Recent developments concerning generic spacelike singularities

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Abstract

This is a review of recent progress concerning generic spacelike singularities in general relativity. For brevity the main focus is on singularities in vacuum spacetimes, although the connection with, and the role of, matter for generic singularity formation is also commented on. The paper describes recent developments in two areas and show how these are connected within the context of the conformally Hubble-normalized state space approach. The first area is oscillatory singularities in spatially homogeneous cosmology and the connection between asymptotic behaviour and heteroclinic chains. The second area concerns oscillatory singularities in inhomogeneous models, especially spike chains and recurring spikes. The review also outlines some underlying reasons for why the structures that are the foundation for generic oscillatory behaviour exists at all, which entails discussing how underlying physical principles and applications of solution generating techniques yield hierarchical structures and connections between them. Finally, it is pointed out that recent progress concerning generic singularities motivates some speculations that suggest that a paradigm shift concerning their physical role, and what mathematical issues to address, might be in order.

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1 Introduction

This review describes recent results about generic spacelike singularities, but this is not its only purpose; the goal is to also provide a mathematical and physical context to these developments. This context will, at least partly, explain why one has been able to obtain any results at all, but it also suggests that future progress to some extent requires revising some cherished concepts, as well as challenging some widely held beliefs, values and goals. To provide a reasonably short and accessible overall picture, it is necessary to make some restrictions, which are chosen as follows: (i) 4-dimensional General Relativity (GR). (ii) Vacuum. (iii) Generic oscillatory spacelike singularities. (iv) Focus on main ideas in order to provide an overall picture. To accomplish this, we refer to the literature for some of the details while other details and open issues, of different levels of difficulty, are yet to be worked out and solved, although outlines for how to do this will sometimes be given.

Any investigation about the detailed nature of generic spacelike singularities has to take into account the work by Belinskiˇı, Khalatnikov and Lifshitz (BKL) [1, 2, 3]. This work began by considering Einstein’s field equations in synchronous coordinates and by dropping all spatial derivatives, which geometrically corresponds to neglecting the Ricci 3-curvature of the spatial surfaces of the synchronous coordinate system, as well as all matter terms [1]. This procedure leads to a set of ordinary differential equations (ODEs) that are identical to those obtained in the vacuum case by imposing spatial homogeneity and an associated simply transitive Abelian symmetry group, which results in the vacuum Bianchi type I models whose solution is the well-known Kasner solution. But in the general inhomogeneous context the constants of integration that appear in the Kasner solution are replaced by spatially dependent functions, leading to a ”generalized Kasner solution,” even though it is not a solution to Einstein’s field equations at all. Instead the relevance of the generalized Kasner solution is as a building block when one attempts to construct generic asymptotic solutions; however, the generalized Kasner solution itself does not have a sufficient number of spatial functions to be a generic solution, something that originally led Lifshitz and Khalatnikov to conclude that singularities are not a generic feature of GR [1].

This state of affairs changed, however, when the singularity theorem of Penrose appeared in 1965 [4]. According to this theorem, and later related ones, singularities occur generically under quite general circumstances. But singularity theorems of ‘Penrose type’ say little about the nature of generic singularities, and it should also be pointed out that there are generic spacetimes without singularities. Prompted by this state of affairs, BKL noted that the generalized Kasner solutions are unstable, which heuristically follows from inserting them into the terms of the Einstein field equations that drives the evolution and studying these terms temporal behaviour. Further insights were gained from considerations of spatially homogeneous (SH) models, especially Bianchi type IX. For these latter models they found that generic solutions could be heuristically approximated asymptotically by piecewise joining different Kasner states by means of different Bianchi type II vacuum solutions in a manner that resulted in infinite Kasner oscillations, a result that was also reached independently by Misner by means of Hamiltonian methods [6, 7]. Moreover, the Bianchi type II solutions allowed for a discretization of the dynamics in terms of sequences of Kasner states by means of a map [2, 3, 8] that turned out to be connected with chaotic properties, something that has inspired many subsequent papers, see e.g. [9, 10] for references.

Coming back to the general inhomogeneous case, armed with the insights from the SH case, and by studying the effects of inserting the generalized Kasner solution into the terms of the Einstein field equations that drives the evolution, BKL claimed that certain terms in the Einstein field equations could be dropped towards a generic spacelike singularity. In particular, this held for the perfect fluid terms in the Einstein field equations if the equations of state was sufficiently soft, e.g. dust or radiation. Hence, according to BKL, ”matter does not matter” for the spacetime geometry in the vicinity of a generic spacelike singularity, even though the energy density blows up, which makes it natural to start investigations about generic spacelike singularities by considering the vacuum equations, as is done here. Furthermore, by dropping terms BKL obtained effective ODEs that resulted in ‘generalized’ Bianchi type II vacuum ‘solutions’ that temporally joined dif-

\footnote{This is illustrated by the global stability of the Minkowski spacetime [5].}
different generalized Kasner states (in analogy with the type IX case) along *individual* timelines.\(^2\)

The resulting BKL picture for the ‘*vacuum dominated*’ case can be summed up in the following ‘*locality conjecture*’: The dynamics towards a generic spacelike singularity for general *inhomogeneous* models can be described as being ‘*local,*’ in the sense that each spatial point is assumed to evolve towards the singularity individually and independently of its neighbors with its evolution described by a system of ODEs; moreover, asymptotically the evolution along individual timelines is described by a sequence of Bianchi type II solutions that connect different Kasner states in an oscillatory manner.

The locality conjecture leads to the question: Why should generic solutions behave in this way? Presumably it has to do with asymptotic causal features. Causal properties constitute an important feature in GR, notably in the derivation of the singularity theorems of Penrose and Hawking.\(^3\)\(^4\)\(^5\)

A generic spacelike singularity is expected to be a scalar curvature singularity, since a generic singularity presumably is associated with ultra-strong gravity and since a non-scalar curvature singularity requires fine tuning. Moreover, it seems reasonable that increasing ultra-strong gravity may lead to the formation of particle horizons that shrink to zero size in all directions along any time line that approaches the singularity, thus increasingly prohibiting communication, a phenomenon that was called *asymptotic silence* in\(^6\)\(^7\)\(^8\)\(^9\). This would then ‘explain’ why asymptotic locality happens, but unfortunately things do not seem to be that simple. Presumably asymptotic silence is a necessary requirement for asymptotic locality, but it is not sufficient. As we will see, generic singularities are not only associated with asymptotic locality, but also with *recurring spikes, non-local evolution* along certain timelines that is described by PDEs\(^10\)\(^11\)\(^12\), and it seems that this type of behaviour is also associated with asymptotic silence (i.e., ultra-strong gravity does not always lead to asymptotic locality, nor to asymptotic silence as illustrated by null singularities); for various examples of asymptotic silence and asymptotic silence-breaking, see\(^13\).

Although we will focus on vacuum in this paper, it is worth pointing out that causal properties of matter also seem to be important for if and how matter influences the spacetime geometry in the vicinity of a generic spacelike singularity. Towards such a singularity there is a competition between how matter and gravity nonlinearly generate spacetime curvature. Examples suggest that ”matter does not matter” for matter sources with characteristics with speed less than the speed of light; such models are therefore said to be *asymptotically vacuum dominated*. But if the matter equations have characteristics with a speed that is equal to the speed of light, then the tug of war between matter and pure gravity generating gravity results in a draw; the Ricci and Weyl scalars obtain generic asymptotic amplitude magnitudes of the same order. Moreover, in this case there exist interesting connections between spin and spacetime curvature in the vicinity of a generic spacelike singularity. A massless scalar field leads to asymptotic locality and convergence to specific limits, a phenomenon that can be characterized as *asymptotic local self-similarity*. An electromagnetic field on the other hand leads to non-convergent infinite oscillatory behaviour. Finally, a perfect fluid, for which the equation of state in the infinite energy density limit leads to a speed of sound that is greater than that of light, yields an asymptotic locally self-similar and isotropic singularity for which the Ricci scalar dominates over the Weyl scalar\(^14\)\(^15\)\(^16\). These tantalizing hints, that ultra-strong gravity seems to be connected with some of the main properties of matter, is a fairly unexplored area of research that deserves further study, and it is also of interest to also go beyond GR in this context, as exemplified by the work of Damour \textit{et al.} \(^17\).

This latter work makes use of Hamiltonian methods that were developed by Chitré and Misner for Bianchi type IX\(^18\)\(^19\). It is worth pointing out that this heuristic approach, which is closely connected to that of BKL, gives rise to asymptotic constants of the motion, which are difficult to obtain by other methods. Moreover, these methods have revealed a remarkable connections between asymptotic dynamics of generic spacelike singularities and Kac-Moody algebras\(^20\).

Finally, the BKL picture, although likely to describe many asymptotic features of generic spacelike singularities, contains several highly non-trivial assumptions. Moreover, consistency arguments for the remarkably simple BKL picture do not exclude other behaviour as regards singularities, special or generic. Indeed, during the last few years there has been a number of developments, analytical,

\(^2\)In addition BKL linearly perturbed the generalized Bianchi type I and II ‘solutions’ and obtained new effective ODEs, which resulted in statements about ‘rotation and freezing of Kasner axes’\(^9\).
numerical, and heuristic, that all point at that the BKL scenario is part of a bigger startlingly subtle picture, with a web of intriguing hierarchical structures, which furthermore hints at that you have to go beyond BKL in order to fully understand the BKL picture.

The outline of the paper is as follows. The next section describes the conformally ‘Hubble-normalized’ asymptotically regularized state space picture. This framework leads to a new context for models with symmetries, which makes it possible to provide a rigorous and simple description of asymptotic locality and the BKL picture; moreover, this state space picture also makes it possible to describe the structures that give rise to non-local recurring spikes. Section 3 describes and discusses the hierarchy of invariant subsets in the Hubble-normalized state space picture that is the foundation for describing and understanding generic spacelike singularities. Section 4 outlines how the BKL picture can be transformed to a rigorous description within the context of the Hubble-normalized state space picture in terms of attractors, transitions, concatenation and heteroclinic chains. Furthermore, the gauge-invariant discrete representation of these structures in terms of Kasner sequences naturally leads to a description of recent progress concerning generic asymptotics in Bianchi types VIII and IX, which via the Hubble-normalized state space picture becomes tied to the generic BKL picture. Section 5 goes beyond the BKL picture and outlines recent progress concerning spike chains and recurring spikes. In addition a comparison of some statistical properties that arise from asymptotic BKL and spike dynamics, respectively, is given. The paper concludes with a discussion about the physical role of generic singularities, and the dangers and possibilities of special models in contexts such as cosmic censorship.

2 The conformally Hubble-normalized state space formulation

Currently there are no theorems about generic inhomogeneous oscillatory spacelike singularities. In this paper it is argued that it is a reasonable proximate goal to construct an asymptotic solution in a small spatiotemporal neighborhood in the vicinity of part of a generic inhomogeneous singularity, and to postpone ultimate goals concerning issues such as cosmic censorship. With the proximate goal in mind:

- Consider a 4-dimensional manifold $M$ endowed with a metric $g$ of signature $(- + + +)$ (the ‘physical spacetime’) that satisfies Einstein’s vacuum equations. By convention set $c = 1$, where $c$ is the speed of light.

- Due to the present cultural prominence of BKL, the singularity is chosen to be located in the past. Assume that there exists a small spatiotemporal neighborhood of the singularity that can be foliated with spacelike leaves that asymptotically coincide with the singularity.

- Assume that the expansion $\theta$ of the timelike future directed unit normal vector field $u$ of the chosen foliation is positive in the small neighborhood and that $\theta \to \infty$ towards the singularity, i.e., the generic singularity is assumed to be a crushing singularity.

- Use the characteristic scale of the problem associated with the expansion $\theta$, or rather, due to historical reasons, the Hubble variable $H = \frac{4}{3} \theta$, to conformally blow up the small neighborhood, as follows:
  \[ G = H^2 g, \]  (1)
  where the unphysical metric $G$ is dimensionless since $g$ has dimension $(\text{length})^2$, or, equivalently, $(\text{time})^2$, and $H$ has dimension $(\text{length})^{-1}$.

- Introduce an orthonormal frame of $G$, $\{ \Omega^a = E^a_{\mu} dx^\mu \}$, $a = 0, 1, 2, 3$, $\mu = 0, 1, 2, 3$ (alternatively interpreted as a conformal orthonormal frame of $g$), according to:
  \[ G = \eta_{ab} \Omega^a \Omega^b = H^2 \eta_{ab} \omega^a \omega^b = H^2 g, \]  (2)
  where $\eta_{ab} = \text{diag}[-1, 1, 1, 1]$, and where $\{ \omega^a = e^a_{\mu} dx^\mu \}$ is the associated orthonormal frame of $g$. 

2 THE CONFORMALLY HUBBLE-NORMALIZED STATE SPACE FORMULATION

- Adap the frame to \( u \) by choosing a set of dual frame vectors \( \partial_\alpha = E_{\alpha}^\mu \partial_{x^\mu} = H^{-1} e_\alpha = H^{-1} e_a^\alpha \partial_{x^a} \), where \( e_\alpha \) (\( e_a \)) are the frame vectors dual to \( \omega^a \) (\( \Omega^\alpha \)) \( (e_a^\mu \partial_{x^a} = \delta^\alpha_a, \quad E_{\alpha}^\mu \partial_{x^\mu} = \delta^a_\alpha) \), by setting \( e_0 = u \). Furthermore, choose \( e_0 = u = H^{-1} u \) to be tangential to the timelines, i.e., \( \partial_0 = (HN)^{-1} \partial_{t^0} =: N^{-1} \partial_{t^0} \) (i.e. set the shift vector to zero), where \( N \) is the lapse function and \( N \) the conformal lapse function. Hence \( G = \eta_{ab} \Omega^a \Omega^b = -N^2(dx^0)^2 + \delta_{\alpha\beta} E_\gamma^a E_\delta^b dx^\gamma dx^\delta = -N^2(dx^0)^2 + G_{ij} dx^i dx^j, \alpha = 1, 2, 3 \).

- Introduce a set of variables given by the inverse Hubble variable \( H^{-1} \), the Hubble-normalized frame variables \( N, E_\alpha \), and the Hubble-normalized commutator (equivalently, connection) variables \( q, \Sigma_{\alpha\beta}, R_\alpha, N^{\alpha\beta}, A_\alpha \), defined by the commutators

\[
[\partial_0, \partial_\alpha] = (\partial_\alpha \log N) \partial_0 + F^\alpha_\beta \partial_\beta, \quad [\partial_\alpha, \partial_\beta] = (2A_{[\alpha} \delta_{\beta]} + \epsilon_{\alpha\beta\delta} N^\delta\gamma) \partial_\gamma,
\]

where \([\ldots]\) corresponds to anti-symmetrization and

\[
F^\alpha_\beta := q \delta^\alpha_\beta - \Sigma^\alpha_\beta - \epsilon^\alpha_\beta_\gamma R^\gamma,
\]

where \(-q, \Sigma_{\alpha\beta}\) are the conformally Hubble-normalized Hubble variable and the trace-free shear of \( \partial_0 \), respectively, where \( q \) also happens to be the physical deceleration parameter; \( R_\alpha \) is the Fermi rotation, which describes how the frame rotates with respect to a Fermi propagated frame; \( N^{\alpha\beta} \) and \( A_\alpha \) are spatial commutator functions that describe the \( \delta \)-curvature of the spatial part of the metric \( G \).

- Impose the commutator equations, Jacobi identities and Einstein field equations on \( H^{-1} \) and the conformally Hubble-normalized frame and commutator variables.

The quantities \( N \) and \( R_\alpha \) represent remaining gauge freedoms since: (i) The above assumptions and choices do not uniquely fix the foliation and there is therefore remaining freedom in choosing \( N \) and hence also \( N \) (for some analytic purposes there is no need to impose any explicit restriction on \( N \), and hence we do not do so below, but \( N \) needs to be specified for numerical investigations as well as for describing asymptotic non-local phenomena such as recurring spikes). (ii) The quantity \( R_\alpha \) can be chosen freely since the spatial frame is only determined up to arbitrary rotations, however, to obtain a deterministic system of equations \( R_\alpha \) needs to be specified; below we will use so-called Fermi and Iwasawa frames, although other choices will be discussed as well. Keeping the choice of \( N \) and \( R_\alpha \) open leads to the following (not fully gauge reduced) state space:

\[
X = (E_{\alpha}^i, H^{-1}, \Sigma_{\alpha\beta}, A_\alpha, N_{\alpha\beta}) = (E_{\alpha}^i) \oplus H^{-1} \oplus S, \quad \text{where} \quad S = (\Sigma_{\alpha\beta}, A_\alpha, N_{\alpha\beta}).
\]

The commutator equations, Jacobi identities and Einstein field equations yield the following equations for the state space variables (where the choices of \( N \) and \( R_\alpha \) are left unspecified):

**Evolution equations**:

\[
\partial_\alpha E_{\alpha}^i = F^\alpha_\beta E_\beta^i,
\]

\[
\partial_0 H^{-1} = (1 + q) H^{-1},
\]

\[
\partial_0 \Sigma_{\alpha\beta} = -(2 - q) \Sigma_{\alpha\beta} - 2 \epsilon^\delta_{\alpha(} \Sigma_{\beta)\gamma} R_\delta - 3 \Sigma_{\alpha\beta} + (I\Sigma)_{\alpha\beta},
\]

\[
\partial_0 N^{\alpha\beta} = (3q \delta^\alpha_{(\alpha} - 2F_{(\alpha}^{\gamma(} \Sigma_{\gamma)}^{\beta)} N^{\delta\gamma}) + (I_N)^{\alpha\beta},
\]

\[
\partial_0 A_\alpha = F^\alpha_\beta A_\beta + (I_A)_{\alpha}.
\]

**Constraint equations**:

\[
0 = (\epsilon^\gamma_{\beta(} (\partial_\gamma - A_\gamma) - N_{\alpha\beta}) E_\beta^i, \quad (I_{J\omega})_{\alpha},
\]

\[
0 = (I_G)_{\alpha} + (I_C)_{\alpha},
\]

\[
0 = -3 A_\beta \Sigma_{\alpha\beta} + \epsilon^\gamma_{\alpha\beta} \Sigma_\delta^\delta N_{\delta\gamma} + (I_C)_{\alpha},
\]

\[
0 = A_\beta N_{\alpha\beta} + (I_I)_{\alpha},
\]

\[
\]

---

3There is also gauge freedom associated with changes of spatial coordinates, which affects \( E_{\alpha}^i \), however, since this only ‘passively’ enters the discussion about evolution we focus on here, we will not be concerned with this freedom further in this paper.
where

\[
\begin{align*}
\Sigma^2 &:= \frac{1}{6} \Sigma_{\alpha\beta} \Sigma^{\alpha\beta}, & q &= 2\Sigma^2 + (I_q), \\
\Omega_k &:= -\frac{1}{3} \mathcal{R}, & 3\mathcal{R} &= -6A_\alpha A^\alpha - \frac{1}{2} B_{\alpha\beta}^{\alpha\beta} + (I_K), \\
B_{\alpha\beta} &:= 2N_\alpha \gamma N_{\gamma\beta} - N_{\gamma\gamma} N_{\alpha\beta}, & 3\mathcal{S}_{\alpha\beta} &= B_{(\alpha\beta)} - 2\epsilon_{\gamma\beta}(\alpha N_{\beta\gamma} A_\delta) + (I_S)_{\alpha\beta}.
\end{align*}
\]

In the above equations \((...)\) and \((...)\) denote spatial symmetrization and trace-free symmetrization, respectively; \(3\mathcal{R}, 3\mathcal{S}_{\alpha\beta}\) are the conformal scalar 3-curvature and trace-free Ricci 3-curvature, respectively; for brevity we omit the expressions for \((I_*)_{**}\), which are given in [13, 23], but all involve terms of the type \(\partial_i (\log N, \log H^{-1}, S, R_\alpha)\), and if these quantities are zero so are all \((I_*)_{**}\).

### 3 Invariant subset hierarchies

#### 3.1 Symmetry based invariant subset hierarchies

Due to that Killing vector fields are Ricci collinations, i.e., the Lie derivative of the Ricci tensor with respect to a Killing vector is zero, such symmetries are compatible with Einstein’s (vacuum) equations and hence lead to invariant subsets of those equations. This is also true for homothetic Killing vectors, but not for proper conformal Killing vector fields, which explains the rarity of solutions with such symmetries. In the present context the most interesting isometry groups are those that act on spacelike orbits. Imposing conditions on the Killing vectors and the algebras they form result in a hierarchy of invariants subsets that play a key role for asymptotic dynamics. This is especially true for the SH case, whose system of equations can be derived straightforwardly from the conformally Hubble-normalized picture.

**The SH case in the conformally Hubble-normalized state space picture**

To obtain the SH case in the conformally Hubble-normalized picture, choose a symmetry compatible frame \(\{e_\alpha\}\) such that \(e_0 = N^{-1} \partial_{x^0}\) is orthogonal to the spatial symmetry surfaces, and hence that \(e_\alpha = e_{\alpha i} \partial_{x^i}\) are tangential to the symmetry surfaces, and similarly for \(\{\partial_\alpha\}\). As a consequence the lapse \(N\) and the coordinate scalars \((H^{-1}, S, R_\alpha)\), and hence also \(N\), are functions of time only, and therefore \(\partial_\gamma (N, H^{-1}, S, R_\alpha) = 0\) (recall that \(\partial_\gamma = E_{\gamma i} \partial_{x^i}\), which leads to that all \((I_*)_{**}\) become zero. As a consequence, the evolution equation \((\text{5a})\) for \(E_{\alpha i}\) decouples from the other equations, leaving a system for \(H^{-1} \oplus S\), where the equation \((\text{5b})\) for \(H^{-1}\) also decouples, which leads to a *reduced coupled dimensionless system* for \(S\), given by the evolution equations \((\text{5c})\), \((\text{5d})\), \((\text{5e})\), and the constraints \((\text{6a})\), \((\text{6b})\), \((\text{6c})\). Since \(q = 2\Sigma^2\), it follows from \((\text{5f})\) that \(H^{-1} > 0\) is monotonically decreasing towards the past towards zero (see, e.g. [9]), which leads to a crushing singularity that is a scalar curvature singularity in the generic case, although there are special cases that result in e.g. a coordinate singularity.

If we in addition add general matter terms to the Einstein field equations, as done in [23, 18], we obtain the general Einstein field equations for the SH case, although these equations have to be complemented with equations for the matter degrees of freedom. Specializing to (i) an orthogonal (‘non-tilted’) perfect fluid (i.e. the fluid 4-velocity is given by \(e_0\)), and a linear equation of state, (ii) the SH class A invariant subset \(A_\alpha = 0\), (iii) and choose a frame for which \(N^{\alpha\beta}\) and \(\Sigma_{\alpha\beta}\) are both diagonal, which is possible for this particular set of SH models, \(3\) leads to the dynamical system introduced by Wainwright and Hsu [23], i.e., that approach is a special case of the present geometrical framework.

**The \(G_2\) case in the conformally Hubble-normalized state space picture**

Models with two commuting spacelike Killing vectors, the so-called \(G_2\) models, turn out to be important for asymptotic dynamics, moreover, they are the simplest inhomogeneous vacuum models...
that (conjecturally) admit oscillatory singularities. These models can be expressed in terms of a general so-called spatial Iwasawa frame. Within the conformally Hubble-normalized framework, such a spatial frame can be implemented by setting

\[ E_1^2 = E_2^2 = E_3^2 = 0 \Rightarrow N_3 = 0, \quad (R_1, R_2, R_3) = (-\Sigma_{23}, \Sigma_{31}, -\Sigma_{12}), \quad (8) \]

where in the \( G_2 \) case one can choose local coordinates and a symmetry adapted Iwasawa frame so that all variables depend on \( x^0 \) and \( x^3 \) only, which in addition gives \( N_{31} = N_{23} = N_{22} = A_1 = A_2 = 0 \), and where one of the Codazzi constraints can be used to also impose \( R_2 = 0 = \Sigma_{31} \). Moreover, due to the imposed symmetries, only \( \partial_0 \) and \( \partial_3 \) occur in the Einstein field equations and the Jacobi identities. As a consequence the evolution equations for all the remaining \( E_\alpha \) variables, except the one for \( E_3^3 \), decouple, leaving a reduced coupled system of PDEs in \( x^0 \) and \( x^3 \) for the state vector \( E_3^3 \oplus H^{-1} \oplus S \), for details, see [13].

Invariant subsets induced by conditions on spatial Killing vectors

For models with spacelike Killing vectors, further specialized invariant subset structures can be obtained by imposing conditions on the geometrical properties of the Killing vector fields. Notable conditions are demands such as *hypersurface orthogonal* individual Killing vectors, and, in the \( G_2 \) case, requiring that the 2-spaces orthogonal to the orbits of the \( G_2 \) symmetry group are surface forming. This latter case yields the important invariant *orthogonally transitive* (OT) subset, for which \( E_3^1 = E_3^2 = 0 \) and \( R_1 = 0 = \Sigma_{23} \). The OT case turns out to be a key subset for the building blocks needed to describe oscillatory singularities. In addition special subsets are obtained by taking intersections of the above subsets and by introducing additional isotropies. Conditions on Killing vector fields is, however, not the only way of obtaining invariant subsets from symmetries, invariant subsets also arise if one imposes discrete isometries. Taken together, symmetry conditions lead to hierarchies of invariant subsets with different state space dimensions, see [26, 16] for examples.

3.2 Symmetries and invariant boundary correspondence hierarchies

The local boundary

The system of PDEs given by (5) and (6) admits an unphysical invariant boundary subset given by setting \( E_\alpha \) to zero except \( E_3^3 \), and hence also \( \partial_1 = \partial_2 = 0 \), which corresponds to setting all components of the spatial covariant metric \( G^{ij} \) except \( G^{33} \) identically to zero. It follows immediately that the equations for the remaining state space \( E_3^3 \oplus H^{-1} \oplus S \) are identical to those in the SH case, and that the equation for \( H^{-1} \) decouple, leaving a coupled system for the dimensionless Hubble-normalized commutator variables \( S \) that form the *dimensionless local boundary state space*. Once the equations are solved on this reduced state space, they yield a quadrature for \( H^{-1} \), but note that \( H^{-1} \to 0 \), where \( H^{-1} = 0 \) is an invariant boundary subset of the state space \( H^{-1} \oplus S \), towards the past, as in the SH case. The only difference with the SH case is that the state space now consists of an infinite set of copies, one for each spatial point. As a consequence constants of the motion in the SH case are now replaced with spatially dependent functions.

Partially local boundaries

The system of PDEs given by (5) and (6) admits an unphysical invariant boundary subset given by setting all \( E_\alpha \) to zero except \( E_3^3 \), and hence also \( \partial_1 = \partial_2 = 0 \), which corresponds to setting all components of the spatial covariant metric \( G^{ij} \) except \( G^{33} \) identically to zero. It follows immediately that the equations for the remaining state space \( E_3^3 \oplus H^{-1} \oplus S \) are identical to those of the
G_2 case. We will refer to this subset as the partially local G_2 boundary. The difference with the G_2 case is that functions of integration are no longer just functions of x^3 but also of x^1 and x^2. The situation is therefore completely analogous to the SH case and the local boundary. A similar statement also holds for models with one spacelike Killing vector, the so-called G_1 models, which hence give rise to the partially local G_1 boundary.

As a consequence of the identification of the reduced equation systems for models with isometry groups and those on corresponding boundary subsets, it follows that the hierarchy of state spaces induced by imposing further conditions on isometries, such as the OT G_2 case, and by imposing discrete isometries, translates to the boundary subset state spaces. The boundary subsets turn out to be essential for past asymptotic dynamics, and hence the conformally Hubble-normalized state space picture provides a new context for models with spacelike isometry groups.

### 3.3 The Lie contraction hierarchy on the local boundary

To simplify the discussion about the local boundary, we use the identification with the SH equations and therefore talk about SH models instead. BKL is associated with generic structures, and hence it is the most general SH models that are of primary interest. These models are the class B (defined by A_0 ≠ 0) general Bianchi type VI_{−1/2} models and the class A (defined by A_0 = 0) Bianchi type VIII and IX models, where the two latter, particularly Bianchi type IX, are the models that have attracted most interest, and for which we have most rigorous results.

The equations on the dimensionless reduced state space S form a hierarchical invariant subset structure based upon the Bianchi type classification, where each Bianchi type is associated with an invariant subset on S. Furthermore, the Bianchi types that can be obtained from another Bianchi type by means of Lie contractions, i.e., by setting structure constants to zero, correspond to invariant boundary subsets of the more general Bianchi type that can be obtained by setting the corresponding Hubble-normalized spatial commutator variables to zero. To be more specific, we will first focus on the class A models.

#### Class A models

In the case of the class A vacuum Bianchi models the Codazzi constraint \( \epsilon_{\alpha \beta \gamma} N_{\beta \gamma} N_{\alpha} = 0 \) takes the form \( \epsilon_{\alpha \beta \gamma} \Sigma_{\beta \gamma} N_{\alpha} = 0 \), from which it follows that \( N^{\alpha \beta} \) and \( \Sigma_{\alpha \beta} \) can be simultaneously diagonalized (neglecting global topological issues), moreover, it is possible to diagonalize them in a Fermi frame. This gives a particularly convenient representation of these models, and we hence set \( N^{\alpha \beta} = \text{diag}(N_1, N_2, N_3) \), \( \Sigma_{\alpha \beta} = \text{diag}(\Sigma_1, \Sigma_2, \Sigma_3) \) and \( R_\alpha = 0 \). Once results are obtained in this frame they can be transformed to an arbitrary Fermi frame by a general constant orthogonal transformation (a temporally constant transformation in the local boundary case). This leads to the following dynamical system on \( S \) [9]:

\[
\begin{align*}
\partial_0 \Sigma_\alpha &= 2(1 - \Sigma^2) \Sigma_\alpha + \frac{1}{3} \left[ N_\alpha (2 N_\alpha - N_\beta - N_\gamma) - (N_\beta - N_\gamma)^2 \right], \\
\partial_0 N_\alpha &= -2 (\Sigma^2 + \Sigma_\alpha) N_\alpha \\&= 1 - \Sigma^2 + \frac{1}{12} \left[ N_1^2 + N_2^2 + N_3^2 - 2 (N_1 N_2 + N_2 N_3 + N_3 N_1) \right],
\end{align*}
\]

where \( (\alpha \beta \gamma) \in \{(123), (231), (312)\} \) in [9], and where \( \Sigma_1 + \Sigma_2 + \Sigma_3 = 0 \), \( \Sigma^2 = \frac{1}{6} (\Sigma_1^2 + \Sigma_2^2 + \Sigma_3^2) \).

The different class A models are characterized as follows: Bianchi type IX (VIII) has \( N_1 N_2 N_3 > 0 \) \( (N_1 N_2 N_3 < 0) \) and a reduced dimensionless 4-dimensional state space; Bianchi type VII_0 (VI_0) \( N_\alpha N_\beta > 0 \) \( (N_\alpha N_\beta < 0) \), with a 3-dimensional state space; Bianchi type II \( N_\alpha = N_\beta = 0 \), \( N_\gamma \neq 0 \) (\( (\alpha \beta \gamma) = (123) \)), and cycle), leading to a 2-dimensional state space, and finally Bianchi type I is characterized by \( N_1 = N_2 = N_3 = 0 \), which leads to the 1-dimensional so-called Kasner circle \( K_\circ \) of fixed points, discussed below. Hence a Bianchi type IX (VIII) model has three Bianchi type VII_0 (two VI_0 and one VII_0) invariant boundary subsets, while each of those subsets have two Bianchi type II invariant boundary subsets, and, finally, each
3 INVARIANT SUBSET HIERARCHIES

Bianchi type II subset in this hierarchy of subsets has the Bianchi type I Kasner circle K\(^0\) as an invariant boundary subset.

Setting a spatial commutator variable \(N_\alpha\) to zero corresponds to setting an associated structure constant to zero, which leads to a lower Bianchi type via a so-called Lie contraction [27]. Thus the above dynamical system exhibits a 'Lie contraction boundary subset hierarchy.' The hierarchical Lie contraction boundary subset structure is central for the asymptotic dynamics towards the initial singularity. Setting a variable \(N_\alpha\) to zero is associated with increasing the dimension of the automorphism group with one. The kinematical consequence of this is that a given Lie contracted boundary subset of (9) describes the true degrees of freedom of the associated Bianchi type, where integrating \(\partial_t H^{-1} = (1 + 2\Sigma^2)H^{-1}\), once a solution of (9) has been obtained, yields the scale parameter of the solution. Due to this, the metric can be algebraically constructed from the solution on \(H^{-1} \oplus S\) by means of group theoretical methods, see [28].

More importantly, however, are the dynamical implications of the group of automorphisms and scale transformations. As explicitly shown in [29], on each level in the Lie contraction boundary subset hierarchy the combined scale-automorphism group induces monotone functions, and even constants of the motion at the bottom of the hierarchy. The resulting hierarchy of monotone functions pushes the dynamics towards the past singularity to boundaries of boundaries in the hierarchy, where the solutions at the bottom of the hierarchy, i.e., those of Bianchi types I and II, are completely determined by the symmetries induced by the scale-automorphism groups of these models. The dynamical evolution towards the initial singularity is therefore to a large extent governed by structures induced by the scale-automorphism groups on the different levels in the Lie contraction boundary subset hierarchy. Since the automorphisms in the present SH context correspond to the spatial diffeomorphism freedom that respects the symmetries of the various Bianchi models, it therefore follows that the dynamical evolution towards the initial singularity is partly determined by physical first principles, namely scale-invariance and general covariance.

In the case of matter sources, hierarchies become even more important than in the vacuum case. Then, in addition to Lie contractions, one also have source contractions, where vacuum is at the bottom of the source hierarchy. For each level of the source contraction hierarchy the scale-automorphism group yield different structures, such as monotone functions, leading to restrictions on asymptotic dynamics; see [29] where this general feature is exemplified explicitly for an orthogonal perfect fluid with a linear equation of state in the case of diagonal class A Bianchi models.

**Class B models: The general Bianchi type VI–1/9 case**

The class B Bianchi models satisfy

\[
A_\beta N_\alpha^\beta = 0, \quad A_\alpha A^\alpha = \frac{1}{2}h \left((N_\alpha^\alpha)^2 - N_\alpha^\alpha N^\beta_\beta\right),
\]

where \(h\) is a constant parameter that characterizes the Bianchi type VI\(h\) and VII\(h\) models, which follows from the analogous structure constant relation. But \(h\) can also be viewed as a constant of the motion on the state space \(S\), and hence \(h\) is a spatial function on the class B local boundary. In class B the vacuum Codazzi constraint \([64]\) gives \(3A_\beta \Sigma^\alpha_\gamma = \epsilon_\alpha^\beta\gamma\Sigma^\delta_\beta N^\gamma_\delta\). For all class B models, except type V and some special type VI\(h\) models (including Bianchi type III), \(3A_\beta \Sigma^\alpha_\gamma = \epsilon_\alpha^\beta\gamma\Sigma^\delta_\beta N^\gamma_\delta \neq 0\), and as a consequence, in contrast to class A, \(\Sigma^\alpha_\gamma\) and \(N_\alpha^\alpha\) are not in general simultaneously diagonalizable. Kinematically the Bianchi type VI\(h\) and VII\(h\) models are the most general class B models, but for \(h = -1/9\) the Codazzi constraints become degenerate, which permits \(\Sigma^\alpha_\beta\) to have one more independent component than for the other type VI\(h\) and VII\(h\) models, making the state space 4-dimensional, i.e., type VI–1/9 is as general as Bianchi types VIII and IX [30] [26] [31]. Moreover, the extra degree of freedom these models admit leads to an oscillatory singularity, in contrast to all other class B vacuum models, which are past asymptotically self-similar. As regards class B, it is thus the general Bianchi type VI–1/9 vacuum models that are of interest for oscillatory singularities, furthermore, they exhibit some other oscillatory features than Bianchi types VIII and IX, making them interesting as toy models for general oscillatory singularities.

---

\(^5\)In the present context automorphism transformations are the linear constant transformations of a symmetry adapted spatial frame that leave the structure constants unchanged.
The reduced dimensionless dynamical system for the general Bianchi type VI\textsubscript{-1/9} vacuum models (and its local boundary analogue) can be written as follows (choosing an Iwasawa frame so that \((R_1, R_2, R_3) = - (\Sigma_{21}, 0, \Sigma_{12})\) and so that the remaining non-zero variables are \(\Sigma_\alpha = \Sigma_{\alpha\alpha}, A_3 = A, N_{11} = N_1, N_{12} = 3A,\) see \[13\]):

**Evolution equations:**

\[
\begin{align*}
\theta_0 \Sigma_1 &= -2\Omega_k \Sigma_1 + 2R_1^2 - \frac{2}{3} N_1^2, \\
\theta_0 \Sigma_2 &= -2\Omega_k \Sigma_2 - 2(R_2^2 - R_1^2) + \frac{1}{3} N_1^2 - 12 A^2, \\
\theta_0 \Sigma_3 &= -2\Omega_k \Sigma_3 - 2R_3^2 + \frac{1}{3} N_1^2 + 12 A^2, \\
\theta_0 R_1 &= (-2\Omega_k + \Sigma_3 - \Sigma_2) R_1, \\
\theta_0 R_3 &= (-2\Omega_k + \Sigma_2 - \Sigma_1) R_3 + 4 N_1 A, \\
\theta_0 N_1 &= 2(\Sigma^2 + \Sigma_1) N_1 - 12 R_3 A, \\
\theta_0 A &= (2\Sigma^2 - \Sigma_3) A.
\end{align*}
\]

**Constraint equations:**

\[
\begin{align*}
0 &= 1 - \Sigma^2 - \Omega_k, \\
0 &= 6\Sigma_1 A + R_3 N_1, \\
0 &= \Sigma_1 + \Sigma_2 + \Sigma_3,
\end{align*}
\]

where \(\Sigma^2 = \frac{1}{6}(\Sigma_1^2 + \Sigma_2^2 + \Sigma_3^2 + 2R_1^2 + 2R_2^2)\) and \(\Omega_k = \frac{1}{12} N_1^2 + 4 A^2.\)

To obtain the dynamical hierarchical structure induced by the scale-automorphism group hierarchy in class A, Hamiltonian techniques were used in [23], but this was only for practical reasons, the results do not depend on them since they are consequences of the scale-automorphism group hierarchy. The class B models do not, in general, have a Hamiltonian formulation (without extra non-potential forces). For this reason an exploration of the dynamical consequences of the scale-automorphism group would have to take a more direct approach. Moreover, due to that the models are non-diagonal, especially the general Bianchi type VI\textsubscript{-1/9} models, these models, in contrast to the class A vacuum models, also involve off-diagonal automorphisms. A natural step in a systematic analysis of the class B models, and especially the general Bianchi type VI\textsubscript{-1/9} models, would therefore be an analysis of the dynamical implications of the scale-automorphism group, a task that remains to be done.

### 4 BKL, transitions, concatenation, chains and discretized maps

#### 4.1 The generalized Kasner solution

The BKL results in [3] were obtained by means of a synchronous coordinate frame, and using the coordinate components of the metric and its synchronous time derivative as basic variables as starting point. However, BKL also emphasizes the role of spatial scale factors, obtained by diagonalizing the spatial frame. As discussed below, such diagonalizations can be accomplished in many different ways, which lead to different descriptions of asymptotic dynamics, although, presumably, the local gauge-invariant features are the same. In all cases, the BKL analysis starts with inserting the generalized Kasner solutions (obtained by dropping all spatial derivatives in a diagonalized spatial frame and solving for the scale factors) into the different terms in the full Einstein field equations in the chosen spatial frame. It is then found that certain terms grow and eventually become non-negligible ‘triggering transitions’ (in the present notation), leading to Kasner oscillations.

The BKL picture thus relies on the central importance of the Kasner solutions. In the SH case these are the Bianchi type I solutions which are obtained by setting \(N^{\alpha\beta} = 0\) and \(A_\alpha = 0.\) Doing
this, in the SH case as well as on the local boundary, leads to that \( \hat{\Sigma} \) takes the form

\[ \partial_{\alpha} \Sigma_{\alpha\beta} = 2\epsilon^{\eta\delta} (\Sigma_{\beta\gamma} R_{\delta}), \]  

(13)
since the Gauss constraint \( \hat{\Sigma} \) gives \( \Sigma^2 = 1 \) and hence \( q = 2\Sigma^2 = 2 \).

In the present framework the metric scale factors of BKL are associated with temporal integration of the diagonal components of the conformally Hubble-normalized expansion tensor \( \Theta_{\alpha\beta} \), which can be expressed in terms of the deceleration parameter \( q \) and the Hubble-normalized shear \( \Sigma_{\alpha\beta} \) as follows:

\[ \Theta_{\alpha\beta} = -q\delta_{\alpha\beta} + \Sigma_{\alpha\beta}. \]  

(14)

Furthermore, from the above equation it follows that if one focusses on evolution that quotes out overall evolution as regards scale factors, as done in ‘asymptotic billiard approaches’ [21, 22, 20], then this is associated with the diagonal shear variables \( (\Sigma_1, \Sigma_2, \Sigma_3) := (\Sigma_{11}, \Sigma_{22}, \Sigma_{33}) \).

In all ‘metric scale factor’ approaches (as explicitly illustrated by spatial Fermi and Iwasawa frames below), the equation \( \Sigma^2 \) admits a special solution characterized by \( R_1 = R_2 = R_3 = 0 \), for which furthermore \( \Sigma_{\alpha\beta} \) is diagonal, i.e., they all admit a special case for which the spatial frame is a Fermi frame in which the Hubble-normalized shear is diagonalized. As a consequence \( \partial_{\alpha}\Sigma_{\alpha} = 0 \), and hence \( \Sigma_{\alpha} = \hat{\Sigma}_{\alpha} \), where \( \hat{\Sigma}_{\alpha} \) are constants in the SH case and temporal constants on the local boundary, i.e. spatial functions. This feature is equivalent to the statement that the Kasner solutions appear as fixed points on the dimensionless state space \( S \) (fixed points are also often referred to as equilibrium points, places where the flow of a dynamical system is zero). Since the Kasner solutions are fixed points on the scale-invariantly dimensionless state space \( S \), it follows that they are scale-invariant and admit a spacetime transitive homothety group, as was formally shown in [32, 33, 34]. This is one of the advantages of the Hubble-normalized state space approach to SH cosmology. However, this feature becomes even more pertinent in the conformally Hubble-normalized approach to general inhomogeneous models, since the SH/local boundary correspondence leads to a correspondence between models with spacetime transitive self-similar symmetry groups and fixed points on the local boundary. Special models that asymptotically approach such a self-similar model in the SH case are said to be asymptotically self-similar, leading to that special inhomogeneous models that approach fixed points on the local boundary are naturally characterized as being locally asymptotically self-similar (in the Kasner case they are often called, less geometrically, asymptotically velocity dominated).

The \( \hat{\Sigma}_{\alpha} \) values can be expressed as follows:

\[ \hat{\Sigma}_{\alpha} = 3p_{\alpha} - 1, \]  

(15)

where \( p_{\alpha} \) are the Kasner parameters in the SH case and the so-called generalized Kasner parameters in the local boundary case. These parameters, which satisfy \( p_1 + p_2 + p_3 = 1 = p_1^2 + p_2^2 + p_3^2 \), as a consequence of that \( \Sigma_{\alpha\beta} \) is trace-free and that \( \Sigma^2 = \Sigma_{11}^2 + \Sigma_{22}^2 + \Sigma_{33}^2 = 1 \), describe the Kasner circle \( K_{\circ} \) on \( S \). It follows that a shear diagonalized Fermi frame yields

\[ \partial_{\alpha} H^{-1} = 3H^{-1}, \quad \partial_{\alpha} E_\alpha^{\iota} = 3(1 - p_{\alpha}) E_\alpha^{\iota} \Rightarrow \partial_{\alpha} e_\iota^{\alpha} = 3p_{\alpha} e_\iota^{\alpha} \]  

(no sum over \( \alpha \)),

(16)

where these equations are obtained by inserting the ‘diagonalized’ Kasner solution on the reduced state space \( S \) into the equation (15) for \( H^{-1} \) and into that of (15) for \( E_\alpha^{\iota} \), which can be viewed as a perturbation of the local boundary into the physical state space. Choosing a synchronous time variable \( t \) defined by \( N = 1 \) gives \( \dot{N} = H \) and \( \partial_{\alpha} = H^{-1} \partial_{\alpha} \), which leads to \( \partial_{\alpha} H^{-1} = 3t \) and hence \( H^{-1} = 3t + C \), where \( C \) is a constant in the SH case, while it is a spatial function in the local boundary case. Using the gauge freedom to choose a foliation such that the singularity occurs simultaneously at \( t = 0 \) gives \( H^{-1} = 3t \). This leads to \( \partial_{\alpha} e_\iota^{\alpha} = p_{\alpha} e_\iota^{\alpha} \), which gives \( e_\iota^{\alpha} = \tilde{e}_\iota^{\alpha} t^{p_{\alpha}} \), where \( \tilde{e}_\iota^{\alpha} \) are arbitrary spatial functions in the perturbed local boundary case. This results in the line element

\[ ds^2 = -dt^2 + \sum_{\alpha} t^{2p_{\alpha}} (\tilde{e}_\iota^{\alpha} dx^i)(\tilde{e}_\iota^{\alpha} dx^i), \]  

(17)

which gives the BKL result of the generalized Kasner metric where \( \tilde{e}_\iota^{\alpha} \) can be identified with the ‘BKL Kasner axes.’
There is another way this result can be derived. Consider a given timeline and choose a time variable $\tau$ along this timeline so that $\mathcal{N} = 1$, i.e., $\mathcal{N} = H^{-1}$, and $\partial_0 = \partial_{\tau}$. Then (10) yields $\partial_i H^{-1} = 3H^{-1}$ and $\partial_i e^\alpha_i = 3p_\alpha e^\alpha_i$, which results in $H^{-1} = H^{-1} \exp(3\tau)$, $e^\alpha_i = e^\alpha_i \exp(3p_\alpha \tau)$ where $e^\alpha_i$ are temporal constants. It follows that the singularity occurs at $\tau \to -\infty$ and that $t - \xi = H^{-1} \exp(3\tau)/3$. By choosing a gauge in which the temporal constant $\xi$ is set to zero, so that the singularity occurs at $t = 0$, and by scaling $e^\alpha_i$ with the temporal constant $H^{-1}$ appropriately, we obtain the previous result (17).

Due to axis permutations, the Kasner circle $K^C$ is naturally divided into six equivalent sectors, denoted by permutations of the triple (123) where sector $\alpha\beta\gamma$ is defined by $p_\alpha \in (-1/3, 0) < p_\beta < p_\gamma \in (2/3, 1)$. The boundaries of the sectors are six special points that in the SH case exhibit multiply transitive symmetry groups and belong to the class of locally rotationally symmetric (LRS) solutions (they are even plane symmetric, and hence also axially symmetric, a property that will feature in the discussion in Section 6),

$$Q_\alpha : (p_\alpha, p_\beta, p_\gamma) = \left(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right),$$

$$T_\gamma : (p_\alpha, p_\beta, p_\gamma) = (0, 0, 1).$$

The $Q_\alpha$, $\alpha = 1, 2, 3$, points yield three equivalent LRS solutions with non-flat geometry, while the Taub points $T_\gamma$, $\gamma = 1, 2, 3$, correspond to the Taub representation of Minkowski spacetime.

Finally, note that $\partial_0 E_\alpha^i = 3(1 - p_\alpha)E_\alpha^i$ leads to that $E_\alpha^i \to 0$ towards the past, except at the Taub points, which suggests that if solutions are ‘dominated’ by non-Taub Kasner states they might approach the local boundary, if $E_\alpha^i \to 0$ faster than spatial derivatives might grow, which turns out not to be the case for the recurring spikes discussed later.

To obtain further results require an explicit choice of spatial frame. We begin with a Fermi frame.

### 4.2 Transitions, concatenation and chains in a Fermi frame

In this subsection we pursue outlining how the BKL results can be obtained in the conformally Hubble-normalized state space picture in a Fermi frame. Choosing a Fermi frame, i.e., $R_\alpha = 0$, leads to that (13) results in $\partial_0 \Sigma_{\alpha\beta} = 0$, and hence that $\Sigma_{\alpha\beta} = \tilde{\Sigma}_{\alpha\beta}$, where $\tilde{\Sigma}_{\alpha\beta}$ are constants in the SH case and temporal constants on the local boundary, i.e. spatial functions. The symmetric matrix $\Sigma_{\alpha\beta}$ can therefore be diagonalized by making an appropriate (temporally) constant orthogonal transformation of the spatial frame, which results in the eigenvalues $\tilde{\Sigma}_\alpha = 3p_\alpha - 1$, which leads to the results derived above.

The instability result of BKL of the generalized Kasner solution follows from perturbing $K^C$ on the dimensionless state space $S$ on the local boundary; a linearization in a shear diagonalized Fermi frame yields

$$\partial_0 A_\gamma = 3(1 - p_\gamma)A_\gamma, \quad (19a)$$

$$\partial_0 N_\alpha = 6p_\alpha N_\alpha \quad \text{where} \quad N_\alpha := N_{\alpha\alpha}, \quad (19b)$$

$$\partial_0 N_{\alpha\beta} = 3(1 - p_\gamma)N_{\alpha\beta} \quad \text{where} \quad (\alpha\beta\gamma) = (123) \quad \text{and cycle} \quad (19c)$$

(there also are zero eigenvalues due to that $\partial_0 \Sigma_{\alpha\beta} = 0$, which correspond to a center subspace). All the eigenvalues associated with (19) are strictly positive, and thus stable towards the past, except at (i) the Taub points, which reflects that they are not transversally hyperbolic fixed points of the dynamical system on the local boundary, and (ii) at the arc of the Kasner circle, denoted by $K^C_{\alpha}$, determined by $p_\alpha < 0$ in eq. (19b), which corresponds to the union of sectors $(\alpha\beta\gamma)$ and $(\alpha\gamma\beta)$ and the point $Q_\alpha$, which leads to that $N_\alpha$ is unstable towards the past; hence $K^C$ consists of three equivalent arcs, $K^C_{\alpha}$, $K^C_{\beta}$ and $K^C_{\gamma}$, separated from each other by the Taub points, where each arc is associated with a towards the past unstable $N_\alpha$ variable, which in turn corresponds to a 1-dimensional past unstable subspace on $S$ at each fixed point on $K^C_{\alpha}$. Neglecting the problems the Taub points pose for the moment, it follows that asymptotically all conformally Hubble-normalized spatial connection variables tend to zero at $K^C_{\alpha}$ except for a single diagonal component of $N^{\alpha\beta}$ at each arc $K^C_{\alpha}$, a situation that corresponds to that in a Fermi frame precisely one of the eigenvalues is unstable towards the past.
of $N^{\alpha \beta}$ is always unstable towards the past. Following the nomenclature of [13], unstable variables towards the past are referred to as trigger variables.\footnote{The BKL procedure corresponds to inserting the generalized Kasner solution into the spatial 3-curvature and studying how it destabilizes the generalized Kasner solutions. In the present approach the analogous leading order expression for the conformally Hubble-normalized spatial 3-curvature is obtained by solving [19].}

### Bianchi type II transitions on the local boundary

Setting all spatial commutator variables to zero except for a single trigger variable $N_\alpha$ yields an invariant subset on the local boundary that corresponds to the Bianchi type II models in the SH case. The equations for these models, on $S$ in a shear diagonalized Fermi frame, are given by [26] by setting $N_\alpha \neq 0$, $N_\beta = N_\gamma = 0$, where $(\alpha \beta \gamma) = (123)$ or a permutation thereof, where we refer to the associated subset by $B_{N_\alpha}$. These equations are easily solved, see e.g. [13], and yield a 1-parameter set of heteroclinic orbits, i.e. solution trajectories that connect two distinct fixed points (in the present case, two different Kasner fixed points). More precisely, consider an equilateral triangle that circumscribes $K^\circ$ in such a way so that it is tangential to the three Taub points. Then each Bianchi type II solution, projected onto the Hubble-normalized shear plane (since $\Sigma_1 + \Sigma_2 + \Sigma_3 = 0$) of $B_{N_\alpha}$, is a straight line inside $K^\circ$ that can be extended outside $K^\circ$ through the corner of the equilateral triangle closest to $Q_\alpha$, see e.g. [26].

In [13] the Bianchi type II heteroclinic orbits were referred to as $T_{N_\alpha}$ (Kasner) transitions since their past and future limits are two distinct points on $K^\circ$, thus yielding a map between different Kasner states. To describe this map it is convenient to parameterize the (generalized) Kasner parameters $p_\alpha$ on $K^\circ$ in terms of a (generalized) Kasner parameter $u$. Instead of following BKL and defining it via $p_\alpha \leq p_\beta \leq p_\gamma$ it can be defined in a gauge-invariant way as follows: Consider the quantity

$$p_1 p_2 p_3 = - \frac{u^2 (1 + u)^2}{(1 + u + u^2)^3}. \quad (20)$$

Due to that $p_1 p_2 p_3$ is monotone in $u$ this relation defines $u$ implicitly, and since it can be shown that $p_1 p_2 p_3$ can be constructed from the Weyl scalars it follows that $u$ gauge-invariantly describes the different Kasner states. The parameter $u \in [1, \infty)$, where $u = 1$ corresponds to the $Q_\alpha$ Kasner state while $u = \infty$ yields the Taub Kasner state.

Towards the past, the transition $T_{N_\alpha}$ gives rise to a map between two different Kasner points on $K^\circ$ (where the map is defined to be the identity at $T_\beta$ and $T_\gamma$), see e.g. [13, 10]; expressing the result in the gauge-invariant Kasner parameter $u$ yields the BKL Kasner map [8, 13]:

$$u_+ = \begin{cases} 
  u_+ - 1 & \text{if } u_+ \geq 2 \\
  (u_+ - 1)^{-1} & \text{if } 1 \leq u_+ < 2 
\end{cases}, \quad (21)$$

where $u = u_-$ and $u_+$ are the initial and final Kasner states, respectively, in the direction towards the past.

The ‘generalized’ Bianchi type II solution can be obtained by taking the dimensionless Bianchi type II solution on the local boundary on the state space $S$ and inserting it into the equations for $H^{-1}$ and $E_\alpha^i$, i.e. [26] and [53], respectively, thus perturbing the local boundary, in a similar way as the generalized Kasner solution was obtained above. One can continue the perturbative expansion by linearly perturbing this solution away further from the local boundary, which yields new ODEs, since partial spatial derivatives act only ‘passively’ on the spatial functions of integration that are obtained in solving the equations that arise from the perturbative expansion, order by order. This gives e.g. the BKL results concerning the rotation and asymptotic freezing of Kasner axes, which are associated with the following feature. On the local boundary one can choose a frame in the vacuum case so that for all of class A, including Bianchi types I and II, $N^{\alpha \beta}$ and $\Sigma_\alpha^\beta$ both become diagonal, a feature that is due to that the Codazzi constraint [63] in this case is given by $\epsilon_\alpha^\beta \Sigma^\delta_\beta N_{\delta \gamma} = 0$. Perturbations of the Bianchi type II solutions on the local boundary into the physical state space give rise to non-zero Hubble-normalized spatial frame derivatives, resulting in
$\epsilon_\alpha^{\beta\gamma} \Sigma_\delta^\beta N_\delta^\gamma \neq 0$, which prevent simultaneous diagonalization. BKL assumes that solutions can be described as increasingly small perturbations of sequences of Bianchi type II solutions on the local boundary, which leads to $\epsilon_\alpha^{\beta\gamma} \Sigma_\delta^\beta N_\delta^\gamma \to 0$, which hence explains the BKL asymptotic freezing effect of the Kasner axes. A similar statement holds for the electric and magnetic Weyl tensors, which offers a more geometric way of stating the rotation and asymptotic freezing of Kasner axes properties.

Next, we describe the recent progress that has been achieved as regards SH dynamics (and hence also for dynamics on the local boundary) for Bianchi types VIII and IX in a $\Sigma_{\alpha\beta}$ and $N_{\alpha\beta}$ diagonalized Fermi frame.

**Global past dynamics results for Bianchi types VIII and IX**

Although the Lie contraction hierarchy limits asymptotic dynamics, it does not uniquely determine it, thus making the endeavor of producing theorems about generic initial singularities in SH models a non-trivial task. The first theorems in this area concerning oscillatory singularities, based on earlier work in [26] and by results obtained by Rendall [35], were produced by Ringström in 2000 [36] and 2001 [37]. In the first of these papers it was shown that a generic Bianchi type VIII or IX solution cannot converge to a Taub point on $K^\circ$, and that the past limit set contains at least two distinct points on $K^\circ$, of which at least one is not a Taub point. As a consequence the past singularity in these models must be oscillatory and a scalar curvature singularity. In the second paper it was proved that the past attractor for the vacuum Bianchi type IX models, $A_{IX}$, resides on the union of $K^\circ$ and the Bianchi type II subsets, i.e.,

$$A_{IX} \in \overline{B_{II}} := K^\circ \cup B_{N_1} \cup B_{N_2} \cup B_{N_3},$$

(22)

which, alternatively, can be expressed as that a generic solution satisfies

$$N_1 N_2 + N_2 N_3 + N_3 N_1 \to 0$$

(23)

towards the initial singularity in Bianchi type IX. Furthermore, it was shown in [37] that $A_{IX}$ contains at least three distinct non-Taub points on $K^\circ$. It is noteworthy that there exists, so far, no such theorem for Bianchi type VIII, especially since both BKL and the Hamiltonian billiard approach, used in e.g. [20], assume that (23) holds generically. This is clearly a highly non-trivial, although plausible, assumption, which presumably requires elaboration on how to measure ‘generic.’ Moreover, as discussed in [9], the above ‘attractor theorem’ says nothing about how a generic solution asymptotically approaches $A_{IX}$, nor if all of $A_{IX}$ is really the past attractor. The results of Ringström therefore say e.g. nothing about if the map (21) has any relevance for generic singularities. The proof of Ringström’s attractor theorem does not fully use the structure of the Lie contraction hierarchy, a shorter proof, making more use of these structures as well as using different bounded variables, is given in [38]. To proceed further, however, it is necessary to take a closer look at the structures on $B_{II}$, which leads to the concepts of concatenation and heteroclinic chains.

**Concatenation and heteroclinic chains on $B_{II}$**

In the diagonalized Bianchi types VIII and IX cases the transitions $T_{N_1}$, $T_{N_2}$ and $T_{N_3}$ on $S$ can be uniquely concatenated on $B_{II}$ towards the past by identifying the ‘final’ fixed point of one transition with the ‘initial’ fixed point of another transition. Concatenating a sequence of such orbits towards the past, obtained by means of the above described equilateral triangle circumscribing $K^\circ$, see e.g. [26] [9], yields a heteroclinic chain, which, in general, is infinite. We refer to the heteroclinic chains in the diagonalized class A case on $B_{II}$ as Mixmaster chains.

Note that heteroclinic chains, obtained by joining solutions by means of their asymptotics into chains of solutions connected via fixed points, are not solutions to the Einstein equations themselves. Instead they are the rigorous dynamical systems formulation of the heuristic BKL concept.
of piecewise joined solutions. Note also that since the local boundary is effectively a finite dimensional system, although it really is an infinite set of copies of the same dynamical system, one copy for each spatial point, this concept is still valid on this boundary subset. However, a solution on the local boundary is in general described by several heteroclinic chains, one for each spatial point.

4.3 Heteroclinic chain discretization: Kasner maps

The Mixmaster chains induce iterations of the gauge-invariant map \( [27] \). Let \( l = 0, 1, 2, \ldots \) and let \( u_l \) denote the initial Kasner state of the \( l \)th transition (time direction towards the past), then the iterated BKL Kasner map is given by:

\[
\begin{align*}
\quad u_l \xrightarrow{\text{ith transition}} u_{l+1} : \\
&
\quad \begin{cases} 
\quad u_l - 1 & \text{if } u_l < 2, \\
\quad (u_l - 1)^{-1} & \text{if } u_l \in [2, \infty), \\
\quad \frac{1}{u_l - 1} & \text{if } u_l \in [1, 2]. 
\end{cases}
\end{align*}
\]

In a sequence \((u_l)_l=0,1,2,\ldots\) that is generated by (24), each Kasner state \( u_l \) is called a Kasner epoch. Every sequence \((u_l)_l=0,1,2,\ldots\) possesses a natural partition into pieces called Kasner eras with a finite number of epochs. An era consists of a sequence of monotonically decreasing values of \( u \) that begins with a maximal value \( u_{l_{\text{in}}} \) generated from \( u_{l_{\text{out}}} \) by \( u_{l_{\text{in}}} = (u_{l_{\text{out}}} - 1)^{-1} \), and continues with a sequence of Kasner parameters obtained via \( u_l \rightarrow u_{l+1} = u_l - 1 \); it ends with a minimal value \( u_{l_{\text{out}}} \) that satisfies \( 1 < u_{l_{\text{out}}} < 2 \), so that \( u_{l_{\text{out}}+1} = (u_{l_{\text{out}}} - 1)^{-1} \) begins a new era \([2]\), as exemplified by:

\[
\begin{align*}
\text{era} & : 3.41 \rightarrow 2.41 \rightarrow 1.41 \rightarrow 2.44 \rightarrow 1.44 \rightarrow 2.27 \rightarrow 1.27 \rightarrow \ldots
\end{align*}
\]

Denoting the initial and maximal value of the Kasner parameter \( u \) in era number \( s \) (where \( s = 0, 1, 2, \ldots \)) by \( u_s \), and decomposing \( u_s \) into its integer \( k_s = [u_s] \) and fractional \( x_s = \{u_s\} \) parts, gives \([2][8]\)

\[
u_s = k_s + x_s,
\]

where \( k_s \) represents the discrete length and number of Kasner epochs of era \( s \). The final and minimal value of the Kasner parameter in era \( s \) is given by \( 1 + x_s \), which implies that era number \( s+1 \) begins with

\[
u_{s+1} = \frac{1}{x_s} = \frac{1}{\{u_s\}}.
\]

The map \( u_s \mapsto u_{s+1} \) is the so-called BKL ‘era map.’ Starting from \( u_0 = u_0 \) it recursively determines \( u_s, s = 0, 1, 2, \ldots \), and thereby the complete Kasner sequence \((u_l)_l=0,1,\ldots\).

The era map admits an interpretation in terms of continued fractions. Applying the Kasner map to the continued fraction representation of the initial value \( u_0 \),

\[
u_0 = k_0 + \frac{1}{k_1 + \frac{1}{k_2 + \cdots}} = [k_0; k_1, k_2, k_3, \ldots],
\]

gives

\[
u_0 = u_0 = [k_0; k_1, k_2, \ldots] \rightarrow [k_0 - 1; k_1, k_2, \ldots] \rightarrow \ldots \rightarrow [1; k_1, k_2, \ldots]
\]
\[
\rightarrow u_1 = [k_1; k_2, k_3, \ldots] \rightarrow [k_1 - 1; k_2, k_3, \ldots] \rightarrow \ldots \rightarrow [1; k_2, k_3, \ldots]
\]
\[
\rightarrow u_2 = [k_2; k_3, k_4, \ldots] \rightarrow [k_2 - 1; k_3, k_4, \ldots] \rightarrow \ldots ,
\]

and hence the era map is simply a shift to the left in the continued fraction expansion,

\[
u_s = [k_s; k_{s+1}, k_{s+2}, \ldots] \rightarrow u_{s+1} = [k_{s+1}; k_{s+2}, k_{s+3}, \ldots].
\]

\footnote{The heteroclinic Mixmaster chains also induce iterations of the map that takes one Kasner point to another on \( K^\circ \), for which the permutation freedom has not been quoted out; for an analytic description, see \([10]\).}
Some of the era and Kasner sequences are periodic, notably $u_0 = [(1)] = [1; 1, 1, 1, \ldots] = (1 + \sqrt{5})/2$, which is the golden ratio, gives $u_\omega = (1 + \sqrt{5})/2 \forall \omega$, and hence the Kasner sequence is also a sequence with period 1,

$$(u_n)_{n \in \mathbb{N}} : \frac{1}{2}(1 + \sqrt{5}) \rightarrow \frac{1}{2}(1 + \sqrt{5}) \rightarrow \frac{1}{2}(1 + \sqrt{5}) \rightarrow \frac{1}{2}(1 + \sqrt{5}) \rightarrow \ldots,$$

while this yields two heteroclinic cycles of period 3 in the state space picture (since the axis permutations are not quotiented out in the state space), see the figures in [9]. The discretized description of the Mixmaster chains is suitable for discussing the recent results that have been obtained by perturbing the heteroclinic chains on $S$, results that can be denoted as asymptotic chain theorems, which, in contrast to the past attractor theorems, can be regarded as ‘non-global state space results’ (although the two types of theorems can, of course, be combined).

### 4.4 Asymptotic chain theorems

Although BKL [2] [3] [8], as well as Misner [6] [7], conjectured that the asymptotic dynamics of Bianchi type IX is governed by the Kasner and era maps, it was only recently that rigorous results were obtained that relate these maps to asymptotic dynamics in Bianchi types VIII and IX. To describe these results, it is convenient to use the following classification scheme of Kasner sequences and associated Mixmaster chains [10] [38] (as usual the time direction is towards the past):

(i) $u_0 = [k_0; k_1, k_2, \ldots, k_n]$, i.e., $u_0 \in \mathbb{Q}$. The associated Kasner sequence is finite with $n$ eras and have an associated Mixmaster chain that terminates at one of the Taub points. It has been proven that these sequences are not asymptotically realized in the generic non-LRS case since a Taub point is not the $\omega$-limit set of any non-LRS solution [36] [37] [9].

(ii) $u_0 = [k_0; k_1, \ldots]$ such that the sequence of partial quotients of its continued fraction representation is bounded, with or without periodicity, which corresponds to that the associated Mixmaster chains avoid a neighborhood of the Taub points. In the case of no periodicity and no cycles, Béguin proved that a family of solutions of codimension one converges to each associated chain [39], where cycles must be excluded to avoid resonances in order for the proof to work. By using different techniques, and different differentiability conditions, Liebscher et al [40] proved explicitly that a family of solutions of codimension one converges to each of the 3-cycles associated with $u_0 = [(1)]$. The authors also gave arguments for how their methods could be extended to the present general case. This was explicitly proved in [41], where the authors introduced a new technique that involves the invariant Bianchi type I and II subset structure, which tie the results to the Lie contraction hierarchy, and hence also implicitly to basic physical principles.

(iii) $u_0 = [k_0; k_1, \ldots]$ is an unbounded sequence of partial quotients, which is the generic case. The associated Kasner sequence is unbounded and the associated Mixmaster chain enters every neighborhood of the Taub points infinitely often. As argued in [42], a subset of these chains is relevant to the description of the asymptotic dynamics of actual solutions: for each $u_0$ such that the sequence $(k_n)_{n \in \mathbb{N}}$ can be bounded by a function of $n$ with a prescribed growth rate, there exists an actual solution that converges to the chain determined by $u_0$. On the other hand, chains associated with initial values $u_0 = [k_0; k_1, \ldots]$ with rapidly increasing partial quotients $k_n$, $n \in \mathbb{N}$ are perhaps less relevant for the description of the asymptotic dynamics of actual solutions; if a solution shadows a finite part of such a chain it may be thrown off course at the point where the chain enters a too small neighborhood of the Taub points. The prescribed bound on the growth rate is weak enough to yield generic continued fraction representations, but these results do not say anything about how many solutions actually converge to a given chain, nor if the asymptotic dynamics of a generic initial data set is represented by a heteroclinic chain.

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8Ref. [42] uses different mathematical techniques than the other rigorous papers in this area. As a consequence the results, although plausible, seem to be somewhat controversial in the research community. Due to this, and due to an intrinsic value, it would be of interest if the results could be confirmed, or preferably even extended, with some other independent methods.
The above results imply that $A_{19} = B_4$ is indeed the global past attractor for Bianchi type IX, but $B_4$ is not necessarily the global past attractor for type VIII, since it still has not been excluded that the type VIII attractor also involves the vacuum Bianchi type VI$_0$ subset.

A spatial Fermi frame is not the only frame that is useful. Damour and coworkers [20] have used spatial Iwasawa frames to produce "cosmological billiards" with a Chitre and Misner [21, 22] procedure that relies on assumptions similar to those of BKL. Iwasawa frames lead to a somewhat different picture of generic past dynamics than Fermi frames, as discussed next.

4.5 Transitions, concatenation and chains in an Iwasawa frame

The Iwasawa approach entails diagonalizing and parameterizing the metric by a Gram-Schmidt orthogonalization of the spatial coordinate coframe $\{dx^i\}$ (corresponding to a Cholesky decomposition of a symmetric matrix $A$ into a product $R^T R$, where the diagonal elements of the triangular matrix $R$ are factored out), which in turn yields a natural conformally Hubble-normalized orthogonal frame [13], obtained by setting

$$E_1^2 = E_1^3 = E_2^3 = 0 \Rightarrow N_3 = 0, \quad (R_1, R_2, R_3) = (-\Sigma_{23}, \Sigma_{31}, -\Sigma_{12}).$$

This leads to a dynamical system on $S$ that admits a Kasner circle $K^\circ$ of fixed points given by setting all variables to zero except the diagonal shear variables, which again are given by $\Sigma_\alpha = \Sigma_\beta = 3p_\alpha - 1$. However, by choosing a different frame than the Fermi frame the linearisation of $K^\circ$ on the dimensionless state space $S$ on the local boundary leads to a different result than [19]:

- $\partial_0 A_\alpha = 3(1 - p_\alpha)A_\alpha$ (no sum over $\alpha$),
- $\partial_0 N_{\alpha\beta} = 6(1 - p_\alpha)N_{\alpha\beta}$ $\alpha \neq \beta \neq \gamma \neq \alpha$,
- $\partial_0 N_1 = 6p_1 N_1$,
- $\partial_0 N_2 = 6p_2 N_2$,
- $\partial_0 R_1 = 3(p_3 - p_2)R_1$,
- $\partial_0 R_2 = 3(p_3 - p_1)R_2$,
- $\partial_0 R_3 = 3(p_2 - p_1)R_3$,

where $N_1 = N_{11}$ and $N_2 = N_{22}$ (recall that $N_3 = N_{33} = 0$). As in the Fermi case, the variables $N_{\alpha\beta}$ ($\alpha \neq \beta$) and $A_\alpha$ belong to the past stable subspace of each fixed point of $K^\circ$ (except at the Taub points). In contrast, the variables $(R_1, R_2, R_3)$ and $(N_1, N_2)$ are stable or unstable depending on where the point $(\Sigma_1, \Sigma_2, \Sigma_3)$ is located on $K^\circ$. Finally, the variables $\Sigma_\alpha = \Sigma_{\alpha\alpha}$ belong to the center subspace, i.e., they are temporally constant to first order.

As argued in [13], generically $N_{22} \to 0$ toward the singularity, even though it is past unstable on part of $K^\circ$. Since $N_{33} = 0$, it therefore follows that $T_{N_1}$ are the only $T_{N_{\alpha\beta}}$ transitions. The $T_{N_1}$ and $T_{N_{\alpha\beta}}$ transitions are instead replaced with the frame transitions $T_{R_1}$ and $T_{R_3}$ (as argued in [13], generically also $R_2 \to 0$), which correspond to rotations of Kasner states with $\pi/2$ in the 2-3-plane and 1-2-plane, respectively; note that these rotations should not be confused with the BKL effect of rotation (and eventual freezing) of Kasner axes [9]. The $T_{R_1}$ and $T_{R_3}$ frame transitions correspond to spatial frame rotating versions of the ‘generalized’ Kasner solutions on the local boundary, with $\Sigma_1$ and $\Sigma_2$ being temporal constants, respectively, as shown in the Bianchi type VI$_{-1/9}$ case below.

As in the case of SH class A models, transitions can be concatenated to yield heteroclinic chains [13, 17]. In the present case this means that $T_{N_1}$, $T_{R_1}$, and $T_{R_3}$ transitions form (BKL) ‘Iwasawa chains,’ which describe the oscillatory evolution along individual timelines.

We refer to [13] for more details for the general case. Instead we here take a closer look at the Bianchi type VI$_{-1/9}$ vacuum models. As stated above, the general Bianchi type VI$_{-1/9}$ models are as general as the Bianchi type VIII and IX models and also exhibits an oscillatory singularity (although there so far exists no formal proof of this). Since these models can be expressed in terms of a symmetry adapted spatial Iwasawa frame, they act as toy models for the general field equations.

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9 Note that the arguments in [13] for the suppression of $N_2$ and $R_2$, as well as of the so-called ‘multiple transitions,’ were temporally non-local in character, using the heteroclinic chains on the so-called oscillatory subsets and its associated statistical features. Note that [13] therefore does not only rely on local stability analysis, but have in common with the more recent (rigorous) papers [35, 40, 41] that it makes use of heteroclinic chains.
when these are expressed in such a frame; moreover, the generic past asymptotic behaviour of the Bianchi type VI$_{-1/9}$ models is expected to be described by an attractor that also describes the generic past asymptotic BKL behaviour of the general inhomogeneous case in an Iwasawa frame [13].

**Bianchi type VI$_{-1/9}$ models**

There are two invariant boundary subsets of (11) and (12) that are of particular interest for the past dynamics: The Bianchi type I (Kasner) subset given by $N_1 = A = 0$, $\Sigma^2 = 1$ and the Bianchi type II subset given by $A = R_3 = 0$. In general both subsets are associated with frame rotation for which $R_1^2 + R_3^2 \neq 0$, although they both admit the standard diagonalized Fermi frame representations described in the class A case, where the Kasner case yields the Kasner circle of fixed points $K^O$. Rotations of the spatial frame are just gauge transformations whose effect can be obtained by means of frame invariants. The first two are given by that the shear tensor is trace-free, i.e., $\Sigma_1 + \Sigma_2 + \Sigma_3 = 0$, and that $\Sigma^2$ is determined by the Gauss constraint, i.e., $1 - \Sigma^2 - \frac{1}{12} N_1^2 = 1 - \Sigma^2 - \Omega_k = 0$.

In the Kasner case the determinant $\det \Sigma_{\alpha\beta}$ is preserved yielding

$$\det \Sigma_{\alpha\beta} = \Sigma_1(\Sigma^2_1 - 3 + R_3^2) - \Sigma_3 R_2^2 = \Sigma_3(\Sigma^2_3 - 3 + R_3^2) - \Sigma_1 R_1^2 = 2 + 2T_{p1p2p3}, \quad (33)$$

where $p_1p_2p_3 = -u^2(1 + u)^2/(1 + u + u^2)$ where $u$ gauge-invariantly describes the Kasner state, which, together with the other invariants, describes the orbits on the Kasner subset. We refer to the orbits as multiple $T_{R_iR_3}$ frame transitions, since they excite more than one degree of freedom; the special cases $R_3 = 0$ ($R_1 = 0$) gives the single $T_{R_1}$ ($T_{R_3}$) frame transitions for which $\Sigma_1$ ($\Sigma_3$) is conserved, which follows directly from [13].

In the Bianchi type II case rotations only occur in the 2-3-plane since $R_3 = 0$. It follows that both $\Sigma_1$ and $N_1$ (and hence $\Omega_k$, which is more convenient to use than $N_1$ in this case) are invariant under such transformations, and since the vacuum Bianchi type II case only has a 2-dimensional true dimensionless state space it follows that the equations for $\Sigma_1$ and $\Omega_k$ form a dynamical system invariant under rotations in the 2-3-plane, as can be seen explicitly since

$$\partial_0 \Sigma_1 = -2\Omega_k(4 + \Sigma_1), \quad \partial_0 \Omega_k = -4\Omega_k[\Omega_k - (1 + \Sigma_1)]. \quad (34)$$

It follows that $[(1 + \Sigma_1)^2 + 3\Omega_k]/(4 + \Sigma_1)^2$ is a conserved quantity that describes the orbits on the subset, which we refer to as (multiple) $T_{R_iR_3}$ Bianchi type II transitions; the special case of a Fermi frame $R_1 = 0$ yields the $T_{R_1}$ Bianchi type II transitions encountered in class A. The conserved quantity can be used to reduce the dynamical system to a $u$-parameterized 1-dimensional problem; following [13] and defining $\zeta$ according to

$$\Sigma_1 = -4 + (1 + u^2) \zeta, \quad \Omega_k = \frac{3}{\zeta_+ \zeta_-} (\zeta_+ - \zeta)(\zeta - \zeta_-), \quad (35)$$

where $\zeta_\pm = 3/(1 \mp u + u^2)$ and $0 < \zeta_- < 1$, $0 < \zeta_+ < 3$, where $u = u_-$ characterizes the initial Kasner state (time direction towards the past), yields

$$\partial_0 \zeta = -2\Omega_k\zeta, \quad (36)$$

whose solution, by taking the asymptotic limits, gives the BKL Kasner map (21). The $T_{N_1}$, $T_{R_1}$ and $T_{R_3}$ transitions are associated with the towards the past unstable subsets of $K^O$, which follows from restricting [12] to the present models; $N_1$ is a trigger of instability towards the past on the Kasner arc $K_1^\gamma$; $R_1$ is a past trigger on the arc $(132) \cup T_2 \cup (312) \cup Q_3 \cup (321)$, while $R_3$ is a past trigger on the arc $(321) \cup T_1 \cup (231) \cup Q_2 \cup (231)$. In contrast to the class A case, where there were only single triggers everywhere on $K^O$ (except at the Taub points), which correspond to 1-dimensional unstable subsets towards the past at each unstable fixed point on $K^O$, sectors (321) and (132) have two triggers, $R_1$, $R_3$ and $R_1$, $N_1$, respectively, corresponding to 2-dimensional unstable subsets towards the past, given by the $T_{R_1R_3}$ and $T_{R_1N_1}$ transitions, respectively. It has
have been argued [31,13] that asymptotically, in some generic sense, solutions follow the 1-dimensional subsets only, i.e., towards the past \( R_1 R_3 \to 0 \) and \( R_1 N_1 \to 0 \).\(^{10}\) Finally, note that it is only the \( T_{N_1} \) (and \( T_{R_1 N_1} \)) transitions that result in a change of \( u \) according to the BKL Kasner map \(^{24}\)

Based on the stability analysis of \( K^\circ \), the past attractor of the vacuum type VI\(_{1/9}\) on \( S \) is conjectured to satisfy

\[
A_{VI_{1/9}} \in K^\circ \cup K_{R_3 R_5} \cup B_{R_3 N_1},
\]

(37)

where \( B_{R_3 N_1} \) is the Bianchi type II subset with \( R_1 N_1 \neq 0 \) and \( K_{R_3 R_5} \) is the Kasner subset with \( R_1 R_3 \neq 0 \) (note the difference with the Bianchi type VIII and IX cases, although the ‘attractor subsets’ are gauge-invariantly the same, i.e., they consist of the union of the Bianchi type I and II subsets), while the temporally non-local analysis of \( [13] \), as well as the numerical analysis in \( [31] \), suggests that

\[
A_{VI_{1/9}} = K^\circ \cup K_{R_1} \cup K_{R_3} \cup B_{N_1},
\]

(38)

or even the stronger ‘billiard’ conjecture:

\[
A_{VI_{1/9}} = K^\circ \cup K_{R_1} \cup K_{R_3} \cup B_{N_1}.
\]

(39)

Note that since the BKL Kasner map \(^{24}\) and the associated Kasner era map are described in terms of changes of the gauge-invariant Kasner parameter \( u \), they still give a gauge-invariant characterization of the heteroclinic chains on e.g. \( K^\circ \cup K_{R_1} \cup K_{R_3} \cup B_{N_1} \), although the chain structure looks quite different than in the SH class A case. However, as in that latter case, it should be noted that none of the above attractor statements say anything about how solutions asymptotically follow the heteroclinic chains that are formed by concatenation of the various transitions on the present representation of \( B_1 = K \) and \( B_{11} \).

### 4.6 Other spatial frame choices

Inclusion of matter generates new challenges. In that case \( \epsilon_{\alpha \beta \gamma} \Sigma_{\beta \delta} N^{\delta \gamma} \neq 0 \) in general, and hence \( N^{\alpha \beta} \) and \( \Sigma_{\alpha \beta} \) cannot be be simultaneously diagonalized. In such cases the spatial Fermi and Iwasawa frames are not the only possible useful spatial frame choices, as is illustrated by the SH models (and hence, due to the SH/local boundary correspondence, this suggests that other spatial frame choices might be useful also in inhomogeneous contexts). In the SH case, the general metric of each Bianchi type in a symmetry adapted spatial frame can be diagonalized in a preferred manner by means of a time dependent off-diagonal (special) automorphism transformation \(^{13,27}\). Thus, for example, the general Bianchi type IX (VIII) metric can be diagonalized by means of the \( SO(3) \) (\( SO(2,1) \)) group, which can be implemented by using time dependent Euler angles \(^{13,27}\) in order to diagonalize the spatial metric in a symmetry compatible frame. In the Hamiltonian billiard picture this leads to that the asymptotic ‘big billiard’ in the diagonal Bianchi type IX case is divided into six equivalent ‘small billiards,’ to use the language in \(^{44,45}\), obtainable from each other by means of axis permutations. In the Hubble-normalized state space picture this corresponds to that two of the \( T_{N_1} \) transition degrees of freedom are replaced by two single frame transition degrees of freedom (single frame transitions correspond to ‘centrifugal bounces’ in Hamiltonian approaches \(^{27,20}\) to BKL dynamics, see \(^{13}\)). The situation for a small billiard is therefore completely analogous, modulo axis permutations, to the Iwasawa case. There is however a difference. In the Iwasawa case there is only one small billiard while there are six in the type IX case. Thus, in the general type IX case, relevant when one for example has a general ‘tilted’ perfect fluid, the conjectural ‘BKL attractor,’ projected onto \( S \) in the Hubble-normalized picture, consists of the union of all Iwasawa like attractors that can be obtained by means of axis permutations.\(^{11}\)

Note that since the attractor is composed by the union of several ‘oscillatory attractor subsets,’ this further complicates the goal of tying initial data to specific heteroclinic chains. Moreover, the above just describes the conjectural asymptotic dynamics, perturbations thereof, which are

\(^{10}\)As discussed in \(^{13}\), the assumption that multiple transitions are generically suppressed asymptotically is also an underlying assumption in the Iwasawa billiard approach used by Damour et al \(^{20}\).

\(^{11}\)Perfect fluids with soft equation of state lead to asymptotic vacuum dominance, but, in the state space picture, the spatial velocity of the fluid becomes a test field that generates ‘tilt’ transitions, as illustrated in \(^{12}\).
necessary for investigations of connections with e.g. heteroclinic chains, are also affected by the different models different automorphism groups since these groups give different expressions for ‘centrifugal walls,’ although these expressions are all of the same type in the asymptotic limit.

Another complication arises from the inclusion of matter. In this case there are not only geometric automorphism structures one can make use of, but also structures associated with the matter content. For example, one might want to align a spatial direction with a spatial eigendirection of the stress-energy tensor. However, in general such a direction is not compatible with the automorphism structure, and one is forced to make a choice for what one thinks is most important for the issue one has chosen to address. Nor is the connection between different choices in general a simple one, instead different choices are related by PDEs, even in the SH case.

In the general case without symmetries, there are really no preferred spatial frame choices from a local BKL perspective, however, global issues might make a spatial frame globally preferred. Nevertheless, note that all choices seem to share the same gauge-invariant description of asymptotic BKL dynamics in terms of the gauge-invariant representation of the past attractor as the union of the (gauge-invariant) Bianchi type I and II vacuum subsets on the local boundary (assuming asymptotic vacuum dominance) and the associated changes in the gauge-invariant generalized Kasner parameter \( u \).

### 4.7 Comments on BKL in the conformally Hubble-normalized state space picture

The BKL analysis is based on a synchronous coordinate frame and an ad hoc procedure for producing the generalized Kasner metric as its starting point. This metric is subsequently used to identify Kasner instabilities that are associated with terms that involve spatial coordinate derivatives. However, this is purely due to the use of a coordinate frame. One of many advantages of the present approach, is that all BKL behaviour is associated with the local boundary (and perturbations thereof) for which all conformally Hubble-normalized spatial frame derivatives are zero. Moreover, the local boundary provides the natural link between ‘generalized’ solutions and their connection with the SH case via the SH/local boundary correspondence. As a consequence it is not only possible to recover the BKL results in the present formalism, one can tie them to rigorous concepts and developments in the SH case, and one can go beyond the BKL picture, which further shed light on it.\(^{12}\)

The present Hubble-normalized state space picture permit us, with hindsight, to assess the remarkable results of BKL. They basically consist of two pieces: (i) ‘Local state space’ results that correspond to perturbations of \( K^0 \) and the Bianchi type II solution on the dimensionless local boundary state space \( S \). (ii) The map \( (21) \) and properties associated with its iteration.\(^{13}\)

To do justice to the recent developments in SH cosmology, it should be pointed out that they involve much more that just contextualizing, shedding light and bringing rigor to BKL related issues; some of the recent developments reveal problems and mechanisms that BKL never even discussed, due to that the BKL analysis is limited to a heuristic local state space analysis and impressive intuitive insights. Moreover, as has been discussed, underlying reasons for why BKL-like behaviour occurs at all are being revealed. Nevertheless, much work remains.

A clarification and extension of the BKL picture requires:

1. Identification of the past attractor on the local boundary (since this has not yet been accomplished for e.g. Bianchi type VIII, this is still an open issue), and the relationship between its

\(^{12}\) Above we have shown and outlined how many of the heuristic BKL results fit and can be derived in the present framework. However, details concerning e.g. rotating and freezing Kasner axes have been left to the reader; either one can use the present formalism to derive these results directly by perturbing the generalized Bianchi type II solutions, or one can simply translate the BKL results to it.

\(^{13}\) In addition BKL have numerous ‘local’ results concerning various matter sources, using similar ‘local state space techniques’ as in the vacuum case, but since this is not the focus in this paper we refrain from discussing those results.
detailed heteroclinic chain structure and past asymptotic dynamics (as discussed previously, there are still many open issues tying e.g. initial data to specific heteroclinic chains).

(ii) Perturbations of the past attractor heteroclinic chain structure on the local boundary into the physical state space.

(iii) Contextualization of (i) and (ii) in terms of the full conformally Hubble-normalized state space picture, which e.g. requires also taking into account the non-local, and hence ‘non-BKL,’ recurring spikes, discussed next.

5 Beyond BKL: Recurring spikes

The simplest inhomogeneous vacuum models that (conjecturally) admit oscillatory singularities are the general $G_2$ models, which conveniently can be expressed in terms of a Hubble-normalized spatial Iwasawa frame. Being the simplest inhomogeneous models with oscillatory singularities, they form the natural testing ground for BKL locality, as well as for breaking BKL locality. Furthermore, the $G_2$ models also occur naturally in another context, namely solution generating techniques.

5.1 Hierarchical solution generating structures

There are several solution generating algorithms for models with two commuting Killing vectors. In 2001 Rendall and Weaver developed and applied one of these techniques to asymptotic expansions [46], obtained by means of Fuchsian methods [47][48], in $T^3$ Gowdy vacuum models, i.e., OT $G_2$ vacuum models with spatial $T^3$ topology. They found that the $T^3$ Gowdy models exhibited both ‘true’ and ‘false spikes’, where false spikes were shown to be gauge artifacts while true spikes corresponded to gauge-invariant asymptotic non-uniformities, not explainable in the BKL picture, even though the evolution of each spatial point approaches $K^0$ on the local boundary in the present framework.

In the conformally Hubble-normalized Iwasawa based formalism, the solution generating technique of Rendall and Weaver corresponds to alternatively performing certain frame rotations and so-called Gowdy-to-Ernst transformations to OT $G_2$ models in a foliation in which the area of the symmetry orbits is purely time dependent, the so-called timelike area gauge [49]. In 2008 Lim applied the solution generating algorithm of Rendall and Weaver to explicit solutions instead of asymptotic expansions, using the 1-parameter family of Kasner solutions as the initial seed solutions [50]. This leads to an infinite sequence of 1-parameter solutions that contains the Bianchi type II solutions; their frame rotated version, which is an example of ‘false spike’ solutions, and the 1-parameter family of inhomogeneous ‘spike solutions,’ which are expressible in terms of elementary functions. It turns out that the solutions obtained in this way, combined with axis permutations, form the building blocks for local BKL and non-local ‘non-BKL’ oscillatory behaviour, and hence the building blocks for all known generic oscillatory behaviour are linked hierarchically to each other by means of a solution generating algorithm, and by means of the symmetry based subset/local boundary correspondence.

5.2 Concatenation and permanent spikes in the OT case

The OT $G_2$ models in the timelike area gauge [49], which, e.g., contain the class A Bianchi type VIIo, VIo, II and I models, have non-oscillatory past singularities due to that $K^0$ for each of these models admit a past stable arc. However, it is more convenient to treat the $G_2$ models in an Iwasawa frame rather than in a Fermi frame, especially since this gives particularly simple expressions for the OT spike solutions. In the spatially Hubble-normalized context these inhomogeneous explicit solutions are described by a 1-parameter family of trajectories, one for each value of $|x^3|$, since the solutions admit a discrete symmetry associated with changing the sign of the spatial coordinate $x^3$. Furthermore, the value $x^3 = 0$ is associated with that $N_1$ goes through a zero (it is of course
possible to make a translation and locate the ‘spike surface’ at any value of \(x^3\), which leads to that the Hubble-normalized spatial frame derivatives obtain \(O(1)\) amplitudes for the spike solutions and that they therefore are of similar size as the Hubble-normalized variables. Hence, the Hubble-normalized state space is not only asymptotically bounded and regular on the local boundary towards the past, but also for the spiky non-local behaviour described by the spike solutions.

The spike solutions come in two kinds: the so-called high and low velocity solutions. The trajectories of the high velocity solutions all originate from a common point on \(K^\circ\) and end at a different common point on \(K^\circ\) on \(S\). For this reason, the high velocity solutions were referred to as high velocity spike transitions, \(T_{Hi}\), in [17]. The trajectories of the low velocity solutions also originate from a common point on \(K^\circ\) (as usual the time direction is towards the past), but all ‘non-spike trajectories’ (for which \(|N_1| > 0\) end at a common point on \(K^\circ\) which differs from that of the spike trajectories (those that correspond to a value \(x^3\) for which \(N_1 = 0\)). This gives rise to ‘permanent spikes’, i.e., non-uniformities in e.g. the curvature scalars, which cannot be explained by the BKL picture, even though the evolution along all timelines end at \(K^\circ\) on the local boundary. This reveals an implicit assumption in the BKL picture, asymptotic differentiability; if the BKL picture is to hold for an open set of timelines, avoiding non-local ‘spiky features,’ then this requires \(|N_1| > 0\) in an Iwasawa frame (\(|N_1N_2N_3| > 0\) in a Fermi frame) for those timelines\(^{14}\).

When it comes to the spike solutions, each solution can be viewed as describing the evolution along an \([x^3]\)-parameterized family of timelines, which complicates the issue of concatenation (joining solutions to each other on their asymptotic properties towards the past and future). However, in the case that the evolution along all timelines of a solution share an asymptotic limit, in the present case a fixed point on \(K^\circ\), then one can proceed in the same manner as when concatenating heteroclinic orbits to heteroclinic chains and match solutions to each other at these Kasner points, thus yielding a concatenated chain of solutions. High velocity spike transitions \(T_{Hi}\) and individual frame transitions \(T_{Rj}\) have this feature and can therefore be alternately concatenated by identifying the ‘final’ Kasner point of one transition with the ‘initial’ Kasner point of another transition, see the figures in [17]. Note that this is possible because the entire family of curves representing a \(T_{Hi}\) transition converges to a point on \(K^\circ\) towards the future and to another point on \(K^\circ\) towards the past. Alternate concatenation of high velocity spike transitions \(T_{Hi}\) and individual frame transitions \(T_{Rj}\) towards the past singularity yields finite chains, which were called high velocity chains in [17], however, concatenation away from the singularity, yields infinite high velocity chains that converge to the Taub point \(T_3\).

Due to the absence of \(R_3\) in the OT models, the sector (312) becomes past stable. As a consequence heteroclinic chains on the local boundary end at this sector, a situation that is similar to that of the Bianchi type VI\(_0\) and VII\(_0\) models in a diagonalized Fermi frame where one of the trigger variables \(N_\alpha\) is missing, which leads to a stable arc on \(K^\circ\). Eventually a high velocity chain either end at the sector (312), or it is joined with a low velocity solution (or a Bianchi type II spiky feature, the latter half of a low velocity spike solution), which is possible since these solutions can be joined at a common Kasner point. This yields a final non-uniform state towards the past with the evolution along non-spike timelines ending at sector (312) while ‘spike timelines,’ associated with \(N_1 = 0\), end at sector (132), as depicted in fig. 8 in [17]. It should be pointed out that not all OT solutions end in this way, e.g., some high velocity solutions end at sectors (231), (213), (321) towards the past. However, as can be expected, a local analysis of \(K^\circ\) shows that this requires fine tuning since \(R_3 \neq 0\) in general, which leads to a subsequent \(T_{Ri}\) frame transition (note that zero values of triggers do not in general lead to invariant subsets in the inhomogeneous case, in contrast to the situation in the SH case). Thus the past evolution along each timeline in the OT case have a specific limit associated with \(K^\circ\), and hence these models can be regarded as being past asymptotically pointwise self-similar, even though some of them exhibit non-uniform features not explainable by the BKL picture.

An impressive array of mathematically rigorous results have been accomplished as regards the \(T^3\),

\(^{14}\)There are other non-uniformities that arise from so-called Bianchi type II spiky features, but since these can be viewed as part of low velocity spike solutions we will refrain from discussing them here; there are also false spike solutions associated with zeroes of \(R_3\) or \(R_3\), which we likewise refrain from discussing. Instead we refer to [17] and references therein for further details.
5 BEYOND BKL: RECURRING SPIKES

Gowdy vacuum models, see [51] and references therein. Not unexpectedly, a lot of terminology therefore exists to describe past asymptotic behaviour for these models. However, the success of that terminology is based upon that the evolution along each timeline end at K^∞ (although, unfortunately, one has historically not used K^∞ to describe this, see [17]). The situation changes dramatically in the general case where R_1 is no longer identically zero. Instead of being asymptotically pointwise self-similar, the singularity becomes oscillatory, both as regards BKL behaviour and as regards spiky features. As a consequence, the nomenclature and tools for describing asymptotics in T^3 Gowdy vacuum models are no longer adequate, but, on the other hand, the conformally Hubble-normalized framework is ideal.

5.3 The general G_2 case: Concatenation and infinite spike chains

When expressed in an Iwasawa frame, the general G_2 case have the same Kasner and B_{II} subsets on S as the Bianchi type VI_{1/9} models, and we therefore expect that the asymptotic evolution along timelines with BKL evolution is connected with the same past attractor as in that case, and presumably the models also have the same asymptotic connection with the heteroclinic chains on that attractor. However, in contrast to the SH Bianchi type VI_{1/9} case, the inhomogeneous general G_2 models also exhibit spike chains, which in general are infinite, and hence associated with infinitely recurring spikes.

In contrast to the OT case, low velocity spike solutions do not yield permanent spikes in the general case. The non-uniform structures in the OT case are a result of these models special features, and are not describing generic features. They are a consequence of the stability induced on K^∞ because R_1 = 0, i.e., because of a lack of a trigger degree of freedom and an associated past unstable subset. The OT models are therefore the analogues of special non-oscillatory SH models, which show a few but not the key properties of the general oscillatory SH models, although the difference is even greater due to an infinite dimensional the state space.

The introduction of R_1 leads to that each low velocity spike solution is combined and transported by means of a T_{R_1} frame transition and then joined with part of a high-velocity solution to form a joint low/high velocity spike transition, T_{Jo}, so that the one-parameter family of curves that form a T_{Jo} transition, one for each |x|^3, all begin and end at two distinct fixed points on K^∞ on S. Thus T_{Jo} form a ‘concatenation block’ that can be joined with T_{Hi} and the frame transitions to form spike chains. The name spike chain is appropriate since the Hubble-normalized spatial frame derivatives have the same O(1) magnitude as the Hubble-normalized variables during T_{Hi} and T_{Jo} transitions (which is to be contrasted with BKL evolution for which the Hubble-normalized spatial frame derivatives are identically zero in the asymptotic limit); however, the Hubble-normalized spatial frame derivatives become negligible at K^∞, and hence spike chains yield oscillating recurring spikes, as well as oscillating Kasner states. For similar reasons, the spike chains, just like the BKL heteroclinic chains, are in general infinite, and hence infinite oscillatory recurring spikes are to be expected generically, just like BKL behavior yields infinite oscillatory evolution.

In e.g. Bianchi types VI_0 and VII_0 solutions end at the stable Kasner arc. Solutions can reach this stable part by means of shadowing finite heteroclinic chains of T_{N_o} transitions, or they can ‘drop down’ on this stable arc more directly, due to different initial conditions. However, in Bianchi type IX all orbits are squeezed down onto B_{II} on S and the associated heteroclinic chains, and the BKL picture assumes that the same is going to happen in Bianchi type VIII. There seems to be an analogous situation for the OT contra the general G_2 case, both for BKL and spike dynamics. For example, the Bianchi type II spiky features in the OT case, see [17], become part of the spike chains in the general G_2 case, which follows from that they are described by the latter part of the low velocity spike solutions. Moreover, since the 1-parameter family of trajectories of T_{Jo} all begin and end at two distinct fixed points on K^∞, it follows that the non-uniform permanent spike mechanism in the OT case is gone in a general setting; they are an artefact of that the evolution of the natural

[15]The name of the spike solutions is a compromise. Instead of characterizing the solutions in terms of their past complicated asymptotic features, the nomenclature reflects their simpler asymptotic future behaviour, namely the Kasner point the solutions originate from towards the past, see [17]. However, it would probably have been better to name them after properties that reflect their role for generic oscillatory singularities.
concatenation block $\mathcal{T}_{S_0}$ has been interrupted, i.e., the type of asymptotic non-uniformities found in OT models such as the $T^3$ Gowdy models are consequences of ‘mutilated’ generic structures. Instead new types of non-uniformities are generated in the general case.

In [17] it was shown that each spike transition leads to a rapid narrowing of the spatial size of the recurring spike, and hence an infinite spike chain lead to a recurring spike that has zero spatial size asymptotically, surrounded by BKL evolution since the spatial limit $|x^3| \to \infty$ of the spike is described by heteroclinic BKL chains. Hence timelines for which $N_1 = 0$ (again we refrain from discussing asymptotic gauge features associated with that the frame variables $R_1$ or $R_3$ go through a zero value) exhibit asymptotic oscillatory evolution (characterized by a purely electric Weyl tensor), for which Hubble-normalized spatial frame derivatives are essential, while a different type of oscillation takes place for surrounding ‘BKL timelines’ (oscillations involving both the electric and magnetic parts of the Weyl tensor), which hence leads to spatial non-uniformities, ‘asymptotic gravitational defects,’ that are very different from the permanent spike features in the OT case.

5.4 The past attractor

Based on analytical and numerical results, we expect that we in the general $G_2$ case only need to take into account the structure that is generated by the heteroclinic chains that form the past attractor on the local boundary and the subset that is associated with the spike chains, asymptotically restricted to only describe the evolution along the timelines that form the spike surfaces for which $N_1 = 0$, since we expect this to be the asymptotic limit of the shrinking spatial size of generic infinitely recurring spikes. In the general $G_2$ case in an Iwasawa frame this motivates the following conjecture for the associated past attractor $\mathcal{A}_{G_2}$, describing where generic asymptotic evolution resides along non-spike and spike timelines:

$$\mathcal{A}_{G_2} \in \begin{cases} K^\circ \cup K_{R,R_3} \cup B_{R_1 N_1} & \text{if } N_1 \neq 0 \\ K \cup S_{T_{\tiny N_1}} \cup S_{T_{\tiny 3}} & \text{if } N_1 = 0 \end{cases},$$

or even the stronger ‘generalized billiard’ conjecture

$$\mathcal{A}_{G_2} = \begin{cases} K^\circ \cup K_{R_1} \cup K_{R_3} \cup B_{N_1} & \text{if } N_1 \neq 0 \\ K \cup S_{T_{\tiny N_1}} \cup S_{T_{\tiny 3}} & \text{if } N_1 = 0 \end{cases},$$

where $S_{T_{\tiny N_1}}$ and $S_{T_{\tiny 3}}$ refers to the evolution along timelines for which $N_1 = 0$, described by the subsets associated with $T_{S_0}$ and $T_{S_b}$; furthermore, it is assumed that $N_1 = 0$ only occurs at isolated values of $x^3$, since $N_1 = 0$ for a continuous range of $x^3$ would require non-generic fine tuning. The above also assumes that $R_1 R_3 \neq 0$ for all $x^3$ that are of relevance for the small spatiotemporal neighborhood of the singularity that is under scrutiny, i.e., the above description does not take into account ‘false’ (i.e. gauge) recurring spikes induced by zeroes in the frame rotation variables.

5.5 Maps and statistics

The BKL picture assumes that the evolution of an open, asymptotically differentiable, set of spatial points is attracted to the union of the Kasner and Bianchi type II subsets on the local boundary, and that the evolution is asymptotically described by (generic) heteroclinic chains on that subset.

In terms of the gauge-invariant Kasner parameter $u$, such heteroclinic chains, irrespective if e.g. a Fermi frame or an Iwasawa frame is used, lead to a discrete representation of the dynamics, given by the BKL Kasner map (24) and its associated era map (27), (30). Since $u_s \in [k_s, k_s + 1)$ (cf. (30)), the number $k_s$ describes the discrete length and the number of Kasner epochs of era $s$. Therefore, passing on to the stochastical interpretation of (generic) Kasner sequences of epochs, it follows that the probability that a randomly chosen era $s$ of a Kasner sequence $(u_i)_{i=0,1,2,\ldots}$ of epochs has length $m \in \mathbb{N}$ corresponds to the probability that $k_s = m$, or, equivalently, to the probability that $u_s \in [m, m + 1)$. Since the sequence $(k_0, k_1, k_2, \ldots)$ arises as the continued fraction expansion of $u_0$ this probability corresponds to the probability that a randomly chosen partial quotient in the continued fraction expansion is equal to $m$. This results in Khinchin’s law [22], which states
that the partial quotients of the continued fraction representation of a generic real number are distributed like a random variable whose probability distribution is given by

\[ K(m) = \log_2 \left( \frac{m + 1}{m + 2} \right) - \log_2 \left( \frac{m}{m + 1} \right), \]  

which leads to

\[ \text{Probability}(\text{length of era} = n) = L(n) = K(n), \]

see Table II.

In [50] and [54] it was shown that \( T_{\text{Hi}} \) yields a map between different Kasner states which is obtained by applying the BKL Kasner map (24) twice. Remarkably the same result was obtained in [17] for \( T_{\text{Jo}} \). Thus a spike transition \( T_{\text{Hi}} \) or \( T_{\text{Jo}} \) that takes place as part of a spike chain leads to a change in Kasner state, which in terms of the gauge-invariant Kasner parameter \( u \) results in the following spike (Kasner) map:

\[
\begin{align*}
  u_+ &= \begin{cases} 
    u_- - 2 & u_- \in [3, \infty), \\
    (u_- - 2)^{-1} & u_- \in [2, 3], \\
    ((u_- - 1)^{-1} - 1)^{-1} & u_- \in [3/2, 2], \\
    (u_- - 1)^{-1} - 1 & u_- \in [1, 3/2].
  \end{cases}
\end{align*}
\]

Iterations of this spike map generate, from every initial value \( u_0 \in [1, \infty) \), a finite or infinite recurring spike-generated sequence of Kasner epochs \((u_i)_{i=0,1,2,...}\). This sequence can be partitioned into recurring spike-induced eras, which we denote as spike (Kasner) eras, or, for brevity, eras. As in the usual case of eras an era is naturally defined by a sequence of monotonically decreasing values of \( u \), and hence \( u_l \) and \( u_{l+1} \) belong to the same era if \( u_{l+1} = u_l - 2 \). If \( u_{l+1} \) arises from \( u_l \) by one of the other three laws of (44), we speak of a change of era. This is exemplified by

\[
\begin{align*}
  \underbrace{7.29}_{\text{era}} \to \underbrace{5.29}_{\text{era}} \to \underbrace{3.29}_{\text{era}} \to \underbrace{1.29}_{\text{era}} \to \underbrace{2.45}_{\text{era}} \to \underbrace{2.24}_{\text{era}} \to \underbrace{4.16}_{\text{era}} \to \underbrace{2.16}_{\text{era}} \to \underbrace{6.14}_{\text{era}} \to \underbrace{4.14}_{\text{era}} \to \underbrace{2.14}_{\text{era}} \to \ldots
\end{align*}
\]

Denote the initial (= maximal) value of the Kasner parameter \( u \) in era number \( s \) (where \( s = 0, 1, 2, \ldots \)) by \( u_s \). The spike map induces an era map \( u_s \mapsto u_{s+1} \), which recursively determines \((u_s)_{s \in \mathbb{N}}\) from \( u_0 = u_0 \), and thereby the complete spike induced sequence \((u_i)_{i=0,1,2,...}\) of Kasner epochs. The length of an era \( s \) is determined by the value of \( u_s \): If \( u_s \in [m, m+1) \) for some \( m \in \mathbb{N} \), then the length of the era \( s \) is \( m/2 \), if \( m \) is even, and \((m+1)/2\), if \( m \) is odd. In the stochastic context, in analogy with (42), let \( K(m) \) denote the probability that a randomly chosen element of an era sequence \((u_s)_{s=0,1,2,...}\) lies in the interval \([m, m+1)\). Let \((u_s)_{s \in \mathbb{N}}\) be a generic spike-induced sequence of eras. Then, as shown in [53], the probability that a randomly chosen era of \((u_s)_{s \in \mathbb{N}}\) lies in the interval \([m, m+1)\) is given by

\[ K(m) = \log_3 \left( \frac{m + 2}{m + 3} \right) - \log_3 \left( \frac{m}{m + 1} \right). \]

Moreover, as also shown in [53], it follows that if \((u_s)_{s \in \mathbb{N}}\) is a generic spike induced sequence of eras, then the probability that a randomly chosen era in this sequence possesses length \( n \) is given by

\[ \text{Probability}(\text{length of era} = n) =: L(n) = \log_3 \left( \frac{2n + 1}{2n + 3} \right) - \log_3 \left( \frac{2n - 1}{2n + 1} \right). \]

It is of interest to compare some consequences of the probability distribution (42), which determines the probabilities for prescribed lengths of BKL eras in BKL sequences of Kasner epochs according to eq. (43), and the probability distribution (45), which determines the probabilities for prescribed lengths of eras in spike-induced sequences of Kasner epochs according to eq. (46). As seen in Table I the eras have the tendency of being shorter than BKL eras. The probability that an era contains one epoch is larger than 50%, while the probability that an era consists of \( n > 1 \) epochs is smaller than that of a BKL era. Asymptotically, for \( n \gg 1 \),

\[
\begin{align*}
  \text{Probability}(\text{length of era} = n) &= (\log 2)^{-1} n^{-2} \left( 1 - 2n^{-1} + O(n^{-2}) \right), \\
  \text{Probability}(\text{length of era} = n) &= (\log 3)^{-1} n^{-2} \left( 1 - n^{-1} + O(n^{-2}) \right).
\end{align*}
\]
and hence the two probabilities are asymptotically proportional with a proportionality factor \( \log 2 / \log 3 \).

It is also of interest to follow [13, 17] and introduce \textit{small and large curvature phases} (the nomenclature comes from the properties of the curvature tensor on \( K^\circ \), where we recall that the Taub points correspond to the flat spacetime). To do so, let \( \Upsilon > 3 \) (although \( \Upsilon \gg 1 \) is most interesting).

A small curvature phase of a BKL or spike induced sequence of Kasner epochs \( (u_l)_{l \in \mathbb{N}} \) is defined as a connected and inextendible piece \( \mathcal{L} \subset \mathbb{N} \) such that \( u_l > \Upsilon \ \forall l \in \mathcal{L} \). During a small curvature phase \( u_l \) is thus monotonically decreasing from a maximal value by the BKL map to a minimal value in the interval \( (\Upsilon, \Upsilon + 1] \), while the spike map yields a monotonic decrease from a maximum value to a minimal value in the interval \( (\Upsilon, \Upsilon + 2] \). The complement of the concept of a small curvature phase is a large curvature phase, which is defined as an inextendible piece of the sequence of Kasner epochs such that \( u_l \leq \Upsilon \) for all \( l \).

While a small curvature phase can be viewed as an era/\( \text{era} \) that is terminated prematurely at \( \Upsilon \), a large curvature phase typically consists of many eras/\( \text{era} \), where small and large curvature phases occur alternately. In the following BKL example, where the choice \( \Upsilon = 4 \) has been made, the large curvature phase contains two and a half eras.

\[
\begin{align*}
\text{small curvature phase} & \quad \text{large curvature phase} \\
7.29 \rightarrow 6.29 \rightarrow 5.29 \rightarrow 4.29 & \quad 3.29 \rightarrow 2.29 \rightarrow 1.29 \rightarrow 3.45 \rightarrow 2.45 \rightarrow 1.45 \rightarrow 2.24 \rightarrow 1.24 \rightarrow \ldots
\end{align*}
\]

Combining the probabilistic viewpoint with the concept of small/large curvature phases lead to a fundamental result in the description of the BKL and spike induced Kasner sequences. As shown in [53], for generic Kasner sequences \( (u_l)_{l \in \mathbb{N}} \), small curvature phases dominate over large curvature phases in the following sense: Let \( (u_l)_{l \in \mathbb{N}} \) be a generic BKL or spike-induced Kasner sequence and let \( \Upsilon \) be arbitrarily large. Then for a randomly chosen epoch \( u \) the probability for the event \( u > \Upsilon \) is one and the probability for the event \( u \leq \Upsilon \) is zero. The underlying reason for the dominance of small curvature phases is the failure of the probability distributions (42), (45); (43), (46), to generate finite expectation values, since

\[
\sum_{m=1}^{\infty} m K(m) = \sum_{m=1}^{\infty} m L(m) = \infty, \quad \sum_{m=1}^{\infty} m \bar{K}(m) = \infty = \sum_{m=1}^{\infty} m \bar{L}(m),
\]

which is due to the infinite tail of the distributions. Accordingly, the average length of an era/\( \text{era} \) is ill-defined.

### 5.6 More about spikes

A repeated application of the solution generating algorithm [46, 50] yields solutions with an increasing number of spikes, and applying it infinitely many times presumably lead to solutions with

| Sequence | 1     | 2     | 3     | 4     | 5     | 10    | 100   | 500   |
|----------|-------|-------|-------|-------|-------|-------|-------|-------|
| era      | 41.50 | 16.99 | 9.31  | 5.89  | 4.06  | 1.20  | 1.4 \times 10^{-2} | 5.7 \times 10^{-4} |
| era      | 36.91 | 16.60 | 9.60  | 6.28  | 4.44  | 1.39  | 1.8 \times 10^{-2} | 7.2 \times 10^{-4} |

**Table 1:** This table first describes probabilities (in %) that a randomly chosen element of a BKL/spike-generated Kasner sequence of Kasner epochs \( (u_l)_{l \in \mathbb{N}} \), is in the interval \([m, m+1)\), \( m = 1, 2, 3, \ldots \), see (42) and (45). The table then describe the probabilities (in %) that a randomly chosen era/\( \text{era} \) of a BKL/spike generated sequence of Kasner epochs, is of a prescribed length. These results are obtained from the probability distributions \( L(m) = K(m) \) and \( \bar{L}(m) \), see (43) and (46).

ininitely many spikes. All these solutions belong to the OT case. However, in the general $G_2$ case the single spikes of a multiple spike solution asymptotically become part of spike chains described by the individual spike solutions (and the frame transitions). The reason for this is that the different spikes in a multiple spike solution asymptotically become causally disconnected due to asymptotic silence$^{16}$ and as a consequence they can individually be described by the single spike solutions towards the singularity $^{54}$. Thus the single spike solutions, and the associated spike chains, suffice to describe asymptotic spike evolution in the general $G_2$ case.

Due to the Hubble-normalized state space structure that has been revealed by the solution generating algorithm, and due to numerical investigations, it seems reasonable to assume that there exist solutions in the inhomogeneous OT case without spikes, with a finite number of spikes, and with infinitely many spikes. These solutions are connected with BKL and spike chains in the general $G_2$ case, which suggests that there in some sense might exist: (i) a generic set of solutions without spikes and pure BKL behavior, (ii) a generic set of solutions with a finite set of recurring spikes, (iii) a generic set of solutions with infinitely many recurring spikes, possibly forming a dense set (a first step to prove this would be to investigate if the solution generating algorithm, when combined with e.g. scalings in $x^3$, can lead to solutions with a dense set of permanent spikes in the OT case).

The presently discussed recurring spikes are located at fixed spatial locations due to the choice of initial data. In general, recurring spikes are moving in space. It may be that they asymptotically freeze, but this is an open issue. If they do freeze, then our present knowledge about recurring spikes form the first step in understanding general recurring spike behavior, otherwise perhaps not.

It should also be pointed out that the analytic results obtained from the solution generating technique, as well as the numerical explorations that have been undertaken so far, all involve special initial data that excludes or explicitly includes spikes. It is not known how spikes might form in a dynamical situation, nor is it known if spikes can annihilate each other in such situations, or if they can trigger additional spike formation, i.e., it is an open issue if destructive or constructive spike interference, respectively, can occur. It is also unknown if there are boundary conditions associated with special physical conditions that explain the existence of recurring spikes.

5.7 The general inhomogeneous case

The primary importance of the $G_2$ models regarding generic singularities is not the models themselves, but that they appear in the context of the partially local $G_2$ boundary subset. Although there is a one-to-one correspondence with the $G_2$ solutions and those on the partially local $G_2$ boundary, the interpretation of the solutions on this latter subset is quite different. In contrast to the $G_2$ case for which the symmetries impose a type of ‘stiffness’ to the spike surfaces, these surfaces are more ‘flexible’ in the general case without symmetries. As a consequence, spike surfaces can a priori intersect in curves that in turn can intersect at points in the general case, which possibly lead to new spike dynamics. At present it is not known whether such intersections persist or recur, although weak numerical evidence suggests that intersections only occur momentarily. If this is correct, it follows that spike intersections may be irrelevant asymptotically, at least in some generic sense. This would then imply that the BKL picture in combination with $G_2$ spike oscillations may capture the essential features of generic spacelike singularities $^{17}$.

$^{16}$Assuming that the heteroclinic BKL chains and spike chains are both ‘dominated’ by the non-Taub Kasner states that they generate on $K^2$, and for which $E_\alpha^i \to 0$, it seems plausible that $E_\alpha^i \to 0$ in both cases, which leads to an associated formation of shrinking particle horizons along both BKL and spike timelines, a conclusion that also has numerical support in the case of $G_2$ models.
6 Discussion

6.1 The physical context of generic spacelike singularities

Generic spacelike singularities are traditionally referred to as being cosmological singularities, but it is not clear that this is their natural physical interpretation, although one can give the following argument: The Big Bang singularity must have been a generic one, and presumably also a spacelike singularity, unless the pre-inflationary universe was fine-tuned, which in turn would require an explanation. On the other hand, there are several arguments that suggest that generic spacelike singularities describe (part of) the singularities that form due to black hole formation. Due to the complexity of the real universe, one expects that an open set of initial data lead to black hole formation. Seen from the outside the black holes gravitationally radiate away their individual properties leading to external asymptotic states described by mass and angular momentum only (charge also remains, but in the real universe this is unimportant), but the spacetime inside a physical, non-idealized, black hole is both unique and complex, where increasing gravity eventually lead to a generic singularity that reflects the black hole’s formation. This scenario obtains further support from the structure we have obtained for generic spacelike singularities. As previously discussed, generic singularities are dominated by small curvature phases, both as regards BKL behaviour and recurring spikes, i.e., they are dominated by a Taub state. The Taub subset is in turn highly symmetric, in particular it is axially symmetric. One might hence ask, why would a cosmological singularity be dominated by almost axisymmetry? On the other hand, this is precisely what one would expect for a black hole singularity.

Another interesting clue for the physical nature of generic singularities comes from the following observation. Consider the SH Kantowski-Sachs region of the Schwarzschild solution in the usual Schwarzschild coordinates, where $r$ then is a timelike coordinate. Then the spacelike singularity is asymptotically self-similar and described by the locally rotationally symmetric state given by $u = 1$, i.e., the singularity corresponds to a $Q_\alpha$ state, while the horizon corresponds to a coordinate singularity described by a Taub state $u = \infty$. As seen from the stability properties of $K^O$, the Schwarzschild singularity is unstable and so is the horizon structure, and due to that it is connected with the Taub state, in a highly nonlinear fashion. The Taub points turn out to belong to a larger Taub subset on and off the local boundary. When considering properties of explicit solutions, this subset is connected with horizons, caustics, and e.g. (weak) null singularities, see [14]. Due to their ‘Taub dominance’, are generic spacelike singularities therefore in some vague sense almost weak null singularities? If so, speculations that black hole singularities might have a generic null singularity connected with a generic spacelike singularity therefore gains support, from the explicit solution examples in [14] as well as from the structure of generic spacelike singularities.

Moreover, gravitational collapse may provide an explanation for why spikes occur; perhaps they are consequences of increasingly nonlinear interactions of gravitational waves due to increasing amplitudes during the final stages of gravitational collapse (this is to be contrasted with the cosmological context: What physical motivation can cosmology give for the structure of generic singularities, including recurring spike formation?). Furthermore, shifting focus from cosmology to black hole formation brings weak cosmic censorship into light in an area that has been dominated by closed spatial topologies and strong cosmic censorship. The above is admittedly speculative, but so is the statement that generic singularities are cosmological in nature. Can we afford to not take the possibility that generic spacelike singularities might have to do with gravitational collapse seriously?

\[17\] A multiverse scenario would also provide possible arguments for that cosmological singularities might be generic in nature.

\[18\] The use of solution generating algorithms is far from exhausted when it comes to generic singularities, and perhaps they could shed light on the possible connection between generic spacelike and null singularities; at least it seems to be an area that has not been explored.
6.2 The dangers and possibilities of special models

Much recent mathematically rigorous work on singularities has been focused on establishing strong cosmic censorship within the context of models with quite high symmetry and closed spatial topology, such as e.g., the $T^3$ Gowdy models. However, as discussed in this paper, these models, which exhibit pointwise asymptotic self-similarity, are highly misleading for generic singularities, and it is really a generic context that is of relevance for cosmic censorship. Moreover, a generic context puts topological issues in a completely different light than when one deals with highly special models. The relevance of special models for the generic cases is not the models themselves, but their one-to-one correspondence with the local and partially local boundary subsets, and these subsets are located in a state space picture for generic models for which the topological issues are completely different than for the symmetric models that are associated with these boundaries. Furthermore, there are only a few solutions on these subsets that are relevant in a general context, and their role is as spatiotemporally local (e.g., connected with the particle horizon scale) building blocks for the asymptotic description of generic singularities, and hence their topology is completely irrelevant. This is illustrated by the following: vacuum Bianchi type I and II models are essential building blocks for the understanding of the singularities of the most general Bianchi models such as Bianchi type IX, irrespective of the fact that e.g. the Bianchi type IX models have a spatial topology that is completely different from the spatial topologies Bianchi types I and II might exhibit.

The key thing is hence one of ‘topological timing’, i.e., when to address topological issues. At this stage of affairs, there are a number of well formulated mathematical issues that probably are of more pressing importance than topological questions. To eventually address ultimate goals such as cosmic censorship, one first needs to solve several proximate goals, e.g., the asymptotic construction of a generic spacetime in a small spatiotemporal vicinity (i.e., in a small domain of dependence) of a generic singularity. The present Hubble-normalized state space picture offers a way to start addressing this issue by considering initial data in the state space that are close to the expected attractor (‘small initial attractor data’ that leads to a small spatiotemporal domain of dependence region); first within the context of generic $G_2$ models, then for models with one or none spatial Killing vectors.

Thus it is clear that focusing too much on properties such as the topology of special models might sometimes be detrimental for the progress of some issues (although it might be useful for other contexts) such as the character of generic singularities and cosmic censorship. However, some models offer the possibility of providing manageable problems that might shed light on some issues that are of relevance for quite general circumstances. In particular one can construct a hierarchy of toy models that mimic various properties of subsets that are related to the attractor for generic spacelike singularities, in different spatial frame representations, thus providing a mathematically and physically progressive research program. Examples of such toy models can be obtained from various ‘billiard problems,’ see [21] [22] [20] [13]. Toy models are also of interest as an intermediate step for an attempt to quantize gravity, and their structures, especially when combined with structures revealed by solution generating algorithms, may also provide clues and asymptotic observables that could be used to asymptotically quantize gravity where it really needs to be quantized, in the extreme gravity region of generic spacetime singularities.

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