On the $\tau$-functions of the Degasperis–Procesi equation

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Abstract

The Degasperis–Procesi (DP) equation is investigated from the point of view of determinant–Pfaffian identities. The reciprocal link between the DP equation and the pseudo 3-reduction of the $C_\infty$ two-dimensional Toda system is used to construct the $N$-soliton solution of the DP equation. The $N$-soliton solution of the DP equation is presented in the form of Pfaffian through a hodograph (reciprocal) transformation. The bilinear equations, the identities between determinants and Pfaffians, and the $\tau$-functions of the DP equation are obtained from the pseudo 3-reduction of the $C_\infty$ two-dimensional Toda system.

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1. Introduction

In this paper, we investigate the $N$-soliton solution of the Degasperis–Procesi (DP) equation [1]

$$u_t + 3\kappa^3 u^3 - u_{txx} + 4uu_x = 3u_xu_{xx} + uu_{xxx}, \quad (1.1)$$

which has received much attention in recent years. Since its discovery in 1999, many interesting mathematical properties of the DP equation have been found [2–8]. Matsuno derived the $N$-soliton solution of the DP equation when $\kappa \neq 0$ through the $\tau$-function of the CKP hierarchy [6, 7].

In recent years, we proposed integrable discretizations of various integrable systems, such as the Camassa–Holm (CH) equation [9, 10], the short wave limit of the CH equation [11], the short pulse equation [12], the WKI elastic beam equation and the Dym equation [13]. Since these equations, as well as the DP equation, are transformed to some integrable systems through hodograph (reciprocal) transformations, constructing integrable discretizations is not an easy task. In our previous studies, we constructed the integrable discretizations of these equations by using Hirota’s bilinear approach and an approach based on discrete differential geometry. In our studies, we found that the key of discretizations for these equations is discretizations of hodograph (reciprocal) transformations. Recently, we found that integrable discretizations
of some of these systems have a geometric formulation which verifies the discrete differential geometric meaning of discrete hodograph (reciprocal) transformations.

The objective of this paper is to establish a useful formulation which can be used in the integrable discretization of the DP equation. Although the DP equation is very similar to the CH equation, there is an obstacle for constructing a discrete analogue of the DP equation. Matsuno derived the $N$-soliton solutions of the DP equation in the form of a Pfaffian through the CKP equation, but bilinear equations (or Pfaffian identities) of the DP equation is still unclear. To construct a discrete analogue of the DP equation through Hirota’s bilinear approach, we need to understand bilinear identities of Pfaffians which consist of the DP equation. Moreover, we need to understand clearly how to obtain the DP’s $\tau$-functions from the ones of the KP hierarchy and the two-dimensional Toda system. Thus, we investigate the DP equation from the point of view of determinant–Pfaffian identities. In the same motivation, we recently investigated the reduced Ostrovsky equation [14]. It is known that the reduced Ostrovsky equation can be obtained as a short wave limit of the DP equation [15].

In this paper, we establish the reciprocal link between the DP equation and the pseudo 3-reduction of the $C_\infty$ two-dimensional Toda system and investigate the relations of $\tau$-functions. Using this reciprocal link and the relations of $\tau$-functions, we construct the $N$-soliton solution of the DP equation in the form of a Pfaffian. The bilinear equations, the identities between determinants and Pfaffians, and the $\tau$-functions of the DP equation are easily obtained from the pseudo 3-reduction of the $C_\infty$ two-dimensional Toda system.

Note that Matsuno obtained the $N$-soliton solution through the $N$-soliton solution of the CKP hierarchy. Since the CKP hierarchy and the $C_\infty$ two-dimensional Toda (2D-Toda) system share the same $\tau$-function, it is natural to obtain the same soliton solution by both methods. However, by using the $C_\infty$ 2D-Toda system, we can include a negative time variable in the $\tau$-functions, so there is an advantage for obtaining the bilinear equations of determinants and the relations of determinants and Pfaffians.

2. The DP equation and the pseudo 3-reduction of the $C_\infty$ 2D-Toda system

The 2D-Toda system of $A_\infty$-type, which is also called the Toda field equation or the two-dimensional Toda lattice, is given as follows [16–19]:

$$\frac{\partial^2 \theta_n}{\partial x_1 \partial x_{-1}} = -\sum_{m \in \mathbb{Z}} a_{n,m} e^{-\theta_m}, \quad n \in \mathbb{Z},$$ (2.1)

where the matrix $A = (a_{n,m})$ is the transpose of the Cartan matrix for the infinite-dimensional Lie algebra $A_\infty$, which is the infinite tridiagonal matrix

$$A = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ \vdots & -1 & 2 & -1 \\ -1 & 2 & -1 & \ddots \\ \ddots & \ddots & \ddots & \ddots \\ \vdots & -1 & 2 & -1 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$ (2.2)

The $A_\infty$ 2D-Toda system (2.1) with (2.2) may be written as

$$\frac{\partial^2 \theta_n}{\partial x_1 \partial x_{-1}} = e^{-\theta_{n-1}} - 2e^{-\theta_n} + e^{-\theta_{n+1}}.$$ (2.3)
The $A_\infty$ 2D-Toda system (2.1) with (2.2) is transformed into the bilinear equation

$$-(\frac{1}{2}D_x D_{x-1} - 1) \tau_n \cdot \tau_m = \tau_{n-1} \tau_{m+1},$$

(2.4)

through the dependent variable transformation

$$\theta_n = -\ln \frac{\tau_{n+1} \tau_{n-1}}{\tau_n^2}.$$  

(2.5)

Here, $D_x$ is the Hirota $D$-operator which is defined as

$$D_x^a a(x) \cdot b(x) = (\partial_x - \partial_x')^a a(x) b(x)|_{x \to x'}.$$  

(2.6)

**Lemma 2.1** (Ueno–Takasaki [20], Babich–Matveev–Sall [21], Hirota [22], Nimmo–Willox [23]). The bilinear equation (2.4) and other bilinear equations of the members of the 2D-Toda lattice hierarchy have the following Gram-type determinant solution:

$$\tau_n = \det(\psi_{i,j}^{(n)})_{1 \leq i,j \leq M},$$

where

$$\psi_{i,j}^{(n)} = c_{i,j} + (-1)^n \int_{-\infty}^{x_i} \phi_{i}^{(n)} \phi_{j}^{(-n)} \, dx_1.$$  

Here, $c_{i,j}$ are constants, and $\phi_{i,j}^{(n)}$ and $\phi_{i,j}^{(-n)}$ must satisfy $\frac{\partial \phi_{i,j}^{(n)}}{\partial n} = \phi_{i,j}^{(n+k)}$ and $\frac{\partial \phi_{i,j}^{(-n)}}{\partial n} = (-1)^{k-1} \phi_{i,j}^{(n+k)}$ for $k = \pm 1, \pm 2, \pm 3, \ldots$.

For example, the following linear independent set of functions $\{\phi_{i,j}^{(n)}, \phi_{i,j}^{(-n)}\}$ for $i, j = 1, 2, \ldots, M$ gives the $M$-soliton solution of the $A_\infty$ 2D-Toda system:

$$\phi_{i,j}^{(n)} = \frac{p_i}{q_i} e^{\xi_i}, \quad \phi_{i,j}^{(-n)} = \frac{q_i}{p_i} e^{\eta_i},$$

where $\xi_i = p_i x_1 + \frac{1}{p_i} x_1 \cdot x_1 + p_i x_2 + \frac{1}{p_i} x_2 \cdot x_2 + \frac{1}{p_i} x_3 + \cdots + \xi_0$ and $\eta_i = q_i x_1 + \frac{1}{q_i} x_1 \cdot x_1 - q_i^2 x_2 - \frac{1}{q_i} x_2 \cdot x_2 + q_i^3 x_3 + \cdots + \eta_0$.

**Proof.** See [22].

We impose the $C_\infty$-reduction $\theta_n = \theta_{-n} (n \geq 0)$ to the $A_\infty$ 2D-Toda system, i.e. fold the infinite sequence $\{\cdots, \theta_{-2}, \theta_{-1}, \theta_0, \theta_1, \theta_2, \cdots\}$ in $\theta_0$ [24, 20, 23, 25–27]. Then, we have $\theta_{-1} = \theta_1, \theta_{-2} = \theta_2, \theta_{-3} = \theta_3, \cdots$.

For $n = 0$,

$$\frac{\partial^2 \theta_0}{\partial x_1 \partial x_{-1}} = e^{-\theta_{-1}} - 2 e^{-\theta_0} + e^{-\theta_1} = -2 e^{-\theta_0} + 2 e^{-\theta_1}.$$  

For $n = 1$,

$$\frac{\partial^2 \theta_1}{\partial x_1 \partial x_{-1}} = e^{-\theta_0} - 2 e^{-\theta_1} + e^{-\theta_0}.$$  

Thus, we obtain the $C_\infty$ 2D-Toda system [16, 19]

$$\frac{\partial^2 \theta_n}{\partial x_1 \partial x_{-1}} = \sum_{m \in \mathbb{Z}_{\geq 0}} d_{n,m+1} e^{-\theta_m}, \quad n \in \mathbb{Z}_{\geq 0}.$$  

(2.7)
where the matrix $A = (a_{n,m})$ is the transpose of the Cartan matrix for the infinite-dimensional Lie algebra $C_\infty$, which is the semi-infinite tridiagonal matrix

$$A = \begin{bmatrix}
2 & -2 & & & \\
-1 & 2 & -1 & & \\
& -1 & 2 & -1 & \\
& & \ddots & \ddots & \ddots \\
& & & -1 & 2 & -1 \\
& & & & \ddots & \ddots & \ddots \\
& & & & & \ddots & \ddots & \ddots \\
\end{bmatrix}. \quad (2.8)
$$

The $C_\infty$ 2D-Toda system (2.7) with (2.8) may be written as

$$\frac{\partial^2 \theta_0}{\partial x_1 \partial x_{-1}} = -2 e^{-\theta_0} + 2 e^{-\hat{\theta}_0}, \quad (2.9)$$
$$\frac{\partial^2 \theta_n}{\partial x_1 \partial x_{-1}} = e^{-\theta_{n-1}} - 2 e^{-\theta_n} + e^{-\theta_{n+1}}, \quad n \geq 1. \quad (2.10)
$$

The $C_\infty$ 2D-Toda system (2.7) with (2.8) is transformed into the bilinear equations

$$- (\frac{1}{2}D_x D_{x_{-1}} - 1) \tau_0 \cdot \tau_0 = \tau_1^2, \quad (2.11)$$
$$- (\frac{1}{2}D_x D_{x_{-1}} - 1) \tau_n \cdot \tau_n = \tau_{n-1} \tau_{n+1}, \quad \text{for } n \geq 1, \quad (2.12)$$

through the dependent variable transformation

$$\theta_0 = - \ln \frac{\tau_1}{\tau_0}, \quad \text{and} \quad \theta_n = - \ln \frac{\tau_{n+1} \tau_{n-1}}{\tau_n^2} \quad \text{for } n \geq 1. \quad \text{Lemma 2.2.}
$$

**The bilinear equations of the $C_\infty$ 2D-Toda system (2.11) and (2.12) have the $N$-soliton solution which is expressed as**

$$\tau_n = \text{det} (\psi_{i,j}^{(n)})_{1 \leq i,j \leq 2N^*},$$

where

$$\psi_{i,j}^{(n)} = c_{i,j} + (-1)^n \int_{-\infty}^{x_i} \psi_0^{(n)} \psi_j^{(-n)} \, dx_1,$$
$$\psi_j^{(n)} = p_i^n e^{\xi_i}, \quad \xi_i = p_i x_1 + \frac{1}{p_i} x_{-1} + p_i^3 x_3 + \frac{1}{p_i^3} x_{-3} + \cdots + \xi_0,$$

and $c_{i,j} = c_{j,i}$.

**Proof.** Imposing the $C_\infty$ reduction $\tau_n = \tau_{-n}$, i.e. folding the sequence of the $\tau$-functions $[\ldots, \tau_{-2}, \tau_{-1}, \tau_0, \tau_1, \tau_2, \ldots]$ in $t_0$, we have $\tau_{-1} = \tau_1, \tau_{-2} = \tau_2, \tau_{-3} = \tau_3, \ldots [20, 23, 24, 25-27]$. Thus, we obtain the bilinear equations (2.11) and (2.12) from the 2D-Toda bilinear equation (2.4).

To impose the $C_\infty$ reduction to the Gram-type determinant solution of the $A_\infty$ 2D-Toda system, we impose the constraint $\psi_j^{(n)} = \psi_j^{(-n)}, c_{i,j} = c_{j,i}, M = 2N$ and $x_{2k} \equiv 0$ for every integer $k$. With this constraint, each element of the Gram-type determinant has the following property:

$$\psi_{i,j}^{(n)} = c_{i,j} + (-1)^n \int_{-\infty}^{x_i} \psi_0^{(n)} \psi_j^{(-n)} \, dx_1$$
$$= c_{j,i} + (-1)^n \int_{-\infty}^{x_i} \psi_j^{(-n)} \psi_0^{(n)} \, dx_1$$
$$= \psi_{j,i}^{(-n)}.$$

Then, the $\tau$-function satisfies $\tau_n = \tau_{-n}$. Therefore, the $N$-soliton solution of the $C_\infty$ 2D-Toda system is expressed by the above Gram-type determinant. \qed
Lemma 2.4. The bilinear equations

\[ - \left( \frac{1}{2} D_y D_{x_{r+1}} \right) \tau_0 \cdot \tau_0 = \tau_1^2, \quad (2.13) \]

\[ - \left( \frac{1}{2} D_y D_{x_{r+1}} \right) \tau_1 \cdot \tau_1 = \tau_0^2, \quad (2.14) \]

are satisfied by the \( \tau \)-functions which are obtained by the pseudo 3-reduction of the \( \mathbb{C}_\infty \) 2D-Toda system. The \( \tau \)-functions are given by

\[ \tau_n = \det(\psi_{i,j}^{(n)})_{1 \leq i, j \leq 2N}, \]

where

\[ \psi_{i,j}^{(n)} = c_{i,j} + (-1)^n \int_{-\infty}^{\infty} \psi_{j}^{(n)} \psi_{i}^{(-n)} \, dx, \]

\[ \psi_i^{(n)} = p_i^n \epsilon^0, \quad \xi_i = p_i x_1 + \frac{1}{p_i} x_{-1} + p_i^3 x_3 + \frac{1}{p_i} x_{-3} + \cdots + \xi_0 \]

and \( c_{i,j} = \delta_{i,2N+1 - \alpha_i}, \quad \alpha_i = \alpha_{2N+1 - \alpha_i}, \quad p_i^1 - p_i p_{2N+1-i} + p_{2N+1-i}^2 = 1 \).

Proof. To impose the pseudo 3-reduction to the \( \tau \)-function in lemma 2.2, we add a constraint \( p_i^1 + p_{2N+1-i}^1 = p_i + p_{2N+1-i}, \quad p_i \neq -p_{2N+1-i}, \) i.e. \( p_i^2 - p_i p_{2N+1-i} + p_{2N+1-i}^2 = 1 \), and \( c_{i,j} = \delta_{i,2N+1 - \alpha_i}, \quad \alpha_i = \alpha_{2N+1 - \alpha_i} \) [28]. \( \square \)

Lemma 2.4. The \( \tau \)-function \( \tau_1 \) of the bilinear equations (2.13) and (2.14) with the pseudo 3-reduction constraint satisfies the following relation:

\[ \tau_1 = \frac{1}{c} g_1 g_2. \quad (2.15) \]

with

\[ g_1 = \pf \left( 2\alpha_i \left( p_i - 1 \right) \left( p_j - 1 \right) \delta_{j,2N+1-i} + \frac{p_i - p_j}{p_i + p_j} \epsilon_i^{+}\epsilon_j^{+} \right)_{1 \leq i, j \leq 2N}, \quad (2.16) \]

\[ g_2 = \pf \left( 2\alpha_i \left( p_i + 1 \right) \left( p_j + 1 \right) \delta_{j,2N+1-i} + \frac{p_i - p_j}{p_i + p_j} \epsilon_i^{+}\epsilon_j^{+} \right)_{1 \leq i, j \leq 2N}, \quad (2.17) \]

where \( \xi_i = p_i^{-1} x_{-1} + p_i x_1 + \xi_0 \), \( p_i^2 - p_i p_{2N+1-i} + p_{2N+1-i}^2 = 1 \), \( \alpha_i = \alpha_{2N+1-i} \) and \( c = 2^{2N} \prod_{k=1}^{2N} p_k \).

Proof. Suppose that \( \alpha_i = \alpha_{2N+1-i} \) and \( p_i^1 - p_i p_{2N+1-i} + p_{2N+1-i}^2 = 1 \) are satisfied. Then, we can rewrite \( \tau_1 \) as follows:

\[ \tau_1 = \det(\psi_{i,j}^{(1)})_{1 \leq i, j \leq 2N} = \det \left( \delta_{j,2N+1-i} \alpha_i + \frac{1}{p_i} \left( \frac{p_i}{p_j} \right) \epsilon_i^{+}\epsilon_j^{+} \right)_{1 \leq i, j \leq 2N} \]

\[ = \det \left( \delta_{j,2N+1-i} \alpha_i \frac{p_i p_{2N+1-i} - p_{2N+1-i}^2}{p_i (p_i - p_j)} - \frac{1}{p_i + p_j} \epsilon_i^{+}\epsilon_j^{+} \right)_{1 \leq i, j \leq 2N} \]

\[ = \det \left( \delta_{j,2N+1-i} \alpha_i \frac{p_i^2 - 1}{p_i (p_i - p_j)} - \frac{1}{p_i + p_j} \epsilon_i^{+}\epsilon_j^{+} \right)_{1 \leq i, j \leq 2N} \]

\[ = \frac{1}{c} \det \left( 2\delta_{j,2N+1-i} \alpha_i \frac{p_i^2 - 1}{p_i - p_j} - \frac{2p_i}{p_i + p_j} \epsilon_i^{+}\epsilon_j^{+} \right)_{1 \leq i, j \leq 2N} \]

\[ = \frac{1}{c} \det \left( 2\delta_{j,2N+1-i} \alpha_i \frac{p_i^2 - 1}{p_i - p_j} - \frac{(p_i - p_j)(p_i + 1)}{(p_i + p_j)(p_i - 1)} \frac{1}{p_i - 1} \epsilon_i^{+}\epsilon_j^{+} \right)_{1 \leq i, j \leq 2N} \]

\[ \]
Lemma 2.5.\[\xi = \prod_{i,j=1}^{c} p_i + p_j \in \mathbb{Z} \quad \text{with} \quad c = 2^{2N} \prod_{k=1}^{2N} P_k.\]

where $c_{i,j} = 2\delta_{i,j} + \alpha, (p_i - 1)(p_j - 1) - \delta_{i,j} + \alpha$, and $c = 2^{N} \prod_{k=1}^{2N} P_k$. Using formula (B.1), we obtain

\[
\tau_1 = \frac{1}{c} \det \left( \begin{array}{cccc}
\Psi_{1,1} & \Psi_{1,2} & \cdots & \Psi_{1,2N} \\
\vdots & \vdots & \ddots & \vdots \\
\Psi_{2N,1} & \Psi_{2N,2} & \cdots & \Psi_{2N,2N} \\
\epsilon^\xi & \epsilon^\xi & \cdots & \epsilon^\xi
\end{array} \right)_{1 \leq i,j \leq 2N},
\]

where $\Psi_{i,j} = c_{i,j} + \frac{p_i - p_j}{p_i + p_j} e^{\epsilon^\xi}$. Then, we note

\[
c_{i,j} = 2\delta_{i,j} + \alpha, (p_i - 1)(p_j - 1) = -2\delta_{i,j} + \alpha, (p_i - 1)(p_j - 1) = -c_{i,j}.
\]

Introducing $c_i = 2\alpha, (p_i - 1)(p_j - 1)$, we can write as $c_{i,j} = \delta_{i,j} + c_i$, and $c_j = -c_{2N+1,1}$. Note that the $2N \times 2N$ matrix $(\Psi_{i,j})_{1 \leq i,j \leq 2N}$ is skew-symmetric. Thus, we can use formula (D.7):

\[
\tau_1 = \frac{1}{c} \det \left( \begin{array}{cccc}
\psi_{i,j} & \psi_{i,j+1} & \cdots & \psi_{i,2N} \\
\vdots & \vdots & \ddots & \vdots \\
\psi_{2N,j} & \psi_{2N,j+1} & \cdots & \psi_{2N,j+2N} \\
\epsilon^\xi & \epsilon^\xi & \cdots & \epsilon^\xi
\end{array} \right)_{1 \leq i,j \leq 2N},
\]

where $\psi_{i,j} = \psi_{i,j} + \frac{p_i - p_j}{p_i + p_j} e^{\epsilon^\xi}$. Then, we note

\[
\psi_{i,j} = 2\delta_{i,j} + \alpha, (p_i - 1)(p_j - 1) = -2\delta_{i,j} + \alpha, (p_i - 1)(p_j - 1) = -c_{i,j}.
\]

Introducing $\psi_{i,j} = 2\alpha, (p_i - 1)(p_j - 1)$, we can write as $c_{i,j} = \delta_{i,j} + c_i$, and $c_j = -c_{2N+1,1}$. Note that the $2N \times 2N$ matrix $(\psi_{i,j})_{1 \leq i,j \leq 2N}$ is skew-symmetric. Thus, we can use formula (D.7):

\[
\tau_1 = \frac{1}{c} \det \left( \begin{array}{cccc}
\psi_{i,j} & \psi_{i,j+1} & \cdots & \psi_{i,2N} \\
\vdots & \vdots & \ddots & \vdots \\
\psi_{2N,j} & \psi_{2N,j+1} & \cdots & \psi_{2N,j+2N} \\
\epsilon^\xi & \epsilon^\xi & \cdots & \epsilon^\xi
\end{array} \right)_{1 \leq i,j \leq 2N},
\]

where $\psi_{i,j} = \psi_{i,j} + \frac{p_i - p_j}{p_i + p_j} e^{\epsilon^\xi}$. Then, we note

\[
\psi_{i,j} = 2\delta_{i,j} + \alpha, (p_i - 1)(p_j - 1) = -2\delta_{i,j} + \alpha, (p_i - 1)(p_j - 1) = -c_{i,j}.
\]

Introducing $\psi_{i,j} = 2\alpha, (p_i - 1)(p_j - 1)$, we can write as $c_{i,j} = \delta_{i,j} + c_i$, and $c_j = -c_{2N+1,1}$. Note that the $2N \times 2N$ matrix $(\psi_{i,j})_{1 \leq i,j \leq 2N}$ is skew-symmetric. Thus, we can use formula (D.7):

\[
\tau_1 = \frac{1}{c} \det \left( \begin{array}{cccc}
\psi_{i,j} & \psi_{i,j+1} & \cdots & \psi_{i,2N} \\
\vdots & \vdots & \ddots & \vdots \\
\psi_{2N,j} & \psi_{2N,j+1} & \cdots & \psi_{2N,j+2N} \\
\epsilon^\xi & \epsilon^\xi & \cdots & \epsilon^\xi
\end{array} \right)_{1 \leq i,j \leq 2N},
\]

where $\psi_{i,j} = \psi_{i,j} + \frac{p_i - p_j}{p_i + p_j} e^{\epsilon^\xi}$. Then, we note

\[
\psi_{i,j} = 2\delta_{i,j} + \alpha, (p_i - 1)(p_j - 1) = -2\delta_{i,j} + \alpha, (p_i - 1)(p_j - 1) = -c_{i,j}.
\]
\textbf{Proof.} Suppose that $\alpha_i = \alpha_{2N+1-i}$ and $p_i^2 - p_i p_{2N+1-i} + p_{2N+1-i}^2 = 1$ are satisfied. Then, we can rewrite $\tau_0$ as follows:

$$
\tau_0 = \det (\Phi_{1,i})_{1 \leq i, j \leq 2N} = \det \left( \delta_{j,2N+1-i} \alpha_i + \frac{1}{p_i + p_j} \epsilon^{\delta + \xi} \right)_{1 \leq i, j \leq 2N}
$$

$$
= \det \left( \delta_{j,2N+1-i} \alpha_i \frac{p_i p_{2N+1-i} - p_{2N+1-i}^2}{p_j (p_i - p_j)} + \frac{1}{p_i + p_j} \epsilon^{\delta + \xi} \right)_{1 \leq i, j \leq 2N}
$$

$$
= \frac{1}{c} \det \left( \delta_{j,2N+1-i} \alpha_i \frac{p_i^2 - 1}{p_i - p_j} + \frac{2p_j}{p_i + p_j} \epsilon^{\delta + \xi} \right)_{1 \leq i, j \leq 2N}
$$

$$
= \frac{1}{c} \det \left( \delta_{j,2N+1-i} \alpha_i \frac{p_i^2 - 1}{p_i - p_j} + \left( \frac{p_i - p_j}{p_j + p_i} \right) (p_i + 1) + \frac{2p_j}{p_i + p_j} \epsilon^{\delta + \xi} \right)_{1 \leq i, j \leq 2N}
$$

where $c = 2^{2N} \prod_{k=1}^{2N} p_k$. Using formula (B.1), we can rewrite $\tau_0$ as follows:

\begin{align*}
\tau_0 &= \frac{1}{c} \begin{vmatrix}
\Phi_{1,1} & \Phi_{1,2} & \cdots & \Phi_{1,2N} & -2 \epsilon_i (p_i + 1) \epsilon_i \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\Phi_{2N,1} & \Phi_{2N,2} & \cdots & \Phi_{2N,2N} & -2 \epsilon_{2N} (p_{2N} + 1) \epsilon_{2N} \\
\end{vmatrix} \\
&= \frac{1}{c} \begin{vmatrix}
\phi_{1,1} & \phi_{1,2} & \cdots & \phi_{1,2N} & (p_i - 1) \epsilon_i (p_i + 1) \epsilon_i \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\phi_{2N,1} & \phi_{2N,2} & \cdots & \phi_{2N,2N} & (p_{2N} - 1) \epsilon_{2N} (p_{2N} + 1) \epsilon_{2N} \\
\end{vmatrix} \\
&= \frac{1}{c} \begin{vmatrix}
\psi_{1,1} & \psi_{1,2} & \cdots & \psi_{1,2N} & p_i \psi_{1,1} + p_i^2 - p_i p_j \psi_{1,1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\psi_{2N,1} & \psi_{2N,2} & \cdots & \psi_{2N,2N} & p_{2N} \psi_{2N,1} + p_{2N}^2 - p_i p_j \psi_{2N,1} \\
\end{vmatrix}
\end{align*}

where $\phi_{i,j} = 2 \delta_{j,2N+1-i} \alpha_i \frac{p_i^2 - 1}{p_i - p_j} + \left( \frac{p_i - p_j}{p_j + p_i} \right) (p_i + 1) + \frac{2p_j}{p_i + p_j} \epsilon_i \epsilon_i$. Then, we can further simplify as follows:

$$
\tau_0 = \prod_{k=1}^{2N} \frac{p_{k+1}}{p_k} \frac{\psi_{1,1}}{p_{1,1} + 1} \epsilon_{1,i} \epsilon_i \\
$$

where $\psi_{i,j} = \psi_{i,1} \psi_{i,2} \cdots \psi_{i,2N}$. Then, we can further simplify as follows:
where $\Psi_{i,j} = 2\delta_{j,2N+1-i} \alpha_{e^{j-1}(p_j-1)} \frac{p_j-1}{p_j}$ $+ \frac{p_j-p_l}{p_j} e^{j+l}$, Note that the $2N \times 2N$ matrix $(\Psi_{i,j})_{1 \leq i,j \leq 2N}$ is skew-symmetric. Thus, we can use formula (A.1):

$$r_0 = \frac{\prod_{k=1}^{2N} \frac{p_k+1}{n_k-1}}{c}$$

$$\begin{vmatrix}
\Psi_{1,2} & \Psi_{1,3} & \cdots & \Psi_{1,2N} & e^{e_1} & e^{e_2} \\
\Psi_{2,3} & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\Psi_{2N-1,2N} & e^{e_{2N-1}} & e^{e_{2N-1}} & \cdots & e^{e_{2N}} & e^{e_{2N}} \\
\end{vmatrix}$$

$$\times$$

$$\begin{vmatrix}
p_1 - 1 & e^{e_1} & p_1 e^{e_1} \\
p_1 + 1 & e^{e_2} & p_1 e^{e_2} \\
p_2 - 1 & e^{e_2} & p_2 e^{e_2} \\
p_2 + 1 & \cdots & \cdots \\
p_{2N-1} - 1 & e^{e_{2N-1}} & p_{2N-1} e^{e_{2N-1}} \\
p_{2N-1} + 1 & e^{e_{2N}} & p_{2N} e^{e_{2N}} \\
\end{vmatrix}$$

Let

$$g_1 = \text{pf}(\Psi_{i,j})_{1 \leq i,j \leq 2N} =$$

$$\begin{vmatrix}
\Psi_{1,2} & \Psi_{1,3} & \cdots & \Psi_{1,2N} & e^{e_1} & e^{e_1} \\
\Psi_{2,3} & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\Psi_{2N-1,2N} & e^{e_{2N-1}} & e^{e_{2N-1}} & \cdots & e^{e_{2N}} & e^{e_{2N}} \\
\end{vmatrix}$$
Lemma 2.6. The τ-function \( \tau_2 \) of the bilinear equations (2.13) and (2.14) with the pseudo 3-reduction constraint satisfies the relation

\[
\tau_2 = \frac{1}{c} (g_1 g_2 - D_{\lambda_1} g_1 \cdot g_2),
\]

(2.19)
Proof. Suppose that $\alpha_i = \alpha_{2N+1-i}$ and $p_i^2 - p_i p_{2N+1-i} + p_{2N+1-i}^2 = 1$, then we can rewrite $\tau_2$ as follows:

\[
\tau_2 = \det \left( g_{1,2(j)}^{(2)} \right)_{1 \leq i,j \leq 2N} = \det \left( \delta_{j,2N+1-i} - \alpha_i - \frac{p_{2N+1-i}(p_i - p_{2N+1-i})}{p_j(p_i - p_j)} + \frac{1}{p_i + p_j} \frac{p_j^2}{p_j^2} e^{\xi_j} \right)_{1 \leq i,j \leq 2N}
\]

where $\xi_j = p_i^{-1}x_i + p_j x_i + \xi_i^0$, $p_i^2 - p_i p_{2N+1-i} + p_{2N+1-i}^2 = 1$, and $c = 2^{2N} \prod_{k=1}^{2N} p_k$. Using formula (B.1), we can rewrite $\tau_2$ as follows:

\[
\tau_2 = \frac{1}{c} \det \left( \delta_{j,2N+1-i} - \alpha_i - \frac{p_{2N+1-i}(p_i - p_{2N+1-i})}{p_j(p_i - p_j)} + \frac{1}{p_i + p_j} \frac{p_j^2}{p_j^2} e^{\xi_j} \right)_{1 \leq i,j \leq 2N}
\]

where $c = 2^{2N} \prod_{k=1}^{2N} p_k$. Using formula (B.1), we can rewrite $\tau_2$ as follows:

\[
\tau_2 = \frac{1}{c} \det \left| \begin{array}{cccc} \Phi_{1,1} & \Phi_{1,2} & \cdots & \Phi_{1,2N} \\ \vdots & \ddots & \vdots & \vdots \\ \Phi_{2N,1} & \Phi_{2N,2} & \cdots & \Phi_{2N,2N} \\ \frac{p_1}{p_1^2 - 1} e^{\xi_1} & \frac{p_2}{p_2^2 - 1} e^{\xi_2} & \cdots & \frac{p_{2N}}{p_{2N}^2 - 1} e^{\xi_{2N}} \\ \frac{p_1^{-1}}{p_1^2 - 1} e^{\xi_1^0} & \frac{p_2^{-1}}{p_2^2 - 1} e^{\xi_2^0} & \cdots & \frac{p_{2N}^{-1}}{p_{2N}^2 - 1} e^{\xi_{2N}^0} \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 1 \\ \end{array} \right| 
\]
\[
\begin{array}{cccc}
\hat{\Phi}_{1,1} & \hat{\Phi}_{1,2} & \cdots & \hat{\Phi}_{1,2N} \\
\vdots & \vdots & \ddots & \vdots \\
\hat{\Phi}_{2N,1} & \hat{\Phi}_{2N,2} & \cdots & \hat{\Phi}_{2N,2N}
\end{array}
\]

\[
= \frac{1}{c} \begin{pmatrix}
\Phi_{1,1} & \Phi_{1,2} & \cdots & \Phi_{1,2N} \\
\vdots & \vdots & \ddots & \vdots \\
\Phi_{2N,1} & \Phi_{2N,2} & \cdots & \Phi_{2N,2N}
\end{pmatrix}
\]

\[
\begin{pmatrix}
(p_1^{-1} - 1) e^{\delta_1} & (p_1^{-1} + 1) e^{\delta_1} \\
\vdots & \vdots \\
(p_{2N}^{-1} - 1) e^{\delta_{2N}} & (p_{2N}^{-1} + 1) e^{\delta_{2N}}
\end{pmatrix}
\]

where \( \hat{\Phi}_{i,j} = 2 \delta_{j,2N+1-i} \alpha \frac{p_i^{-1} - 1}{p_i^{-1} + 1} + \frac{(p_i^{-1} - 1)(p_i^{-1} + 1)}{p_i^{-1} + p_j^{-1}} e^{\delta_i} \). Then, we can further simplify as follows:

\[
\tau_2 = \frac{\prod_{k=1}^{2N} \frac{p_k^{-1} + 1}{p_k^{-1} - 1}}{c} \begin{pmatrix}
\tilde{\Psi}_{1,1} & \tilde{\Psi}_{1,2} & \cdots & \tilde{\Psi}_{1,2N} \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{\Psi}_{2N,1} & \tilde{\Psi}_{2N,2} & \cdots & \tilde{\Psi}_{2N,2N}
\end{pmatrix}
\]

\[
\begin{pmatrix}
p_1^{-1} - 1 & p_1^{-1} + 1 \ e^{\delta_1} \\
\vdots & \vdots \\
(p_{2N}^{-1} - 1) e^{\delta_{2N}} & p_{2N}^{-1} + 1 \ e^{\delta_{2N}}
\end{pmatrix}
\]

where \( \tilde{\Psi}_{i,j} = 2 \delta_{j,2N+1-i} \alpha \frac{(p_i^{-1} - 1)(p_i^{-1} + 1)}{p_i^{-1} + p_j^{-1}} e^{\delta_i} \). Note that the \( 2N \times 2N \) matrix \( (\tilde{\Psi}_{i,j})_{1 \leq i,j \leq 2N} \) is skew-symmetric and \( \tilde{\Psi}_{i,j} = -\tilde{\Psi}_{j,i} \). Thus, we can use formula (E.1):
\[ g_1 = \text{pf} \left( \begin{array}{cccc} | & \tilde{\Psi}_{1,2} & \tilde{\Psi}_{1,3} & \cdots & \tilde{\Psi}_{1,2N} \\ \tilde{\Psi}_{2,3} & \tilde{\Psi}_{2,4} & \cdots & \tilde{\Psi}_{2,2N} \\ & \ddots & \ddots & \ddots \\ \tilde{\Psi}_{2N-1,2N} & & & \tilde{\Psi}_{2N-1,2N} \end{array} \right) = \begin{array}{cccc} \psi_{1,2} & \psi_{1,3} & \cdots & \psi_{1,2N} \\ \psi_{2,3} & \psi_{2,4} & \cdots & \psi_{2,2N} \\ & \ddots & \ddots & \ddots \\ \psi_{2N-1,2N} & & & \psi_{2N-1,2N} \end{array} \right) \]

\[ g_2 = \text{pf} \left( \begin{array}{cccc} \left( p_i + 1 \right) \left( p_j + 1 \right) & \delta_{j,2N+1-i} & p_i - p_j & 1 \\ \delta_{j,2N+1-i} & \left( p_i + 1 \right) \left( p_j + 1 \right) & p_i - p_j & \delta_{i,j} \\ 1 & 1 & \delta_{i,j} & \left( p_i + 1 \right) \left( p_j + 1 \right) \end{array} \right) \]

Then

\[ \partial_{\epsilon_{ij}} g_1 = \begin{array}{cccc} \psi_{1,2} & \psi_{1,3} & \cdots & \psi_{1,2N} \\ \psi_{2,3} & \psi_{2,4} & \cdots & \psi_{2,2N} \\ & \ddots & \ddots & \ddots \\ \psi_{2N-1,2N} & & & \psi_{2N-1,2N} \end{array} \right) \]
Theorem 2.7. The \( g_1 \) and \( g_2 \) satisfy relation (2.19).

Thus, \( \tau_2, g_1 \) and \( g_2 \) satisfy relation (2.19).

Letting \( F = \tau_0, G = \tau_1 \) and \( H = \tau_2 \), we obtain the following equations:

\[
\begin{align*}
- \left( \frac{1}{2} D_{x_1} D_{x_{-1}} - 1 \right) F \cdot F &= G^2, \\
- \left( \frac{1}{2} D_{x_1} D_{x_{-1}} - 1 \right) G \cdot G &= FH,
\end{align*}
\]

\( cG = g_1 g_2, \) \( cF = g_1 g_2 - D_{x_1} g_1 \cdot g_2, \) \( cH = g_1 g_2 - D_{x_{-1}} g_1 \cdot g_2. \)

from the bilinear equations (2.13) and (2.14), and the relations between determinants and Pfaffians (2.15), (2.18) and (2.19).

Theorem 2.7. The \( \tau \)-functions \( g_1 \) (2.16) and \( g_2 \) (2.17) of equations (2.20), (2.21), (2.22), (2.23) and (2.24) give the \( N \)-soliton solution of the DP equation

\[
u_t + 3 \nu_x - \nu_{xxx} + 4 \nu u_x = 3 \nu_x u_{xx} + \nu_{xxxx},
\]

through the dependent variable transformation

\[
u = - \left( \ln \frac{g_1}{g_2} \right)_{x_{-1}},
\]

and the hodograph (reciprocal) transformation

\[
\begin{align*}
x &= x_1 + \int_{-\infty}^{x_{-1}} u(x_1, x'_{-1}) \, dx'_{-1} \\
&= x_1 - \ln \frac{g_1}{g_2}
\end{align*}
\]

(2.26)
Proof. From (2.22), (2.23) and (2.24), we have the relations
\[- \left( \ln \frac{g_1}{g_2} \right)_{x_1} = \frac{F}{G} - 1, \quad (2.27)\]
\[- \left( \ln \frac{g_1}{g_2} \right)_{x_{-1}} = \frac{H}{G} - 1. \quad (2.28)\]

Let
\[\rho = \frac{G}{F}, \quad u = - \left( \ln \frac{g_1}{g_2} \right)_{x_{-1}}. \quad (2.29)\]

Differentiating (2.27) with respect to \(x_{-1}\), we obtain
\[u_{x_1} = \left( \frac{1}{\rho} \right)_{x_{-1}}. \quad (2.30)\]

This is rewritten as
\[(\ln \rho)_{x_{-1}} = - \rho u_{x_1}. \quad (2.31)\]

Equation (2.28) leads to
\[\frac{H}{G} = 1 + u. \quad (2.32)\]

The bilinear equations (2.20) and (2.21) are written as
\[- \left( \ln F \right)_{x_1, x_{-1}} + 1 = \rho^2, \quad (2.33)\]
\[- \left( \ln G \right)_{x_1, x_{-1}} + 1 = \frac{1}{\rho} (1 + u). \quad (2.34)\]

Subtracting (2.33) from (2.34), we obtain
\[- \left( \ln \rho \right)_{x_1, x_{-1}} = \frac{1}{\rho} (1 + u) - \rho^2, \quad (2.35)\]

which leads to
\[\rho^3 = 1 + u + \rho (\ln \rho)_{x_1, x_{-1}}. \quad (2.36)\]

Using (2.31), it becomes
\[\rho^3 = 1 + u - \rho (\rho u_{x_1})_{x_1}. \quad (2.37)\]

Let us consider the hodograph (reciprocal) transformation
\[\begin{align*}
x &= x_1 + \int_{x_1}^{t_{-1}} u(x_1, x_{-1}') \, dx_{-1}' \\
x_1 &= - \ln \frac{g_1}{g_2}, \\
t &= x_{-1}.
\end{align*} \quad (2.38)\]

This yields
\[\begin{align*}
\frac{\partial x}{\partial x_1} &= 1 - \left( \ln \frac{g_1}{g_2} \right)_{x_1} = \rho^{-1}, \\
\frac{\partial x}{\partial x_{-1}} &= - \left( \ln \frac{g_1}{g_2} \right)_{x_{-1}} = u.
\end{align*} \quad (2.39)\]
and
\begin{align}
\begin{cases}
\partial u = \frac{1}{\rho} \partial \xi, \\
\partial_{x_1} = \partial_t + u \partial_x.
\end{cases}
\end{align}
(2.40)
Applying the hodograph (reciprocal) transformation to (2.31) and (2.37), we obtain
\begin{align}
\begin{cases}
(\partial_t + u \partial_x) \ln \rho = -u_x, \\
\rho^2 = 1 + u - u_{xx}.
\end{cases}
\end{align}
(2.41)
This is equivalent to
\begin{align}
(\partial_t + u \partial_x) \ln (1 + u - u_{xx}) = -3u_x,
\end{align}
(2.42)
which can be written as
\begin{align}
(\partial_t + u \partial_x)(1 + u - u_{xx}) = -3u_x(1 + u - u_{xx}).
\end{align}
(2.43)
This is nothing but the DP equation (2.25).
\[\square\]
Remark 2.8. Applying the scale transformation \( u \rightarrow \frac{1}{\kappa} u \), \( t \rightarrow \kappa^2 t \) to (2.25), we obtain the DP equation (1.1).

Remark 2.9. Setting \( u = 0 \) in equation (2.35), we obtain the Tzitzéica equation [29–31, 25]
\begin{align}
\rho_{x_1} = \rho^2 - \frac{1}{\rho}.
\end{align}
(2.44)
Thus, equations (2.35) and (2.30) can be considered as an extension of the Tzitzéica equation.

Let \( k_i = p_i + p_{2N+1-i} \). From \( p_i^2 - p_ip_{2N+1-i} + p_{2N+1-i}^2 = 1 \), we obtain \( p_i = \frac{1}{2} (3k_i + \sqrt{3(4 - k_i^2)}) \), \( p_{2N+1-i} = \frac{1}{2} (3k_i - \sqrt{3(4 - k_i^2)}) \), \( p_ip_{2N+1-i} = \frac{k_i^2-1}{3} \) and \( \rho + \frac{1}{p_ip_{2N+1-i}} = \frac{3k_i}{k_i^2-1} \).
Thus,
\[\xi_1 + \xi_{2N+1-i} = k_ix_1 + \frac{3k_i}{k_i^2-1} x_{-1} + \xi_{i0} + \xi_{2N+1-i0}.\]
In the Pfaffian solution, all phase functions can be expressed by the summation of \( \xi_i + \xi_{2N+1-i} \). So the phase functions can be expressed by the parameters \( \{k_i\} (i = 1, 2, \ldots, N) \). Each coefficient of exponential functions can be normalized to 1 after absorption into phase constants or can be rewritten by the parameters \( \{k_i\} \). Thus, it is possible to rewrite the above \( \tau \)-function by using the parameters \( \{k_i\} \) instead of \( \{p_i\} \).

Example. Soliton solutions.
For \( N = 1 \),
\begin{align}
g_1 &= 2\alpha_1 \frac{(p_1 - 1)(p_2 - 1)}{p_1 - p_2} + \frac{p_1 - p_2}{p_1 + p_2} e^{\xi_1 + \xi_2} \\
&= \left(2\alpha_1 + \frac{(p_1 - p_2)^2}{(p_1 + p_2)(p_1 - 1)(p_2 - 1)}\right) e^{\xi_1 + \xi_2}.
\end{align}
\begin{align}
g_2 &= 2\alpha_1 \frac{(p_1 + 1)(p_2 + 1)}{p_1 - p_2} + \frac{p_1 - p_2}{p_1 + p_2} e^{\xi_1 + \xi_2} \\
&= \left(2\alpha_1 + \frac{(p_1 - p_2)^2}{(p_1 + p_2)(p_1 + 1)(p_2 + 1)}\right) e^{\xi_1 + \xi_2}.
\end{align}
Letting \( \alpha_1 = \frac{1}{4} \) and \( e^{\phi_1} = \frac{(p_1 - p_2)^2}{(p_1 + p_2)^2} \), the \( \tau \)-functions can be rewritten as
\begin{align}
g_1 &= 1 + e^{\xi_1 + \xi_2 + \phi_1 + \gamma_1}, \quad g_2 = 1 + e^{\xi_1 + \xi_2 - \phi_1 + \gamma_1}.
\end{align}
where
\[ \xi_1 + \xi_2 = k_1x_1 + \frac{3k_1}{k_1^2 - 1} x_{-1} + \xi_{10} + \xi_{20} \]
and
\[ e^{\phi} = \sqrt{\frac{(p_1 + 1)(p_2 + 1)}{(p_1 - 1)(p_2 - 1)}} = \sqrt{\frac{k_1^2 + 3k_1 + 2}{k_1^2 - 3k_1 + 2}} = \sqrt{\frac{(k_1 + 2)(k_1 + 1)}{(k_1 - 2)(k_1 - 1)}}. \]

Here, \( \gamma_1 \) can be absorbed into a phase constant.

For \( N = 2 \),
\[ g_1 = \frac{p_1 - p_2}{p_1 + p_2} e^{\xi_1 + \xi_2} \frac{p_3 - p_4}{p_3 + p_4} e^{\xi_3 + \xi_4} \frac{p_5 - p_6}{p_5 + p_6} e^{\xi_5 + \xi_6} \]
\[ = \frac{(p_1 - 1)(p_4 - 1)}{p_1 - p_4} \cdot \frac{(p_2 - 1)(p_3 - 1)}{p_2 - p_3} \left( 2\alpha_1 \cdot 2\alpha_2 + 2\alpha_3 \cdot (p_1 - p_4) \frac{p_2 - p_3}{p_1 + p_4} (p_1 - 1)(p_4 - 1) \right) \cdot \]
\[ + 2\alpha_1 \frac{(p_2 - p_3)^2}{(p_1 + p_4)(p_1 - 1)} e^{\xi_3 + \xi_4} \frac{p_1 - p_4}{p_1 - p_3} + \frac{p_1 - p_4}{p_1 - p_3} \cdot \frac{p_2 - p_3}{p_2 - p_4} \]
\[ \times \left( \frac{p_1 - p_2 p_3 - p_4}{p_1 + p_2 p_3 + p_4} - \frac{p_1 - p_3 p_2 - p_4}{p_1 + p_3 p_2 + p_4} + \frac{p_1 - p_4 p_2 - p_3}{p_1 + p_4 p_2 + p_3} \right) e^{\xi_5 + \xi_6 + \xi_5 + \xi_6}. \]

\[ g_2 = \frac{p_1 - p_2}{p_1 + p_2} e^{\xi_1 + \xi_2} \frac{p_3 - p_4}{p_3 + p_4} e^{\xi_3 + \xi_4} \frac{p_5 - p_6}{p_5 + p_6} e^{\xi_5 + \xi_6} \]
\[ = \frac{(p_1 + 1)(p_4 + 1)}{p_1 - p_4} \cdot \frac{(p_2 + 1)(p_3 + 1)}{p_2 - p_3} \left( 2\alpha_1 \cdot 2\alpha_2 + 2\alpha_3 \cdot (p_1 - p_4) \frac{p_2 - p_3}{p_1 + p_4} (p_1 + 1)(p_4 + 1) \right) \cdot \]
\[ + 2\alpha_1 \frac{(p_2 - p_3)^2}{(p_1 + p_4)(p_1 + 1)} e^{\xi_3 + \xi_4} \frac{p_1 - p_4}{p_1 - p_3} + \frac{p_1 - p_4}{p_1 - p_3} \cdot \frac{p_2 - p_3}{p_2 - p_4} \]
\[ \times \left( \frac{p_1 - p_2 p_3 - p_4}{p_1 + p_2 p_3 + p_4} - \frac{p_1 - p_3 p_2 - p_4}{p_1 + p_3 p_2 + p_4} + \frac{p_1 - p_4 p_2 - p_3}{p_1 + p_4 p_2 + p_3} \right) e^{\xi_5 + \xi_6 + \xi_5 + \xi_6}. \]

Letting \( \alpha_1 = \alpha_2 = \frac{1}{2} \cdot \frac{e^{\phi}}{(p_1 - p_2)^2 (p_3 - p_4)^2 (p_5 - p_6)^2 (p_7 - p_8)^2 (p_9 - p_{10})^2 (p_{11} - p_{12})^2 (p_{13} - p_{14})^2 (p_{15} - p_{16})^2 \sqrt{(p_1 - 1)(p_1 + 1)(p_2 + 1)(p_3 + 1)(p_4 + 1)(p_5 + 1)(p_6 + 1)(p_7 + 1)}} \cdot \frac{1}{(p_1 + p_4)(p_1 + p_2)(p_1 + p_3)(p_1 + p_5)(p_1 + p_6)(p_1 + p_7)(p_1 + p_8)(p_1 + p_9)(p_1 + p_{10})(p_1 + p_{11})(p_1 + p_{12})(p_1 + p_{13})(p_1 + p_{14})}, \]
\[ e^{\tau} = \frac{(p_1 - p_2)^2 (p_3 - p_4)^2 (p_5 - p_6)^2 (p_7 - p_8)^2 (p_9 - p_{10})^2 (p_{11} - p_{12})^2 (p_{13} - p_{14})^2 (p_{15} - p_{16})^2 \sqrt{(p_1 - 1)(p_1 + 1)(p_2 + 1)(p_3 + 1)(p_4 + 1)(p_5 + 1)(p_6 + 1)(p_7 + 1)}}{(p_1 + p_4)(p_1 + p_2)(p_1 + p_3)(p_1 + p_5)(p_1 + p_6)(p_1 + p_7)(p_1 + p_8)(p_1 + p_9)(p_1 + p_{10})(p_1 + p_{11})(p_1 + p_{12})(p_1 + p_{13})(p_1 + p_{14})}, \]
the above \( \tau \)-functions become
\[ g_1 = 1 + e^{\nu_1 + \nu_2 + \nu_3 + \nu_4 + b_{12} e^{\xi_1 + \xi_2 + \xi_3 + \xi_6 + \phi_1 + \phi_2}} \]
\[ g_2 = 1 + e^{\nu_1 + \nu_2 + \nu_3 + \nu_4 + b_{12} e^{\xi_1 + \xi_2 + \xi_3 + \xi_6 + \phi_1 + \phi_2}}, \]
where
\[ b_{12} = \frac{p_1 - p_2}{p_1 + p_2} p_3 p_4 p_5 p_6 + \frac{p_3}{p_3 + p_4} p_1 + p_2 + p_3^2 - \frac{p_5}{p_5 + p_6} p_1 + p_4 + p_6 + p_3 + 1 \]
\[ = \frac{(k_1 - k_2)^2 (k_1^2 - k_1 k_2 + k_2^2 - 3)}{(k_1 + k_2)^2 (k_1^2 + k_1 k_2 + k_2^2 - 3)} \]
\[ \xi_1 + \xi_4 = k_1 x_1 + \frac{3k_1}{k_1^2 - 1} x_{-1} + \xi_{10} + \xi_{20}, \quad \xi_2 + \xi_3 = k_2 x_1 + \frac{3k_2}{k_2^2 - 1} x_{-1} + \xi_{20} + \xi_{30}. \]
Here, \( \gamma_1 \) and \( \gamma_2 \) were absorbed into phase constants.

The \( N \)-soliton solution of (2.25) is written in the following form:

\[
ge_1 = \sum_{\mu=0,1} \exp \left[ \sum_{i=1}^{N} \mu_i (\eta_i + \phi_i) + \sum_{i<j}^{(N)} \mu_i \mu_j \ln b_{ij} \right],
\]

\[
ge_2 = \sum_{\mu=0,1} \exp \left[ \sum_{i=1}^{N} \mu_i (\eta_i - \phi_i) + \sum_{i<j}^{(N)} \mu_i \mu_j \ln b_{ij} \right],
\]

\[
b_{ij} = \frac{(k_i - k_j)^2 (k_i^2 - k_k j + k_j^2 - 3)}{(k_i + k_j)^2 (k_i^2 + k_k j + k_j^2 - 3)} \text{ for } i < j,
\]

\[
\eta_i = \xi_i + \xi_{2N+1-i} = k_i x_1 + \frac{3k_i}{k_i^2 - 4} - 1 + \eta_0,
\]

\[
e_\phi = \sqrt{\frac{k_i^2 + 3k_i + 2}{k_i^2 - 3k_i + 2}} = \sqrt{\frac{(k_i + 2)(k_i + 1)}{(k_i - 2)(k_i - 1)}}.
\]

where \( \sum_{\mu=0,1} \) means the summation over all possible combinations of \( \mu_i = 0 \) or 1 for \( i = 1, 2, \ldots, N \), and \( \sum_{i<j}^{(N)} \) means the summation over all possible combinations of \( N \) elements under the condition \( i < j \).

Applying \( u \rightarrow \frac{1}{\kappa} u, t \rightarrow \kappa^2 t, x_{-1} \rightarrow \kappa^3 x_{-1}, x_1 \rightarrow \frac{3}{8}, \kappa k_i = p_i + p_{2N+1-i} \), we obtain the \( N \)-soliton solution of the DP equation (1.1).

**Theorem 2.10.** The \( N \)-soliton solution of the DP equation (1.1) is given as follows:

\[
u = -\left( \ln \frac{g_1}{g_2} \right)_{x_{-1}},
\]

\[
ge_1 = \sum_{\mu=0,1} \exp \left[ \sum_{i=1}^{N} \mu_i (\eta_i + \phi_i) + \sum_{i<j}^{(N)} \mu_i \mu_j \ln b_{ij} \right],
\]

\[
ge_2 = \sum_{\mu=0,1} \exp \left[ \sum_{i=1}^{N} \mu_i (\eta_i - \phi_i) + \sum_{i<j}^{(N)} \mu_i \mu_j \ln b_{ij} \right],
\]

\[
b_{ij} = \frac{(k_i - k_j)^2 ((k_i^2 - k_k j + k_j^2) \kappa^2 - 3)}{(k_i + k_j)^2 ((k_i^2 + k_k j + k_j^2) \kappa^2 - 3)} \text{ for } i < j,
\]

\[
\eta_i = \xi_i + \xi_{2N+1-i} = k_i x_1 + \frac{3k_i}{\kappa^2 k_i^2 - 4} - 1 + \eta_0,
\]

\[
e_\phi = \sqrt{\frac{\kappa^2 k_i^2 + 3\kappa k_i + 2}{\kappa^2 k_i^2 - 3\kappa k_i + 2}} = \sqrt{\frac{(\kappa k_i + 2)(\kappa k_i + 1)}{(\kappa k_i - 2)(\kappa k_i - 1)}}.
\]
and the hodograph (reciprocal) transformation
\[
\begin{align*}
  x &= \frac{x_1}{\kappa} + \int_{-\infty}^{x_1} u(x_1, x'_{-1}) \, dx'_{-1} \\
  t &= \frac{x_1}{\kappa} - \ln \frac{g_1}{g_2},
\end{align*}
\]

This is consistent with the result in [6, 7].

**Remark 2.11.** There are three regions in which the above soliton solution becomes regular: (i) \( \frac{\kappa}{2} < k_i \), (ii) \(-\frac{1}{3} < k_i < \frac{1}{3}\) and (iii) \( k_i < -\frac{2}{3} \). (Note that this is obtained by the reality condition of \( e^{\phi_i} \.) In region (ii), the graph of the soliton solution shows smooth solitons. In regions (i) and (iii), the graph of the soliton solution shows loop solitons.

In figures 1–4, we show examples of two-soliton interactions.
3. Conclusions

The DP equation is investigated from the point of view of determinant–Pfaffian identities. We have established the reciprocal link between the DP equation and the pseudo 3-reduction of the $C_\infty$ two-dimensional Toda system and investigated the determinant–Pfaffian identities (i.e. the relations of $\tau$-functions). We have shown that the $\tau$-functions of the DP equation satisfy the identities of determinants and Pfaffians. The result in this paper is consistent with the one obtained by Matsuno [6, 7]. Although we have obtained the same $\tau$-functions as Matsuno, we have proved several relations of the DP’s $\tau$-functions from the point of view
of determinant–Pfaffian identities and established a formulation which can be applied to the problem of integrable discretization of the DP equation.

The result in this paper is useful for constructing an integrable discrete analogue of the DP equation. For the CH equation, we constructed an integrable discrete analogue of the CH equation by discretizing the determinant solutions and bilinear equations of the CH equation. For the DP equation, we can construct an integrable discrete analogue of the DP equation by discretizing Pfaffian solutions and identities of Pfaffians which have been given in this paper. We will report the detail in our forthcoming paper.

Appendix A

Let \( A = (a_{i,j})_{1 \leq i,j \leq 2N} \) be a \( 2N \times 2N \) skew-symmetric matrix, i.e. \( a_{i,j} = -a_{j,i} \). The Pfaffian of \( A \), i.e. \( pf(A) \), is defined as follows:

\[
\text{pf}(A) = \text{pf}(a_{i,j})_{1 \leq i,j \leq 2N} = \sum_{\sigma} \text{sgn}(\sigma) \prod_{k=1}^{N} a_{i_{2k-1},i_{2k}}
\]

where the summation is taken over all permutations

\[
\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix},
\]

satisfying \( i_1 < i_2, i_3 < i_4, \ldots, i_{2N-1} < i_{2N} \) and \( i_1 < i_3 < \cdots < i_{2N-1}, \) and \( \text{sgn}(\sigma) \) denotes the parity of the permutation \( \sigma \).

The Pfaffian can be computed recursively by

\[
\text{pf}(A) = \sum_{i=2}^{2N} (-1)^i a_{i} \text{pf}(A_{ij}),
\]

where \( A_{ij} \) denotes the matrix \( A \) with both the first and \( i \)th rows and columns removed.

The determinant of a skew-symmetric matrix \( A \) is the square of the Pfaffian of \( A \):

\[
\text{det}(A) = |pf(A)|^2.
\]

Appendix B

For a bordered determinant, we have the following identity:

\[
\begin{vmatrix}
a_{1,1} & a_{1,2} & \cdots & a_{1,2N-1} & a_{1,2N} & a_1 \\
a_{2,1} & a_{2,2} & \cdots & a_{2,2N-1} & a_{2,2N} & a_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
a_{2N,1} & a_{2N,2} & \cdots & a_{2N,2N-1} & a_{2N,2N} & a_{2N} \\
b_1 & b_2 & \cdots & b_{2N-1} & b_{2N} & \delta
\end{vmatrix} = \delta^{1-2N} \text{det}[\delta a_{i,j} - a_i b_j]_{1 \leq i,j \leq 2N},
\]

where \( \delta \neq 0 \). This is obtained by adding the \((2N+1)\)st row multiplied by \(-a_i/\delta\) to the \(i\)th row.
Appendix C

Let $A = (a_{ij})_{1 \leq i,j \leq 2N+2}$ be a $(2N + 2) \times (2N + 2)$ skew-symmetric matrix. Assume $a_{2N+1,2N+2} \neq 0$. Adding the $(2N+2)$st row multiplied by $a_{i,2N+1}/a_{2N+1,2N+2}$ and the $(2N+1)$st row multiplied by $-a_{i,2N+2}/a_{2N+1,2N+2}$ to the $i$th row, we obtain

$$\det(A) = \det(a_{ij})_{1 \leq i,j \leq 2N+2}$$

$$= (a_{2N+1,2N+2})^{2-2N} \det(a_{2N+1,2N+2}a_{i,j} - a_{2N+1,2N+2}a_{2N+2,j} + a_{2N+2,2N+1,j}),$$

and

$$\operatorname{pf}(A) = \operatorname{pf}(a_{ij})_{1 \leq i,j \leq 2N+2}$$

$$= (a_{2N+1,2N+2})^{1-N} \operatorname{pf}(a_{2N+1,2N+2}a_{i,j} - a_{2N+1,2N+2}a_{2N+2,j} + a_{2N+2,2N+1,j}).$$

(See, e.g., [32].) Thus, we have the formulae

\[
\begin{pmatrix}
0 & \alpha_{1,2} & \cdots & \alpha_{1,2N-1} & \alpha_{1,2N} & a_1 & b_1 \\
-a_{1,2N-1} & 0 & \cdots & \alpha_{2,2N-1} & \alpha_{2,2N} & a_2 & b_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
-a_{1,2N} & -a_{2,2N} & \cdots & -a_{2N-1,2N} & a_{2N} & b_{2N} \\
-a_{1} & -a_{2} & \cdots & -a_{2N-1} & -a_{2N} & 0 & \delta \\
-b_{1} & -b_{2} & \cdots & -b_{2N-1} & -b_{2N} & -\delta & 0 \\
\end{pmatrix}
\]

$$= \delta^{2-2N}$$

and

\[
\begin{pmatrix}
\alpha_{1,2} & \alpha_{1,3} & \cdots & \alpha_{1,2N} & a_1 & b_1 \\
\alpha_{2,3} & \cdots & \alpha_{2,2N} & a_2 & b_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\alpha_{2N-1,2N} & a_{2N-1} & \cdots & a_{2N} & b_{2N-1} & b_{2N} \\
\end{pmatrix}
\]

$$= \delta^{1-N}$$

\[
\begin{pmatrix}
\beta_{1,2} & \beta_{1,3} & \cdots & \beta_{1,2N} \\
\beta_{2,2} & \cdots & \beta_{2,2N} \\
\vdots & \vdots & \ddots & \vdots \\
\beta_{2N-1,2N} \\
\end{pmatrix}
\]

(\ref{appC})

where $\beta_{i,j} = \delta a_{i,j} - a_i b_j + a_j b_i$ and $\delta \neq 0$.

Appendix D

For any determinant $A$ of order $n$, we have the Jacobi identity

$$A_{ij}A_{pq} - A_{iq}A_{pj} = AA_{ip,jq},$$

(\ref{appD})
where $A_{ij}$ is a first minor which can be obtained by deleting the $i$th row and the $j$th column from $A$, and $A_{ij,pq}$ is a second minor which can be obtained by deleting the $i$th and $j$th rows and the $p$th and $q$th columns. Setting $r = i = j$ and $s = p = q$, we have

$$A_{rr}A_{ss} - A_{rs}A_{sr} = AA_{rs,rs}. \tag{D.2}$$

Let $A$ be a skew-symmetric determinant. If $n$ is even, then $A_{rr} = A_{ss} = 0$ and $A_{rs} = -A_{sr}$. Therefore, we obtain

$$A_{rs}^2 = AA_{rs,rs}, \tag{D.3}$$

which leads to

$$A_{rs} = pf(A)pf(A_{rs,rs}). \tag{D.4}$$

If $n$ is odd, then $A = 0$ and $A_{rs} = A_{sr}$. Therefore, we obtain

$$A_{rs}^2 = A_{rr}A_{ss}, \tag{D.5}$$

which leads to

$$A_{rs} = pf(A_{rr})pf(A_{ss}). \tag{D.6}$$

(See \[33, 34\].)

Let

$$A = \begin{vmatrix} 0 & \alpha_{1,2} & \cdots & \alpha_{1,2N-1} & \alpha_{1,2N} & a_1 & b_1 \\ -\alpha_{1,2} & 0 & \cdots & \alpha_{2,2N-1} & \alpha_{2,2N} & a_2 & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ -\alpha_{1,2N-1} & -\alpha_{2,2N-1} & \cdots & 0 & \alpha_{2N-1,2N} & a_{2N-1} & b_{2N-1} \\ -\alpha_{1,2N} & -\alpha_{2,2N} & \cdots & -\alpha_{2N-1,2N} & 0 & a_{2N} & b_{2N} \\ -a_1 & -a_2 & \cdots & -a_{2N-1} & -a_{2N} & 0 & \delta \\ -b_1 & -b_2 & \cdots & -b_{2N-1} & -b_{2N} & -\delta & 0 \end{vmatrix}. $$

Using (D.4) and (C.1), we obtain the formula

$$= pf(\alpha_{i,j})_{1 \leq i,j \leq 2N} \begin{vmatrix} \alpha_{1,2} & \cdots & \alpha_{1,2N} & a_1 & b_1 \\ \alpha_{1,3} & \cdots & \alpha_{1,2N} & a_2 & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{2,1} & \cdots & \alpha_{2,2N} & a_{2N-1} & b_{2N-1} \\ \alpha_{2,2N} & \cdots & \alpha_{2,2N} & a_{2N} & b_{2N} \\ a_1 & \cdots & a_2 & -a_{2N-1} & -a_{2N} & \delta \\ \alpha_{1,3} & \cdots & \alpha_{1,2N} & a_1 & b_1 \\ \alpha_{2,3} & \cdots & \alpha_{2,2N} & a_2 & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{2N-1,2N} & \cdots & \alpha_{2N-1,2N} & a_{2N-1} & b_{2N-1} \\ \alpha_{2N-1,2N} & \cdots & \alpha_{2N-1,2N} & a_{2N} & b_{2N} \\ a_{2N-1} & \cdots & a_{2N} & -a_{2N-1} & -a_{2N} & \delta \end{vmatrix}. \tag{D.7}$$
Using the Jacobi identity (D.1), we obtain

\[ A = \begin{vmatrix}
0 & \alpha_{1,2} & \cdots & \alpha_{1,2N-1} & \alpha_{1,2N} & a_1 & b_1 \\
-\alpha_{1,2} & 0 & \cdots & \alpha_{2,2N-1} & \alpha_{2,2N} & a_2 & b_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
-\alpha_{1,2N-1} & -\alpha_{2,2N-1} & \cdots & 0 & \alpha_{2N-1,2N} & a_{2N-1} & b_{2N-1} \\
-\alpha_{1,2N} & -\alpha_{2,2N} & \cdots & -\alpha_{2N-1,2N} & 0 & a_{2N} & b_{2N} \\
c_1 & c_2 & \cdots & c_{2N-1} & c_{2N} & \alpha & \beta \\
d_1 & d_2 & \cdots & d_{2N-1} & d_{2N} & \gamma & \delta
\end{vmatrix}.\]
From (D.7), we obtain

\[
\begin{vmatrix}
0 & a_{1,2} & \cdots & a_{1,2N-1} & a_{1,2N} & a_1 \\
-a_{1,2} & 0 & \cdots & a_{2,2N-1} & a_{2,2N} & a_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
-a_{1,2N-1} & -a_{2,2N-1} & \cdots & 0 & a_{2N-1,2N} & a_{2N} \\
-a_{1,2N} & -a_{2,2N} & \cdots & -a_{2N-1,2N} & 0 & a_{2N} \\
d_1 & d_2 & \cdots & d_{2N-1} & d_{2N} & \gamma \\
\end{vmatrix}
\]

\[
= \begin{vmatrix}
0 & a_{1,2} & \cdots & a_{1,2N-1} & a_{1,2N} & a_1 & b_1 \\
-a_{1,2} & 0 & \cdots & a_{2,2N-1} & a_{2,2N} & a_2 & b_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
-a_{1,2N-1} & -a_{2,2N-1} & \cdots & 0 & a_{2N-1,2N} & a_{2N} & b_{2N-1} \\
-a_{1,2N} & -a_{2,2N} & \cdots & -a_{2N-1,2N} & 0 & a_{2N} & b_{2N} \\
c_1 & c_2 & \cdots & c_{2N-1} & c_{2N} & \alpha & \beta \\
d_1 & d_2 & \cdots & d_{2N-1} & d_{2N} & \gamma & \delta \\
\end{vmatrix}
\]

\[
\begin{vmatrix}
\alpha_{1,2} & \alpha_{1,3} & \cdots & a_{1,2N} & a_1 & c_1 \\
\alpha_{2,3} & \cdots & a_{2,2N} & a_2 & c_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\alpha_{2N-1,2N} & \alpha_{2N-1} & \cdots & a_{2N} & c_{2N} & \alpha \\
\end{vmatrix}
\]

\[
\begin{vmatrix}
\alpha_{1,2} & \alpha_{1,3} & \cdots & a_{1,2N} & b_1 & d_1 \\
\alpha_{2,3} & \cdots & a_{2,2N} & b_2 & d_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\alpha_{2N-1,2N} & \alpha_{2N-1} & \cdots & b_{2N} & d_{2N} & \delta \\
\end{vmatrix}
\]

\[
\begin{vmatrix}
\alpha_{1,2} & \alpha_{1,3} & \cdots & a_{1,2N} & b_1 & c_1 \\
\alpha_{2,3} & \cdots & a_{2,2N} & b_2 & c_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\alpha_{2N-1,2N} & \alpha_{2N-1} & \cdots & b_{2N} & c_{2N} & \beta \\
\end{vmatrix}
\]

\[
\begin{vmatrix}
\alpha_{1,2} & \alpha_{1,3} & \cdots & a_{1,2N} & a_1 & d_1 \\
\alpha_{2,3} & \cdots & a_{2,2N} & a_2 & d_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\alpha_{2N-1,2N} & \alpha_{2N-1} & \cdots & a_{2N} & d_{2N} & \gamma \\
\end{vmatrix}
\]

(E.1)

References

[1] Degasperis A and Procesi M 1999 Asymptotic integrability Symmetry and Perturbation Theory ed A Degasperis and G Gaeta (Singapore: World Scientific) pp 23–37
[2] Degasperis A, Holm D D and Hone A N W 2002 A new integrable equation with peakon solutions Theor. Math. Phys. 133 1463–74
[3] Hone A N W and Wang J P 2003 Prolongation algebras and Hamiltonian operators for peakon equations Inverse Problems 19 129–45
[4] Lundmark H and Szmigielski J 2003 Multi-peakon solutions of the Degasperis–Procesi equation Inverse Problems 19 1241–5
[5] Lundmark H 2007 Formation and dynamics of shock waves in the Degasperis–Procesi equation J. Nonlinear Sci. 17 169–98
[6] Matsuno Y 2005 Multisoliton solution of the Degasperis–Procesi equation and their peakon limit Inverse Problems 21 1553–70
[7] Matsuno Y 2005 The $N$-soliton solution of the Degasperis–Procesi equation Inverse Problems 21 2085–101
[8] Constantin A and Lannes D 2009 The hydrodynamical relevance of the Camassa–Holm and Degasperis–Procesi equations Arch. Ration. Mech. Anal. 192 165–86
[9] Ohta Y, Maruno K and Feng B F 2008 An integrable semi-discretization of the Camassa–Holm equation and its determinant solution J. Phys. A: Math. Theor. 41 355205
[10] Feng B F, Maruno K and Ohta Y 2010 A self-adaptive moving mesh method for the Camassa–Holm equation J. Comput. Appl. Math. 235 229–43
[11] Feng B F, Maruno K and Ohta Y 2010 Integrable discretizations for the short-wave model of the Camassa–Holm equation J. Phys. A: Math. Theor. 43 265202
[12] Feng B F, Maruno K and Ohta Y 2010 Integrable discretizations of the short pulse equation J. Phys. A: Math. Theor. 43 085203
[13] Feng B F, Inoguchi J, Kajiwara K, Maruno K and Ohta Y 2011 Discrete integrable systems and hodograph transformations arising from motions of discrete plane curves J. Phys. A: Math. Theor. 44 395201
[14] Feng B F, Maruno K and Ohta Y 2012 On the $\tau$-functions of the reduced Ostrovsky equation and the $A_2^{(2)}$ two-dimensional Toda system J. Phys. A: Math. Theor. 45 355203
[15] Matsuno Y 2006 Cusp and loop soliton solutions of short-wave models for the Camassa–Holm and Degasperis–Procesi equations Phys. Lett. A 359 451–7
[16] Leznov A N and Saveliev M V 1979 Representation of zero curvature for the system of nonlinear partial differential equations $x_a, z, \bar{z} = \exp(kx)A$ and its integrability Lett. Math. Phys. 3 489–94
[17] Mikhailov A V 1979 On the integrability of a two-dimensional generalization the Toda lattice JETP Lett. 30 414–8
[18] Mikhailov A V 1981 The reduction problem and the inverse scattering method Physica D 3 73–117
[19] Mikhailov A V 1981 Two-dimensional generalized Toda lattice Commun. Math. Phys. 79 473–88
[20] Ueno K and Takasaki K 1984 Toda lattice hierarchy Adv. Stud. Pure Math. 4 1–95
[21] Babich V M, Matveev V B and Sall M A 1986 Binary Darboux transformation for the Toda lattice J. Math. Sci. 35 2582–9
[22] Hirota R 2004 The Direct Method in Soliton Theory (Cambridge: Cambridge University Press)
[23] Nimmo J J C and Willox R 1997 Darboux transformations for the two-dimensional Toda system Proc. R. Soc. A 453 2497–525
[24] Jimbo M and Miwa T 1983 Solitons and infinite dimensional Lie algebras Publ. RIMS. Kyoto Univ. 19 943–1001
[25] Willox R 2005 On a generalized Tzitzeica equation Glasgow Math. J. A 47 221–31
[26] Date E, Jimbo M, Kashiwara M and Miwa T 1981 KP hierarchies of orthogonal and symplectic type—transformation groups for soliton equations VI J. Phys. Soc. Japan 50 3813–8
[27] Date E, Jimbo M, Kashiwara M and Miwa T 1982 Transformation groups for soliton equations—Euclidean Lie algebras and reduction of the KP hierarchy Publ. RIMS. Kyoto Univ. 18 1077–110
[28] Hirota R 1986 Reduction of soliton equations in bilinear form Physica D 18 161–70
[29] Tzitzeica G 1910 Sur une nouvelle classe de surfaces C. R. Acad. Sci. 150 955–6
[30] Tzitzeica G 1910 Sur une nouvelle classe de surfaces C. R. Acad. Sci. 150 1227–9
[31] Nimmo J J C and Ruijsenaars S N M 2009 Tzitzeica solitons versus relativistic Calogero–Mosser three-body clusters J. Math. Phys. 50 043511
[32] Satate I 1975 Linear Algebra (New York: Dekker)
[33] Muir T 1960 A Treatise on The Theory of Determinants (New York: Dover)
[34] Vein R and Dale P 1999 Determinants and Their Applications in Mathematical Physics (New York: Springer)