Interval colorings of complete bipartite graphs and trees

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A translation from Russian of the work of R.R. Kamalian "Interval colorings of complete bipartite graphs and trees", Preprint of the Computing Centre of the Academy of Sciences of Armenia, Yerevan, 1989. (Was published by the decision of the Academic Council of the Computing Centre of the Academy of Sciences of Armenian SSR and Yerevan State University from 7.09.1989)

In the work interval colorings [1] of complete bipartite graphs and trees are investigated. The obtained results were announced in [2]. Non defined concepts can be found in [3, 4].

Let $G = (V(G), E(G))$ be an undirected graph without multiple edges and loops. The degree of a vertex $x$ in $G$ is denoted by $d_G(x)$, the greatest degree of vertices – by $\Delta(G)$, the chromatic index of $G$ – by $\chi'(G)$.

Interval $t$-coloring of a graph $G$ is a proper coloring of edges of $G$ by the colors $1, \ldots, t$, at which by each color $i, 1 \leq i \leq t$, at least one edge $e_i \in E(G)$ is colored, and edges incident with each vertex $x \in V(G)$ are colored by $d_G(x)$ consecutive colors.

A graph $G$ is called interval colorable if there is $t \geq 1$ for which $G$ has an interval $t$-coloring. For an interval colorable graph $G$, we denote by $w(G)$ and $W(G)$, respectively, the least and the greatest value of $t$, for which $G$ has an interval $t$-coloring.

If $\alpha$ is a proper edge coloring of a graph $G$, then the color of an edge $e \in E(G)$ at this coloring is denoted by $\alpha(e, G)$ or, if it is clear which graph is spoken about, by $\alpha(e)$.

Let $k$ and $l$ be positive integers. Let us denote by $\sigma(k, l)$ the greatest common divisor of $k$ and $l$. The algorithm of Euclid for finding of $\sigma(k, l)$ consists of the construction of sequences $(F_i(k, l)), (f_i(k, l)), i = 1, 2, \ldots$, defined as follows: $F_1(k, l) = \max\{k, l\}$, $f_1(k, l) = \min\{k, l\}$; if $F_i(k, l) = f_i(k, l)$ then the construction of the sequences is finished, and if $F_i(k, l) > f_i(k, l)$ then $F_{i+1}(k, l) = \max\{F_i(k, l) - f_i(k, l), f_i(k, l)\}$, $f_{i+1}(k, l) = \min\{F_i(k, l) - f_i(k, l), f_i(k, l)\}$, $i = 1, 2, \ldots$.

The algorithm is completed at the finding of such $j$ (let us denote it by $s(k, l)$) for which $F_j(k, l) = f_j(k, l) = \sigma(k, l)$.

Let $H(\mu, \nu)$ be a $(0, 1)$-matrix with $\mu$ rows, $\nu$ columns, and with elements $h_{ij}$, $1 \leq i \leq \mu$, $1 \leq j \leq \nu$. The $i$-th row of the matrix $H(\mu, \nu)$, $1 \leq i \leq \mu$, is called collected, if $h_{ip} = h_{iq} = 1$, $p \leq t \leq q$ imply $h_{it} = 1$, and the inequality $\sum_{j=1}^{\nu} h_{ij} \geq 1$ holds. Similarly, the $j$-th column of the matrix $H(\mu, \nu)$, $1 \leq j \leq \nu$, is called collected, if $h_{pj} = h_{qj} = 1$, $p \leq t \leq q$ imply $h_{tj} = 1$, and the inequality $\sum_{i=1}^{\mu} h_{ij} \geq 1$ holds.

For the $i$-th row of the matrix $H(\mu, \nu)$, all rows and columns of which are collected, define a number $\varepsilon(i, H(\mu, \nu)) = \min_{h_{ij} = 1} j$, $i = 1, \ldots, \mu$. For the $j$-th column of the matrix $H(\mu, \nu)$, all rows and columns of which are collected, define a number $\xi(j, H(\mu, \nu)) = |\{i/ \varepsilon(i, H(\mu, \nu)) = j, 1 \leq i \leq \mu\}|$, $j = 1, \ldots, \nu$. $H(\mu, \nu)$ is called an $r$-regular ($r \geq 1$) matrix, if $\sum_{j=1}^{\nu} h_{ij} = r$, $i = 1, \ldots, \mu$. $H(\mu, \nu)$ is called a collected matrix, if all its rows and columns are collected, $h_{11} = h_{\mu\nu} = 1$, and the inequality $\varepsilon(1, H(\mu, \nu)) \leq \cdots \leq \varepsilon(\mu, H(\mu, \nu))$ holds. $(0, 1)$-matrices $A(\alpha, \gamma)$ and $B(\beta, \gamma)$ with elements $a_{ij}$, $1 \leq i \leq \alpha$, $1 \leq j \leq \gamma$ and $b_{ij}$, $1 \leq i \leq \beta$, $1 \leq j \leq \gamma$, respectively, are called equivalent, if $\sum_{i=1}^{\alpha} a_{ij} = \sum_{i=1}^{\beta} b_{ij}$, $j = 1, \ldots, \gamma$. 

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An \( r' \)-regular \((r' \geq 1)\) matrix \( H'(\mu', \nu') \) and an \( r'' \)-regular \((r'' \geq 1)\) matrix \( H''(\mu'', \nu'') \) are called mutually conformed, if \( r' = \mu'' \) and \( r'' = \mu' \).

**Lemma 1.** If a collected \( n \)-regular \((n \geq 1)\) matrix \( P(m, w) \) with elements \( p_{ij}, \ 1 \leq i \leq m, \ 1 \leq j \leq w \) is equivalent to a collected \( m \)-regular \((m \geq 1)\) matrix \( Q(n, w) \) with elements \( q_{ij}, \ 1 \leq i \leq n, \ 1 \leq j \leq w \), then \( w \geq m + n - \sigma(m, n) \).

**Proof** by induction on \( s(m, n) \). If \( s(m, n) = 1 \), then \( m = n = \sigma(m, n) \), and, clearly, the lemma is true. Let

\[
s(m, n) = z_0 > 1 \tag{1}
\]

and the lemma is supposed to be true for mutually conformed equivalent an \( n' \)-regular \((n' \geq 1)\) matrix and an \( m' \)-regular \((m' \geq 1)\) matrix with \( s(m', n') < z_0 \). Assume, in opposite to the desired, that

\[
w < m + n - \sigma(m, n) \tag{2}
\]

and, for definition,

\[
m \geq n \tag{3}
\]

Let us note that \( \varepsilon(n, Q(n, w)) + m - 1 = w < m + n - \sigma(m, n) \leq m + n - 1 \), which implies

\[
\varepsilon(n, Q(n, w)) < n \tag{4}
\]

From (3) we conclude:

\[
\sum_{i=1}^{m} p_{ij} = \sum_{r=1}^{j} \xi(r, P(m, w)), \quad j = 1, \ldots, n \tag{5}
\]

\[
\sum_{i=1}^{n} q_{ij} = \sum_{r=1}^{j} \xi(r, Q(n, w)), \quad j = 1, \ldots, n \tag{6}
\]

From the equivalence of the matrices \( P(m, w) \) and \( Q(n, w) \), and from the relations (3) – (6), it follows that:

\[
\xi(j, P(m, w)) = \xi(j, Q(n, w)), \quad j = 1, \ldots, n \tag{7}
\]

\[
\sum_{i=1}^{m} p_{im} = \sum_{i=1}^{n} q_{in} = n \tag{8}
\]

Let us form from matrices \( P(m, w) \) and \( Q(n, w) \), respectively, matrices \( P_1(m - n, w - n) \) and \( Q_1(n, w - n) \) of smaller dimensions by the following way: form \( P_1(m - n, w - n) \) from \( P(m, w) \) by removing that and only that elements \( p_{ij} \), for which at least one of the inequalities \( i \leq n, j \leq n \) holds; form \( Q_1(n, w - n) \) from \( Q(n, w) \) by zeroing that and only that elements \( q_{ij} \), for which \( j < \varepsilon(i, Q(n, w)) + n \), and further removing of all elements of first \( n \) columns.

From (1) and (3) it follows that

\[
m > n \tag{9}
\]

From the construction of matrices \( P_1(m - n, w - n), Q_1(n, w - n) \) and from the relations (1), (3), (7), (8) it follows that \( P_1(m - n, w - n) \) is a collected \( n \)-regular \((n \geq 1)\) matrix, \( Q_1(n, w - n) \) is an equivalent to it collected \((m - n)\)-regular \((m - n \geq 1)\) matrix. Clearly, \( P_1(m - n, w - n) \)
and \( Q_1(n, w-n) \) are mutually conformed, \( s(m-n, n) < z_0 \). From here, by the assumption of induction, we have the inequality \( w - n \geq (m - n) + n - \sigma(m - n, n) \), or

\[
w \geq m + n - \sigma(m - n, n)
\]

From (9) we conclude \( \sigma(m - n, n) = \sigma(m, n) \), and, taking (10) into account, we obtain the inequality \( w \geq m + n - \sigma(m, n) \), which contradicts the assumption (2).

The Lemma is proved.

Lemma 2. For arbitrary positive integers \( m \) and \( n \), \( K_{m,n} \) has an interval \((m + n - 1)\)-coloring.

Proof. For obtaining of an interval \((m + n - 1)\)-coloring of the graph \( K_{m,n} \), color the edge \((x_i, y_j)\), \( 1 \leq i \leq m \), \( 1 \leq j \leq n \), by the color \( i + j - 1 \).

The Lemma is proved.

Theorem 1. For arbitrary positive integers \( m \) and \( n \),

1) \( K_{m,n} \) is interval colorable,

2) \( w(K_{m,n}) = m + n - \sigma(m, n) \),

3) \( W(K_{m,n}) = m + n - 1 \),

4) if \( w(K_{m,n}) \leq t \leq W(K_{m,n}) \), then \( K_{m,n} \) has an interval \( t \)-coloring.

Proof. The proposition 1) of the theorem immediately follows from the lemma 2. From the already proved proposition 1) and from the corollary of the theorem 1 of the work \([1]\) we have \( W(K_{m,n}) \leq |V(K_{m,n})| - 1 = m + n - 1 \). From here and from the lemma 2 the proposition 3) of the theorem follows.

Now let us be convinced of \( w(K_{m,n}) \geq m + n - \sigma(m, n) \). Consider an interval \( w(K_{m,n}) \)-coloring of the graph \( K_{m,n} \). For \( v \in V(K_{m,n}) \), let us denote by \( \lambda(v) \) the least among colors of edges incident with \( v \). Clearly, without loss of generality, we can assume that

\[
\lambda(x_1) \leq \cdots \leq \lambda(x_m); \quad \lambda(y_1) \leq \cdots \leq \lambda(y_n)
\]

Define a matrix \( X = (x_{ij}) \) with \( m \) rows and \( w(K_{m,n}) \) columns:

\[
x_{ij} = \begin{cases} 1, & \text{if there is an edge colored by } j \text{ incident with the vertex } x_i \\ 0 & \text{otherwise} \end{cases}
\]

\( 1 \leq i \leq m, \ 1 \leq j \leq w(K_{m,n}). \)

Define a matrix \( Y = (y_{ij}) \) with \( n \) rows and \( w(K_{m,n}) \) columns:

\[
y_{ij} = \begin{cases} 1, & \text{if there is an edge colored by } j \text{ incident with the vertex } y_i \\ 0 & \text{otherwise} \end{cases}
\]

\( 1 \leq i \leq n, \ 1 \leq j \leq w(K_{m,n}). \)

From properties of the considered coloring and inequalities \([1]\) it follows that \( X \) is a \( n \)-regular \((n \geq 1)\) collected matrix, and \( Y \) is an equivalent to it \( m \)-regular \((m \geq 1)\) collected.
matrix. It is also clear that $X$ and $Y$ are mutually conformed. It follows from the lemma that $w(K_{m,n}) \geq m + n - \sigma(m,n)$.

Evidently, for the completion of the proof of the theorem it is suffice to show, that, if $m + n - \sigma(m,n) \leq t \leq m + n - 1$, then $K_{m,n}$ has an interval $t$-coloring.

Let $t = m + n - \sigma(m,n) + \mu$, where

$$0 \leq \mu \leq \sigma(m,n) - 1 \quad (12)$$

Let us denote by $G_1$ the subgraph of the graph $K_{m,n}$ induced by the vertices $x_1, \ldots, x_{\sigma(m,n)}, y_1, \ldots, y_{\sigma(m,n)}$.

Let $p = \frac{m}{\sigma(m,n)}$, $q = \frac{n}{\sigma(m,n)}$.

$G_1$ is a regular complete bipartite graph. From the proposition 2 of the work [1] it follows that

$$\chi'(G_1) = \Delta(G_1) = w(G_1) = \sigma(m,n) \quad (13)$$

From the already proved proposition 3) of the theorem we have

$$W(G_1) = 2\sigma(m,n) - 1 \quad (14)$$

From the relations (12) – (14) we obtain

$$\Delta(G_1) = w(G_1) \leq \sigma(m,n) + \mu \leq W(G_1) \quad (15)$$

Since $G_1$ is a regular graph then from (13), (15) and the proposition 2 of the work [1] it follows that there exists an interval $(\sigma(m,n) + \mu)$-coloring $\alpha$ of the graph $G_1$. Now, in order to receive an interval $t$-coloring of the graph $K_{m,n}$, it is suffice for $\tau = 1, \ldots, p - 1$ and $\varepsilon = 1, \ldots, q - 1$ to color the edge $(x_i + \tau\sigma(m,n), y_j + \varepsilon\sigma(m,n))$ of the graph $K_{m,n}$ by the color $(\tau + \varepsilon) \cdot \sigma(m,n) + \alpha((x_i, y_j), G_1)$, $1 \leq i \leq \sigma(m,n)$, $1 \leq j \leq \sigma(m,n)$.

The Theorem is proved.

**Corollary 1.** If $\sigma(m,n) = 1$, then $K_{m,n}$ has an interval $t$-coloring iff $t = m + n - 1$.

Let $D$ be a tree, $V(D) = \{b_1, \ldots, b_\beta\}$, $\beta \geq 1$. Let us denote by $L(b_i, b_j)$ the path connecting the vertices $b_i$ and $b_j$, by $VL(b_i, b_j)$ and $EL(b_i, b_j)$ - the sets of vertices and edges of this path, respectively, $1 \leq i \leq \beta$, $1 \leq j \leq \beta$. For the path $L(b_i, b_j)$, $1 \leq i \leq \beta$, $1 \leq j \leq \beta$, let us introduce a notation:

$$ML(b_i, b_j) = |EL(b_i, b_j)| + |\{(x,y) / (x,y) \in E(D), x \in VL(b_i, b_j), y \notin VL(b_i, b_j)\}|.$$ 

Let

$$M(D) = \max_{1 \leq i \leq \beta, 1 \leq j \leq \beta} ML(b_i, b_j).$$

**Lemma 3.** If a tree $D$ is interval colorable, then $W(D) \leq M(D)$.

**Proof.** Without loss of generality, we can assume that $|E(D)| > 1$ (otherwise the lemma is evident). Consider an interval $W(D)$-coloring $\alpha$ of the tree $D$. Let $\alpha(e_1) = 1$, $\alpha(e_2) = w(D)$, $e_1 = (x', y')$, $e_2 = (x'', y'')$. Without loss of generality we can assume that $|EL(x', x'')| > |EL(y', y'')|$. Let us number the vertices of the set $VL(x', x'')$ in the direction from $x'$ to $x'': x' = z_0, z_1, \ldots, z_s, z_{s+1} = x''$, where $s \geq 1$.

Let us note that $\alpha((z_i, z_{i+1})) \leq 1 + \sum_{j=i}^{i} (d_D(z_j) - 1)$, $i = 1, \ldots, s$. Consequently, $W(D) = \alpha(e_2) = \alpha((z_s, z_{s+1})) \leq 1 + \sum_{j=1}^{s} (d_D(z_j) - 1) = ML(x', x'') \leq M(D)$.

The Lemma is proved.
Lemma 4. If $D$ is a tree, and $\Delta(D) \leq t \leq M(D)$, then $D$ has an interval $t$-coloring.

Proof by induction on $|E(D)|$. If $|E(D)| = 1$, then, clearly, the lemma is true. Let $|E(D)| = k > 1$, and assume that the lemma is true for all trees $D'$ with $|E(D')| < k$.

Case 1. $M(D) < |E(D)|$.

In this case there is a pendent edge $e = (x, y) \in E(D)$, $d_D(x) = 1$, such, that its removing from $D$ gives a tree $D'$ with $M(D') = M(D)$. Since $|E(D)| > 1$, then $d_D(y) \neq 1$. Clearly, $d_{D'}(y) = d_D(y) - 1$, $\Delta(D') \leq \Delta(D) - 1$, $|E(D')| = |E(D)| - 1 < k$, $\Delta(D') \leq t \leq M(D')$. By the assumption of induction, there exists an interval $t$-coloring of the tree $D'$. Suppose that the edges of $E(D')$ incident with the vertex $y$ are colored in this coloring by the colors $\lambda_1(1), \lambda_1(2), \ldots, \lambda_1(d_{D'}(y))$, where $1 \leq \lambda_1(1) < \ldots < \lambda_1(d_{D'}(y)) \leq t$. If $\lambda_1(1) > 1$, we shall color the edge $e$ by the color $\lambda_1(1) - 1$ and obtain an interval $t$-coloring of the tree $D$. If $\lambda_1(1) = 1$, then $\lambda_1(d_{D'}(y)) = d_{D'}(y) = d_D(y) - 1$. We shall color the edge $e$ by the color $d_D(y)$ and obtain an interval $t$-coloring of the tree $D$.

Case 2. $M(D) = |E(D)|$.

Case 2a). $t \leq M(D) - 1$.

Let $e = (x, y)$ be a pendent edge in $D$, and $d_D(x) = 1$. Since $|E(D)| > 1$, then $d_D(y) \neq 1$. Let us denote by $D'$ the tree which is obtained from the tree $D$ by removing of the edge $e$. Clearly, $d_{D'}(y) = d_D(y) - 1$, $\Delta(D') \leq \Delta(D) - 1$, $M(D) - 1 \leq M(D') \leq M(D)$, hence, $\Delta(D') \leq \Delta(D) \leq t \leq M(D')$. Since $|E(D')| = |E(D)| - 1 < k$, then, by the assumption of induction, there exists an interval $t$-coloring of the tree $D'$. Suppose that the edges of $E(D')$ incident with the vertex $y$, are colored in this coloring by the colors $\lambda_2(1), \lambda_2(2), \ldots, \lambda_2(d_{D'}(y))$, where $1 \leq \lambda_2(1) < \ldots < \lambda_2(d_{D'}(y)) \leq t$. If $\lambda_2(1) > 1$, we shall color the edge $e$ by the color $\lambda_2(1) - 1$ and obtain an interval $t$-coloring of the tree $D$. If $\lambda_2(1) = 1$, then $\lambda_2(d_{D'}(y)) = d_{D'}(y) = d_D(y) - 1$. We shall color the edge $e$ by the color $d_D(y)$ and obtain an interval $t$-coloring of the tree $D$.

Case 2b). $t = M(D)$.

Clearly, without loss of generality, we can assume that $ML(b_1, b_2) = M(D)$. Clearly, $d_D(b_1) = d_D(b_2) = 1$. Let us number the vertices of the path $L(b_1, b_2)$ in the direction from $b_1$ to $b_2$: $b_1 = z_0, z_1, \ldots, z_s, z_{s+1} = b_2$, where $s \geq 1$. Let us construct an interval $t$-coloring of the tree $D$. We shall color the edge $(z_0, z_1)$ by the color 1, the edge $(z_i, z_{i+1})$, $i = 1, \ldots, s$ - by the color $1 + \sum_{j=1}^{i}(d_D(z_j) - 1)$. $d_D(z_1) - 2$ edges without a color incident with the vertex $z_1$, will be colored by the colors $2, \ldots, d_D(z_1) - 1$. $d_D(z_i) - 2$ edges without a color incident with the vertex $z_i$, $i = 2, \ldots, s$, will be colored by the colors $(1 + \sum_{j=1}^{i-1}(d_D(z_j) - 1)) + 1, \ldots, (1 + \sum_{j=1}^{i}(d_D(z_j) - 1)) - 1$.

The Lemma is proved.

From lemmas 3 and 4 we obtain

Theorem 2. Let $D$ be a tree. Then

1) $D$ is interval colorable,

2) $w(D) = \Delta(D)$,

3) $W(D) = M(D)$,

4) if $w(D) \leq t \leq W(D)$, then $D$ has an interval $t$-coloring.

I thank A.S. Asratian for advices and attention to the work.
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