DONALDSON INVARIANTS FOR SOME GLUED MANIFOLDS

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Abstract. We prove that every suitable 4-manifold with \( b_1 = 0 \) and with an embedded Riemann surface of genus 2 is of simple type. We find a relationship between the basic classes of two of these 4-manifolds and those of the connected sum along the Riemann surface.

1. Notations and facts

This is a second paper on the problem of studying the behaviour of the basic classes of a 4-manifold under a determined surgical operation. In [7] we studied how the Seiberg-Witten invariants behaved under connected sum of two 4-manifolds along a Riemann surface. Here we try to find the Donaldson invariants (to a reasonable extent) under the same surgery. We also check the agreement with the conjecture of Witten [9] about the relation between the Seiberg-Witten and the Donaldson invariants. The differences in our results appear regarding two aspects:

- The computations in the instanton case are much more cumbersome which hampers us to extend our results beyond the case \( g = 2 \).
- On the other hand, the gluing theory is more developed in the instanton case than in the Seiberg-Witten case, so we can obtain information also for basic classes \( \kappa \) with \( |\kappa \cdot \Sigma| < 2g - 2 \).

Before embarking in the statement of the results, we gather briefly some notation which will be useful. This follows mainly [5]. We say that a four-manifold \( X \) is suitable when \( b^+ - b_1 \) is odd and \( b^+ > 1 \). We have defined for any \( w \in H^2(X; \mathbb{Z}) \) and \( p_1 \in H^4(X; \mathbb{Z}) \cong \mathbb{Z} \) with \( p_1 \equiv w^2 \) (mod 4) the Donaldson invariant \( D^{w,d}_X \) as a symmetric polynomial on \( \mathcal{A}(X) = \text{Sym}^*(H_{\text{even}}(X)) \otimes \wedge^*(H_{\text{odd}}(X)) \) of degree \( 2d = -2p_1 - 3(1 - b_1 + b^+) \) (whenever this quantity is positive). The degree of the elements

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in $H_i$ is $4 - i$. We fix $w$. There are several ways of wrapping up all the information about the different $p_i$ in a single series, defining so (the class of the point is $x$):

$$D_X^w(e^{t\alpha}) = \sum \frac{D_X^{w,d}(\alpha^d)}{d!} t^d$$

$$D_X^w(\alpha) = D_X^w(e^{t\alpha + \lambda x}) = \sum \frac{D_X^{w,d+2a}(\alpha^d x^a)}{d! a!} t^d \lambda^a$$

for $\alpha \in H^2(X)$ (our main concern will be the invariants for the second homology as we will soon suppose $b_1 = 0$).

We say that $X$ is of $w$-simple type when $x^2 - 4$ annihilates the Donaldson invariant $D_X^w$, that is when $D_X^w((x^2 - 4)z) = 0$ for all $z \in A(X)$. It is a fact that when $X$ is of $w$-simple type for some $w$, it is so for every $w'$, and it is called of simple type for brevity. In this case

$$D_X^w(\alpha) = D_X^w(e^{t\alpha}) \cosh 2\lambda + D_X^w\left(\frac{x}{2}e^{t\alpha}\right) \sinh 2\lambda$$

Kronheimer and Mrowka defined in [5] another series containing the same information

$$D_X^w(\alpha) = D_X^w(e^{t\alpha}) + D_X^w\left(\frac{x}{2}e^{t\alpha}\right)$$

**Theorem 1.** Let $X$ be a suitable manifold with $b_1 = 0$ and of simple type. Then we have

$$D_X^w(\alpha) = e^{Q(\alpha)/2} \sum a_{i,w} e^{K_i \cdot \alpha}$$

for finitely many cohomology classes $K_i$ (called basic classes) and rational numbers $a_{i,w}$. These classes are lifts to integral homology of $w_2(X)$. Moreover, for any embedded surface $S \hookrightarrow X$ of genus $g$ and positive self-intersection, one has $2g - 2 \geq S^2 + |K_i \cdot S|$. Also the rational numbers satisfy $a_{i,w} = (-1)^{K_i \cdot w + w^2} a_i$, for some $a_i \in \mathbb{Q}$.

**Definition 2.** Here we are going to have a situation with $w, \Sigma \in H^2(X; \mathbb{Z})$ satisfying $w \cdot \Sigma \equiv 1 \pmod{2}$ and $\Sigma^2 = 0$. Then we define

$$D_X = D_X^w + D_X^{w+\Sigma}$$

Obviously, $D_X$ depends on $w \mod \Sigma$. Note that since $(w + \Sigma)^2 \equiv w^2 + 2 \pmod{4}$, we can recuperate $D_X^w$ and $D_X^{w+\Sigma}$ from the series $D_X$. Analogously we define $D_X = D_X^w + D_X^{w+\Sigma}$. 
Proposition 3. Suppose $X$ is a suitable manifold of simple type with $b_1 = 0$ and $D^w_X = e^{Q/2} \sum a_i e^{K_i}$. Then setting $d_0 = -w^2 - \frac{3}{2}(1 + b^+)$ we have

$$D_X(e^\alpha) = e^{Q(\alpha)/2} \sum_{K_i, \Sigma \equiv 2 \pmod{4}} a_i e^{K_i} \alpha + e^{Q/2} \sum_{K_i, \Sigma \equiv 0 \pmod{4}} i^{-d_0} a_i e^{K_i} \alpha$$

So it is equivalent giving $D^w_X$ to giving $D_X$.

Proof. Since $((w + \Sigma)^2 + K_i \cdot (w + \Sigma)) - (w^2 + K_i \cdot w)) = 2(w \cdot \Sigma + K_i \cdot \Sigma/2)$ we have

$$D^{w+\Sigma}_X = e^{Q/2} \sum_{K_i, \Sigma \equiv 2 \pmod{4}} a_i e^{K_i} - e^{Q/2} \sum_{K_i, \Sigma \equiv 0 \pmod{4}} a_i e^{K_i}$$

Now $D^{w}_X(e^\alpha) = \frac{1}{2}(D^{w+\Sigma}_X(\alpha) + i^{-d_0}D^{w+\Sigma}_X(i\alpha))$ and $D^{w+\Sigma}_X(e^\alpha) = \frac{1}{2}(D^{w+\Sigma}_X(\alpha) - i^{-d_0}D^{w+\Sigma}_X(i\alpha))$, so

$$D_X(e^\alpha) = e^{Q(\alpha)/2} \sum_{K_i, \Sigma \equiv 2 \pmod{4}} a_i e^{K_i} \alpha + i^{-d_0} e^{Q/2} \sum_{K_i, \Sigma \equiv 0 \pmod{4}} a_i e^{K_i} \alpha$$

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2. Statement of results

There are many ways of cooking different four-manifolds out of old ones. It is an interesting problem to relate the invariants of the new manifolds with those of the given ones (and to find the basic classes when appropriate) (for very nice examples see [8]).

The case this paper is concerned with is the following. Let $\bar{X}_1$ and $\bar{X}_2$ be two suitable manifolds with $\Sigma_i \subset \bar{X}_i$ embedded surfaces of the same genus $g$ and self-intersection zero. If we remove a tubular neighbourhood of $\Sigma_i$ we end up with a manifold $X_i$ with boundary $Y = S^1 \times \Sigma$. Choosing an identification between the boundaries of $X_1$ and $X_2$ (there is an element of choice here which produces different four-manifolds [4]), we can construct another manifold $X = X_1 \cup_Y X_2$. The (closure of the) tubular neighbourhood removed is always diffeomorphic to $A = D^2 \times \Sigma$. Obviously $\bar{X}_i = X_i \cup_Y A$.

Our starting hypotheses are that $b_1 = 0$ for both $\bar{X}_i$ and that there exists $w_i \in H^2(\bar{X}_i; \mathbb{Z})$ with $w_i \cdot \Sigma_i \equiv 1 \pmod{2}$, equivalently the cohomology class of $\Sigma_i$ is an odd multiple of a non-torsion primitive class. We fix $w_i$’s in all the manifolds involved ($\bar{X}_1$, $\bar{X}_2$ and $X$) once and for all in a compatible way (i.e. the restriction of $w$ to $X_i$ coincides with the restriction of $w_i$ to $X_i$) with $w^2 = w_1^2 + w_2^2 \pmod{4}$ and such that $w_i \cdot \Sigma_i \equiv 1 \pmod{2}$. We drop the subindices, so we will not differentiate these $w$’s, since the context makes always clear to which manifold they refer.
The series $D^w_X(\alpha)$ is determined by its action on elements $\alpha$ lying in:

- $H^2(X_1) \oplus H^2(X_2) \subset H^2(X)$.
- elements $D$ such that $D = D_1 + D_2$ with $\partial D_1 = -\partial D_2$ being $S^1 \subset Y = S^1 \times \Sigma$ (or a multiple of $S^1$).
- elements $D_\gamma$ such that $D_\gamma = D_1 + D_2$ with $\partial D_1 = -\partial D_2$ representing a cycle $\gamma$ in the subgroup $H_1(\Sigma;\mathbb{Z}) \subset H_1(S^1 \times \Sigma;\mathbb{Z})$.

For studying the behaviour on elements of the first kind we can suppose that $\alpha \in H^2(X_1)$ without loss of generality. As explained in [1] [3] we have Floer homology groups $HF_*(Y)$, cooked out of the flat connections on $Y$. The relevant issues we must be careful with are the appearance of reducibles on $X_i$ (but in our case they are ruled out since the moduli space $M_\Sigma$ of flat connections on $\Sigma$ consists entirely of irreducibles) and of flat connections on $X_i$ (they present no problem using the blow-up trick of [6]).

One has an invariant $D^w_X(\alpha^d) \in HF_*(Y)$ and another invariant $D^w_Y(1) \in HF_*(Y)$ such that $D_X(\alpha^d) = \langle D^w_X(\alpha^d), D^w_Y(1) \rangle$. Looking at it more closely we realize that the pairing above gives either $D^w_X(\alpha^d)$ or $D^w_X(1)$ depending on $d \mod 4$.

For dealing with the other two cases there is an analogous approach (see [1]). Let $D = D_1 + D_2$ with $\partial D_1 = -\partial D_2 = \gamma \in H_1(S^1 \times \Sigma;\mathbb{Z})$. Then we have Fukaya-Floer groups $HFF_*(Y,\gamma)$ such that the invariants are $D^w_{X_1}(D_1) \in HFF_*(Y,\gamma)$ and $D^w_{X_2}(D_2) \in HFF_*(Y,\gamma)$ with

$$D_X(D^m) = \sum \binom{m}{i} \langle D^w_{X_1}(D^i), D^w_{X_2}(D^{m-i}) \rangle$$

We aim to prove that:

**Theorem 4.** Let $X$ have $b_1 = 0$ and an embedded surface of genus 2 and self-intersection zero. Then $X$ is of simple type.

So let $\bar{X}_i$ be two suitable manifolds as above with embedded surfaces $\Sigma_i \subset \bar{X}_i$ of genus two and self-intersection zero (and being an odd multiple of a non-torsion class in homology). Form $X = \bar{X}_1 \# \Sigma \bar{X}_2$ removing tubular neighbourhoods of $\Sigma_i$ and identifying the boundaries $Y = S^1 \times \Sigma$ (recall that the diffeomorphism type of the resulting manifold depends on the homotopy class of the gluing identification). The manifold $X$ is of simple type by the last theorem.

**Theorem 5.** Suppose that $\bar{X}_i$ have $b_1 = 0$ and are of simple type. Then $X$ is of simple type and every basic class $\kappa$ of $X$ intersects $Y$ in $nS^1$ where $n = 0, 2, -2$. Moreover the sum of the coefficients $c_\kappa$ of the different basic classes $\kappa$ agreeing
with \((K, L) \in H^2(\tilde{X}_1; \mathbb{Z})/\mathbb{Z}[\Sigma_1] \oplus H^2(\tilde{X}_2; \mathbb{Z})/\mathbb{Z}[\Sigma_2]\) is zero unless \((K, L)\) comes from \((K_i, L_j) \in H^2(\tilde{X}_1; \mathbb{Z}) \oplus H^2(\tilde{X}_2; \mathbb{Z})\) where \(K_i\) is a basic class for \(\tilde{X}_1\), \(L_j\) is a basic class for \(\tilde{X}_2\) and \(K_1 \cdot \Sigma_1 = L_j \cdot \Sigma_2 = \pm 2\), in which case it is \(\pm 32\) times the product of the coefficients of \(K_i\) and \(L_j\).

For \(g > 2\) we have the obvious following

**Conjecture 6.** Let \(\tilde{X}_i\) have \(b_1 = 0\) and be of simple type. Suppose that there are embedded surfaces \(\Sigma_i \subset \tilde{X}_i\) of genus \(g\) and self-intersection zero. Form \(X = \tilde{X}_1 \#_\Sigma \tilde{X}_2\). Then \(X\) is of simple type and every basic class \(\kappa\) of \(X\) intersects \(Y\) in \(nS^1\) where \(n\) is an even integer with \(-(2g - 2) \leq n \leq (2g - 2)\). Moreover the sum of the coefficients \(c_\kappa\) of the different basic classes \(\kappa\) agreeing with \((K, L) \in H^2(\tilde{X}_1; \mathbb{Z})/\mathbb{Z}[\Sigma_1] \oplus H^2(\tilde{X}_2; \mathbb{Z})/\mathbb{Z}[\Sigma_2]\) is zero unless \((K, L)\) comes from \((K_i, L_j) \in H^2(\tilde{X}_1; \mathbb{Z}) \oplus H^2(\tilde{X}_2; \mathbb{Z})\) where \(K_i\) is a basic class for \(\tilde{X}_1\), \(L_j\) is a basic class for \(\tilde{X}_2\) and \(K_1 \cdot \Sigma_1 = L_j \cdot \Sigma_2 = \pm (2g - 2)\), in which case it is \(\pm 2^{7g-9}\) times the product of the coefficients of \(K_i\) and \(L_j\).

Here we would like to point out the close relationship between this result and the results in [7] about Seiberg-Witten invariants. Witten [9] has conjectured that for a simply connected manifold the condition of being simple type and Seiberg-Witten simple type are equivalent, that in that case the basic classes are the same as the Seiberg-Witten basic classes and that the shape of the Donaldson series is

\[
\mathcal{D}_X^w = e^{Q/2} \sum a_i e^{K_i}
\]

where

\[
a_i = (-1)^{\frac{K_i \cdot w + w^2}{2}} 2^{2 + \frac{1}{4}(7g + 11\sigma)} SW_X(K_i)
\]

We proved in [7] that for basic classes \(K_i\) such that \(K_1 \cdot \Sigma = K_2 \cdot \Sigma = \pm (2g - 2)\) one has

\[
\sum_{L \in \pi^{-1}(K_1, K_2)} SW_X(L) = SW_{\tilde{X}_1}(K_1) SW_{\tilde{X}_2}(K_2)
\]

Now the topological numbers are (see [7]) \(\chi_X = \chi_{\tilde{X}_1} + \chi_{\tilde{X}_2} + 4g - 4\) and \(\sigma_X = \sigma_{\tilde{X}_1} + \sigma_{\tilde{X}_2}\), so for \(g = 2, 2 + \frac{1}{4}(7g + 11\sigma)\) is increased in \(5\). Also from equation (7) any basic class \(K \in \pi^{-1}(K_1, K_2)\) with \(K \cdot \Sigma = \pm (2g - 2)\) has \(K \cdot w = K_1 \cdot w_1 + K_2 \cdot w_2 \pm 2\) (mod 4) (and we chose \(w^2 = w_1^2 + w_2^2\) (mod 4)). Therefore the sum of the coefficients of the different basic classes \(K\) agreeing with \((K_1, K_2) \in H^2(\tilde{X}_1; \mathbb{Z})/\mathbb{Z}[\Sigma_1] \oplus H^2(\tilde{X}_2; \mathbb{Z})/\mathbb{Z}[\Sigma_2]\) is \(\pm 32\) times the product of the coefficients of \(K_i\) and \(L_j\).
Note that this result is more restricted in the sense that it is only valid for the case \( g = 2 \) but it is more general in the sense that it also gives information about basic classes \( K \) with \( |K \cdot \Sigma| < 2g - 2 \).

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3. The case \( g = 2 \)

We are specially interested in the case in which the genus is 2 since the computations can then be carried out quite explicitly. For this we recall the Floer homology of \( S^1 \times \Sigma \) when \( g = 2 \). In general, there is the isomorphism \( HF^*(S^1 \times \Sigma) \cong QH^*(M_\Sigma) \) with the quantum cohomology of \( M_\Sigma \), the moduli space of rank-2 stable bundles over \( \Sigma \) with odd determinant. The universal bundle yields a map \( \mu : H_1(\Sigma) \to H^3(M_\Sigma) \) given by slanting with the first Pontrjagin class. \( M_\Sigma \) can be described algebraically as the intersection of two quadrics in \( \mathbb{CP}^5 \). From this description, Donaldson [2] found the space of lines in \( M_\Sigma \), the necessary input for finding the quantum corrections. \( QH^*(M_\Sigma) \) has (integral) homology equal to \( \mathbb{Z} \) in degrees 0, 2, 4 and 6. The generators are \( 1, h, l \) and \( p \) which correspond in the description above to the fundamental class, a plane, a line and a point in \( M_\Sigma \). The map \( \mu \) gives an isomorphism \( \mu : H_1(\Sigma) \to H^3(M_\Sigma) \), describing the other non-zero bit of the homology of \( M_\Sigma \). We have

\[
\begin{align*}
h \ast h &= 4l + 4 \\
h \ast h \ast h &= 4p + 12h \\
h \ast h \ast h \ast h &= h \ast h
\end{align*}
\]

In general we will drop the \( \ast \) symbol for denoting the quantum product. \( QH^*(M_\Sigma) \) has a natural \( \mathbb{Z}/4 \)-grading coming from reducing the \( \mathbb{Z} \)-grading above. The standard basis is given by \( e_i = h^i \), \( 0 \leq i \leq 3 \), and elements \( \mu(\alpha) \), where \( \alpha \) runs through a basis of \( H_1(\Sigma) \). Note that the matrix \( < e_i, e_j > \) is

\[
\begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 16 \\
1 & 0 & 16 & 0
\end{pmatrix}
\]

The pairings \( < \mu(\alpha), e_i > \) are all zero, so \( QH^3(M_\Sigma) \) is orthogonal to the “even” part of \( QH^*(M_\Sigma) \). This is an important remark since it will allow us to ignore the “odd” part in later computations.
4. The first case

Here we suppose \( g \geq 2 \). First we deal with the invariants for classes \( \alpha \in H_2(X_1) \subset H_2(X) \) and \( \beta \in H_2(X_2) \subset H_2(X) \). We fix elements \( z_l = \sum a^x \alpha_1 \cdots \alpha_r \in \Lambda(\Sigma) \), \( \alpha_i \in H_1(\Sigma) \), such that the corresponding \( \epsilon_i = \mu(\Sigma)^a \mu(x)^b \mu(\alpha_1) \cdots \mu(\alpha_r) \) form a basis for \( HF_*(M_\Sigma) \) (quantum multiplication is understood throughout). Then we have the following:

\[
D^w_A(z_l) = \epsilon_i
\]

For the open manifold \( X_1 \) we write \( D^w_{X_1}(z) = D^w_{X_1}(z), \epsilon_l > \epsilon_i \), where \( \{ \epsilon_i \} \) is the dual basis of \( \{ \epsilon_l \} \). Transferring the cycles contained in \( z_l \) to \( X_1 \) we get for \( z \in \Lambda(X_1) \).

\[
D^w_{X_1}(z) = D^w_{X_1}(z), D^w_A(z_l) > \epsilon_i = D^w_{X_1}(z_l), D^w_A(1) > \epsilon_i = D^w_{X_1}(z_l)\epsilon_i
\]

Then

\[
D_X(e^{(\alpha + \beta)} = D^w_X(e^{\alpha}), D^w_{X_2}(e^{\beta}) = D^w_{X_2}(e^{\beta}) < D^w_{X_1}(e^{\alpha})D^w_{X_2}(e^{\beta}) < e_i^*, e_m^* >
\]

Therefore the series for \( X \) is determined by the series for both sides. We can work out \( D_X(\alpha + \beta) \) adding the class of the point to either side. Note that on cycles of the type \( T_\gamma = S^1 \times \gamma \subset S^1 \times \Sigma \) the series is constant, which agrees with the case of simple type, in which tori of self-intersection zero have intersection zero with all basic classes and so the invariants are zero on such tori.

When both of \( \bar{X}_i \) are of simple type (and \( b_1 = 0 \)), we have that \( X \) is of simple type (on cycles of the first kind) and writing

\[
D^w_{X_1}(e^\alpha) = e^{Q(\alpha)/2} \sum a\epsilon^{K_1 \cdot \alpha}, D^w_{X_2}(e^\beta) = e^{Q(\beta)/2} \sum b\epsilon^{L_2 \cdot \beta}
\]

(we drop the dependence of \( w \) on the coefficients \( a_i \) and \( b_j \)), we have that \( D^w_{X_1}(e^\alpha) = e^{Q(\alpha)/2} \sum a \epsilon^{K_1 \cdot \alpha}, D^w_{X_2}(e^\beta) = e^{Q(\beta)/2} \sum b \epsilon^{L_2 \cdot \beta} \). Also

\[
D^w_{X_1}(e^\alpha) = e^{Q(\alpha)/2} \sum a_i \epsilon^{K_1 \cdot \alpha}, D^w_{X_2}(e^\beta) = e^{Q(\beta)/2} \sum b_j \epsilon^{L_2 \cdot \beta}
\]

where \( a'_i = a_i \) and \( K'_i = K_i \) when \( K_i \cdot \Sigma \equiv 2 \) (mod 4) and \( a'_i = i^{-d_0}a_i \) and \( K_i = iK_i \) when \( K_i \cdot \Sigma \equiv 0 \) (mod 4) (and analogously for \( \bar{X}_2 \)). Now

\[
D_X(e^{\alpha + \beta}) = e^{Q(\alpha + \beta)/2} \sum_{l,m,r=r'} \epsilon^{K_1 \cdot \alpha + L_2 \cdot \beta} (K'_i \cdot \Sigma)^2b^b \epsilon^{e_i^*, e_m^*} < e_i^*, e_m^* >
\]

\[
D^w_X(e^{\alpha + \beta}) = e^{Q(\alpha + \beta)/2} \sum_{l,m,r=r'} (a'_i b'_j) \epsilon^{K_1 \cdot \alpha + L_2 \cdot \beta} (K'_i \cdot \Sigma)^2b^b \epsilon^{e_i^*, e_m^*} < e_i^*, e_m^* >
\]
here \( a, b \) and \( r \) correspond to \( l \) and \( a', b' \) and \( r' \) correspond to \( m \). Using \( \alpha = \Sigma, \beta = 0 \) and \( \alpha = 0, \beta = \Sigma \) we see that \( K_i \cdot \Sigma = L_j \cdot \Sigma \). Moreover from theorem 1 this number is even and less or equal than \( 2g - 2 \) in absolute value.

**The genus 2 case.** When \( g = 2 \), \( K_i \cdot \Sigma \) has to be \(-2, 0 \) or \( 2 \). For the case \( K_i \cdot \Sigma = 0 \) the number in the right hand side vanishes so we have:

**Theorem 7.**

\[
\mathbb{D}_X(e^{\alpha+\beta}) = e^{Q(\alpha+\beta)/2} \left( \sum_{K_i \cdot \Sigma = L_j \cdot \Sigma = 2} 32a_ib_j e^{K_i \cdot \alpha + L_j \cdot \beta} + \sum_{K_i \cdot \Sigma = L_j \cdot \Sigma = -2} -32a_ib_j e^{K_i \cdot \alpha + L_j \cdot \beta} \right)
\]

**Proof.** We use the standard basis for \( QH^*(M_{\Sigma}) \) and note that \( h \) corresponds to \( 2\Sigma \) by lemma 11. So in the above expression \( b = b' = 0 \) and \( 0 \leq a, a' \leq 3 \). The matrix \( <e_i^*, e_m^*> \) is

\[
\frac{1}{4} \begin{pmatrix}
0 & -16 & 0 & 1 \\
-16 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
\]

so when \( K_i \cdot \Sigma = 2 \), the coefficient is computed to be 32 and when \( K_i \cdot \Sigma = -2 \) it will be \(-32 \).

**Remark 8.** The reason for the signs is easy to work out. First, \( w^2 \) for \( X \) is congruent (mod 2) with the sum of both of \( w^2 \) for \( X_i \). Also \( b^+(X) = b^+(\tilde{X}_1) + b^+(\tilde{X}_2) + (2g - 1) \), so \(-\frac{3}{2}(1 + b^+(X)) = -\frac{3}{2}(1 + b^+(\tilde{X}_1)) - \frac{3}{2}(1 + b^+(\tilde{X}_2)) - 3(g - 1) \). Recalling that \( d_0 = -w^2 - \frac{3}{2}(1 + b^+) \) and \( g = 2 \), we have \( d_0(X) \equiv d_0(\tilde{X}_1) + d_0(\tilde{X}_2) + 1 \) (mod 2). Now the sign comes from the fact that the coefficient for the basic class \(-K_i\) is \((-1)^{d_0}a_i\), being \( a_i \) the coefficient for the basic class \( K_i \).

**5. The second case**

Let now have a homology class \( D \) such that \( D = D_1 + D_2 \) with \( D_i \in H_2(X_i; \partial X_i; \mathbb{Z}) \) and \( \partial D_1 = -\partial D_2 = S^1 \subset S^1 \times \Sigma \) (or a multiple of \( S^1 \), e.g. when \( \Sigma \) is non-primitive). We can suppose \( D^2 = 0 \) (by adding a suitable multiple of \( \Sigma \) to \( D \)). This will do no harm to the argument and is notationally convenient. In this section we need to work with the Fukaya-Floer homology version of the gluing theory. This appears as the limit of a spectral sequence whose \( E_2 \) term is \( HF_*(Y) \otimes H_*(\mathbb{C}P^\infty) \). Since all maps in this spectral sequence are \( H_{odd}(M_{\Sigma}) \to H_{even}(M_{\Sigma}) \) and \( H_{even}(M_{\Sigma}) \to H_{odd}(M_{\Sigma}) \) and all commute with the action of \( \text{Diff}(\Sigma) \) (because the boundary cycle is \( S^1 \) and
therefore invariant under that group), they are zero. Thus the spectral sequence
degenerates in the third term and

\[ D^w_{X_1}(D) = (\phi_0, \phi_1, \phi_2, \ldots) \in HF_*(Y) \otimes H_*(\CP^\infty) \]

where we can interpret \( \phi_k = D^w_{X_1}(D^k) \in QH_*(M_\Sigma) \), which appears in a similar fashion
as the invariant for the closed manifold but using the moduli space of connections for
the open manifold with a cylindrical end. The pairing formula reads

\[ D_X(D^m) = \sum \left( \binom{m}{i} \right) < D^{w}_{X_1}(D^i), D^w_{X_2}(D^{m-i}) > \]

Now we use the trick of transferring \( \Sigma \) from \( X_1 \) to \( X_2 \).

\[ D_X(D^m\Sigma) = \sum \left( \binom{m}{i} \right) < D^{w}_{X_1}(D^i\Sigma), D^w_{X_2}(D^{m-i}) > = < D^{w}_{X_1}(D^i), P_\Sigma D^w_{X_2}(D^{m-i}) > \]

for some symmetric map \( P_\Sigma : H_*(M_\Sigma) \to H_*(M_\Sigma) \). We have the following result
about the structure of \( P_\Sigma \) which we do not prove here but hope to return to it later.
In any case, one can avoid using it (see remark 16).

**Lemma 9.** \( P_\Sigma \) is quantum multiplication by \( \mu(\Sigma) \) plus \( i(D \cdot \Sigma) \) times the identity, i.e. \( D^w_{X_1}(D^i\Sigma) = \mu(\Sigma) * D^w_{X_1}(D^i) + i(D \cdot \Sigma)D^w_{X_1}(D^{i-1}) \).

**Corollary 10.** As a consequence of the above lemma we get

\[ D_X(e^{s\Sigma+tD}) = < D^w_{X_1}(e^{tD_1}e^{s\Sigma}), D^w_{X_2}(e^{tD_2}) > = < D^w_{X_1}(e^{tD_1}), e^{(s\Sigma)(tD_1)}e^{s\mu(\Sigma)} * D^w_{X_2}(e^{tD_2}) > \]

**The genus 2 case.** First we start off with a simple result

**Lemma 11.** For \( g = 2 \) one has \( \mu(\Sigma) = \frac{1}{2}h \) and \( \mu(x) = -4h^2 + 2 \). In particular
\( \mu(x)^2 - 4 = 0 \).

**Proof.** Take \( X \) to be a K3 surface blown-up in two points. Consider a tight surface
\( S \) of self-intersection 2 (and therefore of genus 2) in the K3 surface (which existence
is guaranteed by [5]). Let \( E_1 \) and \( E_2 \) be the exceptional divisors in \( X \) and let \( \Sigma \)
be the proper transform of the tight surface, i.e. \( \Sigma = S - E_1 - E_2 \). Put \( w = E_1 \),
so \( w \cdot \Sigma = 1 \) and \( \Sigma \) has genus 2 and self-intersection zero. Then \( X \) is of simple
type, \( DF^w_X = e^{Q/2} \cosh E_2 \sinh E_1 \). So \( DF^w_X(t\Sigma) = \cosh t \sinh t \). The moduli spaces
of connections on \( X \) are of dimensions \( 2d \equiv 6 \) (mod 8). From all of this we get
\( DF^w_X(\Sigma^{n+6}) = 2^{4+4n} \). Write \( X = X_1 \cup_Y A \). Then \( \mu(\Sigma) \) is a multiple of \( h \), say \( ah \).
Now \( DF^w_X(\Sigma^{n+6}) = < D^w_{X_1}(\Sigma^n), (ah)^6 > = < D^w_{X_1}(\Sigma^n), a^616^2h^2 > = a^416^2DF^w_X(\Sigma^{n+2}) \) (for
\( n \equiv 1 \) (mod 4)), from where \( a = \frac{1}{2} \).
For computing $\mu(x)$ we put $\mu(x) = ah^2 + b$. We have $0 = D_X^w((x^{2n} - 2^{2n})\Sigma^3) = < D_X^w(\Sigma^3), (ah^2 + b)^{2n} - 2^{2n} >$, for $n > 0$. The first factor in the pairing is of the form $ch + d \neq 0$, since the expression is non-zero for $n = 0$. Therefore all the second factors (for different $n$) are proportional. This is impossible if $b \neq 2$ since $(ah^2 + b)^{2n} - 2^{2n} = \frac{1}{16}((16a + b)^{2n} - b^{2n})h^2 + (b^{2n} - 2^{2n})$. So $b = 2$, $\mu(x) = ah^2 + 2$ and $\mu(x)^2 - 4 = (16a^2 + 4a)h^2$ implying that $0 = D_X^w((x^2 - 4)z) = (16a^2 + 4a)D_X^w(\Sigma^2 z)$. So either $a = 0$ or $a = -4$. The first case is impossible since $\mu(x)$ has a non-trivial component in $H^4$. □

**Corollary 12.** Let $\tilde{X}_1$ have $b_1 = 0$ and an embedded surface of genus 2 and self-intersection zero (as supposed so far). Then $\tilde{X}_1$ is of simple type. □

Easily we obtain

\[ e^{s\mu(\Sigma)} = 1 + \frac{s}{2}h + \frac{\cosh 2s - 1}{16}h^2 + \frac{\sinh 2s - 2s}{64}h^3 \]

(2)

Let us write $D_{X_1}^w(e^{tD_1}) = (\phi_0, \phi_1, \phi_2, \phi_3)$, $D_{X_2}^w(e^{tD_2}) = (\psi_0, \psi_1, \psi_2, \psi_3)$ and $D_A^w(e^{t\Delta}) = (a_0, a_1, a_2, a_3)$ with respect to the standard basis of section 3 ($\Delta = D^2 \times pt \subset D^2 \times \Sigma = A$). We do not consider the odd part of the Floer homology since when one of the manifolds (say $\tilde{X}_1$) has $b_1 = 0$, one has $< D_{X_1}^w(e^{tD_1}), (\mu(\alpha))^* > = 0$ for $\alpha \in H_1(\Sigma)$, as the dual of $\mu(\alpha)$ is of the form $\mu(\beta)$. We suppose that $D \cdot \Sigma = 1$ although the general case $D \cdot \Sigma \neq 0$ is much the same as this one. We have

\[ D_X(e^{s\Sigma + tD}) = \]

\[ = e^{ts}(\phi_0, \phi_1, \phi_2, \phi_3) \]

\[ = e^{ts}(\phi_0, \phi_1, \phi_2, \phi_3) \]

\[ = e^{ts}(\phi_0, \phi_1, \phi_2, \phi_3) \]
Call the square matrix in the middle $B$. Now we can separate according to coefficients corresponding to functions on $s$.

\[
\begin{pmatrix}
\text{coeff. of } e^{2s}e^{ts} \\
\text{coeff. of } e^{-2s}e^{ts} \\
\text{coeff. of } e^{ts} \\
\text{coeff. of } se^{ts}
\end{pmatrix} = \frac{1}{4}
\begin{pmatrix}
0 & 0 & \frac{1}{128}(4\psi_3 + 16\psi_2) & \frac{1}{128}(4\psi_2 + \psi_3) \\
0 & 0 & \frac{1}{128}(4\psi_3 - 16\psi_2) & \frac{1}{128}(4\psi_2 - \psi_3) \\
\psi_3 - 16\psi_1 & \psi_2 - 16\psi_0 & \psi_1 - \frac{1}{16}\psi_3 & \psi_0 - \frac{1}{16}\psi_2 \\
0 & \psi_3 - 16\psi_1 & 0 & \psi_1 - \frac{1}{16}\psi_3
\end{pmatrix}
\begin{pmatrix}
\phi_0 \\
\phi_1 \\
\phi_2 \\
\phi_3
\end{pmatrix}
\]

Calling the new matrix in the middle $A_\psi$, one has $(e^{2s}, e^{-2s}, 1, s)A_\psi = (\psi_0, \psi_1, \psi_2, \psi_3)B$. We call $A$ to the matrix $A_a$ corresponding to $A = D^2 \times \Sigma$.

\[
\frac{1}{4}
\begin{pmatrix}
0 & 0 & \frac{1}{128}(4a_3 + 16a_2) & \frac{1}{128}(4a_2 + a_3) \\
0 & 0 & \frac{1}{128}(4a_3 - 16a_2) & \frac{1}{128}(4a_2 - a_3) \\
a_3 - 16a_1 & a_2 - 16a_0 & a_1 - \frac{1}{16}a_3 & a_0 - \frac{1}{16}a_2 \\
a_3 - 16a_1 & a_2 - 16a_0 & 0 & a_1 - \frac{1}{16}a_3
\end{pmatrix}
\begin{pmatrix}
\phi_0 \\
\phi_1 \\
\phi_2 \\
\phi_3
\end{pmatrix}
\]

**Lemma 13.** The matrix $A$ is invertible.

**Proof.** That the determinant vanishes would imply that either $a_3 = 4a_2$, $a_3 = -4a_2$ or $a_3 = 16a_1$. The first two cases give that the first or second row of $A$ is zero respectively, which is contradictory as there are examples where the left hand side of (3) has non-zero two first entries (see remark 16). The case $a_3 = 16a_1$ implies that the series for any such $X$ is always of the form $e^{ts}(f_1(t)e^{2s} + f_2(t)e^{-2s} + f_3(t))$. This is also valid for $X = \mathbb{CP}^1 \times \Sigma = A \cup_Y A$ (see remark 16). Particularizing for $t = 0$, $D_X(e^{s\Sigma})$ is a linear combination of $e^{2s}$, $e^{-2s}$ and 1. But from (1) and (2) we get that

\[
D_X(e^{s\Sigma}) = \frac{1}{16}(\sinh 2s - 2s)
\]

$\square$

As a corollary, the $\phi_i$ are determined by the series $D_{X_1}(e^{tD_1})$ ($D_1 = D_1 \cup \Delta$). We have

\[
D_X(e^{tD}) = \langle (\phi_0, \phi_1, \phi_2, \phi_3), (\psi_0, \psi_1, \psi_2, \psi_3) \rangle =
\]

\[
(v_1)^T
\begin{pmatrix}
\frac{32}{(a_3+4a_2)^2} & 0 & 0 & 0 \\
0 & \frac{32}{(a_3-4a_2)^2} & 0 & 0 \\
0 & 0 & \frac{4}{(a_3-16a_1)^2} & 0 \\
0 & 0 & 0 & \frac{4}{(a_3-16a_1)^2}
\end{pmatrix}
(v_2)
\]
where

$$v_i = \begin{pmatrix}
\text{coef. of } e^{2s}e^{ts} \\
\text{coef. of } e^{-2s}e^{ts} \\
\text{coef. of } e^{ts} \\
\text{coef. of } se^{ts}
\end{pmatrix}
$$

for the manifold $\tilde{X}_i$. When $\tilde{X}_1$ is of simple type we can use $D_{\tilde{X}_1}^w$ instead of $D_{\tilde{X}_1}$. We write $\bar{D}_i = D_i \cup \Delta$. We have some freedom, so we impose $\bar{D}_i^2 = 0$. Then we can write

$$v_1 = \begin{pmatrix}
\sum_{K_i, \Sigma = 2} a_i e^{tK_i \cdot D_1} \\
\sum_{K_i, \Sigma = -2} a_i e^{tK_i \cdot \bar{D}_1} \\
\sum_{K_i, \Sigma = 0} a_i e^{tK_i \cdot \bar{D}_1} \\
0
\end{pmatrix}
$$

When both of $\tilde{X}_i$ are of simple type, we have

$$\mathbb{D}_{\tilde{X}}^w(tD) =
\begin{pmatrix}
0 & 0 & 0 \\
0 & \frac{32}{(a_3 + 4a_2)^2} & 0 \\
0 & 0 & \frac{32}{(a_3 - 4a_2)^2}
\end{pmatrix}
\begin{pmatrix}
\sum_{L_j, \Sigma = 2} b_j e^{tL_j \cdot \bar{D}_2} \\
\sum_{L_j, \Sigma = -2} b_j e^{tL_j \cdot \bar{D}_2} \\
\sum_{L_j, \Sigma = 0} b_j e^{tL_j \cdot \bar{D}_2}
\end{pmatrix}
$$

Here we use the fact that the third row and third column are zero.

**Theorem 14.** If we have $D^2 = \bar{D}_1^2 + \bar{D}_2^2$, $D \cdot \Sigma = 1$ then the square matrix in equation (5) is

$$\begin{pmatrix}
32e^{2t} & 0 & 0 \\
0 & -32e^{-2t} & 0 \\
0 & 0 & 0
\end{pmatrix}
$$

So

$$\mathbb{D}_{\tilde{X}}(e^{tD}) = e^{Q(tD)/2}(\sum_{K_i, \Sigma = L_j, \Sigma = 2} 32a_i b_j e^{(K_i \cdot \bar{D}_1 + L_j \cdot \bar{D}_2 + 2)t} + \sum_{K_i, \Sigma = L_j, \Sigma = -2} -32a_i b_j e^{(K_i \cdot \bar{D}_1 + L_j \cdot \bar{D}_2 - 2)t})$$

**Proof.** Here it is enough to find examples of manifolds $\tilde{X}_1$, $\tilde{X}_2$ and $X$ whose basic classes are known. Instead we use an indirect argument. Since all the manifolds involved are of simple type, the non-zero entries of the matrix are finite sums of exponentials, i.e.

$$\begin{pmatrix}
\sum c_n e^{nt} & 0 & 0 \\
0 & \sum d_n e^{nt} & 0 \\
0 & 0 & 0
\end{pmatrix}$$
Now we evaluate the series on $tD + r_1\alpha_1 + r_2\alpha_2$, for $\alpha_i \in H^2(X_i; \mathbb{Z})$, put $t = 0$ and use theorem 7 to get $\sum c_n = 32$ and $\sum d_n = -32$. Let $S = \mathbb{CP}^2 \# 10\mathbb{CP}^2$ the rational elliptic surface blown-up once. Denote by $E_1, \ldots, E_{10}$ the exceptional divisors and let $T_1 = C - E_1 - \cdots - E_9$, $T_2 = C - E_1 - \cdots - E_8 - E_{10}$ where $C$ is the cubic curve in $\mathbb{CP}^2$. So $T_1$ and $T_2$ can be represented by smooth tori of self-intersection zero and with $T_1 \cdot T_2 = 1$. We can glue two copies of $S$ along $T_1$. The result is a K3 surface $S \# T_1 S$ blown-up twice. The $T_2$ pieces glue together to give a genus 2 Riemann surface of self-intersection zero $\Sigma_2$ which intersects $T_1$ in one point. Now set $X = (S \# T_1 S) \# \Sigma_2 (S \# T_2 S)$, call $\Sigma = \Sigma_2$ and get $D$ piecing together both $T_1$ in $S \# T_1 S$. So (choose for instance $w = E_1$ in $S$)

$$D_X(e^{tD+s\Sigma}) = e^{Q(tD+s\Sigma)/2} \left( \sum_{K_i, \Sigma = L_j, \Sigma = 2} c_n a_i b_j e^{2s+nt} + \sum_{K_i, \Sigma = L_j, \Sigma = -2} d_n a_i b_j e^{-2s+nt} \right) =$$

$$= e^{ts} \left( \sum \frac{c_n}{16} e^{2s+nt} + \sum \frac{d_n}{16} e^{-2s+nt} \right)$$

since $T_1$ evaluates 0 on basic classes being a torus of self-intersection zero (the coefficient $\frac{1}{16}$ appears from the explicit computation of the basic classes of the K3 surface blown-up in two points). The trick is now to use the symmetricity fact that $X = (S \# T_1 S) \# \Sigma_1 (S \# T_2 S)$ where $\Sigma_1$ comes from gluing together both $T_1$. Under this diffeomorphism $D = \Sigma_1$ and $\Sigma$ comes from piecing together both $T_2$ in $S \# T_2 S$. Hence

$$D_X(e^{tD+s\Sigma}) = e^{ts} \left( \sum \frac{c_n}{16} e^{2t+ns} + \sum \frac{d_n}{16} e^{-2t+ns} \right)$$

From here we deduce that $c_n = 0$ unless $n = 2$ and $d_n = 0$ unless $n = -2$. Hence the result. $\Box$

**Remark 15.** Note that when $X_1$ is of simple type there is not summand in $D_X^w$ corresponding to $s$. Therefore $0 = \phi_1 (a_3 - 16a_1) + \phi_3 (a_1 - \frac{1}{16} a_3)$ and then $(\phi_3 - 16\phi_1)(a_3 - 16a_1) = 0$, so $\phi_3 = 16\phi_1$. In particular, for any $X_2$, the manifold $X = X_1 \cup Y$ $X_2$ is of simple type and, coherently, has a series without the coefficient corresponding to $s$. In this case we have

$$D_X(e^{tD+s\Sigma}) =$$

$$= \frac{1}{4} \left( 0, -16\psi_0 + \psi_2, \frac{1}{4} \psi_2 \sinh 2s + \frac{1}{16} \psi_3 \cosh 2s, \psi_0 - \frac{1}{16} \psi_2 + \frac{1}{16} \psi_2 \cosh 2s + \frac{1}{64} \psi_3 \sinh 2s \right) \left( \begin{array}{c} \phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_3 \end{array} \right) =$$

$$= \frac{1}{64} (\psi_3 - 16\psi_1)(16\phi_0 - \phi_2) + \frac{1}{256} (\psi_2, \psi_3) \left( \begin{array}{cc} 16 \sinh 2s & 4 \cosh 2s \\ 4 \cosh 2s & \sinh 2s \end{array} \right) \left( \begin{array}{c} \phi_2 \\ \phi_3 \end{array} \right)$$
So if both $\bar{X}_1$ and $\bar{X}_2$ are of simple type, we get

$$D^w_X(tD + s\Sigma) = \frac{1}{32}(\psi_1, \psi_2) \begin{pmatrix} 16 \sinh 2s & 4 \cosh 2s \\ 4 \cosh 2s & \sinh 2s \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

**Remark 16.** Suppose we do not want to use lemma 9 about the explicit description of $P_\Sigma$. Then we could argue as follows. We have, instead of corollary 10,

$$D_X(e^{s\Sigma + tD}) = <D^{w}_{X_1}(e^{tD_1}), Q_\Sigma D^{w}_{X_2}(e^{tD_2})>$$

for some symmetric map $Q_\Sigma$. We go through the same steps as before without the knowledge of the matrices to reach the matrix $A$. For proving its invertibility it is enough to find four linearly independent vectors in the left hand side of formula (3) for $\psi_i = a_i$ (i.e. $X_2 = A$). For this we use

- **X** a K3 surface, $\Sigma$ a tight torus with an added trivial handle to make it of genus 2. The vector we get is $(0, 0, 1, 0)$.
- **X** = $\mathbb{CP}^1 \times \Sigma$ to get a vector with non-zero last component. Since this manifold has $b_1 \neq 0$ we have to use the odd part of the Floer homology in the computations. This produces an extra term in the series $D^w_X(s\Sigma + tD)$ of the form $f(t)$. The only fact that we need is that when we set $t = 0$ there is a summand which is a multiple of $s$.
- **X** a K3 surface blown-up twice, $\Sigma = S - E_1 - E_2$ for $S$ a tight surface of genus 2 in K3, $w = E_1$, $D$ a cohomology class coming form the K3 part such that $D \cdot S = 1$. We get $(1, 1, something, 0)$.
- **X**, $\Sigma$, $D$ as before, $w = R + E_1$, where $R$ comes from the K3, has self-intersection 1 and is orthogonal to $D$ and $S$. We get $(1, -1, something, 0)$.

Having reached this point we know of the existence of a universal matrix as in (4). For the simple type case we get something like in (5) with an unknown 3x3-matrix. This matrix is diagonal since obviously it is always the case $K_{i} \cdot \Sigma = L_{j} \cdot \Sigma$. Now consider the case in which both $\bar{X}_i$ and $\Sigma_i$ are as in the first example above. Then $X = \bar{X}_1 \# \bar{X}_2$ splits off a $S^2 \times S^2$, so the invariants are zero. Therefore the third diagonal entry is zero and the rest of the argument remains intact.

**The case of genus $g > 2$.** Here we propose a way of tackling conjecture 6.

Call $HF$ to the Floer homology of $M_\Sigma$ and let $u = \mu(pt)$, $h = \mu(\Sigma)$ and $\Gamma = \Sigma \mu(\alpha_2)\mu(\alpha_{2i+1})$ be the generators of the invariant part of $HF$. Actually this invariant part is generated as a vector space by $u^i h^j \Gamma^p$ with $i + 2p < g$ and $j + 2p < g$. Now we define $I$ to be the ideal in $HF$ generated by the image of $H^1(\Sigma)$ under $\mu$. The
space $HF/I$ is generated by elements of the form $u^i h^j$ with $i < g, j < g$ (in principle they might not be linearly independent). Consider $V$ any subspace of $HF$ containing the orthogonal complement $I^\perp$ of $I$ such that it has generators $e_{ij} = u^i h^j \mod I$, $i < g, j < g$. The dimension of $V$ is $N = g^2$. We decompose $HF = V \oplus W$ with $W = V^\perp \subset I$.

Now we write

$$E = e^{sh + \lambda u + a \Gamma} = \sum f_{ijp}(s, \lambda, \alpha) u^i h^j \Gamma^p$$

we have that for every relation $R(h, u, \Gamma) = 0$ it is $R(\frac{\partial}{\partial s}, \frac{\partial}{\partial \lambda}, \frac{\partial}{\partial \alpha}) E = 0$ and so $R(\frac{\partial}{\partial s}, \frac{\partial}{\partial \lambda}, \frac{\partial}{\partial \alpha}) f_{ijp} = 0$. Note also that $\frac{\partial}{\partial s} f_{ijp'}(0, 0, 0) = \delta_{ijp'}$ for $i + 2p < g$ and $j + 2p < g$. So the $f_{ijp}$ are linearly independent functions. $E$ defines a map from $V$ to $V$ (which we keep on calling $E$) by multiplication followed by orthogonal projection. This map is of the form $N_c g_c(s, \lambda, \alpha)$, where $N_c$ are constant endomorphisms of $V$, $c = (i, j)$, $0 \leq i, j < g$ and $g_c(s, \lambda, \alpha)$ are linearly independent functions.

Let $X_1$ be open manifold with cylindrical end $Y = S^1 \times \Sigma$ and $D \in H^2(X_1, \partial X_1; \mathbb{Z})$ with $\partial D = S^1$. Then $D_{X_1}(D^i) \in HF$ has component $D_{X_1}(D^i)_V$ in $V$ and $D_{X_1}(D^i)_W$ in $W$. When $X_1$ has $b_1 = 0$ one has $<D_{X_1}(D^i), \mu(\alpha)> = 0$ for any 1-homology class $\alpha$. So $D_{X_1}(D^i) \in V \subset HF$. So $D_X(e^{tD}) = <D_{X_1}(e^{tD_1}), D_{X_2}(e^{tD_2})>_V$.

Now when either of $X_1$ has $b_1 = 0$ one has $D_X(e^{tD + s\Sigma + \lambda x}) = D_X(e^{tD + s\Sigma + \lambda x + \alpha \Gamma}) = e^{ts} <D_{X_1}(e^{tD_1})_V, e_{ij}> <e_{ij}, e_{ij}'>^{-1} <e_{ij'}, E * D_{X_2}(e^{tD_2})>_V$. Write $\phi_a = \phi_a(t)$ for the components of $D_{X_1}(e^{tD_1})$ in $V$ and $\psi_a = \psi_a(t)$ for the components of $D_{X_2}(e^{tD_2})$. Then $D_{X}(e^{tD + s\Sigma + \lambda x}) = e^{ts} \phi_a(t) M_{abc} \psi_b(t) g_c(s, \lambda, \alpha)$ for some matrices $M_{abc}$. Now we can decompose $D_{X}(e^{tD + s\Sigma + \lambda x}) = e^{ts} D_{X,c} g_c(s, \lambda, \alpha)$ so $D_{X,c} = \phi_a(M_{abc} \psi_b)$ (note that the $g_c$ corresponding to non-vanishing $D_{X,c}$ are independent of $\alpha$).

When $X_2 = A = D^2 \times \Sigma$, we put $a_b = \psi_b$. So we have constructed a map

$$V \otimes \mathcal{F}(t) \to \mathbb{R}^N \otimes \mathcal{F}(t)$$

$$\phi_a \mapsto (\phi_a M_{abc} a_b)_c$$

where $\mathcal{F}(t)$ is the vector space of (Laurent) formal power series. To see that $D_{X_1}$ determines $\phi_a$ we need to prove the injectivity of this linear map between vector spaces of the same dimension. If this were proved, the map would be an isomorphism and hence we could mimic the argument of the case $g = 2$ in this situation to arrive to the existence of some universal $N \times N$ matrix $P$ whose coefficients depend on $t$ and $\lambda$ such that

$$D_{X}(e^{tD + \lambda x}) = (D_{X_1}(e^{tD_1})a)(P_{ab}(t, \lambda))(D_{X_2}(e^{tD_2})b)$$
When \( X \) is of simple type, \( D_X(e^{tD+\alpha\Sigma+\lambda x}) \) is a linear combination of the functions \( e^{2\lambda}e^{(2+4n)s} (-\lfloor \frac{g}{2} \rfloor \leq n \leq \lfloor \frac{g-1}{2} \rfloor) \) and \( e^{-2\lambda}e^{4ns} (-\lfloor \frac{g-1}{2} \rfloor \leq n \leq \lfloor \frac{g-1}{2} \rfloor) \). So these functions are among the \( g_c \) (or they are combinations of them) and without loss of generality we can suppose they are the first \( 2g-1 \) of the lot. With arguments as in this paper and one non-trivial example of the gluing where the basic classes were known, we would get the \( (2g-1)x(2g-1) \) minor to be (conjecturally)

\[
\begin{pmatrix}
\text{coef } e^{2t\pm2\lambda} & 0 & \cdots & 0 \\
0 & \text{coef } e^{-2t\pm2\lambda} & 0 \\
\vdots & & \ddots \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

We can further obtain more information from \( X = \mathbb{C}P^1 \times \Sigma \), but this gives us a total of \( (2g)x(2g) \) coefficients (very far from the \( (g^2)x(g^2) \) we seek for).

### 6. Proof of main Theorem

Now we have all the information to prove theorem 5. As it was explained in [7], we have an exact sequence

\[
0 \to H^1(Y;\mathbb{Z}) \to H^2(X;\mathbb{Z}) \xrightarrow{\zeta} G \oplus H^1(\Sigma;\mathbb{Z})
\]

where we call \( G \) to the subgroup of \( H^2(X_1;\mathbb{Z})/\mathbb{Z}[\Sigma_1] \oplus H^2(X_2;\mathbb{Z})/\mathbb{Z}[\Sigma_2] \) consisting of elements \((\alpha_1, \alpha_2)\) such that \( \alpha_1 \cdot \Sigma_1 = \alpha_2 \cdot \Sigma_2 \) (for interpretations of this sequence see [7]). Suppose now that both \( \bar{X}_i \) (and hence \( X \)) are of simple type. Write \( \mathbb{D}_{X_1} = e^{Q/2} \sum a_i e^{K_i} \), \( \mathbb{D}_{X_2} = e^{Q/2} \sum b_j e^{L_j} \) and \( \mathbb{D}_{X} = e^{Q/2} \sum c_i e^{K_i} \). Put \( H \) for the subgroup of \( H^2(X) \) generated by the image of \( H^2(X_1) \oplus H^2(X_2) \) and a (fixed) class \( D = D_1 + D_2 \) such that \( \partial D_1 = -\partial D_2 = S^1 \) (or a non-zero multiple of \( S^1 \)).

First we consider classes \( \bar{D}_i = D_i + \Delta \in H^2(\bar{X}_i;\mathbb{Z}) \). There is an indeterminacy as \( \bar{D}_i \) is defined up to addition of multiples of \( \Sigma \). So we can impose \( D^2 = D_1^2 + D_2^2 \) (so \( (\bar{D}_1, \bar{D}_2) \) is defined up to addition of multiples of \( (\Sigma, -\Sigma) \)). Now we glue \( K_i \) and \( L_j \) (whenever \( K_i \cdot \Sigma = L_j \cdot \Sigma = \pm(2g-2) \)), to get \( \kappa_{ij} \). The indeterminacy comes this time from adding elements in \( H^1(Y;\mathbb{Z}) \). We impose

\[
\kappa_{ij} \cdot D = K_i \cdot \bar{D}_1 + L_j \cdot \bar{D}_2 + 2
\]

depending on whether \( K_i \cdot \Sigma = L_j \cdot \Sigma = \pm(2g-2) \) (when \( g = 1 \) both possibilities must be considered). This prevents additions of multiples of \( \Sigma \) (note that the right hand side of (7) makes sense). So there is still an indeterminacy coming from addition of elements in \( H^1(\Sigma;\mathbb{Z}) \otimes H^0(S^1;\mathbb{Z}) \subset H^1(Y;\mathbb{Z}) \).
Remark 17. The natural condition to impose on $\kappa_{ij}$ is that $(\kappa_{ij})^2 = K_i^2 + L_j^2 + 8(g-1)$. This is equivalent to the above whenever $g > 1$. Instead, the one used above is the correct one even when $g = 1$.

Remark 18. Note that $\kappa_{ij}$ lies in $\pi^{-1}(K_i, L_j)$ by construction.

For a pair $(K_i, L_j)$, all the possible $\kappa_{ij}$ restrict to the same function $f$ in $H$. So

$$D_X^w|_H = e^{Q/2} \sum_{f \in H^*} \left( \sum_{\kappa \mid \kappa \in f} c_\kappa \right) e^f$$

Note that two cohomology classes restrict to the same $f$ if and only if they have the same image under $\pi$ followed by projection to $G$ and they have the same pairing with $D$.

Now we return to the case of $g = 2$. From what we learn in theorems 7 and 14 we have

$$D_X^w|_H = e^{Q/2} \sum_{K_i \Sigma \in L_j \Sigma = 2} 32a_i b_j e^{\kappa_{ij}} + \sum_{K_i \Sigma \in L_j \Sigma = -2} -32a_i b_j e^{\kappa_{ij}}$$

from where we get

$$\left\{ \begin{array}{ll}
\sum_{\kappa \in \pi^{-1}(K_i, L_j)} c_\kappa = \pm 32a_i b_j & \text{if } K_i \cdot \Sigma = L_j \cdot \Sigma = \pm 2 \\
\sum_{\kappa \in \pi^{-1}(K_i, L_j)} c_\kappa = 0 & \text{if } K_i \cdot \Sigma = L_j \cdot \Sigma = 0 \\
\sum_{\kappa \in \pi^{-1}(K, L)} c_\kappa = 0 & \text{if } (K, L) \neq (K_i, L_j)
\end{array} \right.$$  \hspace{1cm} (8)

Note that for any basic class $\kappa$ one has $\kappa \cdot T_\gamma = 0$ (where $T_\gamma = S^1 \times \gamma \subset S^1 \times \Sigma$ for $\gamma \in H_1(\Sigma; \mathbb{Z})$). Therefore $\pi(\kappa)$ lies in $G \subset G \oplus H^1(\Sigma; \mathbb{Z})$. Also note that the condition $\kappa^2 = K_i^2 + L_j^2 + 8$ in the first case means that the possible $\kappa$ in the sum differ by addition of elements in $H^1(\Sigma; \mathbb{Z}) \otimes H^0(S^1; \mathbb{Z}) \subset H^1(Y; \mathbb{Z})$.

Corollary 19. Let $\bar{X}_i$ as before and $g = 2$. Suppose that for every cycle $\gamma \in H^1(\Sigma; \mathbb{Z})$ there exists a $(-1)$-embedded disc in both $X_i$ bounding $\gamma$; then the basic classes $\kappa$ of $X$ are in one-to-one correspondence with pairs of basic classes $(\kappa_1, \kappa_2)$ for $\bar{X}_1$ and $\bar{X}_2$ respectively, such that $\kappa_1 \cdot \Sigma_1 = \kappa_2 \cdot \Sigma_2 = \pm 2$. Moreover, $\kappa$ is determined in an explicit way.

Proof. As explained in [7], there is a splitting $H^2(X; \mathbb{Z}) = V \oplus V^\perp$ where $V$ is generated by tori $T_\gamma = S^1 \times \gamma \subset S^1 \times \Sigma$ (for $\gamma \in H_1(\Sigma; \mathbb{Z})$) and spheres $D_\beta$ of self-intersection $-2$ (with the property $D_\beta \cap Y = \beta$). There is an exact sequence

$$0 \to \mathbb{Z}[\Sigma] \to V^\perp \overset{\pi}{\to} G$$

where

$$0 \to \mathbb{Z}[\Sigma] \to V^\perp \overset{\pi}{\to} G$$
Now let $\kappa$ be a basic class for $X$. We argue as in [5] that $\kappa \in V^\perp$. Thus in the summation of (8) only one term is non-vanishing when $K_i \cdot \Sigma = L_j \cdot \Sigma = \pm 2$. $\kappa$ is characterised as the only class orthogonal to all $T_{\gamma}$ and $D_{\beta}$ such that $\kappa^2 = K_i^2 + L_j^2 \pm 8$. When $K_i \cdot \Sigma = L_j \cdot \Sigma = 0$ there is only one term in the summation since the condition (7). Therefore there are no basic classes such that $\kappa \cdot \Sigma = 0$. 

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