On a Generalization of the Arcsine Law

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March 13, 2022

Abstract

The gaps between the times, in a Weyl chamber, at which the sum of the increments of independent Brownian motions attains its maximum has a Dirichlet distribution.

AMS 2000 Subject Classification: 60C05, 60E05, 60J65, 60K25, 60K35
Keywords: Maximal Brownian Functionals, Queues in Series, GUE, Maximal Eigenvalue, Arcsine Law, Dirichlet Distribution.

The Brownian functional $D_m$, $m \in \mathbb{N}$, introduced by Glynn and Whitt [3], in the context of a queuing problem, is defined as:

$$D_m = \max_{0=t_0 \leq t_1 \leq \cdots \leq t_{m-1} \leq t_m = 1} \sum_{i=1}^{m} [B^i(t_i) - B^i(t_{i-1})],$$  \hspace{1cm}(1)$$

where $(B^1(t), \ldots, B^m(t))$ is an $m$-dimensional standard Brownian motion. The asymptotic behavior, for $m \to +\infty$, of this functional is well understood, in particular, since Baryshnikov [1] as well as Gravner, Tracy and Widom [4] identified its law as that of the law of the maximal eigenvalue of an $m \times m$ element of the Gaussian Unitary Ensemble (GUE). (In fact, [1]

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showed that the whole sequence \((D_m)_{m \geq 1}\) is identical in law to the sequence \((\lambda_m^*)_{m \geq 1}\), where each \(\lambda_m^*\) is the largest eigenvalue of the \(m \times m\) principal minor of an infinite GUE matrix. Moreover, there is equality in law between all the spectra of all the principal minors of an element of the GUE and some multidimensional Brownian functionals, naturally appearing as limits in subsequence problems; see [2] and the references therein.)

The purpose of this note is to describe the distribution of the (a.s. unique) parameters \((\theta_1, \theta_2, \ldots, \theta_{m-1})\) which maximize the functional \(D_m\). Our result is the following generalization of the arcsine law:

**Theorem 1.** The density \(f_m, m \geq 2,\) of the parameters \(\tilde{\theta}^{(m)} := (\theta_1, \theta_2, \ldots, \theta_{m-1}),\) \(0 = \theta_0 \leq \theta_1 \leq \theta_2 \leq \cdots \leq \theta_{m-1} \leq \theta_m = 1,\) which maximize the functional \(D_m\) is given by

\[
f_m(\tilde{\theta}^{(m)}) = \frac{\Gamma\left(\frac{m}{2}\right)}{\pi^{m/2}} \theta_1^{-1/2} (1 - \theta_{m-1})^{-1/2} \prod_{i=2}^{m-1} (\theta_i - \theta_{i-1})^{-1/2}. \tag{2}
\]

That is, the \(m\)-vector of gap lengths \((\Delta_1, \Delta_2, \ldots, \Delta_m),\) where \(\Delta_i = \theta_i - \theta_{i-1},\) \(1 \leq i \leq m,\) has a Dirichlet distribution with parameters \((1/2, 1/2, \ldots, 1/2).\)

**Remark 1.** The gap distribution \(D(1/2, 1/2, \ldots, 1/2)\) shows that the optimal parameters are essentially evenly spaced, in the basic sense that \(E\theta_i = i/m,\) for \(1 \leq i \leq m - 1,\) as well as in the more universal sense that the empirical distribution function of \(\tilde{\theta}^{(m)}\) tends strongly towards the uniform distribution on [0, 1] as \(m \to \infty.\)

**Remark 2.** Consecutive parameter values do not exhibit a strong repulsion-type property, unlike the behavior observed, for example, in GOE/GUE/GSE eigenvalues. The factors \((\theta_i - \theta_{i-1})^{-1/2}\) in (2), in fact, introduce the bimodal phenomenon of a Beta\((1/2, 1/2)\) distribution, whose quartiles are found at \((2 - \sqrt{2})/4 \approx 0.146, 0.5,\) and \((2 + \sqrt{2})/4 \approx 0.854.\) In effect, the gaps thus tend to be either quite small or large with respect to the natural length scale \(1/m.\)

**Remark 3.** In his paper on the joint distribution of the location of the maximum of a Brownian motion and related quantities, Shepp [3] provides a simple argument proving the a.s. uniqueness of the location parameter, an argument that generalizes to the a.s. uniqueness of \((\theta_1, \theta_2, \ldots, \theta_{m-1}).\)
Proof. We first note, and show, that that the parameters enjoy a time-reversal symmetry described by

$$(\theta_1, \theta_2, \ldots, \theta_{m-1}) \overset{d}{=} (1 - \theta_{m-1}, 1 - \theta_{m-2}, \ldots, 1 - \theta_1). \quad (3)$$

Indeed, to prove (3), observe that for any fixed $0 = t_0 \leq t_1 \leq \cdots \leq t_{m-1} \leq t_m = 1$, we have

$$\sum_{i=1}^{m} [B^i(t_i) - B^i(t_{i-1})] \overset{d}{=} \sum_{i=1}^{m} [B^i(1-t_{i-1}) - B^i(1-t_i)]
\overset{d}{=} \sum_{i=1}^{m} [B^{m-i+1}(1-t_{i-1}) - B^{m-i+1}(1-t_i)]
\overset{d}{=} \sum_{i=1}^{m} [B^i(1-t_{m-i}) - B^i(1-t_{m-i+1})], \quad (4)$$

where the first two distributional equalities follow respectively from the stationarity of the increments and from the independence of the coordinates of standard multidimensional Brownian motion. Thus, (3) clearly follows.

Next, we note that since

$$D_2 = \max_{0 \leq t_1 \leq 1} [B^1(t_1) - B^1(0) + B^2(1) - B^2(t_1)]
= B^2(1) + \max_{0 \leq t_1 \leq 1} [B^1(t_1) - B^2(t_1)],$$

the problem reduces to finding the location $\theta_1$ of the maximum of a single standard Brownian motion over $[0, 1]$. The solution to this problem, given by one of Paul Lévy’s arcsine law results, is well-known (see, e.g., Shepp’s more general result in [5]), and has density $f_2$ given by

$$f_2(\theta_1) = \frac{1}{\pi} \frac{1}{\sqrt{\theta_1(1-\theta_1)}}, \quad 0 \leq \theta_1 \leq 1, \quad (5)$$
i.e., $\theta_1$ follows a Beta distribution with parameters $1/2$ and $1/2$. Equivalently, $(\Delta_1, \Delta_2) := (\theta_1, 1 - \theta_1)$ exhibits a $\mathcal{D}(1/2, 1/2)$ distribution.

To complete the proof, we proceed by induction on $m$. Assume that the result is true for $2, \ldots, m$. Then, conditioning on $\theta_m$, and using scaling and the induction hypothesis, we see that

$$f_{m+1}(\theta_1, \ldots, \theta_m) = f_{m+1}(\theta_1, \ldots, \theta_{m-1} \mid \theta_m)g_{m+1}(\theta_m)
= \left\{ f_m \left( \frac{\theta_1}{\theta_m}, \ldots, \frac{\theta_{m-1}}{\theta_m} \right) \frac{1}{(\theta_m)^{m-1}} \right\} g_{m+1}(\theta_m), \quad (6)$$
where $g_{m+1}$ is the marginal density of $\theta_m$. Similarly, conditioning this time on $\theta_1$, and again using scaling and the induction hypothesis, we find that

$$f_{m+1}(\theta_1, \ldots, \theta_m) = f_{m+1}(\theta_2, \ldots, \theta_m | \theta_1) g_{m+1}(1 - \theta_1)$$

$$= \left\{ f_m \left( \frac{\theta_2 - \theta_1}{1 - \theta_1}, \ldots, \frac{\theta_m - \theta_1}{1 - \theta_1} \right) \frac{1}{(1 - \theta_1)^{m-1}} \right\} g_{m+1}(1 - \theta_1), \quad (7)$$

where we have also made use of the time-reversal symmetry property (3). Equating (6) and (7), we can solve for $g_{m+1}$ as follows:

$$f_m \left( \frac{\theta_1}{\theta_m}, \ldots, \frac{\theta_{m-1}}{\theta_m} \right) \frac{1}{(\theta_m)^{m-1}} g_{m+1}(\theta_m)$$

$$= \frac{\Gamma \left( \frac{m}{2} \right)}{\pi^{m/2}} \left( \frac{\theta_1}{\theta_m} \right)^{-1/2} \left( \frac{\theta_m - \theta_{m-1}}{\theta_m} \right)^{-1/2} \prod_{i=2}^{m-1} \left( \frac{\theta_i - \theta_{i-1}}{\theta_m} \right)^{-1/2} g_{m+1}(\theta_m) \quad (8)$$

$$= \frac{\Gamma \left( \frac{m}{2} \right)}{\pi^{m/2}} \left( \frac{\theta_2 - \theta_1}{1 - \theta_1}, \ldots, \frac{\theta_m - \theta_1}{1 - \theta_1} \right) \frac{1}{(1 - \theta_1)^{m-1}} g_{m+1}(1 - \theta_1)$$

$$= \frac{\Gamma \left( \frac{m}{2} \right)}{\pi^{m/2}} \left( \frac{1 - \theta_m}{1 - \theta_1} \right)^{-1/2} \left( \frac{1 - \theta_{m-1}}{1 - \theta_1} \right)^{-1/2} \prod_{i=3}^{m} \left( \frac{\theta_i - \theta_{i-1}}{1 - \theta_1} \right)^{-1/2} g_{m+1}(1 - \theta_1)$$

$$= \frac{\Gamma \left( \frac{m}{2} \right)}{\pi^{m/2}} (1 - \theta_m)^{-1/2} \prod_{i=2}^{m} (\theta_i - \theta_{i-1})^{-1/2} g_{m+1}(1 - \theta_1) \quad (9)$$

and, after cancellation of terms, we have

$$\frac{g_{m+1}(\theta_m)}{(1 - \theta_m)^{1/2-1}} = \frac{g_{m+1}(1 - \theta_1)}{\theta_1^{1/2-1} (1 - \theta_1)^{m/2-1}},$$

for all $0 \leq \theta_1 \leq \theta_m \leq 1$. But this implies that for all $0 \leq t \leq 1$,

$$g_{m+1}(t) \propto t^{\frac{m}{2}-1} (1 - t)^{-\frac{1}{2}},$$

and hence, after normalization, we have

$$g_{m+1}(t) = \frac{\Gamma \left( \frac{m+1}{2} \right)}{\Gamma \left( \frac{m}{2} \right) \Gamma \left( \frac{1}{2} \right)} t^{\frac{m}{2}-1} (1 - t)^{-\frac{1}{2}}, \quad (9)$$
the density of a Beta($\frac{m}{2},\frac{1}{2}$) distribution. Plugging (9) back into (6), and recalling that $\Gamma(1/2) = \sqrt{\pi}$, we find that

$$f_{m+1}(\theta_1, \ldots, \theta_m) = f_m \left( \frac{\theta_1}{\theta_m}, \ldots, \frac{\theta_{m-1}}{\theta_m} \right) \frac{1}{(\theta_m)^{m-1}} g_{m+1}(\theta_m)$$

$$= \frac{\Gamma \left( \frac{m}{2} \right)}{\pi^{m/2}} \theta_1^{-1/2} \prod_{i=2}^{m} (\theta_i - \theta_{i-1})^{-1/2} \frac{1}{(\theta_m)^{m-1}}$$

$$\times \frac{\Gamma \left( \frac{m+1}{2} \right)}{\Gamma \left( \frac{m}{2} \right) \Gamma \left( \frac{1}{2} \right)} (1 - \theta_m)^{-1/2}$$

$$= \frac{\Gamma \left( \frac{m+1}{2} \right)}{\pi^{(m+1)/2}} \theta_1^{-1/2} (1 - \theta_m)^{-1/2} \prod_{i=2}^{m} (\theta_i - \theta_{i-1})^{-1/2}, \quad (10)$$

which proves the result.

This above result suggests further explorations of other properties of closely related Brownian functionals.

Shepp [5] derived the joint density of the location $\theta$ of the maximum, the maximum $M_T$, and its final value $B(T)$ of a single Brownian motion over a finite interval $[0, T]$. Similarly, in our problem, it would be of interest to determine the joint distribution of the location parameters $\tilde{\theta}^{(m)}$, $D_m$, and the terminal values $(B^1(1), \ldots, B^m(1))$.

As already mentioned, [2] obtains maximal Brownian functionals representations for the spectra of all the principal minors of a GUE matrix. It would also be of interest to determine the law of the locations where these maxima are attained.

Another interesting question is the following. Given a random sample of parameter locations $\{\tilde{\theta}^{(m,j)}\}_{j=1}^{n}$ from a $D(1/2, \ldots, 1/2)$ distribution, how do $D_m$ and its empirical counterpart

$$D^n_m := \max_{1 \leq j \leq n} \sum_{i=1}^{m} \left[ B^i \left( \tilde{\theta}^{(m,j)} \right) - B^i \left( \tilde{\theta}^{(m,j)}_{i-1} \right) \right],$$

compare (in expectation, asymptotically, etc.)?

Finally, instead of defining $D^n_m$ in terms of random samples $\{\tilde{\theta}^{(m,j)}\}_{j=1}^{n}$, one might ask what deterministic values of $\{\tilde{\theta}^{(m,j)}\}_{j=1}^{n}$ would maximize $E D^n_m$, an idea analogous to the numerical integration task of finding points for performing Gaussian quadrature.
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