On a question of Luca and Schinzel over Segal–Piatetski-Shapiro sequences

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Abstract
We extend to Segal–Piatetski-Shapiro sequences previous results on the Luca–Schinzel question. Namely, we prove that for any real \( c \) larger than 1, the sequence \( \left( \sum_{m \leq n} \phi(\lfloor mc \rfloor)/\lfloor mc \rfloor \right)_{n \geq 1} \) is dense modulo 1, where \( \phi \) denotes Euler’s totient function. The main part of the proof consists in showing that when \( R \) is a large integer, the sequence of the residues of \( \lfloor mc \rfloor \) modulo \( R \) contains any block of consecutive residues of a given length.

Keywords Segal–Piatetski-Shapiro sequences · Euler’s totient function · Distribution modulo 1

Mathematics Subject Classification 11N64 · 11K31 · 11K38 · 11B50

1 Introduction
At the Czech–Slovak Number Theory Conference in Smolenice in August 2007, F. Luca asked whether the sequences of arithmetic and geometric means of the first val-
ues of the Euler totient function are uniformly distributed modulo 1; A. Schinzel asked whether the weaker statement that those sequences are dense modulo 1 was known; this question was positively answered in [5]. This opened the way to extensions considering different multiplicative functions with constant mean value, or mean values over different sequences of integers as well as the original question of Luca in special cases (cf. [2, 4, 5, 7, 13]). One of the latest results is that of the first and third named authors of this paper, namely the following

**Theorem 1** (Theorem 1 of [6]) Let \( \varphi \) denote the Euler function and \( G \) be a non constant polynomial with integral coefficients and taking positive values at positive arguments. Then the sequence

\[
\left( \sum_{m \leq n} \frac{\varphi(G(m))}{G(m)} \right)_{n \geq 1}
\]

is dense modulo 1.

Here, we extend this result to Segal–Piatetski-Shapiro sequences. We recall that those sequences, which depend on a parameter \( c \) in \((1, +\infty) \setminus \mathbb{N}\), are the sequences \((\lfloor n^c \rfloor)_{n \geq 1}\), where \([u]\) denotes the integral part of the real number \( u \). They have been introduced by Segal [14] who studied their additive properties; in 1953, Piatetski-Shapiro [13] proved that for \( 1 < c < 12/11 \), those sequences contain infinitely many primes, with the expected density.

**Theorem 2** Let \( \varphi \) denote the Euler function and \( c \) be a real number larger than 1. Then the sequence

\[
\left( \sum_{m \leq n} \frac{\varphi([mc])}{[mc]} \right)_{n \geq 1}
\]

is dense modulo 1.

The case when \( c \) is an integer is but a special case of Theorem 1. In this paper, we only consider the case when \( c > 1 \) is not an integer, i.e., the case of Segal–Piatetski-Shapiro sequences.

A way to tackle the Luca–Schinzel question on a sequence \( (a(n) = \sum_{m \leq n} f(m))_n \) is to find, for each given \( \varepsilon \), consecutive values \( f(m+1), f(m+2), \ldots, f(m+H) \) which are all less than \( \varepsilon \) but the sum of which is larger than 1. In our case where \( f \) is a multiplicative function, a natural idea would be to expect some sort of arithmetical independence of the elements \( ([m+1]^c], \ldots, ([m+H]^c] \). Unfortunately, this is not the case: it is proved in [1] that the sequence of the residues of \([mc]\) modulo \( k \) is normal for no \( k \geq 2 \). It is even shown that in the sequence of the residues of \([mc]\) modulo \( k \), there are finite blocks of consecutive values which do not occur, as soon as \( k \) is large enough.

In Sect. 3, we are however going to prove that \( ([m+h]^c] \) can be locally approximated by a linear polynomial, more precisely, denoting by \( \| c \| \) the so-called "distance to the nearest integer" of the real number \( c \), we have
Theorem 3 Let $\beta = 2^{(c+2)}(c + 2)^2\|c\|^{-1}$ and let $H$ be a positive integer. If $R$ is a sufficiently large integer, for any residue $r$ modulo $R$, there exists $m \leq R^\beta$ such that

$$\forall h \in [1, H]: \lfloor (m + h)^c \rfloor \equiv r + h \pmod{R}.$$  

In connection with [1], we notice that this result implies that for any positive integer $H$ and any non-integral $c$ larger than 1, if $R$ is large enough, the sequence of the residues of $[n^c]$ modulo $R$ contains the block $(1, 2, \ldots, H)$.

In Sect. 4, we shall derive Theorem 2 from this result.

We see this paper as an archetype of the more general study of the distribution modulo 1 of mean values of sequences $(\sum_{m \leq n} a(f(m)))_{n \geq 1}$, where $a$ is a regular multiplicative function with constant mean value and $f$ is a function in a Hardy field with polynomial growth.

Another challenging question is to determine the values of $c$ for which the sequence $(\sum_{m \leq n} \varphi([m^c])/[m^c])_{n \geq 1}$ is uniformly distributed modulo 1.

## 2 General notation and results

### 2.1 Notation

For a real number $u$, we can write in a unique way $u = [u] + \{u\}$, where $[u] \in \mathbb{Z}$ and $\{u\} \in [0, 1)$. We further let $\|u\| = \min(\{u\}, 1 - \{u\}) = \min(\{|u - m| : m \in \mathbb{Z}\})$, and $[u]$ denote the least integer greater than or equal to $u$. For $x = (x^{(1)}, \ldots, x^{(s)})$ in $\mathbb{R}^s$, we let $\{x\} = (\{x^{(1)}\}, \ldots, \{x^{(s)}\})$.

For an $s$-tuple of real numbers $k = (k_1, \ldots, k_s)$, we let $\|k\|_\infty$ denote the maximum of the absolute values of its components.

For a real number $t$, we let $e(t) = \exp(2\pi it)$.

The letters $p, q$ denote prime numbers, the letter $h$ a non-negative integer.

We use Vinogradov’s notation: if $f$ and $g$ are two real functions defined on some interval $[a, +\infty)$, $g$ taking positive values, we write $f \ll g$ for $f = O(g)$ and, when a parameter $u$ is involved, we write $f \ll_u g$ for $f = O_u(g)$. Similarly, we define $f \gg g$ (resp. $f \gg_u g$) if there exists a positive constant $C$, absolute, (resp. which may depend on $u$) such that $f(x) \geq C g(x)$ on $[a, +\infty)$.

For $c$ real and $\ell$ a positive integer, we define the binomial coefficient $\binom{\ell}{c} = \frac{c(c-1) \cdots (c-\ell+1)}{\ell!}$; we further let $\binom{\ell}{0} = 1$. By Taylor’s expansion, we have, for $c > 0$, $m > 0$ and $h \geq 0$

$$ (m + h)^c = \sum_{\ell=0}^{[c]} \binom{c}{\ell} h^\ell m^{c-\ell} + r_c(m, h),$$

with $0 \leq r_c(m, h) \leq \binom{c}{[c]+1} h^{[c]+1} m^{[c]-1} \leq h^{[c]+1} m^{[c]-1}$. 

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2.2 The Erdős–Turán–Koksma–Szüsz inequality

This inequality permits one to give an upper bound for the discrepancy of a sequence of elements in \( \mathbb{R}^s \) in terms of finitely many trigonometrical sums. The case when \( s = 1 \) has been given by Erdős and Turán in [9] and generalized to higher dimensions independently by Koksma [11] and Szüsz [15]. We quote here the formulation given by Drmota and Tichy [7]. We first recall the definition of the discrepancy.

An interval \( I \) in \([0, 1)^s\) is a cartesian product \( \prod_{1 \leq i \leq s} [a_i, b_i) \) with \( 0 \leq a_i \leq b_i \leq 1 \) for \( 1 \leq i \leq s \); its Lebesgue measure \( \prod_{1 \leq i \leq s} (b_i - a_i) \) is denoted by \( \lambda(I) \). For an interval \( I \subset [0, 1)^s \), we denote by \( \chi_I \) its indicator (also called characteristic) function.

The discrepancy of a finite set \( X = \{x_1, x_2, \ldots, x_N\} \) of elements of \( \mathbb{R}^k \) is defined by

\[
D_N(X) = D_N(x_1, x_2, \ldots, x_N) = \sup_{I \subset [0, 1)^s} \left| \frac{1}{N} \sum_{n=1}^{N} \chi_I(\{x_n\}) - \lambda(I) \right|. \tag{3}
\]

This definition makes sense because it does not depend on the ordering of the elements of the set \( X \).

**Lemma 1** (ETKS inequality) Let \( s \geq 1 \), \( X = \{x_1, x_2, \ldots, x_N\} \) be a finite set of elements of \( \mathbb{R}^s \) and \( K \) an arbitrary positive integer. We have

\[
D_N(X) \leq \left( \frac{3}{2} \right)^s \left( \frac{2}{K + 1} + \sum_{0 < \|k\|_{\infty} \leq K} \frac{1}{r(k)} \left| \frac{1}{N} \sum_{n=1}^{N} e(k \cdot x_n) \right| \right), \tag{4}
\]

where, for \( k = (k_1, k_2, \ldots, k_s) \in \mathbb{Z}^s \), we let \( r(k) = \prod_{i=1}^{s} \max\{1, |k_i|\} \) and \( u \cdot v \) denote the usual scalar product of two elements \( u \) and \( v \) in \( \mathbb{R}^s \).

2.3 Upper bound for some trigonometrical sums

The following lemma provides us with an upper bound for the trigonometric sums useful in the application of the ETKS inequality. The exponent in (5) is not very strong, but the uniformity in the statement makes it very convenient for our purpose. It is a rather straightforward consequence of the van der Corput inequalities.

**Lemma 2** Let \( c \) be a real number in \((1, \infty) \setminus \mathbb{N}\). There exist two constants \( A \) and \( B \) such that for any \( N \) larger than \( A \) and any non-zero \((\lfloor c \rfloor + 1)\)-tuple \((a_0, \ldots, a_{\lfloor c \rfloor})\) satisfying

\[
\forall \ell \in [0, \lfloor c \rfloor]: \text{ either } a_\ell = 0 \text{ or } N^{-\|c\|/2} \leq |a_\ell| \leq N^{\|c\|/2}, \tag{5}
\]

one has

\[
S := \left| \sum_{n=N+1}^{2N} e \left( \sum_{\ell=0}^{\lfloor c \rfloor} a_\ell n^{c-\ell} \right) \right| \leq BN^{1-\theta}.
\]
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where θ = 2^{-(c+2)}\|c\|.

**Proof** We start with the case when all the coefficients are 0 except \(a_\lfloor c \rfloor\). In this case, we use the Kusmin–Landau inequality (cf. Theorem 2.1 of [10]). If \(N\) is large enough, we have

\[
\| \frac{d}{dt}(a_\lfloor c \rfloor t^{\{c\}}) \| \gg N^{-\|c\|/2 + \{c\} - 1} \quad \text{when } t \text{ is in } (N, 2N)
\]

and thus

\[
S \ll N^{1 + \|c\|/2 - \{c\}} \ll N^{1 - \|c\|/2} \leq N^{1 - \theta}.
\]

For the other cases, we use the following lemma, due to van der Corput, which one can find as a combination of Theorem 2.2 (for \(q = 0\)) and Theorem 2.8 (for \(q \geq 1\)) of [10].

**Lemma 3** There exists an absolute constant \(C\) satisfying the following. Let \(q\) be a non-negative integer, \(f\) a real valued function defined on \([N, 2N]\), with \(q + 2\) continuous derivatives satisfying

\[
\lambda \leq |f^{(q+2)}(t)| \leq \alpha \lambda,
\]

for some \(\lambda > 0\) and some \(\alpha \geq 1\). Then, one has

\[
\sum_{n=N+1}^{2N} e(f(n)) \leq C \left( N(\alpha^2 \lambda)^{\frac{1}{2q-2}} + N^{1 - \frac{1}{2q}} \alpha^{\frac{1}{2q}} + N^{\left(\frac{Q-1}{Q}\right)^2 \lambda^{\frac{1}{2q}}} \right),
\]

(6)

where \(Q = 2^q\).

Let \(\ell_0\) be the smallest integer \(\ell\) such that \(a_\ell\) is non-zero. The function of interest is thus \(f(n) = \sum_{\ell=\ell_0}^{\lfloor c \rfloor} a_\ell n^{c-\ell}\). Since the non-zero coefficients \(a_\ell\) including \(a_{\ell_0}\) are small powers of \(N\), the term \(a_{\ell_0} n^{c-\ell_0}\) dominates \(f\) and its derivatives dominate the derivatives of \(f\). In the application of Lemma 3, we take \(q = \lfloor c \rfloor - \ell_0 - 1\); we first notice that \(q\) is non-negative (we already treated the case \(\ell_0 = \lfloor c \rfloor\)); we also notice that \(q \leq \lfloor c \rfloor - 1 \leq c - 1\) so that \(1 \leq Q \leq 2^{c-1}\). Furthermore, \(\alpha\) depends only of \(c\), and (since our constants may depend on \(c\), it plays no role in our application of (6)); moreover, \(\lambda\) is of the order \(|a_{\ell_0}| N^{\lfloor c \rfloor - 1}\), so that

\[
\lambda \ll_c N^{\|c\|/2 + \{c\} - 1} \ll_c N^{-\|c\|/2} \quad \text{and} \quad \lambda^{-1} \ll_c N^{\|c\|/2 + 1 - \{c\}} \ll_c N^3\|c\|/2.
\]

By (6), we have

\[
S \ll_c N^{1 - \frac{\|c\|}{8^{Q-1}}} + N^{-\frac{2}{Q}} + N^{\left(\frac{Q-1}{Q}\right)^2 + \frac{3\|c\|}{4Q}}.
\]

We easily notice that each of the three terms in the RHS of the last relation is less than \(N^{1 - \theta}\). \(\square\)
2.4 Contribution of large primes to $\varphi(n)/n$

Lemma 4. Let $\alpha > 0$. For any $C$ in $(0, \min(\alpha, 1))$, there exists $N$ such that for $n \geq N$, one has

$$\prod_{p|n \atop p \geq (\log n)^\alpha} \left(1 - \frac{1}{p}\right) \geq C. \quad (7)$$

Proof. This is a simple consequence of classical results on the distribution of prime numbers. Explicit value for $N$ in terms of $\alpha$ and $C$, which is not relevant here, can be obtained from the evaluations given in [8].

Since the LHS of (7) is an increasing function of $\alpha$, it is enough to consider the case when $\alpha \in (0, 1)$. Let $n$ be a sufficiently large integer. We denote by $p_0, p_1, \ldots, p_r, p_{r+1}$ the consecutive prime numbers such that

$$p_0 < \log^\alpha n \leq p_1 < \cdots < p_{r+1} \quad \text{and} \quad p_1 \cdots p_r \leq n < p_1 \cdots p_{r+1}. \quad (8)$$

By a direct lower bound and by Mertens’ Theorem, for $n$ large enough, we have

$$\prod_{p|n \atop p \geq (\log n)^\alpha} \left(1 - \frac{1}{p}\right) \geq \prod_{j=1}^r \left(1 - \frac{1}{p_j}\right) = (1 + o(1)) \frac{\log p_0}{\log p_r}.$$  

From the first part of (8), we have $p_0 = (1 + o(1)) \log^\alpha n$. Taking logarithms in the second part of (8), we have $\log n = (1 + o(1))\log(p_r) - \theta(p_0))$. The last relations we obtained imply $p_r = (1 + o(1)) \log n$ and so we have

$$\frac{\log p_0}{\log p_r} = (1 + o(1))\alpha \frac{\log \log n}{\log \log n} = \alpha + o(1).$$

Thus, for any $0 < C < \min(\alpha, 1)$, Relation (7) holds true when $n$ is large enough. \hfill \Box

2.5 Elementary relations concerning integral and fractional parts

For convenience, we recall here some elementary relations.

Lemma 5. Let $\ell$ denote a non-negative integer and $x, x_1, \ldots, x_\ell$ real numbers. We have

$$[\ell x] \geq \ell [x] \quad \text{and} \quad \{\ell x\} \leq \ell \{x\}, \quad (9)$$

if $\{x_1\} + \cdots + \{x_\ell\} < 1$, then $[x_1 + \cdots + x_\ell] = [x_1] + \cdots + [x_\ell], \quad (10)$

if $\ell \{x\} < 1$, then $[\ell x] = \ell [x]. \quad (11)$
Proof We multiply $x = \lfloor x \rfloor + \{ x \}$ by $\ell$, which leads to

\[
\ell x = \ell \lfloor x \rfloor + \ell \{ x \} = \ell \lfloor x \rfloor + \lfloor \ell \{ x \} \rfloor + \{ \ell \{ x \} \},
\]

which leads in turn to $\lfloor \ell x \rfloor = \ell \lfloor x \rfloor + \lfloor \ell \{ x \} \rfloor$ and $\{ \ell x \} = \ell \{ x \} = \ell \{ x \} - \lfloor \ell \{ x \} \rfloor$, whence (9). We have

\[
x_1 + \cdots + x_\ell = (\lfloor x_1 \rfloor + \cdots + \lfloor x_\ell \rfloor) + (\{ x_1 \} + \cdots + \{ x_\ell \}).
\]

The first term in the RHS is an integer and the second term is in $[0, 1)$, which implies (10). Relation (11) is just the special case of (10) in which all the summands are equal. □

Lemma 6 Let $R$ and $r$ be integers with $0 \leq r < R$, $u$ be a real number in $[0, 1]$, and $x$ be a real number such that

\[
\{ x \} \in \left[ r \frac{R}{R}, r + u \frac{R}{R} \right).
\]

We have

\[
\lfloor x \rfloor \equiv r \pmod{R} \text{ and } \{ x \} < u.
\]

Proof The hypothesis implies that there exists a rational integer $t$ such that

\[
\frac{x}{R} - t \in \left[ r \frac{R}{R}, r + u \frac{R}{R} \right).
\]

This implies

\[
t R + r \leq x < t R + r + u, \text{ whence } \lfloor x \rfloor = t R + r \text{ and } 0 \leq \{ x \} < u.
\]

□

3 Local approximation by a linear polynomial

In this section, we keep the notation of Theorem 3, assuming, without loss of generality that $0 \leq r < R$.

3.1 A lemma on fractional parts

Lemma 7 Let $m$ be an integer satisfying

(i) $m^{1-\{c\}} > 4(c H)^{c+1},$

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(ii) \( \left\{ \frac{mc}{R} \right\} \in \left[ \frac{c}{R}, \frac{c}{R} + \frac{1}{4R} \right), \)

(iii) \( \left\{ \frac{cmc^{-1}}{R} \right\} \in \left[ \frac{c}{R}, \frac{c}{R} + \frac{1}{4RH} \right), \)

(iv) \( \forall \ell \in [2, \lfloor c \rfloor]: \left\{ \frac{\ell mc^{-\ell}}{R} \right\} \in \left[ 0, \frac{1}{4cRH} \right). \)

Then, for any integer \( h \) in \([1, H]\), we have

\[ \lfloor (m + h)c \rfloor \equiv r + h \pmod{R}. \]

**Proof** Let \( h \) be an integer in \([1, H]\).

By Relation (i), we have

\[ 0 \leq cch \lfloor c \rfloor + 1 \frac{mc}{R} - 1 < 1/4, \]

which implies \( \{ cch \lfloor c \rfloor + 1 \frac{mc}{R} - 1 \} < 1/4. \)

By Lemma 6, (ii) implies \( \lfloor mc \rfloor \equiv r \pmod{R} \) and \( \{ mc \} < 1/4. \)

By Lemma 6, (iii) implies \( \lfloor cmc^{-1} \rfloor \equiv 1 \pmod{R} \) and \( \{ cmc^{-1} \} < 1/4H; \) thanks to (9) and (11), we obtain \( \lfloor hcmc^{-1} \rfloor \equiv h \lfloor cmc^{-1} \rfloor \equiv h \pmod{R} \) and \( h \lfloor cmc^{-1} \rfloor < 1/4. \)

Using Taylor’s expansion (2) and (10), we end the proof of Lemma 7. \( \Box \)

### 3.2 The use of Erdos–Turán–Koksma–Szüsz inequality

In this subsection, we let \( R \) be a (large) integer,

\[ \beta = 2^{c+2} (c + 2)^2 \| c \|^{-1}, \quad N = \lfloor R^\beta / 2 \rfloor, \]

\[ s = \lfloor c \rfloor + 1, \quad x_n = \left( \frac{(N + n)^c}{R}, \frac{c(N + n)^{c-1}}{R}, \ldots, \frac{\binom{c}{\lfloor c \rfloor} (N + n)^{\lfloor c \rfloor}}{R} \right), \]

and \( X = \{ x_1, x_2, \ldots, x_N \} \).

Our aim is to show the following

**Lemma 8** As \( R \) tends to infinity, we have

\[ D_N(x_1, x_2, \ldots, x_N) = o_c \left( R^{-\lfloor c \rfloor - 1} \right). \] \[ (12) \]

**Proof** We use the ETKS inequality, with

\[ K = \lceil N^{\theta/(c+2)} \rceil, \] \( \theta = 2^{-(c+2)} \| c \|. \)

Up to a slight change of notation, the indices of the components of \( k \) running from 0 to \( \lfloor c \rfloor \) and the indices of the trigonometric sums running from \( N + 1 \) to \( 2N \), the trigonometric sums we have to consider are

\[ S(N; k, R) = \sum_{n=N+1}^{2N} e \left( R^{-1} \sum_{\ell=0}^{\lfloor c \rfloor} k_\ell \binom{c}{\ell} n^{c-\ell} \right). \]
which are of the type studied in Lemma 2, with \( a_\ell = R^{-1}k_\ell (c) \). We have \( 1 \leq (\frac{c}{k}) \ll c \) and thus

\[
\text{either } a_\ell = 0 \text{ or } R^{-1} \leq |a_\ell| \ll c R^{-1} K.
\]

When \( a_\ell \neq 0 \), we have

\[
|a_\ell| \ll c R^{-1} K \ll c N^{\theta/(c+2)} \quad \text{and} \quad |a_\ell^{-1}| \leq R \ll c N^{\theta/(c+2)}.
\]

If \( R \) is sufficiently large, we can use Lemma 2 and obtain

\[
\forall k \text{ with } 0 < \|k\|_\infty \leq K : S(N; k, R) \leq N^{1-\theta}.
\]

We can now apply Lemma 1, with the most trivial bound \( r(k) \geq 1 \). We obtain, when \( R \), or equivalently \( N \), is large enough

\[
D_N(X) \ll c K^{-1} + K^{\lfloor c \rfloor + 1} N^{-\theta} \ll c N^{\theta/(c+2)} \ll c R^{-c-2} = o_c \left( R^{-\lfloor c \rfloor - 1} \right).
\]

\[\Box\]

3.3 Completion of the proof of Theorem 3

We first prove that, when \( R \) is sufficiently large, there exists an integer \( m \in [N+1, 2N] \) which satisfies Relations (ii), (iii), and (iv) of Lemma 7. For doing so, we wish to find an element \( x_m \) in \( X \) which belongs to the interval

\[
I = \left[ \frac{r}{R}, \frac{r}{R} + \frac{1}{4R} \right] \times \left[ \frac{1}{R}, \frac{1}{R} + \frac{1}{4RH} \right] \times \prod_{\ell=2}^{\lfloor c \rfloor} \left[ 0, \frac{1}{4c RH^{\ell}} \right],
\]

an interval of Lebesgue measure

\[
\lambda(I) = \frac{1}{4R} \times \frac{1}{4RH} \times \prod_{\ell=2}^{\lfloor c \rfloor} \frac{1}{4c RH^{\ell}} \gg c H^{-c-2} R^{-\lfloor c \rfloor - 1}.
\]

By the definition of the discrepancy recalled in (3), any interval in \([0, 1]^s\) with Lebesgue measure larger than \( D_N(X) \) contains an element from \( X \). By (12) and (13), if \( H \) is given and \( R \) is large enough, there exists \( m \) in \([N+1, 2N]\) which satisfies Relations (ii), (iii), and (iv) of Lemma 7. To complete the proof of Theorem 3, it is enough to notice that when \( R \) is large enough, the integer \( m \) we have produced satisfies also Relation (i) and to apply Lemma 7.

\[\Box\]
4 Proof of Theorem 2

We follow a strategy similar to that of [5]: for an arbitrarily large integer $H$, we construct an integer $m$ such that

$$\forall h \in [1, H] : \frac{1}{H} \leq \frac{\varphi([(m + h)^c])}{[(m + h)^c]} \leq \frac{3}{H}. \quad (14)$$

Theorem 2 easily follows from (14).

Without loss of generality, we may assume that $H \geq 21$ which implies that for any $h$ in $[1, H]$, one has $\frac{\varphi(h)}{h} > \frac{3}{H}$ (notice that $\varphi(h)$ is always at least 1 and is larger than 3 for $h \geq H/3 \geq 7$). Since $\varphi(p)/p$ tends to 1 when the prime $p$ tends to infinity and the infinite product of those terms diverges to 0 (by Euler), we can find finite families of prime numbers $(P_h)_h$ such that

(i) for $1 \leq h < k \leq H$ the sets $P_h$ and $P_k$ are disjoint,
(ii) for any $h$ all the elements of $P_h$ are larger than $H$,
(iii) for any $h : 2/H \leq (\varphi(h)/h)(\varphi(P_h)/P_h) \leq 3/H$,

where $P_h$ denotes the product of the primes in $P_h$.

We now let $L$ be a prime number which is larger than all the elements of all the $P_h$ and we let

$$R = H! \prod_{H < p \leq L} p.$$ 

By the Chinese Remainder Theorem, we can find a residue $r$ modulo $R$ satisfying

$$r \equiv -1 \pmod{P_1}$$

$$\vdots$$

$$r \equiv -H \pmod{P_H}$$

$$r \equiv 0 \pmod{R/(P_1 P_2 \cdots P_H)}.$$ 

By Theorem 3, if $R$ is large enough, i.e., if $L$ is large enough, there exists $m \leq R^\beta$ such that

$$\forall h \in [1, H] : [(m + h)^c] \equiv r + h \pmod{R},$$

and thus $\gcd([(m + h)^c], R) = h P_h$. In other words, $[(m + h)^c]$ is an integer which is less than $R^{\beta c + 1}$, divisible by $h P_h$ and such that any of its prime factors which does not divide $h P_h$ is larger than $L$. We thus have, using Relation (iii) above

$$\frac{2}{H} \prod_{\substack{p|[(m+h)^c] \atop p > L}} \frac{\varphi(p)}{p} \leq \frac{\varphi([(m + h)^c])}{[(m + h)^c]} \leq \frac{3}{H}. \quad \Box$$
In order to prove (14), and thus Theorem 2, it is enough to prove that for $L$ large enough, we have

$$
\prod_{\substack{p \mid [(m+h)^c] \text{ and } p > L}} \left( 1 - \frac{1}{p} \right) \geq \frac{1}{2}.
$$

(15)

By the definition of $R$ and the prime number theorem, we have, when $L$ is large enough

$$
\log [(m + h)^c] \leq (\beta c + 1) \left( \sum_{p \leq L} \log p + \log H! \right) \leq (\beta c + 2)L.
$$

When $L$ is large enough, and thus $m$ too, we have

$$
\prod_{\substack{p \mid [(m+h)^c] \text{ and } p > L}} \left( 1 - \frac{1}{p} \right) \geq \prod_{\substack{p \mid [(m+h)^c] \text{ and } p > (\beta c + 2)^{-1} \log [(m+h)^c]}} \left( 1 - \frac{1}{p} \right) \geq \prod_{\substack{p \mid [(m+h)^c] \text{ and } p > (\log [(m+h)^c])^{3/4}}} \left( 1 - \frac{1}{p} \right)
$$

by the previous inequality since $m$ is large enough

$$
\geq \frac{1}{2}
$$

by Lemma 4.

Thus, (15) is proved, as well as Theorem 2.

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