Abstract. We develop a quenched thermodynamic formalism for open random dynamical systems generated by finitely branched, piecewise-monotone mappings of the interval. The openness refers to the presence of holes in the interval, which terminate trajectories once they enter; the holes may also be random. Our random driving is generated by an invertible, ergodic, measure-preserving transformation \( \sigma \) on a probability space \((\Omega, \mathcal{F}, m)\). For each \( \omega \in \Omega \) we associate a piecewise-monotone, surjective map \( T_\omega : I \to I \), and a hole \( H_\omega \subset [0,1] \); the map \( T_\omega \), the random potential \( \varphi_\omega \), and the hole \( H_\omega \) generate the corresponding open transfer operator \( L_\omega \). For a contracting potential, under a condition on the open random dynamics in the spirit of Liverani–Maume-Deschamps \([29]\), we prove there exists a unique random probability measure \( \nu_\omega \) supported on the survivor set \( X_{\omega,\infty} \) satisfying \( \nu_\sigma(\omega) (L_\omega f) = \lambda_\omega \nu_\omega (f) \). Correspondingly, we also prove the existence of a unique (up to scaling and modulo \( \nu \)) random family of functions \( q_\omega \) that satisfy \( L_\omega q_\omega = \lambda_\omega q_{\sigma(\omega)} \). Together, these provide an ergodic random invariant measure \( \mu = \nu q \) supported on the global survivor set \( X_\infty \), while \( q \) combined with the random closed conformal measure yields a unique random absolutely continuous conditional invariant measure (RACCIM) \( \eta \) supported on \([0,1]\). Further, we prove quasi-compactness of the transfer operator cocycle generated by \( L_\omega \) and exponential decay of correlations for \( \mu \). The escape rates of the random closed conformal measure and the RACCIM \( \eta \) coincide, and are given by the difference of the expected pressures for the closed and open random systems. Finally, we prove that the Hausdorff dimension of the surviving set \( X_{\omega,\infty} \) is equal to the unique zero of the expected pressure function for almost every fiber \( \omega \in \Omega \). We provide several examples of our general theory. In particular, we apply our results to random \( \beta \)-transformations and random Lasota-Yorke maps, but our results also apply to the random non-uniformly expanding maps that are treated in \([1]\), such as intermittent maps and maps with contracting branches.

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1. Introduction

Deterministic transitive dynamics $T : [0, 1] \rightarrow [0, 1]$ with enough expansivity enjoy a “thermodynamic formalism”: the transfer operator with a sufficiently regular potential $\varphi$ has a unique absolutely continuous invariant measure (ACIM) $\mu$, absolutely continuous with respect to the conformal measure $\nu$. Furthermore, $\mu$ arises as an equilibrium state, i.e. a maximiser of the sum of the integral of the potential $\varphi$ and the metric entropy $h(\mu)$. Classical results in this direction include [35, 8, 32] for shifts of finite type and smooth dynamics, and [21, 30, 11] for piecewise smooth dynamics.

Continuing with the deterministic setting, if one introduces a hole $H \subset [0, 1]$, which terminates trajectories when they enter, the situation becomes considerably more complicated. In the simplest case where the potential is the usual geometric potential $\varphi = -\log |T'|$, because of the lack of mass conservation, one expects at best an absolutely continuous conditional invariant measure (ACCIM) $\mu$, conditioned according to survival from the infinite past. Absolutely continuity is again with respect to a conformal measure $\nu$, which is supported on the survivor set $X_\infty$, the set of points whose infinite forward trajectories remain in $[0, 1]$. Early work on the existence of the ACCIM and exponential convergence of non-equilibrium densities, includes [38, 12]. The paper [29] handles general potentials that are contracting [30] for the closed system, demonstrating exponential decay for $\mu$. There has been further work on the Lorentz gas and billiards [14, 15], intermittent maps [16, 18], and multimodal maps [17, 19]. In the setting of diffeomorphisms with SRB measures, following the introduction of a hole, relations between escape rates and the pressures have been studied in [20].

Looking to the fractal dimension of the surviving set $X_\infty$, the machinery of thermodynamic formalism was first employed by Bowen [9] to find the Hausdorff dimension of the
limit sets of quasi-Fuchsian groups in terms of the pressure function, and then pioneered in the setting of open dynamical systems in [37].

In the random setting, repeated iteration of single deterministic map is replaced with composition of maps $T_\omega : X \to X$ drawn from a collection $\{T_\omega\}_{\omega \in \Omega}$. A driving map $\sigma : \Omega \to \Omega$ on a probability space $(\Omega, \mathcal{F}, m)$ creates a map cocycle $T^n_\omega := T_{\sigma^{n-1}\omega} \circ \cdots \circ T_{\sigma\omega} \circ T_\omega$. Thermodynamic formalism for random dynamical systems has been largely restricted to $T_\omega$ that are subshifts [7, 25], distance expanding [31, 34], or continuous [27, 6, 26, 36] ([36] considers the sequential case where there is a single orbit of $\omega$). Countable Markov shifts have been treated in a series of papers beginning with [22] (see [1] for further references). The authors recently developed a complete, quenched thermodynamic formalism for random, countably-branched, piecewise monotonic interval maps [1], enabling the treatment of discontinuous, non-Markov $T_\omega$.

The situation of random open dynamics is relatively untouched. For a single piecewise expanding map $T : [0, 1] \to [0, 1]$ with holes $H_\omega$ randomly chosen in an i.i.d. fashion, [4] consider escape rates for the annealed (averaged) transfer operator in the small hole limit (the Lebesgue measure of the $H_\omega$ goes to zero). In a similar setting, now assuming $T$ to be Markov and considering non-vanishing holes, [3] show existence of equilibrium states, again for the annealed transfer operator. In [2], the authors consider random, full-branched interval maps with negative Schwarzian derivative. The maps are allowed to have critical points, but the partition of monotonicity and holes, made up of finitely many open intervals, are fixed and non-random. In this setting the existence of a unique invariant random probability measure is proven as well as a formula for the Hausdorff dimension of the surviving set. In our current setting, we do not allow the existence of critical points, however our maps may have non-full branches, and our partitions of monotonicity as well as our holes are allowed to vary randomly from fiber to fiber.

Sequential systems with holes have been considered in [24], where a cocycle $T_\omega$ of open maps is generated by a single $\omega$ orbit. The maps (which include the hole) must be chosen in a small neighborhood of a fixed map (with hole), in contrast to our setting where our cocycle may include very different maps. Moreover in [24], Lebesgue is used as a reference measure and the specific potential $-\log |\det DT|$ is used. The theory is developed for uniformly expanding maps in higher dimensions and the main goal is to establish the “conditional memory loss”, a concept analogous to exponential decay of correlations for closed dynamics.

In the present paper, we make a considerable advance over prior work by establishing a full, quenched thermodynamic formalism for piecewise monotonic random dynamics with general potentials and general driving—the random driving $\sigma$ can be any invertible ergodic process on $\Omega$. We begin with the random closed dynamics dealt with in [1]: piecewise monotonic interval maps satisfying a random covering condition; we have no Markovian assumptions, our maps may have discontinuities and may lack full branches. The number of branches of our maps need not be uniformly bounded above in $\omega$ and our potentials $\varphi_\omega$ need not be uniformly bounded below or above in $\omega$. To this setting we introduce random holes $H_\omega$ and formulate sufficient conditions that guarantee a random conformal measure $\nu_\omega$ and
corresponding equivariant measure $\mu_\omega$ supported on the random survivor set $X_{\omega,\infty}$, and a random ACCIM $\eta_\omega$ supported on $H^c_\omega$. These augment the notion of a random contracting potential [1] with accumulation rates of contiguous “bad” intervals (with zero conformal measure), and extend similar constructions [29] to the random situation.

To establish the existence of the family of measures $(\nu_\omega)_{\omega \in \Omega}$, we follow the limiting functional approach of [30, 29] by defining a random functional $\Lambda_\omega$ which is a limit of ratios of transfer operators and then showing that $\Lambda_\omega$ may be identified with the open conformal measure $\nu_\omega$. This technique improves on the approach of [1], which uses the Schauder-Tichonov Fixed Point Theorem to prove the existence of $\nu_{\omega,c}$, by eliminating the extra steps necessary to show that the family $(\nu_\omega)_{\omega \in \Omega}$ is measurable with respect to $m$. We establish exponential decay of correlations for $\mu$ and show that the escape rate of the closed conformal measure coincides with that of the RACCIM, and equals the difference of the expected pressures of the closed and open random systems. We define the expected pressure function $t \mapsto E\mathcal{P}(\varphi_t)$ for the potential $\varphi_{\omega,t} = -t \log |T^c_\omega|$ and any $t \geq 0$, and then show that this function has a unique zero $h \in [0,1]$. Furthermore, we show that the Hausdorff dimension of the survivor set $X_{\omega,\infty}$ is equal to $h$ for $m$-a.e. $\omega \in \Omega$.

We apply our general theory to a large class of random $\beta$-transformations with random holes as well as a general random Lasota-Yorke maps with random holes. In particular, for the first time we allow both the maps and the holes to be random. In fact, our theory applies to all of the finitely-branched examples discussed in [1] (this includes maps which are non-uniformly expanding or have contracting branches which appear infrequently enough that we still maintain on-average expansion) when suitable conditions are put on the holes $H_\omega$. This includes the case where $H_\omega$ is composed of finitely many intervals and the number of connected components of $H_\omega$ is log-integrable with respect to $m$.

An outline of the paper is as follows. In Section 2 we present formal definitions, properties, and assumptions concerning the closed and open random dynamics! The notion of random absolutely continuous conditionally invariant probability measures is introduced in Section 3. In Section 4 we introduce the random functional $\Lambda_\omega$ which we eventually show can be identified with the conformal measure on the open system. In this section we also present our main assumptions on the open dynamics as well as state our main results. Section 5 contains background material on Birkhoff cone techniques and the construction of our random cones. In Section 6 we develop several random Lasota-Yorke type inequalities in terms of the variation and the random functional $\Lambda_\omega$. Section 7 sees the construction of a large measure set of “good” fibers $\Omega_G \subseteq \Omega$ for which we obtain cone invariance at a uniform time step, and in Section 8 we show that the remaining “bad” fibers occur infrequently and behave sufficiently well. In Section 9 we collect further properties of the random functional $\Lambda$, which are then used in Section 10 to construct a large measure set of fibers $\Omega_F \subseteq \Omega$ for which we obtain cone contraction with a finite diameter image in a random time step. Using Hilbert metric contraction arguments, Section 11 collects together the fruits of Sections 7-10 to prove our main technical lemma (Lemma 11.1), which is then used to (i) obtain the existence of a random density $q$, (ii) prove the existence of a unique non-atomic random conformal measure $\nu$, and (iii) a random $T$-invariant measure $\mu$ which is absolutely
continuous with respect to \( \nu \). In Section 12 we use the results of Section 11 to show that the convergence to the random density \( q \) happens exponentially quickly, which then implies that the random \( T \)-invariant measure \( \mu \) satisfies an exponential decay of correlations. We also establish the existence of a unique random conditionally invariant measure \( \eta \) which is absolutely continuous with respect to the closed conformal measure \( \nu_c \), and establish the uniqueness of the measures \( \nu \) and \( \mu \) as well as the density \( q \). The escape rate of the measure \( \nu_c \) is given in Section 13 in terms of the closed and open expected pressures. The expected pressure function is further developed and used to prove a Bowen’s formula type result for the Hausdorff dimension of the survivor set \( X_{\omega,\infty} \) for \( m \)-a.e. \( \omega \in \Omega \) in Section 14. Finally, in Section 15 we present detailed calculations for a large classes of random \( \beta \)-transformations with random holes and random Lasota-Yorke maps with random holes for which our results apply.

2. Preliminaries of Random Interval Maps with Holes

Given a probability space \((\Omega, \mathcal{F}, m)\), we begin by considering an invertible, ergodic map \( \sigma : \Omega \to \Omega \) which preserves the measure \( m \), i.e.

\[
m \circ \sigma^{-1} = m.
\]

We will refer to the tuple \((\Omega, \mathcal{F}, m, \sigma)\) as the base dynamical system. Take \( I \) to be a compact interval in \( \mathbb{R} \) and for each \( \omega \in \Omega \) we consider the map \( T_\omega : I \to I \) such that there exists a finite partition \( Z_\omega \) of \( I \) such that

(T1) \( T_\omega : I \to I \) is surjective,

(T2) \( T_\omega(Z) \) is an interval for each \( Z \in Z_\omega \),

(T3) \( T_\omega|_Z \) is continuous and strictly monotone for each \( Z \in Z_\omega \).

In addition, we will assume that

(LIP) \( \log \# Z_\omega \in L^1(m) \).

The maps \( T_\omega \) induce the skew product map \( T : \Omega \times I \to \Omega \times I \) given by

\[
T(\omega, x) = (\sigma(\omega), T_\omega(x)).
\]

For each \( n \in \mathbb{N} \) we consider the fiber dynamics of the maps \( T^n_\omega : I \to I \) given by the compositions

\[
T^n_\omega(x) = T_{\sigma^{n-1}(\omega)} \circ \cdots \circ T_\omega(x).
\]

We let \( Z^{(n)}_\omega \), for \( n \geq 2 \), denote the monotonicity partition of \( T^n_\omega \) on \( I \) which is given by

\[
Z^{(n)}_\omega := \bigvee_{j=0}^{n-1} T^{-j}_\omega(Z_{\sigma^j(\omega)}).
\]

Given \( Z \in Z^{(n)}_\omega \), we denote by

\[
T^{-n}_{\omega,Z} : T^n_\omega(Z) \to Z
\]
the inverse branch of $T^*_\omega$ which takes $T^*_\omega(x)$ to $x$ for each $x \in Z$. We will assume that the partitions $Z_\omega$ are generating, i.e.

\[ \bigvee_{n=1}^{\infty} Z_\omega^{(n)} = \mathcal{B}, \]

where $\mathcal{B} = \mathcal{B}(I)$ denotes the Borel $\sigma$-algebra of $I$. Let $\mathcal{B}(I)$ denote the set of all bounded real-valued functions on $I$ and for each $f \in \mathcal{B}(I)$ and each $A \subseteq I$ let

\[ \text{var}_A(f) := \sup \left\{ \sum_{j=0}^{k-1} |f(x_{j+1}) - f(x_j)| : x_0 < x_1 < \ldots < x_k, x_j \in A \text{ for all } k \in \mathbb{N} \right\}, \]

denote the variation of $f$ over $A$. We let

\[ \text{BV}(I) := \{ f \in \mathcal{B}(I) : \text{var}_I(f) < \infty \} \]

denote the set of functions of bounded variation on $I$. Let $\|f\|_\infty$ and

\[ \|f\|_{\text{BV}} := \text{var}(f) + \|f\|_\infty \]

be norms on the respective Banach spaces $\mathcal{B}(I)$ and $\text{BV}(I)$. Given a function $f : \Omega \times I \rightarrow \mathbb{R}$, by $f_\omega : I \rightarrow I$ we mean

\[ f_\omega(\cdot) := f(\omega, \cdot). \]

**Definition 2.1.** We say that a function $f : \Omega \times I \rightarrow \mathbb{R}$ is random bounded if

(i) $f_\omega \in \mathcal{B}(I)$ for each $\omega \in \Omega$,

(ii) for each $x \in I$ the function $\Omega \ni \omega \mapsto f_\omega(x)$ is measurable,

(iii) the function $\Omega \ni \omega \mapsto \|f_\omega\|_\infty$ is measurable.

Let $\mathcal{B}_\Omega(I)$ denote the collection of all random bounded functions on $\Omega \times I$.

**Definition 2.2.** We say that a function $f \in \mathcal{B}_\Omega(I)$ is of random bounded variation if $f_\omega \in \text{BV}(I)$ for each $\omega \in \Omega$. We let $\text{BV}_\Omega(I)$ denote the set of all random bounded variation functions.

For functions $f : \Omega \times I \rightarrow \mathbb{R}$ and $F : \Omega \rightarrow \mathbb{R}$ we let

\[ S_n,T(f_\omega) := \sum_{j=0}^{n-1} f_{\sigma^j(\omega)} \circ T^*_\omega \quad \text{and} \quad S_{n,\sigma}(F) := \sum_{j=0}^{n-1} F \circ \sigma^j \]

denote the Birkhoff sums of $f$ and $F$ with respect to $T$ and $\sigma$ respectively. We will consider a potential of the form $\varphi_c : \Omega \times I \rightarrow \mathbb{R}$ and for each $n \geq 1$ we consider the weight $g^{(n)}_c : \Omega \times I \rightarrow \mathbb{R}$ whose disintegrations are given by

\[ g^{(n)}_{\omega,c} := \exp(S_n,T(\varphi_{\omega,c})) \]
for each \( \omega \in \Omega \).\footnote{Throughout the text we will use the subscript \(c\) notation to denote a quantity related to the closed dynamical system.} We will often denote \( g_{\omega,c}^{(1)} \) by \( g_{\omega,c} \). We define the (Perron-Frobenius) transfer operator, \( \mathcal{L}_{\omega,c} : \mathcal{B}(I) \to \mathcal{B}(I) \), with respect to the potential \( \varphi_c : \Omega \times I \to \mathbb{R} \), by
\[
\mathcal{L}_{\omega,c}(f)(x) := \sum_{y \in T_\omega^{-1}(x)} g_{\omega,c}(y)f(y); \quad f \in \mathcal{B}(I), \ x \in I.
\]
Inducting on \( n \) gives that the iterates \( \mathcal{L}_{\omega,c}^n : \mathcal{B}(I) \to \mathcal{B}(I) \) are given by
\[
(2.2) \quad \mathcal{L}_{\omega,c}^n(f)(x) = (\mathcal{L}_{\sigma^{-1}(\omega),c} \circ \cdots \circ \mathcal{L}_{\omega,c})(f)(x) = \sum_{y \in T_\omega^{-n}(x)} g_{\omega,c}^{(n)}(y)f(y); \quad f \in \mathcal{B}(I), \ x \in I.
\]
For each \( \omega \in \Omega \) we let \( \mathcal{B}^*(I) \) and \( \text{BV}^*(I) \) denote the respective dual spaces of \( \mathcal{B}(I) \) and \( \text{BV}(I) \). We let \( \mathcal{L}_{\omega,c}^* : \mathcal{B}^*(I) \to \mathcal{B}^*(I) \) denote the dual transfer operator.

**Definition 2.3.** We will say that a measurable potential \( \varphi_c : \Omega \times I \to \mathbb{R} \) is *admissible* if for \( m \)-a.e. \( \omega \in \Omega \) we have

(A1) \( \inf \varphi_{\omega,c}, \sup \varphi_{\omega,c} \in L^1(m) \),

(A2) \( g_{\omega,c} \in \text{BV}(I) \).

**Remark 2.4.** Note that if \( \varphi_{\omega,c} \in \text{BV}(I) \) for each \( \omega \in \Omega \) then (A2) is immediate. Furthermore, as \( \varphi_{\omega,c} \in \mathcal{B}(I) \) we also have that \( \inf g_{\omega,c}^{(n)} > 0 \) for \( m \)-a.e. \( \omega \in \Omega \) and each \( n \in \mathbb{N} \).

As an immediate consequence of (A1) we have that
\[
(2.3) \quad \log \inf g_{\omega,c}, \log \|g_{\omega,c}\|_\infty \in L^1(m).
\]
Note that since we can write
\[
g_{\omega,c}^{(n)} := \prod_{j=0}^{n-1} g_{\sigma^j(\omega),c} \circ T_\omega^j.
\]
for each \( n \in \mathbb{N} \) we must have that \( g_{\omega,c}^{(n)} \in \text{BV}(I) \). Clearly we have that the sequence \( \|g_{\omega,c}^{(n)}\|_\infty \) is submultiplicative, i.e.
\[
\|g_{\omega,c}^{(n+m)}\|_\infty \leq \|g_{\omega,c}^{(n)}\| \cdot \|g_{\sigma^n(\omega),c}^{(m)}\|_\infty.
\]
Similarly we see that the sequence \( \inf g_{\omega,c}^{(n)} \) is supermultiplicative. Submultiplicativity and supermultiplicativity of \( \|g_{\omega,c}^{(n)}\|_\infty \) and \( \inf g_{\omega,c}^{(n)} \) together with (2.3) gives that
\[
(2.4) \quad \log \|g_{\omega,c}^{(n)}\|_\infty, \log \inf g_{\omega,c}^{(n)} \in L^1(m)
\]
for each \( n \in \mathbb{N} \). Our assumptions (T1) and (LIP) combined with (2.4) implies that
\[
(2.5) \quad \log \|\mathcal{L}_{\omega,c}^n \mathbbm{1}\|_\infty, \log \inf \mathcal{L}_{\omega,c}^n \mathbbm{1} \in L^1(m)
\]
for each \( n \in \mathbb{N} \).
2.1. Random Measures. Given a measurable space $Y$, we let $\mathcal{P}(Y)$ denote the collection of all Borel probability measures on $Y$. Recall that $\mathcal{B}$ denotes the Borel $\sigma$-algebra of $I$. Let $\mathcal{F} \otimes \mathcal{B}$ denote the product $\sigma$-algebra of $\mathcal{B}$ and $\mathcal{F}$ on $\Omega \times I$.

Let $\mathcal{P}_m(\Omega \times I)$ denote the set of all probability measures $\mu$ on $\Omega \times I$ that have marginal $m$, i.e.

$$\mathcal{P}_m(\Omega \times I) := \{ \mu \in \mathcal{P}(\Omega \times I) : \mu \circ \pi^{-1}_\Omega = m \},$$

where $\pi_\Omega : \Omega \times X \to \Omega$ is the projection onto the first coordinate.

**Definition 2.5.** A map $\mu : \Omega \times \mathcal{B} \to [0, 1]$ with $\Omega \times \mathcal{B} \ni (\omega, B) \mapsto \mu_\omega(B)$ is said to be a random probability measure on $I$ if

1. for every $B \in \mathcal{B}$, the map $\Omega \ni \omega \mapsto \mu_\omega(B) \in [0, 1]$ is measurable,
2. for $m$-a.e. $\omega \in \Omega$, the map $\mathcal{B} \ni B \mapsto \mu_\omega(B) \in [0, 1]$ is a Borel probability measure.

We let $\mathcal{P}_\Omega(I)$ denote the set of all random probability measures on $I$. We will frequently denote a random measure $\mu$ by $(\mu_\omega)_{\omega \in \Omega}$.

The following proposition, which summarizes results of Crauel [13], shows that the collection $\mathcal{P}_m(\Omega \times I)$ can be canonically identified with the collection $\mathcal{P}_\Omega(I)$ of all random probability measures on $I$.

**Proposition 2.6** ([13], Propositions 3.3, 3.6). For each $\mu \in \mathcal{P}_m(\Omega \times I)$ there exists a unique random probability measure $(\mu_\omega)_{\omega \in \Omega} \in \mathcal{P}_\Omega(I)$ such that

$$\int_{\Omega \times I} f(\omega, x) \, d\mu(\omega, x) = \int_\Omega \int_I f(\omega, x) \, d\mu_\omega(x) \, dm(\omega)$$

for every bounded measurable function $f : \Omega \times I \to \mathbb{R}$.

Conversely, if $(\mu_\omega)_{\omega \in \Omega} \in \mathcal{P}_\Omega(I)$ is a random probability measure on $I$, then for every bounded measurable function $f : \Omega \times I \to \mathbb{R}$ the function

$$\Omega \ni \omega \mapsto \int_I f(\omega, x) \, d\mu_\omega(x)$$

is measurable and

$$\mathcal{F} \otimes \mathcal{B} \ni A \mapsto \int_\Omega \int_I \mathbb{1}_A(\omega, x) \, d\mu_\omega(x) \, dm(\omega)$$

defines a probability measure in $\mathcal{P}_m(\Omega \times I)$.

2.2. Further Assumptions on the Closed System. In addition to the assumptions (T1)-(T3), (A1), (A2), (GP), and (LIP) above, we assume the closed system satisfies the following:

(M) The map $T : \Omega \times I \to \Omega \times I$ is measurable.

(C) There exist a random probability measure $\nu_c$ supported on $\Omega \times I$ such that

$$\nu_{\sigma_n(\omega), c} \left( \mathcal{L}_{\omega, c}^n f \right) = \lambda_{\omega, c}^n \nu_{\omega, c}(f)$$

for $m$-a.e. $\omega \in \Omega$, all $n \in \mathbb{N}$, and all $f \in \text{BV}(I)$, where

$$\lambda_{\omega, c}^n := \nu_{\sigma_n(\omega), c} \left( \mathcal{L}_{\omega, c}^n \mathbb{1} \right) = \prod_{j=0}^{n-1} \nu_{\sigma_j(\omega), c} \left( \mathcal{L}_{\sigma_j(\omega), c} \mathbb{1} \right).$$
Furthermore, \( \log \lambda_{\omega,c} \in L^1(m) \).

**Remark 2.7.** Note that (M) together with (A2) implies that for \( f \in BV_\Omega(I) \) we have \( \mathcal{L}_c f \in BV_\Omega(I) \).

**Remark 2.8.** Examples which satisfy the conditions (T1)-(T3), (A1), (A2), (GP), (LIP), (M), and (C) can be found in [1].

2.3. Random Maps with Holes. We now wish to introduce holes into the class of finite branched random weighted covering systems.

Let \( H \subseteq \Omega \times I \) be measurable with respect to the product \( \sigma \)-algebra \( \mathcal{F} \otimes \mathcal{B} \) on \( \Omega \times I \). For each \( \omega \in \Omega \) the sets \( H_\omega \subseteq I \) are uniquely determined by the condition that

\[
\{ \omega \} \times H_\omega = H \cap (\{ \omega \} \times I).
\]

Equivalently we have

\[
H_\omega = \pi_I(H \cap (\{ \omega \} \times I)),
\]

where \( \pi_I : \Omega \times I \rightarrow I \) is the projection onto the second coordinate. By definition we have that the sets \( H_\omega \) are \( \nu_{\omega,c} \)-measurable. Suppose that \( 0 < \nu_c(H) < 1 \), that is we have

\[
0 < \int_\Omega \nu_{\omega,c}(H_\omega) \, dm(\omega) < 1.
\]

Now define

\[
I_\omega := I \setminus H_\omega.
\]

and denote

\[
1_\omega := 1_{I_\omega}.
\]

We then let

\[
J := H^c = \bigcup_{\omega \in \Omega} \{ \omega \} \times I_\omega.
\]

For each \( \omega \in \Omega \) and \( n \geq 0 \) we define

\[
X_{\omega,n} := \{ x \in I : T^j_\omega(x) \notin H_{\sigma^j(\omega)} \text{ for all } 0 \leq j \leq n \} = \{ x \in I_\omega : T^j_\omega(x) \in I_{\sigma^j(\omega)} \text{ for all } 0 \leq j \leq n \} = \bigcap_{j=0}^n T^{-j}_\omega(I_{\sigma^j(\omega)})
\]

to be the set of points in \( I_\omega \) which survive for at least \( n \) iterations, i.e. the set of points whose orbit does not land in a hole for at least \( n \) iterates. We then naturally define

\[
X_{\omega,\infty} := \bigcap_{n=0}^{\infty} X_{\omega,n} = \bigcap_{n=0}^{\infty} T^{-n}_\omega(I_{\sigma^n(\omega)})
\]
to be the set of points which will never land in a hole under iteration of the maps $T_\omega$. We call $X_{\omega,\infty}$ the $\omega$-surviving set. By definition, for each $n \in \mathbb{N}$, we have that

$T_\omega(X_{\omega,n}) \subseteq X_{\sigma(\omega),n-1}$ and $T_\omega(X_{\omega,\infty}) \subseteq X_{\sigma(\omega),\infty}$.

Note, however, that these survivor sets are, in general, only forward invariant and not backward invariant. For notational convenience for any $0 \leq \alpha \leq \infty$ we set

$\hat{X}_{\omega,\alpha} := \mathbb{1}_{X_{\omega,\alpha}}$.

The global surviving set is defined as

$X_\alpha := \bigcup_{\omega \in \Omega} \{\omega\} \times X_{\omega,\alpha}$

for each $0 \leq \alpha \leq \infty$. Then $X_\infty \subseteq \mathcal{I}$ is precisely the set of points that survive under forward iteration of the skew-product map $T$.

For each $\omega \in \Omega$ we let $\varphi_\omega = \varphi_{\omega,c}|_{I_\omega}$, and thus for each $n \in \mathbb{N}$ this gives

$g^{(n)}_{\omega} = g^{(n)}_{\omega,c}|_{X_{\omega,n-1}} = \exp(S_{n,T}(\varphi_\omega)) = \prod_{j=0}^{n-1} g_{\sigma_j(\omega)} \circ T_j^{\omega}$.

Now define the open operator $\mathcal{L}_\omega : L^1(\nu_{\omega,c}) \to L^1(\nu_{\sigma(\omega),c})$, where $\nu_c$ comes from our assumption (C) on the closed system, by

$(2.6) \quad \mathcal{L}_\omega(f) := \mathcal{L}_{\omega,c}(f \cdot \mathbb{1}_{\omega}), \quad f \in L^1(\nu_{\omega,c}), \ x \in I$.

As a consequence of $(2.6)$, we have that

$\mathcal{L}_\omega \mathbb{1} = \mathcal{L}_\omega \mathbb{1}_\omega$.

Iterates of the open operator $\mathcal{L}^n_\omega : L^1(\nu_{\omega,c}) \to L^1(\nu_{\sigma^n(\omega),c})$ are given by

$\mathcal{L}^n_\omega := \mathcal{L}_{\sigma^{n-1}(\omega)} \circ \cdots \circ \mathcal{L}_\omega$,

which, using induction, we may write in terms of the closed operator $\mathcal{L}_{\omega,c}$ as

$\mathcal{L}^n_\omega(f) = \mathcal{L}^n_{\omega,c} \left(f \cdot \hat{X}_{\omega,n-1}\right), \quad f \in L^1(\nu_{\omega,c})$.

We define the sets $D_{\omega,n}$ to be the support of $\mathcal{L}^n_{\sigma^{-n}(\omega)} \mathbb{1}_{\sigma^{-n}(\omega)}$, that is, we set

$(2.7) \quad D_{\omega,n} := \left\{ x \in I : \mathcal{L}^n_{\sigma^{-n}(\omega)} \mathbb{1}_{\sigma^{-n}(\omega)}(x) \neq 0 \right\}$.

Note that, by definition, we have

$D_{\omega,n+1} \subseteq D_{\omega,n}$

for each $n \in \mathbb{N}$, and we similarly define

$D_{\omega,\infty} := \bigcap_{n=0}^{\infty} D_{\omega,n}$.

From this moment on we will assume that for $m$-a.e. $\omega \in \Omega$ we have that

$(D) \quad D_{\omega,\infty} \neq \emptyset$. 

We let
\[ \hat{D}_{\omega,\alpha} := \mathbb{1}_{D_{\omega,\alpha}} \]
for each \( 0 \leq \alpha \leq \infty \). Since \( D_{\omega,n} \) is the support of \( L_{\bar{\sigma}^n(\omega)} \mathbb{1}_{\sigma} \), using the notation of (2.8) we may write
\[ L^n_\omega(f) = \hat{D}_{\sigma^n(\omega),n} f, \]
More generally, we have that, for \( k > j \), \( D_{\sigma^k(\omega),j} \) is the support of \( L^j_{\sigma^k(\omega)} \mathbb{1} \), i.e.
\[ L^j_{\sigma^k(\omega)}(f) = \hat{D}_{\sigma^k(\omega),j} L^j_{\sigma^k(\omega)}(f) \]
for \( f \in L^1(\nu_{\sigma^k(\omega),c}) \). Note that
\[ D_{\omega,n} = T^m_{\bar{\sigma}^n(\omega)}(X_{\bar{\sigma}^n(\omega),n-1}). \]
Finally, we note that since \( g_{\omega}(n) := g_{\omega}(n) \mathbb{1}_{X_{\omega,n-1}} \), for each \( n \in \mathbb{N} \) we have that
\[ \inf_{X_{\omega,n-1}} g_{\omega}(n) \leq \inf_{X_{\omega,n-1}} g_{\omega}(n) \leq \| g_{\omega}(n) \|_{\infty} \leq \| g_{\omega}(n) \|_{\infty} \]
and
\[ \inf_{D_{\sigma^n(\omega),\infty}} L^n_{\omega} \mathbb{1} \leq \| L^n_{\omega} \mathbb{1} \|_{\infty} \leq \| L^n_{\omega} \mathbb{1} \|_{\infty}, \]
which, in conjunction with (2.4) and (2.5), imply that
\[ \log \inf_{X_{\omega,n-1}} g_{\omega}(n), \log \| g_{\omega}(n) \|_{\infty}, \log \inf_{D_{\sigma^n(\omega),\infty}} L^n_{\omega} \mathbb{1}, \log \| L^n_{\omega} \mathbb{1} \|_{\infty} \in L^1(m). \]
Note that in light (2.13), the Birkhoff Ergodic Theorem implies that the quantities in (2.11) and (2.12) are tempered, e.g.
\[ \lim_{|k| \to \infty} \frac{1}{|k|} \log \inf_{X_{\omega,n-1}} g_{\sigma^k(\omega)}(n) = 0 \]
for \( m \)-a.e. \( \omega \in \Omega \) and each \( n \in \mathbb{N} \).

3. Random Conditionally Invariant Probability Measures

In this section we introduce the notion of a random conditionally invariant measure and give suitable conditions for their existence. We begin with the following definition.

**Definition 3.1.** We say that a random probability measure \( \eta \in \mathcal{P}_\Omega(I) \) is a random conditionally invariant probability measure (RCIM) if
\[ \eta_{\omega}(T^m_{\omega}(A) \cap X_{\omega,n}) = \eta_{\sigma^n(\omega)}(A) \eta_{\omega}(X_{\omega,n}) \]
for all \( n \geq 0, \omega \in \Omega \), and all Borel sets \( A \subseteq I \). If a RCIM \( \eta \) is absolutely continuous with respect to a random probability measure \( \zeta \) we call \( \eta \) a random absolutely continuous conditionally invariant probability measure (RACCIM) with respect to \( \zeta \).

Straight from the definition of a RCIM we make the following observations.
Observation 3.2. Note that if we plug \( A = I_\omega = X_{\omega,0} \) into (3.1) with \( n = 0 \), we have that
\[
\eta_\omega(I_\omega) = \eta_\omega^2(I_\omega),
\]
which immediately implies that \( \eta_\omega(I_\omega) \) is either 0 or 1. If \( \eta_\omega(H_\omega) = 0 \) then we have that \( \eta_\omega \) is supported in \( I_\omega \).

Observation 3.3. Note that since
\[
T_{\omega}^{-n}(X_{\sigma^n(\omega),m}) \cap X_{\omega,n} = X_{\omega,n+m}
\]
for each \( n, m \in \mathbb{N} \) and \( \omega \in \Omega \) we have that if \( \eta \) is a RCIM then
\[
\eta_\omega(X_{\omega,n+m}) = \eta_\omega(X_{\omega,n})\eta_{\sigma^n(\omega)}(X_{\sigma^n(\omega),m})
\]
for each \( n, m \in \mathbb{N} \). In particular, we have that
\[
\eta_\omega(X_{\omega,n}) = \prod_{j=0}^{n-1} \eta_{\sigma^j(\omega)}(X_{\sigma^j(\omega),1}).
\]

In light of Observation 3.3, given a RCIM \( \eta \), for each \( \omega \in \Omega \) we let
\[
\alpha_\omega := \eta_\omega(X_{\omega,1}).
\]
Thus we have
\[
\alpha_\omega^n := \prod_{j=0}^{n-1} \alpha_{\sigma^j(\omega)} = \eta_\omega(X_{\omega,n}).
\]

We now prove a useful identity.

Lemma 3.4. Given any \( f, h \in BV(I) \), any \( \omega \in \Omega \), and \( n \in \mathbb{N} \) we have that
\[
\int_{I_{\sigma^n(\omega)}} h \cdot \mathcal{L}_{\omega}^n f \, d\nu_{\sigma^n(\omega),c} = \int_I h \cdot \mathbbm{1}_{\sigma^n(\omega)} \cdot \mathcal{L}_{\omega,c}^n \left( f \cdot \hat{X}_{\omega,n-1} \right) \, d\nu_{\sigma^n(\omega),c}.
\]

Proof. To prove the identity we calculate the following:
\[
\int_{I_{\sigma^n(\omega)}} h \cdot \mathcal{L}_{\omega}^n f \, d\nu_{\sigma^n(\omega),c} = \int_I h \cdot \mathbbm{1}_{\sigma^n(\omega)} \cdot \mathcal{L}_{\omega,c}^n \left( f \cdot \hat{X}_{\omega,n-1} \right) \, d\nu_{\sigma^n(\omega),c}
\]
\[
= \int_I \mathcal{L}_{\omega}^n \left( f(h \circ T_{\omega}^n)(\mathbbm{1}_{\sigma^n(\omega)} \circ T_{\omega}^n) \cdot \hat{X}_{\omega,n-1} \right) \, d\nu_{\sigma^n(\omega),c}
\]
\[
= \lambda_{\omega,c}^n \int_I f(h \circ T_{\omega}^n) \cdot \hat{X}_{\omega,n} \, d\nu_{\omega,c}
\]
\[
= \lambda_{\omega,c}^n \int_{I_{\omega,n}} f \cdot h \circ T_{\omega}^n \, d\nu_{\omega,c}.
\]

The following lemma gives a useful characterization of RACCIM (with respect to \( \nu_c \)) in terms of the transfer operators \( \mathcal{L}_\omega \).
Lemma 3.5. Suppose \( \eta = \mathbb{1}_I h_\nu \) is a random probability measure on \( I \) absolutely continuous with respect to \( \nu_c \), whose disintegrations are given by

\[
\eta_\omega = \mathbb{1}_\omega h_\nu \nu_{\omega,c}.
\]

Then \( \eta \) is a RACCIM (with respect to \( \nu_c \)) if and only if there exists \( \alpha_\omega > 0 \) such that

\[
L_\omega h_\omega = \lambda_{\omega,c} \alpha_\omega h_{\sigma(\omega)}
\]

for each \( \omega \in \Omega \).

Proof. Beginning with the “reverse” direction, we first suppose (3.3) holds for all \( \omega \in \Omega \). Let \( A \in \mathcal{B} \) (Borel \( \sigma \)-algebra). Using Lemma 3.4 gives

\[
\eta_\omega(T^{-n}_\omega(A) \cap X_{\omega,n}) = \int_{X_{\omega,n}} (1_A \circ T^n_\omega) \cdot h_\omega \, d\nu_{\omega,c}
\]

\[
= (\lambda^n_{\omega,c})^{-1} \int_{I_{\sigma^n(\omega)}} 1_A \cdot L^n_\omega h_\omega \, d\nu_{\sigma^n(\omega),c}
\]

\[
= \int_{I_{\sigma^n(\omega)}} 1_A \cdot \alpha^n_\omega h_{\sigma^n(\omega)} \, d\nu_{\sigma^n(\omega),c}
\]

(3.4)

Inserting \( A = I_{\sigma^n(\omega)} \) into (3.4) gives

\[
\eta_\omega(T^{-n}_\omega(I_{\sigma^n(\omega)}) \cap X_{\omega,n}) = \alpha^n_\omega \eta_{\sigma^n(\omega)}(I_{\sigma^n(\omega)}).
\]

Observation 3.2 implies that \( \eta_{\sigma^n(\omega)}(I_{\sigma^n(\omega)}) = 1 \), and thus

\[
\alpha^n_\omega = \eta_\omega(T^{-n}_\omega(I_{\sigma^n(\omega)}) \cap X_{\omega,n}) = \eta_\omega(X_{\omega,n}),
\]

since \( T^{-n}_\omega(I_{\sigma^n(\omega)}) \cap X_{\omega,n} = X_{\omega,n} \). Thus, for \( A \in \mathcal{B} \) we have

\[
\eta_\omega(T^{-n}_\omega(A) \cap X_{\omega,n}) = \eta_{\sigma^n(\omega)}(A) \eta_\omega(X_{\omega,n})
\]

as desired.

Now to prove the opposite direction, suppose \( \eta_\omega(\mathbb{1}_\omega h_\nu) \) is a RACCIM. Then by the definition of a RCIM there exists \( \alpha_\omega \) such that for any \( A \in \mathcal{B} \) we have

\[
\eta_\omega(T^{-n}_\omega(A) \cap X_{\omega,n}) = \alpha^n_\omega \eta_{\sigma^n(\omega)}(A).
\]

So we calculate

\[
(\lambda^n_{\omega,c})^{-1} \int_{I_{\sigma^n(\omega)}} 1_A \cdot L^n_\omega h_\omega \, d\nu_{\sigma^n(\omega),c} = \int_{X_{\omega,n}} (1_A \circ T^n_\omega) h_\omega \, d\nu_{\omega,c} = \eta_\omega(T^{-n}_\omega(A) \cap X_{\omega,n})
\]

\[
= \alpha^n_\omega \eta_{\sigma^n(\omega)}(A) = \alpha^n_\omega \int_{I_{\sigma^n(\omega)}} 1_A \cdot h_{\sigma^n(\omega)} \, d\nu_{\sigma^n(\omega),c}.
\]

So we have

\[
L^n_\omega h_\omega = \lambda^n_{\omega,c} \alpha^n_\omega h_{\sigma^n(\omega)}
\]

which completes the proof. \( \square \)
4. Functionals and Partitions

In this section we follow [30, 29] and introduce the random functional $\Lambda_\omega$ that we will later show is equivalent to the conformal measure for the open system. We also introduce certain refinements of the partition of monotonicity which are used to define “good” and “bad” intervals and are needed to state our main assumptions on the open system. Following the statement of our main hypotheses, we state our main results.

We begin by defining the functional $\Lambda_\omega: \text{BV}(I) \to \mathbb{R}$ by

$$\Lambda_\omega(f) := \lim_{n \to \infty} \inf_{x \in D_{\sigma(n),n}} \frac{\mathcal{L}_\omega^n(f)(x)}{\mathcal{L}_\omega^n(1)(x)}, \quad f \in \text{BV}(I).$$

We note that this limit exists as the sequence is bounded and increasing. Indeed, we have that

$$-\|f\|_\infty \leq \inf_{x \in D_{\sigma(n),n}} \frac{\mathcal{L}_\omega^n(f)(x)}{\mathcal{L}_\omega^n(1)(x)} \leq \|f\|_\infty,$$

and to see that the ratio is increasing we note that

$$\inf_{x \in D_{\sigma(n),n+1}} \frac{\mathcal{L}_\omega^{n+1}(f)(x)}{\mathcal{L}_\omega^{n+1}(1)(x)} = \inf_{x \in D_{\sigma(n),n+1}} \frac{\mathcal{L}_{\sigma(n)}(\hat{D}_{\sigma(n),n} \cdot \mathcal{L}_\omega^n(f))}{\mathcal{L}_\omega^{n+1}(1)(x)}$$

$$\geq \inf_{x \in D_{\sigma(n),n}} \frac{\mathcal{L}_\omega^n(f)(x)}{\mathcal{L}_\omega^n(1)(x)} \cdot \inf_{x \in D_{\sigma(n),n+1}} \frac{\mathcal{L}_{\sigma(n)}(\hat{D}_{\sigma(n),n} \cdot \mathcal{L}_\omega^n(1))}{\mathcal{L}_\omega^{n+1}(1)(x)}$$

$$= \inf_{x \in D_{\sigma(n),n}} \frac{\mathcal{L}_\omega^n(f)(x)}{\mathcal{L}_\omega^n(1)(x)}.$$

In particular, (4.2) of the above argument gives that

$$-\|f\|_\infty \leq \inf f \leq \Lambda_\omega(f) \leq \|f\|_\infty.$$

**Observation 4.1.** One can easily check that the functional $\Lambda_\omega$ has the following properties.

1. $\Lambda_\omega(1) = \Lambda_\omega(\mathbb{I}_\omega) = 1.$
2. $\Lambda_\omega$ is continuous with respect to the supremum norm.
3. $f \geq h$ implies that $\Lambda_\omega(f) \geq \Lambda_\omega(h)$.
4. $\Lambda_\omega(cf) = c\Lambda_\omega(f)$.
5. $\Lambda_\omega(f + h) \geq \Lambda_\omega(f) + \Lambda_\omega(h)$.
6. $\Lambda_\omega(f + a) = \Lambda_\omega(f) + a$ for all $a \in \mathbb{R}$.
7. If $A \cap X_{\omega,n} = \emptyset$ for some $n \in \mathbb{N}$ then $\Lambda_\omega(\mathbb{1}_A) = 0$.

Furthermore, we note that the homogeneity (4) and super-additivity (5) imply that $\Lambda_\omega$ is convex. In the sequel, we will show that $\Lambda_\omega$ is in fact linear, and can thus be associated with a unique probability measure on $I_\omega$ via the Riesz Representation Theorem.
Remark 4.2. Let $f \in BV(I)$, then for all $x, y \in I_\omega$ we have
\[ f(x) \leq f(y) + \text{var}(f). \]
Using property (2) of $\Lambda_\omega$, together with (4.2), implies
\[ f(x) \leq \inf f + \text{var}(f) \leq \Lambda_\omega(f) + \text{var}(f) \leq \|f\|_\infty + \text{var}(f). \]

We set
\[ \rho_\omega := \Lambda_{\sigma(\omega)}(\mathcal{L}_\omega(\mathbb{1}_\omega)). \]
The following propositions concern various estimates of $\rho_\omega$. We begin by setting
\[ \rho_{\omega}^{(n)} := \inf_{x \in D_{\sigma_{n+1}(\omega), n}} \frac{\mathcal{L}_{\sigma(\omega)}^n(\mathcal{L}_\omega(\mathbb{1}_\omega))(x)}{\mathcal{L}_{\sigma(\omega)}^n(\mathbb{1}_{\sigma(\omega)})(x)}. \]
Then, by the definition and (4.3), we have that
\[ \rho_{\omega}^{(n)} \nearrow \Lambda_{\sigma(\omega)}(\mathcal{L}_\omega(\mathbb{1}_\omega)) = \rho_\omega \]
as $n \to \infty$.

Remark 4.3. Note that (4.6) and the definition of $\mathcal{L}_\omega$ together immediately imply that
\[ \inf_{I_\omega} g_\omega \leq \rho_\omega \leq \|\mathcal{L}_\omega \mathbb{1}_\omega\|_\infty, \]
and similarly, (4.7) implies that
\[ \inf_{I_\omega} g_\omega \leq \inf_{D_{\sigma(\omega), 1}} \mathcal{L}_\omega(\mathbb{1}_\omega) = \rho_\omega^{(0)} \leq \rho_{\omega}^{(n)} \leq \|\mathcal{L}_\omega \mathbb{1}_\omega\|_\infty \leq \#\mathcal{Z}_\omega \|g_\omega\|_\infty, \]
for all $\omega \in \Omega$ and $n \geq 0$. Furthermore, (4.9), together with (2.13), gives that
\[ \log \rho_\omega \in L^1(m). \]
The ergodic theorem then implies that
\[ \lim_{n \to \infty} \frac{1}{n} \log \rho_{\omega}^n = \int_{\Omega} \log \rho_\omega \, dm(\omega), \]
where
\[ \rho_{\omega}^n := \prod_{j=0}^{n-1} \rho_{\sigma^j(\omega)}. \]

Proposition 4.4. There exists a measurable and finite $m$-a.e. function $N_\infty : \Omega \to [1, \infty)$ such that
\[ D_{\omega, n} = D_{\omega, \infty} \]
for all $n \geq N_\infty(\omega)$. Furthermore, this implies that
\[ \inf_{D_{\omega, \infty}} \mathcal{L}_{\sigma^{-n}(\omega)}^n \mathbb{1}_{\sigma^{-n}(\omega)} > 0 \]
for all $n \geq N_\infty(\omega)$. 

Proof. We proceed via contradiction, assuming that there is a sequence \((n_k)_{k=1}^{\infty}\) in \(\mathbb{N}\) such that
\[
D_{\sigma^{n_k+1}(\omega),n_k+1} \subseteq D_{\sigma^{n_k+1}(\omega),n_k}.
\]
Let \(x_{n_k} \in D_{\sigma^{n_k+1}(\omega),n_k} \setminus D_{\sigma^{n_k+1}(\omega),n_k+1}\). Then, we have that
\[
\rho_{\omega}^{(n_k)} = \inf_{x \in D_{\sigma^{n_k+1}(\omega),n_k}} \frac{\mathcal{L}_{\sigma^{n_k}}(\mathbb{1}_\Omega)(x)}{\mathcal{L}_{\sigma^{n_k}}(\mathbb{1}_{\sigma(\Omega)})(x)} \leq \frac{\mathcal{L}_{\sigma^{n_k}}(\mathbb{1}_\Omega)(x_{n_k})}{\mathcal{L}_{\sigma^{n_k}}(\mathbb{1}_{\sigma(\Omega)})(x_{n_k})}.
\]
By our choice of \(x_{n_k}\) and by the definition (2.7), we have that the numerator of the quantity on the right is zero, while its denominator is strictly positive. As this holds for each \(k \in \mathbb{N}\), this implies that \(\rho_\omega = 0\) for m.a.e. \(\omega \in \Omega\), which contradicts (4.11). Thus, we are done. \(\square\)

Remark 4.5. Note that our assumption (D), that \(D_{\omega,\infty} \neq \emptyset\), is satisfied if \(T_\omega(I_\omega) \supseteq I_{\sigma(\omega)}\) for m.a.e. \(\omega \in \Omega\). Moreover, this also implies that \(\rho_\omega \geq \inf I_{\sigma(\omega)} \mathcal{L}_\omega \mathbb{1}_\omega > 0\). This occurs, for example if for m.a.e. \(\omega \in \Omega\) there exists a full branch, i.e. there exists \(Z \in \mathcal{Z}_\omega\) with \(T_\omega(Z) = I\), outside of the hole \(H_\omega\), in which case we would have that \(D_{\omega,\infty} = I\) for m.a.e. \(\omega \in \Omega\).

We now describe various partitions, which depend on the functional \(\Lambda_\omega\), that we will used to obtain a Lasota-Yorke inequality in Section 6. Recall that \(\mathcal{Z}_\omega^{(n)}\) denotes the partition of monotonicity of \(T_\omega^n\). Now, for each \(n \in \mathbb{N}\) and \(\omega \in \Omega\) we let \(\mathcal{A}_\omega^{(n)}\) be the collection of all finite partitions of \(I\) such that
\[
\var_{\mathcal{A}}(g_\omega^{(n)}) \leq 2 \|g_\omega^{(n)}\|_\infty
\]
for each \(\mathcal{A} = \{A_i\} \in \mathcal{A}_\omega^{(n)}\).

Given \(\mathcal{A} \in \mathcal{A}_\omega^{(n)}\), let \(\tilde{\mathcal{Z}}_\omega^{(n)}(\mathcal{A})\) be the coarsest partition amongst all those finer than \(\mathcal{A}\) and \(\mathcal{Z}_\omega^{(n)}\) such that all elements of \(\tilde{\mathcal{Z}}_\omega^{(n)}(\mathcal{A})\) are either disjoint from \(X_{\omega,n-1}\) or contained in \(X_{\omega,n-1}\). Now, define the following subcollections:
\[
\mathcal{Z}_\omega^{(n)}_{*,b} := \left\{ Z \in \tilde{\mathcal{Z}}_\omega^{(n)}(\mathcal{A}) : Z \subseteq X_{\omega,n-1} \right\},
\]
\[
\mathcal{Z}_\omega^{(n)}_{*,b} := \left\{ Z \in \tilde{\mathcal{Z}}_\omega^{(n)}(\mathcal{A}) : Z \subseteq X_{\omega,n-1} and \Lambda_\omega(\mathbb{1}_Z) = 0 \right\},
\]
\[
\mathcal{Z}_\omega^{(n)}_{*,d} := \left\{ Z \in \tilde{\mathcal{Z}}_\omega^{(n)}(\mathcal{A}) : Z \subseteq X_{\omega,n-1} and \Lambda_\omega(\mathbb{1}_Z) > 0 \right\}.
\]

Remark 4.6. Note that in light of (4.1) and (4.3), for every \(Z \in \mathcal{Z}_\omega^{(n)}_{*,d}\) we may define the (open) covering time \(M_\omega(Z) \in \mathbb{N}\) to be the least integer such that
\[
\inf_{x \in D_{\sigma^{M_\omega(Z)}(\omega),M_\omega(Z)}} \frac{\mathcal{L}_\omega^{M_\omega(Z)}(\mathbb{1}_Z)(x)}{\mathcal{L}_\omega^{M_\omega(Z)}(\mathbb{1}_\Omega)(x)} > 0
\]
which is finite since the ratio in (4.17) increases to \(\Lambda_\omega(\mathbb{1}_Z) > 0\). Conversely, given that the ratio in (4.17) is increasing by (4.3) we see that, for \(Z \in \mathcal{Z}_\omega^{(n)}_{*,*}\), if there exists any \(N \in \mathbb{N}\) such that
\[
\inf_{x \in D_{\sigma^{N}(\omega),N}} \frac{\mathcal{L}_\omega^{N}(\mathbb{1}_Z(x))}{\mathcal{L}_\omega^{N}(\mathbb{1}_\Omega)(x)} > 0,
\]
then we must have that $\Lambda_\omega(1_Z) > 0$ and equivalently $Z \in \mathcal{Z}_{\omega,b}^{(n)}$.

We adapt the following definition from [29].

**Definition 4.7.** We say that two elements $W, Z \in \mathcal{Z}_{\omega,b}^{(n)}$ are contiguous if either $W$ and $Z$ are contiguous in the usual sense, i.e. they share a boundary point, or if they are separated by a connected component of $\bigcup_{j=0}^{n-1} T^{-j}_\omega(H_{sr}(\omega))$.

We will consider random open systems that satisfy the following conditions.

(Q1) For each $\omega \in \Omega$ and $n \in \mathbb{N}$ we let $\xi_\omega^{(n)}$ denote the maximum number of contiguous elements of $\mathcal{Z}_{\omega,b}^{(n)}$. We assume

$$\lim_{n \to \infty} \frac{1}{n} \log \|g_\omega^{(n)}\|_\infty + \limsup_{n \to \infty} \frac{1}{n} \log \xi_\omega^{(n)} < \lim_{n \to \infty} \frac{1}{n} \log \rho_n = \int_\Omega \log \rho_\omega \, dm(\omega)$$

(Q2) We assume that for each $n \in \mathbb{N}$ we have $\log \xi_\omega^{(n)} \in L^1(m)$.

(Q3) Let

$$\delta_{\omega,n} := \min_{Z \in \mathcal{Z}_{\omega,b}^{(n)}} \Lambda_\omega(1_Z).$$

We assume that, for each $n \in \mathbb{N}$, $\log \delta_{\omega,n} \in L^1(m)$.

**Remark 4.8.**

1. Note that since $\lim_{n \to \infty} \frac{1}{n} \log \xi_\omega^{(n)} \geq 0$, assumption (Q1) implies that

$$\lim_{n \to \infty} \frac{1}{n} \log \|g_\omega^{(n)}\|_\infty < \lim_{n \to \infty} \frac{1}{n} \log \rho_n.$$ 

2. Since $\|g_\omega\|_\infty \leq \|g_\omega,c\|_\infty$ and $\inf_{D_{\sigma(\omega)},\infty} \mathcal{L}_\omega 1_\omega \leq \rho_\omega$ to check (Q1) it suffices to have

$$\lim_{n \to \infty} \frac{1}{n} \log \|g_\omega^{(n)}\|_\infty + \lim_{n \to \infty} \frac{1}{n} \log \xi_\omega^{(n)} < \lim_{n \to \infty} \frac{1}{n} \log \inf_{D_{\sigma(\omega)},\infty} \mathcal{L}_n 1_\omega.$$ 

3. One can use the open covering times defined in (4.17) to check (Q3). Indeed, note that if

$$N \geq M_{\omega,n} := \max \{ M_\omega(Z) : Z \in \mathcal{Z}_{\omega,b}^{(n)} \}$$

then we have that

$$\delta_{\omega,n} \geq \min_{Z \in \mathcal{Z}_{\omega,b}^{(n)}} \inf_{x \in D_{\sigma(\omega),N}} \frac{\mathcal{L}_n 1_Z(x)}{\mathcal{L}_\omega 1_Z(x)} \geq \min_{Z \in \mathcal{Z}_{\omega,b}^{(n)}} \|\mathcal{L}_n 1_Z\|_\infty \geq \inf_{x \in D_{\sigma(\omega),N}} \frac{\mathcal{L}_n 1_Z(x)}{\mathcal{L}_\omega 1_Z(x)} \geq \frac{\inf_{x_{\omega,N-1}} g_\omega^{(N)}}{\|\mathcal{L}_\omega 1\|_\infty} \geq \frac{\inf g_\omega^{(N)}}{\|\mathcal{L}_\omega,c 1\|_\infty} > 0.$$
Thus (Q3) holds if \( \log \inf q_{\omega,c}^{n(M_{\omega,c}^{n})}, \log \left\| L_{\omega,c}^{n} \text{1}_{\omega} \right\|_{\infty} \in L^{1}(m) \) for each \( n \in \mathbb{N} \).

**Remark 4.9.** In Section 15 we give several alternate hypotheses to our assumptions (Q1)-(Q3) that are more restrictive, but much simpler to check.

**Definition 4.10.** If the assumptions (T1)-(T3), (LIP), (GP), (A1)-(A2), (M), (C), (D), and (Q1)-(Q3) are satisfied then we call the tuple \((\Omega, \sigma, m, I, T, \varphi_{c}, H)\) a random weighted open system.

### 4.1. Main Results

The following are simplified versions of the main results of this paper.

**Theorem A.** Given a random open weighted covering system \((\Omega, \sigma, m, I, T, \varphi_{c}, H)\), the following hold.

1. There exists a unique random probability measure \( \nu \in \mathcal{P}_{\Omega}(I) \) supported in \( X_{\infty} \) such that

\[
\nu_{\sigma(\omega)}(L_{\omega}f) = \lambda_{\omega} \nu_{\omega}(f),
\]

for each \( f \in \text{BV}(I) \), where

\[
\lambda_{\omega} := \nu_{\sigma(\omega)}(L_{\omega}1_{\omega}).
\]

Furthermore, we have that \( \log \lambda_{\omega} \in L^{1}(m) \).

2. There exists a function \( q \in \text{BV}_{\Omega}(I) \) such that \( \nu(q) = 1 \) and for \( m \)-a.e. \( \omega \in \Omega \) we have

\[
L_{\omega}q_{\omega} = \lambda_{\omega}q_{\sigma(\omega)}.
\]

Moreover, \( q \) is unique modulo \( \nu \).

3. The measure \( \mu := q\nu \) is a \( T \)-invariant and ergodic random probability measure supported in \( X_{\infty} \).

4. There exists a unique random absolutely continuous conditionally invariant probability measure \( \eta \), which is supported on \( \mathcal{I} \), and whose disintegrations are given by

\[
\eta_{\omega}(f) := \frac{\nu_{\omega,c}(1_{\omega}q_{\omega}f)}{\nu_{\omega,c}(1_{\omega}q_{\omega})}
\]

for all \( f \in \text{BV}(I) \).

Theorem A follows from results in Sections 11, 12, and 13. We also show that the operator cocycle is quasi-compact.

**Theorem B.** With the same hypotheses as Theorem A, for each \( f \in \text{BV}(I) \) there exists a measurable function \( \Omega \ni \omega \mapsto D(\omega) \in (0, \infty) \) and \( \kappa \in (0, 1) \) such that for \( m \)-a.e. \( \omega \in \Omega \) and all \( n \in \mathbb{N} \) we have

\[
\left\| (\lambda_{\omega}^{n})^{-1} L_{\omega}^{n}f - \nu_{\omega}(f)q_{\sigma^{n}(\omega)} \right\|_{\infty} \leq D(\omega) \|f\|_{\infty} \kappa^{n}.
\]

Furthermore, for all \( A \in \mathcal{B} \) we have

\[
\left| \nu_{\omega,c}(T_{\omega,c}^{-n}(A) | X_{\omega,n}) - \eta_{\sigma^{n}(\omega)}(A) \right| \leq D(\omega)\kappa^{n},
\]
and
\[ |\eta_\omega(A \mid X_{\omega,n}) - \mu_\omega(A)| \leq D(\omega)\kappa^n. \]

For the proof of Theorem B, as well as a more general statement, see Theorem 12.2 and Corollary 12.8. From quasi-compactness we easily deduce the exponential decay of correlations for the invariant measure \( \mu \).

**Theorem C.** With the same hypotheses as Theorem A, there exists a measurable function \( \Omega \ni \omega \mapsto C(\omega) \in (0, \infty) \) such that for every \( h \in BV(I) \), every \( f \in L^1(\mu) \), every \( x \in (\kappa, 1) \), with \( \kappa \) as in Theorem B, every \( n \in \mathbb{N} \), and for \( m \)-a.e. \( \omega \in \Omega \) we have
\[ |\mu_\omega((f_{\sigma^n(\omega)} \circ T^n_\omega) h) - \mu_{\sigma^n(\omega)}(f_{\sigma^n(\omega)} h)| \leq C(\omega) \| f_{\sigma^n(\omega)} \|_{L^1(\mu_{\sigma^n(\omega)})} \| h \|_\infty x^n. \]

Theorem C is proven in Section 12.

**Definition 4.11.** Given a random probability measure \( \varrho \) on \( I \), for each \( \omega \in \Omega \), we define the lower and upper escape rates respectively by the following:
\[ \underline{R}(\varrho_\omega) := -\liminf_{n \to \infty} \frac{1}{n} \log \varrho_\omega(X_{\omega,n}) \quad \text{and} \quad \overline{R}(\varrho_\omega) := -\limsup_{n \to \infty} \frac{1}{n} \log \varrho_\omega(X_{\omega,n}). \]

If \( \underline{R}(\varrho_\omega) = \overline{R}(\varrho_\omega) \), we say the escape rate exists and denote the common value by \( R(\varrho_\omega) \).

**Definition 4.12.** Given a potential \( \varphi_c \) on the closed system we define the expected pressure of the closed and open systems respectively by
\[ \mathcal{E}P(\varphi_c) := \int \log \lambda_{\omega,c} \, dm(\omega) \quad \text{and} \quad \mathcal{E}P(\varphi) := \int \log \lambda_\omega \, dm(\omega). \]

**Theorem D.** With the same hypotheses as Theorem A, for \( m \)-a.e. \( \omega \in \Omega \) we have that
\[ R(\nu_{\omega,c}) = R(\eta_\omega) = \mathcal{E}P(\varphi_c) - \mathcal{E}P(\varphi). \]

Theorem D is proven in Section 13.

**Definition 4.13.** We will say that the weight function \( g_c \) has the Bounded Distortion Property if for each \( \omega \in \Omega \) there exists \( K_\omega \geq 1 \) such that for all \( n \in \mathbb{N} \), all \( Z \in \mathcal{Z}_{\omega,n} \), and all \( x, y \in Z \) we have that
\[ \frac{g_{\omega,c}(x)}{g_{\omega,c}(y)} \leq K_\omega. \]

**Definition 4.14.** We will say that the map \( T \) has large images if for each \( \omega \in \Omega \) we have
\[ \inf_{n \in \mathbb{N}} \inf_{Z \in \mathcal{Z}_{\omega,n}} \nu_{\sigma^n(\omega),c}(T^n_{\omega}(Z)) > 0. \]

\( T \) is said to have large images with respect to \( H \) if for each \( \omega \in \Omega \), each \( n \in \mathbb{N} \), and each \( Z \in \mathcal{Z}_{\omega,n} \) with \( Z \cap X_{\omega,\infty} \neq \emptyset \) we have
\[ T^n_{\omega}(Z \cap X_{\omega,n-1}) \supseteq X_{\sigma^n(\omega),\infty}. \]

Finally we prove a formula for the Hausdorff dimension of the survivor set \( X_{\omega,\infty} \) for \( m \)-a.e. \( \omega \in \Omega \) in the spirit of Bowen.
Theorem E. Given a random open weighted covering system \((\Omega, \sigma, m, I, T, \varphi_c, H)\) such that \(g_c = 1/|T'|\) has bounded distortion, then there exists a unique \(h \in [0, 1]\) such that \(E_P(t) > 0\) for all \(0 \leq t < h\) and \(E_P(t) < 0\) for all \(h < t \leq 1\).

Furthermore, if \(T\) has large images and large images with respect to \(H\), then for \(m\)-a.e. \(\omega \in \Omega\)

\[
\text{HD}(X_{\omega, \infty}) = h,
\]

where \(	ext{HD}(A)\) denotes the Hausdorff dimension of the set \(A\).

The proof of Theorem E appears in Section 14.

5. Random Birkhoff Cones and Hilbert Metrics

In this section we first recall the theory of convex cones first used by Birkhoff in [5], and then present the random cones on which our operator \(L_\omega\) will act as a contraction. We begin with a definition.

**Definition 5.1.** Given a vector space \(V\), we call a subset \(C \subseteq V\) a **convex cone** if \(C\) satisfies the following:

1. \(C \cap -C = \emptyset\),
2. for all \(\alpha > 0\), \(\alpha C = C\),
3. \(C\) is convex,
4. for all \(f, h \in C\) and all \(\alpha_n \in \mathbb{R}\) with \(\alpha_n \to \alpha\) as \(n \to \infty\), if \(h - \alpha_n f \in C\) for each \(n \in \mathbb{N}\), then \(h - \alpha f \in C \cup \{0\}\).

**Lemma 5.2 (Lemma 2.1 [30]).** The relation \(\leq\) defined on \(V\) by

\[
f \leq h \text{ if and only if } h - f \in C \cup \{0\}
\]

is a partial order satisfying the following:

1. \(f \leq 0 \leq f \implies f = 0\),
2. \(\lambda > 0\) and \(f \geq 0 \iff \lambda f \geq 0\),
3. \(f \leq h \iff 0 \leq h - f\),
4. for all \(\alpha_n \in \mathbb{R}\) with \(\alpha_n \to \alpha\), \(\alpha_n f \leq h \implies \alpha f \leq h\),
5. \(f \geq 0\) and \(h \geq 0 \implies f + h \geq 0\).

The Hilbert metric on \(C\) is given by the following definition.

**Definition 5.3.** Define a distance \(\Theta(f, h)\) by

\[
\Theta(f, h) := \log \frac{\beta(f, h)}{\alpha(f, h)},
\]

where

\[
\alpha(f, h) := \sup \{a > 0 : af \leq h\} \quad \text{and} \quad \beta(f, h) := \inf \{b > 0 : bf \geq h\}.
\]
Note that $\Theta$ is a pseudo-metric as two elements in the cone may be at an infinite distance from each other. Furthermore, $\Theta$ is a projective metric because any two proportional elements must be zero distance from each other. The next theorem, which is due to Birkhoff [5], shows that every positive linear operator that preserves the cone is a contraction provided that the diameter of the image is finite.

**Theorem 5.4 ([5]).** Let $V_1$ and $V_2$ be vector spaces with convex cones $C_1 \subseteq V_1$ and $C_2 \subseteq V_2$ and a positive linear operator $L : V_1 \to V_2$ such that $L(C_1) \subseteq C_2$. If $\Theta_i$ denotes the Hilbert metric on the cone $C_i$ and if $\Delta = \sup_{f,h \in C_1} \Theta_2(Lf, Lh)$, then

$$\Theta_2(Lf, Lh) \leq \tanh \left( \frac{\Delta}{4} \right) \Theta_1(f, h).$$

for all $f, h \in C_1$.

Note that it is not clear whether $(C, \Theta)$ is complete. The following lemma of [30] addresses this problem by linking the metric $\Theta$ with a suitable norm $\|\cdot\|$ on $V$.

**Lemma 5.5 ([30], Lemma 2.2).** Let $\|\cdot\|$ be a norm on $V$ such that for all $f, h \in V$ if $-f \leq h \leq f$, then $\|h\| \leq \|f\|$, and let $\varrho : C \to (0, \infty)$ be a homogeneous and order-preserving function, which means that for all $f, h \in C$ with $f \leq h$ and all $\lambda > 0$ we have

$$\varrho(\lambda f) = \lambda \varrho(f) \quad \text{and} \quad \varrho(f) \leq \varrho(h).$$

Then, for all $f, h \in C$ $\varrho(f) = \varrho(h) > 0$ implies that

$$\|f - h\| \leq (e^{\Theta(f, h)} - 1) \min\{\|f\|, \|h\|\}.$$

**Remark 5.6.** Note that the choice $\varrho(\cdot) = \|\cdot\|$ satisfies the hypothesis, however from this moment on we shall make the choice of $\varrho = \Lambda_\omega$.

**Definition 5.7.** For each $a > 0$ and $\omega \in \Omega$ let

$$C_{\omega,a} := \{ f \in BV(I) : f \geq 0, \text{var}(f) \leq a\Lambda_\omega(f) \}. \quad (5.1)$$

To see that this cone is non-empty, we note that the function $f + c \in C_{\omega,a}$ for $f \in BV(I)$ and $c \geq a^{-1}\text{var}(f) - \inf_{x \in I} f$. We also define the cone

$$C_{\omega,+} := \{ f \in BV(I) : f \geq 0 \}.$$ 

Let $\Theta_{\omega,a}$ and $\Theta_{\omega,+}$ denote the Hilbert metrics induced on the respective cones $C_{\omega,a}$ and $C_{\omega,+}$. For each $\omega \in \Omega$, $a > 0$, and any set $Y \subseteq C_{\omega,a}$ we let

$$\text{diam}_{\omega,a}(Y) := \sup_{x,y \in Y} \Theta_{\omega,a}(x, y)$$

and

$$\text{diam}_{\omega,+}(Y) := \sup_{x,y \in Y} \Theta_{\omega,+}(x, y)$$
denote the diameter of $Y$ in the respective cones $C_{\omega,a}$ and $C_{\omega,+}$ with respect to the respective metrics $\Theta_{\omega,a}$ and $\Theta_{\omega,+}$. The following lemma collects together the main properties of these metrics.

**Lemma 5.8** ([30], Lemmas 4.2, 4.3, 4.5). For $f, h \in C_{\omega,+}$ the $\Theta_{\omega,+}$ distance between $f, h$ is given by

$$\Theta_{\omega,+}(f, h) = \log \sup_{x,y \in X_{\omega}} \frac{f(y)h(x)}{f(x)h(y)}$$

If $f, h \in C_{\omega,a}$, then

$$\Theta_{\omega,+}(f, h) \leq \Theta_{\omega,a}(f, h), \quad (5.2)$$

and if $f \in C_{\omega,\eta a}$, for $\eta \in (0, 1)$, we then have

$$\Theta_{\omega,a}(1, f) \leq \log \frac{\|f\|_{\infty} + \eta \Lambda_{\omega}(f)}{\min \{\inf_{X_{\omega}} f, (1 - \eta)\Lambda_{\omega}(f)\}}.$$  

6. **Lasota-Yorke Inequalities**

The main goal of this section is to prove a Lasota-Yorke type inequality. We adopt the strategy of [1], where we first prove a less-refined Lasota-Yorke inequality with (random) coefficients that behave in a difficult manner, and then, using the first inequality, prove a second inequality with measurable random coefficients and uniform decay on the variation as in [10].

We now prove a Lasota-Yorke type inequality following the approach of [29] utilizing the “good” and “bad” interval partitions defined in (4.14)-(4.16).

**Lemma 6.1.** For all $\omega \in \Omega$, all $f \in BV(I)$, and all $n \in \mathbb{N}$ there exist positive, measurable constants $A_{\omega}^{(n)}$ and $B_{\omega}^{(n)}$ such that

$$\text{var}(L_{\omega}^{n} f) \leq A_{\omega}^{(n)} \text{var}(f) + B_{\omega}^{(n)} \Lambda_{\omega}(|f|),$$

where

$$A_{\omega}^{(n)} := (9 + 16\xi_{\omega}^{(n)}) \|g^{(n)}_{\omega}\|_{\infty}$$

and

$$B_{\omega}^{(n)} := 8(2\xi_{\omega}^{(n)} + 1) \|g^{(n)}_{\omega}\|_{\infty} \delta_{\omega,n}^{-1}.$$  

**Proof.** Since $L_{\omega}^{n}(f) = L_{\omega,0}^{n}(f \cdot \tilde{X}_{\omega,n-1})$, if $Z \subseteq Z_{\omega}^{(n)}(A) \setminus Z_{\omega}^{(n)*}$, then $Z \cap X_{\omega,n-1} = \emptyset$, and thus, we have $L_{\omega}^{n}(f1_{Z}) = 0$ for each $f \in BV(I)$. Thus, considering only intervals $Z$ in $Z_{\omega}^{(n)*}$, we are able to write

$$L^{n}_{\omega}f = \sum_{Z \subseteq Z_{\omega}^{(n)*}} (1_Z f g^{(n)}_{\omega}) \circ T^{-n}_{\omega,Z} \quad (6.1)$$

where

$$T^{-n}_{\omega,Z} : T^{n}_{\omega}(I_{\omega}) \to Z$$
is the inverse branch which takes $T^n_\omega(x)$ to $x$ for each $x \in Z$. Now, since
$$1_Z \circ T^n_{\omega,Z} = 1_{T^n_\omega(Z)},$$
we can rewrite (6.1) as
\begin{equation}
\mathcal{L}^n_\omega f = \sum_{Z \in \mathcal{Z}^{(n)}_\omega} 1_{T^n_\omega(Z)} ((f g^{(n)}_\omega) \circ T^{-n}_{\omega,Z}).
\end{equation}
So,
\begin{equation}
\var(\mathcal{L}^n_\omega f) \leq \sum_{Z \in \mathcal{Z}^{(n)}_\omega} \var(1_{T^n_\omega(Z)} ((f g^{(n)}_\omega) \circ T^{-n}_{\omega,Z})).
\end{equation}
Now for each $Z \in \mathcal{Z}^{(n)}_\omega$, using (4.13), we have
$$\var(1_{T^n_\omega(Z)} ((f g^{(n)}_\omega) \circ T^{-n}_{\omega,Z})) \leq \var_Z(f g^{(n)}_\omega) + 2 \sup_Z |f g^{(n)}_\omega|$$
\begin{align*}
& \leq 3 \var_Z(f g^{(n)}_\omega) + 2 \inf_Z |f g^{(n)}_\omega| \\
& \leq 3 \left\| g^{(n)}_\omega \right\|_\infty \var_Z(f) + 3 \sup_Z |f| \var_Z(g^{(n)}_\omega) + 2 \left\| g^{(n)}_\omega \right\|_\infty \inf_Z |f| \\
& \leq 3 \left\| g^{(n)}_\omega \right\|_\infty \var_Z(f) + 6 \left\| g^{(n)}_\omega \right\|_\infty \sup_Z |f| + 2 \left\| g^{(n)}_\omega \right\|_\infty \inf_Z |f| \\
& \leq 9 \left\| g^{(n)}_\omega \right\|_\infty \var_Z(f) + 8 \left\| g^{(n)}_\omega \right\|_\infty \inf_Z |f|.
\end{align*}
Now, using (6.4), we may further estimate (6.3) as
\begin{align*}
\var(\mathcal{L}^n_\omega f) & \leq \sum_{Z \in \mathcal{Z}^{(n)}_\omega} \left(9 \left\| g^{(n)}_\omega \right\|_\infty \var_Z(f) + 8 \left\| g^{(n)}_\omega \right\|_\infty \inf_Z |f| \right) \\
& \leq 9 \left\| g^{(n)}_\omega \right\|_\infty \var(f) + 8 \left\| g^{(n)}_\omega \right\|_\infty \left(\sum_{Z \in \mathcal{Z}^{(n)}_{\omega,g}} \inf_Z |f| + \sum_{Z \in \mathcal{Z}^{(n)}_{\omega,b}} \inf_Z |f| \right).
\end{align*}
In order to investigate each of the two sums in the line above, we first note that as $Z^{(n)}_{\omega,g}$ is finite then, by definition, there exists a constant $\delta_{\omega,n} > 0$ (defined by (4.18)) such that
$$\inf_{Z \in \mathcal{Z}^{(n)}_{\omega,g}} \Lambda_\omega(1_Z) \geq 2 \delta_{\omega,n} > 0.$$
So, we may choose $N_{\omega,n} \in \mathbb{N}$ such that for $x \in D_{\sigma_{N_{\omega,n}(\omega),N_{\omega,n}}}$ we have
\begin{equation}
\inf_{Z \in \mathcal{Z}^{(n)}_{\omega,g}} \frac{\mathcal{L}^{N_{\omega,n}}_\omega(1_Z)(x)}{\mathcal{L}^{N_{\omega,n}}_\omega(1_Z)(x)} \geq \delta_{\omega,n}.
\end{equation}
Note that since this ratio is increasing we have that (6.6) holds for all $\tilde{N} \geq N_{\omega,n}$. Then for each $x \in D_{\sigma_{N_{\omega,n}(\omega),N_{\omega,n}}}$ and $Z \in \mathcal{Z}^{(n)}_{\omega,g}$ we have
$$\mathcal{L}^{N_{\omega,n}}_\omega(|f|1_Z)(x) \geq \inf_{Z} |f| \mathcal{L}^{N_{\omega,n}}_\omega(1_Z)(x) \geq \inf_{Z} |f| \delta_{\omega,n} \mathcal{L}^{N_{\omega,n}}(1_{\omega})(x).$$
In particular, for each \( x \in D_{n,n}^{\omega,n} \), we see that

\[
(6.7) \quad \sum_{Z \in \mathcal{Z}_{\omega,g}^{(n)}} \inf_{Z} |f| \leq \delta_{\omega,n}^{-1} \sum_{Z \in \mathcal{Z}_{\omega,g}^{(n)}} \frac{\mathcal{L}_{\omega,n}^{N}(\mathbb{1}_Z)(x)}{\mathcal{L}_{\omega,n}^{N}(1_\omega)(x)} \leq \delta_{\omega,n}^{-1} \frac{\mathcal{L}_{\omega,n}^{N}(|f|)(x)}{\mathcal{L}_{\omega,n}^{N}(1_\omega)(x)}.
\]

We are now interested in finding appropriate upper bounds for the sum

\[
\sum_{Z \in \mathcal{Z}_{\omega,b}^{(n)}} \inf_{Z} |f|.
\]

However, we must first be able to associate each of the elements of \( \mathcal{Z}_{\omega,b}^{(n)} \) with one of the elements of \( \mathcal{Z}_{\omega,g}^{(n)} \). To that end, let \( Z_* \) and \( Z^* \) denote the elements of \( \mathcal{Z}_{\omega,*}^{(n)} \) that are the furthest to the left and the right respectively. Now, enumerate each of the elements of \( \mathcal{Z}_{\omega,*}^{(n)} \), \( Z_1, \ldots, Z_k \) (clearly \( k \) depends on \( \omega, n \), and \( A \)), such that \( Z_{j+1} \) is to the right of \( Z_j \) for \( j = 1, \ldots, k-1 \). Given \( Z_j \in \mathcal{Z}_{\omega,g}^{(n)} \), \( 1 \leq j \leq k \), with \( Z_j \neq Z^* \) let \( J_{\omega,+}(Z_j) \) be the union of all contiguous elements \( Z \in \mathcal{Z}_{\omega,b}^{(n)} \) which are to the right of \( Z_j \) and also to the left of \( Z_{j+1} \). In other words, \( J_{\omega,+}(Z_j) \) is the union of all elements of \( \mathcal{Z}_{\omega,b}^{(n)} \) between \( Z_j \) and \( Z_{j+1} \). Similarly, for \( Z_j \neq Z_* \), we define \( J_{\omega,-}(Z_j) \) be the union of all contiguous elements \( Z \in \mathcal{Z}_{\omega,b}^{(n)} \) which are to the left of \( Z_j \) and also to the right of \( Z_{j-1} \). Now, we note that our assumption (Q1) implies that each \( J_{\omega,-}(Z) \) and \( J_{\omega,+}(Z) \) \((Z \in \mathcal{Z}_{\omega,g}^{(n)})\) is the union of at most \( \zeta_\omega \) contiguous elements of \( \mathcal{Z}_{\omega,b}^{(n)} \). For \( Z \in \mathcal{Z}_{\omega,g}^{(n)} \) let

\[
J_{\omega,-}^*(Z) = Z \cup J_{\omega,-}(Z) \quad \text{and} \quad J_{\omega,+}^*(Z) = Z \cup J_{\omega,+}(Z).
\]

Then for \( W \subseteq J_{\omega,-}^*(Z) \) we have

\[
(6.8) \quad \inf_{W} |f| \leq \inf_{Z} |f| + \text{var}_{J_{\omega,-}^*(Z)}(f).
\]

We obtain a similar inequality for \( W \subseteq J_{\omega,+}^*(Z) \). We now consider the following two cases.

\begin{itemize}
  \item [Case 1:] At least one of the intervals \( Z_* \) and \( Z^* \) is an element of \( \mathcal{Z}_{\omega,g}^{(n)} \).
  \item [Case 2:] Neither of the intervals \( Z_* \), \( Z^* \) is an element of \( \mathcal{Z}_{\omega,g}^{(n)} \).
\end{itemize}

If we are in the first case, we assume without loss of generality that \( Z_1 = Z_* \), and thus every element \( Z \in \mathcal{Z}_{\omega,b}^{(n)} \) is contained in exactly one union \( J_{\omega,+}(Z_j) \) for some \( Z_j \in \mathcal{Z}_{\omega,g}^{(n)} \) for some \( 1 \leq j \leq k \). If \( Z_1 \neq Z_* \) and instead we have that \( Z_k = Z^* \) we could simply replace \( J_{\omega,+}(Z_j) \) with \( J_{\omega,-}(Z_j) \) in the previous statement. In view of (6.8), Case 1 leads to the conclusion that

\[
\sum_{Z \in \mathcal{Z}_{\omega,b}^{(n)}} \inf_{Z} |f| \leq \xi_\omega^{(n)} \left( \sum_{Z \in \mathcal{Z}_{\omega,g}^{(n)}} \inf_{Z} |f| + \text{var}_{J_{\omega,-}^*(Z)}(f) \right).
\]

If we are instead in the second case, then for each \( Z \in \mathcal{Z}_{\omega,b}^{(n)} \) to the left of \( Z_k \) there is exactly one \( Z_j \), \( 1 \leq j \leq k \), such that \( Z \subseteq J_{\omega,-}(Z_j) \). This leaves each of the elements \( Z \in \mathcal{Z}_{\omega,b}^{(n)} \) to
the right of $Z_k$ uniquely contained in the union $J_{\omega,+}(Z_k)$. Thus, Case 2 yields
\[
\sum_{Z \in \mathcal{Z}_{\omega,b}^{(n)}} \inf_{Z_k} |f| \leq \xi_{\omega}^{(n)} \left( \inf_{Z_k} |f| + \var J_{\omega,+}^*(Z_k) + \sum_{Z \in \mathcal{Z}_{\omega,g}^{(n)}} \inf_{Z_k} |f| + \var J_{\omega,-}^*(Z) \right)
\leq 2\xi_{\omega}^{(n)} \left( \var(f) + \sum_{Z \in \mathcal{Z}_{\omega,g}^{(n)}} \inf_{Z} |f| \right).
\]

Hence, either case gives that
\[
(6.9) \quad \sum_{Z \in \mathcal{Z}_{\omega,b}^{(n)}} \inf_{Z} |f| \leq 2\xi_{\omega}^{(n)} \left( \var(f) + \sum_{Z \in \mathcal{Z}_{\omega,g}^{(n)}} \inf_{Z} |f| \right).
\]
Inserting (6.7) and (6.9) into (6.5) gives
\[
\var(\mathcal{L}_\omega^n f) \leq 9 \|g_\omega^{(n)}\|_\infty \var(f)
+ 8 \|g_\omega^{(n)}\|_\infty \left( 2\xi_{\omega}^{(n)} \left( \var(f) + \sum_{Z \in \mathcal{Z}_{\omega,g}^{(n)}} \inf_{Z} |f| \right) + \delta_{\omega,n}^{-1} \frac{\mathcal{L}_{\omega,n}^N(\mathcal{L}_\omega^n f)(x)}{\mathcal{L}_{\omega,n}^N(\mathcal{L}_\omega^n \mathbf{1}_{\omega})(x)} \right)
\leq (9 + 16\xi_{\omega}^{(n)}) \|g_\omega^{(n)}\|_\infty \var(f) + 8(2\xi_{\omega}^{(n)} + 1) \|g_\omega^{(n)}\|_\infty \delta_{\omega,n}^{-1} \mathcal{L}_{\omega,n}^N(\mathcal{L}_\omega^n f)(x).
\]
In view of (4.3), taking the infimum over $x \in D_{\sigma_n \omega,n}(\omega,N_{\omega,n})$ allows us to replace the ratio \( \frac{\mathcal{L}_{\omega,n}^N(\mathcal{L}_\omega^n f)(x)}{\mathcal{L}_{\omega,n}^N(\mathcal{L}_\omega^n \mathbf{1}_{\omega})(x)} \) with \( \Lambda_\omega(|f|) \), that is, we have
\[
\var(\mathcal{L}_\omega^n f) \leq (9 + 16\xi_{\omega}^{(n)}) \|g_\omega^{(n)}\|_\infty \var(f) + 8(2\xi_{\omega}^{(n)} + 1) \|g_\omega^{(n)}\|_\infty \delta_{\omega,n}^{-1} \Lambda_\omega(|f|).
\]
Setting
\[
(6.10) \quad A_\omega^{(n)} := (9 + 16\xi_{\omega}^{(n)}) \|g_\omega^{(n)}\|_\infty \quad \text{and} \quad B_\omega^{(n)} := 8(2\xi_{\omega}^{(n)} + 1) \|g_\omega^{(n)}\|_\infty \delta_{\omega,n}^{-1}
\]
finishes the proof. \( \square \)

**Remark 6.2.** As a consequence of Lemma 6.1 we have that
\[
(6.11) \quad \mathcal{L}_\omega(\mathcal{C}_{\omega,+}) \subseteq \mathcal{C}_{\sigma(\omega),+},
\]
and thus \( \mathcal{L}_\omega \) is a weak contraction on \( \mathcal{C}_{\sigma(\omega),+} \).

Define the random constants
\[
(6.12) \quad Q_\omega^{(n)} := \frac{A_\omega^{(n)}}{\rho_\omega^n} \quad \text{and} \quad K_\omega^{(n)} := \frac{B_\omega^{(n)}}{\rho_\omega^n}.
\]
In light of our assumption (Q1) on the potential and number of contiguous bad intervals, we see that \( Q_\omega^{(n)} \to 0 \) exponentially quickly for each \( \omega \in \Omega \).

The following proposition now follows from (2.13), (4.11), and assumptions (Q2)-(Q3).
Proposition 6.3. For each \( n \in \mathbb{N} \), \( \log^+ Q^{(n)}_\omega \), \( \log K^{(n)}_\omega \) \( \in L^1_m(\Omega) \).

Lemma 6.4. For each \( f \in \text{BV}(I) \) and each \( n, k \in \mathbb{N} \) we have
\[
\Lambda_{\sigma^k(\omega)}(\mathcal{L}^k_\omega f) \geq \Lambda_{\sigma^k(\omega)}(\mathcal{L}^k_\omega 1_\omega) \cdot \Lambda_\omega(f).
\]

Furthermore, we have that
\[
\rho^\circ_\omega \cdot \Lambda_\omega(f) \leq \Lambda_{\sigma^n(\omega)}(\mathcal{L}^n_\omega f).
\]

In particular, this yields
\[
\rho^\circ_\omega \leq \Lambda_{\sigma^n(\omega)}(\mathcal{L}^n_\omega 1_\omega).
\]

Proof. For each \( f \in \text{BV}(I) \) with \( f \geq 0 \), \( k \in \mathbb{N} \), and \( x \in D_{\sigma^{n+k}(\omega),n} \) we have
\[
\mathcal{L}^n_{\sigma^k(\omega)}(\mathcal{L}^k_\omega f)(x) = \mathcal{L}_{\sigma^k(\omega)}^n(\mathcal{L}^k_\omega f)(x) \cdot \inf_{D_{\sigma^n(\omega),n}} \mathcal{L}^n_\omega(\mathcal{L}^n_\omega 1_\omega).
\]

Taking the infimum over \( x \in D_{\sigma^{n+k}(\omega),n} \) and letting \( n \to \infty \) gives
\[
\Lambda_{\sigma^k(\omega)}(\mathcal{L}^k_\omega f) \geq \Lambda_{\sigma^k(\omega)}(\mathcal{L}^k_\omega 1_\omega) \cdot \Lambda_\omega(f),
\]
proving the first claim. Now to see the second claim we note that as (6.15) holds for all \( \omega \in \Omega \) with \( k = 1 \), we must also have
\[
\Lambda_{\sigma^{n+1}(\omega)}(\mathcal{L}^{n+1}_\omega f) \geq \Lambda_{\sigma^{n+1}(\omega)}(\mathcal{L}^{n+1}_\omega 1_{\sigma^{n+1}(\omega)}) \cdot \Lambda_{\sigma^n(\omega)}(f) = \rho_{\sigma^n(\omega)} \cdot \Lambda_{\sigma^n(\omega)}(f)
\]
for any \( f \in \text{BV}(I) \) and each \( n \in \mathbb{N} \). Proceeding via induction, using (6.15) as the base case, we now suppose that
\[
\Lambda_{\sigma^n(\omega)}(\mathcal{L}^n_\omega f) \geq \rho^n_\omega \cdot \Lambda_\omega(f)
\]
holds for \( n \geq 1 \). Using (6.16) and (6.17), we see
\[
\Lambda_{\sigma^{n+1}(\omega)}(\mathcal{L}^{n+1}_\omega f) \geq \rho_{\sigma^n(\omega)} \cdot \Lambda_{\sigma^n(\omega)}(\mathcal{L}^n_\omega f) \geq \rho^{n+1}_\omega \cdot \Lambda_\omega(f).
\]

Considering \( f = 1_\omega \) proves the final claim, and thus we are done. \( \square \)

Define the normalized operator \( \mathcal{L}_\omega : L^1(\nu_{\omega,c}) \to L^1(\nu_{\sigma(\omega,c)}) \) by
\[
\tilde{\mathcal{L}}_\omega f := \rho^{-1}_\omega \mathcal{L}_\omega f; \quad f \in L^1(\nu_{\omega,c}).
\]

In light of Lemma 6.4, for each \( \omega \in \Omega \), \( n \in \mathbb{N} \), and \( f \in \text{BV}(I) \) we have that
\[
\Lambda_\omega(f) \leq \Lambda_{\sigma^n(\omega)}(\tilde{\mathcal{L}}^n_\omega f)
\]
Now, considering the normalized operator, we arrive at the following immediate corollary.
Corollary 6.5. For all $\omega \in \Omega$, all $f \in \BV(X_\omega)$, and all $n \in \mathbb{N}$ we have

$$\var(\tilde{L}_n^\omega f) \leq Q_\omega^{(n)} \var(f) + K_\omega^{(n)} \Lambda_\omega(|f|).$$

Definition 6.6. Since $Q_\omega^{(n)} \to 0$ exponentially fast by our assumption (Q1), we let $N_* \in \mathbb{N}$ be the minimum integer $n \geq 1$ such that

$$(6.20) \quad -\infty < \int_\Omega \log Q_\omega^{(n)} dm(\omega) < 0,$$

and we define the number

$$(6.21) \quad \theta := -\frac{1}{N_*} \int_\Omega \log Q_\omega^{(N_*)} dm(\omega).$$

Remark 6.7. As we are primarily interested in pushing forward in blocks of length $N_*$, we are able to weaken two of our main hypotheses. In particular, we may replace (Q2) and (Q3) with the following:

(Q2') We have $\log \xi_\omega^{(N_*)} \in L^1(m)$.

(Q3') We have $\log \delta_{\omega,N_*} \in L^1(m)$, where $\delta_{\omega,n}$ is defined by (4.18).

In light of Corollary 6.5 we may now find an appropriate upper bound for the BV norm of the normalized transfer operator.

Lemma 6.8. There exists a measurable function $\omega \mapsto L_\omega \in (0, \infty)$ with $\log L_\omega \in L^1_m(\Omega)$ such that for all $f \in \BV(I)$ and each $1 \leq n \leq N_*$ we have

$$(6.22) \quad \left\| \tilde{L}_n^\omega f \right\|_{\BV} \leq L_\omega^n \left( \var(f) + \Lambda_{\sigma^n(\omega)} \left( \tilde{L}_n^\omega f \right) \right),$$

where

$$L_\omega^n = L_\omega L_{\sigma(\omega)} \cdots L_{\sigma^{n-1}(\omega)} \geq 6^n.$$ 

Proof. Corollary 6.5 and (6.19) give

$$\left\| \tilde{L}_n^\omega f \right\|_{\BV} = \var(\tilde{L}_n^\omega f) + \left\| \tilde{L}_n^\omega f \right\|_{\infty} \leq 2\var(\tilde{L}_n^\omega f) + \Lambda_{\sigma^n(\omega)} \left( \tilde{L}_n^\omega f \right)$$

$$\leq 2 \left( Q_\omega^{(n)} \var(f) + K_\omega^{(n)} \Lambda_\omega(|f|) \right) + \Lambda_{\sigma^n(\omega)} \left( \tilde{L}_n^\omega f \right)$$

$$\leq 2Q_\omega^{(n)} \var(f) + (2K_\omega^{(n)} + 1) \Lambda_{\sigma^n(\omega)} \left( \tilde{L}_n^\omega f \right)$$

Now, set

$$\tilde{L}_\omega^{(n)} := \max \{ 6, 2Q_\omega^{(n)}, 2K_\omega^{(n)} + 1 \}$$

Finally, setting

$$(6.23) \quad L_\omega := \max \left\{ \tilde{L}_\omega^{(j)} : 1 \leq j \leq N_* \right\}$$

and

$$L_\omega^n := \prod_{j=0}^{n-1} L_{\sigma^j(\omega)}$$
for all $n \geq 1$ suffices. The log-integrability of $L^n_\omega$ follows from Proposition 6.3.

We now define the number $\zeta > 0$ by
\begin{equation}
\zeta := \frac{1}{N_s} \int_\Omega \log L^{N_\ast}_\omega d\mu(\omega).
\end{equation}

The constants $B^{(n)}_\omega$ and $K^{(n)}_\omega$ in the Lasota-Yorke inequalities from Lemma 6.1 and Corollary 6.5 grow to infinity with $n$, making them difficult to use. Furthermore, the rate of decay of the $Q^{(n)}_\omega$ in Corollary 6.5 may depend on $\omega$. To remedy these difficulties we prove another, more useful, Lasota-Yorke inequality in the style of Buzzi [10].

**Proposition 6.9.** For each $\varepsilon > 0$ there exists a measurable, $m$-a.e. finite function $C_\varepsilon(\omega) > 0$ such that for $m$-a.e. $\omega \in \Omega$, each $f \in BV(I)$, and all $n \in \mathbb{N}$ we have
\begin{equation}
\text{var}(\tilde{L}^{\sigma - n}_\omega f) \leq C_\varepsilon(\omega)e^{-(\theta - \varepsilon)n}\text{var}(f) + C_\varepsilon(\omega)\Lambda_\omega(\tilde{L}^{\sigma - n}_\omega f).
\end{equation}

As the proof of Proposition 6.9 follows similarly to that of Proposition 4.8 of [1], using (6.19) to obtain $\Lambda_\omega(\tilde{L}^{\sigma - n}_\omega f)$ rather than $\Lambda_{\sigma - n}(\omega)f$, we leave it to the dedicated reader.

### 7. Cone Invariance on Good Fibers

In this section we follow Buzzi’s approach [10], and describe the good behavior across a large measure set of fibers. In particular, we will show that, for sufficiently many iterates $R_\ast$, the normalized transfer operator $\tilde{L}^{R_\ast}_\omega$ uniformly contracts the cone $C_{\omega,a}$ on “good” fibers $\omega$ for cone parameters $a > 0$ sufficiently large. Recall that the numbers $\theta$ and $\zeta$ are given by
\begin{equation}
\theta := -\frac{1}{N_s} \int_\Omega \log Q^{(N_\ast)}_\omega d\mu(\omega) > 0 \quad \text{and} \quad \zeta := \frac{1}{N_s} \int_\Omega \log L^{N_\ast}_\omega d\mu(\omega) > 0.
\end{equation}

Note that Lemma 6.8 and the ergodic theorem imply that
\begin{equation}
\log 6 \leq \zeta = \lim_{n \to \infty} \frac{1}{nN_\ast} \sum_{k=0}^{n-1} \log L^{N_\ast}_{\sigma^kN_\ast}(\omega).
\end{equation}

The following definition is adapted from [10, Definition 2.4].

**Definition 7.1.** We will say that $\omega$ is *good* with respect to the numbers $\varepsilon$, $a$, $B_\ast$, and $R_\ast = q_\ast N_\ast$ if the following hold:

\begin{enumerate}
\item[(G1)] $B_\ast q_\ast e^{-\frac{\theta}{2} R_\ast} \leq \frac{1}{3}$.
\item[(G2)] $\frac{1}{R_\ast} \sum_{k=0}^{r_\ast - 1} \log L^{N_\ast}_{\sigma^kN_\ast}(\omega) \in [\varepsilon - \zeta, \zeta + \varepsilon]$.
\end{enumerate}

Now, we denote
\begin{equation}
\varepsilon_0 := \min \left\{ 1, \frac{\theta}{2} \right\}.
\end{equation}
The following lemma describes the prevalence of the good fibers as well as how to find them.

**Lemma 7.2.** Given $\varepsilon < \varepsilon_0$ and $a > 0$, there exist parameters $B_*$ and $R_*$ (both of which depend on $\varepsilon$) such that there is a set $\Omega_G \subseteq \Omega$ of good $\omega$ with $m(\Omega_G) \geq 1 - \varepsilon/4$.

**Proof.** We begin by letting

$$\Omega_1 = \Omega_1(B_*) := \{\omega \in \Omega : C_\varepsilon(\omega) \leq B_*\},$$

where $C_\varepsilon(\omega) > 0$ is the m.a.e. finite measurable constant coming from Proposition 6.9. Choose $B_*$ sufficiently large such that $m(\Omega_1) \geq 1 - \varepsilon/8$. Noting that $\varepsilon < \theta/2$ by (7.2), we set $R_0 = q_0 N_*$ and choose $q_0$ sufficiently large such that

$$B_* q_0 e^{-(\theta - \varepsilon) R_0} \leq B_* q_0 e^{-\frac{\theta}{2} R_0} \leq \frac{1}{3}.$$ 

Now let $q_1 \geq q_0$ and define the set

$$\Omega_2 = \Omega_2(q_1) := \{\omega \in \Omega : (G2) holds for the value $R_1 = q_1 N_*$ \}.$$

Now choose $q_a \geq q_1$ such that $m(\Omega_2(q_a)) \geq 1 - \varepsilon/8$. Set $R_a := q_a N_*$. Set

$$\Omega_G := \Omega_2 \cap \sigma^{-R_a}(\Omega_1).$$

Then $\Omega_G$ is the set of all $\omega \in \Omega$ which are good with respect to the numbers $B_*$ and $R_*$, and $m(\Omega_G) \geq 1 - \varepsilon/4$. \hfill $\Box$

In what follows, given a value $B_*$, we will consider cone parameters

$$a \geq a_0 := 6B_*$$

and we set

$$q_* = q_{a_0} \quad \text{and} \quad R_* := R_{a_0} = q_* N_*.$$ 

Note that (G1) together with Proposition 6.9 implies that, for $\varepsilon < \varepsilon_0$ and $\omega \in \Omega_G$, we have

$$\text{var}(\tilde{L}^*_{\omega}(f)) \leq B_* e^{-(\theta - \varepsilon)R_*} \text{var}(f) + B_* \Lambda_{\sigma R_*}(\omega)(\tilde{L}^*_{\omega} f)$$

$$\leq B_* q_* e^{-\frac{\theta}{2} R_*} \text{var}(f) + B_* \Lambda_{\sigma R_*}(\omega)(\tilde{L}^*_{\omega} f)$$

$$\leq \frac{1}{3} \text{var}(f) + B_* \Lambda_{\sigma R_*}(\omega)(\tilde{L}^*_{\omega} f).$$

(7.7)

The next lemma shows that the normalized operator is a contraction on the fiber cones $C_{\omega,a}$ and that the image has finite diameter.

**Lemma 7.3.** If $\omega$ is good with respect to the numbers $\varepsilon$, $a_0$, $B_*$, and $R_*$, then for each $a \geq a_0$ we have

$$\tilde{L}^*_{\omega}(C_{\omega,a}) \subseteq C_{\sigma R_*(\omega), a/2} \subseteq C_{\sigma R_*(\omega), a}.$$
Proof. For $\omega$ good and $f \in \mathcal{C}_{\omega,a}$, (7.7) and (7.5) give
\[
\text{var}(\tilde{L}_R^* f) \leq \frac{1}{3} \text{var}(f) + B_\ast \Lambda R_\ast(\omega)(\tilde{L}_R^* f)
\leq \frac{a}{3} \Lambda_\omega(f) + \frac{a}{6} \Lambda R_\ast(\omega)(\tilde{L}_R^* f)
\leq \frac{a}{2} \Lambda R_\ast(\omega)(\tilde{L}_R^* f).
\]
Hence we have
\[
\tilde{L}_R^* (\mathcal{C}_{\omega,a}) \subseteq \mathcal{C}_{\sigma R_\ast(a/2)} \subseteq \mathcal{C}_{\sigma R_\ast(\omega)},
\]
as desired. □

8. DENSITY ESTIMATES AND CONE INVARIANCE ON BAD FIBERS

In this section we recall the notion of "bad" fibers from [1, 10]. We show that for fibers in the small measure set, $\Omega_B := \Omega \setminus \Omega_G$, the cone $\mathcal{C}_{\omega,a}$ of positive functions is invariant after sufficiently many iterations for sufficiently large parameters $a > 0$. We accomplish this by introducing the concept of bad blocks (coating intervals), which we then show make up a relatively small portion of an orbit. As the content of this section is adapted from the closed dynamical setting of Section 7 of [1], we do not provide proofs.

Recall that $R_\ast$ is given by (7.6). Following Section 7 of [1], and using the same justifications therein, we define the measurable function $y_\ast : \Omega \to \mathbb{N}$ so that
\[
0 \leq y_\ast(\omega) < R_\ast
\]
is the smallest integer such that for either choice of sign $+$ or $-$ we have
\[
\lim_{n \to \infty} \frac{1}{n} \# \left\{ 0 \leq k < n : \sigma^{\pm kR_\ast + y_\ast(\omega)}(\omega) \in \Omega_G \right\} > 1 - \varepsilon,
\]
\[
\lim_{n \to \infty} \frac{1}{n} \# \left\{ 0 \leq k < n : C_\varepsilon(\sigma^{\pm kR_\ast + y_\ast(\omega)}(\omega)) \leq B_\ast \right\} > 1 - \varepsilon.
\]
Clearly, $y_\ast : \Omega \to \mathbb{N}$ is a measurable function such that
\[
y_\ast(\sigma y_\ast(\omega)) = 0,
\]
\[
y_\ast(\sigma R_\ast(\omega)) = y_\ast(\omega).
\]
In particular, (8.4) and (8.5) together imply that
\[
y_\ast(\sigma^{y_\ast(\omega) + kR_\ast}(\omega)) = 0
\]
for all $k \in \mathbb{N}$. Let
\[
\Gamma(\omega) := q_\ast \prod_{k=0}^{q_\ast-1} L_{\sigma^{kN_\ast}(\omega)},
\]
where $q_\ast$ is given by (7.6), and for each $\omega \in \Omega$, given $\varepsilon > 0$, we define the coating length $\ell(\omega) = \ell_\varepsilon(\omega)$ as follows:
- if $\omega \in \Omega_G$ then set $\ell(\omega) := 1,$
• if $\omega \in \Omega_B$ then

\begin{equation}
\ell(\omega) := \min \left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{0 \leq k < n} (\mathbb{1}_{\Omega_B} \log \Gamma (\sigma^k R*(\omega))) \leq \log q_\ast + \zeta R_\ast \sqrt{\varepsilon} \right\},
\end{equation}

where $\zeta$ is as in (6.24). If the minimum is not attained we set $\ell(\omega) = \infty$.

Since $L^N_{\omega} \geq 6^N$ by Lemma 6.8, we must have that

\begin{equation}
\Gamma(\omega) \geq q_\ast 6^{R_\ast}
\end{equation}

for all $\omega \in \Omega$. It follows from Lemma 6.8 (applied repeatedly to $q_\ast$ blocks of length $N_\ast$) that for all $\omega \in \Omega$ we have

\begin{equation}
\text{var}(\tilde{\mathcal{L}}_{\omega} R_\ast f) \leq \left( \prod_{k=0}^{q_\ast-1} L^N_{\sigma^k N_\ast(\omega)} \right) \text{var}(f) + \sum_{j=0}^{q_\ast-1} \left( \prod_{k=j}^{q_\ast-1} L^N_{\sigma^k N_\ast(\omega)} \right) \Lambda_{\sigma R_\ast(\omega)}(f)
\end{equation}

\begin{equation}
\leq \Gamma(\omega) (\text{var}(f) + \Lambda_{\sigma R_\ast(\omega)}(f)).
\end{equation}

Furthermore, if $\omega \in \Omega_C$ it follows from (G2) that

\begin{equation}
\log q_\ast + R_\ast (\zeta - \varepsilon) \leq \log \Gamma(\omega) \leq \log q_\ast + R_\ast (\zeta + \varepsilon).
\end{equation}

The following proposition collects together some of the key properties of the coating length $\ell(\omega)$.

**Proposition 8.1.** For all $\varepsilon > 0$ sufficiently small the number $\ell(\omega)$ satisfies the following.

(i) For $m$-a.e. $\omega \in \Omega$ such that $y_\ast(\omega) = 0$ we have $\ell(\omega) < \infty$,

(ii) If $\omega \in \Omega_B$ then $\ell(\omega) \geq 2$.

**Remark 8.2.** Given $\omega_0 \in \Omega$, for each $j \geq 0$ let $\omega_{j+1} = \sigma^{\ell(\omega_j) R_\ast(\omega_j)}$. As a consequence of Proposition 8.1 (i) and (8.5), we see that for $m$-a.e. $\omega_0 \in \Omega$ with $y_\ast(\omega_0) = 0$, we must have that $\ell(\omega_j) < \infty$ for all $j \geq 0$.

**Definition 8.3.** We will call a (finite) sequence $\omega, \sigma(\omega), \ldots, \sigma^{\ell(\omega) R_\ast-1(\omega)} \ast R_\ast$ fibers a good block (originating at $\omega$) if $\omega \in \Omega_C$ (which implies that $\ell(\omega) = 1$). If, on the other hand, $\omega \in \Omega_B$ we call such a sequence a bad block, or coating interval, originating at $\omega$.

For $\varepsilon > 0$ sufficiently small we have that $\sqrt{\varepsilon} / \log 6 < 1$, and so we may define the number

$$
\gamma(\varepsilon) := \frac{\log q_\ast + \zeta R_\ast \sqrt{\varepsilon}}{\log q_\ast + R_\ast \log 6} < 1.
$$

Since $q_\ast \to \infty$ as $\varepsilon \to 0$ (since $q_\ast$ was chosen in (7.6) after Lemma 7.2 depending on $\varepsilon$) and since $R_\ast = q_\ast N_\ast$, for $\varepsilon > 0$ sufficiently small there exists $\gamma < 1$ such that

\begin{equation}
\gamma(\varepsilon) < \gamma < 1.
\end{equation}

We now wish to show that the normalized operator $\tilde{\mathcal{L}}_{\omega}$ is weakly contracting (i.e. non-expanding) on the fiber cones $\mathcal{C}_{\omega, a}$ for sufficiently large values of $a > a_0$. We obtain this cone invariance on blocks of length $\ell(\omega) R_\ast$, however in order to obtain cone contraction.
with a finite diameter image we will have to travel along several such blocks. For this reason we introduce the following notation.

Given \( \omega \in \Omega \) with \( y_\tau(\omega) = 0 \) for each \( k \geq 1 \) we define the length

\[
\Sigma^{(k)}_\omega := \sum_{j=0}^{k-1} \ell(\omega_j) R_*
\]

where \( \omega_0 := \omega \) and for each \( j \geq 1 \) we set \( \omega_j := \sigma^{\Sigma^{(j-1)}}(\omega) \). This construction is justified as we recall from Proposition 8.1 that for \( m \text{-a.e. } \omega \in \Omega \) with \( y_\tau(\omega) = 0 \) we have that \( \ell(\omega) < \infty \). The next lemma was adapted from Lemma 7.5 of [1].

**Lemma 8.4.** For \( \varepsilon > 0 \) sufficiently small, each \( N \in \mathbb{N} \), and \( m \text{-a.e. } \omega \in \Omega \) with \( y_\tau(\omega) = 0 \) we have that

\[
\var \left( \tilde{\mathcal{L}}^{\Sigma^{(N)}}_\omega f \right) \leq \left( \frac{1}{3} \right)^{\Sigma^{(N)}/R_*} \var(f) + \frac{a_*}{6} \Lambda_{\sigma^{\Sigma^{(N)}}(\omega)}(\tilde{\mathcal{L}}^{\Sigma^{(N)}}_\omega f).
\]

Moreover, we have that

\[
\tilde{\mathcal{L}}^{\Sigma^{(N)}}_\omega(\mathcal{C}_{\omega,a_*}) \subseteq \mathcal{C}_{\sigma^{\Sigma^{(N)}}(\omega), a_*/2},
\]

where

\[
a_* = a_*(\varepsilon) := 2a_0 q_* e^{c R_* \sqrt{\varepsilon}} = 12 B_* q_* e^{c R_* \sqrt{\varepsilon}}.
\]

**Proof.** Throughout the proof we will denote \( \ell_i = \ell(\omega_i) \) and \( L_i = \sum_{k=0}^{i-1} \ell_k \) for each \( 0 \leq i < N \). Then \( \Sigma^{(N)} = L_N R_* \). Using (7.7) on good fibers and (8.10) on bad fibers, for any \( p \geq 1 \) and \( f \in \mathcal{C}_{\omega,+} \) we have

\[
\var(\tilde{\mathcal{L}}^{p R_*}_\omega f) \leq \left( \prod_{j=0}^{p-1} \Phi^{(R_*)}_{\sigma^{j R_*}(\omega)} \right) \var(f) + \sum_{j=0}^{p-1} \left( D^{(R_*)}_{\sigma^j R_*}(\omega) \cdot \prod_{k=j+1}^{p-1} \Phi^{(R_*)}_{\sigma^k R_*}(\omega) \right) \Lambda_{\sigma^i R_*}(\omega)(\tilde{\mathcal{L}}^{p R_*}_\omega f),
\]

where

\[
\Phi^{(R_*)}_\tau = \begin{cases} 
B_* e^{-(\theta-\varepsilon)R_*} & \text{for } \tau \in \Omega_G \\
\Gamma(\tau) & \text{for } \tau \in \Omega_B
\end{cases}
\]

and

\[
D^{(R_*)}_\tau = \begin{cases} 
B_* & \text{for } \tau \in \Omega_G \\
\Gamma(\tau) & \text{for } \tau \in \Omega_B.
\end{cases}
\]

For any \( 0 \leq i < N \) and \( 0 \leq j < \ell_i \) we can write

\[
\sum_{0 \leq k < \ell_i} (\mathbb{I}_{\Omega_B} \log \Gamma)(\sigma^{k R_*}(\omega_i)) = \sum_{0 \leq k < j} (\mathbb{I}_{\Omega_B} \log \Gamma)(\sigma^{k R_*}(\omega_i)) + \sum_{j \leq k < \ell_i} (\mathbb{I}_{\Omega_B} \log \Gamma)(\sigma^{k R_*}(\omega_i)).
\]
The definition of \( \ell(\omega_i) \), (8.8), then implies that
\[
\frac{1}{j} \sum_{0 \leq k < j} (1_{\Omega_B} \log \Gamma) (\sigma^{kR_+}(\omega_i)) > \log q_* + \zeta R_+ \sqrt{\varepsilon},
\]
and consequently that
\[
\frac{1}{\ell_i - j} \sum_{j \leq k < \ell_i} (1_{\Omega_B} \log \Gamma) (\sigma^{kR_+}(\omega_i)) \leq \log q_* + \zeta R_+ \sqrt{\varepsilon}.
\] (8.19)

Now, using (8.9), (8.19), and (8.12) we see that for \( \varepsilon \) sufficiently small, the proportion of bad blocks is given by
\[
\frac{1}{\ell_i - j} \# \{ j \leq k < \ell_i : \sigma^{kR_+}(\omega_i) \in \Omega_B \} = \frac{1}{\ell_i - j} \sum_{j \leq k < \ell_i} (1_{\Omega_B}) (\sigma^{kR_+}(\omega_i)) \leq \frac{1}{(\ell_i - j) R_* \log 6} \sum_{j \leq k < \ell_i} (1_{\Omega_B} \log \Gamma) (\sigma^{kR_+}(\omega_i)) \leq \gamma.
\] (8.20)

In view of (8.17), using (8.19), (8.20), for any \( 0 \leq i < N \) and \( 0 \leq j < \ell_i \) we have
\[
\prod_{k=j}^{\ell_i-1} \Phi_{\sigma^{kR_+}(\omega_i)}^{(R_+)} = \prod_{j \leq k < \ell_i} \prod_{\sigma^{kR_+}(\omega_i) \in \Omega_B} B_* e^{-(\theta - \varepsilon)R_*} \cdot \prod_{j \leq k < \ell_i} \prod_{\sigma^{kR_+}(\omega_i) \in \Omega_B} \Gamma(\sigma^{kR_+}(\omega_i)) \\
\leq \left( B_* e^{-(\theta - \varepsilon)R_*} \right)^{(1-\gamma)(\ell_i-j)} \cdot \exp \left( (\log q_* + \zeta R_+ \sqrt{\varepsilon}) (\ell_i - j) \right)
\leq \left( B_*^{1-\gamma} q_* \exp \left( (\zeta \sqrt{\varepsilon} - (\theta - \varepsilon)(1-\gamma)) R_* \right) \right)^{\ell_i-j}.
\] (8.21)

Now for any \( 0 \leq j < L_N \) there must exist some \( 0 \leq i_0 < N \) and some \( 0 \leq j_0 < \ell_{i_0+1} \) such that \( L_{i_0+1} + j_0 = j < L_{i_0} \). Thus, using (8.21) we can write
\[
\prod_{k=j}^{L_N-1} \Phi_{\sigma^{kR_+}(\omega_i)}^{(R_+)} = \prod_{k=j_0}^{\ell_{i_0+1} - 1} \Phi_{\sigma^{kR_+}(\omega_{i_0})}^{(R_+)} \cdot \prod_{i=i_0+1}^{N-1} \prod_{k=0}^{\ell_i - 1} \Phi_{\sigma^{kR_+}(\omega_i)}^{(R_+)} \\
< \left( B_* q_* \exp \left( (\zeta \sqrt{\varepsilon} - (\theta - \varepsilon)(1-\gamma)) R_* \right) \right)^{\ell_{i_0+1} - j_0} \cdot \prod_{i=i_0+1}^{N-1} \left( B_* q_* \exp \left( (\zeta \sqrt{\varepsilon} - (\theta - \varepsilon)(1-\gamma)) R_* \right) \right)^{\ell_i}
\leq \left( B_* q_* \exp \left( (\zeta \sqrt{\varepsilon} - (\theta - \varepsilon)(1-\gamma)) R_* \right) \right)^{L_N - j}.
\] (8.22)

Now, since \( B_* \Gamma(\omega) \geq 1 \) for all \( \omega \in \Omega \), using (8.18) and (8.19), we have that for \( 0 \leq i < N \) and \( 0 \leq j < \ell_i \)
\[
D_{\sigma^{jR_+}(\omega_i)}^{(R_+)} \leq B_* \Gamma(\sigma^{jR_+}(\omega_i)) \leq B_* \cdot \prod_{j \leq k < \ell_i} \Gamma(\sigma^{kR_+}(\omega_i)) \leq B_* \left( q_* e^{\zeta R_+ \sqrt{\varepsilon}} \right)^{(\ell_i-j)}.
\] (8.23)
Similarly to the reasoning used to obtain (8.22), for any \(0 \leq j < L_N\) we see that we can improve (8.23) so that we have
\[
D_{\sigma j R_N(\omega)}^{(R_\ast)} \leq B_\ast \left( q_\ast e^{\xi R_N \sqrt{\varepsilon}} \right)^{L_N - j}.
\]
Thus, inserting (8.22) and (8.24) into (8.16) (with \(p = L_N\)) we see that
\[
\var \left( \tilde{L}^{\Sigma(N)}_\omega f \right) \leq \left( \prod_{j=0}^{L_N-1} \Phi_{\sigma j R_N(\omega)}(\varphi) \right) \var(f) + \sum_{j=0}^{L_N-1} \left( D_{\sigma j R_N(\omega)}^{(R_\ast)} \cdot \prod_{k=j+1}^{L_N-1} \Phi_{\sigma k R_N(\omega)}(\varphi) \right) \Lambda_{\sigma \Sigma(N)}(\omega) \left( \tilde{L}^{\Sigma(N)}_\omega f \right),
\]
\[
\leq \left( B_\ast q_\ast \exp \left( \left( \zeta \sqrt{\varepsilon} - (\theta - \varepsilon)(1 - \gamma) \right) R_\ast \right) \right)^{L_N} \var(f)
\]
\[
- \Lambda_{\sigma \Sigma(N)}(\omega) \left( \tilde{L}^{\Sigma(N)}_\omega f \right) \sum_{j=0}^{L_N-1} \left( B_\ast q_\ast \exp \left( \left( \zeta \sqrt{\varepsilon} - (\theta - \varepsilon)(1 - \gamma) \right) R_\ast \right) \right)^{L_N - j - 1}
\]
\[
= \left( B_\ast q_\ast \exp \left( \left( \zeta \sqrt{\varepsilon} - (\theta - \varepsilon)(1 - \gamma) \right) R_\ast \right) \right)^{L_N} \var(f)
\]
\[
+ B_\ast q_\ast e^{\xi R_N \sqrt{\varepsilon}} \cdot \Lambda_{\sigma \Sigma(N)}(\omega) \left( \tilde{L}^{\Sigma(N)}_\omega f \right) \sum_{j=0}^{L_N-1} \left( B_\ast q_\ast \exp \left( \left( 2 \zeta \sqrt{\varepsilon} - (\theta - \varepsilon)(1 - \gamma) \right) R_\ast \right) \right)^{L_N - j - 1}.
\]
Therefore, taking \(\varepsilon > 0\) sufficiently small\(^2\) in conjunction with (G1), we have that
\[
\var \left( \tilde{L}^{\Sigma(N)}_\omega f \right) \leq \left( \frac{1}{3} \right)^{L_N} \var(f) + B_\ast q_\ast e^{\xi R_N \sqrt{\varepsilon}} \cdot \Lambda_{\sigma \Sigma(N)}(\omega) \left( \tilde{L}^{\Sigma(N)}_\omega f \right) \sum_{j=0}^{L_N-1} \left( \frac{1}{3} \right)^{L_N - j - 1},
\]
and so we must have that
\[
\var \left( \tilde{L}^{\Sigma(N)}_\omega f \right) \leq \left( \frac{1}{3} \right)^{L_N} \var(f) + \frac{2 B_\ast q_\ast e^{\xi R_N \sqrt{\varepsilon}} \Lambda_{\sigma \Sigma(\omega)}(\omega) \left( \tilde{L}^{\Sigma(N)}_\omega f \right)}{6},
\]

\[
= \left( \frac{1}{3} \right)^{L_N} \var(f) + \frac{a_\ast}{6} \Lambda_{\sigma \Sigma(N)}(\omega) \left( \tilde{L}^{\Sigma(N)}_\omega f \right),
\]
which proves the first claim. Thus, for any \(f \in C_{\omega,a_\ast}\) we have that
\[
\var(\tilde{L}^{\Sigma(N)}_\omega f) \leq \left( \frac{1}{3} \right)^{L_N} \Lambda_\omega(f) + \frac{a_\ast}{6} \Lambda_{\sigma \Sigma(N)}(\omega) \left( \tilde{L}^{\Sigma(N)}_\omega f \right)
\]
\[
\leq \frac{a_\ast}{3} \Lambda_{\sigma \Sigma(N)}(\omega) \left( \tilde{L}^{\Sigma(N)}_\omega f \right) + \frac{a_\ast}{6} \Lambda_{\sigma \Sigma(N)}(\omega) \left( \tilde{L}^{\Sigma(N)}_\omega f \right),
\]
where we have used the fact that \(a_\ast > 1\), and consequently we have
\[
\tilde{L}^{\Sigma(N)}_\omega(\mathcal{C}_{\omega,a_\ast}) \subseteq \mathcal{C}_{\sigma \Sigma(N)(\omega),a_\ast/2} \subseteq \mathcal{C}_{\sigma \Sigma(N)(\omega),a_\ast}.
\]

\(^2\)Any \(\varepsilon < \min \left\{ \left( \frac{\log 6}{\lambda_\ast} \right)^2, \left( \frac{a_\ast}{\lambda_\ast} \right)^2 \right\} \) such that \(\frac{\xi}{2 \log 6} \leq \gamma\), which implies \(-\frac{a_\ast}{2} > 2 \sqrt{\varepsilon} \zeta - (\theta - \varepsilon)(1 - \gamma) > \sqrt{\varepsilon} \zeta - (\theta - \varepsilon)(1 - \gamma)\), will suffice; see Observation 7.4 of [1] for details.
as desired.

The next lemma shows that the total length of the bad blocks take up only a small proportion of an orbit, however before stating the result we establish the following notation. For each \( n \in \mathbb{N} \) we let \( K_n \geq 0 \) be the integer such that
\[
(8.25) \quad n = K_n R* + h(n)
\]
where \( 0 \leq h(n) < R* \) is a remainder term. Given \( \omega_0 \in \Omega \), let
\[
(8.26) \quad \omega_j = \sigma^{\ell(\omega_{j-1})} R* (\omega_{j-1})
\]
for each \( j \geq 1 \). Then for each \( n \in \mathbb{N} \) we can break the \( n \)-length \( \sigma \)-orbit of \( \omega_0 \) in \( \Omega \) into \( k_{\omega_0}(n) + 1 \) blocks of length \( \ell(\omega_j) R* \) (for \( 0 \leq j \leq k_{\omega_0}(n) \)) plus some remaining block of length \( r_{\omega_0}(n) R* \) where \( 0 \leq r_{\omega_0}(n) < \ell(\omega_{k_{\omega_0}(n)+1}) \) plus a remainder segment of length \( h(n) \), i.e. we can write
\[
(8.27) \quad n = \sum_{0 \leq j \leq k_{\omega_0}(n)} \ell(\omega_j) R* + r_{\omega_0}(n) R* + h(n);
\]
see Figure 1. We also note that (8.25) and (8.27) imply that
\[
(8.28) \quad K_n = \sum_{0 \leq j \leq k_{\omega_0}(n)} \ell(\omega_j) + r_{\omega_0}(n).
\]

The proof of the following lemma is nearly identical to that of Lemma 7.6 of [1], and

![Figure 1. The decomposition of \( n = \sum_{0 \leq j \leq k_{\omega_0}(n)} \ell(\omega_j) R* + r_{\omega_0}(n) R* + h(n) \) and the fibers \( \omega_j \).](image)

therefore it shall be omitted.

**Lemma 8.5.** There exists a measurable function \( N_0 : \Omega \to \mathbb{N} \) such that for all \( n \geq N_0(\omega_0) \) and for \( m \text{-} a.e. \ \omega_0 \in \Omega \) with \( y_*(\omega_0) = 0 \) we have
\[
E_{\omega_0}(n) := \sum_{0 \leq j \leq k_{\omega_0}(n)} \ell(\omega_j) + r_{\omega_0}(n) < Y \cdot \varepsilon K_n \leq \frac{Y}{R*} \varepsilon n
\]
where
\[
(8.29) \quad Y = Y_\varepsilon := \frac{2(\log q* + (2 + \zeta) R*)}{\log q* + \zeta R* \sqrt{\varepsilon}},
\]
and where $K_n$ is as in (8.25), $\omega_j$ is as in (8.26), and $k_{\omega}(n)$ and $r_{\omega}(n)$ are as in (8.27).

To end this section we note that

\begin{equation}
(8.30) \quad \varepsilon \cdot Y_\varepsilon \to 0
\end{equation}

as $\varepsilon \to 0$. For the remainder of the document we will assume that $\varepsilon > 0$ is always taken sufficiently small such that the results of Section 8 apply.

9. Further Properties of $\Lambda_\omega$

In this section we prove some additional properties of the functional $\Lambda_\omega$ that will be necessary in Section 10 to obtain cone contraction with finite diameter. In particular, in the main result of this section, which is a version of Lemma 3.11 of [29] and dates back to [28, Lemma 3.2], we show that for a function $f \in \mathcal{C}_{\omega,a}$ there exists a partition element on which the function $f$ takes values at least as large as $\Lambda_\omega(f)/4$.

Now we prove the following upper and lower bounds for $\Lambda_{\sigma^{N(k)}_{\omega}}(\tilde{\mathcal{L}}_{\omega}^{N(k)} f)$.

**Lemma 9.1.** For m.a.e. $\omega \in \Omega$ such that $y_* (\omega) = 0$, and each $k \in \mathbb{N}$ we have that

\[
\Lambda_{\sigma^{N(k)}_{\omega}}(\tilde{\mathcal{L}}_{\omega}^{N(k)} \mathbb{1}_\omega) \Lambda_\omega(f) \leq \Lambda_{\sigma^{N(k)}_{\omega}}(\tilde{\mathcal{L}}_{\omega}^{N(k)} f) \leq a_* \Lambda_{\sigma^{N(k)}_{\omega}}(\tilde{\mathcal{L}}_{\omega}^{N(k)} \mathbb{1}_\omega) \Lambda_\omega(f).
\]

**Proof.** From Lemma 6.4 we already see that the first inequality holds. Now, fix $\omega \in \Omega$ (with $y_* (\omega) = 0$) and $k \in \mathbb{N}$. To see the other inequality we first let $n, N \in \mathbb{N}$ and $x \in D_{\sigma^{N(k)}(\omega),N+n}$, then

\[
\frac{\tilde{\mathcal{L}}_{\omega}^{N+n}(f)(x)}{\tilde{\mathcal{L}}_{\sigma^{N(k)}(\omega)}^{N+n}(\mathbb{1}_{\sigma^{N(k)}(\omega)})(x)} = \frac{\tilde{\mathcal{L}}_{\omega}^{N+n}(f)(x)}{\tilde{\mathcal{L}}_{\sigma^{N(k)}(\omega)}^{N+n}(\mathbb{1}_{\sigma^{N(k)}(\omega)})(x)} = \frac{\tilde{\mathcal{L}}_{\omega}^{N+n}(f)(x)}{\tilde{\mathcal{L}}_{\omega}^{N+n}(\mathbb{1}_{\omega})(x)} \cdot \frac{\tilde{\mathcal{L}}_{\sigma^{N(k)}(\omega)}^{N+n}(\mathbb{1}_{\sigma^{N(k)}(\omega)})(x)}{\tilde{\mathcal{L}}_{\sigma^{N(k)}(\omega)}^{N+n}(\mathbb{1}_{\sigma^{N(k)}(\omega)})(x)} \leq \frac{\tilde{\mathcal{L}}_{\omega}^{N+n}(f)(x)}{\tilde{\mathcal{L}}_{\omega}^{N+n}(\mathbb{1}_{\omega})(x)} \cdot \| \tilde{\mathcal{L}}_{\sigma^{N(k)}(\omega)}^{N+n}(\mathbb{1}_{\sigma^{N(k)}(\omega)})(x) \|_{\infty}.
\]

Now taking the infimum over $x \in D_{\sigma^{N(k)}(\omega),N+n}$ and letting $n \to \infty$ gives

\begin{equation}
(9.1) \quad \Lambda_{\sigma^{N(k)}(\omega)}(\tilde{\mathcal{L}}_{\omega}^{N} f) \leq \| \tilde{\mathcal{L}}_{\omega}^{N} \mathbb{1}_{\omega} \|_{\infty} \Lambda_\omega(f).
\end{equation}

Now, set $N = \Sigma_{\omega}^{(k)}$. Since $\mathbb{1}_{\omega} \in \mathcal{C}_{\omega,a}$, (8.14) from Lemma 8.4 implies that

\[
\| \tilde{\mathcal{L}}_{\omega}^{\Sigma_{\omega}^{(k)}} \mathbb{1}_{\omega} \|_{\infty} \leq \var(\tilde{\mathcal{L}}_{\omega}^{\Sigma_{\omega}^{(k)}}(\mathbb{1}_{\omega})) + \Lambda_{\sigma^{\Sigma_{\omega}^{(k)}(\omega)}}(\tilde{\mathcal{L}}_{\omega}^{\Sigma_{\omega}^{(k)}}(\mathbb{1}_{\omega})) \leq \left( \frac{a_*}{2} + 1 \right) \Lambda_{\sigma^{\Sigma_{\omega}^{(k)}(\omega)}}(\tilde{\mathcal{L}}_{\omega}^{\Sigma_{\omega}^{(k)}}(\mathbb{1}_{\omega})) \leq a_* \Lambda_{\sigma^{\Sigma_{\omega}^{(k)}(\omega)}}(\tilde{\mathcal{L}}_{\omega}^{\Sigma_{\omega}^{(k)}}(\mathbb{1}_{\omega}))
\]

\begin{equation}
(9.2) \quad \leq a_* \Lambda_{\sigma^{\Sigma_{\omega}^{(k)}(\omega)}}(\tilde{\mathcal{L}}_{\omega}^{\Sigma_{\omega}^{(k)}}(\mathbb{1}_{\omega})),
\end{equation}

\end{equation}
where we have used the fact that \( a_* > 2 \) which follows from (8.15). Combining (9.2) with (9.1), we see that

\[
\Lambda_{\sigma^n(\omega)}(\mathcal{L}_\omega^{\Sigma^{(k)}} f) \leq a_* \Lambda_{\sigma^n(\omega)}(\mathcal{L}_\omega^{\Sigma^{(k)}} 1_\omega) \Lambda_\omega(f)
\]

completing the proof. 

\[ \square \]

**Lemma 9.2.** For each \( \delta > 0 \) and each \( \omega \in \Omega \) there exists \( N_{\omega,\delta} \) such that for each \( n \geq N_{\omega,\delta} \),

\[
\sup_{z \in \mathcal{Z}_\omega^{(n)}} \Lambda_\omega(1_Z) \leq \delta.
\]

**Proof.** Choose \( N_{\omega,\delta} \in \mathbb{N} \) such that

\[
\left\| g^{(n)}_\omega \right\|_\infty \leq \delta
\]

for each \( n \geq N_{\omega,\delta} \). Now, fix some \( n \geq N_{\omega,\delta} \) and let \( m \in \mathbb{N} \). Then, for \( Z \in \mathcal{Z}_\omega^{(n)} \) we have

\[
\mathcal{L}_\omega^{n} 1_Z(x) \leq \left\| g^{(n)}_\omega \right\|_\infty \leq \delta \rho^n_\omega.
\]

For each \( x \in D_{\sigma^{n+m}(\omega),n+m} \subseteq D_{\sigma^{n+m}(\omega),m} \) we have

\[
\frac{\mathcal{L}_\omega^{n+m} 1_Z(x)}{\mathcal{L}_\omega^{n+m} 1_\omega(x)} \leq \frac{\mathcal{L}_\omega^{n} 1_Z \mathcal{L}_\omega^{m} 1_{\sigma^n(\omega)}(x)}{\mathcal{L}_\omega^{n+m} 1_\omega(x)} \leq \frac{\delta \rho^n_\omega \mathcal{L}_\omega^{m} 1_{\sigma^n(\omega)}(x)}{\mathcal{L}_\omega^{n+m} 1_{\sigma^n(\omega)}(x)} \leq \delta \rho^n_\omega \cdot \frac{1}{\inf_{y \in D_{\sigma^{n+m}(\omega),m}} \mathcal{L}_\omega^{m} 1_{\sigma^n(\omega)}(y)}.
\]

In view of Lemma 6.4, taking the infimum over \( x \in D_{\sigma^{n+m}(\omega),n+m} \) and letting \( m \to \infty \) gives

\[
\Lambda_\omega(1_Z) \leq \delta \rho^n_\omega \cdot \frac{1}{\Lambda_{\sigma^n(\omega)}(\mathcal{L}_\omega^{n} 1_\omega)} \leq \delta \rho^n_\omega (\rho^n_\omega)^{-1} = \delta.
\]

\[ \square \]

We are now ready to prove the main result of this section, a random version of Lemma 3.11 in [29]. Let

\[
\delta_0 := \frac{1}{8a_*^3}.
\]

\[ (9.3) \]

**Lemma 9.3.** For \( m \text{-a.e. } \omega \in \Omega \) with \( y_\omega(\omega) = 0 \), for all \( \delta < \delta_0 \), all \( n \geq N_{\omega,\delta} \) (where \( N_{\omega,\delta} \) is as in Lemma 9.2), and all \( f \in \mathcal{C}_{\omega,a_*} \) there exists \( Z_f \in \mathcal{Z}_\omega^{(n)} \) such that

\[
\inf_{Z_f} f \geq \frac{1}{4} \Lambda_\omega(f).
\]

**Proof.** We shall prove the lemma via contradiction. To that end suppose that the conclusion is false, that is we suppose that

\[
\inf_{Z} f < \frac{\Lambda_\omega(f)}{4}
\]

\[ (9.4) \]
for all $Z \in \mathcal{Z}_{\alpha,b}^{(n)}$. Then, for each $n \geq N_{\alpha,b}$ and each $k \in \mathbb{N}$ such that $n < \Sigma^{(k)}$, using (9.4) we can write

$$
\tilde{L}_{\omega}^{\Sigma^{(k)}} f = \sum_{Z \in \mathcal{Z}_{\omega}^{(n)}} \tilde{L}_{\omega}^{\Sigma^{(k)}} (f 1_Z) = \sum_{Z \in \mathcal{Z}_{\omega}^{(n)}} \tilde{L}_{\omega}^{\Sigma^{(k)}} (f 1_Z) = \sum_{Z \in \mathcal{Z}_{\omega,b}^{(n)}} \tilde{L}_{\omega}^{\Sigma^{(k)}} (f 1_Z) + \sum_{Z \in \mathcal{Z}_{\omega,b}^{(n)}} \tilde{L}_{\omega}^{\Sigma^{(k)}} (f 1_Z) \leq \frac{\Lambda_{\omega}(f)}{4} \sum_{Z \in \mathcal{Z}_{\omega,b}^{(n)}} \tilde{L}_{\omega}^{\Sigma^{(k)}} (1_Z) + \sum_{Z \in \mathcal{Z}_{\omega,b}^{(n)}} \tilde{L}_{\omega}^{\Sigma^{(k)}} (1_Z) \var_Z(f) + \|f\|_\infty \sum_{Z \in \mathcal{Z}_{\omega,b}^{(n)}} \tilde{L}_{\omega}^{\Sigma^{(k)}} (1_Z).
$$

(9.5)

Now for $Z \in \mathcal{Z}_{\omega,b}^{(n)}$, Lemma 9.1 implies that

$$
\Lambda_{\sigma^{\Sigma^{(k)}}(\omega)} (\tilde{L}_{\omega}^{\Sigma^{(k)}} (1_Z)) \leq a_\star \Lambda_{\sigma^{\Sigma^{(k)}}(\omega)} (\tilde{L}_{\omega}^{\Sigma^{(k)}} 1_\omega) \Lambda_{\omega}(1_Z) = 0.
$$

(9.6)

Thus, for $Z \in \mathcal{Z}_{\omega,b}^{(n)}$, using (9.6) and (8.13), applied along the blocks $\Sigma^{(k)}$, we have that

$$
\tilde{L}_{\omega}^{\Sigma^{(k)}} (1_Z) \leq \Lambda_{\sigma^{\Sigma^{(k)}}(\omega)} (\tilde{L}_{\omega}^{\Sigma^{(k)}} (1_Z)) + \var (\tilde{L}_{\omega}^{\Sigma^{(k)}} (1_Z)) = \var (\tilde{L}_{\omega}^{\Sigma^{(k)}} (1_Z)) \leq 2 \left( \frac{1}{3} \right)^{\Sigma^{(k)}/R_\star} + a_\star \Lambda_{\sigma^{\Sigma^{(k)}}(\omega)} (\tilde{L}_{\omega}^{\Sigma^{(k)}} 1_\omega) = 2 \left( \frac{1}{3} \right)^{\Sigma^{(k)}/R_\star}.
$$

(9.7)

Note that the right-hand side above goes to zero as $k \to \infty$. On the other hand, for $Z \in \mathcal{Z}_{\omega,g}^{(n)}$ we again use (8.13) in conjunction with Lemma 9.1 to get that

$$
\tilde{L}_{\omega}^{\Sigma^{(k)}} (1_Z) \leq \Lambda_{\sigma^{\Sigma^{(k)}}(\omega)} (\tilde{L}_{\omega}^{\Sigma^{(k)}} (1_Z)) + \var (\tilde{L}_{\omega}^{\Sigma^{(k)}} (1_Z)) \leq 2 \left( \frac{1}{3} \right)^{\Sigma^{(k)}/R_\star} + a_\star \Lambda_{\sigma^{\Sigma^{(k)}}(\omega)} (\tilde{L}_{\omega}^{\Sigma^{(k)}} 1_\omega) = 2 \left( \frac{1}{3} \right)^{\Sigma^{(k)}/R_\star}.
$$

(9.8)

Substituting (9.8) and (9.7) into (9.5), applying the functional $\Lambda_{\sigma^{\Sigma^{(k)}}(\omega)}$ to both sides yields

$$
\Lambda_{\sigma^{\Sigma^{(k)}}(\omega)} (\tilde{L}_{\omega}^{\Sigma^{(k)}} f) \leq \Lambda_{\sigma^{\Sigma^{(k)}}(\omega)} (\tilde{L}_{\omega}^{\Sigma^{(k)}} 1_\omega) \cdot \frac{\Lambda_{\omega}(f)}{4} + \sum_{Z \in \mathcal{Z}_{\omega,g}^{(n)}} \left( a_\star \Lambda_{\omega}(1_Z) + 2 \left( \frac{1}{3} \right)^{\Sigma^{(k)}/R_\star} + a_\star^2 \Lambda_{\omega}(1_Z) \right) \var_Z(f) \Lambda_{\sigma^{\Sigma^{(k)}}(\omega)} (\tilde{L}_{\omega}^{\Sigma^{(k)}} 1_\omega).
$$
\[(9.9)\]
\[+ \sum_{Z \in \mathcal{Z}_{\omega,b}} 2 \left( \frac{1}{3} \right)^{\Sigma_{\omega}^{(k)}} \|f\|_\infty \Lambda_{\sigma \Sigma_{\omega}^{(k)}}(\omega) \left( \tilde{L}_\omega \Sigma_{\omega}^{(k)} \mathbb{1}_\omega \right).\]

Dividing (9.9) on both sides by \(\Lambda_{\sigma \Sigma_{\omega}^{(k)}}(\omega) \left( \tilde{L}_\omega \Sigma_{\omega}^{(k)} \mathbb{1}_\omega \right)\), letting \(k \to \infty\), and using Lemmas 9.1, 9.2, and 8.4 gives us

\[
\Lambda_{\omega}(f) \leq \frac{\Lambda_{\omega}(f)}{4} + \sum_{Z \in \mathcal{Z}_{\omega,b}} (a_* + a_*^2) \Lambda_{\omega}(1_Z) \text{var}_Z(f)
\leq \frac{\Lambda_{\omega}(f)}{4} + (a_* + a_*^2) \text{var}(f) \sup_{Z \in \mathcal{Z}_{\omega,b}} \Lambda_{\omega}(1_Z)
\leq \left( \frac{1}{4} + 2a_*^2 \delta \right) \Lambda_{\omega}(f).
\]

Given our choice (9.3) of \(\delta < \delta_0\) we arrive at the contradiction

\[
\Lambda_{\omega}(f) \leq \frac{1}{2} \Lambda_{\omega}(f),
\]
and thus we are done. \(\Box\)

10. Finding Finite Diameter Images

We now find a large measure set of fibers \(\omega \in \Omega_F \subseteq \Omega\) for which the image of the cone \(C_{\omega,a_*}\) has a finite diameter image after sufficiently many iterates of the normalized operator \(\tilde{L}_\omega\). Towards accomplishing this task we first recall that for each \(\omega \in \Omega\) with \(y_*(\omega) = 0\) and each \(k \geq 0\)

\[
\Sigma_{\omega}^{(k)} := \sum_{j=0}^{k-1} \ell(\omega_j) R_*
\]

where \(\omega_0 := \omega\) and for each \(j \geq 1\) we set \(\omega_j := \sigma^{\Sigma_{\omega}^{(j-1)}}(\omega)\). For each \(\omega \in \Omega\) with \(y_*(\omega) = 0\), we define the number

\[
(10.1) \quad \Sigma_{\omega} := \min \left\{ \Sigma_{\omega}^{(k)} : \inf_{x \in \mathcal{P}_{\sigma \Sigma_{\omega}^{(k)}}(\omega), \Sigma_{\omega}^{(k)}} \frac{\mathcal{L}_{\omega} \Sigma_{\omega}^{(k)} \mathbb{1}_Z(x)}{\mathcal{L}_{\omega} \Sigma_{\omega}^{(k)} \mathbb{1}_\omega(x)} \geq \frac{\Lambda_{\omega}(1_Z)}{2} \text{ for all } Z \in \mathcal{Z}_{\omega,g}^{(N_{\omega,\delta_0})} \right\}.
\]

Note that by definition we must have that \(\Sigma_{\omega} \geq N_{\omega,\delta_0}\). Recall from the proof of Lemma 7.2 that the set \(\Omega_1 = \Omega_1(B_*)\) is given by

\[
(10.2) \quad \Omega_1 := \{ \omega \in \Omega : C_\varepsilon(\omega) \leq B_* \},
\]

where \(C_\varepsilon(\omega)\) comes from Proposition 6.9 and \(B_*\) was chosen sufficiently large such that \(m(\Omega_1) \geq 1 - \varepsilon/s\). For \(\alpha_* > 0\) and \(C_* \geq 1\) we consider the following

\[
(F1) \quad \Lambda_{\sigma \Sigma_{\omega}^{(k)}}(\omega) \left( \tilde{L}_\omega \Sigma_{\omega} \mathbb{1}_Z \right) \geq \alpha_* \text{ for all } Z \in \mathcal{Z}_{\omega,g}^{(N_{\omega,\delta_0})},
\]
(F2) \( C_*^{-1} \leq \inf_{D_\sigma^{\Sigma_\omega}, \Sigma_\omega} \tilde{L}_\omega^{\Sigma_\omega} \mathbb{1}_\omega \leq \left\| \tilde{L}_\omega^{\Sigma_\omega} \mathbb{1}_\omega \right\|_\infty \leq C_* \).

Now, we define the set \( \Omega_3 \), depending on parameters \( S_* = kR_* \) for some \( k \in \mathbb{N} \), \( \alpha_* > 0 \), and \( C_* \geq 1 \), by

\[
\Omega_3 = \Omega_3(S_*, \alpha_*, C_*) := \{ \omega \in \Omega : \Sigma_\omega \leq S_* \text{, and (F1) – (F2) hold } \}
\]

and choose \( S_* = kR_* \), \( \alpha_* > 0 \), and \( C_* \geq 1 \) such that \( m(\Omega_3) \geq 1 - \varepsilon/s \). Finally, we define

\[
\Omega_F := \Omega_1 \cap \Omega_3,
\]

which must of course have measure \( m(\Omega_F) \geq 1 - \varepsilon \). Furthermore, in light of the definition of \( \Omega_\tau \) from (7.4), we have that

\[
\sigma^{-R_*}(\Omega_F) \subseteq \Omega_\tau.
\]

**Lemma 10.1.** For all \( \omega \in \Omega_F \) such that \( y_\omega(\omega) = 0 \) we have that

\[
\tilde{L}_\omega^{\Sigma_\omega} \mathcal{C}_{\omega,a_*} \subseteq \mathcal{C}_{\sigma^{\Sigma_\omega}(\omega),a_*/2} \subseteq \mathcal{C}_{\sigma^{\Sigma_\omega}(\omega),a_*}
\]

with

\[
\text{diam}_{\sigma^{\Sigma_\omega}(\omega),a_*} \left( \tilde{L}_\omega^{\Sigma_\omega} \mathcal{C}_{\omega,a_*} \right) \leq \Delta := 2 \log \frac{8C_*^2a_* (3 + a_*)}{\alpha_*} < \infty.
\]

**Proof.** The invariance follows from Lemma 8.4. To show that the diameter is finite we first note that for \( 0 \neq f \in \mathcal{C}_{\omega,a_*} \) we must have that \( \Lambda_\omega(f) > 0 \) by definition. Now, Lemma 5.8 implies that for \( f \in \mathcal{C}_{\omega,a_*} \) we have

\[
\Theta_{\sigma^{\Sigma_\omega}(\omega),a_*} \left( \tilde{L}_\omega^{\Sigma_\omega} f, \mathbb{1}_{\sigma^{\Sigma_\omega}(\omega)} \right) \leq \log \frac{\left\| \tilde{L}_\omega^{\Sigma_\omega} f \right\|_\infty + \frac{1}{2} \Lambda_{\sigma^{\Sigma_\omega}(\omega)} \left( \tilde{L}_\omega^{\Sigma_\omega} f \right)}{\min \left\{ \inf_{D_\sigma^{\Sigma_\omega}(\omega), \Sigma_\omega} \tilde{L}_\omega^{\Sigma_\omega} f, \frac{1}{2} \Lambda_{\sigma^{\Sigma_\omega}(\omega)} \left( \tilde{L}_\omega^{\Sigma_\omega} f \right) \right\}}.
\]

Using Lemmas 8.4 and 9.1 and (F2) we bound the numerator by

\[
\left\| \tilde{L}_\omega^{\Sigma_\omega} f \right\|_\infty + \frac{1}{2} \Lambda_{\sigma^{\Sigma_\omega}(\omega)} \left( \tilde{L}_\omega^{\Sigma_\omega} f \right) \leq \text{var} \left( \tilde{L}_\omega^{\Sigma_\omega} f \right) + \frac{3}{2} \Lambda_{\sigma^{\Sigma_\omega}(\omega)} \left( \tilde{L}_\omega^{\Sigma_\omega} \mathbb{1}_\omega \right) \Lambda_\omega(f) \leq \frac{3a_* + a_*^2}{2} \Lambda_{\sigma^{\Sigma_\omega}(\omega)} \left( \tilde{L}_\omega^{\Sigma_\omega} \mathbb{1}_\omega \right) \Lambda_\omega(f)
\]

\[
\leq C_* a_* (3 + a_*) \Lambda_\omega(f).
\]

To find a lower bound for the denominator we first note that for each \( f \in \mathcal{C}_{\omega,a_*} \), by Lemma 9.3 there exists \( Z_f \in \mathcal{Z}_{\omega,g}^{(N_\omega,a_0)} \) such that

\[
\inf f|_{Z_f} \geq \frac{\Lambda_\omega(f)}{4}.
\]

Thus, using (10.8), for each \( x \in D_{\sigma^{\Sigma_\omega}(\omega), \Sigma_\omega} \) we have that

\[
\inf_{x \in D_{\sigma^{\Sigma_\omega}(\omega), \Sigma_\omega}} \tilde{L}_\omega^{\Sigma_\omega} f(x) \geq \inf_{x \in D_{\sigma^{\Sigma_\omega}(\omega), \Sigma_\omega}} \tilde{L}_\omega^{\Sigma_\omega} (f \mathbb{1}_{Z_f})(x) \geq \inf_{Z_f} \inf_{x \in D_{\sigma^{\Sigma_\omega}(\omega), \Sigma_\omega}} \tilde{L}_\omega^{\Sigma_\omega} \mathbb{1}_{Z_f}(x)
\]
\[
\frac{\Lambda_\omega(f)}{4} \inf_{x \in D_{\sigma^k, \sigma^l}} \mathcal{L}_\omega \mathbb{1}_{Z_f}(x) \geq 0.
\]

In light of conditions (F1)-(F2) we in fact have that

\[
\inf_{D_{\sigma^k, \sigma^l}} \mathcal{L}_\omega f \geq \frac{\alpha_\star \Lambda_\omega(f)}{8C_\star} > 0.
\]

Combining the estimates (10.7) and (10.9) with (10.6) gives

\[
\Theta_{\sigma^k, \sigma^l} (\mathcal{L}_\omega f, \mathbb{1}_{\sigma^k, \sigma^l}) \leq \log \frac{8C^2_\alpha \alpha_\star^2 (3 + \alpha_\star)}{\alpha_\star} < \infty.
\]

Taking the supremum over all functions \( f \in \mathcal{C}_{\omega, \alpha_\star} \) and applying the triangle inequality finishes the proof. \(\square\)

To end this section we recall from Section 8 that \( 0 \leq y_\star(\omega) < R_\star \) is chosen to be the smallest integer such that for either choice of sign or \(-\) we have

\[
\lim_{n \to \infty} \frac{1}{n} \# \{ 0 \leq k < n : \sigma^{\pm kR_\star + y_\star(\omega)}(\omega) \in \Omega_G \} > 1 - \varepsilon,
\]

\[
\lim_{n \to \infty} \frac{1}{n} \# \{ 0 \leq k < n : C_\varepsilon (\sigma^{\pm kR_\star + y_\star(\omega)}(\omega)) \leq B_\star \} > 1 - \varepsilon.
\]

In light of the definition of \( \Omega_F \) (10.4) and using the same reasoning as in Section 8 for the existence of \( y_\star \) (see Section 7 of [1]), for each \( \omega \in \Omega \), we now let \( 0 \leq v_\star(\omega) < R_\star \) be the least integer such that for either choice of sign or \(-\) we have that the following hold:

\[
\lim_{n \to \infty} \frac{1}{n} \# \{ 0 \leq k < n : \sigma^{\pm kR_\star + v_\star(\omega)}(\omega) \in \Omega_G \} > 1 - \varepsilon,
\]

\[
\lim_{n \to \infty} \frac{1}{n} \# \{ 0 \leq k < n : \sigma^{\pm kR_\star + v_\star(\omega)}(\omega) \in \Omega_F \} > 1 - \varepsilon.
\]

Two significant properties of \( v_\star \) are the following:

\[(10.14) \quad v_\star (\sigma^{v_\star(\omega)}(\omega)) = 0, \]

\[(10.15) \quad \text{if } v_\star(\omega) = 0, \text{ then } y_\star(\omega) = 0. \]

11. CONFORMAL AND INVARIANT MEASURES

We are now ready to bring together all of the results from Sections 6-10 to establish the existence of conformal and invariant measures supported in the survivor set \( X_{\omega, \infty} \). We follow the methods of [1] and [29], and we begin with the following technical lemma from which the rest of our results will follow.

Lemma 11.1. Let \( f, h \in BV_\Omega(I) \), let \( \varepsilon > 0 \) sufficiently small such that the results of Section 8 apply, and let \( V : \Omega \to (0, \infty) \) be a measurable function. Suppose that for each \( n \in \mathbb{N} \), each \( |p| \leq n \), each \( l \geq 0 \), and for \( m \)-a.e. \( \omega \in \Omega \) we have \( f_{\sigma^l(\omega)} \in \mathcal{C}_{\sigma^l(\omega), +} \) with \( \text{var}(f_{\sigma^l(\omega)}) \leq \varepsilon n \text{V}(\omega) \) and \( h_{\sigma^l(\omega)} \in \mathcal{C}_{\sigma^{l-1}(\omega), +} \) with \( \text{var}(h_{\sigma^{l-1}(\omega)}) \leq \varepsilon (n^{l+1}) \text{V}(\omega) \). Then there
exists $\vartheta \in (0,1)$ and a measurable function $N_3 : \Omega \to \mathbb{N}$ such that for all $n \geq N_3(\omega)$, all $l \geq 0$, and all $|p| \leq n$ we have

$$\Theta_{\sigma^{n+p}(\omega)} \left( \tilde{L}_{\sigma^{n+l}(\omega)} f_{\sigma^p(\omega)}, \tilde{L}_{\sigma^{n-l}(\omega)} h_{\sigma^{-l}(\omega)} \right) \leq \Delta \vartheta^n.$$  

Furthermore, $\Delta$, defined in (10.5), and $\vartheta$ do not depend on $V$.

**Proof.** We begin by noting that by (6.11) for each $l \geq 0$ we have that $\tilde{L}_{\sigma^{n-l}(\omega)} h_{\sigma^{-l}(\omega)} \in \mathcal{C}_{\sigma^{n-l}(\omega),+}$ for each $h_{\sigma^{-l}(\omega)} \in \mathcal{C}_{\sigma^{n-l}(\omega),+}$, and let

$$h_l = \tilde{L}_{\sigma^{n-l}(\omega)} h_{\sigma^{-l}(\omega)}.$$  

Set $v_* = v_*(\sigma^p(\omega))$ (defined in Section 8) and let $d_* = d_*(\sigma^p(\omega)) \geq 0$ be the smallest integer that satisfies

$$v_* + d_* R_* \geq \frac{\varepsilon n + \log V(\omega)}{\theta - \varepsilon},$$

(11.2)  

and let $n \geq N_1(\omega)$. Now using (11.4) to write

$$\frac{4\varepsilon n}{\theta} = \frac{\varepsilon n + \varepsilon n}{\theta/2} \geq \frac{\varepsilon n + \log V(\omega)}{\theta/2}$$

and then using (7.2) and (8.1), we see that (11.2) is satisfied for any $d_* R_* \geq 4\varepsilon n/\theta$. Using (10.13), the construction of $v_*$, and the ergodic decomposition of $\sigma^{R_*}$ following (10.13), we have for $m$-a.e. $\omega \in \Omega$ there is an infinite, increasing sequence of integers $d_j \geq 0$ satisfying (11.3). Furthermore, (10.13) implies that

$$\lim_{n \to \infty} \frac{1}{n/R_*} \# \left\{ 0 \leq k < \frac{n}{R_*} : \sigma^{\pm k R_* + v_*(\omega)}(\omega) \notin \Omega_F \right\} < \varepsilon,$$

and thus for $n \in \mathbb{N}$ sufficiently large (depending measurably on $\omega$), say $n \geq N_2(\omega) \geq N_1(\omega)$, we have that

$$\# \left\{ 0 \leq k < \frac{n}{R_*} : \sigma^{\pm k R_* + v_*(\omega)}(\omega) \notin \Omega_F \right\} < \frac{\varepsilon n}{R_*}.$$  

Thus the smallest integer $d_*$ satisfying (11.2) and (11.3) also satisfies

$$d_* R_* \leq \frac{4\varepsilon n}{\theta} + \varepsilon n = \frac{4 + \theta}{\theta} \varepsilon n.$$  

(11.5)  

Let

$$\hat{v}_* = v_* + d_* R_*.$$  

(11.6)  

Now, we wish to examine the iteration of our operator cocycle along a collection $\Sigma R_*$ of blocks, each of length $\ell(\omega) R_*$, so that the images of $\tilde{L}_{\omega R_*}^{\ell(\omega) R_*}$ are contained in $\mathcal{C}_{\sigma^l(\omega) R_*}^{\sigma^l(\omega) R_*}$ as in Lemma 8.4; see Figure 2.
We begin by establishing some simplifying notation. To that end, set $\tau = \sigma^p(\omega)$, $\tau_{-l} = \sigma^{p-l}(\omega)$, and $\tau_0 = \sigma^{h+\hat{v}_*}(\omega)$; see Figure 2. Note that in light of (10.14), (10.15), and (11.6) we have that

$$v_*(\tau_0) = y_*(\tau_0) = 0.$$  

Now, by our choice of $d_*$, we have that if $f_\tau \in C_{\tau,+}$ with $\text{var}(f_\tau) \leq e^{\epsilon n}V(\omega)$, then

$$\tilde{L}^{\hat{v}_*} f_\tau \in C_{\tau_0, a_*}.$$  

Indeed, applying Proposition 6.9, (11.2), (11.3), and the definition of $\Omega_F$ (10.4), we have

$$\text{var}\left(\tilde{L}^{\hat{v}_*} f_\tau\right) \leq C_\epsilon (\sigma^{\hat{v}_*}(\tau)) e^{-(\theta-\epsilon)\hat{v}_*} \text{var}(f_\tau) + C_\epsilon (\sigma^{\hat{v}_*}(\tau)) \Lambda_{\sigma^{\hat{v}_*}(\tau)} \left(\tilde{L}^{\hat{v}_*} f_\tau\right)$$

$$\leq B_* e^{-(\theta-\epsilon)\hat{v}_*} \text{var}(f_\tau) + B_* \Lambda_{\sigma^{\hat{v}_*}(\tau)} \left(\tilde{L}^{\hat{v}_*} f_\tau\right)$$

$$\leq B_* \frac{\text{var}(f_\tau)}{e^{\epsilon n}V(\omega)} + B_* \Lambda_{\sigma^{\hat{v}_*}(\tau)} \left(\tilde{L}^{\hat{v}_*} f_\tau\right)$$

$$\leq B_* + B_* \Lambda_{\sigma^{\hat{v}_*}(\tau)} \left(\tilde{L}^{\hat{v}_*} f_\tau\right)$$

$$\leq 2B_* \Lambda_{\sigma^{\hat{v}_*}(\tau)} \left(\tilde{L}^{\hat{v}_*} f_\tau\right) \leq \frac{a_*}{6} \Lambda_{\sigma^{\hat{v}_*}(\tau)} \left(\tilde{L}^{\hat{v}_*} f_\tau\right),$$

where we recall that $a_* > 12B_*$ is defined in (8.15). A similar calculation yields that if $h_{\tau_{-l}} \in C_{\tau_{-l},+}$ with $\text{var}(h_{\tau_{-l}}) \leq e^{e(n+l)}V(\omega)$, then $\tilde{L}^{l_{-l}} h_{\tau_{-l}} \in C_{\tau_0, a_*}$.

We now set $\tau_1 = \sigma^{\Sigma_{\tau_0}}(\tau_0)$ and for each $j \geq 2$ let $\tau_j = \sigma^{l(\tau_{j-1})}R_*(\tau_{j-1})$. Note that since $\tau_0 \in \Omega_F$, we have that $\Sigma_{\tau_0} \leq S_*$. 

As there are only finitely many blocks (good and bad) that will occur within an orbit of length $n$, let $k \geq 1$ be the integer such that

$$\hat{v}_* + \Sigma_{\tau_0} + \sum_{j=1}^{k-1} l(\tau_j)R_* \leq n < \hat{v}_* + \Sigma_{\tau_0} + \sum_{j=1}^k l(\tau_j)R_*,$$

Figure 2. The fibers $\tau_j$ and the decomposition of $n = \hat{v}_* + \Sigma_{\tau_0} + \Sigma R_* + \hat{h}(n)$. 

\[ \begin{array}{cccccccc}
\tau_{-l} & \tau & \sigma^p(\omega) & \tau_0 & \tau_1 & \tau_2 & \ldots & \tau_{k-1} & \tau_k & \sigma^n(\tau) & \tau_{k+1} \\
1 & 2 & \ldots & n & l & v & \ldots & \ell(\tau_{k-1})R_* & \hat{v}_0(n)R_* & \hat{h}(n) & \Sigma C R_* & \ell(\tau_k)R_* & \Sigma R_* \\
\end{array} \]
and let
\[ \Sigma_C := \sum_{j=1}^{k-1} \ell(\tau_j) \quad \text{and} \quad \hat{r}_{\tau_0}(n) := r_{\tau_0}(n - \hat{v}_*) \]
where \( r_{\tau_0}(n - \hat{v}_*) \) is the number defined in (8.27). Finally setting
\[ \Sigma = \Sigma_C + \hat{r}_{\tau_0}(n), \quad \hat{h}(n) := n - \hat{v}_* - \Sigma_{\tau_0} - \Sigma R_*, \quad \text{and} \quad \tau^*_k := \sigma^{\hat{r}_{\tau_0}(n)}(\tau_k), \]
we have the right decomposition of our orbit length \( n \) into blocks which do not expand distances in the fiber cones \( \mathcal{C}_{\omega,a_*} \) and \( \mathcal{C}_{\omega,+} \). Now let
\[ \tag{11.9} n \geq N_3(\omega) := \max \left\{ N_2(\omega), \frac{R_*}{\varepsilon}, \frac{S_*}{\varepsilon} \right\}. \]
Since \( v_*, \hat{h}(n) \leq R_* \), by (11.9), (11.5), and for
\[ \tag{11.10} \varepsilon < \frac{\theta}{8(1 + \theta)} \]
sufficiency small, we must have that
\[ \Sigma R_* = n - \hat{v}_* - \Sigma_{\tau_0} - \hat{h}(n) \geq n - v_* - d_* R_* - \Sigma_{\tau_0} - \hat{h}(n) \geq n - \left( \frac{4 + \theta}{\theta} \right) \varepsilon n - 2R_* - S_* \geq n - \left( \frac{4 + \theta}{\theta} \right) \varepsilon n - 3\varepsilon n \]
\[ \tag{11.11} \geq n \left( 1 - 4\varepsilon \left( \frac{1 + \theta}{\theta} \right) \right) > \frac{n}{2}. \]
Now we note that since \( \tilde{\mathcal{L}}_{\tau^*_k}(\Theta_{\tau^*_k, +}) \subseteq \mathcal{C}_{\sigma^{n+p}(\omega), +} \) we have that \( \tilde{\mathcal{L}}_{\tau^*_k} \) is a weak contraction, and hence, we have
\[ \tag{11.12} \Theta_{\sigma^{n+p}(\omega), +} \left( \tilde{\mathcal{L}}_{\tau^*_k}^{\hat{h}(n)}(f'), \tilde{\mathcal{L}}_{\tau^*_k}^{\hat{h}(n)}(h') \right) \leq \Theta_{\tau^*_k, +}(f', h'), \quad f', h' \in \Theta_{\tau^*_k, +}. \]
Recall that \( E_{\tau_1}(n - \hat{v}_* - \Sigma_{\tau_0}), \) defined in Lemma 8.5, is the total length of the bad blocks of the \( n - \hat{v}_* \) length orbit starting at \( \tau_0 \), i.e.
\[ E_{\tau_1}(n - \hat{v}_* - \Sigma_{\tau_0}) = \sum_{1 \leq j < k \atop \tau_j \in \Omega_\mu} \ell(\tau_j) + r_{\tau_0}(n - \hat{v}_*). \]
Lemma 8.5 then gives that
\[ \tag{11.13} E_{\tau_1}(n - \hat{v}_* - \Sigma_{\tau_0}) < Y \varepsilon \Sigma. \]
We are now poised to calculate (11.1), but first we note that we can write
\[ n = \hat{v}_* + \Sigma_{\tau_0} + \Sigma R_* + \hat{h}(n) \]
\[ \tag{11.14} = \hat{v}_* + \Sigma_{\tau_0} + \Sigma C R_* + \hat{r}_{\tau_0}(n) + \hat{h}(n) \]
and that the number of good blocks contained in the orbit of length \( n - \hat{v}_* - \Sigma_{\tau_0} \) is given by
\[ \tag{11.15} \Sigma_G := \# \{ 1 \leq j \leq k : \tau_j \in \Omega_G \} = \Sigma - E_{\tau_1}(n - \hat{v}_* - \Sigma_{\tau_0}) \leq \Sigma_C. \]
Now, using (11.14) we combine (in order) (11.12), (5.2), and Theorem 5.4 (repeatedly) in conjunction with the fact that \( \tau_0 \in \Omega_F \) to see that

\[
\Theta_{\sigma^{n+p}(\omega),+} \left( \tilde{L}^n_\tau(f_\tau), \tilde{L}^{n+l}_{\tau-1}(h_{\tau-1}) \right) \\
= \Theta_{\sigma^{n+p}(\omega),+} \left( \tilde{L}^h_{\tau_k}(n) \circ \tilde{L}^{\Sigma R^*} \circ \tilde{L}^{\Sigma F^0} \circ \tilde{L}^{\Sigma R^*} (f_\tau), \tilde{L}^{\Sigma R^*} \circ \tilde{L}^{\Sigma F^0} \circ \tilde{L}^{\Sigma R^*} (h_l) \right) \\
\leq \Theta_{\tau\kappa,\alpha} \left( \tilde{L}^{\Sigma R^*} \circ \tilde{L}^{\Sigma F^0} \circ \tilde{L}^{\Sigma R^*} (f_\tau), \tilde{L}^{\Sigma R^*} \circ \tilde{L}^{\Sigma F^0} \circ \tilde{L}^{\Sigma R^*} (h_l) \right) \\
\leq \Theta_{\tau\kappa,\alpha} \left( \tilde{L}^{\Sigma R^*} \circ \tilde{L}^{\Sigma F^0} \circ \tilde{L}^{\Sigma R^*} (f_\tau), \tilde{L}^{\Sigma R^*} \circ \tilde{L}^{\Sigma F^0} \circ \tilde{L}^{\Sigma R^*} (h_l) \right) \\
\leq \left( \tanh \left( \frac{\Delta}{4} \right) \right)^{\Sigma_G} \Theta_{\tau_1,\alpha} \left( \tilde{L}^{\Sigma F^0} \circ \tilde{L}^{\Sigma R^*} (f_\tau), \tilde{L}^{\Sigma F^0} \circ \tilde{L}^{\Sigma R^*} (h_l) \right) .
\]

(11.16)

Now since \( \tau_0 \in \Omega_F \) and in light of (11.8), applying Lemma 10.1 allows us to estimate the \( \Theta_{\tau_1,\alpha} \) term in the right hand side of (11.16) to give

\[
(11.17) \quad \Theta_{\sigma^{n+p}(\omega),+} \left( \tilde{L}^n_\tau(f_\tau), \tilde{L}^{n+l}_{\tau-1}(h_{\tau-1}) \right) \leq \left( \tanh \left( \frac{\Delta}{4} \right) \right)^{\Sigma_G} \Delta.
\]

Using (11.15), the fact that \( E_{\tau_1}(n-\hat{v}_s-\Sigma_0) \geq 1 \), (11.13), and (11.11), we see that

\[
\Sigma_G = \Sigma - E_{\tau_1}(n-\hat{v}_s-\Sigma_0) \\
\geq \Sigma - Y \varepsilon \Sigma \\
= (1 - Y \varepsilon) \Sigma \\
\geq \frac{(1 - Y \varepsilon) n}{2R_*}.
\]

(11.18)

In light of (8.30), for all \( \varepsilon > 0 \) sufficiently small we have that \( 1 - Y \varepsilon > 0 \). Finally, inserting (11.17) and (11.18) into (11.16) gives

\[
\Theta_{\sigma^{n+p}(\omega),+} \left( \tilde{L}^n_\tau(f_\tau), \tilde{L}^{n+l}_{\tau-1}(h_{\tau-1}) \right) \leq \Delta \vartheta^n,
\]

where

\[
\vartheta := \left( \tanh \left( \frac{\Delta}{4} \right) \right)^{\frac{(1 - Y \varepsilon) n}{2R_*}} < 1,
\]

which completes the proof. \( \square \)

Combining Lemma 11.1 together with Lemma 5.5 gives the following immediate corollary.

**Corollary 11.2.** Suppose \( \varepsilon > 0 \), \( V : \Omega \to (0, \infty) \), \( f_{\sigma^p(\omega)} \in C_{\sigma^p(\omega),+} \), and \( h_{\sigma^{p-1}(\omega)} \in C_{\sigma^{p-1}(\omega),+} \) all satisfy the hypotheses of Lemma 11.1. Then there exists \( \kappa \in (0, 1) \) such that for \( m \)-a.e. \( \omega \in \Omega \), all \( n \geq N_3(\omega) \), all \( l \geq 0 \), and all \( |p| \leq n \) we have

\[
\left\| \tilde{L}_{\sigma^p(\omega)} f_{\sigma^p(\omega)} - \tilde{L}_{\sigma^{p-1}(\omega)}^{n+l} h_{\sigma^{p-1}(\omega)} \right\|_{\infty} \leq \left\| \tilde{L}_{\sigma^p(\omega)} f_{\sigma^p(\omega)} \right\|_{\infty} \left( e^{\Delta \vartheta^n} - 1 \right).
\]
Notice that if we wish to apply Lemma 11.1 (or Corollary 11.2) repeatedly iterating in the forward direction, i.e. taking \( p = 0 \) so that we push forward starting from the \( \omega \) fiber, then we only need that \( f \in \mathcal{C}_{\omega,+} \) and do not need to be concerned with the assumption on the variation. Indeed, as \( p = 0 \) is fixed, then we will have \( \text{var}(f) \leq \text{var}(f) \cdot e^{\varepsilon n} \) for any \( n \geq 1 \). However, if we wish to apply Lemma 11.1 repeatedly with \( p = -n \) for \( n \) increasing to \( \infty \), then we will need to consider special functions \( f \).

**Definition 11.3.** We let the set \( \mathcal{D} \) denote the set of functions \( f \in \text{BV}_\Omega \) such that for each \( \varepsilon > 0 \) there exists a measurable function \( V_{f,\varepsilon} : \Omega \to (0, \infty) \) such that the following hold for all \( n \in \mathbb{Z} \) with \( |n| \) sufficiently large:

\begin{align*}
(\mathcal{D}1) \quad & \var(f_{\sigma^n(\omega)}) \leq V_{f,\varepsilon}(\omega)e^{\varepsilon |n|}, \\
(\mathcal{D}2) \quad & \Lambda_{\sigma^n(\omega)}(|f_{\sigma^n(\omega)}|) \geq V_{f,\varepsilon}^{-1}(\omega)e^{-\varepsilon |n|}.
\end{align*}

Let \( \mathcal{D}^+ \subseteq \mathcal{D} \) denote the collection of all functions \( f \in \mathcal{D} \) such that \( f_\omega \geq 0 \) for each \( \omega \in \Omega \).

**Remark 11.4.** Note that the space \( \mathcal{D} \) is nonempty. In particular, \( \mathcal{D} \) contains any function \( f : \Omega \times I \to \mathbb{R} \) such that \( f_\omega \) is equal to some fixed function \( f \in \text{BV}(I) \) with \( 0 < \inf |f| \). More generally, \( \mathcal{D} \) contains any functions \( f : \Omega \times I \to \mathbb{R} \) such that \( \log \text{var}(f_\omega) \), \( \log \Lambda(f_\omega) \in L^1(m) \).

**Remark 11.5.** Note that if \( f \in \mathcal{D} \) then taking \( V_f(\omega) = V_{f,\varepsilon}(\omega) \) measurable and \( \varepsilon' = \varepsilon/2 \) we have that

\[
\frac{\var(f_{\sigma^{-n}(\omega)})}{\Lambda_{\sigma^{-n}(\omega)}(f_{\sigma^{-n}(\omega)})} \leq V_f(\omega)\frac{\var(f_{\omega})}{\Lambda_\omega(f_{\omega})}e^{\varepsilon'n}.
\]

In the following corollary we establish the existence of an invariant density.

**Corollary 11.6.** There exists a function \( q \in \text{BV}_\Omega \) and a measurable function \( \lambda : \Omega \to \mathbb{R}^+ \) such that for \( m\text{-a.e.} \ \omega \in \Omega \)

\begin{align}
\mathcal{L}_\omega q_\omega &= \lambda_\omega q_{\sigma(\omega)} \quad \text{and} \quad \Lambda_\omega(q_\omega) = 1. 
\end{align}

Furthermore, we have that \( \log \lambda_\omega \in L^1(m) \) and for \( m\text{-a.e.} \ \omega \in \Omega \), \( \lambda_\omega \geq \rho_\omega \).

**Proof.** First we note that for any \( f \in \mathcal{D}^+ \), Lemma 6.4 and Remark 11.5 give that

\[
\var(f_{\omega,n}) = \frac{\rho_{\sigma^{-n}(\omega)}}{\Lambda_\omega(\mathcal{L}_{\sigma^{-n}(\omega)} f_{\sigma^{-n}(\omega)})} \var(f_{\sigma^{-n}(\omega)}) \leq \frac{\var(f_{\sigma^{-n}(\omega)})}{\Lambda_{\sigma^{-n}(\omega)}(f_{\sigma^{-n}(\omega)})} \leq V_f(\omega)\frac{\var(f_{\omega})}{\Lambda_\omega(f_{\omega})}e^{\varepsilon n}
\]

for all \( n \in \mathbb{N} \) sufficiently large, say for \( n \geq N_4(\omega) \), and some measurable \( V_f : \Omega \to (0, \infty) \), where

\[
f_{\omega,n} := \frac{f_{\sigma^{-n}(\omega)}\rho_{\sigma^{-n}(\omega)}}{\Lambda_\omega(\mathcal{L}_{\sigma^{-n}(\omega)} f_{\sigma^{-n}(\omega)})} \in \mathcal{C}_{\sigma^{-n}(\omega),+}.
\]

Thus, Corollary 11.2 (with \( p = -n \) and \( V(\omega) = V_f(\omega)\var(f_\omega)/\Lambda_\omega(f_\omega) \)) gives that

\[
(\mathcal{L}_{\sigma^{-n}(\omega)} f_{\omega,n})_{n \in \mathbb{N}} = \left( \frac{\mathcal{L}_{\sigma^{-n}(\omega)} f_{\sigma^{-n}(\omega)}}{\Lambda_\omega(\mathcal{L}_{\sigma^{-n}(\omega)} f_{\sigma^{-n}(\omega)})} \right)_{n \in \mathbb{N}}
\]
forms a Cauchy sequence in $C_{\omega,+}$, and therefore there must exist some $q_{\omega,f} \in C_{\omega,+}$ with

$$q_{\omega,f} := \lim_{n \to \infty} \frac{L_{\sigma^{-n}(\omega)}^{n} f_{\sigma^{-n}(\omega)}}{\Lambda_\omega \left( L_{\sigma^{-n}(\omega)}^{n} f_{\sigma^{-n}(\omega)} \right)}.$$  

By construction we have that $\Lambda_\omega(q_{\omega,f}) = 1$. Now, in view of calculating $L_\omega q_{\omega,f}$, we note that (9.1) (with $N = 1$ and $f = L_{\sigma^{-n}(\omega)}^{n} f_{\sigma^{-n}(\omega)}$) gives that

$$\frac{\Lambda_\sigma(\omega) \left( L_{\sigma^{-n}(\omega)}^{n+1} f_{\sigma^{-n}(\omega)} \right)}{\Lambda_\omega \left( L_{\sigma^{-n}(\omega)}^{n} f_{\sigma^{-n}(\omega)} \right)} \leq \frac{\|L_\omega 1_\omega\|_\infty \Lambda_\omega \left( L_{\sigma^{-n}(\omega)}^{n} f_{\sigma^{-n}(\omega)} \right)}{\Lambda_\omega \left( L_{\sigma^{-n}(\omega)}^{n} f_{\sigma^{-n}(\omega)} \right)} = \|L_\omega 1_\omega\|_\infty.$$  

Lemma 6.4 (with $k = 1$ and $f = L_{\sigma^{-n}(\omega)}^{n} f_{\sigma^{-n}(\omega)}$) implies that

$$\frac{\Lambda_\sigma(\omega) \left( L_{\sigma^{-n}(\omega)}^{n+1} f_{\sigma^{-n}(\omega)} \right)}{\Lambda_\omega \left( L_{\sigma^{-n}(\omega)}^{n} f_{\sigma^{-n}(\omega)} \right)} \geq \frac{\rho_\omega \Lambda_\omega \left( L_{\sigma^{-n}(\omega)}^{n} f_{\sigma^{-n}(\omega)} \right)}{\Lambda_\omega \left( L_{\sigma^{-n}(\omega)}^{n} f_{\sigma^{-n}(\omega)} \right)} = \rho_\omega,$$  

and thus, together with (11.21), we have

$$\frac{\Lambda_\sigma(\omega) \left( L_{\sigma^{-n}(\omega)}^{n+1} f_{\sigma^{-n}(\omega)} \right)}{\Lambda_\omega \left( L_{\sigma^{-n}(\omega)}^{n} f_{\sigma^{-n}(\omega)} \right)} \in [\rho_\omega, \|L_\omega 1_\omega\|_\infty].$$  

Thus there must exist a sequence $(n_k)_{k \in \mathbb{N}}$ along which this ratio converges to some value $\lambda_{\omega,f}$, that is

$$\lambda_{\omega,f} := \lim_{k \to \infty} \frac{\Lambda_\sigma(\omega) \left( L_{\sigma^{-n_k(\omega)}}^{n_k+1} f \right)}{\Lambda_\omega \left( L_{\sigma^{-n_k(\omega)}}^{n_k} f \right)}.$$  

Hence we have

$$L_\omega q_{\omega,f} = \lim_{k \to \infty} \frac{L_{\sigma^{-n_k(\omega)}}^{n_k+1} f}{\Lambda_\omega \left( L_{\sigma^{-n_k(\omega)}}^{n_k} f \right)} = \lim_{k \to \infty} \frac{L_{\sigma^{-n_k(\omega)}}^{n_k+1} f}{\Lambda_\sigma(\omega) \left( L_{\sigma^{-n_k(\omega)}}^{n_k+1} f \right)} \cdot \frac{\Lambda_\sigma(\omega) \left( L_{\sigma^{-n_k(\omega)}}^{n_k+1} f \right)}{\Lambda_\omega \left( L_{\sigma^{-n_k(\omega)}}^{n_k} f \right)} = \lambda_{\omega,f} q_{\sigma(\omega),f}.$$  

From (11.23) it follows that $\lambda_{\omega,f}$ does not depend on the sequence $(n_k)_{k \in \mathbb{N}}$, and in fact we have

$$\lambda_{\omega,f} = \lim_{n \to \infty} \frac{\Lambda_\sigma(\omega) \left( L_{\sigma^{-n}(\omega)}^{n+1} f \right)}{\Lambda_\omega \left( L_{\sigma^{-n}(\omega)}^{n} f \right)},$$  

and thus,

$$L_\omega q_{\omega,f} = \lambda_{\omega,f} q_{\sigma(\omega),f}.$$  

(11.24)
To see that \( q_{\omega,f} \) and \( \lambda_{\omega,f} \) do not depend on \( f \), we apply Lemma 11.1 (with \( p = -n, l = 0 \), and \( V(\omega) = \max \{ V_f(\omega) \var(f_\omega)/\Lambda_\omega(f_\omega), V_h(\omega) \var(h_\omega)/\Lambda_\omega(h_\omega) \} \)) to functions \( f, h \in \mathcal{D}_+ \) to get that

\[
\Theta_{\omega,+}(q_{\omega,f}, q_{\omega,h}) \leq \Theta_{\omega,+}(q_{\omega,f}, f_{\omega,n}) + \Theta_{\omega,+}(f_{\omega,n}, h_{\omega,n}) + \Theta_{\omega,+}(q_{\omega,h}, h_{\omega,n}) \leq 3\Delta \vartheta^n
\]

for each \( n \geq N_3(\omega) \). Thus, inserting (11.25) into Lemma 5.5 yields

\[
\|q_{\omega,f} - q_{\omega,h}\|_\infty \leq \|q_{\omega,f}\|_\infty \left( e^{(\Theta_{\omega,+}(q_{\omega,f}, q_{\omega,h})) - 1} \right) \leq \|q_{\omega,f}\|_\infty \left( e^{3\Delta \vartheta^n} - 1 \right),
\]

which converges to zero exponentially fast as \( n \) tends towards infinity. Thus we must in fact have that \( q_{\omega,f} = q_{\omega,h} \) for all \( f, h \). Moreover, in light of (11.24), this implies that \( \lambda_{\omega,f} = \lambda_{\omega,h} \). We denote the common values by \( q_\omega \) and \( \lambda_\omega \) respectively. It follows from (4.11) and (11.22) that

\[
0 < \rho_\omega \leq \lambda_\omega \leq \|\mathcal{L}_\omega \mathbb{1}_\omega\|_\infty.
\]

Measurability of the map \( \omega \mapsto \lambda_\omega \) follows from the measurability of the sequence

\[
\left( \frac{\Lambda_{\sigma(\omega)}(\mathcal{L}_{\sigma^-n(\omega)}^{n+1} \mathbb{1}_{\sigma^-n(\omega)})}{\Lambda_\omega(\mathcal{L}_{\sigma^-n(\omega)}^n \mathbb{1}_{\sigma^-n(\omega)})} \right)_{n \in \mathbb{N}}.
\]

The log-integrability of \( \lambda_\omega \) follows from the log-integrability of \( \rho_\omega \) and (11.26). Finally, measurability of the maps \( \omega \mapsto \inf q_\omega \) and \( \omega \mapsto \|q_\omega\|_\infty \) follows from the fact that we have

\[
q_\omega = \lim_{n \to \infty} \frac{\mathcal{L}_{\sigma^-n(\omega)}^n \mathbb{1}_{\sigma^-n(\omega)}}{\Lambda_\omega(\mathcal{L}_{\sigma^-n(\omega)}^n \mathbb{1}_{\sigma^-n(\omega)})},
\]

which is a limit of measurable functions, and thus finishes the proof.

\[
\square
\]

**Remark 11.7.** For each \( k \in \mathbb{N} \), inducting on (11.24) for any \( f \in \mathcal{D}_+ \) yields

\[
\mathcal{L}_\omega^k(q_\omega) = \lim_{n \to \infty} \frac{\mathcal{L}_{\sigma^-n(\omega)}^{n+k} f_{\sigma^-n(\omega)}}{\Lambda_\omega(\mathcal{L}_{\sigma^-n(\omega)}^n f_{\sigma^-n(\omega)})} = \lim_{n \to \infty} \frac{\mathcal{L}_{\sigma^-n(\omega)}^{n+k} f_{\sigma^-n(\omega)}}{\Lambda_{\sigma^k(\omega)}(\mathcal{L}_{\sigma^-n(\omega)}^{n+k} f_{\sigma^-n(\omega)})} \cdot \frac{\Lambda_{\sigma^k(\omega)}(\mathcal{L}_{\sigma^-n(\omega)}^{n+k} f_{\sigma^-n(\omega)})}{\Lambda_\omega(\mathcal{L}_{\sigma^-n(\omega)}^n f_{\sigma^-n(\omega)})} = q_{\sigma^k(\omega)} \cdot \lim_{n \to \infty} \frac{\Lambda_{\sigma^k(\omega)}(\mathcal{L}_{\sigma^-n(\omega)}^{n+k} f_{\sigma^-n(\omega)})}{\Lambda_\omega(\mathcal{L}_{\sigma^-n(\omega)}^n f_{\sigma^-n(\omega)})}.
\]

The final limit in (11.27) telescopes to give us

\[
\lim_{n \to \infty} \frac{\Lambda_{\sigma^k(\omega)}(\mathcal{L}_{\sigma^-n(\omega)}^{n+k} f_{\sigma^-n(\omega)})}{\Lambda_\omega(\mathcal{L}_{\sigma^-n(\omega)}^n f_{\sigma^-n(\omega)})} = \frac{\Lambda_{\sigma(\omega)}(\mathcal{L}_{\sigma^-n(\omega)}^{n+1} f_{\sigma^-n(\omega)})}{\Lambda_\omega(\mathcal{L}_{\sigma^-n(\omega)}^n f_{\sigma^-n(\omega)})} \cdot \frac{\Lambda_{\sigma(\omega)}(\mathcal{L}_{\sigma^-n(\omega)}^{n+1} f_{\sigma^-n(\omega)})}{\Lambda_{\sigma^k(\omega)}(\mathcal{L}_{\sigma^-n(\omega)}^{n+1} f_{\sigma^-n(\omega)})} \cdot \frac{\Lambda_{\sigma^k(\omega)}(\mathcal{L}_{\sigma^-n(\omega)}^{n+1} f_{\sigma^-n(\omega)})}{\Lambda_{\sigma^{k-1}(\omega)}(\mathcal{L}_{\sigma^-n(\omega)}^{n+1} f_{\sigma^-n(\omega)})} = \lambda_\omega \lambda_{\sigma(\omega)} \cdots \lambda_{\sigma^{k-1}(\omega)},
\]
For each \( k \geq 1 \) we denote
\[
\lambda^k_\omega := \lambda_\omega \lambda_{\sigma(\omega)} \cdots \lambda_{\sigma^{k-1}(\omega)}.
\]
Rewriting (11.27) gives
\[
\mathcal{L}^k_\omega q_\omega = \lambda^k_\omega q_{\sigma^k(\omega)}.
\]

The following proposition shows that the density \( q_\omega \) coming from Corollary 11.6 is in fact supported on the set \( D_{\omega, \infty} \).

**Proposition 11.8.** For m.a.e. \( \omega \in \Omega \) we have that
\[
\inf_{D_{\omega, \infty}} q_\omega > 0.
\]

**Proof.** First we note that since \( \Lambda_\omega(q_\omega) = 1 > 0 \) for m.a.e. \( \omega \in \Omega \), using the definition of \( \Lambda_\omega \) (4.1), we must in fact have that
\[
\inf_{D_{\sigma^n(\omega), n}} \mathcal{L}^n_\omega(q_\omega) > 0
\]
for \( n \in \mathbb{N} \) sufficiently large, which, in turn implies that
\[
\inf_{\lambda_{\omega, n-1}} q_\omega > 0
\]
for all \( n \in \mathbb{N} \) sufficiently large. Next, for m.a.e. \( \omega \in \Omega \) and all \( n \in \mathbb{N} \) we use (11.19) to see that
\[
\inf_{D_{\omega, \infty}} q_\omega = \left( \lambda^n_{\sigma^{-n}(\omega)} \right)^{-1} \inf_{D_{\omega, \infty}} \mathcal{L}^n_\omega(q_{\sigma^{-n}(\omega)})
\]
for \( n \in \mathbb{N} \) sufficiently large by (11.29), (11.26), and (4.12); thus we are finished.

**Lemma 11.9.** For each \( \omega \in \Omega \) the functional \( \Lambda_\omega \) is linear, positive, and enjoys the property that
\[
\Lambda_{\sigma(\omega)}(\mathcal{L}_\omega f) = \lambda_\omega \Lambda_\omega(f)
\]
for each \( f \in \text{BV}(I) \). Furthermore, for each \( \omega \in \Omega \) we have that
\[
\lambda_\omega = \rho_\omega = \Lambda_{\sigma(\omega)}(\mathcal{L}_\omega 1_\omega).
\]

**Proof.** Positivity of \( \Lambda_\omega \) follows from the initial properties of \( \Lambda_\omega \) shown in Observation 4.1. To prove the remaining claims we first prove a more robust limit characterization of \( \Lambda_\omega \) than the one given by its definition, (4.1). Now, for any two sequences of points \((x_n)_{n \geq 0}\) and \((y_n)_{n \geq 0}\) with \( x_n, y_n \in D_{\sigma^n(\omega), n} \) we have
\[
\lim_{n \to \infty} \frac{\mathcal{L}^n_\omega f(x_n) - \mathcal{L}^n_\omega f(y_n)}{\mathcal{L}^n_\omega 1_\omega(x_n) - \mathcal{L}^n_\omega 1_\omega(y_n)} = \lim_{n \to \infty} \frac{\mathcal{L}^n_\omega f(y_n)}{\mathcal{L}^n_\omega 1_\omega(y_n)} - 1
\]
\[
\leq \|f\|_{\infty} \limsup_{n \to \infty} \left| \exp(\Theta_{\sigma^n(\omega), n}(\mathcal{L}^n_\omega f, \mathcal{L}^n_\omega 1_\omega)) - 1 \right| = 0.
\]
Thus, we have shown that we may remove the infimum from (4.1), which defines the functional \( \Lambda_\omega \), that is now we may write
\[
(11.34) \quad \Lambda_\omega(f) = \lim_{n \to \infty} \frac{\mathcal{L}_\omega^n f}{\mathcal{L}_\omega^n \mathbb{1}_\omega}(x_n)
\]
for all \( f \in C_{\omega,+} \) and all \( x_n \in \mathcal{D}_{\sigma(n),n} \). Moreover, this identity also shows that the functional \( \Lambda_\omega \) is linear. To extend (11.34) to all of \( \text{BV}(I) \), we simply write \( f = f_+ - f_- \) so that \( f_+, f_- \in C_{\omega,+} \) for each \( f \in \text{BV}(I) \) so that we have
\[
(11.35) \quad \Lambda_\omega(f) = \Lambda_\omega(f_+) - \Lambda_\omega(f_-) = \lim_{n \to \infty} \frac{\mathcal{L}_\omega^n f_+}{\mathcal{L}_\omega^n \mathbb{1}_\omega} - \lim_{n \to \infty} \frac{\mathcal{L}_\omega^n f_-}{\mathcal{L}_\omega^n \mathbb{1}_\omega} = \lim_{n \to \infty} \frac{\mathcal{L}_\omega^n f}{\mathcal{L}_\omega^n \mathbb{1}_\omega}.
\]
To prove (11.31) and (11.32) we use (11.35) to note that
\[
(11.36) \quad \Lambda_{\sigma(\omega)}(\mathcal{L}_\omega f) = \lim_{n \to \infty} \frac{\mathcal{L}_\omega^{n+1} f}{\mathcal{L}_\sigma(\omega) \mathbb{1}_\sigma}(x_{n+1})
\]
\[
= \lim_{n \to \infty} \frac{\mathcal{L}_\omega^{n+1} f}{\mathcal{L}_\omega^{n+1} \mathbb{1}_\omega}(x_{n+1}) \cdot \frac{\mathcal{L}_\omega^n \mathbb{1}_\omega}{\mathcal{L}_\sigma(\omega) \mathbb{1}_\sigma}(x_{n+1})
\]
\[
= \Lambda_\omega(f) \cdot \Lambda_{\sigma(\omega)}(\mathcal{L}_\omega \mathbb{1}_\omega).
\]
Considering the case where \( f = q_\omega \) in (11.36) in conjunction with the fact that \( \Lambda_\omega(q_\omega) = 1 \) and \( \mathcal{L}_\omega q_\omega = \lambda_\omega q_{\sigma(\omega)} \) gives
\[
(11.37) \quad \rho_\omega := \Lambda_{\sigma(\omega)}(\mathcal{L}_\omega \mathbb{1}_\omega) = \Lambda_\omega(q_\omega)\Lambda_{\sigma(\omega)}(\mathcal{L}_\omega \mathbb{1}_\omega) = \Lambda_{\sigma(\omega)}(\mathcal{L}_\omega q_\omega) = \Lambda_{\sigma(\omega)}(\lambda_\omega q_{\sigma(\omega)}) = \lambda_\omega,
\]
which finishes the proof. \( \square \)

**Remark 11.10.** In light of the fact that \( \log \rho_\omega \in L^1(m) \) by (4.11), Lemma 11.9 implies that
\[
(11.38) \quad \log \lambda_\omega \in L^1(m).
\]

In the next lemma we are finally able to show that the functional \( \Lambda_\omega \) can be thought of as Borel probability measure for the random open system.

**Lemma 11.11.** There exists a non-atomic Borel probability measure \( \nu_\omega \) on \( I_\omega \) such that
\[
\Lambda_\omega(f) = \int_{I_\omega} f \, d\nu_\omega
\]
for all \( f \in \text{BV}(I) \). Consequently, we have that
\[
(11.39) \quad \nu_{\sigma(\omega)}(\mathcal{L}_\omega f) = \lambda_\omega \nu_\omega(f)
\]
for all \( f \in \text{BV}(I) \). Furthermore, we have that \( \text{supp}(\nu_\omega) \subseteq X_{\omega,\infty} \).

**Proof.** The proof that the functional \( \Lambda_\omega \) can be equated to a non-atomic Borel probability measure \( \nu_\omega \) goes exactly like the proof of Lemma 4.3 in [29]. Thus, we have only to prove that \( \text{supp}(\nu_\omega) \subseteq X_{\omega,\infty} \). To that end, suppose \( f \in L^1(\nu_{\omega,c}) \) with \( f \equiv 0 \) on \( X_{\omega,n-1} \). Then
\[
\int_I f \, d\nu_\omega = (\lambda_\omega^n)^{-1} \int_I \mathcal{L}_\omega^n f \, d\nu_{\sigma^n(\omega)} = (\lambda_\omega^n)^{-1} \int_I \mathcal{L}_\omega^n (\hat{X}_{\omega,n-1} \cdot f) \, d\nu_{\sigma^n(\omega)} = 0.
\]
As $0 < \lambda_\omega^n < \infty$ for each $n \in \mathbb{N}$, we must have that $\text{supp}(\nu_\omega) \subseteq X_{\omega,\infty}$. □

**Remark 11.12.** We can immediately see, cf. [23, 2], that the conformality of the family $(\nu_\omega)_{\omega \in \Omega}$ produced in Lemma 11.11 enjoys the property that for each $n \geq 1$ and each set $A$ on which $T^n_\omega|_A$ is one-to-one we have

$$
\nu_{\sigma^n(\omega)}(T^n_\omega(A)) = \lambda_\omega^n \int_A e^{-S_n,\tau(\nu_\omega)} \, d\nu_\omega.
$$

In particular, this gives that for each $n \geq 1$ and each $Z \in Z^{(n)}_\omega$ we have

$$
\nu_{\sigma^n(\omega)}(T^n_\omega(Z)) = \lambda_\omega^n \int_Z e^{-S_n,\tau(\nu_\omega)} \, d\nu_\omega.
$$

**Remark 11.13.** In light of Lemmas 11.9 and 11.11, the normalized operator $\tilde{L}_\omega$ is given by $\tilde{L}_\omega(\cdot) := \rho_\omega^{-1}L_\omega(\cdot) = \lambda_\omega^{-1}L_\omega(\cdot)$. Furthermore, $\tilde{L}_\omega$ enjoys the properties

$$
\tilde{L}_\omega q_\omega = q_{\sigma(\omega)} \quad \text{and} \quad \nu_{\sigma(\omega)}(\tilde{L}_\omega(f)) = \nu_\omega(f)
$$

for all $f \in \text{BV}(I)$.

For each $\omega \in \Omega$ we may now define the measure $\mu_\omega \in \mathcal{P}(I)$ by

$$
\mu_\omega(f) := \int_{X_{\omega,\infty}} f q_\omega \, d\nu_\omega, \quad f \in L^1(\nu_\omega). \tag{11.40}
$$

Lemma 11.11 and Proposition 11.8 together show that, for $m$-a.e. $\omega \in \Omega$, $\mu_\omega$ is a non-atomic Borel probability measure with $\text{supp}(\mu_\omega) \subseteq X_{\omega,\infty}$, which is absolutely continuous with respect to $\nu_\omega$. Furthermore, in view of Proposition 11.8, for $m$-a.e. $\omega \in \Omega$, we may now define the fully normalized transfer operator $\hat{L}_\omega : \text{BV}(I) \to \text{BV}(I)$ by

$$
\hat{L}_\omega f := \frac{1}{\lambda_\omega q_{\sigma(\omega)}} \tilde{L}_\omega(f q_\omega) = \frac{1}{\lambda_\omega q_{\sigma(\omega)}} \lambda^{-1}_\omega L_\omega(f q_\omega), \quad f \in \text{BV}(I). \tag{11.41}
$$

As an immediate consequence of Remark 11.13 and (11.41), we get that

$$
\hat{L}_\omega 1_\omega = 1_{\sigma(\omega)}. \tag{11.42}
$$

We end this section with the following proposition which shows that the family $(\mu_\omega)_{\omega \in \Omega}$ of measures is $T$-invariant.

**Proposition 11.14.** The family $(\mu_\omega)_{\omega \in \Omega}$ defined by (11.40) is $T$-invariant in the sense that

$$
\int_{X_{\omega,\infty}} f \circ T_\omega \, d\mu_\omega = \int_{X_{\sigma(\omega),\infty}} f \, d\mu_{\sigma(\omega)}. \tag{11.43}
$$

for $f \in L^1(\mu_{\sigma(\omega)}) = L^1(\nu_{\sigma(\omega)})$.

The proof of Proposition 11.14 goes just like the proof of Proposition 8.11 of [1], and has thus been omitted.
12. Decay of Correlations

We are now ready to show that images under the normalized transfer operator \( \tilde{L}_\omega \) converge exponentially fast to the invariant density as well as the fact that the invariant measure \( \mu_\omega \) established in Section 11 satisfies an exponential decay of correlations. Furthermore, we show that the families \((\nu_\omega)_\omega \in \Omega \) and \((\mu_\omega)_\omega \in \Omega \) are in fact random measures as defined in Section 2.1 and then introduce the RACCIM \( \eta \) supported on \( \mathcal{J} \).

To begin this section we state a lemma which shows that the BV norm of the invariant density \( q_\omega \) does not grow too much along a \( \sigma \)-orbit of fibers by providing a measurable upper bound. In fact, we show that the BV norm of \( q_\omega \) is tempered. As the proof of the following lemma is the same as the proof of Lemma 8.5 in the closed dynamical setting of [1], its proof is omitted.

**Lemma 12.1.** For all \( \delta > 0 \) there exists a measurable random constant \( C(\omega, \delta) > 0 \) such that for all \( k \in \mathbb{Z} \) and \( m \)-a.e. \( \omega \in \Omega \) we have

\[
\|q_{\sigma^k(\omega)}\|_{BV} = \|q_{\sigma^k(\omega)}\|_\infty + \text{var}(q_{\sigma^k(\omega)}) \leq C(\omega, \delta)e^{\delta|k|}.
\]

Consequently, we have that \( q \in \mathcal{D} \).

We are now able to prove the following theorem which completes the proof of Theorem B.

**Theorem 12.2.** There exists a measurable, \( m \)-a.e. finite function \( D : \Omega \to \mathbb{R} \) and \( \kappa < 1 \) such that for each \( f \in \mathcal{D} \), each \( n \in \mathbb{N} \), and each \( |p| \leq n \) we have

\[
(12.1) \quad \left\| \tilde{L}_{\sigma^p(\omega)}f_{\sigma^p(\omega)} - \nu_{\sigma^p(\omega)}(f_{\sigma^p(\omega)})q_{\sigma^{p+n}(\omega)} \right\|_\infty \leq D(\omega) \left\| f_{\sigma^p(\omega)} \right\|_\infty \kappa^n
\]

and

\[
(12.2) \quad \left\| \tilde{L}_{\sigma^p(\omega)}f_{\sigma^p(\omega)} - \mu_{\sigma^p(\omega)}(f_{\sigma^p(\omega)})\mathbb{1}_{\sigma^{p+n}(\omega)} \right\|_\infty \leq D(\omega) \left\| f_{\sigma^p(\omega)} \right\|_\infty \kappa^n.
\]

**Proof.** We first note that for \( m \)-a.e. \( \omega \in \Omega \), all \( n \geq N_3(\omega) \), all \( |p| \leq n \), and all \( f \in \mathcal{D}_+ \) we may use Lemma 11.1 to get that

\[
\Theta_{\sigma^{p+n}(\omega),+} \left( \tilde{L}_{\sigma^p(\omega)}f_{\sigma^p(\omega)}, \nu_{\sigma^p(\omega)}(f_{\sigma^p(\omega)})q_{\sigma^{p+n}(\omega)} \right) = \Theta_{\sigma^{p+n}(\omega),+} \left( \tilde{L}_{\sigma^p(\omega)}f_{\sigma^p(\omega)}, \nu_{\sigma^p(\omega)}(f_{\sigma^p(\omega)})\tilde{L}_{\sigma^p(\omega)}q_{\sigma^p(\omega)} \right) \leq \Delta \theta^n.
\]

Applying Lemma 5.5 with \( \varrho = \nu_{\sigma^{p+n}(\omega)} \) and \( \|\cdot\| = \|\cdot\|_\infty \), together with Lemma 11.1 then gives

\[
(12.3) \quad \left\| \tilde{L}_{\sigma^p(\omega)}f_{\sigma^p(\omega)} - \nu_{\sigma^p(\omega)}(f_{\sigma^p(\omega)})q_{\sigma^{p+n}(\omega)} \right\|_\infty \leq \Delta \theta^n \leq \left\| f_{\sigma^p(\omega)} \right\|_\infty \left\| q_{\sigma^{p+n}(\omega)} \right\|_\infty \left( e^{\Delta \theta^n} - 1 \right).
\]
for some \( \kappa \in (0, 1)^3 \). Since \( \|q_\omega\|_\infty \) is tempered, as a consequence of Lemma 12.1, for each \( n \in \mathbb{N} \) and each \( \delta > 0 \) we can find a tempered function \( A_{n,\delta} : \Omega \to \mathbb{R} \) such that

\[
\|q_{\sigma^{p+n}(\omega)}\|_\infty \leq A_{n,\delta}(\omega)e^{(p+n)\delta} \|q_\omega\|_\infty \leq A_{n,\delta}(\omega)e^{2n\delta} \|q_\omega\|_\infty ,
\]

where we have used the fact that \( |p| \leq n \). For each \( \delta > 0 \) we let

\[
B_\delta(\omega) := \max_{1 \leq n \leq N_3(\omega)} \{A_{n,\delta}(\omega)\} \kappa^{-N_3(\omega)}.
\]

Combining (12.3) - (12.5), for any \( n \in \mathbb{N} \) we see that

\[
\| \hat{L}_n^{\sigma p(\omega)} f_{\sigma p(\omega)} - \nu_{\sigma p(\omega)}(f_{\sigma p(\omega)}) q_{\sigma^{p+n}(\omega)} \|_\infty \leq 2B_\delta(\omega)e^{2n\delta} \| f_{\sigma p(\omega)} \|_\infty \| q_\omega \|_\infty \kappa^n
\]

where here we have fixed \( \delta > 0 \) sufficiently small such that

\[
e^{2\delta} \kappa =: \kappa < 1,
\]

and we have set \( B(\omega) = B_\delta(\omega) \).

Now, to extend (12.6) to all of \( \mathcal{D} \) we write a function \( f \in \mathcal{D}_+ \) as \( f = f_+ - f_- \), where \( f_+ , f_- \in \mathcal{D}_+ \). Applying the triangle inequality and using (12.6) twice gives

\[
\| \hat{L}_n^{\sigma p(\omega)} f_{\sigma p(\omega)} - \nu_{\sigma p(\omega)}(f_{\sigma p(\omega)}) q_{\sigma^{p+n}(\omega)} \|_\infty \leq 4B(\omega) \| f_{\sigma p(\omega)} \|_\infty \| q_\omega \|_\infty \kappa^n.
\]

Setting \( D(\omega) := 4B(\omega) \| q_\omega \|_\infty \) finishes the proof of (12.1). To prove the second claim follows easily from the first claim in a similar fashion as in Theorem 10.4 of [1].

From the previous result we easily deduce that the invariant measure \( \mu \) satisfies an exponential decay of correlations. The following theorem, whose proof is exactly the same as Theorem 11.1 of [1], completes the proof of Theorem C.

**Theorem 12.3.** For \( m \text{-a.e. } \omega \in \Omega \), every \( n \in \mathbb{N} \), every \( |p| \leq n \), every \( f \in L^1(\mu) \), and every \( h \in \mathcal{D} \) we have

\[
|\mu_\tau((f_{\sigma^n(\tau)} \circ T^n_\tau) h_\tau) - \mu_{\sigma^n(\tau)}(f_{\sigma^n(\tau)}h_\tau)| \leq D(\omega) \| f_{\sigma^n(\tau)} \|_{L^1(\mu_{\sigma^n(\tau)})} \| h_\tau \|_\infty \kappa^n,
\]

where \( \tau = \sigma^p(\omega) \).

**Remark 12.4.** Note that Theorem 12.2 implies a stronger limit characterization of the measure \( \nu_\omega \) than what is concluded in (11.34). Indeed, Theorem 12.2 implies that

\[
\nu_\omega(f) = \lim_{n \to \infty} \frac{\mathcal{L}_{\omega}^n f(x_n)}{\mathcal{L}_{\omega}^n 1_\omega(y_n)}
\]

for any pair of sequences \( (x_n)_{n \in \mathbb{N}} \) and \( (y_n)_{n \in \mathbb{N}} \) with \( x_n , y_n \in D_{\sigma^n(\omega), \infty} \), which further implies that

\[
\nu_\omega(f) = \lim_{n \to \infty} \| \mathcal{L}_{\omega}^n f \|_\infty = \lim_{n \to \infty} \inf \mathcal{L}_{\omega}^n 1_\omega = \inf \mathcal{L}_{\omega}^n 1_\omega.
\]

Furthermore, the same holds for \( \nu_{\omega,c} \); see Lemma 9.2 of [1].

---

\(^3\)Any \( \kappa > 0 \) will work for \( n \) sufficiently large.
We now address the uniqueness of the families of measures $\nu = (\nu_\omega)_{\omega \in \Omega}$ and $\mu = (\mu_\omega)_{\omega \in \Omega}$ as well as the invariant density $q$.

**Proposition 12.5.**

1. The family $\nu = (\nu_\omega)_{\omega \in \Omega}$ is a random probability measure which is uniquely determined by (11.39).
2. The global invariant density $q \in \mathcal{D}$ produced in Corollary 11.6 is the unique element of $L^1(\nu)$ (modulo $\nu$) such that
   \[ \tilde{L}_\omega q_\omega = q_{\sigma(\omega)}. \]
3. The family $\mu = (\mu_\omega)_{\omega \in \Omega}$ is a unique random $T$-invariant probability measure which is absolutely continuous with respect to $\nu$.

**Proof.** The fact that the family $(\nu_\omega)_{\omega \in \Omega}$ is a random measure as in Definition 2.5 follows from the limit characterization given in (12.7), as we have that $\nu_\omega$ is a limit of measurable functions. Indeed, for every interval $J \subset I$, the measurability of the function $\omega \mapsto \nu_\omega(J)$ follows from the fact that it is given by the limit of measurable functions by (12.7) applied to the characteristic function $f_\omega = 1_J$. Since $\mathcal{B}$ is generated by intervals, $\omega \mapsto \nu_\omega(B)$ is measurable for every $B \in \mathcal{B}$. Furthermore, $\nu_\omega$ is a Borel probability measure for $m$-a.e. $\omega \in \Omega$ from Proposition 11.11.

The remainder of the proof Proposition 12.5 follows along exactly like the proofs of Propositions 9.4 and 10.5 of [1], and is therefore left to the reader.

The proof of the following proposition is the same as the proof of Proposition 4.7 of [31], and so it is omitted.

**Proposition 12.6.** The random $T$-invariant probability measure $\mu$ defined in (11.40) is ergodic.

In the following lemma we establish the existence of the unique random absolutely continuous conditionally invariant probability measure $\eta$.

**Lemma 12.7.** The random measure $\eta \in \mathcal{P}_\Omega(I)$, whose disintegrations are given by
\[
\eta_\omega(f) := \frac{\nu_{\omega,c}(f \cdot 1_\omega \cdot q_\omega)}{\nu_{\omega,c}(1_\omega \cdot q_\omega)},
\]
is a unique random absolutely continuous conditionally invariant probability measure with respect to $\nu_c$ and is supported on $J$.

**Proof.** The fact that $\eta$ is an RACCIM follows from Lemma 3.5 and uniqueness follows from Proposition 12.5.

As a corollary of Theorem 12.2, the following results gives the exponential convergence of the closed conformal measure $\nu_{\omega,c}$ conditioned on the survivor set to the RACCIM $\eta_\omega$. 

Corollary 12.8. For m-a.e. every \( \omega \in \Omega \), every \( n \in \mathbb{N} \), every \( |p| \leq n \), and every \( A, B \in \mathcal{B} \) we have

\[
\left| \nu_{\sigma^p(\omega), c} \left( T_{\sigma^p(\omega)}^{-n} (A) \big| X_{\sigma^p(\omega), n} \right) - \eta_{\sigma^{p+n}(\omega)} (A) \right| \leq \frac{D(\omega)}{\nu_{\sigma^{p+n}(\omega), c} \left( 1_{\sigma^{p+n}(\omega)} q_{\sigma^{p+n}(\omega)} \right)} \kappa^n
\]

and

\[
\left| \eta_{\sigma^p(\omega)} (A \big| X_{\sigma^p(\omega), n}) - \mu_{\sigma^p(\omega)} (A) \right| \leq D(\omega) \kappa^n.
\]

Proof. For \( A \in \mathcal{B} \), Lemma 3.4 allows us to write

\[
\nu_{\sigma^p(\omega), c} \left( T_{\sigma^p(\omega)}^{-n} (A) \cap X_{\sigma^p(\omega), n} \right) = \int_{X_{\sigma^p(\omega), n}} 1_{\sigma^p(\omega)} 1_A \circ T_{\sigma^p(\omega)}^{-n} d\nu_{\sigma^p(\omega), c}
\]

\[
= (\lambda^n_{\sigma^p(\omega), c})^{-1} \int_{I_{\sigma^{p+n}(\omega)}} 1_A \mathcal{L}^n_{\sigma^p(\omega)} 1_{\sigma^p(\omega)} d\nu_{\sigma^{p+n}(\omega), c}
\]

\[
= \nu_{\sigma^{p+n}(\omega), c} \left( 1_A 1_{\sigma^{p+n}(\omega)} \mathcal{L}^n_{\sigma^p(\omega)} (1_{\sigma^p(\omega)}) \right).
\]

So, if \( A = I \), then we have

\[
\nu_{\sigma^p(\omega), c} \left( X_{\sigma^p(\omega), n} \right) = \nu_{\sigma^{p+n}(\omega), c} \left( 1_{\sigma^{p+n}(\omega)} \mathcal{L}^n_{\sigma^p(\omega)} (1_{\sigma^p(\omega)}) \right).
\]

Thus, we apply (12.1) of Theorem 12.2 together with elementary calculation to get that

\[
\left| \nu_{\sigma^p(\omega), c} \left( T_{\sigma^p(\omega)}^{-n} (A) \big| X_{\sigma^p(\omega), n} \right) - \eta_{\sigma^{p+n}(\omega)} (A) \right| = \frac{\nu_{\sigma^{p+n}(\omega), c} \left( 1_A 1_{\sigma^{p+n}(\omega)} \mathcal{L}^n_{\sigma^p(\omega)} (1_{\sigma^p(\omega)}) \right)}{\nu_{\sigma^{p+n}(\omega), c} \left( 1_{\sigma^{p+n}(\omega)} q_{\sigma^{p+n}(\omega)} \right)} \kappa^n.
\]

To see the second claim we note that for \( A \in \mathcal{B} \)

\[
\eta_{\sigma^p(\omega)} (A \big| X_{\sigma^p(\omega), n}) = \frac{\nu_{\sigma^p(\omega), c} \left( 1_A \tilde{X}_{\sigma^p(\omega), n} 1_{\sigma^p(\omega)} q_{\sigma^p(\omega)} \right)}{\nu_{\sigma^p(\omega), c} \left( \tilde{X}_{\sigma^p(\omega), n} 1_{\sigma^p(\omega)} \right)}.
\]

\[
= \frac{\nu_{\sigma^{p+n}(\omega), c} \left( 1_{\sigma^{p+n}(\omega)} \tilde{L}^n_{\sigma^p(\omega)} (1_{\sigma^p(\omega)}) \right)}{\nu_{\sigma^{p+n}(\omega), c} \left( 1_{\sigma^{p+n}(\omega)} q_{\sigma^{p+n}(\omega)} \right)}.
\]

\[
= \eta_{\sigma^{p+n}(\omega)} (1_{\sigma^p(\omega)} \tilde{L}^n_{\sigma^p(\omega)} (1_{\sigma^p(\omega)})).
\]
Thus, applying (12.2) of Theorem 12.2, we have
\[ |\eta_{\sigma^p}(\omega)(A|X_{\sigma^p(\omega),n}) - \mu_{\sigma^p(\omega)}(A)| = |\eta_{\sigma^{p+n}(\omega)}\left(\hat{L}_{\sigma^p(\omega)}^n(1_A)\right) - \mu_{\sigma^p(\omega)}(A)| \leq D(\omega)\kappa^n, \]
which finishes the proof. \(\square\)

13. Expected Pressures and Escape Rates

We now establish the rate at which mass escapes through the hole with respect to the closed conformal measure \(\nu_{\omega,c}\) and the RACCIM \(\eta_{\omega}\) in terms of the open and closed expected pressures. Given a (closed) potential \(\varphi_c\), which generates the (open) potential \(\varphi\), in light of (11.38), we recall the definition of the expected pressures \(\mathcal{E}P(\varphi_c)\), \(\mathcal{E}P(\varphi) \in \mathbb{R}\), of \(\varphi_c\) and \(\varphi\) respectively:

(13.1) \[ \mathcal{E}P(\varphi_c) := \int_{\Omega} \log \lambda_{\omega,c} dm(\omega) \quad \text{and} \quad \mathcal{E}P(\varphi) := \int_{\Omega} \log \lambda_{\omega} dm(\omega). \]

Since \(\log \lambda_{\omega}, \log \lambda_{\omega,c} \in L^1(m)\), Birkhoff’s Ergodic Theorem gives that

(13.2) \[ \mathcal{E}P(\varphi) = \lim_{n \to \infty} \frac{1}{n} \log \lambda_{\omega}^n = \lim_{n \to \infty} \frac{1}{n} \log \lambda_{\sigma^{-n}(\omega)}^n \]
and

(13.3) \[ \mathcal{E}P(\varphi_c) = \lim_{n \to \infty} \frac{1}{n} \log \lambda_{\omega,c}^n = \lim_{n \to \infty} \frac{1}{n} \log \lambda_{\sigma^{-n}(\omega),c}^n. \]

The following lemma, which is the open analogue of Lemma 10.1 of [1], gives an alternate method for calculating the expected pressure.

**Lemma 13.1.** For \(m\text{-a.e. } \omega \in \Omega\) we have that

(13.4) \[ \lim_{n \to \infty} \left\| \frac{1}{n} \log \mathcal{L}^n_{\sigma^{-n}(\omega)} 1_{\sigma^{-n}(\omega)} - \frac{1}{n} \log \lambda_{\sigma^{-n}(\omega)}^n \right\|_{\infty} = 0 \]
and

(13.5) \[ \lim_{n \to \infty} \left\| \frac{1}{n} \log \mathcal{L}^n_{\omega} 1_\omega - \frac{1}{n} \log \lambda_{\omega}^n \right\|_{\infty} = 0. \]

As the proof of the previous lemma is exactly the same as the proof of Lemma 10.1 of [1], the proof is left to the reader. Now, in view of the fact that \(\mathcal{L}_{\omega} 1_{\omega} \leq \mathcal{L}_{\omega,c} 1_{\omega}\), Lemma 13.1 and Lemma 10.1 of [1], together with (13.2) and (13.3) imply that

(13.6) \[ \mathcal{E}P(\varphi) \leq \mathcal{E}P(\varphi_c). \]

We now prove the following corollary of Lemma 13.1.

**Lemma 13.2.** For \(m\text{-a.e. } \omega \in \Omega\) we have that

(13.7) \[ \lim_{n \to \infty} \frac{1}{n} \log \inf_{D_{\sigma^n(\omega),\infty}} q_{\sigma^n(\omega)} = 0. \]
Proof. We first note that Lemma 13.1 immediately implies that
\begin{equation}
\lim_{n \to \infty} \frac{1}{n} \log \inf_{D(n, \omega) \in \mathbb{N}} \tilde{L}_{n}^{\sigma_{n} \omega} = \lim_{n \to \infty} \frac{1}{n} \log \|\tilde{L}_{n}^{\omega} \|_{\infty} = 0
\end{equation}
and
\begin{equation}
\lim_{n \to \infty} \frac{1}{n} \log \inf_{D, n} \tilde{L}_{\sigma_{n} \omega}^{\omega} = \lim_{n \to \infty} \frac{1}{n} \log \|\tilde{L}_{\sigma_{n} \omega}^{\omega} \|_{\infty} = 0.
\end{equation}

It follows from Theorem 12.2 that for m-a.e. \( \omega \in \Omega \) and all \( n \in \mathbb{N} \) sufficiently large
\[
\inf_{D(\sigma(n), \omega)} q_{\sigma(n)} \geq \inf_{D(\sigma(n), \omega)} \tilde{L}_{\sigma(n) \omega}^{\omega} - D(\omega) \kappa^{n},
\]
and thus
\[
\frac{1}{n} \log \inf_{D(\sigma(n), \omega)} q_{\sigma(n)} \geq \frac{1}{n} \log \left( \inf_{D(\sigma(n), \omega)} \tilde{L}_{\sigma(n) \omega}^{\omega} - D(\omega) \kappa^{n} \right).
\]

In light of (13.7), we have that the right hand side goes to zero as \( n \to \infty \), and thus the proof is complete.

Recall that the escape rate of a random probability measure \( \varrho \) is given by
\[
R(\varrho) := - \lim_{n \to \infty} \frac{1}{n} \log q_{\omega}(X_{\omega, n})
\]
The previous results allow us to calculate the following escape rates, thus proving Theorem D.

Proposition 13.3. For m-a.e. \( \omega \in \Omega \) we have that
\[
R(\nu_{\omega, c}) = R(\eta_{\omega}) = \mathcal{E} P(\varphi) - \mathcal{E} P(\varphi).
\]

Proof. We begin by noting that
\[
\nu_{\omega, c}(X_{\omega, n-1}) = \left( \lambda_{n, \omega}^{n} \right)^{-1} \nu_{\sigma_{n}(\omega), c} \left( L_{\omega, n-1}^{n} \right) = \frac{\lambda_{n, \omega}^{n}}{\lambda_{n, \omega}^{n}} \nu_{\sigma_{n}(\omega), c} \left( \tilde{L}_{\omega}^{n} \right)
\]
\[
= \frac{\lambda_{n, \omega}^{n}}{\lambda_{n, \omega}^{n}} \left( \nu_{\sigma_{n}(\omega), c}(q_{\sigma_{n}(\omega)}) - \nu_{\sigma_{n}(\omega), c}(\tilde{L}_{\omega}^{n} - q_{\sigma_{n}(\omega)}) \right).
\]

Using Theorem 12.2 gives that
\[
-R(\nu_{\omega, c}) = \lim_{n \to \infty} \frac{1}{n} \log \frac{\lambda_{n, \omega}^{n}}{\lambda_{n, \omega}^{n}} + \lim_{n \to \infty} \frac{1}{n} \log \nu_{\sigma_{n}(\omega), c}(q_{\sigma_{n}(\omega)}).
\]
Thus, the temperedness of \( \inf_{D(n, \omega)} q_{\omega} \) and \( \|q_{\omega}\|_{\infty} \), coming from Lemmas 12.1 and 12.2 respectively, imply that
\[
0 = \lim_{n \to \infty} \frac{1}{n} \log \inf_{D(\sigma(n), \omega)} q_{\sigma(n)} \leq \lim_{n \to \infty} \frac{1}{n} \log \nu_{\sigma(n)(\omega), c}(q_{\sigma(n)}) \leq \lim_{n \to \infty} \frac{1}{n} \log \|q_{\sigma(n)}\|_{\infty} = 0,
\]
which, when combined with (13.2) and (13.3), completes the proof of the first claim. As the second equality follows similarly to the first, we are done.
Remark 13.4. If there exists a $T$-invariant measure $\mu_c$ on the closed system which is absolutely continuous with respect to $\nu_c$ then the proof of Proposition 13.3, with minor adjustments, also shows that for $m$-a.e. $\omega \in \Omega$ we have that

$$R(\nu_{\omega,c}) = R(\mu_{\omega,c}) = \mathcal{E}P(\varphi_c) - \mathcal{E}P(\varphi).$$

14. Bowen’s Formula

This section is devoted to proving a formula for the Hausdorff dimension of the survivor set in terms of the expected pressure function, which was first proven by Bowen in [9] in the setting quasi-Fuchsian groups. In this section we will consider geometric potentials of the form $\varphi_{c,t}(\omega, x) = -t \log |T_{\omega}'(x)|$ for $t \in [0, 1]$. We denote the expected pressure of $\varphi_{c,t}$ by $\mathcal{E}P_c(t)$ and the expected pressure of the open potential $\varphi_t$ by $\mathcal{E}P(t)$. In the case that $t = 1$, the fiberwise closed conformal measures $\nu_{\omega,c,1}$ are equal to Lebesgue measure and $\lambda_{\omega,c,1} = 1$. Furthermore, we note that for any $t \geq 0$ we have that

$$(14.1)\quad g_{\omega,c,t}^{(n)} = \left(g_{\omega,c,1}^{(n)}\right)^{t} = \frac{1}{|T_{\omega}^{n}|^{t}}.$$ 

Definition 14.1. We will say that the weight function $g_{\omega,c}$ has the Bounded Distortion Property if for $m$-a.e. $\omega \in \Omega$ there exists $K_\omega \geq 1$ such that for all $n \in \mathbb{N}$, all $Z \in \mathcal{Z}_{\omega}^{(n)}$, and all $x, y \in Z$ we have that

$$\frac{g_{\omega,c}^{(n)}(x)}{g_{\omega,c}^{(n)}(y)} \leq K_\omega.$$

We now adapt the following definitions from [29] to the random setting.

Definition 14.2. We will say that the map $T$ has large images if for $m$-a.e. $\omega \in \Omega$ there exists $K_\omega \geq 1$ such that for all $n \in \mathbb{N}$, all $T^m_{\omega}$, and all $x, y \in Z$ we have that

$$\frac{g_{\omega,c}^{(n)}(x)}{g_{\omega,c}^{(n)}(y)} \leq K_\omega.$$

Remark 14.3. If $T$ has large images with respect to $H$ then it follows from Remark 11.12 that $\text{supp}(\nu_{\omega,t}) = X_{\omega,\infty}$ for any $t \in [0, 1]$.

We now prove a formula for the Hausdorff dimension of the surviving set, à la Bowen, proving Theorem E.

Theorem 14.4. There exists a unique $h \in [0, 1]$ such that $\mathcal{E}P(t) > 0$ for all $0 \leq t < h$ and $\mathcal{E}P(t) < 0$ for all $h < t \leq 1$. Furthermore, if $T$ has large images and large images with respect to $H$, then for $m$-a.e. $\omega \in \Omega$

$$\text{HD}(X_{\omega,\infty}) = h.$$ 

Proof. We will prove this theorem in a series of lemmas.
Lemma 14.5. The function $\mathcal{E}P(t)$ is strictly decreasing and there exists $h \in [0, 1]$ such that $\mathcal{E}P(t) > 0$ for all $0 \leq t < h$ and $\mathcal{E}P(t) < 0$ for all $h < t \leq 1$.

Proof. We first note that, using (14.1), for any $n \in \mathbb{N}$ and $s < t \in [0, 1]$ we can write
\[
\mathcal{L}_{\omega,t}^n \mathbb{1}_\omega \leq \left\| g_{\omega,1}^{(n)} \right\|_{\infty}^{t-s} \mathcal{L}_{\omega,t}^n \mathbb{1}_\omega.
\]
This immediately implies that
\[
\mathcal{E}P(t) < \mathcal{E}P(s)
\]
since
\[
\lim_{n \to \infty} \frac{1}{n} \log \left\| g_{\omega,1}^{(n)} \right\|_{\infty} < 0.
\]
Now since $\mathcal{E}P(0) \geq 0$ and $\mathcal{E}P(1) \leq \mathcal{E}P_c(1) = 0$, there must exist some $h \in [0, 1]$ such that for all $s < h < t$ we have
\[
\mathcal{E}P(t) < 0 < \mathcal{E}P(s).
\]

To prove the remaining claim of Theorem 14.4, we now suppose that $T$ has large images and large images with respect to $H$.

Lemma 14.6. If $T$ has large images and large images with respect to $H$, and $\nu_{\omega,t}(Z) > 0$ for all $t \in [0, 1]$, all $n \in \mathbb{N}$, and all $Z \in \mathcal{Z}_n^{(n)}$, then for all $x \in Z$ we have
\[
K_w^{-1} \leq \frac{\left( g_{\omega,c,1}^{(n)} \right)^t(x)}{\lambda_{\omega,t}^n \nu_{\omega,t}(Z)} \leq K_w \quad \text{and} \quad K_w^{-1} \leq \frac{g_{\omega,c,1}^{(n)}(x)}{\text{Leb}(Z)} \leq K_w.
\]

Proof. In light of Remark 14.3, $\text{supp}(\nu_{\omega,t}) = X_{\omega,\infty}$, and thus for any $Z \in \mathcal{Z}_n^{(n)}$ for any $n \geq 1$, $\nu_{\omega,t}(Z) > 0$ if and only if $Z \cap X_{\omega,\infty} \neq \emptyset$. Furthermore, since $T$ has large images with respect to $H$, we have that
\[
\nu_{\sigma^n(\omega),t}(T_w^n(Z)) = 1
\]
for any $Z \in \mathcal{Z}_n^{(n)}$ with $Z \cap X_{\omega,\infty} \neq \emptyset$. Thus, we may write
\[
\nu_{\omega,t}(Z) = \int_{X_{\omega,\infty}} \mathbb{1}_Z d\nu_{\omega,t} = \left( \lambda_{\omega,t}^n \right)^{-1} \int_{X_{\sigma^n(\omega),\infty}} \mathcal{L}_{\omega,t}^n \mathbb{1}_Z d\nu_{\sigma^n(\omega),t}
\]
\[
= \left( \lambda_{\omega,t}^n \right)^{-1} \int_{X_{\sigma^n(\omega),\infty}} \mathcal{L}_{\omega,c,t}^n \mathbb{1}_Z \hat{X}_{\omega,n-1} d\nu_{\sigma^n(\omega),t}
\]
\[
= \left( \lambda_{\omega,t}^n \right)^{-1} \int_{T_w^n(Z)} \left( \left( g_{\omega,c,1}^{(n)} \right)^t \hat{X}_{\omega,n-1} \right) \circ T_w^{-n} d\nu_{\sigma^n(\omega),t}
\]
\[
= \left( \lambda_{\omega,t}^n \right)^{-1} \int_{T_w^n(Z)} \left( g_{\omega,c,1}^{(n)} \right)^t \circ T_w^{-n} d\nu_{\sigma^n(\omega),t}.
\]
The Bounded Distortion Property implies that for \( x \in Z \) we have
\[
K_\omega^{-1} \nu_{\sigma^n(\omega),t}(T^n_\omega(Z)) \left( g^{(n)}_{\omega,c,1} \right)^t(x) \leq \int_{T^*_\omega(Z)} \left( g^{(n)}_{\omega,c,1} \right)^t \circ T^{-n}_\omega d\nu_{\sigma^n(\omega),t}
\]
\[
\leq K_\omega \nu_{\sigma^n(\omega),t}(T^n_\omega(Z)) \left( g^{(n)}_{\omega,c,1} \right)^t(x).
\]
Thus
\[
K_\omega^{-1} \frac{\left( g^{(n)}_{\omega,c,1} \right)^t(x)}{\nu_{\sigma^n(\omega),t}(T^n_\omega(Z))} \leq K_\omega \frac{\left( g^{(n)}_{\omega,c,1} \right)^t(x)}{\nu_{\sigma^n(\omega),t}(T^n_\omega(Z))},
\]
which then implies that
\[
K_\omega^{-1} \frac{1}{\nu_{\sigma^n(\omega),t}(T^n_\omega(Z))} \leq \frac{\left( g^{(n)}_{\omega,c,1} \right)^t(x)}{\nu_{\sigma^n(\omega),t}(T^n_\omega(Z))} \leq K_\omega \frac{1}{\nu_{\sigma^n(\omega),t}(T^n_\omega(Z))}.
\]
The first claim follows from (14.2). The proof of the second claim involving the Lebesgue measure follows similarly noting that \( \nu_{\omega,c,1} = \text{Leb} \) and \( \lambda_{\omega,c,1} = 1 \).

Let \( \varepsilon > 0 \) and \( n \in \mathbb{N} \) such that \( \text{diam}(Z) < \varepsilon \) for all \( Z \in Z^{(n)}_\omega \). Denote
\[
\mathcal{F}_\omega := \{ Z \in Z^{(n)}_\omega : Z \cap X_{\omega,\infty} \neq \emptyset \},
\]
which is a cover of \( X_{\omega,\infty} \) by sets of diameter less than \( \varepsilon \). Then, letting \( x_Z \) be any element of \( Z \) and using Lemma 14.4 twice (first with respect to \( \text{Leb} \) and then with respect to \( \nu_{\omega,t} \)), we have
\[
\sum_{Z \in \mathcal{F}_\omega} \lambda^n_{\omega,t}(Z) \leq \sum_{Z \in \mathcal{F}_\omega} \text{Leb}'(Z) \leq K_\omega^t \sum_{Z \in \mathcal{F}_\omega} \left( g^{(n)}_{\omega,c,1} \right)^t(x_Z)
\]
\[
\leq K_\omega^{2t} \sum_{Z \in \mathcal{F}_\omega} \lambda^n_{\omega,t}(Z) = K_\omega^{2t} \lambda^n_{\omega,t}(X_{\omega,\infty}) = K_\omega^{2t} \lambda^n_{\omega,t}.
\]
(14.4)

Now, if \( t > h \) we have
\[
\lim_{n \to \infty} \frac{1}{n} \log \lambda^n_{\omega,t} = \mathcal{E}P(t) < 0,
\]
and thus, for \( \delta > 0 \) sufficiently small and all \( n \in \mathbb{N} \) sufficiently large,
\[
\lambda^n_{\omega,t} \leq e^{-n\delta}.
\]
Consequently, we see that the right-hand side of (14.4) must go to zero, and thus we must have that \( \text{HD}(X_{\omega,\infty}) \leq h \).

For the lower bound we turn to the following result of Young.

**Proposition 14.7** (Proposition, [39]). Let \( X \) be a metric space and \( Z \subseteq X \). Assume there exists a probability measure \( \mu \) such that \( \mu(Z) > 0 \). For any \( x \in Z \) we define
\[
d_{\mu}(x) := \liminf_{\varepsilon \to 0} \frac{\log \mu(B(x,\varepsilon))}{\log \varepsilon}.
\]
If \( d_{\mu}(x) \geq d \) for each \( x \in Z \), then \( \text{HD}(Z) \geq d \).
We will use this result to prove a lower bound for the dimension, thus completing the proof of Theorem 14.4. Let \( x \in X_{\omega,\infty} \), let \( \varepsilon > 0 \), and in light of Lemma 14.6, let \( n_{\omega,0} + 1 \) be the least positive integer such that there exists \( y_0 \in B(x, \varepsilon) \) such that
\[
g^{(n_{\omega,0}+1)}_{\omega,c}(y_0) \leq 2\varepsilon K_\omega.
\]
Note that as \( \varepsilon \to 0 \) we must have that \( n_{\omega,0} \to \infty \). So we must have
\[
g^{(n_{\omega,0})}_{\omega,c}(y_0) g_{\sigma^{n_{\omega,0}}(\omega),c}(T^{n_{\omega,0}}_{\omega}(y_0)) = g^{(n_{\omega,0}+1)}_{\omega,c}(y_0) \leq 2\varepsilon K_\omega.
\]
Thus, using (14.5) and the definition of \( n_{\omega,0} \) we have that
\[
2\varepsilon K_\omega < g^{(n_{\omega,0})}_{\omega,c}(y_0) \leq \frac{2\varepsilon K_\omega}{\inf g_{\sigma^{n_{\omega,0}}(\omega),c}}.
\]
Now let \( Z_0 \in \mathcal{Z}_{\omega}^{(n_{\omega,0})} \) be the partition element containing \( y_0 \). Then Lemma 14.6 gives that
\[
\text{diam}(Z_0) \leq K_\omega g^{(n_{\omega,0})}_{\omega,c}(y_0) \leq \frac{2\varepsilon K_\omega^2}{\inf g_{\sigma^{n_{\omega,0}}(\omega),c}},
\]
and
\[
\text{diam}(Z_0) \geq K_\omega^{-1} g^{(n_{\omega,0})}_{\omega,c}(y_0) > 2\varepsilon.
\]
Combining (14.6) and (14.7) gives
\[
2\varepsilon < \text{diam}(Z_0) \leq \frac{2\varepsilon K_\omega^2}{\inf g_{\sigma^{n_{\omega,0}}(\omega),c}}.
\]
Now, we define
\[
B_{\omega,1} := B(x, \varepsilon) \setminus Z_0,
\]
which may be empty. If \( B_{\omega,1} \neq \emptyset \), then we let \( n_{\omega,1} + 1 \) be the least positive integer such that there exists \( y_1 \in B_{\omega,1} \) such that
\[
g^{(n_{\omega,1}+1)}_{\omega,c}(y_1) \leq 2\varepsilon K_\omega.
\]
Following the same line of reasoning to derive (14.8), we see that
\[
2\varepsilon < \text{diam}(Z_1) \leq \frac{2\varepsilon K_\omega^2}{\inf g_{\sigma^{n_{\omega,1}}(\omega),c}},
\]
where \( Z_1 \in \mathcal{Z}_{\omega}^{(n_{\omega,1})} \) is the partition element containing \( y_1 \). Note that by definition we have that \( n_{\omega,1} \geq n_{\omega,0} \) and \( Z_0 \cap Z_1 = \emptyset \). This immediately implies that
\[
B(x, \varepsilon) \subseteq Z_0 \cup Z_1,
\]
as otherwise using the same construction we could find some \( y_2 \in B_{\omega,1} \setminus Z_1 \), some \( n_{\omega,2} \geq n_{\omega,1} \) and a partition element \( Z_2 \in \mathcal{Z}_{\omega}^{(n_{\omega,2})} \) containing \( y_2 \) with diameter greater than \( 2\varepsilon \). But this would produce three disjoint intervals each with diameter greater than \( 2\varepsilon \) all of which intersect \( B(x, \varepsilon) \), which would obviously be a contradiction.
Now, using \((14.3)\) and Lemma 14.6 gives that
\[
\nu_{\omega,t}(Z_j) = (\lambda_{\omega,t}^{n_{\omega,j}})^{-1} \int_{T_{\omega,j}^{n_{\omega,j}}(Z)} \left( g_{\sigma_{\omega,j}(\omega),c}^{(n_{\omega,j})} \right)^t \circ T_{\omega,j}^{n_{\omega,j}} \, dv_{\sigma_{\omega,j}(\omega),t} \leq K^{\frac{t}{\lambda_{\omega,t}^{n_{\omega,j}}}} \text{diam}^t(Z_j)
\]
for \(j \in \{0, 1\}\). Using this we see that
\[
\log \nu_{\omega,t}(B(x, \varepsilon)) \leq \log (\nu_{\omega,t}(Z_0) + \nu_{\omega,t}(Z_1)) \\
\leq t \log K + \log \left( \lambda_{\omega,t}^{n_{\omega,0}} \text{diam}^t(Z_0) + \lambda_{\omega,t}^{n_{\omega,1}} \text{diam}^t(Z_1) \right) \\
\leq t \log K + \log \left( \lambda_{\omega,t}^{n_{\omega,0}} \left( \frac{2\varepsilon K^2}{\inf g_{\sigma_{\omega,0}(\omega),c}} \right)^t + \lambda_{\omega,t}^{n_{\omega,1}} \left( \frac{2\varepsilon K^2}{\inf g_{\sigma_{\omega,1}(\omega),c}} \right)^t \right) \\
\leq t \log K + t \log 2\varepsilon K^2 + \log \left( \lambda_{\omega,t}^{n_{\omega,0}} \left( \inf g_{\sigma_{\omega,0}(\omega),c} \right)^{-t} + \lambda_{\omega,t}^{n_{\omega,1}} \left( \inf g_{\sigma_{\omega,1}(\omega),c} \right)^{-t} \right) \\
\leq t \log K + t \log 2\varepsilon K^2 + \log \left( \lambda_{\omega,t}^{n_{\omega,0}} \left( \inf g_{\sigma_{\omega,0}(\omega),c} \right)^{-t} + \lambda_{\omega,t}^{n_{\omega,1}} \left( \inf g_{\sigma_{\omega,1}(\omega),c} \right)^{-t} \right).
\]
(14.10)

Now since \(\log \inf g_{\sigma,c} \in L^1(m)\), \(\inf g_{\sigma,c}\) is tempered and thus for each \(\delta > 0\) and all \(n \in \mathbb{N}\) sufficiently large we have that
\[
e^{-nt\delta} \leq \inf (g_{\sigma n(\omega),c})^t.
\]
(14.11)

From (13.2) we get that there for all \(n \in \mathbb{N}\) sufficiently large
\[
\lambda_{\omega,t}^n \geq e^{n(\varepsilon P(t) - \delta)}.
\]
(14.12)

Thus combining (14.11) and (14.12) with (14.10) gives
\[
\log \nu_{\omega,t}(B(x, \varepsilon)) \leq t \log K + t \log 2\varepsilon K^2 + \log \left( e^{n_{\omega,0}\delta(t+1) - n_{\omega,0}\varepsilon P(t)} + e^{n_{\omega,1}\delta(t+1) - n_{\omega,1}\varepsilon P(t)} \right) \\
\leq t \log K + t \log 2\varepsilon K^2 + \log \left( e^{2n_{\omega,0}\delta - n_{\omega,0}\varepsilon P(t)} + e^{2n_{\omega,1}\delta - n_{\omega,1}\varepsilon P(t)} \right),
\]
(14.13)

where we have used the fact that \(t \in [0, 1]\). Then for \(\delta > 0\) sufficiently small and \(n_{\omega,0}\) and \(n_{\omega,1}\) sufficiently large (which requires \(\varepsilon > 0\) to be sufficiently small) we have that
\[
\log \left( e^{2n_{\omega,0}\delta - n_{\omega,0}\varepsilon P(t)} + e^{2n_{\omega,1}\delta - n_{\omega,1}\varepsilon P(t)} \right) < 0,
\]
(14.14)

since for all \(t < h\) we have that \(\varepsilon P(t) > 0\). Dividing both sides of (14.13) by \(\log \varepsilon < 0\) and using (14.14) yields
\[
\frac{\log \nu_{\omega,t}(B(x, \varepsilon))}{\log \varepsilon} \geq \frac{t}{\log \varepsilon} \log K + \frac{t}{\log \varepsilon} \log 2\varepsilon K^2 + \frac{\log \left( e^{2n_{\omega,0}\delta - n_{\omega,0}\varepsilon P(t)} + e^{2n_{\omega,1}\delta - n_{\omega,1}\varepsilon P(t)} \right)}{\log \varepsilon} \\
\geq \frac{t}{\log \varepsilon} \log K + \frac{t}{\log \varepsilon} \log 2\varepsilon K^2 + t.
\]
\]
(14.15)

Taking a liminf of (14.15) as \(\varepsilon\) goes to 0 gives that \(d_{\nu_{\omega,t}}(x) \geq t\) for all \(x \in X_{\omega,\infty}\). As this holds for all \(t < h\), we must in fact have that \(d_{\nu_{\omega,t}}(x) \geq h\). In light of Proposition 14.7, we have proven Theorem 14.4. \(\square\)
15. Examples

In this final section we present several examples of our general theory of open random systems. In particular, we show that our results hold for a large class of random $\beta$-transformations with random holes which have uniform covering times as well as a large class of random Lasota-Yorke maps with random holes. However, we note that in principle any of the finitely branched classes of maps treated in [1] will satisfy our assumptions given a suitable choice of hole. This includes random systems where we allow non-uniformly expanding maps, or even maps with contracting branches to appear with positive probability. We also note that the examples we present allow for both random maps and random hole, which, to the authors' knowledge, has not appeared in literature until now. Before presenting our examples, we first give alternate hypotheses (to our assumptions (Q1)-(Q3)) that are more restrictive but simpler to check.

We begin by recalling the definitions of the various partitions constructed in Section 4 which are used in producing our main Lasota-Yorke inequality (Lemma 6.1) and are implicitly apart of our main assumptions (Q1)-(Q3). Recall that $\omega(n)$ denotes the partition of monotonicity of $T^n$, and $\mathcal{A}(n)$ denotes the collection of all finite partitions of $I$ such that

\begin{equation}
\text{var}_A(g(n)) \leq 2 \|g(n)\|_\infty
\end{equation}

for each $A = \{A_i\} \in \mathcal{A}(n)$. Given $A \in \mathcal{A}(n)$, $\omega(n)(A)$ denotes the coarsest partition amongst all those finer than $A$ and $\omega(n)$ such that all elements of $\omega(n)(A)$ are either disjoint from $X_{\omega,n-1}$ or contained in $X_{\omega,n-1}$. From $\omega(n)$ we recall the subcollections $\omega(n)$, $\omega(n)$, and $\omega(n)$ defined in (4.14)-(4.16).

For the purposes of showing that examples easily satisfy our conditions, we take the more general approach to partitions found in Section 2.2 of [1], and instead now set, for $\hat{\alpha} \geq 0$, \( \omega(n)(\hat{\alpha}) \) to be the collection of all finite partitions of $I$ such that

\begin{equation}
\text{var}_A(g(n)) \leq \hat{\alpha} \|g(n)\|_\infty
\end{equation}

for each $A = \{A_i\} \in \omega(n)(\hat{\alpha})$. Note that for some $\hat{\alpha} \leq 1$ the collection $\omega(n)(\hat{\alpha})$ may be empty, but such partitions always exist for any $\hat{\alpha} > 1$, and may exist even with $\hat{\alpha} = 0$ if the weight function $g$ is constant; see [33] Lemma 6. We now suppose that we can find $\hat{\alpha} \geq 0$ sufficiently large such that

(Z) $\omega(n)(\hat{\alpha}) \in \omega(n)(\hat{\alpha})$ for each $n \in \mathbb{N}$ and each $\omega \in \Omega$.

Now we set $\omega(n)$ be the coarsest partition such that all elements of $\omega(n)$ are either disjoint from $X_{\omega,n-1}$ or contained in $X_{\omega,n-1}$. Note that $\omega(n) = \omega(n)(\hat{\alpha})$. Now, define the following subcollections:

- $\omega(n) := \left\{ Z \in \omega(n) : Z \subseteq X_{\omega,n-1} \right\}$,
- $\omega(n) := \left\{ Z \in \omega(n) : Z \subseteq X_{\omega,n-1} \text{ and } \Lambda_\omega(\chi_Z) = 0 \right\}$,
- $\omega(n) := \left\{ Z \in \omega(n) : Z \subseteq X_{\omega,n-1} \text{ and } \Lambda_\omega(\chi_Z) > 0 \right\}$.
Consider the collection $\overline{\mathcal{Z}}^{(n)}_{\omega,F} \subseteq \overline{\mathcal{Z}}^{(n)}_{\omega,*}$ such that for $Z \in \overline{\mathcal{Z}}^{(n)}_{\omega,F}$ we have $T^{(n)}_{\omega}(Z) = I$. We will elements of $\overline{\mathcal{Z}}^{(n)}_{\omega,F}$ “full intervals”. We let $\overline{\mathcal{Z}}^{(n)}_{\omega,U} := \overline{\mathcal{Z}}^{(n)}_{\omega,*} \setminus \overline{\mathcal{Z}}^{(n)}_{\omega,F}$. Since for any $Z \in \overline{\mathcal{Z}}^{(n)}_{\omega,F}$ we have that $T^{(n)}_{\omega}(Z) = I$, and hence

$$\Lambda_{\omega}(\mathbb{1}_Z) \geq \inf_{D_{\sigma^{(n)}(\omega),n}} \mathcal{L}^{(n)}_{\omega} \mathbb{1}_Z \sup_{D_{\sigma^{(n)}(\omega),n}} \mathcal{L}^{(n)}_{\omega} \mathbb{1}_\omega \geq \inf g^{(n)}_{\omega,c} \| \mathcal{L}^{(n)}_{\omega,c} \mathbb{1}_\omega \|_\infty > 0. \tag{15.3}$$

Consequently, we have that $\overline{\mathcal{Z}}^{(n)}_{\omega,F} \subseteq \overline{\mathcal{Z}}^{(n)}_{\omega,b}$, and thus

$$\overline{\mathcal{Z}}^{(n)}_{\omega,b} \subseteq \overline{\mathcal{Z}}^{(n)}_{\omega,U}. \tag{15.4}$$

We let $\zeta^{(n)}_{\omega} \geq 0$ denote the maximum number of contiguous non-full intervals for $T^{(n)}_{\omega}$ in $\overline{\mathcal{Z}}^{(n)}_{\omega,*}$ for each $\omega \in \Omega$ and $n \in \mathbb{N}$. Note that $\zeta^{(1)}_{\omega}$ may be equal to 0, but $\zeta^{(n)}_{\omega} \geq 1$ for all $n \geq 2$, and so it follows from (15.4) that

$$0 \leq \log \zeta^{(n)}_{\omega} \leq \log \zeta^{(n)}_{\omega} \tag{15.5}$$

for all $n \geq 2$. In the interest of having assumptions that are easier to check than (Q1)-(Q3) we introduce the following simpler assumptions which use the collections $\overline{\mathcal{Z}}^{(n)}_{\omega,F}$ and $\overline{\mathcal{Z}}^{(n)}_{\omega,U}$ rather than $\overline{\mathcal{Z}}^{(n)}_{\omega,g}$ and $\overline{\mathcal{Z}}^{(n)}_{\omega,b}$. We assume the following:

$$(Q0) \quad \overline{\mathcal{Z}}^{(1)}_{\omega,F} \neq \emptyset \text{ for } m\text{-a.e. } \omega \in \Omega,$$

$$(Q1) \quad \lim_{n \to \infty} \frac{1}{n} \log \|g^{(n)}_{\omega}\|_\infty + \limsup_{n \to \infty} \frac{1}{n} \log \zeta^{(n)}_{\omega} < \lim_{n \to \infty} \frac{1}{n} \log \rho^{(n)}_{\omega} = \int_\Omega \log \rho_{\omega} \, dm(\omega),$$

$$(Q2) \text{ for each } n \in \mathbb{N} \text{ we have } \log \zeta^{(n)}_{\omega} \in L^1(m),$$

$$(Q3) \text{ for each } n \in \mathbb{N}, \log \delta_{\omega,n} \in L^1(m),$$

where

$$\delta_{\omega,n} := \min_{Z \in \overline{\mathcal{Z}}^{(n)}_{\omega,F}} \Lambda_{\omega}(\mathbb{1}_Z). \tag{15.6}$$

Our assumptions (Q1)-(Q3) are used exclusively in Section 6, and primarily in Lemma 6.1. In the proof of Lemma 6.1, the good and bad interval collections $\overline{\mathcal{Z}}^{(n)}_{\omega,g}$ and $\overline{\mathcal{Z}}^{(n)}_{\omega,b}$ are used only to estimate the variation of a function and can easily be replaced by the collections $\overline{\mathcal{Z}}^{(n)}_{\omega,F}$ and $\overline{\mathcal{Z}}^{(n)}_{\omega,U}$. Therefore, we can easily replace the assumptions (Q1)-(Q3) with (Q0)-(Q3) without any changes. In particular, we are still able to construct the number $N_*$ which is defined in (6.20). Note that by replacing the 2 in (15.1) with the $\hat{\alpha} \geq 0$ that appears in (15.2), the constant coefficients which appear in the definitions of $A^{(n)}_{\omega}$ and $B^{(n)}_{\omega}$ in (6.10) at the end of Lemma 6.1 may be different, consequently changing the value of $N_*$. This ultimately does not affect our general theory as we only care that such a value exists, though if one wishes to check the simplified assumptions (Q2') and (Q3') a value $\hat{\alpha} < 2$, and thus a smaller value for $N_*$, is helpful.
Remark 15.1. As in Remark 6.7, we again note that checking $(Z)$, $(\hat{Q}2)$, and $(\hat{Q}3)$ for all $n \in \mathbb{N}$ could be difficult and that it suffices to instead check these conditions only for $n = N_*$. Thus we may replace $(Z)$, $(\hat{Q}2)$, and $(\hat{Q}3)$ with the following:

$(Z')$ there exists $\hat{\alpha} \geq 0$ such that $Z^{(N_*)}_\omega \in \mathcal{A}^{(N_*)}_\omega(\hat{\alpha})$ for each $\omega \in \Omega$,

$(\hat{Q}2')$ $\log^+ \zeta^{(N_*)}_\omega \in L^1(m)$,

$(\hat{Q}3')$ $\log \delta_{\omega,N_*} \in L^1(m)$.

The following proposition gives that assumption $(\hat{Q}3)$ is always satisfied, and thus that we really only need to assume $(\hat{Q}1)$ and $(\hat{Q}2)$.

Proposition 15.2. Assumption $(\hat{Q}3)$ is trivially satisfied.

Proof. As the right hand side of (15.3) is log-integrable by (2.4) and (2.5), we must also have $\log \delta_{\omega,n} \in L^1(m)$. □

Recall that two elements $W, Z \in Z^{(n)}_{\omega,*}$ are said to be contiguous if either $W$ and $Z$ are contiguous in the usual sense, i.e. they share a boundary point, or if they are separated by a connected component of $\cup_{j=0}^{n-1} T^{-j}_{\omega}(H_{\sigma(\omega)})$. The following proposition gives an upper bound for the exponential growth of the number $\zeta^{(n)}_\omega$ which will be useful in checking our assumption $(\hat{Q}1)$, which implies $(Q1)$.

Proposition 15.3. The following inequality holds for $\zeta^{(n)}_\omega$, the largest number of contiguous non-full intervals for $T^n_{\omega}$.

(15.7) $\zeta^{(n)}_\omega \leq n \prod_{j=0}^{n-1} (\zeta^{(1)}_{\sigma(\omega)} + 2)$.

Consequently, using (15.5) and the ergodic theorem, we have that

(15.8) $\lim_{n \to \infty} \frac{1}{n} \log \zeta^{(n)}_\omega \leq \lim_{n \to \infty} \frac{1}{n} \log \zeta^{(n)}_\omega \leq \int \log(\zeta^{(1)}_{\omega} + 2) \, dm(\omega)$.

Proof. This is a random version of [29, Lemma 6.3]. We sketch the argument here. To upper bound $\zeta^{(n+1)}_\omega$, we observe that the largest number of contiguous non-full intervals for $T^{n+1}_\omega$ is given by

(15.9) $\zeta^{(n+1)}_\omega \leq \zeta^{(1)}_{\sigma(\omega)} (\zeta^{(n)}_{\sigma(\omega)} + 2) + 2 \zeta^{(n)}_{\sigma(\omega)}$.

Indeed, the first term on the right hand side accounts for the (worst case) scenario that all non-full branches of $T^n_{\sigma(\omega)}$ are pulled back inside contiguous non-full intervals for $T_\omega$.

For each non-full interval of $T_\omega$, there at most $\zeta^{(n)}_{\sigma(\omega)} + 2$ contiguous non-full intervals for $T^{n+1}_\omega$, as in addition to the $\zeta^{(n)}_{\sigma(\omega)}$ non-full intervals pulled back from $T^n_{\sigma(\omega)}$, there may be full branches of $T^n_{\sigma(\omega)}$ to the left and right of these which are only partially pulled back inside the corresponding branch of $T_\omega$. The second term in (15.9) accounts for an extra (at most) $\zeta^{(n)}_{\sigma(\omega)}$ non full branches of $T^n_{\sigma(\omega)}$ pulled back inside the full branches of $T_\omega$ neighboring the cluster of $\zeta^{(1)}_{\sigma(\omega)}$ non-full branches.
Rearranging (15.9) yields \( \zeta^{(n+1)}_{\omega} \leq \zeta^{(n)}_{\sigma(\omega)}(\zeta^{(1)}_{\omega} + 2) + 2\zeta^{(1)}_{\omega}. \) The claim follows directly by induction. \( \square \)

Let \( h_{\omega} \) denote the number of connected components of \( H_{\omega}. \) The following lemma shows that the conditions

**Lemma 15.4.** If assumption \((Z')\) holds as well as

\[
\log h_{\omega} \in L^1(m),
\]

then \( \log \zeta^{(1)}_{\omega} \in L^1(m). \) Consequently, we have that \((Q2')\) and \((\dot{Q}2)\) hold.

**Proof.** Since condition \((Z')\) holds, we see that \((LIP)\) in conjunction with \((CCH)\) gives that \( \log \# Z_{\omega} \in L^1(m), \) and thus we must have that \( \log \zeta^{(1)}_{\omega} \in L^1(m). \) In light of (15.7) we see that \((Q2')\) and \((\dot{Q}2)\) hold if

\[
\log(\zeta^{(1)}_{\omega} + 2) \in L^1(m),
\]

and thus we are done. \( \square \)

For each \( n \in \mathbb{N} \) define

\[
F^{(n)}_{\omega} := \min_{y \in [0,1]} \# \{ T^{-n}(y) \}. \tag{15.10}
\]

Since the sequences \( \{ \omega \mapsto \| g^{(n)}_{\omega} \|_{\infty} \}_{n \in \mathbb{N}} \) and \( \{ \omega \mapsto \inf L^n_{\omega} \} \) are submultiplicative and supermultiplicative, respectively, the subadditive ergodic theorem implies that the assumption that

\[
\lim_{n \to \infty} \frac{1}{n} \log \| g^{(n)}_{\omega} \|_{\infty} < \lim_{n \to \infty} \frac{1}{n} \log \inf_{D_{\omega}^{n(\omega)}(n),n} L^n_{\omega} \| \omega \|
\]

is equivalent to the assumption that there exist \( N_1, N_2 \in \mathbb{N} \) such that

\[
\frac{1}{N_1} \int_{\Omega} \log \| g^{(N_1)}_{\omega} \|_{\infty} dm(\omega) < \frac{1}{N_2} \int_{\Omega} \log \inf_{D_{\omega}^{N_2(\omega),N_2}} L^{N_2}_{\omega} dm(\omega). \tag{15.11}
\]

A useful lower bound for the right hand side is the following:

\[
\inf_{D_{\omega}^{N_2(\omega),N_2}} L^{N_2}_{\omega} \geq \inf_{\chi_{\omega},N_2-1} g^{(N_2)}_{\omega} F^{(N_2)}_{\omega} \geq \inf_{\chi_{\omega},N_2} g^{(N_2)}_{\omega} F^{(N_2)}_{\omega}. \tag{15.12}
\]

The next lemma, which offers a sufficient condition to check assumptions \((Q1)\) and \((\dot{Q}1)\), follows from (15.8), (15.11), (15.12), and the calculations from the proof of Lemma 13.16 in [1].

**Lemma 15.5.** If there exists \( N_1 \geq N_2 \in \mathbb{N} \) such that

\[
\frac{1}{N_1} \int_{\Omega} \sup S_{N_1,T}(\varphi_{\omega,c}) - \inf S_{N_1,T}(\varphi_{\omega,c}) + N_1 \log(\zeta^{(1)}_{\omega} + 2) dm(\omega) < \frac{1}{N_2} \int_{\Omega} \log F^{(N_2)}_{\omega} dm(\omega),
\]

then \((\dot{Q}1)\) (and thus \((Q1)\)) holds.

The following definition will be useful in checking our measurability assumptions for examples.
**Definition 15.6.** We say that a function \( f : \Omega \to \mathbb{R}_+ \) is \( m \)-continuous function if there is a partition of \( \Omega \) (mod \( m \)) into at most countably many Borel sets \( \Omega_1, \Omega_2, \ldots \) such that \( f \) is constant on each \( \Omega_j \), say \( f|_{\Omega_j} = f_j \).

We now give specific classes of random maps with holes which meet our assumptions. In principle, any of the classes of finitely branched maps discussed in Section 13 of [1] (including random non-uniformly expanding maps) will fit our current assumptions given a suitable hole \( H \).

### 15.1. **Random \( \beta \)-Transformations With Holes.**

For this first example we consider the class of maps described in Section 13.2 in [1]. These are \( \beta \)-transformations for which the last (non-full) branch is not too small so that each branch in the random closed system has a uniform covering time. In particular we assume there is some \( \delta > 0 \) such that for \( m \)-a.e. \( \omega \in \Omega \) we have

\[
\beta_\omega \in \bigcup_{k=1}^{\infty} [k + \delta, k + 1].
\]

Further suppose that the map \( \omega \mapsto \beta_\omega \) is \( m \)-continuous. We consider the random \( \beta \)-transformation \( T_\omega : [0, 1] \to [0, 1] \) given by

\[
T_\omega(x) = \beta_\omega x \pmod{1}.
\]

and the potential

\[
\varphi_{\omega,c} = -t \log |T_\omega'| = -t \log \beta_\omega
\]

for \( t \geq 0 \). In addition, we assume that

\[
(15.13) \quad \int_{\Omega} \log \lfloor \beta_\omega \rfloor \, dm(\omega) > \log 3
\]

and

\[
(15.14) \quad \int_{\Omega} \log \lceil \beta_\omega \rceil \, dm(\omega) < \infty.
\]

Note that we allow \( \beta_\omega \) arbitrarily large. It follows from Lemma 13.6 of [1] that our assumptions \((T1)-(T3)\), \((\text{LIP})\), \((\text{GP})\), \((A1)-(A2)\), \((M)\), \((C)\), and \((Z)\) are satisfied.

To check the remainder of our assumptions we must now describe the choice of hole \( H_\omega \). For our holes \( H_\omega \) we will consider intervals of length at most \( 1/\beta_\omega \) so that \( H_\omega \) may not intersect more than two monotonicity partition elements. To ensure that \((\hat{Q}0)\) is satisfied we assume there is a full-branched element \( Z \in \mathcal{Z}_{\omega,F}^{(1)} \) such that \( Z \cap H_\omega = \emptyset \) for each \( \omega \in \Omega \), and thus, in light of Remark 4.5, we also have that assumption \((D)\) is satisfied with \( D_{\omega,\infty} = I \) for each \( \omega \in \Omega \).

Now, we note that since \((15.14)\) implies our assumption \((\text{LIP})\), Lemma 15.4 implies that assumption \((\hat{Q}2)\) is satisfied. Thus, we have only to check the condition \((\hat{Q}1)\). Depending on \( H_\omega \) we may have that

\[
\inf \mathcal{L}_\omega \mathcal{H}_\omega = \frac{|\beta_\omega| - 1}{\beta_\omega^t},
\]
for example if $H_\omega$ is the last full branch. To ensure that ($\hat{Q}_1$) holds, note that (15.11) holds with $N = 1$, and thus it suffices to have
\[
\int_\Omega \log(\lfloor \beta_\omega \rfloor - 1) \, dm(\omega) > \int_\Omega \log(\zeta_1(\omega) + 2) \, dm(\omega),
\]
since
\[
\int_\Omega \log \inf L_\omega 1_\omega - \log \|g_\omega\|_\infty \, dm(\omega) \geq \int_\Omega \log \frac{\lfloor \beta_\omega \rfloor - 1}{\beta_t^\omega} + \log \beta_t^\omega \, dm(\omega).\]

Depending on the placement of $H_\omega$ we may have $\zeta_1(\omega) = i$ for any $i \in \{0, 1, 2, 3\}$. Thus, we obtain the following lemma assuming the worst case scenario, i.e. assuming $\zeta_1(\omega) = 3$ for $m$-a.e. $\omega \in \Omega$.

**Lemma 15.7.** If $H_\omega \subseteq I$ is such that $\zeta_1(\omega) \leq 3$ for $m$-a.e. $\omega \in \Omega$, then Theorems A-D if
\[
\int_\Omega \log(\lfloor \beta_\omega \rfloor - 1) \, dm(\omega) > \log 5.
\]

On the other hand, if we have that $H_\omega$ is equal to the monotonicity partition element which contains 1, then $\zeta_1(\omega) = 0$ and
\[
\inf L_\omega 1_\omega = \frac{\lfloor \beta_\omega \rfloor}{\beta_t^\omega}.
\]
Furthermore, the additional hypotheses necessary for Theorem E are satisfied. In particular, the fact that $T$ has large images follows from the fact that these maps have a uniform covering time; see Lemma 13.5 of [1]. Thus, we thus have the following lemma.

**Lemma 15.8.** If $H_\omega = Z_1$, where $1 \in Z_1 \in Z_\omega$, for $m$-a.e. $\omega \in \Omega$ then Theorems A-E hold.

More generally, we can consider general potentials, non-linear maps, and holes which are unions of finitely many intervals so that condition (CCH) holds.

15.2. Random Open Lasota-Yorke Maps. We now present an example of a large class of random Lasota-Yorke maps with holes. The following lemma summarizes the closed setting for this particular class of random maps was treated in Section 13.6 of [1].

**Lemma 15.9.** Let $\Phi_c : \Omega \to BV(I)$ be an $m$-continuous function, and let $\varphi_{c,c} : \Omega \times I \to \mathbb{R}$ be given by $\varphi_{c,c} := -t \log |T'_\omega| = \Phi_c(\omega)$ for $t \geq 0$. Then $g_{\omega,c} = e^{\varphi_{c,c}} = 1/|T'_\omega|^t \in BV(I)$ for $m$ a.e. $\omega \in \Omega$. We further suppose the system satisfies the following:

1. $\log \# Z_\omega \in L^1(m)$,
2. there exists $M(n) \in \mathbb{N}$ such that for any $\omega \in \Omega$ and any $Z \in Z_\omega^{(n)}$ we have that $T^M(n) Z = I$,
3. for each $\omega \in \Omega$, $Z \in Z_\omega$, and $x \in Z$
   (a) $T_\omega Z \in C^2$, ...
(b) there exists $K \geq 1$ such that
\[
\frac{|T''_\omega(x)|}{|T'_\omega(x)|} \leq K,
\]

(4) there exist $1 < \lambda \leq \Lambda < \infty$ and $n_1 \geq n_2 \in \mathbb{N}$ such that
(a) $|T'_\omega| \leq \Lambda$ for $m\text{-a.e. } \omega \in \Omega$,
(b) $(T''_\omega(x))' \geq \lambda^{n_1}$ for $m\text{-a.e. } \omega \in \Omega$
(c) $\frac{1}{n_2} \int_{\Omega} \log F^{(n_2)} dm(\omega) > t \log \frac{\Lambda}{\lambda}$,

(5) for each $n \in \mathbb{N}$ there exists
\[\varepsilon_n := \inf_{\omega \in \Omega} \min_{Z \in Z_n(\omega)} \text{diam}(Z) > 0.\]

Then Theorems 2.19-2.23 of [1] hold, and in particular, our assumptions (T1)-(T3), (LIP), (GP), (A1)-(A2), (M), and (C) hold.

The following lemma gives a large class of random Lasota-Yorke maps with holes for which our results apply. In particular, we allow our hole to be composed of finitely many intervals which may change depending on the fiber $\omega$, provided the number of connected components of the hole is log-integrable over $\Omega$ (CCH).

**Lemma 15.10.** Let $\varphi_{\omega,c} = -t \log |T'_\omega|$ and suppose the hypotheses of Lemma 15.9 hold. Additionally we suppose that $H \subseteq \Omega \times I$ such that (CCH) holds as well as the following:

(6) for $m\text{-a.e. } \omega \in \Omega$ there exists $Z \in Z_\omega$ with $Z \cap H_\omega = \emptyset$ such that $T_\omega(Z) = I$,
(7) $\frac{1}{n_2} \int_{\Omega} \log F^{(n_2)} dm(\omega) > t \log \frac{\Lambda}{\lambda} + \int_{\Omega} \log(\zeta^{(1)} + 2) dm(\omega)$.

Then the hypotheses of Theorems A-D hold. If in addition we have that

(8) there exists $M : \mathbb{N} \to \mathbb{N}$ such that $T_\omega^{M(n)}(Z) = I$ for $m\text{-a.e. } \omega \in \Omega$ and each $Z \in Z_\omega^{(n)}$, i.e. there is a uniform covering time,
(9) for $m\text{-a.e. } \omega \in \Omega$ there exists $Z_1, \ldots, Z_k \in Z_\omega$ such that $H_\omega = \bigcup_{j=1}^k Z_j$ and $T_\omega(Z) = I$ for all $Z \in Z_\omega$ such that $Z \cap H_\omega = \emptyset$,

then the hypotheses of Theorem E also hold.

**Proof.** The conclusion of Lemma 15.9 leaves only to check assumptions (D) and (Q0)-(Q3). But in light of Proposition 15.2 we see that (Q3) holds, and hypothesis (6) implies (D) (by Remark 4.5) and (Q0) hold.

To check our remaining hypotheses on the open system we first show that (Z') holds. To see this we note that equation (13.20) of [1], together with the fact that hypothesis (2) of Lemma 15.9 implies that $\Lambda^{-kn_1t} \leq g^{(kn_1)}_{\omega,c} \leq \lambda^{-kn_1t} < 1$, gives that for any $\omega \in \Omega$ and $Z \in Z_\omega^{(kn_1)}$ we have
\[
\text{var}_Z(g^{(kn_1)}_{\omega,c}) \leq 2 \|g^{(kn_1)}_{\omega,c}\|_\infty + \frac{tK}{\Lambda - 1} \left(\frac{\Lambda}{\lambda^t}\right)^{kn_1} \leq 2 \|g^{(kn_1)}_{\omega,c}\|_\infty + \frac{tK}{\Lambda - 1} \left(\frac{\Lambda}{\lambda^t}\right)^{kn_1} \cdot \Lambda^{kn_1t} \|g^{(kn_1)}_{\omega,c}\|_\infty.
\]
\[
\leq \hat{\alpha}_k \|g^{(k_n)}_{\omega,c}\|_\infty,
\]

where

\[
\hat{\alpha}_k := 2 + \frac{tK}{\Lambda - 1} \cdot \left( \frac{\Lambda^{2t}}{\lambda^t} \right)^{k_n_1}.
\]

Taking \(k_\ast\) so large that

\[
\int_\Omega \log Q^{(k_\ast n_1)}_{\omega} d\mu(\omega) < 0,
\]

where \(Q^{(k_n)}_{\omega}\) is defined in (6.12), and setting \(N_\ast = k_\ast n_1\), we then see that \((Z')\) holds, that is we have that \(Z^{(N_\ast)}_{\omega} \in \overline{\mathcal{M}^{(N_\ast)}_{\omega}}(\hat{\alpha}_{k_\ast})\) for each \(\omega \in \Omega\). Thus, Lemma 15.4 together with (CCH) ensures that \((Q2)\) holds. Now taking (7) in conjunction with Lemma 15.5 implies assumption \((Q1)\).

The second claim holds since the assumptions (8) and (9) together imply that \(T\) has large images and large images with respect to \(H\), and assumptions (3) and (4)(b) gives the bounded distortion condition for \(g_{\omega,c}\).

\[\square\]

**Remark 15.11.** If one wishes to work with general potentials rather than the geometric potentials in Lemmas 15.9 and 15.10 then one could replace (4) of Lemma 15.9 with (15.11) and (7) of Lemma 15.10 with Lemma 15.5.

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