SYMMETRIC CUBICAL SETS

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Abstract. We introduce a new cubical model for homotopy types. More precisely, we’ll define a category $Q\Sigma$ with the following features: $Q\Sigma$ is a PROP containing the classical box category as a subcategory; the category $q\Sigma Set$ of presheaves of sets on $Q\Sigma$ models the homotopy category; and combinatorial symmetric monoidal model categories with cofibrant unit have homotopically well behaved $q\Sigma Set$ enrichments.

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1. Introduction

Classically we have two models for the homotopy category: one can start with the category \( \mathbf{Top} \) of (compactly generated weak Hausdorff) topological spaces, associate a \( \text{cw} \) approximation \( \gamma X \to X \) to each space \( X \), and take \( \text{Ho} \mathbf{Top}(X, Y) \) to be the homotopy classes of maps between \( \gamma X \) and \( \gamma Y \). The Whitehead theorem implies that weak equivalences between \( \text{cw} \) complexes are homotopy equivalences, so \( \text{Ho} \mathbf{Top} \) is the localization of \( \mathbf{Top} \) at the category of weak homotopy equivalences. Alternatively, one can use the category \( \mathbf{sSet} \) of simplicial sets, Kan approximations, and \( \Delta[1] \)-homotopy—with the proviso that Kan approximations are “on the right”—to construct a homotopy category of simplicial sets. The geometric realization-singular set adjunction

\[
|−| : \mathbf{sSet} \longrightarrow \mathbf{Top} : \text{Sing}
\]

is a Quillen equivalence: it descends to an equivalence of homotopy categories (and preserves the homotopy types of mapping spaces). Any homotopy-theoretic result true in \( \mathbf{Top} \) is thus true in \( \mathbf{sSet} \), and vice versa, so one can view \( \mathbf{Top} \) and \( \mathbf{sSet} \) as two presentations of the same \((\infty, 1)\)-category, using whichever is more convenient for the application at hand.

One advantage of simplicial sets is that the category \( \mathbf{sSet} \) is a presheaf topos, unlike \( \mathbf{Top} \) (the obvious disadvantage is that almost no space comes “in nature” as a simplicial set, and many geometric constructions rely on \( \mathbf{Top} \)). In fact, the category \( \Delta \) of finite nonempty totally ordered sets is not the only site upon which we may model the homotopy category. For example, the cubical category \( \mathbf{2} \)—the category of posets \( \{0 < 1\}^n \), \( n \geq 0 \) with maps those maps given by deleting coordinates or inserting 0s and 1s—also models spaces via the associated category \( \mathbf{qSet} \) of presheaves on \( \mathbf{2} \). This result, in the language of model categories, is relatively recent. Denis-Charles Cisinski and Georges Maltsiniotis, building on conjectures of Grothendieck, have given a unified perspective of categorical homotopy theory and presheaf models for the homotopy category in [11, 35, 25] (see [30] as well)—one side benefit is a straightforward demonstration that \( \mathbf{qSet} \) is a model for the homotopy category.

One advantage of the cubical category \( \mathbf{2} \) is that the product of two cubes is again a cube: in \( \mathbf{sSet} \), the product of two representable functors (i.e., two simplices) is not itself representable. This considerably simplifies the project of finding a spatial enrichment in an arbitrary homotopical category: the “\( n \)-cubes” of a cubical mapping space are simply the \( n \)-fold homotopies. Of course, cubical sets come with their own disadvantages: without adding extra degeneracies, the analogous Dold-Kan correspondence fails [9]; and the convolution monoidal structure is not symmetric. In order to remedy these there is a menagerie of cubical categories containing \( \mathbf{2} \) as a subcategory [24]. In this paper, we’ll add one more category \( \mathbf{2}_\Sigma \) to the zoo, with some useful features:

1. \( \mathbf{2}_\Sigma \) is symmetric monoidal—in fact, it is a PROP in \( \mathbf{Set} \)—and hence the category \( \mathbf{q}_\Sigma \mathbf{Set} = \mathbf{Set}^{2_\Sigma^{op}} \) is symmetric monoidal.

2. There are left Quillen equivalences \( i : \mathbf{qSet} \to \mathbf{q}_\Sigma \mathbf{Set} \) and \( |−|_\Sigma : \mathbf{q}_\Sigma \mathbf{Set} \to \mathbf{Top} \). These are strong monoidal and strong symmetric monoidal, respectively (Theorem 5.18).

3. Any combinatorial symmetric monoidal model category with cofibrant unit may be enriched over \( \mathbf{q}_\Sigma \mathbf{Set} \). (Theorem 10.1)
In a future paper, we’ll show how (3) may be leveraged to show that combinatorial monoidal model categories may be realized as localizations of categories of presheaves of spaces with a convolution model structure, giving a special case of a theorem of Daniel Dugger’s in [14].

The plan of this paper is as follows. In part 1, we’ll describe the category \( q\Sigma \) as a prop and show how to lift the model structure on \( q\text{Set} \) to \( q\Sigma\text{Set} \) via a left Kan extension \( i : q\text{Set} \to q\Sigma\text{Set} \). Along the way, we’ll describe how to decompose presheaves in \( q\Sigma\text{Set} \) as colimits of their skeleta; this requires some careful combinatorial work, but is straightforward. This relies in large part on the methods of Cisinski [11]. In part 2, we’ll discuss \( q\Sigma\text{Set} \) enrichments and some miscellaneous results: we’ll show that the cubical mapping spaces associated with \( q\Sigma\text{Set} \) enrichments have the correct homotopy type, and we’ll use an argument of Schwede and Shipley to show that every combinatorial symmetric monoidal model category with cofibrant unit has a \( q\Sigma\text{Set} \) enrichment.

### 1.1. Acknowledgments

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Part 1. Modeling spaces

2. Day convolution

Before we introduce cubical sets, we briefly review some enriched category theory and introduce some notation. Suppose \( \mathcal{V} \) is a closed symmetric monoidal category with all small limits and colimits and \( \mathcal{I} \) a small \( \mathcal{V} \)-category [32]. Write \( [-] : \mathcal{I} \to \hat{\mathcal{I}} \) for the enriched Yoneda embedding, \( \hat{\mathcal{I}} = \mathcal{V}^\mathcal{I}_{\text{op}} \). Recall that \( [-] \) displays \( \hat{\mathcal{I}} \) as the “free cocompletion” of \( \mathcal{I} \): the category of indexed colimit-preserving functors out of \( \hat{\mathcal{I}} \) is equivalent to the category of functors out of \( \mathcal{I} \) [32, 33]. Now suppose \( (\mathcal{I}, \otimes, e) \) is monoidal. By the universal property of the Yoneda embedding we just mentioned, there is a monoidal structure on \( \hat{\mathcal{I}} \), unique up to unique isomorphism, with the following properties:

1. \( [i] \otimes [j] = [i \otimes j] \) for \( i, j \in \mathcal{I} \).
2. \(- \otimes -\) is cocontinuous (i.e., preserves all indexed colimits) in each variable.

The canonical presentation of a presheaf as a colimit of representables gives the coend formula

\[
(X \otimes Y)_k \cong \int^{i, j \in \mathcal{I}} \mathcal{I}(k, i \otimes j) \otimes (X_i \otimes Y_j).
\]

The unit is the representable presheaf \([e]\). For fixed \( X \in \hat{\mathcal{I}} \), the functors \( X \otimes -\) and \(- \otimes X\) both have right adjoints. If \( \mathcal{I} \) is symmetric, then the product on \( \hat{\mathcal{I}} \) is closed symmetric monoidal. The hom functor \( [-, -] \) is given by the end

\[
[X, Y]_i = \int_{\ell : I} \mathcal{V}(X_\ell, Y_{i \otimes \ell}).
\]
The product $\otimes$ on $\mathcal{J}$ is known as Day convolution; it was introduced in Day’s thesis [13]. Im and Kelly prove the following result in [28]; it is an application of the Yoneda lemma.

**Proposition 2.1.** Suppose $(\mathcal{C}, \otimes, e)$ is a monoidal $\mathcal{V}$-category with small indexed colimits so that $\dashv \otimes \dashv$ preserves indexed colimits in each variable. Given a strong monoidal functor $F : \mathcal{I} \to \mathcal{C}$, the unique cocontinuous extension $\hat{F} : \mathcal{J} \to \mathcal{C}$ is strong monoidal. If $\mathcal{C}$ and $\mathcal{I}$ are symmetric monoidal categories and $F$ is symmetric strong monoidal, then $\hat{F}$ is symmetric strong monoidal as well.

3. The Symmetric Cubical Site

Historically, there are several cubical categories, each generated by a selection of face, degeneracy and possibly symmetry maps. Grandis and Mauri give a zoology of cubical sites in [24]; our site $\mathcal{D} \Sigma$, defined below, is a novel addition. The cubical category with the fewest maps has as objects the posets $\{0 < 1\}^n$, $n \geq 0$; a map $\{0 < 1\}^n \to \{0 < 1\}^m$ may erase coordinates (degeneracies) and insert 0 or 1, but may not repeat coordinates or change their order. This is the classical “box category”; we denote it by $\mathcal{D}$ and write $q\text{Set}$ for the associated category of presheaves of sets on $\mathcal{D}$. We write $\square^n$ for the representable presheaf $\mathcal{D}(\_, [n])$. The category $\mathcal{D}$ has a monoidal structure given by concatenation. Viewed as a PRO [6], its algebras in a monoidal category $(\mathcal{C}, \otimes, e)$ are diagrams

$$e \otimes e \xrightarrow{id \otimes id} I \xrightarrow{id} e.$$  

(3.1)

This may be found in [11]. We’ll call diagrams of the shape (3.1) intervals.

Brown and Higgins introduced in [8, 7] a cubical site with an extra degeneracy called a “connection”; the connection maps are generated by the logical conjunction

$$\land : \{0 < 1\}^2 \to \{0 < 1\}$$

with $x \land y = 1$ if and only if $x = y = 1$. Since the term “connection” is widely established in differential geometry, we’ll call these maps conjunction maps instead. Imposing the structure of a conjunction map on an interval motivates the following definition:

**Definition 3.1.** Suppose $(\mathcal{C}, \otimes, e)$ is a monoidal category. A cubical monoid in $\mathcal{C}$ is a diagram

$$e \otimes e \xrightarrow{d_0 \otimes id} I \xrightarrow{id} e.$$  

(3.3)

together with a map $\mu : X \otimes X \to X$ so that

1. The map $\mu$ makes $X$ an associative monoid with unit $d_1$.
2. The map $s$ is a monoid map.
3. The map $d_0 : e \to X$ is absorbing, i.e., the diagram

$$X \otimes e \xrightarrow{id_X \otimes d_0} X \otimes X \xrightarrow{d_0 \otimes id_X} e \otimes X$$

$$s \otimes id_e \xrightarrow{d_0} e \xrightarrow{d_0} X \xrightarrow{d_0} e$$

(3.4)

commutes.
We’ll sometimes abuse notation and simply say that $I$ is a cubical monoid. A map of cubical monoids $I \to J$ is a map in $\mathcal{C}$ commuting with all the structure data. We write $\text{qMon}(\mathcal{C})$ for the category of cubical monoids in $\mathcal{C}$.

We have the following examples:

**Example 3.2.** Suppose $(\mathcal{C}, \otimes, e)$ is a monoidal category.

1. The unit $e$ is a cubical monoid with $d_0 = d_1 = s = \text{id}_e$ and $\mu : e \otimes e \to e$ given by the coherence isomorphisms of $\mathcal{C}$. This is the terminal cubical monoid in $\mathcal{C}$.
2. The coproduct $e I e$ is a cubical monoid with $d_0$ and $d_1$ given by the inclusion of each summand. The multiplication $\mu$ and degeneracy $s$ are forced. This is the initial cubical monoid in $\mathcal{C}$.
3. The 1-simplex $\Delta[1] \in s\text{Set}$ is a cubical monoid via the conjunction map (3.2).
4. If $F : \mathcal{C} \to \mathcal{D}$ is lax symmetric monoidal and $I$ is a cubical monoid, then $FI$ is a cubical monoid; so, for example, the normalized chains on $\Delta[1]$ are a cubical monoid in chain complexes.

There is an alternative description of cubical monoids pointed out to the author by Reid Barton. Note that the category $([1], \wedge, 1)$ (here $[1] = \{0 < 1\}$) has the structure of a monoidal category. Suppose $(\mathcal{C}, \otimes, e)$ is a monoidal category in which $\mathcal{C}$ has all small colimits and $- \otimes -$ preserves colimits in each variable. We may then equip the category $\mathcal{C}^{[1]}$ of arrows in $\mathcal{C}$ with the Day convolution model structure. If $f : A \to B$ and $g : X \to Y$ are arrows in $\mathcal{C}$, their product $f \circ g$ is the usual pushout-product

$$f \circ g : A \otimes Y \amalg_{A \otimes X} B \otimes X \to B \otimes Y.$$ 

The unit is the unique map $\emptyset \to e$. Note that $\emptyset \to e$ and $\text{id}_e : e \to e$ are both monoids in $\mathcal{C}^{[1]}$.

**Proposition 3.3.** The category $\text{qMon}(\mathcal{C})$ of cubical monoids in $\mathcal{C}$ is equivalent to the category of monoids of the form $d_0 : e \to X$ intervening in a diagram

$$\begin{array}{ccc}
\emptyset & \xrightarrow{e} & e \\
\downarrow & & \downarrow \\
e & \xrightarrow{id_e} & e \\
\downarrow & & \downarrow \\
d_0 & \xrightarrow{id_e} & d_0 \\
\downarrow & & \downarrow \\
e & \xrightarrow{d_1} & X & \xrightarrow{s} & e \\
\downarrow & & \downarrow \\
 & \xrightarrow{d_0} & X & \xrightarrow{id_e} & e
\end{array}$$

of monoids in $\mathcal{C}^{[1]}$.

Note that the condition that $d_0 : e \to X$ be absorbing, in the language of the product $\circ$, becomes the commutativity of the diagram

$$X \otimes e \amalg_{e \otimes e} e \otimes X \xrightarrow{d_0 \circ d_0} X \otimes X$$

It is forced by requiring $d_0 : e \to X$ to be a monoid in $\mathcal{C}^{[1]}$. 


Definition 3.4. The category $\mathcal{S}_\Sigma$ is the PROP whose category of algebras in a symmetric monoidal category $(\mathcal{C}, \otimes, e)$ is the category $\text{qMon}(\mathcal{C})$.

Since each cubical monoid yields an interval by forgetting structure, we have a strict monoidal functor $i : \mathcal{S} \to \mathcal{S}_\Sigma$. This definition of $\mathcal{S}_\Sigma$ is fairly opaque; below, we will give fairly explicit description of its maps. Note that cubical monoids are not abelian. The PRO whose algebras are cubical monoids in an arbitrary monoidal category is straightforward to describe: it is the cubical site obtained from $\mathcal{S}$ by adjoining conjunction maps and the appropriate relations (see [24] or below). The construction of $\mathcal{S}_\Sigma$ is analogous to the symmetrization of a non-$\Sigma$ operad [36]. Note however that $\mathcal{S}_\Sigma$ is not freely generated by an operad, since it includes a $1$–$0$ operation corresponding to the degeneracy $s : X \to e$. This makes symmetrization more complicated: as we’ll see below, permutations can be moved past the map $s$, but not past connections.

3.1. The category $\mathcal{S}_\Sigma$.

Definition 3.5. Suppose $S$ is a set of symbols not containing $0$ or $1$. A formal cubical product on $S$ is either

(1) an ordered conjunction of elements of $S$, none occurring more than once (i.e., a list of symbols in $S$ separated by $\wedge$); or

(2) the numeral $0$ or $1$.

A formal cubical $(m,n)$-product is an $n$-tuple of formal cubical products on $\{x_1, \ldots, x_m\}$ so that no symbol $x_i$ occurs more than once in its concatenation. Write $\mathcal{S}_\Sigma([m], [n])$ for the set of all formal cubical $(m,n)$ products. By convention, $\mathcal{S}_\Sigma([m], [0])$ is a single point.

For example, the following are formal cubical $(3,2)$-products:

$$(x_1, 0) \quad (1, x_3 \wedge x_2) \quad (1, 1) \quad (x_1 \wedge x_3, x_2).$$

However, $(x_1, x_1)$ is not a formal cubical product as the symbol $x_1$ occurs more than once.

Definition 3.6. The identity formal $(n,n)$-product is the $n$-tuple $(x_1, \ldots, x_n)$. Suppose $X$ and $Y$ are formal cubical $(\ell,m)$- and $(m,n)$-products, respectively. The composition $Y \circ X$ is defined as follows:

(1) Replace any occurrence of the symbol $x_i$ in $Y$ with the $i$th entry of $X$.

(2) Delete each occurrence of the symbol $1$ in each conjunction of length at least two.

(3) Replace each conjunction containing $0$ with the numeral $0$.

For example, we have the following compositions:

$$(x_3, x_1 \wedge x_2) \circ (0, x_1, x_5) = (x_5, 0) \quad (x_2 \wedge x_1) \circ (x_1 \wedge x_2, x_3) = (x_3 \wedge x_1 \wedge x_2) \quad (x_1 \wedge x_2) \circ (1, 1) = (1) \quad (0, x_1 \wedge x_4) \circ (0, 1, x_3) = (0, x_{10}).$$

This makes $\mathcal{S}_\Sigma$ a category with objects $[n]$, $n \geq 0$ and maps $[m] \to [n]$ formal cubical $(m,n)$-products. We call $\mathcal{S}_\Sigma$ the extended cubical category and presheaves on $\mathcal{S}_\Sigma$ extended cubical sets; we notate the category of extended cubical sets as $q\Sigma\text{Set}$. We write $\mathcal{C}^\Sigma$ for the representable presheaf $\mathcal{S}_\Sigma(-, [n])$. To complete the description of $\mathcal{S}_\Sigma$ as a PROP we need its symmetric strict monoidal structure:
**Definition 3.7.** Define \([m] \oplus [n] = [m+n]\). Suppose \(X_i\) is a formal cubical \((m_i, n_i)\)-product, \(i = 1, 2\). Define \(X_1 \oplus X_2\) as follows:

1. Replace each \(x_j\) in \(X_2\) by \(x_{j+m_1}\) to form \(X'_2\).
2. Concatenate \(X_1\) and \(X'_2\).

The symmetry \([m] \oplus [n] \rightarrow [n] \oplus [m]\) is the formal product

\[(x_{m+1}, x_{m+2}, \ldots, x_{m+n}, x_1, x_2, \ldots, x_m)\]

For example,

\[(x_1 \land x_2) : [2] \oplus ((0, x_1) : [1] \rightarrow [2]) = (x_1 \land x_2, 0, x_3) : [3] \rightarrow [3].\]

3.2. **Generators and relations in** \(\mathcal{Q}\). In order to describe skeletal filtrations on extended cubical sets, we need a presentation of \(\mathcal{Q}\). The relations we list are similar to those in \([24]\).

**Definition 3.8.** Suppose \(n > 0\), \(1 \leq i \leq n + 1\), and \(\varepsilon = 0, 1\). Define maps \(\delta^i_n\) and \(\sigma^i_n\) by the formal products

\[
\delta^i_n = (x_1, \ldots, x_{i-1}, \varepsilon, x_i, \ldots, x_n) : [n] \rightarrow [n+1]
\]

\[
\sigma^i_n = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}) : [n+1] \rightarrow [n].
\]

For \(n \geq 1\) and \(i \leq n\), we define \(\gamma^i_n\) to be

\[
\gamma^i_n = (x_1, \ldots, x_{i-1}, x_i \land x_{i+1}, x_{i+2}, \ldots, x_{n+1}) : [n+1] \rightarrow [n].
\]

Finally, for \(p \in \Sigma_n\), we let

\[
\pi_p = (x_{p^{-1}(1)}, \ldots, x_{p^{-1}(n)}) : [n] \rightarrow [n].
\]

Note that \(\mathcal{Q}\) is isomorphic to the subcategory of \(\mathcal{Q}_\Sigma\) generated by the coface and codegeneracy maps \(\delta^i_n\) and \(\sigma^i_n\). The inclusion \(\mathcal{Q} \rightarrow \mathcal{Q}_\Sigma\) is the strict monoidal functor we described above in terms of forgetting structure.

**Proposition 3.9.** The codegeneracy and coface maps satisfy the following relations:

\[
\delta^{i-1} \delta^i = \delta^i \delta^{j-1} \quad \text{if } i < j
\]

(3.8)

\[
\sigma^j \delta^i = \begin{cases} 
\delta^i \sigma^j & \text{if } i < j \\
\id & \text{if } i = j \\
\delta^{i-1} \varepsilon \sigma^j & \text{if } i > j 
\end{cases}
\]

\[
\sigma^j \sigma^i = \sigma^i \sigma^{j+1} \quad \text{if } i \leq j
\]

The conjunction maps satisfy the following relations \([24]\):

\[
\gamma^j \gamma^i = \begin{cases} 
\gamma^j \gamma^{i+1} & \text{if } j > i \\
\gamma^j \gamma^i & \text{if } j = i
\end{cases}
\]

\[
\sigma^j \gamma^i = \begin{cases} 
\gamma^{i-1} \sigma^j & \text{if } j < i \\
\sigma^i \gamma^i & \text{if } j = i \\
\gamma^i \sigma^{j+1} & \text{if } j > i
\end{cases}
\]

\[
\gamma^j \delta^i = \begin{cases} 
\delta^{i-1} \varepsilon \gamma^j & \text{if } j < i - 1 \\
\delta^i \sigma^j & \text{if } j = i - 1, i \text{ and } \varepsilon = 0 \\
\id & \text{if } j = i - 1, i \text{ and } \varepsilon = 1 \\
\delta^i \varepsilon \gamma^{j-1} & \text{if } j > i
\end{cases}
\]
Definition 3.10. Let $\mathcal{Q}_\Sigma^+$ be the subcategory of $\mathcal{Q}_\Sigma$ generated by coface maps $\delta_i^\varepsilon$ and cosymmetry maps $\pi_p$; let $\mathcal{Q}_\Sigma^-$ be the subcategory of $\mathcal{Q}_\Sigma$ generated by codegeneracy maps $\sigma_i^\ell$, conjunctions $\gamma_i^n$, and cosymmetry maps $\pi_p$.

Proposition 3.11. Every map $f$ in $\mathcal{Q}_\Sigma$ admits a unique factorization of the form

$$f = \delta_{i_1}^{i_1} \cdots \delta_{i_n}^{i_n} \gamma_{k_1}^{k_1} \cdots \gamma_{k_r}^{k_r} \sigma_{j_1}^{j_1} \cdots \sigma_{j_m}^{j_m}$$

with

$$i_1 > i_2 > \cdots > i_n \quad j_1 < j_2 < \cdots < j_m \quad k_1 < k_2 < \cdots < k_r$$

and $p \in \Sigma_\ell$. If $f \in \ar \mathcal{Q}_\Sigma^+$, then $r = m = 0$; if $f \in \ar \mathcal{Q}_\Sigma^-$, then $n = 0$. If $f \in \ar \mathcal{Q}$, then $r = 0$ and $\pi_p = \text{id}$.

We can read the decomposition (3.10) off of a formal cubical $(a, b)$-product as follows: the indices $j_1, \ldots, j_m$ correspond to the symbols $x_{j_1}, \ldots, x_{j_m}$ omitted from the formal cubical product. The concatenation of the remaining indices determines $\pi_p$ uniquely; the list $k_1, \ldots, k_r$ corresponds to the positions in which a concatenation is performed, and the list $i_1, \ldots, i_n$ corresponds to the positions containing a 0 or 1. For example, the $(5, 4)$-product

$$(x_3, 1, x_1 \land x_5 \land x_2, 0)$$

decomposes uniquely as

$$\delta_5^0 \delta_2^1 \gamma_3^2 \pi_{(1243)} \sigma^4.$$ 

This is analogous to the decompositions given by Grandis and Mauri [24, Theorem 8.3]. However, Grandis and Mauri’s extended cubical category $\mathbb{K}$ has an additional degeneracy operation given by disjunction $\lor$ and some additional relations. More importantly, in $\mathbb{K}$, the operations $\land$ and $\lor$ are commutative. As a result, the permutation $p$ in factorizations of the form (3.10) in $\mathbb{K}$ is uniquely determined up to multiplication of a possibly nontrivial subgroup of $\Sigma_\ell$. This has the upshot that the vertices functor $\mathbb{K}([\{0\}], \cdot) : \mathbb{K} \to \text{Set}$ is faithful. However, the analogous vertices functor $\mathcal{Q}_\Sigma([\{0\}], \cdot) : \mathcal{Q}_\Sigma \to \text{Set}$ is not faithful. This is a marked departure from most cubical sites.

Corollary 3.12. Suppose $f : [m] \to [n]$ is a map in $\mathcal{Q}_\Sigma$. Then $f$ admits a factorization $f = \delta \sigma$ with $\delta \in \mathcal{Q}_\Sigma^+$ and $\sigma \in \mathcal{Q}_\Sigma^-$. Given any other factorization $f = \delta' \sigma'$, there is a unique map $\pi$ such that

$$\sigma \quad \delta \quad \pi$$

commutes; in fact $\pi$ is always a cosymmetry isomorphism.
In this section, we’ll present some machinery that allows us to decompose cubical sets and symmetric cubical sets as colimits of their skeleta. We’ll first need a workable definition of skeleton. There is a general theory due to Cisinski of skeletal decompositions generalizing the classical theory for simplicial sets in [20, 22]—see [11, Chapitre 8]. Moerdijk and Berger also discuss an apparatus for skeletal decomposition in [5]—the theory of Eilenberg-Zilber categories—which we’ll apply to \( \mathcal{Q} \) and \( \mathcal{Q}_\Sigma \).

4.1. Eilenberg-Zilber categories and decompositions. Suppose \( X \) is a simplicial set. Given any \( n \)-simplex \( f : \Delta[n] \to X \), we may take a factorization

\[
\begin{array}{ccc}
\Delta[n] & \xrightarrow{f} & X \\
\downarrow{s} & & \downarrow{g} \\
\Delta[r] & \xleftarrow{s} & \Delta[r]
\end{array}
\]

so that \( s : [n] \to [r] \) is an epimorphism (i.e., a degeneracy map) and \( r \) is minimal among all such factorizations. Of course, the minimality of \( r \) implies that simplex \( g \) is nondegenerate—it does not factor through another degeneracy. One feature of the combinatorics of \( \Delta \) is that this factorization is unique: if \( f = g's' \) is another factorization with \( g' \) nondegenerate and \( s' \) a degeneracy map, then \( g' = g \) and \( s' = s \). This seemingly innocuous observation allows us to identify the \( m \)-skeleton of \( X \) (usually given as the counit of the left Kan extension/restriction adjunction along \( \Delta \leq m \to \Delta \)) as the subsheaf of \( X \) whose \( n \)-simplices are those \( n \)-simplices \( f \) so that \( r \leq m \) in the Eilenberg-Zilber decomposition (4.1) of \( f \). A simple induction argument then shows that the maps \( \partial \Delta[n] \to \Delta[n] \), \( n \geq 0 \) comprise a cellular model for \( s \text{Set} \).

These sort of arguments also work in \( q \text{Set} \) (as we’ll see below), but not in \( q \mathcal{S} \text{et} \) without some modification. The identity map \( \square^n \to \square^n \) is nondegenerate, in the sense that it does not factor through any non-invertible degeneracies, but any symmetry \( \pi \) yields a factorization \( \pi^{-1} \pi \). Also, the maps \( \iota_n : \partial \square^n \to \square^n \) do not comprise a cellular model for \( q \mathcal{S} \text{et} \): there is no way to form, e.g., \( \Sigma_2 \setminus \square^n \) with iterated cobase changes, transfinite compositions, and retracts of the maps \( \iota_n \). As it turns out, these are the only two complications that arise when we try to apply Gabriel and Zisman’s theory to \( \mathcal{D}_C \). We need to replace uniqueness with a properly categorical notion—contractible groupoids—and we need to keep track of the action of \( \text{Aut}(\square^n) \) on \( \square^n \). The appropriate generalization of \( \Delta \) is the notion of an \( Eilenberg-Zilber \) category, which we introduce below. This generalization is due to Berger, Moerdijk and Cisinski [5, 11].

**Definition 4.1.** Suppose \( \mathcal{C} \) is a category and \( \mathcal{I} \) a small category. Suppose further that \( X : \mathcal{I} \to \mathcal{C} \) is a diagram and \( X \to c_\mathcal{Y} \) is a cocone on \( X \) (here \( c_\mathcal{Y} \) is the constant \( \mathcal{I} \)-diagram on \( Y \)). We say \( X \to c_\mathcal{Y} \) is an absolute colimit if \( F X \to c_{F \mathcal{Y}} \) is a colimit for all functors \( F : \mathcal{C} \to \mathcal{D} \).

Split coequalizers are examples of absolute colimits [33]. In Definition 4.1, it is necessary and sufficient to check that \( [X] \to c_{[Y]} \) is a colimit; this is due to Paré [37].
Definition 4.2 ([5 Definition 6.6]). An Eilenberg-Zilber category (briefly EZ category) is a small category $\mathcal{R}$ together with a degree function $\text{deg} : \text{ob} \mathcal{R} \rightarrow \mathbb{Z}_{\geq 0}$ such that

(EZ1) Monomorphisms preserve the degree if and only if they are invertible; they raise the degree if and only if they are non-invertible.

(EZ2) Every morphism factors as a split epimorphism followed by a monomorphism.

(EZ3) Suppose

$$s_1 \overset{\sigma_1}{\longrightarrow} r \overset{\sigma_2}{\longrightarrow} s_2$$

is a pair of split epimorphisms. Then there is an absolute pushout square

$$\sigma_1$$

$$\sigma_1$$

$$\sigma_2$$

in $\mathcal{R}$ in which $\tau_1$ and $\tau_2$ are split epimorphisms.

Suppose $\mathcal{R}$ is an EZ category whose only isomorphisms are identity maps. The factorization provided by EZ2 is then unique by EZ3. Moreover, since the section of a split epimorphism is monic, non-identity split epimorphisms lower degree. In this special case, $\mathcal{R}$ is an example of a Reedy category:

Definition 4.3. Suppose $\mathcal{C}$ is a category and $\mathcal{D}$ a subcategory of $\mathcal{C}$; we say $\mathcal{D}$ is lluf if $\text{ob} \mathcal{D} = \text{ob} \mathcal{C}$ (this terminology is due to Peter Freyd). A Reedy category [16, 26, 27] is a small category $\mathcal{R}$ together with a degree function $\text{deg} : \text{ob} \mathcal{R} \rightarrow \mathbb{Z}_{\geq 0}$ and two lluf subcategories $\mathcal{R}^+$ and $\mathcal{R}^-$ so that

(R1) Non-identity morphisms in $\mathcal{R}^+$ raise the degree; non-identity morphisms in $\mathcal{R}^-$ lower the degree.

(R2) Every morphism $f \in \text{ar} \mathcal{R}$ factors uniquely as $f = gh$ with $g \in \text{ar} \mathcal{R}^+$ and $h \in \text{ar} \mathcal{R}^-$.  

Not all Reedy categories are EZ though. As expected, $\Delta$ is both. The main result of this section is the following:

Proposition 4.4. The categories $\mathcal{Q}$ and $\mathcal{Q}_\Sigma$ are EZ categories.

The proof of this, especially the verification of EZ3, is rather technical and we postpone it to the end of the section. Before we get to it, we’ll continue with a discussion of the properties of EZ categories.

4.2. Skeleta, coskeleta, and cellular models.

Definition 4.5. Let $\mathcal{R}$ be an EZ category and suppose $X \in \mathcal{R}$. We say a section $x \in X_r$ is degenerate if there is a map $\sigma : s \rightarrow r$ in $\mathcal{R}^-$ and $y \in X_s$ so that $\sigma^*y = x$ and $\text{deg} s < \text{deg} r$.

Proposition 4.6 ([5 Proposition 6.7]). Let $\mathcal{R}$ be an EZ category.
(1) Suppose \( X \in \mathring{\mathcal{R}} \). Let \( x \in X_r, r \in \text{ob} \mathcal{R} \). The category of factorizations

\[
\begin{array}{c}
|r| \\
\downarrow \sigma \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
x \\
\downarrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \\
[y] \\
\end{array}
\]

with \( \sigma \in \mathcal{R}^- \) and \( y \) nondegenerate is a contractible groupoid.

(2) If \( f : r \to s \) is an arrow in \( \mathcal{R} \), the category of factorizations

\[
\begin{array}{c}
r \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
f \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
t \\
\end{array}
\]

with \( f^- \) a split epimorphism and \( f^+ \) a monomorphism is a contractible groupoid.

Following [5, 22], we call any such factorization an ez decomposition of \( x \). The Proposition implies in particular that ez decompositions exist.

**Definition 4.7.** Suppose \( \mathcal{R} \) is an ez category and \( n \geq -1 \). Write \( \mathcal{R}_{\leq n} \) for the full subcategory of \( \mathcal{R} \) with objects those of degree at most \( n \). The inclusion \( j_n : \mathcal{R}_{\leq n} \to \mathcal{R} \) yields adjunctions

\[
\mathcal{R}_{\leq n} \xrightarrow{(j_n)_!} \mathcal{R} \xleftarrow{(j_n)^*} \mathcal{R}_{\leq n}
\]

given by left and right Kan extension. We define the \( n \)-skeleton and \( n \)-coskeleton of \( X \in \mathcal{R} \) to be

\[
\text{sk}_n X = (j_n)_!(j_n)^* X \quad \text{and} \quad \text{ck}_n X = (j_n)_*(j_n)^* X
\]

respectively. The counit and unit of the adjunctions in (4.2) yield natural maps

\[
\text{sk}_n X \longrightarrow X \longrightarrow \text{ck}_n X.
\]

We say \( X \) is \( n \)-skeletal if \( \text{sk}_n X \to X \) is an isomorphism and \( n \)-coskeletal if \( X \to \text{ck}_n X \) is an isomorphism.

In a precise sense, the \( n \)-skeleton of \( X \in \mathcal{R} \), \( \mathcal{R} \) an EZ category, is the subsheaf generated by the non-degenerate sections of \( X \) of degree at most \( n \).

**Proposition 4.8 ([5]).** Suppose \( \mathcal{R} \) is an EZ category and \( X \in \mathcal{R} \). The map \( \text{sk}_n X \to X \) is a monomorphism; its image in \( X_r, r \in \text{ob} \mathcal{R} \) is the set of sections

\[
\{ f \in X_r \mid f \text{ has a factorization } [r] \to [s] \to X \text{ with } \deg s \leq n \}.
\]

**Definition 4.9.** Suppose \( \mathcal{R} \) is an ez category and \( r \in \text{ob} \mathcal{R} \). We define the boundary \( \partial [r] \) of \( [r] \) to be the \((n-1)\)-skeleton of \( [r] \), where \( \deg r = n \).

**Proposition 4.10.** Suppose \( \mathcal{R} \) is an EZ category whose only isomorphisms are identity maps. Suppose \( X \in \mathcal{R} \) and \( n \geq 0 \). Let \( S \) be the set of maps \( f : [r] \to X \)
with $\deg r = n$ and $f$ nondegenerate. The square

$$
\begin{array}{ccc}
\coprod_{f:[r] \to X \in S} \partial [r] & \longrightarrow & \text{sk}_{n-1} X \\
\downarrow & & \downarrow \\
\coprod_{f:[r] \to X \in S[r]} & \longrightarrow & \text{sk}_{n} X
\end{array}
$$

(4.3)

is a pushout.

Proof. This proof is a straightforward generalization of [22, §II.3.8]. Since every object in (4.3) is $n$-skeletal, it is sufficient to check that the restriction of (4.3) to $R \leq n$ is a pushout square. In a presheaf topos, pushouts are computed pointwise, so it is sufficient to prove that the square (4.3) is a pushout after evaluation at $s$ for all $s \in \text{ob} \mathcal{R}_{\leq n}$. If $\deg s < n$ and $\deg r = n$, the maps

$$
(\partial [r])_s \to [r]_s \quad \text{and} \quad (\text{sk}_{n-1} X)_s \to (\text{sk}_{n} X)_s,
$$

are isomorphisms. Thus we are reduced to checking that

$$
\begin{array}{ccc}
\coprod_{f:[r] \to X \in S} (\partial [r])_s & \longrightarrow & (\text{sk}_{n-1} X)_s \\
\downarrow & & \downarrow \\
\coprod_{f:[r] \to X \in S[r]} & \longrightarrow & (\text{sk}_{n} X)_s
\end{array}
$$

(4.4)

is a pushout when $\deg s = n$.

Suppose $\deg s = n$. The complement of $(\text{sk}_{n-1} X)_s$ in $(\text{sk}_{n} X)_s$ is the set of all nondegenerate $s$-simplices $[s] \to X$. Since $\mathcal{R}$ has no nontrivial isomorphisms, if $r \neq s$ has degree $n$, each map $s \to r$ factors through an object of lower degree, so

$$(\partial [r])_s \to [r]_s$$

is an isomorphism. On the other hand, the complement of the image of

$$(\partial [s])_s \to [s]_s$$

is the identity map $s \to s$, so the complement of the image of

$$
\prod_{f:[r] \to X \in S} (\partial [r])_s \to \prod_{f:[r] \to X \in S} [r]_s
$$

is the set of nondegenerate $s$-simplices $[s] \to X$. Hence (4.3) is a pushout. $\square$

We can reinterpret Proposition 4.10 as a statement about saturated classes of maps. We first introduce the following definition, using Cisinski’s terminology [11]:

**Definition 4.11.** Suppose $\mathcal{R}$ is a small category. We say that a set of arrows $S \subseteq \text{ar} \mathcal{R}$ is a cellular model for $\mathcal{R}$ if $\text{Cell} S = \text{mono}$. Here, $\text{Cell} S$ is the closure of the set $S$ under transfinite composition, cobase change, coproduct, and retract.

Any topos has a cellular model [11, 4]. In the case of a presheaf topos, all inclusions of subobjects of (regular) quotients of representables form a cellular model. In EZ categories, we have the expected simplification:

**Corollary 4.12.** Under the assumptions of Proposition 4.10, the arrows $\partial [r] \to [r]$, $r \in \text{ob} \mathcal{R}$, comprise a cellular model for $\mathcal{R}$.
This corollary may seem slightly weaker than Proposition 4.10 because it does not say anything about the dimension of the attaching maps (brining to mind the distinction between cellular and CW complexes in $\textbf{Top}$). In $\hat{\mathcal{R}}$ however, every map $\partial[r] \to X$ automatically factors through $\text{sk}_{\text{deg}r-1}X \to X$.

**Proof of Corollary 4.12.** Let $C$ temporarily denote the class of arrows

$$\text{Cell}(\partial[r] \to |r| \mid r \in \text{ob}\mathcal{R}).$$

Since $\hat{\mathcal{R}}$ is a topos, $C \subseteq \text{mono}$. Recall that $\text{sk}_{-1}A = \text{sk}_{-1}B = \emptyset$. Suppose $f : A \to B$ is a monomorphism in $\hat{\mathcal{R}}$. Let $\text{sk}_n f$ be the pushout $\text{sk}_n B \amalg \text{sk}_n A A$ and let $p_n : \text{sk}_n f \to B$ be the corner map. Note that the square

$$\begin{array}{ccc}
\text{sk}_n A & \to & \text{sk}_n B \\
\downarrow & & \downarrow \\
A & \to & B
\end{array}$$

is a pullback. Since $\hat{\mathcal{R}}$ is a topos, $\text{sk}_n f$ is the effective union of $\text{sk}_n B$ and $A$ in $\text{sk}_n f$ and $p_n$ is a monomorphism. The square

$$\begin{array}{ccc}
\text{sk}_n B & \to & \text{sk}_{n+1} B \\
\downarrow & & \downarrow \\
\text{sk}_n f & \to & \text{sk}_{n+1} f
\end{array}$$

is a pushout, so $\text{sk}_n f \to \text{sk}_{n+1} f$ is in $C$. Now $\text{colim}_n \text{sk}_n f \to B$ is an isomorphism. Since $\text{sk}_{-1}f = A$, we’ve realized $f$ as a transfinite composition of maps in $C$, so $f \in C$. Hence $C = \text{mono}$. □

Note that Proposition 4.10 is false if we allow objects in $\mathcal{R}$ to have nontrivial automorphisms. An easy example is the one-object category associated to a group $G$. This is an EZ category with $\text{deg} \ast = 0$. An object $X \in \hat{G}$ is a right $G$-set; were Proposition 4.10 true, it would imply that all $X \in \hat{G}$ are free as $G$-sets. We’ll now prove a generalization of Proposition 4.10 for categories $\mathcal{R}$ containing nontrivial isomorphisms.

**Definition 4.13.** Suppose $\mathcal{R}$ is an EZ category, $X \in \hat{\mathcal{R}}$, and $f : [r] \to X$ is a nondegenerate $r$-simplex of $X$. Note that $\text{Aut}(r)$ acts on $X_r$ on the right. The *isotropy of $f$, denoted $\text{Stab}(f)$, is the stabilizer of $f \in X_r$ in $\text{Aut}(r)$, i.e., the subgroup of $g \in \text{Aut}(r)$ with $g^r f = f$.

In the following, note that the left action of $\text{Aut}(r)$ on $[r]$ restricts to an action on $\partial[r]$. If $H \leq \text{Aut}(r)$, then $H \setminus (\partial[r]) \to H \setminus [r]$ is a monomorphism (here $H \setminus X$ denotes the $H$-orbits of $X$).

**Proposition 4.14.** Suppose $\mathcal{R}$ is a skeletal EZ category, i.e., two objects are isomorphic if and only if they are equal. Let $n \geq 0$ and let $S$ be a set of isomorphism
classes of \( f : [r] \to X \) with \( \text{deg } r = n \) and \( f \) nondegenerate. Then the square

\[
\begin{array}{ccc}
\prod_{f : [r] \to X \in S} \text{Stab} f \setminus \partial [r] & \longrightarrow & \text{sk}_{n-1} X \\
\downarrow & & \downarrow \\
\prod_{f : [r] \to X \in S} \text{Stab} f \setminus [r] & \longrightarrow & \text{sk}_n X
\end{array}
\]

is a pushout.

**Proof.** Note that since \( \text{sk}_n \) is cocontinuous, if \( Y \in \mathcal{R} \) is \( n \)-skeletal and a group \( G \) acts on \( Y \), \( G \setminus Y \) is \( n \)-skeletal as well. Just as in the proof of Proposition 4.10, it is sufficient to check that

\[
\prod_{f : [r] \to X \in S} \left( \text{Stab} f \setminus \partial [r] \right) \rightarrow \left( \text{sk}_{n-1} X \right)_s
\]

(4.5)

is a pushout when \( \text{deg } s = n \). The complement of \( (\text{sk}_{n-1} X)_s \) in \( (\text{sk}_n X)_s \) is the right \( \text{Aut}(s) \)-set of all nondegenerate \( s \)-simplices \( f : [s] \to X \).

Now we have two possibilities. Suppose \( r \neq s \) has degree \( n \). If \( f : [r] \to X \) is a nondegenerate \( r \)-simplex and \( r \neq s \),

\[
\left( \text{Stab} f \setminus \partial [r] \right)_s \rightarrow \left( \text{Stab} f \setminus [r] \right)_s
\]

is an isomorphism: any map \( s \to r \) must factor through an object of lower degree since a degree-preserving map \( s \to r \) is necessarily an isomorphism (recall that we have assumed \( \mathcal{R} \) is skeltal). On the other hand, if \( H \leq \text{Aut}(s) \), the complement of the image of

\[
(H \setminus \partial [s])_s \rightarrow (H \setminus [s])_s
\]

is the right \( \text{Aut}(s) \)-set \( H \setminus \text{Aut}(s) \). Thus the complement of the image of

\[
\prod_{f : [r] \to X \in S} \left( \text{Stab} f \setminus \partial [r] \right)_s \rightarrow \prod_{f : [r] \to X \in S} \left( \text{Stab} f \setminus [r] \right)_s
\]

is the right \( \text{Aut}(s) \)-set

\[
\prod_{g : [s] \to X \in S} \text{Stab}(g) \setminus \text{Aut}(s)
\]

This decomposition maps isomorphically onto \( (\text{sk}_n X)_s \setminus (\text{sk}_{n-1} X)_s \) via the map sending the coset \( \text{Stab}(g) \) to \( g \). Hence (4.5) is a pushout. □

**Corollary 4.15.** Under the assumptions of Proposition 4.14, the set

\[
\left\{ H \setminus \partial [r] \to H \setminus [r] \mid r \in \text{ob } \mathcal{R} \text{ and } H \leq \text{Aut } r \right\}
\]

is a cellular model for \( \mathcal{R} \).

As we’ll see below, \( \mathcal{Q} \) and \( \mathcal{Q}_\Sigma \) are EZ categories, so we obtain the following cellular models. We write \( \square^n \) and \( \square_\Sigma^n \) for the representables \( \mathcal{Q}(\cdot, [n]) \) and \( \mathcal{Q}_\Sigma(\cdot, [n]) \) respectively.
Proposition 4.16. The sets
\[ I = \{ \partial \square^n \to \square^n \mid n \geq 0 \} \]
\[ I\Sigma = \{ H\setminus \partial \square^n \to H\setminus \square^n \mid n \geq 0 \text{ and } H \leq \Sigma_n \} \]
are cellular models for \( q\text{-Set} \) and \( q\Sigma\text{-Set} \), respectively.

4.3. Comparing skeletal filtrations. In this section we’ll prove a base-change theorem that allows us to compare skeleta of cubical and extended cubical sets. We begin with a slightly modified definition from [11, Chapitre 8]:

Definition 4.17. Suppose \( i : \mathcal{R} \to \mathcal{I} \) is a functor. We say that \( i \) is a thickening if
1. \( i \) is an isomorphism on objects.
2. For all \( r, r' \in \text{ob} \mathcal{R} \), the map
   \[ \text{Aut}_{\mathcal{I}}(ir) \times \mathcal{R}(r, r') \to \mathcal{I}(ir, ir') \]
   is a bijection of sets.

Crossed \( \Delta \)-modules, and more generally crossed \( \mathcal{R} \)-modules for a Reedy category \( \mathcal{R} \), are examples of thickenings [5, 21]. Note that \( Q \to Q\Sigma \) is not a thickening, but \( Q^+ \to Q\Sigma^+ \) is a thickening by Proposition 3.11. We start with a simple observation:

Lemma 4.18. Suppose \( i : \mathcal{R} \to \mathcal{I} \) is a thickening and
\[
\begin{array}{ccc}
\sigma & \xrightarrow{i} & i_1 \\
\downarrow & & \downarrow \\
ir_2 & \xrightarrow{i} & ir_3
\end{array}
\]
is a diagram in which \( \sigma \) is an arrow in \( \mathcal{I} \). Then \( \sigma \) is in the image of \( i \) and the triangle may be lifted to one in \( \mathcal{R} \).

Proof. Since \( i \) is a thickening, there is a (unique) factorization of \( \sigma \) as a composition \((ih) \circ \tau \), where \( \tau \in \text{Aut}_{\mathcal{I}}(ir_1) \) and \( h \) is a map \( r_1 \to r_2 \). Then \( i(gh) \circ \tau = if \), so by the uniqueness of factorizations of this form, \( \tau = \text{id}_{ir_1} \) and hence \( \sigma = ih \). \( \square \)

Proposition 4.19. Suppose \( i : \mathcal{R} \to \mathcal{I} \) is a functor between EZ categories \( \mathcal{R} \) and \( \mathcal{I} \) so that \( i \) preserves degree. Then \( i \) preserves monomorphisms; suppose that moreover, the resulting functor \( i^+ : \mathcal{R}^+ \to \mathcal{I}^+ \) is a thickening. The natural base-change transformation \( ij^* \to i^*i_1 \) of functors \( \mathcal{R} \to \mathcal{I}_{\leq n} \) induced by the square of functors
\[
\begin{array}{ccc}
\mathcal{R}_{\leq n} & \xrightarrow{j} & \mathcal{R} \\
\downarrow & & \downarrow \\
\mathcal{I}_{\leq n} & \xrightarrow{j} & \mathcal{I}
\end{array}
\]
is a natural isomorphism.

Proof. The functors \( i_1 \) and \( j^* \) preserve colimits, so it is sufficient to check that \( ij^* \to i^*i_1 \) is an isomorphism on all representables \( [r] \in \mathcal{R}, r \in \text{ob} \mathcal{R} \). Suppose \( r \in \text{ob} \mathcal{R} \) and \( s \in \text{ob} \mathcal{I}_{\leq n} \). Note that \( j^*[r] \cong \text{colim}_r [r'] \) where the colimit is taken
over all $r'$ with $\deg r' < n$, so $i_* j^*[r] \cong \colim_{r' \leq n} [ir']$. As a result, on the level of sets, the map $\varphi: (i_* j^*[r])_s \to (j^* i_* [r])_s$ is given by

$$\colim_{r' \leq n} \{ s \to ir' \in \ar \} \to \{ s \to ir \in \ar \}$$

Now suppose $g: s \to ir$ is a map in $\mathcal{I}$. Since $\mathcal{I}$ is an EZ category and $i^+$ is a thickening, there is a factorization

$$s \xrightarrow{g^-} ir' \xrightarrow{ig^+} ir$$

in which $g^-$ is a split epimorphism in $\mathcal{I}$ and $g^+$ is a monomorphism in $\mathcal{R}$. Since $\deg s \leq n$, $\deg r' \leq n$ as well. Hence $\varphi$ is a surjection. We can assume, moreover, that the degree of $r'$ is minimal among all such factorizations (in fact, there is only one possible degree).

Now suppose we have maps $h: s \to ir_1$ in $\mathcal{I}$ and $\ell: r_1 \to r$ in $\mathcal{R}$ so that $\deg r_1 \leq n$ and $i\ell \circ h = g$. We must show that the pair $(\ell, h)$ is identified with $(g^+, g^-)$ in the colimit in the source of $\varphi$. By repeated factorizations, we can produce a diagram

$$s \xrightarrow{h^-} ir_3 \xrightarrow{i\ell^-} ir \xrightarrow{i\ell^+} ir_2$$

in which $h^-$ is a split epimorphism in $\mathcal{I}$, $\ell^-$ is a split epimorphism in $\mathcal{R}$, and both $\ell^+$ and $h^+$ are monomorphisms in $\mathcal{R}$. The pairs

$$(\ell, h) \ (\ell^+, h \circ i\ell^-) \ (\ell^+ h^+, h^-)$$

are identified in the colimit in the source of $\varphi$. (Note that $\deg r_3 \leq \deg r_1$.) Without loss of generality, then, we can assume that $h$ is a split epimorphism and $\ell$ is a monomorphism. But split epi-monic factorizations in $\mathcal{I}$ are essentially unique (Proposition 4.10), so there is an isomorphism $\sigma$ making

$$s \xrightarrow{g^-} ir' \xrightarrow{ig^+} ir$$

commute. Since $i^+$ is a thickening, the map $\sigma$ must be in the image of $i$ (Lemma 4.18), so $(\ell, h)$ and $(g^+, g^-)$ are identified in the colimit in the source of $\varphi$. Hence $\varphi$ is a bijection of sets. □

**Corollary 4.20.** Suppose $i: \mathcal{R} \to \mathcal{I}$ is a functor between EZ categories satisfying the assumptions of Proposition 4.19. If $X \in \mathcal{R}$, then there is a natural isomorphism $sk_n i_! X \to i_! sk_n X$. 
Proof. With the notation of Proposition 4.19 there is a natural isomorphism $i_! j^* X \to j_! i^* X$. Now apply the functor $j_!$; we obtain a natural isomorphism $j_! i_! j^* X \to j_! j^* i_! X$. Since $j_! i_! \cong i_! j_!$, we obtain a natural isomorphism $i_! j_! j^* X \to j_! j^* i_! X$. □

4.4. The cubical sites. In the remainder of this section, we’ll prove Proposition 4.4. We begin with the definition $\text{deg}[n] = n$. Axioms EZ1 and EZ2 are routine; the difficult axiom to verify will be EZ3.

Lemma 4.21. All epimorphisms of $Q$ and $Q_\Sigma$ are split. The epimorphisms of $Q$ (respectively $Q_\Sigma$) correspond to the arrows of $Q^{-}$ (respectively $Q^{-}_\Sigma$).

Proof. Both the degeneracies $\sigma^i$ and $\gamma^i$ have sections, so they are categorical epimorphisms. Since the cosymmetry maps $\pi^p$ are isomorphisms, we may conclude that the arrows of $Q^{-}_\Sigma$ and $Q^{-}$ are split epimorphisms in $Q_\Sigma$ and $Q$, respectively.

Suppose $f$ is an arrow in $Q_\Sigma$. We may factor $f$ as $f = \delta_i^{s_1} \cdot \epsilon_1 \cdot \cdot \cdot \delta_i^{s_n} \epsilon_n$ with $s \in \ar Q^{-}_\Sigma$. Suppose $n > 0$. Then $f = \delta_i^{s_1} \sigma^i f$ by the relations in Proposition 3.9. However, $\delta_i^{s_1} \sigma^i \neq \text{id}$, so $f$ is not an epimorphism. Hence the epimorphisms of $Q_\Sigma$ are precisely the maps of $Q^{-}_\Sigma$, which are all split. The proof for $Q$ is identical. □

Definition 4.22. Suppose

\[
\begin{array}{c}
\begin{array}{c}
A \\
\downarrow a_2 \\
C \\
\downarrow p_1 \\
P
\end{array} \quad \begin{array}{c}
\begin{array}{c}
B \\
\downarrow p_2 \\
\end{array}
\end{array}
\end{array}
\]

is a commutative square in a category $\mathcal{C}$. If there exist maps $d_0 : P \to B$, $d_1 : C \to A$, $d'_1 : B \to A$, $d'_2 : B \to A$ so that

\[
\begin{align*}
d_0 p_1 &= a_1 d_1 \\
a_2 d_1 &= \text{id}_C \\
p_2 d_0 &= \text{id}_P \\
\end{align*}
\]

and

\[
\begin{align*}
d_0 p_2 &= a_1 d'_1 \\
a_2 d'_1 &= a_2 d'_{2} \\
a_1 d'_2 &= \text{id}_B \\
\end{align*}
\]

then we call (4.7) a split pushout.

Lemma 4.23. Split pushouts are absolute pushouts.

Proof. This is an example of a general criterion by Paré, who classifies all absolute pushouts in [37, Proposition 5.5] (the cited paper also classifies all absolute colimits in general). It is sufficient to check that an split pushout of the shape (4.7) is a pushout square, since split pushouts are manifestly preserved by all functors. Suppose $f : B \to X$ and $g : C \to X$ are given so that $fa_1 = ga_2$. Define $h : P \to X$ to be $h = fd_0$. Then

\[
h p_1 = f d_0 p_1 = f a_1 d_1 = g a_2 d_1 = g
\]

and

\[
h p_2 = f d_0 p_2 = f a_1 d'_1 = g a_2 d'_1 = g a_2 d'_2 = f a_1 d'_2 = f.
\]
Since \( p_2 \) is a split epimorphism, \( h \) is the unique map making

\[
\begin{array}{ccc}
A & \xleftarrow{a_1} & B \\
\downarrow{a_2} & & \downarrow{p_2} \\
C & \xrightarrow{p_1} & P \\
\downarrow{f} & & \downarrow{h} \\
X & & \\
\end{array}
\]

commute.

In all the split pushouts we compute below, we will always set \( d_1' = d_2' d_0 p_2 \). This reduces the relations that we need to verify to the following five:

\[
\begin{align*}
a_2 d_1 &= \text{id}_C \\
p_2 d_0 &= \text{id}_P \\
d_0 p_1 &= a_1 d_1 \\
a_2 d_2' &= a_2 d_0 p_2 \\
a_1 d_2' &= \text{id}_B
\end{align*}
\]

Lemma 4.24. The diagram

\[(4.8) \quad [n-1] \xleftarrow{\sigma^i} [n] \xrightarrow{\sigma^i} [n-1]\]

has an absolute pushout

\[
\begin{array}{ccc}
[n] & \xrightarrow{\sigma^i} & [n-1] \\
\downarrow{\sigma^i} & & \downarrow{\tau_2} \\
[n-1] & \xrightarrow{\tau_1} & [\ell] \\
\end{array}
\]

in \( \mathcal{D} \) with both \( \tau_1 \) and \( \tau_2 \) in \( \mathcal{D}^- \) and \( n - 2 \leq \ell \leq n - 1 \).

Proposition 4.25. The category \( \mathcal{D} \) satisfies axiom EZ3 of Definition 4.2.

Corollary 4.26. The diagram

\[
\begin{array}{ccc}
[n-1] & \xleftarrow{\sigma^i} & [n] \\
\downarrow{\sigma^i} & & \downarrow{\tau_2} \\
[n-1] & \xrightarrow{\tau_1} & [\ell] \\
\end{array}
\]

in \( \mathcal{D}_\Sigma \) with both \( \tau_1 \) and \( \tau_2 \) in \( \mathcal{D}_\Sigma^- \) and \( n - 2 \leq \ell \leq n - 1 \).

Proof. The functor \( i : \mathcal{D} \to \mathcal{D}_\Sigma \) preserves absolute pushouts. \( \Box \)
Proof of Lemma 4.24. If \( i = j \), then the pushout of (4.8) is \([n-1]\) and is preserved by any functor \( \mathcal{D} \to \mathcal{C} \). Suppose \( i < j \). Then the square

\[
\begin{array}{ccc}
[n] & \xrightarrow{\gamma^i} & [n-1] \\
\downarrow{\sigma^i} & & \downarrow{\sigma^{j-1}} \\
[n-1] & \xrightarrow{\gamma^j} & [n-2]
\end{array}
\]  

(4.9)

is a split pushout in \( \mathcal{D} \). Using the notation of Definition 4.22 we define sections

\[
d_0 = \delta_{n-2}^{j-1,0}, \quad d_1 = \delta_{n-1}^{j,0}, \quad d_2' = \delta_{n-1}^{i,0}.
\]

Observe that these maps are well-defined, since \( 1 \leq i < j \leq n \). Using the cubical relations in Proposition 3.9, we verify the five relations:

\[
\sigma^j \delta_{n-1}^{j,0} = \text{id}_{[n-1]} \quad (a_2 d_1 = \text{id}_C)
\]

\[
\sigma^j \delta_{n-2}^{j-1,0} = \text{id}_{[n-2]} \quad (p_2 d_0 = \text{id}_{P})
\]

\[
\sigma^i \delta_{n-1}^{i,0} = \text{id}_{[n-1]} \quad (a_1 d_2' = \text{id}_B)
\]

\[
\delta_{n-1}^{i-1,0} \sigma^i = \sigma^i \delta_{n-1}^{i,0} \quad (d_0 p_1 = a_1 d_1)
\]

\[
\delta_{n-1}^{i-1,0} \sigma^j = \delta_{n-1}^{i-1,0} \delta_{n-2}^{j,0} = \delta_{n-1}^{i,0} \sigma^j = \sigma^j \delta_{n-1}^{i,0} \quad (a_2 d_2' d_0 p_2 = a_2 d_2')
\]

Hence (4.9) is an absolute pushout. □

Lemma 4.27. Suppose \( n \geq 2 \). The diagram

\[
\begin{array}{ccc}
[n-1] & \xrightarrow{\gamma^i} & [n] \\
\downarrow{\gamma^j} & & \downarrow{\gamma^j} \\
[n-1] & \xrightarrow{\gamma^j} & [n-1]
\end{array}
\]  

(4.10)

has an absolute pushout \([\ell]\) in \( \mathcal{D}_\Sigma \) with maps \([n-1] \to [\ell]\) in \( \mathcal{D}_\Sigma^- \) and \( n-2 \leq \ell \leq n-1 \).

Proof. The proof of this lemma is similar to that of Lemma 4.24. Without loss of generality, we may assume that \( j \geq i \). We then have three special cases:

1. \( (j = i) \) The pushout of (4.10) is \([n-1]\) and is absolute.
2. \( (j = i+1) \) The square

\[
\begin{array}{ccc}
[n] & \xrightarrow{\gamma^i} & [n-1] \\
\downarrow{\gamma^{i+1}} & & \downarrow{\gamma^i} \\
[n-1] & \xrightarrow{\gamma^i} & [n-2]
\end{array}
\]

is a split pushout: define sections

\[
d_0 = \delta_{n-2}^{i+1,1}, \quad d_1 = \delta_{n-1}^{i+2,1}, \quad d_2' = \delta_{n-1}^{i,1}.
\]
Note that \(d_1: [n - 1] \to [n]\) is well-defined, since \(i + 2\) is at most \(n\). Now we verify the five relations:

\[
\begin{align*}
\gamma^{i+1}\delta^{i+2,1} &= \text{id}_{[n-1]} \\
\gamma^i\delta^{i+1,1} &= \text{id}_{[n-2]} \\
\gamma^i\delta^{i,1} &= \text{id}_{[n-1]} \\
\delta^{i+1,1}\gamma^i &= \gamma^i\delta^{i+2,1} \\
\gamma^{i+1}\delta^{i+1,1}\delta^{i+1,1}\gamma^i &= \gamma^{i+1}\delta^{i+2,1}\delta^{i,1}\gamma^i = \delta^{i,1}\gamma^i = \gamma^{i+1}\delta^{i,1} \\
\end{align*}
\]

\(a_2d_1 = \text{id}_C\) \(p_2d_0 = \text{id}_P\) \(a_1d'_2 = \text{id}_B\) \(d_0p_1 = a_1d_1\) \(a_2d'_2d_0p_2 = a_2d'_2\)

(3) \((j > i + 1)\) The square

\[
\begin{array}{ccc}
[n] & \xrightarrow{\gamma^i} & [n-1] \\
\gamma^i & \downarrow & \downarrow \\
[n-1] & \xrightarrow{\gamma^j} & [n-2]
\end{array}
\]

is a split pushout: we define sections

\[
d_0 = \delta^{i-1,1} \quad d_1 = \delta^{i,1} \quad d'_2 = \delta^{i,1}.
\]

The five relations are verified:

\[
\begin{align*}
\gamma^i\delta^{i,1} &= \text{id}_{[n-1]} \\
\gamma^j\delta^{j-1,1} &= \text{id}_{[n-2]} \\
\gamma^i\delta^{i,1} &= \text{id}_{[n-1]} \\
\delta^{j-1,1}\gamma^i &= \gamma^i\delta^{j,1} \\
\gamma^j\delta^{j-1,1}\gamma^{j-1} &= \gamma^j\delta^{j,1}\delta^{i-1,1}\gamma^{j-1} = \delta^{i,1}\gamma^{j-1} = \gamma^j\delta^{i,1} \\
\end{align*}
\]

\(a_2d'_2d_0p_2 = a_2d'_2\)

\(\square\)

**Lemma 4.28.** Suppose \(n \geq 2\). The diagram

\[
\begin{array}{ccc}
[n-1] & \xrightarrow{\gamma^i} & [n] \\
\gamma^j & \downarrow & \downarrow \\
[n-1] & \xrightarrow{\sigma^j} & [n-2]
\end{array}
\]

has an absolute pushout \([\ell]\) in \(D_\Sigma\) with maps \([n-1] \to [\ell]\) in \(D_\Sigma^-\) and \(n - 2 \leq \ell \leq n - 1\).

**Proof.** We have four possibilities:

(1) \((i > j)\) The square

\[
\begin{array}{ccc}
[n] & \xrightarrow{\sigma^j} & [n-1] \\
\gamma^i & \downarrow & \downarrow \\
[n-1] & \xrightarrow{\sigma^j} & [n-2]
\end{array}
\]

is an split pushout with sections

\[
d_0 = \delta^{i-1,1} \quad d_1 = \delta^{i,1} \quad d'_2 = \delta^{i,1}.
\]
The relations are satisfied:
\[ \gamma^i \delta^{i:1} = \text{id}_{[n-1]} \]
\[ \gamma^{i-1} \delta^{i-1} = \text{id}_{[n-2]} \]
\[ \sigma^j \delta^{j:1} = \text{id}_{[n-1]} \]
\[ \delta^{i-1} \sigma^j = \sigma^j \delta^{i:1} \]
\[ \gamma^i \delta^{j:1} \delta^{i-1} \gamma^{i-1} = \delta^{j:1} \gamma^{i-1} = \gamma^i \delta^{j:1} \]

(2) \((i = j)\) The square

\[
\begin{array}{c}
[n] \xrightarrow{\gamma^i} [n-1] \\
\sigma^i \downarrow \downarrow \sigma^i \\
[n-1] \xrightarrow{\sigma^i} [n-2]
\end{array}
\]

is an split pushout with sections
\[ d_0 = \delta^{i:0} \quad d_1 = \delta^{i:0} \quad d'_2 = \delta^{i+1:1} \]

The relations are satisfied:
\[ \sigma^i \delta^{i:0} = \text{id}_{[n-1]} \]
\[ \sigma^i \delta^{i:0} = \text{id}_{[n-2]} \]
\[ \gamma^i \delta^{i:1} = \text{id}_{[n-1]} \]
\[ \delta^{i:0} \sigma^i = \gamma^i \delta^{i:0} \]
\[ \sigma^i \delta^{i+1:1} \delta^{i:0} \sigma^i = \sigma^i \delta^{i+1:1} \]
\[ \sigma^i \delta^{i:1} \delta^{i:0} \sigma^i = \sigma^i \delta^{i:1} \sigma^i = \sigma^i \delta^{i:1} \]

(3) \((i + 1 = j)\) The square

\[
\begin{array}{c}
[n] \xrightarrow{\gamma^i} [n-1] \\
\sigma^i \downarrow \downarrow \sigma^i \\
[n-1] \xrightarrow{\sigma^i} [n-2]
\end{array}
\]

is an split pushout with sections
\[ d_0 = \delta^{i:0} \quad d_1 = \delta^{j:0} \quad d'_2 = \delta^{i:1} \]

The relations are satisfied:
\[ \sigma^j \delta^{j:0} = \text{id}_{[n-1]} \]
\[ \sigma^j \delta^{j:0} = \text{id}_{[n-2]} \]
\[ \gamma^j \delta^{i:1} = \text{id}_{[n-1]} \]
\[ \delta^{i:0} \sigma^i = \gamma^j \delta^{j:0} \]
\[ \sigma^j \delta^{i:1} \delta^{i:0} \sigma^i = \sigma^j \delta^{i:1} \delta^{i:0} \sigma^i = \delta^{i:1} \sigma^i = \sigma^j \delta^{i:1} \]

\[ (a_2 d_1 = \text{id}_C) \]
\[ (a_2 d_0 = \text{id}_P) \]
\[ (a_1 d'_2 = \text{id}_B) \]
\[ (d_0 p_1 = a_1 d_1) \]
\[ (a_2 d'_2 d_0 p_2 = a_2 d'_2). \]
(4) \( i + 1 < j \) The square

\[
\begin{array}{ccc}
[n] & \xrightarrow{\gamma^i} & [n - 1] \\
\sigma^i \downarrow & & \sigma^{j-1} \downarrow \\
[n - 1] & \xrightarrow{\gamma^j} & [n - 2]
\end{array}
\]

is an split pushout with sections

\[
d_0 = \delta^{i-1,0}, \quad d_1 = \delta^{j,0}, \quad d_2' = \delta^{i,1}.
\]

The relations are satisfied:

\[
\begin{align*}
\sigma^j \delta^{i,0} &= \text{id}_{[n-1]} \\
\sigma^{j-1} \delta^{i-1,0} &= \text{id}_{[n-2]} \\
\gamma^i \delta^{i-1} &= \text{id}_{[n-1]} \\
\delta^{j-1,0} \gamma^i &= \gamma^i \delta^{i,0} \\
\sigma^j \delta^{i-1,0} \sigma^{j-1} &= \sigma^i \delta^{i,0} \delta^{j-1} \sigma^{j-1} = \delta^{i,1} \\
(a_2 d_1 &= \text{id}_C) \\
p_2 d_0 &= \text{id}_P \\
(a_1 d_2' &= \text{id}_B) \\
d_0 p_1 &= a_1 d_1 \\
(\sigma_2 d_0 p_2 = a_2 d_2'). & \square
\end{align*}
\]

**Corollary 4.29.** The category \( \mathcal{D}_\Sigma \) satisfies axiom E3 of Definition 4.12.

5. The symmetric cubical site models the homotopy category

In this section, we’ll equip \( q\Sigma \text{Set} \) with a model structure Quillen equivalent to \( s\text{Set} \). This is the heart of the paper. We’ll start by describing a spatial model structure on \( q\text{Set} \). We will then lift the model structure from \( q\text{Set} \) along the restriction functor \( i^* : q\Sigma \text{Set} \to \text{Set}. \) In order to do this, we need to check that cell complexes in \( q\Sigma \text{Set} \) built out of the representable functors are well-behaved homotopically. The outline of the argument is standard; as usual, it requires some work to verify. The resulting Quillen pair \( i_i \dashv i^* \) is then readily shown to be a Quillen equivalence. Finally, we’ll discuss the monoidal properties of the lifted model structure on \( q\Sigma \text{Set}. \)

### 5.1. The homotopy theory of cubical sets.

In [11], Cisinski proves that \( \mathcal{D} \) is a test category and thus the category \( q\text{Set} = \text{Set}^{\Delta^{op}} \) models spaces. Jardine gives a summary of cubical homotopy theory from Cisinski’s perspective in [30]. We’ll summarize their results here. Recall that \( \partial \square^n \) is the subpresheaf of \( \square^n \) given by

\[
(\partial \square^n)_m = \{ f : [m] \to [n] \in \mathcal{D} \mid f \text{ factors as } f : [m] \to [k] \to [n], \ k < n \}.
\]

This comes equipped with a monomorphism \( \partial \square^n \to \square^n \). Put another way, \( \partial \square^n \) is the union of the \((n - 1)\)-dimensional faces of \( \square^n \). We define the \( i, \varepsilon \)-cap

\[
(\cap_{i,\varepsilon}^n)_m = \{ f : [m] \to [n] \in \mathcal{D} \mid f \text{ factors as } f : [m] \to [n - 1] \xrightarrow{d} [n], \ d \neq \delta_{n}^{i,1} \}
\]

for \( 1 \leq i \leq n \). This comes equipped with a monomorphism \( \cap_{i,\varepsilon}^n \to \partial \square^n \).

**Definition 5.1.** We say a functor \( F : \mathcal{B} \to \mathcal{C} \) of small categories is a Thomason equivalence if \( F \) induces a weak equivalence \( \text{NF} : \mathcal{B} \to \mathcal{N} \mathcal{C} \) on nerves. Let \( \mathcal{A} \) be a small category. We say a map \( f : X \to Y \) in \( \mathcal{A} \) is an \( \infty\)-equivalence if \( f \) induces a Thomason equivalence

\[
\mathcal{A} \downarrow f : \mathcal{A} \downarrow X \to \mathcal{A} \downarrow Y
\]

of categories.
**Definition 5.2.** The *simplicial realization* of a cubical set $X ∈ q\text{-}\text{Set}$ is the colimit $|X| = \operatorname{colim} (\Delta[1])^n$ of simplicial sets.

Note that simplicial realization is the unique cocontinuous functor $q\text{-}\text{Set} → s\text{-}\text{Set}$ taking $\square^n$ to $(\Delta[1])^n$. Since its restriction to $\mathcal{Q}$ is strong monoidal, it is strong monoidal on $q\text{-}\text{Set}$. We can now state the following theorem:

**Theorem 5.3 ([11, Théorème 8.4.38]).**

1. The category $q\text{-}\text{Set}$ forms a proper model category with cofibrations monomorphisms and weak equivalences the $\infty$-equivalences. We call this model structure the *spatial model structure*. It is cofibrantly generated with generating cofibrations

$$\{\partial \square^n → \square^n \mid n ≥ 0\}$$

and generating acyclic cofibrations

$$\{\square^n_\varepsilon → \square^n \mid 1 ≤ i ≤ n, \varepsilon = 0, 1\}.$$

2. The spatial model structure is monoidal: if $i : A → B$ and $j : K → L$ are cofibrations,

$$i ∘ j : A ⊗ L \amalg K B ⊗ K → B ⊗ L$$

is a cofibration, acyclic if either $i$ or $j$ is.

3. Simplicial realization is a left Quillen equivalence $q\text{-}\text{Set} → s\text{-}\text{Set}$.

Theorem 5.3 is the basis of everything that follows. We will take it for granted. Jardine also gives a proof of it in the survey [30] following Cisinski’s methods.

### 5.2. Homotopy and asphericity

Recall that the categories $\mathcal{Q}$ and $\mathcal{Q}_\Sigma$ are related by an inclusion functor $i : \mathcal{Q} → \mathcal{Q}_\Sigma$. This produces an adjoint pair

$$i_! : q\text{-}\text{Set} ⥲ q\Sigma\text{-}\text{Set} : i^*$$

given by left Kan extension and restriction.

**Proposition 5.4.** The functors $i_!$ and $i^*$ are strong and lax monoidal, respectively.

**Proof.** That $i_!$ is strong monoidal is a consequence of Proposition [11] since the square

$$\begin{array}{ccc}
\mathcal{Q} & \xrightarrow{i} & \mathcal{Q}_\Sigma \\
[-] & \downarrow & [-] \\
q\text{-}\text{Set} & \xleftarrow{i_!} & q\Sigma\text{-}\text{Set}
\end{array}$$

commutes up to natural isomorphism and $i$ is strong monoidal, the extension $i_!$ is strong monoidal. Now suppose $K$ and $L$ are extended cubical sets. The counit of the adjunction $i_! − i^*$ together with the monoidalness of $i_!$ yields a natural map

$$ii_!(i^* K ⊗ i^* L) → ii^* K ⊗ ii^* L → K ⊗ L$$

The adjoint is a natural transformation

$$i^* K ⊗ i^* L → i^*(K ⊗ L)$$

making $i^*$ lax monoidal. □
**Definition 5.5.** Suppose \( n > 0 \). Let \( \{\varepsilon\} \) denote the formal \((0,n)\)-product
\[
(\varepsilon, \ldots, \varepsilon)
\]
\(n\) entries
for \( \varepsilon = 0, 1 \). Note that \( \{\varepsilon\} = (d^1, \varepsilon)^n \). Suppose \( f, g : X \rightarrow Y \) are two maps in \( q\Sigma \text{Set} \). We say \( f \) and \( g \) are \( \Box^n \Sigma \)-homotopic if there is a filler \( h \) in the diagram
\[
\begin{array}{ccc}
X \otimes \Box^n \Sigma & \xrightarrow{f} & Y \\
\downarrow \text{id} \otimes \{0\} & & \\
X \otimes \Box^0 \Sigma & \xrightarrow{g} & Y
\end{array}
\]
We call \( h \) a \( \Box^n \Sigma \)-homotopy from \( f \) to \( g \). By abuse of terminology, we’ll sometimes simply call \( f \) and \( g \) homotopic. We say a map \( k \) is a homotopy equivalence if there is a map \( \ell \) so that \( k\ell \) and \( \ell k \) are both homotopic to the identity. We define \( \Box^n \Sigma \)-homotopy in \( q\Sigma \text{Set} \) similarly.

Note that \( \Box^n \Sigma \)-homotopy is not an equivalence relation for arbitrary \( Y \)—the (extended) cubical set \( Y \) must possess a sort of homotopy extension property, i.e., \( Y \) must be fibrant. This is precisely the same reason that \( \Delta[1] \)-homotopy is not an equivalence relation on maps in \( s\text{Set} \) unless the maps have a Kan complex as their target. Using the spatial model structure on \( q\Sigma \text{Set} \), we have the following standard result:

**Proposition 5.6.** Suppose \( k : X \rightarrow Y \) is a homotopy equivalence in \( q\Sigma \text{Set} \). Then \( k \) is an \( \infty \)-equivalence.

**Lemma 5.7.** Suppose \( f \) and \( g : X \rightarrow Y \) are \( \Box^n \Sigma \)-homotopic maps in \( q\Sigma \text{Set} \). Then \( i^* f \) and \( i^* g \) are \( \Box^n \)-homotopic.

**Proof.** Let \( h : X \otimes \Box^n \Sigma \rightarrow Y \) be a homotopy from \( f \) to \( g \). By Proposition 5.4, \( i^* \) is lax monoidal, so we have a diagram
\[
\begin{array}{ccc}
i^* X \otimes \Box^0 \Sigma & \xrightarrow{\sim} & i^* X \otimes i^* \Box^n \Sigma & \xrightarrow{\sim} & i^* (X \otimes \Box^n \Sigma) \\
\downarrow \text{id} \otimes \{0\} & & \downarrow \text{id} \otimes i^* \{0\} & & \downarrow i^* (\text{id} \otimes \{0\}) \\
i^* X \otimes \Box^n & \xrightarrow{i^* X} & i^* X \otimes i^* \Box^n \Sigma & \xrightarrow{i^* h} & i^* Y \\
\downarrow \text{id} \otimes \{1\} & & \downarrow \text{id} \otimes i^* \{1\} & & \downarrow i^* (\text{id} \otimes \{1\}) \\
i^* X \otimes \Box^0 & \xrightarrow{\sim} & i^* X \otimes i^* \Box^n \Sigma & \xrightarrow{\sim} & i^* (X \otimes \Box^n \Sigma) \\
\downarrow \text{id} \otimes i^* \{1\} & & \downarrow i^* (\text{id} \otimes \{1\}) & & \downarrow i^* g
\end{array}
\]
The unit \( \Box^0 \rightarrow i^* \Box^n \Sigma \) is an isomorphism since \( \Box^n \Sigma \) and \( \Box^0 \) are terminal and \( i^* \) is a right adjoint. Hence the top and bottom horizontal arrows are isomorphisms. As a result, the horizontal chain of arrows is a \( \Box^n \)-homotopy between \( i^* f \) and \( i^* g \).

**Corollary 5.8.** *The functor* \( i^* \) *preserves homotopy equivalences.*
Lemma 5.9. The inclusion functor \( i : \mathcal{D} \to \mathcal{D}_\Sigma \) is aspherical, i.e., for all \( n \geq 0 \),

\[
i \downarrow [n] \to \mathcal{D}_\Sigma \downarrow [n]
\]
is a Thomason equivalence.

Proof. We define a map \( H : [2n] \to [n] \) as the formal product

\[
(x_1 \wedge x_{n+1}, x_2 \wedge x_{n+2}, \ldots, x_n \wedge x_{2n}).
\]

This is an application of a symmetry followed by \( n \) conjunctions. The map \( H \) gives a homotopy between \( \{0\} \) and the identity map on \( \square^m_\Sigma \):

\[
\begin{array}{ccc}
\square^m_\Sigma & \xrightarrow{H} & \square^n_\Sigma \\
\downarrow{id \otimes (0)} & & \downarrow{id} \\
\square^m_\Sigma \otimes \square^n_\Sigma & & \\
\downarrow{id \otimes (1)} & & \\
\square^m_\Sigma & & \\
\end{array}
\]

Hence the inclusion \( \{0\} \) is a homotopy equivalence in \( q\Sigma\text{Set} \). By Corollary 5.8, \( i^* \{0\} : \square^0 \to i^* \square^n_\Sigma \) is a homotopy equivalence and hence \( \infty \)-equivalence in \( \text{qSet} \). Thus

\[
\mathcal{D} \to \mathcal{D}_\Sigma \downarrow i^* \square^n_\Sigma
\]
is a Thomason equivalence and \( N(i^* \square^n_\Sigma) \) is contractible. Since \( \mathcal{D} \downarrow i^* \square^n_\Sigma \) is equivalent to \( i \downarrow \square^n_\Sigma \), we conclude that

\[
i \downarrow [n] \to \mathcal{D}_\Sigma \downarrow [n]
\]
is a Thomason equivalence. \( \square \)

Proposition 5.10. Suppose \( X \in q\Sigma\text{Set} \). Then the functor

\[
\mathcal{D}_\Sigma \downarrow i^*X \to \mathcal{D}_\Sigma \downarrow X
\]
induced by \( i \) induces an equivalence of nerves.

Proof. This is a special case of [35 Proposition 1.2.9]. Suppose \( s : \square^m_\Sigma \to X \) is a cube of \( X \). Consider the category \( (\mathcal{D}_\Sigma \downarrow i^*X) \downarrow s \):

\[
\begin{array}{ccc}
\square^m_\Sigma & \xrightarrow{s} & X \\
\downarrow{(\mathcal{D}_\Sigma \downarrow i^*X)} & & \\
\square^m_\Sigma & & \\
\end{array}
\]

this is the category of triangles

\[
\begin{array}{ccc}
\square^m_\Sigma & \xrightarrow{i(f)} & \square^m_\Sigma \\
\downarrow{s_0} & & \downarrow{s} \\
\square^m_\Sigma & & X
\end{array}
\]

with morphisms diagrams of the shape
The functor 
\[(\mathcal{D} \downarrow i^*X) \downarrow s \to \mathcal{D} \downarrow \square^n_S\]
forgetting the map to \(X\) has a left adjoint given by composition with \(s\), so it is a Thomason equivalence. By Lemma 5.14, we may conclude that
\[(\mathcal{D} \downarrow i^*X) \downarrow s\]
has contractible nerve, so by Quillen’s Theorem A [39],
\[\mathcal{D} \downarrow i^*X \to \mathcal{D} \downarrow X\]
is a Thomason equivalence. □

Corollary 5.11. The functor \(i^*\) reflects \(\infty\)-equivalences; i.e., \(X \to Y\) in \(q\Sigma\text{Set}\) induces a Thomason equivalence
\[\mathcal{D}_\Sigma \downarrow X \to \mathcal{D}_\Sigma \downarrow Y\]
if and only if
\[\mathcal{D} \downarrow i^*X \to \mathcal{D} \downarrow i^*Y\]
is a Thomason equivalence.

5.3. A Quillen equivalence. We will show that \(i_! \dashv i^*\) is a Quillen equivalence simultaneously with the construction of the spatial model structure on \(q\Sigma\text{Set}\).

Proposition 5.12. The functor \(i_! : q\text{Set} \to q\Sigma\text{Set}\) preserves monomorphisms.

Before we embark on this, recall that \(\partial\square^n_S\) is the subpresheaf of \(\square^n_S\) given by
\[\partial\square^n_S([m]) = \{ f \in \mathcal{D}_\Sigma([m],[n]) \mid \text{f factors } [m] \to [k] \to [n], k < n \in \mathcal{D}_\Sigma \}\].

Another description of \(\partial\square^n_S([m])\) is as the set of formal \((m,n)\)-products with at least one entry 0 or 1. As we’ll describe below, \(\partial\square^n_S\) is the union of the faces of \(\square^n_S\).

Proof of Proposition 5.12. By Corollary 4.20, the map \(i_! \sk_{n-1} \square^n \to \sk_{n-1} \square^n_S\) is an isomorphism. This implies that \(i_!(\partial \square^n \to \square^n_S)\) is, up to isomorphism, the map \(\partial \square^n_S \to \square^n_S\). Since
\[\text{mono} = \text{Cell}\{\partial \square^n \to \square^n \mid n \geq 0\},\]
we may conclude that \(i_!\) preserves all monomorphisms in \(q\text{Set}\). □

Lemma 5.13. Suppose \(Y\) is an \(n\)-skeletal cubical set. The map
\[Y \amalg_{\sk_{n-1}} Y i^*i_! \sk_{n-1} Y \to i^*i_! Y\]
is a monomorphism.

Proof. First note that the corner map
\[i^*\partial \square^n_S \amalg_{\partial \square^n} \square^n \to i^*\square^n_S\]
is a monomorphism. This is a consequence of the fact that \(i\) is faithful: for \([m]\), we have
\[(i^*\partial \square^n_S \amalg_{\partial \square^n} \square^n)_m = \{ f : [m] \to [n] \mid \text{f factors through } [k], k < n, \text{ or } f \in \operatorname{ar} \mathcal{D} \}.\]
Let $S$ be the set of nondegenerate $n$-simplices of $Y$. By Proposition 4.10 and Proposition 4.14 we may write $Y$ as a pushout

$$\coprod_S \partial \Box^n \to \sk_{n-1} Y \to \coprod_S \Box^n \to Y$$

where $S$ is the set of nondegenerate $n$-simplices of $Y$. Write $\eta : \id \to i^* i_!$ for the unit of the adjunction $i_! \dashv i^*$. Consider the cube

$$\coprod_S i^* i_! \partial \Box^n \to i^* i_! \sk_{n-1} Y \to \coprod_S i^* i_! \Box^n \to i^* i_! Y.$$  

(5.1)

The functors $i^*$ and $i_!$ both preserve colimits, so the front and back faces are both pushouts. As a result, the square

$$\coprod_S (i^* \partial \Box^n \coprod \partial \Box^n \Box^n) \to i^* i_! \sk_{n-1} Y \to \coprod_S i^* i_! \Box^n \to i^* i_! Y$$

is a pushout; since $q\Set$ is a topos, the arrow $g$ is a monomorphism. 

**Lemma 5.14.** An arbitrary small coproduct of $\infty$-equivalences in $q\Set$ is an $\infty$-equivalence.

**Proof.** This is a standard model category result (Ken Brown’s lemma [27], together with the fact that everything in $q\Set$ is cofibrant). Alternatively, observe that $|-|$ reflects weak equivalences and preserves small coproducts; arbitrary small coproducts of weak equivalences in $s\Set$ are themselves weak equivalences. 

**Proposition 5.15.** The unit $\eta : \id \to i^* i_!$ of the adjunction $i_! \dashv i^*$ is a natural $\infty$-equivalence in $q\Set$.

**Proof.** We’ll first prove that $\eta_X$ is an $\infty$-equivalence for skeletal $X$ by induction on the dimension. If $X$ is $0$-skeletal, then $X = \coprod_S \Box^n$ and $\eta_X$ is an isomorphism. Let $n > 0$ and suppose $\eta_X$ is an $\infty$-equivalence for all $(n-1)$-skeletal $X$. In particular, $\eta_{\partial \Box^n}$ is an $\infty$-equivalence since $\partial \Box^n$ is the $(n-1)$-skeleton of $\Box^n$. Suppose $Y$ is $n$-skeletal. From Corollary 5.11 we know that $\Box^n \to i^* \Box^n$ is an $\infty$-equivalence. Recall the cube (5.1) in the proof of Lemma 5.13. The front and back faces are both pushout squares. The arrows $j$ and $i^* i_! j$ are both monomorphisms by Proposition 5.12. Every object of $q\Set$ is cofibrant, so these pushout squares are both homotopy cocartesian. By Lemma 5.14 and our assumptions, the diagonal arrows
Suppose $X$ is an arbitrary cubical set. Now consider the ladder

$$
\begin{array}{cccccccc}
sk_0 X & \rightarrow & \sk_1 X & \rightarrow & \cdots & \rightarrow & \sk_n X & \rightarrow & \cdots \\
\eta_0 & & \eta_1 & & \cdots & & \eta_n & & \\
i^* i! \sk_0 X & \rightarrow & i^* i! \sk_1 X & \rightarrow & \cdots & \rightarrow & i^* i! \sk_n X & \rightarrow & \cdots \\
\end{array}
$$

By Lemma 5.13, this map is an acyclic Reedy cofibration: the map $\eta_0$ is a cofibration, the corner maps $i^* i! \sk_{n-1} X \amalg_{\sk_n X} \sk_n X \rightarrow i^* i! \sk_n X$ are cofibrations, and each $\eta_i$ is an $\infty$-equivalence. Hence the colimit $\eta^X : X \rightarrow i^* i! X$ is an $\infty$-equivalence.\[\square\]

We could have avoided using Lemma 5.13 by Reedy’s Theorem C [40].

**Corollary 5.16.** The counit $\varepsilon : i^* i! \rightarrow \text{id}$ of the adjunction $i! \dashv i^*$ is a natural $\infty$-equivalence in $Q \Sigma$.

**Proof.** Suppose $X \in q\Sigma \text{Set}$. Consider the triangle

$$
\begin{array}{ccc}
i^* X & \xrightarrow{\eta^X} & i^* i^* X \\
\downarrow \text{id} & & \downarrow i^* \varepsilon_X \\
i^* X & & \\
\end{array}
$$

The map $\eta^X$ is an $\infty$-equivalence by Proposition 5.15 so $i^* \varepsilon_X$ is an $\infty$-equivalence. By Proposition 5.10, $\varepsilon_X$ is an $\infty$-equivalence. \[\square\]

We can finally prove the main theorem of this paper.

**Theorem 5.17.** The category $q\Sigma \text{Set}$ forms a left proper cofibrantly generated model category known as the spatial model structure with weak equivalences the $\infty$-equivalences and fibrations the maps $p : X \rightarrow Y$ which are fibrations in $q\text{Set}$ upon application of the restriction $i^*$. The set of generating cofibrations is

$$I = \{ \partial \square_n^m \rightarrow \square_n^m \mid n \geq 0 \}$$

and the set of generating acyclic cofibrations is

$$J = \{ i_! \cap_n^m \rightarrow \square_n^m \mid 1 \leq j \leq n \text{ and } \varepsilon = 0, 1 \}.$$  

**Proof.** This is a consequence of a standard result on lifting model structures along an adjunction—see, for example, [42]. The key point is the following: suppose

$$
\begin{array}{ccc}
i_! \cap_n^m & \rightarrow & A \\
i_! \varepsilon & \downarrow & \downarrow f \\
i_! \square_n^m & \rightarrow & B
\end{array}
$$

By Lemma 5.13, this map is an acyclic Reedy cofibration: the map $\eta_0$ is a cofibration, the corner maps $i_! i_\varepsilon \square_{n-1}^m \amalg \square_n^m \rightarrow i_! i_\varepsilon \square_n^m$ are cofibrations, and each $\eta_i$ is an $\infty$-equivalence. Hence the colimit $\eta^X : X \rightarrow i_! i_\varepsilon X$ is an $\infty$-equivalence. \[\square\]
is a pushout in $q\Sigma\text{Set}$. The functor $i^*$ preserves all colimits and limits, so the right square in

$$
\begin{array}{ccc}
\Box^n & \xrightarrow{i^*i|_j^n} & i^*A \\
\downarrow & \downarrow & \downarrow \\
i^n & \xrightarrow{i^*f} & i^*B
\end{array}
$$

is a pushout in $q\text{Set}$. But by Proposition 5.15, $i^*i|_e$ is a weak equivalence. By Proposition 5.12, $i^*i|_e$ is a monomorphism. Hence $i^*f$ is an $\infty$-equivalence in $q\text{Set}$, so by Corollary 5.11, $f$ is an $\infty$-equivalence. Since $q\Sigma\text{Set}$ is locally presentable, we can use the small object argument to factor every arrow in $q\Sigma\text{Set}$ as a map in $\text{Cell}_J$ followed by a $J$-injective map $[4]$. But by the above discussion—together with the fact that $i^*$ preserves filtered colimits—the maps in $\text{Cell}_J$ are acyclic cofibrations and the maps in $\text{Inj}_J$ are fibrations. For left properness, apply the functor $i^*$ to the necessary diagram and note that $i^*$ preserves cofibrations. □

Since $\Delta[1]$ is a cubical monoid, we may define the extended geometric realization functor $|-|_\Sigma$ to be the unique cocontinuous strong monoidal functor $\mathcal{E}_\Sigma \to s\text{Set}$ taking $\Box^1$ to $\Delta[1]$.

**Theorem 5.18.** The functors $i_!$ and $|-|_\Sigma$ are both left Quillen equivalences. The diagram

$$
\begin{array}{ccc}
q\text{Set} & \xrightarrow{i_!} & q\Sigma\text{Set} \\
\downarrow \  & \downarrow \  & \downarrow \  \\
s\text{Set} & \xrightarrow{|-|_\Sigma} & s\text{Set}
\end{array}
$$

commutes up to natural isomorphism.

**Proof.** We’ve proved that the unit and counit $\eta: \text{id} \to i^*i_!$ and $\varepsilon: i_!i^* \to \text{id}$ are natural $\infty$-equivalences (5.15 and Corollary 5.10). Strictly speaking, this is not the right condition for Quillen equivalences, as we’d need to use the derived left and right adjoints. However, $i^*$ and $i_!$ coincide with their derived functors: $i_!$ is left Quillen, but everything in $q\text{Set}$ is cofibrant, so it preserves all $\infty$-equivalences. The rights adjoint $i^*$ preserves all $\infty$-equivalences as well (Proposition 5.10). Two-out-of-three ensures that $|-|_\Sigma$ is a left Quillen equivalence. □

**Remark 5.19.** A few notes about Theorem 5.17 are in order. Not all monomorphisms in the spatial model structure on $q\Sigma\text{Set}$ are cofibrations. For example, the $\Sigma_n$-orbits of $\partial\Box^n_2$ to $\Box^n_2$ is a monomorphism, but if $n > 1$, it is not a cofibration. As a result $|-|_\Sigma$ may not reflect weak equivalences—however, its left derived functor $L_!|-|_\Sigma$ preserves and reflects weak equivalences.

We’ve shown that $i^*$ is a left and right Quillen functor. On the level of homotopy categories, since $i_! \dashv i^*$ induces an equivalence of $\text{Ho}q\text{Set}$ with $\text{Ho}q\Sigma\text{Set}$, the adjoint pair $i^* \dashv Ri_!$ must also induce an equivalence of $\text{Ho}q\text{Set}$ with $\text{Ho}q\Sigma\text{Set}$, and so $Ri_!$ and $i_!$ coincide.

5.4. The extended product. We’ve now shown that $q\Sigma\text{Set}$ models spaces. In the remainder of this section, we’ll prove that the monoidal structure on $q\Sigma\text{Set}$ is compatible with the spatial model structure.
Lemma 5.20. Suppose $X$ is an extended cubical set and $n \geq 0$. The map $\pi : X \otimes □_n \rightarrow X$ given by the product of the identity map on $X$ and the unique map $\square_n \rightarrow □_n$ is an $\infty$-equivalence.

Proof. This is essentially the same as the proof of Lemma 5.9. It is sufficient to prove for all $X$ when $n = 1$. Let $I X = X \otimes □_1$ and let $s$ be the map

$$s = \text{id}_X \otimes \{0\} : X \rightarrow IX$$

Then $\pi s = \text{id}_X$. We have a homotopy

$$\begin{array}{ccc}
X \otimes □_1 & \rightarrow & X \otimes □_1 \\
\downarrow \text{id}_{IX} \otimes \{0\} & & \downarrow \text{id}_{IX} \\
X \otimes □_1 & \rightarrow & X \otimes □_1 \\
\downarrow \text{id}_{IX} \otimes \{1\} & & \downarrow \text{id}_{IX} \\
X \otimes □_1 & \rightarrow & X \otimes □_1 \\
\end{array}$$

between $\text{id}_IX$ and $\pi s$, so $\pi$ is a homotopy equivalence. By Corollary 5.8, $i^*\pi$ is a homotopy equivalence and hence $\infty$-equivalence, so $\pi$ is an $\infty$-equivalence. □

Lemma 5.21. Suppose $i : A \rightarrow B$ and $j : K \rightarrow L$ are monomorphisms in $qΣ\text{Set}$. Then the pushout-product

$$i \circ j : A \otimes L \amalg A \otimes K B \otimes K \rightarrow B \otimes L$$

is a monomorphism.

Proof. Recall from Proposition 4.16 that

$$I_Σ = \{ (\partial □_n)H \rightarrow (□_n)H \mid n \geq 0 \text{ and } H \leq Σ_n \}$$

is a cellular model for $qΣ\text{Set}$. First, we’ll show that the pushout-product of any two maps in $I_Σ$ is a monomorphism. Suppose $n, m \geq 0$ and $H_n, H_m$ subgroups of $Σ_n$ and $Σ_m$, respectively. Let $H = H_n \times H_m$ be the subgroup of $Σ_{n+m}$ generated by the images of $H_n$ and $H_m$ under the homomorphism $Σ_n \times Σ_m \rightarrow Σ_{n+m}$. Then

$$((\partial □_n \rightarrow □_n)H_n \circ (\partial □_m \rightarrow □_m)H_m)H_n \cong ((\partial □_{n+m} \rightarrow □_{n+m})H).$$

This map is a monomorphism. A standard deduction lets us upgrade this to deduce that the pushout-product of any two monomorphisms is a monomorphism: by the small object argument applied to $I_Σ$, we know that $j$ is a monomorphism if and only if $j \pitchfork p$ for all $p \in \text{Inj} I_Σ$. □

Theorem 5.22. The spatial model structure on $qΣ\text{Set}$ is monoidal and satisfies the Schwede-Shipley monoid axiom.

Proof. That the spatial model structure is monoidal is a straightforward consequence of the fact that the generating cofibrations and acyclic cofibrations are given by left Kan extension along $i$, which is itself strong monoidal. That is, if $f$ and $g$ are cofibrations in $q\text{Set}$, then $i^*f \circ i^*g \cong i^*(f \circ g)$ is a cofibration in $qΣ\text{Set}$, acyclic if either $f$ or $g$ is.

For the monoid axiom, first note that it is sufficient to check that

$$\text{Cell} \{ X \otimes □_n \rightarrow X \otimes □_n \mid X \in \text{ob} qΣ\text{Set}, 1 \leq i \leq n \text{ and } ε = 0, 1 \}$$
comprises ∞-equivalences by \cite[Lemma 3.5 (2)]{12}. Let ε = 0 or 1 and let n > 0. For arbitrary extended cubical sets Y, the map
\[ \text{id}_Y \otimes \{\varepsilon\} : Y \otimes \Box^n \to Y \otimes \Box^n \]
is a section of an ∞-equivalence by Lemma 5.20, so it is itself a weak equivalence.

Now consider the pushout
\[ \begin{array}{c}
X \otimes \partial \Box^{n-1} \\
\downarrow g \\
X \otimes \Box^{n-1}
\end{array} \quad \begin{array}{c}
\downarrow \text{id} \otimes \{1-\varepsilon\} \\
\downarrow k \\
X \otimes \Box^{n-1}
\end{array} \quad \begin{array}{c}
\downarrow \text{id} \otimes \{1-\varepsilon\} \\
\downarrow \ell \\
X \otimes \Box^n
\end{array} \]

By the two-out-of-three axiom and Lemma 5.20 we know that ℓ is an ∞-equivalence if and only if k is an ∞-equivalence. But g is a monomorphism by Lemma 5.21 so i∗g is a monomorphism. Since i∗k is the coface change of an ∞-equivalence along a cofibration and q\Set is left proper, i∗k is an ∞-equivalence, so k is an ∞-equivalence. The cosymmetry maps allow us to permute the lower cap coordinate n.

6. Diagrams of Extended Cubical Sets and Regularity

Recall that if X is a simplicial set, there is a weak equivalence
\[ \text{hocolim} \Delta[n] \to X \]
induced by the identification of X with the colimit of its simplices. There are various ways to prove this; one method uses Reedy model structures to show that the honest colimit of the diagram of simplices of X computes the homotopy colimit.

In this section, we’ll prove an analogous formula for (extended) cubical sets: sets: these can be decomposed as the homotopy colimit of their cubes.

Suppose \mathcal{R} is a Reedy category and \mathcal{C} is a model category. The category \mathcal{C}^{\mathcal{R}} of \mathcal{R}\text{-diagrams in } \mathcal{C} may be equipped with Reedy model structure \cite{27, 26, 16}. This by now is a well-known construction; we’ve implicitly used it in describing directed colimits and pushouts of weak equivalences. We’ll give a brief overview here.

**Definition 6.1** (\cite[Chapter 15]{26}). Suppose r ∈ ob \mathcal{R}.

1. We define \partial(\mathcal{R}^+ \downarrow r) to be the full subcategory of \mathcal{R}^+ \downarrow r consisting of non-identity arrows s → r. Let F ∈ \mathcal{C}^{\mathcal{R}}. The rth latching object of F is the colimit
\[ L_rF = \text{colim}_{\partial(\mathcal{R}^+ \downarrow r)} F \in \mathcal{C}. \]

This is functorial in F. Note there is a natural map L_rF → F_r. Suppose f : F → G is an arrow in \mathcal{C}^{\mathcal{R}}. We say f is a Reedy cofibration if each corner map
\[ F_r \sqcup_{L_r(F)} L_r(G) \to G_r, \]
r ∈ ob \mathcal{R}, is a cofibration in \mathcal{C}.
(2) We define $\partial(r \downarrow \mathcal{R}^-)$ to be the full subcategory of $r \downarrow \mathcal{R}^-$ consisting of non-identity arrows $r \to s$. The $r$th matching object of $F$ is the limit

$$M_r F = \lim_{\partial(r \downarrow \mathcal{R}^-)} F \in \mathcal{C}.$$ 

This is functorial in $F$ and there is a natural transformation $(-)_r \to M_r$.

An arrow $f : F \to G$ in $\mathcal{C}^\mathcal{R}$ is a Reedy fibration if each corner map

$$F_r \to M_r F \times_{M_r G} G_r,$$

$r \in \text{ob} \mathcal{R}$, is a fibration in $\mathcal{C}$.

(3) We call a map $f : F \to G$ in $\mathcal{C}^\mathcal{R}$ an objectwise weak equivalence if $f_r : F_r \to G_r$ is a weak equivalence for all $r \in \text{ob} \mathcal{R}$.

Theorem 6.2 (26, Theorems 15.3.4, 15.3.15, 15.6.27). Suppose $\mathcal{C}$ is a model category.

(1) The category $\mathcal{C}^\mathcal{R}$ of diagrams has a model category structure with cofibrations the Reedy cofibrations, fibrations the Reedy fibrations, and weak equivalences the objectwise weak equivalences.

(2) If $\mathcal{C}$ is cofibrantly generated, the Reedy model structure on $\mathcal{C}^\mathcal{R}$ is cofibrantly generated as well.

(3) In $\mathcal{C}^\mathcal{R}$, an arrow $f : F \to G$ is an acyclic cofibration if and only if each corner map $L_r G \amalg_{L_r F} F_r \to G_r$ is an acyclic cofibration for all $r \in \text{ob} \mathcal{R}$. Dually, $f$ is an acyclic fibration if and only if each corner map $F_r \to G_r \times_{M_r F} M_r G$ is an acyclic fibration.

Recall that if $\mathcal{R}$ is a Reedy category and $X \in \hat{\mathcal{R}}$, then $\mathcal{R} \downarrow X$ is a Reedy category as well.

Proposition 6.3. Suppose $\mathcal{R}$ is an EZ Reedy category and $X \in \hat{\mathcal{R}}$. Then if $Z \in \text{ob} \mathcal{C}$ is fibrant, the constant diagram $\mathcal{R} \downarrow X \to \mathcal{C}$ on the object $Z$ is Reedy fibrant.

Proof. This is a straightforward generalization of [26, Proposition 15.10.4]. Let $c_Z : \mathcal{R} \downarrow X \to \mathcal{C}$ denote the constant diagram on $Z$. We need to check that for all $f : [r] \to X$, $Z \to M_f(c_Z)$ is a fibration in $\mathcal{C}$. Recall that the $f$th matching object is computed by a limit indexed on $\mathcal{I} = \partial(f \downarrow (\mathcal{R} \downarrow X^-))$. If $\mathcal{I}$ is empty, then $M_f(c_Z) = *$ and $Z \to *$ is a fibration by assumption. Suppose $\mathcal{I}$ is nonempty. Using the notation of Section 4, suppose $[r] \to [s_1] \to X$ are two arrows in $\mathcal{I}$. We may take the absolute pushout of $r \to s_1$ and $r \to s_2$ in $\mathcal{R}$:

$$
\begin{array}{ccc}
[r] & \xrightarrow{\sigma_1} & [s_1] \\
\sigma_2 & \downarrow & \tau_2 \\
[s_2] & \xrightarrow{\tau_1} & [f] \\
& \downarrow & \downarrow \\
& X \\
\end{array}
$$

Note that $\tau_1$ and $\tau_2$ are in $\mathcal{R}^-$. Hence $N_\mathcal{I}$ is connected. (In fact, $\mathcal{I}$ has a terminal object given by the EZ decomposition of $f$.) Write $\pi : \mathcal{I} \to *$; the functor $\pi$ is thus left cofinal, so $id \to \pi^* \pi$ is a natural isomorphism [33]. Hence $Z \to M_f(c_Z)$ is isomorphic to the identity map on $Z$, so it is a fibration. □
Let \( \partial(\mathcal{R} \downarrow r) \) denote the category of \( \mathcal{R} \)-simplices \( \mathcal{R} \downarrow \partial[r] \). This is the full subcategory of \( \mathcal{R} \downarrow r \) spanned by the objects those arrows \( x \to r \) factoring through some object \( s \), \( \deg s < \deg r \).

**Lemma 6.4** ([26, Proposition 15.2.8]). Suppose \( \mathcal{R} \) is a Reedy category and \( r \in \text{ob} \mathcal{R} \). The inclusion functor

\[
j : \partial(\mathcal{R}^+ \downarrow r) \to \partial(\mathcal{R} \downarrow r)
\]

is homotopy right cofinal.

**Proof.** For (1), let \( f : x \to r \) be a non-identity map in \( \mathcal{R} \). We factor \( f \) as \( f = f^+ f^- \), \( f^+ \in \mathcal{R}^+ \), \( f^- \in \mathcal{R}^- \). Suppose

\[
\begin{array}{ccc}
x & \xrightarrow{f} & r \\
\downarrow{k} & & \downarrow{t^+ \in \text{ar} \mathcal{R}^+} \\
\downarrow{s} & & \downarrow{t^+}
\end{array}
\]

is an object in \( f \downarrow j \). We factor \( k = k^+ k^- \), so \( f = (t^+ k^+) k^- \). Since \( \mathcal{R} \) is Reedy, \( k^- = f^- \) and \( t^+ k^+ = f^+ \), so the triangle

\[
\begin{array}{ccc}
x & \xrightarrow{f} & r \\
\downarrow{f^-} & & \downarrow{f^+} \\
\downarrow{s'} & & \downarrow{s'}
\end{array}
\]

is terminal in \( f \downarrow j \). Hence \( N(f \downarrow j) \) is contractible. \( \square \)

**Proposition 6.5.** Suppose \( \mathcal{R} \) is an EZ Reedy category and \( X \in \mathcal{R} \). Let \( \tilde{X} \) be the diagram

\[
\mathcal{R} \downarrow X \xrightarrow{\pi} \mathcal{R} \xrightarrow{[-]} \mathcal{R}
\]

For an \( r \)-simplex \( f : [r] \to X \), the Reedy map \( L_f \tilde{X} \to \tilde{X}_f \) is isomorphic to the inclusion \( \partial[r] \to [r] \).

**Proof.** The forgetful functor

\[
u : \partial((\mathcal{R} \downarrow X) \downarrow f) \to \partial(\mathcal{R} \downarrow r)
\]

has a left adjoint sending \( j : s \to r \) to

\[
[s] \xrightarrow{[j]} [r] \xrightarrow{f} X,
\]

so \( u \) is (homotopy) right cofinal and the map

\[
\text{colim} \ (\partial(\mathcal{R} \downarrow X) \downarrow f) \to \text{colim} [-]
\]

is an isomorphism. By Lemma 6.4, the Reedy map \( L_f \tilde{X} \to \tilde{X}_f \) is thus isomorphic to the map

\[
\int_{s \in \text{ob} \mathcal{R}} (\partial[r])(s) \times [s] \to \int_{s \in \text{ob} \mathcal{R}} [r](s) \times [s]
\]

This is precisely the map \( \partial[r] \to [r] \). \( \square \)
Corollary 6.6. Suppose \( X \in \text{qSet} \). Then the natural map

\[
\text{hocolim} \square^n \to X
\]

is an \( \infty \)-equivalence.

Proof. Recall that \( \mathcal{Q} \) is ez and Reedy. By Proposition 6.3, the adjoint pair

\[
\text{colim} : \text{qSet}^{\mathcal{Q} \downarrow X} \rightleftarrows \text{qSet} : c
\]

is a Quillen adjunction, so we may use the Reedy model structure on \( \text{qSet}^{\mathcal{Q} \downarrow X} \) to compute homotopy colimits. The canonical diagram taking \( \square^n \to X \) to \( \square^n \) is Reedy cofibrant by Proposition 6.5. Hence

\[
\text{hocolim} \square^n \to c \text{olim} \square^n \cong X
\]

is an \( \infty \)-equivalence. \( \square \)

Corollary 6.6 records one of the most important properties of \( \text{qSet} \): every cubical set is the homotopy colimit of its cubes. Using Cisinski’s terminology, the spatial model structure on \( \text{qSet} \) is regular. As we’ll see below, \( \text{qSet} \) is regular as well, but this is significantly more difficult to prove.

6.1. Regularity in \( \text{qSet} \). In the remainder of this section, we’ll show that

\[
\text{hocolim} \square^n_{\text{qSet}} \to X
\]

is an \( \infty \)-equivalence for all extended cubical sets \( X \), i.e., that all extended cubical sets are regular. Our proof uses the internal nerve construction of Cisinski [11, 30]:

Definition 6.7. Suppose \( \mathcal{I} \) is a small category and \( \mathcal{C} \) is a cofibrantly generated model category. The internal nerve of \( \mathcal{I} \) in \( \mathcal{C} \) at an object \( X \) is the homotopy colimit \( \text{hocolim}_{\mathcal{I}} X \) of the constant diagram at \( X \). We denote this by \( N_{\mathcal{C},X} \mathcal{I} \). Writing \( p \) for the projection \( \mathcal{I} \to \star \), we have \( N_{\mathcal{C},X} \mathcal{I} = \text{L}p p^* X \). When \( X \) is the terminal object \( \star \), we’ll abbreviate \( N_{\mathcal{C},\star} \mathcal{I} = N_{\mathcal{C},\star} \).

Example 6.8. In \( \text{sSet} \), \( N_{\text{sSet}} \mathcal{I} \) is weakly equivalent to the nerve of \( \mathcal{I} \). Using the bar resolution, these are isomorphic.

Remark 6.9. Internal nerve, as we’ve defined it, is not functorial. What we have is the following: suppose \( f : \mathcal{A} \to \mathcal{B} \) is a functor between small categories. The triangle

\[
\begin{tikzcd}
\mathcal{A} & & \mathcal{B} \\
\star & & \\
\end{tikzcd}
\]

yields a natural transformation \( \text{L}p p^* \to \text{L}q q^* \) since \( q_! f_1 \cong p_! \) and \( p^* = f^* q^* \). This may be used to give \( N_{\mathcal{C},X} \) the structure of a suitably weak 2-functor. We won’t need that here; we’ll write \( N_{\mathcal{C}} f : N_{\mathcal{C}} \mathcal{A} \to N_{\mathcal{C}} \mathcal{B} \) below, but we’ll be careful not to compose maps.

Proposition 6.10. Suppose \( f : \mathcal{A} \to \mathcal{B} \) is a functor between small categories. Then \( N_{\text{qSet}} f \) and \( N_{\text{qsSet}} f \) are \( \infty \)-equivalences if and only if \( f \) is a Thomason equivalence.
Proof. In $q\text{Set}$, $|-|$ coincides with its left derived functors as everything is cofibrant. This may not be the case in $q\Sigma\text{Set}$. However, $|-|_{q\Sigma\text{Set}}^L$ preserves and reflects weak equivalences, where $|-|_{q\Sigma\text{Set}}^L$ denotes the left derived functor of extended realization.

We have squares

\[
\begin{array}{ccc}
N_{\text{Set}} f & \to & N_{\text{Set}} \mathcal{B} \\
\sim & & \sim \\
|N_{\text{Set}} f| & \to & |N_{\text{Set}} \mathcal{B}|
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
N_{\text{Set}} f & \to & N_{\text{Set}} \mathcal{B} \\
\sim & & \sim \\
|N_{\text{Set}} f| & \to & |N_{\text{Set}} \mathcal{B}|
\end{array}
\]

commuting up to natural weak equivalence, so $N_{\text{Set}} f$ and $N_{\text{Set}} f$ are $\infty$-equivalences if and only if $f$ is a Thomason equivalence. □

Remark 6.11. Proposition 6.10 is part of a general yoga of categorical homotopy theory due to Cisinski [11, 30]: the homotopy theory of categories (i.e., spaces) intervenes in every model category via the internal nerve.

Proposition 6.12. Suppose $X$ is an extended cubical set. The natural map

\[
\text{hocolim}_{\mathcal{R} \to X} N_{\Sigma} \to X
\]

is a natural $\infty$-equivalence.

Proof. By Corollary 6.6, the map

\[
\text{hocolim}_{\mathcal{R} \to X} \to i^* X
\]

is an $\infty$-equivalence in $q\text{Set}$. Since $i_!$ is left Quillen and all cubical sets are cofibrant, the map

\[
\text{hocolim}_{\mathcal{R} \to X} \to i_! i^* X
\]

is an $\infty$-equivalence in $q\Sigma\text{Set}$. Let $G$ denote the canonical diagram of cubes of $X$:

\[
\mathcal{Q}_\Sigma \downarrow X \xrightarrow{\pi} \mathcal{Q}_\Sigma \xrightarrow{r} q\Sigma\text{Set}.
\]

Recall that $i$ induces a functor $j : \mathcal{Q} \downarrow i^* X \to \mathcal{Q}_\Sigma \downarrow X$ and that $j$ is a Thomason equivalence by Proposition 5.10. Note that $F = Gj$ is roughly the diagram of cubes of $i^* X$: it is the functor

\[
\mathcal{Q} \downarrow i^* X \xrightarrow{\pi} \mathcal{Q} \xrightarrow{r} q\Sigma\text{Set}.
\]

The natural transformation $L j j^* \to \text{id}$ induces the left arrow in

\[
\begin{array}{ccc}
\text{hocolim}_{\mathcal{Q} \downarrow i^* X} F & \to & \text{colim}_{\mathcal{Q} \downarrow i^* X} F \\
\downarrow & & \downarrow i_! i^* X \\
\text{hocolim}_{\mathcal{Q} \downarrow i^* X} G & \to & \text{colim}_{\mathcal{Q} \downarrow i^* X} G
\end{array}
\]

which commutes up to natural $\infty$-equivalence. Thus it is sufficient to show that

\[
\begin{array}{ccc}
\text{hocolim}_{\mathcal{Q} \downarrow i^* X} F & \to & \text{hocolim}_{\mathcal{Q} \downarrow i^* X} G
\end{array}
\]
Let $*$ denote the constant diagram on the terminal object in $q\Sigma\text{Set}$; then
\[
hocolim_{\partial I^n} F \longrightarrow \hocolim_{\partial I^n} X * \]
\[
hocolim_{\partial \Sigma^1} G \longrightarrow \hocolim_{\partial \Sigma^1} X * \]
commutes up to natural $\infty$-equivalence. The horizontal arrows are $\infty$-equivalences since $\hocolim$ is a homotopy functor and $\square^n \rightarrow *$ is an $\infty$-equivalence. The right vertical arrow is $N_{q\Sigma\text{Set}}$; this is an $\infty$-equivalence by Proposition 6.10.

Part 2. Extended cubical enrichments

7. Enriched model categories

Suppose $(\mathcal{V}, \otimes, e)$ is a closed symmetric monoidal model category. We assume that the monoidal structure in $\mathcal{V}$ is compatible with the model structure by requiring the usual axiom: the product $\otimes$ to be a left Quillen bifunctor, i.e., if $k : A \rightarrow B$ and $\ell : X \rightarrow Y$ are cofibrations in $\mathcal{V}$, then the pushout-product
\[
k \circ \ell = A \otimes Y \amalg_{A \otimes X} B \otimes X \rightarrow B \otimes Y \]
is a cofibration, acyclic if either $k$ or $\ell$ is. We have the following fundamental definition [27, 2]:

**Definition 7.1.** Suppose $\mathcal{C}$ is a category enriched over $\mathcal{V}$. Write $\mathcal{C}_0$ for the underlying $\text{Set}$-category of $\mathcal{C}$. We say $\mathcal{C}$ is a $\mathcal{V}$-model category if

(\*M1) $\mathcal{C}_0$ is a model category.

(\*M2) $\mathcal{C}$ has all $\mathcal{V}$-indexed limits and colimits [32].

(\*M3) The tensor functor $- \otimes - : \mathcal{V} \otimes \mathcal{C} \rightarrow \mathcal{C}$ is a left Quillen bifunctor.

There are several standard reductions of $\mathcal{V}$M3: the existence of $\mathcal{V}$-indexed limits and colimits grants adjunctions
\[
\mathcal{C}(A \otimes X, Y) \cong \mathcal{V}(A, \mathcal{C}(X, Y)) \cong \mathcal{C}(X, Y^A)
\]
where $A \in \mathcal{V}$ and $X, Y \in \mathcal{C}$. For example, we can replace $\mathcal{V}$M3 with the axiom that $\mathcal{C}(-, -)$ be a right Quillen bifunctor, i.e., if $k$ is a cofibration and $f$ a fibration, $\mathcal{C}(k, f)$ is a fibration, acyclic if either $k$ or $f$ is; in the case of cofibrant generation, we need only check axiom $\mathcal{V}$M3 for generating (acyclic) cofibrations.

Suppose $\mathcal{C}$ is a model category with a functorial “cylinder object,” i.e., for every $X$, a natural factorization of the fold map
\[
(7.1) \quad X \amalg X \xrightarrow{d_0 \amalg d_1} \text{Cyl}(X) \xrightarrow{id \amalg id} X
\]
into a cofibration followed by a weak equivalence. Then $\mathcal{C}$ is naturally enriched over $q\text{Set}$ by setting $\mathcal{C}(X, Y)_n = \text{Hom}_{q\text{Set}}(\text{Cyl}^n(X), Y)$ [11, 30]. Dually, if $X$ has a natural “path object”—a factorization
\[
(7.2) \quad X \xrightarrow{\Delta} PX \xrightarrow{\partial} X \times X
\]
of the diagonal map into a weak equivalence followed by a fibration—we may define a cubical mapping complex $\mathcal{C}(X, Y)_n = \text{Hom}_{q\text{Set}}(X, P^nY)$. As we’ll discuss in Section 9, the cubical realization $|\mathcal{C}(X, Y)|$ is a model for the Dwyer-Kan mapping.
space between $X$ and $Y$. However, the monoidal structure on $q\text{Set}$ is not symmetric, so $C$ cannot possibly be a $q\text{Set}$-model category in the sense we described above. We might try to remedy this by upgrading the enrichment of $C$ from a $q\text{Set}$-category to a $q\Sigma\text{Set}$-category. This isn’t possible in general, but we have the following principle:

**Theorem 7.2.** Let $C$ be a symmetric monoidal model category. Suppose $C$ possesses a cubical monoid

$$e \xrightarrow{d_0 \amalg d_1} I \xrightarrow{s} e$$

so that $d_0 \amalg d_1$ is a cofibration and $s$ a weak equivalence. Then $C$ is a $q\Sigma\text{Set}$-model category, with $C(X, Y)_n = \text{Hom}_C([n] \otimes X, Y)$. Moreover, the monoidal structure on $C$ is given by $q\Sigma\text{Set}$-functors.

One example is furnished by $\text{Ch}(R)$, $R$ a commutative ring: the normalized $R$-chains of $\Delta[1]$ give a cubical monoid with the appropriate homotopical properties. This Theorem amounts to the fact that the $C$-algebras of the prop $Q\Sigma$ are precisely cubical monoids. In the remainder of this section, we’ll show that the $q\Sigma\text{Set}$ mapping spaces given by Theorem 7.2 have the correct homotopy type (i.e. the homotopy type of the Dwyer-Kan mapping space) and that every combinatorial symmetric monoidal model category with cofibrant unit has an extended cubical enrichment.

### 8. Virtual cofibrance and diagram categories

As in Section 7, let $(\mathcal{V}, \otimes, e, [-, -])$ be a closed symmetric monoidal model category. The fundamental example of a $\mathcal{V}$-model category is $\mathcal{V}$ itself. In order to discuss $\mathcal{V}$-diagram categories, we need to introduce some technical model categorical material first.

**Definition 8.1.** Suppose $C$ is a $\mathcal{V}$-model category. We say an arrow $k \in \text{ar} \mathcal{V}$ is a $C$-virtual cofibration if $k \otimes f$ is an (acyclic) cofibration for all (acyclic) cofibrations $f$ in $\text{ar} C$. We say $k$ is a virtual cofibration if it is a $C$-virtual cofibration for all $\mathcal{V}$-model categories $C$.

The following Proposition is straightforward:

**Proposition 8.2.**

1. All cofibrations are virtual cofibrations.
2. The class of virtual cofibrations in $C$ is closed under coproduct, cobase change, transfinite composition, and retract.
3. Virtual (acyclic) cofibrations and (acyclic) cofibrations coincide if and only if $\emptyset \rightarrow e$ is a cofibration in $\mathcal{V}$.

Note that $\emptyset \rightarrow e$ is always a virtual cofibration, but it need not be a cofibration.

**Definition 8.3.** Suppose $\mathcal{I}$ is a small $\mathcal{V}$-category. We say $\mathcal{I}$ has virtually cofibrant mapping spaces if $\emptyset \rightarrow \mathcal{I}(x, y)$ is a cofibration for all $x, y \in \mathcal{I}$. If, furthermore, $e \rightarrow \mathcal{I}(x, x)$ is a cofibration for all $x \in \mathcal{I}$, we say $\mathcal{I}$ is well based.

**Proposition 8.4.** Suppose $\mathcal{V}$ is combinatorial (i.e., cofibrantly generated and locally presentable; see [14, 1, 34]). Let $\mathcal{I}$ is a small $\mathcal{V}$-category with virtually cofibrant mapping spaces, e.g., $\mathcal{I}$ is the free $\mathcal{V}$-category generated by a $\text{Set}$-category.
(1) The \(\mathcal{V}\)-category \(\mathcal{F} = \mathcal{V}^{\mathcal{F}^\text{op}}\) admits a cofibrantly generated model structure, known as the projective model structure, in which \(f : X \to Y\) is a fibration (respectively weak equivalence) if and only if \(f_i : X_i \to Y_i\) is a fibration (respectively weak equivalence) for all \(i \in \mathcal{I}\).

(2) Suppose further that \(\mathcal{I}\) is symmetric monoidal; then the category \(\mathcal{F}\) admits a closed symmetric monoidal structure given by Day convolution. The category \(\mathcal{F}\) with the projective model structure is then a symmetric monoidal model category. If \(\mathcal{V}\) satisfies the Schwede-Shipley monoid axiom, then so does \(\mathcal{F}\).

Proof. For 1, note that if \(K \to L\) is an acyclic cofibration in \(\mathcal{V}\) and \(i \in \mathcal{I}\), the left Kan extension

\[\mathcal{F}(-, x) \otimes K \to \mathcal{F}(-, x) \otimes L\]

must be a weak equivalence in the projective model structure on \(\mathcal{F}\) (indeed, an acyclic cofibration). The virtual cofibrance assumption for \(\mathcal{I}\) guarantees this. The combinatoriality of \(\mathcal{V}\) ensures that we may run the small-object argument.

For 2, suppose that \(\mathcal{I}\) is symmetric monoidal. Since the convolution product makes Yoneda strong monoidal, and the product on \(\mathcal{V}\) preserves colimits in each variable, for arbitrary arrows \(K \to L, A \to B\) in \(\mathcal{V}_0\) and objects \(x, y \in \mathcal{F}\), we have

\[
\left(\mathcal{F}(-, x) \otimes K \to \mathcal{F}(-, x) \otimes L\right) \odot \left(\mathcal{F}(-, y) \otimes A \to \mathcal{F}(-, y) \otimes B\right)
\]

\[
\cong \mathcal{F}(-, x \otimes y) \otimes (K \otimes B \amalg \otimes A) \otimes A \to L \otimes B).
\]

As a result, the monoidalness of the model structure on \(\mathcal{V}\) lifts to show that \(\mathcal{F}\) is a monoidal model category.

To show that \(\mathcal{F}\) satisfies the Schwede-Shipley monoid axiom, it is sufficient to check that the arrows in

\[\text{Cell}\ \left\{\mathcal{F}(-, x) \otimes F \otimes k \mid x \in \mathcal{F}, F \in \mathcal{F} \text{ and } k \text{ an acyclic cofibration}\right\}\]

are weak equivalences in \(\mathcal{F}\). Since cobase change, transfinite composition, retract and coproduct all commute with the evaluation functors \(\mathcal{F} \to \mathcal{V}\) and \(\mathcal{F}\) has the projective model structure, it is sufficient to check that

\[\text{Cell}\ \left\{(\mathcal{F}(-, x) \otimes F)_z \otimes k \mid x \in \mathcal{F}, F \in \mathcal{F} \text{ and } k \text{ an acyclic cofibration}\right\}\]

consists of weak equivalences in \(\mathcal{V}\) for all \(z \in \mathcal{F}\). This is guaranteed by the monoid axiom for \(\mathcal{V}\). \(\square\)

9. Cubical models for mapping spaces

In the series of papers [17, 19, 18], Dwyer and Kan introduced the simplicial localization of a category at a subcategory of weak equivalences. In the case of a model category \(\mathcal{C}\), the simplicial localization of \(\mathcal{C}\) at its weak equivalences \(\mathcal{W}\) associates a simplicial set \(\mathbf{F}(x, y)\) to each pair of objects \(x, y\) so that \(\pi_0 \mathbf{F}(x, y)\) corresponds to the set \(\text{Ho} \mathcal{C}(x, y)\) in a natural way. When \(\mathcal{C}\) is a simplicial model category and \(\mathbf{F}(x, y)\) is the derived mapping space \(\mathcal{C}(x', y')\) \((x'\text{ and } y'\text{ are cofibrant-fibrant replacements for } x\text{ and } y, \text{ respectively})\). In Section 7 we discussed cubical enrichments of model categories (following Cisinski). In this section, we’ll show that those enrichments have the correct homotopy type, i.e., coincide with the space \(\mathbf{F}(x, y)\) up to weak equivalence.
9.1. Quillen adjunctions between Reedy categories. The fundamental technical tool we’ll need is a comparison between cubical and simplicial framings. Conversion between the two is essentially obtained by cubical realization. Most of the material in this section has a straightforward generalization to enriched categories; we won’t need that here. Recall that a small-cocomplete and small-complete category \( \mathcal{C} \) is tensored and cotensored over \( \text{Set} \) by the copower and power operations: there are adjunctions

\[
\mathcal{C}(S \times X, Y) \cong \text{Set}(S, \mathcal{C}(X, Y)) \cong \mathcal{C}(X, Y^S)
\]

where \( S \in \text{ob} \text{Set}, X, Y \in \text{ob} \mathcal{C} \).

**Definition 9.1.** Suppose \( \mathcal{A} \) and \( \mathcal{B} \) are small categories. A distributor from \( \mathcal{A} \) to \( \mathcal{B} \) is a functor

\[
K : \mathcal{A} \times \mathcal{B}^{\text{op}} \to \text{Set}.
\]

This definition is due to Benabou (see [12]). We sometimes denote \( K \) by a dashed arrow \( \mathcal{A} \rightrightarrows \mathcal{B} \). The data of a distributor \( K : \mathcal{A} \rightrightarrows \mathcal{B} \), via the universal property of the Yoneda embedding, is equivalent to an adjunction

\[
L_K : \mathcal{C} \mathcal{A}^{\text{op}} \to \mathcal{C} \mathcal{B}^{\text{op}} : R_K.
\]

Now suppose \( \mathcal{C} \) has all small colimits and limits and \( K : \mathcal{A} \rightrightarrows \mathcal{B} \) is a distributor. Given diagrams \( X \in \mathcal{C} \mathcal{A}^{\text{op}} \) and \( Y \in \mathcal{C} \mathcal{B}^{\text{op}} \), we define

\[
(L_K, \mathcal{C} X)_b = \int_{a \in \text{ob} \mathcal{A}} K^a_b \times X_a \quad \text{and} \quad (R_K, \mathcal{C} Y)_a = \int_{b \in \text{ob} \mathcal{B}} (Y_b)_b^{K^a_b}.
\]

**Proposition 9.2.** Suppose \( K : \mathcal{A} \rightrightarrows \mathcal{B} \) is a distributor. If \( \mathcal{C} \) is small-cocomplete and complete, then

\[
L_K, \mathcal{C} : \mathcal{C} \mathcal{A}^{\text{op}} \rightrightarrows \mathcal{C} \mathcal{B}^{\text{op}} : R_K, \mathcal{C}
\]

is an adjoint pair.

This boils down to (9.1). Note that Proposition 9.2 does not have a converse in general—not all adjunctions between \( \mathcal{C} \mathcal{A}^{\text{op}} \) and \( \mathcal{C} \mathcal{B}^{\text{op}} \) are given by distributors.

Suppose \( \mathcal{I} \) is a small category. The copower operation induces a bifunctor

\[
- \otimes - : \mathcal{I} \times \mathcal{C} \rightarrow \mathcal{C} \mathcal{I}^{\text{op}}
\]

for small \( \mathcal{I} \) with \( (A \otimes X)_a = A_a \times X \). These functor is divisible on both sides [31]: abusing notation a bit, there are adjunctions

\[
\mathcal{C}(X, Y^A) \cong \mathcal{C}(A \otimes X, Y) \cong \mathcal{F}(A, [X, Y])
\]

with

\[
Y^A = \int_{a \in \mathcal{I}} (Y_a)^{A_a} \quad \text{and} \quad [X, Y]_a = \mathcal{C}(X, Y_a).
\]

In fact, the \( \otimes \) bifunctor is really part of an \( \mathcal{F} \)-enrichment on \( \mathcal{C} \mathcal{I}^{\text{op}} \) known as the external \( \mathcal{F} \)-enrichment. Since we won’t need the full power of the external enrichment, we’ve only defined \( A \otimes - \) for constant diagrams in \( \mathcal{C} \mathcal{I}^{\text{op}} \) and we’ve taken global sections in defining its right adjoint \( -^A \). Note that if \( [a] \) is the presheaf represented by \( a \), then \( Y^{[a]} \) is naturally isomorphic to \( Y_a \). The pushout-product

\[
- \odot - : \text{ar} \mathcal{F} \times \text{ar} \mathcal{C} \rightarrow \text{ar} \mathcal{C} \mathcal{I}^{\text{op}}
\]
has adjoints
\[ \text{ar}(f, (i\circ g)) \cong \text{ar}(\mathcal{C}^\text{op}(i \circ f, g) \cong \mathcal{F}(i, (g/f)). \]

Now suppose \( \mathcal{I} \) is an EZ Reedy category and \( \mathcal{C} \) is a model category. (In the remainder of this paragraph, the EZ assumption can be weakened by redefining \( \partial[i] \); see \text{[26]}.) For \( a \in \text{ob} \mathcal{I} \), recall we defined presheaves \( [a] \) and \( \partial[a] \) in Section \text{[3]} in the case of \( \mathcal{I} \). \( \partial^n = [[n]] \) and \( \partial \cap^n = \partial([n]) \). Given \( Y \in \mathcal{C}^\text{op} \), we may realize the Reedy matching object \( M_a Y \) as \( Y^\partial[a] \). A map \( g : Y \rightarrow Z \) in \( \mathcal{C}^\text{op} \) is thus a Reedy fibration if and only if
\[ \langle i_a \rangle : Y[a] \rightarrow Y^\partial[a] \times_{Z^\partial[a]} Z[a], \quad i_a = \partial[a] \rightarrow [a] \]
is a fibration in \( \mathcal{C} \) for all \( a \in \text{ob} \mathcal{I} \). What’s not obvious, but still true, is that \( g \) is a Reedy acyclic fibration if and only if \( \langle i_a \rangle \) is an acyclic fibration for all \( a \in \text{ob} \mathcal{I} \)—recall that weak equivalences in the Reedy model structure are defined objectwise, not in terms of mapping or latching objects. Hirschhorn proves this result in \text{[25] Theorem 15.3.15]; we quoted it earlier as Theorem \text{[6.2]} \). We’ll use it to compare Reedy model structures in Proposition \text{[9.4]} below.

**Lemma 9.3.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be small categories and \( K : \mathcal{A} \rightarrow \mathcal{B} \) a distributor. Suppose \( \mathcal{C} \) has all small colimits and limits. There is an isomorphism \( (R_{K,\mathcal{C}} Y)^A \cong Y^{LK A} \) natural in \( A \in \text{ob} \mathcal{A} \) and \( Y \in \text{ob} \mathcal{C}^\text{op} \).

This is an exercise in adjunctions.

**Proposition 9.4.** Suppose \( \mathcal{C} \) is a model category and both \( \mathcal{A} \) and \( \mathcal{B} \) are EZ Reedy categories. Suppose \( K : \mathcal{A} \rightarrow \mathcal{B} \) is a distributor so that \( L_K : \mathcal{A} \rightarrow \mathcal{B} \) preserves monomorphisms. Then
\[ L_K : \mathcal{C}^\mathcal{A} \rightarrow \mathcal{C}^\mathcal{B} \]
is a Quillen pair when each category is equipped with the Reedy model structure.

**Proof.** We’ll check that \( R_{K,\mathcal{C}} \) is right Quillen. Suppose \( p : Y \rightarrow Z \) is a Reedy fibration in \( \mathcal{C}^\text{op} \). By the computation in Lemma \text{[9.3]} we need \( (L_K i_a \circ p) \) to be a fibration for all \( a \), acyclic if \( p \) is acyclic (as above, \( i_a \) is the inclusion \( \partial[a] \rightarrow [a] \)). Suppose \( f : A \rightarrow B \) is a cofibration in \( \mathcal{C} \) and one of \( p \), \( f \) is a weak equivalence. It is sufficient to check that \( f \cap (L_K i_a \circ p) \), but this occurs if and only if \( (L_K i_a \cap f) \cap p \), which occurs in turn if and only if \( L_K i_a \cap (p/f) \). Since \( p \) is a Reedy fibration, \( f \cap (i_b \circ p) \) for all \( b \in \mathcal{B} \). Thus \( i_b \cap (p/f) \) for all \( b \). By Corollary \text{[4.12]} the collection of maps \( i_b \) form a cellular model for \( \mathcal{B} \): we have
\[ \text{Cell}(i_b \mid b \in \text{ob} \mathcal{B}) = \text{mono}_{\mathcal{B}}. \]

Thus \( \text{mono}_{\mathcal{B}} \cap (p/f) \). By assumption, \( L_K i_a \) is a monomorphism, so \( L_K i_a \cap (p/f). \) \( \square \)

### 9.2. Cubical and simplicial resolutions

We are now in a position to use the machinery we developed in the last section with the distributor \( K : \mathcal{Q} \rightarrow \Delta \) associated to cubical realization; this is given by
\[ K[a^n_m] = s\text{Set}(\Delta[m], \square^n) \cong (s\text{Set}(\Delta[m], \Delta[1]))^n. \]
For an arbitrary model category \( \mathcal{C} \), Proposition [Proposition 9.4] yields a Quillen adjunction between cubical and simplicial objects in \( \mathcal{C} \):

\[
L_{K,\mathcal{C}} : q^* \mathcal{C} \longrightarrow s^* \mathcal{C} : R_{K,\mathcal{C}}.
\]

**Definition 9.5 ([20 Definition 16.1.2]).** Suppose \( \mathcal{C} \) is a model category and \( X \in \mathcal{C} \) is fibrant. Write \( \text{cs} X \) and \( \text{cq} X \) for the constant simplicial and cubical diagrams, respectively, with value \( X \) in \( \mathcal{C} \). A simplicial resolution (resp. cubical resolution) of \( X \) is a weak equivalence \( \text{cs} X \rightarrow \overline{X} \) (resp. \( \text{cq} X \rightarrow X \)) in which \( \overline{X} \) (resp. \( X \)) is Reedy fibrant.

We’ll need this quick geometric lemma later:

**Lemma 9.6.** Let \( \mathcal{C} \) be a small-complete and small-cocomplete category. Suppose \( \text{cs} Y \) is the constant diagram \( \Delta^{op} \rightarrow \mathcal{C} \) on \( Y \). Then \( R_{K,\mathcal{C}} \text{cs} Y \) is isomorphic to the constant diagram \( \text{cq} Y : \mathcal{D}^{op} \rightarrow \mathcal{C} \) on \( Y \).

**Proof.** This amounts to a verification that the representable functors \( \square^n \) have connected geometric realization. First, fix \( [n] \in \text{ob} \mathcal{D} \). We have a functor \( K^{[n]} : \Delta^{op} \rightarrow \text{Set} \). Suppose that this takes values in discrete categories instead and form the Grothendieck construction \( C_n^{op} = \Delta^{op} \int K^{[n]} \). The category \( C_n \) is isomorphic to the category \( \Delta \downarrow \square^n \) of simplices of \( \square^n \). There is a zig-zag of weak equivalences from \( \text{NC}_n \) to \( \square^n \), so \( C_n \) has a weakly contractible nerve. Note that the \( C_n \) taken together form a cocubical object \( \mathcal{D} \rightarrow \text{Cat} \).

Recall that \( R_{K,\mathcal{C}} \text{cs} Y \) is the functor

\[
R_{K,\mathcal{C}} Y([n]) = \int_{[m] \in \Delta} Y^{K^{[m]}} \cong \lim_{[m] \in \Delta} Y^{K^{[m]}}.
\]

Let \( \pi : C_n \rightarrow \Delta \) denote projection and write \( c_Y : C_n \rightarrow \mathcal{C} \) for the constant functor on \( Y \). The chain of functors

\[
C_n \xrightarrow{\pi} \Delta \xrightarrow{\ast}
\]

induces an isomorphism \( \lim_{C_n} c_Y \cong \lim_{\Delta} \pi_* c_Y \), where \( \pi_* : \mathcal{C}^{C_n} \rightarrow \mathcal{C}^{\Delta} \) is right Kan extension along \( \pi \). Suppose \( [m] \in \Delta \). Viewing \( K^{[m]} \) as a discrete category, there is a functor \( \iota : K^{[m]} \rightarrow [m] \downarrow \pi \) practically by definition sending the simplex \( f : \Delta[m] \rightarrow \square^n \) to the pair \( (f, \text{id}_{[m]}) \). The functor \( \iota \) has a right adjoint sending the solid arrows in the diagram

\[
\Delta[m] \xrightarrow{\Delta[r]} \square^n
\]

to the (uniquely determined) dotted arrow: the entire triangle displays the counit of the adjunction. Thus \( \iota \) is left cofinal [33], so there is a chain of isomorphisms

\[
(\pi_* c_Y)[m] \cong \lim_{[m] \in \Delta} c_Y \cong \lim_{K^{[m]}} \iota^* c_Y \cong Y^{K^{[m]}}
\]

natural in \( Y, [m] \), and \([n] \). Hence there is an isomorphism

\[
\lim_{[m] \in \Delta} Y^{K^{[m]}} \cong \lim c_Y
\]
natural in \([n]\) and \(Y\). Since \(NC_n\) is contractible, it is nonempty and connected, so the latter limit is simply \(Y\).

The Reedy model structure on \(s\mathcal{C}\) is not compatible with the external simplicial enrichment: if \(Y\) is a Reedy fibrant simplicial object in \(\mathcal{C}\), then \(Y^n \Delta(n) \to Y^\Lambda_i(n)\) need not be an objectwise weak equivalence. However, if \(Y\) is a homotopically constant Reedy fibrant diagram, then as we’ll see below, \(Y^n \Delta(n) \to Y^\Lambda_i(n)\) is a weak equivalence. One (circular) way to think about this is that the homotopically constant Reedy fibrant diagrams comprise the fibrant objects in a Bousfield localization of \(s\mathcal{C}\) that is both compatible with the simplicial enrichment and Quillen equivalent to \(\mathcal{C}\) \([15, 41]\).

**Proposition 9.7** ([26 Theorem 16.5.7]). Suppose \(\tilde{Y}\) is a Reedy fibrant diagram in \(s\mathcal{C}\).

1. If \(A \to B\) is a monomorphism of simplicial sets, then \(\tilde{Y}^B \to \tilde{Y}^A\) is a fibration in \(\mathcal{C}\).
2. Suppose \(\tilde{Y}\) is homotopically constant, i.e., that each map \(\Delta(n) \to \Delta(m)\) in \(\Delta\) induces a weak equivalence \(\tilde{Y}_m \to \tilde{Y}_n\). If \(A \to B\) is an acyclic cofibration of simplicial sets, then \(\tilde{Y}^B \to \tilde{Y}^A\) is an acyclic fibration.

**Remark 9.8.** Proposition [9.7] is also true for homotopically constant Reedy fibrant diagrams in \(q\mathcal{C}\).

We’ll take the following definition as a sort of black box: given \(X\) cofibrant and \(Y\) fibrant in \(\mathcal{C}\), the Dwyer-Kan mapping space can be constructed by the following process: we construct a simplicial resolution \(csY \to \tilde{Y}\) and define \(F(X, Y)\) to be the simplicial set \([X, \tilde{Y}]\). (Recall from the previous section that \([X, Y]_n = \mathcal{C}(X, Y_n)\). Hirschhorn shows in [26] that this is well-defined up to weak equivalence and that it has the appropriate functorial properties. This construction is by no means the only way of getting at the homotopy type of \(F(X, Y)\). The following lemma, in a simplicial guise, is found in [26 Proposition 16.1.17].

**Lemma 9.9.** Suppose \(X \in \text{ob} \mathcal{C}\) is cofibrant and \(Y \in \text{ob} \mathcal{C}\) is fibrant. Suppose \(j_i : cqY \to \tilde{Y}_i, i = 1, 2\) are objectwise weak equivalences and each \(\tilde{Y}_i\) is Reedy fibrant. Then there is a zig-zag of weak equivalences in \(q\text{Set}\) joining \([X, \tilde{Y}_1]\) to \([X, \tilde{Y}_2]\).

**Proof.** It is sufficient to show that \([X, f]\) is a weak equivalence in \(q\text{Set}\) if \(f\) is an acyclic fibration joining Reedy-fibrant resolutions of \(cqY\) (this amounts to Ken Brown’s lemma and some other standard model-category theoretic moves). It is tempting to conclude by arguing that \([X, -]\) is a right Quillen functor \(q\mathcal{C} \to q\text{Set}\). Unfortunately, \([X, -]\) is not right Quillen as its left adjoint \(- \otimes X\) does not preserve acyclic cofibrations of cubical sets. Fortunately, we only need \(- \otimes X\) to preserve cofibrations. Let \(i_n : \partial \Box^n \to \Box^n\) be the usual inclusion. Observe that

\[
\text{a} \uplus [X, f] \text{ if and only if } \text{a} \otimes (\emptyset \to X) \uplus f \text{ if and only if } (\emptyset \to X) \uplus \langle a \setminus f \rangle.
\]

But since \(q\) is an acyclic Reedy fibration, \(\langle a \setminus f \rangle\) is an acyclic fibration in \(\mathcal{C}\). Since \(X\) is cofibrant, the equivalent conditions \([9.2]\) hold, so \([X, f]\) is an acyclic fibration in \(q\text{Set}\). \(\Box\)
Lemma 9.10. Suppose $Y \in \text{ob} \mathcal{C}$ is fibrant and $g : \text{cs} Y \to \overline{Y}$ is a simplicial resolution of $Y$. Then $R_{K, \mathcal{C}} g$ is a cubical resolution of $Y$.

Proof. Note that $R_{K, \mathcal{C}} \text{cs} Y \congcq Y$ by Lemma 9.6 and $R_{K, \mathcal{C}} \overline{Y}$ is Reedy fibrant since $R_{K, \mathcal{C}}$ is right Quillen (Proposition 9.4). What we need to check is that $R_{K, \mathcal{C}} g$ is an objectwise weak equivalence. Let $q : \square_0 \to \square^n$ be any inclusion. Consider the square

\[
\begin{array}{ccc}
(R_{K, \mathcal{C}} \text{cs} Y)_{\square^n} & \xrightarrow{(R_{K, \mathcal{C}} g)_{\square^n}} & (R_{K, \mathcal{C}} \overline{Y})_{\square^n} \\
q^* & & q^* \\
(R_{K, \mathcal{C}} \text{cs} Y)_{\square^0} & \xrightarrow{(R_{K, \mathcal{C}} g)_{\square^0}} & (R_{K, \mathcal{C}} \overline{Y})_{\square^0}.
\end{array}
\]

By our computations, this is isomorphic to

\[
\begin{array}{ccc}
Y & \xrightarrow{\overline{Y}} & \overline{Y}_{\square^n} \\
|q^*| & & |q^*| \\
Y & \xrightarrow{\overline{Y}_0} & \overline{Y}_{\square^0}.
\end{array}
\]

The bottom arrow is a weak equivalence since $\text{cs} Y \to \overline{Y}$ is a resolution; the arrow $|q^*|$ is a weak equivalence since $|q|$ is an acyclic cofibration and $\overline{Y}$ is homotopically constant. Hence the top arrow is a weak equivalence. □

Recall that Theorem 5.18 gives a triangle of Quillen equivalences

\[
\begin{array}{ccc}
\text{qSet} & \xrightarrow{i} & \text{q}_\ast \text{Set} \\
|\cdot| & & |\cdot|_\Sigma \\
\text{sSet}. & \xrightarrow{i} & \text{s}_\ast \text{Set}.
\end{array}
\]

Let’s write $\text{Sing}$ and $\text{Sing}_\Sigma$ for the right adjoints of $|\cdot|$ and $|\cdot|_\Sigma$, respectively.

Theorem 9.11. Suppose $\mathcal{C}$ is a $\text{q}_\ast \text{Set}$-model category and that $X$ and $Y$ are cofibrant and fibrant objects of $\mathcal{C}$, respectively. There is a zig-zag of natural weak equivalences joining $\text{Sing}_\Sigma F(X, Y)$ to $\mathcal{C}(X, Y)$.

Proof. Define a cubical object $\overline{Y} : \mathcal{Z} \to \mathcal{C}$ by $\overline{Y}_{[n]} = Y_{\square^n}$. Then $\overline{Y}$ is manifestly Reedy fibrant: the map $\overline{Y}([n]) \to M_{[n]} \overline{Y}$ is the fibration $Y_{\square^n} \to Y_{\partial \square^n}$. Moreover, the map $\text{cq} Y \to \overline{Y}$ induced by the projection maps $Y_{\square^n} \to Y_{\square^n}$ is a weak equivalence. Hence $\text{cq} Y \to \overline{Y}$ is a cubical resolution of $Y$. Choose a simplicial resolution $g : \text{cq} Y \to \overline{Y}$. By Lemma 9.10, $R_{K, \mathcal{C}} g$ is a cubical resolution of $Y$, so there is a zig-zag of weak equivalences joining the cubical sets $[X, R_{K, \mathcal{C}} \overline{Y}]$ and $[X, \overline{Y}]$. Observe that

\[
\text{qSet}(A, [X, R_{K, \mathcal{C}} \overline{Y}]) \cong \mathcal{C}(A \otimes X, R_{K, \mathcal{C}} \overline{Y}) \\
\cong \text{sSet}([A] \otimes X, \overline{Y}) \\
\cong \text{sSet}([A], [X, \overline{Y}]),
\]

is natural in $A \in \text{qSet}$, so there is an isomorphism

\[
[X, R_{K, \mathcal{C}} \overline{Y}] \cong \text{Sing}[X, \overline{Y}] \cong i^* \text{Sing}_\Sigma[X, \overline{Y}].
\]
Now, \([X, Y] \cong i^* \mathcal{C}(X, Y)\), so \(i^* \mathcal{C}(X, Y)\) and \(i^* \text{Sing}_\Sigma [X, Y]\) are weakly equivalent. Hence \(\mathcal{C}(X, Y)\) and \(\text{Sing}_\Sigma \mathbf{F}(X, Y)\) are weakly equivalent. □

**Remark 9.12.** We’ve used the \(q_\Sigma \text{Set}\) enrichment in order for the notational convenience. However, the assiduous reader can check that if \(\mathcal{C}\) has functorial path objects, there is a natural cotensor functor \(Y^{A} \in q_\Sigma \text{Set}\), \(X \in \mathcal{C}\) so that \(Y\) sends (acyclic) cofibrations to (acyclic) fibrations for fibrant \(Y\). The proof of Theorem 9.11 indicates that the mapping space given by \(\mathcal{C}(X, Y)_n = \text{Hom}_\mathcal{C}(X, Y^\square^n)\) has the correct homotopy type.

**10. Enrichments for symmetric monoidal model categories**

In Section 7, we gave a criterion for a symmetric monoidal model category \(\mathcal{C}\) to have an extended cubical enrichment, namely that it possesses a cubical monoid satisfying some homotopical properties. In this section, we’ll show that all combinatorial symmetric monoidal model categories with cofibrant unit have extended cubical enrichments.

**Theorem 10.1.** Suppose \((\mathcal{C}, \otimes, e, [-, -])\) is a combinatorial symmetric monoidal model category satisfying the Schwede-Shipley monoid axiom \([42]\). Suppose further that \(\emptyset \to e\) is a cofibration. Then \(\mathcal{C}\) has a cubical monoid

\[
\begin{array}{c}
e \Pi e \\
d_0 \Pi d_1 & f & \Pi \\
\text{id} \Pi \text{id} & & \mathcal{C}
\end{array}
\]

so that \(d_0 \Pi d_1\) is a cofibration and \(s\) a weak equivalence.

**Proof.** Recall that \([1] = \{0 < 1\}\) has a symmetric monoidal structure given by conjunction. We identify \([1]\) with the free \(\mathcal{C}\)-category it generates; then by Proposition 8.3 \(\mathcal{C}[1]\) is a closed symmetric monoidal model category with the convolution product. In this case, a map \(i \to j\) of arrows

\[
\begin{array}{c}
A \\
\downarrow^i
\end{array} \quad \begin{array}{c}
f \quad \downarrow^j \\
C
\end{array} \quad B \\
\downarrow^g \quad \begin{array}{c}
j \\
D
\end{array}
\]

is a cofibration if and only if both \(f\) and \(g \Pi j : B \Pi A C \to D\) are cofibrations. In particular, note that \(\emptyset \to e\) is cofibrant in \(\mathcal{C}[1]\). Now consider the category \(\text{Mon}(\mathcal{C}[1])\) of monoids in \(\mathcal{C}[1]\). This admits a model structure by \([42]\) Theorem 4.1 (3) lifted from the projective model structure on \(\mathcal{C}[1]\). One important property of the model structure on \(\text{Mon}(\mathcal{C}[1])\) is that a cofibration whose source is cofibrant in \(\mathcal{C}[1]\) is a cofibration in \(\mathcal{C}[1]\) (loc. cit.). We may thus produce a factorization

\[
\begin{array}{c}
\emptyset \\
\downarrow^i \\
e \\
\downarrow^f
\end{array} \quad \begin{array}{c}
f \quad \downarrow^e \\
X
\end{array} \quad \begin{array}{c}
\downarrow^r \\
Y \\
\downarrow^g \quad \begin{array}{c}
j \\
\text{id}_e
\end{array}
\end{array}
\]

(10.1)

of monoids in which \(f\) and \(g\) are weak equivalences, \(X\) is cofibrant, and \(j \Pi r : X \Pi e \to Y\) is a cofibration. Now consider this as a diagram in \(\mathcal{C}\). We take the
pushout of $f$ and $j$:

\[
\begin{array}{c}
\emptyset & \xrightarrow{e} & X & \xrightarrow{f} & e & \xrightarrow{id_e} & e \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\ & & \downarrow & & \downarrow & & \\
e & \xrightarrow{j} & Y & \xrightarrow{\pi} & I & \xrightarrow{s} & e \\
\downarrow & & \downarrow & & \downarrow & & \\
e & \xrightarrow{d_0} & e & \xrightarrow{id_e} & e & \xrightarrow{d_1} & \rightarrow \\
\end{array}
\] (10.2)

Let $d_1 = \pi r$. We have two things to verify:

1. $I$ is a cubical monoid. By Proposition 3.3, it is sufficient to show that $e \to I$ is a monoid in $\mathcal{C}^{[1]}$, that $d_1$ is the unit, and that $(id_e, s)$ is a monoid map. This is entirely formal.

2. $d_0 \Pi d_1$ is a cofibration and $s$ a weak equivalence. First, note $j$ is a cofibration with cofibrant source and $f$ a weak equivalence, so $\pi$ is a weak equivalence [20 Proposition 13.1.2] (see also [40]). Since $s \pi = g$ is a weak equivalence, $s$ is a weak equivalence. Now observe that

\[
\begin{array}{c}
X \xrightarrow{f \Pi id_e} Y \\
\downarrow & & \downarrow \\
e \Pi e & \xrightarrow{id_0 \Pi id_1} I \\
\end{array}
\]

is a pushout square; since $j \Pi r$ is a cofibration, $d_0 \Pi d_1$ is a cofibration as well.

\[\square\]

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