Research Article

Symmetry Classification and Exact Solutions of a Variable Coefficient Space-Time Fractional Potential Burgers’ Equation

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We investigate the symmetry properties of a variable coefficient space-time fractional potential Burgers’ equation. Fractional Lie symmetries and corresponding infinitesimal generators are obtained. With the help of the infinitesimal generators, some group invariant solutions are deduced. Further, some exact solutions of fractional potential Burgers’ equation are generated by the invariant subspace method.

1. Introduction

Partial differential equations (PDEs) have a wide range of applications in many fields, such as physics, engineering, and chemistry, which are fundamental for the mathematical formulation of continuum models [1, 2]. Burgers’ equation is a one-dimensional nonlinear partial differential equation, which is a simple form of the one-dimensional Navier-Stokes equation. It was presented for the first time in a paper in 1940s by Burgers. Later, Burgers’ equation was studied by Cole [3] who gave a theoretical solution, based on Fourier series analysis, using the appropriate initial and boundary conditions. Burgers’ equation has a large variety of applications in the modelling of water in unsaturated soil, dynamics of soil water, statistics of flow problems, mixing and turbulent diffusion, cosmology, and seismology [3, 4].

Recently, fractional differential equations have found extensive applications in many fields. Many important phenomena in viscoelasticity, electromagnetics, material science, acoustics, and electrochemistry are elegantly described with the help of fractional-order differential equations [4–9]. It has been revealed that the nonconservative forces can be described by fractional differential equations. Therefore, as most of the processes in the real physical world are nonconservative, the fractional calculus can be used to describe them. Fractional integrals and derivatives also appear in the theory of control of dynamical systems, when the controlled system and/or the controller is described by a fractional differential equation [10, 11]. In the last few decades, the subject of the fractional calculus has caught the consideration of many researchers who contributed to its development. Recently, some analytical and numerical methods have been introduced to solve a fractional-order differential equation [7, 8, 11–24]. However, all the methods have insufficient development as they allow one to find solutions only in case of linear equations and for some isolated examples of nonlinear equations [4, 7, 8, 11, 23, 24].

It is very well known that the Lie group method is the most effective technique in the field of applied mathematics to find exact solutions of ordinary and partial differential equations [25, 26]. However, this approach is not yet applied much to investigate symmetry properties of fractional differential equations (FDEs). To the best of our knowledge, there are a few papers (e.g., [17, 26–32]) in which Lie symmetries and similarity solutions of some fractional differential equations have been discussed by some researchers. More recently, Jumarie [33] proposed the modified Riemann-Liouville derivative and Jumarie-Lagrange method [34], after which a generalized fractional characteristic method and a fractional Lie group method have been introduced by Wu.
In this paper, we intend to apply Lie group method to solve the space-time fractional potential Burgers' equation of the form
\[ \partial_x D_x^\alpha u = f(t) \partial_x D_x^\beta u + g(t) \left( \partial_x D_x^\gamma u \right)^2, \]
where \( \partial_x D_x^\alpha u \) and \( \partial_x D_x^\beta u \) are the modified Riemann-Liouville derivatives with respect to time and space variables, respectively, and \( f(t) \) and \( g(t) \) are arbitrary functions of \( t \). Equation (1) is connected to the fractional Burgers' equation by the well-known Hopf-Cole transformations and is a generalization of the time-fractional Burgers' equation with constant coefficients examined by Wu [32]. This work is based on some basic elements of fractional calculus, with special emphasis on the modified Riemann-Liouville type derivative. The paper is organized as follows. In Section 2, we briefly describe some definitions and properties of fractional calculus. In Section 3, we obtain the symmetries for Burgers' equation (1). Finally, a conclusion is given in Section 6.

### 2. Some Concepts from Fractional Calculus

In this paper, the modified Riemann-Liouville derivative proposed by Jumarie [33] has been adopted. Some definitions are given which have been used in this work.

#### 2.1. Fractional Riemann-Liouville Integral

The fractional Riemann-Liouville integral of a continuous (but not necessarily differentiable) real valued function \( f(x) \) with respect to \((dx)^\alpha\) is defined as
\[ \mathcal{I}_x^\alpha f(x) = \frac{1}{\Gamma (\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) \, dt \]
\[ = \frac{1}{\Gamma (1+\alpha)} \int_0^x f(t) \, (dt)^\alpha, \quad 0 < \alpha \leq 1. \]
The fractional integral with respect to \((dt)^\alpha\) was introduced by Jumarie [36] in order to study the fractional derivative of nondifferentiable functions in modified Riemann-Liouville sense. Here we are fully in Leibniz framework; that is to say, finite increment in fractional sense. As a result, we shall be able to duplicate, in a straightforward manner, most of the known standard formulae by merely making the substitution \((dx)^\alpha \rightarrow dx\).

#### 2.2. Riemann-Liouville Fractional Derivative

The fractional Riemann-Liouville derivative of \( f(x) \) is defined as
\[ \partial_x D_x^\alpha f(x) = \left\{ \frac{d^n}{dx^n} \int_0^x (x-t)^{\alpha-n-1} f(t) \, dt \right\}, \quad n-1 < \alpha < n, \quad n \in \mathbb{N}. \]

### 2.3. Modified Riemann-Liouville Derivative

Through the fractional Riemann-Liouville integral, Jumarie [33] proposed the modified Riemann-Liouville derivative of \( f(x) \) as
\[ \partial_x D_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-t)^{n-\alpha-1} f(t) \, dt, \quad n-1 < \alpha < n, \quad n \in \mathbb{N}. \]

#### 2.4. Some Useful Formulae

Here, some properties of modified Riemann-Liouville derivative are given which have been used in this paper:

(i) \( df(x) = \partial_x D_x^\alpha f(x)(dx)^\alpha / \Gamma (1+\alpha), \quad \alpha > 0. \)

(ii) \( \partial_x D_x^\alpha f(u(t)v(t)) = (\partial_x D_x^\alpha u(t))v(t) + u(t)(\partial_x D_x^\alpha v(t)), \quad 0 < \alpha < 1. \)

(iii) \( \partial_x D_x^\alpha f(x(t)) = (df/dx)(\partial_x D_x^\alpha x(t)), \quad 0 < \alpha < 1, \quad \text{given that} \ df/dx \ \text{exists.} \)

(iv) \( \partial_x D_x^\alpha x^\beta = (\Gamma(1+\beta)/\Gamma(1+\beta-\alpha))x^{\beta-\alpha}, \quad 0 < \alpha < 1, \quad x > 0, \quad \text{and} \beta > -1. \)

Wu [32, 35] extended the characteristic method of first-order linear partial differential equation to a linear fractional differential equation of the form
\[ a(x,t) \frac{\partial^\beta u(x,t)}{\partial x^\beta} + b(x,t) \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = c(x,t), \quad 0 < \alpha, \beta \leq 1 \]
and introduced the fractional characteristic equations [32] as given herein:

\[
\frac{(dx)\beta}{\Gamma(1 + \beta)(x,t)} = \frac{(dt)\alpha}{\Gamma(1 + \alpha)(x,t)} = \frac{du}{c(x,t)}.
\]

(7)

On solving the characteristic equations (7) for various infinitesimal generators of (1), one can obtain the similarity variables and reduction of (1) to an ODE.

### 3. Symmetry Classification of (1)

Herein, we investigate the symmetries and reductions of space-time fractional potential Burgers’ equation (1). A fractional Lie symmetry of (1) is a continuous group of point transformations of independent and dependent variables which leaves (1) invariant.

Let us assume that (1) admits the Lie symmetries of the form

\[
\begin{align*}
\tilde{x} & = x + \epsilon \xi (x,t,u) + o(\epsilon^2), \\
\tilde{t} & = t + \epsilon \tau (x,t,u) + o(\epsilon^2), \\
\tilde{u} & = u + \epsilon \eta (x,t,u) + o(\epsilon^2),
\end{align*}
\]

where \( \epsilon \) is the group parameter and \( \xi, \tau, \) and \( \eta \) are the infinitesimals of the transformations for the independent and dependent variables, respectively. A group invariant solution of space-time fractional potential Burgers’ equation (1) is a solution which can be mapped into another solution of (1) under the point transformations (8). The associated Lie algebra of infinitesimal symmetries of (1) is then the fractional vector field of the form

\[
V = \xi (x,t,u) \frac{\partial}{\partial x} + \tau (x,t,u) \frac{\partial}{\partial t} + \eta (x,t,u) \frac{\partial}{\partial u}.
\]

(9)

The fractional second-order prolongation of (9) is given by

\[
\begin{align*}
pr^{(2)}V & = \xi (x,t,u) \frac{\partial}{\partial x} + \tau (x,t,u) \frac{\partial}{\partial t} + \eta (x,t,u) \frac{\partial}{\partial u} \\
+ \eta \tilde{x} \frac{\partial}{\partial \tilde{x}} + \eta \tilde{t} \frac{\partial}{\partial \tilde{t}} + \eta \tilde{u} \frac{\partial}{\partial \tilde{u}} \\
+ \eta^x \frac{\partial}{\partial (\partial x \tilde{u})} + \eta^t \frac{\partial}{\partial (\partial t \tilde{u})} \\
+ \eta^x \frac{\partial}{\partial (\partial x \tilde{u})} + \eta^t \frac{\partial}{\partial (\partial t \tilde{u})}.
\end{align*}
\]

(10)

Now, for the invariance of (1) according to [26] under (8), we must have

\[
pr^{(2)}V \left([\Delta u]\right)|_{[\Delta u]=0} = 0,
\]

where \([\Delta u] = aD_x^\beta u - f(t) aD_x^{2\beta} u - g(t)(aD_x^{\beta} u)^2\) or equivalently where

\[
\begin{align*}
\left(\eta^x - f(t) \eta^{xx} - 2g(t)(aD_x^{\beta} u) \eta^x
\right.
\end{align*}
\]

\[
- \tau (aD_x^\beta f(t)) aD_x^{\beta} u
\]

(12)

The generalized fractional prolongation vector fields \( \eta^x, \eta^t, \) and \( \eta^{xx} \) are given by

\[
\begin{align*}
\eta^x & = aD_x^\beta \eta + (aD_x^{2\beta} u) \eta_u - (aD_x^\beta \xi + (aD_x^{2\beta} u) \xi_u) aD_x^\beta u \\
& - (aD_x^{2\beta} \tau) \tau_u aD_x^\beta u, \\
\eta^t & = aD_x^\beta \eta + (aD_x^{2\beta} u) \eta_u - (aD_x^\beta \xi + (aD_x^{2\beta} u) \xi_u) aD_x^\beta u \\
& - (aD_x^{2\beta} \tau) \tau_u aD_x^\beta u, \\
\eta^{xx} & = aD_x^{2\beta} \eta + (aD_x^{2\beta} u) (aD_x^{2\beta} \eta_u) - (aD_x^\beta \xi \\
& + (aD_x^{2\beta} u) (aD_x^{2\beta} \xi_u)) aD_x^{2\beta} u - (aD_x^\beta \tau) \\
& + (aD_x^{2\beta} \tau \tau_u) aD_x^{2\beta} u - 2(aD_x^{2\beta} \tau \\
& + (aD_x^{2\beta} \tau_u) aD_x^{2\beta} u + (aD_x^{2\beta} u) \eta_u \\
& - (aD_x^{2\beta} \xi_u - (aD_x^{2\beta} \eta_u) \tau_u) - 2(aD_x^{2\beta} \xi \\
& + (aD_x^{2\beta} \xi_u) aD_x^{2\beta} u + (aD_x^{2\beta} \xi_u) aD_x^{2\beta} u \\
& - (aD_x^{2\beta} \xi_u + (aD_x^{2\beta} u) \xi_u) aD_x^{2\beta} u \\
& - (aD_x^{2\beta} \xi_u + (aD_x^{2\beta} u) \xi_u) aD_x^{2\beta} u.
\end{align*}
\]

(13)

Now, using the above generalized fractional prolongation vector fields in (12) and equating the coefficient of various derivative terms to zero, we get the following simplified set of determining equations:

\[
\begin{align*}
\tau_u & = 0, \\
aD_x^\beta \tau & = 0, \\
\xi_u & = 0, \\
2g(t) aD_x^{2\beta} \xi - g(t) aD_x^\beta \tau - \tau (aD_x^{2\beta} g(t)) - g(t) \eta_u \\
& - f(t) \eta_u = 0, \\
2f(t) aD_x^{2\beta} \xi - f(t) aD_x^\beta \tau - \tau (aD_x^{2\beta} f(t)) = 0, \\
f(t) aD_x^\beta \xi - 2g(t) aD_x^\beta \eta \\
& - 2f(t) aD_x^{2\beta} \eta = 0, \\
aD_x^\beta \eta - f(t) aD_x^{2\beta} \eta = 0.
\end{align*}
\]

(14) - (20)
On solving (20) by using fractional Lie group method, we obtain a particular solution as

\[
\eta = a_1 \frac{x^\beta}{\Gamma(1 + \beta)} + a_2 \left( D_t^\alpha F(t) \right) + a_2 \frac{x^{2\beta}}{\Gamma(1 + 2\beta)} + a_3, \tag{21}
\]

where \( D_t^{2\beta} + a_6 \) are arbitrary constants. Using this value of \( \eta \) in (17) and (18), we get

\[
2 \left( g(t) - f(t) \right) D_x^{2\beta} \xi = 0, \tag{22}
\]

which brings forth the following possibilities:

(i) \( D_x^{2\beta} \xi = 0 \).

(ii) \( g(t) = f(t) \).

Case (i). In this case, from the determining equations, we get

\[
\xi = -2a_1 G(t) - 2a_2 G(t) \frac{x^\beta}{\Gamma(1 + \beta)} + a_2 \frac{x^\beta}{\Gamma(1 + \beta)} + a_5,
\]

\[
\tau = \frac{1}{D_t^{2\beta}} \left[ 4a_2 H(t) + 2a_4 \left(D_t^\alpha F(t) \right) + a_6 \right],
\]

\[
\eta = a_1 \frac{x^\beta}{\Gamma(1 + \beta)} + a_2 \left(D_t^\alpha F(t) \right) + a_2 \frac{x^{2\beta}}{\Gamma(1 + 2\beta)} + a_3,
\]

where \( D_x^{2\beta} \xi = 0 \) and \( a_1, a_2, a_3, a_4, a_5, a_6 \) are six arbitrary constants. Using (23) in (17), we also get \( g(t) = k f(t) \), where \( k \) is an arbitrary constant. This covers case (ii); as for \( f(t) = g(t) \) solving the determining equations, we get \( D_x^{2\beta} \xi = 0 \). Further, for \( f(t) = g(t) = 1 \) and \( \beta = 1 \), the infinitesimals can be reduced to those reported in [32] by setting the coefficients \( a_1 = -c_3, a_2 = -2c_6, a_3 = c_1, a_4 = c_4, a_5 = c_1, \) and \( a_6 = c_2 \).

Hence, the fractional point symmetry generators admitted by (1) are given by

\[
V_1 = -2G(t) \frac{\partial}{\partial x^\beta} + \frac{x^\beta}{\Gamma(1 + \beta)} \frac{\partial}{\partial u},
\]

\[
V_2 = -2G(t) \frac{x^\beta}{\Gamma(1 + \beta)} \frac{\partial}{\partial x^\beta} - \frac{2H(t)}{D_t^\alpha} \frac{\partial}{\partial u} + a_6 \frac{x^{2\beta}}{\Gamma(1 + 2\beta)} \frac{\partial}{\partial u},
\]

\[
V_3 = \frac{\partial}{\partial u},
\]

\[
V_4 = \frac{x^\beta}{\Gamma(1 + \beta)} \frac{\partial}{\partial x^\beta} + 2 \left( D_t^\alpha F(t) \right) \frac{\partial}{\partial u},
\]

\[
V_5 = \frac{\partial}{\partial x^\beta},
\]

\[
V_6 = \frac{1}{D_t^\alpha F(t)} \frac{\partial}{\partial u}.
\]

These infinitesimals can be used to determine a six-parameter fractional Lie group of point transformation acting on \((x, t, u)\)-space. It can be verified easily that the set \( \{ V_1, V_2, V_3, V_4, V_5, V_6 \} \) forms a six-dimensional Lie algebra under the Lie bracket \([X, Y] = XY - YX\), which reduces to the well-known generalized Galilea algebra [26] for \( \alpha = \beta = 1 \). The commutator table is as given in Table 1.

Further, from the commutator table, it can be seen that the sets \( \{ V_3, V_5, V_6 \} \) form solvable subalgebra. Also, \( V_3 \) is the centre of the six-dimensional Lie algebra as it commutes with every element of the Lie algebra. The group transformation generated by the infinitesimal generators

\( V_i, i = 1, 2, 3, \ldots, 6 \), is found by solving the system of ordinary differential equations:

\[
\frac{d(x^\beta)}{d\epsilon} = \xi (x, t, u),
\]

\[
\frac{d(t^\alpha)}{d\epsilon} = \tau (x, t, u),
\]

\[
\frac{d(u)}{d\epsilon} = \eta (x, t, u),
\]

with the initial conditions

\[
\xi|_{\epsilon = 0} = x,
\]

\[
\tau|_{\epsilon = 0} = t,
\]

\[
u|_{\epsilon = 0} = u.
\]

Exponentiating the infinitesimal symmetries of (1), we get the one-parameter groups \( \gamma_i(\epsilon) \) generated by \( V_i, i = 1, 2, \ldots, 6 \):

\[
g_1 : \left( x^\beta \frac{t^\alpha}{\Gamma(1 + \alpha) \Gamma(1 + \beta)} \right) \rightarrow \left( x^\beta \frac{t^\alpha}{\Gamma(1 + \beta)} \right),
\]

\[
- \frac{2G}{\Gamma(1 + \alpha)} \frac{t^\alpha}{\Gamma(1 + \beta)} + \frac{\epsilon x^\beta}{\Gamma(1 + \alpha)} \left( x^\beta \frac{t^\alpha}{\Gamma(1 + \beta)} \right),
\]

\[
g_2 : \left( x^\beta \frac{t^\alpha}{\Gamma(1 + \alpha) \Gamma(1 + \beta)} \right) \rightarrow \left( x^\beta \frac{t^\alpha}{\Gamma(1 + \beta)} \right),
\]

\[
- \frac{2G}{\Gamma(1 + \alpha)} \frac{t^\alpha}{\Gamma(1 + \beta)} + \frac{\epsilon x^\beta}{\Gamma(1 + \beta)} \left( x^\beta \frac{t^\alpha}{\Gamma(1 + \beta)} \right).
\]
Now, since \( g_i \) is a symmetry, if \( u = \chi(x^{\beta}/\Gamma(1+\beta), \xi^{\alpha}/\Gamma(1+\alpha)) \) is a solution of (1), following \( u_i \) are also solutions of (1):

\[
g_1: \frac{x^{\beta}}{\Gamma(1+\beta)} t^{\alpha}/\Gamma(1+\alpha) \rightarrow \left( \frac{x^{\beta}}{\Gamma(1+\beta)} t^{\alpha}/\Gamma(1+\alpha) + \epsilon \right) u,
\]

\[
g_2: \frac{x^{\beta}}{\Gamma(1+\beta)} t^{\alpha}/\Gamma(1+\alpha) \rightarrow \left( \frac{x^{\beta}}{\Gamma(1+\beta)} t^{\alpha}/\Gamma(1+\alpha) + \epsilon \right) u.
\]

(27)

Now, since \( g_i \) is a symmetry, if \( u = \chi(x^{\beta}/\Gamma(1+\beta), \xi^{\alpha}/\Gamma(1+\alpha)) \) is a solution of (1), following \( u_i \) are also solutions of (1):

\[
u_1 = \chi \left( \frac{x^{\beta}}{\Gamma(1+\beta)} - \frac{2eG}{\Gamma(1+\alpha)} \right) - \frac{e x^{\beta}}{\Gamma(1+\alpha)},
\]

\[
u_2 = \chi \left( \frac{x^{\beta}}{\Gamma(1+\beta)} e^{-2eG} \right) - \frac{e (\xi^{\beta} F(1))}{(1+2eG)}.
\]

(28)

4. Some Exact Solutions of the Space-Time Fractional Burgers’ Equation

In this section, we investigate some exact solutions of (1) corresponding to following infinitesimal generators:

(i) \( V_1 \)

(ii) \( V_4 \)

(iii) \( nV_5 + mV_3 \)

(iv) \( rV_5 + V_6 \) and

(v) \( sV_3 + V_6 \)

\( r, s, m, \) and \( n \) are nonzero arbitrary constant parameters.

**Theorem 1.** Under the group of transformations \( T(x, t) = t^{\alpha}/\Gamma(1+\alpha) \) and \( \phi(T) = x^{\beta}/\Gamma(1+\beta) + 2G(t)u(x, t) \), Burgers’ equation (1) reduces to a linear differential equation of first order: \( \phi'(T) - H_1(T)\phi(T) = -H_2(T) \), where \( H_1(T) = \partial^\alpha T G(t)/G(t) \) and \( H_2(T) = \partial^\beta T F(t) \), which admits a solution given by \( u(x, t) = (1/2G(t))[e^{H_1dT} (k_1 - \int H_2 e^{-H_1dT} dT) - x^{\beta}/\Gamma(1+2\beta)] \), where \( k_1 \) is an arbitrary constant.

**Proof.** Consider the infinitesimal generator \( V_1 \), given by

\[
V_1 = -2G(t) \frac{\partial}{\partial x^{\beta}} + \frac{x^{\beta}}{\Gamma(1+\beta)} \frac{\partial}{\partial u}.
\]

(29)

We find the resulting invariant solution by reducing (1) to a linear ordinary differential equation (32) using differential invariants. The fractional characteristic equations for \( V_1 \) are

\[
(\frac{dx}{\Gamma(1+\beta)})^{\beta}/\Gamma(1+\beta) = (\frac{dt}{\Gamma(1+\alpha)})^{\alpha}/\Gamma(1+\alpha) = \frac{du}{x^{\beta}/\Gamma(1+\beta)}.
\]

(30)

On solving the above fractional characteristic equations, we obtain two functionally independent invariants as \( T = t^{\alpha}/\Gamma(1+\alpha) \) and \( \nu = x^{\beta}/\Gamma(1+2\beta) + 2G(t)u \).

Now the solution of the fractional characteristic equations will be of the form \( \nu = \phi(T) \); therefore

\[
u = \frac{1}{2G(T)} \left[ \phi(T) - \frac{x^{\beta}}{\Gamma(1+2\beta)} \right].
\]

(31)

Substituting this value of \( \nu \) in (1), we get the reduced linear ordinary differential equation as

\[
\phi'(T) - H_1(T)\phi(T) = -H_2(T),
\]

where \( T = t^{\alpha}/\Gamma(1+\alpha), H_1(T) = \partial^\alpha T G(t)/G(t), \) and \( H_2(T) = \partial^\beta T F(t) \).

On solving (32), we obtain \( \phi(T) = e^{H_1dT} (k_1 - \int H_2 e^{-H_1dT} dT) - x^{\beta}/\Gamma(1+2\beta) \), where \( k_1 \) is an arbitrary constant. This gives

\[
u(x, t) = \frac{1}{2G(t)} \left[ e^{H_1dT} (k_1 - \int H_2 e^{-H_1dT} dT) - \frac{x^{\beta}}{\Gamma(1+2\beta)} \right].
\]

(33)

**Theorem 2.** The similarity transformations \( \nu(x, t) = \psi(X) \) along with the similarity variable \( X(x, t) = x^{\beta}/\Gamma(1+\beta)^2 \) \( (\partial^\alpha T F(t)) \) reduce fractional Burgers’ equation (1) to a nonlinear
ordinary differential equation $u'(x) + g(t)/f(t))(u'(X))^2 + (1/4)u'(X) + (1/2X)u'(X) = 0$, which leads to the solution $u(x,t) = (1/k) \log [k_2 + 2\sqrt{k}k \text{erf}(x^2/2\Gamma(1 + \beta) \sqrt{t_0D_t^\alpha}(t)) + k_3$, where $k_2$ and $k_3$ are arbitrary constants.

Proof. Let us consider the infinitesimal generator

$$V_4 = \frac{x^{\beta}}{\Gamma(1 + \beta)} \frac{\partial^{\beta}}{\partial x^{\beta}} + \frac{2}{\Gamma(1 + \beta)} \frac{\partial^{\alpha}}{\partial t^{\alpha}}.$$ (34)

Here the fractional characteristic equations give the invariants $X(x,t) = x^{2\beta}/(\Gamma(1 + \beta))^2 \circ D_t^{\alpha}(f(t))$ and $u(x,t) = \psi(X)$. On substituting the above invariants in (1), it becomes a nonlinear ordinary differential equation of second order:

$$\psi''(X) + \frac{\psi'(X)}{f(t)} \psi'(X)^2 + \frac{1}{4} \psi'(X) + \frac{1}{2X} \psi'(X) = 0,$$ (35)

which leads to the solution

$$u(x,t) = \frac{1}{k} \log \left[ k_2 + 2\sqrt{k}k \text{erf} \left( \frac{x^\beta}{2\Gamma(1 + \beta) \sqrt{(\circ D_t^{\alpha}(f(t))} \right) + k_3, \right.$$ (36)

for $\alpha$, $\beta$, and $k_3$ being arbitrary constants.

Theorem 3. Under the transformations $\zeta(x,t) = t^\alpha/\Gamma(1 + \alpha)$ and $\phi(\zeta) = x^{\beta}/\Gamma(1 + \beta) - (m/n)m u(x,t)$, fractional Burgers’ equation (1) reduces to an ordinary fractional differential equation $\circ D_t^{\alpha}(f(t)) + (m/n)m u(x,t)$, which has the general solution $u(x,t) = (m/n)[x^{\beta}/\Gamma(1 + \beta) + (m/n)G(t) - k_4]$, with $k_4$ being an arbitrary constant.

Proof. In this case, we study the infinitesimal generator

$$V = nV_5 + mV_3 = \frac{n}{\partial \phi/\partial x^\beta} + \frac{m}{\partial \phi/\partial u}.$$ (37)

The following invariants can be derived easily: $\zeta(x,t) = t^\alpha/\Gamma(1 + \alpha)$ and $\nu = x^{\beta}/\Gamma(1 + \beta) - (m/n)m u(x,t)$ and the reduced form of (1) as

$$\circ D_t^{\alpha}(f(t)) + \frac{m}{n} \circ D_t^{\alpha}(G(t)) = 0.$$ (38)

This easily yields the solution

$$u(x,t) = \frac{m}{n} \left[ \frac{x^{\beta}}{\Gamma(1 + \beta)} + \frac{m}{n} G(t) - k_3 \right].$$ (39)

Theorem 4. Under the transformations $\mu(x,t) = (1/r) (x^{\beta}/\Gamma(1 + \beta)) - \circ D_t^{\alpha}(f(t))/\Gamma(1 + \alpha)$ and $\omega(\mu) = u(x,t)$, fractional Burgers’ equation (1) reduces to a nonlinear ordinary differential equation $\omega''(\mu) + k\omega^2(\mu) + (r^2/\Gamma(1 + \alpha)) \omega(\mu) = 0$, which admits the solution $u(x,t) = (1/k) \log \left[ \text{erf}(\sqrt{t_0D_t^{\alpha}(f(t))}/\Gamma(1 + \alpha) - k_4 \right] -(r^2/\Gamma(1 + \alpha)) [x^{\beta}/\Gamma(1 + \beta) - \circ D_t^{\alpha}(f(t))/\Gamma(1 + \alpha)] + c_2$, where $c_4$ and $c_5$ are arbitrary constants.

Proof. Results can be easily derived by solving the fractional characteristic equations for the infinitesimal generator $rV_5 + V_6$. \Box

Theorem 5. The similarity transformations $\gamma(x,t) = x^{\beta}/\Gamma(1 + \beta)$ and $\rho(\gamma) = \circ D_t^{\alpha}(f(t)) - (1/(1 + \alpha)) s \circ D_t^{\alpha}(u(x,t))$ reduce Burgers’ equation (1) to a nonlinear ordinary differential equation $\circ D_t^{\alpha}(f(t)) = (ks/\Gamma(1 + \alpha))(\varphi(t)) - (m/n)u(x,t)$. The general solution as $u(x,t) = (s/\Gamma(1 + \alpha))(\circ D_t^{\alpha}(f(t)) + (1/k) \log \text{erf}(\sqrt{k}c_3 + x^{\beta}/\Gamma(1 + \beta)) - c_4$, where $c_3$ and $c_4$ are arbitrary constants.

Proof. The proof is very much similar to the previous theorems. \Box

5. Some Exact Solutions of Fractional Potential Burgers’ Equation by the Invariant Subspace Method

The invariant subspace method was introduced by Galaktionov and Kasatkin [39] and Sahadevan [40] to some nonlinear fractional-order differential equations. Here, we give a brief description of the method.

Consider the fractional evolution equation $\circ D_t^{\alpha}(u) = F[u]$, where $u = u(x,t)$ and $F[u]$ is a nonlinear differential operator.

The $n$-dimensional linear space $W_n = \langle f_1(x), \ldots, f_n(x) \rangle$ is called invariant under the operator $F[u]$, if $\circ D_t^{\alpha}(u) \in W_n$ for any $u \in W_n$, which means there exist $n$ functions $\phi_1, \ldots, \phi_n$ such that $\circ D_t^{\alpha}(u) = \phi_1(f_1), \ldots, \phi_n(f_n)$, where $C_1, \ldots, C_n$ are arbitrary constants and $\{\phi_i\}$ are the expansion coefficients of $F[u] \in W_n$ in the basis $\{f_i\}$. It follows that an exact solution of fractional evolution equation can be obtained as $u(x,t) = \sum_{i=1}^n \alpha_i(t)f_i(x)$, where the coefficient functions $\alpha_1(t), \ldots, \alpha_n(t)$ satisfy a system of fractional ODEs:

$$\circ D_t^{\alpha}(\alpha_i(t)) = \phi_i(\alpha_1(t), \ldots, \alpha_n(t)), \quad i = 1, 2, \ldots, n.$$ (40)

For further details, the reader is referred to [40].

For (1), $F[u] = \circ D_t^{\alpha}(u) - f(t) x^{\beta}/\Gamma(1 + \beta) - g(t) \circ D_t^{\alpha}(u)^2$. We have the space $W_3 = \langle 1, x^{\beta}/\Gamma(1 + \beta), x^{2\beta}/\Gamma(1 + 2\beta) \rangle$ as invariant under $F[u]$, if only if, for any $u(x,t) = a_1(t) + \ldots + a_n(t)$. For further details, the reader is referred to [40].
\( a_2(t)(x^\beta/\Gamma(1+\beta)) + a_3(t)(x^{2\beta}/\Gamma(1+2\beta)) \in W_1, F[u] \in W_2 \) or equivalently if
\[
\begin{align*}
a_3(t)f(t) + (a_1(t))^2 g(t) &= b_1, \\
(a_1(t))^2 g(t) &= b_2, \\
a_2(t)a_3(t)g(t) &= b_3,
\end{align*}
\]
(41)
where \( b_1, b_2, \) and \( b_3 \) are arbitrary constants. This allows us to consider an exact solution of (1) as
\[
\begin{align*}
u(x,t) &= a_1(t) + a_2(t) x^\beta/\Gamma(1+\beta) + a_3(t) x^{2\beta}/\Gamma(1+2\beta).
\end{align*}
\]
(42)
Substituting the value of \( u(x,t) \) from (42) into (1) and equating the coefficients of the elements of space \( W_1 = \{1,x^\beta/\Gamma(1+\beta),x^{2\beta}/\Gamma(1+2\beta)\} \), we get the following system of fractional differential equations:
\[
\begin{align*}
\alpha D_\alpha^\alpha a_3(t) &= \frac{\Gamma(1+2\beta)}{\Gamma(1+\beta)} g(a_3(t))^2, \\
\alpha D_\alpha^\alpha a_2(t) &= 2 a_2(t) a_3(t) g, \\
\alpha D_\alpha^\alpha a_1(t) &= f a_3(t) + g(a_2(t))^2.
\end{align*}
\]
(43)
Equations (43) can be readily solved to yield
\[
\begin{align*}
\frac{1}{a_3(t)} &= \frac{\Gamma(1+2\beta)}{\Gamma(1+\beta)} \left( a_1(t)^2 g \right), \\
\log a_2(t) &= 2 \left( a_1(t)^2 g \right), \\
a_1(t) &= a_1(t)^2 \left( f + (a_2(t))^2 g \right),
\end{align*}
\]
(44)
where the functions \( f(t) \) and \( g(t) \) are integrable in the sense of Riemann-Liouville and \( a_1(t)^2 g(t) \neq 0 \).

On solving (44), we get
\[
\begin{align*}
a_3(t) &= \frac{B}{g(t)}, \\
a_2(t) &= \left( a_1(t)^2 g(t) \right)^{2B}, \\
a_1(t) &= \left( a_1(t)^2 g(t) \right)^{4B} + \frac{Bf(t)}{g(t)}.
\end{align*}
\]
(45)
where \( B = -(\Gamma(1+\beta)^2/\Gamma(1+2\beta)) \).

Using (42) and (45), one can easily obtain an exact solution of (1).

### 6. Conclusion

The main purpose of Lie symmetry method is to reduce PDEs to ODEs by introducing suitable similarity variable and similarity solutions. Here, in this paper, the authors show that the fractional potential Burgers’ equation possesses similarity solutions exactly as its counterparts with integer-order derivatives. By using conventionally defined similarity variables, fractional potential Burgers’ equation reduces to ordinary differential equations which are further solved to derive some group invariant solutions. The authors also utilised the invariant subspace method to deduce some exact solutions of fractional potential Burgers’ equation. Software like Mathematica and Maple has been utilised in solving some ordinary differential equations.

### Competing Interests

The authors declare that they have no competing interests.

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