Time to Extinction in Subcritical Two-Sex Branching Processes

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Abstract Lower and upper bounds for the cumulative distribution function (cdf) of the time to extinction in a subcritical two-sex branching process are derived. A recursive procedure for approximating this cdf is also utilized. The results are illustrated with some simulations.

Keywords Time to extinction · Bisexual branching process · Subcritical process · Discrete time

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1 Introduction

There exists a significant literature regarding extinction probabilities in bisexual Galton-Watson branching processes (BGWP). A detailed introduction to the BGWP model and the physical properties associated with its variables can be found in Hull (2003) and Molina (2010). Necessary and sufficient conditions for certain extinction have been known for over 25 years. This raises the question—“If extinction is certain, when will it occur?”. Here we make the first attempt to give a meaningful response
to this question. Daley et al. (1986), Lemma 4.3, introduced the concept of “stopping time” in the context of two-sex processes. A version of stopping time is used in the present work. Our approach will be to apply known results concerning time-to-extinction in the standard Galton-Watson branching process to estimate time-to-extinction probabilities in the two-sex process. Our intent is to produce the first paper on time-to-extinction for the two-sex model and thus lay a foundation for future research efforts on this subject.

Let \( \{(f_{n,j}, m_{n,j}) : n \geq 0; \ j \geq 1\} \) be a sequence of integer-valued i.i.d. bivariate random variables. A BGWP \( \{Z_n\}_{n \geq 0} \) is defined by the recursion:

\[
Z_0 = i \geq 1,
\]

\[
Z_n = \zeta\left(\sum_{j=1}^{Z_{n-1}} f_{n,j}, \sum_{j=1}^{Z_{n-1}} m_{n,j}\right), \quad n \geq 1,
\]

where \( \zeta(x, y) \) is a deterministic mating function. We assume

\begin{align*}
(A1) \quad & \zeta(x, y) \text{ is superadditive, i.e., for any } x_1, x_2, y_1 \text{ and } y_2 \in [0, \infty) \\
& \zeta(x_1 + x_2, y_1 + y_2) \geq \zeta(x_1, y_1) + \zeta(x_2, y_2), \\
(A2) \quad & \zeta(x, y) \leq x.
\end{align*}

Both hypotheses are natural, at least from a population dynamics outlook. For an intuitive interpretation of assumption (A1), think of a two-sex population where all of the males and females are able to communicate and interact with one another without class distinctions. Superadditivity implies that the number of mating units in that scenario will not be smaller than in a scenario where the population is partitioned into a number of non-communicating groups and the mating takes place in each of these groups separately. The assumption (A2) reflects the fact that in many human and animal populations a female is allowed only one mate while a male may mate with several females. While this is not universally true, it is a common practice due to male dominance and the greater effort that females generally must make in the reproduction process. Notice also that the equality \( \zeta(x, y) = x \) in assumption (A2) yields the standard asexual Galton-Watson process.

Daley (1968) suggested two mating functions relevant to human and animal mating. He called the first completely promiscuous mating, \( \zeta(x, y) = x \min\{1, y\} \), where \( x \) is the number of females and \( y \) is the number of males in a given generation. In this case a champion male arises in each generation and has the capabilities to mate with every female in that generation. All other males are not allowed to mate. The other mating function is polygamous mating with perfect fidelity, \( \zeta(x, y) = \min\{x, ky\} \), where \( k \) is a positive integer. It is assumed that each individual will mate if a mate is available. Females may have no more than one mate and males may have up to \( k \) mates. Both mating functions, as well as most mating functions considered in the literature, satisfy (A1) and (A2).

Let \( T := \min\{n \geq 1 : Z_n = 0\} \) be the \textit{time-to-extinction} in a BGWP. That is, \( T \) counts the number of generations up to and including the generation in which extinction (absorption at zero) occurs. Let us call \( T_a \) the time to extinction in the standard (asexual) Galton-Watson process with \( i \geq 1 \) ancestors. Due to the independence of lines of descent (additive property, Athreya and Kaplan 1978), we