On the Camacho-Lins Neto regularity

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Abstract

We work with codimension one foliations in the projective space $\mathbb{P}^n$, given a differential one form $\omega \in H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(e))$, such differential form verifies the Frobenius integrability condition $\omega \wedge d\omega = 0$.

In this work we show that the Camacho-Lins Neto regularity, applied for $\omega$, is equivalent to the fact that every first order unfolding of $\omega$ is trivial up to isomorphism. We do this by computing the Castelnuovo-Mumford regularity of the ideal $I(\omega)$ of first order unfoldings. With this result, we are also showing that the only regular projective foliations, with reduced singular locus, are the ones that have singular locus only Kupka type singularities.

At last we use these results to show that every foliation $\varpi \in \Omega^1_{\mathbb{P}^n+1}$, with initial form $\omega$ regular and dicritical, is isomorphic to $\omega$.

1 Introduction

In [CLNS2] the authors introduced a notion of regularity associated to a foliation that gives a stability criterion of deformations of foliations. Such notion of regularity is defined in homological terms in the following way: Let $\omega$ be an integrable, homogeneous, differential 1-form of degree $e$, in $\mathbb{C}^{n+1}$. It is said that $\omega$ is regular if, for every $a < e$, the homology in degree 1 of the following complex is trivial

$$
\begin{align*}
\cdots & \longrightarrow \Omega^3_{\mathbb{C}^{n+1}}(a) \longrightarrow \Omega^1_{\mathbb{C}^{n+1}}(a+e) \\
& \uparrow X \downarrow L_X(\omega) \downarrow \eta \\
& \omega \wedge d\eta + d\omega \wedge \eta 
\end{align*}
\tag{1}
$$

where we denote the homogeneous component of the given degree in parenthesis, and $L_X(\omega)$ denotes the Lie derivative of $\omega$ with respect to $X$.

We stress the fact that, in degree equal to $e$, if $\omega$ and $\eta$ are taken to descend to the projective space $\mathbb{P}^n$, the formula above $\omega \wedge d\eta + d\omega \wedge \eta$ defines the Zariski tangent space at $\omega$ of the space of codimension one foliations in $\mathbb{P}^n$, $\mathcal{F}(\mathbb{P}^n, e)$. Also, the image of the first application, in the degree equal to $e$, defines the module of trivial deformations.
For us, it was a mystery why the regularity assumption was to check the exactness of that complex up to degree $e - 1$ and no further and that for the geometrical significance of such condition was unclear. In [Mol16, Corollary 6.9, p. 1609] it is shown that the homology of $L^\bullet(\omega, a)$ for every $a \neq e$, see (3). Even more so, in [Mol16, Section 3, p. 1598] the relation between this complex and the first order unfoldings is made explicit. In particular, it is shown that the 1-cycles of the complex $L^\bullet(\omega)$ are isomorphic to the ideal of first order unfoldings

$$I(\omega) = \{ h \in \mathbb{C}[x_0, \ldots, x_n] : \text{bd} \omega = \omega \wedge \eta \text{ for some } \eta \in \Omega^1_{\mathbb{C}^{n+1}} \},$$

By computing the Castelnuovo-Mumford regularity of $I(\omega)$ we believe to have found a satisfying geometrical interpretation for the regularity of Camacho and Lins Neto, showing that, since the Castelnuovo-Mumford regularity of $I(\omega)$ equals $e - 1$, see Theorem 1 for a complete statement, the notion of Camacho-Lins Neto regularity coincides with the fact that every unfolding is trivial up to isomorphism, see Theorem 2 for a complete statement. The classes of isomorphism of first order unfoldings can be computed with the quotient

$$I(\omega)/J(\omega) = 0,$$

where $J(\omega) = \{ i_X \omega : X \in T\mathbb{P}^n \}$ denotes the trivial first order unfoldings, see Remark 3.2.

With this last result, we are generalizing [Mol16] Theorem 6.10, p. 1609] which was referred to rational and logarithmic foliations only.

Following [MMQ15] Corollary 4.19, p. 16] we are showing that the only regular projective foliations, with singular locus, $\text{Sing}(\omega)$, when reduced, are the ones that have $\text{Sing}(\omega)$ with only Kupka type singularities.

Finally, we show that every foliation $\varpi \in \Omega^1_{\mathbb{C}^{n+1}}$ such that $\varpi = \sum_{j \geq k} \omega_j$ where the initial term $\omega := \omega_k$ is regular and descends to projective space, is isomorphic to $\omega$, see Theorem 3 for a complete statement. Using Theorem 3 we can generalize to arbitrary dimension Théorème A, (3) of [CLNRV16] in $\mathbb{C}^3$ and also we can answer a question posed on Section 2, p. 23 of [CLNS82] in the dicritical case.

1.1 Statement of the results

Let $\omega \in H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(e))$ be an integrable 1-differential form. Our first result computes the Castelnuovo-Mumford regularity of the ideal of first order unfoldings $I(\omega)$.

**Theorem 1.** Let $\omega \in H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(e))$ be an integrable one differential form of degree $e$, then the Castelnuovo-Mumford regularity of the ideal $I(\omega)$ is equal to $e - 1$.

As a corollary of the result in the previous Theorem we can state our main Theorem.

**Theorem 2.** Let $\omega \in H^0(\Omega^1_{\mathbb{P}^n}(e))$ be an integrable one differential form, then $\omega$ is regular in the sense of Camacho-Lins Neto if and only if $I(\omega)/J(\omega) = 0$. 

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This last result has, as a corollary, the following Theorem.

**Theorem 3.** Let \( \varpi \) be an integrable 1-form on \( \mathbb{C}^{n+1} \), with a singularity on \( 0 \in \mathbb{C}^{n+1} \), \( \varpi = \sum_{j \geq k} \omega_j \) with \( \omega_j \) homogeneous polynomial form of degree \( j \). Suppose \( \omega := \omega_k \) is dicritical and regular. Then \( \varpi \) is isomorphic to \( \omega \).

## 2 Preliminaries

### 2.1 Integrable 1-forms in \( \mathbb{P}^n \)

We would like to state here that we will always consider \( \mathbb{P}^n \) with \( n \geq 3 \).

We consider a section \( \omega \in H^0(\mathbb{P}^n, \Omega^1(e)) \) as a 1-form in \( \mathbb{C}^{n+1} \)

\[
\omega = A_0 \, dx_0 + \cdots + A_n \, dx_n
\]

where \( A_0, \ldots A_n \) are homogeneous of degree \( e - 1 \) and satisfying

\[
i_R \omega = \sum_{j=0}^n x_j \, A_j = 0,
\]

where \( R = \sum_{j=0}^n x_j \frac{\partial}{\partial x_j} \) is the radial vector field and we denote \( S = \mathbb{C}[x_0, \ldots, x_n] \) to the homogeneous coordinate ring of \( \mathbb{P}^n \).

We recall that a codimension one foliation in \( \mathbb{P}^n \) is given by a differential one form \( \omega \in H^0(\mathbb{P}^n, \Omega^1(e)) \) satisfying the Frobenius integrability condition which is given by the formula

\[
\omega \wedge d\omega = 0,
\]

such \( \omega \) will be called integrable from now on.

**Remark 2.1.** We would like to state here that we will always consider foliations \( \omega \) such that \( \text{codim} (\text{Sing}(\omega)) \geq 2 \). By doing this, we know that if \( \omega \wedge \eta = 0 \) then \( \eta \) is a multiple of \( \omega \). See, for example, [MMQ15, p. 5].

**Definition 2.2.** We define the graded ideals of \( S \) associated to \( \omega \in H^0(\Omega^1(e)) \) as

\[
I(\omega) := \{ h \in S : \ h \, d\omega = \omega \wedge \eta \text{ for some } \eta \in \Omega^1_S \} \\
J(\omega) := \{ i_X(\omega) \in S : \ X \in T_S \}.
\]

\( I(\omega) \) and \( J(\omega) \) are the ideals of first order unfoldings and the ideal of trivial first order unfoldings, respectively. We will also denote \( I = I(\omega) \) and \( J = J(\omega) \) if no confusion arises.

**Remark 2.3.** Note that by contracting \( \omega \) with the vector fields \( \frac{\partial}{\partial x_i} \), for \( i = 0, \ldots, n \), we get that \( J(\omega) \) defines the singular locus of \( \omega \).

Also, notice that by contracting the integrability condition by a vector field \( X \), one can see that \( J(\omega) \subset I(\omega) \). This implies that the variety defined by the ideal \( I(\omega) \) has codimension \( \geq 2 \).
2.2 Arithmetically Cohen-Macaulay subschemes and Castelnuovo-Mumford regularity

We would like to state the following property relative to arithmetically Cohen-Macaulay varieties, aCM from now on: from [Har09, Proposition 8.6, p. 63] we know that a subvariety $Y \subset \mathbb{P}^n$ is aCM if and only if

$$H^i_*(I_Y) = 0,$$  \hspace{1cm} \text{for } 1 \leq i \leq \dim(Y), \hspace{1cm} (2)$$

where we are using the notation $H^i_*(F) := \bigoplus \ell \in \mathbb{Z} H^i(\mathbb{P}^n, F(\ell))$ and we are denoting $I_Y$ as the sheaf of ideals associated to the variety $Y$.

The Castelnuovo-Mumford regularity of a scheme $Y \subset \mathcal{O}_{\mathbb{P}^n}$ is defined as the smallest integer $r$ such that it is $r$-regular, meaning that

$$H^i(I_Y(r - i)) = 0$$

for all $i > 0$, where, as before, we are denoting $I_Y$ as the ideal sheaf of the variety $Y$.

At this point we would like to recall the following useful proposition:

**Proposition 2.4.** If $r$ is the Castelnuovo-Mumford regularity of a saturated ideal $I$, then $I(k) = \mathbb{C}[x_0, \ldots, x_n](k - r).I(r)$, i.e., the ideal is generated by the homogeneous components of degree $r$.

**Proof.** See [BS87] Definition (1.1) and paragraphs below, p. 3. \hfill \Box

3 Main Theorem

In this section we will develop our main result, which is that the Camacho-Lins Neto regularity is equivalent to the fact that every unfolding is trivial up to isomorphism. For that, we first need to recall some results that can be found in [CAMQ16].

We recall to the reader that we will always consider $\mathbb{P}^n$ with $n \geq 3$.

From [CAMQ16] Proposition 3.2, p. 7 we have that the unfoldings ideal $I$ is aCM. For that, let us denote $\mathcal{I}$ as the sheafification of such ideal:

**Theorem 3.1.** Let $\omega \in \mathcal{F}(\mathbb{P}^n, e)$, then $\mathcal{I}$ is aCM.

For the demonstration of the following, which is one of our main results, in the first part, we will recall the proof of the previous Theorem.

**Theorem 1.** Let $\omega \in \mathcal{F}(\mathbb{P}^n, e)$ then the Castelnuovo-Mumford regularity of $I(\omega)$ is $e - 1$.

**Proof.** From Theorem 3.1 and by Section 2.2 it remains to see when

$$H^{n-1}(\mathcal{I}(r - (n - 1))) = 0 \quad \text{and} \quad H^n(\mathcal{I}(r - n)) = 0.$$

We recall [CAMQ16] Proposition 3.1, p. 7 from where we know that the ideal $I$ is saturated which implies that $H^0(\mathcal{I}) = I$. As we just said, we will follow
the proof of Theorem 3.1 from [CAMQ16, Proposition 3.2, p. 7], that is, first we take an $r$-Čech cocycle $\{h_R\}$ in the usual covering of $\mathbb{P}^n$ given by the open sets $U_i := \{(x_0 : \cdots : x_n) \mid x_i \neq 0\}$, with $h_R$ in the localization $I_{x_i}(\omega)(e')$ where we are denoting as $x_R = \prod_{r \in R} x_r$ and $U_R = \{(x_0 : \cdots : x_n) \mid x_r \neq 0, \text{ for } r \in R\}$. Then we relate this cocycle with a cocycle in the homology of $\mathcal{O}_{\mathbb{P}^n}(e')$ which is trivial. Because of that, we know that it has a $r - 1$-cocycle $\{h_Q\}$ that goes to $\{h_R\}$, but it remains to attach a differential 1-form $\eta_Q$ such that

$$h_Q d\omega = \omega \wedge \eta_Q$$

to see that $h_Q \in I_{x_Q}(\omega)(e')$.

Since $h_Q|_{\mathcal{O}_{\mathbb{P}^n}(e')} = h_R$, where $Q \cup \{r\} = R$, there exists an $n_r \in \mathbb{N}$ such that $x_r^{n_r} h_Q = h_R$, then by using the equation

$$x_r^{n_r} h_Q d\omega = \omega \wedge \eta_R,$$

we can take $\eta_r := \frac{1}{n_r} \eta_R$ on $\Omega^1_S(e')(U_R)$, where the symbol $\sim$ denotes the sheafification of the given module, and we get a family of differential 1-forms $\{\eta_r\}_{r \in \{0, \ldots, n\}}$ that defines a Čech $r$-cocycle on the sheaf $\Omega^1_S(\omega)$. Note that, since $\text{codim}(\text{Sing}(\omega)) \geq 2$ there exists a 1-1 relation between $h_R \in I_{x_R}(e')$ and the $\eta_R \in \left(\Omega^1_{S_R}(\omega)\right)$ such that $h_R d\omega = \omega \wedge \eta_R$, see Remark 2.1. Then it only remains to compute the $H^i(\Omega^1_S(\omega))$. To do so, we consider the exact sequence

$$0 \longrightarrow \mathcal{O}(-e) \xrightarrow{\partial} \Omega^1_S \xrightarrow{\omega} \Omega^1_S(\omega) \longrightarrow 0$$

and taking the long exact sequence of cohomology after twisting the complex by $e'$, we get that $H^i(\Omega^1_S(\omega)(e')) \simeq H^i(\Omega^1_S(e')) = 0$, for $1 \leq i \leq n - 2$. To compute $H^i(\Omega^1_S(e'))$ we use that $\Omega^1_S$ is a free sheaf, that can be written as $\Omega^1_S \simeq \bigoplus^{n+1} \mathcal{O}_{\mathbb{P}^n}$.

Finally, in the last row of the cohomology sequence we have that

$$\delta H^{n-1}(\mathcal{O}(e' - e)) \xrightarrow{\omega} H^{n-1}(\Omega^1_S(e')) \xrightarrow{\delta} H^{n-1}(\Omega^1_S(\omega)(e')) \xrightarrow{\delta} H^n(\mathcal{O}(e' - e)) \xrightarrow{\omega} H^n(\Omega^1_S(e')) \xrightarrow{\delta} H^n(\Omega^1_S(\omega)(e')) \xrightarrow{\delta} 0$$

and what we want to see is when $H^n(\mathcal{O}_{\mathbb{P}^n}(e' - e)) = 0$. Since then there exists an $\eta_Q$ that is related to the $h_Q$ showing that $h_Q \in I_{x_Q}(\omega)(e')$ and concluding that $H^{n-1}(\mathcal{O}(\mathcal{F}(r))) = 0$. Taking $e' = r - (n - 1)$ and by [Har77, Theorem 5.1, d, p. 225], we know that

$$H^n(\mathcal{O}_{\mathbb{P}^n}(-\ell - n - 1)) = H^n(\mathcal{O}_{\mathbb{P}^n}(r - (n - 1) - e))$$

is in a perfect pairing with $H^0(\mathcal{O}_{\mathbb{P}^n}(\ell))$. Then, if $\ell = e - r - 2 < 0$ we are done, meaning that $e - 1 \leq r$.

To see when $H^n(\mathcal{F}(r - n)) = 0$ we proceed as follows: we consider the short exact sequence

$$0 \longrightarrow \mathcal{F} \xrightarrow{\iota} \mathcal{O}_{\mathbb{P}^n} \xrightarrow{\mathcal{F}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^n}/\mathcal{F} \longrightarrow 0$$

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from where we get the long exact sequence of cohomology

\[ \cdots \rightarrow H^{n-1}(\mathcal{I}(r - n)) \rightarrow H^{n-1}(O_{P^n}(r - n)) \xrightarrow{\pi} H^{n-1}(O_{P^n}/(r - n)) \rightarrow \]

\[ \rightarrow H^n(\mathcal{I}(r - n)) \rightarrow H^n(O_{P^n}(r - n)) \xrightarrow{\pi} H^n(O_{P^n}/(r - n)) \]

Since \( \dim(O_{P^n}/\mathcal{I}) \leq n - 2 \), by Remark 2.3, then \( H^{n-1}(O_{P^n}/(r - n)) = 0 \) and \( H^n(O_{P^n}/(r - n)) = 0 \). Now, to see when \( H^n(O_{P^n}(r - n)) = 0 \), as before, we look at its perfect pairing with \( H^0(O_{P^n}(\ell)) \), where

\[ -\ell - n - 1 = r - n. \]

From this equality, we get that \( \ell = -r - 1 \) and if \(-r - 1 < 0\) then \( H^0(O_{P^n}(\ell)) = 0 \), from where we see that \( 0 \leq r \).

Now, the smallest \( r \) such that \( H^i(\mathcal{I}(r - i)) = 0 \), for all \( i \geq 1 \), is \( r = e - 1 \). \( \square \)

Before stating Theorem 2, we should recall the complex \( L^\bullet(\omega) \):

\[ L^\bullet(\omega) : \quad T_S \xrightarrow{d\omega \wedge} \Omega_S^1 \xrightarrow{d\omega \wedge} \Omega_S^2 \xrightarrow{d\omega \wedge} \cdots \]

where \( L^\bullet(\omega) = \Omega_S^{2s-1} \) for \( s \geq 0 \) and the 0-th differential is defined as \( d\omega \wedge X := i\chi d\omega \). The grading of \( L^\bullet(\omega) \) is given by the decomposition \( L^\bullet(\omega) = \bigoplus_{a \in \mathbb{N}} L^\bullet(\omega)(a) \), where \( L^\bullet(\omega)(a) \) is the complex of finite vector spaces

\[ L^\bullet(\omega, a) : \quad T_S(a - e) \xrightarrow{d\omega \wedge} \Omega_S^1(a) \xrightarrow{d\omega \wedge} \Omega_S^2(a + e) \xrightarrow{d\omega \wedge} \cdots \quad (3) \]

Remark 3.2. The homology in first degree of that complex is easily seen to be isomorphic to the classes of isomorphism of first order unfoldings, that can be seen in terms of the quotient of the ideals \( I(\omega) \) and \( J(\omega) \), this follows from [Mol16 Corollary 3.10, p. 1600].

From [Mol16 Theorem 6.8, p. 1608, Corollary 3.10, p. 1600] we know that the complex defined in (1) has the homology isomorphic to \( (I(\omega)/J(\omega))(a) \), provided that \( a \neq e \). Furthermore, by [Mol16 Proposition 3.6, p. 1599, Theorem 3.9, p. 1600], the borders of the first application are isomorphic to \( J(\omega)(a) \) and the cycles of the second application are isomorphic to \( I(\omega)(a) \), always in the case \( a \neq e \).

Theorem 2. Let \( \omega \in \mathcal{F}(P^n, e) \). Then \( \omega \) is regular in the sense of Camacho-Lins Neto if and only if

\[ I(\omega)/J(\omega) = 0. \]

Proof. If \( I(\omega)/J(\omega) = 0 \), then, by the above results, we have that \( \omega \) must be regular.

Reciprocally, if we suppose that \( \omega \) is regular, then \( (I(\omega)/J(\omega))(a) = 0 \), for \( a \leq e - 1 \). Since this last degree coincides with the Castelnuovo-Mumford regularity of the ideal \( I(\omega) \), and \( J(\omega)(e - 1) = I(\omega)(e - 1) \), then, following Proposition 2.4, we have that \( I(\omega)/J(\omega) = 0 \). \( \square \)
4 Determinacy of foliations with regular initial form

The problem of determinacy of a complex analytic foliation germ is addressed in numerous works, see for instance Suw95, CLN82, CLNRV16 and references therein. The general problem of determinacy is to decide if an integrable form ω is isomorphic to a given finite jet \( J^k(\omega) \). A particular question is whether an integrable 1-form \( \tilde{ω} \) is isomorphic to the initial form of the Taylor development of \( \omega \). The naive idea behind much of the work in this direction is that if \( \omega = \sum_{j \geq k} \omega_k \) is the Taylor development then

\[
\tilde{ω} := \omega_k + t\omega_{k+1} + \cdots + t^j\omega_{k+j} + \ldots
\]

is a family of 1-forms depending on a parameter \( t \) such that \( \tilde{ω}(x, 0) = \omega_k \) and \( \tilde{ω}(x, 1) = \omega \), so that many of the features of forms that are stable within families should be the same for \( \omega \) and \( \omega_k \). Here we are going to make this idea rigorous and use it together with our results on regularity of forms to prove that \( \omega \) is isomorphic to \( \omega_k \) in the case where \( \omega_k \) descends to projective space and is regular.

First we write down some elementary results on analytic forms whose proof we include for the sake of completeness.

**Lemma 4.1.** Let \( ω \) be a homogeneous polynomial form of degree \( k \) on \( \mathbb{C}^{n+1} \), and let \( \tilde{ω} \) be a germ of differential 1-form on \( (\mathbb{C}^{n+1}, 0) \). Suppose \( \omega = \omega + \sum_{j > k} \omega_j \) with \( \omega_j \) homogeneous of degree \( j \). Then the series \( \tilde{ω} := \omega + \sum_{j > k} t^j\omega_j \) converges in an open set \( U \times V \subseteq \mathbb{C}^{n+1} \times \mathbb{C} \) where \( U \) is some neighborhood of \( 0 \in \mathbb{C}^{n+1} \) and \( V \) contains the unit disk \( D = \{ z \in \mathbb{C} : |z| < 1 \} \).

**Proof.** Let \( \omega_k = \omega \) and denote by \( \omega_j = \sum_{0 \leq i \leq n} a_j^i(x_i)dx_i \) the polynomial coordinates of the forms \( \omega_j \) \( (j \geq k) \), so each \( a_j^i \) is a homogeneous polynomial of degree \( j \). As \( \omega \) is a germ of 1-form there is a ball \( B_r = \{ x \in \mathbb{C}^{n+1} : |x| < r \} \) such that \( \sum_{j \geq k} a_j^i(x) \) is absolutely convergent for all \( 0 \leq i \leq n \) and all \( x \in B_r \). We define the nonnegative real number \( L := \sup_{x \in B_r} \limsup_j \sqrt{|a_j^i(x)|} \). In particular we have \( L \leq 1 \). Now, as the \( a_j^i \) are homogeneous of degree \( j \), if we write \( x = r \cdot \theta \) with \( r = |x| \) and \( \theta \in S^{2(n+1)-1} \), where \( S^{2(n+1)-1} \) is the \( 2(n+1)-1 \)-sphere, we have continuous functions \( C_j^i(\theta) \) such that \( |a_j^i(x)| = r^j \cdot C_j^i(\theta) \). So

\[
\max_{x \in B_r} \limsup_j \sqrt{|a_j^i(x)|} = \max_{\theta \in S^{2(n+1)-1}} \limsup_j \left( r^j \cdot C_j^i(\theta) \right).
\]

If \( L = 1 \) we can take \( r' < r \) and \( B_{r'} = \{ x \in \mathbb{C}^{n+1} : |x| < r' \} \) so

\[
L' := \sup_{x \in B_{r'}} \limsup_j \sqrt{|a_j^i(x)|} = \frac{r'}{r} L < 1.
\]

By the root test on convergence of series, the last inequality implies that \( \tilde{ω}(x, t) \) converges absolutely for \( x \in B_{r'} \) and \( |t| < 1/L' \).

**Lemma 4.2.** Let \( \omega \), \( \tilde{ω} \) and \( \omega_k \) be as before, if \( \omega \) (and therefore \( \omega_k \)) is integrable so is \( \tilde{ω} \).
Proof. A straightforward calculation shows that, if $\omega \land d\omega = 0$, then
\[ \tilde{\omega} \land d\tilde{\omega} = (\tilde{\omega} \land L_{\frac{\partial}{\partial t}}\tilde{\omega}) \land dt = 0. \]

In other words, what lemma 4.1 and 4.2 are saying is that a germ of integrable 1-form $\omega$ with initial form $\omega$ parametrized by an open set $V \subseteq C$ containing the unit disk such that $\tilde{\omega}(x,1) = \omega(x)$. 

**Proposition 4.3.** Let $\omega \in \Omega^1_S$ be a homogeneous polynomial 1-form of degree $k$. Suppose $\omega$ is dicritical i.e.: $i_R\omega = 0$ where $R$ is the radial vector field of $\mathbb{C}^{n+1}$. Then $\omega$ is regular in the sense of Camacho-Lins Neto if and only if every unfolding of $\omega$ as a local foliation in $(\mathbb{C}^{n+1}, 0)$ is trivial.

**Proof.** By Theorem 2 since $\omega$ descends to the projective space $\mathbb{P}^n$, $\omega$ is regular if and only if the graded $S$-module $I(\omega)/J(\omega)$ is trivial. Now let $O_{\mathbb{C}^{n+1}, 0}$ denote the ring of germs of holomorphic functions on $(\mathbb{C}^{n+1}, 0)$. We define the following ideals of $O\mathbb{C}^{n+1,0}$
\[
\tilde{I}(\omega) := \left\{ h \in O\mathbb{C}^{n+1,0} : h \cdot d\omega = \omega \land \eta \text{ for some } \eta \in \Omega^1_{\mathbb{C}^{n+1,0}} \right\} \\
\tilde{J}(\omega) := \{ i_X(\omega) \in O\mathbb{C}^{n+1,0} : X \in T\mathbb{C}^{n+1,0} \}.
\]

By [Suw93] Proposition 4.14, p. 830, the set of isomorphism classes of first order unfoldings of $\omega$ as a local form can be naturally identified with the vector space $\tilde{I}(\omega)/\tilde{J}(\omega)$. As $\omega$ is polynomial homogeneous, if $h \in \tilde{I}(\omega)$ then the initial part $in(h)$ of $h$ (that is the homogeneous part of lesser degree of the Taylor expansion of $h$) is in the graded ideal $I(\omega)$. Also note that $I(\omega) \subset \tilde{I}(\omega)$, so if $\tilde{I}(\omega) = (h_1, \ldots, h_r)$ then $(h_1, \ldots, h_r) = (in(h_1), \ldots, in(h_r)) = O\mathbb{C}^{n+1,0} \cdot I(\omega)$. And, as we clearly have $\tilde{J}(\omega) = O\mathbb{C}^{n+1,0} \cdot J(\omega)$, then $\tilde{I}(\omega)/\tilde{J}(\omega) = 0$. The above implies that the trivial unfolding of $\omega$ as a local form on $(\mathbb{C}^{n+1}, 0)$ is infinitesimally versal, by [Suw93] Theorem 4.19, p. 832 this in turn implies the trivial unfolding is versal, so every unfolding of $\omega$ as a local form is trivial. 

As a corollary of this proposition we have the following statement about the determinacy of germs of foliations in $\mathbb{C}^{n+1}$

**Theorem 3.** Let $\omega$ be an integrable 1-form on $\mathbb{C}^{n+1}$, with a singularity on $0 \in \mathbb{C}^{n+1}$, $\omega = \sum_{j \geq k} \omega_j$ with $\omega_j$ a homogeneous polynomial form of degree $j$. Suppose $\omega := \omega_k$ is dicritical and regular. Then $\omega$ is isomorphic to $\omega$.

**Proof.** By Lemmas 4.1 and 4.2 there is an unfolding $\tilde{\omega}$ of $\omega$ parametrized by an open set $V \subseteq \mathbb{C}$ containing the unit disk and such that $\tilde{\omega}(x,1) = \omega(x)$. By Proposition 4.3 such an unfolding must be trivial, hence the foliation defined by $\omega$ is isomorphic to that defined by $\omega$.

Notice that, as a foliation on $\mathbb{P}^2$ without centers or nilpotent singularities have only Kupka singularities, it is defined by a regular dicritical form in $\mathbb{C}^3$. In particular Theorem 3 generalizes [CLNR16] Théorème A, (3)] to arbitrary dimensions.

Also, Theorem 3 answers a question posed in [CLNS2] Section 2, p. 23 for the case of forms with regular and dicritical initial form.
Corollary 4.4. If an integrable 1-form germ $\varpi$ in $\mathbb{C}^{n+1}$ ($n > 1$) is such that its initial form $\omega$ is regular, dicritical and can be reduced to $l$ variables, then $\varpi$ can be reduced to $l$ variables as well.

Proof. The corollary follows immediately as $\varpi$ and $\omega$ are isomorphic. 

In conclusion we believe these results shed light to the intimate relation between the Camacho-Lins Neto regularity, the theory of first order unfoldings and the problem of determinacy of local foliations.

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