Open String Amplitudes in Various Gauges

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Abstract

Recently, Schnabl constructed the analytic solution of the open string tachyon. Subsequently, the absence of the physical states at the vacuum was proved. The development relies heavily on the use of the gauge condition different from the ordinary one. It was shown that the choice of gauge simplifies the analysis drastically. When we perform the calculation of the amplitudes in Schnabl gauge, we find that the off-shell amplitudes of the Schnabl gauge is still very complicated. In this paper, we propose the use of the propagator in the modified Schnabl gauge and show that this modified use of the Schnabl gauge simplifies the computation of the off-shell amplitudes drastically. We also compute the amplitudes of open superstring in this gauge.
1 Introduction

One of the recent striking achievements in string field theory is the analytic proof of Sen’s conjectures [1, 2]. In [3], Schnabl constructed an analytic solution for the equation of motion

\[ Q_B \Phi + \Phi \ast \Phi = 0, \quad (1.1) \]

in Witten’s cubic string field theory [4] and proved that the height of the tachyon potential at the vacuum is related to the tension of the D-brane [1]. The consistency of the solutions has been checked in [5, 6]. Subsequently, the Sen’s third conjecture which states that there is no physical state at this vacuum was proved analytically [7].

The equation of motion (1.1) is a highly non-linear equation with an infinite numbers of degrees of freedom. In the Siegel gauge \( b_0 \Phi = 0 \), which is traditionally used for the most of the computation, the equation can be solved by tedious numerical calculation such as level truncation [8, 9, 10]. In this gauge, the calculations of the amplitudes are also formidable task [11, 12].

Recent developments of the string field theory rely heavily on the use of the proper gauge for the calculation. Schnabl realized that the gluing rule of string field theory does not match with the Siegel gauge and used another gauge which is more useful in the star operation [3]. Subsequent proof of the absence of the physical degree of freedom also relies heavily on the use of this gauge [7]. In [13], the technique has been generalized to obtain solutions for a ghost number zero string field equation.

Another problem in the string field theory is the complicated expression of off-shell amplitudes [11, 12]. Recent developments suggest that this gauge simplifies the analysis of the amplitudes. However, the propagator in this gauge turns out not to be convenient for explicit calculation which was stated in [3].

In this paper, we propose the use of another gauge for the quantum fluctuation fields. We will show that this choice of gauge for the propagator drastically simplifies the calculation of the off-shell amplitudes in Witten’s cubic string field theory. We also show that
the modified use can also be applied to WZW-like action [14] of open superstring field theory.

This paper is organized as follows. In the next section, we will review the star calculus in \( \tilde{z} \) coordinate and how to calculate four point amplitudes. In section 3, we will compute the expression of four-tachion off-shell amplitudes in the Schnabl gauge. We will find that the amplitudes is rather complicated although it reduces to the known amplitudes at on-shell. In section 4, We will propose the use of the modified Schnabl gauge for the propagator. We will show that the modified use of the gauge simplifies the amplitudes drastically. We will give a formula for four point amplitudes in this gauge. It will be shown that four point amplitudes of non-Abelian gauge theories reduce the quartic terms which are expected from the Yang-Mills action [15]. In section 5, we show how to use the modified gauge in WZW-like action of the superstring to compute the off-shell string amplitudes. In section 6, we compute the four point amplitude for tachyons in GSO(\(-\)) sector and the effective quartic terms of the gauge fields in the zero momentum limit. The final section is devoted to some discussions.

2 Witten’s cubic interaction in \( \tilde{z} \) coordinate and amplitudes

In Witten’s open string field theory, the gluing condition simplifies in the coordinates \( \tilde{z} = \arctan z \). In this coordinate, the primary state \( \phi(z) \) of dimension \( h \) is given by [3]

\[
\tilde{\phi}(\tilde{z}) = \left( \frac{dz}{d\tilde{z}} \right)^h \phi(\tan \tilde{z}) = (\cos \tilde{z})^{-2h} \phi(\tan \tilde{z}).
\] (2.1)

The scaling generator can be written by the energy momentum tensor in this coordinate as

\[
\mathcal{L}_0 = \oint \frac{d\tilde{z}}{2\pi i} \tilde{z} T_{\tilde{z}\tilde{z}}(\tilde{z}) = L_0 + \sum_{k=0}^{\infty} \frac{2(-1)^{k+1}}{4k^2 - 1} L_{2k}.
\] (2.2)

The scaling operator \( U_r \) can be defined as

\[
U_r = \left( \frac{2}{r} \right)^{c_0}.
\] (2.3)
The action of $U_r$ with the field of conformal dimension $h$ is simply given by

$$U_r\tilde{\phi}(\tilde{z})U_r^{-1} = (\frac{2}{r})^h\tilde{\phi}(\frac{2}{r}\tilde{z}). \quad (2.4)$$

A non-trivial property of this operator is

$$e^{-\beta\hat{L}_0} = U_{2+2\beta}^\dagger U_{2+2\beta}, \quad (2.5)$$

where

$$\hat{L}_0 = L_0 + L_0^\dagger. \quad (2.6)$$

Because of this property, it seems convenient to define an operator

$$\hat{U}_r = U_r^\dagger U_r. \quad (2.7)$$

We can easily find $\hat{U}_2 = 1$ and the product rule is

$$\hat{U}_r\hat{U}_s = \hat{U}_{r+s-2}. \quad (2.8)$$

Three vertex of string field theory defines a mapping gluing two fields into one state. In the coordinate system, the mapping can be simply given by

$$\tilde{\phi}_1(0)\tilde{\phi}_2(0)|0\rangle = \hat{U}_3\tilde{\phi}_1\left(\frac{\pi}{4}\right)\tilde{\phi}_2\left(-\frac{\pi}{4}\right)|0\rangle. \quad (2.9)$$

Since the BPZ conjugate of the states are given by $z' = -1/z$, the conjugation can be expressed as $\tilde{z}' = \tilde{z} \pm \pi/2$ in $\tilde{z}$ coordinate. For example, the conjugate of the above state is given by

$$(\hat{U}_3\tilde{\phi}_1\left(\frac{\pi}{4}\right)\tilde{\phi}_2\left(-\frac{\pi}{4}\right)|0\rangle)^\dagger = \langle 0|\tilde{\phi}_1\left(-\frac{\pi}{4}\right)\tilde{\phi}_2\left(-\frac{\pi}{4}\right)\tilde{\phi}_1\left(\frac{\pi}{4}\right)\tilde{\phi}_2\left(\frac{\pi}{4}\right)|0\rangle. \quad (2.10)$$

More generally, the gluing of the states of the form $\hat{U}\tilde{\phi}$ takes a simple form

$$\hat{U}_r\tilde{\phi}_1(\tilde{x})\tilde{\phi}_2(\tilde{y})|0\rangle = \hat{U}_{r+s-1}\tilde{\phi}_1(\tilde{x} + \frac{\pi}{4}(s-1))\tilde{\phi}_2(\tilde{y} - \frac{\pi}{4}(r-1))|0\rangle. \quad (2.11)$$

In order to obtain the exact solutions, Schnabl used a gauge [3]

$$\mathcal{B}_0\Phi = 0, \quad (2.12)$$
where $B_0$ is the zero mode of the $b$ ghost in the $\tilde{z}$ coordinate

$$B_0 = \oint \frac{d\tilde{z}}{2\pi i} \tilde{z} b(\tilde{z}) = b_0 + \sum_{k=0}^{\infty} \frac{2(-1)^{k+1}}{4k^2 - 1} b_{2k}. \quad (2.13)$$

Its anti-commutator with BRST charge is given by $\{Q_B, B_0\} = L_0$. For later convenience, we define

$$\hat{B}_0 = B_0 + B_0^\dagger, \quad (2.14)$$

from which we find a relation $\{Q_B, \hat{B}_0\} = \hat{L}_0$. The advantage of the Schnabl gauge is that the form of the fields including ghosts and $\hat{L}_0, B_0$ close under the star operations and the action of the BRST charge [3]. This choice turns out to be crucial for obtaining exact solution of eq. (1.1).

Useful identities including $\hat{B}_0$ are

$$\hat{B}_0 \hat{U}_r = \hat{U}_r \hat{B}_0, \quad (2.15)$$

$$\hat{B}_0 U_r^\dagger = \frac{2}{r} U_r^\dagger \hat{B}_0, \quad (2.16)$$

$$\hat{B}_0 (\phi_1 \ast \phi_2) = \frac{\pi}{2} (-1)^{gh(\phi_1)} (B_1 \phi_1) \ast \phi_2 + \phi_1 \ast (\hat{B}_0 \phi_2), \quad (2.17)$$

where $B_1 = b_1 + b_{-1}$. We find that anti-commutators

$$\{B_0, \tilde{c}(\tilde{z})\} = \tilde{z},$$

$$\{B_1, \tilde{c}(\tilde{z})\} = 1, \quad (2.18)$$

are also useful for the calculation of the amplitudes. Correlation functions of the fields in the $\tilde{z}$ coordinates are summarized as

$$\langle \partial \bar{X}^\mu (\tilde{z}) \partial \bar{X}^\nu (\tilde{w}) \rangle = -\frac{\alpha'}{2} \eta^{\mu\nu} \frac{1}{\sin^2 (\tilde{z} - \tilde{w})},$$

$$\langle \tilde{c}(\tilde{x}) \tilde{c}(\tilde{y}) \tilde{c}(\tilde{w}) \rangle = \sin (\tilde{x} - \tilde{y}) \sin (\tilde{y} - \tilde{w}) \sin (\tilde{w} - \tilde{x}). \quad (2.19)$$

In this section, we will show how the above relations will be used to obtain the amplitudes.

Witten’s cubic action is given by

$$S = -\frac{1}{g^2} \left[ \frac{1}{2} \langle \Phi, Q_B \Phi \rangle + \frac{1}{3} \langle \Phi \ast \Phi \ast \Phi \rangle \right]. \quad (2.20)$$
To find the effective action, we will use the background field method [15]. We separate the field into the background field \( \phi_b \) and quantum fluctuation \( R \) as
\[
\Phi = \phi_b + R. \tag{2.21}
\]

We consider the path integral of the field \( R \). The contribution of \( R \) to the action is
\[
S = -\frac{1}{g^2} \left[ \frac{1}{2} \langle R, Q_B R \rangle + \langle \phi_b \ast \phi_b \ast R \rangle + \langle R \ast \phi_b \ast R \rangle + \frac{1}{3} \langle R \ast R \ast R \rangle \right]. \tag{2.22}
\]

To find the effective action, we need to consider the propagator of the quantum fluctuation field \( R \) and fix the gauge. We can obtain the four point process by shifting the field \( R \) [15]
\[
R \to R - P \phi_b \ast \phi_b, \tag{2.23}
\]
where \( P \) is the propagator which satisfies a relation
\[
Q_B P = 1. \tag{2.24}
\]

As a result of the shift of the quantum fluctuation field \( R \), we find that the quartic interaction term is given in \( \tilde{z} \) coordinates as
\[
A_4 = \frac{1}{2} \langle 0 | \tilde{\phi}_b(-\frac{\pi}{4} - \frac{\pi}{2}) \tilde{\phi}_b(\frac{\pi}{4} + \frac{\pi}{2}) \hat{U}_b, P \hat{U}_b \tilde{\phi}_b(\frac{\pi}{4}) \tilde{\phi}_b(-\frac{\pi}{4}) | 0 \rangle. \tag{2.25}
\]

Since the propagator \( P \) depends on the gauge of the field \( R \), the four point off-shell amplitudes also depend on the gauge condition. In the following, we shall show that the dependence of the gauge choice vanishes at on-shell.

### 3 Four point amplitudes in the Schnabl gauge

We are now going to compute the amplitudes in the Schnabl gauge. Let us consider the fields of the form \( \phi_b = c(z)V_1(z) \). For example, the tachyon and photon vertices are simply given by \( \phi_b = c(z)e^{ikX(z)} \) and \( \phi_b = \epsilon_{\mu}(k)c(z)\sqrt{\frac{2}{\alpha'}}\partial X^\mu(z)e^{ikX(z)}. \)
In the Schnabl gauge (2.12), the propagator is given by
\[ \mathcal{P} = \frac{B_0}{L_0} Q_B B_0^\dagger. \] (3.1)

For the computation of the amplitude in this gauge, we need two Schwinger parameters for this propagator
\[ \mathcal{P} = B_0 \int_0^\infty dt_1 e^{-t_1 L_0} Q_B B_0^\dagger \int_0^\infty dt_2 e^{-t_2 L_0^\dagger} \]
\[ = \int_0^\infty dt_1 \int_0^\infty dt_2 B_0 U_{T_1} Q_B B_0^\dagger U_{T_2} \] (3.2)
where \( T_1 = 2e^{t_1}, T_2 = 2e^{t_2} \).

Using the commutation rules, we find
\[ \hat{U}_3 B_0 U_{T_1} Q_B B_0^\dagger U_{T_2} \hat{U}_3 = \hat{U}_3 B_0 U_{T_1} \hat{U}_3 - \hat{U}_3 B_0 U_{T_1} B_0^\dagger U_{T_2} \hat{U}_3 Q_B. \] (3.3)

Thus in the Schnabl gauge, the four point amplitude (2.25) is written as follows.
\[ A_4 = \frac{1}{2} \int_0^\infty dt_1 \int_0^\infty dt_2 (\pi_3 B_0 \hat{U}_3 Q_B B_0^\dagger U_{T_2} \hat{U}_3 B_0 U_{T_1} B_0^\dagger U_{T_2} \hat{U}_3 Q_B |0\rangle \] (3.4)
\[ \times \left( \frac{2T_1}{3T_1 + 2} \right)^{h_1 + h_2} \left( \frac{4}{3T_1 + 2} \right)^{h_3 + h_4}, \] (3.6)
where
\[ \tilde{z}_1 = \frac{\pi(2T_1 + 1)}{3T_1 + 2}, \quad \tilde{z}_2 = \frac{\pi}{3T_1 + 2}. \] (3.7)
As an example, let us consider the four point amplitude of tachyons. Substituting into the tachyon vertex operator, the first term of eq. (3.4) yields

$$\frac{1}{2}(2\pi)^{26} \delta^{26}(\sum k_i) \int_0^{1/2} dy \, y^{-2-\alpha'}(1-y)^{-2-\alpha'} \left( \frac{2T_1}{3T_1 + 2} \right)^{\alpha'(k_1^2 + k_2^2) - 2} \left( \frac{4}{3T_1 + 2} \right)^{\alpha'(k_3^2 + k_4^2) - 2} \times \sin(2\tilde{z}_1)^{2-\alpha'(k_1^2 + k_2^2)} \sin(2\tilde{z}_2)^{2-\alpha'(k_3^2 + k_4^2)} \left( \sin(\tilde{z}_1 + \tilde{z}_2) \right)^{4-\alpha'(k_1^2 + k_2^2 + k_3^2 + k_4^2)} \times \left( \sin(\tilde{z}_1 - \tilde{z}_2) \right)^{4-\alpha'(k_1^2 + k_2^2 + k_3^2 + k_4^2)},$$

(3.8)

where new variable $y$ is introduced such as

$$y = -\frac{\sin(2\tilde{z}_1) \sin(2\tilde{z}_2)}{\sin^2(\tilde{z}_1 - \tilde{z}_2)}$$

$$1 - y = \frac{\sin^2(\tilde{z}_1 + \tilde{z}_2)}{\sin^2(\tilde{z}_1 - \tilde{z}_2)},$$

$$y = 1/2 \quad \text{(for } t = 0), \quad y = 0 \quad \text{(for } t = \infty).$$

(3.9)

The second term of (3.4) which vanishes for on-shell amplitude is rather complicated. Using

$$U_3^\dagger U_2 B_0 U T_1 \frac{B_0}{T_2} U_3^\dagger U_3 = -\frac{4}{3T_1 + 3T_2 - 4} \frac{U_3^\dagger U_2^\dagger + \eta_{T_2} - 1}{T_1} B_0 B_2 U \frac{T_2 + \eta_{T_1} - 1}{T_2},$$

(3.10)

and evaluating correlator, we get

$$-\frac{1}{2} \int_0^{\infty} dt_1 \int_0^{\infty} dt_2 \frac{4(-\alpha' k_1^2 - \alpha' k_2^2) + 3}{3T_1 + 3T_2 - 4} \left( \frac{2}{V} \right)^{-2-\alpha'(k_1^2 + k_2^2)} \left( \frac{2}{W} \right)^{-2+\alpha'(k_3^2 + k_4^2)} \times (2\pi)^{26} \delta^{26}(\sum k_i) \sin(\frac{\pi}{2V})^{2\alpha'(k_1-k_2)} \sin(\frac{\pi}{2W})^{2\alpha'(k_3-k_4)} \cos(\frac{\pi}{V} - \frac{\pi}{W})^{2\alpha'(k_1-k_3+k_2-k_4)} \times \cos(\frac{\pi}{2V} + \frac{\pi}{2W})^{2\alpha'(k_1+k_2-k_3)} \left( \cos(\frac{\pi}{V}) + \cos(\frac{\pi}{W}) \right) \left( \frac{\pi}{V} \cos(\frac{\pi}{W}) - \sin(\frac{\pi}{W}) \right),$$

(3.11)

where

$$\frac{2}{V} = \frac{2T_1}{3(T_1 + T_2) - 4}, \quad \frac{2}{W} = \frac{2T_2}{3(T_1 + T_2) - 4}.$$  

(3.12)

4 Four point amplitudes of tachyons and gauge fields in the modified Schnabl gauge

In the previous section, we have seen that the four point amplitudes in the Schnabl gauge is very complicated. The complication stems from the form of the propagator in
this gauge. To avoid these difficulties, we will use the propagator in the gauge $\tilde{B}_0 R = 0$ not in the Schnabl gauge $B_0 R = 0$ for the quantum fluctuation field. In this gauge, the propagator can be written as

$$\mathcal{P} = \frac{\tilde{B}_0}{L_0},$$  \hspace{1cm} (4.1)$$

which is manifestly self-conjugate and commutes with $\hat{U}$ which appears in the amplitudes.

The first question is whether the choice of gauge conditions may influence the physical observables. We will argue that on-shell amplitudes does not depend on the choice of the gauge conditions.

Suppose we have modified $b_0$ as

$$b'_0 = b_0 + \sum_{n=-\infty,n\neq 0}^{\infty} a_n b_n$$  \hspace{1cm} (4.2)$$

with some parameters $a_n$. We use a gauge for the state $b'_0 R = 0$, the propagator of this gauge is given by

$$\mathcal{P}' = \frac{b'_0}{L'_0},$$  \hspace{1cm} (4.3)$$

where

$$L'_0 = \{Q_B, b'_0\} = L_0 + \sum_{n=-\infty,n\neq 0}^{\infty} a_n L_n.$$  \hspace{1cm} (4.4)$$

We will compute the variation of the propagator with respect to $a_n$:

$$\frac{\partial \mathcal{P}}{\partial a_n} = b_n \frac{1}{L'_0} - b'_0 \frac{1}{L_0} L_n \frac{1}{L'_0}.$$  \hspace{1cm} (4.5)$$

Because of the relations $L'_0 = \{Q_B, b'_0\}$ and $L_n = \{Q_B, b_n\}$, this expression can be rewritten as

$$\frac{\partial \mathcal{P}}{\partial a_n} = \{Q_B, b'_0 \frac{1}{L'_0} b_n \frac{1}{L_0}\}.$$  \hspace{1cm} (4.6)$$

Eq. (4.6) implies that the dependence on the parameters is BRST closed and decouples from the correlation functions of BRST exact states. This statement may be an extension
of the propagator of the usual gauge fields. That is to say, even though the propagator of gauge fields contains the gauge parameters, the total amplitudes do not depend on the parameters.

Since the off-shell states themselves are not BRST exact states, the off-shell amplitudes depend on the gauge conditions. However, the final physical quantities obtained by the use of the off-shell amplitudes should be BRST exact. Therefore, it is expected that the choice of gauge for the internal states will not affect the physical observables.

Let us compute the off-shell amplitudes in the modified Schnabl gauge. We find that the quartic interaction term is given by

$$A_4 = \frac{1}{2} \langle 0 | \bar{\phi}_1 (-\frac{\pi}{4} - \frac{\pi}{2}) \bar{\phi}_2 (\frac{\pi}{4} + \frac{\pi}{2}) \tilde{U}_3, \frac{\hat{B}_0}{\hat{L}_0} \hat{U}_3 \bar{\phi}_3 (\frac{\pi}{4}) \phi_4 (-\frac{\pi}{4}) | 0 \rangle. \quad (4.7)$$

We will use Schwinger parametrization as

$$\hat{B}_0 \hat{U}_2 = \int_0^\infty d\beta \hat{B}_0 e^{-\beta \hat{U}_2} = \int_0^\infty d\beta \hat{B}_0 \hat{U}_{2\beta+2}. \quad (4.8)$$

Using the manipulation rule $\hat{U}_s \hat{U}_r = \hat{U}_{s+r}$ and the fact that $\hat{B}_0$ and $\hat{U}_r$ commute with each other, we easily find

$$A_4 = \frac{1}{2} \int_0^\infty d\beta \langle 0 | \bar{\phi}_1 (-\frac{\pi}{4} - \frac{\pi}{2}) \bar{\phi}_2 (\frac{\pi}{4} + \frac{\pi}{2}) \hat{B}_0 \hat{U}_{2\beta+4} \bar{\phi}_3 (\frac{\pi}{4}) \phi_4 (-\frac{\pi}{4}) | 0 \rangle. \quad (4.9)$$

By expressing $\hat{U}_{2\beta+4} = U_{2\beta+4}^\dagger U_{2\beta+4}$ and relations (2.4) and (2.16), we can move $U_{2\beta+4}^\dagger$ to the left and $U_{2\beta+4}$ to the right to find

$$A_4 = \frac{1}{2} \int_0^\infty \frac{d\beta}{(\beta+2)^{h_i+1}} \langle 0 | \bar{\phi}_1 (-\frac{\pi}{4} - \frac{\pi}{2}) \bar{\phi}_2 (\frac{\pi}{4} + \frac{\pi}{2}) \hat{B}_0 \phi_3 (\frac{1}{\beta+2}) \phi_4 (-\frac{\pi}{4} - \frac{\pi}{2}) | 0 \rangle. \quad (4.10)$$

By changing variable by $t = 1/(\beta + 2)$, we arrive at the following formula of four point amplitudes;

$$A_4 = \frac{1}{2} \int_0^{1/2} dt \sum_{h_i} \langle 0 | \bar{\phi}_1 (-\frac{\pi}{4} - \frac{\pi}{2}) \bar{\phi}_2 (\frac{\pi}{4} + \frac{\pi}{2}) (\mathcal{B}_0 + \mathcal{B}_0^\dagger) \phi_3 (\frac{\pi}{4}) \phi_4 (-\frac{\pi}{4}) | 0 \rangle. \quad (4.11)$$

Let us apply this formula to the four point tachyon amplitudes. The tachyon vertex operators are $\phi_i = c e^{ik_i \cdot X}$, and the following correlator included in the formula (4.11) is
easily evaluated by using eqs. (2.18) and (2.19)
\[
\langle 0 | \tilde{c} e^{i k_1 \cdot \hat{X}} \left( \frac{\pi}{4} t + \frac{\pi}{2} \right) \tilde{c} e^{i k_2 \cdot \hat{X}} (-\frac{\pi}{4} t - \frac{\pi}{2}) (B_0 + B_0^\dagger) \tilde{c} e^{i k_3 \cdot \hat{X}} \left( \frac{\pi}{4} t \right) \tilde{c} e^{i k_4 \cdot \hat{X}} (-\frac{\pi}{4} t) | 0 \rangle = (2\pi)^{26} \delta^{26}(\sum_i k_i) \pi t \sin \left( \frac{\pi}{2} t \right) \cos \left( \frac{\pi}{2} t \right) \frac{\alpha'}{2} \left( \sum_i k_i^2 \right)^{\alpha'} \left( 1 - \frac{\alpha'}{2} \right) \left( \sum_i k_i^2 / 2 (1 - y)^{-\alpha' u - \alpha' \sum_i k_i^2 / 2} \right).
\]

(4.12)

Changing variable \( y = \sin^2 \frac{\pi}{2} t \), we find the four tachyon amplitude as
\[
A_4 = \frac{1}{2} (2\pi)^{26} \delta^{26}(\sum_i k_i) \int_0^{1/2} dy t(y)^{\alpha' \sum_i k_i^2 - 4} y^{-\alpha' s - \alpha' \sum_i k_i^2 / 2} (1 - y)^{-\alpha' u - \alpha' \sum_i k_i^2 / 2}.
\]

(4.13)

where \( t(y) = \frac{2}{\pi} \arcsin \sqrt{y} \). Note that the integral appearing in the amplitudes are very simple although we cannot get any analytic expression for generic values of momenta.

To get the full off-shell tachyon amplitude, we must consider all 4! permutations of external momenta \( k_i \) \((i = 1, 2, 3, 4)\). The sum of these 4! permutations give six different terms, and each of these has a factor 4
\[
A_4 = 2(2\pi)^{26} \delta^{26}(\sum_i k_i) [I(s, u) + I(u, s) + I(u, t) + I(t, u) + I(t, s) + I(s, t)],
\]

(4.14)

where
\[
I(s, u) = \int_0^{1/2} dy t(y)^{\alpha' \sum_i k_i^2 - 4} y^{-\alpha' s - \alpha' \sum_i k_i^2 / 2} (1 - y)^{-\alpha' u - \alpha' \sum_i k_i^2 / 2}.
\]

(4.15)

Let us consider the case of on-shell amplitudes where \( \alpha' k_i^2 = 1 \). Using this on-shell condition, \( I(s, u) \) becomes
\[
I(s, u) = \int_0^{1/2} dy y^{-\alpha' s - 2} (1 - y)^{-\alpha' u - 2}.
\]

(4.16)

Therefore, \((s \leftrightarrow u)\) term contributes to the range \( \frac{1}{2} < y < 1 \) after changing variable \( y \rightarrow 1 - y \), so that
\[
I(s, u) + I(u, s) = \int_0^1 dy y^{-\alpha' s - 2} (1 - y)^{-\alpha' u - 2} = B(-1 - \alpha' s, -1 - \alpha' u),
\]

(4.17)

where \( B(a, b) \) is the Euler beta function
\[
B(a, b) = \int_0^1 dy y^{a-1} (1 - y)^{b-1}.
\]

(4.18)
Other four terms can be combined in the same way. Totally we have the following well-known expression for the tachyon amplitudes:

\[ A_4 = 2(2\pi)^{26}\delta^{26}(\sum_i k_i)[B(-\alpha(s), -\alpha(u)) + B(-\alpha(u), -\alpha(t)) + B(-\alpha(t), -\alpha(s))]. \tag{4.19} \]

where \( \alpha(s) = 1 + \alpha's \).

In the case of non-Abelian gauge fields, the procedures are quite similar. However, even in the off-shell amplitude, we have to impose the transversality condition \( \epsilon_i \cdot k_i = 0 \) on the vector vertex operators \( \phi_i = \epsilon_{\mu i} c \partial X^\mu e^{ik_i \cdot X} \) \( (i = 1, 2, 3, 4) \), since the formula (4.11) is applicable only to the primary operators.

The off-shell four vector amplitude is

\[
A_4 = \frac{1}{2}(2\pi)^{26}\delta^{26}(\sum_i k_i) \left[ J(s, u) \text{Tr} \left( \lambda^{a_1} \lambda^{a_2} \lambda^{a_3} \lambda^{a_4} \right) + J(u, s) \text{Tr} \left( \lambda^{a_1} \lambda^{a_4} \lambda^{a_3} \lambda^{a_2} \right) \\
+ J(u, t) \text{Tr} \left( \lambda^{a_1} \lambda^{a_3} \lambda^{a_2} \lambda^{a_4} \right) + J(t, u) \text{Tr} \left( \lambda^{a_1} \lambda^{a_4} \lambda^{a_2} \lambda^{a_3} \right) \\
+ J(t, s) \text{Tr} \left( \lambda^{a_1} \lambda^{a_2} \lambda^{a_4} \lambda^{a_3} \right) + J(s, t) \text{Tr} \left( \lambda^{a_1} \lambda^{a_3} \lambda^{a_4} \lambda^{a_2} \right) \right] \mathcal{F}(y; \epsilon, k), \tag{4.20} \]

where

\[
J(s, u) = \int_0^{1/2} dy \frac{t(y)}{y^{\alpha's-\alpha' \sum_i k_i^2/2}} (1 - y)^{-\alpha'u-\alpha' \sum_i k_i^2/2}, \tag{4.21} \]

and the explicit form of function \( \mathcal{F}(y; \epsilon, k) \) which includes polarization vectors \( \epsilon_i \) \( (i = 1, 2, 3, 4) \) is given in Appendix A.

Imposing on-shell conditions \( k_i^2 = 0 \), we find

\[
A_4 = (2\pi)^{26}\delta^{26}(\sum_i k_i) \int_0^1 dy \left[ y^{-\alpha's}(1 - y)^{-\alpha'u} \text{Tr} \left( \lambda^{a_1} \lambda^{a_2} \lambda^{a_3} \lambda^{a_4} + \lambda^{a_1} \lambda^{a_4} \lambda^{a_3} \lambda^{a_2} \right) \\
+ y^{-\alpha'u}(1 - y)^{-\alpha't} \text{Tr} \left( \lambda^{a_1} \lambda^{a_3} \lambda^{a_2} \lambda^{a_4} + \lambda^{a_1} \lambda^{a_4} \lambda^{a_2} \lambda^{a_3} \right) \\
+ y^{-\alpha't}(1 - y)^{-\alpha's} \text{Tr} \left( \lambda^{a_1} \lambda^{a_2} \lambda^{a_4} \lambda^{a_3} + \lambda^{a_1} \lambda^{a_3} \lambda^{a_4} \lambda^{a_2} \right) \right] \mathcal{F}(y; \epsilon, k). \tag{4.22} \]

When we take the limit \( k \to 0 \), this expression reduces to the four point interactions required for Yang-Mills action [15].
5  Effective quartic interaction for open superstring

In the previous section, we have shown how the modified use of the Schnabl gauge simplifies the computation of open string amplitudes. In this section, we will extend the analysis to the open superstrings. We are going to derive the formula of the effective quartic coupling for the superstring using WZW-like action [14]

\[
S = \frac{1}{4g^2} \left\langle (e^{-\Phi Q_B e^\Phi})(e^{-\eta_0 e^\Phi}) - \int_0^1 dt (e^{-t\Phi \partial_t e^{t\Phi}} \{ (e^{-t\Phi Q_B e^{t\Phi}}, (e^{-t\Phi \eta_0 e^{t\Phi}}) \}) \right\rangle. \quad (5.1)
\]

The ordinary choice of the gauge is

\[
b_0 \Phi = 0, \quad \xi_0 \Phi = 0. \quad (5.2)
\]

The cubic terms in this action are extracted as

\[
S_3 = \frac{1}{6g^2} \left[ \langle (Q_B \Phi) (\eta_0 \Phi) \rangle - \langle (Q_B \Phi) (\eta_0 \Phi) \Phi \rangle \right]. \quad (5.3)
\]

Expanding the field around the background (\(\Phi \rightarrow \Phi + R\)), we get the terms linear in \(R\)

\[
\begin{align*}
\langle (Q_B \Phi) R(\eta_0 \Phi) \rangle & - \langle (Q_B \Phi)(\eta_0 \Phi) R \rangle + \langle (Q_B R) \Phi(\eta_0 \Phi) \rangle - \langle (Q_B R)(\eta_0 \Phi) \Phi \rangle \\
+ \langle (Q_B \Phi) \Phi(\eta_0 R) \rangle - \langle (Q_B \Phi)(\eta_0 R) \Phi \rangle \\
& = 3 \left[ \langle (Q_B \Phi) R(\eta_0 \Phi) \rangle - \langle (Q_B \Phi)(\eta_0 \Phi) R \rangle \right]. \quad (5.4)
\end{align*}
\]

Therefore, the total action is

\[
S = -\frac{1}{2g^2} \langle \eta_0 R, Q_B R \rangle - \frac{1}{2g^2} \left\langle (Q_B \Phi)(\eta_0 \Phi) R \right\rangle + \langle (Q_B R) \Phi(\eta_0 \Phi) \rangle - \langle (Q_B R)(\eta_0 \Phi) \Phi \rangle + \cdots \\
= -\frac{1}{2g^2} \langle \eta_0 R, Q_B R \rangle + \langle \eta_0 R, \xi_0 \left\{ (Q_B \Phi)(\eta_0 \Phi) + (\eta_0 \Phi)(Q_B \Phi) \right\} \rangle + \cdots.
\]

(5.5)

where we have used \(\xi_0 R = 0\). Shifting the quantum fluctuation field \(R\) by

\[
R \rightarrow R - \frac{1}{2} b_0 L_0 \xi_0 \left\{ (Q_B \Phi)(\eta_0 \Phi) + (\eta_0 \Phi)(Q_B \Phi) \right\} \quad (5.6)
\]

to eliminate the terms linear in \(R\), we get the effective quartic coupling

\[
S^{(4)} = -\frac{1}{2g^2} \frac{1}{4} \langle Q_B \frac{b_0}{L_0} \xi_0 \Phi^{(2)}, \eta_0 \frac{b_0}{L_0} \xi_0 \Phi^{(2)} \rangle \\
= -\frac{1}{2g^2} \frac{1}{4} \langle \Phi^{(2)}, \frac{b_0}{L_0} \xi_0 \Phi^{(2)} \rangle,
\]

(5.7)
where
\[
\Phi^{(2)} = (Q_B \Phi) \ast (\eta_0 \Phi) + (\eta_0 \Phi) \ast (Q_B \Phi).
\] (5.8)

In particular, when the on-shell condition \(\eta_0 Q_B \Phi = 0\) is satisfied, we can rewrite \(\Phi^{(2)}\) as
\[
\Phi^{(2)} = -\eta_0 \left\{ (Q_B \Phi) \ast \Phi - \Phi \ast (Q_B \Phi) \right\}.
\]

Therefore, the effective quartic coupling is given by [15]
\[
S^{(4)} = -\frac{1}{2g^2} \frac{1}{4} \left( Q_B \frac{b_0}{L_0} (\Phi \ast Q_B \Phi - Q_B \Phi \ast \Phi), \eta_0 \frac{b_0}{L_0} (\Phi \ast Q_B \Phi - Q_B \Phi \ast \Phi) \right). \] (5.9)

Again we are going to use \(\tilde{z}\) coordinates in the same way as in Witten’s cubic action. Gauge conditions (5.2) are transformed by \(U_{\tan}\) into
\[
\mathcal{B}_0 \tilde{\Phi} = 0, \quad \tilde{\xi}_0 \tilde{\Phi} = 0. \] (5.10)

Therefore, states which satisfy eq. (5.2) automatically satisfy these conditions in \(\tilde{z}\) coordinates. The first one is what we call the Schnabl gauge condition. We will next consider the modified Schnabl gauge. We define the modified Schnabl gauge for the superstring as
\[
\hat{\mathcal{B}}_0 R = 0, \quad \hat{\tilde{\xi}}_0 R = 0. \] (5.11)

Imposing this modified Schnabl gauge for the fluctuation \(R\), the effective quartic term (5.7) is rewritten as
\[
S^{(4)} = -\frac{1}{2g^2} \frac{1}{4} \left( (Q_B \tilde{\Phi}) \ast (\eta_0 \tilde{\Phi}) + (\eta_0 \tilde{\Phi}) \ast (Q_B \tilde{\Phi}), \hat{\mathcal{B}}_0 \hat{\tilde{\xi}}_0 \left\{ (Q_B \tilde{\Phi}) \ast (\eta_0 \tilde{\Phi}) + (\eta_0 \tilde{\Phi}) \ast (Q_B \tilde{\Phi}) \right\} \right). \] (5.12)

6 Four point amplitude of tachyons and gauge fields

We are now going to compute the off-shell four point amplitude of tachyons in the modified Schnabl gauge. In order to deal with GSO(−) sector that tachyons appear, we
consider the string field action for the non-BPS D-brane:

\[
S = \frac{1}{4g^2} \left\langle \left( e^{-\phi} \hat{Q}_B e^\phi \right) (e^{-\phi} \hat{\eta}_0 e^\phi) - \int_0^1 dt (e^{-t\phi} \partial_t e^\phi \{(e^{-t\phi} \hat{Q}_B e^t\phi), (e^{-t\phi} \hat{\eta}_0 e^t\phi)\} \right) \right\rangle.
\]

(6.1)

In this action, \(2 \times 2\) internal Chan-Paton factors are added both to the vertex operators and to \(Q_B\) and \(\eta_0\). The tachyon vertex operator is then written as

\[
\hat{\phi} = \xi c e^{-\phi} e^{ik \cdot \vec{X}} \otimes \sigma_1.
\]

(6.2)

\(Q_B\) and \(\eta_0\) are tensored with \(\sigma_3\)

\[
\hat{Q}_B = Q_B \otimes \sigma_3, \quad \hat{\eta}_0 = \eta_0 \otimes \sigma_3.
\]

(6.3)

Since the algebraic structure of this non-BPS action is completely identical to that of BPS action (5.1), we can get the same formula for the quartic coupling as eq. (5.12) up to a factor 2 which compensate the trace of the internal CP matrices. Finally, we find the effective quartic coupling

\[
S^{(4)} = -\frac{1}{4g^2} \frac{1}{4} \left\langle \left( \hat{Q}_B \hat{\Phi} \right) \ast (\hat{\eta}_0 \hat{\Phi}) + (\hat{\eta}_0 \hat{\Phi}) \ast (\hat{Q}_B \hat{\Phi}) \right\rangle \frac{\hat{B}_0}{\mathcal{L}_0} \xi_0 \left\{ (\hat{Q}_B \hat{\Phi}) \ast (\hat{\eta}_0 \hat{\Phi}) + (\hat{\eta}_0 \hat{\Phi}) \ast (\hat{Q}_B \hat{\Phi}) \right\}.
\]

(6.4)

Corresponding amplitude is given by the same procedure as in the bosonic string

\[
A_4 = -\frac{1}{4g^2} \frac{1}{4} \int_0^\infty d\beta \left\langle \left( \hat{\phi}_Q(-\frac{\pi}{4} - \frac{\pi}{2}) + \hat{\phi}_Q(-\frac{\pi}{4} + \frac{\pi}{2}) \right) \right\rangle \times \hat{B}_0 U_{2\beta} + \xi_0 \left( \hat{\phi}_Q(-\frac{\pi}{4}) + \hat{\phi}_Q(-\frac{\pi}{4}) \right) \}
\]

\[
= -\frac{1}{4g^2} \frac{1}{4} \int_0^{1/2} dt \Sigma^{h_{1_{-1}}} \left\langle \left( \hat{\phi}_Q(-\frac{\pi}{4} t - \frac{\pi}{2}) + \hat{\phi}_Q(-\frac{\pi}{4} t + \frac{\pi}{2}) \right) \right\rangle \times \hat{B}_0 \xi_0 \left( \hat{\phi}_Q(-\frac{\pi}{4} t) + \hat{\phi}_Q(-\frac{\pi}{4} t) \right),
\]

(6.5)

where we have defined

\[
\hat{\phi}_Q = (Q_B \otimes \sigma_3)(\hat{\phi} \otimes \sigma_1)
\]

\[
= \left[ (-\alpha' k^2 + \frac{1}{2}) \partial \hat{c} \hat{c} \hat{e} e^{-\phi} e^{ik \cdot \vec{X}} + \sqrt{2} \alpha' k^4 \hat{\psi}_\mu \hat{\psi}_\mu \hat{\psi} e^{ik \cdot \vec{X}} - \hat{\eta} e^{-\phi} e^{ik \cdot \vec{X}} \right] \otimes i \sigma_2,
\]

(6.6)

\[
\hat{\phi}_\eta = (\eta_0 \otimes \sigma_3)(\hat{\phi} \otimes \sigma_1)
\]

\[
= \hat{\eta} e^{-\phi} e^{ik \cdot \vec{X}} \otimes i \sigma_2.
\]

(6.7)
Evaluating all correlation functions in eq. (6.5), we get

\[ A_4 = -\frac{1}{4g^2} \frac{2}{4} \int_0^{1/2} dt \sum \left( \alpha' k_i^2 - \frac{1}{2} \right) (2\pi)^d \delta^d(\sum k_i) y^{-\alpha' s - \alpha' \sum k_i^2/2} (1-y)^{-\alpha' t - \alpha' \sum k_i^2/2} \]

\[ \times \left[ \eta_{\mu\nu} \pi t \sin \frac{\pi}{2} \left( 2\alpha'(k_1^\mu k_3^\nu + k_2^\mu k_4^\nu) - 2\alpha'(k_1^\mu k_4^\nu + k_2^\mu k_3^\nu) \cos^2 \frac{\pi}{2} t \right) \right. \]

\[ + \left( -\alpha' k_1^2 + \frac{1}{2} \right) \left( \frac{\pi}{2} \cos \frac{\pi}{2} t + \frac{\pi}{2} \cos \frac{\pi}{2} \right) \left( 2\alpha'(k_1^\mu k_3^\nu + k_2^\mu k_4^\nu) - 2\alpha'(k_1^\mu k_4^\nu + k_2^\mu k_3^\nu) \cos^2 \frac{\pi}{2} t \right) \]

\[ - \left( -\alpha' k_3^2 + \frac{1}{2} \right) \left( \frac{\pi}{2} \cos \frac{\pi}{2} t - \frac{\pi}{2} \cos \frac{\pi}{2} \right) \left( 2\alpha'(k_1^\mu k_3^\nu + k_2^\mu k_4^\nu) - 2\alpha'(k_1^\mu k_4^\nu + k_2^\mu k_3^\nu) \cos^2 \frac{\pi}{2} t \right) \]

\[ - \left( -\alpha' k_4^2 + \frac{1}{2} \right) \left( \frac{\pi}{2} \cos \frac{\pi}{2} t + \frac{\pi}{2} \cos \frac{\pi}{2} \right) \left( 2\alpha'(k_1^\mu k_3^\nu + k_2^\mu k_4^\nu) - 2\alpha'(k_1^\mu k_4^\nu + k_2^\mu k_3^\nu) \cos^2 \frac{\pi}{2} t \right) \]

\[ + \left( -\alpha' k_2^2 + \frac{1}{2} \right) \left( \frac{\pi}{2} \cos \frac{\pi}{2} t - \frac{\pi}{2} \cos \frac{\pi}{2} \right) \left( 2\alpha'(k_1^\mu k_3^\nu + k_2^\mu k_4^\nu) - 2\alpha'(k_1^\mu k_4^\nu + k_2^\mu k_3^\nu) \cos^2 \frac{\pi}{2} t \right) \left( 1/2 \right) \frac{\pi}{2} \left( 1 - \frac{\pi}{2} \right) \left( 1 - \frac{\pi}{2} \right) \left( \alpha' t + 1 \right) \right]. \] \( (6.8) \)

It was shown that the WZW-like action reproduces the on-shell four point amplitudes correctly [16]. Here we will see the consistency of the above off-shell amplitude at on-shell. Imposing \( \alpha' k_i^2 = 1/2 \) \( i = 1, 2, 3, 4 \), we find that on-shell four tachyon amplitude is obtained as

\[ A_4 = -\frac{1}{4g^2} (2\pi)^d \delta^d(\sum k_i) \int_0^{1/2} dy \left[ -y^{-\alpha' s - 1} (1 - y)^{\alpha' u - 2} (\alpha' u + 1) \right. \]

\[ \left. + y^{-\alpha' s - 1} (1 - y)^{\alpha' u - 1} (\alpha' t + 1) \right]. \] \( (6.9) \)

We rewrite the first term of eq. (6.9) by partial integration

\[ \int_0^{1/2} dy y^{-\alpha' s - 1} (1 - y)^{\alpha' u - 2} (\alpha' u + 1) = \int_0^{1/2} dy y^{-\alpha' s - 1} \partial_y (1 - y)^{-\alpha' u - 1} \]

\[ = (1/2)^{-\alpha' s - 1} (1/2)^{-\alpha' u - 1} \]

\[ + (\alpha' s + 1) \int_0^{1/2} dy y^{-\alpha' s - 2} (1 - y)^{-\alpha' u - 1}. \] \( (6.10) \)

Summation over 4! permutations of the momenta yields to Euler beta functions. Using
identity

\[-(\alpha' s - 1)B(-\alpha' s - 1, -\alpha' u) = -(\alpha' s - 1) \frac{\Gamma(-\alpha' s - 1) \Gamma(-\alpha' u)}{\Gamma(-\alpha' s - \alpha' u - 1)} \]
\[= -\frac{\Gamma(-\alpha' s) \Gamma(-\alpha' u)}{\Gamma(-\alpha' s - \alpha' u)}(-\alpha' s - \alpha' u - 1) \]
\[= B(-\alpha' s, -\alpha' u)(\alpha' t + 1), \]

we find

\[A_4 = -\frac{1}{g^2}(2\pi)^d \delta^d(\sum k_i) \left[ 2(1 + \alpha' u)B(-\alpha' s, -\alpha' t) - (1/2)^{-\alpha' s-1}(1/2)^{-\alpha' t} \]
\[+ 2(1 + \alpha' t)B(-\alpha' s, -\alpha' u) - (1/2)^{-\alpha' s-1}(1/2)^{-\alpha' u} \]
\[+ 2(1 + \alpha' s)B(-\alpha' t, -\alpha' u) - (1/2)^{-\alpha' t}(1/2)^{-\alpha' u} \right]. \quad (6.11)\]

In order to get the complete four point amplitude, we have to consider summation over the permutations of momenta. In addition, since the super string field action (6.1) is non-polynomial, quartic coupling which describes contact interaction of four string fields

\[S_4 = \frac{1}{4! \cdot 2g^2} \left[ -2(\hat{Q}_B \hat{\Phi})(\hat{\eta}_0 \hat{\Phi}) + \langle(\hat{Q}_B \hat{\Phi})\hat{\Phi}^2(\hat{\eta}_0 \hat{\Phi}) \rangle + \langle(\hat{Q}_B \hat{\Phi})(\hat{\eta}_0 \hat{\Phi})\hat{\Phi}^2 \rangle \right], \quad (6.12)\]

also contributes to the four point amplitudes. We expect that the on-shell four point tachyon amplitude in total agrees with the first quantization result [17]. The contributions from the contact interaction (6.12) is given by

\[A_4^{\text{contact}} = -\frac{1}{g^2}(2\pi)^d \delta^d(\sum k_i) \left[ (1/2)^{-\alpha' s-1}(1/2)^{-\alpha' u} + (1/2)^{-\alpha' u-1}(1/2)^{-\alpha' t} \right. \]
\[\left. + (1/2)^{-\alpha' t}(1/2)^{-\alpha' s-1} \right]. \quad (6.13)\]

This contribution just cancels the extra terms in eq. (6.11) and the result agrees with that of the first quantization [17]

\[A_4 + A_4^{\text{contact}} = -\frac{2}{g^2}(2\pi)^d \delta^d(\sum k_i) \left[ (1 + \alpha' u)B(-\alpha' s, -\alpha' t) \right. \]
\[\left. + (1 + \alpha' t)B(-\alpha' s, -\alpha' u) + (1 + \alpha' s)B(-\alpha' t, -\alpha' u) \right]. \quad (6.14)\]
Four point amplitude of gauge fields is obtained in the similar way. For simplicity, we only consider the gauge fields with zero momenta. The vertex operator is given by

$$ \phi = \tilde{c}\tilde{\xi}e^{-\tilde{\phi}\tilde{\eta}^\mu}. $$

From eq. (5.9) on-shell four point amplitude is given by

$$ A_4 = -\frac{1}{2g^2} \left((\phi \ast \phi_Q - \phi_Q \ast \phi), \eta_0 \frac{\tilde{B}_0}{\tilde{L}_0} (\phi \ast \phi_Q - \phi_Q \ast \phi)\right) $$

$$ = -\frac{1}{8g^2} \int_0^{1/2} \frac{dt}{t} \left(\phi_Q(-\frac{\pi}{4}t - \frac{\pi}{2})\phi(\frac{\pi}{4}t + \frac{\pi}{2}) - \phi(-\frac{\pi}{4}t - \frac{\pi}{2})\phi_Q(\frac{\pi}{4}t + \frac{\pi}{2})\right) $$

$$ \times \eta_0 \tilde{B}_0 \left(\phi(\frac{\pi}{4}t)\phi_Q(-\frac{\pi}{4}t) - \phi_Q(\frac{\pi}{4}t)\phi(-\frac{\pi}{4}t)\right), \quad (6.15) $$

where

$$ \phi_Q = Q_B\phi $$

$$ = i\sqrt{\frac{2}{\alpha'}}\tilde{c}\tilde{\partial}\tilde{X}^\mu. \quad (6.16) $$

After the evaluation of this correlation functions, integration can be done explicitly

$$ A_4 = \frac{1}{8g^2} \int_0^{1/2} dy \left(2\eta^{\nu\sigma}\eta^{\mu\rho} + 2\frac{\eta^{\nu\rho}\eta^{\mu\sigma}}{(1-y)^2}\right) $$

$$ = \frac{1}{4g^2} \left(\frac{1}{2}\eta^{\nu\rho}\eta^{\mu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\right). \quad (6.17) $$

Contribution from eq. (6.12) is

$$ A_4^{\text{contact}} = \frac{1}{g^2} \left(\frac{1}{8}\eta^{\mu\rho}\eta^{\nu\sigma} - \frac{1}{2}\eta^{\mu\sigma}\eta^{\nu\rho}\right). \quad (6.18) $$

Then the total four point amplitude is just the Yang-Mills quartic coupling.

$$ A_4 = \frac{1}{g^2} \left(\frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma} - \frac{1}{4}\eta^{\mu\sigma}\eta^{\nu\rho}\right), $$

which was first pointed out in [15] via the Siegel gauge.
7 Discussions

We have obtained the formula for four point amplitudes in $\tilde{z}$ coordinates. Even in the Schnabl gauge, the off-shell amplitudes are very complicated. In this paper, we proposed the use of the modified version of the Schnabl gauge for intermediate states and showed that the most of the calculations can be achieved efficiently in this gauge. We also applied this modified gauge for the open superstring field theory.

More interesting question is whether the Schnabl gauge is effective for obtaining closed string amplitudes. In the closed string field theory, the calculations of the amplitudes are so difficult even at on-shell. It would be interesting to investigate whether the amplitudes of the closed string fields could be obtained in the Schnabl gauge.

We have postponed the arguments about the physical meaning of modified use of the Schnabl gauge in the external fields. It is not clear whether this condition fixes the gauge uniquely. At the linearized level, it might be shown that this gauge condition is consistent by the method used for the Siegel gauge.

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A Kinematical factor $\mathcal{F}$

In this appendix, we list the kinetic factor which appeared in eq. (4.20).

$$\mathcal{F}(y; \epsilon, k) = \epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4 y^{-2} + \epsilon_1 \epsilon_3 \epsilon_2 \epsilon_4 + \epsilon_1 \epsilon_4 \epsilon_3 \epsilon_3 (1 - y)^2$$

$$-2 \alpha' \left[ \epsilon_1 \epsilon_2 (k_2 \cdot \epsilon_3) \left\{ (k_2 \cdot \epsilon_4)(1 - y)^{-1} + (k_3 \cdot \epsilon_4) y^{-1} (1 - y)^{-1} \right\} ight]$$

$$- \epsilon_1 \epsilon_2 (k_4 \cdot \epsilon_3) \left\{ (k_2 \cdot \epsilon_4) y^{-1} + (k_3 \cdot \epsilon_4) y^{-2} \right\}$$

$$+ \epsilon_1 \epsilon_3 (k_3 \cdot \epsilon_2) \left\{ (k_3 \cdot \epsilon_4) y^{-1} (1 - y)^{-1} + (k_2 \cdot \epsilon_4) (1 - y)^{-1} \right\}$$

$$- \epsilon_1 \epsilon_3 (k_4 \cdot \epsilon_2) \left\{ (k_3 \cdot \epsilon_4) y^{-1} + (k_2 \cdot \epsilon_4) \right\}$$

$$- \epsilon_1 \epsilon_4 (k_4 \cdot \epsilon_3) \left\{ (k_4 \cdot \epsilon_2) y^{-1} + (k_3 \cdot \epsilon_2) y^{-1} (1 - y)^{-1} \right\}$$

$$+ \epsilon_1 \epsilon_4 (k_2 \cdot \epsilon_3) \left\{ (k_4 \cdot \epsilon_2) (1 - y)^{-1} + (k_3 \cdot \epsilon_2) (1 - y)^{-2} \right\}$$

$$- \epsilon_2 \epsilon_3 (k_2 \cdot \epsilon_1) \left\{ (k_2 \cdot \epsilon_4) y^{-1} + (k_1 \cdot \epsilon_4) y^{-1} (1 - y)^{-1} \right\}$$

$$+ \epsilon_2 \epsilon_3 (k_4 \cdot \epsilon_1) \left\{ (k_2 \cdot \epsilon_4) (1 - y)^{-1} + (k_1 \cdot \epsilon_4) (1 - y)^{-2} \right\}$$

$$+ \epsilon_2 \epsilon_4 (k_2 \cdot \epsilon_3) \left\{ (k_2 \cdot \epsilon_1) y^{-1} (1 - y)^{-1} + (k_3 \cdot \epsilon_1) (1 - y)^{-1} \right\}$$

$$+ \epsilon_2 \epsilon_4 (k_1 \cdot \epsilon_3) \left\{ (k_2 \cdot \epsilon_1) y^{-1} + (k_3 \cdot \epsilon_1) \right\}$$

$$- \epsilon_3 \epsilon_4 (k_3 \cdot \epsilon_2) \left\{ (k_3 \cdot \epsilon_1) (1 - y)^{-1} + (k_2 \cdot \epsilon_1) y^{-1} (1 - y)^{-1} \right\}$$

$$+ \epsilon_3 \epsilon_4 (k_1 \cdot \epsilon_2) \left\{ (k_3 \cdot \epsilon_1) y^{-1} + (k_2 \cdot \epsilon_1) y^{-2} \right\}$$

$$+ 4 \alpha'^2 \left\{ - \epsilon_1 k_3 \epsilon_2 k_4 \epsilon_3 k_2 \epsilon_4 k_2 + \epsilon_1 k_3 \epsilon_2 k_4 \epsilon_3 k_4 \epsilon_4 k_2 (1 - y) y^{-1} ight\}$$

$$+ \epsilon_1 k_3 \epsilon_2 k_3 \epsilon_3 k_4 \epsilon_4 k_2 y^{-1} + \epsilon_1 k_4 \epsilon_2 k_4 \epsilon_3 k_4 \epsilon_4 k_2 y^{-1}$$

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\[-\epsilon_1 \cdot k_3 \epsilon_2 \cdot k_4 \epsilon_3 \cdot k_2 \epsilon_4 \cdot k_3 y^{-1} + \epsilon_1 \cdot k_3 \epsilon_2 \cdot k_4 \epsilon_3 \cdot k_4 \epsilon_4 \cdot k_3 (1 - y)y^{-2}\]
\[+\epsilon_1 \cdot k_3 \epsilon_2 \cdot k_3 \epsilon_3 \cdot k_4 \epsilon_4 \cdot k_3 y^{-2} + \epsilon_1 \cdot k_4 \epsilon_2 \cdot k_4 \epsilon_3 \cdot k_4 \epsilon_4 \cdot k_3 y^{-2}\]
\[-\epsilon_1 \cdot k_3 \epsilon_2 \cdot k_3 \epsilon_3 \cdot k_2 \epsilon_4 \cdot k_2 (1 - y)^{-1} - \epsilon_1 \cdot k_4 \epsilon_2 \cdot k_4 \epsilon_3 \cdot k_2 \epsilon_4 \cdot k_2 (1 - y)^{-1}\]
\[+\epsilon_1 \cdot k_4 \epsilon_2 \cdot k_3 \epsilon_3 \cdot k_4 \epsilon_4 \cdot k_2 y^{-1} (1 - y)^{-1} - \epsilon_1 \cdot k_3 \epsilon_2 \cdot k_3 \epsilon_3 \cdot k_2 \epsilon_4 \cdot k_3 y^{-1} (1 - y)^{-1}\]
\[-\epsilon_1 \cdot k_4 \epsilon_2 \cdot k_4 \epsilon_3 \cdot k_2 \epsilon_4 \cdot k_3 y^{-1} (1 - y)^{-1} + \epsilon_1 \cdot k_4 \epsilon_2 \cdot k_3 \epsilon_3 \cdot k_4 \epsilon_4 \cdot k_3 y^{-2} (1 - y)^{-1}\]
\[-\epsilon_1 \cdot k_4 \epsilon_2 \cdot k_3 \epsilon_3 \cdot k_2 \epsilon_4 \cdot k_2 (1 - y)^{-2} - \epsilon_1 \cdot k_4 \epsilon_2 \cdot k_3 \epsilon_3 \cdot k_2 \epsilon_4 \cdot k_3 y^{-1} (1 - y)^{-2}\]\(\text{(A.1)}\)
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