RAMIFICATION FILTRATION IN GALOIS MODULES

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Abstract. Let \( K = k_0((t_0)) \) where \([k_0 : \mathbb{F}_p] < \infty \). Denote by \( K_{tr} \) the maximal tamely ramified extension of \( K \) in \( K_{sep} \) and let \( I \subset \Gamma_K = \text{Gal}(K_{sep}/K) \) be the wild inertia subgroup, i.e. \( K_{sep}^I = K_{tr} \). Introduce the category \( M^{\text{Lie}}_{\Gamma_K} \) of finite \( \mathbb{F}_p[\Gamma_K] \)-modules \( H \) such that the image \( I(H) \) of \( \Gamma_K \) in \( \text{Aut}_{\mathbb{F}_p}(H) \) such that \( L(H)^p = 0 \). We construct a differential form \( \omega(H) = F dt_0/t_0 \) on \( L(H) \) with coefficients in \( K_{rad} = K_p^{-\infty} \) satisfying the following condition. Suppose \( F = \sum_{r \in \mathbb{Q}} t_0^{-r} l_r \), where all \( l_r \in L(H) \otimes \mathbb{F}_p \). For \( v > 0 \), consider the minimal ideal \( L(H)^{(v)} \) in \( L(H) \) such that \( L(H)^{(v)} \otimes \mathbb{F}_p \) contains all \( l_r \) with \( r \geq v \). Then the image of the ramification subgroup in the upper numbering \( \Gamma_K^{(v)} \) of \( \Gamma_K \) in \( \text{Aut}_{\mathbb{F}_p}(H) \) coincides with \( \exp(L(H)^{(v)}) \). In particular, \( \Gamma_K^{(v)} \) acts trivially on \( H \) if \( v_0(F) > -v_0 \). The form \( \omega(H) \) is defined in terms of the matrices of Frobenius and connection on the \( \phi \)-module associated with \( H \).

In the end of paper we sketch how the same result can be obtained for arbitrary \( \mathbb{Z}_p[\Gamma_K] \)-modules.

Introduction

0.1. Basic notation. Let \( K \) be a complete discrete valuation field of characteristic \( p \) with finite residue field \( k_0 \), \( K_{sep} \) – a separable closure of \( K \) and \( \Gamma_K = \text{Gal}(K_{sep}/K) \). For any \( a \in K_{sep} \), we set \( \sigma(a) = a^p \).

If \( V \) is a vector space over a field \( E \), \( \text{End}_E(V) \) is the \( E \)-algebra of linear endomorphisms of \( V \). This \( E \)-algebra also can be considered as a Lie algebra over \( E \). Similarly, \( \text{Aut}_E(V) \) is a group of \( E \)-automorphisms of \( V \). If \( F/E \) is a field extension we use the notation \( V_F := V \otimes_E F \).

We set \( \mathbb{Z}^+(p) = \{ a \in \mathbb{N} \mid \gcd(a, p) = 1 \} \) and \( \mathbb{Z}^0(p) = \mathbb{Z}^+(p) \cup \{0\} \).

0.2. Basic categories. Consider the category \( M^{\text{Lie}}_{\Gamma_K} \) of finite \( \mathbb{F}_p[\Gamma_K] \)-modules. Its objects are finite dimensional \( \mathbb{F}_p \)-modules \( H \) provided with continuous action of \( \Gamma_K \). The objects of \( M^{\text{Lie}}_{\Gamma_K} \) carry out “arithmetic” information, say, about the fields of definition of \( h \in H \). Much more detailed information, cf. [4], can be obtained if we know how the ramification subgroups in upper numbering \( \Gamma_K^{(v)}, v > 0 \), of \( \Gamma_K \) act on...
$H$. For example, the knowledge of a rational number $v_0(H)$ such that the groups $\Gamma_K^{(v)}$ act trivially on $H$ for all $v > v_0(H)$ will give upper estimates for the discriminants of the fields of definition of $h \in H$.

Let $MF_K$ be the category of $\phi$-modules. Its objects are finite dimensional $K$-vector spaces $M$ together with a $\sigma$-linear morphism $\phi : M \rightarrow M$ such that its $K$-linear extension $\phi_K : M \rightarrow M \otimes_{K,\sigma} K$ is an isomorphism.

The categories $M \Gamma_K$ and $MF_K$ are closely related. The correspondence $H \mapsto M(H) := (H \otimes_{F_p} K_{sep})^{\Gamma_K}$, where $\phi$ comes from the action of $\sigma$ on $K_{sep}$, determines equivalence of these categories. The $\phi$-module $M(H)$ carries out “analytic” information about $H$, e.g. it can be provided with a connection $\nabla : M(H)_K \rightarrow M(H)_K \otimes \Omega^1_{K}$. \cite{8}.

In this paper we describe explicitly the images of all ramification subgroups in upper numbering $\Gamma_K^{(v)}$, \cite{12}, of $\Gamma_K$ in $\text{Aut}_{F_p}(H)$ in terms related to the $\phi$-module $M(H)$. This description is done in terms of a specially constructed differential form $\omega(H)$ associated with the $\phi$-module $M(H)$, and is obtained under the following assumption:

\textbf{(Lie)} the image $I(H) \subset \text{Aut}_{F_p}(H)$ of the wild inertia subgroup $I \subset \Gamma_K$ is of the "Lie type", i.e. there is a Lie subalgebra $L(H) \subset \text{End}_{F_p}(H)$ such that $L(H)^p = 0$ and $\exp(L(H)) = I(H)$.

\textbf{Remark.} a) $L(H)^p = \{l_1 \cdots l_p \mid l_1, \ldots, l_p \in L(H)\} \subset \text{End}_{F_p}(H)$ ;

b) For any $l \in L(H)$, $\exp(l) = \sum_{0 \leq i < p} l^i/i! \in \text{Aut}_{F_p}(H)$ (here and everywhere below $\exp$ is the truncated exponential due to the assumption $L(H)^p = 0$);

c) The set $L(H)$ can be provided with the group composition law via the Campbell-Hausdorff formula (here and everywhere below log is the truncated logarithm)

$$(l_1, l_2) \mapsto l_1 \circ l_2 = \log(\exp(l_1) \cdot \exp(l_2)) = l_1 + l_2 + (1/2)[l_1, l_2] + \cdots$$

d) If $G(L(H))$ is the $p$-group from c) then $\exp : G(L(H)) \rightarrow I(H)$ is a group isomorphism.

e) If $\dim_{F_p} H \leq p$ then $H$ is of the Lie type.

Denote by $M \Gamma_K^{\text{Lie}}$ the full subcategory in $M \Gamma_K$ consisting of $F_p[\Gamma_K]$-modules $H$ which satisfy the \textbf{(Lie)} condition.

0.3. \textbf{Statement of the main result.} Denote by $K_{tr}$ the maximal tamely ramified extension of $K$ in $K_{sep}$. Then $I = \text{Gal}(K_{sep}/K_{tr})$ is the wild inertia subgroup of $\Gamma_K$.

Consider $H \in M \Gamma_K^{\text{Lie}}$ and suppose the group homomorphism $\pi_H : \Gamma_K \rightarrow \text{Aut}_{F_p} H$ determines the structure of the $\Gamma_K$-module on $H$. The main advantage of working with the objects of $M \Gamma_K^{\text{Lie}}$ is that the group $G(L(H))$ and its subgroups can be described in terms of the
Lie algebra $L(H)$ and its ideals. When studying the behaviour of the ramification filtration on $H$ we need to determine the ideals $L(H)^{(v)}$, $v > 0$, of $L(H)$ such that $\pi_H(\Gamma^{(v)}_K) = \exp(G(\ell^v_H))$. (Note that all ramification subgroups $\Gamma^{(v)}_K$ with $v > 0$ are subgroups of $I$.)

Our main result, cf. below, describes the ideals $L(H)^{(v)}$ in terms related to an explicitly constructed differential form on $L(H)$. In particular, this allows us to obtain an explicit characterisation of the biggest upper ramification number $v_0(H)$.

Fix a choice of a uniformising element $t_0 \in K$. Now $K$ appears as the field of formal Laurent series $k(t_0)$. Let $K_{rad} = K_{tr}^{\overline{p}-\infty} = K_{tr}\{[t_0^{\overline{p}-n} \mid n \in \mathbb{N}]\}$ be the radical closure of $K$.

**Theorem 0.1.** Suppose $H \in \text{MT}_K^{\text{lie}}$. Then there is $\omega_H = Fdt_0/t_0 \in L(H)_{K_{rad}} \otimes_K \Omega^1_K$ such that:

- if $F = \sum_{r \in \mathbb{Q}} t^{-r}l_r$, where $l_r \in L(H)_{F_p}$, then the ideal $L(H)^{(v)}$ appears as the minimal ideal in $L(H)$ such that its extension of scalars $L(H)_{F_p}$ contains all $l_r$ with $r \geq v$.

**Corollary 0.2.** If the normalised $t_0$-adic valuation $v_{t_0}(F) > -v_0$ then the ramification subgroup $\Gamma^{(v_0)}_K$ acts trivially on $H$. In particular, the biggest upper ramification number $v_0(H) = -v_{t_0}(F)$.

**Remark.** Describe briefly the construction of $\omega(H)$. The group morphism $I \rightarrow I(H)$ admits an explicit description via the nilpotent version of Artin-Schreier theory from [1, 2, 5]. This allows us to specify a choice of a basis $\hat{m} = (m_1, \ldots, m_N)$ of $M(H)_K$, where $M(H)$ is the $\phi$-module associated with $H$. Here $K$ is some finite tamely ramified extension of $K$. This will give us the Frobenius matrix $A \in \text{Aut}_{H_K}$ such that $\phi(\hat{m}^t) = A\hat{m}^t$. Then we obtain the connection differential form $B = -A^{-1}dA \in I(H) \otimes \Omega^1_K$. Finally, if for $m \in \mathbb{N}$, $C_m = \sigma^{-1}A \cdots \sigma^{-m}(A)$ then $\omega(H) = \lim_{m \rightarrow \infty} C_m^{-1}BC_m$. The form is related to the behaviour of ramification subgroups under the map $\pi_H : I \rightarrow I(H)$ via explicit description of the ramification filtration from [1], cf. also [5].

1. **Ramification filtration modulo $p$-th commutators**

1.1. **Algebras $L_k$ and $L$.** Let $K = k((t))$ be a complete discrete valuation field of characteristic $p$ with finite residue field $k \simeq F_p^{N_0}$, $N_0 \in \mathbb{N}$, and fixed uniformiser $t$. Let $G = \text{Gal}(K_{sep}/K)$ and let $K_{<p}$ be the maximal $p$-extension of $K$ in $K_{sep}$ with the Galois group $\text{Gal}(K_{<p}/K) = \Gamma_K/\Gamma^{p}_{K}C_p(\Gamma_K) := G_{<p}$ of nilpotence class $< p$ and exponent $p$.

Consider the decreasing filtration by ramification subgroups in the upper numbering $\{G_{<p}^{(v)}\}_{v \geq 0}$ of $G_{<p}$.

Fix $\alpha_0 \in k$ such that $\text{Tr}_{K/F_p}(\alpha_0) = 1$. 

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Let \( \tilde{L}_k \) be a profinite free Lie \( \mathbb{F}_p \)-algebra with the set of topological generators \( \{D_0\} \cup \{D_{an} \mid a \in \mathbb{Z}^+ \langle p \rangle, n \in \mathbb{Z}/N_0 \} \).

Let \( L_k = \tilde{L}_k/C_p(\tilde{L}_k) \) where \( C_p(\tilde{L}_k) \) is the ideal of \( p \)-th commutators. Define the \( \sigma \)-linear action on \( L_k \) via \( D_{an} \mapsto D_{a,n+1} \) and set \( L = L_k|_{\sigma = \text{id}} \).

For any \( n \in \mathbb{Z}/N_0 \), set \( D_{0n} = (\sigma^n(\alpha_0))D_0 \).

1.2. **Equivalence of \( p \)-groups and Lie algebras.** [10]. Let \( L \) be a Lie \( \mathbb{F}_p \)-algebra of nilpotent class \( < p \), i.e. the ideal of \( p \)-th commutators \( C_p(L) = 0 \).

Let \( A \) be an enveloping algebra of \( L \), then the elements of \( L \subset A \) generate the augmentation ideal \( J \) of \( A \). There is a morphism of \( \Delta : A \to A \otimes A \) uniquely determined by the condition \( \Delta(l) = l \otimes 1 + 1 \otimes l \) for all \( l \in L \). Then the set \( \exp(L) \mod J^p \) is identified with the set of \( " \)diagonal elements modulo degree \( p \)" consisting of \( a \in 1 + J \mod J^p \) such that \( \Delta(a) \equiv a \otimes a \mod (J^2 + 1 \otimes J + J^2)^p \).

With the above notation the functor \( L \mapsto \exp(L) \mod J^p \) determines equivalence of the categories of \( p \)-periodic groups and \( \mathbb{F}_p \)-Lie algebras of nilpotence class \( < p \).

In particular, there is a natural embedding \( L \subset A/J \) and (as earlier) the Campbell-Hausdorff formula appears as

\[
(l_1, l_2) \mapsto l_1 \circ l_2 = l_1 + l_2 + \frac{1}{2} [l_1, l_2] + \ldots, \quad l_1, l_2 \in L,
\]

from the identity \( \exp(l_1) \cdot \exp(l_2) \equiv \exp(l_1 \circ l_2) \mod J^p \). This composition law provides the set \( L \) with a group structure and we denote this group by \( G(L) \). Clearly \( G(L) \simeq \exp(L) \mod J^p \). In addition, a subset \( I \subset L \) is an ideal in \( L \) iff \( G(I) \) is a normal subgroup in \( G(L) \).

1.3. **Identification** \( \eta_L : G_{\leq p} \simeq G(L) \). Let \( L \) be a finite Lie \( \mathbb{F}_p \)-algebra of nilpotent class \( < p \) and set \( L_{\text{sep}} := L_{K_{\text{sep}}} \). The elements of \( G = \text{Gal}(K_{\text{sep}}/K) \) and \( \sigma \) act on \( L_{\text{sep}} \) through the second factor, \( L_{\text{sep}}|_{\sigma = \text{id}} = L \) and \( (L_{\text{sep}})^G = L_K \). The covariant nilpotent Artin-Schreier theory states that for any \( e \in G(L_K) \), the set

\[
\mathcal{F}(e) = \{ f \in G(L_{\text{sep}}) \mid \sigma(f) = e \circ f \}
\]

is not empty and for any fixed \( f \in \mathcal{F}(e) \), the map \( \tau \mapsto (-f) \circ \tau(f) \) is a continuous group homomorphism \( \pi_f(e) : G \to G(L) \). The correspondence \( e \mapsto \pi_f(e) \) has the following properties:

a) if \( f' \in \mathcal{F}(e) \) then \( f' = f \circ l \), where \( l \in G(L) \), and \( \pi_f(e) \) and \( \pi_{f'}(e) \) are conjugated via \( l \);

b) for any continuous group homomorphism \( \pi : G \to G(L) \), there are \( e \in G(L_K) \) and \( f \in \mathcal{F}(e) \) such that \( \pi_f(e) = \pi \);

c) for appropriate elements \( e, e' \in G(L_K) \) and \( f, f' \in G(L_{\text{sep}}) \), we have \( \pi_f(e) = \pi_{f'}(e') \) iff there is an \( x \in G(L_K) \) such that \( f' = x \circ f \) and, therefore, \( e' = \sigma(x) \circ e \circ (-x) \).
Now we apply a profinite version of this theory to the Lie algebra $\mathcal{L}$ and the element $e_\mathcal{L} = \sum_{a \in \mathbb{Z}^0(p)} t^{-a} D_{a0}$. If we fix $f_\mathcal{L} \in \mathcal{F}(e_\mathcal{L})$ then the map $\pi_\mathcal{L}(f_\mathcal{L})$ induces the group isomorphism $\eta_\mathcal{L} : \mathcal{G}_{<p} \simeq G(\mathcal{L})$.

The above special choice of $e_\mathcal{L}$ appears at the finite level as follows.

Recall that we have already assumed that $K$ has a fixed uniformiser $t$ and chosen $\alpha_0 \in k$ such that $\text{Tr}_{k/F_p} \alpha_0 = 1$.

**Definition.** An element $e \in L_K$ is special if $e = \sum_{a \in \mathbb{Z}^0(p)} t^{-a} l_{a0}^t$, where $l_{00} \in \alpha_0 L$ and for all $a \in \mathbb{Z}^+(p)$, $l_{a0} \in L_k$.

**Lemma 1.1.** Suppose $e \in L_K$. Then there is $x \in G(L_K)$ such that $(\sigma x) \circ e \circ (-x)$ is special.

**Proof.** Use induction on $s$ to prove lemma modulo the ideals of $s$-th commutators $C_s(\mathcal{L})_K$.

If $s = 1$ there is nothing to prove.

Suppose lemma is proved modulo $C_s(L_k)$.

Then there is $x \in L_K$ such that $(\sigma x) \circ e \circ (-x) = \sum_{a \in \mathbb{Z}^0(p)} t^{-a} l_{a0} + l$, where $l \in C_s(L_k)$. Using that

$$\mathcal{K} = (\sigma - \text{id}_K) \mathcal{K} \oplus (\mathbb{F}_p \alpha_0) \oplus \left( \sum_{a \in \mathbb{Z}^0(p)} kt^{-a} \right)$$

we obtain the existence of $\sigma x_s \in C(L_K)$ such that $l = \sigma(x_s) - x_s + \sum_{a \in \mathbb{Z}^0(p)} t^{-a} l_a$, where $l_0 \in \alpha_0 L$ and all remaining $l_a \in L_k$. Then we can take $x' = x - x_s$ to obtain the statement of our lemma modulo $C_{s+1}(L_K)$.

**Lemma 1.2.** Suppose $e \in L_K$ is special and $x \in L_K$. Then the element $(\sigma x) \circ e \circ (-x)$ is special iff $x \in L$ (or, equivalently, if $\sigma x = x$).

**Proof.** Use the relation cf. [2]

$$\exp(X) \exp(Y) \exp(-X) = \exp \left( \sum_{n \geq 0} \frac{1}{n!} \text{ad}^n(X)(Y) \right) \mod(\text{deg } p)$$

to prove the IF part. When proving the inverse statement we can use induction modulo the ideals $C_s(L)_K$ as follows.

Assume the lemma is proved modulo $C_s(L)_K$. Then using the IF part we can assume that $x \in C_s(L)_K$. Therefore, $e + \sigma(x) - x$ is special modulo $C_{s+1}(L)_K$, i.e. $\sigma(x) - x \in \alpha_0 C_s(L) + \sum_{a \in \mathbb{Z}^0(p)} t^{-a} C_s(L)_k$ modulo $C_{s+1}(K)$. By (1.1) it is possible only if $\sigma(x) \equiv x \mod C_{s+1}(L)_K$, i.e. $x \in C_s(L) \mod C_{s+1}(L)_K$. The lemma is proved.

The following property is obvious.

**Proposition 1.3.** Suppose $e \in L_K$ is special and given with notation of the above definition. Then the map $\pi_f(e) : \mathcal{G}_{<p} \hookrightarrow G(L)$ is given via the correspondences $D_{a0} \mapsto l_{a0}$, $a \in \mathbb{Z}^0(p)$.
1.4. The ramification ideal \( L^{(v_0)} \).

**Definition.** Let \( \vec{n} = (n_1, \ldots, n_s) \) with \( s \geq 1 \). Suppose there is a partition \( 0 = i_0 < i_1 < \cdots < i_r = s \) such that if \( i_j < u \leq i_{j+1} \) then \( n_u = m_{j+1} \) and \( m_1 > m_2 > \cdots > m_r \). Then set

\[
\eta(\vec{n}) = \frac{1}{(i_1 - i_0) \cdots (i_r - i_{r-1})!}
\]

If such a partition does not exist we set \( \eta(\vec{n}) = 0 \).

For \( \vec{a} = (a_1, \ldots, a_s) \), \( \vec{n} = (n_1, \ldots, n_s) \), set

\[
[D_{\vec{a} \vec{n}}] = [\ldots [D_{a_1 n_1}, D_{a_2 n_2}], \ldots, D_{a_s n_s}] .
\]

For \( \alpha \geq 0 \) and \( N \in \mathbb{Z}_{\geq 0} \), introduce \( F^{0}_{\alpha, -N} \in L_k \) such that

\[
F^{0}_{\alpha, -N} = \sum_{1 \leq s < p} a_1 \eta(\vec{n})[D_{\vec{a} \vec{n}}].
\]

Here:

1. \( \vec{a} = (a_1, \ldots, a_s) \), \( n_1 = 0 \) and all \( n_i \geq -N \);  
2. \( \gamma(\vec{a}, \vec{n}) = a_1 p^{n_1} + a_2 p^{n_2} + \cdots + a_s p^{n_s} \).

Note that non-zero terms in the above expression for \( F^{0}_{\alpha, -N} \) can appear only if \( 0 = n_1 \geq n_2 \geq \ldots \geq n_s \) and \( \alpha \) has at least one presentation in the form \( \gamma(\vec{a}, \vec{n}) \).

Our result about explicit generators of the ideal \( L^{(v_0)} \) such that \( \eta_L(G_{<p}^{(v_0)}) = G(L^{(v_0)}) \) can be stated in the following form.

**Theorem 1.4.** There is \( \tilde{N}(v_0) \in \mathbb{N} \) such that if \( N \geq \tilde{N}(v_0) \) is fixed then \( L^{(v_0)} \) is the minimal ideal in \( L \) such that for all \( \alpha \geq v_0 \), \( F^{0}_{\alpha, -N} \in L^{(v_0)} \).

1.5. Ramification filtration \( \{L^{(v)}\}_{v \geq 1} \). The above Theorem describes the ramification ideal for one value \( v_0 \) and requires a choice of sufficiently large natural number \( \tilde{N}(v_0) \). The following result allows us to describe the whole filtration \( \{L^{(v)}\}_{v \geq 1} \) under the assumption that we know all ramification breaks.

Suppose \( 1 = v_1 < v_2 < \ldots < v_r < \ldots \) are all jumps of the ramification filtration \( \{G_{<p}^{(v)}\}_{v \geq 1} \). (This set is discrete because the set of ramification jumps in any abelian \( p \)-extension is discrete.) Then:

- \( G_{<p}^{(v_1)} \not\supseteq \cdots \not\supseteq G_{<p}^{(v_r)} \not\supseteq \cdots \);
- \( G_{<p}^{(1)} \) is the ramification subgroup in \( G_{<p} \), \( (G_{<p} : G_{<p}^{(1)}) = p \);
- if \( r \geq 2 \) and \( v_{r-1} < v \leq v_r \) then \( G_{<p}^{(v)} = G_{<p}^{(v_r)} \).

Use the above identification \( \eta_L : G_{<p} \cong G(L) \). Then the ramification filtration appears as the ideals \( L^{(v_1)} \not\supseteq L^{(v_2)} \not\supseteq \cdots \not\supseteq L^{(v_r)} \not\supseteq \cdots \) of \( L \), where \( L^{(1)} \) is generated by all \( D_{an} \), \( a \in \mathbb{Z}^+(p) \).
Suppose $u \geq 2$. Consider the elements $F_{v_u - M_u}$, where for each $u$, $M_u$ satisfies the following inequality
\[ p^{M_u + 1}(v_u - v_{u-1}) > (p - 1)v_{u-1}. \]

**Theorem 1.5.** For $r \geq 2$, $L^{(v_r)}$ is the minimal ideal in $L$ such that $L_k^{(v_r)}$ contains all $F_{v_u - M_u}$ with $u \geq r$.

1.6. **Some relations.** Consider $e \in L_K$ from Sect. 1.3. Let $A(L)$ be the enveloping algebra of $L$ and $\tilde{A}(L) = A(L)/J(L)^p$, where $J(L)$ is the augmentation ideal in $A(L)$.

Let $A_L = \exp(e_L) \in \tilde{A}(L)_K$. Use the element $f_L \in F(e_L)$ (chosen in Sect. 1.3 to fix the isomorphism $\eta_L$) to set $\exp(f_L) = M_L \in \tilde{A}(L)_{\text{sep}}$. Note that the relation $\sigma f_L = e_L \circ f_L$ implies $\sigma(M_L) = A_L \cdot M_L$.

We set $B_L = -\exp(-e_L) \cdot d(\exp(e_L))$

**Proposition 1.6.** For $m \in \mathbb{N}$, set $\sigma^{-1}A_L \cdot \ldots \cdot \sigma^{-m}A_L = C_m$. Then we have the following relations:

(1.2) $B_L = -\exp(-e_L) \cdot d(\exp(e_L)) = -\sum_{s \geq 1} a_1 \eta(\delta_s)[D_{\tilde{a}_s}t^{-\gamma(\tilde{a}_s)}](dt/t)$

(1.3) $C_m^{-1} \cdot B_L \cdot C_m = -\sum_{\alpha > 0} F_{\alpha^{-1}}^{(m)} t^{-\alpha}$

**Proof.** For (1.2) use, cf. [7], theorem 4.22, to obtain $d \exp(e_L) = \exp(e_L) \sum_{k \geq 1} \frac{1}{k!}(-\text{ad}e_L)^{k-1}(de_L)$.

and note that $(-\text{ad}e_L)^{k-1}(de_L) = (-1)^{k-1}[e_L, \ldots, [e_L, de_L], \ldots] = [\ldots [de_L, e_L], \ldots, e_L]$ $k - 1$ times

For (1.3) we need the following relation cf. [7], Sect. 4.4

$\exp(X) \cdot Y \cdot \exp(-X) = \sum_{n \geq 0} \frac{1}{n!} \text{ad}^n(X)(Y)$.

After applying this relation to the case with $m = 1$ we obtain $\exp(-\sigma^{-1}e_L) \cdot B_L \cdot \exp(\sigma^{-1}e_L) = \sum_{s \geq 0} \eta(-1, \ldots, -1)(-1)^s \text{ad}^s(\sigma^{-1}e_L)(B_L) = \sum_{\alpha > 0} F_{\alpha^{-1}}^{(m)} t^{-\alpha}$.

Repeating this procedure we obtain relation (1.3).
2. Proof of Theorem 0.1

Let $H \in \text{MF}_K^{\text{Lie}}$ and the structure of the $\Gamma_K$-module on $H$ is given via $\pi_H : \Gamma_K \rightarrow \text{Aut}_F H$. Then $\pi_H(I) = I(H) \subset \text{Aut}_F H$, $I(H)$ is a finite group of period $p$ and nilpotency class $< p$. In addition, there is a Lie algebra $L(H) \subset \text{End}_F H$ such that $\exp : G(L(H)) \rightarrow I(H)$ is a group isomorphism.

Let $\mathcal{K} \subset K_\nu$ be a finite extension of $K$ such that $\pi_H(\Gamma_{\mathcal{K}}) = I(H)$. We can assume that $\mathcal{K} = k((t))$, where $k \cong \mathbb{F}_{p^{\nu_0}}$ and $t^{\nu_0} = t_0$. (Here $e_0$ is the ramification index for the field extension $\mathcal{K}/K$.)

By nilpotent Artin-Schreier theory for the homomorphism $\log(\pi_H) : \Gamma_{\mathcal{K}} \rightarrow G(L(H))$ there are elements $e_{\mathcal{K}} \in L(H)_\mathcal{K}$ and $f_{\mathcal{K}} \in L(H)_{\mathcal{K}}$ such that $\sigma f_{\mathcal{K}} = e_{\mathcal{K}} \circ f_{\mathcal{K}}$ and for any $\tau \in \Gamma_{\mathcal{K}}$, $\log(\pi_H(\tau)) = (f_{\mathcal{K}} \circ \tau)(f_{\mathcal{K}})$.

We can use these data to recover the $\phi$-module $M(H)_{\mathcal{K}}$ associated with $H|_{\Gamma_{\mathcal{K}}}$. Indeed, let $M_{\mathcal{K}} = \exp(f_{\mathcal{K}})$ and $A_{\mathcal{K}} = \exp(e_{\mathcal{K}})$. Then $M_{\mathcal{K}} \in \exp(L(H)_{\mathcal{K}}) \subset \text{End}_{L_{\mathcal{K}}} H_{\mathcal{K}}$ and $A_{\mathcal{K}} \in \exp(L(H)_{\mathcal{K}}) \subset \text{End}_{L_{\mathcal{K}}} H_{\mathcal{K}}$. After choosing an $\mathbb{F}_p$-basis in $H$ they both can be considered as non-degenerated matrices with coefficients in $K_{\mathcal{K}}$ and, resp., $\mathcal{K}$. Clearly, they satisfy the relations $\sigma(M_{\mathcal{K}}) = A_{\mathcal{K}} \cdot M_{\mathcal{K}}$ and for any $\tau \in \Gamma_{\mathcal{K}}$, $\tau(M_{\mathcal{K}}) = M_{\mathcal{K}} \cdot \pi_H(\tau)$. Therefore, the columns of $M_{\mathcal{K}}$ give a $\mathcal{K}$-basis of the $\phi$-module $M(H)_{\mathcal{K}}$. In particular, $A_{\mathcal{K}}$ is the matrix of Frobenius and $B_{\mathcal{K}} = -A_{\mathcal{K}} \cdot dA_{\mathcal{K}}$ is the corresponding connection.

Now we can proceed with choosing a special basis for $M(H)_{\mathcal{K}}$.

By Lemma 1.1 we can assume that $e_{\mathcal{K}}$ is special, i.e., it can be presented in the form $\sum_{a \in \mathbb{Z}^+(p)} t^{-a} l^{a}_0$, where for all $a \in \mathbb{Z}^+(p)$, $l^{a}_0 \in L(H)_{\mathcal{K}}$ and $l^{a}_{00} \in \alpha_{00} L(H)$. In particular, if we use the identification $\eta_\mathcal{L}$ from Sect.1.3 then by Prop.1.3 $\log(\pi_H|_{\Gamma_{\mathcal{K}}}) : \Gamma_{\mathcal{K}} \rightarrow G(L(H))$ comes from the morphism of Lie algebras $L_k \rightarrow L(H)_k$ such that $D_{a0} \mapsto l^{a}_0$ (and for any $n \in \mathbb{Z}/N_0$, $D_{an} \mapsto l^{a}_n := \sigma^n(l^{a}_0)$).

This implies that $\log(\pi_H)$ transforms the elements $A_{\mathcal{K}}$ and $B_{\mathcal{K}}$ from Prop.1.6 to $A_{\mathcal{K}}$ and $B_{\mathcal{K}}$. As a result, $\log(\pi_H)$ transforms the differential form $C^{-1}_m B_m C_m$ from Prop.1.6 to $\omega_m(H) = -\sum_{r > 0} t^{(m)}_r t^{-r}$, where $l_r = \log \pi_H(F_{r, -m})$. If $m \gg 0$ the elements $t^{(m)}_r$ with $r \geq v$ can be taken as generators for the ideals $L(H)^{[(v)}$ and the corresponding differential form $\omega(H) := \omega_m(H)$ describes the images of the ramification subgroups $\Gamma_{\mathcal{K}}^{[(v)}$.

It remains to note that for any $v > 0$, $\Gamma_{\mathcal{K}}^{[e_0]} = \Gamma_{\mathcal{K}}^{[(v)}$ and $t^{e_0} = t_0$.

Theorem 0.1 is proved.

Remark. a) The conjugacy class of the differential form $\omega(H)$ does not depend on a presentation of $e_{\mathcal{K}}$ in the form $\sum_{a \in \mathbb{Z}^+(p)} D_{a0} t^{-a}$. This follows directly from Lemma 1.2.

b) The above exposition appears as an interpretation of the description of the image of ramification filtration in the quotient $\Gamma_{\mathcal{K}}/\Gamma_{\mathcal{K}}^{p} C(p)(\Gamma_{\mathcal{K}})$. The corresponding Galois modules are $\mathbb{F}_p[\Gamma_{\mathcal{K}}]$-modules which satisfy
quite restrictive condition (Lie). Our approach admits a generalisation to the case of \( \mathbb{Z}_p[\Gamma_K]\)-modules, which satisfy an analogue of the (Lie) condition. Essential ingredient of this generalisation is the description of the image of ramification filtration in \( \Gamma_K / \Gamma_K^{pM} C_p(\Gamma_K) \) in [9]. We sketch the proof in Sect. 3 below.

c) It would be very interesting to verify whether our results could be established in the case of \( \Gamma_K \)-modules which do not satisfy the (Lie) condition, e.g. for the \( \Gamma_K \)-module from [9] (the case \( n = p \) in the notation of that paper).

3. The case of \( \mathbb{Z}_p[\Gamma_K]\)-modules

This case generalises the case of \( \mathbb{F}_p[\Gamma_K]\)-modules and goes very closely to it. We agree to use the notation from Sect. 1 and 2 by explaining in due course their new meaning.

Our aim is to establish Theorem 0.1 in the context of \( \mathbb{Z}_p[\Gamma_K]\)-modules \( H \) if the satisfy an analogue of the condition (Lie).

3.1. The (Lie) condition. Recall that \( K = k_0((t_0)) \), \( I \subset \Gamma_K \) is the wild inertia, \( H \) will be now \( \mathbb{Z}_p[\Gamma_K]\)-module, we have a Lie algebra \( L = L(H) \subset \text{End}_{\mathbb{Z}_p} H \) such that \( L(H)^p = 0 \), i.e. we still have the (Lie) condition, and the group epimorphism \( \pi_H : I \longrightarrow I(H) := \exp L \subset \text{Aut}_{\mathbb{Z}_p} H \) determines the \( \Gamma_K \)-module structure on \( H \).

3.2. The lifts to characteristic 0. We use the uniformiser \( t_0 \) as a \( p \)-basis for any field extension \( E \) of \( K \) in \( K_{\text{sep}} \) to define a compatible system of lifts of the fields \( E \) to characteristic 0 as follows. This is a special case of the modulo \( p^M \)-lifts from [2], where we use the \( p \)-basis of \( K \) coming from the uniformizer \( t_0 \) of \( K \).

For all \( M \in \mathbb{N} \), set \( O_M(E) = W_M(\sigma^{M-1}E)[\tilde{t}_0] \), where \( \tilde{t}_0 = [t_0] \) is the Teichmuller representative of \( t_0 \) in the ring of Witt vectors \( W_M(E) \). The algebras \( O_M(E) \) are flat \( \mathbb{Z}/p^M \)-algebras and \( O(E) = \lim_{M} O_M(E) \) is the required lift of \( E \) to characteristic 0.

Note that \( O(K) = W(k)((t)) \) is a faithfully flat \( \mathbb{Z}_p \)-algebra of Laurent series in \( \tilde{t}_0 \) with coefficients in \( W(k) \). It is easy to see that \( O(K_{\text{sep}}) |_{\sigma = \text{id}} = W(k) \subset O(K) \) and \( O(K_{\text{sep}})^{G_K} = O(K) \).

3.3. The \( \phi \)-module \( M(H) \) and the differential form \( \omega(H) \). Choose \( \mathcal{K} \subset K_{\text{er}} \) such that \( \pi_H(\Gamma_K) = I(H) \), \( \mathcal{K} = k((t)) \) with \( \hat{t}^{o_0} = t_0 \). Note that \( O(K) = W(k)((\hat{t})) \), where \( \hat{t}^{o_0} = \tilde{t}_0 \).

Then the "modulo \( p^M \)" version of the nilpotent Artin-Schreier theory from [2] implies the existence of \( e_L \in L_{O(K)} \) and \( f_L \in L_{O(K_{\text{sep}})} \) such that \( \sigma(f_L) = e_L \circ f_L \) and for any \( \tau \in \Gamma_K \), \( \tau(f_L) = f_L \circ \log(\pi_H(\tau)) \).

As earlier, the relation \( \exp(f_L) = \exp(e_L) \cdot \exp(f_L) \) determines the \( \phi \)-module \( M(H)_{O(K)} \) associated with \( H |_{\Gamma_K} \).
The connection $\nabla : M(H)_{O(\mathcal{K})} \rightarrow M(H)_{O(\mathcal{K})} \otimes \Omega^1_{\mathcal{K}}$ is determined by the condition $\phi \cdot \nabla = \nabla \cdot (\phi \otimes \phi)$. If we set $\exp(e) = A_L$ then $\nabla$ acts on $M(H)_{O(\mathcal{K})}$ via the operator $B_L$ such that

$$B_L = (\text{id} - \text{Ad}(A_{L}^{-1})p\phi)^{-1}(D_L),$$

where $D_L = -A_{L}^{-1} \cdot dA_L$ and $\text{Ad}(X)(Y) = X \cdot Y \cdot X^{-1}$. More explicitly,

$$B_L = D_L + pA_{L}^{-1} \cdot \sigma(D_L) \cdot A_L + p^2 A_{L}^{-1} \cdot \sigma(A_L)^{-1} \cdot \sigma^2(D_L) \cdot \sigma(A_L) \cdot A_L + \ldots.$$  

As earlier, set for $m \in \mathbb{N}$, $C_m = \sigma^{-1}(A_L) \cdots \sigma^{-m}(A_L)$ and

$$\omega(H) = \lim_{m \to \infty} C_m^{-1} \cdot B_L \cdot C_m.$$

3.4. Identification $\eta_L : \mathcal{G}_{<p} \simeq G(\mathcal{L})$. Recall $\mathcal{K} = k((t))$, $k = \mathbb{F}_p^{N_0}$ with $N_0 \in \mathbb{N}$. Consider a free pro-finite Lie algebra $\tilde{\mathcal{L}}'$ over $W(k)$ with the set of free (profinite) generators

$$\{D_{an} \mid a \in \mathbb{Z}^+(p), n \in \mathbb{Z}/N_0\} \cup \{D_0\}.$$

We agree to use the same symbol $\sigma$ for its extension to an automorphism of $\tilde{\mathcal{L}}'$ such that $\sigma(D_{an}) = D_{a,n+1}$ and $\sigma D_0 = D_0$. Then $\tilde{\mathcal{L}} := \tilde{\mathcal{L}}' |_{\sigma = \text{id}}$ is a free Lie algebra over $\mathbb{Z}_p$ and $\tilde{\mathcal{L}}_{W(k)} = \tilde{\mathcal{L}}'$.

Set $\mathcal{L} = \tilde{\mathcal{L}}/C_p(\tilde{\mathcal{L}})$.

Note that the images of $D_{an}$ and $D_0$ in $\mathcal{L}_{W(k)}$ (they will be denoted by the same symbols) give a fixed system of (topological) generators of $\mathcal{L}_{W(k)}$. We also choose $\alpha_0 \in W(k)$ such that $\text{Tr} \alpha_0 = 1$ (the trace for the field extension $W(k)[1/p]/\mathbb{Q}_p$). Set for $n \in \mathbb{Z}/N_0$, $D_{0n} = \sigma^n(\alpha_0)D_0$.

Consider the element $e_L = \sum_{a \in \mathbb{Z}^+(p)} \xi^{-a} D_{an} \in \mathcal{L}_{O(\mathcal{K})}$.

Fix $f_L \in G(\mathcal{O}_{\mathcal{K}, sep})$ such that $\sigma f_L = e_L \circ f_L$ and consider the map $\eta_L$ from $\Gamma_{\mathcal{K}}$ to $G(\mathcal{L})$ defined by $\eta_L : \tau \mapsto (-f_L) \circ \tau(f_L)$. Due to the nilpotent version of the Artin-Schreier theory from [2], the map $\eta_L$ provides us with the induced identification $\eta_L : \mathcal{G}_{<p} \simeq G(\mathcal{L})$, where $\mathcal{G}_{<p} := \Gamma_{\mathcal{K}}/C_p(\Gamma_{\mathcal{K}})$.

3.5. The ramification ideals $\mathcal{L}^{(\upsilon)}$. Denote by $\mathcal{G}_{<p}^{(\upsilon)}$ the image of $\Gamma_{\mathcal{K}}^{(\upsilon)}$ in $\mathcal{G}_{<p}$. Then $\eta_L(\mathcal{G}_{<p}^{(\upsilon)}) = \mathcal{L}^{(\upsilon)}$ is an ideal in $\mathcal{L}$. The ideals $\mathcal{L}^{(\upsilon)}$ were explicitly described in [3] as follows.

For $\alpha > 0$ and $N \in \mathbb{N}$, introduce $\mathcal{F}_{\alpha,-N}^0 \in \mathcal{L}_k$ such that

$$\mathcal{F}_{\alpha,-N}^0 = \sum_{\begin{smallmatrix} 1 \leq s < p \\ \bar{a}, \bar{n} \end{smallmatrix}} a_1 p^{n_1} \eta(\bar{n})[D_{\bar{a} \bar{n}}].$$

Here:

- $\bar{a} = (a_1, \ldots, a_s)$, $\bar{n} = (n_1, \ldots, n_s) \in \mathbb{Z}^s$, $n_1 \geq 0$, all $n_i \geq -N$;

- $\alpha = \alpha(\bar{a}, \bar{n}) = a_1 p^{n_1} + a_2 p^{n_2} + \cdots + a_s p^{n_s}$. 
Theorem 3.1. For any \( v \geq 0 \), there is \( \tilde{N}(v) \) such that if \( N \geq \tilde{N}(v) \) is fixed then the ideal \( \mathcal{L}^{(v)} \) is the minimal ideal in \( \mathcal{L} \) such that its extension of scalars \( \mathcal{L}^{(v)}_{W(k)} \) contains all \( F_{0,\alpha} \) with \( \alpha \geq v \).

3.6. Relation between \( \omega(H) \) and ramification ideals \( L(H)^{(v)} \).

Proceeding similarly to the proof of Prop. 1.6 we deduce:

1) if \( A = \exp(e_L) \) and \( D = -A^{-1} \cdot dA \) then
\[
B := D_L + pA_L^{-1} \cdot \sigma(D_L) \cdot A_L + p^2 A_L^{-1} \cdot \sigma(A_L)^{-1} \cdot \sigma^2(D_L) \sigma(A_L) \cdot A_L + \ldots
\]
\[
= \sum_{1 \leq s < p} a_1 p^{n_1} \eta(\bar{n}) \left[ D_{\bar{a} \bar{n}} \right].
\]

Here:
- \( \bar{a} = (a_1, \ldots, a_s), \bar{n} = (n_1, \ldots, n_s) \in \mathbb{Z}_{\geq 0}^s; \)
- \( \alpha = \alpha(\bar{a}, \bar{n}) = a_1 p^{n_1} + a_2 p^{n_2} + \cdots + a_s p^{n_s}. \)

2) For any \( m \in \mathbb{N} \), let \( C_m = \sigma^{-1}(A_L) \ldots \sigma^{-m}(A_L) \). Then the second analogue of the calculation from Prop. 1.6 shows that
\[
C_m^{-1} B_{L} C_m = \sum_{r > 0} t^{-r} \frac{F^{0}_{\bar{a} \bar{n} - m}}{l}.
\]

As a result \( \eta_L \) maps \( \sum_r t^{-r} F^{0}_{\bar{a} \bar{n} - m} d\bar{l}/\bar{l} \) to \( \omega_m(H) = \sum_r t^{-r} l_r d\bar{l}/\bar{l} \).

It remains to note that \( \bar{t}_0 = \bar{t}^{e_0} \) and for any \( v > 0, \Gamma^{(v)}_K = \Gamma^{(e_0,v)}_K \).

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