Squeezing of a coupled state of two spinors

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Abstract. The notion of spin squeezing involves reduction in the uncertainty of a component of the spin vector \(\vec{S}\) below a certain limit. This aspect has been studied earlier [1, 2] for pure and mixed states of definite spin. In this paper, this study has been extended to coupled spin states which do not possess sharp spin value. A general squeezing criterion has been obtained by requiring that a direct product state for two spinors is not squeezed. The squeezing aspect of entangled states is studied in relation to their spin-spin correlations.

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1. Introduction

The notion of squeezing which involves reduction in the variance (uncertainty) of an observable below a standard quantum limit has been studied [3, 4] in the literature for an oscillator and a bosonic field. This notion has been extended to spin systems with arbitrary but sharp spin values [5, 3, 7].

Quite recently, we have [1, 2] analysed in detail the notion of spin squeezing and looked into several aspects of squeezing in the case of oriented and non-oriented systems, and in the case of a coupled spin \( s \) system composed of \( 2s \) spinor states. Generalizing the squeezing criterion given by Kitagawa and Ueda [5], we have made a detailed study of pure as well as mixed states. Kitagawa and Ueda have suggested in their paper that the occurrence of squeezing is a consequence of quantum spin-spin correlations that exist among the \( 2s \) spinors states which together constitute the spin \( s \) state. Our study reveals that all oriented states are not squeezed but non-oriented states exhibit squeezing. In the case of pure spin-1 state, our claim is that the notion of non-oriented is synonymous with the notion of squeezing.

In the light of the above studies on spin systems it becomes relevant to extend the ideas of squeezing to bipartite systems which do not possess a sharp value of spin. Such systems can arise due to coupling of two systems with sharp angular momenta. An additional aspect that arises in such a coupled state is whether a given state is entangled or not [8]. An entangled state cannot be written as a product of the spin states of the individual systems but only as a linear combination of such products. Further, it follows that the self and mutual spin-spin correlations will be present in an entangled state. It is therefore necessary to look for possible relationships among the three aspects viz, squeezing, entanglement and spin-spin correlations.

The present paper which addresses these intrinsic notions is organized as follows. In section 2, we look at the properties of coupled states and discuss the conditions to be satisfied by a coupled state to be a direct product state and an entangled state. In the next section, we take up the discussion on the squeezing aspect for a coupled state which may be either entangled or not. In the case of uncoupled state, the mutual spin-spin correlation will be absent and consequently this has to be taken into consideration while defining the squeezing criterion. Taking this into consideration we propose a generalization of the squeezing criterion for a coupled state of two spin systems. A detailed presentation of this is given in section 3. The dependence of squeezing on entanglement and on correlations is explored. Section 4 deals with the time evolution of a coupled pure spin state in the presence of a spin-spin interaction. Our aim here is to show that a coupled pure spin state undergoing such interaction develops squeezing as time elapses, even though, it may not have squeezing initially. We also look at the manner in which squeezing depends on spin-spin correlations. The last section is devoted for comments and revision.
2. Properties of coupled states

The $s = \frac{1}{2}$ states enjoy an exalted status in quantum theory since an arbitrary spin $\frac{1}{2}$ state can always be looked upon either as a $|\frac{1}{2}\frac{1}{2}\rangle$ state or as $|\frac{1}{2} - \frac{1}{2}\rangle$ state with respect to an appropriately chosen Cartesian frame. Thus a spinor is always oriented with respect to some $z$-axis. Following the Schwinger construction [9], any higher spin $s > \frac{1}{2}$ can be realized in terms of $2s$ spin $\frac{1}{2}$ states, but it cannot always be looked upon as spin $|ss\rangle$ or $|s - s\rangle$ state with respect to any choice of the Cartesian frame.

For $s > \frac{1}{2}$ an oriented state is identified as an $|sm\rangle$ state in an appropriately chosen Cartesian frame, with $m$ taking any one of values $-s, \ldots, s$. Such a state is cylindrically symmetric with respect to $\hat{z}$-axis which is also referred to as the axis of quantization. All other states are referred to as non-oriented states. In other words a non-oriented state cannot be looked upon as an $|sm\rangle$ state with respect to any frame. A discussion of this idea for a system with sharp angular momentum has been done earlier [1, 10]. We now would like to characterize a coupled spin system based on these aspects. If $\mathcal{H}_1$ and $\mathcal{H}_2$ are the spin spaces of two systems with spins $s_1$ and $s_2$ respectively, then, a coupled state of these two systems is a state belonging to $\mathcal{H}_1 \otimes \mathcal{H}_2$ and can be expressed as

$$|\psi_{12}\rangle = \sum_{ij} a_{ij} |\phi_i\rangle \otimes |\zeta_j\rangle; \quad \sum_{ij} |a_{ij}|^2 = 1 \quad ,$$

(1)

where $|\phi_i\rangle (i = -s_1, \ldots, s_1)$ and $|\zeta_j\rangle (j = -s_2, \ldots, s_2)$ are the angular momentum basis states of the subsystems $s_1$ and $s_2$ respectively. In the study of coupling of two angular momenta, the basis states are usually chosen with respect to some common axis of quantization. For a coupled state, one possible characterization could be to relax this usual choice and ask whether a state $|\psi_{12}\rangle$ is an eigen state for the four mutually commuting operators $J_1^2$, $\hat{J}_1 \cdot \hat{Q}_1$, $J_2^2$ and $\hat{J}_2 \cdot \hat{Q}_2$ where $\hat{Q}_1$, $\hat{Q}_2$ refer to two arbitrary axes one for each system. While every state $|\psi_{12}\rangle$ is an eigen state of $J_1^2$ and $J_2^2$, the eigen states for other two operators form a subset of the space $\mathcal{H}_1 \otimes \mathcal{H}_2$.

It is well known that in the case of two spinors the individual eigen states for $J_1^2$ and $\hat{J}_1 \cdot \hat{Q}_1$, expressed in terms of basis vectors referred to a common axis of quantization $\hat{z}_0$ of a frame $x_0y_0z_0$ can be written in the form

$$|\psi_i\rangle = \begin{pmatrix} \cos \frac{\theta_i}{2} \\ \sin \frac{\theta_i}{2} e^{i\phi_i} \end{pmatrix}_{z_0} , \quad 0 \leq \theta_i \leq \pi , \quad 0 \leq \phi_i \leq 2\pi , \quad i = 1, 2 \quad ,$$

(2)

where $\theta_i, \phi_i$ are the polar angles of $\hat{Q}_i$, with respect to the common frame. From this it is clear that the direct product state

$$|\psi_{12}^{(a)}\rangle = \begin{pmatrix} \cos \frac{\theta_1}{2} \\ \sin \frac{\theta_1}{2} e^{i\phi_1} \end{pmatrix}_{z_0} \otimes \begin{pmatrix} \cos \frac{\theta_2}{2} \\ \sin \frac{\theta_2}{2} e^{i\phi_2} \end{pmatrix}_{z_0} = \begin{pmatrix} \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \\ \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{i\phi_2} \\ \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{i\phi_1} \\ \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{i(\phi_1 + \phi_2)} \end{pmatrix}_{z_0}$$

(3)

is an eigen state of $J_1^2$, $\hat{J}_1 \cdot \hat{Q}_1$, $J_2^2$ and $\hat{J}_2 \cdot \hat{Q}_2$ satisfying (with $\hbar = 1$)

$$J_1^2 |\psi_{12}^{(a)}\rangle = \frac{1}{2} \left( \frac{1}{2} + 1 \right) |\psi_{12}^{(a)}\rangle$$

(4)
Squeezing of a coupled state of two spinors

\[ \vec{J}_1 \cdot \hat{Q}_1 |\psi^{(a)}_{12}\rangle = \frac{1}{2} |\psi^{(a)}_{12}\rangle \]  
\[ \vec{J}_2^2 |\psi^{(a)}_{12}\rangle = \frac{1}{2} \left( \frac{1}{2} + 1 \right) |\psi^{(a)}_{12}\rangle \]  
\[ \vec{J}_2 \cdot \hat{Q}_2 |\psi^{(a)}_{12}\rangle = \frac{1}{2} |\psi^{(a)}_{12}\rangle \]  

The other three common eigen states are

\[ |\psi^{(b)}_{12}\rangle = \begin{pmatrix} 
\cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \\
-\cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{i\phi_2} \\
\sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{i\phi_1} \\
-\sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{i(\phi_1+\phi_2)} 
\end{pmatrix} \]  
\[ |\psi^{(c)}_{12}\rangle = \begin{pmatrix} 
\sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \\
\sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{i\phi_2} \\
-\cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{i\phi_1} \\
-\cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{i(\phi_1+\phi_2)} 
\end{pmatrix} \]  
\[ |\psi^{(d)}_{12}\rangle = \begin{pmatrix} 
\sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \\
-\sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{i\phi_2} \\
-\cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{i\phi_1} \\
\cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{i(\phi_1+\phi_2)} 
\end{pmatrix} \]  

From this it is clear that every direct product state is a common eigen state of \( J_1^2 \), \( \vec{J}_1 \cdot \hat{Q}_1 \), \( J_2^2 \) and \( \vec{J}_2 \cdot \hat{Q}_2 \) for some \( \hat{Q}_1 \) and \( \hat{Q}_2 \) and that given one such state, three orthogonal eigen states can be constructed, which of course span the direct product space \( \mathcal{H}_1 \otimes \mathcal{H}_2 \). Conversely it follows that the common eigen states of the above four operators have to be direct product states only.

In the case of a sharp spin \( s \) state we have characterized a pure state as oriented if it happens to be an angular momentum state \( |sm\rangle \) with respect to some quantization axis. This property can be extended to a bipartite system of two spins \( s_1 \) and \( s_2 \). A pure state \( |\psi_{12}\rangle \) of a bipartite system can be regarded as oriented if

\[ \vec{J}_1 \cdot \hat{Q}_1 \vec{J}_2 \cdot \hat{Q}_2 |\psi_{12}\rangle = m_1 m_2 |\psi_{12}\rangle \]  

for some \( \hat{Q}_1 \) and \( \hat{Q}_2 \). If \( s_1 \) and \( s_2 \) are arbitrary, then every direct product state \( |\xi_1\rangle \otimes |\xi_2\rangle \) is not necessarily oriented. However if \( s_1 = s_2 = \frac{1}{2} \), every direct product state is always oriented. This follows from the homomorphism between SU(2) and O(3).

A direct product state of two spinors is thus non-entangled, oriented and a common eigenstate of four operators \( J_1^2 \), \( \vec{J}_1 \cdot \hat{Q}_1 \), \( J_2^2 \), \( \vec{J}_2 \cdot \hat{Q}_2 \). This implies that the rest of the states in the Hilbert space \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) do not share the above properties. They are not only non-oriented but also possess entanglement. Let us now look at the nature of spin-spin correlations present in these coupled states.
Squeezing of a coupled state of two spinors

The spin-spin correlations according to Kitagawa and Ueda are responsible for the existence of squeezing. In their paper [5] they suggest that every state of a spin $s$ system can be visualized as a coupled state of $2s$ spinor states and claim that the squeezing behaviour of the spin $s$ state is due to the correlations that exist among the $2s$ spinor states. In our earlier paper [1, 2], we have indeed given an explicit method of construction of a general pure spin $s$ state in terms of $2s$ spinor states. Based on this construction we have analysed the nature of a squeezed spin $s$ state and in the case of $s = 1$, we have shown that the squeezing aspect is intimately connected with the pair-wise correlations defined through

$$C_{ij}^{\mu\nu}(s) = \langle S_i^\mu S_j^\nu \rangle - \langle S_i^\mu \rangle \langle S_j^\nu \rangle,$$

where $S_i^\mu$ is the $\mu$th component of the $i$th spin $\vec{S}_i$. We wish to call these as self correlations.

While these are absent in the case of a single spinor, there would be large number of such correlations for large $s$. These coupled states of arbitrary spin $s_1$ and $s_2$ not only possess the above ‘self correlations’ $C_{ij}^{\mu\nu}(s_1)$ and $C_{ij}^{\mu\nu}(s_2)$ but also the ‘mutual correlations’ between $s_1$ and $s_2$. These mutual correlations can be defined in an analogous way as

$$D_{12}^{\mu\nu} = \langle S_{1\mu} S_{2\nu} \rangle - \langle S_{1\mu} \rangle \langle S_{2\nu} \rangle$$

where $S_{1\mu}(S_{2\nu})$ is the $\mu$th ($\nu$th) component of the spin vector $\vec{S}_1$ ($\vec{S}_2$). For a direct product state of two spinors it is easy to see that both $C_{ij}^{\mu\nu}$ and $D_{12}^{\mu\nu}$ are zero. A direct product state with $s_1$ or $s_2$ exceeding $\frac{1}{2}$ may possess self correlations but there are no mutual correlations. An entangled pure state on the other hand certainly possess mutual correlations always, although there may or may not be self correlations. Our work in this paper is limited to the discussion of a coupled state of two spinors and we will be using the ideas presented here to arrive at the right criterion for the squeezing of such a bipartite state.

It may be mentioned here that the most general state of two spins $s_1$ and $s_2$ is a mixed state which is not only entangled but also rich in both self and mutual correlations. In addition, it may possess statistical correlations owing to the distribution of the spins in it.

3. Squeezing criterion for a general bipartite state

The concept of squeezing of a spin $s$ system as emphasized by Kitagawa and Ueda [5] is associated with quantum correlations. According to Kitagawa and Ueda a spin state $|\phi\rangle$ is said to be squeezed if in that state

$$\Delta(\vec{S} \cdot \hat{n}_\perp)^2 < \frac{[\langle \vec{S} \cdot \hat{n}\rangle]^2}{2}$$

where $\hat{n}$ is a unit vector along $\langle \vec{S} \rangle$, called the mean spin direction and $\hat{n}_\perp$ is orthogonal to $\hat{n}$. This condition clearly distinguishes a squeezed state from other states in an intrinsic way.

The task now is to extend this to the case of a bipartite system which in general does not possess a sharp value of the total spin. For example, a coupled state of two spinors can be a superposition of triplet ($s = 1$) as well as the singlet states ($s = 0$). A possible choice for the criterion is to consider the spin components $S_{1\mu} + S_{2\mu}$ of the total spin $\vec{S} = \vec{S}_1 + \vec{S}_2$ with respect to a common frame $x_0y_0z_0$ and define the criterion exactly as in (Eq.14) with the
Squeezing of a coupled state of two spinors

understanding that \( \hat{n} \) denotes the direction of \( \langle \vec{S}_1 + \vec{S}_2 \rangle \) and \( \vec{S}_\perp \), \( \vec{S}_\parallel \) denote the components of \( \vec{S} = \vec{S}_1 + \vec{S}_2 \) along and perpendicular to \( \hat{n} \) respectively. We have looked into this choice which leads to a conclusion that certain direct product states such as

\[
|\psi\rangle = \left( \begin{array}{c}
\frac{\sqrt{3}}{2} \\
\frac{1}{2}
\end{array} \right) \otimes \left( \begin{array}{c}
\frac{\sqrt{3}}{2} \\
-\frac{1}{2}
\end{array} \right) = \left( \begin{array}{c}
\frac{3}{4} \\
-\frac{\sqrt{3}}{4} \\
-\frac{\sqrt{3}}{4} \\
\frac{1}{4}
\end{array} \right),
\]

(15)

will possess squeezing if we consider such a choice. This is undesirable since the two subsystems may be totally independent, non squeezed and noninteracting and if we formally define direct product states of the two, these would be squeezed if we employ the criterion suggested above.

We would therefore like to search for an appropriate criterion for squeezing of bipartite states taking the above aspect into consideration. As an aid in this direction, it may be noted here that in problems aimed at determining correlations in a bipartite system, when the subsystems are physically separated, two observers make measurements in frames of their own choice. This freedom of choice of frame, in the context of entanglement, arises from the property that the entanglement of a bipartite system is invariant under local rotation of frames (local unitary transformations on individual states) describing the subsystems. It is therefore necessary to allow for this freedom of choice of local frames for discussing squeezing and this is done as follows.

Suppose a bipartite state \( |\psi_{12}\rangle \) (Eq.1) of two spins is specified with respect to some frame \( x_0 y_0 z_0 \) (say). Suppose \( \{\hat{n}_1, \hat{n}_1\perp, \hat{n}_{1\perp,} = \hat{n}_1 \times \hat{n}_1\perp\} \) and \( \{\hat{n}_2, \hat{n}_2\perp, \hat{n}_{2\perp,} = \hat{n}_2 \times \hat{n}_2\perp\} \) denote two sets of mutually orthogonal directions. Now it is easy to see that the components of spin operators \( \vec{S}_1 \) and \( \vec{S}_2 \) with respect to these triplets satisfy the usual angular momentum commutation relations,

\[
[\vec{S}_1, \hat{n}_1\perp + \vec{S}_2, \hat{n}_2\perp, \vec{S}_1, \hat{n}_{1\perp,} + \vec{S}_2, \hat{n}_{2\perp,}] = i \left( \vec{S}_1, \hat{n}_1 + \vec{S}_2, \hat{n}_2 \right)
\]

(16)

and the uncertainty relationship for these operators takes the form

\[
\Delta \left( \vec{S}_1, \hat{n}_1 + \vec{S}_2, \hat{n}_2 \right)^2 \Delta \left( \vec{S}_1, \hat{n}_{1\perp,} + \vec{S}_2, \hat{n}_{2\perp,} \right)^2 \geq \frac{\left( \langle \vec{S}_1, \hat{n}_1 \rangle + \langle \vec{S}_2, \hat{n}_2 \rangle \right)^2}{4},
\]

(17)

where,

\[
\Delta \left( \vec{S}_1, \hat{a} + \vec{S}_2, \hat{b} \right)^2 = \langle \psi_{12} \left| \left( \vec{S}_1, \hat{a} + \vec{S}_2, \hat{b} \right) \right| \psi_{12} \rangle - \langle \psi_{12} \left| \left( \vec{S}_1, \hat{a} + \vec{S}_2, \hat{b} \right) \right| \psi_{12} \rangle^2
\]

(18)

Suppose now \( \hat{n}_1 \) and \( \hat{n}_2 \) denote the mean spin directions of the individual spinor states, defined through

\[
\hat{n}_i = \frac{\langle \psi_{12} | \vec{S}_i | \psi_{12} \rangle}{\langle \psi_{12} | \vec{S}_i | \psi_{12} \rangle}, \quad i = 1, 2
\]

(19)

and \( \hat{n}_{i\perp} \) are directions such that

\[
\hat{n}_{i\perp} \cdot \hat{n}_i = 0, \quad i = 1, 2
\]

(20)
Squeezing of a coupled state of two spinors

These directions \((\hat{n}_i, \hat{n}_{i\perp}, \hat{n}_{i\perp'})\) define the individual Lakin frames \([I]\) as we have
\[
\langle \vec{S}_i \cdot \hat{n}_{i\perp} \rangle = 0, \quad i = 1, 2. \tag{21}
\]
The criterion we adopt is as follows. Given a bipartite state there are two mean spin directions \(\hat{n}_1\) and \(\hat{n}_2\) defined through equation (19). A bipartite state with mean spin directions \(\hat{n}_1, \hat{n}_2\) is said to be squeezed in a perpendicular component \(\vec{S}_1 \cdot \hat{n}_{i\perp} + \vec{S}_2 \cdot \hat{n}_{2\perp}\), if the variance of this operator in the given state is less than half the sum of the absolute values of the expectation values of the spin vectors in that state.

Expressed mathematically, the criterion becomes
\[
\Delta (\vec{S}_1 \cdot \hat{n}_{1\perp} + \vec{S}_2 \cdot \hat{n}_{2\perp})^2 < \frac{|\langle \vec{S}_1 \rangle| + |\langle \vec{S}_2 \rangle|}{2} \tag{22}
\]
This can be further written in the form
\[
\Delta (\vec{S}_1 \cdot \hat{n}_{1\perp})^2 + \Delta (\vec{S}_2 \cdot \hat{n}_{2\perp})^2 + 2\langle \vec{S}_1 \cdot \hat{n}_{1\perp} \otimes \vec{S}_2 \cdot \hat{n}_{2\perp} \rangle < \frac{|\langle \vec{S}_1 \cdot \hat{n}_1 \rangle| + |\langle \vec{S}_2 \cdot \hat{n}_2 \rangle|}{2}. \tag{23}
\]
Before we provide some justification for this criterion, it must be noted that it is in an invariant form so that given any frame, the criterion can be expressed in terms of the spin components referred to that frame, once the directions \(\hat{n}_i, \hat{n}_{i\perp}\) are determined in that frame. Instead, one can also transform the frame by appropriate rotations and obtain the individual Lakin frames. Suppose the so obtained frames \(x_1y_1z_1\) and \(x_2y_2z_2\) are named such that \(\hat{z}_i = \hat{n}_i, \hat{x}_i = \hat{n}_{i\perp}\) and \(\hat{y}_i = \hat{n}_{i\perp'}\), the criterion takes the forms
\[
\Delta (S_{1x_1})^2 + \Delta (S_{2x_2})^2 + 2\langle S_{1x_1} \otimes S_{2x_2} \rangle < \frac{|\langle \vec{S}_{1z_1} \rangle| + |\langle \vec{S}_{2z_2} \rangle|}{2} \tag{24}
\]
\[
\Delta (S_{1y_1})^2 + \Delta (S_{2y_2})^2 + 2\langle S_{1y_1} \otimes S_{2y_2} \rangle < \frac{|\langle \vec{S}_{1z_1} \rangle| + |\langle \vec{S}_{2z_2} \rangle|}{2} \tag{25}
\]
The given state would therefore be squeezed in the components \(S_{1x_1} + S_{2x_2}\) or \(S_{1y_1} + S_{2y_2}\) if equation (24) or (25) is satisfied. For a bipartite system of two spinors, the criterion reduces to a simpler form since,
\[
\Delta S_{1x_1}^2 = \Delta S_{2x_2}^2 = \Delta S_{1y_1}^2 = \Delta S_{2y_2}^2 = \frac{1}{4} \tag{26}
\]
always and therefore the squeezing criterion along the individual x and y-axes reduces to
\[
\langle S_{1x_1} \otimes S_{2x_2} \rangle < \frac{|\langle \vec{S}_{1z_1} \rangle| + |\langle \vec{S}_{2z_2} \rangle| - 1}{4} \tag{27}
\]
and in y-direction
\[
\langle S_{1y_1} \otimes S_{2y_2} \rangle < \frac{|\langle \vec{S}_{1z_1} \rangle| + |\langle \vec{S}_{2z_2} \rangle| - 1}{4} \tag{28}
\]
We wish to now apply this criterion to bipartite states of interest and see whether they are squeezed or not. To begin with let us consider a bipartite state which is a direct product state of two states with the first being \(|\frac{1}{2} \rangle\) with respect to \(\hat{z}_1\) while the second \(|\frac{1}{2} \rangle\) with respect
Squeezing of a coupled state of two spinors

When expressed in terms of a common frame \(x_0 y_0 z_0\) and in individual Lakin frames it has the structure

\[
|\psi_{12}\rangle = |\xi_1\rangle \otimes |\xi_2\rangle = \begin{pmatrix}
\cos^2 \theta/2 \\
-\sin \theta \cos \theta/2 \\
\cos \theta/2 \sin \theta/2 \\
-\sin^2 \theta/2
\end{pmatrix}
\]

\[
_{z_0} \begin{pmatrix}
1 \\
0
\end{pmatrix}_{z_1} \otimes \begin{pmatrix}
1 \\
0
\end{pmatrix}_{z_2}
\]

(29)

Since

\[
\Delta(\vec{S}_1 \cdot \hat{n}_1) = \Delta(\vec{S}_2 \cdot \hat{n}_2) = \frac{1}{4},
\]

\[
\langle \vec{S}_1 \cdot \hat{n}_1 \rangle \otimes \langle \vec{S}_2 \cdot \hat{n}_2 \rangle = \langle \vec{S}_1 \rangle \otimes \langle \vec{S}_2 \rangle = 0,
\]

\[
|\langle \vec{S}_1 \cdot \hat{z}_1 \rangle| = |\langle \vec{S}_2 \cdot \hat{z}_2 \rangle| = \frac{1}{2},
\]

(30), (31), (32)

the criterion in the form (23) or in the forms (24), (25) is not satisfied at all and hence a direct product state of two spinors is never squeezed. This is in perfect agreement with Kitagawa and Ueda [5] in that the squeezing arises due to correlations and a direct product state which has neither self nor mutual correlations is therefore not squeezed.

3.1. Squeezing of entangled pure state

It is clear from the previous discussion that entanglement is necessary in the case of a two spinor bipartite state for squeezing to occur. However it is to be seen whether entanglement is sufficient also. We therefore begin this study by considering a general pure state which is entangled. Such a state can be expressed with respect to a basis \(|m_1 m_2\rangle_{z_0}\) referred to a common frame \(x_0 y_0 z_0\) in the form

\[
|\psi_{12}\rangle = \begin{pmatrix}
a_{11} \\
a_{12} \\
a_{21} \\
a_{22}
\end{pmatrix}; \sum_{ij} |a_{ij}|^2 = 1
\]

(33)

where of course \(a_{11}a_{22} \neq a_{12}a_{21}\). This latter condition ensures [8] that \(|\psi_{12}\rangle\) is entangled. Since in the general case the frame \(x_0 y_0 z_0\) may not be a Lakin frame for either spinor, we consider the rotation

\[
R_{12} = R_1 (\phi_1, \theta_1, 0) \otimes R_2 (\phi_2, \theta_2, 0)
\]

(34)

on this state so that

\[
|\psi_{12}\rangle \rightarrow |\psi'_{12}\rangle = R_{12} |\psi_{12}\rangle = \begin{pmatrix}
c_{11} \\
c_{12} \\
c_{21} \\
c_{22}
\end{pmatrix}
\]

(35)
A glance at the squeezing criterion implies that for the state to exhibit squeezing, first of all, the overall phase. The individual rotations on the coordinate system $x_1y_1z_1$ and $x_2y_2z_2$ if the Euler angles of rotation are chosen to satisfy

$$\tan \phi_i = \frac{\langle S_{iy} \rangle}{\langle S_{iz} \rangle}, \quad i = 1, 2$$

$$\tan \theta_i = \frac{(\langle S_{iy} \rangle^2 + \langle S_{ix} \rangle^2)^{\frac{1}{2}}}{\langle S_{iz} \rangle}, \quad i = 1, 2$$

With these transformations we now obtain

$$\langle S_{1x_1} \rangle = \langle R_{12} S_{1x} R_{12}^\dagger \rangle = \frac{1}{2} (c_{11} c_{21} + c_{12}^* c_{22} + c_{22}^* c_{12} + c_{21}^* c_{11}) = 0$$

$$\langle S_{2x_2} \rangle = \langle R_{21} S_{2x} R_{21}^\dagger \rangle = \frac{1}{2} (c_{11} c_{21} + c_{12}^* c_{22} + c_{22}^* c_{12} + c_{21}^* c_{11}) = 0$$

$$\langle S_{1y_1} \rangle = \langle R_{12} S_{1y} R_{12}^\dagger \rangle = \frac{i}{2} (-c_{11} c_{21} - c_{12}^* c_{22} + c_{22} c_{11} + c_{21} c_{12}) = 0$$

$$\langle S_{2y_2} \rangle = \langle R_{21} S_{2y} R_{21}^\dagger \rangle = \frac{i}{2} (-c_{11} c_{21} - c_{12}^* c_{22} + c_{22} c_{11} + c_{21} c_{12}) = 0$$

$$\langle S_{1z_1} \rangle = \langle R_{12} S_{1z} R_{12}^\dagger \rangle = \frac{1}{2} (c_{11}^2 + |c_{12}|^2 - |c_{21}|^2 - |c_{22}|^2)$$

$$\langle S_{2z_2} \rangle = \langle R_{21} S_{2z} R_{21}^\dagger \rangle = \frac{1}{2} (c_{11}^2 - |c_{12}|^2 + |c_{21}|^2 - |c_{22}|^2)$$

A glance at the squeezing criterion implies that for the state to exhibit squeezing, first of all, $\langle S_{1z_1} \rangle, \langle S_{2z_2} \rangle \neq 0$. Further the first four equations referring to Lakin frame yield

$$c_{11} c_{21} = -c_{12}^* c_{22}$$

$$c_{11} c_{22} = -c_{21}^* c_{12}$$

which lead to

$$c_{11} c_{22} (|c_{21}|^2 - |c_{12}|^2) = 0$$

and

$$c_{12}^* c_{21} (c_{11}^2 - |c_{22}|^2) = 0$$

There arise several cases satisfying these conditions

**case 1**: $c_{ij} = \delta_{i0} \delta_{j0}$ for fixed $i_0, j_0$

**case 2**: $|c_{21}| = |c_{12}| \neq 0, \quad |c_{11}| = |c_{22}|, \quad \phi_{12} = \pi + \phi_{22} - \phi_{21}$

**case 3**: $c_{11} = c_{22} = 0, \quad c_{12}, c_{21} \neq 0$

**case 4**: $|c_{12}| = |c_{21}| = 0$

The first case refers to direct product states. The second case implies $\langle S_{1z_1} \rangle = \langle S_{2z_2} \rangle = 0$ and hence although it is entangled, it is not squeezed. This state is actually a singlet state with total spin $s = 0$ and $|\langle S_{1z_1} \rangle| + |\langle S_{2z_2} \rangle| = 0$. 

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_Squeezing of a coupled state of two spinors_
Squeezing of a coupled state of two spinors

It is this latter value which makes a singlet state not squeezed although there is entanglement. This state therefore implies that the entanglement is necessary but not sufficient. Consider now the state in case (3), which has the form

\[ |\xi\rangle = \begin{pmatrix} 0 \\ c_{12} \\ c_{21} \\ 0 \end{pmatrix}. \]  

(52)

Under the rotation \( R_1(0, \frac{\pi}{2}, 0) \otimes I \), which is a local rotation, this state changes to

\[ |\xi'\rangle = (R_1 \otimes I) |\xi\rangle = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ c_{12} \\ c_{21} \\ 0 \end{pmatrix} = \begin{pmatrix} c_{21} \\ 0 \\ 0 \\ -c_{12} \end{pmatrix}. \]  

(53)

which shows that it belongs to case (4). It is therefore enough to consider only the states belonging to case (4). Here too, if we use the degree of freedom for the overall phase, the normalization condition and the freedom of rotation about the respective \( z_i \) axes, the state can be reduced to the simple form with its elements parametrized as

\[ |\chi\rangle = \begin{pmatrix} \cos \frac{\theta}{2} \\ 0 \\ 0 \\ \sin \frac{\theta}{2} \end{pmatrix}, \quad 0 < \theta < \pi. \]  

(54)

This is the simplest matrix form of an entangled state with non zero mean values for its individual spin vectors. The relevant quantities needed to determine the squeezing behaviour are

\[ \Delta S^2_{1x_1} = \Delta S^2_{2x_2} = \Delta S^2_{1y_1} = \Delta S^2_{2y_2} = \frac{1}{4} \]

\[ \langle S_{1x_1} \otimes S_{2x_2}\rangle = \frac{1}{4} \sin \theta = -\langle S_{1y_1} \otimes S_{2y_2}\rangle \]

\[ |\langle S_{1x_1}\rangle| = |\langle S_{2x_2}\rangle| = \frac{1}{2} |\cos \theta|. \]

Substituting these in the squeezing condition given by equations (23) or (24) we obtain

\[ 1 + \sin \theta < |\cos \theta| \quad \text{for } S_{1x_1} + S_{2x_2} \]  

(55)

and

\[ 1 - \sin \theta < |\cos \theta| \quad \text{for } S_{1y_1} + S_{2y_2} \]  

(56)

These conditions are certainly satisfied for a wide range of \( \theta \) which indicates that a wide variety of states of the form (Eq.54) with different values of \( \theta \) exhibit squeezing. The variation of squeezing with respect to \( \theta \) is shown in the graph below figure (1) where we have plotted the difference between the right hand and left hand sides of equation (55) and (56)

\[ Q_x = |\cos \theta| - \sin \theta - 1 \]  

(57)

\[ Q_y = |\cos \theta| + \sin \theta - 1 \]  

(58)
as a function of $\theta$. Positive (negative) values of $Q_x$, $Q_y$ show the presence (absence) of squeezing. It may be noted from the figure (1) that $\theta = 90^\circ$, corresponds to the singlet state referred to in case (2).

At this stage it is relevant to see how the correlations account for squeezing. As mentioned earlier, a bipartite system of two spinors has no self correlations. The mutual correlations which exist between the two spins are,

$$D^{12}_{\mu_1\nu_2} = \langle S_{1\mu_1} S_{2\nu_2} \rangle - \langle S_{1\mu_1} \rangle \langle S_{2\nu_2} \rangle, \quad \mu, \nu = x, y, z. \quad (59)$$

If these are zero, then the state has no mutual spin-spin correlation. All direct product states fall into this category. For the squeezed states defined by (Eq.54) these correlations with respect to the Lakin frames turn out to be

$$D^{12}_{x_1x_2} = -D^{12}_{y_1y_2} = \frac{1}{4} \sin \theta \quad (60)$$

$$D^{12}_{z_1z_2} = D^{12}_{z_1z_2} = D^{12}_{y_1y_2} = 0 \quad (61)$$

The graphs shown in figure (2) in which we have plotted both the squeezing and mutual correlation functions reveal that squeezing is absent whenever there are no mutual correlations and when the mutual correlations assume their extreme values. Indeed these extreme values correspond to a singlet state which confirms that it is a maximally entangled state. However, it has both $\langle \vec{S}_1 \rangle = \langle \vec{S}_2 \rangle = 0$ due to which it does not exhibit squeezing.

![Figure 1. Variation of squeezing $Q_x(\bigcirc)$ and $Q_y(\times)$ with respect to $\theta$.](image-url)
Squeezing of a coupled state of two spinors

Figure 2. Variation of spin-spin correlations $D^{12}_{xx}(\triangle)$, $D^{12}_{yy}(\star)$ and $D^{12}_{zz}(\circ)$, The plot also show squeezing in $Q_x(\diamond)$ and $Q_y(\times)$ with respect to $\theta$.

4. Generation of squeezing

We have so far discussed the nature of spin squeezing for a general coupled pure state in terms of a new criterion which is in some sense a generalization of the earlier criterion. It is of interest to know whether squeezing can be generated by subjecting a spin system to external interactions. Indeed such an attempt has been done by Kitagawa and Ueda for sharp spin state [5] earlier where they consider a Hamiltonian quadratic in the spin operators and show that the state generated during evolution under such a Hamiltonian possesses squeezing. We follow their prescription to see whether squeezing can be generated in the case of a coupled state too. It is clear that an Hamiltonian linear in the spin variables can only lead to a rotation of the state or of the coordinate system. Such an action does not introduce any squeezing. It is therefore necessary that the Hamiltonian should be at least quadratic in the spin variables. We therefore consider below a spin-spin interaction Hamiltonian

$$H = i\xi [S_{1+}S_{2+} - S_{1-}S_{2-}]$$

(63)

where,

$$S_{i\pm} = S_{ix} \pm iS_{iy}, \quad i = 1, 2$$

(64)

and $\xi$ is any real number. If the initial coupled state $|\psi\rangle$ of these two spinors is chosen to be a direct product state $|\psi\rangle$ as in (Eq.29) where the basis vectors and the Hamiltonian are referred to the frame $x_0y_0z_0$, the evolution in time given by

$$|\psi(t)\rangle = \exp(-iHt)|\psi\rangle = \exp[\xi t(S_{1+}S_{2+} - S_{1-}S_{2-})]|\psi\rangle$$

(65)
leads to the explicit form

\[
|\psi(t)\rangle = \begin{pmatrix}
\cos \theta \cos^2 \frac{\theta}{2} - \sin \theta \sin^2 \frac{\theta}{2} \\
- \cos \frac{\theta}{2} \sin \frac{\theta}{2} \\
\cos \frac{\theta}{2} \sin \frac{\theta}{2} \\
- \sin \theta \cos^2 \frac{\theta}{2} - \cos \theta \sin^2 \frac{\theta}{2}
\end{pmatrix} ; \quad \tau = \xi t. \quad (66)
\]

This state is in general entangled. As in any general case, we find here also that the frame \(x_0y_0z_0\) is not the common Lakin frame for the two spinors, since

\[
\langle S_{1x_0} \rangle = \frac{\sin \theta}{2} [\cos \tau + \sin \tau \cos \theta] = -\langle S_{2x_0} \rangle \quad (67)
\]

\[
\langle S_{1y_0} \rangle = \langle S_{2y_0} \rangle = 0 \quad (68)
\]

\[
\langle S_{1z_0} \rangle = \langle S_{2z_0} \rangle = \frac{1}{2} \left[ \cos 2\tau \cos \theta - \frac{1}{2} \sin 2\tau \sin^2 \theta \right] \quad (69)
\]

It is easier to analyse the squeezing behaviour if we go over to the individual Lakin frames \(x_1y_1z_1\) and \(x_2y_2z_2\) via the rotations through

\[
\alpha_i = \tan^{-1} \left( \frac{\langle S_{1x_0} \rangle}{\langle S_{1z_0} \rangle} \right) ; \quad i = 1, 2 \quad (70)
\]

of \(x_0y_0z_0\) about \(y_0\)-axis.

In these Lakin frames the expectation values of the various spin operators are given by

\[
\langle S_{1x_1} \rangle = \langle S_{1x_0} \rangle \cos \alpha_1 - \langle S_{1z_0} \rangle \sin \alpha_1 = 0 \quad (71)
\]

\[
\langle S_{1y_1} \rangle = \langle S_{1y_0} \rangle = 0 \quad (72)
\]

\[
\langle S_{1z_1} \rangle = \langle S_{1z_0} \rangle \sin \alpha_1 + \langle S_{1z_0} \rangle \cos \alpha_1 \quad (73)
\]

\[
\Delta S_{1x_1}^2 + \Delta S_{2x_2}^2 + 2 \langle S_{1x_1} \otimes S_{2x_2} \rangle = \frac{1}{2} (1 - A \cos^2 \alpha_1 - B \sin 2\alpha_1 \sin \theta - \cos^2 \theta \sin^2 \alpha_1) \quad (74)
\]

\[
\Delta S_{1y_1}^2 + \Delta S_{2y_2}^2 + 2 \langle S_{1y_1} \otimes S_{2y_2} \rangle = \frac{1}{2} + \frac{1}{2} \left[ \sin 2\tau \cos \theta - \sin^2 \theta \sin^2 \tau \right] \quad (75)
\]

\[
|\langle S_{1x_1} \rangle| + |\langle S_{2x_2} \rangle| = (\sin^2 \theta \left( \cos \tau + \sin \tau \cos \theta \right)^2 + \left( \cos \theta \cos 2\tau - \frac{1}{2} \sin 2\tau \sin^2 \theta \right)^2) \frac{1}{2} \quad (76)
\]

where \(A = (\cos \theta \sin 2\tau + \sin^2 \theta \cos^2 \tau)\) and \(B = \sin \tau - \cos \theta \cos \tau\). The quantities \(Q_x, Q_y\) defined earlier now become function of time and are given by

\[
Q_x(t) = \left( \sin^2 \theta \left( \cos \tau + \sin \tau \cos \theta \right)^2 + \left( \cos \theta \cos 2\tau - \frac{1}{2} \sin 2\tau \sin^2 \theta \right)^2 \right)^{\frac{1}{2}} - (1 - A \cos^2 \alpha_1 - B \sin 2\alpha_1 \sin \theta - \cos^2 \theta \sin^2 \alpha_1) \quad (77)
\]

\[
Q_y(t) = \left( \sin^2 \theta \left( \cos \tau + \sin \tau \cos \theta \right)^2 + \left( \cos \theta \cos 2\tau - \frac{1}{2} \sin 2\tau \sin^2 \theta \right)^2 \right)^{\frac{1}{2}} - (1 + \sin 2\tau \cos \theta - \sin^2 \theta \sin^2 \tau) \quad (78)
\]
We infer that the state is squeezed if either $Q_x(t)$ or $Q_y(t)$ is positive. The graphs of $Q_x(t)$ and $Q_y(t)$ plotted below in the figure (3) show that the squeezing is observed for a wide range of values of $\theta$ and $\tau$, except at certain points. These points are at $\tau = 90^\circ$, $\theta = \frac{n\pi}{2}$, $n = 0, 1, 2, \ldots$ and for $\tau = 45^\circ$, $\theta = 0$ etc, and correspond to either direct product or singlet states.

We now look at the mutual correlations that exist between the two spinors at various instants of time evolution. These are explicitly given by

$$D_{x_1 x_2}^{12} = -\frac{1}{4}(A \cos^2 \alpha_1 + B \sin 2\alpha_1 \sin \theta + \cos^2 \theta \sin^2 \alpha_1)$$  \hspace{1cm} (79)

$$D_{y_1 y_2}^{12} = \frac{1}{4}(\sin 2\tau \cos \theta - \sin^2 \theta \sin^2 \tau)$$  \hspace{1cm} (80)

$$D_{z_1 z_2}^{12} = \frac{1}{4}[A \sin^2 \alpha_1 - B \sin \theta \sin 2\alpha_1 + \cos^2 \theta \cos^2 \alpha_1 - (\sin^2 \theta (\cos \tau + \sin \tau \cos \theta))^2$$

$$+ 4 (\cos \theta \cos 2\tau - \frac{1}{2} \sin 2\tau \sin^2 \theta)^2)]$$  \hspace{1cm} (81)

$$D_{x_1 y_2}^{12} = D_{y_1 z_2}^{12} = 0.$$  \hspace{1cm} (83)

Plotting these correlations together with squeezing functions, we observe from the figure (4) below that whenever there is squeezing the state necessarily possesses spin-spin correlations. The graphs lead to similar conclusions arrived earlier in the discussion of the general case.
Figure 3. Variation of squeezing $Q_x(\diamond)$ and $Q_y(\triangle)$ with respect to $\theta$ with $\tau = 90^\circ$ in the upper plot and $\tau = 45^\circ$ in the lower plot.
Squeezing of a coupled state of two spinors

Figure 4. Variation of spin-spin correlations $D_{x_1,x_2}^{12}$ ($\times$), $D_{y_1,y_2}^{12}$ ($\bullet$) and $D_{z_1,z_2}^{12}$ ($\ast$). The plot also show squeezing in $Q_x$ ($\diamond$) and $Q_y$ ($\triangle$) with respect to $\theta$ with $\tau=90$.

5. Summary

We have looked into the squeezing aspect of a pure bipartite state consisting of two spinors. A suitable criterion for squeezing of such states has been obtained which is a generalization of the squeezing criterion for states of sharp spin. While squeezing is established in the case of sharp spins due to self correlations, that for a bipartite state occurs due to the presence of both self and mutual correlations. The existence of mutual correlations also implies entanglement. This raises the question whether every entangled state is squeezed. We have shown that all entangled states of two spinors are squeezed except the singlet state which is an exception. This state lacks squeezing since it has both $\langle \vec{S}_1 \rangle = 0$, $\langle \vec{S}_2 \rangle = 0$. A direct product of two spinors, on the other hand has neither self nor mutual correlations and hence is never squeezed. However, if $s_1$ or $s_2 > \frac{1}{2}$, then a direct product state can indeed possess self correlations and such a bipartite state may show squeezing. An example for this could be the direct product state of spin 1 squeezed state with a spin $\frac{1}{2}$ state. These situations indicate that while entanglement stems out from only mutual correlations, squeezing arises because due to both of them and when there is net mean spin value for either of the subsystems.

Our study in this paper gives some justification to some of the claims made by Kitagawa and Ueda earlier regarding what exactly causes squeezing and how squeezed states can be generated. We have shown that spin- spin interactions can lead to entangled as well as squeezed states.

Further studies on these aspects are under progress where we are also planning to analyze the squeezing of mixed states of bipartite systems in which there are not only quantum correlations among and within the spins but also correlations arising due to the nature of statistical distribution of the spin assemblies.
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