SMALL DILATATION PSEUDO-ANOSOV MAPPING CLASSES COMING FROM THE SIMPLEST HYPERBOLIC BRAID.

ERIKO HIRONAKA

Abstract. In this paper we study the minimum dilatation pseudo-Anosov mapping classes arising from fibrations over the circle of a single 3-manifold, namely the mapping torus for the "simplest hyperbolic braid". The dilatations that occur include the minimum dilatations for orientable pseudo-Anosov mapping classes for genus $g = 2, 3, 4, 5,$ or $8$. In particular, we obtain the "Lehmer example" in genus $g = 5$, and Lanneau and Thiffeault's conjectural minima in the orientable case for all genus $g$ satisfying $g = 2$ or $4(\text{mod} \ 6)$. Our examples show that the minimum dilatation for orientable mapping classes is strictly greater than the minimum dilatation for non-orientable ones when $g = 4, 6,$ or $8$. We also prove that if $\delta_g$ is the minimum dilatation of pseudo-Anosov mapping classes on a genus $g$ surface, then

$$\limsup_{g \to \infty} (\delta_g)^g \leq \frac{3 + \sqrt{5}}{2}.$$ 

1. Introduction

Let $S_g$ be a closed oriented surface of genus $g \geq 1$, and let $\text{Mod}_g$ be the mapping class group, the group of orientation preserving homeomorphisms of $S_g$ to itself up to isotopy. A mapping class $\phi \in \text{Mod}_g$ is called pseudo-Anosov if $S_g$ admits a pair of $\phi$-invariant, transverse measured, singular foliations on which $\phi$ acts by stretching transverse to one foliation by a constant $\lambda(\phi) > 1$ and contracting transverse to the other by $\lambda(\phi)^{-1}$. The constant $\lambda(\phi)$ is called the (geometric) dilatation of $\phi$. A mapping class is pseudo-Anosov if it is neither periodic nor reducible [Thu2, FLP, CB]. Denote by $\text{Mod}^{pA}_g$ the set of pseudo-Anosov mapping classes in $\text{Mod}_g$.

A pseudo-Anosov mapping class $\phi$ is defined to be orientable if its invariant foliations are orientable. We will denote the set of orientable pseudo-Anosov mapping classes by $\text{Mod}^{pA+}_g$. Let $\lambda_{\text{hom}}(\phi)$ be the spectral radius of the action of $\phi$ on the first homology of $S$. Then

$$\lambda_{\text{hom}}(\phi) \leq \lambda(\phi),$$

with equality if and only if $\phi$ is orientable (see, for example, [LT] (p. 5) and [KS] (Theorem 1.4)).

The dilatations $\lambda(\phi)$, for $\phi \in \text{Mod}^{pA}_g$ satisfy reciprocal monic integer polynomials of degree bounded from above by $6g - 6$ [Thu2]. If $\phi$ is orientable the degree is bounded by $2g$. For fixed $g$, it follows that $\lambda(\phi)$ achieves a minimum $\delta_g > 1$ on $\text{Mod}^{pA}_g$ (see also, [AY, Iva]). Let $\delta_g^+$ be the minimum dilatation among orientable elements of $\text{Mod}^{pA+}_g$.

In this paper, we address the following question (cf. [Pen, McM1, Far]):

**Question 1.1.** What is the behavior of $\delta_g$ and $\delta_g^+$ as functions of $g$?
So far, exact values of $\delta_g$ have only been found for $g \leq 2$. For $g = 1$, $\text{Mod}_1 = \text{SL}(2; \mathbb{Z})$, and
\[
\delta_1 = \frac{3 + \sqrt{5}}{2}.
\]
For $g = 2$, Cho and Ham [CH] show that $\delta_2$ is the largest real root of
\[
t^4 - t^3 - t^2 - t + 1 = 0
\]
($\approx 1.72208$).

In the orientable case more is known due to recent results of Lanneau and Thiffeault [LT]. Given $(a, b) \in \mathbb{Z} \oplus \mathbb{Z}$ with $0 < a < b$, let
\[
\text{LT}_{(a, b)}(t) = t^{2b} - t^b(1 + x^a + x^{-a}) + 1,
\]
and let $\lambda_{(a, b)}$ be the largest real root of $\text{LT}_{(a, b)}(t)$.

**Theorem 1.2** (Lanneau-Thiffeault [LT] Theorems 1.2 and 1.3). For $g = 2, 3, 4, 6, \text{ and } 8,$
\[
\lambda_{(1,g)} \leq \delta_g^+
\]
with equality when $g = 2, 3, \text{ or } 4$.

For $g = 2$, the value of $\delta_g^+$ was first determined by Zhirov [Zhi]. For $g = 5$, Lanneau and Thiffeault show that $\delta_5^+$ equals Lehmer’s number ($\approx 1.17628$) [Leh]. This dilatation is realized as a product of multi-twists along a curve arrangement dual to the $E_{10}$ Coxeter graph (see [Lei, Hir2]), and as the monodromy of the (-2,3,7)-pretzel knot (see [Hir1]). Lanneau and Thiffeault also find a lower bound for $\delta_7^+$. An example realizing this bound can be found in [AD] (p.4) and [KT2] (Theorem 1.12).

Based on their results, Lanneau and Thiffeault ask:

**Question 1.3** ([LT] Question 6.1). Is $\delta_g^+ = \lambda_{(1,g)}$ for all even $g$?

For convenience, we will call the affirmative answer to their question the LT-conjecture.

In our first result, we improve on the following previous best bounds for the minimum dilatation of infinite families
\[
(\delta_g)^g \leq (\delta_g^+)^g \leq 2 + \sqrt{3}
\]
found in [Min, HK].

**Theorem 1.4.** If $g = 0, 1, 3, \text{ or } 4(\text{mod } 6)$, $g \geq 3$, then
\[
\delta_g \leq \lambda_{(3,g+1)},
\]
and if $g = 2, \text{ or } 5(\text{mod } 6)$ and $g \geq 5$, then
\[
\delta_g \leq \lambda_{(1,g+1)}.
\]

For the orientable case, our results complement those of Lanneau and Thiffeault for $g = 2, \text{ or } 4(\text{mod } 6)$.

**Theorem 1.5.** Let $g \geq 3$. Then
\[
\delta_g^+ \leq \lambda_{(3,g+1)} \quad \text{if } g = 1, \text{ or } 3(\text{mod } 6),
\delta_g^+ \leq \lambda_{(1,g)} \quad \text{if } g = 2, \text{ or } 4(\text{mod } 6), \text{ and}
\delta_g^+ \leq \lambda_{(1,g+1)} \quad \text{if } g = 5(\text{mod } 6).
Putting Theorem 1.5 together with Lanneau and Thiffeault’s lower bound for $g = 8$ gives:

**Corollary 1.6.** For $g = 8$, we have

$$\delta_8^+ = \lambda_{(1,8)}.$$ 

The following is a table of the minimal dilatations that arise in this paper’s examples for genus 1 through 12. All numbers in the table are truncated to 5 decimal places. An asterisk * marks the numbers that have been verified to equal $\delta_g^+$ (resp., $\delta_g^-$). For singularity-type, we use the convention that $(a_1, \ldots, a_k)$ means that the singularities of the invariant foliations have degrees $a_1, \ldots, a_k$ (see Lanneau and Thiffeault’s notation [LT] p.3). The singularity-types for our examples are derived from the formula given in Proposition 3.5.

| $g$ | orientable | degrees of singularities | unconstrained | degrees of singularities |
|-----|-------------|--------------------------|---------------|--------------------------|
| 1   | 2.61803*    | no sing.                 | 2.61803*      | no sing.                 |
| 2   | 1.72208*    | (4)                      | 1.72208*      | (4)                      |
| 3   | 1.40127*    | (2, 2, 2, 2)             | 1.40127       | (2, 2, 2, 2)             |
| 4   | 1.28064*    | (10, 2)                  | 1.26123       | (3, 3, 3)                |
| 5   | 1.17628*    | (16)                     | 1.17628       | (16)                     |
| 6   | -           |                           | 1.1617        | (5, 5, 5, 5)             |
| 7   | 1.13694     | (6, 6, 6, 6)             | 1.13694       | (6, 6, 6, 6)             |
| 8   | 1.12876*    | (22, 6)                  | 1.1135        | (25, 1, 1, 1)            |
| 9   | 1.1054      | (8, 8, 8, 8)             | 1.1054        | (8, 8, 8)                |
| 10  | 1.10149     | (28, 8)                  | 1.09466       | (9, 9, 9)                |
| 11  | 1.08377     | (34, 2, 2, 2)            | 1.08377       | (34, 2, 2, 2)            |
| 12  | -           |                           | 1.07874       | (11, 11, 11, 11)         |

Table 1: Minimal orientable and unconstrained dilatations coming from $M_{ab}$

For $g = 2, 3, 4$, and 5, our orientable examples agree both in dilatation and in singularity-type with the previously known minimizing examples (see [LT] §§3, §4, §6). For $g = 8$, our example agrees with the singularity-type anticipated by Lanneau and Thiffeault [LT] (6.4). We prove that the known minimal dilatation examples for $g = 2, 3, 4, 5$, and 8 arise as the monodromy of fibrations of a single 3-manifold $M_{ab}$. For $g = 7$, our minimal example gives a strictly larger dilatation than $\delta_7^+$. (The dilatation $\delta_7^+$ is realized in [KT2] and [AD].)

Lanneau and Thiffeault show that $\delta_5^+ \leq \delta_6^+$, and hence $\delta_g^+$ is not strictly monotone decreasing (cf. [Far] Question 7.2). Theorem 1.5 implies the following stronger statement.

**Proposition 1.7.** If the LT-conjecture is true, then $\delta_g^+ \leq \delta_{g+1}^+$, whenever $g = 5(\text{mod } 6)$.

Another consequence concerns the question of whether the inequality $\delta_g \leq \delta_g^+$ is strict for any or all $g$. In [KT2] and [AD] it is shown that $\delta_5 < \delta_5^+$. Table 1 shows the following.

**Corollary 1.8.** For $g = 4, 6$, and 8 we have

$$\delta_g < \delta_g^+.$$ 

If the LT-conjecture is true, then Theorem 1.4 and Proposition 4.4 imply that the phenomena revealed in Corollary 1.8 repeats itself periodically.
Proposition 1.9. If the LT-conjecture is true, then for all even $g \geq 4$ we have
\[\delta_g < \delta_g^+.\]

For large $g$, it is known that $\delta_g$ and $\delta_g^+$ converges to 1. Furthermore,
\[\log(\delta_g) \asymp \frac{1}{g}\] and \[\log(\delta_g^+) \asymp \frac{1}{g}\] (see [Pen, McM1, Min, HK]). The LT-conjecture together with (1) leads to the natural question:

**Question 1.10** ([McM1] p.551, [Far] Problem 7.1). Do the sequences $(\delta_g)^g$ and $(\delta_g^+)^g$ converge as $g$ grows? What is the limit?

Theorem 1.11 and Theorem 1.4 imply the following.

**Theorem 1.11.**
\[\limsup_{g \to \infty} (\delta_g)^g \leq \frac{3 + \sqrt{5}}{2}\] and
\[\limsup_{g \not\equiv 0 \pmod{6}} (\delta_g^+)^g \leq \frac{3 + \sqrt{5}}{2}.\]

This leads to the following question.

**Question 1.12** (Golden Mean Question). Do the sequences $(\delta_g)^g$ and $(\delta_g^+)^g$ satisfy
\[\lim_{g \to \infty} (\delta_g)^g = \lim_{g \to \infty} (\delta_g^+)^g = \frac{3 + \sqrt{5}}{2} = 1 + \text{golden mean}?\]

For any pseudo-Anosov mapping class $\phi$, let $M(\phi)$ be the mapping torus of $\phi$. Conversely, given a compact hyperbolic 3-manifold with torus boundary components $M$, let $\Phi(M)$ be the collection of pseudo-Anosov mapping classes $\phi$ such that $M = M(\phi)$.

We prove Theorem 1.4 and Theorem 1.5 by exhibiting a family of mapping classes $\phi(a,b) \in \Phi(M_{sb})$ for a single hyperbolic 3-manifold $M_{sb}$.

Let $\Sigma$ be the suspensions of singularities of the stable and unstable foliations of $\phi$ and let $M^*(\phi) = M(\phi) \setminus \Sigma$.

**Theorem 1.13** ([FLM] Theorem 1.1). The set
\[T_P = \{M^*(\phi) : \phi \in \text{Mod}_{PA}^P, \lambda(\phi) \leq P\frac{1}{2}\}\] is finite for any $P > 1$.

The asymptotic equations (1) and Theorem 1.13 imply that
\[T = \{M^*(\phi) : \phi \in \text{Mod}_{PA}^A, \lambda(\phi) = \delta_g\}\] and
\[T^+ = \{M^*(\phi) : \phi \in \text{Mod}_{PA}^{A+}, \lambda(\phi) = \delta_g^+\}\] are finite.

This invites the question:
Question 1.14. How large are the sets $T$ and $T^+$?

If the LT-conjecture is true, then our results imply that a single 3-manifold would realize $\delta_g^+$ for all $g = 2, 4(\text{mod } 6)$. The manifold we study in this paper $M_{sb}$ is the complement of the $6_2^2$ braid (see Rolfsen’s tables [Rol], and Figure 1). Another 3-manifold that produces small dilatation mapping classes is the complement $M_{-2,3,8}$ of the $(-2,3,8)$-pretzel link in $S^3$. These have been studied independently by Kin and Takasawa [KT2] and Aaber and Dunfield [AD]. For certain genera the mapping classes in $\Phi(M_{-2,3,8})$ have smaller dilatation than the minima realized by $M_{sb}$, but the asymptotic behavior of the minimal dilatations for large genus, supports the affirmative to Question 1.12. Both $M_{-2,3,8}$ and $M_{sb}$ can be obtained from the magic manifold [MP]. The pseudo-Anosov braid monodromies with smallest known dilatations found in [HK] are also realized on the magic manifold [KT1].

Section 2 contains a brief review of Thurston norms, fibered faces and the Teichmüller polynomial. These are the basic tools used in this paper. In Section 3 we describe our family of examples, and in Section 4 we prove Theorem 1.4 and Theorem 1.5.

Acknowledgments: The author thanks Curt McMullen and Thomas Koberda for many helpful conversations, and thanks Eiko Kin, Spencer Dowdall and the referee of this article for corrections to earlier versions. The author is grateful for the hospitality of the Harvard Mathematics Department, where she wrote this paper as a visiting scholar in Fall 2009.

2. Background and tools

In this section we give a brief review of invariants and properties of fibrations of a hyperbolic 3-manifold $M$, emphasizing the tools that we will use in the rest of the paper. For more details see, for example, [Thu1, FLP, McM1, McM2].

The theory of fibered faces of the Thurston norm ball and the existence of Teichmüller polynomials provides a way to study in a single picture a collection of pseudo-Anosov mapping classes defined on surfaces of different Euler characteristics and genera. Assume $M$ is a compact hyperbolic 3-manifold with boundary. Given an embedded surface $S$ on $M$, let $\chi_-(S)$ be the sum of $|\chi(S_\ell)|$, where $S_\ell$ are the connected components of $S$ with negative Euler characteristic. The Thurston norm of $\psi \in H^1(M; \mathbb{Z})$ is defined to be

$$||\psi||_T = \min \chi_-(S),$$

where the minimum is taken over oriented embedded surfaces $(S, \partial S) \subset (M, \partial M)$ such that the class of $S$ in $H_2(M, \partial M; \mathbb{Z})$ is dual to $\psi$.

Elements of $H^1(M; \mathbb{Z})$ are canonically associated with epimorphisms

$$\pi_1(M; \mathbb{Z}) \to \mathbb{Z}.$$ 

We thus make the following natural identification:

$$H^1(M; \mathbb{Z}) = \text{Hom}(\pi_1(M), \mathbb{Z}) = \text{Hom}(H_1(M; \mathbb{Z}), \mathbb{Z}).$$

This can be considered as a lattice $\Lambda_M$ inside $\mathbb{R}^{b_1(M)}$. If $\psi \in \Lambda_M$ corresponds to a fibration

$$\psi : M \to S^1$$
we say that ψ is fibered. In this case the Thurston norm of ψ is given by

\[ ||\psi||_T = \chi_-(S), \]

where S is homeomorphic to the fiber of ψ. Let

\[ \Psi(M) = \{ \psi : M \to S^1 : \psi \text{ is a fibration} \}. \]

The monodromy φ of ψ ∈ Ψ(M) is the mapping class φ : S → S, such that M is the mapping torus of φ, and ψ is the natural projection to S^1. Since M is hyperbolic, φ is automatically pseudo-Anosov.

Let B be the unit ball in \( \mathbb{R}^{b_1(M)} \) with respect to the extended Thurston norm.

**Theorem 2.1 (Thu1).** The Thurston norm ball B is a convex polyhedron and for any top-dimensional open face F of B, \( (F \cdot \mathbb{R}^+) \cap \Psi(M) \) is either empty or equal to \( (F \cdot \mathbb{R}^+) \cap \Lambda_M \).

If \( (F \cdot \mathbb{R}^+) \cap \Psi(M) \neq \emptyset \), we say F is a fibered face of B. An element of Ψ(M) is called primitive if its fiber is connected. The elements of Λ_M project to the rational points on the boundary of B. If F is a fibered face, then each rational point x on F corresponds to a unique primitive element \( \psi_x \in \Psi(M) \), namely the element of \( (x \cdot \mathbb{R}^+) \cap \Psi(M) \) that lies closest to the origin.

**Corollary 2.3.** For each fibered face F,

\[ \lambda(\psi) = \lambda(\phi_\psi)||\psi||_T, \]

extends to a continuous function on \( F \cdot \mathbb{R}^+ \) that is constant on rays through the origin, and \( \lambda \) achieves a unique minimum on F.

Let G be a group and \( \psi : G \to \mathbb{Z} \) a homomorphism. If \( f \in \mathbb{Z}[G] \),

\[ f = \sum_{g \in G} \alpha_g g, \]

then the specialization of f at ψ is the polynomial in \( \mathbb{Z}[t] \) defined by

\[ f^\psi(t) = \sum_{g \in G} \alpha_g \psi(g), \]

where we think of \( \psi(g) \in \mathbb{Z} = \langle t \rangle \) as a power of t.

For a monic integer polynomial \( p(x) \), the house of \( p(x) \), written \( h(p) \), is the absolute value of the largest root of \( p \).

**Theorem 2.4 (McM1).** Let F be a fibered face for a 3-manifold M, and let \( G = H_1(M; \mathbb{Z}) \).

Then there is an element \( \theta_F \in \mathbb{Z}[G] \) such that for all integral lattice points \( \psi \) in the fibered cone of F,

\[ \lambda(\phi_\psi) = h(\theta_F^\psi). \]

The polynomial \( \theta_F \) is called the Teichmüller polynomial of M for the fibered face F.
3. The Mapping Torus for the Simplest Hyperbolic Braid

We now look at a particular 3-manifold, and study properties of its fibrations. This example has also been studied in [McM1] §11, and the first part of this section will be a review of what is found there.

Let \( M = S^3 \setminus N(L) \), where \( L \) is the link drawn in two ways in Figure 1 and \( N(L) \) is a tubular neighborhood. As seen from the left diagram in Figure 1, \( M \) fibers over the circle with fiber a sphere with four boundary components \( S_{0,4} \). Let \( \psi_0 : M \to S^1 \) be the corresponding fibration, and let \( \phi_0 : S_{0,4} \to S_{0,4} \) be the monodromy. Then \( \phi_0 \) is the mapping class associated to the braid written with respect to standard generators as \( \sigma_1 \sigma_2^{-1} \) (see Figure 2) and its dilatation is given by

\[
\lambda(\phi_0) = \frac{3 + \sqrt{5}}{2}.
\]

The braid \( \sigma_1 \sigma_2^{-1} \) has also been called the “simplest pseudo-Anosov braid” (see [McM1] §11).
Let $K_1$ and $K_2$ be the components of $L$ as drawn in Figure 1. Let $\mu_1$ be the meridian of $K_1$ and $\mu_2$ be the meridian of $K_2$. Then any element $\psi$ of $H^1(M;\mathbb{Z})$ is determined by its values

$$(a, b) = (\psi(\mu_1), \psi(\mu_2)) \in \mathbb{Z} \times \mathbb{Z}.$$ 

With respect to these coordinates, the Thurston norm and the Alexander norm both are given by

$$||\psi|| = \max\{2|a|, 2|b|\}.$$ 

The lattice points $\Lambda_M$ in the fibered cone $F \cdot \mathbb{R}^+$ defined by $\psi = (0, 1)$ is the set

$$\Psi = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : b > 0, -b < a < b\}$$ 

as shown in Figure 3. For the rest of this paper, we will only be concerned with the subset $\Psi_{\text{prim}} \subset \Psi$ consisting of elements of $\Psi$ with connected fibers, i.e., the primitive elements. Thus,

$$\Psi_{\text{prim}} = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : b > 0, -b < a < b, \gcd(a, b) = 1\}.$$ 

![Fibered cone Ψ containing ψ = (0, 1).](image)

The Alexander polynomial for $L$ is given by

$$\Delta_L(x, u) = u^2 - u(1 - x - x^{-1}) + 1$$

(see Rolfsen’s table [Roll]), and the Teichmüller polynomial is given by

$$\Theta_L(x, u) = u^2 - u(1 + x + x^{-1}) + 1$$

(see [McM1] p.47).

Specialization to the element $(a, b) \in H^1(M;\mathbb{Z})$ is the same as plugging $(t^a, t^b)$ into the equations for the Alexander and Teichmüller polynomials (see Section 2).
Proposition 3.1. If \((a, b) \in \Psi_{\text{prim}}\), then the associated monodromy \(\phi_{(a,b)}\) is pseudo-Anosov with geometric dilatation equal to the largest root \(\lambda_{(a,b)}\) of
\[
\Theta_L(t^a, t^b) = t^{2b} - t^b(1 + t^a + t^{-a}) + 1,
\]
and homological dilatation the maximum norm among roots of the polynomial
\[
\Delta_L(t^a, t^b) = t^{2b} - t^b(1 - t^a - t^{-a}) + 1.
\]

Corollary 3.2. If \((a, b) \in \Psi_{\text{prim}}\), then the associated monodromy \(\phi_{(a,b)}\) is orientable if \(a\) is odd and \(b\) is even.

Proof. If \(a\) is odd and \(b\) is even, then the roots of \(\Theta_L(t^a, t^b)\) are the negatives of the roots of \(\Delta_L(t^a, t^b)\). This implies that the geometric and homological dilatations of \(\phi_{(a,b)}\) are equal, and therefore \(\phi_{(a,b)}\) is orientable.

Later in this section, we prove the converse of Corollary 3.2. First we consider how the monodromy behaves near the boundary of \(S_{(a,b)}\).

Proposition 3.3. Let \(\phi_{(a,b)} : S_{(a,b)} \to S_{(a,b)}\) be the monodromy associated to \((a, b) \in \Psi_{\text{prim}}\). The boundary components of \(S_{(a,b)}\) has \(\gcd(3, a)\) components coming from \(T(K_1)\) and \(\gcd(3, b)\) coming from \(T(K_2)\). Thus, the total number of boundary components of \(S_{(a,b)}\) is given by
\[
\begin{cases}
2 & \text{if } \gcd(3, ab) = 1 \\
4 & \text{if } \gcd(3, ab) = 3
\end{cases}
\]

Proof. The number of components in \(T(K_i) \cap S_{(a,b)}\) is the index of the image of \(\pi_1(T(K_i))\) in \(\mathbb{Z}\) under the composition of maps
\[
\pi_1(T(K_i)) \to \pi_1(M) \to \mathbb{Z}
\]
induced by inclusion and \(\psi_{(a,b)}\).

For \(i = 1, 2\), let \(\ell_i\) be the longitude of \(K_i\) that is contractible in \(S^3 \setminus K_i\). Then, for \(T(K_1)\) we have
\[
\psi_{(a,b)}(\mu_1) = a \quad \text{and} \quad \psi_{(a,b)}(\ell_1) = 3\psi_{(a,b)}(\mu_2) = 3b,
\]
so the number of boundary components contributed by \(T(K_1)\) is
\[
\gcd(a, 3b) = \gcd(3, a),
\]
since we are assuming that \(\gcd(a, b) = 1\). The contribution of \(T(K_2)\) is computed similarly.

Proposition 3.4. The genus of \(S_{(a,b)}\), for \((a, b) \in \Psi_{\text{prim}}\) is given by
\[
g(S_{(a,b)}) = \begin{cases}
|b| + \left(1 - \frac{\gcd(3, a) + \gcd(3, b)}{2}\right) & \\
|b| - 1 & \text{if } \gcd(3, ab) = 3
\end{cases}
\]

Proof. From (2) we have
\[
2|b| = \chi_-(S_{(a,b)}) = 2g - 2 + \gcd(3, a) + \gcd(3, b).
\]
Figure 4. Train track for $\phi : S \to S$.

**Proposition 3.5.** Let $(a,b) \in \Psi_{\text{prim}}$, and let $\mathcal{F}$ be a $\phi(a,b)$-invariant foliation. Then $\mathcal{F}$

1. has no interior singularities,
2. is $(3b/\gcd(3,a))$-pronged at each of the $\gcd(3,a)$ boundary components coming from $T(K_1)$, and
3. is $(b/\gcd(3,b))$-pronged at each of the $\gcd(3,b)$ boundary components coming from $T(K_2)$.

**Proof.** Let $\mathcal{L}$ be the lamination of $M$ defined by suspending $\mathcal{F}$ over $M$ considered as the mapping torus of $\phi$. From the train track for $\phi$ (Figure 4), one sees that each of the boundary components of $S$ are one-pronged, and that there are no other singularities. It follows that $\mathcal{L}$ has no singularities outside a neighborhood of the $K_i$, and near each $K_i$ the leaves of $\mathcal{L}$ come together at a simple closed curve $\gamma_i \in H_1(T(K_i))$. Write

$$\gamma_i = r_i \mu_i + s_i \ell_i$$

for $i = 1, 2$.

For $(a,b) \in \Psi_{\text{prim}}$, the number of intersections of $\gamma_i$ with $S_{(a,b)}$ is the image of $\gamma_i$ under the epimorphism

$$\psi_{(a,b)} : \pi_1(M) \to \mathbb{Z}$$

defining the fibration. Figure 4 shows that $s_1 = 1$ and $r_2 = 1$. Using the identities

$$s_1 = 1 \quad \ell_1 = 3\mu_2,$$
$$r_2 = 1 \quad \ell_2 = 3\mu_1,$$

we have

$$\psi_{(a,b)}(\gamma_1) = r_1 \psi_n(\mu_1) + 3\psi_n(\mu_2) = r_1a + 3b$$
$$\psi_{(a,b)}(\gamma_2) = \psi_n(\mu_2) + 3s_2\psi_n(\mu_1) = 3s_2a + b.$$
In general, if \( f : \Sigma \to \Sigma \) is pseudo-Anosov on a compact oriented surface \( \Sigma \) with genus \( g \) and and \( n_1, \ldots, n_k \) are the number of prongs at the singularities and boundary components, then by the Poincaré-Hopf theorem

\[
\sum_{i=1}^{k} (n_i - 2) = 4g - 4. \tag{5}
\]

For \((a, b) = (1, n)\), \( n \) not divisible by 3, we have two singularities with number of prongs given by:

\[
\psi_n(\gamma_1) = r_1 + 3n \\
\psi_n(\gamma_2) = 3s_2 + n.
\]

Plugging into (5) gives

\[ r_1 + 3s_2 = 0. \]

The mapping class \( \phi_{(1,2)} \) is the unique genus 2 pseudo-Anosov mapping class with dilatation equal to \( \lambda_2 \) [CH, LT], and has one 6-pronged singularity [HK]. Thus, \( r_1 = s_2 = 0 \) and

\[ \gamma_1 = \ell_1 = 3\mu_2 \]

and

\[ \gamma_2 = \mu_2. \]

The claim follows.

**Corollary 3.6.** The map \( \phi_{(a,b)} \) has singularities with number of prongs (or prong-type) given by

\[
\begin{cases}
(3b, b) & \text{if } gcd(3, ab) = 1 \\
(3b, b/3, b/3, b/3) & \text{if } gcd(3, b) = 3 \\
(b, b, b, b) & \text{if } gcd(3, a) = 3
\end{cases}
\]

**Corollary 3.7.** If \( b \) is odd, then \( \phi_{(a,b)} \) is not orientable.

**Corollary 3.8.** For \((a, b) \in \Psi_{\text{prim}}\), \( \phi_{(a,b)} \) is 1-pronged at one or more boundary components of \( S_{(a,b)} \) if and only if \((a, b) \in \{(0, 1), (\pm 1, 3), (\pm 2, 3)\}\).

**Corollary 3.9.** If \((a,b) \notin \{(0,1), (\pm 1, 3), (\pm 2, 3)\}\), then \( \phi_{(a,b)} \) extends to the closure of \( S_{(a,b)} \) over the boundary components to a mapping class \( \bar{\phi}_{(a,b)} \) with the same dilatation as \( \phi_{(a,b)} \).

**Proposition 3.10.** Table 2 below describes the pairs \((a,b) \in \Psi_{\text{prim}}\) that give rise to an orientable (or non-orientable) genus \( g \) pseudo-Anosov mapping class. (Here \( g \geq 4 \).)

| \( g \mod 6 \) | orientable | non-orientable |
|---------------|------------|----------------|
| 0             | no example | \( b = g + 1, a = 0 \mod 3 \) |
| 1             | \( b = g + 1, a = 3 \mod 6 \) | \( b = g, a = 1, 2 \mod 3 \) |
| 2             | \( b = g, a = 1, 5 \mod 6 \) | \( b = g + 1, a = 1, 2 \mod 3 \) |
| 3             | \( b = g + 1, a = 3 \mod 6 \) | no example |
| 4             | \( b = g, a = 1, 5 \mod 6 \) | \( b = g + 1, a = 0 \mod 3 \) |
| 5             | \( b = g + 1, a = 1, 5 \mod 6 \) | \( b = g, a = 1, 2 \mod 3 \) |

**Table 2:** Fibrations of \( M \) according to genus.
4. Minimal dilatations for the fibered face.

Let $\Psi_{\text{prim}}$ be the primitive elements of the fibered cone discussed in Section 3. Let
\[ d_g = \min \{ \lambda(\psi) : \psi \in \Psi_{\text{prim}}, \text{genus of } \psi \text{ is } g \}, \]
and
\[ d^+_g = \min \{ \lambda(\psi) : \psi \in \Psi_{\text{prim}}, \text{genus of } \psi \text{ is } g, \text{ the monodromy of } \psi \text{ is orientable} \}. \]

In this section, we finish the proofs of Theorem 1.4 and Theorem 1.5 and their consequences by determining $d_g$ and $d^+_g$.

**Proposition 4.1.** Let $(a, b) \in \Psi_{\text{prim}}$. Then
\[ \lambda(a, b) < \lambda(a', b') \]
if either
1. $|a| < |a'|$ and $|b| = |b'|$; or
2. $|a| = |a'|$ and $|b| > |b'|$.

**Proof.** One compares the slopes of rays from the origin to $(a, b)$ and $(a', b')$. The claim follows from Theorem 2.2. \( \square \)

**Proposition 4.2.** For $b \geq 3$, we have
\[ \lambda_{(1, b)} \geq \lambda_{(3, b+1)}, \]
with equality when $b = 3$.

**Proof.** Let $\lambda = \lambda_{(3, b+1)}$. We will show that $LT_{(1, b)}(\lambda) < 0$. Multiplying by $\lambda^2$ and using the fact that $LT_{(3, b+1)}(\lambda) = 0$ gives
\[ \lambda^2 LT_{(1, b)}(\lambda) = \lambda^2 LT_{(1, b)}(\lambda) - LT_{(3, b+1)}(\lambda) \]
\[ = \lambda^{b+4} - \lambda^{b+3} - \lambda^{b+2} + \lambda^{b-2} + \lambda^2 - 1 \]
\[ = (\lambda - 1)(\lambda^{b+3} - \lambda^{b-2}(\lambda^3 + \lambda^2 + \lambda + 1) + \lambda + 1) \]
\[ = (\lambda - 1)\lambda^{b-2}[\lambda^5 - \lambda^3 - \lambda^2 - \lambda - 1 + \lambda^{2-b}(\lambda + 1)]. \]

Thus, it is enough to show that
\[ \lambda^5 - \lambda^3 - \lambda^2 - \lambda - 1 + \lambda^{2-b}(\lambda + 1) < 0. \]

Let $C$ be the quantity on the left side of this inequality. Since $\lambda > 1$ and $b \geq 3$, we have
\[ C < \lambda^5 - \lambda^3 - \lambda^2 = \lambda^2(\lambda^3 - \lambda - 1). \]

One can check that the right hand side is negative for
\[ 1 < \lambda < 1.3. \]

By Proposition 4.1, $\lambda$ decreases as $b$ increases. A check shows that
\[ 1 < \lambda_{(3, 5)} < 1.3, \]
and hence $C < 0$ for $b \geq 4$. For $b = 3$, one checks directly that
\[ \lambda_{(1, 3)} = \lambda_{(3, 4)}. \]

\( \square \)
Remark 4.3. The mapping class $\phi_{(1,3)}$ is defined on a genus 2 surface with four boundary components, with prong-type $(3,1,1,1)$ and is not orientable. The mapping class $\phi_{(3,4)}$ is defined on a genus 3 surface with prong-type $(4,4,4,4)$ and is orientable. By Proposition 4.2 these two examples have the same dilatation.

Putting together Proposition 4.1 and Proposition 4.2, we have the following.

Proposition 4.4. The sequences $\lambda_{(1,b)}$ and $\lambda_{(3,b)}$ satisfy:
\[
\lambda_{(1,b)} > \lambda_{(3,b+1)} > \lambda_{(1,b+1)}.
\]

Proposition 4.5. For $n \geq 2$,
\[
\lim_{n \to \infty} (\lambda_{(a,n)})^n = \frac{3 + \sqrt{5}}{2},
\]
for any fixed $a$.

Proof. The rays through the lattice points $(a, n) \in \Lambda_M$ on the fibered face of $\psi$ converge to the ray through $(0, 1)$. \hfill \Box

Corollary 4.6. For the minimal dilatations $d_g$ and $d_g^+$ that are realized on $M$, we have
\[
\lim_{g \to \infty} (d_g)^g = \frac{3 + \sqrt{5}}{2},
\]
and
\[
\lim_{g \to \infty} \left(\frac{d_g^+}{d_g}\right)^g = \frac{3 + \sqrt{5}}{2}.
\]

g \not\equiv 0 \pmod{6}

Proposition 4.7. The following table describes the pairs $(a, b) \in \Psi_{\text{prim}}$ that give rise to the minima $d_g$ and $d_g^+$ realized on $M$.

| $g \pmod{6}$ | $\lambda(\phi(a,b)) = d_g^+$, $\phi(a,b)$ orientable | $\lambda(\phi(a,b)) = d_g$ |
|--------------|-------------------------------------------------|-----------------|
| 0            | no example                                      | $(3, g + 1)$    |
| 1            | $(3, g + 1)$                                    | $(3, g + 1)$    |
| 2            | $(1, g)$                                        | $(1, g + 1)$    |
| 3            | $(3, g + 1)$                                    | $(3, g + 1)$    |
| 4            | $(1, g)$                                        | $(3, g + 1)$    |
| 5            | $(1, g + 1)$                                    | $(1, g + 1)$    |

Table 3: Pairs $(a, b)$ giving smallest dilatations for $\phi \in \Phi(M_{sb})$.

Proposition 4.7 and Corollary 3.9 complete the proofs of Theorem 1.4 and Theorem 1.5. A pictorial view of how the elements of $\Psi$ giving the least dilatations for each genus up to 12 lie on a fibered cone of $M$ is shown in Figure 5.

Putting together the results of this paper with those in [AD, KT2, LT], we see that for genus $g = 2, 3, 4, 5, 7,$ and 8,
\[
\delta_g^+ = \lambda_{(a,b)}
\]
Figure 5. Minima for $d_g$ and $d_g^+$ in genus $g = 1, \ldots, 12$.

where

$$(a, b) = \begin{cases} 
(1, g) & \text{if } g = 2, 3, 4, \text{ or } 8 \\
(1, g + 1) & \text{if } g = 5 \\
(2, g + 2) & \text{if } g = 7
\end{cases}$$

and

$$\delta_6^+ \geq \lambda(1,6).$$

These results suggest the following generalization to Question 1.3.

**Question 4.8.** For every $g \geq 2$, is it true that

$$\delta_g^+ = \lambda(a, b)$$

for some $a, b$ with $b \geq g \geq a \geq 1$?

**References**

[AD] J. Aaber and N. Dunfield. Closed surface bundles of least volume. Preprint (2010).

[AY] P. Arnoux and J. Yoccoz. Construction de difféomorphismes pseudo-Anosov. C. R. Acad. Sci. Paris 292 (1980), 75–78.

[CB] A. Casson and S. Bleiler. Automorphisms of surfaces after Nielsen and Thurston. Cambridge University Press, 1988.

[CH] J. Cho and J. Ham. The minimal dilatation of a genus two surface. Experiment. Math. 17 (2008), 257–267.
REFERENCES

[Far] B. Farb. Some Problems on mapping class groups and moduli space. In Problems on Mapping Class Groups and Related Topics, volume 74 of Proc. Symp. Pure and Applied Math., pages 10–58. 2006.
[FLM] B. Farb, C. Leininger, and D. Margalit. Small dilatation pseudo-Anosovs and 3-manifolds. Preprint (2009).
[FLP] A. Fathi, F. Laudenbach, and V. Poenaru. Travaux de Thurston sur les surfaces, volume 66-67. Société Mathématique de France, Paris, 1979.
[Fri] D. Fried. Growth Rate of surface homeomorphisms and flow equivalence. Comment. Math. Helvetici 57 (1982), 237–259.
[Hir1] E. Hironaka. The Lehmer Polynomial and Pretzel Knots. Bulletin of Canadian Math. Soc. 44 (2001), 440–451.
[Hir2] E. Hironaka. Chord diagrams and Coxeter links. J. London Math. Soc. 69 (2004), 243–257.
[HK] E. Hironaka and E. Kin. A family of pseudo-Anosov braids with small dilatation. Algebr. Geom. and Topol. 6 (2006), 699–738.
[Iva] N.V. Ivanov. Coefficients of expansion of pseudo-Anosov braids with small dilatation. J. Soviet Math. (transl) 52 (1990), 2819–2822.
[KT1] E. Kin and M. Takasawa. Pseudo-Anosov braids with small entropy and the magic 3-manifold. Preprint (2008).
[KT2] E. Kin and M. Takasawa. Pseudo-Anosovs on closed surfaces having small entropy and the Whitehead sister link exterior. Preprint (2010).
[KS] T. Koberda and A.M. Silberstein. Representations of Galois groups on the homology of surfaces. Preprint (2009).
[LT] E. Lanneau and J-L Thiffeault. On the minimum dilatation of pseudo-Anosov homeomorphisms on surfaces of small genus. Ann. de l’Inst. Four. (to appear).
[Leh] D. H. Lehmer. Factorization of certain cyclotomic functions. Ann. of Math. 34 (1933), 461–469.
[Lei] C. Leininger. On groups generated by two positive multi-twists: Teichmuller curves and Lehmer’s number. Geometry & Topology 88 (2004), 1301–1359.
[MP] B. Martelli and C. Petronio. Dehn filling of the "magic" 3-manifold. Comment. Anal. Geom. 14 (2006), 969–1026.
[McM1] C. McMullen. Polynomial invariants for fibered 3-manifolds and Teichmüller geodesics for foliations. Ann. Sci. École Norm. Sup. 33 (2000), 519–560.
[McM2] C. McMullen. The Alexander polynomial of a 3-manifold and the Thurston norm on cohomology. Ann. Scient. Éc. Norm. Sup. 35 (2002), 153–171.
[Min] H. Minakawa. Examples of pseudo-Anosov homeomorphisms with small dilatations. J. Math. Sci. Univ. Tokyo 13 (2006), 95–111.
[Pen] R. Penner. Bounds on least dilatations. Proceedings of the A.M.S. 113 (1991), 443–450.
[Rol] D. Rolfsen. Knots and Links. Publish or Perish, Inc, Berkeley, 1976.
[Thu1] W. Thurston. A norm for the homology of 3-manifolds. Mem. Amer. Math. Soc. 339 (1986), 99–130.
[Thu2] W. Thurston. On the geometry and dynamics of diffeomorphisms of surfaces. Bull. Amer. Math. Soc. (N.S.) 19 (1988), 417–431.
[Zhi] A. Y. Zhirov. On the minimum dilatation of pseudo-Anosov diffeomorphisms on a double torus. Uspekhi Mat. Nauk 50 (1995), 297–198.

Eriko Hironaka
Department of Mathematics
Florida State University
Tallahassee, FL 32306-4510
U.S.A.