The direction of Coulomb’s force and the direction of gravitational force in the 4-dimensional space-time

Huaiyang Cui
Department of physics, Beihang University, Beijing, 100083, China.
E-mail: hycui@buaa.edu.cn

March 31, 2022

Abstract

In this paper, we point out that the 4-vector force acting on a particle is always in the direction orthogonal to the 4-vector velocity of the particle in the 4-dimensional space-time, rather than along the line joining the particle and the action source. This inference is obviously supported from the fact that the magnitude of the 4-vector velocity is a constant. This orthogonality brings out many new aspects for force concept. In this paper it is found that the Maxwell’s equations can be derived from classical Coulomb’s force and the orthogonality, some gravitational effects such as the perihelion advance of planet can also be explained in terms of the orthogonality.

1 Introduction

In the world, almost everyone knows that Coulomb’s force (or gravitational force) acts along the line joining a couple of particles, but sometime this knowledge is incorrect in the theory of relativity. Consider a particle moving at the 4-vector velocity \( u \) in an inertial Cartesian coordinate system \( S : (x_1, x_2, x_3, x_4 = ict) \), the magnitude of the 4-vector velocity \( u \) is given by

\[
|u| = \sqrt{u_\mu u_\mu} = \sqrt{-c^2} = ic
\] (1)

The above equation is valid so that any force can never change \( u \) in its magnitude but can change \( u \) in its direction. We therefore conclude that the Coulomb’s force or gravitational force on a particle always acts in the direction orthogonal to the 4-vector velocity of the particle in the 4-dimensional space-time, rather than along the line joining the particle and action source. Strictly, any 4-vector force \( f \) satisfies the following orthogonality.

\[
u_\mu f_\mu = u_\mu m \frac{du_\mu}{d\tau} = \frac{m}{2} \frac{d(u_\mu u_\mu)}{d\tau} = 0
\] (2)

Where the Cartesian coordinate system \( S \) is a frame of reference whose axes are orthogonal to one another, there is no distinction between covariant and contravariant components, only subscripts need to be used.

Although the orthogonality has occasionally appeared in some textbooks\[2\], in the present paper, Eq.(2) has been elevated to an essential requirement for the definition of a force, which brings out many new aspects for the Coulomb’s force and gravitational force.

2 The Coulomb’s force and Maxwell’s equations

2.1 the 4-vector Coulomb’s force

Suppose there are two charged particle \( q \) and \( q' \) locating at the positions \( x \) and \( x' \) respectively in the Cartesian coordinate system \( S \), and moving at the 4-vector velocities \( u \) and \( u' \) respectively, as shown in Fig.1 where we have used \( X \) to denote \( x - x' \). The Coulomb’s force \( f \) acting on the particle \( q \) is orthogonal to the velocity direction of \( q \), as illustrated in Fig.1 using the Euclidian geometry to represent the complex space-time. Like a centripetal force, the force \( f \) should make an attempt to rotate itself about the particle path center. We think that the path center, \( q, q' \) and the force \( f \) all should be in the plane of \( u' \) and \( X \), so that we make an expansion to \( f \) as

\[
f = Au' + BX
\] (3)

where \( A \) and \( B \) are unknown coefficients, the possibility of this expansion will be further clear in the next section in which the expansion is not an assumption [see Eq.23].

Using the orthogonality \( f \bot u \), we get

\[
u \cdot f = A(u \cdot u') + B(u \cdot X) = 0
\] (4)

By eliminating the coefficient \( B \), we rewrite Eq.3 as

\[
f = \frac{A}{u \cdot X}[(u \cdot X)u' - (u \cdot u')X]
\] (5)
It follows from the direction of Eq. (5) that the unit vector \( u' \) has the classical form

\[
|f| = \frac{k q q'}{r^2}
\]

Combination of Eq. (6) with Eq. (11), we obtain a modified Coulomb's force

\[
f = \frac{kqq'}{c^2r^3}[(u \cdot X)u' - (u \cdot u')X]
\]

By using the relation

\[
\partial_{\mu}\left(\frac{1}{r}\right) = -\frac{R_{\mu}}{r^3}
\]

we obtain

\[
f_{\mu} = q\left[-(u_{\nu}\partial_{\nu}\left(\frac{kq'}{c^2r^3}\right))u'_{\mu} + (u_{\nu}u'_{\nu}\partial_{\mu}\left(\frac{kq'}{c^2r^3}\right))\right]
\]

The force can be rewritten in terms of 4-vector components as

\[
f_{\mu} = qF_{\mu\nu}u_{\nu}
\]

\[
F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}
\]

\[
A_{\mu} = \frac{kq' u'_{\mu}}{c^2r}
\]

Thus \( A_{\mu} \) expresses the 4-vector potential of the particle \( q' \). It is easy to find that Eq. (11) contains the Lorentz force.

### 2.2 The Lorentz gauge condition and the Maxwell’s equations

From Eq. (13), because of \( u' \perp R \), i.e. \( u'_{\mu}R_{\mu} = 0 \), we have

\[
\partial_{\mu}A_{\mu} = \frac{kq' u'_{\mu}}{c^2} \partial_{\mu}\left(\frac{1}{r}\right) = -\frac{kq' u'_{\mu}}{c^2}\left(\frac{R_{\mu}}{r^3}\right) = 0
\]

It is known as the Lorentz gauge condition.

To note that \( R \) has three degrees of freedom under the condition of \( R \perp u' \), so we have

\[
\partial_{\mu}R_{\mu} = 3
\]

\[
\partial_{\mu}\partial_{\nu}\left(\frac{1}{r}\right) = -4\pi \delta(R)
\]

From Eq. (12), we have

\[
\partial_{\nu}F_{\mu\nu} = \partial_{\nu}\partial_{\mu}A_{\nu} - \partial_{\nu}\partial_{\nu}A_{\mu} = -\partial_{\nu}\partial_{\nu}A_{\mu}
\]

\[
= -\frac{kq' u'_{\mu}}{c^2}\partial_{\nu}\partial_{\nu}\left(\frac{1}{r}\right)
\]

\[
= \frac{kq' u'_{\mu}}{c^2}4\pi \delta(R) = \mu_0 J'_{\nu}
\]

where we define \( J'_{\nu} = q' u'_{\nu} \delta(R) \) as the current density of the source. From Eq. (12), by exchanging the indices and taking the summation of them, we have

\[
\partial_{\lambda}F_{\mu\lambda} + \partial_{\mu}F_{\nu\lambda} + \partial_{\nu}F_{\lambda\mu} = 0
\]

The Eq. (11) and (13) are known as the Maxwell’s equations. For continuous media, they are valid as well as.

### 2.3 The Lienard-Wiechert potential

From the Maxwell’s equations, we know there is a retardation time for the electromagnetic action to propagate between the two particles, as illustrated in Fig. 1, the retardation effect is measured by the distance from point P to the point C.

\[
r = c\Delta t = c\frac{|PC|}{ic} = c\frac{u^0 \cdot X}{ic} = c\frac{u'_{\mu}(x'_\mu - x_\mu)}{c}
\]

Then

\[
A_{\mu} = \frac{kq' u'_{\mu}}{c^2} = \frac{kq' u'_{\mu}(x'_\mu - x_\mu)}{c}
\]

Obviously, Eq. (20) is known as the Lienard-Wiechert potential for a moving particle.

The above formalism clearly shows that the Maxwell’s equations can be derived from the classical Coulomb’s force and the orthogonality of 4-vector force and 4-vector velocity. In other words, the orthogonality is hidden in
the theory of gravity. In analogy with the Maxwell’s equation. Specially, Eq. (1) directly accounts for the geometrical meanings of the curl of vector potential, the curl contains the orthogonality. The orthogonality of force and velocity is one of consequences from the relativistic mechanics.

3 Gravitational force

3.1 the 4-vector gravitational force

The above formalism has a significance on guiding how to develop the theory of gravity. In analogy with the modified Coulomb’s force of Eq. (8), we directly suggest a modified universal gravitational force as

$$f = -\frac{Gm m'}{c^2 r^3}[(u \cdot X)u' - (u \cdot u')X]$$

$$= -\frac{Gm m'}{c^2 r^3}[(u \cdot R)u' - (u \cdot u')R]$$

for a couple of particles with masses $m$ and $m'$, respectively, the gravitational force must satisfy the orthogonality of 4-vector force and 4-vector velocity.

Even though the gravitation is treated as a field in the theory of general relativity, here we emphasize that gravitational force must hold the orthogonality if it can be defined as a force. It follows from Eq. (21) that we can predict that there exist the gravitational radiation and magnetism-like components for a gravitational force. Particularly, the magnetism-like components would play an important role in geophysics and atmosphere physics.

If we have not any knowledge but know there exists the classical universal gravitation $f$ between two particles $m$ and $m'$, what form will take the 4-vector gravitational force $f$? Suppose that $m'$ is at rest at the origin, using $u = (u, u_4), u' = (0, 0, 0, ic)$ and $u \cdot f = 0$, we have

$$f_4 = \frac{u_4 f_4}{u_4} = \frac{u \cdot f - u \cdot f}{u_4} = -\frac{u \cdot f}{u_4}$$

$$f = (f, f_4) = (f, 0) + (0, 0, 0, f_4)$$

$$= (f, 0) + f_4 (0, 0, 0, ic)$$

$$= \frac{|f|}{|R|} R, 0 + f_4 \frac{u'}{ic} = \frac{|f|}{|R|} R, 0 - \left(\frac{u \cdot f}{u_4}\right) \frac{u'}{ic}$$

$$= \frac{|f|}{|R|} icu_4 (0, 0, 0, 0) - (u \cdot R)u'$$

$$= \frac{|f|}{|R|} icu_4 [(u' \cdot u)(R, 0) - (u \cdot R)u']$$

$$= \frac{|f|}{|R|} \left(\frac{1}{u' \cdot u}\right) [(u' \cdot u)R - (u \cdot R)u']$$

where $R \perp u'$, $R = (R, 0), r = |R|$. If we "rotate" (the Lorentz transformations) our frame of reference in the 4-dimensional space-time to make $m'$ not to be at rest, Eq. (28) will still valid because of its covariance. Then we find the 4-vector gravitational force goes back to the form of Eq. (21), like Lorentz force, having magnetism-like components.

Consider a planet $m$ moving about the sun $M$ with the 4-vector velocity $u = (u, u_4), u = u_r + u_4$ in the polar coordinate system $S : (r, \varphi, icr)$. We assume the sun is at rest at the origin, $u' = (0, 0, 0, ic)$, then from Eq. (24), the motion of the planet is governed by

$$\frac{du_4}{d\tau} = -\frac{GM}{c^2 r^3} (ru_r) ic$$

$$\frac{du}{d\tau} = \frac{GM}{c^2 r^3} (ic u_4) r$$

From the above equations we obtain their solutions

$$u_4 = ic (\varepsilon + \frac{s}{2r})$$

$$u_r^2 + u_\varphi^2 + u_4^2 = -c^2$$

$$ru_\varphi = h$$

where $\varepsilon$ and $h$ are two integral constants, $s = 2GM/c^2$ is called the Schwarzschild’s radius, $s/r$ is a small term. Eq. (27) agrees with the relation $u_r^2 = -c^2$. Consider the turning point at which the planet is at rest, $u_4 = ic$, the radius of the turning point is $r_{tp}$, then

$$\varepsilon = 1 - \frac{s}{2r_{tp}}$$

Eq. (28) tells us the relation between the time interval $dt$ and the proper time interval $d\tau$, i.e.

$$u_4 = \frac{ic dt}{d\tau} = ic (\varepsilon + \frac{s}{2r})$$

Hence, we know that the proper time interval $d\tau$ at position $r$ and the proper time interval $d\tau_\infty$ at $r = \infty$ have the relation

$$\frac{d\tau}{d\tau_\infty} = \frac{ic \varepsilon}{ic (\varepsilon + s/(2r))} \approx 1 - \frac{s}{2r}$$

Since $s/r$ (or $s/(2h^2/c^2)$) is a very small quantity, we neglect all but the first-order terms.

In the following subsections, we show that Eq. (28), (27) and (28) can give out the same results as the theory of general relativity for gravitational problems.

3.2 The gravitational red shift

According to Eq. (31), the period of oscillation of an atom at $r$ is related to the period of oscillation of an atom at $r = \infty$ by

$$\frac{T_r}{T_\infty} \approx 1 - \frac{s}{2r}$$

This effect is called the gravitational red shift.
3.3 The perihelion advance of a planet

Consider the planet again, substituting Eq. (31) into Eq. (24) and Eq. (25), we obtain

\[
[(1 + \frac{s}{2r}) \frac{dr}{d\tau_\infty}]^2 + [(1 + \frac{s}{2r}) \frac{dx}{d\tau_\infty}]^2 + [ic(\varepsilon + \frac{s}{2r})]^2 = -c^2
\]

(33)

\[r(1 + \frac{s}{2r}) \frac{dx}{d\tau_\infty} = h
\]

(34)

They also can be rewritten as

\[
\left(\frac{dr}{d\tau_\infty}\right)^2 + h^2 \left(1 - \frac{s}{r}\right) \varepsilon^2 [(\varepsilon^2 - 1) + \varepsilon \frac{s}{r}] (1 - \frac{s}{r})
\]

\[= 0
\]

(35)

\[r(1 + \frac{s}{2r}) \frac{dx}{d\tau_\infty} = h
\]

(36)

In order to pursue the covariance of Eq. (36), we define the angular displacement \(d\varphi_\infty = [1 + s/(2r)]d\varphi\), then we have

\[
\left(\frac{dr}{d\varphi_\infty}\right)^2 + r^2 (1 - \frac{s}{r}) (\varepsilon - \varepsilon^2 + 1) \frac{sc^2}{r} = (\varepsilon^2 - 1)c^2
\]

(37)

\[\varepsilon^2 \frac{d\varphi_\infty}{d\tau_\infty} = h
\]

(38)

Eliminating \(d\tau_\infty\) by dividing Eq. (37) by Eq. (38), we obtain

\[
\left(\frac{dr}{d\varphi_\infty}\right)^2 + r^2 (1 - \frac{s}{r}) (\varepsilon - \varepsilon^2 + 1) \frac{sc^2}{h^2} = (\varepsilon^2 - 1) \frac{c^2}{h^2}
\]

(39)

Now making the change of variable \(U = 1/r\) gives

\[
\left(\frac{dU}{d\varphi_\infty}\right)^2 + U^2 (1 - sU) (\varepsilon - \varepsilon^2 + 1) \frac{sc^2}{h^2} = (\varepsilon^2 - 1) \frac{c^2}{h^2}
\]

(40)

Differentiating with respect to \(\varphi_\infty\) gives

\[
\frac{d^2U}{d\varphi_\infty^2} + U - \frac{3s}{2} U^2 = (\varepsilon - \varepsilon^2 + 1) \frac{sc^2}{2h^2}
\]

(41)

The last term on the left side of the equation is relativistic correction, in the absence of this term, the solution is

\[
U = (\varepsilon - \varepsilon^2 + 1) \frac{c^2 s}{2h^2} [1 + \cos(\varphi_\infty - \varphi_0)]
\]

(42)

where \(c\) and \(\varphi_0\) are the constants of integration. Writing \(U = U_0 + U_1\), we find that Eq. (41) becomes

\[
\frac{d^2U_1}{d\varphi_\infty^2} + (1 - 3sU_0) U_1 = \frac{3s}{2} (U_0^2 + U_1^2)
\]

(43)

By inspection we see that \(U_1\) is on oscillatory function of \(\varphi_\infty\) with the frequency

\[
\omega = \sqrt{1 - 3sU_0}
\]

(44)

From \(\omega \phi = 2\pi\) for a cycle, we know that

\[
\phi \approx 2\pi (1 + \frac{3}{2}sU_0)
\]

(45)

Thus the perihelion advance of the planet is given by

\[
\Delta \phi = \phi - 2\pi = 3\pi sU_0
\]

(46)

This result is the same as that in the theory of general relativity.

3.4 The bending of light rays

Since the above solutions are independent from the planet mass \(m\), if the planet is a photon without mass, it also satisfies the above motion like a planet. For a photon, \(d\tau = 0\), Eq. (25) becomes

\[
h = \nu h \frac{d\varphi}{d\tau} = \infty
\]

(47)

Then Eq. (41) becomes

\[
\frac{d^2U}{d\varphi_\infty^2} + U - \frac{3s}{2} U^2 = 0
\]

(48)

The solution of this equation gives out the path of the photon, it is the same as that of the theory of general relativity. So in its flight past the massive object \(M\) of radius \(R\) the photon is deflected through an angle

\[
\delta = \frac{2s}{R}
\]

(49)

in agreement with experimental observations.

3.5 The bending of space

If there is not the nonlinear effect of photon flying, any straight line in the space-time is of the path of photon. Since the light is bent in the 4-dimensional space-time for an infinite distance observer, we consequently conclude that the space will bend due to the gravitational effect mentioned above near a massive object.

3.6 The Lorentz transformation

In classical mechanics, we have the impression that force yields acceleration, here we make an attempt to express that force yields the rotation of 4-vector velocity in the 4-dimensional space time.

When a particle is accelerated from a rest state at the \(r\) position to a speed \(v\) state at the \(r'\) position by a gravitational field \(\Phi\), the 4-th axis of the frame \(S\) fixed at the
particle will rotate an angle $\theta$ (towards the direction of $v$), the frame $S$ will become a new frame $S'$ in the viewpoint on the ground, the angle is given by $u_4 = ic \cosh \theta$. From Eq. (26), we get

$$u_4|_r = ic = ic(\varepsilon + \frac{s}{2r})$$ 
$$u_4|_{r'} = ic \cosh \theta = ic(\varepsilon + \frac{s}{2r'})$$

$$\cosh \theta = \frac{u_4|_{r'}}{u_4|_r} = 1 + \frac{s}{2\pi r'} - \frac{s}{2\pi r} \approx 1 + \frac{\Phi}{c^2}$$

where $\Phi = c^2 s/(2r') - c^2 s/(2r)$ is the potential difference between $S$ and $S'$. Since the particle is accelerated by $\Phi$, then the rotation is given by

$$\frac{1}{2}v^2 = \Phi$$

$$\cosh \theta \approx 1 + \frac{\Phi}{c^2} \approx \frac{1}{\sqrt{1 - v^2/c^2}}$$

$$\sinh \theta = \sqrt{1 - \cosh^2 \theta} = \frac{iv/c}{\sqrt{1 - v^2/c^2}}$$

Because the origins of $S$ and $S'$ coincide at the time $t = 0$, the coordinate transformations between them are

$$dr = dr' \cosh \theta + dx'_4 \sinh \theta$$

$$dx_4 = -dr' \sinh \theta + dx'_4 \cosh \theta$$

$$dr = \frac{1}{\sqrt{1 - v^2/c^2}}dr' + \frac{iv/c}{\sqrt{1 - v^2/c^2}}dx'_4$$

$$dx_4 = \frac{-iv/c}{\sqrt{1 - v^2/c^2}}dr' + \frac{1}{\sqrt{1 - v^2/c^2}}dx'_4$$

It is known as the Lorentz transformations.

### 4 Discussion

To note that the orthogonality of 4-vector force and 4-vector velocity is valid for any force: strong, electromagnetic, weak and gravitational interactions, therefore many new aspects of force concept in mechanics remain for physics to explore.

### 5 Conclusions

This paper points out that the 4-vector force acting on a particle is always in the direction orthogonal to the 4-vector velocity of the particle in the 4-dimensional spacetime, rather than along the line joining the particle and the action source. This inference is obviously supported from the fact that the magnitude of the 4-vector velocity is a constant. This orthogonality brings out many new aspects for force concept. In this paper it is found that the Maxwell’s equations can be derived from classical Coulomb’s force and the orthogonality, some gravitational effects such as the perihelion advance of planet can also be explained in terms of the orthogonality.

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