Infinite Range Correlations in Non-Equilibrium Quantum Systems and Their Possible Experimental Realizations

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We consider systems that start from and/or end in thermodynamic equilibrium while experiencing a finite rate of change of their energy density or of other intensive quantities \( q \) at intermediate times. We demonstrate that at these times, during which the global intensive quantities \( q \) vary at a finite rate, the size of the associated covariance, the connected pair correlator \( |G_{ij}| = \langle (q_i - \langle q_i \rangle)(q_j - \langle q_j \rangle) \rangle \), between any two (arbitrarily far separated) sites \( i \) and \( j \) may, on average, become finite. Such non-vanishing connected correlations between asymptotically distant sites are general and may also appear in theories that only have local interactions. In simple models, these correlations may be traced to the generic macroscopic entanglement of finite temperature states. Once the global mean \( q \) no longer changes, the average of \( |G_{ij}| \) over all spatial separations \( |i - j| \) may tend to zero. However, when the equilibration times are significant (e.g., as in a glass that is not in true thermodynamic equilibrium yet in which the energy density (or temperature) reaches a final steady state value), these long range correlations may persist also long after \( q \) ceases to change. We explore viable experimental implications of our findings and speculate on their potential realization in glasses (where a prediction of a theory based on the effect that we describe here suggests a universal collapse of the viscosity that agrees with all published viscosity measurements over sixteen decades) and non-Fermi liquids. We briefly comment on parallels between quantum measurements and thermalization.

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I. INTRODUCTION

In theories with local interactions, the connected correlations between two different sites \( i \) and \( j \) often decay with their spatial separation \( |i - j| \). Indeed, connected correlations decay exponentially with distance in systems with finite correlation lengths. In massless (or critical) theories, this exponential decay is typically replaced by an algebraic drop. The detailed understanding of these decays was achieved via numerous investigations that primarily focused on venerable systems with fixed control parameters, e.g., [1–6]. In the current work, we wish to build on these notions and ask what occurs in a general non-relativistic system, when an intensive parameter such as the average energy density \( \langle E/N \rangle \) will largely couch these for theories residing on \( d \)-dimensional hypercubic lattices of \( N = L^d \) sites; the average energy density \( \epsilon = E/N \) with \( E \) the total energy. In theories with local interactions, we may express (in a variety of ways) the Hamiltonian \( H \) as a sum of \( N' = \mathcal{O}(N) \) terms \( \{H_i\}_{i=1}^{N'} \) that are each of finite range, \( H = \sum_{i=1}^{N'} H_i \). Our results apply to both lattice and continuum systems. In either of these cases, our principal interest lies in the thermodynamic \( (N \gg 1) \) limit. We denote the probability density matrix by \( \rho \); we stress that since our focus is on general non-equilibrium systems, \( \rho \) need not be equal to the any of the standard density matrices describing equilibrium systems.

II. SKETCH OF MAIN RESULT

We now provide a synopsis of the considerations that underlie our work and its main result. A central function that we will focus on is that of the probability density of global energy density,

\[ P(\epsilon') \equiv Tr[\rho \delta(\epsilon' - H/N)]. \tag{1} \]
An initial high temperature adiabatic cooling. By contrast, in method (ii), we keep corresponding to its final energy density after the non-recouple the system to a lower temperature heat bath by applying external fields, etc. We will then subsequently alter its Hamiltonian by applying external fields, etc. We will then subsequently couple the system from the high temperature heat bath (Section VIII). In procedure (i), we may decrease the energy density does not vary at a finite rate as a function of time. During this cooling process, as it is being driven, \( P(\epsilon') \) obtains a finite standard deviation (even for macroscopic systems). The demonstration of such a generic widening of the distribution is a principal objective of this paper. A finite standard deviation of \( P(\epsilon') \) implies that the system was cooled. Similar broadening may occur for general intensive quantities \( q \).

In equilibrium, the energy density (similar to all other intensive thermodynamic variables) is sharply defined; regardless of the specific equilibrium ensemble employed, the distribution of Eq. (1) is a Dirac delta-function,

\[ P(\epsilon') = \delta(\epsilon' - \epsilon). \]

This is schematically illustrated in the left and righthand sides of Figure 1. The chief goal of the current article is to demonstrate that when a system that was initially in equilibrium is driven at intermediate times (by, e.g., rapid cooling) such that its energy density varies at a finite rate as a function of time, the distribution \( P(\epsilon') \) need not remain a delta-function (see, e.g., the central panel of Figure 1). Since the variance of \( P(\epsilon') \) is a sum of pair correlators \( G_{ij} \equiv \langle H_i H_j \rangle - \langle H_i \rangle \langle H_j \rangle \), this latter finite width of \( P(\epsilon') \) of the system when it is driven implies (as we will explain in depth) that the correlations \( G_{ij} \) extend over macroscopic length scales that are of the order of the system size. Whenever the formerly driven system re-equilibrates, \( P(\epsilon') \) becomes a delta-function once again (right panel of Figure 1). We will investigate the driving as implemented by either (i) endowing the Hamiltonian with non-adiabatic transient time dependence \( H(t) \) deviating from the equal initial and final Hamiltonian \( H = \sum_i H_i \) (Sections (VI, VII)) or (ii) introducing no time dependence in the Hamiltonian and then allowing for (a time independent) coupling to an external bath (Section VIII). In procedure (i), we may decouple the system from the high temperature heat bath and then non-adiabatically alter its Hamiltonian by applying external fields, etc. We will then subsequently recouple the system to a lower temperature heat bath corresponding to its final energy density after the non-adiabatic cooling. By contrast, in method (ii), we keep the system in contact with an external bath throughout.

If sufficiently slow cooling is achieved such that the energy density does not vary at a finite rate as a function of time then general inter-related lower bounds that we will derive in case (ii) will vanish. These lower bounds relate to (a) the variance of the distribution \( P(\epsilon') \) and (b) the magnitude of the pair correlator \( G_{ij} \) for sites \( i \) and \( j \) that are separated by a distance that is of the order of the system size [12]. A similar broadening of the distribution \( P(q') \) (and ensuing lower bounds on the associated pair correlators) may arise for general intensive quantities \( q \) (that include the energy density \( \epsilon \) only as a special case).

**III. OUTLINE**

The remainder of this paper is organized as follows: In Section [IV] we explain why, in spite of its striking nature, our main finding of large variances (even in systems with local interactions) and the infinite range correlations that they imply is quite natural. By infinite range, we refer, in any \( N \gg 1 \) site system, to correlations that span the entire macroscopic system size. Next, in Section [V] we discuss special situations in which our results do not hold—those of product states. Notwithstanding their simplicity and appeal, product states do not generally describe systems above their ground state energy density. In Section [VI] we turn to more generic situations such as those appearing in two dual (highly entangled systems) on lattices in arbitrary dimensions for which a class of finite en-
ergy density eigenstates can be exactly constructed: (1) general rotationally symmetric spin models in an external magnetic field and (2) systems of itinerant hard-core bosons with attractive interactions. We investigate the effects of “cooling/heating” and “doping” protocols on these systems and illustrate that, regardless of the system size, after a finite amount of time, notable energy or carrier density fluctuations will appear. Armed with these proof of principle demonstrations of the energy density and number density fluctuations, we examine in Section VII the anatomy of the Magnus expansion to see how generic these large fluctuations may be. Straightforward calculations illustrate that although there exist fine tuned situations in which the variance of intensive quantities such as the energy density remain zero (e.g., the product states of Section V) in rapidly driven systems, such circumstances are exceedingly rare. General non-adiabatic evolutions that change the expectation values of various intensive quantities may, concomitantly, lead to substantial standard deviations. In Section VIII we go one step further and establish that under a rather mild set of constraints, infinite range connected fluctuations are all but inevitable. We next illustrate (Section IX) how general expectation values in these systems relate to equilibrium averages. Our effect has broad experimental implications: common systems undergoing heating/cooling and/or other evolutions of their intensive quantities may exhibit long range correlations. The remainder of the paper largely discussing candidate experimental and in silico realizations of our effect is more speculative than the detailed exact solutions and derivations presented in its earlier Sections. In Sections X and XI we turn to two prototypical systems and ask whether our findings may rationalize experimental (and numerical) results. In particular, in Section X we discuss glasses and show a universal collapse of the viscosity data that was inspired by considerations similar to those that we describe in the current work. In Section XI we ask whether the broadened distributions that we find may lead to “non-Fermi” liquid type behavior in various electronic systems. In Section XII we discuss adiabatic quantum processes and demonstrate how these may maintain thermal equilibrium. We further speculate on possible offshoots of this result that suggest certain similarities between quantum measurements and thermalization. We conclude in Section XIII with a synopsis of our results. In Appendix A, we show that using entangled states (similar to those analyzed in Section VI) reproduces the finite temperature correlators of an Ising chain. In Appendix B we demonstrate that the entanglement entropy of symmetric entangled states is logarithmic in the system size; this latter calculation will further illustrate that the entangled spin states studied in Section VI display such macroscopic entanglement. These examples are meant to underscore that, even in closed systems, eigenstates of an energy density larger than that of the ground state can very naturally exhibit a macroscopic entanglement. As we will explain, this macroscopic entanglement may enable the existence of long range correlations. In Appendices C and D we discuss specific aspects related to Sections VI and VII respectively. Appendix E motivates the appearance of long time Gaussian distributions (such as those invoked in our derivation of the 16 decade viscosity collapse of supercooled liquids and glasses).

IV. INTUITIVE ARGUMENTS

To make our more abstract discussions clear, we first try to motivate why our central claim might not, at all, be surprising and expand on the basis premise outlined in Section III. Consider a system that is, initially, in thermodynamic equilibrium with a sharp energy density \( \epsilon \). For an initial closed equilibrium system (described by the microcanonical ensemble), the standard deviation of \( \epsilon \) scales as \( 1/N \) while in open systems connected to a heat bath, the standard deviation of \( \epsilon \) is \( O(1/\sqrt{N}) \). In either of these two cases, the standard deviation of \( \epsilon \) vanishes in the thermodynamic limit (similar results apply to any intensive thermodynamic variable), see, e.g., the right-hand panel of Figure I. Now imagine cooling the system. As the system is cooled, its energy density \( \epsilon \) drops. Various arguments hint that as \( \epsilon \) drifts downwards in value, its associated standard deviation also increases (see the central panel of Figure I). This is analogous to the increase in width of an initially localized “wave packet” with a non-trivial evolution (with the energy density itself playing the role of the packet location). On a rudimentary level, it might be hardly surprising that the energy density obtains a finite standard deviation when it continuously varies in time. A finite standard deviation of the energy density implies long range correlations of the local energy terms. This is so since the variance of the energy density

\[
0 < \sigma^2_\epsilon = \frac{1}{N^2} \sum_{i,j} (\langle \mathcal{H}_i \mathcal{H}_j \rangle - \langle \mathcal{H}_i \rangle \langle \mathcal{H}_j \rangle) \]

\[
= \frac{1}{N^2} \sum_{i,j} G_{ij} \leq \frac{1}{N^2} \sum_{i,j} |G_{ij}| = \frac{1}{|G|}.
\]  

Thus, if \( \tau_\epsilon \) is finite then the average \( |G| \) of \( |G_{ij}| \) over all separations \( |i-j| \) will be non-vanishing. More broadly, similar considerations apply to intensive quantities of the form \( q = \frac{1}{N} \sum_i q_i \) that must have a sharp value in thermodynamic equilibrium. Thus, generally, if \( q \) broadens as some parameters are varied, there must be finite connected correlations \( \langle q_i q_j \rangle - \langle q_i \rangle \langle q_j \rangle \) even when \( |i-j| \to \infty \). Identical conclusions to the ones presented above may be drawn for systems that end in thermodynamic equilibrium (instead of starting from equilibrium) while experiencing a finite rate of change of their energy density at earlier times at which Eq. 2 will hold. Empirically, in cases of experimental relevance, as in, e.g., cooling or heating a material, if the rate of change of its temperature (or energy density) is finite then Eq. 2
V. PRODUCT STATES

Prior to demonstrating that energy density broadening naturally accompanies a cooling or heating of the system, we first regress to a “classical” situation of states that may be associated with individually decoupled local subsystems between which no entanglement exists. For a density matrix \( \rho \) that is a direct tensor product of local density matrices \( \{ \rho_i \}_{i=1}^{M} \) that act on disjoint spaces, with \( M = O(N) \),

\[
\rho = \rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_M, \tag{3}
\]

the standard deviation \( \sigma_E \) of the total energy will, in accord with the central limit theorem, be \( O(\sqrt{N}) \) even when the rate of change of the energy \( dE/dt \) may be extensive (i.e., \( \propto N \)). As a case in point, we may consider the initial (spin \( S = 1/2 \)) state \( |\psi^0_{\text{Ising}}\rangle = |s_1 s_2 \cdots s_N\rangle \) to be a low energy eigenstate of an Ising model \( H_I = -J_{ij} S_i^z S_j^z \) that is a direct tensor product of local spin operators \( S_i^z \). A uni-


Even classically, algebraic power law correlations appear in such nonuniformly driven systems \[45\]. The existence of a spatially non-uniform profile of the local energy density may further enhance the large fluctuations that we find in the current work without assuming nonuniformity. We will briefly touch on related aspects towards the end of Section \[5\]. In classical systems with local interactions, broad distributions of various observables may also occur in the thermodynamic limit when these systems are disordered. This phenomenon is known as “non-self-averaging”, e.g., \[46\]. In these disordered classical systems, an ensemble average of a physical observable computed over different disorder realizations may differ significantly from the expectation value of the same quantity in any single member of the ensemble. To put our work in a broader context, we note that the systems that we will focus on in the current work need not be disordered nor critical. However, given the absence of self-averaging in such disordered classical systems, we remark that the broadening that we find will also apply to various systems when the (“ensemble of”) eigenstates of the density matrix effectively describe these different disorder realizations of classical critical systems. This is so since, in such cases, an average computed with the probability density matrix \( \rho \) will reproduce the average associated with an ensemble of disordered classical states.

In what follows, we first turn to product states where no broad distributions of intensive quantities arise. In the sections thereafter, we will demonstrate that in general quantum systems (not constrained to a product state structure), broadening may be quite prevalent.
states of the Ising chain mirror those in equilibrated Ising chains at positive temperatures. In the one dimensional Ising model and other equilibrium systems at temperatures $T > 0$, the high degree of entanglement and mixing between individual product states leads to contributions to the two point correlation functions that alternate in sign and ultimately lead to the usual decay of correlations with distance. Our central thesis is that an external driving Hamiltonian (such as that present in cooling/heating of a system) may lead to large extensive fluctuations. While the appearance of such extensive fluctuations may seem natural for non-local operators (such as (Heisenberg picture) time evolved local Hamiltonian terms in various examples), these generic fluctuations may also appear for local quantities (e.g., the local operators $\{H_i\}$ in Eq. (2)). In Section VI A we will study systems for which the relevant (Heisenberg picture) operators $\{H_i\}$ are, indeed, local.

When all of the eigenvectors of the density matrix are trivial classical local product states that do not exhibit entanglement, the system described by $\rho$ is a classical system (with different classical realizations having disparate probabilities). In general, there are both “classical” and inherently “quantum” contributions [45] to the variance $\sigma^2$ of the energy density of Eq. (2). In the next sections, we will demonstrate that large fluctuations of any observable may naturally arise for all system sizes (including systems in their thermodynamic limit). The calculations in these examples will be for single quantum mechanical states. Any density matrix (also that capturing a system having a mixed state in any region $S'$) may be expressed as $\rho = |\psi\rangle\langle\psi|$ with a pure state $|\psi\rangle$ that extends over a volume $V \supset S'$ [18, 17].

Our analysis will naturally allow for entangled quantum states. To highlight this aspect, we will typically keep all factors of $\hbar$ explicit.

VI. TWO DUAL EXAMPLES

The existence of finite connected correlations $|G_{ij}|$ (Eq. 2) for far separated sites $|i - j| \rightarrow \infty$ is at odds with common lore. Before turning to more formal general aspects, we illustrate how this occurs in two classes of archetypical systems- (i) any globally $SU(2)$ symmetric (arbitrary graph or lattice) spin $S = 1/2$ model in an external magnetic field (discussed next in Section VI A) and (ii) dual hard core Bose systems on the same graphs or lattices (Section VII B). Although (i) and (ii) constitute two well known (and very general) intractable many-body theories, as we will demonstrate, the analysis of the fluctuations becomes identical to that associated with an integrable one body problem. In the context of example (i), this effective single body problem will be associated with the total system spin $S_{\text{tot}}$. This simplification will enable us to arrive at exact results.

A. Rotationally invariant spin models on all graphs (including lattices in general dimensions)

In what follows, we consider a general rotationally symmetric spin model ($H_{\text{symm}}$) of local spin-$S$ moments augmented by a uniform magnetic field.

$$H_{\text{spin}} = H_{\text{symm}} - B_2 \sum_i \vec{S}_i^z.$$  (4)

Amongst many other possibilities, the general rotationally symmetric Hamiltonian $H_{\text{symm}}$ may be a conventional spin interaction of the type

$$H_{\text{Heisenberg}} = -\sum_{ij} J_{ij} \vec{S}_i \cdot \vec{S}_j$$
$$- \sum_{ijkl} W_{ijkl} (\vec{S}_i \cdot \vec{S}_j) (\vec{S}_k \cdot \vec{S}_l) + \cdots,$$  (5)

with arbitrary Heisenberg spin exchange couplings $\{J_{ij}\}$ (whether these are of the nearest neighbor type or of longer range or spatially uniform or not) augmented by conventional higher order rotationally symmetric terms. We wish to reiterate that the model of Eq. (4) is defined on any graph (including lattices in any number of spatial dimensions). We label the eigenstates of $H_{\text{spin}}$ (and their energies by $\{\phi_\alpha\}$) (having, respectively, energies $E_\alpha$). We next briefly discuss the total spin operator $\vec{S}_{\text{tot}} = \sum_{i=1}^N \vec{S}_i$. Since $[\vec{S}_{\text{tot}}, H_{\text{spin}}] = 0$, it follows that all eigenstates of $H_{\text{spin}}$ may be simultaneously diagonalized with $S_{\text{tot}}^z$ (with eigenvalue $m\hbar$) and $S_{\text{tot}}^2$ (with eigenvalue $S_{\text{tot}}(S_{\text{tot}} + 1)\hbar^2$). Thus any eigenstate of Eq. (4) may be written as $|\phi_\alpha\rangle = |\mu_\alpha; S_{\text{tot}}, S_{\text{tot}}^z, S_{\text{tot}}^x, S_{\text{tot}}^y\rangle$ with $\mu_\alpha$ denoting all additional quantum numbers labeling the eigenstates of $H_{\text{spin}}$ in a given sector of $S_{\text{tot}}$ and $S_{\text{tot}}^z$ [43]. Although our results apply for local spins of any size $S$, in order to elucidate certain aspects, we will often allude to spin $S = 1/2$ systems. For any eigenstate having a general $S_{\text{tot}}^z \neq \pm S_{\text{max}} = \pm NS$, the associated density matrix is not of the local tensor product form of Eq. (8). Rather, any such eigenstate is a particular superposition of spin $S = 1/2$ product states having a total fixed value of $S_{\text{tot}}^z$. The state of maximal total spin $S_{\text{tot}} = S_{\text{max}}$ (which can be trivially shown to be a non-degenerate eigenstate for any value of $S_{\text{tot}}^z$; see Appendix C) corresponds to a symmetric equal amplitude superposition of all such product states of a given $S_{\text{tot}}^z$ (i.e., such a sum of all product states of the type $|\uparrow_1\uparrow_2\uparrow_3\uparrow_4\uparrow_6\cdots\uparrow_N\downarrow_N\rangle$ in which there are a total of $(N/2 \pm S_{\text{tot}}^z/\hbar)$ single spin of up/down polarizations along the $z$ axis). We set an arbitrary eigenstate $|\phi_0\rangle$ to be the initial state (at time $t = 0$) of the system $|\psi_{\text{spin}}\rangle$. The energy density (and the global energy itself) will have a vanishing standard deviation in any such initially chosen eigenstate, $\sigma_\epsilon = 0$. We next evolve this initial ($t = 0$) state via a “cooling/heating process” wherein the energy (as measured by $H_{\text{spin}}$) is varied by non-adiabatically replacing, during the period of time in which the system is cooled or heated, the Hamiltonian of
Eq. (4) by a time dependent transverse field Hamiltonian

$$H_{tr}(t) = -B_y(t) \sum_i S_i^y. \tag{6}$$

Once this “cooling/heating process” terminates at a final time ($t = t_f$), the system Hamiltonian becomes, once again, the original Hamiltonian of Eq. (4). During the evolution with $H_{tr}$, the spins globally precess about the $y$ axis. Thus, after a time $t$, the energy per lattice site is changed (relative to its initial value) by an amount $B_z(S_{tot}^z/\hbar)\int dt' \langle \cos(\int_0^t B_y(t') dt') \rangle$. Employing the shorthand $w \equiv S_{tot}^z/\hbar$, the standard deviation of the energy density of $\frac{\mathcal{H}}{\hbar}$ is \[ \sigma_e(t_f) = \frac{B_z S_{tot}^z}{N} \sin(\int_0^{t_f} B_y(t') dt') \frac{N/2}{\sqrt{1 + \frac{1}{S_{tot}^z} - w^2}} . \tag{7} \]

A comparable standard deviation appears not only for a single eigenstate of $H_{\text{spin}}$, but also for any other initial states having an uncertainty in the total energy that is not extensive. When $w = 1$ (or $-1$) with the total spin being maximal, $S_{tot} = S_{\text{max}}$, the initial state $|\psi_0^{\text{spin}}\rangle$ is a product state of all spins being maximally up (or all spins pointing maximally down) \[ |\psi_0^{\text{spin}}\rangle = |\pm\hbar S_{\text{max}}\rangle \text{ spatial long range entanglement develops. When } (1-w) = O(1), \text{ the scaled standard deviation of the energy density is, for general times, } (\frac{1}{\hbar B_z}) \sigma_e = O(1) \text{ and, as we will elucidate in Appendix B, a macroscopic (logarithmic in system size) entanglement entropy appears. A comparable standard deviation } \sigma_e \text{ appears not only for the eigenstate but also for states initial having an energy uncertainty of order } O(1) \text{ (in units of } B_z\hbar) \text{ (e.g., } c_1|S_{tot}^z, S_{tot}^z\rangle + c_2|S_{tot}^z, S_{tot}^z - \hbar\rangle \text{ with } c_{1,2} \equiv O(1)) \text{. In the following, we briefly remark on the simplest case of a constant (time independent) } B_y. \text{ Here, the time required to first achieve } \frac{1}{\hbar B_z} \sigma_e = O(1) \text{ starting from an eigenstate of } H_{\text{spin}} \text{ is } O(1/B_y). \text{ This requisite waiting time is independent of the system size (as it must be in this model where a finite } \sigma_e \text{ is brought about by the sum of local decoupled transverse magnetic field terms in } H_{tr}. \text{ The large standard deviation implies (Eq. (5)) that long range connected correlations of } S_i^z \text{ emerge once the state is rotated under the evolution with } H_{tr}. \text{ This large standard deviation of } \frac{1}{N} \sum_{i=1}^{N} S_i^z \text{ appears in the rotated state displaying (at all sites } i) \text{ a uniform value of } \langle S_i^z \rangle. \text{ Even though there are no connected correlations of the energy densities themselves in the initial state, the nonlocal entanglement enables long range correlations of the local energy densities once the system is evolved with a transverse field. The variance } \sigma_e \text{ should not, of course, be confused with the spread of energy densities that the system assumes as it evolves (e.g., for the } S_{tot}^z = 0 \text{ state, } \sigma_e = O(1) \text{ while the energy density } \epsilon(t) \text{ does not vary with time). We nonetheless remark that the standard deviation } \sigma_e \text{ vanishes at the discrete times } t_k = k\pi/B_y \text{ (with } k \text{ an integer)-the very same times where the rate of change of the energy density } \epsilon(t) \text{ is zero. Indeed, in our model system, up to important time independent multiplicative factors, } \sigma_e \propto \frac{1}{\hbar B_y} \frac{d\epsilon(t)}{dt}. \text{ We now turn to the higher order moments of the fluctuations, } \langle \left(\Delta \epsilon^2\right)^p \rangle = \frac{1}{\hbar^p} \langle (H_{\text{spin}}^p - \langle H_{\text{spin}}^p \rangle)^p \rangle \text{ with } p > 2. \text{ (The standard deviation of Eq. (7) corresponds to } p = 2.) \text{ Here, } H_{\text{spin}}(t) = \epsilon \int \int_{\text{tor}} \langle H_{\text{spin}} \rangle e^{-i \int \int_{\text{tor}} H_{\text{spin}}(\epsilon) dt'} = \langle H_{\text{spin}}(t) \rangle \text{ is the Heisenberg picture Hamiltonian and the expectation value is taken in the initial state } |\psi_0^{\text{spin}}\rangle. \text{ If } N \gg 1 \text{ and } 1 > |w| \text{ then } S_{\pm}^z|\psi_0^{\text{spin}}\rangle = \hbar \sqrt{S_{\text{tot}}(S_{\text{tot}} + 1) - m(m \pm 1)}|S_{\text{tot}}, m \pm 1\rangle \sim S_{\text{tot}} \hbar \sqrt{1 - w^2}|S_{\text{tot}}, m \pm 1\rangle \text{ where } S_{\text{tot}}^z = mh. \text{ Trivially, for all } m \text{ and } m', \text{ the matrix element of } \delta S_{\pm}^z \equiv S_{\pm}^z - (S_{\pm}^z)^\dagger \text{ between any two eigenstates, } \langle S_{\pm}^z | \delta S_{\pm}^z | S_{\pm}^z \rangle = 0. \text{ Thus, the only nonvanishing contributions to } \langle (\Delta \epsilon^2)^p \rangle \text{ stem from } \langle (\delta \epsilon^2)^p \rangle. \text{ This expectation value may be finite only for even } p. \text{ Thus, in what follows, we set } p = 2g \text{ with } g \text{ being a natural number. For } S_{\text{tot}} = O(N), \text{ expressing the expectation value of } \langle (\Delta \epsilon^2)^2g \rangle \text{ longhand in terms of spin raising and lowering operators, one notices that, in this large } N \text{ limit, each individual term containing an equal number of raising and lowering operators yields an identical contribution (proportional to } (S_{\text{tot}} \hbar \sqrt{1 - w^2})^{2g}) \text{ to the expectation value } \langle (\Delta \epsilon^2)^2g \rangle. \text{ Since there are } (2^g) \text{ such contributions, we see that for all } g \ll N \text{ in the thermodynamic } (N \rightarrow \infty) \text{ limit, the expectation value } \langle (\Delta \epsilon^2)^2g \rangle = (2^g) \frac{\hbar^2 g^2}{N^2} \text{. We write the final (Schroedinger picture) state at time } t = t_f \text{ as } |\psi_{\text{spin}}\rangle = \sum_{\alpha} c_{\alpha} |\phi_{\alpha}\rangle. \text{ The probability distribution of the energy density of Eq. (4) reads}

$$P(\epsilon) = \sum_{\alpha} |c_{\alpha}|^2 \delta(\epsilon - \frac{E_{\alpha}}{N}). \tag{8}$$

In this example, the Heisenberg picture Hamiltonian $H_{\text{spin}}^H$ remains local for all times. Thus, the associated operators $H_i$ are local at any time $t$. In other systems, the time evolved Heisenberg picture Hamiltonian need not be spatially local. Eq. (8) describes the probability distribution associated with the “wave packet” intuitively discussed in Section IV (a “packet” that is now given by the amplitudes $|c_{\alpha}\rangle$ in our eigenvalue decomposition of the final state $|\psi_{\text{spin}}\rangle$). The moments of $\Delta \epsilon \equiv (\epsilon - \epsilon')$ are $\langle (\Delta \epsilon)^2g \rangle = \int d\epsilon' P(\epsilon') (\epsilon - \epsilon')^{2g}$. Here, as throughout, $\epsilon = \frac{1}{\hbar} \langle H_{\text{spin}} \rangle_{\psi_{\text{spin}}} = -\frac{\sum_{ij} J_{ij} + B_z S_{\text{tot}}^z \cos(\int_0^{t_f} B_y(t') dt')}{N}$ is the energy density in the final state. More generally, the expectation value of a general function $f(\frac{H_{\text{spin}}^H}{\hbar})$ in the state $|\psi_{\text{spin}}\rangle$ (or, equivalently, of $f(\frac{H_{\text{spin}}^H}{\hbar})$ in the above defined final Schrodinger picture state $|\psi_{\text{spin}}\rangle$) is given by $\langle f(\frac{H_{\text{spin}}^H}{\hbar}) \rangle = \int d\epsilon' P(\epsilon') f(\epsilon')$. The mean value of each Fourier component $e^{i q \Delta \epsilon}$ when evaluated with $P(\epsilon')$ is $e^{i q \frac{\Delta \epsilon'}{2}} = J_0(q_{\alpha} \sqrt{2})$ \[ \text{[15] where } J_0 \text{ is a Bessel function.} \]
An inverse Fourier transformation then yields

$$P(\epsilon') = \frac{\theta(\sigma\sqrt{2} - |\Delta\epsilon'|)}{\pi \sigma\sqrt{2} - (\Delta\epsilon')^2}. \tag{9}$$

The Heaviside function $\theta(z)$ in Eq. (9) captures the bounded spectrum of $H_{\text{spin}}$. Similar results apply to boundary couplings \[52\]. Comparing our result of Eq. (9) to known cases, we remark that, where it is non-vanishing, the distribution of Eq. (9) is the reciprocal of the Wigner’s semi-circle law governing the eigenvalues of random Hamiltonians and the associated distributions of Eq. (8), e.g., \[54, 55\]. We stress that Eq. (9) is exact for the general spin Hamiltonians of Eqs. (4) and does not hinge on assumed eigenvalue distributions of effective random matrices.

We find qualitatively similar results for other similar “cooling/heating” protocols. For instance, one may consider, at intermediate times $0 \leq t \leq t_f$, the Hamiltonian governing the system to be that of a time independent $H_{\text{tr}}$ (i.e., one with a constant $B_y(t) = B_y$) augmenting $H_{\text{spin}}$ instead of replacing it. That is, we may consider, at times $0 \leq t \leq t_f$, the total Hamiltonian to be

$$H_a = H_{\text{spin}} + H_{\text{tr}}. \tag{10}$$

For such an augmented total Hamiltonian $H_a$, the total spin $S_{\text{tot}}$ precesses around direction of the applied external field $(B_x\hat{e}_x + B_y\hat{e}_y) \equiv \hat{e}_z$. An elementary calculation analogous to that leading to Eq. (1) then demonstrates that the corresponding standard deviation $\sigma_\epsilon^2$ of the energy density at $t = t_f$,

$$\sigma_\epsilon^2 = \frac{B_y B_y S_{\text{tot}} \hbar}{NB \sqrt{2}} \sqrt{1 + \frac{1}{S_{\text{tot}}} - w^2} \times \sqrt{\sin^2(B t_f) + \frac{B_z^2(1 - \cos(B t_f))^2}{B_2^2}}. \tag{11}$$

We wish to stress that if $S_{\text{tot}} = O(N)$ and $|w| < 1$ then, as in Eq. (7), the standard deviation $\sigma_\epsilon^2 = O(N)$ for general times $t_f$. The distribution of the energy density following an evolution with this augmented Hamiltonian will, once again, be given by Eq. (7) for macroscopic systems of size $N \rightarrow \infty$. The reader can readily see how such spin model calculations may be extended to many other exactly solvable cases. The central point that we wish to underscore is that a broad distribution of the energy density, $\sigma_\epsilon \neq 0$, is obtained in all of these exactly solvable spin models in general dimensions.

### B. Itinerant hard core Bose systems

Our spin model of Section VLA can be defined for local spins of any size $S$. The function $P(\epsilon')$ of Eq. (9) characterizing our investigated states in this system is not a very typical probability distribution. However, the non-local entangled character of states having a finite energy density relative to the ground state is pervasive for thermal states. This model can be recast in different ways. In what follows we focus on the spin $S = 1/2$ realization of Eq. (1). The Matsubara-Matsuda transformation \[53, 55\] maps the algebra of spin $S = 1/2$ operators onto that of hard core bosons. Specifically, the bosonic number operator at site $i$ is $n_i = b_i^\dagger b_i = 0, 1$ with $b_i$ and $b_i^\dagger$ the annihilation and creation operators of hard core bosons $(b_i^\dagger)^2 = b_i^2 = 0)$. Following this transformation, the spin Hamiltonian of Eq. (4) is converted into its hard core bosonic dual,

$$H_{\text{Bose}} = -\sum_{ij} J_{ij} ([b_i^\dagger b_j + h.c.] + n_in_j) - \sum_i (B_z - \sum_j J_{ij}) n_i. \tag{12}$$

The above Hamiltonian describes hard core bosons hopping (with amplitudes $J_{ij}$) on the same $d$-dimensional lattice, featuring attractive interactions and a chemical potential set by $(B_z - \sum_j J_{ij})$. Here, the transverse field cooling/heating Hamiltonian $H_{\text{tr}}$ transforms into $H_{\text{Bose-doping}} = -\frac{B_y}{2} \sum_i (b_i^\dagger - b_i)$ - a Hamiltonian that alters the number of the bosons (thereby “doping” the system). The hard core Bose states are symmetric under all permutations $P_{ij}$ of the bosons at occupied sites. The bosonic dual of, e.g., the specific spin product state $|\uparrow_1 \uparrow_2 \uparrow_3 \uparrow_4 \uparrow_5 \uparrow_6 \cdots \uparrow_N \downarrow_1 \downarrow_2 \cdots \downarrow_{N-1} \downarrow_N\rangle$ corresponds to the symmetrized state of a fixed total number of hard core bosons that are placed on the graph (or lattice) sites $(1, 2, 4, 6, \cdots, (N-1))$. Thus, the bosonic dual of an initial spin state $|\psi_{\text{spin}}^0\rangle$ with a total spin $S_{\text{tot}} = S_{\text{max}} = N/2$ is an initial hard core Bose state $|\psi_{\text{Bose}}^0\rangle$ that is an equal amplitude superposition of all real space product states with the same total number of hard core bosons $(\sum_{i=1}^N n_i = m + \frac{N}{2})$ distributed over the $N$ lattice sites (an eigenstate of $H_{\text{Bose}}$ that adheres to the fully symmetric bosonic statistics). Evolving (during times $0 \leq t \leq t_f$) this initial state with $H_{\text{doping}}$, the standard deviation of Eq. (7) and the distribution of Eq. (9) are left unchanged, apart from a trivial rescaling by $\hbar$ (e.g., $\sigma_\epsilon^2 = \frac{B_z |\sin(B_0 t_f)|}{2\sqrt{2}} \sqrt{1 + \frac{N}{2} - w^2}$ for $S_{\text{tot}} = N/2$). Similar to our discussion of the dual spin system of the previous subsection, the finite standard deviation in this energy density (and of the associated particle density $n = \frac{1}{N} \sum_i n_i$) does not imply that the “doping” is, explicitly, spatially inhomogeneous (indeed, at all times, the expectation value of the particle number $\langle n_i \rangle$ stays uniform for all lattice sites $i$).

We conclude with two weaker statements regarding viable extensions of the rigorous results that we derived thus far for hard core bosonic systems on general graphs (these graphs include lattices in general dimensions).

(a) We may pass from the above lattice theory to a continuum scalar field theory in the usual way. On doing so, it is readily seen that for a continuous scaled $\varphi(x)$ field...
replacing \((b_i + \bar{b}_i)\), the canonical Hamiltonian density

\[
\mathcal{H}[\varphi] = \frac{1}{2} (m^2 \varphi^2 + (\nabla \varphi)^2) + u \varphi^4
\]  

(13)

qualitatively constitutes a lowest order continuum rendition of the hard core Bose lattice model of Eq. (12) for a system with uniform nearest neighbor couplings \(J_{ij}\). A large value of the constant \( u \) in generic bosonic field theories of the type of Eq. (13) yields a large local repulsion between the bosonic fields endowing them with hard core characteristics. The continuum analog of \( H_{\text{Bose-doping}} \) is the volume integral of the momentum conjugate to \( \varphi(x) \). Thus, during various continuous changes of the Hamiltonian, such generic scalar field theories (and myriad lattice system described by them) may exhibit the broad \( \sigma \) that we derived for some of their lattice counterpart in this subsection.

(b) The models of Eqs. (4,12) were defined on arbitrary graphs (including lattices in general spatial dimensions). Identical results apply for spinless fermions on one dimensional chains with non-negative nearest neighbor hopping amplitudes/coupling constants \( \{ J_{ij} \} \) and analogs of \( H_{\text{Bose-doping}} \) capturing a non-local coupling of the system to the external bath. These spinless Fermi systems may be trivially engineered by applying the Jordan-Wigner transformation \([56]\) to Eq. (4).

**VII. THE MAGNUS EXPANSION FOR GENERAL EVOLUTIONS**

To make progress beyond intuitive arguments and specific tractable systems, we next compute the standard deviation of the energy density (and, by trivial extension, any other intensive quantity \( q \)). Towards this end, we employ the Magnus expansion for a general non-adiabatic \([12]\) time dependent Hamiltonian \( H(t) \) (of which the piecewise constant Hamiltonians \( H_{\text{spin}} \) and \( H_{tr} \) (or \( H_{\text{Bose}} \) and \( H_{\text{doping}} \)) are particular instances). Our analysis will demonstrate that in general situations, a finite \( \sigma \) will arise. The general evolution operator, the time ordered exponential \( \mathcal{U}(t) = T \exp(-\frac{i}{\hbar} \int_0^t H(t') dt') \), may be written as \( \mathcal{U} = \exp(\Omega(t)) \) with \( \Omega(t) = \sum_{k=1}^{\infty} \Omega_k(t), \)

\[
\begin{align*}
\Omega_1(t) &= -\frac{i}{\hbar} \int_0^t dt_1 H(t_1), \\
\Omega_2(t) &= -\frac{1}{2\hbar^2} \int_0^t dt_1 \int_0^{t_1} dt_2 [H(t_1), H(t_2)], \\
&\quad \ldots.
\end{align*}
\]  

(14)

For a time independent \( H(t) \), applying the Magnus expansion to a Heisenberg picture operator \( A^H(t) = \mathcal{U}^H \mathcal{A} \mathcal{U}^H \), with \( A \) an arbitrary fixed operator, reproduces the celebrated Baker-Campbell-Hausdorff formula. If no change occurs at intermediate times \( t \) and the Hamiltonian is that of the initial system (i.e., \( H(t) = H \)) then, of course, the standard deviation \( \sigma(t) \) will remain unchanged when computed with the (time independent) equilibrium density matrix for which it trivially vanishes. Similarly, if the evolution of \( H(t) \) is adiabatic at all times then no broadening of the distribution \( P(\epsilon) \) will arise. Our interest, however, lies in the Hamiltonians \( H(t) \neq H \) necessary to elicit a change of the energy density \( dc/dt \neq 0 \) in a macroscopic system. In particular, we wish to examine the variance of the total energy density \( \sigma^2(t) = \frac{1}{\hbar^2} \left( Tr(\rho(H(t))^2) - (Tr(\rho H(t))^2) \right) \) and \( \rho \) the initial density matrix the system (time \( t = 0 \)) when cooling or heating commences. (In the dual examples considered in Section [VI], \( \rho = |\psi_0\rangle\langle\psi_0| \) with \( |\psi_0\rangle \) the initial spin or hard core Bose wavefunction.) Expanding in powers of \( H(t) \),

\[
\sigma^2(t) = \sigma^2(0) + \frac{1}{N^2} \left( \langle [H^2, \Omega_1] \rangle + 2E_0 \langle [\Omega_1, H] \rangle \right) \\
+ \frac{1}{N^2} \left( \langle [\Omega_2 + \frac{\Omega_1^2}{2}, H^2] \rangle - \langle \Omega_1 H^2 \Omega_1 \rangle \right) \\
-2E_0 \langle [\{\Omega_2 + \frac{\Omega_1^2}{2}, H\}], \langle \Omega_1 H\Omega_1 \rangle \rangle + \cdots,\]  

(15)

where \((-\langle \cdot \rangle\) denotes an average computed with \( \rho \) and \( E_0 \) is the initial energy \( (H) \). The brackets \( \langle \cdot \rangle \) represent an anticommutator. If, at all times \( t \), the standard deviation vanishes identically for the heated/cooled system with the time dependent Hamiltonian, then the sum of all terms of a given order in \( H(t) \) in the expansion of Eq. (15) must vanish. In the special case \( \rho = |\phi_n\rangle\langle\phi_n| \) with \( |\phi_n\rangle \) an eigenstate of \( H \), the initial variance of the total energy \( \sigma_E(0) = 0 \) and the sum of the two terms in the first set of brackets of Eq. (15) also vanishes identically. In this case, however, the sums appearing in the higher order terms (the last two lines of Eq. (15) and beyond) do not cancel identically (even if \( H(t) \) is local) for non-adiabatic systems. That a resulting \( \sigma \) cannot appear identically is also evident from our spin and hard core bosonic examples of Section [VI]. The non-vanishing series of Eq. (15) illustrates that when the system starts from an equilibrium state with a sharp energy density \( \sigma(0) = 0 \), then notwithstanding any locality of the Hamiltonian, \( \sigma \) may be \( \mathcal{O}(1) \) at later times \( t \). These rather trivial considerations are not limited to the energy density \( \epsilon \) (similar results hold for any other intensive quantity \( q \)) nor to specific continuum or lattice systems. Thus, broad distributions will generally arise in systems displaying an evolution of their intensive quantities. Of course, constrained solutions to the equation \( \sigma(t) = 0 \), at all times \( t \), may be engineered. Indeed, particular solutions (Appendix [D]) bring to life the intuitive analogy that we made with wave packets (Section [V]) as well as the special character of product states (Section [V]).
VIII. GENERALIZED TWO-HAMILTONIAN UNCERTAINTY RELATIONS

We next turn to a more specific demonstration that, in other settings, \( \sigma_\epsilon > 0 \) when the energy density exhibits a finite rate of change. We consider non-relativistic systems \( S \) of size \( N \) that satisfy certain general conditions:

(1) When combined with their environment (or “heat bath”) \( \mathcal{E} \), these systems constitute a larger isolated closed system \( \mathcal{I} = S \cup \mathcal{E} \).

(2) The fundamental interactions and their sums appearing in the global Hamiltonian \( \hat{H} \) describing \( \mathcal{I} \) (including the subset of these interactions appearing in the Hamiltonian \( H \) describing \( S \)) are time independent.

The two Heisenberg picture Hamiltonians \( H^H(t) = e^{iHt/\hbar} e^{-iHt/\hbar} \) and \( \hat{H}^H = \hat{H} \) describe, respectively, the open system \( S \) and the larger closed system \( \mathcal{I} \) at time \( t \). The energy of the system \( S \) is \( E(t) = \text{Tr}_\mathcal{I}(\hat{\rho}H^H(t)) \) where \( \hat{\rho} \) is the density matrix of \( \mathcal{I} \). By the uncertainty relations \[ (16) \]
\[
\sigma_\epsilon(t)\sigma_{\hat{H}} \geq \frac{1}{2} \left| \text{Tr}_\mathcal{I}(\hat{\rho}\frac{H^H(t)}{N}, \hat{H}) \right|,
\]
with \( \sigma_\epsilon(t) \) the uncertainty associated with \( H^H(t)/N \). Combined with the Heisenberg equations of motion for the time independent \( \hat{H} \) and \( H \) (assumption (2)), we obtain an extension of the time-uncertainty relations for this two Hamiltonian realization,
\[
\sigma_\epsilon(t)\sigma_{\hat{H}} \geq \frac{\hbar}{2N} \left| \frac{dE}{dt} \right|.
\]
Eq. (2) then implies a bound on the product of the average infinite range correlators in the subsystem \( S \) and the full closed system \( \mathcal{I} \),
\[
\langle G \rangle_S \langle G \rangle_{\mathcal{I}} \geq \frac{\hbar^2}{4N^2} \left| \frac{dE}{dt} \right|^2.
\]

The lower bound in Eq. (18) scales as \( \mathcal{O}(N^2) \) if the energy \( E(t) \) of \( S \) changes at a rate proportional to the size of \( S \) (i.e., if the energy density changes at a finite rate). Eqs. (16) will remain valid if condition (1) is relaxed, i.e., if \( \mathcal{I} \) is an open system with a time independent Hamiltonian \( \hat{H} \) that, itself, is in contact with a yet larger system. We will next further impose a slightly more restrictive condition \[ (3) \]

(3) The closed system \( \mathcal{I} \) equilibrates at long times. Stated more precisely (and automatically accounting for Poincare recurrence/Loschmidt echo type events), the asymptotic long time average of the probability density \( \rho_{\mathcal{I}} \) in the larger closed system \( \mathcal{I} \) veers towards the equilibrium density matrix of the closed system (and thus microcanonical (mc) averages apply). That is,
\[
\rho_{\text{mc}\mathcal{I}} = \lim_{T \to \infty} \frac{1}{T} \int_0^T \rho_{\mathcal{I}}(t')dt',
\]
with \( \rho_{\text{mc}\mathcal{I}} \) the microcanonical ensemble density matrix for the closed system \( \mathcal{I} \).

One may readily make pedagogical contact with the spin models of Section VI by providing an explicit example in which conditions (1-3) are satisfied \[ (20) \]. We now briefly and more generally discuss these conditions and their implications. Conditions (1) and (3) are often employed in standard textbook derivations of the canonical ensemble for open systems \( S \) by applying the microcanonical ensemble averages for the larger equilibrated closed systems \( \mathcal{I} \) that include the relevant environments \( \mathcal{E} \) that are in contact (or “entangled”) with \( S \). The initial state of \( \mathcal{I} \) will, generally, not be an eigenstate of \( \hat{H} \) and thus the system will typically be non-stationary. If, as evinced by measurements in prototypical states in the composite system \( \mathcal{I} \) at asymptotically long times, ergodicity and equilibrium set in, then the microcanonical ensemble may be invoked. In particular, in any such case described by the microcanonical ensemble, the uncertainty in the energy of \( \mathcal{I} \) at asymptotically long times will be system size independent,
\[
\sigma_{\hat{H}} = \mathcal{O}(1).
\]
Thus, since all expectation values are stationary under evolution in the larger closed system with the time independent Hamiltonian \( \hat{H} \), a corollary of (1-3) is that the standard deviation \( \sigma_{\hat{H}} \) of the time independent Hamiltonian \( H \), at all times \( t \), is of order unity, i.e., \( \sigma_{\hat{H}} = \mathcal{O}(1) \).

We next turn to the scales of the righthand sides of Eqs. (16) and their consequence for systems that are cooled/heat at finite rate. By Heisenberg’s equation, \( \frac{dT^H}{dt} = \frac{i}{\hbar}[\hat{H}, H^H] \). Therefore, in order to obtain a finite \( dE/dt \) (or an extensive rate \( dE/dt \)), the total Hamiltonian \( \hat{H} \) of the large system \( \mathcal{I} \) must have a commutator with the Hamiltonian \( H \) of \( S \) that is of order \( N \), i.e., \( T_{\mathcal{I}}(\hat{\rho}(\hat{H}, H^H)) = \mathcal{O}(N) \). Hence, to achieve a finite global rate of cooling/heating, \( \hat{H} \) must couple to an extensive number of sites in the volume of \( S \) - it is not possible to obtain an extensive cooling/heating rate by a bounded strength coupling that extends over an infinitesimal fraction of the system size (see also the discussion at the end of Section V). Effectively, a finite fraction of the sites lying in the volume of \( S \) must couple to \( \hat{H} \) whenever \( dE/dt = \mathcal{O}(1) \). In diverse situations (i.e., when cooling/heating leads to a finite rate of change of the system energy density or measured temperature), photons and/or other particles/quasiparticles emitted/absorbed by an extensive volume of the surrounding heat bath effectively couple to the system bulk. (Similar considerations apply to any intensive quantity \( q \) that is changing at a finite rate \( dq/dt \)). In the rotationally symmetric spin model of Section V, a time independent (for all times \( t > 0 \)) transverse field (\( B_y \)) Hamiltonian plays the role of a Hamiltonian \( \hat{H} \) acting on all \( N \) sites (so as to have \( \hat{H}, H^H \) be of order \( \mathcal{O}(N) \)). In the examples of Section V the standard deviation \( \sigma_{\hat{H}} \) in the initial state (at
time \( t = 0^+ \) when the transverse field was applied) was \( \mathcal{O}(\sqrt{N}) \) when evaluated with the wave function on \( S \). If \( \hat{H} \) emulates the effect of a uniform transverse field in \( S \) (and does not change as a result of the interaction with the spins) then memory of the initial state will never be lost. In particular, the correlation functions (even those for sites that are arbitrarily far apart) will periodically assume the same values.

We wish to underscore yet another simple point. The initial state of the system \( S \) prior to its cooling/heating (or variation in its other parameters) may have a well defined energy density \( \epsilon \) and other state variables yet nonetheless still be far from a typical equilibrium state. The experimentalist may introduce various probes, clocks, etc., that start the cooling/heating process in a particular way. Thus, the initial state that the experimentalist prepares need not be in equilibrium but may rather be specially crafted. The external fields that act from \( \mathcal{E} \) on \( S \) will generally lead to an effective time dependent Heisenberg picture Hamiltonian \( H'_{\text{eff}}(t) = T_{\mathcal{E}}(\rho e^{iHt/\hbar} H e^{-iHt/\hbar}) \) (akin to our earlier analysis in the current work). However, if all of the above three conditions are met then at asymptotic long times, memory of the special initial state will be lost and all observables may be computed via the microcanonical ensemble with its few thermodynamic state variables determining all physical properties. In particular, the defining feature of closed equilibrated systems holds: \( \sigma_H = \mathcal{O}(1) \).

With all of the above listed caveats, we now turn to our result. If the energy density varies at a finite rate (i.e., if \( dE/dt = \mathcal{O}(N) \)) then, Eqs. (17,20) imply that the standard deviation of the energy density of \( S \),

\[
\sigma_{\epsilon(t)} = \mathcal{O}(1).
\]  

Thus, we discern from Eq. (2) that long range correlations must appear during the cooling or heating period at which the energy density of the system (\( S \)) is varied. Eq. (21) is also valid for any other intensive quantity \( q \) that is varied at a finite rate.

By a simple “proof by contradiction” argument, we next demonstrate that if we consider an initial open thermal system composite \( \mathcal{I} \) at an assumed temperature \( T \) (instead of condition (1)), then we will obtain a bound on the cooling/heating rate beyond which equilibrium is impossible. The bound that we will present shortly encompasses the physical situation of, e.g., a general uniform medium that is heated or cooled via contact with an external environment. In the latter case, in an experimental setting, the open system \( \mathcal{I} \) (including \( S \)) may be taken to lie deep in the uniform medium (i.e., the open system \( \mathcal{I} \) does not involve the external experimental heat bath (or contacts to it)) that changes the temperature of \( \mathcal{I} \) (and thus of \( S \)). The result that we will derive pertains to what transpires if the subsystem \( S \) and the larger open system \( \mathcal{I} \) containing it are in equilibrium with one another at a temperature \( T \). To derive this result, we will set, in Eq. (17), the equilibrium values of standard deviations of the respective Hamiltonians in the appropriate (fixed temperature or canonical) ensemble describing the open systems \( \mathcal{I} \) and \( S \). That is, we set \( \sigma_H = \sqrt{k_B T^2 C_{v,S}(T)} \) (with \( k_B \) the Boltzmann constant and \( C_{v,S}(T) \) the constant volume heat capacity of the large system composite \( \mathcal{I} \)) to be the standard deviation of the large open system \( \mathcal{I} \), and equate \( \sigma_H = \sqrt{k_B T^2 C_{v,S}(T)} \) (where \( C_{v,S}(T) \) is the heat capacity of the small system at temperature \( T \)) to be the standard deviation of the smaller subsystem \( S \). Repeating, mutatis mutandis, the steps that led to Eq. (21) when \( \mathcal{I} \) was a closed system, we now discover that if the cooling/heating rate exceeds a threshold value for an equilibrated open system \( \mathcal{I} \) (and any subsystem \( S \subset \mathcal{I} \) that is in equilibrium with it),

\[
\left| \frac{dE}{dt} \right| > \frac{2}{\hbar} k_B T^2 \sqrt{C_{v,I}(T) C_{v,S}(T)},
\]  

then a simple contradiction will be obtained. That is, an assumption of having a sharp equilibrium energy density state variable (by coupling \( I \) to a larger external bath at a well defined temperature) \( [67] \) becomes inconsistent once Eq. (22) is satisfied. At sufficiently fast cooling or heating rates (given by Eq. (22)), the inequality of Eq. (17) will be violated when we substitute the equilibrium open system values of \( \sigma_{H/N} \) and \( \sigma_{H_I} \).

Our two results of Eqs. (17,22) for, respectively, the closed and open composites \( \mathcal{I} \) apply for any rate of the energy change \( dE/dt \). These include situations in which \( dE/dt \) scales as the surface area of the system \( \mathcal{O}(N^{(d-1)/d}) \) for which an extension of Eq. (2) will in turn imply that \( |G| \geq \mathcal{O}(N^{-2/d}) \). The central results of Eqs. (21,22) hold for any function \( f(q) \) of an intensive quantity \( q \) that is varied at a finite rate. In particular, setting \( f(q) = q^4 \), we find that the uncertainties in all moments of \( q \) are, typically, finite if the rate \( dq/dt \) is finite. With a formal proof at our disposal, we now briefly reflect back on the arguments of Section IV in which we explained why a varying quantity energy density (or any other intensive quantity \( q \)) with a finite rate of change \( dq/dt \) (or general \( dq/dt \)) naturally suggests an uncertainty. The arguments of Section IV provide an intuitive basis for the time-energy uncertainty type relations that we derived and employed in this section for our two Hamiltonian system and, more generally for other intensive quantities \( q \) elsewhere.

We end this Section by connecting our results concerning uncertainties in intensive quantities to conventional (non-weak [61,64]) quantum measurements. Qualitatively, interactions with the environment might be expected to mimic rapid repeated measurements that collapse the wavefunction and not allow Schrodinger type mixing states of significantly different energies to exist. Such a colloquial “paradox” is somewhat ill formed as we now explain. Continuous measurements by an environment will indeed not enable large uncertainties to appear. However, the putative existence of continuous collapses will also not allow for any change in the energy density or other intensive quantities. The situation is somewhat reminiscent to the well-known “Quantum Zeno Effect"
and its popularized idioms such as that of “a watched pot may never boil”, e.g. [66]. Progressively weaker continuous measurements [61,62] may allow for a more rapid evolution of various quantities hand in hand with larger uncertainties. We will return to discuss adiabatic process, quantum measurements, and thermalization in Section [XII].

IX. DEVIATION FROM EQUILIBRIUM AVERAGES

In the earlier Sections, we demonstrated that fully varying the set of intensive (typical state variable) parameters \( \{ q' \} \) characterizing the eigenstates of \( H \) (such as the energy and particle number densities) at a finite rate generally leads to a widening of the distributions \( P(\{ q' \}) \) of these quantities. In this Section, we wish to underscore that such a widening allows, within a quantum mechanical framework, for a natural departure from equilibrium behaviors. That is, even if the expectation values of general observables in individual eigenstates coincide with equilibrium averages [65,66] and \( H \) has no special many body localized eigenstates [44,42], once a broad distribution \( P(\{ q' \}) \) is present, all averages differ from those in true equilibrium ensembles. This will occur since the broad probability distribution \( P(\{ q' \}) \) describing the driven system is different from the corresponding probability distribution in equilibrium systems (where all intensive quantities have vanishingly small fluctuations); thus the broad distribution \( P(\{ q' \}) \) will give rise to expectation values of typical observables that are different from those found in equilibrium. We write the equilibrium averages of quantities \( \mathcal{O}_c \) that commute with the Hamiltonian \( \{ \mathcal{O}_c, H \} = 0 \) in a general equilibrium ensemble \( \mathcal{W} \) for large systems of arbitrary finite size,

\[
\langle \mathcal{O}_c \rangle_{\text{eq}(q'):\mathcal{W}} = \int dq' P_{\text{eq}(q')}(\{ q' \}) : \mathcal{W} \mathcal{O}_c(\{ q' \}) : \mathcal{W} \rangle \tag{23}
\]

Here, the integration is performed over the full set of intensive variables \( \{ q' \} \) and the function \( P_{\text{eq}(q')}(\{ q' \}) : \mathcal{W} \) denotes the probability distribution in an equilibrium ensemble \( \mathcal{W} \) for which the average of the various quantities \( q = \int dq' (q' P_{\text{eq}(q')}(\{ q' \}) : \mathcal{W} \rangle) \). Lastly, \( \mathcal{O}_c(\{ q' \}) : \mathcal{W} \equiv \langle \phi(\{ q' \}) : \mathcal{W} | \mathcal{O}_c | \phi(\{ q' \}) : \mathcal{W} \rangle \). Augmenting the set of intensive quantities \( \{ q' \} \) defining any of the standard equilibrium ensemble probability distributions, the index \( \mathcal{W} \) may specify any additional quantum numbers (possibly associated with symmetries). For instance, in Ising spin systems, the probability distribution \( P_{\text{eq}(q')}(\{ q' \}) : \mathcal{W} \) may be non-vanishing only for states with a positive magnetization \( \sum_{i=1}^{N} S_i^z \) as it is in these systems at temperatures below the ordering temperatures once time reversal symmetry is spontaneously broken. An essential feature of all systems in equilibrium is that they exhibit well defined thermodynamic state variables \( \{ q' \} \). For instance, as we alluded to in earlier Sections, the energy density exhibits \( O(N^{-1/2}) \) fluctuations in the open systems described by the canonical ensemble while it displays \( O(N^{-1}) \) fluctuations in closed systems described by the microcanonical ensemble. In all equilibrium ensembles, the width \( \sigma_q \) of any intensive quantity \( q \) vanishes as \( N \to \infty \). This sharp delta-function like characteristic of the probability distribution \( P_{\text{eq}(q')}(\{ q' \}) : \mathcal{W} \) is diametrically opposite of \( P(\{ q' \}) \) for which \( \sigma_q \) is finite. Consequently, the expectation value in the driven system \( \langle \mathcal{O}_c \rangle_{\text{driven}} \) during the period in which \( \{ q' \} \) are made to vary with time (that will be given by Eq. (23) with the replacement of the equilibrium probability distribution \( P_{\text{eq}} \) by its non-equilibrium counterpart with \( P(\{ q' \}) \)) will generally differ from the equilibrium average \( \langle \mathcal{O}_c \rangle_{\text{eq}(q'):\mathcal{W}} \).

We now discuss how to relate, computationally, the equilibrium and non-equilibrium expectation values. Because the equilibrium distribution \( P_{\text{eq}(q')}(\{ q' \}) : \mathcal{W} \) is, for large systems sizes, essentially a delta-function in \( \{ q' \} \) (and all additional numbers \( \mathcal{W} \)), we may explicitly write the expectation values in the driven system as

\[
\langle \mathcal{O}_c \rangle_{\text{driven}} = \int dq' P(q'; \mathcal{W}) \mathcal{O}_c(\{ q' \}) : \mathcal{W} \rangle \tag{24}
\]

That is, the expectation values of the observables \( \mathcal{O}_c \) in the driven system may be expressed as weighted sums of the equilibrium averages \( \mathcal{O}_c(\{ q' \}) : \mathcal{W} \) with the weights given by the finite width \( \sigma_q \) distribution \( P(q'; \mathcal{W}) \) that we focused on in the earlier Sections [69]. The equilibrium expectation values \( \langle \mathcal{O}_c \rangle_{\text{eq}(q'):\mathcal{W}} \) of Eq. (23) are experimentally known in many cases. Thus, to predict the expectation values in the driven system, we need to know \( P(q'; \mathcal{W}) \). In Eq. (21), we allowed the probability distribution of the driven system to depend both on the general state variables characterizing the eigenstates of \( H \) along with any additional quantum numbers \( \mathcal{W} \) that might be selected to define various equilibrium ensembles (e.g., the sectors of positive and negative magnetization in low temperature Ising systems or qualitatively similar sectors describing the broken translational and rotational symmetries of an equilibrium low temperature crystal).

We next consider what occurs if driven systems fail to equilibrate at times \( t' > t_f \) (when the parameters \( \{ q \} \) are no longer forcefully varied at a finite rate) and the system is effectively governed by the time independent Hamiltonian \( H \). In such instances, the distribution \( P(\varepsilon') \) of energy densities as measured by the Hamiltonian \( H \) will identically remain unchanged at all times \( t' > t_f \). Towards this end, we remark that, for a system with any fixed time independent Hamiltonian \( H \), the long time average of a general bounded operator \( \mathcal{O} \) (that, unlike \( \mathcal{O}_c \), need not commute with the Hamiltonian) is given by

\[
\mathcal{O}_{\text{t.a.}} = \text{Tr} \left( \frac{\rho(t_f)}{\mathcal{T}} \int_{t_f}^{t_f + \mathcal{T}} dt' \mathcal{O}^H(t') \right). \tag{25}
\]

Here, \( \rho(t_f) \) the density matrix at the final time \( t_f \) at which the system is driven, the Heisenberg picture \( \mathcal{O}^H(t') \equiv e^{iH(t'-t_f)}/\hbar \mathcal{O} e^{-iH(t'-t_f)}/\hbar \) and \( \mathcal{T} \) is...
the said long averaging time. Now, by the Heisenberg equations of motion, for bounded operators \( \mathcal{O} \), as \( T \to \infty \), the commutator \( [H, \frac{1}{T} \int_{t_f}^{t_f+T} dt' \mathcal{O}^H(t')] = -\frac{i\hbar}{T} \int_{t_f}^{t_f+T} dt' \mathcal{O}^H(t') = -\frac{i\hbar}{T} (\mathcal{O}^H(t_f+T) - \mathcal{O}^H(t_f)) = 0 \).

In other words, \( \mathcal{O}_{l.t.a.} \) is trivially diagonal in the eigenbasis of the Hamiltonian \([23]\). Since \( \mathcal{O}_{l.t.a.} \) commutes with the Hamiltonian, we may apply Eqs. (23,24). In particular, with the substitution \( \mathcal{O}_c = \mathcal{O}_{l.t.a.} \), Eq. (24) will provide the long time averages of arbitrary observables \( \mathcal{O} \). For any ergodic system in equilibrium, the thermal average of the operator of any long time average \( \mathcal{O}_{l.t.a.} \) of Eq. (25) is the equilibrium average. Substituting in Eq. (24), one thus explicitly has

\[
\mathcal{O}_{l.t.a.} = \int dq' P(q'; W) (\langle \mathcal{O} \rangle_{eq}; q') ; W \quad (26)
\]

Along somewhat different lines, a similar conclusion was drawn in \([11]\). For bounded operators \( \mathcal{O} \) and finite, yet very large, averaging times \( T \), the long time average \( \mathcal{O}_{l.t.a.} \) will only have minute off-diagonal corrections (and thus only very small deviations from Eq. (26) may arise). In the next Section, we discuss whether the initially driven system effectively saturates, for such very long finite times \( T \), to a distribution \( P(c') \) that relative to time independent Hamiltonian \( H \) exhibits a vanishingly narrow \( (\sigma_c = 0 \text{ as in equilibrium systems}) \) or to a finite width \( (\sigma_c \neq 0) \) distribution.

X. “TO THERMALIZE OR TO NOT THERMALIZE”

Our focus thus far has largely been on intermediate times \( 0 \leq t \leq t_f \) during which the energy density (or any other intensive quantity \( q \)) varied. We showed that during these times, the standard deviation of \( q \) may be finite, \( \sigma_q = \mathcal{O}(1) \). Thus, the variation of general quantities \( q \) (including, notably, the energy density or temperature) may trigger long range correlations. As we explained towards the end of Section V this effect may be further exacerbated by “non-self-averaging” \([16][19]\) in classical theories of disordered media. Our inequalities of Eqs. \([21][22]\) hold for general fluctuations (regardless of the magnitude of their “classical” and “quantum” contributions \([15]\) to the variance). In most systems coupled to an external bath, after the temperature or field no longer changes (e.g., when \( |d\epsilon/dt| \) vanishes at times \( t > t_f \) thermalization rapidly ensues already at short times after \( t_f \). Indeed, there are arguments (including certain rigorous results) that “typical” states \([70]\) might thermalize on times set by Planck’s constant and the temperature, viz. the “Planckian” time scale \( \mathcal{O}(\frac{\hbar}{\epsilon}) \). Generally, other, exceedingly short (as well as long), equilibration time scales may be present \([71]\). Planckian time scales appear in a host of interacting systems, e.g., \([72][77]\). Various reaction rates are often given by such minimal Planckian time scales multiplied by \( e^{\Delta G/(k_BT)} \) with \( \Delta G \) the effective Gibbs free energy barrier for the reaction or relaxation to occur, e.g., \([77][78]\). However, some systems such as glasses do not achieve true equilibrium: measurements on viable experimental time scales differ from the predictions of the microcanonical or canonical ensemble averages. (The difference between the microcanonical and canonical ensembles is irrelevant for all intensive quantities in the absence of long range interactions for which “ensemble inequivalence” is known to appear \([79][82]\).) In such cases, the system may, as a result of the rapid cooling, effectively exhibit self-generated disorder. As is well known, structural glasses are disordered relative to their truly thermalized crystalline counterparts. It is important to stress, however, that both structural glasses and crystalline solids are governed by the same (disorder free) Hamiltonian. The effective disorder that glasses exhibit is not intrinsic but merely self-generated by the rapid supercooling protocol of non-disordered liquids. Thus, as hinted in Section VIII, the question remains as to whether, once the energy density or other intensive quantity no longer varies (e.g., once the glass is formed and its temperature is no longer lowered), the system will thermalize on experimental time scales (and display the rightmost distribution of Figure 1 or not be able to do so. Similar to our assumption (3) in Section VIII, starting from a glassy state, supercooled liquids will thermalize and may crystallize only at asymptotically long times \([88]\). In systems that do not thermalize on experimental time scales, the discrepancy between equilibrium ensemble averages and empirical observables hints that the width \( \sigma_c \) of the energy density might become smaller than it was during the cooling process yet is not vanishingly small. Indeed, if \( \sigma_c = 0 \) and no special “many body localized” states \([34][30]\) exist (especially in the most relevant physical situation of more than one spatial dimension \([41][42]\) then the long time averages of all observables must be equal their microcanonical expectation values. Specifically, similar to Eq. (20), the time average of a general quantity \( \mathcal{O} \) over a long (finite) time \( T \) during which the probability distribution \( P(q'; \mathcal{W}) \) is nearly stationary will be identical to the equilibrium average, i.e., \( \mathcal{O}_{l.t.a.} = \langle \mathcal{O} \rangle_{eq}; \epsilon, q' ; \mathcal{W} \) when the distribution \( P(q'; \mathcal{W}) \) is of a delta-function type nature in the energy density \( \epsilon \) and all other intensive quantities \( q \). If the expectation values of the thermodynamic equilibrium observables depend on the temperature or energy density (and are the same for all states related by symmetries of the Hamiltonian) then a deviation of the long time average values of observables \( \mathcal{O} \) from their true equilibrium average values,

\[
\mathcal{O}_{l.t.a.} \neq \langle \mathcal{O} \rangle_{eq}; \epsilon, q' , \mathcal{W} \quad (27)
\]

will imply that the width of the energy density may remain finite even after the system is no longer driven, \( \sigma_c > 0 \). In glassy systems that, by their defining character, cannot achieve true equilibrium (and thus satisfy Eq. (27) on relevant experimental time scales, the link to the external bath is effectively excised since the dy-
FIG. 2. (Color Online.) Reproduced from [89]. On the vertical axis, we plot the experimentally measured viscosity data divided by its value at the liquidus temperature, \( \eta(T_{\text{L}}) \), as a function of \( x \), as defined along the abscissa. The viscosities of 45 liquids of diverse classes/bonding types (metallic, silicate, organic, and others) collapse on a single curve. The underlying continuous “curve” (more clearly visible at high viscosities end where fewer data are present) is predicted by Eq. (29). Since \( A \) varies from fluid to fluid (albeit weakly [89]), the shown collapse does not imply a corresponding collapse of the viscosity as a function of \( T_{\text{L}}/T \) nor as a function of \( T_{\text{L}}/(\Delta T) \) (due, relative to the latter, to an additional shift along the \( x \) axis that is set by \( -1/(\sqrt{2}A) \)).

namics are so slow that little flow may appear. If, in such instances, the probability density becomes time independent on measurable time scales then only an effective equilibrium (different from the true equilibrium defined by an equilibrium ensemble for the Hamiltonian defining the system) may be reached. That is, in systems with an effective equilibrium, at sufficiently long times, the probability density \( P(\epsilon') \) may be history independent and be a function of only a few global state variables yet differ from the conventional equilibrium statistical mechanics probability density in which the standard deviations of all intensive quantities vanish, e.g., \( \sigma_{\epsilon} = 0 \). Since the probability density determines all observable properties of the system, interdependences between the state variables (i.e., equations of state) may result [11]. Such a nearly static effective long time equilibrium distribution bears some resemblance to “prethermalization” in perturbed, nearly-integrable, models and other systems, e.g., [83–87]. We remark that by applying the Mastubara-Matsuda transformation [54] (similar to that invoked in Section VI B), we may map the prethermalized three-dimensional spiral spin states of [57] to establish the existence of long-lived effective equilibrium crystals of hard-core bosons. As stated above, at asymptotic long times, systems such as glasses finally truly thermalize to true equilibrium solids [88]. However, prior to reaching the true equilibrium defined by any of the canonical ensembles for the full system Hamiltonian, over very long finite times, the supercooled liquid/glass may display a nearly static distribution \( P \) and thus obeys equations of state, absence of memory effects and other hallmarks of effective equilibrium. In [11], we introduced this notion of an effective long time distribution \( P \) of finite \( \sigma_{\epsilon} \) and employed it to predict the viscosity of all glass formers. This prediction was later tested [89] for the published viscosity data of all known glass formers when they are supercooled below their melting temperature. A related classical theory was further advanced. Figure 2 reproduces the result. Here, the (finite) energy density width was set to be

\[
\sigma_{\epsilon} = \frac{A T (\epsilon_{\text{melt}} - \epsilon)}{T_{\text{melt}} - T}.
\]

Here, \( A > 0 \) is a liquid dependent constant \((0.05 \lesssim A \lesssim 0.12)\) for all liquids with published viscosity data [89]. The energy densities \( \epsilon \) and \( \epsilon_{\text{melt}} \) in Eq. (29) are, respectively, those of the supercooled liquid or glass at temperature \( T < T_{\text{melt}} \) and at the melting (or “liquidus”) temperature \( T_{\text{melt}} \). The wide distribution of Eq. (28) mirrors that in non-self-averaging disordered classical systems with an approximately linear in \( T \) standard deviation and energy density \( \epsilon(T) \). (In quantum systems, all
eigenstates of the density matrix may share the same energy while displaying a finite standard deviation $\sigma$. We briefly make further contact between our finite width distributions (Eq. \([8]\)) for the energy density with usual thermodynamics. In the models that we investigated in Section \([VI]\) (with the distribution of Eq. \([9]\) that was far from the canonical normal form of equilibrium systems), the systems were driven by an external source whose effect on general quantities was cyclic in time. The situation may be radically different when the system is no longer forcefully driven out of equilibrium yet is still unable to fully equilibrate. In such instances, in the language of Section \([IV]\) the energy density distribution ($P(\epsilon')$) is no longer made to have its average go up or down with time (i.e., the system is not heated or cooled). The distribution may alter and narrow its profile as the system attempts to veer towards equilibrium. If, as in equilibrium thermodynamics, the final state maximizes the Shannon entropy given a particular final energy then the resulting distribution for the energy density will be a Gaussian of width $\sigma = \frac{\Delta E}{N}$. Since that heat capacity is extensive, $C_v = O(N)$, the standard $\frac{1}{\sqrt{\sigma}}$ fluctuations result in this equilibrium setting. For systems of measured temperature $T$ that have not fully equilibrated, we may (as we have illustrated in some detail in the earlier Sections) find finite width energy distributions $P_T(\epsilon')$ for the energy. If these distributions minimally differ in form from those in equilibrium then they may, similar to those in equilibrium systems, be normal distributions of width $\sigma \propto T$. The general distribution that maximizes the Shannon entropy given that it has a given finite standard deviation $\sigma$ is a Gaussian. The sole difference between the known Gaussian distribution of the energy density in equilibrium systems and that assumed for systems that have reached the final temperature energy density (or corresponding temperature $T$) yet have not achieved equilibrium is that in the latter systems $\sigma = O(1)$ (while $\sigma = 0$ for equilibrium systems in their thermodynamic limit.) Non-rigorous considerations further suggest a Gaussian distribution once the system is no longer further cooled (or heated), see Appendix \([E]\) as \([90]\) \([91]\). Assuming a normal distribution $P_T(\epsilon')$ that is of width $\sigma$, the viscosity $\eta$ of supercooled liquids at temperatures $T \leq T_{\text{melt}}$ was predicted to be \([11]\).

$$\eta(T) = \frac{\eta_{s.c.}(T_{\text{melt}})}{erfc\left(\frac{t_{\text{melt}} - T}{\sigma\sqrt{2}}\right)} = \frac{\eta_{s.c.}(T_{\text{melt}})}{erfc\left(\frac{T_{\text{melt}} - T}{\Delta T \sqrt{2}}\right)}.$$  \([29]\)

Eq. \([29]\) is a direct consequence of Eqs. \([24]\) \([25]\). This prediction is indicated by the continuous curve in Figure \([2]\). At the so-called “glass transition temperature” $T_g$, the viscosity $\eta(T_g) = 10^{12}$ Pascal $\times$ second \([22]\). At lower temperatures $T < T_g$, the viscosity is so large that it is hard to measure it on experimental time scales. The very same distribution $P_T(\epsilon')$ invoked in deriving Eq. \([29]\) may relate other properties of supercooled liquids and glasses to those of equilibrium systems. For instance, the experimentally measured thermal emission from supercooled fluids may differ in a subtle manner from one that is typical of equilibrium fluids. This deviation may be found by replacing Planck’s law for the spectral radiance $I$ for photons of frequency $\nu$ in a system with well defined equilibrium temperature $T$ by a weighted average of Planck’s law over effective equilibrium temperatures $T'$ that are associated with internal energy densities of equilibrium systems that are equal to $\epsilon'$.

$$I(\nu, T) = \frac{2h\nu^3}{c^2} \int dT' \frac{\hat{P}_T(T')}{e^{h\nu/(k_B T') - 1}} + I_{\text{PTEI}}(\nu, T).$$

Here, $\hat{P}_T(T') = P_T(\epsilon')c^3$ is the distribution of equilibrium temperatures $T'$ associated with the probability distribution $P_T(\epsilon')$ of the energy densities. The second term, $I_{\text{PTEI}} = \frac{2h\nu}{c^2} \int dT' \frac{d\epsilon' P_T(\epsilon')}{e^{h\nu/(k_B T') - 1}}$, captures contributions from any “Phase Transition Energy Interval” (wherein the energy density $\epsilon'$ of an equilibrium system may vary by an amount set by the latent heat without concomitant changes in the corresponding equilibrium temperature $T'$). We highlight that this prediction for the emission spectrum $I(\nu, T)$ is determined by the same distribution predicting the viscosity collapses of Fig. \([2]\) (the Gaussian $P_T(\epsilon')$ of the width given by Eq. \([28]\)). As such, this prediction may, in principle, be experimentally tested. Similarly, the temperature dependence of other observables (including various response functions) may be expected to have the same increase in the time scale as that characterizing the viscosity. Indeed, the time dependent heat capacity response follows exhibits a dynamical time that increases with temperature in a manner similar to the viscosity, e.g., \([23]\). We speculate that this increase in the relaxation time scale as the temperature is dropped may account very naturally for the experimentally observed static specific heat jump \([24]\) near the glass transition temperature $T_g$. This is so since, at temperatures $T \leq T_g$, on the time scales of the experiment, the system is essentially static (e.g., the viscosity of the Eq. \([29]\) and the associated measured relaxation times are large). Consequently, the relatively stable nearly static structures that appear once the glass is formed need not significantly respond to a small amount of external heat. The situation is somewhat reminiscent of the extensive latent heat that is required to melt equilibrium crystals. Pronounced thermodynamic changes appear at the transition between equilibrium fluids and crystals. Once the supercooled liquid or glass becomes effectively static at $T_g$, it may weakly emulate the latent heat signature of the equilibrium liquid to solid transition.

An energy density distribution of a finite width $\sigma$ also allows for a superposition of low energy density solid type eigenstates (that may break continuous translational and rotational symmetries) and higher energy density liquid type eigenstates \([11]\). Such a general combination of eigenstates does not imply experimentally discernible equilibrium solid (crystalline) order. Sharp Bragg peaks
need not appear in states formed by superposing eigenstates that, individually, display order. This absence of ordering mirrors the possible lack of clear structure when, e.g., randomly superposing different Fourier modes with each Fourier mode displaying its defining periodic order. The mixing of eigenstates of different energy densities over a range set by $\sigma_c$ further suggests the appearance of non-uniform local dynamics. Indeed, dynamical heterogeneities appear in supercooled fluids \cite{103,107}; these large fluctuations remain even when the fluids remain in contact with an external bath for a long time. A simple calculation indeed illustrates that the long time fluctuations in the local energy density given a general initial state relate to the width of this state in the eigenbasis of the Hamiltonian governing the system \cite{104}. We remark that a spatially non-uniform energy density is very natural during the heating or cooling process of general system (e.g., the exterior parts of a system being supercooled may be colder than its interior, see also the discussion towards the end of Section \text{IV}). Once supercooling stops, heat may diffuse through the system yet heterogeneities (that are borne in our framework from a distribution of finite $\sigma_c$) may still persist for a long time.

\text{XI. POSSIBLE EXTENSIONS TO ELECTRONIC AND LATTICE SYSTEMS}

The spin and hard core Bose models of Section \text{VI} are lattice systems. We now further discuss possible extensions to other, more common, theories and lattice systems. The electronic properties of many materials are well described by Landau Fermi Liquid Theory \cite{101–104}. This theory is centered on the premise of well-defined quasiparticles leading to universal predictions. Recent decades have seen the discovery of various unconventional materials displaying rich phases \cite{101,105,124} that often defy Fermi liquid theory. Given the results of the earlier Sections, it is natural to posit that as these systems are prepared by doping or the application of external pressure and fields (in which case the varied parameter $q$ may be the carrier density, specific volume, or magnetization), a widening $\sigma_q$ will appear during the process. This wide distribution might persist also once the samples are no longer experimentally altered. In such cases, the density matrices (and associated response functions) describing these systems may exhibit finite standard deviations $\sigma_q > 0$. The broad distribution may trigger deviations from the conventional behaviors found in systems having sharp energy and number densities ($\sigma_n = 0$) or, equivalently, sharp chemical potentials and other intensive quantities. Theoretically, non-Fermi liquid behavior may be generated by effectively superposing different density Fermi liquids (with each Fermi liquid having a sharp carrier concentration $n$) in an entangled state. Systems harboring such an effective distribution $P(\mu)$ of chemical potentials may be described by a mixture of Fermi liquids of different particle densities.

Any non-anomalous Green’s function is manifestly diagonal in the total particle number. Thus, the value of any such Green’s function may be computed in each sector of fixed particle number and then subsequently averaged over the distribution of total particle numbers in order to determine its expected value when $\sigma_n \neq 0$. In particular, this implies that the conventional jump (set by the quasiparticle weight $Z_{\vec{k},\mu'}$) of the momentum space occupancy \cite{101–104}, in the coherent part of the Green’s function ($G = G_{\text{coh}} + G_{\text{incoh}}$) will be “smeread out” when $\sigma_n \neq 0$. Similar to Eq. \text{(24)}, a distribution of chemical potentials will lead to the replacement of the coherent Green’s function of ordinary Fermi liquids by

$$G_{\text{coh}}(\vec{k},\omega) = \int d\mu' P(\mu') \frac{Z_{\vec{k},\mu'}}{\omega - \epsilon_{\vec{k}} + \mu' + i/\tau_{\vec{k},\mu'}}. \quad (30)$$

Here, $\tau_{\vec{k},\mu'}$ is the quasi-particle lifetime in a system with sharp $\mu'$ at wave-vector $\vec{k}$. The denominator in Eq. \text{(30)} corresponds to the coherent part of the Green’s function of a Fermi liquid of a particular chemical potential $\mu'$ and quasi-particle weight $Z = 1$ \cite{101–104}. Qualitatively, Eq. \text{(30)} is consistent with indications of the very poor Fermi liquid type behavior reported in \cite{126}. The effective shift of the chemical potential in Eq. \text{(30)} is equivalent to a change in the frequency dependence while holding the chemical potential $\mu$ fixed; the resulting nontrivial dependence of the correlation function on the frequency (with little corresponding additional change in the momentum) is, qualitatively similar to that advanced by theories of “local Fermi liquids”, e.g., \cite{101,125}. Our considerations suggest a similar smearing with the distribution $P(\mu')$ will appear for any quantity (other than the Green’s function of Eq. \text{(30)}) that is diagonal in the particle number. Analogous results will appear for a distribution of other intensive quantities. The prediction of Eq. \text{(30)} (and similar others \cite{11} in different arenas) may be tested to see whether a single consistent probability distribution function $P$ accounts for multiple observables. General identities relate expectation values in interacting Fermi systems to a weighted average of the same expectation values in free fermionic systems \cite{127}. These relations raise the possibility of further related smeared averages, akin to those in Eq. \text{(30)}, in numerous systems.

Numerically, in various models of electronic systems that display non-Fermi liquid type behaviors, the energy density differences between contending low energy states $\{\left|\psi^{\alpha}\right\}$ (not necessarily exact eigenstates) are often exceedingly small, e.g., \cite{128}. Since these states globally appear to be very different from one another, the matrix element of any local Hamiltonian between any two such orthogonal states vanishes, $\langle \psi^\beta | H | \psi^\alpha \rangle = 0$ for $\alpha \neq \beta$. We notice that, given these results, arbitrary superpositions of these nearly degenerate states, $\sum a_n | \psi^{\alpha} \rangle$, will also have similar energies. Thus, for various many body Hamiltonians modeling these systems, a superposition of different eigenstates may also be natural from energetic considerations.
Towards the end of Section \textbf{X} we remarked on the viable disordered character of the states formed by superposing eigenstates that break continuous symmetries. We now briefly speculate on the corresponding situation for eigenstates in electronic lattice systems that break discrete point group symmetries on a fixed size unit cell. Here, due to the existence of a finite unit cell in reciprocal space, a superposition of eigenstates that are related to each other by a finite number of discrete symmetry operations may not eradicate all Bragg weights. In other words, order may partially persist when superposing states on the lattice that, individually, display different distinct structures.

**XII. ADIABATIC PROCESSES, THERMALIZATION, AND QUANTUM MEASUREMENTS**

As demonstrated in the current work, rapidly driven systems may exhibit uncertainties in their energy and/or other densities. We now close our circle of ideas and focus on the diametrically opposite case of slow adiabatic processes (for which, obviously, $dq/dt = 0$). We will further speculate on relations concerning thermalization that superficially emulate those of quantum measurements.

A basic tenet of quantum mechanics is that a measurement will project or "collapse" a measured system onto an eigenstate of the operator being measured. A natural question to ask is whether such effective projections may merely emerge as a consequence of an effective very rapid thermalization of microscopic systems. To motivate this query and more generally examine effectively adiabatic processes, we consider the Hamiltonian

$$H_{A∪B}(t) = H_A + H_{AB}(t) + H_B \quad (31)$$

describing the combined system of two systems and the coupling between them ($H_{AB}$). We first consider $H_{AB}(t)$ that changes adiabatically from zero. Consider the situation wherein, initially, at times $t \leq 0$, systems $A$ and $B$ were in respective eigenstates $|\phi_{n_A}\rangle$ and $|\phi_{n_B}\rangle$ of $H_A$ and $H_B$. That is, at times $t \leq 0$, the state of the combined system $A∪B$ was described by the product state of these two eigenstates. We further assume that at times $t < 0$, the coupling $H_{AB}(t) = 0$ and for times $t \geq 0$ an adiabatic change of $H_{AB}(t)$ ensues. Under these conditions, by the adiabatic theorem, at any later time $t$, the initial state has evolved into a particular eigenstate $|\phi_{n_{AB}}(t)\rangle$ of $H_{A∪B}(t)$, where we have $|\phi_{n_A}\rangle|\phi_{n_B}\rangle \rightarrow |\phi_{n_{AB}}(t)\rangle$.

We may expand the density matrices $\rho_{A∪B}$ of the initial system $A$ and $B$ in terms of the eigenvectors of $H_A$ and $H_B$. Explicitly expressing the density matrix $\rho_{A∪B}(t)$ of the combined system at time $t$ in the eigenbasis of $H_{\text{measure}}(t)$, i.e., in the evolution from $t = 0$ to a time $t > 0$, the density matrix trivially evolves as

$$\sum_{n_{A∪B}} \rho_{n_{A∪B}}|\phi_{n_A}\rangle|\phi_{n_B}\rangle\langle\phi_{n_A}|\langle\phi_{n_B}|$$

$$\rightarrow \sum_{n_{A∪B}} \rho_{n_{A∪B}}|\phi_{n_{AB}}(t)\rangle\langle\phi_{n_{AB}}(t)|. \quad (32)$$

Thus, if initially both systems $A$ and $B$ started from equilibrium (and thus had sharp energy densities—i.e., if at time $t = 0$ the eigenstates of $H_A$ and $H_B$ that had significant amplitude were clustered around a given energy density) then an adiabatic evolution of $H_{AB}(t)$ will yield a density matrix $\rho_{A∪B}$ that also displays sharp energy density, $\sigma_{\epsilon_{A∪B}}(t) = 0$. Thus, the notion that sufficiently slow processes enable systems to remain in equilibrium is indeed consistent with the adiabatic theorem of quantum mechanics.

We next comment on how such adiabatic processes (and later briefly discuss more general thermalization events that need not be adiabatic) may superficially emulate certain features of a wavefunction collapse. Towards that end, we consider the extreme case of a microscopic system $A$ ("being measured") and a macroscopic system $B$ that we may regard as an environment that includes a coupling to an experimental probe at the measurement time $t_{\text{measure}}$. As earlier, for a general adiabatic evolution, $|\phi_{A∪B}(0)\rangle \rightarrow |\phi_{n_{AB}}(t_{\text{measure}})\rangle$. We now allow the coupling $H_{AB}(t)$ to be non-vanishing at all times $t$ (i.e., also including times $t \leq 0$) and, due to its ease, first briefly discuss the case when its evolution is adiabatic.

Under these circumstances, by the adiabatic theorem, $|\phi_{A∪B}(t_{\text{measure}})\rangle$ must be an eigenstate of $H_{A∪B}(t_{\text{measure}})$. Thus, such an adiabatic evolution emulates an effective "collapse" onto an eigenstate of the Hamiltonian that measures the state of the microscopic system $A$. We emphasize that the state $|\phi_{A∪B}(t_{\text{measure}})\rangle$, describing both the microscopic system $A$ and the large system $B$, will be in an eigenstate of $H_{A∪B}(t_{\text{measure}})$—i.e., not only the small system $A$ will be altered by the measurement. While, at any time $t$, the state $|\phi_{A∪B}(t)\rangle$ is an eigenstate of $H_{A∪B}(t_{\text{measure}})$, its highly entangled content largely remains unknown. It may be noted that certain "realistic" setups involving quantum measurements often entail higher energy "thermal" states of the measurement device (e.g., the reaction between silver ions and the screen that they strike). The collapsed system is in an excited state.

The effective "collapse" brought about by such an adiabatic process may be nearly immediate for microscopic systems $A$. Typical lower bounds on time scales for adiabatic processes defined by an energy difference $\Delta E$ are set by $\hbar/\Delta E$ (for precise bounds see, e.g., [133]). Such scales are consistent with the uncertainty relations and our bounds of Section \textbf{VIII}. For small energy splittings $\Delta E$, this adiabatic time scale may become large. The above discussion of a hypothetical adiabatic evolution is merely illustrative. A potentially more realistic question concerning realistic $H_{AB}(t)$ is that of the thermalization
of the full system. At room temperature, the “Planckian time” scale for the equilibrium thermalization of random initial states \( \tau_0 \) is \( \hbar/(k_B T) \sim 10^{-13} \) seconds (e.g., the typical period of a thermal photon). The latter time scale may be smaller than that required for an adiabatic evolution yet is still finite; one may attempt to probe for such an effective finite time collapse borne by thermalization (cf., any such deviations from the textbook “instantaneous collapse”) only at extremely low temperatures. The very rapid thermalization evolution suggested here allows for multiple measurement outcomes with different probabilities. A measurement provides only partial information on the many-body entangled state \( |\phi_{A:B}(t)\rangle \) formed by \( A \) and \( B \); it does not specify it. Conditional probabilities may be assigned to the possible future evolutions of this entangled state (and thus of future measurement outcomes thereof). In this regard, our suggestion concerning thermalization is somewhat similar to existing frameworks including “Quantum Bayesianism” \cite{134} and others relying on entanglement, e.g., \cite{135}.

We close this Section by further noting that parallels between equilibration and certain features of an effective collapse in quantum measurements are partially motivated by the eigenstate thermalization hypothesis \cite{25–33}. When valid, this hypothesis allows us to effectively equate the results of local measurements of general observables \( O \) in pure eigenstates \( \{ |\phi_n\rangle \} \) (of energies \( \{ E_n \} \)) of a general Hamiltonian \( H \) (including Hamiltonians describing a coupling between a measurement device and a microscopic system) with expectation values in equilibrated thermal systems defined by the full system Hamiltonian \( H \),

\[
\langle \phi_n | O | \phi_n \rangle = Tr(\rho(E_n)O).
\]

(33)

Here, \( \rho(E_n) \) is an equilibrium density matrix associated with the energy \( E = E_n \) (and, when applicable, any other conserved quantities defining the state \( |\phi_n\rangle \) and the thermal system). Taken to the extreme, this putative equivalence suggests that we may interchange (i) quantum measurement processes with (ii) outcomes of “canonical” equilibration. More general than adiabatic processes alone, thermalization shares other commonalities with quantum measurements. Just as a real quantum measurement (and ensuing collapse) is, generally, not a time reversal invariant operation \cite{130} so, too, is a typical finite \( T \) thermalization process in a highly entangled many body system. The second law of thermodynamics is consistent with an evolution of the entangled \( A:B \) system that displays a non-decreasing entropy when performing consecutive measurements (compatible with indeterminate outcomes for other subsequent measurements thereafter).

XIII. CONCLUSIONS

We illustrated that a finite rate variation of general intensive quantities can lead to non-local correlations. In the simplest variant of our results, in experimental systems with varying intensive observables \( q \) (such as the energy density \( \epsilon \) for which \( dq/\epsilon = \mathcal{O}(1) \), the associated average connected two site correlation functions do not vanish even for sites are arbitrarily far apart. Trivial extensions of this result also hold for weaker variation of the intensive quantities. For instance, if only narrow range couplings to the environment appear and, consequently, the rate \( dq/\epsilon = \mathcal{O}(N^{-1/d}) \) then the average absolute value of the connected two point correlation function for an arbitrary pair \( (i,j) \) of far separated sites will be asymptotically bounded as \(|G| \geq \mathcal{O}(N^{-2/d}) \).

The general non-local correlations that we found originate from the macroscopic entanglement present in typical thermal states. Our results highlight that, even in seemingly trivial thermal systems, one cannot dismiss the existence of long range correlations. Our analysis of non-equilibrium systems does not appeal to conventional coarsening and spinodal decomposition phenomena (although the departure from a spatially uniform true equilibrium state in spinodal systems is very naturally consistent with a distribution of low energy solid like and higher energy fluid like states). Cold atom systems may provide a controlled testbed for our approach. We speculate that our results may also appear in naturally occurring non-equilibrium systems. The peculiar effect that we find may rationalize the unconventional behaviors of glasses and supercooled fluids. Our effect might further appear in electronic systems that do not feature Fermi liquid behavior. Here, a broad distribution of effective energy densities and/or chemical potentials may appear. The validity of weighted averages such as that of Eq. (33) may be assessed by examining whether a unique distribution \( P \) simultaneously accounts for all measurable quantities. The long range correlations that we derived do not hinge on the existence of non-local interactions; the examples that we studied in Section \( \text{VI} \) embodied quintessential local interactions. In Section \( \text{XIII} \) we illustrated how adiabatic processes maintain sharp thermodynamic quantities and speculated that a nearly instantaneous equilibration of small systems with macroscopic ones may emulate certain features of quantum measurements. We hope our results will be further pursued in light of their transparent mathematical generality and ability to suggest new experimental behaviors (e.g., the universal viscosity collapse of supercooled liquids that it predicted and is indeed empirically obeyed over sixteen decades as seen in Figure 2).

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Appendix A: Entangled Ising chain eigenstate expectation values produce thermal averages

In order to explicitly illustrate how macroscopic entanglement may naturally appear in typical thermal states (even those of closed systems that have no explicit contact with an external bath), we turn to a simple example: that of the uniform coupling one dimensional Ising model (the Hamiltonian \(H_I\) of Section \(\text{[V]}\) on an open chain with uniform nearest neighbor coupling- \(J_{ij} = J\). In these appendices, we will dispense with factors of \(\hbar/2\) and use the conventional definition of the Ising model Hamiltonian with the spin at any site \(r\) being \(S^z_r = \pm 1\) (i.e., the diagonal elements of the Pauli matrix \(\sigma^z\)). In each Ising state product state, the value of \(\langle S^z_r S^z_{r'} \rangle\) is either 1 or -1. This single Ising product state expectation value differs from that of the equilibrium system at finite temperatures. It is only if we compute the expectation value within a state formed by a superposition of many such product states (i.e., an expectation value within such a highly entangled state) or if we average under uniform translations of the origin (i.e., entangle with equal weights all states related by translation) that we will obtain the equilibrium result.

The Ising operators \(S^z_r\) are diagonal in the product basis; different product states are orthogonal to each other. In a superposition of different product states, only the diagonal (i.e., weighted Ising product expectation values) terms are of importance when computing \(\langle S^z_r S^z_{r'} \rangle\).

We consider a highly entangled eigenstate \(|\Psi\rangle\) of the one-dimensional Ising model. Such an entangled state emulates, in real space, entangled eigenstates \(|\psi_\mu; S_{\text{tot}}, S^z_{\text{tot}}\rangle\) with (for systems in their thermodynamic limit) \(|S^z_{\text{tot}}/S_{\text{max}}| < 1\) (i.e., not product states of all spins maximally polarized up or down along the field direction) of the spin models discussed in Section \(\text{[V]}\). For an Ising model \(H_I\) on a one dimensional chain of length \(L\), given an eigenstate of energy \(E\), the frequency of low energy nearest neighbor bonds (namely, \(S^z_i = S^z_{i+1} = \pm 1\) (\(\uparrow\uparrow\) or \(\downarrow\downarrow\))) is \(p\) and that of having higher energy bonds (i.e., \(\uparrow\downarrow\) or \(\downarrow\uparrow\)) is \(q\). Clearly, \(p + q = 1\) and \((q - p) = E/(LJ)\) where \(J\) is the Ising model exchange constant and \(E\) is the total energy. In the one dimensional Ising model there is no constraint on the nearest neighbor bonds \(S^z_i S^z_{i+1}\) (these products are all independent variables that are \(+1\) or “-1” that sum to the scaled total energy \(E/J\)). Consider a spin at site \(r\) which is, say, \(\uparrow\uparrow\). We may now ask what is the average value of a spin at another site \(r'\). Evidently, if there is an even number of domain walls (or even number of energetic bonds) between sites \(r\) and \(r'\) then the spin at site \(r'\) is \(\uparrow\uparrow\) while if there is an odd number of domain walls between the two sites then the spin at site \(r'\) is \(\downarrow\downarrow\). The average \(\langle S^z_r S^z_{r'} \rangle = (p - q)|r-r'|\). That is, if we have an even number of bad domain walls (corresponding to an even power of \(q\)) then the contribution to the correlation function will be positive while if we have an odd number of domain walls (odd power of \(q\)) then the contribution to the correlation function will be negative. The prefactors in the binomial expansion of \((p - q)|r-r'|\) account for all of the ways in which domain walls may be placed in the interval \((r, r')\). However, \((p - q) = (-E)/(LJ)\).

Thus, the correlator \(\langle S^z_i S^z_{i+1} \rangle = ((-E)/(LJ))|r-r'|\). This single eigenstate result using the binomial theorem indeed matches with the known results for correlations in the Ising chain in the canonical ensemble at an inverse temperature \(\beta = 1/k_BT\) where \(E = -J(L - 1)\tan^2(\beta J)/2\). The agreement of the spatially long distance correlator result in one eigenstate with the prediction of the fixed energy micro canonical ensemble is obvious. The above probabilistic derivation for general sites \(r\) and \(r'\) will hold so long as the eigenstate \(|\psi\rangle\) is a sum of numerous Ising product states (all having the same energy or, equivalently, the same number of domain walls). If this result holds for all site pairs \((r, r')\) then the entanglement entropy is expected to scale monotonically in the size (or “volume”) of this one dimensional system. Indeed, a rather simple calculation (outlined in Appendix \(\text{[B]}\)) illustrates that if the \(L\) site system is partitioned into subregions \(A\) and \(B\) of “volumes” \(L_A\) and \(L_B\) (with \(L = L_A + L_B\)) then if, e.g., \(|\psi\rangle\) is an equal amplitude superposition \(|\Psi_+\rangle\) of all Ising product states (i.e., an equal amplitude superposition of the product states \(|s_1 s_2 \cdots \rangle\) of Section \(\text{[V]}\) that all have a given fixed energy then the entanglement entropy between regions \(A\) and \(B\) scales as \(\ln L_A \ln L_B\).

More broad than the specific example of this Appendix, the coincidence between the single (entangled) eigenstate expectation values with the equilibrium ensemble averages is expected to hold for general classical systems in arbitrary dimensions. To see why this is so consider the expectation value of a general observable (including any correlation functions) that is diagonal in the basis of degenerate classical product states. When computed in a state formed by a uniform modulus superposition of classical degenerate states (e.g., the equal amplitude sum of all local product states of the same energy), the expectation value of such an observable may naturally emulate the microcanonical ensemble average of this observable over all classical states of the same energy. Finite energy density states (i.e., states whose energy density is larger than that of the ground state) formed by a uniform amplitude superposition of all classical product states generally exhibit macroscopic entanglement. As we have elaborated on in this Appendix, this anticipation is realized for the classical Ising chain. For the classical Ising chains discussed above, the below
states (of energies $\epsilon_i$) on a Hilbert space endowed with a simple Hamiltonian $H$. However, not all eigenstates that display the equilibrium value of the correlators $\langle O \rangle$ may think of $H$ as quasi-particle operators (i.e., colloquially, $H$ may be maximal in one dimensional systems albeit being logarithmic in the "volume.

### Appendix B: Entanglement entropies of a uniform amplitude superposition of simple classical states

We next discuss the reduced density matrices and entanglement entropies associated with (1) the eigenstates $|\phi_\alpha\rangle = |\nu_{\alpha}; S_{\text{tot}}, S_{\text{tot}}^z\rangle$ of Section [VI] when $S_{\text{tot}}$ happens to be maximal ($S_{\text{tot}} = S_{\text{max}}$), (2) the symmetric quantum states described of Appendix A, and a generalization thereof that we now describe. Specifically, we will consider general Hamiltonians that may be expressed as a sum of decoupled commuting local terms, $H = \sum_{i=1}^{L} H_i$ (i.e., $N' = L$ in the notation of the Introduction) on a Hilbert space endowed with a simple local tensor product structure. We denote the eigenstates (of energies $\epsilon_{n_i}$) of each of the local operators $H_i$ by $\{|\nu_{\alpha_i}\rangle\}$. For such systems, any product state $|c\rangle = |\nu_{1}^{(n_1)}\rangle \otimes |\nu_{2}^{(n_2)}\rangle \otimes \cdots \otimes |\nu_{L}^{(n_L)}\rangle$ is, trivially, an eigenstate of $H$ (of total energy $E_c = \sum_{i=1}^{L} \epsilon_{n_i}$). Formally, one may think of $H_i$ as decoupled independent commuting "quasi-particle" operators (i.e., colloquially, $H$ describes "an ideal gas" of such quasi-particles). We now explicitly write the states that are equal amplitude superpositions of all such product states $|c\rangle$ of a given total energy,

$$|\Psi_+\rangle \equiv \frac{1}{\sqrt{N(E)}} \sum_{E_c=E} |c\rangle,$$  \hspace{1cm} (B1)

Similar to the discussion of Appendix A, for observables $O_d$ that are diagonal in the $\{|c\rangle\}$ basis, the single eigenstate expectation values $\langle \Psi_+ | O_d | \Psi_+ \rangle$ are equal to the microcanonical equilibrium averages of $\langle O_d \rangle_{\text{eqmc}}$ in which the energy $E$ is held fixed. In Eq. (B1), $N(E) = e^{S(E)/k_B}$ is the number of product states $|c\rangle$ that have a total energy $E$ (and $S(E)$ is the associated Boltzmann entropy). The states of Eq. (B1) describe those of the Ising spin states alluded to in Appendix A. Such states rear their head also in other realms. For instance, since, in a many body spin system, the state of maximal total spin $S_{\text{tot}} = S_{\text{max}}$ is a uniform amplitude superposition of all product states having a given value of $S_{\text{tot}}^z$ (i.e., a uniform amplitude superposition of all states of decoupled spins in a uniform magnetic field that share the same energy), states of the type $|\Psi_+\rangle$ include the eigenstates that we analyzed in Section [VI] (when these states are those of maximal total spin). The entanglement entropy that we will compute for $|\Psi_+\rangle$ will thus have implications for these and other systems. We partition the $L$ site system into two disjoint regions $A$ and $B$ and examine the entanglement between these two subvolumes. To facilitate the calculation, we will employ the symmetric combinations

$$|E_A\rangle_+ \equiv \frac{1}{\sqrt{N_A(E_A)}} \sum_{E_c(\{c_A\})=E_A} |\{c_A\}\rangle,$$

$$|E_B\rangle_+ \equiv \frac{1}{\sqrt{N_B(E_B)}} \sum_{E_c(\{c_B\})=E_B} |\{c_B\}\rangle.$$  \hspace{1cm} (B2)

In the first of Eqs. (B2), the sum is over all product states $\{|\{c_A\}\rangle\}$ having their support on the sites $1 \leq i \leq L_A$ that are of fixed energy $E_A$. Similarly, the symmetric state $|E_B\rangle_+$ extends over the sites $L_A+1 \leq i \leq L$. With these definitions, we rewrite Eq. (B1) as

$$|\Psi_+\rangle = \sum_{E} \frac{N_A(E_A)N_B(E-E_A)}{N(E)} |E_A\rangle_+ |E_B = E-E_A\rangle_+.$$  \hspace{1cm} (B3)

The density matrix associated with this state is $\rho_+\equiv |\Psi_+\rangle\langle\Psi_+|$. To compute the entanglement entropy, we next write the reduced density matrix

$$\rho_{B,+} = \text{Tr}_A \rho_+ = \frac{1}{N(E)} \sum_{E} \left( N_A(E_A)N_B(E-E_A) \right)$$

$$\times |E_B = E-E_A\rangle_+ |E_B = E-E_A\rangle_+.$$  \hspace{1cm} (B4)

If a given system is partitioned into two non-interacting subsystems $A$ and $B$ then the sole relation linking the two subsystems will be the constraint of total energy $E = E_A + E_B$. Of all possible ways of partitioning the total energy $E = E_A + E_B$, one pair of energies $E_A$ and $E_B$ will yield the highest value of $S_A(E_A) + S_B(E_B)$. The ratios appearing in Eq. (B4),

$$\frac{N_A(E_A)N_B(E-E_A)}{N(E)} = e^{(S_A(E_A)+S_B(E-E_A)-S(E))/k_B},$$  \hspace{1cm} (B5)

follow, upon Taylor expanding the ratio to quadratic order about its maximum at $E_A$ and $E_B = E - E_A$, a Gaussian distribution with a standard deviation set by

$$\sigma_B = \sqrt{k_B T^2 C^{gf}}.$$  \hspace{1cm} (B6)

In Eq. (B6),

$$C^{gf}(T) \equiv \frac{C^{(A)}(T)C^{(B)}(T)}{C^{(A)}(T) + C^{(B)}(T)}.$$  \hspace{1cm} (B7)

The latter Taylor expansion may be carried out for energy densities associated with finite temperatures. (In
the vicinities of either the ground state value of the energy density or the highest energy density, the derivatives of the entropy relative to the energy diverge and the Taylor expansion becomes void.) The entropies $S_A(E_A)$ and $S_B(E_B)$ appearing in Eq. (B5) are those of subsystems $A$ and $B$ that, as emphasized above for this system of non-interacting particles, are merely constrained by the condition that $E_A + E_B = E$. For this non-interacting system,

$$e^{S(E)/k_B} = \sum_{E_A} e^{S_A(E_A)/k_B} e^{S_B(E_B=E-E_A)/k_B}, \quad (B8)$$

and thus, trivially, $S(E) \geq S_A(E_A) + S_B(E_B)$. In Eqs. (B5), $T$ is the temperature (set by the condition that the canonical ensemble equilibrium internal energy $\text{Tr}(H e^{-\beta H})/\text{Tr}(e^{-\beta H})$ is equal to the total energy $E$). The entropy of the Gaussian distribution scales as the logarithm of its width. Specifically, for the saddle point Gaussian approximation of Eqs. (B5, B6, B7),

$$S_{\text{ent,}+} = -\text{Tr}(\rho_{B,+} \ln \rho_{B,+}) = \frac{1}{2} \ln(2\pi \sigma_{B}^2 + 1) \sim \ln \sigma_B, \quad (B9)$$

where in the last line, we made manifest the extensive $L_{A,B} \gg 1$ (and thus $\sigma_B \gg 1$). If $S_A(E_A) = O(L_A)$ and $S_B(E_B) = O(L_B)$ when $L_{A,B} \gg 1$ then, from Eqs. (B5, B6, B7, B9), the entanglement entropy for states of finite temperature (i.e., states exhibiting a finite energy density above that of the ground state value),

$$S_{\text{ent,}+} = O(\min\{\ln L_A, \ln L_B\}). \quad (B10)$$

We reiterate that generic states of fixed total energy will exhibit an entanglement entropy proportional to the system volume (see, e.g., the considerations of (44)). Even though a system of non-interacting particles is trivial and its properties may, generally, be exactly computed, its entanglement entropy may be macroscopic. Beyond systems of non-interacting particles, more profound counterparts to this well known maxim are realized in bona fide interacting one-dimensional quantum systems in which the entropy associated with thermofield double states does not scale with the system indicating that the system may be efficiently represented via a matrix-product representation [6] while, as we highlighted, extensive entanglement entropies may appear. We next discuss two specific realizations of Eqs. (B5, B6, B7).

1. Maximal total spin eigenstates

As we noted above, for any fixed $S^z_{\text{tot}}$, the eigenstates of Eq. (44), $|\Psi_+\rangle$ corresponds to a maximal total spin ($S_{\text{tot}} = S_{\text{max}}$) state of the $L = N$ spins (with the given value of $S^z_{\text{tot}}$). In order to relate this to our general results of Eqs. (B5, B6, B7) for the entanglement entropy, we consider the local Hamiltonians $H_i$ forming the Hamiltonian $H = \sum_{i=1}^N H_i$ to be given by $H_i = -J z_i S_i^z$. With this, $|\Psi_+\rangle$ of Eq. (B11) is an eigenstate of $S^z_{\text{tot}}$ (with each product state $|c\rangle$ being an eigenstate of all $\{S_i^z\}$ operators). We consider what occurs if the $N$ spins are partitioned into the two groups $A$ and $B$ of approximately equal numbers $L_A$ and $L_B$, and $|w\rangle \equiv |S^z_{\text{tot}}/(hS_{\text{tot}})| < 1$. In this case, the saddle point approximation of Eq. (B5) yields, as before, a Gaussian distribution and, consequent entanglement entropy that is logarithmic in the system size,

$$S_{\text{ent,}+} = O(\log N). \quad (B11)$$

Thus, as highlighted in Section VI, initial states $|\psi_0^{\text{spin}}\rangle$ of maximal total spin and $|w\rangle < 1$ feature logarithmic in volume entanglement entropies.

2. Ising chains

Returning to the considerations of Appendix A and the notation introduced in Section VI we now consider the symmetric sum of all Ising product states that share the same energy (as measured by the Ising Hamiltonian $H_I$ of Section VI). As in Section VI, we can transform the problem of computing the entanglement entropy of such symmetric states $|\Psi_+\rangle$ into that involving eigenstates of decoupled local Hamiltonians $H_i$ that led to Eqs. (B10). Towards this end, we focus on the nearest neighbor spin products that were crucial to our analysis in Appendix A and define the operators

$$1 \leq i \leq L - 1: \quad R_i \equiv S_i^z S_{i+1}^z, \quad R_L \equiv S_L^z. \quad (B12)$$

The Ising Hamiltonian now explicitly becomes a sum of the above defined decoupled commuting operators, $H_I = -J \sum_{i=1}^{L-1} R_i$. In terms of the vocabulary that we employed earlier, the “quasi-particle” operators $\{R_i\}_{i=1}^{L-1}$ are associated with the existence ($R_i = 1$) or absence ($R_i = 0$) of domain walls between neighboring Ising spins. On the two subregions $A$ and $B$, we define $H_{AI} = -J \sum_{i=1}^{L_i-1} R_i$ and $H_{BI} = -J \sum_{i=L_i+1}^{L-1} R_i$. The equal amplitude superposition of all Ising Hamiltonian product states that are of fixed energy can now be rewritten as

$$|\Psi_I^+\rangle = \frac{1}{2^{L/2}} \sum_{r_1, r_2, \cdots, r_L} |r_1 r_2 \cdots r_L\rangle, \quad (B13)$$

where $r_i = \pm 1$ denote the eigenvalues of $R_i$. When evaluating the reduced density matrix $\rho_{BI+} = Tr_A|\Psi_I^+\rangle \langle \Psi_I^+|$, the trace over all Ising spins $\{s_{i\leq L_A}\}$ that lie in the spatial region $A$ is replaced by that over $\{r_{i\leq L_A}\}$. Repeating the earlier calculations we find, once again, the entanglement entropy of Eqs. (B5, B6, B7, B9) is given by

$$\frac{1}{k_B T} = -\text{tanh}^{-1}\left(\frac{E}{L J}\right). \quad (B14)$$
In Eq. \textbf{(B7)}, the heat capacities of the Ising chain subsystems $A$ and $B$ (when $L_{A,B} \gg 1$) are

\[ C^{(A,B)}(T) = k_B L_{A,B} \times \left( (\beta J)^2 - \left( \frac{\beta E_{A,B}}{L_{A,B}} \right)^2 \right). \]  

Eq. \textbf{(B10)} provides the asymptotic scale of the entanglement entropy; similar to Eq. \textbf{(B11)}, if $L_A$ and $L_B$ are both of order of the system size, $L_{A,B} = O(N)$ then the entanglement entropy $S_{\text{ent}}$ of the symmetric state will scale logarithmically in $N$. General eigenstates may exhibit larger entanglement entropies \textsuperscript{138}.

**Appendix C: The total spin of large systems**

In this brief Appendix, we discuss the total spin sectors that may appear in the spin model of Section VI A. Our aim is to highlight both statistical and physical aspects of the total spin and its scaling with the system size $N$.

All states with maximal total spin and definite eigenvalues of the total $S_{\text{tot}}^z$ operator are eigenstates of the general Hamiltonian $H_{\text{spin}}$ of Eq. \textbf{(4)}. (These eigenstates span the basis of all ferromagnetic spin states with spins uniformly polarized along different directions.) This assertion may be explicitly proven by the following simple observations: (i) For any spin $S = 1/2$ operators, the scalar product $S_i \cdot S_j = \hbar^2 (\frac{3}{2}P_{ij} - \frac{1}{2})$ where $P_{ij}$ is the operator permuting the two spins, (ii) Any state of maximum total spin ($S_{\text{tot}} = S_{\text{max}} = Nh/2$) is a symmetric state that is invariant under all permutations $\{P_{ij}\}$.

From properties (i) and (ii), it follows that any state $\{S_{\text{tot}} = S_{\text{max}} = Nh/2, S_{\text{tot}}^z\}$ is an eigenstate of both the first and second terms of Eq. \textbf{(4)} and therefore of the full Hamiltonian $H_{\text{spin}}$. Thus, any state of maximal total spin $S_{\text{tot}} = S_{\text{max}}$ that is an eigenstate of $S_{\text{tot}}^z$ is automatically an eigenstate of $H_{\text{spin}}$ of Eq. \textbf{(4)}. In general, when $S_{\text{tot}} < S_{\text{max}}$, only some linear combinations of the multiple states of given values of $S_{\text{tot}}$ and $S_{\text{tot}}^z$ are eigenstates of $H_{\text{spin}}$ (hence the appearance of additional quantum numbers $\xi_\alpha$ defining general eigenstates $|\phi_\alpha\rangle$). To make this clear, we can explicitly write down the total spin for a system of $N$ spin $S = 1/2$ particles. That is,

\[ N = 2 : \quad \frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0, \]

\[ N = 3 : \quad \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{2} \oplus \frac{1}{2}, \]

\[ N = 4 : \quad \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = 2 \oplus 1 \oplus 1 \oplus 0 \oplus 0, \]

\[ \ldots. \]

\textbf{(C1)}

The first equality of Eq. \textbf{(C1)} states that singlet ($S = 0$) and triplet ($S = 1$) total spin combinations may be formed by $N = 2$ spin $S = 1/2$ particles. Similar well known relations are tabulated for higher $N$. Since $H_{\text{spin}}$ is defined on a $(2S + 1)^N$ dimensional Hilbert space, its eigenstates span all states in the direct product basis on the lefthand side of Eq. \textbf{(C1)}. For each $N$, the sector of maximal spin ($S_{\text{tot}} = S_{\text{max}} = NS$) is unique. However, for $N > 2$, all other total spin sectors in Eq. \textbf{(C1)} exhibit a multiplicity $M_{S_{\text{tot}}}$ larger than one. While it is, of course, possible to simultaneously diagonalize the Hamiltonian of Eq. \textbf{(4)} with the two operators $S_{\text{tot}}$ and $S_{\text{tot}}^z$, there are multiple states that share the same eigenvalues of $S_{\text{tot}}$ and $S_{\text{tot}}^z$. One needs, of course, to diagonalize $H_{\text{spin}}$ in every subspace of given $S_{\text{tot}}$ and $S_{\text{tot}}^z$ in order to find its eigenstates in each such subspace. This task is not necessary for the calculations of Section VI A. Rather, the total spin is the quantity of greatest relevance.

Using the characters of the SU(2) group, we find that in Eq. \textbf{(C1)} there are

\[ M_{S_{\text{tot}}} = \frac{N!(2S_{\text{tot}} + 1)}{(\frac{N}{2} + S_{\text{tot}} + 1)!(\frac{N}{2} - S_{\text{tot}})!} \]  

\textbf{(C2)}

sectors of total spin $0 \leq S_{\text{tot}} \leq \frac{N}{2}$ on the righthand side of Eq. \textbf{(C1)}. The decomposition into characters of SU(2) has a transparent physical content. Consider a global rotation by of all spins an arbitrary angle $\theta$ about the $z$ axis. The trace of the operator that implements this rotation is the same into the different basis appearing in Eq. \textbf{(C1)}: (1) the product basis (the lefthand side of Eq. \textbf{(C1)}) and (2) the basis comprised of the total spin sectors (the righthand side of Eq. \textbf{(C1)}). When expressing the basis invariant trace of the arbitrary rotation evolution operator in terms of the Laurent series in $e^{i\theta/2}$ that arises when taking the trace of the rotation operator, the series must identically match in both of these bases of Eq. \textbf{(C1)}. Equating the coefficients leads to Eq. \textbf{(C2)}.

Perusing this equation, it is observed that for large $N$, the highest values of $M_{S_{\text{tot}}}^N$ occur for small $S_{\text{tot}}$: in Eq. \textbf{(C1)}, a “randomly” (“infinite temperature”) chosen state of $N \gg 1$ spins is most likely to have $S_{\text{tot}} \leq O(\sqrt{N})$. Specifically, if we approximate, for fixed $N \gg S_{\text{tot}} \gg 1$, the distribution of binomial coefficients in Eq. \textbf{(C2)} by a Gaussian, we will trivially obtain that

\[ M_{S_{\text{tot}}}^N \sim \frac{2^{N+\frac{1}{2}}e^{-\frac{S_{\text{tot}}^2}{N}}}{\sqrt{\pi N^2}}S_{\text{tot}}. \]  

\textbf{(C3)}

Even though, when $N \gg 1$, states of low $S_{\text{tot}}/N \leq O(N^{-1/2})$ are statistically preferred in the entries of Eq. \textbf{(C1)}, physically finite $S_{\text{tot}}/N$ ratios are, of course, mandatory in numerous instances (including the ability to cool/heat the energy density of the system at a finite rate). For instance, sans symmetry breaking fields, in low temperature ferromagnetic states (having a finite magnetization density or, equivalently, an extensive total spin), the total spin value $S_{\text{tot}} = O(N)$. In the presence of the applied symmetry breaking field in $H_{\text{spin}}$ of Eq. \textbf{(4)}, such a finite average of $(S_{\text{tot}}^z/N)$ arises at general finite temperatures. Furthermore, as noted above, in order to have a finite rate of change of the energy density by applying the transverse field $B_y$ of Eq. \textbf{(6)}, the total spin $S_{\text{tot}}$ must indeed be extensive (i.e., be of order $O(N)$).
Appendix D: Non-adiabatic evolutions with a sharp energy density

In this Appendix, we discuss limited situations in which the energy density may change at a finite rate yet the standard deviation $\sigma_t = 0$. These comments expand on the framework of Section VII.

If the variance of the energy density remains zero at all times $t$, then the derivative of the variance of the energy density (which all too explicitly now write) will vanish, i.e., $\frac{d}{dt} \sigma_t^2 = \frac{1}{N} \text{Tr} (\rho(t))((D, H) - 2 \text{Tr}(\rho(t)H)D)) \equiv J = 0$ where $D \equiv [H(t), H]$ and $\rho(t) \equiv U(t) \rho U(t)$. Careful attention may be paid to the scaling with $N$ for local Hamiltonians $H(t)$ (such as those in the examples of Section VII). In such instances, at small $t > 0$ (where the derivative of the energy density is finite), the operator norm $||D||$ is bounded from below by $O(N)$ contributions and the derivative $\frac{d}{dt} \sigma_t^2$ for general $N$ (including the $N \to \infty$ limit) is of order unity. Whenever the energy density experiences a finite rate of change $\frac{d\epsilon}{dt}$ for a Hamiltonian $H(t)$ which is constant during the cooling/heating period, then the average of $D$ with the probability density matrix $\rho(t)$ will be equal to $-i\hbar \frac{d\epsilon}{dt}$. A vanishing $J$ implies that, when evaluated with $\rho(t)$, the two operators $H/N$ and $D/N$ have a vanishing connected correlation function $\text{Tr}(\rho(t)(HD)) - \text{Tr}(\rho(t)H)\text{Tr}(\rho(t)D))$. A solution for $J = 0$ is afforded by Hamiltonians $H(t)$ that during the cooling/heating time have a commutator $D(t)$ with $H$ that is a non-vanishing c-number (i.e., formally, $D(t) = D(t) \mathbb{1}$ with $D$ a number). When the system is cooled/heated such that the energy density decreases/increases, the time derivative of the energy density $\frac{d\epsilon}{dt} = -\frac{i}{N} \text{Tr}(\rho(t)D(t))$ cannot vanish at all times $t$ during which cooling/heating occurs and thus $\frac{d\epsilon}{dt}$ is finite. Under these circumstances, the evolution operator $U$ has the form of a shift operator for the energy (shifting during each interval of time $[t, t + dt]$ the Heisenberg picture Hamiltonian by $H^H(t) \rightarrow (H^H(t) - \frac{i}{\hbar} \text{Tr}(D(t))\mathbb{1})$ so that $H^H(t) = H - \int_0^t dt' \text{Tr}(D(t'))\mathbb{1}$. The expectation value of the commutator of $(H - E)^2$ (where $E = \text{Tr}(\rho(t)H)$ is the energy at time $t$) with the transverse field cooling/heating Hamiltonian must vanish (so that the derivative of the variance vanishes and the energy density width remains zero as it was in the initial equilibrium state prior to cooling/heating) while the commutator of $H$ with the cooling/heating Hamiltonian must be finite (in order to enable cooling/heating). The expressions for the variance and its time derivative make the similarity with wave packets (Section IV) more vivid. In order to obtain a general shift of the energy without widening the width of the energy density distribution, one may apply a shift operator (an evolution with a “momentum” conjugate to $H$). Indeed, the above evolution with $D(t)$ leads to a shift of the energy density with no additional changes. We allude here to an evolution during the cooling/heating period with a Hamiltonian $H(t)$ that is the momentum conjugate to the full original Hamiltonian $\hat{H}$; this intermediate period Hamiltonian $H(t)$ it is not conjugate to the basic fields $\phi$ of Eq. (13). More comprehensive (yet, physically, exceptionally uncommon) solutions to the equation $\frac{d}{dt} \sigma_t = 0$ at all times $t$ (and thus solutions to $\sigma_t(t) = 0$ at all $t$) given that $\sigma_t = 0$ at time $t = 0$ are afforded by combining multiple “shifts” of the above $D(t)$ type with the product states of Section VII. That is, we may set the initial state to be a general product of decoupled density matrices afforded by Eq. (3) with general values of $1 \leq M \leq N$. If all of the probability density matrices are local (have their support on regions of size $O(1)$) then any Hamiltonian evolution is possible. Conversely, as discussed above, if the density matrices cannot be factorized beyond a region of size $O(N)$ then only an innocuous shift with a constant $D$ will be possible. General hybrids (where for all non-local density matrices, an innocuous shift appears while the evolution of any local density matrices is arbitrary) constitute solutions to the equation $\sigma_t(t) = 0$ at all times $t$. Other non-adiabatic solutions to the equation $J = 0$ are also possible by engineering, in the eigenbasis of the original Hamiltonian $H(t = 0)$, specific matrix forms of Hamiltonians $H(t > 0)$ and probability densities $\rho$ that satisfy the requisite conditions. All of the above particular situations are very special. As is evident from the general finite Eq. (15), when the system is rapidly cooled or heated, an evolution from an initial sharp energy density may not only shift the initial delta-function distribution of the energy density but also lead to a non-vanishing $\sigma_t$.

Appendix E: Intuitive arguments for the appearance of long time Gaussian distributions

The prediction of Eq. (29) for the viscosity of quintessential non-equilibrium liquids (supercooled liquids and glasses) that yielded the 16 decade collapse of Fig. 2 was first derived [11] by invoking a Gaussian distribution of width $\sigma_t$. At the other extreme, equilibrium systems also display a Gaussian distribution of their energy density $P(\epsilon')$. In [11], we motivated the presence of a Gaussian distribution by maximizing the Shannon entropy for a given $\sigma_t$. In this Appendix, we suggest that the appearance of long time normal distributions might also be natural from other considerations.

In general, the probability distribution $P(\epsilon')$ may be calculated along lines similar to those that led to Eq. (9) in our toy example of Eq. (1) where the system was continuously driven by an external transverse field. However, unlike the models studied in Section VII at long times, supercooled liquids and glasses are no longer driven by an external bath $H_{tr}$ that continuously cools/heats them in a predetermined fashion. Instead, for supercooled liquids and glasses, at long times, the external heat bath (similar to the situation in equilibrium thermodynamics), becomes a source of stochastic noise (whose strength is set by its temperature $T$). Thus, the initially driven (i.e., continuously cooled) supercooled
Thus, similar to the calculation that led to Eq. (9), only even moments \( p = 2g \) may be finite. Now, however, the number of non-vanishing contributions (the number of ways in which the elements of \( H^\dagger \) may be matched in complex conjugate pairs) will scale as \( \left( \frac{2g!}{2^{g!}} \right) \). This, in turn, prompts us to consider the possibility that, approximately, \( \langle (\Delta\epsilon)^{2g} \rangle \sim \left( \frac{2g!}{2^{g!}} \right) \sigma^{2g} \). (This is especially the case if the initial state \( |\psi^0\rangle \) corresponds to a single eigenstate of the Hamiltonian \( H \), i.e., \( c_n^{(0)} = \delta_{n,0} \) and \( c_n^{(0)} = \delta_{n,p,n} \). If, for all \( g \), these moments of \( \Delta\epsilon \) are equal to those evaluated with a Gaussian distribution (as follows from the standard application of Wick’s theorem—the combinatorics of which essentially reappeared in the above), then it will follow that the probability distribution \( P(\epsilon') \) for obtaining different energy densities in the final state is indeed a Gaussian. We underscore that for thermal fluctuations in standard (“canonical”) systems, the resulting Gaussian distribution is defined by its average and a standard deviation that is linear in the temperature \( (\sigma \propto T) \) suggestive of Eq. (28). It is important to stress that the assumption of random phases in the above derivation of the Gaussian form does not, of course, imply small variances; the standard deviation of the energy density \( \sigma \) (possibly still linear in the temperature) may be finite. As emphasized in Sections VIII and X, in thermal systems the standard deviation characterizing the distribution \( P(\epsilon') \) is given by \( \sigma = \sqrt{k_B T^2 c'/N} \sim N^{-1/2} \).

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[12] If in case (i) of Section II the Hamiltonian is varied adiabatically such that both its initial and final forms are equal to \( \sum \hat{H}_i \), then, by the adiabatic theorem, no change in \( P(\epsilon') \) (including the cooling sketched in Figure I) can occur between the final and initial states.
[13] Given radiation traveling at a speed \( c \), during a time interval \( \Delta t \), an extensive (i.e., volume proportional) amount of radiative heat \( \Delta Q_{\text{rad}} \) may flow into a hypercubic \( d \) dimensional system of spatial dimensions \( L \times L \times \cdots \times L \) whenever \( L \lesssim c(\Delta t) \). (The linear system scale \( L \) is far larger than the typical inter-atomic separation.) Thus, bulk effects from radiative heat exchange may only be present only after a sufficiently long time \( t \gtrsim L/c \) from the instant at which radiative heating or cooling begins. Similarly, if the effective radiative ab-
The media comprising the system and the surrounding heat bath both satisfy $\ell_{S,B} \gtrsim L$ then the total radiative heat flow rate out of or into the system may be proportional to the volume of the system, $dQ_{rad}/dt = O(V)$. Stated equivalently, when $\ell_{S,B} \gtrsim L$, the heat current (the local heat flux) $J_Q = O(L)$. By contrast, for short absorption lengths $\ell_{S,B} \ll L$ and/or during times $t < L/c$, the radiative contribution to the rate of change of the energy density $d\epsilon/\partial t$ will scale as $O(L^{-1})$. Our primary focus in this current article will be on the empirical situations in which there is a measurable rate of change of the energy density $d\epsilon/\partial t = O(1)$ (or, more generally, measurable rates of change $d\epsilon/\partial t = O(1)$ of other intensive quantities $q$). In the context of the rate of change of the energy density, the heat transfer conditions are the ones of empirical relevance in bulk cooling/heating/experiments, e.g., those on supercooled liquids, wherein the heat absorption length is indeed larger or comparable to the linear dimension of the supercooled liquid sample, $\ell_{S,B} \gtrsim L$; it is indeed experimentally well known that during typical heating/cooling $d\epsilon/\partial t = O(1)$. For completeness, we will, nonetheless, comment throughout our work also on cases in which the total heat transferred to (or given off by) the system is bounded by its surface area (e.g., for systems of very large linear dimensions $L$, the derivatives $d\epsilon/\partial t$ and $d\partial t$ vanish as $L^{-1}$).

As an aside, along nearly identical lines, for conventional thermal systems, the standard deviation scales as $\sigma_i \sim N^{-\alpha}$ with $\alpha = 1/2$ implying that the average of the pair correlator $\langle G_{ij} \rangle \sim O(N^{-\alpha}) = O(1/N)$. At temperatures $T \rightarrow \infty$, this $1/N$ scaling merely amounts to the existence of self-correlations (of typical magnitude $O(1)$) $i = j$ and the absence of pair correlations when the two lattice sites are different, $i \neq j$. Analogously, for critical systems wherein $G_{ij} \sim r_{ij}^p$ with $p = d + 2 + \eta^*$ with $\eta^*$ the anomalous exponent, the average correlator (over all pairs of lattice sites) scales as $\langle G_{ij} \rangle \sim N^{-p/d}$.

[14] As an aside, along nearly identical lines, for conventional thermal systems, the standard deviation scales as $\sigma_i \sim N^{-\alpha}$ with $\alpha = 1/2$ implying that the average of the pair correlator $\langle G_{ij} \rangle \sim O(N^{-\alpha}) = O(1/N)$. At temperatures $T \rightarrow \infty$, this $1/N$ scaling merely amounts to the existence of self-correlations (of typical magnitude $O(1)$) $i = j$ and the absence of pair correlations when the two lattice sites are different, $i \neq j$. Analogously, for critical systems wherein $G_{ij} \sim r_{ij}^p$ with $p = d + 2 + \eta^*$ with $\eta^*$ the anomalous exponent, the average correlator (over all pairs of lattice sites) scales as $\langle G_{ij} \rangle \sim N^{-p/d}$.

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[20] As a concrete example if, on a hypercubic lattice of $N$ sites in $d$ dimensions that is endowed with periodic boundary conditions, the exchange couplings $J_{ij} = J$ for near neighbor pairs $(ij)$ with $J_{ij} = 0$ otherwise, and the initial state was a ground state of uniform polarization (i.e., $s_i = s = \pm 1$) then the difference in energies between the final and initial states will be given by $E_f - E_i = \epsilon_i$. In this example, the standard deviation of the energy in the final state $\langle \chi \rangle$ is equal to $J/\sqrt{N_d}$.

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It follows that the entropy of entanglement $S_{ent}$ of an equilibrium system with its environment obeys a “volume” law (i.e., the entanglement entropy will be proportional to the spatial volume of $S_{th}$).

[45] We denote the eigenvectors of a density matrix $\rho$ by $|\psi_i\rangle$ and their corresponding eigenvalues by $\{\rho_i\}$. The expectation values $E_l = \langle \psi_i | H | \psi_i \rangle$ mark the energies of given density matrix eigenstates. The variance of the energy density $\sigma_l^2 = \frac{Tr[(H - E)^2 \rho]}{Tr(\rho)}$ with $E = Tr(\rho H)$ the system energy may then be expressed as a sum of “classical” and “quantum” contributions,

$$\sigma_l^2 = (\sigma_l^{\text{class}})^2 + (\sigma_l^{\text{quant}})^2.$$  

(E3)

Here,

$$\sigma_l^{\text{class}} = \frac{1}{N^2} \sum_i^N \rho_i E_i^2 - E^2,$$

(E4)

(note that, generally, $\langle \psi_i | H^2 | \psi_i \rangle \neq E_i^2$) and

$$\sigma_l^{\text{quant}} = \frac{1}{N^2} \sum_i^N \rho_i \sigma_i^2,$$

(E5)

where the variances $\sigma_l^2 = \langle \psi_i | (H - E)^2 | \psi_i \rangle$, (E6) stem from the fluctuations of Hamiltonian in given eigenstates of the density matrix. (Similar classical and quantum standard deviations may be defined for quantities $q$ different from the energy density $\epsilon_l$.) In classical systems where $H$ and $\rho$ commute, $(\sigma_l^{\text{quant}})^2 = 0$ and the sole contribution to the variance is that of Eq. (E4) as it is in, e.g., equilibrium classical systems in the canonical ensemble. The standard deviation of $(H/N)$ (i.e., $\sigma_l$) should not, of course, be confused with $\sigma_l^{\text{class}}$. Indeed, in a general single pure state (a situation for which $\sigma_l^{\text{class}} = 0$), the standard deviation of the energy density may be finite, $\sigma_l = O(1)$. Concrete realizations of this maxim will be provided in the examples of Section VI. Physically, one generally anticipates that when reaching a quasi-static state, different eigenstates of the density matrix will exhibit similar energy densities, viz. $\sigma_l^{\text{class}} = 0$ (for otherwise heat may be rapidly exchanged between states $|\psi_i\rangle$ that are of higher/lower energy densities $(E_i/N)$ and their surrounding heat bath).

A rather example of a classical broad distribution mixing various classical product states leading to a nonvanishing standard deviation is afforded by mixing in the ground state sector of the fermionic Ising model. An equal probability mixing of the two fully spin polarized product states of the classical Ising model, $|\uparrow \cdots \uparrow \rangle$ and $|\downarrow \cdots \downarrow \rangle$ exhibits a finite standard deviation of the value of the local magnetization density $m = \frac{1}{N} \sum_{i=1}^N s_i$ (i.e., $\sigma_m = 1$ irrespective of the system size $N$). An equal amplitude superposition of these two (product) ground states of the fermionic Ising model, viz., the ground state $|+\rangle \equiv \frac{1}{\sqrt{2}} (|\uparrow \downarrow \cdot \cdots \uparrow \downarrow \rangle + |\downarrow \uparrow \cdot \cdots \downarrow \uparrow \rangle)$, features a standard deviation $\sigma_m = 1$. In the latter two (classical and quantum) examples concerning the ground states of the fermionic Ising model, the uncertainty in the energy of the ground states of the fermionic Ising model is, of course, zero ($\sigma_m = 0$).
(\zeta_{\alpha}^{\gamma}) \text{ with } (\zeta_{\alpha}^{\gamma} \zeta_{\alpha'}^{\delta}) = \delta_{\alpha \alpha'} \text{ having its support in the environment } \mathcal{E}', \text{ i.e., } S' \cup \mathcal{E}' = \mathcal{I}' \text{ and } S' \cap \mathcal{E}' = \{0\} \text{ then } \rho = \text{Tr}_{\mathcal{E}'}(\psi \langle \psi \rangle) \text{ with } \langle \psi \rangle = \sum_{\alpha} \sqrt{\nu_{\alpha}} \zeta_{\alpha} \otimes \zeta_{\alpha}^{'}. \)

[48] When adding the spin of \( N \) particles, as in, e.g., Eq. (C1), there is a unique sector of maximal total spin \( S_{\text{tot}} = S_{\text{max}} \). For lower values of \( S_{\text{tot}} \), the eigenstates \( |\phi_n \rangle \) of the Hamiltonian \( H_{\text{spin}} \) need to be specified by an additional index \( (\mu_n) \).

[49] The result of Eq. (7) may readily be seen by inspection. The applied transverse field leads to a global Larmor precession of the spins about the \( y \)-axis. While the first term of Eq. (4) is manifestly invariant under rotations, the second term (that of \(-Bz \sum_{i} S_i^z\)) will change. In the Heisenberg picture after the evolution with the transverse field, each such local operator \( Bz S_i^z \) transforms into \( Bz (S_i^z \cos(\int_0^t B_y(t')dt') + S_i^z \sin(\int_0^t B_y(t')dt')) \). Since the expectation value of \( S_i^z \) in any eigenstate of \( S_i^z \) (including \(|\psi_{\text{spin}}^{0}\rangle\)) is zero, the only non-vanishing contributions to the variance of the Hamiltonian of Eq. (4) will originate from the expectation value of the square of the second term of \( H_{\text{spin}} \) and thus (up to a trivial prefactor of \( B_z^2 \text{sin}^2(\int_0^t B_y(t')dt') \)) from

\[
\sigma_{S_{\text{tot}}}^2 = \langle (S_{\text{tot}}^x)^2 \rangle = \frac{1}{2} \langle (S_{\text{tot}}^x)^2 + (S_{\text{tot}}^y)^2 \rangle = \frac{1}{2} \langle (S_{\text{tot}}^z)^2 \rangle = \frac{1}{2} \langle B_z^2 \text{sin}^2(\int_0^t B_y(t')dt') \rangle.
\]

[50] Even in the state of maximal spin \( S_{\text{tot}} = S_{\text{max}} \), so long as \(|\alpha| < 1\), the standard deviation will generally be \( \sigma = O(1) \). Furthermore, although they are statistically preferable values for \( S_{\text{tot}} \) when adding angular momenta in the large \( N \) limit (e.g., Appendix C), regardless of the form of \( H_{\text{symm}} \) (for instance, irrespective of the specific couplings in Eq. (5)), in this \( N \gg 1 \) limit, states of vanishingly small \( \sigma \) will not allow for a for a finite change of the energy density, \( \Delta E = B_z \frac{S_{\text{tot}}}{N} (1 - \cos(\int_0^t B_y(t')dt')) \), via the application of the transverse field (as embodied by the Hamiltonian \( H_{\text{spin}} \)). Indeed, the central point that we wish to emphasize and is evident in our example of Eq. (4) is that, generally, when the energy density \( \Delta E \) does change at a non-vanishing rate, a finite \( \sigma > 0 \) is all but inevitable.

[51] Explicitly,

\[
\langle \psi \rangle = \sum_{g=0}^{\infty} \langle \psi \rangle^{2g} \sigma_{2g}^2 = \sum_{g=0}^{\infty} \frac{(-1)^g}{(2g)!} \langle \psi \rangle^{2g} \sigma_{2g}^2 = J_0(q_{\psi}, \sqrt{2}).
\]

[52] We quickly comment, in the spirit of the discussion appearing at the end of Section [VII] on what transpires if the forms of the Hamiltonian \( H_{\text{spin}} \) and, notably, the coupling of the spins to the transverse field as embodied by the Hamiltonian \( H_{\text{tr}} \), apply only for spins on the surface of the system (the number of these spins \( N_{\text{surf}} = O(N^{d-1/d}) \)). In such situations, the standard deviation in Eq. (4) would naturally scale as \( \sigma = O(N^{-1/d}) \). The distribution of Eq. (9) will retain its form. If this circumstance arises then, in the discussions and equations in the main text concerning \( H_{\text{spin}} \) and \( H_{\text{tr}} \), the total number of spins \( N \) will be replaced by the number of surface spins \( N_{\text{surf}} \).

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[54] T. Matsubara and H. Matsuda, “A Lattice Model of Liquid Helium, I”, Prog. Theor. Physics 16, 569 (1956).

[55] Specifically, the Matsubara-Matsuda transformation \([54]\) from spin \( S = 1/2 \) raising and lowering operators to the creation and annihilation operators of hard core bosons is given by \( \sigma = b_1^1 \) and \( \sigma = b_1^1 \) (from which it follows that \( \sigma = b_1^1 ) \) is the hard core boson number operator).

[56] With \( c_i \) and \( c_i^\dagger \) denoting respectively, the spinless Fermi annihilation and creation operators (and with \( n_i = c_i^\dagger c_i \), the Jordan-Wigner dual of the initial spin Hamiltonian of Eq. \([4]\) reads

\[
H_{\text{Fermi}} = \sum_{|i-j|=1} J_{ij} ((c_i^\dagger c_j + h.c.) + n_i n_j) - \sum_{i=1}^N (B_z - \frac{1}{2}) n_i.
\]

We may consider the initial spinless fermion system of Eq. \([8]\) as the one that is governed, at intermediate times, by the (now non-local) Jordan-Winger dual of \( H_{\text{tr}} \) or, equivalently, the meronized dual of \( H_{\text{symm}} \). To make contact with our previous analysis of Eq. \([4]\), we may represent the spinless fermion dual of the particular spin \( S = 1/2 \) state \(|\uparrow_1 \downarrow_2 \downarrow_3 \downarrow_4 \uparrow_6 \cdots \rangle \) describes an antisymmetric superposition of all product states in which the fixed total number of spinless fermions \( \sum n_i \) are distributed over the specific sites \( (1, 2, 4, 6, \cdots, (N-1)) \).

[57] For a discussion of the standard uncertainty relations for mixed states see, e.g., L. Ballentine, “Quantum Mechanics” (World Scientific, Singapore, 1998).

[58] We remark that a shorter, one line, proof of the uncertainty relations for mixed states follows from the fact that any density matrix may be expressed in terms of pure states \([26, 47]\). If, in the notation of \([47]\), we take \( \mathcal{I} \subset \mathcal{I}' \), then the uncertainty relation for pure states implies the corresponding uncertainty relations for general mixed states with arbitrary density matrices.

[59] These conditions need not always hold. For instance, one may concoct states \(|\psi \rangle \) (or density matrices \( \rho \)) in \( \mathcal{I} \) that exhibit arbitrarily large variances \( \sigma^2_{\psi} = \langle \psi | H^2 | \psi \rangle - \langle \psi | H | \psi \rangle^2 \) when the system size increases. (This may be achieved by superposing eigenstates that span an extensive range of \( H \) eigenvalues.) The microcanonical ensemble (for which \( \sigma^2_{\psi} \) is bounded) will be rendered incompatible in states like these for which the standard deviation scales with the system size.

[60] Although the conditions \([1-3]\) of Section [VIII] are not always satisfied (e.g., \([29]\)), rather general systems do obey them and further exhibit a finite rate of change of the energy density \(|\Delta E|\). To make contact with our earlier examples of Section [VII] as a case in point, we may take \( H \) to be \( H_{\text{spin}} \) of Eq. \([4]\) having a short range \( H_{\text{symm}} \) and in which the magnetic field \( B_z \) is
along the $z$ direction over the entire volume $T$; in a similar spirit, we consider the Hamiltonian $H$ in the region $S$ to be given by Eq. (1) with an internal applied field $B_y$ parallel to the $y$ direction. Thus, in this example (apart from “surface terms” arising from the short range interactions in $H_{symm}$), the interaction between the system $S$ and its external environment $E$ is dominated by an external field $(\vec{B} = (0, -B_y, B_z))$ applied from sites external to $S$. This external field couples to all spins in $S$; when added together with $H$, this field will reproduce the terms of $H$ appearing in $S$. In this example, the density matrix $\rho = |\psi\rangle\langle\psi|$ where we will choose $|\psi\rangle$ to correspond to an equal amplitude superposition of two eigenstates of maximal total spin $S_{tot} = O(N)$ that differ by one quantum of $h$ in their $S^z$ eigenvalues. That is, in the convention of Section VI, $|\psi\rangle = \frac{1}{\sqrt{N}}(|\mu_0; S_{tot}, S^z_{tot}| + |\mu_0; S_{tot}, S^z_{tot} - h|)$. It is readily verified that the conditions (1-3) are satisfied.

VI. The Rate of Change of the Internal Energy Density

The rate of change of the energy density is $\frac{d\epsilon}{dt} = B_y \epsilon + B_z \sqrt{1 - \frac{S_{tot}^2}{N}}$. Thus, $\frac{d\epsilon}{dt} = O(1)$ for finite $B_y$ and $B_z$ when $S_{tot} = O(N)$ and $|w| < 1$.

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[67] Strictly speaking, a finite interval of the internal energy densities corresponds to the single melting (and other coexistence) temperature(s). This width of this interval is set by the latent heat of fusion at the melting temperature. At other temperatures in which no phase coexistence appears, there is a unique internal energy density (a sharp thermodynamic state variable) $\epsilon(T)$ associated with every temperature $T$.

If the spectrum of $H$ exhibits degeneracies then we will explicitly choose the states $\{|\phi((q^i);W)\rangle\}$ to be common eigenstates of both $H$ and $\sigma$.

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In this brief comment, we consider an initial state $|\psi\rangle = \sum_n c_n |\phi_n\rangle$ where $|\phi_n\rangle$ are eigenstates of a local Hamiltonian $H = \sum_{i=1}^{N'}\mathcal{H}_i$ where $H|\phi_n\rangle = E_n|\phi_n\rangle$. The system evolves with the time independent Hamiltonian $H$. We define the long time average of the local energy density fluctuations by

$$\sigma_R \equiv \lim_{T \to \infty} \frac{1}{T} \int_0^T dt (\langle H_R(t) \rangle - \langle H_R \rangle)^2,$$

where $\langle H_R \rangle \equiv \frac{1}{T} \int_0^T dt \langle H_R(t) \rangle = \sum_n |c_n|^2 \langle \phi_n | H_R | \phi_n \rangle$ (E10) is the long time average of $H_R$. Here, $\langle H_R(t) \rangle$ denotes the Heisenberg picture expectation value of the local energy term $H_R(t)$ in the initial state $|\psi\rangle$. In the following, we consider the situation when level spacings of $H$ are relatively incommensurate (any equation of the type $(E_m - E_m') = (E_{m'} - E_m)$ may only be satisfied if $m' = n$ and $m' = m$). A simple calculation (along the lines of that performed in Ref. 11 for the long time average of general observables) then yields

$$\sigma_R^2 = \sum_{n\neq m} |c_n|^2 |c_m|^2 \langle \phi_n | H_R | \phi_m \rangle^2.$$
ations of the local energy density may naturally appear in unison with a larger spread of the global energies in the spectral decomposition of $|\psi\rangle$ in the eigenbasis of $H$ (i.e., a larger standard deviation $\sigma_e$).

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eral case, weights $P_Q$ that are of non-uniform sign (a property related to the NP hard [129] “minus sign problem” of Quantum Monte Carlo). Here, $\langle O \rangle_Q$ denotes the average of any quantity $O$ (including the pair correlator $G$) in a sector of fixed $Q'$ for a free fermionic system [130–132]. If the auxiliary variables $Q'$ are not translationally invariant (so that the resulting free Fermi system for a particular $Q'$ realization is not diagonal in $k$-space), any discontinuity in $G_{coh}$ may be further diminished.

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[138] As we underscored earlier, typical “thermal states” exhibit an entanglement entropy that scales with the system volume [14], not its logarithm. The eigenstates $|\Psi_+\rangle$ examined in Appendix B were special in two different ways: (i) The eigenstates were constructed as an equal weight symmetric combination of all local product states and (ii) The systems that we examined were endowed with a local “quasi-particle” structure embodied by the independent commuting operators $\{H_i\}$ (and associated local product eigenstates). In general, even when only property (i) is violated, larger entropies may arise. It is instructive to see why this is so and how the state $|\Psi_+\rangle$ is special inasmuch as the calculation of its entanglement entropy is concerned. In the space spanned by all product states $|c\rangle$ that given energies $E_B$ (instead of that performed in Appendix B) in the basis of the symmetric basis of Eqs. (B2), the reduced density matrix $\rho_{B,+}$ becomes block diagonal. Repeating the calculation of Appendix B in this basis, we find that in each region of fixed energy $E_B$, the block matrix is equal to

$$ One = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix} \tag{E13}$$

multiplied by the factor $e^{(S_A(E-B)-S(E))/k_B}$. The dimensions of the matrix $One$ are determined by the number $N_B(E_B) = e^{S_A(E_B)/k_B}$ of degenerate states $\{ |E_B, j\rangle \}_{j=1}^{N_B(E_B)}$ that have an energy $E_B$ on the spatial region $B$. We may perform a unitary transformation to the discrete Fourier basis (spanned by the states $|E_B, k_{EB}, j\rangle \equiv (N_B(E_B))^{-1/2} \sum_{E,B} e^{ik_{EB}e} |E_B, j\rangle$ with the wavenumber $k = 2\pi m/N_B(E_B)$ where $m = 0, 1, 2, \cdots, N_B(E_B) - 1$). This transformation reduces (the generally large) matrices of the form of Eq. (E13) to a single non-vanishing entry. Indeed, up to a constant prefactor of $(N_B(E_B))$, the matrix $One$ is equal to the outer product $|E_B, k_{EB}, 0\rangle |E_B, k_{EB}, 0\rangle$. To make the contact with Appendix B lucid, we remark that in the notation of Eqs. (B2) the single non-vanishing Fourier mode $|E_B, k_{EB}, 0\rangle = |E_B\rangle$. In each block of fixed energies $E_B$, all other discrete Fourier $|k_{EB} \neq 0\rangle$ modes have a vanishing amplitude. Such a Fourier transformation immediately yields the eigenvalue spectrum, $Spec\{One\} = \{N_B(E_B), 0, 0, \cdots, 0\}$. $\tag{E14}$

Hence, upon the unitary transformation to the Fourier basis, the block diagonal matrix $\rho_{B,+}$ becomes sparser (each vanishing eigenvalue of the reduced density matrix $\rho_{B,+}$ does not contribute to the entropy) and only the completely symmetric states of the form of Eq. (B2) become important. If the equal amplitude eigenstates $|\Psi_+\rangle$ are replaced by a general linear combination $|\Psi_{\alpha}\rangle = \sum_{E} \alpha_{E} |c\rangle$ (with $\sum_{E} |\alpha_{E}|^2 = 1$), then the associated reduced density matrix $\rho_{\alpha B, \alpha} = Tr_A |\Psi_{\alpha}\rangle \langle \Psi_{\alpha}|$ will still remain block diagonal. However, the block matrices that span each region of fixed energy $E_B$ will, generally, look very different from $One$. Intuition concerning the larger entanglement entropies that may generally result can be gained by suggestive arguments. Towards this end, we may consider what occurs if each diagonal block of $\rho_{\alpha B, \alpha}$ of the type $One$ is replaced by other block diagonal matrices with a wider distribution of the eigenvalues such that, e.g., each of the non-vanishing eigenvalues of $\rho_{B,+}$ for energies $E_B$ close to the energy $E_B$ that maximizes the sum $S_A(\Phi_A = E - E_B) + S_B(\Phi_B)$ is, effectively, split into $K$ equal parts. In such a case, the entanglement entropy $S_{ent, \langle\alpha\rangle}$ will be larger than $S_{ent, +}$ by an additive contribution of $\ln K$. If, for $L_B < L_A$, the logarithm $(\ln K) = O(\mathcal{O}(\Phi_B) = O(L_B)$ then this additive contribution to the entanglement entropy may be linear in the volume $L_B$ of subsystem $B$. 

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