Diameter two properties in some vector-valued function spaces

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Abstract
We introduce a vector-valued version of a uniform algebra, called the vector-valued function space over a uniform algebra. The diameter two properties of the vector-valued function space over a uniform algebra on an infinite compact Hausdorff space are investigated. Every nonempty relatively weakly open subset of the unit ball of a vector-valued function space $A(K, (X, \tau))$ over an infinite dimensional uniform algebra has diameter two, where $\tau$ is a locally convex Hausdorff topology on a Banach space $X$ compatible to a dual pair. Under the assumption of $X$ equipped with the norm topology being uniformly convex and the additional condition that $A \otimes X \subset A(K, X)$, it is shown that Daugavet points and $\Delta$-points on $A(K, X)$ over a uniform algebra $A$ are the same, and they are characterized by the norm-attainment at a limit point of the Shilov boundary of $A$. In addition, a sufficient condition for the convex diametral local diameter two property of $A(K, X)$ is also provided. Similar results also hold for an infinite dimensional uniform algebra.

Keywords Diameter two property · Uniform algebra · Urysohn-type lemma · Shilov boundary

Mathematics Subject Classification Primary 46B04; Secondary 46B20 · 46E30 · 47B38

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1 Introduction

In this article, we study the diameter two properties of a uniform algebra and the vector-valued function space \( A(K, (X, \tau)) \) for a complex Banach space \( X \) with a locally convex Hausdorff topology \( \tau \) compatible with a dual pair by using the Urysohn-type lemma. The diameter two properties have gained a lot of attention recently and have been studied in various Banach spaces. Exploring these properties involves weakly open subsets and slices of the unit ball \( B_X \) of a Banach space \( X \). Let \( S_X \) be the unit sphere of a Banach space \( X \) and \( X^* \) be the dual space of \( X \). For \( x^* \in S_{X^*} \) and \( \epsilon > 0 \), a slice \( S(x^*, \epsilon) \) of the unit ball \( B_X \) is defined by
\[
S(x^*, \epsilon) = \{ x \in B_X : \text{Re} x^* x > 1 - \epsilon \}.
\]
Here we recall the definitions of the diameter two properties as introduced in [2].

Definition 1.1

(i) A Banach space \( X \) has the strong diameter two property (SD2P) when every convex combination of slices of the unit ball \( B_X \) has diameter two.

(ii) A Banach space \( X \) has the diameter two property (D2P) when every nonempty relatively weakly open subset of the unit ball \( B_X \) has diameter two.

(iii) A Banach space \( X \) has the local diameter two property (LD2P) when every slice of the unit ball \( B_X \) has diameter two.

Since every weakly open subset of the unit ball contains a convex combination of slices [13, Lemma II.1], the SD2P implies the D2P. The D2P implies the LD2P because every slice is a relatively weakly open subset of the unit ball. It is well-known that these properties are weaker than the Daugavet property [2, Theorem 4.4] and that these properties are the other extremes of the Radon–Nikodym property which states, geometrically, that there exist nonempty slices with arbitrarily small diameter. The D2Ps have been examined in various spaces. It is well-known that \( c_0 \) and \( \ell_\infty \) have the SD2P. The infinite dimensional uniform algebra over a compact Hausdorff space also has the SD2P [2,25], in fact, even stronger property called the symmetric strong diameter two property (SSD2P) [2,3]. Some known examples with the D2P are interpolation spaces \( L_1 + L_\infty \) and \( L_1 \cap L_\infty \) equipped with certain norms [4]. These spaces do not satisfy the Daugavet property but they have the D2P. For a Banach space \( X \), the space \( C(K, (X, \tau)) \) of \( X \)-valued continuous functions over a compact Hausdorff space \( K \), where \( \tau \) is a locally convex Hausdorff topology compatible to a dual pair, is also known to have the D2P [5]. More recent examples of the D2Ps concern tensor products of Banach spaces [18,29] and free Banach lattices generated by a Banach space [10]. Here we mention that almost squareness (ASQ) of Banach spaces, which is a stronger property than the SD2P, and the duality between various octahedralities and the D2Ps have played important roles to study these examples. It is also shown that polyhedrality is not related to the D2Ps at all [22].

Not only the D2Ps mentioned above, the “diametral” version of the D2Ps will be explored. Studying these properties requires us to have a firm grasp on the \( \Delta \)-points and the Daugavet points. For given \( x \in S_X \) and \( \epsilon > 0 \), let \( \Delta_\epsilon (x) = \{ y \in B_X : \| x - y \| \geq 2 - \epsilon \} \). Then a point \( x \in S_X \) is said to be a \( \Delta \)-point if \( x \in \overline{\text{conv}} \Delta_\epsilon (x) \) for every \( \epsilon > 0 \). We denote the set of all \( \Delta \)-points by \( \Delta_X \). Similarly, a point \( x \in S_X \) is said to be a Daugavet point if \( B_X = \overline{\text{conv}} \Delta_\epsilon (x) \) for every \( \epsilon > 0 \). Notice that every Daugavet point is a \( \Delta \)-point. We recall the definitions of the diametral diameter two properties introduced in [1,6,15].

Definition 1.2 ("Diametral" D2Ps)

(i) A Banach space \( X \) has the convex diametral local diameter two property (convex-DLD2P) if \( B_X = \overline{\text{conv}} \Delta_X \), that is, the unit ball is the closed convex hull of the set of all \( \Delta \)-points.
A Banach space \( X \) has the diametral local diameter two property (DLD2P) if for every slice \( S \) of \( B_X \), every \( x \in S_X \cap S \), and every \( \epsilon > 0 \) there exists \( y \in S \) such that \( \|x - y\| \geq 2 - \epsilon \).

A Banach space \( X \) has the diametral diameter two property (DD2P) if for every relative weakly open subset \( U \) of \( B_X \), every \( x \in S_X \cap U \), and every \( \epsilon > 0 \), there exists \( y \in U \) such that \( \|x - y\| \geq 2 - \epsilon \).

There is also a stronger version of the DD2P called the diametral strong diameter two property (DSD2P), which uses a convex combination of slices, but this property is now known to be equivalent to the Daugavet property [16]. It was shown in [6] that the Daugavet property implies the DD2P and that the DD2P implies the DLD2P. The convex-DLD2P is stronger than the LD2P and weaker than the DLD2P [1]. Moreover, we can see that the D2P is weaker than the LD2P from their definitions. If one wishes to see a clearer picture, we refer to a nice visualization of the relationship between various diameter two properties in the dissertation by Pirk (see [27, pp 84]). These properties have been also examined on several Banach spaces. For example, the spaces \( c_0 \) of convergent sequences and \( l_\infty \) do not have the DLD2P but they have the convex-DLD2P [1, Corollary 5.4, Remark 5.5]. Moreover, \( c_0 \) even fails to have the convex-DLD2P [1, pp 18]. In [19], it is shown that the diametral D2Ps are inherited by F-ideals (e.g. M-ideals) and stable under Köthe–Bochner spaces \( E(X) \) when we assume that the simple functions are dense in \( E(X) \). The diametral D2Ps of a projective tensor product of two Banach spaces are also considered in the same article.

The DLD2P and the Daugavet property have equivalent definitions in terms of \( \Delta \)-points and Daugavet points, respectively. A Banach space \( X \) having the DLD2P is equivalent to the fact that every point in \( S_X \) is a \( \Delta \)-point (see [15, Theorem 1.4] and [31, Open problem (7)]). Similarly, a Banach space \( X \) has the Daugavet property if and only if every point in \( S_X \) is a Daugavet point [31, Corollary 2.3].

The organization of this article consists of four parts. In Sect. 2, we provide necessary information about the uniform algebra \( A \), the space \( A(K, (X, \tau)) \), and the Urysohn-type lemma, which is the key ingredient to prove our results throughout this article. In Sect. 3, we make a few remarks on the symmetric strong diameter two property of a scalar-valued function algebra. In Sect. 4, we show that the vector-valued function space \( A(K, (X, \tau)) \) over a uniform algebra also satisfies the D2P. By the Gelfand transform we show that a function space \( A(\Omega, X) \) on a Hausdorff space \( \Omega \) is isometrically isomorphic to a function space \( A(M_A, (X**, w^*)) \) over a uniform algebra, so it is shown that \( A(\Omega, X) \) also has the D2P if the base (function) algebra is infinite dimensional. This result shows the D2P of some function spaces like \( A_b(B_X, Y) \) (resp. \( A_u(B_X, Y) \)) consisting of \( Y \)-valued functions which are bounded (resp. uniformly continuous) on \( B_X \) and are holomorphic on the interior of \( B_X \) when \( X \) and \( Y \) are Banach spaces. In Sect. 5, we present some results on the Daugavet and \( \Delta \)-points in a vector-valued function space \( A(K, X) \) over a uniform algebra. Under the condition of \( X \) being uniformly convex and the additional assumption that \( A \otimes X \subset A(K, X) \), we show that these points are the same and characterized by the norm-attainment at a limit point of the Shilov boundary. Also under the same assumptions, a sufficient condition for the convex-DLD2P of \( A(K, X) \) is provided.

2 Preliminaries

In this article, we assume that \( K \) is a compact Hausdorff space and \( X, Y \) are nontrivial complex Banach spaces unless specified. We work with a Banach space \( X \) equipped with a
locally convex Hausdorff (LCH) topology $\tau$ compatible to the dual pair $(X, X^*)$ or $(X^*, X)$. Recall that $(X, \tau)$ is said to be compatible to the dual pair $(X, X^*)$ if $(X, \tau)^* = X^*$ and a LCH topology $\tau$ on $X^*$ is said to be compatible to the dual pair $(X^*, X)$ if $(X^*, \tau)^* = X$. In this paper, we say that a LCH topology $\tau$ on a Banach space $X$ is compatible to a dual pair if $(X, \tau)^* = X$ or $(X, \tau)^* = X_*$, where $X_*$ is the predual of $X$.

The weak topology on $X$ is denoted by $(X, \omega)$, the weak* topology on $X^*$ by $(X^*, w^*)$, and the norm topology on $X$ by $(X, \| \cdot \|)$. The Mackey topology for $X$ (resp. $X^*$) is the finest topology where every linear functional $x^* \in X^*$ (resp. linear functional defined by $x \in X$) is continuous. The Mackey–Arens theorem [24, Theorem 8.7.4] says that a locally convex topology $\tau$ for $X$ (resp. $X^*$) compatible to a dual pair includes the weak (resp. weak*-) topology and is included in the Mackey topology on $X$ (resp. $X^*$). The Mackey topology of $X$ (resp. $X^*$) is known to be coarser than the norm topology on $X$ [30, IV.3.4] (resp. the norm topology on $X^*$ [24, Example 8.5.5 and Example 8.8.9]). Hence, for any LCH topology $\tau$ compatible to a dual pair $(X, X^*)$ (resp. $(X^*, X)$), we have $(X, \omega) \subset \tau \subset (X, \| \cdot \|)$ (resp. $(X^*, w^*) \subset \tau \subset (X^*, \| \cdot \|)$). Also, for such topology $\tau$ on $X$ compatible to $(X, X^*)$, $\tau$-bounded sets are weakly bounded and therefore norm-bounded. Similarly, for a LCH topology $\tau$ on $X^*$ compatible to $(X^*, X)$, $\tau$-bounded sets are weakly*-bounded and therefore norm-bounded in $X^*$. For more details on the theory of topological vector spaces, we refer to [24,30].

Now we show that the space $C(K, (X, \tau))$, which consists of all continuous functions from a compact Hausdorff space $K$ to $(X, \tau)$, equipped with the supremum norm is a Banach space. Even though this fact is already known in [5], we include the proof for completeness.

**Proposition 2.1** Let $K$ be a compact Hausdorff space, let $X$ be a Banach space and let $\tau$ be a locally convex Hausdorff topology compatible to a dual pair. Then the space $C(K, (X, \tau))$ equipped with the supremum norm is a Banach space.

**Proof** Let us consider when $\tau$ is a LCH topology compatible to either a dual pair $(X, X^*)$ or $(X, X_*)$. Since $f(K)$ is $\tau$-compact for every $f \in C(K, (X, \tau))$, the range $f(K)$ is bounded in the norm topology of $X$. This makes the norm $\| f \| = \sup \{ \| f(t) \|_X : t \in K \}$ well-defined.

To show the completeness of $C(K, (X, \tau))$, let $(f_n)_{n=1}^\infty \subset C(K, (X, \tau))$ be a Cauchy sequence. Then for any $\varepsilon > 0$, there exists $N$ such that for every $n, m \geq N$, we have $\| f_n - f_m \| < \varepsilon$. Notice that for every $t \in K$,

$$\| f_n(t) - f_m(t) \|_X \leq \| f_n - f_m \| < \varepsilon,$$

and so $(f_n(t))_{n=1}^\infty$ is a Cauchy sequence in $X$ with respect to the norm topology. Since $X$ is complete with the norm topology, $(f_n(t))_{n=1}^\infty$ converges in norm. So for each $t \in K$, let $f(t) = \lim_{n \to \infty} f_n(t)$. Then by (1), for any $\varepsilon > 0$ there exists $N$ such that for all $n \geq N$,

$$\sup_{t \in K} \| f_n(t) - f(t) \|_X < \frac{\varepsilon}{2}.$$ 

Now we have to show that $f \in C(K, (X, \tau))$. First, let $t_0$ be an element of $K$ and let $V$ be a $\tau$-neighborhood of $0$. Choose a balanced, convex $\tau$-neighborhood $W$ of $0$ such that $W + W + W \subset V$. Since $W$ is also open in the norm topology, there exists $\varepsilon B_X \subset W$ and $N \in \mathbb{N}$ such that for all $n \geq N$ and $t \in K$, $f(t) - f_n(t) \in \varepsilon B_X$. Hence, $f(t) - f(t_0) = (f(t) - f_N(t)) + (f_N(t) - f_N(t_0)) + (f_N(t_0) - f(t_0))$

$$\in \varepsilon B_X + (f_N(t) - f_N(t_0)) + \varepsilon B_X$$

$$\subset W + (f_N(t) - f_N(t_0)) + W.$$
Moreover, \( f_N \) is continuous with respect to \( \tau \). So there exists a neighborhood \( U \) of \( t_0 \) such that for every \( t \in U \), \( f_N(t) - f_N(t_0) \in W \). Then, we see that \( f(t) - f(t_0) \in W + W + W \subset V \) for all \( t \in U \). This shows that \( f \in C(K, (X, \tau)) \). Therefore, \( C(K, (X, \tau)) \) is a Banach space. \( \square \)

Let \( \Omega \) be a Hausdorff space and \( C_b(\Omega) \) be the Banach algebra of all bounded continuous functions over \( \Omega \) equipped with the supremum norm. A function algebra \( A(\Omega) \) on \( \Omega \) is a closed subalgebra of \( C_b(\Omega) \) that separates the points of \( \Omega \) and contains constant functions. Here separating points means for each pair \( (s, t) \in \Omega \times \Omega \) with \( s \neq t \), there exists \( f \in A(\Omega) \) such that \( f(s) \neq f(t) \). If \( \Omega \) is a compact Hausdorff space, a function algebra is called a uniform algebra.

We introduce \((X, \tau)\)-valued function space over a uniform algebra for a Banach space \( X \) as follows.

**Definition 2.2** Let \( X \) be a Banach space and \( \tau \) be a LCH topology on \( X \) compatible to a dual pair. A closed subspace \( A(K, (X, \tau)) \) of \( C(K, (X, \tau)) \) is said to be the \((X, \tau)\)-valued function space over a uniform algebra \( A \) if the following two conditions are satisfied:

(i) The base algebra, defined by \( A := \{ y^* \circ f : f \in A(K, (X, \tau)), y^* \in (X, \tau)^* \} \), is a uniform algebra over \( K \).

(ii) \( \phi f \in A(K, (X, \tau)) \) for every \( \phi \in A \) and \( f \in A(K, (X, \tau)) \) where \( (\phi f)(t) = \phi(t) f(t) \) for \( t \in K \).

Notice that if \( X = \mathbb{C} \), we have the usual uniform algebra \( A \) on \( K \). When \( \tau \) is the norm topology, we call the space \( A(K, (X, \tau)) \) an \( X \)-valued function space over a uniform algebra and denote the space by \( A(K, X) \). Similar vector-valued version of a uniform algebra has been studied in [7].

A point \( t_0 \in K \) is said to be a strong boundary point for a uniform algebra \( A \) if for every neighborhood \( U \) containing \( t_0 \), there exists \( f \in A \) such that \( f(t_0) = \|f\| = 1 \) and \( \sup_{t \in K \setminus U} |f(t)| < 1 \). A subset \( S \subset X \) is said to be a boundary if for each \( f \in A \), there exists an element \( t \in S \) such that \( |f(t)| = \|f\|_\infty \). The Shilov boundary \( \Gamma(A) \) of \( A \) is the smallest closed boundary of \( A \) and it is known that it is the intersection of all closed boundaries of \( A \). For a uniform algebra, let \( K_A = \{ \lambda \in A^* : \|\lambda\| = \lambda(1_A) = 1 \} \) and denote the set of its extreme points by \( \text{ex} K_A \). The set \( \text{ex} K_A \) is called the Choquet boundary of \( A \). It is well-known that each \( \lambda \in \text{ex} K_A \) is associated with some elements \( x \in K \) and uniquely represented by the Dirac measure \( \delta_x \) [9, Lemma 4.3.2]. For this reason, we use \( \Gamma_0(A) \) to denote the set of such elements \( x \in K \) corresponding to the elements in the Choquet boundary. Also, the set of strong boundary points and \( \Gamma_0(A) \) coincide when we consider a uniform algebra \( A \) over a compact Hausdorff space \( K \) [9, Theorem 4.3.5]. It is known that the Shilov boundary \( \Gamma(A) \) of a uniform algebra \( A \) is the closure of its Choquet boundary \( \Gamma_0(A) \) [9, Corollary 4.3.7.a].

Now we recall the Urysohn-type lemma that is extensively used throughout this article.

**Lemma 2.3** [8, Lemma 2.5] Let \( A \subset C(K) \) be a uniform algebra and \( \Gamma_0 \) be its Choquet boundary. Then, for every open set \( U \subset K \) with \( U \cap \Gamma_0 \neq \emptyset \) and \( 0 < \epsilon < 1 \), there exists \( f \in A \) and \( t_0 \in U \cap \Gamma_0 \) such that \( f(t_0) = \|f\|_\infty = 1 \), \( |f(t)| < \epsilon \) for every \( t \in K \setminus U \) and

\[
|f(t)| + (1 - \epsilon)|1 - f(t)| \leq 1 \quad \text{for all} \ t \in K. \tag{2}
\]

The following observation is useful for later.

**Lemma 2.4** Let \( X \) be a Banach space and let \( \tau \) be a LCH topology compatible to a dual pair. Suppose that \( A(K, (X, \tau)) \) is a \((X, \tau)\)-valued function space over the base algebra \( A \) and
$L$ is a closed boundary for $A$. The space of restrictions of elements of $A(K, (X, \tau))$ to $L$ is denoted by $A(L, (X, \tau))$ and the restrictions of elements of $A$ to $L$ is denoted by $A(L)$.

Then $A(L, (X, \tau))$ is a $(X, \tau)$-valued function space over the base algebra $A(L)$ and it is isometrically isomorphic to $A(K, (X, \tau))$.

**Proof** Given $f \in A(K, (X, \tau))$, let $f|_L$ be the restriction of $f$ to $L$. Then we have

$$\|f\| = \sup\{\|f(t)\| : t \in K\} = \sup\{|x^* f(t)| : t \in K, x^* \in B_{(X, \tau)^*}\}$$

$$= \sup\{|x^* f(t)| : t \in L, x^* \in B_{(X, \tau)^*}\} = \sup\{\|f(t)\| : t \in L\} = \|f|_L\|$$

So $A(L, (X, \tau))$ is a surjective isometric image of $A(K, (X, \tau))$ by the map $f \mapsto f|_L$ and it is a closed subspace of $C(L, (X, \tau))$. It is easy to see that $A(L, (X, \tau))$ satisfies the condition (i) and (ii) in the Definition 2.2 and its base algebra is $A(L)$. This completes the proof. \(\square\)

Since the Shilov boundary $\Gamma$ of a uniform algebra is a closed boundary of $K$, a uniform algebra is isometric to a uniform algebra of its restrictions to $\Gamma$ by Lemma 2.4 or [21, Theorem 4.1.6]. We need another useful lemma on the isolated point of the Shilov boundary.

**Lemma 2.5** Let $A$ be a uniform algebra on a compact Hausdorff space $K$ and let $t_0$ be an isolated point of the Shilov boundary $\Gamma$ of $A$. Then there exists a function $\phi \in A$ such that $\phi(t_0) = \|\phi\| = 1$ and $\phi(t) = 0$ for $t \in \Gamma \backslash \{t_0\}$.

**Proof** Since $t_0$ is an isolated point of $\Gamma$, there is an open set such that $U \cap \Gamma = \{t_0\}$. The set of strong boundary points $\Gamma_0$ is dense in $\Gamma$. So the point $t_0 \in \Gamma_0$ is in fact a strong boundary point of $A$. By definition of a strong boundary point, there exists a function $\psi \in A$ such that $\psi(t_0) = \|\psi\| = 1$ and $\sup_{t \in K \backslash U} |\psi(t)| < 1$. Note that $(\psi^n)_{n=1}^\infty$ converges uniformly in $C(\Gamma)$. By the isometry of the map from $f \in A$ to its restriction $f|_{\Gamma}$ to $\Gamma$, $(\psi^n)_{n=1}^\infty$ is Cauchy in $A$ and converges uniformly to $\phi$ in $A$. Now it is clear that $\phi(t) = 0$ for $t \in \Gamma \backslash \{t_0\}$ and $\phi(t_0) = \|\phi\| = 1$. Thus, we obtain the desired function. \(\square\)

### 3 A short remark on the SSD2P of function algebras

First we start with what is known as the symmetric strong diameter two property (SSD2P). The SSD2P is stronger than the SD2P and also gained more attention recently. The dual version of SSD2P called the $w^*$-SSD2P has been also considered, and it is shown that if $X^*$ has the $w^*$-SSD2P then $X$ has a property called the decomposable octahedrality [26]. The converse in general is still unknown, but in the case of Lipschitz function spaces, the duality between the $w^*$-SSD2P of Lipschitz function spaces and the decomposable octahedrality of their preduals, which are known as Lipschitz-free spaces, has been shown [26]. For more details on the SSD2P and decomposable octahedrality, we refer the readers to [3,14,26].

**Definition 3.1** [3, Definition 1.3] A Banach space $X$ has the symmetric strong diameter two property (SSD2P) if whenever $n \in \mathbb{N}$, $S_1, \ldots, S_n$ are slices of $B_X$, and $\epsilon > 0$, there exists $x_i \in S_i$, $i = 1, \ldots, n$, and $y \in B_X$ such that $x_i \pm y \in S_i$ for every $i \in \{1, \ldots, n\}$ and $\|y\| > 1 - \epsilon$.

It has been already known from [2, Theorem 4.2] that the infinite dimensional uniform algebra over a compact Hausdorff space has the SSD2P and their proof method is extended to show the SSD2P of somewhat regular linear subspaces of the space $C_0(L)$ of continuous functions over a locally compact Hausdorff space $L$ vanishing at infinity [3, Theorem 2.2].
**Definition 3.2** [3, Definition 2.1] Let $L$ be a locally compact Hausdorff space and $C_0(L)$ be the space of continuous functions over $L$ vanishing at infinity. A linear subspace $Y$ of $C_0(L)$ is somewhat regular, if whenever $V$ is a non-empty open subset of $L$ and $0 < \epsilon < 1$, there is an $f \in Y$ such that 

$$
\|f\| = 1 \quad \text{and} \quad |f(x)| \leq \epsilon \quad \text{for every } x \in L \setminus V.
$$

We make a few remarks here. In view of Lemma 2.3 we can easily verify that the infinite dimensional uniform algebra $A$ over a compact Hausdorff space $K$ is an example of somewhat regular linear subspace of $C(K)$ due to the fact that $A$ is isometric to $A(K)$ and that $\Gamma_0 = \Gamma_0(A)$ is dense in $\Gamma = \Gamma(A)$ because $K$ is compact. In addition, Lemma 2.3 is a special case of the Urysohn-type lemma for somewhat regular linear subspaces (see [3, Lemma 2.4]) that targets uniform algebras.

**Theorem 3.3** [3, Theorem 2.2] The infinite dimensional uniform algebra $A$ over a compact Hausdorff space $K$ has the SSD2P.

Now let $C_b(B_X)$ be the space of bounded, complex-valued, continuous functions on the unit ball $B_X$ of a Banach space $X$. The space $A_b(B_X)$ is a closed subalgebra of $C_b(B_X)$ that consists of holomorphic functions on the interior of $B_X$. The space $A_u(B_X)$ is a closed subalgebra of $C_b(B_X)$ that consists of functions in $A_b(B_X)$ that are uniformly continuous on $B_X$. From the fact that function algebras $A_b(B_X)$ and $A_u(B_X)$ are isometric to a uniform algebra on a compact Hausdorff space via the Gelfand transformation (see [20, Proposition 2] or Theorem 4.5), we can deduce the following fact on the SSD2Ps of $C_b(B_X)$, $A_b(B_X)$ and $A_u(B_X)$.

**Corollary 3.4** Let $X$ be a Banach space. The function algebras $C_b(B_X)$, $A_b(B_X)$ and $A_u(B_X)$ have the SSD2P.

**4 The D2P of vector-valued function spaces over a uniform algebra**

Now we consider the space $A(K, (X, \tau))$ where $X$ is a Banach space with a locally convex Hausdorff topology $\tau$ compatible with a dual pair. Here the norm of the space $A = A(K)$ is denoted by $\|\cdot\|_{\infty}$. First we recall a useful fact about the D2P of a direct sum of two Banach spaces.

**Lemma 4.1** [5, Lemma 2.2] Let $X$ be a Banach space satisfying the diameter two property. Then for any arbitrary Banach space $Y$, $X \oplus Y$ has the diameter two property.

Here we present the main result of this section.

**Theorem 4.2** Let $K$ be a compact Hausdorff space and $X$ be a Banach space endowed with a locally convex Hausdorff topology $\tau$ compatible to a dual pair. If the base algebra $A$ is infinite dimensional, then the space $A(K, (X, \tau))$ has the D2P.

**Proof** In view of Lemma 2.4, we only have to show the D2P for $A(\Gamma, (X, \tau))$ where $\Gamma = \Gamma(A)$ is a Shilov boundary of a uniform algebra. Since $A$ is infinite dimensional, $\Gamma$ is infinite. We write $A(\Gamma, X)$ instead of $A(\Gamma, (X, \tau))$ for convenience.

Now let $W$ be a nonempty weakly open subset of $A(\Gamma, X)$. Three cases will be considered.
(i) Assume that $\Gamma$ is perfect. Let $a \in W \cap S_{A(\Gamma, X)}$. Then for every $\delta > 0$, there exists $t_0 \in \Gamma$ such that $\|a(t_0)\|_X > 1 - \delta$. Also we can find $f \in S_{(X, \tau)'}$ such that $\text{Re} f(a(t_0)) > 1 - \delta$ because $(X, \tau)^* = X^*$ or $(X, \tau)^* = X_*$, where $X_*$ is a predual of $X$.

Define $U = \{ t \in \Gamma : \text{Re} f(a(t)) > 1 - \delta \}$. Notice that $U$ contains $t_0$. Since $t_0$ is a limit point of $\Gamma$, we can construct pairwise disjoint open subsets $U_n \subset U$ for $n \in \mathbb{N}$. From the fact that $\Gamma_0 = \Gamma_0(A)$ is dense in $\Gamma$, we know $U_n \cap \Gamma_0 \neq \emptyset$ for all $n \in \mathbb{N}$.

So by Lemma 2.3, for each $n \in \mathbb{N}$ there exist $\phi_n \in A$ and $t_n \in U_n \cap \Gamma_0$ such that $\phi_n(t_n) = \|\phi_n\|_X = 1$, $|\phi_n(t)| < \frac{\delta}{2 \sqrt{n+2}}$ for $t \in \Gamma \setminus U_n$, and

$$|\phi_n(t)| + \left(1 - \frac{\delta}{2 \sqrt{n+2}}\right)|1 - \phi_n(t)| \leq 1 \quad \text{for } t \in \Gamma.$$  

Define $g_n = 1 - 2\phi_n$. First we want to show that $\|g_n\| \leq 1 + \frac{\delta}{2\sqrt{n}}$. For every $t \in \Gamma$,

$$|1 - 2\phi_n(t)| = \left|1 - 2\phi_n(t) + \frac{\delta}{2 \sqrt{n+2}} - \frac{\delta}{2 \sqrt{n+2}} + \frac{\delta}{2 \sqrt{n+1}} \phi_n(t) - \frac{\delta}{2 \sqrt{n+1}} \phi_n(t)\right|$$

$$\leq \left|1 - \frac{\delta}{2 \sqrt{n+2}}\right|(1 - 2\phi_n(t)) + \frac{\delta}{2 \sqrt{n+2}} + \frac{\delta}{2 \sqrt{n+1}} \phi_n(t)$$

$$\leq \left|1 - \frac{\delta}{2 \sqrt{n+2}}\right|(1 - \phi_n(t)) - \left(1 - \frac{\delta}{2 \sqrt{n+2}}\right) \phi_n(t) + \frac{\delta}{2 \sqrt{n}}$$

$$\leq 1 - |\phi_n(t)| + |\phi_n(t)| + \frac{\delta}{2 \sqrt{n}}$$

$$\leq 1 + \frac{\delta}{2 \sqrt{n}}.$$  

Hence $\|g_n\|_X \leq 1 + \frac{\delta}{2 \sqrt{n}}$. Define $h_n = \frac{g_n}{1 + \frac{\delta}{2 \sqrt{n}}}$. Notice that $h_n \in B_A$ for all $n \in \mathbb{N}$ and that $h_n(t)$ converges to 1 pointwise.

Now, let $F \in A(\Gamma, X)^*$ be a bounded linear functional on $A(\Gamma, X)$. If we define $\varphi(h) = F(ha)$ for $h \in A$, $\varphi$ is a bounded linear functional on $A$ because $|\varphi(h)| \leq \|F\|_{A(\Gamma, X)^*} \|h\|_X \|a\| = \|h\|_X \|F\|_{A(\Gamma, X)^*} < \infty$ for all $h \in A$. By the Hahn–Banach extension theorem and the Riesz representation theorem, $\varphi$ is represented by a regular complex Borel measure $\mu$ and $\varphi(h_n) = \int_{\Gamma} h_n d\mu$. Moreover, $\varphi(h_n) = \int_{\Gamma} h_n d\mu \rightarrow \int_{\Gamma} 1d\mu = \varphi(1)$ by the bounded convergence theorem. Hence $F(h_n a) = \varphi(h_n) \rightarrow \varphi(1) = F(a)$ and so the functions $h_n a$ converges to $a$ weakly in $A(\Gamma, X)$. Furthermore, we have

$$\|a - h_n a\| \geq \|a(t_n) - h_n(t_n)a(t_n)\|_X = |1 - h_n(t_n)| \|a(t_n)\|_X$$

$$\geq \frac{2 + \delta/2^n}{1 + \delta/2^n} \cdot \text{Re} f(a(t_n))$$

$$\geq \left(2 - \frac{\delta/2^n}{1 + \delta/2^n}\right)(1 - \delta)$$

$$\geq (2 - \delta)(1 - \delta)$$

$$\geq 2 - 3\delta.$$  

Since $\delta > 0$ is arbitrary, the diameter of $W \cap B_{A(\Gamma, X)}$ is 2.

(ii) Now we consider when $\Gamma$ has infinitely many isolated points. Let $(t_n)_{n=1}^\infty$ be a sequence of isolated points and $t_0$ be the accumulation point of the sequence. Let $a \in W \cap S_{A(\Gamma, X)}$. Let $U$ be a $\tau$-open neighborhood around 0 and let $V$ be an $\tau$-open neighborhood around $a(t_0) \neq 0$ such that $U \cap V = \emptyset$. Notice that every $\tau$-open neighborhood is also an
open neighborhood in the norm topology on $X$. So by the continuity of $a$, $a^{-1}(V)$ contains infinitely many $t_n$’s. Since $U$ is norm-open, there exists a suitable $\delta > 0$ such that $\delta B_X \subset U$. So, we may assume that $a(t_n) \in V \setminus \delta B_X$ for every $n \in \mathbb{N}$, in other words, $\|a(t_n)\|_X > \delta$ for all $n \in \mathbb{N}$. Since $\Gamma_0$ is dense in $\Gamma$, every isolated point $t_n \in \Gamma$ is a strong boundary point. Hence by Lemma 2.5, there exists $\phi_n \in A$ such that $\phi_n(t_n) = \|\phi_n\| = 1$ and $\phi_n(t) = 0$ for all $t \in \Gamma \setminus \{t_n\}$.

Define $g_n = 1 + \left(\frac{1}{\|a(t_n)\|_X} - 1\right) \phi_n + \left(\frac{1}{\|a(t_{n+1})\|_X} - 1\right) \phi_{n+1}$. We see that $g_n(t)$ converges to 1 pointwise, $\sup_{n \in \mathbb{N}} \|g_n\|_\infty \leq 1 + \frac{1}{\delta}$ and $\|g_n a\| = 1$ for every $n \in \mathbb{N}$. Hence, $g_n \to 1$ weakly in $A$. Moreover, $g_n a \to a$ weakly in $A(\Gamma, X)$ by the same argument as in (i).

When we compute the diameter of the relatively weakly open set of $B_{A(\Gamma, X)}$, we have

$$\text{diam} (W \cap B_{A(\Gamma, X)}) \geq \|g_{n+1} a - g_n a\| \geq \|g_{n+1} (t_{n+1}) a(t_{n+1}) - g_n (t_{n+1}) a(t_{n+1})\|_X = \frac{2}{\|a(t_{n+1})\|_X} \|a(t_{n+1})\|_X = 2.$$

(iii) Finally, let us consider when $\Gamma$ has only finitely many isolated points $t_1, \ldots, t_n$. Denote $\Gamma_1 = \{t_1, \ldots, t_n\}$ and the perfect subset of $\Gamma$ by $\Gamma_2 = \Gamma \setminus \Gamma_1$. Since the set of strong boundary points is dense in $\Gamma$, we see that $t_1, \ldots, t_n$ are strong boundary points and for each $i = 1, \ldots, n$, once again by Lemma 2.5, there exist $\phi_i \in A(\Gamma)$ such that $\phi_i(t_i) = 1$ and $\phi_i(t) = 0$ for all $t \neq t_i$. Define a bounded linear operator $P : A(\Gamma, X) \to A(\Gamma, X)$ by $P f = \sum_{i=1}^n \phi_i f$. Then it is easy to check that $P$ is a norm-one projection. Let $B = P(A(\Gamma, X))$ and $C$ be the restrictions of $f \in A(\Gamma, X)$ to $\Gamma_2$. Then $C$ is an $(X, \tau)$-valued function space over an infinite dimensional uniform algebra on $\Gamma_2$. We will show that $A(\Gamma, X)$ is isometrically isomorphic to $B \oplus_\infty C$. Then by (i), $C$ has the diameter two property and Lemma 4.1 completes the proof.

For $f \in A(\Gamma, X)$, define $\Phi f = (P f, |f|_{\Gamma_2})$. Then $\Phi : A(\Gamma, X) \to B \oplus_\infty C$ is well-defined and an isometry because

$$\|\Phi(f)\| = \max\{\|P f\|, \|f\|_{\Gamma_2}\|\} = \max\left\{\sup_{t \in \Gamma_1} \|f(t)\|_X, \sup_{t \in \Gamma_2} \|f(t)\|_X\right\} = \sup_{t \in \Gamma} \|f(t)\|_X = \|f\|.$$ Given $(f, g) \in B \oplus_\infty C$, there exist $f_1, f_2 \in A(\Gamma, X)$ such that $f = P f_1$ and $g = f_2 |_{\Gamma_2}$. Let $h = P f_1 + f_2 - P f_2 \in A(\Gamma, X)$ and it is clear that $\Phi(h) = (P f_1, f_2 |_{\Gamma_2}) = (f, g)$. So $\Phi$ is surjective. This shows that $A(\Gamma, X)$ is isometrically isomorphic to $B \oplus_\infty C$ and completes the proof.

The norm, weak, and weak-* topologies are compatible to a dual pair and we get the following.

**Corollary 4.3** Let $X$ be a Banach space and let $K$ be an infinite compact Hausdorff space. Then the spaces $A(K, X)$, $A(K, (X, w))$, and $A(K, (X^*, w^*))$ have the D2P if their corresponding base algebras are infinite dimensional.

Now we introduce a Banach space $X$-valued function space (over a function space) on a Hausdorff space and we show that this space is isometrically isomorphic to a $(X^{**}, w^*)$-valued function space over a uniform algebra. Let $\Omega$ be a Hausdorff space and let $X$ be a Banach space equipped with the uniform norm. The space $C_b(\Omega, X)$ is the Banach space of all bounded $X$-valued continuous functions over $\Omega$ equipped with the supremum norm.
Definition 4.4 Let $X$ be a Banach space. An $X$-valued function space over a function algebra $A$ on $\Omega$, denoted by $A(\Omega, X)$, is a closed subspace of $C_b(\Omega, X)$ satisfying

(i) the base algebra $A := \{x^* \circ f : f \in A(\Omega, X), x^* \in X^*\}$ is a function algebra on $\Omega$.
(ii) $f g \in A(\Omega, X)$ for every $f \in A$ and $g \in A(\Omega, X)$.

Given a function algebra $A$ on a Hausdorff space $\Omega$, let $M_A$ be its maximal ideal space consisting of all nonzero algebraic homomorphisms from $A$ to $\mathbb{C}$. Then $M_A$ is a compact Hausdorff space with the Gelfand topology [28, Theorem 11.9]. The Gelfand transform $\hat{g}: M_A \to \mathbb{C}$ of $g \in M_A$ is defined by $\hat{g}(x^*) = \phi(x^* \circ g)$ for every $x^* \in X^*$ and $\phi \in M_A$. From the fact that $\hat{g}(\phi)(x^*) = \phi(x^* \circ g)$, the mapping $\hat{g}$ is continuous on $M_A$ if we consider the weak*-topology on $X^*$. Let $A(M_A, (X^*, w^*))$ be the set of such $\hat{g}$'s.

Theorem 4.5 Let $X$ be a Banach space and $A(\Omega, X)$ be an $X$-valued function space over a function algebra $A$ on a Hausdorff space $\Omega$. Then $A(\Omega, X)$ is isometrically isomorphic to an $(X^*, w^*)$-valued function space over a uniform algebra $\hat{A}$ on the compact Hausdorff space $M_A$. In fact, $M_A$ is the maximal ideal space of $A$ with the Gelfand topology.

We split the proof of Theorem 4.5 into several parts for readability. First we show that the mapping $g \mapsto \hat{g}$ is an isometry.

Proposition 4.6 Let $X$ be a Banach space, let $A(\Omega, X)$ be an $X$-valued function space over a function algebra $A$ on a Hausdorff space $\Omega$, and let $M_A$ be the maximal ideal space of $A$. Then for every $g \in A(\Omega, X)$ and $x^* \in X^*$, the mapping $g \mapsto \hat{g}$, where $\hat{g} \in A(M_A, (X^*, w^*))$ is defined by $\hat{g}(\phi)(x^*) = \phi(x^* \circ g)$, is an isometry.

Proof By the fact that $\|\phi\|_\infty = 1$ we obtain

$$
\|\hat{g}\| = \sup_{\phi \in M_A} \|\hat{g}(\phi)\| = \sup_{\phi \in M_A} \sup_{x^* \in B_{X^*}} \|\hat{g}(\phi)(x^*)\| = \sup_{\phi \in M_A} \sup_{x^* \in B_{X^*}} \|\phi(x^* \circ g)\|
$$

$$
\leq \sup_{x^* \in B_{X^*}} \|x^* \circ g\|_\infty
$$

$$
= \sup_{x^* \in B_{X^*}} \sup_{t \in \Omega} \|x^* \circ g(t)\| = \sup_{t \in \Omega} \|g(t)\| X = \|g\|.
$$

To show the reverse inequality, let $\phi_t \in M_A$ be an evaluation functional at $t \in \Omega_0 = \{t \in \Omega : g(t) \neq 0\}$. Then we have

$$
\|g\| = \sup_{t \in \Omega_0} \|g(t)\| X = \sup_{t \in \Omega_0} \sup_{x^* \in B_{X^*}} \|(x^* \circ g(t))\| = \sup_{t \in \Omega_0} \sup_{x^* \in B_{X^*}} \|\phi_t(x^* \circ g)\|
$$

$$
\leq \sup_{\phi \in M_A} \sup_{x^* \in B_{X^*}} \|\phi(x^* \circ g)\|
$$

$$
= \sup_{\phi \in M_A} \sup_{x^* \in B_{X^*}} \|\hat{g}(\phi)(x^*)\|
$$

$$
\leq \sup_{\phi \in M_A} \|\hat{g}(\phi)\| = \|\hat{g}\|,
$$

so the mapping $g \mapsto \hat{g}$ is an isometry.
Now we show in the following two results that the space $A(M_A, (X^{**}, w^*))$ satisfies the condition (i) and (ii) in the Definition 2.2 for a $(X, \tau)$-valued function space over a uniform algebra on a compact Hausdorff space.

**Proposition 4.7** Let $X$ be a Banach space, let $A(\Omega, X)$ be a $X$-valued function space over a function algebra $A$ on a Hausdorff space $\Omega$, and let $M_A$ be the maximal ideal space of $A$. Then the space

$$\hat{A} = \{x^* \circ \hat{g} : \hat{g} \in A(M_A, (X^{**}, w^*)), \; x^* \in X^*\}$$

is a uniform algebra over $M_A$.

**Proof** Let $\hat{A} = \{x^* \circ \hat{g} : \hat{g} \in A(M_A, (X^{**}, w^*)), \; x^* \in X^*\}$. From the definition of the Gelfand transformation, notice that $x^* \circ \hat{g}(\phi) = \phi(x^* \circ g) = x^* \circ g(\phi)$. Hence we can rewrite $\hat{A} = \{x^* \circ g : g \in A(\Omega, X) \text{ and } x^* \in X^*\}$. We see that the set $\hat{A}$ is, in fact, the image of the Gelfand transformation on $A$. Thus $A$ is isometrically isomorphic to $\hat{A}$ and so $\hat{A}$ is a closed subalgebra of $C(M_A)$ that separates the points of $M_A$ and contains constant functions. This shows that $\hat{A}$ is a uniform algebra over $M_A$. \hfill \Box

**Proposition 4.8** Suppose that $X$ is a Banach space and $A(\Omega, X)$ is an $X$-valued function space over a base algebra $A$ on $\Omega$. Let $M_A$ be the maximal ideal space of $A$ and let

$$\hat{A} = \{x^* \circ \hat{g} : \hat{g} \in A(M_A, (X^{**}, w^*)) \text{ and } x^* \in X^*\}.$$

Then $\phi \cdot \hat{g} \in A(M_A, (X^{**}, w^*))$ for every $\phi \in \hat{A}$ and $\hat{g} \in A(M_A, (X^{**}, w^*))$.

**Proof** Let $\hat{f}, \hat{g} \in A(M_A, (X^{**}, w^*))$ and let $x^* \circ \hat{f} \in \hat{A}$ where $x^* \in (X^{**}, w^*)^* = X^*$. We claim that $(x^* \circ \hat{f}) \cdot \hat{g} \in A(M_A, (X^{**}, w^*))$. Note first that $\hat{h} = (x^* \circ \hat{f}) \cdot \hat{g} \in A(\Omega, X)$. We need to show that $\hat{h} = (x^* \circ \hat{f}) \cdot \hat{g}$.

Since $x^* \circ \hat{f}(\phi) = x^* \circ \hat{f}(\phi) = \phi(x^* \circ f)$ for every nonzero algebra homomorphism $\phi \in M_A$, we have

$$\hat{h}(\phi)(y^*) = (x^* \circ \hat{f} \cdot \hat{g})(\phi)(y^*) = (x^* \circ \hat{f})(\phi) \cdot \hat{g}(\phi)(y^*) = ((x^* \circ \hat{f}) \cdot \hat{g})(\phi)(y^*)$$

for $y^* \in X^*$. Therefore, $\hat{h} = (x^* \circ \hat{f}) \cdot \hat{g}$ and this proves our claim. \hfill \Box

Now, Theorem 4.5 immediately leads to the following conclusion.

**Corollary 4.9** Suppose that $X$ is a Banach space and $A(\Omega, X)$ is an $X$-valued function space over a function algebra $A$ on a Hausdorff space $\Omega$. If the base algebra $A$ is infinite dimensional, then $A(\Omega, X)$ has the D2P.

For Banach spaces $X$ and $Y$, let $C_b(B_X, Y)$ be the bounded $Y$-valued continuous functions on the unit ball $B_X$. The space $A_b(B_X, Y)$ is defined to be a closed subspace of $C_b(B_X, Y)$ that consists of holomorphic functions on the interior of $B_X$ and the space $A_u(B_X, Y)$ is a closed subspace of $A_b(B_X, Y)$ consisting of uniformly continuous functions on $B_X$. It is easy to check that $C_b(B_X, Y), A_b(B_X, Y)$ and $A_u(B_X, Y)$ are $Y$-valued function spaces in Definition 4.4. Hence we have the following consequence.

**Corollary 4.10** Let $X$ and $Y$ be Banach spaces. Then $C_b(B_X, Y), A_b(B_X, Y)$ and $A_u(B_X, Y)$ have the D2P.
5 Daugavet points and Δ-points on vector-valued function spaces over a uniform algebra

In this section, we present another application of Lemma 2.3 to the Daugavet points and Δ-points of $A(K, X)$ with respect to its Shilov boundary of the base algebra. Recall that a Banach space $X$ has the Daugavet property if every rank-one operator $T : X \to X$ satisfies $\|I + T\| = 1 + \|T\|$. Well-known examples are $L_1(\mu)$ and $L_\infty(\mu)$ where $\mu$ is a nonatomic measure [12,23] and $C(K)$ where $K$ does not have isolated points [11]. An infinite dimensional uniform algebra over a compact Hausdorff space satisfies the Daugavet property if the Shilov boundary does not contain isolated points [32,33]. More generally, if a locally compact Hausdorff space $L$ does not have isolated points, somewhat regular subspaces of $C_0(L)$ also have the Daugavet property [3]. Similar results are extended to vector-valued spaces. The space $C(K, X)$ has the Daugavet property if $K$ does not have isolated points [31]. A subspace $A^X$ of $C(K, X)$, defined by $A^X = \{ f \in C(K, X) : x^* \circ f \in A \}$ where $A$ is a uniform algebra, is known to satisfy the Daugavet property from the fact that it satisfies the polynomial Daugavet property [7].

We mentioned earlier that we can investigate the Daugavet property and the DLD2P of $\Gamma$-spaces and we may assume that $\Gamma$ is a closed, compact, Hausdorff space. By Lemma 2.4, we see that for $g \in A(K, (X, \tau))$, the mapping defined by $g \mapsto g|\Gamma$ is an isometry. So it is enough to show that $f|\Gamma$ is a Daugavet point of $A(\Gamma, (X, \tau))$ and we may assume that $K = \Gamma$. Denote the base algebra of $A(K, (X, \tau))$ as $A$.

Now fix $g \in B_{A(K, (X, \tau))}$ and $\epsilon > 0$. Let $U$ be an open set containing $t_0$ such that $\|f(t) - f(t_0)\|_X < \epsilon$ for every $t \in U$. By the Hausdorff condition and the fact that $t_0$ is a limit point of $U$, we have $g|U \subset A(K, (X, \tau))$ and $g|\Gamma = f|\Gamma$. If $f$ is a Daugavet point of $A(\Gamma, (X, \tau))$, then $g$ is a Daugavet point of $A(K, (X, \tau))$.

Lemma 5.1 [1, Lemma 2.1] Let $X$ be a Banach space. The following are equivalent:

(i) $x \in S_X$ is a Daugavet point.

(ii) for every $x^* \in X^*$ with $x^*(x) = 1$, the projection $P = x^* \otimes x$ satisfies $\|I - P\| \geq 2$.

We show that if $f \in S_{A(K, (X, \tau))}$ attains its norm at a limit point on the Shilov boundary of its base algebra with the additional condition $A \otimes X \subset A(K, (X, \tau))$, it is a Daugavet point. For a function algebra $A$ on $\Omega$ and a Banach space $X$, $A \otimes X$ means the set of all functions $f \otimes x (f \in A, x \in X)$ defined by $(f \otimes x)(t) = f(t) \cdot x$, where $t \in \Omega$.

Theorem 5.2 Let $K$ be a compact Hausdorff space and let $X$ be a Banach space endowed with a locally convex Hausdorff topology $\tau$ compatible to a dual pair. Suppose that a function space $A(K, (X, \tau))$ over a uniform algebra $A$ satisfies the additional condition that $A \otimes X \subset A(K, (X, \tau))$ and let $\Gamma$ be the Shilov boundary of $A$. If $f$ is a norm-one element of $A(K, (X, \tau))$ and there is a limit point $t_0$ of $\Gamma$ such that $\|f\| = \|f(t_0)\|_X$, then $f$ is a Daugavet point.

Proof Suppose that $\|f\| = \|f(t_0)\|_X = 1$ for a limit point $t_0$ of $\Gamma$. Since $\Gamma$ is closed, $t_0$ is an element of $\Gamma$. Let $A(\Gamma, (X, \tau))$ be the set of all restrictions of $f \in A(K, (X, \tau))$ to the Shilov boundary $\Gamma$. By Lemma 2.4, we see that for $g \in A(K, (X, \tau))$, the mapping defined by $g \mapsto g|\Gamma$ is an isometry. So it is enough to show that $f|\Gamma$ is a Daugavet point of $A(\Gamma, (X, \tau))$ and we may assume that $K = \Gamma$. Denote the base algebra of $A(K, (X, \tau))$ as $A$.

Now fix $g \in B_{A(K, (X, \tau))}$ and $\epsilon > 0$. Let $U$ be an open set containing $t_0$ such that $\|f(t) - f(t_0)\|_X < \epsilon$ for every $t \in U$. By the Hausdorff condition and the fact that $t_0$ is a limit point of $U$, we have $g|U \subset A(K, (X, \tau))$ and $g|\Gamma = f|\Gamma$. If $f$ is a Daugavet point of $A(\Gamma, (X, \tau))$, then $g$ is a Daugavet point of $A(K, (X, \tau))$. 

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Since the collection $\{U_n\}_{n=1}^\infty$ consists of pairwise disjoint nonempty open subsets of $U$, we obtain that for each $t \in \bigcup_{n=1}^m U_n$, 

$$
\left\| g(t) - \frac{1}{m} \sum_{n=1}^m g_n(t) \right\|_X \leq \frac{1}{m} \left( 2 + \left( \frac{\epsilon}{2} + \frac{\epsilon}{4} + \cdots + \frac{\epsilon}{2^{n-1}} + \frac{\epsilon}{2^n+1} + \cdots + \frac{\epsilon}{2^m} \right) \right)
$$

and for each $t \in K \setminus \bigcup_{n=1}^m U_n$, 

$$
\left\| g(t) - \frac{1}{m} \sum_{n=1}^m g_n(t) \right\|_X \leq \frac{1}{m} \left( \frac{\epsilon}{2} + \frac{\epsilon}{4} + \cdots + \frac{\epsilon}{2^m} \right) < \frac{1}{m} \left( 1 - \frac{1}{2^m} \right) \epsilon.
$$

Hence, we get 

$$
\left\| g - \frac{1}{m} \sum_{n=1}^m g_n \right\| \leq \frac{1}{m} \left( 2 + \left( 1 - \frac{1}{2^m} \right) \epsilon \right).
$$

By taking the limit $m \to \infty$, we show that $g \in \overline{\text{conv}} \Delta_\epsilon (f)$. Hence $B_X = \overline{\text{conv}} \Delta_\epsilon (f)$, in other words, $f$ is a Daugavet point. \hfill \Box

From this, we obtain a sufficient condition for the Daugavet property of $A(K, X)$ since the map $t \mapsto \|f(t)\|$ is continuous on a compact space and attains its maximum on $K$ if $\tau$ is the norm topology and $f \in A(K, X)$.
Corollary 5.3 Let $K$ be a compact Hausdorff space and let $X$ be a Banach space. Suppose that a function space $A(K, X)$ over a base algebra $A$ satisfies the additional condition that $A \otimes X \subset A(K, X)$. If the Shilov boundary $\Gamma$ of the base algebra $A$ does not have isolated points, then $A(K, X)$ has the Daugavet property.

Furthermore, when $X$ is uniformly convex, Daugavet points and $\Delta$-points are the same in $A(K, X)$ and characterized by the norm-attainment at a limit point of $\Gamma$. Recall that the modulus of convexity $\delta_X(\epsilon)$ of a Banach space $X$ for $0 < \epsilon < 2$ is defined by

$$\delta_X(\epsilon) = \inf \left\{ 1 - \frac{x + y}{2} : x, y \in B_X, \|x - y\| \geq \epsilon \right\}.$$ 

A Banach space $X$ is said to be uniformly convex if $\delta_X(\epsilon) > 0$ for each $0 < \epsilon < 2$.

Theorem 5.4 Let $X$ be a uniformly convex Banach space, $K$ be a compact Hausdorff space, $\Gamma$ be the Shilov boundary of the base algebra $A$ of $A(K, X)$, and $f \in S_{A(K, X)}$. Then (i) $\implies$ (ii) $\implies$ (iii) holds, where

(i) $f$ is a Daugavet point.

(ii) $f$ is a $\Delta$-point.

(iii) there is a limit point $t_0$ of $\Gamma$ such that $\|f\|_{X} = \|f(t_0)\|_{X}$.

Moreover, if we assume the additional condition that $A \otimes X \subset A(K, X)$, then (i), (ii) and (iii) are equivalent.

Proof In view of Lemma 2.4, $A(K, X)$ is isometric to $A(\Gamma, X)$ and so it is enough to prove our claim with respect to $f|\Gamma \in S_{A(\Gamma, X)}$. So, we may assume that $K = \Gamma$. (i) $\implies$ (ii) is clear from definitions. (iii) $\implies$ (i) has been already shown in Theorem 5.2 with the additional assumption that $A \otimes X \subset A(K, X)$. Hence we only need to show (ii) $\implies$ (iii).

Assume to the contrary that $f \in S_{A(K, X)}$ is a $\Delta$-point but $\|f\| \neq \|f(t)\|_X$ for every limit point of $K$. Let $F = \{t \in K : \|f(t)\|_X = 1\}$. Then $F$ only contains isolated points of $K$ by the assumption and is nonempty. Since $K$ is compact, we also see that $|F| < \infty$, where $|F|$ is the number of elements of $F$. For each $t \in F$ we can always find $x^*_t \in S_{X^*}$ such that $x^*_t(f(t)) = 1$.

Since $X$ is uniformly convex, let $\delta = \delta_X \left( \frac{1}{2|F|} \right) > 0$. If $\text{Re} x^*_t(x) \geq 1 - \delta$ and $x \in B_X$, then

$$\left\| \frac{x + f(t)}{2} \right\|_X \geq \text{Re} x^*_t \left( \frac{x + f(t)}{2} \right) \geq 1 - \frac{1}{2}\delta > 1 - \delta \delta_X \left( \frac{1}{2|F|} \right).$$

This means that $\|x - f(t)\|_X < \frac{1}{2|F|}$ by the definition of $\delta_X$. Hence

$$\text{if Re} x^*_t(x) \geq 1 - \delta \text{ and } x \in B_X, \text{ then } \|x - f(t)\|_X \leq \frac{1}{2|F|} \text{ for every } t \in F. \quad (3)$$

Let $\epsilon = 1 - \max_{t \in K \setminus F} \|f(t)\|_X > 0$. Define a bounded linear functional $\psi$ on $A(K, X)$ by

$$\psi(g) = \frac{1}{|F|} \sum_{t \in F} x^*_t(g(t)),$$

where $g \in A(K, X)$. Notice that $\|\psi\| = \psi(f) = 1$ and this defines a bounded projection $P : A(K, X) \to A(K, X)$ by $P(g) = \psi(g)f$ for $g \in A(K, X)$. It is clear that $\|P\| = \|P(f)\| = \|f\| = 1$. So, $P$ is a norm-one projection on $A(K, X)$. Then in view of Lemma 5.1, we should have $\|I - P\| \geq 2$. But we show that in fact $\|I - P\| < 2.$
For a given $g \in B_{A(K,X)}$, if $t \in K \setminus F$ we have
\[
\|g(t) - P g(t)\|_X = \|g(t) - \psi(g) f(t)\|_X \leq 1 + \|f(t)\|_X \leq 2 - \epsilon.
\] (4)

Now we divide $F$ into two parts. Let $t_0 \in F_1 = \{t \in F : |x_t^*(g(t))| \geq 1 - \delta\}$. Then there exists a scalar $\lambda_0 \in B_C$ such that $|x_{t_0}^*(g(t_0))| = \lambda_0 x_{t_0}^*(g(t_0)) = x_{t_0}^*(\lambda_0 g(t_0)) \geq 1 - \delta$. By (3), we have $\|\lambda g(t_0) - f(t_0)\|_X \leq \frac{1}{2|F|}$. Then
\[
\|g(t_0) - P g(t_0)\|_X = \left\|g(t_0) - \frac{1}{|F|} \sum_{t \in F} x_t^*(g(t)) f(t)\right\|_X
\leq \left\|g(t_0) - \frac{1}{|F|} x_{t_0}^*(g(t_0)) f(t_0)\right\|_X + \frac{|F| - 1}{|F|}
= \left\|\lambda_0 g(t_0) - \frac{1}{|F|} x_{t_0}^*(\lambda_0 g(t_0)) f(t_0)\right\|_X + \frac{|F| - 1}{|F|}.
\]

From the fact that $x_{t_0}^*(\lambda_0 g(t_0)) \leq 1$, we have $1 - \frac{1}{|F|} x_{t_0}^*(\lambda_0 g(t_0)) \geq 0$. Hence,
\[
\left\|\lambda_0 g(t_0) - \frac{1}{|F|} x_{t_0}^*(\lambda_0 g(t_0)) f(t_0)\right\|_X
\leq \|\lambda_0 g(t_0) - f(t_0)\|_X + \|f(t_0)\|_X \left(1 - \frac{1}{|F|} x_{t_0}^*(\lambda_0 g(t_0))\right)
\leq \frac{1}{2|F|} + 1 - \frac{1}{|F|} (1 - \delta),
\]
and so
\[
\|g(t_0) - P g(t_0)\|_X \leq \frac{1}{2|F|} + 1 - \frac{1}{|F|} (1 - \delta) + \frac{|F| - 1}{|F|} \leq 2 - \frac{1}{2|F|}.
\] (5)

When $t_0 \in F_2 = F \setminus F_1$, we have $|x_{t_0}^*(g(t_0))| < 1 - \delta$. So we see that
\[
\|g(t_0) - P g(t_0)\|_X \leq \left\|\lambda_0 g(t_0) - \frac{1}{|F|} x_{t_0}^*(\lambda_0 g(t_0)) f(t_0)\right\|_X + \frac{|F| - 1}{|F|}
\leq 1 + \frac{1}{|F|} (1 - \delta) + \frac{|F| - 1}{|F|} \leq 2 - \frac{\delta}{|F|}.
\]
Hence, with (4) and (5) we obtain $\|g - P g\| \leq \max\{2 - \frac{1}{2|F|}, 2 - \frac{\delta}{|F|}, 2 - \epsilon\} < 2$. Since $g \in B_{A(K,X)}$ is chosen arbitrarily, $\|I - P\| < 2$, and we get a contradiction.

Since $\mathbb{C}$ is uniformly convex, we have the following consequences for a scalar-valued uniform algebra.

**Corollary 5.5** Let $K$ be a compact Hausdorff space and let $\Gamma$ be the Shilov boundary of $A(K)$. Then for $f \in S_{A(K)}$, the following are equivalent:

(i) $f$ is a Daugavet point.
(ii) $f$ is a $\Delta$-point.
(iii) there is a limit point $t_0$ of $\Gamma$ such that $\|f\|_\infty = \|f(t_0)\|$. 

By the characterization given in Theorem 5.4, we show when the Daugavet property, DD2P, and DLD2P of $A(K,X)$ are equivalent to each other.
Theorem 5.6 Let $X$ be a uniformly convex Banach space, let $K$ be a compact Hausdorff space, and let $\Gamma$ be the Shilov boundary of the base algebra $A$ of $A(K, X)$. Then (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (iv) holds, where

(i) $A(K, X)$ has the Daugavet property.
(ii) $A(K, X)$ has the DD2P.
(iii) $A(K, X)$ has the DLD2P.
(iv) The Shilov boundary $\Gamma$ does not have isolated points.

Moreover, if we assume the additional condition that $A \otimes X \subset A(K, X)$, then (i), (ii), (iii) and (iv) are equivalent.

Proof Since $A(K, X)$ is isometric to $A(\Gamma, X)$ by Lemma 2.4, showing the equivalence for $A(\Gamma, X)$ is enough. It is well-known that (i) $\implies$ (ii) $\implies$ (iii) [6]. By Corollary 5.3, we have (iv) $\implies$ (i) with the additional assumption that $A \otimes X \subset A(K, X)$.

Finally we prove (iii) $\implies$ (iv). Assume that $A(\Gamma, X) \simeq A(K, X)$ has the DLD2P but the Shilov boundary $\Gamma$ of the base algebra $A$ of $A(K, X)$ has an isolated point $t_0$. From the fact that $A$ is isometrically isomorphic to the uniform algebra $A(\Gamma)$, we can also assume that $A = A(\Gamma)$. Since the set of strong boundary points $\Gamma_0$ is dense in $\Gamma$, this $t_0$ is also in $\Gamma_0$. Then by Lemma 2.5 there is $\phi \in A$ such that $\phi(t_0) = 1 = \|\phi\|$ and $\phi(t) = 0$ for all $t \in \Gamma \setminus \{t_0\}$. Since $\phi \in A$, there is an element $g \in A(\Gamma, X)$ such that $\phi = x^* \circ g$ for some $x^* \in X^*$. Then $g(t_0) \neq 0$ and let $h = \frac{\phi}{\|g(t_0)\|_X} g$. It is easy to see that $h$ is an element of $A(\Gamma, X)$. Since $\|h\| = \|h(t_0)\| = 1$ and $\|h(t)\| = 0$ for every point $t$ of $\Gamma$ other than $t_0$. So $\|h(t)\| = 0$ for every limit point $t$ of $\Gamma$. Hence, in view of Theorem 5.4, we see that $h \in S_{A(\Gamma, X)}$ is not a $\Delta$-point, and this is a contradiction to our assumption that $A(\Gamma, X)$ has the DLD2P, which states that every element in $S_{A(\Gamma, X)}$ is a $\Delta$-point. Therefore (iii) $\implies$ (iv) is proved.

As we mentioned before, it is well-known that the nonexistence of isolated points on the Shilov boundary implies that the scalar-valued uniform algebra has the Daugavet property [32,33]. Here we have something even further, namely that the Daugavet property, the DD2P, and the DLD2P are equivalent to each other.

Corollary 5.7 Let $K$ be a compact Hausdorff space and let $\Gamma$ be the Shilov boundary of a uniform algebra $A$. Then the following are equivalent:

(i) $A$ has the Daugavet property.
(ii) $A$ has the DD2P.
(iii) $A$ has the DLD2P.
(iv) The Shilov boundary $\Gamma$ does not have isolated points.

We conclude this article with showing when the space $A(K, X)$ has the convex-DLD2P. But here we take a slightly different approach. To use Lemma 2.3 we need to find a limit point of $\Gamma$ that is also a strong boundary point, but the existence of such a point does not seem to be guaranteed in general. So here we use a different version of the Urysohn-type lemma for strong peak points. For a uniform algebra $A$ over a compact Hausdorff space $K$, an element $t_0 \in K$ is said to be a strong peak point if there exists a function $f \in A$ such that $|f(t_0)| = \|f\|_\infty$ and $\|f\|_\infty > \sup\{|f(t)| : t \in K \setminus U\}$ for every open neighborhood $U$ of $t_0$. Let us denote the set of all strong peak points for $A$ by $\rho(A)$.

Lemma 5.8 [17, Lemma 3] Let $\Omega$ be a Hausdorff space and $A$ be a subalgebra of $C_b(\Omega)$. Suppose that $U$ is an open subset of $\Omega$ and $t_0 \in U \cap \rho(A)$. Then given $0 < \epsilon < 1$, there exists $\phi \in A$ such that $\|\phi\| = 1 = \phi(t_0)$, $\sup_{t \in \Omega \setminus U} |\phi(t)| < \epsilon$, and for all $t \in \Omega$,

$$|\phi(t)| + (1 - \epsilon)|1 - \phi(t)| \leq 1.$$
The original statement of Lemma 5.8 is given with \( \|\phi\| = 1 = |\phi(t_0)| \). However, if we observe the proof of Lemma 3 in [17] carefully, it is easy to see that the statement without the absolute value also holds.

**Theorem 5.9** For a compact Hausdorff space \( K \) and a uniformly convex Banach space \( X \), suppose that a function space \( A(K, X) \) over a uniform algebra \( A \) satisfies the additional condition that \( A \otimes X \subset A(K, X) \). Let \( \Gamma' \) be the set of limit points in the Shilov boundary \( \Gamma \) of the base algebra \( A \) and let \( \rho(A) \) be the set of all strong peak points for \( A \). If \( \rho(A) \cap \Gamma' \neq \emptyset \), then the space \( A(K, X) \) has the convex-DLD2P.

**Proof** By the isometry in Lemma 2.4, we may assume that \( K = \Gamma \). We need to show that \( S_{A(K, X)} \subset \text{conv}\Delta \), where \( \Delta \) is the set of all \( \Delta \)-points of \( A(K, X) \).

Fix \( f \in S_{A(K, X)} \). Pick \( t_0 \in \rho(A) \cap K' \) and let \( \lambda = \frac{1 + \|f(t_0)\|}{2} \). For \( \epsilon > 0 \), let \( U \) be an open neighborhood containing \( t_0 \) such that \( \|f(t) - f(t_0)\| < \epsilon \). By Lemma 5.8, there exists a function \( \phi \in A \) such that \( \phi(t_0) = \|\phi\| = 1 \), \( \sup_{t \in K \setminus U} |\phi(t)| < \epsilon \) and

\[
|\phi(t)| + (1 - \epsilon)|1 - \phi(t)| \leq 1.
\]

for every \( t \in K \). Choose a norm-one vector \( v_0 \in X \) and set

\[
x_0 = \begin{cases} \frac{f(t_0)}{\|f(t_0)\|_X} & \text{if } f(t_0) \neq 0, \\ v_0 & \text{if } f(t_0) = 0. \end{cases}
\]

Now define two functions

\[
f_1(t) = (1 - \epsilon)(1 - \phi(t))f(t) + \phi(t)x_0 \\
f_2(t) = (1 - \epsilon)(1 - \phi(t))f(t) - \phi(t)x_0, \quad t \in K.
\]

Since \( A \otimes X \subset A(K, X) \), we see that the functions \( f_1, f_2 \in A(K, X) \). Notice that

\[
\|f_1(t)\|_X = \|(1 - \epsilon)(1 - \phi(t))f(t) + \phi(t)x_0\|_X \\
\leq (1 - \epsilon)|1 - \phi(t)| + |\phi(t)| \leq 1,
\]

for every \( t \in K \), in particular, \( \|f_1(t_0)\|_X = 1 \). Hence, \( \|f_1\| = \|f_1(t_0)\|_X = 1 \). We can also show \( \|f_2\| = \|f_2(t_0)\|_X = 1 \) by the same argument. Thus, by Theorem 5.4, we see that \( f_1, f_2 \in \Delta \). Let \( g(t) = \lambda f_1(t) + (1 - \lambda)f_2(t) \). We consider two cases.

(i) First, consider when \( f(t_0) \neq 0 \). Then \( g(t) = (1 - \epsilon)(1 - \phi(t))f(t) + \phi(t)f(t_0) \). We see that

\[
\|g(t) - f(t)\|_X = \|(1 - \epsilon)(1 - \phi(t))f(t) + \phi(t)f(t_0) - f(t)\|_X \\
= \|(1 - \epsilon)(1 - \phi(t))f(t) + \phi(t)f(t_0) - (1 - \epsilon)f(t) - \epsilon f(t)\|_X \\
= \|(1 - \epsilon)(\phi(t)f(t) + (1 - \epsilon)\phi(t)f(t_0) + \epsilon f(t)f(t_0) - \epsilon f(t)) - f(t)\|_X \\
= \|(1 - \epsilon)\phi(t)(f(t_0) - f(t)) + \epsilon \phi(t)f(t_0) - \epsilon f(t))\|_X \\
\leq (1 - \epsilon)|\phi(t)| \cdot \|f(t) - f(t_0)\|_X + \epsilon |\phi(t)| \cdot \|f(t_0)\|_X + \epsilon \|f(t)\|_X \\
\leq (1 - \epsilon)|\phi(t)| \cdot \|f(t) - f(t_0)\|_X + 2\epsilon.
\]

For \( t \in U \), we have \( (1 - \epsilon)|\phi(t)| \cdot \|f(t) - f(t_0)\|_X \leq (1 - \epsilon)\epsilon < \epsilon \). Since \( |\phi(t)| < \epsilon \) for \( t \in K \setminus U \), we have \( (1 - \epsilon)|\phi(t)| \cdot \|f(t) - f(t_0)\|_X \leq 2(1 - \epsilon)\epsilon < 2\epsilon \) by the triangle inequality. Therefore, \( \|g - f\| < 4\epsilon \), and this implies that \( f \in \text{conv}\Delta \).
(ii) Now consider when \( f(t_0) = 0 \). Then for all \( t \in U \), \( \| f(t) \|_X < \epsilon \). Notice that \( \lambda = \frac{1}{2} \) and 
\[
g(t) = (1 - \epsilon)(1 - \phi(t)) f(t).
\]
Hence,
\[
\| g(t) - f(t) \|_X = \|(1 - \epsilon)(1 - \phi(t)) f(t) - (1 - \epsilon) f(t) - \epsilon f(t) \|_X \\
\leq (1 - \epsilon)|\phi(t)| \cdot \| f(t) \|_X + \epsilon \| f(t) \|_X \leq (1 - \epsilon)|\phi(t)| \cdot \| f(t) \|_X + \epsilon.
\]
For \( t \in U \), we have \( (1 - \epsilon)|\phi(t)| \cdot \| f(t) \|_X \leq (1 - \epsilon)\epsilon < \epsilon \). Since \( \sup_{t \in K \setminus U} |\phi(t)| < \epsilon \) for \( t \in K \setminus U \), we have \( (1 - \epsilon)|\phi(t)| \cdot \| f(t) \|_X \leq (1 - \epsilon)\epsilon < \epsilon \). Therefore, \( \| g - f \| < 2\epsilon \), and this also implies that \( f \in \mathfrak{conv} \Delta \).

Since \( f \in S_{A(K,X)} \) is arbitrary, we see that \( S_{A(K,X)} \subset \mathfrak{conv} \Delta \), and so \( A(K,X) \) has the convex-DLD2P.

\[ \square \]

**Corollary 5.10** Let \( K \) be a compact Hausdorff space and \( \Gamma' \) be the set of limit points in the Shilov boundary \( \Gamma \) of a uniform algebra \( A \). If \( \rho(A) \cap \Gamma' \neq \emptyset \), then the uniform algebra \( A \) has the convex-DLD2P.

**Remark 5.11** The sufficient condition \( \rho(A) \cap \Gamma' \neq \emptyset \) in Corollary 5.10 does not guarantee the DLD2P. For example, let \( K = \{ \frac{1}{n} : n \in \mathbb{N} \} \cup \{ 0 \} \) in the real line and let \( A = C(K) \) be the space of continuous functions over \( K \) equipped with the supremum norm. Notice that \( K = \rho(A) \) is its Shilov boundary and \( \rho(A) \cap \Gamma' = \{ 0 \} \). So, in view of Theorem 5.7 and Theorem 5.10, the space \( C(K) \) has the convex-DLD2P, while it does not have the DLD2P. In fact, \( C(K) \) is isometric to the space \( c \) of convergent sequences. The same results in the space \( c \) were shown in [1, Corollary 5.4, Remark 5.5].

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