SPECTRAL CHARACTERIZATIONS OF ANTI-REGULAR GRAPHS

CESAR O. AGUILAR, JOON-YEOB LEE, ERIC PIATO, AND BARBARA J. SCHWEITZER

Abstract. We study the eigenvalues of the unique connected anti-regular graph $A_n$. Using Chebyshev polynomials of the second kind, we obtain a trigonometric equation whose roots are the eigenvalues and perform elementary analysis to obtain an almost complete characterization of the eigenvalues. In particular, we show that the interval $\Omega = \left[\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$ contains only the trivial eigenvalues $\lambda = -1$ or $\lambda = 0$, and any closed interval strictly larger than $\Omega$ will contain eigenvalues of $A_n$ for all $n$ sufficiently large. We also obtain bounds for the maximum and minimum eigenvalues, and for all other eigenvalues we obtain interval bounds that improve as $n$ increases. Moreover, our approach reveals a more complete picture of the bipartite character of the eigenvalues of $A_n$, namely, as $n$ increases the eigenvalues are (approximately) symmetric about the number $-\frac{1}{2}$. We also obtain an asymptotic distribution of the eigenvalues as $n \to \infty$. Finally, the relationship between the eigenvalues of $A_n$ and the eigenvalues of a general threshold graph is discussed.

1. Introduction

Let $G = (V, E)$ be an $n$-vertex simple graph, that is, a graph without loops or multiple edges, and let $\deg_G(v)$ denote the degree of $v \in V$. It is an elementary exercise to show that $G$ contains at least two vertices of equal degree. If $G$ has all vertices with equal degree then $G$ is called a regular graph. We say then that $G$ is an anti-regular graph if $G$ has only two vertices of equal degree. If $G$ is anti-regular it follows easily that the complement graph $\overline{G}$ is also anti-regular since $\deg_G(v) = (n - 1) - \deg_{\overline{G}}(v)$. It was shown in [2] that up to isomorphism, there is only one connected anti-regular graph on $n$ vertices and that its complement is the unique disconnected $n$-vertex anti-regular graph. Let us denote by $A_n$ the unique connected anti-regular graph on $n \geq 2$ vertices. The graph $A_n$ has several interesting properties. For instance, it was shown in [3] that $A_n$ is universal for trees, that is, every tree graph on $n$ vertices is isomorphic to a subgraph of $A_n$. Anti-regular graphs are threshold graphs [4] which have numerous applications in computer science and psychology. Within the family of threshold graphs, the anti-regular graph is uniquely defined by its independence polynomial [7]. Also, the eigenvalues of the Laplacian matrix of $A_n$ are all distinct integers and the missing eigenvalue from $\{0, 1, \ldots, n\}$ is $\lfloor (n + 1)/2 \rfloor$. In [6], the characteristic and matching polynomial of $A_n$ are studied and several recurrence relations are obtained for these polynomials, along with some spectral properties of the adjacency matrix of $A_n$.

2000 Mathematics Subject Classification. Primary 05C50, 15B05; Secondary 05C75, 15A18.

Key words and phrases. adjacency matrix; threshold graph; antiregular graph; Chebyshev polynomials; Toeplitz matrix.
In this paper, we study the eigenvalues of the adjacency matrix of $A_n$. If $V(G) = \{v_1, \ldots, v_n\}$ is the vertex set of the graph $G$ then the adjacency matrix of $G$ is the $n \times n$ symmetric matrix $A$ with entry $A(i, j) = 1$ if $v_i$ and $v_j$ are adjacent and $A(i, j) = 0$ otherwise. From now on, whenever we refer to the eigenvalues of a graph we mean the eigenvalues of its adjacency matrix. It is known that the eigenvalues of $A_n$ have algebraic multiplicity equal to one and take on a bipartite character [6] in the sense that if $n$ is even then half of the eigenvalues are negative and the other half are positive, and if $n$ is odd then $\lambda = 0$ is an eigenvalue and half of the remaining eigenvalues are positive and the other half are negative.

Our approach to studying the eigenvalues of $A_n$ relies on a natural labeling of the vertices that results in a block triangular structure for the inverse adjacency matrix. The blocks are tridiagonal pseudo-Toeplitz matrices and Hankel matrices. We are then able to employ the connection between tridiagonal Toeplitz matrices and Chebyshev polynomials to obtain a trigonometric equation whose roots are the eigenvalues. Performing elementary analysis on the roots of the equation we obtain an almost complete characterization of the eigenvalues of $A_n$. In particular, we show that the only eigenvalues contained in the closed interval $\Omega = [-\frac{1-\sqrt{2}}{2}, -\frac{1+\sqrt{2}}{2}]$ are the trivial eigenvalues $\lambda = -1$ or $\lambda = 0$, and any closed bounded interval strictly larger than $\Omega$ will contain eigenvalues of $A_n$ for all $n$ sufficiently large. This improves a result in [10] obtained for general threshold graphs and we conjecture that $\Omega$ is a forbidden eigenvalue interval for all threshold graphs (besides the trivial eigenvalues $\lambda = 0$ or $\lambda = -1$). We also obtain bounds for the maximum and minimum eigenvalues, and for all other eigenvalues we obtain interval bounds that improve as $n$ increases. Moreover, our approach reveals a more complete picture of the bipartite character of the eigenvalues of $A_n$, namely, as $n$ increases the non-trivial eigenvalues are (approximately) symmetric about the number $-\frac{1}{2}$. Lastly, we obtain an asymptotic distribution of the eigenvalues as $n \to \infty$. We conclude the paper by arguing that a characterization of the eigenvalues of $A_n$ will shed light on the broader problem of characterizing the spectrum of general threshold graphs.

2. Main results

It is known that the eigenvalues of $A_n$ are simple and that $\lambda = -1$ is an eigenvalue if $n$ is even and $\lambda = 0$ is an eigenvalue if $n$ is odd [6]. In either case, we will call $\lambda = -1$ or $\lambda = 0$ the trivial eigenvalue of $A_n$ and will be denoted by $\lambda_0$. Throughout this paper, we denote the positive eigenvalues of $A_n$ as

$$\lambda_1^+ < \lambda_2^+ < \cdots < \lambda_k^+$$

and the negative eigenvalues (excluding $\lambda_0$) as

$$\lambda_{k-1}^- < \lambda_{k-2}^- < \cdots < \lambda_1^-$$
if \( n = 2k \) is even and
\[
\lambda_k^- < \lambda_{k-1}^- < \cdots < \lambda_1^-
\]
if \( n = 2k + 1 \) is odd. The eigenvalues are labeled this way because \( \{\lambda_j^+, \lambda_j^-\} \) should be thought of as a pair for \( j \in \{1, 2, \ldots, k-1\} \). In [10], it is proved that a threshold graph has no eigenvalue in the interval \((-1,0)\). Our first result supplies a forbidden interval for the non-trivial eigenvalues of \( A_n \).

**Theorem 2.1.** Let \( A_n \) denote the connected anti-regular graph with \( n \) vertices. The only eigenvalue of \( A_n \) in the interval \( \Omega = [-\frac{1-\sqrt{2}}{2}, -\frac{1+\sqrt{2}}{2}] \) is \( \lambda_0 \in \{-1, 0\} \).

Based on numerical experimentation, and our observations in Section 8, we make the following conjectures.

**Conjecture 2.1.** For any \( n \), the anti-regular graph \( A_n \) has the smallest positive eigenvalue and has the largest non-trivial negative eigenvalue among all threshold graphs on \( n \) vertices.

By Theorem 2.1, a proof of the previous conjecture would also prove the following.

**Conjecture 2.2.** Other than the trivial eigenvalues \( \{0, -1\} \), the interval \( \Omega = [-\frac{1-\sqrt{2}}{2}, -\frac{1+\sqrt{2}}{2}] \) does not contain an eigenvalue of any threshold graph.

Our next result establishes the asymptotic behavior of the eigenvalues of smallest magnitude as \( n \to \infty \).

**Theorem 2.2.** Let \( A_n \) be the connected anti-regular graph with \( n = 2k \) if \( n \) is even and \( n = 2k + 1 \) if \( n \) is odd. Let \( \lambda_1^+(k) \) denote the smallest positive eigenvalue of \( A_n \) and let \( \lambda_1^-(k) \) denote the negative eigenvalue of \( A_n \) closest to the trivial eigenvalue \( \lambda_0 \). The following hold:

(i) The sequence \( \{\lambda_1^+(k)\}_{k=1}^\infty \) is strictly decreasing and converges to \( -\frac{1+\sqrt{2}}{2} \).
(ii) The sequence \( \{\lambda_1^-(k)\}_{k=1}^\infty \) is strictly increasing and converges to \( -\frac{1-\sqrt{2}}{2} \).

As a result, the interval \( \Omega = [-\frac{1-\sqrt{2}}{2}, -\frac{1+\sqrt{2}}{2}] \) in Theorem 2.1 is best possible in the sense that any closed bounded interval strictly larger than \( \Omega \) will contain eigenvalues of \( A_n \) (other than the trivial eigenvalue) for all sufficiently large \( n \).

Our next main result says that \( \lambda_j^+ + \lambda_j^- + 1 \approx 0 \) for almost all \( j \in \{1, 2, \ldots, k-1\} \) provided that \( k \) is sufficiently large. In other words, the eigenvalues are approximately symmetric about the number \( -\frac{1}{2} \).

**Theorem 2.3.** Let \( A_n \) be the connected anti-regular graph where \( n = 2k \) or \( n = 2k + 1 \). Fix \( r \in (0,1) \) and let \( \varepsilon > 0 \) be arbitrary. Then for \( k \) sufficiently large,
\[
|\lambda_j^+ + \lambda_j^- + 1| < \varepsilon
\]

\[\text{1During the publication process of this paper, we were notified that both conjectures have been proved by E. Ghorbani; see } \\https://arxiv.org/abs/1807.10302\]
for all \( j \in \{1, 2, \ldots, k - 1\} \) such that \( \frac{2j}{2k-1} \leq r \) if \( n \) is even and \( \frac{j}{k} \leq r \) if \( n \) is odd.

Note that the proportion of integers \( j \in \{1, 2, \ldots, k - 1\} \) that satisfy the inequality in Theorem 2.3 is \( r \). Hence, Theorem 2.3 implies that as \( k \) increases a larger proportion of the eigenvalues are (approximately) symmetric about the point \(-\frac{1}{2}\). Lastly, we obtain an asymptotic distribution of the eigenvalues of all anti-regular graphs.

**Theorem 2.4.** Let \( \sigma(n) \) denote the set of the eigenvalues of \( A_n \), let \( \sigma = \bigcup_{n \geq 1} \sigma(n) \), and let \( \overline{\sigma} \) denote the closure of \( \sigma \). Then

\[
\overline{\sigma} = (-\infty, -\frac{1-\sqrt{2}}{2}] \cup \{0, -1\} \cup \left[-\frac{1+\sqrt{2}}{2}, \infty\right).
\]

It turns out that if we restrict \( n \) to even then \( \overline{\sigma} = (-\infty, -\frac{1-\sqrt{2}}{2}] \cup \{-1\} \cup \left[-\frac{1+\sqrt{2}}{2}, \infty\right) \), and if we restrict \( n \) to odd then \( \overline{\sigma} = (-\infty, -\frac{1-\sqrt{2}}{2}] \cup \{0\} \cup \left[-\frac{1+\sqrt{2}}{2}, \infty\right) \).

### 3. Eigenvalues of tridiagonal Toeplitz matrices

Our study of the eigenvalues of \( A_n \) relies on the relationship between the eigenvalues of tridiagonal Toeplitz matrices and Chebyshev polynomials \([8, 9]\), and so we briefly review the necessary background. The *Chebyshev polynomial of the second kind* of degree \( m \), denoted by \( U_m(x) \), is the unique polynomial such that

\[
U_m(\cos \theta) = \frac{\sin((m+1)\theta)}{\sin(\theta)}.
\]

The first several \( U_m \)'s are \( U_0(x) = 1 \), \( U_1(x) = 2x \), \( U_2(x) = 4x^2 - 1 \), and \( U_3(x) = 8x^3 - 4x \). The sequence of polynomials \( \{U_m\}_{m=0}^{\infty} \) satisfies the three-term recurrence relation

\[
U_m(x) = 2xU_{m-1}(x) - U_{m-2}(x)
\]

for \( m \geq 2 \). From (1), the zeros \( x_1, x_2, \ldots, x_m \) of \( U_m(x) \) are easily determined to be

\[
x_j = \cos \left( \frac{j\pi}{m+1} \right), \quad j = 1, 2, \ldots, m.
\]

Chebyshev polynomials are used extensively in numerical analysis and differential equations and the reader is referred to \([8]\) for a thorough introduction to these interesting polynomials.

A real tridiagonal Toeplitz matrix is a matrix of the form

\[
T = \begin{pmatrix}
a & c & & \\
b & \ddots & \ddots & \\
& \ddots & \ddots & c \\
& & b & a
\end{pmatrix}
\]
for \(a,b,c \in \mathbb{R}\). For our purposes, and to simplify the presentation, we assume that \(c = b\). We can then write \(T = aI + bM\) where \(I\) is the identity matrix and

\[
M = \begin{pmatrix}
0 & 1 & & & \\
1 & \ddots & \ddots & & \\
& \ddots & \ddots & \ddots & \\
& & 1 & \ddots & \\
& & & \ddots & 1
\end{pmatrix}.
\]

If \(\lambda\) is an eigenvalue of \(M\) then clearly \(a + b\lambda\) is an eigenvalue of \(T\). Let \(\phi_m(t) = \text{det}(tI - M)\) denote the characteristic polynomial of the \(m \times m\) matrix \(M\). The Laplace expansion of \(\phi_m(t)\) along the last row produces the recurrence relation

\[
\phi_m(t) = t\phi_{m-1}(t) - \phi_{m-2}(t)
\]

for \(m \geq 2\), with \(\phi_0(t) = 1\) and \(\phi_1(t) = t\). It then follows that \(\phi_m(t) = U_m(t/2)\). Indeed, we have that \(U_0(t/2) = 1\) and \(U_1(t/2) = 2(t/2) = t\), and from the recurrence (2) we have

\[
U_m(t/2) = 2(t/2)U_{m-1}(t/2) - U_{m-2}(t/2) = tU_{m-1}(t/2) - U_{m-2}(t/2).
\]

4. The Anti-Regular Graph \(A_n\)

As already mentioned, the anti-regular graph \(A_n\) is an example of a threshold graph. Threshold graphs were first studied independently by Chvátal and Hammer [11] and by Henderson and Zalcstein [12]. There exists an extensive literature on the applications and algorithmic aspects of threshold graphs and the reader is referred to [4, 5] for a thorough introduction. A threshold graph \(G\) on \(n \geq 2\) vertices can be obtained via an iterative procedure as follows. One begins with a single vertex \(v_1\) and at step \(i \geq 2\) a new vertex \(v_i\) is added that is either connected to all existing vertices (a dominating vertex) or not connected to any of the existing vertices (an isolated vertex). The iterative construction of \(G\) is best encoded with a binary creation sequence \(b = (b_1, b_2, \ldots, b_n)\) where \(b_1 = 0\) and, for \(i \in \{2, \ldots, n\}\), \(b_i = 1\) if \(v_i\) was added as a dominating vertex or \(b_i = 0\) if \(v_i\) was added as an isolated vertex. The resulting vertex set \(V(G) = \{v_1, v_2, \ldots, v_n\}\) that is consistent with the iterative construction of \(G\) will be called the canonical labeling of \(G\). In the canonical labeling, the adjacency matrix of \(G\) takes the form

\[
A = \begin{pmatrix}
0 & b_2 & b_3 & \cdots & b_{n-1} & b_n \\
b_2 & 0 & b_3 & \cdots & & \\
b_3 & b_3 & 0 & \cdots & & \\
& \ddots & \ddots & \ddots & \ddots & \\
& & b_{n-1} & \ddots & 0 & b_n \\
b_n & \cdots & \cdots & \cdots & b_{n-1} & 0
\end{pmatrix}.
\]
Figure 1. The connected anti-regular graph $A_8$ in the canonical labeling

For the anti-regular graph $A_n$, the associated binary sequence is $b = (0, 1, 0, 1, \ldots, 0, 1)$ if $n$ is even and is $b = (0, 0, 1, 0, 1, \ldots, 0, 1)$ if $n$ is odd. In what follows, we focus on the case that $n$ is even. In Section 7 we describe the details for the case that $n$ is odd.

Example 4.1. When $n = 8$ the graph $A_n$ in the canonical labeling is shown in Figure 1 and the associated adjacency matrix is

$$A = \begin{pmatrix}
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0
\end{pmatrix}.$$  

As will be seen, a distinct labeling of the vertex set of $A_n$ results in a block structure for $A$. Let $J = J_k$ denote the $k \times k$ all ones matrix and let $I = I_k$ denote the $k \times k$ identity matrix.

Lemma 4.1. The adjacency matrix of $A_{2k}$ can be written as

$$A = \begin{pmatrix}
0 & B \\
B & J - I
\end{pmatrix}.$$
where $B$ is the $k \times k$ Hankel matrix

\[
B = \begin{pmatrix}
1 & & & & & & & \\
& 1 & & & & & & \\
& & \ddots & & & & & \\
& & & 1 & & & & \\
& & & & \ddots & & & \\
& & & & & \ddots & & \\
& & & & & & \ddots & \\
1 & 1 & \cdots & \cdots & 1
\end{pmatrix}.
\]

Proof. Recall that $A_n$ is the unique connected graph on $n$ vertices that has exactly only two vertices of the same degree. Moreover, it is known [2] that the repeated degree of $A_{2k}$ is $k$, that is, the degree sequence of $A_{2k}$ in non-increasing order is

\[
d(A_n) = (n-1, n-2, \ldots, \frac{n}{2}, \frac{n}{2}, \frac{n}{2} - 1, \ldots, 2, 1).
\]

It is clear that the degree sequence of the graph with adjacency matrix (4) is also (5). Since $A_n$ is uniquely determined by its degree sequence the claim holds.

Remark 4.1. Starting with the canonically labelled vertex set of $A_n$, the permutation

\[
\sigma = \begin{pmatrix}
v_1 & v_2 & v_3 & \cdots & v_{n-2} & v_{n-1} & v_n \\
v_{\frac{n}{2}} & v_{\frac{n}{2}+1} & v_{\frac{n}{2}-1} & \cdots & v_{n-1} & v_1 & v_n
\end{pmatrix}
\]

relabels the vertices of $A_n$ so that its adjacency matrix is transformed from (3) to (4) via the permutation matrix associated to $\sigma$. The newly labelled graph is such that $\deg(v_i) \leq \deg(v_{i+1})$. For example, when $n = 8$ the adjacency matrix (4) is

\[
A = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0
\end{pmatrix}.
\]

To study the eigenvalues of $A$ we will obtain an eigenvalue equation for $A^{-1}$. Expressions for $A^{-1}$ involving sums of certain matrices are known when the vertex set of $A_n$ is canonically labelled [1]. On the other hand, our choice of vertex labels for $A_n$ produces a closed-form expression for $A^{-1}$. The proof of the following is left as a straightforward computation.

Lemma 4.2. Consider the adjacency matrix (4) of $A_n$ where $n = 2k$. Then

\[
A^{-1} = \begin{pmatrix}
V & W \\
W & 0
\end{pmatrix}
\]
where \( W = B^{-1} \) and \( V = -B^{-1}(J - I)B^{-1} \). Explicitly,

\[
W = \begin{pmatrix}
-1 & 1 \\
\ddots & \ddots \\
-1 & 1
\end{pmatrix}
\quad \text{and} \quad
V = \begin{pmatrix}
2 & -1 & & \\
-1 & \ddots & & \\
& & \ddots & 2 & -1 \\
& & & -1 & 0
\end{pmatrix}.
\]

Notice that \( W \) is a Hankel matrix and the \((k - 1) \times (k - 1)\) leading principal submatrix of \( V \) is a tridiagonal Toeplitz matrix.

Example 4.2. For our running example when \( n = 8 \) we have

\[
A^{-1} = \begin{pmatrix} V & W \\ W & 0 \end{pmatrix} = \begin{pmatrix}
2 & -1 & 0 & 0 & 0 & 0 & -1 & 1 \\
-1 & 2 & -1 & 0 & 0 & -1 & 1 & 0 \\
0 & -1 & 2 & -1 & -1 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

5. The eigenvalues of \( A_n \)

Suppose that \( z = (x, y) \in \mathbb{R}^{2k} \) is an eigenvector of \( A^{-1} \) with eigenvalue \( \alpha \in \mathbb{R} \), where \( x, y \in \mathbb{R}^k \). From \( A^{-1}z = \alpha z \) we obtain the two equations

\[
Vx + Wy = \alpha x \\
Wx = \alpha y
\]

and after substituting \( y = \frac{1}{\alpha}Wx \) into the first equation and re-arranging we obtain

\[
(\alpha^2I - \alpha V - W^2)x = 0.
\]

Clearly, we must have \( x \neq 0 \). Let \( R(\alpha) = \alpha^2I - \alpha V - W^2 \) so that \( \det(R(t)) = \det(tI - A^{-1}) \) is the characteristic polynomial of \( A^{-1} \). It is straightforward to verify that

\[
R(\alpha) = \begin{pmatrix}
f(\alpha) & \alpha + 1 & & \\
\alpha + 1 & \ddots & & \\
& & \ddots & f(\alpha) & \alpha + 1 \\
& & \ddots & \ddots & \cdots \\
& & & \ddots & f(\alpha) & \alpha + 1 \\
& & & & & \alpha + 1 & \alpha^2 - 1
\end{pmatrix}.
\]
where \( f(\alpha) = \alpha^2 - 2\alpha - 2 \). Since it is already known that \( \alpha = -1 \) is an eigenvalue of \( A^{-1} \) (this can easily be seen from the last column or row of \( R(\alpha) \)), we consider instead the matrix

\[
S(\alpha) = \frac{1}{(\alpha + 1)} R(\alpha) = \begin{pmatrix}
h(\alpha) & 1 \\
1 & \ddots & \ddots \\
& \ddots & \ddots & \ddots \\
& & \ddots & \ddots & 1 \\
& & & 1 & h(\alpha)
\end{pmatrix}
\]

where \( h(\alpha) = \frac{\alpha^2 - 2\alpha - 2}{\alpha + 1} \). Hence, \( \alpha \neq -1 \) is an eigenvalue of \( A^{-1} \) if and only if \( \det(S(\alpha)) = 0 \).

We now obtain a recurrence relation for \( \det(S(\alpha)) \). To that end, notice that the \((k-1)\times(k-1)\) leading principal submatrix of \( S(\alpha) \) is a tridiagonal Toeplitz matrix. Hence, for \( m \geq 1 \) define

\[
\phi_m(\alpha) = \det \begin{pmatrix}
h(\alpha) & 1 \\
1 & \ddots & \ddots \\
& \ddots & \ddots & \ddots \\
& & \ddots & \ddots & 1 \\
& & & 1 & h(\alpha)
\end{pmatrix}_{m \times m}
\]

A straightforward Laplace expansion of \( \det(S(\alpha)) \) along the last row yields

\[
\det(S(\alpha)) = (\alpha - 1)\phi_{k-1}(\alpha) - \phi_{k-2}(\alpha).
\]

Hence, \( \alpha \neq -1 \) is an eigenvalue of \( A^{-1} \) if and only if

\[
(\alpha - 1)\phi_{k-1}(\alpha) - \phi_{k-2}(\alpha) = 0.
\]

On the other hand, for \( m \geq 2 \) the Laplace expansion of \( \phi_m(\alpha) \) along the last row produces the recurrence relation

\[
\phi_m(\alpha) = h(\alpha)\phi_{m-1}(\alpha) - \phi_{m-2}(\alpha)
\]

with \( \phi_0(\alpha) = 1 \) and \( \phi_1(\alpha) = h(\alpha) \). We can therefore conclude that \( \phi_m(\alpha) = U_m\left(\frac{h(\alpha)}{2}\right) \) and thus \( \alpha \neq -1 \) is an eigenvalue of \( A^{-1} \) if and only if

\[
(\alpha - 1)U_{k-1}\left(\frac{h(\alpha)}{2}\right) - U_{k-2}\left(\frac{h(\alpha)}{2}\right) = 0.
\]

Substituting \( \alpha = \frac{1}{\lambda} \) into (7) and re-arranging yields

\[
\lambda = \frac{U_{k-1}(\beta(\lambda))}{U_{k-1}(\beta(\lambda)) + U_{k-2}(\beta(\lambda))}
\]

where \( \beta(\lambda) = \frac{h(1/\lambda)}{2} = \frac{1-2\lambda-2\lambda^2}{2\lambda(\lambda+1)} \). Recalling the definition (11) of \( U_m(x) \), we have proved the following.
Theorem 5.1. Let \( n = 2k \) and let \( A_n \) denote the connected anti-regular graph with \( n \) vertices. Then \( \lambda \) is an eigenvalue of \( A_n \) if and only if
\[
\lambda = \frac{\sin(k\theta)}{\sin(k\theta) + \sin((k-1)\theta)}
\]
where \( \theta = \arccos\left(\frac{1-2\lambda^2-2\lambda}{2\lambda(\lambda+1)}\right) \).

Remark 5.1. In [6, Theorem 3], recurrence relations for the characteristic polynomial of the adjacency matrix of \( A_n \) involving Chebyshev polynomials are obtained using combinatorial methods.

We now analyze the character of the solution set of (8). To that end, first define the function
\[
\theta(\lambda) = \arccos\left(\frac{1-2\lambda^2-2\lambda}{2\lambda(\lambda+1)}\right).
\]
Using the fact that the domain and range of \( \arccos \) is \([-1, 1]\) and \([0, \pi]\), respectively, it is straightforward to show that the domain and range of \( \theta(\lambda) \) is \((-\infty, -\frac{1-\sqrt{2}}{2}] \cup [-1+\sqrt{2}, \infty) \) and \([0, \pi]\), respectively. The graph of \( \theta(\lambda) \) is displayed in Figure 2. Next, define the function
\[
F(\theta) = \frac{\sin(k\theta)}{\sin(k\theta) + \sin((k-1)\theta)}.
\]
In the interval \((0, \pi)\), the function \(F\) has vertical asymptotes at
\[
\gamma_j = \frac{2j\pi}{2k-1}, \quad j = 1, 2, \ldots, k - 1.
\]
This follows from the trigonometric identity
\[
\sin(k\theta) + \sin((k - 1)\theta) = 2\sin\left(\frac{(2k - 1)\theta}{2}\right)\cos\left(\frac{\theta}{2}\right).
\]
For notational consistency we define \(\gamma_0 = 0\). Hence, \(F\) is continuously differentiable on the set \((0, \gamma_1) \cup (\gamma_1, \gamma_2) \cup \cdots \cup (\gamma_k, \pi)\). Moreover, using l'Hôpital’s rule it is straightforward to show that
\[
\lim_{\theta \to 0} F(\theta) = \frac{k}{2k - 1}
\]
and
\[
\lim_{\theta \to \pi} F(\theta) = k.
\]
Hence, there is no harm in defining \(F(0) = \frac{k}{2k - 1}\) and \(F(\pi) = k\) so that we can take \(D = [0, \gamma_1) \cup (\gamma_1, \gamma_2) \cup \cdots \cup (\gamma_{k-1}, \pi]\) as the domain of continuity of \(F\).

We can now prove Theorem 2.1.

**Proof of Theorem 2.1.** The domain of \(\theta(\lambda)\) does not contain any point in the interior of \(\Omega\) and therefore no solution of (8) is in the interior of \(\Omega\). At the boundary points of \(\Omega\) we have
\[
\theta(-\frac{1}{\sqrt{2}}) = \theta(-\frac{1+\sqrt{2}}{\sqrt{2}}) = 0.
\]
On the other hand, \(F(0) = \frac{k}{2k - 1}\) and thus the boundary points of \(\Omega\) are not solutions to (8) either. The case that \(n\) is odd is similar and will be dealt with in Section 7. \(\square\)

We now analyze solutions to (8) by treating \(\theta\) as the unknown variable and expressing \(\lambda\) in terms of \(\theta\). To that end, solving for \(\lambda\) from the equation \(\theta = \arccos\left(\frac{1 - 2\lambda^2 - 2\lambda}{2\lambda(\lambda + 1)}\right)\) yields the two solutions
\[
\lambda = f_1(\theta) = \frac{-(\cos \theta + 1) + \sqrt{(\cos \theta + 1)(\cos \theta + 3)}}{2(\cos \theta + 1)}
\]
(10)
\[
\lambda = f_2(\theta) = \frac{-(\cos \theta + 1) - \sqrt{(\cos \theta + 1)(\cos \theta + 3)}}{2(\cos \theta + 1)}.
\]
Notice that
\[
f_1(\theta) + f_2(\theta) = -1,
\]
a fact that will be used to show the bipartite character of large anti-regular graphs. Both \(f_1\) and \(f_2\) are continuous on \([0, \pi]\), continuously differentiable on \((0, \pi)\), and \(\lim_{\theta \to \pi^-} f_1(\theta) = \infty\) and \(\lim_{\theta \to \pi^-} f_2(\theta) = -\infty\). In Figures 3-4 we plot the functions \(f_1(\theta), f_2(\theta),\) and \(F(\theta)\) for the values \(k = 8\) and \(k = 16\) in the interval \(0 \leq \theta \leq \pi\). A dashed line at the value \(\lambda = -\frac{1}{2}\) is included to emphasize that it is a line of symmetry between the graphs of \(f_1\) and \(f_2\).
Figures 3-4 show that the graphs of $F$ and $f_1$ intersect exactly $k$ times, say at $\theta_1^+, \ldots, \theta_k^+$, and thus $\lambda_j^+ = f_1(\theta_j^+)$ for $j = 1, 2, \ldots, k$ are the positive eigenvalues of $A_n$. Similarly, $F$ and $f_2$ intersect exactly $(k - 1)$ times, say at $\theta_1^-, \ldots, \theta_{k-1}^-$, and thus $\lambda_j^- = f_2(\theta_j^-)$ for $j = 1, 2, \ldots, k - 1$ are the negative eigenvalues of $A_n$ besides the eigenvalue $\lambda = -1$. The following theorem formalizes the above observations and supplies interval estimates for the eigenvalues.

**Theorem 5.2.** Let $A_n$ be the connected anti-regular graph with $n = 2k$ vertices. Let $F(\theta)$ be defined as in (9) and let $f_1(\theta)$ and $f_2(\theta)$ be defined as in (10), and recall that $\gamma_j = \frac{2\pi j}{2k - 1}$ for $j = 0, 1, \ldots, k - 1$.

(i) The functions $F(\theta)$ and $f_1(\theta)$ intersect exactly $k$ times in the interval $0 < \theta < \pi$. If $\theta_1^+ < \theta_2^+ < \cdots < \theta_k^+$ are the intersection points then the positive eigenvalues of $A_n$ are

$$f_1(\theta_1^+) < f_1(\theta_2^+) < \cdots < f_1(\theta_k^+).$$

Moreover, for $j = 1, 2, \ldots, k - 1$ it holds that

$$f_1(\gamma_{j-1}) < f_1(\theta_j^+) < f_1(\gamma_j).$$

(ii) The functions $F(\theta)$ and $f_2(\theta)$ intersect exactly $(k - 1)$ times in the interval $0 < \theta < \pi$. If $\theta_1^- < \theta_2^- < \cdots < \theta_{k-1}^-$ are the intersection points then the negative eigenvalues of $A_n$ are

$$f_2(\theta_{k-1}^-) < \cdots < f_2(\theta_2^-) < f_2(\theta_1^-) < -1.$$
Moreover, for \( j = 1, 2, \ldots, k - 1 \) it holds that
\[
f_2(\gamma_j) < f_2(\theta_j^-) < f_2(\gamma_{j-1}).
\]

Proof. One computes that
\[
f'_1(\theta) = \frac{\sin \theta}{2 (\cos \theta + 1) \sqrt{(\cos \theta + 1)(\cos \theta + 3)}}
\]
and thus \( f'_1(\theta) > 0 \) for \( \theta \in (0, \pi) \). Therefore, \( f_1 \) is strictly increasing on the interval \((0, \pi)\). Since \( f_2(\theta) = -1 - f_1(\theta) \) it follows that \( f_2 \) is strictly decreasing on the interval \((0, \pi)\). On the other hand, using basic trigonometric identities and the relation \( \sin(\theta)U_{k-1}(\cos \theta) = \sin(k\theta) \), we compute that
\[
F'(\theta) = \frac{[-k + U_{k-1}(\cos \theta) \cos((k-1)\theta)] \sin \theta}{[\sin(k\theta) + \sin((k-1)\theta)]^2}.
\]
It is known that \( \max_{x \in [-1,1]} |U_m(x)| = (m + 1) \) and the maximum occurs at \( x = \pm 1 \). Therefore, \( F'(\theta) < 0 \) for all \( \theta \in D \setminus \{0, \pi\} \). It follows that \( F \) is a strictly decreasing function on \( D \), and when restricted to the interval \((\gamma_j, \gamma_{j+1})\) for any \( j = 1, \ldots, k - 2 \), \( F \) is a bijection onto \(( -\infty, \infty) \). Now, since \( f_1 \) is a strictly increasing continuous function on \([\gamma_j, \gamma_{j+1}]\) for \( j = 1, 2, \ldots, k - 2 \), the graphs of \( F \) and \( f_1 \) intersect at exactly one point inside the interval \((\gamma_j, \gamma_{j+1})\). A similar argument applies to \( f_2 \) and \( F \) on each interval \((\gamma_j, \gamma_{j+1})\) for \( j = 1, 2, \ldots, k - 2 \). Now consider the leftmost interval \([0, \gamma_1]\). We have that \( f_1(0) < F(0) \) and since \( f_1 \) is strictly increasing and continuous on \([0, \gamma_1]\), and \( F \) is strictly decreasing and
\[
t_k = \frac{(\theta_k^+ - \gamma_{k-1})}{(\pi - \gamma_{k-1})}
\]
for \(k = 125, 250, 500, \ldots, 32000\).

Table 1. The ratio \(t_k = \frac{(\theta_k^+ - \gamma_{k-1})}{(\pi - \gamma_{k-1})}\) for \(k = 125, 250, 500, \ldots, 32000\)

\[
\begin{array}{|c|c|}
\hline
n = 2k & t_k \\
\hline
250 & 0.5020031290 \\
500 & 0.5010007838 \\
1000 & 0.5005001962 \\
2000 & 0.5002500492 \\
4000 & 0.5001250123 \\
8000 & 0.5000625018 \\
16000 & 0.5000312567 \\
32000 & 0.5000156204 \\
\hline
\end{array}
\]

\[\lim_{\theta \to \gamma_{k-1}} F(\theta) = -\infty, \] \(F\) and \(f_1\) intersect only once in the interval \((0, \gamma_1)\). A similar argument holds for \(f_2\) and \(F\) on the interval \((0, \gamma_1)\). Finally, on the interval \((\gamma_{k-1}, \pi]\), we have \(f_2(\gamma_{k-1}) < F(\pi)\) and since \(f_2\) decreases and \(F\) is strictly increasing on the interval \((\gamma_{k-1}, \pi]\) then \(f_2\) and \(F\) do not intersect there. On the interval \((\gamma_{k-1}, \pi]\), \(f_2\) and \(F\) do not intersect there. This completes the proof.

Theorem 2.2 now follows from the fact that \(\lim_{k \to \infty} f_1(\theta_k^+) = f_1(0) = -\frac{1+\sqrt{2}}{2}\) and that \(\lim_{k \to \infty} f_2(\theta_k^-) = f_2(0) = -\frac{1-\sqrt{2}}{2}\). We also obtain the following corollary.

**Corollary 5.1.** Let \(\lambda_{\text{max}} > 0\) and \(\lambda_{\text{min}} < 0\) denote the largest and smallest eigenvalues, respectively, of the connected anti-regular graph \(A_n\) where \(n\) is even. Then

\[F(\pi) = \frac{n}{2} < \lambda_{\text{max}}\]

and

\[f_2 \left(\frac{2(n/2-1)\pi}{n-1}\right) < \lambda_{\text{min}}.\]

Through numerical experiments, we have determined that the mid-point of the interval \((\gamma_{k-1}, \pi]\), which is \(\frac{(4k-3)\pi}{2(2k-1)}\), is a good approximation to \(\theta_k^+ \in (\gamma_{k-1}, \pi]\), that is,

\[\lambda_{\text{max}} \approx F \left(\frac{(4k-3)\pi}{2(2k-1)}\right).\]

In Table 1 we show the results of computing the ratio \(t_k = \frac{(\theta_k^+ - \gamma_{k-1})}{(\pi - \gamma_{k-1})}\) for \(k = 125, 250, \ldots, 32000\) which shows that possibly \(\lim_{k \to \infty} t_k = \frac{1}{2}\).

### 6. The eigenvalues of large anti-regular graphs

A graph \(G\) is called bipartite if there exists a partition \(\{X, Y\}\) of the vertex set \(V(G)\) such that any edge of \(G\) contains one vertex in \(X\) and the other in \(Y\). It is known that the eigenvalues of a bipartite graph \(G\) are symmetric about the origin. Figures 3-4 reveal that
for the connected anti-regular graph $A_{2k}$ a similar symmetry property about the point $-\frac{1}{2}$ is approximately true. Specifically, if $\lambda \neq \lambda_{\max}$ is a positive eigenvalue of $A_{2k}$ then $-1 - \lambda$ is approximately an eigenvalue of $A_{2k}$, and moreover the proportion $r \in (0,1)$ of the eigenvalues that satisfy this property to within a given error $\varepsilon > 0$ increases as the number of vertices increases.

Recall that if $\lambda_1^+ < \lambda_2^+ < \cdots < \lambda_k^+$ denote the positive eigenvalues of $A_{2k}$ then there exists unique $\theta_1^+ < \theta_2^+ < \cdots < \theta_k^+$ in the interval $(0, \pi)$ such that $\lambda_j^+ = f_1(\theta_j^+)$, and if $\lambda_{k-1}^- < \lambda_{k-2}^- < \cdots < \lambda_1^- < -1$ denote the negative eigenvalues of $A_{2k}$ there exists unique $\theta_1^- < \theta_2^- < \cdots < \theta_{k-1}^-$ in $(0, \pi)$ such that $\lambda_j^- = f_2(\theta_j^-)$ for $j = 1, 2, \ldots, k - 1$. With this notation we now prove Theorem 2.3.

Proof of Theorem 2.3: Both $f_1(\theta)$ and $f_2(\theta)$ are continuous on $[0, \pi)$ and therefore are uniformly continuous on the interval $[0, r\pi]$. Hence, there exists $\delta > 0$ such that if $\theta, \gamma \in [0, r\pi]$ and $|\theta - \gamma| < \delta$ then $|f_1(\theta) - f_1(\gamma)| < \varepsilon/2$ and $|f_2(\theta) - f_2(\gamma)| < \varepsilon/2$. Let $k$ be such that $\frac{2\pi}{2k-1} \leq \delta$ and let $j^* \in \{1, \ldots, k-1\}$ be the largest integer such that $\frac{2j^*}{2k-1} \leq r$. Then for all $j \in \{1, \ldots, j^*\}$ it holds that $[\gamma_{j-1}, \gamma_j] \subset [0, r\pi]$. Let $c_j \in [\gamma_{j-1}, \gamma_j]$ be arbitrarily chosen for each $j \in \{1, \ldots, j^*\}$. Then $\theta_j^+, \theta_j^-, c_j \in [\gamma_{j-1}, \gamma_j]$ implies that $|f_1(\theta_j^+) - f_1(c_j)| < \varepsilon/2$ and $|f_2(\theta_j^-) - f_2(c_j)| < \varepsilon/2$ for $j \in \{1, \ldots, j^*\}$. Therefore, if $j \in \{1, \ldots, j^*\}$ then

$$|\lambda_j^+ + \lambda_j^- + 1| = |f_1(\theta_j^+) + f_2(\theta_j^-) + 1|$$

$$= |f_1(\theta_j^+) - f_1(c_j) + f_1(c_j) + f_2(\theta_j^-) - f_2(c_j) + f_2(c_j) + 1|$$

$$\leq |f_1(\theta_j^+) - f_1(c_j)| + |f_2(\theta_j^-) - f_2(c_j)| + |f_1(c_j) + f_2(c_j) + 1|$$

$$= |f_1(\theta_j^+) - f_1(c_j)| + |f_2(\theta_j^-) - f_2(c_j)|$$

$$< \varepsilon$$

where we used the fact that $f_1(c_j) + f_2(c_j) + 1 = 0$. This completes the proof for the even case. As discussed in Section 7, the odd case is similar.

Note that the proportion of $j \in \{1, 2, \ldots, k - 1\}$ such that $\frac{2j}{2k-1} \leq r$ is approximately $r$. In the next theorem we obtain estimates for $|\lambda_j^+ + \lambda_j^- + 1|$ using the Mean Value theorem.

Theorem 6.1. Let $A_n$ be the connected anti-regular graph where $n = 2k$. Then for all $1 \leq j \leq k - 1$ it holds that

$$|\lambda_j^+ + \lambda_j^- + 1| \leq \frac{4\pi f_1(\gamma_j)}{2k - 1}.$$ 

In particular, for fixed $r \in (0,1)$ and a given arbitrary $\varepsilon > 0$, if $k$ is such that $\frac{4\pi f_1(r\pi)}{2k - 1} < \varepsilon$ then

$$|\lambda_j^+ + \lambda_j^- + 1| < \varepsilon$$
for all $1 \leq j \leq \frac{(2k-1)r}{2}$.

**Proof.** First note that since $f_1(\theta) + f_2(\theta) = -1$ it follows that $f_2'(\theta) = -f_1'(\theta)$. The derivative $f_1'$ vanishes at $\theta = 0$, is non-negative and strictly increasing on $[0, \pi)$. Therefore, by the Mean value theorem, on any closed interval $[a, b] \subset [0, \pi)$, both $f_1(\theta)$ and $f_2(\theta)$ are Lipschitz with constant $K = f_1'(b)$. Hence, a similar computation as in the proof of Theorem 2.3 shows that

$$|\lambda_j^+ + \lambda_j^- + 1| \leq \frac{4\pi f_1'(\gamma_j)}{2k-1},$$

for $j = 1, 2, \ldots, k - 1$. Therefore, if $k$ is such that $\frac{4\pi f_1'(\gamma)}{2k-1} < \varepsilon$ then for $1 \leq j \leq \frac{(2k-1)r}{2}$ we have that $\frac{2\pi j}{2k-1} \leq r\pi$ and therefore

$$|\lambda_j^+ + \lambda_j^- + 1| \leq \frac{4\pi f_1'(\gamma_j)}{2k-1} \leq \frac{4\pi f_1'(r\pi)}{2k-1} < \varepsilon.$$

\(\square\)

A similar proof gives the following estimates for the eigenvalues with error bounds.

**Theorem 6.2.** Let $A_n$ be the connected anti-regular graph where $n = 2k$. For $1 \leq j \leq k - 1$ it holds that

$$|\lambda_j^+ - f_1(\gamma_j)| \leq \frac{2\pi f_1'(\gamma_j)}{2k-1}$$

and

$$|\lambda_j^- - f_2(\gamma_j)| \leq \frac{2\pi f_1'(\gamma_j)}{2k-1}.$$

We now prove Theorem 2.4.

**Proof of Theorem 2.4.** It is clear that $\{-1, 0\} \subset \sigma \subset \bar{\sigma}$. Let $\varepsilon > 0$ be arbitrary and let $y \in \left[-\frac{1+\sqrt{2}}{2}, \infty\right)$. Then $y \in \bar{\sigma}$ if there exists $\mu \in \sigma$ such that $|\mu - y| < \varepsilon$. If $y \in \sigma$ the result is trivial, so assume that $y \notin \sigma$. Since $f_1 : [0, \pi) \to \left[-\frac{1+\sqrt{2}}{2}, \infty\right)$ is a bijection, there exists a unique $\theta' \in [0, \pi)$ such that $y = f_1(\theta')$. Let $c \in [0, \pi)$ be such that $\theta' < c < \pi$. For $k$ sufficiently large, there exists $j \in \{1, \ldots, k - 1\}$ such that $\theta' \in [\gamma_j-1, \gamma_j]$ and $\frac{2j\pi}{2k-1} \leq c$. Increasing $k$ if necessary, we can ensure that also $\frac{2\pi f_1'(c)}{2k-1} < \varepsilon$. Then by the Mean value theorem applied to $f_1$ on the interval $I = [\min\{\theta', \theta_j^+\}, \max\{\theta', \theta_j^+\}]$, there exists $c_j \in I$ such that

$$|\lambda_j^+ - y| = |f_1(\theta_j^+) - f_1(\theta')| \leq |\theta_j^+ - \theta'| f_1'(c_j) < \frac{2\pi}{2k-1} f_1'(c) < \varepsilon,$$

where in the penultimate inequality we used the fact that $f_1'$ is increasing and $c_j < c$. This proves that $y$ is a limit point of $\sigma$ and thus $y \in \bar{\sigma}$. A similar argument can be performed in the case that $y \in (-\infty, -\frac{1+\sqrt{2}}{2}]$ using $f_2$. \(\square\)
7. The odd case

In this section, we give an overview of the details for the case that \( A_n \) is the unique connected anti-regular graph with \( n = 2k + 1 \) vertices. In the canonical labeling of \( A_n \), the partition \( \pi = \{\{v_1, v_2\}, \{v_3\}, \ldots, \{v_n\}\} = \{C_1, C_2, \ldots, C_{2k}\} \) is an equitable partition of \( A_n \) [14]. In other words, \( \pi \) is the degree partition of \( A_n \) (we note that this is true for any threshold graph). The quotient graph \( A_n/\pi \) has vertex set \( \pi \) and its \( 2k \times 2k \) adjacency matrix is

\[
A/\pi = \begin{pmatrix}
0 & 1 & 0 & 1 & \cdots & 0 & 1 \\
2 & 0 & 0 & 1 & \cdots & 0 & 1 \\
0 & 0 & 0 & 1 & \cdots & \vdots & \vdots \\
2 & 1 & 1 & 0 & \cdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \\
2 & 1 & 1 & \cdots & \cdots & 1 & 0
\end{pmatrix}.
\]

In other words, \( A/\pi \) is obtained from the adjacency matrix of the anti-regular graph \( A_{2k} \) (in the canonical labeling) with the 1’s in the first column replaced by 2’s. It is a standard result that all of the eigenvalues of \( A/\pi \) are eigenvalues of \( A_n \) [14]. At this point, we proceed just as in Section 4. Under the same permutation \( (6) \) of the vertices of \( A_n/\pi \), the quotient adjacency matrix \( A/\pi \) takes the block form

\[
A/\pi = \begin{pmatrix} 0 & B \\ C & J - I \end{pmatrix}
\]

where

\[
C = \begin{pmatrix} 2 \\ 1 & 2 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 1 & 2 \end{pmatrix}
\]

Then

\[
(A/\pi)^{-1} = \begin{pmatrix} -C^{-1}(J - I)B^{-1} & C^{-1} \\ B^{-1} & 0 \end{pmatrix}.
\]

After computations similar to the even case, the analogue of (7) is

\[
\frac{(\alpha^2 - 1/2)}{\alpha + 1} U_{k-1} \left( \frac{h(\alpha)}{2} \right) - \frac{1}{2} U_{k-2} \left( \frac{h(\alpha)}{2} \right) = 0.
\]

After making the substitution \( \alpha = \frac{1}{\lambda} \) and simplifying one obtains

\[
\frac{(2 - \lambda^2)}{\lambda(\lambda + 1)} U_{k-1}(\beta(\lambda)) - U_{k-2}(\beta(\lambda)) = 0
\]
or equivalently
\[
\frac{(2 - \lambda^2)}{\lambda(\lambda + 1)} = \frac{U_{k-2}(\beta(\lambda))}{U_{k-1}(\beta(\lambda))} = \frac{\sin((k-1)\theta)}{\sin(k\theta)}.
\]
The analogue of Theorem 5.1 in the odd case is the following.

**Theorem 7.1.** Let \( n = 2k + 1 \) and let \( A_n \) denote the connected anti-regular graph with \( n \) vertices. Then \( \lambda \neq 0 \) is an eigenvalue of \( A_n \) if and only if
\[
\frac{(2 - \lambda^2)}{\lambda(\lambda + 1)} = \frac{\sin((k-1)\theta)}{\sin(k\theta)}
\]
where \( \theta = \arccos\left(\frac{1-2\lambda-2\lambda^2}{2\lambda(\lambda+1)}\right) \).

Define the function \( g(\lambda) = \frac{(2 - \lambda^2)}{\lambda(\lambda + 1)} \). Changing variables from \( \lambda \) to \( \theta \) as in the even case, and defining \( g_1(\theta) = g(f_1(\theta)) \), \( g_2(\theta) = g(f_2(\theta)) \), and in this case \( F(\theta) = \frac{\sin((k-1)\theta)}{\sin(k\theta)} \), we obtain the two equations
\[
g_1(\theta) = F(\theta) \\
g_2(\theta) = F(\theta).
\]
The explicit expressions for \( g_1 \) and \( g_2 \) are
\[
g_1(\theta) = 2 + 3\cos(\theta) + \sqrt{(\cos(\theta + 1)(\cos(\theta + 3))}
\]
\[
g_2(\theta) = 2 + 3\cos(\theta) - \sqrt{(\cos(\theta + 1)(\cos(\theta + 3))}.
\]
The graphs of \( g_1, g_2, \) and \( F \) on the interval \([0, \pi]\) are shown in Figure 5. In this case, the singularities of \( F \) occur at the equally spaced points
\[
\gamma_j = \frac{j\pi}{k}, \quad j = 1, 2, \ldots, k.
\]
If \( \theta_1^+ < \theta_2^+ < \cdots < \theta_k^+ \) denote the unique points where \( F \) and \( g_1 \) intersect then \( f_1(\theta_1^+) < f_1(\theta_2^+) < \cdots < f_1(\theta_k^+) \) are the positive eigenvalues of \( A_{2k+1} \). Similarly, if \( \theta_1^- < \theta_2^- < \cdots < \theta_k^- \) denote the unique points where \( F \) and \( g_2 \) intersect then \( f_2(\theta_1^-) < f_2(\theta_2^-) < \cdots < f_2(\theta_k^-) \) are the negative eigenvalues of \( A_{2k+1} \).

Theorems 2.1-2.3 hold for the odd case with now \( \lambda = 0 \) being the trivial eigenvalue. Theorem 5.2, Theorem 6.1, and Theorem 6.2 proved for the even case hold almost verbatim for the odd case; the only change is that the ratio \( \frac{2\pi}{2k-1} \) is now \( \frac{\pi}{k} \).

8. THE EIGENVALUES OF THRESHOLD GRAPHS

In this section, we discuss how a characterization of the eigenvalues of \( A_n \) could be used to characterize the eigenvalues of general threshold graphs. Let \( G \) be a threshold graph with binary creation sequence \( b = (0^{s_1}, 1^{t_1}, \ldots, 0^{s_k}, 1^{t_k}) \), where \( 0^{s_i} \) is short-hand for \( s_i \geq 0 \) consecutive zeros, and similarly for \( 1^{t_i} \). Let \( V(G) = \{v_1, v_2, \ldots, v_n\} \) denote the associated
canonical labeling of $G$ consistent with $b$. The set partition $\pi = \{C_1, C_2, \ldots, C_{2k}\}$ of $V(G)$ where $C_1$ contains the first $s_1$ vertices, $C_2$ contains the next $t_1$ vertices, and so on, is an equitable partition of $G$. The $2k \times 2k$ quotient graph $G/\pi$ has adjacency matrix
\[ A_{\pi} = A_{2k} + \text{diag}(0, \beta_1, \ldots, 0, \beta_k) \]
where $A_{2k}$ is the adjacency matrix of the connected anti-regular graph with $2k$ vertices and $\beta_i = 1 - \frac{1}{t_i}$, see for instance [13]. The eigenvalues of $G$ other than the trivial eigenvalues $\lambda = -1$ and/or $\lambda = 0$ are exactly the eigenvalues of $A_{\pi}$. Presumably, the characterization of the eigenvalues of $A_{2k}$ that we have done in this paper will be useful in characterizing the eigenvalues of $A_{\pi}$. We leave this investigation for a future paper.

9. Acknowledgements

The authors acknowledge the support of the National Science Foundation under Grant No. ECCS-1700578.

References

[1] R. Bapat. On the adjacency matrix of a threshold graph. *Linear Algebra and its Applications*, 439(10):3008–3015, 2013.
[2] M. Behzad and G. Chartrand. No graph is perfect. *The American Mathematical Monthly*, 74(8):962–963, 1967.
[3] R. Merris. Antiregular graphs are universal for trees. *Publikacije Elektrotehničkog fakulteta. Serija Matematika*, pages 1–3, 2003.
[4] N.V.R. Mahadev and U.N. Peled. *Threshold graphs and related topics*. Vol. 56. Elsevier, 1995.
[5] Martin C. Golumbic. *Algorithmic graph theory and perfect graphs*. Vol. 57. Elsevier, 2004.

[6] E. Munarini. Characteristic, admittance, and matching polynomials of an antiregular graph. *Applicable Analysis and Discrete Mathematics*, 3(1):157–176, 2009.

[7] V.E. Levit and E. Mandrescu. On the Independence Polynomial of an Antiregular Graph. *Carpathian Journal of Mathematics*, 28(2): 279–288, 2012.

[8] J.C. Mason and D.C. Handscomb. *Chebyshev polynomials*. Chapman and Hall/CRC, 2002.

[9] D. Kulkarni, D. Schmidt, and S.D. Tsui. Eigenvalues of tridiagonal pseudo-Toeplitz matrices. *Linear Algebra and its Applications*, 297: 63–80, 1999.

[10] D. Jacobs, V. Trevisan, and F. Tura. Eigenvalues and energy in threshold graphs. *Linear Algebra and its Applications*, 465: 412–425, 2015.

[11] V. Chvátal and P.L. Hammer. Aggregation of Inequalities in Integer Programming. *Annals of Discrete Mathematics*, 1: 145–162, 1977.

[12] P.B. Henderson and Y. Zalcstein. A graph-theoretic characterization of the PV class of synchronizing primitives. *SIAM Journal on Computing*, 6(1): 88–108, 1977.

[13] A. Banerjee and R. Mehatari. On the normalized spectrum of threshold graphs. *Linear Algebra and its Applications*, 530: 288–304, 2017.

[14] C. Godsil G. Royle. *Algebraic Graph Theory*. Springer, New York, 2001.

**Department of Mathematics, State University of New York, Geneseo**

*E-mail address: aguilars@geneseo.edu*

**Department of Mathematics, State University of New York, Geneseo**

*E-mail address: jl56@geneseo.edu*

**Department of Mathematics, State University of New York, Geneseo**

*E-mail address: esp6@geneseo.edu*

**Department of Mathematics, State University of New York, Geneseo**

*E-mail address: bjs22@geneseo.edu*