Research Article

Abhijit Sen and Zurab K. Silagadze*

Trigonometric identities inspired by the atomic form factor

https://doi.org/10.1515/gmj-2019-2083
Received May 23, 2018; revised January 31, 2019; accepted February 20, 2019

Abstract: We prove some trigonometric identities involving Chebyshev polynomials of the second kind. The identities were inspired by atomic form factor calculations. Generalizations of these identities, if found, will help to increase the numerical stability of atomic form factor calculations for highly excited states.

Keywords: Trigonometric identities, mathematical physics, special functions

MSC 2010: 33C45, 33B10, 81U05

1 Introduction

The knowledge of discrete-discrete atomic form factors is important in computations of transition probabilities between two different atomic states when a hydrogen-like elementary atom (for example, π⁺ π⁻ or μ⁺ μ⁻) collides with an atom of target material [29]. A comprehensive review of atomic form factor calculations can be found in [15].

Recently, it has become evident that the production and study of the never before observed true muonium (dimuonium) atom are possible at modern electron-positron colliders [7, 10, 12, 28], in fixed-target experiments [6, 9, 21, 26, 35], in relativistic heavy ion collisions [19, 37], in a quark-gluon plasma [14], in elementary particle decays [17, 23–25, 27, 31], or by using ultra-slow muon beams [22, 30]. The observation of a true muonium signal from astrophysical sources is also of considerable interest [16].

As a by-product of atomic form factor calculations, which were initiated by a modern experimental proposal [11] in this field, some interesting trigonometric identities emerged, which we report in this short note.

2 Integral related to the atomic form factor

The integral

\[ I_n^m = \int_0^\infty e^{-x} \sin(\sigma x)x^m[L_n^m(x)]^2 \, dx \]  \hspace{1cm} (2.1)

arises naturally when calculating certain atomic form factors [2, 4]. Here

\[ L_n^m(x) = (n + m)! \sum_{k=0}^{n} \frac{(-1)^k}{k!(n-k)!(k+m)!} x^k \]  \hspace{1cm} (2.2)

are the associated Laguerre polynomials.

*Corresponding author: Zurab K. Silagadze, Novosibirsk State University and Budker Institute of Nuclear Physics, 630 090, Novosibirsk, Russia, e-mail: silagadze@inp.nsk.su. https://orcid.org/0000-0003-0704-4199
Abhijit Sen, Novosibirsk State University, Novosibirsk 630 090, Russia, e-mail: abhijit913@gmail.com
In [4], this integral is calculated by using the formula
\[
\int_0^\infty e^{-bx} x^n [L_n^a(\lambda x) L_m^a(\mu x)] \, dx = \frac{\Gamma(m + n + a + 1)}{\Gamma(m + 1) \Gamma(n + 1)} \frac{(b - \lambda)^n (b - \mu)^m}{b^n \mu^m - a - 1} F_1(-m, -n; -m - n - a; \frac{b(b - \lambda - \mu)}{(b - \mu)(b - \lambda)}),
\]
which can be found in [20, entry 7.414.4].

As a result, equation (2.4) can be rewritten in the form
\[
I_n^m = \frac{(2n + m)!}{n!} \cos^{m+1} \phi \sin[(2n + m + 1)\phi] F_1(-n, n + m + 1; 1; \sin^2\phi).
\]

This result can be expressed in terms of Jacobi polynomials defined by the relation
\[
P_n^{(\alpha, \beta)}(x) = \frac{(\alpha + 1)n!}{n!} F_1(-n, n + \alpha + \beta + 1; 1 + \alpha; \frac{1 - x}{2}).
\]

Finally, we obtain
\[
I_n^m = \frac{(n + m)!}{n!} \cos^{m+1} \phi \sin[(2n + m + 1)\phi] P_n^{(0, m)}(\cos 2\phi).
\]

### 3 Trigonometric identities

However, we can calculate integral (2.1) in a different way. By using Howell’s identity (see [8, 13])
\[
[I_n^m(x)]^2 = \frac{(n + m)!}{2^{2n} n!} \sum_{k=0}^n \frac{(2k)! [2(n - k)]!}{k! [(n - k)]!} I_{2k}^{2m}(2x),
\]
we get
\[
I_n^m = \frac{(n + m)!}{2^{2n} n!} \sum_{k=0}^n \frac{(2k)! [2(n - k)]!}{k! [(n - k)]!} I_{m, k},
\]
where
\[
I_{m, k} = \int_0^\infty e^{-x} x^{1/2} (\sigma x)^{x^{m+1/2} L_{2k}^{2m}(2x)} \, dx.
\]
Here $J_k(x)$ is the Bessel function of the first kind and we have used

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$$

to express $\sin(\alpha x)$ in terms of the Bessel function.

Integrals of type (3.2) were evaluated in [5] with the result

$$\int_0^{\infty} e^{-\alpha x} J_r(\mu x) x^r L_n^m(\beta x) \, dx = \sum_{k=0}^{n} \frac{(-\beta)^k \mu^r \Gamma(n + a + 1) \Gamma(v + y + k + 1)}{k! \Gamma(n - k + 1) \Gamma(\alpha + k + 1) 2^r \Gamma(\nu + 1) \sigma^n \nu^r k+1}$$

$$\times {}_2F_1\left(\frac{v + y + k + 1}{2}, \frac{v + y + k + 2}{2}; 1 + v; -\frac{\mu^2}{\sigma^2}\right).$$

By applying this general result to our integral (3.2), we get

$$I_{m,k} = \cos^{m+1} \phi \left(\begin{array}{c} 2k \\ r \end{array}\right) \frac{(-2)^r [2k + m]! [m + r]!}{r! (2k - r)! (2m + r)!} \cos^r \phi \sin[(m + r + 1)\phi],$$

where we have taken into account

$$\sum_{k=0}^{n} \frac{(-2)^k [2k + m]! [m + r]!}{r! ([n - k]!)^2 (m + k)! (2k - r)! (2m + r)!} \cos^r \phi \sin[(m + r + 1)\phi].$$

Comparing this result with (2.5), we get the following trigonometric identity:

$$\sum_{k=0}^{n} \frac{(-2)^r (k + 1)_m (2n - k)}{(m + r + 1)_m (n - k) r (2k + m + k)} \cos^r \phi U_{m+r}(\phi)$$

$$= 2^{2n} U_{2n+m}(\cos \phi) P_n^{(0,0)}(\cos 2\phi),$$

where

$$U_n(\phi) = \frac{\sin[(n + 1)\phi]}{\sin \phi}$$

are Chebyshev polynomials of the second kind.

In particular, when $m = 0$ we get

$$\sum_{k=0}^{n} \frac{(-2)^r (2n - k)}{n - k r} \cos^r \phi U_r(\cos \phi) = 2^{2n} U_{2n}(\cos \phi) P_n(\cos 2\phi).\tag{3.6}$$

Here $P_n(x) = P_n^{(0,0)}(x)$ are Legendre polynomials.

### 4 Further trigonometric identities

One more possibility to calculate (2.1) and obtain other trigonometric identities is to expand one of the Laguerre polynomials in (2.1) according to (2.2) and then use (3.3). In this way we get

$$I_m^n = \left[\frac{(n + m)!}{n!}\right]^2 \cos^{m+1} \phi \sum_{k=0}^{n} \frac{(-1)^k r (k + r + 1)_m}{(k + 1)_m (r + 1)_m}$$

$$\times \left(\begin{array}{c} n \\ k \end{array}\right) \left(\begin{array}{c} n \\ r \end{array}\right) \cos^r \phi \sin[(m + k + 1)\phi].$$
In the light of (2.5), this result implies the validity of the following trigonometric identity:

\[
\sum_{k=0}^{n} \sum_{r=0}^{m} \frac{(-1)^{k+r}(n+1)_m(k+r+1)_m}{(k+1)_m(r+1)_m} \binom{n}{k} \binom{n}{r} \binom{k+r}{k} \cos^{k+r} \varphi U_{k+r+m}(\cos \varphi) = U_{2n+m}(\cos \varphi) P_n^{(0,m)}(\cos 2\varphi).
\] (4.1)

Its special case when \( m = 0 \) can be written as follows:

\[
\sum_{k=0}^{n} \sum_{r=0}^{m} (-1)^{k+r} \binom{n}{k} \binom{n}{r} \binom{k+r}{k} \cos^{k+r} \varphi U_{k+r}(\cos \varphi) = U_{2n}(\cos \varphi) P_n(\cos 2\varphi).
\] (4.2)

Feldheim identities [18, 32] provide yet another possibility to generate further trigonometric identities. According to these identities,

\[
[L_n^m(x)]^2 = \sum_{k=0}^{2n} \sum_{r=0}^{k} (-1)^k \binom{k}{r} \binom{n+m}{n-k+r} \binom{n+m}{n-r} L_k^{2m}(x)
\]

and

\[
[L_n^m(x)]^2 = \sum_{k=0}^{2n} \sum_{r=0}^{k} (-1)^k \binom{k}{r} \binom{n+m}{n-k+r} \binom{n+m}{n-r} x^k k!
\] (4.3)

The first expansion in a similar way to that described above leads to the identity

\[
\sum_{k=0}^{2n} \sum_{r=0}^{k} (-1)^{k+s} \binom{n+1}{m+s+1}_m \binom{n}{k-r} \binom{n}{n-k+r} \binom{2m+k}{m+s}_m \cos^s \varphi U_{m+k}(\cos \varphi) = U_{2n+m}(\cos \varphi) P_n^{(0,m)}(\cos 2\varphi). \] (4.4)

The second identity (4.3) can be used in conjunction with the integral (which follows from [20, entry 6.621.1])

\[
\int_0^\infty e^{-x} J_{1/2}(\sigma x) x^{m+k+1/2} = \frac{\sigma (m+k+1)!}{2^{1/2} \Gamma(3/2)} F_1(a, a + \frac{1}{2}, \frac{3}{2}, -\sigma^2),
\]

where \( a = (m+k+2)/2 \). As a result, we end up with

\[
\sum_{k=0}^{2n} \sum_{r=0}^{k} (-1)^k \binom{k+1}{r} \binom{n+m}{n-k+r} \binom{n+m}{n-r} \cos^k \varphi U_{m+k}(\cos \varphi) = U_{2n+m}(\cos \varphi) P_n^{(0,m)}(\cos 2\varphi). \] (4.5)

The \( m = 0 \) special cases of these new identities are

\[
\sum_{k=0}^{2n} \sum_{r=0}^{k} (-1)^k \binom{n}{r} \binom{k}{k} \binom{k}{k} \cos^k \varphi U_k(\cos \varphi) = U_{2n}(\cos \varphi) P_n(\cos 2\varphi)
\] (4.6)

and

\[
\sum_{k=0}^{2n} \sum_{r=0}^{k} (-1)^k \binom{k}{r} \binom{n}{n-k+r} \binom{n}{n-r} \cos^k \varphi U_k(\cos \varphi) = U_{2n}(\cos \varphi) P_n(\cos 2\varphi).
\] (4.7)

5 Concluding remarks

Trigonometric identities (3.5), (4.1), (4.4) and (4.5), as well as their particular cases (3.6), (4.2), (4.6) and (4.7), are the main results of this study. These identities are equivalent to the statement that the integral (2.1) can be calculated in the compact form (2.5) which is very convenient for numerical evaluation because Jacobi polynomials can be calculated recursively.
While calculating more general atomic form factors, the integral (2.1) generalizes to

\[
I_n^{m,k} = \int_0^\infty e^{-x} j_k(x) x^{m+1-k} [L_n^m(x)]^2 dx,
\]

where \(j_k(x)\) is the spherical Bessel function. Note that \(j_0(x) = \sin x/x\) and, accordingly, \(I_n^{m,0} = I_n^m\). If we expand one of the Laguerre polynomials in this integral according to (2.2) and then use (3.3), we get

\[
I_n^{m,k} = \frac{2^k k! \sigma^{k+1}}{(2k+1)!} \left[ \frac{(n+m)!}{n!} \right]^2 \sum_{m_1=0}^n \sum_{m_2=0}^n \frac{(-1)^{m_1+m_2} (m_1 + m_2 + 1)!}{(m_1 + 1)! (m_2 + 1)!} \times \left( \binom{n}{m_1} \binom{n}{m_2} \right) \left( \binom{m_1 + m_2}{m_1} \right) (m + m_1 + m_2 + 1) \, _2F_1 \left( a, a + \frac{1}{2}; k + \frac{3}{2}; -\sigma^2 \right),
\]

(5.1)

where \(a = 1 + (m + m_1 + m_2)/2\). In principle, such kind of expressions can be used in atomic transition form factor evaluation, and the general formulas for the atomic transition form factor along these lines were found in [2–4]. The calculations were based on the linearization formulas for the product of two Laguerre polynomials [33]. The final results were expressed in terms of either Jacobi polynomials [2, 4] or generalized Gegenbauer polynomials [3] and their computation was implemented by using the corresponding recurrence relations. However, due to numerical problems related to a near cancellation of large numbers in the alternating sum, in practice this method gives reliable results for all form factors only for relatively small principal quantum numbers \(n \sim 10\) (see [34]).

Calculations of the atomic form factors in [2–4] were motivated by the needs of the DIRAC experiment [1]. Accordingly, the authors of [2–4] never had any numerical problems in the computation of the specified atomic form factors because such computation was required only for the values of the principal quantum number \(n \leq n_{\text{max}} \sim 7 - 10\) (see [3]). Anyway, for the DIRAC experiment a more pressing issue of increasing the accuracy of the Monte-Carlo simulation of the passage of dimesoatoms through the matter was not the inclusion of the highly excited states, but the inclusion of interference effects between different dimesoatomic states during their passage through the matter [36].

Nevertheless, it will be of practical interest if the results of this note can be generalized for sums of type (5.1). Such a generalization, if found, will allow increasing the numerical stability of atomic form factor calculations for highly excited states for which the direct computation by the existing methods might fail [15]. The trigonometric identities obtained in this note are not of direct use in studies of the true muonium and other elementary atoms. However, the method by which they originated allows an alternative numerical algorithm for the calculation of atomic form factors and its realization will be useful in various correctness tests of computer programs designed for atomic form factor calculations.

Acknowledgment: The authors thank the anonymous referee for constructive remarks.

Funding: This work is supported by the Ministry of Education and Science of the Russian Federation and in part by RFBR grant 20-02-00697-a.

References

[1] L. Afanasyev, Last results of dirac experiment on study hadronic hydrogen-like atoms at ps cern, Nucl. Part. Phys. Proc. 273–275 (2016), 1997–2002.

[2] L. Afanasyev and A. Tarasov, Elastic form factors of hydrogenlike atoms in ns-states, JINR Preprint E4-93-293, 1993; also in the book http://www1.jinr.ru/Books/book_Tarasov.pdf, 90–93.

[3] L. Afanasyev and A. Tarasov, Breakup of relativistic \(n^n\) atoms in matter, Phys. At. Nucl. 59 (1996), 2130–2136.

[4] L. Afanasyev, A. Tarasov and O. Voskresenskaya, Total interaction cross sections of relativistic \(n^n\) - atoms with ordinary atoms in the eikonal approach, J. Phys. G 25 (1999), no. 8, B7–B10.

[5] R. S. Alassar, H. A. Mavromatis and S. A. Sofianos, A new integral involving the product of Bessel functions and associated Laguerre polynomials, Acta Appl. Math. 100 (2008), no. 3, 263–267.
[6] N. Arteaga-Romero, C. Carimalo and V. Serbo, Production of bound triplet $\mu^+\mu^−$ system in collisions of electrons with atoms, Phys. Rev. D 35 (1987), no. 7, 2124–2129.

[7] V. Baier and V. Synakh, Bimuonium production in electron-positron collisions, Sov. Phys. JETP 14 (1962), no. 5, 1122–1125.

[8] W. Bailey, On the product of two laguerre polynomials, Quart. J. Math. 10 (1939), no. 1, 60–66.

[9] A. Banburski and P. Schuster, Production and discovery of true muonium in fixed-target experiments, Phys. Rev. D 86 (2012), no. 9, Article ID 093007.

[10] S. Bilenky, V. Nguyen, L. Nemenov and F. Tkebuchava, Production and decay of $\mu^+\mu^−$-atoms, Yad. Fiz. 10 (1969), 812–814.

[11] A. Bogomyagkov, V. Druzhinin, E. Levichev, A. Milstein and S. Sinyatkin, Low-energy electron-positron collider to search and study $\mu^+\mu^−$ bound state, EPJ Web Conf. 181 (2018), Article ID 01032.

[12] S. Brodsky and R. Lebed, Production of the smallest qed atom: True muonium ($\mu^+\mu^−$), Phys. Rev. Lett. 102 (2009), no. 21, Article ID 213401.

[13] L. Carlitz, On the product of two Laguerre polynomials, J. Lond. Math. Soc. 36 (1961), 399–402.

[14] Y. Chen and P. Zhuang, Dimuonium $\mu^+\mu^−$ production in a quark-gluon plasma, preprint (2012), https://arxiv.org/abs/1204.4389.

[15] I. Ginzburg, U. Jentschura, S. Karshenboim, F. Krauss, V. Serbo and G. Soff, Production of bound $\mu^+\mu^−$ systems in relativistic heavy ion collisions, Phys. Rev. C 58 (1998), no. 6, 3565–3573.

[16] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products, 4th ed., Academic Press, New York, 1965.

[17] E. Feldheim, Expansions and integral-transforms for products of Laguerre and Hermite polynomials, J. Math. Oxford Ser. 11 (1940), 18–29.

[18] I. Ginzburg, U. Jentschura, S. Karshenboim, F. Krauss, V. Serbo and G. Soff, Production of bound $\mu^+\mu^−$ systems in relativistic heavy ion collisions, Phys. Rev. C 58 (1998), no. 6, 3565–3573.

[19] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products, 4th ed., Academic Press, New York, 1965.

[20] E. Holvik and H. Olsen, Creation of relativistic fermionium in collisions of electrons with atoms, Phys. Rev. A 62 (2000), no. 3, Article ID 032501.

[21] T. Itahashi, H. Sakamoto, A. Sato and K. Takahisa, Low energy muon apparatus for true muonium production, JPS Conf. Proc. 8 (2015), Article ID 025004.

[22] Y. Ji and H. Lamm, Discovring true muonium in $k_1 → (\mu^+\mu^−)γ$, Phys. Rev. D 98 (2018), no. 5, Article ID 053008.

[23] Y. Ji and H. Lamm, Scouring meson decays for true muonium, preprint (2018), https://arxiv.org/abs/1810.00233.

[24] Y. Ji and H. Lamm, Discovering true muonium in $k_1 → (\mu^+\mu^−)γ$, Phys. Rev. D 98 (2018), no. 5, Article ID 053008.

[25] Y. Ji and H. Lamm, Scouring meson decays for true muonium, preprint (2018), https://arxiv.org/abs/1810.00233.

[26] G. Kozlov, On the problem of production of relativistic lepton bound states in the decays of light mesons, Sov. J. Nucl. Phys. 48 (1988), 167–171.

[27] P. Krachkov and A. Milstein, High-energy $\mu^+\mu^−$ electroproduction, Nucl. Phys. A 971 (2018), 71–82.

[28] H. Lamm and Y. Ji, Predicting and discovering true muonium ($\mu^+\mu^−$), EPJ Web Conf. 181 (2018), Article ID 01016.

[29] J. Moffat, Does a heavy positronium atom exist?, Phys. Rev. Lett. 35 (1975), no. 24, 1605–1606.

[30] S. Mroczynski, Interaction of elementary atoms with matter, Phys. Rev. https://arxiv.org/abs/1204.438933 (1986), no. 3, 1549–1555.

[31] K. Nagamine, Past, present and future of ultra-slow muons, JPS Conf. Proc. 2 (2014), Article ID 010001.

[32] L. Nemenov, Atomic decays of elementary particles, Yad. Fiz. 15 (1972), 1047–1050.

[33] B. S. Popov and H. M. Srivastava, Linearization of a product of two polynomials of different orthogonal systems, Facta Univ. Ser. Math. Inform. (2003), no. 18, 1–8.

[34] A. P. Prudnikov, Yu. A. Brychkov and O. I. Marichev, Integrals and Series. Special Functions, “Nauka”, Moscow, 1983.

[35] C. S. Rios and J. S. Silva, An implementation of atomic form factors, Comp. Phys. Commun. 151 (2003), no. 1, 79–88.

[36] K. Sakimoto, Theoretical study of true-muonium $\mu^+\mu^−$ formation in muon collision processes $\mu^− + \mu^+ e^−$ and $\mu^+ + p\mu^−$, Eur. Phys. J. D 69 (2015), DOI 10.1140/epjd/e2015-60427-6276.

[37] O. Voskresenskaya, A density-matrix kinetic equation describing the passage of fast atomic systems through matter, J. Phys. B At. Mol. Opt. Phys. 36 (2003), no. 15, 3293–3302.

[38] G. M. Yu and Y. D. Li, Photoproduction of large transverse momentum dimuonium $\mu^+\mu^−$ in relativistic heavy ion collisions, Chin. Phys. Lett. 30 (2013), no. 1, Article ID 011201.