THE FOURIER COEFFICIENTS OF A METAPLECTIC EISENSTEIN DISTRIBUTION ON THE DOUBLE COVER OF SL(3) OVER $\mathbb{Q}$.

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Abstract. We compute the Fourier coefficients of a minimal parabolic Eisenstein distribution on the double cover of SL(3) over $\mathbb{Q}$. Two key aspects of the paper are an explicit formula for the constant term, and formulas for the Fourier coefficients at the ramified place $p = 2$. Additionally, the unramified non-degenerate Fourier coefficients of this Eisenstein distribution fit into the combinatorial description provided by Brubaker-Bump-Friedberg-Hoffstein [5].

1. Introduction

The study of metaplectic Eisenstein series goes back at least to Maass [14] who computed the Fourier expansion of half-integral weight Eisenstein series and found quadratic Dirichlet series among the non-degenerate coefficients. Kubota [12], inspired by the works of Hecke [6], Selberg [19], and Weil [22], took the next step by computing the Fourier coefficients of an Eisenstein series on the $n$-fold cover of GL(2) over a field containing the $2n$-th roots of unity. Using the Fourier expansion Kubota was able to study the automorphic residues of the metaplectic Eisenstein series. In the case of the double cover of GL(2) the Jacobi $\theta$-function is such a residue. Subsequent works continued to include the hypothesis that the base field contains the $2n$-th roots of unity. (The construction of $n$-fold covering groups requires the base field to contain the $n$-th roots of unity.) Kazhdan-Patterson [10] developed a general theory of automorphic forms on $n$-fold covers of GL($r + 1$), computed the Fourier coefficients of metaplectic Eisenstein series, and studied their residues. Brubaker-Bump-Friedberg-Hoffstein [5] and Brubaker-Bump-Friedberg [4] provided a combinatorial description of the non-degenerate Fourier coefficients of Eisenstein series on an $n$-fold cover of GL($r + 1$). However, certain arithmetic applications require working over $\mathbb{Q}$.

In this paper we use the results established in [9] to compute all of the Fourier coefficients of a minimal parabolic Eisenstein series on the double cover of SL(3) over $\mathbb{Q}$, as opposed to a number field containing $\mathbb{Q}(i)$. We will now describe some of the notable features of this work. Theorem 52 includes an explicit formula for the constant term. Proposition 51 contains the formula for the non-degenerate Fourier coefficient and the results of Subsection 3.7 express the unramified $p$-parts in the style of [5]. Finally, Subsection 3.5 contains formulas for the ramified parts ($p = 2$) of the non-degenerate Fourier coefficients. It would be interesting to know if these ramified coefficients also admit a combinatorial description in the style of [5].

We would also like to mention that this computation is performed in the context of automorphic distributions. This should be advantageous for studying certain Archimedean integrals as demonstrated by Miller and Schmid [17].

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We will now provide a brief description of the contents of this paper. Section 2 collects notation and basic computations. Subsection 2.7 introduces the Banks-Levy-Sepanski 2-cocycle [1] and collects some computations involving this 2-cocycle. Subsection 2.8 reviews some results from [9] about an arithmetic splitting function s.

Section 3 contains the computation of the exponential sums associated with the big Bruhat cell appearing in the Fourier coefficients of the metaplectic Eisenstein distribution. The symmetries of s studied in Subsection 3.1 induce enough symmetries of the exponential sums to reduce the general computation to a more manageable special case. The most important symmetry is described in Proposition 20. This symmetry shows how the Dirichlet series appearing in the non-degenerate Fourier coefficients, which does not possess an Euler product, can be reconstructed from its p-parts. Similar twisted multiplicativity appears in the work of Brubaker-Bump-Friedberg-Hoffstein [5].

Section 4 contains the computation of the Fourier coefficients of the minimal parabolic Eisenstein distribution on the double cover of SL(3) over Q.

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2. Notation

2.1. SL(3, R) and \( \tilde{\text{SL}}(3, \mathbb{R}) \). This section contains the notation and basic computations that will be used throughout this paper.

Let \( \tilde{\text{SL}}(3, \mathbb{R}) \) be the double cover of SL(3, R). As a set \( \tilde{\text{SL}}(3, \mathbb{R}) \cong \text{SL}(3, \mathbb{R}) \times \{ \pm 1 \} \). As a topological space \( \tilde{\text{SL}}(3, \mathbb{R}) \) is given the nontrivial covering space topology. (Recall that \( \pi_1(\text{SL}(3, \mathbb{R})) = \mathbb{Z}/2\mathbb{Z} \)).

The Banks-Levy-Sepanski 2-cocycle \( \sigma : \text{SL}(3, \mathbb{R}) \times \text{SL}(3, \mathbb{R}) \rightarrow \{ \pm 1 \} \), constructed in [1] and recalled in Subsection 2.7, defines the group multiplication on SL(3, R) as follows:

\[
(g_1, \epsilon_1)(g_2, \epsilon_2) = (g_1g_2, \epsilon_1\epsilon_2\sigma(g_1, g_2)).
\] (2.1)

The following list establishes some notation for some subgroups of SL(3, R) and \( \tilde{\text{SL}}(3, \mathbb{R}) \):

\[
\begin{align*}
G &= \text{SL}(3, \mathbb{R}), & \tilde{G} &= \tilde{\text{SL}}(3, \mathbb{R}) \\
B &= \left\{ \begin{pmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & 1 \end{pmatrix} \right| a, e \neq 0 \right\}, & \tilde{B} &= \left\{ \begin{pmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & 1 \end{pmatrix}, \pm 1 \right| a, e \neq 0 \} \\
N &= \left\{ \begin{pmatrix} x & y \\ 0 & y \\ 0 & 0 \end{pmatrix} \right\}, & \tilde{N} &= \left\{ \begin{pmatrix} x & y \\ 0 & y \\ 0 & 0 \end{pmatrix}, \pm 1 \right\} \\
T &= \left\{ \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| t_i \in \mathbb{R}^\times \right\}, & \tilde{T} &= \left\{ \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \pm 1 \right| t_i \in \mathbb{R}^\times \} \\
A &= \left\{ \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 1 \end{pmatrix} \right| a, b > 0 \right\}, & \tilde{A} &= \left\{ \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 1 \end{pmatrix}, \pm 1 \right| a, b > 0 \} \\
M &= \left\{ \begin{pmatrix} 0 & e_1 & 0 \\ 0 & 0 & e_2 \\ 0 & 0 & \epsilon_1\epsilon_2 \end{pmatrix} \right| \epsilon_1, \epsilon_2 = \pm 1 \right\}, & \tilde{M} &= \left\{ \begin{pmatrix} 0 & e_1 & 0 \\ 0 & 0 & e_2 \\ 0 & 0 & \epsilon_1\epsilon_2 \end{pmatrix}, \pm 1 \right| \epsilon_1, \epsilon_2 = \pm 1 \} \\
K &= \text{SO}(3), & \tilde{K} &= \text{Spin}(3)
\end{align*}
\]

\[ \Gamma = \Gamma_1(4) = \{ \gamma \in \text{SL}(3, \mathbb{Z}) | \gamma \equiv \begin{pmatrix} 1 & x & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \pmod{4} \} \quad , \quad \Gamma_\infty = N \cap \text{SL}(3, \mathbb{Z}). \] (2.2)
Occasionally, to simplify notation \(( \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & \frac{1}{ab} \end{pmatrix}, 1 \)\) will be written \(( \begin{pmatrix} a & 0 \\ 0 & b \\ 0 \end{pmatrix}, 1 \)\), and \(( \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, 1 \)\). Furthermore, we will use the following notation for elements of \(T\) and \(N\) respectively: \(t(a, b, c) = \begin{pmatrix} a & b \\ c \end{pmatrix} \) and \(n(x, y, z) = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \).

Now we list particular representatives of elements of the Weyl group of \(SL(3, \mathbb{R})\):

\[
\begin{align*}
w_{\alpha_1} &= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & w_{\alpha_2} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} ,
\end{align*}
\]

and \( w_\ell = w_{\alpha_1} w_{\alpha_2} w_{\alpha_1} = w_{\alpha_2} w_{\alpha_1} w_{\alpha_2} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \).

These representatives of the Weyl group are those defined in Section 3 of [1]. They are used in the formula for the 2-cocycle, which can be found in Section 4 of the previously cited paper.

If \(g \in SL(3, \mathbb{R})\), let \(^tg\) denote the transpose of \(g\). If \(H\) is a subgroup of \(B\), then \(H_\sim\) or \(H^{\op}\) will denote the set \(\{ h | h \in H \} \).

Let \(sl(3, \mathbb{R})\) be the real Lie algebra of \(SL(3, \mathbb{R})\) and let \(a\) be the subalgebra of diagonal matrices. For \(X \in sl(3, \mathbb{R})\), let \(\exp(X) = \sum_{n=0}^{\infty} \frac{X^n}{n!}\) be the exponential map and let \(\log\) denote its inverse on \(a\). The Iwasawa decomposition states that the map \(K \times A \times N \to SL(3, \mathbb{R})\), given by \((k, a, n) \mapsto kan\), is a diffeomorphism. Define the maps \(\kappa : SL(3, \mathbb{R}) \to K, H : SL(3, \mathbb{R}) \to a\), and \(\nu : SL(3, \mathbb{R}) \to N\) such that \(g \mapsto (\kappa(g), \exp(H(g)), \nu(g))\) is the inverse of the map \((k, a, n) \mapsto kan\).

2.2. The \(A_2\) Root System. This subsection establishes notation having to do with the root system of \(SL(3, \mathbb{R})\). Let \(X(T) = \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}(e_1 + e_2 + e_3)\) be the group of rational characters of \(T\) (written additively), where \(e_i(t_1, t_2, t_3) = t_i\). The Weyl group \(W\) acts on \(X(T)\) through its action on \(T\) by conjugation. The action of \(W\) on \(T\) provides an isomorphism of \(W\) with \(S_3\) as follows: let \(w \in W\) and \(\sigma_w \in S_3\) such that \(w(t_1, t_2, t_3)w^{-1} = t(t_{\sigma_w(1)}, t_{\sigma_w(2)}, t_{\sigma_w(3)})\). The Weyl group action extends to the vector space \(X(T) \otimes \mathbb{C}\) which will be embedded into \(\mathbb{C}^3\) via the map \(\alpha_1e_1 + a_2e_2 + a_3e_3 \mapsto (\frac{2}{3}a_1 - \frac{1}{3}a_2 - \frac{1}{3}a_3, -\frac{1}{3}a_1 + \frac{2}{3}a_2 - \frac{2}{3}a_3, -\frac{1}{3}a_1 - \frac{1}{3}a_2 + \frac{2}{3}a_3)\). Now \(w\) acts on the indices of \((\lambda_1, \lambda_2, \lambda_3)\) by \(\sigma_w^{-1}\).

Let \(\Phi = \Phi(SL(3, \mathbb{R})), T) = \Phi^+ \cup \Phi^-\), where \(\Phi^+ = \{(e_1 - e_2), (e_2 - e_3), (e_1 - e_3)\}\) is the set of positive roots of \(SL(3, \mathbb{R})\) with respect to \(B\) and \(\Phi^- = -\Phi^+\) is the set of negative roots. To each root \(\alpha \in \Phi\) there is a canonically defined element \(h_\alpha \in a\). The map is given by

\[\pm(e_i - e_j) \mapsto \pm(e_{ij}),\]

where \(e_{ij}\) is the \(3 \times 3\) matrix with 1 in the \((i, j)\)-position and 0 elsewhere; details can be found in [7]. The elements of \(X(T) \otimes \mathbb{C}\) act on these matrices by the formula \(\langle (\lambda_1, \lambda_2, \lambda_3), \pm(e_{ij} - e_{jj}) \rangle = \pm(\lambda_i - \lambda_j)\). This action shows that \(X(T) \otimes \mathbb{C}\) is isomorphic to \(a^\vee\), the space of complex valued linear functionals of \(a\).

2.3. Plücker Coordinates. Given \(\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \in SL(3, \mathbb{R})\), define six parameters, called Plücker coordinates, as follows:

\[
\begin{align*}
A_1' &= -g, & B_1' &= -h, & C_1' &= -i \\
A_2' &= -(dh - eg), & B_2' &= (di - fg), & C_2' &= -(ei - fh).
\end{align*}
\]
Theorem 1. ([3, Ch 5]) The map taking \( \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \in \text{SL}(3, \mathbb{R}) \) to \((A'_1, B'_1, C'_1, A'_2, B'_2, C'_2)\) defines a bijection between the coset space \( N \backslash \text{SL}(3, \mathbb{R}) \) and the set of all 
\((A'_1, B'_1, C'_1, A'_2, B'_2, C'_2) \in \mathbb{R}^6\) such that: \( A'_1C'_2 + B'_1B'_2 + C'_1A'_2 = 0 \), not all of \( A'_1, B'_1, C'_1 \) equal 0, and not all of \( A'_2, B'_2, C'_2 \) equal 0. Furthermore, a coset in \( N \backslash \text{SL}(3, \mathbb{R}) \) contains an element of \( \text{SL}(3, \mathbb{Z}) \) if and only if \( A'_1, B'_1, C'_1 \) are coprime integers and \( A'_2, B'_2, C'_2 \) are coprime integers.

Versions of this result hold for other congruence subgroups. Let \( A'_1 = 4A_1, A'_2 = 4A_2, B'_1 = 4B_1, B'_2 = 4B_2, C'_1 = C_1, C'_2 = C_2. \) The coset space \( \Gamma_\infty \backslash \Gamma_1(4) \) can be identified with

\[
\begin{aligned}
\left\{ (4A_1, 4B_1, C_1, 4A_2, 4B_2, C_2) \in \mathbb{Z}^6 | & A_1C_2 + 4B_1B_2 + C_1A_2 = 0, \\
& (A_i, B_i, C_i) = 1, C_j \equiv -1 \text{ (mod 4)} \right\} .
\end{aligned}
\]  

(2.4)

**Coset Representatives:** The following table lists coset representatives of \( \Gamma_\infty \backslash \Gamma \) following Bump [3].

| Cell   | Constraints | \( \Gamma_\infty \backslash \Gamma \) Representative |
|--------|-------------|-----------------------------------------------|
| \( B \) | \( C_1, C_2 \neq 0 \) | \[
\begin{pmatrix}
\frac{-C_2}{C_1} \\
\frac{-C_1}{C_2}
\end{pmatrix}
\] |
| \( Nw_{\alpha_1}B \) | \( A_1, B_1, A_2 = 0, \) \( C_1, B_2 \neq 0 \) | \[
\begin{pmatrix}
1 & -1 \\
-1 & 1
\end{pmatrix}
\begin{pmatrix}
\frac{4B_1}{C_1} & \frac{C_1}{4B_2} \\
\frac{1}{C_2} & \frac{C_1}{C_2}
\end{pmatrix}
\] |
| \( Nw_{\alpha_2}B \) | \( A_1, A_2, B_2 = 0, \) \( B_1, C_2 \neq 0 \) | \[
\begin{pmatrix}
1 & -1 \\
-1 & 1
\end{pmatrix}
\begin{pmatrix}
\frac{1}{C_2} \\
\frac{4B_1}{C_1}
\end{pmatrix}
\] |
| \( Nw_{\alpha_1}w_{\alpha_2}B \) | \( A_1 = 0, \) \( B_1, A_2 \neq 0 \) | \[
\begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
\frac{4B_1}{C_1} & \frac{C_1}{4B_2} \\
\frac{1}{C_2} & \frac{C_1}{C_2}
\end{pmatrix}
\] |
| \( Nw_{\alpha_2}w_{\alpha_1}B \) | \( A_2 = 0, \) \( A_1, B_2 \neq 0 \) | \[
\begin{pmatrix}
1 & -1 \\
-1 & 1
\end{pmatrix}
\begin{pmatrix}
\frac{-4A_1}{C_1} & \frac{4B_2}{C_1} \\
\frac{4B_1}{C_1} & \frac{C_1}{4B_2}
\end{pmatrix}
\] |
| \( Nw_{\ell}B \) | \( A_1, A_2 \neq 0 \) | \[
\begin{pmatrix}
1 & -1 \\
-1 & 1
\end{pmatrix}
\begin{pmatrix}
\frac{4A_1}{C_1} & \frac{4B_2}{C_1} \\
\frac{4B_1}{C_1} & \frac{C_1}{4B_2}
\end{pmatrix}
\] |

**Table 1.** \( \Gamma_\infty \backslash \Gamma \) representatives

The next proposition collects some symmetries satisfied by the Plücker coordinates.

**Proposition 2.** Let \( g \in \text{SL}(3, \mathbb{R}) \) with Plücker coordinates \((4A_1, 4B_1, C_1, 4A_2, 4B_2, C_2)\), \( n = n(x, y, z) \in N, S_2 = t(1, -1, 1), \) and \( S_3 = t(1, 1, -1) \). Then:

1. The matrix \( ngn^{-1} \) has Plücker coordinates 
   \((4A_1, 4B_1 - 4A_1x, C_1 - 4B_1y + 4A_1(xy - z), 4A_2, 4B_2 + 4A_2y, C_2 + 4B_2x + 4A_2z)\).
2. The matrix \( s_3gs_3^{-1} \) has Plücker coordinates \((-4A_1, -4B_1, C_1, -4A_2, 4B_2, C_2)\).
3. The matrix \( s_2gs_2^{-1} \) has Plücker coordinates \((4A_1, -4B_1, C_1, 4A_2, -4B_2, C_2)\).
4. The matrix \( wt_1g^{-1}w_t \) has Plücker coordinates \((4A_2, -4B_2, C_2, 4A_1, -4B_1, C_1)\).
5. Let \( g \in \Gamma_1(4) \). If \( D \) divides \((A_1, A_2)\), \( D_1 = (D, B_1), D = D_1D_2, \) and \( T = t(1, D_2^{-1}, D^{-1}) \) then \( TgT^{-1} \in \text{SL}(3, \mathbb{Z}) \) has Plücker coordinates 
   \((4A_1/D, 4B_1/D_1, C_1, 4A_2/D, (4B_2)/D_2, C_2)\).
Furthermore, \( TgT^{-1} \in \Gamma_1(4) \) if and only if \( D_2 \) divides \( B_2 \).

The proof is straightforward matrix algebra and will be omitted.

Let

\[ S(A_1, A_2) = \{ \gamma \in \Gamma_1(4) | \gamma \text{ has Plücker coordinates of the form } (4A_1, *, *, 4A_2, *, *) \}, \]  

and let

\[ S(A_1, A_2) = \{(4A_1, 4B_1, C_1, 4A_2, 4B_2, C_2) \in \mathbb{Z}^6 | A_1C_2 + 4B_1B_2 + C_1A_2 = 0, \]

\[ (A_1, B_1, C_1) = 1, \quad C_j \equiv -1(\text{mod } 4), \quad \frac{B_1}{A_1}, \frac{B_2}{A_2}, C_2 \in [0, 1) \}. \]

(2.6)

The maps of Proposition 2 induce maps on the double coset spaces \( \Gamma_\infty \backslash S(A_1, \mu A_2) / \Gamma_\infty \cong S(A_1, A_2) \). (This bijection follows from the first item in Proposition 2.) Additionally, these double cosets are multiplicative.

**Proposition 3** ([9]). Let \( A_1, \alpha_1 > 0 \), \( A_2, \alpha_2 \neq 0 \), suppose that \( (A_1 \alpha_1, A_2, \alpha_2) = 1 \), \( A_1, A_2 \) are odd, and suppose that \( A_1 \alpha_1 + A_2 \alpha_2 \equiv 0 \pmod{4} \). Let \( \mu = (-\frac{1}{A_1\alpha_2}) \). Then

\[ \Gamma_\infty \backslash S(A_1\alpha_1, A_2\alpha_2) / \Gamma_\infty \cong \Gamma_\infty \backslash S(A_1, \mu A_2) / \Gamma_\infty \times \Gamma_\infty \backslash S(\alpha_1, -\mu \alpha_2) / \Gamma_\infty. \]

The bijection is induced by the map

\[ ((4A_1\alpha_1, 4B_1, C_1, 4A_2\alpha_2, 4B_2, C_2) \mapsto (4A_1\alpha_1, 4B_1, C_1, 4A_2, 4B_2, \gamma C_2), \]

\[ (4A_1\alpha_1, 4B_1, (\frac{-1}{A_2}) A_2 C_1, -\mu \alpha_2, -\left( \frac{-1}{A_2} \right) \mu A_2, -\mu (\frac{-1}{A_2}) A_1 C_2)), \]

(2.7)

where:

1. \( C_1 = \frac{-A_1\gamma C_2 - 4B_1 B_2}{\mu A_2} \).

2. \( \gamma \) is the smallest positive integer such that \( \gamma \equiv 1 \pmod{4} \) and \( \gamma \equiv \alpha_1 \pmod{A_2} \).

2.4. Kronecker Symbol. If \( a, b \in \mathbb{R}^\times \), then the Hilbert Symbol \( (a, b)_\mathbb{R} \), is equal to 1 if \( a \) or \( b \) is positive and \(-1\) if \( a \) and \( b \) are negative. Let \( n \in \mathbb{Z}_{\neq 0} \) with prime factorization

\[ n = \epsilon p_1^{e_1} \ldots p_\ell^{e_\ell}, \]

where \( \epsilon = \pm 1 \). If \( k \in \mathbb{Z} \) the Kronecker Symbol is defined by

\[ (\frac{k}{n}) = \left( \frac{k}{p_1} \right)^{e_1} \ldots \left( \frac{k}{p_\ell} \right)^{e_\ell}, \]

where \( \left( \frac{k}{p_i} \right) \) is the Legendre symbol when \( p_i \) is an odd prime, \( \left( \frac{k}{\ell} \right) = (k, \ell)_\mathbb{R} \), and \( \left( \frac{k}{2} \right) \) is equal to 0 if 2 divides \( k, 1 \) if \( k \equiv \pm 1 \pmod{8} \), and \(-1\) if \( k \equiv \pm 3 \pmod{8} \). The formula can be extended to \( n = 0 \) by setting \( \left( \frac{k}{0} \right) \) equal to 1 if \( k = \pm 1 \), and 0 otherwise.

The following proposition collects some facts about the Kronecker Symbol. These results can be found in [8].

**Proposition 4.** (Properties of the Kronecker Symbol) Let \( a, b, m, n \in \mathbb{Z}, \epsilon = \pm 1 \), and let \( n' \) and \( m' \) denote the odd part of \( n \) and \( m \), respectively.

1. If \( ab \neq 0 \), then \( \left( \frac{a}{n} \right) \left( \frac{b}{n} \right) = \left( \frac{ab}{n} \right) \).
2. If \( mn \neq 0 \), then \( \left( \frac{a}{m} \right) \left( \frac{n}{m} \right) = \left( \frac{a}{nm} \right) \).
3. Let \( m \) be equal to \( 4n \) if \( n \equiv 2(\text{mod } 4) \), otherwise let \( m \) be equal to \( n \). If \( n \neq 0 \) and \( a \equiv b \pmod{m} \), then \( \left( \frac{a}{n} \right) = \left( \frac{b}{n} \right) \).
4. Let \( b \) be equal to \( 4a \) if \( a \equiv 2(\text{mod } 4) \), otherwise let \( b \) be equal to \( |a| \). If \( a \neq 3(\text{mod } 4) \) and \( m \equiv n \pmod{b} \), then \( \left( \frac{a}{m} \right) = \left( \frac{a}{n} \right) \).
(5) \( \left( \frac{-1}{n} \right) = (-1)^{\frac{n'-1}{2}} \) and \( \left( \frac{2}{n} \right) = (-1)^{\frac{n'^2-1}{8}}. \)

(6) (Quadratic Reciprocity) If \( \gcd(m, n) = 1 \), then \( \left( \frac{m}{n} \right) \left( \frac{n}{m} \right) = (n, m)_{\mathbb{R}} (-1)^{\frac{(m'-1)(n'-1)}{4}}. \)

(7) \( \left( \frac{-1}{n} \right) = (-1)^{\left( \frac{n'-1}{2} \right)} \). \( \left( \frac{2}{n} \right) = \left( \frac{2}{n} \right)_{\mathbb{R}}. \)

2.5. Exponential Sums. This section collects some basic identities involving Gauss and Ramanujan sums. The proofs are elementary and will be omitted.

When \( d \) divides \( n \), let
\[
g(d, m, n) = \sum_{x \in \mathbb{Z}/n\mathbb{Z}}^{} \left( \frac{x}{d} \right) e^{2\pi i \frac{mx}{n}}. \tag{2.8}\]

When \( n = 2^k \), \( k \geq 3 \), and \( \epsilon \equiv \pm 1 \pmod{4} \), let
\[
g_{\epsilon}(2^i, m, 2^k) = \sum_{x \in \mathbb{Z}/2^k\mathbb{Z}}^{} \left( \frac{x}{2^i} \right) e^{2\pi i \frac{m \epsilon x}{2^k}}. \tag{2.9}\]

Note that if \( i \) is even then \( g_{\epsilon}(2^i, m, 2^k) \) is also well defined for \( k = 2 \).

The following is a useful bit of notation. Let \( P \) be a statement. Let \( \delta_P = \)
\[
\begin{cases} 
1 & \text{If } P \text{ is true}, \\
0 & \text{If } P \text{ is false}. 
\end{cases}
\]

Lemma 5. Let \( p \) be an odd prime, let \( a, b, m \in \mathbb{Z} \), and let \( j, k, l \in \mathbb{Z}_{\geq 0} \).

(1) If \( c \in (\mathbb{Z}/d\mathbb{Z})^\times \), then \( g(d, cm, n) = \left( \frac{c}{n} \right) g(d, m, n). \)

(2) If \( k \neq 0 \), then
\[
g(p^1, p^j, p^k) = \begin{cases} 
0, & k \neq j + 1 \\
p^{k-1}g(p^1, p^0, p^1), & k = j + 1, 
\end{cases} \quad \text{and} \quad g(p^0, p^j, p^k) = \begin{cases} 
\phi(p^k), & k - j \leq 0 \\
-p^{k-1}, & k - j = 1 \\
0, & k - j \geq 2. 
\end{cases}
\]

(3) \( p^l g(p^i, \pm p^j, p^k) = g(p^i, \pm p^{j+l}, p^{k+l}). \)

(4) \( 2^l g_{\epsilon}(2^i, \pm 2^j, 2^k) = g_{\epsilon}(2^i, \pm 2^{j+l}, 2^{k+l}). \)

(5) Suppose that \( p \) does not divide \( a \) and \( j > 0 \), then
\[
\sum_{x \in \mathbb{Z}/p^j\mathbb{Z}}^{} \left( \frac{ax + b}{p} \right) e^{2\pi i \frac{ma}{p^j}} = e^{2\pi i \frac{ab}{p^j}} \left( \frac{a}{p} \right)^k g(p^k, m, p^j) - \left( \frac{b}{p} \right)^k \sum_{l=0}^{p^j-1} e^{2\pi i \frac{ml}{p^j}}. \tag{2.10}\]

(6) \[
\sum_{0 \leq \ell \leq k} g(p^\ell, m, p^k) = \delta_{p^k | m} p^k. \tag{2.11}\]
2.6. Zeta Functions. This section collects some definitions and identities involving zeta functions. The prototypical zeta function is the Riemann zeta function

$$\zeta(s) = \prod_{p \text{ prime}} (1 - p^{-s})^{-1} = \sum_{n>0} n^{-s},$$  

(2.12)

where \(\text{Re}(s) > 1\). Occasionally the 2-part will be missing and in this case let

$$\zeta_2(s) = \prod_{p \neq 2} (1 - p^{-s})^{-1} = \zeta(s)(1 - 2^{-s}).$$  

(2.13)

Let \(\phi\) be the Euler \(\phi\)-function. The next identity will be useful during the computation of the degenerate Fourier coefficients:

$$\sum_{k>0} \frac{\phi(4k^2)}{2} (2k)^{-2s} = \frac{2^{-2s}}{1 - 2^{-2s-2}} \prod_{p \neq 2} \frac{1 - p^{-1-2s}}{1 - p^{-2-2s}} = 2^{-2s} \frac{\zeta(2s - 2)}{\zeta_2(2s - 1)}.$$  

(2.14)

The following L-functions also appear in the computation of the semi-degenerate Fourier coefficients of the SL(3) metaplectic Eisenstein series. Let

$$\varepsilon_d = \begin{cases} 1, & \text{if } d \equiv 1 (4) \\ i, & \text{if } d \equiv -1 (4) \\ 0, & \text{otherwise}, \end{cases}$$

and

$$K_n(n; 4c) = \sum_{d \in \mathbb{Z}/(4c)\mathbb{Z}} \varepsilon_d^{-\kappa} \left(\frac{4c}{d}\right) e\left(\frac{nd}{4c}\right),$$  

(2.15)

and

$$a_{\epsilon,\nu}(n) = \epsilon i^{4-\nu-1} \zeta_2(2\nu + 1) \sum_{c \in \mathbb{Z}_{>0}} c^{-\nu-1} K_c(-n; 4c),$$  

(2.16)

where \(\text{Re}(\nu) > 1\). Bate [2, pg 28, 36] shows this last quantity is a Fourier coefficient of a metaplectic Eisenstein series on the double cover of SL(2, \(\mathbb{R}\)) and evaluates it in terms of quadratic L-functions when \(n \neq 0\) and \(\zeta\) when \(n = 0\).

We record an additional identity used in the calculation of the constant term. Let \(n \in \mathbb{Z}_{>0}\), then

$$K_n(0, 4n) = \begin{cases} 0, & \text{if } n \text{ is not a square,} \\ (1 + i^{-\kappa}) \frac{\phi(4n)}{2}, & \text{if } n \text{ is a square.} \end{cases}$$  

(2.17)

2.7. 2-cocycle. This section begins with a formula for the Banks-Levy-Sepanski 2-cocycle [1] along with some of its properties. Then we isolate some computations that will be useful for the computation of the Fourier coefficients.

Let \(g \in \text{SL}(3, \mathbb{R})\) with Plücker coordinates \((A'_1, B'_1, C'_1, A'_2, B'_2, C'_2)\). Let \(X_1(g) = \det(g)\), let \(X_2(g)\) be the first non-zero element of the list \(-A_2, B_2, -C_2\), let \(X_3(g)\) be the first nonzero element of the the list \(-A_1, -B_1, -C_1\), and let \(\Delta(g) = \frac{X_1(g)}{X_2(g)} X_3(g)\).

If \(g_1, g_2 \in G\) such that \(g_1 = naw_1 \ldots w_k n'\) is the Bruhat decomposition of \(g_1\), then in Section 4 of [1], Banks-Levy-Sepanski show that the 2-cocycle \(\sigma\) satisfies the formula

$$\sigma(g_1, g_2) = \sigma(a, w_1 \ldots w_k n' g_2) \sigma(w_1, w_2 \ldots w_k n' g_2) \ldots \sigma(w_{k-1}, w_k n' g_2) \sigma(w_k, n' g_2).$$  

(2.18)
where each factor can be computed using the following rules: let \( h \in G, a \in T \), then

\[
\sigma(t(a_1, a_2, a_3), t(b_1, b_2, b_3)) = (a_1, b_2)(a_1, b_3)(a_2, b_3),
\]

\[
\sigma(a, h) = \sigma(a, \Delta(h)),
\]

and \( \sigma(w_\alpha, h) = \sigma(\Delta(w_\alpha h) \Delta(h), -\Delta(h)) \).

The next lemma collects some simple identities involving \( \sigma \).

**Lemma 6.** (Banks-Levy-Sepanski [1]) Let \( n, n_1, n_2 \in N \) and let \( a \in A \), Then \( \sigma(n_1 g_1, g_2 n_2) = \sigma(g_1, g_2) \), \( \sigma(g_1 n, g_2) = \sigma(g_1, n g_2) \), \( \sigma(n, g) = \sigma(g, n) = 1 \), and \( \sigma(g, a) = 1 \).

Finally we will collect several identities that amount to an explicit Bruhat decomposition for certain elements of \( \widetilde{SL}(3, \mathbb{R}) \). We will only include the proof of one of the identities since the others are similar.

**Proposition 7.** Let \( \gamma \in \Gamma_1(4) \), \( s(\gamma) = \pm 1 \), and \( \gamma' = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 4A_1 & 4B_1 & C_1 \\ A_2 & B_2 & B_1 \end{pmatrix} \). Then

\[
(\gamma', s(\gamma))^{-1}(w_\ell, 1) = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} w_\ell, 1 \begin{pmatrix} 1 & \frac{B_1}{A_1} & \frac{C_1}{A_2} \\ 0 & 1 & -\frac{B_2}{A_2} \end{pmatrix} \begin{pmatrix} |4A_1| & 0 & 0 \\ 0 & |\frac{A_2}{A_1}| & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \times \begin{pmatrix} \text{sign}(A_1) & 0 & 0 \\ 0 & \text{sign}(\frac{A_2}{A_1}) & 0 \\ 0 & 0 & \text{sign}(\frac{1}{A_2}) \end{pmatrix},
\]

where \( \epsilon = (A_1, -A_2) s(\gamma) \).

In later computations \( \gamma' \) will be the representative of the coset \( \Gamma_\infty \gamma \) given in Table 1.

**Proof:** Using Lemma 6 it is equivalent to prove the identity

\[
(w_\ell, 1)^{-1}(\gamma', s(\gamma)) \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} w_\ell, 1 = \begin{pmatrix} 1 & \frac{4A_1(-B_1)}{A_2} & \frac{4A_2C_1}{A_1} \\ 0 & 1 & \frac{4A_2B_2}{A_1} \end{pmatrix} \begin{pmatrix} |4A_1| & 0 & 0 \\ 0 & |\frac{A_2}{A_1}| & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} \text{sign}(A_1) & 0 & 0 \\ 0 & \text{sign}(\frac{A_2}{A_1}) & 0 \\ 0 & 0 & \text{sign}(\frac{1}{A_2}) \end{pmatrix},
\]

Note that \( (w_\ell, 1)^{-1} = (w_\ell, -1) \). The matrix component of this identity follows directly from matrix multiplication. Thus we may now focus on the identity in the second component. By using the group law for \( \widetilde{SL}(3, \mathbb{R}) \) and Lemma 6 it follows that

\[
\epsilon = -s(\gamma) \sigma(w_\ell, \gamma') \left( \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} w_\ell \right).
\]

**Lemma 6** and formula (2.18) in conjunction with the identities

\[
\begin{pmatrix} 4A_1 & 4B_1 & C_1 \\ A_2 & B_2 & B_1 \end{pmatrix} = \begin{pmatrix} 4A_1 \\ A_2 \end{pmatrix} \begin{pmatrix} 1 & \frac{B_1}{A_1} & \frac{C_1}{A_2} \\ 0 & 1 & -\frac{B_2}{A_2} \end{pmatrix} \begin{pmatrix} \frac{A_2}{A_1} & \frac{B_2}{A_2} \\ 1 \end{pmatrix},
\]

\[
8
\]
and

\[
\begin{pmatrix}
1 & -\frac{C_1}{A_1} \\
1 & -\frac{B_2}{A_2}
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
= \begin{pmatrix}
1 & -\frac{C_1}{A_1} \\
1 & -\frac{B_2}{A_2}
\end{pmatrix},
\]

imply that

\[
\sigma(w_{\ell}, \gamma') = \sigma\left(\begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}, \frac{4A_1}{A_1}, \frac{\frac{B_2}{A_2}}{1}
\right),
\]

and

\[
\sigma\left(\begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0
\end{pmatrix}, \frac{4A_1}{A_1}, \frac{\frac{B_2}{A_2}}{1}
\right), \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0
\end{pmatrix} = (-1, -A_2).
\]

Combining the results yields an expression for \(\epsilon\).

\[
\epsilon = (-1)(-1)(-A_1, -A_2)(-1, -A_2)s(\gamma) = (A_1, -A_2)\sigma(\gamma)
\]

(2.22)

Proposition 8. Let \(\gamma \in \Gamma_1(4), s(\gamma) = \pm 1\), and \(\gamma' = \begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix}\left(\begin{pmatrix}
\frac{A_2}{m_1} & \frac{C_2}{m_1} \\
-\frac{B_1}{m_1} & \frac{B_2}{m_1}
\end{pmatrix}\right).\) Then

\[
(\gamma', s(\gamma))^{-1}(w_{\ell}, 1) = \begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix}w_{\ell}, 1
\begin{pmatrix}
1 & 0 & \frac{B_1}{m_1} \\
0 & 1 & \frac{C_2}{m_2}
\end{pmatrix}
\times
\begin{pmatrix}
|\frac{-4B_1}{m_1}| & 0 & 0 \\
0 & |\frac{\frac{B_2}{m_2}}{1} | & 0
\end{pmatrix}^{-1}
\begin{pmatrix}
\text{sign}(\frac{-B_1}{m_1}) & 0 & 0 \\
0 & \text{sign}(\frac{\frac{B_2}{m_2}}{1}) & 0
\end{pmatrix}, \epsilon
\]

where \(\epsilon = (-B_1, -A_2)s(\gamma)\).

Proposition 9. Let \(\gamma \in \Gamma_1(4), s(\gamma) = \pm 1\), and \(\gamma' = \begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix}\left(\begin{pmatrix}
-\frac{4A_1}{m_1} & -\frac{B_1}{m_1} \\
-\frac{C_1}{m_1} & -\frac{B_2}{m_2}
\end{pmatrix}\right).\) Then

\[
(\gamma', s(\gamma))^{-1}(w_{\ell}, 1) = \begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix}w_{\ell}, 1
\begin{pmatrix}
1 & \frac{C_1}{m_1} \\
0 & 0
\end{pmatrix}
\times
\begin{pmatrix}
|\frac{-4A_1}{m_1}| & 0 & 0 \\
0 & |\frac{\frac{B_2}{m_2}}{1} | & 0
\end{pmatrix}^{-1}
\begin{pmatrix}
\text{sign}(\frac{-A_1}{m_1}) & 0 & 0 \\
0 & \text{sign}(\frac{\frac{B_2}{m_2}}{1}) & 0
\end{pmatrix}, \epsilon
\]

where \(\epsilon = (-A_1, -B_2)s(\gamma)\).
Proposition 10. Let \( \gamma \in \Gamma_1(4) \), \( s(\gamma) = \pm 1 \), and let \( \gamma' = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{4B_1}{c_1} & -\frac{C_2}{c_1} \\ \frac{1}{4B_2} & -1 \end{pmatrix} c_1 \). Then

\[
(\gamma', s(\gamma))^{-1}(w, 1) = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} w, 1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{C_2}{4B_2} \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \text{sign}(C_1) & 0 & 0 \\ 0 & \text{sign}(\frac{B_2}{c_1}) & 0 \\ 0 & 0 & \text{sign}(\frac{1}{4B_2}) \end{pmatrix} \epsilon^{-1},
\]

where \( \epsilon = -(C_1, -B_2)s(\gamma) \).

Proposition 11. Let \( \gamma \in \Gamma_1(4) \), \( s(\gamma) = \pm 1 \), and let \( \gamma' = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{c_2} & 4B_1 \\ -B_2 & c_2 \end{pmatrix} \). Then

\[
(\gamma', s(\gamma))^{-1}(w, 1) = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} w, 1 \begin{pmatrix} \frac{4B_1}{c_1} & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \text{sign}(B_1) & 0 & 0 \\ 0 & \text{sign}(\frac{C_1}{c_2}) & 0 \\ 0 & 0 & \text{sign}(\frac{1}{4B_2}) \end{pmatrix} \epsilon^{-1},
\]

where \( \epsilon = (B_1, -C_2)s(\gamma) \).

2.8. The Splitting. In [15], Miller constructs a group homomorphism \( S : \Gamma_1(4) \rightarrow \widetilde{SL}(3, \mathbb{R}) \) such that \( \widetilde{S}(\gamma) = (\gamma, s(\gamma)) \), where \( s(\gamma) \in \{ \pm 1 \} \). This map \( S \) is a splitting of \( \Gamma_1(4) \) into \( \widetilde{SL}(3, \mathbb{R}) \). Now we will describe a formula for \( s \), which by abuse of terminology will also be called the splitting, in terms of Plücker coordinates. These results are proved in [9].

Theorem 12 ([9]). Let \( \gamma \in \Gamma_1(4) \) with Plücker coordinates \( \{A_1, B_1, C_1, A_2, B_2, C_2\} \) such that \( A_1 > 0 \), and \( A_2/(A_1, A_2) \equiv 1 \pmod{2} \). Let \( D = (A_1, A_2) \), \( D_1 = (D, B_1) \), \( D_2 = D/D_1 \), and let \( \epsilon = \frac{-1}{B_1/D_1} \). Then

\[
s(\gamma) = \epsilon \begin{pmatrix} \frac{A_1}{D} \\ \frac{A_2}{D} \end{pmatrix} \begin{pmatrix} B_1/D_1 \\ A_1/D \end{pmatrix} \begin{pmatrix} 4B_2/D_2 \\ \text{sign}(A_2)A_2/D \end{pmatrix} \begin{pmatrix} D_1 \\ C_1 \end{pmatrix} \begin{pmatrix} D_2 \\ C_2 \end{pmatrix}.
\]

Proposition 13 ([9]). Let \( \gamma \in \Gamma_1(4) \) with Plücker Coordinates \( \{A_1, B_1, C_1, A_2, B_2, C_2\} \). Then:

| Cell       | \((A_1, B_1, C_1, A_2, B_2, C_2)\) | \(s(\gamma)\) |
|------------|----------------------------------|----------------|
| \(B\)      | \((0, 0, -1, 0, 0, -1)\)         | 1              |
| \(Bw_\alpha B\) | \((0, 0, -1, 0, B_2, C_2)\)     | \(-B_2/C_2\)  |
| \(Bw_\alpha B\) | \((0, B_1, C_1, 0, 0, -1)\)     | \(-B_1/C_1\)  |
| \(Bw_\alpha B\) | \((0, B_1, C_1, A_2, B_2, C_2)\) | \(-A_1/B_2\)  |
| \(Bw_\alpha B\) | \((A_1, B_1, C_1, 0, B_2, C_2)\) | \(-B_2/C_2\)  |
| \(Bw_\alpha B\) | \((A_1, B_1, C_1, A_2, B_2, C_2)\) | \(-A_1/B_2\)  |
| \(Bw_\alpha B\) | \((A_1, B_1, C_1, A_2, B_2, C_2)\) | \(-B_2/C_2\)  |

Equation (2.27)
The splitting also satisfies some symmetries with respect to the maps described in Proposition 2.

**Proposition 14 ([9]).** Let \( \gamma \in \Gamma_1(4) \) with Plücker coordinates \((4A_1, 4B_1, C_1, 4A_2, 4B_2, C_2)\). Consider the involution \( \psi : \gamma \mapsto w^\dagger_1 \gamma^{-1} w_1^{-1} \). When \( A_1 \) and \( A_2 \) are not equal to 0
\[
s(\psi(\gamma)) = (-A_1, -A_2)s(\gamma).
\]
When \( A_1, B_2 \neq 0 \) and \( A_2 = 0 \),
\[
s(\psi(\gamma)) = (-A_1, B_2)s(\gamma).
\]

**Proposition 15 ([9]).** Let \( \gamma \in \Gamma_1(4) \) and \( n \in \Gamma_\infty \). Then \( s(n\gamma) = s(\gamma n) = s(\gamma) \).

Next we consider conjugation by the elements \( S_2 = t(1, -1, 1) \) and \( S_3 = t(1, 1, -1) \).

**Proposition 16 ([9]).** Let \( \gamma \in \Gamma \). If \( A_1, A_2 \neq 0 \). Then
\[
s(S_2 \gamma S_2) = -\text{sign}(A_1 A_2)s(\gamma),
\]
\[
s(S_3 \gamma S_3) = s(\gamma).
\]

We conclude this subsection by recalling the twisted multiplicativity of \( s \).

**Proposition 17 ([9]).** Let \( A_1, \alpha_1 \in \mathbb{Z}_{>0} \), \( A_2, \alpha_2 \in \mathbb{Z} \) such that \( A_1, A_2 \) are odd,
\[
(A_1 A_2, \alpha_1 \alpha_2) = 1, \ A_1 \alpha_1 + A_2 \alpha_2 \equiv 0 (\text{mod} \ 4), \ \text{and} \ \frac{\alpha_2}{(\alpha_1, \alpha_2)} \equiv 1 (\text{mod} \ 2).
\]
Let \( \mu = \left( \frac{-1}{-A_1 A_2} \right) \).
Then with respect to the map from Proposition 3,
\[
\varphi : S(A_1 \alpha_1, A_2 \alpha_2) \rightarrow S(A_1, \mu A_2) \times S(\alpha_1, -\mu \alpha_2), \text{ the following holds:}
\]
\[
s(\gamma) = s(\pi_1(\varphi(\gamma))) s(\pi_2(\varphi(\gamma))) \left( \frac{\alpha_2}{(\alpha_1, A_1)} \right) \left( \frac{\alpha_1}{A_2} \right), \quad (2.28)
\]
where \( \pi_i \) is projection onto the \( i \)-th factor.

2.9. **Principal Series.** The following discussion establishes the preliminaries needed for the definition of the Eisenstein distribution. A complete treatment of automorphic distributions can be found in [18] and [16]. Bate [2] provides an explicit exposition of some of these ideas in the context of \( SL(2, \mathbb{R}) \) principal series representations. This paper will primarily be concerned with \( SL(3, \mathbb{R}) \) principal series representations and the following discussion introduces notation and addresses some technical points.

To construct the metaplectic principal series we need the following representation. Let \( \phi : \widetilde{M} \rightarrow SL(2, \mathbb{C}) \) be the representation defined by
\[
\left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right), \pm 1 \quad \mapsto \quad \pm \left( \begin{array}{c} 1 \\ 1 \end{array} \right)
\]
\[
\left( \begin{array}{ccc} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{array} \right), \pm 1 \quad \mapsto \quad \pm \left( \begin{array}{c} -i \\ -i \end{array} \right)
\]
\[
\left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{array} \right), \pm 1 \quad \mapsto \quad \pm \left( \begin{array}{c} -i \\ i \end{array} \right)
\]
\[
\left( \begin{array}{ccc} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{array} \right), \pm 1 \quad \mapsto \quad \pm \left( \begin{array}{c} 1 \\ -1 \end{array} \right).
\]
\[
(2.29)
\]
The map \( \phi \) can be written more succinctly as
\[
\phi \left( \left( \begin{array}{ccc} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_1 \epsilon_2 \end{array} \right), \pm 1 \right) = \pm \left( \begin{array}{c} 1 \\ -1 \end{array} \right)^{\frac{1-\epsilon_1}{2}} \left( \begin{array}{c} -i \\ i \end{array} \right)^{\frac{1-\epsilon_2}{2}}.
\]
\[11\]
It follows that \( \tilde{M} \) is isomorphic to
\[
\left\{ (\pm^1 \pm^1), (\pm^i \pm^i), (\pm^1 \pm^1), (\pm^i \pm^i) \right\}.
\]
This group is isomorphic to the quaternion group \( Q_8 = \{ \pm 1, \pm i, \pm j, \pm k \} \) and the representation \( \phi \) is the unique irreducible complex two dimensional representation of \( Q_8 \).

Now we can define the principal series representations. Let \( \lambda, \rho \in \mathfrak{a}_C^\ast \), where \( \rho = (1, 0, -1) \). Let
\[
\tilde{V}_{\Lambda, \rho}^\infty = \left\{ f \in C^\infty(\widetilde{SL}(3, \mathbb{R}))^2 | f(\tilde{g}\tilde{m}an_\rho) = \exp((\lambda - \rho)(H(a^{-1})))\phi(\tilde{m}^{-1})f(\tilde{g}),
\right. \\
\left. \text{for all } \tilde{m}an_\rho \in \tilde{M}A\tilde{N}_- \right\},
\]
\[
\tilde{V}_{\Lambda, \rho} = \left\{ f \in L^2_{\text{loc}}(\widetilde{SL}(3, \mathbb{R}))^2 | f(\tilde{g}\tilde{m}an_\rho) = \exp((\lambda - \rho)(H(a^{-1})))\phi(\tilde{m}^{-1})f(\tilde{g}),
\right. \\
\left. \text{for all } \tilde{m}an_\rho \in \tilde{M}A\tilde{N}_- \right\},
\]
\[
\tilde{V}_{\Lambda, \phi}^{-\infty} = \left\{ f \in C^{-\infty}(\widetilde{SL}(3, \mathbb{R}))^2 | f(\tilde{g}\tilde{m}an_\phi) = \exp((\lambda - \rho)(H(a^{-1})))\phi(\tilde{m}^{-1})f(\tilde{g}),
\right. \\
\left. \text{for all } \tilde{m}an_\phi \in \tilde{M}A\tilde{N}_- \right\}.
\]

The group \( \tilde{G} \) acts on each of the three preceding spaces by \( \pi(h)(f)(g) = f(h^{-1}g) \). These spaces are called the smooth, locally \( L^2 \), and distributional principal series representation spaces, respectively. In the definition of \( \tilde{V}_{\Lambda, \phi}^{-\infty} \) the equality is understood in the sense of distributions. Note that
\[
\tilde{V}_{\Lambda, \rho}^\infty \subset \tilde{V}_{\Lambda, \rho} \subset \tilde{V}_{\Lambda, \phi}^{-\infty}.
\]

Let \( f_1 \in \tilde{V}_{-\lambda, \tau \phi^{-1}} \) and \( f_2 \in \tilde{V}_{\Lambda, \phi} \), define the pairing \( \langle \cdot, \cdot \rangle_{\lambda, \phi} : V_{-\lambda, \tau \phi^{-1}} \times V_{\Lambda, \phi} \to \mathbb{C} \), by
\[
\langle \left[ \begin{array}{c} f_{1,1} \\ f_{1,2} \end{array} \right], \left[ \begin{array}{c} f_{2,1} \\ f_{2,2} \end{array} \right] \rangle_{\lambda, \phi} = \int_{\tilde{K}} (f_{1,1}(\tilde{k})f_{2,1}(\tilde{k}) + f_{1,2}(\tilde{k})f_{2,2}(\tilde{k}))d\tilde{k},
\]
(2.30)
where \( \tilde{K} \cong SU(2) \cong \text{Spin}(3) \) and \( dk \) is the Haar measure of \( \tilde{K} \). By a slight modification to [13, Theorem 3], if \( h \in \tilde{G} \), \( f_1 \in \tilde{V}_{-\lambda, \tau \phi^{-1}} \), and \( f_2 \in \tilde{V}_{\Lambda, \phi} \), then \( \langle \pi(h)f_1, \pi(h)f_2 \rangle_{\lambda, \phi} = \langle f_1, f_2 \rangle_{\lambda, \phi} \).

For us, distributions will be dual to smooth measures, and thus can be thought of as generalized functions in which the action of the distribution on the measure is given by integration of their product over the full space. Thus, the pairing can be extended to \( \tilde{V}_{\Lambda, \phi}^{-\infty} \) on the right. Restriction from \( \tilde{V}_{-\lambda, \tau \phi^{-1}} \) to its smooth vectors results in a pairing
\( \tilde{V}_{-\lambda, \tau \phi^{-1}} \times \tilde{V}_{\Lambda, \phi}^{-\infty} \to \mathbb{C} \). Under this pairing \( \tilde{V}_{\Lambda, \phi}^{-\infty} \) may be identified with the dual of \( \tilde{V}_{-\lambda, \tau \phi^{-1}} \); this duality is to be understood in the context of topological vector spaces, thus some comments about topology are in order.

The map induced by restriction to \( \tilde{K} \) defines a vector space isomorphism between \( \tilde{V}_{-\lambda, \tau \phi^{-1}}^\infty \) and \( C^\infty(\tilde{K}) \). The family of norms \( ||\partial^n f||_u = \sup_{k \in \tilde{K}} \{ ||\partial^n f(k)|| \} \) defines a topology on \( C^\infty(\tilde{K}) \) which can be transferred to \( \tilde{V}_{-\lambda, \tau \phi^{-1}}^\infty \) via the previous isomorphism. The dual \( \tilde{V}_{\Lambda, \phi}^{-\infty} \) can be given the strong topology [21, §19]. With respect to these topologies \( \tilde{V}_{\Lambda, \phi}^{-\infty} \) can be identified with the continuous dual of \( \tilde{V}_{-\lambda, \tau \phi^{-1}}^\infty \). Additionally, \( \tilde{V}_{\Lambda, \phi}^{-\infty} \) is dense in \( \tilde{V}_{-\lambda, \phi}^{-\infty} \), and sequential convergence in \( \tilde{V}_{\Lambda, \phi}^{-\infty} \) with respect to the strong topology is equivalent to sequential convergence with respect to the weak topology [21, §34.4].
The pairing just described focuses on the compact model of the principal series representations. The Eisenstein distribution considered in this paper will be more amenable to study using the non-compact model of the principal series representation which we describe presently.

Let \( w \in W \). As \( w\tilde{N}\tilde{B}_- \) is open and dense in \( \tilde{G} \), restriction from \( \tilde{G} \) to \( w\tilde{N} \) defines an injection \( \tilde{V}_{\lambda,\phi}^\infty \hookrightarrow C^\infty(w\tilde{N}) \) and the pairing is compatible with this injection in the following sense. Let \( F : \tilde{G} \to \mathbb{C} \), be a smooth function such that \( F(gb_-) = e^{2\rho H(b_-)}F(g) \). Then, by a slight modification of Consequence 7 in [11],

\[
\int_{\tilde{K}} F(k)\,dk = \int_{\tilde{N}} F(wn)\,dn. \tag{2.31}
\]

For \( f_1 = \begin{bmatrix} f_{1,1} \\ f_{1,2} \end{bmatrix} \in \tilde{V}_{-\lambda,\tau^\phi}^{-1} \) and \( f_2 = \begin{bmatrix} f_{2,1} \\ f_{2,2} \end{bmatrix} \in \tilde{V}_{\lambda,\phi} \), define

\[
(f_1 \cdot f_2)(g) = f_{1,1}(g)f_{2,1}(g) + f_{1,2}(g)f_{2,2}(g).
\]

Then for \( b_- \in \tilde{B}_- \) we have

\[
(f_1 \cdot f_2)(gb_-) = \exp(2\rho H(b_-))(f_1 \cdot f_2)(g).
\]

This identity allows us to apply Equation (2.31) to establish a bridge between the pairings of principal series in the compact and noncompact pictures. Specifically,

\[
\langle f_1, f_2 \rangle_{\lambda,\phi} = \int_{\tilde{N}} (f_1 \cdot f_2)((w, 1)n)\,dn, \tag{2.32}
\]

and so the pairing can be realized as an integration over the non-compact space \( \tilde{N} \).

We must exercise some care when using this identity with distributions; restricting distributions to embedded submanifolds need not be meaningful. However, we will be working with an element of \( \tilde{V}_{\lambda,\phi}^{-\infty} \) with support contained in the set \((w_\ell, 1)\tilde{N}\tilde{B}_- \). In this case Equation (2.32) may be applied with \( w = w_\ell \).

The element \( \tilde{\tau} \in \tilde{V}_{\lambda,\phi}^{-\infty} \), characterized by

\[
\tilde{\tau} \left( (w_\ell, 1) \begin{bmatrix} \frac{1}{x} & 0 & y \\ 0 & 0 & 1 \end{bmatrix} \right) \tilde{m}an_- = \exp((\lambda - \rho)(H(a^{-1})))\phi(\tilde{m}^{-1}) \chi_{(0,0,0)}(x,y,z), \tag{2.33}
\]

will be used to construct a metaplectic Eisenstein distribution on \( \tilde{\text{SL}}(3, \mathbb{R}) \).

**Proposition 18.** Let \( \tilde{\tau} \in \tilde{V}_{\lambda,\phi}^{-\infty} \) be as above. Then:

1. \( \tilde{\tau} \) is right \( \tilde{N}_- \)-invariant.
2. \( \text{supp}(\tilde{\tau}) = (w_\ell, 1)\tilde{B}_- = \tilde{B}(w_\ell, 1) \).
3. \( \tilde{\tau} \) is left \( \tilde{N} \)-invariant.

**Proof:** The first two properties follow immediately from the definition of \( \tilde{\tau} \). For the final claim let \( f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in \tilde{V}_{-\lambda,\tau^\phi}^{-1} \). By the definition of \( \tilde{\tau} \),

\[
\langle f, \tau \rangle_{\lambda,\phi} = \int_{\tilde{N}} (f \cdot \tau)((w_\ell, 1)n(x,y,z))\,dx\,dy\,dz = f_1(w_\ell).
\]

On the other hand,

\[
\langle f, \pi(n)\tilde{\tau} \rangle_{\lambda,\phi} = \langle \pi(n^{-1})f, \tilde{\tau} \rangle_{\lambda,\phi} = f_1(nw_\ell) = f_1(w_\ell),
\]

where \( \pi(n) \) is a weight operator on \( \pi(nw_\ell) \).
where the last equality follows as $n(w, 1) = (w, 1)n$, since the 2-cocycle $\sigma$ is trivial on $\tilde{N}$. Thus, $\pi(n)\tau = \tau$.

Now we can define the metaplectic Eisenstein distribution as

$$
\tilde{E}(\tilde{g}, \lambda) = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \pi(S(\gamma)^{-1})\tilde{\tau}(\tilde{g})
$$

(2.34)

where $\tilde{\tau}$ is as in line (2.33) and $\lambda = (\lambda_1, \lambda_2, \lambda_3)$. Standard arguments show that the sum is convergent when the real part of $\lambda_1 - \lambda_2$ and $\lambda_2 - \lambda_3$ is sufficiently large.

3. Exponential Sums

3.1. Preliminaries. Let $A_1, A_2 \in \mathbb{Z}_{\neq 0}$. This section begins a study of the exponential sums, $\Sigma(A_1, A_2; m_1, m_2)$ (defined below), that appear as the coefficients of the Dirichlet series that make up the Fourier coefficients of the metaplectic Eisenstein distribution associated with the big cell. The primary focus of this section is to reduce the general computation of $\Sigma(A_1, A_2; m_1, m_2)$ to the case where $A_1$ and $A_2$ are prime powers.

We begin by describing how the symmetries of $s$, described in subsection 2.3, affect

$$
\Sigma(A_1, A_2; m_1, m_2) \overset{\text{def}}{=} \sum_{\gamma \in \Gamma_\infty \setminus S(A_1, A_2)/\Gamma_\infty} s(\gamma) e^{2\pi i (m_1 \frac{A_1}{A_1} + m_2 \frac{A_2}{A_2})}.
$$

(3.1)

Proposition 19. Let $A_1, A_2 \in \mathbb{Z}_{\neq 0}$ and let $m_1, m_2 \in \mathbb{Z}$. Then:

1. $\Sigma(A_1, A_2; m_1, m_2) = \Sigma(-A_1, -A_2; m_1, -m_2)$
2. $\Sigma(A_1, A_2; m_1, m_2) = (-A_1, -A_2)\Sigma(A_2, A_1; -m_2, -m_1)$
3. If $A_1A_2 > 0$, then $\Sigma(A_1, A_2; 0, 0) = 0$.

Proof: We will prove the second equality. Consider the automorphism $\psi$ (the Cartan involution followed by conjugation by $w_\ell$). The map $\psi$ defines a bijection between $\Gamma_\infty \setminus S(A_1, A_2)/\Gamma_\infty$ and $\Gamma_\infty \setminus S(A_2, A_1)/\Gamma_\infty$. In terms of Plücker coordinates $\psi$ is given by $(A_1, B_1, C_1, A_2, B_2, C_2) \mapsto (A_2, -B_2, C_2, A_1, -B_1, C_1)$. Since $s(\psi(\gamma)) = (-A_1, -A_2)s(\gamma)$, by Proposition 14, it follows that

$$
\Sigma(A_1, A_2; m_1, m_2) = (-A_1, -A_2)\Sigma(A_2, A_1; -m_2, -m_1).
$$

The proof of the first equality follows similarly by considering conjugation by $t(1, 1, -1)$. The third statement of the proposition is just a consequence of the first two results.

Continuing with this theme, we can also apply the twisted multiplicativity of Proposition 17 to study $\Sigma(A_1, A_2; m_1, m_2)$.

Proposition 20. Let $A_1, \alpha_1 \in \mathbb{Z}_{>0}$, $A_2, \alpha_2 \in \mathbb{Z}$ such that $A_1, A_2$ are odd, $(A_1A_2, \alpha_1\alpha_2) = 1$, and $\alpha_2$ is divisible by fewer powers of 2 than $\alpha_1$. Let $\mu = \left(\frac{-1}{-A_1A_2}\right)$. Then with respect to the map from Proposition 3, $S(A_1\alpha_1, A_2\alpha_2) \to S(A_1, \mu A_2) \times S(\alpha_1, -\mu\alpha_2)$, the following holds:

$$
\Sigma(A_1\alpha_1, A_2\alpha_2; m_1, m_2)
$$

$$
= \left(\frac{\alpha_2}{\alpha_1}\right)_{A_1} \left(\frac{\alpha_1}{A_2}\right) \Sigma(A_1, \mu A_2; (\alpha_1)_{mod A_1}^{-1} m_1, (\alpha_2)_{mod A_2}^{-1} m_2)
$$

$$
\times \Sigma(\alpha_1, -\mu\alpha_2; (A_1)_{mod A_1}^{-1} m_1, \left(\frac{-1}{A_1}\right) (A_2)_{mod A_2}^{-1} m_2).
$$
Proof: This follows from Proposition 17.

The function \( \Sigma(A_1, A_2; m_1, m_2) \) also exhibits a multiplicativity in the variables \( m_1, m_2 \).

**Proposition 21.** Let \( A_1, A_2, m_1, m_2, c_1, c_2 \in \mathbb{Z} \) such that, \( A_1 > 0, \ A_2, m_1, m_2 \neq 0 \), and \((c_1c_2, A_1A_2) = 1\). Then

\[
\Sigma(A_1, A_2; c_1m_1, c_2m_2) = \left( \frac{c_1}{A_1} \right) \left( \frac{c_2}{A_2} \right) \Sigma(A_1, A_2; m_1, m_2).
\]

This result is the byproduct of Proposition 20 and our computations in subsections 3.4 and 3.5 so no direct proof will be given.

Proposition 20 and Proposition 21 show that the exponential sums \( \Sigma(A_1, A_2; m_1, m_2) \) are built from those of the form \( \Sigma(\pm p^k, \pm p^\ell, \pm r^1, \pm r^2) \), where \( p \) is a prime.

### 3.2. Explicit Description of Double Cosets: \( A_1, A_2 \) Odd.

The sum \( \Sigma(A_1, A_2; m_1, m_2) \) is indexed by the set \( \Gamma_\infty \setminus S(A_1, A_2) \Gamma_\infty \cong S(A_1, A_2) \) (Recall line (2.5) and line (2.6)). In this section we provide a description of the sets \( S(A_1, A_2) \) when \( A_1, A_2 \neq 0 \). We begin with a few simple observations.

First, note that as \( C_j \equiv -1 \pmod{4} \), it follows that \( A_1 \equiv -A_2 \pmod{4} \). Thus, if \( A_1 \neq -A_2 \pmod{4} \), then \( S(A_1, A_2) = \emptyset \); this implies that \( \Sigma(A_1, A_2; m_1, m_2) = 0 \). Second, by appealing to the symmetries described in Proposition 19, it suffices to consider the case in which \( A_1 > 0 \). By Proposition 3 it suffices to study \( S(p^k, \pm p^\ell) \), where \( p \) is prime. However, a description of \( S(A_1, A_2) \), with \((A_1, A_2) = 1\), will be included as it provides a clean presentation of the boundary case \( A_1 = p^k, \ A_2 = \pm 1 \). The proofs of the results of this subsection are straightforward and will be omitted.

**Proposition 22.** Let \( A_1, A_2 \in \mathbb{Z} \) such that \((A_1, A_2) = 1, \ A_1 + A_2 \equiv 0 \pmod{4}\). Then \( S(A_1, A_2) \) consists precisely of the elements:

\[
(A_1, B_1, \frac{A_1C_2 + 4B_1B_2}{-A_2}, A_2, B_2, C_2),
\]

where:

1. \( 0 \leq B_1 < A_1 \) and \((B_1, A_1) = 1\).
2. \( 0 \leq -B_2 < -A_2 \) and \((B_2, A_2) = 1\).
3. \( C_2 \equiv -(A_2)^{-1}(4B_1B_2) \pmod{A_2}, \ C_2 \equiv -1 \pmod{4}, \ \text{and} \ 0 \leq \text{sign}(A_2)C_2 < 4|A_2| \).

In particular, \(|S(A_1, A_2)| = \phi(A_1)\phi(A_2)\).

The description of \( S(p^k, -p^\ell) \) will be broken into several cases: \( k > \ell > 0, k = \ell > 0, \) and \( 0 < k < \ell \). However, the symmetries of Proposition 2 reveal that the last case is redundant.

**Proposition 23.** Let \( p \) be an odd prime, \( \mu = \pm 1, \) and let \( k > \ell > 0 \) be integers such that \( p^k \equiv -\mu p^\ell \pmod{4} \). Then \( S(p^k, \mu p^\ell) \) consists precisely of the elements:

1. \((p^k, B_1, p^{k-\ell}C_2, \mu p^\ell, 0, C_2)\) where:
   - \( 0 \leq B_1 < p^k \) and \((B_1, p^k) = 1\).
   - \( 0 \leq \mu C_2 < 4p^\ell, \ C_2 \equiv -1 \pmod{4}, \ \text{and} \ (C_2, p^\ell) = 1\).

2. \((p^k, B_1p^\ell, p^{k-\ell}C_2 + 4B_1b_2, \mu p^\ell, b_2, C_2)\) where:
   - \( 0 < b_1 < p^{k-\ell}, \ (b_1, p^k) = 1\).
   - \( 0 < \mu b_2 < p^\ell, \ (b_2, p^{k-\ell}) = 1\).
   - \( 0 \leq \mu C_2 < 4p^\ell, \ \text{and} \ C_2 \equiv -1 \pmod{4}\)
Proposition 24. Let $p$ be an odd prime and let $k \in \mathbb{Z}_{>0}$. Then $S(p^k, -p^k)$ consists precisely of the elements:

1. $(p^k, 0, C_2, -p^k, 0, C_2)$ where:
   - $0 \leq -C_2 < 4p^k$, $C_2 \equiv -1 (4)$, and $(C_2, p^k) = 1$
2. $(p^k, 0, C_2, -p^k, B_2, C_2)$ where:
   - $0 < -B_2 < p^k$
   - $0 \leq -C_2 < 4p^k$, $C_2 \equiv -1 (4)$, and $(C_2, p^k) = 1$
3. $(p^k, B_1, C_2, -p^k, 0, C_2)$ where:
   - $0 < B_1 < p^k$
   - $0 \leq -C_2 < 4p^k$, $C_2 \equiv -1 (4)$, and $(C_2, p^k) = 1$
4. $(p^k, b_1p^i, C_2 + 4b_1b_2, -p^k, b_2p^i, C_2)$ where:
   - $0 < i < k$, $0 < j < k$, and $k = i + j$
   - $0 < b_1 < p^{k-i}$, and $(b_1, p^k) = 1$
   - $0 < -b_2 < p^{k-j}$, and $(b_2, p^k) = 1$
   - $0 \leq -C_2 < 4p^k$, $C_2 \equiv -1 (4)$, and $(C_2, p^k) = 1$
   - $C_2 + 4b_1b_2 \not\equiv 0 \pmod{p}$
5. $(p^k, b_1p^i, C_2 + 4b_1b_2, -p^k, b_2p^i, C_2)$ where:
   - $0 < i < k$, $0 < j < k$, and $k < i + j$
   - $0 < b_1 < p^{k-i}$, and $(b_1, p^k) = 1$
   - $0 < -b_2 < p^{k-j}$, and $(b_2, p^k) = 1$
   - $0 \leq -C_2 < 4p^k$, $C_2 \equiv -1 (4)$, and $(C_2, p^k) = 1$

3.3. Explicit Description of Double Cosets: $A_1, A_2$ Even. It remains to consider $S(A_1, A_2)$ where $A_1 = 2^k$ and $A_2 = \pm 2^\ell$. Before describing $S(2^k, \pm 2^\ell)$ a few comments are in order. First, as $A_i$ is a power of 2 and $C_i$ must be odd, the condition $(A_i, B_i, C_i) = 1$ is vacuous. Second, when $k > \ell$ the equation

$$2^k - \ell C_2 + 2^2 - \ell B_1B_2 \pm C_1 = 0,$$

implies that $\ell \geq 2$. Once again the following results are straightforward and the proofs are be omitted.

Proposition 25. Let $k > \ell \geq 2$ be integers and let $\mu = \pm 1$. Then $S(2^k, \mu 2^\ell)$ consists precisely of the elements:

1. $(2^k, 2\mu b_1, 2^k - \ell C_2 + b_1b_2, \mu 2^\ell, 2\mu b_2, C_2)$ where:
   - $0 \leq i < k$, and $0 \leq j < \ell$ such that $i + j = \ell - 2$.
   - $0 \leq \mu C_2 < 4(2^\ell)$, $C_2 \equiv -1 (4)$.
   - $0 \leq b_1 < 2^{k-i}$ such that $(2, b_1) = 1$.
   - $0 \leq \mu b_2 < 2^{\ell-j}$ such that $(2, b_2) = 1$, and $b_1b_2 \equiv \mu + 2^{k-\ell} \pmod{4}$.

Proposition 26. Let $k \in \mathbb{Z}_{>0}$. Then $S(2^k, -2^k)$ consists precisely of the elements:

1. $(2^k, 0, C_2, -2^k, 0, C_2)$
• $0 \leq -C_2 < 4(2^k), C_2 \equiv -1(4)$.

(2) $(2^k, 0, C_2, -2^k, B_2, C_2)$
• $0 \leq -C_2 < 4(2^k), C_2 \equiv -1(4)$.
• $0 < -B_2 < 2^k$.

(3) $(2^k, B_1, C_2, -2^k, 0, C_2)$
• $0 \leq -C_2 < 4(2^k), C_2 \equiv -1(4)$.
• $0 < B_1 < 2^k$.

(4) $(2^k, 2b_1, C_2 + 2^{2+i+j-k}b_1b_2, -2^k, 2^i b_2, C_2)$ where:
• $0 \leq i < k$, and $0 \leq j < k$ such that $i + j \geq k$.
• $0 \leq -C_2 < 4(2^k), C_2 \equiv -1(4)$.
• $0 \leq b_1 < 2^{k-i}$ such that $(2, b_1) = 1$.
• $0 \leq -b_2 < 2^{k-j}$ such that $(2, b_2) = 1$.

**Proposition 27.** Let $k \in \mathbb{Z}_{>0}$. Then $S(2^k)$ consists precisely of the elements:

1. $(2^k, 2b_1, -C_2 - 2b_1b_2, 2^k, 2^i b_2, C_2)$ where:
   • $0 \leq i < k$, and $0 \leq j < k$ such that $i + j = k - 1$.
   • $0 \leq C_2 < 4(2^k), C_2 \equiv -1(4)$.
   • $0 \leq b_1 < 2^k$ such that $(2, b_1) = 1$.
   • $0 \leq -b_2 < 2^{k-j}$ such that $(2, b_2) = 1$.

3.4. **Exponential Sum:** $\Sigma(p^k, \pm p^j; m_1, m_2)$. In this subsection and the next we use the descriptions of the sets $S(A_1, A_2)$ contained in sections 3.2 and 3.3, the formula for the splitting Theorem 12, and Equation (2.27) to evaluate the exponential sums $\Sigma(A_1, A_2; m_1, m_2)$ defined on line (3.1). In particular, by Proposition 20 the computations can be reduced to the case where $A_1$ and $A_2$ are powers of a fixed prime. We begin with $p$ an odd prime.

**Proposition 28.** Let $p$ be an odd prime and let $k, m_1, m_2 \in \mathbb{Z}$ such that $k > 0$. Then

$$
\begin{align*}
\Sigma(1, -1; m_1, m_2) &= 1, \\
\Sigma(1, 1; m_1, m_2) &= 0.
\end{align*}
$$

If $p^k \equiv \mu \pmod{4}$, then

$$
\Sigma(p^k, \mu; m_1, m_2) = 0.
$$

If $p^k \equiv -\mu \pmod{4}$, then

$$
\Sigma(p^k, \mu; m_1, m_2) = g(p^k, m_1, p^k).
$$

**Proof:** The set $S(1, -1)$ consists of the element $(1, 0, -1, -1, 0, -1)$. By equation (2.27) $\Sigma(1, -1; m_1, m_2) = 1$.

The set $S(1, 1)$ is empty thus $\Sigma(1, 1; m_1, m_2) = 0$.

If $p^k \equiv \mu \pmod{4}$, then the set $S(p^k, \mu)$ is empty. Thus, $\Sigma(p^k, \mu) = 0$.

Now suppose that $p^k \equiv -\mu \pmod{4}$. In this case, the set $S(p^k, \mu)$ contains the elements $(p^k, B_1, -\mu p^k, \mu, 0, -1)$, where $(B_1, p) = 1$, and $0 < B_1 < p^k$. Let $\gamma \in \Gamma$ be a representative with Plücker coordinates $(p^k, B_1, -\mu p^k, \mu, 0, -1)$. Using equation (2.27) it follows that $s(\gamma) = \left(\frac{B_1}{p^k}\right)$. Thus $\Sigma(p^k, \mu; m_1, m_2) = \sum_{B_1 \in (\mathbb{Z}/p^k\mathbb{Z})^\times} \left(\frac{B_1}{p^k}\right) e^{\frac{2\pi i (m_1 B_1)}{p^k}} = g(p^k, m_1, p^k)$. □

**Proposition 29.** Let $p$ be an odd prime, let $k, \ell \in \mathbb{Z}$ such that $k > \ell > 0$, and let $m_1, m_2 \in \mathbb{Z}$. If $p^{k-\ell} \equiv \mu \pmod{4}$, then

$$
\Sigma(p^k, \mu p^\ell; m_1, m_2) = 0.
$$
Now suppose that \( p^{k-\ell} \equiv -\mu \pmod{4} \). Then

\[
\Sigma(p^k, \mu p^\ell; m_1, m_2) = p^\ell g(p^\ell, m_2, p^\ell) g(p^k, m_1, p^{k-\ell}) \\
+ \phi(p^\ell) \sum_{0 \leq i < \ell, i \equiv \ell \pmod{2}} g(p^i, m_2, p^i) g(p^k, m_1, p^{k-i}).
\]

**Proof:** Recall the description of \( S(p^k, \mu p^\ell) \) contained in Proposition 23. This set will be empty when \( p^{k-\ell} \equiv \mu \pmod{4} \), so in this case the sum must be equal to 0.

From now on assume that \( p^{k-\ell} \equiv -\mu \pmod{4} \). The summands will be collected into three cases to parallel the structure set forth in Proposition 23.

Case 1: \( B_2 = 0 \). Using equation (2.27) it follows that if \( \gamma \in S(p^k, \mu p^\ell) \) such that \( B_2 = 0 \), then

\[
s(\gamma) = \left( \frac{B_1}{p^{k-\ell}} \right) \left( \frac{p^\ell}{C_2} \right).
\]

By Proposition 23 we see that this case corresponds to the sum

\[
\sum_{B_1 \in (\mathbb{Z}/p^k \mathbb{Z})^\times} \sum_{C_2 \in (\mathbb{Z}/4p^\ell \mathbb{Z})^\times} \left( \frac{p^\ell}{C_2} \right) \left( \frac{B_1}{p^{k-\ell}} \right) e^{2\pi i \left( \frac{m_1 B_1}{p^k} \right)} = \left( \frac{1 + (-1)^\ell}{2} \right) \phi(p^\ell) g(p^k, m_1, p^k). \tag{3.2}
\]

Case 2: \( B_1 B_2 \neq 0 \), \( i + j = \ell \), and \( j = 0 \), where \( B_1 = p^i b_1 \) and \( B_2 = p^j b_2 \) such that \( (b_1 b_2, p) = 1 \). For \( \gamma \in S(p^k, \mu p^\ell) \) subject to these conditions, equation 2.27 states that

\[
s(\gamma) = \left( \frac{p^\ell}{C_1} \right) \left( \frac{b_1}{p^{k-\ell}} \right) = \left( \frac{\mu b_2}{p^\ell} \right) \left( \frac{b_1}{p^k} \right).
\]

By Proposition 23 the sum in this case is

\[
\sum_{b_1 \in (\mathbb{Z}/p^k \mathbb{Z})^\times} \sum_{b_2 \in (\mathbb{Z}/p^\ell \mathbb{Z})^\times} \sum_{\gamma \in (\mathbb{Z}/4p^\ell \mathbb{Z})^\times} \left( \frac{\mu b_2}{p^\ell} \right) \left( \frac{b_1}{p^{k-\ell}} \right) e^{2\pi i \left( \frac{m_1 b_1 + \mu m_2 b_2}{p^k} \right)} = p^\ell g(p^k, m_1, p^{k-\ell}) g(p^\ell, m_2, p^\ell). \tag{3.3}
\]

Case 3: \( B_1 B_2 \neq 0 \), \( i + j = \ell \), and \( 0 < j < \ell \), where \( B_1 = p^i b_1 \) and \( B_2 = p^j b_2 \) such that \( (b_1 b_2, p) = 1 \). Using equation (2.27) it follows that if \( \gamma \in S(p^k, \mu p^\ell) \) subject to these conditions, then

\[
s(\gamma) = \left( \frac{b_1}{p^{k-\ell}} \right) \left( \frac{p^{\ell-i}}{C_2} \right) = \left( \frac{b_1}{p^{k-\ell-i}} \right) \left( \frac{\mu b_2}{p^\ell} \right) \left( \frac{C_2}{p^\ell} \right).
\]

By Proposition 23, this case yields the sum

\[
\sum_{b_1 \in (\mathbb{Z}/p^{k-\ell-i} \mathbb{Z})^\times} \sum_{b_2 \in (\mathbb{Z}/p^\ell \mathbb{Z})^\times} \sum_{\gamma \in (\mathbb{Z}/4p^\ell \mathbb{Z})^\times} \left( \frac{b_1}{p^{k-\ell-i}} \right) \left( \frac{\mu b_2}{p^\ell} \right) \left( \frac{C_2}{p^{\ell-i}} \right) e^{2\pi i \left( \frac{m_1 b_1 + \mu m_2 b_2}{p^k} \right)}
= \left( \frac{1 + (-1)^{\ell-i}}{2} \right) \phi(p^\ell) g(p^k, m_1, p^{k-i}) g(p^i, m_2, p^i). \tag{3.4}
\]

Note that if \( i = 0 \) this expression matches the expression from Case 1. Finally, we sum (3.4) over \( 0 < i < \ell \) and add this to (3.2) and (3.3). \( \square \)
Proposition 30. Let \( p \) be an odd prime and let \( k, m_1, m_2 \in \mathbb{Z} \) such that \( k > 0 \).

\[
\Sigma(p^k, -p^k; m_1, m_2) = \sum_{i=1}^{k-1} p^{k-i} g(p^i, m_2, p^i) g(p^{k-i}, -m_2, p^i) g(p^k, m_1, p^{k-i}) + \delta_{2|k} \phi(p^k)(\delta_{p^k|m_1} p^k + \delta_{p^k|m_2} p^k - \delta_{p^{k-1}|m_1} p^{k-1}).
\]

**Proof:** Recall the description of \( S(p^k, -p^k) \) contained in Proposition 24. The sum will be broken into five cases in accordance with this proposition. However, it will be more convenient to include Case 1 in both Case 2 and Case 3, and then subtract those terms that are double counted.

Case 2: \( B_1 = 0 \). For \( \gamma \in S(p^k, -p^k) \) such that \( B_1 = 0 \), equation (2.27) implies that

\[
s(\gamma) = \left( \frac{p^k}{C_1} \right) = \left( \frac{p^k}{-C_1} \right) = \left( \frac{-C_1}{p^k} \right) = \left( \frac{-C_2}{p^k} \right).
\]

The equation \( A_1C_2 + 4B_1B_2 + C_1A_2 = 0 \) implies the last equality. By Proposition 24 the sum under consideration becomes

\[
\delta_{2|k} \delta_{p^k|m_1} p^k \phi(p^k).
\]

Case 3: \( B_2 = 0 \). For \( \gamma \in S(p^k, -p^k) \) such that \( B_2 = 0 \), equation (2.27) implies that

\[
s(\gamma) = \left( \frac{D_1}{C_1} \right) \left( \frac{D_2}{C_2} \right) = \left( \frac{D}{C_2} \right) = \left( \frac{p^k}{C_2} \right) = \left( \frac{-C_2}{p^k} \right),
\]

where the second equality follows from the identity \( A_1C_2 + 4B_1B_2 + C_1A_2 = 0 \). By Proposition 24, the sum corresponding to this case is given by

\[
\delta_{2|k} \delta_{p^k|m_1} p^k \phi(p^k).
\]

Note that the case \( B_1 = B_2 = 0 \) is counted in both Case 2 and Case 3. To compensate for this we subtract \( \delta_{2|k} \phi(p^k) \).

Case 4: \( B_1B_2 \neq 0, i, j < k, \) where \( B_1 = p^i b_1 \) and \( B_2 = p^j b_2 \) such that \( (b_1 b_2, p) = 1 \). As \( 0 \leq i, j < k \) it follows that \( 0 < i, j \). We apply equation (2.27) to see that if \( \gamma \in S(p^k, -p^k) \) and \( \gamma \) satisfies the conditions of case 4, then

\[
s(\gamma) = \left( \frac{p^i}{C_1} \right) \left( \frac{p^{k-i}}{C_2} \right) = \left( \frac{-C_1}{p^i} \right) \left( \frac{-C_2}{p^{k-i}} \right) \left( \frac{-C_2 - 4b_1 b_2}{p^i} \right) \left( \frac{-C_2}{p^{k-i}} \right).
\]

Now we can sum over the elements of this case described in Proposition 24 with \( i \) fixed to get

\[
\sum_{C_2 \in (\mathbb{Z}/p^{k-i} \mathbb{Z})^*} \sum_{b_1 \in (\mathbb{Z}/p^i \mathbb{Z})^*} \sum_{b_2 \in (\mathbb{Z}/p^i \mathbb{Z})^*} \left( \frac{-C_2 - 4b_1 b_2}{p^i} \right) \left( \frac{-C_2}{p^{k-i}} \right) e^{2\pi i (m_1 \frac{b_1}{p^{k-i}} - m_2 \frac{b_2}{p^{k-i}})}.
\]

Now we will simplify this sum. First consider the sum indexed by \( b_2 \) in line (3.7). By Lemma 5,

\[
\sum_{b_2} \left( \frac{-C_2 + (-4b_1)(b_2)}{p} \right)^i e^{2\pi i (-m_2 \frac{b_2}{p})} = 19
\]
\[ = e^{2\pi i \left( \frac{m_2 \beta - C_2}{p^i} \right)} \left( \frac{b_1}{p} \right)^i g(p^i, m_2, p^i) - \left( \frac{-C_2}{p} \right)^i \delta_{p^i-1|m_2}p^{i-1}. \] (3.8)

Next using equation (3.8) we will consider the sums indexed by \( b_2 \) and \( C_2 \) in line (3.7) to get
\[
\left( \frac{b_1}{p} \right)^i g(p^i, m_2, p^i)p^{k-i} \sum_{C_2 \equiv 1 (\text{mod } 4)} \left( \frac{-C_2}{p^{k-i}} \right) e^{2\pi i \left( \frac{m_2 \beta - C_2}{p^i} \right)} - \sum_{C_2 \equiv 1 (\text{mod } 4)} \left( \frac{-C_2}{p} \right)^i \delta_{p^i-1|m_2}p^{i-1}
\]
\[
= \left( \frac{b_1}{p} \right)^i g(p^i, m_2, p^i)p^{k-i} \sum_{c \in \mathbb{Z}/4p^j \mathbb{Z}} c \left( \frac{c}{p^{k-i}} \right) e^{2\pi i \left( \frac{m_2 \beta - C_2}{p^i} \right)} - \phi(p^j)\delta_{2|k}\delta_{p^i-1|m_2}p^{i-1}, \] (3.9)

where the previous equality follows from an application of the Chinese Remainder Theorem. We simplify the result to get
\[
(3.9) = \left( \frac{b_1}{p} \right)^k g(p^i, m_2, p^i)p^{k-i} \sum_{c \in \mathbb{Z}/4p^j \mathbb{Z}} c \left( \frac{c}{p^{k-i}} \right) e^{2\pi i \left( \frac{m_2 \beta - C_2}{p^i} \right)} - \phi(p^j)\delta_{2|k}\delta_{p^i-1|m_2}p^{i-1}
\]
\[
= \left( \frac{b_1}{p} \right)^k g(p^i, m_2, p^i)p^{k-i}g(p^{k-i}, -m_2, p^j) - \phi(p^j)\delta_{2|k}\delta_{p^i-1|m_2}p^{i-1}. \] (3.10)

Now consider (3.7) in its entirety and apply the simplifications that resulted in line (3.10) to see that (3.7) is equal to
\[
\sum_{b_1} \left( \frac{b_1}{p} \right)^k g(p^i, m_2, p^i)p^{k-i}g(p^{k-i}, -m_2, p^j)e^{2\pi i \left( \frac{m_1 \beta - C_1}{p^i} \right)} - \phi(p^j)\delta_{2|k}\delta_{p^i-1|m_2}p^{i-1} e^{2\pi i \left( \frac{m_1 \beta - C_1}{p^j} \right)}
\]
\[
= g(p^i, m_2, p^i)p^{k-i}g(p^{k-i}, -m_2, p^j)g(p^k, m_1, p^{k-i}) - g(p^2, m_1, p^{k-i})\delta_{2|k}\phi(p^j)\delta_{p^i-1|m_2}p^{i-1}. \] (3.11)

This must be summed over \( 0 < i < k \).

Case 5: \( B_1B_2 \neq 0, i + j > k \), where \( B_1 = p^i b_1 \) and \( B_2 = p^j b_2 \) such that \( (b_1 b_2, p) = 1 \). Using the formula for the splitting, equation (2.27), it follows that if \( \gamma \in S(p^k, -p^k) \) satisfies the conditions of case 5, then
\[
s(\gamma) = \left( \frac{p^i}{C_1} \right) \left( \frac{p^{k-i}}{C_2} \right) = \left( \frac{p^i}{-C_1} \right) \left( \frac{p^{k-i}}{-C_2} \right) = \left( \frac{-C_1}{p^i} \right) \left( \frac{-C_2}{p^{k-i}} \right) = \left( \frac{-C_2}{p^k} \right).
\]

The last equality follows as \( C_1 \equiv C_2 \pmod{p} \). Thus the sum under consideration becomes
\[
\sum_{b_1 \in \mathbb{Z}/p^{k-i} \mathbb{Z}} \sum_{b_2 \in \mathbb{Z}/p^{k-j} \mathbb{Z}} \sum_{C_2 \equiv 1 (\text{mod } 4)} \sum_{C_2 \equiv 1 (\text{mod } 4)} \left( \frac{-C_2}{p^k} \right) e^{2\pi i \left( \frac{m_1 \beta - C_1}{p^i} - m_2 \beta - C_2}{p^{k-j}} \right)
\]
\[
= \delta_{2|k}\phi(p^j)g(p^2, m_1, p^{k-i})g(p^2, m_2, p^{k-j}).
\]

This must be summed over \( 0 < i < k \), and \( k - i + 1 \leq j \leq k - 1 \), and then can be simplified using equation (2.11). Specifically,
\[ \delta_{2|k} \sum_{0<i<k} \sum_{1 \leq j-i \leq 1} \phi(p^k)g(p^2, m_1, p^{k-i})g(p^2, m_2, p^{k-j}) \]

\[ = \delta_{2|k} \phi(p^k) \sum_{0<i<k} g(p^2, m_1, p^{k-i})(-1 + \delta_{p^{i-1}|m_2}p^{i-1}) \]

\[ = \delta_{2|k} \phi(p^k)(-\sum_{0<i<k} g(p^2, m_1, p^{k-i}) + \sum_{0<i<k} g(p^2, m_1, p^{k-i})\delta_{p^{i-1}|m_2}p^{i-1}) \]

\[ = \delta_{2|k} \phi(p^k)(1 - \delta_{p^{k-1}|m_1}p^{k-1}) + \sum_{0<i<k} g(p^2, m_1, p^{k-i})\delta_{p^{i-1}|m_2}p^{i-1}). \quad (3.12) \]

Finally we add (3.5), (3.6), (3.11), and (3.12) together remembering to sum (3.11) over 0 < i < k and to remove what was double counted in cases 1 and 2. \qed

3.5. **Exponential Sum:** \( \Sigma(2^k, \pm 2^l; m_1, m_2) \). In this subsection we consider the bad prime \( p = 2 \). Recall that an explicit parameterization of the index of summation of \( \Sigma(2^k, \pm 2^l; m_1, m_2) \) is described in Section 3.3.

**Proposition 31.** Let \( k, \ell \in \mathbb{Z} \) such that \( k > \ell \geq 0 \), let \( m_1, m_2 \in \mathbb{Z} \), and let \( \mu = \pm 1 \).

If \( \ell = 0, 1 \), then

\[ \Sigma(2^k, \mu 2^\ell; m_1, m_2) = 0. \]

From now on assume that \( \ell \geq 2 \).

If \( k - \ell \geq 3 \), then

\[ \Sigma(2^k, \mu 2^\ell; m_1, m_2) = 2^\ell \sum_{0 \leq i \leq \ell-2 \atop i \equiv \ell (\text{mod } 2)} (g_1(2^k, m_1, 2^{k-i})g_1(2^\ell, m_2, 2^{i+2}) \]

\[ + (-\mu)g_{-1}(2^k, m_1, 2^{k-i})g_{-1}(2^\ell, m_2, 2^{i+2})). \]

If \( k - \ell = 2 \), then

\[ \Sigma(2^k, \mu 2^\ell; m_1, m_2) = (-1)^\ell 2^\ell \sum_{0 \leq i \leq \ell-2 \atop i \equiv \ell (\text{mod } 2)} (g_1(2^k, m_1, 2^{k-i})g_1(2^\ell, m_2, 2^{i+2}) \]

\[ + (-\mu)g_{-1}(2^k, m_1, 2^{k-i})g_{-1}(2^\ell, m_2, 2^{i+2})). \]

If \( k - \ell = 1 \), then

\[ \Sigma(2^k, \mu 2^\ell; m_1, m_2) = (-\mu)^\ell 2^\ell \sum_{0 \leq i \leq \ell-2 \atop i \equiv \ell (\text{mod } 2)} (g_1(2^k, m_1, 2^{k-i})g_{-1}(2^\ell, m_2, 2^{i+2}) \]

\[ + (-\mu)g_{-1}(2^k, m_1, 2^{k-i})g_1(2^\ell, m_2, 2^{i+2})). \]

**Proof:** Recall the description of \( S(2^k, \mu 2^\ell) \) contained in Proposition 25. Note that the set will be empty unless \( \ell \geq 2 \). Thus if this is the case the sum must be equal to 0. From now on suppose that \( \ell \geq 2 \).
We will begin with the formula for the splitting. Let $B_1 = 2^i b_1$, $B_2 = 2^j b_2$, and $\epsilon = (\frac{1}{b_1})$.

Using equation (2.27) it follows that if $\gamma \in S(2^k, \mu 2^\ell)$, then

$$s(\gamma) = \left( \frac{\epsilon}{-\mu 2^{k-\ell}} \right) \left( \frac{b_1}{2^{k-\ell}} \right) \left( \frac{2^i}{\mu (2^{k-\ell} C_2 + b_1 b_2)} \right) \left( \frac{2^{\ell-i}}{C_2} \right).$$

This computation will be broken into three cases, $k - \ell \geq 3$, $k - \ell = 2$, $k - \ell = 1$.

First consider the case $k - \ell \geq 3$. By equation (2.27),

$$s(\gamma) = \left( \frac{-\mu}{b_1} \right) \left( \frac{b_1}{2^{k-\ell}} \right) \left( \frac{2^i}{b_1 b_2} \right) \left( \frac{2^{\ell-i}}{C_2} \right).$$

Now we compute the exponential sum. By Proposition 25, $\Sigma(2^k, \mu 2^\ell; m_1, m_2) = \sum_{i=0}^{\ell-2} \sum_{b_1 \in (\mathbb{Z}/2^{k-i}\mathbb{Z})^\times} \sum_{b_2 \in (\mathbb{Z}/2^{k-i}\mathbb{Z})^\times} \sum_{C_2 \in (\mathbb{Z}/2^{k-i}\mathbb{Z})^\times} \left( -\mu \right) \left( \frac{b_1}{2^{k-i}} \right) \left( \frac{2^i}{b_1 b_2} \right) \left( \frac{2^{\ell-i}}{C_2} \right) e^{2\pi i (\frac{m_1 b_1}{2^k} + \frac{m_2 b_2}{2^\ell}).}$

First sum over $C_2$. If this is to be nonzero it must be that $\ell \equiv i (\mod 2)$. Next consider the sum over $b_2$. By definition 2.9 this is equal to $g_{\mu b_1}(2^i, \mu m_2, 2^{i+2}) = g_{b_1}(2^i, m_2, 2^{i+2})$. Finally consider the sum over $b_1$. Begin by splitting this into two pieces based on the residue of $b_1$ mod 4. By definition (2.9), the summand in line (3.5) corresponding to a fixed $i$ is equal to

$$g_1(2^k, m_1, 2^{k-i})g_1(2^i, m_2, 2^{i+2}) + (-\mu) g_{-1}(2^k, m_1, 2^{k-i})g_{-1}(2^i, m_2, 2^{i+2}).$$

Putting everything together yields

$$\Sigma(2^k, \mu 2^\ell; m_1, m_2) = 2^\ell \sum_{0 \leq i \leq \ell-2 \atop i \equiv i (\mod 2)} (g_1(2^k, m_1, 2^{k-i})g_1(2^i, m_2, 2^{i+2}) + (-\mu) g_{-1}(2^k, m_1, 2^{k-i})g_{-1}(2^i, m_2, 2^{i+2})).$$

Now we will consider the case $k - \ell = 2$. By equation (2.27),

$$s(\gamma) = \left( \frac{-\mu}{b_1} \right) \left( \frac{2^i}{4 C_2 + b_1 b_2} \right) \left( \frac{2^{\ell-i}}{C_2} \right) = (-1)^i \left( \frac{-\mu}{b_1} \right) \left( \frac{2^i}{b_1 b_2} \right) \left( \frac{2^{\ell-i}}{C_2} \right).$$

Now we compute the exponential sum. By Proposition 25, $\Sigma(2^k, -\mu 2^\ell; m_1, m_2) = \sum_{i=0}^{\ell-2} \sum_{b_1 \in (\mathbb{Z}/2^{k-i}\mathbb{Z})^\times} \sum_{b_2 \in (\mathbb{Z}/2^{k-i}\mathbb{Z})^\times} \sum_{C_2 \in (\mathbb{Z}/2^{k-i}\mathbb{Z})^\times} \sum_{b_1 b_2 \equiv \mu (\mod 4)} \sum_{C_2 \equiv -1 (\mod 4)} (-1)^i \left( \frac{-\mu}{b_1} \right) \left( \frac{2^i}{b_1 b_2} \right) \left( \frac{2^{\ell-i}}{C_2} \right) e^{2\pi i \left( \frac{m_1 b_1}{2^{k-i}} + \frac{m_2 b_2}{2^{\ell-i}} \right)}.$

First sum over $C_2$. If this is to be nonzero it must be that $\ell \equiv i (\mod 2)$. Next consider the sum over $b_2$. By definition (2.9) this is equal to $g_{\mu b_1}(2^i, \mu m_2, 2^{i+2}) = g_{b_1}(2^i, m_2, 2^{i+2})$. Finally consider the sum over $b_1$. Begin by splitting this into two pieces based on the residue of $b_1$ mod 4. By definition (2.9) the sum is equal to

$$g_1(2^{i}, m_1, 2^{k-i})g_1(2^{i}, m_2, 2^{i+2}) + g_{-1}(2^{i}, m_1, 2^{k-i})g_{-1}(2^{i}, m_2, 2^{i+2}).$$

Putting everything together and using that $k - \ell = 2$ yields
\[ \Sigma(2^k, -2^l; m_1, m_2) = (-1)^{\ell} 2^\ell \sum_{\substack{0 \leq i \leq \ell - 2 \\ i \equiv \ell \pmod{2}}} \left( g_1(2^k, m_1, 2^{k-i}) g_1(2^\ell, m_2, 2^{i+2}) \right) \\
+ (-\mu) g_{\ell-1}(2^k, m_1, 2^{k-i}) g_{\ell-1}(2^\ell, m_2, 2^{i+2}) . \]

Lastly we will consider the case \( k - \ell = 1 \). Note that since \( b_1 b_2 \equiv -\mu \pmod{4} \) it follows that \( \frac{2^i}{6 + b_1 b_2} = (-\mu) \left( \frac{2^i}{b_1 b_2} \right) \). Thus, by equation (2.27),

\[
s(\gamma) = \left( \frac{-\mu}{b_1} \right) \left( \frac{b_1}{2} \right) \left( \frac{2^i}{2C_2 + b_1 b_2} \right) \left( \frac{2\ell - i}{C_2} \right) = (-\mu)^i \left( \frac{-\mu}{b_1} \right) \left( \frac{b_1}{2} \right) \left( \frac{2^i}{b_1 b_2} \right) \left( \frac{2\ell - i}{C_2} \right) .
\]

Now we compute the exponential sum. By Proposition 25, \( \Sigma(2^k, -2^\ell; m_1, m_2) = \)

\[
\sum_{\substack{\ell - 2 \\ i \equiv \ell \pmod{2}}} \sum_{b_1 \in (\mathbb{Z}/2^{k-1} \mathbb{Z})^\times} \sum_{b_2 \in (\mathbb{Z}/2^{l+2} \mathbb{Z})^\times} \sum_{\substack{C_2 \in (\mathbb{Z}/2^{l+2} \mathbb{Z})^\times \\ C_2 \equiv -1 \pmod{4}}} (-\mu)^i \left( \frac{-\mu}{b_1} \right) \left( \frac{b_1}{2} \right) \left( \frac{2^i}{b_1 b_2} \right) \left( \frac{2\ell - i}{C_2} \right) e^{2\pi i \left( \frac{m_1 b_1 + \mu m_2 b_2}{2^{k-1}} \right)} .
\]

First sum over \( C_2 \). If this is to be nonzero it must be that \( \ell \equiv i \pmod{2} \). Next consider the sum over \( b_2 \). By definition (2.9) this is equal to \( g_{-\mu b_1}(2^i, \mu m_2, 2^{i+2}) = g_{-b_1}(2^i, m_2, 2^{i+2}) \). Finally consider the sum over \( b_1 \). Begin by splitting this into two pieces based on the residue of \( b_1 \pmod{4} \). By definition (2.9) the sum is equal to \( (-\mu)^i g_{\ell+1}(2^{\ell+1}, m_1, 2^{k-i}) g_{\ell}(2^\ell, m_1, 2^{i+2}) + (-\mu)^i g_{\ell+1}(2^{\ell+1}, m_1, 2^{k-i}) g_{\ell}(2^\ell, m_1, 2^{i+2}) \).

Putting everything together yields \( \Sigma(2^k, -2^\ell; m_1, m_2) = (-1)^{\ell} 2^\ell \sum_{\substack{0 \leq i \leq \ell - 2 \\ i \equiv \ell \pmod{2}}} \left( g_1(2^k, m_1, 2^{k-i}) g_{\ell-1}(2^\ell, m_2, 2^{i+2}) \right) \\
+ (-\mu) g_{\ell-1}(2^k, m_1, 2^{k-i}) g_{\ell-1}(2^\ell, m_2, 2^{i+2}) . \)

\[ \square \]

**Proposition 32.** Let \( k \in \mathbb{Z}_{>0} \), and let \( m_1, m_2 \in \mathbb{Z} \). Then

\[
\Sigma(2^k, -2^k; m_1, m_2) = 2^k \delta_{2k}(\delta_{2k}|m_1 2^k + \delta_{2k}|m_2 2^k - \delta_{2k-1}|m_1 2^{k-1} \\
+ \sum_{i=1}^{k-1} (-1)^i g(2^i, m_1, 2^{k-i}) g(2^2, m_2, 2^i) + \sum_{i=1}^{k-1} \delta_{2^{i-1}|m_2 2^{i-1}} g(2^2, m_1, 2^{k-i}) .
\]

**Proof:** Recall the description of \( S(2^k, -2^k) \) from Proposition 26. The computation of \( \Sigma(2^k, -2^k; m_1, m_2) \) will utilize the cases discussed in that proposition. We begin by computing the splitting. If \( B_1 \) and \( B_2 \neq 0 \), let \( B_1 = 2^i b_1 \), \( B_2 = 2^j b_2 \). Let \( \epsilon = \left( \frac{-1}{b_1} \right) \). By equation (2.27),

\[
s(\gamma) = \left( \frac{2^i}{C_2 + 2^{2i+j-k} b_1 b_2} \right) \left( \frac{2^{k-i}}{C_2} \right) .
\]
If \( B_1 \) or \( B_2 = 0 \), then \( s(\gamma) = \left( \frac{\gamma}{C_2} \right) \). In case 1 the sum is given by

\[
\sum_{C_2 \in \mathbb{Z}/2^{2+k}\mathbb{Z}} \left( \frac{2k}{C_2} \right) = 2^k \delta_{2|k}.
\]

We will incorporate case 1 into both case 2 and case 3 and subtract the duplicated terms later.

With the previous remark in mind, case 2 yields the sum

\[
\sum_{B_2 \in \mathbb{Z}/2^k\mathbb{Z}} \sum_{C_2 \in \mathbb{Z}/2^{2+k}\mathbb{Z}} \left( \frac{2k}{C_2} \right) e^{2\pi i (-B_2 B_2)} = 2^{2k} \delta_{2|k} \delta_{2k|m_2}.
\]

Next we consider case 3.

\[
\sum_{B_1 \in \mathbb{Z}/2^k\mathbb{Z}} \sum_{C_2 \in \mathbb{Z}/2^{2+k}\mathbb{Z}} \left( \frac{2k}{C_2} \right) e^{2\pi i (m_1 B_1)} = 2^{2k} \delta_{2|k} \delta_{2k|m_1}.
\]

In cases 2 and 3 we counted case 1. Thus we must subtract case 1 once to account for this. Specifically, the sum of the first three cases is given by

\[
2^k \delta_{2|k} (2^k \delta_{2k|m_1} + 2^k \delta_{2k|m_2} - 1).
\]

Finally we must consider case 4. The cases \( i + j = k \) and \( i + j > k \) must be dealt with separately. First consider \( i + j = k \). In this case the sum is given by

\[
\sum_{i=1}^{k-1} \sum_{b_1 \in (\mathbb{Z}/2^{k-i}\mathbb{Z})} \sum_{b_2 \in (\mathbb{Z}/2^i\mathbb{Z})} \sum_{C_2 \in \mathbb{Z}/2^{2+k}\mathbb{Z}} (-1)^i \left( \frac{\gamma}{C_2} \right) e^{2\pi i (m_1 \frac{b_1}{2^i} - m_2 \frac{b_2}{2^i})} = 2^k \delta_{2|k} \sum_{i=1}^{k-1} (-1)^i g(2^k, m_1, 2^{k-i}) g(2^k, -m_2, 2^i).
\]

Now suppose that \( i + j > k \). In this case the sum is given by

\[
\sum_{1 \leq i \leq k-1} \sum_{b_1 \in (\mathbb{Z}/2^{k-i}\mathbb{Z})} \sum_{b_2 \in (\mathbb{Z}/2^{k-j}\mathbb{Z})} \sum_{C_2 \in \mathbb{Z}/2^{2+k}\mathbb{Z}} \left( \frac{2k}{C_2} \right) e^{2\pi i (m_1 \frac{b_1}{2^i} - m_2 \frac{b_2}{2^j})} = 2^k \delta_{2|k} \sum_{1 \leq i \leq k-1} \sum_{1 \leq j \leq k-1} \sum_{i+j > k} g(2^k, m_1, 2^{k-i}) g(2^k, -m_2, 2^{k-j}).
\]

This expression can be simplified using line (2.11) in Lemma 5. Specifically,

\[
\sum_{1 \leq k-j \leq i-1} g(2^2, m_2, 2^{k-j}) = \delta_{2|1} m_2 2^{i-1} - 1.
\]
This can be summed over $i$ to get
\[
2^k \delta_{2|k} \sum_{1 \leq i < k, 1 \leq j \leq k} \delta_{2i-1|m_2} 2^{i-1} g(2^k, m_1, 2^{k-i}) - \sum_{1 \leq k-i \leq k-1} g(2^k, m_1, 2^{k-i})
\]

\[
= 2^k \delta_{2|k} \left( \sum_{1 \leq i < k} \delta_{2i-1|m_2} 2^{i-1} g(2^k, m_1, 2^{k-i}) - \sum_{1 \leq k-i \leq k-1} g(2^k, m_1, 2^{k-i}) \right)
\]

\[
= 2^k \delta_{2|k} \left( \sum_{1 \leq i < k} \delta_{2i-1|m_2} 2^{i-1} g(2^k, m_1, 2^{k-i}) - (\delta_{2k-1|m_2} 2^{k-1} - 1) \right). \quad (3.16)
\]

Finally we add (3.14), (3.15), and (3.16) together. \hfill \Box

**Proposition 33.** Let $k \in \mathbb{Z}_{>0}$, and let $m_1, m_2 \in \mathbb{Z}$. Then

\[
\Sigma(2^k, 2^k; m_1, m_2)
\]

\[
= 2^k \delta_{2|k} \sum_{i=0}^{k-1} \left( g(2^2, m_1, 2^{k-i}) + (-1)^{i+1} g_{-1}(2^2, m_1, 2^{k-i}) \right)
\]

\[
\times \left( g(2^2, m_2, 2^{i+1}) + (-1)^i g_{-1}(2^2, m_2, 2^{i+1}) \right)
\]

**Proof:** Recall the description of $S(2^k, 2^k)$ from Proposition 27. We begin by computing the splitting. Let $\gamma \in S(2^k, 2^k)$. Let $B_1 = 2^i b_1$, $B_2 = 2^i b_2$, and let $e = \left( \frac{-1}{b_1} \right)$. By equation (2.27),

\[
s(\gamma) = \left( \frac{-1}{b_1} \right) \left( \frac{2^i}{C_2 + 2b_1 b_2} \right) \left( \frac{2^{k-i}}{C_2} \right).
\]

Since $C_2 \equiv -1 \pmod{4}$ and $(b_1, 2) = 1$, it follows that $C_2 + 2b_1 b_2 \equiv 1$ or 5 (mod 8). Thus we will split the sum into two pieces based on the residue of $b_1 b_2$ mod 4. First assume that $b_1 b_2 \equiv 1 \pmod{4}$. In this case the sum is given by

\[
\sum_{i=0}^{k-1} \sum_{b_1 \in (\mathbb{Z}/2^{k-i})^\times} \sum_{b_2 \in (\mathbb{Z}/2^{i+1})^\times} \sum_{C_2 \in (\mathbb{Z}/2^{i+2})^\times} \frac{1}{b_1} \left( \frac{2^i}{C_2 + 2} \right) \left( \frac{2^{k-i}}{C_2} \right) e^{2\pi i \left( \frac{m_1 b_1 + m_2 b_2}{2^{k-i}} \right)}
\]

\[
= \sum_{i=0}^{k-1} \sum_{b_1 \in (\mathbb{Z}/2^{k-i})^\times} \sum_{b_2 \in (\mathbb{Z}/2^{i+1})^\times} \sum_{C_2 \in (\mathbb{Z}/2^{i+2})^\times} \frac{1}{b_1} \left( \frac{2^i}{C_2} \right) e^{2\pi i \left( \frac{m_1 b_1 + m_2 b_2}{2^{i+2}} \right)}.
\]

The sum over $C_2$ is nonzero if and only if $k$ is even. The sum over $b_2$ is equal to $g_{b_1}(2^2, m_2, 2^{i+1})$. The sum over $b_1$ will be split into two cases based on the residue of $b_1 \pmod{4}$. Putting this together gives

\[
\sum_{i=0}^{k-1} \sum_{b_1 \in (\mathbb{Z}/2^{k-i})^\times} \sum_{b_2 \in (\mathbb{Z}/2^{i+1})^\times} \sum_{C_2 \in (\mathbb{Z}/2^{i+2})^\times} \frac{1}{b_1} \left( \frac{2^i}{C_2 + 2} \right) \left( \frac{2^{k-i}}{C_2} \right) e^{2\pi i \left( \frac{m_1 b_1 + m_2 b_2}{2^{k-i}} \right)}
\]
\[= \sum_{i=1}^{k-1} 2^k \delta_{2|k}(g_1(2^2, m_1, 2^{k-i})g_1(2^2, m_2, 2^{i+1}) - g_{-1}(2^2, m_1, 2^{k-i})g_{-1}(2^2, m_2, 2^{i+1})).\]

Next assume that \(b_1b_2 \equiv -1 (\text{mod } 4)\). By computations similar to the previous case we have

\[
\sum_{i=0}^{k-1} \sum_{b_1 \in (\mathbb{Z}/2^{k-i}\mathbb{Z})\times} \sum_{b_2 \in (\mathbb{Z}/2^{i+1}\mathbb{Z})\times} \sum_{C_2 \in (\mathbb{Z}/2^{i+2}\mathbb{Z})} (-1) \left( \frac{2^i}{b_1} \right) \left( \frac{2^{k-i}}{C_2} \right) e^{2\pi i \left( \frac{b_1 m_1 + b_2 m_2}{2^{k+i+1}} \right)}
\]

\[
= \sum_{i=0}^{k-1} (-1)^i 2^k \delta_{2|k}(g_1(2^2, m_1, 2^{k-i})g_{-1}(2^2, m_2, 2^{i+1}) - g_{-1}(2^2, m_1, 2^{k-i})g_1(2^2, m_2, 2^{i+1})).
\]

Putting everything together yields

\[
\Sigma(2^k, 2^k; m_1, m_2) = 2^k \delta_{2|k} \sum_{i=0}^{k-1} (g_1(2^2, m_1, 2^{k-i})g_1(2^2, m_2, 2^{i+1}) - g_{-1}(2^2, m_1, 2^{k-i})g_{-1}(2^2, m_2, 2^{i+1})
+ (-1)^i g_1(2^2, m_1, 2^{k-i})g_{-1}(2^2, m_2, 2^{i+1}) - (-1)^i g_{-1}(2^2, m_1, 2^{k-i})g_1(2^2, m_2, 2^{i+1})).
\]

Finally note that the summands can be factored. \(\square\)

3.6. Big Cell Constant Term. We can specialize the results of the preceding subsections by setting \(m_1 = m_2 = 0\) to determine the big cell’s contribution to the constant term Fourier coefficient. We assemble the complete constant term in Subsection 4.7, but for now it will be convenient to record only these preliminary calculations.

**Proposition 34.** Let \(p\) be an odd prime and let \(k \in \mathbb{Z}\) such that \(k > 0\). Then

\[
\Sigma(1, -1; 0, 0) = 1.
\]

\[
\Sigma(1, 1; 0, 0) = 0.
\]

\[
\Sigma(p^k, 1; 0, 0) = 0.
\]

*If \(k\) is odd, then*

\[
\Sigma(p^k, -1; 0, 0) = 0.
\]

*If \(k\) is even, then*

\[
\Sigma(p^k, -1; 0, 0) = \phi(p^k).
\]

**Proof:** This result follows directly from Proposition 28 by setting \(m_1 = m_2 = 0\). \(\square\)

**Proposition 35.** Let \(p\) be an odd prime and let \(k, l \in \mathbb{Z}\) such that \(k > l > 0\).

*If \(p^{k-l} \equiv -1 (\text{mod } 4)\), then*

\[
\Sigma(p^k, p^l; 0, 0) = 0.
\]

\[
\Sigma(p^k, -p^l; 0, 0) = 0.
\]
Now suppose that \( p^{k-\ell} \equiv 1 \pmod{4} \). If \( k \) or \( \ell \) is odd, then
\[
\Sigma(p^k, -p^\ell; 0, 0) = 0.
\]
If \( k \) and \( \ell \) are even, then
\[
\Sigma(p^k, -p^\ell; 0, 0) = \phi(p^k)\phi(p^\ell)[(\frac{k + 2}{2})p - (\frac{\ell - 2}{2})].
\]

**Proof:** This result follows directly from Proposition 29 by setting \( m_1 = m_2 = 0 \). \( \square \)

**Proposition 36.** Let \( p \) be an odd prime and let \( k, \ell \in \mathbb{Z} \) such that \( k > 0 \).
\[
\Sigma(p^k, p^\ell; 0, 0) = 0.
\]
If \( k \) is odd, then
\[
\Sigma(p^k, -p^\ell; 0, 0) = 0.
\]
If \( k \) is even, then
\[
\Sigma(p^k, -p^\ell; 0, 0) = \phi(p^k)p^{k-2}(\frac{k - 1}{2}p - (\frac{\ell - 2}{2})).
\]

**Proof:** This result follows directly from Proposition 30 by setting \( m_1 = m_2 = 0 \). \( \square \)

**Proposition 37.** Let \( k, \ell \in \mathbb{Z} \) such that \( k > \ell \geq 0 \), and let \( \mu = \pm 1 \). If \( \ell = 0, 1 \), then
\[
\Sigma(2^k, \mu 2^\ell; 0, 0) = 0.
\]
If \( \ell \geq 2 \), then
\[
\Sigma(2^k, \mu 2^{\ell+1}; 0, 0) = \delta_{k=\ell=0, \pmod{2}}(\frac{1 + (-\mu)}{2})2^{k+\ell-1, \pmod{2}}.
\]

**Proof:** This result follows directly from Proposition 31 by setting \( m_1 = m_2 = 0 \). \( \square \)

**Proposition 38.** Let \( k \in \mathbb{Z}_{>0} \). Then
\[
\Sigma(2^k, \mu 2^k; 0, 0) = \delta_{2|k}(\frac{1 + (-\mu)}{2})(k + 4)2^{k-2}.
\]

**Proof:** This result follows directly from Proposition 32 and Proposition 33 by setting \( m_1 = m_2 = 0 \). \( \square \)

### 3.7. The Formula of Brubaker-Bump-Friedberg-Hoffstein

In this section the formulas for \( \Sigma(p^k, \pm p^\ell, m_1, m_2) \), computed in Section 3.4, are manipulated to resemble those contained in Brubaker-Bump-Friedberg-Hoffstein [5].

**Proposition 39.** Let \( p \) be an odd prime, let \( \mu = \pm 1 \), and let \( k, \ell, r_1, r_2 \in \mathbb{Z} \) such that \( k > \ell \geq 0 \), \( p^{k-\ell} \equiv -\mu \pmod{4} \), and \( r_1, r_2 \geq 0 \). Then
\[
\Sigma(p^k, \mu p^\ell; p^{r_1}, p^{r_2}) = \sum_{i=0}^\ell g(p^i, p^\ell, p^\ell)g(p^{\ell-i}, p^{r_2}, p^{r_2-i})g(p^k, p^{r_1+i-\ell}, p^k). \tag{3.17}
\]

**Proof:** In Proposition 29 it was shown that
\[
\Sigma(p^k, \mu p^\ell; p^{r_1}, p^{r_2}) = p^\ell g(p^\ell, p^{r_2}, p^\ell)g(p^k, p^{r_1}, p^{k-\ell}) + \phi(p^\ell) \sum_{0 \leq i \leq \ell} \frac{1 + (-1)^{\ell-i}}{2}g(p^i, p^{r_2}, p^i)g(p^k, p^{r_1}, p^{k-i}). \tag{3.18}
\]
First note that in line (3.17) the terms with \( l \equiv i + 1 \) (mod 2) vanish. Now the \( i \)th term on the right hand side of line (3.17) will be paired with the \((l - i)\)th term in the right hand side of line (3.18) where \( p'g(p', p'^2, p')g(p^k, p^{r_1}, p^{k-l}) \) is considered to be the \( i = l \) term. Now compute the terms explicitly using Lemma 5 and note that they agree.

\[ \Sigma(p^k, -p^k; p'^1, p'^2) = \sum_{0 \leq i \leq k-1} g(p^i, -p^{r_2}, p^i)g(p^k, p^{r_1+i}, p^k)g(p^{k-i}, p^{r_2+k-2i}, p^{k-i}) + \delta_{k \leq r_2}p^k g(p^k, p^k, p^k). \]

**Proof:** First suppose that \( k \) is odd. In this case the claim is that

\[
\begin{align*}
\sum_{i=1}^{k-1} g(p^i, -p^{r_2}, p^i)p^{k-i}g(p^{k-i}, p^{r_2}, p^i)g(p^k, p^{r_1}, p^{k-i}) \\
= \sum_{0 \leq i \leq k-1} g(p^i, -p^{r_2}, p^i)g(p^k, p^{r_1+i}, p^k)g(p^{k-i}, p^{r_2+k-2i}, p^{k-i}) + \delta_{k \leq r_2}p^k g(p^k, p^k, p^k),
\end{align*}
\]

by the computation of Proposition 30. By Lemma 5,

\[
\begin{align*}
g(p^i, -p^{r_2}, p^i)p^{k-i}g(p^{k-i}, p^{r_2}, p^i)g(p^k, p^{r_1}, p^{k-i}) \\
= \sum_{0 \leq i \leq k-1} g(p^i, -p^{r_2}, p^i)g(p^k, p^{r_1+i}, p^k)g(p^{k-i}, p^{r_2+k-2i}, p^{k-i}) + \delta_{k \leq r_2}p^k g(p^k, p^k, p^k).
\end{align*}
\]

Note that \( k \equiv 1 \) (mod 2) was not needed for this. Thus it remains to prove that

\[
0 = g(p^0, -p^{r_2}, p^0)g(p^k, p^{r_1}, p^k)g(p^k, p^{r_2+k}, p^k) + \delta_{k \leq r_2}p^k g(p^k, p^k, p^k).
\]

As \( k \) is odd \( \delta_{k \leq r_2}g(p^k, p^k, p^k) = 0 \) and \( g(p^k, p^{r_2+k}, p^k) = 0 \). Thus the desired equality holds.

Now suppose that \( k \) is even. In this case it must be shown that

\[
\begin{align*}
\sum_{i=1}^{k-1} g(p^i, -p^{r_2}, p^i)p^{k-i}g(p^{k-i}, p^{r_2}, p^i)g(p^k, p^{r_1}, p^{k-i}) + \phi(p^k)(\delta_{k \leq r_1}p^k + \delta_{k \leq r_2}p^k - \delta_{k-1 \leq r_1}p^{k-1}) \\
= \sum_{0 \leq i \leq k-1} g(p^i, -p^{r_2}, p^i)g(p^k, p^{r_1+i}, p^k)g(p^{k-i}, p^{r_2+k-2i}, p^{k-i}) + \delta_{k \leq r_2}p^k g(p^k, p^k, p^k),
\end{align*}
\]

again by Proposition 30. As was mentioned in the previous case

\[
g(p^i, -p^{r_2}, p^i)p^{k-i}g(p^{k-i}, p^{r_2}, p^i)g(p^k, p^{r_1}, p^{k-i}) \\
= \sum_{0 \leq i \leq k-1} g(p^i, -p^{r_2}, p^i)g(p^k, p^{r_1+i}, p^k)g(p^{k-i}, p^{r_2+k-2i}, p^{k-i})g(p^k, p^{r_1+i}, p^k).
\]

First note that since \( k \) is even

\[
\phi(p^k)p^k \delta_{k \leq r_2} = \delta_{k \leq r_2}p^k g(p^k, p^k, p^k).
\]

Thus it remains to show that

\[
\phi(p^k)(\delta_{k \leq r_1}p^k - \delta_{k-1 \leq r_1}p^{k-1}) = \delta_{k \leq r_2}p^k g(p^k, p^{r_2+k}, p^k).
\]

Now \( g(p^0, -p^{r_2}, p^0) = 1 \) and as \( k \) is even \( g(p^k, p^{r_2+k}, p^k) = \phi(p^k) \). Thus it remains to show that
\[
(\delta_{k \leq r_1} p^k - \delta_{k-1 \leq r_1} p^{k-1}) = g(p^k, p^{r_1}, p^k).
\]
This can be proved directly by considering the cases \( r_1 < k - 1, r_1 = k - 1 \) and \( r_1 \geq k \).

4. Metaplectic Eisenstein Distribution

Recall the definition of the metaplectic Eisenstein distribution

\[
\tilde{E}(\tilde{g}, \lambda) = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \pi(S(\gamma)^{-1}) \tilde{\tau}(\tilde{g})
\]
first introduced on line (2.34). In this section we compute the Whittaker distributions of \( \tilde{E}(\tilde{g}, \lambda) \).

The Eisenstein series may be split into six parts depending on the Bruhat cell in which \( \gamma \) resides. Let

\[
\tilde{E}_w(\tilde{g}, \lambda) = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma \cap (NwB)} \pi(S(\gamma)^{-1}) \tilde{\tau}(\tilde{g}).
\]

Thus,

\[
\tilde{E}(\tilde{g}, \lambda) = \sum_{w \in W} \tilde{E}_w(\tilde{g}, \lambda).
\]

As usual, the Fourier coefficient computation is accomplished by addressing each \( \tilde{E}_w \) individually.

The subsections 4.1-4.6 contain the Fourier coefficient computations. For each Bruhat cell the computations are similar, so we will only include the details for \( B, Nw_\alpha, w_{\alpha_2}B, \) and \( Nw_1B \). Let \( \psi_{m_1, m_2} \left( \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right) = e^{2\pi i (m_1 x + m_2 y)} \). When \( m_1 \) and \( m_2 \) are fixed the subscript will be suppressed. In the computations that follow, \( f \in \tilde{V}_{\lambda, \tau}^\infty \) and thus may be paired against elements of \( \tilde{V}_{\lambda, \phi}^{-\infty} \) as described in Section 2.9. In particular, \( f \) may be paired against \( \tilde{E} \).

The Whittaker distribution is defined to be

\[
\int_{\Gamma_\infty \setminus N} \tilde{E}(n\tilde{g}, \lambda) \psi_{m_1, m_2}(n) dn.
\]

In what follows the parameters \( m_1, m_2, \) and \( \lambda \) will be suppressed.

4.1. Bruhat Cell: \( B \).

**Proposition 41.** If \( f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in \tilde{V}_{\lambda, \tau}^\infty \), then

\[
\langle f(\tilde{g}), \int_{\Gamma_\infty \setminus N} \tilde{E}_{id}(n\tilde{g}) \psi(n) dn \rangle_{\lambda, \phi} = \int_{\Gamma_\infty \setminus N} \psi(n) f_1((w_\ell, 1))
\]

This distribution is nonzero if and only if \( m_1 = m_2 = 0 \).

Proof: The distribution \( \tilde{E}_{id}(\tilde{g}) = \tilde{\tau}(\tilde{g}) \) is left \( N \)-invariant. As it is being integrated over a character on \( N \) it will be nonzero precisely when the character is trivial. Now the result follows from the definition of \( \tilde{\tau} \).
4.2. Bruhat Cell: \( Nw_{\alpha_1}B \). Recall the definition of \( K_\kappa(n;4c) \) from line (2.15) and let \( w = \left( \left( \begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right) \right) \). We begin by computing the effect of each summand of \( \tilde{E}_{w_{\alpha_1}} \) on a test vector.

**Proposition 42.** Let \( \gamma \in Nw_{\alpha_1}B \) with Plücker coordinates \((0,0,-1,0,4B_2,C_2)\) and let \( f = [f_1,f_2] \in \tilde{V}_{-\lambda,\tau,-1} \). Then:

If \( B_2 > 0 \),

\[
\langle f, \pi(\gamma^{-1})\tau \rangle_{\lambda,\phi} = -i \left( \frac{B_2}{-C_2} \right) |4B_2|^{-1 - \lambda_2 + \lambda_3} f_2\left( wn(0, \frac{-C_2}{4B_2}, 0) \right).
\]

If \( B_2 < 0 \),

\[
\langle f, \pi(\gamma^{-1})\tau \rangle_{\lambda,\phi} = \left( \frac{B_2}{-C_2} \right) |4B_2|^{-1 - \lambda_2 + \lambda_3} f_2\left( wn(0, \frac{-C_2}{4B_2}, 0) \right).
\]

Now we compute the Fourier coefficients of \( \tilde{E}_{w_{\alpha_1}} \).

**Proposition 43.** Let \( f = [f_1,f_2] \in \tilde{V}_{-\lambda,\tau,-1} \).

If \( m_2 \neq 0 \), then

\[
\langle f(\tilde{g}), \int_{\Gamma_\infty \setminus N} \tilde{E}_{w_{\alpha_1}}(n\tilde{g})\psi(n)dn \rangle_{\lambda,\phi} = 0.
\]

If \( m_2 = 0 \), then

\[
\langle f(\tilde{g}), \int_{\Gamma_\infty \setminus N} \tilde{E}_{w_{\alpha_1}}(n\tilde{g})\psi(n)dn \rangle_{\lambda,\phi} = \left( \sum_{B_2 \in \mathbb{Z}_0} (4B_2)^{-1 - \lambda_2 + \lambda_3} K_{-1}(m_1; 4B_2) \right) \times \int_{\mathbb{R}} f_2(n(x,0,0)w)e^{-2\pi im_1x} dx. \tag{4.6}
\]

This L-function is studied by Shimura [20] and Bate [2]. Bate introduces the L-function in Section 4 and provides an explicit description of it in terms of quadratic L-functions in Proposition 5.2.

4.3. Bruhat Cell: \( Nw_{\alpha_2}B \). Let \( w = \left( \left( \begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right) \right) \) \( \left( \left( \begin{array}{cc} -1 & 1 \\ 1 & -1 \end{array} \right) \right) \). We begin by computing the effect of each summand of \( \tilde{E}_{w_{\alpha_2}} \) on a test vector.

**Proposition 44.** Let \( \gamma \in Nw_{\alpha_2}B \) with Plücker coordinates \((0,4B_1,C_1,0,0,-1)\) and let \( f = [f_1,f_2] \in \tilde{V}_{-\lambda,\tau,-1} \). Then:

If \( B_1 > 0 \),

\[
\langle f, \pi(\gamma^{-1})\tau \rangle_{\lambda,\phi} = i \left( \frac{-B_1}{-C_1} \right) |4B_1|^{-1 - \lambda_1 + \lambda_2} f_1\left( wn\left( \frac{C_1}{4B_1}, 0, 0 \right) \right).
\]

If \( B_1 < 0 \),

\[
\langle f, \pi(\gamma^{-1})\tau \rangle_{\lambda,\phi} = \left( \frac{-B_1}{-C_1} \right) |4B_1|^{-1 - \lambda_1 + \lambda_2} f_2\left( wn\left( \frac{C_1}{4B_1}, 0, 0 \right) \right).
\]

Now we compute the Fourier coefficients of \( \tilde{E}_{w_{\alpha_2}} \).
Proposition 45. Let $f = [f_1, f_2] \in \tilde{V}_{\infty}^{\infty,-1,1,1,\phi^{-1}}$. If $m_1 \neq 0$, then

$$\langle f(\tilde{g}), \int_{\Gamma_{\infty}\backslash N} \tilde{E}_{w_{\alpha_2}}(n\tilde{g})\psi(n)dn \rangle_{\lambda,\phi} = 0.$$ 

If $m_1 = 0$, then

$$\langle f(\tilde{g}), \int_{\Gamma_{\infty}\backslash N} \tilde{E}_{w_{\alpha_2}}(n\tilde{g})\psi(n)dn \rangle_{\lambda,\phi}$$

$$= i \left( \sum_{B_1 \in \mathbb{Z}_{>0}} (4B_1)^{-1-\lambda_1+\lambda_2} \left( \frac{K_1(m_2;4B_1)}{2} \right) \right) \times \int_{\mathbb{R}} f_1(n(0,y,0)(w,1)) \psi^{-1}(n(0,y,0))dy$$

$$+ i \left( \sum_{B_1 \in \mathbb{Z}_{>0}} (4B_1)^{-1-\lambda_1+\lambda_2} \left( \frac{K_1(m_2;4B_1)}{2} \right) \right) \times \int_{\mathbb{R}} f_2(n(0,y,0)(w,1)) \psi^{-1}(n(0,y,0))dy.$$

4.4. Bruhat Cell: $Nw_{\alpha_1}w_{\alpha_2}B$. Let $w = \left( \begin{array}{cc} 1 & -1 \\ 1 & 1 \end{array} \right)$, then $\left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right) w_{\ell}, 1$. We begin by computing the effect of each summand of $\tilde{E}_{w_{\alpha_1}w_{\alpha_2}}$ on a test vector.

Proposition 46. Let $\gamma \in Nw_{\alpha_1}w_{\alpha_2}B$ with Plücker coordinates $(0,4B_1,C_1,4A_2,4B_2,C_2)$ and $f = [f_1, f_2] \in \tilde{V}_{\infty}^{\infty,-1,1,1,\phi^{-1}}$, then:

If $B_1, A_2 > 0$, then

$$\langle f, \pi(\gamma^{-1})\tilde{\tau} \rangle_{\lambda,\phi} = |4B_1|^{1-\lambda_1+\lambda_2} |4A_2|^{1-\lambda_2+\lambda_3} (-1) \left( \frac{A_2/B_1}{-C_2} \right) \left( \frac{-B_1}{-C_1} \right) f_2 \left( \frac{C_2}{4A_2}, \frac{B_2}{A_2} \right).$$

If $B_1 > 0, A_2 < 0$, then

$$\langle f, \pi(\gamma^{-1})\tilde{\tau} \rangle_{\lambda,\phi} = \left( \frac{A_2/B_1}{-C_2} \right) \left( \frac{-B_1}{-C_1} \right) f_2 \left( \frac{C_2}{4A_2}, \frac{B_2}{A_2} \right).$$

If $B_1, A_2 < 0$, then

$$\langle f, \pi(\gamma^{-1})\tilde{\tau} \rangle_{\lambda,\phi} = |4B_1|^{1-\lambda_1+\lambda_2} |4A_2|^{1-\lambda_2+\lambda_3} (-i) \left( \frac{A_2/B_1}{-C_2} \right) \left( \frac{-B_1}{-C_1} \right) f_2 \left( \frac{C_2}{4A_2}, \frac{B_2}{A_2} \right).$$

Proof: By the definition of $\tilde{\tau}$ we have

$$\langle f, \pi(\gamma^{-1})\tilde{\tau} \rangle_{\lambda,\phi} = \langle \pi(\gamma)f, \tilde{\tau} \rangle_{\lambda,\phi} = \int_{\mathbb{R}^3} \pi(\gamma) f \left( \begin{array}{c} w_{\ell} \\ 1 \end{array} \right) \left( \begin{array}{cc} 1 & x \\ 0 & 1 \end{array} \right) \tilde{\tau} \left( \begin{array}{c} w_{\ell} \\ 1 \end{array} \right) \left( \begin{array}{cc} 1 & z \\ 0 & 1 \end{array} \right) dxdydz.$$ (4.7)
By the definition of $\tilde{\tau}$, line (4.7) is equal to the first component of the $f((\gamma, s(\gamma))^{-1}(w_1, 1))$. We can compute this quantity using Proposition 8, the formula for $s(\gamma)$ from Proposition 13, and the definition of $\phi$ from equation (2.29).

Now we compute the Fourier coefficients of $\tilde{E}_{w_{\alpha_1}w_{\alpha_2}}$.

**Proposition 47.** Let $f = [f_1 \ f_2] \in \tilde{V}_{-\lambda, \tau^{-1}}^\infty$. If $m_1 \neq 0$, then

$$\langle f(\tilde{g}), \int_{\Gamma_\infty \setminus \mathbb{N}} \tilde{E}_{w_{\alpha_1}w_{\alpha_2}}(n\tilde{g})\psi(n)dn\rangle_{\lambda, \phi} = 0.$$  

If $m_1 = 0$, then

$$\langle f(\tilde{g}), \int_{\Gamma_\infty \setminus \mathbb{N}} \tilde{E}_{w_{\alpha_1}w_{\alpha_2}}(n\tilde{g})\psi(n)dn\rangle_{\lambda, \phi} = \left(1 - i \right) \sum_{B_1 \in \mathbb{Z}_{>0}} |4B_1|^{-1-\lambda_1+\lambda_3} \left( \frac{K_{-1}(m_2; 4B_1) - K_1(m_2; 4B_1)}{2} \right) \times \left( 2^{-2(-1-\lambda_2+\lambda_3)} \frac{\zeta(2\lambda_2 - 2\lambda_3)}{\zeta(2\lambda_2 - 2\lambda_3 + 1)} \right) \int_{\mathbb{R}^2} f_1(n(0, y, z)) e^{-2\pi im_2 y} dy dz$$

$$+ \left(1 + i \right) \sum_{B_1 \in \mathbb{Z}_{>0}} |4B_1|^{-1-\lambda_1+\lambda_3} \left( \frac{K_{-1}(m_2; 4B_1) + K_1(m_2; 4B_1)}{2} \right) \times \left( 2^{-2(-1-\lambda_2+\lambda_3)} \frac{\zeta(2\lambda_2 - 2\lambda_3)}{\zeta(2\lambda_2 - 2\lambda_3 + 1)} \right) \int_{\mathbb{R}^2} f_2(n(0, y, z)) e^{-2\pi im_2 y} dy dz.$$

**Proof:** We begin with a bit of notation. In what follows, the summation over $\gamma$ (as opposed to over $\gamma'$) will be described by Pücker coordinates for the set $\Gamma_\infty \setminus \Gamma$ such that the matrix representative $\gamma$ is in $N w_{\alpha_1} w_{\alpha_2} B$. In terms of the Plücker coordinates, this set can be described as

$$S_{\alpha_1\alpha_2} = \{(0, 4B_1, C_1, 4A_2, 4B_2, C_2) \in \mathbb{Z}^6 | B_1, A_2 \neq 0, (B_1, C_1) = 1, 4B_1 B_2 = -C_1 A_2, (\frac{A_2}{B_1}, C_2) = 1, C_j \equiv -1 \pmod{4}\}.$$

(4.8)

The summation over $\gamma'$ will consist of distinct representatives of the double coset space $\Gamma_\infty \setminus \Gamma / (\Gamma_\infty \cap \omega \Gamma_\infty \omega^{-1})$. Proposition 2 shows that this double coset space is in bijection with the subset of $S_{\alpha_1\alpha_2}$ such that $0 \leq C_1 < |4B_1|$ and $0 \leq C_2 < |4A_2|$. The switch between $\gamma$ and $\gamma'$ occurs when the integral over $\Gamma_\infty \setminus \mathbb{N}$ is unfolded.

Let $n$ denote $n(x, y, z)$, and define $\epsilon_\gamma \in \{\pm 1, \pm i\}$ and $f_\gamma = f_j$, with $j \in \{1, 2\}$, such that

$$\langle \pi(n)f, \pi(S(\gamma))^{-1}\tilde{\tau})\rangle_{\lambda, \phi} = |4B_1|^{-1-\lambda_1+\lambda_2}|4A_2|^{-1-\lambda_2+\lambda_3} \epsilon_\gamma s(\gamma) f_j \left( n^{-1} w_0(0, \frac{C_2'}{A_2'}, \frac{B_2'}{A_2'}) \right),$$

in accordance with Proposition 46. Observe that $\epsilon_\gamma$, $s(\gamma)$, and $j$ only depend on the double coset $\Gamma_\infty \setminus \Gamma / (\Gamma_\infty \cap \omega \Gamma_\infty \omega^{-1})$. By Proposition 46,

$$\langle f(\tilde{g}), \int_{\Gamma_\infty \setminus \mathbb{N}} \tilde{E}_{w_{\alpha_1}w_{\alpha_2}}(n\tilde{g})\psi(n)dn\rangle_{\lambda, \phi} = \int_{\Gamma_\infty \setminus \mathbb{N}} \sum_{\gamma \in \Gamma_\infty \setminus \Gamma / (N w_{\alpha_1} w_{\alpha_2} B)} \langle f, \pi((\gamma n)^{-1})\tilde{\tau}\rangle_{\lambda, \phi} \psi(n)dn$$
Perform the change of variables

\[ n \mapsto n(0, \frac{B_2}{A_2}, \frac{C_2}{A_2}) \]

and recall that \( \frac{B_2}{A_2} = -\frac{C_1}{2A_1} \) to see that

\[ \frac{C_2}{A_2} \]

is trivial when one entry is an element of \( N \), we have

\[ wn(0, \frac{C_2}{A_2}, -\frac{C_1}{2A_1}) = n(0, \frac{B_2}{A_2}, -\frac{C_1}{2A_1})w. \]

Thus,

\[ (4.9) = \int_{\Gamma_\infty \backslash N} \sum_{\gamma} |4B_1|^{-1-\lambda_1+\lambda_2}|4A_2|^{-1-\lambda_2+\lambda_3}e_{\gamma}S(\gamma)f_{\gamma}\left(\left(n(0, \frac{-B_2}{A_2}, \frac{C_2}{4A_2})n^{-1}w\right) \psi(n) dn. \]

(4.9)

Since the 2-cocycle \( \sigma \) is trivial when one entry is an element of \( N \), we have \( wn(0, \frac{C_2}{A_2}, \frac{C_1}{2A_2}) = n(0, \frac{B_2}{A_2}, \frac{C_1}{2A_2})w. \)

Thus,

\[ \int_{\Gamma_\infty \backslash N} \sum_{\gamma} |4B_1|^{-1-\lambda_1+\lambda_2}|4A_2|^{-1-\lambda_2+\lambda_3}e_{\gamma}S(\gamma)f_{\gamma}\left(\left(n(0, \frac{-B_2}{A_2}, \frac{C_2}{4A_2})n^{-1}w\right) \psi(n) dn. \]

(4.10)

The next step requires unfolding the integral. An element \( \gamma \in Nw_{\alpha_1}w_{\alpha_2}B \cap \Gamma \) representing a coset in \( \Gamma_\infty \backslash SL(3, \mathbb{Z}) \) can be factored as \( \gamma = \gamma'\gamma'' \), where \( \gamma' \in Nw_{\alpha_1}w_{\alpha_2}B \cap \Gamma \) representing a double coset in \( \Gamma_\infty \backslash \Gamma/\langle \Gamma_\infty \cap w\Gamma_\infty w^{-1} \rangle \) and \( \gamma'' \in \langle \Gamma_\infty \cap w\Gamma_\infty w^{-1} \rangle \cong \mathbb{Z}^2 \). If a set of distinct representatives of the double coset space is identified, then the factorization is unique. The integral over \( \Gamma_\infty \backslash N \) is unfolded with respect to the sum over \( \Gamma_\infty \cap w\Gamma_\infty w^{-1} \); the result is an integral over \( \Gamma_\infty \backslash w\Gamma_\infty w^{-1} \backslash N \). Thus

\[ (4.10) = \int_{\Gamma_\infty \backslash N} \sum_{\gamma'} \sum_{j,k \in \mathbb{Z}} |4B_1|^{-1-\lambda_1+\lambda_2}|4A_2|^{-1-\lambda_2+\lambda_3}e_{\gamma'}S(\gamma') \]

\[ \times f_{\gamma'}\left(\left(n(0, \frac{-B_2}{A_2}, \frac{C_2}{4A_2})n(x, y + z)w\right) \psi(n(x, y + z)) dn. \]

(4.11)

Perform the change of variables \( n \mapsto nn(0, \frac{B_2}{A_2}, \frac{C_2}{4A_2}) \) and recall that \( \frac{B_2}{A_2} = -\frac{C_1}{2A_1} \) to see that

\[ (4.11) = \sum_{\gamma'} |4B_1|^{-1-\lambda_1+\lambda_2}|4A_2|^{-1-\lambda_2+\lambda_3}e_{\gamma'}S(\gamma')f_{\gamma'}\left(\left(n(0, \frac{-B_2}{A_2}, \frac{C_2}{4A_2})nn(0, \frac{B_2}{A_2}, -\frac{C_2}{4A_2})n^{-1}w\right) \psi(n) dn. \]

(4.12)

The previous change of variables is advantageous as elements of the form \( nn'n^{-1} \) in \( N \) can be written in the form \( n'z \), where \( z \in [N, N] \) is an element in the derived subgroup. In this case,

\[ n(0, \frac{B_2}{A_2}, \frac{C_2}{4A_2})n(x, y) = n(0, \frac{B_2}{A_2}, -\frac{C_2}{4A_2})n(x, y) = n(0, 0, \frac{B_2}{A_2}x). \]

Thus another change of variables \( n(x, y) \mapsto n(x, y) \) pushes the element of the derived subgroup into the character where it contributes trivially. Thus

\[ (4.12) = \sum_{\gamma'} |4B_1|^{-1-\lambda_1+\lambda_2}|4A_2|^{-1-\lambda_2+\lambda_3}e_{\gamma'}S(\gamma')f_{\gamma'}\left(\left(n(0, 0, \frac{B_2}{A_2}x)n^{-1}w\right) \psi(n(x, y, z)) dn. \]

(4.13)
We now perform the change of variables $n \mapsto n^{-1}$ and apply the right $N_-$-invariance to remove the $x$-variable to see that

\begin{equation}
(4.14) = \sum_{\gamma'} |4B_1|^{-1-\lambda_1+\lambda_2} |4A_2|^{-1-\lambda_2+\lambda_3} e^{2\pi i (m_2 \frac{C_1}{4B_1})} \int_{(\Gamma_\infty \cap \mathbb{Z} w \Gamma_{\infty N} w^{-1}) \setminus N} \epsilon_{\gamma'} s(\gamma') f_{\gamma'} (n^{-1} w) \psi (n) \, dn.
\end{equation}

If $m_1 \neq 0$, then this expression is 0, as can be seen by integrating over $x$. Thus suppose $m_1 = 0$. In this case,

\begin{equation}
\int_{\Gamma_\infty \setminus N} \sum_{\gamma} \langle f, \pi((\gamma n)^{-1}) \tau \rangle \psi(n) \, dn = \sum_{\gamma'} |4B_1|^{-1-\lambda_1+\lambda_2} |4A_2|^{-1-\lambda_2+\lambda_3} e^{2\pi i (m_2 \frac{C_1}{4B_1})} \int_{\mathbb{R}^2} \epsilon_{\gamma'} s(\gamma') f_{\gamma'} (n(0, y, z) w) e^{2\pi i (m_2 y)} \, dydz.
\end{equation}

We have just shown that

\begin{align*}
&\langle f(\tilde{g}), \int_{\Gamma_\infty \setminus N} \tilde{E}_{w_{a_1} w_{a_2}} (n \tilde{g}) \psi(n) \rangle_{\lambda, \phi} \\
&= \sum_{B_1 \in \mathbb{Z}_{>0}} |4B_1|^{-1-\lambda_1+\lambda_2} |4A_2|^{-1-\lambda_2+\lambda_3} \sum_{C_1 \text{ (mod } 4B_1)} \sum_{C_2 \text{ (mod } 4A_2)} (\text{mod } 4) \left( \frac{A_2 / B_1}{-C_1} \right) \left( \frac{B_1}{-C_1} \right) e^{2\pi i m_2 \left( \frac{C_1}{4B_1} \right)} \\
&\times \int_{\mathbb{R}^2} f_2 (n(0, y, z) w) e^{-2\pi i m_2 y} \, dydz \\
&+ \sum_{B_1 \in \mathbb{Z}_{>0}} |4B_1|^{-1-\lambda_1+\lambda_2} |4A_2|^{-1-\lambda_2+\lambda_3} \sum_{C_1 \text{ (mod } 4B_1)} \sum_{C_2 \text{ (mod } 4A_2)} (\text{mod } 4) \left( -i \right) \left( \frac{A_2 / B_1}{-C_2} \right) \left( \frac{B_1}{-C_2} \right) e^{2\pi i m_2 \left( \frac{-C_1}{4B_1} \right)} \\
&\times \int_{\mathbb{R}^2} f_2 (n(0, y, z) w) e^{-2\pi i m_2 y} \, dydz \\
&+ \sum_{B_1 \in \mathbb{Z}_{<0}} |4B_1|^{-1-\lambda_1+\lambda_2} |4A_2|^{-1-\lambda_2+\lambda_3} \sum_{C_1 \text{ (mod } 4B_1)} \sum_{C_2 \text{ (mod } 4A_2)} (\text{mod } 4) \left( -i \right) \left( \frac{A_2 / B_1}{-C_2} \right) \left( \frac{B_1}{-C_2} \right) e^{2\pi i m_2 \left( \frac{-C_1}{4B_1} \right)} \\
&\times \int_{\mathbb{R}^2} f_1 (n(0, y, z) w) e^{-2\pi i m_2 y} \, dydz \\
&+ \sum_{B_1 \in \mathbb{Z}_{<0}} |4B_1|^{-1-\lambda_1+\lambda_2} |4A_2|^{-1-\lambda_2+\lambda_3} \sum_{C_1 \text{ (mod } 4B_1)} \sum_{C_2 \text{ (mod } 4A_2)} (\text{mod } 4) \left( \frac{A_2 / B_1}{-C_2} \right) \left( \frac{B_1}{-C_2} \right) e^{2\pi i m_2 \left( \frac{-C_1}{4B_1} \right)}
\end{align*}
This is followed by the change of variables $k = \frac{-A}{B}$.

(4.17) = \sum_{B_1 \in \mathbb{Z}_{>0}} |4B_1|^{-1-\lambda_1+\lambda_2} |4A_2|^{-1-\lambda_2+\lambda_3} \sum_{C_1 \equiv 1 \pmod{4B_1}} \left( \frac{4k}{-C_2} \right) \left( \frac{B_1}{C_1} \right) e^{2\pi i m_2 C_1 \frac{k}{4B_1}}.

Since the character $\left( \frac{4k}{-C_2} \right)$ only depends on $C_2$ modulo $4k$ we have

(4.17) = \sum_{B_1 \in \mathbb{Z}_{>0}} |4B_1|^{-1-\lambda_1+\lambda_3} |4k|^{-1-\lambda_2+\lambda_3} \sum_{C_1 \equiv 1 \pmod{4B_1}} \left( \frac{4k}{-C_2} \right) \left( \frac{B_1}{C_1} \right) e^{2\pi i m_2 C_1 \frac{k}{4B_1}}.

Now the sum may be factored.

(4.19) = \left( \sum_{B_1 \in \mathbb{Z}_{>0}} |4B_1|^{-1-\lambda_1+\lambda_3} \sum_{C_1 \equiv 1 \pmod{4B_1}} \left( \frac{B_1}{C_1} \right) e^{2\pi i m_2 C_1 \frac{k}{4B_1}} \right) \times \left( \sum_{k \in \mathbb{Z}_{>0}} |4k|^{-1-\lambda_2+\lambda_3} \sum_{C_2 \equiv 1 \pmod{4k}} \left( \frac{4k}{-C_2} \right) \right)

The first factor in line (4.20) can be rewritten as

\sum_{B_1 \in \mathbb{Z}_{>0}} |4B_1|^{-1-\lambda_1+\lambda_3} \sum_{C_1 \equiv 1 \pmod{4B_1}} \left( \frac{B_1}{C_1} \right) e^{2\pi i m_2 C_1 \frac{k}{4B_1}}
\[= (-i) \sum_{B_1 \in \mathbb{Z}_{>0}} |4B_1|^{-1-\lambda_1+\lambda_3} \left( \frac{K_{-1}(m_2; 4B_1) - K_1(m_2; 4B_1)}{2} \right). \quad (4.21)\]

As for the second factor in line (4.20), the exponential sum will be nonzero precisely when \( k \) is a square. Thus by the identity in line (2.14)

\[\sum_{k \in \mathbb{Z}_{>0}} |4k|^{-1-\lambda_2+\lambda_3} \sum_{C_2 \equiv -1 \text{(mod 4)}} \left( \frac{4k}{-C_2} \right) = 2^{-2(-1-\lambda_2+\lambda_3)} \frac{\zeta(2\lambda_2 - 2\lambda_3)}{\zeta_2(2\lambda_2 - 2\lambda_3 + 1)}. \quad (4.22)\]

By combining lines (4.21) and (4.22), we see that

\[(4.16) = (-i) \left( \sum_{B_1 \in \mathbb{Z}_{>0}} |4B_1|^{-1-\lambda_1+\lambda_3} \left( \frac{K_{-1}(m_2; 4B_1) - K_1(m_2; 4B_1)}{2} \right) \right) \times 2^{-2(-1-\lambda_2+\lambda_3)} \frac{\zeta(2\lambda_2 - 2\lambda_3)}{\zeta_2(2\lambda_2 - 2\lambda_3 + 1)}. \quad (4.23)\]

4.5. **Bruhat Cell:** \( Nw_{\alpha_2}w_{\alpha_1}B \). Let \( w = \left( \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, 1 \right) = \left( \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right) w_\ell, 1 \). We begin by computing the effect of each summand of \( \hat{E}_{w_{\alpha_2}w_{\alpha_1}} \) on a test vector.

**Proposition 48.** Let \( \gamma \in Nw_{\alpha_2}w_{\alpha_1}B \) with Plücker coordinates \( (4A_1, 4B_1, C_1, 0, 4B_2, C_2) \) and \( f = [f_1 f_2] \in \hat{V}_{-\lambda,1}^{\infty} \). Then:

If \( A_1, B_2 > 0, \) then

\[\langle f, \pi(\gamma^{-1}) \bar{\tau} \rangle_{\lambda,\phi} = |4A_1|^{-1-\lambda_1+\lambda_2} |4B_2|^{-1-\lambda_2+\lambda_3} \left( \frac{-A_1/B_2}{-C_1} \right) \left( \frac{B_2}{-C_2} \right) f_2 \left( \text{wn} \left( \frac{C_1}{4A_1}, 0, -\frac{B_1}{A_1} \right) \right).\]

If \( A_1 > 0, B_2 < 0, \) then

\[\langle f, \pi(\gamma^{-1}) \bar{\tau} \rangle_{\lambda,\phi} = |4A_1|^{-1-\lambda_1+\lambda_2} |4B_2|^{-1-\lambda_2+\lambda_3} \left( \frac{-A_1/B_2}{-C_1} \right) \left( \frac{B_2}{-C_2} \right) f_2 \left( \text{wn} \left( \frac{C_1}{4A_1}, 0, -\frac{B_1}{A_1} \right) \right).\]

If \( A_1, B_2 < 0, \) then

\[\langle f, \pi(\gamma^{-1}) \bar{\tau} \rangle_{\lambda,\phi} = |4A_1|^{-1-\lambda_1+\lambda_2} |4B_2|^{-1-\lambda_2+\lambda_3} \left( \frac{-A_1/B_2}{-C_1} \right) \left( \frac{B_2}{-C_2} \right) f_1 \left( \text{wn} \left( \frac{C_1}{4A_1}, 0, -\frac{B_1}{A_1} \right) \right).\]

If \( A_1 < 0, B_2 > 0, \) then

\[\langle f, \pi(\gamma^{-1}) \bar{\tau} \rangle_{\lambda,\phi} = |4A_1|^{-1-\lambda_1+\lambda_2} |4B_2|^{-1-\lambda_2+\lambda_3} \left( \frac{-A_1/B_2}{-C_1} \right) \left( \frac{B_2}{-C_2} \right) f_1 \left( \text{wn} \left( \frac{C_1}{4A_1}, 0, -\frac{B_1}{A_1} \right) \right).\]

Now we compute the Fourier coefficients of \( \hat{E}_{w_{\alpha_2}w_{\alpha_1}} \).

**Proposition 49.** Let \( f = [f_1 f_2] \in \hat{V}_{-\lambda,1}^{\infty} \).

If \( m_2 \neq 0, \) then

\[\langle f(\bar{g}), \int_{\Gamma_\infty N} \hat{E}_{w_{\alpha_2}w_{\alpha_1}}(n\bar{g})\psi(n)dn \rangle_{\lambda,\phi} = 0.\]
If \( m_2 = 0 \), then
\[
\langle f(\tilde{g}), \int_{\Gamma_\infty \setminus N} \tilde{E}_{w, z_1^1}(n\tilde{g})\psi(n) dn \rangle_{\lambda, \phi}
= i \left( 2^{-2(-1-\lambda_1 + \lambda_2)} \frac{\zeta(2\lambda_1 - 2\lambda_2)}{\zeta_2(2\lambda_1 - 2\lambda_2 + 1)} \right) \left( \sum_{B_2 \in \mathbb{Z}_{<0}} |4B_2|^{-1-\lambda_1 + \lambda_3} K_{-1}(m_1; 4B_2) \right)
\times \int_{\mathbb{R}^2} f_1(n(x, 0, z) \left( \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, 1 \right)) \psi^{-1}(n(x, 0, z)) dx dz
+ \left( 2^{-2(-1-\lambda_1 + \lambda_2)} \frac{\zeta(2\lambda_1 - 2\lambda_2)}{\zeta_2(2\lambda_1 - 2\lambda_2 + 1)} \right) \left( \sum_{B_2 \in \mathbb{Z}_{>0}} |4B_2|^{-1-\lambda_1 + \lambda_3} K_1(m_1; 4B_2) \right)
\times \int_{\mathbb{R}^2} f_2(n(x, 0, z) \left( \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, 1 \right)) \psi^{-1}(n(x, 0, z)) dx dz.
\]

4.6. Bruhat Cell: Let \( w \in Nw_\ell B \) with Plücker coordinates \((4A_1, 4B_1, C_1, 4A_2, 4B_2, C_2)\) and \( f = [f_1, f_2] \in \tilde{V}_{\lambda, \tau_{\phi^{-1}}}^\infty \), then:

If \( A_1, A_2 > 0 \), then
\[
\langle f, \pi(\gamma^{-1})\tilde{\tau} \rangle_{\lambda, \phi} = |4A_1|^{-1-\lambda_1 + \lambda_2} |4A_2|^{-1-\lambda_2 + \lambda_3} s(\gamma) f_1 \left( \varpi \left( \frac{B_1}{A_1}, -\frac{B_2}{A_2}, \frac{C_2}{4A_2} \right) \right).
\]

If \( A_1 > 0, A_2 < 0 \), then
\[
\langle f, \pi(\gamma^{-1})\tilde{\tau} \rangle_{\lambda, \phi} = |4A_1|^{-1-\lambda_1 + \lambda_2} |4A_2|^{-1-\lambda_2 + \lambda_3} i s(\gamma) f_1 \left( \varpi \left( \frac{B_1}{A_1}, -\frac{B_2}{A_2}, \frac{C_2}{4A_2} \right) \right).
\]

If \( A_1 < 0, A_2 > 0 \), then
\[
\langle f, \pi(\gamma^{-1})\tilde{\tau} \rangle_{\lambda, \phi} = |4A_1|^{-1-\lambda_1 + \lambda_2} |4A_2|^{-1-\lambda_2 + \lambda_3} (-i) s(\gamma) f_2 \left( \varpi \left( \frac{B_1}{A_1}, -\frac{B_2}{A_2}, \frac{C_2}{4A_2} \right) \right).
\]

If \( A_1, A_2 < 0 \), then
\[
\langle f, \pi(\gamma^{-1})\tilde{\tau} \rangle_{\lambda, \phi} = |4A_1|^{-1-\lambda_1 + \lambda_2} |4A_2|^{-1-\lambda_2 + \lambda_3} (-1) s(\gamma) f_2 \left( \varpi \left( \frac{B_1}{A_1}, -\frac{B_2}{A_2}, \frac{C_2}{4A_2} \right) \right).
\]

Recall that the formula for \( s(\gamma) \) in this case is contained in Theorem 12.

Now we compute the Fourier coefficients of \( \tilde{E}_{w_\ell} \).

Proposition 51. Let \( f = [f_1, f_2] \in \tilde{V}_{\lambda, \tau_{\phi^{-1}}}^\infty \).
\[
\langle f(\tilde{g}), \int_{\Gamma_\infty \setminus N} \tilde{E}_{w_\ell}(n\tilde{g})\psi(n) dn \rangle_{\lambda, \phi}
\]

\[
\sum_{A_1>0 \atop A_2>0} |4A_1|^{1-\lambda_1+\lambda_2} |4A_2|^{1-\lambda_2+\lambda_3} (\Sigma(A_1, A_2; -m_1, m_2) + i \Sigma(A_1, -A_2; -m_1, m_2))
\times \int_{\mathbb{R}^3} f_1 \left( n(x, y, z) \left( \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, 1 \right) \right) \psi^{-1}(n(x, y, z)) dx dy dz
\]
\[
+ \sum_{A_1>0 \atop A_2>0} |4A_1|^{1-\lambda_1+\lambda_2} |4A_2|^{1-\lambda_2+\lambda_3} (-1)(\Sigma(A_1, A_2; -m_1, -m_2) + i \Sigma(A_1, -A_2; -m_1, -m_2))
\times \int_{\mathbb{R}^3} f_2 \left( n(x, y, z) \left( \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, 1 \right) \right) \psi^{-1}(n(x, y, z)) dx dy dz
\]

The terms \(\Sigma(A_1, A_2; m_1, m_2)\) satisfy a twisted multiplicativity in \(A_1\) and \(A_2\), stated in Proposition 20; a form of twisted multiplicativity in \(m_1\) and \(m_2\), stated in Proposition 21; and the symmetries of Proposition 19. Thus the computation of \(\Sigma(A_1, A_2; m_1, m_2)\) may be reduced to that of \(\Sigma(p^k, \mu p^t; p^s, p^{t'})\). Formulas for these expressions may be found in Section 3.7.

**Proof:** In what follows, the summation over \(\gamma\) will be described by Pucker coordinates for the set \(\Gamma_\infty \setminus \Gamma \cap (N w t B)\). The summation over \(\gamma'\) will consist of elements of the double coset space \(\Gamma_\infty \setminus \Gamma \cap (N w t B)/\Gamma_\infty\).

Let \(n\) denote \(n(x, y, z)\). Define \(\epsilon_\gamma \in \{\pm 1, \pm i\}\) and \(f_\gamma = f_j\), where \(j \in \{1, 2\}\) be defined so that

\[
\langle \pi(n)f, \pi(S(\gamma)^{-1})\pi_\gamma \rangle_{\lambda, \phi} = |4A_1|^{1-\lambda_1+\lambda_2} |4A_2|^{1-\lambda_2+\lambda_3} \epsilon_\gamma s(\gamma) f_\gamma \left( \frac{B_1}{A_1}, \frac{-B_2}{A_2}, \frac{C_2}{4A_2} \right),
\]

in accordance with Proposition 50. Observe that \(\epsilon_\gamma, s(\gamma),\) and \(j\) only depend on the double coset \(\Gamma_\infty \setminus \Gamma/\Gamma_\infty\).

Begin with the change of variables \(n \mapsto n^{-1}\) and then apply Proposition 50 to see that

\[
\langle f, \int_{\Gamma_\infty \setminus N} E_w(n g) \psi(n) dn \rangle_{\lambda, \phi}
\]

\[
= \int_{N/\Gamma_\infty} \sum_\gamma |4A_1|^{1-\lambda_1+\lambda_2} |4A_2|^{1-\lambda_2+\lambda_3} \epsilon_\gamma s(\gamma) f_\gamma \left( n wn \left( \frac{B_1}{A_1}, \frac{-B_2}{A_2}, \frac{C_2}{4A_2} \right) \right) \psi^{-1}(n) dn. \quad (4.24)
\]

Since the 2-cocycle \(\sigma\) is trivial on \(N\) we have \(wn \left( \frac{B_1}{A_1}, \frac{-B_2}{A_2}, \frac{C_2}{4A_2} \right) = n \left( \frac{-B_1}{A_1}, \frac{B_2}{A_2}, \frac{C_2}{4A_2} \right) w\). Thus

\[
(4.24) = \int_{N/\Gamma_\infty} \sum_\gamma |4A_1|^{1-\lambda_1+\lambda_2} |4A_2|^{1-\lambda_2+\lambda_3} \epsilon_\gamma s(\gamma) f_\gamma \left( n n \left( \frac{-B_1}{A_1}, \frac{B_2}{A_2}, \frac{C_2}{4A_2} \right) w \right) \psi^{-1}(n) dn. \quad (4.25)
\]

The next step is to unfold the integral. For details recall the analogous step in Proposition 47. In this case,

\[
(4.25) = \sum_\gamma |4A_1|^{1-\lambda_1+\lambda_2} |4A_2|^{1-\lambda_2+\lambda_3} \epsilon_\gamma s(\gamma') \int_N f_\gamma \left( n n \left( \frac{-B_1}{A_1}, \frac{B_2}{A_2}, \frac{C_2}{4A_2} \right) w \right) \psi^{-1}(n) dn. \quad (4.26)
\]
Finally, apply the change of variables \( n \mapsto nn(\frac{-p_1}{A_1}, \frac{-p_2}{A_2}, \frac{C_2}{4\lambda_2})^{-1} \) to get

\[
(4.26) = \sum_{\gamma'} |4A_1|^{-1-\lambda_1+\lambda_2} |4A_2|^{-1-\lambda_2+\lambda_3} \epsilon_{\gamma'} S(\gamma') e^{2\pi i (m_1 \frac{p_1}{A_1} + m_2 \frac{p_2}{A_2})} \int_{\mathbb{N}} f(nw) \psi^{-1}(n) dn.
\]

We can complete the proof by applying Proposition 19 and by using Proposition 50 to evaluate \( \epsilon_{\gamma}, S(\gamma) \), and \( f_{\gamma} \). In particular,

\[
\sum_{\gamma'} |4A_1|^{-1-\lambda_1+\lambda_2} |4A_2|^{-1-\lambda_2+\lambda_3} S(\gamma') e^{2\pi i (m_1 \frac{p_1}{A_1} + m_2 \frac{p_2}{A_2})} = \sum_{A_1, A_2 \in \mathbb{Z}_{\neq 0}} |4A_1|^{-1-\lambda_1+\lambda_2} |4A_2|^{-1-\lambda_2+\lambda_3} \sum(A_1, A_2; -m_1, m_2). \quad (4.28)
\]

\[\square\]

4.7. Constant Term. The computations of the previous section can be specialized \((m_1 = m_2 = 0)\) to produce the constant term.

**Theorem 52.** Let \( f = [f_1] \in \tilde{V}_{-\lambda, T, \phi}^{-\infty} \).

\[
\langle f(\tilde{g}), \int_{\tilde{\Gamma}_{-\lambda, T, \phi}} E(n\tilde{g}) dn \rangle_{\lambda, \phi} = f_1((w_\ell, 1))
\]

\[
+ (1 + i)2^{-2(1+\lambda_2-\lambda_3)} \zeta^2(2\lambda_2 - 3\lambda_3) \int_{\mathbb{R}} f_2(n(x, 0, 0) \left( \begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right) w_\ell, 1) dx
\]

\[
+ 2^{-2(1+\lambda_1-\lambda_3)} \zeta^2(2\lambda_1 - 3\lambda_3) \int_{\mathbb{R}} (if_1 - f_2) \left( n(0, y, 0) \left( \begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right) w_\ell, 1) dy
\]

\[
+ (1 + i)2^{-2(1+\lambda_1-\lambda_3)}2^{-2(1+\lambda_2-\lambda_3)} \zeta^2(2\lambda_1 - 3\lambda_3) \zeta^2(2\lambda_2 - 3\lambda_3) \times \int_{\mathbb{R}^2} (f_1 + f_2) \left( n(0, y, z) \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) w_\ell, 1) dydz
\]

\[
+ (1 - i)2^{-2(1+\lambda_1-\lambda_3)}2^{-2(1+\lambda_2-\lambda_3)} \zeta^2(2\lambda_1 - 3\lambda_3) \zeta^2(2\lambda_2 - 3\lambda_3) \times \int_{\mathbb{R}^2} (-f_1 + f_2) \left( n(x, 0, z) \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) w_\ell, 1) dxdz
\]

\[
+ i2^{-2(1+\lambda_1-\lambda_2)}2^{-2(1+\lambda_2-\lambda_3)} \zeta^2(2\lambda_1 - 3\lambda_3) \zeta^2(2\lambda_2 - 3\lambda_3) \times \int_{\mathbb{R}^2} (f_1 - f_2) \left( n(x, y, z) \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) w_\ell, 1) dx dy dz
\]

This formula can also be written more succinctly as follows: Let

\[
F_{\ell, 2}(\lambda_1, \lambda_2, \lambda_3) = 2^{2+2(\lambda_1-\lambda_3)} (1 - 2^{-2(\lambda_1-\lambda_2)} - 2^{-2(\lambda_2-\lambda_3)} + 6(2^{-2(\lambda_1-\lambda_3+1)})).
\]
as a rational function using the computations of Proposition 3.6. Show us that if \( \Sigma(2^i) \) is a prime. We can write line (4.29) as a product of \( \Sigma(p^k, \pm p^\ell; 0, 0) \) for \( p \) prime. The computations of Subsection 3.6 show us that if \( \Sigma(p^k, \pm p^\ell; 0, 0) \neq 0 \), then \( k \) and \( \ell \) are even and \( \epsilon = -1 \). Thus the twisted multiplicativity of Proposition 17 is a true multiplicity and

\[
\sum_{A_1 > 0 \atop A_2 > 0} |4A_1|^{a_1 + \lambda_2} |4A_2|^{1 - \lambda_2 + \lambda_3} \Sigma(A_1, -A_2; 0, 0)
\]
The result follows once we identify the Euler product as a product of zeta functions. □

References

[1] Banks, William D.; Levy, Jason; Sepanski, Mark R. Block-Compatible Metaplectic Cocycles, *Journal für die reine und angewandte Mathematik*, Volume: 507 (1999)

[2] Bate, Brandon. Automorphic Distributions and the Functional Equation for Metaplectic Eisenstein Distributions.

[3] Bump, Daniel. *Automorphic Forms on GL(3, R)*. Springer-Verlag, New York, 1984.

[4] Brubaker, Ben; Bump, Daniel; Friedberg, Solomon. Weyl group multiple Dirichlet series, Eisenstein series and crystal bases. *Ann. of Math.* (2) 173 (2011), no. 2, 1081–1120.

[5] Brubaker, B.; Bump, D.; Friedberg, S.; Hoffstein, J., Weyl group multiple Dirichlet series. III. Eisenstein series and twisted unstable $A_r$. *Ann. of Math.* (2) 166 (2007), no. 1, 293-316.

[6] Hecke, Erich. Vorlesungen über die Theorie der algebraischen Zahlen. (German) Second edition of the 1923 original, with an index. Chelsea Publishing Co., Bronx, N.Y., 1970.

[7] Humphreys, J. E., *Introduction to Lie Algebras and Representation Theory*. Springer-Verlag, New York, 1972.

[8] Iwaniec, H; Kowalski, E., *Analytic Number Theory*. American Mathematical Society, Rhode Island, 2004.

[9] Karasiewicz, Edmund. A Splitting into the Double Cover of $SL(3, \mathbb{R})$. Submitted.

[10] Kazhdan, D. A.; Patterson, S. J. Metaplectic forms, *Inst. Hautes Études Sci. Publ. Math.*, 59, 1984, 35–142.

[11] Knapp, A. W., *Representation Theory of Semisimple Groups*. Princeton University Press, Princeton, 1986.

[12] Kubota, T. On automorphic functions and the reciprocity law in a number field. *Lectures in Mathematics*, Department of Mathematics, Kyoto University, No. 2, Kinokuniya Book-Store Co., Ltd., Tokyo, 1969.

[13] Lang, S., SL(2, R) Springer-Verlag, New York, 1985.

[14] Maass, H. Konstruktion ganzer modulformen halbzahliger dimension mit $\vartheta$-multiplikatoren in einer und zwei variablen, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg, 12, 1937, 1, 133-162.

[15] Miller, S. D., Guide for the Metaplexed. Unpublished Notes.

[16] Miller, S. D.; Schmid, W., Automorphic Distributions, L-functions and Voronoi Summation for GL(3). *Ann. of Math.*, 164 (2006), 423-488.

[17] Miller, Stephen D.; Schmid, Wilfried The Archimedean theory of the exterior square L-functions over $\mathbb{Q}$. *J. Amer. Math. Soc.* 25 (2012), no. 2, 465-506.

[18] Schmid, Wilfried. Automorphic Distributions for SL(2, R). Conference Moshé Flato 1999, Vol. I (Dijon), 345-387, Math. Phys. Stud., 21, Kluwer Acad. Publ., Dordrecht, 2000.

[19] Selberg, Atle. Discontinuous groups and harmonic analysis. 1963 Proc. Internat. Congr. Mathematicians (Stockholm, 1962) pp. 177–189 Inst. Mittag-Leffler, Djursholm

[20] Shimura, Goro. On the holomorphy of certain Dirichlet series. *Proc. London Math. Soc.* (3) 31 (1975), no. 1, 79-98.

[21] Treves, F. *Topological Vector Spaces, Distributions and Kernels*. Academic Press, New York, 1967.

[22] Weil, André. Sur certains groupes d’opérateurs unitaires. (French) *Acta Math.* 111 1964 143–211.

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41