Coherent States with SU(2) and SU(3) Charges

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Abstract
We define coherent states carrying SU(2) charge by exploiting Schwinger boson representation of SU(2) Lie algebra. These coherent states satisfy continuity property and provide resolution of identity on $S^3$. We further generalize these techniques to construct the corresponding SU(3) charge coherent states. The SU(N) extension is also discussed.

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1 Introduction

The concept of coherent states was introduced by Schrödinger [1] in the context of a harmonic oscillator. These harmonic oscillator coherent states, also called canonical coherent states, have been useful and studied extensively in physics [2]. The next most important coherent states are spin coherent states or SU(2) coherent states which are associated with angular momentum or SU(2) group. Like canonical coherent states, they too have found wide applications in different branches of physics such as quantum optics, statistical mechanics, nuclear physics and condensed matter physics [2]. It is known that these spin coherent states can also be constructed using harmonic oscillators by exploiting either Holstein Primakov or Schwinger boson representation of SU(2) Lie algebra [3, 4, 5]. This harmonic oscillator formulation of spin coherent states is appealing because it is simple and analogous to canonical coherent state construction. Further, it bypasses the action of group elements [6] to get the spin coherent states. Motivated by the resulting simplifications, we recently generalized this harmonic oscillator formulation of coherent states to SU(N) group [7]. In this work, we further exploit the above ideas to construct SU(2) and SU(3) charge coherent states defined on $S^3$ and $S^5$ respectively. The coherent states carrying SU(2) and SU(3) (non-abelian) charges in two and three mode Fock spaces have been discussed in the past [5, 8, 9, 10]. However they are defined on full complex planes and are different from the SU(2) and SU(3) charge coherent states discussed in this paper which are defined on the compact manifolds $S^3$ and $S^5$ respectively. We will further elaborate on these differences as we proceed (section 2.1 and section 3).

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The plan of the paper is as follows: We start with a brief description of harmonic oscillator coherent states. The coherent states discussed later will have their roots in this simple construction. In section 2, using two harmonic oscillators, we exploit Schwinger boson representation to construct SU(2) coherent states. This construction is known and is contained in [3]. However, we have included this section to make the presentation self contained. In section 3 we define SU(2) charge coherent states which satisfy resolution of identity over the SU(2) group manifold $S^3$. In section 4 we generalize these ideas to SU(3) group. In section 5 we give SU(N) construction.

The harmonic oscillator coherent states are defined as:

$$ |z⟩ = \sum_{n=0}^{∞} \frac{(za^\dagger)^n}{n!} |0⟩ = \sum_{n=0}^{∞} \frac{z^n}{\sqrt{n!}} |n⟩ $$

These coherent states are associated with Heisenberg-Weyl group whose Lie algebra is given by:

$$ [a, a^\dagger] = I, \ [a, I] = 0, \ [a^\dagger, I] = 0. $$

In (2), $I$ is the identity operator. The manifold corresponding to the Heisenberg-Weyl group is the complex $z$ plane. The coherent states in (1) are analytic over this group manifold $\mathcal{M}$ and satisfy the resolution of identity:

$$ \int_{\mathcal{M}} dμ(z)|z⟩ ∫_{-∞}^{∞}⟨z| ≡ I. $$

In (3), $dμ(z) ≡ exp(-|z|^2) dz d\bar{z}$ is the measure over $\mathcal{M}$. We now give the construction of SU(2) coherent state which is similar to (1).

## 2 SU(2) Coherent States

The SU(2) group involves 3 angular momentum generators, $J_1, J_2$ and $J_3$ and the Lie algebra is:

$$ [J^a, J^b] = i \epsilon_{abc} J^c, \ a, b, c = 1, 2, 3. $$

The SU(2) Casimir operator is $\vec{J}.\vec{J}$ with eigenvalue $j(j+1)$ where $j$ is integer or half integer spin. The angular momentum algebra in (4) can be realized in terms of a doublet of harmonic oscillator annihilation creation operators $\vec{a} \equiv (a_1, a_2)$ and $\vec{a}^\dagger \equiv (a^\dagger_1, a^\dagger_2)$ respectively [3]. The number operators are $\hat{N}_1 \equiv a_1^\dagger a_1$ and $\hat{N}_2 \equiv a_2^\dagger a_2$ They satisfy the bosonic commutation relations:

$$ [a_i, a_j^\dagger] = δ_{ij}, \ i, j = 1, 2. $$
The vacuum state is $|0, 0>$ and the number operator basis is written as $|n_1, n_2>$. It satisfies:

$$\hat{N}_1 |n_1, n_2> = n_1 |n_1, n_2>, \quad \hat{N}_2 |n_1, n_2> = n_2 |n_1, n_2>.$$  \hspace{1cm} (6)

We can now define the angular momentum operators in (4) as:

$$J^a \equiv \frac{1}{2} a_i^\dagger (\sigma^a)_{ij} a_j,$$  \hspace{1cm} (7)

where $\sigma^a$ denote the Pauli matrices. It is easy to check that the operators in (7) satisfy (4). Further, as they involve one creation and one annihilation operator, the Casimir ($C$) is:

$$C = \hat{N}_1 + \hat{N}_2.$$  \hspace{1cm} (8)

One can also explicitly check that $\vec{J} \cdot \vec{J} \equiv \frac{1}{4} C (C + 2) = \frac{1}{4} \hat{a}^\dagger \cdot \hat{a} (\hat{a}^\dagger \cdot \hat{a} + 2)$. Thus the representations of $SU(2)$ can be characterized by the eigenvalues of the total occupation number operator and the spin value $j$ is equal to $n/2 \equiv (n_1 + n_2)/2$. With the SU(2) Schwinger representation (7), we can directly generalize (1) and write down the spin coherent states as:

$$|\vec{z} >_n = \sqrt{n!} \sum_{n_1, n_2 = 0}^n \frac{z_1^{n_1} z_2^{n_2}}{\sqrt{n_1! n_2!}} |n_1, n_2>$$

$$= \sqrt{n!} \sum_{m=0}^n \frac{z_1^{n-m} z_2^{m}}{\sqrt{(n-m)! m!}} |n-m, m>$$  \hspace{1cm} (9)

In (9), the $t$ over the summation sign implies $n_1 + n_2 = n = 2j$. It is easy to see that the states $|n_1, n_2>$ with the above constraint form $(2j+1)$ dimensional representation of SU(2) group. Further, $(z_1, z_2)$ is a doublet of complex numbers satisfying the constraint:

$$|z_1|^2 + |z_2|^2 = 1.$$  \hspace{1cm} (10)

Thus the coherent states in (9) are defined over the sphere $S^3$. It is easy to check:

$$\int_{S^3} d^2z_1 \ d^2z_2 \delta(|z_1|^2 + |z_2|^2 - 1)|\vec{z} >_n < \vec{z}| = I_n$$  \hspace{1cm} (11)

In (11), $I_n$ is $(n+1) \times (n+1)$ dimensional unit matrix. By construction, the coherent states in (9) satisfy:

$$C|\vec{z} >_n = n \ |\vec{z} >_n.$$  \hspace{1cm} (12)

Thus the SU(2) coherent states are smoothly defined over the SU(2) group manifold ($M = S^3$) and are eigenstates of total angular momentum operator ($J^2 = J_1^2 + J_2^2 + J_3^2$).

\[\text{3In [4] the SU(2) coherent state construction is through Holstein Primakov representation of SU(2) Lie algebra.}\]
In terms of angular momentum basis, $|j; m\rangle \equiv |n_1 = j + m, n_2 = j - m\rangle$, the spin coherent states can be written as:

$$|\vec{z}_{s=2j}\rangle = \sqrt{2j!} \sum_{m=-j}^{j} \frac{1}{(j + m)! (j - m)!} z_1^{j+m} z_2^{j-m} |j; m\rangle \quad (13)$$

We can also obtain (13) by directly operating the SU(2) group element $U(\theta, \phi, \psi) \equiv \exp(i\phi J_3) \exp(i\theta J_2) \exp(i\psi J_3)$ on the highest weight state in the $j^{th}$ representation [6]:

$$|\theta, \phi, \psi\rangle_j = U(\theta, \phi, \psi) |j; j\rangle,$$

$$= \exp(j\psi) \sum_{m=-j}^{+j} C_m(\theta, \phi) |j; m\rangle \quad (14)$$

In (14),

$$C_m(\theta, \phi) = \sqrt{\frac{2j!}{(j + m)! (j - m)!}} \exp(im\phi) \left(\sin\frac{\theta}{2}\right)^{j-m} \left(\cos\frac{\theta}{2}\right)^{j+m} \quad (15)$$

The identification:

$$z_1 \equiv e^{i\frac{\phi}{2}} e^{i\frac{\psi}{2}} \cos\frac{\theta}{2}, \quad z_2 \equiv e^{i\frac{\phi}{2}} e^{-i\frac{\psi}{2}} \sin\frac{\theta}{2} \quad (16)$$

shows the equivalence of the harmonic oscillator (9) and group action (14) constructions.

We now exploit Schwinger boson representation (7) to construct new types of coherent states which cannot be generated by simple group action.

### 2.1 SU(2) Charge Coherent States

So far the SU(2) coherent states were defined with fixed angular momentum $j$ and they were linear combinations of the states $|j; m\rangle$ with $m$ varying from $-j$ to $+j$. The corresponding weight factors were definite analytic functions on SU(2) manifold. We now define new types of SU(2) coherent states which carry fixed charge ($q=2m$) and are linear combinations of the states $|j; m\rangle$ with $j$ varying from $|m|$ to $\infty$. Again the corresponding weight factors are certain analytic functions on SU(2) manifold. For convenience, we define the SU(2) charge operator $Q$ to be twice of the third component of the angular momentum, i.e,

$$Q = a_1^\dagger a_1 - a_2^\dagger a_2. \quad (17)$$

The SU(2) fixed charge $(q=2m)$ coherent states are defined as:

$$|\vec{z}_q\rangle \equiv |z_1, z_2 >_q = \sum_{n_1, n_2=0}^{\infty} \sqrt{\frac{(n_1 + n_2 + 1)!}{n_1! n_2!}} z_1^{n_1} z_2^{n_2} |n_1, n_2 >$$

$$= \sum_{r=0}^{\infty} \sqrt{\frac{(q + 2r + 1)!}{(q + r)! r!}} z_1^{(q+r)} z_2^{r} |q + r, r > \quad (18)$$
In (18), the \( \sum \) implies that \( n_1 - n_2 = q \). Therefore, the fixed charge coherent states in (18) satisfy:

\[
Q|\vec{z} >_q = q |\vec{z} >_q
\]  

(19)

Further, as the total number operator \( C \) and \( a_1a_2 \) commute with \( Q \), it is easy to check:

\[
f(C)a_1a_2|\vec{z} >_q = z_1z_2 |\vec{z} >_q
\]  

(20)

where \( f(C) \equiv \frac{1}{\sqrt{C(C+2)}} \). We note that in the context of canonical coherent states, the states satisfying \( f(a^\dagger a)|z > = z|z > \) are known as non-linear coherent states and have been extensively studied (see [9] and references therein).

The charge coherent states satisfy resolution of identity over the SU(2) group manifold \( S^3 \) (see appendix):

\[
\int_{S^3} d^2z_1 d^2z_2 \delta(|z_1|^2 + |z_2|^2 - 1)|\vec{z} >_q < \vec{z} | = I_q
\]  

(21)

In (21), \( I_q \) is the infinite dimensional unit matrix. Thus we have constructed the charge coherent states on \( S^3 \). Therefore, using resolution of identity (11), we can express them in terms of fixed angular momentum coherent states (9): The expansion is:

\[
|z_1, z_2> = \sum_{j=|m|} \int_{S^3} d^2w_1 d^2w_2 \ n<\vec{w}_1, \vec{w}_2|z_1, z_2>_q |\vec{w}_1, \vec{w}_2>_n
\]  

(22)

Putting \( n = 2j \) in (9) and \( q = 2j_3 \) in (18), it is easy to see that \( |\vec{z} >_{q=2j_3} \) has non-zero overlap with \( |\vec{z} >_{n=2j} \) iff \( j \geq |j_3| \). More precisely:

\[
\sqrt{n!(n+1)!} (\bar{w}_1 z_1)^{p+q} (\bar{w}_2 z_2)^p; \quad \text{if } n = 2p + q
\]  

\[
= 0 \quad \text{otherwise.}
\]  

(23)

In (23), \( p \) is a non-negative integer such that \( n \) is positive. It is interesting to write (18) in the angular momentum basis \(|j;m>\):

\[
|\vec{z}\rangle_{q=2m} \equiv \sum_{j=|m|} \sqrt{(2j+1)! (j+m)! (j-m)!} z_1^{j+m} z_2^{-m} |j;m>
\]  

(24)

Note that the fixed charge coherent state (24), unlike spin coherent states (13), cannot be obtained by a simple group action like in (14) and they are not eigenstates of \( J^2 \).

At this stage it is interesting to compare our formulation of SU(2) charge coherent states with the already existing formulations [5, 8]. In [8], the SU(2) charge coherent states are defined as:

\[
|\zeta >_q = N_q \sum_{n=0}^{\infty} \frac{\zeta^n}{\sqrt{n!(n+q)!}} |n + q, n>
\]  

(25)
In (25) $N_q$ is the normalization factor and $\zeta$ are the co-ordinates of complex plane. Further, the above coherent states satisfy resolution of identity:

$$\int_{R^2} \frac{d^2 \zeta}{\pi} \phi_q(\zeta) |\zeta >_q, q < | = I_q$$

(26)

In (26), $\phi_q(\zeta) = J_q(2i|\zeta|)K_q(2|\zeta|)$ where $J_q$, $K_q$ are the Bessel and modified Bessels functions respectively. In [5], SU(2) fixed charge coherent states $|z\rangle_{j,q}$ are constructed on the complex plane $z$ which are eigenstates of $J^2$ as well as the charge operator $J^3$. This has been possible because of the use of three harmonic oscillators $(a_1, a_2, a_3)$ to construct SU(2) Lie algebra:

$$J^i = -i\epsilon^{ijk}a_j^\dagger a_k.$$  (27)

Denoting $|n_1, n_2, n_3> = (n_1!n_2!n_3!)^{-\frac{1}{2}}(a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}(a_3^\dagger)^{n_3}|0, 0, 0>$, the SU(2) charged coherent states are defined as:

$$|\zeta >_{j,q} = \sum_{n=0}^{\infty} \left( \frac{2^{j}(n+j)}{(2n+2j+1)!n!} \right)^{\frac{1}{2}} \zeta^n |j, q, j + 2n>$$

(28)

They satisfy [5]:

$$J.J|\zeta >_{j,q} = j(j+1)|\zeta >_{j,q}, \quad Q|\zeta >_{j,q} = q|\zeta >_{j,q}, \quad a.a|\zeta >_{j,q} = \zeta|\zeta >_{j,q}$$

(29)

where $a.a = a_1a_1 + a_2a_2 + a_3a_3$. The resolution of identity is given by:

$$\sum_{j=0}^{\infty} \sum_{q=-j}^{j} \int_{R^2} \frac{d^2 \zeta}{2\pi} \Phi_j(|\zeta|) |\zeta >_{j,q}, q < | = I$$

(30)

In (30), $\Phi_j(|\zeta|)$ are related to modified Bessel functions [5]. Thus the charged coherent states $|z >_q$ defined in (18) or (24) are different from the ones discussed earlier in [5, 8]. In particular, unlike $|z >_q$ which are defined on the SU(2) group manifold $S^3$, $|\zeta >_{j,q}$ in (25) and $|\zeta >_{j,q}$ in (28) are defined on non-compact manifold $R^2$. In the next section, we will instead use the three oscillators to define SU(3) coherent states.

3 SU(3) Charge Coherent States

We will now generalize the fixed SU(2) charge coherent state ideas to the group SU(3). For the sake of simplicity, we will restrict ourselves to SU(3) representations which are completely symmetric. We, therefore, define a triplet of harmonic oscillator creation and annihilation operators satisfying:

$$[a_i, a_j^\dagger] = \delta_{ij} ; i, j = 1, 2, 3.$$  (31)
Let $\frac{\lambda_a}{2}$, $a = 1, 2, \ldots, 8$ be the generators of $SU(3)$ in the fundamental representation; they satisfy the $SU(3)$ Lie algebra \[ [\frac{\lambda_a}{2}, \frac{\lambda_b}{2}] = i f^{abc} \frac{\lambda_c}{2}. \] Let us define the following operators \[ (32): \]

$$Q^a = \frac{1}{2} a_i^\dagger \lambda^a_{ij} a_j ,$$

More explicitly\(^4\):

$$Q^3 = \frac{1}{2} (a_1^\dagger a_1 - a_2^\dagger a_2), \quad Q^8 = \frac{1}{2\sqrt{3}} (a_1^\dagger a_1 + a_2^\dagger a_2 - 2a_3^\dagger a_3)$$
$$Q^1 = \frac{1}{2} (a_1^\dagger a_2 + a_2^\dagger a_1), \quad Q^2 = -\frac{i}{2} (a_1^\dagger a_2 - a_2^\dagger a_1) ,$$
$$Q^4 = \frac{1}{2} (a_1^\dagger a_3 + a_3^\dagger a_1), \quad Q^5 = -\frac{i}{2} (a_1^\dagger a_3 - a_3^\dagger a_1)$$
$$Q^6 = \frac{1}{2} (a_2^\dagger a_3 + a_3^\dagger a_2), \quad Q^7 = -\frac{i}{2} (a_2^\dagger a_3 - a_3^\dagger a_2). \quad (33)$$

It is clear that the total number operator $C = a_1^\dagger a_1 + a_2^\dagger a_2 + a_3^\dagger a_3$ commutes with all the $SU(3)$ generators in (33).

The $SU(3)$ coherent state analogous to (9) are:

$$|\vec{z} >_n = |z_1, z_2, z_3 >_n = \sqrt{n!} \sum_{n_1,n_2,n_3=0}^{n} \frac{z_1^{n_1} z_2^{n_2} z_3^{n_3}}{\sqrt{n_1! n_2! n_3!}} |n_1, n_2, n_3 > \quad (34)$$

In (34), the $t$ over the summation sign implies $n_1 + n_2 + n_3 = n$. With this constraint, the states $|n_1, n_2, n_3 >$ form all the symmetric representations of $SU(3)$. They are of dimensions $\frac{(n+1)(n+2)}{2}$ and the coherent states $|\vec{z} >_n$ satisfy:

$$C |\vec{z} >_n = n |\vec{z} >_n . \quad (35)$$

Further in (34), $(z_1, z_2, z_3)$ is a triplet of complex numbers satisfying the constraint:

$$|z_1|^2 + |z_2|^2 + |z_3|^2 = 1. \quad (36)$$

Note that the $SU(3)$ coherent states in a mixed representation of $SU(3)$ can be defined by introducing a second independent set of oscillators $(b_1^\dagger, b_2^\dagger, b_3^\dagger)$ and defining \[ (7): \]

$$Q^a = \frac{1}{2} a_i^\dagger \lambda^a_{ij} a_j - \frac{1}{2} b_i^\dagger \lambda^a_{ij} b_j , \quad (37)$$

leading to second Casimir $C' = b^\dagger b = b_1^\dagger b_1 + b_2^\dagger b_2 + b_3^\dagger b_3$. However, in this work, to keep the discussion simple and analogous to $SU(2)$ group, which has only symmetric representations, we will be interested only in the symmetric representations of $SU(3)$ with a single Casimir given in (35) and $C' = 0$. Now the $SU(3)$ construction in (34), (35) and\(^7\)

\(^4\)Note that in [5], $J^1 = 2Q^7, J^2 = -2Q^5, J^3 = 2Q^2$ are used to define coherent states with fixed $J^3$ and $\vec{J}$. 

\(^7\)
The SU(3) charge coherent states are given by:

\[
|z_1, z_2, z_3\rangle_{q, l} = \sum_{p=0}^{\infty} \frac{(3p + 2l - q + 2)!}{(p + l)!(p + l - q)!p!} z_1^{p+l} z_2^{p+l-q} z_3^p |p + l, p + l - q, p\rangle
\]  

They satisfy:

\[
Q_1 |z_1, z_2, z_3\rangle_{q, l} = q |z_1, z_2, z_3\rangle_{q, l}, \quad Q_2 |z_1, z_2, z_3\rangle_{q, l} = (2l - q) |z_1, z_2, z_3\rangle_{q, l},
\]

where \(q = q_1\) and \(2l = q_1 + q_2\). The coherent states in (40) are generalization of the corresponding SU(2) coherent states in (18) and satisfy resolution of identity property (see Appendix). We can also expand (40) in terms of (34):

\[
|z_1, z_2, z_3\rangle_{q, l} = \sum_{n=0}^{\infty} \int_{S^5} d^3 w_1 d^3 w_2 d^3 w_3 \ n(w_1, w_2, w_3 |z_1, z_2, z_3\rangle_{q, l} |w_1, w_2, w_3\rangle_n
\]

The overlap is given by:

\[
n(w, \bar{w})_{q, l} = \sqrt{n!(n+2)!} \frac{(\bar{w}_1 z_1)^{p+l}(\bar{w}_2 z_2)^{p+l-q}(\bar{w}_3 z_3)^p}{(p + l)!(p + l - q)!p!} \quad if \ n = 3p + 2l - q
\]

\[
= 0 \quad otherwise
\]  

In (43), \(p\) is any non-negative integer such that \(n\) is +ve. The above expression for the overlap is analogous to (23) in the case of SU(2).

We now compare our SU(3) charge coherent state construction with the ones given in [9, 10]. These coherent states are defined as:

\[
|\zeta >_{q, y} = N_{q, y} \sum_{m=0}^{\infty} \frac{\zeta^m}{m!(m + y + \bar{q})!(m + 2y - \bar{q})!} |m + y + \bar{q}, m + 2y - \bar{q}, m >
\]  

Again, like in the SU(2) case, the SU(3) charge coherent states \(|\zeta >_{q, y}\) are defined over \(R^2\) and thus different from the ones in (40) defined on the compact manifold \(S^5\).

\(^5\)We are following the notations of [9]. The charges in (40) and (44) are related by: \(q = 2\bar{q} - y\) and \(l = \bar{q} + y\).
4 SU(N) Charge Coherent States

We now briefly discuss SU(N) construction by using N-plet of harmonic oscillators. The coherent states analogous to (9) and (34) are:

\[ |z_1, z_2, ..., z_N >_n = \sqrt{n!} \sum_{n_1, n_2, ..., n_N=0}^{n} \frac{z_1^{n_1} z_2^{n_2} ... z_N^{n_N}}{n_1! ... n_N!} |n_1, n_2, ..., n_N > \] (45)

In (45), \( n_1 + n_2 + ... + n_N = n \) and

\[ |z_1|^2 + |z_2|^2 + ... + |z_N|^2 = 1. \] (46)

Thus the coherent states in (45) are defined on \( S^{2N-1} \). Note that, like in SU(3) case, we have again restricted ourselves to only symmetric representations of SU(N). Now within the symmetric representations, the (N-1) charge operators can be chosen as \( Q_l = a_l^\dagger a_l - a_l^\dagger a_{l+1}^\dagger a_{l+1} ; l=1,2,..,(N-1) \) and the corresponding eigenvalues will be denoted by \( q_l \). Now we can easily generalize the charge coherent states in (18) and (40) to SU(N) group:

\[ |z_1, z_2, ..., z_N >_{q_1, q_2, ..., q_{N-1}} = \sum_{n_N=0}^{\infty} L \frac{z_1^{n_1} z_2^{n_2} ... z_N^{n_N}}{\sqrt{n_1! ... n_N!}} |n_1, n_2, ..., n_N > \] (47)

where

\[ L = \sqrt{(n_1 + n_2 + ... + n_N + (N-1))!} \] (48)

and the N occupation numbers \( n_i = n_N + \sum_{j=1}^{N-1} q_j \) and \( i = 1, 2, .., (N-1) \). Like (45), the charge coherent states in (47) satisfy resolution of identity on \( S^{2N-1} \).

5 Summary and Discussion

We have constructed SU(2) charge coherent states which are defined and satisfy resolution of identity over SU(2) group manifold \( S^3 \). Further, they are eigenstates of \( J^3 \) and not of \( \vec{J} . \vec{J} \). We then defined SU(3) and SU(N) charge coherent states on \( S^5 \) and \( S^{2N-1} \) respectively. The spin coherent states have been extensively used to study the partition functions of SU(2) spin models leading to useful semiclassical descriptions [12]. The SU(2) spin model Hamiltonian is given by \( H = \sum_{<i,j>} \vec{J}(i).\vec{J}(j) \) where \( i \) and \( j \) are site indices and \( <i,j> \) denotes the nearest neighbours. This Hamiltonian commutes with \( \vec{J}(i).\vec{J}(i) \), \( \forall i \). Therefore, the the fixed angular momentum coherent state basis in (9) has been exploited for the path integral formulation of the partition function [12]. Instead, let us consider anisotropic SU(2) spin models with Hamiltonians of the form \( H = \sum_{<i,j>} \left[ J^3(i)J^3(j) + K(h(i), h(j)) \right] \) where \( h(i) \equiv a_1^\dagger(i)a_2^\dagger(i) \) and K is a hermitian operator depending on \( h(i) \).
and \( h(j) \). This Hamiltonian does not commute with \( \vec{J}(i) \cdot \vec{J}(i) \) because of the presence of \( a_1^\dagger a_2^\dagger \) and \( a_1 a_2 \) terms. These terms change the corresponding spin value by \( \pm 1 \) respectively as is clear from (8). However, these terms do not change the value of \( J^3 = \frac{1}{2} (a_1^\dagger a_1 - a_2^\dagger a_2) \). Therefore, to study such Hamiltonians, the fixed charge \( (J^3) \) coherent states in (18) or (24) should be useful.

We note that the construction of SU(N) \( (N \geq 3) \) charge coherent states involved only symmetric representations of SU(N). This was the reason we were led to a simple SU(N) generalization of SU(2) results. It would be interesting to consider all the irreducible representations of SU(N) and define the most general SU(N) charge coherent states. This can be done by including \((N-1)\) sets of harmonic oscillators belonging to all the \((N-1)\) fundamental representations of SU(N) [7]. The work in this direction is in progress and will be reported elsewhere.

**Appendix**

In this appendix we prove that the SU(3) charge, hyper-charge coherent states in (40) satisfy resolution of identity property.

\[
I_{q,l} = \int d^2 z_1 d^2 z_2 d^2 z_3 \delta(|z_1|^2 + |z_2|^2 + |z_3|^2 - 1)|z_1, z_2, z_3\rangle_{q,l} \langle z_1, z_2, z_3| \quad (49)
\]

To solve the \( \delta \) function constraints, it is convenient to define

\[
z_1 = r_1 e^{i\theta_1}, \quad z_2 = r_2 e^{i\theta_2}, \quad z_3 = r_3 e^{i\theta_3}.
\]

After integrating out the angles \( \theta_1, \theta_2 \) and \( \theta_3 \), we get:

\[
I_{q,l} = C \sum_{n=0}^{\infty} \frac{(3n + 2l - q + 2)!}{(n + l)! (n + l - q)! n!} \int r_1 dr_1 r_2 dr_2 r_3 dr_3 \delta(r_1^2 + r_2^2 + r_3^2 - 1)(r_1^{n+l})(r_2^{n+l-q})(r_3^n)|n + l, n + l - q, n\rangle \langle n + l, n + l - q, n| \quad (50)
\]

In (50), \( C \) is a constant. In terms of the above radial co-ordinates the \( \delta \) function constraint can be solved by polar decomposition, i.e;

\[
r_1 = r \sin \theta \cos \phi, \quad r_2 = r \sin \theta \sin \phi, \quad r_3 = r \cos \theta
\]

In (51), since \( r_1, r_2, r_3 \geq 0 \), both \( \theta \) and \( \phi \) vary between 0 and \( \frac{\pi}{2} \). Now \( r \) can be trivially integrated off because of the \( \delta \) function and we are left with:

\[
I_{q,l} = C \sum_{n=0}^{\infty} \frac{(3n + 2l - q + 2)!}{(n + l)! (n + l - q)! n!} \int_0^{\frac{\pi}{2}} \sin \theta \cos \theta d\theta (\sin^2 \theta)^{2n+2l-q+1} (\cos^2 \theta)^n \int_0^{\frac{\pi}{2}} \sin \phi \cos \phi d\phi (\sin^2 \phi)^{n+l-q} (\cos^2 \phi)^{n+l}|n + l, n + l - q, n\rangle \langle n + l, n + l - q, n| \quad (52)
\]
We now set: $x = \sin^2 \theta$, $y = \sin^2 \phi$ to get

\[ I_{q,l} = C \sum_{n=0}^{\infty} \frac{(3n + 2l - q + 2)!}{(n + l)! (n + l - q)! n!} \int_0^1 dx x^{2n+2l-q+1} (1-x)^n \int_0^1 dy y^{n+l-q} (1-y)^{n+l} |n+l, n+l-q, n\rangle \langle n+l, n+l-q, n| \]

Using the fact:

\[ B(m, n) = \int_0^1 dx x^{m-1} (1-x)^{n-1} = \frac{(m-1)! (n-1)!}{(m+n-1)!} \]

we get:

\[ I_{q,l} = C \sum_{n=0}^{\infty} |n+l, n+l-q, n\rangle \langle n+l, n+l-q, n| \]

Therefore, $I_{q,l}$ is an identity operator in the Fock space of 3 harmonic oscillators with fixed SU(3) charge and hyper-charge. In the full Hilbert space of 3 harmonic oscillators, $I_{q,l}$ is the projection operator which projects out the fixed charge component characterized by $q$ and $l$. The above proof of resolution of identity can be easily generalized to SU(N) by using polar decomposition analogous to (51) on $S^{2N-1}$.

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