LOXODROMES ON TWISTED SURFACES IN EUCLIDEAN 3-SPACE

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Abstract. In the present paper, loxodromes, which cut all meridians and parallels of twisted surfaces (that can be considered as a generalization of rotational surfaces) at a constant angle, have been studied in Euclidean 3-space and some examples have been constructed to visualize and support our theory.

1. General Information and Basic Concepts

Loxodromes (also known as rhumb lines) correspond to the curves which intersect all of the meridians at a constant angle on the Earth (see Figure 1). An aircraft flying and a ship sailing on a fixed magnetic compass course move along a curve. Here the course is a rhumb and the curve is a loxodrome. Generally, a loxodrome is not a great circle, thus it does not measure the shortest distance between two points on the Earth. However loxodromes are important in navigation and they should be known by aircraft pilots and sailors [1].

If the shape of the Earth is approximated by a sphere, then the loxodrome is a logarithmic spiral that cuts all meridians at the same angle and asymptotically approaches the Earth’s poles but never meets them. Since maritime surface navigation defines the course as the angle between the current meridian and the longitudinal direction of the ship, it may be concluded that the loxodrome is the curve of the constant course, which means that whenever navigating on an unchanging course we are navigating according to a loxodrome [II].

Figure 1. Loxodrome on Earth

In this context, there are lots of studies about loxodromes in Euclidean and Minkowskian spaces. For instance, the differential equations of loxodromes on a sphere, spheroid, rotational

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surface, helicoidal surface and canal surface in Euclidean 3-space have been given in [11], [13], [12], [2] and [3], respectively. Also, in [4] and [5], spacelike and timelike loxodromes on rotational surface and in [6], differential equations of the spacelike loxodromes on the helicoidal surfaces in Lorentz-Minkowski 3-space have been given.

Now, let we recall some basic notions about curves and twisted surfaces in Euclidean 3-space \( E^3 \).

For two vectors \( \vec{u} = (u_1, u_2, u_3) \) and \( \vec{v} = (v_1, v_2, v_3) \) in \( E^3 \), the inner product of these vectors and the norm of the vector \( \vec{u} \) are defined by

\[
\langle \vec{u}, \vec{v} \rangle = u_1 v_1 + u_2 v_2 + u_3 v_3
\]

(1.1)

and

\[
\| \vec{u} \| = \sqrt{\langle \vec{u}, \vec{u} \rangle},
\]

(1.2)

respectively. We say that \( \vec{u} \) is a unit vector, if it satisfies \( \| \vec{u} \| = 1 \).

The arc-length of a regular curve \( \alpha : I \subset \mathbb{R} \rightarrow E^3, s \rightarrow \alpha(s) \), between \( s_0 \) and \( s \) is

\[
t(s) = \int_{s_0}^{s} \| \alpha'(s) \| \, ds.
\]

(1.3)

Then the parameter \( t \in J \subset \mathbb{R} \) is determined as \( \| \alpha'(t) \| = 1 \).

Also, the angle \( \phi \) \( (0 < \phi < \pi) \) between the vectors \( u \) and \( v \) is

\[
\cos \phi = \frac{\langle \vec{u}, \vec{v} \rangle}{\| \vec{u} \| \| \vec{v} \|}.
\]

(1.4)

Here, we recall the definition and parametrization of twisted surfaces in \( E^3 \). (For detail, see [8].)

A twisted surface in \( E^3 \) is obtained by rotating a planar curve \( \alpha \) in its supporting plane while this plane itself is rotated about some containing straight line. Without loss of generality, the coordinate system can be chosen in such a way that the \( xz \)-plane corresponds with the plane supporting the planar curve with the \( z \)-axis as its containing rotation axis and that the straight line through the point \((a, 0, 0)\) parallel with the \( y \)-axis acts as rotation axis for the planar curve.

Firstly, let we apply the rotation about the straight line through the point \((a, 0, 0)\) parallel with the \( y \)-axis to the profile curve \( \alpha(y) = (f(y), 0, g(y)) \) \( (f \) and \( g \) are real-valued functions) and next apply the rotation about the \( z \)-axis to the obtained surface. Then, up to a transformation, we get the parametrization of the twisted surface in \( E^3 \) as

\[
T(x, y) = \left( \begin{array}{c}
(a + f(y) \cos(bx) - g(y) \sin(bx)) \cos x, \\
(a + f(y) \cos(bx) - g(y) \sin(bx)) \sin x, \\
f(y) \sin(bx) + g(y) \cos(bx)
\end{array} \right).
\]

(1.5)

Here, the presence of the factor \( b \in \mathbb{R} \) allows for differences in the rotation speed of both rotations and it is obvious from the construction that, if we take \( b = 0 \), then the twisted surface reduces to a surface of revolution. Thus, the twisted surfaces can be considered as generalizations of surfaces of revolution.

After giving the definition of the twisted surfaces, twisted surfaces with null rotation axis in Minkowski 3-space have been studied in [9] and twisted surfaces with vanishing curvature in Galilean 3-space have been classified in [7]. Also, in [10], the twisted surfaces in pseudo-Galilean space have been studied.
2. Loxodromes on Twisted Surfaces in $E^3$

In this section, we obtain the equations of loxodromes on the twisted surfaces in $E^3$.

Let $T$ be the twisted surface which parametrized as (1.5). Then the coefficients of first fundamental form of the twisted surface $T$ are obtained by

\[
\begin{align*}
    g_{11} &= \frac{1}{2} \left\{ 2a^2 + (1 + 2b^2 + \cos(2bx))f^2 + (1 + 2b^2 - \cos(2bx))g^2 \right\}, \\
    g_{12} &= g_{21} = b \{ fg' - f'g \}, \\
    g_{22} &= f'^2 + g'^2,
\end{align*}
\]

where $g_{11} = \langle (T_x, T_x) \rangle$, $g_{12} = g_{21} = \langle (T_x, T_y) \rangle$, $g_{22} = \langle (T_y, T_y) \rangle$, $f = f(y)$, $g = g(y)$, $f' = \frac{df}{dy}$ and $g' = \frac{dg}{dy}$.

Also, we know that, the first fundamental form in the base $\{M_x, M_y\}$ for a surface $M(x, y)$ is given by

\[
ds^2 = g_{11}dx^2 + 2g_{12}dxdy + g_{22}dy^2, \tag{2.2}
\]

where $g_{ij}$ are the coefficients of the first fundamental form of $M$. So, from (2.1) and (2.2), we can write the first fundamental form of the twisted surface (1.5) as

\[
ds^2 = \left\{ a^2 - 2ag \sin(bx) + 2f \cos(bx)(a - g \sin(bx)) \right\} dx^2 + 2b \{ fg' - f'g \} dxdy + \left( f'^2 + g'^2 \right) dy^2 \tag{2.3}
\]

and from (2.3), the arc-length of any curve on the twisted surface between $x_1$ and $x_2$ is given by

\[
s = \left| \int_{x_1}^{x_2} \sqrt{\left\{ a^2 - 2ag \sin(bx) + 2f \cos(bx)(a - g \sin(bx)) \right\} dx} \right|. \tag{2.4}
\]

Furthermore, a curve $\gamma$ is called a loxodrome on the twisted surface $T$ in $E^3$ if it cuts all meridians ($y$ constant) (or parallels ($x$ constant)) of $T$ at a constant angle.

Now, let us suppose that $\gamma(t) = T(x(t), y(t))$; i.e. $\gamma(t)$ is a curve on the twisted surface $T$. With respect to the local base $\{T_x, T_y\}$, the vector $\gamma'(t)$ has the coordinates $(x', y')$ and the vector $T_x$ has the coordinates $(1, 0)$. At the point $p = T(x, y)$; where the loxodrome cuts the meridians at a constant angle; we get

\[
\cos \phi = \frac{\langle \gamma'(t), T_x \rangle}{\| \gamma'(t) \| \| T_x \|} = \frac{g_{11}dx + g_{12}dy}{\sqrt{g_{11}^2dx^2 + 2g_{11}g_{12}dxdy + g_{11}g_{22}dy^2}}. \tag{2.5}
\]

Therefore, from (2.5) we get

\[
g_{11}^2 \sin^2 \phi \ dx^2 + 2g_{11}g_{12} \sin^2 \phi \ dx dy + (g_{12}^2 - g_{11}g_{22} \cos^2 \phi) \ dy^2 = 0 \tag{2.6}
\]

and so,

\[
\frac{dy}{dx} = -\frac{2g_{11}g_{12} \sin^2 \phi + g_{11} \sqrt{g_{11}g_{22} - g_{12}^2} \sin(2\phi)}{2(g_{12}^2 - g_{11}g_{22} \cos^2 \phi)}. \tag{2.7}
\]
Hence from (2.1) and (2.7), the loxodrome on the twisted surface (1.5) must satisfy the following equation

\[
\frac{dy}{dx} = -\left(\begin{array}{c}
2b\sin\phi(fg' - fg') \\
\mp 2\cos\phi \sqrt{-b^2(gf' - fg')^2 + \left\{\begin{array}{l}
a^2 - 2ag\sin(bx) + 2f\cos(bx)(a - g\sin(bx)) \\
+ \frac{1}{2} \left(1 + 2b^2 + \cos(2bx)\right)f^2 + (1 + 2b^2 - \cos(2bx))g^2 \end{array}\right\} (f'^2 + g'^2)}
\end{array}\right)
\]

(2.8)

Here, let us construct two examples, with the aid of Mathematica, to visualize and support our theory.

**Example 1.** Taking the profile curve as \(\alpha(y) = (y, 0, 0)\), the twisted surface (1.1) becomes

\[T(x, y) = ((a + y\cos(bx))\cos x, (a + y\cos(bx))\sin x, y\sin(bx)).\]  

(2.9)

From (2.8) (we take \(\mp\) in this equation as \(-\)) , we get the differential equation of the loxodrome on the twisted surface (2.9) for \(a = 0\) and \(b = \frac{1}{2}\) as

\[
\frac{dy}{dx} = \frac{y\sqrt{2}\cos x + 3\tan\phi}{2}.
\]

(2.10)

So, we have

\[
\frac{dy}{y} = \frac{\sqrt{2}\cos x + 3\tan\phi}{2} dx
\]

and by integrating both sides of this equation, we get

\[
\ln y = \int_{x_0}^{x} \frac{\sqrt{2}\cos x + 3\tan\phi}{2} dx.
\]

(2.11)

Putting \(x_0 = 0\) in (2.11), we reach that

\[
y = y(x) = e^{\sqrt{2}\text{EllipticE}[\frac{x}{2}, \frac{3}{2}]\tan\phi}.
\]

(2.12)

Now, if we take \(\phi = \frac{\pi}{6}\) and \(x \in (-2\pi, 2\pi)\), we get \(y \in (0.0476989, 20.9649)\). Thus, the loxodrome which lies on the twisted surface (2.9) is obtained as

\[
\gamma(x) = e^{\sqrt{2}\text{EllipticE}[\frac{x}{2}, \frac{3}{2}]}, \left(\cos \left(\frac{x}{2}\right) \cos x, \cos \left(\frac{x}{2}\right) \sin x, \sin \left(\frac{x}{2}\right)\right).
\]

(2.13)

Also, the arc-length of our loxodrome (2.13) is approximately equal to 41.8343. The twisted surface (2.9), meridian for \(y = 15\) and the loxodrome (2.13) can be seen in Figure 2.
Example 2. For the profile curve $\alpha(y) = (\cos y, 0, \sin y)$, the twisted surface (1.5) is

$$T(x, y) = \left( (a + \cos y \cos(bx) - \sin y \sin(bx)) \cos x, (a + \cos y \cos(bx) - \sin y \sin(bx)) \sin x, \ight.$$
$$\left. \cos y \sin(bx) + \sin y \cos(bx) \right).$$

If we take $a = 1$ and $b = 0$, then from (2.8) (we take $\mp$ in this equation as $-$), we have

$$\frac{dy}{dx} = 2 \cos^2 \left( \frac{y}{2} \right) \tan \phi.$$  

(2.15)

So, we have

$$\frac{dy}{2 \cos^2 \left( \frac{y}{2} \right)} = \tan \phi dx$$

and by integrating both sides of this equation, we get

$$\tan \left( \frac{y}{2} \right) = \int_{x_0}^{x} \tan \phi \, dx.$$  

(2.16)

Taking $x_0 = 0$ in (2.16), we reach that

$$y = y(x) = 2 \left( \arctan (x \tan \phi) + c \pi \right), \ c \in \mathbb{Z}.$$  

(2.17)

Here, if we take $c = 0$, $\phi = \frac{\pi}{4}$ and $x \in (-\pi, \pi)$, we have $y \in (-2.52525, 2.52525)$. Thus, the loxodrome which lies on the twisted surface (2.14) is obtained as

$$\gamma(x) = ((1 + \cos(2 \arctan x)) \cos x, (1 + \cos(2 \arctan x) \sin x, \sin(2 \arctan x)).$$  

(2.18)

Also, the arc-length of our loxodrome (2.18) is approximately equal to 7.1425. One can see the twisted surface (2.14), meridian for $y = 1$ and the loxodrome (2.18) in Figure 3.
Furthermore, from the definition of the angle $\theta$ between the loxodrome and any parallel $(x=\text{constant})$, we have

\[
\cos \theta = \frac{\langle \gamma'(t), T_y \rangle}{\|\gamma'(t)\| \|T_y\|} = \frac{g_{12}dx + g_{22}dy}{\sqrt{g_{11}g_{22}dx^2 + 2g_{12}g_{22}dxdy + g_{22}^2dy^2}}
\]  

(2.19)

From (2.19), we get

\[
(g_{11}g_{22} \cos^2 \theta - g_{12}^2)dx^2 - 2g_{12}g_{22} \sin^2 \theta dxdy - g_{22}^2 \sin^2 \theta dy^2 = 0
\]  

(2.20)

and so,

\[
\frac{dx}{dy} = \frac{-2g_{12}g_{22} \sin^2 \theta \mp g_{22} \sqrt{g_{11}g_{22} - g_{12}^2} \sin(2\theta)}{2(g_{22}^2 - g_{11}g_{22} \cos^2 \theta)}. 
\]  

(2.21)

Therefore, the loxodrome on the twisted surface must satisfy the following equation

\[
\frac{dx}{dy} = \left( \begin{array}{c} 2b \sin \theta (f'g' - fg') \\ -b^2(gf' - g')^2 \\ + \frac{2}{b^2} \left( (1 + 2b^2 + \cos(2bx))f^2 + (1 + 2b^2 - \cos(2bx))g^2 \right) \end{array} \right) \left( f'^2 + g'^2 \right) \sin \theta \\
\mp 2 \cos \theta \left( \begin{array}{c} 2b^2(gf' - g')^2 \\ - \left( 2a^2 - 4ag \sin(bx) + 4f \cos(bx)(a - g \sin(bx)) \right) \\ + \frac{1}{2} \left( (1 + 2b^2 + \cos(2bx))f^2 + (1 + 2b^2 - \cos(2bx))g^2 \right) \end{array} \right) \left( f'^2 + g'^2 \right) \cos \theta
\]  

(2.22)

Now, let us give an example for the loxodrome which cuts the parallels of the twisted surface at a constant angle.
Example 3. Let us take the profile curve as \( \alpha(y) = (\cos^2 y, 0, \sin^2 y) \). Then, the twisted surface is

\[
T(x, y) = \left( \begin{array}{c}
(a + \cos^2 y \cos(bx) - \sin^2 y \sin(bx)) \cos x, \\
(a + \cos^2 y \cos(bx) - \sin^2 y \sin(bx)) \sin x, \\
\cos^2 y \sin(bx) + \sin^2 y \cos(bx)
\end{array} \right).
\] (2.23)

Putting \( a = -1 \) and \( b = 0 \), from (2.22) (we take \( \mp \) in this equation as \( - \)), we get

\[
\frac{dx}{dy} = 2\sqrt{2} \cot y \tan \theta.
\] (2.24)

Thus, by integrating both sides of the equation

\[
dx = 2\sqrt{2} \cot y \tan \theta dy
\]

we have

\[
x = \int_{y_0}^{y} 2\sqrt{2} \cot y \tan \theta dy.
\] (2.25)

For \( y_0 = \frac{\pi}{2} \), we reach that

\[
x = x(y) = 2\sqrt{2} \ln(\sin y) \tan \theta.
\] (2.26)

Here, by taking \( \theta = \frac{\pi}{3} \) and \( y \in (\frac{\pi}{16}, \frac{2\pi}{3}) \), we have \( x \in (-8.00637, -0.704674) \). Therefore, the loxodrome which lies on the twisted surface (2.23) is obtained as

\[
\delta(y) = \left( -\sin^2 y \cos(2\sqrt{6} \ln(\sin y)), -\sin^2 y \sin(2\sqrt{6} \ln(\sin y)), \sin^2 y \right).
\] (2.27)

Also, the arc-length of the loxodrome (2.27) is approximately equal to 3.42788. One can see the twisted surface (2.23), meridian for \( x = -1 \) and the loxodrome (2.27) in Figure 4.

Figure 4. Twisted surface (2.23), Meridian (red) and Loxodrome (blue)
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