AN EXTENDED DEMAZURE PRODUCT ON INTEGER PERMUTATIONS VIA MIN-PLUS MATRIX MULTIPLICATION

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Abstract. Coxeter groups possess an associative operation, called variously the Demazure, greedy, or 0-Hecke product. For symmetric groups, this product has an amusing formulation as matrix multiplication in the min-plus (tropical) semiring of two matrices associated to the permutations. We prove that this min-plus formulation extends to furnish a Demazure product on a much larger group of integer permutations, consisting of all permutations that change the sign of finite many integers. We prove several alternative descriptions of this product and some useful properties of it. These results were developed in service of future applications to Brill–Noether theory of algebraic and tropical curves; the connection is surveyed in an appendix.

1. Introduction

The symmetric group $S_d$, and more generally any Coxeter group, possesses a useful associative operation $\star$, variously called the Demazure, 0-Hecke, or greedy product. If $\alpha \in S_d$ and $\sigma_n$ is the simple transposition of $n$ and $n + 1$, then

$$\alpha \star \sigma_n = \begin{cases} \alpha \sigma_n & \text{if } \alpha(n) < \alpha(n + 1), \\ \alpha & \text{otherwise}. \end{cases}$$

Since $\star$ is associative and every $\beta \in S_d$ factors into adjacent transpositions, Equation (1) is enough to compute any Demazure product in $S_d$. A similar formulation is valid in any Coxeter group.

The Demazure product on $S_d$ has an equivalent formulation that is transparently independent of choice of factorization. To state it, first define, for any permutation $\alpha$, a counting function

$$s_\alpha(a, b) = \# \{ \ell \geq b : \alpha(\ell) < a \}. $$

See Figure 1 for an illustration. The Demazure product on $S_d$ has the following characterization.

$$s_{\alpha \star \beta}(a, b) = \min_{1 \leq \ell \leq d + 1} \left[ s_\alpha(a, \ell) + s_\beta(\ell, b) \right], \quad \text{for all } 1 \leq a, b \leq d + 1.$$
An equivalent form of Equation (2) in the $S_d$ case appears in [CP19, Fact 2.4], with a sketch of a proof. A complete proof is in [Pfl21a]. It seems likely that it was known earlier, and the formula does not appear to surprise experts, but I have been unable to find an earlier reference in the literature. Therefore I hope this paper will draw attention to this amusing formula, and I would be very pleased if any readers can direct me to an earlier reference.

Functions like $s_\alpha$ (perhaps with reversed inequalities or other minor modifications) are sometimes called “rank functions;” they provide a convenient way to state the strong Bruhat order on $S_d$ [BB05, Theorem 2.1.5] and are used to define Schubert varieties in flag varieties [Ful97, §10]. If we view the functions $s_\alpha, s_\beta$ as $(d + 1) \times (d + 1)$ matrices, Equation (2) says that $s_{\alpha \star \beta}$ is obtained by matrix multiplication in the min-plus (tropical) semiring. It has a nice geometric interpretation in terms of relative position of flags; see Appendix A.

We will follow the convention that elements of $S_d$ are extended to permutations of $\mathbb{Z}$ that fix all $n \notin [1, d]$. This does not affect the values $s_\alpha(a, b)$ for $1 \leq a, b \leq d + 1$, but it does allow an alternative form of Equation (2) that does not depend on the choice of $d$:

$$s_{\alpha \star \beta}(a, b) = \min_{\ell \in \mathbb{Z}} \left[ s_\alpha(a, \ell) + s_\beta(\ell, b) \right], \quad \text{for all } a, b \in \mathbb{Z}. \tag{3}$$

This equation generalizes nicely to larger groups than $S_d$. For example, the affine symmetric groups $\tilde{S}_k$ (see Example 2.4 or Section 8.4 for a definition) are Coxeter groups embedded as subgroups of the permutations of $\mathbb{Z}$, and Equation (3) holds, with $\sigma_n$ replaced by different generators, in $\tilde{S}_k$ as well. A simple proof is in [Pfl21a]; this also follows from this paper’s results.

The present paper uses the min-plus description of $\star$ to generalize it to a much wider class of permutations. This generalization makes it possible to combine elements of $S_d$ and elements of $\tilde{S}_k$ for different values of $k$. Though this may appear at first glance to be an idle curiosity, such “mixed modulus” Demazure products occur in the theory of divisors on graphs and algebraic curves. Indeed, the impetus to write this paper was originally to provide a self-contained reference for various facts needed in ongoing work in tropical in algebraic geometry. A preview of the intended applications may be found in Appendix B. See also the short paper [Pfl21a], which is designed as a simple proof-of-concept that does not require the material developed here.

The group considered herein is the group of almost-sign-preserving permutations, which are the largest class of permutations $\alpha : \mathbb{Z} \to \mathbb{Z}$ for which $s_\alpha$ and $s_{\alpha^{-1}}$ both take only finite values. This group is uncountable, and defining $\star$ in terms of a generating set is not feasible. We prove the existence of an associative product $\star$ in this context, defined by the min-plus formulation (Theorem A), and demonstrate that it shares several nice aspects of the Demazure product on $S_d$, including a characterization as a “greedy product” in Bruhat order (Theorem B) and a “reduction theorem” (Theorem C) for simplifying inequalities of the form $\alpha \star \beta \geq \gamma$ (in Bruhat order).

We also study a slightly broader context than ASP, by studying operations $\star, \langle, \rangle$ on a set SF of functions which we call slipface functions $s : \mathbb{Z}^2 \to \mathbb{N}$. Then ASP is embedded in SF by $\alpha \mapsto s_\alpha$; the image of this embedding consists of submodular slipface functions. The non-submodular slipface functions occur in tropical geometry, so there is some utility in this further generalization.

It is plausible that many of this paper’s results can be deduced from existing theory of Coxeter groups, via limit arguments. This would certainly be interesting to carry out, and indeed earlier drafts of this paper follows this approach in some parts (for example, Theorem B may be proved by multiplying simple transpositions one at a time and applying Zorn’s lemma). However, this paper avoids this approach, and instead frames everything in terms of the functions $s_\alpha$ and Equation (3). This reveals some intriguing combinatorics hiding in the min-plus matrix multiplication expression, and also means that this paper is entirely self-contained.
1.1. Conventions and key definitions. The symbol \( \mathbb{N} \) denote the set of nonnegative numbers. The symbol \( \delta \) is always used for an indicator function, equal to 1 if the statement within is true, and 0 otherwise. A permutation always refers to a bijective function \( \mathbb{Z} \rightarrow \mathbb{Z} \). Whenever we refer to a permutation of \( \{1, \ldots, d\} \), or any other finite set, we implicitly extend this to a permutation of all of \( \mathbb{Z} \) that fixes every other integer; so \( S_d \) is always implicitly understood to be embedded in the full permutation group of \( \mathbb{Z} \).

Call a permutation \( \alpha : \mathbb{Z} \rightarrow \mathbb{Z} \) almost-sign-preserving if there are only finitely many \( n \) such that exactly one of \( n, \alpha(n) \) is negative. We denote the group of almost-sign-preserving permutations by ASP. For any \( \alpha \in \text{ASP} \), define a function \( s_\alpha : \mathbb{Z}^2 \rightarrow \mathbb{N} \) by

\[
s_\alpha(a, b) = \# \{ n \geq b : \alpha(n) < a \}.
\]

This function will be called (somewhat whimsically) the slipface function of \( \alpha \); see Section 2.

The shift of a permutation \( \alpha \in \text{ASP} \) is the number

\[
\chi_\alpha = s_\alpha(0, 0) - s_{\alpha^{-1}}(0, 0).
\]

The importance of this number will become clear as we develop our results; some intuition about it is provided by observing that elements of \( S_d \) have shift 0, and more generally a permutation with finitely many inversions satisfies \( \alpha(n) = n - \chi_\alpha \) for all but finitely many \( n \in \mathbb{Z} \).

The Bruhat order on ASP is the partial order \( \leq \), where

\[
\alpha \leq \beta \text{ means that } s_\alpha(a, b) \leq s_\beta(a, b) \text{ for all } a, b \in \mathbb{Z}.
\]

In principle this definition involves infinitely many inequalities, but in practice it often suffices to check only those \((a, b)\) in the essential set; this is examined in Section 7.

Restricted to \( S_d \), this is the usual “strong” Bruhat order [BB05, Theorem 2.1.5]. We discuss an analog in ASP of the “weak” Bruhat orders in Section 5. We will also use the following shorthand: \( \alpha \leq_{\chi} \beta \) means that \( \alpha \leq \beta \) and \( \chi_\alpha = \chi_\beta \).

1.2. Main results. The following existence theorem combines Theorems 4.4 and 8.7.

**Theorem A.** There is an associative operation \( \ast \) on ASP, characterized by

\[
s_{\alpha \ast \beta}(a, b) = \min_{\ell \in \mathbb{Z}} \left[ s_\alpha(a, \ell) + s_\beta(\ell, b) \right]
\]

for all \( \alpha, \beta \in \text{ASP} \) and \( a, b \in \mathbb{Z} \). We call \( \ast \) the Demazure product on ASP. For all \( \alpha \in \text{ASP} \) and \( n \in \mathbb{Z} \), \( \alpha \ast_{\chi} \sigma_n \) satisfies Equation (1), so \( \ast \) extends the standard Demazure product on \( S_d \).

The \( S_d \) case has a geometric interpretation in terms of the relative position of three flags, which we summarize in Appendix A.

The Demazure product coincides with the ordinary product in an important special case. We will see (Lemma 6.1) that \( \alpha \ast \beta \geq \alpha \beta \), and that equality holds if and only if \( \alpha \) and \( \beta^{-1} \) have no inversions in common. In this situation, we will say later that \( \alpha \ast \beta = \alpha \beta \) is a reduced product and write \( \alpha_{\text{red}} \beta \). I encourage the reader to attempt to prove this before reading on, as it provides useful intuition about Equation (5).

We also prove a second characterization of this operation. It generalizes [He09, Lemma 1], [BM15, Lemma 3.1(e)] from \( S_d \) to ASP, and is proved in Section 6.

**Theorem B.** For all \( \alpha, \beta \in \text{ASP} \), the following maxima in Bruhat order all exist, and equal \( \alpha \ast \beta \).

\[
\begin{align*}
\alpha \ast \beta &= \max \{ \alpha_1 \beta_1 : \alpha_1 \leq \alpha, \beta_1 \leq \beta \} \\
\alpha \ast \beta &= \max \{ \alpha \beta_1 : \beta_1 \leq_{\chi} \beta \} \\
\alpha \ast \beta &= \max \{ \alpha_1 \beta : \alpha_1 \leq_{\chi} \alpha \}.
\end{align*}
\]
One way to understand Equation (1) is that for \( \beta \in S_d \), factoring \( \beta \) into adjacent transpositions allows you to find the \( \beta_1 \) in Equation (7) by a greedy algorithm.

The last main objective of this paper is a “reduction theorem” for inequalities of the form \( \alpha \ast \beta \geq \gamma \). We will prove that such an inequality implies an equation \( \alpha_1 \ast \beta_1 = \gamma \). Furthermore, we will want these \( \alpha_1, \beta_1 \) to satisfy \( \alpha_1 \beta_1 = \alpha \ast \beta_1 \), and we wish to show that if \( \alpha, \beta, \gamma \) are from a suitable subgroup of ASP, then we may choose \( \alpha_1, \beta_1 \) from the same subgroup. Several examples of such subgroups are discussed in Section 8, including symmetric and affine symmetric groups. To state our result, we first require an auxiliary operation that will prove to be a useful accomplice to \(*\). This operation was identified by He in [He07, Lemma 1.4] for Coxeter groups.

**Theorem 1.1.** For all \( \alpha, \beta \in ASP \), the following Bruhat-minimum exists:

\[
\alpha \triangleleft \beta^{-1} = \min \{ \gamma \in ASP : \gamma \ast \beta \geq \alpha \}.
\]

The resulting operation \( \triangleleft \) on ASP is characterized by the equation

\[
s_{\alpha \triangleleft \beta}(a, b) = \max_{\ell \in \mathbb{Z}} \left[ s_{\alpha}(a, \ell) - s_{\beta^{-1}}(b, \ell) \right].
\]

For all \( \alpha \in ASP \) and \( n \in \mathbb{Z} \), \( \alpha \triangleleft \sigma_n = \begin{cases} \alpha \sigma_n & \text{if } \alpha(n) > \alpha(n+1), \\ \alpha & \text{otherwise.} \end{cases} \)

Most of this theorem is proved in Theorem 4.10, except the statement about \( \sigma_n \), which is proved in Theorem 8.7. The operation \( \triangleleft \) also has a characterization analogous to Theorem B stated in Theorem 6.1, which is closer to He’s definition in [He07].

There is also a dual version of \( \triangleleft \), which we denote \( \triangleright \), characterized by

\[
\alpha^{-1} \triangleright \beta = \min \{ \gamma \in ASP : \alpha \ast \gamma \geq \beta \}.
\]

It suffices to focus on \( \triangleleft \), since we will prove that \( \alpha \triangleright \beta = (\beta^{-1} \triangleleft \alpha^{-1})^{-1} \) (Theorem 4.10).

We mention the case \( \beta = \sigma_n \) in the statement above to support the following intuition: where \( * \) is a “stingy multiplication,” \( \triangleleft \) is a “stingy multiplication;” \( \beta \) attempts to make \( \alpha \) as Bruhat-small as possible using part of itself. This is made precise in Theorem 6.1 in analogy to Theorem B.

Our reduction theorem for inequalities \( \alpha \ast \beta \geq \gamma \) is the following. It is proved in Section 6 along with a generalization to products of three or more permutations (Theorem 6.5).

**Theorem C.** Let \( G \leq ASP \) be a subgroup that is closed under \( \triangleleft \). For all \( \alpha, \beta, \gamma \in G \) such that \( \chi_\alpha + \chi_\beta = \chi_\gamma \), \( \alpha \ast \beta \geq \gamma \) if and only if there exist \( \alpha_1, \beta_1 \in G \) such that \( \alpha_1 \leq_{\chi} \alpha, \beta_1 \leq_{\chi} \beta \), and \( \alpha_1 \ast \beta_1 = \alpha_1 \beta_1 = \gamma \).

More specifically, for all \( \alpha, \beta, \gamma \in ASP \), if \( \alpha \ast \beta \geq \gamma \), then \( \alpha \triangleleft \beta^{-1} = \beta^{-1} \triangleright \gamma \) satisfy \( \alpha_1 \ast \beta_1 = \alpha_1 \beta_1 = \gamma \) and \( \alpha_1 \leq_{\chi} \alpha, \beta_1 \leq_{\chi} \beta \).

In addition to Theorems A, B, and C we also examine some other properties of Bruhat order and \( \ast \) on certain subgroups of ASP that will useful in our applications; see Sections 7 and 8.

### 1.3. Background on Demazure products

We use the name Demazure product in reference to [Dem74], in which a collection of operators \( L_\alpha \) are defined such that, in our notation, \( L_\alpha L_\beta = L_{\alpha \ast \beta} \) (see §5.6 of that paper). The same operation occurred around the same time in [BGG73]. This point of view, realizing the Demazure product in terms of composition of operators, occurs in many other contexts, e.g. [KM04, Definition 3.1], which considers operators \( \overline{\partial}_i \), \( 1 \leq i \leq n \), on a polynomial ring \( R[x_1, \cdots, x_n] \) defined by

\[
\overline{\partial}_i(f) = \frac{x_{i+1} f - x_i(s_i f)}{x_{i+1} - x_i},
\]
and considers the Demazure algebra generated by these operators. These difference operators are related, but not identical to, the difference operators used to define Schubert polynomials, e.g. as in [Ful97, p. 165]. I do not know if there is a useful way to relate the min-plus formulation of $\star$ to this operator formulation, or whether the Demazure product on ASP can be understood in a similar way.

The other principal point of view on $\star$ is from the Hecke algebra, in which it arises (up to sign) by setting the parameter $q$ to 0, hence called the 0-Hecke product. See [Hum90, §7] or [BB05, §6].

Applications of the Demazure product to algebraic geometry typically relate to the geometry of Schubert varieties. Another interesting example of such applications is [BM15], which shows that a curve neighborhood of a Schubert variety results in another Schubert variety, indexed by permutation given by the Demazure product.

1.4. Outline of the paper. We begin with some preliminary facts about ASP and the functions $s_\alpha$ in Section 2. Section 3 introduces a slightly broader class of functions, called slipfaces, formulates $\star$ and $\triangleright$ in that context, and proves a few basic facts that can be done at that level of generality. Section 4 identifies the functions $s_\alpha$ as those slipfaces that are submodular, thereby constructs $\star$ and $\triangleright$ on ASP and proves Theorem A, and analyzes the locations of inversions after these operations. Sections 5 and 6 leverage the earlier results to relate $\star$ and $\triangleright$ to reduced products and weak Bruhat orders, to prove Theorems B and C. Section 7 generalizes a concept called the essential set to ASP, in order to study Bruhat order. Section 8 considers a number of subgroups of ASP of special interest, and identifies some aspects of $\star$ and $\triangleright$ specific to those subgroups. Finally, two appendices follow that provide some context to motivate the contents of this paper. Appendix A provides a concrete geometric interpretation of the “min-plus matrix multiplication” formula in the $S_d$ case, in terms of relative positions of flags. Appendix B briefly summarizes the intended applications of the Demazure product on ASP to divisor theory on algebraic curves and (metric) graphs.

2. Preliminaries on almost-sign-preserving permutations

This section collects a few preliminary facts about almost-sign-preserving permutations and their slipface functions, and also attempts to explain my somewhat whimsical choice of the word “slipface.” First observe the following characterizations of finite differences of $s_\alpha$. Note that Equation (14) shows that the slipface function $s_\alpha$ uniquely determines the permutation $\alpha$.

\begin{align}
\alpha(a, b) - \alpha(a, b + 1) &= \delta(\alpha(b) < a) \\
\alpha(a + 1, b) - \alpha(a, b) &= \delta(\alpha^{-1}(a) \geq b) \\
s_\alpha(a + 1, b) - s_\alpha(a, b) - s_\alpha(a + 1, b + 1) + s_\alpha(a, b + 1) &= \delta(a(b) = a)
\end{align}

I will now attempt to convince you that the word “slipface” does indeed evoke the basic properties of these functions $s_\alpha$. The function $s_\alpha$ may be usefully visualized as a sequence of functions $f(n) = s_\alpha(n, b)$ for various choices of $b$. For $b$ fixed, this function is a nondecreasing function, beginning at 0, and eventually stabilizing to have slope 1. When $b$ increases, Equation (12) shows that part of the function drops, namely all points with $n > \alpha(b)$, so the slope-1 line that the graph approaches shifts one unit to the right. Some examples are shown in Figure 2. In these three examples, we consider three permutations in $S_9$. To me at least, this sequence of graphs evokes a dune of sand which, over time, drops in height as sand slips and falls down the face, and thereby appears to move to the right.

2.1. Some properties of the shift $\chi_\alpha$. Equations (12) and (13) show that the function $f(a, b) = s_\alpha(a, b) - s_\alpha^{-1}(b, a) - a + b$ is constant. Since $\chi_\alpha = f(0, 0)$ by definition, we have the following
duality between \( s_\alpha \) and \( s_{\alpha^{-1}} \), reminiscent (not coincidentally!) of the Riemann-Roch formula.

\[
(15) \quad s_\alpha(a, b) - s_{\alpha^{-1}}(b, a) = \chi_\alpha + a - b.
\]

Equation (15) shows that \( s_\alpha(a, b) \geq \max \{0, \chi_\alpha + a - b\} \), with equality if and only if either \( s_\alpha(a, b) = 0 \) or \( s_{\alpha^{-1}}(b, a) = 0 \). For \( a \) fixed and \( b \ll 0 \), or \( b \) fixed and \( a \gg 0 \), \( s_{\alpha^{-1}}(b, a) = 0 \), so \( s_\alpha(a, b) = \chi_\alpha + a - b \).

In this sense, the shift \( \chi_\alpha \) governs the asymptotic behavior of \( s_\alpha \). We mention a few other immediate consequences of Equation (15).

**Lemma 2.1.** For all \( \alpha, \beta \in \text{ASP} \), \( \alpha \leq \chi \beta \) if and only if \( \alpha^{-1} \leq \chi^{-1} \).

**Proof.** If \( \chi_\alpha = \chi_\beta \), then Equation (15) implies \( s_\alpha(a, b) - s_{\beta^{-1}}(b, a) = s_{\alpha^{-1}}(b, a) - s_{\beta^{-1}}(b, a) \).

**Definition 2.2.** For any \( \chi \in \mathbb{Z} \), denote by \( \iota_\chi \) the increasing permutation \( \iota_\chi(n) = n - \chi \), of shift \( \chi \).

**Lemma 2.3.** For any \( \chi \in \mathbb{Z} \) and \( \alpha \in \text{ASP} \), if \( \chi_\alpha \geq \chi \) then \( \alpha \geq \iota_\chi \). In particular,

\[
\iota_\chi = \min \{ \alpha \in \text{ASP} : \chi_\alpha = \chi \}.
\]

**Proof.** For all \( a, b \in \mathbb{Z} \), \( s_\alpha(a, b) = s_{\alpha^{-1}}(b, a) + \chi + a - b \geq \max \{0, \chi + a - b\} = s_\chi(a, b) \).

If we define \( \text{sgn}(n) \) to be 1 for \( n \geq 0 \) and -1 for \( n < 0 \), then

\[
\chi_\alpha = \frac{1}{2} \sum_{n \in \mathbb{Z}} \left( \text{sgn}(n) - \text{sgn}(\alpha(n)) \right).
\]

It follows from this description, rearranging a sum and cancelling, that \( \chi \) is a homomorphism:

\[
(16) \quad \chi_{\alpha \beta} = \chi_\alpha + \chi_\beta.
\]

Thus shift sorts \( \text{ASP} \) into cosets of a homomorphism \( \chi : \text{ASP} \to \mathbb{Z} \). This homomorphism has a nice splitting, given by \( \chi \mapsto \iota_\chi \). We will see that \( \chi \) is also a monoid homomorphism for the Demazure product: \( \chi_{\alpha \ast \beta} = \chi_\alpha + \chi_\beta \) (Theorem 4.4). Furthermore, \( \iota_m \ast \alpha = \iota_m \alpha \) and \( \alpha \ast \iota_m = \alpha \iota_m \) for all \( \alpha \in \text{ASP} \) (Lemma 5.1). This means that essentially everything we might wish to know about \( \ast \) on \( \text{ASP} \) is determined by its restriction to \( \text{ASP}_0 = \{ \alpha \in \text{ASP} : \chi_\alpha = 0 \} \).

**Example 2.4.** For any \( k \geq 2 \), the group of permutations \( \alpha : \mathbb{Z} \to \mathbb{Z} \) such that \( \alpha(n + k) = \alpha(n) + k \) for all \( n \in \mathbb{Z} \) is called an extended affine symmetric group. For such a permutation \( \alpha \), sorting \( \mathbb{Z} \) into cosets modulo \( k \) and counting sign changes shows that \( \chi_\alpha = -\frac{1}{k} \sum_{n=0}^{k-1} (\alpha(n) - n) \). One standard definition of the (unextended) affine symmetric group \( \mathfrak{S}_k \) is that its elements satisfy

![Figure 2](image-url)

Figure 2. Superimposed plots of the graphs \( y = s_\alpha(x, b) \) for \( 0 \leq b \leq 20 \), and three different permutations \( \alpha \) of \( \{1, \cdots, 9\} \). Each permutation is written in one-line notation \( \alpha(1) \alpha(2) \cdots \alpha(9) \). The graph in Figure 1 may be useful in studying the middle example.
\( \alpha(n + k) = \alpha(n) + k \) for all \( n \) and \( \sum_{n=0}^{k-1} (\alpha(n) - n) = 0 \). So in our terminology, this is the shift-0 subgroup of the extended affine symmetric group.

2.2. **Inversions, reduced products, and the weak orders.** Much of our analysis will be based on careful study of inversions.

**Definition 2.5.** For a permutation \( \alpha \in \text{ASP} \), the set of inversions of \( \alpha \) is

\[
\text{Inv} \alpha = \{(u,v) \in \mathbb{Z}^2 : u < v \text{ and } \alpha(u) > \alpha(v)\}.
\]

The set of inversions is the basis for two other partial order on \( \text{ASP} \). The terms below are chosen to mirror terminology from Coxeter groups; see e.g. [BB05], especially §1.4, Exercise 1.4.13, §3.1 and Proposition 3.1.3.

**Definition 2.6.** The **left weak order** on \( \text{ASP} \) is the partial order \( \leq_L \), where \( \alpha \leq_L \beta \) if and only if \( \text{Inv} \alpha \subseteq \text{Inv} \beta \). The **right weak order** \( \leq_R \) is defined by \( \alpha \leq_R \beta \) if and only if \( \text{Inv}^{-1} \alpha \subseteq \text{Inv}^{-1} \beta \), i.e. \( \alpha^{-1} \leq_L \beta^{-1} \).

Closely related to the weak orders is the notion of a reduced product.

**Definition 2.7.** A product \( \alpha \beta \) of two permutations is called **reduced** if \( \text{Inv}(\alpha) \cap \text{Inv}(\beta^{-1}) = \emptyset \). We write “\( \alpha \star_{\text{red}} \beta \)” as a shorthand for “\( \alpha \beta \) is a reduced product.” The symbol \( \star_{\text{red}} \) will sometimes be used in a larger statement, in which case it denotes the same thing as \( \star \) and simultaneously asserts that the product is reduced; so \( \alpha \star_{\text{red}} \beta = \gamma \) abbreviates “\( \alpha \star \beta = \gamma \) and \( \text{Inv}(\alpha) \cap \text{Inv}(\beta^{-1}) = \emptyset \).”

In fact, we prove in Lemma 5.1 that \( \alpha \beta \) is a reduced product if and only if \( \alpha \star \beta = \alpha \beta \), which is why we have chosen the notation \( \star_{\text{red}} \).

The two weak orders and the notion of reduced products are closed related.

**Lemma 2.8.** For \( \alpha, \beta \in \text{ASP} \), the following are equivalent.

1. \( \alpha \star_{\text{red}} \beta \)
2. \( \alpha \leq_R \alpha \beta \)
3. \( \beta \leq_L \alpha \beta \)

**Proof.** A pair \( (u,v) \in \mathbb{Z}^2 \) belongs to \( \text{Inv} \alpha \cap \text{Inv} \beta^{-1} \) if and only if \( (\alpha(v), \alpha(u)) \in \text{Inv} \alpha^{-1} \setminus \text{Inv} \beta^{-1} \alpha^{-1} \), and similarly \( (u,v) \in \text{Inv} \alpha \cap \text{Inv} \beta^{-1} \) if and only if \( (\beta^{-1}(v), \beta^{-1}(u)) \in \text{Inv} \beta \setminus \text{Inv} \alpha \beta \). So the three sets \( \text{Inv} \alpha \cap \text{Inv} \beta^{-1} \), \( \text{Inv} \alpha^{-1} \setminus \text{Inv} \beta^{-1} \alpha^{-1} \), \( \text{Inv} \beta \setminus \text{Inv} \alpha \beta \) are either all empty or all nonempty. [1]

**Remark 2.9.** Lemma 4.9 and Theorem C will imply the following alternative description of the weak orders: \( \alpha \leq_L \beta \) (resp. \( \alpha \leq_R \beta \)) if and only if there exists some \( \gamma \in \text{ASP} \) such that \( \gamma \star \alpha = \beta \) (resp. \( \alpha \star \gamma = \beta \)). This gives a way to remember which is “left” and “right:” it depends on which side the extra permutation \( \gamma \) is placed.

**Warning 2.10.** The Bruhat order on \( \text{ASP} \) has some counterintuitive features when used to compare permutations of different shift. For example, Lemma 2.1 is false without the assumption \( \chi_\alpha = \chi_\beta \); a counterexample is \( t_0 \leq t_1 \) but \( t_0 \geq t_{-1} \). In contrast to the situation in \( S_d \) or other Coxeter subgroups of \( \text{ASP} \), and indeed seemingly against basic decency of word choice, \( \alpha \leq_L \beta \) does **not** imply that \( \alpha \leq \beta \) in \( \text{ASP} \), so \( \leq_L \) is not actually a “weaker” partial order than \( \leq \). For example, \( t_m \leq_L t_n \) for all \( m, n \in \mathbb{Z} \). However, this implication is valid if we assume \( \chi_\alpha = \chi_\beta \) (Corollary 5.3).

### 3. Slipface functions

In this section, we define a broader class of functions \( \mathbb{Z}^2 \to \mathbb{N} \), possessing several of the features ascribed to the slipface functions \( s_\alpha \). We call this broader class **slipfaces** as well, and will characterize those slipface functions equal to \( s_\alpha \) for some \( \alpha \in \text{ASP} \) as **submodular slipfaces**. There are two reasons
to step back to this more general setting. First, many of our proofs are naturally organized by first reasoning about general slipfaces and then adding the submodularity condition. Second, in our intended application to metric graphs, non-submodular slipfaces naturally occur.

**Definition 3.1.** Let \( \chi \) be an integer. A slipface function of shift \( \chi \) is a function \( s : \mathbb{Z}^2 \to \mathbb{N} \) such that for all \( a, b \in \mathbb{Z} \),

\[
\begin{align*}
(S1) & \quad 0 \leq s(a + 1, b) - s(a, b) \leq 1 \text{ and } 0 \leq s(a, b) - s(a, b + 1) \leq 1; \\
(S2) & \quad s(a, b) \geq \max\{0, \chi + a - b\}; \\
(S3) & \quad \text{For all but finitely many } a', b' \in \mathbb{Z}, \ s(a', b) = \max\{0, \chi + a - b\}. \text{ Similarly, for all but finitely many } b' \in \mathbb{Z}, \ s(a, b') = \max\{0, \chi + a - b'\}
\end{align*}
\]

A function \( s \) is called a slipface if it is a slipface of some shift; the shift is uniquely determined by \( s \) and denoted \( \chi_s \). Denote by \( \text{SF}_\chi \) the set of slipfaces of shift \( \chi \), and \( \text{SF} \) the set of all slipfaces.

**Definition 3.2.** Partially order \( \text{SF} \) as follows: \( s \leq t \) means that \( s(a, b) \leq t(a, b) \) for all \( a, b \in \mathbb{Z} \). We call this the Bruhat order on \( \text{SF} \).

**Example 3.3.** For \( \alpha \in \text{ASP} \), the discussion in Section 2 shows that \( s_\alpha \in \text{SF}_{\chi_\alpha} \).

Slipfaces possess a duality, analogous to Equation (15).

**Definition 3.4.** If \( s \in \text{SF}_\chi \), then the dual slipface is the function \( s^\vee \in \text{SF}_{-\chi} \) characterized by

\[
s(a, b) - s^\vee(b, a) = \chi + a - b.
\]

Substituting into Definition 3.1 shows \( s^\vee \) is indeed a slipface, with shift \( -\chi \).

**Example 3.5.** Equation (15) demonstrates that, for all \( \alpha \in \text{ASP} \),

\[
s_\alpha^\vee = s_{\alpha - 1}.
\]

One convenient aspect of duality is that it provides a streamlined criterion for checking that a given function is indeed a slipface, and determining its shift, in many situations. For a function \( f : \mathbb{Z}^2 \to \mathbb{Z} \), we consider the following two criteria on \( f \).

- **(D1)** For all \( a, b \in \mathbb{Z} \), \( f(a + 1, b) \geq f(a, b) \) and \( f(a, b + 1) \leq f(a, b) \).
- **(D2)** For fixed \( a \) and any \( b \gg 0 \), \( f(a, b) = 0 \); similarly, for fixed \( b \) and any \( a \ll 0 \), \( f(a, b) = 0 \).

**Lemma 3.6.** Suppose that \( s, t \) are two functions \( \mathbb{Z}^2 \to \mathbb{Z} \) and \( \chi \) is an integer, satisfying the equation

\[
s(a, b) - t(b, a) = \chi + a - b
\]

for all \( a, b \in \mathbb{Z} \). Then \( s \in \text{SF}_{\chi} \) if and only if both \( s \) and \( t \) satisfy conditions (D1) and (D2). If so, then also \( t \in \text{SF}_{-\chi} \) and \( t = s^\vee \).

**Proof.** Criterion (D1) is equivalent to the lower bounds in (S1) when applied to \( s \), and the upper bounds in (S1) when applied to \( t \). Criterion (S2) is equivalent to both \( s \) and \( t \) being nonnegative, and equality holds in (S2) whenever either \( s \) or \( t \) vanishes, so (S3) follows from (D2) applied to both \( s \) and \( t \). Conversely, (D1) follows from (S1) and (D2) follows from (S3). \( \square \)

### 3.1. The operations \( \star, \bowtie, \triangleright \) on \( \text{SF} \)

We are concerned with the following operations on \( \text{SF} \).

**Definition 3.7.** Let \( s, t \in \text{SF} \). Define functions \( s \star t, s \bowtie t \) and \( s \triangleright t \) as follows.

\[
\begin{align*}
s \star t(a, b) &= \min_{\ell \in \mathbb{Z}} \left[ s(a, \ell) + t(\ell, b) \right] \\
s \bowtie t(a, b) &= \max_{\ell \in \mathbb{Z}} \left[ s(a, \ell) - t^\vee(b, \ell) \right] \\
s \triangleright t(a, b) &= \max_{\ell \in \mathbb{Z}} \left[ t(\ell, b) - s^\vee(\ell, a) \right]
\end{align*}
\]
To see that the maxima defining $<$ and $>$ exist, note that as $k \to \pm \infty$, the fact that $s, t$ are slipfaces means that the expressions $s(a, k) - t^\vee(b, k)$ and $s^\vee(k, b) - t(k, a)$ converge, with one of the two limits converging to 0. This implies that the maxima are well-defined, and nonnegative.

The following compatibilities with $\leq$ are immediate from this definition.

Lemma 3.8. The product $s \star t$ is nondecreasing in both $s$ and $t$ (in Bruhat order). That is, if $s_1 \leq s_2$ and $t_1 \leq t_2$, then $s_1 \star t_1 \leq s_2 \star t_2$. On the other hand, $s \triangleleft t^\vee$ and $t^\vee \triangleright s$ are both nondecreasing in $s$ and nonincreasing in $t$.

Proposition 3.9. Let $s, t \in SF$. Then $s \star t$, $s \triangleleft t$ and $s \triangleright t$ are all slipfaces of shift $\chi_s + \chi_t$. These operations satisfy the identities $(s \star t)^\vee = t^\vee \star s^\vee$ and $(s \triangleleft t)^\vee = t^\vee \triangleright s$.

Proof. Consider first the operation $\star$. Note first that $s \star t(a, b)$ is well-defined and nonnegative for all $a, b$ since $s, t$ are nonnegative. Writing $s(a, \ell) = s^\vee(t, a) + \chi_s + a - \ell$ and $t(\ell, b) = t^\vee(b, \ell) + \chi_t + \ell - b$ and unwinding definitions shows that

$$s \star t(a, b) = t^\vee \star s^\vee(b, a) + \chi_s + \chi_t + a - b$$

for all $a, b \in \mathbb{Z}$. By Lemma 3.6 it suffices to verify that $s \star t$ and $t^\vee \star s^\vee$ satisfy (D1) and (D2). We need only check this for $s \star t$ since the same argument will also apply to $t^\vee \star s^\vee$. Criterion (D1) follows from the fact that $s, t$ and their duals satisfy (D1); (D2) requires a bit of casework. To verify (D2) for $s \triangleleft t$, let $L(a, b)$ denote the set of integers $\ell$ such that $s(a, \ell) > t^\vee(b, \ell)$. Observe that $L(a, b + 1) \subseteq L(a, b)$ and $L(a - 1, b) \subseteq L(a, b)$; we must show that if one of $a, b$ is fixed and the other is decreased or increased (respectively) the set $L(a, b)$ eventually shrinks to $\emptyset$. This follows from two observations. First, $L(a, b)$ is finite provided that $\chi_s + \chi_t + a - b \leq 0$, by considering the limit of $s(a, \ell) - t^\vee(b, \ell)$ as $\ell$ tends to $\pm \infty$. Second, for any particular $a, b, \ell \in \mathbb{Z}$, $\ell$ eventually drops out of $L(a, b)$ as $a$ is decreased or $b$ is increased. More precisely, for $a, \ell$ fixed and $b$ sufficiently large, $s(a, \ell) - t^\vee(b, \ell) = s(a, \ell) - \chi_t - b + \ell \leq 0$ and thus $\ell \notin L(a, b)$; similarly for $b$ fixed and $a$ sufficiently small, $s(a, \ell) - t^\vee(b, \ell) = 0 - t^\vee(b, \ell) \leq 0$ and thus $\ell \notin L(a, b)$. Therefore $s \triangleleft t$ satisfies (D2). A similar argument shows that $s \triangleright t$ satisfies (D2): we may define $L(a, b) = \{ \ell \in \mathbb{Z} : t(\ell, b) < s^\vee(a, \ell)\}$; then $L(a, b)$ is finite provided that $\chi_s + \chi_t + a - b \leq 0$, and any particular $\ell \in L(a, b)$ eventually drops out as $a$ is decreased or $b$ is increased.

We now claim that $s \triangleleft t$ and $t^\vee \triangleright s^\vee$ are dual slipfaces, with $\chi_{s \triangleleft t} = \chi_s + \chi_t$. Writing $s(a, \ell) = s^\vee(\ell, a) + \chi_s + a - \ell$ and $t(\ell, b) = t(\ell, b) - \chi_t + b - \ell$ and unwinding definitions shows that

$$s \triangleleft t(a, b) = t^\vee \triangleright s^\vee(b, a) + \chi_s + \chi_t + a - b$$

for all $a, b \in \mathbb{Z}$. Since $s \triangleleft t$ and $t^\vee \triangleright s^\vee$ both satisfy (D1) and (D2), Lemma 3.6 implies that they are dual slipfaces with $\chi_{s \triangleleft t} = \chi_s + \chi_t$. Replacing $s, t$ by $t^\vee, s^\vee$ in this argument shows that $s \triangleright t$ is also a slipface of shift $\chi_s + \chi_t$ and dual $t^\vee \triangleleft s^\vee$.

Lemma 3.10. Let $s, t, u \in SF$. The following are equivalent.

$$(1) \quad s \star t \geq u \quad (2) \quad s \geq u \triangleleft t^\vee \quad (3) \quad t \geq s^\vee \triangleright u$$

In other words, the operators $\triangleleft, \triangleright$ are the following minima in Bruhat order.

$$u \triangleleft t^\vee = \min\{s \in SF : s \star t \geq u\}, \quad \text{and} \quad s^\vee \triangleright u = \min\{t \in SF : s \star t \geq u\}.$$
Then properties (1) and (2) above imply that
\[ t \leq s \] holds exactly as stated for this set \( M \).

Let \( s \leq t \) bounded above since
\[ t \in M \text{ } \text{ and } \text{ the same argument shows that } \text{the minimum occurs, i.e.} \]
\[ s \leq t \] of \( M \).

Lemma 3.11. The operations \( \ast, \triangleleft, \triangleright \) on SF satisfy the following identities.

(1) \( s \ast (t \ast u) = (s \ast t) \ast u \) (\( \ast \) is associative).

(2) \( s \triangleleft (t \triangleleft u) = s \triangleleft (t \ast u) \).

(3) \( s \triangleright (t \triangleright u) = (s \ast t) \triangleright u \).

Proof. Part (1) follows by observing that for all \( a, b \in \mathbb{Z} \),
\[ s \ast (t \ast u)(a, b) = \min_{\ell_1, \ell_2 \in \mathbb{Z}} \left[ s(a, \ell_1) + t(\ell_1, \ell_2) + u(\ell_2, b) \right] = (s \ast t) \ast u(a, b). \]

Parts (2) and (3) follow from combining part (1) with the characterization of \( \triangleleft, \triangleright \) in Lemma 3.10. For part (2), note that \( (s \triangleleft t) \triangleleft u \) is the minimum \( v \in SF \) such that \( v \ast u \geq s \triangleleft t \), which is the minimum such that \( (v \ast u)^\ast t \geq s \). By associativity of \( \ast \) and the equation \( u^\ast t = (t \ast u)^\ast \), this is equivalent to \( v \ast (t \ast u) \geq s \), so \( v = s \triangleleft (t \ast u) \). An analogous argument proves part (3). \( \square \)

3.2. Example computations. We now illustrate the definitions of \( \ast \) and \( \triangleleft \) in two particularly simple cases: when one permutation is increasing, or a disjoint product, possibly infinite, of adjacent transpositions. In these and other computations, it is useful to shrink the set on which the extremum defining \( \ast, \triangleleft, \triangleright \) is computed. The following lemma provides one way to do so (by no means the only way). For convenience this lemma is stated only for \( \ast \) and \( \triangleleft \), but it can be applied to \( \triangleright \) as well using the identity \( s \triangleright t = (t^\triangleright \triangleleft s^\ast)^\ast \).

Lemma 3.12. Fix \( s, t \in SF \) and \( a, b \in \mathbb{Z} \). Define a set
\[ L = \{ \ell \in \mathbb{Z} : t(\ell - 1, b) = t(\ell, b) < t(\ell + 1, b) \}. \]

Then the minimum defining \( s \ast t(a, b) \), and the maximum defining \( s \triangleleft t(a, b) \), each occur for some \( \ell \in L \). That is, \( s \ast t(a, b) = \min_{\ell \in L} \left[ s(a, \ell) + t(\ell, b) \right] \), and \( s \triangleleft t(a, b) = \max_{\ell \in L} \left[ s(a, \ell) - t^\ast (b, \ell) \right] \).

The set \( L \) is finite. In the case \( t = s^\beta \) for some \( \beta \in ASP \), it is also equal to
\[ L = \{ \ell \in \mathbb{Z} : \beta^{-1}(\ell - 1) < b \leq \beta^{-1}(\ell) \}. \]

Proof. The finiteness of \( L \) follows from criterion (S3), and the alternative formula in the case \( t = s^\beta \) follows from Equation (13).

First consider \( s \ast t \). Let \( M \) be the set of integers \( \ell \) where the minimum occurs, i.e.
\[ M = \{ \ell \in \mathbb{Z} : s \ast t(a, b) = s(a, \ell) + t(\ell, b) \}. \]

It suffices to show that \( L \cap M \) is nonempty. Criterion (S1) implies the following two facts.

(1) If \( \ell \in M \) and \( t(\ell - 1, b) < t(\ell, b) \), then \( \ell - 1 \in M \).

(2) If \( \ell \in M \) and \( t(\ell + 1, b) = t(\ell, b) \), then \( \ell + 1 \in M \).

Let \( M_1 \subset M \) denote the subset on which \( t(\ell, b) \) is minimized among all elements of \( M \). This is bounded above since \( t(\ell, b) \) tends to infinity as \( \ell \) grows. Let \( \ell \) be the maximum element of \( M \).

Then properties (1) and (2) above imply that \( \ell \in L \). So \( L \) and \( M \) have nonempty intersection.

Now consider \( \triangleleft \), and redefine \( M = \{ \ell \in \mathbb{Z} : s \triangleleft t(a, b) = s(a, \ell) - t^\ast (b, \ell) \} \). Using the fact that \( t^\ast (b, \ell + 1) < t^\ast (b, \ell) \) if and only if \( t(\ell + 1, b) = t(\ell, b) \), it follows that properties (1) and (2) above hold exactly as stated for this set \( M \), and the same argument shows that \( M \) intersects \( L \). \( \square \)

Lemma 3.13. Let \( s \in SF \) and \( n \in \mathbb{Z} \). For all \( a, b \in \mathbb{Z} \),
\[ s \ast s_{in}(a, b) = s \triangleleft s_{in}(a, b) = s(a, b - n). \]

in particular, \( s_{in} \) is the identity element for \( \ast \) on SF.
Lemma 3.16. Let \( s \) be the slipface associated to the permutation defined above. Let \( s \) be the involutions \( \sigma \). In the special case where \( b \neq n \), let \( s \) be the (possibly infinite) product of the simple transpositions \( \{ \sigma_n : n \in S \} \).

Proof. Let \( t = s_{t_0} \) in Lemma 3.12. Then \( t(a, b) = \max \{0, n + a - b\} \) for all \( a, b \in \mathbb{Z} \), so \( L = \{b - n\} \). Therefore \( s \ast s_{t_0}(a, b) = s(a, b - n) + s_{t_0}(b, n) \), and \( s \ast s_{t_0}(a, b) = s(a, b - n) - s_{t_0}(b, b - n) \). Using \( s_{t_0} = s_{t-n} \), both \( s_{t}(b, n) \) and \( s_{t}(b, b - n) \) are zero, and the displayed equation follows. So \( s \ast s_{t_0} = s \); it also follows that \( s_{t_0} \ast t = (t' \ast s_{t_0})' = (t')' = t \) for all \( t \in SF \), so \( s_{t_0} \ast t \) is both a left-identity and a right-identity for \( \ast \).

\( \square \)

Corollary 3.17. If \( s, t \in SF \) and \( \chi_t \geq 0 \), then \( s \ast t \geq s \) and \( s \ast t \leq s \).

Proof. Lemma 3.8 implies that \( s \ast t \geq s \ast s_{t_0} = s \) and \( s \ast t \leq s \ast s_{t_0} = s \).

\( \square \)

Definition 3.15. Let \( S \subseteq \mathbb{Z} \) be a set of integers that contains no two consecutive integers. Let \( \sigma_S : \mathbb{Z} \to \mathbb{Z} \) be the permutation exchanging \( n \) and \( n+1 \) for all \( n \in S \), and fixing all other integers. In other words, \( \sigma_S \) is the (possibly infinite) product of the simple transpositions \( \{ \sigma_n : n \in S \} \).

These include the simple transpositions \( \sigma_n = \sigma_{\{n\}} \) used to generate symmetric groups, as well as the involutions \( \bar{\sigma}_n = \sigma_{n+k\mathbb{Z}} \) used to generate the affine symmetric group \( \bar{S}_k \).

Lemma 3.16. Let \( S \subseteq \mathbb{Z} \) be a set containing no two consecutive integers, and let \( t = s_{\sigma_S} \) be the slipface associated to the permutation defined above. Let \( s \) be any other slipface. For all \( a, b \in \mathbb{Z} \),

\[
\begin{align*}
s \ast s_{\sigma_S}(a, b) &= s(a, b) + \delta \left[ b - 1 \in S \text{ and } s(a, b - 1) > s(a, b) = s(a, b + 1) \right], \quad \text{and} \\
s \ast s_{\sigma_S}(a, b) &= s(a, b) - \delta \left[ b - 1 \in S \text{ and } s(a, b - 1) = s(a, b) > s(a, b + 1) \right].
\end{align*}
\]

In the special case where \( s = s_{\alpha} \) for some \( \alpha \in ASP \),

\[
\begin{align*}
s_{\alpha} \ast s_{\sigma_S}(a, b) &= s_{\alpha}(a, b) + \delta \left[ b - 1 \in S \text{ and } s_{\alpha}(b - 1) < \alpha \leq s_{\alpha}(b) \right], \quad \text{and} \\
s_{\alpha} \ast s_{\sigma_S}(a, b) &= s_{\alpha}(a, b) - \delta \left[ b - 1 \in S \text{ and } \alpha < b \leq s_{\alpha}(b - 1) \right].
\end{align*}
\]

Proof. The second two equations, in the special case \( s = s_{\alpha} \), follow from the general case and Equation (12), so we focus on the general case. Fix \( a, b \in \mathbb{Z} \). Note that \( \sigma_S^{-1} = \sigma_S \) and thus \( s_{\sigma_S} = s_{\sigma_S} \). The set \( L \) from Lemma 3.12 is

\[
L = \{ \ell \in \mathbb{Z} : \sigma_S(\ell - 1) < b \leq \sigma_S(\ell) \} = \begin{cases} \{b\} & \text{if } b - 1 \notin S, \\ \{b - 1, b + 1\} & \text{if } b - 1 \in S. \end{cases}
\]

If \( b - 1 \notin S \), then \( s_{\sigma_S}(b, b) = 0, \) so \( s \ast s_{\sigma_S}(a, b) = s \ast s_{\sigma_S}(a, b) = s(a, b) \), as desired. On the other hand, if \( b - 1 \in S \), then \( s_{\sigma_S}(b - 1, b) = s_{\sigma_S}(b, b + 1) + 0 \) and \( s_{\sigma_S}(b + 1, b) = s_{\sigma_S}(b, b - 1) = 1 \), so \( s \ast s_{\sigma_S}(a, b) = \min\{s(a, b - 1), s(a, b + 1) + 1\} \), and \( s \ast s_{\sigma_S}(a, b) = \max\{s(a, b - 1) - 1, s(a, b + 1)\} \). The result now follows from criterion (S1).

\( \square \)

Corollary 3.17. If \( \alpha \in ASP \) has \( \alpha(n) > \alpha(n + 1) \) for all \( n \in S \), then \( s_{\alpha} \ast s_{\sigma_S} = s_{\alpha} \). In particular, \( s_{\sigma_S} \ast s_{\sigma_S} = s_{\sigma_S} \). On the other hand, if \( \alpha(n) < \alpha(n + 1) \) for all \( n \in S \), then \( s_{\alpha} \ast s_{\sigma_S} = s_{\alpha} \).

A more general formula for \( s \ast s_{\sigma_S} \) is proved in Theorem 3.7.

4. The image of ASP in SF: submodular slipfaces

The map \( \alpha \mapsto s_{\alpha} \) embeds ASP \( \hookrightarrow SF \). We demonstrate in this section that the image is closed under the operations \( \ast, \ll, \gg \), and thereby obtain corresponding operations on ASP. A key tool is a characterization of the image of this inclusion as the set of submodular slipfaces.

The following shorthand will prove useful. For \( s \in SF \), define \( \Delta s : \mathbb{Z}^2 \to \{-1, 0, 1\} \) by

\[
\Delta s(a, b) = s(a + 1, b) - s(a, b) - s(a + 1, b + 1) + s(a, b + 1).
\]
The fact that $\Delta s(a, b) \in \{-1, 0, 1\}$ follows from Criterion (S1). The function $s$ can be reconstructed from $\Delta s$. Observe that for $a, b$ fixed, there are only finitely many $(a', b') \in \mathbb{Z}$ such that $a' \leq a, b' \geq b$ and $s(a', b') > 0$. From this and a telescoping sum, it follows that

\begin{equation}
(17) \quad s(a, b) = \sum_{a' \leq a, b' \geq b} \Delta s(a', b').
\end{equation}

Here and throughout this section, an equation involving a sum of infinitely many integers is understood to mean implicitly that only finitely many terms are nonzero; equivalently, the sum converges absolutely. The definition of $s^\vee$ and some cancellation shows that

\begin{equation}
(18) \quad \Delta s^\vee(a, b) = \Delta s(b, a),
\end{equation}

and then Equation (17) implies that

\begin{equation}
(19) \quad s^\vee(b, a) = \sum_{a' \geq a, b' < b} \Delta s(a', b').
\end{equation}

Therefore any slipface $s$ can be reconstructed from the knowledge of the points $(a, b)$ where $\Delta s(a, b) = 1$ and the points where $\Delta(a, b) = -1$. These two sets are fairly constrained, however. Since $\lim_{b \to \infty} (s(a + 1, b) - s(a, b)) = 1$, and $\lim_{a \to \infty} (s^\vee(b + 1, a) - s^\vee(b, a)) = 1$, we deduce

\textbf{Lemma 4.1.} For fixed $a \in \mathbb{Z}$, $\sum_{n \in \mathbb{Z}} \Delta s(a, n) = 1$. For fixed $b \in \mathbb{Z}$, $\sum_{n \in \mathbb{Z}} \Delta s(n, b) = 1$.

\textbf{Definition 4.2.} A slipface $s$ is submodular if $\Delta s(a, b) \geq 0$ for all $a, b \in \mathbb{Z}$.

\textbf{Proposition 4.3.} A slipface $s$ is submodular if and only if there exists $\alpha \in \text{ASP}$ such that $s = s_\alpha$. Therefore $\alpha \mapsto s_\alpha$ gives a bijection from $\text{ASP}$ to the set of submodular slipfaces.

\textbf{Proof.} Equation (14) shows that $s_\alpha$ is submodular for all $\alpha \in \text{ASP}$. Conversely, suppose that $s \in \text{SF}$ is submodular. Let $\Gamma = \{(a, b) : \Delta s(a, b) = 1\}$. Lemma 4.1 implies that $\Gamma$ contains a unique element for each value of $a$, and a unique element for each value of $b$. Therefore there exists a (unique) permutation $\alpha : \mathbb{Z} \to \mathbb{Z}$ such that $(a, b) \in \Gamma$ if and only if $\alpha(b) = a$. Equations (17) and (19) may be written $s(a, b) = s_\alpha(a, b)$ and $s^\vee(b, a) = s_{\alpha^{-1}}(b, a)$; since these are finite, $\alpha \in \text{ASP}$. \hfill \Box

4.1. The Demazure product $\star$ on $\text{ASP}$. This subsection proves the following theorem and develops a few combinatorial properties of $\star$ on $\text{ASP}$.

\textbf{Theorem 4.4.} The set of submodular slipfaces is closed under $\star$. Therefore there is a well-defined associative product $\star$ on $\text{ASP}$, uniquely characterized by $s_{\alpha \star \beta} = s_\alpha \star s_\beta$, which we call the Demazure product on $\text{ASP}$. It is associative, and satisfies $(\alpha \star \beta)^{-1} = \beta^{-1} \star \alpha^{-1}$ and $\chi_{\alpha \star \beta} = \chi_\alpha + \chi_\beta$.

\textbf{Definition 4.5.} Let $\alpha, \beta \in \text{ASP}$ and $a, b \in \mathbb{Z}$. Define

$$M_{\alpha \star \beta}(a, b) = \max \{ \ell \in \mathbb{Z} : s_\alpha \star s_\beta(a, b) = s_\alpha(a, \ell) + s_\beta(\ell, b) \}.$$ 

We will use the following elementary lemma several times. We omit the straightforward proof.

\textbf{Lemma 4.6.} For a function $f : \mathbb{Z} \to \mathbb{Z}$ that is bounded below, denote $\min f = \min \{ f(\ell) : \ell \in \mathbb{Z} \}$ and $M_f = \max \{ \ell \in \mathbb{Z} : f(\ell) = \min f \}$. Suppose that $A$ is any integer, and $g : \mathbb{Z} \to \mathbb{Z}$ is defined by

$$g(\ell) = f(\ell) + \delta(\ell \leq A).$$

Then

$$\min g = \min f + \delta(M_f \leq A), \quad M_g \geq M_f.$$
Lemma 4.7. For all $a, b \in \mathbb{Z}$, $s_\alpha \star s_\beta(a + 1, b) = s_\alpha \star s_\beta(a, b) + \delta \left[ M_{\alpha \star \beta}(a, b) \leq \alpha^{-1}(a) \right]$, and $M_{\alpha \star \beta}(a + 1, b) \geq M_{\alpha \star \beta}(a, b)$.

Proof. Let $f(\ell) = s_\alpha(a, \ell) + s_\beta(\ell, b)$ and $g(\ell) = s_\alpha(a + 1, \ell) + s_\beta(\ell, b)$. By Equation (13),

$$g(\ell) = f(\ell) + \delta(\ell \leq \alpha^{-1}(a)).$$

Observe $\min f = s_\alpha \star s_\beta(a, b)$ and $\min g = s_\alpha \star s_\beta(a + 1, b)$, and apply Lemma 4.6.

Lemma 4.8. For all $a, b \in \mathbb{Z}$, $s_\alpha \star s_\beta(a + b + 1) = s_\alpha \star s_\beta(a, b) - \delta \left[ M_{\alpha \star \beta}(a, b) > \beta(b) \right]$, and $M_{\alpha \star \beta}(a + b + 1) \geq M_{\alpha \star \beta}(a, b)$.

Proof. Let $f(\ell) = s_\alpha(a, \ell) + s_\beta(\ell, b) - 1$ and $g(\ell) = s_\alpha(a, \ell) + s_\beta(\ell, b + 1)$. Equation (12) implies that

$$g(\ell) = f(\ell) + 1 - \delta(\ell > \beta(b)) = f(\ell) + \delta(\ell \leq \beta(b)).$$

Observe that $\min f = s_\alpha \star s_\beta(a, b) - 1$ and $\min g = s_\alpha \star s_\beta(a, b + 1)$, and apply Lemma 4.6.

Proof of Theorem 4.4. In light of Proposition 4.3, we must show that for any two $\alpha, \beta \in \text{ASP}$, the slipface $s_\alpha \star s_\beta$ is submodular. Using property (S1), it suffices to show that, if $s_\alpha \star s_\beta(a + 1, b) = s_\alpha \star s_\beta(a, b + 1)$, then also $s_\alpha \star s_\beta(a, b) = s_\alpha \star s_\beta(a + 1, b + 1)$. By Lemma 4.8, we must show that if $M_{\alpha \star \beta}(a, b) \leq \beta(b)$, then also $M_{\alpha \star \beta}(a, b) \leq \beta(b)$. This follows from $M_{\alpha \star \beta}(a, b) \leq M_{\alpha \star \beta}(a + 1, b)$ (Lemma 4.7). This establishes that submodular slipfaces are closed under $\star$, so $\star$ is well-defined on ASP. The remaining claims follow from the identities $s_\alpha = s_{\alpha - 1}$ and $\chi(s_\alpha) = \chi(\alpha)$ (Examples 3.3 and 3.5) and the corresponding identities on slipfaces (Proposition 3.9 and Lemma 3.11).

The discussion above also provides information about the inverses of $\alpha \star \beta$.

Lemma 4.9. For any $\alpha, \beta \in \text{ASP}$, $\beta \leq L \alpha \star \beta$ and $\alpha \leq R \alpha \star \beta$.

Proof. It suffices to prove only the first statement; the second is equivalent to $\alpha^{-1} \leq L \beta^{-1} \star \alpha^{-1}$. We must prove that $\text{Inv}(\alpha \star \beta) \subseteq \text{Inv}(\beta)$. We argue by contrapositive. Suppose that $u, v \in \mathbb{Z}$ satisfy $u < v$ and $\alpha \star \beta(u) < \alpha \star \beta(v)$. We will prove that $\beta(u) < \beta(v)$.

Choose any integer $a$ such that $\alpha \star \beta(u) < a \leq \alpha \star \beta(v)$. By Equation (12), this is equivalent to $s_{\alpha \star \beta}(a, u + 1) = s_{\alpha \star \beta}(a, u) - 1$ and $s_{\alpha \star \beta}(a, v + 1) = s_{\alpha \star \beta}(a, v)$. By Lemma 4.8, this means that $M_{\alpha \star \beta}(a, u) > \beta(u)$ and $M_{\alpha \star \beta}(a, v) \leq \beta(v)$. Lemma 4.8 also implies that $M_{\alpha \star \beta}(a, u) \leq M_{\alpha \star \beta}(a, v)$. Chaining these inequalities implies that $\beta(u) < \beta(v)$.

4.2. The operations $\prec, \triangleright$ on ASP. This subsection parallels the previous. We will prove

Theorem 4.10. The set of submodular slipfaces is closed under $\prec$ and $\triangleright$. Therefore there are well-defined binary operations $\prec, \triangleright$ on ASP characterized by $s_\alpha \prec s_\beta = s_{\alpha \triangleright \beta}$ and $s_\alpha \triangleright s_\beta = s_{\alpha \prec \beta}$. These operations satisfy the following identities: $(\alpha \prec \beta) \prec \gamma = \alpha \prec (\beta \star \gamma)$, $\alpha \triangleright (\beta \triangleright \gamma) = (\alpha \star \beta) \triangleright \gamma$, $(\alpha \triangleright \beta)^{-1} = \beta^{-1} \triangleright \alpha^{-1}$, and $\chi_{\alpha \triangleright \beta} = \chi_{\alpha \star \beta} = \chi_{\alpha} + \chi_{\beta}$, as well as Equations (9) and (11).

Definition 4.11. Let $\alpha, \beta \in \text{ASP}$ and $a, b \in \mathbb{Z}$. Define

$$M_{\alpha \triangleright \beta}(a, b) = \max \{ \ell \in \mathbb{Z} : s_\alpha \prec s_\beta(a, b) = s_{\alpha}(a, \ell) - s_{\beta^{-1}}(b, \ell) \}.$$

Lemma 4.12. For all $a, b \in \mathbb{Z}$, $s_\alpha \prec s_\beta(a + 1, b) = s_\alpha \prec s_\beta(a, b) + \delta(M_{\alpha \triangleright \beta}(a + 1, b) \leq \alpha^{-1}(a))$, and $M_{\alpha \triangleright \beta}(a + 1, b) \leq M_{\alpha \triangleright \beta}(a, b)$.

Proof. Define $f(\ell) = -s_\alpha(a + 1, \ell) + s_{\beta^{-1}}(b, \ell)$ and $g(\ell) = -s_\alpha(a, \ell) + s_{\beta^{-1}}(b, \ell)$. By Equation (13),

$$g(\ell) = f(\ell) + \delta(\ell \leq \alpha^{-1}(a)).$$

Observe $\min g = -s_\alpha \prec s_\beta(a, b)$ and $\min f = -s_\alpha \prec s_\beta(a + 1, b)$, and apply Lemma 4.6.
Lemma 4.13. For all \(a, b \in \mathbb{Z}\), \(s_{\alpha} \triangleleft s_{\beta}(a, b + 1) = s_{\alpha} \triangleleft s_{\beta}(a, b) - \delta(M_{\alpha \triangleleft \beta}(a, b) \leq \beta^{-1}(b))\), and \(M_{\alpha \triangleleft \beta}(a, b + 1) \geq M_{\alpha \triangleleft \beta}(a, b)\).

Proof. Define \(f(\ell) = -s_{\alpha}(a, \ell) + s_{\beta}(b, \ell)\) and \(g(\ell) = -s_{\alpha}(a, \ell) + s_{\beta}(b + 1, \ell)\). By Equation (13),
\[
g(\ell) = f(\ell) + \delta(\ell \leq \beta^{-1}(b)).
\]
Observe that for all \(\ell\) implies \(\ell \leq \beta^{-1}(b)\).

Proof of Theorem 4.14. We first check closure. By Proposition 4.13 it suffices to show that \(s_{\alpha} \triangleleft s_{\beta}\) and \(s_{\alpha} \triangleright s_{\beta}\) are submodular for all \(\alpha, \beta \in \text{ASP}\). Since \(s_{\alpha} \triangleright s_{\beta} = (s_{\beta - 1} \triangleleft s_{\alpha - 1})^{\vee}\), it suffices to check that \(s_{\alpha} \triangleleft s_{\beta}\) is submodular for all \(\alpha, \beta \in \text{ASP}\). Fix permutations \(\alpha, \beta \in \text{ASP}\).

As in the proof of Theorem 4.13 it suffices to verify that if \(s_{\alpha} \triangleleft s_{\beta}(a+1, b) = s_{\alpha} \triangleleft s_{\beta}(a, b+1)\), then also \(s_{\alpha} \triangleleft s_{\beta}(a, b) = s_{\alpha} \triangleleft s_{\beta}(a, b+1)\). Suppose that \(s_{\alpha} \triangleleft s_{\beta}(a+1, b) = s_{\alpha} \triangleleft s_{\beta}(a+1, b+1)\). Lemma 4.13 implies \(M_{\alpha \triangleleft \beta}(a+1, b) > \beta^{-1}(b)\). Lemma 4.12 implies that \(M_{\alpha \triangleleft \beta}(a, b) \geq M_{\alpha \triangleleft \beta}(a+1, b) > \beta^{-1}(b)\); applying Lemma 4.13 again shows that \(s_{\alpha} \triangleleft s_{\beta}(a, b) = s_{\alpha} \triangleleft s_{\beta}(a, b+1)\), as desired.

This establishes that \(\triangleleft\) and \(\triangleright\) are well-defined on \(\text{ASP}\). The identities follow from the identities \(s_{\alpha}^{\vee} = s_{\alpha-1}\) and \(\chi_{s_{\alpha}} = \chi_{\alpha}\) (Examples 3.3 and 3.5) and the corresponding identities on slipfaces, established in Proposition 3.9 and Lemma 4.11. Equations (9) and (11) follow from Lemma 4.10.

The analog of Lemma 4.9 for the operations \(\triangleleft, \triangleright\) is the following.

Lemma 4.14. For all \(\alpha, \beta \in \text{ASP}\), \((\alpha \triangleleft \beta)^{\ast_{\text{red}}} \beta^{-1}\), and \(\alpha \triangleleft \beta \leq R \alpha\).

Proof. First, suppose that \((u, v) \in \text{Inv}(\alpha \triangleleft \beta)\). We will show that \((u, v) \notin \text{Inv}(\beta)\). There exists an integer \(a\) such that \(\alpha \triangleleft \beta(v) < a \leq \alpha \triangleleft \beta(u)\). By Equation (12), \(s_{\alpha \triangleleft \beta}(a, v + 1) = s_{\alpha \triangleleft \beta}(a, v) - 1\) and \(s_{\alpha \triangleleft \beta}(a, u) = s_{\alpha \triangleleft \beta}(a, u + 1)\), whence Lemma 4.13 implies that \(M_{\alpha \triangleleft \beta}(a, v) \leq \beta(v)\), \(M_{\alpha \triangleleft \beta}(a, u) > \beta(u)\). Since \(u < v\), Lemma 4.13 also implies that \(M_{\alpha \triangleleft \beta}(a, v) \geq M_{\alpha \triangleleft \beta}(a, u)\). Chaining these inequalities implies \(\beta(u) < \beta(v)\), as desired.

Now suppose that \((u, v) \in \text{Inv}((\alpha \triangleleft \beta)^{-1})\). We will show that \((u, v) \in \text{Inv}(\alpha^{-1})\). Similarly to above, there exists \(b\) such that \(s_{\alpha \triangleleft \beta}(v + 1, b) = s_{\alpha \triangleleft \beta}(v, b)\) and \(s_{\alpha \triangleleft \beta}(u + 1, b) = s_{\alpha \triangleleft \beta}(u, b) + 1\). Lemma 4.12 implies that \(M_{\alpha \triangleleft \beta}(v + 1, b) > \alpha^{-1}(v)\) and \(M_{\alpha \triangleleft \beta}(u + 1, b) \leq \alpha^{-1}(u)\). Since \(u < v\), Lemma 4.12 also implies that \(M_{\alpha \triangleleft \beta}(v + 1, b) \leq M_{\alpha \triangleleft \beta}(u + 1, b)\). Chaining these inequalities gives \(\alpha^{-1}(u) > \alpha^{-1}(v)\), as desired.

Corollary 4.15. For all \(\alpha, \beta \in \text{ASP}\), \((\alpha^{-1})^{\ast_{\text{red}}} (\alpha \triangleright \beta)\) and \(\alpha \triangleright \beta \leq L \beta\).

Proof. These statements are equivalent to \((\beta^{-1} \triangleleft \alpha^{-1})^{\ast_{\text{red}}} \alpha\) and \(\beta^{-1} \triangleleft \alpha^{-1} \leq R \beta^{-1}\).

5. When \(*, \triangleleft, \triangleright*\ ARE ORINARY PRODUCTS

This section examines the relation between \(*, \triangleleft, \triangleright*\ on \text{ASP} and the ordinary product.

Lemma 5.1. For all \(\alpha, \beta \in \text{ASP}\), \(\alpha \star \beta \geq \alpha \beta\), and \(\alpha \star \beta = \alpha \beta\ if and only if \alpha^{\ast_{\text{red}}} \beta\).

Proof. Observe that for all \(a, b, \ell \in \mathbb{Z}\),
\[
s_{\alpha}(a, \ell) + s_{\beta}(\ell, b) = \#\{n \geq \ell : \alpha(n) < a\} + \#\{n < \ell : \beta^{-1}(n) \geq b\}.
\]
The two sets counted on the right side are disjoint. Partitioning their union a different way, the right side is equal to
\[
\#\{n \in \mathbb{Z} : \alpha(n) < a and \beta^{-1}(n) \geq b\} + \#\{n \geq \ell : \alpha(n) < a and \beta^{-1}(n) < b\} + \#\{n < \ell : \alpha(n) \geq a and \beta^{-1}(n) \geq b\}.
\]
The first of these three terms is equal to \(s_{\alpha \beta}(a, b)\); it follows from this that \(s_{\alpha}(a, \ell) + s_{\beta}(\ell, b) \geq s_{\alpha \beta}(a, b)\) for all \(a, b, \ell \in \mathbb{Z}\).
In other words, $\alpha \star \beta \geq \alpha \beta$. This argument also shows how to describe the equality case: $\alpha \star \beta = \alpha \beta$ if and only if for all $a, b \in \mathbb{Z}$, there exists an integer $\ell \in \mathbb{Z}$ such that the second and third terms in the sum above vanish. This amounts to saying that every element of \{ $n \in \mathbb{Z} : \alpha(n) < a$ and $\beta^{-1}(n) < b$ \} is less than every element of \{ $n \in \mathbb{Z} : \alpha(n) \geq a$ and $\beta^{-1}(n) \geq b$ \}.

Put another way, $\alpha \star \beta > \alpha \beta$ if and only if there exist two integers $a, b$ and two integers $m < n$ such that $\alpha(n) < a \leq \alpha(m)$ and $\beta^{-1}(n) < b \leq \beta^{-1}(m)$. For fixed $m, n$, there exist $a, b$ satisfying these chains if and only if $\alpha(n) < \alpha(m)$ and $\beta^{-1}(n) < \beta^{-1}(m)$. Since we require $m < n$, this simply means $(m, n) \in \text{Inv}(\alpha) \cap \text{Inv}(\beta^{-1})$. So $\alpha \star \beta > \alpha \beta$ if and only if $\text{Inv}(\alpha) \cap \text{Inv}(\beta^{-1})$ is nonempty.

**Lemma 5.2.** For all $\alpha, \beta \in \text{ASP}$, $\alpha \triangleleft \beta \leq \alpha \beta$ and $\alpha \triangleright \beta \leq \alpha \beta$. Furthermore, $\alpha \triangleleft \beta = \alpha \beta$ if and only if $\beta^{-1} \leq_L \alpha$ and $\alpha \triangleright \beta = \alpha \beta$ if and only if $\alpha^{-1} \leq_R \beta$.

**Proof.** We will prove that $\alpha \triangleleft \beta \leq \alpha \beta$, with equality if and only if $\text{Inv}(\beta^{-1}) \subseteq \text{Inv}(\alpha)$. The corresponding statements about $\triangleright$ will then follow from the identity $\alpha \triangleright \beta = (\beta^{-1} \triangleleft \alpha)^{-1}$. The strategy is analogous to that of Lemma 5.1, although the counting argument is slightly more subtle.

Observe that for all $a, b, \ell \in \mathbb{Z}$,

$$s_\alpha(a, \ell) - s_{\beta^{-1}}(b, \ell) = \# \{ n \geq \ell : \alpha(n) < a \} - \# \{ n \geq \ell : \beta^{-1}(n) < b \} = \# \{ n \geq \ell : \alpha(n) < a, \beta^{-1}(n) \geq b \} - \# \{ n \geq \ell : \alpha(n) \geq a, \beta^{-1}(n) < b \}.$$  

In the second line, the intersection of the two sets on the right side has been removed from both. From here, we may rewrite the first term in the last line as

$$\# \{ n \geq \ell : \alpha(n) < a, \beta^{-1}(n) \geq b \} = \{ n \in \mathbb{Z} : \alpha(n) < a, \beta^{-1}(n) \geq b \} - \{ n \leq \ell : \alpha(n) < a, \beta^{-1}(n) \geq b \}.$$  

Therefore

$$s_\alpha(a, \ell) - s_{\beta^{-1}}(b, \ell) = s_{\alpha \beta}(a, b) - \# \{ n \leq \ell : \alpha(n) \geq a, \beta^{-1}(n) < b \}.$$  

Therefore $s_\alpha(a, \ell) - s_{\beta^{-1}}(b, \ell) \leq s_{\alpha \beta}(a, b)$ for all $a, b, \ell \in \mathbb{Z}$, with equality case characterized by two sets being empty, much as in the proof of Lemma 5.1. Therefore $\alpha \triangleleft \beta \leq \alpha \beta$. Studying the equality case as in Lemma 5.1, $\alpha \triangleleft \beta < \alpha \beta$ if and only if there exist integers $m < n$ and integers $a, b$ such that $\alpha(m) < a \leq \alpha(n)$ and $\beta^{-1}(m) \geq b > \beta^{-1}(n)$; this condition is equivalent to $\text{Inv}(\beta^{-1}) \setminus \text{Inv}(\alpha)$ being nonempty. So $\alpha \triangleleft \beta = \alpha \beta$ if and only if $\text{Inv}(\beta^{-1}) \subseteq \text{Inv}(\alpha)$.

Among other things, this clarifies relationship between the weak and strong Bruhat orders.

**Corollary 5.3.** If either $\alpha \leq_L \beta$ or $\alpha \leq_R \beta$, and $\chi_\alpha \leq \chi_\beta$, then $\alpha \leq \beta$.

**Proof.** If $\alpha \leq_L \beta$ and $\chi_\alpha \leq \chi_\beta$, then Lemma 2.8 implies $\beta = (\alpha \alpha^{-1}) \star \text{red} \alpha$. Corollary 3.14 implies that $\beta \geq \alpha$, since $\chi_{\beta \alpha^{-1}} = \chi_\beta - \chi_\alpha \geq 0$. Similarly, if $\alpha \leq_R \beta$, then $\beta = \alpha \star \text{red}(\alpha^{-1} \beta) \geq \alpha$. \hfill \Box

6. Greediness, stinginess, and the reduction theorem

The tools of the previous section furnish proofs of the greedy characterization of $\star$, the “stingy” characterizations of $\triangleleft$ and $\triangleright$, and the reduction theorem.

**Proof of Theorem 13.** Fix $\alpha, \beta \in \text{ASP}$. Lemmas 3.8 and 5.1 imply that if $\alpha_1 \leq \alpha, \beta_1 \leq \beta$, then

$$\alpha \star \beta \geq \alpha_1 \star \beta_1 \geq \alpha_1 \beta_1.$$  

It remains to exhibit some equality cases. First, let $\alpha_1 = (\alpha \star \beta) \beta^{-1}$ and $\beta_1 = \beta$. Since $\beta \leq_L \alpha \star \beta$ (Lemma 4.9), Lemma 5.2 implies that $\alpha_1 = (\alpha \star \beta) \triangleleft \beta^{-1}$. By Lemma 5.10, $\alpha_1$ is the minimum
element of \( \{ \gamma \in \text{ASP} : \gamma \ast \beta \geq \alpha \ast \beta \} \). Of course \( \alpha \) belongs to this set, so \( \alpha_1 \leq \alpha \). The shift of \( \alpha_1 \) is \( \chi_{\alpha_1} = \chi_{\alpha} \), by Theorem 4.11 and Equation (16). Lemma 5.2 implies \( \alpha_1 \red \beta = \alpha_1 \beta = \alpha \ast \beta \). So equality is obtained in Equation (20) in a case where \( \beta_1 = \beta \) and \( \chi_{\alpha_1} = \chi_{\alpha} \); this proves Equations (8) and (13) from the theorem. The remaining Equation (7) follows by applying Equation (8) to \( \beta^{-1} \ast \alpha^{-1} \), plus the identity \( (\beta_1^{-1} \ast \alpha^{-1})^{-1} = \alpha \ast \beta_1 \) and Lemma 2.1.

The analogy of the “greedy theorem” for \( \ast \) is the following “stingy theorem” for \( \prec \).

**Theorem 6.1.** For all \( \alpha, \beta \in \text{ASP} \), the following minima exist in Bruhat order, and equal \( \alpha \prec \beta^{-1} \).

\[
\begin{align*}
(21) & \quad \alpha \prec \beta^{-1} = \min \{ \alpha_1 \beta_1^{-1} : \alpha_1 \geq \alpha, \beta_1 \leq \beta \} \\
(22) & \quad = \min \{ \alpha_1 \beta_1^{-1} : \beta_1 \leq \beta \} \\
(23) & \quad = \min \{ \alpha_1 \beta_1^{-1} : \alpha_1 \geq \chi \alpha \}. 
\end{align*}
\]

**Proof.** Fix \( \alpha, \beta \in \text{ASP} \). Lemmas 3.8 and 5.2 imply that if \( \alpha_1 \geq \alpha \) and \( \beta_1 \leq \beta \), then

\[
\alpha \prec \beta^{-1} \leq \alpha_1 \beta_1^{-1} \leq \alpha_1 \beta_1^{-1}. 
\]

We must now find some equality cases. First, define \( \alpha_1 = \alpha \) and \( \beta_1 = (\beta \circ \alpha^{-1}) \alpha \). So \( \alpha \prec \beta^{-1} = \alpha \beta_1^{-1} \) and \( \chi \beta_1 = \chi \beta \). We must show that \( \beta_1 \leq \beta \). We may also write \( \beta_1 = (\beta \circ \alpha^{-1}) \alpha \), by Corollary 4.15 and Lemma 5.2. So \( \beta_1 \) is the minimum permutation such that \( \beta_1 \ast \alpha^{-1} \geq \beta \circ \alpha^{-1} \). But \( \beta \) is another such permutation, since \( \beta_1 \ast \alpha^{-1} \geq \beta_1 \alpha^{-1} = (\alpha \prec \beta^{-1})^{-1} = \beta \circ \alpha^{-1} \). Therefore \( \beta \geq \beta_1 \).

For the second equality case, define \( \alpha_1 = (\alpha \prec \beta^{-1}) \beta \). Lemma 4.14 implies \( (\alpha \prec \beta^{-1}) \red \beta = \alpha_1 \), and Lemma 3.10 implies \( (\alpha \prec \beta^{-1}) \ast \beta \geq \alpha \). Therefore \( \alpha_1 \geq_{\chi} \alpha \) and \( \alpha_1 \beta = \alpha \prec \beta^{-1} \).

The reduction theorem from the introduction follows by similar techniques.

**Proof of Theorem C.** We will prove only the second paragraph of the theorem statement, since the first follows directly from it, along with the monotonicity property in Lemma 3.8.

Let \( \alpha, \beta, \gamma \in \text{ASP} \) satisfy \( \alpha \ast \beta \geq \gamma \), and define \( \alpha_1 = \gamma \prec \beta^{-1} \) and \( \beta_1 = \alpha_1^{-1} \circ \gamma \). We apply Lemma 3.10 (specialized to ASP) several times in a row. The assumed bound \( \alpha \ast \beta \geq \gamma \) implies \( \alpha \geq \gamma \prec \beta^{-1} = \alpha_1 \). The bound \( \alpha_1 \geq \gamma \prec \beta^{-1} \) implies \( \alpha \ast \beta \geq \gamma \), which implies \( \beta \geq \alpha_1^{-1} \circ \gamma = \beta_1 \). So \( \alpha_1, \beta_1 \) are bounded above by \( \alpha, \beta \), respectively. They also have the same shifts, by Theorem 4.10.

Lemma 4.14 says \( \alpha_1 \leq_{R} \gamma \), so \( \beta_1 = \alpha_1^{-1} \circ \gamma = \alpha_1^{-1} \gamma \) by Lemma 5.2. The inequality \( \alpha_1 \leq_{R} \gamma \) also implies via Lemma 2.8 that \( \alpha_1 \red (\alpha_1^{-1} \gamma) = \gamma \). Combining these equations gives \( \alpha_1 \red \beta_1 = \gamma \).

In some applications, it is convenient to use a form of Theorem C for products of more than two permutations; we give such a statement here for convenience. To state it, we must first clarify what is meant by a reduced product of three or more permutations; the definition below is based on the standard definition of a reduced word in a Coxeter group, adapted both to products of non-generators and to permutations that may have infinitely many inversions.

**Lemma 6.2.** Let \( (\alpha_1, \ldots, \alpha_{\ell}) \) be an \( \ell \)-tuple of permutations in \( \text{ASP} \). Denote by \( \pi_n \) the product of the suffix \( \alpha_{n+1} \cdots \alpha_{\ell} \) (\( \pi_\ell \) is the identity). The following are equivalent.

1. For all \( u, v \in \mathbb{Z} \) with \( u \neq v \), the sign of \( \pi_n(u) - \pi_n(v) \) changes at most once as \( n \) decreases from \( \ell \) to 0.

2. The set \( \text{Inv}(\pi_0) = \text{Inv}(\alpha_1 \cdots \alpha_{\ell}) \) is the disjoint union of the sets

\[
I_n = \{(\pi_n^{-1}(u), \pi_n^{-1}(v)) : (u, v) \in \text{Inv}(\alpha_n)\} \quad \text{for} \quad 1 \leq n \leq \ell.
\]

3. For each \( 1 \leq n \leq \ell \), \( \alpha_n \red \pi_n \).
Proof. (1) $\Rightarrow$ (2): for all $u < v$, $(u, v) \in \text{Inv}(\pi_n)$ if and only if $\pi_n(u) - \pi_n(v)$ changes sign an odd number of times; assuming (1), this is equivalent to changing signs once, and in turn to belonging to exactly one (and necessarily only one) set $I_s$.

(2) $\Rightarrow$ (3): this follows since $\text{Inv}(\alpha_n) \cap \text{Inv}(\pi_n^{-1}) = \{(\pi_n(u), \pi_n(v)) : (u, v) \in I_n\}$ and $\text{Inv}(\pi_n^{-1})$ is contained in the union of $\{(\pi_n(u), \pi_n(v)) : (u, v) \in I_m\}$ for $n + 1 \leq m \leq \ell$, which are disjoint.

(3) $\Rightarrow$ (1): if (1) is false, then there exists some $u < v$ and $n$ such that $\pi_n(u) > \pi_n(v)$ but $\alpha_n\pi_n(u) < \alpha_n\pi_n(v)$; this implies that $(\pi_n(v), \pi_n(u)) \in \text{Inv}(\alpha)_n \cap \text{Inv}(\pi_n^{-1})$, so (3) is false. \hfill $\square$

**Definition 7.3.** An $\ell$-tuple $(\alpha_1, \ldots, \alpha_\ell)$ is **reduced** if the three equivalent conditions in Lemma 6.2 hold.

**Remark 6.4.** Although we don’t need it here, the following recursive criterion for reducedness is sometimes useful. For any $1 \leq i \leq \ell - 1$, a tuple $(\alpha_1, \ldots, \alpha_i)$ is reduced if and only if both $(\alpha_1, \ldots, \alpha_i)$ and $(\alpha_{i+1}, \ldots, \alpha_\ell)$ are reduced tuples, and $(\alpha_1 \cdots \alpha_\ell)_{\text{red}}(\alpha_{i+1} \cdots \alpha_\ell)$. This follows from criterion (1) in Lemma 6.2.

**Theorem 6.5.** Let $G \leq \text{ASP}$ be a subgroup that is closed under $\ast$. For any $\alpha_1, \ldots, \alpha_\ell, \gamma \in G$, if $\alpha_1 \ast \cdots \ast \alpha_\ell \geq \gamma$, then there exists a reduced tuple $(\beta_1, \ldots, \beta_\ell)$ such that $\beta_1 \in G$ and $\beta_1 \leq \chi \alpha_i$ for all $i$, and $\beta_1 \ast \cdots \ast \beta_\ell = \beta_1 \cdots \beta_\ell = \gamma$.

**Proof.** By induction on $\ell$. The base case $\ell = 1$ is tautological. If $\ell \geq 2$, then Theorem C implies that there exist $\beta_1, \pi \in G$ with $\beta_1 \ast \cdots \ast \beta_\ell = \gamma$, $\beta_1 \leq \chi \alpha_1$, and $\alpha_2 \ast \cdots \ast \alpha_\ell \geq \chi \pi$. By inductive hypothesis, there exist $\beta_2 \leq \chi \alpha_2, \ldots, \beta_\ell \leq \chi \alpha_\ell$ such that $(\beta_2, \ldots, \beta_\ell)$ reduced and $\beta_2 \ast \cdots \ast \beta_\ell = \beta_2 \cdots \beta_\ell = \pi$. Then $\beta_1 \ast \cdots \ast \beta_\ell = \gamma$, and $(\beta_1, \ldots, \beta_\ell)$ is reduced by criterion (3) of Lemma 6.2. \hfill $\square$

7. Bounded-difference permutations and the essential set

Having proved our main theorems, this section addresses a somewhat different question that will prove useful in applications: when comparing two permutations in Bruhat order, or more generally two slipfaces, does it suffice to check the inequality $s(a, b) \leq t(a, b)$ for some small subset of $\mathbb{Z} \times \mathbb{Z}$? There is a standard tool for this in $S_{\mathbb{Z}}$, called the **essential set**, introduced in [Ful92] for applications to degeneracy loci, which we adapt herein to SF and ASP. Unfortunately, a subtle finiteness issue restricts the use of this tool to certain slipfaces, which we call **Clifford slipfaces**. In the submodular case, these correspond to a bounded-difference permutations, which we define below.

7.1. The essential set of a slipface. The key definition is the following.

**Definition 7.1.** For any $s \in \text{SF}$, the **essential set** of $s$ is

$$\text{Ess}(s) = \{(a, b) \in \mathbb{Z}^2 : s(a - 1, b) < s(a, b) = s(a + 1, b) \text{ and } s(a, b + 1) < s(a, b) = s(a, b - 1)\}.$$ 

As in the theory of degeneracy loci, the purpose of Ess$(s)$ is to identify a smaller, often finite, set on which to verify inequalities $s(a, b) \leq t(a, b)$. The basic observation is

**Lemma 7.2.** Suppose that $s, t$ are slipfaces such that $s \not\leq t$, and let $E = \{(a, b) : s(a, b) > t(a, b)\}$ be the set witnessing this fact. If $(a, b) \in E$ and $(a, b) \not\in \text{Ess}(s)$, then there exists

$$(a', b') \in \{(a - 1, b), (a + 1, b), (a, b - 1), (a, b + 1)\}$$

such that $(a', b') \in E$, $s(a', b') \geq s(a, b)$, $s'(b', a') \geq s'(b, a)$, and

$$s(a', b') + s'(b', a') > s(a, b) + s'(b, a).$$

**Proof.** Define $f_s(a, b) = s(a, b) + s'(b, a) + \chi_s$; equivalently $f_s(a, b) = 2s(a, b) - a + b$. Then $s(a, b) > t(a, b)$ if and only if $f_s(a, b) > f_t(a, b)$. When one of $a, b$ is increased or decreased by one, one of $s(a, b), s'(b, a)$ increases or decreased by 1, while the other stays the same. So for any
Lemma 7.2 almost implies that \( s \leq t \) if and only if \( s(a, b) \leq t(a, b) \) for all \( (a, b) \in \text{Ess}(s) \). The only catch is that the process of \((a, b)\) by \((a', b')\) might not terminate. A quick, if perhaps slightly unsatisfying, fix is simply to restrict the classes of slipfaces \( s \) considered. Fortunately this restriction still encompasses all slipfaces that occur in our intended applications. The name Clifford refers to Clifford’s theorem from the theory of algebraic curves.

Definition 7.3. A Clifford slipface is a slipface \( s \) such that the following set is bounded above.

\[
\{s(a, b) + s'(b, a) : s(a, b) > 0 \text{ and } s'(b, a) > 0\}
\]

Proposition 7.4. If \( s \) is a Clifford slipface and \( t \) is any slipface with \( \chi_s \leq \chi_t \), then \( s \leq t \) if and only if \( s(a, b) \leq t(a, b) \) for all \( (a, b) \in \text{Ess}(s) \).

Proof. It suffices to prove that if \( s \not\leq t \), then \( s(a, b) > t(a, b) \) for some \( (a, b) \in \text{Ess}(s) \). Define \( E \) as in Lemma 7.2. For all \( a, b \in E, s(a, b) > 0 \) and \( s'(b, a) > 0 \), because otherwise \( \chi_s \leq \chi_t \) implies \( s(a, b) = \min\{\chi_s + a - b, 0\} \leq t(a, b) \). Since \( s \) is Clifford, there exists \( (a, b) \in E \) maximizing \( s(a, b) + s'(b, a) \) on \( E \); Lemma 7.2 shows that \((a, b)\) must be in \( \text{Ess}(s) \).

Example 7.5. The slipface \( s(a, b) = s_{\chi}(a, b) = \max\{\chi + a - b, 0\} \) has \( \text{Ess}(s) = \emptyset \). So Proposition 7.4 recovers again the fact that \( s \) is the Bruhat-minimal slipface of shift \( \chi_s \).

7.2. The essential set of a permutation. Now consider the submodular case.

Definition 7.6. Let \( \alpha \in \text{ASP} \). The essential set of \( \alpha \) is

\[
\text{Ess}(\alpha) = \{(a, b) \in \mathbb{Z}^2 : \alpha^{-1}(a - 1) \geq b > \alpha^{-1}(a), \alpha(b - 1) \geq a > \alpha(b)\}.
\]

Equations (12) and (13) imply that

\[
\text{Ess}(s_{\alpha}) = \text{Ess}(\alpha).
\]

The expression in Definition 7.6 matches the definition in [Ful92], used in the study of degeneracy loci. For this to be useful, we must understand which \( \alpha \in \text{ASP} \) have Clifford slipfaces \( s_{\alpha} \). If \{\( \alpha(n) - n : n \in \mathbb{Z} \}\} is bounded, we will say that \( \alpha \) has bounded difference.

Proposition 7.7. Let \( \alpha \in \text{ASP} \). The following are equivalent.

1. The permutation \( \alpha \) has bounded difference.
2. There exists \( N \in \mathbb{Z} \) such that \( s_{\alpha}(a, b) = \max\{\chi_{\alpha} + a - b, 0\} \) whenever \( |a - b| \geq N \).
3. The slipface \( s_{\alpha} \) is Clifford, i.e. \( \{s_{\alpha}(a, b) + s_{\alpha^{-1}}(b, a) : s_{\alpha}(a, b) > 0 \text{ and } s_{\alpha^{-1}}(b, a) > 0\} \) is bounded.

Lemma 7.8. Let \( \alpha \in \text{ASP} \). The following are equal.

\[
M_{\alpha} = \sup\{s_{\alpha^{-1}}(b, a) : s_{\alpha}(a, b) > 0\} \\
M'_{\alpha} = \sup\{n - \alpha(n) : n \in \mathbb{Z}\} - \chi_{\alpha} \\
M''_{\alpha} = \sup\{b - a : s_{\alpha}(a, b) > 0\} - \chi_{\alpha} + 1
\]

Proof. We will prove that \( M_{\alpha} \geq M'_{\alpha} \geq M''_{\alpha} \geq M_{\alpha} \). First, fix any \( n \in \mathbb{Z} \). Let \( m = \max\{m \geq n : \alpha(m) \leq \alpha(n)\} \).

Then \( s_{\alpha}(\alpha(m) + 1, m) = 1 \) and Equation (15) implies

\[
s_{\alpha^{-1}}(\alpha(m) + 1, m) = m - \alpha(m) - \chi_{\alpha}.
\]
Therefore \( M_\alpha \geq m - \alpha(m) - \chi_\alpha \geq n - \alpha(n) - \chi_\alpha \). This shows that \( M'_\alpha \leq M_\alpha \).

Next, fix any \( a, b \in \mathbb{Z} \) such that \( s_\alpha(a, b) > 0 \). Choose the maximum \( n \geq b \) such that \( \alpha(n) < a \).

Then \( b - a + 1 \leq n - \alpha(n) \leq M'_\alpha \). This implies \( M''_\alpha \leq M'_\alpha \). On the other hand, choosing \( a' = \alpha(n) + 1 \) and \( b' = n \), we have \( s_\alpha(a', b') = 1 \) (by maximality of \( n \)) so \( s_{\alpha^{-1}}(b', a') = 1 - \chi_\alpha + b' - a' \leq M'_\alpha \). This implies that \( s_{\alpha^{-1}}(b, a) \leq s_{\alpha^{-1}}(b', a') \leq M''_\alpha \). Hence \( M_\alpha \leq M''_\alpha \).

\[ \square \]

**Proof of Proposition 7.7.** By Equation \([13]\), statement (2) is equivalent to saying that \( s_\alpha(a, b) = 0 \) if \( b - a > 0 \) and \( s_{\alpha^{-1}}(b, a) = 0 \) if \( a - b > 0 \). By Lemma \([7,8]\) both statements (1) and (2) are equivalent to saying that both \( M_\alpha \) and \( M_{\alpha^{-1}} \) are finite. Now consider statement (3). The proposition is clear when \( \alpha = \chi_\gamma \), so assume \( \alpha \) has at least one inversion. Define

\[
C_\alpha = \sup\{s_\alpha(a, b) + s_{\alpha^{-1}}(b, a) : s_\alpha(a, b) > 0 \text{ and } s_{\alpha^{-1}}(b, a) > 0\}.
\]

So statement (3) is equivalent to \( C_\alpha < \infty \). Observe that we have bounds

\[
1 + \max\{M_\alpha, M_{\alpha^{-1}}\} \leq C_\alpha \leq M_\alpha + M_{\alpha^{-1}}.
\]

So \( C_\alpha \) is finite if and only if both \( M_\alpha \) and \( M_{\alpha^{-1}} \) are finite. \[ \square \]

**Corollary 7.9.** If \( \alpha, \beta \in \text{ASP} \) have the same shift and \( \alpha \) has bounded difference, then \( \alpha \leq \beta \) if and only if \( s_\alpha(a, b) \leq s_\beta(a, b) \) for all \( (a, b) \in \text{Ess}(\alpha) \).

**Example 7.10.** Observe that if \( \alpha \in S_d \), then \( \alpha \) has bounded difference. Both \( \alpha \) and \( \alpha^{-1} \) are increasing on \( (-\infty, 1] \cap \mathbb{Z} \) and \( [d, \infty) \cap \mathbb{Z} \), so

\[
\text{Ess}(\alpha) \subseteq \{2, \ldots, d\} \times \{2, \ldots, d\}.
\]

This is not surprising: for a permutation moving only finite many values, the Bruhat order depends on only a finite subset of \( \mathbb{Z}^2 \).

**Example 7.11.** Fix two nonnegative integers \( m, n \). Consider the permutation \( \gamma = \gamma_m^m \) depicted in Figure 3. More precisely, \( \gamma \) is the unique permutation mapping \((-\infty, -m - 1] \cap \mathbb{Z} \) to \((-\infty, -n] \cap \mathbb{Z} \), \([-m, -1] \cap \mathbb{Z} \) to \([1, m] \cap \mathbb{Z} \), \([0, n - 1] \cap \mathbb{Z} \) to \([-n + 1, 0] \cap \mathbb{Z} \), and \([n, \infty) \cap \mathbb{Z} \) to \([m + 1, \infty) \cap \mathbb{Z} \), each via the order-preserving bijection.

Examining the adjacent inversions of \( \gamma \) shows that

\[
\text{Ess}(\gamma) = \{(1, 0)\}.
\]

Furthermore, \( s_\gamma(1, 0) = n \) and \( s_{\gamma^{-1}}(0, 1) = m \), so \( \chi_\gamma = n - m - 1 \). The set \( \{\gamma(n) - n : n \in \mathbb{Z}\} \) is finite, so it is certainly bounded, and \( s_\gamma \) is Clifford. It follows that for any \( \alpha \in \text{ASP} \) of shift \( \chi_\gamma \),

\[
s_\alpha(1, 0) \geq n \text{ if and only if } \alpha \geq \gamma.
\]

In a similar manner, any inequality of the form \( s_\alpha(a, b) \geq N \) may be expressed equivalently by a Bruhat inequality \( \alpha \geq \gamma \) for some \( \gamma \) of the the same shift as \( \alpha \). This construction is useful in the intended applications to curves and graphs; see Appendix B.
One additional benefit of bounded-difference permutations is that we can state a slightly simplified criterion for checking that a given function is the slipface function of a bounded-difference permutation (or equivalently, a submodular Clifford slipface).

**Proposition 7.12.** A function \( s : \mathbb{Z}^2 \rightarrow \mathbb{Z} \) is the slipface of a bounded-difference permutation (equivalently, a submodular Clifford slipface) if and only if the following two criteria hold.

1. There exists integers \( M, \chi \) such that \( a - b \leq -M \) implies \( s(a,b) = 0 \) and \( a - b \geq M \) implies \( s(a,b) = \chi + a - b \).
2. For all \( a, b \in \mathbb{Z} \), \( s(a+1,b) - s(a,b) - s(a+1,b+1) + s(a,b+1) \geq 0 \) (\( s \) is submodular).

The shift of \( \alpha \) is the the number \( \chi \) mentioned in criterion (1).

**Proof.** If \( s \) is a submodular Clifford slipface, then (2) holds by definition and the boundedness of \( M_\alpha' \) and \( M_{\alpha-1}' \) implies (1). Conversely, suppose that \( s \) is a function satisfying these two criteria. We first check that \( s \) is a slipface; we check the criteria of Lemma 5.6. Submodularity implies that the first finite differences \( s(a+1,b) - s(a,b) \) and \( s(a,b) - s(a,b+1) \) are nondecreasing in \( a \) and nonincreasing in \( b \). Criterion (1) implies that both differences are 0 for \( a - b \) sufficiently small, so both differences are nonnegative for all \( a, b \), and achieve 0 in each row and column. This implies (D1) and (D2) for \( s \). The same argument applies to \( s' \). So \( s \) is a submodular slipface, hence \( s = s_\alpha \) for a unique permutation \( \alpha \in \text{ASP} \). Proposition 7.7 implies that \( \alpha \) has bounded difference. \( \square \)

8. SOME SUBGROUPS OF ASP CLOSED UNDER \( \star \) AND \( \triangleleft \)

We conclude this paper by examining a few types of commonly-encountered subgroups of ASP that are closed under \( \star \) and \( \triangleleft \) (and therefore also \( \triangleright \)), and pointing out some notable special features on the operations on these groups. Several of these examples have the following feature, which is a convenient way to verify closure under \( \star \) and \( \triangleleft \).

**Definition 8.1.** Call a subgroup \( G \leq \text{ASP} \) downward-closed if for all \( \alpha, \beta \in \text{ASP} \), if \( \alpha \leq \chi \) \( \beta \) and \( \beta \in G \), then \( \alpha \in G \) as well.

**Lemma 8.2.** If \( G \) is downward-closed, then it is closed under \( \star \) and \( \triangleleft \).

**Proof.** Suppose \( \alpha, \beta \in G \). The “greedy” Theorem 13 implies that \( \alpha \star \beta = \alpha_1 \beta \) for some \( \alpha_1 \leq \chi \) \( \alpha \). Since \( G \) is downward-closed, \( \alpha_1 \in G \), hence so is \( \alpha \star \beta \). The “stingy” Theorem 6.1 implies \( \alpha \triangleleft \beta = \alpha \beta_1 \) where \( \beta_1 \leq \chi \) \( \beta \). Then \( \beta_1 \in G \), so \( \alpha \triangleleft \beta \in G \) as well. \( \square \)

8.1. **Permutations of bounded difference.** As in Section 7, we say that \( \alpha \) has bounded difference if there exists an integer \( M \) such that \( |\alpha(n) - n| \leq M \) for all \( n \in \mathbb{Z} \). These permutations naturally occur in the our application to algebraic curves and graphs (see Appendix 13), where the bound \( M \) is determined by the genus, and we have seen that they have the virtue of being more easily compared in Bruhat order by using the essential set.

A permutation has difference bounded by \( M \) if and only if \( s_\alpha(a - M, a) = s_{\alpha-1}(a - M, a) = 0 \) for all \( a \in \mathbb{Z} \). Denote the group of such permutations by \( B_M \). Then using again the fact that, if \( \chi_\alpha = \chi_\beta \), then \( \alpha \leq \beta \) is equivalent to \( \alpha^{-1} \leq \beta^{-1} \), it follows that \( B_M \) is closed downwards. Thus so is the union of all the \( B_M \). The group of permutations of bounded difference is closed under \( \star \) and \( \triangleleft \).

In fact, we can say a bit more: if \( \alpha \in B_{M_1} \) and \( \beta \in B_{M_2} \), then the permutations \( \alpha_1, \beta_1 \) in the proof of Lemma 5.2 lie in \( B_{M_1} \) and \( B_{M_2} \), respectively. It follows that \( \alpha \star \beta = \alpha_1 \beta_1 \) and \( \alpha \triangleleft \beta = \alpha_1 \beta_1 \) both must lie in \( B_{M_1+M_2} \). So the bound on the difference cannot grow too quickly under \( \star \) and \( \triangleleft \).
8.2. Symmetric groups. To study symmetric groups, it is useful to first examine somewhat larger
groups, of which symmetric groups may be formed as intersections.

**Definition 8.3.** Let \( F_M \subseteq \text{ASP} \) consist of all permutations that send \( \{n \in \mathbb{Z} : n \geq M \} \) bijectively
to itself (and thus also send \( \{n \in \mathbb{Z} : n < M \} \) bijectively to itself).

Equivalently, \( \alpha \in F_M \) if and only if \( s_\alpha(M, M) = s_{\alpha^{-1}}(M, M) = 0 \). This implies

**Lemma 8.4.** For any \( M \in \mathbb{Z} \), \( F_M \) is downward-closed, hence closed under \( * \) and \( \triangleleft \).

Now, \( S_d = \bigcap_{M \leq 1} F_M \cap \bigcap_{M \geq d+1} F_M \) and therefore

**Corollary 8.5.** The symmetric group \( S_d \) is downward-closed, and thus closed under \( * \) and \( \triangleleft \).

As promised in the introduction, we can conveniently restrict arguments to \( S_d \) when working in \( S_d \).

**Corollary 8.6.** If \( \alpha, \beta \in S_d \), then for all \( 1 \leq a, b \leq d + 1 \),

\[
s_{\alpha \beta}(a, b) = \min_{1 \leq \ell \leq d+1} \left[ s_\alpha(a, \ell) + s_\beta(\ell, b) \right].
\]

**Proof.** Define \( L \) as in Lemma 3.12 and let \( \ell \) be any element of \( L \). Since \( \beta \), and thus \( \beta^{-1} \), fixes all
integers greater than \( d \), \( \beta^{-1}(\ell - 1) < b \leq d + 1 \) implies \( \ell - 1 \leq d \). Similarly, since \( \beta^{-1} \) fixes all
nonpositive integers, \( \beta^{-1}(\ell) \geq b \geq 1 \) implies \( \ell \geq 1 \). Hence \( L \subseteq \{1, \cdots, d + 1\} \), and the corollary follows from Lemma 3.12.

Of course, one often want to work with elements of \( S_d \) via their reduced words, i.e. factorization
into simple transpositions (or ”bubblesorts”). We can formulate the necessary tools here somewhat
generally, so we do so for later use. In the following statements, recall the notation \( \sigma_S \) from
Definition 3.15 for a product of simple transpositions.

**Theorem 8.7.** Let \( \alpha \in \text{ASP} \), and let \( S \) be a set with no two consecutive integers. Define \( S_1 = \{n \in S : \alpha(n) < \alpha(n + 1)\} \) and \( S_2 = \{n \in S : \alpha(n) > \alpha(n + 1)\} \). Then

\[
\alpha * \sigma_S = \alpha \sigma_{S_1}, \text{ and }
\alpha \triangleleft \sigma_S = \alpha \sigma_{S_2}.
\]

**Proof.** By definition, \( \sigma_S = \sigma_{S_1} \sigma_{S_2} = \sigma_{S_2} \sigma_{S_1} \). Since \( S_1, S_2 \) are disjoint, both these products are
reduced. Corollary 3.17 implies that \( * \sigma_{S_2} = \alpha \) and \( \triangleleft \sigma_{S_1} = \alpha \). Lemma 5.1 implies that
\( \alpha * \sigma_{S_1} = \alpha \sigma_{S_1} \) and Lemma 5.2 implies that \( \triangleleft \sigma_{S_2} = \alpha \sigma_{S_2} \). Putting this together and using
associativity and Lemma 3.11, \( \alpha \sigma_S = (\alpha * \sigma_{S_2}) * \sigma_{S_1} = \alpha * \sigma_{S_1} = \alpha \sigma_{S_1} \), and \( \triangleleft \sigma_S = (\triangleleft \sigma_{S_1}) * \sigma_{S_2} = \triangleleft \sigma_{S_2} = \alpha \triangleleft \sigma_{S_2} = \alpha \sigma_{S_2} \), as desired.

So Demazure products, and well as the operations \( \triangleleft \) and \( * \), may be conveniently computed by
factoring one of the permutations into adjacent transpositions, and this can also be slightly
”parallelized” by considering many non-overlapping adjacent transpositions at once. Theorem 8.7
implies the last sentences of Theorems A and 1.1 from the introduction, by taking \( S = \{n\} \).

8.3. Permutations with finitely many inversions. Let \( G \) denote the denote the group of
permutations \( \alpha \) such that \( \text{Inv}(\alpha) \) is finite. Alternatively, \( G \) is the group generated by the shift
permutations together with all symmetric groups \( S_d \). Then \( G \) is closed under \( \triangleleft \) in a strong sense:
for any \( \alpha \in G \) and \( \beta \in \text{ASP} \) (not necessarily in \( G \)), \( \alpha \triangleleft \beta \in G \). This is because \( \alpha \triangleleft \beta \leq \alpha \) (Lemma 4.14), so \( \text{Inv}(\alpha \triangleleft \beta^{-1}) \subseteq \text{Inv}(\alpha^{-1}) \) is finite.

This group is also closed downward, and therefore closed under \( * \). This can be seen from a
quick induction argument. If \( \alpha \leq \beta \), and \( \text{Inv}(\beta) \neq \emptyset \), then there exists some adjacent inversion
(n,n + 1) ∈ Inv(β). Then α ⪯ σ_n ≤ β ⪯ σ_n. Theorem B implies that β ⪯ σ_n = βσ_n has one fewer inversion, and α ⪯ σ_n is either α or ασ_n, which either both have finitely many inversions or both do not. In the base case, if β has no inversions, then β = t_χβ, so α = t_χβ as well.

In fact, with just a bit more care the inductive argument above shows the subword property [BIB05, Theorem 2.2.2]: if β = t_χσ_{n_1} · · · σ_{n_k}, where ℓ = # Inv(β), and α ⪯_χ β, then there exists a subsequence 1 ≤ i_1 < · · · < i_m ≤ ℓ such that α = t_χσ_{i_1} · · · σ_{i_m} and m = # Inv(α). In particular, # Inv(α) ≤ # Inv(β). This fact provides quick proofs of the following two inequalities. If both α, β have finitely many inversions, then

\[ # \text{Inv}(α * β) \leq # \text{Inv}(α) + # \text{Inv}(β) \] and # Inv(α ⪯ β) ≥ # Inv(α) − # Inv(β).

For the first, write α * β = α_1β, with α_1 ≤ α (Theorem B). Equality holds if and only if α*redβ. For the second, write α ⪯ β = αβ_1 with β_1 ≤ β (Theorem 6.1). Equality holds if and only if β−1 ⪯_L α.

The stronger form of closure under ⪯ allows the following strong form of the reduction theorem. This follows from the second paragraph of Theorem C.

**Proposition 8.8.** Suppose that α * β ≥ γ, where γ (but not necessarily α or β) has finitely many inversions. Then there exist α_1 ⪯_χ α, β_1 ⪯_χ β, with finitely many inversions, such that α_1⋆redβ_1 = γ.

Observe also that permutations with finitely many inversions must also have bounded difference and finite essential set.

### 8.4. The (extended) affine symmetric groups.

Fix an integer k ≥ 2. The extended affine symmetric group of modulus k is the group of permutations α such that α(n + k) = α(n) + k for all n ∈ Z. This condition is equivalent to t_kαt_{−k} = α, and therefore is equivalent to the condition

\[ s_α(a, b) = s_α(a + k, b + k) \text{ for all } a, b ∈ \mathbb{Z}. \]

It is immediate from the definitions of * and ⪯ that if s_α, s_β satisfy this condition, then so does s_{α*β}. So the extended affine symmetric group is closed under * and ⪯. As mentioned in Example 2.4, the affine symmetric group Ș_k is the subgroup of the shift-0 permutations in the extended affine symmetric group. Since shift-0 permutations are closed under * and ⪯, so is Ș_k.

The affine symmetric group Ș_k is generated by the permutations Ș_n = σ_{n+kZ} (one quick way to see this is to define the length of α ∈ Ș_k to be the number of inversions (m,n) with 0 ≤ m < k, and show that this is finite and, if positive, can be reduced by multiplying by some Ș_n). Therefore the operations * and ⪯ can be easily computed via factorization into the elements Ș_n, using Theorem 8.7. If S = n + kZ in the theorem statement, and α ∈ Ș_k, then one of S_1, S_2 is empty, so one of α * Ș_n, α ⪯ Ș_n is α Ş_n, and the other is α, according to whether or not α(n) < α(n + 1).

Since Ș_k is a Coxeter group, this description of * and ⪯ is known, so there is nothing novel here except verifying that the standard definition of these operations in a Coxeter group accords with the definition on ASP via Equation 5.

**APPENDIX A. A GEOMETRIC INTERPRETATION OF THE MIN-PLUS FORMULA FOR * **

The Demazure product on S_d arises naturally in the geometry of flag varieties and Schubert calculus; we briefly summarize a situation that provides useful intuition for Equation 5. This discussion parallels the discussion in [CP19, §2.3], although the notation is different.

Elements of S_d can be used to measure the “distance” between two complete flags in an n-dimensional vector space; the identity permutation corresponds to identical flags, and the descending permutation to transverse flags. Let H be a d-dimensional vector space. Let U, =
$(U_a)_{0 \leq a \leq d}, V = (V_b)_{0 \leq b \leq d}$ be two complete flags in $H$, indexed by dimension. There exists a unique permutation $\sigma = \sigma(U_*, V_*)$ such that for all $0 \leq a, b \leq d$,
\begin{equation}
\dim U_a \cap V_b = \# \{ n \leq b : \sigma(n) \leq a \} = a - s_\sigma(a + 1, b + 1).
\end{equation}

Note that this definition of $\sigma(U_*, V_*)$ differs from the one used in [CP19] and some other sources; we use it here to fit better the notation of this paper. Another description of $\sigma$ is via adapted bases. A basis $B$ is adapted to a flag $U_*$ if $B \cap U_a$ is a basis for $U_a$, for all $0 \leq a \leq d$. It is always possible to find a basis $B$ that is adapted to both of two given flags $U_*, V_*$: this can be proved by an argument in Gaussian elimination. If this basis is ordered $\{u_1, \cdots, u_d\}$ such that $\{u_1, \cdots, u_b\}$ is a basis of $V_b$ for all $b$, then there is a unique permutation $\sigma$ of $\{1, \cdots, d\}$ such that for all $a$, $\{u_{\sigma(1)}, \cdots, u_{\sigma(a)}\}$ is a basis for $U_a$. Then $U_a \cap V_b$ has basis $\{u_i : i \leq b \text{ and } \sigma(i) \leq a\}$, and Equation (28) follows.

For example, if $U_* = V_*$, then dim $U_a \cap V_b = \min(a, b)$, so $s_\sigma(a, b) = a - 1 - \min\{a - 1, b - 1\} = \max\{0, a - b\}$, and $\sigma(U_*, V_*)$ is the identity permutation. At the other extreme, if the flags are transverse then dim $U_a \cap V_b = \max\{0, a + b - d\}$ and $\sigma(U_*, V_*)$ is the descending permutation.

The Demazure product answers the question: given three flags $U_*, V_*, W_*$ and two permutations $\alpha, \beta$, if $\sigma(U_*, V_*) = \alpha$ and $\sigma(V_*, W_*) = \beta$, then what is the Bruhat-maximum of all possible $\sigma(U_*, W_*)$?

To see why the Demazure product answers this question, note that for all $a, b, \ell \in \{0, \cdots, d\}$,
\begin{equation}
\dim U_a \cap W_b \geq \dim U_a \cap V_\ell \cap W_b \geq \dim U_a \cap V_\ell + \dim V_\ell \cap W_b - \dim V_\ell.
\end{equation}
In the notation of slipface functions, this may be rewritten, after subtracting both sides from $a$, as
\begin{equation}
s_\sigma(U_*, W_*)(a + 1, b + 1) \leq s_\sigma(U_*, V_*)(a + 1, \ell + 1) + s_\sigma(V_*, W_*)(\ell + 1, b + 1).
\end{equation}

Taking the maximum over $\ell \in \{0, \cdots, d\}$, Corollary [8.0] gives $\sigma(U_*, W_*) \leq \sigma(U_*, V_*) \circ \sigma(V_*, W_*)$. In fact, for “generic” choices of flags, equality is obtained (see e.g. [CP19] Theorem 1.1)).

The discussion in this section can be generalized, with some care, from $S_d$ to ASP if one considers infinite flags, indexed by $Z$, in an infinite-dimensional vector space. Rather than expressing the dimension of $U_a \cap V_b$ as $a - s_\sigma(a + 1, b + 1)$, we may write rank $(U_a \to H/V_b) = s_\sigma(a + 1, b + 1)$, which is applicable in this infinite-dimensional setting provided that these ranks are all finite.

**Appendix B. Intended applications to curves and graphs**

This paper originated with applications to Brill-Noether theory of curves and graphs in mind. We summarize the basic ideas without full details in this appendix to motivate the content of this paper. Most of these applications are intended for future work; some of the simple applications to graphs are done in the short paper [P121a] that is independent of this one. This appendix assumes vocabulary from algebraic curves and the divisor theory of metric graphs. The basic goal in both settings is to provide a toolkit for inductive arguments, in which the geometry of a curve or graph with two marked points is reduced to the geometry of two such curves/graphs that are glued at a single point.

Let $C$ be a smooth projective algebraic curve over an algebraically closed field, and let $p, q$ be two distinct points on $C$. Then a line bundle $\mathcal{L}$ on $C$ determines a permutation $\tau = \tau^C_{p,q}$ by
\begin{equation}
h^0(C, \mathcal{L}(ap - bq)) = \# \{ n \geq b : \tau(n) \leq a \}
\end{equation}
\begin{equation}
= s_\tau(a + 1, b).
\end{equation}

The existence of $\tau$ follows from Proposition [7.12] which also proves that $\tau$ has bounded difference, as we now explain. Letting $\chi = d - g$, the Riemann–Roch formula implies that
\begin{equation}
h^0(C, \mathcal{L}(ap - bq)) = \max\{ \chi + a - b, 0 \} \text{ for } |a - b| \gg 0,
\end{equation}
which verifies hypothesis (1) of the proposition. To prove hypothesis (2), submodularity, we must show that for any line bundle $\mathcal{L}$,

$$h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-p)) - h^0(C, \mathcal{L}(-q)) + h^0(C, \mathcal{L}(-p - q)) \geq 0,$$

If $H = H^0(C, \mathcal{L})$ and $P, Q \subseteq H$ are the subspaces of sections vanishing at $p, q$ respectively, then

$$\dim H - \dim P - \dim Q + \dim P \cap Q = \dim H/(P + Q) \geq 0,$$

which proves the desired inequality and the submodularity of $h^0(C, \mathcal{L}(ap - bq))$.

Call $\tau^{p,q}_\mathcal{L}$ the transmission permutation of $\mathcal{L}$ with respect to $p, q$. The shift of $\tau^{p,q}_\mathcal{L}$ is given by the Euler characteristic $\chi(C, \mathcal{L}(-p)) = d - g$, where $d = \deg \mathcal{L}$ and $g$ is the genus of $C$. Reversing this process, we may define for any $\tau \in \text{ASP}$ a transmission locus

$$T^\tau(C, p, q) = \{[\mathcal{L}] \in \text{Pic}^{g+r}(C) : \tau^{p,q}_\mathcal{L} \geq \tau\}.$$

Assuming that $\tau$ has bounded difference, then making use of Lemma 7.3 this may be rewritten

$$T^\tau(C, p, q) = \{[\mathcal{L}] \in \text{Pic}^{g+r}(C) : h^0(C, \mathcal{L}((a - 1)p - bq)) \geq s_\tau(a, b) \text{ for all } (a, b) \in \text{Ess}(\tau)\}.$$

In particular, for any choice of $r, d$, we may consider the permutation $\gamma = \gamma^{g-d+r}_r$ from Example 7.11 for which $\chi_\gamma = d - g$ and $\text{Ess}(\gamma) = \{(1, 0)\}$. Therefore

$$T^\gamma(C, p, q) = W^\gamma_{C}(C)$$

$$:= \{[\mathcal{L}] \in \text{Pic}^{d}(C) : h^0(C, \mathcal{L}) \geq r + 1\}.$$

This locus is called a Brill–Noether locus. These loci play a central role in the theory of algebraic curves. So these transmission loci provide a reframing, and generalization, of Brill–Noether loci.

In another direction, there has been substantial recent interest in Hurwitz–Brill–Noether theory, which concerns the classification of line bundles on a general $k$-gonal curve of genus $g$. See for example [LLV20] and the references therein. The principle objects of study are splitting loci $W^{r\vec{e}}(C)$, for $k$-gonal curves $C$. Transmission loci are a useful tool for studying these loci in a special case: if $(C, p, q)$ is a $k$-gonal curve with $p, q$ points of total ramification for the degree-$k$ cover $\pi : C \to \mathbb{P}^1$, then for every splitting type $\vec{e}$ there exists an extended $k$-affine permutation $\gamma_{r\vec{e}}^\vec{e}$ such that

$$T^{\gamma_{r\vec{e}}^\vec{e}}(C, p, q) = W^{r\vec{e}}_{\mathcal{L}}(C).$$

The utility of this reframing is that transmission loci are very well-suited to induction arguments, once the Demazure product is in hand. We sketch the reason here without proofs or references, as it is the subject of forthcoming work (I am happy to provide a draft to any interested readers).

The construction of transmission permutations and transmission loci can be generalized in a natural way to chains of smooth curves. An analogous construction, of splitting loci on certain chains of $k$-gonal curves, is described in [LLV20] §3. Briefly, this generalization is defined in such a way that, in a one-parameter family of smooth curves degenerating to a chain, we require that dimension bounds hold for all extensions of the line bundle from the general fiber. The permutation $\tau^{p,q}_\mathcal{L}$ is upper semicontinuous (“upper” in the Bruhat order) in families. The Demazure product provides a “gluing” operation. Precisely, if $(C_1, p_1, q_1), (C_2, p_2, q_2)$ are two twice-marked smooth curves, $X = C_1 \cup C_2$ is the nodal curve obtained by gluing $q_1$ to $p_2$, and $\mathcal{L}$ is a line bundle on $X$ restricting to $\mathcal{L}_1$ on $C_1$ and $\mathcal{L}_2$ on $C_2$, then

$$\tau^{p_1,q_2}_\mathcal{L} = \tau^{p_1,q_1}_\mathcal{L}_1 \ast \tau^{p_2,q_2}_\mathcal{L}_2.$$

Remark B.1. The apparently strange definition of $\tau^{p,q}_\mathcal{L}$, in which $s_\tau(a + 1, b)$ appears rather than $s_\tau(a, b)$, was chosen so that (33) would be as simple and intuitive as possible.
In other words, the transmission locus on a union of two curves (or chains of curves) decomposes:

\[
T^\tau(C, p_1, q_2) = \bigcup_{\alpha \ast \beta \geq \tau} T^\alpha(C_1, p_1, q_1) \times T^\beta(C_2, p_2, q_2).
\]

The notation above sweeps one detail under the rug: we should also choose the degrees \(d_1, d_2\) of the line bundles on \(C_1, C_2\), and add the requirement \(\chi_\alpha = d_1 - g_1, \chi_\beta = d_2 - g_2\), where \(g_i\) is the genus of \(C_i\). This union is indexed over quite a large set: all pairs of permutations \(\alpha, \beta \in \text{ASP}\) with these shifts such that \(\alpha \ast \beta \geq \tau\). The utility of the reduction Theorem \([C]\) is to shrink this to a finite union of pieces that all have the same expected dimension. Namely,

\[
T^\tau(C, p_1, q_2) = \bigcup_{\alpha \ast \beta = \tau} T^\alpha(C_1, p_1, q_1) \times T^\beta(C_2, p_2, q_2).
\]

Once again, we have omitted the constraints on \(\chi_\alpha, \chi_\beta\) from the notation for simplicity. Equation (35) now provides a convenient inductive tool for analyzing both

1. local geometry of transmission loci, and thereby Brill–Noether loci, and
2. enumerative aspects of transmission loci, and thereby Brill–Noether loci.

As a simple example, Equation (35) gives a quick proof, by induction on genus, that for a general twice-marked curve \((C, p, q)\),

\[
\dim T^\tau(C, p, q) \leq g - \# \text{Inv}(\tau).
\]

In fact, the reverse inequality can also be proved, for any twice-marked curve, by degeneracy locus techniques (see e.g. [Ph21b]), so it is possible to deduce in this way that

\[
\dim T^\tau(C, p, q) = g - \# \text{Inv}(\tau),
\]

for a general \((C, p, q)\). When \(\tau = g_{r + 1} - d + r\), we have \#Inv\((g_{r + 1} - d + r) = (r + 1)(g - d + r)\), and this recovers the classical Brill–Noether theorem. Taking \(\tau\) to be extended \(k\)-affine, and \((C, p, q)\) to be a \(k\)-gonal curve with two points of total ramification, a completely analogous argument furnishes the main dimension upper bound of Hurwitz–Brill–Noether theory.

A completely parallel story may be developed for finite graphs, and also for metric graphs. Let \(G\) be either a graph or metric graph, and \(p, q\) two distinct vertices on \(G\). If \(D\) is a divisor on \(G\), then we may associate a slipface function to \(D\), by

\[
s^p,q_D(a, b) = r(D + (a + 1)p - bq) + 1.
\]

Here \(r\) is the Baker-Norine rank. As in the curve case, the shift of this slipface is \(d - g\). Unfortunately, this slipface need not be submodular (though it is always Clifford), so there may not be a well-defined “transmission permutation.” The problem, compared to algebraic curves, is that the Baker-Norine rank is not the dimension of a vector space, and there is no concrete description of \(\Delta s^p,q_D\) available like the one given above in the algebraic curves case. However, it appears that in many cases, such as chains of loops, all divisors do have submodular slipfaces.

In either case, this slipface still satisfies an easy-to-remember gluing equation. Let \(G_1\) be a (metric) graph with two points \(p_1, q_1\), and \(G_2\) a (metric) graph with two points \(p_2, q_2\). Let \(G\) be obtained by gluing \(q_1\) to \(p_2\). Then for any pair of divisors \(D_1\) on \(G_1\), \(D_2\) on \(G_2\), if we let \(D\) denote the sum of these two divisors on \(G\), then the analog of Equation (33) holds for (metric) graphs:

\[
s^p,q_D = s^p,q_{D_1} \ast s^p,q_{D_2}.
\]

This is proved in [Ph21a]. Equation (39) suggest the following strong form of Brill-Noether generality, for twice-marked metric graphs: \((G, p, q)\) is Brill-Noether general if, for every divisor \(D\) on \(G\), the function \(s^p,q_D\) is equal to \(s_\alpha\) for some permutation \(\alpha\) such that \#Inv \(\alpha \leq g\), where \(g\) is the
genus of \(G\). This notion is studied in \([\text{Pfl21a}]\), and called “0-general transmission.” The same paper considers also “\(k\)-general transmission” for \(k \geq 2\), with applications to Hurwitz–Brill–Noether theory in mind.

When studying finite graphs in particular, the element \([p - q] \in \text{Jac}(G)\) always has some torsion order \(k\), which implies that \(s^{p,q}_D\) has the periodicity property \(s^{p,q}_D(a + k, b + k) = s^{p,q}_D(a, b)\) and all transmission permutations (when they exist) are extended \(k\)-affine. When chaining finite graphs together, then, one considers Demazure products of extended affine permutations. The paper \([\text{Pfl21a}]\) studies this situation in a simple case: chains of twice marked loops, all of the same torsion order. This situation is somewhat simple, in that all transmission permutations involved belong to the same extended affine symmetric group. The stronger results in this paper will make somewhat more subtle arguments possible, in which different torsion orders are used in a chain, and therefore it is necessary to intermingle extended affine permutations of different moduli.

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