Regularization of an ill-posed problem for elliptic equation with nonlinear source

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Abstract

This paper studies the Cauchy problem for elliptic equations with nonlinear source term. This semilinear problem arises in many application problems, such as Helmholtz equation, elliptic sine-Gordon equation, and other equations formed by, for example, Lane-Emden equation and Poisson equation. However, it is severely ill-posed in the sense of Hadamard. Therefore, we consider theoretical aspects of regularization of the problem by a method of integral equation. Under some priori assumptions on the exact solution, we obtain convergence estimates in many cases. A numerical test is presented that validate the applicability and efficiency of the theoretical result.

Keywords and phrases: Cauchy problem; Nonlinear elliptic equation; Ill-posed problem; Error estimates.

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1 Introduction

Let \( H \) be a Hilbert space with the inner product \( \langle ., . \rangle \) and the norm \( \| . \| \), and let \( A : D(A) \subset H \rightarrow H \) be a positive-definite, self-adjoint operator with compact inverse on \( H \). For \( T \) be a positive number, we consider the problem of finding a function \( u : H \rightarrow H \) from the system

\[
\begin{aligned}
    u_{tt} &= Au + f(t, u(t)), t \in (0, T) \\
    u(0) &= \varphi, \\
    u_t(0) &= g,
\end{aligned}
\]

where the data \( g, \varphi \) are given in \( H \) and the source function \( f : H \rightarrow H \) is defined later. Such problem is not well posed; that is, its solution is not unique and it also does not depend continuously on the "noise" Cauchy data \( \varphi, g \). Hence, a regularization is in order.

The problem (1.1) is a generalization of many well-known equations. For a simple example, if \( A = -\Delta \) (Laplace operator) and \( f(x, t, u) = -k^2 u \) then the problem (1.1) is called Helmholtz equation. The Helmholtz-type equations arise in many engineering applications related to propagating waves in different environments, such as acoustic, hydrodynamic and electromagnetic waves (See [2] [5]). For a general linear example of (1.1) with \( f(x, t, u) = f(x, t) \), let \( A \) be the second-order differential operator
defined in $H^{1}_{0}(\Omega)$

$$Au = - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial u}{\partial x_j}) + a(x)u,$$

with $a_{ij}, a \in L^{\infty}(\Omega)$ satisfying

$$\sum_{i,j=1}^{n} a_{ij} \xi_i \xi_j \leq \nu \sum_{i=1}^{n} |\xi_i|^2,$$

where $\nu > 0$ is given, $a(x) \geq 0$. Such a problem, called Poisson equation, arises in many practical situations; see e.g. [14]. If $f(x, t, u) = \sin u$, then the problem (1.1) is called elliptic-sine Gordon equation. From the point of view of the modelling of physical phenomena, the motivation for the study of elliptic-sine Gordon equation comes from its applications in several areas of mathematical physics including the theory of Josephson effects, superconductors and spin waves in ferromagnets, see e.g. [10, 18]. Furthermore, the Lane-Emden equation

$$\Delta u = -u^p,$$

implying $f(u) = -u^p, p > 1$ plays a vital role in describing the structure of the polytropic stars where $p$ is called the polytropic index. Many abstract studies about this equation are the platform for the system Lane-Emden-Fowler arising in molecular biology, that received considerable mathematical attention, such as Pohozaev-type identities, moving plane method. For more details, the Lane-Emden equation can be referred to the book by Chandrasekhar [28] and Emden et al. [17].

The problem of stability for ill-posed elliptic Cauchy problems plays an important role in the fields of inverse problems governed by elliptic PDEs. They arises from many physical and engineering problems such as geophysic, carsiology (see [3, 4]). It is well-known to be ill-posed in the sense of Hadamard. In fact, a small perturbation in the given Cauchy data may result in a very large error on the solution. Therefore, it is very difficult to solve it using classic numerical methods. In order to overcome this difficulty, the regularization methods are required [15, 19]. In the past, there are many studies on the homogeneous problem, i.e. $f = 0$ in Eq. (1.1). For instance, Elden and Berntsson [7] used the logarithmic convexity method to obtain a stability result of Hölder type. Alessandrini et al [1] provided essentially optimal stability results, in wide generality and under substantially minimal assumptions. Réginska and Tautenhahn [22] presented some stability estimates and regularization method for a Cauchy problem for Helmholtz equation. The homogeneous problems were also investigated by some earlier papers, such as [3, 4, 9, 14, 21, 23, 26]. Recently, a linear inhomogeneous version of elliptic equation has been considered in [25].

To the authors’ knowledge, the results on regularization for the Cauchy problem of an elliptic equation with nonlinear source, (for example Problem (1.1)), are very rare.

In the present paper, we propose a method of integral equation to regularize Problem (1.1). This method is applied to some other ill-posed problems, for example [11]. Moreover, we establish some error estimations between the regularized solution and exact solution. Especially, the convergence of the approximate solution at $t = T$ is also proved. The paper is organized as follows. In Section 2, the regularization method-integral equation method is introduced. In Section 3, a stability estimate is proved under a priori condition of the exact solution. In Section 4, a generalized case of the nonlinear problem with a special type of non Lipschitzian functions is remarkable. Then, a numerical example is shown in Section 5.
2 Mathematical Problem and Mild solution

From now on, suppose that $A : D(A) \subset H \rightarrow H$ is a positive-definite, self-adjoint operator with compact inverse on $H$. As a consequence, $A$ admits an orthonormal eigenbasis $\{\phi_p\}_{p \geq 1}$ in $H$, associated with the eigenvalues such that

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \lim_{p \rightarrow \infty} \lambda_p = \infty.$$ 

In practice, we get the data $(\varphi, g) \in H \times H$ by measuring at discrete nodes. Hence, instead of $(\varphi, g)$, we shall get an inexact data $(\varphi^\epsilon, g^\epsilon) \in H \times H$ satisfying

$$\|\varphi^\epsilon - \varphi\|_H \leq \epsilon, \|g^\epsilon - g\|_H \leq \epsilon,$$  \hspace{1cm} \text{(2.2)}

where the constant $\epsilon > 0$ represents a bound on the measurement error. We can divide the problem into three cases: homogeneous linear problem, inhomogeneous linear problem and nonlinear problem.

2.1 Homogeneous linear problem

Let $g_0, \varphi_0 \in H$, we first consider the problem of finding a function $u : [0, T] \rightarrow H$ satisfying

$$u_{tt} = Au,$$ \hspace{1cm} \text{(2.3)}

subject to conditions

$$\begin{cases} u(0) = \varphi_0, \\ u_t(0) = g_0. \end{cases}$$ \hspace{1cm} \text{(2.4)}

Let $u(t) = \sum_{p=1}^{\infty} \langle u(t), \phi_p \rangle \phi_p$ be the Fourier series of $u$ in the Hilbert space $H$. From (2.3), we get the homogeneous second order differential equation as follows

$$\frac{d^2}{dt^2} \langle u(t), \phi_p \rangle - \lambda_p \langle u(t), \phi_p \rangle = 0.$$ 

Solving this equation, we get

$$\langle u(t), \phi_p \rangle = A_p e^{\sqrt{\lambda_p} t} + B_p e^{-\sqrt{\lambda_p} t}. $$

It follows from (2.4) that $\langle u(0), \phi_p \rangle = \langle \varphi, \phi_p \rangle$ and $\frac{d}{dt} \langle u(0), \phi_p \rangle = \langle g, \phi_p \rangle$. It is easy to compute $A_p$ and $B_p$ and we obtain

$$u(t) = \sum_{p=1}^{\infty} \left[ \cosh \left( \sqrt{\lambda_p} t \right) \langle \varphi, \phi_p \rangle + \frac{\sinh \left( \sqrt{\lambda_p} t \right)}{\sqrt{\lambda_p}} \langle g, \phi_p \rangle \right] \phi_p.$$ \hspace{1cm} \text{(2.5)}

From F. Browder terminology, as in [Geometric Theory of Semilinear Parabolic Equations, Dan Henry, Springer-Verlag, Berlin Heidelberg, Berlin, 1981], the latter form can be called the mild solution of (2.3).
2.2 Inhomogeneous linear problem and nonlinear problem

For the case when the equation is inhomogeneous, i.e., \( u_{tt} = Au + f(t) \), its solution has the Fourier series

\[
\sum_{p=1}^{\infty} \langle u(t), \phi_p \rangle \phi_p
\]

where \( \langle u(t), \phi_p \rangle \) satisfies homogeneous second order differential equation

\[
d^2 \langle u(t), \phi_p \rangle - \lambda_p \langle u(t), \phi_p \rangle = f(t).
\]

Solving this equation and using (2.4), we obtain the exact solution

\[
\sum_{p=1}^{\infty} \left[ \cosh \left( \sqrt{\lambda_p} t \right) \varphi_p + \frac{\sinh \left( \sqrt{\lambda_p} t \right)}{\sqrt{\lambda_p}} g_p + \int_0^t \frac{\sinh \left( \sqrt{\lambda_p} (t-s) \right)}{\sqrt{\lambda_p}} f(s) ds \right] \phi_p,
\]

\[\tag{2.6}\]

where

\[
g_p = \langle g, \phi_p \rangle, \varphi_p = \langle g, \phi_p \rangle, f_p(s) = \langle f(s), \phi_p \rangle.
\]

\[\tag{2.7}\]

Recently, Tuan and his group \[25\] regularized a simple version of the equation (2.6) by truncation method and quasi-boundary value method. In this case \( Au = -\Delta u + k^2 u \) and we get a modified Helmholtz equation with inhomogeneous source. Until now, some results on numerical regularization for nonlinear case is limited. For the nonlinear problem

\[ u_{tt} = Au + f(t, u(t)), \]

subjects to conditions (2.4), we say that \( u \in C([0, T]; H) \) is a mild solution of (2.8) if \( u \) satisfies the integral equation

\[
\sum_{p=1}^{\infty} \left[ \cosh \left( \sqrt{\lambda_p} t \right) \varphi_p + \frac{\sinh \left( \sqrt{\lambda_p} t \right)}{\sqrt{\lambda_p}} g_p + \int_0^t \frac{\sinh \left( \sqrt{\lambda_p} (t-s) \right)}{\sqrt{\lambda_p}} f_p(u(s)) ds \right] \phi_p,
\]

\[\tag{2.9}\]

where \( f_p(u)(s) = \langle f(s, u(s)), \phi_p \rangle \). The transformation (2.8) into (2.9) is easily proved by a separation method which is similar above process. From now on, to regularize Problem (1.1), we only consider the integral equation (2.9) and find a regularization method for it. The main idea of integral equation method can be found in a paper \[24\] on nonlinear backward heat equation. In next Section, we introduce a method of integral equation to regularize Problem (2.9).

3 The nonlinear problem : regularization and error estimate

In this section, let \( f : \mathbb{R} \times H \rightarrow H \) be a function satisfying

\[
\|f(t, w) - f(t, v)\| \leq K\|w - v\|,
\]

\[\tag{3.10}\]

for a positive constant \( K \) independent on \( w, v \in H, t \in \mathbb{R} \).

Since \( t > 0 \), we know from (2.9) that, when \( p \) becomes large, the terms

\[
\cosh \left( \sqrt{\lambda_p} t \right), \sinh \left( \sqrt{\lambda_p} t \right), \sinh \left( \sqrt{\lambda_p} (t-s) \right),
\]



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increases rather quickly. Thus, these terms are the unstability causes. Hence, to regularize the problem, we have to replace these terms by some better terms (called stability terms). Naturally, they will be replaced by some better terms. In particular, \( \cosh(\sqrt{\lambda_p} t) \), \( \sinh(\sqrt{\lambda_p} t) \) and \( \sinh(\sqrt{\lambda_p}(t-s)) \) are replaced by \( \cosh'(\sqrt{\lambda_p} t) \), \( \sinh'(\sqrt{\lambda_p} t) \) and \( \sinh'(\sqrt{\lambda_p}(t-s)) \), respectively. These new terms are defined as follows.

\[
\begin{align*}
\cosh'(\sqrt{\lambda_p} t) &= \frac{Q(\epsilon, \lambda_p) e^{\sqrt{\lambda_p} t} + e^{-\sqrt{\lambda_p} t}}{2}, \\
\sinh'(\sqrt{\lambda_p} t) &= \frac{Q(\epsilon, \lambda_p) e^{\sqrt{\lambda_p} t} - e^{-\sqrt{\lambda_p} t}}{2}, \\
\sinh'(\sqrt{\lambda_p}(t-s)) &= \frac{Q(\epsilon, \lambda_p) e^{\sqrt{\lambda_p}(t-s)} - e^{-\sqrt{\lambda_p}(t-s)}}{2}, \\
Q(\epsilon, \lambda_p) &= \frac{e^{-\sqrt{\lambda_p}T}}{\epsilon + e^{-\sqrt{\lambda_p}T}}.
\end{align*}
\]

Then, we shall approximate (2.9) by the regularized solution \( v^\epsilon \) defined by

\[
v^\epsilon(t) = \sum_{p=1}^{\infty} \left[ \cosh'(\sqrt{\lambda_p} t) \varphi_p + \frac{\sinh'(\sqrt{\lambda_p} t)}{\sqrt{\lambda_p}} g_p + \frac{\sinh'(\sqrt{\lambda_p}(t-s))}{\sqrt{\lambda_p}} f_p(v^\epsilon(s)) ds \right] \phi_p.
\]

We have the following lemma which will be useful in the next results.

**Lemma 3.1.** For \( 0 \leq s \leq t \leq T \), we have the following inequalities

\[
\begin{align*}
\cosh'(\sqrt{\lambda_p} t) &\leq \epsilon^{-\frac{t}{T}}, \\
\frac{\sinh'(\sqrt{\lambda_p} t)}{\sqrt{\lambda_p}} &\leq \frac{\epsilon^{-\frac{t}{T}}}{\sqrt{\lambda_1}}, \\
\frac{\sinh'(\sqrt{\lambda_p}(t-s))}{\sqrt{\lambda_p}} &\leq \frac{\epsilon^{-\frac{s-t}{T}}}{\sqrt{\lambda_1}}.
\end{align*}
\]

**Proof.** First, we can deduce the following inequality.

\[
Q(\epsilon, \lambda_p)e^{\sqrt{\lambda_p} t} = \frac{e^{-\sqrt{\lambda_p}(T-t)}}{\epsilon + e^{-\sqrt{\lambda_p}T}} = \frac{e^{-\sqrt{\lambda_p}(T-t)}}{\left(\epsilon + e^{-\sqrt{\lambda_p}T}\right)^{\frac{t}{T}} \left(\epsilon + e^{-\sqrt{\lambda_p}T}\right)^{\frac{T}{T}}} \leq \left(\epsilon + e^{-\sqrt{\lambda_p}T}\right)^{\frac{t}{T}} \leq \epsilon^{-\frac{t}{T}}.
\]

Similarly, this leads to

\[
Q(\epsilon, \lambda_p)e^{\sqrt{\lambda_p}(t-s)} = \frac{e^{-\sqrt{\lambda_p}(T-t+s)}}{\epsilon + e^{-\sqrt{\lambda_p}T}} = \frac{e^{-\sqrt{\lambda_p}(T-t+s)}}{\left(\epsilon + e^{-\sqrt{\lambda_p}T}\right)^{\frac{t-s}{T}} \left(\epsilon + e^{-\sqrt{\lambda_p}T}\right)^{\frac{t}{T}}} \leq \left(\epsilon + e^{-\sqrt{\lambda_p}T}\right)^{\frac{s-t}{T}} \leq \epsilon^{-\frac{s-t}{T}}.
\]

We note that \( \epsilon^\frac{t}{T} \leq 1 \leq e^{\sqrt{\lambda_p} t} \), then we will obtain the proof simultaneously. Indeed, we see that

\[
\cosh'(\sqrt{\lambda_p} t) = \frac{Q(\epsilon, \lambda_p)e^{\sqrt{\lambda_p} t} + e^{-\sqrt{\lambda_p} t}}{2} \leq \epsilon^{-\frac{t}{T}}.
\]
and
\[ \frac{\sinh^\epsilon(\sqrt{\lambda_p} t)}{\sqrt{\lambda_p}} = \frac{Q(\epsilon, \lambda_p) e^{\sqrt{\lambda_p} t} - e^{-\sqrt{\lambda_p} t}}{2\sqrt{\lambda_p}} \leq \frac{\epsilon^2}{\sqrt{\lambda_1}}, \]

and
\[ \frac{\sinh^\epsilon(\sqrt{\lambda_p} (t-s))}{\sqrt{\lambda_p}} = \frac{Q(\epsilon, \lambda_p) e^{\sqrt{\lambda_p} (t-s)} - e^{-\sqrt{\lambda_p} (t-s)}}{2\sqrt{\lambda_p}} \leq \frac{\epsilon^2}{\sqrt{\lambda_1}}. \]

In the next theorem, we shall study the existence, the uniqueness of a (weak) solution of Problem (3.11).

**Theorem 3.1.** The integral equation (3.11) has a unique solution \( v^\epsilon \in C([0; T]; H) \).

**Proof.** For \( w \in C([0; T]; H) \), we consider the following function
\[
F(w)(t) = \sum_{p=1}^{\infty} \left[ \cosh^\epsilon(\sqrt{\lambda_p} t) \varphi_p + \sinh^\epsilon(\sqrt{\lambda_p} t) g_p + \int_0^t \frac{\sinh^\epsilon(\sqrt{\lambda_p} (t-s))}{\sqrt{\lambda_p}} f_p(w)(s) ds \right] \phi_p.
\]

We claim that, for every \( w, v \in C([0, T]; H), m \geq 1 \), we have
\[
\| F^m(w)(t) - F^m(v)(t) \|^2 \leq \left( \frac{K^2 \epsilon^{-2}}{\lambda_1} \right)^m \frac{t^m C^m}{m!} \| w - v \|^2,
\]
where \( C = \max\{T, 1\} \) and \( \| . \| \) is supremum norm in \( C([0, T]; H) \). We are going to prove this inequality by induction. For \( m = 1 \), using the inequality \( \frac{\sinh^\epsilon(\sqrt{\lambda_p} (t-s))}{\sqrt{\lambda_p}} \leq \frac{1}{\epsilon \sqrt{\lambda_1}} \), we obtain the following estimate
\[
\| F(w)(., t) - F(v)(., t) \|^2 = \sum_{p=1}^{\infty} \left[ \int_0^t \frac{\sinh^\epsilon(\sqrt{\lambda_p} (t-s))}{\sqrt{\lambda_p}} \langle f(s, w(s)) - f(s, v(s)), \phi_p \rangle ds \right]^2 \\
\leq \sum_{p=1}^{\infty} \int_0^t \left( \frac{\sinh^\epsilon(\sqrt{\lambda_p} (t-s))}{\sqrt{\lambda_p}} \right)^2 ds \int_0^t |\langle f(s, w(s)) - f(s, v(s)), \phi_p \rangle|^2 ds \\
\leq \frac{1}{\lambda_1 \epsilon^2} t \int_0^t \| f(s, w(s)) - f(s, v(s)) \|^2 ds \\
\leq \frac{1}{\lambda_1 \epsilon^2} K^2 t \int_0^t \| w(s) - v(s) \|^2 ds \leq \frac{K^2 C t}{\lambda_1 \epsilon^2} \| w - v \|^2.
\]

Thus (3.17) holds.

Next, suppose that (3.17) holds for \( m = k \), then we prove that (3.17) holds for \( m = k + 1 \). We
have

\[
\|F^{k+1}(w)(., t) - F^{k+1}(v)(., t)\|^2
\]

\[
= \sum_{p=1}^{\infty} \left[ \int_0^t \frac{\sinh^e(\sqrt{\lambda_p}(t-s))}{\sqrt{\lambda_p}} \left( f(F^k(w))(s) - f(F^k(v))(s), \phi_p \right) \, ds \right]^2
\]

\[
\leq \sum_{p=1}^{\infty} \int_0^t \left( \frac{\sinh^e(\sqrt{\lambda_p}(t-s))}{\lambda_p} \right)^2 \, ds \int_0^t \left| \left( f(F^k(w))(s) - f(F^k(v))(s), \phi_p \right) \right|^2 \, ds
\]

\[
\leq \frac{t}{\lambda_1 \epsilon^2} \int_0^t \| f(F^k(w))(s) - f(F^k(v))(s) \|^2 \, ds
\]

\[
\leq \frac{K^2 t}{\lambda_1 \epsilon^2} \int_0^t \left( \frac{K^2}{\lambda_1 \epsilon^2} \right)^k \frac{k! C^k}{k!} ||w - v||^2
\]

\[
\leq \left( \frac{K^2}{\lambda_1 \epsilon^2} \right)^{k+1} \frac{C^k+1}{(k+1)!} ||w - v||^2.
\]

Therefore, by the induction principle, we have

\[
\|F^m(w)(., t) - F^m(v)(., t)\| \leq \sqrt{\left( \frac{K^2}{\lambda_1 \epsilon^2} \right)^m \frac{m! t^m C^m}{m!}} ||w - v||,
\]

for all \( w, v \in C([0, T]; H) \). We consider \( F : C([0, T]; H) \rightarrow C([0, T]; H) \). Since

\[
\lim_{m \to \infty} \sqrt{\left( \frac{K^2}{\lambda_1 \epsilon^2} \right)^m \frac{m! t^m C^m}{m!}} = 0,
\]

there exists a positive integer number \( m_0 \) such that \( \sqrt{\left( \frac{K^2}{\lambda_1 \epsilon^2} \right)^{m_0} \frac{m_0! t^{m_0} C^{m_0}}{m_0!}} < 1 \), and \( F^{m_0} \) is a contraction.

It follows that the equation \( F^{m_0}(w) = w \) has a unique solution \( v^\epsilon \in C([0, T]; H) \). We claim that \( F(v^\epsilon) = v^\epsilon \). In fact, we have \( F^{m_0}(F(v^\epsilon)) = F(v^\epsilon) \) because \( F(F^{m_0}(v^\epsilon)) = F(v^\epsilon) \). Then, the uniqueness of the fixed point of \( F^{m_0} \) leads to \( F(v^\epsilon) = v^\epsilon \); i.e., the equation \( F(w) = w \) has a unique solution \( v^\epsilon \in C([0, T]; H) \). Hence, the result is proved completely.

In the next Theorem, we make a study on the stability of solution of the considered problem.

**Theorem 3.2.** Let \((\varphi, g) \in H \times H\). Then the function \( u^\epsilon \in C([0, T]; H) \) as in (3.11) depends continuously on \((\varphi, g)\) for any \( \epsilon > 0 \). Let \( u^{1,\epsilon} \) and \( u^{2,\epsilon} \) be two solutions of (3.11) corresponding to the values \((\varphi^1, g^1)\) and \((\varphi^2, g^2)\) respectively, then

\[
\|u^{1,\epsilon}(., t) - u^{2,\epsilon}(., t)\|^2 \leq 3 \exp \left\{ \frac{3TK^2 t}{\lambda_1} \right\} \epsilon^2 \left( \|\varphi^1 - \varphi^2\|^2 + \|g^1 - g^2\|^2 \right),
\]

**Proof.** For \( i = 1, 2 \), we have

\[
u^{i,\epsilon}(t) = \sum_{p=1}^{\infty} \left[ \cosh^e(\sqrt{\lambda_p} t) \varphi^i_p + \frac{\sinh^e(\sqrt{\lambda_p} t)}{\sqrt{\lambda_p}} g^i_p + \int_0^t \frac{\sinh^e(\sqrt{\lambda_p}(t-s))}{\sqrt{\lambda_p}} f_p(u^{i,\epsilon}(s)) \, ds \right] \phi_p.
\]
It follows from (3.19) that
\[
\|u^{1,\epsilon}(t) - u^{2,\epsilon}(t)\|^2 = \sum_{p=1}^{\infty} \left[ \cosh(\sqrt{\lambda_p}t)(\varphi_p^1 - \varphi_p^2) + \frac{\sinh(\sqrt{\lambda_p}t)}{\sqrt{\lambda_p}}(g_p^1 - g_p^2) \right.
\]
\[
+ \int_0^t \frac{\sinh(\sqrt{\lambda_p}(t-s))}{\sqrt{\lambda_p}} \left( f_p(u^{1,\epsilon})(s) - f_p(u^{2,\epsilon})(s) \right) ds \right]^2
\]
\[
\leq \sum_{p=1}^{\infty} \left[ 3\cosh(\sqrt{\lambda_p}t)^2(\varphi_p^1 - \varphi_p^2)^2 + 3\left( \frac{\sinh(\sqrt{\lambda_p}t)}{\sqrt{\lambda_p}} \right)^2(g_p^1 - g_p^2)^2 \right.
\]
\[
+ 3t \int_0^t \left( \frac{\sinh(\sqrt{\lambda_p}(t-s))}{\sqrt{\lambda_p}} \right)^2 \left( f_p(u^{1,\epsilon})(s) - f_p(u^{2,\epsilon})(s) \right)^2 ds \right].
\]
Using the Lipschitzian property of $f$, we get the following inequality.
\[
\|u^{1,\epsilon}(t) - u^{2,\epsilon}(t)\|^2 \leq 3\epsilon^2 \frac{2t}{T} \|\varphi^1 - \varphi^2\|^2 + \frac{3}{\lambda_1} \epsilon^2 \frac{2t}{T} \|g^1 - g^2\|^2
\]
\[
+ \frac{3t}{\lambda_1} \int_0^t \epsilon^2 \frac{2s - 2t}{T} \|f(s, u^{1,\epsilon}(s)) - f(s, u^{2,\epsilon}(s))\|^2 ds
\]
\[
\leq 3\epsilon^2 \frac{2t}{T} \left( \|\varphi^1 - \varphi^2\|^2 + \frac{\|g^1 - g^2\|^2}{\lambda_1} \right) + \frac{3tK^2}{\lambda_1} \int_0^t \epsilon^2 \frac{2s - 2t}{T} \|u^{1,\epsilon}(s) - u^{2,\epsilon}(s)\|^2 ds.
\]
This implies that
\[
\epsilon^2 \frac{2t}{T} \|u^{1,\epsilon}(t) - u^{2,\epsilon}(t)\|^2 \leq 3 \left( \|\varphi^1 - \varphi^2\|^2 + \frac{\|g^1 - g^2\|^2}{\lambda_1} \right) + \frac{3tK^2}{\lambda_1} \int_0^t \epsilon^2 \frac{2s - 2t}{T} \|u^{1,\epsilon}(., s) - u^{2,\epsilon}(., s)\|^2 ds.
\]
Applying Gronwall’s inequality, we have
\[
\epsilon^2 \frac{2t}{T} \|u^{1,\epsilon}(t) - u^{2,\epsilon}(t)\|^2 \leq 3 \exp \left\{ \frac{3tK^2t}{\lambda_1} \right\} \left( \|\varphi^1 - \varphi^2\|^2 + \frac{\|g^1 - g^2\|^2}{\lambda_1} \right).
\]
Thus, it turns out that
\[
\|u^{1,\epsilon}(t) - u^{2,\epsilon}(t)\|^2 \leq 3 \exp \left\{ \frac{3tK^2t}{\lambda_1} \right\} e^{\frac{2t}{T}} \left( \|\varphi^1 - \varphi^2\|^2 + \frac{\|g^1 - g^2\|^2}{\lambda_1} \right).
\]

**Theorem 3.3.** Suppose that Problem (1.1) has a weak solution $u$ which satisfies
\[
\sum_{p=1}^{\infty} e^{2\sqrt{\lambda_p}(T-t)} \left( \langle u(t), \phi_p \rangle + \frac{\langle u(t), \phi_p \rangle}{\sqrt{\lambda_p}} \right)^2 \leq P^2.
\]
for a positive number $P$. Let $(\varphi^\epsilon, g^\epsilon) \in H \times H$ be measured data such that (2.2). Then, we can construct a regularized solution $U^\epsilon$ such that
\[
\begin{align*}
\|U^\varepsilon(t) - u(t)\| &\leq C e^{t - \frac{T}{4}}, \quad t \in [0, T) \\
\|U^\varepsilon(T) - u(T)\| &\leq (D + C) \sqrt{\frac{T}{\ln(\frac{1}{\varepsilon})}}, \quad t = T,
\end{align*}
\] (3.22)

where

\[
C = 2P \exp \left\{ \frac{K^2 T^2 t}{\lambda_1} \right\} \sqrt{3(1 + \frac{1}{\lambda_1})} \exp \left\{ \frac{3TK^2}{2\lambda_1} \right\},
\]

\[
D = \sup_{0 \leq t \leq T} \|u_t(t)\|.
\]

Proof. Differentiating (2.9) with respect to \( t \) gives

\[
\langle u_t(t), \phi_p \rangle = \sqrt{\lambda_p} \left[ \sinh \left( \sqrt{\lambda_p} t \right) \varphi_p + \cosh \left( \sqrt{\lambda_p} t \right) g_p + \int_0^t \cosh \left( \sqrt{\lambda_p} (t - s) \right) \frac{f_p(u(s))}{\sqrt{\lambda_p}} ds \right].
\] (3.23)

Dividing (3.23) by \( \sqrt{\lambda_p} \) and adding the result obtained to (2.9), we get

\[
\langle u(t), \phi_p \rangle + \frac{\langle u_t(t), \phi_p \rangle}{\sqrt{\lambda_p}} = e^{\sqrt{\lambda_p} t} \varphi_p + \frac{e^{\sqrt{\lambda_p} t}}{\sqrt{\lambda_p}} g_p + \int_0^t \frac{e^{\sqrt{\lambda_p} (t - s)}}{\sqrt{\lambda_p}} f_p(u(s)) ds.
\] (3.24)

Combining (2.9), (3.11) and (3.24) yields

\[
\langle v^\varepsilon(t) - u(t), \phi_p \rangle =
\]

\[
= \left( Q(\varepsilon, \lambda_p) - 1 \right) \left( e^{\sqrt{\lambda_p} t} \varphi_p + \frac{e^{\sqrt{\lambda_p} t}}{\sqrt{\lambda_p}} g_p + \int_0^t \frac{e^{\sqrt{\lambda_p} (t - s)}}{\sqrt{\lambda_p}} f_p(u(s)) ds \right)
\]

\[
+ \int_0^t \frac{\sinh \left( \sqrt{\lambda_p} (t - s) \right)}{\sqrt{\lambda_p}} f_p(v^\varepsilon)(s) ds - \int_0^t \frac{\sinh \left( \sqrt{\lambda_p} (t - s) \right)}{\sqrt{\lambda_p}} f_p(u(s)) ds
\]

\[
= \left( Q(\varepsilon, \lambda_p) - 1 \right) \left( \langle u(t), X_p \rangle + \frac{\langle u_t(t), \phi_p \rangle}{\sqrt{\lambda_p}} \right)
\]

\[
+ \int_0^t \frac{\sinh \left( \sqrt{\lambda_p} (t - s) \right)}{\sqrt{\lambda_p}} \left( f_p(v^\varepsilon)(s) - f_p(u(s)) \right) ds.
\] (3.25)
Applying Gronwall's inequality, we deduce that

\[
\left| \langle v'(t) - u(t), \phi_p \rangle \right|^2 \\
\leq 2 \left( Q(\epsilon, \lambda_p) - 1 \right) ^2 \left( \langle u(t), \phi_p \rangle + \frac{\langle u'_t(t), \phi_p \rangle}{\sqrt{\lambda_p}} \right) ^2 \\
+ 2t^2 \int_0^t \left( \sinh' \left( \frac{\sqrt{\lambda_p} (t - s)}{\lambda_p} \right) \right) ^2 \left( f_p(v')(s) - f_p(u)(s) \right) ^2 ds \\
\leq 2\epsilon^2 \left( e^{-\sqrt{\lambda_p}(T-t)} \right) ^2 e^{2\sqrt{\lambda_p}(T-t)} \left( \langle u(t), \phi_p \rangle + \frac{\langle u'_t(t), \phi_p \rangle}{\sqrt{\lambda_p}} \right) ^2 \\
+ 2t^2 \int_0^t \left( \sinh' \left( \frac{\sqrt{\lambda_p} (t - s)}{\lambda_p} \right) \right) ^2 \left| f_p(v')(s) - f_p(u)(s) \right|^2 ds.
\]

At this time, applying Lemma 3.1 leads to

\[
\left| \langle v'(t) - u(t), \phi_p \rangle \right|^2 \\
\leq 2\epsilon^2 t^{2T-2t} e^{2\sqrt{\lambda_p}(T-t)} \left( \langle u(t), \phi_p \rangle + \frac{\langle u'_t(t), \phi_p \rangle}{\sqrt{\lambda_p}} \right) ^2 \\
+ 2t^2 \int_0^t \frac{e^{2s-2t}}{\lambda_1} \left| f_p(v')(s) - f_p(u)(s) \right|^2 ds.
\]

Using Lipschitz property of \( f \) and the priori assumption (3.21), we get

\[
\| v'(t) - u(t) \|^2 = \sum_{p=1}^{\infty} \left| \langle v'(t) - u(t), \phi_p \rangle \right|^2 \\
\leq 2\epsilon^2 t^{2T-2t} \sum_{p=1}^{\infty} e^{2\sqrt{\lambda_p}(T-t)} \left( \langle u(t), \phi_p \rangle + \frac{\langle u'_t(t), \phi_p \rangle}{\sqrt{\lambda_p}} \right) ^2 \\
+ 2t^2 \int_0^t \frac{e^{2s-2t}}{\lambda_1} \sum_{p=1}^{\infty} \left| f_p(v')(s) - f_p(u)(s) \right|^2 ds \\
\leq 2\epsilon^2 t^{2T-2t} P^2 + \frac{2T^2}{\lambda_1} \int_0^t e^{2s-2t} \| f'(v')(s) - f(u)(s) \|^2 ds \\
\leq 2\epsilon^2 t^{2T-2t} P^2 + \frac{2K^2T^2}{\lambda_1} \int_0^t e^{2s-2t} \| v'(s) - u(s) \|^2 ds.
\]

Multiplying \( e^{2t} \) by both sides, we thus have

\[
e^{2t} \| v'(., t) - u(., t) \|^2 \leq 2\epsilon^2 P^2 + \frac{2K^2T^2}{\lambda_1} \int_0^t e^{2s} \| v'(s) - u(s) \|^2 ds.
\]

Applying Gronwall's inequality, we deduce that

\[
e^{2t} \| v'(t) - u(t) \|^2 \leq 2P^2 \exp \left\{ \frac{2K^2T^2}{\lambda_1} \right\} \epsilon^2.
\]
By simplification, we conclude that
\[
\|v(t) - u(t)\| \leq 2P \exp \left\{ \frac{K^2T^2t}{\lambda_1} \right\} \epsilon^{1-\frac{3}{2}}. \tag{3.26}
\]

On the other hand, notice that the following integral equation
\[
\phi(t) = \sum_{p=1}^{\infty} \left[ \cosh(\sqrt{\lambda_p}t) \varphi_p^t + \frac{\sinh(\sqrt{\lambda_p}t)}{\sqrt{\lambda_p}} g_p^t + \int_{0}^{t} \frac{\sinh(\sqrt{\lambda_p}(t-s))}{\sqrt{\lambda_p}} f_p(u^t)(s)ds \right] \varphi_p.
\tag{3.27}
\]

has a unique solution \( u^t \in C([0,T]; H) \). Then, by using Lemma 3.1, we obtain
\[
\|u^t(t) - v^t(t)\| \leq \sqrt{3} \exp \left\{ \frac{3TK^2t}{\lambda_1} \right\} \epsilon^{\frac{3}{2}} \left( \|\varphi^t - \varphi\|^2 + \|g^t - g\|^2 \right) \lambda_1
\leq \sqrt{3(1 + \frac{1}{\lambda_1})} \exp \left\{ \frac{3TK^2t}{2\lambda_1} \right\} \epsilon^{1-\frac{3}{2}}. \tag{3.28}
\]

Combining (3.26) and (3.28) and using the triangle inequality, we have
\[
\|u^t(t) - u(t)\| \leq \|u^t(t) - v^t(t)\| + \|v^t(t) - u(t)\|
\leq 2P \exp \left\{ \frac{K^2T^2t}{\lambda_1} \right\} \epsilon^{1-\frac{3}{2}} + \sqrt{3(1 + \frac{1}{\lambda_1})} \exp \left\{ \frac{3TK^2t}{2\lambda_1} \right\} \epsilon^{1-\frac{3}{2}}
\leq C \epsilon^{1-\frac{3}{2}}.
\]

Moreover, we get
\[
\|u(T) - u^t(t)\| \leq \|u(T) - u(t)\| + \|u(t) - u^t(t)\|
\leq \sup_{0 \leq t \leq T} \|u_t(t)\|(T-t) + C \epsilon^{1-\frac{3}{2}}.
\]

For every \( \epsilon > 0 \), there exists a unique \( t_\epsilon \in (0,T) \) such that
\[
(T-t_\epsilon)^2 = \epsilon^{2-\frac{3}{2}}. \tag{3.29}
\]

It implies that \( \frac{\ln(T-t_\epsilon)}{T-t_\epsilon} = \frac{\ln \epsilon}{T} \). Using the inequality \( \ln t > -\frac{1}{t} \) for every \( t > 0 \), we obtain \( T-t_\epsilon < \frac{T}{\ln(\frac{1}{t})} \).

Hence, we get
\[
\|u(t) - u^t(t_\epsilon)\| \leq (D+C) \sqrt{\frac{T}{\ln(\frac{1}{t})}},
\]

where \( D = \sup_{0 \leq t \leq T} \|u_t(t)\| \). Let \( U^\epsilon \) be defined as follows
\[
U^\epsilon(x,t) = \begin{cases} u^\epsilon(x,t), & t \in [0,T), \\ u^\epsilon(x,t_\epsilon), & t = T. \end{cases}
\tag{3.30}
\]

For \( 0 \leq t < T \), then claim that
\[
\|U^\epsilon(t) - u(t)\| = \|u^\epsilon(t) - u(t)\| \leq C \epsilon^{1-\frac{3}{2}}.
\]

and
\[
\|U^\epsilon(T) - u(T)\| = \|u^\epsilon(t_\epsilon) - u(t)\| \leq (D+C) \sqrt{\frac{T}{\ln(\frac{1}{t})}}.
\]

\( \square \)
Remark 3.1. The condition in (3.21) is accepted and natural. If the source function \( f = 0 \) then from (3.34), we have

\[
\langle u(t), \phi_p \rangle + \frac{\langle u_t(t), \phi_p \rangle}{\sqrt{\lambda_p}} = e^{\sqrt{\lambda_p}t} \varphi_p + e^{\sqrt{\lambda_p}t} g_p. \tag{3.31}
\]

By letting \( t = T \), we have

\[
\langle u(T), \phi_p \rangle + \frac{\langle u_t(T), \phi_p \rangle}{\sqrt{\lambda_p}} = e^{\sqrt{\lambda_p}T} \varphi_p + e^{\sqrt{\lambda_p}T} g_p. \tag{3.32}
\]

Combining (3.31) and (3.32), we obtain

\[
e^{\sqrt{\lambda_p}(T-t)} \left( \langle u(T), \phi_p \rangle + \frac{\langle u_t(T), \phi_p \rangle}{\sqrt{\lambda_p}} \right) = \langle u(T), \phi_p \rangle + \frac{\langle u_t(T), \phi_p \rangle}{\sqrt{\lambda_p}}. \tag{3.33}
\]

Then, it follows that

\[
\sum_{p=1}^{\infty} e^{2\sqrt{\lambda_p}(T-t)} \left( \langle u(t), \phi_p \rangle + \frac{\langle u_t(t), \phi_p \rangle}{\sqrt{\lambda_p}} \right)^2 = \sum_{p=1}^{\infty} \left( \langle u(T), \phi_p \rangle + \frac{\langle u_t(T), \phi_p \rangle}{\sqrt{\lambda_p}} \right)^2.
\]

Remark 3.2. In the case \( f(x, t, u) = f(x, t) \), we have the some different error estimates between the exact solution and regularized solution.

Proof of Remark 3.2.

Before proving this, we have to obtain some results in the following lemma.

Lemma 3.2. Let \( s > 0, X \geq 0 \). Then for all \( 0 \leq t \leq T \) and \( 0 < \epsilon < 1 \), we have

\[
\frac{\epsilon}{(1 + X)^s(\epsilon + e^{-TX})} \leq C(s) \left( \frac{T}{\ln(1/\epsilon)} \right)^s.
\]

where \( C(s) = s^s e^{1-s}(1 + T^{-s}) \).

Proof.

Case 1. \( X \in [0, \frac{1}{T}] \). It is clear to see that

\[
\frac{\epsilon}{(1 + X)^s(\epsilon + e^{-TX})} \leq \frac{\epsilon}{(1 + X)^s e^{-TX}} \leq \epsilon e^{TX} \leq \epsilon.
\]

From the inequality \( \epsilon \leq \left( \frac{s}{e} \right)^s \left( \frac{1}{\ln(1/\epsilon)} \right)^s \), we get

\[
\frac{\epsilon}{(1 + X)^s(\epsilon + e^{-TX})} \leq s^s e^{1-s} \left( \frac{1}{\ln(1/\epsilon)} \right)^s \leq s^s e^{1-s}(1 + T^{-s}) \left( \frac{T}{\ln(1/\epsilon)} \right)^s. \tag{3.34}
\]

Case 2. \( X > \frac{1}{T} \). Set \( e^{-TX} = \epsilon Y \). Then, we obtain

\[
\frac{\epsilon}{(1 + X)^s(\epsilon + e^{-TX})} = \frac{\epsilon}{\epsilon + \epsilon Y} \left( \frac{T}{T - \ln(\epsilon Y)} \right)^s = \frac{1}{1 + Y} \left( \frac{T}{T - \ln(\epsilon Y)} \right)^s = \frac{1}{1 + Y} \left( \frac{T}{\ln(1/\epsilon)} \right)^s \frac{1}{1 + Y} \left( \frac{T - \ln(\epsilon Y)}{T - \ln(\epsilon Y)} \right)^s. \tag{3.35}
\]
We continue to estimate the term \( \frac{1}{1 + Y} \left( \frac{-\ln(\epsilon)}{T - \ln(\epsilon)Y} \right)^s \).

If \( 0 < Y \leq 1 \) then \( 0 < -\ln(\epsilon) < -\ln(\epsilon Y) \), thus
\[
\frac{1}{1 + Y} \left( \frac{-\ln(\epsilon)}{T - \ln(\epsilon)Y} \right)^s < 1, \tag{3.36}
\]

else if \( Y > 1 \) then \( \ln Y > 0 \) and \( \ln(\epsilon Y) = -TX < -1 \) due to the assumption \( X \in (\frac{1}{T}, \infty) \). Therefore \( \ln Y(1 + \ln(\epsilon Y)) \leq 0 \). This implies that
\[
0 < \frac{-\ln \epsilon}{1 + \ln(\epsilon Y)} < \frac{-\ln \epsilon}{-\ln(\epsilon Y)} < 1 + \ln Y.
\]

Hence, in this case, we get
\[
\frac{1}{1 + Y} \left( \frac{-\ln(\epsilon)}{T - \ln(\epsilon)Y} \right)^s < \frac{(1 + \ln Y)^s}{Y} = (1 + \ln Y)^s Y^{-1}.
\]

We set \( g(Y) = (1 + \ln Y)^s Y^{-1} \) for \( Y > e^{-1} \). Then, taking the derivative of this function is to get
\[
g'(Y) = (1 + \ln Y)^{s-1} Y^{-2} (s - 1 - \ln Y)
\]

The function \( g \) has maximum at the point \( Y_0 \) such that \( g'(Y_0) = 0 \). This implies that \( Y_0 = e^{s-1} \). Therefore, it leads to the following inequality.
\[
\sup_{Y \geq 1} (1 + \ln Y)^s Y^{-1} \leq g(Y_0) = s^s e^{1-s}. \tag{3.37}
\]

Combining (3.36) and (3.37), we have
\[
\frac{1}{1 + Y} \left( \frac{-\ln(\epsilon)}{T - \ln(\epsilon)Y} \right)^s \leq s^s e^{1-s}.
\]

From (3.35), we will see that
\[
\frac{\epsilon}{(1 + X)^s(\epsilon + e^{-TX})} \leq s^s e^{1-s} \left( \frac{T}{\ln(1/\epsilon)} \right)^s \leq C(s) \left( \frac{T}{\ln(1/\epsilon)} \right)^s \tag{3.38}
\]

Since (3.39), we obtain
\[
\left| \langle v^s(t) - u(t), \phi_p \rangle \right|^2 = \left( Q(\epsilon, \lambda_p) - 1 \right)^2 \left( \langle u(t), \phi_p \rangle + \frac{\langle u_t(t), \phi_p \rangle}{\sqrt{\lambda_p}} \right)^2
\]
\[
= \frac{\epsilon^2}{(1 + \sqrt{\lambda_p})^{2s} (\epsilon + e^{-\sqrt{\lambda_p} T})^2} (1 + \sqrt{\lambda_p})^{2s} \left( \langle u(t), \phi_p \rangle + \frac{\langle u_t(t), \phi_p \rangle}{\sqrt{\lambda_p}} \right)^2.
\]

Using the lemma above, we conclude that
\[
\left\| v^s(t) - u(t) \right\|^2 = \sum_{p=1}^{\infty} \left| \langle v^s(t) - u(t), \phi_p \rangle \right|^2
\]
\[
\leq C^2(s) \left( \frac{T}{\ln(1/\epsilon)} \right)^{2s} \sum_{p=1}^{\infty} (1 + \sqrt{\lambda_p})^{2s} \left( \langle u(t), \phi_p \rangle + \frac{\langle u_t(t), \phi_p \rangle}{\sqrt{\lambda_p}} \right)^2
\]
\[
\leq C^2(s) D \left( \frac{T}{\ln(1/\epsilon)} \right)^{2s} \left( \| u(t) \|_{H^s}^2 + \| u'(t) \|_{H^s}^2 \right).
\]
4 Remark a generalized case of the nonlinear problem with non Lipschitzian function

Section 3 only regularized problems in which \( f \) is a global Lipschitzian function. This condition still makes the applicability of the method limited to a small field of study. We can list some functions such as \( f(x) = \sin x, \arctan x, \frac{1}{x^2+1} \), then observe that the class of space function is very small. From the point of view, we tend to establish the error estimate for a bigger class \( a(t, u)f(t, u) \) where \( a(t, u) \) will be defined later.

We still pay more attention to the problem (1.1). Until now, we did not find any results associated with a non global Lipschitzian function in the right hand side of (1.1). Therefore, in this section, we are going to introduce the main idea to solve a special generalized case of the problem (1.1) with the following form

\[
\begin{cases}
  u_{tt} = Au + a(t, u(t))f(t, u(t)), t \in (0, T) \\
  u(0) = \varphi, \\
  u_t(0) = g
\end{cases}
\]  

(4.39)

where \( a : \mathbb{R} \times H \rightarrow H \) satisfies that \( \|a(x, t, u)\| \leq M \) and the Lipschitzian condition

\[ \|a(t, u) - a(t, v)\| \leq N\|u - v\|, \]

the function \( f \) satisfies the condition (3.10).

Let \( G(t, u(t)) = a(t, u(t))f(t, u(t)) \). A mild solution of (4.39) satisfies the following integral equation

\[
u(t) = \sum_{p=1}^{\infty} \left[ \cosh\left(\sqrt{\lambda_p} t\right) \varphi_p + \frac{\sinh\left(\sqrt{\lambda_p} t\right)}{\sqrt{\lambda_p}} g_p + \int_0^t \frac{\sinh\left(\sqrt{\lambda_p}(t-s)\right)}{\sqrt{\lambda_p}} G_p(u(s)) ds \right] \phi_p
\]

(4.40)

The regularization result is in the next theorem.

**Theorem 4.1.** Let \( u \) be defined as (4.40). Then the problem

\[
v^\epsilon(t) = \sum_{p=1}^{\infty} \left[ \cosh\left(\sqrt{\lambda_p} t\right) \varphi_p^\epsilon + \frac{\sinh\left(\sqrt{\lambda_p} t\right)}{\sqrt{\lambda_p}} g_p^\epsilon + \int_0^t \frac{\sinh\left(\sqrt{\lambda_p}(t-s)\right)}{\sqrt{\lambda_p}} G_p(v^\epsilon(s)) ds \right] \phi_p.
\]

(4.41)

has a unique solution \( v^\epsilon \) and we get

\[ \|u^\epsilon - u\| \leq C\epsilon^{1-\frac{1}{T}} \]

for \( C \) is a constant which not depend on \( \epsilon \).

**Proof.** We can divide the proof into three steps.

**Step 1.** The existence of \( v^\epsilon \).

**Step 2.** Error estimate \( \|v^\epsilon - u\| \) where \( v^\epsilon \) satisfies

\[
v^\epsilon(t) = \sum_{p=1}^{\infty} \left[ \cosh\left(\sqrt{\lambda_p} t\right) \varphi_p^\epsilon + \frac{\sinh\left(\sqrt{\lambda_p} t\right)}{\sqrt{\lambda_p}} g_p^\epsilon + \int_0^t \frac{\sinh\left(\sqrt{\lambda_p}(t-s)\right)}{\sqrt{\lambda_p}} G_p(v^\epsilon(s)) ds \right] \phi_p.
\]

(4.42)
Due to orthonormal eigenbasis $\phi_p$ and the explicit formula of $v^\epsilon$ and $u$, we obtain
\[
\langle v^\epsilon(t) - u(t), \phi_p \rangle = \left( Q(\epsilon, \lambda_p) - 1 \right) \left( e^{\sqrt{\lambda_p}t} \varphi_p + e^{\sqrt{\lambda_p}t} g_p + \int_0^t e^{\sqrt{\lambda_p}(t-s)} G_p(u)(s)ds \right)
\]
\[
+ \int_0^t \sinh' \left( \frac{\sqrt{\lambda_p}(t-s)}{\sqrt{\lambda_p}} \right) G_p(v^\epsilon)(s)ds - \int_0^t \sinh' \left( \frac{\sqrt{\lambda_p}(t-s)}{\sqrt{\lambda_p}} \right) G_p(u)(s)ds
\]
\[
= \left( Q(\epsilon, \lambda_p) - 1 \right) \left( \langle u(t), \phi_p \rangle + \frac{\langle u_t(t), \phi_p \rangle}{\sqrt{\lambda_p}} \right)
\]
\[
+ \int_0^t \sinh' \left( \frac{\sqrt{\lambda_p}(t-s)}{\sqrt{\lambda_p}} \right) \left( G_p(v^\epsilon)(s) - G_p(u)(s) \right)ds.
\]
(4.43)

Using Lipschitz property of $f$ and the priori assumption (3.21), we get
\[
||v^\epsilon(t) - u(t)||^2 = \sum_{p=1}^{\infty} \left| \langle v^\epsilon(t) - u(t), \phi_p \rangle \right|^2
\]
\[
\leq 2\epsilon \frac{2T-2t}{T} \sum_{p=1}^{\infty} e^{2\sqrt{\lambda_p}(T-t)} \left( \langle u(t), \phi_p \rangle + \frac{\langle u_t(t), \phi_p \rangle}{\sqrt{\lambda_p}} \right)^2
\]
\[
+ 2t^2 \int_0^t \frac{\epsilon^{2x+2t}}{\lambda_1} \sum_{p=1}^{\infty} \left| G_p(v^\epsilon)(s) - G_p(u)(s) \right|^2 ds
\]
\[
\leq 2\epsilon \frac{2T-2t}{T} P^2 + \frac{2T^2}{\lambda_1} \int_0^t \epsilon^{2x+2t} \|G(v^\epsilon)(s) - G(u)(s)\|^2 ds.
\]

On the other hand, we have
\[
\|G(v^\epsilon)(t) - G(u)(t)\| = \left\| a(t, v^\epsilon(t)) f(t, v^\epsilon(t)) - a(t, u(t)) f(t, u(t)) \right\|
\]
\[
\leq \left\| a(t, v^\epsilon(t)) \right\| \left\| f(t, v^\epsilon(t)) - f(t, u(t)) \right\| + \left\| f(t, u(t)) \right\| \left\| a(t, v^\epsilon(t)) - a(t, u(t)) \right\|
\]
\[
\leq MK \left\| v^\epsilon(t) - u(t) \right\| + N \left\| f(t, u(t)) \right\| \left\| v^\epsilon(t) - u(t) \right\|.
\]

Notice that, from (3.10) we can have
\[
\|f(t, u(t))\| \leq f(t, 0) + K\|u(t)\| \leq Q + K\|u(t)\|
\]

In addition, another one is basically obtained as follows.
\[
\|u(t)\| = \sqrt{\sum_{p=1}^{\infty} \left| \langle u(t), \phi_p \rangle \right|^2}
\]
\[
\leq \sqrt{\sum_{p=1}^{\infty} e^{2\sqrt{\lambda_p}(T-t)} \left( \langle u(t), \phi_p \rangle + \frac{\langle u_t(t), \phi_p \rangle}{\sqrt{\lambda_p}} \right)^2} \leq P.
\]
Therefore, we deduce that
\[ \|G(v^\epsilon(t)) - G(u(t))\| \leq \left( MK + NQ + NKP \right) \| v^\epsilon(t) - u(t) \|. \]

Then, this follows from that
\[ \|v^\epsilon(t) - u(t)\|^2 \leq 2 \epsilon \frac{2T}{\lambda_1} T^2 + \frac{2T^2}{\lambda_1} \left( MK + NQ + NKP \right) \int_0^t \epsilon \frac{2T}{\lambda_1} \|v^\epsilon(s) - u(s)\|^2 ds. \]

Thus, we have the following inequality.
\[ \epsilon^2 \|v^\epsilon(t) - u(t)\|^2 \leq 2 \epsilon^2 T^2 + \frac{2T^2}{\lambda_1} \left( MK + NQ + NKP \right) \int_0^t \epsilon \frac{2T}{\lambda_1} \|v^\epsilon(s) - u(s)\|^2 ds. \]

Applying Gronwall’s inequality, we obtain
\[ \epsilon^2 \|v^\epsilon(t) - u(t)\|^2 \leq 2 \epsilon^2 T^2 + \frac{2T^2}{\lambda_1} \left( MK + NQ + NKP \right) \int_0^t \epsilon \frac{2T}{\lambda_1} \|v^\epsilon(s) - u(s)\|^2 ds. \]

Hence, we finish the step completely.
\[ \|v^\epsilon(t) - u(t)\| \leq 2 P \exp \left\{ \frac{T^2}{\lambda_1} \left( MK + NQ + NKP \right) \right\} \epsilon^{1 - \frac{T}{T^2}}. \quad (4.44) \]

**Step 3.** Error estimate \( \|v^\epsilon - u^\epsilon\| \). By a similar way, we have
\[
\begin{align*}
\|G(v^\epsilon(t)) - G(u^\epsilon(t))\| &= \left\| a(t, v^\epsilon(t)) f(t, v^\epsilon(t)) - a(t, u^\epsilon(t)) f(t, u^\epsilon(t)) \right\| \\
&\leq \left\| a(t, v^\epsilon(t)) \right\| \left\| f(t, v^\epsilon(t)) - f(t, u^\epsilon(t)) \right\| + \\
&\quad + \left\| f(t, v^\epsilon(t)) \right\| \left\| a(t, v^\epsilon(t)) - a(t, u^\epsilon(t)) \right\| \\
&\leq MK \| v^\epsilon(t) - u^\epsilon(t) \| + N \left\| f(t, v^\epsilon(t)) \right\| \left\| v^\epsilon(t) - u^\epsilon(t) \right\|. \\
\end{align*}
\]

On the other hand, it is similar that from (3.10) we will have
\[
\begin{align*}
\|f(t, v^\epsilon(t))\| &\leq \|f(t, 0)\| + K \| v^\epsilon(t) \| \leq Q + K \| v^\epsilon(t) \| \\
&\leq Q + K \left( \|u(t)\| + 2P \exp \left\{ \frac{T^2}{\lambda_1} \left( MK + NQ + NKP \right) \right\} \epsilon^{1 - \frac{T}{T^2}} \right) \\
&\leq Q + K \left( P + 2P \exp \left\{ \frac{T^3}{\lambda_1} \left( MK + NQ + NKP \right) \right\} \right). \\
\end{align*}
\]

It follows that
\[ \|G(v^\epsilon(t)) - G(u^\epsilon(t))\| \leq R \| v^\epsilon(t) - u^\epsilon(t) \| \]

where
\[ R = MK + NQ + NK \left( P + 2P \exp \left\{ \frac{T^3}{\lambda_1} \left( MK + NQ + NKP \right) \right\} \right). \]
It follows from (3.19) that
\[
\|v^\epsilon(t) - u^\epsilon(t)\|^2 = \sum_{p=1}^{\infty} \left[ \cosh^\epsilon(\sqrt{\lambda_p} t)(\varphi^\epsilon_p - \varphi_p) + \frac{\sinh^\epsilon(\sqrt{\lambda_p} t)}{\sqrt{\lambda_p}}(g^\epsilon_p - g_p) \right] + \int_0^t \frac{\sinh^\epsilon(\sqrt{\lambda_p}(t-s))}{\sqrt{\lambda_p}} \left( G_p(v^\epsilon)(s) - G_p(u^\epsilon)(s) \right) ds^2
\]
\[
\leq \sum_{p=1}^{\infty} \left[ 3 \cosh^\epsilon(\sqrt{\lambda_p} t)(\varphi^\epsilon_p - \varphi_p)^2 + 3 \left( \frac{\sinh^\epsilon(\sqrt{\lambda_p} t)}{\sqrt{\lambda_p}} \right)^2 (g^\epsilon_p - g_p)^2 \right] + 3t \int_0^t \left( \frac{\sinh^\epsilon(\sqrt{\lambda_p}(t-s))}{\sqrt{\lambda_p}} \right)^2 \left( G_p(v^\epsilon)(s) - G_p(u^\epsilon)(s) \right)^2 ds.
\]

Using (9),(10) and the Lipschitzian property of \(f\), we get the following inequality
\[
\|v^\epsilon(t) - u^\epsilon(t)\|^2 \leq 3\epsilon^{-\frac{2t}{\lambda_1}} \|\varphi^\epsilon - \varphi\|^2 + \frac{3}{\lambda_1} \epsilon^{-\frac{2t}{\lambda_1}} \|g^\epsilon - g\|^2 + \frac{3tR^2}{\lambda_1} \int_0^t \epsilon^{\frac{2t-2t}{\lambda_1}} \|v^\epsilon(s) - u^\epsilon(s)\|^2 ds.
\]
This implies that
\[
\epsilon^{\frac{2t}{\lambda_1}} \|v^\epsilon(t) - u^\epsilon(t)\|^2 \leq 3 \left( \|\varphi^\epsilon - \varphi\|^2 + \frac{\|g^\epsilon - g\|^2}{\lambda_1} \right) + \frac{3tR^2}{\lambda_1} \int_0^t \epsilon^{\frac{2t-2t}{\lambda_1}} \|v^\epsilon(s) - u^\epsilon(s)\|^2 ds.
\]
Applying Gronwall’s inequality, we have
\[
\epsilon^{\frac{2t}{\lambda_1}} \|v^\epsilon(t) - u^\epsilon(t)\|^2 \leq 3 \exp \left\{ \frac{3T R^2 t}{\lambda_1} \right\} \left( \|\varphi^\epsilon - \varphi\|^2 + \frac{\|g^\epsilon - g\|^2}{\lambda_1} \right) \leq 3 \exp \left\{ \frac{3T R^2 t}{\lambda_1} \right\} (1 + \frac{1}{\lambda_1}) \epsilon^2.
\]
(4.45)
By simplification, it yields
\[
\|v^\epsilon(t) - u^\epsilon(t)\| \leq \sqrt{3 \exp \left\{ \frac{3T R^2 t}{\lambda_1} \right\} (1 + \frac{1}{\lambda_1}) \epsilon^2}.
\]
(4.46)
Combining (4.44) and (4.46), we thus obtain
\[
\|u^\epsilon(t) - u(t)\| \leq \|v^\epsilon(t) - u^\epsilon(t)\| + \|v^\epsilon(t) - u(t)\|
\]
\[
\leq \sqrt{3 \exp \left\{ \frac{3T R^2 t}{\lambda_1} \right\} (1 + \frac{1}{\lambda_1}) \epsilon^2} + 2P \exp \left\{ \frac{T^2 t}{\lambda_1} \left( MK + NQ + NK P \right) \right\} \epsilon^{\frac{1}{\lambda_1}}.
\]
This completes the proof of Theorem.
5 A numerical example

The core of this paper is to show Section 3. Therefore, we intend to give in this section a very simple example in one-dimensional in order to illustrate how the regularized solution in Section 3 approximates the exact solution for nonlinear elliptic problems. The example is involved with the operator $-\frac{\partial^2}{\partial x^2}$ and the domain $D(A) = H^1_0(0, 1) \subset L^2(0, 1)$. Then, the problem is in the following form.

\[
\begin{aligned}
&u_{tt} + u_{xx} = F(u) + G(x, t), \quad (x, t) \in (0, 1) \times (0, 1); \\
&u(0, t) = u(1, t) = 0, \quad t \in (0, 1); \\
&u(x, 0) = \varphi(x), u_t(x, 0) = g(x), \quad x \in (0, 1);
\end{aligned}
\]

(5.47)

where $F, G, \varphi$ and $g$ are given as follows.

\[
F(u) = \frac{1}{a^3} u^3,
\]

(5.48)

\[
G(x, t) = 2at(1 - 3x) - t^3x^6(1 - x)^3,
\]

(5.49)

\[
\varphi(x) = 0, \quad g(x) = ax^2(1 - x).
\]

(5.50)

It is not too hard to see that the exact solution is $atx^2(1 - x)$ where $a \in \mathbb{R} \setminus \{0\}$. An orthonormal eigenbasis in $L^2(0, 1)$ is $\phi_p(x) = \sqrt{2} \sin(\sqrt{\lambda_p}x)$ and $\lambda_p = p^2\pi^2$ is the corresponding eigenvalue. As a result, choose $a = 1$, we have

\[
u(x, t) = \sum_{p=1}^{\infty} \left[ \cosh(\sqrt{\lambda_p}t) \varphi_p + \frac{\sinh(\sqrt{\lambda_p}t)}{\sqrt{\lambda_p}} g_p + \int_0^t \frac{\sinh(\sqrt{\lambda_p}(t - s))}{\sqrt{\lambda_p}} f_p(u)(s) \, ds \right] \phi_p(x),
\]

(5.51)

where

\[
\varphi_p = \int_0^1 \varphi(x) \phi_p(x) \, dx, \quad g_p = \int_0^1 g(x) \phi_p(x) \, dx, \quad f_p(u)(s) = \int_0^1 [F(u) + G(x, s)] \phi_p(x) \, dx.
\]

(5.52)

Remark 5.1. We will approximate the regularized solution by taking perturbation number in data function by two ways. The perturbation is intended to define as $\epsilon \mathrm{rand}(.)$ where each random term $\epsilon \mathrm{rand}(.)$ will be determined on $[-1, 1]$ uniformly, i.e.

\[
f^\epsilon(\cdot) = f(\cdot) + \epsilon \mathrm{rand}(\cdot), \quad f^\epsilon(\cdot) = f(\cdot) \left(1 + \frac{\epsilon \mathrm{rand}(\cdot)}{\|f\|}\right).
\]

In particular, we let

\[
\varphi^\epsilon(x) = \epsilon \mathrm{rand}(\cdot), \quad g^\epsilon(x) = g(x) \left(1 + \sqrt{105} \epsilon \mathrm{rand}(\cdot)\right).
\]
Figure 1: The exact solution $u_{ex} = tx^2 (1 - x)$.

Remark 5.2. The aim of the numerical experiments is to observe $\epsilon = 10^{-r}$ where $r = 1, 10$. The couple of $(\varphi^\epsilon, g^\epsilon)$ plays as measured data with a random noise. Then, the regularized solution is expected to be closed to the exact solution under a proper discretization.

As we introduced, we proceed to define stability terms. Those are

$$
cosh^\epsilon \left( \sqrt{\lambda_p} t \right) = \frac{Q (\epsilon, \lambda_p) e^{\sqrt{\lambda_p} t} + e^{-\sqrt{\lambda_p} t}}{2},
$$

$$
\sinh^\epsilon \left( \sqrt{\lambda_p} t \right) = \frac{Q (\epsilon, \lambda_p) e^{\sqrt{\lambda_p} t} - e^{-\sqrt{\lambda_p} t}}{2},
$$

$$
\sinh^\epsilon \left( \sqrt{\lambda_p} (t - s) \right) = \frac{Q (\epsilon, \lambda_p) e^{\sqrt{\lambda_p} (t-s)} - e^{-\sqrt{\lambda_p} (t-s)}}{2},
$$

$$
Q (\epsilon, \lambda_p) = \frac{e^{-\sqrt{\lambda_p}}}{e + e^{-\sqrt{\lambda_p}}}. 
$$

Therefore, we have the regularized solution.

$$
v^\epsilon (x, t) = \sum_{p=1}^{\infty} \left[ \cosh^\epsilon \left( \sqrt{\lambda_p} t \right) \varphi_p^\epsilon + \frac{\sinh^\epsilon \left( \sqrt{\lambda_p} t \right) g_p^\epsilon}{\sqrt{\lambda_p}} + \int_0^t \frac{\sinh^\epsilon \left( \sqrt{\lambda_p} (t-s) \right)}{\sqrt{\lambda_p}} f_p (v^\epsilon) (s) \, ds \right] \varphi_p (x). 
$$

After dividing the time $t_i = i \Delta t, \Delta t = \frac{1}{M}, i = 0, M$, the regularized solution can be computed by the following iterative scheme.
\[ v^*_N(x, t_i) = v^*_{N,i}(x) \]
\[ = w^*_1, i \sin(\pi x) + w^*_2, i \sin(2\pi x) + \ldots + w^*_N, i \sin(N\pi x) \]
\[ = \begin{bmatrix} w^*_1, i & w^*_2, i & \cdots & w^*_N, i \end{bmatrix} \begin{bmatrix} \sin(\pi x) \\ \sin(2\pi x) \\ \vdots \\ \sin(N\pi x) \end{bmatrix}, \tag{5.58} \]

where

\[
\frac{1}{2} w^*_{p,i} = \cosh\left(p \pi t_i \right) \int_0^1 \varphi^\epsilon(x) \sin(p \pi x) \, dx + \frac{\sinh\left(p \pi t_i \right)}{p \pi} \int_0^1 g^\epsilon(x) \sin(p \pi x) \, dx \\
+ \frac{1}{p \pi} \sum_{j=1}^i \int_{t_{j-1}}^{t_j} \sinh\left(p \pi (t_i - s) \right) \left( (v^*_N(x, t_{j-1}))^3 + G(x, s) \right) \sin(p \pi x) \, dx \, ds, \tag{5.59} \]

\[
v^*_N(x, t_0) = v^*_{N,0}(x) = \varphi^\epsilon(x). \tag{5.60} \]

Then, let \( x_j = j \Delta x, \Delta x = \frac{1}{K}, j = 0, K \), we have another iterative scheme.

\[
v^*_{N,i}(x_j) = v^*_N(x_j, t_i) = w^*_1, i \sin(\pi x_j) + w^*_2, i \sin(2\pi x_j) + \ldots + w^*_N, i \sin(N\pi x_j). \tag{5.61} \]

The whole process is concluded into four steps.

**Step 1.** Have \( \epsilon \), choose \( N = N_0, K = K_0 \) and \( M = M_0 \) respectively. We get

\[
x_j = j \Delta x, \Delta x = \frac{1}{K}, j = 0, K, \tag{5.62} \]

\[
t_i = i \Delta t, \Delta t = \frac{1}{M}, i = 0, M. \tag{5.63} \]

**Step 2.** Put \( v^*_N(x, t_i) = v^*_{N,i}(x), i = 0, M \) and \( v^*_{N,0}(x) = \varphi^\epsilon(x) \). We find out

\[
V^*_N(x) = [v^*_N(0, x), v^*_N(1, x), \ldots, v^*_N(M, x)]^T \in \mathbb{R}^{M+1}. \tag{5.64} \]

**Step 3.** For \( j = 0, K \), put \( v^*_{N,i}(x_j) = v^*_{N,i,j} \), we present

\[
U^*_{N,M,K} = [v^*_{N,0}(x_j), v^*_{N,1}(x_j), \ldots, v^*_{N,M}(x_j)] \tag{5.65} \]

\[
= \begin{bmatrix} v^*_{N,0,0} & v^*_{N,0,1} & \cdots & v^*_{N,0,K} \\
 v^*_{N,1,0} & v^*_{N,1,1} & \cdots & v^*_{N,1,K} \\
 \vdots & \vdots & \ddots & \vdots \\
 v^*_{N,M,0} & v^*_{N,M,1} & \cdots & v^*_{N,M,K} \end{bmatrix} \in \mathbb{R}^{M+1} \times \mathbb{R}^{K+1}. \tag{5.66} \]

**Step 4.** Calculate the error

\[
E^*_N(t_i) = \sqrt{\sum_{j=0}^{K} \left| v^*_{N}(x_j, t_i) - u_{ex}(x_j, t_i) \right|^2}, \quad i = 0, M. \tag{5.67} \]
\[\epsilon \in \mathbb{E}, \quad \mathbb{E} \in \mathbb{N}, \quad \frac{1}{2}, \quad \frac{1}{4}, \quad \frac{3}{4}\]

| \(\epsilon\) | \(E_N^\epsilon \left(\frac{1}{2}\right)\) | \(E_N^\epsilon \left(\frac{1}{4}\right)\) | \(E_N^\epsilon \left(\frac{3}{4}\right)\) |
|---|---|---|---|
| 1.0E-01 | 3.6697975496E-01 | 2.6038073148E-01 | 6.3238102008E-01 |
| 1.0E-02 | 5.7702326175E-02 | 3.482150118E-02 | 8.6516725190E-02 |
| 1.0E-03 | 1.2897073639E-02 | 1.2765356426E-02 | 3.2397554936E-02 |
| 1.0E-04 | 3.2189707364E-02 | 1.9598409706E-02 | 6.3238102008E-01 |
| 1.0E-05 | 3.01169841E-02 | 6.3238102008E-01 | 1.905362258E-02 |

Table 1: Errors between the regularized solution and exact solution for \(t = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}\).

| \(\epsilon = 10^{-4}\) | \(E_N^\epsilon \left(\frac{1}{2}\right)\) | \(E_N^\epsilon \left(\frac{1}{4}\right)\) | \(E_N^\epsilon \left(\frac{3}{4}\right)\) |
|---|---|---|---|
| \(N = 3\) | 1.0711862180E-02 | 5.5059970016E-03 | 1.5931956961E-02 |
| \(N = 4\) | 3.8073451089E-03 | 1.9328671997E-03 | 5.6915418779E-03 |

Table 2: Errors between the regularized solution and exact solution at \(t = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}\) for \(N = 3, 4\).

Figure 2, Figure 3 and Figure 4 illustrate the regularized solution and exact solution in 2D graph in some cases which are known in each caption. On the other hand, in Figure 1 and Figure 5, these solutions in 3D graph are represented. Although we consider many values of noise in Table 1, showing 2D and 3D graphs is stopped at a reasonable by observation. In details, we only show \(r = \frac{1}{4}\) for both 2D and 3D graphs.

In this example, we simply choose a slightly coarse grid \(M = 12, K = 20\) because we want to reduce computational workloads. However, before deciding to choose those, we make a test with a finer grid \(M = 16, K = 30\) then even it is worse. In particular, for \(\epsilon = 10^{-4}\), \(E_N^\epsilon \left(\frac{1}{2}\right)\) in the coarse grid is \(1.3001169841 \times 10^{-2}\) while \(1.5925046512 \times 10^{-2}\) is for the finer. Also, for \(\epsilon = 10^{-5}\), \(E_N^\epsilon \left(\frac{1}{4}\right)\) in the coarse is \(6.3529040819 \times 10^{-3}\) while \(7.7873931681 \times 10^{-3}\) is for the finer one.

We note that larger \(N\) mostly leads to better approximation. For example, we merely choose \(N = 2\) for the main result, then from Figure 2-Figure 4, the regularized solution still do not fit the exact solution completely. In [26], the authors Hongwu Zhang and Ting Wei show Table 1 to present many results of the large \(N\). Thus, from the point of view, we consider two supplement cases \(N = 3\) and \(N = 4\) which are shown in the Figure 6-Figure 8 and Table 2. They are all extremely better than what we obtain in the Table 1 and Figure 2-Figure 4. However, based on the advice in [26], we should choose \(N = 4\) or \(N = 5\) to not only get the whole desired, but also reduce computational workloads.

6 Conclusion

In this paper, we propose a method of integral equation to solve the Cauchy problem for elliptic equation with nonlinear source. This problem may be difficult and there are few results on the regularized problem. From that point, we aim to consider the regularization method for this problem in theoretical framework. The convergence results have been presented for the cases of \(0 \leq t < T\) and \(t = T\) under some assumptions for the exact solution. However, our method still has a little theoretical range.
Figure 2: The regularized solution (green) and exact solution (red) at $t = \frac{1}{2}$ for $\epsilon = 10^{-r}$ with $r = 1; 2; 3; 4$.

Figure 3: The regularized solution (green) and exact solution (red) at $t = \frac{1}{4}$ for $\epsilon = 10^{-r}$ with $r = 1; 2; 3; 4$. 
Figure 4: The regularized solution (green) and exact solution (red) at $t = \frac{3}{4}$ for $\epsilon = 10^{-r}$ with $r = 1; 2; 3; 4$.

Figure 5: The regularized solution in 3D for $\epsilon = 10^{-r}$ with $r = 1; 2; 3; 4$. 
Figure 6: The regularized solution at $t = \frac{1}{2}$ for $N = 3$ and $N = 4$ with $\epsilon = 10^{-4}$.

Figure 7: The regularized solution at $t = \frac{1}{4}$ for $N = 3$ and $N = 4$ with $\epsilon = 10^{-4}$.

Figure 8: The regularized solution at $t = \frac{3}{4}$ for $N = 3$ and $N = 4$ with $\epsilon = 10^{-4}$.
since the class of function $f$ is still small. This makes the applicability of the method very narrow. Moreover, in the numerical result, there is an issue about choosing the truncation number which plays a role in regularization effect. In our future research, we will, therefore, consider the regularized problem in the case where $f$ is a locally Lipchitzian function and study the theoretical analysis regarding the influence of the truncation term.

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