Classification of solitons for pluriclosed flow on complex surfaces

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Abstract
We give a classification of compact solitons for the pluriclosed flow on complex surfaces. First, by exploiting results from the Kodaira classification of surfaces, we show that the complex surface underlying a soliton must be Kähler except for the possibility of steady solitons on minimal Hopf surfaces. Then, we construct steady solitons on all class 1 Hopf surfaces by exploiting a symmetry ansatz.

1 Introduction

The pluriclosed flow is a geometric evolution equation generalizing the Kähler–Ricci flow to more general complex, non-Kähler manifolds. As shown in [32, Theorem 1.1], this flow is gauge equivalent to a certain natural coupling of the Ricci flow and the heat flow for a closed three-form first appearing in mathematical physics, namely

\[
\frac{\partial}{\partial t} g_{ij} = -2 R c_{ij} + \frac{1}{2} H_{i}^{pq} H_{jpq}, \\
\frac{\partial}{\partial t} H = \Delta_{d} H.
\]  

(1.1)

We will refer to this system of equations informally as “generalized Ricci flow.” As shown in [25, Proposition 3.1], generalized Ricci flow admits a Perelman-type energy monotonicity formula, and is in fact the gradient flow of a certain Schrödinger operator. This indicates that, as in the case of Ricci flow, soliton solutions of (1.1) should be expected as long time limits and singularity models for this flow. The steady gradient soliton equations implied by the gradient formulation take the form
\[
\begin{align*}
\text{Rc} - \frac{1}{4}H^2 + \nabla^2 f &= 0, \\
\frac{1}{2}d^* H + i_1 \nabla_f H &= 0.
\end{align*}
\] (1.2)

Thus a fundamental step in understanding the regularity and long time behavior the pluriclosed flow is to understand the existence and uniqueness of solutions to this system of equations.

The first step is to understand fixed points of the pluriclosed flow, where \( f \) above is a constant function. On the diagonal Hopf surfaces, there is a well-known metric which is a non-Kähler fixed point of pluriclosed flow, which is in fact the unique non-Kähler fixed point on compact complex surfaces up to scaling (cf. Sect. 2). To address genuine, non-trivial soliton solutions, first recall the fundamental fact that any compact steady soliton for the Ricci flow is automatically Einstein [16, Proposition 1], [26, §2.4]. However, the Bianchi type identities/monotonicity formulae behind these proofs do not immediately generalize to the case of pluriclosed flow, and thus one is left with the possibility that nontrivial compact steady solitons may exist. Moreover, natural conjectures on the pluriclosed flow loosely suggest the existence of such objects. In particular, the main existence conjecture for pluriclosed flow [32, Conjecture 5.2] suggests that, on minimal Hopf surfaces, the flow exists on \([0, \infty)\) and is nonsingular. If true, a standard argument using the Perelman \( \mathcal{F} \)-functional referenced above would then imply convergence to a compact steady soliton on such surfaces.

The main result of this paper confirms this expectation, and gives a nearly complete classification of compact pluriclosed solitons on complex surfaces.

**Theorem 1.1** Let \((M^4, J)\) be a compact complex surface. Suppose \((g, f)\) is a pluriclosed soliton on \((M, J)\).

1. If \((g, f)\) is an expanding soliton, then \((M, J)\) is Kähler, \(f \equiv \text{const}\) and \(g\) is the Kähler-Einstein metric.
2. If \((g, f)\) is a shrinking soliton, then \((M, J)\) is Kähler and \((g, f)\) is a Kähler–Ricci shrinking soliton.
3. If \((g, f)\) is a steady soliton, then either
   (a) \((M, J)\) is Kähler, \(f \equiv \text{const}\) and \(g\) is a Calabi–Yau metric, or
   (b) \((M, J)\) is biholomorphic to a minimal Hopf surface.

Furthermore, on minimal Hopf surfaces of class 1, there exists a nontrivial pluriclosed steady soliton.

**Remark 1.2**

1. The rigidity of expanding solitons follows from a Perelman-type expanding entropy formula for generalized Ricci flow, as already observed in [32, Corollary 6.11]. In fact this rigidity holds in any dimension, and moreover any expanding soliton of generalized Ricci flow (i.e. not necessarily pluriclosed) must satisfy \(H \equiv 0\) with the underlying metric Einstein [28, Proposition 6.4].
2. In the shrinking case there is a complete picture of the existence and uniqueness following from a long series of works in Kähler geometry. The complex surface underlying a Kähler–Ricci soliton must be Fano, and so for complex surfaces must be either \( \mathbb{CP}^1 \times \mathbb{CP}^1 \) or \( \mathbb{CP}^2 \# k \mathbb{CP}^2 \), \(1 \leq k \leq 8\). The spaces \( \mathbb{CP}^1 \times \mathbb{CP}^1 \) and \( \mathbb{CP}^2 \).
admit natural Kähler-Einstein metrics coming from their realization as symmetric spaces. The existence of Kähler-Einstein metrics on blowups of $\mathbb{CP}^2$ for the cases $k = 3, 4$ was established by Tian–Yau [37], and for $5 \leq k \leq 8$ by Tian [36]. The cases $k = 1, 2$ cannot admit Kähler-Einstein metrics due to Matsushima’s obstruction [24], but do admit Kähler–Ricci solitons. By exploiting a codimension 1 symmetry reduction, Koiso [21] established the existence of a soliton in the case $k = 1$. Exploiting toric symmetry, Wang–Zhu [41] constructed a soliton in the case $k = 2$. Uniqueness of these solitons was established by Tian–Zhu [38].

(3) The construction of steady solitons on Hopf surfaces of class 1 leaves open the question of existence on Hopf surfaces of class 0. While the loose argument given above suggesting the existence of steady solitons applies in principle to these surfaces, it is possible that a “jumping” phenomenon occurs for pluriclosed flow, where the convergence in Cheeger–Gromov topology allows for the biholomorphism type of the complex structure to change in the limit. As it is known that Hopf surfaces of class 1 occur as the central fiber in a family of complex surfaces, all other fibers of which all biholomorphic to the same class 0 surface, this possibility could certainly occur. Furthermore, we do not address the question of uniqueness. This question is likely difficult, since for instance the proof of uniqueness of Kähler–Ricci solitons [38,43] requires a number of delicate a priori estimates which are difficult to generalize to this setting.

The proof of Theorem 1.1 breaks into two phases. First we establish the rigidity of shrinking solitons, as well as the restriction of the possible biholomorphism types of steady solitons. These require exhibiting a priori structural results for solitons, and then comparing this structure against results from the Kodaira classification of surfaces. An important initial step is to identify a natural holomorphic vector field associated to a steady pluriclosed soliton. As is well-known, for Kähler–Ricci solitons, the gradient vector field associated to the soliton function is holomorphic. This follows directly from the soliton equation for the Riemannian metric together with pointwise identities exploiting the Kähler condition (see Proposition 3.2). Due to the weaker integrability of pluriclosed metrics, we cannot expect the same behavior for pluriclosed solitons. Rather, as we show in Sect. 3, the vector field $\theta^\sharp + \nabla f$, where $\theta^\sharp$ is the Lee vector field (see Sect. 2), will always be a nontrivial holomorphic vector field, unless the metric is already Kähler-Einstein. Another piece of input is a Bochner argument showing that pluriclosed solitons on Kähler manifolds are automatically Kähler–Ricci solitons. With these tools in hand, we exploit deep results from the Kodaira classification of surfaces to rule out backgrounds other than minimal Hopf surfaces.

The second phase of the proof is a nearly explicit construction of solitons on class 1 Hopf surfaces. To begin we use the correspondence between Sasakian 3-manifolds and Hermitian surfaces first introduced by Vaisman [39]. This in particular allows us to construct pluriclosed metrics with a pair of holomorphic Killing fields. We note here that this construction was originally used to construct locally conformally Kähler metrics on complex surfaces [3,13], and while it is natural to imagine that such metrics play a distinguished role for pluriclosed flow, this does not seem to be the case (cf. Remark 5.6). Nonetheless, the pluriclosed flow will preserve invariance by holomorphic vector fields, and thus it is natural to look for solitons in this ansatz. The next
step is to reduce the flow and soliton equations in this setting to equations depending only on the directions transverse to the foliation generated by the holomorphic Killing fields.\(^1\)

We complete the construction by constructing solutions in this symmetry class. Note that, as described so far, the construction is of cohomogeneity 2, and thus one would expect to be faced with solving a PDE. However, a crucial extra symmetry arises in this setting which allows for a further reduction. It is by now a well-known fact [4, p. 241], [5] that the shrinking Ricci soliton equations on Riemann surfaces acquire a natural holomorphic isometry generated by \(J\nabla f\), where \(f\) denotes the soliton function. This feature persists to our setting (Proposition 6.1), which allows us to reduce to an ordinary differential equation in a single parameter generated by \(\nabla f\).

In the case of elliptic Hopf surfaces, the isometry \(J\nabla f\) corresponds to the rotational symmetry of the base orbifold, which is either a so-called “teardrop” or “football” (see Sect. 4.1). In the general, irregular case, the vector field \(J\nabla f\) is a certain explicit holomorphic vector field which is transverse to the underlying Sasaki structure. Hence the problem is now reduced to one parameter, and so we are faced with solving a certain nonlinear system of ODE. Through a series of estimates we produce the necessary solutions as well as a qualitative picture of their behavior, finishing the existence proof.

**Remark 1.3** Our construction is closely related to some classic constructions for Ricci flow. In his study of the Ricci flow on surfaces Hamilton [14] investigated Ricci solitons on surfaces, showing that there are no compact shrinking solitons other than the round metric on \(S^2\). He also mentions nontrivial solitons which exist on orbifolds. A further analysis of these resulting ODEs yielded the existence of Sasaki–Ricci solitons on 3-manifolds in [40]. These connections are however thematic, and our analysis is distinct from that underlying Ricci or Sasaki–Ricci solitons.

We also note that Theorem 1.1 provides an interesting conceptual distinction between generalized Ricci flow and Ricci flow. Theorem 1.1 provides examples of nontrivial gradient steady solitons for the generalized Ricci flow in dimension 4, and by taking products with tori yields such structures in all dimensions \(n \geq 4\). Interestingly, a careful examination of the our construction reveals a product structure, yielding an example of a three-dimensional soliton as well. Thus we obtain:

**Corollary 1.4** There exist nontrivial compact generalized steady solitons in all dimensions \(n \geq 3\).

Here is an outline of the paper. We begin in Sect. 2 with a discussion of relevant background material. In Sect. 3 we prove the first part of Theorem 1.1, restricting the class of complex surfaces which can admit solitons to minimal Hopf surfaces. In Sect. 4 we review the basic correspondence between Sasakian 3-manifolds and associated Hermitian manifolds. Next in Sect. 4.2 we make a more general investigation of pluriclosed metrics on complex surfaces admitting a pair of holomorphic Killing fields. We reduce the pluriclosed flow and soliton equations in this invariant setting in Sect. 5. Lastly, in Sect. 6 we finish the existence proof through a detailed ODE analysis.

\(^1\) Surprisingly, these flow equations reduce to a natural coupling of Ricci flow and Yang–Mills flow (Ricci–Yang–Mills flow) introduced by the author [33] and Young [42] (see Remark 5.4).
2 Background

In this section we provide some background on pluriclosed flow, referring the reader to [30,32] for more details. First, given a complex manifold \((M^{2n}, J)\), a Hermitian metric \(g\) is called pluriclosed if its associated Kähler form \(\omega\) satisfies \(i\partial \bar{\partial} \omega = 0\). Associated to a pluriclosed metric is the Bismut connection, defined by

\[
\nabla^B = \nabla^{LC} + \frac{1}{2} g^{-1} d^c \omega,
\]

where \(\nabla^{LC}\) denotes the Levi-Civita connection and \(d^c = i (\bar{\partial} - \partial)\). This is the unique Hermitian connection whose torsion is skew symmetric, and in this case also closed since \(\omega\) is pluriclosed. Let \(\Omega^B\) denote the curvature of this connection, and let \(\rho_B(X, Y) = \frac{1}{2} \Omega(X, Y, e_i, J e_i)\) denote the Bismut–Ricci form. By general theory \(d \rho_B = 0\), and \(\rho_B \in \Lambda^{1,1}\). However, unlike the Chern–Ricci form, \(\rho_B \notin \Lambda^{1,1}\), and we let \(\rho_B^{1,1}\) denote the \((1, 1)\) piece. The pluriclosed flow equation takes the form

\[
\frac{\partial}{\partial t} \omega = -\rho_B^{1,1}.
\]

As shown in [30, Theorem 1.2], this is a parabolic equation for \(\omega\) which admits short time solutions on compact manifolds.

Connecting the pluriclosed flow to the system (1.1) requires a nontrivial gauge transformation. Associated to a Hermitian metric we have the Lee form \(\theta = d^* \omega \circ J\). Let \(\omega_t\) denote a solution to pluriclosed flow, let \(g_t\) denote the one parameter family of associated metrics, and \(H_t = d^c \omega_t\) the one parameter family of associated torsion tensors. Let \(\phi_t\) denote the one-parameter family of diffeomorphisms generated by \(\theta^\#_t\). Then [32, Theorem 6.5] yields that \((\phi_t^* g_t, \phi_t^* H_t)\) is a solution to (1.1), up to rescaling time. We record one key curvature identity behind this theorem which is relevant to what follows:

**Proposition 2.1** (cf. [32, Theorem 6.5]) Let \((M^{2n}, g, J)\) be a complex manifold with pluriclosed metric \(g\). Then

\[
\rho_B^{1,1}(J \cdot, \cdot) = \text{Rc} - \frac{1}{4} H^2 + \frac{1}{2} L_{\theta^\#} g.
\]

As a final important introductory remark, we note that there is a classification of fixed points of pluriclosed flow on complex surfaces. As it happens there is only one non-Kähler example which we now describe. Define a metric on \(\mathbb{C}^2 \setminus \{0, 0\}\) via

\[
g_{\text{Hopf}} = \frac{g_E}{|z_1|^2 + |z_2|^2},
\]

where \(g_E\) denotes the standard Euclidean metric on \(\mathbb{C}^2\). Note that the metric \(g_{\text{Hopf}}\) is isometric to the metric \(dr^2 \oplus g_{S^3}^1\) on \(\mathbb{R} \times S^3\), with the \(\mathbb{R}\) factor spanned by the radial
direction. Furthermore, $g_{\text{Hopf}}$ is compatible with the standard complex structure, and direct calculations show that it is pluriclosed, and moreover $\rho^{1,1}_{B}(g_{\text{Hopf}}) = 0$. It is also invariant under actions $(z_1, z_2) \to (\alpha z_1, \beta z_2)$, $|\alpha| = |\beta|$, thus yielding a well-defined metric on diagonal Hopf surfaces (cf. Sect. 4.1 for this terminology).

These turns out to be the only compact non-Kähler fixed points of pluriclosed flow. To show this classification, one first exploits a Bochner argument [12, Theorem 2] to show that for any Hermitian surface with $\rho^{1,1}_{B} = 0$, the Lee form is parallel. If the Lee form vanishes, the metric is Calabi–Yau, and if not, the induced deRham splitting shows that the universal cover is isometric, up to scaling, to $\mathbb{R} \times S^3$ with the standard product metric. One still has to identify the complex structure, and further work of Gauduchon (cf. [11, III Lemma 11]) shows that the only possible underlying complex surfaces are the standard Hopf surfaces, as claimed, and so the metrics as described above are the only examples. See [31] for more detail.

3 Classification of complex surfaces admitting solitons

In this section we give a classification of the possible complex surfaces admitting pluriclosed soliton structures. To begin we define the correct notion of pluriclosed soliton, which is delicate due to the fact that it is related to the generalized Ricci flow via a nontrivial, and moreover non-gradient gauge transformation. With this definition in place we show that a compact pluriclosed soliton on a surface is either Kähler-Einstein, or non-Kähler with the associated vector field $\theta^\# + \nabla f$ being a nontrivial holomorphic vector field. With this in place we apply results from the Kodaira classification of surfaces to show that in the non-Kähler case the underlying complex surface must be a Hopf surface.

3.1 Basic definitions

The Ricci soliton equation is in part justified as the critical point equation for Perelman’s $\lambda$-functional, defined for arbitrary Riemannian metrics. Thus, in the Kähler setting, a priori one only expects the Ricci soliton PDE (cf. Definition 3.1) to hold for the Riemannian metric, i.e. not necessarily the Kähler form. However, by exploiting the Kähler condition, an elementary, well-known argument (cf. Proposition 3.2) shows that the gradient of the soliton function $f$ is automatically a holomorphic vector field, which also implies the Kähler form version of the soliton equation. We include this simple argument for convenience as it indicates why an elementary adaptation to the pluriclosed flow setting is not possible.

**Definition 3.1** A Kähler–Ricci soliton is a Kähler manifold $(M^{2n}, g, J)$ together with $\lambda \in \{-1, 0, 1\}$ and a vector field $X$ such that

$$Rc - \lambda g = L_X g.$$
We say that it is *expanding, steady, or shrinking* according to the cases $\lambda = -1, 0, 1$ respectively. Finally, we say that the soliton is *gradient* if there exists $f \in C^\infty(M)$ such that $X = \frac{1}{2} \nabla f$.

**Proposition 3.2** Let $(M^{2n}, g, J, \lambda, f)$ be a gradient Kähler–Ricci soliton. Then

$$\rho - \lambda \omega = L \frac{1}{2} \nabla f \omega,$$

and $\frac{1}{2} \nabla f$ is the real part of a holomorphic vector field.

**Proof** As the Ricci tensor of a Kähler metric is $(1, 1)$, it follows from the soliton equation that $(\nabla^2 f)_{ij} + (\nabla^2 f)_{pq} J_p J_q = 0$. Hence the Hessian is $(1, 1)$, and we can moreover compute that

$$(\nabla^2 f)^{1,1}(\cdot, J \cdot)_{ij} = \frac{1}{2} \left[ (\nabla^2 f)_{ik} + (\nabla^2 f)_{pq} J_p J_q \right] J^k_j$$

$$= \frac{1}{2} \left[ J^k_j (\nabla^2 f)_{ik} - (\nabla^2 f)_{pj} J^p_i \right]$$

$$= \frac{1}{2} \left[ \nabla_i (J^k_j \nabla_k f) - \nabla_j (J^k_i \nabla_k f) \right]$$

$$= -\frac{1}{2} (\partial^c f)_{ij}.$$  

On the other hand we can compute, using the Cartan formula and that $d \omega = 0$,

$$L \frac{1}{2} \nabla f \omega = di \frac{1}{2} \nabla f \omega + i \frac{1}{2} \nabla f d \omega = di \frac{1}{2} \nabla f \omega.$$  

Finally we also have

$$(i \nabla f \omega)_j = \nabla^i f \omega_{ij} = g^{ip} d_p f g_{iq} J^q_j = -(\partial^c f)_j.$$  

Thus (3.1) follows. Now note the equations above imply

$$L \frac{1}{2} \nabla f \omega = (L \frac{1}{2} \nabla f g)(\cdot, J \cdot) = L \frac{1}{2} \nabla f \omega - g(\cdot, L \frac{1}{2} \nabla f J \cdot),$$

and hence $L \frac{1}{2} \nabla f J \equiv 0$, i.e. $\nabla f$ the real part of a holomorphic vector field. \qed

We note that the conclusion of (3.1), together with the holomorphicity of $\nabla f$, is what is taken as the definition of Kähler–Ricci soliton in for instance [38] in the shrinking case. Proposition 3.2 (which adapts easily to the non-gradient soliton case), shows that these points of view are equivalent. We emphasize the point of view of the underlying PDE for the Riemannian metric as it more easily adapts to the pluriclosed case, which differs in a subtle way. As explained in Sect. 2, it is necessary to apply a nontrivial gauge transformation to the pluriclosed flow to yield a solution of (1.1), and it is that system of equations which is the gradient flow of a modified Perelman functional ([25], cf. [28]). Thus self-similar solutions to pluriclosed flow must satisfy the critical point equation for this modified Perelman functional, and this then determines our definition of soliton below.
Definition 3.3 Let $(M^{2n}, J)$ be a complex manifold. A pluriclosed soliton is a pluriclosed metric $g$ together with $\lambda \in \{-1, 0, 1\}$ and a vector field $X$ such that

\[
\begin{align*}
Rc - \frac{1}{4}H^2 - \lambda g &= L_X g, \\
\frac{1}{2}d^*H - \lambda b &= i_X H,
\end{align*}
\]

(3.2)

where $\lambda(H - db) = 0$ (this condition encodes the fact that pluriclosed flow preserves the cohomology class of $H$, and thus if it is to evolve by homothetic shrinking or expansion, the underlying cohomology class must be zero). We say that it is expanding, steady, or shrinking according to the cases $\lambda = -1, 0, 1$ respectively. Finally, we say that the soliton is gradient if there exists $f \in C^\infty(M)$ such that $X = \frac{1}{2} \nabla f$.

Crucially, as the underlying metric is only pluriclosed, it does not follow from direct local calculations as in Proposition 3.2 that the vector field $\nabla f$ is automatically holomorphic in the case of gradient solitons. Nonetheless by applying classification results of Pontecorvo [27] on metrics compatible with several complex structures, we are able to show that a compact soliton on a complex surface, it is either Kähler-Einstein or the associated vector field $\theta^\sharp + \nabla f$ is nontrivial and holomorphic.

Proposition 3.4 Let $(M^4, g, J)$ be a compact pluriclosed gradient steady soliton. Then either

1. $(M^4, g, J)$ is hyperHermitian, i.e. it is biholomorphically isometric to a flat torus, K3 surface with Calabi–Yau metric, or a hyperHermitian Hopf surface.
2. The vector field $\theta^\sharp + \nabla f$ is a nontrivial holomorphic vector field on $M$.

Proof Let $(\tilde{g}_t, \tilde{H}_t)$ denote the unique solution to (1.1) with initial condition $(g, H)$. Let $\psi_t$ denote the one-parameter family of diffeomorphisms generated by $-\nabla_{\tilde{g}_t} f$. By a standard argument using the soliton equations (3.2) we know that $(\tilde{g}_t, \tilde{H}_t) = (\psi_t^* g, \psi_t^* H)$. On the other hand, let $g_t$ be the solution to pluriclosed flow with the given initial condition, with associated torsion $H_t = d^c \omega_t$. Let $\phi_t$ be the one-parameter family of diffeomorphisms generated by $\theta^\sharp_t$. It follows from [32, Theorem 6.5], that $(\phi_t^* g_t, \phi_t^* H_t)$ is the unique solution to (1.1) with initial condition $(g, H)$. Thus

\[
(\phi_t^* g_t, \phi_t^* H_t) = (\tilde{g}_t, \tilde{H}_t) = (\psi_t^* g, \psi_t^* H),
\]

and hence, setting $\Psi_t = \phi_t \circ \psi_t^{-1}$, we see that $(\Psi_t^* g_t, \Psi_t^* H_t) = (g, H)$. Observe that since $g_t$ is a pluriclosed metric with respect to $J$, it is in particular compatible with $J$, and hence $g = \Psi_t^* g_t$ is compatible with $\Psi_t^* J$. In particular, we have shown that the metric $g$ is compatible with a (possibly trivial) one-parameter family of complex structures. If the family of complex structures is not stationary, then this yields a continuous family of complex structures compatible with $g$. It follows from [27, Theorem 5.5] that the metric is hyperHermitian, in particular it is either a flat torus, a K3 surface with Calabi–Yau metric or a hyperHermitian Hopf surface.

2 We thank the referee for pointing out that it follows in general dimension that this vector field is holomorphic by a pointwise computation.
If this family is stationary, then by definition the generating vector field $\theta^\sharp + \nabla f$ is holomorphic. If the vector field is trivial, then one has $\theta = -df$. It follows from the conformal transformation law for the Lee form (cf. [10, §I.13]) that $\theta_{e^f g} = \theta_g + df = 0$. In particular, the conformally related Hermitian metric $e^f g$ is Kähler. But by the uniqueness of the Gauduchon metric in a fixed conformal class [10, Main Theorem], it follows that, after possibly modifying $f$ by a constant, $g = e^f g$ and so $f \equiv 0$. This means that the metric $g$ is Kähler, and Ricci flat, and we have reverted to the first case.

3.2 Classification of compact steady solitons

Proposition 3.5 Let $(M^{2n}, J)$ be a compact Kähler manifold, and suppose $g$ is a pluriclosed steady or shrinking soliton on $M$. Then $g$ is a gradient Kähler–Ricci soliton.

Proof We first address the case of a steady soliton. First we construct a particular 1-form reduction of pluriclosed flow as in [29, §3 and §4]. As the background manifold is Kähler, by a short argument (cf. [1, Proposition 6.1]) using Hodge theory and the result of Demailly–Paun [6], there exists $\alpha \in \Lambda^{1,0}$ and a Kähler metric $\tilde{\omega}$ such that $\omega = \tilde{\omega} + \partial \bar{\alpha} + \partial \alpha$. Next, fix an arbitrary Hermitian metric $h$, and for short time fix a background metric for the flow $\tilde{\omega}_t = \tilde{\omega} - t \rho_C(h)$. Note that in the notation of [29] we have $\mu = 0$. We then apply [29, Lemma 3.2] to construct a 1-parameter family of $(1, 0)$-forms $\alpha_t$ which satisfy

$$\frac{\partial}{\partial t} \alpha = \bar{\partial}^* \omega_t - \frac{1}{2} \partial \log \frac{\det g_t}{\det h}.$$ 

A straightforward calculation shows then that $\omega_t = \tilde{\omega} + \partial \bar{\alpha}_t + \partial \alpha_t$ is the given solution to pluriclosed flow. Furthermore, since $\partial \tilde{\omega}_t = 0$ for all $t$, we can apply [29, Proposition 4.10] with $\eta = 0$ to conclude that $\partial \alpha$ satisfies the evolution equation

$$\left( \frac{\partial}{\partial t} - \Delta^C_{g_t} \right) |\partial \alpha|^2 \leq - \left| T_{g_t} \right|^2,$$ 

(3.3)

where $T_{g_t}$ denotes the torsion of the Chern connection of the evolving metric. But since the solution is a soliton, and hence evolves purely by diffeomorphism, there exists a vector field $X$ such that $\frac{\partial}{\partial t} |\partial \alpha|^2 = X |\partial \alpha|^2$. Thus $|\partial \alpha|^2$ is a subsolution of a strictly elliptic equation with zero constant term, and it follows from the strong maximum principle that it is constant. It thus follows that $| T_{g_t} |^2 = 0$, and so the metric is Kähler, and hence a Kähler–Ricci soliton, as claimed.

The case of a shrinking soliton is essentially the same. We make choices of $\tilde{\omega}$ and $h$ as above, and this time set $\tilde{\omega}_t = \tilde{\omega} - t (\rho_C(h) - \tilde{\omega})$. Following the arguments of [29, §3] it is straightforward to construct a one-parameter family of $(1, 0)$ forms which this time satisfy the 1-form reduction of normalized pluriclosed flow, i.e.

$$\frac{\partial}{\partial t} \alpha = \bar{\partial}^* \omega_t - \frac{1}{2} \partial \log \frac{\det g_t}{\det h} + \alpha.$$
One also easily checks then that then the 1-parameter family of metrics \( \tilde{\omega}_t + \tilde{\partial}_\alpha + \tilde{\partial}_t \) is a solution of

\[
\frac{\partial}{\partial t} \omega = -\rho_B^{1,1} + \omega,
\]

the normalized pluriclosed flow. Furthermore, an elementary adaptation of [29, Proposition 4.10] shows that \( \tilde{\partial}_\alpha \) will still satisfy (3.3). The reason no normalization terms are present is due to the fact that \( |\tilde{\partial}_\alpha|_g^2 \) has zero homogeneity in terms of the metric scaling. As the solution to normalized pluriclosed flow follows a one-parameter family of diffeomorphisms, one can argue using (3.3) exactly as above in the steady case to conclude that \( \omega_t \) is Kähler, and hence a Kähler–Ricci soliton, which is automatically gradient [26].

Proposition 3.6 The following hold:

1. A shrinking pluriclosed soliton \((M^4, J, g)\) is a shrinking gradient Kähler–Ricci soliton.
2. A steady pluriclosed soliton \((M^4, J, g)\) satisfies either
   (a) \((M^4, J, g)\) is Calabi–Yau, and \( f \equiv \text{const} \), or
   (b) \((M^4, J)\) is biholomorphic to a Hopf surface.

Proof If the underlying surface \((M^4, J)\) is Kähler, Proposition (3.5) implies that \( g \) is a gradient Kähler–Ricci soliton (of course Calabi–Yau in the steady case), fitting into cases (1) and (2a) above. Thus we may assume that \((M^4, J)\) is non-Kähler. We begin with the fundamental identity

\[
\text{tr}_{\omega_B^{1,1}} \rho_B^{1,1} = \text{tr}_g \left( \text{Rc} - \frac{1}{4} H^2 + L_{\theta^g} g \right)
= \text{tr}_g \left( L_{\theta^g + X} g + \lambda g \right)
= 2 \text{ div } X + 4\lambda,
\]

since the Lee form is divergence free. But also by [15, Proposition 3.3]

\[
\text{tr}_{\omega_B^{1,1}} \rho_B^{1,1} = s_C - |T|^2,
\]

where \( s_C \) is the scalar curvature of the Chern connection. Thus

\[
c_1 \cdot [\omega] = \int_M s_C \omega^2 = \int_M (|T|^2 + 2 \text{ div } X + 4\lambda) \, dV_g = \int_M |T|^2 \, dV_g + 4\lambda \text{ Vol}(g) > 0,
\]

(3.4)

where the inequality is strict since \( \lambda \geq 0 \) and the metric is not Kähler. It follows from Gauduchon’s plurigenera theorem [9,10] that \( p_m = 0 \) for all \( m > 0 \). Hence \( \text{Kod}(M) = -\infty \), and \((M, J)\) is a Class VII surface.

Next we observe that, since the pluriclosed flow with initial condition \( g \) evolves by diffeomorphism pullback by a family of biholomorphisms and scaling by \( \lambda \), using
Proposition 3.4 one obtains, using that \( s_C = \text{tr}_\omega \rho_C \), where \( \rho_C \) is the Chern–Ricci curvature,
\[
-\lambda = \frac{\partial}{\partial t} \int_M s_C \omega^2 = \frac{\partial}{\partial t} \int_M \rho_C \wedge \omega = -c_1^2.
\] (3.5)

Now note that it follows from [7, Theorem 1.8] that for Class VII surfaces with \( b_2 > 0 \), one has \( c_1^2 = -b_2 \). Since \( \lambda \geq 0 \) this violates (3.5), and hence we conclude \( b_2 = 0 \). This implies moreover that \((M^4, J)\) is minimal, and then it follows that \( c_1^2 \leq 0 \) (cf. [2, §VI]). Thus (3.5) now forces both \( \lambda = 0 \) and \( c_1^2 = 0 \). Note that this has now ruled out the possibility of a shrinking soliton on a non-Kähler surface, finishing the proof of claim (1). To determine the biholomorphism type in the steady case, first note that by the classification of Class VII surfaces with \( b_2 = 0 \) [22,35], \((M, J)\) is either a Hopf surface or an Inoue surface. It was shown in [34, Remark 4.2] that \( c_1 \cdot [\omega] < 0 \) for all metrics on Inoue surfaces, and so these cannot occur by (3.4). Hence \((M^4, J)\) is biholomorphic to a minimal Hopf surface, as claimed. \( \Box \)

4 Invariant geometry on Hermitian cylinders

We now begin our construction of steady solitons on class 1 Hopf surfaces. The construction builds upon a fundamental observation of Vaisman [39] which exhibits a link between three-dimensional Sasakian structures and locally conformally Kähler metrics with parallel Lee form. This shows the existence of pluriclosed metrics admitting two holomorphic Killing fields on certain Hopf surfaces. We then derive formulas for the curvature and torsion of such invariant metrics, which will be used in the next section to reduce the pluriclosed flow and the gradient steady soliton equations.

4.1 Hermitian cylinders

In this section we will briefly recall fundamental notions of Sasaki geometry, Vaisman’s construction, and discuss the relationship between the underlying Sasaki structure and the resulting complex surface.

Definition 4.1 A Sasakian manifold consists of a triple \((M, g, Z)\) where \( g \) is a Riemannian metric and \( Z \) is a unit length Killing field with respect to \( g \), such that \( I \in \text{End}(TM) \) defined via \( I(X) := \nabla_X Z \) satisfies
\[
(\nabla_X I)(Y) = g(Z, Y)X - g(X, Y)Z.
\] (4.1)

Associated to a Sasakian manifold we define \( \eta = Z^b \), which is easily seen to satisfy.
\[
\eta(Z) = 1, \quad d\eta(Z, X) = 0.
\]
The kernel of $\eta$ is the \textit{transverse distribution}, which we will denote by $Q$, which comes equipped with a projection map

$$\pi^T(X) := X - \eta(X)Z.\quad (4.2)$$

**Proposition 4.2** Let $(M, g, Z, \eta, I)$ be a Sasakian manifold. Then

1. $I^2 Y = -Y + \eta(Y)Z$,
2. $IZ = 0$, $\eta(IY) = 0$,
3. $g(X, IY) - g(IX, Y) = 0$, $g(IY, IX) = g(Y, X) - \eta(Y)\eta(X)$,
4. $d\eta(Y, X) = 2g(IY, X)$,
5. $L_Z I \equiv 0$.

**Definition 4.3** Let $(M, g, Z)$ be a Sasakian manifold. The \textit{transverse metric} is defined by

$$g^T(X, Y) = \frac{1}{2}d\eta(X, IY).$$

Similarly, define the \textit{transverse Kähler form} by

$$\omega^T(X, Y) = -\frac{1}{2}d\eta(X, Y).$$

The reason for the factor $\frac{1}{2}$ in both formulas is explained by Proposition 4.2 (4), whereas the sign above is in keeping with the convention that the metric and Kähler form of a Hermitian structure satisfy $\omega(X, Y) = g(X, JY)$. The terminology “transverse” arises from the fact that $g^T$ defines a positive definite metric on the distribution orthogonal to $Z$, whereas $g^T(Z, X) = 0$.

**Definition 4.4** Given $(N, g, Z)$ a Sasakian three-manifold, there is an associated Hermitian manifold $(M, h, J)$ defined as follows. Let $M \cong N \times \mathbb{R}$, where we parameterize $\mathbb{R}$ with coordinate $t$, and set $W = \frac{\partial}{\partial t}$. Choose the metric $h = \pi_1^*g + \pi_2^*dt^2$, and define $J$ via (recall $Q$ denotes the transverse distribution)

$$J_{|Q} = I_{|Q}, \quad J(Z) = W.$$ 

We will refer to this Hermitian cylinder as a \textit{Sasaki-type complex surface}. Moreover, we will refer to a tensor field $\mu$ on such a surface as \textit{invariant} if

$$L_Z \mu = L_W \mu = 0.$$ 

**Proposition 4.5** Given a Sasakian three-manifold $(N, g, Z)$ the triple $(M, h, J)$ of Definition 4.4 satisfies:

1. The transverse projection map is holomorphic, i.e.

$$\pi^T \circ J = I \pi^T.$$ 

2. $(M, h, J)$ is indeed Hermitian

\[ Springer \]
(3) The Kähler form associated to \((h, J)\) satisfies
\[
\omega_h = -\frac{1}{2} d\eta - dt \wedge \eta.
\]

(4) The tensors \(J, h\) and \(\omega_h\), are all invariant.

The cylinder construction above can yield many different complex surfaces, but we are here only interested in Hopf surfaces. To recall, a Hopf surface is a compact complex surface whose universal covering space is \(\mathbb{C}^2 - \{(0, 0)\}\). A Hopf surface is primary if \(\pi_1(M) = \mathbb{Z}\), and is otherwise secondary. For primary Hopf surfaces, Kodaira [19, 20] shows that the fundamental group is generated by a map \(\gamma\) defined by
\[
(z_1, z_2) \rightarrow (\alpha z_1, \beta z_2 + \lambda z_1^m),
\]
where \(\alpha, \beta, \lambda\) are complex numbers satisfying \(0 < |\alpha| \leq |\beta| < 1\) and
\[
(\alpha - \beta^m)\lambda = 0.
\]
We say that a Hopf surface is of class 1 if \(\lambda = 0\), otherwise, it is class 0. Furthermore, we say the surface is diagonal if \(\lambda = 0\) and \(\alpha = \beta\).

We require explicit information on the construction of LCK metrics on Hopf surfaces, thus we briefly recall some elements of [3, §5], where this is carried out. First we determine the holomorphic vector fields on Hopf surfaces. To begin, express the generic holomorphic vector field on \(\mathbb{C}^2 - \{(0, 0)\}\) as \(W = A(z_1, z_2)\partial_{z_1} + B(z_1, z_2)\partial_{z_2}\), where \(A\) and \(B\) are holomorphic functions. By Hartogs’ Theorem \(A\) and \(B\) extend to \(\mathbb{C}^2\), and can be expressed as convergent power series. To see which vector fields descend to the quotient, we must check invariance under the contraction (4.3). For class 1 Hopf surfaces, i.e. \(\lambda = 0\), power series computations show that the general form of \(A\) and \(B\) for a \(\gamma\)-invariant vector field is
\[
\begin{align*}
A(z_1, z_2) &= az_1 + bz_2, \quad B(z_1, z_2) = cz_1 + dz_2, \quad a, b, c, d \in \mathbb{C}, \quad \alpha = \beta \\
A(z_1, z_2) &= az_1 + cz_2^m, \quad B(z_1, z_2) = bz_2, \quad a, b, c \in \mathbb{C}, \quad \alpha^m = \beta \\
A(z_1, z_2) &= az_1, \quad B(z_1, z_2) = bz_2, \quad a, b \in \mathbb{C}, \quad \alpha \neq \beta
\end{align*}
\]
As explained in [3, Proposition 7], the vector field \(W\) playing the role of the lift of the parallel Lee vector field must satisfy the condition that \(JW\) has relatively compact orbits in \(\mathbb{C}^2 - \{(0, 0)\}\). By analyzing the orbits of \((1, 0)\) and \((0, 1)\) and comparing against (4.4), Belgun shows [3, Proposition 8] that the relevant vector fields are
\[
W = \text{Re} \left\{ \ln |\alpha| z_1 \partial_{z_1} + \ln |\beta| z_2 \partial_{z_2} \right\} = \frac{1}{2} \ln |\alpha| (x_1 \partial_{x_1} + y_1 \partial_{y_1}) + \frac{1}{2} \ln |\beta| (x_2 \partial_{x_2} + y_2 \partial_{y_2}).
\]
Of course then one has
\[
Z = -JW = \frac{1}{2} \ln |\alpha| (y_1 \partial_{x_1} - x_1 \partial_{y_1}) + \frac{1}{2} \ln |\beta| (y_2 \partial_{x_2} - x_2 \partial_{y_2}).
\]
As this $Z$ is the Reeb vector field of the associated Sasakian structure, we can derive
the associated contact form. First, for notational simplicity let

$$a = \frac{1}{2} \ln |\alpha|, \quad b = \frac{1}{2} \ln |\beta|, \quad \sigma = a |z_1|^2 + b |z_2|^2.$$ 

It follows from elementary calculations that the associated contact form is

$$\eta_{\alpha, \beta} = \frac{1}{\sigma} \left( x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2 \right).$$

In our construction we also require a certain basis for the contact distribution. To that
end set

$$E_1 = |z_2|^2 \left( y_1 \partial_{x_1} - x_1 \partial_{y_1} \right) - |z_1|^2 \left( y_2 \partial_{x_2} - x_2 \partial_{y_2} \right)$$

$$E_2 = J E_1 = |z_2|^2 \left( x_1 \partial_{x_1} + y_1 \partial_{y_1} \right) - |z_1|^2 \left( x_2 \partial_{x_2} + y_2 \partial_{y_2} \right).$$

Again straightforward calculations yield

$$E_i \in \ker \eta_{\alpha, \beta}, \quad [E_i, Z] = 0.$$ 

Furthermore, there is a natural trichotomy in this construction which will inform
our construction. If the orbits of the Reeb vector field of a Sasaki structure are all
compact, then it generates a circle action. If this action is free, the Sasakian structure
is called regular, and is called quasi-regular if the action is locally free. In case
there is a noncompact orbit of the Reeb vector field, the Sasaki structure is called
irregular. In our setting the only regular Sasaki structure has Reeb vector field tangent
to the standard Hopf action on $S^3$. The corresponding complex surface is the primary
diagonal Hopf surface. Furthermore in this setting the quotient space for the circle
action is a smooth manifold, specifically $\mathbb{C}P^1$. Quasi-regular Sasaki structures arise
when one has $\alpha^m = \beta^n$. Here the corresponding complex surfaces are known as elliptic
Hopf surfaces, and the quotient space has the structure of a bad orbifold, in particular
is biholomorphic to one of the classic “teardrop” or “football” orbifolds (see Fig. 1).
For elliptic Hopf surfaces we can solve for steady solitons on this base orbifold, and
aspects of the construction have clear geometric meaning on this space. In the case

Fig. 1  Bad orbifolds
of Hopf surfaces corresponding to irregular Sasaki structures one is forced to work entirely on the total space. As a final remark here we emphasize that Hopf surfaces of class 0 do not appear via this construction, as exhibited in [3, Proposition 10]. It remains an open question whether class 0 Hopf surfaces admit steady pluriclosed solitons or not, arising from some other construction.

4.2 Invariant metrics on Hermitian cylinders

Despite the link between Sasakian and Hermitian geometry discussed above, we should not expect the pluriclosed flow equation to preserve any underlying connection to Sasakian geometry, due to the extra integrability conditions. However, by standard arguments we obtain that invariance by vector fields is preserved by the flow (see Proposition 5.1). In this section we thus investigate complex manifolds which arise via the Hermitian cylinder construction, equipped with metrics which are invariant, but not necessarily coming from the construction of Sect. 4. As we will see, the local geometry of invariant metrics is identical to that of a Kaluza–Klein type metric on a principal bundle. Of course there is not necessarily a smooth quotient space for the action, and so we must work only on the total space.

Given this one might wonder why we bother to pass through Sasaki geometry in the first place. The reason is that one cannot apply standard averaging arguments to obtain an invariant metric in general for all of the complex surfaces under consideration, especially those corresponding to an irregular Sasaki structure, since the vector fields are not tangent to the action by a compact Lie group. In other words, the construction of any invariant metric at all necessitates the construction via Sasakian geometry. We work within the class of invariant metrics to produce a soliton.

Definition 4.6 Given a Sasaki-type complex surface \( (M^4, J) \), Consider the Lie algebra \( t^2 \) with basis \( \{\xi_1, \xi_2\} \) and complex structure \( J_{t^2} \xi_1 = \xi_2 \). We say that a form \( \mu \in \Lambda^1 \otimes t^2 \) is Hermitian connection if, expressing

\[
\mu(X) = \mu^Z(X)\xi_1 + \mu^W(X)\xi_2,
\]

one has

1. \( \mu^Z(Z) \equiv \mu^W(W) \equiv 1, \mu^Z(W) \equiv \mu^W(Z) \equiv 0, \)
2. \( \mu(JX) = J_{t^2}\mu(X), \)
3. \( L_Z\mu \equiv L_T\mu \equiv 0. \)

We will refer to the subbundle \( \mathcal{V} = \langle V, W \rangle \subset TM \) as the vertical space. Given a Hermitian connection form \( \mu \) there is an associated complementary horizontal space defined by \( \mathcal{H} = \ker \mu \). Observe furthermore that condition (2), expanded out, says

\[
\mu^Z(JX)\xi_1 + \mu^W(JX)\xi_2 = \mu(JX) = J_{t^2}\mu(X) = J_{t^2}(\mu^Z(X)\xi_1 + \mu^W(X)\xi_2) = -\mu^W(X)\xi_1 + \mu^Z(X)\xi_2,
\]

and so

\[
\mu^Z(JX) = -\mu^W(X), \quad \mu^W(JX) = \mu^Z(X).
\]

(4.6)
Also, \( \mu \) defines natural projection operators
\[
\pi_{\mathcal{V}}(X) := \mu^Z(X)Z + \mu^W(X)W, \quad \pi_{\mathcal{H}}(X) := X - \pi_{\mathcal{V}}(X).
\]
Lastly, we endow \( t^2 \) with the unique metric for which \( \{\xi_1, \xi_2\} \) is an orthonormal basis, denoted \( \langle , \rangle \). In particular note that
\[
\langle \mu(X), \mu(Y) \rangle = \mu^Z(X)(\mu^Z(Y) + \mu^W(X)\mu^W(Y)).
\] (4.7)

**Lemma 4.7** Given a Sasaki-type complex surface \((M^4, J)\) and a Hermitian connection form, one has
\[
[V, V] \subset \mathcal{V}, \quad [V, H] \subset \mathcal{H}, \quad J V = V, \quad J H = H.
\]

**Proof** Since \( V \) is spanned by \( Z \) and \( W \), which commute, the first inclusion follows immediately, as does \( J V = V \). To show \( [V, H] \subset \mathcal{H} \), we observe that for \( X \in \mathcal{H} \),
\[
\mu([Z, X]) = \mu(L_Z X) = L_Z(\mu X) - (L_Z \mu)X = 0.
\]
To show that \( J H = H \), since \( J \) is invertible it suffices to show that \( J \mathcal{H} \subset \mathcal{H} \). Suppose that there existed \( X \in \mathcal{H} \) such that \( \pi_{\mathcal{V}} J X \neq 0 \). Then since \( J \) is an invertible endomorphism on \( V \), it follows that \( 0 \neq \pi_{\mathcal{V}} J J X = -\pi_{\mathcal{V}} X \), a contradiction. \( \square \)

**Proposition 4.8** Given a Sasaki-type complex surface \((M^4, J)\), an invariant Hermitian metric \( g \) is equivalent to an invariant triple \((g^T, \mu, \psi)\) where \( g^T \) is a transverse Hermitian metric, \( \mu \) is a Hermitian connection form, and \( \psi \in C^\infty(M) \) is a positive function.

**Proof** This is essentially the standard argument decomposing an invariant metric on a principal bundle, just without the existence of a smooth quotient space. First, given a triple \((g^T, \mu, \psi)\) as in the statement, we recover \( g \) as
\[
g(X, Y) = g^T(X, Y) + \psi \langle \mu(X), \mu(Y) \rangle.
\]
To obtain the decomposition given an invariant metric \( g \), first observe using the Hermitian property that we may define
\[
\psi := g(Z, Z) = g(W, W),
\] (4.8)
which is positive since \( g \) is positive definite. We check that \( \psi \) is invariant using the Leibniz rule for Lie derivatives to obtain
\[
L_Z \psi = L_Z g(Z, Z) = (L_Z g)(Z, Z) + 2g([Z, Z], Z) = 0.
\]
Similarly \( L_W \psi = 0 \).

Next we define a Hermitian connection form via
\[
\mu^Z(X) := \psi^{-1} g(Z, X), \quad \mu^W(X) := \psi^{-1} g(W, X). \tag{4.9}
\]

We check the algebraic conditions for \(\mu\) to define a Hermitian connection form. First, it follows directly from (4.8) and (4.9) that \(\mu^Z(Z) \equiv \mu^W(W) \equiv 1\). Since \(g\) is Hermitian we conclude
\[
\mu^Z(W) \equiv \psi^{-1} g(Z, W) \equiv \psi^{-1} g(Z, JZ) \equiv 0,
\]
and similarly \(\mu^W(Z) \equiv 0\), finishing condition (1). To check condition (2), we choose any vector \(X \in TM\) and then compute
\[
\mu(JX) = \mu^Z(JX)\xi_1 + \mu^W(JX)\xi_2
\]
\[
= \psi^{-1} g(Z, JX)\xi_1 + \psi^{-1} g(W, JX)\xi_2
\]
\[
= -\psi^{-1} g(W, X)\xi_1 + \psi^{-1} g(Z, X)\xi_2
\]
\[
= \mu^W(X)\xi_1 - \mu^Z(X)\xi_2
\]
\[
= Jt^2 \mu(X).
\]

To check the invariance, we build upon the invariance of \(\psi\) to obtain
\[
(L_Z \mu^Z)(X) = \psi^{-1} \left( L_Z (\mu^Z(X)) - \mu^Z(L_Z X) \right)
\]
\[
= \psi^{-1} (L_Z g(Z, X) - g(Z, [Z, X]))
\]
\[
= 0.
\]
The invariance by \(W\) is proved similarly.
Lastly we define
\[
g^T(X, Y) = g(X, Y) - \psi \langle \mu(X), \mu(Y) \rangle.
\]
The invariance follows directly from the previous invariance claims. Next we check that it is indeed transverse, i.e.
\[
g^T(X, Z) = g(X, Z) - \psi \langle \mu(X), \mu(Z) \rangle
\]
\[
= \psi \mu^Z(X) - \psi \mu^Z(X)
\]
\[
= 0.
\]
We also check that \(g^T\) is a \((1, 1)\)-tensor. To that end,
\[
g^T(JX, JY) = g(JX, JY) - \psi \langle \mu(JX), \mu(JY) \rangle
\]
\[
= g(X, Y) - \psi \langle Jt^2 \mu(X), Jt^2 \mu(Y) \rangle
\]
\[
= g^T(X, Y).
\]
\[\square\]
Definition 4.9 Given a Sasaki-type complex surface \((M^4, J)\) and an invariant Hermitian metric \(g\), define the \textit{transverse Kähler form} via

\[\omega^T(X, Y) = g^T(X, JY).\]  

(4.10)

Similarly define the \textit{vertical metric} via

\[g^V(X, Y) = g(X, Y) - g^T(X, Y) = \psi \langle \mu(X), \mu(Y) \rangle,\]

with associated vertical Kähler form

\[\omega^V(X, Y) = g(X, JY) - g^T(X, JY) = \psi(-\mu^Z(X)\mu^W(Y) + \mu^W(X)\mu^Z(Y)) = \psi \mu^W \wedge \mu^Z(X, Y).\]  

(4.11)

4.3 Torsion of invariant metrics

Here we investigate the structure of the torsion of invariant Hermitian metrics. The principal observation, deduced in Lemma 4.11, is that such a metric is pluriclosed if and only if the fiber length function \(\psi\) is constant. We begin with the definition of the curvature of a Hermitian connection form.

Definition 4.10 Given a Sasaki-type complex surface \((M^4, J)\) and a Hermitian connection form \(\mu\), the curvature \(F \in \Lambda^2(M) \otimes t^2\) is

\[F = d\mu = F^Z\xi_1 + F^W\xi_2.\]

Notice that, as sections of \(\Lambda^2(\mathcal{H}^*)\), both \(F^Z\) and \(F^W\) are type \((1, 1)\) since \(\mathcal{H}\) is of real rank 2.

Lemma 4.11 Given a Sasaki-type complex surface \((M^4, J)\), an invariant Hermitian metric \(g\) satisfies

1. \(d\omega^T = 0\),
2. \(d\omega^V = d\psi \wedge \mu^W \wedge \mu^Z + \psi d\mu^W \wedge \mu^Z - \psi \mu^W \wedge d\mu^Z\),
3. \(d^c\omega_g = -d^c\psi \wedge \mu^Z \wedge \mu^W + \psi d\mu^W \wedge \mu^W + \psi \mu^Z \wedge d\mu^Z\),
4. \(dd^c\omega_g = \frac{dd^c\psi}{\psi} \wedge \omega_g\).

In particular, \(g\) is pluriclosed if and only if \(\psi \equiv \text{const}\). In this case the Lee form is

\[\theta = \text{tr}_\omega F^Z \mu^W - \text{tr}_\omega F^W \mu^Z\]  

(4.12)

Proof For item (1), first note that since both \(\mathcal{H}\) and \(\mathcal{V}\) are rank 2, evaluating any three form purely on vectors of one type or the other yields zero. Now choose \(X, Y \in \mathcal{H}\) and \(Z \in \mathcal{V}\) we obtain

\[d\omega^T(X, Y, Z) = X\omega^T(Y, Z) + Y\omega^T(Z, X) + Z\omega^T(X, Y) - \omega^T([X, Y], Z) + \omega^T([X, Z], Y) - \omega^T([Y, Z], X)\]
\[ Z \omega^T (X, Y) + \omega^T ([X, Z], Y) - \omega^T ([Y, Z], X) \]
\[ = L_Z (\omega^T (X, Y)) - \omega^T (L_Z X, Y) - \omega^T (X, L_Z Y) \]
\[ = (L_Z \omega^T) (X, Y) + \omega^T (X, L_Z Y) - \omega^T (X, L_Z Y) \]
\[ = 0, \]
as required. Similarly, choosing \( X \in \mathcal{H} \) and \( Z, W \in \mathcal{V} \) we compute, using that \( \mathcal{V} \) is integrable,
\[ d \omega^T (X, Z, W) \]
\[ = X \omega^T (Z, W) + W \omega^T (X, Z) + Z \omega^T (W, X) \]
\[ - \omega^T ([X, Z], W) + \omega^T ([X, W], Z) - \omega^T ([Z, W], X) \]
\[ = 0, \]
and so \( d \omega^T = 0. \)

We compute \( d \omega^V \) using (4.11) to immediately yield
\[ d \omega^V = d(\psi \mu^W \wedge \mu^Z) = d \psi \wedge \mu^W \wedge \mu^Z + \psi d \mu^W \wedge \mu^Z - \psi \mu^W \wedge d \mu^Z, \]
as claimed.

Next, we compute, using (1), (2), (4.6), and the fact that \( F \) is of type \((1, 1)\),
\[ d^c \omega_g = -d \omega_g (J, J, J) \]
\[ = -d \omega^V (J, J, J) \]
\[ = - \left\{ (d \psi \circ J) \wedge \mu^Z \wedge (-\mu^W) + \psi \mu^W \wedge (-\mu^W) - \psi (\mu^Z) \wedge d \mu^Z \right\} \]
\[ = -d^c \psi \wedge \mu^Z \wedge \mu^W + \psi \mu^W \wedge \mu^W + \psi \mu^Z \wedge d \mu^Z, \]
as claimed.

Differentiating again we obtain, using that \( d \mu^Z, d \mu^W \) and \( \psi \) are invariant and comparing against (4.11),
\[ dd^c \omega_g = d \left[ -d^c \psi \wedge \mu^Z \wedge \mu^W + \psi \mu^W \wedge \mu^W - \psi \mu^Z \wedge d \mu^Z \right] \]
\[ = -dd^c \psi \wedge \mu^Z \wedge \mu^W \]
\[ = \frac{dd^c \psi}{\psi} \wedge \omega^V \]
\[ = \frac{dd^c \psi}{\psi} \wedge \omega_g, \]
where the last line follows because \( dd^c \psi \) is horizontal, and hence \( dd^c \psi \wedge \omega^T = 0. \) This finishes (4).

Using this formula and multiplying by \( \psi > 0 \), we see that \( dd^c \omega_g = 0 \) if and only if \( 0 = dd^c \psi \wedge \omega_g = (\text{tr}_\omega 1 \delta \delta \psi) \omega_g^2 \), if and only if \( \Delta_C \psi = 0. \) It thus follows from a standard maximum principle argument that \( dd^c \omega_g = 0 \) if and only if \( \psi \) is constant,
as claimed. In case $\psi$ is constant, line (4.12) follows using the formula $d\omega = \theta \wedge \omega$ together with items (1) and (2).

\[ \square \]

### 4.4 Bismut curvature of invariant metrics

In this subsection we establish formulas for the Bismut curvature of an invariant pluriclosed metric on a Sasaki-type complex surface. We begin with a basic lemma producing a frame canonically associated to any point.

**Lemma 4.12** Given a pluriclosed invariant Hermitian metric $g$ on a Sasaki-type complex surface $(M^4, J)$, for each $p \in M$ there exist local coordinate vector fields $\{ \partial_{x_1}, \partial_{x_2} \}$ such that

$$\mu(\partial_{x_i})(p) = 0, \quad J\partial_{x_1} = \partial_{x_2}, \quad [\partial_{x_i}, Z] = [\partial_{x_i}, W] = 0.$$  

Furthermore, the vector fields defined by

$$e_i = \partial_{x_i} - \mu Z(\partial_{x_i}) Z - \mu W(\partial_{x_i}) W,$$

satisfy

$$\text{span}\{ e_1, e_2 \} = \mathcal{H}, \quad Je_1 = e_2.$$

Furthermore one has

$$[e_1, e_2] = -FZ(\partial_{x_1}, \partial_{x_2}) Z - FW(\partial_{x_1}, \partial_{x_2}) W, \quad [e_i, Z] = [e_i, W] = 0.$$

**Proof** Given $p \in M$, we fix a Hermitian basis for $\mathcal{H}_p$, then extend this using $\{ Z_p, W_p \}$ to yield a Hermitian basis for $TM_p$. By standard arguments this basis can be extended locally to a complex coordinate frame for $TM$, with the extended basis including $\{ V, W \}$ locally. Choosing the vector fields associated to the initially spanning vectors for $\mathcal{H}_p$ yields $\{ \partial_{x_1}, \partial_{x_2} \}$ which by construction satisfy the three claimed conditions.

It is clear by construction that $e_1, e_2$ are horizontal and linearly independent, hence span $\mathcal{H}$. Moreover, we note using (4.6) that

$$Je_1 = J \left( \partial_{x_1} - \mu Z(\partial_{x_1}) Z - \mu W(\partial_{x_1}) W \right)$$

$$= \partial_{x_2} - \mu Z(\partial_{x_1}) W + \mu W(\partial_{x_1}) Z$$

$$= \partial_{x_2} + \mu Z(J\partial_{x_2}) W - \mu W(J\partial_{x_2}) Z$$

$$= \partial_{x_2} - \mu Z(\partial_{x_2}) Z - \mu W(\partial_{x_2}) W$$

$$= e_2.$$

Now we compute the commutators. Note using invariance of the connection that

$$Z\mu(\partial_{x_i}) = (L_Z\mu)(\partial_{x_i}) + \mu([Z, \partial_{x_i}]) = 0,$$
and similarly for $W$. Thus

$$[e_1, e_2] = \left[ \partial_{x_1} - \mu^Z(\partial_{x_1})Z - \mu^W(\partial_{x_1})W, \partial_{x_2} - \mu^Z(\partial_{x_2})Z - \mu^W(\partial_{x_2})W \right]$$

$$= \left\{ \partial_{x_2} \mu^Z(\partial_{x_1}) - \partial_{x_1} \mu^Z(\partial_{x_2}) + \partial_{x_2} \mu^W(\partial_{x_1}) - \partial_{x_1} \mu^W(\partial_{x_2}) \right\} W$$

$$= -F^Z(\partial_{x_1}, \partial_{x_2})Z - F^W(\partial_{x_1}, \partial_{x_2})W.$$

The remaining vanishing claims are immediate. 

**Remark 4.13** Without further explicitly invoking Lemma 4.12, given a pluriclosed invariant Hermitian metric $g$ we will henceforth ask for “an adapted frame” at a point $p \in (M^4, J)$ a Sasaki-type complex surface. This will mean the vector fields $\{e_i\}$ constructed in Lemma 4.12, augmented with $\{Z, W\}$ to yield a local frame. In computations we will use lowercase Roman letters to refer to the vectors $\{e_i\}$, and Greek letters $\{e_\alpha\}, \alpha = 1, 2$, to refer to the vectors $\{V, W\}$. Furthermore we use uppercase Roman letters to indicate a general element of the overall frame.

Lastly, we note that in the construction of Lemma 4.12, by a linear change of coordinates on $\{e_1, e_2\}$ in the transverse direction, we can assume without loss of generality that $e_i g(e_j, e_k)(p) = 0$. Using this it is easy to see that one then has $e_A g(e_B, e_C)(p) = 0$, so that all first derivatives of the metric with respect to the frame vanish at $p$.

**Lemma 4.14** Given a pluriclosed invariant Hermitian metric $g$ on a Sasaki-type complex surface $(M^4, J), p \in M$, and an adapted frame at $p$, one has

$$g\Gamma_{ijk} = g^T\Gamma_{ijk},$$

$$g\Gamma_{ij\alpha} = -\frac{1}{2} F_{ij\alpha},$$

$$g\Gamma_{i\alpha j} = g\Gamma_{aij} = \frac{1}{2} F_{ij\alpha},$$

and all other Christoffel symbols vanish.

**Proof** Recall the basic formula

$$\langle \nabla_X Y, Z \rangle = \frac{1}{2} \{ X \langle Y, Z \rangle + Y \langle X, Z \rangle - Z \langle X, Y \rangle - \langle [X, Z], Y \rangle - \langle [Y, Z], X \rangle - \langle [Y, X], Z \rangle \}.$$

Using Lemma 4.12, since the Lie bracket of any two horizontal fields in the frame is vertical, the equality $g\Gamma_{ijk} = g^T\Gamma_{ijk}$ follows immediately. To compute $g\Gamma_{ij\alpha}$, using the invariance properties of the metric and the frame the only possible remaining terms are the Lie bracket terms, which using Lemma 4.12 yields

$$g\Gamma_{ij\alpha} = \frac{1}{2} \langle [e_i, e_j], e_\alpha \rangle = -\frac{1}{2} g_{\alpha\beta} F_{ij}^\beta = -\frac{1}{2} F_{ij\alpha}.$$

Similarly

$$g\Gamma_{i\alpha j} = -\frac{1}{2} \langle [e_i, e_j], e_\alpha \rangle = \frac{1}{2} g_{\alpha\beta} F_{ij}^\beta = \frac{1}{2} F_{ij\alpha}.$$
Lemma 4.15 Given a pluriclosed invariant Hermitian metric $g$ on a Sasaki-type complex surface $(M^4, J)$, $p \in M$ and an adapted frame at $p$, one has

$$(d^c \omega)_{ijk} = (d^c \omega)_{i\alpha\beta} = 0, \quad (d^c \omega)_{ij\alpha} = F_{ij\alpha}.$$ 

Proof We recall $H = \frac{1}{2} d^c \omega$. We first make a general calculation. Since $\omega(X, Y) = g(X, JY)$, we see

$$d^c \omega(X, Y, Z) = -d\omega(JX, JY, JZ) = - (JX_\omega(JY, JZ) + JZ_\omega(JX, JY) + JY_\omega(JZ, JX))\omega([JX, JY], JZ) + \omega([JX, JZ], JY) - \omega([JY, JZ], JX)$$

$$= JXg(JY, Z) + JZg(JX, Y) + JYg(JZ, X) - g([JX, JZ], Y) - g([JY, JZ], X).$$

Using this and Lemma 4.12 it is clear that $H_{ijk} = H_{i\alpha\beta} = 0$, and moreover

$$(d^c \omega)_{12\alpha} = -g([Je_1, Je_2], e_\alpha) = g([e_2, e_1], e_\alpha) = F_{12\alpha}.$$

Lemma 4.16 Given a pluriclosed invariant Hermitian metric $g$ on a Sasaki-type complex surface $(M^4, J)$, $p \in M$ and an adapted frame at $p$, one has

$$B^i_{ijk} = \delta^i_{ijk},$$

$$B^i_{aij} = F_{ij\alpha},$$

and all other Christoffel symbols vanish.

Proof These formulas follow directly from the above Lemmas 4.14 and 4.15 and the formula

$$B^i_{ABC} = \delta^i_{ABC} + \frac{1}{2} (d^c \omega)_{ABC}.$$

Proposition 4.17 Given a pluriclosed invariant Hermitian metric $g$ on a Sasaki-type complex surface $(M^4, J)$, $p \in M$, and an adapted frame at $p$, one has

$$\rho_B(e_1, e_2) = \frac{1}{2} \left(-R^T + |F|^2 \right),$$

$$\rho_B(e_1, Z) = \left( d \operatorname{tr}_o F^Z \right)_1,$n

$$\rho_B(e_1, W) = \left( d \operatorname{tr}_o F^W \right)_1,$n

$$\rho_B(e_2, Z) = \left( d \operatorname{tr}_o F^Z \right)_2.$$
\[
\rho_B(e_2, W) = \left( d \text{tr}_\omega F^W \right)_2 \\
\rho_B(Z, W) = 0.
\]

**Proof** We will drop the notation “\( B \)” from the connection coefficients and curvature tensor throughout this proof, and so \( \Gamma \) and \( \Omega \) are associated to the Bismut connection.

To begin with recall the basic formula
\[
\Omega^D_{ABC} = e_A \Gamma^D_{BC} - e_B \Gamma^D_{AC} + \Gamma^E_{BC} \Gamma^D_{AE} - \Gamma^E_{AC} \Gamma^D_{BE} - [e_A, e_B]^E \Gamma^D_{EC}.
\]

(4.13)

Since all first derivatives of the metric coefficients with respect to the frame vanish at \( p \) (cf. Remark 4.13), we obtain the formula
\[
\Omega(A, B, C, D) = e_A \Gamma_{BCD} - e_B \Gamma_{ACD} + \Gamma^E_{BC} \Gamma_{AED} - \Gamma^E_{AC} \Gamma_{BED} - [e_A, e_B]^E \Gamma_{ECD}.
\]

Next we observe the general calculation
\[
\rho_B(X, Y) = \frac{1}{2} \sum_{i=1}^{4} \Omega(X, Y, e_i, J e_i)
= \Omega(X, Y, e_1, e_2) + \Omega(X, Y, Z, W)
= \Omega(X, Y, e_1, e_2),
\]

where the last line follows because every Christoffel symbol of the form \( \Gamma^\alpha_{ABC} \) vanishes by Lemma 4.16. First we can compute
\[
\rho_B(e_1, e_2) = \Omega(e_1, e_2, e_1, e_2)
= e_1 \Gamma_{212} - e_2 \Gamma_{112} + \Gamma^E_{21} \Gamma_{1E2} - \Gamma^E_{11} \Gamma_{2E2} - [e_1, e_2]^E \Gamma_{E12}
= \text{Rm}_{1212}^g + F_{12\alpha} F_{12\alpha}
= \frac{1}{2} \left( -R^T + |F|^2 \right).
\]

Next we see
\[
\rho_B(e_1, Z) = e_1 \Gamma_{Z12} - Z \Gamma_{112} + \Gamma^E_{Z1} \Gamma_{1E2} - \Gamma^E_{11} \Gamma_{ZE2} = e_1(F_{12Z}) = \left( d \text{tr}_\omega F^Z \right)_1.
\]

Similarly
\[
\rho_B(e_1, W) = e_1 \Gamma_{W12} - W \Gamma_{112} + \Gamma^E_{W1} \Gamma_{1E2} - \Gamma^E_{11} \Gamma_{WE2} = e_1(F_{12W}) = \left( d \text{tr}_\omega F^W \right)_1.
\]

Similarly
\[
\rho_B(e_2, Z) = e_2 \Gamma_{Z12} - Z \Gamma_{112} + \Gamma^E_{Z1} \Gamma_{2E2} - \Gamma^E_{21} \Gamma_{ZE2} = e_2(F_{12Z}) = \left( d \text{tr}_\omega F^Z \right)_2.
\]
Similarly
\[ \rho_B(e_2, W) = e_2 \Gamma_{W12} - W \Gamma_{112} + \Gamma_{E W12}^E \Gamma_{2E2} - \Gamma_{21}^E \Gamma_{WE2} = e_2 (F_{12W}) = \left( d \text{tr}_\omega F^W \right)_2. \]

Lastly we see, using the invariance properties and Lemma 4.16,
\[ \rho_B(Z, W) = Z \Gamma_{W12} - W \Gamma_{Z12} + \Gamma_{E W12}^E \Gamma_{ZE2} - \Gamma_{Z1}^E \Gamma_{WE2} = F_{E W} F_{E2Z} - F_{1EZ} F_{E2W} = 0. \]

\[ \square \]

4.5 Lie derivative operators

Lemma 4.18 Given a pluriclosed invariant Hermitian metric \( g \) on a Sasaki-type complex surface \((M^4, J)\), \( p \in M \), and an adapted frame at \( p \), one has
\[
\begin{align*}
(L_{\theta^* g} g)(e_i, e_j) &= 0, \\
(L_{\theta^* g} g)(e_i, Z) &= -e_i (\text{tr}_\omega F^W), \\
(L_{\theta^* g} g)(e_i, W) &= e_i (\text{tr}_\omega F^Z), \\
(L_{\theta^* g} g)(e_\alpha, e_\beta) &= 0.
\end{align*}
\]

**Proof** We first observe the general formula
\[
(L_X g)(Y, Z) = X g(Y, Z) - g(L_X Y, Z) - g(Y, L_X Z).
\]

Using this and the properties of our adapted frame we see
\[
\begin{align*}
(L_{\theta^* g} g)(e_i, e_j) &= \theta^* g(e_i, e_j) - g((\text{tr}_\omega F^Z) W - (\text{tr}_\omega F^W) Z, e_i), e_j) \\
&\quad - g(e_i, [(\text{tr}_\omega F^Z) W - (\text{tr}_\omega F^W) Z, e_j]) \\
&= 0.
\end{align*}
\]

Next we have
\[
\begin{align*}
(L_{\theta^* g} g)(e_i, Z) &= \theta^* g(e_i, Z) - g((\text{tr}_\omega F^Z) W - (\text{tr}_\omega F^W) Z, e_i), Z) \\
&\quad - g(e_i, [(\text{tr}_\omega F^Z) W - (\text{tr}_\omega F^W) Z, Z]) \\
&= -e_i (\text{tr}_\omega F^W).
\end{align*}
\]

Similarly
\[
\begin{align*}
(L_{\theta^* g} g)(e_i, W) &= \theta^* g(e_i, W) - g((\text{tr}_\omega F^Z) W - (\text{tr}_\omega F^W) Z, e_i), W) \\
&\quad - g(e_i, [(\text{tr}_\omega F^Z) W - (\text{tr}_\omega F^W) Z, W]) \\
&= e_i (\text{tr}_\omega F^Z).
\end{align*}
\]
Lastly, using the invariance properties and the general formula for $L_X g(Y, Z)$ above it is clear that $L_\partial g(Z, Z) = L_\partial g(W, W) = L_\partial g(Z, W) = 0$. □

**Lemma 4.19** Given a pluriclosed invariant Hermitian metric $g$ on a Sasaki-type complex surface $(M^4, J)$, $p \in M$, and an adapted frame at $p$, one has for $f$ an invariant function,

\[
\nabla^2 f(e_i, e_j) = (\nabla^T)^2 f(\partial_{x_i}, \partial_{x_j}),
\]

\[
\nabla^2 f(e_i, e_\alpha) = -\frac{1}{2}(\nabla^T)^k f F_{i k \alpha},
\]

\[
\nabla^2 f(e_\alpha, e_\beta) = 0.
\]

**Proof** The result follows directly from the general formula

\[
(\nabla^2 f)(e_A, e_B) = f_{A B} - \Gamma^C_{A B} f_{, C},
\]

and comparison against Lemma 4.14. □

5 Invariant metrics and pluriclosed flow

In this section we investigate the pluriclosed flow in the setting of invariant metrics on Sasaki-type complex surfaces. First in Proposition 5.1 we show that invariance is preserved by the flow equations. Building on this we show in Proposition 5.3 how to decompose the pluriclosed flow equations into a flow of a transverse metric and a Hermitian connection form.

**Proposition 5.1** Let $(M^{2n}, g, J)$ admit a holomorphic Killing field $X$. Let $g_t$ be the solution to pluriclosed flow with this initial condition. Then $X$ is a Killing field for $g_t$ for all $t \geq 0$.

**Proof** Since $J$ is fixed by pluriclosed flow, $X$ certainly remains holomorphic. To show $X$ remains a Killing field it is thus equivalent to show that $L_X \omega_t \equiv 0$ along the flow. First note that as $\omega_t \in \Omega^{1,1}$ is pluriclosed and $X$ is holomorphic, it follows that $L_X \omega_t \in \Omega^{1,1}$, and is also pluriclosed. We note that the linearization of $-\rho_B^{1,1}$ acting on pluriclosed $(1, 1)$ tensors is a linear elliptic operator with symbol that of the Laplacian (cf. [30, Proposition 3.1]), which we denote $\mathcal{L}$. We thus derive a heat equation for $L_X \omega_t$,

\[
\frac{\partial}{\partial t} L_X \omega = L_X (-\rho_B^{1,1})
\]

\[= [D(-\rho_B^{1,1})](L_X \omega)
\]

\[= \mathcal{L}(L_X \omega).
\]

It follows from a standard argument that the condition $L_X \omega \equiv 0$ is preserved by uniqueness of solutions to this linear parabolic system. □
In view of this and Proposition 4.8, we expect the pluriclosed flow to reduce to a flow of triples \((g^T, \mu, f)\). Although, since all of the metrics are pluriclosed, by Lemma 4.11 we expect \(f\) to remain constant, a fact reflected by the vanishing of the Bismut curvature in these directions as in Proposition 4.17. To confirm this we need a preliminary lemma indicating how to vary Hermitian connection forms in a manner analogous to varying a Hermitian metric on a vector bundle and producing the associated Hermitian connections.

Lemma 5.2 Given a Sasaki-type complex surface \((M^4, J)\), a Hermitian connection form \(\mu \in \Lambda^1(t^2)\), and \(f_1, f_2\) invariant functions, the form

\[
\mu_f := (\mu^Z + d^c f_1 + df_2) \otimes Z + (\mu^W + d^c f_2 - df_1) \otimes W
\]

is Hermitian connection.

Proof Since the \(f_i\) are \([Z, W]\)-invariant, conditions (1) and (3) of Definition 4.6 follow. To check condition (2) we compute

\[
\mu_f(JX) - J_i \mu(X) = \mu(JX) - J_i \mu(X) + (d^c f_1(JX) + df_2(JX)) Z
\]

\[
+ (d^c f_2(JX) - df_1(JX)) W
\]

\[
- J_i ((d^c f_1(X) + df_2(X)) Z + (d^c f_2(X) - df_1(X)) W)
\]

\[
= (df_1(X) - df_1(X) + df_2(JX) + d^c f_2(X)) Z
\]

\[
+ (d^c f_2(JX) - df_2(X) - df_1(JX) - d^c f_1(X)) W
\]

\[= 0, \]

since \(d^c f(X) = -df(JX)\). \(\square\)

Proposition 5.3 Given a Sasaki-type complex surface \((M^4, J)\) and \(\mu\) a Hermitian connection form, suppose \((g^T, f_1, f_2)\) is a one-parameter family of invariant transverse metrics and functions satisfying

\[
\frac{\partial g^T}{\partial t} = -\frac{1}{2} \left( R^T - |F_{\mu_f}|^2 \right) g^T,
\]

\[
\frac{\partial}{\partial t} f_1 = \frac{1}{2} \text{tr}_{\omega_T} F_{\mu_f}^Z = \frac{1}{2} \Delta_{g^T} f_1 + \frac{1}{2} \text{tr}_{\omega_T} F_{\mu}^Z,
\]

\[
\frac{\partial}{\partial t} f_2 = \frac{1}{2} \text{tr}_{\omega_T} F_{\mu_f}^W = \frac{1}{2} \Delta_{g^T} f_2 + \frac{1}{2} \text{tr}_{\omega_T} F_{\mu}^W.
\]

Then the one-parameter of associated metrics \(g_t = g(g^T, \mu_f, 1)\) is a solution to pluriclosed flow.

Proof This follows from a direct computation of the evolution of the Kähler form associated to \(g_t\), comparing against Proposition 4.17. \(\square\)
Remark 5.4 Consider the standard Sasakian structure on $S^3$ generated by the Hopf action on $\mathbb{C}^2$, where the resulting complex surface is the standard diagonal Hopf surface, which is a principal $T^2$ fibration over $\mathbb{CP}^1$. One can easily compute that the curvature tensors $F_Z$ and $F_W$ both evolve by the Hodge Laplacian heat flow, i.e. we have the system of equations on $\mathbb{CP}^1$,

$$\frac{\partial}{\partial t} g = -\frac{1}{2} R_g g + \frac{1}{2} |F|^2 g = - Rg + F^2,$$

$$\frac{\partial}{\partial t} F = \frac{1}{2} \Delta_d F .$$

where $F_{ij}^2 = F_{ik} F_{jk}$. These equations are, up to immaterial global scaling factors, a natural coupling of the Ricci and Yang–Mills flows introduced independently by the author [33] and Young [42]. These equations result from studying the Ricci flow of an invariant metric on a principal bundle, but fixing the metric on the fibers. While freezing the bundle metric may seem natural from a Yang–Mills point of view, it is arguably unnatural from the point of view of the geometry of the total space (cf. [23] for a discussion of the Ricci flow of an invariant metric on a principal bundle). Thus it is somewhat surprising that the pluriclosed flow, defined on general complex manifolds with no symmetry considerations in mind, should naturally freeze the metric on the fibers in this invariant setting.

Proposition 5.5 An invariant pluriclosed Hermitian metric $g = g(Tg, \mu)$ on a Sasaki-type complex surface is a steady soliton with defining function $f$ if and only if $f$ is invariant and there exists $\lambda \in \mathbb{R}$ such that

$$(R^T - |F_Z|^2 )g^T + L_{\nabla_f g} g^T = 0,$$

$$e^{-f} \text{tr}_{\omega_T} F^Z = \lambda_1,$$

$$F^W = 0 .$$

Proof To see that $f$ is invariant, we compute using Proposition 4.17 and Lemma 4.18

$$0 = \rho_{B}^{1,1}(JZ, Z) - \frac{1}{2} L_{\partial_Z} g(Z, Z) + \frac{1}{2} L_{\nabla_f g} g(Z, Z)$$

$$= \nabla^2 f(Z, Z)$$

$$= Z(Zf) - (\nabla_Z Z)f$$

$$= Z(Zf) ,$$

where the last line follows since $Z$ is a constant length Killing field, hence $\nabla_Z Z = \nabla |Z|^2 = 0$. We integrate this against $e^{-f} dV_g$ to yield

$$0 = \int_M Z(Zf) e^{-f} dV_g$$

$$= \int_M \left\{ Z \left( e^{-f} Zf \right) + e^{-f} (Zf)^2 \right\} dV_g$$
\[
\int_M L Z \left( e^{-f} Z f dV_g \right) + \int_M (Z f)^2 e^{-f} dV_g = \int_M (Z f)^2 e^{-f} dV_g.
\]

Thus \(Z f \equiv 0\), and similarly \(W f \equiv 0\).

Now note that as a consequence of the definition of soliton and Proposition 2.1 we see that
\[
0 = \text{Rc} - \frac{1}{4} H^2 + \frac{1}{2} L \nabla_f g = \rho_B^{1,1} (J \cdot, \cdot) - \frac{1}{2} L_{\theta \xi} g + \frac{1}{2} L \nabla_f g.
\]

Pairing this equation against two horizontal vectors and using Proposition 4.17 and Lemmas 4.18 and 4.19 yields the first equation of (5.2). Next fix a horizontal vector \(e_i\) and note that the above equation again in conjunction with Proposition 4.17 and Lemmas 4.18 and 4.19 implies
\[
0 = \rho_B^{1,1} (J \cdot, e_i) - \frac{1}{2} L \theta \xi g (Z, e_i) + \frac{1}{2} L \nabla_f g (Z, e_i)
= -\rho_B^{1,1} (e_i, W) + \frac{1}{2} e_i (\text{tr}_\omega F^W) - \frac{1}{2} \nabla_k f F^Z_{ik}
= -\frac{1}{2} e_i \cdot (d^c \text{tr}_\omega T F Z) + \frac{1}{2} e_i \cdot (\nabla f \cdot F Z),
\]

So that
\[
0 = -d^c \text{tr}_\omega T F Z + \nabla f \cdot F Z.
\]

Let us expand this component wise with respect to an adapted frame to observe the equations
\[
0 = \left( -d^c \text{tr}_\omega T F Z + \nabla f \cdot F Z \right)_1
= e_2 \text{tr}_\omega T F Z + e_2 f F_{21}
= e f e_2 \left( e^{-f} \text{tr}_\omega T F Z \right),
\]

and similarly
\[
0 = \left( -d^c \text{tr}_\omega T F Z + \nabla f \cdot F Z \right)_2
= -e_1 \text{tr}_\omega T F Z + e_1 f F_{12}
= -e f e_1 \left( e^{-f} \text{tr}_\omega T F Z \right).
\]

Hence
\[
e^{-f} \text{tr}_\omega T F Z \equiv \text{constant} = \lambda,
\]

which is the second equation of (5.2).
A similar argument shows that $e^{-f} \text{tr}_{\omega^T} F^W = \text{constant} = \lambda^\prime$. Since by construction $W$ is tangent to the product $S^1$ action over $S^3$, it follows that $F^W = da$ for an invariant 1-form $a$. If $\lambda^\prime \neq 0$, then we would obtain $\omega^T = \frac{e^{-f}}{\lambda} da$, contradicting that $\omega^T$ is positive definite. Thus $\lambda^\prime = 0$, and this implies $F^W = 0$.

\[ \square \]

**Remark 5.6** We pause here to note that the only case of a soliton in this ansatz which is also locally conformally Kähler is the standard metric on the diagonal Hopf surfaces. In particular, taking the Hodge star of the second soliton equation yields that $d(e^{-f} \theta) = 0$. If the metric was also locally conformally Kähler, then $d\theta = 0$, and one then concludes $df \wedge \theta = 0$. Since $f$ is basic, comparing against (4.12) we obtain $df = 0$, and so the metric is a fixed point of pluriclosed flow, and thus is the standard metric on a diagonal Hopf surface (see Sect. 2).

### 6 Existence proofs

In this section we complete the proof of Theorem 1.1. Starting with the symmetric ansatz above, we first observe a further a priori Killing field present in this setting. Using this we provide two conceptually distinct but ultimately equivalent reductions of the soliton system to ordinary differential equations. First we consider Hopf surfaces where the underlying Sasaki manifold is quasiregular, in which case the quotient space is an orbifold. In this setting the extra Killing field corresponds to the natural rotational symmetry on this orbifold, and we reduce the soliton to equations to a system of ODEs on a geodesic transverse to this rotational symmetry. In the irregular case will directly reduce to a system of ODEs on an interval, in analogy with the quasiregular case.

### 6.1 A further a priori symmetry

Thus far we have only set up the soliton and flow equations with two real holomorphic Killing fields on a complex surface, and thus it is a “codimension 2” construction, and one would still expect to use methods of partial differential equations to find solutions. However, building upon a fundamental observation in the theory of Ricci solitons on surfaces [4, p. 241], [5], we see that the equations automatically acquire an extra symmetry, and thus we have a codimension 1 symmetry, which can be addressed by ODE methods.

**Proposition 6.1** Let $(g, f)$ be an invariant soliton on a Sasaki-type complex surface $(M^4, J)$. Then

$$L_J \nabla f g^T = L_J \nabla f \omega^T = L_J \nabla f F^Z = 0.$$ 

**Proof** Since the function $f$ is invariant, $\nabla f$ is a horizontal vector field. A direct calculation shows that for an invariant horizontal vector field $W$ one has

$$L_W g^T (Y, Z) = \nabla^T W^\flat (Y, Z) + \nabla^T W^\flat (Z, Y).$$
Moreover, since $J$ is invariant, and preserves $\mathcal{H}$ by Lemma 4.7, it follows that $J \nabla f$ is horizontal and invariant. Since the transverse structure is Kähler, a short calculation shows that

$$\nabla^T (J \nabla f)^b (Y, Z) + \nabla^T (J \nabla f)^b (Z, Y) = \nabla^T \nabla^T f (J Y, Z) + \nabla^T \nabla^T f (J Z, Y)$$

But from the reduced solitons equations (5.2), we know that $\nabla^T \nabla^T f$ is of type $(1, 1)$, hence the above quantity must vanish, as required.

Next, since $d\omega^T = 0$ by Lemma 4.11, we see by the Cartan formula that

$$L_{J \nabla f} \omega^T = d(J \nabla f \lrcorner \omega^T) = -d(df) = 0.$$ 

To show the invariance of $F$, we first note that using the second equation of (5.2)

$$L_{J \nabla f} \text{tr}_{\omega^T} F Z = (J \nabla f) \lrcorner d \text{tr}_{\omega^T} F Z = -\nabla f \lrcorner d \gamma \text{tr}_{\omega^T} F Z = -\nabla f \lrcorner \nabla f \lrcorner F Z = 0.$$

Since we can express $F Z = \text{tr}_{\omega^T} F^2 \omega^T$, the invariance of $F Z$ now follows. □

### 6.2 ODE reduction in quasiregular case

As discussed in Sect. 4.1, the quotient orbifold is only singular at cone points, of which there are no more than two. On the smooth part, we note that Proposition 6.1 implies that a hypothetical soliton has $J \nabla f$ as a Killing field. Moreover it implies that $J \nabla f$ is holomorphic, and in fact must correspond to the natural holomorphic rotational symmetry present on bad orbifolds. Such a metric can be expressed with respect to polar coordinates as

$$g = dr^2 + \phi^2(r)d\theta^2,$$  \hspace{1cm} (6.1)

Note also that the bundle curvature $F Z$ is invariant, and so we may express

$$F Z = \gamma(r)dr \wedge d\theta.$$  \hspace{1cm} (6.2)

It will also be useful to work in terms of the quantity

$$\psi := \text{tr}_{\omega} F Z,$$  \hspace{1cm} (6.3)

which is related to $\phi$ and $\gamma$ using Lemma 6.2 below. Also note that for a metric as in (6.1) to correspond to a metric on an orbifold means that we have $\phi(0) = \phi(L) = 0$ for some $L > 0$, and moreover if our cone points have angles $\beta_1$ and $\beta_2$, then we require $\phi'(0) = \frac{\beta_1}{2\pi}, \phi'(L) = -\frac{\beta_2}{2\pi}$.

**Lemma 6.2** Given the setup above, one has

1. $J \partial_r = \phi^{-1} \partial_\theta, \ J \partial_\theta = -\phi \partial_r,$
2. $R = -2\frac{\phi_x}{\phi}$,
\( \psi = \text{tr}_{\omega} FZ = \frac{\gamma}{\phi}, \quad |FZ|^2 = 2\frac{\gamma^2}{\phi^2}, \)
\( dc \text{tr}_{\omega} FZ = \left( \gamma_{r} - \frac{\gamma \phi_{r}}{\phi} \right) d\theta, \)
\( (5) \text{ For a radial function } f \text{ one has } \nabla f \cdot F = f_{r} \gamma d\theta, \)
\( (6) \text{ For a radial function } f \text{ one has } \nabla^2 f = f_{rr} dr^2 + \phi \phi_{r} f_{r} d\theta^2. \)

**Proof** To determine the complex structure, note that by (6.1) and the fact that \( g \) is compatible with \( J \) we see that \( J \partial_{r} \) must be a multiple of \( \partial_{\theta} \). But then appealing to compatibility again we obtain

\[ 1 = g(\partial_{r}, \partial_{r}) = g(J \partial_{r}, J \partial_{r}), \]

and so it follows that \( J \partial_{r} = \phi^{-1} \partial_{\theta} \), using the standard orientation. The equation \( J \partial_{\theta} = -\phi \partial_{r} \) thus follows via \( J^2 = -\text{Id} \). The scalar curvature in item (2) is a standard calculation we omit. For item (3), note using our conventions that in these coordinates we have

\[ \text{tr}_{\omega} FZ = F_{r} g_{\theta} J_{r} = \frac{\gamma}{\phi}, \]

as claimed. The square norm of \( FZ \) in item (3) follows easily from (6.1) and (6.2).

Using the general formula \( df f(X) = -df (JX) \), by the rotational invariance it is clear that the only nonvanishing component for \( dc \text{tr}_{\omega} FZ \) is a multiple of \( d\theta \). Thus we observe, using (4),

\[ dc \text{tr}_{\omega} FZ (\partial_{\theta}) = -d \text{tr}_{\omega} FZ (-\phi \partial_{r}) = \phi \left( \text{tr}_{\omega} FZ \right)_{r} = \gamma_{r} - \frac{\gamma \phi_{r}}{\phi} \]

For item (6) note first that for a radial function \( f \) one has \( df = f_{r} dr \), and so by (6.1) it follows that \( \nabla f = f_{r} \partial_{r} \). Combining this with (6.2) it is clear that \( \nabla f \cdot F = f_{r} \gamma d\theta \), as claimed.

**Lemma 6.3** Given the setup above, one has a steady soliton if and only if there exists a constant \( A \) such that

\[ J \nabla f = A \partial_{\theta}, \quad 0 = A \phi_{r} - \frac{\phi_{rr}}{\phi} - \frac{\gamma^2}{\phi^2}, \quad \psi_{r} = A \phi \psi. \]

**Proof** Using the metric soliton equation of (5.2) with items (1) and (7) of Lemma 6.2 we obtain

\[ 0 = \frac{1}{2} \left( R - |F|^2 \right) g + \nabla^2 f, \]

\[ = \left( -\frac{\phi_{rr}}{\phi} - \frac{\gamma^2}{\phi^2} \right) \left( dr^2 + \phi^2 d\theta^2 \right) + f_{rr} dr^2 + \phi \phi_{r} f_{r} d\theta^2. \]
Looking at the different components we obtain the equations

\[ 0 = f_{rr} - \frac{\phi_{rr}}{\phi} - \frac{\gamma^2}{\phi^2}, \]
\[ 0 = \frac{\phi_r f_r}{\phi} - \frac{\phi_{rr}}{\phi} - \frac{\gamma^2}{\phi^2}. \]  

(6.4)

Combining these two yields

\[ (\log f_r)_r = \frac{f_{rr}}{f_r} = \frac{\phi_r}{\phi} = (\log \phi)_r \]

Integrating this we see that \( f_r = A\phi \), for an as yet undetermined constant \( A \). Comparing against Lemma 6.2 (1) it follows that \( J\nabla f = A\partial_\theta \), as claimed. Note that a formula of this type was inevitable from the construction, as we already knew that \( J\nabla f \) generated the rotational symmetry. Also plugging \( f_r = A\phi \) into (6.4) yields the second equation of the lemma. For the final equation we simply differentiate the equation \( e^{-f} \psi = \lambda \) to yield

\[ 0 = \psi_r - f_r \psi = \psi_r - A\phi \psi, \]

as claimed.

6.3 ODE reduction in general case

Here we reduce the soliton equations to a system of ODE in the general case. While the presentation is different, this reduction is a generalization of that in Sect. 6.2.

Fix \( \alpha, \beta \in \mathbb{C} \), determining a Hopf surface as described in Sect. 4.1. We adopt the notation of that section here. Note that in the irregular case the generic orbit of the Reeb vector \( Z \) is dense in the standard torus \( T_\lambda := \{|z_1|^2 = \lambda^2, |z_2|^2 = 1 - \lambda^2\} \) containing it, thus it follows that basic functions are constant on such tori. As the vector field \( E_1 \) is tangent to these tori, it follows that basic functions are invariant under \( E_1 \) as well. While this is not true for general invariant functions in the quasiregular case, we nonetheless impose \( E_1 \) invariance as an ansatz. Thus, as \( \{E_1, E_2\} \) span the contact distribution, we expect to reduce the soliton equation to an ODE along the \( E_2 \) direction. In particular, let \( X = \sigma^{-1}E_2 \), and define a parameter

\[ s(t) = \frac{b}{2} \ln (b - t) - \frac{a}{2} \ln (t - a), \]

defined for \( t \in (a, b) \), (note that as in Sect. 4.1 we have \( a < b \)). Observe that

\[ X(s(\sigma)) = \frac{1}{\sigma} \left\{ |z_2|^2 (x_1\partial_{x_1} + y_1\partial_{y_1}) - |z_1|^2 (x_2\partial_{x_2} + y_2\partial_{y_2}) \right\} \left\{ -\frac{a}{2} \ln (\sigma - a) + \frac{b}{2} \ln (b - \sigma) \right\} \]
\[ = \frac{1}{\sigma} \left\{ |z_2|^2 \left( -\frac{a}{\sigma - a} (ax_1^2 + ay_1^2) + \frac{b}{b - \sigma} (-ax_1^2 - ay_1^2) \right) \right\} \]
\begin{align*}
- |z_1|^2 \left( - \frac{a}{\sigma - a} (bx_2^2 + by_2^2) + \frac{b}{b - \sigma} (-bx_2^2 - by_2^2) \right) \\
= \frac{|z_1|^2 |z_2|^2}{\sigma} \left\{ - \frac{a^2}{\sigma - a} - \frac{ab}{b - \sigma} + \frac{ab}{\sigma - a} + \frac{b^2}{b - \sigma} \right\} \\
= 1,
\end{align*}

where the last line follows by applying the identities \( \sigma - a = (b - a) |z_2|^2, \quad b - \sigma = (b - a) |z_1|^2 \).

We now define functions which describe the metric and bundle curvature as in Sect. 6.2. First, for a given invariant metric, define \( \phi \) and \( \xi \geq 0 \) via

\[ \phi^2 = \xi := \frac{1}{2} \omega^T (E, J E). \]

Also, we set

\[ \psi = \text{tr}_\omega F Z. \]

Next we must determine the boundary conditions, i.e. the behavior at the points \( z_2 = 0, \) or \( z_1 = 0, \) corresponding to \( s \to \infty, \) \( s \to -\infty, \) or \( \sigma = a, \sigma = b, \) respectively. First, as a function of \( \sigma, \) it follows from the definition of \( \xi \) and the nondegeneracy of the metric that \( \xi \) must be of order \( |z_2|^2 \) near \( z_2 = 0. \) To convert this to the parameter \( s, \) we observe the formula

\[ (b - a) |z_2|^2 = e^{-\frac{2}{\alpha} s} \left( (b - a) |z_1|^2 \right)^\frac{b}{a}, \]

Hence, expanded as a power series in \( e^{-s}, \) the leading order term is proportional to \( e^{-\frac{2}{\alpha} s}. \) It follows easily that the function \( \phi \) has such an expansion with leading order term \( e^{-\frac{1}{\alpha} s}, \) and thus the \( Y \) derivative of \( \phi \) at this boundary point is \( -\frac{1}{\alpha}. \) An identical argument shows that the \( Y \) derivative of \( \phi \) at the boundary point \( z_1 = 0 \) is \( \frac{1}{b}, \) as claimed.

**Lemma 6.4** Let \((M^4, J)\) be an irregular Sasaki-type complex surface. An invariant metric \((g^T, \mu)\) determines a steady pluriclosed soliton if and only if there exists a constant \( A \) such that

\[ 0 = A \phi_r - \frac{\phi_{rr}}{\phi} - \psi^2, \quad \psi_r = -A \phi \psi, \quad \tag{6.5} \]

where \( \frac{\partial}{\partial r} = \frac{1}{\phi \sigma} E_2. \)

**Proof** Considering the transverse piece of the reduced soliton equations in Proposition 5.5, we obtain the fact that the transverse Hessian of \( f \) is pure trace, i.e.

\[ \nabla^T \nabla^T f = \frac{1}{2} \Delta^T f g^T. \quad \tag{6.6} \]
We will use this condition to first of all determine an explicit relationship between \( f \) and \( g^T \). The calculations can be effectively globalized using the frame \( \{E_1, E_2\} \). Let
\[
E = \sigma^{-1} (E_1 - 1E_2),
\]
and note that \( E \) spans the space of transverse \((1, 0)\)-vector fields everywhere except the points where \( z_1 = 0 \) or \( z_2 = 0 \). Since (6.6) implies that the \((2, 0) + (0, 2)\) piece of the transverse Hessian of \( f \) vanishes, it follows that
\[
EE f - E(\log \xi)Ef = 0.
\]
Since basic functions are also invariant under \( E_1 \) as described above, we see that this implies
\[
-\sigma^{-1}E_2 \left( \sigma^{-1}E_2f \right) + \sigma^{-1}(E_2 \log \xi) \left( \sigma^{-1}E_2f \right),
\]
and thus there exists a constant \( A \) such that
\[
Xf = \sigma^{-1}E_2f = A\xi. \tag{6.7}
\]
Let \( \tilde{X} = \pi_H X \), the horizontal projection, and then note that, since \( f \) is basic,
\[
g^T (\nabla f, \tilde{X}) = Xf = A\xi, \tag{6.8}
\]
and hence
\[
\nabla f = A\tilde{X}. \tag{6.9}
\]
Also, using the \( E_1 \) invariance of \( \xi \), we obtain the formula for the transverse scalar curvature
\[
R^T = -\xi^{-1}XX \log \xi.
\]
Combining these observations we finally obtain that the transverse piece of the soliton equation reduces to
\[
0 = AX\xi - X^2 \log \xi - \psi^2 \xi. \tag{6.10}
\]
To connect this to the point of view in Sect. 6.2, we need to choose an arclength parameter for $X$. In particular, recall that $\phi^2 = \xi$, and let $\frac{\partial}{\partial r} = \phi^{-1} X = \frac{1}{\phi} E_2$ as in the statement. Observe then that (6.10) implies

$$0 = A \left( \phi \frac{\partial}{\partial r} \right) \phi^2 - 2 \left( \phi \frac{\partial}{\partial r} \right) \left( \phi \frac{\partial}{\partial r} \right) \log \phi - \psi^2 \phi^2$$

$$= 2A\phi^2 \phi_r - 2\phi \phi_{rr} - \psi^2 \phi_r,$$

from which the first claimed equation follows by dividing by $\phi^{-2}$. Also, it is elementary to differentiate the equation $e^{-f} \psi = \lambda_1$ with respect to $X$ and apply (6.7) to obtain

$$0 = X(e^{-f} \psi) = e^{-f} (X \psi - A\xi \psi),$$

which in turn directly implies

$$\psi_r = \phi^{-1} X \psi = A\phi^{-1} \xi \psi = A\phi \psi.$$

The lemma follows. $\square$

### 6.4 Solutions

In this subsection, we construct solutions to the system of ODEs derived in Lemma 6.4. First let us address the constant $A$. Observe that if $A = 0$, it follows directly that $\psi_r = 0$, and so after scaling we can assume $\psi \equiv 1$, and we obtain the ODE $\phi_{rr} = -\phi$. Thus the metric on the only possible solution then corresponds to the round metric $S^2$. Indeed the resulting metric corresponds to the Hopf metric (2.1).

Thus, now assuming $A \neq 0$, we perform a change of variables which greatly simplifies the system and causes the parameter $A$ to drop out. In particular, let

$$x = A\phi, \quad y = A\phi_r, \quad z = A^\frac{1}{2} \psi.$$

Then from the system of ODEs one derives

$$x_r = A\phi_r = y$$

$$y_r = A\phi_{rr} = A \left( A\phi_r \phi - \frac{\gamma^2}{\phi^2} \right) = xy - z^2 \quad \text{(6.11)}$$

$$z_r = A^\frac{1}{2} \psi_r = -AA^\frac{1}{2} \left( \frac{\gamma \gamma_r - A \gamma \phi^2}{\phi^2} - \gamma \phi_r \right) = xz.$$

**Proposition 6.5** For every $1 < \rho < \infty$ there exists $z_0$ such that the solution to (6.11) with initial condition $(0, 1, z_0)$ exists (at least) on a finite time interval $[0, T]$ and satisfies

$$x|_{[0,T]} \geq 0, \quad x(T) = 0, \quad y(T) < 0, \quad y(0)/|y(T)| = \rho.$$  

(6.12)
Proof The overall argument consists of finding choices of $z_0$ which give the required behavior first for $\rho$ close to 1, then for $\rho$ large, then arguing by a continuity method that one obtains all values in between. The proof is summarized in Fig. 2.

We first describe solutions with $\rho$ close to 1. In particular, we claim that for $z_0$ sufficiently large the following inequalities are preserved:

1. $x(t) \leq t - \frac{1}{2}(1 + \delta)z_0^2t^2$,
2. $1 - (1 + \delta)z_0^2t \leq y(t) \leq 1 - (1 - \delta)z_0^2t$,
3. $z(t) \leq (1 + \delta)z_0$

These are certainly satisfied at time $t = 0$, so it remains to show that they are preserved up to the first time $T$ such that $x(T) = 0$. First note that, as long as inequality (2) is preserved we see

$$x(t) = x(0) + \int_0^t y(s)ds \leq \int_0^t \left\{ 1 - (1 - \delta)z_0^2s \right\} ds \leq t - \frac{1}{2}(1 - \delta)z_0^2t^2.$$

Thus, the maximal time we must consider satisfies $t \leq \frac{2}{(1 - \delta)z_0^2}$, and this also shows that condition (1) is preserved as long as condition (2) is. This also implies that as long as condition (1) is preserved, we have the overall upper bound

$$x(t) \leq \sup \left( t - \frac{1}{2}(1 - \delta)z_0^2t^2 \right) = \frac{1}{2(1 - \delta)z_0^2}.$$

Thus we can integrate and estimate the differential equation for $z$ to obtain

$$z(t) = z_0 \exp \left( \int_0^t x(s)ds \right) \leq z_0 \exp \left( \frac{t}{2(1 + \delta)z_0^3} \right) \leq z_0 \exp \left( \frac{1}{(1 + \delta)^2z_0^4} \right) < (1 + \delta)z_0.$$

for $z_0$ chosen sufficiently large. Lastly we note that using all the estimates in play and integrating

$$y(t) \leq y(0) + \int_0^t \left( xy - z^2 \right) ds \leq 1 + \int_0^t \left( \frac{1}{2(1 + \delta)z_0^2} \left( 1 - (1 - \delta)z_0^2s \right) - z_0^2 \right) ds \leq 1 + \left( \frac{1}{2(1 + \delta)z_0^2} - z_0^2 \right) t \leq 1 - (1 - \delta)z_0^2t,$$

for $z_0$ chosen sufficiently large. A very similar integration yields the lower bound as well. We have shown that conditions (1), (2), and (3) hold until $x(T) = 0$. We claim that $y(0)/|y(T)|$ approaches 1 for $z_0$ chosen large. To do this we first obtain a lower
bound for the first time $T$ that $x(T) = 0$. In particular, integrating and estimating we obtain
\[
x(t) \geq x_0 + \int_0^t \left( 1 - (1 + \delta)z_0^2 s \right) ds = t - \frac{1}{2}(1 + \delta)z_0^2 t^2
\]
Thus one sees that $T \geq \frac{2}{(1+\delta)z_0}$. Returning to estimate (2) we thus obtain
\[
1 - 2 \frac{1 + \delta}{1 - \delta} \leq y(T) \leq 1 - 2 \frac{1 - \delta}{1 + \delta}.
\]
Thus certainly for $\delta$ chosen sufficiently small $y(T)$ approaches $-1$, as claimed.

We now describe solutions corresponding to large values of $\rho$. This is more involved, requiring describing three phases of the solution which we name the “growth phase,” “control phase,” and “decay phase.” By the “growth phase” we mean that, given $\Lambda > 0$, we can choose $z_0$ sufficiently small that there exists a time $t_0 > 0$ where $x(t_0) \geq \Lambda$.

Fix a small constant $\delta > 0$, and note that, as long as $z \leq \delta$, and $y \geq 0$ we can estimate
\[
y(t) \geq y_0 + \int_0^t (xy - z^2) ds \geq 1 - \delta^2 t,
\]
and so in particular for $\delta$ small we have $\inf_{[0, 1]} y \geq \frac{1}{2}$, and hence $x(1) \geq \frac{1}{2}$. Note then that for times $t \geq 1$, assuming still $z \leq \delta$ sufficiently small, we obtain the elementary estimate $y \geq \frac{1}{4}$, which will be preserved, and hence we conclude $y(t) \geq y(1) = \frac{1}{2}$, and thus $x(t) \geq x(1) + \frac{1}{2} t$, and so there exists a first time $t_1 \leq 2 \Lambda$ such that $x(t_1) = \Lambda$.

It remains to ensure we can choose $z_0$ sufficiently small to guarantee the hypothesis $z \leq \delta$ on a time interval of this length. To that end we integrate the equation for $z$ and estimate on the time interval $[0, t_1]$,
\[
z \leq z_0 \exp \left( \int_0^t x(s) ds \right) \leq z_0 e^{t \Lambda} \leq z_0 e^{2\Lambda^2} < \delta,
\]
provided $z_0 < \delta e^{-2\Lambda^2}$.

Next we have the “control phase.” In particular, we establish that $x$ does not grow without bound, but rather achieves a unique maximum value. Specifically, we claim that there exists a time $t_1$ such that $y(t_1) = 0$. To see this first note that
\[
\frac{d}{dt} \frac{y}{z} = \frac{xy - z^2}{z} - \frac{yxz}{z^2} = -z < -z_0.
\]
Hence by an elementary integration we obtain
\[
y(t) < z(t) \left( \frac{y_0}{z_0} - z_0 t \right) \leq 0
\]
for $t \geq \frac{y_0}{z_0}$, as claimed.
Lastly we have the “decay phase,” wherein we show $x$ returns to zero, and moreover that $y$ becomes very large and negative at that time. Note that $y(t) \leq 0$ is certainly preserved by the ODE, and in fact for $x(t) \geq 0$, $y(t) \leq 0$ one has $y_t \leq -\frac{1}{2} x^2 \leq -\frac{1}{2} y_0^2$, it follows easily that there exists a first time $t_3$ such that $x(t_3) = 0$. We furthermore claim that one has $y(t_3) \leq -\frac{1}{4} x(t_2)^2$. We obtain this again via comparison with the idealized flow lines, in other words, we know that

$$y_t = xy - z^2 \leq xy = \frac{1}{2} (x^2)_t.$$ Integrating the ODE $y_t = \frac{1}{2} (x^2)_t$ yields

$$y(t) = \frac{1}{2} x^2 + C,$$

for some constant of integration $C$. The flow lines are thus parabolas in a standard phase space diagram. Choosing $C = -\frac{1}{4} x(t_2)^2$, we obtain the ideal boundary indicated in Figure 6.4, which intersects the $y$-axis at the point $(0, -\frac{1}{4} x(t_2)^2)$. By comparison we know that the solution to our ODE must lie below this curve, and thus at the time $t_3$ where $x(t_3) = 0$, it follows immediately that $y(t_3) \leq -\frac{1}{4} x(t_2)^2$, as claimed.

\[ \square \]

### 6.5 Main proofs

**Proof of Theorem 1.1** Fix $\alpha, \beta$ such that $|\alpha| < |\beta|$ determining a primary Hopf surface as in Sect. 4.1. Let $\rho = \frac{b}{a} = \frac{|\beta|}{|\alpha|} > 1$, and choose $z_0$ and $(x(r), y(r), z(r))$ according to Proposition 6.5. By rescaling this solution as in (6.11), we can obtain $x'(0) = \frac{1}{\rho}$, which by construction forces $x'(T) = -\frac{1}{a}$. Thus, as explained in Sect. 6.3, the function $\phi = x$ defines a transverse metric $g_T^x$, with transverse Kähler form $\omega_T$. This in turn defines a curvature form $F_Z = z_0 \omega_T$. Furthermore, by construction the soliton function $f$ will only depend on the parameter $\sigma$, and is solved for using $\phi$ via (6.7). By construction the triple $(g^T, F, f)$ solves the system of equations (5.2). To finish we must ensure that $F_Z$ arises as the curvature of a Hermitian connection form. To that end we first note that for a solution to the reduced soliton equations $(g, F, f)$, given $\lambda > 0$ one has that $(\lambda g, \lambda^{-2} F_Z, f)$ is also a solution for any $\lambda > 0$. Thus without loss of generality we can rescale so that $[F_Z]_B = [F_{\mu_0}^Z]$, where $\mu_0$ denotes the connection for associated to any background invariant metric on $M$, for instance the one arising from the original Sasakian structure. By the $\delta_0, \delta_\beta$-lemma [8] there exists an invariant function $\zeta$ such that $F_Z = F_{\mu_0}^Z + i \delta_0 \delta_\beta \zeta$. It follows that $F_Z = F_{\mu_0}^Z$, in the notation of Lemma 5.2. Moreover, it is clear by construction that $F_{W_Z}^Z = 0$. The triple $(g^T, \mu_\zeta, f)$ is the claimed soliton.

Finally, we address the case of secondary Hopf surfaces. As explained in the work of Kato [17,18], for Hopf surfaces of class 1 with $|\alpha| \neq |\beta|$, the fundamental group $\Gamma$ of $M$ is expressed as a semidirect product $\Gamma = \langle \gamma_{\zeta, \beta} \rangle \ltimes H$, where $H \subset U(1) \times U(1)$, the group of diagonal unitary matrices acting in the standard way on $\mathbb{C}^2$, and so it suffices to show that the solitons we have constructed are invariant under this torus,
and hence descend to the quotient. As explained in Sect. 6.3, the functions $\phi$ and $\psi$ are constant on these tori. Since the final metric is determined by these functions, natural operators, and $J$, it follows that holomorphic vector fields tangent to these orbits are Killing, and thus one obtains $U(1) \times U(1)$ invariance.

**Proof of Corollary 1.4** Theorem 1.1 yields nontrivial steady soliton structures on $S^3 \times S^1$. By taking products with flat tori we obtain nontrivial soliton structures on $S^3 \times T^k$ for all $k \geq 1$. To obtain nontrivial solitons in dimension $n = 3$, we note that it follows from the reduced soliton equations of Proposition 5.5 and elementary calculations...
using (4.12) and Lemma 4.18 that the 1-form $e^{-f} \theta$ is closed, and moreover satisfies $L_{e^{-f} \theta} \theta = 0$, and thus $e^{-f} \theta$ is parallel. Therefore the universal cover is isometric to a product $(S^3 \times \mathbb{R}, g' \oplus dt^2)$. Since $H = \star \theta = \star e^f dt$, it follows that $\frac{\partial}{\partial t} \star H = 0$. Thus, setting $H' = i^* H$, where $i$ denotes the inclusion map of an $S^3$ leaf, it follows easily that $(H')^2 = i^* (H^2)$. Also, in the construction of Theorem 1.1, we noted that the function $f$ was $Z, W$-invariant. It follows, setting $f' = f \circ i$, that $(\nabla^2 f')' = i^* (\nabla^2 f)$. Thus we conclude that for the structure $(g', H', f')$ on the $S^3$ leaf,

$$\text{Rc}_{g'} - \frac{1}{4} (H')^2 + (\nabla^2 f')' = i^* \left( \text{Rc}_g - \frac{1}{4} H^2 + \nabla^2 f \right) = 0,$$

as required. \qed

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