On the Cut Number of a 3–manifold

SHELLY L HARVEY

Department of Mathematics
University of California at San Diego
La Jolla, CA 92039-0122, USA

Email: sharvey@math.ucsd.edu
URL: http://math.ucsd.edu/~sharvey

Abstract

The question was raised as to whether the cut number of a 3–manifold $X$ is bounded from below by $\frac{1}{3}\beta_1 (X)$. We show that the answer to this question is “no.” For each $m \geq 1$, we construct explicit examples of closed 3–manifolds $X$ with $\beta_1 (X) = m$ and cut number 1. That is, $\pi_1 (X)$ cannot map onto any non-abelian free group. Moreover, we show that these examples can be assumed to be hyperbolic.

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1 Introduction

Let $X$ be a closed, orientable $n$–manifold. The cut number of $X$, $c(X)$, is defined to be the maximal number of components of a closed, 2–sided, orientable hypersurface $F \subset X$ such that $X - F$ is connected. Hence, for any $n \leq c(X)$, we can construct a map $f: X \to \bigvee_{i=1}^{n} S^1$ such that the induced map on $\pi_1$ is surjective. That is, there exists a surjective map $f_\ast: \pi_1(X) \twoheadrightarrow F(c)$, where $F(c)$ is the free group with $c = c(X)$ generators. Conversely, if we have any epimorphism $\phi: \pi_1(X) \twoheadrightarrow F(n)$, then we can find a map $f: X \to \bigvee_{i=1}^{n} S^1$ such that $f_\ast = \phi$. After making the $f$ transverse to a non-wedge point $x_i$ on each $S^1$, $f^{-1}(X)$ will give $n$ disjoint surfaces $F = \bigcup F_i$ with $X - F$ connected. Hence one has the following elementary group-theoretic characterization of $c(X)$.

**Proposition 1.1** $c(X)$ is the maximal $n$ such that there is an epimorphism $\phi: \pi_1(X) \twoheadrightarrow F(n)$ onto the free group with $n$ generators.

**Example 1.2** Let $X = S^1 \times S^1 \times S^1$ be the 3–torus. Since $\pi_1(X) = \mathbb{Z}^3$ is abelian, $c(X) = 1$.

Using Proposition 1.1, we show that the cut number is additive under connected sum.

**Proposition 1.3** If $X = X_1 \# X_2$ is the connected sum of $X_1$ and $X_2$ then
\[ c(X) = c(X_1) + c(X_2). \]

**Proof** Let $G_i = \pi_1(X_i)$ for $i = 1, 2$ and $G = \pi_1(X) \cong G_1 * G_2$. It is clear that $G$ maps surjectively onto $F(c(X_1)) * F(c(X_2)) \cong F(c(X_1 + X_2))$. Therefore $c(X) \geq c(X_1) + c(X_2)$.

Now suppose that there exists a map $\phi: G \twoheadrightarrow F(n)$. Let $\phi_1: G_i \twoheadrightarrow F(n)$ be the composition $G_i \to G_1 * G_2 \overset{\cong}{\to} G \overset{\phi}{\twoheadrightarrow} F(n)$. Since $\phi$ is surjective and $G \cong G_1 * G_2$, $\text{Im}(\phi_1)$ and $\text{Im}(\phi_2)$ generate $F(n)$. Moreover, $\text{Im}(\phi_1)$ is a subgroup of a free group, hence is free of rank less than or equal to $c(X_i)$. It follows that $n \leq c(X_1) + c(X_2)$. In particular, when $n$ is maximal we have $c(X) = n \leq c(X_1) + c(X_2)$. \hfill $\square$

In this paper, we will only consider 3–manifolds with $\beta_1(X) \geq 1$. Consider the surjective map $\pi_1(X) \to H_1(X) / \{\text{Z–torsion}\} \cong \mathbb{Z}^{\beta_1(X)}$. Since $\beta_1(X) \geq 1$,
we can find a surjective map from $\mathbb{Z}^{\beta_1(X)}$ onto $\mathbb{Z}$. It follows from Proposition 1.1 that $c(X) \geq 1$. Moreover, every map $\phi: \pi_1(X) \to F(n)$ gives rise to an epimorphism $\bar{\phi}: H_1(X) \to H_1 \left( \bigvee_{i=1}^{n} S^1 \right) \cong \mathbb{Z}^n$. It follows that $\beta_1(X) \geq n$ which gives us the well known result:

$$1 \leq c(X) \leq \beta_1(X). \quad (1)$$

It has recently been asked whether a (non-trivial) lower bound exists for the cut number. We make the following observations.

**Remark 1.4** If $S$ is a closed, orientable surface then $c(S) = \frac{1}{2} \beta_1(S)$.

**Remark 1.5** If $X$ has solvable fundamental group then $c(X) = 1$ and $\beta_1(X) \leq 3$.

**Remark 1.6** Both $c$ and $\beta_1$ are additive under connected sum (Proposition 1.3).

Therefore it is natural to ask the following question first asked by A Sikora and T Kerler. This question was motivated by certain results and conjectures on the divisibility of quantum 3–manifold invariants by P Gilmer–T Kerler [2] and T Cochran–P Melvin [1].

**Question 1.7** Is $c(X) \geq \frac{1}{3} \beta_1(X)$ for all closed, orientable 3–manifolds $X$?

We show that the answer to this question is “as far from yes as possible.” In fact, we show that for each $m \geq 1$ there exists a closed, hyperbolic 3–manifold with $\beta_1(X) = m$ and $c(X) = 1$. We actually prove a stronger statement.

**Theorem 3.1** For each $m \geq 1$ there exist closed 3–manifolds $X$ with $\beta_1(X) = m$ such that for any infinite cyclic cover $X_\phi \to X$, rank$_{\mathbb{Z}[t^{\pm 1}]} H_1(X_\phi) = 0$.

We note the condition stated in the Theorem 3.1 is especially interesting because of the following theorem of J Howie [3]. Recall that a group $G$ is large if some subgroup of finite index has a non-abelian free homomorphic image. Howie shows that if $G$ has an infinite cyclic cover whose rank is at least 1 then $G$ is large.

**Theorem 1.8** (Howie [3]) Suppose that $\bar{K}$ is a connected regular covering complex of a finite 2–complex $K$, with nontrivial free abelian covering transformation group $A$. Suppose also that $H_1(\bar{K}; \mathbb{Q})$ has a free $\mathbb{Q}[A]$–submodule of rank at least 1. Then $G = \pi_1(K)$ is large.
Using the proof of Theorem 3.1 we show that the fundamental group of the aforementioned 3–manifolds cannot map onto $F/F_4$ where $F$ is the free group with 2 generators and $F_4$ is the 4th term of the lower central series of $F$.

**Proposition 3.3** Let $X$ be as in Theorem 3.1, $G = \pi_1(X)$ and $F$ be the free group on 2 generators. There is no epimorphism from $G$ onto $F/F_4$.

Independently, A Sikora has recently shown that the cut number of a “generic” 3–manifold is at most 2 [8]. Also, C Leininger and A Reid have constructed specific examples of genus 2 surface bundles $X$ satisfying (i) $\beta_1(X) = 5$ and $c(X) = 1$ and (ii) $\beta_1(X) = 7$ and $c(X) = 2$ [6].

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## 2 Relative Cut Number

Let $\phi$ be a primitive class in $H^1(X;\mathbb{Z})$. Since $H^1(X;\mathbb{Z}) \cong \text{Hom}(\pi_1(X),\mathbb{Z})$, we can assume $\phi$ is a surjective homomorphism, $\phi: \pi_1(X) \rightarrow \mathbb{Z}$. Since $X$ is an orientable 3–manifold, every element in $H_2(X;\mathbb{Z})$ can be represented by an embedded, oriented, 2–sided surface [10, Lemma 1]. Therefore, if $\phi \in H^1(X;\mathbb{Z}) \cong H_2(X;\mathbb{Z})$ there exists a surface (not unique) dual to $\phi$. The **cut number of $X$ relative to $\phi$, $c(X,\phi)$**, is defined as the maximal number of components of a closed, 2–sided, oriented surface $F \subset X$ such that $X - F$ is connected and one of the components of $F$ is dual to $\phi$. In the above definition, we could have required that “any number” of components of $F$ be dual to $\phi$ as opposed to just “one.” We remark that since $X - F$ is connected, these two conditions are equivalent. Similar to $c(X)$, we can describe $c(X,\phi)$ group theoretically.

**Proposition 2.1** $c(X,\phi)$ is the maximal $n$ such that there is an epimorphism $\psi: \pi_1(X) \rightarrow F(n)$ onto the free group with $n$ generators that factors through $\phi$ (see diagram on next page).
It follows immediately from the definitions that $c(X, \phi) \leq c(X)$ for all primitive $\phi$. Now let $F$ be any surface with $c(X)$ components and let $\phi$ be dual to one of the components, then $c(X, \phi) = c(X)$. Hence

$$c(X) = \max \left\{ c(X, \phi) \mid \phi \text{ is a primitive element of } H^1(X; \mathbb{Z}) \right\}. \tag{2}$$

In particular, if $c(X, \phi) = 1$ for all $\phi$ then $c(X) = 1$.

We wish to find sufficient conditions for $c(X, \phi) = 1$. In [5, page 44], T Kerler develops a skein theoretic algorithm to compute the one-variable Alexander polynomial $\Delta_X,\phi$ from a surgery presentation of $X$. As a result, he shows that if $c(X, \phi) \geq 2$ then the Frohman–Nicas TQFT evaluated on the cut cobordism is zero, implying that $\Delta_X,\phi = 0$. Using the fact that $\mathbb{Q}[t^{\pm 1}]$ is a principal ideal domain one can prove that $\Delta_X,\phi = 0$ is equivalent to $\text{rank}_{\mathbb{Z}[t^{\pm 1}]} H_1(X,\phi) \geq 1$.

We give an elementary proof of the equivalent statement of Kerler’s.

**Proposition 2.2** If $c(X, \phi) \geq 2$ then $\text{rank}_{\mathbb{Z}[t^{\pm 1}]} H_1(X,\phi) \geq 1$.

**Proof** Suppose $c(X, \phi) \geq 2$ then there is a surjective map $\psi: \pi_1(X) \to F(n)$ that factors through $\phi$ with $n \geq 2$. Let $\overline{\phi}: F(n) \to \mathbb{Z}$ be the homomorphism such that $\phi = \overline{\phi} \circ \psi$. $\phi$ surjective implies that $\psi|_{\ker \phi}: \ker \phi \to \ker \overline{\phi}$ is surjective. Writing $\mathbb{Z}$ as the multiplicative group generated by $t$, we can consider $\frac{\ker \phi}{[\ker \phi, \ker \phi]}$ and $\frac{\ker \overline{\phi}}{[\ker \overline{\phi}, \ker \overline{\phi}]}$ as modules over $\mathbb{Z}[t^{\pm 1}]$. Here, the $t$ acts by conjugating by an element that maps to $t$ by $\phi$ or $\overline{\phi}$. Moreover, $\psi|_{\ker \phi}: \frac{\ker \overline{\phi}}{[\ker \phi, \ker \phi]} \to \frac{\ker \overline{\phi}}{[\ker \phi, \ker \phi]}$ is surjective hence

$$\text{rank}_{\mathbb{Z}[t^{\pm 1}]} \left( \frac{\ker \phi}{[\ker \phi, \ker \phi]} \right) \geq \text{rank}_{\mathbb{Z}[t^{\pm 1}]} \left( \frac{\ker \overline{\phi}}{[\ker \overline{\phi}, \ker \overline{\phi}]} \right) = n - 1.$$

Since $n \geq 2$, $\text{rank}_{\mathbb{Z}[t^{\pm 1}]} H_1(X,\phi) = \text{rank}_{\mathbb{Z}[t^{\pm 1}]} \left( \frac{\ker \phi}{[\ker \phi, \ker \phi]} \right) \geq 1$. \hfill $\square$

**Corollary 2.3** If $\pi_1(X) \to F/F''$ where $F$ is a free group of rank 2 then there exists a $\phi: \pi_1(X) \to \mathbb{Z}$ such that $\text{rank}_{\mathbb{Z}[t^{\pm 1}]} H_1(X,\phi) \geq 1$. 

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Proof This follows immediately from the proof of Proposition 2.2 after noticing that $F'' \subset [\ker(\bar{\phi}), \ker(\bar{\phi})]$ and $\text{Hom}(F/F'', \mathbb{Z}) \cong \text{Hom}(F, \mathbb{Z})$. 

3 The Examples

We construct closed 3–manifolds all of whose infinite cyclic covers have first homology that is $\mathbb{Z}[t^\pm 1]$–torsion. The 3–manifolds we consider are 0–surgery on an $m$–component link that is obtained from the trivial link by tying a Whitehead link interaction between each two components.

Theorem 3.1 For each $m \geq 1$ there exist closed 3–manifolds $X$ with $\beta_1(X) = m$ such that for any infinite cyclic cover $X_\phi \to X$, $\text{rank}_{\mathbb{Z}[t^\pm 1]} H_1(X_\phi) = 0$.

It follows from Proposition 2.2 that the cut number of the manifolds in Theorem 3.1 is 1. In fact, Corollary 2.3 implies that $\pi_1(X)$ does not map onto $F/F''$ where $F$ is a free group of rank 2. Moreover, the proof of this theorem shows that $\pi_1(X)$ does not even map onto $F/F_4$ where $F_n$ is the $n^{th}$ term of the lower central series of $F$ (see Proposition 3.3).

By a theorem of Ruberman [7], we can assume that the manifolds with cut number 1 are hyperbolic.

Corollary 3.2 For each $m \geq 1$ there exist closed, orientable, hyperbolic 3–manifolds $Y$ with $\beta_1(Y) = m$ such that for any infinite cyclic cover $Y_\phi \to Y$, $\text{rank}_{\mathbb{Z}[t^\pm 1]} H_1(Y_\phi) = 0$.

Proof Let $X$ be one of the 3–manifolds in Theorem 3.1. By [7, Theorem 2.6], there exists a degree one map $f: Y \to X$ where $Y$ is hyperbolic and $f_*$ is an isomorphism on $H_*$. Denote by $G = \pi_1(X)$ and $P = \pi_1(Y)$. It is then well-known that $f$ is surjective on $\pi_1$. It follows from Stallings’s theorem [9, page 170] that the kernel of $f_*$ is $P_\omega \equiv \cap P_n$. Now, suppose $\phi: P \xrightarrow{f_*} G \xrightarrow{\bar{\phi}} \mathbb{Z}$ defines an infinite cyclic cover of $Y$. Then $H_1(Y_\phi) \to H_1(X_\bar{\phi})$ has kernel $P_\omega/\ker \phi \ker \phi$. To show that rank$_{\mathbb{Z}[t^\pm 1]} H_1(Y_\phi) = 0$ it suffices to show that $P_\omega$ vanishes under the map $H_1(Y_\phi) \to H_1(Y_\phi) \otimes_{\mathbb{Z}[t^\pm 1]} \mathbb{Q}[t^\pm 1] \to H_1(Y_\phi) \otimes_{\mathbb{Z}[t^\pm 1]} \mathbb{Q}(t)$ since then rank$_{\mathbb{Z}[t^\pm 1]} H_1(Y_\phi) = \text{rank}_{\mathbb{Z}[t^\pm 1]} H_1(X_\bar{\phi}) = 0$.

Note that $H_1(Y_\phi) \otimes_{\mathbb{Z}[t^\pm 1]} \mathbb{Q}[t^\pm 1] \cong \bigoplus_{i=1}^n \mathbb{Q}[t^\pm 1] \oplus T$ where $T$ is a $\mathbb{Q}[t^\pm 1]$ torsion module. Moreover, $P_n$ is generated by elements of the form $\gamma = \ldots
\[ p_1 [p_2 [p_3, \ldots [p_{n-2}, \alpha]]]] \text{ where } \alpha \in P_2 \subseteq \ker \phi. \text{ Therefore}
\[
\gamma = (\phi(p_1) - 1) \cdots (\phi(p_{n-2}) - 1) [\alpha]
\]
in \( H_1(Y_\phi) \) which implies that \( P_n \subseteq J^{n-2}(H_1(Y_\phi)) \) for \( n \geq 2 \) where \( J \) is the augmentation ideal of \( \mathbb{Z} [t^\pm] \). It follows that any element of \( P_\omega \) considered as an element of \( H_1(Y_\phi) \otimes_{\mathbb{Z}[t^\pm]} \mathbb{Q} [t^\pm] \) is infinitely divisible by \( t - 1 \) and hence is torsion.

**Proof of Theorem 3.1** Let \( L = \sqcup L_i \) be the oriented trivial link with \( m \) components in \( S^3 \) and \( \sqcup D_i \) be oriented disjoint disks with \( \partial D_i = L_i \). The fundamental group of \( S^3 - L \) is freely generated by \( \{x_i\} \) where \( x_i \) is a meridian curve of \( L_i \) which intersects \( D_i \) exactly once and \( D_i \cdot x_i = 1 \). For all \( i,j \) with \( 1 \leq i < j \leq m \) let \( \alpha_{ij} : I \rightarrow S^3 \) be oriented disjointly embedded arcs such that \( \alpha_{ij}(0) \in L_i \) and \( \alpha_{ij}(1) \in L_j \) and \( \alpha_{ij}(I) \) does not intersect \( \sqcup D_i \). For each arc \( \alpha_{ij} \), let \( \gamma_{ij} \) be the curve embedded in a small neighborhood of \( \alpha_{ij} \) representing the class \([x_i, x_j]\) as in Figure 1. Let \( X \) be the 3–manifold obtained performing 0–framed Dehn surgery on \( L \) and \(-1\)–framed Dehn surgery on each \( \gamma = \sqcup \gamma_{ij} \). See Figure 2 for an example of \( X \) when \( m = 5 \).

Denote by \( X_0 \), the manifold obtained by performing 0–framed Dehn surgery on \( L \). Let \( W \) be the 4–manifold obtained by adding a 2–handle to \( X_0 \times I \) along each curve \( \gamma_{ij} \times \{1\} \) with framing coefficient -1. The boundary of \( W \) is \( \partial W = X_0 \sqcup -X \). We note that \( \pi_1(W) = \langle x_1, \ldots, x_m \mid [x_i, x_j] = 1 \text{ for all } 1 \leq i < j \leq m \rangle \cong \mathbb{Z}^m. \)
Figure 2: The surgered manifold $X$ when $m = 5$

Let $\{x_{ik}, \mu_{ijl}\}$ be the generators of $\pi_1(S^3 - (L \sqcup \gamma))$ that are obtained from a Wirtinger presentation where $x_{ik}$ are meridians of the $i^{th}$ component of $L$ and $\mu_{ijl}$ are meridians of the $(i,j)^{th}$ component of $\gamma$. Note that $\{x_{ik}, \mu_{ijl}\}$ generate $G \equiv \pi_1(X)$. For each $1 \leq i \leq m$ let $\overline{x}_i = x_i$ and $\overline{\mu}_{ij}$ be the specific $\mu_{ijl}$ that is denoted in Figure 3. We will use the convention that

$$[a, b] = aba^{-1}b^{-1}$$

and

$$a^b = bab^{-1}.$$  

We can choose a projection of the trivial link so that the arcs $\alpha_{ij}$ do not pass under a component of $L$. Since $\overline{\mu}_{ij}$ is equal to a longitude of the curve $\gamma_{ij}$ in $X$, we have $\overline{\mu}_{ij} = [x_{in_{ij}}, \lambda x_{jn_{ji}} \lambda^{-1}]$ for some $n_{ij}$ and $n_{ji}$ and $\lambda$ where $\lambda$ is a product of conjugates of meridian curves $\mu_{lk}$ and $\mu_{lk}^{-1}$. Moreover, we can find
a projection of $L \sqcup \gamma$ so that the individual components of $L$ do not pass under or over one another. Hence $x_{ij} = \omega x_i \omega^{-1}$ where $\omega$ is a product of conjugates of the meridian curves $\mu_{lk}$ and $\mu_{lk}^{-1}$. As a result, we have

$$\overline{\mu}_{ij} = [x_{in_{ij}}, \lambda x_{jn_{ij}} \lambda^{-1}]$$

$$= [\omega_1 x_i \omega_1^{-1}, \lambda \omega_2 x_j \omega_2^{-1} \lambda^{-1}]$$

$$= [\mu_i, \omega^{-1}_1 \lambda \omega_2 x_j \omega_2^{-1} \lambda^{-1} \omega_1^{-1}]$$

for some $\lambda$, $\omega_1$, and $\omega_2$.

We note that $\overline{\mu}_{ij} = [x_{in_{ij}}, \lambda x_{jn_{ij}} \lambda^{-1}]$ hence $\overline{\mu}_{ij} \in G'$ for all $i < j$. Setting $v = \omega_1^{-1} \lambda \omega_2$ and using the equality

$$[a, bc] = [a, b] [a, c]^b$$

we see that

$$\overline{\mu}_{ij} = [\mu_i, v \mu_j v^{-1}]^\omega$$

$$= [\mu_i, v \mu_j v^{-1}] \mod G''$$

$$= [\mu_i, [v, \mu_j] \mu_j]$$

$$= [\mu_i, [v, \mu_j]]\,[\mu_i, \mu_j][v, \mu_j]$$

$$= [\mu_i, [v, \mu_j]]\,[\mu_i, \mu_j] \mod G''$$

since $\omega_1, v \in G'$.

Consider the dual relative handlebody decomposition $(W, X)$. $W$ can be obtained from $X$ by adding a 0–framed 2–handle to $X \times I$ along each of the
meridian curves $\overline{\mu}_{ij} \times \{1\}$. (3) implies that $\overline{\mu}_{ij}$ is trivial in $H_1(X)$ hence the inclusion map $j: X \to W$ induces an isomorphism $j_*: H_1(X) \cong H_1(W)$. Therefore if $\phi: G \to \Lambda$ where $\Lambda$ is abelian then there exists a $\psi: \pi_1(W) \to \Lambda$ such that $\psi \circ j_* = \phi$.

Suppose $\phi: G \to \langle t \rangle \cong \mathbb{Z}$ and $\psi: \pi_1(W) \to \langle t \rangle$ is an extension of $\phi$ to $\pi_1(W)$. Let $X_\phi$ and $W_\psi$ be the infinite cyclic covers of $W$ and $X$ corresponding to $\psi$ and $\phi$ respectively. Consider the long exact sequence of pairs,

$$
\to H_2(W_\psi, X_\phi) \xrightarrow{\partial} H_1(X_\phi) \to H_1(W_\psi) \to
$$

(6)

Since $\pi_1(W) \cong \mathbb{Z}^m$, $H_1(W_\psi) \cong \mathbb{Z}^{m-1}$ where $t$ acts trivially so that $H_1(W_\psi)$ has rank 0 as a $\mathbb{Z}^m$–module. $H_2(W_\psi, X_\phi) \cong \left(\mathbb{Z}^{t^{\pm 1}}\right)^{\binom{m}{3}}$ generated by the core of each 2–handle (extended by $\overline{\mu}_{ij} \times I$) attached to $X$. Therefore, $\text{Im}(\partial)$ is generated by a lift of $\overline{\mu}_{ij}$ in $H_1(X_\phi)$ for all $1 \leq i < j \leq m$. To show that $H_1(X_\phi)$ has rank 0 it suffices to show that each of the $\overline{\mu}_{ij}$ are $\mathbb{Z}^{t^{\pm 1}}$–torsion in $H_1(X_\phi)$.

Let $F = \langle \overline{x}_1, \ldots, \overline{x}_m \rangle$ be the free group of rank $m$ and $f: F \to G$ be defined by $f(\overline{x}_i) = \overline{x}_i$. We have the following $\binom{m}{3}$ Jacobi relations in $F/F''$ [4, Proposition 7.3.6]. For all $1 \leq i < j < k \leq m$,

$$
[\overline{x}_i, [\overline{x}_j, \overline{x}_k]] [\overline{x}_j, [\overline{x}_k, \overline{x}_i]] [\overline{x}_k, [\overline{x}_i, \overline{x}_j]] = 1 \mod F''.
$$

Using $f$, we see that these relations hold in $G/G''$ as well. From (5), we can write

$$
[\overline{x}_i, \overline{x}_j] = [[v_{ij}, \overline{x}_j], \overline{x}_i] \overline{\mu}_{ij} \mod G''.
$$

Hence for each $1 \leq i < j < k \leq m$ we have the Jacobi relation $J(i, j, k)$ in $G/G''$,

$$
1 = [\overline{x}_i, [\overline{x}_j, \overline{x}_k]] [\overline{x}_j, [\overline{x}_i, \overline{x}_k]]^{-1} [\overline{x}_k, [\overline{x}_i, \overline{x}_j]] \mod G''
$$

$$
= [\overline{x}_i, [[v_{jk}, \overline{x}_k], \overline{x}_j] \overline{\mu}_{jk}] [\overline{x}_j, \overline{\mu}_{jk}^{-1} [\overline{x}_i, [v_{ik}, \overline{x}_k]]]
$$

$$
[\overline{x}_k, [[v_{ij}, \overline{x}_j], \overline{x}_i] \overline{\mu}_{ij}] \mod G''
$$

$$
= [\overline{x}_i, [[v_{jk}, \overline{x}_k], \overline{x}_j]] [\overline{x}_j, \overline{\mu}_{jk}^{-1} [\overline{x}_i, [v_{ik}, \overline{x}_k]]]
$$

$$
[\overline{x}_k, [[v_{ij}, \overline{x}_j], \overline{x}_i]] \mod G''
$$

Moreover, for each component of the trivial link $L_i$ the longitude, $l_i$, of $L_i$ is trivial in $G$ and is a product of commutators of $\overline{\mu}_{ij}$ with a conjugate of $\overline{x}_j$. We
can write each of the longitudes (see Figure 4) as

\[ l_i = \prod_{j<i} \alpha_j \lambda_j^{-1} \mu_{ji}^{-1} \lambda_j \cdot \prod_{k>i} \mu_{ik} \beta_k \mod G'' \]

\[ = \prod_{j<i} \left( \lambda_j^{-1} \gamma_j x_{jnji}^{-1} \mu_{ji} x_{jnji} \lambda_j \right) \lambda_j^{-1} \mu_{ji}^{-1} \lambda_j \cdot \prod_{k>i} \mu_{ik} \left( \lambda_k x_{knki}^{-1} \lambda_k^{-1} \mu_{ik}^{-1} \lambda_k x_{knki} \lambda_k^{-1} \right) \]

\[ = \prod_{j<i} \left[ x_{jnji}^{-1}, \mu_{ji} \right] \lambda_j^{-1} \cdot \prod_{k>i} \left[ \mu_{ik}, \lambda_k x_{knki} \lambda_k^{-1} \right] \mod G''. \tag{8} \]

\[ \text{Figure 4} \]
It follows that

\[ \prod_{j<i} \left[ \overline{\varphi}_{ji}^{-1}, \overline{\mu}_{ji} \right] \cdot \prod_{k>i} \left[ \overline{\mu}_{ik}, \overline{\varphi}_{ik}^{-1} \right] = 1 \mod G''. \]

Since \( G'' \subset [\ker \phi, \ker \phi] \), the relations in (7) and (8) hold in \( H_1(X_\phi) \)
(\( = \ker \phi/ [\ker \phi, \ker \phi] \)) as well. Suppose \( \phi: G \to \mathbb{Z} \) is defined by sending \( \overline{\varphi}_i \mapsto t^{n_i} \). Since \( \phi \) is surjective, \( n_N \neq 0 \) for some \( N \). We consider a subset of \( \binom{m}{2} \)
relations in \( H_1(X_\phi) \) that we index by \( (i, j) \) for \( 1 \leq i < j \leq m \). When \( i = N \) or \( j = N \) we consider the \( m - 1 \) relations

(i) \( R_{iN} = l_i \) and (ii) \( R_{Nj} = l_j^{-1} \).

Rewriting \( l_i \) as an element of the \( \mathbb{Z}[t^\pm 1] \)-module \( H_1(X_\phi) \) generated by \( \{ \overline{\mu}_{ij} | 1 \leq i < j \leq m \} \) from (8) we have

\[
R_{iN} = \sum_{j<i} (t^{n_j} - 1) \overline{\mu}_{ji} + \sum_{k>i} (1 - t^{-n_k}) \overline{\mu}_{ik}
= \sum_{j<i} t^{n_j} (1 - t^{n_j}) \overline{\mu}_{ji} + \sum_{k>i} t^{-n_k} (t^{n_k} - 1) \overline{\mu}_{ik}
= \sum_{j<i} \left[ (1 - t^{n_j}) + (t^{n_j} - 1)(1 - t^{n_j}) \right] \overline{\mu}_{ji} + \sum_{k>i} \left[ (t^{n_k} - 1) + (t^{-n_k} - 1)(1 - t^{n_k}) \right] \overline{\mu}_{ik}.
\]

Similarly, we have

\[
R_{Nj} = \sum_{i<j} \left[ (t^{n_i} - 1) + (t^{-n_i} - 1)(1 - t^{n_i}) \right] \overline{\mu}_{ij} + \sum_{k>j} \left[ (1 - t^{n_k}) + (t^{-n_k} - 1)(1 - t^{n_k}) \right] \overline{\mu}_{jk}.
\]

For the other \( \binom{m-1}{3} \) relations, we use the Jacobi relations from (7). Define \( R_{ij} \)
to be

\[
R_{ij} = \begin{cases} 
J(N, i, j) & \text{for } N < i < j \\
J(i, N, j)^{-1} & \text{for } i < N < j \\
J(i, j, N) & \text{for } i < j < N
\end{cases}
\]

We can write these relations as

\[
R_{ij} = \begin{cases} 
(t^{n_j} - 1) \overline{\mu}_{Ni} + (1 - t^{n_i}) \overline{\mu}_{Nj} + (t^{n_N} - 1) \overline{\mu}_{ij} + (t^{n_i} - 1)(t^{n_j} - 1)(\overline{v}_{ij} + \overline{v}_{Nj} - \overline{v}_{Nj}) & \text{for } N < i < j \\
(t^{n_N} - 1) + (t^{n_i} - 1)(t^{n_j} - 1)(\overline{v}_{ij} + \overline{v}_{Nj} - \overline{v}_{Nj}) & \text{for } i < N < j \\
(t^{n_N} - 1) \overline{\mu}_{ij} + (1 - t^{n_j}) \overline{\mu}_{Nj} + (t^{n_i} - 1)(t^{n_N} - 1) \overline{\mu}_{IN} + (t^{n_N} - 1)(t^{n_i} - 1)(\overline{v}_{ij} + \overline{v}_{Nj} - \overline{v}_{Nj}) & \text{for } i < j < N
\end{cases}
\]
where $\tilde{v}_{ij}$ is a lift of $v_{ij}$.

For $1 \leq i < j \leq m$ order the pairs $ij$ by the dictionary ordering. That is, $ij < lk$ provided either $i < l$ or $j < k$ when $i = l$. The relations above give us an $\binom{m}{2} \times \binom{m}{2}$ matrix $M$ with coefficients in $\mathbb{Z}[t^{\pm 1}]$. The $(ij, kl)^{th}$ component of $M$ is the coefficient of $\overline{\mu}_{kl}$ in $R_{ij}$. We claim for now that

$$M = (t^{n_N} - 1) I + (t - 1) S + (t - 1)^2 E$$

(12)

for some “error” matrix $E$ where $I$ is the identity matrix and $S$ is a skew-symmetric matrix. For an example, when $m = 4$ and $N = 1$, $M$ is the matrix

$$
\begin{bmatrix}
  t^{n_1} - 1 & 0 & 0 & 1 - t^{n_3} & 1 - t^{n_4} & 0 \\
  0 & t^{n_1} - 1 & 0 & t^{n_2} - 1 & 0 & 1 - t^{n_4} \\
  0 & 0 & t^{n_1} - 1 & 0 & t^{n_2} - 1 & t^{n_3} - 1 \\
  t^{n_3} - 1 & 1 - t^{n_2} & 0 & t^{n_1} - 1 & 0 & 0 \\
  t^{n_4} - 1 & 0 & 1 - t^{n_2} & 0 & t^{n_1} - 1 & 0 \\
  0 & t^{n_4} - 1 & 1 - t^{n_2} & 0 & 0 & t^{n_1} - 1
\end{bmatrix} + (t - 1)^2 E.
$$

The proof of (12) is left until the end.

We will show that $M$ is non-singular as a matrix over the quotient field $\mathbb{Q}(t)$. Consider the matrix $A = \frac{1}{t - 1} M$. We note that $A$ is a matrix with entries in $\mathbb{Z}[t^{\pm 1}]$ and $A(1)$ evaluated at $t = 1$ is

$$A(1) = NI + S(1).$$

To show that $M$ is non-singular, it suffices to show that $A(1)$ is non-singular.

Consider the quadratic form $q: \mathbb{Q}(\binom{m}{2}) \rightarrow \mathbb{Q}(\binom{m}{2})$ defined by $q(z) \equiv z^T A(1) z$ where $z^T$ is the transpose of $z$. Since $A(1) = NI + S(1)$ where $S(1)$ is skew-symmetric we have,

$$q(z) = N \sum z_i^2.$$

Moreover, $N \neq 0$ so $q(z) = 0$ if and only if $z = 0$. Let $z$ be a vector satisfying $A(1) z = 0$. We have $q(z) = z^T A(1) z = z^T 0 = 0$ which implies that $z = 0$. Therefore $M$ is a non-singular matrix. This implies that each element $\overline{\mu}_{ij}$ is $\mathbb{Z}[t^{\pm 1}]$–torsion which will complete the the proof once we have established the above claim.

We ignore entries in $M$ that lie in $J^2$ where $J$ is the augmentation ideal of $\mathbb{Z}[t^{\pm 1}]$ since they only contribute to the error matrix $E$. Using (9), (10), and (11) above we can explicitly write the entries in $M \pmod{J^2}$. Let $m_{ij,lk}$ denote the $(ij, lk)$ entry of $M \pmod{J^2}$.

\textit{Geometry & Topology}, Volume 6 (2002)
Case 1 ($j = N$): From (9) we have
\[ m_{iN,li} = 1 - t^{ni}, \quad m_{iN,ik} = t^{nk} - 1, \]
and $m_{iN,lk} = 0$ when neither $l$ nor $k$ is equal to $N$.

Case 2 ($i = N$): From (10) we have
\[ m_{Nj,lj} = t^{nl} - 1, \quad m_{Nj,jk} = 1 - t^{nk}, \]
and $m_{Nj,lk} = 0$ when neither $l$ nor $k$ is equal to $N$.

Case 3 ($N < i < j$): From (11) we have
\[ m_{ij,Ni} = t^{nj} - 1, \quad m_{ij,Nj} = 1 - t^{ni}, \quad m_{ij,ij} = t^{nN} - 1, \]
and $m_{ij,lk} = 0$ otherwise.

Case 4 ($i < N < j$): From (11) we have
\[ m_{ij,iN} = 1 - t^{nj}, \quad m_{ij,ij} = t^{nN} - 1, \quad m_{ij,Nj} = 1 - t^{ni}, \]
and $m_{ij,lk} = 0$ otherwise.

Case 5 ($i < j < N$): From (11) we have
\[ m_{ij,ij} = t^{nN} - 1, \quad m_{ij,iN} = 1 - t^{nj}, \quad m_{ij,jN} = t^{ni} - 1, \]
and $m_{ij,lk} = 0$ otherwise.

We first note that in each of the cases, the diagonal entries $m_{ij,ij}$ are all $t^{nN} - 1$. Next, we will show that the off diagonal entries have the property that $m_{ij,lk} = -m_{lk,ij}$ for $ij < lk$. This will complete the proof of the claim since we see that each entry is divisible by $t - 1$.

We verify the skew symmetry in Cases 1 and 3. The other cases are similar and we leave the verifications to the reader.

Case 1 ($j = N$):
\[ m_{iN,li} = 1 - t^{ni} = -m_{li,iN} \] (case 5)
and
\[ m_{iN,ik} = t^{nk} - 1 = -m_{ik,iN} \] (case 4).

Case 3 ($N < i < j$):
\[ m_{ij,Ni} = t^{nj} - 1 = -m_{Ni,ij} \] (case 2)
and
\[ m_{ij,Nj} = 1 - t^{ni} = -m_{Nj,ij} \] (case 2).
Proposition 3.3  Let $X$ be as in Theorem 3.1, $G = \pi_1 (X)$ and $F$ be the free group on 2 generators. There is no epimorphism from $G$ onto $F/F_4$.

Proof  Let $F = \langle x, y \rangle$ be the free group and $\phi: F/F_4 \to \langle t \rangle$ be defined by $x \mapsto t$ and $y \mapsto 1$. Suppose that there exists a surjective map $\eta: G \to F/F_4$. Let $N = \ker \phi$ and $H = \ker (\eta \circ \phi)$. Since $\eta$ is surjective we get an epimorphism of $\mathbb{Z} [t^\pm]$–modules $\tilde{\eta}: H/H' \to N/N'$. From (6) we get the short exact sequence

$$0 \to \text{Im}\partial_{t^x} \to H_1 (X_{t^x}) \to H_1 (W_\psi) \to 0.$$  

Let $J$ be the augmentation ideal of $\mathbb{Z} [t^\pm]$. We compute $N/N' \cong \mathbb{Z} [t^\pm] / J^3$ so that $\tilde{\eta}: H_1 (X_{t^x}) \to \mathbb{Z} [t^\pm] / J^3$. Let $\sigma \in H_1 (X_{t^x})$ such that $\tilde{\eta} (\sigma) = 1$. Since every element in $H_1 (W_\psi) \cong \bigoplus_{i=1}^{m-1} \mathbb{Z} [t^\pm] / J^3$ is $(t - 1)$–torsion, $(t - 1) \sigma \in \text{Im}\partial_{t^x}$ hence $t - 1 \in \text{Im}(\tilde{\eta} \circ i)$. Recall that in the proof of the Theorem 3.1, we showed that there exists a surjective $\mathbb{Z} [t^\pm]$–module homomorphism $\rho: P \to \text{Im}\partial_{t^x}$ where $P$ is finitely presented as

$$0 \to \mathbb{Z} [t^\pm] (\frac{t}{2}) (\frac{t - 1}{2}) \mathbb{Z} [t^\pm] (\frac{m}{2}) \to P \to 0.$$  

Let $g: P \to \mathbb{Z} [t^\pm] / J^3$ defined by $g \equiv \tilde{\eta} \circ i \circ \rho$. Since $\rho$ is surjective, $t - 1 \in \text{Im} g$. After tensoring with $\mathbb{Q} [t^\pm]$, we get a map $g: P \otimes_{\mathbb{Z}[t^\pm]} \mathbb{Q} [t^\pm] \to \mathbb{Q} [t^\pm] / J^3$. It is easy to see that either $g$ is surjective or the image of $g$ is the submodule generated by $t - 1$. Note that the submodule generated by $t - 1$ is isomorphic $\mathbb{Q} [t^\pm] / J^2$. Hence, in either case, we get a surjective map $h: P \otimes_{\mathbb{Z}[t^\pm]} \mathbb{Q} [t^\pm] \to \mathbb{Q} [t^\pm] / J^2$.

Consider the $\mathbb{Q} [t^\pm]$–module $P'$ presented by $A$. Let $h': \mathbb{Q} [t^\pm] (\frac{m}{2}) \to \mathbb{Q} [t^\pm] / J^2$ be defined by $h' = (t - 1) h \circ \pi$. Since $h' (A (\sigma)) = (t - 1) h (\pi (A (\sigma))) = h (\pi ((t - 1) A (\sigma))) = h (0) = 0$, this defines a map $h': P' \to \mathbb{Q} [t^\pm] / J^2$ whose image is the submodule generated by $t - 1$. It follows that $P'$ maps onto $\mathbb{Q} [t^\pm] / J$. Setting $t = 1$, the vector space over $\mathbb{Q}$ presented by $A (1)$ maps onto $\mathbb{Q}$. Therefore $\text{det}(A(1)) = 0$. However, it was previously shown that $A (1)$ was non-singular which is a contradiction. 

Corollary 3.4  For any closed, orientable 3–manifold $Y$ with $P/P_4 \cong G/G_4$ where $P = \pi_1 (Y)$ and $G = \pi_1 (X)$ is the fundamental group of the examples in Theorem 3.1, $c(Y) = 1$.

Using Proposition 3.3, it is much easier to show that there exist hyperbolic 3–manifolds with cut number 1.
Corollary 3.5 For each $m \geq 1$ there exist closed, orientable, hyperbolic 3–manifolds $Y$ with $\beta_1(Y) = m$ such that $\pi_1(Y)$ cannot map onto $F/F_4$ where $F$ is the free group on 2 generators.

Proof Let $X$ be one of the 3–manifolds in Theorem 3.1. By [7, Theorem 2.6], there exists a degree one map $f: Y \to X$ where $Y$ is hyperbolic and $f_*$ is an isomorphism on $H_*$. Denote by $G = \pi_1(X)$ and $P = \pi_1(Y)$. It follows from Stallings’s theorem [9] that $f$ induces an isomorphism $f_*: P/P_n \to G/G_n$. In particular this is true for $n = 4$ which completes the proof.

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