ON THE REGULARITY OF GLOBAL ATTRACTORS

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Abstract. This note is focused on a novel technique in order to establish the boundedness in more regular spaces for global attractors of dissipative dynamical systems, without appealing to uniform-in-time estimates. As an application of the abstract result, the semigroup generated by the strongly damped wave equation

\[ u_{tt} - \Delta u_t - \Delta u + \varphi(u) = f \]

with critical nonlinearity is considered, whose attractor is shown to possess the optimal regularity.

1. Introduction

The evolution of many physical phenomena is ruled by a differential equation generating a semigroup of operators \( \{S(t)\}_{t \geq 0} \), otherwise called a dynamical system, acting on a suitable infinite-dimensional Banach space \( \mathcal{H} \). In mathematical terms, the presence of some dissipation mechanism in the model often reflects into the existence of an absorbing set for the semigroup. This is, by definition, a bounded set \( \mathcal{B}_0 \subset \mathcal{H} \) enjoying the following property: for any \( R \geq 0 \), there exists an entering time \( t_R \geq 0 \) such that

\[ S(t)\mathcal{B} \subset \mathcal{B}_0, \quad \forall t \geq t_R, \]

whenever \( \mathcal{B} \subset \mathcal{H} \) with \( \|\mathcal{B}\|_\mathcal{H} \leq R \). An alternative notion is the one of an attracting set, namely, a bounded set \( \mathcal{C} \subset \mathcal{H} \) satisfying the relation

\[ \lim_{t \to \infty} \left[ \text{dist}_\mathcal{H}(S(t)\mathcal{B}, \mathcal{C}) \right] = 0, \]

for all bounded sets \( \mathcal{B} \subset \mathcal{H} \), where \( \text{dist}_\mathcal{H} \) is the Hausdorff semidistance in \( \mathcal{H} \), given by

\[ \text{dist}_\mathcal{H}(\mathcal{B}_1, \mathcal{B}_2) := \sup_{x_1 \in \mathcal{B}_1} \inf_{x_2 \in \mathcal{B}_2} \|x_1 - x_2\|_\mathcal{H}. \]

Clearly, an absorbing set is attracting as well, whereas the existence of an attracting set implies the existence of an absorbing one. On the other hand, an attracting set is more likely to possess nice additional properties, such as compactness and finite fractal dimension. A relevant object providing the ultimate description of the asymptotic dynamics is the global attractor: the unique compact set \( \mathfrak{A} \subset \mathcal{H} \) which is at the same time attracting and fully invariant under the action of \( S(t) \), that is,

\[ S(t)\mathfrak{A} = \mathfrak{A}, \quad \forall t \geq 0. \]

Roughly speaking, \( \mathfrak{A} \) is the smallest possible set where the evolution is eventually confined. Accordingly, any possible further regularity of the attractor is extremely important for a

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better understanding of the longterm behavior of the semigroup. For more details on the theory of dynamical systems and their attractors, we address the reader to the classical textbooks [2, 13, 14, 20] (see also the more recent [3, 4, 15]).

A standard way to prove the existence of the global attractor for a (strongly continuous) semigroup is to exhibit a compact attracting set. In that case, \( S(t) \) is called *asymptotically compact*, and the attractor \( A \) turns out to be the \( \omega \)-limit set of any absorbing set \( B_0 \):

\[
A = \bigcap_{t \geq 0} \overline{\bigcup_{\tau \geq t} S(\tau)B_0}.
\]

This is usually attained through the limit

\[
\lim_{t \to \infty} \left[ \text{dist}_H(S(t)B_0, C(t)) \right] = 0,
\]

where, for every fixed \( t \), \( C(t) \) is a bounded subset (say, a closed ball about the origin) of another Banach space \( V \) compactly embedded into \( H \). Even though (1.1) yields the global attractor, no conclusion can be drawn at this stage on the regularity of \( A \). However, if \( V \) is reflexive, so that closed balls of \( V \) are closed in \( H \), and one is able to produce the uniform-in-time estimate

\[
\sup_{t \geq 0} \| C(t) \|_V = \rho < \infty,
\]

it is immediate to see that \( A \) is norm-bounded in \( V \) by the very constant \( \rho \).

As a rule, verifying (1.1) in concrete situations requires a reasonable effort; on the contrary, showing the uniform bound (1.2), and in turn the \( V \)-boundedness of \( A \), is generally a much harder task, if not out of reach. A paradigmatic example is the semigroup of the damped wave equation with a critical nonlinearity lacking monotonicity properties, whose global attractor in the weak-energy space was found in [1], but its optimal regularity has been obtained only several years later [12, 22].

The aim of this note is to present a new and easy to handle technique apt to establish the boundedness of \( A \) in the higher space \( V \) without making use of the uniform estimate (1.2), proving that, to some extent, (1.1) alone suffices (see Section 3). Actually, we say more: we find a ball \( C \) of \( V \) attracting exponentially fast all bounded subsets of \( H \); precisely,

\[
\text{dist}_H(S(t)B, C) \leq J(\| B \|_H) e^{-\omega t},
\]

for some \( \omega > 0 \) and some increasing function \( J \).

**Remark 1.1.** A sufficiently regular exponentially attracting sets is crucial in order to demonstrate the existence of an *exponential attractor*: a compact set \( E \subset H \) of finite fractal dimension and positively invariant \( (S(t)E \subset E \) for all \( t \geq 0 \)), which attracts bounded subsets of \( H \) at an exponential rate, contrary to the global attractor, whose attraction rate can be arbitrarily slow and not measurable in terms of the structural parameters of the problem [6, 7, 8, 15]. In this respect, an exponential attractor happens to be more helpful than the global one for practical purposes, e.g. numerical simulations.

As an application, in the final Section 4, we consider the dynamical system generated by the strongly damped wave equation with a nonlinearity of critical growth, providing a simple proof of the optimal regularity of the related attractor.
2. A Basic Inequality

We begin with some notation. Given a Banach space $V$ and $R \geq 0$, we set
$$B_V(R) = \{z \in V : \|z\|_V \leq R\}.$$  
We denote by $\mathcal{I}$ the space of continuous increasing functions $J : \mathbb{R}^+ \to \mathbb{R}^+$, and by $\mathcal{D}$ the space of continuous decreasing functions $\beta : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\beta(\infty) < 1$.

**Definition 2.1.** A solution operator on $V$ is a family of maps $U(t) : V \to V$, depending on $t \geq 0$, satisfying the “initial condition”
$$U(0)z = z, \quad \forall z \in V.$$  
The family $U(t)$ is called a semigroup if it fulfills the further property
$$U(t + \tau) = U(t)U(\tau), \quad \forall t, \tau \geq 0.$$  

**Remark 2.2.** The above maps are closely related to the study of differential equations in Banach spaces: suppose that, for all initial data $z \in V$, there is a unique solution (in some weak sense) $\zeta : \mathbb{R}^+ \to V$ to the Cauchy problem
$$\left\{ \begin{array}{l}
\frac{d}{dt}\zeta(t) = A(\zeta(t), t), \\
\zeta(0) = z,
\end{array} \right.$$  
where $A(\cdot, t)$ is a family of operators densely defined on $V$. Then, we can write
$$\zeta(t) = U(t)z.$$  
When the system is autonomous, i.e. $A$ does not depend explicitly on time, $U(t)$ is a semigroup.

The inequality of the following lemma, although not more than a trivial observation, is really the key idea of the paper.

**Lemma 2.3.** Let $U(t)$ be a solution operator on a Banach space $V$. Assume that
$$\|U(t)z\|_V \leq \beta(t)\|z\|_V + J(t), \quad \forall z \in V,$$  
for some $\beta \in \mathcal{D}$, $J \in \mathcal{I}$. Then, for any $t_* > 0$ large enough so that $\beta_* := \beta(t_*) < 1$,
$$\|U(t_*)z\|_V \leq \beta_*\|z\|_V + \frac{1}{2}(1 - \beta_*)R_*,$$  
with $R_* = \frac{2}{1 - \beta_*}J(t_*)$.

Here is a first remarkable consequence of (2.1).

**Corollary 2.4.** If $U(t)$ is also a semigroup, then it possesses an absorbing set.

**Proof.** If $\|z\|_V \leq R_*$, from (2.1) and the semigroup properties we readily get
$$\sup_{n \in \mathbb{N}} \|U(nt_*)z\|_V \leq R_*.$$  
For an arbitrary $t \geq 0$, we write $t = nt_* + \tau$, with $n \in \mathbb{N}$ and $\tau \in [0, t_*)$. This yields
$$\|U(t)z\|_V = \|U(\tau)U(nt_*)z\|_V \leq \beta(\tau)\|U(nt_*)z\|_V + J(\tau) \leq \beta(0)R_* + J(t_*) = \kappa R_*,$$
having set (necessarily, $\beta(0) \geq 1$)

\[ \kappa = \beta(0) + \frac{1}{2}(1 - \beta_*) > 1. \]

Thus, we have proved the implication

\[ \|z\|_V \leq R_* \Rightarrow \sup_{t \geq 0} \|U(t)z\|_V \leq \kappa R_. \]

Conversely, if $\|z\|_V = R > R_*$, we infer from (2.1) that

\[ \|U(t_*)z\|_V \leq \frac{1}{2}(1 + \beta_*)R, \]

and in light of the semigroup properties,

\[ \|U(n_Rt_*)z\|_V \leq R_*, \]

up to taking

\[ n_R = 1 + \left[ \frac{\ln R - \ln R_\ast}{\ln 2 - \ln(1 + \beta_*)} \right]. \]

In summary,

\[ U(t)B_V(R) \subset B_V(\kappa R_\ast), \quad \forall t \geq t_R, \]

with an entering time

\[ t_R = \begin{cases} 
0 & \text{if } R \leq R_*, \\
 n_R t_* & \text{if } R > R_*. 
\end{cases} \]

In other words, the ball $B_V(\kappa R_\ast)$ is an absorbing set for $U(t)$.

\[ \Box \]

3. The Abstract Theorem

Throughout the section, let $S(t)$ be a semigroup of closed operators acting on a Banach space $\mathcal{H}$ (cf. [18]). This is a semigroup for which the implication

\[ x_n \to x, \quad S(t)x_n \to \xi \Rightarrow \xi = S(t)x \]

holds at any fixed time $t \geq 0$, whenever $x_n, x, \xi \in \mathcal{H}$.

**Remark 3.1.** A strongly continuous semigroup, i.e. a semigroup enjoying the continuity

\[ S(t) \in C(\mathcal{H}, \mathcal{H}), \quad \forall t \geq 0, \]

is a particular instance of a semigroup of closed operators.

The semigroup $S(t)$ is also required to possess an absorbing set $\mathfrak{B}_0 \subset \mathcal{H}$. Without loss of generality,

\[ \mathfrak{B}_0 = B_{\mathcal{H}}(R_0), \quad R_0 > 0. \]

Finally, let $\mathcal{V}$ be a reflexive Banach space compactly embedded into $\mathcal{H}$.

Within the above assumptions, our main result reads

**Theorem 3.2.** For every $x \in \mathfrak{B}_0$, let there exist two solution operators $V_x(t)$ and $U_x(t)$ on $\mathcal{H}$ with the following properties:

(i) For any two vectors $y, z \in \mathcal{H}$ satisfying $y + z = x$,

\[ S(t)x = V_x(t)y + U_x(t)z. \]
(ii) There is \( \alpha \in \mathcal{D} \) such that
\[
\sup_{x \in \mathcal{B}_0} \| V_x(t)y \|_\mathcal{H} \leq \alpha(t)\| y \|_\mathcal{H}, \quad \forall y \in \mathcal{B}_0.
\]

(iii) There are \( \beta \in \mathcal{D} \) and \( J \in \mathcal{I} \) such that
\[
\sup_{x \in \mathcal{B}_0} \| U_x(t)z \|_\mathcal{V} \leq \beta(t)\| z \|_\mathcal{V} + J(t), \quad \forall z \in \mathcal{V}.
\]

Then, \( \mathcal{B}_0 \) is exponentially attracted by a closed ball of \( \mathcal{V} \); namely, there exist (strictly) positive constants \( \rho, K, \omega \) such that
\[
(3.1) \quad \text{dist}_\mathcal{H}(S(t)\mathcal{B}_0, \mathcal{B}_\rho) \leq Ke^{-\omega t}.
\]

As a byproduct, we have

Corollary 3.3. The semigroup \( S(t) \) possesses the global attractor \( \mathfrak{A} \) bounded in \( \mathcal{V} \).

Indeed, as mentioned in the introduction, if \( S(t) \) is strongly continuous it is well known that (3.1) implies the existence of the global attractor \( \mathfrak{A} \) subject to the bound \( \| \mathfrak{A} \|_\mathcal{V} \leq \rho \) (see e.g. [20]). Same thing if \( S(t) \) is only a semigroup of closed operators (see [18]).

Proof of Theorem 3.2. Let \( x \in \mathcal{B}_0 \) be arbitrarily fixed, and select \( t_* > 0 \) large enough such that
\[
S(t_*)\mathcal{B}_0 \subset \mathcal{B}_0,
\]
and
\[
\alpha_* := \alpha(t_*) < 1, \quad \beta_* := \beta(t_*) < 1.
\]

For every \( n \in \mathbb{N} \), we claim that the vector
\[
x_n := S(nt_*)x \in \mathcal{B}_0
\]
admits the decomposition
\[
x_n = y_n + z_n,
\]
for some \( y_n, z_n \) satisfying the bounds
\[
\| y_n \|_\mathcal{H} \leq \alpha_*^nR_0, \quad \| z_n \|_\mathcal{V} \leq R_* := \frac{2}{1-\beta_*}J(t_*).
\]

We proceed by induction on \( n \in \mathbb{N} \). The case \( n = 0 \) is verified by \( y_0 = x, z_0 = 0 \). Assume the claim true for all \( n \leq m \in \mathbb{N} \). Choosing
\[
y_{m+1} = V_{x_m}(t_*)y_m \quad \text{and} \quad z_{m+1} = U_{x_m}(t_*)z_m,
\]
we obtain the equality
\[
x_{m+1} = S((m+1)t_*)x = S(t_*)x_m = y_{m+1} + z_{m+1}.
\]

Observing that \( y_m \in \mathcal{B}_0 \), and using (2.1), we derive the estimates
\[
\| y_{m+1} \|_\mathcal{H} = \| V_{x_m}(t_*)y_m \|_\mathcal{H} \leq \alpha_*\| y_m \|_\mathcal{H} \leq \alpha_*^{m+1}R_0,
\]
\[
\| z_{m+1} \|_\mathcal{V} = \| U_{x_m}(t_*)z_m \|_\mathcal{V} \leq \frac{1}{2}(1 + \beta_*)R_* \leq R_*.
\]

This proves the claim. Let then \( t \geq 0 \). Writing \( t = nt_* + \tau \), with \( n \in \mathbb{N} \) and \( \tau \in [0, t_*) \),
\[
S(t)x = S(\tau)x_n = V_{x_n}(\tau)y_n + U_{x_n}(\tau)z_n,
\]
and
\[
\|V_{x_n}(\tau)y_n\|_H \leq \alpha(0)\|y_n\|_H \leq \alpha(0)\alpha_*^{-1}\alpha_*^{1/t_*}R_0,
\]
\[
\|U_{x_n}(\tau)z_n\|_V \leq \beta(0)\|z_n\|_V + J(t_*) \leq \kappa R_*
\]
with \(\kappa > 1\) as in (2.2). Thus, setting
\[
\varrho = \kappa R_*, \quad K = \alpha(0)\alpha_*^{-1}R_0, \quad \omega = t_*^{-1}\ln \alpha_*^{-1},
\]
the required exponential attraction property (3.1) follows. □

Incidentally, Corollary 3.3 is still true under weaker hypotheses.

**Proposition 3.4.** Let \(t_* > 0\) be such that \(S(t_*)\mathfrak{B}_0 \subset \mathfrak{B}_0\). For every \(x \in \mathfrak{B}_0\), let there exist two operators \(V_x\) and \(U_x\) on \(H\) with the following properties:

(i) For any two vectors \(y, z \in H\) satisfying \(y + z = x\),
\[
S(t_*)x = V_xy + U_xz.
\]

(ii) There is \(\alpha_* < 1\) such that
\[
\sup_{x \in \mathfrak{B}_0}\|V_xy\|_H \leq \alpha_*\|y\|_H, \quad \forall y \in \mathfrak{B}_0.
\]

(iii) There are \(\beta_* < 1\) and \(J_* \geq 0\) such that
\[
\sup_{x \in \mathfrak{B}_0}\|U_xz\|_V \leq \beta_*\|z\|_V + J_*, \quad \forall z \in V.
\]

Then, \(S(t\) possesses the global attractor \(\mathfrak{A}\) bounded in \(V\).

**Proof.** Let \(x \in \mathfrak{B}_0\) be fixed. Arguing exactly as in the proof of Theorem 3.2,
\[
S(nt_*)x = y_n + z_n, \quad \forall n \in \mathfrak{N},
\]
with
\[
\|y_n\|_H \leq \alpha_*^nR_0, \quad \|z_n\|_V \leq R_* := \frac{2}{1-\beta_*}J_*.
\]
Therefore,
\[
\text{dist}_H(S(nt_*)\mathfrak{B}_0, B_V(R_*)) \leq \alpha_*^nR_0 \to 0,
\]
which is enough to establish the existence of \(\mathfrak{A}\) (cf. [18]). Since the attractor is fully invariant and contained in the absorbing set \(\mathfrak{B}_0\),
\[
\mathfrak{A} = S(nt_*)\mathfrak{A} \subset S(nt_*)\mathfrak{B}_0.
\]
Hence, letting \(n \to \infty\), we conclude that
\[
\text{dist}_H(\mathfrak{A}, B_V(R_*)) = 0,
\]
yielding the set inclusion \(\mathfrak{A} \subset B_V(R_*)\). □

In concrete cases, a commonly adopted strategy leading to the global attractor \(\mathfrak{A}\) is finding a decomposition
\[
S(t)x = \eta(t; x) + \zeta(t; x), \quad \forall x \in \mathfrak{B}_0,
\]
such that, for some function $\mu$ vanishing at infinity and some $J \in \mathcal{I}$,

\begin{align}
\sup_{x \in \mathcal{B}_0} \| \eta(t;x) \|_{\mathcal{H}} &\leq \mu(t), \\
\sup_{x \in \mathcal{B}_0} \| \zeta(t;x) \|_{\mathcal{V}} &\leq J(t).
\end{align}

(3.3) \hspace{1cm} (3.4)

However, in order to deduce the $\mathcal{V}$-boundedness of $\mathcal{A}$, estimate (3.4) need be uniform in time, same as requiring that

$$\lim_{t \to \infty} J(t) = \rho < \infty.$$ \hspace{1cm} (3.5)

Let us first dwell on a simple, albeit quite interesting, situation.

**Example 3.5.** For two (linear and nonlinear, respectively) operators $A_0, A_1$, assume that the differential equation

$$\frac{d}{dt} \xi = A_0 \xi + A_1(\xi)$$

generates a semigroup $S(t)$ on $\mathcal{H}$. Besides, let the linear semigroup $L(t)$, generated by the equation with $A_1 \equiv 0$, be exponentially stable on both spaces $\mathcal{H}$ and $\mathcal{V}$, i.e.

$$\| L(t)x \|_{\mathcal{H}, \mathcal{V}} \leq Me^{-\delta t} \| x \|_{\mathcal{H}, \mathcal{V}}, \quad \forall x \in \mathcal{H}; \mathcal{V},$$

for some $M \geq 1$, $\delta > 0$. Finally, suppose that (3.2)-(3.4) hold, with $\eta(t;x) = L(t)x$ and $\zeta(t;x)$ solution to

$$\begin{cases}
\frac{d}{dt} \zeta = A_0 \zeta + A_1(\xi), \\
\zeta(0) = 0,
\end{cases}$$

where $\xi(t) = S(t)x$ (actually, (3.3) follows directly from exponential stability). This kind of decomposition has been successfully employed several times (e.g. [10, 11]), and typically works for subcritical problems. Then, setting

$$V_x(t)y = L(t)y \quad \text{and} \quad U_x(t)z = L(t)z + \zeta(t;x),$$

hypotheses (i)-(iii) of Theorem 3.2 are easily verified. Hence, in contrast to the standard procedure, our approach gives at once the $\mathcal{V}$-boundedness of $\mathcal{A}$, with no need of (3.5).

In general, a semigroup decomposition of the form (3.2), complying with (3.3)-(3.4), can be much more complicated (cf. [1, 16]). Nonetheless, whenever (3.2)-(3.4) occur, we have a strong evidence that the conclusions of Theorem 3.2 hold true, as in the quite challenging case of the strongly damped wave equation with critical nonlinearity, discussed below.

### 4. A Concrete Application

Consider the semilinear strongly damped wave equation in a smooth bounded domain $\Omega \subset \mathbb{R}^3$ subject to Dirichlet boundary conditions

$$\begin{cases}
\frac{\partial u}{\partial t} - \Delta u_t - \Delta u + \varphi(u) = f, \\
u_t|_{\partial \Omega} = 0,
\end{cases}$$

(4.1)

where $f \in L^2(\Omega)$ is independent of time, and the nonlinear term $\varphi \in C(\mathbb{R})$ satisfies the critical growth condition

$$|\varphi(u) - \varphi(v)| \leq c|u - v|(1 + |u|^4 + |v|^4),$$

(4.2)
and the standard dissipativity assumption

$$\liminf_{|u| \to \infty} \frac{\varphi(u)}{u} > -\lambda_1.$$  

Here and in the sequel, $c$ denotes some positive constant, while $\lambda_1 > 0$ is the first eigenvalue of the linear operator $A = -\Delta$ on $L^2(\Omega)$ with $\text{dom}(A) = H^2(\Omega) \cap H_0^1(\Omega)$.

**Notation.** For $r \in \mathbb{R}$, we introduce the scale of Hilbert spaces (we will always omit the index $r$ when $r = 0$)

$$H^r = \text{dom}(A^{r/2}), \quad \langle u, v \rangle_r = \langle A^{r/2}u, A^{r/2}v \rangle_{L^2(\Omega)}, \quad \|u\|_r = \|A^{r/2}u\|_{L^2(\Omega)},$$

and we define the product spaces $\mathcal{H}^r = H^{r+1} \times H^r$.

Equation (4.1) generates a strongly continuous semigroup $S(t)$ on $\mathcal{H}$ possessing the global attractor $\mathfrak{A}$ (see [5, 16]). In fact, for a nonlinearity of (critical) growth of polynomial order 5, the existence itself of the attractor remained an open question for a long time.

Replacing (4.3) with the more restrictive assumption

$$\varphi \in C^1(\mathbb{R}), \quad \liminf_{|u| \to \infty} \varphi'(u) > -\lambda_1,$$

the boundedness of $\mathfrak{A}$ in $\mathcal{H}^1$ has been demonstrated in [17], by means of a "parabolic" approach. The same paper indicates the way to obtain the result also within (4.3), by means of a rather complicated scheme borrowed from [22]. This has been recently carried out in detail by some other authors [19, 21].

Our goal is a simpler proof, which does not actually require anything more than what already contained in [16]. To this end, let us first recall some results therein.

$\diamond$ $S(t)$ has an absorbing set $\mathfrak{B}_0 \subset \mathcal{H}$. From now on, $c_0 > 0$, $\nu_0 > 0$ and $J_0 \in \mathcal{I}$ will denote generic constants and a generic function, respectively, depending only on $\mathfrak{B}_0$.

$\diamond$ The following uniform estimate holds:

$$\sup_{x \in \mathfrak{B}_0} \sup_{t \geq 0} \|S(t)x\|_\mathcal{H} \leq c_0.$$

$\diamond$ For every $x \in \mathfrak{B}_0$, the solution $S(t)x = (u(t), u_t(t))$ splits into the sum

$$\hat{\eta}(t) + \hat{\zeta}(t) = (\hat{v}(t), \hat{v}_t(t)) + (\hat{w}(t), \hat{w}_t(t)),$$

where

$$\|\hat{\eta}(t)\|_\mathcal{H} \leq c_0 e^{-\nu_0 t} \quad \text{and} \quad \|\hat{\zeta}(t)\|_{\mathcal{H}^{1/4}} \leq J_0(t).$$

**Remark 4.1.** In the same spirit of [1], the proofs of [16] lean on the decomposition

$$\varphi(u) = \varphi_0(u) + \varphi_1(u),$$

where the continuous functions $\varphi_0$ and $\varphi_1$ satisfy (4.2) and (4.3), respectively, along with

$$\varphi_0(u)u \geq 0 \quad \text{and} \quad |\varphi_1(u)| \leq c(1 + |u|).$$

This is easily obtained noting that, from (4.3), there are $\sigma \geq 0$ and $\lambda < \lambda_1$ such that

$$|u| \geq \sigma \quad \Rightarrow \quad \varphi(u)u \geq -\lambda u.$$
For instance, a compatible choice is
\[ \varphi_0(u) = \gamma(u)[\varphi(u) + \lambda u], \quad \varphi_1(u) = \varphi(u) - \varphi_0(u), \]
for any continuous \( \gamma : \mathbb{R} \to [0, 1] \), with \( \gamma(u) = 0 \) if \( |u| \leq \sigma \) and \( \gamma(u) = 1 \) if \( |u| > \sigma + 1 \).

A standard Gronwall-type lemma will also be needed.

**Lemma 4.2.** Let \( \Lambda : \mathbb{R}^+ \to \mathbb{R}^+ \) be an absolutely continuous function satisfying
\[ \frac{d}{dt} \Lambda(t) + \varepsilon \Lambda(t) \leq ke^{-\varepsilon t} \Lambda(t) + J(t), \]
for some \( \varepsilon, \nu, k > 0 \) and some \( J \in \mathcal{F} \). Then,
\[ \Lambda(t) \leq e^{k/\nu}e^{-\varepsilon t} \Lambda(0) + \varepsilon^{-1}e^{k/\nu}J(t). \]

We are now in a position to state and prove

**Theorem 4.3.** The attractor \( \mathcal{A} \) of the semigroup \( S(t) \) on \( \mathcal{H} \) is bounded in \( \mathcal{H}^1 \).

**Proof.** The first step is to apply the abstract result with \( \mathcal{V} = \mathcal{H}^{1/4} \). To this aim, we decompose \( \varphi \) as in Remark 4.1, choosing \( \sigma \) strictly positive. Accordingly, \( \varphi_0 \) vanishes on the interval \([-\sigma, \sigma]\), which allows us to write
\[ \varphi_0(u) = u\psi(u), \quad |\psi(u)| \leq c|u|^4. \]

For \( y, z \in \mathcal{H} \), we define
\[ V_x(t)y = \eta(t) \quad \text{and} \quad U_x(t)z = \zeta(t), \]
where \( \eta(t) = (v(t), v_t(t)) \) and \( \zeta(t) = (w(t), w_t(t)) \) solve the Cauchy problems
\[ \begin{align*}
&v_{tt} + Av_t + Av = g, \\
&\eta(0) = y,
\end{align*} \quad \begin{align*}
&w_{tt} + Aw_t + Aw = h, \\
&\zeta(0) = z,
\end{align*} \]
having set
\[ g = -v\psi(\hat{v}) \quad \text{and} \quad h = f - \varphi(u) + v\psi(\hat{v}). \]

Hypothesis (i) of Theorem 3.2 holds by construction, whereas verifying (ii)-(iii) requires some passages. By virtue of (4.4)-(4.5), the Hölder inequality with exponents \((5, 5/4)\) and the Sobolev embedding \( H^1 \subset L^6(\Omega) \),
\[ \|g\|_{L^{5/4}(\Omega)} \leq c\|v\|_{1}\|\hat{v}\|_1^4 \leq c_0 e^{-\varepsilon t}\|v\|_1. \]
Due to (4.2), (4.5) and the straightforward equality
\[ h = f - \varphi(u) + \varphi(\hat{v}) + \hat{w}\psi(\hat{v}) - w\psi(\hat{v}) - \varphi_1(\hat{v}), \]
we get
\[ |h| \leq |f| + c|\hat{w}|(1 + |u|^4 + |\hat{v}|^4) + c|w||\hat{v}|^4 + c(1 + |\hat{v}|). \]

Hence, making use of (4.4), the Hölder inequality with exponents \((9, 9/8)\) and the embeddings \( H^{5/4} \subset L^{12}(\Omega) \) and \( H^1 \subset L^6(\Omega) \), we obtain
\[ \begin{align*}
\|h\|_{L^{5/4}(\Omega)} &\leq c\|f\| + c|\hat{w}|_{5/4}(1 + \|u\|_1^4 + |\hat{v}|_1^4) + c\|w\|_{5/4}|\hat{v}|_1^4 + c(1 + |\hat{v}|_1) \\
&\leq c_0 e^{-\varepsilon t}\|w\|_{5/4} + J_0(t).
\end{align*} \]
Then, as in [16], we introduce the energy functionals

\[ \Lambda_0 = \| \eta \|^2_{\dot{H}^1} + \varepsilon \| v \|^2_1 + 2 \varepsilon \langle v_t, v \rangle, \quad \Lambda_1 = \| \zeta \|^2_{\dot{H}^{3/4}} + \varepsilon \| w \|^2_{5/4} + 2 \varepsilon \langle w_t, w \rangle_{1/4}, \]

with \( \varepsilon > 0 \) small enough in order for \( \Lambda_0 \) and \( \Lambda_1 \) to be equivalent to \( \| \eta \|^2_{\dot{H}^1} \) and \( \| \zeta \|^2_{\dot{H}^{3/4}} \), respectively. From the equation for \( v \) and (4.6), we infer

\[ \frac{d}{dt} \Lambda_0 + 2 \varepsilon \| v \|^2_1 + 2 \| v_t \|^2_1 - 2 \varepsilon \| v_t \|^2_1 = 2 \langle g, v_t + \varepsilon v \rangle \leq \| v_t \|^2_1 + c_0 e^{-\varepsilon t} \| v \|^2_1. \]

Clearly, for \( \varepsilon \) sufficiently small,

\[ \frac{d}{dt} \Lambda_0 + \varepsilon \Lambda_0 \leq c_0 e^{-\varepsilon t} \Lambda_0, \]

so that (ii) is a direct consequence of Lemma 4.2, which gives

\[ \| V_\varepsilon(t)y \|^2_{\dot{H}^1} \leq c_0 e^{-\varepsilon t} \| y \|^2_{\dot{H}^1}. \]

Likewise for (iii), exploiting (4.7), the Hölder inequality with exponents \( (4, 4/3) \) and the continuous embedding \( H^{3/4} \subset L^4(\Omega) \), we find

\[ \frac{d}{dt} \Lambda_1 + 2 \varepsilon \| w \|^2_{5/4} + 2 \| w_t \|^2_{5/4} - 2 \varepsilon \| w_t \|^2_{5/4} = 2 \langle h, A^{1/4} w_t + \varepsilon A^{1/4} w \rangle \leq \varepsilon^2 \| w \|^2_{5/4} + \| w_t \|^2_{5/4} + c_0 e^{-\varepsilon t} \| w \|^2_{5/4} + J_0(t). \]

Therefore, taking \( \varepsilon \) small, we end up with the differential inequality

\[ \frac{d}{dt} \Lambda_1 + \varepsilon \Lambda_1 \leq c_0 e^{-\varepsilon t} \Lambda_1 + J_0(t), \]

and a further application of Lemma 4.2 provides the estimate

\[ \| U_\varepsilon(t)z \|^2_{\dot{H}^{1/4}} \leq c_0 e^{-\varepsilon t} \| z \|^2_{\dot{H}^{1/4}} + J_0(t). \]

By means of Theorem 3.2, we conclude that \( \mathfrak{A} \) is bounded in \( \mathcal{H}^{1/4} \), whereas the boundedness in \( \mathcal{H}^1 \) follows from a standard bootstrap procedure. Indeed, on account of the obtained \( \mathcal{H}^{1/4} \)-regularity, the problem becomes in every respect subcritical for initial data on the attractor. In particular, \( \| \varphi(u) \| \) is uniformly bounded, so that the simple decomposition of Example 3.5 applies, and the desired boundedness is drawn in one single step.

\[ \square \]

As a matter of fact, Theorem 3.2 in its full strength, together with the transitivity property of exponential attraction devised in [9], yield a stronger result, whose proof is left to the interested reader.

**Theorem 4.4.** There exist \( \varrho > 0, \omega > 0 \) and \( J \in \mathcal{J} \) such that

\[ \text{dist}_{\mathcal{H}} (S(t) \mathfrak{B}, B_{\mathcal{H}^1}(\varrho)) \leq J(\| \mathfrak{B} \|_{\mathcal{H}}) e^{-\omega t}, \]

for every bounded set \( \mathfrak{B} \subset \mathcal{H} \).

**Remark 4.5.** In light of [18, Lemma 3.6], Theorem 4.4 (and so Theorem 4.3) is easily seen to hold replacing \( \mathcal{H}^1 \) with the more regular space \( H^2 \times H^2 \), provided that \( \varphi \in C^1(\mathbb{R}) \).
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