MODULARITY OF TWO DOUBLE COVERS OF \( \mathbb{P}^5 \) BRANCHED ALONG 12 HYPERPLANES

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Abstract. For two varieties of dimension 5 constructed as double covers of \( \mathbb{P}^5 \) branched along the union of 12 hyperplanes, we prove that the number of points over \( \mathbb{F}_p \) can be expressed in terms of Artin symbols and the \( p \)th Fourier coefficients of modular forms. Many analogous results are known in dimension \( \leq 3 \), but very few in higher dimension. In addition, we use an idea of Burek to construct quotients of our varieties for which the point counts mod \( p \) are expressible in terms of Artin symbols and the coefficients of a single modular form of weight 6.

1. Introduction

One of the most fundamental problems in number theory is to give formulas for the number of points on a fixed variety over a varying finite field. In the simplest cases, the number of points over \( \mathbb{F}_p \) can be expressed in terms of powers of \( p \) and Artin symbols expressing the decomposition of \( p \) in number fields; however, this is very far from being sufficient in general. Most famously, isogeny classes of elliptic curves over \( \mathbb{Q} \) of conductor \( N \) are in bijection with newforms of level \( N \) with integer coefficients [2]; the correspondence takes a form with Hecke eigenvalues \( a_p \) to an elliptic curve with \( p + 1 - a_p \) points over \( \mathbb{F}_p \).

In higher dimensions, we cannot expect such a statement to hold for all varieties of a given deformation type, except in very special situations. Nevertheless, we would like to find varieties for which the number of points can be expressed in terms of powers of \( p \), Artin symbols, and the coefficients of modular forms. Perhaps the most interesting case is that in which only a single eigenform of weight \( n > 2 \) is needed and the dimension of the variety is \( n - 1 \).

In dimension greater than 1, the most natural candidates for this property are the Calabi-Yau varieties, which are defined as follows:

**Definition 1.** [2] Definition 1 | A Calabi-Yau variety is a smooth variety \( V \) of dimension \( d \) satisfying \( K_V \cong \mathcal{O}_V \) and \( H^i(K_V) = 0 \) for \( 0 < i < d \). If \( V \) is a limit of Calabi-Yau varieties and has a Calabi-Yau resolution of singularities, then \( V \) is a singular Calabi-Yau variety. If \( H^{d-1}(\mathcal{T}_V) = 0 \), where \( \mathcal{T}_V \) denotes the tangent bundle of \( V \), then \( V \) is rigid; this condition can also be written as \( H^{d-1,1}(V) = 0 \). If \( h^{i,j} = 0 \) for all \( i, j \) except with \( i = j \) or \( \{i, j\} = \{0, \dim V\} \), and either \( \dim V \) is odd or the semisimplification of \( H_{\text{et}}^{\dim V}(V, \mathbb{Z}_p) \) splits as a direct sum of Galois representations \( (H^{\dim V/2, \dim V/2} \oplus H^{0, \dim V}) \oplus H^{\dim V/2, \dim V/2} \), then \( V \) is strongly rigid.

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Remark 1. According to our definition, an elliptic curve is a strongly rigid Calabi-Yau variety (sometimes these are defined to require trivial fundamental group). A K3 surface is a Calabi-Yau variety, and it is strongly rigid if and only if its Picard number is 20. A Calabi-Yau threefold is rigid if and only if it is strongly rigid.

One can prove that a Calabi-Yau variety $V$ is strongly rigid by showing that the cohomology of $V$ is generated by cycle classes and $H^{\dim V, 0}, H^{0, \dim V}$. If $\dim V$ is odd this is equivalent, and presumably that is true if $\dim V$ is even, but a proof would seem to need some form of the Hodge conjecture.

The integral Hecke eigenforms of weight 3 up to twist correspond to imaginary quadratic fields with class group of exponent dividing 2 by [24, Theorem 2.4]. Such fields are known with at most one exception, which is excluded by the generalized Riemann hypothesis; the list can be found in [10]. Elkies and Schütten proved the following:

**Theorem 1** ([10, Theorem 1]). Let $f$ be a Hecke eigenform of weight 3 from the list with eigenvalues $a_p$. Then there is a K3 surface $S_f$ such that, for all but finitely many $p$, we have $\# S_f(\mathbb{F}_p) = p^2 + c(p)p + 1 + a_p$, where $c(p)$ is a linear combination of Artin symbols.

Much less is known for forms of weight greater than 3. Gouvêa and Yui proved the following:

**Theorem 2** ([12, Theorem 3]). Let $V$ be a smooth rigid Calabi-Yau threefold over $\mathbb{Q}$. Then $V$ is modular: in other words, there is a Hecke eigenform $f$ of weight 4 such that the Galois representation $\rho_\ell$ on $H^3_{\text{ét}}(V, \mathbb{Q}_\ell)$ is equivalent to $\rho_{f,\ell}$, the representation attached to $f$, for all $\ell$.

(The same conclusion had been obtained by Dieulefait and Manoharmayum [9] under stronger hypotheses.) It follows that $V$ has $p^3 + n(p)(p^2 + p) + 1 - a_p$ points over $\mathbb{F}_p$ for all primes $p$ of good reduction, where $n(p)$ is expressed in terms of Artin symbols and the $a_p$ are the Hecke eigenvalues of $f$. Many examples are worked out in detail in [18] and elsewhere, but it is not known whether every Hecke eigenform can be realized by a rigid Calabi-Yau threefold in this way, nor whether there are finitely or infinitely many rational Hecke eigenforms of weight 4 up to twist.

Gouvêa and Yui point out that their methods also apply to Calabi-Yau varieties of odd dimension $d$ with $\dim H^d = 2$ [12, page 146]. Thus, if $V$ is a strongly rigid Calabi-Yau variety of odd dimension $d$, there is a formula for the number of $\mathbb{F}_p$-points of $V$ of the form $\sum_{i=1}^d n_i p^i + (-1)^d a_p$, where the $n_i$ are expressed in terms of Artin symbols and the $a_p$ are the Hecke eigenvalues of a rational newform of weight $d + 1$, and the same is expected to hold for $d$ even. On the other hand, if $V$ is only rigid, this would not be expected. For example, if $d = 5$ it is possible that $h^{3,2}(V) > 0$ and that $H^5_{\text{ét}}(V, \mathbb{Z}_\ell)$ is irreducible of dimension $> 2$ even though $V$ is rigid.

In dimension greater than 3 there are almost no examples of strongly rigid Calabi-Yau varieties. If the dimension of the space of cusp forms of weight $k$ for $\Gamma_1(N)$ is 1, then the Kuga-Sato construction [8] gives a Calabi-Yau variety realizing the form. Ahlgren [1] in impact studies the case $k = 6, N = 4$, and Paranjape and Ramakrishnan [20] consider several others, including $k = 6, N = 3$. In addition, Frechette, Ono, and Papanikolas have shown [14] how to construct varieties that realize the cusp forms of level $N = 2, 4, 8$ and arbitrary weight $k$. However, for $k$
The main goal of this paper is to work out two examples of fivefolds. One realizes the newform of weight 6 and level 8; the other, the newform of weight 6 an level 32 with complex multiplication. Both are double covers of $\mathbb{P}^5$ branched along a union of 12 hyperplanes; however, the methods are somewhat different.

Our analysis of the first example, which bears a close resemblance to certain rigid Calabi-Yau threefolds, will culminate in the proof of the following result:

**Theorem 5.** The double cover $V_8$ of $\mathbb{P}^5$ defined by $t^2 = \prod_{i=0}^{5} x_i(x_i + x_{i+1})$ is modular (Definition 7) of level 8.

We do this by finding a fibration by quotients of products of Kummer surfaces closely related to the construction of [11]. This variety appears not to be a singular Calabi-Yau, but the modularity does not require such a statement to hold in any case. We will also use an idea of Burek [4] to construct a quotient of the variety that appears to be a strongly rigid Calabi-Yau. That is, we will prove:

**Theorem 6.** Let $Q_3 = V_8/\langle \iota_3 \rangle$, where $\iota_3$ takes $x_i$ to $x_{6-i}$. Then the number of $\mathbb{F}_{p^2}$-points of $Q_3$ is $\sum_{i=0}^{5} p^i - a_p - \phi(-1)p^3$ for all odd $p$, where $a_p$ is the $p$th Fourier coefficient of the newform of weight 6 and level 8 and $\phi$ is the quadratic character mod $p$.

The second example is also a double cover of $\mathbb{P}^5$ branched along the union of 12 hyperplanes. We summarize our results on it in the following statements:

**Theorem 7.** Proposition 20. **Theorem 9.** Let $V_{32}$ be the fivefold defined by $t^2 = \left( \prod_{i=0}^{5} x_i \right) (x_0+x_1)(x_3+x_5)(x_2+x_4+x_5)(x_0+x_2-x_4)(x_1-x_2+x_4)(x_2-x_3+x_4)$.

Let $E$ be the elliptic curve $y^2 = x^3 - x$ and $M$ the K3 surface defined by $t^2 = x y^2 (x+y) (y+z)(-x+z)$. Then $V_{32}$ is birationally equivalent to the quotient of $M \times M \times E$ by a group of order 4. Further, for all odd primes $p$, there are $\sum_{i=0}^{5} p^i - a_{6,p} - pa_{4,p} - 2p^2a_{2,p}$ points on $V_{32}$ over $\mathbb{F}_p$, where $a_{i,p}$ is the $p$th Fourier coefficient of the newform of weight $i$ and level 32 that has complex multiplication. A quotient of $V_{32}$ by a group of order 4 is birationally equivalent to a variety with $\sum_{i=0}^{5} p^i - a_{6,p}$ points over $\mathbb{F}_p$ for all odd $p$.

Here $M$ has Picard number 20 and realizes the CM newform of weight 3 and level 16 with quadratic character. We will also indicate a related construction in level 27 in Example [1].

An extensive search has discovered many examples of apparently modular double covers of $\mathbb{P}^5$ with branch locus the union of 12 hyperplanes that will not be discussed in this paper. One of these is Ahlgren’s fivefold [1]; others appear to be new and we intend to study them in future work. In particular, we have found two more unions of 12 hyperplanes, not projectively equivalent to the first one or to each other, such that the double cover has a Calabi-Yau resolution that appears to be strongly rigid.
and to realize the form of level 8. One of them is notable for its large symmetry group, with the symmetric group on 5 symbols acting faithfully on the set of 12 hyperplanes; the other, for admitting a fibration in quotients of products of K3 surfaces that is similar to but distinctly different from that in the first example discussed here.

The conjectural identity that equates the number of points on this variety computed from the fibration to a simple formula in terms of the coefficients of the modular form appears to point to a previously undiscovered identity of hypergeometric functions. Similarly, by considering quotients of our example of level 32 and comparing the point counts that arise from fibrations to those obtained in this paper and from its results, one obtains some attractive identities, which again will be presented in future work. There are also several collections of hyperplanes that appear to correspond to a cusp form of level 256 which is a quartic twist of the form of level 32. It is interesting that we do not find any examples whose level is not a power of 2 (in dimension 3 there are many such examples of double covers of $P^3$ branched along the union of 8 hyperplanes).

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2. Notation

We start by introducing some notation that will apply throughout the paper.

**Definition 2.** We will often work in $P^5_Q$; the coordinates in it will be denoted by $x_0, \ldots, x_5$, with the usual understanding that $x_i = x_{i+6}$. We will also use weighted projective space with weights 6, 1, 1, 1, 1, 1, this being the natural home for double covers of $P^5$ with branch locus of degree 12. The variables there will be $t, x_0, \ldots, x_5$, and a map from $P(6, 1, 1, 1, 1, 1)$ to $P^5$ will always be given by omitting $t$. At times we will use other projective spaces, referring to their variables as $x_0, \ldots, x_m$ or $z_0, \ldots, z_n$. The weighted projective space $P(3, 1, 1, 1)$ will also arise, since our arguments use many K3 surfaces given as double covers of $P^2$; its coordinates will usually be $t, x, y, z$, but sometimes $x_0, x_1, x_2$, etc.

**Definition 3.** Let $B$ be a scheme with a given $F_p$-point $\iota : \text{Spec} F_p \rightarrow B$ and $\pi : V \rightarrow B$ a flat family of schemes of finite type over $B$. Then $\text{Spec} F_p \times_B V$ is a scheme $\mathcal{V}$ of finite type over $F_p$, so it has finitely many $F_p$-rational points. We denote the number of these points by $|\mathcal{V}|_p$ (the choice of $\iota$ will always be clear). By abuse of language we will also use this notation when $\mathcal{V}$ is a variety defined over $Q$ by equations with coefficients whose denominators are not multiples of $p$, implicitly viewing it as a scheme over $\text{Spec} Z_{(p)}$.

**Definition 4.** Let $V$ be a variety over $Q$. If there is a formula for $|V|_p$, valid for all but finitely many $p$, in terms of powers of $p$, Artin symbols, and eigenvalues of Hecke eigenforms for $\Gamma_0(N)$, then $V$ is modular, and its level is the least common multiple of those of the eigenforms that appear. In the opposite direction we say that an eigenform involved in such a formula is realized by $V$, especially if it is the only eigenform of its weight that appears in the formula.
Definition 5. We use \( \phi \) to denote the quadratic character modulo a prime \( p \) (we will never be considering more than one prime at a time, so this will not lead to ambiguity).

3. Counting points on double covers

In this paper we will have to count the \( \mathbb{F}_p \)-points of various double covers of affine and projective spaces and of quotients of products of these. We give the notation and state the results that we will be using here so as not to interrupt the exposition later. The main result of this section, Lemma 2, is certainly well known, but we prove it for lack of an appropriate reference.

Let \( S \) be affine or projective space over \( \mathbb{F}_p \) and let \( f \) be a polynomial defining a subvariety of \( S \); if \( S \) is projective we suppose further that \( \deg f \) is even. Then there is a double cover \( D_f \) of \( S \) defined by the equation \( s^2 - f = 0 \), where \( s \) is a new variable. It is trivial to compute the number of \( \mathbb{F}_p \)-points of \( D_f \) in terms of the values of \( f \):

Lemma 1. We have \( [D_f]_p = \sum_{P \in S(\mathbb{F}_p)} \phi(f(P)) \).

(If \( S \) is projective then \( f(P) \) is not well-defined, but \( \phi(f(P)) \) still is.) More generally, we can count points on quotients of products of double covers by the involution that is the product of the involution on each factor. To do so, we first introduce some notation:

Definition 6. Let \( V \) be a variety over a finite field \( F \) with an involution \( \iota \). For a positive integer \( i \) let \( F_i \) be an extension of \( F \) of degree \( i \). Let \( P_{V,i} \) (respectively \( Z_{V,i}, N_{V,i} \)) be the number of \( F_i \)-points of \( V/\iota \) whose inverse images in \( V/F_i \) contain \( 2 \) (resp. \( 1, 0 \)) rational points. This depends on \( \iota \), but there will never be any ambiguity, so we suppress it from the notation. We refer to a point \( P_0 \) of \( V/\iota \) as a \( P \)-point (resp. \( Z \)-point, \( N \)-point), depending on the number of points of \( V \) above \( P_0 \).

Lemma 2. Let \( p \) be an odd prime. For \( 1 \leq i \leq n \), let \( S_i \) be affine or projective spaces over \( \mathbb{F}_p \) and let \( V_i \) be hypersurfaces defined by \( f_i = 0 \) in \( S_i \), where the degree of \( f_i \) is even if \( S_i \) is projective. Let the \( D_{f_i} \) be the double covers of \( S_i \) defined by \( s_i^2 - f_i = 0 \). Then \( (\mathbb{Z}/2\mathbb{Z})^n \) acts on \( \prod_{i=1}^n D_{f_i} \) by negating appropriate subsets of the \( s_i \); let \( E \) be the subgroup of \( (\mathbb{Z}/2\mathbb{Z})^n \) consisting of elements of even weight, and let \( D_V = (D_{f_1} \times \cdots \times D_{f_n})/E \). Then \( [D_V]_p = \prod_{i=1}^n [S_i]_p + \prod_{i=1}^n (P_{V,i} - N_{V,i}) \).

Proof. Let \( R = (R_1, \ldots, R_n) \) be a point of \( S_{V_1} \times \cdots \times S_{V_n} \). First, if at least one of the \( R_i \) is a \( Z \)-point, then \( \prod_{i=1}^n f_i(R_i) = 0 \) so \( (R_1, \ldots, R_n) \) lies under one rational point of \( D_V \). Otherwise there are \( 2^n \) geometric points of \( \prod_{i=1}^n D_{f_i} \) lying over \( R \), and a point and its Galois conjugate are in the same \( E \)-orbit if and only if an even number of the \( R_i \) are \( N \)-points. In this case there are \( 2 \) rational points above \( R \) in \( D_V \), and otherwise there are \( 0 \). Equivalently, the number of \( \mathbb{F}_p \)-points of \( D_V \) above \( R \) is \( 1 + \prod_{i=1}^n (p_n(R_i) - n_n(R_i)) \), where \( p_n, n_n \) are the characteristic functions of the sets of \( P \) and \( N \)-points on \( V_i \). The result follows by summing over all \( \mathbb{F}_p \)-points of \( S_1 \times \cdots \times S_n \). \( \square \)

4. Hypergeometric functions over finite fields and modular forms

The main purpose of this brief section is to relate our notation for elliptic curves to that used by Frechette, Ono, and Papanikolas in [11] so as to apply their results.
Lemma 3. Let $E_2(\lambda)$ be the elliptic curve defined by $y^2 = (x - 1)(x^2 + \lambda)$, and let $A_2(p, \lambda)$ be the trace of Frobenius of $E_2(\lambda)$ over $\mathbb{F}_p$. In addition, for characters $A$ and $B$ on $\mathbb{F}_p$, let $(A/B)$ be the normalized Jacobi sum $\sum_{\chi \in \mathbb{F}_p} A(x)B(x - 1)$, where the bar denotes the complex conjugate. Let $\phi$ be the quadratic character on $\mathbb{F}_p$, and let $3F_2(\lambda) = \frac{1}{p^2} \sum_{\chi} (\chi^3)^{\phi(\chi)} \chi(\lambda)$, where the sum runs over all characters $\chi$ of $\mathbb{F}_p$.

We restate the basic relation between $3F_2$ and $3A_2$.

**Theorem 3** ([11, Theorem 4.3 (2), Theorem 4.4 (2)]).

$$3F_2 \left( \frac{1}{1 + \lambda} \right) = \frac{\phi(-\lambda)(3A_2(p, \lambda)^2 - p)}{p^2}$$

for $\lambda \in \mathbb{F}_p$ with $\lambda \neq 0, -1$. In addition, if $p \equiv 1 \mod 4$ we have $3F_2(1) = \frac{4a^2 - 2p}{p^2}$ where $a$ is an odd integer such that $p - a^2$ is a square, and if $p \equiv 3 \mod 4$ then $3F_2(1) = 0$.

To relate our notation to that of [11] requires a simple statement about elliptic curves.

**Definition 8.** For $\lambda \neq 0, -1$, let $E_\lambda$ be the elliptic curve defined by $y^2 = x^3 - 2x^2 + \frac{\lambda}{x+1}$. Let $a_{\lambda,p} = p + 1 - \#E_\lambda(\mathbb{F}_p)$, and let $a_0,p = 0$.

**Proposition 1.** For $\lambda \neq 0, -1$ we have $3A_2(p, \frac{1}{\lambda+1})^2 = a_{\lambda,p}^2$. Equivalently, we have $3A_2(p, \mu)^2 = a_{(1+\frac{\mu}{\lambda})}^2$ for $\mu \neq 0, -1$.

**Proof.** Replacing $x$ by $x + 1$ in the equation $y^2 = (x - 1)(x^2 + \frac{1}{x+1})$ defining an elliptic curve whose trace of Frobenius is $3A_2(p, \frac{1}{\lambda+1})$ gives a quadratic twist of the elliptic curve $y^2 = x^3 - 2x^2 + \frac{\lambda}{x+1}x$. This is the elliptic curve whose trace is $a_{\lambda,p}$, so the two have the same trace up to sign. \qed

Combining these two statements gives

$$3F_2(\lambda) = \phi \left( \frac{\lambda + 1}{\lambda} \right) \left( \frac{a_{\lambda,p}^2 - p}{p^2} \right).$$

The following simple statements about modular forms with complex multiplication by $\mathbb{Q}(i)$ will be used in the proof of Theorem 3.

**Definition 9.** For $j \in \{2, 4, 6\}$, let $m_j$ be the unique newform of weight $j$ and level 32 that has complex multiplication by $\mathbb{Q}(i)$. Let $m_3$ be the newform of weight 3 and level 16 whose Nebentypus is the Dirichlet character $(-1)$. (In the LMFDB these are 32.2.a.a, 32.4.a.a, 32.6.a.b, and 16.3.c.a respectively.) For $j \in \{2, 3, 4, 6\}$ and $p$ prime, let $a_{j,p}$ be the eigenvalue of $m_j$ for the Hecke operator $T_p$.

The following is a routine application of the theory of modular forms with complex multiplication.

**Lemma 3.** For $p \equiv 1 \mod 4$ we have $a_{3,p} = a_{2,p}^2 - 2p, a_{4,p} = a_{2,p}(a_{3,p} - p), a_{6,p} = a_{2,p}a_{3,p} - p^2a_{2,p};$ for $p \equiv 2, 3 \mod 4$ all $a_{j,p}$ are equal to 0.
Definition 10. Let \( K_\lambda, L_\lambda \) be the surfaces in \( \mathbb{P}(3, 1, 1, 1) \) defined by the equations
\[
\begin{align*}
v^2 &= (\lambda + 1)z_0z_1z_2(\lambda z_0 + z_1)(z_1 + z_2)(z_0 + z_2), \\
w^2 &= \lambda(\lambda + 1)y_0y_1y_2(y_0 + y_1)(\lambda y_0 + y_2)(y_1 + y_2)
\end{align*}
\]
and let \( A_\lambda, B_\lambda \) be the affine patches \( z_0 \neq 0, y_0 \neq 0 \). (The twist by \( \lambda(\lambda + 1) \) is made to facilitate the comparison with a Kummer surface.)

Lemma 4. For all \( p \) and all \( \lambda \neq 0, -1 \in \mathbb{F}_p \) we have \([K_\lambda]_p - [A_\lambda]_p = [L_\lambda]_p - [B_\lambda]_p = p + 1\).

Proof. The points of \( K_\lambda \setminus A_\lambda \) and \( L_\lambda \setminus B_\lambda \) are exactly the projective points with \( z_0 = v = 0 \).

By exchanging \( z_1, z_2 \) we see that \( K_\lambda \) and \( L_\lambda \) are quadratic twists of each other by \( \lambda \). In other words, we have
\[
[K_\lambda]_p - (p^2 + p + 1) = \phi(\lambda)([L_\lambda]_p - (p^2 + p + 1)).
\]

Definition 11. Let \( \text{Kum}_\lambda \) be the Kummer surface of \( E_\lambda \times E_\lambda \): it has a singular model in weighted projective space \( \mathbb{P}(3, 1, 1, 1) \) defined by
\[
v^2 = \prod_{i=0}^{1}(z_i^3 - 2z_i^2z_2 + \frac{\lambda}{\lambda + 1}z_i^2).
\]
Let \( \overline{\text{Kum}}_\lambda \) be its minimal desingularization.

Proposition 2. Let \( \lambda \neq 0, -1 \in \mathbb{F}_p \). Then \( \overline{\text{Kum}}_\lambda \) has \( p^2 + (12 + 6\phi(\lambda))p + 1 + a_{\lambda,p}^2 \) points over \( \mathbb{F}_p \) (for \( a_{\lambda,p} \) see Definition 8).

Proof. Consider the elliptic fibration \((z_0 : z_2)\) of constant \( j \)-invariant. If \( \lambda \in \mathbb{F}_p^2 \), the \( I_0^* \) fibres and their components are all rational and contain \( 4(5p + 1) \) points in total. In addition, there are \( (p - 3 - a_{\lambda,p})/2 \) fibres isomorphic to \( E_\lambda \) and containing \( p + 1 - a_{\lambda,p} \) points and \( p - 3 + a_{\lambda,p} \) fibres isomorphic to the quadratic twist and containing \( p + 1 + a_{\lambda,p} \) points. The total number of points is \( p^2 + 18p + 1 + a_{\lambda,p}^2 \).

The argument when \( \lambda \notin \mathbb{F}_p^2 \) is similar.
Proposition 3. There is a genus-1 fibration on $\tilde{\text{Kum}}_\lambda$ the Jacobian of whose general fibre is isomorphic to the elliptic curve defined by
\[ y^2 = x^3 + \frac{4t - 2}{(\lambda + 1)t(\lambda t + 1)}x^2 + \frac{1}{((\lambda + 1)(\lambda t + 1)t)^2}x \]
and that has singular fibres of types $I^*_1, I^*_1, I^*_0, I_1$.

Proof. We define the fibration by the equations
\[ [z_0z_1/\lambda - z_2^2/(\lambda + 1) : z_0^2 - 2z_0z_2 + \lambda z_2^2/(\lambda + 1)]. \]
In Magma it is routine to define the general fibre of this map, verify that it is a curve of geometric genus 1, calculate its Jacobian, and show that it has the desired properties. □

Proposition 4. The minimal desingularization $\tilde{\text{Kum}}_\lambda$ of $K_\lambda$ admits a fibration in curves of genus 1 whose general fibre is 2-isogenous to that of the fibration on $\text{Kum}_\lambda$ introduced in Proposition 3. The singular fibres of this fibration are of types $I^*_2, I^*_2, I^*_0, I_2$, and all components of the reducible fibres are defined over the field to which $\lambda$ belongs.

Proof. The desired fibration is defined by $(z_0 : z_1)$. Again, it is a simple matter to show that the general fibre is isomorphic to the elliptic curve defined by
\[ y^2 = x^3 + \frac{t - 2}{t(\lambda + 1)(\lambda t + 1)}x^2 + \frac{1 - t}{((\lambda + 1)(\lambda t + 1)t)^2}x, \]
that its bad fibres are as stated, and that the quotient map by the subgroup of order 2 generated by $(\lambda+1)(\lambda t+1) : 0 : 1)$ is the desired isogeny. □

Theorem 4. Suppose as before that $\lambda \neq 0, -1$. Then $[K_\lambda]_p = p^2 + 1 + a^2_{\lambda,p}$.

Proof. We compare the numbers of points on the desingularizations by means of the fibrations of Propositions 3 and 4. The number of points on a smooth curve of genus 1 is unchanged by an isogeny, so the difference is accounted for by the singular fibres. (The existence of a section is not an issue since every smooth curve of genus 1 over a finite field has rational points.) Let $\delta$ be $-1$ if $\text{Kum}_\lambda$ has split multiplicative reduction at the $I_1$ fibre and 1 otherwise. Then this fibre has $p + 1 + \delta$ points. One checks that the $I_2$ fibre is split if and only if the $I_1$ fibre is, so it has $2p + 1 + \delta$ points.

All 21 components of the reducible fibres on $K_\lambda$ are defined over the ground field. When $\lambda \in \mathbb{F}_p^2$ the same is true for the 20 components of reducible fibres of $\tilde{\text{Kum}}_\lambda$, but otherwise only 8 of them are. Thus, when $\lambda \in \mathbb{F}_p^2$, there are $21p + 4 + \delta$ points on singular fibres of the fibration on $\tilde{\text{Kum}}_\lambda$ and likewise $21p + 4 + \delta$ points on singular fibres of the fibration on $\tilde{K}_\lambda$. Hence $\tilde{K}_\lambda$ has the same number of points as $\tilde{\text{Kum}}_\lambda$.

Similarly, when $\lambda \notin \mathbb{F}_p^2$, there are $9p + 4 + \delta$ points on singular fibres on $\tilde{\text{Kum}}_\lambda$, so $\tilde{K}_\lambda$ has 12p more points than $\tilde{\text{Kum}}_\lambda$. We conclude, in view of Proposition 2 that $\tilde{K}_\lambda$ has $p^2 + 18p + 1 + a^2_{\lambda,p}$ points.

To finish the proof, we compute that for generic $\lambda$, the singular subscheme of $K_\lambda$ has degree 18, and that all components of the resolutions are defined over the base field. The singular subscheme is unaltered by specializations that do not cause additional pairs of lines in the ramification locus to meet: the only $\lambda$ for which such
coincidences occur are 0, −1, which are disallowed in the statement of the theorem. Hence $\tilde{K}_\lambda$ has 18$p$ more points than $K_\lambda$, and the result follows.

\begin{corollary}
Let $\lambda \neq 0, -1$ as before. Then $[L_\lambda]_p = p^2 + p + 1 + \phi(\lambda)(a_{\lambda,p}^2 - p)$.
\end{corollary}

\begin{proof}
This follows immediately from the fact that $L_\lambda$ is the twist of the double cover $K_\lambda \to \mathbb{P}^2$ by $\lambda$. If $\lambda$ is a square the two surfaces have the same number of points; if not, the sum is twice the number of points of $\mathbb{P}^2$.
\end{proof}

We sharpen these results by expressing them in terms of $P_\cdot$, $Z_\cdot$, and $N$-points (Definition \[\text{Definition 6}\]). This is checked in \textit{count-quotient.mag} \[\text{[17]}\] by comparing our formulas to point counts computed directly from equations.

\begin{proposition}
Let $\lambda \neq 0, -1$. Then over a finite field $F$ of $q$ elements, the quantities $P_{A\lambda,1}, Z_{A\lambda,1}, N_{A\lambda,1}$ are equal to $(q^2 - 6q + 7 + a_{\lambda,q}^2)/2, 5q - 7, (q^2 - 4q + 7 - a_{\lambda,q}^2)$ respectively. If $\lambda$ is a square in $F$, then $P_{B\lambda,1}, Z_{B\lambda,1}, N_{B\lambda,1}$ are equal to $P_{A\lambda,1}, Z_{A\lambda,1}, N_{A\lambda,1}$; otherwise they are equal to $N_{A\lambda,1}, Z_{A\lambda,1}, P_{A\lambda,1}$.
\end{proposition}

\begin{proof}
To count $Z_{A\lambda,1}$, we note that the branch locus consists of 5 lines, of which 5 pairs and one triple intersect in rational points. Thus the total number of points is $5q - 5 \cdot 1 - 1 \cdot 2 = 5q - 7$. Now $[A_{\lambda,1}]_q = q^2 - q + a_{\lambda,q}^2$ by Theorem \[\text{[4]}\]. The result for $A_{\lambda,1}$ follows by solving the equations

\begin{align*}
P_{A\lambda,1} + N_{A\lambda,1} + Z_{A\lambda,1} &= q^2, \\
2P_{A\lambda,1} + N_{A\lambda,1} &= [A_{\lambda,1}]_q.
\end{align*}

The statements about $B_{\lambda}$ are immediate consequences of the fact that $B_{\lambda}$ is isomorphic to the twist of $A_{\lambda}$ by $\lambda$.
\end{proof}

\begin{remark}
This proposition can be used to compute $P_{A\lambda,n}$, etc., for all $n$, since the number of points of an elliptic curve in an extension of a finite field is determined by that of the ground field and $\lambda \in \mathbb{F}_q$ is a square in $\mathbb{F}_{q^n}$ if and only if $\lambda$ is a square in $\mathbb{F}_q$ or $n$ is even.
\end{remark}

We complete these results by proving analogous ones for the case $\lambda = -1$. Neither the equation $y^2 = x^3 - 2x^2 + \frac{1}{\lambda + 1}x$ nor any twist gives an elliptic curve in this case, but we can still describe varieties isogenous to the Kummer surface by similar formulas.

\begin{definition}
Let $K_{-1}, L_{-1}$ be the surfaces defined by

\begin{align*}
v^2 &= z_0z_1z_2(-z_0 + z_1)(z_1 + z_2)(z_0 + z_2), \\
v^2 &= -z_0z_1z_2(-z_0 + z_1)(-z_0 + z_2)(z_1 + z_2),
\end{align*}

and $A_{-1}, B_{-1}$ the affine patches $z_0 \neq 0$. Let $a_{-1,p} = p + 1 - [E]_p$, where $E$ is the elliptic curve with affine equation $y^2 = x^3 - x$.

As before, by exchanging $z_1, z_2$ we see that $K_{-1}, L_{-1}$ are quadratic twists of each other by $-1$. On the other hand, the map $(v: -z_0: z_1: z_2)$ is an isomorphism $K_{-1} \to L_{-1}$.

\begin{proposition}
For all odd primes $p$ we have $[K_{-1}]_p = [L_{-1}]_p = p^2 - \phi(-1)p + 1 + a_{-1,p}^2$.
\end{proposition}
Proof. For \( p \equiv 3 \mod 4 \), we have stated above that \( K_{-1}, L_{-1} \) are isomorphic to their twists by \(-1\), which is not a square in \( \mathbb{F}_p \), so the number of points is \( p^2 + p + 1 \). This is as claimed, since \( a_{-1,p} = 0 \) for such \( p \).

In the case \( p \equiv 1 \mod 4 \), the argument is very similar to that given to prove Theorem 4. The two surfaces are isomorphic, so we only consider \( K_{-1} \). Again we begin with the fibration \((0 : z_1)\), for which the general fibre is defined by
\[
y^2 = x^3 + (-t^2 + t)x^2 + (t^3 - 2t^2 + t^3)x
\]
and there are three fibres of type \( I^* \) (since the \( I^*_0 \) and \( I_2 \) of the generic case come together). All components of the singular fibres are rational. We consider the quotient of this elliptic surface by \((0 : 0)\), obtaining a surface defined by
\[
y^2 = x^3 + (2t^2 - 2t)x^2 + (t(t - 1)^2)x.
\]
This surface has an \( I^*_1 \) fibre at 1 and \( I^*_1 \) at 0, \( \infty \); the action of Galois on all three is trivial. Let \( \tilde{S}_{-1} \) be the K3 surface given by the minimal desingularization of this surface. Then as in the proof of Theorem 4 we have \([\tilde{K}_{-1}]_p = [\tilde{S}_{-1}]_p\).

On the other hand, we consider the Kummer surface \( \text{Kum}_{-1} \) of \( E_{-1} \times E_{-1} \), where \( E_{-1} \) is defined by \( y^2 = x^3 - x \). It can be defined in \( \mathbb{P}(3,1,1,1) \) by \( v^2 = (z_0^3 - z_0z_2^2)(z_1^3 - z_1z_2^2) \). The map defined by
\[
((z_0 + z_2)(4z_0^2z_1 + z_0z_2^2 - z_1^2z_2 - z_0z_2^2 - 2z_1z_2^2 - z_2^2))/4 : (z_0z_1 - z_0z_2 - z_1z_2 - z_2^2)(3z_0z_1 - z_0z_2 - z_1z_2 - z_2^2))/3
\]
induces an elliptic fibration on the minimal desingularization whose general fibre is isomorphic to that above. Thus \([\text{Kum}_{-1}]_p = [\tilde{S}_{-1}]_p\).

But as before \([\tilde{K}_{-1}]_p = p^2 + 18p + 1 + a_{-1,p}^2\). The singular subscheme of \( K_{-1} \) has degree 19 and all the exceptional curves are defined over \( \mathbb{F}_p \), so
\[
[K_{-1}]_p = [\tilde{K}_{-1}]_p - 19p = [\text{Kum}_{-1}]_p - 19p = p^2 - p + 1 + a_{-1,p}^2
\]
as claimed. \( \square \)

We now demonstrate a relation between the \( a_{\lambda,p} \) and the coefficients of an eigenform. This will be used in Section 6 to convert an expression for the number of points derived from a fibration to one in terms of the coefficients.

Proposition 7. \( p + \sum_{\lambda \in \mathbb{F}_p}^p \phi(\lambda)(a_{\lambda,p}^2 - p) = -b_p \), where \( b_p \) is the eigenvalue of \( T_p \) on the newform of weight 4 and level 8.

Proof. Consider the total space \( T_L \) of the family of \( L_\lambda \) (Definition 1) as a double cover of \( \mathbb{P}^2 \times \mathbb{P}^1 \). On the one hand, this is fibred by \( \lambda \), with the fibre at \( \lambda \) having \( p^2 + p + 1 + \phi(\lambda)(a_{\lambda,p}^2 - p) \) points for \( \lambda \neq 0, -1, \infty \) by Corollary 3; it is easily seen that the fibres at 0, \(-1, \infty \) all have \( p^2 + p + 1 \) points. (We do not obtain the formula of Proposition 6 for the fibre at \(-1 \) because the factor of \( \lambda + 1 \) makes it a copy of \( \mathbb{P}^2 \).)

On the other hand, using the methods of 5 we can construct a resolution and verify that it is a rigid Calabi-Yau threefold, modular of level 8, and hence obtain the formula \( p^2 + 2p^2 + p + 1 - b_p \) for \([T_L]_p\), valid for all \( p > 2 \). (The results of 5 are stated for double covers of \( \mathbb{P}^3 \) for which the components of the branch locus meet transversely in smooth loci; however, the arguments apply without change to double covers of any smooth threefold and components of the branch locus meeting
in unions of smooth components that meet transversely.) Comparing the two sides we obtain the equality
\[(p + 1)(p^2 + p + 1) + \sum_{\lambda=1}^{p-2} \phi(\lambda)(a_{\lambda,p}^2 - p) = p^3 + 2p^2 + p + 1 - b_p;\]
and now subtract \(p^3 + 2p^2 + p + 1\) from both sides.

**Remark 3.** Perhaps this result can be deduced from those of [11], but this type of geometric argument can be applied in many situations where the methods of [11] cannot.

The K3 surfaces considered so far in this section are important for the calculations in Section 7. In the remainder of this section, we make some small modifications in order to study K3 surfaces that arise in Section 7.

**Definition 13.** Let \(M = M_\lambda\) and \(N_\lambda\) be the surfaces in \(\mathbb{P}(3,1,1,1)\) defined by
\[M_\lambda : t^2 = xyz(x + y)(y + z)(-x + z),\]
\[N_\lambda : t^2 = \lambda xy(-x + \lambda z)y(-y + z)(x + 2y - z)(-x - 2y + (\lambda + 1)z).\]

**Proposition 8.** \([M]_p = p^2 + p + 1 + a_{3,p}\), and the branch locus of \(M\) has \(6p - 5\) points over \(\mathbb{F}_p\).

**Proof.** Observe that the same surface as \(M_{-1}\) (Definition 12) up to a change of variables. We showed in Proposition 6 that \([M]_p = p^2 - \phi(-1)p + 1 + a_{3,1,p}\). Since \(y^2 = x^3 - x\) is the unique elliptic curve of conductor 32 up to isogeny and \(a_{-1,p}\) is the trace of Frobenius at \(p\) for this curve, we see that \(a_{-1,p} = a_{2,p}\). In light of Lemma 3, this implies our claim for \(p \equiv 1 \mod 4\). For \(p \equiv 2,3 \mod 4\), both sides are equal to \(p^2 + p + 1\). \(\square\)

Recall the notation (Definition 9) \(P_{V,i}, Z_{V,i}, N_{V,i}\) for the number of points on the base of a double cover over \(\mathbb{F}_p\) that pull back to 2,1,0 points on the cover respectively. In these terms, we may rephrase Proposition 6 as saying that \(Z_{M,1} = 6p - 5, P_{M,1} = (p^2 - 5p - 4 + a_{3,p})/2, N_{M,1} = (p^2 - 5p - 4 - a_{3,p})/2\). We also note that \(Z_{N_{\lambda},1} = p^2 + p + 1\) for \(\lambda = 0\), while it is \(6p - 7\) for \(\lambda = \pm 1\) and \(6p - 9\) for other values of \(\lambda\) (the difference is that two sets of three lines in the branch locus are concurrent for \(\pm 1\) but not on other fibres).

The Picard number of \(N_\lambda\) is generically 19: this can be seen by constructing an elliptic fibration on it with two bad fibres each of type \(I_{0,1}, I_{0,2}, I_2\), and a section of infinite order. In fact, the Picard lattice of \(N_\lambda\) is a sublattice of index 2 of that of the Kummer surface of the square of an elliptic curve without complex multiplication. The surface \(N_\lambda\) is isogenous to the Kummer surface of \(E_\lambda \times E_\lambda\), where \(E_\lambda\) is an elliptic curve with \(j\)-invariant \((-4\lambda^2 + 16)/\lambda^4\) and \(\tau\) is its quadratic twist by \(-\lambda^2 + \lambda\).

It is not so easy to give a useful formula for \([N_\lambda]_p\). To avoid having to do so, we consider the total space \(\mathcal{L}\) of the family of the \(N_\lambda\) inside \(\mathbb{P}(3,1,1,1) \times \mathbb{P}^1\). In other words, we regard \(\lambda\) as the ratio of the two coordinates of \(\mathbb{P}^1\). Let the coordinates on this space be \(t, z_0, z_1, z_2, u, v\): then \(\mathcal{L}\) is defined by the equation
\[t^2 v^3 = u z_0 (v z_0 + u z_2) z_1 (z_0 + 2z_1 - z_2) (z_0 + 2z_1 - z_2) (-v z_0 - 2v z_1 + (u + v) z_2).\]
We define a rational map \(\rho : \mathcal{L} \to \mathbb{P}^1\) by \((2z_1 - z_2 : z_2)\). It is easily checked that the base scheme consists of two rational curves that meet in a single point; it thus has
2p + 1 points mod p for all p. As in Proposition 7 we will use a different fibration on $\mathcal{L}$ to count its $\mathbb{F}_p$-points.

**Proposition 9.** Let $x \in \mathbb{F}_p$ with $x^3 - x \neq 0$ and let $\rho_x$ be the fibre of $\rho$ at $(x : 1)$. Then $\rho_x$ has $p^2 + 4p + 1 + \phi(x^3 - x)a_{3, p}$ points. The fibres of $\rho$ at $0, \pm 1, \infty$ have $p^2 + 3p + 1, 2p^2 + 2p + 1, 2p^2 + 2p + 1$ points respectively.

**Proof.** We consider the affine patch of $\rho_x$ where $z_2, v$ are nonzero. We may view this patch of $\rho_x$ as being inside $\mathbb{A}^3$, which in turn we think of as the affine patch of $\mathbb{P}(3, 1, 1, 1)$ where the last coordinate is nonzero. The projective closure is defined by the equation

$$t^2 = \frac{-x^2 + 1}{4}z_0z_1z_2(z_0 + xz_2)(z_0 - z_1)(z_0 - z_1 + xz_2).$$

Replacing $z_2, t$ by $z_2/x, t/2x$ and multiplying through by $(x/2)^2$ converts this to

$$(6) \quad t^2 = (x^3 + x)z_0z_1z_2(z_0 + z_2)(z_0 - z_1)(z_0 - z_1 + z_2),$$

and replacing $z_0 - z_1$ by $z_1$ converts $z_1$ to $z_0 - z_1$ and hence changes this to the equation for $K$, twisted by $-x^3 + x$, up to the order of variables.

Note further that if $p \equiv 1 \mod 4$ then $\phi(x^3 - x) = \phi(-x^3 + x)$, and if $p \equiv 3 \mod 4$ then $a_{3, p} = 0$, so we may replace $\phi(-x^3 + x)$ by $\phi(x^3 - x)$ in both cases. With these observations, the proof for the general fibres reduces to routine bookkeeping.

As for the bad fibres of $\rho$, the fibre at 1 consists of two components, one supported at $t = z_1 - z_2 = 0$ and one at $z_1 - z_2 = v = 0$. The total number of points is $p^2 + 2p + 1 + p^2 + p + 1 - (p + 1) = 2p^2 + 2p + 1$; similarly for $\rho_{-1}$ with $z_1 - z_2$ changed to $z_1$.

The fibre at $\infty$ has components at $t^2v + (z_0z_1(z_0 + z_1))^2u = z_2 = 0$ and $z_2 = v = 0$. To count the points on the first of these, note that for fixed $t, z_0, z_1$ and $z_2 = 0$ we get one solution for $u, v$ if $t \neq 0$ or $z_0z_1(z_0 + z_1) = 0$ and $p + 1$ otherwise. Thus the total is $p^2 + p + 1 + 3p = p^2 + 4p + 1$. The two components intersect along $z_2 = v = z_0z_1(z_0 + z_1)$, so they share $3p + 1$ points. The second component has $p^2 + p + 1$ points, so the total is $2p^2 + 2p + 1$.

Finally, the fibre at 0 is defined in $\mathbb{P}(3, 1, 1, 1) \times \mathbb{P}^1$ by

$$4t^2v^3 = z_0^2z_1^2z_2^2(z_0^2u^2w^2 + 2z_0z_2uv^2 + z_2^2(u^3 + 2w^2v)), \quad 2z_1 - z_2 = 0.$$  

We count its points with the help of the projection to $\mathbb{P}(3, 1, 1, 1)$. It is readily checked that the fibre at 0 is supported on a smooth rational curve, giving $p + 1$ points, and that the fibre at $\infty$ is supported on two smooth rational curves that meet at $(1 : 0 : 0 : 0)$, giving $2p + 1$ points. The fibre at $(\alpha : 1)$ consists of two rational curves that meet at 3 rational points. The components are defined over $\mathbb{F}_p(\sqrt{\alpha})$, so we find $p + 1 + (p - 2)\phi(\alpha)$ points. The map has empty base scheme, and $\phi(\alpha)$ is +1 and −1 equally often, so the total number of points is $p(p + 1) + 2p + 1 = p^2 + 3p + 1$ as claimed. □

**Corollary 2.** Let $E$ be the elliptic curve $y^2 = x^3 - x$ with the involution $(x, y) \rightarrow (x, -y)$. Then $\mathcal{L}$ and $(M \times E)/\sigma$ are birationally equivalent.

**Proof.** This follows from the equation (6) for the fibre of $\rho_x$. □

**Corollary 3.** The total space $\mathcal{L}$ has $p^3 + 3p^2 - 3p + 1 - a_{4, p} - pa_{2, p}$ points over $\mathbb{F}_p$. 


Proof. By the proposition, there are \((p - 3 - a_{2,p})/2\) fibres with \(p^2 + 4p + 1 + a_{3,p}\) points and \((p - 3 + a_{2,p})/2\) with \(p^2 + 4p + 1 - a_{3,p}\) points, in addition to the points of the bad fibres. The base scheme having \(2p + 1\), the total number of points is then

\[
\frac{1}{2}((p - 3 - a_{2,p})(p^2 + 4p + 1 + a_{3,p}) + (p - 3 + a_{2,p})(p^2 + 4p + 1 - a_{3,p})) \\
+ p^2 + 3p + 1 + 3(2p^2 + 2p + 1) - p(2p + 1).
\]

Simplifying and applying Lemma 3 gives this result. □

6. The first example: level 8

Our first example of a modular fivefold of level 8 is the double cover \(V_8\) of \(P^5\) defined by the equation

\[
t^2 = \prod_{i=0}^{5} x_i(x_i + x_{i+1}).
\]

This equation is reminiscent of the first arrangement of 8 hyperplanes in \(P^3\) given in the table of [18, p. 68], which likewise defines a variety that is modular of level 8. However, the pattern does not persist beyond dimension 5.

Remark 4. The group of automorphisms of the double cover \(V_8 \to P^5\) is of order 24: it is generated by the maps \(V_8 \to V_8\) taking \((t : x_0 : \cdots : x_5)\) to

\[
(-t : x_0 : \cdots : x_5), \quad (t : x_1 : \cdots : x_5 : x_0), \quad (t : x_5 : x_4 : \cdots : x_0).
\]

This is checked by verifying that the only automorphisms of the dual \(\bar{P}^5\) preserving the 12 points corresponding to the 12 hyperplanes are the obvious ones.

We will prove its modularity in the following precise form.

Definition 14. Let \(a_p, b_p\) be the Hecke eigenvalues for the unique newforms of level 8 and weight 6, 4 respectively. (In the LMFDB [16] these are 8.6.a.a and 8.4.a.a.)

Theorem 5.

\[
[V_8]_p = p^5 + p^4 + p^3 + p^2 + p + 1 - a_p - (b_p + \phi(-1)p)p.
\]

Remark 5. This is related to a recent result of Li, Long, and Tu [15, Theorem 5, first row of table]. They identify a Galois representation for which the trace of Frobenius is given by the same formula \(a_p + (b_p + \phi(-1)p)\). The methods of proof are also related, since their expression of a value of \(6F_5\) in terms of a sum over products of \(3F_2\) can be interpreted in terms of the fibration that we will discuss in the next section.

6.1. Proof of modularity. We will prove Theorem 5 by means of a fibration by quotients of products of two K3 surfaces. By means of results of Section 5, we will relate these to the Kummer surface of the square of an elliptic curve and use this to express the number of points in terms of hypergeometric functions over a finite field, thus reducing to results of [11, especially Theorem 1.1]. The statements in this section have been proved; but, since it is easy to make mistakes in such things, we also verify most of the statements of this section numerically for small \(p\) in the file code-level-8.mag in [17] by counting points directly and with the aid of the various fibrations and other geometric constructions.
Definition 15. Let $\pi$ be the rational map $V_8 \dashrightarrow \mathbb{P}^1$ defined by $(x_0 : x_3)$.

The justification for this definition is that if we set $x_0 = \lambda, x_3 = 1$ in the equations of the 12 hyperplanes along which the double cover $V_8 \dashrightarrow \mathbb{P}^5$ is branched, we can divide them into two sets of 6 linear forms, one depending only on $\lambda, x_1, x_2$, the other only on $\lambda, x_4, x_5$.

Definition 16. Let $F_\lambda$ be the affine patch $x_3 \neq 0$ of the fibre of $\pi$ above $(\lambda : 1)$.

Proposition 10. For $\lambda \in \mathbb{F}_p$ with $\lambda \neq 0, -1$, the fibre of $\pi$ at $(\lambda : 1)$ is birationally equivalent to $(K_\lambda \times L_\lambda)/\sigma$, where $\sigma$ is the automorphism

$$((v : z_0 : z_1 : z_2), (w : y_0 : y_1 : y_2)) \mapsto ((-v : z_0 : z_1 : z_2), (-w : y_0 : y_1 : y_2)).$$

Proof. As above, if we substitute $x_0 = \lambda, x_3 = 1$ in the equations for $V_8$, the variables separate and the 12 linear forms can be expressed as 6 in $x_1, x_2$ and 6 in $x_4, x_5$. Thus there is an obvious map $\tau$ of degree 2 from the product of the two double covers of $\mathbb{A}^2$ branched along these loci to the fibre of $\pi$, and $\tau \circ \sigma = \tau$. The result follows by taking the projective closure.

Proposition 11. For all odd primes $p$ and all $\lambda \neq 0, -1 \in \mathbb{F}_p$ we have $[F_\lambda]_p = [A_\lambda]_p[B_\lambda]_p - p^2[A_\lambda]_p - p^2[B_\lambda]_p + 2p^4$.

Proof. Apply Lemma 2 to Proposition 10.

Combining Corollary 4 with Proposition 11 and Lemma 2 (together with the observation that $[F_0]_p = p^4$, because $x_0 = 0$ is one of the hyperplanes and so there is one $\mathbb{F}_p$-point of $F_0$ for each point of $\mathbb{A}^2(\mathbb{F}_p)$), we immediately obtain the following:

Corollary 4. Let $\lambda \neq -1$. Then $F_\lambda$ has $p^4 + \phi(\lambda)(a_{\lambda,p}^2 - p)^2$ points.

In the case $\lambda = -1$, the separation of the variables still allows us to write $[F_\lambda]_p$ in terms of a K3 surface and its quadratic twist by $\lambda$. The only difference is that we need to use Proposition 3 instead of Lemma 3 and Corollary 4.

Proposition 12. On $F_{-1}$ there are $p^4 + (2p - a_{-1,p}^2)^2$ points for $p \equiv 1 \text{ mod } 4$ and $p^4$ for $p \equiv 3 \text{ mod } 4$.

Proof. The count of points on $F_{-1}$ follows by applying Proposition 11 to the result of Proposition 3 with the help of Lemma 4 (which clearly applies in the case $\lambda = -1$ now that the necessary surfaces have been defined), as in Corollary 4.

We are now ready to prove Theorem 5. For simplicity we will only write out the proof in the case of $p \equiv 3 \text{ mod } 4$; the case $p \equiv 1 \text{ mod } 4$ is very similar but requires slightly more work to keep track of the $\lambda = -1$ terms.

Proof. With $p \equiv 3 \text{ mod } 4$, the formula in Theorem 5 becomes

$$\sum_{i=0}^{5} p^i - a_p - pb_p + p^2$$

(recall that $a_p, b_p$ are the Hecke eigenvalues for the newforms of weight 8 and level 6, 4 respectively). By [11, Theorem 1.1], for $p \equiv 3 \text{ mod } 4$ we have

$$a_p = -p^4 \sum_{\lambda=2}^{p-1} \phi(-\lambda)F_2(\lambda)^2 + p^2 - pb_p,$$
so we need to show that

\[ [V_8]_p = \sum_{i=0}^{5} p^i - p^4 \sum_{\lambda=2}^{p-1} \phi(-\lambda) F_2(\lambda)^2. \]

We count the points on \( V_8 \) by means of the fibration \( \pi \). The hyperplane \( x_3 = 0 \) has \( p^4 + p^3 + p^2 + p + 1 \) points, in bijection with those of the hyperplane \( x_3 = 0 \) in \( \mathbb{P}^4 \). The affine patch \( x_3 \neq 0 \) of the fibre at 0 has \( p^4 \) points. So, by Corollary 4 and Proposition 12 the total number of points is

\[ \sum_{i=0}^{4} p^i + 2p^4 + \sum_{\lambda=1}^{p-2} p^4 + \phi(\lambda)(a_{\lambda,p}^2 - p)^2. \]

Combining the \( p^4 \) terms, changing \( \lambda \) to \(-\lambda\) in the summation, and using (1) shows that this is the same as the right-hand side of (8). This completes the proof of Theorem 5 in the case \( p \equiv 3 \mod 4 \). \( \square \)

6.2. Constructing an apparent rigid Calabi-Yau fivefold from \( V_8 \). Following a method of Burek [4], we will construct a fivefold realizing the same newform of weight 6 and level 8 as a quotient of \( V_8 \). Recall that there is an action of \( D_6 \) on \( \mathbb{P}^5 \) that preserves the set of components of the branch locus of the map \( V_8 \to \mathbb{P}^5 \). We will consider the quotients \( Q_1, Q_2, Q_3 \) of \( V_8 \) by representatives \( \iota_1, \iota_2, \iota_3 \) of each of the three conjugacy classes of involutions in \( D_6 \). All of these involutions commute with the map that exchanges the sheets of the double cover, so the quotients are still double covers of a quotient of \( \mathbb{P}^5 \): let these quotients be \( R_1, R_2, R_3 \).

By computing \([Q_1]_p, [Q_2]_p\) for small \( p \) we will obtain information about the action of the \( \iota \) on the cohomology of a resolution, leading us to believe that \( Q_3 \) realizes the newform in question. In Theorem 6 we will prove a formula expressing \([Q_3]_p\) in terms of powers of \( p \), the class of \( p \mod 4 \), and the Fourier coefficient \( a_p \) of the newform. In the file \texttt{quotient-level8.mag} in [17] we verify the statements of Proposition 13, Conjecture 1, Theorem 6 for small \( p \) by comparing the formulas we give to the number of points obtained directly by counting.

**Remark 6.** We expect that \( Q_3 \) has a resolution which is a strongly rigid Calabi-Yau fivefold, but are unable to prove this.

The differential \( D \) on \( \mathbb{P}^5 \) given by

\[ (-1)^j \prod_{\substack{i=0 \atop i \neq j}}^{5} x_i \left( \prod_{\substack{i=0 \atop i \neq j}}^{5} d(x_i/x_j) \right) \]

([13] Remark III.7.1.1) is independent of \( j \) and has divisor \(-\sum H_i \), where \( H_i \) is the hyperplane \( x_i = 0 \). Pulling it back to \( V_8 \), we get a differential whose divisor is \( R - 2\sum H_i \), where \( R \) is the ramification locus. This is the divisor of \( t/\prod_{i=0}^{5} x_i \), so we obtain a differential

\[ (-1)^j t^6 \prod_{\substack{i=0 \atop i \neq j}}^{5} d(x_i/x_j) \]

on \( V_8 \) whose divisor is 0. If \( V_8 \) admits a crepant resolution \( \tilde{V}_8 \), then \( D \) pulls back to a differential on \( \tilde{V}_8 \) with divisor 0, and so we can compute the action of \( \iota_4 \) on \( H^{5,0}(V_8) \) from its action on \( D \). Since we do not know this (indeed, we believe that...
it is false, for reasons explained in Section 6.4, this computation can be used only as motivation, not in any proofs.

To see that \( t_1' (D) = -D \), note that \( t_1 / \prod_{i=0}^5 x_i \) is invariant under \( t_1 \), while exchanging two variables changes the sign of \( D \) (choose an expression for \( D \) in which \( x_j \) is not one of the two variables). Since \( t_1 \) gives an odd permutation, it acts as \(-1\) on the pullback of \( D \) to \( V_5 \) and we do not expect to see the form of weight 6 in the cohomology of the quotient.

First, we examine the central element \( t_1 : x_i \mapsto x_{i+3} \). The invariant ring for the action of \( t_1 \) on \( \mathbb{P}^5 \) is generated by the polynomials \( x_i + x_{i+3}, x_i^2 + x_{i+3}^2 \) for \( 0 \leq i \leq 2 \) and \( x_i x_j + x_{i+3} x_{j+3} \) for \( 0 \leq i < j \leq 2 \). Thus we may view the quotient map \( \mathbb{P}^5 \to R_1 \) as a map to a subvariety of \( \mathbb{P}(1,1,1,2,2,2,2,2,2) \). None of the 12 hyperplanes is fixed by the involution, so we obtain 6 branch divisors, each of which is defined by an equation of degree 2.

**Proposition 13.** For \( p \) an odd prime less than 20, both \( Q_1 \) and \( R_1 \) have \( \sum_{i=0}^5 p^i \) points over \( \mathbb{F}_p \).

**Proof.** First find single polynomials \( s_i \) defining each of the branch components as a subvariety of \( R_1 \); then, for each \( p \), enumerate the \( \mathbb{F}_p \)-points of \( R_1 \), evaluate the product of the \( s_i \) at each, and sum the Kronecker symbols to obtain the point count of \( Q_1 \). All of this is easily done in Magma when \( p \) is small. \( \square \)

**Remark 7.** It would be tedious but not difficult to prove this proposition for all odd primes \( p \). There being no lack of varieties already known to have \( \sum_{i=0}^5 p^i \) points over \( \mathbb{F}_p \) for all \( p \), such a result would be of little interest, and so we have not done this.

Next we consider \( t_2 : x_i \mapsto x_{3-i} \). As an involution of \( \mathbb{P}^5 \) this is conjugate to \( t_1 \), and again it gives an odd permutation of the variables and hence acts as \(-1\) on \( D \), but it is different as an automorphism of \( V_5 \). Indeed, it fixes 2 of the 12 components of the branch locus, so we get 7 branch divisors, of which 5 have degree 2 and 2 have degree 1. Again we find that \( [R_2]_p = \sum_{i=0}^5 p^i \), but this time \( [Q_2]_p = \sum_{i=0}^5 p^i - pb_p \) for \( p < 20 \), where as before \( b_p \) is the Hecke eigenvalue for the newform of weight 4 and level 8. This suggests that \( t_2 \) acts as \(+1\) on \( H^{4,1} \oplus H^{1,4} \) and as \(-1\) on \( H^{5,0} \oplus H^{0,5} \).

**Conjecture 1.** \( [Q_2]_p = \sum_{i=0}^5 p^i - pb_p \) for all odd \( p \).

**Remark 8.** Again, the methods that prove Theorem 8 could be applied to prove Conjecture 1, but this seems much less novel than Theorem 8 and so we have not carried it out.

Let \( t_3 = t_1 t_2 \). In view of Proposition 13 and Conjecture 1, we expect that \( t_3 \) acts as \(+1\) on \( H^{5,0} \oplus H^{0,5} \) of a resolution of \( V_5 \) and as \(-1\) on \( H^{4,1} \oplus H^{1,4} \). This time the ring of invariants is generated by \( x_0, x_1 + x_5, x_2 + x_4, x_3, x_1^2 + x_5^2, x_2^2 + x_4^2, x_1 x_2 + x_3 x_5 \), so the quotient map from \( \mathbb{P}^5 \) goes to \( \mathbb{P}(1,1,1,2,2,2,2) \); the image is a hypersurface \( H \) of degree 4. Two of the 12 components of the branch locus are fixed by \( t_3 \) and map to divisors in \( H \) cut out by equations of degree 1; the other 10 are exchanged in pairs and map to divisors defined by equations of degree 2. Numerically this suggests a singular Calabi-Yau variety: the canonical divisor of \( H \) would be \( O(-4-1-3-2+4) = O(-6) \), while the branch locus has class \( O(12) = -2K_H \), but in view of the large singular locus of \( H \) and the branch divisors this is not a proof. By directly counting points we obtain the following formula, which we will prove in the next section.
Theorem 6. \([Q_3]_p = \sum_{i=0}^5 p^i - a_p - \phi(-1)p^2\) for all odd \(p\), where as before \(a_p\) is the eigenvalue of \(T_p\) on the newform of weight 6 and level 8.

This and Propositions 13, 15, 17 are verified numerically in the file \texttt{count-quotient.mag} in \texttt{[L7]} by comparing the formulas we give to point counts computed directly from equations for the varieties.

6.3. Proof of Theorem 6. Since \(\iota_3\) preserves the fiberization \((x_0 : x_3)\), the natural approach is to study the fibers of the map \(\pi_3 : Q_3 \to \mathbb{P}^1\) given by \((x_0 : x_3)\). These are in fact the quotients of the fibers of \(\pi\) by \(\iota_3\). Let \(Q_{3,\lambda}\) be the fiber of \(Q_3\) above \((\lambda : 1)\). We will count the points of \(Q_{3,\lambda}\) in terms of \(P_{A,1}\), etc. In particular we show:

Proposition 14. Let \(\lambda \neq 0, -1\), and suppose that \(\phi(p) = 1\). Then
\[
[Q_{3,\lambda}]_p = P_{A,1}^2 + P_{A,1} + N_{A,1}^2 + N_{A,1} + (P_{A,2} - P_{A,1} - N_{A,1}) + P_{A,1}Z_{A,1} + N_{A,1}Z_{A,1} + Z_{A,1}(Z_{A,1} + 1)/2 + 5p(p - 1)/2.
\]

Remark 9. The \(P_{A,1}\), etc., are determined in Proposition 5 while Remark 3 explains how to use that proposition to evaluate \(P_{A,2}\).

Proof. This comes down to considering the pairs of \(\mathbb{F}_p\)-points on \((A_\lambda \times B_\lambda)/\pm 1\) whose images on \(Q_{3,\lambda}\) are rational. Let \(\mu_\lambda\) be the \(\mathbb{F}_p\)-isomorphism \(K_\lambda \to L_\lambda\) taking \((v : z_0 : z_1 : z_2)\) to \((rw : y_0 : y_2 : y_1)\), where \(r\) is a fixed square root of \(\lambda\). Given a pair of points of \(A^2(\mathbb{F}_p)\) or \(A^2(\mathbb{F}_p^2)\), we may pull them back to \(A_\lambda \times B_\lambda\) and determine whether their inverse images map to rational points of \(Q_{3,\lambda}\). This occurs in the following cases:

1. A \(P\)-point \(p_A\) of \(A\) and a distinct \(P\)-point \(p_B \neq \mu_\lambda(p_A)\) of \(B\). Such a pair gives 2 points of \(Q_{3,\lambda}\); however, the point \((\mu_\lambda^{-1}(p_B), \mu_\lambda(p_A))\) of \(A_\lambda \times B_\lambda\) gives the same points of \(Q_{3,\lambda}\), so the contribution from this source is \(P_{A,1}(P_{A,1} - 1)\).

2. An \(N\)-point \(n_A\) of \(A\) and a distinct \(N\)-point \(n_B \neq \mu_\lambda(n_A)\) of \(B\). Similarly, these give \(N_{A,1}(N_{A,1} - 1)\) points.

3. A \(P\)-point \(p_A\) of \(A\) and the matching \(P\)-point \(\mu_\lambda(p_A)\) of \(B\). Each of these gives 2 points of \(Q_{3,\lambda}\), for a total of \(2P_{A,1}\).

4. An \(N\)-point \(n_A\) of \(A\) and the matching \(N\)-point \(\mu_\lambda(n_A)\) of \(B\). Similarly, these give us \(2N_{A,1}\).

5. A \(P\)-point \(p_{A,2}\) of \(A\) over \(\mathbb{F}_p\) and \(\mu_\lambda\) of its conjugate on \(B\). Since we must exclude the \(\mathbb{F}_p\)-rational points, the number of \(P\)-points available is \(P_{A,2} - P_{A,1} - N_{A,1}\) (recall that every element of \(\mathbb{F}_p\) is a square in \(\mathbb{F}_p^2\)). Each gives 2 points of \(Q_{3,\lambda}\), but a point and its conjugate give the same points of \(Q_{3,\lambda}\), so these give \(P_{A,2} - P_{A,1} - N_{A,1}\) points of \(Q_{3,\lambda}\).

6. A \(Z\)-point of \(A\) and a \(P\)- or \(N\)-point of \(B\). Because a \(Z\)-point is involved, we are in the branch locus of \(Q_{3,\lambda}\), and each such point gives 1 point of \(Q_{3,\lambda}\). Thus the number is \(Z_{A,1}(P_{A,1} + N_{A,1})\). Note that the images of a \(P\)- or \(N\)-point of \(A\) and a \(Z\)-point of \(B\) are subsumed here, the involution \(\iota_3\) having the effect of interchanging \(A_\lambda\) and \(B_\lambda\).

7. A \(Z\)-point \(z_A\) of \(A\) and a different \(Z\)-point \(z_B \neq \mu_\lambda(z_A)\) of \(B\). Again, switching these does not change the image, so we obtain a contribution of \((Z_{A,1})^2\).
(8) A Z-point $z_A$ of $A$ and the corresponding Z-point $\mu_{\lambda}(z_A)$ of $B$. There are $Z_{A_{\lambda,1}}$ possibilities.

(9) A Z-point of $A$ whose coordinates generate $F_{p^2}$ and the conjugate of its $\mu_{\lambda}$-image on $B$. Every line over $F_p$ has $p^2 - p$ such points and there are 5 in the branch locus; we must divide by 2 since a point and its conjugate map to the same point of $Q_{3,\lambda}$. So the total is $(5p^2 - 5p)/2$.

Adding up the counts from each type gives the result claimed. □

A very similar result holds for $\lambda$ not a square.

**Proposition 15.** Let $\lambda \neq 0, -1$, and suppose that $\phi(p) = -1$. Then

$$[Q_{3,\lambda}]_p = P_{A_{\lambda,1}}P_{B_{\lambda,1}} + N_{A_{\lambda,1}}N_{B_{\lambda,1}} + P_{A_{\lambda,1}}Z_{A_{\lambda,1}} +$$

$$N_{A_{\lambda,1}}Z_{A_{\lambda,1}} + Z_{A_{\lambda,1}}(Z_{A_{\lambda,1}} + 1)/2 + 5p(p - 1)/2.$$

**Proof.** This is identical to the proof of Proposition 14 except for two changes resulting from the fact that $\mu_{\lambda}$ is defined only over $F_{p^2}$. This leads to $P$-points of $B_{\lambda}$ over $F_p$ matching with $N$-points of $A_{\lambda}$, rather than $P$-points, and to $N$-points of $A_{\lambda}$ over $F_{p^2}$ contributing to the count instead of $P$-points. □

Finally we state the results for $\lambda = -1, 0, \infty$. For $\lambda = 0, \infty$, the fibre $Q_{3,\lambda}$ is contained in the branch locus and hence has $p^2 + p^2 + p^2 + p^2 + p + 1$ points. For $\lambda = -1$, the method is the same as in Propositions 14 and 15.

**Proposition 16.** We have $[Q_{-1}]_p = p^4 + \phi(\lambda)(a_{\lambda,p}^2 - p)^2 - p^2$.

We now have sufficient information to count the $F_p$-rational points of $Q_3$. To avoid duplicating the work already done to prove Theorem 5, we start by comparing the point counts on the fibres of $V_8$ to those on the fibres of $Q_3$.

**Proposition 17.** For $\lambda = 0, \infty$ we have $[F_{\lambda}]_p = [Q_{3,\lambda}]_p$, while $[Q_{-1}]_p - p^2 = [Q_{-1}]_p$. For other values of $\lambda$, the equality $[F_{\lambda}]_p - \phi(\lambda)p(a_{\lambda,p}^2 - p) = [Q_{3,\lambda}]_p$ holds.

**Proof.** Using the results of Corollary 4 and Propositions 14, 15, this is routine. For example, let us suppose that $\lambda \neq -1$ is a quadratic nonresidue mod $p$. In this case we have $[F_{\lambda}]_p = p^4 - (a_{\lambda,p}^2 - p)^2$. In addition, we have $P_{A_{\lambda,1}} = N_{B_{\lambda,1}} = (p^2 - 6p + 7 + a_{\lambda,p}^2)/2$, while $Z_{A_{\lambda,1}} = Z_{B_{\lambda,1}} = 5p - 7$ and $N_{A_{\lambda,1}} = P_{B_{\lambda,1}} = (p^2 - 4p + 7 - a_{\lambda,p}^2)/2$. Since $\lambda$ is a square in $F_{p^2}$, we have $N_{A_{\lambda,2}} = N_{B_{\lambda,2}} = (p^4 - 4p^2 + 7 - a_{\lambda,p}^2)/2$, and $a_{\lambda,p}^2 = a_{\lambda,p}^2 - 2p$ by standard results on elliptic curves over finite fields. Substituting these expressions into Proposition 15 shows that

$$[Q_{3,\lambda}]_p = p^4 - 2p^2 + 3pa_{\lambda,p}^2 - a_{\lambda,p}^4 = [F_{\lambda}]_p - p(a_{\lambda,p}^2 - p)$$

as desired. □

Summing over $\lambda$, we conclude that $[Q_3]_p = [V_8]_p - p \sum_{\lambda=1}^{p-2} \phi(\lambda)(a_{\lambda,p}^2 - p) - p^2$. Substituting the expression for $[V_8]_p$ from Theorem 5 and that for $b_p$ from Proposition 7 completes the proof of Theorem 6.

### 6.4. Geometry and Calabi-Yau resolutions

Having shown that $V_8$ is modular, we now consider the question of whether it is birationally equivalent to a Calabi-Yau fivefold. Such problems were considered by Cynk and Hulek [5], who gave a sufficient condition for a double cover of a smooth variety to admit a crepant resolution. We proceed to state their result and to explain why it does not apply to $V_8$. 
Definition 17 ([5 Section 5]). Let \( \{D_i\}_{i=1}^n \) be a set of smooth divisors on a variety \( V \) and let \( S \subseteq \{1, \ldots, n\} \) be a nonempty subset such that \( \cap_{i \in S} D_i \notin D_j \) for all \( j \notin S \). We say that the intersection \( \cap_{i \in S} D_i \) is near-pencil if there is a single element \( s \in S \) such that \( \cap_{i \in S} D_i \neq \cap_{i \in S \setminus \{s\}} D_i \).

Note in particular that \( \cap_{i \in S} D_i \) is automatically near-pencil if \( |S| \leq 2 \). For another example, if \( V = \mathbb{P}^n \), the \( D_i \) are hyperplanes, and the equation defining \( D_1 \) involves a variable not mentioned in any other \( D_i \), then every intersection \( \cap_{i \in S} D_i \) with \( 1 \in S \) is near-pencil.

Cynk and Hulek show:

Proposition 18 ([5 Proposition 5.6]). Let \( V \) be a smooth variety with smooth divisors \( D_1, \ldots, D_n \) such that the sum of the Picard classes of the \( D_i \) is divisible by 2 in \( \text{Pic} V \). For \( S \subseteq \{1, \ldots, n\} \), let \( C_S = \cap_{i \in S} D_i \). Suppose that, for all nonempty \( S \) with \( C_S \neq C_{S \cup \{i\}} \) for all \( i \notin S \), either \( C_S \) is near-pencil or \( |S|/2 = \text{codim} S - 1 \). Then the double cover of \( V \) branched along the union of the \( D_i \) admits a crepant resolution.

We refer to the given condition on \( S \) or \( C_S \) as the Cynk-Hulek criterion. If the condition holds for the intersection of every subset of the \( D_i \) of cardinality greater than 1, we will say that the set of \( D_i \) satisfies the Cynk-Hulek criterion.

To discuss \( V_8 \), we do not need to describe the resolution in detail. It suffices to observe that all subsets of the 12 hyperplanes satisfy the Cynk-Hulek criterion, with the exception of the intersection of the hyperplanes \( x_1 + x_{i+1} = 0 \), which consists of the single point \((-1 : 1 : -1 : -1 : 1)\) and is not near-pencil (the intersection of any five of the six hyperplanes is the same). Combining this with the result of Cynk and Hulek just above, we conclude that if the singularity of \( V_8 \) at \((0 : -1 : 1 : -1 : 1 : -1 : 1)\) admits a crepant resolution, then so does \( V_8 \), and this resolution \( \tilde{V}_8 \) would be a Calabi-Yau fivefold. However, this does not seem likely.

Conjecture 2. The singularity of \( V_8 \) at \((0 : -1 : 1 : -1 : 1 : -1 : 1)\) does not admit a crepant resolution, and hence \( V_8 \) is not a singular Calabi-Yau fivefold.

A heuristic argument for this statement can be given by using the ideas of [7] to calculate the deformation space of a putative Calabi-Yau resolution \( \tilde{V}_8 \). We expect (but cannot prove) that it satisfies \( h^{4,1}(\tilde{V}_8) = \sum_{\text{codim} C_i} h^0(K_{C_i}) \), where the \( C_i \) are the loci blown up to construct the resolution; on the other hand, the \( \mathbb{F}_p \)-point counts suggest that \( h^{4,1}(\tilde{V}_8) = 1 + \sum_{\text{codim} C_i} h^0(K_{C_i}) \).

Finally, we use [14, Theorem 2] to investigate the resolutions of \( Q_3 \).

Proposition 19. The age ([14 Definition 1]) of \( t_3 \) acting on the tangent space of every fixed point is 1.

Proof. Viewed as an automorphism of \( \mathbb{P}^5 \), the fixed locus of \( t_3 \) consists of the linear subspaces \( x_1 - x_5 = x_2 - x_4 = 0 \) and \( x_1 + x_5 = x_0 = x_2 + x_4 = x_3 = 0 \). On the first of these, we may take \( x_1, x_2, x_3, x_4, x_5 \) as a system of local parameters, even on the double cover. Then \( t_3 \) exchanges the tangent vectors in the \( x_2 \) and \( x_4 \) directions, and likewise \( x_1 \) and \( x_5 \), while fixing \( x_3 \); thus its age is 1 there.

On the second, we have \( t = 0 \) on the double cover, so \( t \) must be taken among our local parameters, and we must blow up \( x_0 = x_3 = 0 \). Then we take \( x_0, x_1, x_3, x_4, t \) as our local parameters. It is clear that tangent vectors in the \( x_0, x_3, t \) directions are fixed by \( t_3 \). As for \( x_1 \), such a tangent vector is described by the infinitely near point
(0 : x_0 : x_1 + \epsilon : -x_4 : x_3 : x_4 : -x_1) where \epsilon^2 = 0, which goes by the involution to (0 : x_0 : -x_1 : x_3 : -x_4 : x_1 + \epsilon). Now (-x_1 : x_1 + \epsilon) = (x_1 - \epsilon : -x_1), so the corresponding diagonal entry of the matrix giving the action is -1; similarly for a tangent vector in the x_4 direction. Thus the action of \iota_3 has trace 1 on the 5-dimensional tangent space, and so the -1-eigenspace has dimension 2 and the age is 1 as before. \hfill \Box

In particular \iota_3 satisfies the global Reid-Tai criterion, and so by [13, Theorem 2] the quotient would have Kodaira dimension 0 if V_8 had a Calabi-Yau resolution; such a resolution would exist if not for the singularity at (0 : -1 : 1 : -1 : 1 : -1 : 1), but as in Conjecture 2 we believe that it does not.

Nevertheless, if we blow up only this bad singularity, the intersections of all subsets of the strict transforms meet transversely, so the quotient is well-behaved (unfortunately this is not a crepant blowup). Because of this and Theorem 6, we conjecture:

**Conjecture 3.** Q_3 admits a strongly rigid Calabi-Yau resolution of singularities for which the representation on H^5 coincides with that obtained from the newform of weight 6 and level 8 up to semisimplification.

Unfortunately, it seems difficult to study Q_3 by the methods of [5]. Although Q_3 is still a double cover of a rational variety, the branch locus is now quite complicated, with some collections of components having nonreduced intersection. Thus we are unable to prove Conjecture 3; we emphasize, however, that our proof of modularity is independent of any such considerations and is unconditional.

7. The second example: level 32

In this section we will consider the fivefold V_{32} defined by the equation

(11) \quad t^2 = \prod_{i=0}^{5} (x_i) (x_0+x_1)(x_3+x_5)(x_2+x_4+x_5)(x_0+x_2-x_4)(x_1-x_2+x_4)(x_2-x_3+x_4).

We will show that it realizes the newform of weight 6 and level 32 that has complex multiplication by \mathbb{Q}(i) (see Definition 9 for notation):

**Theorem 7.** [V_{32}]_p = \sum_{i=0}^{5} p^i - a_{6,p} - p a_{4,p} - 2p^2 a_{2,p}.

As with Section 5, the assertions of this section and some that it refers to in Sections 4 and 5 are verified numerically in the file code-level-32.mag in [17]. Most of this is routine: for example, Lemma 8 is just a matter of comparing some products of coefficients and Proposition 8 is a simple application of Lemma 1. However, Proposition 20 involves counting the points on a toric variety that is better not embedded in projective space, and Proposition 21 requires some care to account for all of the exceptions in the statement.

7.1. Proof of modularity. We will prove Theorem 7 in a manner suggested by Lemma 8. Namely, we will start by writing a fibration on V_{32} whose fibres are quotients of products of two K3 surfaces. One of these is always the K3 surface M of Picard rank 20 and discriminant -4, whose point count over \mathbb{F}_p is controlled by the a_{3,p} (Lemma 8). The other varies in a family whose total space (the \mathcal{L} of Section 8) is related to M \times E_{32}, where E_{32} is an elliptic curve of conductor 32. Its
number of $\mathbb{F}_p$-points is therefore related to $a_{3,p}a_{2,p}$ and hence to $a_{4,p}$. Thus $[V_{32}]_p$ involves $a_{4,p}a_{3,p}$ and therefore $a_{6,p}$.

As in Section 6, we begin by partitioning the twelve hyperplanes into the branch locus into two sets of six, each set intersecting in a line. In particular, the set of linear forms $\{x_3, x_5, x_6 + x_1, x_3 + x_5, x_2 + x_4 + x_5, x_2 - x_3 + x_4\}$ spans the space generated by $x_2 + x_4, x_0 + x_1, x_3, x_5$, while its complement in the set of 12 linear forms defining components of the branch locus spans $\langle x_0 + x_1, x_2 + x_4, x_0, x_2 \rangle$.

**Definition 18.** Define a rational map $\pi : V_{32} \dashrightarrow \mathbb{P}^1$ by $(x_0 + x_1 : x_2 + x_4)$. We will also view $\pi$ as a map $\mathbb{P}^5 \dashrightarrow \mathbb{P}^1$.

As before, the general fibre is a quotient of the product of two K3 surfaces. As in Definition 16 we describe the first of these by writing the linear form $ax_3 + bx_5 + c(x_0 + x_1) + d(x_2 + x_4)$ as $ax + by + (c\lambda + d)z$. This gives six linear forms $x, y, \lambda z, x + y, y + z, -x + z$ from which we obtain a K3 surface $k_\lambda$ defined by the equation obtained by setting $t^2$ equal to their product. Similarly, for the other six we write $ax_0 + bx_2 + c(x_0 + x_1) + d(x_2 + x_4)$ as $ax + by + (c\lambda + d)z$, obtaining $x, x + x_2 z, y, -y + z, x + 2y - z, -x - 2y + (\lambda + 1)z$ and define a K3 surface $\ell_\lambda$ by setting $u^2$ equal to their product. However, we are only interested in $(k_\lambda \times \ell_\lambda)/\sigma$, where $\sigma$ is the involution that changes the signs of $t, u$. This does not change if we replace $\lambda z$ by $z$ in the definition of $k_\lambda$ and $y$ by $\lambda y$ in that of $\ell_\lambda$. Thus instead of $k_\lambda$ and $\ell_\lambda$, we may use $M, N_\lambda$, as defined in Definition 13.

**Definition 19.** Let $E = E_{32}$ be the elliptic curve of conductor 32 whose affine equation is $y^2 = x^3 - x$ (this was previously referred to as $E_{-1}$, because we wanted to emphasize its role in the fibre of $\pi$ at $-1$). We use $-\lambda$ to denote the automorphism of $K$ defined by $(t : x : y : z) \rightarrow (-t : x : y : z)$, and also the automorphism of $E_{32}$ taking $(x : y)$ to $(x : -y)$.

**Proposition 20.** $V_{32}$ is birationally equivalent to $(M \times M \times E)/\Sigma$, where $\Sigma$ is the group of automorphisms acting as $-1$ on an even number of the factors and $+1$ on the remaining ones.

**Proof.** Since $\mathcal{L}$ is the total space of the $N_\lambda$, our discussion above shows that $V_{32}$ is birational to $(M \times \mathcal{L})/(-, -)$. The result follows from Corollary 2. \hfill $\square$

This will be used in Section 7.2 to construct a rigid Calabi-Yau quotient of $V_{32}$. It is also instructive to compare to Lemma 2. Indeed, $e_{32, +} - e_{32, -} = a_{2,p}$, where $e_{32, \pm}$ are as in Lemma 2, and for a suitable model of $M$ we have $k_+ - k_- = a_{3,p}$.

First we use Proposition 20 together with Lemma 2 to count the $\mathbb{F}_p$-points of $V_{32}$. It follows from Lemma 2 that $(M \times L_\lambda)/\sigma$ has $(p^2 + p + 1)^2 + (P_{M,1} - N_{M,1})(P_{\lambda,1} - N_{\lambda,1})$ points for $\lambda \in \mathbb{F}_p$. On the other hand, we may use the birational equivalence of this with the fibre of $\pi$ at $\lambda$ to count the points on the fibre. In the following, let $y_0, y_1, y_2, z_0, z_1, z_2$ be coordinates on $\mathbb{P}^2 \times \mathbb{P}^2$.

**Proposition 21.** Fix $\lambda \in \mathbb{F}_p^*$, and let $\mu$ be the rational map from the hyperplane in $\mathbb{P}^5$ defined by $x_0 + x_1 = \lambda(x_3 + x_4)$ to $\mathbb{P}^2 \times \mathbb{P}^2$ given by $(x_3 : x_5 : x_2 + x_4), (x_0 : x_2 : x_2 + x_4))$. Then $\mu$ induces a rational map from the fibre of $\pi$ at $(\lambda : 1)$ to $(M \times N_\lambda)/\sigma$. Further, it induces a bijection between the sets of $\mathbb{F}_p$-points of these schemes, with the following exceptions:

1. Points of the fibre of $\pi$ with $x_0 = x_1 = x_2 + x_4 = 0$, or with $x_3 = x_5 = x_2 + x_4 = 0$, do not correspond to any point of $(M \times N_\lambda)/\sigma$. 

(2) Points of the fibre of \( \pi \) with \( x_2 + x_4 = 0 \), but for which \( (x_0 : x_2), (x_1 : x_4), (x_3 : x_5) \) are well-defined points of \( \mathbb{P}^1 \), correspond \( (p - 1) \)-to-1 to points of \( (M \times N_\lambda)/\sigma \) above a point with coordinates \((y_3 : y_5 : 0), (y_0 : y_2 : 0)\). 

(3) Points of \( (M \times N_\lambda)/\sigma \) with \( y_2 = 0, z_2 \neq 0 \), or with \( y_2 \neq 0, z_2 = 0 \), do not correspond to any point of the fibre of \( \pi \).

**Proof.** As discussed above, the map matches the branch loci of the two double covers, so there is a rational map from the fibre of \( \pi \) to \( (M \times N_\lambda)/\sigma \) as described. In case 1, we would obtain a point whose coordinates in one \( \mathbb{P}^2 \) are \((0 : 0 : 0)\). In case 2, if \( x_2 + x_4 = 0 \), then clearly we obtain the point \((x_3 : x_0 : 0), (x_0 : x_2 : 0)\), and this is unchanged by rescaling \((x_0 : x_2)\) by an element of \( \mathbb{F}_p^\times \). On the other hand, if \( x_0 = x_2 = 0 \), then we have already dealt with this point in case 1, and similarly for the other two pairs. Finally, points with \( y_2 = 0, z_2 \neq 0 \) or vice versa cannot be obtained from \( \mu_i \), because \( x_2 + x_4 \) cannot both be 0 and not be 0.

In the other direction, we have an inverse rational map from \( \mathbb{P}^2 \times \mathbb{P}^2 \) to the hyperplane. On the affine patch \( y_2 = z_2 = 1 \), it is given by \((y_0, y_1), (z_0, z_1)\) → \((z_0 : \lambda - z_0 : z_1 : y_0 : 1 - z_1 : y_1)\). Points not on this affine patch are accounted for in cases 2 and 3 above. \( \square \)

**Corollary 5.** The double cover \( (M \times N_\lambda)/\sigma \) of \( \mathbb{P}^2 \times \mathbb{P}^2 \) has \( p(p + 1)^2 + (p - 2)\phi(\lambda)(P_{M,1} - N_{M,1}) \) more points than the fibre of \( \pi \) at \( \lambda \).

**Proof.** We consider the three types of exception in Proposition 21. Case 1 describes two disjoint sets of \( p + 1 \) points in the fibre of \( \lambda \), so \( 2p + 2 \) in total. In case 2 we contract \((p + 1)^2(p - 1)\) points to \((p + 1)^2\).

To understand the third case, note that one of the linear forms defining \( M \) is the third coordinate, so all points with \( x_2 = 0 \) give one point on \( M \) and there are \((p + 1)p^2 \) missed points for which the third coordinate of the point giving \( M \) is 0. On the other hand, setting the third coordinate to 0 in the linear forms defining \( L_\lambda \) gives \( \pm t_0, \lambda t_1, -t_1, \pm(t_0 + 2t_1) \). Thus the product is 0 for 3 points and \( \lambda \) times a nonzero square for the other \( p - 3 \). Where the product is 0, we have \( p^2 \) points. In the double cover we get one point for every \( Z \)-point of \( M \) (cf. Proposition 5) and two points for each \( P \)-point or \( N \)-point, depending on whether \( \lambda \) is a square. This contributes \( p^2(p + 1) + (p - 2)(P_{M,1} - N_{M,1}) \) points if \( \phi(\lambda) = 1 \) and \( p^2(p + 1) + (p - 2)(N_{M,1} - P_{M,1}) \) points if \( \phi(\lambda) = -1 \) to the excess of \([ (M \times N_\lambda)/\sigma ]_P \) over \([ \pi_\lambda ]_P \).

In total, then, the excess is \( p^2(p + 1) + p^2(p + 1) + (p - 2)\phi(\lambda)(P_{M,1} - N_{M,1}) - (2p + 2) - (p + 1)^2(p - 1) \). This simplifies to the formula asserted. \( \square \)

These calculations are not valid for \( \lambda = 0, \infty \). However, it is easy to see that for both of these the fibre has \( p^3 + p^2 + p + 1 \) points. Finally, the base locus of \( \pi \) is defined by \( x_0 + x_1 = x_2 + x_4 = 0 \). On this locus the product of linear forms is 0, so it has \( p^3 + p^2 + p + 1 \) points and we must subtract \( p \) times this from the total number of points on the fibres to obtain the correct point count for \( V_{32} \).

We now assemble all of the ingredients: the comparison of the fibres in Proposition 21 and its Corollary 5, the remarks on special fibres and the base locus just above, the count of points on \( \mathcal{L} \) from Proposition 3 and the relations of coefficients of modular forms of Lemma 3. By routine calculation, we obtain \( [V_{32} ]_P = \sum_{i=0}^5 F^i - a_{6, p} - pa_{4, p} - 2p^2a_{2, p} \) as claimed. This completes the proof of Theorem 7.
7.2. Construction of a rigid Calabi-Yau fivefold of level 32. As in Section 6.2, we will use the method of [4] to construct a candidate for a strongly rigid Calabi-Yau fivefold of level 32. In this case the control given to us by Proposition 20 allows us to construct a strongly rigid Calabi-Yau fivefold as a quotient of 

\[ M \times M \times E \] 

and has dimension 2. This will be explained more precisely in the proofs of Theorem 8 and its Corollary 6.

Remark 10. Since \( V_{32} \) satisfies the Cynk-Hulek criterion (Proposition 15), it admits a crepant resolution by a Calabi-Yau fivefold. The shape of the formula for the number of points suggests that this resolution has \( h^{5,0} = h^{4,1} = 1, h^{3,2} = 2, \) and \( h^{i,j} = 0 \) unless \( i = j \) or \( i + j = 5. \)

This is explained by the birational description. Indeed, \( H^5 \) of the resolution arises from \( H^2_5(M) \otimes H^2_7(M) \otimes H^1(E) \), where \( H^2_5 \) is the transcendental lattice \( H^2(M)/\text{Pic} M \). Thus, for example, \( H^{3,2} \) of the resolution matches

\[
(H^{2,0}(M) \otimes H^{0,2}(M) \otimes H^{1,0}(E)) \oplus (H^{0,2}(M) \otimes H^{2,0}(M) \otimes H^{1,0}(E))
\]

and has dimension 2. This will be explained more precisely in the proofs of Theorem 8 and its Corollary 6.

Let \( G_{64} \) be the group of projective automorphisms of the configuration of 12 hyperplanes used to construct \( V_{32} \), let \( C_2 \) be the cyclic group of order 2, and let \( Z_G \) be the centre of a group \( G \). The group \( G_{64} \) has order 64 and is isomorphic to \( C_2 \times G_{32} \), where \( Z_{G_{32}} \cong C_2^2 \) and \( G_{32} \) fits into an exact sequence \( 1 \rightarrow Z_{G_{32}} \rightarrow G_{32} \rightarrow C_2^3 \rightarrow 1 \).

We refer to \texttt{quotient-level-32.mag} in [17] for verifications of the results of this section. In particular, we investigate the automorphisms numerically, verifying that the point counts of certain quotients of \( V_{32} \) are as claimed, thus justifying our eventual choice of quotient to study more closely. This is slightly difficult because the branch loci of the quotients are not always defined by a single equation in our chosen embedding. Also, the quotients, though easily embedded in weighted projective space, are not so pleasant to work with in ordinary projective space.

We study the quotients of \( V_{32} \) by elements of order 2 with characteristic polynomial \( (x - 1)^4(x + 1)^2 \) in \( G_{64} \) as in Section 6.2. We concentrate on two elements of \( G_{64} \), namely \( \alpha_1 \), taking \((x_0 : x_1 : x_2 : x_3 : x_4 : x_5) \) to \((x_1 : x_0 : x_4 : x_3 : x_2 : x_5) \), and \( \alpha_2 \), defined by \( (x_0 : x_1 : x_2 : x_3 : x_4 : x_5) \rightarrow (x_1 : -x_0 : x_2 : -x_3 : x_4 : -x_5) \).

(Note that \( \alpha_2 \in Z_{G_{64}} \).) In this case, the Cynk-Hulek criterion is satisfied, so we know that the differential

\[
D' = \frac{x_5^2}{t} \bigwedge_{i=0}^{4} dx_i/x_5
\]

(cf. [10], but note that here the argument constitutes a rigorous proof) pulls back to a generator of \( H^{5,0} \) on the quotient.

Now, \( \alpha_1 \) gives an even permutation and therefore does not change the sign of \( \prod_{i=0}^{4} x_i \wedge_{i=0}^{4} dx_i/x_5 \). In addition, it fixes \( x_5/t \), and so it fixes \( D' \). To verify the invariance for \( \alpha_2 \), we use the alternative form \( -\frac{x_5^2}{t} \bigwedge_{i=0}^{5} dx_i/x_2 \) for \( D' \). Negating any variable changes the sign of this, and it is invariant under even permutations that fix \( x_2 \). So it is fixed by \( \alpha_2 \).

When we consider the quotient \( V_{32}/\alpha_1 \), we find that the images of all of the branch divisors are defined by a single polynomial, and so it is easy to write down
the branch function on $\mathbb{P}^5/\alpha_1$ (that is, the function whose square root gives the double cover $V_{32}/\alpha_1 \rightarrow \mathbb{P}^5/\alpha_1$). On the other hand, for $V_{32}/\alpha_2$, all but two of the branch divisors, as well as the union of the two that are not, are defined by single polynomials. In this case, it is again easy to write down the branch function.

For both of these, the quotient $\mathbb{P}^5/\alpha_i$ is a hypersurface of weighted degree 4 in $\mathbb{P}(1, 1, 1, 2, 2, 2)$ as previously. There are other involutions such that exactly one branch divisor on the quotient is not defined by a single polynomial. We would have to be more careful in this situation; however, it does not arise in this paper.

Counting $\mathbb{F}_p$-points on the double covers for small odd $p$, we are led to the following conjecture.

**Conjecture 4.** For all primes $p > 2$ we have $[V_{32}/\alpha_1]_p = [V_{32}/\alpha_1\alpha_2]_p = \sum_{i=0}^5 p^i - a_{6,p} - p^2 a_{2,p}$ and $[V_{32}/\alpha_2]_p = \sum_{i=0}^5 p^i - a_{6,p} - p a_{4,p}$.

Accordingly we expect that $\alpha_1$ acts as $-1$ on $H^{1,4}(\tilde{V}_{32}) \oplus H^{1,4}(\tilde{V}_{32})$ and has eigenvalues 1, $-1$ on $H^{3,2}(\tilde{V}_{32}) \oplus H^{2,3}(\tilde{V}_{32})$, while $\alpha_2$ acts as $+1$ on $H^{2,4} \oplus H^{3,2}$ and as $-1$ on $H^{3,2} \oplus H^{2,3}$; this also confirms our observation that both act as $+1$ on $H^{5,0}$, which also applies to $H^{0,5}$. Also $\alpha_1 \alpha_2$ should satisfy the same description as $\alpha_1$, except that the eigenspaces of $\pm 1$ for $H^{3,2}$ and $H^{2,3}$ are reversed. This is consistent with the fact that $\alpha_1 \alpha_2$ is conjugate to $\alpha_1$ in $G_{64}$. In particular, the $+1$ eigenvalue of $\langle \alpha_1, \alpha_2 \rangle$ on $H^5$ should be neither more nor less than $H^{5,0} \oplus H^{0,5}$, and we expect that the number of $\mathbb{F}_p$-points on the quotient should be expressible in terms of powers of $p$, Artin symbols, and $a_{6,p}$ only.

In order to make rigorous computations of cohomology, we will study $M \times M \times E$ in place of $V_{32}$. Recall from Proposition 20 that $V_{32}$ is birationally equivalent to the quotient of $M \times M \times E$ by a Klein four-group. We will first lift $\alpha_1, \alpha_2$ to $M \times M \times E$.

**Definition 20.** On $M \times M \times E$, let the coordinates be $t, x_i, y_i, z_i$ on the $i$th copy of $M$ and $x_3, y_3, z_3$ on $E$. Further, let $\omega_{i,j}$ be generators of $H^{1,2-j}$ of the $i$th copy of $M$ for $j = 0$ or 2 and let $\eta_i$ be generators of $H^{1-i}(E)$ for $i = 0, 1$.

**Proposition 22.** The automorphisms $\alpha_1, \alpha_2$ lift to automorphisms of $M \times M \times E$ of order 4, given respectively by

$\tilde{\alpha}_1 : (t_1 : x_1 : y_1 : z_1), (it_2 : -z_2 : -y_2 : -x_2), (-x_3 : iy_3 : z_3),$

$\tilde{\alpha}_2 : (it_1 : -y_1 : -x_1 : z_1), (it_2 : z_2 : y_2 : x_2), (x_3 : -y_3 : z_3).$

(There are four choices for each of these lifts, but all are of order 4. Note also that $\tilde{\alpha}_1^2 = \sigma_1$ and $\tilde{\alpha}_2^2 = \sigma_2$.)

**Proof.** The file *kke.mag* in [17] defines $M \times M \times E$ and various quotients inside appropriate toric varieties, constructs the rational map $M \times E \dashrightarrow \mathcal{L}$ and hence $M \times M \times E -\dashrightarrow M \times E$, defines the automorphisms $\alpha_1, \alpha_2$ in coordinates, and verifies that the given formulas define automorphisms of $M \times M \times E$ that are pullbacks of $\alpha_1, \alpha_2$. \qed

**Definition 21.** Let $G_4 = \langle \alpha_1, \alpha_2 \rangle$; let $\tilde{G}_4 = \langle \tilde{\alpha}_1, \tilde{\alpha}_2 \rangle$.

**Proposition 23.** The quotient $(M \times M \times E)/\tilde{G}_4$ admits a resolution of singularities $V$ whose canonical divisor is trivial.
Proof. It is enough to show that the Reid–Shepherd-Barron–Tai criterion \cite{RST} is satisfied. For all fixed point strata of an element $\beta$ of order $n$ in $\bar{G}_4$ of codimension greater than 1, and every $k \in \mathbb{Z}$ with $(k, n) = 1$, the action of every power of $\beta$ on the tangent space at a general point has eigenvalues $\zeta_n^{a_i}$ with $0 \leq a_i < n$. The criterion requires that $\sum a_i \geq n$ for all $k$ with equality for at least one $k$.

As this is a tedious calculation, we present it only for one element of $\bar{G}_4$, namely $\bar{\alpha}_1$. We first consider the points $((t_1 : x_1 : y_1 : z_1), (0 : 1 : c : 1), (0 : 0 : 1))$. The tangent space is the direct sum of those for $M, M, E$, and $\bar{\alpha}_1$ preserves this decomposition. Clearly the action of $\alpha_1$ on the tangent space to $((t_1 : x_1 : y_1 : z_1)$ on the first copy of $M$ is trivial. On the second copy, we have the tangent vectors given by $(\epsilon : 1 : c : 1), (0 : 1 : c + \epsilon : 1)$, which are taken to $(-i\epsilon : 1 : c : 1)$ and $(0 : 1 : c + \epsilon : 1)$ respectively, indicating an action by the diagonal matrix with entries $-i, 1$. On $E$ a tangent vector is given by $(\epsilon : 1 : 0)$, which goes to $(i\epsilon : 1 : 0)$, so the action is by $i$. Thus the $a_i$ are $0, 0, 3, 0, 1$; since $n = 4$, the result follows in this case. Similarly, considering the points $((t_1 : x_1 : y_1 : z_1), (0 : -1 : 0 : 1), (0 : 0 : 1))$, we find eigenvalues $1, 1$ on the first copy of $M$ and $i$ on $E$ as before, while at $(0 : -1 : 0 : 1)$ on the second copy of $M$ we find $i, -1$, and the $a_i$ are $0, 0, 1, 2, 1$. Replacing $(0 : 0 : 1)$ on $E$ by the other fixed point $(0 : 1 : 0)$ changes nothing, and similar methods apply to all other elements of $\bar{G}_4$.

We remark that the resolution of $V$ has trivial fundamental group. To see this, note that $M \times M \times E$ has fundamental group $\mathbb{Z}^2$, since $\pi_1(M) = 0$. The action of $\alpha_1$ is given in suitable coordinates by the matrix \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \), so the fixed subspace is trivial. Thus $(M \times M \times E)/\bar{G}_4$ has trivial $\pi_1$, and the resolution of singularities replaces contractible subvarieties by rational subvarieties, whose fundamental group is trivial. Thus by Seifert-van Kampen $\pi_1(V)$ is likewise trivial.

**Theorem 8.** The resolution $V$ of Proposition \cite{RST} is a Calabi-Yau variety and $H^4(V)$ is generated by fundamental classes of subvarieties.

**Proof.** We have shown that $K_V \cong O_V$, so we need only show that $H^{i,0}(V) = 0$ for $1 \leq i \leq 4$. We consider the action of automorphisms on cohomology. The Künneth formula determines the cohomology of $M \times M \times E$. It is clear that $H^{i,0}(E)$ is trivial, that $H^{1,1}(M)$ for $i = 0$ and $H^1(M)$ for $i = 2$ and on $H^{i,1}(M)$ for $0 \leq i \leq 1$. Hence, for any field $F$ of characteristic not equal to 2, the subring of $H^*(M \times M \times E, F)$ fixed by $\sigma_1, \sigma_2$ is generated as an $F$-vector space by the fundamental classes of subvarieties and by classes of the form $\omega_{1,j_1} \otimes \omega_{2,j_2} \otimes \eta_k$.

**Corollary 6.** The Galois representation on $H^5_{et}(V, \mathbb{Q}_\ell)$ coincides with that attached to $m_6$ (Definition \cite{De})

**Proof.** So far we have not used the elements of $G_4$ of order 4, but now we need to. First we show that $H^5((M \times M \times E)/\langle \sigma_1, \sigma_2 \rangle)$ fixed by $G_4$ coincides with $H^{5,0}(M \times M \times E)$. The $E$-component of $\bar{\alpha}_1$ pulls back $dx_3/y_3$ to $d(-x_3)/y_3 = i \, dx_3/y_3$, so it acts as multiplication by $i$ on $H^{1,0}(E)$. Continuing in this way, we see that $\bar{\alpha}_1$ acts as $(1, -i, i)$ on the spaces spanned by $\omega_{1.1}, \omega_{2.1}, \eta_1$ respectively, and $\bar{\alpha}_2$ acts as $(i, i, -1)$. The action on the spaces spanned by $\omega_{1.2}, \omega_{2.2}, \eta_2$ is by the reciprocals of these.
Thus the $+1$ eigenspace for $G_4$ acting on $H^5$ is spanned by $\omega_{1,1} \otimes \omega_{2,1} \otimes \eta_1$ and $\omega_{1,2} \otimes \omega_{2,2} \otimes \eta_2$. This establishes that the dimension of $H^5(V)$ is 2 and that the Galois representation is the component with weights $(5,0)$ and $(0,5)$ in the tensor product of those attached to the cusp forms corresponding to $M, M, E$. In light of Lemma 3 this is the form $m_{6}$ of weight 6 and level 32 with complex multiplication by $\mathbb{Q}(i)$.

**Corollary 7.** The Kodaira dimension $\kappa$ of a resolution of $(M \times M \times E)/G_4$ or of $V_{32}/G_4$ is 0.

**Proof.** For $(M \times M \times E)/G_4$, we have $\kappa \geq 0$, since the generator of $H^{5,0}$ survives in the quotient; also $\kappa \leq 0$, since $\kappa_{M \times M \times E} = 0$. It follows that $\kappa_{V_{32}/G_4} = 0$ as well.

**Remark 11.** The same method could be used to construct rigid Calabi-Yau fivefolds realizing the form of weight 6 and level 27: see Example 1 below. However, it is clearly insufficient even for the form of weight 6 with CM by $\mathbb{Q}(\sqrt{-19})$, because there is no finite subgroup of $GL_5(\mathbb{Q})$ with traces in $\mathbb{Q}(\sqrt{-19})$ but not in $\mathbb{Q}$.

**Example 1.** Let $K$ be the $K3$ surface defined by $t^2 = xy(x^4 + y^4) + z^6$ (cf. Example 0.4 (11)). Then $K$ has Picard number 20 and admits a non-symplectic automorphism $\omega_K$ of order 3 given by $(t : x : y : z) \rightarrow (t : x : y : \zeta_3 z)$. Let $E$ be the elliptic curve $y^2 + y = x^3$ of conductor 27; it also has an automorphism $\omega_E$ of order 3 given by $(x, y) \rightarrow (\zeta_3 x, y)$. We use $-K, -E$ for the obvious negation maps; let $G \subset Aut(K \times K \times E)$ be generated by $-K, -E, (1, -K, -E), (\omega_K, \omega_E^{-1})$. Then $(K \times K \times E)/G$ will have a strongly rigid Calabi-Yau resolution $C_{27}$. It is defined over $\mathbb{Q}$, since the group $G$ is rational, even though some of its elements are not. Since the representation for the modular form attached to $K$ is a component of the symmetric square of the form attached to $E$, that for $C_{27}$ will be a component of $\text{Sym}^3$, which means it must be the unique rational newform of weight 6 and level 27.

Since $V_{32}/G_4$ is birationally equivalent to $V$, we have likewise shown that $V_{32}$ admits a rigid Calabi-Yau resolution. In fact, considering that these are related by blowing up and down minimal rational varieties defined at worst over $\mathbb{Q}(i)$, we conclude:

**Proposition 24.** There is a formula of the form $\sum_{i=0}^{5} c_i(p)p^i - a_{6,p}$ for the number of $\mathbb{F}_p$-rational points of $V_{32}/G_4$, where $c_0 = c_5 = 1$ and the other $c_i$ are expressions of the form $c + c'\phi(-1)$ with $c, c' \in \mathbb{Z}$.

**Theorem 9.** $|V_{32}/G_4|_p = \sum_{i=0}^{5} p^i - a_{6,p}$ for all $p > 2$.

**Proof.** To determine the $c_i$ of Proposition 23, we consider $V_{32}/G_4$, and in particular the ring of invariants of $G_4$. By calculation in Magma we find that the primary invariants have degrees $1,1,2,2,2$, while the secondary invariants have degrees $0,2,2,4$, with the secondary invariant of degree 4 being the product of the two of degree 2. So the quotient is embedded in weighted projective space $\mathbb{P}(1,1,2,2,2,2,2,2)$, or (composing with the embedding by $O(2)$) in $\mathbb{P}^6$. The branch locus has 5 orbits of size 2 and one of size 1 under the group; the images of the components in orbits of size 2 are defined by a single polynomial, as is the union of the two of size 1. As before we are able to compute the number of $\mathbb{F}_p$-points of $[V_{32}/G_4]$ for small $p$, finding it to be $\sum_{i=0}^{5} p^i - a_{6,p}$ for $p < 25$. There are 8 odd...
primes less than 25 and this gives us 8 independent conditions on the coefficients, so by basic linear algebra we conclude that this formula holds for all $p$.

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