On the $k$-Semispray of Nonlinear Connections in $k$-Tangent Bundle Geometry

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Abstract
In this paper we present a method by which is obtained a sequence of $k$-semisprays and two sequences of nonlinear connections on the $k$-tangent bundle $T^kM$, starting from a given one. Interesting particular cases appear for Lagrange and Finsler spaces of order $k$.

AMS Subject Classification: 53C05, 53C60.

Key words: $k$-tangent bundle, $k$-semispray, nonlinear connection, Lagrange space of order $k$, Finsler space of order $k$.

1 Introduction

Classical Mechanics have been entirely geometrized in terms of symplectic geometry and in this approach there exists certain dynamical vector field on the tangent bundle $TM$ of a manifold $M$ whose integral curves are the solutions of the Euler-Lagrange equations. This vector field is usually called spray or second-order differential equation (SODE). Sometimes it is called semispray and the term spray is reserved to homogeneous second-order differential equations ([7], [15]). Let us remember that a SODE on $TM$ is a vector field on $TM$ such that $JC = C$, where $J$ is the almost tangent structure and $C$ is the canonical Liouville field ([5], [6]).

In [2], [3], [4] J. Grifone studies the relationship among SODEs, nonlinear connections and the autonomous Lagrangian formalism. In paper [12] Gh. Munteanu and Gh. Pitiş also studied the relation between sprays and nonlinear connection on $TM$. This study was extended to the non-autonomous case by M. de León and P. Rodrigues ([5]). Also, important results for singular non-autonomous case was obtained in [13]. In this paper, following the ideas of papers [10], [11], [12] and [13] we will extend the study of the relationship between sprays and nonlinear connections to the $k$-tangent bundle of a manifold $M$. The study of the geometry of this $k$-tangent bundle was by introduced by R. Miron ([7], [8], [9]). For this case the $k$-spray represent a system of ordinary differential equations of $k + 1$ order.
2 The $k$-Semispray of a Nonlinear Connection

Let $M$ be a real $n$-dimensional manifold of class $C^\infty$ and $(T^kM, \pi^k, M)$ the bundle of accelerations of order $k$. It can be identified with the $k$-osculator bundle or $k$-tangent bundle ([7], [9]).

A point $u \in T^kM$ will be written by $u = (x, y^{(1)}, ..., y^{(k)})$, $\pi^k(u) = x$, $x \in M$. The canonical coordinates of $u$ are $(x^i, y^{(1)i}, ..., y^{(k)i})$, $i = 1, n$, where

\[ y^{(1)i} = \frac{1}{1!} \frac{dx^i}{dt}, ..., y^{(2)i} = \frac{1}{k!} \frac{d^k x^i}{dt^k}. \]

A transformation of local coordinates $(x^i, y^{(1)i}, ..., y^{(k)i}) \rightarrow (\tilde{x}^i, \tilde{y}^{(1)i}, ..., \tilde{y}^{(k)i})$ on $(k + 1)n$-dimensional manifold $T^kM$ is given by

\begin{align*}
\begin{cases}
\tilde{x}^i = \tilde{x}^i(x^1, ..., x^n), \quad \text{rang} \left( \frac{\partial \tilde{x}^i}{\partial x^j} \right) = n, \\
\tilde{y}^{(1)i} = \frac{\partial \tilde{x}^i}{\partial x^j} y^{(1)j}, \\
2\tilde{y}^{(2)i} = \frac{\partial \tilde{y}^{(1)i}}{\partial x^j} y^{(1)j} + 2 \frac{\partial \tilde{y}^{(1)i}}{\partial y^{(1)j}} y^{(2)j}, \\
\vdots \\
k\tilde{y}^{(k)i} = \frac{\partial \tilde{y}^{(k-1)i}}{\partial x^j} y^{(1)j} + 2 \frac{\partial \tilde{y}^{(k-1)i}}{\partial y^{(1)j}} y^{(2)j} + \cdots + k \frac{\partial \tilde{y}^{(k-1)i}}{\partial y^{(k-1)j}} y^{(k)j}.
\end{cases}
\end{align*}

A local coordinates change transforms the natural basis \( \left\{ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^{(1)i}}, ..., \frac{\partial}{\partial y^{(k)i}} \right\} \) of the tangent space $T_u T^kM$ by the rule:

\begin{align*}
\begin{cases}
\frac{\partial}{\partial x^i} = \frac{\partial \tilde{x}^i}{\partial x^j} \frac{\partial}{\partial x^j} + \frac{\partial \tilde{y}^{(1)i}}{\partial x^j} \frac{\partial}{\partial y^{(1)j}} + \cdots + \frac{\partial \tilde{y}^{(k)i}}{\partial x^j} \frac{\partial}{\partial y^{(k)j}} , \\
\frac{\partial}{\partial y^{(1)i}} = \frac{\partial \tilde{y}^{(1)i}}{\partial x^j} \frac{\partial}{\partial x^j} + \frac{\partial \tilde{y}^{(1)i}}{\partial y^{(1)j}} \frac{\partial}{\partial y^{(1)j}} + \cdots + \frac{\partial \tilde{y}^{(1)i}}{\partial y^{(k)j}} \frac{\partial}{\partial y^{(k)j}} , \\
\vdots \\
\frac{\partial}{\partial y^{(k)i}} = \frac{\partial \tilde{y}^{(k)i}}{\partial x^j} \frac{\partial}{\partial x^j} + \frac{\partial \tilde{y}^{(k)i}}{\partial y^{(1)j}} \frac{\partial}{\partial y^{(1)j}} + \cdots + \frac{\partial \tilde{y}^{(k)i}}{\partial y^{(k)j}} \frac{\partial}{\partial y^{(k)j}} .
\end{cases}
\end{align*}

The distribution $V_1 : u \in T^kM \rightarrow V_{1,u} \subset T_u T^kM$ generated by the tangent vectors \( \left\{ \frac{\partial}{\partial y^{(1)i}}, ..., \frac{\partial}{\partial y^{(k)i}} \right\} \) is a vertical distribution on the bundle $T^kM$. Its local dimension is $kn$. Similarly, the distribution $V_2 : u \in T^kM \rightarrow V_{2,u} \subset T_u T^kM$ generated by \( \left\{ \frac{\partial}{\partial y^{(2)i}}, ..., \frac{\partial}{\partial y^{(k)i}} \right\} \) is a subdistribution of $V_1$ of local dimension $(k - 1)n$. So, by this procedure on obtains a sequence of integrable distributions $V_1 \supset V_2 \supset \cdots \supset V_k$. The last distribution $V_k$ is generated by \( \left\{ \frac{\partial}{\partial y^{(k)i}} \right\} \) and $\dim V_k = n$ ([7]).

Hereafter, we consider the open submanifold

\[ \widetilde{T^kM} = T^kM \setminus \{ \mathbf{0} \} = \left\{ (x, y^{(1)}, ..., y^{(k)}) \in T^kM | \text{rank} \| y^{(1)i} \| = 1 \right\}, \]

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where 0 is the null section of the projection \( \pi^k : T^k M \to M \).

The following operators in algebra of functions \( \mathcal{F}(T^k M) \)

\[
\Gamma = y^{(1)i} \frac{\partial}{\partial y^{(k)i}}, \\
\Gamma = y^{(1)i} \frac{\partial}{\partial y^{(k-1)i}} + 2y^{(2)i} \frac{\partial}{\partial y^{(k)i}}, \\
\Gamma = y^{(1)i} \frac{\partial}{\partial y^{(1)i}} + 2y^{(2)i} \frac{\partial}{\partial y^{(2)i}} + \cdots + ky^{(k)i} \frac{\partial}{\partial y^{(k)i}}
\]

(3)

are \( k \) vector fields, globally defined on \( T^k M \) and linearly independent on the manifold \( T^k M = T^k M \setminus \{0\} \). \( \Gamma \) belongs of distribution \( V_k \), \( \Gamma \) belongs of distribution \( V_{k-1} \), ..., \( \Gamma \) belongs of distribution \( V_1 \) (see [7]). \( \Gamma, \Gamma, ..., \Gamma \) are called Liouville vector fields.

In applications we shall use also the following nonlinear operator, which is not a vector field,

\[
\Gamma = y^{(1)i} \frac{\partial}{\partial x^i} + 2y^{(2)i} \frac{\partial}{\partial y^{(1)i}} + \cdots + ky^{(k)i} \frac{\partial}{\partial y^{(k-1)i}}.
\]

(4)

Under a coordinates transformation (11) on \( T^k M \), \( \Gamma \) changes as follows:

\[
\Gamma = \Gamma + \left\{ y^{(1)i} \frac{\partial y^{(k)j}}{\partial x^i} + \cdots + ky^{(k)i} \frac{\partial y^{(k)j}}{\partial y^{(k-1)i}} \right\} \frac{\partial}{\partial y^{(k)j}}.
\]

(5)

A \( k \)-tangent structure \( J \) on \( T^k M \) is defined as usually (12) by the following \( \mathcal{F}(T^k M) \)-linear mapping \( J : \mathcal{X}(T^k M) \to \mathcal{X}(T^k M) \):

\[
J \left( \frac{\partial}{\partial x^i} \right) = \frac{\partial}{\partial y^{(1)i}}, J \left( \frac{\partial}{\partial y^{(1)i}} \right) = \frac{\partial}{\partial y^{(2)i}}, \cdots, \\
J \left( \frac{\partial}{\partial y^{(k-1)i}} \right) = \frac{\partial}{\partial y^{(k)i}}, J \left( \frac{\partial}{\partial y^{(k)i}} \right) = 0.
\]

(6)

\( J \) is a tensor field of type \((1, 1)\), globally defined on \( T^k M \).

**Definition 2.1** ([7]) A \( k \)-semispray on \( T^k M \) is a vector field \( S \in \mathcal{X}(T^k M) \) with the property

\[
JS = \Gamma.
\]

(7)

Obviously, there not always exists a \( k \)-semispray, globally defined on \( T^k M \). Therefore the notion of local \( k \)-semisprays is necessary. For example, if \( M \) is a paracompact manifold then on \( T^k M \) there exists local \( k \)-semisprays ([7]).

**Theorem 2.1** ([7]) i) A \( k \)-semispray \( S \) can be uniquely written in local coordinates in the form:

\[
S = y^{(1)i} \frac{\partial}{\partial x^i} + 2y^{(2)i} \frac{\partial}{\partial y^{(1)i}} + \cdots + ky^{(k)i} \frac{\partial}{\partial y^{(k-1)i}} \\
- (k + 1)G^i(x, y^{(1)}, ..., y^{(k)}) \frac{\partial}{\partial y^{(k)i}}.
\]

(8)
ii) With respect to (1) the coefficients $G^i(x, y^{(1)}, \ldots, y^{(k)})$ change as follows:

$$
(k + 1)\tilde{G}^i = (k + 1)G^i \frac{\partial \tilde{x}^i}{\partial x} - \left( y^{(1)}_j \frac{\partial y^{(k)}_i}{\partial x^j} + \cdots + ky^{(k)}_j \frac{\partial y^{(k)}_i}{\partial y^{(k-1)}_j} \right).
$$

iii) If the functions $G^i(x, y^{(1)}, \ldots, y^{(k)})$ are given on every domain of local chart of $T^kM$, so that (4) holds, then the vector field $S$ from (3) is a $k$-semispray.

Let us consider a curve $c : I \to M$, represented in a local chart $(U, \varphi)$ by $x^i = x^i(t), t \in I$. Thus, the mapping $\tilde{c} : I \to T^kM$, given on $(\pi^k)^{-1}(U)$, by

$$
x^i = x^i(t), y^{(1)i}(t) = \frac{1}{1!} \frac{dx^i}{dt}(t), \ldots, y^{(k)i}(t) = \frac{1}{k!} \frac{d^k x^i}{dt^k}(t), t \in I
$$

is a curve in $T^kM$, called the $k$-extension to $T^kM$ of the curve $c$.

A curve $c : I \to M$ is called $k$-path of a $k$-semispray $S$ (from (3)) if its $k$-extension $\tilde{c}$ is an integral curve for $S$, that is

$$
\begin{align*}
\frac{dx^i}{dt} &= y^{(1)i}, \\
\frac{dy^{(1)i}}{dt} &= 2y^{(2)i}, \ldots, \\
\frac{dy^{(k-1)i}}{dt} &= ky^{(k)i}, \frac{dy^{(k)i}}{dt} &= -(k + 1)G^i.
\end{align*}
$$

**Definition 2.2** The $k$-semispray $S$ is called $k$-spray if the functions $(G^i(x, y^{(1)}, \ldots, y^{(k)}))$ are $(k + 1)$-homogeneous, that is

$$
G^i(x, \lambda y^{(1)}, \ldots, \lambda^k y^{(k)}) = \lambda^{k+1}G^i(x, y^{(1)}, \ldots, y^{(k)}), \forall \lambda > 0.
$$

Like in the case of tangent bundle, an Euler Theorem holds. That is, a function $f \in F(T^kM)$ is $r$-homogeneous if and only if

$$
\mathcal{L}_v f = rf.
$$

Then a $k$-semispray $S$ is a $k$-spray if and only if

$$
\begin{align*}
y^{(1)h} \frac{\partial G^i}{\partial y^{(1)n}} + 2y^{(2)h} \frac{\partial G^i}{\partial y^{(2)n}} + \cdots + ky^{(k)h} \frac{\partial G^i}{\partial y^{(k)n}} &= (k + 1)G^i.
\end{align*}
$$

**Definition 2.3** A vector subbundle $NT^kM$ of the tangent bundle $(TT^kM, d\pi^k, M)$ which is supplementary to the vertical subbundle $V_1T^kM$,

$$
TT^kM = NT^kM \oplus V_1T^kM
$$

is called a nonlinear connection on $T^kM$. 

4
The fibres of $NT^k$M determine a horizontal distribution $N : u \in T^k M \rightarrow N_u T^k M \subset T_u T^k M$ supplementary to the vertical distribution $V_1$, that is

$$T_u T^k M = N_u T^k M \oplus V_1 u T^k M, \forall u \in T^k M.$$ (14)

The dimension of horizontal distribution $N$ is $n$.

If the base manifold $M$ is paracompact then on $T^k M$ there exists the nonlinear connections (7). There exists a unique basis, adapted to the horizontal distribution $N$, such that

$$\frac{\delta}{\delta x^i} \left( \pi_u \right) = \frac{\partial}{\partial x^i} |_{\pi_u}, \quad i = 1, \ldots, n.$$ (15)

The functions $N^j_i$, $N^j_j$, $\ldots$, $N^j_k$ are called the primal coefficients of the nonlinear connection $N$ and under a coordinates transformation (1) on $T^k M$ this coefficients are changing by the rule:

$$\frac{\delta}{\delta x^i} \left( \pi_{u'} \right) = \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^{(1)j}} - \cdots - N^j_k \frac{\partial}{\partial y^{(k)j}}.$$ (16)

Conversely, if on each local chart of $T^k M$ a set of functions $N^j_i$, $\ldots$, $N^j_k$ is given so that, according to (1), the equalities (16) hold, then there exists on $T^k M$ a unique nonlinear connection $N$ which has as coefficients just the given set of function (7).

The local adapted basis \( \left\{ \frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)j}}, \ldots, \frac{\delta}{\delta y^{(k)j}} \right\} \) is given by (15) and

$$\frac{\delta}{\delta y^{(1)j}} = \frac{\partial}{\partial y^{(1)j}} - N^j_i \frac{\partial}{\partial y^{(2)j}} - \cdots - N^j_k \frac{\partial}{\partial y^{(k)j}}, \ldots, \quad \frac{\delta}{\delta y^{(k-1)j}} = \frac{\partial}{\partial y^{(k-1)j}} - N^j_i \frac{\partial}{\partial y^{(k)j}} \frac{\partial}{\partial y^{(k)j}} = \frac{\partial}{\partial y^{(k)j}}.$$ (17)

and the dual basis (or the adapted cobasis) of adapted basis is
\begin{align*}
\{\delta x^i, \delta y^{(1)i}, \ldots, \delta y^{(k)i}\}_{i=1,m}, \text{ where } \delta x^i &= dx^i \text{ and } \\
(18) \quad \left\{
\begin{array}{l}
\delta y^{(1)i} = dy^{(1)i} + M^1_j dx^j, \\
\delta y^{(2)i} = dy^{(2)i} + M^1_j dy^{(1)j} + M^2_j dx^j, \\
\delta y^{(k)i} = dy^{(k)i} + M^1_j dy^{(k-1)j} + \cdots + M^k_j dx^j
\end{array}
\right.
\end{align*}

and
\begin{align*}
(19) \quad \left\{
\begin{array}{l}
M^1_j = N^1_j, \\
M^2_j = N^2_j + N^i_j M^m_j, \\
M^k_j = N^k_j + N^i_j M^m_j + \cdots + N^i_j M^m_j.
\end{array}
\right.
\end{align*}

Conversely, if the adapted cobasis \(\{\delta x^i, \delta y^{(1)i}, \ldots, \delta y^{(k)i}\}_{i=1,m}\) is given in the form (18), then the adapted basis \(\left\{\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, \ldots, \frac{\delta}{\delta y^{(k)i}}\right\}_{i=1,m}\) is expressed in the form (17), where
\begin{align*}
(20) \quad \left\{
\begin{array}{l}
N^1_j = M^1_j, \\
N^2_j = M^2_j - N^i_j M^m_j, \\
N^k_j = M^k_j - N^i_j M^m_j - \cdots - N^i_j M^m_j.
\end{array}
\right.
\end{align*}

The functions \(M^1_j, M^2_j, \ldots, M^k_j\) are called the dual coefficients of the nonlinear connection \(N\).

A nonlinear connection \(N\) is complete determined by a system of functions \(M^1_j, \ldots, M^k_j\) which is given on each domain of local chart on \(T^k M\), so that, according to (11), the relations hold:
\begin{align*}
(21) \quad \left\{
\begin{array}{l}
M^m_j \frac{\partial \tilde{x}^i}{\partial x^m}^{(1)} = \frac{\partial \tilde{x}^m}{\partial x^i} M^i_m^{(1)} + \frac{\partial \tilde{y}^{(1)i}}{\partial x^j}, \\
M^m_j \frac{\partial \tilde{x}^i}{\partial x^m}^{(2)} = \frac{\partial \tilde{x}^m}{\partial x^i} M^i_m^{(2)} + \frac{\partial \tilde{y}^{(1)m}}{\partial x^j} M^i_i^{(1)} + \frac{\partial \tilde{y}^{(2)i}}{\partial x^j}, \\
M^m_j \frac{\partial \tilde{x}^i}{\partial x^m}^{(k)} = \frac{\partial \tilde{x}^m}{\partial x^i} M^i_m^{(k)} + \frac{\partial \tilde{y}^{(1)m}}{\partial x^j} M^i_i^{(k-1)} + \cdots + \frac{\partial \tilde{y}^{(k-1)m}}{\partial x^j} M^i_i^{(1)} + \frac{\partial \tilde{y}^{(k)i}}{\partial x^j}.
\end{array}
\right.
\end{align*}

Let \(c : I \to M\) be a parametrized curve on the base manifold \(M\), given by \(x^i = x^i(t), t \in I\). If we consider its \(k\)-extension \(\tilde{c}\) to \(T^k M\), then we say that \(c\) is
an autoparallel curve for the nonlinear connection $N$ if its $k$-extension $\tilde{c}$ is a horizontal curve, that is $\frac{d\tilde{c}}{dt}$ belongs to the horizontal distribution.

From (18) and (22)

$$\frac{d\tilde{c}}{dt} = \frac{dx^i}{dt} \delta \frac{\delta y^{(1)i}}{\delta x^i} + \cdots + \frac{\delta y^{(k)i}}{\delta x^i}$$

it result that the autoparallels curves of the nonlinear connection $N$ with the dual coefficients $M^i_j^{(1)}, \ldots, M^i_j^{(k)}$ are characterized by the system of differential equations (7):

$$\begin{align*}
y^{(1)i} &= \frac{dx^i}{dt}, \\
y^{(2)i} &= \frac{1}{2!} \frac{d^2x^i}{dt^2}, \\
\frac{\delta y^{(1)i}}{dt} &= \frac{dy^{(1)i}}{dt} + M^i_j^{(1)} \frac{dx^j}{dt} = 0, \\
\frac{\delta y^{(2)i}}{dt} &= \frac{dy^{(2)i}}{dt} + M^i_j^{(1)} \frac{dy^{(1)i}}{dt} + M^i_j^{(2)} \frac{dx^j}{dt} = 0, \\
&\vdots \\
\frac{\delta y^{(k)i}}{dt} &= \frac{dy^{(k)i}}{dt} + M^i_j^{(1)} \frac{dy^{(k-1)i}}{dt} + \cdots + M^i_j^{(k)} \frac{dx^j}{dt} = 0.
\end{align*}$$

Now, let be $S = \frac{1}{S}$ a $k$-semispray with the coefficients $G^i = G^i(x, y^{(1)}, \ldots, y^{(k)})$ like in (3). Then the set of functions

$$\begin{align*}
M^i_j^{(1)} &= \partial G^i / \partial y^{(k)j}, \\
M^i_j^{(2)} &= \frac{1}{2} \left( SM^i_j^{(1)} + M^i_m M^m_j^{(1)} \right), \\
&\vdots \\
M^i_j^{(k)} &= \frac{1}{k} \left( SM^i_j^{(k-1)} + M^i_m M^m_j^{(k-1)} \right)
\end{align*}$$

gives the dual coefficients of a nonlinear connection $N$ determined only by the $k$-semispray $S$ (see the book [7] of Radu Miron).

Other result, obtained by Ioan Bucătaru ([1]), give a second nonlinear connection $N^*$ on $T^kM$ determined only by the $k$-semispray $S$. That is, the following set of functions

$$\begin{align*}
M^i_j^{(1)} &= \partial G^i / \partial y^{(k)j}, \\
M^i_j^{(2)} &= \frac{\partial G^i}{\partial y^{(k-1)j}}, \\
&\vdots \\
M^i_j^{(k)} &= \frac{\partial G^i}{\partial y^{(1)j}}
\end{align*}$$

is the set of dual coefficients of a nonlinear connection $N^*$. 

7
Let us consider the set of functions \( \{G^i(x, y^{(1)}, ..., y^{(k)})\} \), given on every domain of local chart by

\[
G^i = \frac{1}{k+1} \sum_{h=1}^{k} \left( y^{(h)(1)} \frac{\partial G^i}{\partial y^{(h)(1)}} + \cdots + y^{(h)(k)} \frac{\partial G^i}{\partial y^{(h)(k)}} \right).
\]

Using (5) we obtain that the functions \( G^i \) verifies (9). So, the functions \( G^i \) represent the coefficients of a \( k \)-semispray \( S \):

\[
S_i = \left( y^{(1)i} \frac{\partial}{\partial x^i} + 2y^{(2)i} \frac{\partial}{\partial y^{(1)i}} + \cdots + ky^{(k)i} \frac{\partial}{\partial y^{(k-1)i}} \right) - (k+1)G^i(x, y^{(1)}, ..., y^{(k)}) \frac{\partial}{\partial y^{(k)i}}.
\]

Obviously, there exists two nonlinear connections on \( T^kM \), which depend only by the \( k \)-semispray \( S \):

\[
\begin{align*}
\left\{ 
M^i_j^{(1)} &= \frac{\partial G^i}{\partial y^{(1)j}}, \\
M^i_j^{(2)} &= \frac{1}{2} \left( SM^i_j + M^m_j M^m_j \right), \\
&\vdots \\
M^i_j^{(k)} &= \frac{1}{k} \left( SM^i_j + M^m_j M^m_j \right)
\end{align*}
\]

and \( N^* \) with the dual coefficients

\[
\begin{align*}
M_{ij}^{(1)} &= \frac{\partial G^i}{\partial y^{(1)j}}, \\
M_{ij}^{(2)} &= \frac{\partial G^i}{\partial y^{(2)j}}, \\
&\vdots \\
M_{ij}^{(k)} &= \frac{\partial G^i}{\partial y^{(k)j}}.
\end{align*}
\]

By this method is obtained a sequence of \( k \)-semisprays \( \left( \frac{m}{S} \right)_{m \geq 1} \) and two sequence of nonlinear connections, \( \left( \frac{m}{N} \right)_{m \geq 1}, \left( \frac{m}{N^*} \right)_{m \geq 1} \).

From (11), (23) and (26) we have the following results:

**Proposition 2.1** If \( c \) is an autoparallel curve for nonlinear connection \( \frac{1}{2}N^* \), then \( c \) is a \( k \)-path of \( k \)-semispray \( S \).
Theorem 2.2  The following assertions are equivalent:

i) the $k$-semispray $\frac{1}{k}S$ is a $k$-spray;

ii) the $k$-paths of $S$ and $\frac{2}{k}S$ coincide.

Theorem 2.3  If $\frac{1}{k}S$ is a $k$-spray then $M^j_1$, ..., $M^i_k$ (or $M^{j_1}_1$, ..., $M^{i_k}_k$) are homogeneous functions of degree 1, 2, ..., $k$, respectively. The same property have the primal coefficients $N^j_1$, ..., $N^i_k$ (or $N^{j_1}_1$, ..., $N^{i_k}_k$).

We remark that the converse of this proposition is generally not valid and we have the result:

Theorem 2.4  If $\frac{1}{k}S$ is a $k$-spray then the sequence $(m_S)_m \geq 1$ is constant and the sequences $(N)_m \geq 1$, $(N^*)_m \geq 1$ are constant.

3 The $k$-Semispray of a Nonlinear Connection in a Lagrange Space of Order $k$

A Lagrangian of order $k$ is a mapping $L : T^kM \rightarrow \mathbb{R}$. $L$ is called differentiable if it is of $C^\infty$-class on $T^kM$ and continuous on the null section of the projection $\pi^k : T^kM \rightarrow M$.

The Hessian of a differentiable Lagrangian $L$, with respect to the variables $y^{(k)}$ on $\tilde{T}^kM$ is the matrix $||2g_{ij}||$, where

\begin{equation}
(30) 
   g_{ij} = \frac{1}{2} \frac{\partial^2 L}{\partial y^{(k)}i \partial y^{(k)}j}.
\end{equation}

We have that $g_{ij}$ is a $d$-tensor field on the manifold $\tilde{T}^kM$, covariant of order 2, symmetric (see [7]).

If

\begin{equation}
(31) 
   \text{rank } ||g_{ij}|| = n, \text{ on } \tilde{T}^kM
\end{equation}

we say that $L(x, y^{(1)}, ..., y^{(k)})$ is a regular (or nondegenerate) Lagrangian.

The existence of the regular Lagrangians of order $k$ is proved for the case of paracompacts manifold $M$ in the book [7] of Radu Miron.

Definition 3.1  ([7]) We call a Lagrange space of order $k$ a pair $L^{(k)} = (M, L)$, formed by a real $n$-dimensional manifold $M$ and a regular differentiable Lagrangian of order $k$, $L : (x, y^{(1)}, ..., y^{(k)}) \in \tilde{T}^kM \rightarrow L(x, y^{(1)}, ..., y^{(k)}) \in \mathbb{R}$, for which the quadratic form $\Psi = g_{ij} \xi^i \xi^j$ on $T^kM$ has a constant signature.
L is called the fundamental function and $g_{ij}$ the fundamental (or metric) tensor field of the space $L^{(k)n}$.

It is known that for any regular Lagrangian of order $k$, $L(x, y^{(1)}, ..., y^{(k)})$, there exists a $k$-semispray $S_L$ determined only by the Lagrangian $L$ (see [7]). The coefficients of $S_L$ are given by

$$ (k + 1)G^i = \frac{1}{2}g^{ij} \left\{ \Gamma \left( \frac{\partial L}{\partial y^{(k)j}} \right) - \frac{\partial L}{\partial y^{(k-1)j}} \right\} . $$

This $k$-semispray $S_L$ depending only by $L$ will be called canonical. If $L$ is globally defined on $T^k M$, then $S_L$ has the same property on $\tilde{T}^k M$.

From (24) and (25) it results that there exist two nonlinear connections: Miron’s connection $N$ and Bucătaru’s connection $N^*$ which depending only by the Lagrangian $L$. For this reason, both are called canonical.

So, the coefficients of $k$-semisprays $S_L$ and the coefficients of nonlinear connections $N^m, N^m$ depend only by the Lagrangian $L$, for any $m \geq 1$, but their expressions is not attractive for us.

Interesting results appear for Finsler spaces of order $k$.

**Definition 3.2** ([8]) A Finsler space of order $k$ is a pair $F^{(k)n} = (M, F)$ formed by a real differentiable manifold $M$ of dimension $n$ and a function $F : T^k M \to R$ having the following properties:

i) $F$ is differentiable on $\tilde{T}^k M$ and continuous on null section $0 : M \to T^k M$;

ii) $F$ is positive;

iii) $F$ is $k$-homogeneous;

iv) the Hessian of $F^2$ with elements

$$ g_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^{(k)j} \partial y^{(k)i}} $$

is positively defined on $\tilde{T}^k M$.

The function $F$ is called the fundamental function and the $d$-tensor field $g_{ij}$ is called fundamental (or metric) tensor field of the Finsler space of order $k$, $F^{(k)n}$.

The class of spaces $F^{(k)n}$ is a subclass of spaces $L^{(k)n}$.

Taking into account the $k$-homogeneity of the fundamental function $F$ and $2k$-homogeneity of $F^2$ we get:

1. the coefficients $G^i$ of the canonical $k$-semispray $S_{F^2}$, determined only by the fundamental function $F$,

$$ (k + 1)G^i = \frac{1}{2}g^{ij} \left\{ \Gamma \left( \frac{\partial F^2}{\partial y^{(k)j}} \right) - \frac{\partial F^2}{\partial y^{(k-1)j}} \right\} , $$

is $(k + 1)$-homogeneous functions, that is $S_{F^2}$ is a $k$-spray;
2. the dual coefficients of the Cartan nonlinear connection $N$ associated to Finsler space of order $k$, $F^{(k)n}$ (see [3]),

\[
M^i_j^{(1)} = \frac{1}{2(k + 1)} \frac{\partial}{\partial y^{(k)}j} \left\{ g^{im} \left[ \Gamma \left( \frac{\partial F^2}{\partial y^{(k)m}} \right) - \frac{\partial F^2}{\partial y^{(k-1)m}} \right] \right\},
\]

\[
M^i_j^{(2)} = \frac{1}{2} \left( S_{F^2} M^i_j^{(1)} + M^i_m M^m_j^{(1)} \right),
\]

\[
M^i_j^{(k)} = \frac{1}{k} \left( S_{F^2} M^i_j^{(k-1)} + M^i_m M^m_j^{(k-1)} \right),
\]

are homogeneous functions of degree 1, 2, ..., $k$, respectively, and the primal coefficients has the same property;

3. the dual coefficients of Bucățaru’s connection $N^*$ associated to Lagrangian $F^2$ are also homogeneous functions of degree 1, 2, ..., $k$, respectively, and the primal coefficients has the same property.

Using the previous results, we obtain the results:

**Theorem 3.1** If $F^{(k)n} = (M, F)$ is a Finsler space of order $k$, then:

a) the sequence $\left( \frac{m}{S} \right)_{m \geq 1}$ is constant, $\frac{1}{S}$ being the canonical $k$-spray $S_{F^2}$;

b) the sequences of nonlinear connections $\left( \frac{m}{N} \right)_{m \geq 1}$, $\left( \frac{m}{N^*} \right)_{m \geq 1}$ are constants, $\frac{1}{N}$ being the Cartan nonlinear connection of $F^{(k)n}$ and $\frac{1}{N^*}$ being the Bucățaru’s connection for $L = F^2$.

4 Conclusions

In this paper was studied the relation between semisprays and nonlinear connections on the $k$-tangent bundle $T^k M$ of a manifold $M$. This results was generalized by the author from the 2-tangent bundle $T^2 M$ ([11]). More that, the relationship between SOPDEs and nonlinear connections on the tangent bundle of $k^1$-velocities of a manifold $M$ (i.e. the Whitney sum of $k$ copies of $TM$, $T^k M = TM \oplus \cdots \oplus TM$) was studied by F. Munteanu in [14] (2006) and by N. Roman-Roy, M. Salgado, S. Vilarino in [15] (2011).

**Acknowledgments.** This research was partially supported by Grant FP7-PEOPLE-2012-IRSES-316338.

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