I-Love-Q: Unexpected Universal Relations for Neutron Stars and Quark Stars

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Neutron stars and quark stars are not only characterized by their mass and radius but also by how fast they spin, through their moment of inertia, and how much they can be deformed, through their Love number and quadrupole moment. These depend sensitively on the star’s internal structure and thus on unknown nuclear physics. We find universal relations between the moment of inertia, the Love number, and the quadrupole moment that are independent of the neutron and quark star’s internal structure. These can be used to learn about neutron star deformability through observations of the moment of inertia, break degeneracies in gravitational wave detection to measure spin in binary inspirals, distinguish neutron stars from quark stars, and test general relativity in a nuclear structure-independent fashion.

One of the largest uncertainties in nuclear physics is the relation between energy density ($\rho$) and pressure ($p$) at very high densities, the so-called equation of state (EoS). The interior structure of very compact stars, like neutron stars (NSs) and quark stars (QSs), depends sensitively on their EoS. This, in turn, determines their exterior properties, such as their mass and radius; their rotation rate, characterized by their moment of inertia; and their deformability, characterized by their quadrupole moment and tidal Love number ($I$, $Q$, $K_{tid}$).

Some astrophysical observations allow us to infer properties of the EoS of compact stars (3–5). None of these observations, however, is currently accurate enough to select between the many different EoSs that have been proposed or to distinguish between NSs and QSs, which then leads to degeneracies in the extraction of information from observations. For example, gravitational wave (GW) observations of NS binary inspirals may have difficulty in extracting the individual spins, because these are degenerate with the quadrupole moment for nonprecessing binaries (6). Similarly, GWs from NS binary inspirals cannot be easily used to test general relativity (GR) because of EoS degeneracies (7, 8). We here find a way to uniquely break these degeneracies through universal I-Love-Q relations between the reduced moment of inertia, $\tilde{I}$; tidal Love number, $\tilde{K}_{tid}$; and quadrupole moment, $\tilde{Q}$, that are essentially insensitive to the star’s EoS (9).

Consider an isolated, slowly rotating NS or QS described by its mass, $M*$; the magnitude of its spin angular momentum, $J$, and angular velocity, $\Omega$; its (spin-induced) quadrupole moment, $Q$, and its moment of inertia, $I = J/\Omega$. Let us introduce dimensionless quantities $\tilde{I} = I/M^*_c$ and $\tilde{Q} = -Q/(M^*_c)^{3/2}$, where $\chi = J/M^*_c$ is the dimensionless spin parameter (10). $I$ determines how fast a body can spin given a fixed $J$, whereas $Q$ encodes the amount of stellar quadrupolar deformation. They are determined by numerically solving the perturbed Einstein equations for realistic EoSs in a slow-rotation expansion ($\chi < 1$) to first and second order in spin, respectively (9, 11).

The slow-rotation approximation requires that $\chi$ be small enough such that all equations can be expanded in $\chi \ll 1$. In this approximation, the neglected corrections to $I$ and $Q$ are of $O(\chi^2)$ smaller than the leading-order contributions. Thus, demanding that any subleading terms be less than 10% of the leading-order ones means that $\chi < 0.3$ or equivalently spin frequencies $< 600$ Hz or spin periods $> 1.7$ ms. “True” millisecond pulsars (with periods $\sim 1$ ms) cannot be modeled in a slow-rotation expansion. However, double NS binary pulsars are expected to be spinning much more slowly, and the slow-rotation approximation should be allowed.

Fig. 1. I-Love and Q-Love relations. (A and B) (Top) The neutron star (NS) and quark star (QS) universal I-Love and Love-Q relations for various EoSs, together with fitting curves (solid and dashed curves). On the top axis, we show the corresponding NS mass with an APR EoS. The thick vertical lines show the stability boundary for

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In the presence of a companion, a NS or a QS will also be quadrupolarly deformed. The quadrupole moment tensor determines the magnitude of this deformation: $Q_{ij} = -\lambda^{(ad)} \varepsilon_{ij}$, where $\lambda^{(ad)}$ is the tidal Love number and $\varepsilon_{ij}$ is the quadrupole (gravitoelectric) tidal tensor that characterizes the source of the perturbation. The dimensionless tidal Love number, $\lambda^{(ad)} = \lambda^{(ad)} / M^2$, characterizes the tidal deformability of a star in the presence of a companion, and it can be calculated by treating the tidal effects as a perturbation to an isolated (nonrotating) NS or QS solution.

One might have expected universal relations between $I$, $Q$, and $\lambda^{(ad)}$ because $I \propto C^{-2}$ and $\lambda^{(ad)} \propto C^{-5}$ for polytropic EoSs in the Newtonian limit, where $C = M/R$ is the compactness parameter with $R$ the stellar radius. In the slow-rotation and small-deformation approximations, these barred quantities depend on spin only quadratically, and thus the relations are essentially spin-independent.

To find these relations, consider the following realistic EoSs for NSs: APR (14), SLy (15), Lattimer-Swesty with nuclear incompressibility of 220 MeV (LS220) (16), Shen (17), PS (18), PCL2 (19), and a simple $n = 1$ polytropic EoS with $p = K \rho^{1+1/n}$. For the LS220 and Shen EoSs, we adopt a temperature of 0.1 MeV and assume they are neutrino-less and in equilibrium. For QSs, we consider the EoSs: SQM1, SQM2, and SQM3 (19). We assume the stars are uniformly rotating, with isotropic pressure.

As shown in Fig. 1, the I-Love-Q relations hold universally for each NS and QS sequence, essentially independently of their EoSs for each class. Such relations can be numerically fitted with a polynomial on a log-log scale (9), shown in Fig. 1 with solid and dashed black curves:

$$\ln y_i = a_i \ln x_i + b_i (\ln x_i)^2 + c_i (\ln x_i)^3 + d_i (\ln x_i)^4 + e_i (\ln x_i)^5$$

where the coefficients are summarized in Table 1.

The data in Fig. 1 were obtained by numerically solving the perturbed Einstein equations, which is unavoidable for realistic EoSs. For very simple polytropic EoSs, the equations can be solved analytically in the Newtonian limit, and we obtain similar universal and analytic relations (9).

Two possible reasons may explain the I-Love-Q relations. First, the mathematical relations that define $I$, $\lambda^{(ad)}$, and $Q$ seem to depend mostly on the star’s internal structure near its outer layer, where nuclear physics constrains realistic EoSs. The integral that defines $I$ and $Q$ in the Newtonian limit accumulates the most near the NS surface (9). This suggests that the relations should lose their universality for unrealistic EoSs that modify the star’s internal structure near its surface. We have verified this explicitly by computing these relations for NSs with $n = 2, 2.5$, and 3 polytropic EoSs: the I-Love-Q curves deviate away from those in Fig. 1 as $n$ increases. The NS and QS I-Love-Q relations present different universal behavior because their EoSs are drastically different in the low-density stellar region.

The second reason is related to the no-hair theorems of GR. Figure 1 shows that the NS and the QS I-Love-Q relations approach each other as compactness is increased and approach the expected I-Love-Q relations for black holes (BHs), that is, $I \rightarrow 4$, $\lambda^{(ad)} \rightarrow 0$, and $Q \rightarrow 1$ (9). For BHs, all multipole moments of the exterior spacetime are related to the BH mass and spin ($21, 22$) because of the no-hair theorems ($23, 24$). But for NSs and QSs, such relations do not hold because of the lack of no-hair theorems for nonvacuum spacetimes. Our results suggest the existence of NS universal relations that are similar to the BH ones and perhaps hint at the existence of non-hair-like relations for nonvacuum spacetimes.

The I-Love-Q relations suggest an effacing of internal structure. This is not a consequence of the effacement principle (25) in GR, because the latter applies only to the motion of BHs. The I-Love-Q relations relate different multipole components of the exterior gravitational field of isolated, nonvacuum compact objects and say nothing about their relative motion.

Double NS binary pulsars may allow measuring $I$ to 10% accuracy in the near future (26, 27). This is because $I$ induces additional periastron precession, as well as precession of the angular momentum vector and the NS spin vectors. The former translates into a time-dependent inclination angle, whereas the latter may force the pulsar beams to sweep in and out of Earth’s line of sight. Alternatively, this precession may only cause a change in the observed average pulse shape, as in the Hulse-Taylor binary pulsar, in which case direct measurement may be more difficult.

Given a measurement of $I$, $M$, and $\Omega$, the I-Love-Q relations automatically provide the value of $\lambda^{(ad)}$ and $Q$, assuming the star is either a NS or a QS. The latter would not be easily observable with binary pulsars directly; although $Q$ and $\lambda^{(ad)}$ do induce additional precession, their effect is greatly suppressed relative to that of $I$. The I-Love-Q relations refer to reduced (barred) quantities, which must be appropriately normalized by the mass and the spin period. A small error in the latter could induce a large error in derived quantities. Such an error is smaller than the nonuniversality of the I-Love-Q relations if the NS spin period is much greater than 8.5 ms, which is the case for the double pulsar binary and a NS binary in the Laser Interferometer Gravitational Wave Observatory (LIGO) band.

### Table 1. Fit parameters

| $y_i$ | $x_i$ | $a_i$ | $b_i$ | $c_i$ | $d_i$ | $e_i$ |
|-------|-------|-------|-------|-------|-------|-------|
| $I$   | $\lambda^{(ad)}$ | 1.47  | 0.0817| 0.0149| $2.87 \times 10^{-4}$| $-3.64 \times 10^{-5}$|
| $Q$   | $\lambda^{(ad)}$ | 1.35  | 0.697 | -0.143| $9.94 \times 10^{-2}$| $-1.24 \times 10^{-2}$|
| $\lambda^{(ad)}$ | $\lambda^{(ad)}$ | 0.194 | 0.0936| 0.0474| $-4.21 \times 10^{-3}$| $1.23 \times 10^{-4}$|

*Fig. 2. Breaking spin degeneracies. Measurement accuracy of spin parameters $\beta$, $\chi_a$, and $\chi_\sigma$ using a Fisher analysis with Advanced LIGO, given a detection at a distance of $D_0 = 100$ Mpc with signal-to-noise ratio $\approx 30$. The spin-dependent part of the waveform phase is parameterized with the dimensionless averaged spin, $\chi_a$, and spin difference, $\chi_\sigma$, when using the Love-Q relation with and without the effective spin parameter $\beta$, constructed from a certain combination of the individual spins, when not using the relation. We consider three different NS binaries [(ii), (iii)], with parameters $(m_1, m_2) = (1.45, 1.35)M_\odot$ and $\chi_\sigma = \chi_a$, $(m_1, m_2) = (1.45, 1.35)M_\odot$, and $\chi_a = 2 \chi_\sigma$, and $(m_1, m_2) = (1.4, 1.35)M_\odot$, and $\chi_a = 2 \chi_\sigma$, respectively. $\Delta \beta$ is computed without use of the NS Love-Q relation, whereas $\Delta \chi_\sigma$ are computed by using this relation. $\Delta \beta$ and $\Delta \chi_\sigma$ are almost identical for all systems, because they are dominated by their priors ($|\beta| < 0.2$ and $|\chi_\sigma| < 0.1$). Thanks to the Love-Q relation, $\chi_a$ can be measured to $\sim 0.01$.  *}
Given independent measurements of any two members of the I-Love-Q trio, one could also distinguish NSs from QSs, assuming GR is correct. These relations are different for NSs and QSs, even though they are universal within their class of EoSs, as shown in Fig. 1. Given two such measurements with sufficiently small uncertainties, one could determine whether the observed compact object is a NS or a QS.

Interferometric GW detectors are most sensitive to the GW phase. For waves emitted during NS binary inspirals, the phase contains a term proportional to the NSs’ spin-induced quadrupole moments, $Q_1$ and $Q_2$, and another term proportional to their tidally induced quadrupolar deformations, $\lambda_1^{\text{tid}}$ and $\lambda_2^{\text{tid}}$. The former enters with a factor proportional to $v/c^3$, where $v$ is the orbital velocity of binary constituents (28), whereas the latter is proportional to $c^4/\Delta c^3 (12)$, relative to the leading-order term. Because by Kepler’s third law $v \propto \sqrt{G M/R^3}$ (where $G$ is the gravitational constant) and $c$ has a distinct GW frequency dependence that makes them nondegenerate.

The NS quadrupole moment is degenerate with the NSs’ individual spins, because there is a spin-spin interaction term in the GW phase that enters at the same order in $v/c$ as the quadrupole term (28).

Such a degeneracy may prevent us from simultaneously extracting the quadrupole moment and the individual spins from a GW detection. However, the Love-Q relation can be used to break this degeneracy by rewriting $Q$ as a function of $\lambda^{\text{tid}}$. If the Love number can be measured with a GW detection, then one can also separately measure the spins, as shown in Fig. 2.

Pulsar observations allow GR tests (8) when the gravitational field is much stronger than that in the Solar System (7). Unfortunately, these tests are not effective most of the time because of degeneracies between modified gravity effects and the EoS. The I-Love-Q relations can be used to break this degeneracy.

A robust GR test would require at least two independent measurements of any two quantities in the I-Love-Q trio. Given a single measurement, the I-Love-Q relations give us the other two for a NS or a QS in GR. A second independent measurement can then be used as a redundancy test.

Depending on where the two observed values lie in the I-Love-Q plane, different tests are possible. If the two observables lie on the GR NS (QS) I-Love-Q line, then

1) Case a. The object is a NS (QS), and any modified gravity effect must be small enough to fit this observation.

2) Case b. The object is a QS (NS) in a modified gravity theory with the right coupling parameters to still fit this observation.

If the two observables lie on neither GR I-Love-Q line, then the observations would indicate a GR deviation, provided the assumptions made here are correct. An example of such a GR test is given in Fig. 3. One can carry out a similar test by using the Love-C relation (shown in Fig. 3), which is effectively universal.

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5. For example, the observation of x-ray bursters and low-mass x-ray binaries has allowed for the simultaneous determination of the star’s mass and radius to 10% accuracy (4). Observations of double NS pulsars, such as J0737-3039 (29), may allow for the measurement of the moment of inertia to the same accuracy (26, 27).

6. Gravitational wave (GW) observations from binary NS inspirals with second-generation ground-based detectors, such as Advanced LIGO, Advanced Virgo, and KAGRA, may allow for the measurement of the tidal Love number (32, 30, 31).

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Holographic Vortex Liquids and Superfluid Turbulence

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Superfluid turbulence is a fascinating phenomenon for which a satisfactory theoretical framework is lacking. Holographic duality provides a systematic approach to studying such quantum turbulence by mapping the dynamics of a strongly interacting quantum liquid into the dynamics of classical gravity. We use this gravitational description to numerically construct turbulent flows in a holographic superfluid in two spatial dimensions. We find that the superfluid kinetic energy spectrum obeys the Kolmogorov $\sim k^{-5/3}$ scaling law, with energy injected at long wavelengths undergoing a direct cascade to short wavelengths where dissipation by vortex annihilation and vortex drag becomes efficient. This dissipation has a simple gravitational interpretation as energy flux across a black hole event horizon.

Superfluid turbulence is a non-equilibrium phenomenon dominated by the dynamics of quantized vortices (1–7), which drive the system outside the hydrodynamic regime of normal turbulent fluids. Powerful numerical simulations of phenomenological models of vortex dynamics have produced considerable insight. However, a complete theoretical framework remains lacking, and an ab initio study is desirable.

Here, we use holographic duality to study superfluid turbulence in two spatial dimensions. Holographic duality is a precise equivalence between certain systems of quantum matter without gravity and classical gravitational systems in a curved spacetime with one additional spatial dimension (8–16). This allows a first-principles study in these systems of superfluid dynamics, including turbulent flows, by using the corresponding gravity description of the superfluid phase. In this framework, dissipation in the gravitational description can be understood in terms of excitations falling through a black hole event horizon. This provides a direct measure of the rate of energy dissipation and its spectrum.

We focus on “non-counterflow” superfluid turbulence (17–25). Experimentally, these systems appear to obey Kolmogorov’s $\sim k^{-5/3}$ scaling law in the kinetic energy spectrum, which suggests a remnant similarity between quantum and classical turbulence. In classical turbulence, this scaling behavior can be understood as a consequence of an energy cascade in which the injected energy is passed from one scale to another without substantial loss. Whether quantum turbulence admits a similar cascade picture—and, if so, whether or when the cascade drives energy to long or to short wavelengths—remains an important open question. Several recent numerical studies of the phenomenological Gross-Pitaevskii equation in a two-dimensional (2D) superfluid (with dissipation put in by hand) observed Kolmogorov scaling but came to conflicting conclusions regarding the direction of the cascade (26–29).

To set up our superfluid, consider a quantum field theory in two spatial dimensions with a complex scalar operator, $\psi(x)$, carrying charge $q$ under a global $U(1)$ symmetry. Let $j^\mu(x)$ denote the conserved current operator of this global $U(1)$ symmetry. To induce a superfluid condensate for $\psi$, we will turn on a chemical potential $\mu$ for the $U(1)$ charge. For sufficiently large $\mu$, $\psi$ can develop a nonzero expectation value $\langle \psi \rangle \neq 0$ when the temperature falls below a critical temperature $T_c$, spontaneously breaking the global $U(1)$ symmetry and driving the system into a superfluid phase.

We now construct a simple holographic model of our 2D superfluid. We begin with a classical field theory in an asymptotically anti–de Sitter spacetime with three spatial dimensions (AdS$_3$). Under the standard holographic dictionary, the conserved current $j^\mu(x)$ is mapped to a dynamical $U(1)$ gauge field $A_{\mu}(x, z)$ in the gravitational bulk, and the scalar operator $\psi(x)$ is mapped to a bulk scalar field $\Phi(x, z)$ carrying charge $q$ under the gauge field $A_{\mu}$ (where $z$ is the radial coordinate of AdS$_3$) (30). A nonzero temperature corresponds to adding to the bulk spacetime a black hole whose horizon is a 2D plane extended in boundary spatial directions. Adding a chemical potential corresponds to imposing a boundary condition on the bulk gauge field $A_\mu = \mu$ at the boundary of AdS$_3$ (31). If the charge $q$ and scaling dimension $\Delta$ of $\psi$ lie in certain range (32, 33), a sufficiently large $\mu$ drives the bulk scalar field $\Phi$ to condense through the Higgs mechanism. Different values of the charge $q$ and potential for $\Phi$ define different holographic quantum theories, each with a low-temperature superfluid phase (34, 35). We choose a quadratic potential with a mass for $\Phi$ corresponding to $\psi$ having scaling dimension $\Delta = 2$, and we work in the limit of large $\mu$ [the probe limit of (33); see supplementary text].

A superfluid state generally has gapped vortex excitations, which play an important role in our discussion below. Around a vortex, the fluid circulation is quantized. Introducing the (unnormalized) superfluid velocity

$$u = \frac{j^\mu}{|\psi|^2}, \quad j^\mu = \frac{i}{2} \left(\langle \psi^* \nabla \psi \rangle - \langle \psi \nabla \psi^* \rangle\right)$$

the winding number $W$ of a vortex is determined by

$$W = \frac{1}{2\pi} \oint \Phi \cdot dx \cdot u$$

where the path $\Gamma$ encloses a single vortex and is oriented counterclockwise. Boldface symbols denote vectors along boundary spatial directions; $x = \{x_1, x_2\}$, where $x_1$ denotes the two spatial directions and $\nabla = \left[\partial / \partial x_1, \partial / \partial x_2\right]$. Vortices map into the gravitational bulk as flux tubes extending along the AdS radial direction from the boundary, where they have a characteristic size $1/\mu$, to the horizon. Inside the flux tube, the condensate goes to zero, effectively pushing a hole through the bulk scalar condensate. Explicit gravity solutions corresponding to a static vortex of arbitrary winding number were previously constructed numerically (36–38).

The gravity dual thus provides a first-principles description of superfluid flows involving vortices, as well as tools to describe dissipation in the system. Consider turning on a perturbation of $j^\mu$ in the boundary theory, which on the gravity side co-

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Erratum for the Report: “I-Love-Q: Unexpected universal relations for neutron stars and quark stars” by K. Yagi and N. Yunes

In the Report “I-Love-Q: Unexpected universal relations for neutron stars and quark stars,” the results for the quark-star (QS) tidal Love numbers contained an error. When calculating such quantities, one must carefully account for the density continuity at the surface, as pointed out around equation 15 of T. Hinderer et al., Phys. Rev. D81, 123016 (2010). The QS-corrected I-Love-Q relations are presented in Fig. 1 (below). The QS relations now lie on top of the neutron-star (NS) ones. Although one cannot distinguish NSs and QSs using such relations, the new results strengthen the universality. The fit of the I-Love-Q relations for NSs presented in the original paper now applies to both NSs and QSs.

Figure 3 of the original manuscript is also modified and should be replaced by Fig. 2 (next page). The left panel shows the I-Love relation in dynamical Chern-Simons gravity only for NSs. The right panel shows the QS-corrected Love-compactness relations. The QS relations are now outside of the hypothetical error box. This means that one has to assume that the star is a NS in order to use the Love-compactness relation to test general relativity (GR).

**Fig. 1. I-Love and Q-Love relations. (Top left and right)** The universal I-Love and Love-Q relations for various EoSs, together with fitting curves (solid curves). Observe that both NS and QS-corrected relations lie on top of each other. **(Bottom left and right)** Fractional errors between the fitting curves and numerical results.

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Erratum for the Report: “I-Love-Q: Unexpected universal relations for neutron stars and quark stars” by K. Yagi and N. Yunes

Fig. 2. Testing general relativity. (Left) Error box (shaded region) in the I-Love plane, given two independent observations of the moment of inertia and tidal Love number consistent with GR (black star) to 10% accuracy with a binary pulsar observation and to 60% with a gravitational-wave observation, respectively. The black solid line shows the GR I-Love relation, whereas other lines show the NS relations in dynamical Chern-Simons gravity. (Right) Error box (shaded region) in the Love-compactness plane, given two independent observations of the tidal Love number and compactness consistent with GR (black star) to 60% accuracy with a gravitational-wave observation and to 5% with a future low-mass x-ray binary observation, respectively. The NS and QS-corrected Love-compactness relations are dependent on the equation of state, as shown by the spread in the curves. The difference between these curves for NSs is smaller than the error box, thus allowing for a generic test of GR. Such a test, however, requires the assumption that the observed object is a NS and not a QS, because the latter has a different Love-compactness relation.