An efficient Monte Carlo scheme for Zakai equations

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Abstract

In this paper we develop a numerical method for efficiently approximating solutions of certain Zakai equations in high dimensions. The key idea is to transform a given Zakai SPDE into a PDE with random coefficients. We show that under suitable regularity assumptions on the coefficients of the Zakai equation, the corresponding random PDE admits a solution random field which, for almost all realizations of the random coefficients, can be written as a classical solution of a linear parabolic PDE. This makes it possible to apply the Feynman–Kac formula to obtain an efficient Monte Carlo scheme for computing approximate solutions of Zakai equations. The approach achieves good results in up to 25 dimensions with fast run times.

Keywords: Zakai equation, nonlinear filtering problems, stochastic partial differential equations, Doss–Sussmann transformation, Feynman–Kac representation

1 Introduction

The goal of stochastic filtering is to estimate the conditional distribution of a not directly observable stochastic process blurred by measurement noise. The process of interest is usually called signal process, while the observed process is referred to as observation process. Whereas the signal process follows a hidden dynamic, probing the system only reveals the observation process, which, in general, might depend nonlinearly on the signal process and, in addition, is blurred by measurement noise. Stochastic filtering problems were first studied in connection with tracking and signal processing (see the seminal works by Kalman [26] and Kalman & Bucy [27]) but soon turned out to also be relevant in a variety of other applications in finance, the natural sciences and engineering. Among others, nonlinear filtering problems naturally arise in e.g., financial engineering ([3, 10, 13, 18, 20, 21]), weather forecasting ([8, 9, 11, 17, 19, 33]) or chemical engineering ([7, 12, 34, 35, 36, 38]). For further applications of nonlinear filtering, we refer to the survey paper [31]. Stochastic filtering problems are naturally related to stochastic partial differential equations (SPDEs) since in continuous time, the (unnormalized) density of the unobserved signal process given the observations is described by a suitable SPDE, such as the Zakai equation [39] or Kushner equation [30]. The SPDEs arising in this context can typically not be solved explicitly but instead, have to be computed numerically. Moreover, they often are high-dimensional as the number of dimensions corresponds to the state space dimension of the filtering problem.

In this paper, we focus on Zakai equations with coefficients that satisfy certain regularity conditions. Let us assume the signal follows the \(d\)-dimensional dynamics

\[ Y_t = Y_0 + \int_0^t \mu(Y_s) \, ds + \sigma W_t \]
for a $d$-dimensional random vector $Y_0$ with density $\varphi: \mathbb{R}^d \to [0, \infty)$, a sufficiently regular function $\mu: \mathbb{R}^d \to \mathbb{R}^d$, a constant $d \times d$-matrix $\sigma$ and a $d$-dimensional Brownian motion $(W_t)_{t \in [0,T]}$ independent of $Y_0$, while we observe a $k$-dimensional process of the form

$$Z_t = \int_0^t h(Y_s) ds + V_t$$

for a sufficiently regular function $h: \mathbb{R}^d \to \mathbb{R}^k$ and a $k$-dimensional Brownian motion $(V_t)_{t \in [0,T]}$ independent of $Y_0$ and $(W_t)_{t \in [0,T]}$. Then the solution of the corresponding Zakai equation

$$X_t(x) = \varphi(x) + \int_0^t \frac{1}{2} \text{Trace}_{\sigma \sigma^T} \left[ \text{Hess}(X_s(x)) \right] - \text{div}(\mu X_s(x)) ds + \int_0^t X_s(x) \langle h(x), dZ_s \rangle_{\mathbb{R}^d}$$

(1)

describes the evolution of an unnormalized density of the conditional distribution of $Y_t$ given observations of $Z_s$, $s \leq t$; that is,

$$P[Y_t \in A | Z_s, s \in [0,t]] = \frac{\int_A X_t(x) dx}{\int_{\mathbb{R}^d} X_t(x) dx} \text{ for every Borel subset } A \subseteq \mathbb{R}^d.$$ Our numerical method is based on a transformation which converts a Zakai SPDE of the form (1) into a PDE with random coefficients. We show that under suitable conditions on the coefficients of the Zakai SPDE, the solution of the resulting random PDE is $\omega$-wise a classical solution of a linear parabolic PDE. This makes it possible to apply the Feynman–Kac formula to obtain an efficient Monte Carlo scheme for the numerical approximation of solutions of high-dimensional Zakai equations. The following is this paper’s main theoretical result.

**Theorem 1.** Let $T \in (0, \infty)$, $d, k \in \mathbb{N}$, $\sigma \in \mathbb{R}^{d \times d}$, and consider functions $\varphi \in C^2(\mathbb{R}^d, [0, \infty))$, $\mu \in C^3(\mathbb{R}^d, \mathbb{R}^d)$ and $h \in C^4(\mathbb{R}^d, \mathbb{R}^k)$ such that $\varphi$ has at most polynomially growing derivatives up to the second order, $\mu$ has bounded derivatives up to the third order and $h$ has bounded derivatives up to the fourth order. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, P)$ be a filtered probability space satisfying the usual conditions, which supports standard $(\mathcal{F}_t)_{t \in [0,T]}$-Brownian motions $W, U: [0,T] \times \Omega \to \mathbb{R}^d$ and $V: [0,T] \times \Omega \to \mathbb{R}^k$ with continuous sample paths such that $W$ and $V$ are independent. Let $Y: [0,T] \times \Omega \to \mathbb{R}^d$ and $Z: [0,T] \times \Omega \to \mathbb{R}^k$ be $(\mathcal{F}_t)_{t \in [0,T]}$-adapted stochastic processes such that $P(Y_0 \in A) = \int_A \varphi(x) dx$ for every Borel subset $A \subseteq \mathbb{R}^d$ and

$$Y_t = Y_0 + \int_0^t \mu(Y_s) ds + \sigma W_t, \quad Z_t = \int_0^t h(Y_s) ds + V_t \quad \text{for } t \in [0,T].$$

(2)

For all $z \in C([0,T], \mathbb{R}^k)$, $t \in [0,T]$ and $x \in \mathbb{R}^d$, let $R^{z,t,x}_s: [0,t] \times \Omega \to \mathbb{R}^d$ be an $(\mathcal{F}_s)_{s \in [0,t]}$-adapted stochastic processes satisfying

$$R^{z,t,x}_s = x + \int_0^s [\sigma \sigma^T D_h(R^{z,t,x}_r)] T z(t-r) - \mu(R^{z,t,x}_r)] dr + \sigma U_s \quad \text{for all } s \in [0,t].$$

(3)

Moreover, let for all $z \in C([0,T], \mathbb{R}^k)$, the functions $B_z, u_z: [0,T] \times \mathbb{R}^d \to \mathbb{R}$ be given by

$$B_z(t,x) = \frac{1}{2} \langle \sigma \sigma^T D_h(x), z(t) \rangle_{\mathbb{R}^d} + \frac{1}{2} \langle h(x), h(x) \rangle_{\mathbb{R}^k} \quad \text{and}$$

$$u_z(t,x) = \mathbb{E} \left[ \varphi(R^{z,t,x}_t) \exp \left( \int_0^t B_z(t-s, R^{z,t,x}_s) ds \right) \right].$$

(4)

(5)

1. A filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, P)$ is said to satisfy the usual conditions if for all $t \in (0,T)$, one has $\cup_{s \in \mathbb{F}, \mathcal{P}(\mathcal{A})=0} (B \subseteq \Omega: B \subseteq \mathcal{A}) \subseteq \mathcal{F}_t = \cap_{s \in [t,T]} \mathcal{F}_s$.

3. For $d \in \mathbb{N}$, we denote by $\langle , \rangle_{\mathbb{R}^d} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ the standard scalar product given by $\langle x, y \rangle_{\mathbb{R}^d} = \sum_{i=1}^d x_i y_i$ and by $||.||_{\mathbb{R}^d} : \mathbb{R}^d \to [0, \infty)$ the corresponding norm $||x||_{\mathbb{R}^d} = \sqrt{(x,x)}$. 
Then
\[ X_t(x, \omega) = u_{Z(\omega)}(t, x) \exp((h(x), Z_t(\omega))_{\mathbb{R}^d}), \quad t \in [0, T], \ x \in \mathbb{R}^d, \ \omega \in \Omega, \] (6)
is, up to indistinguishability, the unique random field \( X : [0, T] \times \mathbb{R}^d \times \Omega \to \mathbb{R} \) satisfying the following properties:

(i) for all \( t \in [0, T] \) and \( x \in \mathbb{R}^d \), the mapping \( X_t(x) : \Omega \to \mathbb{R} \) is \( \mathcal{F}_t \)-measurable,

(ii) for all \( \omega \in \Omega \), the mapping \( (t, x) \mapsto X_t(x, \omega) \) is in \( C^{0,2}([0, T] \times \mathbb{R}^d, \mathbb{R}) \) and there exist constants \( a(\omega), c(\omega) \geq 0 \) such that
\[
\sup_{t \in [0, T]} |X_t(x, \omega)| \leq a(\omega)e^{c(\omega)\|x\|_{\mathbb{R}^d}} \quad \text{for all } x \in \mathbb{R}^d,
\]

(iii) \( X_t(x) = \varphi(x) + \int_0^t \left[ \frac{1}{2} \text{Trace}_{\mathbb{R}^d}((\sigma \sigma^T \text{Hess}(X_s))(x)) - \text{div}(\mu X_s)(x) \right] ds + \int_0^t X_s(x) \langle h(x), dZ_s \rangle_{\mathbb{R}^d} \) (7)
P-\text{a.s. for all } t \in [0, T] \text{ and } x \in \mathbb{R}^d.

Representation \( (\ref{representation}) \) makes it possible to approximate the solution \( X_t(x, \omega) \) of the Zakai equation \( (\ref{zakai_equation}) \) along a realization of the observation process \( (Z_s(\omega))_{s \in [0, T]} \) by averaging over different Monte Carlo simulations of the process \( R^{Z(\omega), t, x} \) given in \( (\ref{random_process}) \). We provide numerical results for a Zakai equation of the form \( (\ref{zakai_equation}) \) for dimensions \( d \in \{1, 2, 5, 10, 20, 25\} \) in Section \( \ref{numerical_experiments} \) below. The proof of Theorem \( \ref{main_theorem} \) is given in the Appendix.

The idea of transforming a stochastic differential equation into an ordinary differential equation with random coefficients goes back to Doss \( \cite{doss} \) and Sussmann \( \cite{sussmann} \). An extension to SPDEs was used by Buckdahn and Ma \( \cite{buckdahn1, buckdahn2} \) to introduce a notion of stochastic viscosity solution for SPDEs and show existence and uniqueness results as well as connections to backward doubly stochastic differential equations. The same approach was employed by Buckdahn and Ma \( \cite{buckdahn3} \) and Boufoussi et al. \( \cite{boufoussi} \) to study stochastic viscosity solutions of stochastic Hamilton–Jacobi–Bellman (HJB) equations. In this paper, we analyze the regularity properties of such transformations and use them to develop a Monte Carlo method for approximating solutions of Zakai equations. The numerical results in Section \( \ref{numerical_experiments} \) below show that it produces accurate results in high dimensions with fast run times. For different numerical approximation methods for Zakai equations, see e.g., \( \cite{buckdahn1, buckdahn2, boufoussi, douglas, tuoc} \).

2 Numerical experiments

Together with time-discretization, the trapezoidal rule and Monte Carlo sampling, Theorem \( \ref{main_theorem} \) can be used to approximate the solution of a given Zakai equation of the form \( (\ref{zakai_equation}) \) along a realization of the observation process \( Z \). We illustrate this in the following example: Choose \( T, \alpha \in (0, \infty), \beta \in \mathbb{R}, d \in \mathbb{N}, \) and let \( \sigma \in \mathbb{R}^{d \times d} \) be given by \( \sigma_{ij} = d^{-1/2} \) for all \( i, j \in \{1, \ldots, d\} \). Consider a \( d \)-dimensional signal process with dynamics
\[ Y_t = Y_0 + \int_0^t \frac{\beta Y_s}{1 + \|Y_s\|_{\mathbb{R}^d}^2} ds + \sigma W_t, \quad t \in [0, T], \] (8)
for an \( \mathcal{F}_0 \)-measurable random initial condition \( Y_0 : \Omega \to \mathbb{R}^d \) with density
\[ \varphi(x) = \left( \frac{\alpha}{\pi} \right)^{d/2} \exp \left( -\frac{\alpha}{2} \|x\|_{\mathbb{R}^d}^2 \right), \quad x \in \mathbb{R}^d, \]
defined on a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})\) satisfying the usual conditions and a standard \((\mathcal{F}_t)_{t \in [0, T]}\)-Brownian motion \( W : [0, T] \times \Omega \to \mathbb{R}^d \) with continuous sample paths. Assume the observation process is of the form
\[ Z_t = \int_0^t \gamma Y_s ds + V_t, \quad t \in [0, T], \] (9)

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for a constant $\gamma \in \mathbb{R}$ and a standard $(\mathcal{F}_t)$-Brownian motion $V : [0, T] \times \Omega \to \mathbb{R}^d$ with continuous sample paths independent of $W$. Let $U : [0, T] \times \Omega \to \mathbb{R}^d$ be another standard Brownian motion with continuous sample paths and consider stochastic processes $R^{z,t,x} : [0, t] \times \Omega \to \mathbb{R}^d$, $z \in C([0, T], \mathbb{R}^d)$, $t \in [0, T]$, $x \in \mathbb{R}^d$ satisfying

$$R^{z,t,x}_s = x + \int_0^s \left[ \gamma \sigma^T z(t-r) - \frac{\beta R^{z,t,x}_{r,t,x}}{1 + \|R^{z,t,x}_{r,t,x}\|_{\mathbb{R}^d}} \right] dr + \sigma U_s$$

for all $z \in C([0, T], \mathbb{R}^d)$, $t \in [0, T]$, $s \in [0, t]$ and $x \in \mathbb{R}^d$. Let the mappings $B_z, u_z : [0, T] \times \mathbb{R}^d \to \mathbb{R}$, $z \in C([0, T], \mathbb{R}^d)$ be given by

$$B_z(t, x) = \frac{\sigma^T z(t) + x}{\sqrt{d}} - \frac{\beta(1 + \|x\|_{\mathbb{R}^d}^2)^{-1} \langle x, z(t) \rangle_{\mathbb{R}^d}}{1 + \|x\|_{\mathbb{R}^d}} - d\beta(1 + \|x\|_{\mathbb{R}^d}^2)^{-1} + 2\beta\|x\|_{\mathbb{R}^d}(1 + \|x\|_{\mathbb{R}^d}^2)^{-2}$$

and

$$u_z(t, x) = \mathbb{E} \left[ \phi \left( R^{z,t,x}_t \right) \exp \left( \int_0^t B_z(t-s, R^{z,t,x}_s) \, ds \right) \right],$$

$z \in C([0, T], \mathbb{R}^d)$, $t \in [0, T]$, $x \in \mathbb{R}^d$. By Theorem 1

$$X_t(x, \omega) = u_{Z(\omega)}(t, x) \exp \left( \langle \gamma x, Z(\omega) \rangle_{\mathbb{R}^d} \right), \quad t \in [0, T], x \in \mathbb{R}^d, \omega \in \Omega,$$

is, up to indistinguishability, the unique random field $X : [0, T] \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ satisfying (8)–(9) of Theorem 1 and solving the Zakai equation

$$X_t(x) = \varphi(x) + \int_0^t X_s(x) \langle \gamma x, dZ_s \rangle_{\mathbb{R}^d}$$

$$+ \int_0^t \left[ \frac{1}{2} \sum_{i,j=1}^d \frac{\partial^2}{\partial x_i \partial x_j} X_s(x) - \sum_{i=1}^d \frac{\partial}{\partial x_i} \left( \frac{\beta x_i X_s(x)}{1 + \|x\|_{\mathbb{R}^d}^2} \right) \right] ds \quad \mathbb{P}\text{-a.s.,}$$

$t \in [0, T]$, $x \in \mathbb{R}^d$, corresponding to the dynamics (8)–(9).

We use representation (10) to approximate $X_T(x)$ for a given realization $(z(t))_{t \in [0, T]}$ of $(Z_t)_{t \in [0, T]}$. For numerical purposes, we generate a discrete realization of the observation process by choosing an $N \in \mathbb{N}$ and considering the following discretized versions of (8)–(9):

$$Y_0 \sim N(0, \frac{1}{N} I_d), \quad Y_n = Y_{n-1} + \frac{\beta Y_{n-1}}{1 + \|Y_{n-1}\|_{\mathbb{R}^d}} \frac{T}{N} + \sigma(W_{nT/N} - W_{(n-1)T/N}),$$

$$Z_0 = 0, \quad Z_n = Z_{n-1} + \gamma Y_{n-1} + \frac{Y_n T}{N} + (V_{nT/N} - V_{(n-1)T/N}), \quad n \in \{1, \ldots, N\}.$$

Let $U^{(i)} : [0, T] \times \Omega \to \mathbb{R}^d$, $i \in \mathbb{N}$, be i.i.d. standard Brownian motions independent of $Y_0$, $W$, $V$, and consider $R^{(x,i)}_n : \Omega \to \mathbb{R}^d$, $n \in \{0, 1, \ldots, N\}$, $x \in \mathbb{R}^d$, $i \in \mathbb{N}$, given by $R^{(x,i)}_0 = x$ and

$$R^{(x,i)}_n = R^{(x,i)}_{n-1} + \left( \gamma \sigma^T Z_{N-n+1} - \frac{\beta R^{(x,i)}_{n-1}}{1 + \|R^{(x,i)}_{n-1}\|_{\mathbb{R}^d}^2} \right) \frac{T}{N} + \sigma \left( U^{(i)}_{nT/N} - U^{(i)}_{(n-1)T/N} \right),$$

for $x \in \mathbb{R}^d$, $i \in \mathbb{N}$ and $n \in \{1, 2, \ldots, N\}$. Define the mappings $B_n, \mathcal{X}^M : \Omega \times \mathbb{R}^d \to \mathbb{R}$, $n \in \{0, 1, \ldots, N\}$,

$$M \in \mathbb{N}, \text{ by}$$

$$B_n(x) = \frac{\sigma^T Z_n + x}{\sqrt{d}} - \frac{\beta(1 + \|x\|_{\mathbb{R}^d}^2)^{-1} \langle x, Z_n \rangle_{\mathbb{R}^d}}{1 + \|x\|_{\mathbb{R}^d}} - d\beta(1 + \|x\|_{\mathbb{R}^d}^2)^{-1} + 2\beta\|x\|_{\mathbb{R}^d}(1 + \|x\|_{\mathbb{R}^d}^2)^{-2}, \quad n \in \{1, \ldots, N\}, x \in \mathbb{R}^d,$$
and
\[ X^M(x) = \frac{1}{M} \sum_{i=1}^{M} \varphi(R^{(x,i)}_N) \times \exp \left( \sum_{n=1}^{N} \frac{T}{2N} \left[ B_{N-n}(R^{(x,i)}_n) + B_{N-n+1}(R^{(x,i)}_{n-1}) \right] + \langle \gamma x, Z_N \rangle_{\mathbb{R}^d} \right), \quad M \in \mathbb{N}, \ x \in \mathbb{R}^d. \]

It follows from the law of large numbers that, for \( M \to \infty, \)
\[ X^M(x) \xrightarrow{\mathbb{P}-a.s.} \mathbb{E}[X^1(x) \mid Z], \]
which approximates \( X_T(x) \).

Table II below shows point estimates and 95% confidence intervals for \( \mathbb{E}[X^1(x) \mid Z] \) for different realizations of \( (Y, Z) \), \( \alpha = 2\pi, \beta = 1/4, \gamma = 1, T = 1/2, N = 100 \) and \( x \in \{Y_N, 2Z_N\} \). For every \( d \in \{1, 2, 5, 10, 20, 25\} \) we simulated five realizations of \( (Y, Z) \) and computed estimates of \( \mathbb{E}[X^1(x) \mid Z] \) for \( x \in \{Y_N, 2Z_N\} \) by computing realizations of \( X^M(x), x \in \{Y_N, 2Z_N\} \), for \( M = 4,096,000 \). Note that in a typical application, the signal \( Y_N \) is not directly observable, while, in view of (9), \( 2Z_N = Z_N/(\gamma T) \) is a naive estimate of \( Y_N \) based on the observation process \( Z \). As expected, with a few exceptions, the values \( X^M(x) \) reported in Table II are higher for \( x = Y_N \) than for \( x = 2Z_N \). The 95% confidence intervals were approximated, using the central limit theorem, with
\[ \left[ X^M(x) - \frac{s_M(x)}{\sqrt{M}} q_{0.975}, X^M(x) + \frac{s_M(x)}{\sqrt{M}} q_{0.975} \right], \]
where \( q_{0.975} \) is the 97.5%-quantile of the standard normal distribution and \( s_M^2(x) \) the sample variance of (13) given by
\[ s_M^2(x) = \frac{1}{M-1} \sum_{i=1}^{M} \left\{ \varphi \left( R^{(x,i)}_N \right) \times \exp \left( \sum_{n=1}^{N} \frac{T}{2N} \left[ B_{N-n}(R^{(x,i)}_n) + B_{N-n+1}(R^{(x,i)}_{n-1}) \right] + \langle \gamma x, Z_N \rangle_{\mathbb{R}^d} \right) - X^M(x) \right\}^2. \]

The reported runtimes are averages of the ten times needed to compute \( X^M(x), x \in \{Y_N, 2Z_N\} \), for five different realizations of \( (Y, Z) \).

To approximate the whole function \( x \mapsto X_T(x) \), our algorithm can be run simultaneously for different \( x \in \mathbb{R}^d \). To produce the plots in Figures 1–3 we divided the time interval into \( N = 20 \) subintervals and computed \( X^M(x) \) for a given realization of \( (Y, Z) \) and different \( x \in \mathbb{R}^d \) based on \( M = 102,400 \) independent copies of (12), which we generated simultaneously for different \( x \in \mathbb{R}^d \) using the same simulated Brownian increments. The first plot in Figure 1 shows \( X^M(x) \) for \( x \) on a regular grid with 1024 grid points in the interval \( [Y_N-5, Y_N+5] \), while the second plot in Figure 2 shows \( X^M(x) \) for \( x \) on a regular grid with 128^2 grid points in the square \( [Y_{N,1}-5, Y_{N,1}+5] \times [Y_{N,2}-5, Y_{N,2}+5] \). Figures 2–3 show \( X^M(x, Y_{N,d}), \ldots, Y_{N,d} \) for \( x \) on a regular grid with 1024 grid points in the interval \( [Y_{N,1}-5, Y_{N,1}+5] \) for \( d \in \{5, 10, 20, 25\} \). The computation times for the results depicted in Figures 1–3 were 1.1s, 28.4s, 7.4s, 14.5s, 28.1s, 34.1s, respectively.

The numerical experiments presented in this section were implemented in Python using TensorFlow on a NVIDIA GeForce RTX 2080 Ti GPU. The underlying system was an AMD Ryzen 9 3950X CPU with 64 GB DDR4 memory running Tensorflow 2.1 on Ubuntu 19.10. The Python source codes can be found in the GitHub repository [https://github.com/seb-becker/zakai](https://github.com/seb-becker/zakai).

## 3 Conclusion

In this paper we have introduced a Monte Carlo method for approximating solutions of certain Zakai equations in high dimensions. It is based on a Doss–Sussmann-type transformation which transforms
Table 1: Estimates of $X_{1/2}(Y_N)$ and $X_{1/2}(2Z_N)$ together with 95% confidence intervals for the Zakai equation \[\text{(11)}\] for five different realizations of $(Y, Z)$ in each of the cases $d \in \{1, 2, 5, 10, 20, 25\}$.
Figure 1: Approximations of $X_{1/2}(x)$ for $x \in [Y_N - 5, Y_N + 5]$ (left) and $x \in [Y_{N,1} - 5, Y_{N,1} + 5] \times [Y_{N,2} - 5, Y_{N,2} + 5]$ (right). The vertical line in the picture at the left shows the location of $Y_N$.

Figure 2: Approximations of $X_{1/2}(x,Y_{N,2},\ldots,Y_{N,d})$, $x \in [Y_{N,1} - 5, Y_{N,1} + 5]$, for $d \in \{5,10\}$. The vertical lines show the location of $Y_{N,1}$.
Figure 3: Approximations of $X_{1/2}(x, Y_{N,2}, \ldots, Y_{N,d})$, $x \in [Y_{N,1} - 5, Y_{N,1} + 5]$, for $d \in \{20, 25\}$. The vertical lines show the location of $Y_{N,1}$.

A Zakai SPDE into a PDE with random coefficients. This makes it possible to apply the Feynman–Kac formula to obtain a Monte Carlo approximation of the solution of a given Zakai equation. The numerical experiments in Section 2 show that the proposed method achieves good results in up to 25 dimensions with fast run times.

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Appendix. Proof of the main result

In this appendix we derive approximation, stability, integrability as well as regularity results and use them to prove Theorem 1.

A.1 Approximation and mollification results for at most polynomially growing functions

Lemma 2. Let $c, p \in [0, \infty)$, $d \in \mathbb{N}$, $\alpha, T \in (0, \infty)$, and consider at most polynomially growing functions $G \in C([0,T] \times \mathbb{R}^d, \mathbb{R})$ and $H \in C(\mathbb{R}^d, \mathbb{R})$. Moreover, assume that

$$|G(t, x) - G(s, x)| \leq c (1 + \|x\|_{\mathbb{R}^d})^p |t - s|^\alpha \quad \text{for all } s, t \in [0, T] \text{ and } x \in \mathbb{R}^d,$$

and let $G_n \in C([0,T] \times \mathbb{R}^d, \mathbb{R})$ and $H_n \in C(\mathbb{R}^d, \mathbb{R})$ for all $n \in \mathbb{N}$, $t \in [0, T]$ and $x \in \mathbb{R}^d$ be given by

$$G_n(t, x) = (\frac{n}{2\pi})^{1/2} \int_{-\infty}^{\infty} G(\min\{T, \max\{s, 0\}\}, x) \exp\left( -\frac{n}{2}(t-s)^2 \right) ds$$

and

$$H_n(x) = (\frac{n}{2\pi})^{d/2} \int_{\mathbb{R}^d} H(y) \exp\left( -\frac{n}{2}\|x-y\|_{\mathbb{R}^d}^2 \right) dy.$$

Then
(i) \( \limsup_{n \to \infty} \sup_{t \in [0,T]} \sup_{x \in \mathbb{R}^d} \frac{|G_n(t,x) - G(t,x)|}{(1 + \|x\|_{\mathbb{R}^d})^p} = 0 \) and

(ii) \( \limsup_{n \to \infty} \sup_{x \in [-q, q]^d} |H_n(x) - H(x)| = 0 \) for all \( q \in (0, \infty) \).

**Proof.** From (14) and the fact that \( \min\{T, \max\{s, 0\} - t\} \leq |s - t| \) for all \( s \in \mathbb{R} \) and \( t \in [0,T] \), we obtain

\[
\frac{|G_n(t,x) - G(t,x)|}{(1 + \|x\|_{\mathbb{R}^d})^p} \leq \left( \frac{n}{2\pi} \right)^{1/2} \int_\mathbb{R} \frac{|G(\min\{T, \max\{s, 0\}\}, x) - G(t,x)|}{(1 + \|x\|_{\mathbb{R}^d})^p} \exp\left(-\frac{n(t-s)^2}{2}\right) ds
\]

\[
\leq \left( \frac{n}{2\pi} \right)^{1/2} \int_\mathbb{R} c |\min\{T, \max\{s, 0\}\} - t|^\alpha \exp\left(-\frac{n(t-s)^2}{2}\right) ds
\]

\[
\leq \left( \frac{n}{2\pi} \right)^{1/2} \int_\mathbb{R} c |s - t|^\alpha \exp\left(-\frac{n(t-s)^2}{2}\right) ds = c \left( \frac{1}{2\pi n^\alpha} \right)^{1/2} \int_\mathbb{R} |z|^\alpha \exp\left(-\frac{z^2}{2}\right) dz
\]

for all \( t \in [0,T] \) and \( x \in \mathbb{R}^d \). In particular, \( \limsup_{n \to \infty} \sup_{t \in [0,T]} \sup_{x \in \mathbb{R}^d} \frac{|G_n(t,x) - G(t,x)|}{(1 + \|x\|_{\mathbb{R}^d})^p} = 0 \), which shows (i). Next, note that one has

\[
|H_n(x) - H(x)| \leq \left( \frac{n}{2\pi} \right)^{1/2} \int_\mathbb{R}^d |H(y) - H(x)| \exp\left(-\frac{n}{2}\|x - y\|_{\mathbb{R}^d}^2\right) dy
\]

\[
= \left( \frac{1}{2\pi} \right)^{1/2} \int_\mathbb{R}^d \left| H(x + \frac{z}{\sqrt{n}}) - H(x) \right| \exp\left(-\frac{1}{2}\|z\|_{\mathbb{R}^d}^2\right) dz \quad \text{for all } n \in \mathbb{N} \text{ and } x \in \mathbb{R}^d,
\]

and the assumption that \( H \in C(\mathbb{R}^d, \mathbb{R}) \) is at most polynomially growing implies that

\[
\sup_{n \in \mathbb{N}} \int_\mathbb{R} \sup_{x \in [-q,q]^d} \left| H(x + \frac{z}{\sqrt{n}}) - H(x) \right|^q \exp\left(-\frac{1}{2}\|z\|_{\mathbb{R}^d}^2\right) dz < \infty \quad \text{for all } q \in (0, \infty).
\]

Combining (15), (16), the assumption that \( H \in C(\mathbb{R}^d, \mathbb{R}) \), the de la Vallée Poussin theorem (cf., e.g., [28, Corollary 6.2]), and the Vitali convergence theorem (cf., e.g., [28, Theorem 6.25]) yields that \( \limsup_{n \to \infty} \sup_{x \in [-q,q]^d} |H_n(x) - H(x)| = 0 \) for all \( q \in (0, \infty) \). This establishes (ii) and completes the proof of the lemma. \( \square \)

**Lemma 3.** Let \( d, m \in \mathbb{N} \), \( T \in (0, \infty) \), and consider two families of functions \( f^{n,t,x} \in C([0,t], \mathbb{R}^m) \) and \( g_n \in C(\mathbb{R}^m, \mathbb{R}) \), \( n \in \mathbb{N}_0 \), \( t \in [0,T] \), \( x \in \mathbb{R}^d \), such that for all \( q \in (0, \infty) \),

\[
\sup_{t \in [0,T]} \sup_{x \in [0,t]} \sup_{s \in [-q,q]^d} \|f^{0,t,x}(s)\|_{\mathbb{R}^m} < \infty,
\]

\[
\lim_{n \to \infty} \sup_{t \in [0,T]} \sup_{x \in [-q,q]^d} \|f^{n,t,x}(s) - f^{0,t,x}(s)\|_{\mathbb{R}^m} = 0
\]

\[
\lim_{n \to \infty} \sup_{t \in [0,T]} \sup_{x \in [-q,q]^d} \|g_n(y) - g_0(y)\| = 0.
\]

Then, one has for all \( q \in (0, \infty) \),

\[
\lim_{n \to \infty} \sup_{t \in [0,T]} \sup_{x \in [-q,q]^d} |g_n(f^{n,t,x}(s)) - g_0(f^{0,t,x}(x))| = 0.
\]

**Proof.** It follows from (17) and (18) that there exist \( n_q, N_q \in \mathbb{N} \), \( q \in (0, \infty) \) such that

\[
\sup_{n \in \mathbb{N}_0} \sup_{t \in [0,T]} \sup_{x \in [0,t]} \sup_{s \in [-q,q]^d} \|f^{n,t,x}(s)\|_{\mathbb{R}^m} \leq N_q < \infty
\]

for all \( q \in (0, \infty) \). So one obtains from the assumption that \( g_0 \in C(\mathbb{R}^m, \mathbb{R}) \) and (18) that

\[
\lim_{n \to \infty} \sup_{t \in [0,T]} \sup_{x \in [-q,q]^d} |g_0(f^{n,t,x}(s)) - g_0(f^{0,t,x}(s))| = 0 \quad \text{for all } q \in (0, \infty).
\]

Combining this with (19), (20), and the triangle inequality yields

\[
\lim_{n \to \infty} \left[ \sup_{t \in [0,T]} \sup_{s \in [0,t]} \sup_{x \in [-q,q]^d} |g_n(f^{n,t,x}(s)) - g_0(f^{0,t,x}(s))| \right]
\]

\[
\leq \lim_{n \to \infty} \sup_{t \in [0,T]} \sup_{s \in [0,t]} \sup_{x \in [-q,q]^d} |g_n(f^{n,t,x}(s)) - g_0(f^{n,t,x}(s))| + \lim_{n \to \infty} \sup_{t \in [0,T]} \sup_{s \in [0,t]} \sup_{x \in [-q,q]^d} |g_0(f^{n,t,x}(s)) - g_0(f^{0,t,x}(s))|
\]

\[
\leq \lim_{n \to \infty} \sup_{t \in [0,T]} \sup_{s \in [0,t]} \sup_{y \in [-N_q,N_q]^m} |g_n(y) - g_0(y)| = 0
\]

for all \( q \in (0, \infty) \), which completes the proof of the lemma. \( \square \)
A.2 Integrability and regularity properties for solutions of ODEs with additive noise

The following is a consequence of [32, Corollary 3.2]. For more details; see e.g., [25, Lemma 6.3].

**Lemma 4.** Let $T, c \in [0, \infty)$, $\alpha \in [0, 2)$ and $d \in \mathbb{N}$. Let $W : [0, T] \times \Omega \rightarrow \mathbb{R}^d$ be a standard Brownian motion with continuous sample paths defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then

(i) the mapping $\Omega \ni \omega \mapsto \sup_{t \in [0,T]} \|W(t,\omega)\|_{\mathbb{R}^d}^\alpha \in \mathbb{R}$ is $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable

(ii) $\mathbb{E} \left[ \exp \left( c \sup_{t \in [0,T]} \|W(t,\omega)\|_{\mathbb{R}^d}^\alpha \right) \right] < \infty$.

**Lemma 5.** Let $T, c \in (0, \infty)$, $d \in \mathbb{N}$ and $\sigma \in \mathbb{R}^{d \times d}$. Let $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a Borel measurable function satisfying $\|b(t,x)\|_{\mathbb{R}^d} \leq c(1 + \|x\|_{\mathbb{R}^d})$ for all $t \in [0,T]$ and $x \in \mathbb{R}^d$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space supporting a standard Brownian motion $U : [0, T] \times \Omega \rightarrow \mathbb{R}^d$, $x \in \mathbb{R}^d$, $t \in [0, T]$ satisfying

$$R^t_{s,x} = x + \int_0^s b(t, \sigma U_r, x) \, dr + \sigma U_r, \quad \text{for all } t \in [0, T], \, s \in [0,t] \text{ and } x \in \mathbb{R}^d.$$ \hspace{1cm} (21)

Then

(i) $\mathbb{E} \left[ \exp \left( p \sup_{t \in [0,T]} \sup_{s \in [0,t]} \|R^t_{s,x}\|_{\mathbb{R}^d} \right) \right] < \infty \quad \text{for all } p \in (0, \infty) \text{ and } x \in \mathbb{R}^d,$

(ii) $\mathbb{E} \left[ \exp \left( p \sup_{t \in [0,T]} \sup_{s \in [0,t]} \sup_{x \in [-p,p]^d} \|R^t_{s,x}\|_{\mathbb{R}^d} \right) \right] < \infty \quad \text{for all } p \in (0, \infty).$

**Proof.** It follows from [21], the triangle inequality and the assumption that $\|b(t, x)\|_{\mathbb{R}^d} \leq c(1 + \|x\|_{\mathbb{R}^d})$ for all $t \in [0, T]$ and $x \in \mathbb{R}^d$ that

$$\|R^t_{s,x}\|_{\mathbb{R}^d} \leq \|x\|_{\mathbb{R}^d} + \int_0^s c(1 + \|R^t_{r,x}\|_{\mathbb{R}^d}) \, dr + \sup_{r \in [0,T]} \|\sigma U_r\|_{\mathbb{R}^d}$$

$$\leq \|x\|_{\mathbb{R}^d} + cT + \sup_{r \in [0,T]} \|\sigma U_r\|_{\mathbb{R}^d} + c \int_0^s \|R^t_{r,x}\|_{\mathbb{R}^d} \, dr$$

for all $x \in \mathbb{R}^d$, $t \in [0, T]$ and $s \in [0, t]$. Hence, one obtains from Gronwall’s integral inequality (cf., e.g., [24, Lemma 2.11]) that

$$\|R^t_{s,x}\|_{\mathbb{R}^d} \leq \left( \|x\|_{\mathbb{R}^d} + cT + \sup_{r \in [0,T]} \|\sigma U_r\|_{\mathbb{R}^d} \right) e^{cT}$$

for all $x \in \mathbb{R}^d$, $t \in [0, T]$ and $s \in [0, t]$. So it follows from Lemma 4 that

$$\mathbb{E} \left[ \exp \left( p \sup_{t \in [0,T]} \sup_{s \in [0,t]} \|R^t_{s,x}\|_{\mathbb{R}^d} \right) \right] \leq \exp(pcT \|x\|_{\mathbb{R}^d}) \exp(pT e^{cT}) \mathbb{E} \left[ \exp \left( p \sup_{r \in [0,T]} \|\sigma U_r\|_{\mathbb{R}^d} \right) \right] < \infty$$

for all $p \in (0, \infty)$ and $x \in \mathbb{R}^d$, as well as

$$\mathbb{E} \left[ \exp \left( p \sup_{t \in [0,T]} \sup_{x \in [-p,p]^d} \sup_{s \in [0,t]} \|R^t_{s,x}\|_{\mathbb{R}^d} \right) \right] \leq \exp(pcT \sqrt{dp}) \exp(pT e^{cT}) \mathbb{E} \left[ \exp \left( p \sup_{r \in [0,T]} \|\sigma U_r\|_{\mathbb{R}^d} \right) \right] < \infty$$

for all $p \in (0, \infty)$, which proves the lemma. \hfill \square
Lemma 6. Let $T \in (0, \infty)$, $d \in \mathbb{N}$ and $\sigma \in \mathbb{R}^{d \times d}$. Denote by $e_i$, $i \in \{ 1, 2, \ldots, d \}$, the standard unit vectors in $\mathbb{R}^d$. Let $b \in C^{0,2}([0, T] \times \mathbb{R}^d, \mathbb{R}^d)$ have bounded partial derivatives of first and second order with respect to the $x$-variables. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space supporting a standard Brownian motion $U : [0, T] \times \Omega \to \mathbb{R}^d$ with continuous sample paths, and let $R^{i,x} = (R^{i,x}_s)_{s \in [0, t]} : [0, t] \times \Omega \to \mathbb{R}^d$, $t \in [0, T]$, $x \in \mathbb{R}^d$, be stochastic processes satisfying

$$R^{i,x}_s = x + \int_0^s b(t - r, R^{i,x}_r) \, dr + \sigma U_s \quad \text{for all } t \in [0, T], \; s \in [0, t] \text{ and } x \in \mathbb{R}^d. \tag{22}$$

Then

(i) for all $t \in [0, T]$ and $\omega \in \Omega$, the mapping $(s, x) \mapsto R^{i,x}_s(\omega)$ is in $C^{0,2}([0, t] \times \mathbb{R}^d, \mathbb{R}^d)$,

(ii) $\frac{\partial}{\partial x_i} R^{i,x}_s = e_i + \int_0^s \left[ D_x b(t - r, R^{i,x}_r) \right] \left( \frac{\partial}{\partial x_j} R^{i,x}_r \right) \, dr$

for all $i \in \{ 1, 2, \ldots, d \}$, $t \in [0, T]$, $s \in [0, t]$ and $x \in \mathbb{R}^d$,

(iii) $\sup_{t \in (0, T]} \sup_{s \in [0, t]} \sup_{x \in \mathbb{R}^d} \| \frac{\partial}{\partial x_i} R^{i,x}_s \|_{\mathbb{R}^d} \leq \exp \left( T \sup_{(t, x) \in [0, T] \times \mathbb{R}^d} \| D_x b(t, x) \|_{L(\mathbb{R}^d, \mathbb{R}^d)} \right) < \infty$

for all $i \in \{ 1, 2, \ldots, d \}$,

(iv) $\frac{\partial^2}{\partial x_i \partial x_j} R^{i,x}_s = \int_0^s \left[ D_x^2 b(t - r, R^{i,x}_r) \right] \left( \frac{\partial}{\partial x_i} R^{i,x}_r \right) \left( \frac{\partial}{\partial x_j} R^{i,x}_r \right) \, dr + \int_0^s \left[ D_x b(t - r, R^{i,x}_r) \right] \left( \frac{\partial^2}{\partial x_i \partial x_j} R^{i,x}_r \right) \, dr$

for all $i, j \in \{ 1, 2, \ldots, d \}$, $t \in [0, T]$, $s \in [0, t]$ and $x \in \mathbb{R}^d$,

(v) $\sup_{t \in (0, T]} \sup_{s \in [0, t]} \sup_{x \in \mathbb{R}^d} \| \frac{\partial^2}{\partial x_i \partial x_j} R^{i,x}_s \|_{\mathbb{R}^d} \leq T \sup_{(t, x) \in [0, T] \times \mathbb{R}^d} \| D_x^2 b(t, x) \|_{L(\mathbb{R}^d, \mathbb{R}^d)} \times \exp \left( 3T \sup_{(t, x) \in [0, T] \times \mathbb{R}^d} \| D_x b(t, x) \|_{L(\mathbb{R}^d, \mathbb{R}^d)} \right) < \infty$

for all $i, j \in \{ 1, 2, \ldots, d \}$.

Proof. Since, by assumption, $b \in C^{0,2}([0, T] \times \mathbb{R}^d, \mathbb{R}^d)$ has bounded partial derivatives with respect to the $x$-variables, it follows from \cite{22}, the de la Vallée Poussin theorem (cf., e.g., \cite{28} Corollary 6.21) and the Vitali convergence theorem (cf., e.g., \cite{28} Theorem 6.25) that for all $i \in \{ 1, 2, \ldots, d \}$, $t \in [0, T]$ and $\omega \in \Omega$, the mapping $(s, x) \mapsto R^{i,x}_s(\omega)$ is in $C^{0,1}([0, t] \times \mathbb{R}^d, \mathbb{R}^d)$ and satisfies

$$\frac{\partial}{\partial x_i} R^{i,x}_s = e_i + \int_0^s \left[ D_x b(t - r, R^{i,x}_r) \right] \left( \frac{\partial}{\partial x_i} R^{i,x}_r \right) \, dr \quad \text{for all } s \in [0, t], \; x \in \mathbb{R}^d. \tag{23}$$

(cf. also, e.g., \cite{29} Theorem 4.6.5). This shows (ii).

Now, (i) and the assumption that $b \in C^{0,2}([0, T] \times \mathbb{R}^d, \mathbb{R}^d)$ has bounded partial derivatives with respect to the $x$-variables imply that for all $i \in \{ 1, 2, \ldots, d \}$, $x \in \mathbb{R}^d$, $t \in [0, T]$ and $s \in [0, t]$, one has

$$\| \frac{\partial}{\partial x_i} R^{i,x}_s \|_{\mathbb{R}^d} \leq 1 + \int_0^s \sup_{(w, y) \in [0, T] \times \mathbb{R}^d} \| D_x b(w, y) \|_{L(\mathbb{R}^d, \mathbb{R}^d)} \| \frac{\partial}{\partial x_i} R^{i,x}_r \|_{\mathbb{R}^d} \, dr,$n

which, by Gronwall’s integral inequality (cf., e.g., \cite{24} Lemma 2.11), yields

$$\sup_{t \in (0, T]} \sup_{s \in [0, t]} \sup_{x \in \mathbb{R}^d} \| \frac{\partial}{\partial x_i} R^{i,x}_s \|_{\mathbb{R}^d} \leq \exp \left( T \sup_{(t, x) \in [0, T] \times \mathbb{R}^d} \| D_x b(t, x) \|_{L(\mathbb{R}^d, \mathbb{R}^d)} \right) < \infty,$n

for all $i \in \{ 1, 2, \ldots, d \}$. This establishes (iii).
and satisfies

\[ \frac{\partial^2}{\partial x_i \partial x_j} R_{t,x}^s = \int_0^s \left[ D_2^2 b(t-r, R_{t,x}^r) \left( \frac{\partial}{\partial x_i} R_{t,x}^r, \frac{\partial}{\partial x_j} R_{t,x}^r \right) dr + \int_0^r \left[ D_x b(t-r, R_{t,x}^r) \right] \left( \frac{\partial^2}{\partial x_i \partial x_j} R_{t,x}^r \right) dr \right. \]

for all \( s \in [0,t] \) and \( x \in \mathbb{R}^d \) (cf. also, e.g., [29] Theorem 4.6.5)). This shows \([i]\) and \([iv]\).

Since \( b \in C^{0,2}([0,T] \times \mathbb{R}^d, \mathbb{R}^d) \) has bounded partial derivatives of second order with respect to the \( x \)-variables, it follows from \([iii]\)–\([v]\) that for all \( i, j \in \{1,2,\ldots,d\} \), \( x \in \mathbb{R}^d \), \( t \in [0,T] \) and \( s \in [0,t] \), one has

\[
\left\| \frac{\partial^2}{\partial x_i \partial x_j} R_{t,x}^s \right\|_{\mathbb{R}^d} \leq T \left[ \sup_{(w,y) \in [0,T] \times \mathbb{R}^d} \left\| D_2^2 b(w,y) \right\|_{L^2(\mathbb{R}^d, \mathbb{R}^d)} \right] \\
\times \exp \left( 2T \sup_{(w,y) \in [0,T] \times \mathbb{R}^d} \left\| D_x b(w,y) \right\|_{L(\mathbb{R}^d, \mathbb{R}^d)} \right) \\
+ \int_0^s \left[ \sup_{(w,y) \in [0,T] \times \mathbb{R}^d} \left\| D_x b(w,y) \right\|_{L(\mathbb{R}^d, \mathbb{R}^d)} \right] \left\| \frac{\partial^2}{\partial x_i \partial x_j} R_{t,x}^r \right\|_{\mathbb{R}^d} dr,
\]

which, by Gronwall’s integral inequality (cf., e.g., [24] Lemma 2.11)), implies that

\[
\sup_{t \in [0,T]} \sup_{s \in [0,t]} \sup_{x \in \mathbb{R}^d} \left\| \frac{\partial^2}{\partial x_i \partial x_j} R_{t,x}^s \right\|_{\mathbb{R}^d} \leq T \left[ \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \left\| D_2^2 b(t,x) \right\|_{L^2(\mathbb{R}^d, \mathbb{R}^d)} \right] \exp \left( 3T \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \left\| D_x b(t,x) \right\|_{L(\mathbb{R}^d, \mathbb{R}^d)} \right) < \infty,
\]

for all \( i, j \in \{1,2,\ldots,d\} \). This establishes \([v]\) and completes the proof of the lemma. \( \square \)

A.3 Stability properties of solutions of ODEs with additive noise

**Lemma 7.** Let \( T \in (0, \infty) \), \( d \in \mathbb{N} \), \( \sigma \in \mathbb{R}^{d \times d} \), and consider two mappings \( b, \sigma \in C^{0,2}([0,T] \times \mathbb{R}^d, \mathbb{R}^d) \) that have bounded partial derivatives of first and second order with respect to the \( x \)-variables. Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space supporting a standard Brownian motion \( U : [0,T] \times \Omega \to \mathbb{R}^d \) with continuous sample paths, and consider stochastic processes \( R_{t,x}^t, \mathcal{R}_{t,x}^t : [0,T] \times \Omega \to \mathbb{R}^d \), \( t \in [0,T] \), \( x \in \mathbb{R}^d \), satisfying

\[
R_{t,x}^s = x + \int_0^s b(t-r, R_{t,x}^r) \, dr + \sigma U_s \quad \text{and} \quad \mathcal{R}_{t,x}^s = x + \int_0^s b(t-r, R_{t,x}^r) \, dr + \sigma U_s \quad (23)
\]

for all \( t \in [0,T] \), \( s \in [0,t] \) and \( x \in \mathbb{R}^d \). Then

(i) \[
\sup_{s \in [0,t]} \left\| R_{t,x}^s - R_{t,x}^t \right\|_{\mathbb{R}^d} \leq T \sup_{(r,y) \in [0,T] \times \mathbb{R}^d} \left\| b(r,y) - b(r,y) \right\|_{\mathbb{R}^d} \exp \left( T \sup_{(r,y) \in [0,T] \times \mathbb{R}^d} \left\| D_x b(r,y) \right\|_{L(\mathbb{R}^d, \mathbb{R}^d)} \right),
\]

for all \( t \in [0,T] \) and \( x \in \mathbb{R}^d \),

(ii) \[
\sup_{s \in [0,t]} \left\| \frac{\partial}{\partial x_i} R_{t,x}^s - \frac{\partial}{\partial x_i} R_{t,x}^t \right\|_{\mathbb{R}^d} \leq T \sup_{s \in [0,t]} \left\| D_x b(t-s, R_{t,x}^s) - D_x b(t-s, \mathcal{R}_{t,x}^t) \right\|_{L(\mathbb{R}^d, \mathbb{R}^d)} \times \exp \left( T \sup_{(r,y) \in [0,T] \times \mathbb{R}^d} \left\| D_x b(r,y) \right\|_{L(\mathbb{R}^d, \mathbb{R}^d)} + \sup_{(r,y) \in [0,T] \times \mathbb{R}^d} \left\| D_x b(r,y) \right\|_{L(\mathbb{R}^d, \mathbb{R}^d)} \right),
\]

for all \( i \in \{1,2,\ldots,d\} \), \( t \in [0,T] \), \( x \in \mathbb{R}^d \), and
(iii) \[ \sup_{s \in [0,t]} \| \frac{\partial^2}{\partial x_i \partial x_j} R^{t,x}_s - \frac{\partial^2}{\partial x_i \partial x_j} R^{t,x}_s \|_{\mathbb{R}^d} \]
\[ \leq \exp \left( T \sup_{(r,y) \in [0,T] \times \mathbb{R}^d} \| D_x b(r,y) \|_{L(\mathbb{R}^d,\mathbb{R}^d)} \right) \]
\[ \times \left[ 2T^2 \sup_{(r,y) \in [0,T] \times \mathbb{R}^d} \| D_x^2 b(r,y) \|_{L(\mathbb{R}^d,\mathbb{R}^d)} \right] \]
\[ \times \left[ 3T \sup_{(r,y) \in [0,T] \times \mathbb{R}^d} \max \left\{ \| D_x b(t-s, R^{t,x}_s) - D_x b(t-s, R^{t,x}_s) \|_{L(\mathbb{R}^d,\mathbb{R}^d)} \right\} \right] \]
\[ + T \exp \left( 2T \sup_{(r,y) \in [0,T] \times \mathbb{R}^d} \| D_x b(r,y) \|_{L(\mathbb{R}^d,\mathbb{R}^d)} \right) \]
\[ \times \sup_{s \in [0,t]} \| D_x^2 b(t-s, R^{t,x}_s) - D_x^2 b(t-s, R^{t,x}_s) \|_{L(\mathbb{R}^d,\mathbb{R}^d)} \]
\[ + T^2 \sup_{(r,y) \in [0,T] \times \mathbb{R}^d} \| D_x^2 b(r,y) \|_{L(\mathbb{R}^d,\mathbb{R}^d)} \]
\[ \times \left[ 3T \sup_{(r,y) \in [0,T] \times \mathbb{R}^d} \| D_x b(r,y) \|_{L(\mathbb{R}^d,\mathbb{R}^d)} \right] \]
\[ \times \sup_{s \in [0,t]} \| D_x b(t-s, R^{t,x}_s) - D_x b(t-s, R^{t,x}_s) \|_{\mathbb{R}^d} \] .

for all \( i, j \in \{1, 2, \ldots, d\} \), \( t \in [0,T] \) and \( x \in \mathbb{R}^d \).

Proof. Throughout this proof we assume without loss of generality that

\[ \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \| b(t,x) - b(t,x) \|_{\mathbb{R}^d} < \infty. \] (24)

From (23) and the triangle inequality we obtain

\[ \| R^{t,x}_s - R^{t,x}_s \|_{\mathbb{R}^d} \leq \int_0^s \| b(t-r, R^{t,x}_r) - b(t-r, R^{t,x}_r) \|_{\mathbb{R}^d} dr \]
\[ \leq \int_0^s \left[ \| b(t-r, R^{t,x}_r) - b(t-r, R^{t,x}_r) \|_{\mathbb{R}^d} + \| b(t-r, R^{t,x}_r) - b(t-r, R^{t,x}_r) \|_{\mathbb{R}^d} \right] dr \]

for all \( t \in [0,T] \), \( s \in [0,t] \) and \( x \in \mathbb{R}^d \). Combining this with (24) and the assumption that \( b \) has bounded partial derivatives with respect to the \( x \)-variables yields

\[ \| R^{t,x}_s - R^{t,x}_s \|_{\mathbb{R}^d} \leq T \sup_{(r,y) \in [0,T] \times \mathbb{R}^d} \| b(r,y) - b(r,y) \|_{\mathbb{R}^d} \]
\[ + \sup_{(r,y) \in [0,T] \times \mathbb{R}^d} \| D_x b(r,y) \|_{L(\mathbb{R}^d,\mathbb{R}^d)} \int_0^s \| R^{t,x}_r - R^{t,x}_r \|_{\mathbb{R}^d} dr \]

for all \( t \in [0,T] \), \( s \in [0,t] \) and \( x \in \mathbb{R}^d \). Therefore, we obtain from Gronwall’s integral inequality (cf., e.g., [24] Lemma 2.11) that

\[ \sup_{s \in [0,t]} \| R^{t,x}_s - R^{t,x}_s \|_{\mathbb{R}^d} \]
\[ \leq T \sup_{(r,y) \in [0,T] \times \mathbb{R}^d} \| b(r,y) - b(r,y) \|_{\mathbb{R}^d} \exp \left( T \sup_{(r,y) \in [0,T] \times \mathbb{R}^d} \| D_x b(r,y) \|_{L(\mathbb{R}^d,\mathbb{R}^d)} \right) \]

for all \( t \in [0,T] \) and \( x \in \mathbb{R}^d \), which establishes (3). 

Next, observe that (23), Lemma 6.13 and the triangle inequality imply that

\[ \| \frac{\partial}{\partial x_i} R^{t,x}_s - \frac{\partial}{\partial x_i} R^{t,x}_s \|_{\mathbb{R}^d} \leq \int_0^s \left\| \left[ D_x b(t-r, R^{t,x}_r) \right] \left( \frac{\partial}{\partial x_i} R^{t,x}_r \right) - \left[ D_x b(t-r, R^{t,x}_r) \right] \left( \frac{\partial}{\partial x_i} R^{t,x}_r \right) \right\|_{\mathbb{R}^d} dr \]
\[ \leq \int_0^s \| D_x b(t-r, R^{t,x}_r) \|_{L(\mathbb{R}^d,\mathbb{R}^d)} \| \frac{\partial}{\partial x_i} R^{t,x}_r - \frac{\partial}{\partial x_i} R^{t,x}_r \|_{\mathbb{R}^d} dr \]
\[ + \int_0^s \| D_x (t-r, R^{t,x}_r) - D_x b(t-r, R^{t,x}_r) \|_{L(\mathbb{R}^d,\mathbb{R}^d)} \| \frac{\partial}{\partial x_i} R^{t,x}_r \|_{\mathbb{R}^d} dr \]
for all $i \in \{1, 2, \ldots, d\}$, $t \in [0, T]$, $s \in [0, t]$ and $x \in \mathbb{R}^d$. Therefore, we obtain from another application of Lemma 6 (iii) that

$$
\frac{\partial}{\partial x_i} R_{t,x}^{l,s} - \frac{\partial}{\partial x_i} R_{s}^{l,s} \leq \sup_{r(y) \in [0,T] \times \mathbb{R}^d} \| D_x b(r, y) \|_{L(\mathbb{R}^d, \mathbb{R}^d)} \int_0^s \| \frac{\partial}{\partial x_i} R_{t,x}^{l,r} - \frac{\partial}{\partial x_i} R_{r}^{l,x} \|_{\mathbb{R}^d} \, dr
$$

$$
+ T \sup_{r(y) \in [0,T] \times \mathbb{R}^d} \| D_x b(t - r, R_{t,x}^{l,r}) - D_x b(t - r, R_{r,x}^{l,r}) \|_{L(\mathbb{R}^d, \mathbb{R}^d)}
$$

$$
\times \exp \left( T \sup_{r(y) \in [0,T] \times \mathbb{R}^d} \| D_x b(r, y) \|_{L(\mathbb{R}^d, \mathbb{R}^d)} \right)
$$

for all $i \in \{1, 2, \ldots, d\}$, $t \in [0, T]$, $s \in [0, t]$ and $x \in \mathbb{R}^d$. Gronwall’s inequality (cf., e.g., [24, Lemma 2.11]) hence ensures that

$$
\frac{\partial}{\partial x_i} R_{t,x}^{l,s} - \frac{\partial}{\partial x_i} R_{s}^{l,s} \leq \sup_{r(y) \in [0,T] \times \mathbb{R}^d} \| D_x b(t - r, R_{t,x}^{l,r}) - D_x b(t - r, R_{r,x}^{l,r}) \|_{L(\mathbb{R}^d, \mathbb{R}^d)}
$$

$$
\times \exp \left( T \sup_{r(y) \in [0,T] \times \mathbb{R}^d} \| D_x b(r, y) \|_{L(\mathbb{R}^d, \mathbb{R}^d)} + \sup_{r(y) \in [0,T] \times \mathbb{R}^d} \| D_x b(r, y) \|_{L(\mathbb{R}^d, \mathbb{R}^d)} \right)
$$

for all $i \in \{1, 2, \ldots, d\}$, $t \in [0, T]$, $s \in [0, t]$ and $x \in \mathbb{R}^d$. This shows (iii). Now, note that it follows from (23) and Lemma 6 (iv) that

$$
\frac{\partial^2}{\partial x_i \partial x_j} R_{t,x}^{l,s} - \frac{\partial^2}{\partial x_i \partial x_j} R_{s}^{l,s} = \int_0^s \left[ D_x^2 b(t - r, R_{t,x}^{l,r}) \left( \frac{\partial}{\partial x_i} R_{t,x}^{l,r} - \frac{\partial}{\partial x_j} R_{t,x}^{l,r} \right) \right] \, dr
$$

$$
+ \int_0^s \left[ D_x^2 b(t - r, R_{t,x}^{l,r}) \left( \frac{\partial}{\partial x_i} R_{t,x}^{l,r} - \frac{\partial}{\partial x_j} R_{t,x}^{l,r} \right) \right] \, dr
$$

$$
+ \int_0^s \left[ D_x^2 b(t - r, R_{t,x}^{l,r}) \left( \frac{\partial}{\partial x_i} R_{t,x}^{l,r} - \frac{\partial}{\partial x_j} R_{t,x}^{l,r} \right) \right] \, dr
$$

$$
+ \int_0^s \left[ D_x^2 b(t - r, R_{t,x}^{l,r}) \left( \frac{\partial}{\partial x_i} R_{t,x}^{l,r} - \frac{\partial}{\partial x_j} R_{t,x}^{l,r} \right) \right] \, dr
$$

for all $i, j \in \{1, 2, \ldots, d\}$, $t \in [0, T]$, $s \in [0, t]$ and $x \in \mathbb{R}^d$. (ii) together with (iii) and (v) of Lemma 6 therefore yield

$$
\left\| \frac{\partial^2}{\partial x_i \partial x_j} R_{t,x}^{l,s} - \frac{\partial^2}{\partial x_i \partial x_j} R_{s}^{l,s} \right\|_{\mathbb{R}^d}
$$

$$
\leq 2T^2 \sup_{r(y) \in [0,T] \times \mathbb{R}^d} \| D_x^2 b(r, y) \|_{L(\mathbb{R}^d, \mathbb{R}^d)}
$$

$$
\times \exp \left( 3T \sup_{r(y) \in [0,T] \times \mathbb{R}^d} \max \left\{ \| D_x b(r, y) \|_{L(\mathbb{R}^d, \mathbb{R}^d)}, \| D_x b(r, y) \|_{L(\mathbb{R}^d, \mathbb{R}^d)} \right\} \right)
$$

$$
\times \sup_{r \in [0,t]} \| D_x b(t - r, R_{t,x}^{l,r}) - D_x b(t - r, R_{r,x}^{l,r}) \|_{L(\mathbb{R}^d, \mathbb{R}^d)}
$$

$$
+ T \exp \left( 2T \sup_{r(y) \in [0,T] \times \mathbb{R}^d} \| D_x b(r, y) \|_{L(\mathbb{R}^d, \mathbb{R}^d)} \right)
$$

$$
\times \sup_{r \in [0,t]} \| D_x b(t - r, R_{t,x}^{l,r}) - D_x b(t - u, R_{t,x}^{l,r}) \|_{\mathbb{R}^d}
$$

$$
+ T^2 \sup_{r(y) \in [0,T] \times \mathbb{R}^d} \| D_x^2 b(r, y) \|_{L(\mathbb{R}^d, \mathbb{R}^d)}
$$

$$
\times \exp \left( 3T \sup_{r(y) \in [0,T] \times \mathbb{R}^d} \| D_x b(r, y) \|_{L(\mathbb{R}^d, \mathbb{R}^d)} \right)
$$

$$
\times \sup_{r \in [0,t]} \| D_x b(t - r, R_{t,x}^{l,r}) - D_x b(t - u, R_{t,x}^{l,r}) \|_{\mathbb{R}^d}
$$

$$
+ \sup_{r(y) \in [0,T] \times \mathbb{R}^d} \| D_x b(r, y) \|_{L(\mathbb{R}^d, \mathbb{R}^d)} \int_0^s \left\| \frac{\partial^2}{\partial x_i \partial x_j} R_{t,x}^{l,r} - \frac{\partial^2}{\partial x_i \partial x_j} R_{r}^{l,x} \right\|_{\mathbb{R}^d} \, dr
$$

for all $i, j \in \{1, 2, \ldots, d\}$, $t \in [0, T]$, $s \in [0, t]$ $x \in \mathbb{R}^d$. So it follows from Gronwall’s integral inequality.
(cf., e.g., [21 Lemma 2.11]) that
\[
\frac{\partial^2}{\partial x_i \partial x_j} R^{i,x}_s - \frac{\partial^2}{\partial x_i \partial x_j} R^{i,x}_t \leq \exp \left( T \sup_{(r,y) \in [0,T] \times \mathbb{R}^d} \| D_x b(r, y) \|_{L([\mathbb{R}^d], \mathbb{R}^d))} \right) \\
\times \left[ 2T^2 \sup_{(r,y) \in [0,T] \times \mathbb{R}^d} \| D^2_2 b(r, y) \|_{L(2)([\mathbb{R}^d], \mathbb{R}^d))} \right] \\
\times \left[ \exp \left( 3T \sup_{(r,y) \in [0,T] \times \mathbb{R}^d} \max \left\{ \| D_x b(r, y) \|_{L([\mathbb{R}^d], \mathbb{R}^d))}, \| D_b b(r, y) \|_{L([\mathbb{R}^d], \mathbb{R}^d))} \right\} \right) \\
\times \sup_{s \in [0,t]} \| D_x b(t-s, R^{i,x}_s) - D_x b(t-s, R^{i,x}_t) \|_{L([\mathbb{R}^d], \mathbb{R}^d))} + T \exp \left( 2T \sup_{(r,y) \in [0,T] \times \mathbb{R}^d} \| D_x b(r, y) \|_{L([\mathbb{R}^d], \mathbb{R}^d))} \right) \\
\times \sup_{r \in [0,t]} \| D^2_2 b(t-r, R^{i,x}_r) - D^2_2 b(t-r, R^{i,x}_t) \|_{L(2)([\mathbb{R}^d], \mathbb{R}^d))} \\
\times \exp \left( 3T \sup_{(r,y) \in [0,T] \times \mathbb{R}^d} \| D_x b(r, y) \|_{L([\mathbb{R}^d], \mathbb{R}^d))} \right) \\
\times \sup_{r \in [0,t]} \| D_x b(t-r, R^{i,x}_r) - D_x b(t-r, R^{i,x}_t) \|_{\mathbb{R}^d} \right]
\]
for all \( i, j \in \{1, 2, \ldots, d\} \), \( t \in [0,T] \), \( s \in [0,t] \) and \( x \in \mathbb{R}^d \). This shows \( \text{(iii)} \) and completes the proof of the lemma.

\[ \square \]

### A.4 Differentiability properties of certain random fields defined in terms of ODEs with additive noise

**Lemma 8.** Let \( T \in (0, \infty) \), \( d \in \mathbb{N} \), \( \sigma \in \mathbb{R}^{d \times d} \), and consider a function \( b \in C^{0,2}([0,T] \times \mathbb{R}^d, \mathbb{R}^d) \) with bounded partial derivatives of first and second order with respect to the \( x \)-variables. Let \( \varphi \in C^2([\mathbb{R}^d, [0, \infty)) \) and \( B \in C^{0,2}([0,T] \times \mathbb{R}^d, \mathbb{R}) \) such that \( \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \frac{B(t,x)}{1+\|x\|_{\mathbb{R}^d}} < \infty \) and the first and second order partial derivatives of \( B \) with respect to the \( x \)-variables are at most polynomially growing. Let \( (\Omega, \mathcal{F}, P) \) be a probability space supporting a standard Brownian motion \( U : [0,T] \times \Omega \to \mathbb{R}^d \) with continuous sample paths. Consider stochastic processes \( R^{i,x} : [0,T] \times \Omega \to \mathbb{R}^d \), \( t \in [0,T] \), \( x \in \mathbb{R}^d \), satisfying
\[
R^{i,x}_s = x + \int_0^s b(t-r, R^{i,x}_r) \, dr + \sigma U_r
\]
for all \( x \in \mathbb{R}^d \), \( t \in [0,T] \) and \( s \in [0,t] \). Let the function \( \mathcal{U} : [0,T] \times \mathbb{R}^d \times \Omega \to \mathbb{R} \) be given by
\[
\mathcal{U}(t, x) = \varphi(R^{i,x}_t) \exp \left( \int_0^t B(t-s, R^{i,x}_s) \, ds \right)
\]
for all \( t \in [0,T] \) and \( x \in \mathbb{R}^d \). Then

(i) for all \( \omega \in \Omega \), the mapping \((t,x) \mapsto \mathcal{U}(t,x,\omega)\) is in \( C^{0,2}([0,T] \times \mathbb{R}^d, \mathbb{R}) \),

(ii) \[
\frac{\partial}{\partial x_i} \mathcal{U}(t, x) = \exp \left( \int_0^t B(t-s, R^{i,x}_s) \, ds \right) \\
\times \left[ \nabla \varphi(R^{i,x}_t) \left( \frac{\partial}{\partial x_i} R^{i,x}_t \right) + \varphi(R^{i,x}_t) \int_0^t \left[ D_x B(t-s, R^{i,x}_s) \right] \frac{\partial}{\partial x_i} R^{i,x}_s \, ds \right]
\]
for all \( i \in \{1, 2, \ldots, d\} \), \( t \in [0,T] \), \( x \in \mathbb{R}^d \), and

(iii)
\[ \frac{\partial^2}{\partial x_i \partial x_j} U(t,x) = \exp \left( \int_0^t B(t-s,R^t_s,x) \text{ ds} \right) \left[ D^2 \varphi(R^t_i) \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) \right] \\
+ \left[ \nabla \varphi(R^t_i) \right]^T \left( \frac{\partial^2}{\partial x_i \partial x_j} R^t_i \right) \\
+ \left[ \nabla \varphi(R^t_i) \right]^T \left( \frac{\partial}{\partial x_i} R^t_i \right) \int_0^t \left[ D_x B(t-s,R^t_s,x) \right] \left( \frac{\partial}{\partial x_j} R^t_s \right) \text{ ds} \\
+ \left[ \nabla \varphi(R^t_i) \right]^T \left( \frac{\partial}{\partial x_i} R^t_i \right) \int_0^t \left[ D_x B(t-s,R^t_s,x) \right] \left( \frac{\partial}{\partial x_j} R^t_s \right) \text{ ds} \\
+ \varphi(R^t_i) \int_0^t \left[ D_x B(t-s,R^t_s,x) \right] \left( \frac{\partial}{\partial x_i} R^t_s \right) \text{ ds} \\
\times \int_0^t \left[ D_x B(t-s,R^t_s,x) \right] \left( \frac{\partial}{\partial x_j} R^t_s \right) \text{ ds} \\
+ \varphi(R^t_i) \int_0^t \left[ D_x^2 B(t-s,R^t_s,x) \right] \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} R^t_s \right) \text{ ds} \\
+ \varphi(R^t_i) \int_0^t \left[ D_x^2 B(t-s,R^t_s,x) \right] \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} R^t_s \right) \text{ ds} \right] \\
\text{ for all } i,j \in \{ 1, 2, \ldots, d \}, t \in [0,T] \text{ and } x \in \mathbb{R}^d. \]

**Proof.** It follows from Lemma 6.1, our assumptions on \( b, \varphi \) and \( B \), the chain rule, the fundamental theorem of calculus, (25) and (26) that for all \( t \in [0,T] \) and \( \omega \in \Omega \), the mapping \( x \mapsto U(t,x,\omega) \) belongs to \( C^1(\mathbb{R}^d, \mathbb{R}) \) and

\[ \frac{\partial}{\partial x_i} U(t,x) = \left[ \nabla \varphi(R^t_i) \right]^T \left( \frac{\partial}{\partial x_i} R^t_i \right) \exp \left( \int_0^t B(t-s,R^t_s,x) \text{ ds} \right) \\
+ \varphi(R^t_i) \exp \left( \int_0^t B(t-s,R^t_s,x) \text{ ds} \right) \int_0^t \left[ D_x B(t-s,R^t_s,x) \right] \left( \frac{\partial}{\partial x_i} R^t_s \right) \text{ ds} \]

for all \( i \in \{ 1, 2, \ldots, d \}, t \in [0,T] \) and \( x \in \mathbb{R}^d \). This shows (iii).

Similarly, it follows from Lemma 6.2, (29), and the assumptions that for all \( t \in [0,T] \) and \( \omega \in \Omega \), the mapping \( x \mapsto U(t,x,\omega) \) is in \( C^2(\mathbb{R}^d, \mathbb{R}) \) and

\[ \exp \left( - \int_0^t B(t-s,R^t_s,x) \text{ ds} \right) \frac{\partial^2}{\partial x_i \partial x_j} U(t,x) \]

\[ = \left[ D^2 \varphi(R^t_i) \right] \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) + \left[ \nabla \varphi(R^t_i) \right]^T \left( \frac{\partial^2}{\partial x_i \partial x_j} R^t_i \right) \\
+ \left[ \nabla \varphi(R^t_i) \right]^T \left( \frac{\partial}{\partial x_i} R^t_i \right) \int_0^t \left[ D_x B(t-s,R^t_s,x) \right] \left( \frac{\partial}{\partial x_j} R^t_s \right) \text{ ds} \\
+ \left[ \nabla \varphi(R^t_i) \right]^T \left( \frac{\partial}{\partial x_i} R^t_i \right) \int_0^t \left[ D_x B(t-s,R^t_s,x) \right] \left( \frac{\partial}{\partial x_j} R^t_s \right) \text{ ds} \\
+ \varphi(R^t_i) \int_0^t \left[ D_x B(t-s,R^t_s,x) \right] \left( \frac{\partial}{\partial x_i} R^t_s \right) \text{ ds} \int_0^t \left[ D_x B(t-s,R^t_s,x) \right] \left( \frac{\partial}{\partial x_j} R^t_s \right) \text{ ds} \\
+ \varphi(R^t_i) \int_0^t \left[ D_x^2 B(t-s,R^t_s,x) \right] \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} R^t_s \right) \text{ ds} \\
\text{ for all } i,j \in \{ 1, 2, \ldots, d \}, t \in [0,T] \text{ and } x \in \mathbb{R}^d. \] This shows (i) and (iii) and completes the proof of the lemma.

**Lemma 9.** Let \( T \in (0,\infty), d \in \mathbb{N}, \sigma \in \mathbb{R}^{d \times d} \), and consider a function \( b \in C^{0,2}([0,T] \times \mathbb{R}^d, \mathbb{R}^d) \) with bounded partial derivatives of first and second order with respect to the \( x \)-variables. Let \( \varphi \in \mathbb{R}^{d \times d} \).
$C^2(\mathbb{R}^d, [0, \infty))$ and $B \in C^{0,2}([0, T] \times \mathbb{R}^d, \mathbb{R})$ have at most polynomially growing partial derivatives of first and second order with respect to the $x$-variables and assume that $\sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \frac{B(t,x)}{1 + \|x\|^2_{\mathbb{R}^d}} < \infty$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space supporting a standard Brownian motion $U: [0, T] \times \Omega \rightarrow \mathbb{R}$ with continuous sample paths. Consider stochastic processes $R^{t,x}: [0, T] \times \Omega \rightarrow \mathbb{R}^d$, $t \in [0, T]$, $x \in \mathbb{R}^d$, satisfying

$$R^{t,x}_s = x + \int_0^t b(t - r, R^{t,x}_r) \, dr + \sigma U_s \quad \text{for all } t \in [0, T], \ s \in [0, t] \text{ and } x \in \mathbb{R}^d.$$ 

Let the functions $U: [0, T] \times \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}$ and $u: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ be given by

$$U(t,x) = \phi(R^{t,x}_t) \exp\left(\int_0^t B(t - s, R^{t,x}_s) \, ds\right) \quad \text{and} \quad u(t,x) = \mathbb{E}[U(t,x)].$$

for $x \in \mathbb{R}^d$ and $t \in [0, T]$. Then

(i) $u \in C^{0,2}([0, T] \times \mathbb{R}^d, \mathbb{R}),$

(ii) $\frac{\partial}{\partial t} u(t,x) = \mathbb{E}\left[\frac{\partial}{\partial t} U(t,x)\right]$ for all $t \in [0, T]$, $x \in \mathbb{R}^d$, and

(iii) $\frac{\partial^2}{\partial t^2} u(t,x) = \mathbb{E}\left[\frac{\partial^2}{\partial t^2} U(t,x)\right]$ for all $t \in [0, T]$ and $x \in \mathbb{R}^d$.

Proof. Since, by assumption, $\varphi \in C^2(\mathbb{R}^d, [0, \infty))$ has at most polynomially growing partial derivatives of first and second order, we obtain from Lemma 5 that

$$\mathbb{E}\left[\sup_{t \in [0,T]} \sup_{s \in [0,t]} \sup_{x \in [-p,p]^d} |\varphi(R^{t,x}_s)|^p + \|\nabla \varphi(R^{t,x}_s)\|_{L(\mathbb{R}^d, \mathbb{R})}^p + \|D^2 \varphi(R^{t,x}_s)\|_{L_{2}(\mathbb{R}^d, \mathbb{R})}^p \right] < \infty$$

for all $p \in (0, \infty)$. Similarly, it follows from the assumption that $B \in C^{0,2}([0, T] \times \mathbb{R}^d, \mathbb{R})$ has at most polynomially growing partial derivatives of first and second order with respect to the $x$-variables and Lemma 5 that

$$\mathbb{E}\left[\sup_{t \in [0,T]} \sup_{s \in [0,t]} \sup_{x \in [-p,p]^d} |B(t - s, R^{t,x}_s)|^p + \|D_x B(t - s, R^{t,x}_s)\|_{L(\mathbb{R}^d, \mathbb{R})}^p + \|D_x^2 B(t - s, R^{t,x}_s)\|_{L(\mathbb{R}^d, \mathbb{R})}^p \right] < \infty$$

for all $p \in (0, \infty)$. The assumption that $\sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \frac{B(t,x)}{1 + \|x\|^2_{\mathbb{R}^d}} < \infty$ and Lemma 5 guarantee that

$$\mathbb{E}\left[\sup_{t \in [0,T]} \sup_{s \in [0,t]} \sup_{x \in [-p,p]^d} \left|\exp\left(\int_0^s B(t - r, R^{t,x}_r) \, dr\right)\right|^p \right] < \infty$$

for all $p \in (0, \infty)$. Combining (27)–(29) with Hölder’s inequality shows that

$$\mathbb{E}\left[\sup_{t \in [0,T]} \left|\sup_{x \in \mathbb{R}^d} U(t,x)\right|^p \right] < \infty \quad \text{for all } p \in (0, \infty),$$

which together with the de la Vallée Poussin theorem (cf., e.g., [28 Corollary 6.21]), the Vitali convergence theorem (cf., e.g., [28 Theorem 6.25]), and Lemma 8(i) implies that $u \in C([0, T] \times \mathbb{R}^d, \mathbb{R})$. Next, we note that (27)–(29), Lemma 8(ii), Lemma 8(iii) and Hölder’s inequality yield that

$$\mathbb{E}\left[\sup_{t \in [0,T]} \sup_{x \in [-p,p]^d} \left|\frac{\partial}{\partial t} U(t,x)\right|^p_{L(\mathbb{R}^d, \mathbb{R})} \right] < \infty.$$
(a) for all $t \in [0, T]$, the mapping $x \mapsto u(t, x)$ is in $C^1(\mathbb{R}^d, \mathbb{R})$, 
(b) for all $i \in \{1, 2, \ldots, d\}$, the mapping $(t, x) \mapsto \frac{\partial}{\partial x_i}u(t, x)$ is in $C([0, T] \times \mathbb{R}^d, \mathbb{R})$, and 
(c) $\frac{\partial}{\partial t}u(t, x) = \mathbb{E}[(\frac{\partial}{\partial x}U(t, x))]$ for all $x \in \mathbb{R}^d$ and $t \in [0, T]$.

This establishes (iii).

Next, note that it follows from (27)–(29), Lemma 8 (iii), items (vi) and (v) of Lemma 6 and Hölder’s inequality that

$$\mathbb{E} \left[ \sup_{t \in [0, T]} \sup_{x \in [-p, p]} \| \frac{\partial^2}{\partial x^2}u(t, x) \|_{L^p(\mathbb{R}^d, \mathbb{R})}^p \right] < \infty$$

for all $p \in (0, \infty)$. Hence, one obtains from Lemma 8 (i), (a)–(b) above, the de la Vallée Poussin theorem (cf., e.g., [28, Corollary 6.21]), the Vitali convergence theorem (cf., e.g., [28, Theorem 6.25]), and the fundamental theorem of calculus that

(A) for all $t \in [0, T]$, the mapping $x \mapsto u(t, x)$ is in $C^2(\mathbb{R}^d, \mathbb{R})$,

(B) for all $i, j \in \{1, 2, \ldots, d\}$, the mapping $(t, x) \mapsto \frac{\partial^2}{\partial x_i \partial x_j}u(t, x)$ is in $C([0, T] \times \mathbb{R}^d, \mathbb{R})$,

(C) $\frac{\partial}{\partial x}u(t, x) = \mathbb{E}[\frac{\partial}{\partial x}U(t, x)]$ for all $x \in \mathbb{R}^d$ and $t \in [0, T]$.

This directly establishes (iii). Moreover, (a)–(c), (A)–(C), and the fact that $u \in C([0, T] \times \mathbb{R}^d, \mathbb{R})$ imply (iv), which completes the proof of the lemma. □

A.5 Feynman–Kac representations for linear PDEs

The following lemma is a stepping stone towards the proof of the Feynman–Kac representation in Proposition 11 below. It makes stronger regularity assumptions on the coefficients. Proposition 11 can be derived from it by mollifying the coefficients.

Lemma 10. Let $T \in (0, \infty)$, $d \in \mathbb{N}$, $\sigma \in \mathbb{R}^{d \times d}$, and consider a function $b \in C^{1,2}([0, T] \times \mathbb{R}^d, \mathbb{R}^d)$ with a bounded partial derivative with respect to $t$ and bounded partial derivatives of first and second order with respect to the $x$-variables. Let $\varphi \in C^3([\mathbb{R}^d, [0, \infty)])$ be at most polynomially growing partial derivatives of first, second and third order. Let $B \in C^{1,2}([0, T] \times \mathbb{R}^d, \mathbb{R})$ have an at most polynomially growing partial derivative with respect to $t$ and at most polynomially growing partial derivatives of first and second order with respect to the $x$-variables. In addition, assume that $\sup_{(t, x) \in [0, T] \times \mathbb{R}^d} \frac{|B(t, x)|}{\sqrt{x^T \varphi(x) x}} < \infty$, and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space supporting a standard Brownian motion $U : [0, T] \times \Omega \to \mathbb{R}^d$ with continuous sample paths. Consider stochastic processes $R^{t \times} : [0, t] \times \Omega \to \mathbb{R}^d$, $t \in [0, T]$, $x \in \mathbb{R}^d$, satisfying

$$R^{t \times}_s = x + \int_0^s b(t - r, R^{t \times}_r) \, dr + \sigma U_s \quad \text{for all } t \in [0, T], \ s \in [0, t] \text{ and } x \in \mathbb{R}^d, \quad (30)$$

and let the mappings $U : [0, T] \times \mathbb{R}^d \to \mathbb{R}$, $u : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ be given by

$$U(t, x) = \varphi(R^{t \times}_t) \exp\left( \int_0^t B(t - s, R^{t \times}_s) \, ds \right) \quad \text{and} \quad u(t, x) = \mathbb{E}[U(t, x)] \quad (31)$$

for $t \in [0, T]$ and $x \in \mathbb{R}^d$. Then

(i) $u \in C^{1,2}([0, T] \times \mathbb{R}^d, \mathbb{R})$, and 

(ii) $u(t, x) = \varphi(x) + \int_0^t \left[ \frac{1}{2} \text{Trace}_{\mathbb{R}^d}(\sigma \sigma^T \text{Hess}_x(u)(s, x)) + (b(s, x), \nabla_x u(s, x))_{\mathbb{R}^d} + B(s, x)u(s, x) \right] ds$

for all $t \in [0, T]$ and $x \in \mathbb{R}^d$.  

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Proof. Consider the mapping $\mathcal{V} : [0, T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ given by

$$
\mathcal{V}(t, x) = \varphi(x) + \int_0^t \exp\left(\int_0^s B(t-r, R_r^{l,x}) \, dr\right) \left[\frac{1}{2} \text{Tr} \mathbb{E}_{t, r} \left(\sigma^T \text{Hess}(\varphi)(R_r^{l,x})\right) + \langle \nabla \varphi(R_r^{l,x}), b(t-s, R_s^{l,x}) \rangle_{\mathbb{R}^d} + B(t-s, R_s^{l,x}) \varphi(R_s^{l,x}) \right] \, ds \quad \text{for } t \in [0, T] \text{ and } x \in \mathbb{R}^d.
$$

(32)

It follows from (30) by Itô’s formula that

$$
\varphi(R_s^{l,x}) = \varphi(x) + \int_0^s \langle \nabla \varphi(R_r^{l,x}), \sigma \, dU_r \rangle_{\mathbb{R}^d} + \frac{1}{2} \text{Tr} \mathbb{E}_{s, r} \left(\sigma^T \text{Hess}(\varphi)(R_r^{l,x})\right) \, dr \quad \mathbb{P}\text{-a.s.}
$$

(33)

for all $t \in [0, T]$, $s \in [0, t]$ and $x \in \mathbb{R}^d$. In addition, one has

$$
\exp\left(\int_0^s B(t-r, R_r^{l,x}) \, dr\right) = 1 + \int_0^s \exp\left(\int_0^r B(t-w, R_w^{l,x}) \, dw\right) B(t-r, R_r^{l,x}) \, dr
$$

(34)

for all $t \in [0, T]$, $s \in [0, t]$ and $x \in \mathbb{R}^d$. (31)-(34) and another application of Itô’s formula give

$$
U(t, x) = \varphi(R_t^{l,x}) \exp\left(\int_0^t B(t-s, R_s^{l,x}) \, ds\right)
$$

$$
= \varphi(x) + \int_0^t \exp\left(\int_0^s B(t-r, R_r^{l,x}) \, dr\right) \langle \nabla \varphi(R_r^{l,x}), \sigma \, dU_s \rangle_{\mathbb{R}^d} + \frac{1}{2} \text{Tr} \mathbb{E}_{t, r} \left(\sigma^T \text{Hess}(\varphi)(R_r^{l,x})\right) \, ds
$$

$$
+ \int_0^t \exp\left(\int_0^s B(t-r, R_r^{l,x}) \, dr\right) \langle \nabla \varphi(R_r^{l,x}), b(t-s, R_s^{l,x}) \rangle_{\mathbb{R}^d} \, ds
$$

$$
+ \int_0^t \exp\left(\int_0^s B(t-r, R_r^{l,x}) \, dr\right) B(t-s, R_s^{l,x}) \varphi(R_s^{l,x}) \, ds
$$

(35)

$$
= \mathcal{V}(t, x) + \int_0^t \exp\left(\int_0^r B(t-w, R_w^{l,x}) \, dw\right) B(t-r, R_r^{l,x}) \, dr \quad \mathbb{P}\text{-a.s.}
$$

for all $t \in [0, T]$ and $x \in \mathbb{R}^d$. Now, observe that Lemma 5 and the assumption that $\varphi \in C^3(\mathbb{R}^d, [0, \infty))$ has at most polynomially growing partial derivatives of first, second and third order imply that

$$
\mathbb{E}\left[\sup_{t \in [0, T]} \sup_{s \in [0, t]} \sup_{x \in [-p, p]^d} \left(\|\varphi(R_s^{l,x})\|^p + \|\nabla \varphi(R_s^{l,x})\|^p_{L^p(\mathbb{R}^d, \mathbb{R})} + \|D^2 \varphi(R_s^{l,x})\|^p_{L^p(\mathbb{R}^d, \mathbb{R})} + \|D^3 \varphi(R_s^{l,x})\|^p_{L^p(\mathbb{R}^d, \mathbb{R})}\right) \right] < \infty \quad \text{for all } p \in (0, \infty).
$$

(36)

Moreover, Lemma 5 and the assumption that the partial derivative of $B \in C^{1,2}([0, T] \times \mathbb{R}^d, \mathbb{R})$ with respect to $t$ as well as its first and second order partial derivatives with respect to the $x$-variables are at most polynomially growing ensure that

$$
\mathbb{E}\left[\sup_{t \in [0, T]} \sup_{s \in [0, t]} \sup_{x \in [-p, p]^d} \left(\|B(t-s, R_s^{l,x})\|^p + \|\frac{\partial}{\partial t} B(t-s, R_s^{l,x})\|^p + \|D_x B(t-s, R_s^{l,x})\|^p_{L^p(\mathbb{R}^d, \mathbb{R})} + \|D^2_x B(t-s, R_s^{l,x})\|^p_{L^p(\mathbb{R}^d, \mathbb{R})}\right) \right] < \infty \quad \text{for all } p \in (0, \infty).
$$

(37)

Similarly, Lemma 5 and the assumption $\sup_{(t,x) \in [0, T] \times \mathbb{R}^d} \frac{B(t,x)}{1 + \|x\|_{\mathbb{R}^d}} < \infty$ imply that

$$
\mathbb{E}\left[\sup_{t \in [0, T]} \sup_{s \in [0, t]} \sup_{x \in [-p, p]^d} \left(\exp\left(\int_0^s B(t-r, R_r^{l,x}) \, dr\right) \right) \right] < \infty \quad \text{for all } p \in (0, \infty).
$$

(38)
From (36), (38) and Hölder’s inequality one obtains
\[
\mathbb{E}\left[\int_0^t \left\| \exp\left(\int_0^s B(t-r, R_{t,s}^{i,x}) \, dr \right) \sigma^T \nabla \varphi(R_{s,x}^t) \right\|_{\mathbb{R}^d}^2 \, ds \right] < \infty \quad \text{for all } t \in [0, T] \text{ and } x \in \mathbb{R}^d. \tag{39}
\]

Next, note that it follows from the assumptions on \(b \in C^{1,2}([0, T] \times \mathbb{R}^d, \mathbb{R}^d)\) that it grows at most linearly. Therefore, we obtain from (31) and (35)–(39) together with Fubini’s theorem that
\[
u(t, x) = \mathbb{E}[\mathcal{U}(t, x)] = \mathbb{E}[\mathcal{V}(t, x)] = \varphi(x) + \int_0^t \mathbb{E}\left[ \exp\left(\int_0^s B(t-r, R_{t,s}^{i,x}) \, dr \right) \langle \nabla \varphi(R_{s,x}^t), b(t-s, R_{s,x}^t) \rangle_{\mathbb{R}^d} \right] \, ds
\]

\[
+ \frac{1}{2} \int_0^t \mathbb{E}\left[ \exp\left(\int_0^s B(t-r, R_{t,s}^{i,x}) \, dr \right) \text{Trace}_{\mathbb{R}^d}(\sigma \sigma^T \text{Hess}(\varphi)(R_{s,x}^t)) \right] \, ds
\]

\[
+ \int_0^t \mathbb{E}\left[ \exp\left(\int_0^s B(t-r, R_{t,s}^{i,x}) \, dr \right) B(t-s, R_{s,x}^t) \varphi(R_{s,x}^t) \right] \, ds \quad \text{for all } t \in [0, T] \text{ and } x \in \mathbb{R}^d.
\tag{40}
\]

It follows from (30), the assumption that \(b \in C^{1,2}([0, T] \times \mathbb{R}^d, \mathbb{R}^d)\) has bounded partial derivatives, the de la Vallée Poussin theorem (cf., e.g., [28, Corollary 6.21]), and the Vitali convergence theorem (cf., e.g., [28, Theorem 6.25]) that for all \(\omega \in \Omega\) and \(s \in [0, T]\), the mapping \((t, x) \mapsto R_{s,x}^t\) is in \(C^{1,0}([s, T] \times \mathbb{R}^d, \mathbb{R}^d)\) with
\[
\frac{\partial}{\partial t} R_{s,x}^t = \int_0^s \left[ \frac{\partial}{\partial t} b(t-r, R_{t,s}^{i,x}) + D_x b(t-r, R_{t,s}^{i,x}) \left( \frac{\partial}{\partial x} R_{t,s}^{i,x} \right) \right] \, dr
\tag{41}
\]
(cf. also, e.g., [29, Theorem 4.6.5]). This and the assumption that \(b \in C^{1,2}([0, T] \times \mathbb{R}^d, \mathbb{R}^d)\) has bounded partial derivatives imply that
\[
\left\| \frac{\partial}{\partial t} R_{s,x}^t \right\|_{\mathbb{R}^d} \leq T \sup_{(r,y) \in [0,T] \times \mathbb{R}^d} \left\| \frac{\partial}{\partial t} b(r, y) \right\|_{\mathbb{R}^d}
\]

\[
+ \sup_{(r,y) \in [0,T] \times \mathbb{R}^d} \left\| \frac{\partial}{\partial x} b(r, y) \right\|_{L(\mathbb{R}^d, \mathbb{R}^d)} \int_0^s \left\| \frac{\partial}{\partial x} R_{t,s}^{i,x} \right\|_{\mathbb{R}^d} \, dr \quad \text{for all } t \in [0, T], \, s \in [0, t] \text{ and } x \in \mathbb{R}^d,
\]

which, by Gronwall’s integral inequality (cf., e.g., [24, Lemma 2.11]), yields
\[
\sup_{t \in [0,T]} \sup_{s \in [0,t]} \sup_{x \in \mathbb{R}^d} \left\| \frac{\partial}{\partial t} R_{s,x}^t \right\|_{\mathbb{R}^d} \leq T \left[ \sup_{(r,y) \in [0,T] \times \mathbb{R}^d} \left\| \frac{\partial}{\partial t} b(r, y) \right\|_{\mathbb{R}^d} \right] \exp\left( T \sup_{(r,y) \in [0,T] \times \mathbb{R}^d} \left\| \frac{\partial}{\partial x} b(r, y) \right\|_{L(\mathbb{R}^d, \mathbb{R}^d)} \right) < \infty.
\tag{42}
\]

Next observe that (41), Lemma 5, our assumptions on \(b, \varphi, B\), the de la Vallée Poussin theorem (cf., e.g., [28, Corollary 6.21]), and the Vitali convergence theorem (cf., e.g., [28, Theorem 6.25]) imply that
\[
\left(\frac{d}{dt}\right) \left( \int_0^t \exp\left(\int_0^s B(t-r, R_{t,s}^{i,x}) \, dr \right) \left[ \frac{\partial}{\partial t} \varphi(R_{s,x}^t) \right](b(t-s, R_{s,x}^t)) \, ds \right)
\]

\[
= \exp\left(\int_0^t B(t-r, R_{t,s}^{i,x}) \, dr \right) \left[ \frac{\partial}{\partial t} \varphi(R_{t,s}^{i,x}) \right](b(0, R_{0,s}^{i,x}))
\]

\[
+ \int_0^t \exp\left(\int_0^s B(t-r, R_{t,s}^{i,x}) \, dr \right) \left[ \frac{\partial}{\partial x} \varphi(R_{s,x}^t) \right](b(t-s, R_{s,x}^t))
\]

\[
\times \int_0^s \left[ \frac{\partial}{\partial t} B(t-r, R_{t,s}^{i,x}) + \left[ \frac{\partial}{\partial x} B(t-r, R_{t,s}^{i,x}) \right] \left( \frac{\partial}{\partial x} R_{t,s}^{i,x} \right) \right] \, dr \, ds
\]

\[
+ \int_0^t \exp\left(\int_0^s B(t-r, R_{t,s}^{i,x}) \, dr \right) \left[ \frac{\partial^2}{\partial x^2} \varphi(R_{s,x}^t) \right](b(t-s, R_{t,s}^{i,x})) \, ds
\]

\[
+ \int_0^t \exp\left(\int_0^s B(t-r, R_{t,s}^{i,x}) \, dr \right) \times \left[ \frac{\partial}{\partial t} \varphi(R_{s,x}^t) \right] \left[ \frac{\partial}{\partial t} b(t-s, R_{t,s}^{i,x}) \right] \, ds
\]

\[
+ \int_0^t \exp\left(\int_0^s B(t-r, R_{t,s}^{i,x}) \, dr \right) \times \left[ \frac{\partial}{\partial t} \varphi(R_{s,x}^t) \right] \left[ \frac{\partial}{\partial x} b(t-s, R_{t,s}^{i,x}) \right] \left( \frac{\partial}{\partial x} R_{t,s}^{i,x} \right) \, ds
\]

for all \(t \in [0, T]\) and \(x \in \mathbb{R}^d\),
(b) \[
\frac{d}{dt} \left( \int_0^t \exp\left( \int_0^s B(t, r, R^t_{r,x}) dr \right) \text{Trace}_{\mathbb{R}^d} \left( \sigma \sigma^T \text{Hess}(\varphi)(R^t_{r,x}) \right) ds \right) \\
= \exp\left( \int_0^t B(t, r, R^t_{r,x}) dr \right) \text{Trace}_{\mathbb{R}^d} \left( \sigma \sigma^T \text{Hess}(\varphi)(R^t_{r,x}) \right) \\
+ \int_0^t \exp\left( \int_0^s B(t, r, R^t_{r,x}) dr \right) \text{Trace}_{\mathbb{R}^d} \left( \sigma \sigma^T \text{Hess}(\varphi)(R^t_{r,x}) \right) \\
\times \left[ \int_0^s \left( \frac{\partial}{\partial r} B(t, r, R^t_{r,x}) + \left[ \frac{\partial}{\partial r} B(t, r, R^t_{r,x}) \right] \left( \frac{\partial}{\partial r} R^t_{r,x} \right) \right) dr \right] ds \\
+ \int_0^t \exp\left( \int_0^s B(t, r, R^t_{r,x}) dr \right) \text{Trace}_{\mathbb{R}^d} \left( \sigma \sigma^T \text{Hess}(\varphi)(R^t_{r,x}) \right) \left( \frac{\partial}{\partial r} R^t_{r,x} \right) ds
\]
for all \( t \in [0, T] \) and \( x \in \mathbb{R}^d \), and

(c) \[
\frac{d}{dt} \left( \int_0^t \exp\left( \int_0^s B(t, r, R^t_{r,x}) dr \right) B(t-s, R^t_{r,x}) \varphi(R^t_{r,x}) ds \right) \\
= \exp\left( \int_0^t B(t, r, R^t_{r,x}) dr \right) B(0, R^t_{r,x}) \varphi(R^t_{r,x}) \\
+ \int_0^t \exp\left( \int_0^s B(t, r, R^t_{r,x}) dr \right) B(t-s, R^t_{r,x}) \varphi(R^t_{r,x}) \\
\times \left[ \int_0^s \left( \frac{\partial}{\partial r} B(t, r, R^t_{r,x}) + \left[ \frac{\partial}{\partial r} B(t, r, R^t_{r,x}) \right] \left( \frac{\partial}{\partial r} R^t_{r,x} \right) \right) dr \right] ds \\
+ \int_0^t \exp\left( \int_0^s B(t, r, R^t_{r,x}) dr \right) \varphi(R^t_{r,x}) \\
\times \left[ \frac{\partial}{\partial r} B(t, s, R^t_{r,x}) + \left[ \frac{\partial}{\partial r} B(t, s, R^t_{r,x}) \right] \left( \frac{\partial}{\partial r} R^t_{r,x} \right) \right] ds \\
+ \int_0^t \exp\left( \int_0^s B(t, r, R^t_{r,x}) dr \right) B(t-s, R^t_{r,x}) \left[ \frac{\partial}{\partial r} \varphi(R^t_{r,x}) \right] \left( \frac{\partial}{\partial r} R^t_{r,x} \right) ds
\]
for all \( t \in [0, T] \) and \( x \in \mathbb{R}^d \).

Combining (32), (36), (38), and (39) and the assumption that \( \varphi \in C^1([0, T] \times \mathbb{R}^d, \mathbb{R}^d) \) has bounded partial derivatives with respect to \( \varphi \) and Hölder’s inequality shows that

\[
\mathbb{E}\left[ \sup_{t \in [0, T]} \sup_{\delta \in [0, p]^{d}} \left| \frac{\partial}{\partial r} \varphi(t, x) \right|^p \right] < \infty \quad \text{for all} \ p \in (0, \infty).
\]

This, (40), (41), and Hölder’s inequality ensure that

(A) for all \( x \in \mathbb{R}^d \), the mapping \( t \mapsto u(t, x) \) is in \( C^1([0, T], \mathbb{R}) \),

(B) the mapping \( (t, x) \mapsto \frac{\partial}{\partial r} u(t, x) \) is in \( C([0, T] \times \mathbb{R}^d, \mathbb{R}) \), and

(C) \( \frac{\partial}{\partial r} u(t, x) = \mathbb{E}\left[ \frac{\partial}{\partial r} \varphi(t, x) \right] \) for all \( t \in [0, T] \) and \( x \in \mathbb{R}^d \),

which, together with Lemma 9, implies 10.

Next, note that it follows from the Markov property of \( (R^t_{r,x})_{s \in [0, \delta]} \) that

\[
\mathbb{E}\left[ G((R^t_{r,x})_{s \in [0, \delta]}) \mathbb{1}_A(R^t_{r,x}) \right] = \int_A \mathbb{E}\left[ G((R^{t-\delta,y}_{s \in [0, \delta]}) \mathbb{1}_A(R^{t-\delta,y}_{r,x}) \right] R^t_{r,x}(\mathbb{P})(dy)
\]
for all Borel subsets $A \subseteq \mathbb{R}^d$, $t \in [0, T]$, $\delta \in [0, t]$, $x \in \mathbb{R}^d$ and bounded functions $G \in C(C([0, t - \delta], \mathbb{R}^d), \mathbb{R})$, which together with $(31)$, implies that

$$u(t, x) = \mathbb{E}[u(t, x)] = \mathbb{E}\left[\varphi(R^{t,x}_r) \exp\left(\int_0^t B(t - s, R^{t,x}_s) \, ds\right)\right]$$

$$= \mathbb{E}\left[\exp\left(\int_0^\delta B(t - s, R^{t,x}_s) \, ds\right) \mathbb{E}\left[\varphi(R^{t,x}_r) \exp\left(\int_\delta^t B(t - s, R^{t,x}_s) \, ds\right) \left| (R^{t,x}_s)_{0 \leq s \leq \delta}\right]\right]$$

$$= \mathbb{E}\left[\exp\left(\int_0^\delta B(t - s, R^{t,x}_s) \, ds\right) \mathbb{E}\left[\varphi(R^{t,x}_r) \exp\left(\int_\delta^t B(t - s, R^{t,x}_s) \, ds\right) R^{t,x}_\delta\right]\right]$$

$$= \mathbb{E}\left[\exp\left(\int_0^\delta B(t - s, R^{t,x}_s) \, ds\right) \mathbb{E}\left[\varphi(R^{t,x}_{t-\delta}) \exp\left(\int_0^{t-\delta} B(t - s, R^{t,x}_s) \, ds\right) \left| R^{t,x}_s\right| = y\right]\right]$$

Furthermore, $(30)$, $(31)$, and Itô’s formula assure that

$$u(t - \delta, R^{t,x}_\delta) = u(t, x) - \int_0^\delta \frac{\partial}{\partial t} u(t, R^{t,x}_s) \, ds + \int_0^\delta (\nabla_x u(t, R^{t,x}_s), b(t, s, R^{t,x}_s)) \, ds$$

$$+ \int_0^\delta (\nabla_x u(t, R^{t,x}_s), \sigma \, du_s) + \frac{1}{2} \int_0^\delta \text{Trace}_\mathbb{R}^d\left(\sigma \sigma^T \text{Hess}_x(u)(t, s, R^{t,x}_s)\right) \, ds \quad \mathbb{P}\text{-a.s.}$$

for all $t \in [0, T]$, $\delta \in [0, t]$ and $x \in \mathbb{R}^d$. Combining this, $(34)$, and Itô’s formula gives

$$\exp\left(\int_0^\delta B(t - s, R^{t,x}_s) \, ds\right) = u(t, x) - \int_0^\delta \exp\left(\int_0^s B(t - r, R^{t,x}_r) \, dr\right) \frac{\partial}{\partial t} u(t, R^{t,x}_s) \, ds$$

$$+ \int_0^\delta \exp\left(\int_0^s B(t - r, R^{t,x}_r) \, dr\right) (\nabla_x u(t - s, R^{t,x}_s), b(t, s, R^{t,x}_s)) \, ds$$

$$+ \int_0^\delta \exp\left(\int_0^s B(t - r, R^{t,x}_r) \, dr\right) (\nabla_x u(t - s, R^{t,x}_s), \sigma \, du_s)$$

$$+ \frac{1}{2} \int_0^\delta \exp\left(\int_0^s B(t - r, R^{t,x}_r) \, dr\right) \text{Trace}_\mathbb{R}^d\left(\sigma \sigma^T \text{Hess}_x(u)(t - s, R^{t,x}_s)\right) \, ds$$

$$+ \int_0^\delta \exp\left(\int_0^s B(t - r, R^{t,x}_r) \, dr\right) B(t - s, R^{t,x}_s) \, u(t - s, R^{t,x}_s) \, ds \quad \mathbb{P}\text{-a.s.}$$

for all $t \in [0, T]$, $\delta \in [0, t]$ and $x \in \mathbb{R}^d$. Moreover, it follows from Lemma 5, Lemma 8, Lemma 9, $(30)$, $(31)$, $(A)$–$(C)$, and our assumptions on $b$, $\varphi$, and $B$ that

$$\mathbb{E}\left[\sup_{t \in [0, T]} \sup_{s \in [0, t]} \sup_{x \in [-p, p]^d} \left(\|u(t, s, R^{t,x}_s)\|^p + \|\nabla_x u(t, s, R^{t,x}_s)\|_{L(\mathbb{R}^d, \mathbb{R})} + \|D^2u(t, s, R^{t,x}_s)\|_{L(2)(\mathbb{R}^d, \mathbb{R})}\right)\right] < \infty \quad \text{for all } p \in (0, \infty).$$

(45)

This ensures that

$$\mathbb{E}\left[\int_0^t \left\|\exp\left(\int_0^s B(t - r, R^{t,x}_r) \, dr\right) \sigma^T \nabla_x u(t - s, R^{t,x}_s)\right\|^2 \, ds\right] < \infty \quad \text{for all } t \in [0, T] \text{ and } x \in \mathbb{R}^d,$$
which, together with (43), (44), (45) and Fubini’s theorem yields

\[
0 = \mathbb{E}\left[ \exp\left( \int B(t-s, R^t_{s,x}) \, ds \right) u(t-\delta, R^t_{\delta,x}) - u(t, x) \right]
= \int_0^\delta \mathbb{E}\left[ \exp\left( \int B(t-r, R^t_{s,x}) \, dr \right) \left( -\frac{\partial}{\partial t}u(t-s, R^t_{s,x}) + \frac{1}{2} \text{Trace}_{\mathbb{R}^d}(\sigma\sigma^T \text{Hess}_x(u)(t-s, R^t_{s,x})) \right. \\
\left. + \langle \nabla_x u(t-s, R^t_{s,x}), b(t-s, R^t_{s,x}) \rangle_{\mathbb{R}^d} + B(t-s, R^t_{s,x}) u(t-s, R^t_{s,x}) \right) \right] ds
\]

for all \( t \in (0, T] \), \( \delta \in [0, t] \) and \( x \in \mathbb{R}^d \). This, (1), (36)–(38), (45), Lemma 6 and Lemma 8 imply that

\[
0 = \lim_{\delta \to 0} \frac{1}{\delta} \mathbb{E}\left[ \exp\left( \int B(t-s, R^t_{s,x}) \, ds \right) u(t-\delta, R^t_{\delta,x}) - u(t, x) \right] = -\frac{\partial}{\partial t}u(t, x) + \frac{1}{2} \text{Trace}_{\mathbb{R}^d}(\sigma\sigma^T \text{Hess}_x(u)(t,x)) + \langle b(t,x), \nabla_x u(t,x) \rangle_{\mathbb{R}^d} + B(t,x)u(t,x)
\]

for all \( t \in (0, T] \) and \( x \in \mathbb{R}^d \). Moreover, it follows from (30)–(31) that \( u(0, x) = \varphi(x) \) for all \( x \in \mathbb{R}^d \). Combining this with (46) and (1) proves (3), which completes the proof.

\[\square\]

**Proposition 11.** Let \( T, c, \alpha \in (0, \infty) \), \( d \in \mathbb{N} \), \( \sigma \in \mathbb{R}^{d \times d} \), and consider a function \( b \in C^{0,2}([0, T] \times \mathbb{R}^d, \mathbb{R}^d) \) with bounded partial derivatives of first and second order with respect to the \( x \)-variables. Let \( \varphi \in C^2(\mathbb{R}^d, [0, \infty)) \) have at most polynomially growing partial derivatives of first and second order and \( B \in C^{0,2}([0, T] \times \mathbb{R}^d, \mathbb{R}) \) at most polynomially growing partial derivatives of first and second order with respect to the \( x \)-variables. In addition, assume that

\[
\|b(t,x) - b(s,x)\|_{\mathbb{R}^d} + \|\frac{\partial}{\partial x}b(t,x) - \frac{\partial}{\partial x}b(s,x)\|_{L^\infty(\mathbb{R}^d, \mathbb{R}^d)} \leq c|t-s|^\alpha, \tag{47}
\]

\[
B(t,x) \leq c(1 + \|x\|_{\mathbb{R}^d}), \tag{48}
\]

\[
|B(t,x) - B(s,x)| + \|\frac{\partial}{\partial x}B(t,x) - \frac{\partial}{\partial x}B(s,x)\|_{L^\infty(\mathbb{R}^d, \mathbb{R})} \leq c(1 + \|x\|_{\mathbb{R}^d})|t-s|^\alpha \tag{49}
\]

for all \( n \in \{1, 2\} \), \( s, t \in [0, T] \) and \( x \in \mathbb{R}^d \). Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space supporting a standard Brownian motion \( U : [0, T] \times \Omega \to \mathbb{R}^d \) with continuous sample paths. Consider stochastic processes \( R^{t,x} : [0, t] \times \Omega \to \mathbb{R}^d, t \in [0, T], x \in \mathbb{R}^d \), satisfying

\[
R^t_{s,x} = x + \int_0^s b(t-r, R^t_{r,x}) \, dr + \sigma U_s
\]

for all \( t \in [0, T] \), \( s \in [0, t] \) and \( x \in \mathbb{R}^d \), and let the function \( u : [0, T] \times \mathbb{R}^d \to \mathbb{R} \) be given by

\[
u(t,x) = \mathbb{E}\left[ \varphi(R^t_{t,x}) \exp\left( \int_0^t B(t-s, R^t_{s,x}) \, ds \right) \right] \quad \text{for} \quad t \in [0, T] \quad \text{and} \quad x \in \mathbb{R}^d.
\]

Then

(i) \( u \in C^{1,2}([0, T] \times \mathbb{R}^d, \mathbb{R}) \), and

(ii) \( u(t,x) \)

\[
= \varphi(x) + \int_0^t \left[ \frac{1}{2} \text{Trace}_{\mathbb{R}^d}(\sigma\sigma^T \text{Hess}_x(u)(s,x)) + \langle b(s,x), \nabla_x u(s,x) \rangle_{\mathbb{R}^d} + B(s,x)u(s,x) \right] ds
\]

for all \( t \in [0, T] \) and \( x \in \mathbb{R}^d \).

**Proof.** Throughout this proof we fix a \( q \in [1, \infty) \) such that

\[
sup_{t \in [0, T]} \sup_{x \in \mathbb{R}^d} \left[ \left| \varphi(t,x) \right| + \|\nabla \varphi(t,x)\|_{L^\infty(\mathbb{R}^d, \mathbb{R})} + \|D^2 \varphi(t,x)\|_{L^2(\mathbb{R}^d, \mathbb{R})} \right] < \infty \tag{50}
\]

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and
\[
\sup_{t \in [0,T]} \sup_{x \in \mathbb{R}^d} \left[ \frac{|B(t,x)| + \|D_x B(t,x)\|_{L(\mathbb{R}^d,\mathbb{R})} + \|D_x^2 B(t,x)\|_{L(\mathbb{R}^d,\mathbb{R})}}{(1 + \|x\|_{\mathbb{R}^d})^q} \right] < \infty. \tag{51}
\]

Let the mappings \( \varphi_n : \mathbb{R}^d \to [0, \infty) \) and \( b_n, B_n : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d, n \in \mathbb{N} \), be given by
\[
\varphi_n(x) = \left( \frac{n}{2\pi} \right)^{d/2} \int_{\mathbb{R}^d} \varphi(y) \exp\left(-\frac{n}{2} \|x-y\|_{\mathbb{R}^d}^2\right) dy, \tag{52}
\]
\[
b_n(t,x) = \left( \frac{n}{2\pi} \right)^{1/2} \int_{-\infty}^{\infty} b\left(\min\{T, \max\{s,0\}, x\}\right) \exp\left(-\frac{n(t-s)^2}{2}\right) ds, \tag{53}
\]
\[
B_n(t,x) = \left( \frac{n}{2\pi} \right)^{1/2} \int_{-\infty}^{\infty} B\left(\min\{T, \max\{s,0\}, x\}\right) \exp\left(-\frac{n(t-s)^2}{2}\right) ds \tag{54}
\]
for all \( n \in \mathbb{N}, t \in [0,T] \) and \( x \in \mathbb{R}^d \). Consider the stochastic processes \( R_{n,t,x} : [0,t] \times \Omega \to \mathbb{R}^d, n \in \mathbb{N}, t \in [0,T], x \in \mathbb{R}^d \), and mappings \( U_n : [0,T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}, n \in \mathbb{N}, u_n : [0,T] \times \mathbb{R}^d \to \mathbb{R}, n \in \mathbb{N} \), given by
\[
R_{n,t,x} = x + \int_{0}^{s} b_n(t-r, R_{n,t,x}^r) dr + \sigma U_s, \quad U_0(t,x) = \varphi(R_{t,x}^0) \exp\left(\int_{0}^{t} B(t-r, R_{t,x}^r) ds\right), \quad U_n(t,x) = \varphi_n(R_{t,x}^0) \exp\left(\int_{0}^{t} B_n(t-r, R_{n,t,x}^r) ds\right),
\]
and
\[
u_n(t,x) = \mathbb{E}[U_n(t,x)]
\]
for \( n \in \mathbb{N}, t \in [0,T] \) and \( x \in \mathbb{R}^d \). \([50]\) and \([52]\) imply that \((\varphi_n)_{n \in \mathbb{N}} \subseteq C^\infty(\mathbb{R}^d, \mathbb{R})\) and
\[
\sup_{n \in \mathbb{N}} \sup_{t \in [0,T]} \sup_{x \in \mathbb{R}^d} \left[ \frac{|\varphi_n(t,x)| + \|\nabla \varphi_n(t,x)\|_{L(\mathbb{R}^d,\mathbb{R})} + \|\nabla^2 \varphi_n(t,x)\|_{L(\mathbb{R}^d,\mathbb{R})}}{\left(1 + \|x\|_{\mathbb{R}^d}\right)^q} \right] < \infty. \tag{55}
\]
Furthermore, since \( \exp(-s^2/2) \) is even in \( s \in \mathbb{R} \), it follows from \([53]\) that
\[
\frac{\partial}{\partial t} b_n(t,x) = \int_{-\infty}^{\infty} b\left(\min\{T, \max\{s,0\}, x\}\right) n(s-t) \exp\left(-\frac{n(t-s)^2}{2}\right) ds \nonumber
\]
\[
= \left( \frac{n}{2\pi} \right)^{1/2} \int_{-\infty}^{\infty} \left[b\left(\min\{T, \max\{s,0\}, x\}\right) - b(t,x) \right] n(s-t) \exp\left(-\frac{n(t-s)^2}{2}\right) ds. \nonumber
\]
Moreover, one obtains from \([47]\) and the fact that \( |\min\{T, \max\{s,0\}\} - t| \leq |s-t| \) for all \( t \in [0,T] \) and \( s \in \mathbb{R} \) that
\[
\left|\frac{\partial}{\partial t} b_n(t,x)\right| \leq \left( \frac{n}{2\pi} \right)^{1/2} \int_{-\infty}^{\infty} c |\min\{T, \max\{s,0\}\} - t| \alpha n|s-t| \exp\left(-\frac{n(t-s)^2}{2}\right) ds \nonumber
\]
\[
\leq c \left( \frac{n}{2\pi} \right)^{1/2} \int_{-\infty}^{\infty} n|s-t|^{1+\alpha} \exp\left(-\frac{n(t-s)^2}{2}\right) ds \leq \frac{c}{\sqrt{2\pi}} n^{1-\alpha/2} \int_{-\infty}^{\infty} |z|^{1+\alpha} \exp\left(-\frac{|z|^2}{2}\right) dz \nonumber
\]
for all \( n \in \mathbb{N}, t \in [0,T] \) and \( x \in \mathbb{R}^d \). Combining this with the assumption that \( b \in C^{0,2}([0,T] \times \mathbb{R}^d, \mathbb{R}^d) \) has bounded partial derivatives of first and second order with respect to the \( x \)-variables and \([53]\) ensures that
\((a)\) for all \( n \in \mathbb{N}, b_n \in C^{1,2}([0,T] \times \mathbb{R}^d, \mathbb{R}^d) \) has a bounded partial derivative with respect to \( t \) and bounded partial derivatives of first and second order with respect to the \( x \)-variables,
\[(b)\]
\[
\sup_{n \in \mathbb{N}} \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \|D_x b_n(t,x)\|_{L(\mathbb{R}^d,\mathbb{R})} \leq \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \|D_x b(t,x)\|_{L(\mathbb{R}^d,\mathbb{R})} < \infty, \nonumber
\]
\[(c)\]
\[
\sup_{n \in \mathbb{N}} \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \|D_x^2 b_n(t,x)\|_{L(\mathbb{R}^d,\mathbb{R})} \leq \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \|D_x^2 b(t,x)\|_{L(\mathbb{R}^d,\mathbb{R})} < \infty. \nonumber
\]
Next, note that it follows from (54) and the assumption that $B(t, x) \leq c(1 + \|x\|_{\mathbb{R}^d})$ for all $t \in [0, T]$ and $x \in \mathbb{R}^d$ that
\[
\sup_{n \in \mathbb{N}} \sup_{t \in [0, T]} \sup_{x \in \mathbb{R}^d} \left[ \frac{B_n(t, x)}{1 + \|x\|_{\mathbb{R}^d}} \right] \leq \sup_{n \in \mathbb{N}} \sup_{t \in [0, T]} \sup_{x \in \mathbb{R}^d} \left[ \frac{B(t, x)}{1 + \|x\|_{\mathbb{R}^d}} \right] < \infty, \tag{56}
\]
In addition, one obtains from (51) and (54) that $(B_n)_{n \in \mathbb{N}} \subseteq C^{1,2}([0, T] \times \mathbb{R}^d, \mathbb{R})$ and
\[
\sup_{n \in \mathbb{N}} \sup_{t \in [0, T]} \sup_{x \in \mathbb{R}^d} \left[ \frac{|B_n(t, x)| + \|D_x B_n(t, x)\|_{L(\mathbb{R}^d, \mathbb{R})} + \|D_x^2 B_n(t, x)\|_{L(\mathbb{R}^d, \mathbb{R})}}{(1 + \|x\|_{\mathbb{R}^d})^q} \right] < \infty. \tag{57}
\]
Now, note that it follows from Lemma $9$ and the assumptions on $\varphi$, $b$, and $B$ that
\begin{enumerate}[(a')]\item $u \in C^{0,2}([0, T] \times \mathbb{R}^d, \mathbb{R}),$
\item $\frac{\partial}{\partial t} u(t, x) = \mathbb{E} \left[ \frac{\partial}{\partial t} U_0(t, x) \right]$ for all $t \in [0, T]$ and $x \in \mathbb{R}^d$, and
\item $\frac{\partial^2}{\partial x^2} u(t, x) = \mathbb{E} \left[ \frac{\partial^2}{\partial x^2} U_0(t, x) \right]$ for all $t \in [0, T]$ and $x \in \mathbb{R}^d$.
\end{enumerate}
By Lemma $2$, Lemma $10$, (a)–(c) and (55)–(57), one has
\begin{enumerate}[(A)]\item $u_n \in C^{1,2}([0, T] \times \mathbb{R}^d, \mathbb{R})$ for all $n \in \mathbb{N}$,
\item $\frac{\partial}{\partial t} u_n(t, x) = \mathbb{E} \left[ \frac{\partial}{\partial t} U_n(t, x) \right]$ for all $n \in \mathbb{N}$, $t \in [0, T]$ and $x \in \mathbb{R}^d$, and
\item $\frac{\partial^2}{\partial x^2} u_n(t, x) = \mathbb{E} \left[ \frac{\partial^2}{\partial x^2} U_n(t, x) \right]$ for all $n \in \mathbb{N}$, $t \in [0, T]$ and $x \in \mathbb{R}^d$, and
\item $u_n(t, x) = \varphi_n(x) + \int_0^t \left[ \frac{1}{2} \text{Trace}_{\mathbb{R}^d}(\sigma \sigma^* (\text{Hess}_x u_n)(s, x)) + \langle b_n(s, x), (\nabla_x u_n)(s, x) \rangle_{\mathbb{R}^d} + B_n(s, x) u_n(s, x) \right] ds \tag{58}
\end{enumerate}
for all $n \in \mathbb{N}$, $t \in [0, T]$ and $x \in \mathbb{R}^d$. From Lemma $2$ together with (47), (53), (b) and (c) one obtains
\begin{enumerate}[(A')]\item $\lim_{n \to \infty} \sup_{t \in [0, T]} \sup_{x \in \mathbb{R}^d} \|b_n(t, x) - b(t, x)\|_{\mathbb{R}^d} = 0,$
\item $\lim_{n \to \infty} \sup_{t \in [0, T]} \sup_{x \in \mathbb{R}^d} \|D_x b_n(t, x) - D_x b(t, x)\|_{L(\mathbb{R}^d, \mathbb{R})} = 0,$ and
\item $\lim_{n \to \infty} \sup_{t \in [0, T]} \sup_{x \in \mathbb{R}^d} \|D_x^2 b_n(t, x) - D_x^2 b(t, x)\|_{L(\mathbb{R}^d, \mathbb{R})} = 0.$
\end{enumerate}
Combining Lemma $7$ with (A) gives
\[
\lim_{n \to \infty} \sup_{t \in [0, T]} \sup_{x \in \mathbb{R}^d} \|R^{n,t,x}_s - R^{t,x}_s\|_{\mathbb{R}^d} 
\leq T \exp(T \sup_{(r,y) \in [0, T] \times \mathbb{R}^d} \|D_x b(r, y)\|_{L(\mathbb{R}^d, \mathbb{R})}) \times \lim_{n \to \infty} \sup_{t \in [0, T]} \sup_{y \in \mathbb{R}^d} \|b_n(t, x) - b(t, y)\|_{\mathbb{R}^d} = 0. \tag{59}
\]
Hence, we obtain from Lemma $5$ that
\[
\sup_{n \in \mathbb{N}} \mathbb{E} \left[ \exp(p \sup_{t \in [0, T]} \sup_{x \in [-p, p]^d} \|R^{n,t,x}_s\|_{\mathbb{R}^d}) \right] 
\leq \sup_{n \in \mathbb{N}} \mathbb{E} \left[ \exp(p \sup_{t \in [0, T]} \sup_{x \in [-p, p]^d} \|R^{n,t,x}_s - R^{t,x}_s\|_{\mathbb{R}^d}) \times \exp(p \sup_{t \in [0, T]} \sup_{x \in [-p, p]^d} \|R^{t,x}_s\|_{\mathbb{R}^d}) \right] < \infty \quad \text{for all } p \in (0, \infty). \tag{60}
\]
Moreover, Lemma $3$, (b), and (59) yield
\[
\lim_{n \to \infty} \sup_{t \in [0, T]} \sup_{x \in [-p, p]^d} \|D_x b_n(t, x) - R^{n,t,x}_s\|_{L(\mathbb{R}^d, \mathbb{R})} = 0 \tag{61}
\]
for all $p \in (0, \infty)$, which together with Lemma $[\text{III,} (b)]$ and (66) shows that
\[
\limsup_{n \to \infty} \sup_{t \in [0,T]} \sup_{s \in [0,t]} \sup_{x \in [-p,p]^d} \| \frac{\partial}{\partial x_1} R_i^n(t, x) - \frac{\partial}{\partial x_1} R_i(t, x) \|_{\mathbb{R}^d} = 0
\] (62)
for all $i \in \{1, 2, \ldots, d\}$ and $p \in (0, \infty)$. Furthermore, it follows from Lemma $[\text{I,} (i)$ and (b) that
\[
\sup_{n \in \mathbb{N}} \sup_{t \in [0,T]} \sup_{s \in [0,t]} \sup_{x \in \mathbb{R}^d} \left[ \left\| \frac{\partial}{\partial x_1} R_i^n(t, x) \right\|_{\mathbb{R}^d}^2 + \left\| \frac{\partial^2}{\partial x_1 \partial x_2} R_i^n(t, x) \right\|_{\mathbb{R}^d}^2 \right] \leq 2 \exp \left( T \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \| D_x b(t, x) \|_{L(\mathbb{R}^d, \mathbb{R}^d)} \right) \exp \left( 3T \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \| D_x b(t, x) \|_{L(\mathbb{R}^d, \mathbb{R}^d)} \right)
\] (63)
By Lemma $[\text{I,} (i)$, (b), and (59), we obtain that
\[
\limsup_{n \to \infty} \sup_{t \in [0,T]} \sup_{s \in [0,t]} \sup_{x \in [-p,p]^d} \left\| \frac{\partial}{\partial x_1} R_i^n(t, x) - \frac{\partial}{\partial x_1} R_i(t, x) \right\|_{\mathbb{R}^d}^2 = 0
\] (64)
for all $i, j \in \{1, 2, \ldots, d\}$ and $p \in (0, \infty)$. Lemma $[\text{II,} (iii)$ (b), (c), and (61) hence assure that
\[
\limsup_{n \to \infty} \sup_{t \in [0,T]} \sup_{s \in [0,t]} \sup_{x \in [-p,p]^d} \left\| \frac{\partial^2}{\partial x_1 \partial x_2} R_i^n(t, x) - \frac{\partial^2}{\partial x_1 \partial x_2} R_i(t, x) \right\|_{\mathbb{R}^d}^2 = 0
\] (65)
Next, note that Lemma $[\text{II,} (ii)$, Lemma $[\text{III,} (b)$, (c), and (60) ensure that
\[
\limsup_{n \to \infty} \sup_{t \in [0,T]} \sup_{x \in [-p,p]^d} \left\| \mathcal{U}_n(t, x) - \mathcal{U}_0(t, x) \right\|_p = 0 \quad \text{for all } p \in (0, \infty).
\] (66)
Moreover, it follows from Lemma $[\text{V,} (iii)$ together with (50), (55), (56), and (60) that
\[
\sup_{n \in \mathbb{N}} \mathbb{E} \left[ \sup_{t \in [0,T]} \left| \mathcal{U}_n(t, x) - \mathcal{U}_0(t, x) \right|^p \right] < \infty \quad \text{for all } x \in \mathbb{R}^d \text{ and } p \in (0, \infty).
\]
Combining this and (66) with the de la Vallée Poussin theorem (cf., e.g., [28, Corollary 6.21]) and the Vitali convergence theorem (cf., e.g., [28, Theorem 6.25]) implies that
\[
\limsup_{n \to \infty} \sup_{t \in [0,T]} \left| u_n(t, x) - u(t, x) \right| \leq \limsup_{n \to \infty} \mathbb{E} \left[ \sup_{t \in [0,T]} \left| \mathcal{U}_n(t, x) - \mathcal{U}_0(t, x) \right| \right] = 0
\] (67)
for all $x \in \mathbb{R}^d$. Next, note that Lemma $[\text{II,} (i)$, Lemma $[\text{III,} (i)$, (49), (50), (59), and (62) ensure that
\[
\limsup_{n \to \infty} \sup_{t \in [0,T]} \sup_{x \in [-p,p]^d} \left\| \frac{\partial}{\partial x} \mathcal{U}_n(t, x) - \frac{\partial}{\partial x} \mathcal{U}_0(t, x) \right\|_{L(\mathbb{R}^d, \mathbb{R})} = 0 \quad \text{for all } p \in (0, \infty).
\] (68)
Moreover, (50), (55), (56), (60), (63), Lemma $[\text{V,} (iii)$ and Lemma $[\text{VIII,} (ii)$ imply that
\[
\sup_{n \in \mathbb{N}} \mathbb{E} \left[ \sup_{t \in [0,T]} \left\| \frac{\partial}{\partial x} \mathcal{U}_n(t, x) - \frac{\partial}{\partial x} \mathcal{U}_0(t, x) \right\|_{L(\mathbb{R}^d, \mathbb{R})}^p \right] < \infty \quad \text{for all } x \in \mathbb{R}^d \text{ and } p \in (0, \infty).
\]
It therefore follows from (a)–(b), (A)–(B), (68), the de la Vallée Poussin theorem (cf., e.g., [28, Corollary 6.21]), and the Vitali convergence theorem (cf., e.g., [28, Theorem 6.25]) that
\[
\limsup_{n \to \infty} \sup_{t \in [0,T]} \left\| \frac{\partial}{\partial x} u_n(t, x) - \frac{\partial}{\partial x} u(t, x) \right\|_{\mathbb{R}^d} = 0 \quad \text{for all } x \in \mathbb{R}^d.
\] (69)
Similarly, we obtain from Lemma 2, Lemma 3, Lemma 8(iii), (49), (50), (59), (62), and (64) that
\[
\limsup_{n \to \infty} \sup_{t \in [0,T]} \sup_{x \in [-p_n,p_n]^d} \| \frac{\partial^2}{\partial x^2} U_n(t, x) - \frac{\partial^2}{\partial x^2} U_0(t, x) \|_{L^2([0,T], \mathbb{R}^d)} = 0 \quad \text{for all } p \in (0, \infty).
\] (70)
Moreover, observe that (50), (55), (56), (60), (63), (65), Lemma 5, and Lemma 8(iii) show that
\[
\sup_{n \in \mathbb{N}} \mathbb{E} \left( \sup_{t \in [0,T]} \| \frac{\partial^2}{\partial x^2} U_n(t, x) - \frac{\partial^2}{\partial x^2} U_0(t, x) \|_{L^p([0,T], \mathbb{R}^d)}^p \right) < \infty \quad \text{for all } x \in \mathbb{R}^d \text{ and } p \in (0, \infty).
\]
\[\] (A), (C), (A), (C), (70), the de la Vallée Poussin theorem (cf., e.g., [28, Corollary 6.21]), and the Vitali convergence theorem (cf., e.g., [28, Theorem 6.25]) hence imply that
\[
\limsup_{n \to \infty} \sup_{t \in [0,T]} \| \frac{\partial^2}{\partial x^2} u_n(t, x) - \frac{\partial^2}{\partial x^2} u(t, x) \|_{L^2([0,T], \mathbb{R}^d)} = 0 \quad \text{for all } x \in \mathbb{R}^d.
\] (71)
Moreover, note that (52) and Lemma 2(ii) ensure that
\[
\limsup_{n \to \infty} | \varphi_n(x) - \varphi(x) | = 0 \quad \text{for all } x \in \mathbb{R}^d.
\]

It therefore follows from (12), (67), (69), and (71) that
\[
u(t, x) = \varphi(x) + \int_0^t \frac{1}{2} \text{Trace}_{\mathbb{R}^d} (\sigma \sigma^T \text{Hess}(u)(s, x)) + \langle u(s, x), \nabla u(s, x) \rangle_{\mathbb{R}^d} + B(s, x) u(s, x) \rangle \ ds
\]
for all \( t \in [0, T] \) and \( x \in \mathbb{R}^d \), which establishes (ii). Moreover, (ii) and (a) imply (i), which finishes the proof of the proposition. \( \square \)

A.6 Proof of Theorem 1

Consider the random field
\[
I(t, x) = \langle h(x), Z_t \rangle, \quad t \in [0, T], \ x \in \mathbb{R}^d,
\]
and define the operator \( \mathcal{L} : C^2(\mathbb{R}^d, \mathbb{R}) \to C(\mathbb{R}^d, \mathbb{R}) \) by
\[
\mathcal{L} w(x) = \frac{1}{2} \text{Trace}_{\mathbb{R}^d} (\sigma \sigma^T \text{Hess}(w)(x)) - \langle \mu(x), \nabla w(x) \rangle_{\mathbb{R}^d}, \quad w \in C^2(\mathbb{R}^d, \mathbb{R}), \ x \in \mathbb{R}^d.
\] (72)
Let the mappings \( b_z : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d, \ z \in C([0, T], \mathbb{R}^d), \) be given by
\[
b_z(t, x) = \sigma \sigma^T [Dh(x)]^T (z(t) - \mu(x), \quad z \in C([0, T], \mathbb{R}^k), \ t \in [0, T], \ x \in \mathbb{R}^d,
\]
and define the random field
\[
u = (u(t, x))_{t \in [0,T], x \in \mathbb{R}^d} = (u(t, x, \omega))_{t \in [0,T], x \in \mathbb{R}^d, \omega \in \Omega} : [0, T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}
\]
by
\[
u(t, x, \omega) = u_{Z(\omega)}(t, x), \quad t \in [0, T], \ x \in \mathbb{R}^d, \ \omega \in \Omega,
\]
where \( u_z \) is given by (7). Observe that it follows from (2)–(6) that for all \( t \in [0, T] \) and \( x \in \mathbb{R}^d, \)
\[
X_t(x) = u(t, x) e^{\frac{I}{2}} \text{is } \mathcal{F}_t/\mathcal{B}(\mathbb{R})-\text{measurable. So it satisfies (i).}
\]
Next, note that it follows from the assumptions on \( \mu \in C^3(\mathbb{R}^d, \mathbb{R}^d) \) and \( h \in C^4(\mathbb{R}^d, \mathbb{R}^d) \) together with (4) that
\[
(a) \ \text{for every } z \in C([0, T], \mathbb{R}^k), \ b_z \in C^{0,2}([0, T] \times \mathbb{R}^d, \mathbb{R}^d) \text{ has bounded partial derivatives of first and}
\]
second order with respect to the \( x \)-variables,
Moreover, note that it follows from (2) that

\[ \inf_{\alpha \in (0, 1)} \sup_{s \neq t \in [0, T]} \frac{\|Z_t - Z_s\|_{L^k}}{|t - s|^{\alpha}} < \infty \quad \text{P-a.s.} \]

This, (4) and (73) assure that for P-a.a. \( \omega \in \Omega \) the following hold:

\[ \inf_{\alpha \in (0, 1)} \sup_{s \neq t \in [0, T]} \sup_{x \in \mathbb{R}^d} \frac{\|b_{Z(\omega)}(t, x) - b_{Z(\omega)}(s, x)\|_{L^d}}{|t - s|^{\alpha}} < \infty, \]
\[ \inf_{\alpha \in (0, 1)} \sup_{s \neq t \in [0, T]} \sup_{x \in \mathbb{R}^d} \frac{\|\partial_t b_{Z(\omega)}(t, x) - \partial_t b_{Z(\omega)}(s, x)\|_{L^1(\mathbb{R}^d, \mathbb{R})}}{|t - s|^{\alpha}} < \infty, \]
\[ \inf_{\alpha \in (0, 1)} \sup_{s \neq t \in [0, T]} \sup_{x \in \mathbb{R}^d} \frac{|B_{Z(\omega)}(t, x) - B_{Z(\omega)}(s, x)|}{(1 + \|x\|_{L^d}) |t - s|^{\alpha}} < \infty, \]
\[ \inf_{\alpha \in (0, 1)} \sup_{s \neq t \in [0, T]} \sup_{x \in \mathbb{R}^d} \frac{\|\partial_t Z_{Z(\omega)}(t, x) - \partial_t Z_{Z(\omega)}(s, x)\|_{L^1(\mathbb{R}^d, \mathbb{R})}}{(1 + \|x\|_{L^d}) |t - s|^{\alpha}} < \infty, \]
\[ \sup_{(t, x) \in [0, T] \times \mathbb{R}^d} \frac{B_{Z(\omega)}(t, x)}{1 + \|x\|_{L^d}} < \infty. \]

The assumption that \( \varphi \in C^2(\mathbb{R}^d, [0, \infty)) \) has at most polynomially growing derivatives up to the second order, Lemma 5, Lemma 9 and Proposition 11 (with \( b \leftarrow b_{V(\omega)}, R^{t,x} \leftarrow R^V(t,x), \varphi \leftarrow \varphi, B \leftarrow B_{V(\omega), Y(\omega)}, u \leftarrow u_{V(\omega)}, Y(\omega) \)) in the notation of Lemma 9 and Proposition 11 ensure the following:

(A) for all \( \omega \in \Omega \), the mapping \( (t, x) \mapsto u(t, x, \omega) \) is in \( C^{0,2}([0, T] \times \mathbb{R}^d, \mathbb{R}) \) and there exist constants \( a(\omega), c(\omega) \in \mathbb{R} \) such that
\[
\sup_{t \in [0, T]} |u(t, x, \omega)| \leq a(\omega)e^{c(\omega)|x|_{L^d}} \quad \text{for all } x \in \mathbb{R}^d,
\]

(B) for P-a.a. \( \omega \in \Omega \), the mapping \( (t, x) \mapsto u(t, x, \omega) \) is in \( C^{1,2}([0, T] \times \mathbb{R}^d, \mathbb{R}) \), and

(C)\[
\begin{align*}
\frac{\partial_{t} u(t, x, \omega)}{1 + \|x\|_{L^d}} &= \varphi(x) + \int_{0}^{t} \frac{1}{4} \text{Trace}_{\mathbb{R}^d} (\sigma \sigma^{T} \text{Hess}_{x}(u)(s, x, \omega)) \]
\[+ \left( b_{Z(\omega)}(s, x), \nabla_{x} u(s, x, \omega) \right)_{\mathbb{R}^d} + B_{Z(\omega)}(s, x) u(s, x, \omega) \]
\[ds \]
\end{align*}
\]
for all \( t \in [0, T], x \in \mathbb{R}^d \) and P-almost all \( \omega \in \Omega \).

It follows from the assumptions on \( h \in C^4(\mathbb{R}^d, \mathbb{R}^d) \) that for all \( \omega \in \Omega \), the mapping \( (t, x) \mapsto I(t, x, \omega) \) is in \( C^{0,2}([0, T] \times \mathbb{R}^d, \mathbb{R}) \). Moreover, since \( X(t) = u(t, x) T(t, x) \) for all \( t \in [0, T] \) and \( x \in \mathbb{R}^d \), we obtain from (A) that for every \( \omega \in \Omega \), the mapping \( (t, x) \mapsto X(t, x, \omega) \) is in \( C^{0,2}([0, T] \times \mathbb{R}^d, \mathbb{R}) \) and there exist constants \( a(\omega), c(\omega) \geq 0 \) such that
\[
\sup_{t \in [0, T]} |X(t, x, \omega)| \leq a(\omega)e^{c(\omega)|x|_{L^d}} \quad \text{for all } x \in \mathbb{R}^d.
\]

Hence, \( X \) fulfils (B).

Next, note that it follows from
\[
\nabla_{x} I(t, x) = [Dh(x)]^{T} Z_{t} \quad t \in [0, T], x \in \mathbb{R}^d,
\]
(74)
and (C) that
\[ u(t, x) = \varphi(x) + \int_0^t \left[ \mathcal{L}_x u(s, x) + \langle \sigma \sigma^T \nabla_x I(s, x), \nabla_x u(s, x) \rangle_{\mathbb{R}^d} + B_Z(s, x) u(s, x) \right] ds \quad \mathbb{P}\text{-a.s.} \quad (75) \]
for all \( t \in [0, T] \) and \( x \in \mathbb{R}^d \). Now, consider the random field
\[ v(t, x) = e^{-I(t,x)} \mathcal{L}_x X_t(x), \quad t \in [0, T], \ x \in \mathbb{R}^d. \quad (76) \]
Since \( X_t(x) = u(t, x) e^{I(t,x)} \) for all \( t \in [0, T] \) and \( x \in \mathbb{R}^d \), we obtain from (74) and Lemma 12 below (applied for every \( t \in [0, T] \) and \( \omega \in \Omega \) with \( a \leftarrow \sigma \sigma^T, \nu \leftarrow \mu, f \leftarrow I(t,.,\omega), \ g \leftarrow u(t,.,\omega) \) in the notation of Lemma 12) that
\[ v(t, x) = \mathcal{L}_x u(t, x) + \langle \sigma \sigma^T \nabla_x I(t, x), \nabla_x u(t, x) \rangle_{\mathbb{R}^d} \]
\[ + \left[ \frac{1}{2} \langle \sigma \sigma^T \nabla_x I(t, x), \nabla_x I(t, x) \rangle_{\mathbb{R}^d} + \mathcal{L}_x I(t, x) \right] u(t, x) \quad \text{for all} \ t \in [0, T] \ \text{and} \ x \in \mathbb{R}^d. \]
So it follows from (75) that
\[ u(t, x) = \varphi(x) + \int_0^t \left[ v(s, x) + \left\{ B_Z(s, x) - \frac{1}{2} \langle \sigma \sigma^T \nabla_x I(s, x), \nabla_x I(s, x) \rangle_{\mathbb{R}^d} - \mathcal{L}_x I(s, x) \right\} u(s, x) \right] ds \]
\[ \mathbb{P}\text{-a.s. for all} \ t \in [0, T] \ \text{and} \ x \in \mathbb{R}^d. \]
Combining this and (4) with (74) shows that
\[ u(t, x) = \varphi(x) + \int_0^t \left[ v(s, x) - \left\{ \frac{1}{2} \| h(x) \|_{L^2_k}^2 + \text{div}(\mu(x)) \right\} u(s, x) \right] ds \quad \mathbb{P}\text{-a.s.} \quad (77) \]
for all \( t \in [0, T] \) and \( x \in \mathbb{R}^d \). Next, note that since \( I(t, x) = \langle h(x), Z_t \rangle_{\mathbb{R}^k}, \ t \in [0, T], \ x \in \mathbb{R}^d \), we obtain from Itô’s formula that
\[ e^{I(t,x)} = 1 + \int_0^t e^{I(s,x)} \left( \langle h(x), dZ_t \rangle_{\mathbb{R}^k} + \frac{1}{2} \| h(x) \|^2_{L^2_k} \right) ds \quad \mathbb{P}\text{-a.s for all} \ t \in [0, T] \ \text{and} \ x \in \mathbb{R}^d. \quad (78) \]
Since \( X_t(x) = u(t, x) e^{I(t,x)} \) for all \( t \in [0, T] \) and \( x \in \mathbb{R}^d \), we obtain from (78)–(77) and Itô’s formula that
\[ X_t(x) = \varphi(x) + \int_0^t e^{I(s,x)} \left[ v(s, x) - \left\{ \frac{1}{2} \| h(x) \|^2_{L^2_k} + \text{div}(\mu(x)) \right\} u(s, x) \right] ds \]
\[ + \int_0^t u(s, x) e^{I(s,x)} \left( \langle h(x), dZ_t \rangle_{\mathbb{R}^k} + \frac{1}{2} \| h(x) \|^2_{L^2_k} \right) ds \quad \mathbb{P}\text{-a.s.} \]
for all \( t \in [0, T] \) and \( x \in \mathbb{R}^d \). Moreover, it follows from (76) that
\[ X_t(x) = \varphi(x) + \int_0^t \left[ \mathcal{L}_x X_s(x) - X_s(x) \text{div}(\mu(x)) \right] + \int_0^t X_s(x) \langle h(x), dZ_t \rangle_{\mathbb{R}^k} \quad \mathbb{P}\text{-a.s.} \]
for all \( t \in [0, T] \) and \( x \in \mathbb{R}^d \). This, (4), the fact that
\[ \text{div}(\mu X_s(x)) = \langle \mu(x), \nabla X_s(x) \rangle_{\mathbb{R}^d} + X_s(x) \text{div}(\mu(x)) \quad \text{for all} \ s \in [0, T] \ \text{and} \ x \in \mathbb{R}^d, \]
\[ 2 \] and (72) show that
\[ X_t(x) = \varphi(x) + \int_0^t \left[ \frac{1}{2} \text{Trace}_{\mathbb{R}^d}(\sigma \sigma^T \text{Hess}(X_s(x))) - \text{div}(\mu X_s(x)) \right] ds + \int_0^t X_s(x) \langle h(x), dZ_t \rangle_{\mathbb{R}^d} \]
\[ \mathbb{P}\text{-a.s. for all} \ t \in [0, T] \ \text{and} \ x \in \mathbb{R}^d, \] which shows that \( X \) satisfies (iii).
In addition, it can be seen from (84) that the second expectation in (82) is bounded by $ae$ for every continuous function $f$. But, by (83), one has
\[
\sup_{t \in [0,T]} |\tilde{u}(t, x, \omega)| \leq a(\omega)e^{c(\omega)}||\tilde{u}||_{\mathbb{R}^d} \quad \text{for all } x \in \mathbb{R}^d.
\] (79)

Now, fix an $\omega \in \Omega$ and set $z = Z(\omega)$. Then, the function $(t, x) \mapsto \tilde{u}(t, x) = \tilde{u}(t, x, \omega)$ belongs to $C^{1,2}([0, T] \times \mathbb{R}^d, \mathbb{R})$ and satisfies
\[
d[\tilde{u}(t - s, R_{s,t}^{z,t,x}) \exp \left( \int_0^t B_z(t - r, R_{s,t}^{z,t,x}) dr \right)] = \langle b_z(t - s, R_{s,t}^{z,t,x}), \nabla_x \tilde{u}(t - s, R_{s,t}^{z,t,x}) \rangle_{\mathbb{R}^d} \exp \left( \int_0^t B_z(t - r, R_{s,t}^{z,t,x}) dr \right) \sigma dU_t
\] (81)
for all $t \in [0, T]$, $s \in [0, t]$ and $x \in \mathbb{R}^d$. Using the stopping times
\[
\tau_n = \inf\{s \in [0, t] : \|R_{s,t}^{z,t,x}\|_{\mathbb{R}^d} \geq n\}, \quad n \in \mathbb{N},
\]
one obtains from (81) that
\[
\tilde{u}(t, x) = E\left[ \varphi(R_{s,t}^{z,t,x}) \exp \left( \int_0^t B_z(t - r, R_{s,t}^{z,t,x}) dr \right) 1_{\{\tau_n \geq t\}} \right]
+ E\left[ \tilde{u}(t - \tau_n, R_{\tau_n,t}^{z,t,x}) \exp \left( \int_0^\tau B_z(t - r, R_{s,t}^{z,t,x}) dr \right) 1_{\{\tau_n < t\}} \right] \quad \text{for all } n \in \mathbb{N}.
\] (82)

We know from Lemma 3(iii) that
\[
c_p := E\left[ \sup_{s \in [0, t]} \exp (p\|R_{s,t}^{z,t,x}\|_{\mathbb{R}^d}) \right] < \infty \quad \text{for all } p \in (0, \infty).
\] (83)

Moreover, it follows from (79) and (b) that there exist constants $a, c \geq 0$ such that
\[
\sup_{s \in [0, t]} |\tilde{u}(t - s, f(s))| \exp \left( \int_0^s B_z(t - r, f(r)) dr \right) \leq \sup_{s \in [0, t]} ae^{c\|f(s)\|_{\mathbb{R}^d}}
\] (84)
for every continuous function $f : [0, T] \to \mathbb{R}^d$, which together with Lemma 5(ii) and Lebesgue’s dominated convergence theorem implies that the first expectation in (82) converges to
\[
E\left[ \varphi(R_{t}^{z,t,x}) \exp \left( \int_0^t B_z(t - s, R_{s,t}^{z,t,x}) ds \right) \right] \quad \text{for } n \to \infty.
\]
In addition, it can be seen from (84) that the second expectation in (82) is bounded by $ae^{cn}P[\tau_n < t]$. But, by (83), one has
\[
P[\tau_n < t] \leq P\left[ \sup_{s \in [0, t]} \|R_{s,t}^{z,t,x}\|_{\mathbb{R}^d} \geq n \right] \leq c_p e^{-pn} \quad \text{for all } p \in (0, 1).
\]
So for $p > c$, the second expectation in $[82]$ is bounded by $ac_p e^{(c-p)n}$, which converges to 0 for $n \to \infty$. This shows that

$$\tilde{u}(t,x,\omega) = u_{\mathcal{Z}(\omega)}(t,x)$$

and therefore, $\tilde{X}_t(x,\omega) = X_t(x,\omega)$ for all $t \in [0,T]$, $x \in \mathbb{R}^d$ and $\mathbb{P}$-a.a. $\omega \in \Omega$, which completes the proof of Theorem $[1]$. 

**Lemma 12.** Let $d \in \mathbb{N}$, and consider the mapping $\mathcal{L} : C^2(\mathbb{R}^d, \mathbb{R}) \to C(\mathbb{R}^d, \mathbb{R})$ given by

$$\mathcal{L}g(x) = \frac{1}{2} \text{Trace}_{\mathbb{R}^d}(a \text{Hess}(g)(x)) - \langle \nu(x), \nabla g(x) \rangle_{\mathbb{R}^d}, \quad g \in C^2(\mathbb{R}^d, \mathbb{R}), \quad x \in \mathbb{R}^d,$$

where $a$ is a symmetric $d \times d$-matrix and $\nu$ a function in $C^1(\mathbb{R}^d, \mathbb{R}^d)$. Then

$$e^{-f(x)} \mathcal{L}(e^f g)(x) = \mathcal{L}g(x) + \langle a \nabla f(x), \nabla g(x) \rangle_{\mathbb{R}^d} + \left[ \frac{1}{2} \langle a \nabla f(x), \nabla f(x) \rangle_{\mathbb{R}^d} + \mathcal{L}f(x) \right] g(x)$$

for all $f,g \in C^2(\mathbb{R}^d, \mathbb{R})$ and $x \in \mathbb{R}^d$.

**Proof.** If $f,g \in C^2(\mathbb{R}^d, \mathbb{R})$, one obtains from the product and chain rule that

$$\nabla(e^f g)(x) = \nabla e^f(x)g(x) + e^f(x)\nabla g(x) = e^f(x)\nabla f(x)g(x) + e^f(x)\nabla g(x)$$

$$= e^f(x)\left[ \nabla f(x)g(x) + \nabla g(x) \right]$$

for all $x \in \mathbb{R}^d$. Another application of the product and chain rule gives

$$\frac{\partial^2}{\partial x_i \partial x_j}(e^f g)(x) = e^f(x) \frac{\partial}{\partial x_i} f(x) \left[ \frac{\partial}{\partial x_j} f(x) g(x) + \frac{\partial}{\partial x_j} g(x) \right]$$

$$+ e^f(x) \left[ \frac{\partial^2}{\partial x_i \partial x_j} f(x) g(x) + \frac{\partial}{\partial x_j} f(x) \frac{\partial}{\partial x_i} g(x) + \frac{\partial^2}{\partial x_i \partial x_j} g(x) \right]$$

for all $i,j \in \{1,2,\ldots,d\}$ and $x \in \mathbb{R}^d$, which implies

$$e^{-f(x)} \text{Hess}(e^f g)(x)$$

$$= \text{Hess}(g)(x) + \nabla f(x)(\nabla g(x))^T + \nabla g(x)(\nabla f(x))^T + g(x)[\nabla f(x)(\nabla f(x))^T + \text{Hess}(f)(x)]$$

for all $x \in \mathbb{R}^d$. Since $a$ is symmetric, one has $\text{Trace}_{\mathbb{R}^d}(apq^T) = \langle ap, q \rangle_{\mathbb{R}^d} = \langle aq, p \rangle_{\mathbb{R}^d} = \text{Trace}_{\mathbb{R}^d}(aqp^T)$ for all $p,q \in \mathbb{R}^d$. Therefore, one obtains from $[85]$ and $[86]$ that

$$e^{-f(x)} \mathcal{L}(e^f g)(x) = \frac{1}{2} \text{Trace}_{\mathbb{R}^d}(a \text{Hess}(g)(x)) - \langle \nu(x), e^{-f(x)} \nabla (e^f g)(x) \rangle_{\mathbb{R}^d}$$

$$= \frac{1}{2} \text{Trace}_{\mathbb{R}^d}(a \text{Hess}(g)(x)) + a \nabla f(x)(\nabla g(x))^T + a \nabla g(x)(\nabla f(x))^T$$

$$+ g(x)a \nabla f(x)(\nabla f(x))^T + g(x) a \text{Hess}(f)(x) - \langle \nu(x), g(x) \nabla f(x) + \nabla g(x) \rangle_{\mathbb{R}^d}$$

$$= \frac{1}{2} \text{Trace}_{\mathbb{R}^d}(a \text{Hess}(g)(x)) - \langle \nu(x), \nabla g(x) \rangle_{\mathbb{R}^d}$$

$$+ \langle a \nabla f(x), \nabla g(x) \rangle_{\mathbb{R}^d} + \frac{1}{2} \langle a \nabla f(x), \nabla f(x) \rangle_{\mathbb{R}^d} g(x)$$

$$+ \frac{1}{2} \text{Trace}_{\mathbb{R}^d}(a \text{Hess}(f)(x)) g(x) - \langle \nu(x), \nabla f(x) \rangle_{\mathbb{R}^d} g(x)$$

$$= \mathcal{L}g(x) + \langle a \nabla f(x), \nabla g(x) \rangle_{\mathbb{R}^d} + \frac{1}{2} \langle a \nabla f(x), \nabla f(x) \rangle_{\mathbb{R}^d} g(x) + \mathcal{L}f(x) g(x),$$

for all $x \in \mathbb{R}^d$, which proves the lemma. \[\square\]

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