Beyond identical utilities: buyer utility functions and fair allocations

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Abstract

The problem of finding envy-free allocations of indivisible goods can not always be solved; therefore, it is common to study some relaxations such as envy-free up to one good (EF1). Another property of interest for efficiency of an allocation is the Pareto Optimality (PO). Under additive utility functions, it is possible to find allocations EF1 and PO using Nash social welfare. However, to find an allocation that maximizes the Nash social welfare is a computationally hard problem. In this work we propose a polynomial time algorithm which maximizes the utilitarian social welfare and at the same time produces an allocation which is EF1 and PO in a special case of additive utility functions called buyer utility functions. Moreover, a slight modification of our algorithm produces an allocation which is envy-free up to any positively valued good (EFX).

Keywords: Allocation of indivisible goods, envy-free up to one good, efficiency, additive utility function.
1 Introduction

The resource allocation problem has been widely studied in mathematics and economics for almost a century, [1–6]. The main elements in the problem are agents and resources. The goal is to distribute (or allocate) the resources among the agents in a “good manner.” The agents can represent individuals, objects, government institutions, among others, depending on the application. The set of resources or goods to be distributed can be divisible or indivisible. In general, these sets are considered finite. In this work we study the problem of allocating indivisible resources among a group of agents, with the aim of satisfying both the group and each individual in the best possible way.

This topic has many applications, for instance in solving divorce disputes, dividing an inheritance, sharing apartment rents, or even assigning household chores. In the last decade, there has been a considerable interest in computational aspects of this problem. In particular, in Artificial Intelligence and more specifically in MultiAgent Systems, these problems are studied and renamed MultiAgent Resource Allocations (MARA) problems [7, 8].

Finding a correct distribution of resources consists of distributing all the resources among the agents fairly and efficiently.

To establish efficiency and some criterion of fairness, it is necessary to consider the preferences that each agent has over resources. In general, these preferences over resources are established through additive utility functions. Traditionally, fairness is established through properties such as envy-freeness or proportionality. However, there are situations where it is impossible to find allocations that meet any of these properties. Thus, other weaker versions of fairness, such as envy-free up to one good [9] or proportionality up to one good [10] are considered. Although there are results that, under certain conditions, guarantee the existence of allocations with some fairness property (see [5]), finding such allocations is a computationally hard problem [11].

Efficiency, also known as Pareto efficiency or Pareto optimality, is related to the group satisfaction by an allocation. One way to find efficient allocations is through social welfare functions. Caragianis et al, [5], showed that, under additive utility functions, it is always possible to find an allocation that is Pareto optimal and envy-free up to one good. Actually, they prove that some of the allocations that maximize the Nash welfare are Pareto optimal and envy-free up to one good. Unfortunately, finding allocations which maximize the Nash welfare is also an NP-Hard problem [11] (see also [7]).

Searching allocations which maximize the utilitarian social welfare is in general a more tractable problem from the computational point of view and, because of that, commonly used. A well-known result is that, under additive utility functions, allocations that maximize this social welfare are Pareto optimal (see Theorem 2). However, the converse is not true. In Example 1 is proposed an allocation that is Pareto optimal and does not maximize utilitarian social welfare. In general, allocations that maximize this social welfare do not always satisfy fairness properties. In Example 1, we find that no allocation that maximizes utilitarian social welfare is envy free up to one good.
When we consider additive utility functions, it is possible to define a procedure that allows to find all the allocations that maximize utilitarian social welfare, see [6]. Moreover, finding these allocations is a computationally tractable problem. Actually, in this work we propose a very simple algorithm in polynomial time, for scenarios of additive utilities, that finds an allocation which maximizes the utilitarian social welfare.

Unlike the allocations that maximize the Nash social welfare which are EF1, the allocations that maximize the utilitarian social welfare are not, in general, EF1. Moreover, there are additive scenarios in which the property EF1 fails for every allocation which maximizes the utilitarian social welfare (see Example 1). Indeed, in that example we can see that the Nash and the utilitarian social welfare are independent.

However, for some specialized scenarios, there are very simple algorithms in polynomial time which find EF1 even EFX0 allocations. That is the case of identical utilities (see Barman et al. [12]). In this work, we consider a class of additive functions, called the buyer utility functions. This class is more general than the class of identical utilities. In the framework of these utility functions, the following results are established:

1. A characterization for Pareto optimal allocations (Theorem 5).
2. Each allocation that maximizes Nash social welfare also maximizes utilitarian social welfare (Theorem 6).
3. Constructive proofs of the existence of allocations which maximize the utilitarian social welfare, which are PO and respectively EF1 and EFX.

These allocations are obtained in polynomial time (Theorems 7, 8 and 9). Moreover, we propose a basic algorithm in $O(nm)$ which finds, under additive utility functions, an allocation that maximizes the utilitarian social welfare (Theorem 4).

The rest of this work is organized as follows. Section 2 introduces the concepts and problems studied in this paper. In Section 3, we propose a very simple and tractable algorithm for maximizing the utilitarian social welfare and study its justice properties in the case of buyer utilities. In Section 4, we show some experimental results of our algorithm at work and make some comparisons with the algorithms proposed by Bartman et al. [12]. We conclude in Section 5 with some final remarks. The proofs can be found in Appendix A.

2 Preliminaries

The set of agents is denoted by $A = \{1, \ldots, n\}$ and the set of resources is denoted by $R = \{r_1, \ldots, r_m\}$. So, $|R| = m$ and $|A| = n$.

An allocation of resources is a function $F : R \rightarrow A$. For all agent $i$, $F^{-1}(i) = \{r \in R : F(r) = i\}$ is the set of resources (or bundle) assigned to $i$. The set of all possible allocations is denoted by $\mathcal{A}^R$. The number of possible allocations depends on $n$ and $m$, given that $|\mathcal{A}^R| = |A|^{|R|} = n^m$. The set of all subsets of $R$ is denoted by $\mathcal{P}(R)$. The preference of agents over resources is established through utility functions, as defined below:
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Definition 1 A function \( u : \mathcal{P}(\mathcal{R}) \rightarrow \mathbb{R} \) is an additive utility function if:

- \( \forall S \in \mathcal{P}(\mathcal{R}), u(S) \geq 0; \)
- \( u(\emptyset) = 0; \)
- \( \forall S \in \mathcal{P}(\mathcal{R}) \) with \( S \neq \emptyset, u(S) = \sum_{s \in S} u(\{s\}). \)

For each \( i \in \mathcal{A}, u_i \) denotes the additive utility function associated to \( i \). For simplicity, \( u(\{s\}) \) will be denoted by \( u(s) \). If every agent establishes its preferences through an additive utility function in a problem of indivisible resource allocation, we say that it is an additive scenario.

Definition 2 Let \( u \) be a function \( u : \mathcal{P}(\mathcal{R}) \rightarrow \mathbb{R} \). We say that \( u \) is a buyer utility function if:

- \( u \) is an additive utility function;
- for every \( r_k \in \mathcal{R} \) there exists an unique \( p_k > 0 \) such that \( u(r_k) \in \{0, p_k\} \).

We say that we are in a buyer scenario when every agent of the resource allocation problem has a buyer utility function. A buyer scenario in which the value 0 is not taken by the utility functions is called a scenario of identical utility functions. These scenarios were considered by Barman et al. [12].

Note that the buyer scenario is a simplification of the Fisher market model in economy [13].

Clearly, if \( u \) is an additive utility function such that for each \( r_k \in \mathcal{R}, u(r_k) \in \{0, 1\} \), then \( u \) is a buyer utility function. This type of function is known as binary utility function, see [14]. Thus, the buyer scenarios are a generalization of identical scenarios and a generalization of binary scenarios.

2.1 Fairness, efficiency, and social welfare

An attractive fairness criterion in additive scenarios is the absence of envy. If no agent strictly prefers the bundle assigned to another agent instead of its own bundle, the allocation is envy-free. The following is the formal definition:

Definition 3 Let \( F \) be in \( \mathcal{A}^\mathcal{R} \), \( F \) is an envy-free allocation (EF) if:

\[
\forall i, j \in \mathcal{A}, \quad u_i(F^{-1}(i)) \geq u_i(F^{-1}(j)).
\]

If there exists an agent \( i \in \mathcal{A} \) such that \( u_i(F^{-1}(i)) < u_i(F^{-1}(j)) \) for some \( j \in \mathcal{A} \), then the agent \( i \) envies the agent \( j \). The property EF is the most desirable property, but in a simple example of an indivisible resource with two agents it is impossible to find an allocation without envy. In the literature [5, 9, 14], weaker versions of the envy free property can be found. The weakest among these is envy free up to one good. The following definition establishes the main weak envy free notions.
Definition 4 Let $F$ be in $\mathcal{A}^R$, $F$ is an envy-free up to one good (EF1) allocation if
\begin{equation}
\forall i,j \in A, \exists g \in F^{-1}(j) \text{ such that } u_i(F^{-1}(i)) \geq u_i(F^{-1}(j) \setminus \{g\}) \tag{2}
\end{equation}
$F$ is an envy-free up to any positively valued good (EFX) allocation if
\begin{equation}
\forall i,j \in A, \forall g \in F^{-1}(j) \text{ with } u_i(g) > 0, \ u_i(F^{-1}(i)) \geq u_i(F^{-1}(j) \setminus \{g\}) \tag{3}
\end{equation}
$F$ is an envy-free up to any good (EFX0) allocation if
\begin{equation}
\forall i,j \in A, \forall g \in F^{-1}(j), \ u_i(F^{-1}(i)) \geq u_i(F^{-1}(j) \setminus \{g\}) \tag{4}
\end{equation}

It is clear that in the case of additive utilities we have:
\[
EF \Rightarrow EFX_0 \Rightarrow EFX \Rightarrow EF1
\]

The efficiency, also known as Pareto efficiency or Pareto optimality, aims at characterizing when the allocation best satisfies the group.

Definition 5 Let $F$ and $G$ be in $\mathcal{A}^R$.
- $F$ is Pareto dominated by $G$, if:
  \begin{itemize}
  \item $\forall i \in A, u_i(F^{-1}(i)) \leq u_i(G^{-1}(i))$ and
  \item $\exists j \in A$ such that $u_j(F^{-1}(j)) < u_j(G^{-1}(j))$
  \end{itemize}
- $G$ is Pareto optimal (PO), if it is not Pareto dominated by another allocation.

One way to measure the social satisfaction of the agents is through the Nash and utilitarian social welfare functions.

Definition 6 Let $F \in \mathcal{A}^R$.
- The utilitarian social welfare of $F$, denoted by $SW_U(F)$, is defined by
  \[
  SW_U(F) = \sum_{i \in A} u_i(F^{-1}(i)) \tag{5}
  \]
  and $MSW_U = \{F : SW_U(F) \geq SW_U(G), \forall G \in \mathcal{A}^R\}$.
- The Nash social welfare, denoted by $SW_{Nash}(F)$, is defined by
  \[
  SW_{Nash}(F) = \prod_{i \in A} u_i(F^{-1}(i)) \tag{6}
  \]
  and $MSW_{Nash} = \{F : SW_{Nash}(F) \geq SW_{Nash}(G), \forall G \in \mathcal{A}^R\}$.

Fortunately, Caragiannis and colleagues [5] showed that under an additive scenario it is possible to find EF1 and PO resource allocations. Actually, they prove the following theorem:
Theorem 1 [Caragianis et al. [5]] Under additive scenario, every allocation\(^1\) that maximizes Nash social welfare is EF1 and PO.

Another well-known result is that any allocation that maximizes utilitarian social welfare is PO:

Theorem 2 Under additive scenario, let \(F\) be in \(\mathcal{A}^\mathcal{R}\), if \(F \in MSW_U\), then \(F\) is PO.

The converse of Theorem 2 is false. Example 1 shows this.

Example 1 Let \(\mathcal{R} = \{r_1, r_2\}\) be the set of resources and \(\mathcal{A} = \{1, 2\}\) the set of agents where each agent \(i\) establishes its preferences \(u_i\) over every resource, with utility functions according to Table 1.

Table 1: Preference of agents.

|        | \(r_1\) | \(r_2\) |
|--------|---------|---------|
| \(u_1\) | 10      | 10      |
| \(u_2\) | 3       | 2       |

Let \(F\) and \(G\) be the allocations defined by Table 2. The utility by bundle received by each agent and social welfare are showed in Tab 3. It is easy to see that \(F\) is the only allocation which maximizes utilitarian social welfare. However, \(F\) is not EF1 because the agent 2 envies agent 1, \(u_2(F^{-1}(2)) = 0 < 5 = u_2(F^{-1}(1))\), but even after removing \(r_1\) or \(r_2\) from \((F^{-1}(1))\), envy does not disappear:

\[
\begin{align*}
v_2(F^{-1}(2)) &= 0 < 2 = u_2(F^{-1}(1)\backslash \{r_1\}) \text{ and} \\
v_2(F^{-1}(2)) &= 0 < 3 = u_2(F^{-1}(1)\backslash \{r_2\})
\end{align*}
\]

\(^1\)Actually, the Theorem as stated is true when the maximum Nash social welfare is strictly positive. When it is zero it is necessary to impose that the set of agents having a good is a maximal set (see [5]).
The maximum Nash social welfare is reached at 30, so $G$ maximizes the Nash social welfare. By Theorem 1 $G$ is EF1, but it does not maximize the utilitarian social welfare. In the following table, we identify the properties that satisfy $F$ and $G$; if an allocation satisfies a property we use $\checkmark$ and $\times$ otherwise.

**Table 4:** Identify property without buyer utility function.

|   | PO | EF1 | MSW_U | MSW_{Nash} |
|---|----|-----|-------|------------|
| $F$ | $\checkmark$ | $\times$ | $\checkmark$ | $\times$ |
| $G$ | $\checkmark$ | $\checkmark$ | $\times$ | $\checkmark$ |

The following result establishes that if an allocation distributes resources to the agents that maximize them, then this allocation maximizes the utilitarian social welfare. One can find a straightforward proof of this result, using a matrix approach, in [6].

**Theorem 3** Under an additive scenario, if $F \in A^R$ and for all $r \in F^{-1}(i)$, $u_i(r) = \max\{u_j(r) : j \in A\}$, then $F \in MSW_U$.

### 3 An algorithm to find maximum utilitarian social welfare and its behavior in buyer scenarios

In this section, we present first an algorithm to build a resource allocation which maximizes the utilitarian social welfare. We assume an additive scenario, $R = \{r_1, \ldots, r_m\}$ and $A = \{1, \ldots, n\}$. Let $\alpha_1, \ldots, \alpha_m$ be the real numbers defined in the following way: for each $r_k \in R$, $\alpha_k = \max\{u_j(r_k) : j \in A\}$. We will assume that every $\alpha_k$ is strictly positive.

**Remark 1** Note that the requirement that every $\alpha_k$ is strictly positive has no impact. Indeed if there is a resource $r_k$ such that $\alpha_k = 0$, that is, all the agents give 0 as valuation to this particular resource, we can suppress $r_k$ in the set of resources because an allocation that maximizes the utilitarian social welfare in the set of modified resources can be extended in any way to the set of original resources preserving this property of being maximal.

The allocation which maximizes the utilitarian social welfare, denoted $\Gamma$, is built following the next algorithm:

**Definition 7** (Algorithm for building the allocation $\Gamma$)

1. Let $v_0 = (0, \ldots, 0)$ be the initial vector of partial valuations, with size $n$. 

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2. For \( k = 1 \) to \( m \):
   (a) We take \( r_k \in \mathcal{R} \).
   (b) \( \alpha_k = \max \{ u_j(r_k) : j \in \mathcal{A} \} \).
   (c) Let \( P_k \) and \( M_k \) be the sets given by:
      \[
P_k = \{ j \in \mathcal{A} : u_j(r_k) = \alpha_k \}
      \]
      and
      \[
      M_k = \{ i \in P_k : \lfloor v_{k-1} \rfloor_i = \min \{ \lfloor v_{k-1} \rfloor_j : j \in P_k \} \}
      \]
      where \( \lfloor v_{k-1} \rfloor_i \) is the position \( i \) of \( v_{k-1} \).
   (d) Let \( j_k \) be the minimum of \( M_k \).
   (e) Allocation of \( r_k \):
      \[
      \Gamma(r_k) = j_k
      \]
   (f) Updating \( v_k \) the vector of partial utilities, for all \( i \in \mathcal{A} \):
      \[
      [v_k]_i = \begin{cases} 
      [v_{k-1}]_i + u_{j_k}(r_k), & \text{if } i = j_k \\
      [v_{k-1}]_i, & \text{if } i \neq j_k
      \end{cases}
      \]

Notice that, \( P_k \neq \emptyset \). Therefore, \( M_k \neq \emptyset \) and \( M_k \subseteq \mathbb{N} \); so, there exists a minimum for \( M_k \). On the other hand, \( \forall i \in \mathcal{A}, u_i \) is an additive utility function, then each position \( i \) of \( v_k \), \( [v_k]_i \), is the partial valuation given by agent \( i \) to its assigned bundle up to step \( k \).

The interesting feature about \( \Gamma \) is that it maximizes utilitarian social welfare. More precisely, we have the following theorem:

**Theorem 4** Under an additive scenario, \( \Gamma \in MSW_U \) and it is obtained in \( O(nm) \) operations.

From Theorem 4 and Theorem 2 we have the following result:

**Corollary 1** Under an additive scenario, \( \Gamma \) is PO.

### 3.1 The utilitarian social welfare and fairness

In this subsection, we will show that in buyer scenarios it is possible to build an allocation that maximizes utilitarian social welfare and that is also envy-free up to one good. Remember that in general additive scenarios this is not always possible, as we could see in Example 1. Indeed, in these scenarios, the \( \Gamma \) allocation, defined in (9), satisfies EF1 and PO.

Moreover, with a very slight modification of the algorithm for building \( \Gamma \), we will obtain another allocation, \( \Gamma^* \), which is EFX.

The following example shows the behavior of some allocations in a buyer scenario.

**Example 2** Suppose that \( \mathcal{R} = \{r_1, r_2, r_3, r_4, r_5\} \) and that each resource \( r_k \) is valued as \( p_k \) according to Table 5.
Table 5: $p_k$ values for each resource in Example 2.

| $p_k$ | $r_1$ | $r_2$ | $r_3$ | $r_4$ | $r_5$ |
|-------|-------|-------|-------|-------|-------|
| 500   | 200   | 50    | 100   | 250   |

Let $A = \{1, 2, 3\}$ be the set of agents; each agent $i$ establishes its preferences $u_i$ over each resource using Table 6.

Table 6: Preference of agents $u_i$ over resources $r_k$.

| $u_1$  | $r_1$ | $r_2$ | $r_3$ | $r_4$ | $r_5$ |
|--------|-------|-------|-------|-------|-------|
| 500    | 500   | 0     | 50    | 0     |

Note that all agents prefer the resource $r_1$ and they are willing to pay the value of 500. The resource $r_2$ is preferred by agents 1 and 3, and they are willing to pay the value of 200; the resource $r_3$, whose value is 50, is preferred by agents 1 and 2. The value of resource $r_4$ is 100 and is preferred by agents 2 and 3; whereas, resource $r_5$, with a value of 250, is only preferred by agent 2. If for $i = 1, 2, 3$, the function $u_i$ is extended additively over each subset $S \subseteq R$; that is, $\forall i \in A, u_i(S) = \sum_{r \in S} u_i(\{r\})$, then each $u_i$ is a buyer utility function, i.e., it is a buyer scenario.

In this scenario, we will consider four allocations in order to illustrate their behavior with respect to the properties EF1, PO, MSW_U and MSW_Nash. Let $F$, $G$, $\Gamma$ and $J$ be the allocations defined by Table 7, that shows the agent number who received the resource. In Table 8, we show the utility assigned by each agent to its received bundle, and the social welfare in each allocation.

Table 7: Definition of allocations.

|     | $r_1$ | $r_2$ | $r_3$ | $r_4$ | $r_5$ |
|-----|-------|-------|-------|-------|-------|
| $F$ | 1     | 3     | 3     | 3     | 2     |
| $G$ | 1     | 3     | 2     | 3     | 2     |
| $\Gamma$ | 1     | 3     | 2     | 2     |       |
| $J$ | 1     | 1     | 2     | 3     | 2     |

The properties of each allocation are described in Table 9. For more details about the verification of these properties see Appendix.

It is important to note that $F$ is an allocation EF1 and it is not in MSW_U. $G$ satisfies all the properties. $\Gamma$ is EF1, PO, it is in MSW_U but it is not in MSW_Nash. Finally, $J$ is PO and it is in MSW_U but it is neither EF1 nor in MSW_Nash.

The following theorem shows that under a buyer scenario, the converse of Theorem 2 is satisfied; i.e., having the property PO and belonging to MSW_U are equivalent.
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Table 8: Utility by received bundle and social welfare.

|       | 1   | 2   | 3   | SWU  | SW_{Nash} |
|-------|-----|-----|-----|------|-----------|
| $u_i(F^{-1}(i))$ | 500 | 250 | 300 | 1050 | 37500000  |
| $u_i(G^{-1}(i))$ | 500 | 300 | 300 | 1100 | 45000000  |
| $u_i(G^{-1}(i))$ | 500 | 400 | 200 | 1100 | 40000000  |
| $u_i(J^{-1}(i))$ | 700 | 300 | 100 | 1100 | 21000000  |

Table 9: Allocation properties under a buyer scenario.

|       | PO | EF1 | MSWU | MSW_{Nash} |
|-------|----|-----|------|------------|
| $F$   | X  |     | X    | X          |
| $G$   | ✓  | ✓   | ✓    | ✓          |
| $\Gamma$ | ✓ | ✓   | ✓    | X          |
| $J$   | ✓  | X   | ✓    | X          |

Theorem 5 Assume a buyer scenario and let $F$ be an allocation in $A^R$. Then, $F$ is PO if, and only if, $F \in MSW_{U}$.

We have seen in Example 1 that $MSW_{Nash} \nsubseteq MSW_{U}$. However, under buyer scenarios, Theorems 1 and 5 together, tell us that $MSW_{Nash} \subseteq MSW_{U}$. This is important and will be stated in the following result.

Theorem 6 Under a buyer scenario, each allocation that maximizes Nash social welfare also maximizes utilitarian social welfare.

Now, we give a result which guarantees the existence, under buyer scenarios, of allocations maximizing the utilitarian social welfare and satisfying the EF1 property.

Theorem 7 Under a buyer scenario, there exists an allocation that is EF1 and maximizes utilitarian social welfare.

Note that, in general additive scenarios, the previous result does not hold as Example 1 reveals.

Consider the allocation $\Gamma$ from Definition 7. By Corollary 1, $\Gamma$ is PO. Unfortunately, in general additive scenarios, $\Gamma$ does not satisfy the EF1 property. This can be seen through Example 1: the allocation $F$, the unique allocation that maximizes the utilitarian social welfare, is indeed the allocation $\Gamma$, which is not EF1, see Table 4. However, under a buyer scenario, Theorem 7 guarantees the existence of allocations that maximize utilitarian social welfare and satisfy property EF1. Note that the existential proof of Theorem 7 is based in finding an allocation that maximizes the Nash social welfare. In the following theorem, we will show that under a buyer scenario, the allocation $\Gamma$ is EF1 and PO.
Theorem 8 Under a buyer scenario, there exists an allocation which is EF1 and PO and is obtained in \(O(mn)\). Indeed \(\Gamma\) is such an allocation.

A slight modification of the algorithm given by Definition 7 which consists in taking one more step: reordering the resources in descending ordering (after the “prices” \(p_i\)) produces an allocation \(\Gamma^*\) which is furthermore EFX. This is stated in the following result.

Theorem 9 Under a buyer scenario, there exists an allocation which maximizes the utilitarian social welfare and is EFX and PO. Moreover, if \(m\) is the number of resources and \(n\) is the number of agents, this allocation is produced by an algorithm in \(O(m \log m + mn)\).

It is interesting to observe that the algorithm producing \(\Gamma^*\) is quite similar to Barman et al. algorithm [12] producing an allocation EFX, in the case of identical scenarios. In fact, in identical scenarios, the notions EFX and EFX\(_0\) coincide. This is not the case in buyer scenarios. The following example, in the framework of a buyer scenario, shows that \(\Gamma^*\), the allocation given by the algorithm of Definition 7 modified, is EFX but it is not EFX\(_0\).

Example 3 Let’s consider the following Buyer scenario where \(n = 3\), \(m = 8\). Table 10 shows the preferences that each agent gives to each resource.

|   | \(r_1\) | \(r_2\) | \(r_3\) | \(r_4\) | \(r_5\) | \(r_6\) | \(r_7\) | \(r_8\) |
|---|---|---|---|---|---|---|---|---|
| \(u_1\) | 20 | 0 | 10 | 2 | 0 | 0 | 3 | 1 |
| \(u_2\) | 20 | 0 | 10 | 2 | 11 | 19 | 0 | 1 |
| \(u_3\) | 20 | 9 | 0 | 2 | 0 | 19 | 3 | 1 |

Now, using the algorithm of Definition 7 modified, we get the following allocation \(\Gamma^*\). Then, in Table 12 we show the utility assigned by each agent to its received bundle, and the social welfare in each allocation. Notice that agent 3 envies agent 2.

|   | \(r_1\) | \(r_2\) | \(r_3\) | \(r_4\) | \(r_5\) | \(r_6\) | \(r_7\) | \(r_8\) |
|---|---|---|---|---|---|---|---|---|
| \(\Gamma^*\) | 1 | 3 | 1 | 3 | 2 | 2 | 3 | 3 |

| \(u_i(\Gamma^{*-1}(i))\) | 1 | 2 | 3 | \(SW_U\) | \(SW_{Nash}\) |
|---|---|---|---|---|---|
| 30 | 30 | 15 | 75 | 13500 |
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because

\[ u_3(\{r_2, r_5, r_6\}) = 15 \]
\[ u_3(\{r_5, r_6\}) = 19 \]

which means that

\[ u_3(\{r_2, r_5, r_6\}) < u_3(\{r_5, r_6\}). \]

Since \( \Gamma^* \) was obtained using the modified algorithm of Definition 7, we obtain that \( \Gamma^* \) is EFX. Now we check that \( \Gamma^* \) does not satisfy EFX\(_0\). Observe that

\[ u_3(\{r_5, r_6\}\{r_5\}) = 19 \text{ given that } u_3(\{r_5\}) = 0 \]

so, it follows that

\[ u_3(\{r_2, r_5, r_6\}) = 15 < 19 = u_3(\{r_5, r_6\}\{r_5\}) \] (10)

Thus, the property EFX\(_0\) is not satisfied.

In the following result, we will establish conditions to guarantee that \( \Gamma^* \) is EFX\(_0\).

**Theorem 10** Under a buyer scenario, if \( F \) is an allocation EFX and for all \( i, j \in A \) such that \( u_i(F^{-1}(i)) < u_i(F^{-1}(j)) \) we have that \( F^{-1}(j) \subseteq C_{ij} = \{ r \in R : u_i(r) = u_j(r) > 0 \} \), then \( F \) is EFX\(_0\).

4 Efficiency of the proposed algorithm in buyer scenarios with identical valuations

In this section, we compare our proposed algorithm with ALG-IDENTICAL algorithm proposed in [12], which is a greedy algorithm for buyer scenarios with identical valuation functions. In these scenarios, for any good \( k \) and any pair of agents \( i, j \), the valuation of \( i \) to the good \( k \) is equal to the valuation of \( j \): \( u_i(k) = u_j(k) \). We compare both algorithms under identical valuation functions and get their computational cost.

In definition 7, we presented a general algorithm to build the allocation \( \Gamma \); Algorithm 1 presents the respective pseudo-code under buyer scenarios, see Definition 2. Notice that the \( \alpha_k, P_k \), and \( M_k \) sets are implemented through the argmin operator for buyer scenarios. In this algorithm we code the information of agent utilities through a valuation matrix \( V \) in which the entry \([V]_{ij}\) contains the utility that the agent \( i \) gives to the resource \( j \).

We implemented a version of BUYER algorithm to receive identical valuations, called BUYER-IDENTICAL, Algorithm 2 shows the pseudo-code of this version. Note that this version does not check that the utility is different from 0.

ALG-IDENTICAL algorithm follows a greedy approach that starts sorting the resources; later, it allocates each resource to the agent with minimal sum of received utilities. ALG-IDENTICAL guarantees to find EFX allocations, which are also EF1. Both ALG-IDENTICAL and our algorithm, under buyer scenarios with identical valuations, find EF1 and PO allocations.
Algorithm 1 \textsc{Buyer} Maximizing utilitarian social welfare in buyer scenarios.

\textbf{Require:} A buyer scenario $\langle n,m,V \rangle$, $n$ number of agents, $m$ number of indivisible resources, $V$ matrix of resource valuations.

\textbf{Ensure:} An allocation $\Gamma$.

1: $v \leftarrow (0,\ldots,0)$ \hspace{1cm} $\triangleright$ Partial valuations for each agent
2: \textbf{for} $k \leftarrow 1$ to $m$ \textbf{do}
3: \hspace{1cm} $j \leftarrow \text{argmin} \{ [v]_i : [V]_{ij} \neq 0 \}$ \hspace{1cm} $\forall i \in \{1,\ldots,n\}$
4: \hspace{1cm} $\Gamma(k) = j$
5: \hspace{1cm} $[v]_j \leftarrow [v]_j + [V]_{jk}$ \hspace{1cm} $\triangleright$ Received utility
6: \textbf{end for}

Algorithm 2 \textsc{Buyer-Identical} Maximizing utilitarian social welfare in buyer scenarios with identical valuations.

\textbf{Require:} A buyer scenario with identical valuations: $\langle n,m,V \rangle$, $n$ number of agents, $m$ number of indivisible resources, $V$ valuations of resources.

\textbf{Ensure:} An allocation $\Gamma$.

1: $v \leftarrow (0,\ldots,0)$ \hspace{1cm} $\triangleright$ Partial valuations for each agent
2: \textbf{for} $k \leftarrow 1$ to $m$ \textbf{do}
3: \hspace{1cm} $j \leftarrow \text{argmin} \{ [v]_i \}$ \hspace{1cm} $\forall i \in \{1,\ldots,n\}$
4: \hspace{1cm} $\Gamma(k) = j$
5: \hspace{1cm} $[v]_j \leftarrow [v]_j + [V]_j$ \hspace{1cm} $\triangleright$ Received utility
6: \textbf{end for}

Remember that \textsc{Buyer} algorithm runs in $O(nm)$ and \textsc{Alg-Identical} runs in $O(m \log m + nm)$. We implemented the \textsc{Buyer-Identical} and \textsc{Alg-Identical} using Google Colaboratory, the last as is described in [12].

We experimentally compared the implementations in order to visualize which algorithm is faster. With the aim of having a just comparison, we generate random scenarios. Each scenario has different number of resources and agents, this is the problem size $m \times n$. The evaluation set has different combinations varying the size of $m \in \{500,1000,1500,2000,\ldots,9500,10000\}$ and $n \in \{500,1000,1500,2000,\ldots,9500,10000\}$. The evaluations were repeated 30 times with each size, in order to obtain the average measurement of the running time.

Table 13 shows the results on average with their standard deviation only for the problems size $n \times n$ where $n = \{1000,2000,\ldots,9000,10000\}$ we note that for this case $m = n$. The first column has the problem size, which is the number of agents and the number of resources. The next columns show the time of \textsc{Buyer-Identical} and \textsc{Alg-Identical} algorithms. It is important to note that the \textsc{Buyer-Identical} and \textsc{Alg-Identical} algorithm run in a similar time to problems of small size. However, from problems of big size onward, the running time difference grows considerably.
Table 13: Comparison between Buyer, and Alg-Identical Algorithms

| n   | Buyer-Identical [s] | Alg-Identical [s] |
|-----|---------------------|-------------------|
| 1000| 0.0070 ± 0.0020     | 0.0083 ± 0.0007   |
| 2000| 0.0179 ± 0.0142     | 0.0211 ± 0.0110   |
| 3000| 0.0250 ± 0.0028     | 0.0317 ± 0.0019   |
| 4000| 0.0393 ± 0.0141     | 0.0497 ± 0.0159   |
| 5000| 0.0511 ± 0.0127     | 0.0677 ± 0.0111   |
| 6000| 0.0684 ± 0.0179     | 0.0990 ± 0.0155   |
| 7000| 0.0850 ± 0.0191     | 0.1412 ± 0.0157   |
| 8000| 0.1038 ± 0.0222     | 0.1984 ± 0.0195   |
| 9000| 0.1251 ± 0.0222     | 0.2601 ± 0.0184   |
| 10000| 0.1442 ± 0.0218    | 0.3284 ± 0.0188   |

Figure 1 shows the comparison of running time between BUYER-IDENTICAL and ALG-IDENTICAL, varying the problem size. We can see that the extra cost of the sorting step added by ALG-IDENTICAL has a computational cost when the problem size grows.

Fig. 1: Comparison of running time between BUYER-IDENTICAL and ALG-IDENTICAL

5 Concluding remarks

In this work, we have studied fair and efficient allocation for indivisible resources when the valuations are defined through buyer utility functions. We showed that there exists an allocation that is envy-free up to one good (EF1)
and maximizes utilitarian social welfare ($MSW_U$) using buyer utility functions. This result can slightly be modified to produce an allocation that is envy-free up to any positively valued good (EFX). In this framework some relationships between fairness and efficiency notions are characterized. These relationships were foreseen in [Camacho et al. [6]].

Actually, we build algorithms for finding an allocation which are $EF1$ and $EFX$ and and at the same time are $MSW_U$. They run in polynomial-time. However, the algorithm finding an allocation $EF1$ is computationally slightly better than the algorithm finding an allocation $EFX$.

As a matter of fact, the difference between the two algorithms is captured in the experimental study in Section 4.

As future work related to this research is the quest of simple algorithms for finding allocations that maximize the Nash social welfare in the case of buyer scenarios.

**Appendix A  Proofs**

The following observation is very useful in the proofs. Its proof is obvious by equations (9), (7) and (8).

**Remark 2** We assume an additive scenario. Let $\Gamma$ be the allocation of Definition 7. For every $r_k \in R$ and every $j \in A$ such that $\Gamma(r_k) = j$, we have

1. $\forall i (i \in P_k \implies [v_{k-1}]_i \geq [v_{k-1}]_j)$
2. $u_j(r_k) = \alpha_k = \max\{u_i(r_k) : \forall i \in A\}$

**Proof of Theorem 4:** Let $\Gamma$ be the allocation of Definition 7. By Remark 2, part 2, and Theorem 3, we have that $\Gamma \in MSW_U$. The proposed algorithm starts initializing the vector of partial utilities $v_0$ with zeros, which has one position by each one of the $n$ agents. This step demands $O(n)$ operations. The step-2 allocates the $m$ resources finding the agent with minimum partial utility in each iteration, following (8). Allocating $m$ resources, finding the minimum of $n$ agents demands $O(nm)$ operations. In the next steps, the agent receives the resource in (9); for all the resources, this step and the updating of the vector of partial utilities runs in $O(m)$. Building the resource allocation runs in $O(\max(n, nm, m))$, thus, we have that the proposed algorithm is in $O(nm)$.

**Proof of Theorem 5:** Let $F \in A^R$, and we suppose that $F$ does not maximize utilitarian social welfare. We want to show that $F$ is not Pareto optimal. Since all agents consider buyer utility functions and $F$ does not maximize utilitarian social welfare, by Theorem 3, there exist $r^* \in R$ and $j \in A$ such that
Lemma 1 Under a buyer scenario, if $\Gamma$ is the allocation of Definition 7, then for all $i,j \in \mathcal{A}$

$$u_i(\Gamma^{-1}(j)) = u_i(\Gamma^{-1}(j) \cap C_{ij})$$

(A1)

where $C_{ij} = \{r_k \in \mathcal{R} : u_i(r_k) = u_j(r_k) \neq 0\}$.

Proof: Agents have buyer utility functions, then for each $r_k \in \mathcal{R}$, there exists $p_k > 0$ such that $u_i(r_k) \in \{0, p_k\}$. Let $i,j \in \mathcal{A}$ and we define the following sets:

$$A_{ij} = \{r_k \in \mathcal{R} : u_i(r_k) = p_k \text{ and } u_j(r_k) = 0\}$$

$$B_{ij} = \{r_k \in \mathcal{R} : u_i(r_k) = 0 \text{ and } u_j(r_k) = p_k\}$$

$$C_{ij} = \{r_k \in \mathcal{R} : u_i(r_k) = u_j(r_k) = p_k\}$$

$$D_{ij} = \{r_k \in \mathcal{R} : u_i(r_k) = u_j(r_k) = 0\}.$$ 

As $u_i$ and $u_j$ are buyer utility functions, then $\mathcal{R} = A_{ij} \cup B_{ij} \cup C_{ij} \cup D_{ij}$ and, in consequence,

$$\Gamma^{-1}(j) = (\Gamma^{-1}(j) \cap A_{ij}) \cup (\Gamma^{-1}(j) \cap B_{ij}) \cup (\Gamma^{-1}(j) \cap C_{ij}) \cup (\Gamma^{-1}(j) \cap D_{ij}).$$

Proof of Theorem 7: Let $F$ be an allocation that maximizes Nash social welfare. By Theorem 6, $F$ maximizes utilitarian social welfare. And, by Theorem 1, $F$ is EF1.

In order to prove Theorem 8, we establish the following technical lemma.
As $\Gamma$ is defined by equation (9) and $u_i$ is additive, then

$$u_i(\Gamma^{-1}(j) \cap A_{ij}) = u_i(\Gamma^{-1}(j) \cap B_{ij}) = u_i(\Gamma^{-1}(j) \cap D_{ij}) = 0.$$ 

Again, by additivity of $u_i$

$$u_i(\Gamma^{-1}(j)) = u_i(\Gamma^{-1}(j) \cap C_{ij}).$$

Proof of Theorem 8: We suppose that $\forall i \in \mathcal{A}$, $u_i$ is a buyer utility function. We consider the allocation $\Gamma$ defined as (9). By Theorem 4, $\Gamma$ maximizes utilitarian social welfare in $O(mn)$ operations, by Theorem 5, $\Gamma$ is PO. We want to show that $\Gamma$ is EF1. Suppose that there exists $i \in \mathcal{A}$ such that

$$u_i(\Gamma^{-1}(i)) < u_i(\Gamma^{-1}(j))$$

(A2)

for some agent $j$. As $u_i$ is a buyer utility, by Lemma 1, it is enough to show that there exists $r \in \Gamma^{-1}(j) \cap C_{ij}$ such that $u_i(\Gamma^{-1}(i)) \geq u_i(\Gamma^{-1}(j)) - u_i(r)$. Clearly, $\Gamma^{-1}(j) \cap C_{ij} \neq \emptyset$. Otherwise, by (A1) and (A2), we have $u_i(\Gamma^{-1}(j)) < 0$, which is a contradiction because $u_i$ is non negative. Let $\Gamma^{-1}(j) \cap C_{ij} = \{q_{s_1}, \ldots, q_{s_k}\}$, where $q_{s_k}$ is the last resource assigned to $j$ and it is preferred by $i$ and $j$. Since $\Gamma(q_{s_k}) = j$ and $i \in P_{s_k}$, by Remark 2, in part 1,

$$u_i(\Gamma^{-1}(i) \cap \{q_1, \ldots, q_{s_k-1}\}) \geq u_j(\Gamma^{-1}(j) \cap \{q_1, \ldots, q_{s_k-1}\}).$$

By additivity of $u_j$,

$$u_j(\Gamma^{-1}(j) \cap \{q_1, \ldots, q_{s_k-1}\}) \geq u_j(\Gamma^{-1}(j) \cap \{q_1, \ldots, q_{s_k-1}\} \cap C_{ij}).$$

Using the transitivity of $\geq$, we have

$$u_i(\Gamma^{-1}(i) \cap \{q_1, \ldots, q_{s_k-1}\}) \geq u_j(\Gamma^{-1}(j) \cap \{q_1, \ldots, q_{s_k-1}\} \cap C_{ij}).$$

(A3)

As $u_i$ and $u_j$ are buyer utility functions, then the utility that agents $i$ and $j$ assign to each resource in $C_{ij}$ is equal, so

$$u_j(\Gamma^{-1}(j) \cap \{q_1, \ldots, q_{s_k-1}\} \cap C_{ij}) = u_i(\Gamma^{-1}(j) \cap \{q_1, \ldots, q_{s_k-1}\} \cap C_{ij}).$$

(A4)

By equations (A3) and (A4),

$$u_i(\Gamma^{-1}(i) \cap \{q_1, \ldots, q_{s_k-1}\}) \geq u_i(\Gamma^{-1}(j) \cap \{q_1, \ldots, q_{s_k-1}\} \cap C_{ij}).$$

(A5)

Now, by equation (A1) and additivity of $u_i$,

$$u_i(\Gamma^{-1}(j)) = u_i(\Gamma^{-1}(j) \cap C_{ij})$$
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\[ u_i(\Gamma^{-1}(j) \cap \{q_1, \ldots, q_{s_k-1}\} C_{ij}) + u_i(q_{s_k}) \]

and, by equation (A5),

\[ u_i(\Gamma^{-1}(j)) \leq u_i(\Gamma^{-1}(i) \cap \{q_1, \ldots, q_{s_k-1}\}) + u_i(q_{s_k}); \]

so,

\[ u_i(\Gamma^{-1}(i) \cap \{q_1, \ldots, q_{s_k-1}\}) \geq u_i(\Gamma^{-1}(j)) - u_i(q_{s_k}). \]

But, \( u_i(\Gamma^{-1}(i)) \geq u_i(\Gamma^{-1}(i) \cap \{q_1, \ldots, q_{s_k-1}\}) \), and using transitivity,

\[ u_i(\Gamma^{-1}(i)) \geq u_i(\Gamma^{-1}(j)) - u_i(q_{s_k}). \quad \text{(A6)} \]

Therefore, \( \Gamma \) is EF1.

Proof of Theorem 9: The argument of the proof is similar to the proof of the Theorem 8. Let \( i, j \) be in \( A \) such that \( u_i(\Gamma^*(i)) < u_i(\Gamma^*(j)) \). By Lemma 1,

\[ u_i(\Gamma^*(j)) = u_i(\Gamma^*(j) \cap C_{ij}) = \{q_{s_1}, \ldots, q_{s_k}\}. \]

As \( R \) is sorted in descending order after the prices of each resource, then for all \( r \in \Gamma^*(j) \),

\[ u_i(q_{s_k}) \leq u_i(r) \]

by equation (A6)

\[ u_i(\Gamma^*(i)) \geq u_i(\Gamma^*(j)) - u_i(q_{s_k}). \]

So, for all \( r \in \Gamma^*(j) \)

\[ u_i(\Gamma^*(i)) \geq u_i(\Gamma^*(j)) - u_i(r). \]

Thus, \( \Gamma^* \) is EFX.

Proof of Theorem 10: Let \( F \) be EFX allocation. Let \( i \) and \( j \) be in \( A \) such that \( i \) envies \( j \). By hypothesis, for all \( r \in F^{-1}(j) \), we have that \( u_i(r) > 0 \). And, as \( F \) is EFX, then \( F \) is EFX0.

Proof of Properties in Example 2: We give details of the properties fulfilled by the allocations in Table 9.

From Theorem 3 and Table 5, it is easy to see that the maximum utilitarian welfare is reached in 1100. Moreover; \( G, \Gamma \) and \( J \) maximize \( SW_U \). For Theorem 2 and Table 8, we have that \( G, \Gamma, \) and \( J \) are PO. On the other hand, from Table 8, we can observe that agents 1 and 3 in the allocations \( F \) and \( G \), have the same utility for the received bundle; but, in \( G \), agent 2 improves its utility. Then, \( F \) is Pareto dominated by \( G \). Therefore, \( F \) is not PO.
A search determined that the maximum Nash social welfare is reached at 45000000. Then $G$ is a maximum Nash social welfare and, by Theorem 1, $G$ is EF1. Moreover, allocations $F$ and $\Gamma$ are EF1. In fact, $\Gamma$ is EF1 by Theorem 4. For $F$, agents 2 and 3 envy agent 1, however, the envy disappears when eliminating $r_1$. Finally, $J$ is no EF1, because agent 3 envy agent 1, $u_3(J^{-1}(3)) < u_3(J^{-1}(1))$ and

$$u_3(J^{-1}(3)) = 100 < 200 = u_3(J^{-1}(1) \setminus \{r_1\}) < 500 = u_3(J^{-1}(1) \setminus \{r_2\}).$$

\section*{Appendix B \quad Algorithm Proposed by Barman et al.}

\begin{algorithm}
\caption{Alg-Identical Maximizing utilitarian social welfare in buyer scenarios with identical valuations.}
\begin{algorithmic}[1]
\Require A buyer scenario with identical valuations: $(n, m, V)$, $n$ number of agents, $m$ number of indivisible resources, $V$ valuations of resources.
\Ensure An allocation $\Gamma$. \\
1: sort resources descending, i.e., $[V]_1 \geq [V]_2 \geq \cdots [V]_m \geq 0$
2: $v \leftarrow (0, \ldots, 0)$ \hfill \Comment{Partial valuations for each agent}
3: $\Gamma \leftarrow (\emptyset, \ldots, \emptyset)$
4: \For{$k \leftarrow 1$ \textbf{to} $m$}
5: \quad $j \leftarrow \arg\min_{i \in \{1, \ldots, n\}} \{[v]_i\}$
6: \quad $[\Gamma]_j \leftarrow [\Gamma]_j \cup k$
7: \quad $[v]_j \leftarrow [v]_j + [V]_j$ \Comment{Received utility}
8: \EndFor
\end{algorithmic}
\end{algorithm}

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