The Tate Thomason Conjecture
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The Tate Thomason Conjecture

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Abstract

We prove the Tate Thomason conjecture using K where K is the complex topology spectrum. (see the introduction below). Fundamental to our proof is Theorem 2.2. Much of this work is related to the paper [TH89]

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Introduction:

By a spectrum we mean the following: A spectrum X is a collection of simplicial sets X_n for n ≥ 0 together with morphisms of simplicial sets σ_n : ΣX_n → Y_n. A morphism of spectra f : X → Y is a collection of morphisms f_n : X_n → Y_n of simplicial sets that commute with the structure maps σ_n, ie σ_n ◦ Σf_n = f_n+1 ◦ σ_n. We are going to consider the stable homotopy category S derived from the Bousfield-Friedlander model category of simplicial spectra [BF78]. The topological K spectrum is written K and its localization is written L most of the time. The spectra we consider are K(X) where X is a scheme over a finite field F_q with q = p^s where p ≠ l, and K(X) is Quillen’s Algebraic K theory spectra related to X. See [Jar] page 148-150. Our aim is to prove the Tate Thomason conjecture ([TH89]): Let X_∞ be X ×_{Spec(F_q)} Spec( ¯F_q), where X is a smooth projective variety over Spec(F_q), and ¯F_q is the algebraic closure of F_q, then the homotopy group π_{−1}(LK(X_∞)) is reduced. This statement implies the Tate Conjecture, (See Remark 0.2 below and also see [TH89] page 390 diagram 21). The proof is based on theorem 2.2 from section 2.

0. The Tate Conjecture

For the projective smooth variety X_∞ over the finite field F_q, consider the i-th Chow Group CH^i(X_∞) generated by cycles of codimension i on X_∞ modulo rational equivalence. If l is a prime invertible in F_q we can define a Q_l-linear cycle morphism from the i-th Chow group into the 2i-th l-adic etale cohomology with i-th Tate twist coefficient:

\[ γ^{l}_{i} : CH^i(X_∞) ⊗ Q_l \mapsto H^{2i}(X_∞, Q_l(i)) \]
Consider $F_q$ as a finite subfield of $\bar{F}_q$. Let $H^2(X_{\infty}, Q_l(i))^{\text{Gal}(F_q/F_q)}$ be the subspace of $H^2(X_{\infty}, Q_l(i))$ of finite order under the action of $\text{Gal}(F_q/F_q)$, then the Tate Conjecture states:

0.1 **Tate Conjecture**: The image of the cycle class morphism $\gamma^i_{Q_l}$ in the cohomology group $H^2(X_{\infty}, Q_l(i))$ is exactly $H^2(X_{\infty}, Q_l(i))^{\text{Gal}(F_q/F_q)}$.

0.2 **Thomason’s reformulation of the Tate Conjecture**:

Take the smooth projective variety $X_{\infty}$. Let $X_n = X \otimes F_q^n$ where $X$ is a smooth projective variety over $F_q$ as mentioned in the introduction. Let $K$ be the complex topology spectrum. Thomason proves in [Th89] the following theorem:

**Theorem 0.1 (Tate Thomason’s Conjecture)**: The Tate Conjecture is equivalent to the finiteness statement that for all $n \in \mathbb{N}$, see [TH89]

\[ \text{Hom}(Q/Z(l), \pi_{-1}(L_K K(X_n))) = 0 \]

Remark 0.1: observe that in page 390 diagram 21 Thomason states that the arrow $f \otimes Q$ should be 0 for the Tate conjecture to be true. Hence we only need to prove that

\[ \text{Hom}(Q/Z(l), \pi_{-1}(L_K K(X_n))) = 0 \]

Remark 0.2: Though the above Hom being 0 is sufficient to prove the Tate Conjecture, to prove the equivalence with the Tate Conjecture we need the Hom with the $X_n$ [TH89]

**Definition 0.1**: A group $G$ which verifies that $\text{Hom}(Q/Z(l), G) = 0$ is said to be $l$ reduced.

We will simplify all over this work the terminology by saying that a group is reduced when we really mean that it is $l$-reduced.

The main lemmas stated by Thomason in [Th89] to prove the Tate Conjecture are
Lemma 0.1: Let $X$ be a smooth projective variety over $F_q$ with $q = p^n$ and $p$ prime. If $l$ is a prime number different from $p$, and if $K^{Top}(X)$ is in Thomason’s notation the topological $K$-theory spectrum, and $(\ldots)^\wedge$ is the $l$-adic completion of a spectrum, then

$$L_K K(X)^\wedge \cong K^{Top}(X)^\wedge$$

The proof of this lemma follows from Thomason’s descent theorem proved in [TH85, 4.1] which relates algebraic $K$-theory to topological $K$-theory. See also ([TH89], page 388, equation (14) and references therein)

Remark 0.3: It also follows from [TH89] page 388 equation (13) that

$$K^{Top}/l^n(X) \cong L_K(X)/l^n$$

See also the descent problem in ([M96], section 4)

Lemma 0.2: (See [TH89], the lemma of page 387)

The image of $colim K_0^{Top}(X_n) \otimes Q$ in $K_0^{Top}(X_\infty) \otimes Q$ consists precisely of those elements of finite orbit under $Gal(\overline{F}/F_q)$.

Remark 0.4: (See [TH89], page 387 square diagram 10) which is equal to

$$colim(K_0(X_n) \otimes Q_l) = K_0(X_\infty) \otimes Q_l \rightarrow colim K_0^{Top}(X_n) \otimes Q \downarrow K_0^{Top}(X_\infty) \otimes Q$$

Where the equality means an isomorphism and in light of this, lemma 0.2, and the well established fact that $K_0(X_\infty) \otimes Q_l$ is isomorphic to $\bigoplus_{i=1}^d CH^i(X_\infty) \otimes Q_l$ with $d$ the dimension of $X$, and that $K_0^{Top}(X_\infty) \otimes Q$ is isomorphic to $\bigoplus_{i=1}^d H^{2i}_{et}(X_\infty, Q_l(i)).$ (see [TH89], page 390) the above composed map is exactly the cycle map $\gamma(X_\infty)_{Q_l}$ and the Tate Conjecture is equivalent to the conjecture that $K_0(X_\infty) \otimes Q_l$ and $colim(K_0^{Top}(X_n) \otimes Q)$ have the same image in $K_0^{Top}(X_\infty) \otimes Q$. 
Lemma 0.3: is diagram 21 on page 390

Remark 0.5: To our knowledge, Thomason never said anything about the $l$-reducibility of the homotopy group $\pi_{-1}LK(X_\infty)$

1. The Moore Spectrum and an important exact sequence

Remark 1.1: Given $l^\nu$, with $\nu \in N$, and $M(Z/l^\nu)$ the Moore spectrum of the ring $Z/l^\nu$, and if $X$ is a smooth variety over a field $k$ where $k = F_q$ or $k = \bar{F}_q$ with $q = p^n$ and $L$ is the localization functor at the complex K-Theory spectrum $K$, then

$$LK(X)/l^\nu = LK(X) \wedge M(Z/l^\nu) = K(X) \wedge L\Sigma^\infty S^0 \wedge M(Z/l^\nu) = K(X)/l^\nu \wedge L\Sigma^\infty S^0 = L(K(X)/l^\nu).$$

These equalities follow from the smash property for $L$.

Remark 1.2: $\pi_*(L(K(X)/l^\nu))$ is $l^\nu$-torsion.

The claim follows from the exact sequence

$$0 \to \pi_*(LK(X) \otimes Z/l^\nu) \to \pi_*(LK(X) \wedge M(Z/l^\nu)) \to \text{Tor}^1(\pi_{-1}(LK(X)), Z/l^\nu) \to 0$$

which splits (See [TH85] Appendix A, (6))

2. Main Theorem.

Theorem 2.2: The Tate Module (which we call $M$) of the abelian group $\pi_m(LK(X_\infty))$ is trivial for every integer $m$

As we have mentioned above, we have by [TH85] Appendix 6) that the exact sequence below splits

$$0 \to \pi_m(LK(X_\infty) \otimes Z/l^\nu) \to \pi_m(LK(X_\infty) \wedge M(Z/l^\nu)) \to \text{Tor}^1(\pi_{m-1}(LK(X_\infty)), Z/l^\nu) \to 0$$

Taking inverse limit over $\nu$ on the above exact sequence, we get an exact sequence since $\lim$ is left exact, and moreover exact in the case of the above exact sequence since the left extreme of that sequence is a surjective inverse system, and therefore Mittag-Leffler. Then, if we prove that

$$\lim \pi_m(LK(X_\infty) \otimes Z/l^\nu) = (\pi_m(LK(X_\infty)))^l \cong \lim \pi_m(LK(X_\infty)) \wedge M(Z/l^\nu)$$
where \((\pi_m(LK(X_\infty)))^l\) stands for \(l\)-adic completion of that abelian group, with \(l\) prime to \(p\) the characteristic of the base field of our smooth projective variety, being the \(\lim\) exact here, we conclude that the Tate Module of \(\pi_m(LK(X_\infty))\) is trivial since it is equal to (taking inverse limit over \(\nu\) for fixed \(l\))

\[
\lim Tor^1(\pi_{m-1}(LK(X_\infty)), Z/l^\nu)
\]

Again, taking inverse limit with respect to the parameter \(\nu\) with \(l\) fixed,

\[
\lim Tor^1(\pi_{m-1}LK(X_\infty)), Z/l^\nu)) = M = 0
\]

where \(M = \lim Tor^1(\pi_{m-1}LK(X_\infty), Z/l^\nu)\) is equal to

\[
\Pi_{i=1}^\infty \{g_i = l^i - \text{torsion element} \in \pi_{m-1}(LK(X_\infty)) : lg_{i+1} = g_i\}
\]

by definition of the inverse limit and since \(Tor^1(\cdot, Z/l^\nu)\) are the \(l^\nu\)-torsion elements of \(\pi_{m-1}LK(X_\infty)\).

See [Ro79] 8.4 page 223 and also the fact that \(Tor^1(Z, A) = Tor^1(Z, B)\) on page 222).

Now, consider the image of a map \(f\) in \(\text{Hom}(Q/Z(l), \pi_{m-1}(LK(X_\infty)))\)

Since \(Q/Z(l)\) is an \(l\) quasicycle divisible group (see [LF], if \(D\) is the image of \(f\) then \(D\) is an increasing union of groups \(D_n\), of order \(l^n\), where each \(D_n\) is generated by one generator \(g_n\) which is \(l^n\)-torsion, and also \(lg_n = g_{n-1}\), if the Tate module \(M\) trivial, this sequence of \(g_n\) has to be trivial and therefore \(f = 0\)

Therefore we now know that details about the Tate Module \(M\) of the homotopy groups \(\pi_{m-1}LK(X_\infty)\) are of fundamental importance. The above arguments show that the Tate module \(M\) is isomorphic to \(\text{Hom}(Q/Z(l), \pi_{m-1}(LK(X_\infty)))\) by taking the sequence of generators of the increasing sequence of cyclic subgroups of \(Q/Z(l)\) to the sequence of it’s images which belong to \(M\). By [TH89] pp 389 (15) we have an exact sequence

\[
0 \rightarrow Ext(Q/Z(l), \pi_m(LK(X_\infty))) \rightarrow \pi_m((LK(X_\infty))^l) \rightarrow Hom(Q/Z(l), \pi_{m-1}(LK(X_\infty))) \rightarrow 0
\]

We can compare that exact sequence with the above exact sequence after taking \(\lim\) and using the fact that \(\lim \pi_m(LK(X_\infty)) \wedge M(Z/l^\nu) \cong \pi_m((LK(X_\infty))^l)\) by [TH89] (6) pp 387 and (13) pp 388

\[
0 \rightarrow (\pi_m(LK(X_\infty)))^l \rightarrow \pi_m((LK(X_\infty))^l) \rightarrow M = \lim Tor^1(\pi_{m-1}LK(X_\infty), Z/l^\nu) \rightarrow 0
\]
We have the same middle term in both exact sequences and moreover we also proved that the extreme right terms are isomorphic. Therefore by the snake lemma, the extreme left terms must also be isomorphic. By [TH89] pp 389 (16) we have another exact sequence see [TH89] pp 389 (16)

\[ 0 \mapsto \lim^1 \text{Hom}(\mathbb{Z}/l^k \mathbb{Z}, \pi_m((LK(X_\infty)))) \mapsto \text{Ext}(\mathbb{Q}/\mathbb{Z}(l), \pi_m(LK(X_\infty))) \mapsto \lim \pi_m(LK(X_\infty)) \otimes \mathbb{Z}/l^\nu \mapsto 0 \]

Since we have just shown that Ext is isomorphic to the right extreme of the above exact sequence, therefore by [TH89] pp 389 below (17) that means that means that \( \lim^1 \text{Hom}(\mathbb{Z}/l^k \mathbb{Z}, \pi_m(LK(X_\infty))) = 0 \) Thomason states in [TH89] that if the \( l \) torsion of the groups \( \pi_m(LK(X_\infty)) \) are finite, then the Mittag Leffler condition holds for the inverse system of the abelian groups \( \text{Hom}(\mathbb{Z}/l^k \mathbb{Z}, \pi_m((LK(X_\infty)))) \) and then \( \lim^1 = 0 \) for those groups. It is a well known fact that the Mittag Leffler condition for an inverse system of abelian groups implies the statement \( \lim^1 = 0 \) for that system of abelian groups and that is what is implied in Thomason statement in [TH89]. The other way around is not always true, since an inverse system having \( \lim^1 = 0 \) does not in general imply the Mittag Leffler condition for that inverse system. But if the abelian groups \( \pi_m((LK(X_\infty))) \) are at most countable abelian groups then \( \text{Hom}(\mathbb{Z}/l^k \mathbb{Z}, \pi_m((LK(X_\infty)))) \) is an inverse system of at most countable abelian groups. In [E94] it is stated, and in [G66] it is proven, that an inverse system of at most countable abelian groups has \( \text{Lim}^1 = 0 \) if and only if the Mittag Leffler condition holds true for that inverse system. Using this last statement if we show that the \( \text{Hom}(\mathbb{Z}/l^k \mathbb{Z}, \pi_m((LK(X_\infty)))) \) are at most countable, since we know that \( \lim^1 \text{Hom}(\mathbb{Z}/l^k \mathbb{Z}, \pi_m((LK(X_\infty)))) = 0 \) then the Mittag Leffler must hold true for the inverse system \( \text{Hom}(\mathbb{Z}/l^k \mathbb{Z}, \pi_m((LK(X_\infty)))) \) in which case the homotopy groups \( \pi_m((LK(X_\infty))) \) must have bounded \( l \) torsion, since the inverse system of our hom groups, being Mittag Leffler at certain point become stationary. If that is the case, ie if they have bounded \( l \)-torsion, taking an \( f \) in \( \text{Hom}(\mathbb{Q}/\mathbb{Z}(l), \pi_{m-1}((LK(X_\infty)))) \) it’s image is \( l \)-torsion, and hence must have bounded \( l \)-torsion because of what we have just proved. But that is not possible since \( \mathbb{Q}/\mathbb{Z}(l) \) is divisible and therefore the image of \( f \) is an \( l \)-torsion and divisible abelian group. Hence \( f = 0 \) and we conclude that the Tate module \( M \) is trivial as stated in theorem 2.2, and moreover we can state the following theorem:

**Theorem 2.3:** \( \text{Hom}(\mathbb{Q}/\mathbb{Z}(l), \pi_{m-1}((LK(X_\infty)))) = 0 \) for all \( m \)

To end the proof of theorem 2.2 and theorem 2.3, we must show that the inverse system of hom groups \( \text{Hom}(\mathbb{Z}/l^k \mathbb{Z}, \pi_m((LK(X_\infty)))) \) is an inverse system of at most countable abelian groups. Since the image of an \( f \) in \( \text{Hom}(\mathbb{Q}/\mathbb{Z}(l), \pi_m((LK(X_\infty)))) \) can only intersect \( l \)-quasicyclic subgroups of \( \pi_m((LK(X_\infty))) \) we are only concerned with the \( l \) quasicyclic part of \( \pi_m((LK(X_\infty))) \). Since that part is divisible we can quotient out the complement of that divisible group in \( \pi_m((LK(X_\infty))) \) when considering \( \text{Hom}(\mathbb{Q}/\mathbb{Z}(l), \pi_m((LK(X_\infty)))) \), and it also informs us that \( \pi_m((LK(X_\infty))) \) must have at least an \( l \) a quasicyclic direct
summand as a subgroup. Now if $\text{Hom}(\mathbb{Z}/l^k \mathbb{Z}, \pi_m((LK(X_\infty))))$ is uncountable then writing this homotopy group as $A_r \times A_d$ where $A_r$ is the reduced subgroup and $A_d$ the divisible subgroup then either b) $\text{Hom}(\mathbb{Q}/\mathbb{Z}(l), A_r)$ is uncountable or a) $\text{Hom}(\mathbb{Q}/\mathbb{Z}(l), A_d)$ is uncountable. In case a) the $l$ quasicyclic part of $\pi_m((LK(X_\infty)))$ which is included in $A_d$ must have an uncountable number of direct summands or must be an infinite product of infinite $l$ quasicyclic groups.

We will focus first on case a), and later on we will consider case b). In case a) we must analyze if the $l$ quasicyclic part of $\pi_m((LK(X_\infty)))$ can be an infinite product of $l$ quasicyclic groups or an uncountable direct sum of $l$ quasicyclic groups. We might as well consider as $\pi_m((LK(X_\infty)))$ having one of those two structures, since adding a complementary abelian group will basically not change what we will prove in what follows. Also, since both cases are similar we analyze the first case only. Hence, if we consider the case of an infinite product of $l$-quasicyclic groups, we now are going to show that that structure cannot possibly hold.

Consider the already mentioned exact sequence (see below theorem 2)

$$0 \to \pi_m(LK(X_\infty)) \otimes \mathbb{Z}/l^\nu \to \pi_m(LK(X_\infty)) \wedge M(\mathbb{Z}/l^\nu) \to \text{Tor}^1(\pi_{m-1}(LK(X_\infty)), \mathbb{Z}/l^\nu) \to 0$$

The left hand side would be 0 since multiplying that infinite product of quasicyclic groups by a power of $l$ would be equal to the initial infinite product of quasicyclic $l$ groups because of the structure that the $l$ quasicyclic groups have. Therefore since the middle term in the above exact sequence which is $\pi_m(LK(X_\infty)) \wedge M(\mathbb{Z}/l^\nu)$ is finite by page 387 in [TH89] and page 388 (13) in [TH89] it would be isomorphic to the $l^\nu$ torsion elements of $\pi_m(LK(X_\infty))$

Then this torsion group would have to be finite and therefore it cannot have as a structure an infinite product of quasicyclic groups since that group has an $l^\nu$ torsion subgroup which is infinite because of the structure the group has. We can conclude then that case a) is not possible. Now we turn to case b) in which $\pi_m((LK(X_\infty)))$ has as a subgroup included in $A_r$ an infinite product of finite cyclic groups $\mathbb{Z}/l^\nu \mathbb{Z}$ with $\nu$ growing to infinity or not growing to infinity. If $\nu$ does not grow to infinity the proof is exactly as the proof of case a). If $\nu$ grows to infinity We have $L = \prod_{\nu=1}^\infty \mathbb{Z}/l^\nu \mathbb{Z}$ as a subgroup of $A_r$, so that it is sufficient to consider $A_r = L \times H$. Observe that by definition of $L$ the $l$-adic completion of $L$ is 0. Hence we have by [LF] pp 222 that

$$\lim_1 \text{Hom}(\mathbb{Z}/l^\nu \mathbb{Z}, L) \cong \text{Ext}(\mathbb{Q}/\mathbb{Z}(l), L) \cong \prod_{\nu=1}^\infty \text{Ext}(\mathbb{Q}/\mathbb{Z}(l), \mathbb{Z}/l^\nu \mathbb{Z})$$
But observe then that using the observation in [TH89] pp 389 below (17) we have the $\lim^1$ below is equal to 0, so that taking $\lim^1$ over $j$ below

$$0 \mapsto \lim^1 \text{Hom}(Z/l^j Z, Z/l^\nu Z) = 0 \mapsto \text{Ext}(Q/Z(1), Z/l^\nu) \mapsto (Z/l^\nu Z)^{\dagger} = Z/l^\nu Z \mapsto 0$$

and hence $\lim^1 \text{Hom}(Z/l^\nu Z, L) \cong \Pi_{\nu=1}^\infty Z/l^\nu Z$ which is not possible since then

$$\lim^1 \text{Hom}(Z/l^\nu Z, \pi_m(LK(X_\infty))) = \lim^1 (\text{Hom}(Z/l^\nu Z, L) \times \text{Hom}(Z/l^\nu Z, H) \times \text{Hom}(Z/l^\nu Z, A_d))$$

which is not equal to 0 and that is a contradiction. Therefore neither case a) or b) can hold and $\text{Hom}(Z/l^\nu Z, \pi_m(LK(X_\infty)))$ must be at most countable as wanted.

Remark: 2.4: this theorem 2.3 shows what we conjectured above in order to research the Tate module $M$, that homotopy groups of the $l$-adic completion of the $LK(X_\infty)$ spectrum are isomorphic to the $l$-adic completion of the homotopy groups of the $K$ spectrum, which at first sight is not trivial.

Corollary 2.5: $\pi_{-1}(LK(X_\infty))$ is reduced

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