Shape of Traveling Densities with Extremum Statistical Complexity

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ABSTRACT

In this paper, we analyze the behavior of statistical complexity in several systems where two identical densities that travel in opposite direction cross each other. Besides the crossing between two Gaussian, rectangular and triangular densities studied in a previous work, we also investigate in detail the crossing between two exponential and two gamma distributions. For all these cases, the shape of the total density presenting an extreme value in complexity is found.

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1 Introduction

The behavior of the statistical complexity in time-dependent systems has not been broadly investigated. A previous work in this direction is presented in (Calbet and López-Ruiz, 2001) where a gas decaying toward the asymptotic equilibrium state is studied. It was found that this system goes towards equilibrium by approaching the \textit{maximum complexity path}, which is the trajectory in distribution space formed by the distributions with the maximal complexity. Then, from a physical point of view, it can have some interest to study the extremal behavior of statistical magnitudes in time dependent systems.

In this work, we start by studying the statistical complexity $C$ in a simplified time-dependent system $\rho(x, t)$ composed of two one-dimensional (variable $x$) identical densities that travel in opposite directions with the same velocity $v$, one of them, $\rho_+(x, t)$, going to the right and the other one, $\rho_-(x, t)$ going to the left. That is

$$\rho(x,t) = \frac{1}{2} \rho_+(x,t) + \frac{1}{2} \rho_-(x,t),$$

(1.1)
with the normalization condition \( \int_{\mathbb{R}} \rho_\pm(x,t) dx = 1 \) that implies the normalization of \( \rho(x,t) \), and the initial condition \( \rho_+(x,0) = \rho_-(x,0) \). In the next section, we recall the analysis of \( C \) done for two Gaussian, rectangular and triangular traveling densities in (López-Ruiz and Sañudo, 2010). Also, the extension of this study for two exponential and two gamma distributions is performed. Specifically, the shape of \( \rho(x,t) \) presenting the maximum and minimum \( C \) is explicitly shown for all these cases. The final section includes our conclusions.

2 Complexity in Traveling Densities

Let us start by recalling the definition of the statistical complexity \( C \) (López-Ruiz, Mancini and Calbet, 1995), the so-called LMC complexity, that is defined as

\[
C = H \cdot D ,
\]

where \( H \) represents the information content of the system and \( D \) gives an idea of how much concentrated is its spatial distribution. For our purpose, we take a version used in (Catalán, Garay and López-Ruiz, 2002) as quantifier of \( H \). This is the simple exponential Shannon entropy (Dembo, Cover and Thomas, 1991), that takes the form,

\[
H = e^S ,
\]

where \( S \) is the Shannon information entropy (Shannon, 1948),

\[
S = - \int \rho(x) \log \rho(x) \, dx ,
\]

with \( x \) representing the continuum of the system states and \( \rho(x) \) the probability density associated to all those states. We keep for the disequilibrium the form originally introduced in (López-Ruiz et al., 1995; Catalán et al., 2002), that is,

\[
D = \int \rho^2(x) \, dx .
\]

Now we proceed to calculate \( C \) for the system above mentioned (1.1) in the Gaussian, exponential, rectangular, triangular and gamma cases.

2.1 Gaussian traveling densities

Here the two one-dimensional traveling densities that compose system (1.1) take the form:

\[
\rho_\pm(x,t) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left\{ -\frac{(x \mp vt)^2}{2\sigma^2} \right\} ,
\]

where \( \sigma \) is the variance of the density distribution.

The behavior of complexity, \( C_G \), as a function of the adimensional quantity \( 2vt/\sigma \) is given in Fig. 1. Let us observe that \( C_G \) presents a minimum. The shape of system (1.1) for this minimum complexity case is plotted in an adimensional scale in Fig. 2.
Figure 1: Statistical complexity, $C_G$, vs. the adimensional separation, $2vt/\sigma$, between the two traveling Gaussian densities defined in Eq. (2.5). The minimum of $C_G$ is reached when $2vt/\sigma = 2.91$. The dashed line indicates the value of complexity for the normalized Gaussian distribution.

Figure 2: Shape of the density (1.1) in adimensional units that presents the minimum statistical complexity when the two traveling Gaussian densities defined in (2.5) are crossing. Notice that the value of the adimensional separation between the centers of both Gaussian distributions must be 2.91.
Figure 3: Statistical complexity, $C_{Exp}$, vs. the adimensional separation, $2vt/\beta$, between the two traveling exponential densities defined in Eq. (2.6). The minimum of $C_{Exp}$ is reached when $2vt/\beta = 2.94$. The dashed line indicates the value of complexity for the normalized exponential distribution.

Figure 4: Shape of the density (1.1) in adimensional units that presents the minimum statistical complexity when the two traveling exponential densities defined in (2.6) are crossing. Notice that the value of the adimensional separation between the centers of both exponential distributions must be 2.94.
Figure 5: Statistical complexity, $C_R$, vs. the adimensional separation, $2vt/\delta$, between the two traveling rectangular densities defined in Eq. (2.7). The maximum of $C_R$ is reached when $2vt/\delta = 0.557$. Observe that the normalized rectangular distribution has $C_R = 1$.

2.2 Exponential traveling densities

A similar case to the Gaussian former one is the crossing between two exponential distributions. Here the two one-dimensional traveling densities that compose system (1.1) take the form:

$$\rho_{\pm}(x, t) = \frac{1}{2\beta} \exp \left\{ -\left| x \mp vt \right| / \beta \right\}, \quad (2.6)$$

where $\beta$ is the width of the density distribution.

The formula for $C_{Exp}$ can be analytically obtained. The behavior of $C_{Exp}$ as a function of the adimensional quantity $2vt/\beta$ is given in Fig. 3. Let us observe that $C_{Exp}$ presents a minimum. The shape of system (1.1) for this minimum complexity case is plotted in an adimensional scale in Fig. 4.

2.3 Rectangular traveling densities

Now the two one-dimensional traveling densities that compose system (1.1) take the form:

$$\rho_{\pm}(x, t) = \begin{cases} 1/\delta & \text{if } -\delta/2 \leq x \mp vt \leq \delta/2, \\ 0 & \text{if } |x \mp vt| > \delta/2. \end{cases} \quad (2.7)$$

where $\delta$ is the width of each distribution.

For this case, the complexity, $C_R$, can be analytically obtained. Its expression is:

$$C_R(t) = \begin{cases} 2^{2vt/\delta} \left( 1 - \frac{vt}{\delta} \right) & \text{if } 0 \leq 2vt \leq \delta, \\ 1 & \text{if } 2vt > \delta. \end{cases} \quad (2.8)$$
Figure 6: Shape of the density \((1.1)\) in adimensional units that presents the maximum statistical complexity when the two traveling rectangular densities defined in \((2.7)\) are crossing. Notice that the value of the adimensional separation between the centers of both rectangular distributions must be 0.557. Then, the width of the overlapping between both distributions is 0.443.

The behavior of \(C_R\) as a function of the adimensional quantity \(2vt/\delta\) is given in Fig. 5. Let us observe that \(C_R\) presents a maximum. The shape of system \((1.1)\) for this maximum complexity case is plotted in an adimensional scale in Fig. 6.

2.4 Triangular traveling densities

The two one-dimensional traveling densities that compose system \((1.1)\) take the form in this case:

\[
\rho_{\pm}(x, t) = \begin{cases} 
\frac{(x \mp vt)}{\epsilon^2} + \frac{1}{\epsilon} & \text{if } -\epsilon \leq x \mp vt \leq 0, \\
-\frac{(x \mp vt)}{\epsilon^2} + \frac{1}{\epsilon} & \text{if } 0 < x \mp vt \leq \epsilon, \\
0 & \text{if } |x \mp vt| > \epsilon,
\end{cases}
\]  

(2.9)

where \(\epsilon\) is the width of each distribution (isosceles triangle whose base length is \(2\epsilon\)).

The behavior of complexity, \(C_T\), as a function of the adimensional quantity \(2vt/\epsilon\) is given in Fig. 7. Let us observe that \(C_T\) presents a maximum and a minimum. The shape of system \((1.1)\) for both cases, with maximum and minimum complexity, are plotted in an adimensional scale in Figs. 8 and 9 respectively.

2.5 Gamma traveling densities

We study a final case given by the crossing between two gamma distributions. Here the two one-dimensional traveling densities that compose system \((1.1)\) take the form:

\[
\rho_{\pm}(x, t) = \frac{1}{2\theta^k \Gamma(k)} |x \mp vt|^{k-1} \exp \left\{-|x \mp vt|/\theta\right\},
\]  

(2.10)
Figure 7: Statistical complexity, $C_T$, vs. the adimensional separation, $2vt/\varepsilon$, between the two traveling triangular densities given in Eq. (2.9). The maximum and minimum of $C_T$ are reached when $2vt/\varepsilon$ takes the values 0.44 and 1.27, respectively. The dashed line indicates the value of complexity for the normalized triangular distribution.

Figure 8: Shape of the density (1.1) in adimensional units that presents the maximum statistical complexity when the two traveling triangular densities defined in (2.9) are crossing. Notice that the value of the adimensional separation between the centers of both triangular distributions must be 0.44.
where $k$ is a real parameter such that $k \geq 1$ and $\theta$ is a scale parameter related to the width of the distribution for a given $k$. Observe that the exponential case is recovered for $k = 1$. Let us also remark that for a fixed $k$ the parameter $\theta$ modifies the shape of the distribution but the complexity $C$ is conserved.

The behavior of complexity, $C_{Gam}$, as a function of the adimensional quantity $2vt/\theta$ is given in Fig. 10 for $k = 3$. The behavior for other values of $k$ seems to be similar. Let us observe that $C_{Gam}$ presents two local maxima and two local minima. The shape of system (1.1) for all these cases, with local extremal complexity, are plotted in an adimensional scale in Figs. 11 and 12, for the local maxima, and in Figs. 13 and 14, for the local minima.

3 Conclusion

In this work, we have studied the behavior of the statistical complexity as a function of time when two traveling identical densities are crossing each other. Five cases have been analyzed: Gaussian, exponential, rectangular, triangular and gamma densities. The Gaussian and exponential cases present a particular configuration with minimum complexity. The rectangular case displays a particular configuration with maximum complexity. The triangular case shows an intermediate behavior between the two former cases with two particular configurations, one of them with maximum complexity and the other one with minimum complexity. The gamma case displays two configurations with local minima complexity and other two configurations with local maxima complexity. In general, all these configurations with extremal complexity cannot be analytically obtained and a careful computational study is required in order to determine them.
Figure 10: Statistical complexity, $C_{\text{Gam}}$, vs. the adimensional separation, $2\nu t/\theta$, between the two traveling gamma densities given in Eq. (2.10) for $k = 3$. The local maxima of $C_{\text{Gam}}$ are reached when $2\nu t/\theta$ take the values 0.36 and 5.01, and the local minima of $C_{\text{Gam}}$ are on the values 2.32 and 9.99. The dashed line indicates the value of complexity for the normalized gamma distribution for $k = 3$.

Figure 11: Shape of the density (1.1) in adimensional units that presents a local maximum statistical complexity when the two traveling gamma densities defined in (2.10) are crossing. Notice that the value of the adimensional separation between the centers of both gamma distributions must be 0.36.
Figure 12: Shape of the density (1.1) in adimensional units that presents a local maximum statistical complexity when the two traveling gamma densities defined in (2.10) are crossing. Notice that the value of the adimensional separation between the centers of both gamma distributions must be $5.01$.

Figure 13: Shape of the density (1.1) in adimensional units that presents a local minimum statistical complexity when the two traveling gamma densities defined in (2.10) are crossing. Notice that the value of the adimensional separation between the centers of both gamma distributions must be $2.32$. 
Figure 14: Shape of the density in adimensional units that presents a local minimum statistical complexity when the two traveling gamma densities defined in are crossing. Notice that the value of the adimensional separation between the centers of both gamma distributions must be 9.91.

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References

Calbet, X. and López-Ruiz, R. 2001. Tendency toward maximum complexity in a non-equilibrium isolated system, *Phys. Rev. E* 63: 066116(9pp).

Catalán, R., Garay, J. and López-Ruiz, R. 2002. Features of the extension of a statistical measure of complexity to continuous systems, *Phys. Rev. E* 66: 011102(6pp).

Dembo, A., Cover, T. and Thomas, J. 1991. Information theoretic inequalities, *IEEE Trans. Inf. Theory* 37: 1501–1518.

López-Ruiz, R., Mancini, H. and Calbet, X. 1995. A statistical measure of complexity, *Phys. Lett. A* 209: 321–326.

López-Ruiz, R. and Sañudo, J. 2010. Statistical complexity in traveling densities, *Proceedings of the 9th WSEAS International NOLASC’10 Conference, Special Session "Statistical Complexity in Classical and Quantum Systems", Sousse, Tunisia*: 20–23.

Shannon, C. 1948. A mathematical theory of communication, *Bell. Sys. Tech. J. 27*: 379–423.
