Theory of the d.c. Josephson effect in Pb/Sr$_2$RuO$_4$/Pb

Masashi Yamashiro and Yukio Tanaka
Graduate School of Science and Technology, Niigata University, Ikarashi, Niigata 950-2181, Japan

Satoshi Kashiwaya
E. L. Ginzton Laboratory, Stanford University, Stanford, CA 94305-4085, USA
(August 27, 2019)

To clarify the origin of anomalous behaviors in Pb/Sr$_2$RuO$_4$/Pb junctions in terms of the pairing symmetry, a theory of the d.c. Josephson current in s-wave superconductor / p-wave superconductor / s-wave superconductor junctions is developed. Calculated results on the temperature dependence of the critical Josephson current exhibit non-monotonous behaviors when the thickness of the p-wave superconductor is comparable to the coherence length. The consistency between present results with recent experimental measurement supports the possibility of a unitary p-wave pairing state in Sr$_2$RuO$_4$.

Recently, physical properties on Sr$_2$RuO$_4$ have been studied vigorously both in the superconducting states and in the normal state. There have been presented several experimental results as well as theories that support an unconventional superconducting states i.e. triplet p-wave pair potential, in superconducting Sr$_2$RuO$_4$. In order to identify definitively the pairing symmetry, phase sensitive measurements utilizing the Josephson junctions give the most useful information. Standing on this viewpoint, it is an interesting problem to develop a theory of the Josephson effect for junctions which include triplet p-wave superconductors.

One of the most significant differences of unconventional superconductors from conventional ones is the existence of the internal phase in the pair potential. The important role of the internal phase on the current-phase relation of the Josephson current has been pointed out theoretically. In a two-dimensional junction configuration with the internal phase in the pair potential. The important role of the internal phase on the current-phase relation of the Josephson current is to be proportional to $\sin(n\varphi)$ decomposed into the infinite series of $\sin(n\varphi)$. The critical Josephson current $J_c$ via the ZES is shown to be proportional to $T^{-1}$. The interplay of above two effects induces a non-monotonous temperature dependence of the Josephson current depending on the orientation of the junctions. As for the current-phase relation, it can be decomposed into the infinite series of $\sin(n\varphi)$ with integer $n$,

$$J(\varphi) = \sum_n n \tilde{J}_n \sin(n\varphi),$$

where $\varphi$ is the difference of the external phase between two superconductors. It was revealed in previous papers that $\tilde{J}_1$ vanishes in singlet superconductor / triplet superconductor junctions independent of the orientation of the junctions if the spin orbit coupling is absent. This peculiar property originates from the difference in the parities between the two superconductors.

Recent experiment by Jin reported a non-monotonous temperature dependence of the critical Josephson current in Pb/Sr$_2$RuO$_4$/Pb junctions. Since this behavior cannot be understood in terms of conventional theory of the Josephson effect, it is interesting to clarify the origin in terms of the influence of the p-wave pair potentials on the Josephson current lying between two s-wave superconductors. In this paper, we present a theory of the Josephson current in a s-wave / p-wave / s-wave (s/p/s) trilayer superconductor junction where the transition temperature of p-wave superconductor $T_p$ is lower than that of s-wave superconductor $T_s$.

In the following, we assume an s/p/s junction in the clean limit with semi-infinite s-wave superconductor region ($x < 0$ and $x > L$) where the thickness of the p-wave superconductor is $L$. The flat interfaces are perpendicular to the x-axis, and are located at $x = 0$ and $x = L$. The barrier potential at the interface is ignored for simplicity. We also assume that the Fermi wave number $k_F$ and the effective mass $m$ are equal in the all three regions. To express the nearly two-dimensional Fermi surface, the $z$ component of the Fermi momentum $k_F z$ is restricted to the region given by $-\delta < k_F z < \delta$. The two-component wave function $\Psi(x)$ of the quasiparticles is given as the solution of the Bogoliubov-de Gennes (BdG) equation following the quasi-classical approximations. In this approximation, the effective pair potentials felt by the quasiparticles depend only on the direction of the motion of the quasiparticles. The pair potential of the system is assumed to be the stepwise form (see Fig.1)
\[
\Delta_{\sigma \sigma'}(x, \theta, \phi) = \delta_{\sigma, -\sigma'}(\delta_{\sigma, \uparrow} - \delta_{\sigma, \downarrow}) \Delta_s(T) e^{-ix/2} \quad x < 0, \\
= \delta_{\sigma, -\sigma'} \Delta_p(T, \theta, \phi) \quad 0 < x < L, \\
= \delta_{\sigma, -\sigma'}(\delta_{\sigma, \uparrow} - \delta_{\sigma, \downarrow}) \Delta_s(T) e^{ix/2} \quad L < x.
\]

In the above, \(\theta\) is the polar angle and \(\phi\) is the azimuthal angle in the \(x-y\) plane. For the \(p\)-wave pair potential, we assume an unitary state where only \(\Delta_{\uparrow \downarrow}(x, \theta, \phi) = \Delta_{\downarrow \uparrow}(x, \theta, \phi)\) is non-zero for simplicity [20]. The effective pair potential felt by the quasiparticles changes its sign depending on the spin index in the \(s\)-wave region. While in the \(p\)-wave region, the pair potential has the same sign regardless of the spins of quasiparticles. The external phase difference between the \(p\)-wave superconductor and the \(s\)-wave superconductor at each interfaces is \(\phi/2\) as to satisfy the current conservation law [23].

Suppose an electron-like quasiparticle (ELQ) is injected from the left superconductor with the injection angles \(\theta\) and \(\phi\). In the present case, the possible reflection process is only Andreev reflection because of the absence of the barrier potential at the interfaces. The coefficients of the Andreev reflection is determined by solving the BdG equation under the following boundary conditions:

\[
\begin{align*}
\Psi(x)|_{x=0-} &= \Psi(x)|_{x=0+}, \\
\frac{d\Psi(x)}{dx}|_{x=0-} &= \frac{d\Psi(x)}{dx}|_{x=0+}, \\
\Psi(x)|_{x=L-} &= \Psi(x)|_{x=L+}, \\
\frac{d\Psi(x)}{dx}|_{x=L-} &= \frac{d\Psi(x)}{dx}|_{x=L+}.
\end{align*}
\]

The obtained coefficients depend on the direction of the spin of the injected ELQ. For the calculation of Josephson current, we will extend the formula by Furusaki and Tsukada [26] for \(s\)-wave superconductors. In this formula, the Josephson current is expressed by the generalized Andreev coefficients \(\bar{A}_{1(2)}(\varphi, \theta, \phi)\) which are obtained by the analytic continuation from \(E\) to \(i\omega_n\) in \(\alpha_{1(2)}(\varphi, \theta, \phi)\) where \(E\) is the energy of the quasiparticles measured from the Fermi energy \(E_F\) and \(\omega_n = 2\pi k_B T(n + 1/2)\) is the Matsubara frequency. Then, the Josephson current \(J(\varphi)\) is given as

\[
e R_N J(\varphi) = \Delta_s(T) k_B T \frac{\int_{\pi/2 - \delta}^{\pi/2} \int_{\pi/2}^{\pi/2} K(\varphi, \theta, \phi) \sin^2 \theta \cos \phi d\theta d\phi}{\frac{1}{\pi} \int_{\pi/2 - \delta}^{\pi/2} \int_{\pi/2}^{\pi/2} \sin^2 \theta \cos \phi d\theta d\phi}.
\]

\[
K(\varphi, \theta, \phi) = \Sigma_{\omega_n} \bar{a}_1(\varphi, \theta, \phi) + \bar{a}_2(\varphi, \theta, \phi) - \bar{a}_1(-\varphi, \theta, \phi) - \bar{a}_2(-\varphi, \theta, \phi),
\]

\[
\bar{a}_1(\varphi, \theta, \phi) = \frac{1}{i} \left\{ \frac{A_+ + \Gamma_{ns} B_+}{\Gamma_{ns} A_+ - B_+} \right\},
\bar{a}_2(\varphi, \theta, \phi) = \frac{1}{i} \left\{ \frac{A_+ + \Gamma_{ns} B_+}{\Gamma_{ns} A_+ - B_+} \right\},
\]

\[
A_\pm = e^{i\varphi/2} \gamma^* \Gamma_{\eta p}(e^{-2iW} - 1) \pm e^{i\varphi} \Gamma_{ns}(e^{-2iW} + \Gamma_{np}^2),
\]

\[
B_\pm = e^{i\varphi/2} \gamma \Gamma_{\eta p}(1 - e^{-2iW}) \pm (1 + \Gamma_{np}^2) e^{-2iW},
\]

\[
\gamma = \frac{\Delta_p(T, \theta, \phi)}{\Delta_p(T, \theta, \phi)} |\Gamma_{ns(p)}| = \sqrt{\frac{\Omega_{ns(p)} - \omega_n}{\Omega_{ns(p)} + \omega_n}},
\]

\[
\Omega_{ns} = \sqrt{\omega_n^2 + |\Delta_s(T)|^2}, \quad \Omega_{np} = \sqrt{\omega_n^2 + |\Delta_p(T, \theta, \phi)|^2}.
\]

Here \(R_N\) denotes the normal resistance, with \(X = L/\xi_p\) and \(W = \Omega_{np}/|\Delta_p(0) \sin \theta \cos \phi|\). The coherence length in the triplet superconductor \(\xi_p\) is given as \(\xi_p = \hbar k_F / |m \Delta_p(0)|\). In the present paper, we select two kinds of unitary \(p\)-wave pair potentials belonging to the two dimensional \(E_u\) representation which are expressed as \(\Delta_{\uparrow \downarrow}(T, \theta, \phi) = \Delta_p(T, \theta, \phi) = \Delta_p(T) \sin \theta (\sin \phi + \cos \phi)\) and \(\Delta_p(T) e^{i\phi} \sin \theta\). Hereafter, we will call these as \(E_u(U1)\) and \(E_u(U2)\) states, respectively. The quantity \(\gamma\) which is expressed by the internal phase of the triplet superconductor is
given by $\gamma = 1$ for $E_u(U1)$, and $\gamma = e^{i\phi}$ for $E_u(U2)$. By substituting these pair potentials in the above formulas, we can obtain the Josephson current $J(\phi)$ and $J_C(T)$.

In the following calculation, the transition temperatures $T_p$ and $T_x$ are set to be 1.0K and 8.1K, respectively, in order to simulate Pb/Sr$_2$RuO$_4$/Pb junctions. To express the two-dimensional features of the Fermi surface, the quantity $\delta$ is chosen as $0.1\pi$. The temperature dependence of $\Delta_s(T)$ and $\Delta_p(T)$ are assumed to obey the BCS relation. Figures 2 and 3 show the calculated $J_C(T)$'s for $E_u(U1)$ and $E_u(U2)$, respectively. With the decrease of the $T$, the characteristic temperature where $J_C(T)$ begins to decrease is given by $T_p$. With the further decrease of $T$, $J_C(T)$ increases again. Apparently these features are quite distinct from those for conventional s-wave superconductors. To understand this behavior intuitively, it is useful to present analytical forms of $K(\varphi, \theta, \phi)$ in the two limiting cases. For $T > T_p$, where $\Delta_p(T) = 0$, is satisfied, $K(\varphi, \theta, \phi)$ is reduced to be

$$K(\varphi, \theta, \phi) = \Sigma_{\omega_n} \frac{1}{2\Omega_{ns}} \left\{ \frac{4\Gamma_{ns}(1 + \Gamma_{ns}^2)e^{-2\pi XW} \sin \varphi}{|\Gamma_{ns}e^{-2\pi XW} e^{\imath \varphi} + 1|^2} \right\},$$  

which reproduces that of in an ordinary SNS junction [19]. While for the case $L \to \infty$, $K(\varphi, \theta, \phi)$ is calculated as

$$K(\varphi, \theta, \phi) = \Sigma_{\omega_n} \frac{1}{2\Omega_{ns}} \left\{ \frac{-2\gamma^* \Gamma_{ns}(1 + \Gamma_{ns}^2)\Gamma_{np}^2 \sin \varphi}{(1 - \gamma^* 2\Gamma_{ns}^2 \Gamma_{np})^2 + 4\gamma^* 2\Gamma_{ns}^2 \Gamma_{np}^2 \sin^2(\varphi/2)} \right\}.$$  

This result coincides with that of the s-wave / p-wave junction [20]. It should be emphasized that a contribution to $J(\varphi)$ from the former (referred to as s/s-coupling) is positive and the latter is negative (referred to as s/p-coupling) for $0 < \varphi < \pi$. This means that the junction changes so-called "$\pi$-junction" when the s/p-coupling becomes dominant compared to the s/s-coupling. Since these two coupling terms have different $T$ and $L$ dependence, the competition of these two determines the behavior of the junction.

Next, we will examine the thickness ($L$) dependence of the Josephson current. For sufficiently small $L$ ($L \ll \xi_p$), $J(\varphi)$ remains positive independent of temperatures because of the strong s/s-coupling (see upper panel of Fig. 4). When $L \approx \xi_p$, the junction is also governed by the s/s-coupling for $T > T_p$. However, with the decrease of temperatures, the contribution from s/p-coupling term (Eq. (13)) is rapidly enhanced. The sign changes in $J(\varphi)$ occurs when the s/p-coupling becomes dominant at sufficiently low temperature. At temperature just below $T_p$, a crossover occurs between these two terms. This is the origin of the non-monotonous temperature dependence of $J_C(T)$. While for $L \gg \xi_p$, the s/s-coupling at $T > T_p$ is weakened because it is governed by the factor $\exp[-LT/(\hbar v_F)]$. As the temperature is lowered below $T_p$, the junction easily transits to the $\pi$-junction [11]. Then the region, where $dJ_C(T)/dT > 0$ is satisfied, becomes narrower with the increase of $L$. Thus the non-monotonous temperature dependence is reduced and becomes undetectable.

In this paper, we present a theory of the Josephson current in a s/p/s junction and obtain the non-monotonous temperature dependence of $J_C(T)$. This effect originates from the competition between the positive contribution of $K(\varphi, \theta, \phi)$ to $J(\varphi)$ due to the conventional s/s-coupling and the negative one due to the s/p-coupling. The present effect is quite distinct from the non-monotonous $J_C(T)$ predicted in d-wave superconductor junction [13][17] in the point that the present effect is insensitive to the orientation of the junction. Although we have assumed unitary state for Sr$_2$RuO$_4$, it is also possible to perform the similar calculations for several non-unitary pair potentials [3][11][27]. However, only the monotonous increase of $J_C(T)$ is expected because of the absence of the Josephson coupling between s-wave and these non-unitary pair potentials. It is important to note that the present result is consistent with the recent measurements of Josephson current in Pb/Sr$_2$RuO$_4$/Pb junction [23]. We believe that the consistency strongly supports the unitary triplet p-wave pairing states in Sr$_2$RuO$_4$. Since the difference of the parity plays an essential role in this effect, the present trilayer junction configuration can be applied to a novel phase-sensitive test for the unconventional pairing states. For example, if $E_{u1}$ state [28] is also realized in UPt$_3$, similar effect will be detected in Pb/UPt$_3$/Pb junctions. Throughout this paper, we ignored the barrier potential at the interface for the simplicity. However, the essence of the physics will not be changed even if we will take into account of this effect. Details will be presented in the forthcoming paper.

We would like to thank to Y. Maeno for stimulating this topic and for showing their results prior to publications. One of the author (S. K.) would like to thank to M. R. Beasley for fruitful discussion. This work is supported by a Grant-in-Aid for Scientific Research in Priority Areas "Anomalous metallic state near the Mott transition" and "Nissan Science Foundation". The computational aspect of this work has been done for the facilities of the Supercomputer Center, Institute for Solid State Physics, University of Tokyo and the Computer Center, Institute for Molecular Science, Okazaki National Research Institute.
Permanent address: Electrotechnical Laboratory, Tsukuba, Ibaraki 305-9568, Japan.

[1] Y. Maeno et al., J. Low. Temp. Phys. 105, 1577 (1996); J. Phys. Soc. Jpn. 66, 1405 (1997).
[2] S. Nishizaki et al., J. Phys. Soc. Jpn. 67, 560 (1998).
[3] K. Yoshida et al., Physica C 263, 519 (1996).
[4] K. Ishida et al., Phys. Rev. B 56, R505 (1997).
[5] T.M. Rice and M. Sigrist, J. Phys. Condens. Matter. 7, 643 (1995).
[6] G. Baskaran, Physica B 223&224, 490 (1996).
[7] I. I. Mazin and D. Singh, Phys. Rev. Lett. 79, 733 (1997).
[8] M. Sigrist and M.E. Zhitomirsky, J. Phys. Soc. Jpn. 65, 3452 (1996).
[9] K. Machida, et al., J. Phys. Soc. Jpn. 65, 3720 (1996).
[10] V. B. Geshkenbein et al., Phys. Rev. B 36, 235 (1987).
[11] D.J. van Harlingen, Rev. Mod. Phys. 67, 515 (1995).
[12] M. Sigrist and T.M. Rice, J. Phys. Soc. Jpn. 61, 4283 (1992).
[13] Y. Tanaka and S. Kashiwaya, Phys. Rev. B 53, R11957 (1996); ibid., 56, 892 (1997); Physica C 293, 101 (1997).
[14] C.R. Hu, Phys. Rev. Lett. 72, 1526 (1994).
[15] Y. Tanaka and S. Kashiwaya, Phys. Rev. Lett., 74, 3451 (1995); Phys. Rev. B, 53, 9371, (1996).
[16] S. Kashiwaya et al., Phys. Rev. B, 51, 1350, (1995); ibid., 53, 2667, 1996.
[17] Yu. S. Barash et al., Phys. Rev. Lett. 77, 4070 (1996).
[18] M.P. Samanta and S. Datta, Phys. Rev. B, 55, R689 (1997).
[19] C. Ishii, Prog. Theor. Phys. 44, 1525 (1970); 47, 1464 (1972).
[20] S. Yip, J. Low Temp. Phys. 91, 203 (1993).
[21] J. A. Pals et al., Phys. Rev. B 15, 2592 (1977); J. A. Pals and W. van Haeringen, Physica 92B, 360 (1977).
[22] E.W. Fenton, Solid State Commun. 54, 709 (1985); 60, 347 (1986).
[23] R. Jin, et al., unpublished.
[24] C. Bruder, Phys. Rev. B 41, 4017 (1990).
[25] M. Hurd and G. Wendin, Phys. Rev. B 51, 3754 (1995).
[26] A. Furusaki and M. Tsukada, Solid State Commun. 78, 299 (1991).
[27] M. Yamashiro et al., Phys. Rev. B 56, 7847 (1997).
[28] J. A. Sauls, J. Low Temp. Phys. 95, 153 (1994); Adv. Phys. 43, 143 (1994).

FIG. 1. Schematic illustration of s-wave superconductor / p-wave superconductor / s-wave superconductor (s/p/s) junction.
FIG. 2. Normalized critical Josephson current as a function of normalized temperature for $E_u(U1)$ state. $T_p/T_s = 0.13$. $X$ is 0.5 and 1 as indicated in the figure.

FIG. 3. Normalized critical Josephson current as a function of normalized temperature for $E_u(U2)$ state. $T_p/T_s = 0.13$. $X$ is 0.5 and 1 as indicated in the figure.
FIG. 4. Normalized Josephson current as a function of the phase difference $\varphi$ for $E_{u}(U1)$ state. $T_p/T_s = 0.13$. $X$ is 0.5 and 1 as indicated in the figure. For $X = 0.5$, a: $T/T_s = 0.15$, b: $T/T_s = 0.13$, c: $T/T_s = 0.11$, and d: $T/T_s = 0.03$. For $X = 1$, a: $T/T_s = 0.15$, b: $T/T_s = 0.13$, c: $T/T_s = 0.12$, and d: $T/T_s = 0.09$. 
Normalized Maximum Josephson current $eR_N J_C(T) / [\Delta_s(0)\Delta_p(0)]^{1/2}$

Normalized Temperature $T / T_s$
$eR_N J(\varphi) / [\Delta_s(0)\Delta_p(0)]^{1/2}$

Normalized Josephson current

Normalized Phase $\varphi / \pi$

Eu(U1) $X=0.5$

Eu(U1) $X=1$