TORELLI GROUP, JOHNSON KERNEL AND INVARIANTS OF HOMOLOGY SPHERES

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ABSTRACT. In the late 1980’s, it was shown that the Casson invariant appears in the difference between the two filtrations of the Torelli group: the lower central series and the Johnson filtration. This was interpreted as the secondary characteristic class $d_1$ associated with the fact that the first MMM class vanishes on the Torelli group. It is a rational generator of $H^1(K_g;\mathbb{Z})$ where $K_g$ denotes the Johnson subgroup of the mapping class group $\mathcal{M}_g$. Then Hain proved, as a particular case of his fundamental result, that this is the only difference in degree 2. In this paper, we prove that no other invariant than the above gives rise to new rational difference between the two filtrations up to degree 6. We apply this to determine $H_1(K_g;\mathbb{Q})$ explicitly by computing the description given by Dimca, Hain and Papadima. We also show that any finite type rational invariant of homology 3-spheres of degrees up to 6, including the second and the third Ohtsuki invariants, can be expressed by $d_1$ and lifts of Johnson homomorphisms.

1. INTRODUCTION AND STATEMENTS OF THE MAIN RESULTS

Let $\mathcal{M}_g$ be the mapping class group of a closed oriented surface $\Sigma_g$ of genus $g$ and let $I_g \subset \mathcal{M}_g$ be the Torelli subgroup. Namely, it is the subgroup of $\mathcal{M}_g$ consisting of all the elements which act on the homology $H := H_1(\Sigma_g;\mathbb{Z})$ trivially.

There exist two filtrations of the Torelli group. One is the lower central series which we denote by $I_g(k)$ ($k = 1, 2, \ldots$) where $I_g(1) = I_g$, $I_g(2) = [I_g, I_g]$ and $I_g(k + 1) = [I_g(k), I_g]$ for $k \geq 1$. The other is called the Johnson filtration $\mathcal{M}_g(k)$ ($k = 1, 2, \ldots$) of the mapping class group where $\mathcal{M}_g(k)$ is defined to be the kernel of the natural homomorphism

$$\rho_k : \mathcal{M}_g \to \text{Out}(N_k(\pi_1 \Sigma_g)).$$

Here $N_k(\pi_1 \Sigma_g)$ denotes the $k$-th nilpotent quotient of the fundamental group of $\Sigma_g$ and $\text{Out}(N_k(\pi_1 \Sigma_g))$ denotes its outer automorphism group. $\mathcal{M}_g(1)$ is nothing other than the Torelli group $I_g$ so that $\mathcal{M}_g(k)$ ($k = 1, 2, \ldots$) is a filtration of $I_g$. This filtration was originally introduced by Johnson [13] for the case of a genus $g$ surface with one boundary component. The above is the one adapted to the case of a closed surface (see [23] for more details). It can be shown that $I_g(k) \subset \mathcal{M}_g(k)$ for all $k \geq 1$. Johnson showed in [15] that $I_g(2)$ is a finite index subgroup of $\mathcal{M}_g(2)$ and asked whether this will continue to hold for the pair $I_g(k) \subset \mathcal{M}_g(k)$ ($k \geq 3$). He also showed in [14] that
$\mathcal{M}_g(2)$ is equal to the subgroup $\mathcal{K}_g$, called the Johnson subgroup or Johnson kernel, consisting of all the Dehn twists along separating simple closed curves on $\Sigma_g$.

The above question was answered negatively in [20]. More precisely, a homomorphism $d_1 : \mathcal{K}_g \to \mathbb{Z}$ was constructed which is non-trivial on $\mathcal{M}_g(3)$ while it vanishes on $\mathcal{I}_g(3)$ so that the index of the pair $\mathcal{I}_g(3) \subset \mathcal{M}_g(3)$ was proved to be infinite. Furthermore it was shown in [21] that there exists an isomorphism

$$H^1(\mathcal{K}_g; \mathbb{Z})_{\mathcal{M}_g} \cong \mathbb{Z} \quad (g \geq 2)$$

where the homomorphism $d_1$ serves as a rational generator. It is characterized by the fact that its value on a separating simple closed curve on $\Sigma_g$ of type $(h, g-h)$ is $h(g-h)$ up to non-zero constants. This homomorphism was defined as the secondary characteristic class associated with the fact that the first MMM class, which is an element of $H^2(\mathcal{M}_g; \mathbb{Z})$, vanishes in $H^2(\mathcal{I}_g; \mathbb{Z})$. It was also interpreted as a manifestation of the Casson invariant $\lambda$, which is an invariant defined for homology 3-spheres, in the structure of the Torelli group.

Now let us consider the following two graded Lie algebras

$$\text{Gr}\ t_g = \bigoplus_{k=1}^{\infty} t_g(k), \quad t_g(k) = (\mathcal{I}_g(k)/\mathcal{I}_g(k+1)) \otimes \mathbb{Q}$$

$$\text{m}_g = \bigoplus_{k=1}^{\infty} m_g(k), \quad m_g(k) = (\mathcal{M}_g(k)/\mathcal{M}_g(k+1)) \otimes \mathbb{Q}$$

associated to the above two filtrations of the Torelli group. Here $t_g$ denotes the Malcev Lie algebra of the Torelli group and $\text{Gr}\ t_g$ denotes its associated graded Lie algebra. Hain [9] obtained fundamental results about the structure of these Lie algebras. He gave an explicit finite presentation of them (see Theorem 2.1 below) which implies that $\text{Ker}(t_g(2) \to m_g(2)) \cong \mathbb{Q}$. Furthermore, he proved that the natural homomorphism

$$t_g(k) \to m_g(k)$$

is surjective for any $k$ which implies that the index of the pair $\mathcal{I}_g(k) \subset \mathcal{M}_g(k)$ remains to be infinite for any $k \geq 4$ extending the above mentioned result. He also showed that all the higher Massey products of the Torelli group vanish for $g \geq 4$.

On the other hand, Ohtsuki [26] defined a series of invariants $\lambda_k$ for homology 3-spheres the first one being the Casson invariant $\lambda$. He also initiated a theory of finite type invariants for homology 3-spheres in [27]. Then Garoufalidis and Levine [7] studied the relation between this theory and the structure of the Torelli group extending the case of the Casson invariant mentioned above extensively.

In these situations, it would be natural to ask whether there exists any other difference between the two filtrations of the Torelli group than the Casson invariant, in particular whether any finite type rational invariant of homology 3-spheres of degree greater than 2 appears there or not. This is equivalent to asking that the natural homomorphism (1) is an isomorphism for $k = 3, 4, \ldots$ or not.
Now it was proved in [23] that $t_g(3) \cong m_g(3)$. The main theorem of the present paper is the following.

**Theorem 1.1.** For any $k = 4, 5, 6$, we have

$$t_g(k) \cong m_g(k).$$

As a corollary to the above theorem, we obtain the cases $k = 5, 6, 7$ of the following result. The case $k = 3$ follows from Hain’s theorem (Theorem 2.1) combined with a result of [20] and the case $k = 4$ follows from a result of [23] mentioned above.

**Corollary 1.2.** For any $k = 3, 4, 5, 6, 7$, the $k$-th group $I_g(k)$ in the lower central series of the Torelli group is a finite index subgroup of the kernel of the non-trivial homomorphism $d_1 : \mathcal{M}_g(k) \to \mathbb{Z}$.

Recall here that Johnson [15] proved that $I_g(2) = [I_g, I_g]$ is a finite index subgroup of $M_g(2) = K_g$.

Next we present two applications of Theorem 1.1. First, we give the explicit form of the rational abelianization $H_1(K_g; \mathbb{Q})$ of the Johnson subgroup. Dimca and Papadima [5] proved that $H_1(K_g; \mathbb{Q})$ is finite dimensional for $g \geq 4$. Then Dimca, Hain and Papadima [4] gave a description of it. However they did not give the final explicit form. Here we compute their description by combining the case $k = 4$ of Theorem 1.1 and former results concerning the Johnson homomorphisms to obtain the following result.

**Theorem 1.3.** The secondary class $d_1$ together with the refinement $\tilde{\tau}_2$ of the second Johnson homomorphism gives the following isomorphism for $g \geq 6$.

$$H_1(K_g; \mathbb{Q}) \cong \mathbb{Q} \oplus [2^2] \oplus [31^2].$$

Here and henceforth, for a given Young diagram $\lambda = [\lambda_1 \cdots \lambda_h]$, we denote the irreducible representation of $Sp(2g, \mathbb{Q})$ corresponding to $\lambda$ simply by $[\lambda_1 \cdots \lambda_h]$. As for the refinements $\tilde{\tau}_k$ of Johnson homomorphisms, see §7 for details.

By making use of recent remarkable results of Ershov-He [6] and Church-Ershov-Putman [3], we obtain the following.

**Corollary 1.4.**

(i) Two subgroups $[K_g, K_g]$ and $I_g(4)$ of the Torelli group $I_g$ are commensurable for $g \geq 6$.

(ii) $[K_g, K_g]$ is finitely generated for $g \geq 7$.

**Remark 1.5.** Johnson [15] determined the abelianization $H_1(I_g; \mathbb{Z})$ of the Torelli group completely where the Birman-Craggs homomorphisms introduced in [2] played an essential role in its torsion part. Although some of the Birman-Craggs homomorphisms restrict non-trivially on $K_g$, they are mod 2 reductions of integral ones because the Casson invariant defines an integer valued homomorphism on $K_g$. Therefore, no 2-torsion
class in $H_1(I_g; \mathbb{Z})$ can be lifted to $H_1(K_g; \mathbb{Z})$ as a torsion class. Thus, at present, there is no known information about the torsion part of $H_1(K_g; \mathbb{Z})$. It should be an important problem to determine it.

Another application of our main theorem is the following.

**Theorem 1.6.** Any finite type rational invariant of homology 3-spheres of degrees 4 and 6, including the Ohtsuki invariants $\lambda_2$ and $\lambda_3$, can be expressed by $d_1$ and (lifts of) Johnson homomorphisms.

In §7, we give more detailed statements Theorem 7.8 and Theorem 7.2.

Based on the above result, we would like to propose the following conjecture (see Problem 6.2 of [23]).

**Conjecture 1.7.** For any $k \neq 2$, the equality $t_g(k) \cong m_g(k)$ holds so that

$$\text{Ker} (\text{Gr} t_g \to m_g) \cong \mathbb{Q}.$$  

**Remark 1.8.** This is equivalent to the statement that Corollary 1.2 continues to hold for all $k \geq 3$.

In the context of characteristic classes of the mapping class group, the above conjecture can be translated as follows.

**Conjecture 1.9** (another formulation). The Lie algebra $t_g$ is isomorphic to the completion of the central extension of $m_g$ associated to the infinitesimal first MMM class defined in $H^2(m_g)$.

Hain [9] considered the relative completion of the mapping class group with respect to the classical homomorphism $M_g \to \text{Sp}(2g, \mathbb{Z})$. He proved that the kernel of the natural surjective homomorphism $t_g \to u_g$ is isomorphic to $\mathbb{Q}$ where $u_g$ denotes the graded Lie algebra associated to the Lie algebra of his relative completion. In this terminology, the above conjecture is also equivalent to saying that the natural homomorphism

$$\text{Gr} u_g \to m_g,$$

which exists because of the universality of $u_g$, is an isomorphism.

The content of the present paper is roughly as follows. In §2, we relate the difference between the two filtrations of the Torelli group to the second homology group $H_2(m_g)$ of the Lie algebra $m_g$. In §3, we explain the method of proving the main Theorem 1.1. Then in §4, §5 and §6, we prove the vanishing of the weight 4, 5 and 6 parts of $H_2(m_g)$, respectively. Finally in §7, we prove the main results.

In this paper, whenever we mention groups like $M_g, I_g, M_g(k), I_g(k)$, modules like $t_g(k), m_g(k)$ and also homomorphisms like $\tau_g(k)$, which depend on the genus $g$, we always assume that it is sufficiently large, more precisely in a stable range with respect to the property we consider, unless we describe the range of $g$ explicitly.
2. The Second Homology Groups of the Lie Algebras \( t_g, m_g \)

In this section, we reduce the problem of determining the kernel of the surjective homomorphism \( \text{Gr} \ t_g \rightarrow m_g \) to the computation of the second homology group \( H_2(m_g) \) of the Lie algebra \( m_g \). Let us denote \( \text{Ker}(\text{Gr} \ t_g \rightarrow m_g) \) by \( i_g \) so that we have a short exact sequence

\[
0 \rightarrow i_g \rightarrow \text{Gr} \ t_g \rightarrow m_g \rightarrow 0
\]

of the three graded Lie algebras

\[
i_g = \bigoplus_{k=1}^{\infty} i_g(k), \quad \text{Gr} \ t_g = \bigoplus_{k=1}^{\infty} t_g(k), \quad m_g = \bigoplus_{k=1}^{\infty} m_g(k).
\]

Now it is a classical result of Johnson \([12][15]\) that \( t_g(1) \cong m_g(1) \cong \wedge^3 H_Q/H_Q \) and hence \( i_g(1) = 0 \). The module \( \wedge^3 H_Q/H_Q \) is an irreducible representation of \( \text{Sp}(2g, Q) \) corresponding to the Young diagram \([1^3]\).

Hain proved the following fundamental result.

**Theorem 2.1** (Hain \([9]\)). The Lie algebra \( t_g \) (\( g \geq 6 \)) is isomorphic to its associated graded \( \text{Gr} \ t_g \) which has presentation

\[
\text{Gr} \ t_g = \mathcal{L}([1^3])/\langle [1^6] + [1^4] + [1^2] + [2^21^2] \rangle
\]

where \( \mathcal{L}([1^3]) \) denotes the free Lie algebra generated by \( [1^3] \) and \( \langle [1^6] + [1^4] + [1^2] + [2^21^2] \rangle \) is the ideal generated by

\[
[1^6] + [1^4] + [1^2] + [2^21^2] \subset \wedge^2[1^3].
\]

Here we recall a few facts about the homology of graded Lie algebras briefly. Let us consider a graded Lie algebra

\[
g = \bigoplus_{k=1}^{\infty} g(k)
\]

satisfying the condition that \( H_1(g) \cong g(1) \). Namely, we assume that \( g \) is generated by the degree 1 part \( g(1) \) as a Lie algebra. Both the Lie algebras \( t_g, m_g \) satisfy this condition. The homology group of a graded Lie algebra is bigraded, where the first grading is the usual homology degree while the second grading is induced from the grading of the graded Lie algebra. We call the latter grading by weight, denoted simply by the subscript \( w \). In particular, the second homology group has the following direct sum decomposition

\[
H_2(g) = \bigoplus_{w=2}^{\infty} H_2(g)_w
\]

where

\[
H_2(g)_w = \frac{\text{Ker}(\partial : (\wedge^2 g)_w \rightarrow g(w))}{\text{Im}(\partial : (\wedge^3 g)_w \rightarrow (\wedge^2 g)_w)}.
\]

In this terminology, the following is an immediate consequence of Theorem 2.1

**Corollary 2.2.** \( H_2(\text{Gr} \ t_g)_2 \cong [1^6] + [1^4] + [1^2] + [2^21^2] \) and for any \( w \geq 3 \), \( H_2(\text{Gr} \ t_g)_w = 0 \).
Now we consider the short exact sequence (2).

**Proposition 2.3.** The following short exact sequence

\[ 0 \rightarrow H_2(\text{Gr } t_g)_2 \rightarrow H_2(m_g)_2 \rightarrow (H_1(i_g)m_g)_2 \cong \mathbb{Q} \rightarrow 0 \]

holds and for any \( w \geq 3 \)

\[ H_2(m_g)_w \cong (H_1(i_g)m_g)_w. \]

**Proof.** The 5-term exact sequence of the Lie algebra extension (2) is given by

\[ H_2(\text{Gr } t_g)^{\text{Hain}} \cong H_2(\text{Gr } t_g)_2 \rightarrow H_2(m_g) \rightarrow H_1(i_g)m_g \rightarrow H_1(\text{Gr } t_g) \cong H_1(m_g). \]

The result follows from this. \( \Box \)

**Corollary 2.4.**

\[ H_2(m_g)_3 = 0, \quad i_g(4) \cong H_2(m_g)_4 \]

and for any \( w \geq 4 \), we have the following exact sequence.

\[ 0 \rightarrow \bigoplus_{k=3}^{w-1} [i_g(k), t_g(w-k)] \rightarrow i_g(w) \rightarrow H_2(m_g)_w \rightarrow 0. \]

**Proof.** It was shown in [23] (see also a related result of Sakasai [29]) that \( t_g(3) \cong m_g(3) \) and hence \( i_g(3) = 0 \). It follows that \( H_2(m_g)_3 \cong (H_1(i_g)m_g)_3 = 0 \). As for the case \( w = 4 \), we have

\[ H_2(m_g)_4 \cong (H_1(i_g)m_g)_4 = i_g(4) \]

because \( i_g(2) \cong \mathbb{Q} \) is contained in the center of \( t_g \). Finally, for any \( w \geq 4 \), we have

\[ H_2(m_g)_w \cong (H_1(i_g)m_g)_w \cong i_g(w) / \bigoplus_{k=3}^{w-1} [i_g(k), t_g(w-k)]. \]

This completes the proof. \( \Box \)

In general, we have the following

**Proposition 2.5.** Assume \( i_g(k) = 0 \) for \( k = 4, 5, \ldots, w - 1 \) (\( w \geq 4 \)). Then we have

\[ i_g(w) \cong H_2(m_g)_w. \]

**Proof.** This is an immediate consequence of Corollary 2.4. \( \Box \)
3. Method of Proving Theorem 1.1

To prove Theorem 1.1, it is enough to show that

\[ H_2(m_g)_w = 0 \ (w = 4, 5, 6) \]

by Proposition 2.5. We prove this in Sections 4, 5, 6. To do so, for technical reasons regarding computer computations, we have to consider the case of one boundary component.

Let \( \mathcal{M}_{g,1} \) be the mapping class group of a genus \( g \) compact oriented surface with one boundary component and let \( \mathcal{I}_{g,1} \) be its Torelli subgroup. Then we have the Johnson filtration \( \{\mathcal{M}_{g,1}(k)\}_k \) for the former and the lower central series \( \{\mathcal{I}_{g,1}(k)\}_k \) for the latter.

We set

\[ Gr_t_{g,1} = \bigoplus_{k=1}^{\infty} t_{g,1}(k), \quad t_{g,1}(k) = (\mathcal{I}_{g,1}(k)/\mathcal{I}_{g,1}(k+1)) \otimes \mathbb{Q} \]

\[ m_{g,1} = \bigoplus_{k=1}^{\infty} m_{g,1}(k), \quad m_{g,1}(k) = (\mathcal{M}_{g,1}(k)/\mathcal{M}_{g,1}(k+1)) \otimes \mathbb{Q} \]

where \( t_{g,1} \) denotes the Malcev Lie algebra of \( \mathcal{I}_{g,1} \). Define \( i_{g,1} = \text{Ker}(Gr_t_{g,1} \rightarrow m_{g,1}) \) so that we have a short exact sequence

\[ 0 \rightarrow i_{g,1} \rightarrow Gr_t_{g,1} \rightarrow m_{g,1} \rightarrow 0 \]  

(4)

of graded Lie algebras, where the subjectivity of the natural homomorphism \( Gr_t_{g,1} \rightarrow m_{g,1} \) is again due to Hain.

**Proposition 3.1.** The natural homomorphism \( i_{g,1} \rightarrow i_g \) is an isomorphism so that \( i_{g,1}(k) \cong i_g(k) \) for any \( k \).

**Proof.** Recall that the kernel of the natural surjection \( \mathcal{I}_{g,1} \rightarrow \mathcal{I}_g \) is known to be isomorphic to \( \pi_1 T_1 \Sigma_g \) where \( T_1 \Sigma_g \) denotes the unit tangent bundle of \( \Sigma_g \). Thus we have a short exact sequence \( 1 \rightarrow \pi_1 T_1 \Sigma_g \rightarrow \mathcal{I}_{g,1} \rightarrow \mathcal{I}_g \rightarrow 1 \). Let \( p_{g,1} \) be the Malcev Lie algebra of \( \pi_1 T_1 \Sigma_g \) which is a central extension of the Malcev Lie algebra \( p_g \) of \( \pi_1 \Sigma_g \). Hain \[9\] showed that this induces a short exact sequence \( 0 \rightarrow p_{g,1} \rightarrow t_{g,1} \rightarrow t_g \rightarrow 0 \) which in turn induces a short exact sequence of their associated graded Lie algebras as depicted
On the other hand, a result of Asada and Kaneko (and Labute) in [1] implies that the kernel of the natural surjection \( m_{g,1} \rightarrow m_g \) is isomorphic to \( \text{Gr} p_{g,1} \) (see [23][25] for details). Thus the two rows as well as the two columns of the commutative diagram (5) are all short exact sequences. Then it is easy to see from this fact that the homomorphism \( i_{g,1} \rightarrow i_g \) is an isomorphism. This completes the proof.

Now we consider the short exact sequence (4).

**Proposition 3.2.** For any \( w \geq 4 \),

\[
H_2(m_{g,1})_w \cong (H_1(i_{g,1})_{m_{g,1}})_w.
\]

**Proof.** By Labute [17], \( \text{Gr} p_g \) is isomorphic to \( \mathcal{L}([1])/(\omega_0) \) where \( \omega_0 \in \wedge^2[1] \) denotes the symplectic class. On the other hand, \( \text{Gr} p_{g,1} \) is the central extension of \( \text{Gr} p_g \) by the Euler class \( \in H^2(\text{Gr} p_g) \). It follows from these facts that, we have isomorphisms

\[
H_2(\text{Gr} p_{g,1}) = H_2(\text{Gr} p_g)_{3} \cong H_2(T_1\Sigma_g) \cong [1]
\]

where all elements are detected by triple Massey products. If we combine this with Corollary (2.2) and consider the extension

\[
0 \rightarrow \text{Gr} p_{g,1} \rightarrow \text{Gr} t_{g,1} \rightarrow \text{Gr} t_g \rightarrow 0,
\]

then we can conclude that \( H_2(t_{g,1})_w = 0 \) for all \( w \geq 4 \).

Now we consider the 5-term exact sequence of the Lie algebra extension (4) which is given by

\[
H_2(\text{Gr} t_{g,1}) = H_2(\text{Gr} t_{g,1})_{2} \oplus H_2(\text{Gr} t_{g,1})_{3}
\]

\[
\rightarrow H_2(m_{g,1}) \rightarrow H_1(i_{g,1})_{m_{g,1}} \rightarrow H_1(\text{Gr} t_{g,1}) \xrightarrow{\cong} H_1(m_{g,1}).
\]

The result follows from this.

**Proposition 3.3.** The natural homomorphism

\[
H_2(m_{g,1})_w \rightarrow H_2(m_g)_w
\]
is an isomorphism for all \( w \geq 4 \).

**Proof.** By Proposition 3.2 and Proposition 2.3, the natural homomorphism \( H_2(m_{g,1})_w \to H_2(m_g)_w \) can be identified with the natural homomorphism

\[
(H_1(i_{g,1})_{m_{g,1}})_w \to (H_1(i_g)_{m_g})_w
\]

for any \( w \geq 4 \). Then it is easy to see that this is an isomorphism by Proposition 3.1. □

As mentioned in the beginning of this section, to prove Theorem 1.1, it is enough to show that

\[
H_2(m_g)_w = 0 \ (w = 4, 5, 6).
\]

By the above Proposition 3.3, this is equivalent to showing that

\[
H_2(m_{g,1})_w = 0 \ (w = 4, 5, 6).
\]

We consider this case of one boundary component which is best suited for our computer computation because of the following reason. By virtue of the Johnson homomorphisms (see [12][13][22]), the Lie algebra \( m_{g,1} \) can be embedded into the Lie algebra \( h_{g,1} \) consisting of all the symplectic derivations of the free Lie algebra generated by \( \Sigma g = H_1(\Sigma g, \mathbb{Q}) \) as a Lie subalgebra. More precisely, we have an embedding

\[
\tau_{g,1} = \bigoplus_{k=1}^{\infty} \tau_{g,1}(k) : m_{g,1} = \bigoplus_{k=1}^{\infty} m_{g,1}(k) \to h_{g,1} = \bigoplus_{k=1}^{\infty} h_{g,1}(k) \subset \bigoplus_{k=1}^{\infty} H_{Q}^{\otimes (k+2)}.
\]

Thus each graded piece \( m_{g,1}(k) \) can be identified with \( \text{Im} \tau_{g,1}(k) \), which is a submodule of \( H_{Q}^{\otimes (k+2)} \), so that we can make explicit computer computations on this space of tensors.

4. PROOF OF \( H_2(m_g)_4 = 0 \)

In this section, we prove the following.

**Proposition 4.1.** \( H_2(m_g)_4 \cong H_2(m_{g,1})_4 = 0 \).

Let \( g \) be either \( m_g \) or \( m_{g,1} \). Then by equality (3) we have

\[
H_2(g)_4 = \frac{\text{Ker} ((g(1) \otimes g(3)) \oplus \wedge^2 g(2) \to g(4))}{\text{Im} (\wedge^2 g(1) \otimes g(2) \to (g(1) \otimes g(3)) \oplus \wedge^2 g(2))}.
\]

Here the boundary operator

\[
\partial : \wedge^2 g(1) \otimes g(2) \to (g(1) \otimes g(3)) \oplus \wedge^2 g(2)
\]

is given by

\[
\wedge^2 g(1) \otimes g(2) \ni (u \wedge v) \otimes w \mapsto u \otimes [v, w] - v \otimes [u, w] - [u, v] \wedge w \in (g(1) \otimes g(3)) \oplus \wedge^2 g(2).
\]

In our paper [25], we determined \( m_g(k) \cong \text{Im} \tau_g(k) \), \( m_{g,1}(k) \cong \text{Im} \tau_{g,1}(k) \) for all \( k \leq 6 \) as in Table 1.

By using this and applying our techniques described in [24], we can determine the space of 2-cycles for the weight 4 homology group \( H_2(m_g)_4 \)

\[
Z_2(4) = \text{Ker} \left( (m_g(1) \otimes m_g(3)) \oplus \wedge^2 m_g(2) \to m_g(4) \right)
\]
This component of checking this depends on the shape of the Young diagram. The easiest case is the surjective computation which can be applied only to the case of computing and the multiplicity becomes larger. To treat these difficult cases, we use computer increases according as the number of boxes of the Young diagram becomes smaller.

We have computed the boundary operator \( \partial \) explicitly and checked that all the \( - \)-cycles for the weight \( 4 \) homology group \( H_2(m_g,1)_4 \) appear in \( \wedge^2[2^2] \) with multiplicity 1. This component is a boundary because the composition

\[
\wedge^2[1^3] \otimes [2^2] \overset{\partial}{\to} (1^3 \otimes [31^2]) \oplus \wedge^2[2^2] \to \wedge^2[2^2]
\]

is easily seen to be surjective and \([431]\) does not appear in \([1^3] \otimes [31^2]\). The difficulty increases according as the number of boxes of the Young diagram becomes smaller and the multiplicity becomes larger. To treat these difficult cases, we use computer computation which can be applied only to the case of computing \( H_2(m_g,1)_w \) rather than \( H_2(m_g)_w \). Fortunately these two groups are isomorphic to each other for any \( w \geq 4 \) by Proposition 3.3, although the computation of \( H_2(m_g,1)_w \) yields extra difficulty coming from the factor \( p_{g,1} \subset m_{g,1} \). More precisely, the multiplicity becomes considerably larger than the case of \( m_g \).

We have determined the space of 2-cycles for the weight 4 homology group \( H_2(m_{g,1})_4 \)

\[
\tilde{Z}_2(4) = \text{Ker} \left( (m_{g,1}(1) \otimes m_{g,1}(3)) \oplus \wedge^2 m_{g,1}(2) \to m_{g,1}(4) \right)
\]

as in Table 2. Thus we can write

\[ H_2(m_g)_4 \cong \text{Coker} \left( \wedge^2[1^3] \otimes [2^2] \overset{\partial}{\to} Z_2(4) \right). \]

Table 1. List of \( m_g(k), m_{g,1}(k) = p_{g,1}(k) \oplus m_g(k) \)

| \( k \) | \( p_{g,1}(k) \) | \( m_g(k) \) |
|---|---|---|
| 1 | 1 | 1^3 |
| 2 | \( [1^2] \oplus 0 \) | 2^2 |
| 3 | 21 | 31^2 |
| 4 | \([31][21]^2\) | \([42][31]^3\) \oplus [2^1] \oplus 31^2 |
| 5 | \([41][32][31^2][2^1][21^3] \oplus [1^3]\) | 51^2 \oplus [41][32][31^2][2^1][21^3] \oplus [2^1] |
| 6 | \([51][42][41^2][3^2][321] \oplus [2^1][31^2]\) \oplus [2^2][3^2][21^4] | 62[521][51^2][321][41^2][242^2][421^2][41^4] \oplus 2[3^2][321][3^2][2^1][21^4][1^6] |

Table 2. Sp-irreducible decomposition of \( Z_2(4) \)

| \( Z_2(4) \) | \( [431][42^2][421^2][232^1][1^4]\) \oplus [321^3][31^2] |
|---|---|
| \( m_{g}(1) \otimes m_{g}(3) \) | \([2^1][321][232^1][1^4]\) \oplus [321^3][31^2] |
| \( \Lambda^2 m_{g}(2) \) | \([321][3^2][321][1^4] \oplus [31^2][2^1][21^4]\) |
| \( m_{g}(4) \) | \([42][31^2][2^1][31^2]\) |
as in Table 3.

**Table 3. Sp-irreducible decomposition of \( \tilde{Z}_2(4) \) (extra terms)**

| \( Z_2(4) = Z_2(4) \oplus \) | \([41^2][3^2]3[321][2[31^3][2[2^12][21^4][3[3][4][2^2][2][4][1^2]
| \( m_{g,1}(1) \otimes m_{g,1}(3) \) = | \([321][31^4][2^12][21^4][3[31^2][21^4][2^2][2^2][21^4][1^2][2][1^2]
| \( (m_{g,1}(1) \otimes m_{g,1}(3)) \oplus \) | \([41^2][321][31^3][31][21^2][31][2^2][21^2][1^2][2][1^2]
| \( \wedge^2 m_{g,1}(2) = \wedge^2 m_{g,1}(2) \oplus \) | \([3^2][321][2^21^2][31][2^2][21^2][1^2][2^2][21^2][2][1^2]
| \( m_{g,1}(4) = m_{g,1}(4) \oplus \) | \([31][21^2][2]

Thus we can write

\[
H_2(m_g) \cong H_2(m_{g,1}) \cong \text{Coker} \left( \wedge^2([1^3] \oplus [1]) \otimes ([2^2] \oplus [1^2] \oplus [0]) \xrightarrow{\partial} \tilde{Z}_2(4) \right).
\]

As an example of difficult cases, we pick up \([21^2]\). The multiplicity of \([21^2]\) in \( Z_2(4) \) is 2 while that in \( \tilde{Z}_2(4) \) is 7 so that there are 5 extra terms (see Table 2 and Table 3). We have checked that all of these \([21^2]\) components are hit by the boundary operator

\[
\partial : \wedge^2 m_{g,1}(1) \otimes m_{g,1}(2) \to \tilde{Z}_2(4) \subset (m_{g,1}(1) \otimes m_{g,1}(3)) \oplus \wedge^2 m_{g,1}(2).
\]

More precisely, we consider \((m_{g,1}(1) \otimes m_{g,1}(3)) \oplus \wedge^2 m_{g,1}(2)\) to be a subspace of \( H_Q \otimes H_Q \) by the natural embeddings

\[
(m_{g,1}(1) \otimes m_{g,1}(3)) \oplus \wedge^2 m_{g,1}(2) \subset (h_{g,1}(1) \otimes h_{g,1}(3)) \oplus \wedge^2 h_{g,1}(2) \subset (H_Q^{\otimes 3} \otimes H_Q^{\otimes 5}) \oplus \wedge^2 H_Q^{\otimes 4} \subset H_Q^{\otimes 8} \oplus H_Q^{\otimes 8}.
\]

Then we constructed 7 Sp-projections \( D_i : H_Q^{\otimes 8} \oplus H_Q^{\otimes 8} \to [21^2] \) \((i = 1, \ldots, 7)\) which detect the 7 copies of \([21^2]\) in \( \tilde{Z}_2(4) \). We call these Sp-projections *detectors*. Next, we made a computer program yielding vectors of \( \wedge^2 m_{g,1}(1) \otimes m_{g,1}(2) \) in a systematic way. Finally, we computed the boundary operator \( \partial \) on these vectors and then apply the 7 detectors, again systematically. In each step, we computed the rank of the relevant vector space until it reached the maximal value 7. This is the process of proving that the \([21^2]\) component of \( H_2(m_{g,1}) \) vanishes.

In this way, we checked that all the Sp-irreducible components of \( \tilde{Z}_2(4) \) are boundaries. This finishes the proof of \( H_2(m_{g,1}) = 0 \) and hence \( H_2(m_g) = 0 \).

**Remark 4.2.** If an Sp-irreducible component \( \lambda \subset \tilde{Z}_2(4) \) does not appear in \( Z_2(4) \), e.g. the components of \([1^4]\) and \([1^2]\), we do not need to compute that case. This is because \( H_2(m_{g,1}) \otimes \lambda \cong H_2(m_g) \otimes \lambda = 0 \) by Proposition 3.3. This remark applies to the cases of all \( w \geq 4 \).

5. **Proof of \( H_2(m_g) = 0 \)**

In this section, we prove the following.

**Proposition 5.1.** \( H_2(m_g) = 0 \).
Let $g$ be either $m_g$ or $m_g,1$. Then by equality (3) we have

$$H_2(g)_5 = \frac{\text{Ker } ((g(1) \otimes g(4)) \oplus (g(2) \otimes g(3)) \rightarrow g(5))}{\text{Im } ((\wedge^2 g(1) \otimes g(3)) \oplus (g(1) \otimes \wedge^2 g(2))) \rightarrow (g(1) \otimes g(4)) \oplus (g(2) \otimes g(3))}.$$ 

Here the boundary operator

$$(7) \quad \partial : (\wedge^2 g(1) \otimes g(3)) \oplus (g(1) \otimes \wedge^2 g(2)) \rightarrow (g(1) \otimes g(4)) \oplus (g(2) \otimes g(3))$$

is given by

$$\wedge^2 g(1) \otimes g(3) \ni (u \wedge v) \otimes w \mapsto$$

$$u \otimes [v, w] - v \otimes [u, w] - [u, v] \otimes w \in (g(1) \otimes g(4)) \oplus (g(2) \otimes g(3))$$

$$g(1) \otimes \wedge^2 g(2) \ni u \otimes (v \wedge w) \mapsto$$

$$u \otimes [v, w] - v \otimes [u, w] + w \otimes [u, v] \in (g(1) \otimes g(4)) \oplus (g(2) \otimes g(3)).$$

As in the preceding section, by using Table 1 and applying our techniques described in [24], we can determine the space of 2-cycles for the weight 5 homology group $H_2(m_g)_5$

$$Z_2(5) = \text{Ker } ((m_g(1) \otimes m_g(4)) \oplus (m_g(2) \otimes m_g(3)) \rightarrow m_g(5))$$

as in Table 4

| Table 4. Sp-irreducible decomposition of $Z_2(5)$ |
|-----------------------------------------------|
| $Z_2(5)$                                      |
| $m_g(1) \otimes m_g(4)$                       |
| $m_g(2) \otimes m_g(3)$                       |
| $m_g(5)$                                      |
| $2[531]2[521]^2[432]2[421]^2[4]3[421]^4[321]^2[41]4[31]2[321]^4[312]^2[21]^3$ | $531][521]^2[432][421]^2[421]^2[41]4[321]^2[321]^2[31]^2[21]^3$ |
| $m_g(5)$                                      |
| $51^2[431]^4[421]^2[421]^2[41]^2[32]^2[31]^2[21]^3$ | $41][32][31][21]^3[21]^3[21]^3[21]^3$ |

Thus we can write

$$H_2(m_g)_5 \cong \text{Coker } \left( (\wedge^2 [1]^3 \otimes [3]^2] \oplus ([1]^3 \otimes \wedge^2 [2]^2]) \xrightarrow{\partial} Z_2(5) \right).$$

We have computed the boundary operator (7) explicitly and checked that all the 2-cycles (34-types of Young diagrams with multiplicities) are boundaries.

We have also determined the space of 2-cycles for the weight 5 homology group $H_2(m_g,1)_5$

$$\tilde{Z}_2(5) = \text{Ker } ((m_g,1(1) \otimes m_g,1(4)) \oplus (m_g,1(2) \otimes m_g,1(3)) \rightarrow m_g,1(5))$$

as in Table 5

Thus we can write

$$H_2(m_g)_5 \cong H_2(m_g,1)_5 \cong \text{Coker } \left( (\wedge^2([1]^3] + [1]) \otimes (3[1]^2] + [21])) \oplus \cdot \cdot \cdot \rightarrow \tilde{Z}_2(5) \right).$$
TABLE 5. \( \text{Sp-irreducible decomposition of } \tilde{Z}_2(5) \) (extra terms)

| \( Z_2(5) = Z_2(5) \oplus \) | 52 | 2 | 43 | 3 | [421] 3 | [41] 8 | [3] 2 | 421 | [41] 5 | [3] 8 | 2 | 4 | [3] 8 | [21] 4 | [1] 4 |
|--------------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|
| \( m_{g,1}(1) \otimes m_{g,1}(4) = \) | 52 | [43] 2 | 421 | 2 | 41 | 3 | 2 | 32 | 3 | 32 | 2 | 3 | 2 | 2 | 4 | 2 | 1 | 2 | 2 | 4 | 2 | 1 |
| | 5 | [4] 1 | [3] 8 | [3] 3 | 2 | 4 | 2 | 1 | 2 | 2 | 3 | 2 | 2 | 3 | 2 | 4 | 2 | 2 | 3 | 2 | 3 | 2 | 4 |
| \( m_{g,1}(2) \otimes m_{g,1}(3) = \) | [43] 2 | 421 | 2 | 41 | 3 | 2 | 32 | 3 | 32 | 2 | 3 | 2 | 2 | 4 | 2 | 2 | 3 | 2 | 4 | 2 | 2 | 3 |
| | 1 | [2] 4 | 3 | 2 | 6 | 3 | 3 | 2 | 3 | 4 | 1 | 3 | 6 | 3 | 2 | 1 | 2 | 1 | 3 | 2 | 1 |
| \( m_{g,1}(5) = m_{g,1}(5) \oplus \) | 41 | [3] 2 | [3] 8 | [21] 4 | [1] 3 | [2] 2 | [1] 4 | [1] 1 |

In this way, we checked that all the \( \text{Sp-irreducible components of } \tilde{Z}_2(5) \) are boundaries. This finishes the proof of \( H_2(m_{g,1})_5 = 0 \) and hence \( H_2(m_g)_5 = 0 \).

6. PROOF OF \( H_2(m_g)_6 = 0 \)

In this section, we prove the following.

**Proposition 6.1.** \( H_2(m_g)_6 \cong H_2(m_{g,1})_6 = 0 \).

Let \( g \) be either \( m_g \) or \( m_{g,1} \). Then by equality (3) we have

\[
H_2(g)_6 = \frac{Z^g_2(6)}{B^g_2(6)}
\]

where

\[
Z^g_2(6) = \text{Ker} \left( (g(1) \otimes g(5)) \oplus (g(2) \otimes g(4)) \oplus \wedge^2 g(3) \to g(6) \right)
\]

and

\[
B^g_2(6) = \text{Im}((\wedge^2 g(1) \otimes g(4)) \oplus (g(1) \otimes g(2) \otimes g(3)) \oplus \wedge^3 g(2))
\]

\[
\partial : (\wedge^2 g(1) \otimes g(4)) \oplus (g(1) \otimes g(2) \otimes g(3)) \oplus \wedge^3 g(2)
\]

\[
\to (g(1) \otimes g(5)) \oplus (g(2) \otimes g(4)) \oplus \wedge^2 g(3)
\]

is given by

\[
\wedge^2 g(1) \otimes g(4) \ni (u \wedge v) \otimes w \mapsto (u \otimes [v, w] - v \otimes [u, w], -[u, v] \otimes w, 0) \in (g(1) \otimes g(5)) \oplus (g(2) \otimes g(4)) \oplus \wedge^2 g(3)
\]

\[
(g(1) \otimes g(2) \otimes g(3)) \ni u \otimes v \otimes w \mapsto (u \otimes [v, w], -v \otimes [u, w], -[u, v] \wedge w) \in (g(1) \otimes g(5)) \oplus (g(2) \otimes g(4)) \oplus \wedge^2 g(3)
\]

\[
\wedge^3 g(2) \ni u \otimes v \wedge w \mapsto (0, u \otimes [v, w] + v \otimes [u, w] + w \otimes [u, v], 0) \in (g(1) \otimes g(5)) \oplus (g(2) \otimes g(4)) \oplus \wedge^2 g(3).
\]

As in the preceding two sections, by using Table 4 and applying our techniques described in [24], we can determine the space of 2-cycles for the weight 6 homology
group $H_2(m_6)_6$

(9) \[ Z_2(6) = \text{Ker} \left( (m_6(1) \otimes m_6(5)) \oplus (m_6(2) \otimes m_6(4)) \oplus \wedge^2 m_6(3) \xrightarrow{\partial} m_6(6) \right) \]

as in Table 6.

| $Z_2(6)$ | $m_6(1) \otimes m_6(5)$ | $m_6(2) \otimes m_6(4)$ | $\wedge^2 m_6(3)$ | $m_6(6)$ |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| $[64][631]2[62^2][262^2][61^4][541][3532][3531^2][452^1][452^1^2][251^2][34^2^2]$ | $[62^2][61^4][532][531^2][25^2^2][251^2][51^2][4^2^2][4^2^2^2][3432][3431^2]$ | $[64][631][62^2^1][551^2][531^2][52^1^2][52^1^2^2][52^1^2^3][52^1^2^4][51^2^2][4^2^2^2][4^2^2^2^2][3432][3431^2]$ | $[62^1][541][532][531^2][52^1^2][52^1^2^2][52^1^2^3][52^1^2^4][51^2^2][4^2^2^2][4^2^2^2^2][3432][3431^2]$ | $[62][521][51^2][431^4][431^2^1][431^3^1][42^1^2][42^1^2^2][42^1^2^3][42^1^2^4][32^1^2][32^1^2^2][32^1^2^3][32^1^2^4][21^2^1][21^2^2][21^2^3][21^2^4][21^2^5][21^2^6]$ |

Thus we can write

$H_2(m_6)_6 \cong \text{Coker} \left( (\wedge^2[1^3] \otimes ([42] + [31^3] + [2^2] + [31] + [2^1])) \oplus ([1^3] \otimes [2^2] \otimes [31^2]) \oplus \wedge^2[2^2] \xrightarrow{\partial} Z_2(6) \right)$.

We have computed the boundary operator $[\partial]$ explicitly and checked that all the 2-cycles (67-types of Young diagrams with multiplicities) are boundaries.

Here we pick up two examples to illustrate our method of computation for the present case of weight 6. One is the trivial representation. There exists a unique copy of trivial representation [0] in $Z_2(6)$, as shown below, and if this were not a boundary, then it would give a non-trivial element in $H_2(m_6)_6$ and hence in $i_6(6)^{Sp}$. However, we can show that it is a boundary as follows. First, recall that there appears a trivial representation in the Sp-irreducible decomposition of the tensor product $\lambda \otimes \mu$ of two irreducible representations $\lambda$ and $\mu$ if and only if $\lambda = \mu$ and in that case it appears
with multiplicity 1. Keeping this fact in mind as well as an additional fact that the trivial representation in $[31^2] \otimes [31^2]$ appears in $\wedge^2[31^2] \subset [31^2] \otimes [31^2]$, if we compare the equation (9) and Table I we find that

\[(m_g(1) \otimes m_g(5)) \oplus (m_g(2) \otimes m_g(4)) \oplus \wedge^2 m_g(3)^{\text{Sp}} \cong \mathbb{Q}^6_2.\]

On the other hand, the boundary operator $\partial$ in equation (9) is surjective and $m_g(6)$ contains a unique copy $[0]$. Therefore we can conclude that $Z_2(6)^{\text{Sp}} \cong \mathbb{Q}$. Now the restriction, to the subspace $m_g(1) \otimes m_g(2) \otimes m_g(3)$

\[\partial : m_g(1) \otimes m_g(2) \otimes m_g(3) \rightarrow \wedge^2 m_g(3)\]

of the boundary operator (8) for the case $g = m_g$ is surjective because the bracket $m_g(1) \otimes m_g(2) \rightarrow m_g(3)$ is so. Hence the unique copy of $[0]$ in $Z_2(6)$ is a boundary as claimed. Thus, in this case, we do not need computer computation.

Next, we consider the case of $[21^2]$-component of $Z_2(6)$. Here we need computer computation so that we have to consider the case of one boundary component. We have determined the space of 2-cycles for the weight 6 homology group $H_2(m_g,1)_6$

\[\tilde{Z}_2(6) = \ker ((m_{g,1}(1) \otimes m_{g,1}(5)) \oplus (m_{g,1}(2) \otimes m_{g,1}(4)) \oplus \wedge^2 m_{g,1}(3) \rightarrow m_{g,1}(6))\]

as in Table [7]

| Table 7. Sp-irreducible decomposition of $\tilde{Z}_2(6)$ (extra terms) |
|-------------------------------------------------------------|
| $Z_2(6) = Z_2(6)^{\oplus}$                                |
| $m_{g,1}(1) \otimes m_{g,1}(5) = (m_{g,1}(2) \otimes m_{g,1}(5))^{\oplus}$ |
| $\wedge^2 m_{g,1}(3) = \wedge^2 m_{g}(3)^{\oplus}$      |
| $m_{g,1}(6) = m_{g}(6)^{\oplus}$                         |

Thus we can write

\[H_2(m_g)_6 \cong H_2(m_{g,1})_6 \cong \text{Coker} \left( (\wedge^2 [1^3] + [1]) \otimes ([42] + [31^3] + [2^3] + 2[31] + [21^2] + 2[2]) \right) \oplus \left( ([3^1] + [1]) \otimes ([2^2] + [1^2] + [0]) \otimes ([31^2] + [21]) \right) \oplus \wedge^3 ([2^2] + [1^2] + [0]) \rightarrow \tilde{Z}_6(5).\]
The multiplicity of $[21^2]$ in $Z_2(6)$ is 17 while that in $\tilde{Z}_2(6)$ is 56 because there are 39 extra terms (see Table 6 and Table 7). We have checked that all of these $[21^2]$ components are hit by the boundary operator

$$
\partial : (\wedge^2 m_{g,1}(1) \otimes m_{g,1}(4)) + (m_{g,1}(1) \otimes m_{g,1}(2) \otimes m_{g,1}(3)) + \wedge^3 m_{g,1}(2)
$$

$$\rightarrow (m_{g,1}(1) \otimes m_{g,1}(5)) + (m_{g,1}(2) \otimes m_{g,1}(4)) + \wedge^2 m_{g,1}(3).$$

More precisely, we consider the target of the above boundary operator to be a subspace of $H^3 Q^6 \oplus H^3 Q^5 \oplus H^3 Q^10$ by the natural embeddings

$$m_{g,1}(1) \otimes m_{g,1}(5) \subset H^3 Q^6 \otimes H^5 Q^7 = H^6 Q^10$$

$$m_{g,1}(2) \otimes m_{g,1}(4) \subset H^3 Q^4 \otimes H^6 Q^6 = H^6 Q^10$$

$$\wedge^2 m_{g,1}(3) \subset H^5 Q^5 \otimes H^5 Q^5 = H^6 Q^10.$$

Then we constructed 56 $Sp$-projections $E_i : H^3 Q^6 \oplus H^3 Q^5 \oplus H^3 Q^10 \rightarrow [21^2]$ ($i = 1, \ldots, 56$) which detect the 56 copies of $[21^2]$ in $\tilde{Z}_2(6)$. We call these $Sp$-projections detectors. Next, we made a computer program yielding vectors of the domain of the boundary operator

$$\wedge^2 m_{g,1}(1) \otimes m_{g,1}(4) + (m_{g,1}(1) \otimes m_{g,1}(2) \otimes m_{g,1}(3)) + \wedge^3 m_{g,1}(2)$$

in a systematic way. Finally, we computed the boundary operator $\partial$ on these vectors and then apply the 56 detectors, again systematically. In each step, we computed the rank of the relevant vector space until it reached the maximal value 56. This is the process of proving that the $[21^2]$ component of $H_2(m_{g,1})_6$ vanishes.

In this way, we checked that all the $Sp$-irreducible components of $\tilde{Z}_2(6)$ are boundaries. This finishes the proof of $H_2(m_{g,1})_6 = 0$ and hence $H_2(m_{g,1})_6 = 0$.

**Remark 6.2.** The size of our computer computation grows very rapidly with respect to weights and, in particular, the weight 6 case is approximately 1000 times as large as the weight 4 case.

### 7. Proofs of the main results

**Proof of Theorem 1.1** By Corollary 2.4, we have $i_g(4) \cong H_2(m_g)_4$. On the other hand, we have $H_2(m_g)_4 = 0$ by Proposition 4.1. Hence, we can conclude that $i_g(4) = 0$, namely $t_g(4) \cong m_g(4)$. Then, if we combine Proposition 2.5 with Proposition 5.1 and Proposition 6.1, we can conclude that $i_g(5) = i_g(6) = 0$ so that $t_g(5) \cong m_g(5)$ and $t_g(6) \cong m_g(6)$. This finishes the proof.

**Proof of Corollary 1.2** Observe first that, for any $k, \ell$ with $1 \leq k \leq \ell$, the quotient group $\mathcal{M}_g(k)/\mathcal{I}_g(\ell)$ is a finitely generated nilpotent group because it is a subgroup of $\mathcal{I}_g/\mathcal{I}_g(\ell)$ which is finitely generated by Johnson [15] and nilpotent. Hence we can consider the rational form $(\mathcal{M}_g(k)/\mathcal{I}_g(\ell)) \otimes \mathbb{Q}$ of $\mathcal{M}_g(k)/\mathcal{I}_g(\ell)$.

Now the case $k = 3$ is a direct consequence of Theorem 2.1 combined with a result in [20] because of the following reason. Since $\mathcal{I}_g(2)$ is a finite index subgroup of $\mathcal{M}_g(2) = \mathcal{K}_g$ by Johnson, we have a short exact sequence

$$0 \to (\mathcal{M}_g(3)/\mathcal{I}_g(3)) \otimes \mathbb{Q} \to (\mathcal{I}_g(2)/\mathcal{I}_g(3)) \otimes \mathbb{Q} \to (\mathcal{M}_g(2)/\mathcal{M}_g(3)) \otimes \mathbb{Q} \to 0.$$
Here \( (\mathcal{I}_g(2)/\mathcal{I}_g(3)) \otimes \mathbb{Q} = t_g(2) \) and \((\mathcal{M}_g(2)/\mathcal{M}_g(3)) \otimes \mathbb{Q} = m_g(2)\) by definition. Hence we can conclude \((\mathcal{M}_g(3)/\mathcal{I}_g(3)) \otimes \mathbb{Q} \cong i_g(2) \cong \mathbb{Q}\). The result follows from this because we know that the homomorphism \(d_1 : \mathcal{M}_g(3) \to \mathbb{Q}\) is non-trivial whereas its restriction to \(\mathcal{I}_g(3)\) is trivial as shown in \[20\].

Next, we consider the cases \(k \geq 4\). We recall here how the non-triviality of the homomorphism \(d_1 : \mathcal{M}_g(k) \to \mathbb{Q}\) for all \(k \geq 4\) follows immediately from Hain’s result that the homomorphism \(\mathcal{I}_g(3)\) is surjective for any \(k\). Assume that \(d_1 : \mathcal{M}_g(k) \to \mathbb{Q}\) were trivial for some \(k \geq 4\) and take \(m\) to be the smallest one. Then consider the homomorphism

\[
t_g(m - 1) = \mathcal{I}_g(m - 1)/\mathcal{I}_g(m) \to \mathcal{M}_g(m - 1) = \mathcal{M}_g(m - 1)/\mathcal{M}_g(m).
\]

By the assumption, the non-trivial homomorphism \(d_1 : \mathcal{M}_g(m - 1) \to \mathbb{Q}\) factors through \(d_1 : \mathcal{M}_g(m - 1)/\mathcal{M}_g(m) \to \mathbb{Q}\). On the other hand, we know that the restriction of \(d_1\) on \(\mathcal{I}_g(k)\) is trivial for all \(k \geq 3\). We can now conclude that the above homomorphism \(10\) is not surjective which is a contradiction.

Now the case \(k = 4\) follows from the fact \(t_g(3) \cong m_g(3)\) proved in \[23\] as follows. We have the following two exact sequences.

\[0 \to t_g(3) = (\mathcal{I}_g(3)/\mathcal{I}_g(4)) \otimes \mathbb{Q} \to (\mathcal{M}_g(3)/\mathcal{I}_g(4)) \otimes \mathbb{Q} \to (\mathcal{M}_g(3)/\mathcal{I}_g(3)) \otimes \mathbb{Q} \to 0,\]
\[0 \to (\mathcal{M}_g(4)/\mathcal{I}_g(4)) \otimes \mathbb{Q} \to (\mathcal{M}_g(3)/\mathcal{I}_g(4)) \otimes \mathbb{Q} \to \mathcal{m}_g(3) = (\mathcal{M}_g(3)/\mathcal{M}_g(4)) \otimes \mathbb{Q} \to 0.\]

By the case \(k = 3\) above, we have \((\mathcal{M}_g(3)/\mathcal{I}_g(3)) \otimes \mathbb{Q} \cong \mathbb{Q}\). If we put this to the first exact sequence, we obtain

\[\text{rank} (\mathcal{M}_g(3)/\mathcal{I}_g(4)) \otimes \mathbb{Q} = \text{dim} t_g(3) + 1.\]

Here \(\text{rank} (\mathcal{M}_g(3)/\mathcal{I}_g(4)) \otimes \mathbb{Q}\) means the rank of the nilpotent group \((\mathcal{M}_g(3)/\mathcal{I}_g(4)) \otimes \mathbb{Q}\) over \(\mathbb{Q}\). On the other hand, from the second exact sequence, we have

\[\text{rank} (\mathcal{M}_g(3)/\mathcal{I}_g(4)) \otimes \mathbb{Q} = \text{dim} m_g(3) + \text{rank} (\mathcal{M}_g(4)/\mathcal{I}_g(4)) \otimes \mathbb{Q}.\]

Since \(t_g(3) \cong m_g(3)\), we can conclude that

\[\text{rank} (\mathcal{M}_g(4)/\mathcal{I}_g(4)) \otimes \mathbb{Q} = 1\]

finishing the proof of the case \(k = 4\).

The remaining cases \(k = 5, 6, 7\) follow from similar arguments as above by using Theorem \[1.1\].

Next we prove Theorem \[1.3\] Corollary \[1.4\], Theorem \[1.6\] and its refinements. For that, we recall a few facts about the relation between the Torelli group and homology spheres. Let \(S^1 \times D^2\) denote a framed solid torus and let \(H_g = \natural g(S^1 \times D^2)\) (boundary connected sum of \(g\)-copies of \(S^1 \times D^2\)) denote a handle body of genus \(g\). We identify \(\partial H_g\) with \(\Sigma_g\) equipped with a system of \(g\) meridians and longitudes. Let \(i_g \in \mathcal{M}_g\) be the mapping class which exchanges each meridian and longitude curves so that the manifold \(H_g \cup_{i_g} -H_g\) obtained by identifying the boundaries of \(H_g\) and \(-H_g\) by \(i_g\) is \(S^3\).

Now for each element \(\varphi \in \mathcal{I}_g\), we consider the manifold \(M_\varphi = H_g \cup_{i_\varphi} -H_g\) which is a homology \(3\)-sphere. It was shown in \[20\] that any homology sphere can be expressed
as $M_\varphi$ for some $\varphi \in \mathcal{K}_g = \mathcal{M}_g(2)$ and Pitsch [28] further proved that $\varphi$ can be taken in $\mathcal{M}_g(3)$.

Now we recall the relation between the Casson invariant and the structure of the Torelli group as revealed in [20] briefly (see [21] for further results). We defined $\lambda^* : \mathcal{K}_g \to \mathbb{Z}$ by setting $\lambda^*(\varphi) = \lambda(M_\varphi)$ $(\varphi \in \mathcal{K}_g)$ and then proved that it is a homomorphism.

On the other hand, we have the following two abelian quotients of the group $\mathcal{K}_g$

$$\tau_g(2) : \mathcal{K}_g \to \mathfrak{h}_g(2)$$
$$d_1 : \mathcal{K}_g \to \mathbb{Z}$$

where the first one is the second Johnson homomorphism and the second one is constructed in the above cited paper. Then we have the following.

**Theorem 7.1 ([20]).** The homomorphism $\lambda^* : \mathcal{K}_g \to \mathbb{Z}$ is expressed as

$$\lambda^* = \frac{1}{24}d_1 + \bar{\tau}_g(2)$$

where $\bar{\tau}_g(2)$ denotes a certain quotient of the second Johnson homomorphism. Furthermore, the restriction of $\lambda^*$ to the subgroup $\mathcal{M}_g(3) \subset \mathcal{K}_g$ is given by

$$\lambda^* = \frac{1}{24}d_1 : \mathcal{M}_g(3) \to \mathbb{Z}.$$

Ohtsuki initiated a theory of finite type invariants for homology $3$-spheres in [27] and in [26] he constructed a series of such invariants $\lambda_k$ ($k = 1, 2, \ldots$) the first one being (6 times) the Casson invariant. They are now called the Ohtsuki invariants. Garoufalidis and Levine [7] studied the relation between the finite type invariants of homology spheres and the structure of the Torelli group, particularly its lower central series. This work extended the case of the Casson invariant mentioned above extensively.

Now, let $v$ be an invariant of homology spheres of finite type $k$. Then we can define a mapping

$$v^* : \mathcal{I}_g \to \mathbb{Q}$$

by setting $v^*(\varphi) = v(M_\varphi)$. By a result of Garoufalidis and Levine [7], it vanishes on $\mathcal{I}_g(k+1)$. On the other hand, the following result is known.

**Proposition 7.2 (Levine [18, Lemma 5.5]).** Let $v$ be an invariant of homology spheres of finite type $k$. Then for any $\varphi \in \mathcal{I}_g(k_1), \psi \in \mathcal{I}_g(k_2)$ with $k_1 + k_2 > k$, the equality

$$v(M_{\varphi \psi}) = v(M_\varphi) + v(M_\psi)$$

holds.

As a direct corollary, we obtain the following.

**Corollary 7.3.** Let $v$ be an invariant of homology spheres of finite type $k$. Then the mapping

$$v^* : \mathcal{I}_g/\mathcal{I}_g(k+1) \to \mathbb{Q}$$

is induced and its restriction to $\mathcal{I}_g(m)/\mathcal{I}_g(k+1)$

$$v^* : \mathcal{I}_g(m)/\mathcal{I}_g(k+1) \to \mathbb{Q}$$
is a homomorphism if $2m > k$.

Since the Ohtsuki invariant $\lambda_k$ is of finite type $2k$, it induces a homomorphism

$$\lambda_k^*: \mathcal{I}_g(k + 1)/\mathcal{I}_g(2k + 1) \to \mathbb{Q}.$$

**Remark 7.4.** If we put $k = 1$ here, then we obtain that

$$\lambda^*: \mathcal{I}_g(2)/\mathcal{I}_g(3) \to \mathbb{Q}$$

is a homomorphism. However, this follows from a fact already proved in [20] because $\mathcal{I}_g(2)$ is a finite index subgroup of $\mathcal{K}_g$ by Johnson as mentioned above.

In view of Corollary 7.3, it should be meaningful to consider abelian quotients of the group $\mathcal{I}_g(k)$. Here we recall known abelian quotients of a larger group $\mathcal{M}_g(k) \supset \mathcal{I}_g(k)$. First, we have the $k$-th Johnson homomorphism

$$\tau_g(k): \mathcal{M}_g(k) \to \mathfrak{h}_g(k)$$

and secondly we have its lift

$$(11) \quad \tilde{\tau}_g(k): \mathcal{M}_g(k) \to \bigoplus_{i=k}^{2k-1} \mathfrak{h}_g(i)$$

defined as follows. In [22], a homomorphism

$$\tilde{\tau}_{g,1}(k): \mathcal{M}_{g,1}(k) \to H_3(N_k(\pi_1\Sigma^0_g)) \quad (\Sigma^0_g = \Sigma_g \setminus \text{Int } D^2)$$

was defined which is a refinement of $\tau_{g,1}(k)$. Heap [10] studied this homomorphism by giving a geometric construction of it and, in particular, proved that $\text{Ker } \tilde{\tau}_{g,1}(k) = \mathcal{M}_{g,1}(2k)$. Comparing his result with the description of $H_3(N_k(\pi_1\Sigma^0_g))$ by Igusa-Orr [11], we have an embedding

$$\tilde{\tau}_{g,1}(k): \mathcal{M}_{g,1}(k)/\mathcal{M}_{g,1}(2k) \hookrightarrow \bigoplus_{i=k}^{2k-1} \mathfrak{h}_{g,1}(i),$$

though the direct sum decomposition is not canonical except for the lowest part $i = k$. Massuyeau [19] (see also Habiro-Massuyeau [8]) further studied this homomorphism by an infinitesimal approach. The above homomorphism (11) is obtained from this by passing from $\mathcal{M}_{g,1}, \mathfrak{h}_{g,1}$ to $\mathcal{M}_g, \mathfrak{h}_g$ along the line described in [23][25]. Here, if we add the homomorphism $d_1$ and taking modulo smaller subgroup $\mathcal{I}_g(2k) \subset \mathcal{M}_g(2k)$, then we obtain a homomorphism

$$(12) \quad (d_1, \tilde{\tau}_g(k)): \mathcal{M}_g(k)/\mathcal{I}_g(2k) \to \mathbb{Z} \oplus \bigoplus_{i=k}^{2k-1} \mathfrak{h}_g(i) \quad (k \geq 2)$$

which is conjecturally an embedding modulo torsion elements (see Remark 1.8). As far as the authors understand, this homomorphism $(d_1, \tilde{\tau}_g(k))$ gives the known largest free abelian quotient of the group $\mathcal{M}_g(k)$ for $k \geq 2$. The case $k = 2$ gives an abelian quotient

$$(d_1, \tilde{\tau}_g(2)): \mathcal{M}_g(2) = \mathcal{K}_g \to \mathbb{Z} \oplus \mathfrak{h}_g(2) \oplus \mathfrak{h}_g(3) = \mathbb{Z} \oplus [2^2] \oplus [31^2]$$
which is rationally surjective.

Before considering the cases of $k = 3, 4$, here we prove Theorem 1.3 which is equivalent to the statement that the above homomorphism gives the whole rational abelianization of $\mathcal{K}_g$.

**Proof of Theorem 1.3** Dimca, Hain and Papadima [4, Theorem C] proved that there exists an isomorphism

$$H_1(\mathcal{K}_g; \mathbb{Q}) \cong H_1(\langle \text{Gr}_g \rangle).$$

Since the projection to the degree 1 part $\text{Gr}_g \to \text{Gr}_g(1) = [1^3]$ gives the abelianization of $\text{Gr}_g$, we have

$$[\text{Gr}_g, \text{Gr}_g] = \bigoplus_{k=2}^{\infty} \text{Gr}_g(k).$$

Hence we can write

$$H_1(\langle \text{Gr}_g \rangle) = \bigoplus_{k=2}^{\infty} H_1(\langle \text{Gr}_g \rangle)_k.$$

The results of [9] and [23] imply that

$$H_1(\langle \text{Gr}_g \rangle)_2 = t_g(2) \cong \mathbb{Q} \oplus [2^2]$$

$$H_1(\langle \text{Gr}_g \rangle)_3 = t_g(3) \cong m_g(3) \cong [31^2].$$

By the definition of the first homology group of Lie algebras, we can write

$$H_1(\langle \text{Gr}_g \rangle)_4 = \text{Coker} \left( \bigwedge^2 t_g(2) \xrightarrow{\cdot \cdot} t_g(4) \right).$$

On the other hand, it was proved in [30] that the homomorphism

$$[\cdot, \cdot] : \bigwedge^2 m_g(2) \to m_g(4)$$

is surjective. Here we use the case $k = 4$ of our main Theorem 1.1, namely the fact that $t_g(4) \cong m_g(4)$. This is the key point of our proof of Theorem 1.3. Then we can conclude that the homomorphism $[\cdot, \cdot] : \bigwedge^2 t_g(2) \to t_g(4)$ is also surjective. It follows that

$$H_1(\langle \text{Gr}_g \rangle)_4 = 0.$$

To finish the proof, it remains to prove that

$$H_1(\langle \text{Gr}_g \rangle)_k = 0$$

for all $k \geq 5$. In other words, we have to prove that the homomorphism

$$(13) \quad \bigoplus_{i+j=k, i,j > 1} t_g(i) \otimes t_g(j) \to t_g(k)$$

induced by the bracket operation is surjective. Note that the above homomorphism (13) is surjective for $k = 4$ as already mentioned above. We use induction on $k \geq 4$. Assuming that (13) is surjective for $k \geq 4$ and we prove the surjectivity for $k + 1$. Since the Torelli Lie algebra is generated by degree 1 part $t_g(1)$, if we delete the condition $i, j > 1$ in the left hand side of (13), then it is surjective. Hence it is enough to show that any element of the form

$$[\alpha, \xi] \in t_g(k+1) \quad (\alpha \in t_g(1), \xi \in t_g(k))$$
is contained in the image of (13). By the induction assumption, we can write
\[
\xi = \sum_s [\beta_s, \gamma_s] \quad (\beta_s, \gamma_s \in t_g(k_s), 2 \leq k_s \leq k - 2).
\]
Then, by the Jacobi identity, we have
\[
[\alpha, \xi] = \sum_s [\alpha, [\beta_s, \gamma_s]] = -\sum_s ([\beta_s, [\gamma_s, \alpha]] + [\gamma_s, [\alpha, \beta_s]]).
\]
This element is clearly contained in the image of (13) proving that it is surjective for \(k + 1\). This completes the proof. \(\square\)

**Remark 7.5.** Here we mention the relation between our computation and the statement
\[
H_1(K_g; \mathbb{Q}) \cong \mathbb{Q} \oplus \bigoplus_{k=0}^{\infty} \text{Coker } q_k
\]
\[
q_k : \text{Sym}^{k-1}[1^3] \otimes \wedge^3[1^3] \to \text{Sym}^k[1^3] \otimes [2^2]
\]
given by Dimca, Hain and Papadima [4, Theorem B]. The factor \(\mathbb{Q}\) is detected by \(d_1\) and \(\text{Coker } q_0 = [2^2]\) is detected by the second Johnson homomorphism \(\tau_g(2)\). These two summands correspond to \(t_g(2) = \mathbb{Q} \oplus [2^2]\) determined by Hain [9]. The homomorphism \(q_1 : \wedge^3[1^3] \to [1^3] \otimes [2^2]\) appeared already in [23] (Proposition 6.3) and it was proved that \(\text{Coker } q_1 = t_g(3) \cong [3^2]\). Our computation \(H_1([\text{Gr } t_g, \text{Gr } t_g])_4 = 0\) corresponds to the fact that the homomorphism \(q_2\) is surjective, namely \(\text{Coker } q_2 = 0\). Then, by the definition of the homomorphisms \(q_k\), it is easy to see that they are surjective for all \(k \geq 3\) as well.

**Proof of Corollary 1.4** (i) The result of Heap mentioned above implies \(\text{Ker } \tilde{\tau}_g(2) = M_g(4)\). On the other hand, the case \(k = 4\) of Corollary 1.2 shows that \(I_g(4)\) is a finite index subgroup of \(\text{Ker}(d_1 : M_g(4) \to \mathbb{Q}) = \text{Ker}(d_1, \tilde{\tau}_2)\).

Now Theorem 1.3 implies that
\[
\text{Ker}(d_1, \tilde{\tau}_2)/[K_g, K_g] \cong \text{Torsion}(H_1(K_g; \mathbb{Z})).
\]
According to Ershof-He [6] and Church-Ershov-Putman [3], \(K_g\) is finitely generated. It follows that the torsion subgroup \(\text{Torsion}(H_1(K_g; \mathbb{Z}))\) of \(H_1(K_g; \mathbb{Z})\) is a finite group and hence \([K_g, K_g]\) is a finite index subgroup of \(\text{Ker}(d_1, \tilde{\tau}_2)\). Thus both the groups \(I_g(4)\) and \([K_g, K_g]\) are finite index subgroups of the same group \(\text{Ker}(d_1, \tilde{\tau}_2)\). Therefore they are commensurable.

(ii) According to [3], \(I_g(4)\) is finitely generated for \(g \geq 7\). On the other hand, \(I_g(4)\) is a finite index subgroup of \(\text{Ker}(d_1, \tilde{\tau}_2)\) as shown in (i). It follows that \(\text{Ker}(d_1, \tilde{\tau}_2)\) is finitely generated. Since \([K_g, K_g]\) is a finite index subgroup of \(\text{Ker}(d_1, \tilde{\tau}_2)\) as above, we can conclude that it is also finitely generated.

This finishes the proof. \(\square\)
Problem 7.6. For a given $k \geq 3$, determine whether the rationally surjective homomorphisms

$$ (d_1, \tilde{\tau}_g(k)) : M_g(k) \to \mathbb{Q} \oplus \mathop{\bigoplus}_{i=k}^{2k-1} m_g(i) $$

$$ \tilde{\tau}_g(k) : I_g(k) \to \mathop{\bigoplus}_{i=k}^{2k-1} m_g(i) $$

give the whole of $H_1(M_g(k); \mathbb{Q})$ and $H_1(I_g(k); \mathbb{Q})$ or not.

In view of the result of [3] that $M_g(k)$ and $I_g(k)$ are all finitely generated for $g \geq 2k - 1$, a positive solution to the above problem would imply that the subgroups $[M_g(k), M_g(k)], [I_g(k), I_g(k)], I_g(2k)$ of the Torelli group $I_g$ are commensurable for $g \geq 4k - 1$. It then follows that the groups $[M_g(k), M_g(k)], [I_g(k), I_g(k)]$ will be finitely generated in the same range.

Now we go back to the homomorphism (12) for the cases $k = 3, 4$.

Proposition 7.7. There exist isomorphisms

$$ (d_1, p \circ \tilde{\tau}_g(3)) : (M_g(3)/I_g(5)) \otimes \mathbb{Q} \cong \mathbb{Q} \oplus m_g(3) \oplus m_g(4), $$

$$ (d_1, q \circ \tilde{\tau}_g(4)) : (M_g(4)/I_g(7)) \otimes \mathbb{Q} \cong \mathbb{Q} \oplus m_g(4) \oplus m_g(5) \oplus m_g(6) $$

where $p, q$ are the projections

$$ p : m_g(3) \oplus m_g(4) \oplus m_g(5) \to m_g(3) \oplus m_g(4), $$

$$ q : m_g(4) \oplus m_g(5) \oplus m_g(6) \oplus m_g(7) \to m_g(4) \oplus m_g(5) \oplus m_g(6). $$

Proof. This follows by combining the facts that $I_g(5), I_g(7)$ are finite index subgroups of the kernel of the homomorphism $d_1$ on $M_g(5), M_g(7)$, respectively (see Corollary 1.2) and the above homomorphism (12).

Theorem 7.8. Let $v$ be a rational invariant of homology spheres of finite type 4, including the second Ohtsuki invariant $\lambda_2$, and let $v^* : I_g \to \mathbb{Q}$ be the associated mapping. Then we have the following.

(i) The restriction $v^* : I_g(3) \to \mathbb{Q}$, which is a homomorphism, is a certain quotient of the following homomorphism where $p$ denotes the natural projection.

$$ p \circ \tilde{\tau}_g(3) : I_g(3) \to \mathop{\bigoplus}_{i=3}^{5} m_g(i) \xrightarrow{p} m_g(3) \oplus m_g(4). $$

(ii) The restriction $v^* : M_g(3) \to \mathbb{Q}$ can be expressed by the homomorphism

$$ (d_1, p \circ \tilde{\tau}_g(3)) : M_g(3) \to \mathbb{Q} \oplus m_g(3) \oplus m_g(4). $$

(iii) The restriction $v^* : M_g(5) \to \mathbb{Q}$ can be expressed by solely the homomorphism

$$ d_1 : M_g(5) \to \mathbb{Q}. $$
Proof. This follows from Corollary 7.3 and Proposition 7.7.

Theorem 7.9. Let \( \nu \) be a rational invariant of homology spheres of finite type 6, including the third Ohtsuki invariant \( \lambda_3 \), and let \( \nu^* : I_g \to \mathbb{Q} \) be the associated mapping. Then we have the following.

(i) The restriction \( \nu^* : I_g(4) \to \mathbb{Q} \), which is a homomorphism, is a certain quotient of the following homomorphism where \( q \) denotes the natural projection.

\[
q \circ \tilde{\tau}_g(4) : I_g(4) \to \bigoplus_{i=4}^7 m_g(i) \xrightarrow{q} m_g(4) \oplus m_g(5) \oplus m_g(6).
\]

(ii) The restriction \( \nu^* : M_g(4) \to \mathbb{Q} \) can be expressed by the homomorphism

\[
(d_1, q \circ \tilde{\tau}_g(4)) : M_g(4) \to \mathbb{Q} \oplus m_g(4) \oplus m_g(5) \oplus m_g(6).
\]

(iii) The restriction \( \nu^* : M_g(7) \to \mathbb{Q} \) can be expressed by solely the homomorphism

\[
d_1 : M_g(7) \to \mathbb{Q}.
\]

Proof. This also follows from Corollary 7.3 and Proposition 7.7.

Problem 7.10. Determine the precise formulae for the expressions of \( \lambda_2^*, \lambda_3^* \) in terms of \( d_1 \) on the subgroups \( M_g(5), M_g(7) \), respectively. Recall here that \( \lambda^* = \frac{1}{24} d_1 \) on \( M_g(3) \) (Theorem 7.1) and based on this, we call \( d_1 \) the core of the Casson invariant. It seems reasonable to imagine that \( \lambda_2^*, \lambda_3^* \) are constant times \( d_2^1, d_3^1 \), respectively, including the cases where the constant vanishes.

Proof of Theorem 1.6. These invariants are detected by mappings from the nilpotent groups \( I_g/I_g(5) \) and \( I_g/I_g(7) \) to \( \mathbb{Q} \), respectively. On the other hand, by Theorem 1.1 and Corollary 1.2, their building blocks can be described by the Johnson homomorphisms \( \tau_g(k) \ (k \leq 6) \) and the extra \( \mathbb{Z} \) which is detected by \( d_1 \). The restrictions of these mappings to the subgroups \( M_g(3), M_g(4) \) can be analyzed as in Theorem 7.8 and Theorem 7.9, respectively. On the whole group \( I_g \), theoretically these mappings can be described by making use of the machinery given by Massuyeau [19], Habiro-Massuyeau [8] and also Kawazumi and Kuno (see [16] for details of their theory), although the precise formulae should be much more complicated.

We mention here that, in the case of the Casson invariant, the mapping \( \lambda^* : I_g \to \mathbb{Q} \) is not a homomorphism. However, it was shown in [21] that the deviation from a homomorphism can be expressed by the first Johnson homomorphism \( \tau_g(1) \) introduced in [12].

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