DISTANCES BETWEEN MATRIX ALGEBRAS
THAT CONVERGE TO COADJOINT ORBITS

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Abstract. For any sequence of matrix algebras that converge
to a coadjoint orbit we give explicit formulas that show that the
distances between the matrix algebras (viewed as quantum metric
spaces) converges to 0. In the process we develop a general point
of view that is likely to be useful in other related settings.

Introduction

In earlier papers [6, 7, 9] I provided ways to give a precise meaning
to statements in the literature of high-energy physics and string theory
of the kind “Matrix algebras converge to the sphere”. I did this by
equipping the matrix algebras with suitable “Lipschitz seminorms” that
make the matrix algebras into “compact quantum metric spaces”, and
then by defining convergence by means of a suitable “quantum Gromov-
Hausdorff distance” between quantum metric spaces. By now a number
of variations on this approach have been studied [1, 2, 3, 4, 5, 10].

When I then began to examine what consequences the convergence
of quantum metric spaces had for the convergence of “vector bundles”
(i.e. projective modules) over them [8], I found that it is very important
that the Lipschitz seminorms satisfy a suitable Leibniz property. In
[9] I showed that a very convenient source for seminorms that satisfy
this Leibniz property consisted of normed bimodules, and in [9] I also
constructed explicit normed bimodules that worked well for matrix
algebras converging to coadjoint orbits.

However, for our approach to work well, it should be the case that
for a convergent sequence of matrix algebras the quantum Gromov-
Hausdorff distances between the matrix algebras go to 0; but when I

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required that all of the seminorms satisfy the Leibniz property I did not see at first how to show this convergence directly. The purpose of the present paper is to give explicit normed bimodules and corresponding Leibniz Lipschitz seminorms that demonstrate this convergence to 0. In the process we develop a general point of view that is likely to be useful in other related situations. This point of view is motivated by the “nuclear distance” introduced and studied by Hanfeng Li [2, 4, 5], in which all of the bimodules are required to be \( C^* \)-algebras. I have so far not seen how to apply Hanfeng Li’s approach directly to obtain explicit normed bimodules for the matrix-algebra case. But by trying to arrange that all of the normed bimodules that I used were \( C^* \)-algebras I was led to see the path to the explicit bimodules that I sought.

The first section of this paper recalls the setting for matrix algebras converging to coadjoint orbits, reformulates the bimodules from [9] so that they are \( C^* \)-algebras, and then uses these reformulated bimodules to construct candidates for \( C^* \)-bimodules between matrix algebras whose Leibniz Lipschitz seminorms might show that the distances go to 0. In Section 2 we place matters in a general framework, and obtain a basic theorem in this general framework. In Section 3 we prove that the candidate bimodules and corresponding Lipschitz seminorms of Section 1 do indeed show that the distances between the converging matrix algebras go to 0. An important step in the proof comes from the general theorem in Section 2. The full statement of the main theorem is given at the end of Section 3.

1. The bimodules

We recall the setting from [7, 9]. We let \( G \) be a compact connected semisimple Lie group, and we let \( g \) denotes the complexification of the Lie algebra of \( G \). We choose a maximal torus in \( G \), with corresponding Cartan subalgebra of \( g \), its set of roots, and a choice of positive roots. We fix a specific irreducible unitary representation, \( (U, \mathcal{H}) \), of \( G \), and we choose a highest-weight vector, \( \xi \), for \( (U, \mathcal{H}) \) with \( \| \xi \| = 1 \). For any \( n \in \mathbb{Z}_{\geq 1} \) we set \( \xi^n = \xi^\otimes n \) in \( \mathcal{H}^\otimes n \), and we let \( (U^n, \mathcal{H}^n) \) be the restriction of \( U^\otimes n \) to the \( U^\otimes n \)-invariant subspace, \( \mathcal{H}^n \), of \( \mathcal{H}^\otimes n \) that is generated by \( \xi^n \). Then \( (U^n, \mathcal{H}^n) \) is an irreducible representation of \( G \) with highest-weight vector \( \xi^n \), and its highest weight is just \( n \) times the highest weight of \( (U, \mathcal{H}) \). We denote the dimension of \( \mathcal{H}^n \) by \( d_n \).

We let \( B^n = L(\mathcal{H}^n) \). The action of \( G \) on \( B^n \) by conjugation by \( U^n \) will be denoted simply by \( \alpha \). We assume that a continuous length function, \( \ell \), has been chosen for \( G \), and we denote the corresponding
$C^*$-metric on $B^n$ by $L^B_n$. It is defined by

$$L^B_n(T) = \sup\{\|\alpha_x(T) - T\|/\ell(x) : x \notin e_G\}$$

for $T \in B^n$. (The term “$C^*$-metric” is defined in definition 4.1 of [9].) We let $P^n$ denote the rank-one projection along $\xi^n$. Then the $\alpha$-stability subgroup, $H$, for $P = P^1$ will also be the $\alpha$-stability subgroup for each $P^n$. We let $A = C(G/H)$, and we let $L_A$ be the $C^*$-metric on $A$ for $\ell$ and the left-translation action of $G$ on $G/H$, defined much as is $L^B_n$.

Roughly speaking, our goal is to obtain estimates on the distance between $(B^m, L^B_m)$ and $(B^n, L^B_n)$ that show that the distance goes to 0 as $m$ and $n$ go to 0. We want to do this in the setting of [9], where we insist that the Lipschitz seminorms involved satisfy a strong Leibniz property. We require this because of its importance for treating vector bundles (and projective modules), as shown in [8].

But in contrast to [9], our presentation here is influenced by Hanfeng Li’s definition of the “nuclear distance” between quantum metric spaces, although I have not seen how to use his nuclear distance directly. The effect of this influence is that we try to arrange that all of the bimodules that we consider are actually $C^*$-algebras.

To motivate the construction of our bimodules, we first reformulate the corresponding constructions from [9] in terms of $C^*$-algebras. For any given $n$ we form the $C^*$-algebra $A \otimes B^n = C(G/H, B^n)$. There are canonical injections of $A$ and $B^n$ into $A \otimes B^n$, and by means of these we view $A \otimes B^n$ as an $A$-$B^n$-bimodule. Let $\omega_n \in C(G/H, B^n)$ be defined by

$$\omega_n(x) = \alpha_x(P^n).$$

We use the distinguished element $\omega_n$ and the bimodule structure to define a seminorm, $N_n$, on $A \oplus B^n$ by

$$N_n(f, T) = \|f \omega_n - \omega_n T\|.$$ 

This seminorm is easily seen to be the same as the seminorm $N_\sigma$ described by other means in proposition 7.2 of [9]. It is also easy to see that $N_n$ satisfies the strong Leibniz property defined in definition 1.1 of [9], for the reasons discussed in example 2.3 of [9] if $A \otimes B^n$ is viewed as an $(A \otimes B^n)$-bimodule in the evident way.

For a suitable choice of the constant $\gamma$, as discussed in propositions 8.1 and 8.2 of [9], $\gamma^{-1}N_n$ is a bridge, as defined in definition 5.1 of [6]. This implies that the $*$-seminorm $L_n$ on $A \oplus B^n$ defined by

$$L_n(f, T) = L_A(f) \vee L^B_n(T) \vee \gamma^{-1}N_n(f, T) \vee N_n(\bar{f}, T^*)$$

where

$$L_A(f) = \sup\{\alpha_x(f) - f\|/\ell(x) : x \notin e_A\}$$

and

$$L^B_n(T) = \sup\{\|\alpha_x(T) - T\|/\ell(x) : x \notin e_G\}$$

for $f \in A$ and $T \in B^n$. (The term “$C^*$-seminorm” is defined in definition 4.1 of [9].)
is a $C^*$-metric on $A \oplus B^n$ (where $\vee$ means “maximum of”) that has the further property that its quotients on $A$ and $B^n$ agree with $L^A_n$ and $L^B_n$ on self-adjoint elements. (See notation 5.5 and definition 6.1 of [9].) This quotient condition on seminorms is exactly what we required in [6, 7, 9] in order to define distances between $C^*$-algebras such as $A$ and $B^n$. Specifically, for our situation, let $S(A)$ denote the state space of $A$, and similarly for $B^n$ and $A \oplus B^n$. Then $S(A)$ and $S(B^n)$ are naturally viewed as subsets of $S(A \oplus B^n)$. Now $L_n$ defines a metric, $\rho_{L_n}$, on $S(A \oplus B^n)$ by

$$\rho_{L_n}(\mu, \nu) = \sup\{|\mu(f, T) - \nu(f, T)| : L_n(f, T) \leq 1\}.$$  

(By definition this supremum should be taken over just self-adjoint $f$ and $T$, but by the comments made just before definition 2.1 of [6] it can equivalently be taken over all $f$ and $T$ because $L_n$ is a $*$-seminorm. This fact is also used later for other $*$-seminorms.) The corresponding ordinary Hausdorff distance

$$\text{dist}_{\rho_{L_n}}(S(A), S(B^n))$$

gives, by definition, an upper bound for $\text{dist}_q(A, B^n)$ as defined in definition 4.2 of [6] when we don’t require the strong Leibniz condition, and for $\text{prox}(A, B^n)$ as defined in definition 5.6 of [9] when we do require the strong Leibniz condition. It is shown in theorem 4.3 of [6] that $\text{dist}_q$ satisfies the triangle inequality. But $\text{prox}$ probably does not satisfy the triangle inequality, basically because the quotient of a seminorm that satisfies the Leibniz condition need not satisfy the Leibniz condition. We always have $\text{dist}_q(A, B) \leq \text{prox}(A, B)$, so if we can show that $\text{prox}(A, B)$ is “small” than it follows that $\text{dist}_q(A, B)$ is “small” too.

Since for our specific situation $\text{prox}(A, B^n)$ converges to 0 as $n$ goes to $\infty$, as seen in theorem 9.1 of [9], (and similarly for its matricial version, $\text{prox}_s$, by theorem 14.1 of [9]), it is natural to expect that $\text{prox}(B^m, B^n)$ converges to 0 as $m$ and $n$ go to $\infty$. But because we can not invoke the triangle inequality, we need to give a direct proof of this fact. In the process of doing this we will construct a specific seminorm that gives quantitative estimates.

Towards our goal we seek to construct a suitable $B^m$-$B^n$-bimodule. We can, of course, view the $C^*$-algebra $A \otimes B^m$ as being the $B^m$-$A$-bimodule $B^m \otimes A$, and then it is natural to form an “amalgamation” over $A$ of these two $C^*$-algebras, to obtain the $B^m$-$B^n$-bimodule

$$(B^m \otimes A) \otimes_A (A \otimes B^n) = B^m \otimes A \otimes B^n,$$
which we can view as $C(G/H, B^m \otimes B^n)$. Notice that this is again a C*-algebra, and that we have natural injections of $B^m$ and $B^n$ into it. Inside this bimodule we choose a distinguished element, namely $\omega_{mn} = \omega_m \otimes \omega_n$, viewed as defined by

$$\omega_{mn}(x) = \alpha_x(P^m) \otimes \alpha_x(P^n) = \alpha_x(P^m \otimes P^n).$$

In terms of $\omega_{mn}$ we define a seminorm, $N_{mn}$, on $B^m \oplus B^n$ by

$$N_{mn}(S, T) = \|S\omega_{mn} - \omega_{mn}T\|,$$

where the norm is that of the C*-algebra $C(G/H, B^m \otimes B^n)$. We can now hope to find constants $\gamma$ such that $\gamma^{-1}N_{mn}$ is a bridge between $B^m$ and $B^n$. In the next section we describe a more general setting within which to choose such bridge constants.

2. THE BRIDGE CONSTANTS

In this section we consider the following more general setting. We are given three compact C*-metric spaces, $(A, L_A)$, $(B, L_B)$ and $(C, L_C)$. We are also given unital C*-algebras $D$ and $E$ together with injective unital homomorphisms of $A$ and $B$ into $D$, and of $B$ and $C$ into $E$. (Actually, we do not need the unital homomorphisms to be injective, but then we should provide notation for them, and that would clutter our calculations.) Thus we can consider $D$ to be an $A$-$B$-bimodule and $E$ to be a $B$-$C$-bimodule. We assume further that we are given distinguished elements $d_0$ and $e_0$ of $D$ and $E$ respectively. For convenience we assume that $\|d_0\| = 1 = \|e_0\|$. We then define seminorms $N_D$ and $N_E$ on $A \oplus B$ and $B \oplus C$ by

$$N_D(a, b) = \|ad_0 - d_0b\|_D$$

and similarly for $N_E$. We assume that there are constants $\gamma_D$ and $\gamma_E$ such that $\gamma_D^{-1}N_D$ and $\gamma_E^{-1}N_E$ are bridges for $(L_A, L_B)$ and $(L_B, L_C)$ respectively. This means that when we form the *-seminorm

$$L_{AB}(a, b) = L_A(a) \vee L_B(b) \vee \gamma_D^{-1}(N_D(a, b) \vee N_D(a^*, b^*)),$$

its quotients on $A$ and $B$ agree with $L_A$ and $L_B$ on self-adjoint elements, and similarly for $L_B$. Note that $L_{AB}$ and $L_{BC}$ are C*-metrics by theorem 6.2 of [9].

Motivated by Hanfeng Li’s treatment of his nuclear distance [5], we consider any amalgamation, $F$, of $D$ and $E$ over $B$. This means that there are unital injections of $D$ and $E$ into $F$ whose compositions with the injections of $B$ into $D$ and $E$ coincide. We denote the images of $d_0$ and $e_0$ in $F$ again by $d_0$ and $e_0$, and we set $f_0 = d_0e_0$. Unfortunately in this generality it could happen that $f_0 = 0$. (In Hanfeng Li’s
Let notation be as above, and assume that \( f_0 \neq 0 \). View \( F \) as an \( A \)-\( C \)-bimodule in the evident way, and define a seminorm, \( N_F \), on \( A \oplus C \) by

\[
N_F(a, c) = \|af_0 - f_0c\|_F.
\]

Then for any \( \gamma \geq \gamma_D + \gamma_E \) the seminorm \( \gamma^{-1}N_F \) is a bridge for \((L_A, L_C)\).

Proof. It is clear that \( \gamma^{-1}N_F(1_A, 0_C) \neq 0 \) since \( f_0 \neq 0 \), and that \( \gamma^{-1}N_F \) is norm-continuous. Thus the first two conditions of definition 5.1 of [6] are satisfied. We must verify the third, final, condition. To simplify notation, we identify \( A, B, C, D \) and \( E \) with their images in \( F \). For any \( a \in A, b \in B \) and \( c \in C \) we have

\[
N_F(a, c) = \|af_0 - f_0c\|_F \leq \|ad_0e_0 - d_0be_0\|_F + \|d_0be_0 - d_0e_0c\|_F
\]

\[
\leq \|ad_0 - d_0b\|_D\|e_0\|_E + \|d_0\|_D\|be_0 - e_0c\|_E
\]

\[
= N_D(a, b) + N_E(b, c).
\]

Now let \( a \in A \) with \( a = a^* \) be given, and let \( \varepsilon > 0 \) be given. Since \( \gamma_D^{-1}N_D \) is a bridge for \((L_A, L_B)\), there is by definition a \( b \in B \) with \( b^* = b \) such that

\[
L_B(b) \vee \gamma_D^{-1}N_D(a, b) \leq L_A(a) + \varepsilon.
\]

Then since \( \gamma_E^{-1}N_E \) is a bridge for \((L_B, L_C)\), there is a \( c \in C \) with \( c^* = c \) such that

\[
L_C(c) \vee \gamma_D^{-1}N_D(b, c) \leq L_B(b) + \varepsilon.
\]

Consequently

\[
L_C(c) \leq L_B(b) + \varepsilon \leq L_A(a) + 2\varepsilon,
\]

and, from the earlier calculation,

\[
N_F(a, c) \leq N_D(a, b) + N_E(b, c) \leq \gamma_D(L_A(a) + \varepsilon) + \gamma_E(L_B(b) + \varepsilon)
\]

\[
\leq (\gamma_D + \gamma_E)L_A(a) + \varepsilon(\gamma_D + 2\gamma_E).
\]

The situation is basically symmetric between \( A \) and \( C \), so one can make a similar calculation but starting with a \( c \in C \) to obtain a \( b \in B \) and then an \( a \in A \) satisfying the corresponding inequalities. This shows that \((\gamma_D + \gamma_E)^{-1}N_F \) is indeed a bridge. Then also \( \gamma^{-1}N_F \) will be a bridge for any \( \gamma \geq \gamma_D + \gamma_E \).

However, I have so far not seen any good general conditions that yield estimates showing that if the corresponding seminorm

\[
L_{AB} = L_A \vee L_B \vee \gamma^{-1}(N_D \vee N_D^*)
\]
brings \((A, L_A)\) and \((B, L_B)\) close together, and similarly for \(L_{BC}\), then \(L_{AC}\) using \((\gamma_D + \gamma_E)^{-1}N_F\) brings \((A, L_A)\) and \((C, L_C)\) close together, in the sense that \(\text{dist}_{H^L}^{PL-AC}(S(A), S(C))\) is small. In Hanfeng Li’s nuclear distance, in which the distinguished elements are all, implicitly, the identity elements, this aspect works much better. And since the nuclear distance satisfies the triangle inequality, it is clear that \(\text{dist}_{nu}(B^m, B^n)\) converges to 0 as \(m\) and \(n\) go to \(\infty\). But so far I find the nuclear distance to be more elusive, as I discuss briefly in section 6 of [9], though it is certainly attractive. I do not yet see how to obtain for the nuclear distance the kind of quantitative estimates that we will obtain here for prox.

3. The proof and statement of the main theorem

For the context of Section 1 the role of \(F\) of Section 2 is played by \(C(G/H, B^m \otimes B^n)\), while the roles of \(d_0\) and \(e_0\) are played by \(\omega_m\) and \(\omega_n\), with \(f_0\) being \(\omega_{mn}\). Let \(\gamma^A_m\) be defined as in proposition 8.1 of [9] but for \(P = P^m\), and let \(\gamma^B_m\) be defined as in proposition 8.2 of [9] but for \(P = P^m\). Let \(\gamma_m = \max\{\gamma^A_m, \gamma^B_m\}\). All that we need to know here about \(\gamma_m\) is that propositions 8.1 and 8.2 of [9] tell us that, for \(N_m\) as defined in Section 1 above, \(\gamma_m^{-1}N_m\) is a bridge for \((L_A, L_B^m)\), and that propositions 10.1 and 12.1 of [9] tell us that \(\gamma_m\) converges to 0 as \(m\) goes to \(\infty\). From Theorem 2.1 above and from the identifications made above, it follows immediately that for any \(\gamma\) with \(\gamma \geq \gamma_m + \gamma_n\) the seminorm \(\gamma^{-1}N_{mn}\) is a bridge for \((L^m, L^n)\).

We now investigate how close \(S(B^m)\) and \(S(B^n)\) are in the metric from the corresponding seminorm \(L_{mn}\) on \(B^m \oplus B^n\). Given \(\mu \in S(B^m)\), we want a systematic way to find a \(\nu \in S(B^n)\) that is “relatively close” to \(\mu\). For this purpose we use the Berezin symbols \(\sigma^n\) and \(\hat{\sigma}^n\) that we used in [7, 9]. We recall that \(\sigma^n\) is the completely positive unital map from \(B^n\) to \(A\) defined by \(\sigma^n_T(x) = \text{tr}(\alpha_x(P^n)T)\), while \(\hat{\sigma}^n\) is the completely positive unital map from \(A\) to \(B^n\) defined by

\[
\hat{\sigma}^n_f = d_n \int_{G/H} f(x)\alpha_x(P^n)dx,
\]

where we recall that \(d_n\) is the dimension of \(\mathcal{H}^n\), and the \(G\)-invariant measure on \(G/H\) gives \(G/H\) measure 1. Then \(\hat{\sigma}^m \circ \sigma^n\) will be a completely positive unital map from \(B^n\) to \(B^m\), whose transpose will map \(S(B^m)\) into \(S(B^n)\), for any \(m\) and \(n\). For any \(T \in B^n\) we have

\[
\hat{\sigma}^m(\sigma^n_T) = d_m \int_{G/H} \alpha_x(P^m)\text{tr}(\alpha_x(P^n)T)dx.
\]
Let $N_{mn}$ be the seminorm on $B^m \oplus B^n$ determined by $\omega_{mn}$, so that

\[ N_{mn}(S, T) = \|S \omega_{mn} - \omega_{mn}T\| \]

\[ = \sup\{\|S \otimes I_n \alpha_x(P^m \otimes P^n) - \alpha_x(P^m \otimes P^n)(I_m \otimes T)\| : x \in G/H\}. \]

Then $L_{mn}$ is defined on $B^m \oplus B^n$ by

\[ L_{mn}(S, T) = L^B_m(S) \vee L^B_n(T) \vee \gamma^{-1}(N_{mn}(S, T) \vee N_{mn}(S^*, T^*)) \]

for some $\gamma \geq \gamma_m + \gamma_n$. Let $\mu \in S(B^m)$ be given, and as state $\nu \in S(B^n)$ potentially close to $\mu$, we choose $\nu$ to be defined by $\nu(T) = \mu(\hat{\sigma}^m(\sigma^n_T))$. We then want an upper bound on $\rho_{L_{mn}}(\mu, \nu)$. Now

\[ \rho_{L_{mn}}(\mu, \nu) = \sup\{\|\mu(S) - \nu(T)\| : L_{mn}(S, T) \leq 1\}, \]

and

\[ |\mu(S) - \nu(T)| = |\mu(S) - \mu(\hat{\sigma}^m(\sigma^n_T))| \leq ||S - \hat{\sigma}^m(\sigma^n_T)||. \]

So we need to understand what the condition $L_{mn}(S, T) \leq 1$ implies for $||S - \hat{\sigma}^m(\sigma^n_T)||$. This seems difficult to do directly, so we use a little gambit that we have used before, e.g. shortly before notation 8.4 of [9], namely

\[ ||S - \hat{\sigma}^m(\sigma^n_T)|| \leq \|S - \hat{\sigma}^m(\sigma^m_S)\| + \|\hat{\sigma}^m(\sigma^m_S) - \hat{\sigma}^m(\sigma^n_T)\| \]

\[ \leq \delta^B_m L^B_m(S) + \|\sigma^m_S - \sigma^n_T\|_\infty, \]

where for the last inequality we have used theorem 11.5 of [9], which includes the definition of $\delta^B_m$. (We remark that theorem 11.5 of [9] is the same as theorem 6.1 of [7], but [9] gives a simpler proof of this theorem.) Note that $L_{mn}(S, T) \leq 1$ implies that $L^B_m(S) \leq 1$. Thus we see that it is $\|\sigma^m_S - \sigma^n_T\|_\infty$ that we need to control. In preparation for this we establish some additional notation in order to put the situation into a comfortable setting. Notice that $B^m \otimes B^n = L(H^m \otimes H^n)$, and its weight is just the sum of the highest weights of $(U^m \otimes \mathcal{H}^m)$ and $(U^n \otimes \mathcal{H}^n)$, which is just the highest weight of $(U, \mathcal{H})$ multiplied by $m + n$. Thus $\xi^m \otimes \xi^n$ is just the highest-weight vector for a copy of $(U^{m+n}, \mathcal{H}^{m+n})$ inside $\mathcal{H}^m \otimes \mathcal{H}^n$. To simplify notation we now just set $\xi^{m+n} = \xi^m \otimes \xi^n$, and view $\mathcal{H}^{m+n}$ as being the $G$-invariant subspace of $\mathcal{H}^m \otimes \mathcal{H}^n$ generated by $\xi^{m+n}$. Then the rank-1 projection $P^{m+n}$ on $\xi^{m+n}$ is exactly $P^m \otimes P^n$. We let $\Pi^{mn}$ denote the projection from $\mathcal{H}^m \otimes \mathcal{H}^n$ onto $\mathcal{H}^{m+n}$. Our notation will not distinguish between viewing the domain of $P^{m+n}$ as being $\mathcal{H}^m \otimes \mathcal{H}^n$ or as being $\mathcal{H}^{m+n}$, and we will use below the fact that $\alpha_x(P^{m+n}) = \alpha_x(P^m \otimes P^n)\Pi^{mn}$ for any $x \in G$. 

Lemma 3.1. For any $S \in B^m$ and $T \in B^n$ we have

$$\sigma^m_S - \sigma^n_T = \sigma^{m+n}_R$$

where $R = \Pi^{mn}(S \otimes I_n - I_m \otimes T)\Pi^{mn}$, viewed as an element of $B^{m+n}$.

Proof. For any $x \in G$ we have

$$\sigma_S(x) - \sigma_T(x) = \text{tr}^m(\alpha_x(P^m)S) - \text{tr}^n(T\alpha_x(P^n))$$
$$= (\text{tr}^m \otimes \text{tr}^n)(\alpha_x(P^m \otimes P^n)(S \otimes I_n - I_m \otimes T)\alpha_x(P^m \otimes P^n))$$
$$= \text{tr}^{m+n}(\alpha_x(P^{m+n})\Pi^{mn}(S \otimes I_n - I_m \otimes T)\Pi^{mn})$$
$$= \sigma^{m+n}_R(x).$$

□

Notice now that for $R$ defined as just above, because the rank of $P^{m+n}$ is 1, we have for any $x \in G$

$$|\sigma^{m+n}_R(x)| = |\text{tr}^{m+n}(\alpha_x(P^{m+n})\Pi^{mn}(S \otimes I_n - I_m \otimes T)\Pi^{mn})|$$
$$= ||\alpha_x(P^{m+n})(S \otimes I_n - I_m \otimes T)\alpha_x(P^{m+n})||$$
$$\leq ||\alpha_x(P^{m+n})(S \otimes I_n) - (I_m \otimes T)\alpha_x(P^{m+n})||,$$

and consequently

$$\|\sigma^{m+n}_R\| \leq N_{mn}(S^*, T^*).$$

But if $L_{mn}(S, T) \leq 1$, then $N_{mn}(S^*, T^*) \leq \gamma_m + \gamma_n$ if we have taken $\gamma = \gamma_m + \gamma_n$. Thus we find that

$$|\mu(S) - \nu(T)| \leq \delta^B_m + \gamma_m + \gamma_n.$$

Since the situation is symmetric in $m$ and $n$, we conclude that

$$\text{dist}_H^{\rho_{mn}}(S(B^m), S(B^n)) \leq \max\{\delta^B_m, \delta^B_n\} + \max\{\gamma_m^A, \gamma^A_m\} + \max\{\gamma^B_n, \gamma_n^B\}.$$ 

As mentioned in part above, it is shown in proposition 10.1, theorem 11.5, and proposition 12.1 of [9] that, respectively, $\gamma^A_m$, $\delta^B_m$, and $\gamma^B_m$ all converge to 0 as $m$ goes to $\infty$. We thus obtain the main theorem of this paper:

**Theorem 3.2.** With notation as above, for all $m$ and $n$ we have

$$\text{prox}(B^m, B^n) \leq \max\{\delta^B_m, \delta^B_n\} + \max\{\gamma^A_m, \gamma^A_m\} + \max\{\gamma^B_n, \gamma^B_n\},$$

and in particular, $\text{prox}(B^m, B^n)$ converges to 0 as $m$ and $n$ go to $\infty$.

One can also obtain matricial versions of this theorem along the lines discussed in section 14 of [9].
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