Abstract In this paper, we use four-dimensional quaternionic algebra to describe space-time geodesics in curvature form. The transformation relations of quaternionic variables are established with the help of basis-transformations of quaternion algebra. We deduce the quaternionic covariant derivative that explains how the quaternion components vary with scalar and vector fields. The quaternionic metric tensor and the geodesic equation are also discussed to describe the quaternionic line element in curved space-time. We examine a quaternionic metric tensor equation for the Riemannian Christoffel curvature tensor. We present the quaternionic Einstein’s field-like equation, which indicates that quaternionic matter and geometry are equivalent.

Relevance of the work: In recent decades, hypercomplex algebra, viz., quaternion and octonions, has been widely used to explain various branches of physics. In this way, we have investigated quaternionic transformations and field equations in curved space-time. The present novel work will help to explain the characteristics of the curved space-time universe in terms of quaternion algebra. It can also be used to describe quaternionic gravitational waves, the black hole formulation, and so on.

Keywords Quaternion · Curvature space-time · Riemannian Christoffel curvature tensor · Einstein field equation

Mathematics Subject Classification 12Hxx · 35Q76 · 83C10

1 Introduction

In physics, there are four natural forces [1]: gravitational force, caused by the attraction of massive substances, electromagnetic force, caused by the attraction of electrically charged particles, weak force, which causes atoms to disintegrate radioactively, and strong force, which binds fundamental particles together. In the gravitational pull, the weakest force is dominant. The concept of absolute space and time is fundamental for the dynamics of an object in Newtonian physics, yet they are not relative to each other. Einstein developed the theory of relativity with the concept of space-time. In the four-spaces theory of special and general relativity, time is considered the fourth dimension. The general theory of relativity (GTR) [2] is the theory that explains gravity. Gravity does not act as a force in GTR; instead, it refers to the curvature of space-time. Since non-Euclidean geometry is used to describe space-time geometry, the space-time path becomes curved in the presence of a gravitational field. The theory of relativity requires a covariant form that is independent of the coordinate system. The tensor shows the covariant form in all coordinate systems. Besides, for a flat space-time, the Pythagoras theorem is valid, but for a curved space-time, this theorem no longer holds and uses a metric to measure the distance.

Moreover, in GTR, the metric tensor behaves as the gravitational potential of Newtonian gravitation. The Riemannian Christoffel’s curvature tensor captures the notion of parallel transport. It gives the idea of the connections of the parallel transported vector on a curve in space-time.
Further, Riemannian curvature is required for all the changes in the tangent vectors when we transport them around a curved space. The Riemann-Christoffel tensor is the only tensor that is constructed from the second derivative of the metric tensor [3]. The contraction of the Riemann tensor gives Ricci curvature and scalar curvature. Ricci curvature explains the concept in which mass converges and diverges in time corresponding to the part of the curvature of space-time. The scalar curvature yields a single real number that represents the quantity of Riemannian manifold volume that distinguishes a geodesic ball from a normal ball in Euclidean space. The equation of motion of free particles is recognised by the geodesic equation, followed by the analogous to Newton’s equation of motion, which clarifies the equation for the acceleration of particles. The source of the curvature of space-time is generalised by the energy and momentum of the field, expressed as an energy-momentum tensor.

In mathematical algebraic structure, there are four types of division algebra [4], i.e., real algebra, complex algebra, quaternion algebra (Hamilton algebra [5]), and octonion algebra (Cayley algebra) [6–8]. Real numbers are the numbers that are used normally without any imaginary numbers. The numbers written as a mixture of a real number and an imaginary number are the complex numbers. Quaternions are the extension of a complex number with a non-commutative property, used to label the rotations in three dimensions. Cayley algebra is the extension of the quaternions with non-commutative and non-associative as well. There are many applications of quaternionic algebra in various branches of theoretical and computational physics [9–13]. In electrodynamics, Maxwell’s equations in the presence of magnetic monopoles [14] and the classical wave equations of motion [15] have been constructed in terms of quaternionic algebra. The quaternionic form of quantum electrodynamics has been discussed [16, 17]. Chanyal [18, 19] proposed the quaternionic covariant theory of four-dimensional particle dyons in the form of relativistic quantum mechanics and also focused on the quantised Dirac-Maxwell equations for dyons. Recently, in magneto-hydrodynamics, the quaternionic dual field equations for dyonic cold plasma have been analysed [20]. Furthermore, the Dirac-Maxwell, Bernoulli, and Navier Stokes like equations for dyonic fluid-plasma in the generalised quaternionic field have been developed [21]. Currently, a new approach to Dirac’s relativistic field equation for rotating free particles has been investigated in quaternionic form [22]. Beyond the quaternionic algebra, many authors [23–35] have studied the role of higher dimensional hypercomplex division algebras in various fields of modern physics. In GTR, Edmonds [36] discussed the quaternion wave equation in curved space-time by using the curvilinear coordinate system in relativistic quantum theory. The quaternionic form of curvature quantum theory fills the gap between quantum and gravity theory. In the same way, Weng [37] studied the electromagnetic and gravitational field equations in complex curved space with the help of quaternions and octonions. The application of octonion algebra to electrodynamics and geometrodynamics has also been thoroughly researched [38, 39]. Aside from that, there have been numerous studies of topological aspects in 4-dimensional Minkowski space and curved space-time [40–43]. Keeping in view the Riemannian space and its connection with quaternions, we discussed the Einstein field-like equation in quaternionic curvilinear form. Starting with quaternionic basis transformation from one frame to another, we define the transformation of scalar and vector field components of a quaternion curvature variable. From the quaternionic covariant derivative, we expressed the Christoffel symbol, quaternionic metric tensor, and quaternionic geodesic equation. We also formulated the quaternionic Riemann tensor that keeps track of how many scalar and vector components of a quaternion change if we propagate in parallel along with a small parallelogram. Interestingly, if the quaternionic Riemannian Christoffel curvature is zero, then the quaternionic curved space-time is converted into flat space-time. Further, the quaternionic Ricci tensor is an important contraction of the quaternionic Riemannian Christoffel tensor which explains the changes in four space-time when an object parallels transport along a geodesic. We have discussed the quaternionic form of the Einstein field-like-equation in compact form, which shows that the quaternionic value of matter-energy is equivalent to the quaternionic geometry.

2 Preliminaries

A tensor is the generalised form of a vector, defined by the arrangement of numbers, or functions that transform according to certain rules under a change in coordinates [44, 45]. According to the Einstein summation convention, if the same index appears twice in a term, then that index stands for the sum of all terms at the complete range of values. For example, we can write the two forms of a vector $\mathbf{u}$ as

$$\mathbf{u} = \sum_i u^i e_i, \quad \mathbf{u} = \sum_j u^j e^j \quad \forall (i, j = 1, 2, 3),$$

where $i$ and $j$ are the repeated indices or summation indices, while $u^i$ and $u_j$ are the vector components in contravariant and covariant form, respectively. The scalar product of two vectors can be represented in terms of indices with their components, i.e.
On the Quaternion Transformation and Field...

\[ \mathbf{a} \cdot \mathbf{b} = \left( \sum_i a_i e_i \right) \cdot \left( \sum_j b_j e_j \right) = \sum_{ij} a_ib_j (e_i \cdot e_j) = \sum_{ij} a_ib_j \delta_{ij} = \sum_i a_i b_i, \]

(2)

where \( \delta_{ij} \) is the Kronecker delta symbol defined as \( \delta_{ij} = 1 \), for \( (i = j) \) and \( \delta_{ij} = 0 \), for \( (i \neq j) \). Also, the vector product becomes

\[ [\mathbf{a} \times \mathbf{b}]_i = \sum_{jk} \epsilon_{ijk} a_j b_k. \]

(3)

Here \( \epsilon_{ijk} \) is the Levi-Civita symbol with three indices having value \( \epsilon_{ijk} = +1 \) for cyclic permutation, \( \epsilon_{ijk} = -1 \) for non-cyclic permutation, and \( \epsilon_{ijk} = 0 \) for any two repeated indices. The tensors are recognised by their order, viz. vectors are first-order tensors, dyadics are second-order tensors, triadics are third-order tensors, and tetradics are fourth-order tensors. By adding any two tensors of the same rank, it gives the tensor of the same order, i.e.

\[ \mathbf{A}'_u = \mathbf{B}'_u + \mathbf{C}'_u, \]

(4)

where \( \mathbf{A}'_u, \mathbf{B}'_u, \) and \( \mathbf{C}'_u \) are the tensors of same rank, i.e. three-rank tensor. The product of two tensors is given by

\[ \mathbf{B}'_u \mathbf{C}'_{uv} = \mathbf{A}'_{uv}. \]

(5)

In the above representation, \( \mathbf{B}'_u \) is tensor of rank two having indices \( r \) and \( s \), \( \mathbf{C}'_{uv} \) is a tensor of rank three having indices \( t, u \) and \( v \), while \( \mathbf{A}'_{uv} \) is a tensor of rank five. If a magnitude or scalar quantity \( \phi \) transforms from one reference frame to another, it remains invariant, such that [46]

\[ \phi' = \phi. \]

(6)

This invariant quantity is known as the contravariant tensor of rank zero or the covariant tensor of rank zero. On the other hand, a vector quantity is transformed as

\[ \psi'^r : \psi^s \rightarrow \psi'^r = \frac{\partial \psi'^r}{\partial \psi^s} \psi^s. \]

(7)

Equation (7) signifies that the components of vector \( \psi^s \) transformed into components of vector \( \psi'^r \) when the coordinate \( x^s \) transformed to \( x'^r \). We can also define the contravariant tensor components of rank two as

\[ \mathbf{A}'^{tu} : \mathbf{A}^{tu}, \rightarrow \mathbf{A}'^{tu} = \frac{\partial \psi'^r}{\partial \psi^s} \frac{\partial \psi^u}{\partial \psi^s} \mathbf{A}^{tu}. \]

(8)

Similarly, we can transform the tensors for higher rank.

3 The Quaternionic Algebra

In mathematics, a complex number over the real algebra \( \mathbb{R} \) having the imaginary number \( i \), is defined by the algebraic extension of a normal real number. Any complex number \( Z \in \mathbb{C} \) is expressed in the form of its basis \( (1, i) \) as

\[ Z = \xi_1 + i\xi_2, \quad \forall (\xi_1, \xi_2) \in \mathbb{R}^2 \text{ and } Z \in \mathbb{C}. \]

(9)

If \( \text{Re}(Z) = 0 \) then the complex number is called purely imaginary. In the same way, the quaternion is a number system that extends the complex number in a form of algebra having some properties over addition and multiplication. Hamilton [5] extended the complex number over real algebra \( \mathbb{R} \) in terms of four-dimensional norm-division algebra. The quaternions have four unit elements \( (e_0, e_1, e_2, e_3) \) called basis elements, in which \( e_0 \) is the scalar unit and \( e_1, e_2, e_3 \) are the imaginary units. A quaternion \( H \in \mathbb{Q} \) is written as

\[ H = e_0w + e_1x + e_2y + e_3z = e_0w + \sum_{j=1}^{3} e_j r_j, \]

(10)

where \( w, x, y, z \) are the real numbers. The quaternion may also be composed of the scalar and vector parts as

\[ H = S_H + V_H. \]

(11)

Here \( e_0w \) is the scalar part of quaternion denoted by \( S_H \) and \((e_1x + e_2y + e_3z)\) is the vector part of quaternion denoted by \( V_H \). If scalar part is zero in Eq. (11), then

\[ H \rightarrow V_H = e_1x + e_2y + e_3z, \]

(12)

is known as right quaternion or pure quaternion. Although any quaternion can be seen as a vector in a four-dimensional vector space, it is usually referred to as the pure quaternion as a vector. Conditionally, if the vector part is zero, then

\[ H \rightarrow S_H = e_0w, \]

(13)

is known as scalar quaternion. The addition of two quaternions is

\[ A + B = (e_0A_0 + e_1A_1 + e_2A_2 + e_3A_3) + (e_0B_0 + e_1B_1 + e_2B_2 + e_3B_3) = e_0(A_0 + B_0) + e_1(A_1 + B_1) + e_2(A_2 + B_2) + e_3(A_3 + B_3), \]

(14)

where \( H_0 \sim (A_0 + B_0) \) is the scalar part of quaternion, while \( H_j \sim (A_j + B_j) \) is the vector part of quaternion. Therefore, Eq. (14) shows the closure property concerning quaternionic addition. The quaternionic algebra also satisfies the associative and commutative properties of
addition. Furthermore, the multiplication properties of two quaternions is expressed by
\[ \mathbb{A} \circ \mathbb{B} = (e_0 a_0 + e_1 a_1 + e_2 a_2 + e_3 a_3) \circ (e_0 b_0 + e_1 b_1 + e_2 b_2 + e_3 b_3) \]
\[ = e_0 p_0 + e_1 p_1 + e_2 p_2 + e_3 p_3 = p \in \mathbb{Q}, \]
(15)
where \( \circ \) is a symbol used for the quaternionic multiplication and the components of \( p \) given in Eq. (15) are written by
\[ P_0 = (A_0 b_0 - A_1 b_1 - A_2 b_2 - A_3 b_3) \text{(coefficient of } e_0) \]
\[ P_1 = (A_0 b_1 + A_1 b_0 + A_2 b_3 - A_3 b_2) \text{(coefficient of } e_1) \]
\[ P_2 = (A_0 b_2 + A_1 b_3 + A_2 b_0 - A_3 b_1) \text{(coefficient of } e_2) \]
\[ P_3 = (A_0 b_3 + A_3 b_0 + A_1 b_2 - A_2 b_1) \text{(coefficient of } e_3). \]
(16)
Here the quaternionic multiplication follows the given rules for basis \((e_0, e_1, e_2, e_3)\), i.e.,
\[ e_0^2 = 1, \quad e_i^2 = -1, \quad e_0 e_i = e_i e_0 = e_i, \]
\[ e_i e_j = -\delta_{ij} e_0 + e_{ijk} e_k, \quad \forall (i,j,k = 1,2,3). \]
(17)
All indices in Levi-Civita symbol \(\epsilon_{ijk}\) are antisymmetric and satisfy \(e_i e_j = \sum_k \epsilon_{ijk} e_k\). Thus, Eq. (15) can be written in compact form in terms of ordinary dot and cross products as,
\[ \mathbb{A} \circ \mathbb{B} = e_0 \left( A_0 B_0 - \vec{A} \cdot \vec{B} \right) \]
\[ + e_i \left[ A_0 \vec{B} + B_0 \vec{A} + (\vec{A} \times \vec{B}) \right]. \]
(18)
In order to check the non-commutative property, we may write
\[ \mathbb{A} \circ \mathbb{B} = (e_0 a_0 + e_1 a_1 + e_2 a_2 + e_3 a_3) \circ (e_0 b_0 + e_1 b_1 + e_2 b_2 + e_3 b_3) \]
\[ = e_0 (B_0 a_0 - B_1 a_1 - B_2 a_2 - B_3 a_3) \]
\[ + e_1 (B_0 a_1 + B_1 a_0 + B_2 a_3 - B_3 a_2) \]
\[ + e_2 (B_0 a_2 + B_2 a_0 + B_3 a_1 - B_1 a_3) \]
\[ + e_3 (B_0 a_3 + B_3 a_0 + B_1 a_2 - B_2 a_1) \]
\[ = e_0 \left( B_0 a_0 - \vec{B} \cdot \vec{A} \right) \]
\[ + e_j \left[ B_0 \vec{A} + A_0 \vec{B} + (\vec{B} \times \vec{A}) \right]. \]
\( \neq \mathbb{A} \circ \mathbb{B}. \)
(19)
It is clear that the product of two quaternions is non-commutative because the product of two vectors is always non-commutative, so that \(\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}\). On the other hand, the multiplication of quaternions satisfy the associative property i.e. \((\mathbb{A} \circ \mathbb{B}) \circ \mathbb{C} = \mathbb{A} \circ (\mathbb{B} \circ \mathbb{C})\).

Further, the quaternionic conjugate of Eq. (10) can be expressed as
\[ \mathbb{H}^* = e_0 w - (e_1 x + e_2 y + e_3 z), \]
and the product of two quaternions is given by [47]
\[ \mathbb{A} \cdot \mathbb{B} = -\frac{1}{2} (\mathbb{A} \circ \mathbb{B}^* + \mathbb{B} \circ \mathbb{A}^*) = -\frac{1}{2} (\mathbb{A}^* \circ \mathbb{B} + \mathbb{B}^* \circ \mathbb{A}). \]
(21)
The norm of a quaternion \(\mathbb{H}\) denoted by \(|\mathbb{H}|\) can be represented as
\[ |\mathbb{H}| = \sqrt{\mathbb{H} \circ \mathbb{H}^*} = \sqrt{w^2 + x^2 + y^2 + z^2}. \]
(22)
Now, the inverse of a quaternion is defined by
\[ \mathbb{H}^{-1} = \frac{\mathbb{H}^*}{|\mathbb{H}|^2} \equiv \frac{e_0 w - e_1 x - e_2 y - e_3 z}{w^2 + x^2 + y^2 + z^2}. \]
(23)
Moreover, the quotient of two vectors is also known as quaternion which is represented by [48]
\[ \mathbb{H} = \frac{\mathbf{a}}{\mathbf{b}}, \]
(24)
where \(\mathbf{a}\) and \(\mathbf{b}\) are the two vectors. The other approach on a tensor of a quaternion \((\mathbf{T}_{\mathbf{H}})\) is given by Hamilton [48], written by
\[ (\mathbf{T}_{\mathbf{H}})^2 = \mathbf{H} \circ \mathbf{H}^* = w^2 + x^2 + y^2 + z^2, \]
(25)
so that \(T_{\mathbf{H}} = \sqrt{(w^2 + x^2 + y^2 + z^2)}\) is known as the tensor of a quaternion, and if the factor part is zero, then \(\mathbf{T}(\mathbf{S}_{\mathbf{H}}) = w\) is known as the tensor of a scalar quaternion. Further, the versor of a quaternion which indicates direction, is a unit quaternion known as a normalised quaternion \((\mathbf{U}_{\mathbf{H}})\), so that
\[ \mathbf{U}_{\mathbf{H}} = \frac{\mathbf{H}}{|\mathbf{H}|} \equiv \frac{e_0 w + e_1 x + e_2 y + e_3 z}{\sqrt{(w^2 + x^2 + y^2 + z^2)}}. \]
(26)
The scalar and vector parts of a quaternionic versor are expressed by \(\mathbf{S}(\mathbf{U}_{\mathbf{H}}) = w(w^2 + x^2 + y^2 + z^2)^{-1/2}\) and \(\mathbf{V}(\mathbf{U}_{\mathbf{H}}) = (e_1 x + e_2 y + e_3 z)(w^2 + x^2 + y^2 + z^2)^{-1/2}\). A quaternion is also written in terms of tensor of a quaternion and versor of a quaternion as [48]
\[ \mathbf{H} = \mathbf{T}_{\mathbf{H}} \mathbf{U}_{\mathbf{H}} = e_0 w + e_1 x + e_2 y + e_3 z. \]
(27)

### 4 Transformation of Quaternionic Basis Elements

If a coordinate system rotates with respect to its original frame of reference, the unit vectors are changed. Then, the transformation equations are represented by a change in unit vectors. Here, we introduce the following equations
used to transform coordinates from one plane to another plane in a 2-dimensional system, i.e.

\[ x' := x \cos \phi + y \sin \phi, \quad y' := -x \sin \phi + y \cos \phi, \]  

which gives

\[
\begin{align*}
\frac{\partial x'}{\partial x} &= \cos \phi, \quad \frac{\partial x'}{\partial y} = \sin \phi, \\
\frac{\partial y'}{\partial x} &= -\sin \phi, \quad \frac{\partial y'}{\partial y} = \cos \phi.
\end{align*}
\]

Thus, the transformation of basis elements can be written in the matrix form:

\[
\begin{pmatrix} e_1' \\ e_2' \end{pmatrix} = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix},
\]

which leads to following linear transformation equations

\[
\begin{align*}
e_1' &\mapsto \frac{\partial x'}{\partial x} e_1 + \frac{\partial x'}{\partial y} e_2, \\
e_2' &\mapsto \frac{\partial y'}{\partial x} e_1 + \frac{\partial y'}{\partial y} e_2,
\end{align*}
\]

where \(e_1\) and \(e_2\) are the unit vectors corresponding to \(x\) and \(y\)-axis in the \(XY\)-plane, while \(e_1'\) and \(e_2'\) are the unit vectors corresponding to \(x'\) and \(y'\)-axis in \(X'Y'\)-plane, respectively. \(\phi\) is the angle between these two planes. Like in 2-D transformation, we may extend the transformation relations of the basis elements to a 3-D system. Thus, the 3-D transformation matrix \((D)\) can be written by

\[
D = \begin{pmatrix} 
\cos \psi \cos \phi - \sin \psi \sin \phi \cos \theta & -\cos \psi \sin \phi - \sin \psi \cos \theta \cos \phi & \sin \psi \sin \theta \\
\sin \psi \cos \phi + \cos \psi \sin \phi \cos \theta & -\sin \psi \sin \phi + \cos \psi \cos \theta \cos \phi & -\cos \psi \sin \theta \\
\sin \theta \sin \phi & \sin \theta \cos \phi & \cos \theta
\end{pmatrix},
\]

where \(\psi, \theta, \phi\) are the three independent parameters known as Euler angles, which represent the rotation of an axis in a new frame of reference with respect to the original axis. Let \((e_1, e_2, e_3)\) are pure quaternionic unit vectors along the \(x, y, z\)-axis in \(S\)-frame, while \((e_1', e_2', e_3')\) are unit vectors corresponding to \(x', y', z'\)-axis in \(S'\)-frame, then the linear transformation relations for pure quaternionic unit vectors become,

\[
\begin{align*}
e_1' &\mapsto \frac{\partial x'}{\partial x} e_1 + \frac{\partial x'}{\partial y} e_2 + \frac{\partial x'}{\partial z} e_3, \\
e_2' &\mapsto \frac{\partial y'}{\partial x} e_1 + \frac{\partial y'}{\partial y} e_2 + \frac{\partial y'}{\partial z} e_3, \\
e_3' &\mapsto \frac{\partial z'}{\partial x} e_1 + \frac{\partial z'}{\partial y} e_2 + \frac{\partial z'}{\partial z} e_3,
\end{align*}
\]

whereas the derivatives become

\[
\begin{align*}
\frac{\partial x'}{\partial x} &= \cos \psi \cos \phi - \sin \psi \sin \phi \cos \theta, \\
\frac{\partial x'}{\partial y} &= -\cos \psi \sin \phi - \sin \psi \cos \theta \cos \phi, \\
\frac{\partial x'}{\partial z} &= \sin \psi \sin \theta, \\
\frac{\partial y'}{\partial x} &= \sin \psi \cos \phi + \cos \psi \sin \phi \cos \theta, \\
\frac{\partial y'}{\partial y} &= -\sin \psi \sin \phi + \cos \psi \cos \theta \cos \phi, \\
\frac{\partial y'}{\partial z} &= -\cos \psi \sin \theta, \\
\frac{\partial z'}{\partial x} &= \sin \theta \sin \phi, \\
\frac{\partial z'}{\partial y} &= \sin \theta \cos \phi, \\
\frac{\partial z'}{\partial z} &= \cos \theta.
\end{align*}
\]

Interestingly, these transformation equations satisfy the properties of vector algebra. The cross product of two vectors is always non-commutative in pure quaternionic space. Another advantage to using pure quaternionic unit elements is that their properties correspond to the vector algebra basis, i.e.
\[ e'_1 \circ e'_1 = \left( \frac{\partial x'}{\partial x} e_1 + \frac{\partial y'}{\partial y} e_2 + \frac{\partial z'}{\partial z} e_3 \right) \circ \]
\[ \left( \frac{\partial x'}{\partial x} e_1 + \frac{\partial y'}{\partial y} e_2 + \frac{\partial z'}{\partial z} e_3 \right) \]
\[ = - \left( \frac{\partial x'}{\partial x} \right)^2 - \left( \frac{\partial y'}{\partial y} \right)^2 - \left( \frac{\partial z'}{\partial z} \right)^2 \]
\[ = - (\cos \psi \cos \phi - \sin \psi \sin \phi \cos \theta)^2 \]
\[ - (\sin \psi \sin \phi - \sin \psi \cos \phi \cos \theta)^2 \]
\[ - (\sin \psi \sin \theta)^2 = -1. \]

and
\[ e'_2 \circ e'_2 = -1, \quad e'_3 \circ e'_3 = -1. \] (35)

Furthermore,
\[ e'_1 \circ e'_2 = \left( \frac{\partial y'}{\partial x} e_1 + \frac{\partial y'}{\partial y} e_2 + \frac{\partial y'}{\partial z} e_3 \right) \circ \]
\[ \left( \frac{\partial y'}{\partial x} e_1 + \frac{\partial y'}{\partial y} e_2 + \frac{\partial y'}{\partial z} e_3 \right) \]
\[ = - \left( \frac{\partial y'}{\partial x} \right)^2 - \left( \frac{\partial y'}{\partial y} \right)^2 \]
\[ - \left( \frac{\partial y'}{\partial z} \right)^2 \]
\[ e'_1 \]
\[ - \left( \frac{\partial y'}{\partial y} \right)^2 \]
\[ - \left( \frac{\partial y'}{\partial z} \right)^2 \]
\[ = - (\cos \psi \cos \phi - \sin \psi \sin \phi \cos \theta) \]
\[ (\sin \psi \cos \phi + \cos \psi \sin \phi \cos \theta) \]
\[ - (\sin \psi \sin \phi - \sin \psi \cos \phi \cos \theta) \]
\[ - (\sin \psi \sin \phi + \cos \psi \cos \theta \cos \phi) \]
\[ - (\sin \psi \sin \theta) \]
\[ - e_1[(\sin \psi \sin \theta)(- \sin \psi \sin \phi + \cos \psi \cos \theta \cos \phi)] \]
\[ + (\cos \psi \sin \phi - \sin \psi \cos \theta \cos \phi)(- \cos \psi \sin \theta)] \]
\[ - e_2[(\cos \psi \cos \phi - \sin \psi \sin \phi \cos \theta) \]
\[ - (\cos \psi \sin \theta) \]
\[ + (\sin \psi \sin \theta)(\sin \psi \cos \phi + \cos \psi \sin \phi \cos \theta)] \]
\[ - e_3[(- \cos \psi \sin \phi - \sin \psi \cos \theta \cos \phi) \]
\[ (\sin \psi \cos \phi + \cos \psi \sin \phi \cos \theta) \]
\[ + (\cos \psi \cos \phi - \sin \psi \sin \phi \cos \theta) \]
\[ (- \sin \psi \sin \phi + \cos \psi \cos \theta \cos \phi)] \]
\[ = e'_3, \] (36)

As a result, we also get
\[ e'_2 \circ e'_3 = e'_1, \quad e'_3 \circ e'_2 = -e'_1, \]
\[ e'_3 \circ e'_1 = e'_2, \quad e'_1 \circ e'_3 = -e'_2. \] (38)

Now we can easily extend the pure-quaternion to a quaternion by adding the scalar unit element \( e_0 \). It should be noticed that the scalar unit element becomes invariant under transformation given in equation (6) but it plays an important role in quaternionic transformation along with unit elements \( e_1, e_2, e_3 \). Myśzkowski [50] has given the idea of a 4-D transformation similar to the 3-D transformation. Thus, we assume the transformation equations for quaternionic basis elements as
\[ e'_0 : t \rightarrow e_0, \]
\[ e'_1 : t \rightarrow \frac{\partial x'}{\partial t} e_0 + \frac{\partial x'}{\partial x} e_1 + \frac{\partial x'}{\partial y} e_2 + \frac{\partial x'}{\partial z} e_3, \]
\[ e'_2 : t \rightarrow \frac{\partial y'}{\partial t} e_0 + \frac{\partial y'}{\partial x} e_1 + \frac{\partial y'}{\partial y} e_2 + \frac{\partial y'}{\partial z} e_3, \]
\[ e'_3 : t \rightarrow \frac{\partial z'}{\partial t} e_0 + \frac{\partial z'}{\partial x} e_1 + \frac{\partial z'}{\partial y} e_2 + \frac{\partial z'}{\partial z} e_3. \]

(39)

Here, we focused on space-time structure for quaternionic transformation, the quaternionic properties are also satisfied as similar to the 3-D rotation. Now, in the next section, we will interpret these relations in terms of Riemannian geometry.

5 Quaternionic Approach on Riemannian geometry

5.1 Quaternion Transformation

In Riemannian geometry, the motion of objects takes place in a curvature space-time in which a 4-dimensional frame of reference rotates along the curve. The structure of quaternionic space-time coordinates is written as \( H = (P^0, P^1, P^2, P^3) \). The time coordinate may correspond to \( e_0 \) i.e. \( t \rightarrow P^0 \), and the spatial coordinates may correspond to \( e_j \) i.e. \( (x, y, z) \rightarrow (P^1, P^2, P^3) \) where \( j = 1, 2, 3 \). Now the quaternionic basis elements may undergo transformation as

\[ e'_0 = e_0, \quad e'_i = \frac{\partial P^i}{\partial P^0} e_0, \quad \forall (i = 1, 2, 3). \]

(40)

The transformation of scalar and vector field components of a quaternion variable (\( H \)) from \( S \) to \( S' \)-frame can be expressed as

\[ H'^z = H^z + \frac{\partial P^z}{\partial P^i} H^i \] (coefficient of \( e_0 \)),

(41)

\[ H'^i = \frac{\partial P^i}{\partial P^i} H^i \] (coefficient of \( e_j \)).

(42)

In the above transformation relation (41), we assume that the effect of \( H'^z \) on \( H^z \) is linear because it shows the simple transformation on flat space-time. So, we can neglect the component \( H'^z \) to get the quaternionic transformation (Q-transformation) in curved space-time. Therefore, we have

\[ H' : t \rightarrow H = \left( \frac{\partial P^z}{\partial P^i} H^i, \frac{\partial P^i}{\partial P^i} H^i \right). \]

(43)

Thus, Eq. (43) represents the Q-transformation in contravariant form. The aforementioned transformation relations can alternatively be written in quaternionic covariant form as

\[ H'_z = \frac{\partial P^i}{\partial P^z} H^i, \quad H'_j = \frac{\partial P^j}{\partial P^i} H^i. \]

(44)

Now we can write the transformation relation of the quaternionic differential operator \( \Box \cong \{ \partial_\phi, \partial_j \} \) as

\[ \partial_\phi = \frac{\partial P^0}{\partial P^0} \partial_\phi, \quad \partial_j = \frac{\partial P^j}{\partial P^0} \partial_\phi, \quad \forall (j, k = 1, 2, 3). \]

(45)

Since the components of a quaternion are also be represented in the form of coordinates, then we write the transformation of quaternionic coordinates [36],

\[ dP^z = \frac{\partial P^z}{\partial P^i} dP^i, \quad dP^j = \frac{\partial P^j}{\partial P^0} dP^0, \]

\[ \forall (i, j = 1, 2, 3). \]

(46)

5.2 Quaternionic Covariant Derivative

The scalar field derivative of Eq. (44) in the quaternionic form of curvature space-time is expressed as

\[ H'_\phi = H^z + \left\{ \begin{array}{c} \xi \\ \phi \phi \end{array} \right\} H^\phi, \]

(47)

where the Christoffel symbol is denoted by

\[ \left\{ \begin{array}{c} \xi \\ \phi \phi \end{array} \right\} = \frac{\partial P^\phi}{\partial P^m} \left\{ \begin{array}{c} \phi \\ \mu \phi \end{array} \right\}. \]

As such, we have

\[ H'_x,\phi = H_x,\phi - \left\{ \begin{array}{c} \xi \\ \phi \xi \end{array} \right\} H_\phi. \]

(48)

For Riemannian space-time, the importance of the Christoffel symbol is crucial. If \( \left\{ \begin{array}{c} \xi \\ \phi \phi \end{array} \right\} \rightarrow 0 \), then the quaternionic curvature covariant derivative transforms into the quaternionic usual partial derivative, which indicates the linear transformation. Correspondingly, applying quaternionic derivative \( \partial_\phi \) on \( H^j \), we obtain

\[ H'_j = H^z + \left\{ \begin{array}{c} j \\ \phi \phi \end{array} \right\} H^\phi. \]

(49)

As such, we can also operate the quaternionic derivative \( \partial_k \) (\( k = 1, 2, 3 \)) on Eq. (44), so that

\[ H'_k = H^z + \left\{ \begin{array}{c} k m \\ j \end{array} \right\} H^m, \]

(50)

where \( j, k, m = 1, 2, 3 \). For generalised quaternionic fields, Eqs. (49) and (50) are represented by scalar to vector and vector to vector field transformations, respectively [42]. It should be remarked that through the quaternionic derivative, the generalised covariant derivative gives enlargement to the rank of quaternionic tensors, which is important to provide more information about the
transformed coordinate. In general, the quaternionic tensor derivatives are

$$T_{\xi;\phi} = T_{\xi;\phi} - \left\{ \eta \phi_{\xi} \right\} T_{\xi;\eta} - \left\{ \eta \phi_{\xi} \right\} T_{\eta;\xi}, \quad (51)$$

$$T_{\xi;\phi} = T_{\xi;\phi} - \left\{ \frac{i}{k} \right\} T_{\phi;\xi} - \left\{ \frac{i}{k} \right\} T_{\xi;\phi}, \quad (52)$$

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$$T_{\xi;\phi} = T_{\xi;\phi} - \left\{ \frac{i}{k} \right\} T_{\phi;\xi} - \left\{ \frac{i}{k} \right\} T_{\xi;\phi}, \quad (58)$$

where the right-hand side of Eqs. (51)-(58), the first term shows the tensorial transformation of the derivative of the quaternionic tensor and the last two terms show the Christoffel symbols which tell us how the geodesic path changes from point to point.

### 5.3 Quaternionic Metric Tensor

The covariant derivative of the quaternionic tensor is used to describe the quaternionic metric tensor in terms of the Christoffel symbol. As a result, one may write

$$g_{\xi;\phi} = g_{\xi;\phi} - \left\{ \eta \phi_{\xi} \right\} g_{\xi;\eta} - \left\{ \eta \phi_{\xi} \right\} g_{\eta;\xi}, \quad (59)$$

where the line element becomes $ds = \sqrt{-g_{\mu\nu}(x) dx^\mu dx^\nu}$. The quaternionic metric tensor is constant under covariant differentiation, i.e.

$$0 = g_{\xi;\phi} - \left\{ \eta \phi_{\xi} \right\} g_{\xi;\eta} - \left\{ \eta \phi_{\xi} \right\} g_{\eta;\xi}, \quad (60)$$

Notice that the metric tensor $g'_{\eta\xi} = \frac{\partial g_{\eta\xi}}{\partial x^\mu} \frac{\partial x^\mu}{\partial x^\phi} \Omega_{\mu\nu}$, where $\Omega_{\mu\nu}$ is the four-dimensional Minkowski metric defined by

$$\Omega_{\mu\nu} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},$$

and the Christoffel symbol satisfies the following properties:

$$\left\{ \frac{i}{kl} \right\} = \left\{ \frac{i}{lk} \right\}, \quad \left\{ \frac{m}{il} \right\} g_{mk} = \left\{ \frac{k}{il} \right\}. \quad (61)$$

Now, we write the value of the Christoffel symbol corresponding to the quaternionic scalar field as,

$$\left\{ \frac{i}{\xi} \right\} g_{\xi} = \frac{1}{2} g^{\eta\phi} (g_{\phi;\xi} + g_{\phi;\xi} - g_{\xi;\phi}), \quad (62)$$

where $g^{\eta\phi}$ is the inverse matrix of $g_{\eta\phi}$. Correspondingly, in quaternionic pure vectorial field, the Christoffel symbol can be played as

$$\left\{ \frac{i}{jn} \right\} = \frac{1}{2} g^{jk} (g_{kj,n} + g_{kn,j} - g_{jn,k}), \quad (63)$$

and the other diversified quaternionic scalar and vector fields show a mixed form of the quaternionic Christoffel symbol in terms of the metric tensor, i.e.

$$\left\{ \frac{i}{\xi} \right\} = \frac{1}{2} g^{jk} (g_{kj,n} + g_{kn,j} - g_{jn,k}), \quad (64)$$

$$\left\{ \frac{i}{\xi} \right\} = \frac{1}{2} g^{jk} (g_{kj,n} + g_{kn,j} - g_{jn,k}), \quad (65)$$

$$\left\{ \frac{i}{jn} \right\} = \frac{1}{2} g^{jk} (g_{kj,n} + g_{kn,j} - g_{jn,k}), \quad (66)$$

$$\left\{ \frac{\eta}{jn} \right\} = \frac{1}{2} g^{jk} (g_{kj,n} + g_{kn,j} - g_{jn,k}), \quad (67)$$

where $(\xi; \eta; \phi)$ are used for generalised quaternionic scalar field variables, while $(i; j; k; l; m; n)$ are the generalised quaternionic vector field variables.

### 5.4 Quaternionic Geodesic Equation

In the general theory of relativity, the geodesic equation gives the replacement of linear space-time by curved space-time. In other words, a particle moving in a curved space-time follows the path of a geodesic. With the principle of least action, the geodesic path in a curvature space-time is expressed as a four-vector form i.e. $dH^\mu = 0$, see ref. [3]. In the pure quaternionic scalar field, we get
\[ H^i_{\phi} = H^i_{\phi} + \left\{ \frac{\zeta}{\phi} \right\} H^\phi = 0 \]
\[ \Rightarrow dH^i + \left\{ \frac{\zeta}{\phi} \right\} dP^\phi H^\phi = 0 . \]  

(68)

Now, dividing Eq. (68) by \( ds \) (e.g. a scalar parameter of motion as proper time) and substituting \( H^i = \frac{dP^i}{ds} \), \( H^\phi = \frac{dP^\phi}{ds} \), we obtain
\[ \frac{d^2 P^i}{ds^2} + \left\{ \frac{\zeta}{\phi} \right\} \frac{dP^\phi}{ds} \frac{dP^m}{ds} = 0 . \]  

(69)

Similarly, for quaternionic vector fields, we get
\[ \frac{d^2 P^i}{ds^2} + \left\{ j \right\} \frac{dP^k}{ds} \frac{dP^m}{ds} = 0 , \]  

(70)

and for a mixed fields,
\[ \frac{d^2 P^i}{ds^2} + \left\{ j \right\} \frac{dP^\phi}{ds} \frac{dP^m}{ds} = 0 , \]  

(71)

Eqs. (69)-(71) represent the nonlinear equations called the quaternionic geodesic equations that arise due to the effect of Christoffel symbols. In a quaternionic formulation, we can also emphasise that the space-time curvature path is not only followed by vector components but also by the scalar components and mixed components of a quaternion variable. If the Christoffel symbol is zero, then the quaternionic geodesic equations of motion lead to
\[ \frac{d^2 P^\mu}{ds^2} = 0 , \quad \forall (\mu = 0, 1, 2, 3) . \]  

(72)

This means that the generalised quaternionic acceleration will be zero, implying that the uniform velocity of a particle moving in a straight line (or non-Riemannian space-time).

### 5.5 Quaternionic Riemannian Christoffel Curvature Tensor

The Riemannian Christoffel curvature tensor possesses four indices and is obtained by subtracting two covariant derivatives of quaternionic tensors with interchanged indices, such that
\[ T_{\zeta i \phi} - T_{\phi i \zeta} = \left( R^i_{\zeta \phi} \right) H q , \]  

(73)

where
\[ R^i_{\zeta \phi} = \partial_{\zeta} \left( \left\{ \frac{\eta}{\phi} \zeta \right\} \right) - \partial_{\phi} \left( \left\{ \frac{\eta}{\zeta} \phi \right\} \right) + \left\{ \frac{\phi}{\zeta} \right\} \left\{ \frac{\eta}{\phi} \right\} \]  

(74)

is the Riemannian Christoffel curvature tensor for purely quaternionic scalar field. It is a four-index tensor in curved space-time that describes manifold curvature. We may also get the Riemannian Christoffel curvature tensor for various quaternionic components given in Eqs. (52)-(58), i.e.
\[ T_{\zeta i k} - T_{\zeta k i} = \left( R^i_{\zeta k} \right) H_l , \]  

(75)

\[ T_{\zeta \phi j} - T_{\phi j \zeta} = \left( R^j_{\zeta \phi} \right) H_l , \]  

(76)

\[ T_{\zeta \phi k} - T_{\zeta k \phi} = \left( R^k_{\zeta \phi} \right) H_l , \]  

(77)

\[ T_{\phi j k} - T_{\phi k j} = \left( R^j_{\phi k} \right) H_l , \]  

(78)

\[ T_{\phi j k} - T_{\phi k j} = \left( R^k_{\phi j} \right) H_l , \]  

(79)

\[ T_{\phi j m} - T_{\phi m j} = \left( R^m_{\phi j} \right) H_l , \]  

(80)

\[ T_{\phi j k} - T_{\phi k j} = \left( R^k_{\phi j} \right) H_l , \]  

(81)

where the Riemannian Christoffel curvature tensors give rise to the following:
\[ R^i_{\zeta k} = \partial_{\zeta} \left( \left\{ \frac{l}{k \zeta} \right\} \right) - \partial_{k} \left( \left\{ \frac{l}{\zeta} \right\} \right) + \left\{ \frac{i}{k \zeta} \right\} \left\{ \frac{l}{\zeta} \right\} - \left\{ \frac{i}{\zeta} \right\} \left\{ \frac{l}{k \zeta} \right\} , \]  

(82)

\[ R^i_{\eta \phi} = \partial_{\eta} \left( \left\{ \frac{\eta}{\phi} \zeta \right\} \right) - \partial_{\phi} \left( \left\{ \frac{\eta}{\zeta} \right\} \right) + \left\{ \frac{\phi}{\zeta} \right\} \left\{ \frac{\eta}{\phi} \right\} \left\{ \frac{\eta}{\phi} \right\} \left\{ \frac{\eta}{\phi} \right\} , \]  

(83)

\[ R^i_{\phi k} = \partial_{\phi} \left( \left\{ \frac{l}{k \phi} \right\} \right) - \partial_{k} \left( \left\{ \frac{l}{\phi} \right\} \right) + \left\{ \frac{i}{k \phi} \right\} \left\{ \frac{l}{\phi} \right\} - \left\{ \frac{i}{\phi} \right\} \left\{ \frac{l}{k \phi} \right\} , \]  

(84)

\[ R^i_{\zeta j} = \partial_{\zeta} \left( \left\{ \frac{\eta}{\phi j} \zeta \right\} \right) - \partial_{j} \left( \left\{ \frac{\eta}{\zeta \phi} \right\} \right) + \left\{ \frac{\phi}{\zeta} \right\} \left\{ \frac{\eta}{\phi j} \right\} \left\{ \frac{\eta}{\phi j} \right\} \left\{ \frac{\eta}{\phi j} \right\} , \]  

(85)
\[ R^\eta_{\rho\lambda} = \partial_\eta \left( \left\{ \frac{l}{k} \right\} \right) - \partial_k \left( \left\{ \frac{l}{j\eta} \right\} \right) \]

\[ \text{(86)} \]

\[ R^\phi_{\eta\phi} = \partial_\eta \left( \left\{ \frac{n^\phi}{\phi_j} \right\} \right) - \partial_\phi \left( \left\{ \frac{n^\phi}{\eta_j} \right\} \right) \]

\[ \text{+} \left\{ \frac{\phi}{\phi_j} \right\} \left\{ \frac{\eta}{\phi_j} \right\} - \left\{ \frac{\phi}{n_j} \right\} \left\{ \frac{\eta}{\phi_j} \right\} \]

\[ \text{(87)} \]

\[ R^\rho_{\eta\phi} = \partial_\eta \left( \left\{ \frac{n^\phi}{\phi_j} \right\} \right) - \partial_\phi \left( \left\{ \frac{n^\phi}{\eta_j} \right\} \right) \]

\[ \text{+} \left\{ \frac{\phi}{\phi_j} \right\} \left\{ \frac{n^\phi}{\phi_j} \right\} - \left\{ \frac{\phi}{n_j} \right\} \left\{ \frac{n^\phi}{\phi_j} \right\} \]

\[ \text{(88)} \]

Here, the quaternionic Riemannian tensor keeps track of how many scalar and vector components of the quaternion change when we propagate in parallel with a small parallelogram. If the value of the quaternionic Riemannian Christoffel curvature is zero, then the quaternionic curved space-time is converted into flat space-time. The quaternionic Ricci tensor, on the other hand, is a significant contraction of the quaternion Ricci tensor that explains volume changes when an object parallels a geodesic. In this case,

\[ g_{\rho\phi} R^\eta_{\phi\rho} = R^\rho_{\phi\rho\phi} \]

\[ = \partial_\rho \left( g_{\rho\phi} \left\{ \frac{\eta^\phi}{\phi_j} \right\} \right) - \partial_\phi \left( g_{\rho\phi} \left\{ \frac{\eta^\phi}{\eta_j} \right\} \right) \]

\[ \text{+} g_{\rho\phi} \left\{ \frac{\phi}{\phi_j} \right\} \left\{ \frac{n^\phi}{\phi_j} \right\} - g_{\rho\phi} \left\{ \frac{\phi}{n_j} \right\} \left\{ \frac{n^\phi}{\phi_j} \right\} \]

\[ \text{(89)} \]

which can be simplified in terms of a metric tensor as

\[ R^\rho_{\phi\rho\phi} = \frac{1}{2} \left[ \frac{\partial^2 g_{\rho\phi}}{\partial P^\rho \partial P^\phi} + \frac{\partial^2 g_{n\phi}}{\partial P^\rho \partial P^\phi} \right] \]

\[ \text{+} \frac{\partial^2 g_{\phi\rho}}{\partial P^\rho \partial P^\phi} - \frac{\partial^2 g_{\eta\phi}}{\partial P^\rho \partial P^\phi} \]

\[ \text{+} g_{\rho\phi} \left\{ \frac{\phi}{\phi_j} \right\} \left\{ \frac{n^\phi}{\phi_j} \right\} - g_{\rho\phi} \left\{ \frac{\phi}{n_j} \right\} \left\{ \frac{n^\phi}{\phi_j} \right\} \]

\[ \text{(90)} \]

Now, use the metric tensor to contract Eq. (90) as

\[ g^{\rho\phi} R^\rho_{\phi\rho\phi} = R_{\xi\phi}, \text{ where the quaternionic Ricci tensor } R_{\xi\phi} \text{ can be expressed as} \]

\[ R_{\xi\phi} = \partial_\xi \left( \left\{ \frac{n^\phi}{\phi_j} \right\} \right) - \partial_\phi \left( \left\{ \frac{n^\phi}{\eta_j} \right\} \right) \]

\[ \text{+} \left\{ \frac{\phi}{\phi_j} \right\} \left\{ \frac{n^\phi}{\phi_j} \right\} - \left\{ \frac{\phi}{n_j} \right\} \left\{ \frac{n^\phi}{\phi_j} \right\} \]

\[ \text{(91)} \]

Similarly, the Ricci tensors become involved in quaternionic vector fields in addition to scalar fields as well. Therefore,

\[ R^\eta_{\rho\lambda} = \partial_\eta \left( \left\{ \frac{n^\rho}{k_j} \right\} \right) - \partial_k \left( \left\{ \frac{n^\rho}{j\eta} \right\} \right) \]

\[ \text{+} \left\{ \frac{\rho}{k_j} \right\} \left\{ \frac{n^\rho}{k_j} \right\} - \left\{ \frac{\rho}{n_j} \right\} \left\{ \frac{n^\rho}{k_j} \right\} \]

\[ \text{(92)} \]

\[ R^\phi_{\eta\phi} = \partial_\eta \left( \left\{ \frac{n^\phi}{\phi_j} \right\} \right) - \partial_\phi \left( \left\{ \frac{n^\phi}{\eta_j} \right\} \right) \]

\[ \text{+} \left\{ \frac{\phi}{\phi_j} \right\} \left\{ \frac{n^\phi}{\phi_j} \right\} - \left\{ \frac{\phi}{n_j} \right\} \left\{ \frac{n^\phi}{\phi_j} \right\} \]

\[ \text{(93)} \]

We should notice that the Riemannian tenor (R^\rho_{\phi\rho\phi}) is symmetric to the first with third indices and the second with fourth indices [3]. Thus, the Ricci tensor is also symmetric. The quaternionic form of the Ricci tensor can further contract with the metric tensor and obtain a scalar curvature tensor (R) of rank zero, such that g^{\xi\phi} R_{\xi\phi} = R \cdot

### 5.6 Quaternionic Einstein-field like Equation

To express the quaternionic Einstein field like equation, let us start with the Poisson equation as

\[ \nabla^2 \Phi = 4\pi G \rho, \]

where \( \Phi \) is the Newtonian gravitational potential, \( G \) is gravitational constant and \( \rho \) is the mass density. Moreover, in metric tensor [51], we have \( \nabla^2 \Phi = -\frac{1}{2} \nabla^2 g_{00} \). Then,

\[ \nabla^2 g_{00} = -2\pi G \rho. \]

In Eq. (95), the quaternionic form of scalar metric tensor (g_{\xi\phi}) is used instead of Newtonian potential (\( \Phi \)) and quaternionic mass density (T_{\xi\phi}) is adopted instead of (\( \rho \)). The quaternionic energy-momentum tensor is usually represented as

\[ T^{uv} \approx \begin{pmatrix} T_{\xi\xi} & T_{\xi1} & T_{\xi2} & T_{\xi3} \\ T_{1\xi} & T_{11} & T_{12} & T_{13} \\ T_{2\xi} & T_{21} & T_{22} & T_{23} \\ T_{3\xi} & T_{31} & T_{32} & T_{33} \end{pmatrix}. \]

\[ \text{(96)} \]

As a result, Eq. (95) yields \( \nabla^2 g_{\xi\phi} = -8\pi G T_{\xi\phi} \). Because the quaternionic Ricci tensor is expressed as a double derivative of the quaternionic metric tensor, so we have

\[ R_{\xi\phi} = -8\pi G T_{\xi\phi}. \]

\[ \text{(97)} \]

Furthermore, the rest components of quaternionic Riemannian Christoffel curvature tensors can also be expressed as
The curvature of space-time is intimately related to the energy and momentum of whatever matter and radiation are present in Riemannian geometry. Generally, the space-time can be shown by using simple experiments following the free-fall trajectories of different test particles. The result of transporting space-time vectors that can denote a particle’s velocity (time-like vectors) vary with the particle’s trajectory. The origin of the idea of this space-time curvature is a mathematical concept. A quaternionic algebra has thus been used to depict the four-dimensional curvature space-time. The transformation rules for two quaternionic frames have been discussed. We discussed that the effect of curvature space-time arises due to the quaternionic Christoffel symbol. Furthermore, the quaternionic geodesic equation efficiently explains the curved path of freely moving particles. The Christoffel symbol has been discussed in terms of the derivative of the quaternionic metric tensor, while the gravitational potential is represented as the metric tensor in tensorial form. It has been confirmed that the volume element of an object obeying the geodesic path changes due to the contraction of the quaternionic Riemannian tensor, while the contraction of the quaternionic Ricci tensor gives a scalar quantity representing the magnitude of change in volume of the object. The quaternionic Einstein-field-like equation for gravitation, which connects the geometry of quaternionic space-time with the distribution of matter inside it, has also been established. The simplification of quaternionic formalism is achieved by the approximation as quaternionic flat space-time with a small deviation, which leads to the linearised Einstein field equation. Moreover, the analysis of the obtained quaternionic equations may also be used to study various phenomena such as gravitational waves, the formation of black holes.

6 Conclusion

The curvature of space-time is intimately related to the energy and momentum of whatever matter and radiation are present in Riemannian geometry. Generally, the space-time can be shown by using simple experiments following the free-fall trajectories of different test particles. The result of transporting space-time vectors that can denote a particle’s velocity (time-like vectors) vary with the particle’s trajectory. The origin of the idea of this space-time curvature is a mathematical concept. A quaternionic algebra has thus been used to depict the four-dimensional curvature space-time. The transformation rules for two quaternionic frames have been discussed. We discussed that the effect of curvature space-time arises due to the quaternionic Christoffel symbol. Furthermore, the quaternionic geodesic equation efficiently explains the curved path of freely moving particles. The Christoffel symbol has been discussed in terms of the derivative of the quaternionic metric tensor, while the gravitational potential is represented as the metric tensor in tensorial form. It has been confirmed that the volume element of an object obeying the geodesic path changes due to the contraction of the quaternionic Riemannian tensor, while the contraction of the quaternionic Ricci tensor gives a scalar quantity representing the magnitude of change in volume of the object. The quaternionic Einstein-field-like equation for gravitation, which connects the geometry of quaternionic space-time with the distribution of matter inside it, has also been established. The simplification of quaternionic formalism is achieved by the approximation as quaternionic flat space-time with a small deviation, which leads to the linearised Einstein field equation. Moreover, the analysis of the obtained quaternionic equations may also be used to study various phenomena such as gravitational waves, the formation of black holes.

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