Path integral of Bianchi I models in loop quantum cosmology

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Abstract
A path integral formulation of the Bianchi I models containing a massless scalar field in loop quantum cosmology is constructed. Following the strategy used in the homogenous and isotropic case, the calculation is extended to the simplest non-isotropic models according to the $\bar{\mu}$ and $\bar{\mu}'$ schemes. It is proved from the path integral angle that the quantum dynamic lacks the full invariance with respect to fiducial cell scaling in the $\bar{\mu}$ scheme, but it does not in the $\bar{\mu}'$ scheme. The investigation affirms the equivalence of the canonical approach and the path integral approach in loop quantum cosmology.

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1. Introduction

Loop quantum gravity (LQG) [1] and the spinfoam formalism [2, 3] can be considered respectively as canonical quantization and covariant quantization of gravity just like what we did in ordinary quantum field theory. Applying the techniques used in LQG to simple cosmology models, we obtain loop quantum cosmology (LQC) [4], which is a canonical version of quantum cosmology. Naturally, we desire to understand LQC conclusions from a path integral, i.e. a covariant angle. Starting from the Hilbert space of LQC, there are two kinds of path integrals. One is integral over paths in configuration space [5–8], which leads to a spinfoam like ‘vertex summation’ rather than a standard path integral format. The other is integral over phase space paths [9–11] from which we can obtain a typical continuous path integral that features continuous paths with weights given by the exponential of the phases.

In this paper, we will use phase space paths following the methods in the original work of Ashtekar, Campiglia and Henderson [9], and extend the LQC path integral formulation from the homogenous and isotropic FRW model to the simplest homogenous Bianchi type-I models [12]. Up to now there are two kinds of ‘loop regularizations’ of the gravitational part

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of the scalar constraint for the Bianchi I models in the literature, referred to as $\bar{\mu}$ and $\bar{\mu}'$ schemes [13, 14]. Each of them has its own advantages and drawbacks. The $\bar{\mu}$ scheme gives a difference equation in terms of affine variables and therefore the well-developed framework of the spatially flat-isotropic LQC can be straightforwardly adopted. While the $\bar{\mu}'$ scheme has better scaling properties [14], the difference equation of this scheme is very complex. Here, we use both $\bar{\mu}$ and $\bar{\mu}'$ schemes presented in [13] to perform a path integral formulation, and to explore their similarities and differences.

The organization of this paper is as follows. In the section 2, we present the quantum theory of Bianchi I models in the volume and connection representations. In section 3, we construct the path integral formalism of quantum Bianchi I models in phase space using the $\bar{\mu}$ scheme, and the construction in the $\bar{\mu}'$ scheme will be presented in section 4. Finally, we discuss the results obtained from those two schemes in section 5.

2. LQC of the diagonal Bianchi I models

Bianchi cosmologies are homogeneous cosmological models, in which there is a foliation of spacetime $M = \Sigma \times \mathbb{R}$ such that $\Sigma$ is space-like and there is a transitive isometry group freely acting on $\Sigma$. Thanks to these symmetry properties, in classical theory, after the diagonalizing and rescaling processes of the Ashtekar variables (see [12]), the only non-vanishing constraint (scalar constraint) for the Bianchi I cosmology (coupled with a massless scalar field $\phi$) is given by

$$C = - \frac{1}{8\pi G} \rho_\gamma (c^1 p_1 c^2 p_2 + c^1 p_1 c^3 p_3 + c^2 p_2 c^3 p_3) + \frac{1}{2} \rho_\phi^2,$$

(1)

where $p_\phi$ is the momentum of the scalar field, while $c^i$ and $p_j$ are the reduced and rescaled Ashtekar variables. The Poisson brackets of $c^i$ and $p_j$ are

$$\{c^i, p_j\} = 8\pi G \delta^i_j.$$

(2)

In the isotropic models, there is a reflection symmetry $\Pi(p) = -p$, which corresponds to the orientation reversal of triads. These are gauge transformations which leave the Hamiltonian constraint invariant. In Bianchi I case, we have three reflections $\Pi_i$s corresponding respectively to the reversal of one of the triads while leaving the other two fixed. The Hamiltonian flow is invariant under the action of each $\Pi_i$ [14]. Hence it suffices to restrict our attention to one of the octant of three $p_i$s. In this paper, we focus on the positive octant in which all three $p_i$s are positive.

2.1. $\bar{\mu}$ scheme

In the canonical quantization theory, i.e. LQC, there exists a comprehensive construction of the operator corresponding to the equation (1) in the full LQG scheme [12]. The full LQG quantization proposes two kinds of quantum corrections to the scalar constraint, one of which is the inverse triad correction by using a so-called Thiemann trick, while the other one is holonomy correction which means to correct the curvature term by using an $SU(2)$ holonomy. In this paper, we start out setting the lapse function $N = \sqrt{|p_1 p_2 p_3|}$ in the classical theory as in [14]. This treatment does not only make the Hamiltonian constraint simpler, but also avoids the use of inverse triad correction. Hence, we can use the regularized constraint [13] to replace
Then we fix the factor ordering ambiguity of equation (3), by defining the operator
\[ {\ell} \]
then the kinematical Hilbert space of the gravitational part is given by
\[
|\nu\rangle = \exp\left(\frac{\pi}{\gamma \sqrt{|p_i|}}\right) |\nu\rangle
\]
where the index \( i \) does not sum over and their Poisson bracket is given by
\[
\{\hat{\theta}^i, v_j\} = \Theta^i_j
\]
where \( \Theta^i_j \) is the ‘area gap’ [15]. Usually, we can make an algebraic simplification by introducing new phase space variables:
\[
v_i = \frac{p_{3i}}{6\pi G}, \quad b_i = \frac{c_i}{\gamma \sqrt{|p_i|}}
\]
To define the constraint operator, we have to define two kinds of operators first. One is the volume operator defined as
\[
\hat{V} |v_1, v_2, v_3\rangle = 6\pi G \sqrt{\Delta} |v_1 v_2 v_3\rangle^{1/3} |v_1, v_2, v_3\rangle,
\]
where \( \hat{V} = \hat{V}_1 \hat{V}_2 \hat{V}_3 \) is the unitary shift operator corresponding to \( p_i \). The other is the unitary shift operator
\[
\hat{U}_i |v_i\rangle = \exp(i\hat{\mu}_i c^i) |v_i\rangle = |v_i + 2\ell_0 h\rangle,
\]
where \( \ell_0 \) is related to the ‘area gap’ by \( \ell_0^2 = \gamma^2 \Delta \), where \( \gamma \) is the Barbero–Immirzi parameter. Replace the function \( \sin \hat{\mu}_i c^i \) in equation (3) by the operator \( \sin \hat{\mu}_i c^i \) using the unitary shift operator defined above, we obtain
\[
\sin(\hat{\mu}_i c^i) |v_i\rangle = \frac{1}{2i} \left( \exp(i\hat{\mu}_i c^i) - \exp(-i\hat{\mu}_i c^i) \right) |v_i\rangle
= \frac{1}{2i} \left( |v_i + 2\ell_0 h\rangle - |v_i - 2\ell_0 h\rangle \right).
\]
Then we fix the factor ordering ambiguity of equation (3), by defining the operator
\[
\hat{\Theta}_i |v_i\rangle := \frac{1}{2} \left( \sin \frac{\hat{\mu}_i c^i}{\sqrt{\Delta}} \hat{V}_i + \hat{V}_i \sin \frac{\hat{\mu}_i c^i}{\sqrt{\Delta}} \right) |v_i\rangle
= -3\pi G \gamma \left( (v_i + \ell_0 h) |v_i + 2\ell_0 h\rangle - (v_i - \ell_0 h) |v_i - 2\ell_0 h\rangle \right).
\]
Note that \( \{\hat{\Theta}_i, \hat{\Theta}_j\} |v_i, v_j, v_3\rangle = 0 \) for \( i \neq j \). The gravitational part of the scalar constraint operator can be written in terms of these operators as
\[
\hat{C}\text{grav} = -\frac{1}{8\pi G \gamma^2} (\hat{\Theta}_1 \hat{\Theta}_2 + \hat{\Theta}_2 \hat{\Theta}_3 + \hat{\Theta}_3 \hat{\Theta}_1).
\]
The action of this operator on the arbitrary basis element of the kinematical Hilbert space is [16]
\[
\hat{C}^{\text{grav}} |v_1, v_2, v_3\rangle = \frac{9\pi G}{8} \left[ (v_1 + 1)(v_2 + 1) |v_1 + 2, v_2 + 2, v_3\rangle - (v_1 - 1)(v_2 + 1) |v_1 - 2, v_2 + 2, v_3\rangle \\
- (v_1 + 1)(v_2 - 1) |v_1 + 2, v_2 - 2, v_3\rangle + (v_1 - 1)(v_2 - 1) |v_1 - 2, v_2 - 2, v_3\rangle \\
+ (v_1 + 1)(v_3 + 1) |v_1, v_2 + 2, v_3 + 2\rangle - (v_2 - 1)(v_3 + 1) |v_1, v_2 - 2, v_3 + 2\rangle \\
- (v_2 + 1)(v_3 - 1) |v_1, v_2 + 2, v_3 - 2\rangle + (v_2 - 1)(v_3 - 1) |v_1, v_2 - 2, v_3 - 2\rangle \\
+ (v_1 + 1)(v_3 + 1) |v_1 + 2, v_2, v_3 + 2\rangle - (v_1 - 1)(v_3 + 1) |v_1 - 2, v_2, v_3 + 2\rangle \\
- (v_1 + 1)(v_3 - 1) |v_1 + 2, v_2, v_3 - 2\rangle + (v_1 - 1)(v_3 - 1) |v_1 - 2, v_2, v_3 - 2\rangle \right],
\]
where the $\ell_0/\hbar$ terms are omitted for short. From this equation we can see that in the $\tilde{\mu}$ scheme, the space of solutions to the quantum constraint is very similar to the isotropic case. $v_1$, $v_2$, $v_3$ are discrete variables and they are supported on a specific superselections. They can take the values of $\nu$ are discrete variables and they are supported on a specific superselections. They can take the 

2.2. $\tilde{\mu}'$ scheme

We still use the regularized constraint equation (3) in the $\tilde{\mu}'$ scheme. Only $\tilde{\mu}_i$ changes to $\tilde{\mu}'_i$ in this situation, and $\tilde{\mu}'_i$ takes the form
\[
\tilde{\mu}'_i = \sqrt{\frac{|p_i|\Delta}{|p_j|\Delta}}, \quad \tilde{\mu}'_j = \sqrt{\frac{|p_j|\Delta}{|p_i|\Delta}}, \quad \tilde{\mu}'_3 = \sqrt{\frac{|p_3|\Delta}{|p_2|\Delta}}.
\]
Owning to this condition, it is inconvenient to still use $v_i$ as configuration variables. By introducing new variables $\lambda_i$,
\[
\lambda_i = \sqrt[3]{\frac{|p_i|}{(4\pi G)^{1/3}}},
\]
the algebra of the $\tilde{\mu}'$ scheme could be much more simplified [14]. The variable conjugate to $\lambda_i$ is
\[
k^i = \frac{2\sqrt{|p_i|c_i}}{\gamma (4\pi G)^{2/3}}.
\]
We can prove that the resulting Poisson bracket between $k^i$ and $\lambda_j$ is
\[
\{k^i, \lambda_j\} = 2\delta^i_j.
\]
Then the new orthonormal basis is $|\lambda_1, \lambda_2, \lambda_3\rangle$ in $H_{\text{kin}}^{\text{grav}}$, and the fundamental volume operator and unitary shift are given by
\[
\hat{V} |\lambda_1, \lambda_2, \lambda_3\rangle = 2\pi G \gamma \sqrt{\Delta v} |\lambda_1, \lambda_2, \lambda_3\rangle
\]
and
\[
\hat{E}_{\lambda_i}|\lambda_i\rangle = \exp(\sqrt{i\tilde{\mu}_i c_i}) |\lambda_i\rangle = |\lambda_i + \sqrt{i\tilde{\mu}_i c_i}|\lambda_i\rangle.
\]
Here, $v = 2\lambda_1\lambda_2\lambda_3$ is the configuration variable related to the volume of the elementary cell, and the factor 2 ensures that this $v$ reduces to the $v$ used in the isotropic case. We can also use $\lambda_i, \lambda_j$, and $v$ as the basic configuration variables to substitute for $\lambda_1, \lambda_2$ and $\lambda_3$. The indices $i, j, k = 1, 2, 3$ appearing in equation (20) are required to be different from each other, and, as usual, the index $i$ does not sum over. Having these operators in hands, we could obtain the gravitational part of the scalar constraint operator. Its action on the arbitrary basis element of the new kinematical Hilbert space is [14]

$$\hat{C}^{\text{grav}} |\lambda_1, \lambda_2, v\rangle = \frac{\pi G}{2} \sqrt{\nu} |(v + 2)\sqrt{\nu + 4}|\lambda_1, \lambda_2, v\rangle = (v + 2) \sqrt{\nu} |\lambda_1, \lambda_2, v\rangle_0^{+} - (v - 2) \sqrt{\nu} |\lambda_1, \lambda_2, v\rangle_0^{-} + (v - 2) \sqrt{|v - 4|} |\lambda_1, \lambda_2, v\rangle_4^{+}. \quad (21)$$

where

$$|\lambda_1, \lambda_2, v\rangle_4^{\pm} = \left[ \frac{v \pm 4}{v \pm 2} \cdot \lambda_1, \frac{v \pm 4}{v \pm 2} \cdot \lambda_2, v \pm 4 \right] + \left( \frac{v \pm 4}{v \pm 2} \cdot \lambda_1, \frac{v \pm 4}{v \pm 2} \cdot \lambda_2, v \pm 4 \right) + \left( \frac{v \pm 4}{v \pm 2} \cdot \lambda_1, \frac{v \pm 4}{v \pm 2} \cdot \lambda_2, v \pm 4 \right), \quad (22)$$

and

$$|\lambda_1, \lambda_2, v\rangle_0^{\pm} = \left[ \frac{v \pm 4}{v \pm 2} \cdot \lambda_1, \frac{v \pm 4}{v \pm 2} \cdot \lambda_2, v \pm 4 \right] + \left( \frac{v \pm 4}{v \pm 2} \cdot \lambda_1, \frac{v \pm 4}{v \pm 2} \cdot \lambda_2, v \pm 4 \right) + \left( \frac{v \pm 4}{v \pm 2} \cdot \lambda_1, \frac{v \pm 4}{v \pm 2} \cdot \lambda_2, v \pm 4 \right). \quad (23)$$

As what we did in equation (14), the $\ell_0h$ terms are omitted here. From equations (22) and (23), we can see that the variable $v$ is fine, and the wavefunction only involves $(v - 4\ell_0h), v, (v + 4\ell_0h)$ terms. It is exactly the same as in the isotropic case. In contrast, the situation for $\lambda_1$ and $\lambda_2$ is much different. We see that they depend only on the value of $v$. This dependence is through fractional factors whose denominator is two or four units bigger or smaller than the numerator. This leads to the consequence that the iterative action of the constraint operator leads only to states whose quantum numbers $\lambda_i$ are of the form $\lambda_i = \omega_i \lambda_i'$; here $\lambda_i'$ is the initial value of $\lambda_i$, and $\epsilon$ is a constant number that $v = \epsilon + 4n\ell_0h, n \in \mathbb{N}$ and $\omega_i$ belongs to the set [18]

$$\Omega_\epsilon = \left\{ \left( \frac{\epsilon - 2}{\epsilon} \right)^z \prod_{m \in \mathbb{N}} \left( \frac{\epsilon + 2m}{\epsilon + 2m + \epsilon} \right)^{\lambda_i'} \right\}. \quad (24)$$

where $k^m_n \in \mathbb{N}$, and $z \in \mathbb{Z}$ if $\epsilon > 2$, while $z = 0$ when $\epsilon \leq 2$. The discrete set $\Omega_\epsilon$ is countably infinite and turns out to be dense in the positive real line. For the proof of this statement, see appendix D3 of [18]. So, on the one hand, $\lambda_i$ are superselected in separable sectors and have countable numbers of values. On the other hand, the dependence on the fractional factors makes it can take values in entire positive real line.

3. Phase space path integral in the $\tilde{\mu}$ scheme

In this section, we calculate the extraction amplitude $A(\tilde{v}_i, \phi_j; \tilde{v}_i, \phi_j)$ for Bianchi I cosmology (here $\tilde{v}$ is short for $\{v_1, v_2, v_3\}$). As we know, in the `timeless’ framework of LQC, the whole
information of the quantum dynamics is encoded in \( A(\vec{v}_f, \phi_f; \vec{v}_i, \phi_i) \), which is the transition amplitude for our phase space path integral just as in ordinary quantum mechanics. We can express it as [5]

\[
A(\vec{v}_f, \phi_f; \vec{v}_i, \phi_i) = \int d\phi \langle \vec{v}_f, \phi_f | e^{i\hat{C}/\hbar} | \vec{v}_i, \phi_i \rangle,
\]

(25)

where the total constraint operator \( \hat{C} \) is composed of two parts: the gravitational part and the scalar field part

\[
\hat{C} = \hat{C}^{\text{grav}} + \hat{C}^{\text{matt}}.
\]

(26)

The two operators \( \hat{C}^{\text{grav}} \) and \( \hat{C}^{\text{matt}} \) are commutative, so they can act on their own Hilbert space, \( \mathcal{H}^{\text{grav}}_{\text{Kin}} \) and \( \mathcal{H}^{\text{matt}}_{\text{Kin}} \), respectively. Then following the spirit of [9], decomposing the fictitious evolution into \( N \) evolutions of length \( \epsilon = 1/N \) and inserting complete basis in between each factor, we obtain

\[
\langle \vec{v}_f, \phi_f | e^{i\hat{C}/\hbar} | \vec{v}_i, \phi_i \rangle = \sum_{v_0, \ldots, v_N} \int d\phi \langle \vec{v}_N, \phi_N | e^{i\hat{C}/\hbar} | \vec{v}_0, \phi_0 \rangle \ldots \langle \vec{v}_1, \phi_1 | e^{i\hat{C}/\hbar} | \vec{v}_i, \phi_i \rangle,
\]

(27)

where \( \langle \vec{v}_N, \phi_N \rangle = \langle \vec{v}_i, \phi_i \rangle \) and \( \langle \vec{v}_0, \phi_0 \rangle = \langle \vec{v}_i, \phi_i \rangle \).

Firstly, we need to calculate the \( n \)th term appearing in the above expression. Using equation (26), it can be stated as

\[
\langle \phi_{n+1} | e^{i\hat{C}^{\text{grav}}}/\hbar | \phi_n \rangle = \langle \phi_{n+1} | e^{i\hat{C}^{\text{grav}}}/\hbar | \phi_n \rangle = \int \frac{d\phi}{2\pi} e^{i\hat{C}^{\text{grav}}/\hbar} = \langle \phi_{n+1} | \hat{C}^{\text{grav}} | \phi_n \rangle + O(\epsilon^2),
\]

(29)

and the gravitational factor is

\[
\langle v^1_{n+1}, v^2_{n+1}, v^3_{n+1} | e^{i\hat{C}^{\text{grav}}}/\hbar | v^1_n, v^2_n, v^3_n \rangle = \delta_{\nu^1_{n+1}, \nu^1_n} \delta_{\nu^2_{n+1}, \nu^2_n} \delta_{\nu^3_{n+1}, \nu^3_n} + \frac{i}{\hbar} \epsilon \alpha (\nu^1_{n+1}, \nu^2_{n+1}, \nu^3_{n+1}) + \hat{C}^{\text{grav}} | v^1_n, v^2_n, v^3_n \rangle + O(\epsilon^2),
\]

(30)

where we made an expansion in \( \epsilon \), and \( O(\epsilon^2) \) is the high-order term of the expansion. Using equation (3), the matrix element of \( \hat{C}^{\text{grav}} \) is

\[
\langle \nu^1_{n+1}, \nu^2_{n+1}, \nu^3_{n+1} | \hat{C}^{\text{grav}} | \nu^1_n, \nu^2_n, \nu^3_n \rangle = \frac{9\pi G}{8 \epsilon} \left[ \begin{array}{c}
\nu^1_{n+1} + \nu^2_{n+1} + \nu^3_n \\
\nu^1_n + \nu^2_{n+1} + \nu^3_n \\
\nu^1_{n+1} + \nu^2_{n+1} + \nu^3_n
\end{array} \right]
\]

\[
\times \left( \frac{\delta_{\nu^1_{n+1}, \nu^1_n} + \delta_{\nu^2_{n+1}, \nu^2_n} + \delta_{\nu^3_{n+1}, \nu^3_n}}{8} - \delta_{\nu^1_{n+1}, \nu^2_n} - \delta_{\nu^2_{n+1}, \nu^1_n} - \delta_{\nu^1_{n+1}, \nu^3_n} - \delta_{\nu^3_{n+1}, \nu^1_n} - \delta_{\nu^2_{n+1}, \nu^3_n} - \delta_{\nu^3_{n+1}, \nu^2_n}
\right)
\]

\[
+ \frac{\nu^1_{n+1} + \nu^2_{n+1} + \nu^3_n}{2} \left( \delta_{\nu^1_{n+1}, \nu^1_n} - \delta_{\nu^1_{n+1}, \nu^2_n} - \delta_{\nu^2_{n+1}, \nu^1_n} + \delta_{\nu^1_{n+1}, \nu^3_n} - \delta_{\nu^3_{n+1}, \nu^1_n} - \delta_{\nu^1_{n+1}, \nu^3_n}
\right)
\]

\[
+ \frac{\nu^1_{n+1} + \nu^2_{n+1} + \nu^3_n}{2} \left( \delta_{\nu^2_{n+1}, \nu^2_n} - \delta_{\nu^1_{n+1}, \nu^2_n} - \delta_{\nu^1_{n+1}, \nu^2_n} + \delta_{\nu^3_{n+1}, \nu^2_n}
\right)
\]

\[
- \frac{\nu^1_{n+1} + \nu^2_{n+1} + \nu^3_n}{2} \left( \delta_{\nu^3_{n+1}, \nu^3_n} - \delta_{\nu^1_{n+1}, \nu^3_n} - \delta_{\nu^3_{n+1}, \nu^3_n} + \delta_{\nu^1_{n+1}, \nu^3_n}
\right)
\]

\[
+ \frac{\nu^1_{n+1} + \nu^2_{n+1} + \nu^3_n}{2} \left( \delta_{\nu^3_{n+1}, \nu^3_n} - \delta_{\nu^1_{n+1}, \nu^3_n} - \delta_{\nu^3_{n+1}, \nu^3_n} + \delta_{\nu^1_{n+1}, \nu^3_n}
\right)
\]

\[
+ \frac{\nu^1_{n+1} + \nu^2_{n+1} + \nu^3_n}{2} \left( \delta_{\nu^3_{n+1}, \nu^3_n} - \delta_{\nu^1_{n+1}, \nu^3_n} - \delta_{\nu^3_{n+1}, \nu^3_n} + \delta_{\nu^1_{n+1}, \nu^3_n}
\right).
\]

(31)
As demonstrated before, we choose a specific superselection of $v_i$ such that $v_i = 2n_i \ell_0 \hbar$, where $n_i \in \mathbb{N}$. Then we can use the Fourier expansion formula of the Kronecker delta

$$\delta_{\nu' \nu} = \frac{\ell_0}{\pi} \int_0^{\pi/\ell_0} d\nu e^{-i b (\nu' - \nu)/2 \hbar},$$

(32)

where, obviously, $b$ is the conjugate variable to $\nu$ and takes values in the range $(0, \pi/\ell_0)$. By using equation (32), then equation (31) can be expressed as

$$\langle \nu_0 | v_{n+1}^1, v_{n+1}^2, v_{n+1}^3 \rangle e^{i \omega \nu_0} \langle v_{n+1}^1, v_{n+1}^2, v_{n+1}^3 | \nu_0 \rangle = \left( \frac{\ell_0}{\pi} \right)^3 \int d\nu_0 e^{-i \nu_0 (v_{n+1}^1, v_{n+1}^2, v_{n+1}^3)} \cdot e^{-i \nu_0 (v_{n+1}^1, v_{n+1}^2, v_{n+1}^3)} \cdot e^{-i \nu_0 (v_{n+1}^1, v_{n+1}^2, v_{n+1}^3)}
\times \left\{ 1 - \frac{i}{\hbar} \frac{9}{2} \pi G \left[ \frac{v_{n+1}^1 + v_n^1}{2} + v_n^2 \right] \sin \ell_0 b_{n+1}^1 \sin \ell_0 b_{n+1}^2
+ \frac{v_{n+1}^2 + v_n^2}{2} \frac{v_{n+1}^1 + v_n^1}{2} \sin \ell_0 b_{n+1}^2 \sin \ell_0 b_{n+1}^3
+ \frac{v_{n+1}^3 + v_n^3}{2} \frac{v_{n+1}^1 + v_n^1}{2} \sin \ell_0 b_{n+1}^3 \right\} \right\}. \tag{33}

Now according to the calculation before, we could put the scalar field part and gravitational part together, and equation (27) takes the form

$$\langle \hat{\psi}_f, \phi_f | e^{i \omega_0 \mu} | \hat{\psi}_f, \phi_f \rangle = \sum_{v_0, \ldots, v_N} \left( \frac{\ell_0}{\pi} \right)^{3N} \int d\nu_0 \cdots d\nu_1 \cdot \left( \frac{1}{2\pi} \right)^{3N} \int d\nu_N \cdots d\nu_1 e^{i S_N} + O(\epsilon^2),$$

(34)

where

$$S_N = \epsilon \sum_{n=0}^{N-1} \left\{ \frac{p_n}{2} - \frac{9}{2} \pi G \left[ \frac{v_{n+1}^1 + v_n^1}{2} + v_n^2 \right] \sin \ell_0 b_{n+1}^1 \sin \ell_0 b_{n+1}^2
+ \frac{v_{n+1}^2 + v_n^2}{2} \frac{v_{n+1}^1 + v_n^1}{2} \sin \ell_0 b_{n+1}^2 \sin \ell_0 b_{n+1}^3
+ \frac{v_{n+1}^3 + v_n^3}{2} \frac{v_{n+1}^1 + v_n^1}{2} \sin \ell_0 b_{n+1}^3 \right\}. \tag{35}

This is the ‘discrete-time action’ of the Bianchi I cosmology. Just like in [9], the final step is to take the limit $N \to \infty$, and because the variable $v_i$ is discrete, it is impossible to interpret the $(v_{n+1}^i - v_n^i)/\epsilon$ as a derivative, and we should transform the terms $(v_{n+1}^i - v_n^i)/\epsilon$ into $(b_{n+1}^i - b_n^i)/\epsilon$:

$$\frac{1}{2} (v_nb_N - v_nb_1) = \frac{1}{2} \sum_{n=0}^{N-1} (b_{n+1}^i v_{n+1} - v_nb_n). \tag{36}
$$

Then

$$\epsilon \sum_{n=0}^{N-1} \left[ \frac{b_{n+1}^i v_{n+1} - v_n^i}{\epsilon} \right] = \epsilon \sum_{n=0}^{N-1} \left[ \frac{v_n^i b_{n+1} - b_n^i}{\epsilon} \right] + \frac{1}{2} (b_1 v_0 - b_N v_N). \tag{37}
$$
Using this transformation, $S_N$ can be rewritten as

$$S_N = \epsilon \left( p_{n+1} - p_n + \frac{v_{b_n}^2}{2} + \frac{v_{b_n}^2}{2} + \frac{v_{b_n}^2}{2} - \frac{v_{b_n}^2}{2} \right) + \alpha \left[ p_n^2 - 9 \pi G \left( \frac{v_{b_n}^2}{2} + \frac{v_{b_n}^2}{2} + \frac{v_{b_n}^2}{2} \right) \sin \ell_0 b_{n+1} \sin \ell_0 b_{n+1} \right] - \frac{1}{2} (\vec{v}_f \cdot \vec{b}_f - \vec{v}_i \cdot \vec{b}_i).$$

(38)

Finally, taking the limit $N \to \infty$, we have

$$A(\vec{v}_f, \phi_f; \vec{v}_i, \phi_i) = \int \alpha \left[ \int [Dv_q(\tau)][Db_q(\tau)][Dp(\tau)][D\phi(\tau)] e^{\frac{i}{\hbar} S} \right],$$

(39)

where

$$S = \int_0^1 d\tau \left\{ p\phi + \frac{1}{2} \vec{v} \cdot \vec{b} - \alpha \left[ p^2 - 9 \pi G (v^1 v^2 \sin \ell_0 b^1 + v^2 v^3 \sin \ell_0 b^2 + v^3 v^1 \sin \ell_0 b^3 + v^1 v^3 \sin \ell_0 b^3) \right] - \frac{1}{2} (\vec{v}_f \cdot \vec{b}_f - \vec{v}_i \cdot \vec{b}_i) \right\}.$$  

(40)

As in the isotropic case, we use the subscript $q$ here to emphasize that now we can take a sum in the geometrical sector including only the ‘quantum paths’. Also, we use the same trick to transform it into the familiar format

$$\int_0^{\pi/\ell_0} \sum_{n} \int_0^{\infty} db_n \rightarrow \int_{-\infty}^{\infty} dv_n \int_{-\infty}^{\infty} db_n.$$  

(41)

Using equation (41), we obtain the final expression of the extraction amplitude

$$A(\vec{v}_f, \phi_f; \vec{v}_i, \phi_i) = \int \alpha \left[ \int [Dv(\tau)][Db(\tau)][Dp(\tau)][D\phi(\tau)] e^{\frac{i}{\hbar} S} \right],$$

(42)

where

$$S = \int_0^1 d\tau \left\{ p\phi - \frac{1}{2} \vec{b} \cdot \vec{v} - \alpha \left[ p^2 - 9 \pi G (v^1 v^2 \sin \ell_0 b^1 + v^2 v^3 \sin \ell_0 b^2 + v^3 v^1 \sin \ell_0 b^3 + v^1 v^3 \sin \ell_0 b^3) \right] \right\}.$$  

(43)

This is the path integral formulation and its action we desired. Because we used equation (41), the integration about $v_i$ and $b_i$ is taken from $-\infty$ to $\infty$, with the boundary terms in equation (40) canceling each other out. And then we are able to integrate over all paths in the classical phase space as in usual path integrals.

4. Phase space path integral in the $\vec{\mu}'$ scheme

The construction of a phase space path integral in the $\vec{\mu}'$ scheme is more complex and subtler than in the $\vec{\mu}$ scheme. As we illustrated in section 2.2, the new orthonormal basis in $H_{grav}^{Ko}$ can be $|\lambda_1, \lambda_2, \lambda_3\rangle$ or $|\lambda_i, \lambda_j, v\rangle$. In order to compare with the $\vec{\mu}$ scheme situation conveniently, we choose the former one. Following the procedure employed in 3, we are going to calculate the extraction amplitude $A(\vec{\lambda}_f, \phi_f; \vec{\lambda}_i, \phi_i)$. 

8
The matter part of the extraction amplitude we use here is the same as in the $\tilde{\mu}$ scheme, so we ignore it for a little while and directly use it at the end. Decomposing the fictitious evolution into $N$ evolutions of length $\epsilon = \frac{1}{n}$, the $n$th term of the gravitational part is

$$
\langle \tilde{C}^{n+1}_{\epsilon} \mid e^{ip\tilde{C}^{grav}_{\epsilon}} \mid \tilde{C}^{n}_\epsilon \rangle = \delta (\lambda^{n+1}_1 - \lambda^n_1) \delta (\lambda^{n+1}_2 - \lambda^n_2) \delta (\lambda^{n+1}_3 - \lambda^n_3) + \frac{i}{\hbar} \delta (\lambda^{n+1}_1, \lambda^{n+1}_2, \lambda^{n+1}_3) \tilde{C}^{grav}_{\epsilon} \langle \lambda^n_1, \lambda^n_2, \lambda^n_3 \rangle + O(\epsilon^2).
$$

As we know, although the number of possible values of $\lambda_i$ is countable, there does not exist an obvious superselection with respect to it; therefore, we have to work with the entire positive real line it spans. So, for the possibility of performing a path integral, in a sense, we have used an approximation here. The Dirac delta is used rather than Kronecker delta to express the inner product $\langle \lambda^{n+1}_1 \mid \lambda^n_1 \rangle$, which means we will take $\lambda_i$ as continuous variables below. This assumption may generate a little confusion. When we think $\lambda_i$ is continuous, the quantum theory will be changed which is caused by the change of the Hilbert space. But, what we interest in the following is the effective action, which is a semiclassical quantity itself. And considering the properties of $\lambda_i$, it is reasonable to perform a semiclassical approximation that let $\lambda_i$ be continuous in the following calculations and do not affect the quantum theory. It is worthwhile to emphasize that we will focus on the semiclassical situation from now on.

Then, make a algebraic transformation: $v' = v + 2$, $\lambda_i' = v' \lambda_i / (v' - 2)$; $v'' = v - 2$, $\lambda''_i = v'' \lambda_i / (v'' + 2)$. In terms of equations (21)–(23), the matrix element of $\tilde{C}^{grav}_{\epsilon}$ is expressed as

$$
\langle \lambda^{n+1}_1, \lambda^{n+1}_2, \lambda^{n+1}_3 \mid \tilde{C}^{grav}_{\epsilon} \mid \lambda^n_1, \lambda^n_2, \lambda^n_3 \rangle = \frac{\pi G}{2} \sqrt{v' v'' + 1} (v'(A) + v''(B)),
$$

where

$$
(A) = \pm \left( \lambda^{n+1}_1, \lambda^{n+1}_2, \lambda^{n+1}_3 \right) \left( \lambda^n_1 \pm \frac{\epsilon_0 h}{\lambda^n_2 \lambda^n_3} \lambda^n_2, \lambda^n_3 \right) \pm \left( \lambda^{n+1}_1, \lambda^{n+1}_2, \lambda^{n+1}_3 \right) \left( \lambda^n_1 \pm \frac{\epsilon_0 h}{\lambda^n_2 \lambda^n_3} \lambda^n_2, \lambda^n_3 \right)
$$

$$
\pm \left( \lambda^{n+1}_1, \lambda^{n+1}_2, \lambda^{n+1}_3 \right) \left( \lambda^n_1 \pm \frac{\epsilon_0 h}{\lambda^n_2 \lambda^n_3} \lambda^n_2, \lambda^n_3 \right) \pm \left( \lambda^{n+1}_1, \lambda^{n+1}_2, \lambda^{n+1}_3 \right) \left( \lambda^n_1 \pm \frac{\epsilon_0 h}{\lambda^n_2 \lambda^n_3} \lambda^n_2, \lambda^n_3 \right).
$$

and

$$
(B) = \pm \left( \lambda^{n+1}_1, \lambda^{n+1}_2, \lambda^{n+1}_3 \right) \left( \lambda^n_1 \pm \frac{\epsilon_0 h}{\lambda^n_2 \lambda^n_3} \lambda^n_2, \lambda^n_3 \right) \pm \left( \lambda^{n+1}_1, \lambda^{n+1}_2, \lambda^{n+1}_3 \right) \left( \lambda^n_1 \pm \frac{\epsilon_0 h}{\lambda^n_2 \lambda^n_3} \lambda^n_2, \lambda^n_3 \right)
$$

$$
\pm \left( \lambda^{n+1}_1, \lambda^{n+1}_2, \lambda^{n+1}_3 \right) \left( \lambda^n_1 \pm \frac{\epsilon_0 h}{\lambda^n_2 \lambda^n_3} \lambda^n_2, \lambda^n_3 \right) \pm \left( \lambda^{n+1}_1, \lambda^{n+1}_2, \lambda^{n+1}_3 \right) \left( \lambda^n_1 \pm \frac{\epsilon_0 h}{\lambda^n_2 \lambda^n_3} \lambda^n_2, \lambda^n_3 \right).
$$

Employing the identity

$$
\delta (\lambda' - \lambda) = \frac{1}{\pi} \int_{-\infty}^{+\infty} dk \ e^{-ik(\lambda' - \lambda)/2\hbar},
$$

(48)
where $k$ is the variable conjugated to $\lambda$, as we stated in equation (17), we are led to the following expression for equation (44):

\[
\langle \hat{\lambda}^{n+1}|e^{\tau \hat{C}_{\text{grav}}} |\hat{\lambda}^n \rangle = \frac{1}{(\pi)^{3/2}} \int d\vec{k} \hat{e}^{-i\hat{C}_{\text{grav}}(\hat{\lambda}^n)} \frac{1}{2\hbar} \left[ 1 + \frac{i}{\hbar} \epsilon \alpha \pi G \sqrt{v^n s^{n+1}} \right] \times \left[ (u''^a)^n \epsilon e^{i(\hat{\lambda}^{1+}_n - \hat{\lambda}^{1-}_n)} + \epsilon e^{i(\hat{\lambda}^{2+}_n - \hat{\lambda}^{2-}_n)} \cdots + \text{cyclic terms} \right] + \epsilon e^{i(\hat{\lambda}^{2+}_n + \hat{\lambda}^{2-}_n)} \cdots + \text{cyclic terms} \right] + O(\epsilon^2).
\]

Let $\epsilon \hbar k^n\lambda^n / v^n = a_n$, $\epsilon \hbar k^n\lambda^n / (u''^a)^n = (u''^a)^n = a''_n$ for short. The exponential terms of this equation can be expressed as

\[
v''[\cos(a_1' + a_2') - i \sin(a_1' + a_2') - \cos(a_1' - a_2') - i \sin(a_1' - a_2')] + \text{cyclic terms} \]

Finally, taking $N \to \infty$, we obtain

\[
A(\vec{\lambda}', \phi'; \vec{\lambda}, \phi) = \int d\alpha \int [D\vec{\lambda}(\tau)][D\vec{\phi}(\tau)][Dp(\tau)][D\phi(\tau)] e^{iS},
\]

where

\[
S = \int_0^1 d\tau \left\{ p\phi - \frac{1}{2} \hat{\lambda} \cdot \hat{\lambda} - \alpha \left[ p^2 - 4\pi Gv^2 \left( \frac{k_1\lambda_1}{v} \epsilon_0 \sin \frac{k_2\lambda_2}{v} \epsilon_0 + \frac{k_3\lambda_3}{v} \sin \frac{k_3\lambda_3}{v} \epsilon_0 \right) \right] \right\}. \tag{55}
\]

Equations (54) and (55) are the final expressions of the extraction amplitude and its action on the $\mu'$ scheme. Note that $\lambda_{i}$ has already been taken as a continuous variable; therefore, it is
reasonable to interpret \((\bar{x}^{n+1} - \bar{x}^n)/\epsilon\) as a derivative here, and the integration ranges of \(\lambda\) and \(\bar{k}\) are taken from \(-\infty\) to \(\infty\) directly. We no longer need to play the trick which was used in the \(\bar{\mu}\) scheme and the isotropic case to transform the integration range. Integration is over all trajectories in the classical phase space from the beginning.

5. Discussion

In this paper, we extended the phase space path integral formulation of Friedmann spacetimes to anisotropic Bianchi I models, and performed the calculation both in \(\bar{\mu}\) and \(\bar{\mu}'\) schemes. We restricted the matter source to be a massless scalar field and focused on the positive octant in which all three \(p_i\) are positive.

Since \(\bar{\mu}_i = \sqrt{\lambda_i^2/p_i}\), in the \(\bar{\mu}\) scheme, by setting \(v_i \sim p_i^{3/2}\), just as \(v \sim p^{3/2}\) in the isotropic case, the procedure used in [9] could be implemented directly. The formulation of the extraction amplitude then resembles three copies of the isotropic model as we desired. To incorporate the \(\bar{\mu}'\) scheme, we have to overcome a few obstacles. The main problem comes from the fact that the expression of \(\mu'_i\), which was given by equation (15), is much more complicated in the \(\bar{\mu}'\) scheme than in the \(\bar{\mu}\) scheme. This leads to the consequence that we have to make an algebraic simplification by introducing \(\lambda_i \sim p_i^{3/2}\), and then the wavefunction is dragged along the \(\lambda_i\) direction by the unitary shift operator \(E_i\). However, the affine distance involved in this dragging depends on \(\lambda_2, \lambda_3\). From the difference equation of the gravitational part of the total constraint, it can be seen that the new variable \(\lambda_i\) has a discrete spectrum but does not support a specific \(\lambda_i\)-lattice. We cannot write an integral representation for the Kronecker delta by using a standard residue calculation like equation (32).

In order to perform a path integral formulation in the \(\bar{\mu}'\) case, we have used two approximations in section 4. The first one is taking \(\lambda_1\) as continuous variable; then one can use the Dirac delta to express the inner product. The second one is to presume the volume of the elementary cell is large enough such that \((v - 2) \approx v \approx (v + 2)\). Actually they are the same thing. In general, our treatment presupposed that the calculation is away from the Planck regime; in other words, a semiclassical approximation was taken. This is the key point where \(\bar{\mu}'\) is different from \(\bar{\mu}\) and the isotropic case. In the two later situations, the calculation is exact and does not need any additional assumption.

If we compare the two path integrals in \(\bar{\mu}\) and \(\bar{\mu}'\) schemes, it is obviously that the actions have the same formulation as in the isotropic case: a classical action plus a quantum gravity correction term, i.e. the \(\sin\) term. Ignoring the constant terms, from equations (43) and (55), one can write down directly the two quantum gravity correction terms in the two schemes (denoted by \(T\) and \(T'\), respectively). In the \(\bar{\mu}\) scheme,

\[
T = v^4 v^2 \sin \ell_0 b_1^3 \sin \ell_0 b_2^3 \sin \ell_0 b_3^3 + v^4 v^3 \sin \ell_0 b_1 \sin \ell_0 b_2^3 + v^4 v^3 \sin \ell_0 b_1^3 \sin \ell_0 b_2^3. \quad (56)
\]

In the \(\bar{\mu}'\) scheme,

\[
T' = v^2 \left( \sin \frac{k_1 \lambda_1}{\mu} \ell_0 \sin \frac{k_2 \lambda_2}{\mu} \ell_0 \sin \frac{k_3 \lambda_3}{\mu} \ell_0 + \sin \frac{k_1 \lambda_1}{\mu} \ell_0 \sin \frac{k_2 \lambda_2}{\mu} \ell_0 \sin \frac{k_3 \lambda_3}{\mu} \ell_0 + \sin \frac{k_1 \lambda_1}{\mu} \ell_0 \sin \frac{k_2 \lambda_2}{\mu} \ell_0 \right). \quad (57)
\]

These terms do not look the same. By using the explicit expressions of \(b_i, \lambda_i\) and \(k_i\), \(T\) and \(T'\) could be expressed as

\[
T = v^4 v^2 \sin \bar{\mu}_1 c_1 \sin \bar{\mu}_2 c_2 + v^4 v^3 \sin \bar{\mu}_2 c_2 \sin \bar{\mu}_3 c_3 + v^4 \sin \bar{\mu}_1 c_1 \sin \bar{\mu}_3 c_3 \quad (58)
\]

and

\[
T' = v^2 (\sin \bar{\mu}'_1 c_1 \sin \bar{\mu}'_2 c_2 + \sin \bar{\mu}'_2 c_2 \sin \bar{\mu}'_3 c_3 + \sin \bar{\mu}'_1 c_1 \sin \bar{\mu}'_3 c_3). \quad (59)
\]
Note that $\nu_i \sim p_i^{3/2}$ and $v \sim (p_1 p_2 p_3)^{1/2}$. Now we find that $T$ and $T'$ are extremely similar in the formulation. The difference is the immediate cause for the fiducial cell dependence on the semiclassical limit in the $\bar{\mu}$ scheme. Consider the fiducial cell $V$. Its volume is $V_0 = L_1 L_2 L_3$. Note that $c_i \sim L_i \dot{a}_i$ and $p_i \sim L_4 a_i \dot{a}_i$, so if we rescale $V$ as

$$V_0 = L_1 L_2 L_3 \to V'_0 = l_1 L_1 l_2 L_2 l_3 L_3 = l_1 l_2 l_3 V_0,$$

then

$$\bar{\mu}_1 c_1 \to \frac{l_1}{\sqrt{l_2 l_3}} \bar{\mu}_1 c_1$$

and

$$\bar{\mu}'_1 c_1 \to \bar{\mu}'_1 c_1.$$ (62)

This is the reason why the $\bar{\mu}'$ scheme has better scaling properties. These properties are already known in the canonical quantization program. In path integral formulation, we have proved again that the quantum dynamic does depend on the choice of the fiducial cell in the $\bar{\mu}$ scheme, while it does not in the $\bar{\mu}'$ scheme. Furthermore, if we take $p_1 = p_2 = p_3$, and therefore $\nu_i = v$, then we find $T = T' \sim v^2 \sin^2 \bar{\mu} c$. This is precisely the quantum correction term in the Friedmann model. Hence, it is rational to forecast that in the path integral formulation of Bianchi I models the quantum bounce will also replace the big bang singularity due to the quantum gravity correction term. All of these properties provide a strong evidence for the equivalence of the canonical approach and the path integral approach in LQC.

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