Initial data transients in binary black hole evolutions

Nigel Bishop\(^1\), Denis Pollney\(^2\) and Christian Reisswig\(^3\)

\(^1\) Department of Mathematics, Rhodes University, Grahamstown 6139, South Africa
\(^2\) Departament de Física, Universitat de les Illes Balears, Palma de Mallorca, E-07122, Spain
\(^3\) Theoretical Astrophysics Including Relativity, California Institute of Technology, Pasadena, CA 91125, USA

E-mail: n.bishop@ru.ac.za

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Abstract
We describe a method for initializing characteristic evolutions of the Einstein equations using a linearized solution corresponding to purely outgoing radiation. This allows for a more consistent application of the characteristic (null cone) techniques for invariantly determining the gravitational radiation content of numerical simulations. In addition, we are able to identify the ingoing radiation contained in the characteristic initial data, as well as in the initial data of the 3 + 1 simulation. We find that each component leads to a small but long lasting (several hundred mass scales) transient in the measured outgoing gravitational waves.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

It is well known in numerical relativity that current practice for the setting of initial data introduces spurious radiation into the system, in both the 3 + 1 and the characteristic approaches, resulting from the fact that the past history of the spacetime is not known. The error in the initial data leads to the presence of spurious ‘junk’ radiation that results from solving the constraint equations on a single hypersurface, without knowing the past history of the radiation content. The junk radiation includes an initial burst, and common practice regards the signal as physical only after it has settled down following this burst. While it is straightforward to handle the initial burst in this way, a more serious issue is whether the junk radiation content of the initial data leads to long-term transients in the wave signal. This question has been considered before, but previous work on the long-term effect of the initial data is limited [1–4].
Characteristic extraction is a method of invariantly measuring the gravitational wave emission of an isolated source by transporting the data to null infinity \((J^+)\) using a null formulation of the Einstein equations [5–10]. Initial data are needed on a null cone in the far-field region, say \(r > 100M\), where \(M\) is the length scale set by the Schwarzschild mass. Previous work has taken the simplistic approach of setting the null shear \(J = 0\) everywhere [6, 8, 9, 11], although a recent investigation sets \(J\) by the condition that the Newman–Penrose quantity \(\psi_0 = 0\) [10]. Setting shear-free initial data is not necessarily incorrect physically—for example in the Schwarzschild geometry in natural coordinates \(J = 0\) everywhere, and it is possible to construct a radiating solution with \(J = 0\) everywhere at a specific time. However, in the generic case, a radiating solution has \(J \neq 0\), and imposing \(J = 0\) in effect means that the outgoing radiation implied by the boundary data must be matched at the initial time by other radiation. In principle, the spurious incoming content of \(J = 0\) data extends out towards infinity, and so could contaminate the entire evolution. However, previous results comparing characteristic extraction with conventional finite radius extrapolation indicate that it is at most a minor effect [8, 9].

There are two essential ideas on which this paper is built: (a) the radiation content of an inspiral in which the entire past history was known would be purely outgoing (since the sources of ingoing radiation, in particular backscatter, lead to effects that are too small to be observable in these simulations); and (b) characteristic initial data are needed only in the far-field region, where linearized theory provides a suitable approximation. Thus, our aim is to construct characteristic initial data that, at the linearized approximation, represent the physical situation of a gravitational field with purely outgoing radiation produced by sources in a central region [12].

We first used linearized theory to solve the case of two equal mass objects in circular orbit around each other in a Minkowski background, as a model analytic problem. We then develop and implement a procedure to calculate characteristic metric data that contain purely outgoing radiation. This is done within the context of characteristic extraction, so that initial data on the null cone are constructed that are compatible with given data on a worldtube \(\Gamma\) at constant radius. We are then able to compare the waveforms computed by characteristic extraction using as initial data (a) \(J = 0\), and (b) the linearized solution. We find that while the dominant gravitational wave modes are largely unaffected by the choice of initial \(J\), a small residual difference is visible between the two approaches, and can take several hundred \(M\) to be damped below other effects.

While in a regime in which deviations from flatness are weak so that nonlinear effects can be discounted, any mismatch between the linearized solution and the actual data is an indication of an ingoing radiation content. Now, on the worldtube \(\Gamma\), the characteristic metric data are determined entirely by the \(3+1\) data so that any ingoing radiation can be traced back to an ingoing radiation content in the conformally flat \(3+1\) initial data. In this way, we show that the \(3+1\) initial data contain a component of ingoing radiation which results in a long-lasting transient.

The plan of the paper is as follows. Section 2 introduces the notation, and reviews results needed from previous work. Section 3 applies linearized characteristic theory to calculate metric data for two equal mass non-spinning black holes in circular orbit around each other. Section 4 describes the method to construct metric data everywhere from data on a worldtube. Section 5 describes our numerical results, which are obtained within the context of a binary black hole inspiral and merger. Finally, section 6 presents discussion and conclusion.
2. Review of results needed from other work

2.1. The Bondi–Sachs metric

The formalism for the numerical evolution of Einstein’s equations, in null cone coordinates, is well known [11, 13–17]. For the sake of completeness, we give a summary of those aspects of the formalism that will be used here. We start with coordinates based upon a family of outgoing null hypersurfaces. Let \( u \) label these hypersurfaces, \( x^A (A = 2, 3) \) label the null rays and \( r \) be a surface area coordinate. In the resulting \( x^\alpha = (u, r, x^A) \) coordinates, the metric takes the Bondi–Sachs form [13, 18]

\[
\begin{align*}
\text{d}s^2 &= - (e^{2\beta}(1 + W_c r) - r^2 h_{AB} U^A U^B) \, \text{d}u^2 \\
&\quad - 2 e^{2\beta} \, \text{d}u \, \text{d}r - 2r^2 h_{AB} U^A \, \text{d}u \, \text{d}x^B, \\
&\quad + r^2 h_{AB} \, \text{d}x^A \, \text{d}x^B, \quad (1)
\end{align*}
\]

where \( h_{AB} h_{BC} = \delta_A^C \) and \( \det(h_{AB}) = \det(q_{AB}) \), with \( q_{AB} \) a metric representing a unit 2-sphere; \( W_c \) is a normalized variable used in the code, related to the usual Bondi–Sachs variable \( V \) by \( V = r + W_c r^2 \). As discussed in more detail below, we represent \( q_{AB} \) by means of a complex dyad \( q_A \). Then, for an arbitrary Bondi–Sachs metric, \( h_{AB} \) can be represented by its dyad component

\[ J = h_{AB} q^A q^B / 2, \quad (2) \]

with the spherically symmetric case characterized by \( J = 0 \). We also introduce the fields

\[ K = \sqrt{1 + J \bar{J}}, \quad U = U^A q_A, \quad (3) \]

as well as the (complex differential) \( \eth \) operators \( \delta \) and \( \bar{\delta} \) [19].

In the Bondi–Sachs framework, Einstein’s equations \( R_{\alpha\beta} = 8\pi \left( T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T \right) \) are classified as hypersurface equations—\( R_{11}, q^A R_{1A}, h^{AB} R_{AB} \)—forming a hierarchical set for \( \beta, U \) and \( W_c \); evolution equation \( q^A q^B R_{AB} \) for \( J \) and constraints \( R_{0\alpha} \). An evolution problem is normally formulated in the region of spacetime between a timelike or null worldtube \( \Gamma_1 \) and future null infinity (\( \mathcal{J}^+ \)), with (free) initial data \( J \) given on \( u = 0 \), and with boundary data for \( \beta, U, U_c, W_c, J \) satisfying the constraints given on the inner worldtube. We extend the computational grid to \( \mathcal{J}^+ \) by compactifying the radial coordinate \( r \) by means of a transformation

\[ r \rightarrow x = \frac{r}{r + r_{\text{ct}}}. \quad (4) \]

In characteristic coordinates, the Einstein equations remain regular at \( \mathcal{J}^+ \) under such a transformation.

The free initial data for \( J \) essentially determines the ingoing radiation content at the beginning of the evolution. For the case of binary black hole evolutions in a 3 + 1 formalism, the initial Cauchy data are determined by solving the Hamiltonian and momentum constraints, usually under the assumption of conformal flatness. Compatible null data initial solutions are not known, and so we must choose an ansatz for \( J \) which is approximately compatible. Previous work [8, 9] has simply set \( J = 0 \). In Section 3, we propose a refinement whereby \( J \) is set according to a linearized solution which is determined by the Cauchy initial data solution.

2.2. The spin-weighted formalism and the \( \delta \) operator

A complex dyad \( q_A \) is a 2-vector whose real and imaginary parts are unit vectors that are orthogonal to each other. Further, \( q_A \) represents the metric, and has the properties

\[ q^A q_A = 0, \quad q^A \bar{q}_A = 2, \quad q_{AB} = \frac{1}{2} (q_A \bar{q}_B + \bar{q}_A q_B). \quad (5) \]
Note that \( q_A \) is not unique, up to a unitary factor: if \( q_A \) represents a given 2-metric, then so does \( e^{i\alpha}q_A \). Thus, considerations of simplicity are used in deciding the precise form of dyad to represent a particular 2-metric.

Having defined a dyad, we may construct complex quantities representing angular tensor components on the sphere. For instance, given a set of tensors \( T_A, T^{AB}, T^{ABC} \), we can define spin-weighted scalars \( X^1 = T_Aq^A, X^2 = T^{AB}\bar{q}^B, X^3 = T^{ABC}\bar{q}^A\bar{q}^B\bar{q}^C \). Each object has no free (angular) indices, and has associated with it a spin-weight \( s \) defined by means of the transformation behaviour under rotation of dyad through an angle \( \gamma \) using \( X = X' \exp(i\gamma) \); in practice the spin-weight \( s \) of \( X \) is calculated as the number of ‘\( q \)’ factors less the number of ‘\( \bar{q} \)’ factors in its definition. For example, the definition of \( X^1 \) contains one \( q_A \) factor and no \( \bar{q} \) terms so \( s(X^1) = 1 \). Also, \( s(X^2) = 0, s(X^3) = -3 \); and in general, \( s(X) = -s(\bar{X}) \).

We define derivative operators \( /dh \) and \( \bar{\ }/dh \) acting on a quantity \( V \) with spin-weight \( s \)
\[
\frac{\delta V}{dh} = q^A\partial_AV + s\Upsilon V, \quad \bar{\delta}V = \bar{q}^A\partial_AV - s\bar{\Upsilon}V
\]
where the spin-weights of \( \delta V \) and \( \bar{\delta}V \) are \( s+1 \) and \( s-1 \), respectively, and where
\[
\Upsilon = -\frac{1}{2}q^A\bar{q}^B\nabla_Aq_B.
\]

Some commonly used dyad quantities are

| Spherical polars | Stereographic |
|------------------|---------------|
| \( ds^2 = d\theta^2 + \sin^2\theta d\phi^2 \) | \( \frac{4(dq^2 + dp^2)}{(1 + q^2 + p^2)^2} \) |
| \( q^A = (1, i\sin\theta) \) | \( \frac{1}{2}(1 + q^2 + p^2)(1, i) \) |
| \( \Upsilon = -\cot\theta \) | \( q + ip \). |

The spin-weights of the quantities used in the Bondi–Sachs metric are

\[ s(W_c) = s(\beta) = 0, \quad s(J) = 2, \quad s(\bar{J}) = -2, \quad s(K) = 0, \quad s(U) = 1, \quad s(\bar{U}) = -1. \]

It will be necessary to decompose the angular part of the metric quantities into basis functions, and for this purpose we use spin-weighted spherical harmonics \( Y_{\ell m} \), where as above \( s \) denotes the spin-weight. The standard spherical harmonics correspond to the case \( s = 0 \), and in this case the \( s \) will be omitted, i.e. \( Y_{\ell m} = Y_{\ell m} \). Further, it is convenient to make use of the formalism described in [12, 22], and have basis functions, which in the spin-weight 0 case are purely real; following [12], these are denoted as \( Z_{\ell m} \). The \( Z_{\ell m} \) are constructed from the \( Z_{\ell m} \) by applying the \( \delta \) or \( \bar{\delta} \) operator:

\[
\begin{align*}
\delta Z_{\ell m} &= \sqrt{\ell(\ell + 1)}Z_{\ell m}, \\
\delta Z_{\ell m} &= \sqrt{(\ell - 1)(\ell + 1)(\ell + 2)}Z_{\ell m}, \\
\bar{\delta}Z_{\ell m} &= \sqrt{\ell(\ell + 1)}Z_{\ell m}, \\
\bar{\delta}^2 Z_{\ell m} &= \sqrt{(\ell - 1)(\ell + 1)(\ell + 2)}Z_{\ell m}.
\end{align*}
\]

### 2.3. Solutions to the linearized Einstein equations

Solutions to the linearized Einstein equations in Bondi–Sachs form using the ansatz

\[
F(u, r, x^A) = \Re(f_{\ell m}(r) \exp(ivu))Z_{\ell m},
\]

for a metric coefficient \( F \) with spin-weight \( s \) were obtained in [12] (see also [23]). Here, we need the results for linearization about a Minkowski background, in which the spacetime is
vacuum everywhere except on a spherical shell at \( r = r_0 \). Strictly speaking, we should be performing the linearization about a Schwarzschild (or even Kerr) background rather than about Minkowski. In the Kerr case it is not known how to do so, and in the Schwarzschild case the difference is that, in equation (12b), the \( 1/r^3 \) term is replaced by a term whose leading-order behaviour is also \( 1/r^3 \) but which is not representable analytically. We consider the lowest-order case \( \ell = 2 \) and in the exterior of the shell (i.e. \( r > r_0 \)), and describe that part of the solution that represents purely outgoing gravitational radiation:

\[
\beta_{2,\nu}(r) = b_1 \text{ (constant)} \quad (12a)
\]

\[
j_{2,\nu}(r) = (12b_1 + 6ivc_1 + iv^3c_2)\frac{\sqrt{6}}{9} + \frac{2\sqrt{6}c_1 + \sqrt{6}c_2}{3r^3} \quad (12b)
\]

\[
u_2,\nu(r) = \sqrt{6}\left(\frac{v^4c_2 + 6v^2c_1 - 12ivb_1}{3} + \frac{2b_1}{r} + \frac{2c_1}{r^2} - \frac{2ivc_2}{3r^3} - \frac{c_2}{2r^4}\right) \quad (12c)
\]

\[
w_2,\nu(r) = \frac{r^2 12ivb_1 - 6v^2c_1 - v^4c_2}{3} + r\left[-\frac{6b_1 + 12ivc_1 + 2iv^3c_2 + 2ivc_2}{3} + 2v^2c_2 - \frac{2ivc_2}{r} - \frac{c_2}{r^2}\right] \quad (12d)
\]

The solution is determined by setting the constant (real-valued) parameters \( b_1, c_1 \) and \( c_2 \). The gravitational news corresponding to this solution is given by

\[
N = \Re[(n_{2,\nu}\exp(\nu u))_2Z_{2,m}] \quad \text{with } n_{2,\nu} = -iv^3c_2\sqrt{6}/6.
\]

We will also need the solution in the case \( \nu = 0, \ell = 2 \) in the exterior region \( r > r_0 \):

\[
\beta_{2,0}(r) = b_0 \text{ (constant)} \quad (14a)
\]

\[
j_{2,0}(r) = \sqrt{6}\left(\frac{4b_0}{3} + \frac{2c_3}{r} + \frac{2c_4}{r^3}\right) \quad (14b)
\]

\[
u_{2,0}(r) = \sqrt{6}\left(\frac{2b_0}{r} + \frac{2c_3}{r^3} - \frac{3c_4}{r^4}\right) \quad (14c)
\]

\[
w_{2,0}(r) = -2b_0r - \frac{6c_4}{r^3} \quad (14d)
\]

in terms of the additional parameters \( b_0, c_3 \) and \( c_4 \).

3. Black hole binaries in circular orbit: solution in the linearized limit

The linearized solution described in the previous section will be used to set initial data on the null cone. We seek a solution which corresponds roughly to the source of the gravitational radiation which we will eventually measure, namely a binary black hole system. In order to be able to apply the linearized theory, we model each black hole as having a matter density that is described by a Dirac-\( \delta \) function whose location moves uniformly around a spherical shell. More precisely, the matter density \( \rho \) in the spacetime is

\[
\rho = \frac{M}{r_0^2} \delta(r - r_0)\delta(\theta - \pi/2)(\delta(\phi - vu) + \delta(\phi - vu - \pi)),
\]

with respect to Bondi–Sachs coordinates \((u, r, \theta, \phi)\). The mass of each black hole is \( M \), the circular orbit has radius \( r_0 \) and the black holes move with angular velocity \( v \). We next express \( \rho \) in terms of spherical harmonics

\[
\rho = \sum_{\ell,m} \Re(\rho_{\ell,m}\exp(|m|uv))Z_{\ell,m}.
\]
and apply the usual procedure, that is multiplication by $Z_{\ell,m}^*$ followed by integration over the sphere, to determine the coefficients $\rho_{\ell,m}$. We find that, for $\ell \leq 2$, the only nonzero coefficients are

$$\rho_{0,0} = \delta(r - r_0) \frac{M}{r_0^2 \sqrt{\pi}}, \quad \rho_{2,0} = -\delta(r - r_0) \frac{M}{2r_0^2 \sqrt{\frac{5}{\pi}}},$$

$$\rho_{2,2} = \delta(r - r_0) \frac{M}{2r_0^2 \sqrt{\frac{15}{\pi}}}, \quad \rho_{2,-2} = -i \delta(r - r_0) \frac{M}{2r_0^2 \sqrt{\frac{15}{\pi}}}. \quad (17)$$

In linearized form, the $R_{11}$ Einstein equation is

$$\beta, r = 2\pi r \rho v_1^2, \quad (18)$$

where $v_1$ is the covariant component of velocity in the $r$-direction. Imposing the gauge condition that the coordinates should be such that on the worldline of the origin, the metric takes Minkowski form, it follows that $\beta = 0$ there and consequently at all points within $r < r_0$. Expanding $\beta$ in terms of spherical harmonics

$$\beta = \sum_{\ell,m} \Re(b_{\ell,m} \exp(|m| i \nu u)) Z_{\ell,m}, \quad (19)$$

and integrating equation (18), we find the coefficients $b_{\ell,m}$ for $r > r_0$:

$$b_{0,0} = \frac{2Mv_1^2}{r_0} \sqrt{\pi}, \quad b_{2,0} = -\frac{Mv_1^2}{r_0} \sqrt{5\pi},$$

$$b_{2,2} = \frac{Mv_1^2}{r_0} \sqrt{15\pi}, \quad b_{2,-2} = -\frac{Mv_1^2}{r_0} \sqrt{15\pi}. \quad (20)$$

The determination of the remaining metric coefficients depends on the value of $(\ell, m)$. The case $(0, 0)$ is straightforward, and we find for $r > r_0$

$$J = 0, \quad U = 0, \quad \beta = \frac{Mv_1^2}{r_0}, \quad W_c = -\frac{4Mv_1^2}{r^2} + \frac{2Mv_1^2}{rr_0}. \quad (21)$$

The case $(2, 0)$ uses the results for a static shell on a Minkowski background in [12]. We solve the jump conditions across the shell for the various metric quantities\(^4\). The result is, in the interior,

$$b_{2,0} = 0, \quad j_{2,0}(r) = -\frac{4Mv_1^2 r^2 \sqrt{30\pi}}{15r_0^3}, \quad (22)$$

and in the exterior

$$b_{2,0} = -\frac{Mv_1^2 \sqrt{5\pi}}{r_0}, \quad (23a)$$

$$j_{2,0}(r) = \frac{4Mv_1^2 \sqrt{30\pi}}{3r_0^3} \left(-1 + \frac{r_0}{r} - \frac{r_0^3}{5r^3}\right). \quad (23b)$$

The cases $(2, \pm 2)$ use the results for a dynamic shell on a Minkowski background in [12].\(^5\) The script constructs the general solution inside and outside the shell $r = r_0$, and uses the constraint equations, as well as regularity conditions at the origin and at infinity, to eliminate

\(^4\) Maple script for this purpose (nu0_regular_0.map with output in nu0_regular_0.out) is provided in the supplementary data available at stacks.iop.org/CQG/28/155019/mmedia.

\(^5\) The calculation is provided in the Maple script regular_0.map with the output in regular_0.out in the supplementary data available at stacks.iop.org/CQG/28/155019/mmedia.
some of the unknown coefficients. It imposes the jump conditions at the shell, and finds a unique solution for the remaining unknowns. The result in the exterior is

\[ \beta = \Re(b_{2,2} \exp(2i\nu u))Z_{2,2} + \Re(-ib_{2,2} \exp(2i\nu u))Z_{2,-2} \]  

\[ J = \Re((j_{2,2}(r) \exp(2i\nu u))Z_{2,2} + \Re(-ij_{2,2}(r) \exp(2i\nu u))Z_{2,-2} \]  

(24a)

(24b)

where

\[ b_{2,2} = \frac{Mv^2}{r_0} \sqrt{\frac{15\pi}{2}} \]  

(25)

and \( j_{2,2}(r) \) takes the form given in equation (12b). The gravitational news is

\[ N = \Re\left(-i(2\nu)^3 c_{2,2} \sqrt{\frac{\pi}{6}} \exp(i2\nu u)\right)Z_{2,2} + \Re\left(-(2\nu)^3 c_{2,2} \sqrt{\frac{\pi}{6}} \exp(i2\nu u)\right)Z_{2,-2}. \]  

(26)

Although the coefficient \( c_2 \) is complicated, it can be expressed to leading order in \( r_0 \nu \):

\[ c_2 = \frac{4b_{2,2}}{5} \frac{r_0}{v}. \]  

(27)

It follows that, again to leading order in \( r_0 \nu \), the news is

\[ N = Mv^2 r_0^4 \frac{\pi}{5} \sqrt{\frac{2\pi}{5}} \left(\Re(-i \exp(2i\nu u))\right)_2 Z_{2,2} + \Re\left(-\exp(2i\nu u)\right)_2 Z_{2,-2} \]  

(28)

from which it is easy to deduce, via the Bondi relation, that the rate of energy loss of the system is

\[ \frac{dE}{du} = -M^2 v^4 r_0^6 \nu^2 \frac{27}{5}. \]  

(29)

In the limit of a low-velocity circular orbit, \( v_1 = 1, v^2 = M/(4r_0^3) \), the above formula reduces to

\[ \frac{dE}{du} = -\frac{2}{5} \frac{M^3}{r_0^3}. \]  

(30)

which is identical to that found from the standard quadrupole formula [24].

4. Constructing the metric from data on a worldtube

In characteristic extraction, the Cauchy evolution provides the characteristic metric variables \( \beta, J, U \) and \( W_c \) on the worldtube \( \Gamma_c \), decomposed into spherical harmonics \( sY^{\ell,m} \), at every time step. In this section, we develop a method to find coefficients of the linearized solutions that provide a fit to the actual numerical data at the worldtube (to linear order and excluding incoming radiation). Then, we use the linearized solutions with the coefficients just found to predict \( J \) everywhere at some chosen time \( u \), and in this way provide initial data for a numerical characteristic evolution. We restrict attention to the dominant modes \( sY_{2,2}, sY_{2,-2}, sY_{2,0} \). The method uses a Fourier decomposition in the time domain, and works well when the data are approximately sinusoidal, with amplitude and frequency varying slowly. Accordingly, for the binary black hole computation, the Fourier decomposition is applied over a time interval that starts after the burst of junk radiation has passed, and ends before the merger occurs.

A metric variable \( A \) may be written as

\[ A = a_{\ell,2}sY_{2,2} + a_{\ell,0}sY_{2,0} + a_{\ell,-2}sY_{2,-2} \]  

(31)

where * denotes the complex conjugate. The relationship between the coefficients of \( sY_{2,2} \) and \( sY_{2,-2} \) follows theoretically from the requirement that the spin-weight 0 metric components
must be real; and further the metric data has been checked to confirm that it does satisfy the
relationship. Transforming to the $Z_{ℓ,m}$ basis, we find

$$A = a_{Z,2}Z_{2,2} + a_{Z,-2}Z_{2,-2} + a_{Z,0}Z_{2,0} \tag{32}$$

where

$$a_{Z,2} = \sqrt{2}Re(a_{Y,2}), \quad a_{Z,-2} = -\sqrt{2}Im(a_{Y,2}), \quad a_{Z,0} = a_{Y,0} \tag{33}$$

so that the metric data on $\Gamma$ can be re-expressed as coefficients of $Z_{2,2}, Z_{2,-2}$ and $Z_{2,0}$ at
discrete time values. Although the data are oscillatory in time, it is not at constant frequency
but is a superposition of multiple solutions with different frequencies. The linearized solutions
behave as $e^{i\nu u}$ for fixed $\nu$, so for the theory to be applicable the next step is to decompose
the metric data into a superposition of constant frequency components. This is achieved by
making a discrete fast Fourier transform of each metric coefficient

$$a_{Z,2,k} = \sum_{j=1}^{L} a_{Z,2}(u_j) \exp \left( \frac{-2\pi i(k-1)(j-1)}{L} \right) \tag{34}$$

where there are $L$ data points over the time interval $(u_1, u_L)$, counted by $1 \leq k \leq L$. The
frequency $\nu$ is related to $k$ by

$$\nu = \frac{2\pi (L - 1)(k - 1)}{(u_L - u_1)L} \tag{35}$$

We found that $J$ at $J^+$ from the linearized solutions provides a smoother fit to the actual data
if high frequencies are eliminated (compare section 5, figure 1), and so we undertake further
processing only for $k \leq L_1$ (with $1 < L_1 \ll L$); the setting of the Fourier coefficients for
$k > L_1$ is described later.

In the case $k > 1$, for each value of $k$ equations (12a)–(12d) evaluated at the worldtube
are four equations for the three unknowns $b_{1,k}, c_{1,k}, c_{2,k}$. Such an over-determined system
can be tackled by a least-squares-fit algorithm, or alternatively by ignoring one of the equations so
making the system uniquely determined. In the case that the equation for $W_c$ (equation (12d))
was ignored, we found that the reconstructed linearized solution gave a better fit to the actual
data at $J^+$. This then means that a comparison between the actual and reconstructed data
for $W_c$ provides an indication of the error, which is expected because of
(a) incoming radiation in the initial Cauchy data, (b) Fourier transform effects and (c) other
effects. We discuss item (b) at the end of this section, and items (a) and (c) in the next section.

In the case $k = 1$, $\nu = 0$, and equations (14a)–(14d) are four equations for the constants
$b_0, c_3, c_4$. We solved four equations for three unknowns using a least-squares-fit algorithm,
because this approach led to the reconstructed data having a better fit to the actual data than in
the case that equation (14d) was ignored.

In this way, for a given spherical harmonic, say $Z_{2,2}$, we obtain values for the constants
of the frequencies represented by $k = 1, \ldots, L_1$. Now, our purpose is to use the worldtube
data to estimate $J$ off the worldtube. From equation (12b) we can write

$$j_{2,k}(r) = d_{0,k} + \frac{d_{1,k}}{r} + \frac{d_{2,k}}{r^3} \tag{36}$$

where for $2 \leq k \leq L_1$

$$d_{0,k} = \frac{\sqrt{2}}{r}(12b_{1,k} + 6ivc_{1,k} + iv^3c_{2,k}), \tag{37a}$$
$$d_{1,k} = 2\sqrt{6}c_{1,k}, \tag{37b}$$
$$d_{2,k} = \frac{\sqrt{2}}{r}c_{1,k}. \tag{37c}$$
and for \( k = 1 \)
\[
d_{0,1} = \frac{2 \sqrt{6}}{3}, \quad d_{1,1} = d_{2,1} = 2 \sqrt{6}.
\] (38)

We then apply the inverse discrete Fourier transform to find
\[
d_{0}(u) = \frac{1}{L} \sum_{k=1}^{L} d_{0,k} \exp \left( \frac{2\pi i (k - 1)(u - u_1)(L - 1)}{(u_L - u_1)L} \right),
\] (39)

where \( d_{0,k} = 0 \) for \( L_1 + 1 \leq k \leq L - L_1 + 1 \), and
\[
d_{0,L-k+1} = d_{0,k+1}^* \text{ for } k = 1, \ldots, (L_1 - 1).
\] (40)

Equation (40) follows from the condition that \( d_{0}(u) \) (and all other coefficients in the time domain) are real. The functions \( d_{1}(u) \) and \( d_{2}(u) \) are found in a similar way, and so we are able to find the coefficient of \( J(u,r) \) of a given spherical harmonic, say \( Z_{2,2} \). Repeating the calculation for the other spherical harmonics \( Z_{2,-2} \) and \( Z_{2,0} \) leads to a prediction of \( J(u,r,x^A) \) to lowest order \( \ell = 2 \).

The coefficient of \( Z_{2,0} \) is not oscillatory, but can rather be described as slowly varying. While, in principle, such behaviour can be represented by a Fourier decomposition, we found that a better fit was obtained by regarding the solution as almost constant and solving equations (14a)-(14c) at each time step, with equation (14d) used as a measure of the error6.

We investigated the possibility of errors introduced in the Fourier transform and the inverse transform process by comparing on the worldtube the actual and reconstructed values of \( \beta, J \) and \( U \), because the construction is such that they should be identical 7. The following comparison is for the R100 case as specified in the next section. We found that there was essentially no difference between the original and reconstructed data, apart from minor variations over about the first and last \( \pm 30 \) \( M \) of the time interval (of total duration 1290 \( M \)), presumably caused by the cut-off of high frequencies. This test was performed for both \( L_1 = 50 \) and \( L_1 = 100 \) with no visible difference seen in the graphs, indicating that the precise way in which high frequencies are removed is not important.

5. Numerical results

We apply the method outlined in the previous section to the problem of measuring gravitational waves from binary black hole simulations. Following our implementation of characteristic extraction [8, 9], we first evolve a spacetime with a \( 3 + 1 \) (Cauchy) evolution code, recording metric data on a world tube, \( \Gamma_1 \), of fixed coordinate radius. These data are subsequently used as inner boundary data for a null-cone evolution of the Einstein equations, which transport the data to \( J^+ \), where the gravitational waves are measured. The linearized solution allows us to specify data for \( J \) on the initial null cone which (a) represents an outgoing gravitational wave, and (b) is compatible, to the linear level, with the data on \( \Gamma_1 \) (which was determined entirely by the \( 3 + 1 \) evolution).

As a fiducial test case, we return to the well-studied model of an 8-orbit binary system with equal mass non-spinning black holes carried out in [8, 9]. For the Cauchy evolution, we use the Llama multipatch code described in [25, 26]. We output metric data on two worldtubes located at \( R_F = 100 \) \( M \) and \( R_F = 250 \) \( M \) that are used as inner boundary data for a subsequent characteristic evolution. The waveforms at \( J^+ \) should be independent of the worldtube location. Thus, evolutions from different worldtube locations help us validate our

6 The Matlab scripts ft_ut_driver.m, FT_WT.m and nu0.m are provided in the supplementary data available at stacks.iop.org/CQG/28/155019/mmedia.
7 Note that \( W \) is not necessarily identical, since there may be differences due to nonlinearities and incoming radiation.
Table 1. The characteristic evolutions performed based on the same Cauchy evolution of two equal mass non-spinning black holes.

| Dataset       | Worldtube location $R_t$ [M] | Initial time $u_0$ [M] | Initial data $J$ |
|---------------|------------------------------|------------------------|------------------|
| J0-R100-u0    | 100                          | 0                      | $J = 0$          |
| J0-R250-u0    | 250                          | 0                      | $J = 0$          |
| J0-R100-u450  | 100                          | 450                    | $J = 0$          |
| J0-R250-u900  | 100                          | 900                    | $J = 0$          |
| Jlin-R100-u450| 100                          | 450                    | $J = J_{\text{lin}}$ |
| Jlin-R250-u900| 250                          | 900                    | $J = J_{\text{lin}}$ |

results. Table 1 summarizes the various characteristic evolutions that we have performed, all of which are based on the same Cauchy data, but with different characteristic initial data, $J$, and starting points in Bondi time, $u_0$.

The first approach follows the original prescription laid out in [8, 9]. The characteristic evolution is started coincident with the first available Cauchy data, at coordinate time $t_0 = u_0 = 0$. The characteristic variable $J$ is initialized by the shear-free solution $J = 0$. In this prescription, the characteristic evolution begins from the initial Cauchy slice, and these models include the spurious junk radiation contained in the conformally flat constraint solution. Models J0-R100-u0 and J0-R250-u0 listed in table 1 follow this prescription, using data from the worldtubes at $R/G_{\Gamma} = 100$ M and $R/G_{\Gamma} = 250$ M, respectively.

It is interesting to compare the results of the fully nonlinear characteristic Einstein evolution with the corresponding linearized solution. Figure 1 plots the $(\ell, m) = (2, 2)$ spherical harmonic modes of $J_{\text{num}}$, computed by the null Einstein evolution J0-R100-u0. The linearized solution $J_{\text{lin}}$ is computed using linearly reconstructed worldtube data according to the prescription in sections 3 and 4. We compute the linearized solution from the boundary data at $\Gamma$ once the radiative outgoing wave solution has set in, that is after the initial data junk radiation has passed the worldtube radius $R_t$, at around $u = 150$ M, when the system has settled to the expected binary black hole inspiral pattern compatible with the solution construction. The upper panel of figure 1 plots the real and imaginary parts of $J$, evaluated at $J^+$. The centre panel plots the amplitudes of $J$, while the bottom panel shows the relative difference between the linearly estimated $J_{\text{lin}}$ and the fully relativistic result $J_{\text{num}}$ (model J0-R100-u0). The linearized $J_{\text{lin}}$ and numerically evolved $J_{\text{num}}$ differ initially, but eventually, after around $u = 450$ M, differ by less than 1%, which remains consistent for the bulk of the time.

We first discuss whether the difference between the solutions can be due to effects other than ingoing radiation; such effects could include (a) nonlinearity, or (b) the linearization background being Minkowski rather than Schwarzschild. Looking at figures 1 and 2, we see that to lowest order the metric components are slowly varying sinusoidal functions; this statement also applies to the other metric components (not plotted here). We would therefore expect that the magnitude of effects (a) and (b) would be roughly constant, if not with some increase at later times as the merger is approached. We can gauge the growth of nonlinear effects by looking at the Einstein equation for $R_{11}$, which is $\beta_\tau = 0$ to linear order. That is, non-zero values for $\beta_\tau$ are a measure of nonlinearities. We find that this quantity has small oscillations about a mean that is initially of order $10^{-8}$, and the mean slowly increases with time towards the black hole merger, as would be expected. However, figures 1 and 2 show that the difference between the solutions, until about $450$ M, decreases. This implies
Figure 1. Components of the $Y_{22}$ mode of $J$ at $J^+$ estimated from boundary data at $R_G = 100 M$ once using linearized solutions, and once using the full nonlinear characteristic evolution J0-R100-u0. The numerical solution is denoted by $J_{\text{num}}$, while the reconstructed linear solution is $J_{\text{lin}}$. The linearized solution makes use of data starting from a time after which the initial burst of junk radiation has left the system. The two solutions agree reasonably well only after a time $t_2$ (equation (41)), a time after which the incoming radiation content of the Cauchy initial data has essentially settled to zero.

that the initial transient does not result from the growing nonlinearities. Similarly, the initial transient cannot be due to the linearization background, since such an effect would also not be expected to decrease with time. Rather, the linearized solution assumes that the radiation is purely outgoing, whereas the actual data may contain incoming modes originating in either the characteristic or Cauchy initial data—both options being possible since $J$ at $J^+$ is influenced by both datasets. Thus, figure 1 (and also figures 2 and 3) reflect the slow decay of the effect of incoming modes in the initial data, until saturated by other factors (such as nonlinear effects).

A similar effect is seen in the characteristic variable $W_c$, related to the Newtonian potential, plotted in figure 2. In this case, however, there is clarity about the source of the incoming radiation: it must be in the Cauchy data. This is because in characteristic extraction, the characteristic metric at the worldtube is determined entirely by the Cauchy data. Again, the lower panel shows an approximately exponential decay in the differences, until around $u = 400 M$. The residual steady state differences result from other effects, which gradually increase with the strength of the gravitational radiation towards the binary merger.

The above findings indicate that a physically expected purely outgoing inspiral radiation pattern is only present after some time

$$u > u_{\text{incoming}},$$

where $u_{\text{incoming}}$ is the length of a time interval until the incoming radiation content of the Cauchy initial data has settled to a negligible amount at the given worldtube location. Hence, in order to construct physically meaningful and consistent initial data via the outgoing linearized solution, it is optimal to begin the solution at a time $u_0 > u_{\text{incoming}}$ after which both the junk and incoming radiation content of the initial data have subsided. The results of figures 1 and 2
suggest that the linearized solution provides a reasonably good approximation to the data \( u \approx 450 \, M \), at which time the system has settled into an outgoing radiative solution.

For instance, in model Jlin-R100-u450, the exact worldtube location is \( R/\Gamma = 100.8492 \). Allowing 50 \( M \) for the visible outgoing junk radiation to pass, we set the time range over which we use the boundary data for linearized solution construction to \( (u_0, u_f) = (150.192, 1439.856) \), which includes the inspiral, but not the merger and ringdown. The time increment in is given by \( d\,u = 0.144 \), thus comprising 8967 data points. Referring to equation (34), we have \( L = 8957, \, L_1 = 100 \). In order to determine \( J_{lin} \) at the initial time \( u_0 \), we use equations (12b), (14b) and (31) to write:

\[
J = \left( e_0 + \frac{e_1}{r} + \frac{e_2}{r^3} \right) 2Y_{2,2} + \left( e_0 + \frac{e_1}{r} + \frac{e_2}{r^3} \right)^{2} 2Y_{2,-2} + \left( e_0 + \frac{e_1}{r} + \frac{e_2}{r^3} \right)^{2} 2Y_{2,0}.
\]  

The coefficients are determined by comparing with the worldtube data. We perform a Fourier transform on the numerically determined worldtube variables over the time interval \((u_0, u_f)\). The spectrum determines the constants \( b_1, c_1 \) and \( c_2 \) of equations (12) at each fixed frequency \( \nu \). These values are transformed back to the time domain, and evaluated at \( t = 450 \, M \) in order to determine the coefficients of (42):

\[
e_0 = (4.5217 + 3.7702i) \times 10^{-4}, \quad e_1 = -0.04578 - 0.17159i,
\]

\[
e_2 = 12.582 + 42.500i, \quad e_3 = -1.4788 \times 10^{-4},
\]

\[
e_4 = 0.020365, \quad e_5 = -35.563.
\]  

A goal of this paper is to investigate, within the context of characteristic extraction, the effect of the initial data on the calculation of the gravitational news. To this end, we compare waveforms at \( J^+ \) from two characteristic evolutions based on the same Cauchy boundary data, but different initial data constructions: Jlin-R100-u450 and J0-R100-u450. Both evolutions
Figure 3. Time domain differences between the $N_{450}^{(450)}$ and $N_{450}^{lin}$ waveforms of models J0-R100-u450 and Jlin-R100-u450, computed from worldtube location $R = 100 M$ for which the characteristic runs have been initialized by $J = 0$ and linearized solutions at a time $u_0 = 450 M$, respectively. The top panel plots the wave amplitude, the middle shows the phase, and the lower plots the differences between the two solutions. The $N_{450}^{(450)}$ data show notable oscillations in amplitude at early times (inset), which decay exponentially with time. The waveforms have been aligned at the amplitude peak.

use boundary data from $R_f = 100 M$ and begin at the initial time $u_0 = t_2 = 450 M$, which was determined above to be a point where the linearized solution is well matched to the nonlinear evolution. The model Jlin-R100-u450 uses the initial data determined by the linearized solutions, equations (42), (43). In contrast, J0-R100-u450 simply sets $J = 0$, corresponding to the original prescription of [8, 9]. Figure 3 plots the Bondi news at $J^+$ as computed from both evolutions, denoted by $N_{450}^{lin}$ and $N_{450}^{(450)}$ for the linearized and $J = 0$ initial data runs, respectively. Whereas the phase of $N$ shows very little difference between the runs (middle panel), and the amplitude shows visible oscillations for the $N_{450}^{(450)}$ evolution (upper panel and inset). The waveforms agree to within 1% only after a time $u = 400 M$ (which must be added to the $u_0 = 450 M$ starting point of the simulation). Therefore, the influence of the $J = 0$ ansatz for the characteristic initial data has a notable impact over an extended time.

We note, however, that the original prescription of [8, 9] used characteristic initial data $J = 0$ at the initial Cauchy time $u_0 = t_0 = 0$ (that is, including the junk radiation). At that time, the shear-free approximation $J = 0$, for the characteristic initial data, is compatible with the Cauchy initial data solution.

Hence, we also compare the waveforms of model Jlin-R100-u450 against those of model J0-R100-u0, where the latter model uses $J = 0$ initial data at time $u_0 = 0$. The results are plotted in figure 4 with the J0-R100-u0 results labeled by $N_{0}^{(0)}$. We still observe an oscillation in the amplitude in $N_{0}^{(0)}$, though it is drastically reduced compared to the $N_{0}^{(450)}$ of model J0-R100-u450 shown in figure 3. The relative errors in amplitude, plotted in the bottom panel of figure 4, are well below 1% over the entire evolution, and the total dephasing is smaller than $\Delta \phi = 0.04$ rad. We note that the differences are larger than the systematic error reported in [8, 9]. In that work, the error estimate refers to the difference between evolutions starting from the same initial data but different worldtube locations, at a fixed resolution (though the
results converge to the same waveform as the resolution is increased). Our results indicate that at typical current resolutions, the choice of characteristic initial data has a larger influence on the simulation error than the worldtube location. The initial burst of junk as well as spurious incoming transients can alter the measured waveforms by an amount that is of the order of the discretization error of current numerical relativity codes (e.g. [27] and references therein), over a period of several hundred $M$.

An important aspect of our findings is whether the differences in the obtained waveforms have an impact on matched-filtering searches in the upcoming next generation of ground-based gravitational-wave detectors. A measure of the differences as seen by a gravitational-wave detector such as (advanced) LIGO in matched-filtering wave-searches is given by the waveform mismatch $M_{\text{mis}}$ (see, e.g., [28, 29]). This quantity is defined via the match $M$ by

$$M_{\text{mis}} = 1 - M,$$

where

$$M = \max_{t_0} \max_{\Delta \phi} \mathcal{O}[h_1, h_2]$$

is given by a maximization of the overlap $\mathcal{O}[h_1, h_2]$ of two waveforms $h_1$ and $h_2$ over their time of arrival and phase differences. The overlap is defined by

$$\mathcal{O}[h_1, h_2] := \frac{\langle h_1, h_2 \rangle}{\sqrt{\langle h_1, h_1 \rangle \langle h_2, h_2 \rangle}},$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product

$$\langle h_1, h_2 \rangle = 4 \text{Re} \int_0^\infty df \frac{\tilde{h}_1^* \tilde{h}_2}{S_h},$$
In the above expression, a tilde denotes the Fourier-transformed waveforms, and $S_n$ is the detector sensitivity function.

A mismatch of $M_{\text{mis}} = 0$ indicates that the two waveforms are identical. Vice versa, a mismatch of $M_{\text{mis}} = 1$ indicates that the waveforms are completely different. The minimum requirement for detection is given by a mismatch of $M_{\text{mis}} = 3.5 \times 10^{-2}$ between a template signal and a measured signal, corresponding to a loss of no more than 10% of all detected signals [28–30]. In our present case, we can treat one of our waveforms as the ‘template’ signal and the second waveform as the ‘measured’ signal. Ideally, when computing the mismatch above, both waveforms should agree to higher precision than what is given by the minimum requirement for detection. Indeed, this is the case. For instance, we compute the mismatch of the waveforms from models J0-R100-u0 and Jlin-R100-u450 over a mass range $M = [200M_\odot, 500M_\odot]$. The minimum mass is set by twice the initial orbital frequency compatible with the advanced LIGO cut-off frequency of 10 Hz. The maximal mass is given by a resulting maximum frequency that is still overlapping with the detector sensitivity range above 10 Hz. We find a maximal mismatch $M_{\text{mis}} = 1.46 \times 10^{-5}$ between the waveforms of models J0-R100-u450 and Jlin-R100-u450. For models J0-R100-u0 and Jlin-R100-u450, we find a maximal mismatch $M_{\text{mis}} = 4.78 \times 10^{-7}$, which is better than the former case above. This behavior is expected, since $J=0$ initial data at $u_0 = 0M$, as well as $J = J_{\text{lin}}$ initial data at $u_0 = 450M$ are compatible with the Cauchy data at given times, respectively. Therefore, the remaining differences in the two waveforms must be small. Having $J = 0$ initial data at a time $u_0 = 450M$ where the Cauchy data contain non-trivial amounts of outgoing radiation, on the other hand, is clearly incompatible with the Cauchy data at that time, hence leading to large mismatches between the two waveforms. Overall, all waveform mismatches are orders of magnitude better than what is imposed by the minimum detection requirement. Therefore, the differences due to different characteristic initial data are unlikely to affect matched-filtering searches, at least not in the advanced LIGO detectors. Very similar results hold for the other models as well.

The numerical tests described so far were with the extraction radius $R_T = 100M$. All these runs were repeated with the extraction radius re-set to $R_T = 250M$. The results are qualitatively similar to the $R_T = 100M$ case, and the details are not presented here. One interesting feature was the behaviour of $W_c$ at the worldtube, i.e. the analogy to figure 2: it took until about 900 $M$ until the decay in the difference between the linearized and actual data was saturated. This is surprising since one would expect that the radiation that passed $R_T = 100M$ at about $u = 450M$ would pass $R_T = 250M$ at $u = 600M$. We note that at these radii and grid resolutions, the $3+1$ data are several orders of magnitude larger than the truncation error and show little radius-dependent degradation (as we have also observed in previous studies [25, 26]). One possible explanation is that the saturation is due to nonlinear effects, and since they are somewhat weaker at $R_T = 250M$, saturation takes longer. Furthermore, the boundary data amplitudes are one order of magnitude smaller at $R_T = 250M$ than those at $R_T = 100M$, and thus potentially more sensitive to incoming modes. Further studies would be required to fully understand the nature of these effects.

6. Discussion and conclusion

Characteristic evolutions provide a means of determining radiated gravitational energy which is free from the ambiguities associated with local measures, namely nonlinear effects in the near-zone, as well as ambiguities due to gauge and extrapolations to infinity. The principal remaining issue has been to specify appropriate initial data on the null cone, compatible with the data on the worldtube and the Cauchy 3 + 1 spacetime. The strong correlation between
finite radius results and characteristic extraction, as well as the invariance of the results on the worldtube location observed in [8, 9], suggests that even the simplest initial data ansatz, \( J = 0 \), can provide results which are accurate enough for astrophysical estimates, provided the Cauchy 3 + 1 spacetime is essentially free of radiation at the initial characteristic time. However, in scenarios where strong outgoing radiation is present during the initial characteristic time, \( J = 0 \) data effectively represents incoming radiation, and may alter significantly the evolution of the wave signal towards \( J^+ \).

The gravitational wave solution developed in section 3 provides initial data \( J = J_{\text{lin}} \) which is compatible, to the linearized level, with outgoing radiation from a binary system. As such, it provides a more physically motivated starting point than the shear-free, \( J = 0 \), alternative. Importantly, we find that the evolutions which take place from either \( J = J_{\text{lin}} \) initialized at a time \( u_0 = 450 M \) when outgoing radiation is present at the worldtube, or \( J = 0 \) at the initial time \( u_0 = t_0 = 0 \) when the Cauchy 3 + 1 spacetime is conformally flat are very similar (compare figure 4), and indeed very similar to the purely 3 + 1 result which can be obtained by polynomial extrapolating finite radius measurements. That is, for simple choices of initial \( J \), the physical conclusions are not altered dramatically, provided the Cauchy 3 + 1 spacetime does not contain strong amounts of radiation at the worldtube location during the initial characteristic time \( u_0 \). Support for this claim is provided by the computed matched-filtering mismatches between waveforms obtained using different initial data. In the worst case, the mismatch is on the order of \( 10^{-5} \) which is orders of magnitude better than the minimum detection requirement of \( 3.5 \times 10^{-2} \). It is also smaller than the mismatch resulting from the systematic error which arises when using finite radius extrapolation to determine the asymptotic waveforms rather than characteristic extraction [8].

On the other hand, we have demonstrated that the choice of characteristic initial data does result in a small but measurable difference, which decays at a slow exponential rate over a time period of several hundred \( M \). We see this transient in the graphs of gravitational news exhibited in figure 3. Since the linearized initial data contain only an outgoing mode, we conclude that the shear-free characteristic initial data contain incoming radiation. While this is expected, it is interesting that it takes so much time for the effect to decay away.

This study was designed to assess the long-term effect of characteristic initial data, but it also provides information about the incoming radiation in 3 + 1 initial data. The point is that the characteristic data on the worldtube are determined entirely by the 3 + 1 data, and the extent to which this data does not fit the linearized solution is a measure of its incoming radiation content. By construction, the quantities \( \beta, U \) and \( J \) in the linearized solution must fit the data, with the difference in \( W \) being an indication of incoming radiation in the 3 + 1 evolution. Since \( W \) is not a gauge invariant quantity, it is not possible to make a quantitative statement about the magnitude of the incoming radiation, but figure 2 indicates that it takes until at least \( \pm 400 M \) until it is possible to neglect the effect of incoming radiation in the region \( r < 100 M \).

The incoming radiation in both 3 + 1 and characteristic initial data can be regarded as a component of the junk radiation. It decays exponentially, and has a more long-term impact than the outgoing burst component of the junk radiation which passes the \( r = 100 M \) worldtube radius by approximately \( u = 150 M \). The effect of incoming radiation has not received much attention in the previous literature dealing with the problem of initial data construction. It has potential significance for the construction of high-accuracy gravitational-wave templates. Further investigations may also explain properties of the incoming radiation content, giving further insight into the peeling property of more complex spacetimes other than Kerr. In particular, it will be important to determine, using a gauge invariant measure, the long-term effect of incoming radiation in 3 + 1 initial datasets.
The results suggest some promising avenues for future study. Current methods (‘characteristic extraction’) transport data in one direction (from the Cauchy to the characteristic code). Great efficiencies would be possible if the coupling were also carried out in the other direction, so that the characteristic evolution would provide outer boundary data for the Cauchy evolution. The linearized wave solution provides a simple recipe for isolating ingoing versus outgoing modes on the characteristic grid, and may be useful in designing a method for stably transporting data in both directions across the worldtube interface.

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