Microcanonical black hole statistics and the finite infinite range Heisenberg model

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Abstract

The Gelfand pattern of the reduction of the $N$-fold tensor product of the fundamental representation of the special unitary group SU(2) by itself is studied in the framework of a finite Heisenberg model with infinite range, where $N$ spins couple to each other with the same strength. A speculative comment relates the present findings to the microstatistics of black holes for illustrative purposes.

\textbf{Keywords.} Heisenberg model, Quantum Ising model, Lie groups, group theory, tensor products, lattice theory, black holes.

Introduction

Qubits are the elements of the lowest-dimensional non-trivial quantum mechanical two-state space, and from a reductionist point of view one may argue that complex systems should be understandable as compositions obtained from direct products and superpositions of qubit product states.

The special unitary group SU(2) is unusual among the matrix Lie groups in that in the two-fold tensor product decomposition an irreducible representation (irrep) appears at most once. For $n$-fold tensor products with $n \geq 3$, multiplicity does appear, and a fundamental issue is how to deal with the repeated appearance of the same irrep \footnote{This is a reference or citation.}. Of course, the spin group SU(2) acting on an abstract qubit space is not necessarily related to rotational real space degrees of freedom, as the notation below may suggest.

In this paper, the infinite range Heisenberg model \footnote{This is another reference or citation.} or 'quantum Ising model' with a finite number of spins in a homogeneous external field $B$ is investigated, where each of the $N$ spins interacts with every other spin with equal strength given by an antiferromagnetic renormalizable coupling parameter $g(N) > 0$ depending in a non-local manner on the size of the system. The Hamiltonian is given by

$$H_N = -B \sum_{i=1}^{N} \frac{\sigma_i^3}{2} + g(N) \sum_{i=1}^{N} \sum_{j=1}^{N} \left( \frac{\sigma_i^1 \sigma_j^1}{2} + \frac{\sigma_i^2 \sigma_j^2}{2} + \frac{\sigma_i^3 \sigma_j^3}{2} \right),$$

where the spin configuration $|S\rangle$ the Hamiltonian acts upon is an element of the $2^N$-dimensional N-fold tensor product of the complex two-dimensional qubit space $\mathbb{C}_2^2$,

$$\mathcal{H} = (\mathbb{C}_2^2)^\otimes N \cong \mathbb{C}_2^{2N},$$

(1)
since a single spin-$\frac{1}{2}$ state $|s\rangle$ is a ray in the Hilbert space $C^2_0$ and can be represented by a linear combination of orthogonal basis states

$$|s\rangle = u|\uparrow\rangle + d|\downarrow\rangle = u \begin{pmatrix} 1 \\ 0 \end{pmatrix} + d \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad u, d \in \mathbb{C}, \quad |u|^2 + |d|^2 = 1,$$

transforming under the fundamental representation $[\frac{1}{2} \ 0]$ of SU(2) in two complex dimensions. The Pauli matrices $\sigma_{1,2,3}$ and the spin-$\frac{1}{2}$ operators $\vec{s} = (s_1, s_2, s_3)$, $s_{1,2,3} = \frac{1}{2}\sigma_{1,2,3}$, in a single qubit sector are given as usual by

$$\sigma_j = \begin{pmatrix} \delta_{j3} & \delta_{j1} - i\delta_{j2} \\ \delta_{j1} + i\delta_{j2} & -\delta_{j3} \end{pmatrix}, \quad i^2 = -1. \quad (4)$$

Since the square of total angular momentum operator $\vec{J} = \vec{s}^1 + \ldots + \vec{s}^N$ is given by

$$\vec{J}^2 = \sum_{i=1}^{N} \vec{s}^i \vec{s}^i,$$

the Hamiltonian can be written in the form

$$H_N = -BJ_3 + g(N)\vec{J}^2,$$

and the energy eigenvalues are given therefore by

$$E_{J,m} = -Bm + g(N)J(J+1), \quad m = -J, -J+1, \ldots, J, \quad 0 \leq J = \frac{N}{2}, \frac{N}{2} - 1, \ldots. \quad (7)$$

The level-splitting field $B$ will not play any further role in the following.

**Reduction of the N-fold Kronecker product**

In order to find degeneracy of the energy eigenvalues in the absence of an external field, one has to decompose $\mathcal{H}$ into the SU(2)-invariant subspaces according to the Kronecker (tensor, or direct) product reduction with multiplicities $\alpha_{N,K}$

$$\bigotimes_{N}^N \left[ \frac{1}{2} \right] = \left[ \frac{1}{2} \right] \otimes \left[ \frac{1}{2} \right] \otimes \ldots \otimes \left[ \frac{1}{2} \right] = \bigoplus_{K=0}^{N} \alpha_{N,K} \left[ \frac{K}{2} \right]. \quad (8)$$

The following table displays the non-vanishing multiplicities of the irreps $\left[ \frac{K}{2} \right]$ for $N = 1, \ldots 12$ with $J = K/2$

| $N$ | $\frac{1}{2}$ | $1$ | $\frac{3}{2}$ | $2$ | $\frac{5}{2}$ | $3$ | $\frac{7}{2}$ | $4$ | $\frac{9}{2}$ | $5$ | $\frac{11}{2}$ | $6$ |
|-----|--------------|-----|-------------|----|------------|----|-------------|----|------------|----|-------------|----|
| 1   | 1            |     |             |    |            |    |             |    |            |    |             |    |
| 2   | 1            | 1   |             |    |            |    |             |    |            |    |             |    |
| 3   | 2            |     | 1           |    |            |    |             |    |            |    |             |    |
| 4   | 2            | 3   | 1           |    |            |    |             |    |            |    |             |    |
| 5   | 5            |     | 4           | 1  |            |    |             |    |            |    |             |    |
| 6   | 5            | 9   | 5           | 1  |            |    |             |    |            |    |             |    |
| 7   | 14           | 14  | 6           | 1  |            |    |             |    |            |    |             |    |
| 8   | 14           | 28  | 20          | 7  | 1          |    |             |    |            |    |             |    |
| 9   | 42           | 48  | 27          | 8  | 1          |    |             |    |            |    |             |    |
| 10  | 42           | 90  | 75          | 35 | 9          | 1  |            |    |            |    |             |    |
| 11  | 132          | 165 | 110         | 44 | 10         | 1  |            |    |            |    |             |    |
| 12  | 132          | 297 | 275         | 154| 54         | 11 | 1           |    |            |    |             |    |
For \( N = 2 \), the product \( \left[ \frac{1}{2} \right] \otimes \left[ \frac{1}{2} \right] \) decays into one singlet \([0]\) and one tripllet \([1]\), for \( N = 3 \) the product \( \left[ \frac{1}{2} \right] \otimes \left[ \frac{1}{2} \right] \otimes \left[ \frac{1}{2} \right] \) results in

\[
\left[ \frac{1}{2} \right] \otimes \left[ \frac{1}{2} \right] = \left[ 0 \right] \oplus \left[ 1 \right],
\]

and it is clear from the well-known Clebsch-Gordan decomposition of the Kronecker product of two SU(2)- irreps of dimension \( 2j_1 + 1 \) and \( 2j_2 + 1 \) corresponding to angular momenta \( j_1, j_2 \), respectively

\[
[j_1] \otimes [j_2] = \bigoplus_{j = |j_1 - j_2|}^{j_1 + j_2} [j],
\]

how two construct the table inductively, however, there is no simple construction scheme for an explicit expression. Still, a thorough study of the pattern leads to the compact formula for the non-vanishing multiplicities (here with \( J = K/2 \)), which can be proved by induction

\[
\alpha_{N,K} = \frac{2J + 1}{N + 1} \frac{(N + 1)!}{(N/2 - J)!(N/2 + J + 1)!}, \quad J = \begin{cases} 0, 1, \ldots, \frac{N}{2} & : N \text{ even} \\ \frac{1}{2}, \frac{3}{2}, \ldots, \frac{N}{2} & : N \text{ odd} \end{cases}. \tag{11}
\]

Of course one has for the dimension of the full Hilbert space \( \mathcal{H} \)

\[
\sum_K (K + 1)\alpha_{N,K} = 2^N, \tag{12}
\]

where

\[
(K + 1)\alpha_{N,K} = \frac{(2J + 1)^2N!}{(N/2 - J)!(N/2 + J + 1)!} = \frac{(2J + 1)^2}{N + 1} \left( \frac{N + 1}{N/2 - J} \right). \tag{13}
\]

**Large N, J limit**

For large values of \( N \) and values of \( J \) such that the degeneracy eq. (13) is comparatively large, an approximation of binomial coefficients based on their relation to the normal distribution is

\[
\binom{n}{k} \simeq \frac{2^{n+1/2}}{\sqrt{n\pi}} e^{-\frac{(k-n/2)^2}{(n/2)}}, \tag{14}
\]

From this expression one derives the asymptotic formulae

\[
(K + 1)\alpha_{N,K} \simeq (2J + 1)^2 \frac{2^{N+3/2}}{(N + 1)^3\pi} e^{-2(J+1/2)^2/(N+1)} \simeq \frac{2^N}{\sqrt{2^{-7}N^3\pi}} J(J + 1) e^{-2J(J+1)/N}
\]

\[
\simeq \frac{2^N}{\sqrt{2^{-7}N^3\pi}} J^2 e^{-2J^2/N} \tag{15}
\]

Considering \( J \) as a continuous parameter for the moment, the discrete sum rule eq. (12) becomes

\[
\int_0^\infty dJ \frac{2^N}{\sqrt{2^{-7}N^3\pi}} J^2 e^{-2J^2/N} = 2^N \tag{16}
\]

as a rescaled version of the Gaussian identity

\[
\int_0^\infty dx x^2 e^{-x^2} = \frac{\sqrt{\pi}}{4}. \tag{17}
\]
The most abundant irreps are located around

\[ J \simeq \sqrt{N/2}. \]  

(18)

For large \( N \), \( \text{erf}(\sqrt{8}) - 4\sqrt{2/\pi}e^{-8} \simeq 99.8866\% \) of all states are located in the interval \( 0 < J < 2\sqrt{N} \), however, the width of the distribution is not sharp and also of the order of \( \sqrt{N/2} \). From the expressions above, all relevant thermodynamic properties can be calculated in the case of large systems potentially using Hubbard-Stratonovich transformation techniques [3, 4].

**Microcanonical black hole ensemble**

The literature contains a plethora of more or less naive attempts to derive expressions for the thermodynamic behavior of black holes [5, 6] by counting microscopic degrees of freedom of many kinds. The assumption that the horizon of a spherically symmetric black hole of mass \( M \) and corresponding area \( A = 4\pi R_S^2 = 16\pi G^2 M^2/c^4 \) (\( G \) is the gravitational constant, \( c \) the speed of light in vacuo, and \( R_S = 2GM/c^2 \) the Schwarzschild radius) is segmented into area quanta \( a = 4\ln(2)(l_P^2) \) of the order of the Planck length squared \( l_P^2 = \hbar G/c^3 \), inspired Bekenstein, Mukhanov [7] and Kastrup [8] to postulate and study discrete energys levels for such black holes in the spirit of the Bohr-Sommerfeld quantization in atomic physics, with a mass spectrum \( M_n = \mu\sqrt{n}, n \in \mathbb{N}_0 \), and an energy level degeneracy of the form \( \nu(n) = b^n \) with \( \mu = \sigma m_P = \sigma \sqrt{\hbar c/G}, \sigma = o(1), b > 1. \)

Inspired by the Heisenberg spin picture above one may consider the special case with energy spectrum

\[ E(n) = \sqrt{\frac{\ln(2)}{4\pi}} E_P \sqrt{n}, \quad n = 0, 1, 2, 3, \ldots, \quad E_P = \sqrt{\hbar c^3/G}, \]  

(19)

with a degeneracy given by

\[ \nu(n) = 2^n. \]  

(20)

The microcanonical partition function \( \Omega(E) \) is the number of energy eigenstates with an energy below \( E \)

\[ \Omega(E) = \sum_{n=0}^{\frac{E}{E_P} - 1} 2^n = 2^{\frac{E}{E_P}} = 2^{\frac{4\pi}{\ln(2)} \left( \frac{E}{E_P} \right)^2} = e^{4\pi \left( \frac{E}{E_P} \right)^2}. \]  

(21)

The microcanonical temperature becomes

\[ k_B T = \frac{\Omega}{\omega}, \quad \omega = \frac{\partial \Omega}{\partial E} = 8\pi \frac{E}{E_P^2} \Omega, \]  

(22)

hence one arrives at the Bekenstein-Hawking temperature

\[ k_B T = \frac{E_P^2}{8\pi M c^2}, \]  

(23)

which is quantized if the model is taken seriously, and the entropy is

\[ S = k_B \ln \Omega = k_B \frac{A}{4l_P^2}. \]  

(24)

Of course, it is possible to assume that the idealized energy spectrum of a spherically symmetric black hole alone in an infinite asymptotically flat universe is non-degenerate and therefore

\[ n(E) = e^{4\pi \left( \frac{E}{E_P} \right)^2}, \]  

(25)

or

\[ E(n) = E_P \sqrt{\frac{\ln(n)}{4\pi}}. \]  

(26)
Of course, an unstable object is never isolated since it quantum superposes itself with its decay states, thereby blurring its energy spectrum into the complex plane. Even if the ansatz eq. (19) is not valid, it remains tempting to figure the black hole horizon as a network of \( N \) area quanta interacting (locally) as spins with decreasing coupling strength \( g(N) \) as \( N \) increases together with a decreasing surface gravity \( \sim R_S^{-1} \), interpolating the discrete spectrum eq. (19) to something smoother, closer in style to eq. (26). Of course, the exactly solvable Heisenberg model on a complete graph probably is to coarse to display a realistic spectrum adequately. Still, assuming that the number of spins is given by the surface or the energy of the black hole

\[
N = \frac{4\pi R_S^2}{4 \ln(2) l_P^2} = \frac{4\pi G}{\ln(2) h c^5} E^2 \tag{27}
\]

together with the Ansatz

\[
g(N) = \sqrt{\frac{\ln(2)}{\pi}} \frac{E_P}{\sqrt{N}} \tag{28}
\]

the collapse of non-rotating matter with energy \( E \) will lead to the creation of \( N \) qubits coupled to a total qubit momentum \( J \simeq \sqrt{N/2} \), such that

\[
E = g(N) J^2 = \sqrt{\frac{\ln(2)}{\pi}} \frac{E_P}{\sqrt{N}} N = \sqrt{\frac{\ln(2)}{4\pi}} E_P \sqrt{N} \tag{29}
\]
in accordance with eqns. (27) and (18). Such a size of \( J \) corresponds to the radius of the black hole as a fuzzy sphere.

Conclusions

A group theoretical result concerning the \( n \)-fold tensor product of the defining, two-dimensional pseudo-real representation of SU(2) has been presented. The present letter is an illustration for the construction of a theoretical framework mimicking aspects of mainstream theories inspired and dogmatized by string or loop quantum gravity approaches or black hole weather forecasting [9], which have less testable predictive power than Ptolemy’s epicycle theory [10] due to the current lack of relevant experimental data concerning the physics in the immediate vicinity of a low-mass black hole like, e.g., experimental spectra of electromagnetic or muonic black hole radiation. Of course, it would be interesting to investigate the implications of holographic firewall dynamics, decoherence and gravity induced wave function collapse [11] within a Heisenberg model approach in a future work. A simple amusing firewall scenario dating back to 2005 can be found in [12].

Although originally this paper was intended as a hoax, one should keep in mind that no experimental evidence exists for string theory or quantum physics in very strong gravitational fields, but a familiarization process in the scientific community with some mathematical structures during the last decades. This does not depreciate the merits of the concepts mentioned above, however, it should be discussed whether some of them should be considered in a preliminary manner as a part of a new scientific branch classified as, e.g., physico-mathematical speculation or physico-mathematical philosophy. In this view, any Heisenberg spin model of black holes is not more or less realistic than the models in the current literature, provided that the model it is made more or less consistent.

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