An $\tilde{O}(n^{2.5})$-Time Algorithm for Online Topological Ordering

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Abstract

We present an $\tilde{O}(n^{2.5})$-time algorithm for maintaining the topological order of a directed acyclic graph with $n$ vertices while inserting $m$ edges. This is an improvement over the previous result of $O(n^{2.75})$ by Ajwani, Friedrich, and Meyer.

Key words. online algorithm, directed acyclic graph, topological ordering

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1 Introduction

A topological order $T$ of a directed acyclic graph (DAG) $G = (V, E)$ is a linear order of all its vertices such that if $G$ contains an edge $(u, v)$, then $T(u) < T(v)$. In this paper we study an online variant of the topological ordering problem in which the edges of the DAG are given one at a time and we have to update the order $T$ each time an edge is added. Its practical applications can be found in \cite{2,5,7}. In this paper, we give an $\tilde{O}(n^{2.5})$-time\textsuperscript{1} algorithm for online topological ordering.

1.1 Related Work

Alpern et al. \cite{2} gave an algorithm which takes $O(||\delta|| \log ||\delta||)$ time for each edge insertion, where $||\delta||$ is a measure of the change. (For a formal definition of $||\delta||$, please see \cite{2,7,8}.) Pearce and Kelly \cite{7} proposed a different algorithm which needs slightly more time to process an edge insertion in the worst case than the algorithm given by Alpern et al. \cite{2}, but showed experimentally their algorithm perform well on sparse graphs.

Marchetti-Spaccamela et al. \cite{6} gave an algorithm which takes $O(mn)$ time for inserting $m$ edges. Katriel \cite{3} showed that the analysis is tight. Katriel and Bodlaender \cite{4} modified the algorithm proposed by Alpern et al. \cite{2}, which is referred to as the Katriel-Bodlaender algorithm. Katriel and Bodlaender proved that their algorithm has both an $O(\min\{m^{3/2} \log n, m^{3/2} + n^2 \log n\})$ upper bound and an $\Omega(m^{3/2})$ lower bound on runtime for $m$ edge insertions. Katriel and Bodlaender also analyzed the complexity of their algorithm on structured graphs. They showed that the Katriel-Bodlaender algorithm runs in time $O(mk \log^2 n)$ where $k$ is the treewidth of the underlying undirected graph and

\textsuperscript{1}The symbol $\tilde{O}$ means $O$ with log factors ignored. Depending on the implementation, the runtime may vary from $O(n^{2.5} \log^2 n)$ to $O(n^{2.5} \log n)$. 
can be implemented to run in \(O(n \log n)\) time on trees. In [9], we proved that the Katriel-Bodlaender algorithm takes \(\Theta(m^{3/2} + mn^{1/2} \log n)\) time for inserting \(m\) edges. Recently, Ajwani et al. [1] proposed an \(O(n^{2.75})\)-time algorithm, independent of the number of edges \(m\) inserted. To the best of our knowledge, it is the best result for dense DAGs.

2 Algorithm

We keep the current topological order as a bijective function \(T : V \rightarrow [1..n]\). Let \(d(u, v)\) denote \(|T(v) - T(u)|\), \(u \to v\) express that there is an edge from \(u\) to \(v\), \(u \sim v\) express that there is a path from \(u\) to \(v\) and \(u < v\) be a short form of \(T(u) < T(v)\). Let \(n^{0.5} < t_0 < t_1 < t_2 < \ldots < t_p-1 < t_p < t_{p+1} = n\), where \(p = O(\log n)\) is a nonnegative integer. In Section 6, we shall show how to determine the values of these parameters.

Figure 1 gives the pseudo code of our algorithm. \(T\) is initialized with the topological order of the starting graph. Whenever an edge \((u, v)\) is inserted into the graph, \(\text{INSERT}(u, v)\) is called. If \(u < v\), then \(\text{INSERT}(u, v)\) does not change \(T\) and simply insert the edge into the graph. If \(u > v\), then \(\text{INSERT}(u, v)\) calls \(\text{REORDER}(v, u, 0, 0)\) to update \(T\) such that \(T\) is still a valid topological order and \(T(u) > T(v)\). After the call to \(\text{REORDER}(v, u, 0, 0)\), \(\text{INSERT}(u, v)\) can safely insert the edge into the graph.

It remains to explain how the procedure \(\text{REORDER}(u, v, f_1, f_2)\) works. The duty of the procedure \(\text{REORDER}(u, v, f_1, f_2)\) is to update \(T\) such that \(T\) is still a valid topological order and \(T(u) > T(v)\). The flag \(f_1 = 1\) indicates that the set \(A' = \{w : u \to w \text{ and } w \leq v\}\) has been known to be empty. The flag \(f_2 = 1\) indicates that the set \(B' = \{w : w \to v \text{ and } w \geq u\}\) has been known to be empty. If \(T(u) > T(v)\), then we directly exit. Otherwise, there are two cases to consider:

1: \(t_i < d(u, v) \leq t_{i+1}\) for some \(i = 0, \ldots, p\). In this case, we first have to compute \(\hat{A}_i = \{w : u \to w, d(u, w) \leq t_i, \text{ and } w < v\}\) and \(\hat{B}_i = \{w : w \to v, d(w, v) \leq t_i, \text{ and } w > u\}\). If \(\hat{A}_i = \emptyset\) and \(f_1 = 0\), then we still have to compute \(\hat{A}_{i+1} = \{w : u \to w, d(u, w) \leq t_{i+1}, \text{ and } w < v\}\) and set \(A = \hat{A}_{i+1}\); otherwise, we directly set \(A = \hat{A}_i\). Similarly, if \(\hat{B}_i = \emptyset\) and \(f_2 = 0\), then we still have to compute \(\hat{B}_{i+1} = \{w : u \to w, d(u, w) \leq t_{i+1}, \text{ and } w < v\}\) and set \(B = \hat{B}_{i+1}\); otherwise, we directly set \(B = \hat{B}_i\).

2: \(d(u, v) \leq t_0\). In this case we directly set \(A = \hat{A}_0 = \{w : u \to w, d(u, w) \leq t_0, \text{ and } w < v\}\) and \(B = \hat{B}_0 = \{w : w \to v, d(w, v) \leq t_0, \text{ and } w > u\}\).

If both \(A\) and \(B\) are empty, then we directly swap \(u\) and \(v\) and exit the procedure. Otherwise, let \(T_{\text{original}}\) be the topological order at the start of the execution of the procedure. For each \(u' \in \{u\} \cup A\), considered in order of decreasing \(T_{\text{original}}(u')\), we do the following. For each \(v' \in \{v' : v' \in B \cup \{v\}\} \text{ and } T_{\text{original}}(v') > T_{\text{original}}(u')\), considered in order of increasing \(T_{\text{original}}(v')\), recursively call \(\text{REORDER}(u', v', f'_1, f'_2)\). The first flag \(f'_1\) is set to 1 if and only if \(u' = u\) and \(A = \emptyset\), and the second flag \(f'_2\) is set to 1 if and only if \(v' = v\) and \(B = \emptyset\).

The idea behind the algorithm. Our algorithm broadly follows the algorithm by Ajwani et al. [1]. The main difference is that Ajwani et al. always set \(A\) to \(\hat{A}_{i+1}\) and \(B\) to \(\hat{B}_{i+1}\) during the execution of \(\text{REORDER}\) but we set \(A\) to \(\hat{A}_{i+1}\) only if \(\hat{A}_i = \emptyset\) and \(B\) to \(\hat{B}_{i+1}\) only if \(\hat{B}_i = \emptyset\). We shall prove that the total number of calls to \(\text{REORDER}\) won’t increase (bounded above by \(O(n^2)\)) by introducing this modification. Thus intuitively, our algorithm should run faster because in each call to \(\text{REORDER}\) we might only need to compute \(A_i\) and \(B_i\) instead of \(\hat{A}_{i+1}\) and \(\hat{B}_{i+1}\).
**Online Topological Ordering**

**3 Data Structures**

3.1 Main Data Structures

In the following, we shall describe the main data structures used in our algorithm.

The current topological order $T$ and its inverse $T^{-1}$ are stored as arrays. Thus finding $T(i)$ and $T^{-1}(u)$ can be done in constant time.

The DAG $G = (V, E)$ is stored as an array of vertices. For each vertex $u$ we maintain two adjacency lists $InList(u)$ and $OutList(u)$. The backward adjacency list $InList[u]$ contains all vertices $v$ with $(v, u) \in E$. The forward adjacency list $OutList(u)$ contains all vertices $v$ with $(u, v) \in E$. Adjacency lists are implemented by using $n$-bit arrays and support the following operations.

1. **List-Insert**: Given a vertex and a list, add the vertex to the list.
2. **List-Search**: Given a vertex and a list, determine if the vertex is in the list. If yes, return 1. Else, return 0.

Since the adjacency lists are implemented by using $n$-bit arrays, it takes $O(1)$ time per List-Insert or List-Search operation.
3.2 Auxiliary Data Structures

In the following we describe some auxiliary data structures which are used in our algorithm to improve the time complexity. For each vertex \( u \), we maintain two arrays of pails: \( \text{InPails}(u)[0 \cdots p + 1] \) and \( \text{OutPails}(u)[0 \cdots p + 1] \). \( \text{InPails}(u)[i] \) contains all vertices \( v \) with \( 0 < d(v, u) \leq t_i \) and \( (v, u) \in E \). \( \text{OutPails}(u)[i] \) contains all vertices \( v \) with \( 0 < d(u, v) \leq t_i \) and \( (u, v) \in E \). A vertex \( v \) in a pail is stored with its vertex index (and not \( T(v) \)) as its key. Pails are implemented by using balanced binary search trees and support the following operations.

1. \text{Pail-Insert}: Given a vertex and a pail, add the vertex to the pail.
2. \text{Pail-Delete}: Given a vertex and a pail, delete the vertex from the pail.
3. \text{Pail-Collect-All}: Given a pail, report all vertices in the pail.

It takes \( O(\log n) \) time per \text{Pail-Insert} or \text{Pail-Delete} and \( O(1 + \gamma) \) time per \text{Pail-Collect-All}, where \( \gamma \) is the number of vertices in the pail.

3.3 Instructions for Data Structures

Given a DAG \( G \) with a valid topological order and two vertices \( u \) and \( v \) with \( u \not\rightarrow v \), define sorted vertex sets \( \hat{A}_i \) and \( \hat{B}_i \), \( i = 0, \ldots, p + 1 \), as follows:

\[
\hat{A}_i = \{ w : u \rightarrow w \text{ and } d(u, w) \leq t_i \text{ and } w < v \} \text{ sorted by the topological order.}
\]

\[
\hat{B}_i = \{ w : w \rightarrow v \text{ and } d(u, w) \leq t_i \text{ and } w > u \} \text{ sorted by the topological order.}
\]

In the following we discuss how to insert an edge, compute vertex sets \( \hat{A}_i \), and \( \hat{B}_i \), and swap two vertices in terms of the above five basic operations.

a. Inserting an edge \((u, v)\): This means inserting vertex \( v \) to the forward adjacency list of \( u \) and \( u \) to the backward adjacency list of \( v \). This requires two \text{List-Insert} operations and at most \( 2(p + 2) \) \text{Pail-Insert} operations. Thus inserting an edge \((u, v)\) can be done in \( O(p \log n) = \tilde{O}(1) \) time.

b. Computing \( \hat{A}_i \) and \( \hat{B}_i \): \( \hat{A}_i \) can be computed by sorting the vertices in \( \text{OutPail}(u)[i] \) and choosing all \( w \) with \( w < v \). This can be done by first calling \text{Pail-Collect-All} to collect all the vertices in \( \text{OutPail}(u)[i] \) in \( O(t_i) \) time. Note that for all these vertices \( w \), we have \( 0 < T(w) - T(u) \leq t_i \). Thus we can sort these vertices in \( O(t_i) \) time by counting sort and then choose all \( w \) with \( w < v \) in \( O(|A_i| + 1) \) time. The total time required to compute \( \hat{A}_i \) is \( O(t_i + |A_i|) = O(t_i) \). Similarly, the time required to compute \( \hat{B}_i \) is \( O(t_i) \).

c. Computing \( \hat{A}_i \) and \( \hat{B}_i \) when \( t_i < d(u, v) \leq t_{i+1} \): \( \hat{A}_i \) can be computed by sorting the vertices in \( \text{OutPail}(u)[i] \). This can be done by first calling \text{Pail-Collect-All} to collect all the vertices in \( \text{OutPail}(u)[i] \) in \( O(|A_i| + 1) \) time, and then sorting these vertices in \( O((|A_i| + 1) \log n) \) time. Thus the total time required to compute \( \hat{A}_i \) is \( \tilde{O}(|A_i| + 1) \). Similarly, the time required to compute \( \hat{B}_i \) is \( \tilde{O}(|B_i| + 1) \).

d. Swapping \( u \) and \( v \): Without loss of generality assume \( u < v \). When swapping \( u \) and \( v \), we need to update the pails, \( T \), and \( T^{-1} \). We now show how to update the pails. For all vertices \( w \) with \( T(u) - t_i \leq T(w) < \min\{T(u), T(u) - t_i + d(u, v)\} \), we delete \( w \) from \( \text{InPail}(u)[i] \) and delete \( u \) from \( \text{OutPail}(w)[i] \). For all vertices \( w \) with \( \max\{T(v) + t_i - d(u, v), T(v)\} < T(w) \leq T(v) + t_i \), we delete \( w \) from \( \text{OutPail}(v)[i] \) and delete \( v \) from \( \text{InPail}(w)[i] \). It requires total \( O(d(u, v)) \) \text{Pail-Delete} operations for each \( i \). For all \( w \) with \( \max\{T(v), T(u) + t_i\} < T(w) \leq T(u) + t_i + d(u, v) \), if \( w \) is in the forward adjacency list of \( u \), then insert \( w \) into \( \text{OutPail}(u)[i] \) and insert \( u \) into \( \text{InPail}(w)[i] \). For all \( w \) with \( T(v) - t_i - d(u, v) \leq T(w) < \min\{T(u), T(v) - t_i + d(u, v)\} \), if \( w \) is in...
the backward adjacency list of v, then insert w into InPail(v)[i] and insert v into OutPail(w)[i]. It requires total $O(d(u, v))$ List-Search and Pail-Insert operations for each i. In total, we need $O(p \cdot d(u, v))$ List-Search, Pail-Insert, and Pail-Delete operations. Updating T and $T^{-1}$ is trivial and can be done in constant time. Thus the total time is $O(p \cdot d(u, v) \log n) = \tilde{O}(d(u, v))$.

4 Correctness

In this section, we shall argue that our algorithm is correct. We say a call of a recursive procedure leads to an operation "by itself" if and only if this operation is executed during the execution of this call and not during the execution of subsequent recursive calls. Given a DAG G with a valid topological order T and two vertices u, v of G with $u < v$, let $A' = \{w : u \rightarrow w \text{ and } w \leq v\}$ and $B' = \{w : w \rightarrow v \text{ and } w \geq u\}$. We say the flag $f_1$ of the call to Reorder(u, v, $f_1$, $f_2$) is correctly set only if $(f_1 \Rightarrow (A' = \emptyset)) = 1$. That is, if $f = 1$, then $A'$ is empty. We say the flag $f_2$ of the call to Reorder(u, v, $f_1$, $f_2$) is correctly set only if $f_2 \Rightarrow (B' = \emptyset) = 1$. That is, if $f_2 = 1$, then $B'$ is empty.

Lemma 1: Given a DAG G with a valid topological order and two vertices u, v of G with $u < v$, let $A' = \{w : u \rightarrow w \text{ and } w < v\}$ and $B' = \{w : w \rightarrow v \text{ and } w > u\}$. If the flags are correctly set, then in the call of Reorder(u, v, $f_1$, $f_2$), $A := \emptyset$ if and only if $A' = \emptyset$. Similarly, if the flags are correctly set, then in the call of Reorder(u, v, $f_1$, $f_2$), $B := \emptyset$ if and only if $B' = \emptyset$.

Proof: We shall only prove that $A := \emptyset$ if and only if $A' = \emptyset$. It can be proved in a similar way that $B := \emptyset$ if and only if $B' = \emptyset$.

Case 1: $t_i < d(u, v) \leq t_{i+1}$ for some i with $0 < i \leq p$. By the algorithm, $A := \emptyset$ if and only if

$\hat{A}_i = \{w : u \rightarrow w \text{ and } d(u, w) \leq t_i \text{ and } w < v\} = \emptyset$ and

$(\hat{A}_{i+1} = \{w : u \rightarrow w \text{ and } d(u, w) \leq t_{i+1} \text{ and } w < v\} = \emptyset \lor f_1 = 1)$. By $t_i < d(u, v) \leq t_{i+1}$, we have $\hat{A}_i \subseteq \hat{A}_{i+1} = A'$. From $\hat{A}_i \subseteq \hat{A}_{i+1} = A'$ and $(f_1 \Rightarrow (A' = \emptyset)) = 1$, we conclude that $A := \emptyset$ if and only if $A' = \emptyset$.

Case 2: $d(u, v) \leq t_0$. By the algorithm, $A := \emptyset$ if and only if

$\hat{A}_0 = \{w : u \rightarrow w \text{ and } d(u, w) \leq t_0 \text{ and } w < v\} = \emptyset$. By $d(u, v) \leq t_0$, we have $\hat{A}_0 = A'$. From $\hat{A}_0 = A'$, we conclude that $A := \emptyset$ if and only if $A' = \emptyset$. \qed

Lemma 2: Given a DAG G with a valid topological order and two vertices u, v of G with $u < v$, let $A' = \{w : u \rightarrow w \text{ and } w < v\}$ and $B' = \{w : w \rightarrow v \text{ and } w > u\}$. If the flags are correctly set, then Reorder(u, v, $f_1$, $f_2$) leads to a swap by itself if and only if $A' = \emptyset$ and $B' = \emptyset$.

Proof: By the algorithm, the call to Reorder(u, v, $f_1$, $f_2$) leads to a swap by itself if and only if in this call $A := \emptyset$ and $B := \emptyset$. By Lemma 1, $A := \emptyset$ and $B := \emptyset$ if and only if $A' = \emptyset$ and $B' = \emptyset$. \qed

Given a DAG G with a valid topological order, Reorder(u, v, $f_1$, $f_2$) is said to be local if and only if the execution of Reorder(u, v) will not affect $T(w)$ for all w with $w > v$ or $w < u$.

Lemma 3: Given a DAG G with a valid topological order and two vertices u, v with $u \neq v$, if the flags are correctly set, then Reorder(u, v, $f_1$, $f_2$) maintains a valid topological order and stop with $v < u$ and is local.
Proof: We prove the lemma by induction on $T(v) - T(u)$. When $T(u) - T(v) \leq 0$, the lemma is trivially correct.

Assume the lemma to be true when $T(v) - T(u) < k$, where $k > 0$. We shall prove that the lemma is true when $T(v) - T(u) = k$. If $A' = \{w : u \to w \text{ and } w < v\} = \emptyset$ and $B' = \{w : w \to v \text{ and } w > u\} = \emptyset$, then by Lemma 2, line 13 is executed. Thus, Reorder($u, v, f_1, f_2$) maintains a valid topological order, stops with $v < u$, and only $T(u)$ and $T(v)$ are updated, so the lemma follows. If $A' \neq \emptyset$ or $B' \neq \emptyset$, by Lemma 2, the for-loops are executed. Let $T'$ be the initial topological order. By our induction hypothesis and Lemma 1, the following loop invariants hold:

1. $T$ is a valid topological order.
2. At the start of the execution of line 19, $T(v') - T(u') < k$ and $T'(u) \leq T(u') < T(v') \leq T'(v)$.
3. At the start of the execution of line 19, $u' \neq v'$.
4. The flags are correctly set for the recursive call.

By the loop invariants and our induction hypothesis, each recursive call Reorder($u', v', f'_1, f'_2$) in the for-loops stops with $v' < u'$ and is local. Since the last recursive call is Reorder($u, v$), the entire procedure stops with $v < u$. Since each recursive call Reorder($u', v'$) is local and starts with $T'(u) \leq T(u') < T(v') \leq T'(v)$, the topological order of vertices $w$ with $T'(w) > T'(v)$ or $T'(w) < T'(u)$ is not affected. Thus the entire procedure maintains a valid topological order, stops with $v < u$, and is local.

Theorem 1: Given a DAG $G$ with a valid topological order and two vertices $u, v$ of $G$ with $u \not< v$, if the flags are correctly set, then Insert($u, v$) will add an edge $(u, v)$ to $G$ and maintain a valid topological order.

Proof: Because $u \not< v$, we know that $u$ and $v$ are two different vertices and either $u < v$ or $u > v$. If $u < v$ then the theorem is trivially correct. Assume that $v > u$. By Lemma 3, Reorder($v, u, 0, 0$) will stop with $u < v$ and maintain a valid topological order. Thus when line 2 of Insert is ready to be executed, we will have a valid topological order and $u < v$, and adding an edge $(u, v)$ to $G$ won’t affect the validness of the topological order.

In addition to the correctness of the algorithm, we also want to prove that the flags are always correctly set.

Lemma 4: Given a DAG $G$ with a valid topological order and two vertices $u, v$ of $G$ with $u < v$, consider a call to Reorder($u, v, f_1, f_2$). If the flags $f_1$ and $f_2$ are correctly set, then while executing this call, all subsequent calls to Reorder will also own correct flags.

Proof: We prove the lemma by induction on the depth of the recursion tree. By Lemma 3, the call to Reorder($u, v, f_1, f_2$) will stop, so the depth of the recursion tree is finite. If the depth is zero, then no recursive calls are made and the lemma follows.

Assume the lemma to be true when the depth of the recursion tree is less than $k$, where $k > 0$. We shall prove that the lemma is true when the depth of the recursion tree is $k$. Since $k > 0$, there is at least one recursive call. Thus the for-loops are executed. By Lemma 1 and Lemma 2, the following loop invariants hold:

1. $T$ is a valid topological order.
2. At the start of the execution of line 19, $T(u) < T(v)$.
3. At the start of the execution of line 19, $u' \neq v'$.
4. The flags are correctly set for the recursive call.
By the loop invariants and our induction hypothesis, each recursive call to \texttt{Reorder}(u', v', f_1', f_2') in the \texttt{for}-loops, together with all subsequent calls to \texttt{Reorder} in it, own correct flags, and the lemma follows.

**Theorem 2:** Given a DAG \( G \) with a valid topological order and two vertices \( u, v \) of \( G \) with \( u \not\sim v \), while executing \texttt{Insert}(u, v), the flags are correctly set for all calls to \texttt{Reorder}.

**Proof:** If \( u < v \) then there are no calls to \texttt{Reorder} made while executing \texttt{Insert}(u, v), and the lemma follows. If \( u > v \) then \texttt{Insert}(u, v) will call \texttt{Reorder}(v, u, 0, 0). By Lemma 4, the call to \texttt{Reorder}(v, u, 0, 0), together with all subsequent calls to \texttt{Reorder} in it, own correct flags, and the lemma follows.

## 5 Runtime

In this section, we analyze the time required to insert a sequence of edges. By Theorem 2, the flags are always correctly set. To avoid unnecessary discussion, each lemma, theorem, corollary, and proof in this section is state under the assumption that the flags are correctly set. To avoid notational overload, sometimes we shall just write \texttt{Reorder}(u, v) and ignore the flags.

### 5.1 Properties

**Lemma 5:** Given a DAG \( G \) with a valid topological order and two vertices \( u \) and \( v \) with \( u < v \), then during the execution of \texttt{Reorder}(u, v), we have that (1) \( T(x) \) is nondecreasing if \( u \sim x \); (2) \( T(y) \) is nonincreasing if \( y \sim v \); and (3) \( T(z) \) doesn’t change if \( u \not\sim z \) and \( z \not\sim v \).

**Proof:** We prove the lemma by induction on the depth of the recursion tree. By Lemma 3, the call to \texttt{Reorder}(u, v) will stop, so the depth of the recursion tree is finite. If the depth is zero, then no recursive calls are made. It follows that line 13 is executed, so the lemma follows.

Assume the lemma to be true when the depth of the recursion tree is less than \( k \), where \( k > 0 \). We shall prove that the lemma is true when the depth of the recursion tree is \( k \). Since \( k > 0 \), there is at least one recursive call. Thus the \texttt{for}-loops are executed. By Lemma 1 and Lemma 2 the following loop invariants hold:

1. \( T \) is a valid topological order.
2. At the start of the execution of line 19, \( T(u') < T(v') \).
3. At the start of the execution of line 19, \( u' \not\sim v' \).
4. The flags are correctly set for the recursive call.

Note that for each recursive call \texttt{Reorder}(u', v') in the \texttt{for}-loops, we have \( u \sim u' \) and \( v' \sim v \). Let \( u \sim x \). Then we have either \( u' \sim x \) or \( u' \not\sim x \) and \( x \not\sim v' \). By the induction hypothesis, \( T(x) \) is nondecreasing or doesn’t change during the execution of the recursive call. Thus \( T(x) \) is nondecreasing if \( u \sim x \). Let \( y \sim v \). Then we have either \( y \sim v' \) or \( u' \not\sim y \) and \( y \not\sim v' \). By the induction hypothesis, \( T(y) \) is nonincreasing or doesn’t change during the execution of the recursive call. Thus \( T(y) \) is nonincreasing if \( y \sim v \). Let \( u \not\sim z \) and \( z \not\sim v \). Then \( u' \not\sim z \) and \( z \not\sim v' \). By the induction hypothesis, \( T(z) \) doesn’t change during the execution of the recursive call. Thus \( T(z) \) doesn’t change if \( u \not\sim z \) and \( z \not\sim v \).
**Lemma 6:** Given a DAG $G$ with a valid topological order and two vertices $u$ and $v$ with $u < v$, for all $x$ and $y$, $\text{REORDER}(u, v)$ leads to at most one swap of $x$ and $y$.

**Proof:** Suppose that $\text{REORDER}(u, v)$ leads to at least one swap of $x$ and $y$. Without loss of generality we assume that $x < y$ before the first swap occurs. Then the first swap of $x$ and $y$ leads to increase of $T(x)$ and decrease of $T(y)$. Thus by Lemma 5, $T(x)$ is nondecreasing and $T(y)$ is nonincreasing during the execution of $\text{REORDER}(u, v)$. After the first swap, we have $x > y$. Since $T(x)$ is nondecreasing and $T(y)$ is nonincreasing, we know there are no more swaps. \qed

**Theorem 3:** While inserting a sequence of edges, for all vertices $x$ and $y$, after the first swap of $x$ and $y$, the relative order of $x$ and $y$ won’t change.

**Proof:** Suppose that the vertex pair $(x, y)$ is swapped at least once while inserting a sequence of edges. By Lemma 6 and the algorithm $\text{INSERT}$, each edge insertion leads to at most one swap of $x$ and $y$. Let $(u, v)$ be the first edge whose insertion leads to a swap of $x$ and $y$. Without loss of generality we assume that $x < y$ before the first swap occurs. We shall prove that $x > y$ will hold after the first swap of $x$ and $y$. Consider the execution process of $\text{INSERT}(u, v)$. The first swap occurs during the execution of $\text{REORDER}(v, u)$. By Lemma 5 we have $v \sim x$ and $y \sim u$. Since $T(x)$ is nondecreasing and $T(y)$ is nonincreasing, after the first swap of $x$ and $y$, $x > y$ will hold until $\text{REORDER}(v, u)$ returns. After $\text{REORDER}(v, u)$ returns, the edge $(u, v)$ will be added to the graph. Thus we will have $y \sim x$ and $x > y$ just after $\text{INSERT}(u, v)$ returns. By Lemma 3 calls to $\text{REORDER}$ maintain a valid topological order, so $x > y$ will hold hereafter. \qed

**Corollary 1:** While inserting a sequence of edges, for all vertices $x$ and $y$, there is at most one swap of $x$ and $y$.

**Lemma 7:** Given a DAG $G$ with a valid topological order and two vertices $u$ and $v$ with $u < v$, $\text{REORDER}(u, v)$ leads to a swap of $u$ and $v$.

**Proof:** We prove the lemma by induction on the depth of the recursion tree. By Lemma 3 the call to $\text{REORDER}(u, v)$ will stop, so the depth of the recursion tree is finite. If the depth is zero, then no recursive calls are made. It follows that line 13 is executed, so the lemma follows.

Assume the lemma to be true when the depth of the recursion tree is less than $k$, where $k > 0$. We shall prove that the lemma is true when the depth of the recursion tree is $k$. Since $k > 0$, there is at least one recursive call. Thus the for-loops are executed. By Lemma 1 and Lemma 2 the following loop invariants hold:

1. $T$ is a valid topological order.
2. At the start of the execution of line 19, $T(u') < T(v')$.
3. At the start of the execution of line 19, $u' \not\sim v'$.
4. The flags are correctly set for the recursive call.

Note that when executing the last recursive call $\text{REORDER}(u', v')$ in the for-loops, we have $u' = u$ and $v' = v$. By our induction hypothesis and the loop invariants, the last recursive call leads to a swap of $u$ and $v$, and the lemma follows. \qed
Theorem 4: While inserting a sequence of edges, the summation of $|A| + |B|$ over all calls of Reorder is $O(n^2)$.

Proof: Consider arbitrary vertices $u$ and $v$. We shall prove that $v \in B$ occurs at most once over all calls of Reorder$(u, \cdot)$. This proves that the summation of $|B|$ over all calls of Reorder$(u, \cdot)$ is less than or equal to $n$. Therefore the summation of $|B|$ over all calls of Reorder$(\cdot, \cdot)$ is less than or equal to $n^2$.

Consider the execution process of the first call of Reorder$(u, \cdot)$ for which $v \in B$. By the algorithm, a recursive call to Reorder$(u, \cdot)$ is made in the for-loops. Before the recursive call to Reorder$(u, \cdot)$ in the for-loops, at the start of the execution of each recursive call to Reorder$(u', v')$ in the for-loops, we have $u < \hat{v}$ and $(u < u' < v'$ or $u = u' < v' < \hat{v})$. This follows from the order in which we make the recursive calls and the local property (Lemma 3). Since $u < \hat{v}$ and $(u < u' < v'$ or $u = u' < v' < \hat{v})$, by the local property, $u < \hat{v}$ will hold during the execution of the call to Reorder$(u', v')$. Thus before the recursive call to Reorder$(u, \hat{v})$ in the for-loops, all recursive calls to Reorder$(u', v')$ in the for-loops won’t lead to a call to Reorder$(u, \cdot)$ for which $v \in B$; otherwise, the call to Reorder$(u, \cdot)$ for which $v \in B$ will lead to a call to Reorder$(u, \hat{v})$ which will further lead to $\hat{v} > u$ by Lemma 3 leading to a contradiction. Suppose for the contradiction that the recursive call to Reorder$(u, \hat{v})$ in the for-loops leads to a call to Reorder$(u, v'')$ for which $v \in B$. By the order in which we make the recursive calls and the local property, at the start of the execution of the recursive call to Reorder$(u, \hat{v})$ in the for-loops, we have $u < \hat{v}$. Since $\hat{v} \sim v''$, at the start of the execution of the recursive call to Reorder$(u, \hat{v})$ in the for-loops, we have $u < \hat{v} < v''$. Thus by the local property, $v'' > u$ will hold during the execution of the recursive call to Reorder$(u, \hat{v})$ in the for-loops. However, by Lemma 3 Reorder$(u, v'')$ stops with $v'' < u$, which is a contradiction. After the recursive call to Reorder$(u, \hat{v})$ in the for-loops, we have $\hat{v} < u$ by Lemma 3. Since $\hat{v} > u$ before the recursive call to Reorder$(u, \hat{v})$ in the for-loops, by Lemma 7 this recursive call leads to a swap of $u$ and $\hat{v}$. Thus after the recursive call to Reorder$(u, \hat{v})$ in the for-loops, we will have $\hat{v} < u$ and, by Lemma 3 the relative order of $u$ and $\hat{v}$ won’t change hereafter. Since $\hat{v} < u$ holds hereafter, there will be no more calls of Reorder$(u, \cdot)$ for which $\hat{v} \in B$. Putting all things together, it follows that $\hat{v} \in B$ occurs at most once over all calls of Reorder$(u, \cdot)$.

Similarly, we can prove that for arbitrary vertices $\hat{u}$ and $v$, $\hat{u} \in A$ occurs at most once over all calls of Reorder$(\cdot, \cdot)$. It follows that the summation of $|A|$ over all calls of Reorder$(\cdot, \cdot)$ is less than or equal to $n^2$. \hfill \Box

Corollary 2: While inserting a sequence of edges, Reorder is called $O(n^2)$ times.

Proof: By the algorithm, a call to Reorder for which $|A| + |B| = 0$ leads to a swap by itself. By Corollary 1 there are at most $n^2$ swaps. Thus there are at most $n^2$ calls to Reorder for which $|A| + |B| = 0$. By Theorem 4 there are $O(n^2)$ calls to Reorder for which $|A| + |B| > 0$. Therefore, there are $O(n^2)$ calls to Reorder in total. \hfill \Box

Let $S = \{(u, v) : \text{there is a swap of } u \text{ and } v \text{ such that } u < v \text{ while inserting the edges}\}$. Define

$$D(u, v) = \begin{cases} \text{the value of } d(u, v) \text{ while swapping } u \text{ and } v & \forall (u, v) \in S; \\ 0 & \forall (u, v) \not\in S. \end{cases}$$

Since by Corollary 1 each vertex pair is swapped at most once, $D(u, v)$ is well defined. Let $k$ be a number with $1 \leq k \leq n$. Define

$$D_k(u, v) = \begin{cases} D(u, v) & \text{if } D(u, v) \leq k; \\ k & \text{otherwise.} \end{cases}$$
The following theorem is the key to our runtime analysis.

**Theorem 5**: For all $k \in [1, n]$ with $k \geq n^{0.5}$, we have $\sum D_k(u, v) = O(n^2 \cdot \sqrt{k})$.

**Proof**: Let $k = n^r$. Let $T^*$ denote the final topological order. Define $x(T^*(u), T^*(v)) = D(u, v)$ and $z(T^*(u), T^*(v)) = D_k(u, v)$ for all vertices $u$ and $v$. The following linear inequalities are proved to be true by Ajwani et al. [1].

\[
\begin{align*}
(1) \quad & x(i, j) \leq 0 \text{ for all } 1 \leq i \leq n \text{ and } 1 \leq j \leq i, \\
(2) \quad & x(i, j) \leq n \text{ for all } 1 \leq i \leq n \text{ and } i < j \leq n, \\
(3) \quad & \sum_{j > i} x(i, j) - \sum_{j < i} x(j, i) \leq n \text{ for all } 1 \leq i \leq n.
\end{align*}
\]

It is easy to derive the following linear inequalities from the definitions of $x(i, j)$ and $z(i, j)$.

\[
\begin{align*}
(4) \quad & z(i, j) \leq n^r \text{ for all } 1 \leq i, j \leq n, \\
(5) \quad & z(i, j) \leq x(i, j) \text{ for all } 1 \leq i, j \leq n, \\
(6) \quad & 0 \leq z(i, j) \text{ for all } 1 \leq i \leq n \text{ and } 1 \leq j \leq n, \\
(7) \quad & 0 \leq x(i, j) \text{ for all } 1 \leq i \leq n \text{ and } 1 \leq j \leq n.
\end{align*}
\]

We aim to estimate an upper bound on the objective values of the following linear program.

\[
\max \sum_{1 \leq i, j \leq n} z(i, j) \text{ such that }
\begin{align*}
(1) \quad & x(i, j) \leq 0 \text{ for all } 1 \leq i \leq n \text{ and } 1 \leq j \leq i, \\
(2) \quad & x(i, j) \leq n \text{ for all } 1 \leq i \leq n \text{ and } i < j \leq n, \\
(3) \quad & \sum_{j > i} x(i, j) - \sum_{j < i} x(j, i) \leq n \text{ for all } 1 \leq i \leq n, \\
(4) \quad & z(i, j) \leq n^r \text{ for all } 1 \leq i, j \leq n, \\
(5) \quad & z(i, j) \leq x(i, j) \text{ for all } 1 \leq i, j \leq n, \\
(6) \quad & 0 \leq z(i, j) \text{ for all } 1 \leq i \leq n \text{ and } 1 \leq j \leq n, \\
(7) \quad & 0 \leq x(i, j) \text{ for all } 1 \leq i \leq n \text{ and } 1 \leq j \leq n.
\end{align*}
\]

In order to prove the upper bound on the objective values of the above linear program, we consider its dual problem.

\[
\min \left[ \sum_{0 \leq i, j < n} n \cdot Y_{i,n+j} + \sum_{0 \leq i < n} n \cdot Y_{n^2+i} + \sum_{0 \leq i < n} n^r \cdot Z_{i,n+j} \right] \text{ such that }
\begin{align*}
(1) \quad & Y_{i,n+j} - W_{i,n+j} \geq 0 \text{ for all } 0 \leq i \leq n \text{ and } 0 \leq j \leq i, \\
(2) \quad & Y_{i,n+j} - W_{i,n+j} + Y_{n^2+i} - Y_{n^2+j} \geq 0 \text{ for all } 0 \leq i < n \text{ and } j > i, \\
(3) \quad & Z_{i,n+j} + W_{i,n+j} \geq 1 \text{ for all } 0 \leq i < n \text{ and } 0 \leq j < n, \\
(4) \quad & Y_i \geq 0 \text{ for all } 0 \leq i < n^2 + n, \\
(5) \quad & Z_i \geq 0 \text{ for all } 0 \leq i < n^2, \\
(6) \quad & W_i \geq 0 \text{ for all } 0 \leq i < n^2.
\end{align*}
\]

Let $c$ be a large enough constant, e.g. 120, such that $(i + c \cdot n^{r/2})^{r/2} \geq (i^{r/2} + 1)$ for any $1 \leq i \leq n$. The following is a feasible solution to the dual problem.
This feasible solution to the dual problem has an objective value of $O(n \sum_{i=1}^{n} i^{r/2} + n \cdot n^r \cdot c \cdot n^{r/2}) = O(n^{2+r/2} + n^{1+r+r/2}) = O(n^{2+r/2}) = O(n^2 \cdot \sqrt{r})$, which by the primal-dual theorem is an upper bound on the objective values of the original linear program.

Lemma 8: Given a DAG $G$ with a valid topological order and two vertices $u$ and $v$ with $u < v$, consider a call to Reorder$(u, v)$. If $A' = \{w : u \rightarrow w \text{ and } w < v\} = \emptyset$, then when executing this call, we shall have the first flag $f_1 = 1$ for all subsequent calls to Reorder$(u, \cdot)$.

Proof: We prove the lemma by induction on the depth of the recursion tree. By Lemma 3 the call to Reorder$(u, v)$ will stop, so the depth of the recursion tree is finite. If the depth is zero, then no subsequent recursive calls are made, so the lemma follows.

Assume the lemma to be true when the depth of the recursion tree is less than $k$, where $k > 0$. We shall prove that the lemma is true when the depth of the recursion tree is $k$. Since $k > 0$, there is at least one recursive call. Thus the for-loops are executed. By Lemma 1 $A = \emptyset$. By the local property (Lemma 2) and the order in which we make the recursive calls, any subsequent call to Reorder$(u, \cdot)$ must occur during the execution of last iteration of the outer for-loop. Consider any first level recursive call Reorder$(u', v', f_1', f_2')$ in the last iteration of the outer for-loop. Note that we have $u' = u < v'$ and $\{w : u \rightarrow w \text{ and } w < v' \leq v\} \subseteq A' = \emptyset$ when this call is ready to be executed. Since $u' = u$ and $A = \emptyset$, we also have $f_1' = 1$. By the induction hypothesis, we also have the first flag $f_1 = 1$ for all subsequent calls to Reorder$(u, \cdot)$ while executing this call. It completes the proof.

Lemma 9: Given a DAG $G$ with a valid topological order and two vertices $u$ and $v$ with $u < v$, let $A' = \{w : u \rightarrow w \text{ and } w < v\} = \emptyset$. Then a call to Reorder$(u, v)$ will stop with $u$ at the initial position of $v$. That is, letting $T_{before}$ be the topological order just before the call to Reorder$(u, v)$, then the call to Reorder$(u, v)$ will return a topological order $T_{after}$ such that $T_{after}(u) = T_{before}(v)$.

Proof: We prove the lemma by induction on the depth of the recursion tree. By Lemma 3 the call to Reorder$(u, v)$ will stop, so the depth of the recursion tree is finite. If the depth is zero, then no recursive calls are made. It follows that line 13 is executed, so the lemma follows.

Assume the lemma to be true when the depth of the recursion tree is less than $k$, where $k > 0$. We shall prove that the lemma is true when the depth of the recursion tree is $k$. Since $k > 0$, there is at least one recursive call. Thus the for-loops are executed. Let $T_{before}$ be the initial topological order. By Lemma 1, Lemma 2 and the induction hypothesis, the following loop invariants hold:
1. $T$ is a valid topological order.
2. At the start of the execution of line 19, $T(u) = T(u') < T(v') = T_{\text{before}}(v')$.
3. At the start of the execution of line 19, $u' \not\prec v'$.
4. After the execution of line 19, $T(u) = T_{\text{before}}(v')$.
5. The flags are correctly set for the recursive call.

Note that for the last recursive call $\text{REORDER}(u', v')$ in the for-loops, we have $v' = v$. Thus by the loop invariants, we have $T(u) = T_{\text{before}}(v)$ after the last recursive call in the for-loops. \hfill \square

**Lemma 10:** Given a DAG $G$ with a valid topological order and two vertices $u$ and $v$ with $u < v$, when executing a call of $\text{REORDER}(u, v)$ in which both $\hat{A}_i$ and $\hat{A}_{i+1}$ are computed, $u$ will be moved right with distance at least $t_i$.

**Proof:** Since both $\hat{A}_i$ and $\hat{A}_{i+1}$ are computed, by the algorithm, we have $t_i < d(u, v) \leq t_{i+1}$ and $\hat{A}_i = \emptyset$. There are two cases to consider.

Case 1: $\hat{A}_{i+1} = \emptyset$. It follows that $A' = \{w : u \rightarrow w \text{ and } w < v\} = \emptyset$. By Lemma 9, $u$ will be moved right to the initial position of $v$. Since initially $d(u, v) > t_i$, $u$ will be moved right with distance at least $t_i$.

Case 2: $\hat{A}_{i+1} \neq \emptyset$. By the algorithm, the for-loops are executed. Let $\hat{u}$ be the vertex with lowest topological order in $\hat{A}_{i+1}$. Let $T_{\text{initial}}$ be the initial topological order. Since $\hat{A}_i = \emptyset$, we have initially $d(u, \hat{u}) > t_i$, i.e., $T_{\text{initial}}(\hat{u}) - T_{\text{initial}}(v) > t_i$. By Lemma 3 and the order in which we make the recursive calls, before the last iteration of the outer for-loop, $T(v) \geq T_{\text{initial}}(\hat{u})$ will hold. Consider the execution of the last iteration of the outer for-loop. Let $T_{\text{start}}$ be the topological order at the start of this iteration. Then we have $T_{\text{start}}(v) - T_{\text{initial}}(u) \geq T_{\text{initial}}(\hat{u}) - T_{\text{initial}}(u) > t_i$. By Lemma 3 and Lemma 9, the following loop invariants hold.

1. $T$ is a valid topological order.
2. At the start of the execution of the last iteration of line 19, $T(u) = T(u') < T(v') = T_{\text{start}}(v')$.
3. At the start of the execution of line 19, $u' \not\prec v'$.
4. At the start of the execution of line 19, $\{w : u \rightarrow w \text{ and } w < v' \leq v\} = \emptyset$.
5. After the execution of line 19, $T(u) = T_{\text{start}}(v')$.
6. The flags are correctly set for the recursive call.

Thus after this iteration, we will have $T(u) = T_{\text{start}}(v) > T_{\text{initial}}(u) + t_i$, and the lemma follows. \hfill \square

**Theorem 6:** While inserting a sequence of edges, there are at most $O(\frac{n^2 \sqrt{t_{i+1}}}{t_i})$ calls of $\text{REORDER}$ for which both $\hat{A}_i$ and $\hat{A}_{i+1}$ are computed. Similarly, there are at most $O(\frac{n^2 \sqrt{t_{i+1}}}{t_i})$ calls of $\text{REORDER}$ for which both $\hat{B}_i$ and $\hat{B}_{i+1}$ are computed.

**Proof:** We shall only prove that there are at most $O(\frac{n^2 \sqrt{t_{i+1}}}{t_i})$ calls of $\text{REORDER}$ for which both $\hat{A}_i$ and $\hat{A}_{i+1}$ are computed. It can be proved in a similar way that there are at most $O(\frac{n^2 \sqrt{t_{i+1}}}{t_i})$ calls of $\text{REORDER}$ for which both $\hat{B}_i$ and $\hat{B}_{i+1}$ are computed.

Let $C_1(u, v), C_2(u, v), \ldots, C_m(u, v) (u, v)$ be the calls to $\text{REORDER}(u, v)$ for which both $\hat{A}_i$ and $\hat{A}_{i+1}$ are computed for all vertices $u$ and $v$. Let $S_i(u, v) = \{w : C_i(u, v) \text{ leads to a swap of } u \text{ and } w\}$
for \( i = 1, \ldots, m(u,v) \). We shall prove that \( \sum_{u,v} \sum_{i=1}^{m(u,v)} \sum_{w \in S_i(u,v)} D(u, w) = O(n^2 \sqrt{t_{i+1}}) \). By Lemma 10, \( \sum_{w \in S_i(u,v)} D(u, w) \geq t_i \) for all vertices \( u \) and \( v \) and \( i = 1, \ldots, m(u,v) \). It follows that \( \sum_{u,v} \sum_{i=1}^{m(u,v)} 1 = O\left(\frac{n^2}{t_{i+1}}\right) \).

In a call to \textsc{Reorder}(u, v), \( \hat{A}_i \) and \( \hat{A}_{i+1} \) are both computed only if \( t_i < d(u,v) \leq t_{i+1} \). Thus by the local property (Lemma 2), we have \( D(u, w) \leq t_{i+1} \) for all \( w \in S_i(u,v) \), \( i = 1, \ldots, m(u,v) \). By Theorem 5, we have \( \sum_{u,v} D_i(u,v) = O(n^2 \sqrt{t_{i+1}}) \). Thus it suffices to show that in the summation \( \sum_{u,v} \sum_{i=1}^{m(u,v)} \sum_{w \in S_i(u,v)} D(u, w) \), \( D(u, w) \) is counted at most twice for each vertex pair \( (u,w) \).

To show that in the summation \( \sum_{u,v} \sum_{i=1}^{m(u,v)} \sum_{w \in S_i(u,v)} D(u, w) \), \( D(u, w) \) is counted at most twice for each vertex pair \( (u,v) \), we only have to prove that \( \hat{S}_i(u,v) \cap \hat{S}_j(u,v') \cap \hat{S}_k(u,v'') = \emptyset \) if \( C_i(u,v), C_j(u,v'), \) and \( C_k(u,v'') \) are three different calls.

Suppose for the contradiction that \( w \in \hat{S}_i(u,v) \cap \hat{S}_j(u,v') \cap \hat{S}_k(u,v'') \) and \( C_i(u,v), C_j(u,v'), \) and \( C_k(u,v'') \) are three different calls. Without loss of generality, we assume \( C_i(u,v) \) occurs before \( C_j(u,v') \) and \( C_j(u,v') \) occurs before \( C_k(u,v'') \). By Corollary 1 there is only one swap of \( u \) and \( w \), so \( C_j(u,v') \) must be a subsequent recursive call which occurs during the execution of \( C_i(u,v) \) and \( C_k(u,v'') \) must be a subsequent recursive call which occurs during the execution of \( C_j(u,v') \). Consider the execution of \( C_i(u,v) \). By Lemma 8 and the order in which we make the recursive calls in the \texttt{for-loops}, \( C_j(u,v') \) must occur during the last iteration of the outer \texttt{for-loop}. Note that by Lemma 3 before the last iteration of the outer \texttt{for-loop} begins, all vertices in \( \hat{A}_{i+1} = \{ w : u \rightarrow w, d(u,w) \leq t_{i+1} \} \) and \( v < w \} = \{ w : u \rightarrow w \) and \( v < w \} \) are moved to the right of \( v \). Thus when the last iteration of the outer \texttt{for-loop} begins, there are no vertices \( w \) between \( u \) and \( v \) with \( u \rightarrow w \). Therefore, by the local property of \textsc{Reorder}, during the last iteration of the outer \texttt{for-loop}, for each call to \textsc{Reorder}(u,v'), we have \( \{ w : u \rightarrow w \) and \( v < w' \} = \emptyset \). By Lemma 8 we have the first flag \( f_1 = 1 \) for each subsequent call to \textsc{Reorder}(u,·) during the execution of \( C_j(u,v') \). It follows that the first flag \( f_1 \) of the call \( C_k(u,v'') \) is 1. Thus by the algorithm, we don’t compute \( \hat{A}_{i+1} \) in the call \( C_k(u,v'') \), which is a contradiction.

### 5.2 Runtime Analysis

**Lemma 11:** While inserting a sequence of edges, the total time spent on executing line 2 of \textsc{Insert} is \( \hat{O}(n^2) \).

**Proof:** As discussed in Section 3.3 each execution of line 2 of \textsc{Insert} can be done in \( \hat{O}(1) \) time. Since there are at most \( n(n-1)/2 \) edge insertions, the lemma follows.

**Lemma 12:** While inserting a sequence of edges, the total time spent on computing \( \hat{A}_i \) and \( \hat{B}_i \), \( i = 1, \ldots, p \), over all calls of \textsc{Reorder}(u,v) with \( t_i < d(u,v) \leq t_{i+1} \) is \( \hat{O}(n^2) \).

**Proof:** As discussed in Section 3.3 it needs \( \hat{O}(|\hat{A}_i| + |\hat{B}_i| + 1) \) time to compute \( \hat{A}_i \) and \( \hat{B}_i \) in a call of \textsc{Reorder}(u,v) if \( d(u,v) < t_{i+1} \). By Theorem 3 the summation of \( |\hat{A}_i| + |\hat{B}_i|, i = 1, \ldots, p \), over all calls of \textsc{Reorder} is \( O(n^2) \). By Corollary 2 \textsc{Reorder} is called \( O(n^2) \) times. Thus the summation of \( (|\hat{A}_i| + |\hat{B}_i| + 1) \) over all calls of \textsc{Reorder} is \( O(n^2) \), and the lemma follows.

**Lemma 13:** For each \( i \) with \( 1 \leq i \leq p + 1 \), while inserting a sequence of edges, the total time spent on computing \( \hat{A}_i \) and \( \hat{B}_i \), over all calls of \textsc{Reorder}(u,v) with \( d(u,v) \leq t_i \) is \( O\left(\frac{n^2}{t_{i-1}}\right) \).
Proof: If \( d(u, v) \leq t_i \), then by the algorithm, \( \hat{A}_i \) is computed only if \( \hat{A}_{i-1} \) is also computed and is empty. Thus by Theorem 6 there are at most \( O\left(\frac{n^{2t_i^{1/2}}}{t_{i-1}}\right) \) such calls. As discussed in Section 3.3 it needs \( O(t_i) \) time to compute \( \hat{A}_i \) in each of such calls. Thus the total time spent on computing \( \hat{A}_i \) over all calls of \textsc{Reorder}(u, v) with \( d(u, v) \leq t_i \) is \( O\left(\frac{n^{2t_i^{1/2}}}{t_{i-1}}\right) \). Similarly, the total time spent on computing \( \hat{B}_i \) over all calls of \textsc{Reorder}(u, v) with \( d(u, v) \leq t_i \) is \( O\left(\frac{n^{2t_i^{3/2}}}{t_{i-1}}\right) \).

Lemma 14: While inserting a sequence of edges, the total time spent on computing \( \hat{A}_0 \) and \( \hat{B}_0 \) over all calls of \textsc{Reorder} is \( O(n^2 \cdot t_0) \).

Proof: As discussed in Section 3.3 it needs \( O(t_0) \) time to compute \( \hat{A}_0 \) and \( \hat{B}_0 \) in a call of \textsc{Reorder}. By Corollary 2 \textsc{Reorder} is called \( O(n^2) \) times, and the lemma follows.

Lemma 15: While inserting a sequence of edges, the total time spent on swapping vertices is \( \tilde{O}(n^{2.5}) \).

Proof: As discussed in Section 3.3 each swap of vertices \( u \) and \( v \) with \( d(u, v) < t \) can be done in \( \tilde{O}(d(u, v)) \) time. By Theorem 5 \( \sum D_n(u, v) = O(n^{2.5}) \). Thus the total time is \( \tilde{O}(n^{2.5}) \).

Theorem 7: While inserting a sequence of edges, the total time required is \( \tilde{O}\left(\sum_{i=1}^{p+1} \frac{n^{2t_i^{3/2}}}{t_{i-1}} + n^2 \cdot t_0\right) \).

Proof: It follows directly from the above lemmas.

6 Further Discussion

Let \( n^{0.5} < t_0 < t_1 < \ldots < t_p < t_{p+1} = n \) and \( p = O(\log n) \). We have known that the runtime is \( \tilde{O}\left(\sum_{i=1}^{p+1} \frac{n^{2t_i^{3/2}}}{t_{i-1}} + n^2 \cdot t_0\right) \). In this section, we show how to determine the values of these parameters. By letting

\[
\frac{n^3}{\sqrt{t_{p+1}}} = \frac{n^{2t_{p+1}^{3/2}}}{t_p} = \frac{n^{2t_{p-1}^{3/2}}}{t_{p-1}} = \ldots = \frac{n^{2t_1^{3/2}}}{t_0} = n^2 \cdot t_0,
\]

we have

\[
t_1 = t_0^{4/3}, t_i = \left(\frac{t_{i-1}^{5/3}}{t_{i-2}^{2/3}}\right) \quad \text{for all } i = 2, \ldots, p + 1.
\]

Let \( t_i = t_0^{x_i} \) for \( i = 0, \ldots, p + 1 \). we have

\[
x_0 = 1, x_1 = 4/3, \quad \text{and } x_i = \frac{5x_{i-1}}{3} - \frac{2x_{i-2}}{3} \quad \text{for all } i = 2, \ldots, p + 1.
\]

By solving this linear second order recurrence relation, we get \( x_i = 2 - (2/3)^{i} \) for all \( i = 0, 1, 2, \ldots, p + 1 \).

It follows that \( t_{p+1} = t_0^{2 - (2/3)^{p+1}} \). Since \( t_{p+1} = n \), we have \( t_0 = n^{f(p)} \), where \( f(p) = \frac{3^{p+1}}{(2.3^{p+1}-2^{p+1})} \).

Corollary 3: While inserting a sequence of edges, the total time required is \( \tilde{O}(n^{2+f(p)}) \) if \( t_0 = n^{f(p)} \) and \( t_i = t_0^{2-(2/3)^i} \) for all \( i = 1, \ldots, p + 1 \), where \( f(p) = \frac{3^{p+1}}{(2.3^{p+1}-2^{p+1})} \).
Note that \( f(p) = \frac{3p+1}{2^{\frac{3p+1}{2p+1}}} \) = 0.5 + \( \frac{2^{p+1} - 2p-1}{2^{p+1} - 2p+1} \). By letting \( \epsilon(p) = \frac{2^{p+1} - 2p+1}{2^{p+1} - 2p-1} \), we have \( \epsilon(p) < \frac{3p}{2^{p+1}} < \frac{3}{2} \cdot 2^p \). By letting \( p = \log_{3/2} n \), we have \( 1 < n^{\epsilon(p)} < n^{1/n} < 2 \) when \( n > 2 \). Thus \( \tilde{O}(n^{1+f(p)}) = \tilde{O}(n^{2.5+\epsilon(p)}) = \tilde{O}(n^{2.5}) \) if we choose \( p = \lceil \log_{3/2} n \rceil \). The following theorem summarizes our discussion.

**Theorem 8:** There exists an \( \tilde{O}(n^{2.5}) \)-time algorithm for online topological ordering.

### 7 Concluding Remarks

We propose an \( \tilde{O}(n^{2.5}) \)-time algorithm for maintaining the topological order of a DAG with \( n \) vertices while inserting \( m \) edges. By combining this with the result in [4], we get an upper bound of \( \tilde{O}(\min\{m^{3/2}, n^{2.5}\}) \) for online topological ordering. The only non-trivial lower bound is due to Ramalingam and Reps [3], who show that any algorithm need \( \Omega(n \log n) \) time while inserting \( n-1 \) edges in the worst case if all labels are maintained explicitly. Precisely, our algorithm runs in \( O(n^{2.5} \log^2 n) \) time. Choosing a better implementation for the pails, like data structures discussed in Section 5 of [1], can further drop one log factor from the runtime.

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