LONGTIME BEHAVIOR OF THE SEMILINEAR WAVE EQUATION WITH GENTLE DISSIPATION

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ABSTRACT. The paper investigates the well-posedness and longtime dynamics of the semilinear wave equation with gentle dissipation $u_{tt} - \Delta u + \gamma(-\Delta)^\alpha u + f(u) = g(x)$, with $\alpha \in (0, 1/2)$. The main results are concerned with the relationships among the growth exponent $p$ of nonlinearity $f(u)$ and the well-posedness and longtime behavior of solutions of the equation. We show that (i) the well-posedness and longtime dynamics of the equation are of characters of parabolic equations as $1 \leq p < p^\ast \equiv N + \frac{4}{N-2}$; (ii) the subclass $G$ of limit solutions has a weak global attractor as $p^\ast \leq p \leq p^{**} \equiv \frac{N+2}{\frac{N}{2}-2} (N \geq 3)$.

1. Introduction. In this paper, we are concerned with the well-posedness and longtime behavior of solutions for the semilinear wave equation with gentle dissipation

$$u_{tt} - \Delta u + \gamma(-\Delta)^\alpha u_t + f(u) = g(x), \quad (1)$$

$$u|_{\partial \Omega} = 0, \quad u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega, \quad (2)$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with the smooth boundary $\partial \Omega$, $\gamma > 0, \alpha \in (0, 1/2)$ and the assumptions on $f(u)$ and $g$ will be specified later.

Primary attention for Eq. (1) was paid to the case when $\alpha = 0$ (weak damping) and $\alpha = 1$ (strong damping). When $0 < \alpha < 1$, the term $(-\Delta)^\alpha u_t$ is said to be fractional damping (especially, structural damping when $1/2 \leq \alpha < 1$; gentle dissipation when $0 < \alpha < 1/2$), and it plays a dissipative role, which is weaker than that of strong damping but stronger than that of weak damping (cf. [10, 11, 12]).

When $\alpha = 0$, Eq. (1) becomes weakly damped wave equation

$$u_{tt} - \Delta u + \gamma u_t + f(u) = g(x), \quad (3)$$

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There have been extensive researches on the well-posedness and longtime dynamics for Eq. (3) with subcritical and critical nonlinearities (cf. [13, 25, 32] and the references therein), where the growth exponent of nonlinearity $f(u)$, say $p$, is called critical relative to the natural energy space $X = H_0^1(\Omega) \times L^2(\Omega)$ if $p = \tilde{p} \equiv \frac{N}{N-2}$ ($N \geq 3$) because the uniqueness of energy solutions is lost when $p > \tilde{p}$.

In order to establish the existence of global attractor in supercritical nonlinearity case, Ball [1, 2] proposed the concept of generalized semiflows and used it (see [2]) to study longtime dynamics of problem (3), (2). In the supercritical nonlinearity case:

$$|f(s)| \leq C(1 + |s|^p), \quad \text{with} \quad p > \tilde{p},$$

based on an unproved assumption that every weak solution satisfies the energy equation, he showed that the related generalized semiflow possesses in natural energy space a global attractor. But by now, to the best of our knowledge, the improved assumption is still an open question.

In order to avoid the ‘unproved assumption’, Carvalho, Cholewa and Dlotko [9] proposed the concept of ‘the subclass $\mathcal{LS}$ of limit solutions’ and proved that the subclass $\mathcal{LS}$ of Eq. (3) has a weak global attractor in the supercritical nonlinearity case: $\tilde{p} < p < p^{**} = \frac{N+2}{N-2}$ ($N \geq 3$) (see (4)).

Due to the existence of global attractor for Eq. (3) was proved for nonlinearity $f(u)$ satisfying (4), with $1 \leq p < p^{**}$, by Feireisl [18] (in $\mathbb{R}^3$) and Kapitanski [22] (in the case of smooth manifold without boundary) using the estimates of Strichartz type, one would guess that for problem (3)-(2) the critical exponent of nonlinearity $f(u)$ to guarantee the existence of global attractor is $p^{**} = \frac{N+2}{N-2}$ rather than $\tilde{p} = \frac{N}{N-2}$ ($N \geq 3$).

Recently, some advance has been achieved for this conjecture. By utilizing Strichartz type estimates for the linear wave equation in a bounded domain $\Omega \subset \mathbb{R}^3$, Kalantarov, Savostianov and Zelik [21], Savostianov and Zelik [27] establish for the critical quintic nonlinearity ($p^{**} = 5$) the existence and uniqueness of Shatah-Struwe solutions (i.e., energy solution belonging to the space $L^4(0,T; L^{12}(\Omega))$) and show that the (Shatah-Struwe) solution semiflow $S(t)$ associated with problem (3)-(2) possesses a smooth global attractor in natural phase space.

When $1/2 < \alpha < 1$, a more or less complete theory on the well-posedness and the existence of global attractor for problem (1)-(2) in the case: $1 \leq p < p^{**}$ can be found in [5, 6, 7]. For the research on the mentioned problem in the $L^q(\Omega)$ ($2 < q < +\infty$) setting, one can see [8].

Chueshov [14] studied at the abstract level the longtime behavior of Kirchhoff type equations with structural nonlinear damping

$$u_{tt} + \sigma(\|A^{\frac{1}{2}}u\|^2)A^\alpha u_t + \phi(\|A^{\frac{1}{2}}u\|^2)Au + f(u) = h(x),$$

with $\alpha \in [1/2, 1)$. He found the critical exponent $\tilde{p} = \frac{N+2\alpha}{N-2}$ ($N \geq 3$) of nonlinearity $f(u)$. In non-supercritical nonlinearity case:

$$|f'(s)| \leq C(1 + |s|^{p-1}), \quad \text{with} \quad 1 \leq p \leq \tilde{p},$$

he established the existences of global and exponential attractors in natural energy space. In particular, his results hold for problem (1)-(2), with $\alpha \in [1/2, 1)$, because Eq. (5) becomes (1) when $\sigma(s) = \gamma, \phi(s) = 1$.

Recently, in the case of structural damping $\alpha = 1/2$ and $\Omega \subset \mathbb{R}^3$, under the assumption that the nonlinearity $f(u)$ with critical quintic growth ($p = p^{**} = 5$) is
odd, Savostianov and Zelik [28] establish additional regularity of energy solutions and based on this, they obtain the global well-posedness and dissipativity of energy solutions as well as the existences of smooth global and exponential attractors.

Chen and Triggiani [12] proved that the semigroup generated by the linear part of Eq. (1) is of Gevrey class. The importance of this result is that, although the semigroup is not analytic, it is nevertheless regularizing (in particular it is infinitely differentiable). The interpolation between the weakly damped wave equation and the analyticity makes that the linear part of Eq. (1) has parabolic-like characteristics. Therefore, one should expect the corresponding semilinear problem to exhibit parabolic-like characteristics (See for example [19]).

For $0 < \alpha < 1/2$, Savostianov [26, 29] also investigates the well-posedness and longtime dynamics of problem (1)-(2) on a bounded smooth domain $\Omega \subset \mathbb{R}^3$. Under the assumptions:

$$f \in C^1(\mathbb{R}), \quad f(s)s \geq -C, \quad |f'(s)| \leq C(1 + |s|^{p-1}), \quad p \in [1, 5), \quad (6)$$

by using spectral cluster estimates obtained in [30] and adapting technique presented in [4], he establishes Strichartz estimates for the linear damped wave equation

$$u_{tt} - \triangle u + \gamma(-\triangle)^\alpha u_t = h(x), \quad (7)$$

that is, the control of $L^5(0, T; L^{10}(\Omega))$ for the solution of Eq. (7). This allows him to establish the existence and uniqueness of Shatah-Struwe solutions (see Definition 3.2 below) in semilinear case (with $p < 5$). Furthermore, the existences of smooth global attractor as well as exponential attractor for the class of Shatah-Struwe solutions, which is a subclass of energy solutions, are proved based on that Shatah-Struwe solutions of problem (1)-(2) possess parabolic-like smoothing property.

From the works of Kalantarov, Savostianov and Zelik [21, 26, 27, 28, 29], it seems that a major breakthrough in the study of these equations with critical nonlinearity involves the recent availability of Strichartz estimates on bounded domains, as well as the notion of Shatah-Struwe solutions.

But when $\alpha \in (0, 1/2)$, for general bounded smooth domain $\Omega \subset \mathbb{R}^N$, say $N \geq 4$, what about the well-posedness and longtime dynamics of Eq. (1)? Whether or not the energy solutions of problem (1)-(2) still possess parabolic-like smoothing property (that is, the higher global regularity when $t > 0$)? What about the existences of global and exponential attractors for the class of energy solutions? Even if when $N = 3$, although energy solution is Shatah-Struwe solution as $p \leq 3$ because of the uniqueness of energy solutions (this is obvious), what is the critical exponent of the nonlinearity such that energy solution becomes Shatah-Struwe solution? If the uniqueness of energy solutions is lost, what about the longtime behavior of them? These questions remain unsolved.

The purpose of the present paper is to solve these questions. For general bounded smooth domain $\Omega \subset \mathbb{R}^N$, we make a sharp analysis for the relationships among the growth exponent $p$ of nonlinearity $f(u)$ and the well-posedness and longtime behavior of energy solutions of problem (1)-(2) and obtain some results similar to threshold. The contributions of this paper are that we find the critical exponent $p^* = \frac{N+4\alpha}{N-2} \quad (N \geq 3)$ of nonlinearity $f(u)$ (to the best of our knowledge, even for $\Omega \subset \mathbb{R}^3$, the critical exponent of nonlinearity $f(u)$ to guarantee the uniqueness of energy solutions is unknown before) and show that:

1. When $1 \leq p < p^*$ (the subcritical case), the well-posedness and longtime dynamics of problem (1)-(2) are of characteristics of parabolic equations, especially,
the energy solutions (when \( t > 0 \)) and the bounded absorbing set are of higher global regularity (rather than higher partial one as usual) (See Theorem 4.3). Obviously, when \( \Omega \subset \mathbb{R}^3, 1 \leq p < p^* = 3 + 4\alpha, u \) is an energy solution if and only if \( u \) is a Shatah-Struwe solution. So, all the conclusions in [26, 29] hold for energy solutions as long as \( 1 \leq p < 3 + 4\alpha \) rather than \( 1 \leq p \leq 3 \) as known before (see Corollary 5.6).

2. When \( 1 \leq p < p^* \), we establish the existence of an exponential attractor in natural energy space by using the quasi-stability estimate of energy solutions in weaker space \( E_{1-\alpha} \) (rather than in original energy space \( E_1 \) as usual) (See Theorem 5.2).

3. When \( p^* \leq p < p^{**} \equiv \frac{N+2}{N-2} (N \geq 3) \) (the critical and supercritical case), we construct the subclass \( \mathcal{G} \) of the limit solutions (See Theorem 6.3) and show that \( \mathcal{G} \) has a weak global attractor in natural energy space (See Theorem 6.6). In particular, when \( \Omega \subset \mathbb{R}^3, 3 + 4\alpha \leq p < 5 \), the subclass of limit solutions (which are not of uniqueness) exists and it has a weak global attractor.

According to recently developed theory on the dynamics of quasi-stable dissipative systems [16, 17], once the quasi-stability estimate in the sense of strong topology (in the energy level) has been established on an absorbing ball, the existence of the finite dimensional global attractor, as well as its additional regularity follows immediately. In addition, the existence of an exponential attractor follows immediately from the Hölder estimate and the quasi-stability property on any bounded, forward invariant set. Unfortunately, all these known results can not be directly utilized because we can only get the quasi-stability estimate in the sense of weaker topology.

The major technical hurdles in the present paper seem that:

(i) How to establish the higher global regularity of energy solutions when \( t > 0 \) and the uniqueness of energy solutions when \( p > \frac{N}{N-2} \)? What is the critical exponent of the nonlinearity \( f(u) \)?

(ii) How to get the existences of global and exponential attractors in natural energy space by virtue of the quasi-stability estimate in the sense of weaker topology? In other words, how back to the energy level and to get the existences of global and exponential attractors in natural energy space?

(iii) In the critical and supercritical nonlinearity case when the uniqueness of energy solutions is lost (which leads to all the known criterions to guarantee the existence of global attractor cease to be effective), how to establish the existence of global attractor?

The techniques adopted here to overcome the above-mentioned hurdles are that:

(i) By utilizing the multipliers \( A^{-\alpha}u_t + \epsilon u_t \) and \( Au \) and delicate arguments, we get the higher global regularity of \( (u, u_t) \) in space \( V_{1+\alpha} \times V_{1-\alpha} \) (see below for their definitions) when \( t > 0 \). And by using the multiplier \( A^{-\alpha}z_t + \epsilon z \), we obtain the Lipschitz stability of the energy solutions in weaker space \( E_{1-\alpha} \) (rather than in original energy space \( E_1 \)) and find the critical exponent of nonlinearity \( f(u) \) to guarantee the uniqueness of energy solutions.

(ii) By utilizing the technique of recovering the strong topology step by step, we recover the global and exponential attractors of the discrete dynamical system equipped with weaker topology into that of the continuous dynamical system equipped with strong topology.
(iii) By utilizing the simplified operator method, we construct the subclass of limit solutions and a set similar to so-called $\omega$-limit set and prove that it is just the desired weak global attractor.

We mention that there have been some recent results concerned with the parabolic characters of longtime dynamics of the nonlinear strongly damped wave equation (see [15, 20, 23, 24]), in the concrete, the higher partial regularity of both energy solutions (when $t > 0$) and bounded absorbing set, and the asymptotic regularity of solution semigroup have been investigated by the authors in [15, 20, 23, 24]. One of the contributions of the present paper is that we further show that both the energy solutions (rather than only Shatah-Struwe solutions as $N = 3$) (when $t > 0$) and the bounded absorbing set are of the higher global regularity (rather than the higher partial one) for the semilinear wave equation with gentle dissipation (see Theorem 4.3). These facts mean that although the dissipative role of the gentle dissipation is weaker than that of the strong ($\alpha = 1$)/structural ($\alpha = 1/2$) damping, it nevertheless causes the semilinear wave equation to have parabolic-like properties related to both well-posedness and longtime behavior of solutions.

The paper is organized as follows. In Section 2, some preliminaries are stated. In Section 3, the well-posedness of energy solutions as $1 \leq p < p^*$ is discussed. In Sections 4 and 5, the existences of global and exponential attractors are investigated, respectively. In Section 6, the subclass $\mathcal{G}$ of limit solutions is constructed and the existence of weak global attractor is established as $p^* \leq p < p^{**}$.

2. Preliminaries. For brevity, we use the following abbreviations:

$L^p = L^p(\Omega), \ V_1 = H^1_0(\Omega), \ H = L^2, \ V_{-1} = H^{-1}(\Omega), \ \|\cdot\| = \|\cdot\|_{L^2}, \ \|\cdot\|_p = \|\cdot\|_{L^p},$

with $p \geq 1$. The notation $(\cdot, \cdot)$ for the $H$-inner product will also be used for the notation of duality pairing between dual spaces, the sign $H_1 \hookrightarrow H_2$ denotes that the functional space $H_1$ continuously embeds into $H_2$ and $H_1 \hookrightarrow \hookrightarrow H_2$ denotes that $H_1$ compactly embeds into $H_2$, and $C(\cdots)$ stands for positive constants depending on the quantities appearing in the parenthesis.

Obviously, $V_1 \hookrightarrow \hookrightarrow H \hookrightarrow \hookrightarrow V_{-1}$. Define the operator $A : V_1 \rightarrow V_{-1}$,

$$(Au, v) = (\nabla u, \nabla v) \quad \text{for any } \ u, v \in V_1.$$ 

Then, $A$ is self-adjoint in $H$ and strictly positive on $V_1$. So we can define the power $A^s$ of $A$ ($s \in \mathbb{R}$). The spaces $V_s = D(A^\frac{s}{2})$ are Hilbert spaces with the scalar products and the norms

$$(u, v)_s = (A^{\frac{s}{2}} u, A^{\frac{s}{2}} v), \quad \|u\|_{V_s} = \|A^{\frac{s}{2}} u\|.$$ 

We denote the phase spaces

$$E_1 = V_1 \times H, \quad E_{1+s} = V_{1+s} \times V_s,$$

with $s \in \mathbb{R}$, which are equipped with usual graph norms, for example,

$$\|(u, v)\|_{E_1}^2 = \|u\|_{V_1}^2 + \|v\|^2.$$ 

Obviously, they are Hilbert spaces and

$$E_{1+s_1} \leftrightarrow E_{1+s_2} \quad \text{for } s_1 > s_2.$$
Rewriting Eq. (1) as the operator equation, we get the Cauchy problem equivalent to problem (1)-(2):

\[ u_{tt} + Au + \gamma A^\alpha u_t + f(u) = g, \]
\[ u(0) = u_0, \quad u_t(0) = u_1. \tag{8} \]

**Assumption (H)**

1. \( f \in C^1(\mathbb{R}), \)
2. \( \liminf_{s \to \infty} \frac{f(s)}{s} > -\lambda_1, \tag{10} \)

where \( \lambda_1(>0) \) is the first eigenvalue of the operator \( A, \) and when \( N \geq 2, \)
\[ |f'(s)| \leq C(1 + |s|^{p-1}), \quad s \in \mathbb{R}, \]

where \( 1 \leq p < \infty \text{ if } N = 2, \quad 1 \leq p \leq p^* = \frac{N+2}{N-2} \text{ if } N \geq 3; \)
3. \( g \in V_{-1}, (u_0, u_1) \in E_1, \) with \( \|(u_0, u_1)\|_{E_1} \leq R. \)

**Remark 2.1.**

(i) Formula (10) implies that there exists a positive constant \( \theta : 0 < \lambda_1 - \theta << 1 \) such that
\[ f(s)s \geq -\theta s^2 - C, \quad F(s) = \int_0^s f(\tau)d\tau \geq -\frac{\theta}{2}s^2 - C, \quad s \in \mathbb{R}. \]

(ii) Formula (10) is weaker than the formula: \( f(s)s \geq -C, s \in \mathbb{R} \) (see (6)).

**Lemma 2.2.** ([31]). Let \( X, B \) and \( Y \) be Banach spaces, \( X \hookrightarrow B \hookrightarrow Y, \)
\[ W = \{ u \in L^p(0,T;X)|u_t \in L^1(0,T;Y) \}, \text{ with } 1 \leq p < \infty, \]
\[ W_1 = \{ u \in L^\infty(0,T;X)|u_t \in L^r(0,T;Y) \}, \text{ with } r > 1. \]

Then,
\[ W \hookrightarrow L^p(0,T;B), \quad W_1 \hookrightarrow C([0,T];B). \]

For simplicity, we restrict ourselves to the case \( N \geq 3 \) for the cases \( N = 1, 2 \) are easy. But all the results hold for the cases \( N = 1, 2. \)

3. Well-posedness of energy solutions as \( 1 \leq p < p^*. \)

**Definition 3.1.** Function \( u, \) with \( (u, u_t) \in L^\infty(0,T;E_1) \) is said to be an energy solution (weak solution) of problem (8)-(9) if \( u \) solves Eq. (8) in the sense of distributions on \([0,T] \) and \((u(0), u_t(0)) = (u_0, u_1). \)

**Definition 3.2.** ([29]). Let \( \alpha \in (0,1/2) \) and \( \Omega \subset \mathbb{R}^3 \) be a bounded smooth domain. Energy solution \( u \) of problem (8)-(9) is said to be Shatah-Struwe solution on \([0,T] \) if in addition \( u \in L^5(0,T;L^{10}(\Omega)). \)

**Theorem 3.3.** Let Assumption (H) be valid. Then problem (8)-(9) admits an energy solution \( u, \) with \( (u, u_t) \in L^\infty(\mathbb{R}^+, E_1) \cap C_\alpha(\mathbb{R}^+, E_1) \). More precisely, the solution \( u \) possesses the following properties:

(i)
\[ \|(u, u_t)(t)\|_{E_1} + \gamma \int_0^t \|u_t(\tau)\|^2_{E_1} d\tau \leq C(R) + C_1, \quad t \geq 0, \tag{11} \]

where and in the following \( C_1 = C(\|g\|_{V_{-1}}). \)

(ii) When \( 1 \leq p < p^* \equiv \frac{N+4}{N-2} \text{ for any } a > 0, \)
\( (u_t, uu_t) \in L^\infty(a, T; E_{1-a}), \quad uu_t \in L^2(a, T; H), \)
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and

\[ \|u(t)\|_{V^{-\alpha}}^2 + \|u(t)\|_{V^\alpha}^2 + \int_t^{t+1} (\gamma\|u(t)\|^2 + \|u_t(t)\|^2)\,d\tau \leq \frac{t^{1/\alpha} + 1}{t^{1/\alpha}} - C(R)e^{-\kappa t} + C_1, \quad t > 0, \]  

(12)

where and in the following \( \kappa > 0 \) stands for a small constant.

(iii) When \( 1 \leq p < p^* \), the solution is Lipschitz continuous in weaker space \( E_{1-\alpha} \), that is,

\[ \|(z, z_t)(t)\|_{E_{1-\alpha}}^2 \leq C(R,T)\|(z, z_t)(0)\|_{E_{1-\alpha}}^2, \quad 0 \leq t \leq T, \]  

(13)

where \( z = u - v, u \) and \( v \) are two energy solutions of problem (8)-(9) corresponding to the initial data \((u_0, u_1)\) and \((v_0, v_1)\). Furthermore, the solution is quasi-stable in weaker space \( E_{1-\alpha} \), that is,

\[ \|(z, z_t)(t)\|_{E_{1-\alpha}}^2 \leq C(R)e^{-\kappa t}\|(z, z_t)(0)\|_{E_{1-\alpha}}^2 + \int_0^t e^{-\kappa(t-\tau)} \|z(\tau)\|^2\,d\tau. \]  

(14)

Remark 3.4. Formula (13) and Formula (14) are called stability estimate and quasi-stability estimate of energy solutions in weaker space \( E_{1-\alpha} \), respectively.

Proof of Theorem 3.3. We first obtain some a priori estimates to the solutions of problem (8)-(9). Using the multiplier \( u_t \) in Eq. (8), we have

\[ \frac{d}{dt}E(u, u_t) + \gamma\|A^\frac{2}{p}u_t\|^2 = 0, \quad t > 0, \]  

(15)

where

\[ E(u, u_t) = \frac{1}{2}\left(\|u_t\|^2 + \|A^\frac{2}{p}u\|^2\right) + \int_\Omega F(u)\,dx - (g, u) \geq \frac{1}{4}\left(\|u_t\|^2 + (1 - \frac{\theta}{\lambda_1})\|A^\frac{2}{p}u\|^2\right) - C_1, \]

and where we have used Remark 2.1. Integrating (15) over \((0, t)\) yields (11).

It follows from Eq. (8) and (11) that

\[ \|u_t\|_{V^{-\alpha}} \leq C(\|u\|_{V_1} + \|u_t\| + \|f(u)\|_{1+1/p} + \|g\|_{V_{-\alpha}}) \leq C(R, C_1). \]  

(16)

Differentiating Eq. (8) with respect to \( t \), we get that \( v = u_t \) solves

\[ v_{tt} + Av + \gamma A^\alpha v_t + f'(u)v = 0. \]  

(17)

Using the multiplier \( A^{-\alpha}v_t + \epsilon v \) in (17), we get

\[ \frac{d}{dt}H_1(v, v_t) + \gamma\|A^\frac{2}{p}v_t\|^2 + \|v_t\|^2 \leq -(f'(u)v, A^{-\alpha}v_t + \epsilon v), \]  

(18)

where

\[ H_1(v, v_t) = \frac{1}{2}\left[\|A^{-\frac{\alpha}{2}}v_t\|^2 + \|A^{\frac{\alpha}{p}-\frac{\alpha}{2}}u\|^2 + \epsilon\left(\|v\|^2_{V_1} + 2(v, v_t)\right)\right] \sim \|v\|^2_{V_1} + \|v_t\|^2_{V_{-\alpha}} \]

for \( \epsilon > 0 \) suitably small. On account of the Sobolev embeddings: \( V_1 \hookrightarrow L^{p^*+1} \), \( V_{1-\delta} \hookrightarrow L^{\frac{N}{\alpha-2(1-\delta)}} \), \( V_{2\alpha} \hookrightarrow L^{\frac{2N}{\alpha}} \), and by virtue of the interpolation theorem, we
have
\[ |(f'(u)v, A^{-\alpha}v_t + \epsilon v)| \]
\[ \leq C \int_{\Omega} (1 + |u|^{p-1}) |v|(|A^{-\alpha}v_t + \epsilon v)| dx \]
\[ \leq C \epsilon (1 + ||u|^{p-1}) ||v||^2 + C (1 + ||u|^{\frac{p-1}{N(p-1)}}) ||v|| \frac{2N-\alpha}{2(1-\alpha)} ||A^{-\alpha}v_t|| \frac{2N-\alpha}{2} \] 
(19)

where \( \mu = \frac{(p^*)^+(p-1)}{(p^*-1)(p+1)} \) \( \mu < 1 \) and where we have used the fact: \( \frac{N(p-1)}{1-\delta + 2\alpha} \leq \frac{2N}{N-2} \) for \( \delta : 0 < \delta < 1 \) if \( 1 \leq p < p^* \).

Inserting (19) into (18), we get
\[ d \frac{dt}{dt} H_1(v, v_t) + \kappa H_1(v, v_t) + \frac{\epsilon^2}{4} ||v||^2 \leq C(R, \epsilon) ||v||^2. \]
(20)

When \( 0 < t < 1 \), multiplying (20) by \( t^{\frac{\alpha}{2}} \), we get
\[ d \frac{dt}{dt} \left( t^{\frac{\alpha}{2}} H_1(v, v_t) \right) + t^{\frac{\alpha}{2}} H_1(v, v_t) + \frac{\epsilon}{4} t^{\frac{\alpha}{2}} \left( \gamma ||v_t||^2 + ||v||_V^2 \right) \]
\[ \leq C(R) ||v||^2 + C t^{\frac{\alpha}{2}} \left( ||v||_{V_{1-\delta}}^2 + ||v||_V^2 \right). \]
(21)

By virtue of the interpolation theorem,
\[ t^{\frac{\alpha}{2}} \left( ||v||_{V_{1-\delta}}^2 \right) \leq C t^{\frac{\alpha}{2}} ||v||_{V_{1-\delta}}^2 \leq \frac{\epsilon}{8} t^{\frac{\alpha}{2}} ||v||^2 + C ||v||_{V_{1-\delta}}^2, \]
\[ t^{\frac{\alpha}{2}} \left( ||v||_{V_{1-\alpha}}^2 \right) \leq C t^{\frac{\alpha}{2}} ||v||_{V_{1-\alpha}}^2 \leq \frac{\epsilon}{8} t^{\frac{\alpha}{2}} ||v||^2 + C ||v||_{V_{1-\alpha}}^2, \]
where we have used the fact: \( \frac{1-\alpha}{\alpha(1-2\alpha)} > \frac{1}{\alpha} \) and \( 0 < t \leq 1 \). So
\[ d \frac{dt}{dt} \left( t^{\frac{\alpha}{2}} H_1(v, v_t) \right) + \kappa t^{\frac{\alpha}{2}} H_1(v, v_t) + \kappa t^{\frac{\alpha}{2}} \left( \gamma ||v_t||^2 + ||v||_V^2 \right) \]
\[ \leq C(\delta ||v||_{V_{1-\delta}}^2 + ||v||_{V_{1-\alpha}}^2). \]
(22)

Taking account of (11) and (16) and applying the Gronwall lemma to (22), we get
\[ t^{\frac{\alpha}{2}} H_1(v, v_t) + t^{\frac{\alpha}{2}} \int_t^{t+1} \left( \gamma ||u_t(\tau)||^2 + ||v(\tau)||_V^2 \right) d\tau \]
\[ \leq C(R) e^{-\kappa t} + C_1, \]
\[ ||u_t(t)||_{V_{1-\alpha}}^2 + ||u_{tt}(t)||_{V_{1-\alpha}}^2 \leq t^{\frac{\alpha}{2}} \int_t^{t+1} \left( \gamma ||u_t(\tau)||^2 + ||u_t(\tau)||_V^2 \right) d\tau \]
\[ \leq \frac{1}{t^{\frac{\alpha}{2}}} \left( C(R) e^{-\kappa t} + C_1 \right), \quad 0 < t \leq 1. \]
(23)

Applying the Gronwall lemma to (20) over \((1, t)\) and exploiting (23), we get
\[ ||u_t(t)||_{V_{1-\alpha}}^2 + ||u_{tt}(t)||_{V_{1-\alpha}}^2 \leq \int_t^{t+1} \left( \gamma ||u_t(\tau)||^2 + ||u_t(\tau)||_V^2 \right) d\tau \]
\[ \leq C(R) e^{-\kappa t} + C_1, \quad t > 1. \]
(24)
The combination of (23) and (24) gives (12).

On the basis of estimate (11), which obviously holds true for the Galerkin approximations, by exploiting the standard Galerkin method one easily gets that problem (8)-(9) has an energy solution \( u \), with \((u, u_t) \in L^\infty(\mathbb{R}^+; E_1) \cap C_w(\mathbb{R}^+; E_1) \). By the lower semicontinuity of the weak limit, estimate (11) holds for \( u \). Especially, when \( 1 \leq p < p^* \), estimate (12) holds. We omit the process here.

Now, we show that when \( 1 \leq p < p^* \), \( \varphi_u = (u, u_t) \) is Lipschitz continuous in weaker space \( E_{1-\alpha} \).

Let \( u, v \) be two solutions of problem (8)-(9) as shown above corresponding to the initial data \( u_0, u_1 \) and \( v_0, v_1 \), respectively. Then \( z = u - v \) solves
\[
\begin{align*}
  z_t + A z + \gamma A^\alpha z_t + f(u) - f(v) &= 0, \\
  z(0) = u_0 - v_0 &= z_0, \\
  z_t(0) = u_1 - v_1 &= z_1.
\end{align*}
\]

Using the multiplier \( A^{-\alpha} z_t + \varepsilon z \) in (25), we get
\[
\frac{d}{dt} H_2(z, z_t) + \varepsilon \| A^{\frac{3}{2}} z \|^2 + (\gamma - \varepsilon) \| z_t \|^2 = -(f(u) - f(v), A^{-\alpha} z_t + \varepsilon z),
\]
where
\[
H_2(z, z_t) = \frac{1}{2} \left[ \| A^{-\alpha} z_t \|^2 + \| A^{\frac{3}{2}} z \|^2 + \gamma \| z_t \|^2 + 2(z, z_t) \right]
\]
for \( \varepsilon > 0 \) suitably small. Similar to estimate (19), we have
\[
|\langle f(u) - f(v), A^{-\alpha} z_t + \varepsilon z \rangle| \leq C \int_\Omega (1 + |u|^{p-1} + |v|^{p-1}) |z| (|A^{-\alpha} z_t| + \varepsilon |z|) dx
\]
\[
\leq \frac{\varepsilon}{4} \| z \|^2_{V_{1-\alpha}} + \frac{\gamma}{4} \| z_t \|^2 + C(\varepsilon) \| z \|^2,
\]
Inserting (27) into (26), we obtain
\[
\frac{d}{dt} H_2(z, z_t) + \kappa H_2(z, z_t) + \varepsilon \| A^{\frac{3}{2}} z \|^2 + \frac{\gamma}{4} \| z_t \|^2 \leq C(\varepsilon) \| z \|^2,
\]
which implies estimates (13) and (14). Theorem 3.3 is proved. \( \square \)

4. Global attractor as \( 1 \leq p < p^* \). Define the operator
\[
S(t) : E_1 \to E_1, \quad S(t)(u_0, u_1) = (u(t), u_t(t)),
\]
where \( u \) is an energy solution of problem (8)-(9). When \( 1 \leq p < p^* \), by Theorem 3.3, \( S(t) \) composes a semigroup on \( E_1 \), which is continuous in the topology of \( E_{1-\alpha} \).

**Definition 4.1.** A set \( \mathcal{A} \subset E_1 \) is said to be a global attractor of the semigroup \( S(t) \) acting on a Banach space \( E_1 \) if
\begin{enumerate}
  \item \( \mathcal{A} \) is a compact set in \( E_1 \);
  \item \( \mathcal{A} \) is an invariant set, i.e., \( S(t)\mathcal{A} = \mathcal{A}, t \geq 0 \);
  \item \( \mathcal{A} \) attracts the images of all bounded sets in \( E_1 \), i.e.,
  \[
  \text{dist}_{E_1} \{ S(t)B, \mathcal{A} \} \to 0 \quad \text{as} \quad t \to +\infty
  \]
\end{enumerate}
for every bounded set \( B \subset E_1 \).

**Lemma 4.2.** (3) (Gronwall-type lemma). Let \( X \) be a Banach space, and let \( \mathcal{Z} \subset C(\mathbb{R}^+, X) \). Let \( \Phi : X \to \mathbb{R} \) be a function such that
\[
\sup_{t \in \mathbb{R}^+} \Phi(z(t)) \geq -\eta, \quad \Phi(z(0)) \leq K,
\]
for some \(\eta, K \geq 0\) and every \(z \in \mathcal{Z}\). In addition, assume that for every \(z \in \mathcal{Z}\)
the function \(t \mapsto \Phi(z(t))\) is continuously differentiable, and satisfies the differential inequality

\[
\frac{d}{dt} \Phi(z(t)) + \delta \|z(t)\|^2 \leq k
\]

for some \(\delta > 0\) and \(k \geq 0\) independent of \(z \in \mathcal{Z}\). Then, for every \(\gamma > 0\) there exists
a \(t_0 = \frac{\eta + K}{\gamma} > 0\) such that

\[
\Phi(z(t)) \leq \sup_{\zeta \in X} \{-\Phi(\zeta) : \delta \|\zeta\|^2 \leq k + \gamma\}, \quad t \geq t_0.
\]

**Theorem 4.3.** (Longtime dynamics with parabolic characteristics). Let Assumption \((H)\) be valid, with \(1 \leq p < p^*\) and \(q \in H\). Then the energy solution \(u\) of
problem \((8)-(9)\) further possesses the following properties:

(i) (Regularity of \(u\) when \(t > 0\)) For any \(a > 0\), \(u \in L^\infty(a, T; V_{1+\alpha})\), and

\[
\|u(t)\|_{V_{1+\alpha}}^2 \leq \left(1 + \frac{1}{(t - a/2)^{1-\alpha}}\right)C(R, a, \|g\|), \quad t > a/2.
\]  

(ii) The following energy identity holds:

\[
E(u(s), u_t(s)) = E(u(t), u_t(t)) + \gamma \int_s^t \|A^{\frac{\alpha}{2}} u_t(\tau)\|^2 d\tau, \quad t \geq s \geq 0,
\]  

where \(E(u, u_t)\) is as shown in (15).

(iii) (Strong continuity)

\((u, u_t) \in C(\mathbb{R}^+, E_1)\),

and the solution semigroup \(S(t) : E_1 \rightarrow E_1\) is \(1/2\)-Hölder continuous for each \(t \geq 0\).

(iv) (Existence of the bounded absorbing set with higher global regularity) The solution
semigroup \(S(t)\) has an absorbing set \(B_1\), which is bounded in \(V_{1+\alpha} \times V_{1-\alpha}\).

(v) The solution semigroup \(S(t)\) has a global attractor \(A\) in \(E_1\), which has finite
fractional dimension, and \(A\) is of \(V_{1+\alpha} \times V_{1-\alpha}\) regularity.

**Proof.** (i) Using the multiplier \(Au\) in Eq. (8) and making use of estimate (11) and
the interpolation theorem, we get

\[
\frac{\gamma}{2} \frac{d}{dt} \|A^{\frac{1+\alpha}{2}} u\|^2 + \|Au\|^2 = -(f'(u), |\nabla u|^2) + (g - u_{tt}, Au)
\]

\[
\leq C(1 + \|u\|^{p-1}_p)\|\nabla u\|^2 + \frac{1}{4} \|Au\|^2 + \|g\|^2 + \|u_{tt}\|^2
\]

\[
\leq \frac{1}{2} \|u\|_{V_2}^2 + C(R)\|u\|_{V_1}^2 + \|g\|^2 + \|u_{tt}\|^2,
\]

\[
\frac{d}{dt} \|A^{\frac{1+\alpha}{2}} u\|^2 + \kappa \|A^{\frac{1+\alpha}{2}} u\|^2 + \kappa_1 \|Au\|^2 \leq C\|u\|^2_{V_1} + \|g\|^2 + \|u_{tt}\|^2.
\]  

(31)

When \(a/2 < t \leq a/2 + 1\), multiplying (31) by \((t - a/2)^{\frac{1-\alpha}{\alpha}}\) and using the interpolation
theorem, we obtain
\[ \frac{d}{dt} \left( (t - a/2)^{1 - \alpha} \| A^{\frac{1 + \alpha}{2}} u(t) \|^2 \right) + \kappa(t - a/2)^{1 - \alpha} \| A^{\frac{1 + \alpha}{2}} u(t) \|^2 + \kappa_1 (t - a/2)^{1 - \alpha} \| Au \|^2 \leq C \left( \| u \|_{V_1}^2 + \| g \|^2 + \| u_{tt} \|^2 \right) + \frac{1}{1 - \alpha} (t - a/2)^{1 - \alpha} \| A^{\frac{1 + \alpha}{2}} u(t) \|^2 \]

(32)

Integrating Eq. (32) in time and using estimate (12), we have

\[ (t - a/2)^{1 - \alpha} \| A^{\frac{1 + \alpha}{2}} u(t) \|^2 \leq C \int_{a/2}^{t} (\| u(\tau) \|_{V_1}^2 + \| g \|^2 + \| u_{tt}(\tau) \|^2) d\tau \]

\[ \leq C(R, a, \| g \|), \]

(33)

\[ \| u(t) \|_{V_1 + \alpha}^2 \leq \frac{1}{(t - a/2)^{1 - \alpha}} C(R, a, \| g \|), \quad a/2 < t \leq a/2 + 1. \]

(34)

When \( t > a/2 + 1 \), applying the Gronwall lemma to (31) over \((a/2 + 1, t)\) and making use of (33) and (12), we obtain

\[ \| u(t) \|_{V_1 + \alpha}^2 \leq C(R, a, \| g \|), \quad t > a/2 + 1. \]

(35)

The combination (33) and (34) gives (29).

(ii) For any \( a > 0 \), since

\[ \left| \int_{\Omega} f(u)u_{tt} dx \right| \leq C \int_{\Omega} \left( 1 + |u|^{p-1} \right) |u| |u_{tt}| dx \]

\[ \leq C(1 + \|u\|_{L^p(\Omega)}^{p-1}) \|u\|_{L^\frac{2p}{p+1}(\Omega)} \|u_{tt}\|_{L^\frac{2p}{p+1}(\Omega)} \]

\[ \leq C(1 + \|u\|_{L^p(\Omega)}^{p-1}) \|u\|_{V_{1+a}} \|u_{tt}\|_{V_{1-a}} \leq C \|u\|_{V_{1+a}} \|u_{tt}\|_{V_{1-a}}, \]

and \( u(t) \in V_{1+a}, u_{tt}(t) \in V_{1-a} \) for \( t \in [a, T] \) (see (29) and (12)), we get \( f(u)u_{tt} \in L^1(\Omega) \) for \( t \in [a, T] \). This allows us to multiply Eq. (8) by \( u_t \) and get the energy equality

\[ E(u(s), u_t(s)) = E(u(t), u_t(t)) + \gamma \int_s^t \| A^{\frac{1}{2}} u_t(\tau) \|^2 d\tau, \quad t \geq s > 0, \]

(36)

where \( E(u, u_t) \) is as shown in (15). Letting \( s \to 0^+ \) in (36), we have

\[ E^* = \lim_{s \to 0^+} E(u(s), u_{tt}(s)) = E(u(t), u_{tt}(t)) + \gamma \int_0^t \| A^{\frac{1}{2}} u_t(\tau) \|^2 d\tau. \]

Obviously, for the Galerkin approximations \( u^n \), (35) holds for \( s = 0 \), that is,

\[ E(u^n(0), u^n_{tt}(0)) = E(u^n(t), u^n_{tt}(t)) + \gamma \int_0^t \| A^{\frac{1}{2}} u^n_t(\tau) \|^2 d\tau. \]

By (36), the lower semicontinuity of the weak limit and the Fatou lemma (subsequence if necessary)

\[ E^* \leq \liminf_{n \to \infty} \left( E(u^n(t), u^n_{tt}(t)) + \gamma \int_0^t \| A^{\frac{1}{2}} u^n_t(\tau) \|^2 d\tau \right) = \liminf_{n \to \infty} E(u^n(0), u^n_{tt}(0)) = E(u(0), u_{tt}(0)). \]
On the other hand, by the property of the limit inferior, for any \( t_0 \geq 0 \), there exists a sequence \( t_n : t_n \to t_0 \) as \( n \to \infty \) such that
\[
\liminf_{t \to t_0} \int_{\Omega} F(u(t))dx = \lim_{n \to \infty} \int_{\Omega} F(u(t_n))dx.
\]
The fact \((u, u) \in C_\infty(\mathbb{R}^+, E_1)\) implies that
\[
\|u_t(t_0)\|^2 + \|A_1^1 u(t_0)\|^2 \leq \liminf_{t \to t_0} \left( \|u_t(t)\|^2 + \|A_1^1 u(t)\|^2 \right).
\] (37)
Since \((u, u) \in L^\infty(\mathbb{R}^+, E_1)\), by Lemma 2.2, \( u \in C(\mathbb{R}^+, H) \),
\[u(t_n) \to u(t_0) \quad \text{in} \quad H \quad \text{as} \quad n \to \infty.\]
So there exists a subsequence \( t_{n_k} : t_{n_k} \to t_0 \) as \( k \to \infty \) such that
\[u(t_{n_k}) \to u(t_0), \quad F(u(t_{n_k})) \to F(u(t_0)) \quad \text{a.e. in} \quad \Omega \quad \text{as} \quad k \to \infty.\]
By Remark 2.1 and the Fatou lemma,
\[
\int_{\Omega} \left( \int_{\Omega} \left( F(u(t_0)) + \frac{\theta}{2} u^2(t_0) \right) dx \right) \leq \liminf_{k \to \infty} \left( \int_{\Omega} F(u(t_{n_k}))dx + \frac{\theta}{2} \|u(t_{n_k})\|^2 \right)
= \liminf_{k \to \infty} \int_{\Omega} F(u(t))dx + \frac{\theta}{2} \|u(t_0)\|^2.
\] (38)
So, by (37)-(38) (taking \( t_0 = 0 \)),
\[
E(u(0), u_t(0)) 
\leq \liminf_{t \to 0} \frac{1}{2} \left( \|u_t(t)\|^2 + \|A_1^1 u(t)\|^2 \right) + \liminf_{t \to 0} \int_{\Omega} F(u(t))dx - \lim_{t \to 0} (g, u(t))
\leq \liminf_{t \to 0} E(u(t), u_t(t)) = \lim_{t \to 0} E(u(t), u_t(t)) = E^*.
\]
Therefore, \( E^* = E(u(0), u_t(0)) \) and (35) holds for \( t \geq s \geq 0 \). That is, (30) holds.

(iii) We infer from energy identity (30) and (37)-(38) that
\[
\limsup_{t \to t_0} \frac{1}{2} \left( \|u_t(t)\|^2 + \|A_1^1 u(t)\|^2 \right) + \liminf_{t \to t_0} \int_{\Omega} F(u(t))dx
\leq \lim_{t \to t_0} \left[ \frac{1}{2} \left( \|u_t(t)\|^2 + \|A_1^1 u(t)\|^2 \right) + \int_{\Omega} F(u(t))dx \right]
= \frac{1}{2} \left( \|u_t(t_0)\|^2 + \|A_1^1 u(t_0)\|^2 \right) + \int_{\Omega} F(u(t_0))dx
\leq \liminf_{t \to t_0} \frac{1}{2} \left( \|u_t(t)\|^2 + \|A_1^1 u(t)\|^2 \right) + \liminf_{t \to t_0} \int_{\Omega} F(u(t))dx.
\]
Therefore,
\[
\lim_{t \to t_0} \|u(t, u_t(t))\|_{E_1} = \|(u(t_0), u_t(t_0))\|_{E_1}.
\]
By \((u, u) \in C_\infty(\mathbb{R}^+, E_1)\) and the uniform convexity of the Banach space \( E_1 \), we have \((u, u) \in C(\mathbb{R}^+, E_1)\).

Obviously, when \( t = 0 \), \( S(0) = I : E_1 \to E_1 \) is Lipschitz continuous, so \( S(0) \) is \( 1/2 \)-Hölder continuous. When \( t > 0 \), for any \((u_0, u_1), (v_0, v_1) \in E_1\), by (12), (29), the interpolation theorem and (13),
\[
\|S(t)(u_0, u_1) - S(t)(v_0, v_1)\|_{E_1} = \|(z, z_t)(t)\|_{E_1} \leq C\| (z, z_t)(t) \|_{E_1}^{1/2} \| (z, z_t)(t) \|_{E_1}^{1/2}
\leq C\| (z, z_t)(0) \|_{E_1}^{1/2} \leq C\| (z, z_t)(0) \|_{E_1}^{1/2},
\]
where \( z(t) = u(t) - v(t) \).
(iv) Using the multiplier \( u \) in Eq. (8), we have
\[
\frac{d}{dt} (u, u_t) + \frac{\gamma}{2} \| A^\frac{1}{4} u \|^2 + \| A^\frac{1}{4} u_t \|^2 + (f(u), u) = \| u_t \|^2 + (g, u). \tag{39}
\]
(15) + \( \epsilon \times (39) \) yields
\[
\frac{d}{dt} H_3(\varphi_u) + K_3(\varphi_u) = 0, \quad t > 0,
\]
where \( \varphi_u = (u, u_t) \),
\[
H_3(\varphi_u) = \frac{1}{2} \left( \| u_t \|^2 + \| A^\frac{1}{4} u \|^2 \right) + \epsilon \left( (u, u_t) + \frac{\gamma}{2} \| A^\frac{1}{4} u \|^2 \right) + \int_{\Omega} F(u)dx - (g, u)
\geq \frac{1}{4} \left( \| u_t \|^2 + (1 - \frac{\theta}{\lambda_1}) \| A^\frac{1}{4} u \|^2 \right) - C_1,
\]
\[
K_3(\varphi_u) = \gamma \| A^\frac{1}{4} u \|^2 + \epsilon \left( \| A^\frac{1}{4} u_t \|^2 + (f(u), u) - \| u_t \|^2 - (g, u) \right)
\geq \kappa \epsilon \left( \| A^\frac{1}{4} u \|^2 + \| u_t \|^2 \right) - C_1 \epsilon
\]
for \( \epsilon > 0 \) suitably small, and where we have used Remark 2.1. Inserting (41) into (40), we get
\[
\frac{d}{dt} H_3(\varphi_u) + \kappa \epsilon \| \varphi_u \|_{E_1}^2 \leq C_1 \epsilon, \quad t > 0. \tag{42}
\]
Obviously,
\[
\sup_{t \in \mathbb{R}^+} H_3(\varphi_u(t)) \geq -C_1, \quad H_3(\varphi_u(0)) \leq C(\| \varphi_u(0) \|_{E_1}^2 + \| g \|_{V_{-1}}) \leq K.
\]
Applying Lemma 4.2 (taking \( \delta = \kappa \epsilon, \gamma = \epsilon \) there) to (42), we arrive at
\[
H_3(\varphi_u) \leq \sup_{\zeta \in E_1} \left\{ H_3(\zeta) : \| \zeta \|_{E_1}^2 \leq \frac{C_1 + 1}{\kappa} \right\} \equiv K_1, \quad t \geq t_0 \equiv \frac{C_1 + K}{\epsilon}.
\]
Therefore,
\[
\| \varphi_u(t) \|_{E_1}^2 \leq K_1 + C_1 \equiv R, \quad t \geq t_0. \tag{43}
\]
Estimates (43) and (11) show that the solution semigroup \( S(t) \) has a bounded absorbing set \( B_0 \) in \( E_1, S(t)B_0 \subset B_0 \) as \( t \geq t(B_0) \). Let
\[
B_1 = \bigcup_{t \geq t(B_0) + 1} S(t)B_0. \tag{44}
\]
Obviously, \( S(t)B_1 \subset B_1 \) and \( B_1 \) is also an absorbing set of \( S(t) \), and we see from (12) and (29) that \( B_1 \) is bounded in \( V_{1+\alpha} \times V_{1-\alpha} \) for \( B_1 \subset \bigcup_{t \geq 1} S(t)B_0 \).

(v) We see from case (iv) that the dynamical system \((S(t), E_1)\) is dissipative and the semigroup \( S(t) \) is uniformly compact for \( V_{1+\alpha} \times V_{1-\alpha} \hookrightarrow E_1 \), so \( S(t) \) has a global attractor \( \mathcal{A} \) in \( E_1 \), and \( \mathcal{A} \) is of \( V_{1+\alpha} \times V_{1-\alpha} \) regularity. By Theorem 5.2 (see below), the semigroup \( S(t) \) has an exponential attractor \( \mathcal{A}_{exp} \) in \( E_1 \), so
\[
dist_{E_1} \{ \mathcal{A}, \mathcal{A}_{exp} \} = dist_{E_1} \{ S(t)\mathcal{A}, \mathcal{A}_{exp} \} \to 0 \quad \text{as} \quad t \to +\infty,
\]
which means \( \mathcal{A} \subset \mathcal{A}_{exp} \),
\[
dim_f \{ \mathcal{A}, E_1 \} \leq \dim_f \{ \mathcal{A}_{exp}, E_1 \} < +\infty.
\]
Theorem 4.3 is proved. \( \square \)
Corollary 4.4. Under the assumptions of Theorem 4.3, the dynamical system $(S(t), E_1)$ is a gradient system. Any full trajectory $\nu = \{(u(t), u_t(t))|t \in \mathbb{R}\}$ from the global attractor $\mathcal{A}$ is of the following properties: (i)

$$\lim_{t \to -\infty} \text{dist}_{E_1}\{(u(t), u_t(t)), \mathcal{N}\} = 0,$$

$$\lim_{t \to +\infty} \text{dist}_{E_1}\{(u(t), u_t(t)), \mathcal{N}\} = 0,$$

where $\mathcal{N}$ is the set of all the fixed points of $S(t)$, that is,

$$\mathcal{N} = \{(u, 0) \in E_1|Au + f(u) = g\}.$$

Furthermore, for any $U_0 \in E_1$,

$$\lim_{t \to +\infty} \text{dist}_{E_1}\{S(t)U_0, \mathcal{N}\} = 0.$$

(ii) $(u, u_t, u_{tt}) \in L^\infty(\mathbb{R}; V_{1+\alpha} \times V_{1-\alpha} \times V_{-\alpha})$, and

$$\sup_{\nu \in \mathcal{A}} \sup_{t \in \mathbb{R}} (\|u(t)\|_{V_{1+\alpha}} + \|u_t(t)\|_{V_{1-\alpha}} + \|u_{tt}(t)\|_{V_{-\alpha}}) \leq C(\mathcal{A}).$$

(45)

**Proof.** Obviously, the functional $E(u, v)$ (see (15)) is continuous on $E_1$, and the energy identity (30) implies that $E(u, v)$ is a strict Lyapunov function on $E_1$. So the dynamical system $(S(t), E_1)$ is gradient. Therefore, the full trajectory $\nu = \{(u(t), u_t(t))|t \in \mathbb{R}\}$ from the attractor $\mathcal{A}$ is of the property (i) (cf. Theorems 2.28 and 2.31 in [13]).

By Theorem 4.3: (v), $(u, u_t) \in L^\infty(\mathbb{R}; V_{1+\alpha} \times V_{1-\alpha})$, and

$$\sup_{t \in \mathbb{R}} (\|u(t)\|_{V_{1+\alpha}} + \|u_t(t)\|_{V_{1-\alpha}}) \leq C(\mathcal{A}).$$

(46)

Let $z(t) = u(t + l) - u(t), t, l \in \mathbb{R}$. Obviously, estimate (14) holds for $z$, that is,

$$\|z(z_t)(t)\|_{E_{1-\alpha}}^2 \leq C(\mathcal{A}) \left( e^{-\kappa(t-s)} \|z(z_t)(s)\|_{E_{1-\alpha}}^2 + \max_{s \leq \tau \leq t} \|z(\tau)\|^2 \right).$$

Letting $s \to -\infty$ and making use of (46), we get

$$\|z(z_t)(t)\|_{E_{1-\alpha}}^2 \leq C(\mathcal{A}) \max_{-\infty \leq \tau \leq t} \|z(\tau)\|^2.$$

Hence,

$$\|D_t u(t)\|_{V_{1-\alpha}}^2 + \|D_t u_t(t)\|_{V_{-\alpha}}^2 \leq C(\mathcal{A}) \max_{-\infty \leq \tau \leq t} \|D_t u(\tau)\|^2 \leq C(\mathcal{A}) \max_{-\infty \leq \tau \leq t} \|u_t(\tau)\|^2 \leq C(\mathcal{A}).$$

(47)

where $D_t u(t) = (u(t + l) - u(t))/l$. Taking a sequence $\{l_n\} : l_n \to 0$ as $n \to \infty$. We infer from (47) that (subsequence if necessary)

$$(D_{l_n} u, D_{l_n} u_t) \to (u_t, u_{tt}) \text{ weakly}^* \text{ in } L^\infty(\mathbb{R}; E_{1-\alpha}).$$

Therefore,

$$\sup_{t \in \mathbb{R}} (\|u_t(t)\|_{V_{1-\alpha}} + \|u_{tt}(t)\|_{V_{-\alpha}}) \leq C(\mathcal{A}).$$

(48)

The combination of (46) and (48) gives (45). This completes the proof. □
5. Exponential attractor as $1 \leq p < p^*$.

**Definition 5.1.** A set $\mathcal{A}_{exp} \subset E_1$ is said to be an exponential attractor of the semigroup $S(t)$ acting on a Banach space $E_1$ if

1. $\mathcal{A}_{exp}$ is a compact set in $E_1$;
2. $\mathcal{A}_{exp}$ is forward invariant, i.e., $S(t)\mathcal{A}_{exp} \subset \mathcal{A}_{exp}, t \geq 0$;
3. $\mathcal{A}_{exp}$ attracts exponentially the images of all bounded sets in $E_1$, i.e., there exists a positive constant $\gamma > 0$ such that

$$\text{dist}_{E_1}\{S(t)B, \mathcal{A}_{exp}\} \leq C(\|B\|_{E_1})e^{-\gamma t}$$

for all bounded subsets $B \subset E_1$, where $\|B\|_{E_1} = \sup_{\xi \in B} \|\xi\|_{E_1}$;
4. $\mathcal{A}_{exp}$ has finite fractal dimension in $E_1$, i.e., $\dim \{\mathcal{A}_{exp}, E_1\} < +\infty$.

**Theorem 5.2.** Let Assumption (H) be valid, with $1 \leq p < p^*$ and $g \in H$. Then the solution semigroup $S(t)$ has an exponential attractor $\mathcal{A}_{exp}$ in $E_1$.

**Proof.** Let

$$B_R = [B_1]_{E_{1-\alpha}},$$

where $B_1$ is as shown in (44) and the sign $[ \ ]_{E_{1-\alpha}}$ stands for the closure in space $E_{1-\alpha}$. Then $B_R$ is an absorbing set of the semigroup $S(t)$, which is bounded in $V_{1+\alpha} \times V_{1-\alpha}$ and closed in $E_{1-\alpha}$ (Indeed, for any $\xi \in B_R$, there exists a sequence $\{\xi_n\} \subset B_1$ such that $\xi_n \to \xi$ in $E_{1-\alpha}$, and (subsequence if necessary) $\xi_n \to \xi$ weakly in $V_{1+\alpha} \times V_{1-\alpha}$, so $\|\xi\|_{V_{1+\alpha} \times V_{1-\alpha}} \leq \liminf_{n \to \infty} \|\xi_n\|_{V_{1+\alpha} \times V_{1-\alpha}} \leq C(B_1)$). So $B_R$ constitutes a complete metric space (with the $E_{1-\alpha}$ norm) and one sees from (13) that the solution semigroup $S(t)$ is continuous on $B_R$ with respect to $E_{1-\alpha}$ topology, and the system $(S(t), B_R)$, with $E_{1-\alpha}$ topology, constitutes a dissipative dynamical system.

Define the operator

$$V^k = S(kT) : B_R \mapsto B_R, \quad k \in \mathbb{Z}^+.$$

We show that the discrete system $(V^k, B_R)$, with $E_{1-\alpha}$ topology, has an exponential attractor.

We introduce the functional space

$$W(0, T) = \{z \in L^2(0, T; V_{1-\alpha}) | z_t \in L^2(0, T; V_{-\alpha}), \|\xi_z\|_{W^2} < \infty\},$$

equipped with the norm

$$\|\xi_z\|_{W^2} = \int_0^T \|z(z, t)(\tau)\|_{E_{1-\alpha}}^2 d\tau$$

and the functional space

$$X_T = E_{1-\alpha} \times W(0, T),$$

equipped with usual graph norm, that is

$$\|U\|_{X_T} = \|\eta\|_{E_{1-\alpha}} + \|\xi_z\|_{W}, \quad \forall U = (\eta, z) \in X_T.$$

Obviously, the spaces $W(0, T)$ and $X_T$ are Banach spaces.

Let the set

$$B_T = \{\xi_u(0), \xi_u(t), t \in [0, T]) | \xi_u(0) \in B_R, \xi_u(t) = S(t)\xi_u(0)\}.$$

Define the operator $\nabla : B_T \mapsto X_T$,

$$\nabla U = (\xi_u(T), \xi_u(T + \cdot)) = \left(S(T)\xi_u(0), S(T + \cdot)\xi_u(0)\right),$$

where and in the following $U = (\xi_u(0), \xi_u(\cdot)) \in B_T$ and $\xi_u(\cdot)$ means $\xi_u(t), t \in [0, T]$. 
Obviously, the set \( B_T \) is bounded in \( X_T \). We claim that \( B_T \) is closed in \( X_T \). Indeed, for any sequence \( \{U^n\} \subset B_T \),
\[
U^n = (\xi_{u^n}(0), \xi_{u^n}(\cdot)) \to U = (\xi_u(0), \xi_u(\cdot)) \quad \text{in} \quad X_T.
\]
Since \( \xi_{u^n}(0) \in B_R \) and \( B_R \) is closed in \( E_{1-\alpha} \), \( \xi_u(0) \in B_R \). By the Lipschitz continuity of \( S(t) \) in \( E_{1-\alpha} \) (see (13)),
\[
\sup_{0 \leq t \leq T} \|\xi_{u^n}(t) - \xi_u(t)\|_{E_{1-\alpha}}^2 \leq C(R,T)\|\xi_{u^n}(0) - \xi_u(0)\|_{E_{1-\alpha}}^2 \to 0.
\]
By the uniqueness of the limit, \( \xi_u(\cdot) = \xi_u(\cdot), U = (\xi_u(0), \xi_u(\cdot)) \in B_T, B_T \) is closed in \( X_T \).

Therefore, \( B_T \) is complete with respect to the topology of \( X_T \), and the dynamical system \((Y^k, B_T)\) constitutes a discrete dissipative dynamical system. \( \square \)

**Lemma 5.3.** Under the assumptions of Theorem 5.2, the discrete dynamical system \((Y^k, B_T)\) has an exponential attractor \( \mathcal{A} \).

**Proof.** Obviously, \( \forall B_T \subset B_T \). For any
\[
U_1 = (\xi_{u_1}(0), \xi_{u_1}(\cdot)), \quad U_2 = (\xi_{u_2}(0), \xi_{u_2}(\cdot)), \quad z = u_1 - u_2,
\]
the weak quasi-stability estimate (14) holds for \( z \). Integrating (14) over \((T, 2T)\), we get
\[
\int_T^{2T} \|(z, z_t)(t)\|_{E_{1-\alpha}}^2 dt \leq C \int_T^{2T} e^{-\kappa t} d\tau \|(z_0, z_1)\|_{E_{1-\alpha}}^2 + CT \int_0^{2T} \|z(\tau)\|^2 d\tau. \quad (49)
\]
Hence,
\[
\|\nabla U_1 - \nabla U_2\|^2_{X_T} = \|\xi_{u_1}(T) - \xi_{u_2}(T)\|_{E_{1-\alpha}}^2 + \|\xi_t(\cdot + T)\|_{W^1,2}^2
\]
\[
\leq C e^{-\kappa T} \|\xi_{u_1}(0) - \xi_{u_2}(0)\|_{E_{1-\alpha}}^2 + C \int_0^T e^{-\kappa(T-\tau)} \|z(\tau)\|^2 d\tau
\]
\[
+ \int_T^{2T} \|\xi_t(t)\|_{E_{1-\alpha}}^2 dt \quad (50)
\]
\[
\leq \eta_T \|\xi_{u_1}(0) - \xi_{u_2}(0)\|_{E_{1-\alpha}}^2 + K_T \int_0^{2T} \|z(\tau)\|^2 d\tau
\]
\[
\leq \eta_T \|U_1 - U_2\|^2_{X_T} + K_T \left(n_1(U_1 - U_2) + n_1(\nabla U_1 - \nabla U_2)\right),
\]
where \( \xi_t(t) = \xi_{u_1}(t) - \xi_{u_2}(t) = (z(t), z_t(t)) \),
\[
\eta_T = 2Ce^{-\kappa T} + C \int_T^{2T} e^{-\kappa t} d\tau, \quad K_T = C(T + 2),
\]
\[
n_1(U) = \int_0^T \|u(t)\|^2 dt, \quad U = (\xi_u(0), \xi_u(\cdot)) \in B_T.
\]

It follows form (50) and (13) that,
\[
\|\nabla U_1 - \nabla U_2\|^2_{X_T} \leq b_T \|U_1 - U_2\|^2_{X_T},
\]
with \( b_T = \eta_T + CK_T C(R,T) \). By Lemma 2.2, \( W(0, T) \hookrightarrow L^2(0, T; H) \). Therefore, the seminorm \( n_1(U) \) is compact in \( X_T \). Taking \( T : 0 < \eta_T < 1 \) and making use of the standard theory on exponential attractor (cf. Corollary 2.23 in [13]), we get the conclusion of Lemma 5.3. Lemma 5.3 is proved. \( \square \)
Define the project operator
\[ P : B_T \to B_R, \quad PU = \xi_u(0), \quad \forall U = (\xi_u(0), \xi_u(\cdot)) \in B_T. \]

**Lemma 5.4.** Let \( X, Y \) be metric spaces and the mapping \( h : X \mapsto Y \) be \( \beta \)-Hölder continuous on the set \( B \subset X \). Then
\[ \dim_f \{ h(B), Y \} \leq \frac{1}{\beta} \dim_f \{ B, X \}. \]

**Lemma 5.5.** \( A = P\mathcal{A} \) is an exponential attractor of the discrete dynamical system \((V^k, B_R)\).

**Proof.** (i) \( A \) is compact because \( A \) is the image of the compact set \( \mathcal{A} \) under continuous mapping \( P \).
(ii) Since \( V^k \mathcal{A} \subset \mathcal{A} \), we have \( V^k A = P V^k \mathcal{A} \subset P \mathcal{A} = A \).
(iii) Obviously,
\[ \text{dist}_{\mathcal{A}_{1-\alpha}} \{ V^k B_R, A \} \leq \text{dist}_{\mathcal{A}_{1-\alpha}} \{ V^k B_R, \mathcal{A} \} \leq C q^k \]
for some \( 0 < q < 1 \) (cf. Corollary 2.23 in [13]).
(iv) By the Lipschitz continuity of the operator \( P \) and Lemma 5.4,
\[ \dim_f \{ A, \mathcal{A}_{1-\alpha} \} \leq \dim_f \{ \mathcal{A}, X_T \} < \infty. \]

Therefore, \( A \) is the desired exponential attractor. Lemma 5.5 is proved. \( \square \)

Let
\[ A_{exp} = \bigcup_{0 \leq t \leq T} S(t)A. \]
By the standard argument one easily knows that \( A_{exp} \) is an exponential attractor of the dynamical system \((S(t), B_R)\), with \( E_{1-\alpha} \) topology. So there exists a constant \( \kappa > 0 \), such that
\[ \text{dist}_{\mathcal{A}_{1-\alpha}} \{ S(t) B_R, A_{exp} \} \leq C e^{-\kappa t}, \quad t \geq 0. \]
(51)

Since the set \( A_{exp} \subset B_R \) is bounded in \( V_{1+\alpha} \times V_{1-\alpha} \), we claim that \( A_{exp} \) is an exponential attractor of the dynamical system \((S(t), E_1)\).

Indeed, (i) Obviously, \( A_{exp} \) is forward invariant.
(ii) Define the operator
\[ F : [0, T] \times A \to B_R, \quad F(t, \xi_u) = \xi_u(t), \quad \xi_u \in A, \quad t \in [0, T]. \]
By the interpolation theorem, for any \( \xi_u = (u, v) \in A \subset V_{1+\alpha} \times V_{1-\alpha} \), on account of \( V_{1-\alpha} \hookrightarrow V_\alpha \), we have
\[ \| u \|_{V_{1+\alpha}} \leq C \| u \|_{V_1}^{\frac{1}{2}} \| u \|_{V_{1-\alpha}}^{\frac{1}{2}}, \quad \| v \| \leq C \| v \|_{V_1}^{\frac{1}{2}} \| v \|_{V_{1-\alpha}}^{\frac{1}{2}}, \quad \|(u, v)\|_{E_1} \leq C \|(u, v)\|_{E_{1-\alpha}}^{\frac{1}{2}}. \]

Therefore, by estimates (12) and (13),
\[ \| F(t_1, \xi_u) - F(t_2, \xi_u) \|_{E_1} \leq C \| \xi_u(t_1) - \xi_u(t_2) \|_{E_{1-\alpha}} \leq \left( \int_{t_1}^{t_2} \| \xi_u'(\tau) \|_{E_{1-\alpha}} d\tau \right)^{\frac{1}{2}} \]
\[ \leq C \left( \int_{t_1}^{t_2} \| \xi_u'(\tau) \|_{E_{1-\alpha}}^{2} d\tau \right)^{\frac{1}{2}} |t_1 - t_2|^{\frac{1}{2}} \leq C |t_2 - t_1|^{\frac{1}{2}}, \]
\[ \| F(t, \xi_u) - F(t, \xi_u) \|_{E_1} \leq C \| \xi_u(t_1 - \xi_u(t_2) \|_{E_{1-\alpha}} \leq C(R, T) \| \xi_u_1 - \xi_u_2 \|_{E_{1-\alpha}} \]
for any \( \xi_u, \xi_u_1, \xi_u_2 \in A, t, t_1, t_2 \in [0, T] \), which imply that the mapping \( F \) is \( 1/4 \)-Hölder continuous. Therefore, \( A_{exp} = F([0, T] \times A) \) (the image of \([0, T] \times A\) ) is compact in \( E_1 \).
(iii) By Lemma 5.4, 
\[ \dim_f \{ A_{\exp}, E_1 \} \leq 4 \dim_f \{ [0, T] \times A, \mathbb{R}^+ \times E_{1-\alpha} \} \leq 4(1 + \dim_f \{ A, E_{1-\alpha} \}) < \infty. \]

(iv) By (51), 
\[ \text{dist}_{E_1} \{ S(t)B_R, A_{\exp} \} = \sup_{\xi_u \in B_R} \inf_{\xi_v \in A_{\exp}} \| S(t)\xi_u - \xi_v \|_{E_1} \leq C \sup_{\xi_u \in B_R} \inf_{\xi_v \in A_{\exp}} \| S(t)\xi_u - \xi_v \|_{E_{1-\alpha}}^{1/2} \leq C \left( \text{dist}_{E_{1-\alpha}} \{ S(t)B_R, A_{\exp} \} \right)^{1/2} \leq Ce^{-\frac{\mu t}{2}}. \]

Therefore, the claim is valid. Theorem 5.2 is proved.

**Corollary 5.6.** Assume that (i) \( \alpha \in (0, 1/2), \Omega \subset \mathbb{R}^3 \) is a bounded smooth domain, \( f \in C^1(\mathbb{R}) \), 
\[ f(s)s \geq -C, \quad |f'(s)| \leq C(1 + |s|^{p-1}), \quad p \in [1, 3 + 4\alpha); \]
(ii) \((u_0, u_1) \in E_1, g \in H\).

Then problem (8)-(9) admits a unique energy solution (i.e., Shatah-Struwe solution) \( u \), with \((u, u_t) \in C_b(\mathbb{R}^+; E_1) \). Furthermore, the solution possesses the following properties:

(i) For every \( t > 0 \), 
\[ (u(t), u_t(t)) \in E_2 \equiv H^2 \cap H^1_0 \times H^1_0, \quad (u(t), u_t(t)) \in E_1. \]
(ii) The (energy) solution semigroup \( S(t) \) has a finite dimensional global attractor and an exponential attractor in \( E_1 \) which are bounded subsets in \( E_2 \).

**Proof.** Under the assumptions of Corollary 5.6, on account of the equivalence of energy solution and Shatah-Struwe solution, the combination of Theorem 3.3 with the conclusions in [26, 29] gives Corollary 5.6. \( \square \)

6. **Limit solution and weak global attractor** as \( p^* \leq p < p^{**} \). When \( p^* \leq p < p^{**} \) (critical and supercritical case), although problem (8)-(9) still possesses an energy solution (see Theorem 3.3: (i)), the uniqueness of energy solutions is lost. So we can not define the solution semigroup according to standard manner, and all the traditional criteria on the existence of global attractor cease to be effective. In order to overcome this difficulty, we introduce the subclass \( \mathcal{G} \) of limit solutions, define solution semigroup in a new manner, and directly construct weak global attractor of \( \mathcal{G} \).

We first consider the parabolic type perturbations of Eq. (8): 
\[ u_{tt} + Au + \eta A^\frac{1}{2} u_t + \gamma A^\alpha u_t + f(u) = g, \] 
with parameter \( \eta > 0 \).

**Theorem 6.1.** Let Assumption \( (H) \) be valid, with \( p^* \leq p < p^{**} = \frac{N+2}{N-2} \). Then problem (52), (9) admits a unique energy solution \( u \), with \( \varphi_u = (u, u_t) \in L^\infty(\mathbb{R}^+, E_1) \cap C_0(\mathbb{R}^+, E_1) \). Furthermore, estimates (11) and (43) hold (not depending on \( \eta \)) and the solution is of Lipschitz continuity in weaker space \( E_{1/2} \), that is, 
\[ \left\| (z, z_t)(t) \right\|_{E_{1/2}}^2 \leq C(R, \eta, T) \left\| (z, z_t)(0) \right\|_{E_{1/2}}^2, \quad 0 \leq t \leq T, \] 
where \( z = u - v, u \) and \( v \) are two energy solutions of problem (52), (9) corresponding to initial data \((u_0, u_1)\) and \((v_0, v_1)\).
Proof. By the same arguments as in Theorem 3.3 and Theorem 4.3, one can easily get the existence of energy solutions and estimates (11) and (43) (not depending on \( \eta \)). So we only prove (53) here. Obviously, \( z = u - v \) solves

\[
\begin{align*}
z_t + Az + \eta A^{\frac{1}{2}} z_t + \gamma A^\alpha z_t + f(u) - f(v) &= 0, \\
z(0) &= u_0 - v_0 \equiv z_0, \quad z_t(0) = u_1 - v_1 \equiv z_1.
\end{align*}
\]

Using the multiplier \( A^{-\frac{1}{2}} z_t + \epsilon z \) in (54), we get

\[
\begin{align*}
\frac{d}{dt} H_4(z, z_t) + \epsilon \| A^{\frac{1}{2}} z_t \|^2 + \left( \eta - \epsilon \right) \| z_t \|^2 + \gamma \| A^{\frac{2\alpha - 1}{4}} z_t \|^2 \\
= - (f(u) - f(v), A^{-\frac{1}{2}} z_t + \epsilon z),
\end{align*}
\]

where

\[
H_4(z, z_t) = \frac{1}{2} \| A^{-\frac{1}{2}} z_t \|^2 + \frac{1}{2} \| A^{\frac{1}{2}} z_t \|^2 + \epsilon \left( \frac{\gamma}{2} \| z \|^2_{V_1} + \frac{\eta}{2} \| A^{\frac{1}{2}} z \|^2 + (z, z_t) \right)
\]

for \( \epsilon > 0 \) suitably small. Taking account of the Sobolev embedding: \( V_{1-\delta} \hookrightarrow L^{p+1} \) for \( \delta : 0 < \delta < 1 \), and using the interpolation theorem, we have

\[
\begin{align*}
& \| (f(u) - f(v), z + A^{-\frac{1}{2}} z_t) \| \\
& \leq C(1 + \| u \|_{p+1}^{p-1} + \| v \|_{p+1}^{p-1})(\| z \|^2_{V_1} + \| z \|_{p+1} + \| A^{-\frac{1}{2}} z_t \|_{p+1}) \\
& \leq C(\| z \|^2_{V_{1-\delta}} + \| z \|_{V_{1-\delta}} \| z_t \|_{V_{-\delta}}) \\
& \leq C(\| z \|^2_{V_{1-\delta}} \| z \|^{2(1-\delta)}_{V_{1-\delta}} + \| z \|^2_{V_{1-\delta}} \| z \|^{1-\delta}_{V_{-\delta}} \| z_t \|^{1-\delta}_{V_{-\delta}}) \\
& \leq C(\| z \|^2_{V_1} + \| z \|^2 + \| z_t \|^2_{V_{-\delta}}).
\end{align*}
\]

Inserting (56) into (55) and taking \( \epsilon : 0 < \epsilon < \eta/2 \), we obtain

\[
\frac{d}{dt} H_4(z, z_t) \leq C(\epsilon, \eta) H_4(z, z_t).
\]

Applying the Gronwall inequality to (57), we get estimate (53). Theorem 6.1 is proved.

Remark 6.2. One can see from the proof of Theorem 6.1 that the parameter \( \eta > 0 \) is indispensable to guarantee the uniqueness of energy solutions as \( p^* \leq p < p^{**} \).

We define the solution operator \( T_\eta(t) : E_1 \mapsto E_1 \) (with \( \eta > 0 \)),

\[
T_\eta(t) \varphi_0 = \varphi(t) = (u(t), u_t(t)) \quad \text{for every} \quad \varphi_0 \in E_1, \quad t \geq 0,
\]

where \( u \) is an energy solution of problem (52), with \( \varphi(0) = \varphi_0 \). Theorem 6.1 shows that \( \{ T_\eta(t) \} \) constitutes a semigroup on \( E_1 \), which is Lipschitz continuous in weaker space \( E_{1/2} \).

Theorem 6.3. (Existence of limit solutions). Let Assumption (H) be valid, with \( p^* \leq p < p^{**} \), and let \( \varphi_0 = (u_0, u_1) \in E_1, \eta_n \to 0^+, \varphi_0 = (u_0^n, u_1^n) \to \varphi_0 \). Then there exists a subsequence \( \{ n_k \} \) such that

\[
T_{\eta_{n_k}}(t) \varphi_0 \to \varphi(t) = (u(t), u_t(t)), \quad k \to \infty,
\]

where \( u \) is an energy solution of problem (8)-(9) (\( u \) is called limit solution), and the sign “\( \to \)” stands for “weakly convergent in \( E_1 \)”. Moreover, estimates (11) and (43) hold for the limit solution \( u \).
Proof. Let \( \varphi_0^n = (u_0^n, u_t^n) \in E_1 \),
\[ \varphi^n(t) = (u^n(t), u_t^n(t)) = T_n(t)\varphi_0^n, \quad \text{with } \eta_n > 0, \quad \eta_n \to 0^+. \]
Then \( u^n \) solves Eq. (52), with \( \eta = \eta_n, \varphi^n(0) = \varphi_0^n \), that is, for any \( \phi \in V_1 \),
\[ (u_t^n + Au^n + \eta_n A^{\frac{1}{2}} u_t^n + \gamma A^{\alpha} u_t^n + f(u^n), \phi) = (g, \phi), \quad t > 0. \] (59)
Obviously, estimates (11) and (43) hold for \( u^n \). So (subsequence if necessary)
\[ u^n \to u \text{ weakly}^* \text{ in } L^\infty(0, T; V_1); \]
\[ u_t^n \to u_t \text{ weakly}^* \text{ in } L^\infty(0, T; H) \cap L^2(0, T; V_0). \]
By Lemma 2.2,
\[ u^n \to u \text{ in } C([0, T], V_1) \] and a.e. in \( \Omega \) for \( t \in [0, T] \),
\[ f(u^n) \to f(u) \text{ weakly}^* \text{ in } L^\infty(0, T; L_{1+\frac{1}{2}}); \]
\[ \eta_n (A^{\frac{1}{2}} u_t^n, \phi) = \eta_n (u_t^n, A^{\frac{1}{2}} \phi) \to 0. \]
Letting \( n \to \infty \) in (59) we see that the limit function \( u \) is an energy solution of problem (8)-(9). By the lower semi-continuity of the norm of the weak limit one gets that estimates (11) and (43) hold for \( u \). This completes the proof. \( \square \)

Let \( 2^{E_1} \) be the space of all subsets of \( E_1 \). Define the operator \( T(t) : 2^{E_1} \to 2^{E_1}, \)
\[ T(t)\varphi_0 = \varphi(t) = (w) \lim_{k \to \infty} T_{\eta_k}(t)\varphi_0^{\eta_k}, \quad t \geq 0. \] (60)
where and in the following the sign “(w) limit” stands for weak limit in phase space \( E_1 \) and \( \{\varphi_0^{\eta_k}\} \) is as shown in (58).

Theorem 6.3 shows that for every \( \varphi_0 \in E_1 \), problem (8)-(9) possesses at least one “limit solution” \( \varphi(t) \), with \( \varphi(0) = \varphi_0 \). So the operator \( T(t) \) is well defined, \( \varphi(t) = T(t)\varphi_0 \) and \( \varphi(t) \) may be multiple-valued because there may be lots of subsequences of \( \{\varphi_0^{\eta_k}\} \) such that the weak limit on the right hand side of (60) exists and all these weak limits are denoted by \( \varphi(t) \). Let \( \mathcal{G} \) be the set of all these “limit solutions”, that is,
\[ \mathcal{G} = \{ \varphi \in L^\infty(\mathbb{R}^+; E_1)|\varphi(t) = T(t)\varphi_0, \varphi_0 \in E_1 \}. \]

Lemma 6.4. The subclass \( \mathcal{G} \) of limit solutions is of the following properties:
(i) (Existence) For each \( \varphi_0 \in E_1 \), there exists at least one \( \varphi \in \mathcal{G} \) with \( \varphi \in C_w(\mathbb{R}^+; E_1), \varphi(0) = \varphi_0 \).
(ii) (Translates of solutions are solutions) If \( \varphi \in \mathcal{G} \) and \( \tau \geq 0 \), then \( \varphi^\tau \in \mathcal{G} \), where
\[ \varphi^\tau(t) = \varphi(t + \tau), t \in [0, \infty). \]
(iii) (Concatenation) If \( \varphi, \psi \in \mathcal{G} \), with \( \psi(0) = \varphi(t), t \geq 0 \), then \( \theta \in \mathcal{G} \), where
\[ \theta(\tau) := \begin{cases} \varphi(\tau), & 0 \leq \tau \leq t, \\
\psi(\tau - t), & \tau > t. \end{cases} \]
Proof. Repeating the same abstract argument as in [33], one can easily get the conclusions of the lemma. Here we omit the process. \( \square \)

Obviously, for any subset \( B \subset E_1 \),
\[ T(t)B = \{ \varphi(t) = T(t)\varphi_0|\varphi_0 \in B \}. \]
It follows from Lemma 6.4: (ii) that, for any subset \( B \subset E_1 \),
\[ T(0)B = B, \quad T(t + \tau)B = \{ \varphi(t + \tau) = T(t)T(\tau)\varphi_0|\varphi_0 \in B \} = T(t)T(\tau)B. \]
for \( t, \tau \geq 0 \). That is,

\[
T(0) = I, \quad T(t + \tau) = T(t)T(\tau) \quad \text{for} \quad t, \tau \geq 0,
\]

\( \{T(t)\}_{t \geq 0} \) constitutes a semigroup on \( 2E_1 \).

**Definition 6.5.** The set \( A \subset E_1 \) is said to be a weak global attractor \( ((E_1, E_s)(0 \leq s < 1) \) attractor) of the subclass \( G \) of limit solutions, if

(i) \( A \) is closed in \( E_1 \) and compact in \( E_s \);

(ii) \( A \) is an invariant set, i.e., \( T(t)A = A, t \geq 0 \);

(iii) \( A \) attracts all bounded sets in \( E_1 \) in the topology of \( E_s \), that is, for every bounded set \( B \subset E_1 \)

\[
\text{dist}_{E_s}(T(t)B, A) \to 0 \quad \text{as} \quad t \to \infty.
\]

**Theorem 6.6.** Let Assumption \((H)\) be valid, with \( p^* \leq p < p^{**} \). Then the subclass \( G \) of limit solutions has an \( (E_1, E_s)(0 \leq s < 1) \) attractor.

**Proof.** Estimate (43) implies that the ball

\[
B_R = \{ \zeta \in E_1 \mid \| \zeta \|_{E_1} \leq R \}
\]

is an absorbing set of the operator \( T(t) \) for \( R \) suitably large. Obviously, \( B_R \) is also an (uniform) absorbing set of the semigroup \( T_\eta(t) \) \( (\eta > 0) \). That is, for every bounded set \( B \) in \( E_1 \), there exists a time \( t(B) \) independent of \( \eta \in (0, 1] \) such that

\[
T_\eta(t)B \subset B_R \quad \text{for all} \quad t \geq t(B), \eta \in (0, 1].
\]

We construct the set

\[
A = \{ \psi \mid \psi = (w) \lim_{n \to \infty} T_\eta_n(t_n)\psi_n, \text{ where } t_n \to \infty, \eta_n \to 0^+, \psi_n \in B_R \}.
\]

We show that the set \( A \) is the desired \((E_1, E_s)\) \((0 \leq s < 1)\) attractor.

(i) \( A \) is a bounded closed set in \( E_1 \) and a compact set in \( E_s \).

For any \( \psi = (w) \lim_{n \to \infty} T_\eta_n(t_n)\psi_n \in A \), by the lower semicontinuity of the norm of the weak limit and the fact: \( T_\eta_n(t_n)\psi_n \in T_\eta_n(t_n)B_R \subset B_R \), \( t_n \geq t(B_R) \), we get

\[
\| \psi \|_{E_1} \leq \liminf_{n \to \infty} \| T_\eta_n(t_n)\psi_n \|_{E_1} \leq R,
\]

i.e., \( A \) is bounded in \( E_1 \). For any sequence \( \{ \psi^k \} \subset A, \psi^k \rightharpoonup \psi \) in \( E_1 \) as \( k \to \infty \). By the definition of \( A \), for every fixed \( k > 0 \), there exist \( \psi_k \in B_R \) and \( t_k > k, 0 < \eta_k \leq 1/k \) such that

\[
|\langle T_\eta_k(t_k)\psi_k - \psi^k, \xi \rangle| \leq \frac{1}{k} \quad \text{for all } \xi \in E_1^* \text{ (the dual space of } E_1).\]

So

\[
|\langle T_\eta(t_k)\psi - \psi, \xi \rangle| \leq |\langle T_\eta_k(t_k)\psi_k - \psi^k, \xi \rangle| + |\langle \psi^k - \psi, \xi \rangle| \to 0, \quad k \to \infty,
\]

that is, \( \psi \in A \). Therefore, \( A \) is a weakly closed set in \( E_1 \), and hence a closed set in \( E_1 \).

For any sequence \( \{ \psi^n \} \subset A \), we can extract a subsequence \( \{ \psi^{n_k} \} \subset \{ \psi^n \} \) such that

\[
\psi^{n_k} \rightharpoonup \psi \text{ in } E_1, \text{ and } \psi^{n_k} \rightarrow \psi \text{ in } E_s,
\]

because \( A \) is bounded in \( E_1 \) and \( E_1 \hookrightarrow E_s \). Since \( A \) is weakly closed in \( E_1 \), we get \( \psi \in A \). Therefore, \( A \) is a compact set in \( E_s \).

(ii) \( A \) is an invariant set, i.e., \( T(t)A = A, t \geq 0 \).
For any $\varphi(t) \in T(t)A$, there exists a $\varphi_0 = (w) \lim_{n \to \infty} T_{\eta_n}(t_n) \varphi_0^n \in A$ such that
$$\varphi(t) = T(t) \varphi_0 = (w) \lim_{n \to \infty} T_{\eta_n}(t) T_{\eta_n}(t_n) \varphi_0^n = (w) \lim_{n \to \infty} T_{\eta_n}(t + t_n) \varphi_0^n,$$
where $\varphi_0^n \in B_R$. So $\varphi(t) \in A, T(t)A \subset A$.

For any $\varphi_0 \in A$, there exists a sequence $\{\varphi_0^n\} \subset B_R$ such that
$$\varphi_0 = (w) \lim_{n \to \infty} T_{\eta_n}(t_n) \varphi_0^n = (w) \lim_{n \to \infty} T_{\eta_n}(t) T_{\eta_n}(t_n - t) \varphi_0^n, \quad t \geq t \geq 0.$$
For any fixed $t \geq 0$, due to $t_n \to +\infty$ as $n \to \infty$, there exists a $N > 0$ such that $\{T_{\eta_n}(t_n - t) \varphi_0^n\}_{n \geq N} \subset B_R$ is bounded in $E_1$, so one can extract a subsequence $n_k$ such that
$$T_{\eta_{n_k}}(t_{n_k} - t) \varphi_0^{n_k} \to \bar{\varphi} \text{ in } E_1 \text{ as } k \to \infty,$$
which means $\bar{\varphi} \in A$ and
$$\varphi_0 = (w) \lim_{k \to \infty} T_{\eta_{n_k}}(t) T_{\eta_{n_k}}(t_{n_k} - t) \varphi_0^{n_k} = T(t) \bar{\varphi} \subset T(t)A.$$
Therefore, $T(t)A = A, t \geq 0$.

(iii) $A$ attracts every bounded set in $E_1$ in the topology of $E_s$, that is, for every bounded set $B \subset E_1$,
$$\operatorname{dist}_{E_1}(T(t)B, A) \to 0 \quad \text{as} \quad t \to \infty.$$
Or else, there exist a bounded set $B \subset E_1$, a sequence $\{\varphi_n\}_{n=1}^{\infty} \subset B, t_n \to +\infty$ and $\epsilon_0 > 0$ such that
$$\inf_{\varphi \in A} \|T(t_n)\varphi_n - \varphi\|_{E_s} > \epsilon_0. \quad (61)$$
Then $\varphi_n \in E_1 \setminus A$ (or else, if $\varphi_n \in A$, then $T(t_n) \varphi_n \in A$, which violates (61)). For every $n > 0$, by the definition of the operator $T(t)$,
$$\varphi_n(t_n) = T(t_n) \varphi_n = (w) \lim_{k \to \infty} T_{\eta_k}^n(t_n) \varphi_n^k,$$
where $\varphi_n^k \to \varphi_n, \eta_k^k \to 0^+$ as $k \to \infty$. Due to $E_1 \hookrightarrow \hookrightarrow E_s$, (subsequence if necessary)
$$T_{\eta_k}^n(t_n) \varphi_k \to \varphi_n(t_n) \text{ in } E_s \text{ as } k \to \infty.$$
So for every $n > 0$, there exists an element (we denote it by) $T_{\eta_k}^n(t_n) \varphi_n^k \in \{T_{\eta_k}^n(t_n) \varphi_n^k\}$ such that $0 < \eta_k^k < 1/n$,
$$\|\varphi_n(t_n) - T_{\eta_k}^n(t_n) \varphi_n^k\|_{E_s} < \frac{1}{n}, \quad (62)$$
and
$$\|(\varphi_n^k - \varphi_n, \xi)\| \leq \frac{1}{n} \quad \text{for all} \quad \xi \in E_1^{*}.$$
Since $\{\varphi_n\}_{n=1}^{\infty} \subset B$ is bounded in $E_1$, (subsequence if necessary)
$$\varphi_n \to \varphi \text{ in } E_1.$$
So
$$\varphi_n \to \varphi \text{ in } E_1 \text{ and } \{\varphi_n\} \text{ is bounded in } E_1.$$
Therefore, $\{T_{\eta_k}^n(t_n) \varphi_n^k\} \subset B_R \ (t_n \geq t_0)$ is bounded in $E_1(\hookrightarrow \hookrightarrow E_s)$, and (subsequence if necessary)
$$T_{\eta_k}^n(t_n) \varphi_n^k = T_{\eta_k}^n(t_n - t_0) \bar{\varphi}_n \to \bar{\varphi} \text{ in } E_1, \quad \|T_{\eta_k}^n(t_n) \varphi_n^k - \bar{\varphi}\|_{E_s} \to 0, \quad (63)$$
where $\tilde{\phi}_n = T_{y_n}^t(0)\phi_n \in B_R$. By the definition of $\mathcal{A}, \tilde{\phi} \in \mathcal{A}$. The combination of (62) and (63) gives

$$
\|T(t_n)\tilde{\phi}_n - \tilde{\phi}\|_{E_s} = \|\phi_n(t_n) - \tilde{\phi}\|_{E_s} \leq \|\phi_n(t_n) - T_{y_n}^t(0)\phi_n\|_{E_s} + \|T_{y_n}^t(0)\phi_n - \tilde{\phi}\|_{E_s} \to 0, \ n \to \infty,
$$

which violates (61). Theorem 6.6 is proved. \qed

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