Landau theory of multipolar orders in Pr(TM)$_2$X$_{20}$ Kondo materials

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A series of Pr(TM)$_2$X$_{20}$ (with TM=Ti,V,Rh,Ir and X=Al,Zn) Kondo materials, containing non-Kramers Pr$^{3+}$ 4$f^2$ moments on a diamond lattice, have been shown to exhibit intertwined orders such as quadrupolar order and superconductivity. Motivated by these experiments, we propose and study a Landau theory of multipolar order to capture the phase diagram and its field dependence. In zero magnetic field, we show that different quadrupolar states, or the coexistence of quadrupolar and octupolar orderings, may lead to ground states with multiple broken symmetries. Upon heating, such states may undergo two-step thermal transitions into the symmetric paramagnetic phase, with partial restoration of broken symmetries in the intervening phase. For nonzero magnetic field, we show the evolution of these thermal phase transitions strongly depends on the field direction, due to clock anisotropy terms in the free energy. Our findings shed substantial light on experimental results in the Pr(TM)$_2$Al$_{20}$ materials. We propose further experimental tests to distinguish purely quadrupolar orders from coexisting quadrupolar-octupolar orders.

I. INTRODUCTION

Heavy fermion materials with partially filled 4$f$ or 5$f$ shells often exhibit unusual phases attributed to broken symmetries involving higher order multipolar degrees of freedom. Given the challenging task of experimentally probing such broken symmetries, they are generally dubbed “hidden orders”. 1–10 In order to obtain a broad understanding of such systems, it is useful to study families of materials which share similar underlying microscopics and related phenomenology. A particularly useful example is provided by the Pr(TM)$_2$X$_{20}$ intermetallic compounds, with TM=Ti,V,Rh,Ir and X=Al,Zn. 11–23 All these materials have been shown to exhibit quadrupolar orders and superconductivity at lower temperatures. The common ingredient in this family is the local moment degree of freedom provided by the Pr ion. The interplay of strong spin-orbit coupling (SOC) and weaker crystal field splitting leads to a ground state $\Gamma^{(3)}$ non-Kramers doublet on Pr, with a significant gap to the higher order multiplets. This doublet carries no dipole moment, but has nonzero quadrupolar and octupolar moments. 13 A key motivation to explore such materials was the theoretical proposal that conduction electrons scattering off such doublets would lead to non-Fermi liquid behavior associated with the single ion two-channel Kondo model 24,25. The low temperature fate of the Kondo lattice system, however, remains an important open question. An understanding of these ground states is also important for clarifying the possible quantum phase transitions of these heavy fermion materials.26–29

Recent experiments on these Pr(TM)$_2$X$_{20}$ materials have confirmed the existence of quadrupolar order-
Our Landau theory includes uniform and staggered quadrupolar orders which are relevant to the FQ and AFQ states. For FQ order, we find that the Landau theory permits a cubic anisotropy term, which was previously pointed out within a microscopic theory and classical Monte Carlo study of a lattice model.\textsuperscript{39,40} This selects an FQ ordered state which is consistent with experimental results on PrTi\textsubscript{2}Al\textsubscript{20}.\textsuperscript{35} On the other hand, AFQ order is generally accompanied by a “parasitic” FQ order due to a cubic term which couples them. However, previous work found a single transition at which both orders are generated, and it thus does not explain the emergence of the two transitions observed in PrV\textsubscript{2}Al\textsubscript{20} at zero field. Moreover, the octupole moment carried by the doublet is typically ignored in previous studies; however, we have argued in recent work that this might be an important ingredient, and studied an appropriate diamond lattice model which hosts coexisting quadrupolar and octupolar orders.\textsuperscript{41}

Our Landau theory approach, which incorporates quadrupolar as well as octupolar order parameters, and symmetry-allowed clock anisotropies in the free energy, suggests two possible ways to explain the multiple thermodynamically allowed clock anisotropies in the free energy, evolving quadrupolar and octupolar orders.\textsuperscript{41}

Within a purely quadrupolar description, we show that the interplay of AFQ and FQ orders can lead to a second (lower temperature) transition at \( T^* \) within the AFQ phase due to a competition between different clock terms in the free energy. The intermediate phase in this picture preserves an Ising \( S_{1z} \) symmetry, which is further broken for \( T < T^* \).

Alternatively, we consider the more exotic possibility that the lower temperature transition at \( T^* \) might correspond to the ordering of octupolar degrees of freedom within the AFQ phase, which would lead to spontaneous time-reversal symmetry breaking for \( T < T^* \).

We find that both scenarios can potentially lead to similar experimental phase diagrams and their magnetic field evolution while the way that zero and finite temperature transitions are connected may be different in the two cases. We therefore conclude with a discussion of possible further experimental tests to distinguish between these two scenarios.

II. SYMMETRIES

Pr(TM)\textsubscript{2}X\textsubscript{20} (with TM=Ti, V, Rh, Ir and X=Al, Zn) are cage compounds with the space group Fd\textsubscript{3}m. In particular, the \( Pr^{3+} \) ions live on a diamond lattice, with each ion at the center of the Frank Kasper cage formed by 16 neighboring X ions with the local point group \( T_d \).\textsuperscript{13} Strong SOC leads to a total angular momentum \( J = 4 \) on the Pr ion, while crystal field splitting leads to a \( \Gamma_3 \) doublet ground state. (We note that PrRh\textsubscript{2}Zn\textsubscript{20} has the local point group \( T \) due to a further structural transition, and has a \( \Gamma_{23} \) doublet ground state.)\textsuperscript{13,30} The \( \Gamma_3 \) doublet wavefunctions are given by\textsuperscript{12,39}

\[
|\Gamma_3^{(1)}\rangle = \frac{1}{\sqrt{2}} \left( |4\rangle - |0\rangle \right) - \frac{1}{\sqrt{6}} |3\rangle - |2\rangle
\]

\[
|\Gamma_3^{(2)}\rangle = \frac{1}{\sqrt{2}} |2\rangle + \frac{1}{\sqrt{6}} |3\rangle - |4\rangle
\]

In these compounds, the first excited triplet \( \Gamma_4 \) or \( \Gamma_5 \) is separated from the ground doublet by \( \Delta \approx 30-70K \). This allows us to study the broken symmetry phases, which typically have transition temperatures \( \lesssim 5K \), by projecting to the \( \Gamma_3 \) (or \( \Gamma_{23} \)) doublets. Using these doublets, we define pseudospin-1/2 basis as in Ref. [41], namely,

\[
|\uparrow\rangle = \frac{1}{\sqrt{2}} (|\Gamma_3^{(1)}\rangle + i|\Gamma_3^{(2)}\rangle)
\]

\[
|\downarrow\rangle = \frac{1}{\sqrt{2}} (i|\Gamma_3^{(1)}\rangle + |\Gamma_3^{(2)}\rangle).
\]

We identify the corresponding pseudospin operators in terms of Stevens operators\textsuperscript{42,43} \( O_{22} = \frac{1}{2}|J_x^2 - J_y^2\rangle \), \( O_{20} = \frac{1}{4}(3J_z^2 - J_x^2) \), and \( T_{xyz} = \sqrt{\frac{3}{5}} J_x J_y J_z \) (with the overline denoting a fully symmetrized product), as

\[
\tau^x = -\frac{1}{4} O_{22}; \tau^y = -\frac{1}{4} O_{20}; \tau^z = \frac{1}{3\sqrt{5}} T_{xyz}.
\]

Here, the components of the pseudospin \( \vec{\tau} \) are such that \((\tau^x, \tau^y) \equiv \tau^z \) describes a time-reversal invariant quadrupolar moment, while \( \tau^z \) describes a time-reversal odd octupolar moment.

The point group symmetries of \( Pr^{3+} \) ions include \( S_{4z} \) (\( \pi/2 \) rotation about \( z \) axis and inversion about a site), \( C_{31} \) (\( 2\pi/3 \) rotation along (111) direction), \( \sigma_{dl} \) (mirror reflection with a plane perpendicular to (110) direction) and \( \mathcal{I} \) (bond-centered inversion). Under these point group operations and time reversal (\( \Theta \)), the pseudospins transform as:

\[
\Theta: \quad \tau_{A/B}^+ \rightarrow - \tau_{A/B}^-, \quad \tau_{A/B}^- \rightarrow \tau_{A/B}^+
\]

\[
\mathcal{I}: \quad \tau_A \leftrightarrow \tau_B,
\]

\[
S_{4z}: \quad \tau_{A/B}^+ \rightarrow - \tau_{A/B}^-, \quad \tau_{A/B}^- \rightarrow \tau_{A/B}^+
\]

\[
\sigma_{dl}: \quad \tau_{A/B}^+ \rightarrow - \tau_{A/B}^-, \quad \tau_{A/B}^- \rightarrow \tau_{A/B}^+
\]

\[
C_{31}: \quad \tau_{\mu}^\pm \rightarrow e^{\pm 2\pi i/3} \tau_{\mu}^\pm
\]

Since the pseudospins transform in the same manner under \( S_{4z} \) and \( \sigma_{dl} \), we drop the \( \sigma_{dl} \) symmetry in the following analysis. We next use these symmetries in order to construct the Landau theory.
III. LANDAU THEORY

In this paper, we study the simplest scenarios with uniform or two-sublattice orders which do not enlarge the unit cell of the diamond lattice. Thus, we consider FerroQuadrupole (FQ), AntiFerroQuadrupole (AFQ), FerroOctupole (FO) and AntiFerroOctupole (AFO) broken symmetry states. Some of these orders could potentially coexist. Let us introduce uniform and staggered multipolar order parameters

\[ \phi_{u,s} \equiv \langle \tau^u_s \rangle \pm \langle \tau^d_s \rangle \]

\[ m_{u,s} \equiv \langle \tau^u_s \rangle \pm \langle \tau^d_s \rangle. \]

Here, the complex scalars \( \phi_{u,s} \) denote, respectively, the uniform (for FQ) and staggered parts (for AFQ) of the XY quadrupolar order, while the real scalars \( m_{u,s} \) refer to the uniform (for FO) and staggered parts (for AFO) of the Ising octupolar order. The underlying crystal and time-reversal symmetry transformations act on the order parameters \( \phi_{u,s} \) and \( m_{u,s} \) as follows:

\[ \Theta : \phi_{u,s} \rightarrow \phi_{u,s}^*; \quad m_{u,s} \rightarrow -m_{u,s} \]

\[ \mathcal{I} : (\phi_{u,m}) \rightarrow (\phi_{u,m}); \quad (\phi_{s,m}) \rightarrow - (\phi_{s,m}) \]

\[ \mathcal{S}_{4z} : \phi_{u,s} \rightarrow - \phi^*_{u,s}; \quad m_{u,s} \rightarrow - m_{u,s} \]

\[ \mathcal{C}_{31} : \phi_{u,s} \rightarrow e^{i2\pi/3} \phi_{u,s}; \quad m_{u,s} \rightarrow m_{u,s}. \]

The symmetry-allowed terms in the Landau free energy with independent order parameters are thus:

\[ \mathcal{F}_{\phi u} = r_{u\phi} |\phi_u|^2 + iv(\phi_u^3 - \phi_u^*^3) + g_{u\phi} |\phi_u|^4 + \ldots \]

\[ \mathcal{F}_{\phi s} = r_{s\phi} |\phi_s|^2 + g_{s\phi} |\phi_s|^4 + w(|\phi_s|^6 + |\phi_s^*|^6) + \ldots \]

\[ \mathcal{F}_{m u} = r_{u m} m_u^2 + g_{u m} m_u^4 + \ldots \]

\[ \mathcal{F}_{m s} = r_{s m} m_s^2 + g_{s m} m_s^4 + \ldots, \]

where the ellipses denote dropped higher order terms. The important difference between the FQ versus AFQ free energies appears in the “clock” anisotropy terms which break XY symmetry for \( \phi_u, \phi_s \) respectively; this is cubic for FQ and sixth order for AFQ. This free energy must be supplemented by \( \mathcal{F}_{\text{int}} \) which encapsulates interactions between the different order parameters. Symmetry allows for a single cubic interaction,

\[ \mathcal{F}^{(3)}_{\text{int}} = i\lambda (\phi_u^3 \phi_s - \phi_u^* \phi_s^*). \]

This leads to “parasitic” FQ order \( \phi_u \sim \phi_s^* \) in an AFQ state. Additional quartic interactions between order parameters take the form

\[ \mathcal{F}^{(4)}_{\text{int}} = c_1 |\phi_u|^2 |\phi_s|^2 + c_2 m_u^2 m_s^2 + c_3 |\phi_u|^2 m_s^2 + c_4 |\phi_s|^2 m_u^2 + c_5 |\phi_u|^2 m_s^2 + c_6 |\phi_s|^2 m_u^2. \]

Such terms can lead to coexistence of quadrupolar and octupolar order parameters depending on the signs of the coefficients. Below, we will analyze this Landau free energy in various cases, starting from the simplest example.

A. FQ order in PrTi\(_2\)Al\(_20\)

PrTi\(_2\)Al\(_20\) exhibits FQ order, so we can focus on the single term \( \mathcal{F}_{\phi u} \) in Eq. (16) above.\(^{12,30,31} \) For \( r_{u\phi} > 0 \), this describes a paramagnetic (PM) phase with \( \phi_u = 0 \), while \( r_{u\phi} < 0 \) leads to FQ order with \( \phi_u \neq 0 \). The phase of \( \phi_u \equiv \langle \phi_u e^{i\theta_u} \rangle \) is determined by the clock term \( v \). For \( v > 0 \), we favor \( \theta_u = \pi/6 + 2n\pi/3 \) (with integer \( n \)), while \( v < 0 \) pins \( \theta_u = \pi/6 + (2n+1)\pi/3 \). In particular, either sign of \( v \) favors \( O_{20} \) order over \( O_{22} \) order, which is consistent with nuclear magnetic resonance (NMR) experiments\(^{35} \) on PrTi\(_2\)Al\(_20\). In the “hard-spin” limit, the theory for the PM-to-FQ transition is a \( Z_3 \) clock model which is known to exhibit a first-order transition in three dimensions (3D).\(^{14,45} \) However, disorder effects have been shown in certain examples to convert first-order transitions into continuous phase transitions.\(^{46} \) Such effects may be important in understanding experimental observations; this needs further investigation.

B. AFQ with 'parasitic' FQ order

Let us ignore the octupolar orders \( m_u, m_s \), and focus on the free energy \( \mathcal{F}_{\phi u} + \mathcal{F}_{\phi s} + \mathcal{F}^{(3)}_{\text{int}} + c_1 |\phi_u|^2 |\phi_s|^2 \). For an AFQ transition driven by \( r_{s\phi} < 0 \), we get \( \phi_s \neq 0 \). This AFQ transition will happen within mean field theory at \( T_Q \) if we set \( r_{s\phi} = \alpha_s (T - T_Q) \), with \( \alpha_s > 0 \). In this case, even if \( r_{u\phi} > 0 \), the cubic interaction \( \lambda \neq 0 \) in \( \mathcal{F}^{(3)}_{\text{int}} \) leads to \( \phi_u \neq 0 \). It is useful to begin our analysis of the interplay of AFQ and FQ orders by considering the regime where \( r_{u\phi} \) is large. The resulting FQ order is then parasitic, and it will be slaved to the AFQ order. Let us simplify the problem by setting \( (v, g_{u\phi}) \rightarrow 0 \) to leading order, and minimizing the free energy with respect to \( \phi_u \) which leads to

\[ \phi_u = \frac{i \lambda}{r_{u\phi}} \phi_s^2. \]

Substituting back, the full free energy is given by

\[ \mathcal{F}^{\text{eff}}_{\phi u} = r_{s\phi} |\phi_s|^2 + g_{s\phi} |\phi_s|^4 + w^{\text{eff}} (|\phi_s|^6 + |\phi_s^*|^6) + \ldots \]

where

\[ g^{\text{eff}}_{s\phi} = g_{s\phi} - \frac{\lambda^2}{r_{u\phi}} \]

\[ w^{\text{eff}} = w + \frac{\lambda^3}{r_{u\phi}^3}. \]
With $\phi_s = |\phi_s|e^{i\theta_s}$, we find that the clock term with $w^{\text{eff}} > 0$ favors $\theta_s = (2n + 1)\pi/6$, while $w^{\text{eff}} < 0$ would favor $\theta_s = 2n\pi/6$. Now, even if $r_{u\phi} > 0$, it may have a temperature dependence as $r_{u\phi} = r_{u\phi}(0) + \alpha u T$ with $r_{u\phi}(0) > 0, \alpha_u > 0$. Such a (benign) temperature dependence of $r_{u\phi}$ could, nevertheless, lead to a change of sign of $g_{u\phi}^{\text{eff}}$ which could lead to first-order transitions, or a sign change of $w^{\text{eff}}$ (if the product $w v \lambda < 0$) which may modify the competition between the different clock terms. This, admittedly crude, argument suggests that the interplay of AFQ and FQ orders could lead to a rich phase diagram with new phases and phase transitions.

In order to examine this scenario, we numerically minimize the Landau free energy $F_{\phi_s} + F_{\phi_u} + F_{\text{int}}^{(3)}$ as a function of $r_{u\phi}$ and $r_{u\phi}$, while keeping $r_{u\phi} > 0$. For illustrative purposes, we fix $g_{u\phi} = 1$ and $g_{u\phi} = 1/2$, and consider the choice for the coefficients of the clock terms $(w, v, \lambda) \equiv (1/4, -1/4, 1/4)$. The resulting phase diagram is shown in Fig. 1(a), and exhibits five different phases: a paramagnet (PM), a FQ state driven by the cubic term $v$, and three types of AFQ phases (with coexisting FQ order) which result from competition between the different clock terms in the free energy. Fig. 1(c) shows the nature of the different AFQ phases, which are distinguished by the behavior of the quadrupole moment on the two sublattices.

In AFQ-I, the staggered quadrupolar order points along $\tau_x$ ($O_{22}$) while the parasitic uniform component points along $\tau_y$ ($O_{20}$). This phase minimizes the clock anisotropy terms $v$ and $\lambda$. The AFQ-I state depicted in Fig. 1(c) preserves $S_{1z}$ and $\Theta$ symmetries.

In AFQ-III, both the staggered and uniform components favor $O_{30}$ order, so the overall magnitude of the ordered quadrupole moment is different on the two sublattices. This phase minimizes the clock terms $w$ and $\lambda$. Again, the AFQ-III state depicted in Fig. 1(c) preserves $S_{1z}$ and $\Theta$ symmetries.

Finally, AFQ-II is a “frustrated” phase, where the competition of the different clock terms $(w, v, \lambda)$ results in none of them being fully minimized. This phase exhibits a generically complex superposition of $O_{30}$ and $O_{22}$ orders, with unequal magnitude of the ordered moment on the two sublattices, and only preserves $\Theta$. We thus expect the AFQ-II state, which breaks the residual $S_{1z}$ symmetry, and thus has lower symmetry than AFQ-I or AFQ-III, to arise from either one of them upon cooling.

We find that different choices for these clock coefficients, keeping the product $w v \lambda < 0$ yield phase diagrams with the same phases and a roughly similar topology. For instance, when we decrease $\lambda = 1/16$, we find the following differences: (i) the AFQ-I phase shrinks, (ii) the AFQ-II to AFQ-III phase transition becomes first order, and (iii) there is no direct transition from PM into AFQ-I.

To see how this $(r_{u\phi}, r_{u\phi})$ phase diagram might translate into a phase diagram as a function of temperature, consider a cut through Fig. 1(a) at large $r_{u\phi}$. Such a cut will yield a PM to AFQ-III transition, i.e., a single transition into a phase with coexisting AFQ order and parasitic FQ order. This scenario is consistent with what has been previously explored by Hattori and Tsunetsugu.39, 40

However, for smaller $r_{u\phi}$, along the cut shown in Fig. 1(a), we find that the transition splits into two transitions, a PM to AFQ-III transition, and a subsequent AFQ-III to AFQ-II transition. Fig. 1(b) shows the evolution of the order parameters with “temperature”, where going along the cut from PM to AFQ-III to AFQ-II is viewed as corresponding to decreasing temperature. The two thermal transitions in this sce-
nario might potentially explain the two observed zero field thermal transitions in PrV$_2$Al$_{20}$.\textsuperscript{15,21} We note that while there are many possible cuts we could take which would lead to multiple thermal transitions, the one we have chosen seems most promising from the point of view of understanding the magnetic field evolution as discussed in Section IV.

C. Coexisting AFQ and octupolar orders

Finally, let us turn to the most interesting possibility, that the two thermal transitions in PrV$_2$Al$_{20}$ correspond, respectively, to the onset of AFQ and of octupolar order which spontaneously breaks time-reversal symmetry. In previous work, we have considered this possibility within a particular (phenomenological) microscopic Hamiltonian with competing two-spin and four-spin interactions which we studied using classical Monte Carlo simulations.\textsuperscript{41} Here, we revisit this scenario using Landau theory which goes beyond a specific microscopic model. We note the precise type of octupolar order, either ferrooctupolar or antiferrooctupolar, does not change our Landau theory analysis performed below; without loss of generality, we thus consider the case with ferro-octupolar order. This distinction will of course be important when we turn in the end to a discussion of experimental consequences.

To illustrate this interplay of AFQ and octupolar orders, Fig. 2(a) shows a phase diagram obtained using the Landau free energy $\mathcal{F}^{\text{eff}}_{\phi u} + \mathcal{F}_{\phi s} + \mathcal{F}_{\mu u} + \mathcal{F}_{\mu s}$, where we consider having integrated out $\phi_u$ and assumed large $r_{u0}$ so any multiple thermal transitions must arise from additional octupolar order. We pick $c_0 \neq 0$ in $\mathcal{F}^{\text{int}}_{\mu s}$ in Eq. (21); specifically, we chose $c_0 < 0$ to allow for a coexistence phase. As we vary $r_{s0}, r_{um}$, there exist four distinct phases: a paramagnet (PM) ($\phi_s = \phi_u = m_s = 0$), an AFQ phase with parasitic FQ order ($\phi_s \neq 0, \phi_u \neq 0, m_u = 0$), an FO phase ($\phi_s = \phi_u = 0, m_u \neq 0$), and finally a phase with coexisting AFQ and FO orders with parasitic FQ order ($\phi_s \neq 0, \phi_u \neq 0, m_u \neq 0$). Fig. 2(b) shows the temperature dependence of the order parameters as we “cool” from the PM into the phase with coexisting AFQ and FO orders; for simplicity, we consider going along the trajectory indicated in Fig. 2(a), i.e., keeping $r_{sm}$ fixed and varying $r_{s0}$. This clearly shows the double transition, with the upper transition $T_Q$ being associated with AFQ order (with parasitic FQ) and the lower transition at $T^*$ arising from the octupolar order. Fig. 2(c) shows the common origin plots of pseudospin $\tau$ for AFQ and AFQ-FO$^*$ respectively (both with parasitic FQ).

IV. IMPACT OF A MAGNETIC FIELD

We next consider the impact of an applied magnetic field $B$ on the Landau free energy, and its phases and phase transitions. The leading term is a quadratic-in-field coupling to the quadrupolar order; microscopically, this arises via second order perturbation theory in $B \cdot J$, where $J$ is the $J = 4$ angular momentum operator. Projecting to the $\Gamma_3$ doublet, we arrive at the form\textsuperscript{49}

$$H_{\text{field}} = \gamma B^2 (b_1 \tau^x + b_2 \tau^y)$$

(26)
where \( b_1 \equiv \frac{\sqrt{2}}{2}(\hat{b}_x - \hat{b}_y), b_2 \equiv \frac{1}{3}(3\hat{b}_z^2 - 1), \) and \((\hat{b}_x, \hat{b}_y, \hat{b}_z)\) describes the unit vector pointing along \( B. \) The coupling constant
\[
\gamma \Delta \left( -\frac{14}{3\Delta (\Gamma_4)} + \frac{2}{\Delta (\Gamma_5)} \right), \tag{27}
\]
with \( \Delta (\cdot) \) being the energy of the indicated higher energy crystal field multiplets.\(^{41}\)

Note that a magnetic field along the (111) direction does not directly couple to the quadrupolar moment, but even along this direction \( B^2 \) could couple to the energy density via \(|\phi_u|^2\) or \(|\phi_u|^2\), with the coupling to \(|\phi_u|^2\) being less important if the FQ order is parasitic and small. Moreover, along this special (111) direction, the magnetic field can couple to the octupolar moment at cubic order in the field as \( |B|^2 \hat{b}_x \hat{b}_y \hat{b}_z \); however, given that this last term is expected to be much weaker for typical fields, we omit it in the analysis below.

To proceed, it is useful to define a complex scalar \( \psi_B \equiv b_1 + ib_2 \) representing the external magnetic field, which transforms identical to the FQ order parameter \( \phi_u \), and thus couples to it linearly. This leads to terms in the Landau free energy
\[
\mathcal{F}_B = \gamma B^2 (\psi_B^* \phi_u + \phi_u^* \psi_B) + B^2 (\hat{r}_{sB} |\phi_u|^2 + \hat{r}_{uB} |\phi_u|^2), \tag{28}
\]
where we have included extra, symmetry allowed, couplings \( \hat{r}_{sB}, \hat{r}_{uB} \) to the energy density as discussed above. Along key high symmetry directions, \( \psi_B(111) = 0, \psi_B(100) = e^{i\pi/2 + \alpha}, \psi_B(110) = \frac{1}{2} e^{-i\pi/2 + \alpha}. \)

**A. FQ order in PrTi\(_{2}\)Al\(_{20}\)**

As seen from the coupling in \( \mathcal{F}_B \) above, the direction of the magnetic field pins the quadrupolar moment direction, thus explicitly breaking the \( Z_3 \) symmetry associated with the choice of phase of \( \phi_u \). This converts the PM-FQ transition into a smooth crossover for both (001) and (110) field directions, as has also been predicted based on microscopic model studies and confirmed by specific heat measurements on PrTi\(_{2}\)Al\(_{20}\).

**B. AFQ with parasitic FQ order**

For this case, we proceed by considering the Landau free energy \( \mathcal{F}_{\phi_u} + \mathcal{F}_{\phi_u} + \mathcal{F}_{\text{int}} \) supplemented by the field term \( \mathcal{F}_B. \) For simplicity, we set \( \hat{r}_{sB} = 0 \) and \( \hat{r}_{uB} = 0, \) and only consider the impact of the coupling \( \gamma. \) Minimizing this full free energy along the cut shown in Fig. 1(a), we find the strongly direction-dependent field evolution displayed in Fig. 3 for fields along (001) and (110) directions. In both cases, the field couples linearly to \( \phi_u, \) and thus pins its phase as soon as \( B \neq 0. \) We refer to the resulting phase as ‘FQ’ to denote that it is not a symmetry broken FQ state, but rather a field induced FQ state which is thus qualitatively similar to a PM. Along the (001) direction, the entire region of AFQ-III and AFQ-II get replaced by the AFQ-II phase as \( \phi_u \) canters away from pure O\(_{20}\) order, while phase AFQ-I emerges only for nonzero \( B \) from the PM-to-AFQ-III transition point. Along the (110) direction however, all three phases present at zero field and the corresponding two thermal phase transitions survive even for \( B \neq 0. \)

**C. Coexisting AFQ and octupolar orders**

Finally, let us turn to the field evolution in the case where we assume \( r_{u\phi} \) is large and positive and integrated out \( \phi_u, \) but study the interplay of \( \phi_u \) and \( m_u \) as we have done at \( B = 0. \) We thus minimize the free energy \( \mathcal{F}_{\phi_u} + \mathcal{F}_{\phi_u} + \mathcal{F}_{\text{int}} \), and supplement this with
\[
\mathcal{F}_{\phi_u} = i B^2 \gamma (|s|^2 - |s|^2) + B^2 \hat{r}_{sB} |\phi_u|^2, \tag{29}
\]
where the term \( \gamma \) arises from the coupling \( \gamma \) in Eq. 28 upon integrating out \( \phi_u. \) Fig. 4 shows the direction dependent field evolution of multiple transitions for coexisting AFQ and octupolar orders. When a magnetic field is applied, the transition temperature (blue lines in Fig. 4(a) and (b)) between paramagnet (PM) and AFQ with parasitic FQ phase (AFQ-FQ) increases due to field coupling term \( \hat{r}_{sB} \) which is quadratic in \( \phi_u, \) and since the
phase is directly locked to the field direction. On the other hand, the lower transition temperature strongly depends on field direction; it decreases with \( B \) (red line in Fig. 4(a)) and increases with \( B \) (red line in Fig. 4(b)). The decrease of transition temperature with field (001) originates from the competition between the sixth order anisotropy term and field coupling terms for finite magnitudes of \( \phi_4 \). Thus, an anisotropic evolution of the phase diagram in a magnetic field can be also present due to AFQ and octupolar orders.

V. DISCUSSION

In this paper, we have formulated and studied the Landau theory of multipolar orderings in the Pr(TM)\(_2\)Al\(_{20}\) systems, including quadrupolar and octupolar orders. In the absence of any octupolar order, the phases of the Landau theory preserve time-reversal symmetry. In this case, examining the different quadrupolar orders, we find that while a single thermal transition is expected in the case of FQ order, there may be multiple thermal transitions for the case of AFQ orders. Such a scenario involves a higher temperature transition from a paramagnetic phase into an AFQ order which breaks all point group symmetries except \( S_{4z} \), followed by a lower temperature transition into a phase where this residual Ising symmetry is broken. The residual \( S_{4z} \) symmetry in the intermediate phase has implications for \(^{27}\)Al NMR experiments which probe the induced dipole order for a ‘probe’ magnetic field applied along the (111) direction. For a (111) field, there are a set of ‘3c’ Al sites on the Frank-Kasper cage which are symmetry equivalent in the paramagnetic phase, and yield a single NMR line. Based on symmetry, an AFQ-II state with broken \( S_{4z} \) should exhibit six NMR lines with equal intensity. Thus, upon cooling from the AFQ-III state, which preserves \( S_{4z} \) symmetry, into the low temperature AFQ-II state with broken \( S_{4z} \) symmetry, each of the two original high intensity lines should split into two peaks. Alternatively, the lower temperature transition may be from an intermediate AFQ-III state which preserves \( S_{4z} \) and time reversal into a state where time-reversal is broken by the octupolar order. In this case, the NMR should show four lines with a 1:2 intensity ratio in both broken symmetry phases assuming that the octupolar order is only weakly affected by field, but the time reversal breaking or distinctions between FO and AFO could be possibly detectable by \( \mu \)SR. Further work is needed to understand the role of domains and nature of domain walls in systems with such multipolar orders due to possible spin-lattice couplings. Clarifying the nature of these multipolar orders in the Pr(TM)\(_2\)Al\(_{20}\) systems would be a significant step in understanding the phase diagram and quantum critical points of such multipolar Kondo materials.

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