Dissipativity, Convexity and Tight
O’Shea-Zames-Falb Multipliers for Safety
Guarantees

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Abstract: We develop a novel convex parametrization of integral quadratic constraints with a terminal cost for subdifferentials of convex functions, involving general O’Shea-Zames-Falb multipliers. We show the benefit of our results for the reduction of conservatism of existing techniques, and sketch applications to the analysis of optimization algorithms or the stability analysis of neural network controllers. The development is prepared by providing a novel link between the convex integrability of a multivariable mapping and dissipativity theory.

Keywords: Dissipativity, Robustness Analysis, Integral quadratic constraints, Absolute Stability, Linear matrix inequalities.

1. INTRODUCTION

The field of systems and control is dominated by understanding complex dynamic interconnections. It has been a highly successful point-of-view to consider such monolithic systems as an interconnection of individual subsystems, and to characterize their global dynamical characteristics by those of the subsystems and properties of their interconnection. Dissipativity theory as developed by Willems (1972a,b) constitutes a cornerstone in system theory. It not only translates this conceptual idea into a concrete mathematical framework, but also lays ground for the tailored construction of related computational tools in a modular fashion. For robustness analysis, this theme has been surveyed, e.g., in the recent contributions by Arcak et al. (2016) and Scherer (2022), in which one can find many more references to the literature.

The absolute stability analysis of a feedback loop consisting of a linear system and a static nonlinearity, a so-called Lur’e system, has laid ground to this development. As emphasized in Jan Willems’s introduction to the seminal paper by Popov in Basar (2001), the celebrated circle-and Popov-criteria had a substantial impact on the construction of more flexible and less conservative stability results using so-called O’Shea-Zames-Falb (OZF) multipliers (O’Shea (1967); Willems and Brockett (1968) and Zames and Falb (1968)). It is by now well-understood how to generate corresponding computational stability tests based on the main result in Megretski and Rantzer (1997) using integral quadratic constraints (IQCs), which are strongly inspired by the seminal contributions of Yakubovich (1967). For systems described in continuous time, computational aspects are exposed in detail in Chen and Wen (1996) (see also Veenman et al. (2016)), while the discrete-time counterparts have been proposed by Carrasco et al. (2020) and Fetzer and Scherer (2017a).

Most existing stability proofs based on OZF multipliers are functional analytic in nature. Direct dissipativity-based proofs can be extracted from Seiler (2015) and Veenman and Scherer (2014). The latter has lead to the notion of IQCs with a terminal cost, as introduced by Scherer and Veenman (2018) and further developed by Scherer (2022) for continuous time systems. It has been argued that this concept constitutes a seamless link between so-called hard IQCs (which are conservative) and more powerful soft IQCs on the infinite time-horizon. The papers by Fetzer and Scherer (2017a), Fetzer and Scherer (2017b), Iannelli et al. (2019) and Yin et al. (2021) demonstrate the benefit of such IQCs for reducing conservatism with examples.

Discrete-time absolute stability results based on OZF multipliers have recently drawn considerably attention for the analysis of optimization algorithms (Lessard et al. (2016)), the generalization to the design of extremum controllers (Scherer and Ebenbauer (2021)), and for the safety verification of neural network controllers (Yin et al. (2020); Pauli et al. (2021)). However, results about IQCs with a terminal cost for discrete-time systems are missing in the literature, which is partly due to technical challenges concerning the correct formulation of the related algebraic Riccati equations.

To fill this gap, this paper develops a novel convex parametrization of IQCs with a nontrivial terminal cost for subdifferentials of convex functions and involving both causal and anti-causal OZF multipliers. We demonstrate the benefit over existing results by a numerical example. It is conjectured that our construction is tight, in the sense that it constitutes the correct formulation of IQCs in between hard and soft ones; see Fetzer and Scherer (2017b) for a discussion of this issue in continuous-time. We only mention various potential applications to optimization algorithms and stability analysis of neural network con-
trollers, the details of which are left for future research. As a preparation, we reveal a new link between the convex integrability of a multi-valued mapping and dissipativity theory. This is of independent interest on a path towards identifying new classes of functions for which multiplier-based stability tests can be developed more systematically.

The paper is structured as follows. In Section 2, we relate dissipativity theory to convex integration as obtained in a celebrated paper by Rockafellar (1966). After recalling strict dissipativity characterizations for linear systems with quadratic supply rates (Section 3), we formulate a stability result for Lur’ë systems based on IQCs with nonpositive and which satisfy located in the unit disk D can be lifted as k standard unit vectors for respectively. In

\[ x_{t+1} = Ax_t + Bu_t, \quad y_t = Cx_t + Du_t \]

for \( t \in \mathbb{N} \)

let us consider the discrete-time dynamical system

\[ \left( x_{t+1}, y_t \right) \in (a(x_t, u_t), c(x_t, u_t)) \]

for \( t \in \mathbb{N} \) with the time axis \( \mathbb{N} \). We say that the trajectory \( t \mapsto (x_t, y_t) \) is admissible if \( (x_0, y_0) \in X \times U \) holds and if the inclusion (2) is satisfied for all \( t \in \mathbb{N}_0 \). With the left-shift operator \( (\sigma x)(t) := x(t+1) \) for \( t \in \mathbb{N}_0 \), the system (2) is compactly described as \( (\sigma x, y) \in (a(x, u), c(x, u)) \).

Definition 1. System (2) is dissipative with respect to the supply rate \( S : U \times X \rightarrow \mathbb{R} \) if there exists a storage function \( V : X \rightarrow \mathbb{R} \) such that the dissipation inequality (DI)

\[ V(x_{t+1}) \leq V(x_t) + \sum_{t=t_i}^{t_{t+1}} S(u_t, y_t). \]

holds for all admissible trajectories and all time instances \( t_1, t_2 \in \mathbb{N}_0 \) with \( t_1 \leq t_2 \). System (2) is cyclo-dissipative with respect to \( S \) in case that

\[ 0 \leq \sum_{t=t_i}^{t_{t+1}} S(u_t, y_t) \]

holds for all admissible trajectories and all admissible trajectories satisfying \( x_{t_2} = x_{t_1} \). These properties are referred to as passivity or cyclo-passivity for the particular choice \( S(u, y) = u' y \) of the supply-rate.

Note that, in contrast to the original definition in Willems (1972a), it is not required that storage functions are non-negative or bounded from below. Verifying dissipativity requires to come up with a storage function for which the dissipation inequality holds; we also say that storage functions certify dissipativity. Instead, cyclo-dissipativity is characterized directly in terms of system trajectories without the need to have a storage function available. Clearly, dissipative systems are cyclo-dissipative.

The converse implication permits to conclude the existence of a storage function from cyclo-dissipativity. In other words, a mere trajectory-based input-output property of the system guarantees the existence of storage functions which certifies dissipativity. Under a suitable assumption on the richness of state-trajectories, this converse is established by explicitly defining two extremal storage functions through the solution of optimal control problems over admissible system trajectories.

Theorem 2. Suppose there is a ground state \( x_0 \in X \) such that for every \( \xi \in X \) there exists an admissible round trip state trajectory with \( x_0 = \xi, x_{t_1} = x_r \) and \( x_{t_2} = \xi \) for some \( t_1, t_2 \in \mathbb{N}_0 \) with \( t_1 \leq t_2 \). Then (2) is cyclo-dissipative if it is dissipative as certified by the available storage function

\[ V^*_\xi(\xi) := \sup_{T \in \mathbb{N}_0, x_0 = \xi, x_{T-1} = x_T} \left[ \sum_{t=0}^{T-1} S(u_t, y_t) \right] \quad \text{for } \xi \in X, \]

and the required supply function

\[ V^*_\xi(\xi) := \sup_{T \in \mathbb{N}_0, x_0 = x_r, x_{T-1} = x_T} \left[ \sum_{t=0}^{T-1} S(u_t, y_t) \right] \quad \text{for } \xi \in X. \]

This result is proved in complete analogy to the continuous-time counterparts in Hill and Moylan (1975) and van der Schaft (2021). The same can be said for the well-known fact that dissipativity is equivalent to the validity of the local dissipation inequality

\[ V(x_r) - V(x) \leq S(u, y) \]

for all \( (x, u) \in X \times U \), \( (x_r, y) \in (a(x, u), c(x, u)) \).
As a first contribution of this paper, let us now establish an interesting relationship of dissipativity with the integration of multi-valued mappings and convexity. Precisely, we wonder under which conditions a mapping \( F : \mathbb{R}^n \to \mathbb{R}^n \) has a convex primitive, i.e., a convex function \( f : \mathbb{R}^n \to \mathbb{R} \) whose subdifferential satisfies
\[
F(x) \subset \partial f(x) \quad \text{for all } x \in \mathbb{R}^n. \tag{5}
\]
If \( f : \mathbb{R}^n \to \mathbb{R} \) is convex, the subdifferential \( \partial f : \mathbb{R}^n \to \mathbb{R}^n \) satisfies \( f(z + d) - f(z) \geq \partial f(z)^T d \) for all \( z, d \in \mathbb{R}^n \). If (5) is valid, we hence conclude (after the change of variables \( z = x + u \) and \( d = -u \)) that \( F \) is linked to \( f \) by
\[
f(x + u) - f(x) \leq F(x + u)^T u \quad \text{for all } x, u \in \mathbb{R}^n. \tag{6}
\]
This just is the local dissipation inequality for the system
\[
\sigma x = x + u, \quad y \in F(x + u)
\]
and the supply rate \( S(u, y) = u^T y \), with \( f \) being a storage function. Hence (7) is passive. Conversely, passivity of (7) implies (6) for some \( f : \mathbb{R}^n \to \mathbb{R} \). It is then elementary to see that \( f \) is convex, and (5) holds by the mere definition of the subdifferential. This proves the following result.

**Theorem 3.** The mapping \( F : \mathbb{R}^n \to \mathbb{R}^n \) has a convex primitive iff the system \( \sigma x = x + u, \ y \in F(x + u) \) is passive.

Since the linear system \( \sigma x = x + u \) is controllable, passivity is equivalent to cyclo-passivity by Theorem 2. Now recall that \( F \) is said to be cyclically monotone if
\[
\sum_{j=1}^m F(v_j)^T (v_j - v_{j+1}) \geq 0
\]
holds for any choice of points \( v_0, \ldots, v_m \in \mathbb{R}^n \) and with \( v_{m+1} := v_0 \). It is not hard to establish the following link (as shown in Section A.1).

**Lemma 4.** The system \( \sigma x = x + u, \ y \in F(x + u) \) is cyclo-passive iff \( F \) is cyclically monotone.

This leads to Rockafellar’s celebrated result that \( F \) has a convex primitive iff it is cyclically monotone (Rockafellar (1966)), with a proof based on dissipativity theory.

Instead of exploring these links any further, we rather exploit Theorem 3 for the generation of integral quadratic constraints in robustness analysis. As a first step, we note that (7) is dissipative w.r.t. \( (u, y) \to u^T y \) if this holds for the system and the supply rate obtained after the pre-compensation \( u = -x + v \). This transformation leads to the system \( \sigma x = v, \ y \in F(v) \) with supply rate \( (v, y) \to y^T (v - x) \). Thus, passivity of (7) and dissipativity of (1) for all admissible trajectories and all instance times \( t_1, t_2 \in \mathbb{N} \) with \( t_1 \leq t_2 \).

For a linear system
\[
\sigma x = Ax + Bu, \quad y = Cx + Du
\]
and a homogeneous quadratic supply rate
\[
S_p(u, y) = \begin{pmatrix} y \\ u \end{pmatrix}^T P \begin{pmatrix} y \\ u \end{pmatrix}, \quad P = P^T = \mathbb{R}^{(k+m) \times (k+m)},
\]
strict dissipativity with a general storage function is equivalent to strict dissipativity with a homogeneous quadratic storage function. This leads to the following key result.

**Theorem 8.** The linear system (16) is strictly dissipative with respect to the quadratic supply rate (17) iff there exists a symmetric solution \( X \) of the strict LMI
\[
\begin{pmatrix} A & B \\ I & 0 \end{pmatrix}^T \begin{pmatrix} X & 0 \\ 0 & -X \end{pmatrix} \begin{pmatrix} A & B \\ I & 0 \end{pmatrix} = \begin{pmatrix} C & D \\ 0 & I_m \end{pmatrix} P \begin{pmatrix} C & D \\ 0 & I_m \end{pmatrix}.
\]
If \( \text{eig}(A) \cap \partial \mathbb{D} = \emptyset \) and with \( G(z) := C(zI - A)^{-1}B + D \), this is equivalent to the frequency-domain inequality

\[
0 < \left( \frac{G(\lambda)}{I_m} \right)^* P \left( \frac{G(\lambda)}{I_m} \right) \quad \text{for all } \lambda \in \partial \mathbb{D}.
\]  

(19)

In summary, strict dissipativity can be assured by checking the feasibility of the LMI (18) or, under a mild assumption, through the frequency domain inequality (19).

4. ROBUST STABILITY ANALYSIS

Let us now consider the feedback interconnection of

\[
x_{t+1} = Ax_t + Bu_t, \quad z_t = Cx_t + Dw_t
\]

(20)

with the subgradient linearity

\[
w_t \in \partial f(z_t)
\]

(21)

for \( t \in \mathbb{N}_0 \). We assume that \( A \in \mathbb{R}^{n \times n} \) is Schur and that \( f : \mathbb{R}^d \to \mathbb{R} \) is convex with \( f(0) = 0 \) and \( 0 \not\in \partial f(0) \). Hence \( f, f^* \) satisfy the conditions \( f(x) \geq 0 \) and \( f^*(x) \geq f^*(0) = 0 \) for all \( x \in \mathbb{R}^d \).

The current interest in such classical Lur'e system has been emphasized in the introduction, see also Section 7.

The interconnection (20)-(21) is said to be stable if there exists some \( L > 0 \) such that

\[
\sum_{t=0}^{\infty} (\|x_t\|^2 + \|w_t\|^2) \leq L\|x_0\|^2
\]

(22)

holds for all trajectories of (20)-(21). Now note, by (6) for \( F = \partial f \), that the static system (21) is dissipative w.r.t. \( S(z,w) = z^T w \), i.e., it is passive. Therefore, the classical dissipativity theorem guarantees stability of (20)-(21) if the linear system (20) is strictly dissipative with respect to the supply rate \( S(w,z) = -w^T z \) (see, e.g., Brogliato (2004); Brogliato and Tanwani (2020) in continuous-time).

This passivity-based stability test can be substantially improved by imposing dissipativity constraints after passing the input and output signals of (20) or (21) through some

\[
\sigma \xi = A\eta + B\Phi \begin{pmatrix} z \\ w \end{pmatrix}, \quad v = C\xi + D\Phi \begin{pmatrix} z \\ w \end{pmatrix}, \quad \xi_0 = 0
\]

(23)

with \( \xi \) of dimension \( n_\Phi \). With the state \( \eta = \text{col}(\xi, x) \) of dimension \( n + n_\eta \), the interconnection of (23) with (20) is

\[
\sigma \eta = A\eta + Bw, \quad v = C\eta + Dw, \quad \text{where}
\]

(24)

where

(25)

Let us now formulate a discrete-time robust stability result whose continuous-time counterpart has been first presented in Scherer and Veenman (2018).

Theorem 9. Let \( P = P^T \) and suppose the following holds.

a) All trajectories of (23) under the constraint (21) satisfy the following IQC with a quadratic terminal cost defined by some matrix \( Z = Z^T \):

\[
\sum_{t=0}^{T-1} v_t^T P v_t - \xi_t^T Z \xi_T \geq 0 \quad \text{for all } T \in \mathbb{N}.
\]

b) The system (24) is strictly dissipative w.r.t. the supply rate \( S(w,v) = -v^T P v \) as certified by \( \mathcal{X} = \mathcal{X}' \).

c) The certificates \( \mathcal{X} = \begin{pmatrix} X^\Phi + W^T W \end{pmatrix} \in \mathbb{R}^{(n+n_\Phi) \times (n+n_\Phi)} \) and \( Z \in \mathbb{R}^{n \times n} \) are coupled as

\[
\begin{pmatrix} X^\Phi + Z W^T \end{pmatrix} > 0.
\]

(26)

Then there exists some \( \epsilon > 0 \) such that all the trajectories of (20)-(21) satisfy

\[
x_T^T Y x_T + \sum_{t=0}^{T-1} (\|x_t\|^2 + \|w_t\|^2) \leq \epsilon x_0^T X x_0
\]

(27)

for all \( T \in \mathbb{N} \) where \( Y := X - W^T (X^\Phi + Z)^{-1} W \). In particular, the loop (20)-(21) is stable.

The proof is given in Section A.4. Let us contrast Theorem 9 with a stability result based on soft IQCs due to Megretski and Rantzer (1997), which can be proved in discrete-time as in Seiler (2015) and Veenman and Scherer (2014) without relying on homotopy arguments. This requires that (20)-(21) is well-posed in the sense defined in these references. Moreover, \( f \) needs to be differentiable with \( \|\nabla f(z)\| \leq M \|z\| \) for all \( z \in \mathbb{R}^n \) and some \( M \geq 0 \). Finally, \( A_\Phi \) of the filter (23) with the transfer matrix \( \Psi(z) = C\Phi(zI - A\Phi)^{-1}B + D_\Phi \) is assumed to be stable.

Theorem 10. Let \( P = P^T \) and suppose the following holds.

a) All trajectories of (23) with \( z \in l^2_\mathbb{Z} \) and \( w = \nabla f(z) \) satisfy the soft IQC \( \sum_{t=0}^{T-1} v_t^T P v_t \geq 0 \).

b) The system (24) is strictly dissipative with respect to the supply rate \( S(w,v) = -v^T P v \).

c) The left-upper/right-lower \( d \times d \)-block of \( \Psi(\lambda)^* P \Psi(\lambda) \) is positive/negative semi-definite for all \( \lambda \in \partial \mathbb{D} \).

Then the loop (20)-(21) is stable.

In both results, it is a misnomer to talk about an integral quadratic constraint in \( a) \), since the terminology results from the continuous-time counterparts involving integration instead of summation. Moreover, the transfer matrix \( \Psi(\lambda)^* P \Psi(\lambda) \) is referred to as a dynamic multiplier if \( \Psi(\lambda) \) is not constant, and it is said to be static otherwise.

Observe that the IQC in Theorem 10 \( a) \) is called soft since it is formulated on the infinite time-horizon. Although the conditions in \( b) \) are equivalent, Theorem 10 does not involve any sign-constraint on the corresponding certificate. For these reasons, Theorem 10 does not allow to draw conclusions about the transient behaviour of the system state. In contrast, Theorem 9 leads to guaranteed pointwise in time constraints for the future state-trajectory as in (27). In classical dissipation-based stability tests, this is achieved by taking \( Z = 0 \) in Theorem 9 \( a) \), which leads to a so-called hard IQC. Then Theorem 9 \( c) \) requires that the certificate \( \mathcal{X} \) is positive definite, which involves possibly severe conservatism (see Section 6).

A much more detailed discussion of these links and further consequences are provided in Scherer and Veenman (2018) and Scherer (2022).

5. O'SHEA-ZAMES-FALB MULTIPLIERS

We now establish how to construct IQCs with a nontrivial terminal cost \( Z = 0 \) as appearing in Theorem 9 \( a) \). To this end, let \( y = \psi_{k,\nu}(u) \) be defined by the filter

\[
\begin{pmatrix} s x \\ y \end{pmatrix} = \begin{pmatrix} A_v & B_v \\ C_{k,\nu} & 1 \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \quad \text{with } x_0 = 0.
\]

Since \( f \) is nonnegative and if \( k \in \{1, \ldots, \nu\} \), Corollary 6 implies that \( y \in l^2_\mathbb{Z} \) and \( y = (\psi_{k,\nu} \otimes I_\nu)(z) \) with \( \psi \in \partial f(z) \) that the hard IQC \( \sum_{t=0}^{T-1} w_t^T y_t \geq 0 \) is satisfied for all \( T \in \mathbb{N} \).
This is condition a) in Theorem 9 for $\psi$. Let us now conically combine all these IQCs. To this end, fix possibly different values of $\nu, \tilde{\nu} \in \mathbb{N}_0$ and define
\[
\psi_{\nu}(\lambda) := \sum_{k=0}^{\nu} \lambda_k \psi_{k,\nu} \quad \text{for} \quad \lambda := (\lambda_0, \ldots, \lambda_{\nu}) \in \mathbb{R}^{\nu+1}.
\]  
(28)

Note that $\psi_{\nu}$ admits a state-space realization in terms of $(A_{\nu,\nu}, B_{\nu,\nu}, C_{\nu}(\lambda), D_{\nu}(\lambda))$ with $C_{\nu}(\lambda) := \sum_{k=1}^{\nu} \lambda_k C_{k,\nu}$ and $D_{\nu}(\lambda) := \sum_{k=0}^{\nu} \lambda_k$. If the vectors $\lambda \in \mathbb{R}^{\nu+1}$ and $\tilde{\lambda} \in \mathbb{R}^{\nu+1}$ are nonnegative, we then obtain for any $z \in \mathbb{Z}_L, w \in \partial f(z)$ and $y = (\psi_{\nu}(\lambda) \otimes I_d)(z)$, $\tilde{y} = (\psi_{\nu}(\tilde{\lambda}) \otimes I_d)(w)$ the hard IQC
\[
\sum_{t=0}^{T-1} (w_t^T g_t + z_t^T \tilde{g}_t) \geq 0 \quad \text{for all} \quad T \in \mathbb{N}.
\]  
(29)

This is condition a) in Theorem 9 for $Z = 0$. Indeed, with
\[
A_{\Psi} := \begin{pmatrix} A_{\nu} & 0 \\ 0 & A_{\tilde{\nu}} \end{pmatrix} \otimes I_d, \quad B_{\Psi} := \begin{pmatrix} B_{\nu} & 0 \\ 0 & B_{\tilde{\nu}} \end{pmatrix} \otimes I_d,
\]
\[
C_{\Psi} := \begin{pmatrix} I_{\nu+\tilde{\nu}} & 0 \\ 0 & I_{\nu+\tilde{\nu}} \end{pmatrix} \otimes I_d, \quad D_{\Psi} := \begin{pmatrix} 0 \\ 0 \end{pmatrix} \otimes I_d,
\]
and the symmetric matrix
\[
P(\lambda, \tilde{\lambda}) := \begin{pmatrix} 0 & 0 & C_{\nu}(\lambda)^T \\ 0 & 0 & D_{\nu}(\lambda) + D_{\nu}(\tilde{\lambda})^T \end{pmatrix} \otimes I_d.
\]  
(31)

the response of (23) to $z \in \mathbb{Z}_L$ and $w \in \partial f(z)$ satisfies
\[
v_t^T P(\lambda, \tilde{\lambda}) v_t = (w_t^T g_t + z_t^T \tilde{g}_t) + (y_t^T w_t + \tilde{y}_t^T z_t) \quad \text{for all} \quad t \in \mathbb{N}_0.
\]  
(32)

To generate IQCs with $Z \neq 0$, let us introduce the lifted state-space representations of the filters $y = \psi_{\nu}(\lambda)(z)$ and $\tilde{y} = \psi_{\nu}(\tilde{\lambda})(w)$ as in the notation section. Since the filter’s initial conditions are zero, this results in
\[
\begin{pmatrix} x_T \\ y_T \end{pmatrix} = \begin{pmatrix} B_{\nu}^T \\ D_{\nu}(\lambda) \end{pmatrix} \begin{pmatrix} x_T \\ y_T \end{pmatrix} + \begin{pmatrix} B_{\tilde{\nu}}^T \\ D_{\nu}(\tilde{\lambda}) \end{pmatrix} \begin{pmatrix} x_T \\ y_T \end{pmatrix} T.
\]  
(33)

We will show that (25) indeed holds with the filter (30) and the matrix $P(\lambda, \tilde{\lambda})$ in (31) for the terminal cost matrix
\[
Z(E) := \begin{pmatrix} 0 & E^T \otimes I_d \\ E \otimes I_d & 0 \end{pmatrix},
\]  
(34)

if the coefficient vectors $\lambda, \tilde{\lambda}$ and the free matrix $E$ render
\[
M_T(\lambda, \tilde{\lambda}, E) := D_{\nu}^T(\lambda) + D_{\nu}^T(\tilde{\lambda})^T - (B_{\nu}^T)^T E B_{\nu}^T
\]  
(35)
doubly hyperdominant for $T = \nu + \tilde{\nu} + 1$.

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**Theorem 11.** Suppose that $\lambda \in \mathbb{R}^{\nu+1}, \tilde{\lambda} \in \mathbb{R}^{\nu+1}$ and $E \in \mathbb{R}^{\nu \times \nu}$ are chosen with
\[
M_{\nu+\tilde{\nu}+1}(\lambda, \tilde{\lambda}, E) \in \mathcal{H}(\nu+\tilde{\nu}+1)^\times(\nu+\tilde{\nu}+1).
\]

Then the nonlinearity $\partial f$ satisfies the IQC (25) with the filter defined by (30) for the supply rate matrix $P(\lambda, \tilde{\lambda})$ in (31) and the terminal cost matrix $Z(E)$ in (34).

This is the second main result of this paper, whose proof is found in Section A.5. No analogous result is available in continuous-time.

6. A CONCRETE ALGORITHM

Theorems 9 and 11 form the basis for generating various robust performance results with dynamic IQCs by known dissipativity arguments. A collection of such results can be found, e.g., in Fetz et al. (2017) and Scherer (2022).

For example, if $e = C_{\nu} x$ is an output of the interconnection (20)-(21), we can target at the computation of some tight $\gamma > 0$ such that the amplitude bound $\sup_{t \in [0,T]} \| e_t \| \leq \gamma \| x_0 \|$ holds for all initial conditions $x_0$ in the unit ball of $\mathbb{R}^n$ and all trajectories of (20)-(21).

Corollary 12. The loop (20)-(21) is stable and all its trajectories satisfy
\[
\sup_{t \in [0,T]} \| C_{\nu} x_t \| \leq \gamma_s(\nu, \tilde{\nu}) \| x_0 \|
\]
for $\gamma_s(\nu, \tilde{\nu})$ is determined as follows:

a) Choose the dimensions $\nu, \tilde{\nu} \in \mathbb{N}_0$ to fix the multiplier complexity and set $T_0 := \nu + \tilde{\nu} + 1$.

b) With the filter matrices (30) construct (23) and (24).

c) With $P(\cdot), Z(\cdot), M_{T_0}(\cdot)$ defined by (31), (34), (35) and the variables $X = X^T, H = H^T, E \in \mathbb{R}^{\nu \times \nu}, \lambda \in \mathbb{R}^{\nu+1}, \tilde{\lambda} \in \mathbb{R}^{\nu+1}, \gamma \in \mathbb{R}$, introduce the LMIs

\[
\begin{pmatrix} A & B \\ I & 0 \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & -X \end{pmatrix} \begin{pmatrix} A & B \\ I & 0 \end{pmatrix}^T + (C \ D)^T P(\lambda, \tilde{\lambda}) (C \ D) < 0,
\]

\[
\begin{pmatrix} H & 0 \\ C_{\nu}^T & W^T \end{pmatrix}^T X \begin{pmatrix} H & 0 \\ C_{\nu}^T & W^T \end{pmatrix} > 0, \quad X > 0, \quad \gamma > \gamma I, \quad x > 0,
\]

\[
M_{T_0}(\lambda, \tilde{\lambda}, E) \in \mathcal{H}(T_0 \times T_0),
\]

where $X$ is partitioned as in Theorem 9.

d) Then $\gamma_s(\nu, \tilde{\nu})$ denotes the infimum of all $\gamma > 0$ for which the LMIs (36) are feasible.

It is instrumental to observe that the constraints (36) are indeed affine in all variables. Hence, the computation of $\gamma_s(\nu, \tilde{\nu})$ involves solving a standard semi-definite program.
For a numerical example, take the system matrices

\[
\begin{pmatrix}
A & B \\
C & D_L \\
C_e & 0
\end{pmatrix} =
\begin{pmatrix}
0.1 & 1 & 0 & 0 & 0 \\
-0.24 & 0.1 & -0.54 & -0.35 & 0.84 & 0 \\
0 & 0 & 0.54 & -0.24 & 0.59 & 0 \\
0 & 0 & 0 & 0.54 & 1 & 0 \\
0 & 0 & 0 & -0.56 & 0.54 & 1.04 \\
-0.08 & -0.17 & 0.13 & 0.09 & -0.21 & -1/L
\end{pmatrix}
\]

depending on \( L \in (0, 3.5] \), and compute \( \gamma_L(\nu, \tilde{\nu}) \) as in Corollary 12 for \( \nu = \tilde{\nu} \in \{0, 1, 2, 3\} \) using the LMI-solver of MATLAB (2020) and Yalmip (Lofberg (2004)). The results are plotted over \( L \) in Fig. 2 as full lines. They are compared with the values for hard IQCs (dotted curves), which are obtained with \( E = 0 \) in Corollary 12. The blue curve indicates the severe conservatism of results based on static IQCs (\( \nu = \tilde{\nu} = 0 \)). For dynamic IQCs (\( \nu = \tilde{\nu} > 0 \)), the plots permit to quantify the conservatism of hard IQCs if compared to those with a nontrivial terminal cost as newly developed in this paper. For example, the yellow and purple dots in Fig. 2 indicate a reduction of conservatism by about 20% and 35%, respectively.

7. DISCUSSION AND A CONJECTURE

Due to the modularity of results based on dissipativity, we emphasize that Theorems 9 and 11 can be seamlessly merged with the approaches in, e.g., Yin et al. (2020) and Pauli et al. (2021) to reduce the conservatism in the determination of stability margins of neural network controllers. Our simple numerical example demonstrates the striking potential limitation of static and hard dynamic IQCs, as employed in a whole stream of recent papers revolving around the generation of safety guarantees for static maps or dynamic systems involving neural networks.

In an another direction, the analysis of gradient descent algorithms for \( m \)-strongly convex and \( L \)-smooth functions is addressed by Lessard et al. (2016) based on causal OZF multipliers, while the extension to general OZF multipliers is proposed by Michalowsky et al. (2021). In the latter situation, our results permit to guarantee transient properties of algorithms, next to the typically investigated global exponential stability.

We conclude the paper with the conjecture that robustness analysis based on Theorems 9 and 11 does not involve any more conservatism than robustness analysis based on Theorem 10 with general OZF multipliers.

8. CONCLUSIONS

We have developed a novel discrete-time absolute stability result by dissipation techniques and based on the notion of integral quadratic constraints with a nontrivial terminal cost. The benefit over existing results has been shown by a numerical example, and we conjecture that it is a lossless extension of a known robust stability result relying on O’Shea-Zames-Falb multipliers and soft IQCs. The suggested link of dissipativity theory with convex analysis might offer avenues for systematically developing new classes of multipliers in IQC based stability results.

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Let (7) be cyclo-passive. If we are given

$$x_t := v_{m+1-t}$$ and $$x_t + u_t = v_{m-t}$$ for $$t = 0,\ldots,m + 1.$$ With $$y_t := w_{m-t}$$ we get $$y_t^* u_t = w_{m-t}^T (v_{m-t} - v_{m-t+1})$$ for $$t = 0,\ldots,m + 1,$$ and thus

$$m \sum_{j=0}^m w_j^T (v_j - v_{j+1}) = m \sum_{t=0}^m y_t^* u_t. \quad (A.1)$$

If noting $$y_t \in F(x_{m-t}) = F(x_t + u_t)$$ and $$x_0 = v_{m+1} = v_0 = x_{m+1},$$ cyclo-passivity of (7) allows us to conclude $$\sum_{t=0}^m y_t^* u_t \geq 0,$$ which implies (8) due to (A.1).

Conversely, suppose that $$F$$ is cyclically monotone. Let us pick any trajectory of (7) with $$x_0 = x_{m+1}.$$ If defining $$v_j := x_{m+1-j}$$ and $$w_j := y_{m-j}$$ for $$j = 0,\ldots,m + 1,$$ we have $$w_j^T (v_j - v_{j+1}) = y_{m-j}^T (x_{m+1-j} - x_{m-j}) = y_{m-j}^T u_{m+1-j}$$ for $$j = 0,\ldots,m + 1,$$ giving again (A.1). Since $$w_j = y_{m-j} \in F(x_{m+1-j} + u_{m+1-j}) = F(x_{m-j+1}) = F(v_j)$$ and due to (8), we infer that $$\sum_{t=0}^m y_t^* u_t \geq 0.$$ Since the choice of the round trip trajectory of (7) was arbitrary, (7) is cyclo-passive.

A.2 Proof of Theorem 5

We only need to prove the second statement. Pick any $$u \in U$$ and $$v \in \partial f(u).$$ Then $$0$$ is a subgradient of $$w \mapsto v^T w - f(w)$$ at $$w = u.$$ By Fermat’s principle, $$w = u$$ is optimal for the maximization in the definition of $$f^*(v),$$ which implies

$$f^*(v) = v^T u - f(u) < \infty. \quad (A.2)$$

To show the dissipation inequality, take any $$x \in X^*$$ and pick $$\bar{u} \in \mathbb{R}^n$$ with $$x \in \partial f(\bar{u}).$$ Then the value of $$f^*(x)$$ is finite, and the definition of $$f^*(x)$$ implies $$-f^*(x) \leq f(u) - x^T u.$$ Addition to (A.2) gives $$f^*(v) - f^*(x) \leq u^T (v-x).$$ We have proven that

$$f^*(v) - f^*(x) \leq u^T (v-x)$$ for all $$x \in X^*, u \in U, v \in \partial f(u).$$ This is the local dissipation inequality for (11) on $$X^* \times U.$$

A.3 Proof of Corollary 6

If $$(x, u) \in X \times U,$$ then $$A_{q,x} x + B_{q,u} u = \text{col}(x^2, \ldots, x^v, u)$$ and $$C_{q,x} x + u = u - x^k$$ just by (12). This shows

$$V_k(A_{q,x} x + B_{q,u} u) - V_k(x) = \sum_{j=k+1}^\nu f(x^j) + f(u) - \sum_{j=k}^\nu f(x^j) = f(u) - f(x^k) \leq \partial f(u)^T (u-x^k) = \partial f(u)^T (C_{q,x} x + u),$$

where the inequality follows from the dissipation inequality for (10) with storage function $$f.$$ This proves the first claim.

Similarly, $$(x, u) \in X^* \times U$$ implies that there exist $$u_j \in \mathbb{R}^n$$ with $$x \in \text{col}(\partial f(u_1), \partial f(u_2), \ldots, \partial f(u_\nu)).$$ For any $$u \in U$$ and $$v \in \partial f(u),$$ we get $$A_{q,x} x + B_{q,u} u = \text{col}(x^2, \ldots, x^v),$$ thus

$$V_k^*(A_{q,x} x + B_{q,u} u) - V_k^*(x) = \sum_{j=k+1}^\nu f^*(x^j) + f^*(v) - \sum_{j=k}^\nu f^*(x^j) = f^*(v) - f^*(x^k) \leq u^T (v-x^k) = u^T (C_{q,x} x + v),$$

by relying on the dissipation inequality for (11) with the storage function $$f^*.$$ This completes the proof.

A.4 Proof of Theorem 9

Pick any trajectory of the loop (20)-(21). By the dissipativity condition b) and Theorem 8, there exists some $$\epsilon > 0$$
such that the signals $z$ and $w$ of the loop trajectory filtered by (23) fulfill the dissipation inequality

$$
(\xi_T^{x})^T (X^T W^T) (\xi_T^{x}) + \sum_{t=0}^{T-1} v_t^T P v_t^+ + \epsilon \sum_{t=0}^{T-1} \left( \|x_t\|^2 + \|w_t\|^2 \right) \leq (0, \nu) (X^T W^T) (0, \nu)
$$

for all $T \in \mathbb{N}_0$. By (21), we can make use of (25) to infer

$$
(\xi_T^{x})^T (X^T W^T) (\xi_T^{x}) + \epsilon \sum_{t=0}^{T-1} \left( \|x_t\|^2 \right) \leq x_0^T x_0
$$

for all $T \in \mathbb{N}_0$. With (c) and the standard fact

$$
x_T^T Y x_T = \inf_{x \in \mathbb{R}^n} (\xi_T^{x})^T (X^T W^T) (\xi_T^{x})
$$

on top (to the left) of these are obtained by left (up) shifting this row (column) and filling up the last entries with zeros. This shows that the entries in the $i$-th row for $i = 1, \ldots, \nu + 1$ and (j-th column for $j = 1, \ldots, \nu + 1$) sum up to nonnegative values as well. For these structural reasons, $M_{T_{\nu+1}}$ is double hyperdominant.

Now fix $T \in \mathbb{N}$. Due to (33), the lifted representations of $y = (\psi_\nu(\lambda) \otimes I_d)(z)$ and $\tilde{y} = (\tilde{\psi}_\nu(\tilde{\lambda}) \otimes I_d)(w)$ read as

$$
\begin{bmatrix}
(x_T^y) & (y_T^f) \\
(y_T^f) & (z_T^f)
\end{bmatrix} = \begin{bmatrix}
B^T \otimes I_d \\
D^T \otimes I_d
\end{bmatrix} w_T^T.
$$

If $z \in I_{\nu}^{d_2}$ and $w \in \partial f(z)$, we get for these responses that

$$
\begin{align*}
\sum_{t=0}^{T-1} \left( w_t^T y_t + z_t^T \tilde{y}_t \right) - \mu_T^T(E \otimes I_d) x_T &= (w_T^T) (D^T \otimes I_d) z_T + (z_T^T) (D^T \otimes I_d) w_T - (w_T^T) (B^T \otimes I_d) \gamma_T^T (E \otimes I_d) (B^T \otimes I_d)(z_T^T) = \\
&= (w_T^T) (M_T \otimes I_d) z_T. \quad (A.3)
\end{align*}
$$

Note that this relation motivates the definition of $M_T$ after all. Since $M_T$ is double hyperdominant and since $w_t = \cos(|x_t + z_t| \otimes (z_0, \ldots, \partial f(x_T^\nu)))$, Lemma 2 in Mancera and Safonov (2005) guarantees that (A.3) is nonnegative.

Let us finally recall (32) for the response of the filter (23) to the signals $z$ and $w$ and note that $\xi_T^z = \cos(x_T^\nu, x_T^\nu)$. For the matrix (34) and together with (A.3) added to its transposed version, we get

$$
\begin{align*}
\sum_{t=0}^{T-1} v_t^T P(\nu, \nu) v_t - \mu_T^T Z(E) \xi_T^x &= \\
= \sum_{t=0}^{T-1} v_t^T P(\nu, \nu) v_t - \mu_T^T (E \otimes I_d) x_T - \mu_T^T (E^T \otimes I_d) \tilde{x}_T = \\
&= (w_T^T) (M_T \otimes I_d) z_T + (z_T^T) (M_T \otimes I_d) w_T \geq 0.
\end{align*}
$$

Since $T \in \mathbb{N}$ was arbitrary, the proof is concluded.

A.6 Proof of Corollary 12

Take any $\gamma > \gamma_\nu(\nu, \nu) \geq 0$. For this value of $\gamma$, the LMIs (36) are feasible. Taking Schur complements gives

$$
\frac{1}{\gamma} C^T C < C^T C \Lambda^{-1} C < X - W^T (X^T W^T) ^{-1} W^{-1}
$$

as a consequence of the second and third LMI. For any trajectory of the loop (20)-(21), the fourth one implies $x_0^T x_0 \leq \gamma \|x_0\|^2$. On the one hand, Theorem 9 for $T \to \infty$ implies $\|x_T^T \| \leq \frac{1}{\gamma} \|x_0\|^2$ for some $\epsilon > 0$, which shows stability. On the other hand, Theorem 9 combined with (A.4) also guarantees

$$
\frac{1}{\gamma} \|C x_T \| \leq \gamma \|x_0\|^2
$$

for all $T \in \mathbb{N}$, which shows that $\sum_{t=0}^{T-1} \|C x_t \|^2 \leq \gamma \|x_0\|^2$. Since $\gamma > \gamma_\nu(\nu, \nu)$ was chosen arbitrarily, the proof is concluded.