Ample line bundles
on certain toric fibered 3-folds

Shoetsu Ogata

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Abstract

Let $X$ be a projective nonsingular toric 3-fold with a surjective torus equivariant morphism onto the projective line or a nonsingular toric surface not isomorphic to the projective plane. Then we prove that an ample line bundle on $X$ is always normally generated.

Introduction

Let $X$ be a projective algebraic variety and let $\mathcal{L}$ an ample line bundle on it. If the multiplication map

$$\text{Sym}^k \Gamma(X, \mathcal{L}) \longrightarrow \Gamma(X, \mathcal{L}^\otimes k)$$

of the $k$-th symmetric power of the global sections of $\mathcal{L}$ to the global sections of the $k$-th tensor product is surjective for all positive integers $k$, then Mumford \cite{Mumford} calls $\mathcal{L}$ normally generated. A normally generated ample line bundle is always very ample, but not conversely.

If $X$ is a toric variety of dimension $n$ and $\mathcal{L}$ is an ample line bundle on it, then Ewald and Wessels \cite{EwaldWessels} showed that $\mathcal{L}^\otimes k$ is very ample for $k \geq n - 1$, and Nakagawa \cite{Nakagawa} showed that the multiplication map

$$\Gamma(X, \mathcal{L}^\otimes k) \otimes \Gamma(X, \mathcal{L}) \longrightarrow \Gamma(X, \mathcal{L}^\otimes (k+1))$$

is surjective for $k \geq n - 1$. We know that there exists a polarized toric variety $(X, \mathcal{L})$ of dimension $n \geq 3$ such that $\mathcal{L}^\otimes (n-2)$ is not normally generated. We
also know that any ample line bundle on a nonsingular toric variety is always very ample (see [6, Corollary 2.15]). Ogata [7] showed that an ample line bundle $L$ on a nonsingular toric 3-fold $X$ is normally generated if the adjoint bundle $L + K_X$ is not big.

If a toric variety $X$ of dimension $n \geq 2$ has a surjective torus equivariant morphism $\varphi : X \to Y$ onto a toric variety $Y$ of dimension $r$ ($1 \leq r < n$) with connected fibers, then we call $X$ a toric fibered $n$-fold over $Y$.

In this paper we restrict $X$ to a certain class of toric fibered 3-folds.

**Theorem 1** Let $X$ be a nonsingular projective toric fibered 3-fold over the projective line. Then an ample line bundle on $X$ is always normally generated.

Since a nonsingular toric surface not isomorphic to $\mathbb{P}^2$ has a torus equivariant surjective morphism onto $\mathbb{P}^1$, Theorem 1 implies the following corollary.

**Corollary 1** Let $X$ be a nonsingular projective toric fibered 3-fold over a nonsingular toric surface not isomorphic to the projective plane. Then an ample line bundle on $X$ is always normally generated.

On a toric variety $X$ of dimension $n$, the space $\Gamma(X, L)$ of global sections of an ample line bundle $L$ is parametrized by lattice points in a lattice polytope $P$ of dimension $n$ (see, for instance, Oda’s book [6, Section 2.2] or Fulton’s book [3, Section 3.5]). If $X$ has a surjective morphism $\varphi : X \to \mathbb{P}^1$ onto the projective line, then the corresponding polytope $P$ has a special shape. From this fact, we shall prove Theorem 1.

In Section 3, we prove the same statement of Theorem 1 under the assumption that one invariant fiber of $\varphi$ is irreducible. This is given as Proposition 3. Full statement is proved in Section 4 as Proposition 4. In the end of this paper, we remark that nonsingularity condition is necessary by giving an example.

## 1 Toric varieties and lattice polytopes

In this section we recall the fact about toric varieties and ample line bundles on them and corresponding lattice polytopes from Oda’s book [6] or Fulton’s book [3].
Let $N \cong \mathbb{Z}^n$ be a free abelian group of rank $n$ and $M := \text{Hom}(N, \mathbb{Z})$ its dual with the pairing $\langle \cdot, \cdot \rangle : M \times N \to \mathbb{Z}$. By scalar extension to real numbers $\mathbb{R}$, we have real vector spaces $N_\mathbb{R} := N \otimes \mathbb{Z} \mathbb{R}$ and $M_\mathbb{R} := M \otimes \mathbb{Z} \mathbb{R}$. We also have the pairing of $M_\mathbb{R}$ and $N_\mathbb{R}$ by scalar extension, which is denoted by the same symbol $\langle \cdot, \cdot \rangle$.

The group ring $\mathbb{C}[M]$ defines an algebraic torus $T_N := \text{Spec} \mathbb{C}[M] \cong (\mathbb{C}^\times)^n$ of dimension $n$. Then the character group $\text{Hom}_{gr}(T_N, \mathbb{C}^\times)$ of the algebraic torus $T_N$ coincides with $M$. For $m \in M$ we denote the corresponding character by $\mathbf{e}(m) : T_N \to \mathbb{C}^\times$.

Let $\Delta$ be a finite complete fan of $N$ and $X(\Delta)$ denote the toric variety defined by $\Delta$. Set $N_0 := \mathbb{Z}$ and $\Delta_0 := \{ \mathbb{R}_{\leq 0}, \{ 0 \}, \mathbb{R}_{\geq 0} \}$. Then we have $X(\Delta_0) = \mathbb{P}^1$. If a surjective morphism $\varphi : X(\Delta) \to \mathbb{P}^1$ is torus equivariant, then it defines a morphism of fans $\varphi^\natural : (N, \Delta) \to (N_0, \Delta_0)$. Moreover, if fibers of $\varphi$ are connected, then $\varphi^\natural(N) = N_0$. Set $N_0^\vee$ the dual to $N_0$. Then the dual homomorphism $\varphi^\natural : N_0^\vee \to M = N^\vee$ maps $N_0^\vee$ as a saturated submodule in $M$. Thus we have a direct sum decomposition $M \cong M' \oplus N_0^\vee$.

Set $N_f := (\varphi^\natural)^{-1}(0)$ and $\Delta_f := \{ \sigma \in \Delta ; \varphi^\natural(\sigma) = 0 \}$. Then $\Delta_f$ is a fan of $N_f$ and the toric variety $X(\Delta_f)$ is a general fiber of $\varphi : X(\Delta) \to \mathbb{P}^1$.

We define a lattice polytope as the convex hull $P := \text{Conv}\{ m_1, \ldots, m_r \}$ of a finite subset $\{ m_1, \ldots, m_r \}$ of $M$ in $M_\mathbb{R}$. We define the dimension of a lattice polytope $P$ as that of the smallest affine subspace $\mathbb{R}(P)$ containing $P$.

Let $X$ be a projective toric variety of dimension $n$ and $L$ an ample line bundle on $X$. Then there exists a lattice polytope $P$ of dimension $n$ such that the space of global sections of $L$ is described by

$$\Gamma(X, L) \cong \bigoplus_{m \in P \cap M} \mathbb{C}\mathbf{e}(m), \quad (2)$$

where $\mathbf{e}(m)$ is considered as a rational function on $X$ since $T_N$ is identified with the dense open subset (see [6, Section 2.2] or [3, Section 3.5]). Conversely, a lattice polytope $P$ in $M_\mathbb{R}$ of dimension $n$ defines a polarized toric variety $(X, L)$ satisfying the equality (2) (see [6, Chapter 2] or [3, Section 1.5]).

The $k$-th tensor product $L^\otimes k$ of $L$ corresponds to the $k$-th multiple $kP$ of $P$ for a positive integer $k$. The condition that the multiplication map

$$\Gamma(X, L^\otimes k) \otimes \Gamma(X, L) \to \Gamma(X, L^\otimes (k+1))$$

is isomorphic to the multiplication map on the space of global sections of $L^\otimes k$. This is the natural map that respects the torus action, and it is an isomorphism of vector bundles on $X$.

The map $\Gamma(X, L^\otimes k) \otimes \Gamma(X, L) \to \Gamma(X, L^\otimes (k+1))$ is the multiplication map of the global sections of $L^\otimes k$ and $L$. It is a morphism of sheaves on $X$, and it is an isomorphism of vector bundles. This is a consequence of the fact that $X$ is a projective toric variety and $L$ is an ample line bundle.

Moreover, the map $\Gamma(X, L^\otimes k) \otimes \Gamma(X, L) \to \Gamma(X, L^\otimes (k+1))$ is an isomorphism of vector bundles, which means that the map is a bijection on the space of global sections of $L^\otimes k$ and $L$. This is because $X$ is a projective toric variety and $L$ is an ample line bundle.

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is surjective is equivalent to the equality

$$(kP) \cap M + (P \cap M) = ((k + 1)P) \cap M.$$  

A lattice polytope $P$ in $M_\mathbb{R}$ is called \textit{normal} if the equality

$$k \text{ times} \quad (P \cap M) + \cdots + (P \cap M) = (kP) \cap M$$

holds for all positive integers $k$. This is equivalent to the condition that the equality

$$(kP) \cap M + P \cap M = ((k + 1)P) \cap M$$

holds for all positive integers $k$. We note that an ample line bundle $L$ on a toric variety is normally generated if and only if the corresponding lattice polytope $P$ is normal. We also note that the equality \(3\) holds if and only if for each lattice point $v \in (kP) \cap M$, there exists just $k$ lattice points $u_1, \ldots, u_k$ in $P \cap M$ with $v = u_1 + \cdots + u_k$.

In order to prove the normality of lattice polytopes, the following theorem is useful.

\textbf{Theorem 2 (Nakagawa [5])} Let $P$ be a lattice polytope in $M_\mathbb{R}$ of dimension $n$. Then we have the equality

$$(kP) \cap M + P \cap M = ((k + 1)P) \cap M$$

for integer $k \geq n - 1$.

From Theorem 2 we see that for the normality of $P$ of dimension three, it is enough to show the equality

$$(P \cap M) + (P \cap M) = (2P) \cap M.$$  

For a face of a lattice polytope, it is called an \textit{edge} if it is of dimension one and a \textit{facet} if of codimension one. A lattice polytope $P$ of dimension $n$ is called \textit{simple} if at each vertex $v$, just $n$ edges meet, that is, the convex cone $C_v(P) := \mathbb{R}_{\geq 0}(P - v)$ is written as

$$\mathbb{R}_{\geq 0}m_1 + \cdots + \mathbb{R}_{\geq 0}m_n$$

with $m_1, \ldots, m_n \in M$. Moreover, if the set $\{m_1, \ldots, m_n\}$ is a $\mathbb{Z}$-basis of $M$, then $P$ is called \textit{nonsingular}. For a face $F$ of $P$, we call $F$ is \textit{nonsingular} if it is nonsingular with respect to the sublattice $\mathbb{R}(F) \cap M$. We note that a face of a nonsingular lattice polytope is also nonsingular.
2 Polygonal prisms

For two lattice polytopes $P$ and $Q$ in $M_R$, we define the Minkowski sum as

$$P + Q := \{x + y \in M_R; x \in P \text{ and } y \in Q\}.$$  

Then $P + Q$ is also a lattice polytope.

In this section we investigate the normality of a lattice polytope $P$ of dimension three which is the Minkowski sum of a lattice polygon $A$ of dimension two and a lattice line segment $I$. See Figure 1 (b). Here we set $M = M' \oplus L$, rank $M' = 2$, rank $L = 1$ and $A \subset M'_R$.

If $A$ is a parallelogram, that is, if $A$ is the Minkowski sum $J_1 + J_2$ of two not parallel lattice line segments $J_1$ and $J_2$, then $P = A + I$ is normal because it is a parallelootope. See Figure 1 (a).

![Figure 1: A polygonal prism $P$](image)

From this observation we obtain the following proposition.

**Proposition 1** Let $A$ be a nonsingular lattice polygon in $M'_R$ not isomorphic to a basic triangle and $I$ a lattice line segment not contained in $M'_R$. Then $P = A + I$ is normal.

In the case that $A$ is a basic triangle, if $I \subset L_R$, then $P = A + I$ is normal. Here a lattice triangle is called basic if it is isomorphic to the convex hull of the origin and a basis of $M' \cong \mathbb{Z}^2$.
In order to prove Proposition 1, it is enough to show the following lemma.

**Lemma 1** If a nonsingular lattice polygon \( A \subset M'_\mathbb{R} \) is not basic, then it is covered by a union of lattice parallelograms.

*Proof.* Take a coordinates \((x, y)\) in \( M'_\mathbb{R} \). Assume that a lattice polygon \( A \) is \( r \)-gonal, that is, \( A \) has \( r \) edges. By a suitable affine transformation of \( M' \), we may take a vertex \( v_0 \) of \( A \) to be the origin, an edge \( E_1 \) from \( v_0 \) to be on the positive part of the \( x \)-axis and the other edge \( E_r \) from \( v_0 \) on the positive part of the \( y \)-axis. We will prove the lemma by dividing into several steps.

(a): The case of \( A = \text{Conv}\{0,(a,0),(0,1),(b,1)\} \), that is, \( r = 4 \). If \( a = b \), then \( A \) is a parallelogram. Set \( a < b \) and \( s = b - a \). If we set \( A_i = \text{Conv}\{0,(a,0),(i,1),(a+i,1)\} \) for \( i = 0, 1, \ldots, s \), then \( A_i \) is a parallelogram and \( A \) is covered by the union of \( A_i \) with \( i = 0, \ldots, s \). The same is when \( a > b \).

(b): Set \( F(E_1) := A \cap (0 \leq y \leq 1) \). Since \( A \) is nonsingular, \( F(E_1) \) is also a lattice polygon. From (a), we see that \( F(E_1) \) is covered by a union of lattice parallelograms. For all edges \( E_1, \ldots, E_r \) of \( A \), define \( F(E_i) \) in the same way. Then we have

\[ A = A^o \cup \bigcup_{i=1}^{r} F(E_i), \]

where \( A^o \) is the convex hull of \( \text{Int}(A) \cap M' \).

If \( \dim A^o \leq 1 \), then \( A \) is covered by the union of \( F(E_i) \). If \( \dim A^o = 2 \), then \( A^o \) is a nonsingular lattice polygon. If \( A^o \) is not isomorphic to a basic triangle, then we continue this process.

(c): When \( A^o \) is isomorphic to a basic triangle, we may consider \( A \) is the 4 times multiple \( \text{Conv}\{0,(4,0),(0,4)\} \) of a basic triangle, or, a polygon obtained from this by cutting off several basic triangles at vertices. Set \( A' := A \cap (1 \leq y \leq 4) \). Then we have a decomposition \( A = A' \cup F(E_1) \) and we see that \( A' \) is nonsingular and \( \dim(A')^o \leq 1 \). Thus we see that \( A \) is covered by a union of lattice parallelograms in this case.

Since the normalized area of \( A^o \) is an integer less than that of \( A \), this process is stop after several steps. \( \square \)

Next we introduce another direct sum decomposition of \( M = M' \oplus L \) with respect to \( I \) of the Minkowski sum \( P = A + I \).
Set \( L' := (\mathbb{R}I) \cap M \cong \mathbb{Z} \) and \( M = M'' \oplus L' \) with the projection map \( \pi : M \to M'' \). We note that \( M' = (\mathbb{R}A) \cap M \cong \mathbb{Z}^2 \) does not always coincide with \( M'' \). Set \( B := \pi(A) \subset M'' \). Then \( B \) is a lattice polygon in \( M'' \).

Take coordinates \((x, y)\) in \( M'' \) and \( z \) in \( L' \). From a suitable affine transform of \( M \), we may set so that a vertex \( v_0 \) of \( P = A + I \) is the origin and \( P \) is contained in the upper half space \((z \geq 0)\). Then define \( Q(A) \) as the convex hull of \( B \times 0 \) and \( A + I \). The polytope \( Q(A) \) is an upright polygonal prism with the \( r \)-gonal polygon \( B \) as its base and \( A \) as its roof. See Figure 2.

\[\text{Figure 2: An upright polygonal prism } Q(A)\]

**Proposition 2** Let \( A \) be a nonsingular lattice polygon in \( M''_\mathbb{R} \) not isomorphic to a basic triangle. Then \( Q(A) \) defined above is normal.

**Proof.** Decompose the lattice polygon \( B \) into a union of basic lattice triangles \( B_i \) \((i = 1, \ldots, s)\) with vertices in \( B \cap M'' \). For each \( B_i \), define \( R(B_i) \) as the convex hull of \((B_i \times \mathbb{R}_{\geq 0}) \cap Q(A) \cap M \). Then the prism \( R(B_i) \) is normal. We have a cover
\[ Q(A) = (A + I) \cup \bigcup_{i=1}^{s} R(B_i). \]

Since \( A + I \) is normal from Proposition 1, the polytope \( Q(A) \) is normal. \(\Box\)
3 Union of polygonal prisms

In this section we assume that a projective toric fibered 3-fold $X$ over $\mathbb{P}^1$ has one irreducible invariant fiber.

As in Section 1, we set $N_0 := \mathbb{Z}$ and $\Delta_0 := \{\mathbb{R}_{\leq 0}, \{0\}, \mathbb{R}_{\geq 0}\}$. Then $X(\Delta_0) = \mathbb{P}^1$. The torus equivariant morphism $\varphi : X = X(\Delta) \to \mathbb{P}^1$ is defined by the morphism of fans $\varphi^\#: (N, \Delta) \to (N_0, \Delta_0)$ with $\varphi^\#(N) = N_0$. Set $N_0^\vee$ the dual to $N_0$. Denote by $L$ the image of the dual homomorphism $\varphi^* : N_0^\vee \to M$. Then we have a direct sum decomposition $M = M_f \oplus L$, where $M_f^\vee \cong N_f := (\varphi^\#)^{-1}(0)$. The subset $\Delta_f := \{\sigma \in \Delta; \varphi^\#(\sigma) = 0\}$ is a fan of $N_f$. A general fiber of $\varphi$ is the toric surface $X(\Delta_f)$.

Let $\mathcal{L}$ be an ample line bundle on a toric fibered 3-fold $X(\Delta)$ over $\mathbb{P}^1$. Let $P$ be the lattice polytope in $M_\mathbb{R}$ corrsponding to the polarized toric 3-fold $(X(\Delta), \mathcal{L})$. Denote by $\mathcal{L}_f$ the restriction of the ample line bundle $\mathcal{L}$ to $X(\Delta_f)$. The polarized toric surface $(X(\Delta_f), \mathcal{L}_f)$ defines a nonsingular lattice polygon $B \subset (M_f)_\mathbb{R}$. Then the lattice polytope $P$ is contained in the polygonal prism $B \times \mathbb{R} \subset (M_f \oplus L)_\mathbb{R} = M_\mathbb{R}$ and each side wall of the prism contains a facet of $P$.

If one invariant fiber of $\varphi$ is irreducible, then $P$ has a facet isomorphic to $B$. We may draw the picture of $P$ so that it is a polygonal upright prism with $B$ as the base and the roof consists of a collection of lattice polygons. See Figure 3.

![Figure 3: Union of upright polygonal prisms](image)

Proposition 3 Assume that a nonsingular projective toric fibered 3-fold $\varphi : X(\Delta) \to \mathbb{P}^1$ has one irreducible invariant fiber $\varphi^{-1}(1 : 0)$, that is, one
irreducible invariant fiber is isomorphic to a general fiber. Then an ample line bundle on $X(\Delta)$ is always normally generated.

Proof. Take coordinates $(x, y, z)$ in $M_\mathbb{R} = (M_f \oplus L)_\mathbb{R}$ so that $(M_f)_\mathbb{R} = (z = 0)$. Let $P \subset M_\mathbb{R}$ be a lattice polytope corresponding to an ample line bundle $L$ on $X(\Delta)$.

From our assumption, $P$ has the special facet $B$ corresponding to the irreducible fiber of $\varphi^{-1}([1 : 0])$. From a suitable affine transform of $M$, we may assume that $P$ is contained in the upper half space ($z \geq 0$) and $B$ is contained in the plane ($z = 0$).

Set $A_1, \ldots, A_s$ the all facets in the roof of the upright prism $P$. Set $B_i = \pi(A_i)$ the lattice polygon in $(M_f)_\mathbb{R}$ defined as the image of a facet $A_i$ by the projection $\pi : (M_f \oplus L)_\mathbb{R} \to (M_f)_\mathbb{R}$. For each facet $A_i$ define $Q(A_i) := (B_i \times L_\mathbb{R}) \cap P$. Then we have a decomposition of $P$ as a union of polygonal prisms $Q(A_i)$. For each $A_i$, set $M_i := (\mathbb{R}A_i) \cap M \cong \mathbb{Z}^2$. Then $A_i$ is a nonsingular lattice polygon in $(M_i)_\mathbb{R}$.

If $A_i$ is not a basic triangle, then $Q(A_i)$ is normal from Proposition 2. Even if $A_i$ is a basic triangle if it meets a side wall of $P$, then it is normal because $M_i \oplus L \cong M$.

We assume that $A_i$ is a basic triangle and meets no side walls of $P$. Set $v_1, v_2, v_3$ the three vertices of $A_i$ and $E_1, E_2, E_3$ the edges of $P$ from $v_1, v_2, v_3$ outside $A_i$, respectively. Let $w_j$ be the lattice points on the edge $E_j$ nearest $v_j$ for $j = 1, 2, 3$. Set $\tilde{A}_i := \text{Conv}\{w_1, w_2, w_3\}$. Then the lattice triangle $\tilde{A}_i$ is similar and parallel to $A_i$ since $P$ is nonsingular. If $\tilde{A}_i \cong A_i$, then $P = Q(A_i)$. It contradicts the assumption. Thus $\tilde{A}_i$ is not basic. The subset $(\pi(A_i) \times L_\mathbb{R}) \cap P$ of $P$ can be decomposed into a union of the slice $\text{Conv}\{A_i, \tilde{A}_i\}$ of the roof and the rest $Q(\tilde{A}_i)$. Both are normal.

Since $P$ is covered by a union of normal lattice polytopes, it is normal.

\[\square\]

4 General case

Proposition 4 Let $X(\Delta)$ be a projective nonsingular toric fibered 3-fold over $\mathbb{P}^1$. Then an ample line bundle on $X(\Delta)$ is always normally generated.

In this section we assume that two invariant fibers of $\varphi : X(\Delta) \to \mathbb{P}^1$ are reducible.
Let $\mathcal{L}$ be an ample line bundle on $X(\Delta)$ and $P \subset M_\mathbb{R}$ the lattice polytope corresponding to $(X(\Delta), \mathcal{L})$. As in the proof of Proposition 3, take coordinates $(x, y, z)$ in $M_\mathbb{R} = (M_f \oplus L)_\mathbb{R}$ so that $(M_f)_\mathbb{R} = (z = 0)$. From a suitable affine transform of $M$, we may assume that $P$ is contained in the upper half space ($z \geq 0$).

Let $F$ be a side wall, a facet of $P$ parallel to $L$. From a suitable affine transform of $M$, if we set $F$ to be contained in the plane $(y = 0)$, then we define the lattice polytope $P(F) := P \cap (0 \leq y \leq 1)$. Then $P(F)$ is normal by Lemma 2.5 in [6]. Set $P^o := \text{Conv}\{\text{Int}(P) \cap M\}$ the inner polytope of $P$. If $\dim P^o \leq 2$, then $P$ is normal because $P$ is covered by a union of $P(F)$ for all side walls $F$ of $P$.

(S1) We assume $\dim P^o = 3$.

By the projection $\pi : (M_f \oplus L)_\mathbb{R} \rightarrow (M_f)_\mathbb{R}$, we define $G := \pi(P)$, which is a nonsingular lattice polygon. Set $G^o := \text{Conv}\{\text{Int}(G) \cap M_f\}$ the inner polygon of $G$, which is also a nonsingular lattice polygon from the assumption (S1).

Set $A_1, \ldots, A_s$ the all facets in the roof of the upright prism $P$ for $i = 1, \ldots, s$. For a lattice point $m \in M$, denote by $l(m)$ the line through $m$ parallel to the $z$-axis.

**Lemma 2** For a lattice point $m' \in (\partial G^o) \cap M_f$, the line segment $l((m', 0)) \cap P$ has length greater than or equal to two.

**Proof.** We note that both edges of $l((m', 0)) \cap P$ are lattice points. If the length is less than two, then it is one, hence $P$ is contained in $(0 \leq z \leq 1)$. In this case, two invariant fibers of $\varphi$ are both irreducible. This contradicts the assumption in the beginning of this section. \hfill $\square$

Set $A'_i := A_i \cap \pi^{-1}(G^o)$ for $i = 1, \ldots, s$. Set $I := [0, (0, 0, -1)]$ the unit interval on the $z$-axis of negative direction. Define the convex sets as

$$U_1(P) := \text{Conv}\{\bigcup_i (A'_i + I)\} \quad \text{and} \quad U_2(P) := \text{Conv}\{\bigcup_i (A'_i + 2I)\}.$$ 

From Lemma 2, we have $U_1(P) \subset U_2(P) \subset P$.

In order to prove that $P \cap M + P \cap M = (2P) \cap M$, take a lattice point $m$ in $2P$ and consider $\frac{1}{2}m \in P$. If $m$ is located on the boundary of $2P$, then it is a lattice point on a lattice polytope of dimension less than three, hence, there exist two lattice points $m_1, m_2 \in (\partial P) \cap M$ such that $m = m_1 + m_2$. 

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We may assume that $\frac{1}{2}m$ is contained in the interior of $P$ and that $\frac{1}{2}m \notin M$. If $\frac{1}{2}m$ is contained in $P(F)$ for a side wall $F$ of $P$, then there exist $m_1, m_2 \in P(F)$ such that $m = m_1 + m_2$ because $P(F)$ is normal. Here we call that $m$ is 2-normal in $P$ if there exist $m_1, m_2 \in P \cap M$ such that $m = m_1 + m_2$.

**Lemma 3** If $\frac{1}{2}m$ is contained in $U_1(P)$, then $m$ is 2-normal.

**Proof.** If $A_i'$ is not a basic triangle, then $I + A_i'$ is normal. Even if $A_i'$ is basic, as in the proof of Proposition 3 take $\tilde{A}_i'$ a triangle similar and parallel to $A_i'$ so that $I + \tilde{A}_i'$ is normal. Thus if $\frac{1}{2}m$ is contained in a $I + A_i'$, then $m$ is 2-normal in $P$.

Let $\{m'_1, \ldots, m'_r\}$ be the set of vertices of $G$. Set $\tilde{m}_i \in \pi^{-1}(G) \cap M$ the lattice point on the roof of $U_1(P)$. For three $\tilde{m}_i, \tilde{m}_j, \tilde{m}_k$ of $\{\tilde{m}_1, \ldots, \tilde{m}_r\}$, set $F_{ijk} = \text{Conv}\{\tilde{m}_i, \tilde{m}_j, \tilde{m}_k\}$. Decompose the lattice triangle $\pi(F_{ijk})$ into a union of basic lattice triangle $G_l$, $l = 1, \ldots, t$ with vertices in $M_f$. For $G_l$, define $R(G_l)$ the convex hull of $(G_l \times L) \cap P \cap M$. Then the prism $R(G_l)$ is normal. $F_{ijk}$ is covered by a union of $R(G_l)$'s. We take all three of $\{\tilde{m}_1, \ldots, \tilde{m}_r\}$. Then the boundary of $U_1(P)$ is covered by a union of normal polytopes.

Assume that $\frac{1}{2}m$ is not contained in $I + A_i'$ nor $R(G_l)$. If $\pi(\frac{1}{2}m) \in M_f$, then $\frac{1}{2}m$ is contained in a lattice line segment parallel to $L$, hence, $m$ is 2-normal.

Assume that $\pi(\frac{1}{2}m) \notin M_f$. Since $G^o = \cup_l \pi(A'_l)$, we can choose a $\pi(A'_l)$ such that $\pi(\frac{1}{2}m) \in \pi(A'_l)$. Decompose $\pi(A'_l)$ into a union of basic lattice triangle $G_j$ with vertices in $M_f$ such that $G_1 = \text{Conv}\{u_1, u_2, u_3\}$ with $\frac{1}{2}m = \frac{1}{2}(u_1 + u_2)$ since $2\pi(\frac{1}{2}m) \in M_f$.

Set

$$\text{Conv}\{l((u_1, 0)) \cap P \cap M\} = [u_1^-, u_1^+] \quad \text{and} \quad \text{Conv}\{l((u_2, 0)) \cap P \cap M\} = [u_2^-, u_2^+]$$

Here the $z$-coordinates of $u_1^+$ is bigger than those of $u_1^-$. Since $u_1^+$ and $u_2^+$ are contained in $U_1(P)$, the lattice line segment $[u_1^+, u_2^+]$ is contained in $U_1(P)$.

Consider the lattice line segment $[u_1^-, u_2^-]$. Im $(M_f)_{\mathbb{R}}$, the line containing both $u_1$ and $u_2$ meets opposite edges of $G^o$. From two edges we take vertices $\{m'_1, m'_2, m'_3, m'_4\}$ such that the line segment $[u_1, u_2]$ is contained in the interior of the lattice polygon $\text{Conv}\{m'_1, m'_2, m'_3, m'_4\}$. Set $m''_1, m''_2, m''_3, m''_4$ the vertices of $U_1(P)$ on the bottom with $\pi(m''_i) = m'_i$. Consider the convex hull.
of \((\mathbb{R}_{\geq 0}I + \text{Conv}\{m''_1, m''_2, m''_3, m''_4\}) \cap P\). This contains \(u^-_1\) and \(u^-_2\), hence, the line segment \([u^-_1, u^-_2]\) because \(U_2(P) \subset P\).

Consider the lattice polygon \(R = \text{Conv}\{u^-_1, u^+_1, u^-_2, u^+_2\} \subset P\). Since \(R\) contains \(\frac{1}{2}m\), we see that \(m\) is 2-normal in \(P\). □

Proof of Proposition 4. Set \(B_1, \ldots, B_t\) be all facets in the bottom of \(P\). Set \(B'_j := B_j \cap \pi^{-1}(G^o)\). We define

\[
D_1(P) := \text{Conv}\{\bigcup_j (B'_j + (-I))\} \subset P.
\]

As in the proof of Lemma 3 we see that if \(\frac{1}{2}m\) is contained in \(D_1(P)\), then \(m\) is 2-normal.

Assume that \(\frac{1}{2}m\) is not contained in \(U_1(P)\) nor \(D_1(P)\). As in the proof of Lemma 3 we can choose a basic lattice triangle \(\text{Conv}\{u_1, u_2, u_3\} \subset \pi(A'_i)\) with \(\frac{1}{2}m = \frac{1}{2}(u_1 + u_2)\). In this case, since \(u^-_1, u^-_2 \in D_1(P)\), the lattice line segment \([u^-_1, u^-_2]\) is contained in \(D_1(P)\). Thus \(m\) is 2-normal. □

Remark. Nonsingularity condition of a toric fibered 3-fold \(\varphi : X(\Delta) \to \mathbb{P}^1\) in Proposition 4 is necessary. We know a singular toric fibered 3-fold over \(\mathbb{P}^1\) with a very ample but not normally generated line bundle on it. Finally, we will give an example found by Burns and Gubeladze [1, Exercise 2.24].

For a positive integer \(q\), define a lattice tetrahedron as

\[
Q_q := \text{Conv}\{0, (1, 0, 0), (0, 1, 0), (1, 1, q)\}.
\]

If \(q \geq 2\), then \(Q_q\) is not very ample. Set \(I = [0, (0, 0, 1)]\) the unit interval on the z-axis. Define \(P_q := Q_q + I\) as the Minkowski sum. Let \((X, \mathcal{L})\) be the polarized toric 3-fold corresponding to \(P_q\). Then this \(X\) is a singular toric fibered 3-fold over \(\mathbb{P}^1\) and \(\mathcal{L}\) is very ample. If \(q \geq 4\), then \(\mathcal{L}\) is not normally generated. See also [8].

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