RESONANCE TYPE INSTABILITIES
IN THE GASEOUS DISKS
OF THE FLAT GALAXIES

I. The Acoustical Resonance Type Instability and
the Absence of Vortex Sheet Stabilization on Shallow Water.

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Linear analysis of vortex sheet stability in the rotating gaseous disk or shallow water layer shows that presence of a central reflecting surface changes system stability significantly. An effect of absence of vortex sheet stabilization has been found as compressibility exceeds Landau criterion. The properties of multimode short-scale instability of acoustical resonance type are investigated and probability of its influence upon experiments on the rotating shallow water is discussed.

Introduction.

One of urgent problems of astrophysics is clearing of mechanisms, resulting occurrence of a wide spectrum of instabilities in differentially rotating gaseous disks. Given question is most heavily studied in connection with attempts of an explanation of an angular moment tap from accretion and protoplanet disks, as well as reasons of generation of spiral structure of flat galaxies. The important place in these researches occupies consideration of hydrodynamic instabilities, caused purely by differentiability of rotation, that is velocity kink. For non-dissipative supersonic axisymmetric currents of non-selfgravitating gas three mechanisms are known supporting these instabilities for disturbances with wave vectors without \( z \)-componenent in cylindrical coordinate system, namely centrifugal one (Morozov 1977, Morozov 1979, Fridman 1990, Nezlin & Snezhkin 1990), an interaction of modes with opposite signs of energy density and resonant waves amplification on corotation radius (Papaloizou & Pringle 1985, Glatzel 1987, Savonije & Heemskerk 1990).

At the same time, as the real astrophysical systems are characterized by significant radial density gradients, without any reasons one more version of the resonant mechanism of waves amplification is ignored — because of repeated reflection from area of the specified gradient and from a corotation circle with amplification on amplitude and constructive interference of reflected waves (i.e. actually we are dealing unstable waveguide modes). For plainparallel currents this mechanism was analyzed in many works (see, e.g., Blumen, Drazin & Billings 1975, Drazin & Davey 1977, Morozov & Mustsevoy 1991, Morozov, Mustsevaya & Mustsevoj 1991).

In the present work using extremely simplified model it is shown that the latter amplification mechanism is effective enough — so that the instability caused by it is not suppressed even by centrifugal stabilization of disturbances. Thus in considered model the role of an internal reflecting surface plays not a sharp gradient of density, but firm wall. On the one hand, this significantly simplifies the dispersion equation and its analysis, on the other hand, the results received in such model can have the direct application to analogue laboratory modelling of spiral structure of galaxies generation on installation with rotating shallow water (see Nezlin & Snezhkin 1990 and quoted there literature). In a central part of this installation a cylindrical rigid surface (protective casing, interfering penetration of a working liquid in the bearing), is located; we discuss an opportunity of influence of disturbances reflection upon stability of the vortex sheet on shallow water. The situation with reflection from density jump will be considered in the subsequent parts of the article.

1. Model and basic equations
It is investigated a dynamics of Fourier-harmonics of small disturbances in such form:

\[ f = f(r, z) \exp(i m \varphi + i \omega t), \]

where \( \omega \) is complex frequency, \( m = 0, 1, 2, \ldots \) is number of mode on an azimuth, imposed on equilibrium system parameters, representing stationary differentially rotating disk-like configuration of polytropic non-dissipative non-selfgravitating gas with equation of state \( P_0 = \rho_0 c_s^2/\gamma \) (\( c_s \) is adiabatic sound speed, \( \gamma \) is adiabatic index). The model stability is provided by radial balance of centrifugal force and gradients of pressure and external gravitational potential \( \Psi_0 \) and hydrostatic balance along \( z \)-coordinate.

We consider the solutions of equations averaging along \( z \)-coordinate (on opportunity of such approach see Part II). The equations describing dynamics of small disturbances in such system are equivalent to those in rotating shallow water accurate to replacement \( \rho \) on liquid layer depth \( H \), \( c_s \) on \( \sqrt{gH} \) and for \( \gamma = 2 \) (Fridman 1990, Landau & Lifshits 1988). Thus the structure of installation bottom respects a radial structure of gravitational potential.

The angular rotation speed gets out as \( \Omega(r) = \Omega_{in} \) at \( r < R_\Omega \), \( \Omega(r) = \Omega_{ex} \) at \( r > R_\Omega \), and firm reflecting surface situated at \( r = R_\rho \) (equivalence of a firm wall and density jump with significant gradient is shown by Morozov & Mustsevoy 1991). It is necessary to note especially that the discontinuous model was chosen quite consciously, in order to exclude from consideration all instability mechanisms excepting considered. Really, for resonant amplification in a vicinity of corotation non-zero shear layer thickness is necessary (Savonije & Heemskerk 1990), the centrifugal instability does not develop, if we put \( \Omega_{in} < \Omega_{ex} \) (Morozov 1977, Morozov 1979, Fridman 1990, Nezlin & Snezhkin 1990), the Kelvin–Helmholtz instability in this case is stabilized at \( M_s = R(\Omega_{ex} - \Omega_{in})/c_s < M_{crit} = 8^{1/2} \) (Bazdenkov & Pogutse 1983, Torgashin 1986) (the value \( M_{crit} = 8^{1/2} \) corresponds to flat vortex sheet stabilization, Landau 1944), and interaction of waves with opposite signs of energy at such \( M_s \) values in absence of a central wall or density jump results to spontaneous radiation of neutral (\( Im \omega = 0 \)) sound waves by jump fading at \( r \to 0 \) and \( r \to \infty \), and to considered instability if the wall is present.

In view of told, the system of linearized equations of gas dynamics it is possible to reduce to system (Morozov 1977):

\[ \frac{dp}{dr} = \frac{2m\Omega}{r\hat{\omega}} p + \rho_0 (\hat{\omega}^2 - \kappa^2) \xi, \]
\[ \frac{d\xi}{dr} = \left( \frac{\hat{\omega}^2}{c_s^2} - \frac{m^2}{r^2} \right) \frac{p}{\rho_0 \hat{\omega}^2} - \left( \frac{2m\Omega}{r\hat{\omega}} + \frac{1}{r} \right) \xi, \]

where \( \hat{\omega} = \omega - m\Omega \) is disturbances frequency with respect to Doppler shift, \( \kappa^2 = 2\Omega(2\Omega + rd\Omega/dr) \) is squared epicyclic frequency, \( p(r) \) and \( \xi(r) \) are peak functions of perturbed pressure and radial Lagrange displacement accordingly. The correspondence between the latter and radial component of perturbed speed is set by expression \( \tilde{v}_r = d\tilde{\xi}/dt = -i\hat{\omega}\tilde{\xi} \).

In the area of uniformity the solution of system (1), (2) is a linear combination of modified Bessel functions of complex argument of the integer order \( m \):

\[ p(r) = \begin{cases} AI_m(k_{in}r) + BK_m(k_{in}r), & R_\rho < r < R_\Omega, \\ CK_m(k_{ex}r), & r > R_{st}, \end{cases} \]

where

\[ k_i^2 = [4\Omega_i^2 - (\omega - m\Omega_i^2)]/c_s^2. \]
Here the index $i$ takes “in” and “ex” values. In (3) limitation of the solution on infinity is discounted (for uniqueness of solution choice we take $Re \ k_i > 0$ hereafter).

On a velocity jump surface boundary conditions for the solutions (Morozov 1977) should be hold:

$$
\xi(r = R_\Omega - 0) = \xi(r = R_\Omega + 0),
$$

(5)

$$
p(r = R_\Omega - 0) + \rho_0 \Omega_{in}^2 R_\Omega \xi(R_\Omega) = P(r = R_\Omega + 0) + \rho_0 \Omega_{ex}^2 R_\Omega \xi(R_\Omega),
$$

(6)

and non-flow condition $\xi(R_\rho) = 0$ on a wall surface.

Coupling the solution for areas $r < R_\Omega$ and $r > R_\Omega$ in view of conditions (5), (6) and condition at $R_\rho$ and writing out a non-flow condition of received system — the equality to zero of its determinant, — it is possible to receive the dispersion equation of considered model:

$$
\begin{vmatrix}
-\alpha^{(\Omega)}_{ex} & \alpha^{(\Omega)}_{in} & \beta^{(\Omega)}_{in} \\
-k_1^2 R_\Omega^2 + M_{ex}^2 \alpha^{(\Omega)}_{ex} & -k_1^2 R_\Omega^2 - M_{in}^2 \alpha^{(\Omega)}_{in} & -k_1^2 R_\Omega^2 - M_{in}^2 \beta^{(\Omega)}_{in} \\
0 & -\alpha^{(\rho)}_{in} & -F \beta^{(\rho)}_{in}
\end{vmatrix} = 0,
$$

(7)

where the following designations are introduced:

$$
M_i = \frac{R_\Omega \Omega_i}{c_s}, \quad F = \frac{I_m(k_i R_\rho) K_m(k_i R_\Omega)}{I_m(k_i R_\Omega) K_m(k_i R_\rho)},
$$

(8)

$$
\alpha^{(j)}_i = \frac{2m \Omega_i}{\omega - m \Omega_i} - K_i R_j \frac{K'_m(k_i R_j)}{K_m(k_i R_j)},
$$

(9)

$$
\beta^{(j)}_i = \frac{2m \Omega_i}{\omega - m \Omega_i} - K_i R_j \frac{I'_m(k_i R_j)}{I_m(k_i R_j)}.
$$

Here the prime stand for differentiation of Bessel functions on argument and top index shows that the given combination is written on radius of velocity or density jump.

The equation (7) describes permitted eigenfrequencies and the positive $Im \ \omega$ presence (growth rate) means instability.

2. The asymptotic solutions

First of all it is necessary to note, that the equation (7) is satisfied identically at frequencies $\omega_1 \equiv \Omega_{in}(m - 2)$ or $\omega_2 \equiv \Omega_{ex}(m - 2)$. These roots describe gyroscopic modes of fluctuations, having in one of media only an azimuthal component of the wave vector $k_\varphi = \frac{\omega}{\varpi}$ (as $k_{in} \equiv 0$ at $\omega \equiv \omega_1$ and $k_{ex} \equiv 0$ at $\omega \equiv \omega_2$). This cause their neutral character — a nonzero flow of wave energy on radial coordinate in both media and, hence, $k_i \neq 0$ is necessary for considered instability.

In a limit of disappeasing small radius of a wall: $D = (R_\Omega - R_\rho)/R_\Omega \rightarrow 1$, using asymptotic decomposition $I_m(z)$ and $K_m(z)$ at $z \rightarrow 0$ (Handbook of Mathematical Functions... 1964), it is possible to reduce equation (7) to a form:

$$
\alpha^{ex}_i \left[ k_{in}^2 R_\Omega^2 + M_{in}^2 \beta^{in}_i \right] - \beta^{in}_i \left[ k_{ex}^2 R_\Omega^2 + M_{ex}^2 \alpha^{ex}_i \right] \simeq 0.
$$

(10)

The equation (10) accurate to designations coincides with the dispersion equation from Morozov (1977), where a similar problem in absence of a wall was considered.

In an incompressible limit ($|k_{in} R_\Omega| \ll 1$; $|k_{ex} R_\Omega| \ll 1$) from (7) follows the solution:

$$
\omega \simeq \frac{m(\Omega_{ex} - \Omega_{in}) + (\Omega_{ex} - \Omega_{in}) + i|\Omega_{in} - \Omega_{ex}| \sqrt{(1 + m)(1 + ml)}}{1 - l},
$$

(11)
where \( l = [(1-D)^{2m}+1]/[(1-D)^{2m}-1] \), describing development of the Kelvin–Helmholtz instability. If not has places \( D \ll 1 \), (11) is reduced to asymptotic, received before (Morozov 1977):

\[
\omega \simeq \frac{1}{2} \left[ m(\Omega_{in} + \Omega_{ex}) + (\Omega_{ex} - \Omega_{in}) + i|\Omega_{in} - \Omega_{ex}| \sqrt{|m^2 - 1|} \right],
\]

(12)

With reduction \( D \) the presence of wall close to velocity jump suppresses this instability — growth rate monotonously decreases and at \( D \to 0 \) takes place \( \omega \to m\Omega_{in} + i \cdot 0 \). This result, as well as (11), (12), does not depend on velocity jump “direction” on jump (i.e. it is valid both at \( \Omega_{in} > \Omega_{ex} \) and at \( \Omega_{in} < \Omega_{ex} \)).

In applied aspect a case of essential liquid compressibility (\( |k_{in}R\Omega| \gg m; |k_{ex}R\Omega| \gg m \)) is much more interesting. Then (7) is reduced to a kind:

\[
k_{in}R\Omega = \left[ \frac{R\Omega^2(\Omega_{in}^2 - \Omega_{ex}^2)}{c_s^2} - k_{ex}R\Omega \right] \text{th}(k_{in}R\Omega D) \simeq 0.
\]

(13)

When \( D \ll 1 \) is not valid a solution, received by Morozov (1977), valid only in a case \( \Omega_{in} > \Omega_{ex} \) and describing centrifugal instability, follows from (13):

\[
\omega \simeq \frac{1}{2} \left[ m(\Omega_{in} + \Omega_{ex}) + \frac{iR\Omega(\Omega_{in}^2 - \Omega_{ex}^2)}{c_s} \right].
\]

(14)

However as against model considered by Morozov (1977), the presence at system of the acoustic screen (reflecting firm wall) makes possible higher unstable harmonics excitation (former at the flat jets stability analysis, such harmonics were called “reflective” — see, for example, Payne & Cohn 1985) besides the basic unstable mode, described in different limits by expressions (11) and (14). Really, (13) admits the solutions of a kind:

\[
\omega = \omega_0 + \delta \omega,
\]

(15)

where

\[
\delta \omega \simeq \frac{\omega_0 - m\Omega_{in}}{DR\Omega^2(\Omega_{in}^2 - \Omega_{ex}^2)/c_s^2 - 1 - iDR\Omega(\omega_0 - m\Omega_{ex})/c_s},
\]

\[
\omega_0 = m\Omega_{in} + \sqrt{4\Omega_{in}^2 + \frac{n^2\pi^2c_s^2}{D^2R\Omega^2}},
\]

\( n = 1, 2, 3, \ldots \) is harmonic number. The expression (15) is valid if \( \Omega_{in}/|\omega - m\Omega_{in}| \ll 1; \Omega_{ex}/|\omega - m\Omega_{ex}| \ll 1 \) and \( |\delta \omega|/\omega_0 \ll 1 \). As against (14) the expression (15) describes unstable roots both at \( \Omega_{in} > \Omega_{ex} \) and at \( \Omega_{in} < \Omega_{ex} \) and in essentially supersonic (\( R\Omega|\Omega_{in} - \Omega_{ex}|/c_s \gg 1 \)) case, and is similar to analogous asymptotic of reflective harmonics in a plainparallel flow (see Morozov, Mustsevaya & Mustsevoj 1991).

Finally in a strong compressibility limit again (\( |k_{in}R\Omega| \gg m; |k_{ex}R\Omega| \gg m \)) but at a backlash between jump and wall so small, that \( |k_{in}R\Omega D| \ll 1 \), solution describing another resonant mode follows from (13):

\[
\omega \simeq m\Omega_{ex} + i\sqrt{\left[ \frac{c_s}{R\Omega - R\rho} - \frac{R\Omega(\Omega_{in}^2 - \Omega_{ex}^2)}{c_s} \right]^2 - 4\Omega_{ex}^2}.
\]

(16)

It is easy to note that first term under square root sign, resulting to instability in basic, represents the inverted runtime of a sound wave between jump and wall. Second term in a
square bracket defines stabilizing or destabilizing impact of difference of centrifugal force density on jump. Last term describes stabilizing influence of gyroscopic effects.

Comparison (16) with results of Morozov & Mustsevoy (1991) and Morozov, Mustsevaya & Mustsevoj (1991) allows to conclude that (16) is long-wave (along a waveguide layer) limit for a harmonic with \( n = 1 \) described in a short-wave limit expression (15).

### 3. Numerical solution results

On Fig. 1.1–1.2 we bring dispersion curves, constructed on the data of the numerical equation (7) solution. For demonstration of considered effect we have selected a case \( m = 2, \Omega_{in} = 0 \), which is most “rigid” from the point of view of centrifugal stabilization of disturbances. With the purposes of facilitation of comparison with results of Morozov & Mustsevoy (1991) and Morozov, Mustsevaya & Mustsevoj (1991) the frequency normalized on sound \( z = \omega/(k_x c_s) = \omega R_{\Omega}/(mc_s) \) is shown on figures.

The basic results for a situation when gas between jump and wall is at rest are as follows:

1) The presence of a central reflecting surface (acoustic screen) essentially influences on system stability. If in absence of this surface the vortex sheet is stabilized at \( M > 8^{1/2} \) (Bazdenkov & Pogutse 1983, Torgashin 1986), in a considered case the instability takes place even at \( M \gg 1 \).

2) To change of parameters \( M \) and \( D \) for each azimuthal mode \( m \) there is the alternation of stability and instability areas.

3) For disturbances with small azimuthal number \( m \) the basic unstable mode (the Kelvin–Helmholtz one) is stabilized at \( M = M_* < 8^{1/2} \). \( M_* \) grows as \( m \) and at \( M \gg 1 \) the basic mode ceases to be stabilized with Mach number growth, as it takes place in case of flat vortex sheet in vicinity of a reflecting surface (see Morozov & Mustsevoy 1991). The reason of the last effect as already was specified is instability of an acoustic resonance type.

4) As well as predicts analytical equation (7) investigation, alongside with the basic mode the higher reflective unstable harmonics (see (15)) are excited and their quantity grows as \( m \).

5) Differences from a case investigated by Morozov, Mustsevaya & Mustsevoj (1991) for higher harmonics at \( m \gg 1 \) when the curvature effects are insignificant, are as small as well as for the basic mode.

In a case \( \Omega_{in} > \Omega_{ex} \) the results can be generalized as follows:

1) In an incompressible limit \( R_{\Omega}(\Omega_{in} - \Omega_{ex}) \ll c_s \) the disturbances dispersion law will be well coordinated with (11).

2) At supersonic velocity difference on jump both centrifugal and resonant modes develop.

3) The centrifugal instability mode growth rate changes with change \( D \) extremely weak in a wide range of this parameter and is described by (14) successfully; only at \( D \to 0 \) the growth rate quickly decreases up to zero.

4) Resonant modes growth rates are much less than centrifugal one and become comparable to it only at \( D \to 0 \).

### 4. Conclusions

We shall move a total by formulating the basic conclusions about an opportunity of a central reflecting surface influence on results of experiments on rotating shallow water (Nezlin & Snezhkin 1990, Morozov et al. 1984, Morozov et al. 1985):

1) The resonant instability of a considered type could not appreciably affect spiral structure centrifugal instability modeling on shallow water because for parameters of the
unit “Spiral” its development time is significantly greater. Namely, speed jump and liquid-making pipe radii ratio such that $D > 0.8$ (Nezlin & Snezhkin 1990, Morozov et al. 1984, Morozov et al. 1985).

2) In experiments on supersonic jump stabilization at speed difference growing up to $R_{\Omega}(\Omega_{in} - \Omega_{ex}) > 2\sqrt{2gH}$ (Antipov et al. 1983) the resonant instability should result unstable regime renewal at further difference increasing, as against a situation considered by Bazdenkov & Pogutse (1983) and Torgashin (1986). Thus it would be curiously to repeat experiments of Antipov et al. (1983) in a wider range of relative speed of a liquid layers. At the same time even at smooth Mach number increasing the system will quickly pass through narrow domination areas of different harmonics $n$ of various azimuthal modes $m$, which characteristic development time $\sim (R_{\Omega} - R_{\rho})/\sqrt{gH}$, so any ordered wave structures observation will be hardly possible.

The results obtained by us for a considered model situation cannot be applied directly to the analysis of stability of astrophysical objects and represent especially academic interest. However generalizing told it is necessary to specify necessity of further research of influence of a central density gradient (Zasov & Fridman 1987) on stability of galactic gas disks as the offered mechanism of disturbances exciting is rather strong.

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Fig. 1.1. Dimensionless disturbances phase velocity (a) and dimensionless growth rate (b) for $M = 5.5$, $m = 2$, $\Omega_1 = 0$. 
Fig. 1.2. Dimensionless disturbances phase velocity (continuous curves) and growth rate (dashed curves) for $D = 0.5$, $m = 2$, $\Omega_1 = 0$. 