TIGHT LOWER BOUND FOR PATTERN AVOIDANCE SCHUR-POSITIVITY

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Abstract. For a set of permutations (patterns) $\Pi$ in $S_k$, consider the set of all permutations in $S_n$ that avoid all patterns in $\Pi$. An important problem in current algebraic combinatorics is to find pattern sets $\Pi$ such that the corresponding quasi-symmetric function is symmetric for all $n$. Recently, Bloom and Sagan proved that for any $k \geq 4$, the size of such $\Pi$ must be at least 3 unless $\Pi \subseteq \{(1,2,\ldots,k), (k,\ldots,1)\}$, and asked for a general lower bound.

We prove that the minimal size of such $\Pi$ is exactly $k-1$. The proof applies a new generalization of a theorem of Bose from extremal combinatorics. This generalization is proved using the multilinear polynomial approach of Alon, Babai and Suzuki to the extension by Ray-Chaudhuri and Wilson to Bose’s theorem.

Contents

1. Introduction
2. Preliminaries
  2.1. Symmetric and Quasisymmetric Functions
  2.2. Symmetric and Schur-Positive Sets
3. Proof of Theorem 1.7
4. Proof of Theorem 1.4
5. Further Remarks and Open Problems
6. Acknowledgements
References

1. Introduction

Let $S_n$ be the group of permutations on $[n] = \{1,\ldots,n\}$. We write permutations in one-line notation, and denote $\sigma_i := \sigma(i)$. For example, if $\sigma = [5,1,2,4,3] \in S_5$ then $\sigma_1 = 5$ and $\sigma_2 = 1$ (square brackets are used to distinguish one-line notation from disjoint-cycle notation). Furthermore, define the increasing and decreasing permutations in $S_n$: $\iota_n = [1,\ldots,n] = Id_n$, $\delta_n = [n,\ldots,1]$. These permutations are called the monotone elements.

Let $k \leq n$ and let $\pi \in S_k$, $\sigma \in S_n$. We say that $\sigma$ contains the pattern $\pi$, if there exist indices $1 \leq i_1 < \cdots < i_k \leq n$ such that for all $1 \leq j < j' \leq k$,

$$\pi_j < \pi_{j'} \iff \sigma_{i_j} < \sigma_{i_{j'}}.$$  

Otherwise (or if $k > n$), we say that $\sigma$ avoids $\pi$. For example, $[7,2,6,8,1,5,4,3] \in S_8$ contains $[2,4,1,3] \in S_4$, due to the subsequence 2,6,1,4. The set of permutations in $S_n$ that avoid every $\pi \in \Pi$ is denoted $S_n(\Pi)$.

Pattern avoidance has been extensively studied for many years. One of the earliest results in this area is the theorem of Erdős and Szekeres [2], claiming (as a special case) that every $\sigma \in S_{a^2+1}$ contains a monotone subsequence of length $a+1$. That is, $S_n([\iota_{a+1}, \delta_{a+1}]) = \emptyset$ for all $n \geq a^2 + 1$. Enumerative problems involving pattern avoidance were also studied extensively. For example, MacMahon proved [13].

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that \(|S_n(\{[1, 2, 3]\})| = |S_n(\{[3, 2, 1]\})|\) is the Catalan number, and Knuth proved \(\text{[12]}\) that so is \(|S_n(\{\pi\})|\) for every \(\pi \in S_3\).

A set \(S \subseteq S_n\) is called symmetric (Schur-positive), if its quasisymmetric generating function (defined in Subsection 2.2) is symmetric (respectively, Schur-positive). A symmetric function of degree \(n\) corresponds, via the Frobenius characteristic map, to a class function on \(S_n\). It is Schur-positive if and only if the corresponding class function is a proper character (see e.g. \(\text{[1]}\) for details). This point of view gives a rich algebraic structure to symmetric and Schur-positive sets. The search for such sets starts with the seminal papers of Gessel \(\text{[8]}\) and Gessel — Reutenauer \(\text{[9]}\).

In an attempt to connect pattern avoidance with algebraic structures related to Schur-positivity, it is desired to find symmetric pattern avoiding sets of permutations. A remarkable example is the set of arc permutations: the set of permutations \(\sigma\) desired to find symmetric pattern avoiding sets of permutations. A remarkable example is the set of arc singletons \(\Pi = \{\pi\}\) for all \(\pi\in S_n\) avoiding \(\{\pi \in S_4 \mid |\pi(1) - \pi(2)| = 2\}\) \(\text{[3]}\).

**Definition 1.1.** Let \(\Pi \subseteq S_k\). We say that \(\Pi\) is symmetrically avoided, if the pattern avoiding set \(S_n(\Pi)\) is symmetric for all \(n\).

That is, the aforementioned set \(\{\pi \in S_4 \mid |\pi(1) - \pi(2)| = 2\}\) is symmetrically avoided. Motivated by this result, Sagan and Woo posed the problem:

**Question 1.2 (Sagan and Woo \(\text{[16]}\)).** Which \(\Pi \subseteq S_k\) are symmetrically avoided?

Sagan and coauthors \(\text{[16, 10, 3]}\) initiated the study of small symmetrically avoided sets. By the Robinson-Schensted correspondence, the singletons \(\Pi = \{i\}\) and \(\Pi = \{i, j\}\), consisting of the increasing and the decreasing element, respectively, are symmetrically avoided (see \(\text{[1]}\) Section 2 for details). Furthermore, for these sets, \(S_n(\Pi)\) is Schur-positive. Bloom and Sagan \(\text{[3]}\) Theorem 2.3] proved that these are the only symmetrically avoided singletons.

A full classification of 2-element symmetrically avoided sets follows from a combination of \(\text{[1]}\) Theorem 1.2 and \(\text{[3]}\) Theorem 2.6]. In particular, for all \(k \geq 4\), if \(\Pi \subseteq S_k\) has two elements and \(\Pi \neq \{i_k, j_k\}\), then \(\Pi\) is not symmetrically avoided.

The preceding results led Bloom and Sagan to state the following conjecture:

**Conjecture 1.3 \(\text{[3, Conjecture 2.7]}\).** For each \(p \geq 3\), there exists a \(K = K(p)\), such that if \(|\Pi| = p\) and \(\Pi \subseteq S_k\) for some \(k \geq K\), then \(\Pi\) is not symmetrically avoided.

We prove that the conjecture holds for \(K(p) = p + 2\):

**Theorem 1.4.** Let \(p \geq 3\) and \(k \geq p + 2\) be positive integers, and let \(\Pi \subseteq S_k\) with \(|\Pi| = p\). Then \(\Pi\) is not symmetrically avoided.

The bound is tight: For every \(p \geq 3\), there exists a symmetrically avoided set \(\Pi \subseteq S_{p+1}\) with \(|\Pi| = p\). Such a set was found in \(\text{[10]}\) Section 5], and appears in Corollary \(\text{[5, 2]}\).

The proof applies a new generalization of a theorem of Bose \(\text{[4]}\) from extremal combinatorics, which bounds the size of an intersecting family. First of all, let us define an intersecting family:

**Definition 1.5.** Let \(\mathcal{F} = \{A_1, \ldots, A_m\}\) be a family of \(m\) sets.

1. We say that \(\mathcal{F}\) is \(k\)-uniform if for all \(i\), \(|A_i| = k\).
2. We say that \(\mathcal{F}\) is \(\ell\)-intersecting if for all \(i \neq j\), \(|A_i \cap A_j| = \ell\).
3. We say that \(\mathcal{F}\) is \((\ell_1, \ell_2)\)-intersecting if the following holds for all \(i \neq j\):
   - If \(|i - j| = 1\), then \(|A_i \cap A_j| = \ell_1\).
   - If \(|i - j| \geq 2\), then \(|A_i \cap A_j| = \ell_2\).

In 1949, Bose proved the following theorem:

**Theorem 1.6 (Bose \(\text{[4]}\)).** Let \(\mathcal{F} = \{A_1, \ldots, A_m\}\) be a \(k\)-uniform family of \(m\) distinct subsets of \([n]\). If \(\mathcal{F}\) is \(\ell\)-intersecting for some \(\ell \geq 0\), then \(m \leq n\).

This theorem was generalized by Ray-Chaudhuri and Wilson in 1975 \(\text{[14]}\). In 1991, Alon, Babai and Suzuki \(\text{[2]}\) Section 2] presented a new proof to the Ray-Chaudhuri — Wilson theorem, by associating subsets with multilineal polynomials and analyzing their linear structure.

We generalize Bose theorem in a different direction:
Theorem 1.7. Let \( F = \{ A_1, \ldots, A_m \} \) be a \( k \)-uniform family of \( m \) distinct subsets of \([n] \). If \( F \) is \((\ell_1, \ell_2)\)-intersecting for some \( \ell_1, \ell_2 \geq 0 \), then \( m \leq n \).

Our proof applies the multilinear polynomial method of Alon, Babai and Suzuki.

The paper is organized as follows: In the next section we give some relevant background. In section 3 we prove Theorem 1.7. Then in Section 4 we use it to prove Theorem 1.4. Finally, in Section 5 we conclude with further research.

2. Preliminaries

2.1. Symmetric and Quasisymmetric Functions.

Definition 2.1. Given \( n \in \mathbb{N} \), a partition \( \lambda \) of \( n \) (denoted \( \lambda \vdash n \)), is a non-decreasing sequence \( \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k) \) of positive integers, such that \( \lambda_1 + \cdots + \lambda_k = n \).

A composition \( \alpha \) of \( n \) (denoted \( \alpha \vdash n \)) is a sequence \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \) of positive integers, such that \( \alpha_1 + \cdots + \alpha_k = n \). Clearly, every partition is a composition too.

Compositions of \( n \) are in bijection with subsets of \([n-1] := \{1, \ldots, n-1\} \) by

\[
[n-1] \ni \{i_1, i_2, \ldots, i_k\} \mapsto (i_1, i_2 - i_1, \ldots, i_k - i_{k-1}, n - i_k) \vdash n.
\]

For example, for \( n = 10 \), \( \{2, 4, 7, 8\} \mapsto (2, 2, 3, 1, 2) \). The composition associated to a set \( S \) is denoted \( \alpha_S \), and the set associated to a composition \( \alpha \) is denoted \( S_{\alpha} \).

For two compositions \( \alpha, \beta \vdash n \), we say that:

- \( \beta \) is a refinement of \( \alpha \) (denoted \( \beta \leq \alpha \)) if \( S_{\alpha} \subseteq S_{\beta} \).
- The compositions are equivalent (denoted \( \alpha \sim \beta \)), if \( \beta \) is a rearrangement of the elements of \( \alpha \).

Denote the ring of symmetric functions \( \text{Sym} \) and the ring of quasisymmetric functions \( \text{QSym} \) (see e.g. [10, Section 1] for details). Denote the space of homogeneous symmetric (quasisymmetric) functions of degree \( n \) by \( \text{Sym}_n \) (respectively, \( \text{QSym}_n \)).

The space \( \text{Sym}_n \) has a basis consisting of the monomial symmetric functions:

\[
m_{\lambda} := \sum_{i_1, i_2, \ldots, i_k \text{ distinct}} x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \cdots x_{i_k}^{\lambda_k},
\]

for partitions \( \lambda \vdash n \). For example:

\[
m_{3,1} = x_1^3 x_2 + x_2^3 x_1 + x_3^3 x_1 + x_2^3 x_3 + x_3^3 x_2 + \cdots
\]

The space \( \text{Sym}_n \) has another important basis that consists of the Schur functions \( \{ s_{\lambda} \mid \lambda \vdash n \} \), that we will not define here. The reader is referred to [15, Section 4.4] for details.

The space \( \text{QSym}_n \) has a basis consisting of the monomial quasisymmetric functions:

\[
M_{\alpha} := \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k},
\]

for compositions \( \alpha \vdash n \). For example:

\[
M_{1,3} = x_1 x_2^3 + x_1 x_3^3 + x_2 x_3^3 + \cdots, \quad M_{3,1} = x_1^3 x_2 + x_1^3 x_3 + x_2^3 x_3 + \cdots
\]

Observation 2.2. For every \( \lambda \vdash n \):

\[
m_{\lambda} = \sum_{\alpha \sim \lambda} M_{\alpha}
\]

The space \( \text{QSym}_n \) has an additional important basis, consisting of the fundamental quasisymmetric functions

\[
F_S := \sum_{i_1 \leq i_2 \leq \cdots \leq i_n \forall j \in S : i_j < i_{j+1}} x_{i_1} x_{i_2} \cdots x_{i_n}, \quad S \subseteq [n-1].
\]

We also denote \( F_{\alpha} = F_{S_{\alpha}} \) for a composition \( \alpha \vdash n \).
Observation 2.3. 
\[ F_\alpha = \sum_{\beta \leq \alpha} M_\beta. \]

Or equivalently:
\[ F_S = \sum_{\beta : S_\beta \geq S} M_\beta. \]

2.2. Symmetric and Schur-Positive Sets.

Definition 2.4. The descent set of a permutation \( \pi \in S_n \) is
\[ \text{Des}(\pi) := \{i \in [n-1] : \pi_i > \pi_{i+1}\}. \]

Definition 2.5. Let \( S \subseteq S_n \) be a set of permutations. Its quasisymmetric generating function is defined to be
\[ Q_n(S) := \sum_{\pi \in S} F_{\text{Des}(\pi)}. \]

Definition 2.6. Let \( S \subseteq S_n \). We say that \( S \) is symmetric if \( Q_n(S) \in \text{Sym}_n \).

If in addition, the (unique) decomposition into Schur functions \( Q_n(S) = \sum_\lambda c_\lambda s_\lambda \) satisfies \( c_\lambda \in \mathbb{Z}_{\geq 0} \) for all \( \lambda \vdash n \), \( S \) is called Schur-positive.

Symmetric and Schur-positive sets provide new approaches for investigating symmetric functions, and therefore by the Frobenius characteristic map (see \[15, \text{Chapter 4.7}\])—representations of the symmetric group. Many Schur-positive sets were found to correspond to significant representations of the symmetric group, and vice versa. For instance, Gessel showed \[8\] that Knuth classes correspond to irreducible characters. As another example, Gessel and Reutenauer proved \[9\] that conjugacy classes in \( S_n \) correspond to representations of \( S_n \) on a subspace of free Lie algebras, and used this discovery to prove some enumerative results about conjugacy classes.

3. Proof of Theorem 1.7

First, notice that the sets are distinct, so \( 0 \leq \ell_1, \ell_2 < k \).

The theorem obviously holds for \( m \leq 3 \). Thus, assume \( m \geq 4 \). Then:
\[
\ell_1 = |A_1 \cap A_2| \geq |A_1 \cap A_2 \cap A_4| = |A_4| - |A_4 \setminus (A_1 \cap A_2)| \\
= k - |A_1 \cap (A_4 \setminus A_1)| - |A_4 \cap (A_1 \setminus A_2)| \\
= k - (|A_4| - |A_4 \cap A_1|) - (|A_4| - |A_4 \cap A_2|) = k - 2(k - \ell_2) = 2\ell_2 - k,
\]
so \( 2\ell_2 \leq \ell_1 + k \). In addition, \( 2\ell_2 = \ell_1 + k \) if and only if \( |A_1 \cap A_2| = |A_1 \cap A_2 \cap A_4| \) and the union \( (A_4 \cap A_1) \cup (A_4 \cap A_2) \) is disjoint. These conditions are equivalent to the condition \( A_1 \cap A_2 \subseteq A_4 \subseteq A_1 \cup A_2 \).

We will prove the theorem by case analysis:

Case 1: \( 2\ell_2 < \ell_1 + k \). For each \( 1 \leq i \leq m \), define a multivariate polynomial \( f_i : \mathbb{R}^n \to \mathbb{R} \) by
\[ f_i(x) = \sum_{j \in A_i} x_j - \ell_2. \]

Moreover, define the polynomial
\[ g(x) = \sum_{j=1}^n x_j - k. \]

Associate indicator vectors to sets in the obvious way: \( (v_i)_j = 1_{j \in A_i} \). Notice that for every \( i_1, i_2 \in [m] \), we have:
\[ f_{i_1}(v_{i_2}) = \begin{cases} 
    k - \ell_2, & i_1 = i_2 \\
    \ell_1 - \ell_2, & |i_1 - i_2| = 1 \\
    0, & |i_1 - i_2| \geq 2
\end{cases}. \]

Furthermore, notice that \( g(v_i) = 0 \) for all \( i \). Define an arbitrary \( w \in \mathbb{R}^n \) such that \( g(w) \neq 0 \) (e.g. \( w = 0 \)).

Lemma 3.1. The set \( \{f_1, \ldots, f_m, g\} \) is linearly independent over \( \mathbb{R} \).
Proof of the lemma. Let us define a linear transformation $T$ from the polynomial space to $\mathbb{R}^{m+1}$, by:

$$T : f \mapsto \begin{pmatrix} f(v_1) \\ \vdots \\ f(v_m) \\ f(w) \end{pmatrix}.$$ 

It suffices to prove that the set $B = \{T(f_1), \ldots, T(f_m), T(g)\}$ forms a basis of $\mathbb{R}^{m+1}$. Its representing matrix is:

$$
\begin{pmatrix}
 k - \ell_2 & \ell_1 - \ell_2 & 0 & 0 \\
 \ell_1 - \ell_2 & k - \ell_2 & \ddots & 0 \\
 & \ddots & \ddots & \ell_1 - \ell_2 \\
 0 & \ell_1 - \ell_2 & k - \ell_2 & 0 \\
 & & & * & 0 \\
 & & & & * & 0 \\
 & & & & & * & 0 \\
 & & & & & & * & 0
\end{pmatrix}
$$

In order to prove the lemma, we need to prove that this matrix is invertible. It is invertible if and only if the minor obtained by deleting the last row and the last column is invertible too:

$$
\begin{pmatrix}
 k - \ell_2 & \ell_1 - \ell_2 & 0 \\
 \ell_1 - \ell_2 & k - \ell_2 & \ddots \\
 & \ddots & \ddots & \ell_1 - \ell_2 \\
 0 & \ell_1 - \ell_2 & k - \ell_2 & 0
\end{pmatrix}
$$

This is a tri-diagonal matrix, with the value $k - \ell_2$ on the main diagonal, and the value $\ell_1 - \ell_2$ on the two adjacent diagonals. Divide all the entries by ($k - \ell_2$) (this is valid because $\ell_2 < k$) to get:

$$M(\alpha) = \begin{pmatrix} 1 & \alpha & 0 \\
 \alpha & 1 & \ddots \\
 0 & \ddots & \ddots & \alpha \\
 0 & \alpha & 1 \end{pmatrix},$$

for $\alpha = \frac{k - \ell_2}{\ell_1 - \ell_2} \in \mathbb{Q}$. Observe that $-1 < \alpha < 1$ (because $\ell_1 < k$ and $2\ell_2 < \ell_1 + k$). Denote the determinant of $M(\alpha)$ by $d_m(\alpha)$, and notice that it is a polynomial (in $\alpha$) with integer coefficients. In addition, the constant coefficient of $d_m(\alpha)$ is $d_m(0) = \det(I) = 1$. Thus, following The Rational Root Theorem (and the fact that $\alpha \in \mathbb{Q}$), we conclude that the only optional solutions to $d_m(\alpha) = 0$ are $\alpha \in \frac{1}{2}$. Together with $-1 < \alpha < 1$, we get $|\alpha| \leq \frac{1}{2}$.

For the remaining case, $|\alpha| \leq \frac{1}{2}$, we follow the proof of Gershgorin’s circle theorem [11]: Assume by contradiction that $M := M(\alpha)$ is singular. Let $0 \not= v \in \ker M$, i.e. $M \cdot v = 0$. Without loss of generality, there exists $1 \leq i \leq m$, such that $|v_i| = 1$ and for all $j \in [m]$: $|v_j| \leq 1$. Take the minimal such $i$.

If $|v_1| = 1$, we get $0 = (M \cdot v)_1 = v_1 + \alpha v_2$. So $1 = |v_1| = |\alpha| \cdot |v_2| \leq \frac{1}{2}$, in contradiction. Thus, $|v_1| < 1$. The case where $|v_m| = 1$ is handled similarly.

Assume that $1 < i < m$. So $0 = (M \cdot v)_i = v_i + \alpha (v_{i-1} + v_{i+1})$. Recall that $\alpha \not= 0$ because $\alpha \in \frac{1}{2}$, therefore $-\frac{v_i}{\alpha} = v_{i-1} + v_{i+1}$, so $2 \leq \frac{|v_i|}{|\alpha|} \leq |v_{i-1}| + |v_{i+1}| \leq 2$. Thus, we may conclude that all the inequalities are equalities. In particular, $|v_{i-1}| = 1$, contradicting the minimality of $i$.

Let us return to the proof of Theorem [17]. The set $\{f_1, \ldots, f_m, g\}$ has $m + 1$ elements, and it is linearly independent over $\mathbb{R}$. All these polynomials are of degree at most 1, and the linear space of all such polynomials has dimension $n + 1$. Thus, $m + 1 \leq n + 1$, completing the proof of Theorem [17] for this case.

Case 2: $2\ell_2 = \ell_1 + k$. The proof of this case is more technical.

As mentioned before, this equality implies $A_1 \cap A_2 \subseteq A_4 \subseteq A_1 \cup A_3$. Notice that for $k \in \{0, 1, n - 1, n\}$ we have $n \geq \binom{n}{k}$ and the theorem holds trivially. Thus, we need to prove it only for $n = 4$, $k = 2$ and for $n \geq 5$.

For $n = 4$, $k = 2$: In the current case, we assume $2\ell_2 = \ell_1 + k = \ell_1 + 2$. In addition, $\ell_2 < k = 2$. Thus, $\ell_1 = 0$, $\ell_2 = 1$. Without loss of generality, $A_1 = \{1, 2\}$, $A_1 \cap A_2 = \emptyset$, so $A_2 = \{3, 4\}$. Furthermore, $A_2 \cap A_3 = \emptyset$ so $A_3 = \{1, 2\} = A_1$. We get $\ell_2 = |A_1 \cap A_3| = 2$, contradicting the assumption that $\ell_2 = 1$.
Now, assume \( n \geq 5 \). We need to prove that no \( k \)-uniform \((\ell_1, \ell_2)\)-intersecting family satisfies \( m \geq n + 1 \). We will prove a stronger proposition:

**Lemma 3.2.** Let \( \mathcal{F} \) be a \( k \)-uniform family containing 6 sets, and let \( \ell_1, \ell_2 \geq 0 \) such that \( 2\ell_2 = \ell_1 + k \). Then \( \mathcal{F} \) is not \((\ell_1, \ell_2)\)-intersecting.

**Proof.** Assume by contradiction that \( \mathcal{F} \) is \((\ell_1, \ell_2)\)-intersecting. As shown above, it implies that \( A_1 \cap A_2 \subseteq A_4 \). Similarly, for every \( 1 \leq i \leq 5 \) and every \( j \) that is neither adjacent to \( i \) nor adjacent to \( i + 1 \), \( A_i \cap A_{i+1} \subseteq A_j \). Thus, \( A_2 \cap A_3 \subseteq A_5 \), and \( A_2 \cap A_3 \subseteq A_6 \). By symmetry, \( A_2 \cap A_3 \supseteq A_5 \) and \( A_2 \cap A_3 \supseteq A_6 \). Hence, we can see that \( A_1 \cap A_2 = A_5 \cap A_6 \), \( A_1 \cap A_2 = A_2 \cap A_3 \) and thus \( A_1 \cap A_2 \subseteq A_3 \).

To conclude, we proved that \( A_1 \cap A_2 \subseteq A_3 \) for all \( i \), so \( |\bigcup_i A_i| = \ell_1 \). If we remove the elements of \( \bigcup_i A_i \) from all the sets, we get a new intersecting family with \( \ell_1 = 0 \). Therefore, we may assume, without loss of generality, that \( \ell_1 = 0 \). This means that every two adjacent sets are disjoint, and that \( k = 2\ell_2 \).

We can test this case manually: without loss of generality, \( A_1 = (0, 2\ell_2) = \{1, \ldots, 2\ell_2\} \). \( A_1 \cap A_2 = \emptyset \), so, without loss of generality, \( A_2 = (2\ell_2, 4\ell_2) \). \( A_2 \cap A_3 = \emptyset \) and \( |A_1 \cap A_3| = \ell_2 \), so, without loss of generality, \( A_3 = (0, \ell_2) \cup (4\ell_2, 5\ell_2) \). In the same way, we get that \( A_4 = (\ell_2, 3\ell_2) \), \( A_5 = (0, \ell_2) \cup (3\ell_2, 4\ell_2) \). The set \( A_6 \) needs to intersect \( A_1 \) and be disjoint with \( A_5 \), so \( (\ell_2, 2\ell_2) \subseteq A_6 \). For similar reasons, \( (2\ell_2, 3\ell_2) \subseteq A_6 \). Thus, \( A_6 = (\ell_2, 3\ell_2) = A_4 \), in contradiction. □

This completes the proof of Theorem 1.7.

4. **Proof of Theorem 1.4**

The strategy of the proof is motivated by Bloom and Sagan’s proof [3, Section 2] that, for all \( k \geq 3 \), every symmetrically avoided set \( \Pi \subseteq S_k \) with at most 2 elements has only monotone elements (recall that the monotone elements are \( \ell_n = [1, \ldots, n] = Id_n \) and \( \delta_n = [n, \ldots, 1] \)). First, we give some properties of symmetry.

For a set of permutations \( S \subseteq S_n \), the subset of permutations that respect a given composition \( \alpha \vdash n \) is defined as follows:

\[
S(\alpha) := \{ \pi \in S \mid \pi \text{ monotonically increases along the segments of } \alpha \}.
\]

Namely, we split the elements in every permutation into segments of lengths \( \alpha_1, \alpha_2, \ldots \), and check whether the permutation increases along each segment separately. Equivalently, for a given \( \pi \in S \), we have \( \pi \in S(\alpha) \) if and only if \( \text{Des}(\pi) \subseteq I \) for \( I \subseteq [n-1] \) the set corresponding to the composition \( \alpha \) (see Section 2 for the relevant definitions).

For example, take \( \alpha = (1, 3, 2) \vdash 6 \). Then \( \{4, 2, 5, 6, 1, 3\} \in S_6(\alpha) \), because after splitting the permutation into segments with lengths 1, 3, 2, we get \( 4 \mid 2 \mid 5 \mid 6 \mid 1, 3 \), and all the segments increase. On the other hand \( \{2, 5, 4, 6, 1, 3\} \notin S_6(\alpha) \), because its segments are \( 2 \mid 5 \mid 4 \mid 6 \mid 1, 3 \), and the segment 546 does not increase. Indeed, the set \( I = \{\alpha_1, \alpha_1 + \alpha_2\} = \{1, 4\} \) corresponds to the composition \( \alpha = (1, 3, 2) \), and \( \text{Des}(4, 2, 5, 6, 1, 3) = \{1, 4\} \subseteq I \) but \( \text{Des}(2, 5, 4, 6, 1, 3) = \{2, 4\} \notin I \).

As noted in Subsection 2.2 the set \( \{M_\alpha \mid \alpha \vdash n\} \) of monomial quasisymmetric functions forms a basis of \( \text{QSym}_n \). Thus, every \( f \in \text{QSym}_n \) has a unique representation as \( f = \sum_\alpha c_\alpha M_\alpha \). In addition, the set \( \{m_\lambda \mid \lambda \vdash n\} \) forms a basis of \( \text{Sym}_n \), and by Observation 2.2

\[
m_\lambda = \sum_{\alpha \sim \lambda} M_\alpha.
\]

Therefore, we deduce:

**Corollary 4.1.** The quasisymmetric function \( f = \sum_\alpha c_\alpha M_\alpha \) is symmetric if and only if

\[
\forall \alpha, \beta \vdash n : \alpha \sim \beta \implies c_\alpha = c_\beta.
\]

**Corollary 4.2.** A set \( S \subseteq S_n \) is symmetric if and only if \( |S(\alpha)| = |S(\beta)| \) for all \( \alpha \sim \beta \vdash n \).
Proof. By Definition 2.6, $S$ is symmetric if and only if its generating function $Q_n(S)$ is symmetric. We have:

$$Q_n(S) = \sum_{\pi \in S} F_{\text{Des}(\pi)}$$

(Definition 2.5)

$$= \sum_{\pi \in S} \sum_{\alpha : S_{\alpha} \supseteq \text{Des}(\pi)} M_{\alpha}$$

(Observation 2.3)

$$= \sum_{\alpha \neq n} \{ \{ \pi \in S \mid \text{Des}(\pi) \subseteq S_{\alpha} \} \cdot M_{\alpha} \}$$

$$= \sum_{\alpha \neq n} |S(\alpha)| \cdot M_{\alpha}.$$ 

The corollary follows immediately from Corollary 4.1. \hfill \Box

Before proving the theorem, let us state a few lemmas.

The first lemma will help us to connect symmetric sets with intersecting families:

Lemma 4.3. Let $n \in \mathbb{N}$, and assume that a set $S = \{\pi_1, \ldots, \pi_m\} \subseteq S_n$ is symmetric. For every $1 \leq i \leq n - 1$, denote $A_i := \{1 \leq j \leq m \mid i \notin \text{Des}(\pi_j)\}$. Then there exist $k, \ell_1, \ell_2 \geq 0$, such that $\{A_i \mid 1 \leq i \leq n - 1\}$ is a $k$-uniform and $(\ell_1, \ell_2)$-intersecting family (as defined in Definition 1.5).

Proof. According to Definition 1.5, we need to prove three claims: that $|A_i|$ is constant for all $i$, that $|A_i \cap A_{i+1}|$ is constant for all $i$, and that $|A_i \cap A_j|$ is constant for $|i - j| \geq 2$. We prove each claim separately.

Each claim will be proved by considering some equivalent compositions and applying Corollary 4.2.

- By considering the compositions of the form $\alpha_i := (1^{i-1}, 2, 1^{n-i-1})$, we get:

  $$\pi \in S(\alpha_i) \iff \pi \in S \text{ and } i \notin \text{Des}(\pi).$$

  Thus, $S(\alpha_i) = \{\pi_j \mid j \in A_i\}$. The compositions $\{\alpha_i \mid 1 \leq i \leq n - 1\}$ are pairwise equivalent. Therefore, following Corollary 4.2 we deduce that for all $i_1, i_2$: $|A_{i_1}| = |A_{i_2}|$. Denote $k = |A_i|$.

- Consider the composition $\alpha_{i,i+1} := (1^{i-1}, 3, 1^{n-i-2})$ (for $i \leq n - 2$): It satisfies

  $$\pi \in S(\alpha_{i,i+1}) \iff \pi \in S \text{ and } i, i + 1 \notin \text{Des}(\pi).$$

  Therefore, $S(\alpha_{i,i+1}) = \{\pi_j \mid j \in A_i \cap A_{i+1}\}$. Again, by Corollary 4.2 for all $i_1, i_2$: $|A_{i_1} \cap A_{i_1+1}| = |A_{i_2} \cap A_{i_2+1}|$. Denote $\ell_1 = |A_i \cap A_{i+1}|$.

- Finally, by considering $\alpha_{i,j} := (1^{i-1}, 2, 1^{j-i-2}, 2, 1^{n-j-1})$ (for non-adjacent indices $i, j$), we conclude that $|A_i \cap A_j|$ is constant for $|i - j| \geq 2$, denote it by $\ell_2$.

Lemma 4.3 shows that the family $\{A_i \mid 1 \leq i \leq n - 1\}$ is $k$-uniform and $(\ell_1, \ell_2)$-intersecting. In order to apply Theorem 1.7 we need to prove that all the sets in the family are distinct. The following lemma proves this fact for all the relevant cases:

Lemma 4.4. For every $n \geq 5$ and a non-empty set $S \subseteq S_n$ with no monotone element, if there exist $i_1 \neq i_2$ such that $i_1 \in \text{Des}(\pi) \iff i_2 \in \text{Des}(\pi)$ for all $\pi \in S_n$, then $S$ is not symmetric.

Remark. The condition $n \geq 5$ of the lemma is necessary, because the lemma fails at $n = 4$: The Knuth class $S = \{[3, 4, 1, 2], [3, 1, 4, 2]\}$ has $\text{Des}([3, 4, 1, 2]) = \{2\}$, $\text{Des}([3, 1, 4, 2]) = \{1, 3\}$, so every $\pi \in S$ descends in 1 if and only if it descends in 3.

Proof. Assume by contradiction that $S$ is symmetric and that there are two indices $i_1 \neq i_2$ such that $A_{i_1} = A_{i_2}$. By Lemma 4.3 there exist $k, \ell_1, \ell_2 \geq 0$, such that $\{A_i \mid 1 \leq i \leq n - 1\}$ is $k$-uniform and $(\ell_1, \ell_2)$-intersecting. If $i_1, i_2$ are adjacent, we infer that $\ell_1 = |A_{i_1} \cap A_{i_2}| = |A_{i_1}| = k$. Otherwise, we infer that $\ell_2 = k$. In the second case, we get that $|A_{i_1} \cap A_{i_2}| = |A_{i_1}|$ (recall that $n \geq 5$, so there are $n - 1 \geq 4$ sets), so $A_{i_1} = A_4$. Similarly, $|A_2 \cap A_{i_2}| = |A_2| = |A_4|$, so $A_2 = A_4$. Thus, $A_1 = A_2$ and $\ell_1 = \ell_2 = k$.

We may conclude that in all cases $\ell_1 = k$. Therefore, for every $1 \leq i \leq n - 2$ we get that $|A_i \cap A_{i+1}| = |A_i| = |A_{i+1}|$, so $A_i = A_{i+1}$. From here we get that all the sets are equal, and thus every $\pi \in S$ is monotone. This contradicts the assumptions of the lemma. \hfill \Box

Let us combine these lemmas in order to prove that every non-empty symmetric set with no monotone element has at least $n - 1$ elements:
Lemma 4.5. For every \( n \geq 5 \), the size of the smallest non-empty symmetric set \( S \subseteq S_n \) with no monotone element (i.e. neither \( t_n \) nor \( \delta_n \)) is \( n - 1 \).

Remark. It is well known that for \( n \geq 5 \), the minimal non-empty Schur-positive set in \( S_n \) with no monotone element has \( n - 1 \) elements. It is achieved e.g. by a Knuth equivalence class corresponding to the partition \( (n - 1, 1) \) \([8]\). The lemma states that the same bound holds for the weaker property of symmetry too.

The counter-example to the preceding lemma for \( n = 4 \) holds here too, because the Knuth class \( S = \left\{ \{3, 4, 1, 2\}, \{3, 1, 4, 2\}\right\} \) is symmetric (and even Schur-positive).

Proof. Let \( S \subseteq S_n \) with \( |S| = m \leq n - 2 \), and assume by contradiction that \( S \) is symmetric.

Define the sets \( A_i \subseteq [m] \) for \( 1 \leq i \leq n - 1 \) like the definition in Lemma 4.3. By Lemma 4.3 the family \( \{A_i \mid 1 \leq i \leq n - 1\} \) is \( k \)-uniform and \((\ell_1, \ell_2)\)-intersecting. Furthermore, by Lemma 4.4 all the sets \( A_i \) are distinct. Thus, we can apply Theorem 1.7 and deduce that \( n - 1 \leq m \), contradicting the assumption that \( m \leq n - 2 \).

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. We choose \( n = k \), and prove that \( S_n(\Pi) = S_k \setminus \Pi \) is not symmetric.

Assume by contradiction that \( S_k \setminus \Pi \) is symmetric. Therefore, its generating function \( Q_k(S_k \setminus \Pi) \) is symmetric (according to Definition 2.6). By Definition 2.5 clearly \( Q_k(S_k \setminus \Pi) = Q_k(S_k) - Q_k(\Pi) \). The function \( Q_k(S_k) \) is symmetric (see e.g. \([10\), Section 2]), so \( Q_k(\Pi) \) is symmetric. Therefore, \( \Pi \) is symmetric.

The set \( \Pi \) is symmetric if and only if \( \Pi \setminus \{\tau_k, \delta_k\} \) is symmetric. In addition, \( |\Pi| = p \geq 3 \), so \( \Pi \setminus \{\tau_k, \delta_k\} \) is a non-empty symmetric set with no monotone element. By Lemma 4.5 every such a set has at least \( k - 1 \) elements, but \( |\Pi \setminus \{\tau_k, \delta_k\}| \leq p \leq k - 2 \), in contradiction. \( \square \)

5. Further Remarks and Open Problems

We proved in Theorem 1.3 that for every \( p \geq 3 \) and \( k \geq p + 2 \), no \( p \)-element set \( \Pi \subseteq S_k \) is symmetrically avoided. The bound is tight. Namely, for every \( p \geq 3 \) there exists a symmetrically avoided \( p \)-element set \( \Pi \subseteq S_{p + 1} \). This fact follows from a theorem of Hamaker, Pawlowski and Sagan:

Theorem 5.1 (\([10\), Lemma 5.7\]). For any \( J \subseteq [k - 1] \), the inverse descent class \( D_J^{-1} := \{\pi \in S_k \mid \text{Des}(\pi^{-1}) = J\} \) is symmetrically avoided.

Corollary 5.2. Let \( k \in \mathbb{N} \). Then the following subset of \( S_k \), of size \( k - 1 \), is symmetrically avoided:

\[
D_{(k-1)}^{-1} = \{[k, 1, 2, \ldots, k - 1], [1, k, 2, \ldots, k - 1], \ldots, [1, 2, \ldots, k - 2, k, k - 1]\}.
\]

Remark. In fact, Hamaker, Pawlowski and Sagan proved that inverse descent classes satisfy a stronger property, called pattern-Knuth closed. Their result implies Theorem 5.1 because every pattern-Knuth closed set is symmetrically avoided \([10\), Section 5]\).

To conclude, Theorem 1.3 proves that the set \( D_{(k-1)}^{-1} \) presented in Corollary 5.2 is the minimal symmetrically avoided set in \( S_k \) with at least 3 elements. That is, for every set \( \Pi \subseteq S_k \) with \( 3 \leq |\Pi| \leq k - 2 \) and \( k \geq 5 \), there exists \( n \in \mathbb{N} \) such that the set \( S_n(\Pi) \) is not symmetric.

Both the proof of this fact and the proof of Bloom and Sagan \([3\) for \(|\Pi| \leq 2\), begin by picking \( n = k \) and proving that \( S_k(\Pi) \) is not symmetric. Nothing is known about the symmetry of \( S_n(\Pi) \) for \( n > k \). This fact raises the following question:

Definition 5.3. We say that a set \( \Pi \subseteq S_k \) is eventually symmetrically avoided if there is some \( N \in \mathbb{N} \), such that for every \( n > N \), the set \( S_n(\Pi) \) is symmetric.

Question 5.4. For which \( p \in \mathbb{N} \), is there a \( K = K(p) \), such that if \( |\Pi| = p \) and \( \Pi \subseteq S_k \) for \( k \geq K \) and \( \Pi \not\subseteq \{\tau_k, \delta_k\} \), is \( \Pi \) not eventually symmetrically avoided?

As far as we know, this question is still open even for \( p = 1 \). The condition \( \Pi \not\subseteq \{\tau_k, \delta_k\} \) was added, because every \( \Pi \not\subseteq \{\tau_k, \delta_k\} \) is symmetrically avoided \([10\), Section 2]\).

It is worth pointing out that a pattern set which is eventually symmetrically avoided isn’t necessarily symmetrically avoided. In particular, Bloom and Sagan \([3\), Section 6\] have constructed a pattern set \( \Pi \subseteq S_6 \), such that \( S_6(\Pi) \) is not symmetric but \( S_n(\Pi) \) is symmetric for all \( n \geq 7 \). That is, \( \Pi \) is not symmetrically avoided, but it is eventually symmetrically avoided.
In addition, we considered only the case $\Pi \subseteq S_k$ for some $k$. In the area of pattern avoidance, it is common to consider also $\Pi \subseteq \bigcup_{k \in \mathbb{N}} S_k$, i.e. sets that can contain patterns of different sizes. One would like to find a tight bound for this case. Wrong formulation of the problem may lead to trivial solutions. An accurate and precise formulation of this problem is a challenge.

Furthermore, during the proof we also proved in Lemma 4.5 that there is no symmetric set $S \subseteq S_n$ of size $1 \leq |S| \leq n-2$ with no monotone element. The analogous result for Schur-positive sets is known. The fact that the minimal size of such a symmetric set is equal to the minimal size of such a Schur-positive set is inspiring. It raises the question if every minimal symmetric set is Schur-positive. Namely,

**Question 5.5.** Let $S \subseteq S_n$ be a symmetric set of size $n-1$ with no monotone element. Is $S$ necessarily Schur-positive?

Note that following the same strategy we used for bounding the size of symmetric sets will probably not be enough. This is because there are many extremal families to Bose’s theorem (described in Theorem 1.6); and thus, there are many extremal families to Theorem 1.7. The extremal families of Bose’s theorem are called symmetric block designs, and they include projective spaces, Hadamard designs and many other constructions (see e.g. [2], Part II, Section 6) for details). Therefore, for proving that there are no other symmetric sets of size $n-1$ except for the Schur-positive sets, one must consider other properties of symmetry too.

Another interesting question regarding sizes of symmetric sets is the following:

**Question 5.6.** Let $n \in \mathbb{N}$. For which values $p \in \mathbb{N}$, does there exist a symmetric set $S \subseteq S_n$ of size $p$ with no monotone element?

We proved that for $0 < p < n-1$ such a set does not exist (except for $n = 4$, $p = 2$). As mentioned before, symmetric (and even Schur-positive) sets of size $p = n-1$ are known. For $p = n$, no such Schur-positive set exists (as can be shown by the theory of Young diagrams [1]), but there are symmetric sets of size $n$. For example, the set

$$\{[n,1,\ldots,n-1],[1,\ldots,n-2,n,n-1]\}$$

is symmetric. It corresponds to the symmetric function $s_{(n-1,1)} + s_{(n-2,1,1)} - s_{(n-2,2)}$. However, we do not know a general answer to this question.

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