A class of nonassociative algebras and cogebras

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Abstract

We present classes of nonassociative algebras whose associator satisfies invariance conditions given by the action of the 3 order symmetric group. Amongst these algebras we find the wellknown Vinberg algebras, the Pre-Lie algebras, the Lie-admissible algebras and the 3-power associative algebras.

I. $Σ_3$-ASSOCIATIVE ALGEBRAS

A. $Σ_3$-invariant spaces

Let $Σ_3$ be the 3-order symmetric group and $K[Σ_3]$ the associated group algebra, where $K$ is a field of characteristic zero. We denote by $τ_{ij}$ be the transposition exchanging $i$ and $j$, $c_1 = (1, 2, 3)$ and $c_2 = c_1^2$ the two 3-cycles of $Σ_3$. Every $v \in K[Σ_3]$ decomposes as follows:

$$v = a_1id + a_2τ_{12} + a_3τ_{13} + a_4τ_{23} + a_5c_1 + a_6c_2$$

or simply

$$v = \sum_{σ ∈ Σ_3} a_σ σ$$

where $a_σ \in K$.

Consider the natural right action of $Σ_3$ on $K[Σ_3]$:

$$(σ, \sum_i a_i σ_i) \mapsto \sum_i a_i σ_i^{-1} \circ σ_i$$

For every vector $v \in K[Σ_3]$ we denote by $O(v)$ the corresponding orbit of $v$. Let $F_v = K(O(v))$ be the linear subspace of $K[Σ_3]$ generated by $O(v)$. It is an invariant subspace of $K[Σ_3]$. Therefore, using Mashke’s theorem, we deduce that it is a direct sum of irreducible invariant subspaces.

B. $Σ_3$-associative algebras

Let $(A, μ)$ be a $K$-algebra with multiplication $μ$. We denote by $A_μ$ the associator of $μ$ that is

$$A_μ = μ \circ (μ \otimes Id - Id \otimes μ).$$

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Every $\sigma \in \Sigma_3$ defines a linear map denoted by $\Phi_\sigma$ given by

$$\Phi_\sigma : A \otimes^3 \rightarrow A \otimes^3$$

$$x_1 \otimes x_2 \otimes x_3 \rightarrow x_{\sigma^{-1}(1)} \otimes x_{\sigma^{-1}(2)} \otimes x_{\sigma^{-1}(3)}.$$  

If $v = \sum_{\sigma \in \Sigma_3} a_\sigma \sigma$ is a vector of $K[\Sigma_3]$ we define the endomorphism $\Phi_v$ of $A \otimes^3$ by taking:

$$\Phi_v = \sum_{\sigma \in \Sigma_3} a_\sigma \Phi_\sigma.$$  

**Definition 1** An algebra $(A, \mu)$ is a $\Sigma_3$-associative algebra if there exists $v \in K[\Sigma_3]$ such that

$$A_\mu \circ \Phi_v = 0.$$  

C. Lie-admissible algebras. 3-power associative algebras

**Proposition 2** Let $v$ be in $K[\Sigma_3]$ such that $\dim F_v = 1$. Then $v = \alpha V$ or $v = \alpha W$ with $\alpha \in K$ where the vectors $V$ and $W$ are the following vectors:

$$V = Id - \tau_{12} - \tau_{13} - \tau_{23} + c_1 + c_2,$$

$$W = Id + \tau_{12} + \tau_{13} + \tau_{23} + c_1 + c_2.$$  

The first case corresponds to the character of $\Sigma_3$ given by the signature, the second corresponds to the trivial case.

Every algebra $(A, \mu)$ whose associator satisfies

$$A_\mu \circ \Phi_V = 0$$  

is a Lie-admissible algebra. This means that the algebra $(A, [\,])$ whose product is given by the bracket $[x, y] = \mu(x, y) - \mu(y, x)$ is a Lie algebra. Likewise an algebra $(A, \mu)$ whose associator satisfies

$$A_\mu \circ \Phi_W = 0$$  

is 3-power associative that is it satisfies $A_\mu(x, x, x) = 0$ for every $x \in A$.

**II. $G_i$-ASSOCIATIVE ALGEBRAS**

In this section we study the $\Sigma_3$-associative algebras corresponding to the subgroups of $\Sigma_3$.

A. Notations

Let us consider the subgroups of $\Sigma_3$:

$$G_1 = \{Id\},$$

$$G_2 = \{Id, \tau_{12}\},$$

$$G_3 = \{Id, \tau_{23}\},$$

$$G_4 = \{Id, \tau_{13}\},$$

$$G_5 = \{Id, c_1, c_2\} = A_3$$ (the alternating group),

$$G_6 = \sum_{A_3}$$

We keep these notations for the subgroups of $\Sigma_3$ in the following sections.

B. $G_i$-associative algebras

This notion was introduced by E.Remm in her thesis [10].

**Definition 3** Let $G_i$ be a subgroup of $\Sigma_3$. The algebra $(A, \mu)$ is $G_i$-associative if

$$\sum_{\sigma \in G_i} (-1)^{\varepsilon(\sigma)} A_\mu \circ \Phi_\sigma = 0.$$  

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Proposition 4 Every $G_i$-associative algebra is a $\Sigma_3$-associative algebra.

Proof. Every subgroup $G_i$ of $\Sigma_3$ corresponds to an invariant linear space $F(v_i)$ generated by a single vector $v_i \in K[\Sigma_3]$. More precisely we consider $v_1 = Id, v_2 = \tau_12, v_3 = \tau_23, v_4 = \tau_31, v_5 = Id + c_1 + c_2$ and $v_6 = V$ that we have defined in (1). □

Proposition 5 Every $G_i$-associative algebra is a Lie-admissible algebra.

Proof. The vector $V$ belongs to the orbits $O(v_i)$ for every $v_i$. Thus, if $\mu$ is a $G_i$-associative product, it also satisfies

$$A_\mu \circ \Phi_V = 0$$

and $\mu$ is a Lie-admissible multiplication. □

We deduce the following type of Lie-admissible algebras:

1. A $G_1$-associative algebra is an associative algebra.
2. A $G_2$-associative algebra is a Vinberg algebra. If $A$ is finite-dimensional, the associated Lie algebra is provided with an affine structure.
3. A $G_3$-associative algebra is a pre-Lie algebra.
4. If $(A, \mu)$ is $G_4$-associative then $\mu$ satisfies

$$(X.Y).Z - X.(Y.Z) = (Z.Y).X - Z.(Y.X)$$

with $X.Y = \mu(X,Y)$.
5. If $(A, \mu)$ is $G_5$-associative then $\mu$ satisfies the generalized Jacobi condition:

$$(X.Y).Z + (Y.Z).X + (Z.X).Y = X.(Y.Z) + Y.(Z.X) + Z.(X.Y)$$

with $X.Y = \mu(X,Y)$. Moreover if the product is antisymmetric, then it is a Lie algebra bracket.
6. A $G_6$-associative algebra is a Lie-admissible algebra.

III. $G_i$-ASSOCIATIVE COGEBRAS

A. An axiomatic definition of the $G_i$-associative algebras

To define a $G_i$-cogebra it is easier to present an equivalent and axiomatic definition of the notion of $G_i$-associative algebra.

Definition 6 A $G_i$-associative algebra is $(A, \mu, \eta, G_i)$ where $A$ is a vector space, $G_i$ a subgroup of $\Sigma_3$, $\mu : A \otimes A \rightarrow A$ and $\eta : K \rightarrow A$ are linear maps satisfying the following axioms:

1. $(G_i\text{-ass})$: The square

$$\begin{array}{ccc}
A \otimes A \otimes A & \xrightarrow{(\mu \otimes Id)_{G_i}} & A \otimes A \\
(Id \otimes \mu)_{G_i} \downarrow & & \downarrow \mu \\
A \otimes A & \xrightarrow{\mu} & A
\end{array}$$

commutes, where $(Id \otimes \mu)_{G_i}$ is the linear mapping defined by:

$$(Id \otimes \mu)_{G_i} = \sum_{\sigma \in G_i} (-1)^{\ell(\sigma)} (Id \otimes \mu) \circ \Phi_\sigma.$$

2. (Un) The following diagram is commutative:

$$\begin{array}{ccc}
K \otimes A & \xrightarrow{\eta \otimes Id} & A \otimes A \\
& & \xrightarrow{id \otimes \eta} A \otimes K \\
& & \xrightarrow{\mu} A
\end{array}$$
The axiom \((G_i\text{-ass})\) expresses that the multiplication \(\mu\) is \(G_i\)-associative whereas the axiom \((\text{Un})\) means that the element \(\eta(1)\) of \(A\) is a left and right unit for \(\mu\).

**Definition 7** A morphism of \(G_i\)-associative algebras

\[
 f : (A, \mu, \eta, G_i) \to (A', \mu', \eta', G_i)
\]

is a linear map from \(A\) to \(A'\) such that

\[
 \mu' \circ (f \otimes f) = f \circ \mu \quad \text{and} \quad f \circ \eta = \eta'
\]

**B. \(G_i\)-associative cogebras**

We want to dualize the previous diagrams to obtain the notions of corresponding cogebras. Let \(\Delta\) be a comultiplication on a vector space \(C\):

\[
 \Delta : C \to C \otimes C.
\]

We define the bilinear map

\[
 G_i \circ (\Delta \otimes \text{Id}) : C^{\otimes 3} \to C^{\otimes 3}
\]

by

\[
 G_i \circ (\Delta \otimes \text{Id}) = \sum_{\sigma \in G_i} (-1)^{\epsilon(\sigma)} \Phi_{\sigma} \circ (\Delta \otimes \text{Id}).
\]

**Definition 8** A \(G_i\)-cogebras is a vector space \(C\) provided with a comultiplication \(\Delta : C \to C \otimes C\) and a counit \(\epsilon : C \to K\) such that:

1. \((G_i\text{-ass co})\) The following square is commutative:

\[
 \begin{array}{ccc}
 C & \xrightarrow{\Delta} & C \otimes C \\
 \downarrow & & \downarrow \left[ G_i \circ (\text{Id} \otimes \Delta) \right] \\
 C \otimes C & \xrightarrow{G_i \circ (\Delta \otimes \text{Id})} & C \otimes C \otimes C.
\end{array}
\]

2. (Coun) The following diagram is commutative:

\[
 \begin{array}{ccc}
 K \otimes C & \xrightarrow{\epsilon \otimes \text{Id}} & C \otimes C \\
 & \searrow \downarrow \Delta \swarrow & \\
 & C \otimes K \\
 & \downarrow \epsilon \swarrow & \\
 C & \xrightarrow{\epsilon \otimes \text{Id}} & C \otimes K.
\end{array}
\]

**Definition 9** A morphism of \(G_i\)-associative cogebras

\[
 f : (C, \Delta, \epsilon, G_i) \to (C', \Delta', \epsilon', G_i)
\]

is a linear map from \(C\) to \(C'\) such that

\[
 (f \otimes f) \circ \Delta = \Delta' \circ f \quad \text{and} \quad \epsilon = \epsilon \circ f
\]

**C. The dual space of a \(G_i\)-associative cogebras**

For any natural number \(n\) and any \(K\)-vector spaces \(E\) and \(F\), we denote by

\[
 \lambda_n : \text{Hom}(E, F)^{\otimes n} \to \text{Hom}(E^{\otimes n}, F^{\otimes n})
\]

the natural embedding

\[
 \lambda_n(f_1 \otimes \ldots \otimes f_n)(x_1 \otimes \ldots \otimes x_n) = f_1(x_1) \otimes \ldots \otimes f_n(x_n).
\]

**Proposition 10** The dual space of a \(G_i\)-associative cogebras is a \(G_i\)-associative algebra.
Proof. Let \((C, \Delta)\) a \(G_i\)-associative cogebra. We consider the multiplication on the dual vector space \(C^*\) of \(C\) defined by :

\[
\mu = \Delta^* \circ \lambda_2.
\]

It provides \(C^*\) with a \(G_i\)-associative algebra structure. In fact we have

\[
\mu(f_1 \otimes f_2) = \mu_k \circ \lambda_2(f_1 \otimes f_2) \circ \Delta \quad (3)
\]

for all \(f_1, f_2 \in C^*\) where \(\mu_k\) is the multiplication of \(\mathbb{K}\). The equation (3) becomes :

\[
\mu \circ (\mu \otimes Id)(f_1 \otimes f_2 \otimes f_3) = \mu_k \circ (\lambda_2(\mu(f_1 \otimes f_2) \otimes f_3) \circ \Delta)
\]

\[
= \mu_k \circ \lambda_2((\mu_k \circ \lambda_2(f_1 \otimes f_2) \circ \Delta) \otimes f_3) \circ \Delta
\]

\[
= \mu_k \circ (\mu_k \otimes Id) \circ \lambda_3(f_1 \otimes f_2 \otimes f_3) \circ (\Delta \otimes Id) \circ \Delta.
\]

The associator \(A_\mu\) satisfies :

\[
A_\mu = \mu_k \circ (\mu_k \otimes Id) \circ \lambda_3(f_1 \otimes f_2 \otimes f_3) \circ (\Delta \otimes Id) \circ \Delta
\]

\[-\mu_k \circ (Id \otimes \mu_k) \circ \lambda_3(f_1 \otimes f_2 \otimes f_3) \circ (Id \otimes \Delta) \circ \Delta.
\]

and using associativity and commutativity of the multiplication in \(\mathbb{K}\), we obtain

\[
A_\mu = \mu_k \circ (\mu_k \otimes Id) \circ \lambda_3(f_1 \otimes f_2 \otimes f_3) \circ ((\Delta \otimes Id) \circ \Delta - (Id \otimes \Delta) \circ \Delta).
\]

For every \(\sigma \in \Sigma_3\) and any \(\mathbb{K}\)-vector space \(E\), we denote by \(\Phi^E_\sigma\) the action

\[
\Phi^E_\sigma : E^3 \longrightarrow E^3
\]

defined in the first section :

\[
\Phi^E_\sigma(e_1 \otimes e_2 \otimes e_3) = e_{\sigma^{-1}(1)} \otimes e_{\sigma^{-1}(2)} \otimes e_{\sigma^{-1}(3)}.
\]

Thus

\[
\sum_{\sigma \in G_i} (-1)^{s(\sigma)} A_\mu \circ \Phi^E_\sigma = \mu_k \circ (\mu_k \otimes Id) \circ \lambda_3(f_1 \otimes f_2 \otimes f_3) \circ (G_i \circ (\Delta \otimes Id) \circ \Delta - G_i \circ (Id \otimes \Delta) \circ \Delta)
\]

\[= 0.
\]

\(\square\)

D. The dual space of a finite dimensional \(G_i\)-associative algebra

**Proposition 11** The dual vector space of a finite dimensional \(G_i\)-associative algebra has a \(G_i\)-associative cogebra structure.

Proof. Let \(A\) be a finite dimensional \(G_i\)-associative algebra and let \(\{e_i, i = 1, \ldots, n\}\) be a basis of \(A\). If \(\{f_i\}\) is the dual basis then \(\{f_i \otimes f_j\}\) is a basis of \(A^* \otimes A^*\). The coproduct \(\Delta\) on \(A^*\) is defined by

\[
\Delta(f) = \sum_{i,j} f(\mu(e_i \otimes e_j)) f_i \otimes f_j.
\]

In particular

\[
\Delta(f_k) = \sum_{i,j} c_{ij}^{k} f_i \otimes f_j
\]
where $C^G_i$ are the structure constants of $\mu$ related to the basis $\{e_i\}$. Then $\Delta$ is the comultiplication of a $G_i$-associative cogebra.

**E. Associated Lie cogebras**

Since every $G_i$-associative algebra $(A, \mu)$ is Lie-admissible, the bilinear map $\mu(X, Y) - \mu(Y, X)$ is a Lie bracket. Similarly we prove a similar property for $G_i$-associative cogebras. Let us recall the notion of Lie cogebras. Let $C$, $\Delta$ be a Lie cogebra. But the identity $\Delta: C \rightarrow C \otimes C$
a linear map satisfying
1. $\tau \circ \Delta = - \Delta$ where $\tau$ is the permutation $\tau(x \otimes y) = y \otimes x$
2. $\Phi_v \circ (Id \otimes \Delta) \circ \Delta = 0$ where $v = \sum_{\sigma \in G_0} \sigma$.

Then $(C, \Delta)$ is called a Lie cogebra.

**Proposition 12** Let $(C, \Delta)$ be a $G_i$-associative cogebra (non necessary counitary). Let $\Delta_L$ the linear map defined by $\Delta_L = \Delta - \tau \circ \Delta$. Then $(C, \Delta_L)$ is a Lie cogebra.

**Proof.** It is clear that $\tau \circ \Delta_L = - \Delta_L$. As $W$ is in the orbit of the vector $u_i = \sum_{\sigma \in G_0} \sigma$, then any $G_i$-associative cogebra is $G_0$-associative cogebra, that is a Lie admissible cogebra. But the identity

$$\Phi_W \circ (Id \otimes \Delta) \circ \Delta - \Phi_W \circ (\Delta \otimes Id) \circ \Delta = 0$$

is equivalent to

$$\Phi_v \circ (Id \otimes \Delta) \circ \Delta - \tau \circ \Phi_v \circ (Id \otimes \Delta) \circ \Delta$$

$$-\Phi_v \circ (Id \otimes \tau \circ \Delta) \circ \Delta - \tau \circ \Phi_v \circ (Id \otimes \tau \circ \Delta) \circ \Delta = 0$$

with $v = Id + c + c^2$, we deduce the result. □

**IV. THE CONVOLUTION PRODUCT**

Let $(A, \mu, \eta, G_1)$ be an associative algebra and $(C, \Delta, \epsilon, G_1)$ an associative cogebra. The convolution product on the vector space $Hom(C, A)$ is given by

$$f \ast g = \mu \circ \lambda_2 \circ (f \otimes g) \circ \Delta$$

for every $f$ and $g \in Hom(C, A)$ and $(Hom(C, A), \ast, G_1)$ is an associative algebra. We want to generalize this property to the other groups $G_i$, $i \geq 2$. Let us begin by introduce new classes of associative algebras which appear naturally by studying the duality in the corresponding operads (see section V).

**A. The $G_i'$-algebras and cogebras**

**Definition 13** For $i \geq 2$, a $G_i'$-algebra is an associative algebra satisfying :

- for $i = 2$ : $x_1.x_2.x_3 = x_2.x_1.x_3$
- for $i = 3$ : $x_1.x_2.x_3 = x_1.x_3.x_2$
- for $i = 4$ : $x_1.x_2.x_3 = x_3.x_2.x_1$
- for $i = 5$ : $x_1.x_2.x_3 = x_2.x_3.x_1 = x_3.x_1.x_2$
- for $i = 6$ : $x_1.x_2.x_3 = x_{\sigma(1)}.x_{\sigma(2)}.x_{\sigma(3)}$ for all $x_1, x_2, x_3$ and $\sigma \in \Sigma_3$.

**Definition 14** For $i \geq 2$, a $G_i'$-cogebra is a coassociative cogebra, that is a $G_1'$-associative cogebra, satisfying :

$$\Phi_{u_i} \circ (Id \otimes \Delta) \circ \Delta = (Id \otimes \Delta) \circ \Delta$$

where $u_i = \sum_{\sigma \in G_0} \sigma^{-1}$. 

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B. Convolution product on $\text{Hom}(C,A)$

We will provide $\text{Hom}(C,A)$ with a structure of $G_i$-associative algebra.

**Proposition 15** Let $(A, \mu, \eta, G_i)$ be a $G_i$-associative algebra and $(C, \Delta, \varepsilon, G_i')$ a $G_i'$-cogebras. Then $(\text{Hom}(C,A), \star, \eta \circ \varepsilon, G_i)$ is a $G_i$-associative algebra where $\star$ is the convolution product:

$$f \star g = \mu \circ \lambda_2 (f \otimes g) \circ \Delta.$$

**Proof.** Let us compute the associator $A_*$ of the convolution product.

$$(f_1 \star f_2) \star f_3 = \mu \circ \lambda_2 ((f_1 \star f_2) \otimes f_3) \circ \Delta$$

$$= \mu \circ \lambda_2 ((\mu \circ \lambda_2 (f_1 \otimes f_2) \circ \Delta) \otimes f_3) \circ \Delta$$

$$= \mu \circ (\mu \otimes \text{Id}) \circ \lambda_3 (f_1 \otimes f_2 \otimes f_3) \circ (\Delta \otimes \text{Id}) \circ \Delta.$$

Thus

$$A_*(f_1 \otimes f_2 \otimes f_3) = \mu \circ (\mu \otimes \text{Id}) \circ \lambda_3 (f_1 \otimes f_2 \otimes f_3) \circ (\Delta \otimes \text{Id}) \circ \Delta$$

$$- \mu \circ (\text{Id} \otimes \mu) \circ \lambda_3 (f_1 \otimes f_2 \otimes f_3) \circ (\text{Id} \otimes \Delta) \circ \Delta.$$

Therefore

$$\sum_{\sigma \in G_i} (-1)^{(o)} A_* \circ \Phi_{\sigma_{\text{Hom}(C,A)}} (f_1 \otimes f_2 \otimes f_3)$$

$$= \mu \circ (\mu \otimes \text{Id}) \circ (\sum_{\sigma \in G_i} \lambda_3 (\Phi_{\sigma_{\text{Hom}(C,A)}} (f_1 \otimes f_2 \otimes f_3))) \circ (\Delta \otimes \text{Id}) \circ \Delta$$

$$- \mu \circ (\text{Id} \otimes \mu) \circ (\sum_{\sigma \in G_i} \lambda_3 (\Phi_{\sigma_{\text{Hom}(C,A)}} (f_1 \otimes f_2 \otimes f_3))) \circ (\text{Id} \otimes \Delta) \circ \Delta.$$

But

$$\lambda_3 (\Phi_{\sigma_{\text{Hom}(C,A)}} (f_1 \otimes f_2 \otimes f_3)) = \Phi_A^\sigma \circ \lambda_3 (f_1 \otimes f_2 \otimes f_3) \circ \Phi_{\sigma^{-1}}^{C^\ast}.$$

This gives

$$\sum_{\sigma \in G_i} (-1)^{(o)} A_* \circ \Phi_{\sigma_{\text{Hom}(C,A)}} (f_1 \otimes f_2 \otimes f_3)$$

$$= \mu \circ (\mu \otimes \text{Id}) \circ (\sum_{\sigma \in G_i} \Phi_A^\sigma \circ \lambda_3 (f_1 \otimes f_2 \otimes f_3)) \circ \Phi_{\sigma^{-1}}^{C^\ast} \circ (\Delta \otimes \text{Id}) \circ \Delta$$

$$- \mu \circ (\text{Id} \otimes \mu) \circ (\sum_{\sigma \in G_i} \Phi_A^\sigma \circ \lambda_3 (f_1 \otimes f_2 \otimes f_3)) \circ \Phi_{\sigma^{-1}}^{C^\ast} \circ (\text{Id} \otimes \Delta) \circ \Delta.$$

As $\Delta$ is coassociative

$$(\Delta \otimes \text{Id}) \circ \Delta = (\text{Id} \otimes \Delta) \circ \Delta$$

and the $G_i'$-cogebras structure implies

$$\Phi_{C^\ast} \circ (\text{Id} \otimes \Delta) \circ \Delta = \Phi_{C^\ast} \circ (\Delta \otimes \text{Id}) \circ \Delta = (\Delta \otimes \text{Id}) \circ \Delta.$$

Then

$$\sum_{\sigma \in G_i} (-1)^{(o)} A_* \circ \Phi_{\sigma_{\text{Hom}(C,A)}} (f_1 \otimes f_2 \otimes f_3)$$

$$= \mu \circ (\mu \otimes \text{Id}) \circ (\sum_{\sigma \in G_i} \Phi_A^\sigma \circ \lambda_3 (f_1 \otimes f_2 \otimes f_3)) \circ (\Delta \otimes \text{Id}) \circ \Delta$$

$$- \mu \circ (\text{Id} \otimes \mu) \circ (\sum_{\sigma \in G_i} \Phi_A^\sigma \circ \lambda_3 (f_1 \otimes f_2 \otimes f_3)) \circ (\Delta \otimes \text{Id}) \circ \Delta$$

$$= \sum_{\sigma \in G_i} A_* \circ \Phi_A^\sigma \circ \lambda_3 (f_1 \otimes f_2 \otimes f_3) \circ (\Delta \otimes \text{Id}) \circ \Delta$$

$$= 0.$$

This proves the proposition. □
V. TENSOR PRODUCT of $\Sigma_3$-ASSOCIATIVE ALGEBRAS

A. Non monoidal categories We know that the tensor product of associative algebras can be provided with an associative algebra structure. In other words, the category of associative algebras is monoidal and closed for the tensor product. This is generally not true for other categories of $\Sigma_3$-associative algebras.

Proposition 16 Let $(A, \mu_A)$ and $(B, \mu_B)$ be two $\Sigma_3$-associative algebras respectively defined by the relations $A_{\mu_A} \circ \Phi_v = 0$ and $A_{\mu_B} \circ \Phi_w = 0$. Then $(A \otimes_K B, \mu_A \otimes \mu_B)$ is a $\Sigma_3$-associative algebra if and only if $A$ and $B$ are associative algebras (i.e. $G_1$-associative algebras).

Proof. Let $A_\mu$ be the associator of the law $\mu$ and $A_{\mu}(x_1 \otimes x_2 \otimes x_3) = \mu(x_1, x_2, x_3)$ and $A^R_\mu = A^L_\mu - A_\mu$. By hypothesis, $A$ (resp. $B$) is defined by $A_{\mu} \circ \Phi_v = 0$ (resp. $A_{\mu} \circ \Phi_w = 0$). Let us suppose that $A \otimes B$ is a $\Sigma_3$-associative algebra. There exists $v' \in K$ such that $A_{\mu_{A \otimes B}} \circ \Phi_{v'} = 0$. By taking $v' = \sum_{i=1}^6 \gamma_i \sigma_i$ the last condition becomes

$$\sum_{i=1}^6 \gamma_i [A_{\mu_{A \otimes B}} \circ \Phi_{\sigma_i}] = 0 \quad (3)$$

and

$$\sum_{i=1}^6 \gamma_i [A^L_{\mu_A} \circ \Phi_{\sigma_i} \otimes A^L_{\mu_B} \circ \Phi_{\sigma_i} - A^R_{\mu_A} \circ \Phi_{\sigma_i} \otimes A^R_{\mu_B} \circ \Phi_{\sigma_i}] = 0 \quad (4)$$

Let us denote by $e_i = A^L_{\mu_A} \circ \Phi_{\sigma_i}$, $\tilde{e}_i = A^R_{\mu_A} \circ \Phi_{\sigma_i}$, $f_i = A^L_{\mu_B} \circ \Phi_{\sigma_i}$ and $\tilde{f}_i = A^R_{\mu_B} \circ \Phi_{\sigma_i}$. The vectors $e_i$ and $\tilde{e}_i$ belong to $\text{Hom}(A^\otimes^3, A)$ and the vectors $f_i$ and $\tilde{f}_i$ to $\text{Hom}(B^\otimes^3, B)$. Equation (4) becomes:

$$\sum_{i=1}^6 \gamma_i [e_i \otimes f_i - \tilde{e}_i \otimes \tilde{f}_i] = 0. \quad (5)$$

From the definition of the algebras $A$ and $B$ we have:

$$\sum_{i=1}^6 a_i (e_i - \tilde{e}_i) = 0 \quad \text{and} \quad \sum_{i=1}^6 b_i (f_i - \tilde{f}_i) = 0$$

if $v = \sum_{i=1}^6 a_i \sigma_i$ and $w = \sum_{i=1}^6 b_i \sigma_i$. Suppose that $\text{dim} F_v = k$ with $0 \leq k \leq 6$. Then the rank of the vectors $\{f_i, \tilde{f}_i\}$ is equal to $(6 + k)$. We can suppose that $\{f_1, ..., f_6, \tilde{f}_1, ..., \tilde{f}_6\}$ are independent. This implies:

$$\begin{align*}
\tilde{f}_{k+1} &= \rho_{1}^{k+1} f_1 + ... + \rho_{6}^{k+1} f_6 + \tilde{\rho}_{1}^{k+1} \tilde{f}_1 + ... + \tilde{\rho}_{6}^{k+1} \tilde{f}_6 \\
&\vdots \\
\tilde{f}_6 &= \rho_{1}^{6} f_1 + ... + \rho_{6}^{6} f_6 + \rho_{1}^{6} \tilde{f}_1 + ... + \rho_{6}^{6} \tilde{f}_6.
\end{align*}$$

The equation (5) becomes:

$$\sum_{i=1}^6 e_i \otimes f_i + \sum_{i=1}^k e_i'' \otimes \tilde{f}_k = 0$$
and the independence of the vectors \( \{f_1, ..., f_6, \tilde{f}_1, ..., \tilde{f}_6\} \) implies \( e'_1 = ... = e'_6 = e''_1 = ... = e''_6 = 0 \). Then we deduce

\[
\begin{align*}
\gamma_1 e_1 - \gamma_k \rho_{k+1}^{k+1} e_1 - ... - \gamma_6 \rho_6^6 e_6 &= 0 \\
\vdots \\
\gamma_a e_a - \gamma_k \rho_{k+1}^{k+1} e_1 - ... - \gamma_6 \rho_6^6 e_6 &= 0 \\
\gamma_1 e_1 + \gamma_k \rho_{k+1}^{k+1} e_1 + ... + \gamma_6 \rho_6^6 e_6 &= 0 \\
\vdots \\
\gamma_k e_k + \gamma_k \rho_{k+1}^{k+1} e_k + ... + \gamma_6 \rho_6^6 e_6 &= 0.
\end{align*}
\]

Let us suppose that \( k \neq 0 \). Then the conditions on the vector \( v \) are of the type

\[
\sum_{\gamma=0}^6 a_i (e_i - \tilde{e}_i) = 0
\]

and we have successively \( \gamma_1 = ... = \gamma_k = 0 \) and \( \gamma_k \rho_{k+1}^{k+1} = ... = \gamma_6 \rho_6^6 = 0 \) for \( i = 1, ..., 6 \). If one of \( \gamma_j \) for \( j = k+1, ..., 6 \) is not 0 then \( \tilde{f}_j = 0 \) for \( i = 1, ..., 6 \). Thus

\[
\tilde{f}_j = \rho_1 f_1 + ... + \rho_k f_k
\]

and, as for every \( i \) there exists \( \sigma \in \Sigma_3 \) such that \( \tilde{f}_i = \tilde{f}_j \circ \Phi_\sigma \), we have also

\[
\tilde{f}_i = \rho_1 f_1 + ... + \rho_k f_k
\]

for every \( i = 1, ..., 6 \). This implies \( k = 0 \) which is impossible from the hypothesis. Thus \( k = 0 \) and the vectors \( \{f_1, \tilde{f}_i\}_{i=1, ..., 6} \) is of rank 6. The only possible relations are then \( f_i = \tilde{f}_i \) and \( B \) is associative. We deduce the associativity of \( A \). \( \square \)

B. \( G_i \)-algebras

In [7], we have studied the operads corresponding to the \( G_i \)-associative algebras. More precisely, the operad \( G_i - Ass \) is the quadratic operad defined by the relations

\[
\sum_{\sigma \in G_i} (-1)^{\ell(\sigma)}(x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, x_{\sigma^{-1}(3)}) - x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, x_{\sigma^{-1}(3)}).
\]

Then an algebra on the operad \( G_i - Ass \) is a \( G_i \)-associative algebra. In [7], we have also computed the dual operads \( (G_i - Ass)! \) and proved that only \( G_1 - Ass \) (classically noted \( Ass \)) is autodual that is \( (G_i - Ass) = (G_i - Ass)! \). We call a \( G_i \)-algebra an algebra on the quadratic operad \( (G_i - Ass)! \). In definition 13 we have given their structures.

C. Tensor products

**Theorem 17** If \( A \) is a \( G_i \)-associative algebra and \( B \) a \( G_i \)-algebra (with the same indice) then \( A \otimes B \) can be provided with a \( G_i \)-algebra structure for \( i = 1, ..., 6 \).

**Proof.** Let us consider on \( A \otimes B \) the classical tensor product

\[
\mu_A \otimes \mu_B ((a_1 \otimes b_1) \otimes (a_2 \otimes b_2)) = \mu_A(a_1 \otimes a_2) \otimes \mu_B(b_1 \otimes b_2).
\]

To simplify, we denote by \( \mu \) the product \( \mu_A \otimes \mu_B \). As \( B \) is an associative algebra, the associator \( A_\mu \) satisfies:

\[
A_\mu((a_1 \otimes b_1) \otimes (a_2 \otimes b_2) \otimes (a_3 \otimes b_3)) = A_{\mu_A}(a_1 \otimes a_2 \otimes a_3) \otimes \mu_B \circ (\mu_B \otimes Id)(b_1 \otimes b_2 \otimes b_3).
\]

Therefore

\[
\sum_{\sigma \in G_i} (-1)^{\ell(\sigma)} A_\mu \circ \Phi_\sigma((a_1 \otimes b_1) \otimes (a_2 \otimes b_2) \otimes (a_3 \otimes b_3))
\]

\[
= \sum_{\sigma \in G_i} (-1)^{\ell(\sigma)} A_{\mu_A} \circ \Phi_\sigma(a_1 \otimes a_2 \otimes a_3) \otimes \mu_B \circ (\mu_B \otimes Id) \circ \Phi_\sigma(b_1 \otimes b_2 \otimes b_3).
\]

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But $B$ a $G_i^\al$-algebra. Then

$$\mu_B \circ (\mu_B \otimes \text{Id}) \circ \Phi_\sigma(b_1 \otimes b_2 \otimes b_3) = \mu_B \circ (\mu_B \otimes \text{Id})(b_1 \otimes b_2 \otimes b_3).$$

We obtain

$$\sum_{\sigma \in G_i} (-1)^{\varepsilon(\sigma)} A_{\mu} \circ \Phi_\sigma((a_1 \otimes b_1) \otimes (a_2 \otimes b_2) \otimes (a_3 \otimes b_3))$$

$$= (\sum_{\sigma \in G_i} (-1)^{\varepsilon(\sigma)} A_{\mu} \circ \Phi_\sigma(a_1 \otimes a_2 \otimes a_3)) \otimes \mu_B \circ (\mu_B \otimes \text{Id})(b_1 \otimes b_2 \otimes b_3)$$

$$= 0.$$

\[ \square \]

D. The categories $G_i - \text{ASS}$

Let $G_i - \text{ASS}$ and $(G_i - \text{ASS})^\dagger$ the categories whose objects are respectively $G_i$-associative algebras and $G_i^\al$-algebras and the morphisms are the homomorphisms of algebras. The previous theorem can be interpreted as follows: Let $A$ be a $G_i$-associative algebra. Then $A \otimes -$ is a covariant functor

$$A \otimes - : (G_i - \text{ASS})^\dagger \rightarrow G_i - \text{ASS}.$$ 

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