AN OPTIMAL MATCHING PROBLEM

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Abstract. Given two measured spaces \((X, \mu)\) and \((Y, \nu)\), and a third space \(Z\), given two functions \(u(x, z)\) and \(v(x, z)\), we study the problem of finding two maps \(s : X \to Z\) and \(t : Y \to Z\) such that the images \(s(\mu)\) and \(t(\nu)\) coincide, and the integral \(\int_X u(x, s(x))d\mu + \int_Y v(y, t(y))d\nu\) is maximal. We give condition on \(u\) and \(v\) for which there is a unique solution.

1. The main result.

Suppose we are given three goods, \(X, Y\), and \(Z\). They are not homogeneous, but come in different qualities, \(x \in X\), \(y \in Y\), and \(z \in Z\). Goods \(X\) and \(Y\) are used for the sole purpose of producing good \(Z\), which we are interested in. To obtain one piece of good \(Z\), one has to assemble one piece of good \(X\) and one piece of good \(Y\). More precisely, one can obtain a piece of quality \(z\) by assembling one piece of quality \(x\) and one piece of quality \(y\), yielding a benefit of \(u(x, z) + w(y, z)\). Given the distributions \(\mu\) and \(\nu\) of goods \(X\) and \(Y\), one wishes to minimize the total benefit of production.

This translates into the following optimization problem: find maps \(s : X \to Z\) and \(t : Y \to Z\) such that \(s(\mu) = t(\nu)\) (this is the matching condition) and the integral

\[
\int_X u(x, s(x))d\mu + \int_Y w(y, t(y))d\nu
\]

is maximized.

The origins of that problem lie in the economic theory of hedonic pricing (see [5] for an overview). The economic aspects will be developed in another paper [3].

Mathematically speaking, this is related to the classical optimal transportation problem (see the monographs [8] and [9] for accounts of the theory). Recall that this problem consists in minimizing the integral

\[
\int_X u(x, s(x))d\mu
\]

among all maps \(s : X \to Y\) such that \(s(\mu) = \nu\). Here the measured spaces \((X, \mu)\) and \((Y, \nu)\) are given, as well as the function \(u : X \times Y \to R\). A seminal result by Brenier [1] states that, if \(X\) and \(Y\) are bounded open subsets of \(R^n\) endowed with the Lebesgue measure, with \(X\) connected, and \(u(x, y) = ||x - y||\), then there is a

\[\text{Date: August 15, 2003.}\]

1991 Mathematics Subject Classification. Primary 05C38, 15A15; Secondary 05A15, 15A18.

Key words and phrases. optimal transportation, measure-preserving maps.

This research has been supported by NSF grant ***. The author thanks Jim Heckman for introducing him to the economics of hedonic pricing.
unique solution $s$ to the optimal transportation problem, and $s$ is almost everywhere equal to the gradient of a convex function.

Kantorovich introduced into the optimal transportation problem a duality method which will be crucial to our proof. Instead of proving directly existence and uniqueness in the optimal matching problem, we solve in section 3 another optimization problem, and we will show in section 4 that it yields the solution to the original one. This correspondence relies heavily on an extension of the classical duality results in convex analysis (see [4]). This extension has been can be found in [8] and [?]; for the reader’s convenience, we will give the main results in section 2. Finally, in section 5, we will give some consequences of the main result.

From now on, $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$, and $Z \subset \mathbb{R}^n$ will be compact subsets. We are given measures $\mu$ on $X$ and $\nu$ on $Y$, which are absolutely continuous with respect to the Lebesgue measure, and satisfy:

$$\mu (X) = \nu (Y) < \infty$$

We are also given functions $u : \Omega_1 \to \mathbb{R}$ and $v : \Omega_2 \to \mathbb{R}$, where $\Omega_1$ is a neighbourhood of $X \times Z$ and $\Omega_2$ is a neighbourhood of $Y \times Z$. It is assumed that $u$ and $v$ are continuous with respect to both variables, differentiable with respect to $x$ and $y$, and that the partial derivatives $D_x u$ and $D_y v$ are continuous with respect to both variables, and injective with respect to $z$:

\begin{align*}
(1.3) & \quad \forall x \in X, \ D_x u (x, z_1) = D_x u (x, z_2) \implies z_1 = z_2 \\
(1.4) & \quad \forall y \in Y, \ D_y v (y, z_1) = D_y v (y, z_2) \implies z_1 = z_2
\end{align*}

The latter condition is a generalization of the classical Spence-Mirrlees condition in the economics of asymmetric information (see [2]). It is satisfied for $u (x, z) = \| x - z \| \alpha$, provided $\alpha \neq 0$ and $\alpha \neq 1$.

**Theorem 1.** Under the above assumptions, there exists a pair of Borelian maps $(\bar{s}, \bar{t})$ with $\bar{s} (\mu) = \bar{t} (\nu)$, such that for every $(s, t)$ satisfying $s (\mu) = t (\nu)$, we have:

$$\int_X u (x, s (x)) \, d\mu - \int_Y v(y, t(y)) \, dv \leq \int_X u (x, \bar{s} (x)) \, d\mu - \int_Y v(y, \bar{t}(y)) \, dv < \infty$$

This solution is unique, up to equality almost everywhere, and it is described as follows: there is some Lipschitz continuous function $\bar{p} : Z \to \mathbb{R}$ and some negligible subsets $X_0 \subset X$ and $Y_0 \subset Y$ such that, for every $x \notin X_0$ and every $y \notin Y_0$:

\begin{align*}
(1.5) & \quad \forall z \neq \bar{s} (x), \ u (x, z) - \bar{p} (z) < u (x, \bar{s} (x)) - \bar{p} (\bar{s} (x)) \\
(1.6) & \quad \forall z \neq \bar{t} (y), \ v (y, z) - \bar{p} (z) > v (y, \bar{t} (y)) - \bar{p} (\bar{t} (y)) \\
(1.7) & \quad \bar{p} \text{ is differentiable at } \bar{s} (x) \text{ and } \bar{t} (y)
\end{align*}

If in addition $u$ and $v$ are differentiable with respect to $z$, we get, from the minimization \[1.5\] and the maximization \[1.6\]:

$$D_z u (x, \bar{s} (x)) = D_z \bar{p} (\bar{s} (x))$$

$$D_z v (y, \bar{t} (y)) = D_z \bar{p} (\bar{t} (y))$$

Set $\bar{s} (\mu) = \bar{t} (\nu) = \lambda$. It follows from the above that, for $\lambda$-almost every $z \in Z$, there is some $x \notin X_0$ and $y \notin Y_0$ such that $z = \bar{s} (x) = \bar{t} (y)$, and for every such $(x, y)$ we have:

$$D_z u (x, z) = D_z \bar{p} (z) = D_z v (y, z)$$
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Note that there is no reason why $\lambda$ should be absolutely continuous with respect to the Lebesgue measure.

The proof of theorem 1 is deferred to section 4. Meanwhile, let us notice that we have slightly changed the formulation of the optimal matching problem: by setting $w = -v$ we recover the original one. This change will simplify future notations.

2. FUNDAMENTALS OF $u$-CONVEX ANALYSIS.

In this section, we basically follow Carlier [2].

2.1. $u$-convex functions. We will be dealing with functions taking values in $\mathbb{R} \cup \{+\infty\}$.

A function $f : X \to \mathbb{R} \cup \{+\infty\}$ will be called $u$-convex iff there exists a non-empty subset $A \subset \mathbb{Z} \times \mathbb{R}$ such that:

$$(2.1) \quad \forall x \in X, \quad f(x) = \sup_{(z, a) \in A} \{u(x, z) + a\}$$

A function $p : Z \to \mathbb{R} \cup \{+\infty\}$ will be called $u$-convex iff there exists a non-empty subset $B \subset X \times \mathbb{R}$ such that:

$$p(z) = \sup_{(x, b) \in B} \{u(x, z) + b\}$$

2.2. Subconjugates. Let $f : X \to \mathbb{R} \cup \{+\infty\}$, not identically $\{+\infty\}$, be given. We define its subconjugate $f^\flat : Z \to \mathbb{R} \cup \{+\infty\}$ by:

$$(2.3) \quad f^\flat(z) = \sup_x \{u(x, z) - f(x)\}$$

It follows from the definitions that $f^\flat$ is a $u$-convex function on $Z$ (it might be identically $\{+\infty\}$).

Let $p : Z \to \mathbb{R} \cup \{+\infty\}$, not identically $\{+\infty\}$, be given. We define its subconjugate $p^\flat : X \to \mathbb{R} \cup \{+\infty\}$ by:

$$(2.4) \quad p^\flat(x) = \sup_z \{u(x, z) - p(z)\}$$

It follows from the definitions that $p^\flat$ is a $u$-convex function on $X$ (it might be identically $\{+\infty\}$).

Example 1. Set $f(x) = u(x, \bar{z}) + a$. Then

$$f^\flat(\bar{z}) = \sup_x \{u(x, \bar{z}) - u(x, \bar{z}) - a\} = -a$$

Conjugation reverses ordering: if $f_1 \leq f_2$, then $f_1^\flat \geq f_2^\flat$, and if $p_1 \leq p_2$, then $p_1^\flat \geq p_2^\flat$. As a consequence, if $f$ is $u$-convex, not identically $\{+\infty\}$, then $f^\flat$ is $u$-convex, not identically $\{+\infty\}$. Indeed, since $f$ is $u$-convex, we have $f(x) \geq u(x, z) + a$ for some $(z, a)$, and then $f^\flat(z) \leq -a < \infty$.

Proposition 1 (the Fenchel inequality). For any functions $f : X \to \mathbb{R} \cup \{+\infty\}$ and $p : X \to \mathbb{R} \cup \{+\infty\}$, not identically $\{+\infty\}$, we have:

$$\forall (x, z), \quad f(x) + f^\flat(z) \geq u(x, z)$$

$$\forall (x, z), \quad p(z) + p^\flat(x) \geq u(x, z)$$
2.3. **Subgradients.** Let \( f : X \to \mathbb{R} \cup \{+\infty\} \) be given, not identically \( \{+\infty\} \). Take some point \( x \in X \). We shall say that a point \( z \in Z \) is a **subgradient** of \( f \) at \( x \) if the points \( x \) and \( z \) achieve equality in the Fenchel inequality:

\[
(2.5) \quad f(x) + f^*(z) = u(x, z)
\]

The set of subgradients of \( f \) at \( x \) will be called the **subdifferential** of \( f \) at \( x \) and denoted by \( \partial f(x) \).

Similarly, let \( p : Z \to \mathbb{R} \cup \{+\infty\} \) be given, not identically \( \{+\infty\} \). Take some point \( z \in Z \). We shall say that a point \( x \in X \) is a **subgradient** of \( p \) at \( z \) if:

\[
(2.6) \quad p^*(x) + p(z) = u(x, z)
\]

The set of subgradients of \( p \) at \( z \) will be called the **subdifferential** of \( p \) at \( z \) and denoted by \( \partial p(z) \).

**Proposition 2.** The following are equivalent:

1. \( z \in \partial f(x) \)
2. \( \forall x', f(x') \geq f(x) + u(x', z) - u(x, z) \)

   If equality holds for some \( x' \), then \( z \in \partial f(x') \) as well.

**Proof.** We begin with proving that the first condition implies the second one. Assume \( z \in \partial f(x) \). Then, by (2.5) and the Fenchel inequality, we have:

\[
f(x') \geq u(x', z) - f^*(z) = u(x', z) - [u(x, z) - f(x)]
\]

We then prove that the second condition implies the first one. Using the inequality, we have:

\[
f^*(z) = \sup_{x'} \{ u(x', z) - f(x') \} \\
\leq \sup_{x'} \{ u(x', z) - f(x) - u(x', z) + u(x, z) \} \\
= u(x, z) - f(x)
\]

so \( f(x) + f^*(z) \leq u(x, z) \). We have the converse by the Fenchel inequality, so equality holds.

Finally, if equality holds for some \( x' \) in condition (2), then \( f(x') - u(x', z) = f(x) - u(x, z) \), so that:

\[
\forall x'', f(x'') \geq f(x) - u(x, z) + u(x'', z) = f(x') - u(x', z) + u(x'', z)
\]

which implies that \( z \in \partial f(x') \).

There is a similar result for functions \( p : Z \to \mathbb{R} \cup \{+\infty\} \), not identically \( \{+\infty\} \): we have \( x \in \partial p(z) \) if and only if

\[
(2.7) \quad \forall (x', z'), \ p(z') \geq p(z) + u(x, z') - u(x, z)
\]

2.4. **Biconjugates.** It follows from the Fenchel inequality that, if \( p : Z \to \mathbb{R} \cup \{+\infty\} \) is not identically \( \{+\infty\} \):

\[
(2.8) \quad p^{**}(z) = \sup_x \{ u(x, z) - p^*(x) \} \leq p(z)
\]
Example 2. Set \( p(z) = u(\bar{x}, z) + b. \) Then
\[
p^{\#}(z) = \sup_x \left\{ u(x, z) - p^{\#}(x) \right\}
\geq u(\bar{x}, z) - p^{\#}(\bar{x})
= u(\bar{x}, z) + b = p(z)
\]

This example generalizes to all \( u \)-convex functions. Denote by \( C_u(Z) \) the set of all \( u \)-convex functions on \( Z \).

Proposition 3. For every function \( p : Z \to \mathbb{R} \cup \{+\infty\} \), not identically \( \{+\infty\} \), we have
\[
p^{\#}(z) = \sup_{\varphi} \{ \varphi(z) \mid \varphi \leq p, \varphi \in C_u(Z) \}
\]

Proof. Denote by \( \bar{p}(z) \) the right-hand side of the above formula. We want to show that \( p^{\#}(z) = \bar{p}(z) \)

Since \( p^{\#} \leq p \) and \( p^{\#} \) is \( u \)-convex, we must have \( p^{\#} \leq \bar{p} \).

On the other hand, \( \bar{p} \) is \( u \)-convex because it is a supremum of \( u \)-convex functions. So there must be some \( B \subset X \times \mathbb{R} \) such that:
\[
p(z) = \sup_{(x,b) \in B} \{ u(x, z) + b \}
\]

Let \( (x,b) \in B \). Since \( \bar{p} \leq p \), we have \( u(x, z) + b \leq \bar{p}(z) \leq p(z) \). Taking biconjugates, as in the preceding example, we get \( u(x, z) + b \leq p^{\#}(z) \). Taking the suprema over \( (x,b) \in B \), we get the desired result. \( \Box \)

Corollary 1. Let \( p : Z \to \mathbb{R} \cup \{+\infty\} \) be a \( u \)-convex function, not identically \( \{+\infty\} \). Then \( p = p^{\#} \), and the following are equivalent:

1. \( x \in \partial p(z) \)
2. \( p(z) + p^\dagger(x) = u(x, z) \)
3. \( z \in \partial p^{\#}(x) \)

Proof. We have \( p^{\#} \leq p \) always by relation (2.8). Since \( p \) is \( u \)-convex, we have:
\[
p(z) = \sup_{(x,b) \in B} \{ u(x, z) + b \}
\]
for some \( B \subset X \times \mathbb{R} \). By proposition 3, we have:
\[
\sup_{(x,b) \in B} \{ u(x, z) + b \} \leq p^{\#}(z)
\]
and so we must have \( p = p^{\#} \). Taking this relation into account, as well as the definition of the subgradient, we see that condition (2) is equivalent both to (1) and to (2). \( \Box \)

Definition 1. We shall say that a function \( p : Z \to \mathbb{R} \cup \{+\infty\} \) is \( u \)-adapted if it is not identically \( \{+\infty\} \) and there is some \( (x,b) \in X \times \mathbb{R} \) such that:
\[
\forall z \in Z, \ p(z) \geq u(x, z) + b
\]

It follows from the above that if \( p \) is \( u \)-adapted, then so are \( p^\dagger, p^{\#} \) and all further subconjugates. Note that a \( u \)-convex function which is not identically \( \{+\infty\} \) is \( u \)-adapted.

Corollary 2. Let \( p : Z \to \mathbb{R} \cup \{+\infty\} \) be \( u \)-adapted. Then :
\[
p^{\#\#} = p^\dagger
\]
Proof. If \( p \) is \( u \)-adapted, then \( p^\sharp \) is \( u \)-convex and not identically \( \{+\infty\} \). The result then follows from corollary \( \square \)

2.5. Smoothness. Since \( u \) is continuous and \( X \times Z \) is compact, the family \( \{ u (x, \cdot) \mid x \in X \} \) is uniformly equicontinuous on \( Z \). It follows from the definition \( \square \) that all \( u \)-convex functions on \( Z \) are continuous (in particular, they are finite everywhere).

Denote by \( k \) the upper bound of \( \|D_x u (x, z)\| \) for \((x, z) \in X \times Z \). Since \( D_x u \) is continuous and \( X \times Z \) is compact, we have \( k < \infty \), and the functions \( x \to u (x, z) \) are all \( k \)-Lipschitzian on \( X \). Again, it follows from the definition \( \square \) that all \( u \)-convex functions on \( X \) are \( k \)-Lipschitz (in particular, they are finite everywhere). By a theorem of Rademacher, they are differentiable almost everywhere with respect to the Lebesgue measure.

Let \( f : X \to R \) be convex. Since \( f = f^\sharp \), we have:

\[
f (x) = \sup_z \{ u (x, z) - f^\sharp (z) \}
\]

Since \( f^\sharp \) is \( u \)-convex, it is continuous, and the supremum is achieved on the right-hand side, at some point \( z \in \partial f (x) \). This means that all \( u \)-convex functions on \( X \) are subdifferentiable everywhere on \( X \).

Let \( x \) be a point where \( f \) is differentiable, with derivative \( D_x f (x) \), and let \( z \in \partial f (x) \). Consider the function \( \varphi (x') = u (x', z) - f^\sharp (z) \). By proposition \( \square \) we have \( \varphi \leq f \) and \( \varphi (x) = f (x) \), so that \( \varphi \) and \( f \) must have the same derivative at \( x \):

\[
D_x f (x) = D_x u (x, z)
\]

(2.9)

By assumption (see \( \square \)), this equation defines \( z \) uniquely. We shall denote it by \( z = \nabla_u f (x) \). In other words, at every point \( x \) where \( f \) is differentiable, the subdifferential \( \partial f (x) \) reduces to a singleton, namely \( \{ \nabla_u f (x) \} \). Combining all this information, we get:

**Proposition 4.** For every \( u \)-convex function \( f : X \to R \), there is a map \( \nabla_u f : X \to Z \) such that, for almost every \( x \):

\[
D_x f (x) = D_x u (x, z) \iff z = \nabla_u f (x)
\]

The following result will also be useful:

**Proposition 5.** Let \( p : Z \to R \) be \( u \)-adapted, and let \( x \in X \) be given. Then there is some point \( z \in \partial p^\sharp (x) \) such that \( p (z) = p^\sharp^\pi (z) \).

**Proof.** Assume otherwise, so that for every \( z \in \partial p^\sharp (x) \) we have \( p^\sharp^\pi (z) < p (z) \). For every \( z \in \partial p^\sharp (x) \), we have \( x \in \partial p^\sharp^\pi (z) \), so that, by proposition \( \square \) we have

\[
p^\sharp^\pi (z') \geq u (x, z') - u (x, z) + p^\sharp^\pi (z)
\]

for all \( z' \in Z \), the inequality being strict if \( z' \notin \partial p^\sharp (x) \). Set \( \varphi_z (z') = u (x, z') - u (x, z) + p^\sharp^\pi (z) \). We have:

\[
z' \notin \partial p^\sharp (x) \implies \varphi_z (z') < p^\sharp^\pi (z') \leq p (z')
\]

\[
z' \in \partial p^\sharp (x) \implies \varphi_z (z') \leq p^\sharp^\pi (z') < p (z')
\]

so that \( \varphi_z (z') \leq p (z') \) for all \((z, z')\). Since \( Z \) is compact, there is some \( \varepsilon > 0 \) such that \( \varphi_z (z') + \varepsilon \leq p (z') \) for all \((z, z')\). Taking the subconjugate with respect to \( z' \),
we get:
\[
\begin{align*}
p^\sharp(x) & \leq \sup_{z'} \{ u(x, z') - \varphi_z(z') \} - \varepsilon \\
& = \sup_{z'} \{ u(x, z') - u(x, z) + u(x, z) - p^{\sharp\sharp}(z) \} - \varepsilon \\
& = u(x, z) - p^{\sharp\sharp}(z) - \varepsilon = p^\sharp(x) - \varepsilon
\end{align*}
\]
which is a contradiction. The result follows. □

**Corollary 3.** If \( x \) is a point where \( p^\sharp \) is differentiable, then:

\[
p \left( \nabla u p^\sharp(x) \right) = p^{\sharp\sharp} \left( \nabla u p^\sharp(x) \right)
\]
and:

\[
p^\sharp(x) = u(x, \nabla u p^\sharp(x)) - p \left( \nabla u p^\sharp(x) \right)
\]

**Proof.** Just apply the preceding proposition, bearing in mind that \( \partial p^\sharp(x) \) contains only one point, namely \( \nabla u p^\sharp(x) \). This yields equation (2.10). Equation (2.11) follows from the definition of the subgradient:

\[
p^\sharp(x) = u(x, \nabla u p^\sharp(x)) - p^{\sharp\sharp} \left( \nabla u p^\sharp(x) \right)
\]
and equation (2.10). □

2.6. \( v \)-concave functions. Let us now consider the duality between \( Y \) and \( \hat{Z} \). Given \( v : Y \times Z \to \mathbb{R} \), we say that a map \( g : Y \to \mathbb{R} \cup \{-\infty\} \) is \( v \)-concave iff there exists a non-empty subset \( A \subset Z \times \mathbb{R} \) such that:

\[
\forall y \in Y, \quad g(y) = \inf_{(z,a) \in A} \{ v(y, z) + a \}
\]
and a function \( p : Z \to \mathbb{R} \cup \{-\infty\} \) will be called \( v \)-concave iff there exists a non-empty subset \( B \subset X \times \mathbb{R} \) such that:

\[
p(z) = \inf_{(x,b) \in B} \{ v(y, z) + b \}
\]
All the results on \( u \)-convex functions carry over to \( v \)-concave functions, with obvious modifications. The **superconjugate** of a function \( g : Y \to \mathbb{R} \cup \{-\infty\} \), not identically \( \{-\infty\} \), is defined by:

\[
g^\flat(z) = \inf_y \{ v(y, z) - g(y) \}
\]
and the **superconjugate** of a function \( p : Z \to \mathbb{R} \cup \{-\infty\} \), not identically \( \{-\infty\} \), is given by:

\[
p^\flat(y) = \inf_z \{ v(y, z) - p(z) \}
\]
The superdifferential \( \partial p^\flat \) is defined by:

\[
\partial p^\flat(y) = \arg\min_z \{ v(y, z) - p(z) \}
\]
and we have the Fenchel inequality:

\[
p(z) + p^\flat(y) \leq v(y, z) \quad \forall (y, z) \quad \text{with equality iff } z \in \partial p^\flat(y).
\]
Note finally that \( p^{\flat\flat} \geq p \), with equality if \( p \) is \( v \)-concave.
3. The dual optimization problem.

Denote by $\mathcal{A}$ the set of all bounded functions on $\mathbb{Z}$:

$$ p \in \mathcal{A} \iff \sup_z |p(z)| < \infty $$

and consider the minimization problem:

$$ \inf_{p \in \mathcal{A}} \left[ \int_X p^\sharp(x) \, d\mu - \int_Y p^\flat(y) \, d\nu \right] $$

**Proposition 6.** The minimum is attained in problem (P)

Take a minimizing sequence $p_n$:

$$ \int_X p_n^\flat(x) \, d\mu - \int_Y p_n^\flat(y) \, d\nu \to \inf (P) $$

Setting $q_n = p_n + a$, for some constant $a$. Then $q_n^\flat = p_n^\flat - a$ and $q_n^\sharp = p_n^\sharp - a$. Since $\mu(X) = \nu(Y)$, we have:

$$ \int_X q_n^\sharp(x) \, d\mu - \int_Y q_n^\flat(y) \, d\nu = \int_X p_n^\flat(x) \, d\mu - \int_Y p_n^\flat(y) \, d\nu \to \inf (P) $$

Setting $a = -\inf_z p_n(z)$, we find $\inf_z q_n(z) = 0$. So there is no loss of generality in assuming that:

$$ \forall z, \inf_z p_n(z) = 0 $$

which we shall do from now on.

The sequences $p_n^\sharp$ is $k$-Lipschitzian. Since $p_n \geq 0$, we have

$$ p_n^\sharp(x) \leq \sup_z u(x, z) \leq \max u(x, z) $$

Choose $z_n$ such that $p_n(z_n) \leq 1$. We then have:

$$ p_n^\sharp(x) \geq u(x, z_n) - p_n(z_n) \geq \min u(x, z) - 1 $$

So the sequence $p_n^\sharp$ is uniformly bounded. By Ascoli’s theorem, there is a uniformly convergent subsequence. Similarly, after extracting this first subsequence, we extract another one along which $p_n^\flat$ converges uniformly. The resulting subsequence will still be denoted by $p_n$, so that:

$$ p_n^\sharp \to f \text{ uniformly} $$

$$ p_n^\flat \to g \text{ uniformly} $$

Taking limits, we get:

$$ \int_X f(x) \, d\mu - \int_Y g(y) \, d\nu = \inf (P) $$

It is easy to see that $p_n^{\sharp\sharp} \to f^\sharp$ and $p_n^{\flat\flat} \to g^\flat$ uniformly. Since $p_n^{\sharp\sharp} \leq p_n$ and $p_n^{\flat\flat} \geq p_n$, we have $p_n^{\sharp\sharp} \leq p_n^{\flat\flat}$, and hence $f^\sharp \leq g^\flat$. Set:

$$ \bar{p} = \frac{1}{2} \left( f^\sharp + g^\flat \right). $$
Since \( f^1 \) and \( g^1 \) are Lipschitz continuous, so is \( \bar{p} \). Since \( f^2 \leq \bar{p} \leq g^2 \), we must have \( \bar{p}^\sharp \leq f^2 \leq f \) and \( \bar{p}^\flat \geq g^2 \geq g \). Hence:

\[
\int_X \bar{p}^\sharp (x) \, d\mu - \int_Y \bar{p}^\flat (y) \, d\nu \leq \int_X f (x) \, d\mu - \int_Y g (y) \, d\nu = \inf (P)
\]

Since \( f^2 \leq \bar{p} \leq g^2 \), both sides being continuous functions, the function \( \bar{p} \) must be bounded on \( Z \), and the above inequality shows that it is a minimizer. The proof is concluded.

4. Proof of the Main Result.

Let us now express the optimality condition in problem (P). Set:

\[
\bar{s} (x) = \nabla_u p^\sharp (x) \\
\bar{t} (y) = \nabla_v p^\flat (y)
\]

where the gradient maps \( \nabla_u p^\sharp \) and \( \nabla_v p^\flat \) have been defined in proposition \( \text{[4]} \).

**Proposition 7.** \( \bar{s} (\mu) = \bar{t} (\nu) \)

**Proof.** We follow the argument in Carlier \( \text{[??]} \). Take any continuous function \( \varphi : Z \to R \). Since \( \bar{p} \) is a minimizer, we have, for any integer \( n \):

\[
(4.1) \quad \int_X n \left[ \left( \bar{p} + \frac{1}{n} \varphi \right)^\sharp - \bar{p}^\sharp \right] \, d\mu - \int_Y n \left[ \left( \bar{p} + \frac{1}{n} \varphi \right)^\flat - \bar{p}^\flat \right] \, d\nu \geq 0
\]

We deal with the first integral. Set \( \bar{p} + \frac{1}{n} \varphi = p_n \). Since \( p_n^\sharp \) is \( u \)-convex, it differentiable almost everywhere. Take a negligible subset \( X_0 \) such that all the \( p_n, n \in N \), and \( \bar{p} \), are differentiable at every \( x \notin X_0 \). If \( x \notin X_0 \), then \( \nabla_u p_n^\sharp (x) \) is the only point in \( \partial p_n^\sharp (x) \), and we have, by corollary \( \text{[3]} \):

\[
(4.2) \quad u (x, \nabla_x p_n^\sharp (x)) - \bar{p} \left( \nabla_u p_n^\sharp (x) \right) \leq \bar{p}^\sharp (x) - \bar{p} \left( \nabla_u \bar{p}^\sharp (x) \right)
\]

so that:

\[
(4.3) \quad p_n^\sharp (x) + \frac{1}{n} \varphi \left( \nabla_u p_n^\sharp (x) \right) - \bar{p}^\sharp (x) \leq 0
\]

From the definition of \( p_n^\sharp \), we have, using corollary \( \text{[3]} \) again:

\[
(4.4) \quad u (x, \nabla_u \bar{p}^\sharp (x)) - \bar{p} \left( \nabla_u \bar{p}^\sharp (x) \right) - \frac{1}{n} \varphi \left( \nabla_u p_n^\sharp (x) \right) \leq p_n^\sharp (x) - \bar{p} \left( \nabla_u p_n^\sharp (x) \right) - \frac{1}{n} \varphi \left( \nabla_u p_n^\sharp (x) \right)
\]

Rewriting this, we get:

\[
(4.5) \quad p_n^\sharp (x) + \frac{1}{n} \varphi \left( \nabla_u p_n^\sharp (x) \right) - \bar{p}^\sharp (x) = p_n^\sharp (x) + \frac{1}{n} \varphi \left( \nabla_u p_n^\sharp (x) \right) - \bar{p}^\sharp (x)
\]

yielding:

\[
(4.6) \quad -\frac{1}{n} \varphi \left( \nabla_u \bar{p}^\sharp (x) \right) \leq p_n^\sharp (x) - \bar{p}^\sharp (x)
\]
Now let \( n \to \infty \). Using corollary 3, we have:

\[
u(x, \nabla u p_n^\sharp (x)) = p_n \left( \nabla u p_n^\sharp (x) \right) + p_n^\sharp (x)
\]

Since \( Z \) is compact, the sequence \( \nabla u p_n^\sharp (x) \in Z \) has a cluster point \( z \), and since \( p_n \) and \( p_n^\sharp \) converge to \( \bar{p} \) and \( \bar{p}^\sharp \) uniformly, we get in the limit:

\[
u(x, z) = \bar{p} (z) + \bar{p}^\sharp (z)
\]

so that \( z \in \partial \bar{p}^\sharp (x) \). But that subdifferential consists only of the point \( \nabla u \bar{p}^\sharp (x) \), so that \( z = \nabla u \bar{p}^\sharp (x) \). This shows that the cluster point \( z \) is unique, so that the whole sequence must converge:

\[
\nabla u p_n^\sharp (x) \to \nabla u \bar{p}^\sharp (x)
\]

Inequalities (4.7) together give:

\[
-\varphi \left( \nabla u \bar{p}^\sharp (x) \right) \leq n \left( p_n^\sharp (x) - \bar{p}^\sharp (x) \right) \leq \varphi \left( \nabla u p_n^\sharp (x) \right)
\]

Taking limits in the inequalities 1.3 and 1.0, we get:

\[
\forall x \notin X_0, \lim_n \left( p_n^\sharp (x) - \bar{p}^\sharp (x) \right) = -\varphi \left( \nabla u \bar{p}^\sharp (x) \right)
\]

Similarly, we have:

\[
\forall y \notin Y_0, \lim_n \left( \bar{p}^\sharp (x) - \bar{p}^\sharp (x) \right) = -\varphi \left( \nabla u \bar{p}^\sharp (x) \right)
\]

where \( Y_0 \subset Y \) is negligible.

Because of (4.7), we can apply the dominated convergence theorem to inequality (1.4). We get:

\[
-\int_X \varphi \left( \nabla \bar{p}^\sharp (x) \right) \, d\mu + \int_Y \varphi \left( \nabla \bar{p}^\sharp (y) \right) \, d\nu \geq 0
\]

Since the inequality must hold for \( -\varphi \) as well as \( \varphi \), it is in fact an equality. In other words, for any \( \varphi : Z \to R \) with compact support, we have:

\[
\int_X \varphi \left( \nabla \bar{p}^\sharp (x) \right) \, d\mu = \int_Y \varphi \left( \nabla \bar{p}^\sharp (y) \right) \, d\nu
\]

and this means that \( \bar{s} (\mu) = \bar{t} (\nu) \), as announced. \( \square \)

Set \( \bar{s} (\mu) = \bar{t} (\nu) = \lambda \). This is a positive measure on \( Z \), not necessarily absolutely continuous with respect to the Lebesgue measure.

Applying corollary 3, we have:

\[
\bar{p}^\sharp (x) = u(x, \bar{s} (x)) - \bar{p} (\bar{s} (x))
\]

\[
\bar{p}^\sharp (y) = v(y, \bar{t} (y)) - \bar{p} (\bar{t} (y))
\]

and hence:

\[
\int_X \bar{p}^\sharp (x) \, d\mu - \int_Y \bar{p}^\sharp (y) \, d\nu = \int_X [u(x, \bar{s} (x)) - \bar{p} (\bar{s} (x))] \, d\mu - \int_Y [v(y, \bar{t} (y)) - \bar{p} (\bar{t} (y))] = \int_X u(x, \bar{s} (x)) \, d\mu - \int_Y v(y, \bar{t} (y)) \, d\nu - \left[ \int_X \bar{p} (\bar{s} (x)) \, d\mu - \int_Y \bar{p} (\bar{t} (y)) \, d\nu \right]
\]

\[
= \int_X u(x, \bar{s} (x)) \, d\mu - \int_Y v(y, \bar{t} (y)) \, d\nu - \left[ \int_Z \bar{p} (z) \, d\lambda - \int_Z \bar{p} (z) \, d\lambda \right]
\]

\[
= \int_X u(x, \bar{s} (x)) \, d\mu - \int_Y v(y, \bar{t} (y)) \, d\nu
\]

(4.9)
Let \((s, t)\) be a pair of Borelian maps such that \(s(\mu) = t(\nu)\). Then, by the Fenchel inequality:

\[
\int_X u(x, s(x)) \, d\mu - \int_Y v(y, t(y)) \, d\nu \leq \int_X \left[ \tilde{p}^\sharp (x) + \tilde{p}(s(x)) \right] \, d\mu - \int_Y \left[ p^\sharp (y) + \tilde{p} (t(y)) \right] \, d\nu
\]

\[
= \int_X \tilde{p}^\sharp (x) \, d\mu - \int_Y \tilde{p}^\sharp (y) \, d\nu + \int_X \tilde{p}(s(x)) \, d\mu - \int_Y \tilde{p} (t(y)) \, d\nu
\]

The last bracket vanishes because \(s(\mu) = t(\nu)\). Applying inequality 4.9 we get:

\[
\int_X u(x, s(x)) \, d\mu - \int_Y v(y, t(y)) \, d\nu \leq \int_X u(x, \bar{s}(x)) \, d\mu - \int_Y v(y, \bar{t}(y)) \, d\nu
\]

This shows that \((\bar{s}, \bar{t})\) is a maximizer, and proves the existence part of theorem 1.

As for uniqueness, assume that there is another maximizer \((s', t')\). The preceding inequality then becomes an equality:

\[
\int_X u(x, s(x)) \, d\mu - \int_Y v(y, t(y)) \, d\nu = \int_X \left[ \tilde{p}^\sharp (x) + \tilde{p}(s(x)) \right] \, d\mu - \int_Y \left[ p^\sharp (y) + \tilde{p} (t(y)) \right] \, d\nu
\]

which we rewrite as:

\[
\int_X \left[ \tilde{p}^\sharp (x) + \tilde{p}(s(x)) - u(x, s(x)) \right] \, d\mu + \int_Y \left[ v(y, t(y)) - p^\sharp (y) + \tilde{p} (t(y)) \right] \, d\nu = 0
\]

Both integrands are non-negative by the Fenchel inequality. If the sum is zero, each integral must vanish, and since the integrands are non-negative, each integrand must vanish almost everywhere. This means that:

\[
s(x) \in \partial \tilde{p}^\sharp (x) \quad \text{a.e.}
\]

\[
t(y) \in \partial \tilde{p}^\sharp (y) \quad \text{a.e.}
\]

and since \(\partial \tilde{p}^\sharp (x) = \{\bar{s}(x)\}\) and \(\partial \tilde{p}^\sharp (y) = \{\bar{t}(y)\}\) almost everywhere, the result is proved.

5. Some consequences.

We shall now investigate some properties of the function \(\bar{p} : Z \to R\).

Recall that we denote \(\lambda = \bar{s}(\mu) = \bar{t}(\nu)\). It is a positive measure on \(Z\). Its support \(\text{Supp} (\lambda)\) is the complement of the largest open subset \(\Omega \subset Z\) such that \(\lambda = 0\) on \(\Omega\). If for instance \(\mu\) and \(\nu\) have the property that the measure of any open non-empty subset is positive, then:

\[
\text{Supp} (\lambda) = \bar{s}(X) = \bar{t}(Y)
\]

**Proposition 8.** \(\bar{p}^\sharp = \bar{p} = \tilde{p}^b\) on \(\text{Supp} (\lambda)\)

**Proof.** We have seen that \(\bar{p}^\sharp (\bar{s}(x)) = \bar{p}(\bar{s}(x))\) \(\mu\)-almost everywhere. Since \(\bar{s}(\mu) = \lambda\), this means that \(\bar{p}^\sharp (z) = \bar{p}(z)\) \(\lambda\)-almost everywhere. Since \(\bar{p}\) and \(\tilde{p}^b\) are continuous, equality extends to the support of \(\lambda\). Similarly, \(\bar{p} = \tilde{p}^b\) on \(\text{Supp} (\lambda)\), and the result follows.

Let us illustrate this with an example.
5.1. The linear case. Suppose \( X, Y, Z \) are compact subsets of \( \mathbb{R}^n \), and we want to minimize:

\[
\int_X \|x - s(x)\|^2 \, d\mu - \int_Y \|y - t(y)\|^2 \, d\nu
\]

among all maps \((s, t)\) such that \( s(\mu) = t(\nu) \). Developing the squares, this amounts to minimizing:

\[
\left[ \int_X x^2 \, d\mu - \int_Y y^2 \, d\nu \right] + \left[ \int_X s(x)^2 \, d\mu - \int_Y t(y)^2 \, d\nu \right] - \left[ \int_X x' s(x) \, d\mu - \int_Y y' t(y) \, d\nu \right]
\]

The first bracket is a constant (it does not depend on the choice of \( s \) and \( t \)). The second bracket vanishes because \( s(\mu) = t(\nu) \). We are left with the last one. So the problem amounts to maximizing:

\[
\int_X x' s(x) \, d\mu - \int_Y y' t(y) \, d\nu
\]

and it falls within the scope of the theorem by setting \( u(x, z) = x'z \) and \( v(y, z) = y'z \). Then \( u\)-convex functions are convex in the usual sense, \( v\)-concave functions are concave in the usual sense. By proposition \( p \) is linear on \( \text{Supp}(\lambda) \), so we may take:

\[
p(z) = \pi' z
\]

for some vector \( \pi \in \mathbb{R}^n \). We then get \( s(x) \) by maximizing \((x - \pi)' z \) over \( Z \). Similarly, we get \( t(y) \) by minimizing \((y - \pi)' z \) over \( Z \). Note that this implies that \( s(X) = t(Y) \subset \partial Z \), the boundary of \( Z \). Let us summarize:

**Proposition 9.** There is a single map \((\tilde{s}, \tilde{t})\) which minimizes the integral among all maps \((s, t)\) such that \( s(\mu) = t(\nu) \). It is given by:

\[
(5.2) \quad \tilde{s}(x) = \arg \max_z (x - \pi)' z
\]

\[
(5.3) \quad \tilde{t}(y) = \arg \min_z (y - \pi)' z
\]

and the actual value of \( \pi \in \mathbb{R}^n \) is found by substituting (5.2) and (5.3) in the integral and by minimizing the resulting function of \( \alpha \).

This first example is degenerate: the dimension of \( \text{Supp}(\lambda) \) is strictly less than the dimension \( n \) of \( X, Y \) and \( Z \). Let us now go in the opposite direction.

5.2. The non-degenerate case. Suppose \( X, Y, Z \) are compact subsets of \( \mathbb{R}^n \). Let \( \bar{z} \) belong to the interior of \( \text{Supp}(\lambda) \), so that \( \bar{z} = \bar{s}(\bar{x}) = \bar{t}(\bar{y}) \) and suppose there are neighbourhoods \( \Omega_x, \Omega_y, \) and \( \Omega_z \) of \( \bar{x}, \bar{y} \) and \( \bar{z} \) such that the restrictions \( \bar{s} : \Omega_x \to \Omega_z \) and \( \bar{t} : \Omega_y \to \Omega_z \) are invertible, with continuous inverses \( \sigma : \Omega_z \to \Omega_x \) and \( \tau : \Omega_z \to \Omega_y \).

The function \( \bar{p} \) then satisfies two partial differential equations on \( \Omega_z \): a second-order equation of Monge-Ampère type, and a fourth-order equation of Euler-Lagrange type.

5.2.1. A Monge-Ampère equation. Write the definitions of \( \sigma(z) \) and \( \tau(z) \), for \( z \in \Omega_z \):

\[
(5.4) \quad D_z u(\sigma(z), z) = D_z \bar{p}(z)
\]

\[
(5.5) \quad D_z v(\tau(z), z) = D_z \bar{p}(z)
\]
Inverting the first equation expresses \( \sigma(z) \) in terms of \( D_z p(z) \). Inverting the second one expresses \( \tau(z) \) in terms of \( D_z p(z) \). Substituting into the equation \( \sigma^{-1} \circ \tau(\mu) = \nu \) gives a second-order partial differential equation for \( p \).

Let us give an example. Consider the problem of minimizing the integral:

\[
\int_X \frac{\alpha}{2} \|x - s(x)\|^2 \, dx + \int_Y \frac{1}{2} \|y - t(y)\|^2 \, dy
\]

over all maps \((s, t)\) such that \( s(\mu) = t(\nu) \). Here \( \alpha > 0 \) is a given constant. We apply theorem \[\text{H}\] with \( u(x, z) = -\frac{\alpha}{2} \|x - z\|^2 \) and \( v(y, z) = \frac{1}{2} \|y - z\|^2 \).

Assume \( \Omega_x, \Omega_y \) and \( \Omega_z \) are as above. Equations \[\text{5.6}\] and \[\text{5.7}\] become:

\[
\begin{align*}
D_z \tilde{p}(z) &= -\alpha (\sigma(z) - z) \\
D_z \bar{p}(z) &= (\tau(z) - z)
\end{align*}
\]

So \( \sigma(z) = z - \frac{\alpha}{2} D_z \tilde{p}(z) \) and \( \tau(z) = z + D_z \bar{p}(z) \). The map \( \sigma^{-1} \circ \tau \) sends \( \mu \) on \( \nu \).

We must have:

\[
[\det D_z \sigma(z)]^{-1} \det D_z \tau(z) = 1
\]

and this gives a second-order equation for \( \tilde{p} \):

\[
(5.7) \quad \det \left[I + D_z^2 \tilde{p}(z)\right] = \det \left[I - \frac{1}{\alpha} D_z^2 \bar{p}(z)\right]
\]

5.2.2. An Euler-Lagrange equation. Assume that the constraints on \( \tilde{p} \) are not binding on \( \Omega_z \). In other words, there is some \( \varepsilon > 0 \) such that, for every \( h \) such that \( |h| < \varepsilon \), and every \( C^\infty \) function \( \varphi \) with compact support in \( \Omega_z \), the function \( p_h = p + h\varphi \) is still \( u \)-convex and \( v \)-concave.

Recall that \( \tilde{p} \) solves the optimization problem:

\[
\inf_{\tilde{p}} \left[ \int_X \tilde{p}^\ast(x) \, d\mu - \int_Y \tilde{p}^\ast(y) \, d\nu \right]
\]

and this implies that:

\[
\int_{\Omega_x} \tilde{p}_h^\ast(x) \, d\mu - \int_{\Omega_y} \tilde{p}_h^\ast(y) \, d\nu \geq \int_{\Omega_x} \tilde{p}^\ast(x) \, d\mu - \int_{\Omega_y} \tilde{p}^\ast(y) \, d\nu
\]

for every \( h \). Expressing the sub- and superconjugates in terms of the sub- and superdifferentials, and taking advantage of the fact that \( \tilde{p}_h^\ast = p = \bar{p}_h^\ast \), we get, for every \( h \):

\[
(5.8) \quad \int_{\Omega_x} [u(\sigma_h(z), z) - p_h(z)] \, d[\sigma_h^{-1}(\mu)] - \int_{\Omega_x} [v(\tau_h(z), z) - p_h(z)] \, d[\tau_h^{-1}(\nu)]
\]

\[
(5.9) \quad \geq \int_{\Omega_x} [u(\sigma(z), z) - \tilde{p}(z)] \, d[\sigma^{-1}(\mu)] - \int_{\Omega_x} [v(\tau(z), z) - \tilde{p}(z)] \, d[\tau^{-1}(\nu)]
\]

where \( \sigma_h \) and \( \tau_h \) are defined by:

\[
(5.10) \quad D_z u(\sigma_h(z), z) = D_z p_h(z)
\]

\[
(5.11) \quad D_z v(\tau_h(z), z) = D_z p_h(z)
\]
Note that for $h = 0$, we have $\sigma_h = \sigma$, $\tau_h = \tau$ and $p_h = \bar{p}$, while $\sigma(\mu) = \tau(\mu)$. Letting $h \to 0$, we find that $\bar{p}$ must satisfy an Euler-Lagrange equation of the fourth order.

Let us illustrate this with example (5.6).

A function $p : Z \to R$ is $u$-convex iff $D^2zzp(z) \geq -\alpha I$ for every $z$, where $D^2zzp(z)$ is the Hessian matrix at $z$, and it is $v$-concave iff $D^2zzp(z) \leq I$. So there are many functions $p : Z \to R$ which are both $u$-convex and $v$-concave: they must satisfy $-\alpha I \leq D^2zz\bar{p}(z) \leq I$ everywhere.

Assume $\Omega_x$, $\Omega_y$ and $\Omega_z$ are as above, and $-\alpha I < D^2_{zz}\bar{p}(z) < I$ on $\Omega_z$. Then the integral to be maximized with respect to $p = p_h$ is:

$$\int_{\Omega_z} \left[ u \left( z - \frac{1}{\alpha} Dzp(z), z \right) - p(z) \right] \left[ I - \frac{1}{\alpha} D^2zzp(z) \right]^{-1} dz$$

$$- \int_{\Omega_z} \left[ v \left( z + Dzp(z), z \right) - p(z) \right] \left[ I - D^2_{zz}p(z) \right]^{-1} dz =$$

$$- \int_{\Omega_z} \left( \left[ \frac{1}{2\alpha} \|Dzp\|^2 - p \right] \left[ I - \frac{1}{\alpha} D^2zzp \right]^{-1} + \left[ \frac{1}{2} \|Dzp\|^2 - p \right] \left[ I - D^2_{zz}p \right]^{-1} \right) dz$$

So $\bar{p}$, which minimizes the last integral, must satisfy the corresponding fourth order Euler-Lagrange equation. We will not write it down explicitly, although relation (5.7), which is valid for $p = \bar{p}$, would introduce some simplifications.

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