CONFORMAL EDGE CURRENTS IN CHERN-SIMONS THEORIES

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ABSTRACT

We develop elementary canonical methods for the quantization of abelian and non-abelian Chern-Simons actions using well known ideas in gauge theories and quantum gravity. Our approach does not involve choice of gauge or clever manipulations of functional integrals. When the spatial slice is a disc, it yields Witten’s edge states carrying a representation of the Kac-Moody algebra. The canonical expression for the generators of diffeomorphisms on the boundary of the disc are also found, and it is established that they are the Chern-Simons version of the Sugawara construction. This paper is a prelude to our future publications on edge states, sources, vertex operators, and their spin and statistics in 3d and 4d topological field theories.
1. INTRODUCTION

The Chern-Simons or CS action describes a three-dimensional field theory of a connection $A_\mu$. In the absence of sources, the field equations require $A_\mu$ to be a zero curvature field and hence to be a pure gauge in simply connected spacetimes. As the dynamics is gauge invariant as well, it would appear that the CS action is an action for triviality in these spacetimes.

Such a conclusion however is not always warranted. Thus, for instance, it is of frequent interest to consider the CS action on a disc $D \times \mathbb{R}^1$ ($\mathbb{R}^1$ accounting for time) and in this case, as first emphasized by Witten [1], it is possible to contemplate a quantization which eliminates degrees of freedom only in the interior of $D$. In such a scheme, then, gauge transformations relate equivalent fields only in the interior of $D$ whereas on the boundary $\partial D$, they play a role more akin to global symmetry transformations. The residual states localized on the circular boundary $\partial D$ are the CS edge states. As they are associated with gauge transformations on $\partial D = \text{the circle } S^1$, it is natural to expect that the loop or the Kac-Moody group [2] of the gauge group will play a role in their description, the latter being a central extension of the former. Witten [1] in fact outlined an argument to show that the edge states form a conformal family carrying a representation of the Kac-Moody group.

Subsequent developments in the quantum theory of CS action have addressed both its formal [3, 4, 5, 7] and its physical [8, 9, 10] aspects. As regards the former, methods have been invented and refined for its fixed time quantization [3, 4, 5] and for the treatment of its functional integral [3, 4, 6]. They yield Witten’s results and extend them as well. An important achievement of all this research beginning in fact with Witten’s work is the reproduction of a large class of two-dimensional (2d) conformal field theories (CFT’s) from 3d CS theories.
There have been equally interesting developments which establish the significance of
the CS interaction for 2d condensed matter systems which go beyond phase transition
phenomena described by CFT’s [2]. It is now well appreciated for instance that the edge
states of the Fractional Quantum Hall Effect (FQHE) are well described by the CS theory
and its variants [8, 9] and that it is of fundamental importance in the theory of fractional
statistics [10]. Elsewhere, we will also describe its basic role in the theory of London
equations of 2d superconductors.

In this paper, we develop a canonical quantization of the CS action assuming for
simplicity that spacetime is a solid cylinder $D \times \mathbb{R}^1$. A notable merit of our approach is
that it avoids making a gauge choice or delicate manipulations of functional integrals. It
is furthermore based on ideas which are standard in field theories with constraints such as
QCD or quantum gravity [11] and admits easy generalizations, for example, to 4d gauge
theories. In subsequent papers, we will extend this approach to certain gauge field theories
(including the CS theory) with sources. We will establish that an anyon or a Laughlin
quasiparticle is not just a single particle, but is in reality a conformal family (a result
due to Witten [1]) and derive similar results in four dimensions. Simple considerations
concerning spin and statistics of these sources will also be presented using basic ideas
of European schools [12] on “fields localized in space–like cones” and generalizing them
somewhat. A brief account of our work has already appeared elsewhere [13].

In Section 2, we outline a canonical formalism for the U(1) CS action on $D \times \mathbb{R}^1$ and
its relation to certain old ideas in gauge theories or gravity. The observables are shown to
obey an algebra isomorphic to the U(1) Kac–Moody algebra on a circle [2]. The classical
canonical expression for the diffeomorphism (diffeo) generators on $\partial D$ are also found.

In Section 3, the observables are Fourier analyzed on $\partial D$. It is then discovered that the
CS diffeo generators are weakly the same as those obtained by the Sugawara construction
[2]. Quantization is then carried out in a conventional way to find that the edge states
and their observables describe a central charge 1 conformal family \[2\]. We next briefly illustrate our techniques by quantizing a generalized version of the CS action which has proved important in the theory of FQHE \[9\].

The paper concludes with Section 4 which outlines the nonabelian version of the foregoing considerations.

\section{2. THE CANONICAL FORMALISM}

The U(1) CS action on the solid cylinder \(D \times \mathbb{R}^1\) is

\[
S = \frac{k}{4\pi} \int_{D \times \mathbb{R}^1} AdA, \quad A = A_\mu dx^\mu, \quad AdA \equiv A \wedge dA
\]

where \(A_\mu\) is a real field.

The action \(S\) is invariant under diffeos of the solid cylinder and does not permit a natural choice of a time function. As time is all the same indispensable in the canonical approach, we arbitrarily choose a time function denoted henceforth by \(x^0\). Any constant \(x^0\) slice of the solid cylinder is then the disc \(D\) with coordinates \(x^1, x^2\).

It is well known that the phase space of the action \(S\) is described by the equal time Poisson brackets (PB’s)

\[
\{A_i(x), A_j(y)\} = \epsilon_{ij} \frac{2\pi}{k} \delta^2(x - y) \quad \text{for } i, j = 1, 2, \quad \epsilon_{12} = -\epsilon_{21} = 1
\]

(using the convention \(\epsilon^{012} = 1\) for the Levi-Civita symbol) and the constraint

\[
\partial_i A_j(x) - \partial_j A_i(x) \equiv F_{ij}(x) \approx 0
\]

where \(\approx\) denotes weak equality in the sense of Dirac \[11\]. All fields are evaluated at the same time \(x^0\) in these equations, and this will continue to be the case when dealing with the canonical formalism or quantum operators in the remainder of the paper. The connection \(A_0\) does not occur as a coordinate of this phase space. This is because, just
as in electrodynamics, its conjugate momentum is weakly zero and first class and hence eliminates \( A_0 \) as an observable.

The constraint (2.3) is somewhat loosely stated. It is important to formulate it more accurately by first smearing it with a suitable class of “test” functions \( \Lambda^{(0)} \). Thus we write, instead of (2.3),

\[
g(\Lambda^{(0)}) : = \frac{k}{2\pi} \int_D \Lambda^{(0)}(x)dA(x) \approx 0 . \tag{2.4}
\]

It remains to state the space \( \mathcal{L}^{(0)} \) of test functions \( \Lambda^{(0)} \). For this purpose, we recall that a functional on phase space can be relied on to generate well defined canonical transformations only if it is differentiable. The meaning and implications of this remark can be illustrated here by varying \( g(\Lambda^{(0)}) \) with respect to \( A_\mu \):

\[
\delta g(\Lambda^{(0)}) = \frac{k}{2\pi} \left( \int_{\partial D} \Lambda^{(0)} \delta A - \int_D d\Lambda^{(0)} \delta A \right) . \tag{2.5}
\]

By definition, \( g(\Lambda^{(0)}) \) is differentiable in \( A \) only if the boundary term – the first term – in (2.5) is zero. We do not wish to constrain the phase space by legislating \( \delta A \) itself to be zero on \( \partial D \) to achieve this goal. This is because we have a vital interest in regarding fluctuations of \( A \) on \( \partial D \) as dynamical and hence allowing canonical transformations which change boundary values of \( A \). We are thus led to the following condition on functions \( \Lambda^{(0)} \) in \( \mathcal{L}^{(0)} \):

\[
\Lambda^{(0)} \big|_{\partial D} = 0 . \tag{2.6}
\]

It is useful to illustrate the sort of troubles we will encounter if (2.6) is dropped. Consider

\[
q(\Lambda) = \frac{k}{2\pi} \int_D d\Lambda A \tag{2.7}
\]

It is perfectly differentiable in \( A \) even if the function \( \Lambda \) is nonzero on \( \partial D \). It creates fluctuations

\[
\delta A \big|_{\partial D} = d\Lambda \big|_{\partial D}
\]
of $A$ on $\partial D$ by canonical transformations. It is a function we wish to admit in our canonical approach. Now consider its PB with $g(\Lambda(0))$:

$$\{g(\Lambda^0), q(\Lambda)\} = \frac{k}{2\pi} \int d^2 x d^2 y \Lambda(0)(x) \epsilon^{ij} \left[ \partial_j \Lambda(y) \right] \left[ \frac{\partial}{\partial x^i} \delta^2(x - y) \right]$$  \hspace{1cm} (2.8)

where $\epsilon^{ij} = \epsilon_{ij}$. This expression is quite ill defined if

$$\Lambda(0)|_{\partial D} \neq 0.$$  

Thus integration on $y$ first gives zero for (2.8). But if we integrate on $x$ first, treating derivatives of distributions by usual rules, one finds instead,

$$- \int_D d\Lambda^0 d\Lambda = - \int_{\partial D} \Lambda^0 d\Lambda .$$  \hspace{1cm} (2.9)

Thus consistency requires the condition (2.6).

We recall that a similar situation occurs in QED. There, if $E_j$ is the electric field, which is the momentum conjugate to the potential $A_j$, and $j_0$ is the charge density, the Gauss law can be written as

$$\bar{g}(\bar{\Lambda}(0)) = \int d^3 x \bar{\Lambda}(0)(x) \left[ \partial_i E_i(x) - j_0(x) \right] \approx 0 .$$  \hspace{1cm} (2.10)

Since

$$\delta \bar{g}(\bar{\Lambda}(0)) = \int_{r=\infty} r^2 d\Omega \bar{\Lambda}(0)(x) \hat{x}_i \delta E_i - \int d^3 x \partial_i \bar{\Lambda}(0)(x) \delta E_i(x), r = | \hat{x} |, \hat{x} = \frac{\vec{x}}{r}$$  \hspace{1cm} (2.11)

for the variation $\delta E_i$ of $E_i$, differentiability requires

$$\bar{\Lambda}(0)(x)|_{r=\infty} = 0 .$$  \hspace{1cm} (2.12)

[$d\Omega$ in (2.11) is the usual volume form of the two sphere]. The charge, or equivalently the generator of the global U(1) transformations, incidentally is the analogue of $q(\Lambda)$. It is got by partial integration on the first term. Thus let

$$\bar{q}(\bar{\Lambda}) = - \int d^3 x \partial_i \bar{\Lambda}(x) E_i(x) - \int d^3 x \bar{\Lambda}(x) j_0(x) .$$  \hspace{1cm} (2.13)
This is differentiable in $E_i$ even if $\Lambda |_{r=\infty} \neq 0$ and generates the gauge transformation for the gauge group element $e^{i\Lambda}$. It need not to vanish on quantum states if $\Lambda |_{r=\infty} \neq 0$, unlike $\bar{g}(\Lambda^{(0)})$ which is associated with the Gauss law $\bar{g}(\Lambda^{(0)}) \approx 0$. But if $\Lambda |_{r=\infty} = 0$, it becomes the Gauss law on partial integration and annihilates all physical states. It follows that if $(\bar{A}_1 - \bar{A}_2) |_{r=\infty} = 0$, then $\bar{q}(\bar{A}_1) = \bar{q}(\bar{A}_2)$ on physical states which are thus sensitive only to the boundary values of test functions. The nature of their response determines their charge. The conventional electric charge of QED is $\bar{q}(\bar{1})$ where $\bar{1}$ is the constant function with value 1.

The constraints $g(\Lambda^{(0)})$ are first class since

$$
\{g(\Lambda_1^{(0)}), g(\Lambda_2^{(0)})\} = \frac{k}{2\pi} \int_D d\Lambda_1^{(0)} d\Lambda_2^{(0)} \\
= \frac{k}{2\pi} \int_{\partial D} \Lambda_1^{(0)} d\Lambda_2^{(0)} \\
= 0 \quad \text{for } \Lambda_1^{(0)}, \Lambda_2^{(0)} \in \mathcal{T}^{(0)}.
$$

(2.14)

$g(\Lambda^{(0)})$ generates the gauge transformation $A \rightarrow A + d\Lambda^{(0)}$ of $A$.

Next consider $q(\Lambda)$ where $\Lambda |_{\partial D}$ is not necessarily zero. Since

$$
\{q(\Lambda), g(\Lambda^{(0)})\} = -\frac{k}{2\pi} \int_D d\Lambda d\Lambda^{(0)} \\
= \frac{k}{2\pi} \int_{\partial D} \Lambda^{(0)} d\Lambda = 0 \quad \text{for } \Lambda^{(0)} \in \mathcal{T}^{(0)},
$$

(2.15)

they are first class or the observables of the theory. More precisely observables are obtained after identifying $q(\Lambda_1)$ with $q(\Lambda_2)$ if $(\Lambda_1 - \Lambda_2) \in \mathcal{T}^{(0)}$. For then,

$$
q(\Lambda_1) - q(\Lambda_2) = -g(\Lambda_1 - \Lambda_2) \approx 0.
$$

The functions $q(\Lambda)$ generate gauge transformations $A \rightarrow A + d\Lambda$ which do not necessarily vanish on $\partial D$.

It may be remarked that the expression for $q(\Lambda)$ is obtained from $g(\Lambda^{(0)})$ after a partial integration and a subsequent substitution of $\Lambda$ for $\Lambda^{(0)}$. It too generates gauge
transformations like \( g(\Lambda^{(0)}) \), but the test function space for the two are different. The pair \( q(\Lambda), g(\Lambda^{(0)}) \) thus resemble the pair \( \bar{q}(\bar{\Lambda}), \bar{g}(\bar{\Lambda}^{(0)}) \) in QED. The resemblance suggests that we think of \( q(\Lambda) \) as akin to the generator of a global symmetry transformation. It is natural to do so for another reason as well: the Hamiltonian is a constraint for a first order Lagrangian such as the one we have here, and for this Hamiltonian, \( q(\Lambda) \) is a constant of motion.

In quantum gravity, for asymptotically flat spatial slices, it is often the practice to include a surface term in the Hamiltonian which would otherwise have been a constraint and led to trivial evolution [14]. However, we know of no natural choice of such a surface term, except zero, for the CS theory.

The PB’s of \( q(\Lambda) \)'s are easy to compute:

\[
\{q(\Lambda_1), q(\Lambda_2)\} = \frac{k}{2\pi} \int_D d\Lambda_1 d\Lambda_2 = \frac{k}{2\pi} \int_{\partial D} \Lambda_1 d\Lambda_2 .
\]  
(2.16)

Remembering that the observables are characterized by boundary values of test functions, (2.16) shows that the observables generate a U(1) Kac-Moody algebra [2] localized on \( \partial D \). It is a Kac-Moody algebra for “zero momentum” or “charge”. For if \( \Lambda |_{\partial D} \) is a constant, it can be extended as a constant function to all of \( D \) and then \( q(\Lambda) = 0 \). The central charges and hence the representation of (2.16) are different for \( k > 0 \) and \( k < 0 \), a fact which reflects parity violation by the action \( S \).

Let \( \theta \) (mod \( 2\pi \)) be the coordinate on \( \partial D \) and \( \phi \) a free massless scalar field moving with speed \( v \) on \( \partial D \) and obeying the equal time PB’s

\[
\{\phi(\theta), \dot{\phi}(\theta')\} = \delta(\theta - \theta') .
\]  
(2.17)

If \( \mu_i \) are test functions on \( \partial D \) and \( \partial_{\mp} = \partial_{\pm} \phi = \pm v \partial_\theta \), then

\[
\left\{ \frac{1}{v} \int \mu_1(\theta) \partial_{\mp} \phi(\theta), \frac{1}{v} \int \mu_2(\theta) \partial_{\pm} \phi(\theta) \right\} = \pm 2 \int \mu_1(\theta) d\mu_2(\theta) ,
\]  
(2.18)

the remaining PB’s being zero. Also \( \partial_+ \partial_{\mp} \phi = 0 \). Thus the algebra of observables is isomorphic to that generated by the left moving \( \partial_+ \phi \) or the right moving \( \partial_- \phi \).
The CS interaction is invariant under diffeos of $D$. An infinitesimal generator of a
diffeo with vector field $V(0)$ is \[ \delta(V(0)) = -\frac{k}{2\pi} \int_D V^{(0)i} A_i dA. \tag{2.19} \]
The differentiability of $\delta(V(0))$ imposes the constraint
eq 0. \tag{2.20} \]
Hence, in view of (2.4) as well, we have the result
\[ \delta(V(0)) = -\frac{k}{4\pi} \int_D A\mathcal{L}_{V(0)} A \approx 0 \tag{2.21} \]
where $\mathcal{L}_{V(0)} A$ denotes the Lie derivative of the one form $A$ with respect to the vector field $V(0)$ and is given by
\[ (\mathcal{L}_{V(0)} A)_i = \partial_j A_i V^{(0)j} + A_j \partial_i V^{(0)j}. \]
Next, suppose that $V$ is a vector field on $D$ which on $\partial D$ is tangent to $\partial D$,
\[ V^i \mid_{\partial D} (\theta) = \epsilon(\theta) \left( \frac{\partial x^i}{\partial \theta} \right) \mid_{\partial D}, \tag{2.22} \]
$\epsilon$ being any function on $\partial D$ and $x^i \mid_{\partial D}$ the restriction of $x^i$ to $\partial D$. $V$ thus generates a
diffeo mapping $\partial D$ to $\partial D$. Consider next
\[ l(V) = \frac{k}{2\pi} \left( \frac{1}{2} \int_D d(V^i A_i A) - \int_D V^i A_i dA \right) \]
\[ = -\frac{k}{4\pi} \int_D A\mathcal{L}_V A. \tag{2.23} \]
Simple calculations show that $l(V)$ is differentiable in $A$ even if $\epsilon(\theta) \neq 0$ and generates
the infinitesimal diffeo of the vector field $V$. We show in the next Section that it is, in
fact, related to $g(\Lambda)'s$ by the Sugawara construction.

The expression (2.23) for the diffeo generators of observables seems to be new.

As final points of this Section, note that
\[ \{l(V), g(\Lambda^{(0)})\} = g(V^i \partial_i \Lambda^{(0)}) = g(\mathcal{L}_V \Lambda^{(0)}) \approx 0, \tag{2.24} \]
\[
\{ l(V), q(\Lambda) \} = q(V^i \partial_i \Lambda) = q(\mathcal{L}_V \Lambda),
\]
\[
\{ l(V), l(W) \} = l(\mathcal{L}_V W)
\]

where \( \mathcal{L}_V W \) denotes the Lie derivative of the vector field \( W \) with respect to the vector field \( V \) and is given by

\[
(\mathcal{L}_V W)^i = V^j \partial_j W^i - W^j \partial_j V^i.
\]

\( l(V) \) are first class in view of (2.24). Further, after the imposition of constraints, they are entirely characterized by \( \epsilon(\theta) \), the equivalence class of \( l(V) \) with the same \( \epsilon(\theta) \) defining an observable.

3. QUANTIZATION

Our strategy for quantization relies on the observation that if

\[
\Lambda |_{\partial D} (\theta) = e^{iN\theta},
\]

then the PB’s (2.16) become those of creation and annihilation operators. These latter can be identified with the similar operators of the chiral fields \( \partial_\pm \phi \).

Thus let \( \Lambda_N \) be any function on \( D \) with boundary value \( e^{iN\theta} \):

\[
\Lambda_N |_{\partial D} (\theta) = e^{iN\theta}, \quad N \in \mathbb{Z}.
\]

These \( \Lambda_N \)’s exist. All \( q(\Lambda_N) \) with the same \( \Lambda_N |_{\partial D} \) are weakly equal and define the same observable. Let \( \langle q(\Lambda_N) \rangle \) be this equivalence class and \( q_N \) any member thereof. [\( q_N \) can also be regarded as the equivalence class itself.] Their PB’s follow from (2.16):

\[
\{ q_N, q_M \} = -iNk\delta_{N+M,0}.
\]

The \( q_N \)’s are the CS constructions of the Fourier modes of a massless chiral scalar field on \( S^1 \).
The CS construction of the diffeo generators \( l_N \) on \( \partial D \) (the classical analogues of the Virasoro generators) are similar. Thus let

\[
< l(V_N) >
\]

be the equivalence class of \( l(V_N) \) defined by the constraint

\[
V_N^i|_{\partial D} = e^{iN\theta} \left( \frac{\partial x^i}{\partial \theta} \right) |_{\partial D}, \quad N \in \mathbb{Z},
\]

\( (x^1, x^2) |_{\partial D} (\theta) \) being chosen to be \( R(\cos \theta, \sin \theta) \) where \( R \) is the radius of \( D \). Let \( l_N \) be any member of

\[
< l(V_N) >.
\]

It can be verified that

\[
\{ l_N, q_M \} = iMq_{N+M}, \quad (3.4)
\]

\[
\{ l_N, l_M \} = -i(N - M) l_{N+M}. \quad (3.5)
\]

These PB’s are independent of the choice of the representatives from their respective equivalence classes. Equations (3.2), (3.4) and (3.5) define the semidirect product of the Kac-Moody algebra and the Witt algebra (Virasoro algebra without the central term) in its classical version.

We next show that

\[
l_N \approx \frac{1}{2k} \sum_M q_M q_{N-M}, \quad (3.6)
\]

which is the classical version of the Sugawara construction [4].

For convenience, let us introduce polar coordinates \( r, \theta \) on \( D \) (with \( r = R \) on \( \partial D \)) and write the fields and test functions as functions of polar coordinates. It is then clear that

\[
l_N \equiv l(V_N) = \frac{k}{4\pi} \int_{\partial D} d\theta e^{iN\theta} A^2_\theta(R, \theta) - \frac{k}{2\pi} \int_D V_N^l(r, \theta) A_l(r, \theta) dA(r, \theta)
\]

where \( A = A_r dr + A_\theta d\theta \).
Let us next make the choice

\[ e^{jM\theta} \lambda(r), \quad \lambda(0) = 0, \quad \lambda(R) = 1 \]  

(3.8)

for \( \Lambda_M \). Then

\[ q_M = q(e^{jM\theta} \lambda(r)). \]  

(3.9)

Integrating (3.9) by parts, we get

\[ q_M = \frac{k}{2\pi} \left( \int_{\partial D} d\theta e^{jM\theta} A_\theta(R, \theta) - \int_D dr d\theta \lambda(r) e^{jM\theta} F_{r\theta}(r, \theta) \right) \]  

(3.10)

where \( F_{r\theta} \) is defined by \( dA = F_{r\theta} dr \wedge d\theta \). Therefore

\[ \frac{1}{2k} \sum_M q_M q_{N-M} = + \frac{k}{4\pi} \int_{\partial D} d\theta e^{jN\theta} A_\theta^2(R, \theta) \]

\[ - \frac{k}{2\pi} \int_D dr d\theta e^{jN\theta} \lambda(r) A_\theta(R, \theta) F_{r\theta}(r, \theta) \]

\[ + \frac{k}{4\pi} \int_D dr d\theta dr' \lambda(r) \lambda(r') e^{jN\theta} F_{r\theta}(r, \theta) F_{r'\theta}(r', \theta) \]  

(3.11)

where the completeness relation

\[ \sum_N e^{jN(\theta - \theta')} = 2\pi \delta(\theta - \theta') \]

has been used.

The test functions for the Gauss law in the last term in (3.11) involves \( F_{r\theta} \) itself. We therefore interpret it to be zero and get

\[ \frac{1}{2k} \sum_M q_M q_{N-M} \approx \frac{k}{4\pi} \int_{\partial D} e^{jN\theta} A_\theta^2(R, \theta) d\theta - \frac{k}{2\pi} \int_D dr d\theta e^{jN\theta} \lambda(r) A_\theta(R, \theta) F_{r\theta}(r, \theta). \]  

(3.12)

Now in view of (3.3) and (3.8), it is clear that

\[ V_N^l(r, \theta) A_l(r, \theta) - e^{jN\theta} \lambda(r) A_\theta(R, \theta) = 0 \quad \text{on} \quad \partial D. \]  

(3.13)

Therefore

\[ l_N \approx \frac{1}{2k} \sum_M q_M q_{N-M} \]
which proves (3.6).

We can now proceed to quantum field theory. Let $G(\Lambda(0)), Q(\Lambda_N), Q_N$ and $L_N$ denote the quantum operators for $g(\Lambda(0)), q(\Lambda_N), q_N$ and $l_N$. We then impose the constraint

$$G(\Lambda(0))|\cdot\rangle = 0 \quad (3.14)$$

on all quantum states. It is an expression of their gauge invariance. Because of this equation, $Q(\Lambda_N)$ and $Q(\Lambda'_N)$ have the same action on the states if $\Lambda_N$ and $\Lambda'_N$ have the same boundary values. We can hence write

$$Q_N|\cdot\rangle = Q(\Lambda_N)|\cdot\rangle. \quad (3.15)$$

Here, in view of (3.2), the commutator brackets of $Q_N$ are

$$[Q_N, Q_M] = Nk\delta_{N+M,0}. \quad (3.16)$$

Thus if $k > 0 \ (k < 0)$, $Q_N$ for $N > 0 \ (N < 0)$ are annihilation operators (up to a normalization) and $Q_{-N}$ creation operators. The "vacuum" $|0\rangle$ can therefore be defined by

$$Q_N | 0 > = 0 \text{ if } Nk > 0. \quad (3.17)$$

The excitations are got by applying $Q_{-N}$ to the vacuum.

The quantum Virasoro generators are the normal ordered forms of their classical expression [2] :

$$L_N = \frac{1}{2k} : \sum_M Q_M Q_{N-M} : \quad (3.18)$$

They generate the Virasoro algebra for central charge $c = 1$ :

$$[L_N, L_M] = (N - M)L_{N+M} + \frac{c}{12}(N^3 - N)\delta_{N+M,0}, \ c = 1. \quad (3.19)$$

When the spatial slice is a disc, the observables are all given by $Q_N$ and our quantization is complete. When it is not simply connected, however, there are further observables
associated with the holonomies of the connection $A$ and they affect quantization. We will not examine quantization for nonsimply connected spatial slices here.

The CS interaction does not fix the speed $v$ of the scalar field in (2.18) and so its Hamiltonian, a point previously emphasized by Frohlich and Kerler [8] and Frohlich and Zee [9]. This is but reasonable. For if we could fix $v$, the Hamiltonian $H$ for $\phi$ could naturally be taken to be the one for a free massless chiral scalar field moving with speed $v$. It could then be used to evolve the CS observables using the correspondence of this field and the former. But we have seen that no natural nonzero Hamiltonian exists for the CS system. It is thus satisfying that we can not fix $v$ and hence a nonzero $H$.

In the context of Fractional Quantum Hall Effect, the following generalization of the CS action has become of interest [9]:

$$S' = \frac{k}{4\pi} K_{IJ} \int_{D \times \mathbb{R}^1} A^{(I)} dA^{(J)}. \quad (3.20)$$

Here the sum on $I, J$ is from 1 to $F$, $A^{(I)}$ is associated with the current $j^{(I)}$ in the $I^{th}$ Landau level and $K$ is a certain invertible symmetric real $F \times F$ matrix. By way of further illustration of our approach to quantization, we now outline the quantization of (3.20) on $D \times \mathbb{R}^1$.

The phase space of (3.20) is described by the PB’s

$$\{A_i^{(I)}(x), A_j^{(J)}(y)\} = \epsilon_{ij} \frac{2\pi}{k} K^{-1}_{IJ} \delta^2(x - y), \quad x^0 = y^0 \quad (3.21)$$

and the first class constraints

$$g^{(I)}(\Lambda^{(0)}) = \frac{k}{2\pi} \int_D \Lambda^{(0)} dA^{(I)} \approx 0, \quad \Lambda^{(0)} \in \mathcal{C}^{(0)}. \quad (3.22)$$

with zero PB’s.

The observables are obtained from the first class variables

$$q^{(I)}(\Lambda) = \frac{k}{2\pi} \int_D d\Lambda A^{(I)} \quad (3.23)$$
after identifying $q^{(I)}(\Lambda)$ with $q^{(I)}(\Lambda')$ if $(\Lambda - \Lambda') |_{\partial D} = 0$. The PB’s of $q^{(I)}$’s are

$$\{q^{(I)}(\Lambda_1^{(I)}), q^{(J)}(\Lambda_2^{(J)})\} = \frac{k}{2\pi} \mathcal{K}_{IJ}^{-1} \int_{\partial D} \Lambda_1^{(I)} d\Lambda_2^{(J)}. \quad (3.24)$$

Choose a $\Lambda_N^{(I)}$ by the requirement $\Lambda_N^{(I)} |_{\partial D} (\theta) = e^{iN\theta}$ and let $q_N^{(I)}$ be any member of the equivalence class $< q^{(I)}(\Lambda_N^{(I)}) >$ characterized by such $\Lambda_N^{(I)}$. Then

$$\{q_N^{(I)}, q_M^{(J)}\} = -i\mathcal{K}_{IJ}^{-1} N k \delta_{N+M,0}. \quad (3.25)$$

As $\mathcal{K}_{IJ}^{-1}$ is real symmetric, it can be diagonalized by a real orthogonal transformation $M$ and has real eigenvalues $\lambda_\rho (\rho = 1, 2, ..., F)$. As $\mathcal{K}_{IJ}^{-1}$ is invertible, $\lambda_\rho \neq 0$. Setting

$$q_N(\rho) = M_\rho q_N^{(I)} \quad (3.26)$$

we have

$$\{q_N(\rho), q_M(\sigma)\} = -i\lambda_\rho N k \delta_{\rho\sigma} \delta_{N+M,0}. \quad (3.27)$$

(3.27) is readily quantized. If $Q_N(\rho)$ is the quantum operator for $q_N(\rho)$,

$$[Q_N(\rho), Q_M(\sigma)] = \lambda_\rho N k \delta_{\rho\sigma} \delta_{N+M,0}. \quad (3.28)$$

(3.28) describes $F$ harmonic oscillators or edge currents. Their chirality, or the chirality of the corresponding massless scalar fields, is governed by the sign of $\lambda_\rho$.

The classical diffeo generators for the independent oscillators $q_N(\rho)$ and their quantum versions can be written down using the foregoing discussion. The latter form $F$ commuting Virasoro algebras, all for central charge 1.

### 4. THE NONABELIAN CHERN-SIMONS ACTION

Let $G$ be a compact simple group with Lie algebra $\mathfrak{g}$. Let $\gamma$ be a faithful representation of $\mathfrak{g}$. Choose a hermitian basis $\{T_\alpha\}$ for $\gamma$ (more precisely $i\gamma$) with normalization
\[ T \alpha T \beta = \delta_{\alpha \beta}. \] Let \( A_\mu \) define an antihermitean connection for \( G \) with values in \( \gamma \). We define the real field \( A_\mu^\alpha \) by \( A_\mu = i A_\mu^\alpha T_\alpha \). With these conventions, the Chern-Simons action for \( A_\mu \) on \( D \times \mathbb{R}^1 \) is

\[
S = -\frac{k}{4\pi} \int_{D \times \mathbb{R}^1} Tr \left[ AdA + \frac{2}{3} A_3 \right], \quad A = A_\mu dx^\mu
\]

where the constant \( k \) can assume only quantized values for well known reasons. If \( G = SU(N) \) and \( \gamma \) the Lie algebra of its defining representation, then \( k \in \mathbb{Z} \).

Much as for the Abelian problem, the phase space for (4.1) is described by the PB’s

\[
\{ A_i^\alpha(x), A_j^\beta(y) \} = \delta_{\alpha \beta} \epsilon_{ij} \frac{2\pi}{k} \delta^2(x - y), \quad x^0 = y^0
\]

and the Gauss law

\[
g(\Lambda^{(0)}) = -\frac{k}{2\pi} \int_D Tr \left\{ \Lambda^{(0)}(dA + A^2) \right\} = -\frac{k}{2\pi} \int_D Tr \left( \Lambda^{(0)} F \right) \approx 0
\]

where \( F = F_{ij} dx^i dx^j \) is the curvature of \( A \), \( \Lambda^{(0)} = i \Lambda^{(0)}\alpha T_\alpha \) and \( \Lambda^{(0)}\alpha \in \mathcal{T}^{(0)} \). This test function space for \( \Lambda^{(0)} \) ensures that \( g(\Lambda^{(0)}) \) is differentiable in \( A_i^\alpha \). The PB’s between \( g \)'s are

\[
\{ g(\Lambda_1^{(0)}), g(\Lambda_2^{(0)}) \} = g(\{ \Lambda_1^{(0)}, \Lambda_2^{(0)} \}) - \frac{k}{2\pi} \int_{\partial D} Tr \Lambda_1^{(0)} d\Lambda_2^{(0)} = g(\{ \Lambda_1^{(0)}, \Lambda_2^{(0)} \})
\]

so that they are first class constraints.

Next define

\[
q(\Lambda) = \frac{k}{2\pi} \int_D Tr (-d\Lambda A + \Lambda A^2), \quad \Lambda = i \Lambda^{\alpha} T_\alpha.
\]

It is differentiable in \( A_i^\alpha \) even if \( \Lambda|_{\partial D} \neq 0 \). But if \( \Lambda|_{\partial D} \) is zero, it is equal to the Gauss law \( g(\Lambda) \). Further, \( q(\Lambda) \) is first class for any choice of \( \Lambda \) since

\[
\{ q(\Lambda), g(\Lambda^{(0)}) \} = -g(\{ \Lambda, \Lambda^{(0)} \}) \approx 0.
\]

Thus (with \( \Lambda|_{\partial D} \) free), \( q(\Lambda) \)'s define observables, the latter being the same if their test functions are equal on \( \partial D \).
The PB’s of \( q(\Lambda) \)'s are

\[
\{q(\Lambda_1), q(\Lambda_2)\} = -q([\Lambda_1, \Lambda_2] - \frac{k}{2\pi} \int_{\partial D} Tr \ (\Lambda_1 d\Lambda_2) \tag{4.7}
\]

which can be recognized as a Kac-Moody algebra for observables.

The diffeo generators can also be constructed following Section 3. The generators of diffeos which keep \( \partial D \) fixed and vanish weakly are

\[
\delta(V(0)) = \frac{k}{2\pi} \int_D V^{(0)i} Tr A_i F, \quad V^{(0)i}|_{\partial D} = 0 , \tag{4.8}
\]

while those generators which also perform diffeos of \( \partial D \) are

\[
l(V) = \frac{k}{2\pi} \left( \int_D V^i Tr A_i F - \frac{1}{2} \int_D d(V^i Tr A_i A) \right)
= \frac{k}{4\pi} \int_D Tr A \mathcal{L}_V A \tag{4.9}
\]

where \( V^i|_{\partial D}(\theta) = \epsilon(\theta) \left( \frac{\partial V^i}{\partial \theta} \right)|_{\partial D} \). The PB’s involving \( l(V) \) are patterned after (2.24–2.26):

\[
\{l(V), g(\Lambda^{(0)})\} = g(V^i \partial_i \Lambda^{(0)}) = g(\mathcal{L}_V \Lambda^{(0)}) \approx 0 , \tag{4.10}
\]

\[
\{l(V), q(\Lambda)\} = q(V^i \partial_i \Lambda) = q(\mathcal{L}_V \Lambda) , \tag{4.11}
\]

\[
\{l(V), l(W)\} = l(\mathcal{L}_V W) . \tag{4.12}
\]

We can now conclude that \( l(V) \) are first class and define observables, all \( V \) with the same \( \epsilon(\theta) \) leading to the same observable.

Let \( \Lambda_N^\alpha \) be any test function with the feature \( \Lambda_N^\alpha|_{\partial D} = e^{iN\theta}T_\alpha \) and let \( V_N^i \) be defined following Section 3. As in that Section, let us call the set of first class variables weakly equal to \( q(i\Lambda_N^\alpha T_\alpha) \) and \( l(V_N) \) by \( \langle q(i\Lambda_N^\alpha T_\alpha) \rangle \) and \( \langle l(V_N) \rangle \). [ Here there is no sum over \( \alpha \) in \( i\Lambda_N^\alpha T_\alpha \)]. Let \( q_N^\alpha \) and \( l_N \) be any member each from these sets. Their PB’s are

\[
\{q_N^\alpha, q_M^\beta\} \approx f_{\alpha\beta\gamma} q_{N+M}^\gamma - iNk\delta_{N+M,0} \delta_{\alpha\beta} , \tag{4.13}
\]

\[
\{l_N, q_M^\alpha\} \approx iM q_N^\alpha , \tag{4.14}
\]

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\{l_N, l_M\} \approx -i(N - M)l_{N+M}, \quad (4.15)

\[ f_{\alpha\beta\gamma} \text{ being defined by } [T_\alpha, T_\beta] = if_{\alpha\beta\gamma}T_\gamma. \] Furthermore, as in Section 3,

\[ l_N \approx \frac{1}{2k} \sum_{M,\alpha} q_M^\alpha q_{N-M}^\alpha. \quad (4.16)\]

We next go to quantum field theory. In quantum theory, the operators for \(g(\Lambda(0)), q_N^\alpha\) and \(l_N\) are denoted by \(\mathcal{G}(\Lambda(0)), Q_N^\alpha, L_N\) and all states are subjected to the Gauss law

\[ \mathcal{G}(\Lambda(0))|> = 0. \quad (4.17)\]

As a consequence, all the weak equalities can be regarded as strong for the quantum operators. We are thus dealing with a Kac-Moody algebra for a certain level \([2]\). A suitable highest weight representation for it can be constructed in the usual way \([2]\), thereby defining the quantum theory. The expression for the Virasoro generators normalized to fulfill the commutation relations (3.16) is not the normal ordered version of (4.16), but as is well known, it is

\[ L_N = \frac{1}{2k + c_V} \sum_{M,\alpha} : Q_M^\alpha Q_{N-M}^\alpha :, \quad (4.18)\]

\((c_V\) being the quadratic Casimir operator in the adjoint representation). The central charge \(c\) now is not of course 1, but rather,

\[ c = \frac{2k \dim G}{2k + c_V}, \quad \dim G \equiv \text{dimension of } G. \quad (4.19)\]

These results about the Kac-Moody and Virasoro algebras are explained in ref. 2.

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