On the Fundamental Limits of Adaptive Sensing

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Abstract

Suppose we can sequentially acquire arbitrary linear measurements of an $n$-dimensional vector $x$ resulting in the linear model $y = Ax + z$, where $z$ represents measurement noise. If the signal is known to be sparse, one would expect the following folk theorem to be true: choosing an adaptive strategy which cleverly selects the next row of $A$ based on what has been previously observed should do far better than a nonadaptive strategy which sets the rows of $A$ ahead of time, thus not trying to learn anything about the signal in between observations. This paper shows that the folk theorem is false. We prove that the advantages offered by clever adaptive strategies and sophisticated estimation procedures—no matter how intractable—over classical compressed acquisition/recovery schemes are, in general, minimal.

Keywords: sparse signal estimation, adaptive sensing, compressed sensing, support recovery, information bounds, hypothesis tests.

1 Introduction

This paper is concerned with the fundamental question of how well one can estimate a sparse vector from noisy linear measurements in the general situation where one has the flexibility to design those measurements at will (in the language of statistics, one would say that there is nearly complete freedom in designing the experiment). This question is of importance in a variety of sparse signal estimation or sparse regression scenarios, but perhaps arises most naturally in the context of compressive sensing (CS) [4, 5, 11]. In a nutshell, CS asserts that it is possible to reliably acquire sparse signals from just a few linear measurements selected a priori. More specifically, suppose we wish to acquire a sparse signal $x \in \mathbb{R}^n$. A possible CS acquisition protocol would proceed as follows. ($i$) Pick an $m \times n$ random projection matrix $A$ (the first $m$ rows of a random unitary matrix) in advance, and collect data of the form

\[ y = Ax + z, \tag{1.1} \]

where $z$ is a vector of errors modeling the fact that any real world measurement is subject to at least a small amount of noise. ($ii$) Recover the signal by solving an $\ell_1$ minimization problem such as the Dantzig selector [6] or the LASSO [28]. As is now well known, theoretical results guarantee that such convex programs yield accurate solutions. In particular, when $z = 0$, the recovery is exact, and the error degrades gracefully as the noise level increases.

A remarkable feature of the CS acquisition protocol is that the sensing is completely nonadaptive; that is to say, no effort whatsoever is made to understand the signal. One simply selects a collection \{a_i\}
of sensing vectors a priori (the rows of the matrix $A$), and measures correlations between the signal and these vectors. One then uses numerical optimization—e.g., linear programming [6]—to tease out the sparse signal $x$ from the data vector $y$. While this may make sense when there is no noise, this protocol might draw some severe skepticism in a noisy environment. To see why, note that in the scenario above, most of the power is actually spent measuring the signal at locations where there is no information content, i.e., where the signal vanishes. Specifically, let $a$ be a row of the matrix $A$ which, in the scheme discussed above, has uniform distribution on the unit sphere. The dot product is

$$\langle a, x \rangle = \sum_{j=1}^{n} a_j x_j,$$

and since most of the coordinates $x_j$ are zero, one might think that most of the power is wasted. Another way to express all of this is that by design, the sensing vectors are approximately orthogonal to the signal, yielding measurements with low signal power or a poor signal-to-noise ratio (SNR).

The idea behind adaptive sensing is that one should localize the sensing vectors around locations where the signal is nonzero in order to increase the SNR, or equivalently, not waste sensing power. In other words, one should try to “learn” as much as possible about the signal while acquiring it in order to design more effective subsequent measurements. Roughly speaking, one would (i) detect those entries which are nonzero or significant, (ii) progressively localize the sensing vectors on those entries, and (iii) estimate the signal from such localized linear functionals. This is akin to the game of 20 questions in which the search is narrowed by formulating the next question in a way that depends upon the answers to the previous ones. Note that in some applications, such as in the acquisition of wideband radio frequency signals, aggressive adaptive sensing mechanisms may not be practical because they would require near instantaneous feedback. However, there do exist applications where adaptive sensing is practical and where the potential benefits of adaptivity are too tantalizing to ignore.

The formidable possibilities offered by adaptive sensing give rise to the following natural “folk theorem.”

**Folk Theorem.** The estimation error one can get by using a clever adaptive sensing scheme is far better than what is achievable by a nonadaptive scheme.

In other words, learning about the signal along the way and adapting the questions (the next sensing vectors) to what has been learned to date is bound to help. In stark contrast, the main result of this paper is this:

**Surprise.** The folk theorem is wrong in general. No matter how clever the adaptive sensing mechanism, no matter how intractable the estimation procedure, in general it is not possible to achieve a fundamentally better mean-squared error (MSE) of estimation than that offered by a naïve random projection followed by $\ell_1$ minimization.

The rest of this article is mostly devoted to making this claim precise. In doing so, we shall also show that adaptivity does not help in obtaining a fundamentally better estimate of the signal support, which is of independent interest.

### 1.1 Main result

To formalize matters, we assume that the error vector $z$ in (1.1) has i.i.d. $\mathcal{N}(0, \sigma^2)$ entries. Then if $A$ is a random projection with unit-norm rows as discussed above, [6] shows that the Dantzig selector estimate $\hat{x}^{DS}$ (obtained by solving a simple linear program) achieves an MSE obeying

$$\frac{1}{n} \mathbb{E} \| \hat{x}^{DS} - x \|_2^2 \leq C \frac{k}{m} \log(n) \sigma^2,$$  \hspace{1cm} (1.2)
where \( C \) is some numerical constant. The bound holds \textit{universally} over all \( k \)-sparse signals\(^1\) provided that the number of measurements \( m \) is sufficiently large (on the order of at least \( k \log(n/k) \)). Moreover, one can show that this result is essentially optimal in the sense that any possible nonadaptive choice of \( A \) (with unit-norm rows) and any possible estimation procedure \( \hat{x} \) will satisfy
\[
\frac{1}{n} \mathbb{E} \| \hat{x} - x \|_2^2 \geq C' \frac{k}{m} \log(n/k) \sigma^2,
\]
where \( C' \) is a numerical constant \([3]\). The fundamental question is thus: \textit{how much lower can the MSE be} when (i) we are allowed to sense the signal adaptively and (ii) we can use any estimation algorithm we like to recover \( x \).

The distinction between adaptive and nonadaptive sensing can be expressed in the following manner. Begin by rewriting the statistical model (1.1) as
\[
y_i = \langle a_i, x \rangle + z_i, \quad i = 1, \ldots, m,
\]
in which a power constraint imposes that each \( a_i \) is of norm at most 1, i.e., \( \| a_i \|_2 \leq 1 \); then in a nonadaptive sensing scheme the vectors \( a_1, \ldots, a_m \) are chosen in advance and do not depend on \( x \) or \( z \) whereas in an adaptive setting, the measurement vectors may be chosen depending on the history of the sensing process, i.e., \( a_i \) is a (possibly random) function of \( (a_1, y_1, \ldots, a_{i-1}, y_{i-1}) \).

If we follow the principle that “you cannot get something for nothing,” one might argue that giving up the freedom to adaptively select the sensing vectors would result in a far worse MSE. Our main contribution is to show that this is not the case.

\textbf{Theorem 1.} Suppose that \( k < n/2 \) and let \( m \) be an arbitrary number of measurements. Assume that \( x \) is sampled with i.i.d. coordinates such that \( x_j = 0 \) with probability \( 1 - k/n \) and \( x_j = \mu \) with probability \( k/n \) (so that we have \( k \) nonzero entries on the average). Then for \( \mu = \frac{4}{3\sqrt{m}} \), any sensing strategy and any estimate \( \hat{x} \) obey
\[
\frac{1}{n} \mathbb{E} \| \hat{x} - x \|_2^2 \geq \frac{4}{27} \frac{k}{m} \sigma^2 > \frac{1}{7} \frac{k}{m} \sigma^2.
\]

For any \( n \) and \( k \), the number of nonzero entries in a random vector drawn from the Bernoulli prior is between \( k \pm 3\sqrt{k} \) with probability at least 99%. With some additional arguments, and when \( n \) and \( k \) are sufficiently large, we can actually show that the last inequality in (1.5) holds true in a minimax sense when \( x \) is known to have a support size in that range. In order to avoid unnecessary technicalities, we prove a simpler result.

\textbf{Theorem 2.} For any \( n \geq 2 \) and \( k < n/2 \), and any \( m \),
\[
\inf_{\hat{x}} \sup_{|x|_0 \leq k} \frac{1}{n} \mathbb{E} \| \hat{x} - x \|_2^2 \geq C_k \frac{k}{m} \sigma^2,
\]
in which \( \inf_{k \geq 1} C_k \geq 1/33 \). For \( k \geq 10 \) we can take \( C_k \geq 1/15 \), and when \( k \) is sufficiently large we can take \( C_k = 1/7 \).

In short, Theorems 1 and 2 say that if one ignores a logarithmic factor, then \textit{adaptive measurement schemes cannot (substantially) outperform nonadaptive strategies}. While seemingly counterintuitive, we find that precisely the same sparse vectors which determine the minimax rate in the nonadaptive setting are essentially so difficult to estimate that by the time we have identified the support, we will have already exhausted our measurement budget (i.e., we will have acquired all \( m \) measurements).

\(^1\)A signal is said to be \( k \)-sparse if it has at most \( k \) nonzero components. We also occasionally use the notation \( \| x \|_0 \) to denote the number of nonzero components of \( x \).
Before moving on, we should clarify precisely what we mean by a *substantial* improvement. After all, the lower bound in Theorem 2 does improve upon the nonadaptive bound in (1.3) by a factor of $\log(n/k)$. Indeed, we will see in Section 4 that at least in some very special cases (e.g., when $k = 1$), this log factor can in fact be eliminated. However, this is a relatively modest improvement compared to what one might hope to gain by exploiting adaptivity. Specifically, consider a simple adaptive procedure that uses $m/2$ measurements to identify the support of $\mathbf{x}$ and uses the remaining $m/2$ measurements to estimate the values of the nonzeros. If such a scheme identifies the correct support, then it is easy to show that this procedure will yield an estimate satisfying

$$
\frac{1}{n} \mathbb{E} \|\hat{\mathbf{x}} - \mathbf{x}\|_2^2 = \frac{2k}{n} \frac{k}{m} \sigma^2.
$$

Thus, there seems to be room for reducing the error by a factor of $k/n$ beyond the $\log(n/k)$ factor. Theorem 2, however, shows that this gain is not possible in general.

On the one hand, our main result states that one cannot universally improve on bounds achievable via nonadaptive sensing strategies. Indeed, we will see that there are natural classes of sparse signals for which, even after applying the most clever sensing scheme and the most subtle testing procedure, one would still not be sure about where the nonzeros lie. This remains true even after having used up the entirety of our measurement budget. On the other hand, our result does not say that adaptive sensing *never* helps. In fact, there are many instances in which it will. For example, when some or most of the nonzero entries in $\mathbf{x}$ are sufficiently large, they may be detected sufficiently early so that one can ultimately get a far better MSE than what would be obtained via a nonadaptive scheme, see Section 3 for simple experiments in this direction and Section 4 for further discussion.

### 1.2 Connections with testing problems

The arguments we develop to reach our conclusions are quite intuitive, simple, and yet they seem different from the classical Fano-type arguments for obtaining information-theoretic lower bounds (see Section 1.3 for a discussion of the latter methods). Our approach involves proving a lower bound for the Bayes risk under the prior from Theorem 1. To obtain such a lower bound, we make a detour through testing—multiple testing to be exact. Our argument proceeds through two main steps:

- **Support recovery in Hamming distance.** We consider the multiple testing problem of deciding which components of the signal are zero and which are not. We show that no matter which adaptive strategy and tests are used, the Hamming distance between the estimated and true supports is large. Put differently, the multiple testing problem is shown to be difficult. In passing, this establishes that adaptive schemes are not substantially better than nonadaptive schemes for support recovery.

- **Estimation with mean-squared loss.** Any estimator with a low MSE can be converted into an effective support estimator simply by selecting the largest coordinates or those above a certain threshold. Hence, a lower bound on the Hamming distance immediately gives a lower bound on the MSE.

The crux of our argument is thus to show that it is not possible to choose sensing vectors adaptively in such a way that the support of the signal may be estimated accurately.

### 1.3 Differential entropies and Fano-type arguments

Our approach is significantly different from classical methods for getting lower bounds in decision and information theory. Such methods typically rely on Fano’s inequality [9], and are all intimately related to methods in statistical decision theory (see [29, 31]). Before continuing, we would like to point out that
Fano-type arguments have been used successfully to obtain (often sharp) lower bounds for some adaptive methods. For example, the work [8] uses results from [29] to establish a bound on the minimax rate for binary classification (see the references therein for additional literature on active learning). Other examples include the recent paper [26], which derives lower bounds for bandit problems, and [24] which develops an information theoretic approach suitable for stochastic optimization, a form of online learning, and gives bounds about the convergence rate at which iterative convex optimization schemes approach a solution.

Following the standard approaches in our setting leads to major obstacles that we would like to briefly describe. Our hope is that this will help the reader to better appreciate our easy itinerary. As usual, we start by choosing a prior for $\mathbf{x}$, which we take having zero mean. Coming from information theory, one would want to bound the mutual information between $\mathbf{x}$ (what we want to learn about) and $\mathbf{y}$ (the information we have), for any measurement scheme $\mathbf{a}_1, \ldots, \mathbf{a}_m$. Assuming a deterministic measurement scheme, by the chain rule, we have

$$I(\mathbf{x}, \mathbf{y}) = h(\mathbf{y}) - h(\mathbf{y} \mid \mathbf{x}) = \sum_{i=1}^{m} h(y_i \mid y_{[i-1]}) - h(y_i \mid y_{[i-1]}, \mathbf{x}), \quad (1.7)$$

where $y_{[i]} := (y_1, \ldots, y_i)$. Since the history up to time $i - 1$ determines $\mathbf{a}_i$, the conditional distribution of $y_i$ given $y_{[i-1]}$ and $\mathbf{x}$ is then normal with mean $\langle \mathbf{a}_i, \mathbf{x} \rangle$ and variance $\sigma^2$. Hence, $h(y_i \mid y_{[i-1]}, \mathbf{x}) = \frac{1}{2} \log(2\pi e \sigma^2)$. This is the easy term to handle—the challenging term is $h(y_i \mid y_{[i-1]})$ and it is not clear how one should go about finding a good upper bound. To see this, observe that

$$\text{Var}(y_i \mid y_{[i-1]}) = \text{Var}(\langle \mathbf{a}_i, \mathbf{x} \rangle \mid y_{[i-1]}) + \sigma^2.$$

A standard approach to bound $h(y_i \mid y_{[i-1]})$ is to write

$$h(y_i \mid y_{[i-1]}) \leq \frac{1}{2} \mathbb{E} \log(2\pi e \text{Var}(\langle \mathbf{a}_i, \mathbf{x} \rangle \mid y_{[i-1]}) + 2\pi e \sigma^2),$$

using the fact that the Gaussian distribution maximizes the entropy among distributions with a given variance. If we simplify the problem by applying Jensen’s inequality, we obtain

$$I(\mathbf{x}, \mathbf{y}) \leq \sum_{i=1}^{m} \frac{1}{2} \log(\mathbb{E} \langle \mathbf{a}_i, \mathbf{x} \rangle^2 / \sigma^2 + 1). \quad (1.8)$$

The RHS needs to be bounded uniformly over all choices of measurement schemes, which is a daunting task given that $\mathbf{a}_i$ is a function of $y_{[i-1]}$ which is in turn a function of $\mathbf{x}$. We note however that the RHS can be bounded in the nonadaptive setting, which is the approach taken in [3] to establish (1.3). See also [25, 1, 30] for other asymptotic results in this direction.

We have presented the problem in this form to help information theorists see the analogy with the problem of understanding the role of feedback in a Gaussian channel [9]. Specifically, we can view the inner products $\langle \mathbf{a}_i, \mathbf{x} \rangle$ as inputs to a Gaussian channel where we observe the output of the channel via feedback. It is well-known that feedback does not substantially increase the capacity of a Gaussian channel, so one might expect this argument to be relevant to our problem as well. Crucially, however, in the case of a Gaussian channel the user has full control over the channel input—whereas in the absence of a priori knowledge of $\mathbf{x}$, in our problem we are much more restricted in our control over the “channel input” $\langle \mathbf{a}_i, \mathbf{x} \rangle$.

### 1.4 Connections with other works

A number of papers have studied the advantages (or sometimes the lack thereof) offered by adaptive sensing in the setting where one has noiseless data, see for example [11, 23, 16] and references therein. Of course, it is well known that one can uniquely determine a $k$-sparse vector from $2k$ linear nonadaptive
noise-free measurements and, therefore, there is not much to dwell on. The aforementioned works of course
do not study such a trivial problem. Rather, the point of view is that the signal is not exactly sparse,
only approximately sparse, and the question is thus whether one can get a lower approximation error by
employing an adaptive scheme. Whereas we study a statistical problem, this is a question in approximation
theory. Consequently, the techniques and results of this line of research have no bearing on our problem.

There is much research suggesting intelligent adaptive sensing strategies in the presence of noise and
we mention a few of these works. In a setting closely related to ours—that of detecting the locations of
the nonzeros of a sparse signal from noisy point samples (so that \( m > n \))—[14] shows that by adaptively
allocating sensing resources one can significantly improve upon the best nonadaptive schemes [12]. Lower
bounds for nonadaptive and adaptive methods in this context were recently established in [20], with the
adaptive lower bounds established through the sequential probability ratio test (SPRT) [27]. Closer to
home, [15, 13] consider CS schemes (with \( m < n \)) which perform sequential subset selection via the
random projections typical of CS, but which focus in on promising areas of the signal. When the signal is
(i) very sparse (ii) has sufficiently large entries and (iii) has constant dynamic range, the method in [13]
is able to remove a logarithmic factor from the MSE achieved by the Dantzig selector with (nonadaptive)
i.i.d. Gaussian measurements. In a different direction, [7, 18] suggest Bayesian approaches where the
measurement vectors are sequentially chosen so as to maximize the conditional differential entropy of \( y_i \)
given \( y_{i-1} \). Finally, another approach in [17] suggests a bisection method based on repeated measurements
for the detection of 1-sparse vectors, subsequently extended to \( k \)-sparse vectors via hashing. None of these
works, however, establish a lower bound on the MSE of the recovered signal.

1.5 Content

We prove all of our results in Section 2, trying to give as much insight as possible as to why adaptive
methods are not much more powerful than nonadaptive ones for detecting the support of a sparse signal.
We will also attempt to describe the regime in which adaptivity might be helpful via simple numerical
simulations in Section 3. These simulations show that adaptive algorithms are subject to a fundamental
phase transition phenomenon. Finally, we comment on open problems and future research in Section 4.

2 Limits of Adaptive Sensing Strategies

This section establishes nonasymptotic lower bounds for the estimation of a sparse vector from adaptively
selected noisy linear measurements. To begin with, we remind ourselves that we collect possibly adaptive
measurements of the form (1.4) of an \( n \)-dimensional signal \( x \) where \( \|a_i\|_2 \leq 1 \); from now on, we assume for
simplicity and without loss of generality that \( \sigma = 1 \).

In our analysis below, we denote the total-variation metric between any two probability distributions \( P \)
and \( Q \) by \( \|P - Q\|_{TV} \), and their KL divergence by \( K(P, Q) \) [22]. Our arguments will make use of Pinsker’s
inequality, which relates these two quantities via

\[
\|P - Q\|_{TV} \leq \sqrt{K(Q, P)/2}. \tag{2.1}
\]

We shall also use the convexity of the KL divergence, which states that for \( \lambda_i \geq 0 \) and \( \sum_i \lambda_i = 1 \), we have

\[
K\left(\sum_i \lambda_i P_i, \sum_i \lambda_i Q_i\right) \leq \sum_i \lambda_i K(P_i, Q_i) \quad \tag{2.2}
\]

in which \( \{P_i\} \) and \( \{Q_i\} \) are families of probability distributions.

Before proceeding, we argue that when we are given a prior \( \pi(x) \), we can restrict ourselves to determinis-
tic measurement schemes in the sense that \( a_1 \) is a deterministic vector and, for \( i \geq 2 \), \( a_i \) is a deterministic
function of \( y_{[i-1]} = (y_1, \ldots, y_i) \). In the general case we have \( a_i = F_i(y_{[i-1]}, U_i) \), where \( F_i \) is a deterministic function and \( U_i \) is random and independent of \( y_{[i-1]} \) and \( z_i \). With \( U = (U_1, \ldots, U_m) \), it follows from the law of iterated expectation

\[
\mathbb{E} \| \tilde{x} - x \|^2 = \mathbb{E} \left[ \mathbb{E}[\| \tilde{x} - x \|^2 | U] \right]
\]

(the expectation in the left-hand side is taken over \( x, y \) and \( U \) that there exists a fixed realization \( u = (u_1, \ldots, u_m) \) obeying

\[
\mathbb{E}[\| \tilde{x} - x \|^2 | U = u] \leq \mathbb{E} \| \tilde{x} - x \|^2.
\]

Hence, we can construct an estimator based on a deterministic measurement scheme which is as good as any based on a randomized measurement scheme. Note that in a deterministic scheme, letting \( P_x \) be the distribution of \( y_{[i-1]} \) when the target vector is \( x \) and using the fact that \( y_i \) is conditionally independent of \( y_{[i-1]} \) given \( a_i \), we see that the likelihood factorizes as

\[
P_x(y_{[m]}) = \prod_{i=1}^m P_x(y_i | a_i),
\]

which will be of use in our analysis below.

### 2.1 The Bernoulli prior

We begin by studying the model in Theorem 1 which makes our argument most transparent. The proof of Theorem 2 essentially reduces to that of Theorem 1.

In this model, we suppose that \( x \in \mathbb{R}^n \) is sampled from a product prior: for each \( j \in \{1, \ldots, n\} \),

\[
x_j = \begin{cases} 0 & \text{w.p. } 1 - k/n, \\ \mu & \text{w.p. } k/n, \end{cases}
\]

and the \( x_j \)'s are independent. In this model, \( x \) has on average \( k \) nonzero entries, all with known positive amplitudes equal to \( \mu \). This model is easier to study than the related model in which one selects \( k \) coordinates uniformly at random and sets those to \( \mu \). The reason is that in this Bernoulli model, the independence between the coordinates of \( x \) brings welcomed simplifications, as we shall see.

Our goal here is to establish a lower bound on the MSE when \( x \) is drawn from this prior. We do this in two steps. First, we look at recovering the support of \( x \), which is done via a reduction to multiple testing. Second, we show that a lower bound on the error for support recovery implies a lower bound on the MSE, leading to Theorem 1.

#### 2.1.1 Support recovery in Hamming distance

We would like to understand how well we can estimate the support \( S = \{ j : x_j \neq 0 \} \) of \( x \) from the data (1.4), and shall measure performance by means of the expected Hamming distance. Here, the error of a procedure \( \hat{S} \) for estimating the support \( S \) is defined as

\[
\mathbb{E} | \hat{S} \Delta S | = \sum_{j=1}^n P(\hat{S}_j \neq S_j)
\]

where \( \Delta \) denotes the symmetric difference, \( S_j = 1 \) if \( j \in S \) and equals zero otherwise, and similarly for \( \hat{S}_j \). As we can see, this reduces our problem to a sequence of \( n \) independent hypothesis tests. We will obtain a lower bound on the number of errors among these tests by exploiting the following lemma.
Lemma 1. Consider the testing problem of deciding between $H_0 : x \sim P_0$ and $H_1 : x \sim P_1$, where $H_0$ and $H_1$ occur with prior probabilities $\pi_0$ and $\pi_1$ respectively. Under the 0-1 loss, The Bayes risk $B$ obeys

$$B \geq \min(\pi_0, \pi_1) \left( 1 - \|P_1 - P_0\|_{TV} \right).$$

Proof. Assume without loss of generality that $\pi_1 \leq \pi_0$. The test with minimum risk is the Bayes test rejecting $H_0$ if and only if

$$\Lambda = \frac{\pi_1 P_1(x)}{\pi_0 P_0(x)} > 1;$$

that is, if the adjusted likelihood ratio exceeds one; see [19, Pbm. 3.10]. A simple calculation shows that the Bayes risk obeys

$$B = \pi_0 E_0(\min(1, \Lambda)),$$

where $E_0$ denotes expectation under $P_0$. Using the fact that $E_0 \Lambda = \frac{\pi_1}{\pi_0}$ together with

$$\min(1, \Lambda) = 1 + \frac{\Lambda - 1}{2},$$

we obtain

$$B = \frac{1}{2} - \frac{\pi_0}{2} E_0 |\Lambda - 1|. \tag{2.5}$$

Finally,

$$\pi_0 E_0 |\Lambda - 1| = \int |\pi_1 dP_1 - \pi_0 dP_0| \leq \pi_1 \int |dP_1 - dP_0| + \pi_0 - \pi_1$$

$$= 2\pi_1 \|P_1 - P_0\|_{TV} + \pi_0 - \pi_1,$$

which when combined with (2.5) establishes the lemma.

Theorem 3. Suppose that $x$ is sampled according to the Bernoulli prior with $k \leq n/2$, then any estimate $\hat{S}$ obeys

$$E |\hat{S} \Delta S| \geq k \left( 1 - \frac{\mu}{2} \sqrt{\frac{m}{n}} \right). \tag{2.6}$$

Hence, if the amplitude of the signal is below $\sqrt{n/m}$, we expect a large number of errors; indeed, if $\mu = \sqrt{n/m}$, then $E |\hat{S} \Delta S| \geq k/2$.

Proof. Let $\pi_1 = k/n$ and $\pi_0 = 1 - \pi_1$. For any $j$, set $P_{0,j} = P(\cdot | x_j = 0)$ and $P_{1,j} = P(\cdot | x_j \neq 0)$. Let $B_j$ denote the Bayes risk of the decision problem $H_{0,j} : x_j = 0$ versus $H_{1,j} : x_j = 1$. From Lemma 1 we have that

$$E |\hat{S} \Delta S| = \sum_{j=1}^{n} P(\hat{S}_j \neq S_j) \geq \sum_{j=1}^{n} B_j \geq \pi_1 \sum_{j=1}^{n} \left( 1 - \|P_{1,j} - P_{0,j}\|_{TV} \right).$$

Applying the Cauchy-Schwartz inequality, we obtain

$$E |\hat{S} \Delta S| \geq k \left( 1 - \frac{1}{\sqrt{n}} \sqrt{\sum_{j=1}^{n} \|P_{1,j} - P_{0,j}\|_{TV}^2} \right) \tag{2.7}$$

The theorem is a consequence of (2.7) combined with

$$\sum_{j=1}^{n} \|P_{1,j} - P_{0,j}\|_{TV}^2 \leq \frac{\mu^2}{4} m. \tag{2.8}$$

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2The main ideas of our proof are similar to those in that of Assouad’s Lemma, see [2, 29] for instance. Note, however, that our approach yields a sharper constant.
To establish (2.8), we apply Pinsker’s inequality twice to obtain

\[ ||P_{1,j} - P_{0,j}||_{TV}^2 \leq \frac{\pi_0}{2} K(P_{0,j}, P_{1,j}) + \frac{\pi_1}{2} K(P_{1,j}, P_{0,j}) \]  

(2.9)

so that it remains to find an upper bound on the KL divergence between \(P_{0,j}\) and \(P_{1,j}\). Write \(P_0 = P_{0,j}\) for short and likewise for \(P_{1,j}\). Then

\[ P_0(y_{[m]}) = \sum_{x'} P(x') P(y_{[m]}|x_j = 0, x') := \sum_{x'} P(x') P_{0,x'}, \]

where \(x' = (x_1, \ldots, x_j-1, x_{j+1}, \ldots, x_m)\) and \(P_{0,x'}\) is the conditional probability distribution of \(y_{[m]}\) given \(x'\) and \(x_j = 0\); \(P_1(y_{[m]})\) is defined similarly. The convexity of the KL divergence (2.2) gives

\[ K(P_0, P_1) \leq \sum_{x'} P(x') K(P_{0,x'}, P_{1,x'}). \]  

(2.10)

We now calculate this divergence. In order to do this, observe that we have \(y_i = \langle a_i, x \rangle + z_i = c_i + z_i\) under \(P_{0,x'}\) while \(y_i = a_{i,j} \mu + c_i + z_i\) under \(P_{1,x'}\). This yields

\[
K(P_{0,x'}, P_{1,x'}) = E_{0,x'} \log \frac{P_{0,x'}}{P_{1,x'}} \\
= \sum_{i=1}^{m} E_{0,x'} \left( \frac{1}{2} (y_i - \mu a_{i,j} - c_i)^2 - \frac{1}{2} (y_i - c_i)^2 \right) \\
= \sum_{i=1}^{m} E_{0,x'} \left( -z_i \mu a_{i,j} + (\mu a_{i,j})^2/2 \right) \\
= \frac{\mu^2}{2} \sum_{i=1}^{m} E_{0,x'} (a_{i,j}^2).
\]

The first equality holds by definition, the second follows from (2.3), the third from \(y_i = c_i + z_i\) under \(P_{0,x'}\) and the last holds since \(z_i\) is independent of \(a_{i,j}\) and has zero mean. Using (2.10), we obtain

\[ K(P_0, P_1) \leq \frac{\mu^2}{2} \sum_{i=1}^{m} E[a_{i,j}^2|x_j = 0]. \]

Similarly,

\[ K(P_1, P_0) \leq \frac{\mu^2}{2} \sum_{i=1}^{m} E[a_{i,j}^2|x_j = \mu] \]

and, therefore, (2.9) shows that

\[
||P_{1,j} - P_{0,j}||_{TV}^2 \leq \frac{\mu^2}{4} \left( \sum_{i=1}^{m} \pi_0 E[a_{i,j}^2|x_j = 0] + \pi_1 E[a_{i,j}^2|x_j = \mu] \right) = \frac{\mu^2}{4} \sum_{i=1}^{m} E[a_{i,j}^2].
\]

For any particular pair \((i, j)\) with \(j > 1\), we can say very little about \(E[a_{i,j}^2]\) since it can depend on all the previous measurements in a potentially very complicated manner. However, by summing this inequality over \(j\) we can obtain (2.8) by using the only constraint we have imposed on the \(a_i\), namely, \(\|a_i\|_2 = 1\), so that \(\sum_{ij} a_{ij}^2 = m\). This establishes the theorem. \(\square\)
2.1.2 Estimation in mean-squared error

It is now straightforward to obtain a lower bound on the MSE from Theorem 3.

Proof of Theorem 1. Let $S$ be the support of $x$ and set $\hat{S} := \{j : |\hat{x}_j| \geq \mu/2\}$. We have
\[
\|\hat{x} - x\|_2^2 = \sum_{j \in S} (\hat{x}_j - x_j)^2 + \sum_{j \notin S} \hat{x}_j^2 \geq \frac{\mu^2}{4} |S \setminus \hat{S}| + \frac{\mu^2}{4} |\hat{S} \setminus S| = \frac{\mu^2}{4} |\hat{S}\Delta S|
\]
and, therefore,
\[
E\|\hat{x} - x\|_2^2 \geq \frac{\mu^2}{4} E|\hat{S}\Delta S| \geq \frac{\mu^2}{4} k \left(1 - \frac{\mu}{2} \sqrt{\frac{m}{n}}\right),
\]
where the last inequality is from Theorem 3. We then plug in $\mu = \frac{4}{3} \sqrt{\frac{m}{n}}$ and simplify to conclude. \qed

2.2 The conditional Bernoulli prior and minimax bound

To establish Theorem 2, we choose as distribution on $x$ the prior $\nu_{n,k}$ defined as follows: we start with the Bernoulli prior $\pi_{n,ak}$ (2.4) with mean $ak$ (instead of $k$) for some fixed $\alpha \in (0, 1)$, and then condition that distribution to realizations with at most $k$ nonzero entries.

Proposition 1. Suppose that $x$ is sampled according to $\nu_{n,k}$ with $k \leq n/2$, then any estimate $\hat{S}$ obeys
\[
E|\hat{S}\Delta S| \geq ak \left(1 - \gamma_{n,k}(\alpha) - \frac{\mu}{2} \sqrt{\frac{m}{n}}\right),
\]
where
\[
\gamma_{n,k}(\alpha) := \frac{1}{\alpha} \sum_{j=k+1}^{n} (2 + (j - 1)/k) \mathbb{P}(\text{Bin}(n, \alpha k/n) = j).
\]

Proof. We begin by arguing that we can restrict attention to estimates $\hat{S}$ with cardinality at most $2k - 1$. To see why, consider an arbitrary estimate $\hat{S}$ and set
\[
\hat{S}_k = \begin{cases} \hat{S}, & |\hat{S}| \leq 2k - 1, \\ \emptyset, & |\hat{S}| \geq 2k. \end{cases}
\]
Now if $|\hat{S}| \geq 2k$, then for any $S$ with $|S| \leq k$, we have
\[
|\hat{S}\Delta S| \geq |\hat{S} \setminus S| \geq |\hat{S}| - |S| \geq k \geq |\emptyset\Delta S|.
\]
Since $|S| \leq k$ under $\nu_{n,k}$, it follows that $E|\hat{S}\Delta S| \geq E|\hat{S}_k\Delta S|$, which proves the claim. From now on, we assume that $|\hat{S}| < 2k$.

Set $\pi_{n,ak}(k) = \mathbb{P}_{x \sim \pi_{n,ak}}(|S| \leq k)$ and observe the identity
\[
E_{x \sim \nu_{n,k}} |\hat{S}\Delta S| = E_{x \sim \pi_{n,ak}} \left[|\hat{S}\Delta S| | |S| \leq k\right] = \frac{1}{\pi_{n,ak}(k)} E_{x \sim \pi_{n,ak}} \left[|\hat{S}\Delta S| 1_{|S| \leq k}\right].
\]
To conclude, Theorem 3 together with $|\hat{S}\Delta S| \leq |\hat{S}| + |S| \leq 2k - 1 + |S|$ give
\[
\pi_{n,ak}(k) E_{x \sim \nu_{n,k}} |\hat{S}\Delta S| = E_{x \sim \pi_{n,ak}} |\hat{S}\Delta S| - E_{x \sim \pi_{n,ak}} |\hat{S}\Delta S| 1_{|S| \geq k+1}
\geq ak \left(1 - \frac{\mu}{2} \sqrt{\frac{m}{n}}\right) - \sum_{j=k+1}^{n} (2k - 1 + j) \mathbb{P}(\text{Bin}(n, \alpha k/n) = j).
\]
\qed
Note that $\beta > 0$ for any $\alpha < 1$. Applying this inequality, we get

$$\alpha k \gamma_{n,k}(\alpha) \leq 3ke^{-(k+1)\beta} + \frac{e^{-(k+2)\beta}}{1-e^{-\beta}} \leq (3k + 1)e^{-(k+1)\beta},$$

when $\beta \geq \log 2$. This bound yields $\gamma < 1$ for all $\alpha \geq 1$ when $\alpha \leq 0.03$, in which case (2.11) and (2.12) become meaningful. This bound is quite conservative, however. Using the definition of $\gamma$, we can numerically show that that by choosing $\alpha$ appropriately we can obtain $\alpha(1 - \gamma) \geq 2e^{-1/2} - 1 > 0.21$ for any $n \geq 2$ and all $k \leq n/2$. Thus we can always write $\alpha(1 - \gamma) \frac{4}{27} \geq \frac{1}{33}$. While setting $C_k = \frac{1}{33}$ ensures that Theorem 2 holds for any possible choice of $k$, it is somewhat pessimistic in the sense that it is entirely dictated by the special case of $k = 1$ (which could be handled more efficiently by alternative means [10]).

For larger values of $k$, it is possible to obtain an improved constant. For example, when $k \geq 10$ numerical calculations show that we can take $C_k = \frac{1}{15}$. Moreover, in view of the first inequality in (2.14), and the fact that $\beta > 0$ for all $\alpha < 1$, we have $\gamma_{n,k}(\alpha) \to 0$ as $k \to \infty$ and $\alpha$ is held fixed. Thus, for $\alpha$ sufficiently close to 1 we will have that $\alpha(1 - \gamma) \frac{4}{27} \geq \frac{1}{7}$ for $k$ sufficiently large. We have also verified this numerically. Hence, the numerical constant $\frac{1}{7}$ of Theorem 1 is also valid in Theorem 2 provided $k$ is sufficiently large.

## 3 Numerical Experiments

In order to briefly illustrate the implications of the lower bounds in Section 2 and the potential limitations and benefits of adaptivity in general, we include a few simple numerical experiments. To simplify our discussion, we limit ourselves to existing adaptive procedures that aim at consistent support recovery: the adaptive procedure from [7] and the recursive bisection algorithm of [17].

We emphasize that in the case of a generic $k$-sparse signal, there are many possibilities for adaptively estimating the support of the signal. For example, the approach in [13] iteratively rules out indices and could, in principle, proceed until only $k$ candidate indices remain. In contrast, the approaches in [7] and [17] are built upon algorithms for estimating the support of 1-sparse signals. An algorithm for a 1-sparse signal could then be run $k$ times to estimate a $k$-sparse signal as in [7], or used in conjunction with a hashing scheme as in [17]. Since our goal is not to provide a thorough evaluation of the merits of all the different possibilities, but merely to illustrate the general limits of adaptivity, we simplify our discussion and focus exclusively on the simple case of one-sparse signals, i.e., where $k = 1$. 

... (remainder of the text continues)
Note that when \( \mu > 0 \) as in Section 2. Since we are focusing only on the case of \( k = 1 \), the algorithms in [7] and [17] are extremely simple and are shown in Algorithm 1 and Algorithm 2 respectively. Note that in Algorithm 1 the step of updating the posterior distribution \( \mathbf{p} \) consists of an iterative update rule given in [7] and does not require any a priori knowledge of the signal \( \mathbf{x} \) or \( \mu \). In Algorithm 2, we simplify the recursive bisection algorithm of [17] using the knowledge that \( \mu > 0 \), which allows us to eliminate the second stage of the algorithm aimed at detecting negative coefficients. Note that this algorithm proceeds through \( s_{\text{max}} = \log_2 n \) stages and we must allocate a certain number of measurements to each stage. In our experiments we set \( m_s = [\beta 2^{-s}] \), where \( \beta \) is selected to ensure that \( \sum_{s=1}^{\log_2 n} m_s \leq m \).

### 3.1 Evolution of the posterior

We begin by showing the results of a simple simulation that illustrates the behavior of the posterior distribution of \( \mathbf{x} \) as a function of \( \mu \) for both adaptive schemes. Specifically, we assume that \( m \) is fixed and collect \( m \) measurements using each approach. Given the measurements \( \mathbf{y} \), we then compute the posterior distribution \( \mathbf{p} \) using the true prior used to generate the signal, which can be computed using the fact that

\[
p_j \propto \exp \left( -\frac{1}{2\sigma^2} \| \mathbf{y} - \mu \mathbf{A} \mathbf{e}_j \|^2 \right),
\]

where \( \sigma^2 \) is the noise variance and \( \mathbf{e}_j \) denotes the \( j \)th element of the standard basis. What we expect is that once \( \mu \) exceeds a certain threshold (which depends on \( m \)), the posterior will become highly concentrated on the true support of \( \mathbf{x} \). To quantify this, we consider the case where \( j^* \) denotes the true location of the nonzero element of \( \mathbf{x} \) and define

\[
\lambda = \frac{p_{j^*}}{\max_{j \neq j^*} p_j}.
\]

Note that when \( \lambda \leq 1 \), we cannot reliably detect the nonzero, but when \( \lambda \gg 1 \) we can.
Figure 1: Behavior of the posterior distribution as a function of $\mu$ for several values of $m$. (a) shows the results for nonadaptive measurements. (b) shows the results for Algorithm 1. (c) shows the results for Algorithm 2. We see that Algorithm 2 is able to detect somewhat weaker signals than Algorithm 1. However, for both cases we observe that once $\mu$ exceeds a certain threshold proportional to $\sqrt{n/m}$, the ratio $\lambda$ of $p_j^*$ to the second largest posterior probability grows exponentially fast, but that this does not differ substantially from the behavior observed in (a) when using nonadaptive measurements.

In Figure 1 we show the results for a few representative values of $m$ (a) when using nonadaptive measurements, i.e., a (normalized) i.i.d. Rademacher random matrix $A$, compared to the results of (b) Algorithm 1, and (c) Algorithm 2. For each value of $m$ and for each value of $\mu$, we acquire $m$ measurements using each approach and compute the posterior $p$ according to (3.1). We then compute the value of $\lambda$. We repeat this for 10,000 iterations and plot the median value of $\lambda$ for each value of $\mu$ for all three approaches. In our experiments we set $n = 512$ and $\sigma^2 = 1$. We truncate the vertical axis at $10^4$ to ensure that all curves are comparable. We observe that in each case, once $\mu$ exceeds a certain threshold proportional to $\sqrt{n/m}$, the ratio $\lambda$ of $p_j^*$ to the second largest posterior probability grows exponentially fast. As expected, this occurs for both the nonadaptive and adaptive strategies, with no substantial difference in terms of how large $\mu$ must be before support recovery is assured (although Algorithm 2 seems to improve upon the nonadaptive strategy by a small constant).

### 3.2 MSE performance

We have just observed that for a given number of measurements $m$, there is a critical value of $\mu$ below which we cannot reliably detect the support. In this section we examine the impact of this phenomenon on the resulting MSE of a two-stage procedure that first uses $m_d = pm$ adaptive measurements to detect the location of the nonzero with either Algorithm 1 or Algorithm 2 and then reserves $m_e = (1-p)m$ measurements to directly estimate the value of the identified coefficient. It is not hard to show that if we correctly identify the location of the nonzero, then this will result in an MSE of $(m_e n)^{-1} = ((1-p)mn)^{-1}$. As a point of comparison, if an oracle provided us with the location of the nonzero a priori, we could devote all $m$ measurements to estimating its value, with the best possible MSE being $\frac{1}{mn}$. Thus, if we can correctly detect the nonzero, this procedure will perform within a constant factor of the oracle.

We illustrate the performance of Algorithm 1 and Algorithm 2 in terms of the resulting MSE as a function of the amplitude $\mu$ of the nonzero in Figure 2. In this experiment we set $n = 512$ and $m = 128$ with $p = \frac{1}{2}$ so that $m_d = 64$ and $m_e = 64$. We then compute the average MSE over 100,000 iterations for each value of $\mu$ and for both algorithms. We compare this to a nonadaptive procedure which uses a (normalized) i.i.d. Rademacher matrix followed by orthogonal matching pursuit (OMP). Note that in the worst case the MSE of the adaptive algorithms is comparable to the MSE obtained by the nonadaptive algorithm and exceeds the lower bound in Theorem 2 by only a small constant factor. However, when $\mu$ begins to exceed a critical threshold, the MSE rapidly decays and approaches the optimal value of $\frac{1}{m_en}$. 
Figure 2: The performance of Algorithm 1 and Algorithm 2 in the context of a two-stage procedure that first uses $m_d = \frac{m}{2}$ adaptive measurements to detect the location of the nonzero and then uses $m_e = \frac{m}{2}$ measurements to directly estimate the value of the identified coefficient. We show the resulting MSE as a function of the amplitude $\mu$ of the nonzero entry, and compare this to a nonadaptive procedure which uses a (normalized) i.i.d. Rademacher matrix followed by OMP. In the worst case, the MSE of the adaptive algorithms is comparable to the MSE obtained by the nonadaptive algorithm and exceeds the lower bound in Theorem 2 by only a small constant factor. When $\mu$ begins to exceed this critical threshold, the MSE of the adaptive algorithms rapidly decays below that of the nonadaptive algorithm and approaches $\frac{1}{m_e n}$, which is the MSE one would obtain given $m_e$ measurements and a priori knowledge of the support.

Note that when $\mu$ is large we can take $m_e \to m$ and hence can actually get arbitrarily close to $\frac{1}{mn}$ in the asymptotic regime.

4 Discussion

The contribution of this paper is to show that if one has the freedom to choose any adaptive sensing strategy and any estimation procedure no matter how complicated or computationally intractable, we would not be able to universally improve over a simple nonadaptive strategy that simply projects the signal onto a lower dimensional space and perform recovery via $\ell_1$ minimization. This “negative” result should not conceal the fact that adaptivity may help tremendously if the SNR is sufficiently large, as illustrated in Section 3. Hence, we regard the design and analysis of effective adaptive schemes as a subject of important future research. At the methodological level, it seems important to develop adaptive strategies and algorithms for support estimation that are as accurate and as robust as possible. Further, a transition towards practical applications would need to involve engineering hardware that can effectively implement this sort of feedback, an issue which poses all kinds of very concrete challenges. Finally, at the theoretical level, it would be of interest to analyze the phase transition phenomenon we expect to occur in simple Bayesian signal models. For instance, a central question would be how many measurements are required to transition from a nearly flat posterior to one mostly concentrated on the true support.

In closing, we note that after the submission of this paper, a variant of Algorithm 2 was shown to recover the correct support of a 1-sparse vector with high probability provided that the amplitude $\mu$ of the nonzero entry obeys $\mu \geq C \sqrt{n/m}$ for some positive numerical constant $C$ [10, 21]. This implies that for $k = 1$, the lower bound in Theorem 2 is tight up to constant factors. Thus, adaptive methods have the potential to remove the $\log(n/k)$ factor required in the nonadaptive setting.
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