \geq 0$. Let $c$ be the following continuous integer valued function on $\mathbb{R}^n \setminus f(\Sigma)$:

$$c(y) = \begin{cases} \chi(f^{-1}(y)) & \text{if } \chi(f^{-1}(y)) \text{ is even,} \\ \chi(f^{-1}(y)) + 1 & \text{if } \chi(f^{-1}(y)) \text{ is odd.} \end{cases}$$

Recall that under elementary surgery, the Euler characteristic of fibers in adjacent regions is changed by $\pm 2$. From this fact, it follows that $c$ is in integral chessboard function.

7.4. The depth function. Let $f : M \to \mathbb{R}^n$ be a smooth generic map of a closed manifold of dimension $m \geq n$. Suppose that $n > 1$. Given a point $y \in \mathbb{R}^n \setminus f(\Sigma)$, we say that a path $\gamma$ is a path to infinity if one endpoint of $\gamma$ is contained in the unbounded region of $\mathbb{R}^n \setminus f(\Sigma)$. We note that since $n > 1$, the unbounded region is unique. Also, we say that a path $\ell_y$ from $y$ to infinity is a generic path if it intersects each stratum $f(\Sigma')$ of the singular set transversely, and the intersection $\ell_y \cap f(A_{11})$ is empty. We note that a generic curve $\ell_y$ is disjoint from the strata $f(\Sigma')$ of dimension $\leq n - 1$. Consequently, the curve $\ell_y$ only intersects the singular set $f(\Sigma)$ at fold critical values, i.e., the intersection $\ell_y \cap f(\Sigma)$ is a subset of $f(A_1)$.

The depth function $d : \mathbb{R}^n \setminus f(\Sigma) \to \mathbb{Z}_{\geq 0}$ associates with each point $y$, the minimal number of intersection points $\ell_y \cap f(\Sigma)$, where $\ell_y$ ranges over all generic paths from $y$ to infinity. For estimates of the invariant

$$\text{dep}(\Sigma) = \min \{d(y) \mid y \in \mathbb{R}^n \setminus f(\Sigma)\}$$

we refer the reader to [6].

Lemma 7.2. Let $f : M \to \mathbb{R}^n$ be a smooth generic map of a closed manifold of dimension $m \geq n$. Suppose that $n > 1$. Let $\gamma : [-1,1] \to \mathbb{R}^n$ be a smooth embedded
curve with image in \( f(A_i) \cup f(A_j) \). Suppose that \( \gamma \) intersects \( f(A_i) \) transversely at a unique point \( \gamma(0) \), and define \( y = \gamma(1) \) and \( z = \gamma(-1) \). Then \( d(y) = d(z) \pm 1 \).

**Proof.** Let \( X \) denote the set of singular points \( x \in \Sigma(f) \) of types \( I = (m - n), (m - n + 1, 1), (m - n + 1, 1, 1) \). Then \( f(X) \) is a finite union of submanifolds of \( \mathbb{R}^n \) of codimension at least 3 and \( \Sigma(f) \cap X \) is a submanifold of \( \mathbb{R}^n \) of dimension \( n - 1 \).

Indeed, the set \( \Sigma(f) \) is the union of sets \( \Sigma^i(f) \), which consist of points \( x \) at which the kernel rank is \( i \), where \( i = m - n, m - n + 1, \ldots, m \). If \( f \) is generic, then each \( \Sigma^i(f) \subset \mathbb{R}^n \) is a submanifold of codimension \( i(n - m + i) \). In particular, if \( i \geq m - n + 2 \), then the codimension of \( \Sigma^i(f) \) is at least 4. Similarly, by the Boardman formula, the codimension of \( \Sigma^{i_1, i_2, \ldots, i_k} \) is

\[
\nu_{i_1, \ldots, i_k}(m, n) = (n - m + i_1)\mu(i_1, \ldots, i_k) - (i_1 - i_2)\mu(i_2, \ldots, i_k) - \cdots - (i_{k-1} - i_k)\mu(i_k),
\]

where \( \mu(i_1, \ldots, i_k) \) is the number of sequences \( j_1, \ldots, j_k \) of non-negative integers such that \( j_1 \geq j_2 \geq \cdots \geq j_k \), and \( i_1 \geq j_1 > 0 \), \( i_2 \geq j_2 \), \ldots, \( i_k \geq j_k \). Thus, the codimension of \( f(\Sigma^I) \subset \mathbb{R}^n \) is at most 2 if and only if \( I \) is \( (m - n), (m - n + 1, 1), \) or \( (m - n + 1, 1, 1) \).

Now, let \( \gamma \subset \mathbb{R}^n \) be a closed curve intersecting \( f(\Sigma) \) transversely at a unique point. Assume, contrary to the conclusion of Lemma 7.2, that \( y = d(\gamma(1)) \) does not differ from \( z = d(\gamma(-1)) \) by 1. Let \( \ell_y \) and \( \ell_z \) be respective paths from \( y \) and \( z \) to infinity that intersect \( f(\Sigma) \) transversely precisely \( d(y) \) and \( d(z) \) times. Without loss of generality, we may assume that the path \( \ell_y^{-1} * \gamma * \ell_z \) is closed, where \( * \) is path concatenation. It is important to note that this closed path is null-homotopic. Furthermore, without loss of generality, we may assume that \( \ell_y^{-1} * \gamma * \ell_z \) avoids \( X \) for all moments of time during the homotopy to a point. Thus, under the specified generic regular homotopy of \( \ell_y^{-1} * \gamma * \ell_z \), the number of intersection points of \( \ell_y^{-1} * \gamma * \ell_z \) with the stratified manifold \( f(\Sigma \setminus X) \) changes by an even number. Therefore, the number \( d(y) + d(z) + 1 \) of intersection points of \( \ell_y^{-1} * \gamma * \ell_z \) with \( f(\Sigma) \) is even. On the other hand, by definition of the depth function, it is clear that \( d(y) \) differs from \( d(z) \) by at most 1. Thus, \( d(y) \) differs from \( d(z) \) precisely by 1.

The depth function can also be defined for any smooth generic map \( f: M \to N \) of a closed manifold of dimension \( m \) to a pointed manifold of dimension \( n \leq m \). In this case, a path to infinity is a path \( \gamma \) with an endpoint at the distinguished point of \( N \).

We note that the proof of Lemma 7.2 remains valid for maps to simply connected manifolds \( N \).

### 8. The Cumulative Winding Number

We first recall the definition of the Gauss map. Given an immersion \( \gamma: [a, b] \to \mathbb{R}^2 \) of a segment, the **Gauss map** \( G: [a, b] \to S^1 \) associates with a point \( t \in [a, b] \) the unit vector \( \hat{\gamma}(t)/|\hat{\gamma}| \). Let \( \mathbb{R} \to S^1 = [0, 1]/\sim \), where \( \{0\} \sim \{1\} \), be the universal covering
that takes a point \( x \) to its congruence class modulo 1. Let \( \tilde{G} \) denote a lift of \( G \) with respect to the universal covering. We define the \textit{winding number} of \( \gamma \) by \( \tilde{G}(b) - \tilde{G}(a) \). Given two parametrizations \( \gamma' \) and \( \gamma \) of the same immersed curve, it follows that the winding numbers of \( \gamma' \) and \( \gamma \) are the same if and only if the orientations of the curve induced by \( \gamma \) and \( \gamma' \) agree.

Let \( \alpha \) be an arc of the diagram \((\Sigma(f); P, Q)\) associated with a map \( f \). It corresponds to an arc \( \tilde{\alpha} = f(\alpha) \) contained in the set \( f(\Sigma) \). The curve \( \tilde{\alpha} \) is an immersed curve in \( \mathbb{R}^2 \), with possible self-intersection points only on the boundary. By definition, the \textit{winding function} \( \varphi \) is a function on the set of arcs of \( f(\Sigma) \) that associates with an arc \( \alpha \) the winding number \( \varphi(\alpha) \) of the curve \( \tilde{\alpha} \).

\textbf{Definition 8.1.} Suppose that at every self-intersection point the two intersecting segments are perpendicular. Then the real number
\[
\omega(f) := \sum_{\alpha} \varphi(\alpha)
\]
is the \textit{cumulative winding number} of \( f(\Sigma) \), where \( \alpha \) ranges over all arcs of \( \Sigma(f) \).

\textbf{Proposition 8.2.} For a generic smooth map \( f : M \to \mathbb{R}^2 \), we have
\[
\omega(f) \in \frac{1}{2} \mathbb{Z}.
\]

To prove Proposition 8.2 we introduce the notion of a regularization of the singular set. The regularization of the singular set \( f(\Sigma) \) is a smooth embedded closed curve \( \mathcal{R}f(\Sigma) \subset \mathbb{R}^2 \) obtained from \( f(\Sigma) \) by smoothing the curve \( f(\Sigma) \) near the cusp points as in Fig. 8 and modifying \( f(\Sigma) \) near its self-intersection points. Namely, let \( y \) be a self-intersection point of \( \Sigma(f) \). Then near \( y \) the curve \( f(\Sigma) \) consists of two arcs \( \alpha \) and \( \beta \). We remove the two arcs \( \alpha \) and \( \beta \) from \( f(\Sigma) \) and attach them back so that the orientation on \( f(\Sigma) \setminus \{\alpha \cup \beta\} \) extends over the new attached arcs, see Fig. 9, 10 and 11.

The proof of the following lemma is omitted as it is straightforward.

\textbf{Lemma 8.3.} The regularization of a cusp decreases the cumulative winding number by \( \frac{1}{2} \) if the coorientations of \( \alpha \) and \( \beta \) are as indicated in Fig. 10, and increases the cumulative winding number by \( \frac{1}{2} \), otherwise.
Lemma 8.4. For a self-intersection point of \( f(\Sigma) \) of the form \((a, a + 1, a, a + 1)\), there are two regularizations that preserve the orientation of the diagram: \( \mathcal{R}_- \) and \( \mathcal{R}_+ \). The regularizations \( \mathcal{R}_- \) and \( \mathcal{R}_+ \) decrease and increase the cumulative winding.
number by $\frac{1}{2}$ respectively. For a self-intersection point of the form $(a, a + 1, a, a - 1)$, the only possible regularization does not change the cumulative winding number.

**Proof.** For a self-intersection point of the form $(a, a + 1, a, a - 1)$, the only possible regularization does not change the cumulative winding number, see Fig. 11. If a double point is of the form $(a, a+1, a, a+1)$ there are two possible regularizations that preserve orientation. One of the regularizations increases the cumulative winding number by $1/2$, while the other one decreases the cumulative winding number by $1/2$, see Fig. 9 and Fig. 10. The two regularizations are denoted by $\mathcal{R}_+$ and $\mathcal{R}_-$ respectively. □

**Proof of Proposition 8.2.** We note that $\mathcal{R}f(\Sigma(f))$ consists of embedded curves, and therefore its cumulative winding number is an integer. On the other hand, under the regularization, the cumulative winding number is changed by $\pm \frac{1}{2}$ for each regularization of a cusp, and $\pm \frac{1}{2}$ or 0 for each regularization of a self-crossing. □

9. **Changes of the cumulative winding number under homotopy**

We now observe and record how the cumulative winding number is changed under generic homotopy. We will denote an $R_2$ move by $R_2(a_1, a_2, a_3, a_4)$, where the quadruple $(a_1, a_2, a_3, a_4)$ encodes the type of the two self-intersection points that are either being created or removed as a result of the $R_2$ move. We note that the types $(a_1, a_2, a_3, a_4)$ of the two self-intersection points are the same up to permutation and a reflection. For example, $(a_1, a_2, a_3, a_4) \leftrightarrow (a_3, a_2, a_1, a_4)$ does not change the type of the double point. For the remainder of our discussion, we adopt the convention that $a_1$ corresponds to the bounded region. In Fig. 12, this is the region bounded by $a_2 \cup \beta_2$.

We warn the reader that it is possible that $(a_1, a_2, a_3, a_4)$ and $(b_1, b_2, b_3, b_4)$ are double points of the same type, but $R_2(a_1, a_2, a_3, a_4)$ and $R_2(b_1, b_2, b_3, b_4)$ are different.

**Lemma 9.1.** Let $f: M \to \mathbb{R}^2$ be a generic map of a smooth manifold of dimension $\geq 2$. For any integral chessboard function, there are at most five possible types of $R_2$ moves: $R_2(a, a - 1, a - 1, a - 1)$, $R_2(a, a + 1, a + 2, a + 1)$, $R_2(a, a + 1, a, a - 1)$, $R_2(a, a + 1, a, a + 1)$, and $R_2(a, a - 1, a, a - 1)$. The moves $R_2(a, a - 1, a - 2, a - 1)$, $R_2(a, a + 1, a, a + 1)$ and $R_2(a, a + 1, a, a - 1)$ do not change $\omega$. The moves $R_2(a, a + 1, a, a + 1)$ and $R_2(a, a - 1, a, a - 1)$ change the cumulative winding number by 1 and $-1$ respectively.

**Proof.** Consider an $R_2$-move of type $R_2(a_1, a_2, a_3, a_4)$. Since the numbers $a_i$ represent the values of an integral chessboard function, we have $a_i + 1 = a_i \pm 1$ and $a_4 = a_1 \pm 1$. Since up to rotation, the type $R_2(a, a - 1, a, a + 1)$ is the same as $R_2(a, a + 1, a, a - 1)$, the list of $R_2$ moves in the statement of Lemma 9.1 exhausts all possibilities of different types of $R_2$ moves.
It remains to compute the changes of the cumulative winding number $\omega$ under each $R_2$ type move. Denote the two arcs undergoing an $R_2$ move by $\alpha$ and $\beta$. Without loss of generality, we assume that $\beta$ is straight and fixed, so that only $\alpha$ moves under homotopy. After the $R_2$ move, the two new double points partition the diagram into six arcs: $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \text{ and } \beta_3$ (see Fig 12). We notice that for any type of $R_2$ move, the winding numbers of $\beta, \alpha_1, \alpha_3, \beta_1, \beta_2, \text{ and } \beta_3$ are trivial. Thus, the change in the cumulative winding number is the same as the difference of the winding numbers of $\alpha$ and $\alpha_2$. For example, for the move $R_2(a, a - 1, a - 2, a - 1)$, the winding numbers $\varphi(\alpha)$ and $\varphi(\alpha_2)$ are $-1/2$. Therefore, the cumulative winding number does not change under the $R_2$ move of type $R_2(a, a - 1, a - 2, a - 1)$. The changes in the cumulative winding number for the other $R_2$ moves can be calculated similarly.

Next we turn to the case of swallowtail moves. Denote the swallowtail move that creates a self-intersection point of type $(a_1, a_2, a_3, a_4)$ by $ST(a_1, a_2, a_3, a_4)$, where $a_1$ corresponds to the bounded region. In Fig. 13, this is the region entrapped by $\alpha_1, \alpha_2, \alpha_3$. For a $\mathbb{Z}_2$-valued chessboard function, we say that an $R_2$ move or an $ST$ move is even if the value of the chessboard function over the bounded region is 0. Otherwise, we say that the $R_2$ move or $ST$ move is odd.

**Lemma 9.2.** For any $\mathbb{Z}_2$-valued chessboard function, there are at most two $R_2$ moves: even and odd. An even $R_2$ move increases the winding number by 1, while an odd $R_2$ move decreases the winding number by 1.

We omit the proof of Lemma 9.2 since the proofs for even and odd $R_2$ moves are the same as those for $R_2(a, a + 1, a, a + 1)$ and $R_2(a, a - 1, a, a - 1)$ in Lemma 9.1.

**Lemma 9.3.** Let $f: M \to \mathbb{R}^2$ be a generic map of a smooth manifold of dimension $\geq 2$. For any integral chessboard function there are at most four possible types of swallowtail moves. Namely, $ST(a, a + 1, a + 2, a + 1)$, $ST(a, a + 1, a, a + 1)$, $ST(a, a - 1, a, a - 1)$, and $ST(a, a - 1, a - 2, a - 1)$. Moreover, the moves $ST(a, a + 1, a + 2, a + 1)$, $ST(a, a + 1, a, a + 1)$, $ST(a, a - 1, a, a - 1)$, and $ST(a, a - 1, a - 2, a - 1)$ are even.
The moves $ST(a, a + 1, a, a + 1)$ and $ST(a, a - 1, a, a - 1)$ do not change the winding number. The moves $ST(a, a + 1, a, a + 1)$ and $ST(a, a - 1, a, a - 1)$ respectively decrease and increase the winding number by $\frac{1}{2}$.

**Proof.** Given a swallowtail type $ST(a_1, a_2, a_3, a_4)$, the numbers $a_i$ represent the values of a chessboard function and therefore satisfy the relations $a_{i+1} = a_i \pm 1$ and $a_4 = a_1 \pm 1$. Consequently, $ST(a, a+1, a+2, a+1)$, $ST(a, a+1, a, a+1)$, $ST(a, a-1, a, a-1)$, and $ST(a, a - 1, a - 2, a - 1)$ are the only possible types of swallowtail moves.

We now calculate how the winding number is affected by the swallowtail move of type $ST(a_1, a_2, a_3, a_4)$, where $\alpha$ is an arc of the singular set diagram $f(p \Sigma q)$. Without loss of generality, we may assume that $\alpha_1$ corresponds to the arc whose endpoints are both cusps. We may assume that $\alpha$ and $\alpha_1$ are straight, thus $\varphi(\alpha) = \varphi(\alpha_1) = 0$. Then $\varphi(\alpha_2) = \varphi(\alpha_3) = -\frac{1}{2}$, $\varphi(\alpha_4) = \varphi(\alpha_5) = \frac{1}{8}$, and therefore the cumulative winding number of the singular set does not change under the swallowtail move of type $ST(a_1, a_2, a_3, a_4)$.

The change of the winding number for other types of swallowtail moves can be calculated similarly. $\square$

**Lemma 9.4.** For any $\mathbb{Z}_2$-valued chessboard function, there are at most two $ST$ moves: even and odd. An even $ST$ move decreases the winding number by $1/2$, while an odd $ST$ move increases the winding number by $1/2$.

The proofs of Lemma 9.4 are the same as those for $ST(a_1, a_2, a_3, a_4)$ in Lemma 9.3.

It remains to examine how the cumulative winding number $\omega(f)$ is changed under wrinkles, $R_3$ moves, cusp-fold moves, and cusp merges.

**Lemma 9.5.** Let $f : M \to \mathbb{R}^2$ be a generic map of a smooth manifold of dimension $\geq 2$. For any integral chessboard function, the wrinkle and cusp-fold moves do not change the cumulative winding number associated with the diagram $(\Sigma(f); P, Q)$. The same is true for cusp merges for the Euler chessboard function, and the chessboard functions that count path components of regular fibers.
Proof. The statements of Lemma 9.5 for wrinkles and cusp merges are easily verified. Next, we examine how cusp-fold moves affect $\omega$. Label the arcs before and after a cusp-fold move as in Fig. 14. Then the contribution of $\varphi(\alpha)$ is replaced with $\varphi(\alpha_1) + \varphi(\alpha_2)$, the contribution of $\varphi(\beta)$ is replaced with $\varphi(\beta_1) + \varphi(\beta_2)$, and the contribution of $\varphi(\gamma)$ is replaced with $\varphi(\gamma_1) + \varphi(\gamma_2) + \varphi(\gamma_3)$. Consequently, under a cusp-fold move the winding number is modified continuously. Since the cumulative winding number is an element of $\frac{1}{2}\mathbb{Z}$, we conclude that $\omega$ is unchanged under cusp-fold moves. □

Similarly, we can determine changes of cumulative winding number for $\mathbb{Z}_2$-valued chessboard functions.

**Lemma 9.6.** For $\mathbb{Z}_2$-valued chessboard functions, the cusp merge, cusp-fold, wrinkle moves do not change the cumulative winding number.

**Lemma 9.7.** For any integral chessboard function, $R_3$ moves change the cumulative winding number by $\pm 1/2$ or $\pm 1$ or 0. For any $\mathbb{Z}_2$-valued chessboard function, $R_3$-moves change the cumulative winding number by $\pm 1$.

The following proposition summarizes the above calculations for $\mathbb{Z}_2$-valued chessboard functions.

**Proposition 9.8.** Let $f: M \to \mathbb{R}^2$ be a generic map of a smooth manifold of dimension $\geq 2$. For any $\mathbb{Z}_2$-valued chessboard function, under generic homotopy of a stable map $f$, the cumulative winding number $\omega(f)$ may change only under an ST, $R_2$ or $R_3$ move. Under an ST move, the cumulative winding number changes by $\pm 1/2$. Under an $R_2$ or $R_3$ move, the cumulative winding number changes by $\pm 1$.

In the rest of the section we prove Lemmas 9.11 and 9.13. To prove Lemmas 9.11 and 9.13 we will need Lemmas 9.9 and 9.10.

**Lemma 9.9.** Let $f: M \to \mathbb{R}^2$ be a smooth map of a manifold of dimension 3. Then for the integral chessboard function of §7.2, the coorientation of arcs in $(\Sigma(f); P, Q)$ that have a cusp endpoint is as on Fig. 16. The opposite coorientation is not possible.
Proof. Recall that locally a generic map $f : M^3 \to F^2$ is a Morse 2-function. In particular, for a cusp point $p \in A_2(f)$, we may identify a neighborhood $V$ of $f(p)$ with $[0, 1] \times [0, 1]$, and the inverse image $f^{-1}(V)$ with $[0, 1] \times M_0$ in such a way that $f|_{f^{-1}(V)}$ is given by $(t, x) \mapsto (t, g_t(x))$, where $g_t$ is a family of generalized Morse functions such that $g_t$ has no critical points for $t \in [0, 1/2)$, $g_{1/2}$ has a unique critical point, and $g_t$ has two canceling Morse critical points for $t \in (1/2, 1]$, see Fig. 15.

Let $\alpha$ and $\beta$ be two arcs in $f(\Sigma) \cap V$ that share the common cusp endpoint $p \in A_2(f)$. Then the indices $i_\alpha$ and $i_\beta$ of the two critical points of $g_{3/4}$ on the arcs $\alpha$ and $\beta$ satisfy the relation $i_\beta = i_\alpha + 1$. The arcs $\alpha$ and $\beta$ split $V$ into two regions $A$ and $B$ containing the points $(0, 1/2)$ and $(1, 1/2)$ respectively. Both in the case $(i_\alpha, i_\beta) = (0, 1)$ and $(i_\alpha, i_\beta) = (1, 2)$ the number of path-connected components in the
inverse image of any point in $B$ is one less than that of any point in $A$. Therefore, the coorientations of the arcs $\alpha$ and $\beta$ are as on Fig. 16.

\[ \square \]

**Lemma 9.10.** Consider a smooth generic map $f : M \to \mathbb{R}^2$ of a manifold $M$ of even dimension $m \geq 2$. When $c$ is the integral Euler chessboard function, $\Sigma(f)$ does not have self-intersection points of type $(a,a-1,a,a-1)$.

We note that the statement of Lemma 9.10 is not true for the depth chessboard function.

**Proof.** The intersecting strands of $f(\Sigma)$ break a neighborhood of a self-intersection point into four regions, which we denote by $R,T,L$ and $B$, for the right, top, left, and bottom regions, respectively. Note that the diffeomorphism types of the fibers $M_R,M_T,M_L$ and $M_B$ over points in the four respective regions do not depend on the choice of regular values. If the manifold $M_T$ is obtained from $M_R$ by a surgery of index $i$, then $M_L$ is obtained from $M_B$ by a surgery of the same index $i$. Since $M$ is of even dimension, we conclude

$$\chi(M_T) - \chi(M_R) = \chi(M_L) - \chi(M_B) = \pm 2.$$ 

This rules out the existence of double points of type $(a,a-1,a,a-1)$. \[ \square \]

Recall that a cusp-fold move creates or eliminates two double points of the same type. We will henceforth denote cusp-fold moves creating or eliminating double points of type $(a_1,a_2,a_3,a_4)$ by $\text{CF}(a_1,a_2,a_3,a_4)$, and practice the convention that $a_1$ corresponds to the value of a prescribed chessboard function in the bounded region (in Figure 14 this is the region with boundary $\alpha_1 \cup \beta_1 \cup \gamma_2$). In particular, there are at most two types of cusp-fold moves involving self-intersection points of type $(a,a-1,a,a-1)$, namely, $\text{CF}(a,a-1,a,a-1)$ and $\text{CF}(a,a+1,a,a+1)$.

**Lemma 9.11.** Let $f,g : M \to \mathbb{R}^2$ be two homotopic smooth generic maps, where $M$ has odd dimension $m \geq 3$, with $f|_{\Sigma(f)}$ and $g|_{\Sigma(g)}$ embeddings. For the depth chessboard function, the number of cusp-fold moves involving self-intersection points of type $(a,a-1,a,a-1)$ is even. If the dimension of $M$ is 3, the same is true for the chessboard function counting path-connected components of fibers. If the dimension of $M$ is even, the same is true for the Euler chessboard function.

**Proof.** Suppose the dimension of $M$ is 3. Equip $(\Sigma(f); P,Q)$ with the chessboard function counting the number of path-connected components of regular fibers. By Lemma 9.9 all cusps are cooriented as in Fig. 16 and therefore, the value of the chessboard function over the bounded region is maximal. Consequently, the only possible cusp-fold move involving self-intersection points of type $(a,a-1,a,a-1)$ is $\text{CF}(a,a-1,a,a-1)$. Every cusp-fold move changes the parity of self-intersection points of the fold curve where one intersecting segment of the fold curve has an odd
index while the other one has an even index. No other moves change the parity of the number of such self-intersection points. Since \( f(\Sigma) \) and \( g(\Sigma) \) are embedded, we conclude that the number of \( CF(a, a - 1, a, a - 1) \) moves must be even.

The same argument holds for maps \( f : M \to \mathbb{R}^2 \) of a manifold \( M \) of an arbitrary odd dimension \( m \geq 3 \) equipped with the integral depth chessboard function. Indeed, the value of the depth chessboard function over the bounded region on Fig. 16 is greater than or equal to its values over the other regions. Therefore, in this case \( CF(a, a - 1, a, a - 1) \) is the only cusp-fold type move that involves self-intersection points of type \( (a, a - 1, a, a - 1) \).

Now, let \( f : M \to \mathbb{R}^2 \) be a generic map of a manifold \( M \) of arbitrary even dimension \( m \geq 2 \). By Lemma 9.10, there are no self-intersection points of type \( (a, a - 1, a, a - 1) \) with respect to the Euler chessboard function, and therefore, there are no cusp-fold moves involving double points of this type at all. □

**Remark 9.12.** We note that for an arbitrary chessboard function, its value need not be maximal over the bounded region created by a cusp-fold move. In general, there may possibly be six different types of cusp-fold moves: \( CF(a, a - 1, a, a - 1), CF(a, a - 1, a, a + 1), CF(a, a - 1, a - 2, a - 1), CF(a, a + 1, a, a + 1), CF(a, a + 1, a, a - 1), \) and \( CF(a, a + 1, a + 2, a + 1) \).

**Lemma 9.13.** Suppose that \( f : M \to \mathbb{R}^2 \) is a stable map of a manifold of even dimension. Then for the integral Euler chessboard function, the cumulative winding number does not change under homotopy of \( f \). The same is true for stable maps of manifolds of even dimension to \( F \), where \( F \) is the complement to a point in a closed oriented surface.

**Proof.** Consider a map \( f \) to \( \mathbb{R}^2 \). By Lemma 9.10, no double points of type \( (a, a - 1, a, a - 1) \) may occur for the Euler chessboard function. Consequently, the local coorientation of fold arcs defines a global orientation of the curve of fold points as the coorientations of arcs with common endpoints agree, see Fig. 7. Therefore, \( R_3 \) moves do not change the cumulative winding number. The cumulative winding number is preserved by \( R_2 \) and \( ST \) moves by Lemma 9.1 and Lemma 9.3.

In the case where the target space is a punctured surface, we note that the tangent bundle \( TF \) is trivial, and therefore the winding number associated with the integral Euler chessboard function is well-defined. The rest of the proof in this case is similar to one in the case where the target space is \( \mathbb{R}^2 \). □

10. **Proof of Theorem 1.1 and Theorem 1.2**

**Theorem 1.1.** Let \( f \) and \( g \) be two homotopic image simple fold maps from a closed oriented manifold \( M \) of even dimension \( m \geq 2 \) to an oriented surface \( F \) of finite
genus. Then, the number of components of $\Sigma(f)$ is congruent modulo two to the number of components of $\Sigma(g)$.

Proof. To begin with let us assume that the target surface is $\mathbb{R}^2$. Recall that $\#|\Sigma(f)|$ denotes the number of components of $\Sigma(f)$. Let $c$ be the Euler chessboard function as described in §7.3.

By Lemma 9.13

$$\omega(f) \equiv \omega(g) \pmod{2}. $$

Next, utilizing the hypothesis that $f(\Sigma)$ and $g(\Sigma)$ are embedded, we deduce

$$\omega(f) \equiv \#|\Sigma(f)| \pmod{2}. $$

Combining the previous congruences yields the desired result

$$\#|\Sigma(f)| \equiv \omega(f) \equiv \omega(g) \equiv \#|\Sigma(g)| \pmod{2}. $$

This concludes the proof of Theorem 1.1 in the case of maps to $\mathbb{R}^2$.

Now, let $p$ be a point in the closed surface $F$, away from $f(\Sigma(f))$. Then the tangent bundle of $F\setminus\{p\}$ is trivial, and therefore the winding number $w(f)$ is well-defined. Under a generic homotopy of $f$, the curve $f(\Sigma(f))$ may slide through the point $p$. As the curve $f(\Sigma(f))$ slides through the point $p$, the winding number changes by $\pm\chi(F)$, where $\chi(F)$ denotes the Euler characteristic of the surface $F$. Since the surface $F$ is closed and oriented of genus $g$, we have $\chi(F) = 2 - 2g$. Thus, the parity of the winding number is well-defined. Consequently, as in the case when the target surface is $\mathbb{R}^2$, we have

$$\omega(f) \equiv \omega(g) \pmod{2}. $$

This also shows that for every embedded closed curve $\gamma$ on an oriented closed surface $F$, there is a well-defined winding number $\rho(\gamma) \in \mathbb{Z}_2$. The winding number $\rho(\gamma)$ does not depend on the orientation of $\gamma$.

Lemma 10.1. Let $\gamma_1$ and $\gamma_2$ be two embedded closed curves on an oriented closed surface $F$. Suppose that $\gamma_1$ and $\gamma_2$ represent the same homology class in $H_1(F;\mathbb{Z}_2)$. Then

$$\rho(\gamma_1) - \#|\gamma_1| \equiv \rho(\gamma_2) - \#|\gamma_2| \pmod{2},$$

where $\#|\gamma_i|$ denotes the number of components of $\gamma_i$, for $i = 1, 2$.

Proof. We may assume that the surface $F$ is connected.

Recall that an oriented surgery of an embedded closed curve $\gamma$ is embedded if the base of surgery is an embedded strip whose interior is disjoint from the curve $\gamma$, see Fig. 17 and 18. We note that under each oriented embedded surgery the value $\rho(\gamma)$, as well as the modulo two residue class of $\#|\gamma|$, is changed. Thus, the value $\rho(\gamma) - \#|\gamma|$ remains the same.
By performing an appropriate number of elementary surgeries, we may assume \( \gamma_1 \) and \( \gamma_2 \) are path-connected closed embedded curves. Since \( \gamma_1 \) and \( \gamma_2 \) represent the same homology class in \( H_1(F; \mathbb{Z}_2) \), they are either both separating or non-separating.

If the curves are non-separating, then there is a diffeomorphism \( \varphi \) of the target surface \( F \) to itself that takes \( \gamma_1 \) to \( \gamma_2 \). Thus, the parity of \( \rho(\gamma_1) - \#|\gamma_1| \) is the same as the parity of \( \rho(\gamma_2) - \#|\gamma_2| \).

Next, suppose that the curves \( \gamma_1 \) and \( \gamma_2 \) are separating. Without loss of generality, we may assume that \( \gamma_1 \) and \( \gamma_2 \) are disjoint, since there is a diffeomorphism \( \varphi \) of \( F \) such that \( \gamma_1 \) and \( \varphi(\gamma_2) \) are disjoint. We may always construct a Morse function \( h \) on \( F \) such that \( \gamma_1 \) and \( \gamma_2 \) are two regular level sets of \( F \), say \( h^{-1}(0) = \gamma_1 \) and \( h^{-1}(1) = \varphi(\gamma_2) \). Each critical point of \( h \) in \( h^{-1}[0,1] \) corresponds to an elementary oriented embedded surgery. The composition of these elementary oriented embedded surgeries takes \( \gamma_1 \) to a curve isotopic to \( \gamma_2 \). As mentioned above, the value of \( \rho(\gamma_1) \) and the modulo two residue class \( \#|\gamma_1| \) are changed under each elementary oriented embedded surgery. Therefore, the value \( \rho(\gamma_1) - \#|\gamma_1| \) is preserved.

In both cases, we have deduced the desired result. \( \square \)
In view of Lemma 10.1 we conclude that
\[ \#|\Sigma(f)| \equiv \#|\Sigma(g)| \pmod{2}. \]

If \( F \) is not a closed surface, then it admits an embedding \( j \) into a closed surface \( F' \). Then the numbers of path components of \( \Sigma(f) \) and \( \Sigma(g) \) are the same as the numbers of components of \( \Sigma(j \circ f) \) and \( \Sigma(j \circ g) \), respectively. Therefore, the case where \( F \) is an open surface of finite genus follows from the case where \( F \) is a closed surface. \( \square \)

Theorem 1.2. Let \( f \) and \( g \) be two homotopic image simple fold maps from a closed oriented manifold \( M \) of odd dimension \( m \geq 2 \) to an oriented surface \( F \). Suppose that \( F \) is \( \mathbb{R}^2 \) or \( S^2 \) if \( m > 3 \). Suppose that no \( R_3 \) moves occur during the homotopy from \( f \) to \( g \). Then, the number of components of \( \Sigma(f) \) is congruent modulo two to the number of components of \( \Sigma(g) \).

Proof. We will work with the \( \mathbb{Z}_2 \)-valued depth chessboard function if \( m \geq 3 \). If \( m = 3 \), then we will work with the \( \mathbb{Z}_2 \)-valued chessboard function that counts the number modulo 2 of path components in the preimage of a regular value.

Since \( f \) and \( g \) are odd dimensional image simple fold maps to a surface, the homology class of swallowtail singularities of a homotopy of \( f \) to \( g \) is trivial. Therefore, by the argument in [14], there is a formal homotopy of \( f \) to \( g \) with no swallowtail singularities. By the relative h-principle for swallowtail singular points [1], we may assume that the (genuine) homotopy of \( f \) to \( g \) does not have swallowtail singular points.

We claim that the number of \( CF \) moves is even. Indeed, since the dimension of the maps \( f \) and \( g \) is odd, the parity of the index of any fold point does not depend on the local coorientation of the fold curve in the target space. Furthermore, the parities of the indices of two fold strands in a neighborhood of any cusp point are different. Consequently, each \( CF \) move changes the number of intersection points of the fold curve of even index with the fold curve of odd index by 1. No other moves change the parity of the number of such intersection points. Therefore, the number of \( CF \) moves is even.

On the other hand, since there are no swallowtail singular points, the number of pairs of self-intersection points changes under homotopy by
\[ \#|CF| + \#|R_2| \equiv 0 \pmod{2}. \]

Consequently, the number of \( R_2 \) moves is also even.

Suppose now that the surface \( F \) is \( \mathbb{R}^2 \). By Proposition 9.8, only \( ST \), \( R_2 \) and \( R_3 \) moves may change the cumulative winding number. We have assumed that the homotopy of \( f \) to \( g \) does not involve \( ST \) and \( R_3 \) moves. Therefore, since each \( R_2 \) move changes the cumulative winding number by \( \pm 1 \), and the number of \( R_2 \) moves
is even, we conclude that the parities of the cumulative winding numbers for \( f \) and \( g \) are the same. Consequently, the number of components of \( \Sigma(f) \) is congruent modulo two to the number of components of \( \Sigma(g) \).

The argument in the end of the proof of Theorem \([1.1]\) shows that the same conclusion is true in the case where the target surface \( F \) is a sphere if \( m > 3 \), and where \( F \) is an oriented surface when \( m = 3 \). \( \square \)

11. The invariant \( I \) and proof of Theorem \([1.3]\)

In this section we prove Theorem \([1.3]\). The main ingredient of the proof is the \( \mathbb{Z}_4 \)-valued homotopy invariant \( I(f) \) defined in the introduction. We will recall the precise definition of the function \( I(f) \) in the statement of Lemma \([11.1]\).

Let \( M \) be a connected closed oriented manifold of dimension \( m \geq 3 \), and \( f: M \to F \) a smooth stable map to an oriented surface \( F \). Then, the singular set \( \Sigma(f) \) is a closed 1-dimensional submanifold of \( M \), which consists of fold points \( A_1(f) \), and finitely many cusp points \( A_2(f) \). Recall, the number of components of the singular set \( \Sigma(f) \) is denoted by \( \#|\Sigma(f)| \), while the number of cusp points is denoted by \( \#|A_2(f)| \).

We will also consider the modulo two congruence class \( \Delta(f) \) of the number of self-intersections of \( f \). We note that if \( f \) is generic, then the image of cusp points is not at the self-intersection points of \( f(\Sigma) \).

**Lemma 11.1.** Let \( f, g: M \to F \) be two generic maps of a closed orientable manifold of dimension \( m \geq 3 \) into an orientable surface. Suppose that there exists a generic homotopy \( H: M \times [0,1] \to F \times [0,1] \) between \( f \) and \( g \) such that the singular set \( \Sigma(H) \) is an orientable submanifold of \( M \times [0,1] \). Then \( I(f) = I(g) \) where

\[
I \equiv \#|A_2| + 2\Delta + 2\#|\Sigma| \pmod{4}.
\]

**Proof.** Let \( F: M \times [0,1] \to N \times [0,1] \) be a generic homotopy such that \( F(x,0) = f(x) \) and \( F(x,1) = g(x) \). Under the homotopy \( F \), the singular set of \( f \) may be modified by any of the six allowable homotopy moves. Let \( s \) denote the number of \( ST \) moves and their inverses, and \( m \) the number of wrinkles, cusp merges, and their inverses. \( R_3 \) moves do not change the number of self-intersection points. Under \( R_2 \) and \( CF \) moves the number of self-intersection points may change, but the congruence class of \( 2\Delta(f) \) does not change modulo 4. Therefore,

\[
2\Delta(g) \equiv 2\Delta(f) + 2s \pmod{4},
\]

since every \( ST \) move and their inverse change the parity of the number of double points of the image of the singular set. On the other hand, we have

\[
\#A_2(g) \equiv \#|A_2(f)| + 2s + 2m \pmod{4},
\]

since every \( ST \) move, wrinkle, cusp merge and their inverse changes the number of cusps by two. Next, since every wrinkle, cusp merge and their respective inverses
change the parity of $\#|\Sigma(f)|$, we also have the congruence
\[ 2\#|\Sigma(g)| \equiv 2\#|\Sigma(f)| + 2m \pmod{4}. \]

To summarize,
\[ 2\#|\Sigma(g)| + 2\Delta(g) + \#|A_2(g)| \equiv 2\#|\Sigma(f)| + 2\Delta(f) + \#|A_2(f)| + 4s + 4m \pmod{4}, \]
which simplifies to
\[ 2\#|\Sigma(g)| + 2\Delta(g) + \#|A_2(g)| \equiv 2\#|\Sigma(f)| + 2\Delta(f) + \#|A_2(f)| \pmod{4}, \]
yielding
\[ I(g) \equiv I(f) \pmod{4}. \]

Remark 11.2. When the closed orientable source manifold $M$ is even dimensional, the singular set $\Sigma(f)$ is necessarily orientable, by Theorem 3.4. Thus, the function $I(f)$ is a homotopy invariant for generic maps $f : M \to F$ of an orientable closed manifold of even dimension into an orientable surface.

Corollary 11.3. The function
\[ I(f) = \#|A_2(f)| + 2\#|\Sigma(f)| \pmod{4} \]
is a homotopy invariant of image simple maps $f : M \to F$, where $M$ is an even dimensional closed orientable manifold and $F$ is an orientable surface.

Corollary 11.4. The function
\[ I(f) = 2\Delta(f) + 2\#|\Sigma(f)| \pmod{4} \]
is a homotopy invariant of simple stable maps $f : M \to F$, where $M$ is an even dimensional closed orientable manifold and $F$ is an orientable surface. Moreover, if $g$ is a simple stable map obtained from $f$ via generic homotopy, then
\[ \Delta(f) + \#|\Sigma(f)| \equiv \Delta(g) + \#|\Sigma(g)| \pmod{2}. \]

Theorem 1.3 essentially follows from the existence of the invariant $I(f)$.

Theorem 1.3. Let $f$ and $g$ be two homotopic image simple maps from a closed oriented manifold $M$ of dimension $m \geq 2$ to an oriented surface $F$ of finite genus. Suppose the surface $\Sigma(H)$ of singular points of the homotopy $H$ between $f$ and $g$ is orientable. Then, the number of components of $\Sigma(f)$ is congruent modulo two to the number of components of $\Sigma(g)$.

Proof. Consider the homotopy invariant
\[ I(f) = \#|A_2(f)| + 2\Delta(f) + 2\#|\Sigma(f)| \pmod{4}. \]
By assumption, the maps $f$ and $g$ have no cusps and are embedded, therefore $\#|A_2(f)| = \Delta(f) = 0$ and $\#|A_2(g)| = \Delta(g) = 0$. Therefore,
$$I(f) = 2\#|\Sigma(f)| \pmod{4} \quad \text{and} \quad I(g) = 2\#|\Sigma(g)| \pmod{4}.$$ 
By Lemma 11.1 we have $I(f) \equiv I(g)$. Thus,
$$2\#|\Sigma(f)| \equiv 2\#|\Sigma(g)| \pmod{4}$$
which results in
$$\#|\Sigma(f)| \equiv \#|\Sigma(g)| \pmod{2}.$$

12. Low dimensional applications

In this section we consider examples and applications in the cases of maps to surfaces of manifolds of dimension $m = 2, 3$ and 4.

12.1. Maps of Surfaces to Surfaces. Let $f : F_g \to F_h$ be a simple stable map of oriented surfaces of genera $g$ and $h$, respectively. By Theorem 1.1 the number of path components in $\Sigma(f)$ depends only on the homotopy class of $f$. In fact, a stronger statement is true.

**Proposition 12.1** (M.Yamamoto [20]).
$$\#|\Sigma(f)| \equiv \deg(f)(h-1) - (g-1) \pmod{2}.$$ 
The above proposition holds for arbitrary $g, h \geq 0$ and even for non-embedded singular sets of fold maps. For example,
$$\#|\Sigma(f)| \equiv \deg(f) - 1 \pmod{2}$$
if $f$ is a fold map of a sphere into itself.

12.2. Maps of the 3-sphere to the 2-sphere. In [16], Saeki studied fold maps of 3-dimensional manifolds into surfaces and showed that every closed connected oriented 3-manifold admits a stable map to the 2-sphere without definite fold points. In particular, for maps of the 3-sphere to the 2-sphere, Saeki constructs an image simple stable indefinite fold map $f : S^3 \to S^2$ such that $\Sigma(f) = n + 1$, where $n \in \mathbb{Z}$ is the Hopf invariant $H(f)$ of $f$. Saeki poses the following question.

**Problem 12.2.** For an integer $n \in \mathbb{Z} \simeq \pi_3 S^2$, let us consider stable maps $f : S^3 \to S^2$ without definite fold which represent the associated homotopy class and which satisfies that $\Sigma(f) \neq \emptyset$ and $f|_{\Sigma(f)}$ is an embedding, where $\Sigma(f)$ is the set of singular points of $f$. Then, is the number of components of $\Sigma(f)$ congruent modulo two to $n + 1$?
Saeki solved Problem 12.2 in the negative in [17]. The following corollary shows under what conditions Saeki’s problem can be answered in the positive. As a corollary of Theorems 1.2 and 1.3, we prove the following statement related to Saeki’s question.

**Corollary 12.3.** Let \( f: S^3 \to S^2 \) be an image simple indefinite fold map with Hopf invariant \( H(f) = n \) constructed by Saeki in [16]. If \( g: S^3 \to S^2 \) is obtained from \( f \) by a homotopy 
\[
F: S^3 \times [0,1] \to S^2 \times [0,1]
\]
such that \( \Sigma(F) \) is orientable or \( F(\Sigma) \) has no triple self-intersection points, then
\[
\#|\Sigma(g)| \equiv \#|\Sigma(f)| \equiv n + 1 \pmod{2}.
\]

12.3. **Maps of the 4-sphere to the 2-sphere.** A natural progression is to consider Problem 12.2 for maps of \( S^4 \) to \( S^2 \). As a consequence of Theorem 1.1, we obtain a result on the 4-dimensional analog of Problem 12.2.

**Corollary 12.4.** Let \( f: S^4 \to S^2 \) be an image simple fold map of the 4-sphere into the 2-sphere. Then,
\[
\#|\Sigma(f)| \equiv 1 \pmod{2}.
\]

**Proof.** Let us examine an image simple fold map representative of both the trivial and non-trivial elements of \( \pi_4(S^2) \cong \mathbb{Z}_2 \). We respectively denote the equivalence classes of the trivial and non-trivial elements of \( \pi_4(S^2) \) by \([0]\) and \([1]\). The trivial element is constructed via the standard projection to \( \mathbb{R}^2 \), followed by the inclusion into \( S^2 \), i.e. \( f_{[0]}: S^4 \to \mathbb{R}^2 \to S^2 \), where \( f_{[0]}(\Sigma) \) consists of one closed embedded definite fold curve. Therefore, by Theorem 1.1 any image simple fold map \( g \in [0] \) has a singular set such that \( \#|\Sigma(g)| \) is odd.

\[
\times \times \quad \rightarrow \quad \triangledown \lor \triangleleft
\]

**Figure 19.** Replacing Lefschetz critical points with cusp and indefinite fold points.

Next, we examine the non-trivial element of \( \pi_4(S^2) \). Consider the suspension of the Hopf fibration \( H: S^3 \to S^2 \), defined as \( \Sigma H : \Sigma S^3 \to \Sigma S^2 \), which is equivalent to \( \Sigma H: S^4 \to S^3 \). Composition of the suspended Hopf fibration with the Hopf fibration itself results in the map \( f_{[1]}: H \circ \Sigma H: S^4 \to S^3 \). The singular set of
Figure 20. Three cusp merges.

\( f_{[1]} \) consists of a pair of Lefschetz critical points, see [12] for a detailed explanation. Each Lefschetz critical point can be deformed into a component consisting of three cusps and indefinite folds as in Figure 19. For an explicit description of the move in Figure 19 we refer the reader to the third section of [10].

We then obtain an embedding of three indefinite fold components after thrice merging pairs of the recently created cusps, see Figure 20. Now, the singular set of the image simple fold map \( f_{[1]} \) has an odd number of components and thus, by Theorem 1.1 the singular set of any image simple fold map \( h \in [1] \) must also have an odd number of connected components.

Up to homotopy, we have examined the singular set of all image simple fold maps from the 4-sphere to the 2-sphere. In all cases, the singular set has an odd number of connected components. \( \square \)

Remark 12.5. We note that through steps described in [16], every image simple fold map is homotopic to an image simple indefinite fold map.

Combining the statement of Remark 11.2 with Corollary 12.4, we obtain the following corollary.

**Corollary 12.6.** For every smooth stable map \( f : S^4 \to S^2 \), we have

\[
I(f) \equiv 2 \pmod{4}.
\]

We may also combine Corollary 11.4 and Corollary 12.6 to get the following result.

**Corollary 12.7.** For every simple stable map \( f : S^4 \to S^2 \), we have

\[
\Delta(f) \equiv \#|\Sigma(f)| + 1 \pmod{2}.
\]

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