ORTHOSYMPLECTIC LIE SUPERALGEBRAS IN SUPERSPACE ANALOGUES OF QUANTUM KEPLER PROBLEMS

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ABSTRACT. A Schrödinger type equation on the superspace $\mathbb{R}^{D|2n}$ is studied, which involves a potential inversely proportional to the negative of the $osp(D|2n)$ invariant “distance” away from the origin. An $osp(2, D + 1|2n)$ dynamical supersymmetry for the system is explicitly constructed, and the bound states of the system are shown to form an irreducible highest weight module for this superalgebra. A thorough understanding of the structure of the irreducible module is obtained. This in particular enables the determination of the energy eigenvalues and the corresponding eigenspaces as well as their respective dimensions.

1. INTRODUCTION

The quantum Kepler problem and its analogues in higher dimensions are a series of soluble quantum mechanical systems with $-\frac{1}{r}$ potentials. The simplest of such systems is the hydrogen atom. It was discovered in the 60s by McIntosh and Cisneros [MC] and Zwanziger [Z] independently that the quantum system describing the hydrogen atom remained soluble when coupled to a magnetic charge. Since then the quantum Kepler problem has been generalised to include couplings to nonabelian magnetic monopoles in 5-dimensions in [I] and in arbitrary dimensions in [M1]. Such generalisations are referred to as the generalised MICZ-Kepler problem in the literature.

It has long been known that the original quantum Kepler problem had an $so(2, 4)$ dynamical symmetry. In [BB], Barut and Bornzin showed that the dynamical symmetry survived when the system was under the influence of a magnetic charge. They used the dynamical symmetry to give a beautiful solution of the problem. In a recent joint publication [MZ] with Meng, we demonstrated that the generalised MICZ-Kepler problem in odd dimension $D$ has an $so(2, D + 1)$ dynamical symmetry, and gave a solution to the problem by algebraic means using a particular irreducible unitary highest weight representation of the dynamical symmetry. This work has also been extended to even dimensions (but for restricted classes of magnetic monopoles) in [M2].

A natural problem is to introduce supersymmetries into the Kepler problem and its generalisations and to study the resulting supersymmetric quantum mechanical systems. The $\mathcal{N} = 2$ supersymmetric case was studied in [KLPW] in arbitrary dimensions but without magnetic monopoles. An $so(D + 1)$ dynamical symmetry remained in this case, which helped to obtain the bound states spectrum and the multiplicities of the eigenvalues.

In this paper we investigate the Schrödinger equation on the superspace $\mathbb{R}^{D|2n}$ involving a potential inversely proportional to the negative of the $osp(D|2n)$ invariant “distance” away from the origin (see equation (2.3)). We shall refer to the study of this eigenvalue problem as the quantum Kepler problem on the superspace $\mathbb{R}^{D|2n}$ without magnetic monopoles. As we shall see, the system is integrable, and will be solved by

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algebraic means. It will be very interesting to extend the study to include couplings to magnetic monopoles.

Schrödinger equations on superspaces were studied by Delbourgo in the late 80s [D1] as a method to incorporate spin. This developed into a fruitful programme (see, e.g., [DJW] and references therein) on using supermanifolds to describe gauge symmetries and also to explain internal degrees of freedom of elementary particles (see [D2, D3] for recent developments). We hope that the present work and further studies will provide useful mathematical information for the programme of Delbourgo and co-workers.

Our primary interest in this paper is the integrability of the quantum Kepler problem on superspace without magnetic monopoles, and also the relevant representations of its dynamical supersymmetry. We shall give the precise definition of the quantum Kepler problem on the superspace $\mathbb{R}^{D|2n}$ in Section 2 and reformulate the problem algebraically following ideas of Barut and Bornzin [BB]. We show in Theorem 3.1 that the problem has an $osp(2, D + 1|2n)$ dynamical supersymmetry, where the generators of the orthosymplectic Lie superalgebra are constructed explicitly. The bound states are shown to form an infinite dimensional irreducible highest weight module of this Lie superalgebra (Theorem 4.1), and a thorough understanding of the structure of this module is also given. Using this information we obtain the bound state spectrum and also the corresponding eigenspaces in Theorem 4.2.

It is quite remarkable that the Kepler problem remains integrable when generalised to superspaces (see Remarks 2.1 and 4.2). Furthermore, the appearance of the dynamical supersymmetry and the way in which its representation theory enables us to solve the problem all appear to be quite fascinating from the point of view of the theory of Lie superalgebras. It is well known that even the finite dimensional representations of orthosymplectic Lie superalgebras are extremely hard to study and very little is known about them. Thus it is a nice surprise that the infinite dimensional irreducible representation of $osp(2, D + 1|2n)$ appearing in the problem can be understood for all $D > 2n + 1$. Therefore, results in this paper should be of interest to the representation theory of Lie superalgebras as well.

2. Quantum Kepler problem on superspace

In this section we introduce the quantum Kepler problems on superspaces, and also give an algebraic formulation for the problems following the strategy of Barut and Bornzin [BB].

2.1. Generalities. Let $\mathbb{R}^{D|2n}$ denote the superspace with $D$ even dimensions and $2n$ odd dimensions. Denote by $X^a$ with $a = 1, 2, \ldots, D + 2n$ the coordinate of the superspace, where $X^a$ is even if $a \leq D$ and odd if $a > D$. We shall assign to $\mathbb{R}^{D|2n}$ the metric $\eta = (\eta_{ab}) = \begin{pmatrix} I_D & 0 & 0 \\ 0 & 0 & -I_n \\ 0 & I_n & 0 \end{pmatrix}$, where $I_D$ and $I_n$ are the identity matrices of sizes $D \times D$ and $n \times n$ respectively. By a function on $\mathbb{R}^{D|2n}$ we shall mean a map from $\mathbb{R}^D$ to the complex Grassmann algebra $\Lambda_{2n}$ generated by the odd coordinates $X^\nu$ with $\nu = D + 1, D + 2, \ldots, D + 2n$. Denote $\partial_a = \frac{\partial}{\partial X^a}$ which acts on functions from the left. Let $X_a = \sum_b \eta_{ab} X^b$ and $\partial^a = \sum_b \eta^{ab} \partial_b$, where $\eta^{ab}$ are the entries of $\eta^{-1}$. Set $\Delta = \sum_a \partial^a \partial_a$. 
Given a function $V(X)$ on $\mathbb{R}^{D|2n}$ which is assumed to be even in the Grassmann variables, we introduce the operator

$$H = -\frac{1}{2} \Delta + V(X),$$

which will be referred to as a quantum Hamiltonian operator. Our broad aim is to investigate the eigenvalue problem for the quantum Hamiltonian operator, that is, to solve the Schrödinger equation

$$H \Psi = E \Psi, \quad (2.1)$$

where the eigenvalue $E$ is required to belong to $\mathbb{R}$ (thus $V(X)$ has to be even). We shall be particularly interested in systems of the form of (2.1) which are integrable.

For each given $V(X)$, we shall need to specify the class of functions on $\mathbb{R}^{D|2n}$, to which the solutions of the Schrödinger equation belong. For the potential corresponding to the Kepler problem, this will be discussed in some detail in the next subsection. Here we merely point out that the eigenfunction $\Psi$ is a polynomial in the odd coordinates with coefficients being complex valued functions on $\mathbb{R}^D$, which will be referred to as coefficient functions.

**Remark 2.1.** The Schrödinger equation on $\mathbb{R}^{D|2n}$ is equivalent to a system of partial differential equations on $\mathbb{R}^D$ for the coefficient functions.

Note that since the Hamiltonian operator is even, the Schrödinger equation separates into two independent equations for the even and odd parts of $\Psi$ respectively.

### 2.2. Quantum Kepler problem on superspace

Let us now introduce the quantum Hamiltonian operator which we shall study in this paper. Let $\mathcal{R} = (\sum_a X^a X_a)^{\frac{1}{2}}$, which is only defined away from the origin of $\mathbb{R}^D$, and should be interpreted as a polynomial in the odd coordinates. More precisely, let $r^2 = \sum_{i,j=1}^{D+2n} X^i \eta_{ij} X^j$ and $\Theta^2 = \sum_{\mu,\nu=D+1}^{D+2n} X^\mu \eta_{\mu\nu} X^\nu$. Then $R = r \sqrt{1 + \Theta^2}$, where $\sqrt{1 + \Theta^2}$ should be understood as a Taylor expansion in $\frac{\Theta^2}{r^2}$. Since the odd coordinates are Grassmannian, the expansion terminates at order $n$ in $\frac{\Theta^2}{r^2}$, and we have a polynomial in the odd coordinates. We have

$$\partial_a (R) = \frac{X_a}{R}, \quad \Delta (R^2) = 2d, \quad \text{where } d = D - 2n. \quad (2.2)$$

We shall take the following quantum Hamiltonian operator

$$H = -\frac{1}{2} \Delta - \frac{1}{R}. \quad (2.3)$$

Our purpose is to determine the spectrum of $H$ and the corresponding eigenvectors. In the remainder of this paper, we shall only consider the Schrödinger equation (2.1) with this quantum Hamiltonian.

Let us now specify the class of functions on $\mathbb{R}^{D|2n}$ to which the eigenfunctions belong. We should mention that the so-called superanalysis (analysis of functions on superspace) is yet to develop into a coherent theory. It will take us too far astray to investigate superanalysis in any depth here, as we shall adopt an algebraic approach to the quantum Kepler problem on superspace, which by-passes many analytic issues. The rather superfluous discussion below on functions on $\mathbb{R}^{D|2n}$ suffices for us to get by.

The Grassmann algebra $\Lambda_{2n}$ generated by the odd coordinates is $\mathbb{Z}_+$-graded with $X^\mu$ ($1 + D \leq \mu \leq 2n + D$) having degree 1. Let $\zeta_s$ ($0 \leq s \leq 2^{2n} - 1$) be a homogeneous
basis of \( \Lambda_{2n} \) consisting of products of the odd coordinates, and denote by \( \text{deg}(\zeta_s) \) the degree of \( \zeta_s \). We order the basis elements in such a way that \( \text{deg}(\zeta_s) \leq \text{deg}(\zeta_{s+1}) \).

Introduce the conjugate linear algebra automorphism \( \bar{\cdot} : \Lambda_{2n} \rightarrow \Lambda_{2n} \) defined by

\[
\bar{X}^\mu = X_{\mu}, \quad \mu = D + 1, D + 2, \ldots, D + 2n.
\]

As usual, conjugate linear means that for any \( \lambda = c_1 \lambda_1 + c_2 \lambda_2 \) with \( \lambda_1, \lambda_2 \in \Lambda_{2n} \) and \( c_1, c_2 \in \mathbb{C} \), \( \bar{\lambda} = \bar{c}_1 \bar{\lambda}_1 + \bar{c}_2 \bar{\lambda}_2 \), where \( \bar{c}_1 \) and \( \bar{c}_2 \) are the complex conjugates of \( c_1 \) and \( c_2 \). Also, being an algebra automorphism, the map \( \bar{\cdot} \) obeys the rule \( \bar{\lambda_1 \lambda_2} = \bar{\lambda}_1 \bar{\lambda}_2 \). Note that \( \bar{\zeta}_s \zeta_s \neq 0 \) if \( \text{deg}(\zeta_s) \leq n \), but \( \bar{\zeta}_t \zeta_t = 0 \) if \( \text{deg}(\zeta_t) > n \).

The conjugate linear automorphism on \( \Lambda_{2n} \) extends to the superalgebra of functions on \( \mathbb{R}^{D|2n} \) in a natural way, and we shall still denote the resulting map by \( \bar{\cdot} \). More explicitly, write a function \( \Psi \) as \( \Psi = \sum_s \zeta_s \psi_s \) where the \( \psi_s \) are complex valued functions on \( \mathbb{R}^D \). Then \( \bar{\Psi} = \sum_s \bar{\zeta}_s \bar{\psi}_s \), where \( \bar{\psi}_s \) is the usual complex conjugate of \( \psi_s \). For any two functions \( \Phi \) and \( \Psi \) on \( \mathbb{R}^{D|2n} \), we let

\[
\langle \Phi \mid \Psi \rangle = \int_{\mathbb{R}^D} \bar{\Phi} \Psi \text{ if the integral over } \mathbb{R}^D \text{ exists (thus lies in } \Lambda_{2n}).
\]

Let \( \mathcal{F} \) denote the set of functions on \( \mathbb{R}^{D|2n} \) such that for every \( \Psi \in \mathcal{F} \)

1. the integral \( \langle \Psi \mid \Psi \rangle = \int_{\mathbb{R}^D} \overline{\Psi} \Psi \) exists; and
2. the coefficient functions of \( \Psi \) are twice differentiable on \( \mathbb{R}^D \setminus \{0\} \).

Then for any \( \Psi = \sum_s \zeta_s \psi_s \in \mathcal{F} \), the coefficient function \( \psi_0 \), which will be called the body of \( \Psi \), must be square integrable in the usual sense. However note that all functions \( \Psi \) satisfying \( \psi_s = 0 \) for all \( s \leq 2^n - 1 \) belong to \( \mathcal{F} \).

**Remark 2.2.** Even though our definition of \( \mathcal{F} \) imposes conditions on all the coefficient functions \( \psi_s \) with \( s \leq 2^n - 1 \), we have to some extent followed the type of thinking common in physics that appropriate conditions only need to be imposed on the body (if the body is nonzero) of a function and then the supersymmetry of the physical problem will determine properties of the function as a whole.

**Remark 2.3.** When defining \( \mathcal{F} \), one may be tempted to use the more “natural” valuation \( \int_{\mathbb{R}^{D|2n}} \overline{\Psi} \Psi \) instead, where the integration over the odd coordinates is given by the Berezin integral. However, this does not make the body of \( \Psi \) square integrable.

We require that a solution of the Schrödinger equation (2.1) with the quantum Hamiltonian operator (2.3) belongs to the set \( \mathcal{F} \). The complex vector space spanned by all the solutions is evidently stable under the action of the quantum Hamiltonian operator, and a major aim is to understand this vector space. It is also this vector space which the dynamical supersymmetry algebra acts on. As we shall see later, the space in fact forms an irreducible module over the dynamical supersymmetry algebra.

Note that \( \overline{\Theta^2} = \Theta^2 \), and hence \( \overline{R} = R \). Thus for any function \( \Psi \), we have

\[
\overline{H \Psi} = H \Psi.
\]

If \( \Psi \) is a solution of the Schrödinger equation satisfying \( \overline{\Psi} = \Psi \), then the corresponding eigenvalue will necessarily be real. Thus we may regard the quantum Hamiltonian operator as Hermitian in a generalised sense.

We shall further restrict ourselves to considering only the bound states, that is, the eigenvectors of \( \overline{H} \) associated with negative eigenvalues. In order to have bound state solutions in \( \mathcal{F} \), we need to impose the following condition on the superspace:

\[
D > 2n + 1.
\]
In this case, the ground state eigenvalue $E_0$ and the associated eigenvector $\Psi_0$ of the system are given by

$$E_0 = -\frac{1}{2} \left( \frac{2}{d-1} \right)^2, \quad \Psi_0 = \exp \left( -\frac{2}{d-1} R \right).$$

Note that $\int_{\mathbb{R}^d} \bar{\Psi}_0 \Psi_0$ exists thus $\Psi_0 \in \mathcal{F}$. It is also easy to check that $\Psi_0$ indeed satisfies the Schrödinger equation with the energy eigenvalue $E_0$ (also see equation (4.3) and discussions after).

2.3. Algebraic formulation. We shall solve the quantum Kepler problem algebraically by using the representation theory of a dynamical supersymmetry algebra, which will be constructed later. For this purpose we need to reformulate the Schrödinger equation (2.1) algebraically for the quantum Hamiltonian (2.3).

It is well known that the following differential operators

$$J_{ab} = X_a \partial_b - (-1)^{[a][b]} X_b \partial_a \quad \text{(2.4)}$$

form the orthosymplectic Lie superalgebra $osp(D|2n)$ with the commutation relations (see, e.g., [JG, (40)])

$$[J_{ab}, J_{cd}] = \eta_{cb} J_{ad} - (-1)^{[a][b]} \eta_{da} J_{bc} - (-1)^{[c][d]} \eta_{db} J_{ac} - (-1)^{[a][b]} \eta_{ca} J_{bd}. \quad \text{(2.5)}$$

Here $[a] = 0$ if $a \leq D$ and $[a] = 1$ otherwise. As customary, $[A, B]$ represents the usual commutator unless both $A$ and $B$ are odd and in that case $[A, B] = AB + BA$. One can easily check that

$$[J_{ab}, \Delta] = 0, \quad [J_{ab}, R] = 0. \quad \text{(2.6)}$$

Therefore, $[J_{ab}, H] = 0$ for all $a, b$, and the system has an $osp(D|2n)$ symmetry. Let

$$E = \sum_{a=1}^{D+2n} X^a \partial_a, \quad T = E + \frac{d-1}{2},$$

where $E$ is the Euler operator. The following lemma is a generalisation of a result in [BB, Appendix] to the superspace setting.

**Lemma 2.1.** Define the following operators

$$J_{-2} = T, \quad J_{-1} = \frac{i}{2} R (-\Delta - 1), \quad J_0 = \frac{i}{2} R (-\Delta + 1). \quad \text{(2.7)}$$

(1) The operators satisfy the commutation relations of the Lie algebra $so(2,1)$:

$$[J_{-1}, J_0] = J_{-2}, \quad [J_{-2}, J_{-1}] = -J_0, \quad [J_0, J_{-2}] = J_{-1}. \quad \text{(2.8)}$$

(2) This Lie algebra commutes with the Lie superalgebra $osp(D|2n)$ spanned by the operators $J_{ab}$:

$$[J_{ab}, J_0] = [J_{ab}, J_{-1}] = [J_{ab}, J_{-2}] = 0, \quad \forall a, b.$$

**Proof.** Note that the Euler operator satisfies $[E, X^a] = X^a$ and $[E, \partial_a] = -\partial_a$. This immediately leads to $[E, J_{ab}] = 0$. Now part (2) easily follows from this commutation relation and also the commutation relations (2.6). To prove the first relation in (2.8), note that $[J_{-1}, J_0] = \frac{i}{2} R [\Delta, R]$. Using

$$[\Delta, R] = \frac{2}{R} \left( \frac{d-1}{2} + E \right), \quad \text{(2.9)}$$

we obtain the desired result. The other two relations are easily proven by using properties of the Euler operator $E$. \qed
Theorem 2.1. For the bound states (with $E < 0$), let
\[
\Phi = g\Psi, \quad \text{where} \quad g = \exp \left( -T \ln \sqrt{-2E} \right).
\]
Then the Schrödinger equation (2.1) is equivalent to
\[
h_0\Phi = -\frac{1}{\sqrt{-2E}}\Phi \quad \text{where} \quad h_0 = iJ_0.
\]

Proof. Let $\Theta = R(H - E)$. The Schrödinger equation can be re-written as $g\Theta\Psi = 0$. Using the following relations
\[
gRg^{-1} = \frac{1}{\sqrt{-2E}}R, \quad g\Delta g^{-1} = -2E\Delta
\]
in the equation we obtain (2.10). \hfill \Box

In the remainder of this paper we shall consider bound states only. The algebraic formulation of the quantum Kepler problem allows us to obtain the spectrum of the Hamiltonian for the bound states by using the representation theory of the algebras described in Lemma 2.1. However, in order to determine the multiplicities of the eigenvalues and also to construct the corresponding eigenfunctions, we need to explore a larger dynamical symmetry of the problem. This is done in the next section.

3. Dynamical supersymmetry

In this section we shall show that the quantum Kepler problem in the superspace $\mathbb{R}^{D|2n}$ has a dynamical supersymmetry described by the orthosymplectic Lie superalgebra $osp(2, D+1|2n)$. The generators and commutation relations of the superalgebra will be given explicitly. A parabolic subalgebra and some nilpotent subalgebra of the dynamical supersymmetry algebra will be studied in detail, which play crucial roles in solving the quantum Kepler problem.

3.1. Dynamical supersymmetry. We have the following result.

Lemma 3.1. Let $\Gamma_a = R\partial_a$ for all $a = 1, 2, \ldots, D+2n$.

(1) We have $[J_{-2}, \Gamma_a] = 0$, and
\[
A_a := [J_{-1}, \Gamma_a] = \frac{i}{2}X_a(\Delta + 1) - iT\partial_a,
\]
\[
M_a := [J_0, \Gamma_a] = \frac{i}{2}X_a(\Delta - 1) - iT\partial_a.
\]

(2) The operators $\Gamma_a$, $A_a$ and $M_a$ satisfy the following commutation relations
\[
[\Gamma_a, \Gamma_b] = J_{ab}, \quad [\Gamma_a, A_b] = -\eta_{ba}J_{-1}, \quad [\Gamma_a, M_b] = -\eta_{ba}J_0,
\]
\[
[A_a, A_b] = -J_{ab}, \quad [M_a, M_b] = J_{ab}, \quad [A_a, M_b] = -\eta_{ba}J_{-2}.
\]

Proof. The lemma can be proved by straightforward computations. To prove part (1), note that
\[A_a = -\frac{i}{2}R[\Delta, R]\partial_a + \frac{i}{2}R[\partial_a, R](\Delta + 1).\]
Using the first relation in (2.2) and also (2.9), we easily arrive at the result. The formula for $M_a$ can be proved in exactly the same way.
The proof for the first formula in (3.2) is very simple, thus we omit the details. Now consider \([\Gamma_a, A_b]\). We have

\[
[\Gamma_a, A_b] = \frac{i}{2} \eta_{ab} R - i [R \partial_a, T \partial_b] + \frac{i}{2} [R \partial_a, X_b \Delta].
\]

Using the following relations

\[
[R \partial_a, T \partial_b] = -(-1)^{[a][b]} \frac{X_b}{R} T \partial_a,
\]

\[
[R \partial_a, X_b \Delta] = \eta_{ab} R \Delta - (-1)^{[a][b]} \frac{2X_b}{R} T \partial_a,
\]

and also (2.9) we easily arrive at the desired formula. The commutator \([\Gamma_a, M_b]\) can be similarly proved by using relations (3.3) and (2.9).

Let us consider \([A_a, A_b]\). We have

\[
-[A_a, A_b] = [T \partial_a, T \partial_b] - \frac{1}{2} [T \partial_a, X_b (\Delta + 1)]
\]

\[
+ (-1)^{[a][b]} \frac{1}{2} [T \partial_b, X_a (\Delta + 1)]
\]

\[
+ \frac{1}{4} \left( X_{a[\Delta}, X_{b]} - (-1)^{[a][b]} X_b [\Delta, X_a] \right) (\Delta + 1).
\]

To simplify the right hand side of the above equation, we use the following relations

\[
[T \partial_a, T \partial_b] = 0, \quad [\Delta, X_a] = 2 \partial_a,
\]

\[
[T \partial_a, X_b] = T \eta_{ba} + (-1)^{[a][b]} X_b \partial_a,
\]

\[
[T \partial_a, X_b \Delta] = (T \eta_{ba} - (-1)^{[a][b]} X_b \partial_a) \Delta.
\]

Carefully combining similar terms we arrive at \([A_a, A_b] = -J_{ab}\). The commutators \([M_a, M_b]\) and \([A_a, M_b]\) can be calculated in the same way by using the relations in (3.4).

Let us introduce a \((D + 2n + 3) \times (D + 2n + 3)\) matrix \((\eta_{KL}) = \left( \begin{array}{cc} I_{2,1} & 0 \\ 0 & \eta \end{array} \right)\), where \(I_{2,1} = \left( \begin{array}{ccc} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{array} \right)\) and \(\eta\) is the metric of \(\mathbb{R}^{D\vert 2n}\). The indices \(K\) and \(L\) take values \(-2, -1, 0, 1, 2, \ldots, D\) ordered as shown. We set \([K] = 0\) if \(K = i \leq 0\), and \([K] = [a]\) if \(K = a \geq 1\).

**Theorem 3.1.** Introduce the operators \(J_{KL}\) such that \(J_{KL} = -(-1)^{|K||L|} J_{LK}\) with

\[
J_{ij} = \epsilon_{ijk} J_k, \quad \text{for} \ i, j, k \in \{-2, -1, 0\},
\]

\[
J_{-2,a} = \Gamma_a, \quad J_{0,a} = -A_a, \quad J_{-1,a} = M_a,
\]

\[
J_{ab} \text{ is defined by (2.3),} \quad \text{for} \ a, b \in \{1, 2, \ldots, D + 2n\},
\]

where \(\epsilon_{ijk}\) is totally skew symmetric in the three indices and \(\epsilon_{-2,-1,0} = 1\). These operators form a basis for the othosymplectic Lie superalgebra \(osp(2, D + 1\vert 2n)\) with the commutation relations

\[
[J_{KL}, J_{PQ}] = \eta_{PL} J_{KQ} + (-1)^{|K||L| + |P|} \eta_{QL} J_{KP} - (-1)^{|K||L|} \eta_{PK} J_{LQ}. \tag{3.6}
\]
Proof. Lemma 2.4 and Lemma 3.1 have verified all the commutation relations except \([J_{ab}, J_{ic}]\) for \(a, b, c \geq 1\) and \(i \leq 0\), and some cases of \([J_{ij}, J_{kc}]\) with \(i, j, k \leq 0\) and \(c \geq 1\). The relations for \([J_{ij}, J_{kc}]\) easily follow from Lemma 3.1(1) and Lemma 2.1(1). A direct calculation gives
\[
[J_{ab}, J_{-2,c}] = \eta_{cb} J_{-2,a} - (-1)^{|b||c|} \eta_{ca} J_{-2,b}.
\]
Since \(J_{ab}\) commutes with all \(J_{ij}\) by Lemma 2.1(2), we may replace the index \(-2\) in this equation by any \(i \in \{-2, -1, 0\}\). This completes the proof of the theorem. \(\square\)

3.2. Root system. Denote by \(\mathfrak{g}\) the complexification of the real Lie superalgebra \(osp(2, D + 1|2n)\) spanned by the operators in \([3.5]\), then \(\mathfrak{g} \cong osp(D + 3|2n)\). Let us now specify the root system for \(\mathfrak{g}\) that will be used for characterising highest weight representations. We adopt notations and conventions from [K] (see also [S]), but take a nonstandard root system for the Lie superalgebra \(\mathfrak{g}\). Dynkin diagrams corresponding to the root systems of \(\mathfrak{g}\) for even and odd \(D\) are respectively given by

- **If \(D\) is even:**
  [Dynkin diagram for even \(D\)]

- **If \(D\) is odd:**
  [Dynkin diagram for odd \(D\)]

Here the black and grey nodes correspond to odd simple roots.

When \(D = 2m\), there are \(m + 1 + n\) nodes in the Dynkin diagram. Order the nodes from left to right. They respectively correspond to the simple roots
\[
\epsilon_0 - \epsilon_1, \epsilon_1 - \epsilon_2, \ldots, \epsilon_{m-1} - \epsilon_m, \epsilon_m - \delta_1, \delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n, \delta_n.
\]
The element \(h_0\) belongs to the Cartan subalgebra with \(\epsilon_0(h_0) = 1\) and \(\epsilon_i(h_0) = 0 = \delta_j(h_0)\) for all the other \(\epsilon_i\) and all the \(\delta_j\).

When \(D = 2m + 1\), there are \(m + 2 + n\) nodes, respectively corresponding to the simple roots
\[
\epsilon_0 - \epsilon_1, \epsilon_0 - \epsilon_1 - \epsilon_2, \ldots, \epsilon_{m-1} - \epsilon_m, \epsilon_m - \delta_1, \delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n, 2\delta_n.
\]
In this case, \(\epsilon_{-1}(h_0) = 1\) while the evaluations of all the other \(\epsilon_i\) and all the \(\delta_j\) on \(h_0\) are zero.

An element in the weight space of \(\mathfrak{g}\) will be written as an \([D+3\over2] + n\) tuple, which is the coordinate in the basis consisting of the \(\epsilon_i\) and \(\delta_j\). The basis is ordered in such a way that the \(\delta_j\) are positioned after all \(\epsilon_i\), and the \(\epsilon_i\) appear in their natural order, and so do also the \(\delta_j\).

Since \(osp(D + 1|2n)\) is a regular subalgebra of \(\mathfrak{g}\), we shall take for it the root system compatible with that of \(\mathfrak{g}\). This is specified by the Dynkin diagram obtained by removing the left most node from the Dynkin diagram of \(\mathfrak{g}\).

3.3. Parabolic subalgebra. In this subsection we discuss subalgebras of the Lie superalgebra \(\mathfrak{g}\). The main result established is Proposition 3.1, which is of crucial importance for solving the Kepler problem on the superspace. Unfortunately the material presented here is quite technical, so we have relegated some of it to the Appendix.

Define the linear operator \(ad_{h_0}\) on \(\mathfrak{g}\) by \(ad_{h_0}(Y) = [h_0, Y]\) for all \(Y \in \mathfrak{g}\) (recall that \(h_0 = iJ_0\)). Now \(\mathfrak{g}\) decomposes into a direct sum \(\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1}\) of eigenspaces of \(ad_{h_0}\) with eigenvalues \(+1\), \(0\) and \(-1\) respectively. Here

1. \(\mathfrak{g}_{+1}\) is spanned by \(i(J_{-1} - iJ_{-2})\) and \(M_a - i\Gamma_a\) \((a \geq 1)\),
2. \(\mathfrak{g}_{-1}\) is spanned by \(i(J_{-1} + iJ_{-2})\) and \(M_a + i\Gamma_a\) \((a \geq 1)\),
3. \(\mathfrak{g}_0\) is spanned by \(h_0\), \(A_a\) and \(J_{ab}\) \((a, b \geq 1)\).
The subspaces $\mathfrak{g}_{+1}$, $\mathfrak{g}_{-1}$ and $\mathfrak{g}_0$ all form subalgebras of $\mathfrak{g}$. In particular, $\mathfrak{g}_0$ is the subalgebra $\mathfrak{gl}_1 \oplus \mathfrak{osp}(D + 1|2n)$, and $[\mathfrak{g}_{+1}, \mathfrak{g}_{+1}] = [\mathfrak{g}_{-1}, \mathfrak{g}_{-1}] = 0$. Note that if $D$ is even, $\mathfrak{g}_{+1}$ is spanned by the root vectors associated with the positive roots
\[ \epsilon_0, \epsilon_0 \pm \epsilon_i, \forall i > 0; \quad \epsilon_0 \pm \delta_j, \forall j. \]
If $D$ is odd, the subalgebra $\mathfrak{g}_{+1}$ is spanned by the root vectors associated with the positive roots
\[ \epsilon_{-1} \pm \epsilon_i, \forall i \geq 0; \quad \epsilon_{-1} \pm \delta_j, \forall j. \]
Both $\mathfrak{g}_{+1}$ and $\mathfrak{g}_{-1}$ are stable under the adjoint action of $\mathfrak{g}_0$ in the sense that $[\mathfrak{g}_0, \mathfrak{g}_{\pm 1}] = \mathfrak{g}_{\pm 1}$, thus form $\mathfrak{g}_0$-modules. When restricted to modules for the $\mathfrak{osp}(D+1|2n)$ subalgebra of $\mathfrak{g}_0$, both $\mathfrak{g}_{+1}$ and $\mathfrak{g}_{-1}$ are isomorphic to the natural $\mathfrak{osp}(D+1|2n)$-module.

We have the following parabolic subalgebra of $\mathfrak{g}$:
\[ \mathfrak{p} := \mathfrak{g}_0 \oplus \mathfrak{g}_{+1} \]  
with the nilpotent radical $\mathfrak{g}_{+1}$. Furthermore, $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{g}_{-1}$.

Denote by $\mathcal{U}_0$ the complex linear span of products of the operators in $\mathfrak{g}$. Let $\mathcal{U}_\mathfrak{p}$ and $\mathcal{U}_{\mathfrak{g}_{-1}}$ be the sub superalgebra of $\mathcal{U}_\mathfrak{g}$ generated by the elements of $\mathfrak{p}$ and $\mathfrak{g}_{-1}$ respectively. Then the underlying vector space of $\mathcal{U}_\mathfrak{p}$ is isomorphic to $\mathcal{U}_\mathfrak{g}_{-1}\mathcal{U}_\mathfrak{p}$. Note that $\mathcal{U}_\mathfrak{g}_{-1}$ is $\mathbb{Z}_2$-graded commutative.

**Remark 3.1.** The associative superalgebra $\mathcal{U}_\mathfrak{g}$ is a quotient of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g} = \mathfrak{osp}(D + 3|2n)$. Similar comments apply to $\mathcal{U}_\mathfrak{p}$ and $\mathcal{U}(\mathfrak{p})$ etc..

The subalgebra $\mathcal{U}_{\mathfrak{g}_{-1}}$ has a $\mathbb{Z}$-grading $\mathcal{U}_{\mathfrak{g}_{-1}} = \bigoplus_{l=0}^{\infty} \mathcal{U}_{\mathfrak{g}_{-1}}(-l)$. Let $K_0 = i(J_{-1} + iJ_{-2})$ and $K_a = M_a + i \Gamma_a \ (a \geq 1)$, which form a basis of the subalgebra $\mathfrak{g}_{-1}$. Then $\mathcal{U}_{\mathfrak{g}_{-1}}(-l)$ is spanned by all the homogeneous polynomials of order $l$ in the elements $K_A$ with $A = 0, 1, \ldots, D + 2n$. Here we should note that $K_A$ are odd in the $\mathbb{Z}_2$ grading for all $A > D$, and in this case $(K_A)^2 = 0$. However, $K_0$ is even and $(K_0)^l \neq 0$ for all $l$.

Let $\eta_0 = (\eta_{AB})$ with $A, B = 0, 1, \ldots, D + 2n$ be the matrix obtained from $(\eta_{KL})$ by removing the first two rows and first two columns. This is the metric relative to which the $\mathfrak{osp}(D + 1|2n)$ subalgebra of $\mathfrak{g}_0$ is defined. Now the (adjoint) action of $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$ naturally extends to an action on $\mathcal{U}_{\mathfrak{g}_{-1}}(-l)$. Denote by $\eta_0(\mathfrak{g}_{-1}, \mathfrak{g}_{-1})$ the subspace of $\mathfrak{osp}(D + 1|2n)$-invariants in $\mathcal{U}_{\mathfrak{g}_{-1}}(-2)$. Then $\eta_0(\mathfrak{g}_{-1}, \mathfrak{g}_{-1})$ is spanned by
\[ (K)^2 := \sum_{A, B = 0}^{D+2n} \eta^{BA} K_A K_B, \]
where $\eta^{AB}$ are the matrix elements of $\eta_0^{-1}$. We have the following result.

**Lemma 3.2.** The operators $K_A$ satisfy $(K)^2 = 0$.

*Proof.* The proof of this claim involves a considerable amount of calculations. Thus we shall present only the main formulæ needed. Let us first calculate $K_0^2$. We have
\[ K_0^2 = \left( \frac{1}{2} R(\Delta + 1) - T \right)^2 \]
\[ = \frac{1}{4} (R(\Delta + 1))^2 - TR(\Delta + 1) - \frac{1}{2} [R(\Delta + 1), T] + T^2. \]
Using $[R(\Delta + 1), T] = R(\Delta - 1)$, we obtain
\[ K_0^2 = \frac{1}{4} (R(\Delta + 1))^2 - TR(\Delta + 1) - \frac{1}{2} R(\Delta - 1) + T^2. \]  
(3.8)
Now \( \sum_{a=1}^{D+2n} K^a K_a = \sum_a (M^a M_a + i(M^a \Gamma_a + \Gamma^a M_a) - \Gamma^a \Gamma_a) \). Note that
\[
\sum_{a=1}^{D+2n} M^a M_a = -\frac{1}{4} \sum_a X^a(\Delta - 1)X_a(\Delta - 1) \\
+ \frac{1}{2} \sum_a (X^a(\Delta - 1)T\partial_a + T\partial^a X_a(\Delta - 1)) \\
- \sum_a T\partial^a T\partial_a.
\]
The first term can be expressed as
\[
-\frac{1}{4} R^2(\Delta - 1)^2 - \frac{1}{4} \sum_a X^a[(\Delta - 1), X_a](\Delta - 1). 
\]
Using \([[(\Delta - 1), X_a] = 2\partial_a\) we obtain
\[
-\frac{1}{4} R^2(\Delta - 1)^2 - \frac{1}{2} E(\Delta - 1). 
\]
The second term can be simplified to
\[
T(T + 1/2)(\Delta - 1) + \frac{1}{2} E(\Delta + 1), 
\]
while the third yields
\[
-T(T + 1)\Delta. 
\]
Combining these results together we arrive at
\[
\sum_{a=1}^{D+2n} M^a M_a = -\frac{1}{4} R^2(\Delta - 1)^2 - \frac{1}{2} T(\Delta + 1) - T^2 + E. 
\]  
(3.9)

It is easy to show that
\[
\sum_{a=1}^{D+2n} \Gamma^a \Gamma_a = E + R^2 \Delta. 
\]  
(3.10)

Now \(2i \sum_{a=1}^{D+2n} (M^a \Gamma_a + \Gamma^a M_a)\) can be expressed as
\[
- \sum_a X^a(\Delta - 1)R\partial_a - \sum_a R\partial^a X_a(\Delta - 1) + 2 \sum_a (T\partial^a R\partial_a + R\partial^a T\partial_a). 
\]  
(3.11)
The first sum can be rewritten as
\[
\sum_a X^a[(\Delta - 1), R]\partial_a + RE(\Delta - 1). 
\]
Using equation (2.9) we obtain
\[
\sum_a X^a[(\Delta - 1), R]\partial_a = \sum_a 2X^a \frac{1}{R} T\partial_a = 2T\frac{1}{R} E. 
\]
This leads to
\[
\sum_a X^a(\Delta - 1)R\partial_a = 2T\frac{1}{R} E + RE(\Delta - 1). 
\]
We also have
\[
\sum_a R\partial^a X_a(\Delta - 1) = R(d + E)(\Delta - 1). 
\]
Finally the third sum in (3.11) gives
\[
2 \sum_a (T\partial^a R\partial_a + R\partial^a T\partial_a) = 2T\frac{1}{R} E + 4TR\Delta. 
\]
Combining the results together we obtain
\[
i \sum_{a=1}^{D+2n} (M^a \Gamma_a + \Gamma^a M_a) = \frac{1}{2} R(\Delta - 1) + TR(\Delta + 1). 
\]  
(3.12)
Combining equations (3.9), (3.10) and (3.12) we readily obtain
\[
\sum_{a=1}^{D+2n} K^a K_a = -\frac{1}{4} R^2(\Delta - 1)^2 - R^2 \Delta - \frac{1}{2} T(\Delta + 1) \\
+ TR(\Delta + 1) + \frac{1}{2} R(\Delta - 1) - T^2.
\]
Using $R^2(\Delta + 1)^2 = (R(\Delta + 1))^2 - 2T(\Delta + 1)$, we immediately arrive at

$$\sum_{a=1}^{D+2n} K_a^2 = -\frac{1}{4}(R(\Delta + 1))^2 + TR(\Delta + 1) + \frac{1}{2}R(\Delta - 1) - T^2. \quad (3.13)$$

Recalling equation (3.8), we see that $K_0^2 + \sum_{a=1}^{D+2n} K_a^2 = 0$. This completes the proof. \□

With the help of Lemma 3.2, we can prove the following result, which will play a crucial role in understanding the $osp(2, D + 1|2n)$ representation appearing in the quantum Kepler problem.

**Proposition 3.1.** As a module for the subalgebra $osp(D + 1|2n)$ of $\mathfrak{g}_0$, the subspace $U_{g_0}(-l)$ of $U_{g_0}$ is irreducible and isomorphic to the irreducible rank 1 symmetric (in the $\mathbb{Z}_2$-graded sense) tensor of the natural module for $osp(D + 1|2n)$. The $osp(D + 1|2n)$ highest weight of $U_{g_0}(-l)$ is given by $(l, 0, \ldots, 0)$.

**Remark 3.2.** The analogous statement for the subalgebra $U(\mathfrak{g}_{-1})$ of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g} = osp(D + 3|2n)$ is completely false.

**Proof for Proposition 3.1.** We need the general facts about symmetric tensors of the natural module $V$ for $osp(D + 1|2n)$ discussed in Appendix A. Here it is more convenient to choose a basis $\{v^A | A = 0, 1, \ldots, D + 2n\}$ for $V$ such that the non-degenerate invariant bilinear form $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{C}$ gives the metric $(\eta^{AB})$. Now we use this matrix to define $\mathbb{D}$ and $\mathbb{D}^*$. Under the condition that $D > 2n + 1$, every symmetric tensor power $S(V)_l$ of $V$ is semisimple as a $osp(D + 1|2n)$-module by Lemma A.1. Furthermore, $S(V)_l = S(V)_l^0 \oplus S(V)_{l-2} \mathbb{D}^*$, and the harmonic space $S(V)_l^0$ defined by $S(V)_l^0 = \{w \in S(V)_l | \mathbb{D} w = 0\}$ is irreducible with highest weight $(l, 0, \ldots, 0)$.

Now we turn to $U_{g_0}$. The $\mathbb{Z}$-graded superalgebra homomorphism $S(V) \rightarrow U_{g_0}$ is also an $osp(D + 1|2n)$ map. The restriction of this map to any homogeneous component is nonzero. Since $(K)^2 = 0$ by Lemma 3.2, the map sends $S(V)_{l-2} \mathbb{D}^*$ to zero. Thus $U_{g_0}(-l)$ is the image of $S(V)_l^0$. Since $S(V)_l^0$ is irreducible, we must have $U_{g_0}(-l) \cong S(V)_l^0$ as $osp(D + 1|2n)$-modules. \□

4. **Solution of quantum Kepler problem on superspace**

4.1. **Induced representations.** Let $\Phi_0 = g_0 \Psi_0$ with $g_0 = \exp (-T \ln \sqrt{-2\xi_0})$, where $\Psi_0$ is the ground state wave function. Then $\Phi_0 = (-2\xi_0)^{d-1} e^{-R}$.

**Lemma 4.1.** The function $\Phi_0$ spans a 1-dimensional module for the parabolic subalgebra $\mathfrak{p}$ defined by equation (3.7).

**Proof.** It is evident that $J_{ab}(\Phi_0) = 0$ for all $a, b \geq 1$. Also the definition of $\Phi_0$ implies that it is an eigenvector of $h_0$. Therefore the lemma will be valid if $\Phi_0$ is annihilated by $i(J_{-1} - iJ_{-2})$, $A_a$ and $M_a - i\Gamma_a$ ($\alpha \geq 1$).

To proceed further, we need to use the following formulae:

$$\Delta(e^{-R}) = \left(1 - \frac{d-1}{R}\right)e^{-R},$$

$$T(e^{-R}) = \left(\frac{d-1}{2} - R\right)e^{-R}, \quad (4.1)$$

$$T \partial_a(e^{-R}) = X_a \left(1 - \frac{d-1}{2R}\right)e^{-R}.$$
Using the first and third formulae in the following equation

\[ A_a(e^{-R}) = \frac{i}{2}X_a(\Delta + 1)(e^{-R}) - iT \partial_a(e^{-R}), \]

we immediately arrive at \( A_a(e^{-R}) = 0 \). Note that

\[ M_a(e^{-R}) = A_a(e^{-R}) - iX_a e^{-R} = -iX_a e^{-R}. \]

This easily leads to

\[ (M_a - i\Gamma_a)(e^{-R}) = 0, \quad \forall a \geq 1. \]

Finally by using the first and second formulae of (1.1), we can show that

\[ i(J_{-1} - iJ_{-2})(e^{-R}) = 0. \]

Let \( \mathcal{H} = \mathcal{U}_g \Phi_0 \), which has a natural \( g \)-module structure. Then

\[ \mathcal{H} = \mathcal{U}_{g-1} \Phi_0 = \oplus_{i=0}^{\infty} \mathcal{U}_{g-1}(-l) \Phi_0. \]

It follows that the operator \( h_0 \) is diagonalisable in \( \mathcal{H} \) and its eigenvalues must be of the form \(-\frac{d-1}{2} - k \) (\( k \in \mathbb{Z}_+ \)).

If \( L_\lambda^0 \) denotes the irreducible \( p \)-module with highest weight \( \lambda \), we have the generalised Verma module \( V_\lambda = \mathcal{U}(g) \otimes_{\mathcal{U}(p)} L_\lambda^0 \). The \( p \)-module \( L_\lambda^0 \) is 1-dimensional if

\[ \lambda = \left( -\frac{d-1}{2}, 0, \ldots, 0 \right), \]  

(4.2)

and it is evident that \( \mathcal{H} \) is a quotient module of the corresponding generalised Verma module \( V_\lambda \).

The highest weight vector \( \Phi_0 \) of \( \mathcal{H} \) generates a module for the \( so(2, D+1) \) subalgebra of \( osp(2, D+1|2n) \) with a highest weight \( \lambda^0 = (-\frac{d-1}{2}, 0, \ldots, 0) \) (the first \( \left\lceil \frac{D+3}{2} \right\rceil \) entries of (1.2)). This module is necessarily infinite dimensional as the highest weight is not dominant. This in particular implies that

\[ \mathcal{U}_{g-1}(-l) \Phi_0 \neq 0, \quad \text{for all } l. \]

Another fact which can be deduced is that if \( \mathcal{H} \) contains any nontrivial submodule \( \mathcal{H}_1 \), then \( \mathcal{H}/\mathcal{H}_1 \) has to be infinite dimensional as well, as the image of \( \Phi_0 \) generates an infinite dimensional \( so(2, D+1) \)-submodule in the quotient.

Since \( \Phi_0 \) has trivial \( g_0 \)-action and \( \mathcal{U}_{g-1}(-l) \) is irreducible under the action of \( g_0 \) by Proposition 3.1, we immediately see that \( \mathcal{U}_{g-1}(-l) \Phi_0 \) is irreducible as a \( g_0 \)-module. Now let \( \mathcal{H}_1 \) be a non-trivial \( g \)-submodule of \( \mathcal{H} \), then there exists some integer \( l_0 \) such that all \( \mathcal{U}_{g-1}(-l) \) with \( l \geq l_0 \) belong to \( \mathcal{H}_1 \). This contradicts the fact that \( \mathcal{H}/\mathcal{H}_1 \) is infinite dimensional. Thus \( \mathcal{H} \) must be irreducible as a \( g \)-module.

Let \( L_\mu^{D+1|2n} \) denote the irreducible \( osp(D+1|2n) \)-module with highest weight \( \mu \). We have proved the following theorem.

**Theorem 4.1.**

(1) As an \( osp(2, D+1|2n) \)-module \( \mathcal{H} \) is irreducible.

(2) The restriction of \( \mathcal{H} \) is isomorphic to \( \bigoplus_{l=0}^{\infty} L_{l(0,0,\ldots,0)}^{D+1|2n} \) as an \( osp(D+1|2n) \)-module.

**Remark 4.1.** It is known [1] [EHW] that \( \lambda^0 \) does not give rise to a unitarisable highest weight \( so(2, D+1) \)-module. Thus the \( so(2, D+1) \)-submodule of \( \mathcal{H} \) generated by \( \Phi_0 \) is not unitarisable, and this in turn implies that \( \mathcal{H} \) as an \( osp(2, D+1|2n) \)-module is not unitarisable.
4.2. Solution of quantum Kepler problem on superspace. In this subsection we use the results on the irreducible \(osp(D + 3|2n)\)-module \(H\) obtained to determine the bound states of the quantum Kepler problem on the superspace \(\mathbb{R}^{D|2n}\). Let \(\mathcal{H}_k = U_{g,-1}(-k)\Phi_0\), and for \(k \in \mathbb{Z}_+\),

\[
\mathcal{E}_k = -\frac{1}{2} \left( \frac{1}{d-1 + k} \right)^2.
\]

Define \(g_k = \exp \left( -T \ln \sqrt{-2\mathcal{E}_k} \right)\), and set

\[
\tilde{\mathcal{H}} = \bigoplus_{k=0}^{\infty} \tilde{\mathcal{H}}_k, \quad \text{where} \quad \tilde{\mathcal{H}}_k = \{ g_k^{-1} v \mid v \in \mathcal{H}_k \}.
\]

We have the following result.

**Theorem 4.2.**

1. The quantum Hamiltonian operator \(H\) is diagonalisable when acting on \(\tilde{\mathcal{H}}\) and has the eigenvalues \(\mathcal{E}_l\) for \(l \in \mathbb{Z}_+\).

2. The dimension of the subspace \(\mathcal{H}_l\) is given by the formula

\[
\sum_{k=0}^{l} \binom{D + k}{k} \binom{2n}{l-k} - \binom{2n}{l-2-k},
\]

where the binomial coefficient \(\binom{a}{b}\) is assumed to be zero if \(b < 0\) or \(b > a\).

3. The subspace \(\tilde{\mathcal{H}}_I\) (\(I = 0, 1, \ldots\)) is the entire \(\mathcal{E}_I\)-eigenspace of the quantum Hamiltonian operator \(H\).

**Proof.** It follows from Theorem 2.1 that ever nonzero vector in \(\tilde{\mathcal{H}}_l\) is indeed an eigenvector of the quantum Hamiltonian operator \(H\) with eigenvalue \(\mathcal{E}_l\). To prove the formula for the dimension of \(\mathcal{H}_l\), we note that \(\dim \tilde{\mathcal{H}}_I = \dim \mathcal{H}_I\), and by Theorem 4.1, \(\dim \tilde{\mathcal{H}}_I = \dim L^{D+1|2n}_{(l,0,\ldots,0)}\). We have

\[
\dim L^{D+1|2n}_{(k,0,\ldots,0)} = \dim S(V)_l - \dim S(V)_{l-2}.
\]

It is well known that

\[
\dim S(V)_l = \sum_{k=0}^{l} \binom{D + k}{k} \binom{2n}{l-k}.
\]

Hence the formula for \(\dim \tilde{\mathcal{H}}_I\) follows.

To prove the third claim, we need some input from the Schrödinger equation. Let us return to the original form (2.1) of the equation. Because of the \(osp(D|2n)\) symmetry of the equation, the wave functions \(\Psi\) can be written as \(\mathbb{C}\)-linear combinations of functions of the form \(\frac{\omega}{R^n} \chi_l\) by the first part of Lemma A.1, where \(\omega\) is a harmonic polynomial (that is, \(\Delta \omega = 0\)) in the coordinates \(X^a\), which is homogeneous of degree \(l\). The function \(\chi_l\) depends on \(R\) only, and the factor \(R^{-l}\) is introduced for convenience. Then equation (2.1) reduces to the following equation for \(\chi_l\)

\[
-\frac{1}{2} \left( \frac{d^2 \chi_l}{dR^2} + \frac{d-1}{R} \frac{d\chi_l}{dR} - \frac{l(l-2+l)}{R^2} \chi_l \right) - \frac{1}{R} \chi_l = \mathcal{E}_l \chi_l.
\]
This has the same form as the radial part of the Schrödinger equation for the Kepler problem on \( \mathbb{R}^d \), and can be solved in terms of generalised Laguerre polynomials (see, e.g., [Ai (4), (14)]). Such a solution, when its argument \( R \) is replaced by \( r \), is square integrable over the positive half line \( \mathbb{R}_+ \) with respect to the measure \( r^{d-1} dr \). For a fixed \( l \), we denote by \( \chi_{l,j} \) (\( j = 0, 1, 2, \ldots \)) the independent solutions of (4.3).

Consider \( \left\langle \frac{\omega}{R} \chi_{l,j} | \frac{\omega}{R} \chi_{l,j} \right\rangle = \int_{\mathbb{R}_+} \omega (\frac{\chi_{l,j}}{R})^2 \), the integrant of which vanishes exponentially fast as \( r \) goes to infinity. Thus we only need to analyse the \( r \to 0 \) end of the integral to see whether the integral converges. By inspecting the form of the generalised Laguerre polynomials we can see that at the worst, the integral behaves like

\[
\int_{\mathbb{R}_+} \left( \frac{\omega}{R} \right)^{l} \left( \chi_{l,j}(R) \right)^2 \propto (\Xi^2)^n \int_0^\infty (\chi_{l,j}(r))^2 r^{d-1} dr.
\]

It follows from the square integrability of \( \chi_{l,j} \) over \( \mathbb{R}_+ \) with respect to the measure \( r^{d-1} dr \) that all \( \frac{\omega}{R} \chi_{l,j} \) belong to the set \( \mathcal{F} \) defined in subsection 2.2.

For a fixed \( l \), the solutions \( \chi_{l,j} \) of (4.3) respectively correspond, in a one-to-one manner, to the eigenvalues \( \mathcal{E}_{l+j} \) of the quantum Hamiltonian operator (2.3). In particular, the ground state corresponds to the energy eigenvalue \( \mathcal{E}_0 \). Thus for a given non-negative integer \( I \), the eigenspace of \( H \) corresponding to the energy \( \mathcal{E}_I \) is spanned by \( \frac{\omega}{R} \chi_{l,j} \) for all homogeneous harmonic polynomials \( \omega \) of degree \( l \), and for all \( l, j \geq 0 \) with \( l + j = I \). As a module over \( osp(D|2n) \), the \( \mathcal{E}_I \)-eigenspace is isomorphic to \( \bigoplus_{l=0}^I L^{D+2n}_{(l,0,0,\ldots,0)} \). By using the branching rule (4.3), we have

\[
\bigoplus_{l=0}^I L^{D+2n}_{(l,0,0,\ldots,0)} \cong L^{D+1|2n}_{(I,0,0,\ldots,0)} \cong \mathcal{F}_I.
\]

**Remark 4.2.** In view of Remark 2.1, the generalised Kepler problem on the superspace is equivalent to an eigenvalue problem involving a system of partial differential equations. It is a rather non-trivial matter that the problem is still soluble.

### Appendix A. Symmetric superalgebra

For proving Theorem 3.1 we need some general facts about symmetric tensors of the natural module \( V \) for \( osp(M|2n) \), which we briefly discuss in this appendix. In general the symmetric tensors are not completely reducible, and this complicates the matter of decomposing these representations enormously. Fortunately we only need to consider the case \( M = 2n \) for the purpose of this paper. In this case, the symmetric tensor are semi-simple, as we shall show.

We work over the complex field and use the root system described in Section 3 for \( osp(M|2n) \). For the sake of being concrete, we label the basis elements \( \epsilon_i \) and \( \delta_j \) of the weight space by \( i = 1, 2, \ldots, \left[ \frac{M}{2} \right] \) and \( j = 1, 2, \ldots, n \). Let \( v^a \) with \( a = 1, 2, \ldots, M + 2n \) be a weight basis for the natural module \( V \), where the weight \( wt(v^a) \) of \( v^a \) is greater than that of \( v^b \) if \( a < b \). Then the highest weight is \( \epsilon_1 \), and the lowest weight is \( -\epsilon_1 \). We assume that the highest weight vector \( v^i \) is even.

There exists a non-degenerate \( osp(M|2n) \)-invariant bilinear form \( \langle \ , \rangle : V \times V \to \mathbb{C} \), which is unique up to scalar multiples. Note that \( \langle v^a, v^b \rangle \neq 0 \) if and only if \( wt(v^a) = -wt(v^b) \). Let \( \eta^{ab} = \langle v^a, v^b \rangle \) and form the matrix \( \eta^{-1} = (\eta^{ab}) \). We also write \( \eta = (\eta_{ab}) \) for the inverse matrix.

Let \( S(V) \) be the \( \mathbb{Z}_2 \)-graded symmetric superalgebra of \( V \), that is, the superalgebra generated by the \( v^a \) subject to the relations that \( v^a v^b = -v^b v^a \) if both \( v^a \) and \( v^b \) are
odd, and $v^a v^b = v^b v^a$ otherwise. Then this is a $\mathbb{Z}$-graded algebra $S(V) = \bigoplus_{l=0}^{\infty} S(V)_l$, with elements of $V$ having degree 1.

Define operators

\[
\square^* = \sum_{a=1}^{M+2n} v^a \eta_{ab} v^b, \quad \square = \sum_{a=1}^{M+2n} \eta^{ba} \frac{\partial}{\partial v^a} \frac{\partial}{\partial v^b}, \quad T = \frac{M + 2n - 1}{2} + \sum_{a=1}^{M+2n} v^a \frac{\partial}{\partial v^a},
\]

which all commute with $osp(M|2n)$. The operators satisfy the commutation relations

\[
[T, \square^*] = 2\square^*, \quad [T, \square] = -2\square, \quad [\square^*, \square] = -T,
\]

thus their real spanned is isomorphic to the Lie algebra $su(1,1)$.

We consider $S(V)$ as a complex module for $su(1,1)$. The operator $T$ is diagonalisable with eigenvalues $\frac{M + 2n - 1}{2} + l$ for $l \in \mathbb{Z}_+$. Now we need to assume that

\[
M - 2n > 1.
\]

Evidently every submodule of $S(V)$ is of highest weight type. Since the eigenvalues of $-T$ are all strictly negative, the sub-representations are necessarily infinite dimensional. The eigenvalues of the quadratic Casimir of $su(1,1)$ corresponding to distinct highest weights $\lambda_l := -\left(\frac{M + 2n - 1}{2} + l\right)$ with $l \in \mathbb{Z}_+$ are different, thus $S(V)$ decomposes into a direct sum $S(V) = \bigoplus_l C^{(l)}$, where each submodule $C^{(l)}$ has the property that all is irreducible subquotients are isomorphic with the same highest weight $\lambda_l$. An irreducible $su(1,1)$-module with highest weight $\lambda_l$ for $l \in \mathbb{Z}_+$ is unitarisable. It follows that every isotypical component is unitarisable and hence completely reducible. Thus $S(V)$ is completely reducible with respect to $su(1,1)$.

This in particular implies that every weight vector in $S(V)_l$ can be uniquely expressed as $v + w$ with $v \in S(V)_l \cap ker \square$ being a highest weight vector, and $w \in \square^* S(V)_l$. We can also deduce that $\square^* \square S(V)_l = S(V)_l \cap im \square^*$ for each $S(V)_l$ by noting the obvious fact that if $v$ belongs to a weight space in an irreducible $su(1,1)$-module, then $\square^* \square v$ is a nonzero scalar multiple of $v$ unless $v$ is the highest weight vector. Thus we have the following vector space decomposition

\[
S(V) = ker \square \oplus im \square^*.
\]

Since the $su(1,1)$ algebra commutes with $osp(M|2n)$, equation (A.2) is a decomposition of $osp(M|2n)$-modules.

Denote $S(V)_l \cap ker \square$ by $S(V)_l^0$ and call it the harmonic space of $S(V)_l$. It is easy to see that for all $l \leq 2$, $S(V)_l^0$ is isomorphic to the irreducible $osp(M|2n)$-module with highest weight $(l,0,\ldots,0)$. Assume that this is also true for $l > 2$, then the highest weight vector for $S(V)_l^0$ is $(v^1)^l$. Each highest weight vector in $S(V)_{l+1}^0$ must contain a term $(v^1)^l v^b$ for some $b$. In order for the corresponding weight to be dominant, $v^b$ is either $v^1$, $v^2$ or the lowest weight vector of $V$, which respectively have weights $(l + 1,0,\ldots,0)$, $(l,1,0,\ldots,0)$ and $(l - 1,0,\ldots,0)$. We can write down all the vectors of the same weights in $S(V)_{l+1}$, and try to make linear combinations of them to obtain highest weight vectors. Simple calculations show that there can not be any highest weight vector with weight $(l,1,0,\ldots,0)$ as $S(V)$ is the symmetric superalgebra. The highest weight vector corresponding to the weight $(l - 1,0,\ldots,0)$ is $(v^1)^{l-1} \square^*$, which belongs to $im \square^*$ but not $S(V)_{l+1}^0$. This proves that $S(V)_{l+1}^0$ is isomorphic to the irreducible $osp(M|2n)$-module with highest weight $(l + 1,0,\ldots,0)$.

To summarise, we have the following result.

**Lemma A.1.** Keep notations as above.
(1) Under the condition $M > 2n$, $S(V)_l = S(V)_0^0 \oplus S(V)_{l-2}^{\square^*}$ as $\text{osp}(M|2n)$-module, and $S(V)_0^0$ is isomorphic to the irreducible module $L_{(l,0,...,0)}^{M|2n}$ with highest weight $(l, 0, \ldots, 0)$.

(2) As an $\text{su}(1, 1) \times \text{osp}(M|2n)$-module, $S(V) \cong \bigoplus_{L=0}^{\infty} L^{(l)} \otimes L^{M|2n}_{(l,0,...,0)}$ where $L^{(l)}$ is the irreducible $\text{su}(1, 1)$-module with highest weight $-\frac{M+2n-1}{2} - l$.

A further result which can be deduced from the above lemma is the branching rule of the irreducible symmetric tensor module $L^{M|2n}_{(l,0,...,0)}$ to $\text{osp}(M - 1|2n)$-modules. We assume that the condition $M - 1 - 2n > 1$ is satisfied. Then the symmetric tensor powers of the natural module $V'$ for $\text{osp}(M - 1|2n)$ is completely reducible by the lemma. We have the following $\text{osp}(M - 1|2n)$-module isomorphism $S(V)_l \cong \bigoplus_{k=0}^{l} S(V'_{l-k})$. Then by the first part of lemma A.1,

$$S(V)_l \cong \left( \bigoplus_{k=0}^{l} S(V')_{l-k} \right) \bigoplus \left( \bigoplus_{k=0}^{l-2} S(V'_{l-k}) \right),$$

where $S(V')_0$ is the harmonic subspace of $S(V')_i$. Using the first part of lemma A.1 to the left hand side, and also noting that the second term on the right hand side can be re-written as $\bigoplus_{k=0}^{l-2} S(V')_{l-k} \cong S(V)_{l-2}$, we obtain

$$S(V)_0^0 \bigoplus S(V)_{l-2} \cong \left( \bigoplus_{k=0}^{l} S(V')_{l-k} \right) \bigoplus S(V)_{l-2}.$$

That is, the following branching rule holds:

$$L^{M|2n}_{(l,0,...,0)} \cong \bigoplus_{k=0}^{l} L^{M-1|2n}_{(l-k,0,...,0)}$$

as $\text{osp}(M - 1|2n)$-module. \hspace{1cm} (A.3)

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