Two-boundary problems in Euclidean quantum gravity

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Summary. - Recent work in the literature has studied a new set of local boundary conditions for the quantized gravitational field, where the spatial components of metric perturbations, and ghost modes, are subject to Robin boundary conditions, whereas normal components of metric perturbations obey Dirichlet boundary conditions. Such boundary conditions are here applied to evaluate the one-loop divergence on a portion of flat Euclidean four-space bounded by two concentric three-spheres.

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1. - Introduction

After several decades of work by many authors on the problems of quantum gravity [1-11], it seems fair enough to say that the path-integral approach remains an essential ingredient of any attempt to understand the properties of the quantized gravitational field. The crucial point is that quantum mechanics is a physical theory whose predictions are of statistical nature. When one tries to “combine” it with general relativity, one may thus expect to obtain a formalism where statistical concepts as the partition function [12] find a natural place. This is indeed the case for Euclidean field theories. This property is possibly even more important than the opportunity to obtain a space-time covariant approach to quantization, via the sum over suitable classes of (or all) Riemannian four-geometries with their topologies. Moreover, one knows that the effective action provides, in principle, a tool for studying quantum theory as a theory of small disturbances of the underlying classical theory, as well as many non-perturbative properties in field theory [13-15].

The basic object of a space-time covariant formulation of quantum gravity may be viewed as being the path-integral representation of the ⟨out|in⟩ amplitude [13,14], which involves the consideration of ghost fields that reflect the gauge freedom of the classical theory [13,14]. In particular, what seems to emerge is that the consideration of the elliptic boundary-value problems of quantum gravity casts new light on the one-loop semiclassical approximation, which is the “bridge” in between the classical world and the as yet unknown (full) quantum theory [11]. We shall thus focus on this part of the quantum gravity problem, i.e. the boundary conditions on metric perturbations, when a Riemannian four-manifold (say $M$) with boundary is considered (this may be a portion of flat Euclidean four-space, or part of the de Sitter four-sphere, or a more general curved background). To begin, we consider the problem of imposing boundary conditions on the spatial components $h_{ij}$ of metric perturbations. Following ref. [16], we are interested in Robin boundary conditions on $h_{ij}$. They are relevant for the following reasons:
(i) They are part of a set of mixed boundary conditions of local nature which ensure symmetry (and, with some care, self-adjointness) of the elliptic operator acting on metric perturbations [17].

(ii) They admit, as a particular case, the boundary conditions on the linearized magnetic curvature, which have a deep motivation in several branches of classical and quantum gravity [18,19].

For simplicity, we study problems where the background is totally flat, and curvature effects result from the boundary only. All metric perturbations are then expanded on concentric three-spheres of radius $\tau$, with $\tau \in [a, b]$, $a$ and $b$ being the radii of the two bounding three-spheres. This problem lies in between the quantum field-theoretical case, where the boundary surfaces are by no means forced to be three-spheres, and may located in two asymptotic regions, and the quantum cosmological case, where one of the two boundary surfaces shrinks to a point [20]. The Robin-like boundary conditions proposed in ref. [16] read, therefore,

$$\left[ \frac{\partial h_{ij}}{\partial \tau} + \frac{\rho}{\tau} h_{ij} \right] = 0,$$

(1.1)

where $\rho$ is a real parameter. Since an infinitesimal diffeomorphism changes $h_{ij}$ according to the law (hereafter, a vertical stroke denotes three-dimensional covariant differentiation tangentially with respect to the Levi–Civita connection of the boundary, and $K_{ij}$ is the extrinsic-curvature tensor of the boundary)

$$\varphi h_{ij} = h_{ij} + \varphi_{(ij)} + K_{ij} \varphi_0,$$

(1.2)

the request of being able to preserve (1.1) under the transformations (1.2) leads to the following boundary conditions on normal and tangential components of the ghost one-form [16]:

$$\left[ \frac{\partial \varphi_0}{\partial \tau} + \frac{\rho + 1}{\tau} \varphi_0 \right] = 0,$$

(1.3)

$$\left[ \frac{\partial \varphi_i}{\partial \tau} + \frac{\rho}{\tau} \varphi_i \right] = 0.$$

(1.4)
The remaining boundary conditions are of the Dirichlet type on normal components of metric perturbations:

\[ [h_{00}]_{\partial M} = 0, \]  
\[ [h_{0i}]_{\partial M} = 0. \]  

(1.5)  
(1.6)

Regrettably, the invariance of both (1.5) and (1.6) under infinitesimal diffeomorphisms of metric perturbations is incompatible with the boundary conditions (1.3) and (1.4), as was proved in ref. [16]. Thus, we are studying a scheme where only the \( h_{ij} \) sector of the boundary conditions is gauge-invariant. The expansions that we need are [21]

\[ h_{00}(x, \tau) = \sum_{n=1}^{\infty} a_n(\tau)Q^{(n)}(x), \]  
\[ h_{0i}(x, \tau) = \sum_{n=2}^{\infty} \left[ b_n(\tau) \frac{Q^{(n)}_i(x)}{(n^2 - 1)} + c_n(\tau)S^{(n)}_i(x) \right], \]  
\[ h_{ij}(x, \tau) = \sum_{n=3}^{\infty} u_n(\tau) \left[ Q^{(n)}_{ij}(x) + \frac{1}{3} c_{ij}Q^{(n)}(x) \right] + \sum_{n=1}^{\infty} \frac{e_n(\tau)}{3} c_{ij}Q^{(n)}(x) \]

\[ + \sum_{n=3}^{\infty} \left[ f_n(\tau) \left( S^{(n)}_{ij}(x) + S^{(n)}_{ji}(x) \right) + z_n(\tau)G^{(n)}_{ij}(x) \right], \]  

(1.7)  
(1.8)  
(1.9)

where \( x \) are local coordinates on a three-sphere of radius \( \tau \). With a standard notation, \( Q^{(n)}(x), S^{(n)}_i(x) \) and \( G^{(n)}_{ij}(x) \) are scalar, transverse vector and transverse-traceless tensor harmonics on a unit three-sphere, respectively [22].

Section 2 describes the way to implement the \( \zeta \)-function method which is best suited for our analysis of one-loop divergences. Sections 3, 4, 5 and 6 derive in detail the contribution of transverse-traceless, vector, scalar and ghost modes, respectively. Results and open problems are discussed in sect. 7.
2. - ζ-Function method

For a given elliptic operator, say \( A \), the spectral theorem makes it possible to define its complex power \( A^{-s} \), with \( s \in \mathbb{C} \) [11], and the \( L^2 \)-trace of such a power is the generalized \( ζ \)-function for the operator \( A \):

\[
ζ_A(s) ≡ Tr_{L^2}(A^{-s}) = \sum_{λ>0} λ^{-s}.
\]

(2.1)

As is well known, the \( ζ \)-function defined in (2.1) admits an analytic continuation to the complex-\( s \) plane as a meromorphic function which is regular at \( s = 0 \), so that the functional determinant of the operator \( A \) may be defined by the formula [23]

\[
det(A) ≡ e^{-ζ'(0)}.
\]

(2.2)

The value at the origin of the generalized \( ζ \)-function contains all the information about the one-loop divergence and the anomalous scaling factor of the amplitudes [11].

There exist, by now, several powerful algorithms for the evaluation of \( ζ_A(0) \). In particular, we are interested in the technique developed in ref. [24] and applied several times since then (see ref. [11] and references therein). Thus, we say that, denoting by \( f_n \) the function occurring in the equation obeyed by the eigenvalues by virtue of the boundary conditions, after taking out false roots, and writing \( d(n) \) for the degeneracy of the eigenvalues parametrized by the integer \( n \), one defines the function

\[
I(M^2, s) ≡ \sum_{n=n_0}^{∞} d(n)n^{-2s} \log f_n(M^2).
\]

(2.3)

What is crucial is the analytic continuation “\( I(M^2, s) \)” to the complex-\( s \) plane of the function \( I(M^2, s) \), which is a meromorphic function with a simple pole at \( s = 0 \), i.e.

\[
“I(M^2, s)” = \frac{I_{\text{pole}}(M^2)}{s} + I^R(M^2) + O(s).
\]

(2.4)
The function $I_{\text{pole}}$ is the residue at $s = 0$, and makes it possible to obtain the $\zeta(0)$ value as [11,24]
\[ \zeta(0) = I_{\text{log}} + I_{\text{pole}}(M^2 = \infty) - I_{\text{pole}}(M^2 = 0), \tag{2.5} \]
where $I_{\text{log}}$ is the coefficient of the $\log(M)$ term in $I^R$ as $M \to \infty$. The contributions $I_{\text{log}}$ and $I_{\text{pole}}(\infty)$ are obtained from the uniform asymptotic expansions of basis functions as $M \to \infty$ and their order $n \to \infty$, while $I_{\text{pole}}(0)$ is obtained by taking the $M \to 0$ limit of the eigenvalue condition, and then studying the asymptotics as $n \to \infty$. More precisely, $I_{\text{pole}}(\infty)$ coincides with the coefficient of $\frac{1}{n}$ in the asymptotic expansion as $n \to \infty$ of
\[ \frac{1}{2} d(n) \log[\sigma_\infty(n)], \]
where $\sigma_\infty(n)$ is the $n$-dependent term in the eigenvalue condition as $M \to \infty$ and $n \to \infty$. The $I_{\text{pole}}(0)$ value is instead obtained as the coefficient of $\frac{1}{n}$ in the asymptotic expansion as $n \to \infty$ of
\[ \frac{1}{2} d(n) \log[\sigma_0(n)], \]
where $\sigma_0(n)$ is the $n$-dependent term in the eigenvalue condition as $M \to 0$ and $n \to \infty$ [11,24].

3. - Transverse-traceless modes

On using the de Donder gauge-averaging functional:
\[ \Phi_a(h) \equiv \nabla^b \left( h_{ab} - \frac{1}{2} g_{ab} g^{cd} h_{cd} \right), \tag{3.1} \]
the operator on metric perturbations reduces to the Laplacian on symmetric rank-two tensors. Thus, the transverse-traceless (TT) modes in the expansion (1.9) are found to take the form [21]
\[ z_n(\tau) = \alpha_1 \tau I_n(M \tau) + \alpha_2 \tau K_n(M \tau), \tag{3.2} \]
for all $n \geq 3$, the corresponding degeneracy being $2(n^2 - 4)$. The boundary conditions (1.1) lead to the equations

\begin{align*}
\alpha_1 \left( I'_n(M\tau_+) + (\rho + 1) \frac{I_n(M\tau_+)}{M\tau_+} \right) &+ \alpha_2 \left( K'_n(M\tau_+) + (\rho + 1) \frac{K_n(M\tau_+)}{M\tau_+} \right) = 0, \quad (3.3) \\
\alpha_1 \left( I'_n(M\tau_-) + (\rho + 1) \frac{I_n(M\tau_-)}{M\tau_-} \right) &+ \alpha_2 \left( K'_n(M\tau_-) + (\rho + 1) \frac{K_n(M\tau_-)}{M\tau_-} \right) = 0. \quad (3.4)
\end{align*}

This implies that, to get rid of false roots, one has to multiply by $M^2$ the resulting eigenvalue condition; on the other hand, as $M \to \infty$, the eigenvalue condition is proportional to $M^{-1}$. Thus, $I_{\log}$ is found to be

\begin{equation}
I_{\log} = \frac{1}{2} \sum_{n=3}^{\infty} 2(n^2 - 4)(2 - 1) = \zeta_R(-2) - 4\zeta_R(0) + 3 = 5. \quad (3.5)
\end{equation}

Moreover, as $n \to \infty$ and $M \to \infty$, no $n$-dependent term occurs in the eigenvalue condition, which implies

\begin{equation}
I_{\text{pole}}(\infty) = 0. \quad (3.6)
\end{equation}

Last, as $M \to 0$ and $n \to \infty$, the $\sigma_0(n)$ term in the eigenvalue condition reads

\begin{equation}
\sigma_0(n) = n \left( 1 - \frac{(\rho + 1)^2}{n^2} \right), \quad (3.7)
\end{equation}

which implies that no coefficient of $\frac{1}{n}$ occurs in the expansion of $(n^2 - 4) \log \sigma_0(n)$, and hence

\begin{equation}
I_{\text{pole}}(0) = 0. \quad (3.8)
\end{equation}

The results (3.5), (3.6) and (3.8) imply that

\begin{equation}
\zeta_{TT}(0) = 5. \quad (3.9)
\end{equation}

Note that this contribution to $\zeta(0)$ has opposite sign, with respect to the case when $h_{ij}$ perturbations are set to zero at $\tau = \tau_-$ and $\tau = \tau_+$ [21].
4. - Vector modes

In the expansions (1.8) and (1.9) there is a decoupled vector mode, \( c_2(\tau) \), which reads

\[
c_2(\tau) = \varepsilon I_3(M\tau) + \eta K_3(M\tau),
\]

(4.1)

and coupled vector modes, given by [21]

\[
c_n(\tau) = \tilde{\varepsilon}_1 I_{n+1}(M\tau) + \tilde{\varepsilon}_2 I_{n-1}(M\tau) + \eta_1 K_{n+1}(M\tau) + \eta_2 K_{n-1}(M\tau),
\]

(4.2)

\[
f_n(\tau) = \tau \left[ -\frac{1}{(n+2)} \tilde{\varepsilon}_1 I_{n+1}(M\tau) + \frac{1}{(n-2)} \tilde{\varepsilon}_2 I_{n-1}(M\tau)
\right.
\]

\[
- \frac{1}{(n+2)} \eta_1 K_{n+1}(M\tau) + \frac{1}{(n-2)} \eta_2 K_{n-1}(M\tau) \right],
\]

(4.3)

with degeneracy \( 2(n^2 - 1) \). By virtue of (1.1) and (1.6), one has the boundary conditions

\[
c_2(\tau_+ + \tau_-) = 0,
\]

(4.4)

\[
c_n(\tau_+ + \tau_-) = 0 \quad \forall n \geq 3,
\]

(4.5)

\[
\left[ \frac{df_n}{d\tau} + \frac{\rho}{\tau} f_n \right]_{\tau = \tau_+} = \left[ \frac{df_n}{d\tau} + \frac{\rho}{\tau} f_n \right]_{\tau = \tau_-} = 0 \quad \forall n \geq 3.
\]

(4.6)

As is well known, for decoupled (or finitely many) modes, the contribution to \( \zeta(0) \) is given by \( I_{\log} \) only, and for \( c_2 \) this reads

\[
\zeta_{c_2}(0) = \frac{1}{2} 2(4-1)(0-1) = -3,
\]

(4.7)

because no false roots occur in the eigenvalue condition, whereas, as \( n \to \infty \) and \( M \to \infty \), such eigenvalue condition is proportional to \( M^{-1} \), picking up a \( \frac{1}{\sqrt{M}} \) factor from both \( I_3 \) and \( K_3 \).

The eigenvalue condition resulting from the boundary conditions (4.5) and (4.6) for coupled vector modes implies that one has to multiply by \( M^2 \) to get rid of false roots.
Moreover, as \( n \to \infty \) and \( M \to \infty \), there is proportionality to \( M^{-2} \), and hence \( I_{\log} \) is found to vanish:

\[
I_{\log} = \frac{1}{2} \sum_{n=3}^{\infty} 2(n^2 - 1)(2 - 2) = 0. \tag{4.8}
\]

To compute \( I_{\text{pole}}(\infty) \), one first evaluates \( \sigma(\infty) \), which is found to be

\[
\sigma(\infty) = \frac{4n^2}{(n^2 - 4)^2}, \tag{4.9}
\]

and hence

\[
I_{\text{pole}}(\infty) = 0. \tag{4.10}
\]

Last, \( \sigma_0(n) \) is found to be an even function of \( n \)

\[
\sigma_0(n) = \frac{1}{(n^2 - 1)} \left[ \frac{(2n^4 - 8n^2 - 2n^2\rho^2 - 4n^2\rho - 8\rho^2 - 16\rho)}{(n^2 - 4)^2} \right.
+ \left. 2 \frac{(n^2 - \rho^2 - 2 - 2\rho)}{(n^2 - 4)} \right], \tag{4.11}
\]

and hence \( I_{\text{pole}}(0) \) vanishes as well,

\[
I_{\text{pole}}(0) = 0, \tag{4.12}
\]

which implies

\[
\zeta_{c_n,f_n}(0) = 0. \tag{4.13}
\]

5. - Scalar modes

In the expansions (1.7)–(1.9), the scalar modes are \( a_n(\tau), b_n(\tau), u_n(\tau) \) and \( e_n(\tau) \). The modes \( \{a_1(\tau), e_1(\tau)\} \), and \( \{a_2(\tau), b_2(\tau), e_2(\tau)\} \), belong to finite-dimensional subspaces, and read [21]

\[
a_1(\tau) = \frac{1}{\tau} \left[ \gamma_1 I_1(M\tau) + \gamma_4 I_3(M\tau) + \delta_1 K_1(M\tau) + \delta_4 K_3(M\tau) \right], \tag{5.1}
\]

\[
e_1(\tau) = \tau \left[ 3\gamma_1 I_1(M\tau) - \gamma_4 I_3(M\tau) + 3\delta_1 K_1(M\tau) - \delta_4 K_3(M\tau) \right], \tag{5.2}
\]
Moreover, for all \(n \geq 3\), the scalar modes are all coupled, and read [21]

\[
a_n(\tau) = \frac{1}{\tau} \left[ \gamma_1 I_n(\tau) + \gamma_3 I_{n-2}(\tau) + \gamma_4 I_{n+2}(\tau) + \delta_1 K_n(\tau) + \delta_3 K_{n-2}(\tau) + \delta_4 K_{n+2}(\tau) \right],
\]

\[
b_n(\tau) = \gamma_2 I_n(\tau) + (n+1)\gamma_3 I_{n-2}(\tau) - (n-1)\gamma_4 I_{n+2}(\tau) + \delta_2 K_n(\tau) + (n+1)\delta_3 K_{n-2}(\tau) - (n-1)\delta_4 K_{n+2}(\tau),
\]

\[
u_n(\tau) = \tau \left[ -\gamma_2 I_n(\tau) + \frac{(n+1)}{(n-2)}\gamma_3 I_{n-2}(\tau) + \frac{(n-1)}{(n+2)}\gamma_4 I_{n+2}(\tau) - \delta_2 K_n(\tau) + \frac{(n+1)}{(n-2)}\delta_3 K_{n-2}(\tau) + \frac{(n-1)}{(n+2)}\delta_4 K_{n+2}(\tau) \right],
\]

\[
e_n(\tau) = \tau \left[ 3\gamma_1 I_n(\tau) - 2\gamma_2 I_n(\tau) - \gamma_3 I_{n-2}(\tau) - \gamma_4 I_{n+2}(\tau) + 3\delta_1 K_n(\tau) - 2\delta_2 K_n(\tau) - 3\delta_3 K_{n-2}(\tau) - 3\delta_4 K_{n+2}(\tau) \right],
\]

with degeneracy \(n^2\). Of course, it is the choice (3.1) of gauge-averaging functional which leads to full agreement with the formulae found in ref. [21] for the perturbative modes.

For the modes \(a_1(\tau)\) and \(e_1(\tau)\) the boundary conditions resulting from (1.5) and (1.1) are

\[
a_1(\tau_+) = a_1(\tau_-) = 0,
\]

\[
\left[ \frac{de_1}{d\tau} + \frac{\rho}{\tau} e_1 \right]_{\tau=\tau_+] = \left[ \frac{de_1}{d\tau} + \frac{\rho}{\tau} e_1 \right]_{\tau=\tau_-} = 0.
\]
The Eqs. (5.10) and (5.11) lead to an eigenvalue condition where one has to multiply by $M^2$ to get rid of false roots. On the other hand, such eigenvalue condition is proportional to $M^{-2}$ as $M \to \infty$. Thus, the contribution to $\zeta(0)$ is found to vanish:

$$\zeta_{a_1,e_1}(0) = \frac{1}{2}(2 - 2) = 0.$$  \hspace{1cm} (5.12)

The modes $a_2, b_2, e_2$ obey, from sect. 1, the boundary conditions

$$a_2(\tau_+) = a_2(\tau_-) = 0,$$  \hspace{1cm} (5.13)

$$b_2(\tau_+) = b_2(\tau_-) = 0,$$  \hspace{1cm} (5.14)

$$\left[ \frac{de_2}{d\tau} + \frac{\rho}{\tau} e_2 \right]_{\tau=\tau_+} = \left[ \frac{de_2}{d\tau} + \frac{\rho}{\tau} e_2 \right]_{\tau=\tau_-} = 0.$$  \hspace{1cm} (5.15)

In the resulting eigenvalue condition one has to multiply by $M^2$ to get rid of false roots, whereas, as $M \to \infty$, the eigenvalue condition is proportional to $M^{-3}$. This property leads to a non-vanishing contribution to $\zeta(0)$:

$$\zeta_{a_2,b_2,e_2}(0) = \frac{1}{2} \cdot 4(2 - 3) = -2.$$  \hspace{1cm} (5.16)

Coupled scalar modes obey, for all $n \geq 3$, the boundary conditions

$$a_n(\tau_+) = a_n(\tau_-) = 0,$$  \hspace{1cm} (5.17)

$$b_n(\tau_+) = b_n(\tau_-) = 0,$$  \hspace{1cm} (5.18)

$$\left[ \frac{du_n}{d\tau} + \frac{\rho}{\tau} u_n \right]_{\tau=\tau_+} = \left[ \frac{du_n}{d\tau} + \frac{\rho}{\tau} u_n \right]_{\tau=\tau_-} = 0,$$  \hspace{1cm} (5.19)

$$\left[ \frac{de_n}{d\tau} + \frac{\rho}{\tau} e_n \right]_{\tau=\tau_+} = \left[ \frac{de_n}{d\tau} + \frac{\rho}{\tau} e_n \right]_{\tau=\tau_-} = 0.$$  \hspace{1cm} (5.20)

The Eqs. (5.17)–(5.20) lead to an eigenvalue condition expressed by the vanishing of the determinant of an $8 \times 8$ matrix. However, the calculation is considerably simplified if one remarks that, as $M \to \infty$, only $K$ functions at $\tau = \tau_-$ and $I$ functions at $\tau = \tau_+$ give a
non-negligible contribution \[11,21\]. Thus, the desired determinant splits into the product of two determinants, say \(D_1\) and \(D_2\), of \(4 \times 4\) matrices. As \(M \to 0\), \(D_1\) is proportional to \(M^{4n-2}\), and \(D_2\) is proportional to \(M^{-4n-2}\). Thus, one has to multiply by \(M^4\) the full determinant to get rid of false roots. Moreover, both \(D_1\) and \(D_2\) are proportional to \(M^{-2}\) as \(M \to \infty\), and hence the full \(I_{\log}\) vanishes:

\[
I_{\log} = \frac{1}{2} \sum_{n=3}^{\infty} n^2(4 - 4) = 0. \tag{5.21}
\]

To evaluate \(I_{\text{pole}}(\infty)\) and \(I_{\text{pole}}(0)\) we note that, on defining

\[
\kappa \equiv \rho + 1, \tag{5.22}
\]

\[
F_+(n) \equiv I_n'(M\tau_+) + \kappa \frac{I_n(M\tau_+)}{M\tau_+}, \tag{5.23}
\]

\[
G_-(n) \equiv K_n'(M\tau_-) + \kappa \frac{K_n(M\tau_-)}{M\tau_-}, \tag{5.24}
\]

one finds from (5.17)–(5.20) and (5.6)–(5.9) the fundamental formulae

\[
D_1(n) = -\frac{6n}{(n^2 - 4)} F_+(n - 2) F_+(n + 2) I_n^2(M\tau_+) \\
- 3 \frac{(n^2 + 1)}{(n + 2)} F_+(n) F_+(n + 2) I_n(M\tau_+) I_{n-2}(M\tau_+) \\
- 3 \frac{(n^2 + 1)}{(n - 2)} F_+(n) F_+(n - 2) I_n(M\tau_+) I_{n+2}(M\tau_+) \\
- 6n F^2_+(n) I_{n-2}(M\tau_+) I_{n+2}(M\tau_+), \tag{5.25}
\]

\[
D_2(n) = -\frac{6n}{(n^2 - 4)} G_-(n - 2) G_-(n + 2) K_n^2(M\tau_-) \\
- 3 \frac{(n^2 + 1)}{(n + 2)} G_-(n) G_-(n + 2) K_n(M\tau_-) K_{n-2}(M\tau_-) \\
- 3 \frac{(n^2 + 1)}{(n - 2)} G_-(n) G_-(n - 2) K_n(M\tau_-) K_{n+2}(M\tau_-) \\
- 6n G^2_-(n) K_{n-2}(M\tau_-) K_{n+2}(M\tau_-). \tag{5.26}
\]
By virtue of (5.25) and (5.26), the $n$-dependent term $D(n) = D_1(n)D_2(n)$ in the eigenvalue condition as $M \to \infty$ and $n \to \infty$ is

$$\sigma_\infty(n) = \frac{144n^2(n^2 - 1)^2}{(n^2 - 4)^2}. \tag{5.27}$$

This is an even function of $n$, and hence

$$I_{\text{pole}}(\infty) = 0. \tag{5.28}$$

Last, from the limiting form of modified Bessel functions as $M \to 0$, one finds

$$D_1(n) = -\frac{3\Gamma^{-4}(n)(n-1)}{n^3(n+1)(n+2)(n^2-4)} \left[ 4n(n^2-1)(\kappa+n)^2 - 8(n^2+1)(\kappa+n) - 8n \right], \tag{5.29}$$

$$D_2(n) = -\frac{3\Gamma^4(n)n(n+1)}{(n-1)(n-2)(n^2-4)} \left[ 4n(n^2-1)(\kappa-n)^2 + 8(n^2+1)(\kappa-n) - 8n \right]. \tag{5.30}$$

The results (5.29) and (5.30) lead to

$$\sigma_0(n) = D(n) = \frac{9}{n^2(n^2 - 4)^3} H(n), \tag{5.31}$$

where

$$H(n) \equiv 16n^2(n^2 - 1)^2(\kappa^2 - n^2)^2 + 64(n^6 - n^4 - 3n^2 - 1)(\kappa^2 - n^2)$$

$$- 64n^2(n^2 - 1)(\kappa^2 + n^2) + 64n^2(2n^2 + 3), \tag{5.32}$$

which implies

$$I_{\text{pole}}(0) = 0, \tag{5.33}$$

because $\frac{n^2}{2} \log \sigma_0(n)$ is then an even function of $n$. The Eqs. (5.21), (5.28) and (5.33) imply a vanishing contribution to $\zeta(0)$,

$$\zeta_{a_n,b_n,u_n,e_n}(0) = 0. \tag{5.34}$$
6. - Ghost modes

The ghost one-form has a normal component, $\varphi_0(x, \tau)$, and three tangential components, $\varphi_i(x, \tau)$. In our problem, they are expanded on a family of concentric three-spheres according to the relations [21]

$$\varphi_0(x, \tau) = \sum_{n=1}^{\infty} l_n(\tau) Q^{(n)}(x), \quad (6.1)$$

$$\varphi_i(x, \tau) = \sum_{n=2}^{\infty} \left[ m_n(\tau) \frac{Q^{(n)}_i(x)}{(n^2 - 1)} + p_n(\tau) S^{(n)}_i(x) \right]. \quad (6.2)$$

By virtue of (3.1), the ghost operator reduces, in flat space, to $-g_{ab}\Box$, and hence one finds [21]

$$l_1(\tau) = \frac{1}{\tau} \left[ \kappa_1 I_2(M\tau) + \theta_1 K_2(M\tau) \right], \quad (6.3)$$

$$l_n(\tau) = \frac{1}{\tau} \left[ \kappa_1 I_{n+1}(M\tau) + \kappa_2 I_{n-1}(M\tau) + \theta_1 K_{n+1}(M\tau) + \theta_2 K_{n-1}(M\tau) \right], \quad (6.4)$$

$$m_n(\tau) = -(n - 1)\kappa_1 I_{n+1}(M\tau) + (n + 1)\kappa_2 I_{n-1}(M\tau) - (n - 1)\theta_1 K_{n+1}(M\tau) + (n + 1)\theta_2 K_{n-1}(M\tau), \quad (6.5)$$

$$p_n(\tau) = \theta_1 I_n(M\tau) + \theta_2 K_n(M\tau). \quad (6.6)$$

By virtue of (1.3), the decoupled ghost mode $l_1(\tau)$ obeys the boundary conditions

$$\left[ \frac{dl_1}{d\tau} + \frac{\rho + 1}{\tau} l_1 \right]_{\tau=\tau_+} = \left[ \frac{dl_1}{d\tau} + \frac{\rho + 1}{\tau} l_1 \right]_{\tau=\tau_-} = 0. \quad (6.7)$$

The resulting eigenvalue condition is

$$\left[ \frac{I'_2(M\tau_+)}{M\tau_+} + \frac{\rho}{M^2\tau_+^2} I_2(M\tau_+) \right] \left[ \frac{K'_2(M\tau_-)}{M\tau_-} + \frac{\rho}{M^2\tau_-^2} K_2(M\tau_-) \right] = 0. \quad (6.8)$$
Hence one has to multiply by $M^4$ to get rid of false roots, whereas the behaviour of (6.8) as $M \to \infty$ is proportional to $M^{-3}$. This leads to

$$
\zeta_{l_1}(0) = \frac{1}{2}(4 - 3) = \frac{1}{2}.
$$

(6.9)

Coupled ghost modes are $l_n$ and $m_n$, for all $n \geq 2$. In the light of (1.3) and (1.4), they obey the boundary conditions

$$
\left[ \frac{dl_n}{d\tau} + \frac{(\rho + 1)}{\tau} l_n \right]_{\tau=\tau_+} = \left[ \frac{dl_n}{d\tau} + \frac{(\rho + 1)}{\tau} l_n \right]_{\tau=\tau_-} = 0,
$$

(6.10)

$$
\left[ \frac{dm_n}{d\tau} + \frac{\rho}{\tau} m_n \right]_{\tau=\tau_+} = \left[ \frac{dm_n}{d\tau} + \frac{\rho}{\tau} m_n \right]_{\tau=\tau_-} = 0.
$$

(6.11)

In the resulting eigenvalue condition, one has to multiply by $M^4$ to get rid of false roots, whereas the behaviour as $M \to \infty$ is proportional to $M^{-2}$, which implies

$$
I_{\log} = \frac{1}{2} \sum_{n=2}^{\infty} n^2(4 - 2) = \zeta_R(-2) - 1 = -1.
$$

(6.12)

When $n \to \infty$ and $M \to \infty$, the term $\sigma_{\infty}(n)$ in the eigenvalue condition is $4n^2$, and hence

$$
I_{\text{pole}}(\infty) = 0.
$$

(6.13)

Moreover, when $M \to 0$ and $n \to \infty$, the term $\sigma_0(n)$ in the eigenvalue condition is

$$
\sigma_0(n) = \frac{4n^2}{(n^2 - 1)} \left[ (\rho^2 + n^2 - 1)^2 - 4n^2\rho^2 \right].
$$

(6.14)

This is an even function of $n$, which implies

$$
I_{\text{pole}}(0) = 0,
$$

(6.15)

and, from (6.12), (6.13) and (6.15),

$$
\zeta_{l_n, m_n}(0) = -1.
$$

(6.16)
Last, ghost vector modes obey, by virtue of (1.4), the eigenvalue condition

\[
\left[ I'_n(M\tau_+) + \frac{\rho}{M\tau_+} I_n(M\tau_+) \right] \left[ K'_n(M\tau_-) + \frac{\rho}{M\tau_-} K_n(M\tau_-) \right] = 0. \tag{6.17}
\]

The resulting false roots are eliminated upon multiplication by \(M^2\), whereas the behaviour of (6.17) as \(n \to \infty\) and \(M \to \infty\) is proportional to \(M^{-1}\), which implies

\[
I_{\log} = \frac{1}{2} \sum_{n=2}^{\infty} 2(n^2 - 1)(2 - 1) = \zeta(-2) - \zeta(0) = \frac{1}{2}. \tag{6.18}
\]

The Eq. (6.17) has no \(n\)-dependent term when \(n \to \infty\) and \(M \to \infty\), so that

\[
I_{\text{pole}}(\infty) = 0. \tag{6.19}
\]

Last, as \(M \to 0\) and \(n \to \infty\), one finds

\[
\sigma_0(n) = \frac{1}{n} (\rho^2 - n^2), \tag{6.20}
\]

and this leads to

\[
I_{\text{pole}}(0) = 0, \tag{6.21}
\]

\[
\zeta_p(0) = \frac{1}{2}. \tag{6.22}
\]

The full \(\zeta(0)\) value is the sum of the 9 contributions given by Eqs. (3.9), (4.7), (4.13), (5.12), (5.16), (5.34), (6.9), (6.16) and (6.22), i.e.

\[
\zeta(0) = 5 - 3 - 2 - 2 \left( \frac{1}{2} - 1 + \frac{1}{2} \right) = 0, \tag{6.23}
\]

where the round bracket is multiplied by \(-2\) because ghost fields for gravitation are fermionic and complex. Our result agrees completely with the result expected for all two-boundary problems in the presence of a totally flat Euclidean background (see the discussion in ref. [21]).
7. - Concluding remarks

The contribution of our paper is a detailed evaluation of the one-loop divergence for the quantized gravitational field, by studying all perturbative modes which contribute to the one-loop Faddeev–Popov amplitude on a portion of flat Euclidean four-space bounded by two concentric three-spheres. The boundary conditions used are (1.1) and (1.3)–(1.6), first proposed by the authors in ref. [16]. Although a vanishing one-loop divergence might have been expected on general ground, since the background is totally flat, and only the gravitational field is considered, the technical aspects of our analysis remain of some interest. As has been shown in refs. [25,26], completely gauge-invariant boundary conditions in Euclidean quantum gravity are in fact incompatible with the request of strong ellipticity of the boundary-value problem. This is a technical condition, which amounts to requiring that a unique solution should exist of the eigenvalue equation for the leading symbol of the operator of Laplace type on metric perturbations, subject to the boundary conditions and to an asymptotic condition [25,26]. If this uniqueness fails to hold, it is no longer possible to have a well defined form of one-loop divergences, because the heat-kernel diagonal acquires a part which is not integrable near the boundary [26].

Thus, the consideration of boundary conditions which are not completely invariant under infinitesimal diffeomorphisms on metric perturbations acquires new interest, since the lack of tangential derivatives in the boundary operator makes it then possible to satisfy the condition of strong ellipticity of the boundary-value problem [25,26]. As far as we can see, at least three outstanding problems should be now considered:

(i) Local boundary conditions along the lines of (1.1) and (1.3)–(1.6) for curved backgrounds, with one or two boundary surfaces.

(ii) The effect of the Prentki gauge for gravitation on manifolds with boundary [27]. The resulting operator on metric perturbations is no longer of Laplace type, and the corresponding form of heat-kernel asymptotics on manifolds with boundary is largely unexplored.

(iii) Inclusion of boundary operators of the integro-differential type. For example, non-local boundary conditions for the Laplace operator have been studied within the framework
of Bose–Einstein condensation models [28]. The counterpart for the gravitational field remains unknown, but could be studied by using the powerful tools of functional calculus for pseudo-differential boundary problems [29].

∗ ∗ ∗

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REFERENCES

[1] DeWITT B. S., Rev. Mod. Phys., 29 (1957) 377.
[2] MISNER C. W., Rev. Mod. Phys., 29 (1957) 497.
[3] WHEELER J. A., Geometrodynamics (Academic Press, New York) 1962.
[4] FEYNMAN R. P., Acta Phys. Polonica, 24 (1963) 697.
[5] DeWITT B. S., Phys. Rev., 160 (1967) 1113.
[6] DeWITT B. S., Phys. Rev., 162 (1967) 1195.
[7] HAWKING S. W., in General Relativity, an Einstein Centenary Survey, eds. S. W. Hawking and W. Israel (Cambridge University Press, Cambridge) 1979.
[8] ROVELLI C. and SMOLIN L., Nucl. Phys. B, 331 (1990) 80.
[9] ASHTEKAR A., Lectures on Non-Perturbative Canonical Gravity (World Scientific, Singapore) 1991.
[10] GIBBONS G. W. and HAWKING S. W., Euclidean Quantum Gravity (World Scientific, Singapore) 1993.
[11] ESPOSITO G., KAMENSHCHIK A. Yu. and POLLIFRONE G., Euclidean Quantum Gravity on Manifolds with Boundary, in Fundamental Theories of Physics, Vol. 85 (Kluwer, Dordrecht) 1997.
[12] GIBBONS G. W. and HAWKING S. W., Phys. Rev. D, 15 (1977) 2752.
[13] DeWITT B. S., Dynamical Theory of Groups and Fields (Gordon and Breach, New York) 1965.
[14] DeWITT B. S., in Relativity, Groups and Topology II, eds. B. S. DeWitt and R. Stora (North-Holland, Amsterdam) 1984.
[15] JONA-LASINIO G., *Nuovo Cimento*, **34** (1964) 1790.

[16] ESPOSITO G. and KAMENSHCHIK A. Yu., *Class. Quantum Grav.*, **12** (1995) 2715.

[17] AVRAMIDI I. G., ESPOSITO G. and KAMENSHCHIK A. Yu., *Class. Quantum Grav.*, **13** (1996) 2361.

[18] HAWKING S. W., *Phys. Lett. B*, **126** (1983) 175.

[19] ESPOSITO G., *Quantum Gravity, Quantum Cosmology and Lorentzian Geometries*, in *Lecture Notes in Physics, New Series m: Monographs*, Vol. **m12** (Springer-Verlag, Berlin) 1994.

[20] HARTLE J. B. and HAWKING S. W., *Phys. Rev. D*, **28** (1983) 2960.

[21] ESPOSITO G., KAMENSHCHIK A. Yu., MISHAKOV I. V. and POLLIFRONE G., *Phys. Rev. D*, **50** (1994) 6329.

[22] LIFSHITZ E. M. and KHALATNIKOV I. M., *Adv. Phys.*, **12** (1963) 185.

[23] HAWKING S. W., *Commun. Math. Phys.*, **55** (1977) 133.

[24] BARVINSKY A. O., KAMENSHCHIK A. Yu. and KARMAZIN I. P., *Ann. Phys. (N.Y.)*, **219** (1992) 201.

[25] AVRAMIDI I. G. and ESPOSITO G., *Class. Quantum Grav.*, **15** (1998) 1141.

[26] AVRAMIDI I. G. and ESPOSITO G., *Gauge Theories on Manifolds with Boundary* ([hep-th 9710048](https://arxiv.org/abs/9710048)).

[27] ‘t HOOFT G. and VELTMAN G., *Ann. Inst. Henri Poincaré*, **20** (1974) 69.

[28] SCHRÖDER M., *Rep. Math. Phys.*, **27** (1989) 259.

[29] GRUBB G., *Functional Calculus of Pseudodifferential Boundary Problems*, in *Progress in Mathematics*, Vol. **65** (Birkhäuser, Boston) 1996.