Regular Maps on Cartesian Products and Disjoint Unions of Manifolds

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Abstract

A map from a manifold to a Euclidean space is said to be $k$-regular if the image of any distinct $k$ points are linearly independent. For $k$-regular maps on manifolds, lower bounds of the dimension of the ambient Euclidean space have been extensively studied. In this paper, we study the lower bounds of the dimension of the ambient Euclidean space for $2$-regular maps on Cartesian products of manifolds. As corollaries, we obtain the exact lower bounds of the dimension of the ambient Euclidean space for $2$-regular maps and $3$-regular maps on spheres as well as on some real projective spaces. Moreover, generalizing the notion of $k$-regular maps, we study the lower bounds of the dimension of the ambient Euclidean space for maps with certain non-degeneracy conditions from disjoint unions of manifolds into Euclidean spaces.

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1 Introduction

Let $M$ be a smooth manifold and let $\mathbb{F}$ denote the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$. For any $k \geq 2$, a map $f : M \rightarrow \mathbb{F}^N$ is called (real or complex) $k$-regular if for any distinct $k$ points $x_1, \ldots, x_k$ in $M$, $f(x_1), \ldots, f(x_k)$ are linearly independent in $\mathbb{F}^N$. For simplicity, a real $k$-regular map is also called a $k$-regular map. Any (real or complex) $(k+1)$-regular map is (real or complex) $k$-regular, and any $k$-regular map is injective.

Throughout this paper, all maps and functions are assumed to be continuous. We use $S^m$ to denote the $m$-sphere and use $\mathbb{R}P^m$, $\mathbb{C}P^m$ and $\mathbb{H}P^m$ to denote the real, complex and quaternionic projective spaces consisting of real lines through the origin in $\mathbb{R}^{m+1}$, complex lines through the origin in $\mathbb{C}^{m+1}$ and quaternionic lines through the origin in $\mathbb{H}^{m+1}$ respectively. Moreover, we assume $m \geq 2$.

In 1957, The study of $k$-regular maps was initiated by K. Borsuk [3]. Later, the problem attracted additional attention because of its connection with the theory of Cebyshev approximation (cf. [9] and [18, pp. 237-242]):

**Theorem [Haar-Kolmogorov-Rubinstein].** Suppose $M$ is compact and $f_1, \ldots, f_n$ are linearly independent real-valued functions on $M$. Let $F$ be the linear space spanned by $f_1, \ldots, f_n$ over $\mathbb{R}$. Then $(f_1, \ldots, f_n)$ is a $k$-regular map from $M$ to $\mathbb{R}^n$ if and only if for any real-valued function $g$ on $M$, the dimension of the set $\{f \in F \mid \sup_{x \in M} |g(x) - f(x)| = m\}$ is smaller than or equal to $n - k$, where $m$ is the infimum of $\sup_{x \in M} |g(x) - f(x)|$ for all $f$ in $F$.

From 1970’s to nowadays, $k$-regular maps on manifolds have been extensively studied. In 1978, some $k$-regular maps on the plane were constructed by F.R. Cohen and D. Handel [5]:

1. **Example 1.2.** The map from $\mathbb{C}$ to $\mathbb{R}^{2k-1}$ sending $z$ to $(1, z, z^2, \ldots, z^{k-1})$ is $k$-regular.

And in 2016, some $3$-regular maps on surfaces were constructed by P. Blagojević, W. Lück and G. Ziegler [1]:

1. **Example 2.6-(2).** Let $i$ be the standard embedding from $S^m$ to $\mathbb{R}^{m+1}$ and $i$ the constant map with image 1. Then the map $(1, i)$ from $S^m$ to $\mathbb{R}^{m+2}$ is $3$-regular.

On the other hand, generalizing the results of M.E. Chisholm [1] in 1979 and F.R. Cohen and D. Handel [5] in 1978, the lower bounds of $N$ for $k$-regular maps of Euclidean spaces into $\mathbb{R}^N$ were studied by P. Blagojević, W. Lück and G. Ziegler [1] in 2016:

1. **Theorem 2.1.** Let $\alpha(k)$ denote the number of ones in the dyadic expansion of $k$. If there exists a $k$-regular map of $\mathbb{R}^m$ into $\mathbb{R}^N$, then $N \geq m(k - \alpha(k)) + \alpha(k)$.

While the lower bounds of $N$ for complex $k$-regular maps of Euclidean spaces into $\mathbb{C}^N$ were studied by P. Blagojević, F.R. Cohen, W. Lück and G. Ziegler [2] in 2015:
Theorem 1.1 (Main Theorem I). Let \( p \) be an odd prime. If there exists a complex \( p \)-regular map of \( \mathbb{R}^m \) into \( \mathbb{C}^N \), then \( N \geq \frac{m+1}{2}(p-1) + 1 \).

Theorem 5.3. Let \( p \) be an odd prime and let \( \alpha_p(k) \) be the sum of coefficients in the \( p \)-adic expansion of \( k \). If \( m \) is a power of \( p \) and there exists a complex \( k \)-regular map of \( \mathbb{C}^m \) into \( \mathbb{C}^N \), then \( N \geq m(k - \alpha_p(k)) + \alpha_p(k) \).

Compared with \( k \)-regular maps on Euclidean spaces, it is more difficult to study the dimension of the ambient Euclidean space for \( k \)-regular maps on general manifolds. In 1996, D. Handel considered the lower bounds of \( N \) for \( 2k \)-regular maps of manifolds into \( \mathbb{R}^N \). In 2011, R. Karasev pointed out a gap of the proof p. 1611.

The family of \( k \)-regular maps on disjoint unions of manifolds is a particular family of \( k \)-regular maps on general manifolds. Problems concerning \( k \)-regular maps on disjoint unions of manifolds attracted attention since 1980’s and was firstly considered by D. Handel:

Theorem 2.4. Let \( M_1, \ldots, M_k \) be closed, connected manifolds of dimensions \( n_1, \ldots, n_k \) respectively. Suppose for \( 1 \leq i \leq k \), the \( q_i \)-th dual Stiefel-Whitney class of \( M_i \) is non-zero. If there exists a \( 2k \)-regular map of the disjoint union \( \coprod_{i=1}^{k} M_i \) into \( \mathbb{R}^N \), then \( N \geq 2k + \sum_{i=1}^{k} (n_i + q_i) \).

In this paper, our first aim is to give a lower bound of \( N \) for \( 2 \)-regular maps on Cartesian products of spheres and real, complex and quaternionic projective spaces into \( \mathbb{R}^N \). We prove the next theorem.

Theorem 1.1 (Main Theorem I). Suppose there is a \( 2 \)-regular map

\[
f : \prod_{i=1}^{k_1} S^{m_{1,i}} \times \prod_{j=1}^{k_2} \mathbb{R}P^{m_{2,j}} \times \prod_{t=1}^{k_3} \mathbb{C}P^{m_{3,t}} \times \prod_{l=1}^{k_4} \mathbb{H}P^{m_{4,l}} \rightarrow \mathbb{R}^N.
\]

Then

\[
N \geq \sum_{i=1}^{k_1} m_{1,i} + \sum_{j=1}^{k_2} 2^{\lfloor \log_2 m_{2,j} \rfloor + 1} + \sum_{t=1}^{k_3} 2^{\lfloor \log_2 m_{3,t} \rfloor + 2} + \sum_{l=1}^{k_4} 2^{\lfloor \log_2 m_{4,l} \rfloor + 3} - k_2 - 2k_3 - 4k_4 + 2.
\]

The following corollaries follow from Theorem 1.1

Corollary 1.2. Let \( 2^i \leq m < 2^{i+1} \), \( i \geq 1 \).

(a). If there exists a \( 2 \)-regular map of \( \mathbb{R}P^m \) into \( \mathbb{R}^N \), then \( N \geq 2^{i+1} + 1 \).

(b). If there exists a \( 2 \)-regular map of \( \mathbb{C}P^m \) into \( \mathbb{R}^N \), then \( N \geq 2^{i+2} \).

(c). If there exists a \( 2 \)-regular map of \( \mathbb{H}P^m \) into \( \mathbb{R}^N \), then \( N \geq 2^{i+3} - 2 \).

Corollary 1.3. The following are equivalent

(a). there exists a \( 3 \)-regular map of \( S^m \) into \( \mathbb{R}^N \),

(b). there exists a \( 2 \)-regular map of \( S^m \) into \( \mathbb{R}^N \),

(c). \( N \geq m + 2 \).

Remark 1.4. Corollary 1.3 gives a counter-example of 1.4 Theorem 4.4].

Corollary 1.5. Let \( m = 2^i + 1 \), \( i \geq 1 \). Then the following are equivalent

(a). there exists a \( 3 \)-regular map of \( \mathbb{R}P^m \) into \( \mathbb{R}^N \),

(b). there exists a \( 2 \)-regular map of \( \mathbb{R}P^m \) into \( \mathbb{R}^N \),

(c). \( N \geq 2m - 1 \).

Besides considering \( k \)-regular maps, we generalize the notion of \( k \)-regular maps on disjoint unions of manifolds and consider a weaker non-degeneracy condition for maps on disjoint unions of manifolds. We give the following definition.

Definition 1.6. Let \( M_1, \ldots, M_n \) be manifolds. A map

\[
f : \prod_{i=1}^{n} M_i \rightarrow \mathbb{R}^N
\]
is called \((\text{real or complex})\) \((M_1, k_1; M_2, k_2; \cdots; M_n, k_n)\)-regular if for any distinct points \(x_{i,1}, x_{i,2}, \cdots, x_{i,k_i} \in M_i, i = 1, 2, \cdots, n,\) their images

\[
\prod_{i=1}^{n} \{f(x_{i,1}), f(x_{i,2}), \cdots, f(x_{i,k_i})\}
\]

are linearly independent in \(\mathbb{R}^N\). In particular, a real \((M_1, k_1; M_2, k_2; \cdots; M_n, k_n)\)-regular map is called \((M_1, k_1; M_2, k_2; \cdots; M_n, k_n)\)-regular for short.

With the help of Definition 1.6, we re-obtain \([11, \text{Theorem 2.4}]\) in Remark 6.6.

The second aim of this paper is to give a lower bound of \(N\) for the regular maps defined in Definition 1.6 from disjoint unions of planes, spheres and real, complex and quaternionic projective spaces into \(\mathbb{R}^N\). We prove the next theorem.

**Theorem 1.7** (Main Theorem II). Suppose we have a \((\mathbb{R}^2, 2^{d_1}; \cdots; \mathbb{R}^2, 2^{d_{k_0}}; S^{m_1}, 2; \cdots; S^{m_{k_1}}, 2; \mathbb{R}P^{m_{k_2}}, 2; \cdots; \mathbb{C}P^{m_{k_3}}, 2; \mathbb{H}P^{m_{k_4}}, 2; \cdots; \mathbb{H}P^{m_{k_4}}, 2)\)-regular map

\[
f: \prod_{k_0}^{k_1} \left(\mathbb{R}^2\right) \prod_{i=1}^{k_2} (\mathbb{R}P^{m_{i+1}}) \prod_{j=1}^{k_3} (\mathbb{C}P^{m_{j+1}}) \prod_{l=1}^{k_4} (\mathbb{H}P^{m_{l+1}}) \to \mathbb{R}^N.
\]

Then

\[
N \geq \sum_{s=1}^{k_1} 2^{d_s+1} + \sum_{i=1}^{k_2} m_{i+1} + \sum_{j=1}^{k_3} 2^{\log_2 m_{j+1}} + 1
\]

\[
+ \sum_{i=1}^{k_1} 2^{\log_2 m_{i+1}} + 2 + \sum_{j=1}^{k_3} 2^{\log_2 m_{j+1}+3} - k_0 + 2k_1 + k_2 - 2k_4.
\]

The next corollary follows from Theorem 1.7

**Corollary 1.8.** The following are equivalent:

(a). there exists a \((\mathbb{R}^2, 2^{d_1}; \cdots; \mathbb{R}^2, 2^{d_{k_0}}; S^{m_1}, 2; \cdots; S^{m_{k_1}}, 2; \mathbb{R}P^{2^{m_1}+1}, 2; \cdots; \mathbb{R}P^{2^{m_{k_2}+1}}, 2)\)-regular map

\[
f: \prod_{k_0}^{k_1} \left(\mathbb{R}^2\right) \prod_{i=1}^{k_2} (\mathbb{R}P^{2^{m_{i+1}+1}}) \to \mathbb{R}^N;
\]

(b). there exists a \((\mathbb{R}^2, 2^{d_1}; \cdots; \mathbb{R}^2, 2^{d_{k_0}}; S^{m_1}, 3; \cdots; S^{m_{k_1}}, 3; \mathbb{R}P^{2^{m_1}+1}, 3; \cdots; \mathbb{R}P^{2^{m_{k_2}+1}}, 3)\)-regular map

\[
f: \prod_{k_0}^{k_1} \left(\mathbb{R}^2\right) \prod_{i=1}^{k_2} (\mathbb{R}P^{2^{m_{i+1}+1}}) \to \mathbb{R}^N;
\]

(c).

\[
N \geq \sum_{s=1}^{k_1} 2^{d_s+1} + \sum_{i=1}^{k_2} m_{i+1} + \sum_{j=1}^{k_3} 2^{u_{j+1}} - k_0 + 2k_1 + k_2.
\]

As by-products, we give some lower bounds of \(N\) for the complex regular maps from Euclidean spaces, spheres and complex projective spaces into \(\mathbb{C}^N\). In the last section of this paper, we give a generalization of \([2, \text{Theorem 5.2}]):

- For an odd prime \(p\), if there exists a complex \(np\)-regular map from \(\mathbb{R}^m\) into \(\mathbb{C}^N\), then \(N \geq n(\lceil \frac{m+1}{2} \rceil (p-1) + 1)\).

Moreover, we prove the following:

- If there exists a complex \(2\)-regular map of \(S^m\) into \(\mathbb{C}^N\), then \(N \geq \lceil \frac{m}{2} \rceil + 2\).

- For \(m \geq 4\), if there exists a complex \(2\)-regular map of \(\mathbb{C}P^m\) into \(\mathbb{C}^N\), then \(N \geq 2m\).
The paper is organized as follows. In Section 2, we give some examples of $k$-regular maps as well as the regular maps on disjoint unions of manifolds defined in Definition 1.6. In Section 3, we review the cohomology of Grassmannians. In Section 4, we Firstly review the Stiefel-Whitney classes and Chern classes of the canonical vector bundle over configuration spaces. Then we prove some auxiliary lemmas. In Section 5, we prove Theorem 1.7. In Section 6, we give an obstruction for the regular maps on disjoint unions of manifolds defined in Definition 1.6. In Section 7, we prove Theorem 1.7. In Section 8, we prove the lower bounds of $N$ (listed above) for complex regular maps from Euclidean spaces, spheres and complex projective spaces into $\mathbb{C}^N$.

The main results of this paper are Theorem 1.7 and Theorem 1.11.

2 Examples of regular maps

With the help of [1, Example 2.6-(2)], if there exists an embedding of $M$ into $\mathbb{R}^n$, then composed with an embedding of $\mathbb{R}^n$ into $S^n$ and a 3-regular map of $S^n$ into $\mathbb{R}^{n+2}$, we obtain a 3-regular map of $M$ into $\mathbb{R}^{n+2}$. Let $M$ be $\mathbb{R}P^m$. By applying [3] Theorem 4.1, [15] Theorem 5.2, Theorem 5.7 and [17] Theorem 5], we have the following example.

Example 2.1. There exist 3-regular maps of $\mathbb{R}P^m$ into $\mathbb{R}^N$ for the cases listed in Table 1.

| $m = 8q + 3$ or $8q + 5$, $q > 0$ | $N \geq 2m - \min\{5, \alpha(q)\}$ |
| $m = 8q + 1$, $q > 0$ | $N \geq 2m - \min\{7, \alpha(q)\} + 2$ |
| $m = 32q + 7$, $q > 0$ | $N \geq 2m - 6$ |
| $m = 8q + 7$, $q > 1$ | $N \geq 2m - 5$ |
| $m \equiv 3 \mod 8$, $m \geq 19$ | $N \geq 2m - 4$ |
| $m \equiv 1 \mod 4$, $m \neq 2^i + 1$ | $N \geq 2m - 2$ |
| $m = 4q + i$, $i = 0$ or $2$, $q \neq 2^j$ or 0 | $N \geq 2m - 1$ |
| $m = 2^j + 1$, $j \geq 2$ | $N \geq 2m - 1$ |
| $m = 2^j + 2$, $j \geq 3$ | $N \geq 2m$ |

Let $f_i : M_i \rightarrow \mathbb{P}^n$, be a (real or complex) $k_i$-regular map for $i = 1, 2, \cdots, n$. Then we have a (real or complex) $(M_1, k_1; M_2, k_2; \cdots; M_n, k_n)$-regular map

$$f : \prod_{i=1}^{n} M_i \rightarrow \prod_{i=0}^{n} \mathbb{P}N_i \cong \prod_{i=1}^{n} \mathbb{P}^N$$

given by $f(x) = (0, \cdots, 0, f_i(x), 0, \cdots, 0)$ for $x \in M_i, i = 1, 2, \cdots, n$. Consequently, with the help of [5] Example 1.2, [3] Example 2.6-(2) and Example 2.1, we have the following example.

Example 2.2. There exists a $(\mathbb{R}^2, b_1; \cdots; \mathbb{R}^2, b_{k_1}; S^{m_1}, 3; \cdots; S^{m_k}, 3; \mathbb{R}P^{2^{m_1}+1}, 3; \cdots; \mathbb{R}P^{2^{m_k}+1}, 3)$-regular map

$$f : \left(\prod_{i=1}^{k_1} \prod_{j=1}^{k_2} \mathbb{R}^2 \right) \prod_{i=1}^{k_1} S^{m_i} \prod_{j=1}^{k_2} \mathbb{R}P^{2^{m_j}+1} \rightarrow \mathbb{R}^N$$

for any

$$N \geq 2(\sum_{s=1}^{k_2} b_s) + \sum_{i=1}^{k_1} m_i + \sum_{j=1}^{k_2} 2^{m_j+1} - k_0 + 2k_1 + k_2.$$  

3 Cohomology of Grassmannians

Let $A$ be a ring and $a \in A \setminus A^*$ where $A^*$ is the set of invertible elements of $A$. The height of $a$ is defined as the smallest positive integer $n$ such that $a^{n+1} = 0$, or infinity, if such an $n$ does not exist (cf. [2]).

Given positive integers $k$ and $n$ with $n \geq k$, let $G_k(\mathbb{P}^{n+1})$ be the (real or complex) Grassmannian consisting of $k$-dimensional subspaces of $\mathbb{P}^{n+1}$ and $G_k(\mathbb{F}^\infty)$ the direct limit of $G_k(\mathbb{P}^{n+1})$. The canonical inclusion of $\mathbb{P}^{n+1}$ into $\mathbb{F}^\infty$ as the first $(n+1)$-coordinates induces an inclusion

$$i : G_k(\mathbb{P}^{n+1}) \rightarrow G_k(\mathbb{F}^\infty).$$  

(1)
Case 1: $\mathbb{F} = \mathbb{R}$. It is known that $H^*(G_k(\mathbb{R}^\infty); \mathbb{Z}_2) = \mathbb{Z}_2[w_1, w_2, \ldots, w_k]$ where $w_i$ is the $i$-th universal Stiefel-Whitney class with $|w_i| = i$. And

$$H^*(G_k(\mathbb{R}^{n+1}); \mathbb{Z}_2) = \mathbb{Z}_2[w_1, w_2, \ldots, w_k]/(\bar{w}_n-k+2, \bar{w}_n-k+3, \ldots, \bar{w}_{n+1})$$  \hspace{1cm} (2)

where $\bar{w}_j$ is defined as the $j$-th degree term in the expansion of $(1+w_1+\cdots+w_k)^{-1}$ and $(\bar{w}_{n-k+2}, \bar{w}_{n-k+3}, \ldots, \bar{w}_{n+1})$ is the ideal generated by $\bar{w}_{n-k+2}, \bar{w}_{n-k+3}, \ldots, \bar{w}_{n+1}$. The inclusion $[\bar{c}]$ induces an epimorphism on mod 2 cohomology.

Case 2: $\mathbb{F} = \mathbb{C}$. It is known that $H^*(G_k(\mathbb{C}^\infty); \mathbb{Z}) = \mathbb{Z}[c_1, c_2, \ldots, c_k]$ where $c_i$ is the $i$-th universal Chern class with $|c_i| = 2i$. And

$$H^*(G_k(\mathbb{C}^{n+1}); \mathbb{Z}) = \mathbb{Z}[c_1, c_2, \ldots, c_k]/(\bar{c}_n-k+2, \bar{c}_n-k+3, \ldots, \bar{c}_{n+1})$$  \hspace{1cm} (3)

where $\bar{c}_j$ is defined as the $2j$-th degree term in the expansion of $(1+c_1+\cdots+c_k)^{-1}$ and $(\bar{c}_{n-k+2}, \bar{c}_{n-k+3}, \ldots, \bar{c}_{n+1})$ is the ideal generated by $\bar{c}_{n-k+2}, \bar{c}_{n-k+3}, \ldots, \bar{c}_{n+1}$. The inclusion $[\bar{c}]$ induces an epimorphism on integral cohomology.

**Proposition 3.1.** Let $n \geq k \geq 1$. Then in $[\mathbb{C}]$,

$$\text{height}(c_1) = \dim_{\mathbb{C}} G_k(\mathbb{C}^{n+1}).$$

In particular, if $h(n)$ denotes the height of $c_1$ in $H^*(G_2(\mathbb{C}^{n+1}); \mathbb{Z})$, then $h(n) = 2n - 2$.

**Proof.** The Plücker embedding embeds $G_k(\mathbb{C}^{n+1})$ as a subvariety into $\mathbb{C}P^n$, where $r$ is the complex dimension of the $k$-th exterior power of $\mathbb{C}^{n+1}$. It follows that $c_1$ is the pull-back a generator of $H^2(\mathbb{C}P^n; \mathbb{Z})$, which is an ample class. Since the complex dimension of $G_k(\mathbb{C}^{n+1})$ is $k(n+1-k)$, it follows that $c_1^{k(n+1-k)+1}$ is non-zero and $c_1^{k(n+1-k)+1}$ is zero. The assertion follows. \hfill $\square$

**Remark 3.2.** For the backgrounds of the above proof, we may refer to [7].

### 4 Characteristic classes of the canonical vector bundle over configuration spaces

Let $\Sigma_k$ be the permutation group of order $k$ and let the $k$-th configuration space of $M$ be

$$F(M, k) = \{(x_1, \ldots, x_k) \in M \times \cdots \times M \mid \text{for any } i \neq j, x_i \neq x_j\}.$$  \hspace{1cm} (4)

For any $\sigma \in \Sigma_k$, let $\sigma$ act on $F(M, k)$ by

$$(x_1, \ldots, x_k)\sigma = (x_{\sigma(1)}, \ldots, x_{\sigma(k)})$$

and act on $\mathbb{F}^k$ by

$$\sigma(r_1, \ldots, r_k) = (r_{\sigma^{-1}(1)}, \ldots, r_{\sigma^{-1}(k)}).$$

Then we have a space $F(M, k)/\Sigma_k$, called the $k$-th unordered configuration space of $M$, and an $O(\mathbb{F}^k)$-bundle

$$\xi_{M,k}^F : \mathbb{F}^k \rightarrow F(M, k) \times_{\Sigma_k} \mathbb{F}^k \rightarrow F(M, k)/\Sigma_k.$$  \hspace{1cm} (5)

We denote the classifying map of $\xi_{M,k}^F$ as

$$h : F(M, k)/\Sigma_k \rightarrow G_k(\mathbb{F}^\infty).$$

We consider a $\Sigma_k$-invariant subspace $W_k^\mathbb{F}$ of $\mathbb{F}^k$ consisting of vectors $(x_1, x_2, \ldots, x_k)$ such that $\sum_{i=1}^k x_i = 0$. Then we have an $O(\mathbb{F}^{k-1})$-bundle

$$\zeta_{M,k}^F : W_k^\mathbb{F} \rightarrow F(M, k) \times_{\Sigma_k} W_k^\mathbb{F} \rightarrow F(M, k)/\Sigma_k.$$  \hspace{1cm} (6)

Let $e_{M,k}^\mathbb{F}$ be the trivial $\mathbb{F}^1$-bundle over $F(M, k)/\Sigma_k$. Then we have an isomorphism of vector bundles

$$\xi_{M,k}^F \cong \zeta_{M,k}^F \oplus e_{M,k}^\mathbb{F}.$$  \hspace{1cm} (7)
For simplicity, we omit the symbol $F$ in (1) if $F = \mathbb{R}$.

For a vector bundle $\eta$, we denote $w_i(\eta)$ and $w(\eta)$ as its $i$-th Stiefel-Whitney class and its total Stiefel-Whitney class respectively, and $\bar{w}(\eta) = 1/w(\eta)$ the dual Stiefel-Whitney class. For a complex vector bundle $\eta^c$, we denote $c_i(\eta^c)$ and $c(\eta^c)$ as its $i$-th Chern class and its total Chern class respectively, and $\bar{c}(\eta^c) = 1/c(\eta^c)$ the dual Chern class. It follows from (1) that

\begin{align}
    w_k(\xi_{M,k}) &= 0, \\
    c_k(\xi^c_{M,k}) &= 0.
\end{align}

By detecting the dual Stiefel-Whitney class and the dual Chern class, the next two lemmas give obstructions for (real and complex) $k$-regular maps.

**Lemma 4.1.** Let $f : M \to \mathbb{R}^N$ be a $k$-regular map on the manifold $M$. If $\bar{w}_1(\xi_{M,k}) \neq 0$, then $N \geq t + k$.

**Lemma 4.2.** Let $f : M \to \mathbb{C}^N$ be a complex $k$-regular map on the manifold $M$. If $\bar{c}_i(\xi^c_{M,k}) \neq 0$, then $N \geq t + k$.

The next lemma gives the height of the first Stiefel-Whitney class of a closed, connected manifold.

**Lemma 4.3.** Let $M$ be a closed, connected $m$-dimensional manifold. Suppose $q$ is the largest integer such that $\bar{w}_q(M) \neq 0$. Then

\begin{align}
    w_1(\xi_{M,2})^{m+q} &\neq 0, \\
    w_1(\xi_{M,2})^{m+q+1} &= 0.
\end{align}

The next corollary follows from Lemma 4.3.

**Corollary 4.4.** Let $M$ be a closed, connected $m$-dimensional manifold. Suppose $q$ is the largest integer such that $\bar{w}_q(M) \neq 0$. Then

\begin{align*}
    \bar{w}_{m+q}(\xi_{M,2}) &\neq 0, \\
    \bar{w}_{m+q+1}(\xi_{M,2}) &= 0.
\end{align*}

**Proof.** With the help of Lemma 4.3

\[ w(\xi_{M,2}) = (1 + w_1(\xi_{M,2}))^{-1} = 1 + \sum_{l=1}^{m+q} (-1)^l w_1(\xi_{M,2})^l. \]

This implies Corollary 4.4.

For a connected manifold $M$, we notice that $F(M,k)$ is connected and the covering map from $F(M,k)$ to $F(M,k)/\Sigma_k$ induces an epimorphism

\[ \pi : \pi_1(F(M,k)/\Sigma_k) \to \Sigma_k. \]

Let $r : \Sigma_k \to \{ \pm 1 \}$ be the sign representation of $\Sigma_k$. Since $\pi$ is surjective and $r$ is non-trivial, the map $r \circ \pi$ is non-trivial. Moreover, it is direct to verify that there is a bijection between $\text{Hom}(\pi_1(F(M,k)/\Sigma_k), \mathbb{Z}_2)$ and $\text{Vect}_k^+(F(M,k)/\Sigma_k)$, and this bijection sends $r \circ \pi$ to the determinant line bundle of $\xi_{M,k}$. Consequently, the determinant line bundle of $\xi_{M,k}$ is non-trivial. Therefore, $\xi_{M,k}$ is non-orientable. The next lemma follows.

**Lemma 4.5.** Let $M$ be a connected manifold. Then $w_1(\xi_{M,k}) \neq 0$.

The next two theorems give the cohomology ring of the second unordered configuration space of complex projective spaces.

**Theorem 4.6.** As a $H^*(G_2(\mathbb{C}^{m+1}); \mathbb{Z}_2)$-module, the cohomology

\[ H^*(F(\mathbb{C}P^m, 2)/\Sigma_2; \mathbb{Z}_2) \]

has $\{1, v, v^2\}$ as a basis. Moreover, the ring structure of (7) is given by $v^3 = e_1v$. Here $v = w_1(\xi_{\mathbb{C}P^m, 2})$.  

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**Theorem 4.7.** [21, Theorem 4.10] As a $H^*(G_2(C^{m+1});\mathbb{Z})$-module, the cohomology
\[
H^*(F(CP^m,2)/\Sigma_2;\mathbb{Z})
\] (8)
has $\{1,u\}$ as generators with $|u|=2$. Moreover, the ring structure of \[21\] satisfies $2u=0$ and $u^2=c_1u$. Here $c_1=c_1(\xi^C_{S,2})$.

In the remaining part of this section, we prove some auxiliary lemmas.

**Lemma 4.8.** For $i \geq 1$, the largest integer $\lambda$ such that $\tilde{w}_\lambda(\xi_{R^2,2}) \neq 0$ is $\lambda = 2^i - 1$.

**Proof.** By [5, Theorem 3.1], we see that $w_{2^i-1}(\xi_{R^2,2}) \neq 0$. It is also given in [5] that $\tilde{w}(\xi_{R^2,2}) = w(\xi_{R^2,2})$. Thus we have $\tilde{w}_{2^i-1}(\xi_{R^2,2}) \neq 0$. On the other hand, by [5, Example 1.2], we see that $\tilde{w}_{2^i}(\xi_{R^2,2}) = 0$. Consequently, $\lambda = 2^i - 1$.

It follows from a geometric observation that $F(S^m,2)/\Sigma_2$ is homotopy equivalent to $\mathbb{R}P^m$. Consequently,
\[
H^*(F(S^m,2)/\Sigma_2;\mathbb{Z}) = \mathbb{Z}_2[u]/(u^{m+1}), \quad |u| = 1;
\]
\[
H^*(F(S^m,2)/\Sigma_2;\mathbb{Z}) = \left\{ \begin{array}{ll}
\mathbb{Z}[x]/(2x,x^m), & |x| = 2, \text{ if } m \text{ is even}, \\
\mathbb{Z}[x]/(2x,x^{m+1}), & |x| = 2, \text{ if } m \text{ is odd}.
\end{array} \right.
\] (9) (10)

**Lemma 4.9.** (a). The largest integer $q$ such that $\tilde{w}_q(S^m) \neq 0$ is $q = 0$; (b). The largest integer $\lambda$ such that $\tilde{w}_\lambda(S^{m-2}) \neq 0$ is $\lambda = m$.

**Proof.** By [16, Corollary 11.15] and [9], we obtain $w(S^m) = 1$. With the help of Corollary 4.4, we have $\lambda = m$.

**Lemma 4.10.** The largest integer $\tau$ such that $\tilde{c}_\tau(\xi^C_{S,2}) \neq 0$ is $\tau = \lceil \frac{m}{2} \rceil$.

**Proof.** Let $(\xi^C_{S,2})_R$ denote the underlying real vector bundle of $\xi^C_{S,2}$. Then
\[
(\xi^C_{S,2})_R \cong (\xi^S_{S,2})^\oplus 2.
\] (11) Moreover, since $u$ is the unique nonzero element in $H^1(F(S^m,2)/\Sigma_2;\mathbb{Z})$, it follows with the help of Lemma 4.7 that
\[
w_1(\xi_{S,2}) = u.
\] (12) It follows from [5, 11] and [12] that
\[
w((\xi^C_{S,2})_R) = (w(\xi^S_{S,2}))^2 = (1+u)^2 = 1+u^2.
\] (13) Since $m \geq 2$, by [9], $u^2 \neq 0$. Consequently, (13) implies $w((\xi^C_{S,2})_R) \neq 1$. Hence the underlying real vector bundle of $\xi^C_{S,2}$ is not trivial and it follows that $\xi^C_{S,2}$ is not trivial as a complex vector bundle. With the help of (14), we see that as a complex line bundle, $\xi^C_{S,2}$ is not trivial. Since the triviality of a complex line bundle is equivalent to the vanishing of its first Chern class, we have
\[
c_1(\xi^C_{S,2}) = c_1(\xi^C_{S,2}) = 0.
\] (14) Since $x$ is the unique nontrivial element in $H^2(F(S^m,2)/\Sigma_2;\mathbb{Z})$, it follows from (14) that
\[
c_1(\xi^C_{S,2}) = x.
\] (15) From (6) and (15) we obtain
\[
\tilde{c}_t(\xi^C_{S,2}) = (-1)^t x^t.
\] (16) By applying (11) to (16), we finish the proof.
Lemma 4.11. For \(2^j \leq m < 2^{j+1}\) and \(j \geq 1\), (a). the largest integer \(q\) such that \(w_q(\mathbb{R}P^m) \neq 0\) is \(q = 2^{j+1} - m - 1\); (b). the largest integer \(\lambda\) such that \(w_\lambda(\xi_{\mathbb{C}P^{m-2}}) \neq 0\) is \(\lambda = 2^{j+1} - 1\).

Proof. We notice that

\[
H^*(\mathbb{R}P^m; \mathbb{Z}_2) = \mathbb{Z}_2[\alpha]/(\alpha^{m+1})
\]

(17)

where \(\alpha\) is a generator of degree 1. Hence by applying (17) to Corollary 11.15, \(w(\mathbb{R}P^m) = (1 + \alpha)^{m+1}\).

It follows from (18) that

\[
w(\mathbb{R}P^m) = (1 + \alpha)^{-m+1}
\]

(18)

\[= 1 + \sum_{i=1}^{\infty} (-m+1) \alpha^i
\]

(19)

\[= 1 + \sum_{i=1}^{m} (-1)^i \left(\frac{m+i}{m}\right) \alpha^i.
\]

On the other hand, let the dyadic expansions of \(m+i\) and \(m\) be

\[m+i = (a_t, a_{t-1}, \ldots, a_1, a_0)_2,
\]

\[m = (b_t, b_{t-1}, \ldots, b_1, b_0)_2
\]

such that \(a_t = 1\) and \(a_i, b_i \in \{0, 1\}\) for all \(i = 0, 1, \ldots, t\). By the Lucas Theorem (cf. [4]), \((m+i)_m\) is odd if and only if for each \(i = 0, 1, \ldots, t\), \(a_i \geq b_i\). Hence for \(2^j \leq m < 2^{j+1}\), the largest integer \(i\) such that \((m+i)_m\) is odd is \(i = 2^{j+1} - m - 1\). Consequently, it follows from (19) that the largest integer \(q\) such that \(w_q(\mathbb{R}P^m) \neq 0\) is \(q = 2^{j+1} - m - 1\). Hence we obtain (a). By applying Corollary 4.11, we obtain (b).

The following two lemmas could be proved analogously with Lemma 4.11.

Lemma 4.12. For \(2^j \leq m < 2^{j+1}\) and \(j \geq 1\), (a). the largest integer \(q\) such that \(w_q(\mathbb{C}P^m) \neq 0\) is \(q = 2^{j+2} - 2m - 2\); (b). the largest integer \(\lambda\) such that \(w_\lambda(\xi_{\mathbb{C}P^{m-2}}) \neq 0\) is \(\lambda = 2^{j+2} - 2\).

Proof. We note that

\[
H^*(\mathbb{C}P^m; \mathbb{Z}_2) = \mathbb{Z}_2[\beta]/(\beta^{m+1})
\]

where \(\beta\) is a generator of degree 2. Applying the same argument as in the proof of Lemma 4.11, we see that the largest \(i\) such that \(w_i(\mathbb{C}P^m) \neq 0\) is \(i = 2^{j+2} - 2m - 2\). With the help of Corollary 4.11, we finish the proof.

Lemma 4.13. For \(2^j \leq m < 2^{j+1}\) and \(j \geq 1\), (a). the largest integer \(q\) such that \(w_q(\mathbb{H}P^m) \neq 0\) is \(q = 2^{j+3} - 4m - 4\); (b). the largest integer \(\lambda\) such that \(w_\lambda(\xi_{\mathbb{H}P^{m-2}}) \neq 0\) is \(\lambda = 2^{j+3} - 4\).

Proof. We note that

\[
H^*(\mathbb{H}P^m; \mathbb{Z}_2) = \mathbb{Z}_2[\delta]/(\delta^{m+1})
\]

where \(\delta\) is a generator of degree 4. Applying the same argument as in the proof of Lemma 4.11, we see that the largest \(i\) such that \(w_i(\mathbb{H}P^m) \neq 0\) is \(i = 2^{j+3} - 4m - 4\). With the help of Corollary 4.11, we finish the proof.

Lemma 4.14. Let \(m \geq 4\) and \(\kappa\) be the largest integer such that \(c_\kappa(\xi_{\mathbb{C}P^{m-2}}) \neq 0\). Then

\[
\kappa \geq 2m - 2.
\]

Proof. By Theorem 4.7 and 4.11 we obtain

\[c_1(\xi_{\mathbb{C}P^{m-2}}) = c_1(\xi_{\mathbb{C}P^{m-2}}) = au + bc_1
\]

(21)

with \(a, b \in \mathbb{Z}\) and \(u^2 = c_1u\). And from 4.11 and (21) we obtain

\[c_i(\xi_{\mathbb{C}P^{m-2}}) = (-1)^i(au + bc_1)^i
\]

(22)

\[= (-1)^i \left(\sum_{i=1}^{m} a^i b^{i-1} c_1^{i-1} u + (-1)^i b^i c_1^i
\]

(22)
On the other hand, let \((\xi^c_{\mathbb{C}P^m,2})_\mathbb{R}\) denote the underlying real vector bundle of \((\xi^c_{\mathbb{C}P^m,2})\). Then
\[
(\xi^c_{\mathbb{C}P^m,2})_\mathbb{R} \cong (\xi_{\mathbb{C}P^m,2})^\oplus 2. \tag{23}
\]
Let \(v = w_1(\xi_{\mathbb{C}P^m,2})\). It follows from (5) and (23) that
\[
w((\xi^c_{\mathbb{C}P^m,2})_\mathbb{R}) = (w(\xi_{\mathbb{C}P^m,2}))^2
= (1 + v)^2
= 1 + v^2. \tag{24}
\]
By Theorem 4.9 \(v^2 \neq 0\). Hence from (24), \(w((\xi^c_{\mathbb{C}P^m,2})_\mathbb{R}) \neq 1\). Hence \((\xi^c_{\mathbb{C}P^m,2})_\mathbb{R}\) is not trivial. Consequently, as a complex vector bundle, \(\xi^c_{\mathbb{C}P^m,2}\) is not trivial. It follows that \(\xi^c_{\mathbb{C}P^m,2}\) is not trivial as a complex line bundle. Therefore, we see that the first Chern class of \(\xi^c_{\mathbb{C}P^m,2}\) is nonzero, i.e. in (21), either \(a \neq 0\) or \(b \neq 0\).

Case 1: \(b \neq 0\).

Then by (22), we have \(\bar{c}_t(\xi^c_{\mathbb{C}P^m,2}) \neq 0\) whenever \(c^t_1 \neq 0\). Noting that \(c^t_1 = 0\) if and only if \(t \geq h(m) + 1\), we conclude that
\[
k \geq h(m). \tag{25}
\]

Case 2: \(b = 0\).

Then \(a \neq 0\) and by (22),
\[
\bar{c}_t(\xi^c_{\mathbb{C}P^m,2}) = (-1)^t u^t u^t
= (-1)^t a^t c^t_1 u. \tag{26}
\]
We note that \(c_1^t - 1 u = 0\) if and only if \(t - 1 \geq h(m) + 1\). Hence by (26),
\[
k = h(m) + 1. \tag{27}
\]

Summarizing both Case 1 and Case 2, it follows from (25) and (27) that \(k(m) \geq h(m)\). With the help of Proposition 3.1 we obtain (20).

5 Proof of Theorem 1.1

Proof of Theorem 1.1. Firstly, we have
\[
w(\prod_{i=1}^{k_1} S^{m_1,i} \times \prod_{j=1}^{k_2} \mathbb{R} P^{m_2,i} \times \prod_{t=1}^{k_3} \mathbb{C} P^{m_3,t} \times \prod_{l=1}^{k_4} \mathbb{H} P^{m_4,l})
= \prod_{i=1}^{k_1} w(S^{m_1,i}) \times \prod_{j=1}^{k_2} w(\mathbb{R} P^{m_2,i}) \times \prod_{t=1}^{k_3} w(\mathbb{C} P^{m_3,t}) \times \prod_{l=1}^{k_4} w(\mathbb{H} P^{m_4,l}).
\]

It follows that
\[
\bar{w}(\prod_{i=1}^{k_1} S^{m_1,i} \times \prod_{j=1}^{k_2} \mathbb{R} P^{m_2,i} \times \prod_{t=1}^{k_3} \mathbb{C} P^{m_3,t} \times \prod_{l=1}^{k_4} \mathbb{H} P^{m_4,l})
= \prod_{i=1}^{k_1} \bar{w}(S^{m_1,i}) \times \prod_{j=1}^{k_2} \bar{w}(\mathbb{R} P^{m_2,i}) \times \prod_{t=1}^{k_3} \bar{w}(\mathbb{C} P^{m_3,t}) \times \prod_{l=1}^{k_4} \bar{w}(\mathbb{H} P^{m_4,l}). \tag{28}
\]

Secondly, it follows from (28) and Lemma 4.9 - Lemma 4.18 that the largest integer \(q\) such that
\[
\bar{w}_q(\prod_{i=1}^{k_1} S^{m_1,i} \times \prod_{j=1}^{k_2} \mathbb{R} P^{m_2,i} \times \prod_{t=1}^{k_3} \mathbb{C} P^{m_3,t} \times \prod_{l=1}^{k_4} \mathbb{H} P^{m_4,l}) \neq 0
\]
By applying the Kunneth formula, we see that for each $i$

\[
q = \sum_{j=1}^{k_2} (2^{[\log_2 m_{2,j}]+1} - m_{2,j} - 1) + \sum_{t=1}^{k_3} (2^{[\log_2 m_{3,t}]+2} - 2m_{3,t} - 2) + \\
\sum_{l=1}^{k_4} (2^{[\log_2 m_{4,l}]+3} - 4m_{4,l} - 4)\\
= \sum_{j=1}^{k_2} (2^{[\log_2 m_{2,j}]+1} - m_{2,j}) + \sum_{t=1}^{k_3} (2^{[\log_2 m_{3,t}]+2} - 2m_{3,t}) + \\
+ \sum_{l=1}^{k_4} (2^{[\log_2 m_{4,l}]+3} - 4m_{4,l}) - k_2 - 2k_3 - 4k_4.\tag{29}
\]

Since $\prod_{i=1}^{k_1} \mathbb{S}^{m_{1,i}} \times \prod_{j=1}^{k_2} \mathbb{R}^{m_{2,j}} \times \prod_{t=1}^{k_3} \mathbb{C}^{m_{3,t}} \times \prod_{l=1}^{k_4} \mathbb{H}^{m_{4,l}}$ is a closed and connected manifold, by (29) and Corollary 4.3, the largest integer $\lambda$ such that

\[
\overline{w}_\lambda(\prod_{i=1}^{k_1} \mathbb{S}^{m_{1,i}} \times \prod_{j=1}^{k_2} \mathbb{R}^{m_{2,j}} \times \prod_{t=1}^{k_3} \mathbb{C}^{m_{3,t}} \times \prod_{l=1}^{k_4} \mathbb{H}^{m_{4,l}}) \neq 0
\]

is

\[
\lambda = q + \dim \left( \prod_{i=1}^{k_1} \mathbb{S}^{m_{1,i}} \times \prod_{j=1}^{k_2} \mathbb{R}^{m_{2,j}} \times \prod_{t=1}^{k_3} \mathbb{C}^{m_{3,t}} \times \prod_{l=1}^{k_4} \mathbb{H}^{m_{4,l}} \right)\\
= \sum_{i=1}^{k_1} m_{1,i} + \sum_{j=1}^{k_2} 2^{[\log_2 m_{2,j}]+1} + \sum_{t=1}^{k_3} 2^{[\log_2 m_{3,t}]+2} + \\
+ \sum_{l=1}^{k_4} 2^{[\log_2 m_{4,l}]+3} - k_2 - 2k_3 - 4k_4.\tag{30}
\]

Finally, from Lemma 4.1 and (30), we obtain Theorem 1.1.

Corollary 1.2 is obtained from Theorem 1.1 immediately. Corollary 1.3 is obtained from [11 Example 2.6-(2)] and Theorem 1.1. And Corollary 1.5 is obtained from Example 2.1 and Theorem 1.1.

6 An obstruction for regular maps on disjoint unions

The main purpose of this section is to give an obstruction for the regular maps on disjoint unions of manifolds defined in Definition 1.6. As a by-product, we re-obtain [11 Theorem 2.4] in Remark 6.6. Throughout this section, all Chern classes are with mod $p$ coefficients for some primes $p$.

We consider the following canonical projections:

\[
\pi_i : \prod_{i=1}^{n} (F(M_i, k_i)/\Sigma_i) \longrightarrow F(M_i, k_i)/\Sigma_i.
\]

By applying the Kunneth formula, we see that for each $i = 1, 2, \cdots, n$, the map $\pi_i$ induces a monomorphism on cohomology

\[
\pi_i^* : H^*(F(M_i, k_i)/\Sigma_i; k) \longrightarrow H^*(\prod_{i=1}^{n} (F(M_i, k_i)/\Sigma_i; k) \\
\cong \bigotimes_{i=1}^{n} H^*(F(M_i, k_i)/\Sigma_i; k)
\]

where $k$ is an arbitrary field. The tensor product of these maps is an isomorphism

\[
\bigotimes_{i=1}^{n} \pi_i^* : \bigotimes_{i=1}^{n} H^*(F(M_i, k_i)/\Sigma_i; k) \cong \bigotimes_{i=1}^{n} H^*(F(M_i, k_i)/\Sigma_i; k). \tag{31}
\]
Let $\xi_{M_1, k_1; \ldots; M_n, k_n}$ denote the following vector bundle

$$
P_{\sum_{i=1}^n k_i} \cong \prod_{i=1}^n \mathbb{P}^{k_i} \longrightarrow \prod_{i=1}^n (F(M_i, k_i) \times \Sigma_i \mathbb{R}^{k_i})$$

$$\prod_{i=1}^n (F(M_i, k_i) / \Sigma_{k_i}).$$

For simplicity, we write $\xi_{M_1, k_1; \ldots; M_n, k_n}$ as $\xi_{M_1, k_1; \ldots; M_n, k_n}$. For each $i = 1, 2, \ldots, n$, it can be verified immediately that $\pi_i$ induces a pull-back of vector bundles by the following commutative diagram

$$
\begin{array}{ccc}
\mathbb{P}^{k_i} & \longrightarrow & \mathbb{P}^{k_i} \\
\downarrow & & \downarrow \\
\prod_{i \neq 1} (F(X_i, k_i) / \Sigma_{k_i}) \times (F(M_i, k_i) \times \Sigma_i \mathbb{R}^{k_i}) & \longrightarrow & (F(M_i, k_i) \times \Sigma_i \mathbb{R}^{k_i}) \\
\downarrow & & \downarrow \\
\prod_{i=1}^n (F(X_i, k_i) / \Sigma_{k_i}) & \longrightarrow & F(M_i, k_i) / \Sigma_{k_i}.
\end{array}
$$

Moreover, taking the Whitney sum of the vector bundles given in the left column of (32) with $i$ going from 1 to $n$, we can recover the vector bundle $\xi_{M_1, k_1; \ldots; M_n, k_n}$.

**Proposition 6.1.** We have the isomorphism of vector bundles

$$
\xi_{M_1, k_1; \ldots; M_n, k_n} \cong \bigoplus_{i=1}^n \pi_i^* \xi_{M_i, k_i}.
$$

The following corollary follows from Proposition 6.1.

**Corollary 6.2.** (a) Under the identification given by the isomorphism (37) with $\mathbb{Z}_2$-coefficient, the Stiefel-Whitney classes satisfy

$$w(\pi_i^* \xi_{M_i, k_i}) = \pi_i^* w(\xi_{M_i, k_i}) = w(\xi_{M_i, k_i}).$$

(b) Under the identification given by the isomorphism (37) with $\mathbb{Z}_p$-coefficient for any primes $p$, the Chern classes satisfy

$$c(\pi_i^* \xi_{M_i, k_i}) = \pi_i^* c(\xi_{M_i, k_i}) = c(\xi_{M_i, k_i}).$$

The next proposition follows by a straight-forward generalization of [1, Lemma 2.11] or [5, Proposition 2.1].

**Proposition 6.3.** Suppose there is a $(M_1, k_1; M_2, k_2; \ldots; M_n, k_n)$-regular map $f : \prod_{i=1}^n M_i \longrightarrow \mathbb{R}^N$. If

$$\bar{w}_r(\xi_{M_1, k_1; \ldots; M_n, k_n}) \neq 0,$$

then

$$N \geq r + \sum_{i=1}^n k_i.$$

The next proposition follows from Proposition 6.3.

**Proposition 6.4.** Let $\lambda_i$ be the largest integer such that

$$\bar{w}_{\lambda_i}(\xi_{M_i, k_i}) \neq 0.$$

If there is a $(M_1, k_1; M_2, k_2; \ldots; M_n, k_n)$-regular map $f : \prod_{i=1}^n M_i \longrightarrow \mathbb{R}^N$, then

$$N \geq \sum_{i=1}^n (\lambda_i + k_i).$$

**Proof.** It follows from Proposition 6.1 and Corollary 6.2 (a) that

$$w(\xi_{M_1, k_1; M_2, k_2; \ldots; M_n, k_n}) = w(\bigoplus_{i=1}^n \pi_i^* \xi_{M_i, k_i})$$

$$= \prod_{i=1}^n w(\pi_i^* \xi_{M_i, k_i})$$

$$= \prod_{i=1}^n w(\xi_{M_i, k_i}).$$
Therefore,

\[ \bar{w}(\xi_1, \ldots, \xi_n) = \prod_{i=1}^{n} \bar{w}(\xi_i). \]  

(33)

By the definition of \( \lambda_i \)'s, it follows from (33) that

\[ \bar{w}_{\sum_{i=1}^{n} \lambda_i} (\xi_1, \ldots, \xi_n) = \prod_{i=1}^{n} \bar{w}_{\lambda_i}(\xi_i) \neq 0. \]  

(34)

Suppose there is a \((M_1, k_1; M_2, k_2; \cdots; M_n, k_n)\)-regular map \( f : \prod_{i=1}^{n} M_i \to \mathbb{R}^N \). Then by applying (34) to Proposition 6.3 we obtain Proposition 6.3.

**Corollary 6.5.** For \( i = 1, 2, \ldots, n \), let \( M_i \) be a closed, connected \( m_i \)-dimensional manifold and \( h_i \) be the largest integer such that

\[ \bar{w}_{h_i}(M_i) \neq 0. \]

If there is a \((M_1, 2; M_2, 2; \cdots; M_n, 2)\)-regular map of \( \prod_{i=1}^{n} M_i \) into \( \mathbb{R}^N \), then

\[ N \geq \sum_{i=1}^{n} (m_i + h_i) + 2n. \]

**Proof.** By Corollary 6.4 we see that the largest integer \( \lambda_i \) such that

\[ \bar{w}_{\lambda_i}(\xi_i, 2) \neq 0 \]

is \( \lambda_i = m_i + h_i \). By Proposition 6.4, we finish the proof.

**Remark 6.6.** Since any \( 2n \)-regular map on \( \prod_{i=1}^{n} M_i \) is a \((M_1, 2; M_2, 2; \cdots; M_n, 2)\)-regular map, we re-obtain [11, Theorem 2.4] from Corollary 6.5.

The next proposition is a straight-forward generalization of [2, Lemma 5.7].

**Proposition 6.7.** Suppose there is a complex \((M_1, k_1; M_2, k_2; \cdots; M_n, k_n)\)-regular map \( f : \prod_{i=1}^{n} M_i \to \mathbb{C}^N \). If

\[ \bar{c}_r(\xi_{M_1, k_1}; \cdots; M_n, k_n) \neq 0, \]

then \( N \geq r + \sum_{i=1}^{n} k_i \).

The next proposition follows from Proposition 6.7.

**Proposition 6.8.** Let \( \tau_i \) be the largest integer such that

\[ \bar{c}_{\tau_i}(\xi_{M_i, k_i}) \neq 0. \]

If there is a complex \((M_1, k_1; M_2, k_2; \cdots; M_n, k_n)\)-regular map \( f : \prod_{i=1}^{n} M_i \to \mathbb{C}^N \), then

\[ N \geq \sum_{i=1}^{n} (\tau_i + k_i). \]

**Proof.** It follows from Proposition 6.1 and Corollary 6.2 (b) that

\[ c(\xi_{M_1, k_1}; M_2, k_2; \cdots; M_n, k_n) = \prod_{i=1}^{n} c(\pi_i^* \xi_{M_i, k_i}) \]

\[ = \prod_{i=1}^{n} c(\pi_i^* \xi_{M_i, k_i}) \]

\[ = \prod_{i=1}^{n} c(\xi_{M_i, k_i}). \]

Therefore,

\[ c(\xi_{M_1, k_1}; M_2, k_2; \cdots; M_n, k_n) = \prod_{i=1}^{n} c(\xi_{M_i, k_i}) \]

(35)
By the definition of $\tau_i$'s, it follows from (35) that

$$\overline{\sum_{i=1}^{n} \tau_i (\xi^C_{M_i,k_i})} = \prod_{i=1}^{n} \overline{\tau_i (\xi^C_{M_i,k_i})} \neq 0.$$  \hspace{1cm} (36)

Suppose there is a complex $(M_1,k_1;M_2,k_2;\cdots;M_n,k_n)$-regular map $f : \prod_{i=1}^{n} M_i \rightarrow \mathbb{C}^N$. Then by applying (36) to Proposition 6.8, we obtain Proposition 6.9.

7 Proof of Theorem 1.7

Proof of Theorem 1.7. By Lemma 4.8, the largest integer $\lambda_{0,s}$ such that $\bar{w}_{\lambda_{0,s}} (\xi^R_{p^m,2}) \neq 0$ is $\lambda_{0,s} = 2^{d_s} - 1$. By Lemma 4.9, the largest integer $\lambda_{1,i}$ such that $\bar{w}_{\lambda_{1,i},i} (\xi^R_{p^m,1}) \neq 0$ is $\lambda_{1,i} = m_{1,i}$. By Lemma 4.10, the largest integer $\lambda_{2,j}$ such that $\bar{w}_{\lambda_{2,j},j} (\xi^R_{p^m,2}) \neq 0$ is $\lambda_{2,j} = 2^{\lfloor \log_2 \cdot m_{2,j} \rfloor} + 1 - 1$. By Lemma 4.11, the largest integer $\lambda_{3,t}$ such that $\bar{w}_{\lambda_{3,t},t} (\xi^R_{p^m,3}) \neq 0$ is $\lambda_{3,t} = 2^{\lfloor \log_2 \cdot m_{3,t} \rfloor} + 2 - 2$. By Lemma 4.12, the largest integer $\lambda_{4,l}$ such that $\bar{w}_{\lambda_{4,l},l} (\xi^R_{p^m,4}) \neq 0$ is $\lambda_{4,l} = 2^{\lfloor \log_2 \cdot m_{4,l} \rfloor} + 3 - 4$. By applying all the above to Proposition 6.4, we have

$$N \geq \sum_{s=1}^{k_0} (\lambda_{0,s} + 2^{d_s}) + k_1 \sum_{i=1}^{k_1} (\lambda_{1,i} + 2) + \sum_{j=1}^{k_2} (\lambda_{2,j} + 2)
+ \sum_{t=1}^{k_3} (\lambda_{3,t} + 2) + \sum_{l=1}^{k_4} (\lambda_{4,l} + 2)$$

$$= \sum_{s=1}^{k_0} 2^{d_s} + \sum_{i=1}^{k_1} m_{1,i} + \sum_{j=1}^{k_2} 2^{\lfloor \log_2 \cdot m_{2,j} \rfloor} + 1
+ \sum_{t=1}^{k_3} 2^{\lfloor \log_2 \cdot m_{3,t} \rfloor} + 2 + \sum_{l=1}^{k_4} 2^{\lfloor \log_2 \cdot m_{4,l} \rfloor} + 3 - k_0 + 2k_1 + k_2 - 2k_4.$$

The assertion follows.

Corollary 1.8 follows immediately from Theorem 1.7 and Example 2.2.

8 Complex regular maps on Euclidean spaces, spheres and complex projective spaces

Proposition 8.1. For an odd prime $p$, if there exists a complex $(\mathbb{R}^{m_1}, p; \cdots; \mathbb{R}^{m_k}, p)$-regular map

$$f : \prod_{j=1}^{k} \mathbb{R}^{m_j} \rightarrow \mathbb{C}^N,$$

then

$$N \geq \sum_{j=1}^{k} \frac{m_j + 1}{2} (p - 1) + k.$$

Suppose there exists a complex $np$-regular map $f : \mathbb{R}^n \rightarrow \mathbb{C}^N$. Then the composition

$$\prod_{i=1}^{n} \mathbb{R}^m \xrightarrow{\pi} \prod_{i=1}^{n} D^m \xrightarrow{f} \mathbb{C}^N$$

gives a complex $(\mathbb{R}^m, p; \cdots; \mathbb{R}^m, p)$-regular map from $\prod_{i=1}^{n} \mathbb{R}^m$ into $\mathbb{C}^N$ where $D^m$ denotes the unit open ball in $\mathbb{R}^m$. Hence as a consequence of Proposition 8.1, we obtain the next proposition.

Proposition 8.2. Let $p$ be an odd prime. If there exists a complex $np$-regular map from $\mathbb{R}^m$ into $\mathbb{C}^N$, then $N \geq n(\lfloor \frac{m+1}{2} \rfloor (p - 1) + 1)$.
Proof of Proposition 8.1. Throughout the proof, we let the cohomology coefficients to be $\mathbb{Z}_p$, where $p$ is the given odd prime. Let $\nu_j$ be the largest integer such that the Chern class with mod $p$ coefficients

$$\bar{c}_{\nu_j}(\xi_{\mathbb{C}^{m_j},p}^C) \neq 0.$$ 

We notice that $\nu_j$ must be finite. On the other hand, in [2, Proof of Theorem 5.2], it is proved that the Chern class with mod $p$ coefficients satisfies

$$\bar{c}_{\left\lfloor \frac{m_j}{2} \right\rfloor}((\xi_{\mathbb{C}^{m_j},p}^C) \neq 0.$$ 

Hence

$$\nu_j \geq \left\lfloor \frac{m_j}{2} \right\rfloor (p - 1).$$ 

With the help of Proposition 6.8, we have

$$N \geq \sum_{j=1}^{k} (\nu_j + p) \geq \sum_{j=1}^{k} \left\lfloor \frac{m_j + 1}{2} \right\rfloor (p - 1) + k.$$ 

The assertion follows. \hfill \square

The next proposition follows from Lemma 4.2 and Lemma 4.10.

**Proposition 8.3.** If there exists a complex 2-regular map of $S^m$ into $\mathbb{C}^N$, then $N \geq \left\lceil \frac{m}{2} \right\rceil + 2$.

The next proposition follows from Lemma 4.2 and Lemma 4.14.

**Proposition 8.4.** Let $m \geq 4$. If there exists a complex 2-regular map of $\mathbb{C}P^m$ into $\mathbb{C}^N$, then $N \geq 2m$.

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