Expansion in the Width and Collective Dynamics of a Domain Wall

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We show that collective dynamics of a curved domain wall in a (3+1)-dimensional relativistic scalar field model is represented by Nambu-Goto membrane and (2+1)-dimensional scalar fields defined on the worldsheet of the membrane. Our argument is based on a recently proposed by us version of the expansion in the width. Derivation of the expansion is significantly reformulated for the present purpose. Third and fourth order corrections to the domain wall solution are considered. We also derive an equation of motion for the core of the domain wall. Without the (2+1)-dimensional scalar fields this equation would be nonlocal.

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1 Introduction

One of the most interesting branches of field theory is devoted to spatially extended topological solitons in 3+1 dimensional space-time such as domain walls or vortices. By now it is clear that these solitons, mathematically represented by certain nontrivial solutions of classical field equations, play an important role in many phenomena in condensed matter physics [1, 2, 3], particle physics [4], and perhaps also in cosmology [5]. Dynamics of the extended solitons is extremely rich, but also it poses rather hard problems because of its strongly nonlinear character. In contradistinction to the case of solitons in one-dimensional space, there are no methods of constructing generic exact domain wall or vortex solutions. Till now most theoretical results have been obtained with the help of numerical methods, see, e.g. [1, 4, 5], eventhough there are certain remarkable exact analytical results like the discovery of travelling waves on a straightline vortex [9], or of the 90° scattering of straightline parallel vortices [10].

Analytical approaches to evolution of a generic curved vortex/domain wall are scarce. In most of them one attempts to reduce the dynamics to much simpler dynamics of a string/membrane. This line of thought has been initiated in the seminal paper [11], continued in [12, 13] and later on in [14–21]. It has been found already in [11, 12] that in the case of local vortices, e.g., in the Abelian Higgs model, one may approximately describe motion of the vortex using the simple classical Nambu-Goto string model as the leading approximation. Dynamics of internal degrees of freedom of the vortex is completely neglected in this approximation. In a reference frame co-moving with the vortex all fields have by assumption the same form as for a static, straightline vortex. In other words, the field structure of the vortex in the co-moving reference frame is kept frozen. The case of domain walls is strictly analogous — here one has the Nambu-Goto membrane instead of the string.

In the case of a global vortex a long range field is present. In the leading approximation the collective dynamics of the vortex is described in terms of the Nambu-Goto string coupled to a massless 2-form gauge field [22].

Our present paper is devoted to the problem how to compute corrections to the collective dynamics which are due to the internal dynamics of the vortex/domain wall. Specifically, we further develop the approach proposed in [23]. In the literature one can also find other attempts to calculate evolution of the curved vortex/domain wall beyond the leading approximation. Let us
review them briefly.

In the most popular approach [14–21] the string/membrane is identified with the locus of zeros of a scalar field, that is with the core. Influence of the internal dynamics on the collective motion is taken into account by corrections to the Nambu-Goto action for the core. The total action defines the so-called effective classical string/membrane model. It turns out that such effective models are rather unpleasant. Equations of motion contain higher derivatives with respect to time and consequently they have unphysical runaway type solutions. Moreover, it seems that the effective models should be nonlocal, see [24] where caveats concerning this approach are presented and Section 4 of the present paper.

In another approach [25] one derives directly from field equations approximate equations of motion for certain averages, e.g. for an average radius in the case of a ring-like vortex, in an analogy with the Ehrenfest’s approach to evolution of a quantum mechanical wave packet in an external potential. It is clear that the knowledge of time evolution of such averages is not sufficient in order to recover the time evolution of the fields of the vortex/domain wall.

In the polynomial approximation [26] the string/membrane is identified with the core, but it gives only a part of the collective dynamics. The other collective degrees of freedom are represented by coefficients of certain polynomials which approximate the fields inside the vortex/domain wall. They can be regarded as scalar fields defined on the worldsheet of the string/membrane. Here one avoids shortcomings of the previous two approaches. On the other hand, the accuracy of this approximation has been checked only in the simplest cases of static straightlinear vortices and planar domain walls. Moreover, it is likely that this approach should be generalised to include radiation of massive fields from the extended solitons. Such radiation could dump oscillations of the width which in the present formulation of the method can persist indefinitely (this problem is currently under investigation).

The approach proposed in [23] is based on the expansion in the width for the vortex/domain wall solution of the field equations. It is constructed along the lines of Hilbert-Chapman-Enskog method for singularly perturbed evolution equations [27]. In this approach certain consistency conditions play a crucial role. Co-moving string/membrane is of the Nambu-Goto type in all orders, but it does not coincide with the core. In addition to the string/membrane there are also scalar fields defined on the worldsheet of the co-moving string/membrane. We avoid the unpleasant effective classical
models with higher derivatives. In contradistinction to the case of polynomial approximation we have here physically meaningful expansion parameter, that is the width defined as the inverse mass \( M^{-1} \) of the scalar field. More precisely, the expansion parameters are given by dimensionless ratios of the width to local curvature radii of the string/membrane worldsheet. In the case of a domain wall the terms in the expansion have been calculated up to the order \( M^{-2} \). These results have been compared with purely numerical solutions for cylindrical and spherical domain walls \([28]\) — it has turned out that the approximate analytical solution is astonishingly accurate.

Expansion in the width was invoked also in the earlier papers, e.g., in \([14, 15, 17, 21]\) where a version of it was used on intermediate stages of derivations of the effective string/membrane models. However, in those papers the consistency conditions were not taken into account. This is a crucial difference because in our version of the expansion just these conditions make the calculation of the effective classical string/membrane action superfluous. More detailed comparison of our version of the expansion with the other ones can be found in the first of papers \([23]\). See also Section 4 of the present paper.

The construction of the expansion given in \([23]\) has two main elements: a special co-moving coordinate system \([12]\) introduced to maintain the Lorentz invariance order by order in the expansion, and the consistency conditions. While the use of the co-moving coordinates could be regarded as a rather elegant and relatively simple way to secure the Lorentz invariance, the consistency conditions appeared in \([23]\) as formal, purely mathematical observations. In spite of the crucial role they played in determining the vortex/domain wall solution, their field theoretical meaning was hidden. Moreover, the calculations were rather cumbersome and even for the domain wall, which is simpler than the vortex, we were able to obtain only the first and the second order corrections. We found that the core did not coincide with the co-moving string/membrane, and that the co-moving string/membrane obeyed Nambu-Goto equation in all orders, but we did not derive any equation of motion for the core. Such equation would be particularly useful for comparison with the effective string/membrane approach in which the string/membrane is identified with the core.

In the present paper we concentrate on the case of a domain wall. Analogous results can be obtained also for a vortex but the necessary calculations are much longer. We reformulate the derivation of the expansion in such a
way that the (2+1)-dimensional scalar fields defined on the worldsheet of the co-moving membrane, which have been introduced in [23] for purely mathematical reasons, acquire a physical interpretation — they are identified with perturbative contributions to the values of the original scalar field on the membrane. We also find a field theoretical interpretation of the consistency conditions: they coincide with certain Euler-Lagrange equation. Moreover, derivations of the higher order terms in the expansion are significantly simpler in the new formulation. Now we can consider corrections in the next two orders, that is in the $M^{-3}$ and $M^{-4}$ orders. Order by order in the $1/M$ expansion we express profile of the domain wall by the collective degrees of freedom given by the co-moving membrane and the (2+1)-dimensional scalar fields. Finally, we derive an approximate equation of motion for the core. It turns out to be different from equations obtained in the effective action approaches. In particular, it does not contain higher derivatives.

The main conclusion of our paper is that the collective dynamics of the domain wall beyond the leading approximation involves the (2+1)-dimensional scalar fields defined on the worldsheet of the Nambu-Goto membrane. Elimination of those fields would lead to a nonlocal membrane model for the collective dynamics.

The plan of our paper is as follows. In Section 2 we present Euler-Lagrange equations and we discuss certain important consequences of them. In Section 3 we consider the extrinsic curvature corrections to the domain wall solution up to the $M^{-4}$ order. The equation of motion for the core is derived in Section 4. Section 5 is devoted to a discussion of our results. In Appendix A we have collected formulas relevant for the transformation to the co-moving coordinates. In Appendix B we present solutions of certain linear ordinary differential equation. In Appendix C we list functions and constants which appear in Section 3.

## 2 The consistency conditions as Euler – Lagrange equation

We consider a domain wall in the model defined by the following Lagrangian

$$
\mathcal{L} = -\frac{1}{2} \eta_{\mu\nu} \partial^\mu \Phi \partial^\nu \Phi - \frac{\lambda}{2} (\Phi^2 - \frac{M^2}{4\lambda})^2,
$$

(1)
where \( \Phi \) is a single real scalar field, \((\eta_{\mu\nu})=\text{diag}(-1,1,1,1)\) is the space-time metric, and \(\lambda, M\) are positive constants. Two vacuum values of \(\Phi\) are equal to \(\pm M/2\sqrt{\lambda}\). The mass of the corresponding scalar particle is equal to \(M\).

The domain wall appears when a solution \(\Phi\) of the Euler-Lagrange equation obtained from the Lagrangian \(\mathbb{L}\) smoothly interpolates between the two vacuum values. Then, at each instant of time the field \(\Phi\) vanishes somewhere in the interior of the domain wall. The locus of these zeros is assumed to be a smooth connected surface \(\tilde{S}\) in the space \(\mathbb{M}\). It is called the core of the domain wall. The transverse width of such a domain wall is of the order \(M^{-1}\) and energy density is exponentially localised around the core. The world-volume \(\tilde{\Sigma}\) of the core is a 3-dimensional manifold embedded in Minkowski space-time. The core plays an essential role in the effective membrane models mentioned in the Introduction.

In our approach we use another smooth connected surface \(S\) attached to the domain wall. In general it differs from the core, except at the initial instant of time when it coincides with the core by assumption. We shall obtain Nambu-Goto equation for \(S\), so this surface can be regarded as the Nambu-Goto type relativistic membrane co-moving with the domain wall. Let us stress that this co-moving membrane is merely an auxiliary mathematical notion. It is introduced in order to define the co-moving coordinate system. The world-volume of \(S\) is denoted by \(\Sigma\). We shall parametrise it as follows

\[ \Sigma \ni (Y^\mu(u^a)) = (\tau, Y^i(u^a)). \]  

(2)

We use here the notation \((u^a)_{a=0,1,2} = (\tau, \sigma^1, \sigma^2)\), where \(\tau\) coincides with the laboratory frame time \(x^0\), while \(\sigma^1, \sigma^2\) parametrise the co-moving membrane \(S\) at each instant of time. The index \(i = 1, 2, 3\) refers to the spatial components of the four-vector. The points of the co-moving membrane \(S\) at the instant \(\tau\) are given by \((Y^i)(\tau, \sigma^1, \sigma^2)\). The coordinate system \((\tau, \sigma^1, \sigma^2, \xi)\) co-moving with the domain wall is defined by the formula

\[ x^\mu = Y^\mu(u^a) + \xi n^\mu(u^a), \]  

(3)

where \(x^\mu\) are Cartesian laboratory frame coordinates in Minkowski space-time, and \((n^\mu)\) is a normalised space-like four-vector orthogonal to \(\Sigma\) in the

\footnote{We use slightly different notation than in [24] — the present one is more convenient, especially in Section 4.}
covariant sense,
\[ n^\mu (u^a) Y^\mu_{\alpha}(u^a) = 0, \quad n^n n^\mu = 1, \]
where \( Y^\mu_{\alpha} \equiv \partial Y^\mu / \partial u^a \). The three four-vectors \( Y^\mu_{\alpha} \) are tangent to \( \Sigma \). The definition (3) implies that \( \xi \) and \( u^a \) are Lorentz scalars. In the co-moving coordinates the co-moving membrane is described by the simple Lorentz invariant condition \( \xi = 0 \). For points lying on \( S \) the parameter \( \tau \) coincides with the laboratory time \( x^0 \), but for \( \xi \neq 0 \) in general \( \tau \) is not equal to \( x^0 \).

In the co-moving coordinates Lagrangian (1) has the following form
\[ \mathcal{L} = -\frac{M^4}{32\lambda} \left[ \partial_s \phi \partial \phi_s + (\phi^2 - 1)^2 + \frac{4}{M^2} G^{ab} \partial_a \phi \partial_b \phi \right], \]
where \( \partial_s = \partial / \partial s, \partial_a = \partial / \partial u^a \), and \( G^{ab} \) is given in Appendix A. The field \( \Phi \) and the coordinate \( \xi \) have been expressed by dimensionless \( \phi \) and \( s \),
\[ \Phi(x^\mu) = \frac{M}{2\sqrt{\lambda}} \phi(s, u^a), \quad \xi = \frac{2}{M}s. \]

According to papers [23] the core is shifted with respect to the co-moving membrane, that is the scalar field has non-zero values on the membrane. We take this fact as the starting point for the new derivation of the expansion in the width. The idea consists in extracting from the scalar field its component living on the co-moving membrane and treating it separately from the remaining part of the scalar field. To achieve this we use the identity
\[ \phi(s, u^a) = B(u^a) \psi(s) + \chi(s, u^a), \]
where
\[ B(u^a) \overset{df}{=} \phi(0, u^a) \]
is the component of the scalar field living on the co-moving membrane, and
\[ \chi \overset{df}{=} \phi(s, u^a) - B(u^a) \psi(s) \]
is the remaining part. The auxiliary, fixed function \( \psi(s) \) depends on the variable \( s \) only. It is smooth, concentrated around \( s = 0 \), and
\[ \psi(0) = 1. \]
It follows that
\[ \chi(0, u^a) = 0. \]  
(9)

We shall see that the best choice for \( \psi(s) \) is given by formula (22) below.

Formulas (5-7) can be regarded as an invertible change of variables

\[ \phi(s, u^a) \rightarrow (B(u^a), \chi(s, u^a)) \]

in the configuration space of the scalar field. Therefore, it is legitimate to use formula (5) in Lagrangian (4) and to derive Euler-Lagrange equations by taking independent variations of \( B(u^a) \) and \( \chi \). The variation \( \delta \chi \) has to respect the condition (9), hence

\[ \delta \chi(0, u^a) = 0. \]

Because of this condition, variation of the action functional

\[ S = \frac{2}{M} \int dsd^3u \sqrt{-g}h(s, u^a)\mathcal{L} \]

with respect to \( \chi \) gives Euler-Lagrange equation in the regions \( s < 0 \) and \( s > 0 \). It has the following form

\[ \frac{2}{M^2} \frac{1}{\sqrt{-g}} \partial_a [\sqrt{-g}hG^{ab} \partial_b (B\psi + \chi)] \]

\[ + \frac{1}{2} \partial_s[h \partial_s(B\psi + \chi)] + h(B\psi + \chi)[1 - (B\psi + \chi)^2] = 0, \]  
(10)

where \( h, g \) and \( G^{ab} \) are given in Appendix A. At \( s = 0 \) there is no Euler-Lagrange equation corresponding to the variation \( \delta \chi \). Instead, we have the condition (9). Equation (10) should be solved in the both regions separately, with (9) regarded as a part of boundary conditions for \( \chi \). To complete the boundary conditions we also specify the behaviour of \( \chi \) for \( |\xi| \) much larger than the characteristic length \( 1/M \), that is for \( |s| \gg 1 \). In our model (1) the expected behaviour of the domain wall field \( \Phi \) for large \( |\xi| \) is given by an exponential approach to the vacuum values. Therefore, we shall seek a solution such that \( \chi \) is exponentially close to \(+1\) for \( s \gg 1 \), while for \( s \ll -1 \) it is exponentially close to \(-1\).

At this stage of our considerations Eq.(10) should not be extrapolated to \( s = 0 \). For example, the l.h.s. of it could have a \( \delta(s) \)-type singularity. It
would occur if $\chi$ was smooth for $s > 0$ and for $s < 0$ but had a spike at $s = 0$.

In addition to Eq.(10) we also have the Euler-Lagrange equation corresponding to variations of $B(u^a)$. This equation has the following form

$$
\frac{2}{M^2} \int ds \frac{1}{\sqrt{-g}} \partial_a \left[ \sqrt{-g} G^{ab} \partial_b (B\psi + \chi) \right] \psi
$$

$$
- \frac{1}{2} \int ds \ h \partial_s \psi \partial_s (B\psi + \chi)
$$

$$
+ \int ds \ h \psi (B\psi + \chi) \left[ 1 - (B\psi + \chi)^2 \right] = 0.
$$

Here and in the following we use $\int ds$ as a shorthand for the definite integral $\int_{-\infty}^{+\infty} ds$.

Comparing equations (10) and (11) one might think that they are not independent because it seems that multiplying Eq.(10) by $\psi$ and integrating the result over $s$ with the help of integration by parts we obtain Eq.(11). This argument is false, namely it ignores the above mentioned possibility that the l.h.s. of Eq.(10) might have the $\delta$-type singularity at $s = 0$. On the other hand, if the singularity at $s = 0$ is absent then indeed Eq.(11) does follow from Eq.(10).

Actually, just because Eqs.(10) and (11) are independent we can prove that the solution $\chi$ does not have the spike at $s = 0$. To this end, we multiply (10) by $\psi$ and integrate over $s$ in the intervals $(-\infty, \epsilon], [-\epsilon, -\infty)$ with a positive $\epsilon$ which approaches zero. In this manner we avoid the not-excluded-yet singularity at $s = 0$. The l.h.s. of the resulting formula is compared with the l.h.s. of Eq.(11): in the limit $\epsilon \to 0$ they differ by the term

$$
- \frac{1}{2} \psi h \partial_s \chi|_{s=0+} + \frac{1}{2} \psi h \partial_s \chi|_{s=0-},
$$

which has to vanish because the r.h.s.'s of Eqs.(10), (11) vanish. In conclusion,

$$
\lim_{s \to 0+} \partial_s \chi = \lim_{s \to 0-} \partial_s \chi,
$$

that is $\partial_s \chi$ is continuous at $s = 0$. Hence, Eq.(10) and Eq.(11) together imply that the spike at $s = 0$ is absent. It follows that Eq.(10) is obeyed by $\chi$ also at $s = 0$. To summarize, Eq.(11) and the statement that the singularity at $s = 0$ is absent are equivalent.
Let us now solve Eq. (10) in the leading approximation which is obtained by putting $1/M = 0$. The equation is then reduced to
\[ \frac{1}{2} \partial_s^2 \phi^{(0)} + \phi^{(0)}[1 - (\phi^{(0)})^2] = 0, \]  
(12)
where
\[ \phi^{(0)} = B^{(0)} \psi + \chi^{(0)}. \]
In the $1/M$ expansion this is the only nonlinear equation we have to solve. Mathematically, it coincides with a well-known equation for a planar domain wall. It has the following particular solutions
\[ \phi_{\pm}^{(0)}(s, u^a) = \pm \tanh(s - b(u^a)), \]  
(13)
where $b(u^a)$ is an arbitrary function of the indicated variables. In the following we shall take the $+$ sign (the $-$ sign corresponds to an anti-domain wall). The functions $b(u^a)$ can be removed from the solution (13) by a suitable choice of the co-moving membrane. The new membrane is obtained by shifting the points of the worldsheet $\Sigma$ of the original membrane along the direction normal to $S$ by the distance $b(u^a)$. Because the normal direction is given locally by the $n(u^a)$ four-vector, this shift is equivalent to the change $(\xi, u^a) \rightarrow (\xi', u^a)$, $\xi' = \xi - b(u^a)$ of the co-moving coordinates. Thus, we may take as the zeroth order domain wall solution in the co-moving frame
\[ \phi^{(0)} = \tanh s, \]  
(14)
without any loss of generality. This solution together with conditions (8), (9) gives
\[ B^{(0)} = 0, \quad \chi^{(0)} = \tanh s. \]  
(15)
The solution (14) in the co-moving coordinates does not determine the field $\phi$ in the laboratory frame because we do not know yet the position of the co-moving membrane with respect to the laboratory frame. Equations (10), (11) should also yield an equation for the co-moving membrane, otherwise they would not form the complete set of evolution equations for the domain wall. In fact, we shall see that the first order terms in Eq. (10) imply the Nambu-Goto equation for the membrane.

The expansion in the width in the present formulation has the form
\[ \chi(s, u^a) = \tanh s + \frac{1}{M} \chi^{(1)}(s, u^a) + \frac{1}{M^2} \chi^{(2)}(s, u^a) + \frac{1}{M^3} \chi^{(3)}(s, u^a) + ..., \]  
(16)
\[ B(u^a) = \frac{1}{M} B^{(1)}(u^a) + \frac{1}{M^2} B^{(2)}(u^a) + \frac{1}{M^3} B^{(3)}(u^a) + \ldots, \quad (17) \]

where we have taken into account the zeroth order results (15). The expansion parameter is \( 1/M \) and not \( 1/M^2 \) because \( 1/M \) in the first power appears in the \( h \) and \( G^{ab} \) functions after passing to the \( s \) variable, see formulas in Appendix A. In order to obey the condition (9) and to ensure the proper asymptotics of \( \chi \) at large \( |s| \) we assume that for \( n \geq 1 \)

\[ \chi^{(n)}(0, u^a) = 0, \quad \lim_{s \to \pm \infty} \chi^{(n)} = 0. \quad (18) \]

Inserting the perturbative Ansatz (16), (17) in Eqs.(10) and (11), expanding the l.h.s.’s of them in powers of \( 1/M \), and equating to zero coefficients in front of the powers of \( 1/M \) we obtain a sequence of linear, inhomogeneous equations for \( \chi^{(n)}(s, u^a), B^{(n)}(u^a) \) with \( n \geq 1 \).

The first order terms in Eq.(10) give the following equation

\[ \hat{L}\chi^{(1)} = -B^{(1)}\hat{L}\psi + K_a \partial_s \chi^{(0)}, \quad (19) \]

where

\[ \hat{L} \overset{df}{=} \frac{1}{2} \partial_s^2 + 1 - 3(\chi^{(0)})^2, \]

and \( \chi^{(0)} \) is given by the second of formulas (15). General solution of the equation of the form (19) is presented in Appendix B.

The most important point in our derivation of the expansion in the width is the observation that the operator \( \hat{L} \) has a zero-mode, that is the normalizable solution

\[ \psi_0(s) = \frac{1}{\cosh^2 s} \]

of the homogeneous equation

\[ \hat{L}\psi_0 = 0. \quad (20) \]

Notice that \( \psi_0 = \partial_s \chi^{(0)} \) — this means that the zero-mode \( \psi_0 \) is related to the translational invariance of Eq.(12) under \( s \to s + \text{const} \). The presence of the zero-mode implies the consistency condition: we multiply Eq.(19) by \( \psi_0 \) and integrate over \( s \). With the help of integration by parts we show that \( \int \psi_0 \hat{L}\chi^{(1)} \) and \( \int \psi_0 \hat{L}\psi \) vanish because of (20). Finally, we obtain the following condition

\[ K_a \int ds \, \psi_0(s) \partial_s \chi^{(0)}(s) = 0. \]
which is equivalent to

\[ K_a^a = 0. \]  \hspace{1cm} (21)

This condition coincides with the well-known Nambu-Goto equation. It determines the motion of the co-moving membrane, that is the functions \( Y^i(u^a), \) \( i=1,2,3, \) once initial data are fixed. When we know these functions we can calculate the extrinsic curvature coefficients \( K_{ab} \) and the metric \( g_{ab}. \) Review of properties of relativistic Nambu-Goto membranes can be found in, e.g., \cite{29}.

Due to Nambu-Goto equation (21) one term on the r.h.s. of Eq.(19) vanishes. The resulting equation

\[ \hat{L}\chi^{(1)} = -B^{(1)} \hat{L}\psi \]

has the following solution obeying the boundary conditions (18)

\[ \chi^{(1)} = B^{(1)}(u^a)(\psi_0(s) - \psi(s)). \]

It can be obtained from the formulas given in Appendix B, but there is a shorter way: notice that \( \chi^{(1)} + B^{(1)}\psi \) obeys the homogeneous equation (20) which has the general solution \( C(u^a)\psi_0(s) + D(u^a)\psi_1(s); \) the boundary conditions (18) imply that \( D = 0, C = B^{(1)} \). Now, we recall that the function \( \psi(s) \) is an auxiliary mathematical object, like the co-moving membrane in our approach, and we may adjust it in order to simplify formulas. The obvious choice is

\[ \psi(s) = \psi_0(s) = \frac{1}{\cosh^2 s}, \]  \hspace{1cm} (22)

because then the first order correction has the simplest possible form

\[ \chi^{(1)}(s, u^a) = 0. \]  \hspace{1cm} (23)

In the remaining part of our paper we will use \( \psi \) given by formula (22). Notice that vanishing \( \chi^{(1)} \) does not mean that the first order correction to the original field \( \phi \) also vanishes — there is the first order contribution equal to \( B^{(1)}\psi_0/M. \) It does not vanish on the co-moving membrane that is at \( s = 0. \)

Equations (10), (11) in the first order do not give any restriction on the function \( B^{(1)} \). In the next Section we shall see that an equation for \( B^{(1)} \) (Eq.(32) below) follows from the third order terms in Eq.(11). This situation is typical for singular perturbation theories of which the \( 1/M \) expansion is
an example — higher order equations imply restrictions (the consistency conditions) for the lower order contributions \[27\].

With the choice (22) for \( \psi \), Eq.(11) expanded in the powers of \( 1/M \) gives equations for \( B^{(n)} \) coinciding with the consistency conditions found in \[24\]. This follows from the fact that both Eq.(11) and the consistency conditions are obtained by multiplying expanded with respect to \( 1/M \) Eq.(10) by the zero-mode \( \psi_0 \) and next integrating over \( s \). Euler-Lagrange equation (11) can be regarded as the generating equation for the consistency conditions.

3 The higher order corrections

At this point we have the complete set of equations determining the evolution of the domain wall in the \( 1/M \) expansion. Each of Eqs.(10), (11), (21) describes a different aspect of the dynamics of the curved domain wall. We shall see that expanded in the powers of \( 1/M \) Eq.(10) determines dependence of \( \chi \) on \( s \), that is on the distance from the worldsheet \( \Sigma \) of the co-moving membrane along the perpendicular direction \( n(u^a) \) at each point \( Y(u^a) \) of the worldsheet. Because the term \( B\psi \) in formula (5) has explicit dependence on \( s \), see formula (22), we may say that Eq.(10) for \( \chi \) fixes the transverse profile of the domain wall.

Equation (11) determines the \( B^{(n)}(u^a) \) functions, which can be regarded as a (2+1)-dimensional scalar fields defined on the worldsheet \( \Sigma \) and having nontrivial nonlinear dynamics, see Eqs.(32), (35) below. The extrinsic curvature \( K_{ab} \) of \( \Sigma \) acts as an external source for these fields. The fields \( B^{(n)} \) can propagate along \( \Sigma \). One may regard this effect as causal propagation of deformations which are introduced by the extrinsic curvature.

Finally, Nambu-Goto equation (21) for the co-moving membrane determines the evolution of the shape of the domain wall.

Equations (21) and (11) are of the evolution type in the \( 1/M \) expansion — we have to specify initial data for them, otherwise their solutions are not unique. Equations for the perturbative contributions \( \chi^{(n)} \) are of different type — in order to ensure uniqueness of their solutions it is sufficient to adopt the boundary conditions (18). The initial data for \( B(u^a) \) and \( Y^i(u^a) \) follow from initial data for the original field \( \phi \). From such data for \( \phi \) we know the initial position and velocity of the core. As in papers \[23\] we assume that at the initial instant \( \tau_0 \) the co-moving membrane and the core have the same...
position and velocity. Hence,

initial data for the membrane = initial data for the core.

Using formula (5) one can show that then

\[ B^{(n)}(\tau_0, \sigma^1, \sigma^2) = 0, \quad \partial_\tau B^{(n)}(\tau_0, \sigma^1, \sigma^2) = 0. \] (24)

Our perturbative scheme with the initial conditions (24) gives the basic curved domain wall. In Section 5 we discuss more general solutions obtained by adopting more general than (24) initial data for the \(B^{(n)}\) fields.

In order to find the domain wall solution one should first solve the collective dynamics, that is to compute evolution of the co-moving membrane and of the \(B\) field. The profile \(\chi\) of the domain wall is found in the next step from formulas (16) above and (27) below. In our perturbative scheme the profile of the domain wall can not be chosen arbitrarily even at the initial time — it is fixed uniquely once the initial data for the membrane and for the \(B\) field are given. Evolution of the core can be determined afterwards, from the explicit expression for the scalar field \(\phi\). This will be done in Section 4.

Now we shall show that order by order in the \(1/M\) expansion Eq.(10) determines the dependence of the field \(\chi\) on \(s\). \(\chi\) will be explicitly expressed by the perturbative contributions to the \(B\) field and by the geometric characteristics \(g_{ab}, K_{ab}\) of the worldsheet of the co-moving membrane. Therefore, the dynamics of the curved domain wall is reduced to the Nambu-Goto dynamics of the co-moving membrane and to nonlinear dynamics of the \(B^{(n)}\) fields. A convenient starting point for this calculation is obtained by rewriting Eq.(10) in a different form. We substitute

\[
\chi = \chi^{(0)} + \bar{\chi}, \quad h = 1 + \bar{h}, \quad G^{ab} = g^{ab} + G^{ab},
\]

where

\[
\bar{h} = \frac{1}{M^2} h^{(2)} + \frac{1}{M^3} h^{(3)}
\]

with

\[
h^{(2)} = -2s^2 K_a K_b^a, \quad h^{(3)} = -\frac{8}{3} s^3 K_a K_b K_c K_a,
\]

and

\[
\bar{G}^{ab} = \frac{1}{M} G^{(1)ab} + \frac{1}{M^2} G^{(2)ab} + \mathcal{O}(M^{-3}),
\]

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with
\[ G^{(1)ab} = 4sK^{ab}, \quad G^{(2)ab} = -12s^2K^{a}_cK^{ca}. \]

We also use Eqs. (12), (20) and formulas (15). In the end Eq. (10) is written in the following form
\[ \hat{L}\hat{\chi} = -\frac{2}{M^2}\sqrt{-g}\frac{1}{\sqrt{-g}}\partial_a\left[\sqrt{-g}(1 + \tilde{h})(g^{ab} + \tilde{G}^{ab})\partial_b(B\psi_0 + \tilde{\chi})\right] - \tilde{h}\hat{L}\tilde{\chi} \]
\[ -\frac{1}{2}\partial_s\tilde{h}(\psi_0 + \partial_s\tilde{\chi} + B\partial_s\psi_0) + (1 + \tilde{h})(3\psi^{(0)} + B\psi_0 + \tilde{\chi})(B\psi_0 + \tilde{\chi})^2. \]

Expanded in the powers of $1/M$ it gives equations of the type considered in Appendix B, that is
\[ \hat{L}\chi^{(n)} = f^{(n)}. \]

The source term $f^{(n)}$ is determined by the lower order terms in $\tilde{\chi}$ and $B$, namely $f^{(n)}$ contains $\chi^{(k)}$ with $k \leq n - 2$ and $B^{(l)}$ with $l \leq n - 1$. Solution of Eq. (26) is given by the formula
\[ \chi^{(n)}(s) = \int dx \ G(s, x) f^{(n)}(x), \]
where the Green’s function $G(s, x)$ can be found in Appendix B. This solution obeys the boundary conditions (18).

Let us list $f^{(n)}$ with $n=2,3,4$:
\[ f^{(2)} = 2s\psi_0K^a_bK^b_a + 3\chi^{(0)}\psi_0^2(B^{(1)})^2, \]
\[ f^{(3)} = 4s^2\psi_0K^a_bK^b_cK^c_a - 2\psi_0\Box^{(3)}B^{(1)} \]
\[ + f_1^{(3)}(s)K^a_bK^b_aB^{(1)} + f_2^{(3)}(s)(B^{(1)})^3 + 6\chi^{(0)}\psi_0^2B^{(1)}B^{(2)}, \]
\[ f^{(4)} = f_1^{(4)} + f_2^{(4)}, \]
where
\[ f_1^{(4)} = -2\Box^{(3)}(\chi^{(2)} + \psi_0B^{(2)}) - 4s\psi_0\frac{1}{\sqrt{-g}}\partial_a(\sqrt{-g}K^{ab}\partial_bB^{(1)}) \]
\[ -\frac{1}{2}\partial_s\tilde{h}(\partial_s\psi_0B^{(1)} - \frac{1}{2}\partial_s\tilde{h}^{(2)}(\partial_s\psi_0B^{(2)} + \partial_s\chi^{(2)} - \tilde{h}^{(2)}\psi_0) \]
\[ + 3\psi_0^2(B^{(1)})^2(\chi^{(2)} + \psi_0B^{(2)}) + 3\chi^{(0)}[\psi_0^2(B^{(2)})^2 + (\chi^{(2)})^2 + 2\psi_0\chi^{(3)}B^{(1)}], \]
and
\[ f_2^{(4)} = 6\psi_0^2 \chi^{(0)} B^{(1)} B^{(3)}. \]
Here
\[ \Box^{(3)} = \frac{1}{\sqrt{-g}} \partial_a (\sqrt{-g} g^{ab} \partial_b) \]
is the three-dimensional d’Alembertian on the world-volume Σ of the comoving membrane. The functions \( f_{1,2}^{(4)}(s) \) can be found in Appendix C. The formulas for \( f^{(3,4)} \) will become a little bit simpler because we shall show that \( B^{(2)} \) vanishes.

From formula (27) we obtain the explicit dependence of \( \chi^{(n)} \) on \( s \). The integrals over \( x \) which appear on the r.h.s. of that formula after we substitute expressions (28-30) are rather simple. Most of them can be evaluated analytically and the rest numerically by a computer algebra system. From the remark at the end of Appendix B and from formula (28) we see that \( \chi^{(2)} \) is an odd function of \( s \). The correction \( \chi^{(3)} \) is an even and \( \chi^{(4)} \) an odd function of \( s \) provided that \( B^{(2)} \) vanishes.

Equations of motion for the \( (2+1) \)-dimensional fields \( B^{(n)}(u^a) \) are obtained by expanding Eq. (11). It is convenient to rewrite that equation in the following form
\[
\frac{2}{M^2} \int ds \psi_0 \frac{1}{\sqrt{-g}} \partial_a \left[ \sqrt{-g} (1 + \bar{h})(g^{ab} + \bar{G}^{ab}) \partial_b (B\psi_0 + \bar{\chi}) \right] + \frac{1}{2} \int ds \partial_s \bar{h} \left( \psi_0^2 + \bar{\chi} \partial_s \psi_0 + B\psi_0 \partial_s \psi_0 \right) - \int ds \psi_0 (1 + \bar{h})(3\chi^{(0)} + B\psi_0 + \bar{\chi})(B\psi_0 + \bar{\chi})^2 = 0. \tag{31}
\]
It has been obtained with the help of integration by parts. We have also used Eq. (20). From Eq. (31) we see that the first nontrivial equation appears in the order \( 1/M^3 \). It has the form of non-linear, inhomogeneous \( (2+1) \)-dimensional wave equation for \( B^{(1)}(u^a) \) regarded as a field defined on Σ
\[
c_0 \Box^{(3)} B^{(1)} + c_1 K^a_b K^b_a B^{(1)} - c_2 (B^{(1)})^3 = d_0 K^a_b K^b_c K^c_a. \tag{32}
\]
The constants \( c_i, d_0 \) are listed in Appendix C.

The equation for \( B^{(2)} \) follows from the \( 1/M^4 \) terms in Eq. (31). It has the form
\[
\Box^{(3)} B^{(2)} + \frac{1}{2} K^a_b K^b_a B^{(2)} - \frac{c_3}{c_0} (B^{(1)})^2 B^{(2)} = 0. \tag{33}
\]
This equation is homogeneous with respect to $B^{(2)}$. Because also the initial data (24) are homogeneous, $B^{(2)}$ vanishes

$$B^{(2)}(u^a) = 0. \quad (34)$$

The $1/M^5$ terms give the equation for $B^{(3)}$. It can be written in the form

$$c_0 \Box B^{(3)} + c_1 K^a_b K^b_c B^{(3)} - 3c_2 (B^{(1)})^2 B^{(3)} = j^{(5)}, \quad (35)$$

where the source term $j^{(5)}$ does not contain $B^{(3)}$. It is given by the following formula

$$j^{(5)} = -\int ds \, \frac{1}{\sqrt{-g}} \partial_a [\sqrt{-g} g^{ab} \partial_b \chi^{(3)}(s) \psi_0 + G^{(1)ab} \partial_b \chi^{(2)}(s) \psi_0$$

$$+ h^{(2)} g^{ab} \partial_b B^{(1)}(s) \psi_0^2 + G^{(2)ab} \partial_b B^{(1)}(s) \psi_0^2]$$

$$- \frac{1}{4} \int ds \, \partial_s \psi_0 (\partial_s h^{(2)}(s) \chi^{(3)} + \partial_s h^{(2)}(s) \chi^{(3)}) + \frac{1}{2} \int ds \, \psi_0^4 h^{(2)}(B^{(1)})^3$$

$$+ \frac{3}{2} \int ds \, \psi_0^2 [2\chi^{(0)}(s) \chi^{(2)}(s) \chi^{(3)}(s) + \psi_0^2 \chi^{(3)}(s) (B^{(1)})^2 + \psi_0 (\chi^{(2)}(s))^2 B^{(1)}$$

$$+ \psi_0^2 \chi^{(0)}(s) h^{(3)}(B^{(1)})^2 + 2\psi_0 h^{(2)} \chi^{(0)}(s) \chi^{(2)}(s) B^{(1)}]$$

$$+ 3 \int ds \, \psi_0^3 G[f_2^{(4)}(s)] B^{(1)},$$

where

$$G[f_2^{(4)}](s) = \int dx \, G(s, x) f_2^{(4)}(x).$$

The function $B^{(2)}$ is not present in formula (36) because we have taken into account the result (34).

Finally, from Eq.(31) in the $1/M^6$ order we obtain a homogeneous wave equation for the function $B^{(4)}$ and therefore, similarly as in the case of the function $B^{(2)}$ with Eq.(33), we conclude that

$$B^{(4)}(u^a) = 0. \quad (37)$$

Let us summarize our results. We have found that

$$\phi = \chi^{(0)}(s) + \frac{1}{M} B^{(1)}(u^a) \psi_0(s) + \frac{1}{M^2} \chi^{(2)}(s, u^a)$$

$$+ \frac{1}{M^3} B^{(3)}(u^a) \psi_0(s) + \frac{1}{M^3} \chi^{(3)}(s, u^a) + \frac{1}{M^4} \chi^{(4)}(s, u^a) + O(M^{-5}),$$
where $\chi^{(0)}, \psi_0$ are given by formulas (15), (22), $\chi^{(n)}$ are given by formula (27), and $B^{(1)}, B^{(3)}$ are solutions of Eqs.(32), (35) with the initial data (24). Before solving Eqs.(32), (35) one should first solve Nambu-Goto equation (21) for the co-moving membrane. This is necessary in order to determine $g_{ab}$ and $K_{ab}$ which appear in these equations, and also in order to find the explicit form of the transformation (3) which relates the co-moving coordinates to the Cartesian laboratory coordinates.

4 Equation of motion for the core

In the description of the collective dynamics of the domain wall we could use the core instead of the co-moving membrane. This might even seem a better choice because the definition of the core as the locus of zeros of the scalar field $\phi$ is very simple and it directly refers to the domain wall solution. In this Section we derive the equation of motion for the core in our approach. We shall see that this equation is more complicated than the Nambu-Goto equation for the co-moving membrane and that it involves the $B^{(n)}$ fields.

The worldsheet $\tilde{\Sigma}$ of the core $\tilde{S}$ is related to the worldsheet $\Sigma$ of the co-moving membrane by the formula

$$\tilde{\Sigma} \ni \tilde{Y}^{\mu}(u^a) = Y^{\mu}(u^a) + \frac{2\tilde{s}}{M} n^{\mu}(u^a),$$

(39)

where $\tilde{s}$ is determined from the equation

$$\phi(\tilde{s}, u^a) = 0.$$  

(40)

Taking for $\phi$ the expansion (38) we find that

$$\tilde{s} = -\frac{B^{(1)}}{M} + O(M^{-3}).$$

(41)

We parametrise the core by the same coordinates $(u^a)$ as for the co-moving membrane. The induced metric tensor and the extrinsic curvature for $\tilde{\Sigma}$ are defined by the formulas

$$\tilde{g}_{ab} = \tilde{Y}_{,a} \tilde{Y}_{,b}, \quad \tilde{K}_{ab} = -\tilde{n}_{,a} \tilde{Y}_{,b} = \tilde{n} \tilde{Y}_{,ab},$$

(42)
where the normal to \( \tilde{\Sigma} \)
\[
\tilde{n}(u^a) = n(u^a) - \frac{2}{M^2} g^{ab} B^{(1)}{}_{ab} Y_b + \mathcal{O}(M^{-4})
\]
has been determined from the condition
\[
\tilde{n} \tilde{Y}_a = 0 \quad \text{for} \quad a = 0, 1, 2.
\]
We have also used the Nambu-Goto equation \( K^a_a = 0 \). Simple calculations give
\[
\tilde{g}_{ab} = g_{ab} + \frac{4}{M^2} B^{(1)} K_{ab} + \mathcal{O}(M^{-4}), \quad (43)
\]
\[
\tilde{K}_{ab} = K_{ab} + \frac{2}{M^2} B^{(1)} K_{ac} K^c_b - \frac{2}{M^2} \nabla_b B^{(1)}_{,a} + \mathcal{O}(M^{-4}), \quad (44)
\]
where
\[
\nabla_b B^{(1)}_{,a} = B^{(1)}_{,ab} - \Gamma^c_{ab} B^{(1)}_{,c}
\]
is the standard covariant derivative of \( B^{(1)} \) with respect to the metric \( g_{ab} \) on the co-moving membrane. The relations inverse to (43), (44) have the form
\[
g_{ab} = \tilde{g}_{ab} - \frac{4}{M^2} B^{(1)} \tilde{K}_{ab} + \mathcal{O}(M^{-4}), \quad (45)
\]
\[
K_{ab} = \tilde{K}_{ab} - \frac{2}{M^2} B^{(1)} \tilde{K}_{ac} \tilde{K}^c_b + \frac{2}{M^2} \tilde{\nabla}_b B^{(1)}_{,a} + \mathcal{O}(M^{-4}), \quad (46)
\]
where the tilda over \( \nabla \) means that now we use the metric \( \tilde{g}_{ab} \). Using formulas (45), (46) we find that the equation \( K^a_a = 0 \) for the co-moving membrane implies the following equation for the core
\[
\tilde{K}^a_a = -\frac{6}{M^2} B^{(1)} \tilde{K}^a_b \tilde{K}^b_a - \frac{2}{M^2} \Box^{(3)} B^{(1)} + \mathcal{O}(M^{-4}). \quad (47)
\]
Using Eq.(32) we eliminate \( \Box^{(3)} B^{(1)} \). Then
\[
\tilde{K}^a_a = -\frac{2}{M^2} (3 - \frac{c_1}{c_0}) B^{(1)} \tilde{K}^a_b \tilde{K}^b_a - \frac{2}{M^2} \frac{c_2}{c_0} (B^{(1)})^3
\]
\[
- \frac{2}{M^2} \frac{d_0}{c_0} \tilde{K}^a_b \tilde{K}^b_c \tilde{K}^c_a + \mathcal{O}(M^{-4}).
\]
This is the final form of the equation of motion for the core.

The terms on the r.h.s. of Eq.(48) can be regarded as corrections to the Nambu-Goto equation \( \tilde{K}^a_a = 0 \). They vanish if \( \tilde{K}^a_b \tilde{K}^b_c \tilde{K}^c_a = 0 \) because then
also $B^{(1)} = 0$ as it follows from Eq.(32) with the initial conditions (24). In general, Eq.(48) has to be considered together with Eq.(32) for $B^{(1)}$ in which we may replace $K_{ab}$ by $\tilde{K}_{ab}$. It is clear that the preferred by us description of the collective dynamics in terms of the co-moving membrane of the Nambu-Goto type instead of the core is much simpler.

In the literature one can find many attempts to represent the collective dynamics of the domain wall by the core only. In our approach this would amount to expressing the $B^{(1)}$ field in Eq.(48) by the extrinsic curvatures. In principle this is possible because Eq.(32) with the initial data (24) gives a one-to-one relation between $B^{(1)}$ and $\tilde{K}_a^b \tilde{K}_b^c \tilde{K}_c^a$. However, the presence on the l.h.s. of Eq.(32) of the operator $\Box^{(3)}$ acting on $B^{(1)}$ has the consequence that in general $B^{(1)}$ depends on $\tilde{K}_a^b \tilde{K}_b^c \tilde{K}_c^a$ in a nonlocal manner. Hence, the resulting self-contained equation for the core will be nonlocal too.

5 Discussion

1. The (2+1)-dimensional fields $B^{(n)}(u^a)$ are an important component in our version of the expansion in the width. The leading term $B^{(1)}/M$ does not vanish if $K_a^b K_b^c K_c^a \neq 0$. Using formulas from the second of papers [26] one can show that $K_a^b K_b^c K_c^a$ is proportional to the product of the mean and Gaussian curvatures in a local rest frame of an infinitesimal piece of the co-moving membrane. It is also shown in that paper that the evolution governed by the Nambu-Goto equation is sensitive only to the mean curvature in the local rest frame. Therefore, the co-moving membrane, which obeys the Nambu-Goto equation to all orders, can not account for effects which are due to the Gaussian curvature. This is the reason why for the complete local description of the collective dynamics of the domain wall we need the $B$ field too.

2. There is an important assumption we have tacitly made: that the derivatives $\chi^{(n)}_a, B^{(n)}_a$ are of the order $1/M^n$. It is not satisfied, for example, if $\chi$ and $B$ contain modes oscillating with a frequency $\sim M$ which give positive powers of $M$ upon differentiation with respect to $u^a$. If such oscillating components were present the counting of powers of $1/M$ would no longer be so straightforward as in Sections 2 and 3. The assumption excludes radiation modes as well as massive excitations of the domain wall. Therefore, the approximate solution we obtain gives what we may call the basic curved domain wall. To obtain more general domain wall solutions one would have to
change appropriately the approximation scheme. Actually, the fact that such particular radiationless unexcited curved domain wall exists is a prediction coming from the $1/M$ expansion. The expansion yields domain walls of concrete transverse profile — the dependence on $s$ is explicit in the approximate solution we construct even at the initial instant of time. Once we choose the initial position and velocity of points of the membrane the dependence of the scalar field on the variable $s$ at the initial time is given by formulas (16), (27). This unique profile is characteristic for the basic curved domain wall.

3. The approximate domain wall solution we have obtained can be generalised by relaxing the initial conditions (24) for the $B^{(n)}$ fields. These homogeneous initial data forbid appearance of freely propagating along the co-moving membrane waves of the $B^{(n)}$ fields — only the $B^{(n)}$ fields generated by the extrinsic curvature as the source can appear. Suppose however that we admit such free waves, that is that the initial data for $B^{(n)}$ are chosen arbitrarily. Now $B^{(2)}$ and $B^{(4)}$ do not have to vanish. Equations (32) and (33) remain unchanged, but in Eq.(35) new terms containing $B^{(2)}$ will appear. Let us also choose certain initial data for Nambu-Goto equation (21) for the co-moving membrane. Then, substitution of the corresponding unique solutions of Eqs.(21), (32), (33) and of (the modified) Eq.(35) in formula (38) gives an approximate solution $\phi$ of the domain wall type. These more general solutions are not excluded by the assumption formulated in the preceding paragraph because the characteristic frequency of the $B^{(n)}$ fields is not related to the mass parameter M. From Eqs.(32), (33), (35) we see that the $B^{(n)}$ fields have the effective mass $2\sim K^a_bK_b^a$.

Actually, a simple particular case of solutions analogous to the generalised ones we are now discussing has been reported in paper [30]. In that paper certain small amplitude excitations of a straightlinear vortex in the Abelian Higgs model have been considered. Such excitations are represented by fields defined on the axis of the straightlinear vortex. They have been called in [30]

\[ \frac{B^{(n)}}{M^n} \ll 1, \quad \frac{B^{(n+1)}}{M} \ll B^{(n)}, \quad \partial_a \frac{B^{(n)}}{M^n} \ll M. \]

These bounds ensure that order of magnitude of the perturbative contributions, in principle given by a power of $K_{ab}/M$, is not changed because of too large values of the $B^{(n)}$ fields at the initial time or later.
the zero-mode fields. They obey a linear, massless wave equation. It turns out that our $B^{(1)}$ field gives the domain wall counterpart of the zero-mode field of Ref. [30] if we take a planar domain wall for which $K_{ab} = 0$, and if we assume that the $B^{(1)}$ field is so small that one may neglect the nonlinear term on the l.h.s. of Eq. (32).

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7 Appendix A. The co-moving coordinates

The extrinsic curvature coefficients $K_{ab}$ and induced metrics $g_{ab}$ on $\Sigma$ are defined by the following formulas:

$$K_{ab} \equiv n_\mu Y_{ab}^\mu,$$

$$g_{ab} \equiv Y_{a\mu} Y_{b\mu},$$

where $a, b = 0, 1, 2$. The covariant metric tensor in the new coordinates has the following form

$$[G_{\alpha\beta}] = \begin{bmatrix} G_{ab} & 0 \\ 0 & 1 \end{bmatrix},$$

where $\alpha, \beta = 0, 1, 2, 3; \quad \alpha = 3$ corresponds to the $\xi$ coordinate; and

$$G_{ab} = N_{ac} g^{cd} N_{db}, \quad N_{ac} \equiv g_{ac} - \xi K_{ac}.$$ 

Thus, $G_{\xi\xi} = 1, G_{\xi a} = 0$. Straightforward computation gives

$$\sqrt{-G} = \sqrt{-g} \ h(\xi, u^a),$$

where as usual $g \equiv det[g_{ab}], \quad G \equiv det[G_{\alpha\beta}]$, and

$$h(\xi, u^a) = 1 - \xi K_a^a + \frac{1}{2}\xi^2(K_a^b K_b^a - K_a^b K_b^a) - \frac{1}{3}\xi^3 K_a^b K_b^c K_c^a.$$

Also $g$ can depend on $u^a$. For raising and lowering the Latin indices of the extrinsic curvature coefficients we use the induced metric tensors $g^{ab}, \quad g_{ab}$. 

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The inverse metric tensor $G^{\alpha\beta}$ is given by the formula

$$[G^{\alpha\beta}] = \begin{bmatrix} G^{ab} & 0 \\ 0 & 1 \end{bmatrix},$$

where

$$G^{ab} = (N^{-1})^{ac}g_{cd}(N^{-1})^{db}.$$  

Explicit formula for $(N^{-1})^{ac}$ has the following form:

$$(N^{-1})^{ac} = \frac{1}{h} \left\{ g^{ac}[1 - \xi K^b_b + \frac{1}{2} \xi^2(K^b_b K^d_d - K^a_a K^d_d)] ight. \\
\left. + \xi(1 - \xi K^b_b)K^{ac} + \xi^2 K^a_a K^{dc} \right\}.$$  

This is just the matrix inverse to $[N_{ab}]$. It has the upper indices by definition.  

In general, the coordinates $(u^a, \xi)$ are defined locally, in a vicinity of the world-volume $\Sigma$ of the co-moving membrane. Roughly speaking, the allowed range of the $\xi$ coordinate is determined by the smaller of the two main curvature radii of the membrane in a local rest frame. We assume that this curvature radius is sufficiently large so that on the outside of the region of validity of the co-moving coordinates there are only exponential tails of the domain wall, that is the field $\phi$ is exponentially close to one of the two vacuum solutions. More detailed discussion of the region of validity of the co-moving coordinates can be found in the first of papers \cite{23}.

8 Appendix B. Solutions of equation $\hat{L}\chi = f$

Let us consider the equation

$$\frac{1}{2} \partial_s^2 \chi + (1 - 3 \tanh^2 s) \chi = f(s).$$

The presence of the zero mode implies that it has solutions only if $f(s)$ obeys the condition

$$\int ds \psi_0(s)f(s) = 0,$$

where $f \, ds$ is our shorthand for $\int_{-\infty}^{x}$. General solution of our equation can be obtained with the help of the standard procedure consisting of finding
two independent solutions of the corresponding homogeneous equation and constructing a suitable Green’s function, see, e.g., [31]. The two linearly independent solutions of the homogeneous equation have the form

\[
\psi_0(s) = \frac{1}{\cosh^2 s},
\]

\[
\psi_1(s) = \frac{1}{8} \sinh(2s) + \frac{3}{8} \tanh s + \frac{3}{8} s \cosh^2 s.
\]

As the Green’s function we take

\[
G(s, x) = 2 \left[ \psi_1(s)\psi_0(x) - \psi_0(s)\psi_1(x) \right] \Theta(s - x) + 2\psi_0(s)\psi_1(x)\Theta(-x),
\]

where \(\Theta\) is the step function. This Green’s function vanishes at \(s = 0\). The general solution of our equation has the form

\[
\chi = \alpha \psi_0 + \beta \psi_1 + \int dx G(s, x)f(x),
\]

where \(\alpha, \beta\) do not depend on \(s\). If we require that \(\chi\) obeys the conditions (18) we have to put \(\alpha = \beta = 0\). In this case, using the given above formula for the Green’s function we can write the solution \(\chi\) in the following form

\[
\chi(s) = 2\psi_1(s) \int_{-\infty}^{s} dx \psi_0(x)f(x) - 2\psi_0(s) \int_{0}^{s} dx \psi_1(x)f(x).
\]

This specific formula for \(\chi(s)\) is used in Section 3. One can easily check that if \(f\) is even (odd) function then \(\chi\) is even (odd) function too.

9 Appendix C. The list of functions and constants

It is convenient to introduce the notation

\[
G[f](s) = \int dx G(s, x)f(x).
\]

The functions appearing in formula (29) for \(f^{(3)}\) have the form

\[
f_1^{(3)}(s) = 2s \partial_s \psi_0 + 12\chi^{(0)}\psi_0 G[x\psi_0],
\]

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\[ f_2^{(3)}(s) = \psi_0^3 + 18\chi^{(0)}\psi_0 G[\chi^{(0)}\psi_0^2]. \]

The constants present in Eqs. (32), (33), (35) are given by the following formulas:

\[ d_0 = 2 \int ds \, s^2 \psi_0^2, \quad c_0 = \int ds \, \psi_0^2, \]

\[ c_1 = -\frac{1}{2} \int ds \, \psi_0 f_1^{(3)}, \]

\[ c_2 = \frac{1}{2} \int ds \, \psi_0 f_2^{(3)}, \]

\[ c_3 = \frac{3}{2} \int ds \, \psi_0^4 + 18 \int ds \, \psi_0^2 \chi^{(0)} G[\chi^{(0)}\psi_0^2]. \]

References

[1] R.P.Huebener, Magnetic Flux Structures in Superconductors. Springer-Verlag, Berlin - Heidelberg - New-York, 1979.

[2] R.J.Donnelly, Quantized Vortices in HeliumII. Cambridge University Press, Cambridge, 1991.

[3] S.Chandrasekhar and G.S.Ranganath, Adv. Phys. 35, 507 (1986).

[4] See, e.g., M.Baker, J.S.Ball and F.Zachariasen, Phys.Rep. 209, 73 (1991); S. Loh, C. Greiner, U. Mosel and M. H. Thoma, hep-ph/9701363.

[5] T.W.B.Kibble, J.Phys. A9, 1387 (1976); A.L.Vilenkin, Phys. Rep. 121, 263 (1985).

[6] R. A. Matzner, Computers in Physics 2, 51 (1988).

[7] E.P.S.Shellard, Nucl.Phys.B283, 624 (1987).

[8] L.M.Widrow, Phys.Rev.D40, 1002 (1989).

[9] Vachaspati and T. Vachaspati, Phys. Lett. 238B, 41 (1990).

[10] P. J. Ruback, Nucl. Phys. B296, 669 (1988).

[11] H.B.Nielsen and P.Olesen, Nucl.Phys. B61, 45 (1973).
[12] D.Förster, Nucl.Phys. B81, 84 (1974).
[13] J.-L.Gervais and B.Sakita, Nucl.Phys. B91, 301 (1975).
[14] K. Maeda and N. Turok, Phys.Lett. B202, 376 (1988).
[15] R.Gregory, Phys.Lett. B206, 199 (1988).
[16] S.M.Barr and D.Hochberg, Phys.Rev.D39, 2308 (1989).
[17] R.Gregory, D.Haws and D.Garfinkle, Phys.Rev. D42, 343 (1990).
[18] Uri Ben-Ya'acov, Nucl.Phys. B382, 597 (1992).
[19] V.Silveira and M.D.Maia, Phys.Lett. A174, 280 (1993).
[20] A.L.Larsen, Phys.Lett. A181, 369 (1993).
[21] B.Carter and R.Gregory, Phys. Rev. D51, 5839 (1995).
[22] F. Lund and T. Regge, Phys. Rev. D14, 1524 (1976).
[23] H. Arodź, Nucl.Phys. B450, 174 (1995); ibidem, 189.
[24] H.Arodź and P.Węgrzyn, Phys.Lett. B291, 251 (1992).
[25] V.Silveira, Phys.Rev. D41, 1914 (1990).
[26] H.Arodź and A.L. Larsen, Phys.Rev. D49, 4154 (1994); H.Arodź, Phys. Rev. D52, 1082 (1995); H. Arodź and L. Hadasz, Phys. Rev. D54, 4004 (1996).
[27] N.G.van Kampen, Stochastic Processes in Physics and Chemistry. North-Holland Publ.Comp., Amsterdam, 1987. Chapt.8, §7.
[28] J. Karkowski and Z. Świerczyński, Acta Phys. Pol. B 30, 234 (1996).
[29] B. Carter, in "Formation and Interactions of Topological Defects", p.303. A.-Ch. Davis and R. Brandenberger (Eds.). Plenum Press, New York and London, 1995.
[30] N. G. Khariton and V. B. Svetovoy, Phys. Lett. B286, 53 (1992).
[31] G.A. Korn and T.M. Korn, Mathematical Handbook. IInd Edition. McGraw-Hill Book Comp., New York, 1968. Chapt.9.3-3.