A note on invertibility of the Dirac operator twisted with Hilbert-$A$-module coefficients

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Abstract
Given a closed connected spin manifold $M$ with non-negative scalar curvature which is non-zero, we show that the Dirac operator twisted with any flat Hilbert module bundle is invertible.

Let $M$ be a compact spin manifold with Riemannian metric $g$. It is an important and standard fact that the spectrum of the spin Dirac operator is restricted by the scalar curvature of $M$. By the Schrödinger-Lichnerowicz formula, if $\text{scal}_g(x) \geq 4c^2$ for every $x \in M$ then

$$\text{spec}(D) \cap (-c, c) = \emptyset.$$ 

This is a direct consequence of the Schrödinger-Lichnerowicz formula

$$D^2 = \nabla^* \nabla + \frac{\text{scal}_g}{4},$$

where the connection Laplacian $\nabla^* \nabla$ is a non-negative operator, and the operator of multiplication by $\frac{\text{scal}_g}{4}$ is bounded below by $c^2$.

Note that this argument works exactly the same way if we replace the classical Atiyah-Singer Dirac operator $D$ by $D_E$, where we twist with a bundle $E$ with flat connection, because the Schrödinger-Lichnerowicz formula remains unchanged. In particular, this also works for twists with a bundle $E$ of Hilbert-$A$-modules with a $C^*$-algebra $A$, such that $D_E$ is an operator in the Mishchenko-Fomenko calculus. This is important for higher index theory, then $\text{ind}(D_E) \in K_*(A)$, and the invertibility implies of course that $\text{ind}(D_E) = 0 \in K_*(A)$, with its applications to the topology of positive scalar curvature, compare e.g. [1].

There is another easy generalization of the above spectral consideration which is well known: if $M$ is connected and $\text{scal}_g(x) \geq 0$, with $\text{scal}_g(x_0) > 0$ for some $x_0 \in M$, then we can argue as follows:

For the usual untwisted Dirac operator $D$ take $s \in \ker(D)$. Then

$$0 = (Ds, Ds)_{L^2} = (\nabla S, \nabla S)_{L^2} + \int_M \frac{\text{scal}_g(x)}{4} (s(x), s(x))_x \, d\text{vol}_g.$$
The assumptions then imply that $\nabla s = 0$, i.e. $s$ is parallel and in particular $(s(x), s(x))_x$ is constant. Because by assumption $\text{scal}_g(x) > 0$ for $x$ in a neighborhood of $x_0$, this implies $s = 0$. Consequently, $\ker(D) = 0$

As $D$ has discrete spectrum, this implies again that $D$ is invertible, i.e.

$$\text{spec}(D) \cap (-\epsilon, \epsilon) = \emptyset$$

In this note, we now prove that this extends to all operators $D_E$ for flat Hilbert-$A$-module bundles $E$, even though in general the spectrum of $D_E$ is not discrete if $\dim(A) = \infty$. In that case, the argument just given does not work. It only gives $\ker(D_E) = \{0\}$, which does not imply that $D_E$ is invertible.

This question came up in the analysis of the geometry of the space of metrics of non-negative scalar curvature carried out recently in [2].

**Theorem 1.** Let $M$ be a connected closed spin manifold with Riemannian metric $g$, $A$ a $C^*$-algebra and $E \to M$ a flat Hilbert-$A$-module bundle over $M$. Assume that $\text{scal}_g(x) \geq 0$ for all $x \in M$, and $\text{scal}_g(x_0) > 0$ for some $x_0 \in M$.

Then $D_E$ is invertible, i.e. there is $c_0 > 0$ such that $\text{Spec}(D_E) \cap (-c_0, c_0) = \emptyset$.

**Proof.** Instead of arguing with $s \in \ker(D_E)$, consider for $c > 0$ a function $f_c : \mathbb{R} \to [0, 1]$ with $f_c(x) > 0$ if and only if $|x| < c$ and then $f_c(D_E)$. This is a replacement for the spectral projector $P_c := \chi_{[-c, c]}(D_E)$ which in general one can't build because the Hilbert-$A$-module morphisms don't form a von Neumann algebra. We will show that for $c$ sufficiently small, $\text{im}(f_c(D_E)) = \{0\}$. This implies that $f_c(D_E) = 0$ and, by the choice of $f_c$, that $\text{Spec}(D_E) \cap (-c, c) = \emptyset$. By contraposition, assume that $s \in \text{im}(f_c(D_E))$ with $|s|_{L^2} \neq 0$. We will show that this implies that $c > c_0$ for some $c_0 > 0$ depending on the geometry of $(M, g)$.

In the following, the norms and inner products have values in $A$, and the inequalities $a \leq b$ refer to the partial order in $A$.

First, we have $|D_E^k s|_{L^2} \leq c^k$ for all $k > 0$.

By the Sobolev embedding theorem, $s$ is smooth and there are a priori estimates (depending on the geometry of $M$) for the supremum norm of $s$ and all its covariant derivatives.

Next, the Schrödinger-Lichnerowicz formula implies

$$c^2 |s|_{L^2}^2 \geq |D_E s|_{L^2}^2 = |\nabla s|_{L^2}^2 + \int_M \frac{\text{scal}_g(x)}{4} (s(x), s(x))_x \text{dvol}_g.$$

Because $\text{scal}_g(x) \geq 0$ for all $x \in M$ this implies

$$|\nabla s|_{L^2}^2 \leq c^2 |s|_{L^2}^2 \quad \text{and} \quad \int_M \frac{\text{scal}_g(x)}{4} |s(x)|_x^2 \text{dvol}_g \leq c^2 |s|_{L^2}^2. \quad (2)$$

Choose an small open disk $U = B_{2r}(x_0)$ around $x_0$ such that $\text{scal}_g(x) \geq 4a > 0$ for $x \in U$ (note that $U$ and $a$ depend only on $g$).

We obtain then from (2)

$$\int_U |s(x)|_x^2 \leq \int_U \frac{\text{scal}_g(x)}{4a} |s(x)|_x^2 \leq \int_M \frac{\text{scal}_g(x)}{4a} |s(x)|_x^2 \leq \frac{c^2}{a} |s|_{L^2}^2. \quad (3)$$

Choose a smooth cutoff function $\phi : M \to [0, 1]$ which is equal to 1 outside $B_{2r}(x_0)$ and vanishes on $B_r(x_0)$ and consider $\tilde{s} := \phi \cdot s$. Note that there is
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$C_g > 0$ depending only on the geometry of $M$ such that

$$|\nabla \phi(x)|^2 \leq C_g \quad \forall x \in U, \quad \nabla \phi(x) = 0 \forall x \in M \setminus U.$$  \hspace{1cm} (4)

Because $M$ is connected, we can apply the Poincaré inequality [3, Proposition 5.2] (extended to sections of Hilbert $A$-module bundles) for the subset $B_r(x_0)$ inside $M$:

$$|\tilde{s}|^2_{L^2} \leq C_{g,U} |\nabla \tilde{s}|^2_{L^2} = C_{g,U} |(\nabla \phi)s + \phi \nabla s|^2_{L^2} \leq 2C_{g,U}(|(\nabla \phi)s|^2_{L^2} + |\phi \nabla s|^2_{L^2})$$

with constant $C_{g,U}$ depending only on the set $U$ and the geometry $g$ of $M$. Note finally that

$$|\tilde{s}|^2_{L^2} = \int_M |s(x)|^2_{L^2} + \int_U (\phi(x)^2 - 1)|s(x)|^2_{L^2} \geq |\tilde{s}|^2_{L^2} - \int_U |s(x)|^2_{L^2} \geq (1 - \frac{c^2}{a})|\tilde{s}|^2_{L^2},$$

while on the other hand

$$|(\nabla \phi)s|^2_{L^2} = \int_M |(\nabla \phi)x|^2 |s(x)|^2_{L^2} \leq \int_U C_g |s(x)|^2_{L^2} \leq \frac{C_g}{a} c^2 |\tilde{s}|^2_{L^2}. \hspace{1cm} (6)$$

Combining (5) with (6), (7), and (2) we finally obtain

$$(1 - \frac{c^2}{a})|\tilde{s}|^2_{L^2} \leq |\tilde{s}|^2_{L^2} \leq 2C_{g,U} \left(\frac{C_g}{a} c^2 + c^2\right) |\tilde{s}|^2_{L^2}.$$  

As $s \neq 0$ and therefore $|s|^2_{L^2} > 0$ in $A$, this implies that $c$ must be sufficiently large, explicitly

$$c > c_0 := \left(\frac{2C_{g,U} C_g + 1}{a} + 2C_{g,U}\right)^{-1/2}.$$  

Now, the constants $C_{g,U}$, $C_g$, and $a$ depend only on the geometry of $M$ and the assertion follows. \hspace{1cm} \Box

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References
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