Existence, nonexistence, and asymptotic behavior of solutions for $N$-Laplacian equations involving critical exponential growth in the whole $\mathbb{R}^N$

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Abstract
In this paper, we are interested in studying the existence or non-existence of solutions for a class of elliptic problems involving the $N$-Laplacian operator in the whole space. The nonlinearity considered involves critical Trudinger–Moser growth. Our approach is non-variational, and in this way we can address a wide range of problems not yet contained in the literature. Even $W^{1,N}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$ failing, we establish $\|u\|_{L^\infty(\mathbb{R}^N)} \leq C \|u\|_{W^{1,N}(\mathbb{R}^N)}^\Theta$ (for some $\Theta > 0$), when $u$ is a solution. To conclude, we explore some asymptotic properties.

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1 Introduction
Let $W^{1,p}_0(\Omega), \Omega \subseteq \mathbb{R}^N, N \geq 2$, be the Sobolev space endowed with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \left(\int_\Omega (|\nabla u|^p + |u|^p)dx\right)^{1/p}.$$
If \( p < N \), the critical growth \( p^* = Np/(N - p) \) means that \( W^{1,p}_0(\Omega) \hookrightarrow L^q(\Omega) \), \( N \leq p \leq p^* \). The case \( p = N \) is a borderline case in the sense of Sobolev embeddings. As is well-known, one has \( W^{1,N}_0(\Omega) \hookrightarrow L^q(\Omega), q \geq N, \) but a function in \( W^{1,N}(\Omega) \) may have a local singularity and this causes the failure of the embedding \( W^{1,N}(\Omega) \not\subset L^\infty(\Omega). \) If \( \Omega \) is a bounded domain, another kind of maximal growth was proved by Yudovič [40], Pohožaev [37] and Trudinger [38]. They proved, in an independent way, that

\[
\sup_{\|\nabla u\|_{L^N(\Omega)} \leq 1} \int_{\Omega} e^{\alpha |u|^{N'}} \, dx \begin{cases} \leq c |\Omega|, & \text{if } \alpha \leq \alpha_N, \\ = \infty, & \text{if } \alpha > \alpha_N,
\end{cases}
\]

where \( \alpha_N = N \omega ^{(N-1)/(N-1)}_N \), \( c \) is a constant which depends on \( N \), and \( \omega _N \) is the measure of the unit sphere in \( \mathbb{R}^N \). Inequality (2) is now called Moser–Trudinger inequality and the term \( e^{\alpha_N |u|^{N'}} \) is known as critical Moser–Trudinger growth.

This well-known Moser–Trudinger inequality has been generalized in many ways. In the case \( \Omega = \mathbb{R}^N \), Ruf when \( N = 2 \) in [35], and Li and Ruf in [27] for \( N > 2 \), proved the following assertion

\[
\sup_{\|u\|_{W^{1,N}(\mathbb{R}^N)} \leq 1} \int_{\mathbb{R}^N} \phi_N(\alpha |u|^N) \, dx \begin{cases} \leq C(\alpha, N)^N', & \text{if } \alpha \leq \alpha_N, \\ = \infty, & \text{if } \alpha > \alpha_N,
\end{cases}
\]

where \( \alpha_N > 0 \) is a constant and

\[
\phi_N(t) = e^t - \sum_{j=0}^{N-2} \frac{t^j}{j!} = \sum_{j=N-1}^{\infty} \frac{t^j}{j!}, \quad t \geq 0.
\]

**Remark 1** Notice that \( \phi_N(t) \) is a increasing function.

In this paper, we consider the existence, nonexistence, and asymptotic behavior of positive solutions for a class of \( N \)-Laplacian elliptic equations related to the critical growth (3) in the whole space \( \mathbb{R}^N \). More precisely, we are concerned with the following problem:

\[
- \Delta_N u + |u|^{N-2}u = \lambda a(x)|u|^{q-2}u + f(u) \quad \text{in } \mathbb{R}^N,
\]

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where $\lambda > 0$, $2 \leq N$, $1 < q < N$, $a$ is a positive function such that $a \in L^{\frac{N}{N-q}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the growth condition

$$0 \leq f(t)t \leq a_1|t|^p\phi_N(\alpha|t|^\frac{N}{N-1}), \ t \in \mathbb{R},$$

where $N < p < +\infty$, $a_1 > 0$ and $\alpha > 0$ are constants.

In this setting, we mean by a solution of problem (5) any function $u \in W^{1,N}(\mathbb{R}^N) \cap C^1_{loc}(\mathbb{R}^N)$, such that

$$\int_{\mathbb{R}^N} \left( |\nabla u|^{N-2}\nabla u \nabla \phi + |u|^{N-2}u\phi - \lambda a(x)|u|^{q-2}u\phi - f(u)\phi \right) \, dx = 0,$n

for all $\phi \in C^\infty_0(\mathbb{R}^N)$.

Besides existence, this paper is regarding explore some asymptotic properties of the obtained solutions. If $u_\lambda$ is a solution of (5), we are interested in investigating the following properties:

$$u_\lambda(x) \to 0 \ \text{as} \ |x| \to \infty,$$

$$\|u_\lambda\|_{W^{1,N}(\mathbb{R}^N)} \to 0 \ \text{as} \ \lambda \to 0^+,$$

and

$$\|u_\lambda\|_{L^\infty(\mathbb{R}^N)} \to 0 \ \text{as} \ \lambda \to 0^+.$$

Furthermore, concerning the problem (5), we scrutinize the nonexistence of solution for $\lambda$ large enough.

The class of equations in (5) appears as a model for several problems in the fields of electromagnetism, astronomy, and fluid dynamics. They can be used to accurately describe the behavior of electric, gravitational, and fluid potentials. Problems of the form (5) are important in several applications to the study of evolution equations of $N$-laplacian type that appear in non-Newtonian fluids, turbulent flows in porous media and other contexts.

The case $N = 2$ is also related to the stationary problem associated with the following initial value Schrödinger equation

$$i \partial_t u + \Delta u + \mu g(u) = 0, \ \text{in} \ \mathbb{R}_t \times \mathbb{R}^2_x,$$

with data

$$u_0 := u(0, \cdot) \in W^{1,2}(\mathbb{R}^2),$$

where $u := u(t, x)$ is a complex-valued function of $(t, x) \in \mathbb{R} \times \mathbb{R}^2$, and

$$g(u) := u(e^{4\pi|u|^2} - 1).$$
Schrödinger equations involving exponential nonlinearities have several applications, such as the self-trapped beams in plasma [24]. In [8], the author considered Schrödinger equation with decreasing exponential nonlinearity. The stationary problem associated to (10) is given by

\[
\Delta \phi - \phi + g(\phi) = 0, \quad \phi \in W^{1,2}(\mathbb{R}^2).
\] (13)

If \( \phi \in W^{1,2}(\mathbb{R}^2) \) is a solution to (13), then \( e^{i \phi} \) is a solution to (10) called a soliton or a standing wave. Equations of type (13) arise in various other contexts of physics such that, the classical approximation in statistical mechanics, constructive field theory, false vacuum in cosmology, nonlinear optics, laser propagations, etc. They are also called nonlinear Euclidean scalar field equations, see [3,4,9,17,19].

Recently, there has been considerable interest in the study of existence results for problems of the form

\[
\begin{cases}
-\Delta_N u + V(x)|u|^{N-2}u = g(x,u) & \text{in } \mathbb{R}^N, \\
u > 0 & \text{on } \mathbb{R}^N,
\end{cases}
\] (14)

where \( g(x,u) \) is continuous and behaves like \( \phi_N(\alpha|u|^N) \) as \( |u| \to +\infty \), we would like to mention [5,12–14,31].

In general, the potential \( V : \mathbb{R}^N \to \mathbb{R} \) is bounded away from zero, i.e.

\[ V(x) \geq c > 0 \quad x \in \mathbb{R}^N. \]

If \( V \) is large at infinity in some suitable sense, then the loss of compactness due to the unboundedness of the domain \( \mathbb{R}^N \) can be overcome and vanishing phenomena can be ruled out. So, a natural framework for the function space setting of problem (14) is given by the subspace \( E \) of \( W^{1,N}(\mathbb{R}^N) \) defined as

\[ E := \left\{ u \in W^{1,N}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^N dx < +\infty \right\}, \]

endowed with the norm

\[ \|u\|_E := \left( \int_{\mathbb{R}^N} \left( |\nabla u|^N + V(x)|u|^N \right) dx \right)^{\frac{1}{N}} \forall u \in E. \]

Under appropriate assumptions on the potential \( V \), the embedding

\[ E \hookrightarrow L^p(\mathbb{R}^N) \] (15)

turns out to be compact. For instance, if

\[ V^{-1} \in L^{\frac{N}{N-1}}(\mathbb{R}^N), \] (16)
then the embedding (15) is compact for any \( p \geq 1 \) (see e.g. [39, Lemma 2.4]), while assuming the weaker condition
\[
V^{-1} \in L^1 \left( \mathbb{R}^N \right)
\]
the embedding (15) becomes compact only for \( p \geq N \) (see [10], see also [15] for other compactness results).

The authors of [13,14,25,39], considering a potential \( V \) satisfying (16) or (17), obtained existence results for equations of the form (14) and even more general equations. Their proofs rely, crucially, on the compact embeddings (15), given by (16) and (17) (in particular, on the compact embedding of \( E \) into \( L^N \left( \mathbb{R}^N \right) \)).

Most papers treat problem (14) employing variational methods. Then, usually, it is assumed that \( g \) has subcritical or critical growth and sometimes asking
\[
g(s) \geq c|s|^{p-1}, \quad \text{for each } s \geq 0 \text{ where } c > 0 \text{ and } p > N
\]
are constants, see [2,12].

Another common assumption on \( g \) is the so-called Ambrosetti–Rabinowitz condition, that is,
\[
\exists R > 0 \text{ and } \theta > N \text{ such that } 0 < \theta G(x,s) \leq sg(x,s) \forall |s| \geq R \text{ and } x \in \mathbb{R}^N,
\]
where \( G(x,s) = \int_0^s g(x,t)dt \), see [12,13,25,31].

Even when the Ambrosetti–Rabinowitz is dropped, it is usually to be assumed some additional condition to obtain compactness of the Palais–Smale sequences or Cerami sequences. See, for instance, [25] (see also [26] for bounded smooth domains), where the authors assume the set of conditions on the nonlinearity
\[
\begin{align*}
H(x,t) &\leq H(x,s) \text{ for all } 0 < t < s, \forall x \in \mathbb{R}^N, \\
\text{where } H(x,u) &\equiv ug(x,u) - NG(x,u), \\
\text{and } g : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R} \text{ behaves like } \exp \left( \alpha|u|^{N/(N-1)} \right), \\
&\text{when } |u| \rightarrow \infty.
\end{align*}
\]

The case when the potential \( V \) is constant, i.e. \( V(x) = c \) for any \( x \in \mathbb{R}^N \), is much less developed. In such a case, only a few existence results are known, and we should like to mention [31] (and references therein), where the authors proved existence results by means of Variational approach. In [12], the author consider \( V(x) = 1 \) and
\[
g(x,t) = \lambda h(x)|t|^{q-2}t + f(t), \quad x \in \mathbb{R}^N, \quad \text{as in (6)}, \quad \text{with } f \in C^1(\mathbb{R}, \mathbb{R}) \text{ satisfying the Ambrosetti–Rabinowitz condition.}
\]

In the following, we state our main results.

**Theorem 1.1** Suppose that \( f : \mathbb{R} \rightarrow \mathbb{R} \) is a continuous function satisfying assumption (6). Then there exists \( \lambda^* > 0 \) such that for every \( \lambda \in (0, \lambda^*) \) problem (5) admits a positive solution \( u_\lambda \in W^{1,N}(\mathbb{R}^N) \cap C^1_{loc}(\mathbb{R}^N) \). Furthermore,
\[
\|u_\lambda\|_{W^{1,N}(\mathbb{R}^N)} \rightarrow 0.
\]
as $\lambda \to 0^+.$

**Proposition 1** Suppose that $f(t) = |t|^{p-1} \phi_N(\alpha |t|^\frac{N}{N-1})$ and let

$$\lambda^* = \sup\{\lambda > 0; (5) \text{ has a solution } u_\lambda \text{ in } W^{1,N}(\mathbb{R}^N)\}.$$ 

Then $\lambda^* < \infty.$

**Theorem 1.2** Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying assumption (6). Then any solution $u_\lambda$, $\lambda \in (0, \lambda^*)$, given by Theorem 1.1, satisfies $u_\lambda(x) \to 0$ as $|x| \to \infty.$

**Corollary 1** Any solution $u_\lambda$, $\lambda \in (0, \lambda^*)$, given by Theorem 1.1, satisfies

$$\|u_\lambda\|_{L^\infty(\mathbb{R}^N)} \to 0,$$

as $\lambda \to 0^+.$

Notice that in this paper we do not impose any extra hypotheses on $f$ beyond (6). To compare our condition on the nonlinearity of problem (5) with we have found in the literature, we present two examples that follow.

**Example 1** As the first example of a function satisfying (6), we have

$$f(t) := h(t)\phi_N(\alpha |t|^\frac{N}{N-1})$$

where

$$h(t) = |t|^{p-2}t \sin^2(t).$$

Now, define

$$g(x, t) := \lambda a(x)t^{q-1} + h(t)\phi_N(\alpha |t|^\frac{N}{N-1}),$$

where $a \in L^{\frac{N}{N-q}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is a positive function. Notice that (21) satisfies neither (18) nor Ambrosetti–Rabinowitz condition nor (19). First, let us verify that $g$ does not satisfy (18). Let $c > 0$ be a positive fixed constant. Taking the sequence $t_k = 2k\pi$, with $k$ a positive integer, $q < N$ and $a \in L^\infty(\mathbb{R}^N)$, there holds $g(x, t_k) = \lambda a(x)t_k^{q-1} < ct_k^{p-1}$, for $k$ large enough.

Now, let us define

$$H(x, t) = t[\lambda a(x)t^{q-1} + |t|^{p-2}t \sin^2(t)\phi_N(\alpha |t|^\frac{N}{N-1})]$$

$$-N \int_0^t (\lambda a(x)s^{q-1} + |s|^{p-2}s \sin^2(s)\phi_N(\alpha |s|^\frac{N}{N-1}))ds.$$
Still considering the sequence \( t_k = 2k\pi \), with \( k \) a positive integer, \( H \) satisfies

\[
H(x, t_k) = \lambda(x) t_k^q \left( 1 - \frac{N}{q} \right) - N \int_0^{t_k} |s|^{p-2} s |\sin\alpha| s^{\frac{N}{p-2}} ds < 0.
\]

Notice that \( H(x, t_k) > H(x, t_{k+1}) \). Therefore, (21) satisfies neither Ambrosetti–Rabinowitz condition nor (19).

**Example 2** The second example of a function satisfying (6) we would like to mention is

\[
f(t) := |t|^{p-2} t [\sin(t)]_+ \phi_N(\alpha |t|^{\frac{N}{p-2}}), \quad \text{where } z_+ = \max\{z, 0\}. \tag{22}
\]

Notice that \( f \) is a continuous function but it is not derivable. Indeed, since

\[
f'_-(\pi) = \lim_{l \to 0^-} \frac{f(\pi + l)}{l} = \lim_{l \to 0^-} \frac{\pi + l}{l} |\sin(\pi + l)| \phi_N(\alpha |\pi + l|^{\frac{N}{p-2}})
\]

\[
= -\lim_{l \to 0^-} |\pi + l|^{p-2} |\sin(l)| \phi_N(\alpha |l|^{\frac{N}{p-2}})
\]

\[
= -\pi^{p-1} \phi_N(\alpha \pi^{\frac{N}{p-2}}) < 0,
\]

and

\[
f'_+(\pi) = \lim_{l \to 0^+} \frac{f(\pi + l)}{l} = 0,
\]

so \( f'(\pi) \) does not exist. In particular, \( f \) does not belong to \( C^1(\mathbb{R}, \mathbb{R}) \), as it is imposed in [12] to obtain the existence result. Furthermore, if \( N \geq 2 \), (22) satisfies neither Ambrosetti–Rabinowitz condition nor (19). Indeed, let us define

\[
\tilde{H}(t) = t |t|^{p-2} t [\sin(t)]_+ \phi_N(\alpha |t|^{\frac{N}{p-2}})
\]

\[
- N \int_0^t |s|^{p-2} s [\sin(s)]_+ \phi_N(\alpha |s|^{\frac{N}{p-2}}) ds.
\]

Considering the sequence \( t_k = 2k\pi \), with \( k \) being a positive integer, \( \tilde{H} \) satisfies

\[
\tilde{H}(t_k) = -N \int_0^{t_k} |s|^{p-2} s [\sin(s)]_+ \phi_N(\alpha |s|^{\frac{N}{p-2}}) ds < 0.
\]

Notice that \( \tilde{H}(t_k) > \tilde{H}(t_{k+1}) \). Therefore, (22) does not satisfies neither Ambrosetti–Rabinowitz condition nor (19).

By observing these simple examples above, we can see that our results are not included in the previous literature. Therefore, it brings novelty in the study of such equations in the field.
The solution of (5) is obtained as the limit of auxiliary problems in bounded domains. The technique combines the Galerkin method, comparison principle of lower and upper solutions, and regularity scheme. As in [11], we also consider a special class of normed spaces of finite dimension. However, to clarify this approach we show Lemma 2.4, which is a result of independent interest. Such a lemma, together Brouwer’s fixed point theorem, allows us, in the scheme of the Galerkin method, to work in general Banach spaces of finite dimensions with general norms.

Due to the presence of the critical term $\varphi_{N}(\alpha|u|^{N-2})$ and since we are in the whole space $\mathbb{R}^{N}$, we had to overcome several difficulties. One of them is a suitable improvement needed in the estimates of the approximating functions of $f$ by comparing it with [11, Lemma 2.2.] (see Lemma 3.2 below). Another delicate point in our approach is the regularity needed to ensure that the solution, obtained in the limit, does not vanish. In the bounded domain, we obtain regularity up to the boundary for the auxiliary problems considering approximate functions in the sense of Strauss [36]. For the domain-wide solution case, a kind of a priori estimate in the sup norm is presented, see Proposition 4 (and the consequences of that outcome).

Now we proceed to introduce the organization of the rest of the paper. Section 2 presents comparison principles, some preliminaries results that are useful through the text and an important result labeled as Lemma 2.4. Section 3 is concerning approximating functions that enable us to obtain regularity up to the boundary to the approximating solutions. In Sect. 4, we study a class of auxiliary problems in bounded domains. Section 5 is devoted to the proof of our main results.

2 Preliminaries

In this section, we state some preliminaries results which will be necessary throughout the paper.

The following notion will help us to verify strongly convergent subsequences.

**Definition 2.1** Let $X$ be a reflexive Banach space and $V : X \to X^*$. We say that $V$ is of type $(S_+)$ if for every sequence $\{x_k\}_{k \in \mathbb{N}} \subset X$ such that $x_k \rightharpoonup x$ and

$$\limsup_{k \to \infty} \langle V(x_k), x_k - x \rangle \leq 0,$$

one has that $x_k \to x$ in $X$.

**Remark 2** Let $V : W^{1,N}_0(D) \to W^{1,N}_0(D)^*$ be the map defined by

$$\langle V(u), v \rangle = \int_D |\nabla u|^{N-2} \nabla u \nabla v dx, \quad \text{for all } u, v \in W^{1,N}_0(D). \quad (23)$$

By [33, Proposition 3.5], we obtain that $V$ is of type $(S_+).$ In this sense, we say that $-\Delta_N$ has the $(S_+)-$property.
Now, let $u \in W^{1,N}_0(D)$. In what follows, let us denote by $\tilde{u}$ the canonical extension of $u$ by 0 outside $D$, that is,

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x \in D, \\ 0 & \text{if } x \in \mathbb{R}^N \setminus D. \end{cases} \quad (24)$$

It is well known that $u \in W^{1,N}_0(D)$ implies $\tilde{u} \in W^{1,N}(\mathbb{R}^N)$ (see e.g. [6, Proposition 9.18]).

The next technical lemma was proved in [1, Lemma 2.3].

**Lemma 2.2** Let $\alpha > 0$ and $r > 1$. Then, for every $\beta > r$, there exists a constant $C := C(\beta)$ such that

$$\left( \phi_N(\alpha |u|^{N-1}) \right)^r \leq C \phi_N(\beta \alpha |u|^{N-1}),$$

where $\phi_N$ is given in (4).

A complete proof of the next result can be found in [16, Lemma 3].

**Lemma 2.3** Let $1 < q < N$. For any constants $b > 0$, the problem

$$\begin{cases} -\Delta_N u + |u|^{N-2}u = b |u|^{q-2}u & \text{in } D, \\ u > 0 & \text{in } D, \\ u = 0 & \text{on } \partial D, \end{cases} \quad (25)$$

admits a solution $u_b \in C^1_0(D)$ satisfying $\partial u_b / \partial n < 0$ on $\partial D$.

We conclude this section by presenting a lemma, which is a consequence of Brouwer’s Fixed Point Theorem. However, our statement is a subtle (but very useful) generalization by comparing it with the literature. In particular, this result allows us to work in general Banach spaces, with freedom in choosing the norm (see the proof of Lemma 4.2). We will adopt $|x|_2 = \sqrt{\langle x, x \rangle}$ to denote the usual euclidean norm in $\mathbb{R}^d$ and $\|x\|_d$ to denote a general norm in $\mathbb{R}^d$. The proof of next lemma follows some ideas like in Kesavan [22], where the result for the particular case $\|x\|_d := |x|_2$ is presented.

**Lemma 2.4** Let $F : (\mathbb{R}^d, \| \cdot \|_d) \to (\mathbb{R}^d, \| \cdot \|_d)$ be a continuous function such that $\langle F(\xi), \xi \rangle \geq 0$ for every $\xi \in \mathbb{R}^d$ with $\|\xi\|_d = \varrho$ for some $\varrho > 0$, and $\langle \cdot, \cdot \rangle_1^{1/2} = |\cdot|_2$. Then, there exists $z_0$ in the closed ball $\overline{B}_\varrho^d(0) := \{z \in \mathbb{R}^d; \|z\|_d \leq \varrho \}$ such that $F(z_0) = 0$.

**Proof** Firstly, there exists $c(d) > 0$ such that

$$\|x\|_d \leq c(d)|x|_2, \quad \forall x \in \mathbb{R}^d. \quad (26)$$

Suppose, $F(x) \neq 0$ for all $x \in \overline{B}_\varrho^d(0)$. Define

$$g : (\mathbb{R}^d, \| \cdot \|_d) \to (\mathbb{R}^d, \| \cdot \|_d)$$
by

\[ g(x) = -\frac{\varrho}{\| F(x) \|} F(x) \]

which maps \( \overline{B}^d_\varrho(0) \) into itself and is continuous. Hence it has a fixed point \( x_0, \) by Brouwer’s Fixed Point Theorem. Since \( x_0 = g(x_0), \) we have \( \| x_0 \|_d = \| g(x_0) \|_d = \varrho > 0. \) But then by (26)

\[ 0 < \varrho^2 = \| x_0 \|_d^2 \leq c(d)^2 \| x_0 \|_2^2 = c(d)^2 \langle x_0, x_0 \rangle = c(d)^2 \langle x_0, g(x_0) \rangle \]

\[ = -c(d)^2 \frac{\varrho}{\| F(x_0) \|_d} \langle x_0, F(x_0) \rangle \leq 0, \]

by assumptions, which is a contradiction. \( \square \)

### 2.1 Comparison principle

In this section, we assume that \( D \) is a bounded domain in \( \mathbb{R}^N \) with \( C^2 \) boundary \( \partial D. \) In the following, we present a couple of comparison principles for a subsolution and a supersolution of the problem

\[
\begin{cases}
-\Delta_N u + |u|^{N-2}u = g(u) & \text{in } D, \\
u = 0 & \text{on } \partial D,
\end{cases}
\]

where \( g : \mathbb{R} \rightarrow \mathbb{R} \) is a continuous function.

We say that \( u_1 \in W^{1,N}(D) \) is a subsolution of problem (27) if \( u_1 \leq 0 \) on \( \partial D \) and

\[
\int_D (|\nabla u_1|^{N-2}\nabla u_1 \nabla \varphi + |u_1|^{N-2}u_1 \varphi)dx \leq \int_D g(u_1)\varphi dx
\]

for all \( \varphi \in W^{1,N}_0(D) \) with \( \varphi \geq 0 \) in \( D \) provided the integral \( \int_D g(u_1)\varphi dx \) exists. We say that \( u_2 \in W^{1,N}(D) \) is a supersolution of (27) if the reversed inequalities are satisfied with \( u_2 \) in place of \( u_1 \) for all \( \varphi \in W^{1,N}_0(D) \) with \( \varphi \geq 0 \) in \( D. \)

The next comparison results are particular cases of the ones achieved in [16, Theorem 3, Theorem 5].

#### Proposition 2

Let \( g : \mathbb{R} \rightarrow \mathbb{R} \) be a continuous function such that \( g(t)/t^{N-1} \) is decreasing for \( t > 0. \) Assume that \( u_1 \) and \( u_2 \) are a positive subsolution and a positive supersolution of problem (27), respectively. If \( u_2(x) > u_1(x) = 0 \) for all \( x \in \partial D, \) \( u_i \in C^{1,\alpha}(\overline{D}) \) with some \( \alpha \in (0,1), \) \( \Delta_N u_i \in L^\infty(D), \) for \( i, j = 1, 2, \) then \( u_2 \geq u_1 \) in \( D. \)

Whenever \( u_1 \) and \( u_2 \) satisfy the homogeneous Dirichlet boundary condition we can state the following result.

#### Proposition 3

Let \( g : \mathbb{R} \rightarrow \mathbb{R} \) be a continuous function such that \( g(t)/t^{N-1} \) is decreasing for \( t > 0. \) Assume that \( u_1, u_2 \in C^{1,\alpha}_0(\overline{D}) \) with some \( \alpha \in (0,1), \) are...
a positive subsolution and a positive supersolution of problem (27), respectively. If \( \Delta_N u_i \in L_\infty(D) \), for \( i, j = 1, 2, u_1/u_2 \in L_\infty(D) \) and \( u_2/u_1 \in L_\infty(D) \), then \( u_2 \geq u_1 \) in \( D \).

### 3 Approximating functions

To prove Theorem 1.1, we approximate \( f \) by Lipschitz functions \( f_k : \mathbb{R} \to \mathbb{R} \) defined by

\[
\begin{align*}
    f_k(s) &= \begin{cases} 
        -k[G(-k - 1/k) - G(-k)], & \text{if } s \leq -k, \\
        -k[G(s - 1/k) - G(s)], & \text{if } -k \leq s \leq -1/k, \\
        k^2s[G(-2/k) - G(-1/k)], & \text{if } -1/k \leq s \leq 0, \\
        k^2s[G(\frac{1}{k}) - G(\frac{1}{k})], & \text{if } 0 \leq s \leq 1/k, \\
        k[G(s + 1/k) - G(s)], & \text{if } 1/k \leq s \leq k, \\
        k[G(k + 1/k) - G(k)], & \text{if } s \geq k,
    \end{cases}
\end{align*}
\]

(28)

where \( G(s) = \int_0^s f(\xi)d\xi \).

The following approximation result was proved in [36].

**Lemma 3.1** Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function such that \( sf(s) \geq 0 \) for every \( s \in \mathbb{R} \). Then there exists a sequence \( f_k : \mathbb{R} \to \mathbb{R} \) of continuous functions satisfying

(i) \( sf_k(s) \geq 0 \) for every \( s \in \mathbb{R} \);

(ii) \( \forall k \in \mathbb{N} \exists c_k > 0 \) such that \( |f_k(\xi) - f_k(\eta)| \leq c_k|\xi - \eta| \) for every \( \xi, \eta \in \mathbb{R} \);

(iii) \( f_k \) converges uniformly to \( f \) in bounded subsets of \( \mathbb{R} \).

The sequence \( f_k \) of the previous lemma has some additional properties presented below. Here, we present a suitable improvement in the estimates of the approximating functions \( f_k \) by comparing it with [11, Lemma 2.2.].

**Lemma 3.2** Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function satisfying (6) for every \( s \in \mathbb{R} \). Then the sequence \( f_k \) of Lemma 3.1 satisfies

(i) \( \forall k \in \mathbb{N}, 0 \leq sf_k(s) \leq C_1|s|^p\phi_N(2^{N-1}\alpha|s|^{N-1}) \) for every \( |s| \geq \frac{1}{k} \);

(ii) \( \forall k \in \mathbb{N}, 0 \leq sf_k(s) \leq C_2\frac{1}{k^{p+2}}|s|^2 \) for every \( |s| \leq \frac{1}{k} \).

where \( C_1 \) and \( C_2 \) are positive constants independent of \( k \).

**Proof** Everywhere in this proof, the constant \( a_1 \) is the one of (6).

**First step.** Suppose that \( -k \leq s \leq -\frac{1}{k} \).

By the mean value theorem, there exists \( \eta \in (s - \frac{1}{k}, s) \) such that

\[
f_k(s) = -k \left[ G\left(s - \frac{1}{k}\right) - G(s)\right] = -kG'(\eta)\left(s - \frac{1}{k} - s\right) = f(\eta)
\]

and

\[
    sf_k(s) = sf(\eta).
\]
Since $s - \frac{1}{k} < \eta < s < 0$ and $f(\eta) < 0$, we have $sf(\eta) \leq \eta f(\eta)$. Therefore, by using Remark 1, we obtain

$$sf_k(s) \leq \eta f(\eta) \leq a_1 |\eta|^p \phi_N(\alpha |\eta|^{\frac{N}{N-1}})$$

$$\leq a_1 |s - \frac{1}{k}|^p \phi_N(\alpha |s - \frac{1}{k}|^{\frac{N}{N-1}})$$

$$\leq a_1 (|s| + \frac{1}{k})^p \phi_N(\alpha (|s| + \frac{1}{k})^{\frac{N}{N-1}})$$

$$\leq a_1 (2|s|)^p \phi_N(\alpha (2|s|)^{\frac{N}{N-1}})$$

$$= a_1 2^p |s|^p \phi_N(2^{\frac{N}{N-1}} \alpha |s|^{\frac{N}{N-1}}).$$

**Second step.** Assume $\frac{1}{k} \leq s \leq k$.

By the mean value theorem, there exists $\eta \in (s, s + \frac{1}{k})$ such that

$$f_k(s) = k \left[ G \left( s + \frac{1}{k} \right) - G(s) \right] = kG'(\eta) \left( s + \frac{1}{k} - s \right) = f(\eta)$$

and

$$sf_k(s) = sf(\eta).$$

Since $0 < s < \eta < s + \frac{1}{k}$ and $f(\eta) > 0$, we have $sf(\eta) \leq \eta f(\eta)$. Therefore,

$$sf_k(s) \leq \eta f(\eta) \leq a_1 |\eta|^p \phi_N(\alpha |\eta|^{\frac{N}{N-1}})$$

$$\leq a_1 |s + \frac{1}{k}|^p \phi_N(\alpha |s + \frac{1}{k}|^{\frac{N}{N-1}})$$

$$\leq a_1 (2|s|)^p \phi_N(\alpha (2|s|)^{\frac{N}{N-1}})$$

$$= a_1 2^p |s|^p \phi_N(2^{\frac{N}{N-1}} \alpha |s|^{\frac{N}{N-1}}).$$

**Third step.** Suppose that $|s| \geq k$, then

$$f_k(s) = \begin{cases} 
-k[G(-k - \frac{1}{k}) - G(-k)], & \text{if } s \leq -k \\
 k[G(k + \frac{1}{k}) - G(k)], & \text{if } s \geq k. 
\end{cases} \tag{29}$$

If $s \leq -k$, by the mean value theorem, there exists $\eta \in (-k - \frac{1}{k}, -k)$ such that

$$f_k(s) = k \left[ G \left( -k - \frac{1}{k} \right) - G(-k) \right] = -kG'(\eta) \left( -k - \frac{1}{k} - (-k) \right) = f(\eta)$$

and

$$sf_k(s) = sf(\eta).$$
Since $-k - \frac{1}{k} < \eta < -k < 0$ and $k < |\eta| < k + \frac{1}{k}$, we conclude that

$$sf_k(s) = \frac{s}{\eta} f(\eta) \leq \left|\frac{s}{|\eta|}\right| a_1|\eta|^p \phi_N(\alpha |\eta|^{\frac{N}{N-1}}) = a_1|s| |\eta|^{p-1} \phi_N(\alpha |\eta|^{\frac{N}{N-1}})$$

\begin{align*}
&\leq a_1 |s| (|k + \frac{1}{k}|)^{p-1} \phi_N(\alpha (|k + \frac{1}{k}|)^{\frac{N}{N-1}}) \\
&\leq a_1 |s| (|s| + \frac{1}{k})^{p-1} \phi_N(\alpha (|s| + \frac{1}{k})^{\frac{N}{N-1}}) \\
&\leq a_1 |s| (2|s|)^{p-1} \phi_N(\alpha (2|s|)^{\frac{N}{N-1}}) \\
&\leq a_1 2^p |s|^p \phi_N(2^{\frac{N}{N-1}} \alpha |s|^{\frac{N}{N-1}}). \\
\end{align*}

(30)

If $s \geq k$, by the mean value theorem, there exists $\eta \in (k, k + \frac{1}{k})$ such that

$$f_k(s) = k \left[ G\left( k + \frac{1}{k}\right) - G(k) \right] = kG'(\eta) \left( k + \frac{1}{k} - k \right) = f(\eta).$$

By computations similar to conclude (30) one has

$$sf_k(s) = sf(\eta) = \frac{s}{\eta} \eta f(\eta) \leq \left|\frac{s}{|\eta|}\right| a_1|\eta|^p \phi_N(\alpha |\eta|^{\frac{N}{N-1}}) \leq a_1 2^p |s|^p \phi_N(2^{\frac{N}{N-1}} \alpha |s|^{\frac{N}{N-1}}).$$

**Fourth step.** Assume $-\frac{1}{k} \leq s \leq \frac{1}{k}$, then

$$f_k(s) = \begin{cases} k^2 s [G\left( \frac{-2}{k}\right) - G\left( -\frac{1}{k}\right)] , & \text{if } -\frac{1}{k} \leq s \leq 0 \\
 k^2 s [G\left( \frac{2}{k}\right) - G\left( \frac{1}{k}\right)] , & \text{if } 0 \leq s \leq \frac{1}{k}. \end{cases}$$

(31)

If $-\frac{1}{k} \leq s \leq 0$, by the mean value theorem, there exists $\eta \in \left( -\frac{2}{k}, -\frac{1}{k} \right)$ such that

$$f_k(s) = k^2 s \left[ G\left( \frac{-2}{k}\right) - G\left( -\frac{1}{k}\right) \right] = k^2 s G'(\eta) \left( \frac{-2}{k} - \left( -\frac{1}{k}\right) \right) = -ks f(\eta).$$

Therefore

$$sf_k(s) = -ks^2 f(\eta) = -k s^2 \eta f(\eta) \leq k \frac{s^2}{|\eta|} \eta f(\eta)$$

\begin{align*}
&\leq a_1 k |s|^2 |\eta|^{p-1} \phi_N(\alpha |\eta|^{\frac{N}{N-1}}) \leq a_1 k |s|^2 \left( \frac{2}{k} \right)^{p-1} \phi_N(\alpha |\eta|^{\frac{N}{N-1}}) \\
&\leq a_1 2^{p-1} \frac{|s|^2}{k^{p-2}} \phi \left( \alpha \left( \frac{2}{k} \right)^{\frac{N}{N-1}} \right) \\
&\leq a_1 2^{p-1} \exp(2^{\frac{N}{N-1}} \alpha) \frac{1}{k^{p-1}} |s|^2. \end{align*}

(32)

If $0 \leq s \leq \frac{1}{k}$, by the mean value theorem, there exists $\eta \in \left( \frac{1}{k}, \frac{2}{k} \right)$ such that

$$f_k(s) = k^2 s \left[ G\left( \frac{2}{k}\right) - G\left( \frac{1}{k}\right) \right] = k^2 s G'(\eta) \left( \frac{2}{k} - \frac{1}{k} \right) = ks f(\eta).$$
By similar computations to conclude (32) one obtains

\[ sf_k(s) = k s^2 f(\eta) = k \frac{s^2}{|\eta|} \eta f(\eta) \leq a_1 2^{p-1} \exp(2 \frac{N}{p-2} \alpha) \frac{1}{k^{p-2}} |s|^2. \]

The proof of the lemma follows by taking \( C_1 = a_1 2^p \) ad \( C_2 = a_1 2^{p-1} \exp(2 \frac{N}{p-2} \alpha) \), where \( a_1 \) is given in (6).

\[ \square \]

### 4 Solution on bounded domains

From now on, we assume that \( D \) is a bounded domain in \( \mathbb{R}^N \) with \( C^2 \) boundary \( \partial D \).

For \( r \geq 1 \), we denote by \( \|u\|_{L^r(D)} \) the usual norm on the space \( L^r(D) \). We endow \( W_0^{1,N}(D) \) with the norm \( \|u\|_{W_0^{1,N}(D)} = \|\nabla u\|_{L^N(D)} + \|u\|_{L^N(D)} \).

In this section, we focus on the existence of a positive solution for the problem:

\[
\begin{cases}
-\Delta_N u + |u|^{N-2} u = \lambda a(x)|u|^{q-2} u + f(u) & \text{in } D, \\
\quad u > 0 & \text{in } D, \\
\quad u(x) = 0 & \text{on } \partial D,
\end{cases}
\]

where \( a \) is a positive function such that \( a \in L^{\frac{N}{N-q}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \). Notice that, in particular \( a|_{D} \in L^{\frac{N}{N-q}}(D) \cap L^\infty(D) \). Here \( \lambda > 0 \) is a parameter, \( 1 < q < N \), \( f : [0, \infty) \to \mathbb{R} \) is a continuous function satisfying (6).

We say that \( u \in W_0^{1,N}(D) \) is a solution of \( (PD) \) if \( u(x) > 0 \) in \( D \) and

\[
\int_D |\nabla u|^{N-2} \nabla u \nabla \phi dx + \int_D |u|^{N-2} u \phi dx = \lambda \int_D a(x)|u|^{q-2} u \phi dx + \int_D f(u) \phi dx,
\]

for all \( \phi \in W_0^{1,N}(D) \).

The existence of a solution for problem \( (PD) \) is stated below.

**Theorem 4.1** Suppose that \( f : [0, \infty) \to \mathbb{R} \) is a continuous function satisfying (6). Then there exists \( \lambda^* > 0 \) such that for every \( \lambda \in (0, \lambda^*) \) problem \( (PD) \) admits a solution \( u_\lambda \in W_0^{1,N}(D) \) such that \( \partial u_\lambda / \partial \nu < 0 \) on \( \partial D \), where \( \nu \) stands for the outer normal to \( \partial D \).

**4.1 Approximate equation**

In the proof of Theorem 4.1 we utilize the following auxiliary problem:

\[
\begin{cases}
-\Delta_N u + |u|^{N-2} u = \lambda a(x)|u|^{q-2} u + f_n(u) + \frac{\varphi}{n} & \text{in } D, \\
\quad u > 0 & \text{in } D, \\
\quad u(x) = 0 & \text{on } \partial D,
\end{cases}
\]

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with \( n > 0 \) an integer number, \( \varphi \) is a fixed positive function such that \( \varphi \in L^\infty(\mathbb{R}^N) \cap L^N(\mathbb{R}^N) \) and \( f_n \) is given by Lemmas 3.1 and 3.2.

**Lemma 4.2** There exists \( \lambda^* > 0 \) and \( n^* \in \mathbb{N} \) such that \( (PD_n) \) has a solution \( u_n \in C^1_0(\overline{D}) \) such that \( \partial u_n / \partial n < 0 \) on \( \partial D \) for every \( \lambda \in (0, \lambda^*) \) and \( n \geq n^* \). Furthermore,

\[
\| u_n \|_{W^{1,N}(D)} \leq \varrho,
\]

where \( \varrho \) does not depend on \( n \).

**Proof** Let \( B = \{ w_1, w_2, \ldots, w_n, \ldots \} \) be a Schauder basis (see [6,18] for details) for the Banach space \( W^{1,N}_0(D), \| \cdot \|_{W^{1,N}(D)} \). For each positive integer \( m \), let

\[
W_m = [w_1, w_2, \ldots, w_m]
\]

be the \( m \)-dimensional subspace of \( W^{1,N}_0(D) \) generated by \( \{ w_1, w_2, \ldots, w_m \} \) with norm induced from \( W^{1,N}_0(D) \). Let \( \xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m \), notice that

\[
|\xi|_m := \left\| \sum_{j=1}^m \xi_j w_j \right\|_{W^{1,N}(D)},
\]

defines a norm in \( \mathbb{R}^m \) (see [11] for the details).

By using the above notation, we can identify the spaces \( (W_m, \| \cdot \|_{W^{1,N}(D)}) \) and \( (\mathbb{R}^m, | \cdot |_m) \) by the isometric linear transformation

\[
u = \sum_{j=1}^m \xi_j w_j \in W_m \mapsto \xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m.\]

Now, define the function \( F : \mathbb{R}^m \rightarrow \mathbb{R}^m \) such that

\[
F(\xi) = (F_1(\xi), F_2(\xi), \ldots, F_m(\xi)),
\]

where \( \xi = (\xi_1, \xi_2, \ldots, \xi_m) \in \mathbb{R}^m \),

\[
F_j(\xi) = \int_D |\nabla u|^{N-2} \nabla u \nabla w_j dx + \int_D |u|^{N-2} u w_j dx - \lambda \int_D a(x)(u_+)^q w_j dx - \lambda \int_D f_n(u_+) w_j dx - \frac{1}{n} \int_D \varphi w_j dx,
\]

\[ j = 1, 2, \ldots, m, \]

and \( u = \sum_{i=1}^m \xi_i w_i \in W_m \). Therefore,

\[
\langle F(\xi), \xi \rangle = \int_D |\nabla u|^N dx + \int_D |u|^N dx - \lambda \int_D a(x)(u_+)^q dx - \lambda \int_D f_n(u_+) u_+ dx - \frac{1}{n} \int_D \varphi u dx,
\]

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where \( u_+ = \max\{u, 0\}, u_- = u_+ - u \).

Given \( u \in W_m \) we define

\[
D_n^+ = \left\{ x \in D : |u(x)| \geq \frac{1}{n} \right\}
\]

and

\[
D_n^- = \left\{ x \in D : |u(x)| < \frac{1}{n} \right\}.
\]

Thus, we can write (35) as

\[
\langle F(\xi), \xi \rangle = \langle F(\xi), \xi \rangle_P + \langle F(\xi), \xi \rangle_N,
\]

where

\[
\langle F(\xi), \xi \rangle_P = \int_{D_n^+} |\nabla u|^N dx + \int_{D_n^+} |u|^N dx - \lambda \int_{D_n^+} a(x)(u_+)^q dx
\]

\[
- \int_{D_n^+} f_n(u_+)u_+ dx - \frac{1}{n} \int_{D_n^+} \varphi u dx
\]

and

\[
\langle F(\xi), \xi \rangle_N = \int_{D_n^-} |\nabla u|^N dx + \int_{D_n^-} |u|^N dx - \lambda \int_{D_n^-} a(x)(u_+)^q dx
\]

\[
- \int_{D_n^-} f_n(u_+)u_+ dx - \frac{1}{n} \int_{D_n^-} \varphi u dx.
\]

**Step 1.** In what follows, \( C \) denotes a generic real constant. Since the embedding \( W^{1,N}(\mathbb{R}^N) \subset L^\tau(\mathbb{R}^N) \) is continuous for all \( \tau \geq N \) (see [6, Corollary 9.11]), we have

\[
\int_{D_n^+} |a(x)|(u_+)^q dx \leq C \|a\|_{L^{N/(N-\alpha)}(\mathbb{R}^N)} \|\tilde{u}\|_{W^{1,N}(\mathbb{R}^N)}^q \leq \frac{K_1}{2} \|\tilde{u}\|_{W^{1,N}(\mathbb{R}^N)}^q.
\] (36)

By virtue of Lemmas 2.2 and 3.2, we get

\[
\int_{D_n^+} f_n(u_+)u_+ dx \leq C \int_{D_n^+} |u_+|^p \phi_N(2^{N\alpha} \alpha |u_+|^{\frac{N}{N-\alpha}}) dx
\]

\[
\leq \left( \int_{\mathbb{R}^N} |\tilde{u}|^N dx \right)^\frac{1}{N} \left( \int_{\mathbb{R}^N} \left( \phi_N(2^{N\alpha} \alpha |\tilde{u}|^{\frac{N}{N-\alpha}}) \right)^N dx \right)^\frac{1}{N}
\]

\[
\leq K_2 \|\tilde{u}\|_{W^{1,N}(\mathbb{R}^N)}^p \left( \int_{\mathbb{R}^N} (\phi_N(2^{N\alpha} \alpha |\tilde{u}|^{\frac{N}{N-\alpha}})) dx \right)^\frac{1}{N}
\]

\[
\leq K_2 C(\alpha, N) \|\tilde{u}\|_{W^{1,N}(\mathbb{R}^N)}^p,
\] (37)
for $\|\tilde{u}\|_{W^{1,N}(\mathbb{R}^N)}$ small enough, where $\tilde{u}$ is given in (24). Indeed, if

$$\|\tilde{u}\|_{W^{1,N}(\mathbb{R}^N)} \leq \frac{1}{4} \left( \frac{\alpha N}{N\alpha} \right)^{(N-1)/N}, \tag{38}$$

then (3) guarantees the following estimate

$$\left( \int_{\mathbb{R}^N} (\phi_N N^{2N^{-1}} \alpha |\tilde{u}|^{N^{-1}}) \, dx \right)^{1/N} = \left( \int_{\mathbb{R}^N} (\phi_N N^{2N^{-1}} \alpha \|\tilde{u}\|^N_{W^{1,N}(\mathbb{R}^N)} \frac{\tilde{u}}{\|\tilde{u}\|_{W^{1,N}(\mathbb{R}^N)}} \, dx \right)^{1/N} \leq C(\alpha, N). \tag{39}$$

Now, since $\phi \in L^{N'}(\mathbb{R}^N)$, we have

$$\int_{D_n^+} \phi \, u \, dx \leq \|\phi\|_{L^{N'}(\mathbb{R}^N)} \|\tilde{u}\|_{L^N(\mathbb{R}^N)} \leq \frac{K_3}{2} \|\tilde{u}\|_{W^{1,N}(\mathbb{R}^N)}. \tag{40}$$

It follows from (36), (37) and (40) that

$$\langle F(\xi), \xi \rangle_p \geq \int_{D_n^+} \frac{1}{|D_n^+|} |\nabla u|^N \, dx + \int_{D_n^+} |u|^N \, dx - \lambda \frac{K_1}{2} \|\tilde{u}\|^q_{W^{1,N}(\mathbb{R}^N)} - K_2 C(\alpha, N) \|\tilde{u}\|^p_{W^{1,N}(\mathbb{R}^N)} - \frac{K_3}{2N} \|\tilde{u}\|_{W^{1,N}(\mathbb{R}^N)}, \tag{41}$$

Remark 3 Notice that the constants $K_1$, $K_2$ and $K_3$ do depend on neither $|D| = \int_D \, dx$ nor $n$.

Step 2. In a similarly way, we obtain

$$\int_{D_n^+} |a(x) (u_+)^q \, dx \leq \frac{K_1}{2} \|\tilde{u}\|^q_{W^{1,N}(\mathbb{R}^N)}. \tag{42}$$

By virtue of Lemma 3.2 (ii) we obtain

$$\int_{D_n^+} f_n(u_+) \, u_+ \, dx \leq C_2 \frac{1}{n^{p-2}} \int_{D_n^+} |u_+|^2 \, dx \leq C_2 \frac{1}{n^{p-2}} \left( \int_{D_n^+} \, dx \right)^{N-2/N} \left( \int_{\mathbb{R}^N} |\tilde{u}|^N \, dx \right)^{2/N} \leq C_2 |D|^{(N-2)/N} \frac{1}{n^{p-2}} \|\tilde{u}\|^2_{W^{1,N}(\mathbb{R}^N)}. \tag{43}$$
We also have

\[
\int_{D^n} \varphi ud\mathbf{x} \leq \frac{K_3}{2} \|\tilde{u}\|_{W^{1,N}(\mathbb{R}^N)}.
\]  

(44)

It follows from (42), (43) and (44) that

\[
\langle F(\xi), \xi \rangle_N \geq \int_{D^n} |\nabla u|^N d\mathbf{x} + \int_{D^n} |u|^N d\mathbf{x} - \frac{\lambda K_1}{2} \|\tilde{u}\|^q_{W^{1,N}(\mathbb{R}^N)}
\]

\[
- \frac{C_2 |D|^{(N-2)/N}}{n^{p-2}} \|\tilde{u}\|^2_{W^{1,N}(\mathbb{R}^N)} - \frac{K_3}{2n} \|\tilde{u}\|_{W^{1,N}(\mathbb{R}^N)}.
\]  

(45)

Using that \( \|u\|_{W^{1,N}(D)}^N = \|\tilde{u}\|_{W^{1,N}(\mathbb{R}^N)}^N = \|\nabla \tilde{u}\|_{L^N(\mathbb{R}^N)}^N + \|\tilde{u}\|_{L^N(\mathbb{R}^N)}^N \), inequalities (41) and (45) imply

\[
\langle F(\xi), \xi \rangle_N \geq \|\tilde{u}\|_{W^{1,N}(\mathbb{R}^N)}^N - \frac{\lambda K_1}{2} \|\tilde{u}\|^q_{W^{1,N}(\mathbb{R}^N)} - \frac{C_2 |D|^{(N-2)/N}}{n^{p-2}} \|\tilde{u}\|^2_{W^{1,N}(\mathbb{R}^N)} - \frac{K_3}{n} \|\tilde{u}\|_{W^{1,N}(\mathbb{R}^N)}.
\]  

(46)

Now, let \( |\xi|_m = \|\tilde{u}\|_{W^{1,N}(\mathbb{R}^N)} = \varrho \) for some \( \varrho > 0 \) to be chosen later. Thus, we have

\[
\langle F(\xi), \xi \rangle_N \geq \varrho^N - \lambda K_1 \varrho^q - K_2 C(\alpha, N) \varrho^p - \frac{C_2 |D|^{(N-2)/N}}{n^{p-2}} \varrho^2 - \frac{K_3}{n} \varrho.
\]

If \( \varrho \) is such that

\[
\varrho \leq \frac{1}{(2K_2 C(\alpha, N))^{-\frac{1}{p-n}}},
\]

then

\[
\varrho^N - K_2 C(\alpha, N) \varrho^p \geq \frac{\varrho^N}{2}.
\]

Thus, by choosing

\[
\varrho := \min \left\{ \frac{1}{(2K_2 C(\alpha, N))^{-\frac{1}{p-n}}}, \frac{1}{4} \left( \frac{\alpha N}{N \alpha} \right)^{(N-1)/N} \right\},
\]  

(47)

we obtain

\[
\langle F(\xi), \xi \rangle_N \geq \frac{\varrho^N}{2} - \lambda K_1 \varrho^q - \frac{C_2 |D|^{(N-2)/N}}{n^{p-2}} \varrho^2 - \frac{K_3}{n} \varrho.
\]
Now, define $\zeta := \frac{\rho^N}{2} - \lambda K_1 q^q$. If we choose
\[
\lambda^* := \frac{\rho^{N-q}}{4K_1} > 0,
\]
then $\zeta > \frac{\rho^N}{4}$ for all $0 < \lambda < \lambda^*$. Now, we choose $n^* \in \mathbb{N}$ such that
\[
\frac{C_2 |D|^{(N-2)/N}}{n^{p-2}} \rho^2 + \frac{K_3}{n} \rho < \frac{\zeta}{2},
\]
for every $n \geq n^*$. Notice that $n^*$ depends on the domain $D$. Since $\xi \in \mathbb{R}^m$ is such that $|\xi|_m = \rho$, then for $\lambda < \lambda^*$ and $n \geq n^*$ we obtain
\[
\langle F(\xi), \xi \rangle \geq \frac{\zeta}{2} > 0. \tag{48}
\]

Since $f_n$ is a Lipschitz function (for each $n \in \mathbb{N}$), it easy to see that $F : \mathbb{R}^m \to \mathbb{R}^m$ is a continuous function. Thus, for each $\lambda < \lambda^*$ and $n > n^*$ fixed, Lemma 2.4 ensure the existence of $y \in \mathbb{R}^m$ with $|y|_m \leq \rho$ and such that $F(y) = 0$. In other words, there exists $u_m \in W_m$ verifying
\[
\|u_m\|_{W^{1,N}(D)} \leq \rho, \tag{49}
\]
and such that
\[
\int_D |\nabla u_m|^{N-2}_m \nabla u_m \nabla wdx + \int_D |u_m|^{N-2}_m u_m wdx = \lambda \int_D a(x)(u_m+)^{q-1}wdx + \int_D f_n(u_m+)wdx + \frac{1}{n} \int_D \varphi wdx, \tag{50}
\]
for all $w \in W_m$.

**Remark 4** It is important to mention that $\rho$, given in (47), does not depend on the domain $D$, $m$ nor $n$.

Since $W_m \subset W^{1,N}_0(D) \forall m \in \mathbb{N}$ and $\rho$ does not depend on $m$, then $(u_m)_{m \in \mathbb{N}}$ is a bounded sequence in $W^{1,N}_0(D)$. Therefore, for some subsequence, there exists $u \in W^{1,N}_0(D)$ such that
\[
\begin{align*}
u_m & \rightharpoonup u \text{ weakly in } W^{1,N}_0(D), \tag{51} \\
u_m & \to u \text{ in } L^s(D) \text{ } s \geq N, \tag{52} \\
u_m & \to u, \text{ a.e. in } D. \tag{53}
\end{align*}
\]

Thus,
\[
\|u\|_{W^{1,N}(D)} \leq \liminf_{m \to \infty} \|u_m\|_{W^{1,N}(D)} \leq \rho. \tag{54}
\]
We claim that
\[ u_m \to u \quad \text{in } W_0^{1,N}(D). \]  
(55)

Indeed, using the fact that \( B = \{ w_1, w_2, \ldots, w_n, \ldots \} \) is a Schauder basis of \( W_0^{1,N}(D) \), for every \( u \in W_0^{1,N}(D) \) there exists a unique sequence \( (\alpha_n)_{n \geq 1} \) in \( \mathbb{R} \) such that \( u = \sum_{j=1}^{\infty} \alpha_j w_j \), so that

\[ \psi_m := \sum_{j=1}^{m} \alpha_j w_j \to u \quad \text{in } W_0^{1,N}(D) \quad \text{as } m \to \infty. \]  
(56)

Using \( w = (u_m - \psi_m) \in W_m \) as a test function in (50), we obtain

\[
\int_D |\nabla u_m|^{N-2} \nabla u_m \nabla (u_m - \psi_m) \, dx + \int_D |u_m|^{N-2} u_m (u_m - \psi_m) \, dx \\
= \lambda \int_D a(x)(u_{m+})^{q-1} (u_m - \psi_m) \, dx + \int_D f_n(u_{m+})(u_m - \psi_m) \, dx \\
+ {1 \over n} \int_D \varphi (u_m - \psi_m) \, dx.
\]  
(57)

From (51), (52) and (56), it is easy to see that

\[
\int_D \left( |u_m|^{N-1} + |\lambda a(x)(u_{m+})^{q-1}| + {1 \over n} |\varphi| \right) |u_m - \psi_m| \, dx \\
\leq \left( \|u_n\|_{L^N(D)}^{N-1} + \lambda \|a\|_{L^{N/(N-q)}}(D) \|u_n\|_{L^N(D)}^{q-1} + {1 \over n} \|\varphi\|_{L^{N'} (D)} \right) \|u_m - \psi_m\|_{L^N(D)}. \]  
(58)

By continuity of \( f_n \) and (53) we obtain

\[ f_n(u_{m+})^{N'} \to f_n(u_+)^{N'} \quad \text{a.e. in } D. \]

By Lemma 3.1, (49), and by using Hölder inequality we obtain

\[
\int_D f_n(u_{m+})^{N'} \, dx \leq c_n^{N'} \int_D |u_m|^{N'} \, dx \\
\leq c_n^{N'} C |D|^{N/(N-N')} \|u_m\|_{W^{1,N}(D)}^{N'} \\
\leq c_n^{N'} C |D|^{N/(N-N')} \|u_m\|_{L^N(D)}^{N'}. \]  
(59)

Hence, [20, Theorem 13.44] leads to

\[ f_n(u_{m+}) \to f_n(u_+) \quad \text{weakly in } L^{N'}(D). \]  
(60)
Applying (52), (56) and (60), we conclude that
\[ \lim_{m \to \infty} \int_D f_n(u_m +)(u_m - \psi_m)dx = 0. \] (61)

By (49) and (56), we obtain
\[ \lim_{m \to \infty} \int_D |\nabla u_m|^{N-2}(\nabla u_m \nabla (u - \psi))dx = 0. \] (62)

By (57), (58) (61) and (62), we obtain
\[ \lim_{m \to \infty} \int_D |\nabla u_m|^{N-2}(\nabla u_m \nabla (u_m - u))dx = 0. \] (63)

Now it is sufficient to apply the \((S_+)-\text{property of } -\Delta_N\) (see Remark 2) to obtain (55).

Now, for every \(m \geq k\) we obtain
\[ \int_D |\nabla u_m|^{N-2}(\nabla u_m \nabla w_k)dx + \int_D |u_m|^{N-2}u w_k dx \]
\[ = \lambda \int_D a(x)(u_m +)^{q-1}w_k dx + \int_D f_n(u_m +)w_k dx + \frac{1}{n} \int_D \varphi w_k dx, \] (64)

for all \(w_k \in W_k\).

It follows from (55) and (60) that
\[ \int_D |\nabla u|^{N-2}(\nabla u \nabla w_k)dx + \int_D |u|^{N-2}uw_k dx \]
\[ = \lambda \int_D a(x)(u +)^{q-1}w_k dx + \int_D f_n(u +)w_k dx + \frac{1}{n} \int_D \varphi w_k dx, \] (65)

for all \(w_k \in W_k\). Since \([W_k]_{k \in \mathbb{N}}\) is dense in \(W^{1,N}_0(D)\) we conclude that
\[ \int_D |\nabla u|^{N-2}(\nabla u \nabla w)dx + \int_D |u|^{N-2}uw dx \]
\[ = \lambda \int_D a(x)(u +)^{q-1}w dx + \int_D f_n(u +)w dx + \frac{1}{n} \int_D \varphi w dx, \] (66)

for all \(w \in W^{1,N}_0(D)\). Furthermore, \(u \geq 0\) in \(D\). In fact, since \(u_- \in W^{1,N}_0(D)\) then from (66) we obtain
\[ -\|u_\|_{W^{1,N}_0(D)}^N = \int_D |\nabla u|^{N-2}(\nabla u \nabla u_-)dx + \int_D |u|^{N-2}uu_- dx \]
\[ = \lambda \int_D a(x)(u +)^{q-1}u_- dx + \int_D f_n(u +)u_- dx + \frac{1}{n} \int_D \varphi u_- dx \]
\[ \geq 0. \]
Then \( u_\varnothing \equiv 0 \text{ a.e. in } D \), whence \( u \geq 0 \text{ a.e. in } D \). Moreover, \( u \not\equiv 0 \) is valid due to \( \psi_n > 0 \text{ in } D \). By applying the strong maximum principle in [34, Theorem 5.4.1] we obtain \( u > 0 \text{ in } D \), and [34, Theorem 5.5.1] ensure that \( \frac{\partial u}{\partial v} < 0 \text{ on } \partial D \) holds. By Lemma 3.1 and [23, Theorem 7.1] we conclude that \( u \in L^{\infty}(D) \), and [34, Theorem 5.5.1] ensure the regularity up to the boundary \( u \in C^{1,\beta}(\overline{D}) \), for some \( \beta \in (0, 1) \).

Therefore, we conclude that proof of the lemma by taking \( u_n = u \).  

\[ \square \]

### 4.2 Proof of Theorem 4.1

First we show that \((P_D)\) has a positive solution. For each \( n \in \mathbb{N} \), \( n > n^* \), by Lemma 4.2, equation \((P_{D_n})\) has a solution \( u_n \in W^{1,N}_0(D) \cap C^{1,\beta}(\overline{D}) \). Thus

\[
\int_D |\nabla u_n|^{N-2} \nabla u_n \nabla w dx + \int_D |u_n|^{N-2} u_n w dx = \mathcal{L} \int_D a(x) u_n^q w dx + \int_D f_n(u_n) w dx + \frac{1}{n} \int_D \varphi w dx, \tag{67}
\]

for all \( w \in W^{1,N}_0(D) \).

By (54) we have that

\[
\|u_n\|_{W^{1,N}(D)} \leq \varrho, \quad \forall n \in \mathbb{N}, \tag{68}
\]

and \( \varrho \) does not depend on \( n \). Thus, along a subsequence again relabeled as \( u_n \), there exists \( u \in W^{1,N}_0(D) \) such that

\[
u_n \rightharpoonup u \text{ weakly in } W^{1,N}_0(D), \text{ as } n \to \infty. \tag{69}
\]

Thus,

\[
\|u\|_{W^{1,1,N}(D)} \leq \lim \inf_{n \to \infty} \|u_n\|_{W^{1,N}(D)} \leq \varrho. \tag{70}
\]

We claim that

\[
u_n \to u \text{ in } W^{1,N}_0(D), \text{ as } n \to \infty. \tag{71}
\]

In fact, the proof of (71) follows in a similarly way as we did in the previous section. First, notice that Lemma 3.1 and (68) imply

\[
\int_D f_n(u_n)^N dx \leq c_n^N \int_D |u_n|^N dx \leq c_n^N C \varrho^N. \tag{72}
\]

Moreover,

\[
u_n \to u \text{ a.e. in } D, \tag{73}
\]
and by the uniform convergence of Lemma 3.1 (iii) we have

\[ f_n(u_n(\cdot)) \to f(u(\cdot)) \text{ a.e. in } D. \]  

Hence, [20, Theorem 13.44] leads to

\[ f_n(u_n) \to f(u) \text{ weakly in } L^N(D). \]  

On the other hand, taking \( w = (u_n - u) \) as a test function in (67), we get

\[ \int_D |\nabla u_n|^{N-2} \nabla u_n \nabla (u_n - u) dx \leq - \int_D |u_n|^{N-2} u_n (u_n - u) dx + \int_D a(x) u_n^{q-1} (u_n - u) dx \]
\[ + \int_D f_n(u_n) (u_n - u) dx + \frac{1}{n} \int_D \varphi(u_n - u) dx \]
\[ \leq \|u_n\|_{L^N(D)}^{N-1} \|u_n - u\|_{L^N(D)} \]
\[ + \lambda \|a\|_{L^{N/(N-q)}(D)} \|u_n\|_{L^N(D)}^{q-1} \|u_n - u\|_{L^N(D)} \]
\[ + \|f_n(u_n)\|_{L^N(D)} \|u_n - u\|_{L^N(D)} \]
\[ + \frac{1}{n} \|\varphi\|_{L^N(D)} \|u_n - u\|_{L^N(D)} \to 0 \text{ as } n \to \infty. \]  

And then, \( \limsup_{n \to \infty} \int_D |\nabla u_n|^{N-2} \nabla u_n \nabla (u_n - u) dx \leq 0. \) Therefore, (71) follows by \((S_+)\) property.

By using (71) and (74), we can pass to the limit in (67) to obtain

\[ \int_D |\nabla u|^{N-2} \nabla u \nabla w dx + \int_D |u|^{N-2} u w dx = \lambda \int_D a(x) u^{q-1} w dx + \int_D f(u) w dx, \]  

for all \( w \in W^{1,N}_0(D). \) Thus, \( u \) is a solution of \((PD)\).

Now, fix a positive constant \( \lambda \) such that

\[ \lambda < \lambda^* = \frac{\varrho^{N-q}}{4K_1}. \]  

Since \( a > 0 \) is a continuous function, define

\[ a_D = \inf_D a(x). \]
Then, according to Lemma 2.3, there exists a positive solution \( u_0 \) of

\[
\begin{cases}
-\Delta_N u + |u|^{N-2}u = \lambda a_D u^{q-1} & \text{in } D, \\
u > 0 & \text{in } D, \\
u = 0 & \text{on } \partial D.
\end{cases}
\]

Let \( u_n \) be a positive solution of problem \((PD_n)\) obtained by Lemma 4.2. We observe that \( u_0/u_n, u_n/u_0 \in L^\infty(D) \) because \( u_0 \) and \( u_n \) are positive functions belonging to \( C_0^{1,\beta} (\overline{D}) \) and satisfying \( \partial u_n/\partial \nu < 0, \partial u_0/\partial \nu < 0 \) on \( \partial D \). Notice that

\[
\lambda a(x) t^{q-1} + f_n(t) + \frac{q}{n} \geq \lambda a_D t^{q-1} = g(t).
\]

Hence, \( u_1 = u_0 \) and \( u_2 = u_n \) are a positive subsolution and a positive supersolution of problem (25), respectively. Thus, by Proposition 3 we see that \( u_n \geq u_0 \) in \( D \) for every \( n > n^* \). Therefore, by passing to the limit we obtain

\[ u \geq u_0 \quad \text{a.e. in } D. \]

Thus, we conclude that \( u \) is a positive solution of problem \((PD)\).

From now, the solution we just found will be labeled as \( u_\lambda \) with explicit dependence on \( \lambda \). In what follows, we will deduce that \( \|u_\lambda\|_{W^{1,N}(D)} \to 0 \) as \( \lambda \to 0 \). Fix the pair \((\lambda, u_\lambda)\), where \( \lambda \in (0, \lambda^*) \) and \( u_\lambda \) is the corresponding solution of problem \((PD)\). By using \( w = u_\lambda \) as a test function in (76), we obtain

\[
\int_{\mathbb{R}^N} |\nabla \tilde{u}_\lambda|^N dx + \int_{\mathbb{R}^N} \tilde{u}_\lambda^N dx = \int_D |\nabla u_\lambda|^N dx + \int_D u_\lambda^N dx
\]

\[= \lambda \int_D a(x) u_\lambda^q dx + \int_D f(u_\lambda) u_\lambda dx
\]

\[= \lambda \int_{\mathbb{R}^N} a(x) \tilde{u}_\lambda^q dx + \int_{\mathbb{R}^N} f(\tilde{u}_\lambda) \tilde{u}_\lambda dx
\]

\[\leq \lambda K_1 \|\tilde{u}_\lambda\|_{W^{1,N}(\mathbb{R}^N)}^q + K_2 C(\alpha, N) \|\tilde{u}_\lambda\|_{W^{1,N}(\mathbb{R}^N)}^p,
\]

(78)

where \( K_1, K_2 \) are given in (36), (37), respectively.

Since \( \tilde{u}_\lambda \neq 0 \), from (78), we have the following estimate

\[
\|\tilde{u}_\lambda\|_{W^{1,N}(\mathbb{R}^N)}^{N-q} (1 - K_2 C(\alpha, N) \|\tilde{u}_\lambda\|_{W^{1,N}(\mathbb{R}^N)}^{p-N}) \leq \lambda K_1.
\]

(79)

By combining (47) and (68), we obtain

\[
\|\tilde{u}_\lambda\|_{W^{1,N}(\mathbb{R}^N)}^{p-N} \leq \frac{1}{2K_2 C(\alpha, N)}.
\]
Thus,
\[ \|u_\lambda\|_{W^{1,N}(D)} = \|\tilde{u}_\lambda\|_{W^{1,N}(\mathbb{R}^N)} \leq (2\lambda K_1)^{1/(N-q)} \to 0 \quad \text{as} \quad \lambda \to 0. \] (80)

Thus, the proof of the theorem is complete.

5 Proof of the main theorem

5.1 A priori estimates

In this subsection, for convenience, when necessary, we will omit the notation \( \tilde{u} \). In order to prove Theorems 1.1 and 1.2, it will be needed a couple of estimates proven in the following results. It is important to mention that much less is known about the results of regularity for the \( L_p \) operator, as can be seen in [21]. This becomes an obstacle to obtain uniform estimates in the sense of Hölder norm.

Fix \( u \in W^{1,N}(D) \) any positive solution of \( (PD) \) given by Theorem 4.1. Here, we will borrow some ideas from [16]. Define \( u_M := \min\{u, M\} \) for \( M > 0 \). Choose \( \overline{p}^* \) satisfying \( 2N^2 < \overline{p}^* \). For \( R' > R > 0 \), we take a smooth function \( \eta_{R,R'} \) such that \( 0 \leq \eta_{R,R'} \leq 1, \|\eta_{R,R'}\|_\infty \leq 2/(R'-R) \). \( \eta_{R,R'}(t) = 1 \) if \( t \leq R \) and \( \eta_{R,R'} = 0 \) if \( t \geq R' \).

**Lemma 5.1** Let \( x_0 \in \mathbb{R}^N, M > 0, R'>R>0 \), such that \( B(x_0,R') \subset D, \gamma_1 = N/q > 1 \) and \( \gamma'_1 = N/\overline{p} \) such that \( 1/\gamma_1 + 1/\gamma'_1 = 1 \). Denote \( \eta(x) := \eta_{R,R'}(|x-x_0|) \).

Assume that \( 2N \leq \overline{p} \) (in particular \( \gamma'_1 \leq \overline{p} \)) and \( u \in W^{1,N}(D) \) a solution of \( (PD) \), in particular, \( u \in L^{\overline{p}(N+\beta)}(B(x_0,R')) \) with \( \beta \geq 0 \). Then it holds:

\[ \int_{B(x_0,R')} f(u)uu_M^{\beta} \eta^N dx \leq C(q) \|u\|_{L^{\overline{p}(N+\beta)}(B(x_0,R'))}^{\beta} B_{R'} \] (81)

\[ \int_{B(x_0,R')} a(x)u^q u_M^{\beta} \eta^N dx \leq \|a\|_{L^{\gamma_1}(B(x_0,R'))} \|u\|_{L^{\overline{p}(N+\beta)}(B(x_0,R'))}^{q+\beta} B_{R'} \] (82)

where \( B_{R'} := (1 + |B(0,R')|) \) and \( |B(0,R')| \) denotes the Lebesgue measure of the ball \( B(0,R') \).

**Proof** According to Hölder’s inequality, we easily show (82). So, we will prove only (81). By Young’s inequality, Lemmas 2.2, 3.2, and inequality (38), we obtain

\[
\int_{B(x_0,R')} f(u)uu_M^{\beta} \eta^N dx \\
\leq C \int_{B(x_0,R')} |u|^p u_M^{\beta} \phi_N(\alpha|u|^{N/\gamma_1}) dx \\
\leq C \left( \int_{B(x_0,R')} |u|^{Np} u_M^{N\beta} dx \right)^{1/N} \left( \int_{B(x_0,R')} \phi_N(\alpha|u|^{N/\gamma_1}) dx \right)^{1/N'}
\]
\[
\leq C \left( \int_{B(x_0, R')} |u|^{2Np} \, dx \right)^{1/2N} \left( \int_{B(x_0, R')} u_{M}^{2N\beta} \, dx \right)^{1/2N} C(\alpha, N)
\]
\[
\leq C \|u\|^{p}_{L^{2Np}(B(x_0, R'))} \|u\|^{\beta}_{L^{2N\beta}(B(x_0, R'))}
\]
\[
\leq C(\rho) \|u\|^{\beta}_{L^{2N\beta}(B(x_0, R'))},
\]

Because \( \|u\|^{\alpha}_{L^{p}(B(x_0, R'))} \leq C \|\tilde{u}\|_{W^{1,N}(\mathbb{R}^{N})} \leq C\rho \).

Since \( \tilde{\rho}(N+\beta) > 2N\beta \), by Hölder’s inequality we obtain
\[
\left( \int_{B(x_0, R')} u_{M}^{2N\beta} \, dx \right)^{1/2N} \leq \|u\|^{\beta}_{L^{\tilde{\rho}(N+\beta)}(B(x_0, R'))} |B(x_0, R')|^\frac{\tilde{\rho}(N+\beta)-2N\beta}{2N\beta(N+\beta)},
\]

that conclude the inequality (81). \( \square \)

**Lemma 5.2** Let \( x_0 \in \mathbb{R}^{N}, R' > R > 0 \), such that \( B(x_0, R') \subset D, \gamma_1 = \frac{N}{N-q} > 1 \) and \( \gamma_1' = \frac{N}{q} \). Assume that \( 2N \leq \tilde{\rho} \) and \( u \in W^{1,N}(D) \) a solution of (PD). As \( u \in L^{\tilde{\rho}(N+\beta)}(B(x_0, R')) \) with \( \beta \geq 0 \),

\[
\|u\|^{\beta}_{L^{\tilde{\rho}(N+\beta)}(B(x_0, R'))} \leq 2^{N}(N+\beta)^{N} C^{*} \beta_{R'}(C_{R'} + D_{R,R'}) \max\{1, \|u\|^{\beta}_{L^{\tilde{\rho}(N+\beta)}(B(x_0, R'))}\}^{N+\beta} \quad(83)
\]

holds with

\[
B_{R'} := 1 + |B(0, R')|, \quad C_{R'} := C(\rho) + \lambda^{*}\|a\|_{L^{\gamma_1}(B(x_0, R'))}, \quad D_{R,R'} := \frac{N^{N}2^{2N-1} + 2^{N-1}}{(R' - R)^{N}},
\]

where \( C^{*} \) is the positive constant embedding from \( W^{1,N}(\mathbb{R}^{N}) \) to \( L^{\tilde{\rho}^{*}}(\mathbb{R}^{N}) \).

**Proof** Taking \( uu_{M}^{\beta} \eta^{N} \in W^{1,N}_{0}(B(x_0, R')) \) (for \( M > 0 \)) as a test function in (76), where \( \eta(x) = \eta_{R,R'}(|x - x_0|) \), and by Lemma 5.1, we obtain

\[
\begin{aligned}
C(\rho) \|u\|^{\beta}_{L^{\tilde{\rho}(N+\beta)}(B(x_0, R'))} &\leq B_{R'} \\
&+ \lambda^{*}\|a\|_{L^{\gamma_1}(B(x_0, R'))} \|u\|^{q+\beta}_{L^{\tilde{\rho}(N+\beta)}(B(x_0, R'))} B_{R'} \\
&\geq \int_{B(x_0, R')} |\nabla u|^{N} u_{M}^{\beta} \eta^{N} \, dx + \int_{B(x_0, R')} u_{M}^{N+\beta} \eta^{N} \, dx \\
&- \frac{2N}{R' - R} \int_{B(x_0, R')} |\nabla u|^{N-1} u_{M}^{\beta} u \eta^{N-1} \, dx,
\end{aligned}
\]
where we use $|\nabla \eta| \leq 2/(R' - R)$. From Young’s inequality and Hölder’s inequality, we obtain

$$\frac{2N}{R' - R} \int_{B(x_0, R')} |\nabla u|^{N-1} u^\beta_M u^\eta^{N-1} \, dx$$

$$\leq \frac{1}{2} \int_{B(x_0, R')} |\nabla u|^N u^\beta_M u^\eta^N \, dx + \frac{2N N 2^{N-1}}{(R' - R)^N} \int_{B(x_0, R')} u^{N+\beta} \, dx$$

$$\leq \frac{1}{2} \int_{B(x_0, R')} |\nabla u|^N u^\beta_M u^\eta^N \, dx + \frac{N N 2^{N-1}}{(R' - R)^N} \|u\|_{L^p(B(x_0, R'))}^{N+\beta} B_{R'}.$$  (85)

Thus, from (84) and (85) we have

$$B_{R'} \left( C_{R'} + \frac{N N 2^{N-1}}{(R' - R)^N} \right) \max\{1, \|u\|_{L^p(B(x_0, R'))}^{N+\beta}\}$$

$$\geq \frac{1}{2} \int_{B(x_0, R')} |\nabla u|^N u^\beta_M u^\eta^N \, dx + \int_{B(x_0, R')} u^{N+\beta} \, dx.$$  (86)

Moreover, by using

$$\|\nabla (u^1_{M+\beta/N} \eta)\|_{L^N(X)}^N \leq 2^{N-1} \left\{ \|\eta \nabla (u^1_{M+\beta/N})\|_{L^N(X)}^N + \|u^1_{M+\beta/N} \nabla \eta\|_{L^N(X)}^N \right\}$$

$$\leq 2^{N-1} \left( 1 + \frac{\beta}{N} \right) N \int_{B(x_0, R')} |\nabla u|^N u^\beta_M u^\eta^N \, dx + \frac{2^{N-1}}{(R' - R)^N} \int_{B(x_0, R')} u^{N+\beta} \, dx$$

and Hölder’s inequality, due to the embedding from $W^{1,N}(X)$ to $L^{p^*}(X)$, we have

$$\frac{1}{2} \int_{B(x_0, R')} |\nabla u|^N u^\beta_M u^\eta^N \, dx + \int_{B(x_0, R')} u^{N+\beta} \, dx$$

$$\geq 2^{-N} N^N (N + \beta)^{-N} \left\{ \|\nabla (u^1_{M+\beta/N} \eta)\|_{L^N(X)}^N + \|u^1_{M+\beta/N} \nabla \eta\|_{L^N(X)}^N \right\}$$

$$- \frac{2^{N-1}}{(R' - R)^N} \int_{B(x_0, R')} u^{N+\beta} \, dx$$

$$\geq 2^{-N} N^N (N + \beta)^{-N} \|u^1_{M+\beta/N} \eta\|_{W^{1,N}(X)}^N$$

$$- \frac{2^{N-1}}{(R' - R)^N} \|u\|_{L^p(B(x_0, R'))}^{N+\beta} \left( 1 + |B(0, R')| \right)$$

$$\geq 2^{-N} N^N (N + \beta)^{-N} C_{N}^{-N} \|u^1_{M+\beta/N} \eta\|_{L^{p^*}(X)}^N$$

$$- \frac{2^{N-1}}{(R' - R)^N} \|u\|_{L^p(B(x_0, R'))}^{N+\beta} B_{R'}$$

$$\geq 2^{-N} N^N (N + \beta)^{-N} C_{N}^{-N} \|u_M\|_{L^{p^*(N+\beta)/N}(B(x_0, R))}^{N+\beta}$$

$$- \frac{2^{N-1}}{(R' - R)^N} \|u\|_{L^p(B(x_0, R'))}^{N+\beta} B_{R'}.$$  (87)
Consequently, it follows from (86) and (87) that
\[
2^{-N} N^N (N + \beta)^{-N} C_*^{-N} \|uM\|_{L^{p^*(N+\beta)/N}(B(x_0, R))}^{N+\beta} \\
\leq B^*(C R + D_{R,R^*}) \max\{1, \|u\|_{L^p(N+\beta)(B(x_0, R^*))}^{N+\beta}\}. \tag{88}
\]
The conclusion follows by applying Fatou’s lemma and letting \(M \to \infty\) in (88). \(\square\)

**Proposition 4** Assume the assumptions of Lemma 5.2. Let us suppose \(x_0 \in D\) and \(a\) a positive function such that \(a \in L^{N-q}/N\), \(f\) satisfying (6), \(R_0 > 0\) such that \(B(x_0, 2R_0) \subset D\). If \(u \in W^{1,N}(D)\) is a solution of \((PD)\), then \(u \in L^\infty(D)\). Furthermore, \(\|u\|_{W^{1,N}(D)} \leq C\) implies \(\|u\|_{L^\infty(D)} \leq \tilde{C}\).

**Proof** Since \(2N < \tilde{p}^*/N\), we can choose \(\tilde{p}\) such that
\[
2N < \tilde{p} < \frac{\tilde{p}^*}{N}.
\]

Let \(R_0 > 0\) satisfying \(B(x_0, 2R_0) \subset D\). Put
\[
A := \lambda^* \|a\|_{L^1(R^N)}.
\]
Define sequences \(\{\beta_n\}, \{R_n\}\) and \(\{R_n\}\) by
\[
\beta_0 := \frac{\tilde{p}^*}{\tilde{p}} - N > 0, \quad \tilde{p}(N + \beta_{n+1}) = \frac{\tilde{p}^*}{N}(N + \beta_n),
\]
\[
R'_n := (1 + 2^{-n})R_0, \quad R_n := R_{n+1}.
\]
Since \(u \in W^{1,N}(D)\), by using the embedding of \(W^{1,N}(D)\) to \(L^{\tilde{p}^*}(D)\), we see that \(u \in L^{\tilde{p}^*}(D) = L^{\tilde{p}(N+\beta_0)(D)}\).

Let \(x_0 \in D\) fixed. Lemma 5.2 guarantees that if \(u \in L^{\tilde{p}(N+\beta_0)(B(x_0, R'_n))}\), then \(u \in L^{\tilde{p}^*(N+\beta_n)(B(x_0, R_n))} = L^{\tilde{p}(N+\beta_{n+1})(B(x_0, R'_{n+1}))}\). Notice that
\[
B_{R'_n} \leq (1 + |B(0, 2R_0)|) =: B_0,
\]
\[
C_{R'_n} \leq C(q) + A + 1 =: C_0,
\]
\[
D_{R_n, R'_n} = \frac{N^N 2^{N-1} + 2^{N-1}}{R_0} 2^{N(n+1)} = C' 2^{N(n+1)} =: D_n
\]
for any \(n \geq 0\) with \(C'\) independent of \(n\). By setting
\[
b_n := \max\{1, \|u\|_{L^{\tilde{p}(N+\beta_n)}(B(x_0, R'_n))}\},
\]
and by Lemma 5.2 we obtain
\[
b_{n+1} \leq C \frac{1}{\beta_n} (N + \beta_n) \frac{N}{\beta_n} (C_0 + D_n) \frac{1}{\beta_n} b_n \tag{89}
\]
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for every \( n \geq 0 \) with \( C := 2^N (C_* + 1)^N B_0 \). Put \( P := \hat{p} N / \overline{p}^* < 1 \). Then, \( N + \beta_{n+1} = (N + \beta_n) / P \), \( \beta_{n+1} > \beta_n / P > \beta_0 (1 / P)^{n+1} \to \infty \) as \( n \to \infty \). Moreover, we see that

\[
S_1 := \sum_{n=0}^{\infty} \frac{1}{N + \beta_n} = \frac{1}{N + \beta_0} \sum_{n=0}^{\infty} P^n = \frac{1}{(N + \beta_0)(1 - P)} < \infty,
\]

\[
S_2 := \ln \prod_{n=0}^{\infty} (N + \beta_n)^{\frac{N}{N + \beta_n}} = \frac{N}{N + \beta_0} \sum_{n=0}^{\infty} P^n \left( \ln(N + \beta_0) + n \ln P^{-1} \right) < \infty
\]

and

\[
S_3 := \ln \prod_{n=0}^{\infty} (C_0 + D_n)^{\frac{1}{N + \beta_n}} = \sum_{n=0}^{\infty} \frac{P^n}{N + \beta_0} \ln(C_0 + D_n)
\]

\[
\leq \sum_{n=0}^{\infty} \frac{P^n}{N + \beta_0} N(n + 1) \ln(C_0 + C') 2 < \infty.
\]

As a result, by iteration in (89) and \( \hat{p}(N + \beta_0) = \overline{p}^* \), we obtain

\[
\|u\|_{L^p(N+\beta_n)(B(x_0, R_*))} \leq b_n \leq C S_1 e^{S_2} e^{S_3} \max\{1, \|u\|_{L^p^*(B(x_0, 2R_*))}\}
\]

for every \( n \geq 1 \). Letting \( n \to \infty \), this ensures that

\[
\|u\|_{L^\infty(B(x_0, R_*))} \leq C S_1 e^{S_2} e^{S_3} \max\{1, \|u\|_{L^p^*(B(x_0, 2R_*))}\}. \tag{90}
\]

By using the embedding of \( W^{1,N}(D) \) to \( L^{p^*}(D) \), (90) yields that

\[
\|u\|_{L^\infty(B(x_0, R_*))} \leq C S_1 e^{S_2} e^{S_3} \max\{1, \|u\|_{L^p^*(D)}\}
\]

\[
\leq C S_1 e^{S_2} e^{S_3} \max\{1, C_* \|u\|_{W^{1,N}(D)}\} \tag{91}
\]

whence \( u \) is bounded in \( D \) because \( x_0 \in D \) is arbitrary and the constant \( C S_1 e^{S_2} e^{S_3} \) is independent of \( x_0 \).

\( \square \)

5.2 Proof of Theorem 1.1

In this section, we denote \( B_n := B_n(0) \) the open ball centered at the origin with radius \( n \). Throughout this section, we will consider \( \lambda \in (0, \lambda^*) \) fixed. The space \( W^{1,N}(B_n) \) is endowed with the norm

\[
\|u\|_{N,n}^N := \int_{B_n} \left( |\nabla u|^N + |u|^N \right) dx.
\]
By applying Theorem 4.1 with $D = B_n (n \in \mathbb{N})$, we obtain a solution $u_n \in W^{1,N}_0(B_n)$ of the problem

$$
\begin{cases}
-\Delta_N u + |u|^{N-2}u = \lambda a(x)|u|^{q-2}u + f(u) & \text{in } B_n \\
u > 0 & \text{in } B_n \\
u(x) = 0 & \text{on } \partial B_n.
\end{cases} \quad (P_n)
$$

Again, (70) and (80) show the boundedness of $\{u_n\}_{n \in \mathbb{N}}$ in $W^{1,N}_0(D)$. That is,

$$\|u_n\|_{N,n} \leq \tilde{\varrho} \quad \text{for all } n \in \mathbb{N},$$

where

$$\tilde{\varrho} := \min \left\{ (2\lambda C_1)^{1/(N-q)}, \rho \right\}$$

is independent of $B_n$.

If $n \geq m + 1$, notice that

$$\int_{B_{m+1}} (|\nabla u_n|^{N-2}\nabla u_n \nabla \varphi + |u_n|^{N-2}u_n \varphi - \lambda a(x)|u_n|^{q-2}u_n \varphi - f(u_n)\varphi) dx = 0,$$

for all $\varphi \in C_0^\infty(B_{m+1})$. By (92) we obtain

$$\|u_n\|_{N,m+1} \leq \|u_n\|_{N,n} \leq \tilde{\varrho}.$$

Therefore, there exists $u_\lambda \in W^{1,N}(B_{m+1})$ such that

$$u_n \rightharpoonup u_\lambda \quad \text{in } W^{1,N}(B_{m+1}),$$

$$u_n \to u_\lambda \quad \text{in } L^s(B_{m+1}), \quad s \geq N,$$

$$u_n(x) \to u_\lambda(x) \quad \text{a.e. } x \in B_{m+1},$$

as $n \to \infty$.

By inequality (91) in Proposition 4 and (94), we infer that

$$\|u_n\|_{L^\infty(B_{m+1})} \leq C S_1 e^{S_2} e^{S_3} \max\{1, C_\ast \tilde{\varrho}\} := \Theta.$$

Now we use regularity result up to the boundary due to Lieberman [29, Theorem 1.7], to conclude from (98) that

$$\|u_n\|_{C^{1,\beta} (\overline{B_m})} \leq \vartheta,$$

where $\beta \in (0, 1)$ and $\vartheta$ is independent of $n$. Thus, using (99) and Arzelà-Ascoli theorem, we conclude that

$$u_\lambda \in C^{1,\alpha}(\overline{B_m}) \quad \text{for some } \alpha \in (0, \beta).$$
We also have
\[ f(u_n) \to f(u_\lambda) \text{ weakly in } L^{N'}(B_{m+1}), \]  
(101)
as \( n \to \infty \). Indeed, notice that
\[
\int_{B_{m+1}} |f(u_n)|^N \, dx \leq \int_{B_{m+1}} (u_n^{(p-1)} \phi_N(2^{\frac{N}{N-1}} \alpha |u_n|^{\frac{N}{N-1}}))^N \, dx
\]
\[
\leq \Theta (p-1)^N \int_{B_{m+1}} \phi_N(2^{\frac{2N}{N-1}} \alpha \|u_n\|_{W^{1,N}(B_n)}^N) \frac{u_n}{\|u_n\|_{W^{1,N}(B_n)}^{N-1}} \, dx
\]
\[
\leq \Theta (p-1)^N C(\alpha, N)^N < \infty. \tag{102}
\]

Since
\[ u_n \to u_\lambda \text{ a.e. in } B_{m+1}, \]
by the continuity of \( f \) we have
\[ f(u_n(\cdot)) \to f(u_\lambda(\cdot)) \text{ a.e. in } B_{m+1}. \tag{103} \]

Hence, [20, Theorem 13.44] leads to
\[ f(u_n) \to f(u_\lambda) \text{ weakly in } L^{N'}(B_{m+1}). \tag{104} \]

Let us show that \( u_n \) converges to \( u_\lambda \) strongly in \( W^{1,N}(B_m) \). To this end, fix \( l \in \mathbb{N} \) and choose a smooth function \( \psi_l \) satisfying \( 0 \leq \psi_l \leq 1 \), \( \psi_l(r) = 1 \) if \( r \leq m \) and \( \psi_l(r) = 0 \) if \( r \geq m + 1/l \). Setting \( \eta_l(x) := \psi_l(|x|) \), we note that \( (u_n - u_\lambda) \eta_l \in W^{1,N}_0(B_{m+1}) \subset W^{1,N}_0(B_n) \) for any \( n \geq m + 1 \). Denote
\[ V_n = \int_{B_m} |\nabla u_n|^{N-2} \nabla u_n (\nabla u_n - \nabla u_\lambda) \, dx. \]

Using \( (u_n - u_\lambda) \eta_l \) as a test function in (93) and invoking the growth condition (6), we obtain
\[
V_n = \int_{|x|<m+1/l} (\lambda a(x) u_n^{q-1} + f(u_n) - u_n^{N-1}) (u_n - u_\lambda) \eta_l \, dx
\]
\[ - \int_{m \leq |x|<m+1/l} |
\]
\[ - \int_{m \leq |x|<m+1/l} |
\]
\[ - \int_{m \leq |x|<m+1/l} |
\]
\[ \leq \int_{B_{m+1}} (\lambda a(x) u_n^{q-1} + u_n^{p-1} \phi_N(2^{\frac{N}{N-1}} \alpha |u_n|^{\frac{N}{N-1}}) + u_n^{N-1}) |u_n - u_\lambda| \, dx
\]
where \( d_l := \sup_{|x| < m + 1/l} |\nabla \eta_l(x)| \).

By Hölder’s inequality and (92) we have

\[
I^1_n \leq \|u_n - u_\lambda\|_{L^N(B_{m+1})} \left\{ \lambda \|a\|_{L^{N/(N-q)}(B_{m+1})} \|u_n\|_W^{q} \right. \\
+ \|u_n\|^{N-1}_{W^{1,N}(B_{m+1})} \left\} \\
\leq \overline{C} \|u_n - u_\lambda\|_{L^N(B_{m+1})},
\]

where \( \overline{C} \) is a positive constant independent of \( u_n, n, m \) and \( l \).

By Hölder’s inequality, the following estimates follow:

\[
I^2_n \leq \|u_n\|_{W^{1,N}(B_{m+1})}^{N-1} \left( \int_{m \leq |x| < m + 1/l} |\nabla u_\lambda|^N \, dx \right)^{1/N},
\]

\[
I^3_n \leq d_l \|u_n - u_\lambda\|_{L^N(B_{m+1})} \|\nabla u_n\|_{L^N(B_{m+1})}^{N-1}.
\]

Thereby, from (92) and (96) we derive

\[
\limsup_{n \to \infty} V_n \leq \tilde{Q}^{N-1} \left( \int_{m \leq |x| < m + 1/l} |\nabla u_\lambda|^N \, dx \right)^{1/N},
\]

for all \( l \in \mathbb{N} \). Thus, letting \( l \to \infty \), we obtain

\[
\limsup_{n \to \infty} V_n \leq 0. \tag{105}
\]

As known from (95), \( u_n \) weakly converges to \( u_\lambda \) in \( W^{1,N}(B_m) \), so we may write

\[
V_n + o(1) = \int_{B_m} \left( |\nabla u_n|^{N-2} \nabla u_n - |\nabla u_\lambda|^{N-2} \nabla u_\lambda \right) (\nabla u_n - \nabla u_\lambda) \, dx \\
\geq \left( \|\nabla u_n\|_{L^N(B_m)}^{N-1} - \|\nabla u_\lambda\|_{L^N(B_m)}^{N-1} \right) \left( \|\nabla u_n\|_{L^N(B_m)} - \|\nabla u_\lambda\|_{L^N(B_m)} \right) \\
\geq 0. \tag{106}
\]

By (105) and (106), we obtain \( \lim_{n \to \infty} V_n = 0 \), \( \lim_{n \to \infty} \|\nabla u_n\|_{L^N(B_m)} = \|\nabla u_\lambda\|_{L^N(B_m)} \). This implies that \( u_n \) converges to \( u_\lambda \) strongly in \( W^{1,N}(B_m) \) because the spaces \( W^{1,N}(B_m) \) is uniformly convex.

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Now, since \( u_n > 0 \) in \( B_m \), we infer that \( u_\lambda \) is a nonnegative solution of the problem

\[-\Delta_N u_\lambda + u_\lambda^{N-1} = \lambda a(x) u_\lambda^{q-1} + f(u_\lambda) \quad \text{in} \ B_m, \quad u_\lambda \geq 0 \quad \text{on} \ \partial B_m.\]

By using (24) and since \( C_0^\infty(\mathbb{R}^N) \) is dense in \( W^{1,N}(\mathbb{R}^N) \), by a diagonal argument, there exist a relabeled subsequence of \( \{\tilde{u}_n\} \) and a function \( u_\lambda \in W^{1,N}(\mathbb{R}^N) \) such that

\[\tilde{u}_n \to u_\lambda \quad \text{in} \quad W^{1,N}(\mathbb{R}^N),\]

\[\tilde{u}_n(x) \to u_\lambda(x) \quad \text{for a.e } x \in \mathbb{R}^N.\]

These convergence properties, and some iteration process, ensure that \( u_\lambda \) is a solution of problem (5), that belongs to \( C^{1}_{loc}(\mathbb{R}^N) \) (see (100)).

The next step in the proof is to show that \( u_\lambda \) does not vanish in \( \mathbb{R}^N \). Indeed, Lemma 2.3 provides a solution \( u_{\lambda,m} \) of the problem

\[
\begin{cases}
-\Delta_N u + |u|^{N-2} u = \lambda a_m |u|^{q-2} u & \text{in } B_m \\
u > 0 & \text{in } B_m \\
u(x) = 0 & \text{on } \partial B_m,
\end{cases}
\]

where

\[a_m = \inf_{B_m} a(x).\]

Since \( \lambda a(x) t^{q-1} + f(t) \geq \lambda a_m t^{q-1} \) for all \( x \in \mathbb{R}^N \). We are thus in a position to apply Proposition 2 to the functions \( u_{\lambda,m} \) and \( \tilde{u}_n \) with \( n > m \), in place of \( u_1 = u_{\lambda,m} \) and \( u_2 = \tilde{u}_n \), respectively, which renders \( \tilde{u}_n \geq u_{\lambda,m} \) in \( B_m \) for every \( n > m \). This enables us to deduce that \( u_\lambda \geq u_{\lambda,m} \) in \( B_m \), so \( u_\lambda(x) > 0 \) for every \( x \in \mathbb{R}^N \), because \( m \) was arbitrary chosen.

Furthermore, since \( \tilde{u}_n \) weakly converges to \( u_\lambda \) in \( W^{1,N}(\mathbb{R}^N) \), by (94) we have

\[\|u_\lambda\|_{W^{1,N}(\mathbb{R}^N)} \leq \liminf_{n \to \infty} \|\tilde{u}_n\|_{W^{1,N}(\mathbb{R}^N)} = \liminf_{n \to \infty} \|\tilde{u}_n\|_{N,N} \leq \tilde{q}, \quad (107)\]

and then, we obtain \( u_\lambda \in W^{1,N}(\mathbb{R}^N) \), and

\[\|u_\lambda\|_{W^{1,N}(\mathbb{R}^N)} \to 0, \quad \text{as } \lambda \to 0. \quad (108)\]

Thus, the proof of Theorem 1.1 is complete.

### 5.3 Proof of Proposition 1

In this proof, we borrowed some ideas from [7]. For the sake of contradiction, suppose that \( \lambda^* = \infty \). Thus, there is a sequence \( \lambda_n \to \infty \) and corresponding solutions \( u_{\lambda_n} > 0 \)
in \( \mathbb{R}^N \) given by Theorem 1.1. In what follows, define \( a_R = \inf_{B_R(0)} a(r) \), for some \( R > 0 \) fixed. Define also

\[
P(x, t) = \lambda a(x)t^{q-1} + t^{p-1}\phi_N(\alpha|t|^\frac{N}{p-1})
\]

and

\[
P_1(t) = \Lambda t^{q-1} + t^{p-1}\phi_N(\alpha|t|^\frac{N}{p-1}),
\]

where \( \Lambda = \lambda a_R \).

We claim that there exists a constant \( C_\Lambda > 0 \) such that

\[
P(x, t) \geq C_\Lambda t^{N-1} \quad \text{for every} \quad t > 0.
\]

Indeed, define the function \( Q(t) = P_1(t)t^{-(N-1)} \). Then, since \( q < N < p \), \( Q(t) \to \infty \) as \( t \to 0^+ \) and as \( t \to \infty \). The minimum value is \( Q(t_1) = C_\Lambda \), where \( t_1 > 0 \) is the unique solution of

\[
\phi_N(\alpha t^{\frac{N}{N-1}})(p - N)t^{p-q} + \alpha \frac{N}{N-1} t^{\frac{(N-1)(p-q+1)}{N-1}} \phi_{N-1}(\alpha t^{\frac{N}{N-1}}) = \Lambda(N - q).
\]

Indeed, let us define

\[
H(t) := \phi_N(\alpha t^{\frac{N}{N-1}})(p - N)t^{p-q} + \alpha \frac{N}{N-1} t^{\frac{(N-1)(p-q+1)}{N-1}} \phi_{N-1}(\alpha t^{\frac{N}{N-1}}).
\]

Note that \( H(0) = 0 \) and \( H(t) \to \infty \) as \( t \to \infty \). Since

\[
H'(t) = (p - N)(p - q)t^{p-q-1}\phi_N(\alpha t^{\frac{N}{N-1}})
+ \frac{\alpha N(p - N)}{N-1} t^{p-q-1} \phi_{N-1}(\alpha t^{\frac{N}{N-1}})
+ \frac{N\alpha}{N-1} \left[ \frac{(N-1)(p-q+1)+1}{N-1} \right] t^{\frac{(N-1)(p-q+1)+1}{N-1}} \phi_{N-1}(\alpha t^{\frac{N}{N-1}})
+ \frac{\alpha^2 N^2}{(N-1)^2} t^{\frac{(N-1)(p-q+1)+1}{N-1}} \phi_{N-2}(\alpha t^{\frac{N}{N-1}}) > 0, \quad \forall t \geq 0,
\]

hence \( H \) is increasing. Therefore, there is a unique point \( t_1 > 0 \) such that \( H(t_1) = \Lambda(N - q) \). Consequently, by the arguments before, \( t_1 > 0 \) is the unique point of minimum of \( Q \), with \( Q(t_1) = C_\Lambda \).

Denote by \( \sigma_1 > 0 \) and \( \varphi_1 > 0 \), respectively, the principal eigenvalue and corresponding eigenfunction of the eigenvalue problem

\[
\begin{cases}
-\Delta_N(\varphi_1) = \sigma_1|\varphi_1|^{N-2}\varphi_1 & \text{in} \ B_R(0) \\
\varphi_1 = 0 & \text{on} \ \partial B_R(0).
\end{cases}
\]
Since $C_n$ increases as $\lambda_n$ increases, for $\delta > 0$ small enough, there is $\lambda_0$ in this sequence such that the corresponding constant satisfies $C_{\lambda_0} \geq \sigma_1 + \delta + 1$. Hence the solution $u_{\lambda_0}$ of (5) associated to $\lambda_0$ satisfies $u_{\lambda_0} > 0$ in $\mathbb{R}^N$ and
\[
\begin{cases}
-\Delta u_{\lambda_0} \geq (C_{\lambda_0} - 1)|u_{\lambda_0}|^{N-2}u_{\lambda_0} \geq (\sigma_1 + \delta)|u_{\lambda_0}|^{N-2}u_{\lambda_0} & \text{in } B_R(0), \\
u_{\lambda_0} \geq 0 & \text{on } \partial B_R(0).
\end{cases}
\]
Otherwise, taking $\varepsilon > 0$ small enough we obtain $\varepsilon \varphi_1 < u_{\lambda_0}$ in $B_R(0)$ and
\[
\begin{cases}
-\Delta (\varepsilon \varphi_1) = \sigma_1 |\varepsilon \varphi_1|^{N-2}(\varepsilon \varphi_1) \leq (\sigma_1 + \delta)|\varepsilon \varphi_1|^{N-2}(\varepsilon \varphi_1) & \text{in } B_R(0), \\
\varphi_1 = 0 & \text{on } \partial B_R(0).
\end{cases}
\]
By the method of subsolution and supersolution, see [7, Theorem 2.1], there is a solution $\varepsilon \varphi_1 \leq \omega \leq u_{\lambda_0}$ in $B_R(0)$ of
\[
\begin{cases}
-\Delta \omega = (\sigma_1 + \delta)|\omega|^{N-2}\omega & \text{in } B_R(0), \\
\omega = 0 & \text{on } \partial B_R(0).
\end{cases}
\]
Hence there is a contradiction to the fact that $\sigma_1$ is isolated (see [30]). Therefore, $\lambda^* < \infty$.

5.4 Proof of Theorem 1.2

By using the same arguments as in the proof of Theorem 1.1, with $D = B_n$, and considering (107), we can pass to the limit $n \to +\infty$ in (91) to obtain
\[
\|u_\lambda\|_{L^\infty(\mathbb{R}^N)} \leq C S_1 e^{S_2} e^{S_3} \max\{1, C_* \tilde{q}\}. \tag{109}
\]
Since $u_\lambda$ is bounded in $\mathbb{R}^N$, we put $M_0 := \|u_\lambda\|_{L^\infty(\mathbb{R}^N)}$. Then, by Lemma 5.1, we have
\[
\begin{align*}
\int_{B(x_0, R')} f(u_\lambda) u_\lambda^\beta M \eta^N dx & \leq C(\varphi) \|u_\lambda\|_{L^{\tilde{p}(N+\beta)}(B(x_0, R'))}^\beta B_{R'} \tag{110} \\
\int_{B(x_0, R')} a(x) u_\lambda^q M \eta^N dx & \leq \|a\|_{L^{q}(B(x_0, R'))} M_0^q \|u_\lambda\|_{L^{\tilde{p}(N+\beta)}(B(x_0, R'))}^\beta B_{R'}. \tag{111}
\end{align*}
\]
By Hölder inequality, we obtain
\[
\|u_\lambda\|_{L^{\tilde{p}(N+\beta)}(B(x_0, R'))}^{N+\beta} \leq M_0^N \|u_\lambda\|_{L^{\tilde{p}(N+\beta)}(B(x_0, R'))}^\beta B_{R'}. \tag{112}
\]
Now, fix $x_0 \in \mathbb{R}^N$. It follows from the argument as in the proof of Lemma 5.2 with (110), (111) and (112) that
\[ \|u_\lambda\|_{L^{N+\beta}(N+\beta)(B(x_0, R))} \leq 2^N (N + \beta)^N C_\ast B^R(C_R + D_R, R') (M_0 + 1)^N \|u_\lambda\|_{L^{\tilde{p}(N+\beta)}(B(x_0, R'))}^{\tilde{p}} \] 

(113)

provided \( u_\lambda \in L^{\tilde{p}(N+\beta)}(B(x_0, R')) \). Choose \( \gamma_1, \tilde{p} \) and define sequences \( \{\beta_n\} \), \( \{R'_n\} \) and \( \{R_n\} \) as in the proof of Proposition 4. Set

\[ V_n := \|u_\lambda\|_{L^{\tilde{p}(N+\beta)}(B(x_0, R'_n))}^{\beta_n}. \]

The remainder of the proof follows with the same arguments as in in the proof of [16, Theorem 2] to conclude that

\[ V_n^{N+\beta_n-1} \leq C (N + \beta_n-1)^N (C_0 + D_{n-1}) V_{n-1} \]

(114)

with \( C := 2^N C_\ast B_0 (M_0 + 1)^N \). Recall that

\[ \beta_n + N = P^{-1} (N + \beta_{n-1}) \quad \text{and} \quad \frac{N}{N + \beta_0} = P. \]

Define

\[ Q_n := \prod_{k=2}^{n+1} \left( 1 + \frac{P^k}{1 - P^k} \right) = \prod_{k=2}^{n+1} (1 - P^k)^{-1} \quad \text{and} \quad W_n := (C_0 + D_n). \]

Then, the inequality (114) leads to

\[
\ln V_n \leq \frac{\beta_n}{N + \beta_{n-1}} \left( \ln V_{n-1} + \ln[C (N + \beta_{n-1})^N] + \ln W_{n-1} \right) \\
= P^{-1} \left( 1 - P^{n+1} \right) \left( \ln V_{n-1} + N \ln[C P^{-n+1} (N + \beta_0)] + \ln W_{n-1} \right) \\
\leq P^{-1} \left( 1 - P^{n+1} \right) \ln V_{n-1} + N P^{-1} \ln[(C + 1) P^{-n+1} (N + \beta_0)] + P^{-1} \ln W_{n-1} \\
\leq P^{-n} \left( \prod_{k=1}^{n} \left( 1 - P^{k+1} \right) \right) \ln V_0 + N \sum_{k=1}^{n} P^{-k} \ln[(C + 1) P^{-n+k} (N + \beta_0)] \\
+ \sum_{k=1}^{n} P^{-k} \ln W_{n-k} \\
= P^{-n} Q_n^{-1} \ln V_0 + N \sum_{k=1}^{n} P^{-k} \ln[(C + 1) P^{-n+k} (N + \beta_0)] + \sum_{k=1}^{n} P^{-k} \ln W_{n-k}
\]

for every \( n \) because of \( \ln[(C + 1) P^{-n+1} (N + \beta_0)] > 0 \) and \( \ln W_n > 0 \) for all \( n \). Therefore, we have

\( \hdots \) Springer
\[
\ln \| u_\lambda \|_{L^{p(N+\beta_0)}(B(x_0,R^*_n))} \leq \frac{Q_n^{-1} \ln V_0}{N + \beta_0 - N \beta} + \frac{\sum_{l=0}^{n-1} P^l \ln [(C + 1) P^{-l} (N + \beta_0)]}{N + \beta_0 - N \beta} + \frac{\sum_{l=0}^{n-1} P^l \ln W_l}{N + \beta_0 - N \beta}.
\]

(115)

Here, taking a sufficiently large positive constant \( C' \) independent of \( n \), we see that

\[
\sum_{l=0}^{n-1} P^l \ln [(C + 1) P^{-l} (N + \beta_0)] \leq C' \sum_{l=0}^{\infty} P^l (l + 1) =: S_1 < \infty
\]

and

\[
\sum_{l=0}^{n-1} P^l \ln W_l \leq C' \sum_{l=0}^{\infty} P^l (l + 1) \leq C' \sum_{l=0}^{\infty} P^l (l + 1) =: S_2 < \infty.
\]

Next, we shall show that \( \{ Q_n \} \) is a convergent sequence. It is easily to see that \( \{ Q_n \} \) is increasing. Moreover, setting \( d_k := \ln \left( 1 + \frac{p_k}{1 - p_k} \right) \), we see that

\[
\lim_{k \to \infty} \frac{d_{k+1}}{d_k} = \lim_{k \to \infty} \frac{\ln(1 - P^{k+1})}{\ln(1 - P^k)} = \lim_{k \to \infty} \frac{1 - P^k}{1 - P^{k+1}} P = P < 1
\]

by L’Hospital’s rule. This implies that

\[
\ln Q_n = \sum_{k=2}^{n+1} \ln \left( 1 + P^k \right) \leq \sum_{k=1}^{\infty} \ln \left( 1 + P^k \right) < \infty.
\]

Therefore, \( \{ Q_n \} \) is bounded from above, whence \( \{ Q_n \} \) converges and

\[
1 < \frac{1}{1 - P^2} = Q_1 \leq Q_\infty := \lim_{n \to \infty} Q_n < \infty
\]

holds. Consequently, letting \( n \to \infty \) in (115), we have

\[
\| u_\lambda \|_{L^\infty(B(x_0,R^*_n))} \leq (NS_1 S_2)^{\frac{1}{N + \beta_0}} \| u_\lambda \|_{L^{p(N+\beta_0)}(B(x_0,2R^*_n))}.
\]

(116)

This yields our conclusion since \( \| u_\lambda \|_{L^{p(N+\beta_0)}(B(x_0,2R^*_n))} \to 0 \) as \( |x_0| \to \infty \), \( \beta_0 > 0 \) and the constant \( NS_1 S_2 \) is independent of \( x_0 \).

### 5.5 Proof of Corollary 1

Notice that by (109), \( \| u_\lambda \|_{L^\infty(\mathbb{R}^N)} \) is uniformly bounded in the variable \( \lambda \). Hence, the constant \( C \) in (114) is uniformly bounded in the variable \( \lambda \).
By (116), we obtain
\[ \| u_\lambda \|_{L^\infty(B(x_0,R_\ast))} \leq (NS_1S_2)^{\frac{1}{N+\beta_0}} \| u_\lambda \|_{L^\infty(B(\mathbb{R}^N))}^{\frac{\beta_0}{(N+\beta_0)Q_\infty}} \leq \tilde{C} \| u_\lambda \|_{L^\infty(B(\mathbb{R}^N))}^{\frac{\beta_0}{(N+\beta_0)Q_\infty}}, \]

with \( \tilde{C} \) independent of \( \lambda \). Since \( x_0 \) is arbitrary, on has
\[ \| u_\lambda \|_{L^\infty(\mathbb{R}^N)} \leq \tilde{C} \| u_\lambda \|_{W^{1,N}(\mathbb{R}^N)}^{\frac{\beta_0}{(N+\beta_0)Q_\infty}}, \]

Thus, (108) ensure that
\[ \| u_\lambda \|_{L^\infty(\mathbb{R}^N)} \rightarrow 0, \]
as \( \lambda \rightarrow 0. \)

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**Declarations**

**Conflict of interest** On behalf of all authors, the corresponding author states that there is no conflict of interest.

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