EQUIDISTRIBUTION OF SMALL SUBVARIETIES OF AN
ABELIAN VARIETY

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Abstract. We prove an equidistribution result for small subvarieties of an abelian
variety which generalizes the Szpiro-Ullmo-Zhang theorem on equidistribution of
small points.

1. Introduction

1.1. Notation. The following notation and conventions will be used throughout this
paper:

K a number field.

O_K the ring of integers of K.

A an abelian variety defined over K.

We fix, for future use, a choice of an algebraic closure \overline{K} of K and an embedding
of \overline{K} into \mathbb{C}.

1.2. Heights of cycles. Let X be a smooth projective variety over K of dimension
N \geq 1, and let \mathcal{X} be a model for X over \mathcal{O}_K, i.e., an integral scheme projective and
flat over Spec \mathcal{O}_K whose generic fiber is X.

Let \mathcal{L} be a hermitian line bundle on \mathcal{X}. A hermitian metric is always assumed to
be smooth and invariant under complex conjugation. We assume furthermore that
\mathcal{L}_K is ample, and that the curvature form c_1(\mathcal{L}) satisfies c_1(\mathcal{L}) > 0. (See [3] for a
discussion of the curvature form associated to a hermitian line bundle).

By using arithmetic intersection theory, one defines the height of a cycle in Arakelov
geometry as follows (see e.g. [5]):

Definition. The height of a nonzero effective cycle Y (of pure dimension) of X with
respect to \mathcal{L} is

\[ h_{\mathcal{L}}(Y) := \frac{c_1(\mathcal{L}|_Y)^{\dim Y + 1}}{(\dim Y + 1)c_1(\mathcal{L}|_Y)^{\dim Y}}, \]

where \overline{Y} is the (scheme-theoretic) Zariski closure of Y in \mathcal{X}.

For a detailed overview of all the properties of curvature forms, arithmetic Chern
classes, and heights of arithmetic cycles which we will need, see [1] or [5]. Proofs of
the relevant facts can be found in [3].

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1.3. **Canonical heights on abelian varieties.** Let $A/K$ be an abelian variety. Using the choice of an embedding of $\overline{K}$ into $\mathbb{C}$, we view $A(\overline{K})$ as a subset of $A(\mathbb{C})$. Let $\mathcal{A}$ be a model for $A$ over $\text{Spec} \, \mathcal{O}_K$. Let $\mathcal{L}$ be a hermitian line bundle on $\mathcal{A}$ such that $L := \mathcal{L}_K$ is symmetric and ample, and such that $L$ is equipped with the **cubical metric** (see [I]).

Fix a nontrivial multiplication map on $A$ (e.g. multiplication by 2). One can then construct from $(A, \mathcal{L})$ a sequence $(A_n, \mathcal{L}_n)_{n \geq 1}$ of models of $(A, L)$, where each $\mathcal{L}_n$ is equipped with the cubical metric, in such a way that the sequence $h_{\mathcal{L}_n}(Y) = c_1(\mathcal{L}_n|_{\overline{Y}_n})^{\dim Y + 1} (\dim Y + 1)c_1(L|_Y) \dim Y$ converges (uniformly in $Y$) to a nonnegative real number $\hat{h}_L(Y)$. (Here $Y$ is a nonzero effective cycle of pure dimension on $A$, and $\overline{Y}_n$ is the Zariski closure of $Y$ in $A_n$. See [I] or [K] for details.) Note that $\hat{h}_L$ does not depend on the choice of $(A, \mathcal{L})$ or a sequence of models $(A_n, \mathcal{L}_n)_{n \geq 1}$. For $x \in A(\overline{K})$, $\hat{h}_L(x)$ is the Néron-Tate canonical height of $x$ with respect to $L$.

1.4. **Statement of the main theorem.** We need several definitions in order to state our main result. By a *variety* $X$ over a field $k$, we mean an integral separated scheme of finite type over $k$. By a *subvariety* of $X$, we mean an integral closed subscheme.

**Definition.** A *torsion subvariety* of $A$ is a translate of an abelian subvariety of $A$ by a torsion point.

**Definition.** A sequence $(X_n)_{n \geq 1}$ of closed subvarieties of $A$ is *small* if $\hat{h}_L(X_n) \to 0$ (as $n \to \infty$).

**Definition.** A sequence $(X_n)_{n \geq 1}$ of closed subvarieties of $X$ is *generic* if it has no subsequence contained in a proper Zariski closed subset of $X$.

**Definition.** A sequence $(X_n)_{n \geq 1}$ of closed subvarieties of $A$ is *strict* if it has no subsequence contained in a proper torsion subvariety of $A$.

Note that the subvarieties $X_n$ are required to be defined over $\overline{K}$, but not necessarily over $K$.

The following result is a generalization of the Szpiro-Ullmo-Zhang/Ullmo/Zhang equidistribution theorem to sequences of small subvarieties of an abelian variety.

**Theorem 1.1 (Strict Equidistribution).** Let $A/K$ be an abelian variety, let $L$ be a symmetric ample line bundle on $A$, and let $\mathcal{L}$ denote $L$ with the cubical metric. Let $(X_n)_{n \geq 1}$ be a small strict sequence of closed subvarieties of $A$. Then for every real-valued continuous function $f$ on $A(\mathbb{C})$, we have

$$\int_{A(\mathbb{C})} f \, \mu_n \to \int_{A(\mathbb{C})} f \, \mu,$$

as $n \to \infty$, where setting $d_n = \dim X_n$ and $g = \dim A$, we have $\mu_n = \frac{1}{c_1(L|_{X_n})^{d_n}c_1(\mathcal{L})^{d_n} \delta X_n}$ and $\mu = \frac{c_1(\mathcal{L})^g}{c_1(L)^g}$.
Remarks.
1. The first integral is the integral of $f$ against the restriction of $c_1(\mathcal{L})^{d_n}/\deg_L(X_n)$ to $X_n(\mathbb{C})$. The second integral is the integral of $f$ with respect to the Haar measure $\mu$ on $A(\mathbb{C})$, normalized to have total mass 1.

2. If $X_n = x_n$ is a point, i.e., if $d_n = 0$, note that
$$\int_{A(\mathbb{C})} f(x) \mu_n = \frac{1}{\#O(x_n)} \sum_{x \in O(x_n)} f(x),$$
where $O(x_n)$ is the orbit of $x_n$ under the action of $\text{Gal}(\overline{K}/K)$.

3. For notational convenience, we write $\mu_n \xrightarrow{w} \mu$ as $n \to \infty$, and say the sequence $(\mu_n)_{n \geq 1}$ of measures weakly converges to $\mu$, if
$$\int_{A(\mathbb{C})} f \mu_n \to \int_{A(\mathbb{C})} f \mu$$
for every continuous function $f : A(\mathbb{C}) \to \mathbb{R}$. In this case, we say that the $X_n$’s are equidistributed with respect to $\mu$.

4. To get a feeling for what Theorem 1.1 says, consider the following simple example. Let $E$ be an elliptic curve defined over $\mathbb{Q}$ and let $A = E \times E$. For each $n \geq 1$, let $E_n \subset A$ be the graph of the multiplication-by-$n$ map on $E$. Then each $E_n$ is a torsion subvariety of $A$ defined over $\mathbb{Q}$ (in fact, $E_n$ is $\mathbb{Q}$-isogenous to $E$).

It is easy to see that $\deg(E_n) \to \infty$ as $n \to \infty$, and that $\bigcup_{n \geq 1} E_n$ is Zariski dense in $A$. Theorem 1.1 says something stronger than this, namely that as $n \to \infty$, the normalized Haar measure on $E_n$ approximates the normalized Haar measure on $A$ arbitrarily closely.

5. For related equidistribution results, see Theorem 1.1 of [2], Theorem 4.1 of [5], Theorem 2.3 of [6], and Theorem 1.1 of [9]. In addition, Pascal Autissier has recently obtained a proof of Theorem 2.2 of the present paper independently of the authors.

2. Generic Equidistribution

Let $X$ be a closed subvariety of dimension $N \geq 1$ of $A$. The following result is a special case of Zhang’s “Theorem of the successive minima” (see [9] for details):

**Theorem 2.1.** Define
$$\lambda_1(X) := \sup_Z \inf_{x \in Z} \hat{h}_L(x),$$
where $Z$ runs over the set of all proper closed subsets of $X$. Then
$$\lambda_1(X) \geq \hat{h}_L(X) \geq \frac{1}{N+1} \lambda_1(X).$$

**Definition.** Let $\mathcal{L}$ be a hermitian line bundle on $A$. If $f$ is a real-valued $C^\infty$-function on $A(\mathbb{C})$, define
$$\overline{\mathcal{L}}(f) := \mathcal{L} \otimes (\mathcal{O}_A, e^{-f})$$
to be the tensor product of $\mathcal{L}$ with the trivial bundle, endowed with the metric given by $\|1\|(P) = e^{-f(P)}$.

**Theorem 2.2 (Generic Equidistribution).** Let $A/K$ be an abelian variety, and let $L$ be a symmetric ample line bundle on $A$. Let $(X_n)_{n \geq 1}$ be a small generic sequence of closed subvarieties of $A$. Then, for every real-valued continuous function $f$ on $A(\mathbb{C})$, we have

$$\int_{A(\mathbb{C})} f(x) \mu_n \to \int_{A(\mathbb{C})} f(x) \mu$$

as $n \to \infty$, where $d_n = \dim X_n$, $\mu_n = \frac{1}{c_1(L|X_n)^{\cdot n}} c_1(\mathcal{L})^n \delta_{X_n}$, $g = \dim A$, $\mu = \frac{c_1(\mathcal{L})^g}{c_1(\mathcal{L})^\cdot}$, and $\mathcal{L}$ is $L$ with the cubical metric.

**Proof.** Enumerate the countably many subvarieties $(Z_n)_{n \geq 1}$ of $A$ defined over $\overline{K}$. Since $(X_n)_{n \geq 1}$ is generic, we may assume, without loss of generality, $X_n \not\subset Z_1 \cup \cdots \cup Z_n$. By the definition of $\lambda_1(X_n)$, we can find (for each $n \geq 1$) an infinite sequence $(x_{n,m})_{m \geq 1}$ in $X_n$ such that:

(i) for each $m \geq 1$, $x_{n,m} \not\in \bigcup_{1 \leq i \leq n} Z_i$;

(ii) $|\hat{h}_L(x_{n,m}) - \lambda_1(X_n)| < \frac{1}{n}$ for all $m \geq 1$; and

(iii) for each $n \geq 1$, $\lim_{m \to \infty} \hat{h}_L(x_{n,m}) = \lambda_1(X_n)$.

By choosing a bijection between $\mathbb{N}^2$ and $\mathbb{N}$, we may consider the doubly-indexed sequence $(x_{n,m})$ as a sequence indexed by the natural numbers. Property (i) guarantees that the resulting sequence $(x_{n,m})$ is generic. Furthermore, since $\hat{h}_L(X_n) \to 0$ by assumption, it follows from the theorem of the successive minima that $\lambda_1(X_n) \to 0$ as $n \to \infty$. Using this observation, properties (ii) and (iii) easily imply that $\lim_{n,m \to \infty} \hat{h}_L(x_{n,m}) = 0$, i.e., that the sequence $(x_{n,m})$ is small.

Define

$$\alpha_{n,m} := \frac{1}{\#O(x_{n,m})} \sum_{f(x) \in O(x_{n,m})} f(x),$$

where $O(x_{n,m})$ is the orbit of $x_{n,m}$ under the action of $\text{Gal}(\overline{K}/K)$. By choosing a subsequence of $(x_{n,m})_{m \geq 1}$ if necessary, we may assume that $\lim_{m \to \infty} \alpha_{n,m}$ exists for all $n \geq 1$. Note that every subsequence of a small (resp. generic) sequence is small (resp. generic).

Approximating $f$ by $C^\infty$-functions if necessary, we may assume that $f$ is a $C^\infty$-function. Let $\lambda > 0$ be a real number. Note that $c_1(\mathcal{L}(\lambda f)) > 0$ if $\lambda > 0$ is small enough. We then note, for $l \geq 1$, that

$$h_{\mathcal{L}(\lambda f)}(x_{n,m}) = h_{\mathcal{L}(\lambda f)}(x_{n,m}) + \lambda \alpha_{n,m};$$

and

$$\liminf_{m \to \infty} h_{\mathcal{L}(\lambda f)}(x_{n,m}) \geq h_{\mathcal{L}(\lambda f)}(X_n) \quad \text{[5], Proposition 2.1}$$

$$= h_{\mathcal{L}(f)}(X_n) + \lambda \int_{A(\mathbb{C})} f(x) \mu_n + O(\lambda^2),$$

where the last equality follows from [11] Proof of Proposition 2.9]. Here the $O$-constant is independent of $l$ and $n$, and $\lambda > 0$ is sufficiently small.
Fix $n \geq 1$ and $\varepsilon > 0$. Then for $m$ sufficiently large, we have:

$$h_{\Xi}(x_{n,m}) + \lambda \alpha_{n,m} \geq h_{\Xi}(X_n) + \lambda \int_{A(\mathbb{C})} f(x)\mu_n + O(\lambda^2) - \varepsilon.$$  

Letting $l \to \infty$, we have:

$$\hat{h}_L(x_{n,m}) - \hat{h}_L(X_n) + \lambda \alpha_{n,m} \geq \lambda \int_{A(\mathbb{C})} f(x)\mu_n + O(\lambda^2) - \varepsilon.$$  

Now let $m \to \infty$, and we obtain (since $\varepsilon > 0$ is arbitrary):

$$\lambda_1(X_n) - \hat{h}_L(X_n) + \lambda \lim_{m \to \infty} \alpha_{n,m} \geq \lambda \int_{A(\mathbb{C})} f(x)\mu_n + O(\lambda^2). \quad (1)$$

On the other hand, the Szpiro-Ullmo-Zhang/Ullmo/Zhang equidistribution theorem ([5], [6], and [9]), applied to the small generic sequence $(x_{n,m})$, implies that $\lim_{n,m \to \infty} \alpha_{n,m}$ exists, and that

$$\lim_{n,m \to \infty} \alpha_{n,m} = \int_{A(\mathbb{C})} f(x)\mu. \quad (2)$$

Taking $\limsup_{n \to \infty}$ in (1), we have:

$$\lambda \lim_{n,m \to \infty} \alpha_{n,m} \geq \lambda \limsup_{n \to \infty} \int_{A(\mathbb{C})} f(x)\mu_n + O(\lambda^2).$$

Now divide both sides by $\lambda > 0$, and let $\lambda \to 0^+$. We obtain:

$$\lim_{n,m \to \infty} \alpha_{n,m} \geq \limsup_{n \to \infty} \int_{A(\mathbb{C})} f(x)\mu_n. \quad (3)$$

Replacing $f$ by $-f$, we see that:

$$\lim_{n,m \to \infty} \alpha_{n,m} \leq \liminf_{n \to \infty} \int_{A(\mathbb{C})} f(x)\mu_n. \quad (4)$$

It then follows from (3) and (4) that $\lim_{n \to \infty} \int_{A(\mathbb{C})} f(x)\mu_n$ exists and that

$$\lim_{n \to \infty} \int_{A(\mathbb{C})} f(x)\mu_n = \lim_{n,m \to \infty} \alpha_{n,m}.$$

We conclude from (2) that

$$\lim_{n \to \infty} \int_{A(\mathbb{C})} f(x)\mu_n = \int_{A(\mathbb{C})} f(x)\mu,$$  

as desired. \qed
3. Strict Equidistribution

The following result is a consequence of two results of Zhang: the generalized Bogomolov conjecture (see [9]) and the theorem of the successive minima. The proof is similar to the proof of Theorem 2.2.

**Theorem 3.1.** Let $X$ be a nontorsion subvariety of $A$. Then there is an $\epsilon > 0$ such that the set

$$\bigcup \{ Y : \text{$Y$ is a closed subvariety of $X$ such that $\hat{h}_L(Y) \leq \epsilon$} \}$$

is not Zariski dense in $X$.

**Proof.** Suppose, for the sake of contradiction, that $(Y_n)_{n \geq 1}$ is a sequence of distinct closed subvarieties of $X$ which is small (i.e., $\hat{h}_L(Y_n) \to 0$) and generic in $X$ (i.e., no subsequence is contained in a proper Zariski closed subset of $X$). Then, proceeding as in the proof of Theorem 2.2, we can construct an infinite sequence $(y_k)_{k \geq 1}$ of points in $X$ such that $\{ y_k \in X : k \geq 1 \}$ is Zariski dense in $X$ and $\hat{h}_L(y_k) \to 0$. But Corollary 3 of [9] then implies that $X$ is a torsion subvariety of $A$, a contradiction. $\square$

Now we are ready to prove Theorem 1.1 (Strict Equidistribution Theorem).

**Proof (of Theorem 1.1).** By Theorem 2.2, it suffices to show that the small and strict sequence $(X_n)_{n \geq 1}$ is generic. Let $X'$ be the Zariski closure of $\bigcup_k X_{n_k}$ for any subsequence $(X_{n_k})_{k \geq 1}$ of $(X_n)_{n \geq 1}$. By Theorem 3.1, $X$ must be a torsion subvariety of $A$. Since $(X_n)_{n \geq 1}$ is strict, it follows that $X' = A$, so that $(X_n)_{n \geq 1}$ is generic as desired. $\square$

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