Automorphisms in loop quantum gravity

Benjamin Bahr\textsuperscript{1} and Thomas Thiemann\textsuperscript{1,2}

\textsuperscript{1} MPI für Gravitationsphysik, Albert-Einstein Institut, Am Mühlenberg 1, 14467 Golm, Germany
\textsuperscript{2} Perimeter Institute for Theoretical Physics, 31 Caroline St N., Waterloo Ontario N2 L 2Y5, Canada

E-mail: bbahr@aei.mpg.de and thomas.thiemann@aei.mpg.de

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Abstract
We investigate a certain distributional extension of the group of spatial diffeomorphisms in loop quantum gravity. This extension, which is given by the automorphisms $\text{Aut}(\mathcal{P})$ of the path groupoid $\mathcal{P}$, was proposed by Velhinho and is inspired by category theory. These automorphisms have much larger orbits than piecewise analytic diffeomorphisms. In particular, we will show that graphs with the same combinatorics but different generalized knotting classes can be mapped into each other. We describe the automorphism-invariant Hilbert space and comment on how a combinatorial formulation of LQG might arise.

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1. Introduction

Loop quantum gravity is an attempt to quantize the theory of general relativity (see [1–4] and references therein). This is done by casting classical GR into a hamiltonian formulation, in which it becomes a constrained theory with fields on a Cauchy hypersurface $\Sigma$. These fields are an $\mathfrak{su}(2)$-connection $A^I_a(x)$ (the Ashtekar connection) and its canonically conjugate $E^I_a(x)$, which is the analogue of the electric field in $SU(N)$-Yang–Mills theory. Apart from physical constants, which can be absorbed into the definition of $E^I_a(x)$, the Poisson structure reads

$$\{ A^I_a(x), E^b_J(y) \} = \delta^b_a \delta^I_J \delta(x,y).$$

(1.1)

The connection $A$ plays the rôle of the configuration variable, and $E$ is the canonically conjugate momentum. The classical configuration space is the space of smooth connections $\mathcal{A}$. By going over to the quantum regime, the space $\mathcal{A}$ is extended to $\mathcal{A}$, by adding also connections which are ‘distributional’ in the sense that they can e.g. have support on lower dimensional submanifolds of $\Sigma$, or arise as certain limits of smooth connections\textsuperscript{3}. The space $\mathcal{A}$ can be endowed with a topology that turns it into a compact Hausdorff space, and a regular, normalized Borel measure $\mu_{\mathcal{A}}$, the \textit{Ashtekar–Isham–Lewandowski measure}. The quantum

\textsuperscript{3} See [4, 5] for explicit examples.
theory is then formulated in the kinematical Hilbert space $\mathcal{H}_{\text{kin}} = L^2(\mathcal{A}, d\mu_{\text{AL}})$. The space $\mathcal{A}$ is topologically dense in $\mathcal{A}$, but measure theoretically thin: $\mathcal{A}$ is contained in a set of measure zero [6].

This is analogous to the case of quantum field theory: consider the case of the Euclidean free scalar field. Then the path integral formalism proposes to write the partition function $Z([J])$ as an integral over all field configurations

$$Z[J] = Z[0] \int D\phi \ e^{-\int (\frac{1}{2}m^2 + iJ \phi) \phi}$$

(1.2)

using the abstract Wiener measure. In this case, the integration ranges over a set of field configurations $\phi$, in which the smooth fields lie dense. The set of these fields, however, are contained in a set of measure zero [7].

In the Hamiltonian formulation of GR employed in loop quantum gravity, the dynamics is contained in the constraint functions, which are phase space functions that generate a Hamiltonian flow on the constraint hypersurface. Two of the constraints, the Gauss and the diffeomorphism constraint, encode the invariance of the theory under change of $SU(2)$-gauge and diffeomorphisms on $\Sigma$. Consequently, their Hamiltonian flows generate gauge transformations and diffeomorphisms on $\Sigma$, respectively. Classically, the corresponding group of smooth gauge transformations $G$ and diffeomorphisms $\text{Diff}(\Sigma)$ act on $\mathcal{A}$:

$$\alpha_g : \mathcal{A} \longrightarrow \mathcal{A}, \quad g \in G,$$

$$\alpha_\phi : \mathcal{A} \longrightarrow \mathcal{A}, \quad \phi \in \text{Diff}(\Sigma).$$

(1.3)

The action of the groups $G$ and $\text{Diff}(\Sigma)$ can be easily extended to the quantum configuration space $\mathcal{A}$. But also the groups themselves can be extended to groups of generalized gauge transformations and generalized diffeomorphisms, $\mathcal{G}$ and $\mathcal{D}(\Sigma)$, to give

$$\alpha_g : \mathcal{A} \longrightarrow \mathcal{A}, \quad g \in \mathcal{G},$$

$$\alpha_\phi : \mathcal{A} \longrightarrow \mathcal{A}, \quad \phi \in \mathcal{D}(\Sigma).$$

(1.4)

The group of generalized gauge transformation $\mathcal{G}$ is taken to be the group of all maps from $\Sigma$ to $SU(2)$, i.e.

$$\mathcal{G} = SU(2)^\Sigma.$$  

(1.5)

Although this group is much larger than the group of smooth gauge transformations within classical GR (and in fact it does not leave $\mathcal{A}$ invariant), in the quantum theory this arises as a natural candidate for the extension of the smooth gauge transformations, since $\mathcal{A}/G \simeq \mathcal{A}/\mathcal{G}$.

There are reasons to believe that also the diffeomorphisms have to be extended: first, the Hilbert space $\mathcal{H}_{\text{diff}}$ of states invariant under diffeomorphisms is not separable and contains many degrees of freedom (the so-called moduli) that are believed to be unphysical [4, 8]. Also, there are other physical reasons to believe that the group of diffeomorphisms is too small [9].

For an extension of the diffeomorphisms $\text{Diff}(\Sigma)$ to $\mathcal{D}(\Sigma)$, several suggestions have been made.

In [8], it was shown that already a slight extension of the group of diffeomorphisms gives rise to a separable diff-invariant Hilbert space. In [10], Ashtekar and Lewandowski discussed $C^\infty$-diffeomorphisms on $\Sigma$, which are analytic except for lower dimensional subsets of $\Sigma$. The proof of the uniqueness of the diffeomorphism-invariant state $\omega_{\text{AL}}$ on the holonomy-flux algebra for loop quantum gravity uses these for $n \geq 1$ [11]. In [9], the stratified diffeomorphisms, introduced earlier by Fleischhack, have been investigated. In [12] the piecewise analytic diffeomorphisms have been introduced, which are bijections on $\Sigma$ that leave the set of analytical graphs $\Gamma$ invariant. In [13], the graphomorphisms extended this concept to the smooth and other categories.
In [14, 15], it was displayed how the basic ingredients of loop quantum gravity can be formulated naturally as concepts of category theory, i.e. as morphisms, functors and natural transformations. In this language, the connections arise as functors from the path groupoid $\mathcal{P}$ of $\Sigma$ to the suspension of the gauge group $\text{Susp}(SU(2))$, and the generalized gauge transformations are in one-to-one correspondence to the natural transformations of these functors. Furthermore, the diffeomorphisms acting on $\Sigma$ can be naturally interpreted as elements in the automorphism group $\text{Aut}(\mathcal{P})$, i.e. as invertible functors from $\mathcal{P}$ to itself. Velhinho pointed out that in the light of category language, $\text{Aut}(\mathcal{P})$ arises as a candidate for an extension of the diffeomorphisms $\text{Diff}(\Sigma)$, and this extension appears to be natural, at least from the mathematical point of view.

In this paper, we will investigate the consequences of choosing $\text{Diff}(\Sigma) = \text{Aut}(\mathcal{P})$. The automorphisms $\phi \in \text{Aut}(\mathcal{P})$ are invertible functors on the path groupoid $\mathcal{P}$, i.e. they permute points in $\Sigma$, and also the paths between them in a consistent way. We will, however, encounter elements in $\text{Aut}(\mathcal{P})$ that cannot be interpreted as bijections of $\Sigma$. By this, the elements in $\text{Aut}(\mathcal{P})$ will also be able to map graphs into each other that have the same combinatorics, but lie in different generalized knotting classes. By this, a combinatorial picture emerges, which is a desirable feature for a quantum theory of gravity [4, 17, 18].

The emphasis of this paper lies on two topics: first, we will prove that the automorphisms $\text{Aut}(\mathcal{P})$ leave the Ashtekar–Isham–Lewandowski measure $\mu_{\text{AL}}$ invariant, and hence have a well-defined unitary action on the kinematical Hilbert space $\mathcal{H}_{\text{kin}} = L^2(\mathcal{A}, d\mu_{\text{AL}})$. Second, we will have a closer look at the automorphisms and the orbits of its action on $\mathcal{H}_{\text{kin}}$, in order to describe the automorphism-invariant Hilbert space.

We will start in section 2 by reviewing the basic concepts of loop quantum gravity, with emphasis on the categorial formulation, and for the general gauge group $G$. We introduce the concept of a (primitive) metagraph, which will be useful in the investigation of the automorphisms $\text{Aut}(\mathcal{P})$. Section 3 will be devoted to the proof that the automorphisms in fact leave the Ashtekar–Lewandowski measure invariant, where some of the more technical mathematical labour will be transferred to the appendix. We will continue by presenting two kinds of nontrivial automorphisms in section 4, which will both be not induced by a bijection on $\Sigma$, but are most useful in what follows. In particular, with the help of these automorphisms we will be able to show that any two graphs (in fact, hyphs) with the same combinatorics can be mapped into each other by an automorphism. It is in particular this fact which suggests that by using the automorphisms, a combinatorial picture emerges.

In section 5 we will investigate the orbits of vectors in $\mathcal{H}_{\text{kin}}$ under the action of $\text{Aut}(\mathcal{P})$. For a certain choice of rigging map, we will define the Hilbert space of vectors invariant under the action of $\text{Aut}(\mathcal{P})$. For the case of Abelian loop quantum gravity, i.e. $G = U(1)$, an explicit orthonormal basis will be given. For $G = SU(2)$, we will comment on how to obtain such a basis.

In the appendix, we will, after briefly presenting notions from category theory and combinatorial group theory, present a way of how to write the exponentiated fluxes in category language, and present a categorial version of the Weyl- and the holonomy-flux algebra.

2. Basics

Loop quantum gravity rests on the observation that instead of knowing a $g$-connection $A_a$, it is equivalent to know, for every possible path $p$, its parallel transport $A(p) = \mathcal{P} \exp \int_p A_a \, dx^a$ along $p$, which is an element of the gauge group $G$. These parallel transports, or holonomies,
Table 1. A table summing up the most important properties of curves, ways, paths and edges. Note that although the range \( r(...) \) is not well defined for paths in general, it is well defined for the edges \( e \in \mathcal{P} \).

| Concept | Symbol | Space | Equivalence relation | Category? | Groupoid? | \( r(...) \) well-defined? |
|---------|--------|-------|----------------------|-----------|-----------|--------------------------|
| Curves  | \( c \) | \( \mathcal{C} \) | None | No | No | Yes |
| Ways    | \( w = [c] \) | \( \tilde{\mathcal{C}} \) | Reparametrizations | Yes | No | Yes |
| Paths   | \( p = [[w]] = [[[c]]] \) | \( \mathcal{P} \) | Reparametrizations and retracings | Yes | Yes (in general) | No |

Table 2. Different types of paths, the definition of which will be convenient throughout the paper.

| Name   | Symbol | Defining properties | Invariant under |
|--------|--------|---------------------|-----------------|
| Path   | \( p \) | Morphism in \( \mathcal{P} \) | Automorphisms |
| Edge   | \( e \) | Without self-intersection, has analytical representative \( c \in \mathcal{C} \) | Analytic diffeomorphisms |
| Loop   | \( l \) | Without self-intersection, with \( s(l) = t(l) \) | Graphomorphisms |
| Simple loop | \( l \) | Loop \( l \) with \( l = \tilde{l} \Rightarrow n = \pm 1 \) | Graphomorphisms |

Table 3. A summary of the different collections of paths. Here \( \text{Diff}^\omega(\Sigma) \) denotes the analytic diffeomorphisms on \( \Sigma \), and \( \text{Gr}(\Sigma) \) denotes the group of graphomorphisms on \( \Sigma \) (see [5]). Note that, unlike with graphs and metagraphs, the order of the \( p_k \) in a hyph \( v \) is important, which is why it is noted as an \( H \)-tuple rather than a set.

| Name    | Symbol | Space  | Invariant under | Consists of |
|---------|--------|--------|-----------------|-------------|
| Graph   | \( \gamma = \{e_1, \ldots, e_k\} \) | \( \Gamma \) | \( \text{Diff}^\omega(\Sigma) \) | Edges |
| Hyph    | \( v = (p_1, \ldots, p_H) \) | \( \Upsilon \) | \( \text{Gr}(\Sigma) \) | Paths |
| Metagraph | \( \mu = \{p_1, \ldots, p_M\} \) | \( \mathcal{M} \) | \( \text{Aut}(\mathcal{P}) \) | Paths |

form the configuration space \( \mathcal{A} \) of the classical theory, which is extended to \( \mathcal{A} \), the set of generalized connections, in the quantum theory. The close relation to lattice gauge theory is rooted in this formalism.

2.1. Curves, ways and paths

In the following section we will review the basic notions to deal with the set of generalized connections, as well as their categorical formulation\(^5\). The definitions and their respective properties are summarized in tables 1, 2 and 3 to provide an overview of the different concepts. For a more detailed mathematical treatment and the omitted proofs, see e.g. [4, 19, 20]. Also, we will restrict ourselves to the category of piecewise analytic paths, which is usually employed in loop quantum gravity. In all what follows, \( \Sigma \) will denote an analytic, connected, closed manifold of dimension \( n > 2 \).

\(^5\) We follow the notation of [4], where the concatenation of, say, two curves \( c_1 \) and \( c_2 \) is denoted as \( c_1 \circ c_2 \), rather than \( c_2 \circ c_1 \), which is usually employed in category theory texts.
Definition 2.1. Let $\Sigma$ be an analytic, connected, closed manifold of dimension $n > 2$. A (piecewise analytic) curve in $\Sigma$ is a map $c : [0, 1] \to \Sigma$, such that there exist $0 = t_0 < t_1 < \cdots < t_n = 1$, so that $c$ restricted to $[t_k, t_{k+1}]$ is an analytic embedding. Denote the set of all curves as $C$. Write

- $s(c) := c(0)$
- $t(c) := c(1)$
- $r(c) := c([0, 1]) \subset \Sigma$.

Some curves can be concatenated, and all of them can be inverted:

Definition 2.2. Let $c_1, c_2 \in C$ such that $t(c_1) = s(c_2)$. Then

$$c_1 \circ c_2(t) := \begin{cases} c_1(2t) & t \in [0, \frac{1}{2}] \\ c_2(2t - 1) & t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

defines an element in $C$, which is called the product of $c_1$ and $c_2$. Furthermore, for any $c \in C$:

$$c^{-1}(t) := c(1 - t)$$

also defined an element in $c^{-1} \in C$ which is called the inverse of $c$.

All this endows $C$ with some structure, but so far the multiplication $\circ$ is not associative.

Definition 2.3. Define an equivalence relation on curves: $c_1 \sim c_2$ if there is a piecewise analytic, monotonically increasing function $\varphi : [0, 1] \to [0, 1]$ such that $c_1(t) = c_2(\varphi(t))$. So two curves $c_1, c_2$ are equivalent if and only if there is a sequence of curves $d_1, \ldots, d_k \in C$ such that $d_k \sim d_{k+1}$ in the above sense and $d_1 = c_1, d_k = c_2$.

The set of all equivalence classes $[c]$ is called the set of ways, and is denoted by $\hat{C}$. For $w = [c] \in \hat{C}$, $s(w) := s(c), t(w) := t(c)$ and $r(w) := r(c)$ are well defined.

The set $\hat{C}$ carries the structure of a category in the following way: the objects of $\hat{C}$ are points in $\Sigma$: $|\hat{C}| = \Sigma$. For $x, y \in \Sigma$ the set of morphisms $\text{Mor}_\hat{C}(x, y)$ is given by the set of all ways $w$ starting at $x$ and ending at $y$, i.e. $s(w) = x, t(w) = y$. The concatenation $[c_1] \circ [c_2] := [c_1 \circ c_2]$ is well defined and associative, since ways do not depend on a parametrization. Also $[c]^{-1} := [c^{-1}]$ defines an involution on $\hat{C}$. For each $x \in \Sigma$ the identity $\text{id}_x$ is given by the equivalence class $[c]$ of the constant curve $c(t) = x$.

Definition 2.4. Let $x, y \in \Sigma$. On the elements of $\text{Mor}_\hat{C}(x, y)$ define an equivalence relation by the following: let $w_1 \approx w_2$ if there are ways $v, v_1, v_2$ such that $v_1 = v_1 \circ v_2$ and $w_2 = v \circ v^{-1} \circ v_2$. This generates an equivalence relation $\approx$ on $\text{Mor}_\hat{C}(x, y)$. The set of equivalence classes $[[w]]$ are called paths. The set of paths is denoted by $\mathcal{P}$.

The category which is defined by $|\mathcal{P}| := |\hat{C}|$ and $\text{Mor}_\mathcal{P}(x, y) := \text{Mor}_\hat{C}(x, y)/\approx$, as well as $[[w]] \circ [[v]] := [[w \circ v]]$ is called the path groupoid, and will also be denoted by $\mathcal{P}$.

Usually, $\mathcal{P}$ is defined by defining an equivalence relation $\approx$ on $\hat{C}$ by combining the two equivalence relations above, and directly go from the curves $c$ to their equivalence classes $p = [[[c]]]$. However, in this work it will turn out to be more convenient at some points to work with the category $\hat{C}$, so we also define it here.

Also note that $s([[w]]) := s(w)$ and $t([[w]]) := t(w)$ can also be defined in $\mathcal{P}$, but $r([[w]]) = r(w)$ is not well defined. One can define

$$r(p) := \bigcup_{p=[[w]]} r(w), \quad (2.3)$$
but this has the property that \( r(p \circ q) \neq r(p) \cup r(q) \). But \( r(p \circ q) = r(p) \cup r(q) \) is a desirable property if one wants to speak about 'points \( x \in \Sigma \) lying on \( p \). It is this particular fact that makes the set of automorphisms of \( P \) much larger than the set of automorphisms of \( C \), as we see in detail later on.

**Definition 2.5.** Let \( p \) be a path in \( \Sigma \). If there is a representative \( c \in C \) with \( p = [[c]] \), such that from \( c(t) = c(t') \) it follows that either \( t = t' \) or \( t = 0 \) or \( t = 1 \), \( t' = 0 \), then \( p \) is said to have no self-intersections.

Note that for paths without self-intersections it is easier to talk about points lying on that path. Let \( c \) be such a representative of such a path, then \( r(p) := r(c) \) is well defined and gives the same result as (2.3). We will use this notion in some constructions later.

**Definition 2.6.** A path \( e \in P \) is called an edge if there is a representative \( c \in Cyl \) with \( e = [[[c]]] \), such that \( c \) is analytic, and \( e \) has no self-intersections.

The edges are special elements in \( P \). They are the equivalence classes of analytic curves that do not self-intersect (unless they start and end at the same point). By definition, every path \( p \) can be decomposed into finitely many edges, which will be crucial for the rest of the work.

Edges are the key to gain access to the analytic structure of loop quantum gravity.

**Definition 2.7.** Let \( \gamma = \{e_1, \ldots, e_E\} \) be a set of edges in \( P \), such that the following holds: for each \( e_k \) there is a representative \( c_k \in C \) with \( e_k = [[[c_k]]] \), such that \( c_k \) mutually intersect at most in their beginning or endpoints. Then \( \gamma \) is called a graph. Denote the set of all graphs by \( \Gamma \).

**Lemma 2.1.** Given two graphs, \( \gamma_1, \gamma_2 \), such that each edge in \( \gamma_1 \) is a finite product of edges and their inverses of \( \gamma_2 \). Then one writes \( \gamma_1 \leq \gamma_2 \). This defines a partial ordering, in particular for any two graphs \( \gamma_1, \gamma_2 \) there is a graph \( \gamma_3 \) such that \( \gamma_1 \leq \gamma_3 \) and \( \gamma_2 \leq \gamma_3 \).

This lemma has an important corollary: given any finite number of paths \( \{p_1, \ldots, p_n\} \) in \( P \), there is always a graph \( \gamma \) such that each \( p_k \) is a product of edges (and their inverses) in \( \gamma \). For this to hold, the piecewise analyticity of the curves in \( C \) is essential. In particular, the same does not hold if one drops the condition of analyticity and works in the smooth category. However, one can work with the so-called webs, or hyphs, which can also be used in this context and which generalize the concept of graphs. In the analytic category, the definition of hyphs is as follows.

**Definition 2.8.** Let \( v = (p_1, \ldots, p_n) \) be a finite sequence of paths with the following property.

For each \( k \in \{1, \ldots, n\} \) the path \( p_k \) has a segment which is free with respect to the \( p_l, l \in \{1, \ldots, k-1\} \). This means that given a graph \( \gamma \) such that each \( p_k \) is a product of edges in \( \gamma \) and their inverses, then in the decomposition \( p_k = e_{k_1} \circ \cdots \circ e_{k_{m_k}} \), there is one edge \( e_{k_{l}} \) (or its inverse) which appears exactly once, and which does not appear in the decompositions of all \( p_l, l \in \{1, \ldots, k-1\} \).

Denote the set of all hyphs by \( \Upsilon \).

The set of hyphs is also partially ordered. Since all graphs are also hyphs, this is trivial in the analytic category, but stays true in other cases, such as the smooth category as well [20]. In the original definition of a hyph, note the additional condition that each \( p_k \) has no self-intersections. This is a much stronger condition, but we will need the weaker definition presented here later.
In loop quantum gravity, GR is written in terms of a gauge theory, i.e. the notion of a compact, connected Lie Group $G$ is employed. The following definition \textit{a priori} depends on this gauge group. For LQG, the cases $G = SU(2)$ and $G = U(1)$ are the most important.

\textbf{Definition 2.9.} Let $G$ be a compact, connected Lie group. Let furthermore $\mu = \{p_1, \ldots , p_M\}$ be a set of paths in $\mathcal{P}$ that are algebraically independent. This means that for each set $\{g_1, \ldots , g_M\}$, $g_k \in G$, there is a functor $A : \mathcal{P} \to \text{Susp}(G)$ such that

$$A(p_k) = g_k.$$ 

(2.4)

Then $\mu$ is called a metagraph. Denote the set of all metagraphs by $\mathcal{M}$.

Note that this notion depends on $G$: consider for instance a graph with one vertex $x$ and two edges $e_1, e_2 \in \text{Mor}(x, x)$, and $p = e_1 \circ e_2 \circ e_1^{-1} \circ e_2^{-1}$. Then $\{p\}$ is a metagraph for $G = SU(2)$ as we will show later, but is clearly not for $G = U(1)$, as one can see right away.

For the gauge groups in question, however, one can show that $\Gamma_1 \subset \Upsilon \subset \mathcal{M}$. So, each graph is a hyph, and each hyph is a metagraph, but not the other way round. Define a partial ordering on $\mathcal{M}$ by the same rule as for graphs or hyphs: $\mu_1 \leq \mu_2$ iff every path $p \in \mu_1$ can be composed by paths in $\mu_2$, or their inverses. Since for every finite set of paths $\{p_k\}$ one can find a graph $\gamma$ so that each $p_k$ can be composed of edges in $\gamma$ and their inverses, and each graph is again a metagraph; $\leq$ defines a partial ordering on $\mathcal{M}$.

The metagraphs are a useful concept when investigating the automorphisms $\text{Aut}(\mathcal{P})$, due to the following property.

\textbf{Lemma 2.2.} $\text{Aut}(\mathcal{P})$ leaves $\mathcal{M}$ invariant.

\textbf{Proof.} Let $\mu = \{p_1, \ldots , p_M\} \in \mathcal{M}$. Let $g_1, \ldots , g_M \in G$ and find $A \in \mathcal{A}$ such that $A(p_k) = g_k$. Then $\alpha \phi \cdot A \in \mathcal{A}$ has the property that

$$\alpha \phi \cdot A(p_k) = A(\phi^{-1}(\phi(p_k))) = A(p_k) = g_k.$$ 

Therefore, $\phi(\mu) := \{\phi(p_1), \ldots , \phi(p_M)\}$ is also algebraically independent by the above definition. □

In contrast, we will encounter explicit examples of elements $\phi \in \text{Aut}(\mathcal{P})$ that do not leave $\Gamma$ or $\Upsilon$ invariant. So, the mathematical concept of metagraphs will be quite useful in order to investigate the action of $\text{Aut}(\mathcal{P})$ on $\mathcal{P}$.

\section*{2.2. Notions from loop quantum gravity}

We now briefly review how the concept of the path groupoid $\mathcal{P}$ is used in loop quantum gravity, in particular how to define the quantum configuration space $\mathcal{A}$. We will do this in terms staying as close as possible to category language. For a brief introduction to category notions, see appendix A.

\textbf{Definition 2.10.} Let $\mathcal{P}$ be the path groupoid of $\Sigma$, and $G$ a compact, connected Lie group. Then a functor $A : \mathcal{P} \to \text{Susp}(G)$ is called a (generalized) connection\footnote{We will see, for instance, that for every path $p$ with $s(p) \neq t(p)$, $[p]$ is a metagraph.}. This means that $A$ maps paths in $\mathcal{P}$ to elements in $G$ such that

$$A(p \circ q) = A(p) \cdot A(q)$$

$$A(p^{-1}) = A(p)^{-1}.$$ 

(2.5)

The set of all connections is denoted by $\mathcal{A}$.

In the literature, this space is also denoted as $\text{Hom}(\mathcal{P}, G)$, the set of groupoid homomorphisms from $\mathcal{P}$ to $G$.\footnote{In the literature, this space is also denoted as $\text{Hom}(\mathcal{P}, G)$, the set of groupoid homomorphisms from $\mathcal{P}$ to $G$.}
It is clear that every smooth $\mathfrak{su}(2)$-connection $A^I_a(x)$ gives rise to such a functor, by mapping each path $p \in \Sigma$ to the holonomy of $A^I_a$ along $p$:  

$$A(p) := \mathcal{P} \exp \int_p A^I_a \frac{T_I}{2} \, dx^a.$$  

(2.6)

In this sense the set of all smooth connections $A$ is a subset of $\mathcal{A}$.

Under a gauge transformation $g \in \mathcal{G} = C^\infty(\Sigma, SU(2))$, the holonomy $A(p)$ along a path $p$ changes as

$$A(p) \rightarrow g_x A(p) g_y^{-1},$$

if $p$ starts at $x \in \Sigma$ and ends at $y \in \Sigma$, and $g_x$ is the value of the function $g$ at $x \in \Sigma$. This motivates the following definition.

**Definition 2.11.** A natural transformation $g$ on functors $A \in \mathcal{A}$ is called a (generalized) gauge transformation. The set of all such gauge transformations is denoted by $\mathcal{G}$.

Recall that functors $A_1$ and $A_2$ can be related by a natural transformation, if there is for every object $x \in |P| = \Sigma$ a morphism $g_x : A_1(x) \rightarrow A_2(x)$ such that the following diagram commutes for all $p \in \text{Mor}_P(x, y)$:

$$\begin{array}{ccc}
A_1(x) & \xrightarrow{A_1(p)} & A_1(y) \\
g_y \downarrow & & \downarrow g_x \\
A_2(x) & \xrightarrow{A_2(p)} & A_2(y)
\end{array}$$

Since $	ext{Susp}(G)$ has only one object $\ast$, $A(x) = \ast$ for all $A$ and $x$. This amounts to say that for each $x \in \Sigma$ there is an element $g_x \in G$ such that

$$A_1(p) = g_x A_2(p) g_x^{-1}. \quad (2.8)$$

This justifies the name gauge transformation, and shows that the set $\mathcal{G} \simeq G^\Sigma$. Thus, given a functor $A$ and $\{g_x\}_{x \in \Sigma} \in \mathcal{G}$, (2.8) can be seen as a definition of the gauge-transformed functor $\alpha_g A(p) := g_x A(p) g_x^{-1}$. So the set $\mathcal{G}$ acts on $\mathcal{A}$.

This immediately shows that $\mathcal{G} \simeq SU(2)^\Sigma$, i.e. the set of all maps from $\Sigma$ to $SU(2)$, without any smoothness (or continuity, measurability) condition. It is clear that this is a tremendous extension to a symmetry group which is not a symmetry of classical GR anymore. This shows that the quantum theory is in fact invariant under larger groups, having to do with the fact that spacetime becomes discrete in some sense: gauging can happen at each point in space completely independent of each other. The same can be done for the diffeomorphisms; every, say analytical, diffeomorphism $\phi$ acts in the path groupoid $P$ in the following way—points are mapped to points, and paths to paths:

$$x \mapsto \phi(x)$$

$$p \mapsto \phi(p) \quad (2.9)$$

with $\phi([c]) := [[\phi \circ c]]$ for representative curves $c \in \mathcal{C}$. If $p$ starts at $x$ and ends at $y$, then $\phi(p)$ starts at $\phi(x)$ and ends at $\phi(y)$. This means that $\phi$ induces a functor on the path groupoid $P$, which is invertible since $\phi$ is as as a map.

---

8 This is only true when the principal bundle in question is trivial and has at least one covering chart. About this point usually not much care is taken for two reasons: firstly the manifolds encountered in LQG are three dimensional and closed, hence parallelizable, and the spin structure can therefore chosen to be trivial. Secondly, even if the bundle in question is not trivial, the same machinery can be used using a slightly more complicated groupoid instead of $\text{Susp}(G)$, and the result quotient $\mathcal{A}/\mathcal{G}$ does not depend on the explicit choice of the bundle.
Later, there are many automorphisms in\[\text{diffeomorphisms, or 'graphomorphisms'. However, this is a proper subgroup: as we will see.}

Refer to Definition 2.12.

Let \( \phi : \mathcal{P} \rightarrow \mathcal{P} \) be an invertible functor. Then \( \phi \) is called an automorphism of \( \mathcal{P} \). Denote the set of all automorphisms by \( \text{Aut}(\mathcal{P}) \).

Note that \( \text{Aut}(\mathcal{P}) \) also acts on \( \mathcal{A} \), via \( \alpha_{\phi} A(p) := A(\phi^{-1}(p)) \). By this, \( \text{Aut}(\mathcal{P}) \) appears as an extension of \( \text{Diff}(\Sigma) \).

It is this set of automorphisms \( \text{Aut}(\mathcal{P}) \) that we will focus our attention on the rest of the paper. The automorphisms extend the analytic diffeomorphisms, and is the largest possible extension [15]. We will comment on the actual size of \( \text{Aut}(\mathcal{P}) \) in contrast to \( \text{Diff}(\Sigma) \) later in this paper. It should be noted that each invertible functor \( \zeta : \mathcal{C} \rightarrow \mathcal{C} \), i.e. \( \zeta \in \text{Aut} (\mathcal{C}) \) induces also an element \( \phi_{\zeta} \in \text{Aut}(\mathcal{P}) \). In particular, the set \( \{\phi_{\zeta} | \zeta \in \text{Aut}(\mathcal{C})\} \) forms a subgroup of \( \text{Aut}(\mathcal{P}) \), which has been investigated in the literature [13, 17], called 'piecewise analytic diffeomorphisms', or 'graphomorphisms'. However, this is a proper subgroup: as we will see later, there are many automorphisms in \( \text{Aut}(\mathcal{P}) \) which are no graphomorphisms.

The reason for this can be seen as follows: the automorphisms permute points of \( \Sigma \), and also permute the paths between points, in a consistent way. Consistent means that if \( p \in \text{Mor}_{\Sigma}(x, y) \), then \( \phi(p) \in \text{Mor}_{\Sigma}(\phi(x), \phi(y)) \). However, a path itself consists (if it is without self-intersection) of many points. So one could feel that for \( z \) lying on \( p \) (more precisely: \( z \in r(p) \)), then also \( \phi(z) \) should lie on \( \phi(p) \). Even more, since \( p \) can be decomposed into \( p_1 \circ p_2 \), where \( p_1 \) is the part of \( p \) which goes from \( x \) to \( z \), and \( p_2 \) is the remainder, which goes from \( z \) to \( y \). Since \( \phi(p) = \phi(p_1 \circ p_2) = \phi(p_1) \circ \phi(p_2) \), one might think that since \( \phi(p_1) \) ends at \( \phi(z) \), and \( \phi(p_2) \) starts at \( z \), \( \phi(p) \) should pass through \( \phi(z) \). But this is not the case, as figure 1 shows.

Let \( p \) be a path without self-intersections, which is composed of \( p = p_1 \circ p_2 \), i.e. \( p \) passes through \( z := t(p_1) \). The images of \( p_1 \) and \( p_2 \) under \( \phi \) are given by the dashed lines, and the solid line is \( \phi(p) \). \( \phi(p_1) \) ends at \( \phi(z) \) and \( \phi(p_2) \) starts at it, but since \( \phi(p_1) \circ \phi(p_2) \) contains a retracing, \( \phi(p) = \phi(p_1) \circ \phi(p_2) \) does not pass through \( z \). We see that the fact that retracings cancel out in \( \mathcal{P} \), the ill-definedness of \( r(p) \), and the existence of automorphisms which are not induced by maps from \( \Sigma \) to \( \Sigma \), is deeply interrelated to each other.

### 2.3. Automorphisms and connections

In this section we will investigate the metagraphs further, and their relation to the automorphisms. We will extend the LQG notions for graphs [19] to metagraphs, and in particular show that the topology on \( \mathcal{A} \) can also be defined in terms of metagraphs. This will allow us to prove i.e. continuity of the action of automorphisms \( \text{Aut}(\mathcal{P}) \) on the set of connections \( \mathcal{A} \). So, fix a compact, connected Lie group \( G \) for the rest of this section. We will refer to \( G \) as the gauge group.
The partial ordering defined on the set of metagraphs $\mathcal{M}$ has a category theory background, which we will use in the following.

**Definition 2.13.** Let $\mathcal{P}$ be the path groupoid of $\Sigma$, and $\mu \in \mathcal{M}$ a metagraph. Then define $\mathcal{P}_\mu$ to be the subgroupoid of $\mathcal{P}$ which is generated by the elements in $\mu$.

This groupoid contains the elements in $\mu$, their inverses, the identities $\text{id}_{\gamma(p_k)}$, $\text{id}_{\gamma(p_k)}$, and all products that can be formed of them. Thus, all $\mathcal{P}_\mu$ are finitely generated subgroupoids of $\mathcal{P}$.

Note that by this definition, also $\mathcal{P}_\gamma$ for $\gamma \in \Gamma$ and $\mathcal{P}_v$ for $v \in \Upsilon$ are declared.

We immediately conclude the following.

**Corollary 2.1.** For two metagraphs $\mu_1, \mu_2 \in \mathcal{M}$, we have that $\mu_1 \leq \mu_2$ if and only if $\mathcal{P}_{\mu_1}$ is a subgroupoid of $\mathcal{P}_{\mu_2}$.

The same holds for graphs and hyphs. In fact, a topology on the set $\mathcal{A}$ of generalized connections $A$, or equivalently functors $A : \mathcal{P} \to \text{Susp}(G)$, is defined by convergence on the finitely generated subgroupoids $\mathcal{P}_\gamma$ for all $\gamma \in \Gamma$. In [20] it was shown that the same topology is defined if one replaces $\Gamma$ by $\Upsilon$. In fact, defining the topology on $\mathcal{A}$ by using the metagraphs leads to the same result, as we will briefly indicate in the following.

**Definition 2.14.** Let $\mathcal{P}$ be the path groupoid, and $\mathcal{A}$ the set of all morphisms $A : \mathcal{P} \to \text{Susp}(G)$. For a metagraph $\mu$ denote the set of all morphisms from $\mathcal{P}_\mu$ to $\text{Susp}(G)$ by $\mathcal{A}_\mu$. Define the projection

$$\pi_\mu : \mathcal{A} \mapsto \mathcal{A}_\mu$$

$$(\pi_\mu A)(p_k) := A(p_k).$$

(2.10)

For $\mu_1 \leq \mu_2$ define the projections $\pi_{\mu_1, \mu_2} : \mathcal{A}_{\mu_2} \to \mathcal{A}_{\mu_1}$ via

$$(\pi_{\mu_1, \mu_2} A)(p) := A(q_1) \cdots A(q_n)$$

(2.11)

if $p \in \mu_1$ can be written by $p = q_1 \circ \cdots \circ q_n$, $q_k$ being elements in $\mu_2$ or their inverses.

It is easy to show that $\pi_{\mu_1, \mu_2} \circ \pi_{\mu_2} = \pi_{\mu_1}$ for $\mu_1 \leq \mu_2$. By the definition of metagraphs, each $\mathcal{A}_\mu, \mu = \{p_1, \ldots, p_M\}$, comes with a natural bijection $A \mapsto (A(p_1), \ldots, A(p_M))$. Pulling the topology of $G^M$ back to $\mathcal{A}_\mu$ makes all $\mathcal{A}_\mu$ into compact Hausdorff spaces.

**Definition 2.15.** Define the topology on $\mathcal{A}$ to be the weakest topology such that all the projections $\pi_\mu$ are continuous.

One can show that with this topology $\mathcal{A}$ becomes a compact Hausdorff space. The proof goes entirely along the same lines as in the case for graphs or hyphs [4, 13], and rests crucially on the compactness of $G$. We will not repeat the proof here.

In fact, this topology coincides with the topology which is defined by the condition that all $\pi_\gamma$ for $\gamma \in \Gamma$ are continuous.

**Lemma 2.3.** Let $T_1$ be the weakest topology such that for all $\mu \in \mathcal{M}$ the map $\pi_\mu$ is continuous. Let $T_2$ be the weakest topology such that for all graphs $\gamma \in \Gamma$ the map $\pi_\gamma$ is continuous. Then $T_1 = T_2$.

**Proof.** Consider the identity map between the two topological spaces:

$$\text{id} : (\mathcal{A}, T_1) \longrightarrow (\mathcal{A}, T_2).$$

(2.12)

But not the other way round, as the example for $G = U(1)$ above suggests.
Since $\pi^{-1}(U)$ are a basis for $T_2$ and every graph is also a metagraph, all of the $id^{-1}(\pi^{-1}(U))$ are open in $T_1$. So $id$ is a continuous bijection between compact Hausdorff spaces, hence also a homeomorphism. Therefore, the two topologies coincide.

Thus, defining the topology on $\mathcal{A}$ by means of metagraphs is completely analogous to defining it by graphs. The proof that $Aut(\mathcal{P})$ acts continuously on $\mathcal{A}$ runs along similar lines as for graphs, as well.

**Lemma 2.4.** Let $\phi : \mathcal{P} \to \mathcal{P}$ be an invertible functor, i.e. $\phi \in Aut(\mathcal{P})$. Then the map

$$a_\phi : \mathcal{A} \to \mathcal{A}$$

$$(a_\phi A)(p) := A(\phi^{-1}(p))$$

is a homeomorphism.

**Proof.** Let $\mu = \{p_1, \ldots, p_M\}$ be a metagraph. In the following, we deliberately use the homeomorphism $\mathcal{A}_\mu \simeq G^M$. For $A \in \mathcal{A}$ we have

$$\pi_\mu(A) = (A(p_1), \ldots, A(p_M))$$

$$= (a_\phi A(\phi(p_1)), \ldots, a_\phi A(\phi(p_M)))$$

$$= \pi_{\phi(\mu)}(a_\phi A),$$

i.e. we get

$$\pi_\mu \circ a_\phi^{-1} = \pi_{\phi(\mu)}.$$  \hspace{1cm} (2.15)

From this it follows that, for each open $U \in \mathcal{A}_\mu \simeq \mathcal{A}_{\phi(\mu)}$, one has

$$(\pi_{\phi(\mu)})^{-1}(U) = a_\phi((\pi_\mu)^{-1}(U)).$$

But since $(\pi_\mu)^{-1}(U)$ form a basis of the topology on $\mathcal{A}$, preimages of open sets under $a_\phi$ are open; hence $a_\phi$ is continuous. Since this is true for all automorphisms $\phi \in Aut(\mathcal{P})$ and each $\phi$ is invertible, all automorphisms $a_\phi : \mathcal{A} \to \mathcal{A}$ are homeomorphisms. This concludes the proof. \hspace{1cm} $\Box$

### 3. Automorphisms and Ashtekar–Lewandowski measure

In this section, we turn to a specific measure on $\mathcal{A}$, which has been defined by Ashtekar, Lewandowski [1, 19], and which lies at the very heart of the foundation of LQG. During the course of this section, we will show that the automorphism group $Aut(\mathcal{P})$ leaves this measure invariant.

**Definition 3.1.** The Ashtekar–Lewandowski measure $\mu_{AL}$ is the unique regular Borel measure on $\mathcal{A}$ such that, for any graph $\gamma = \{e_1, \ldots, e_E\} \in \Gamma$, one has

$$(\pi_\gamma)_* \mu_{AL} = \mu_H$$

where $\mu_H$ is the normalized Haar measure on $\mathcal{A}_\gamma \simeq G^E$.

One could think that, similar to the case of the topology on $\mathcal{A}$, one could simply replace the graphs $\Gamma$ by the metagraphs $\mathcal{M}$. However, the concept of metagraphs is slightly too broad for this: there is in general no measure $\nu$ on $\mathcal{A}$ such that $(\pi_\mu)_* \nu = \mu_H$ on $\mathcal{A}_\mu$ for all $\mu \in \mathcal{M}$. This can be seen as follows: choose $G = SU(2)$ and $l \in \mathcal{P}$ be a simple loop$^{10}$, which is also an edge, i.e. $l \in Mor_\mathcal{P}(x, x)$ for some $x \in \Sigma$. Then $\mu_1 := |l|$ as well as $\mu_2 := |l^2|$ is

$^{10}$ See definition 5.1 on page 27.
metagraphs, since one can take square roots in $SU(2)$. Now $(\pi_{\mu_1})_{*}\mu_{AL} = \mu_H$ on $SU(2)$, but $(\pi_{\mu_2})_{*}\mu_{AL} \neq \mu_H$ would imply
\[ \int_{SU(2)} d\mu_H(h) \ F(h) = \int_{SU(2)} d\mu_H(h) \ F(h^2) \tag{3.2} \]
for all continuous functions $F$ on $SU(2)$, which is not true. As a result, there can be no measure on $\mathcal{A}$ such that its push-forward by $\pi_{\mu}$ on $\mathcal{A}_{\mu}$ gives the Haar measure, for every metagraph $\mu \in \mathcal{M}$.

As a result, we cannot simply give an equivalent definition of $\mu_{AL}$ in terms of metagraphs, which would have enabled us to immediately show that $\mu_{AL}$ is invariant under the action of $\text{Aut}(\mathcal{P})$, as a consequence that $\mathcal{M}$ is. Some more care is needed to prove that $\text{Aut}(\mathcal{P})$ leaves $\mu_{AL}$ invariant. The notion of the primitive metagraphs will play a crucial role for this.

In the rest of this section we will make extensive use of concepts of combinatorial group theory, the details of which can be found in appendix B.

**Definition 3.2.** Let $\mu = \{p_1, \ldots, p_M\}$ and $\mu' = \{p'_1, \ldots, p'_M\}$ be metagraphs with $\mu \leq \mu'$ (in particular $M \leq M'$). Since every element $p_k \in \mu$ can be composed from elements $p'_l \in \mu'$, one can write
\[ p_k = \theta_k(p'_1, \ldots, p'_M) \tag{3.3} \]
where $\theta_k$ is a word in the letters $p'_l$, i.e. some ordered product of the $p'_l$ and their inverses. Then this defines a homomorphism of free groups
\[ \theta : F_M \longrightarrow F_{M'} \tag{3.4} \]
\[ \theta : X_k \longmapsto \theta_k(Y_1, \ldots, Y_{M'}) \tag{3.5} \]
between the free groups $F_M$ generated by the letters $X_1, \ldots, X_M$ and $F_{M'}$ generated by the letters $Y_1, \ldots, Y_{M'}$.

If there is a group automorphism (i.e. an invertible group homomorphism) $\tilde{\theta} : F_{M'} \rightarrow F_{M'}$ that extends $\theta$, i.e. such that
\[ \tilde{\theta}(Y_k) = \theta(X_k) \quad \text{for} \quad 1 \leq k \leq M, \tag{3.6} \]
we call $\mu$ to be embedded primitively\(^{11}\) into $\mu'$.

The primitive embeddings are of particular importance, since it is exactly those that will assure the cylindrical consistency of the Haar measures. With the notion of primitive embeddings at hand, we are ready to define primitive metagraphs.

**Definition 3.3.** Let $\mu \in \mathcal{M}$ be a metagraph with the following property: for any other metagraph $\mu' \in \mathcal{M}$ with $\mu \leq \mu'$, the embedding $\mu \leq \mu'$ is primitive. Then $\mu$ is called a primitive metagraph. Denote the set of primitive metagraphs by $\mathcal{M}_{pr}$.

Due to the purely combinatorial nature of the definition, it is not hard to show that also $\mathcal{M}_{pr}$ is invariant under the action of $\text{Aut}(\mathcal{P})$.

**Lemma 3.1.** Let $\mu = \{p_1, \ldots, p_M\} \in \mathcal{M}_{pr}$ be a primitive metagraph and $\phi \in \text{Aut}(\mathcal{P})$. Then $\phi(\mu) = \{\phi(p_1), \ldots, \phi(p_M)\}$ is also a primitive metagraph.

**Proof.** Let $\mu' \in \mathcal{M}$ be any (not necessarily primitive) metagraph with $\phi(\mu) \leq \mu'$ then, due to the functorial property of $\phi$, one also has $\mu \leq \phi^{-1}(\mu')$. Since $\mu$ is a primitive metagraph,\(^{11}\)

\(^{11}\) Note that this notion is a slight abuse of language. More precisely, it is the respective subgroupoids generated by $\mu, \mu'$ that are embedded into each other.
the embedding $\mu \leq \phi^{-1}(\mu')$ is primitive. By the very definition of primitive embeddings, it is therefore easy to see that also the embedding $\phi(\mu) \leq \mu'$ must be primitive. Since $\mu'$ was arbitrary, $\phi(\mu)$ is also a primitive metagraph. □

We now come to the crucial part of the proof that the Ashtekar–Lewandowski measure $\mu_{AL}$ is invariant under automorphisms. Since $\mu_{AL}$ is defined in terms of graphs (or hyphs), and those sets are not invariant under automorphisms, we have to make use the set of primitive metagraphs instead. But first we need another lemma.

Lemma 3.2. Let $\gamma, \gamma' \in \Gamma$ be graphs such that $\gamma \leq \gamma'$. Then the embedding $\gamma \leq \gamma'$ is primitive.

Proof. Let $\gamma = \{e_1, \ldots, e_E\}$ and $\gamma' = \{e'_1, \ldots, e'_{E'}\}$. Then every $e_k$ can be expressed as a word in $e'_j$:

$$e_k = \theta_k(e'_1, \ldots, e'_{E'}) . \quad (3.7)$$

But since each edge $e_k$ of $\gamma$ intersects each other edge $e_m$ at most in its endpoints, $e'_j$ occurring in each $\theta_k(e'_1, \ldots, e'_{E'})$ are different ones for each $k$. Furthermore, in each of the $\theta_k(e'_1, \ldots, e'_{E'})$ each edge occurs at most once. Hence, by relabeling edges $e'_j$ in $\gamma'$ we can write

$$e_k = e'^{(k)}_1 \circ \cdots \circ e'^{(k)}_{n_k}, \quad 1 \leq k \leq E, \quad (3.8)$$

which shows the simple form that $\theta_k$ has in this case. Note that $e'^{(k)}_j$ are just a different labeling of $e'_j$. We then define $\hat{\theta} : F_{M'} \rightarrow F_M$ by

$$\hat{\theta} : Y^{(k)}_1 \rightarrow Y^{(k)}_1 Y^{(k)}_2 \cdots Y^{(k)}_{n_k} \quad (3.9)$$

$$\hat{\theta} : Y^{(k)}_l \rightarrow Y^{(k)}_l \quad l > 1. \quad (3.10)$$

This is clearly an invertible group homomorphism of $F_{M'}$, its inverse given by

$$\hat{\theta}^{-1} : Y^{(k)}_1 \rightarrow Y^{(k)}_1 (Y^{(k)}_{n_k})^{-1} \cdots (Y^{(k)}_2)^{-1} \quad (3.11)$$

$$\hat{\theta}^{-1} : Y^{(k)}_l \rightarrow Y^{(k)}_l \quad l > 1. \quad (3.12)$$

We then define $\pi : F_{M'} \rightarrow F_M$ to be the permutation that permutes $Y_k$ into $Y^{(k)}_1$ for all $k = 1, \ldots, M$, and leaves all other $Y_l$ invariant. Then clearly

$$\hat{\theta} := \hat{\theta} \circ \pi \quad (3.13)$$

is the desired extension of $\theta$. It is also clearly a group automorphism, because $\pi$ and $\hat{\theta}$ are. Therefore, the embedding $\gamma \leq \gamma'$ is primitive. □

This leads us to the crucial part of the proof that $\text{Aut}(\mathcal{P})$ leaves the measure $\mu_{AL}$ on $\mathcal{A}$ invariant.

Lemma 3.3. Every graph $\gamma$ is a primitive metagraph, i.e. $\Gamma \subset M_{pr}$.

Proof. Note that this is absolutely not clear a priori from lemma 3.2, since primitivity of a graph $\gamma$ demands primitivity of its embeddings into all finer metagraphs $\mu' \geq \gamma$, and there can be arbitrary wild ones, intersecting itself in complicated manner, so that it is not clear at all whether the embeddings of a graph $\gamma$ into $\mu'$ are nicely behaved.

The proof relies in fact heavily on quite technical results from combinatorial group theory, the proofs of which have been therefore shifted into appendix B.
Let \( \gamma = \{e_1, \ldots, e_E\} \in \Gamma \) and \( \mu = \{p_1, \ldots, p_M\} \in \mathcal{M} \) such that \( \gamma \leq \mu \). Since every \( \mu_k \in \mu \) is a finite product of finitely many piecewise analytic curves in \( \Sigma \), one can find a graph \( \gamma' = \{e'_1, \ldots, e'_E\} \) with the property that \( \mu \leq \gamma' \). The embedding \( \gamma \leq \mu \) induces a group homomorphism \( \theta : F_\gamma \rightarrow F_\mu \), and the embedding induces \( \chi : F_m \rightarrow F_{E'} \). So we have the chain of homomorphisms

\[
F_\gamma \rightarrow F_\mu \rightarrow F_{E'}.
\]

Since \( \gamma \leq \gamma' \), we know by lemma (3.2) that \( \chi \circ \theta : F_\gamma \rightarrow F_{E'} \) is a primitive embedding. By lemma (appendix B.2), which is proven in the appendix, we therefore also know that \( \theta \) has to be a primitive embedding. Since \( \mu \) was arbitrary, we have established that \( \gamma \) embeds primitively into every finer metagraph, hence is itself a primitive metagraph. This finishes the proof. \( \square \)

**Theorem 3.1.** The action of the automorphisms \( \phi \in \text{Aut}(\mathcal{P}) \) on \( \overline{A} \) leaves the measure \( \mu_{AL} \) invariant.

**Proof.** Let \( f : \overline{A} \rightarrow \mathbb{C} \) be a smooth cylindrical function over a graph \( \gamma = \{e_1, \ldots, e_E\} \), i.e. there is a smooth function \( F : G^E \rightarrow \mathbb{C} \) with

\[
f(A) = F(A(e_1), \ldots, A(e_E)).
\]

One can show [4] that the set of all smooth cylindrical functions \( Cyl \) is a dense subset of \( L^2(\overline{A}, d\mu_{AL}) \) in its Hilbert space topology.

Let \( \phi \in \text{Aut}(\mathcal{P}) \) be any automorphism. Although \( \phi(\gamma) \) need not be a graph, there is a finer graph \( \gamma' = \{e'_1, \ldots, e'_{E'}\} \) with \( \phi(\gamma) \leq \gamma' \), i.e.

\[
\phi(e_k) = \theta_k(e'_1, \ldots, e'_{E'}).
\]

for words \( \theta_k \) in \( e'_k \). Therefore, with \( \hat{V}(\phi) f := f \circ \phi^{-1} \), we have

\[
\hat{V}(\phi) f (A) = F(\phi(e_1), \ldots, \phi(e_E))
\]

\[
= F(\theta_1(A(e'_1), \ldots, A(e'_{E'})), \ldots, \theta_{E'}(A(e'_1), \ldots, A(e'_{E'}))).
\]

So \( \hat{V}(\phi) f \) is a smooth cylindrical function over \( \gamma' \), hence integrable, and we get

\[
\int_{\overline{A}} d\mu_{AL}(A) \hat{V}(\phi) f (A)
\]

\[
= \int_{G^E} d\mu_{HE}^E(h_1, \ldots, h_{E'}) F(\theta_1(h_1, \ldots, h_{E'}), \ldots, \theta_{E'}(h_1, \ldots, h_{E'})).
\]

On the other hand, from lemma 3.3 we know that \( \theta \) can be extended to a group automorphism \( \tilde{\theta} : F_{E'} \rightarrow F_{E'} \). So we trivially have

\[
\int_{\overline{A}} d\mu_{AL}(A) \hat{V}(\phi) f (A)
\]

\[
= \int_{G^E} d\mu_{HE}^E(h_1, \ldots, h_{E'}) \tilde{\theta}(h_1, \ldots, h_{E'}))
\]

since \( \tilde{\theta}_k = \tilde{\theta}(X_k) = \theta(X_k) = \theta_k \), as words in \( Y_k \). On the other hand, we have, since the Haar measures on \( G \) are normalized:

\[
\int_{\overline{A}} d\mu_{AL}(A) f (A) = \int_{G^E} d\mu_{HE}^E(h_1, \ldots, h_E) F(h_1, \ldots, h_E)
\]

\[
= \int_{G^E} d\mu_{HE}^E(h_1, \ldots, h_E) F(h_1, \ldots, h_E).
\]
Since \( \tilde{\theta} \) is a group automorphism, by corollary appendix B.1 to Nielsen’s theorem, the two integrals (3.18) and (3.19) are identical:

\[
\int_{\mathcal{A}} d\mu_{\mathcal{A}L}(A) f(A) = \int_{\mathcal{A}} d\mu_{\mathcal{A}L}(A) \hat{V}(\phi) f(A).
\]

(3.20)

As a result, \( \hat{V}(\phi) \) defines, on the dense subset \( \text{Cyl} \) of smooth cylindrical functions, an isometric operator. But \( \alpha_{\phi} \) is also continuous on \( \mathcal{A} \), so the extension of \( \hat{V}(\phi) \) to a unitary operator \( \hat{U}(\phi) \) agrees with the pullback of the action \( \alpha_{\phi} \) on \( \mathcal{A} \), i.e. for any \( \psi \in L^2(\mathcal{A}, d\mu_{\mathcal{A}L}) \), one has

\[
(\hat{U}(\phi)\psi)(A) = \psi(\alpha_{\phi}^{-1}(A)).
\]

(3.21)

We therefore conclude that for \( \phi \in \text{Aut}(\mathcal{P}) \) the action \( \alpha_{\phi} \) on \( \mathcal{A} \) results in a unitary operator \( \hat{U}(\phi) \) on \( L^2(\mathcal{A}, d\mu_{\mathcal{A}L}) \), or equivalently, leaves \( \mu_{\mathcal{A}L} \) invariant. \( \Box \)

4. Examples for automorphisms

In this section, we give some explicit examples for automorphisms that will turn out to come in handy during the later course of the paper, and which will give some insight into the nature of \( \text{Aut}(\mathcal{P}) \).

It is obvious that every analytic diffeomorphism \( \phi : \Sigma \to \Sigma \) induces an automorphism \( \phi \in \text{Aut}(\mathcal{P}) \). In this sense, \( \text{Diff}(\Sigma) \) is a subgroup of \( \text{Aut}(\mathcal{P}) \). But also all invertible functors on \( \mathcal{C} \), which contain e.g. maps from \( \Sigma \) to \( \Sigma \) mapping hyphs to hyphs, descend to automorphisms on \( \mathcal{P} \) by \( \phi([w]) := ([\phi(w)]) \).

But there are many automorphisms \( \phi \in \text{Aut}(\mathcal{P}) \) that are not induced by a map \( \Sigma \to \Sigma \). The reason for this is deeply connected to the groupoid structure of \( \mathcal{P} \). By declaring retracings to be equivalent to the identity \( [[\delta \circ \delta^{-1}]] = [[\text{id}_{\Sigma}]] \), one disconnects the paths from the points: if \( p \) is a path that passes through a point \( x \), i.e. \( x \in r(p) \), the path \( p \circ q \) does not necessarily also have this property. Thus, there is no good notion for a point \( x \) lying on a path \( p \). This is the reason why there are automorphisms \( \phi \in \text{Aut}(\mathcal{P}) \) that have actions on points (objects) and paths (morphisms) which are not compatible with each other. In the following, we will give two extreme examples for this.

The first example is the natural transformations of the identity, which arbitrarily permute the points in \( \Sigma \), while leaving the paths essentially invariant.

The second example for nontrivial automorphisms will be the edge-interchangers, which interchange two edges with identical beginning and endpoints, but leave all objects (points \( x \in \Sigma \) invariant, as well as all paths intersecting the two given edges at most in finitely many points.

These two will be most helpful in determining the size of the orbits of vectors in \( \mathcal{H}_{\text{kin}} \) under the action of \( \text{Aut}(\mathcal{P}) \), in order to compute the automorphism-invariant Hilbert space \( \mathcal{H}_{\text{Aut}} \).

4.1. Natural transformations of the identity

Recall from category theory that, given two functors \( F, G : \mathcal{C} \to \mathcal{D} \), the two are called to be natural transformations from each other, if for each object \( X \) in \( \mathcal{C} \) there is a morphism \( g_X : F(X) \to G(X) \) such that the following diagram commutes:

\[
\begin{array}{ccc}
F(X) & \xrightarrow{F(f)} & F(Y) \\
\downarrow g_X & & \downarrow g_Y \\
G(X) & \xrightarrow{G(f)} & G(Y)
\end{array}
\]

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In the context of functors $A : \mathcal{P} \to \text{Sus}(SU(2))$, two such functors (generalized connections) $A_1, A_2$ are natural transformations of each other if and only if the one is a gauge transformed of the other. But also automorphisms can be natural transformations of each other. In particular, two automorphisms $\phi_1, \phi_2 \in \text{Aut}(\mathcal{P})$ are natural transformations of each other, if and only if there is a bijection $b : \Sigma \to \Sigma$ and for each $x \in \Sigma$ a path $p_x \in \text{Mor}(x, b(x))$, such that for every path $p \in \text{Mor}(x, y)$:

$$\phi_2(p) = p_{\phi_1(x)}^{-1} \circ \phi_1(p) \circ p_{\phi(y)}.$$  \hspace{1cm} (4.1)

Note that this requires $b(x) = \phi_2 \circ \phi_1^{-1}(x)$. Given $\phi_1, b$ and $\{p_x\}_{x \in \Sigma}$, this can also be seen as a definition of the transformed functor $\phi_2$. One special case occurs for $\phi_1 = \text{id}$ being the identity functor.

**Definition 4.1.** Let $b : \Sigma \to \Sigma$ be a bijection and, for every $x \in \Sigma$ a path $p_x \in \text{Mor}(x, b(x))$ be given. The functor $\phi_{b, p}$ defined by

$$\phi_{b, p}(x) := b(x)$$

$$\phi_{b, p}(p) := p_{s(p)}^{-1} \circ p \circ p_{t(p)}$$

is called a natural transformation of the identity. 

Given two bijections $b_1, b_2$ and $p_1^x \in \text{Mor}(x, b_1(x))$ and $p_2^x \in \text{Mor}(x, b_2(x))$, one has

$$\phi_{b_1, p_1} \circ \phi_{b_1, p_1} = \phi_{b, p_2}$$

with $b := b_2 \circ b_1$ and $p_x \in \text{Mor}(x, b_2(b_1(x)))$ given by $p_x := p_2^x \circ p_{b_1(x)}^{-1}$. In particular, by choosing $b_2 = b_1^{-1}$ and $p_2^x = (p_{b_1(x)})^{-1}$, one sees that every such functor is invertible, hence an automorphism.

**Corollary 4.1.** The natural transformations of the identity form a subgroup $N$ of $\text{Aut}(\mathcal{P})$.

The natural transformations of the identity will be of particular importance later on.

4.2. Edge-interchanger

The following example of an automorphism will prove to be most important in order to compute the automorphism-invariant Hilbert space $\mathcal{H}_{\text{Aut}}$. It will be an example of a functor which acts trivially on the objects $\Sigma$, but modifies the morphisms. In particular, it will interchange two edges (or paths without self-intersections) $e_1, e_2$ with the same beginning and endpoint. On the other hand, it will leave every edge which intersects $e_1, e_2$ in at most finitely many points invariant. It will therefore be termed ‘edge-interchanger’. See figure 2 for the action of an edge-interchanger on a generic path.

Let $e_1, e_2$ be two paths in $\text{Mor}(x, y)$ for $x \neq y$ without self-intersection, and such that they do not mutually intersect, apart from their beginning and endpoints. With this we mean that if one chooses representative curves $c_1, c_2 \in \mathcal{C}$ with $[[[c_1]]] = e_1, [[[c_2]]] = e_2$ which contain no retracings, i.e. the maps $c_1, c_2 : [0, 1] \to \Sigma$ are injective, then $r(c_1) \cap r(c_2) \subset [c_1(0), c_1(1), c_2(0), c_2(1)]$. We now choose two such representative curves.

Furthermore, choose for any $t \in (0, 1)$ a path $p_t \in \text{Mor}(r(c_1)(t), r(c_2)(t))$, such that all paths $p_t$ have a representative that intersects $r(c_1), r(c_2)$ only at the respective beginning and endpoints\(^ {12}\). Define $p_0 = \text{id}_{(c_1)}$ and $p_1 = \text{id}_{(c_1)}$ to be the constant paths. For $t_1, t_2 \in [0, 1]$

\(^{12}\) Note that for this to be possible, the dimension of the underlying manifold $\Sigma$ has to be at least 2. In the case of $\dim(\Sigma) > 2$, $\Sigma \setminus (r(c_1) \cup r(c_2))$ is an open, connected subset. Hence, one can always find the appropriate paths $p_t$. 

16
denote by \(c_{1}^{t_{1},t_{2}}\) the curve
\[
c_{1}^{t_{1},t_{2}} : [0, 1] \ni t \mapsto c_{1}(t_{1} + t(t_{2} - t_{1})),
\]
and a similar definition of \(c_{2}^{t_{1},t_{2}}\). Note that this definition also makes sense for \(t_{1} > t_{2}\), in particular \(c_{1,2}^{t_{1},t_{2}} = (c_{1,2}^{t_{2},t_{1}})^{-1}\).

With these data, we now build a functor \(\tilde{\phi} : \tilde{\mathcal{C}} \to \mathcal{P}\) as follows.

**Lemma 4.1.** Let \(w \in Mor_{\Sigma}(x, y)\) be a way in \(\Sigma\). Then divide \(w\) according to the edges \(e_{1}, e_{2}\):
\[
w = w_{1} \circ w_{2} \circ \ldots \circ w_{n},
\]
where \(w_{k}\) falls into either of the following categories:

- \(w_{k} = \left[c_{1}^{t_{1},t_{2}}\right]\) or \(\left[c_{2}^{t_{1},t_{2}}\right]\) for some \(t_{1}, t_{2} \in [0, 1]\)
- \(r(w_{k}) \cap (r(e_{1}) \cup r(e_{2})) \subset \{s(w_{k}), t(w_{k})\}\).

Then the following assignment
\[
\tilde{\phi}(w) := \tilde{\phi}(w_{1}) \circ \ldots \circ \tilde{\phi}(w_{n}) \in Mor_{\mathcal{P}}(x, y)
\]
with
\[
w_{k} = \left[c_{1}^{t_{1},t_{2}}\right] \Rightarrow \tilde{\phi}(w_{k}) := p_{t_{1}} \circ \left[\left[c_{1}^{t_{1},t_{2}}\right]\right] \circ p_{t_{2}^{-1}}
\]
\[
w_{k} = \left[c_{2}^{t_{1},t_{2}}\right] \Rightarrow \tilde{\phi}(w_{k}) := p_{t_{1}}^{-1} \circ \left[\left[c_{2}^{t_{1},t_{2}}\right]\right] \circ p_{t_{2}}
\]
\[
r(w_{k}) \cap (r(e_{1}) \cup r(e_{2})) \subset \{s(w_{k}), t(w_{k})\} \Rightarrow \tilde{\phi}(w_{k}) := [w_{k}].
\]

defines a functor \(\tilde{\phi} : \tilde{\mathcal{C}} \to \mathcal{P}\).

**Proof.** What has to be shown first is that the above assignment is well defined, i.e. does not depend on the manner the way \(w \in Mor_{\Sigma}(x, y)\) is decomposed w.r.t. the edges \(e_{1}, e_{2}\). First we note that, given two decompositions \(w_{1} \circ \ldots \circ w_{n}\) and \(w'_{1} \circ \ldots \circ w'_{m}\) of \(w\), there is a decomposition \(w''_{1} \circ \ldots \circ w''_{n}\) of \(w\) such that each \(w_{k}, w'_{k}\) is a product of \(w''_{k}\). Thus, if we can show that \(\tilde{\phi}(w)\) defined by one decomposition of \(w\) does not change if we decompose the decomposition further, we are done.

So let \(w = w_{1} \circ \ldots \circ w_{n}\) and \(w = w'_{1} \circ \ldots \circ w'_{m}\) be two decompositions of \(w\) w.r.t. the edges \(e_{1}\) and \(e_{2}\), such that each \(w_{k}\) is a product of \(w'_{k}\). We need to show that
\[
\tilde{\phi}(w_{1}) \circ \ldots \circ \tilde{\phi}(w_{n}) = \tilde{\phi}(w'_{1}) \circ \ldots \circ \tilde{\phi}(w'_{m})
\]
where \( \hat{\phi}(w_k) \), \( \hat{\phi}(w'_k) \) are defined according to (4.8). Let now \( w_k = [c_1^{t_i}] \), and \( w_k = w'_k \circ \ldots \circ w_{i+1}' \). Then there are points \( t_1 = t'_1 < t'_{i} < \ldots < t'_{i+1} = t_2 \) such that \( w'_{i+1} = [c_1^{\{t'_i\}}] \). Then we have

\[
\tilde{\phi}(w'_i) \circ \ldots \circ \tilde{\phi}(w'_{i+1}) = p_{c_1} \circ \left[ \left( c_1^{t'_i+1} \right) \right] \circ p_{c_1}^{-1} \circ p_{c_1} \circ \left[ \left( c_1^{t'_i} \right) \right] \circ p_{c_1}^{-1} \circ \cdots \circ p_{c_1} \circ \left[ \left( c_1^{t'_i} \right) \right] \circ p_{c_1}^{-1} = p_{c_1} \circ \left( c_1^{t'_i+1} \right) \circ \cdots \circ \left( c_1^{t'_i} \right) \circ p_{c_1}^{-1} = p_{c_1} \circ \left( c_1^{t'_i+1} \right) \circ p_{c_1}^{-1} = \tilde{\phi}(w_k). \]

Analogous relations hold in the case that \( w_k = \left[ \left( c_1^{t'_i} \right) \right] \).

Let now \( w_k \) be such that \( r(w_k) \cap (r(e_1) \cup r(e_2)) \subset \{s(w_k), t(w_k)\} \), and \( w_k = w'_i \circ \ldots \circ w_{i+1}' \). Then obviously also \( r(w'_{i+1}) \cap (r(e_1) \cup r(e_2)) \subset \{s(w'_{i+1}), t(w'_{i+1})\} \) for all \( i \in \{1, \ldots, l\} \). By (4.8) we immediately see that

\[
\tilde{\phi}(w'_i) \circ \ldots \circ \tilde{\phi}(w'_{i+1}) = \tilde{\phi}(w_k). \]  

Since these are the only cases occurring, we conclude that \( \tilde{\phi} \) is a well-defined map of morphisms in \( \text{Mor}_c(z, z') \) to morphisms in \( \text{Mor}_c(z, z') \).

The properties of a functor remain to be shown. But given two morphisms \( v, w \) in \( \mathcal{C} \) such that \( t(v) = s(w) \), and decompositions \( v = v_1 \circ \ldots \circ v_n \) and \( w = w_1 \circ \ldots \circ w_m \) w.r.t. \( e_1 \) and \( e_2 \), then \( v \circ w = v_1 \circ \ldots \circ v_n \circ w_1 \circ \ldots \circ w_m \) is a decomposition of \( v \circ w \) w.r.t. \( e_1 \) and \( e_2 \). By (4.7) we see that

\[
\tilde{\phi}(v \circ w) = \tilde{\phi}(v) \circ \tilde{\phi}(w). \]

Since \( \phi \) leaves beginning and endpoints of path invariant, we conclude that the map induces a functor \( \tilde{\phi} : \mathcal{C} \to \mathcal{P} \) that acts trivially on the objects \( |\mathcal{C}| = |\mathcal{P}| = \Sigma \).

Lemma 4.2. For \( v, w \) morphisms in \( \mathcal{C} \) with \( [v] = [w] \), one has \( \tilde{\phi}(v) = \tilde{\phi}(w) \). Thus, \( \tilde{\phi} \) descends to a functor \( \phi : \mathcal{P} \to \mathcal{P} \). Furthermore, one has \( \phi^2 = \text{id}_\mathcal{P} \), i.e. \( \phi \in \text{Aut}(\mathcal{P}) \).

Proof. If \( [v] = [w] \), then by the definition one can reach \( w \) by starting with \( v \) and deleting or inserting ways \( u^{-1} \). Hence, we need to show \( \tilde{\phi}(w^{-1}) = \tilde{\phi}(w)^{-1} \). Since for ways \( v, w \) in \( \mathcal{C} \) one has \( v^{-1} \circ w^{-1} = w^{-1} \circ v^{-1} \), we have only to show \( \tilde{\phi}(w^{-1}) = \tilde{\phi}(w)^{-1} \) for \( w \) being of one of the types (4.8). But for \( w = [c_1^{t_i}] \) this follows from the fact that

\[
[c_1^{t_i}]^{-1} = [c_1^{t_i}] = [c_1^{t_i}]. \]

with an analogous relation for \( c_2^{t_i} \). For \( u \) touching the edges \( e_1 \) and \( e_2 \) at most at its beginning and endpoint, the assertion is trivial. We thus get \( \tilde{\phi}(v) = \tilde{\phi}(w) \) for \( [v] = [w] \). Thus, \( \phi \) descends to a functor \( : \mathcal{P} \to \mathcal{P} \).

It remains to show that \( \phi^2 = \text{id}_\mathcal{P} \). Take a path \( p \) in \( \mathcal{P} \). Choose a representative \( w \) in \( \mathcal{C} \), i.e. \( p = [w] \). Then \( \phi(p) = \tilde{\phi}(w) \). Decompose \( w \) w.r.t. the edges \( e_1, e_2 \):

\[
w = w_1 \circ \ldots \circ w_n \]

(14.14)

13 This only holds if \( t_i < t_j \). If \( t_i > t_j \) then the corresponding points have the property that \( t_i > t'_{j+1} > \ldots > t'_{i+1} = t_2 \). The proof is then analogous.
where \( w_k \) are of the type (4.8). Assume \( w_k \) to meet the edges \( e_1 \) and \( e_2 \) at most at their endpoints. Then
\[
\phi^2([[w_k]]) = \phi(\tilde{\phi}(w_k)) = [[w_k]].
\]
(4.15)

Let now \( w_k = [c_{12}^{01}] \). Then
\[
\tilde{\phi}(w_k) = p_{t_1} \circ [c_{21}^{12}] \circ p_{t_2}^{-1}.
\]
(4.16)

By construction, \( p_t \) are such that they have a representative in \( \tilde{C} \) which touches the edges \( e_1, e_2 \) only at their beginning and endpoints \( s(p_t), t(p_t) \). It follows that \( \phi(p_t) = p_t^{-1} \) for all \( t \in [0, 1] \). So we have
\[
\phi^2([[w_k]]) = \phi \left( p_{t_1} \circ \left[ \left[ [c_{21}^{12}] \right] \circ p_{t_2}^{-1} \right] \right)
= p_{t_1} \circ p_{t_1}^{-1} \circ \left[ \left[ [c_{21}^{12}] \right] \circ p_{t_2} \circ p_{t_2}^{-1} \right]
= \left[ \left[ [c_{21}^{12}] \right] \right] = [[w_k]].
\]
(4.17)

analogously for \( w_k = [c_{12}^{01}] \). We conclude that \( \phi^2 = \id_P \). So \( \phi \in \Aut(P) \). □

From the definition it is immediate that \( \phi(e_1) = e_2 \) and vice versa. This justifies the name ‘edge-interchanger’, although not only edges can be interchanged, but also every two paths without self-intersection that only intersect in their respective beginning and endpoints. Note that \( \phi \) is an automorphism that acts trivially on the points in \( \Sigma \), but nontrivially on the paths. Every path, however, that is composed of edges that meet \( e_1, e_2 \) in at most finitely many points is left invariant under \( \phi \). Note further that \( \phi \) depends on a chosen parametrization of \( e_1, e_2 \), as well as a choice \( \{p_t|t \in (0, 1)\} \). This shows that there are many such automorphisms that interchange \( e_1 \) and \( e_2 \). One can convince oneself easily that the condition that \( p_t \) have to be chosen such that they intersect \( e_1, e_2 \) only in their boundary points can be relaxed to the condition that each of \( p_t \) can have finitely many isolated intersections with \( e_1, e_2 \). However, since this would have complicated the cumbersome notation even further, we restricted ourselves to the simpler case of \( p_t \) being well away from \( e_1, e_2 \) apart from their boundary.

5. The automorphism-invariant Hilbert space

The sub-Hilbert space \( \mathcal{H}_{\text{kin}}^{(\mathfrak{g})} \subset \mathcal{H}_{\text{kin}} \) of vectors which are invariant under the pullback of the action of \( \mathfrak{g} \) on \( \mathcal{A} \) is in one-to-one correspondence with vectors in \( L^2(\mathcal{A}\backslash\mathcal{G}, \mu_{\text{AL}}) \). Here \( \mu_{\text{AL}} \) is the push-forward of the Ashtekar–Lewandowski measure \( \mu_{\text{AL}} \) on \( \mathcal{A} \) via the quotient map from \( \mathcal{A} \) to \( \mathcal{A}\backslash\mathcal{G} \), and is also called the Ashtekar–Lewandowski measure—in abuse of both notation and nomenclature.

In LQG, one does not only want to find Hilbert spaces of vectors invariant under \( \mathfrak{g} \), but also under \( \text{Diff}(\Sigma) \) at the same time. Due to the easy commutation relations between the two symmetries, one can equivalently look for states in \( \mathcal{H}_{\text{kin}}^{(\mathfrak{g})} \) which are invariant under \( \text{Diff}(\Sigma) \), i.e. ‘solve one constraint after the other’ [1].

We now review the definition of the \( \text{Diff}(\Sigma) \)-invariant Hilbert space for the case \( G = SU(2) \) and \( \text{Diff}(\Sigma) \) being the group of analytical diffeomorphisms on \( \Sigma \) [1, 10].

Since the authors are not aware of any normalized Borel measure on \( \text{Diff}(\Sigma) \), defining the rigging map naively via the group averaging:
\[
\eta[\psi](\chi) = \int_{\text{Diff}(\Sigma)} d\mu(\phi) \langle \psi | \hat{U}(\phi) \chi \rangle
\]
(5.1)
is not possible. But there are other ways of defining an antilinear map
\[
\eta : D \longrightarrow D_{\text{Diff}}^\ast
\]
(5.2)
from a dense subspace \( D \subset \mathcal{H}_{\text{kin}}^{(g)} \) invariant under \( \text{Diff}(\Sigma) \) into the linear functionals on \( D \) that are invariant under \( \text{Diff}(\Sigma) \). This is usually done referring to an explicit orthonormal basis of \( \mathcal{H}_{\text{kin}}^{(g)} = L^2(\mathcal{A}/\mathcal{G}, \mu_{\text{AL}}) \), which we describe in the following.

**Lemma 5.1.** Let \( G \) be a compact and connected Lie group. Let \( \gamma = \{ e_1, \ldots, e_E \} \) be a graph where each connected component has no one-valent vertices. Let furthermore \( \{ \pi_k \}_{k=1}^{E} \) be a sequence of irreducible nonzero representations of \( G \), and for each vertex \( l \in \{ 1, \ldots, E \} \) a nonzero linear map (an ‘intertwiner’)

\[
I_l : V'_{l_{k_1}} \otimes \cdots \otimes V'_{l_{k_{E-1}}} \otimes V^*_{l_{k_E}} \otimes \cdots \otimes V^*_{l_{k_1}} \longrightarrow \mathbb{C}
\]  

(5.3)

with

\[
I_l \circ [\pi_{l_{k_1}}(g) \otimes \cdots \otimes \pi_{l_{k_{E-1}}}(g) \otimes \pi^*_{l_{k_E}}(g) \otimes \cdots \otimes \pi^*_{l_{k_1}}(g)] = I_l \quad \text{for all } g \in G.
\]  

(5.4)

Here for each vertex \( l \) the numeration is such that the edges \( e_{l_{k_1}}, \ldots, e_{l_{k_{E-1}}} \) and the edges \( e_{l_{k_{E-1}}}^{-1}, \ldots, e_{l_{k_1}}^{-1} \) have \( l \) as the endpoint and the edges \( e_{l_{k_{E-1}}} \), \ldots, \( e_{l_{k_1}} \) have \( l \) as the starting point. Furthermore, \( V'_{l_{k}} \) is the representation space of \( \pi_k \), and \( V^*_{l_{k}} \) and \( \pi^*_{l_{k}} \) are the dual space and the dual representation respectively.

Then the function

\[
T_{\gamma, \vec{\pi}, \vec{I}}(\vec{A}) := \sum_{m_1, m_1, \ldots, m_E, n_E} \left( \prod_{k=1}^{E} \pi_k(A(e_{k}))n_km_k \right) \left( \prod_{l=1}^{V} I_l^{m_{l_{k_1}}, n_{l_{k_1}}, m_{l_{k_{E-1}}}, \ldots, n_{l_{k_{E-1}}}} \right)
\]  

(5.5)

is cylindrical over \( \gamma \) and gauge invariant. Furthermore, they span a dense subset in the gauge-invariant Hilbert-space \( \mathcal{H}_{\text{kin}}^{(g)} \). Two such functions \( T_{\gamma, \vec{\pi}, \vec{I}}, T'_{\gamma', \vec{\pi}, \vec{I}'} \) are orthogonal in \( \mathcal{H}_{\text{kin}}^{(g)} \) whenever the graphs \( \gamma = \{ e_1, \ldots, e_E \}, \gamma' = \{ e'_1, \ldots, e'_E \} \) are inequivalent, where two graphs are declared equivalent whenever

\[
\bigcup_{k=1}^{E} r(e_k) = \bigcup_{k=1}^{E'} r(e'_k).
\]

If the two graphs are equivalent\(^{14}\), then they are orthogonal whenever some of \( \pi_k \) do not coincide. If for every equivalence class of graphs \( \gamma \) and every choice \( \vec{\pi} \) of irreducible representations along the edges of \( \gamma \) one chooses an orthonormal basis within the spaces spanned by the intertwiner (5.3), one obtains an orthonormal basis of \( \mathcal{H}_{\text{kin}}^{(g)} \) this way.

In the case of \( G = SU(2) \), the corresponding functions (5.5) are called spin-network functions (SNF)\(^{15}\), and are denoted by \( T_{\gamma, \vec{j}, \vec{I}} \), where each of \( j_k \) is a half-integer corresponding to an irreducible representation of \( SU(2) \), and the intertwiner \( \vec{I} \) is given by the Clebsch–Gordan coefficients. In the case of \( G = U(1) \), where all irreducible representations are one dimensional, the functions \( T_{\gamma, \vec{j}} \) are called charge-network functions, and \( n_k \in \mathbb{Z} \) denote irreducible representations of \( U(1) \), and there is (up to complex multiples) exactly one intertwiner if and only if at each vertex the sum of the ‘outgoing’ and the ‘ingoing’ \( n_k \)s add up to zero.

For any \( \phi \in \text{Diff}(\Sigma) \) and any graph \( \gamma \), also \( \phi(\gamma) \) is a graph. It follows that in the case of \( G = SU(2) \), which we consider here, spin network functions \( T_{\gamma, \vec{j}, \vec{I}} \) are mapped into

\(^{14}\) If the vertex is two-valent, then there is only a nonzero map if the two representations coincide, and in this case the space of maps (5.3) is one dimensional. As a consequence, two-valent vertices do not contain any information. If two graphs are equivalent, they only differ by removal or adding of such ‘spurious’ vertices.

\(^{15}\) Where one implicitly chooses, once and for all, an orthonormal basis in the intertwiner spaces, where this choice is only dependent on the combinatorics of the graph.
Then there is an automorphism \( \phi \in \text{Diff}(\Sigma) \). Then \( \eta \) is defined by

\[
\eta[T_{v,\tilde{v}}](T_{v',\tilde{v}'}):= \sum_{\phi \in \text{Diff}(\Sigma)/\text{Diff}_{v}} F(GS_{v}) \sum_{\tilde{\phi} \in GS_{\tilde{v}}} \langle \tilde{U}(\phi \circ \tilde{\phi}) T_{v,\tilde{v}} | T_{v',\tilde{v}'} \rangle
\]  

(5.6)

where \( \text{Diff}_{v} \) is the set of diffeomorphisms which leave \( \gamma_{v} \), the subgroupoid generated by the elements in \( \gamma \), invariant. \( GS_{v} \) is the group of graph symmetries, i.e. the quotient of \( \text{Diff}_{v} \) and the subgroup of \( \text{Diff}(\Sigma) \) which leaves \( \gamma \) itself invariant. This is a finite group. \( F \) is a function, which is not fixed by the formalism. This defines an antilinear map from the span of the SNFs to the \( \text{Diff}(\Sigma) \)-invariant linear functionals, which serves as a rigging map, and defines an \( \text{Diff}(\Sigma) \)-invariant inner product via

\[
\langle \eta[\psi] | \eta[\chi] \rangle_{\text{Diff}} := \eta[\bar{\psi}] \langle \chi \rangle.
\]

(5.7)

In the case of \( \text{Diff}(\Sigma) \), an orthogonal basis for \( \mathcal{H}_{\text{kin}} \) is given by the set of equivalence classes of SNFs under the action of \( \text{Diff}(\Sigma) \). The normalization of these vectors is a nontrivial issue [21], and is governed by the function \( F \) in (5.6).

If we now replace the diffeomorphisms \( \text{Diff}(\Sigma) \) by the automorphisms \( \text{Aut}(\mathcal{P}) \), we can try to use the same techniques to define an automorphism-invariant inner product \( \langle \cdot | \cdot \rangle_{\text{Aut}} \) and an automorphism-invariant Hilbert space \( \mathcal{H}_{\text{Aut}} \).

### 5.1. Graph combinatorics

The automorphisms \( \text{Aut}(\mathcal{P}) \) act unitarily on the Hilbert space \( \mathcal{H}_{\text{kin}} \), as was demonstrated in the last section. Furthermore, we have seen that there are many automorphisms \( \phi \in \text{Aut}(\mathcal{P}) \) that do not correspond simply to a piecewise analytic map \( \Sigma \rightarrow \Sigma \). In particular, given any bijection \( b : \Sigma \rightarrow \Sigma \), any choice of paths \( p_{k} \in \text{Mor}(x, b(x)) \) defines a functor \( \phi_{b,p} \in \mathcal{N} \).

On the other hand, each edge-interchanger acts trivially on the objects in \( \Sigma \), but changes morphisms. Thus, the automorphisms allow for a lot of freedom how to change metagraphs \( \mu \in \mathcal{M} \). In this section we will present a lemma that shows how large the orbit of a graph is under the action of \( \text{Aut}(\mathcal{P}) \).

**Lemma 5.2.** Denote the set of vertices of a hyph \( v \) by \( V(v) \). Given any two hyphs \( v_{1} = (p_{1}, \ldots, p_{H}) \) and \( v_{2} = (q_{1}, \ldots, q_{H}) \), such that the two are combinatorially the same, i.e. there is a bijection \( b : V(v_{1}) \rightarrow V(v_{2}) \) and a bijection \( c : \{p_{1}, \ldots, p_{H}\} \rightarrow \{q_{1}, \ldots, q_{H}\} \) such that

\[
\begin{align*}
s(c(p_{k})) = b(s(p_{k})), \\
t(c(p_{k})) = b(t(p_{k})).
\end{align*}
\]

Then there is an automorphism \( \phi \in \text{Aut}(\mathcal{P}) \) such that \( \phi(v_{1}) = v_{2} \).\( ^{16} \)

**Proof.** We will explicitly construct this automorphism as a product of finitely many natural transformations of the identity and edge-interchangers. First we show that there is a sequence of edge-interchangers such that \( v_{1} \) can be mapped to a graph \( \gamma' \), which has the same vertices as \( v_{1} \), as well as the same combinatorics as \( v_{1} \).

Choose a graph \( \gamma' = \{e'_{1}, \ldots, e'_{E}\} \) such that \( v_{1} \leq \gamma' \). Then, by the definition, there is an edge \( e'_{i} \) in \( \gamma' \) such that \( e'_{i} \) meets \( \{p_{1}, \ldots, p_{H-1}\} \) at most in its endpoints and such that

\[
p_{H} = p_{1} \circ e'_{i} \circ p_{2},
\]

(5.9)

for some \( p_{1}, p_{2} \). Choose an analytic curve \( c \) without self-intersections (i.e. \( [[c]] \) is an edge) from \( s(p_{H}) \) to \( t(p_{H}) \) that does not meet any of \( e'_{i} \), apart from the points \( s(p_{H}) \) and \( t(p_{H}) \). Then choose two curves \( c_{1} \) and \( c_{2} \) with the following properties:

\( ^{16} \)Note that this is \textit{a priori} slightly weaker than demanding that every automorphism between the subgroupoids \( \mathcal{P}_{v_{1}} \rightarrow \mathcal{P}_{v_{2}} \) can be extended to an automorphism on \( \mathcal{P} \).
Then by construction the following paths define an automorphism $p_{k < H}$, which may have arbitrary complicated intersection with these segments.

- $s(c_1) = c(\frac{1}{2})$ and $t(c_1) = s(e_i')$
- $s(c_2) = c(\frac{3}{4})$ and $t(c_2) = t(e_i')$
- the curves $c_1, c_2$ are injective and $r(c_1), r(c_2)$ do not intersect, as well as they do not intersect with any of the $r(e_i'), r(e_i'_{l})$, apart from their beginning and endpoints.

For any curve $c$ and $0 \leq t_1, t_2 \leq 1$ denote by $c^{t_1:t_2}$ the curve $c^{t_1:t_2}(t) := c(t_1 + t(t_2 - t_1))$. Then by construction the following paths are paths without self-intersection that mutually intersect only at their beginning and endpoints:

$$e_1 := [[[c_1^{\frac{1}{2}}]]] \circ e_i' \circ [[[c_1^{\frac{1}{2}}]]]^{-1}$$
$$e_2 := [[[c_2^{0\frac{1}{2}}]]]^{-1} \circ [[[c_1^{\frac{1}{2}}]]] \circ [[[c_2^{0\frac{1}{2}}]]]$$

are paths without self-intersection that mutually intersect only at their beginning and endpoints. Furthermore, they intersect with the other edges $e_i', k \neq l$ only at most in $s(e_i'), t(e_i')$. Thus, by construction, the same is true for the paths $p_1, \ldots, p_{H-1}$, since they are built from $e_i', k \neq l$.

Now choose representative curves $d_1$ and $d_2$ for $e_1$ and $e_2$ such that

- $d_1 (\frac{1}{2}) = s(e_i')$ and $d_1 (\frac{3}{4}) = t(e_i')$
- $d_2 (\frac{1}{2}) = c(\frac{3}{4})$ and $d_2 (\frac{3}{4}) = c(\frac{1}{2})$.

Finally, choose for every $t \in (0, 1) \setminus \{\frac{1}{2}, \frac{3}{4}\}$ a path $p_t$ that goes from $d_1(t)$ to $d_2(t)$, and which does not meet $e_1, e_2$ apart from their respective beginning and endpoint. Furthermore, define

$$p^t_1 := (\tilde{p}_1)^{-1} \circ [[[c_1^{\frac{1}{2}}]]]$$
$$p^t_2 := \tilde{p}_2 \circ [[[c_1^{\frac{1}{2}}]]]^{-1}.$$  

Now we have chosen two paths without self-intersections $e_1, e_2$ which meet only in their beginning and endpoint, a parametrization $d_1, d_2$ for each of the paths, and for each $t \in (0, 1)$ a path $p_t \in \text{Mor}(d_1(t), d_2(t))$ that does not meet $e_1, e_2$, apart from its beginning and endpoint. These data define an automorphism $\phi \in \text{Aut}(P)$ by lemma 4.1. See figure 3 for a visualization of the action of this automorphism on $p_{H}$.
Since \( e_1, e_2 \) have been chosen to be such that they meet \( p_1, \ldots, p_{H-1} \) only in at most finitely many points, we conclude that \( \phi(p_k) = p_k \) for \( k = 1, \ldots, H - 1 \). By the definition of \( \phi \) and (5.11), (5.12), we have

\[
\phi(p_H) = \phi(p_1) \circ \phi(e_1') \circ \phi(p_2) = p_1 \circ (p_1 \circ [[[c^{1,1}]]] \circ p_1^{-1}) \circ \bar{p}_2
\]

\[
= [[[c^{0,1}]]] \circ [[[c^{1,1}]]] \circ [[[c^{1,1}]]] = [[[c]]].
\]

(5.13)

So by this construction, all paths \( p_k \) have been left invariant, except for \( p_H \) which has been mapped to an edge \( e_H := [[[c]]] \), which meets \( p_k, k < H \), at most in its beginning and endpoint. For this construction it was crucial that \( p_H \) had a free segment w.r.t. all the other \( p_k, k < H \), but also w.r.t. \( e_H \), by construction. This means that \((e_H, p_1, \ldots, p_{H-1}) \) is also a hyph, i.e. each of \( p_k \) has a free segment w.r.t all \( p_l, l < k \), but also w.r.t. \( e_H \). So the same construction can be carried out once again to obtain an automorphism \( \phi' \in \text{Aut}(\mathcal{P}) \) that leaves \( e_H \) as well as all \( p_k, k > H - 1 \), invariant, but maps \( p_{H-1} \) to an edge \( e_{H-1} \), which meets \( e_H, p_k, k < H - 1 \), at most in its beginning and endpoint. This gives a hyph \((e_{H-1}, e_H, p_1, \ldots, p_{H-2}) \). By repeating this process, we arrive at a hyph \((e_1, \ldots, e_H) \), where all \( e_l \) are edges and meet at most in their beginning and endpoint. So \( \gamma := \{e_1, \ldots, e_H\} \) is a graph. Thus, we have constructed a finite sequence of automorphisms that map the hyph \((p_1, \ldots, p_H) \) into the graph \( \gamma \).

Now let \( \gamma'' := \{e_1'', \ldots, e_H''\} \) be any graph which has the same combinatorics as \( \gamma = \{e_1, \ldots, e_H\} \), but the edges and vertices of which do not intersect with the edges of \( \gamma \): \( V(\gamma) = \{v_1, \ldots, v_{v_1}\} \) and \( V(\gamma'') = \{v_1'', \ldots, v_v''\} \), and

\[
s(e_k) = v_k, \quad t(e_k) = v_k'' \quad \Leftrightarrow \quad s(e_k') = v_k'', \quad t(e_k') = v_k''.
\]

(5.14)

Then construct an automorphism mapping one to the other by the following method: for each \( v_1 \in V(\gamma) \) choose a path \( p_n \in \text{Mor}(v_1, v_1') \) such that all \( p_n \) are without self-intersections, do not intersect each other and intersect \( e_k, e_k'' \) only in the vertices \( v_1, v_1'' \). Then define a natural transformation of the identity \( \phi_{b,p} \in \mathcal{N} \) by the following data:

\[
b(v_1) := v_1'', \quad b(v_1') := v_1, \quad b(x) = x \text{ else}
\]

\[
p_n := p_n, \quad p_n^{-1}, \quad p_n = \text{id}_x \text{ else}.
\]

(5.15)

Then each of \( p_k' := \phi_{b,p}(e_k) \) is a path without self-intersections, and all \( p_k' \) have free segments w.r.t. each other (in particular the \( e_k \)). So \( (p_1', \ldots, p_H') \) is a hyph with vertices \( v_1', \ldots, v_v' \). On the other hand, \( \gamma'' \) is a graph, the edges \( e_k'' \) of which intersect all \( p_k' \) at most in the vertices \( v_1'', v_v'' \). Since for each path \( p_k' \), the edge \( e_k'' \) starts and ends at the same points as \( p_k' \) by construction, there is a sequence of \( H \) edge-interchangers that maps each \( p_k' \) to \( e_k'' \).

So we have started with a hyph \( v_1 = (p_1, \ldots, p_H) \), have mapped \( v_1 \) to \( \gamma = (e_1, \ldots, e_H) \) by a sequence of \( H \) edge-interchangers, have mapped \( \gamma \) by a natural transformation of the identity \( \phi_{b,p} \) to the hyph \( v_1' = (p_1', \ldots, p_H') \) and have mapped \( v_1' \) by another sequence of \( H \) edge-interchangers to some other graph \( \gamma'' \), of which we only demanded that it has the same combinatorics as \( v_1 \), and that it has empty intersection with \( \gamma \). By the same construction, we can show that we can map any two graphs that have the same combinatorics but empty intersection into each other by a sequence of automorphisms. But this shows that one can map any two hyphs \( v_1, v_2 \) having the same combinatorics into each other by a sequence of edge-interchangers and natural transformations of the identity. Thus, the lemma is proven.
5.2. Orbits of the automorphisms

The group $\text{Aut}(\mathcal{P})$ maps gauge-invariant functions to gauge-invariant functions due to

$$\alpha_g \circ \alpha_f = \alpha_{\phi(g)} \circ \alpha_f$$

for $g \in \mathcal{G}$, $\phi \in \text{Aut}(\mathcal{P})$ \hspace{1cm} (5.16)

where $\phi(g)_x := g\phi(x)$ for $g$ seen as map from $\Sigma$ to $G$ and $\phi(x)$ being the action of the functor $\phi \in \text{Aut}(\mathcal{P})$ on the objects $\Sigma$ in $\mathcal{P}$. The action of the automorphisms $\phi \in \text{Aut}(\mathcal{P})$ can therefore be restricted to $\mathcal{H}_\text{kin}^{(\phi)}$.

To compute the space of linear functionals invariant under both the action of $\mathcal{G}$ and $\text{Aut}(\mathcal{P})$, it suffices to investigate the orbits of vectors $\psi \in \mathcal{H}_\text{kin}^{(\phi)}$ under the action of $\text{Aut}(\mathcal{P})$. First we will investigate these orbits in the following for arbitrary gauge groups $G$. Then we will specialize to $G = U(1)$ and $G = SU(2)$ in order to say something about the corresponding spaces $\mathcal{H}_\text{kin}$.

**Lemma 5.3.** Let $f \in \text{Cyl}$ be a gauge-invariant cylindrical function. Then for any natural transformation of the identity $\phi_{b,p} \in \mathcal{N}$ one has

$$\hat{U}(\phi_{b,p})f = f.$$ \hspace{1cm} (5.17)

**Proof.** Let $f$ be cylindrical over the graph $\gamma = \{e_1, \ldots, e_E\}$. For each $v \in V(\gamma)$ and each $A \in \mathcal{A}$ denote $g^A_v := A(p_v) \in G$. With

$$f(A) = F(A(e_1), \ldots, A(e_E))$$

we get

$$(\hat{U}(\phi_{b,p})f)(A) = f(\alpha_{\phi_{b,p}}^{-1}A)$$

$$= F(A(p_{s(e_1)}^{-1} \circ p_1 \circ p_{t(e_1)}) \circ \ldots, A(p_{s(e_E)}^{-1} \circ p_1 \circ p_{t(e_E)}))$$

$$= F((g^A_{s(e_1)})^{-1}A(e_1) g_{t(e_1)}, \ldots, (g^A_{s(e_E)})^{-1}A(e_E) g_{t(e_E)})$$

$$= F(A(e_1), \ldots, A(e_E))$$

$$= f(A),$$ \hspace{1cm} (5.19)

which was the claim. \hfill \Box

Lemma 5.3 shows one important fact: although $\phi_{b,p}$ can change graphs quite arbitrarily, the corresponding functions on that graph remain unchanged. The reason for this is that a function cylindrically over a metagraph $\mu \in \mathcal{M}$ does generally not carry all of the information in order to reconstruct $\mu$ from its dependence $A \mapsto f(A)$.

Consider the following example: given a metagraph $\mu = \{p_1, \ldots, p_4\}$, as in figure 4. Note that the paths $p_1 \circ p_2$ and $p_3 \circ p_4$ each contain a retracing. Now consider the function cylindrical over the metagraph $\mu$ by

$$f(A) = F(A(p_1), A(p_2), A(p_3), A(p_4))$$ \hspace{1cm} (5.20)

with $F(h_1, h_2, h_3, h_4) = \tilde{F}(h_1 h_2, h_3 h_4)$ for some smooth function $\tilde{F}$ on $G^2$. Note that $f$ does not depend on the parallel transports along all $p_1, \ldots, p_4$, but only on the ones along $p_1 \circ p_2$ and $p_3 \circ p_4$. So in particular, $f$ does not depend at all on the parallel transports along the retracings. Consequently, $f$ is cylindrical over the graph $\gamma$, which consists of the two edges shown in figure 5.

So we see that the dependence of the function $f$ is not on all of $\mu$, but only of a certain subgraph. This is of particular importance, since the following graph $\gamma$ has the same combinatorics as $\mu$. So, by lemma 5.2 there is an automorphism $\phi \in \text{Aut}(\mathcal{P})$ mapping one to the other, i.e. $\phi(e_k) = p_k$. Thus, a function $f$ cylindrical over $\gamma$ is mapped into $\hat{U}(\phi)f$, ...
Figure 4. A metagraph $\mu$ (in fact a hyph) consisting of four paths and five vertices.

Figure 5. The function $f$ given by (5.20) is also cylindrical over the graph $\tilde{\gamma} \leq \mu$, which only consists of two edges and four vertices.

Figure 6. The metagraph $\mu$ can be mapped to this graph $\gamma$ by an automorphism.

which is cylindrical over $\mu$. But if $f$ depends only on the parallel transports along $e_1 \circ e_2$ and $e_3 \circ e_4$, then $\hat{U}(\phi)f$ is also cylindrical over $\tilde{\gamma} \leq \mu$, as we have just seen. So, although $\phi$ respects the number of vertices and paths in a metagraph, $\hat{U}(\phi)$ can map a function which is cylindrical over a metagraph to a function cylindrical over another metagraph, which does not have the same combinatorics, such as $\gamma$ and $\tilde{\gamma}$ in the example above.

Although this seems paradoxical at first, it is in fact quite natural: consider the function $f$, which is cylindrical over the graph $\gamma$, as shown in figure 6, and which depends only on the parallel transports along $e_1 \circ e_2$ and $e_3 \circ e_4$. Then the function does not ‘know’ that it is cylindrical over a graph with four edges, in particular it does not know anything about the
middle vertex \( t(e_1) \). For instance, it is automatically gauge invariant w.r.t. gauging at this vertex. This does, of course, not happen to an arbitrary function \( g \) cylindrical over \( \gamma \), whose dependence on the parallel transports along all four edges \( e_1, \ldots, e_4 \) is nontrivial.

We see that the cylindrical functions can carry much less information than the graph that they are cylindrical over. In particular, the functions only carry at most the information about how many independent parallel transports they depend, and which of these start or end at the same points. It is exactly this information that is preserved by the automorphisms.

This is of particular importance for the gauge-invariant functions, since these carry only information about the first fundamental class of the graph, but not about the graph topology itself, which is summarized by the following lemma.

**Lemma 5.4.** Let \( f \in \text{Cyl} \) be a gauge-invariant cylindrical function over a graph \( \gamma \) with \( E \) edges and \( V \) vertices. Then there is a \((E - V + 1)\)-flower graph \( \tilde{\gamma} \) (a graph with one vertex and \( E - V + 1 \) edges all starting and ending at that vertex) and an automorphism \( \phi \in \text{Aut}(\mathcal{P}) \) such that \( \tilde{U}(\phi)f \) is a gauge-invariant function cylindrical over \( \tilde{\gamma} \).

**Proof.** Choose a maximal tree \( \tau \) in the graph \( \gamma \). Choose a vertex \( x \in V(\tau) \) and for each edge \( e_i \in E(\gamma), E(\tau) \) not belonging to the tree denote the unique path from \( x \) to \( s(e_i) \) lying in \( \tau \) as \( p_i \), and from \( v \) to \( t(e_i) \) as \( p_i^2 \). Then define the \( E - V + 1 \) paths

\[
p_i := p_i^1 \circ e_i \circ (p_i^2)^{-1}. \tag{5.21}
\]

Then \( v = (p_1, \ldots, p_{E-V+1}) \) is a hyph, since each path \( p_i \) contains \( e_i \), which is a free segment for all the other paths. Note that, due to gauge invariance, the function \( f \) only depends on the parallel transports along the paths \( p_i \) (see, e.g. [22]). In particular, \( f \) is cylindrical over the hyph \( v := (p_1, \ldots, p_{E-V+1}) : f \in \text{Cyl}(v) \). But since all paths \( p_i \) in \( v \) start and end in the vertex \( x \), by lemma 5.2 there is an automorphism mapping the hyph \( v \) to an \((E - V + 1)\)-flower graph \( \tilde{\gamma} \). So \( f \) gets mapped to a function \( \tilde{U}(\phi)f \) being cylindrical over that flower graph. \( \square \)

Lemma 5.4 shows the tremendous size that the orbits of a vector under the action of \( \text{Aut}(\mathcal{P}) \) have. In the case of the Abelian gauge group \( G = U(1) \), the size is so large that one can in fact compute the orbits, in the sense that one can determine the set of equivalence classes of vectors in \( \mathcal{H}^{(\text{kin})}_{\text{aut}} \) under the action of \( \text{Aut}(\mathcal{P}) \).

In the following section we will need the notion of a simple loop.

**Definition 5.1.** Let \( x \in \Sigma \) and \( l \in \text{Mor}(x, x) \) be a morphism. Then \( l \) is called a simple loop if one has \( l = [[[c]]] \) for a piecewise analytic curve \( c : [0, 1] \to \Sigma \) with the property that \( c(\ell) = c(s) \) means that \( |s - t| \) is an integer.

Simple loops have an important property, which is that they are no proper power of another loop.

**Lemma 5.5.** Let \( l \in \text{Mor}(x, x) \) be a simple loop. If \( l = \tilde{l}^n \) for some \( \tilde{l} \in \text{Mor}(x, x) \), then \( n = \pm 1 \).

**Proof.** Decompose \( \tilde{l} \) into edges \( e_1, \ldots, e_E \) of a graph \( \gamma \) with \( [\tilde{l}] \leq \gamma \). Then \( \tilde{l} \) is a word in \( e_i^1 \):

\[
\tilde{l} = \theta(e_1, \ldots, e_E). \tag{5.22}
\]

Denote by \( N_k \) the numbers of times the edge \( e_k \) appears in the word \( \theta \) (with multiplicity, i.e. \( e_k^{-1} \) counts as \(-1\)). Clearly \( l \) is a word in \( e_k \) as well, and with multiplicity the number of times \( e_k \) appears in the word \( \theta^n \) is \( nN_k \). However,

\[
l = e_k^{\pm 1} \circ e_k^{\pm 1} \circ \ldots \circ e_k^{\pm 1}. \tag{5.23}
\]
with distinct $e_k$, since $l$ is injective (up to the endpoints), and $e_k$ also at most intersect in their respective endpoints, since they form a graph. Therefore, 1 is divisible by $n \in \mathbb{Z}$, hence $n = \pm 1$. □

5.3. The automorphism-invariant Hilbert space for gauge group $G = U(1)$

Cylindrical functions for the gauge group $G = U(1)$ carry particularly few information about the graph they are cylindrical over.

**Lemma 5.6.** Let $\gamma = \{e_1, \ldots, e_E\}$ be an $E$-flower graph (see figure 7). Consider a word $\vartheta(e_2, \ldots, e_E)$ in the edges of $\gamma$, apart from $e_1$. Since $\gamma$ is a flower graph, $\vartheta$ is a path in the groupoid $P_\gamma$, i.e. a path starting and ending at the one vertex in $V(\gamma)$, and going through the edges $e_2, \ldots, e_E$ in an order determined by the word $\vartheta$. Then there is an automorphism $\phi \in \text{Aut}(P)$ with the following properties:

$$
\begin{align*}
\phi : e_1 &\mapsto e_1 \circ \vartheta(e_2, \ldots, e_E) \\
e_k &\mapsto e_k, \quad \text{for } k \in \{2, \ldots, E\}.
\end{align*}
$$

(5.24)

**Proof.** Note that $(e_2, \ldots, e_E, e_1 \circ \vartheta)$ is a hyph, since $e_1 \circ \vartheta(e_2, \ldots, e_E)$ has a free segment w.r.t. $e_2, \ldots, e_E$ (which is $e_1$), and the remaining $(e_2, \ldots, e_E)$ form a graph. Since $(e_1, \ldots, e_E)$ is also a hyph, having the same combinatorics, the assertion follows directly from lemma 5.2. □

**Lemma 5.7.** Let $\gamma$ be an $E$-flower graph, $\vec{n} \in \mathbb{Z}^E$ and $T_{\gamma, \vec{n}}$ be a charge network. Let $(m_2, \ldots, m_E) \in \mathbb{Z}^{E-1}$, then there is an automorphism $\phi \in \text{Aut}(P)$ such that

$$
\hat{U}(\phi)T_{\gamma, \vec{n}} = T_{\gamma, \vec{n}'}
$$

(5.25)

with

$$
\begin{align*}
n'_1 &= n_1 \\
n'_{k} &= n_k + n_1 m_k \quad \text{for } k \in \{2, \ldots, E\}.
\end{align*}
$$

(5.26)

**Proof.** Choose the word

$$
\vartheta(e_2, \ldots, e_E) = e_2^{m_2} \circ \ldots \circ e_E^{m_E}.
$$

(5.27)

Then, by lemma 5.6, there is an automorphism $\phi \in \text{Aut}(P)$ with

$$
\phi(e_1) = e_1 \circ e_2^{m_2} \circ \ldots \circ e_E^{m_E}.
$$

(5.28)
and which leaves all other \( e_k \) invariant. Thus, we get

\[
(\hat{U}(\phi)T_{\gamma,\vec{n}})(A) = \prod_{k=1}^{E} A(\phi(e_k))^n_k
\]

\[
= A(e_1)^{n_1} \prod_{k=2}^{E} A(e_k)^{n_k} = A(e_1)^{n_1} \prod_{k=2}^{E} A(e_k)^{n_k+m_k}
\]

\[
= T_{\gamma,\vec{n}'}(A),
\]

where \( T_{\gamma,\vec{n}'} \) is given by (5.26).

\[
\text{Lemma 5.8.} \quad \text{Let } \gamma \text{ be an E-flower graph, } \vec{n} \in \mathbb{Z}^E \text{ and } T_{\gamma,\vec{n}} \text{ a charge-network function. Then there is an automorphism } \phi \in \text{Aut}(\mathcal{P}) \text{ such that}
\]

\[
\hat{U}(\phi)T_{\gamma,\vec{n}} = T_{\gamma,(p,0,0,\ldots,0)},
\]

where \( p \in \mathbb{Z} \) is the greatest common divisor of \( |n_1|, \ldots, |n_E| \).

\textbf{Proof.} First assume that all \( n_k \geq 0 \). If this is not the case, one can, by lemma 5.2, find an automorphism that reverses the direction of all edges \( e_k \) with \( n_k < 0 \):

\[
\phi(e_k) = e_k^{-1} \quad \text{if} \quad n_k < 0
\]

\[
\phi(e_k) = e_k \quad \text{else}.
\]

The assertion then follows directly from the Euclidean algorithm. \cite{23} This algorithm transforms an \( n \)-tuple of numbers \( (n_1, \ldots, n_E) \) to \( (p,0,0,\ldots,0) \) by successively applying steps of the type (5.26). The unique number \( p \) is the greatest common divisor of \( n_k \).

To each of the steps in the Euclidean algorithm there exists a corresponding automorphism \( \phi^{(k)} \). Then we have constructed a series of automorphisms mapping \( T_{\gamma,\vec{n}} \) to some \( T_{\gamma,(p,0,0,\ldots,0)} \):

\[
\hat{U}(\phi^{(N)})\hat{U}(\phi^{(N-1)}) \cdots \hat{U}(\phi^{(1)})T_{\gamma,\vec{n}} = T_{\gamma,(p,0,0,\ldots,0)}.
\]

(5.31)

This completes the proof.

\text{Lemma 5.9.} \quad \text{Let } \gamma \text{ be a one-flower graph. If for } n, m \in \mathbb{Z} \text{ there is an automorphism } \phi \in \text{Aut}(\mathcal{P}) \text{ such that } \hat{U}(\phi)T_{\gamma,n} = T_{\gamma,m}, \text{ then } n = \pm m.

\textbf{Proof.} We have \( \gamma = \{l\} \) for a simple loop \( l \). \( \hat{U}(\phi)T_{\gamma,n} = T_{\gamma,m} \) means that

\[
A(\phi(l))^n = A(l)^m \quad \text{for all} \quad A \in \mathcal{A}.
\]

(5.32)

Decompose \( l \) as well as \( \phi(l) \) into edges \( e_1, \ldots, e_E \) of a graph \( \tilde{\gamma} \). Denote the number of times each of \( e_k \) appears in the decomposition of \( l \) (with multiplicity, i.e. \( e_k^{-1} \) counts as \( -1 \)) by \( N_k \), and the number of times it appears in \( \phi(l) \) by \( M_k \). W.l.o.g. we numerate \( e_k \) such that \( l = e_1 e_2 \cdots e_K \), i.e. \( N_k = 1 \) for \( k \leq K \), and \( N_k = 0 \) else. For \( A \in \mathcal{A} \) denote \( \lambda_k := A(e_k) \in U(1) \). Then by (5.32) for each \( k \leq K \) one has

\[
\lambda_k^{M_k} = \lambda_k^m \quad \text{for all} \quad k \leq K, \quad \text{and} \quad \lambda_k \in U(1).
\]

(5.33)

We conclude that \( n M_k = m \) for all \( k \). Since \( M_k \in \mathbb{Z} \), \( m \) is divisible by \( n \). However, we also have \( \hat{U}(\phi^{-1})T_{\gamma,m} = T_{\gamma,n} \), and by the analogous argument \( n/m \in \mathbb{Z} \). We conclude that \( n = \pm m \), which was the claim.
By lemma 5.4, every gauge-invariant charge-network function can be mapped by an automorphism to one on a flower graph (on which all charge networks are automatically gauge invariant). In lemma 5.8 we have seen that for the gauge group $G = U(1)$, each charge-network function on a flower graph can be mapped by an automorphism into $T_{y,n}$ for some one-flower graph $y$ and some $n \in \mathbb{Z}$. This $n$ is unique up to a sign, as we have seen in lemma 5.9. We have thus determined the complete set of orbits of charge-network states under the action of $Aut(M)$.

**Lemma 5.10.** Let $D$ be the linear span of all charge-network states $T_{y,\vec{n}}$ in $\mathcal{H}_{\text{kin}}^{(g_1)}$. Then the following map:

$$\eta : D \longrightarrow D_{\text{Aut}(M)}$$

$$\eta[T_{y,\vec{n}}]T_{y',\vec{n}'} := \sum_{[\phi] \in Aut(M)/{\sim_{y,\vec{n},y',\vec{n}'}}} \langle T_{y,\vec{n}} | \hat{U}(\phi)T_{y',\vec{n}'} \rangle$$

and extended antilinearly to all of $D$ define an antilinear map from $D$ to the automorphism-invariant linear functionals over $D$. Here the equivalence relation $\sim_{y,\vec{n},y',\vec{n}'}$ is given by

$$\phi_1 \sim_{y,\vec{n},y',\vec{n}'} \phi_2 \iff \langle T_{y,\vec{n}} | \hat{U}(\phi_1)T_{y',\vec{n}'} \rangle = \langle T_{y,\vec{n}} | \hat{U}(\phi_2)T_{y',\vec{n}'} \rangle.$$  

**Proof.** One needs to show that

$$\eta[T_{y,\vec{n}}]T_{y',\vec{n}'} := \eta[T_{y,\vec{n}}](\hat{U}(\phi)T_{y',\vec{n}'})$$

for all charge-network functions $T_{y,\vec{n}}, T_{y',\vec{n}'}$ and all $\phi \in Aut(M)$. Since for every graph $y'$ and every $\phi \in Aut(M)$ one has that $\phi(y') \supseteq y''$ for a graph $y''$, and $U(1)$ is Abelian, the action of $Aut(M)$ leaves the charge-network functions invariant, so $\hat{U}(\phi)T_{y',\vec{n}'} = T_{y'',\vec{n}''}$. Define a map

$$\Xi_\phi : Aut(M)/{\sim_{y,\vec{n},y',\vec{n}'}} \longrightarrow Aut(M)/{\sim_{y,\vec{n},y'',\vec{n}''}}$$

$$[\chi] \longmapsto [\chi \circ \phi^{-1}].$$

This map is well defined: if $[\chi] = [\rho] \in Aut(M)/{\sim_{y,\vec{n},y',\vec{n}'}}$, then

$$\langle T_{y,\vec{n}} | \hat{U}(\chi)T_{y',\vec{n}'} \rangle = \langle T_{y,\vec{n}} | \hat{U}(\rho)T_{y',\vec{n}'} \rangle$$

hence

$$\langle T_{y,\vec{n}} | \hat{U}(\chi \circ \phi^{-1})T_{y',\vec{n}'} \rangle = \langle T_{y,\vec{n}} | \hat{U}(\rho \circ \phi^{-1})T_{y',\vec{n}'} \rangle,$$

which is equivalent to say that $[\chi \circ \phi^{-1}] = [\rho \circ \phi^{-1}] \in Aut(M)/{\sim_{y,\vec{n},y'',\vec{n}''}}$. Furthermore, the map is injective, since if (5.40) does not hold, so does not (5.39). Due to $\Xi_\phi[\rho \circ \phi] = [\rho]$ the map is also surjective, hence a bijection. We now get

$$\eta[T_{y,\vec{n}}]T_{y',\vec{n}'} = \sum_{[\rho] \in Aut(M)/{\sim_{y,\vec{n},y',\vec{n}'}}} \langle T_{y,\vec{n}} | \hat{U}(\rho)T_{y',\vec{n}'} \rangle$$

$$= \sum_{[\rho] \in Aut(M)/{\sim_{y,\vec{n},y',\vec{n}'}}} \Xi_\phi^{-1}[\rho] \langle T_{y,\vec{n}} | \hat{U}(\rho \circ \phi^{-1})T_{y',\vec{n}'} \rangle$$

$$= \sum_{[\rho] \in Aut(M)/{\sim_{y,\vec{n},y',\vec{n}'}} \Xi_\phi^{-1}[\rho] \langle T_{y,\vec{n}} | \hat{U}(\rho)T_{y',\vec{n}'} \rangle$$

$$= \eta[T_{y,\vec{n}}](\hat{U}(\phi)T_{y',\vec{n}'})$$

This was the claim. □

We have seen that for each charge-network state $T_{y,\vec{n}}$ there is a unique $n \geq 0$ such that

$$[T_{y,\vec{n}}] = [\bigcirc_n]$$

(5.41)
where $\bigcirc$ is a one-flower graph, $\bigcirc_n$ denotes the charge-network function on $\bigcirc$ given by the one charge $n$ and $[\cdot]$ denotes the orbit of $\cdot$ under $\text{Aut}(\mathcal{P})$. Denote this $n$ by $\omega(\tilde{n})$, then we have
\[
\sum_{[\phi] \in \text{Aut}(\mathcal{P})/\sim} \langle T_{\gamma,\tilde{n}} \mid \hat{U}(\phi)T_{\gamma',\tilde{n}'} \rangle = \delta_{\omega(\tilde{n}),\omega(\tilde{n}')},
\]
due to the fact that any $\phi \in \text{Aut}(\mathcal{P})$ maps a charge network into another charge network, so $\langle T_{\gamma,\tilde{n}} \mid \hat{U}(\phi)T_{\gamma',\tilde{n}'} \rangle$ is either 0 or 1.

With this rigging map $\eta$, an inner product $\langle \cdot | \cdot \rangle_{\text{Aut}}$ can be defined on the set of all finite linear combinations of charge-network functions, and we thus see.

**Corollary 5.1.** The automorphism-invariant inner product on $\eta(D)$ given by the rigging map (5.34) can be completed to the automorphism-invariant Hilbert space $H_{\text{Aut}} = \left\{ \infty \sum_{k=0}^{\infty} c_n [\bigcirc_n] \mid \sum_n |c_n|^2 < \infty \right\}$.

**Proof.** This is clear from the fact that the orbit of every finite linear combination $f$ of charge networks is given by
\[
[f] = \sum_{k=0}^{N} c_n [\bigcirc_n]
\]
and
\[
\langle [\bigcirc_n] | [\bigcirc_m] \rangle_{\text{Aut}} = \delta_{nm}.
\]
As we have seen, the automorphism-invariant Hilbert space can be computed directly, for a certain choice of rigging map (5.34). For each charge-network function $T_{\gamma,\tilde{n}}$ on a graph $\gamma$, there is a natural number $n$ and an automorphism $\phi$ such that $\hat{U}(\phi)T_{\gamma,\tilde{n}} = \bigcirc_n$. This shows how tremendously large the orbits of the automorphism group are in the case of $G = U(1)$. The reason for this is the following: given any graph $\gamma$ with $E$ edges and a charge-network function $T_{\gamma,\tilde{n}}$, then $T_{\gamma,\tilde{n}}(A) = e^{i(n_1\phi_1 + \cdots + n_E\phi_E)}$ (5.46)

where $e^{i\phi} = A(e_k)$ is the holonomy of $A$ along the edge $e_k$. Then our previous results show that there is a closed loop $l$ in $\gamma$, i.e. a path consisting of edges in $\gamma$ and their inverses, which starts and ends at the same point, such that each $e_k$ is traversed exactly $n_k$ times (counting going against the orientation of $e_k$ as $-1$). Then
\[
T_{\gamma,\tilde{n}}(A) = e^{i\psi}
\]
with $\psi = n_1\psi_1 + \cdots + n_E\psi_E$, and $e^{i\psi} = A(l)$ is the holonomy along $l$. So the charge-network function $T_{\gamma,\tilde{n}}$ is cylindrical over the metagraph $l$. If $l = \tilde{l}^n$ for a simple loop $\tilde{l}$, then $T_{\gamma,\tilde{n}}$ is also cylindrical over $\tilde{l}$, and we have shown that there is an automorphism $\phi$ mapping the loop $\tilde{l}$ to a (say) circle $\bigcirc$, hence $T_{\gamma,\tilde{n}}$ to $\bigcirc_n$. □

The above consideration rests crucially on the abelianess of $U(1)$. In particular, it does not matter in which order $l$ transverses the paths in $\gamma$, just how many times. This will not be true for non-Abelian gauge groups, such as $G = SU(2)$, as we will see in the following.
5.4. The automorphism-invariant Hilbert space for the gauge group $G = SU(2)$

In this section, we investigate the set of orbits of vectors in $\mathcal{H}_{\text{kin}}^{(g)}$ under the action of $Aut(\mathcal{P})$, in the case of $G = SU(2)$. Ultimately, the goal is to compute the set of linear functionals invariant under $Aut(\mathcal{P})$ with some inner product, such as in the case for $G = U(1)$, for example

$$\eta : D \rightarrow D_{\text{Aut}(\mathcal{P})}$$

$$\eta[f]g := \sum_{[\phi] \in \text{Aut}(\mathcal{P})/\sim} \langle f \mid \hat{U}(\phi)g \rangle.$$  (5.48)

Again, $\phi_1 \sim \phi_2$ if $\langle f \mid \hat{U}(\phi_1)g \rangle = \langle f \mid \hat{U}(\phi_2)g \rangle$.

To compute the set of orbits of vectors in $\mathcal{H}_{\text{kin}}^{(g)}$ under the action of $Aut(\mathcal{P})$ for the case of $G = SU(2)$ is more difficult, due to the fact that for a spin network function $T_{\gamma,j,\vec{I}}$, the transformed $\hat{U}(\phi)T_{\gamma,j,\vec{I}}$ is not necessarily a spin network function anymore: if one chooses, for each graph $\gamma$, an orthonormal basis of intertwiners $\vec{I}$ at each vertex, then any such basis vector can be mapped to a finite linear combination of basis vectors by an automorphism.

From this difficulty the phenomenon arises that for two vectors $T_{\gamma,j,\vec{I}}, T_{\gamma',j',\vec{I}'}$ the set of all possible overlaps

$$\{ \langle T_{\gamma,j,\vec{I}} \mid \hat{U}(\phi) T_{\gamma',j',\vec{I}'} \rangle \mid \phi \in \text{Aut}(\mathcal{P}) \}$$  (5.49)

is not just $\{0, 1\}$, as in the case for piecewise analytic diffeomorphisms, or for automorphisms and gauge group $G = U(1)$. We consider a simple example.

We have already seen in lemma 5.4 that every gauge-invariant function on a graph can be mapped by an automorphism to a gauge-invariant function on a flower graph. Thus, it is sufficient to consider the orbits of gauge-invariant functions on flower graphs. Consider the two-flower graph $\gamma = \{e_1, e_2\}$.

For fixed $j_1, j_2 \in \frac{1}{2}[N]$, the intertwiner space is $j_1 + j_2 - |j_1 - j_2|$-dimensional. Assume $j_1 = j_2 = \frac{1}{2}$. Then the intertwiner space is two-dimensional. If we choose the normalized vector

$$T_1(A) := \text{tr}_\frac{1}{2}(A(e_1)) \text{ tr}_\frac{1}{2}(A(e_2))$$  (5.50)

to be one of the two orthonormal basis vectors in the intertwiner space, and choose some normalized $T_2$ orthogonal to it, we get an orthonormal base of the intertwiner space. On the other hand, consider the one-flower graph (in fact, a Wilson loop $\bigcirc = \{e\}$ having only one edge $e$, and the gauge-invariant function $T_0$ cylindric on $\bigcirc$ given by

$$T_0(A) := \text{tr}_\frac{1}{2}(A(e)).$$  (5.51)

With the spin $j = \frac{1}{2}$ on the edge $e$, there is only one gauge-invariant function on $\bigcirc$ (the intertwiner space is one dimensional), which is given exactly by $T_0$. By lemma 5.2, however, we know that there is an automorphism $\phi$ such that $\phi(e) = e_1 \circ e_2$, so $\hat{U}(\phi)T_0$ is cylindrical over the two-flower graph $\gamma$, and in fact can be decomposed into $T_1, T_2$. Since

$$\hat{U}(\phi)T_0(A) = \text{tr}_\frac{1}{2}(A(e_1) \cdot A(e_2)),$$  (5.52)

a short calculation reveals that

$$\langle T_1 \mid \hat{U}(\phi)T_0 \rangle = \frac{1}{2}.$$  (5.53)

17 Technically, there are many flower graphs, but all of them can be mapped into each other by the automorphisms by lemma 5.2, so in the following we speak of ‘the’ 2-flower graph (see figure 8).
Figure 8. The two-flower graph.

So

$$\hat{U}(\phi)T_0 = \cos \frac{\pi}{3} T_1 + \sin \frac{\pi}{3} T_2.$$  \hspace{1cm} (5.54)

We see that, since there are nontrivial ways to embed an $E$-flower into an $E'$-flower (with $E < E'$) by ‘wrapping loops’, certain basis vectors on the $E$-flower will be mapped to nontrivial combinations of basis vectors on the $E'$-flower. All these nontrivial overlaps will show up in the automorphism-invariant inner product (5.48). To compute all the contributions is now a combinatorial task. We refrain from doing so here, but simply state a conjecture about the nature of the set of orbits of vectors in $\mathcal{H}_{\text{kin}}$ under the action of $\text{Aut}(\mathcal{P})$.

In our example above, we have seen that the function $T_1$ and $\hat{U}(\phi)T_0$ constitute a basis of the intertwiner space for functions on a 2-flower graph with $j_1 = j_2$. Note that this is not an orthonormal basis, but by (5.53) have an angle of $\cos \frac{\pi}{3}$ with respect to each other. Now consider again the two-flower graph $\gamma = \{e_1, e_2\}$, but with $j_1 = \frac{1}{2}$, $j_2 = 1$. Then again, the intertwiner space is two dimensional, and one normalized vector is

$$T_3(A) := \text{tr}_2(A(e_1)) \text{ tr}_1(A(e_2)).$$  \hspace{1cm} (5.55)

On the other hand, consider the automorphism $\phi' \in \text{Aut}(\mathcal{P})$ mapping the closed loop $e$ to $e_1 \circ (e_2)^2$. One can then show that the projection of $\hat{U}(\phi')T_0$ to the intertwiner space on the 2-flower graph with $j_1 = \frac{1}{2}$, $j_2 = 1$ is unequal $T_3$, but has non-vanishing inner product with $T_3$. Thus, this projection (denote it as $\Pi(\hat{U}(\phi')T_0)$), together with $T_3$, forms a basis for the intertwiner space for $j_1 = \frac{1}{2}$, $j_1 = 1$. Again, this is no orthonormal basis, since the two vectors are only linearly independent, not orthogonal.

For each $E$-flower graph $\gamma = \{e_1, \ldots, e_E\}$ with spins $j_1, \ldots, j_E$ there is a normalized vector in the intertwiner space given by

$$T_{E,j}(A) := \prod_{k=1}^{E} \text{ tr}_k(A(e_k)).$$  \hspace{1cm} (5.56)

In a graphical notation similar to the one used in the last section, we write

$$\begin{bmatrix}
\circ_{j_1} & \circ_{j_2} & \cdots \\
\circ_{j_{E-1}} & \circ_{j_E} \\
\cdots & \cdots & \cdots
\end{bmatrix} := [T_{E,j}].$$  \hspace{1cm} (5.57)

Extending what we have just seen in the examples to higher flowers with higher spins suggests that all other elements in the intertwiner space can be composed by certain $\hat{U}(\phi)T_{E',j'}$ with $E' < E$. This would mean that the orbits $[T_{E,j}]$ would constitute a basis for the set of orbits of all smooth cylindrical functions in $\mathcal{H}_{\text{kin}}$. Since these vectors have nontrivial overlap, however, one would have to compute their automorphism-invariant inner product by (5.48) and deduce
an orthonormal basis from them by the Gram–Schmidt procedure. This would provide a way to find an orthonormal basis for $\mathcal{H}_{\text{Aut}}$.

This would suggest that the linear combinations of the form

$$\psi = \sum_{E=0}^{N} \sum_{j_1,\ldots,j_E} c_{E,j} \begin{bmatrix} \circ j_1 \\ \circ j_{E-1} \\ \circ j_{E} \\ \cdots \end{bmatrix}$$

form a dense set in $\mathcal{H}_{\text{Aut}}$. If one could then derive a good formula for the inner product between two such vectors, we would be able to write down an orthonormal basis for $\mathcal{H}_{\text{Aut}}$. So far, this has not been done due to the complicated combinatorics, but we will address this point in a later publication.

6. Summary and outlook

6.1. Summary of the work

In this publication, we have investigated the consequence of extending the group of spatial diffeomorphisms $\text{Diff}(\Sigma_1)$ to path groupoid automorphisms $\text{Aut}(\mathcal{P})$ in loop quantum gravity. This extension is inspired by category theory, and contains many elements that cannot be interpreted as diffeomorphisms from $\Sigma$ to itself. This mimics the extension of smooth to generalized gauge transformations: while the first consists of smooth maps from $\Sigma$ to the gauge group $G$, the latter one consists of all such maps, without continuity or even measurability assumption.

We have delivered a proof that the action of $\text{Aut}(\mathcal{P})$ on $\mathcal{A}$, the set of (distributional) connections leave the Ashtekar–Isham–Lewandowski measure $\mu_{\text{AL}}$ invariant.

An automorphism is given by a permutation of the points in $\Sigma$, and a permutation of paths in $\Sigma$ which are compatible with each other in the sense that if the path $p$ starts at $x$ and ends at $y$, then the transformed path $\phi(p)$ starts at $\phi(x)$ and ends at $\phi(y)$. But by the groupoid structure of $\mathcal{P}$, the set of piecewise analytic paths in $\Sigma$, this does not necessarily restrict what happens to points that lie ‘in the middle’ of $p$. So the notion of a point lying on a path is not invariant under automorphisms, which makes many nontrivial constructions possible. In particular, some automorphisms simply cannot be interpreted as maps from $\Sigma$ to itself.

We have given some explicit examples for automorphisms that do not arise as diffeomorphisms on $\Sigma$. The first example was given by the natural transformations of the identity, which are able to arbitrarily permute the points in $\Sigma$, but keep the paths essentially the same (and which act, in particular, as identity on the gauge-invariant part of the kinematical Hilbert space $\mathcal{H}_{\text{kin}}$). The second example was given by the edge-interchangers, which left all points invariant, but swapped two edges $e_1, e_2$ with the same beginning and endpoints. All paths that meet these two edges at finitely many points are left invariant, however. In this sense, these automorphisms have support only at two edges $e_1, e_2$, and hence can be viewed as distributional.

We have used these two types of automorphisms in order to show that every two graphs with the same combinatorics can be mapped into each other by an automorphism $\phi \in \text{Aut}(\mathcal{P})$. This shows how little information about the differential structure, and in fact even the topology of $\Sigma$, is encoded in the path groupoid $\mathcal{P}$.

In the last part of this work, we have investigated the induced action of the automorphisms $\text{Aut}(\mathcal{P})$ on the kinematical Hilbert space $\mathcal{H}_{\text{kin}}$. In particular, we have shown that due to the size of automorphisms, the only information that is conserved by acting with automorphisms
on a cylindrical function is the combinatorics of paths, over which it is cylindrical. This information is highly redundant in the description of function cylindrical over paths, and it is the use of graphs to provide a way of finding a good representative in the set of collections of paths over which a function is cylindrical. Since the automorphisms do not leave the set of graphs invariant, functions cylindrical over one graph can be mapped to a function cylindrical over another graph, although the graphs themselves are not mapped to each other.

A gauge-invariant function on a graph $\gamma$ does only depend on the holonomies along a number of loops in $\gamma$, which correspond to the first fundamental class $\pi_1(\gamma)$. Consequently, any gauge-invariant function on a graph can be mapped to a gauge-invariant function on some flower graph. This enabled us to gain some control over the size of the orbits of vectors in $H_{\text{kin}}$ under the action of $\text{Aut}(P)$. For some choice of rigging map, we derived the automorphism-invariant Hilbert space $H_{\text{Aut}}$ for the gauge group $G = U(1)$. It was found that the space is infinite dimensional and separable, the generic element being of the form

$$\psi_{\text{Aut}} = \sum_{k=0}^{\infty} c_n [\bigcirc_n],$$

(6.1)

with square-summable coefficients $\{c_n\}_{n\in\mathbb{N}}$, and $[\bigcirc_n]$ being the equivalence class of one Wilson loop with charge $n$.

For $G = SU(2)$, the combinatorics to work out the exact form of the orbits is harder, due to the recoupling scheme of $SU(2)$. However, we argued why we believe an element of the automorphism-invariant Hilbert space would be of the form

$$\psi_{\text{Aut}} = \sum_{E=0}^{\infty} \sum_{j_1,\ldots,j_E} c_{E,j} \left[ \begin{array}{cccc} \bigcirc_{j_1} & & & \\
 & \bigcirc_{j_{E-1}} & & \\
 & & \cdots & \\
 & & & \bigcirc_{j_E} \end{array} \right],$$

(6.2)

where the vectors are equivalence classes of $E$ separate Wilson loops with spin charges $j_k$. These vectors will, however, not be orthonormal to each other. Rather, their inner product will be determined by embedding combinatorics of $E$-flowers into $E'$-flowers for $E < E'$, and the corresponding recoupling scheme. We will return to a detailed analysis of this space in a future publication.

6.2. Further directions

By the form of the vectors (6.1) and (6.2), we see that the information about the degrees of freedom is completely delocalized, as one would have expected from a diffeomorphism-invariant theory. However, in order to get a good physical intuition for the meaning of the states in $H_{\text{Aut}}$, one would have the following possibility: one can repeat the whole analysis, but with matter degrees of freedom coupled to gravity. Note that the automorphism-invariant content of a cylindrical function for pure gravity is just given by the combinatorics of paths, the holonomies of which it depends on, but not on how these paths are embedded into space, i.e. if they intersect, or are partially parallel. So the automorphism-invariant vectors do not know about the vertices of a graph, for instance. Having matter degrees of freedom coupled to gravity, this might change, due to the following reason: consider a, say, fermionic matter field coupled to gravity. This field, when quantized along the lines of loop quantum gravity, would become a field sitting on the vertices of a graph, and is transformed by some nontrivial representation of $SU(2)$. The information about which field is situated at which vertex will still be contained in the gauge-invariant sector of the theory. Shifting these gauge-invariant cylindrical functions for matter plus gravity around by automorphisms would result in graphs
being mapped to other metagraphs, that look like different graphs, since the paths can intersect, or partially overlap. But the fact that some matter is excited at some vertex will not be changed by this, so one would be able to distinguish the real vertices (where matter is excited) from the ones that just appear as vertices, because the paths are embedded into space in some peculiar way. This would provide a natural mechanism of how matter could be used to localize gravitational degrees of freedom, as has been advocated e.g. in [24].

Another important point is to see whether operators corresponding to physical observables can be defined, i.e. such as the volume operator. This would not only allow for an interpretation of the automorphism-invariant states in terms of physical quantities. Also, if one could define a volume operator on the automorphism-invariant Hilbert space, one would be able to define the master constraint operator [25] on the automorphism-invariant Hilbert space. Since the graph-changing version of the master constraint changes in particular the first fundamental group of a graph, and the automorphism-invariant vectors contain exactly this information, this might provide a way to rephrase the quantum dynamics in some combinatorial way in $\mathcal{H}_{\text{Aut}}$. This would be particularly interesting, since one could make contact to [18], where such an operator already exists, leading to a combinatorial version of the dynamics.

One key step in this proof was theorem appendix B.1, a classical result from combinatorial group theory by Nielsen ('36). On the other hand, we also relied on the explicit construction of the kinematical Hilbert space $\mathcal{H}_{\text{kin}}$ for this proof. From an aesthetical point of view, however, this is unsatisfactory, since we feel that the fundamental reason for the automorphisms to leave the measure $\mu_{\text{AL}}$ invariant is exactly Nielsen’s theorem. It states a deep connection between automorphisms of free groups and the symmetries of the Haar measure. One therefore should be able to generalize most of the machinery presented in this paper to more general groupoids. With this one might be able to quantize much more general theories of connections on groupoids, not only on manifolds. We will return to this in a future publication.

The path groupoid $\mathcal{P}$, as a category, is a useful concept when investigating quantizations of Riemannian metrics on 3-manifolds. It was indicated [26–28] that for the investigation of Lorentzian metrics on 4-manifolds the notion of 2-category is an appropriate concept. It would be interesting to investigate, in which sense the analysis presented here could be repeated in such a framework, in order to see spacetime diffeomorphisms as automorphisms of 2-categories.

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Appendix A. Elements from category theory

In this section, we will briefly review the basic notions of category theory that are used in this paper18. Details and more about categories can be found in [29].

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18 Note that in this appendix, we use the standard convention for the composition, i.e. $f : X \to Y$ and $g : Y \to Z$ have a composition $g \circ f : X \to Z$. In the rest of the paper, however, we are using the notation $f \circ g$ for the composition, in order to stay consistent with large parts of the LQG literature.
Definition appendix A.1. A category $\mathcal{C}$ consists of a class of objects $X, Y, Z, \ldots$, denoted by $|\mathcal{C}|$, and for each pair of objects $X, Y \in |\mathcal{C}|$ a class of morphisms $f, g, h, \ldots$, denoted by $\text{Mor}_\mathcal{C}(X, Y)$. For these, the following rules hold.

- For each $f \in \text{Mor}_\mathcal{C}(X, Y)$ and $g \in \text{Mor}_\mathcal{C}(Y, Z)$ there is a morphism $g \circ f \in \text{Mor}_\mathcal{C}(X, Z)$ (the composite).
- Composition is associative, i.e. $h \circ (g \circ f) = (h \circ g) \circ f$.
- For each object $X \in |\mathcal{C}|$ there is a morphism $\text{id}_X \in \text{Mor}_\mathcal{C}(X, X)$ such that for all morphisms $f \in \text{Mor}_\mathcal{C}(X, Y)$ and $g \in \text{Mor}_\mathcal{C}(Z, X)$ one has $f \circ \text{id}_X = f$, $\text{id}_X \circ g = g$.

If $f \in \text{Mor}_\mathcal{C}(X, Y)$, then the source and the target of $f$ are denoted by $s(f) := X$ and $t(f) := Y$ respectively.

There are plenty of examples for categories.

- The category $\text{Set}$, the objects of which are sets, and the morphisms between two sets $X, Y$ are exactly all maps between these sets.
- For a manifold $\Sigma_1$, the category $\text{Hom}(\Sigma_1)$, the objects of which are the points in $\Sigma_1$, and the morphisms of which are homotopy equivalence classes of curves between points.
- For a manifold $\Sigma_1$, the category $\mathcal{W}(\Sigma_1)$, the objects of which are points in $\Sigma_1$, and the morphisms between two points are all curves between these points, modulo reparametrization.
- For a group $G$, the category $\text{Susp}(G)$, called the suspension of $G$. This category has only one object, denoted by $\ast$: $|\text{Susp}(G)| = \{\ast\}$, and the morphisms from $\ast$ to itself are in one-to-one correspondence with elements in $G$: $\text{Mor}_{\text{Susp}(G)}(\ast, \ast) = G$. Composition of morphisms then corresponds to group multiplication in $G$.

A category $\mathcal{C}$ in which every morphism has a right and a left inverse is called a groupoid.

In the above examples, $\text{Set}$ and $\mathcal{W}(\Sigma)$ are no groupoids, but $\text{Hom}(\Sigma)$ and $\text{Susp}(G)$ are.

It is customary in category theory to write statements as diagrams: consider a category $\mathcal{C}$ and objects $X, Y, Z, W \in |\mathcal{C}|$. Then the following diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{g} & & \downarrow{h} \\
Z & \xrightarrow{k} & W
\end{array}
$$

is said to commute, of $f \in \text{Mor}_\mathcal{C}(X, Y), g \in \text{Mor}_\mathcal{C}(X, Z), h \in \text{Mor}_\mathcal{C}(Y, W)$ and $k \in \text{Mor}_\mathcal{C}(Z, W)$, and $k \circ g = h \circ f$, as morphisms in $\text{Mor}_\mathcal{C}(X, W)$. Working with commuting diagrams makes reasoning in category theory fairly intuitive. Since the proofs in this paper are thoroughly analytical; however, commutative diagrams are only used at some points, to point out some connections.

Consider two morphisms $f, g$ in $\mathcal{W}(\Sigma)$ with $s(g) = t(f)$. Then they can be composed, i.e. two curves (up to reparametrization) can be concatenated and again give a curve modulo parametrization. It should be noted that, due to the above definitions, the concatenation is denoted by $g \circ f$. However, in the loop quantum gravity literature, this concatenation is usually denoted as $f \circ g$. The reason is that with this convention the generalized connections are functors from the path groupoid into the suspension of the gauge group $\text{Susp}(G)$, not its opposite category $\text{Susp}(G)^{op}$.

This is just a matter of convention, of course, but it should be noted that, throughout this paper, the composition of morphisms $f, g$ is usually denoted as $f \circ g$, not as $g \circ f$. 

36
Definition appendix A.2. Let $\mathcal{C}, \mathcal{D}$ be two categories. A functor is an assignment $F : |\mathcal{C}| \to |\mathcal{D}|$ and $F : \text{Mor}_\mathcal{C}(X, Y) \to \text{Mor}_\mathcal{D}(F(X), F(Y))$, such that

- $F(g \circ f) = F(g) \circ F(f)$
- $F(id_X) = id_{F(X)}$.

So a functor assigns objects to objects and morphisms to morphisms in a compatible way. As an example, consider the category which has smooth manifolds as objects and smooth maps between them. Then the Cartan differential is a functor from this category in itself. In particular, let $f : M \to N$ and $g : N \to O$ be smooth maps; then the chain rule guarantees that

$$d(g \circ f) = dg \circ df (A.1)$$

as maps $df : TM \to TN$, i.e. $dg : TN \to TO$.

Definition appendix A.3. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$ be two functors. One calls these two functors to be related by a natural transformation (or being natural transformations from each other), if there is, for each object $X \in |\mathcal{C}|$, a morphism $g_X \in \text{Mor}_\mathcal{D}(F(X), G(X))$ such that the following diagram commutes:

\[
\begin{array}{ccc}
F(X) & \xrightarrow{F(f)} & F(Y) \\
g_X \downarrow & & \downarrow g_Y \\
G(X) & \xrightarrow{G(f)} & G(Y)
\end{array}
\]

Note that if $\mathcal{D}$ is a groupoid (i.e. its morphisms can be inverted), then given a functor $F$ and for each $X \in |\mathcal{C}|$ an object $Z_X$ and a morphism $g_X \in \text{Mor}_\mathcal{D}(F(X), Z_X)$, $G(X) := t(g_X)$ for all $X \in |\mathcal{C}|$ $G(f) = g_Y \circ F(f) \circ g_X^{-1}$ for all $f \in \text{Mor}_\mathcal{C}(X, Y)$ defines a functor $G$, which can be related to $F$ by a natural transformation.

Appendix B. Elements from combinatorial group theory

Going over from piecewise analytic diffeomorphisms $\text{Diff}(\Sigma)$ to the automorphisms $\text{Aut}(\mathcal{P})$ is a significant enlargement of the gauge group. While the former preserves notions of (generalized) knotting classes of a graph, the latter one only keeps combinatorial information of the graph, i.e. which vertices are attached to each other by paths and which are not. Consequently, elements of combinatorial group theory enter the description, as soon as automorphisms are considered. In this appendix we review some basic notions from combinatorial group theory (details can be found in [30]) and conclude with a classical result of Nielson, which shows a connection between automorphisms of free groups and the symmetries of Haar measures. This is one key point in proving that the elements of $\text{Aut}(\mathcal{P})$ act unitarily on $\mathcal{H}_{\text{kin}}$.

Definition appendix B.1. Let $E \in \mathbb{N}$, and $\{X_1, \ldots, X_E\}$ be $E$ abstract symbols, called letters. A (finite) word in these letters is a (finite) sequence in these letters, i.e. $X_2, X_4 X_1$ or

19 Sometimes called ‘covariant functor’.

20 Note that there is no problem with allowing $E = \omega$, i.e. considering countably many letters. In this paper, however, we will only consider finitely many different letters.
A word in the $2E$ letters $\{X_1, \ldots, X_E, X_1^{-1}, \ldots, X_E^{-1}\}$ is called reduced, if no $X_k$ occurs next to its inverse $X_k^{-1}$.

It is clear that every word can be put into its reduced form, by successively eliminating $X_kX_k^{-1}$ or $X_k^{-1}X_k$ from it.

**Definition appendix B.2.** Consider the set of all reduced words in the $2E$ letters $\{X_1^\pm 1, \ldots, X_E^\pm 1\}$, together with the ‘empty word’, denoted by $1$, and define multiplication between two words as concatenating (and possibly reducing) them. The set of all these reduced words form a group under this multiplication, which is called the free group in $E$ letters, and are denoted by $F_E$.

**Examples**

\[(X_1X_2X_3) \cdot (X_3X_2X_1) = X_1X_2X_3X_2X_1 \] (B.1)

\[(X_1X_2X_3) \cdot (X_3^{-1}X_2^{-1}X_{17}) = X_1X_{17}. \] (B.2)

Similarly one can define the free group in zero $F_0 = \{1\}$ and in countable many parameters $F_\omega$. Free groups play a prominent rôle in combinatorics, algorithm theory and graph theory. Obviously, for $E < E'$, there is a natural inclusion of $F_E$ as a subgroup $F_{E'}$. What is less intuitive is that the free group in two parameters $F_2$ has a subgroup isomorphic to $F_\omega$. As a consequence, every $F_E$ has a subgroup isomorphic to $F_E$ for $E < E'$. This makes these groups more difficult than their linear counterparts, the $E$-dimensional vector spaces. A homomorphism $\phi$ from a free group $F_E$ into another group is completely determined by its values on the basic letters, $\phi(X_1), \ldots, \phi(X_E)$. An invertible group homomorphism from $F_E$ to itself is called an automorphism, and is completely determined by the $E$ words $\phi(X_k) = \vartheta_k(X_1, \ldots, X_E)$ which are the images of $X_k$ under $\phi$. The automorphisms of a free group are in fact well understood, which is shown by the following theorem.

**Theorem appendix B.1.** (Nielsen) Let $\phi$ be an automorphism on $F_E$. Then $\phi$ can be written as a finite product of ‘elementary’ automorphisms (called elementary Whitehead automorphisms)

\[ \phi = \xi_n \circ \xi_{n-1} \circ \ldots \circ \xi_1 \] (B.3)

where every $\xi_r$ is one of the following.

- A permutation ($l \neq k \in \{1, \ldots, E\}$):
  \[ \xi_r(X_k) = X_l, \quad \xi_r(X_l) = X_r, \quad \xi_r(X_m) = X_m \text{ else.} \] (B.4)

- An inversion ($k \in \{1, \ldots, E\}$):
  \[ \xi_r(X_k) = X_k^{-1}, \quad \xi_r(X_l) = \xi_r(X_l) \text{ else.} \] (B.5)

- A shift ($k \neq l \in \{1, \ldots, E\}$):
  \[ \xi_r(X_k) = X_kX_l, \quad \xi_r(X_l) = \xi_r(X_l), \quad \xi_r(X_m) = X_m \text{ else.} \] (B.6)

This important structural theorem has an immediate consequence.

**Corollary appendix B.1.** Let $G$ be a compact Lie group with bi-invariant Haar measure $d\mu_H$. Let $F \in L^1(G^E, d\mu_H^\otimes E)$, and $\phi$ be an automorphism on $F_E$. Denote $\phi(X_k) = \vartheta_k(X_1, \ldots, X_E)$. Then

\[ \int_{G^E} d\mu_H^\otimes E(h_1, \ldots, h_E) \ F(h_1, \ldots, h_E) \]

\[ = \int_{G^E} d\mu_H^\otimes E(h_1, \ldots, h_E) \ F(\vartheta_1(h_1, \ldots, h_E), \ldots, \vartheta_E(h_1, \ldots, h_E)). \] (B.7)
Proof. Nielson’s theorem tells us that the substitution
\[
\begin{align*}
  h_1 & \mapsto \vartheta_1(h_1, \ldots, h_E) \\
  \vdots & \quad \vdots \\
  h_E & \mapsto \vartheta_E(h_1, \ldots, h_E)
\end{align*}
\]
(B.8)

can be achieved by successively applying the ‘elementary’ substitutions (B.4), (B.5), (B.6) (with the symbols \(h_k\), instead of \(X_k\)). But under these substitutions the Haar measure \(d\mu_E^\otimes(h_1, \ldots, h_E)\) is invariant. The statement (B.7) follows. \(\square\)

Very often throughout this appendix, we will not be confronted with an automorphism of a free group, but an endomorphism between two free groups, i.e. an embedding of one free group into another, but not always is an embedding such that it can be continued to an automorphism\(^{21}\).

Definition appendix B.3. Let \(F_E\) be the free group in the letters \(X_1, \ldots, X_E\) and \(\chi_1, \ldots, \chi_n\) be \(n\) words in \(X_k\). A Nielson transformation on the words \(\{\chi_k\}_{k=1}^n\) is given by either of the following:

- exchange two of \(\chi_l\),
- replace \(\chi_l\) by \(\chi_l^{-1}\),
- replace \(\chi_l\) by \(\chi_l\chi_l'\) for some \(l' \neq l\), leaving the other \(\chi_l\) invariant.

The following is not hard to see: for any \(n\) words \(\chi_1, \ldots, \chi_n\) there is exactly one endomorphism \(\chi : F_n \to F_E\), where \(F_n\) is the group freely generated by the \(n\) letters \(Y_1, \ldots, Y_n\), such that \(\chi(Y_l) = \chi_l\). The image of \(\chi\) is exactly the subgroup of \(F_E\) generated by the words \(\chi_l\). Conversely, to every such endomorphism \(\chi\) correspond \(n\) words \(\chi_l\) in \(X_k\) given by the images of \(Y_l\) under \(\chi\). Applying a Nielson transformation to a collection of words is equivalent to replacing the corresponding endomorphism \(\chi\) by \(\chi \circ \phi\), where \(\phi : F_n \to F_n\) is an elementary Whitehead automorphism.

Definition appendix B.4. Let \(F_E\) be the free group generated by \(X_1, \ldots, X_E\), and let \(\chi_1, \ldots, \chi_n\) be \(n\) words in \(X_k\). The words \(\{\chi_k\}_{k=1}^n\) are called Nielsen reduced iff the following holds.

- For every word \(\vartheta\) in \(\{\chi_k\}_{k=1}^n\) each of the occurring \(\chi_k\) contributes at least one \(X_k^{\pm1}\) to the word \(\vartheta\), considered as a word in \(X_k\) (after it has been freely reduced).
- For any word \(\vartheta\) in \(\{\chi_k\}_{k=1}^n\), its length (considered as word in \(X_k\)) is at least as large as the length of any of \(\chi_l\) (as word in the \(X_k\)), i.e. in \(\vartheta\) at least as many \(X_k\) appear as in any of \(\chi_l\).

Generally, in a set of Nielsen-reduced words, by taking products not too many cancellations can take place. For instance, in \(F_2\) the two words \(X_1X_2, X_2X_1\) are Nielsen reduced, while \(X_1, X_1X_2\) are not.

The following lemma is showing that in each automorphism orbit of a set of words, there is always a set of words that is Nielsen reduced.

Lemma appendix B.1. (Nielsen) Let \(\vartheta_1, \ldots, \vartheta_n\) be words in \(F_E\). Then there is a sequence of Nielsen transformations \(T^{(i)}\) given by elementary Whitehead automorphisms \(\phi^{(i)}\) such that the words
\[
T^{(N)}T^{(N-1)} \ldots T^{(1)} \vartheta_1, \ldots, T^{(N)}T^{(N-1)} \ldots T^{(1)} \vartheta_n
\]
(B.9)

\(^{21}\)This is in contrast to the case of e.g. finite-dimensional vector spaces, where a linear map without kernel \(f : V \to W\) for \(V \subset W\) can always be continued to an isomorphism \(\tilde{f} : W \to W\).
are Nielsen reduced, where the $n$ words (B.9) correspond to the $n$ words given by the endomorphism

$$\vartheta \circ \phi^{(1)} \circ \ldots \circ \phi^{(N-1)} \circ \phi^{(N)}.$$  \hfill (B.10)

**Proof.** See [30] for a proof. \hfill □

The Nielsen-reduced words play a prominent rôle in combinatorial group theory. We will make intensive use of their properties in the following.

**Definition appendix B.5.** Let $E < E'$, $F_E$ the free group generated by $X_1, \ldots, X_E$, $F_{E'}$ the free group generated by $X_1, \ldots, X_{E'}$ (a canonical embedding from $F_E$ into $F_{E'}$ understood implicitly) and $\vartheta : F_E \to F_{E'}$ be an endomorphism. $\vartheta$ is called primitive if it can be continued to an automorphism, e.g. if there is an automorphism $\tilde{\vartheta} : F_{E'} \to F_{E'}$ with

$$\tilde{\vartheta}(X_k) = \vartheta(X_k), \quad 1 \leq k \leq E.$$  \hfill (B.11)

Then $\vartheta$ is called a primitive endomorphism, or a primitive embedding.

With these ingredients, we come to the main lemma of this appendix which comprises the core of the proof of unitarity of $\text{Aut}(P)$ on $\mathcal{H}_{\text{kin}}$ (see section 3).

**Lemma appendix B.2.** Let $K \leq L \leq M$. Let $\vartheta : F_K \to F_L$ and $\chi : F_L \to F_M$ be group homomorphisms. If $\chi \circ \vartheta : F_K \to F_M$ is a primitive embedding of $F_K$ into $F_M$, then $\vartheta$ is a primitive embedding of $F_K$ into $F_L$.

**Proof.** Let $F_K$ be generated by the letters $X_1, \ldots, X_K$, $F_L$ by $X_1, \ldots, X_L$ and $F_M$ by $X_1, \ldots, X_M$. By this, we make implicit use of the apparent canonical embedding of $F_n$ into $F_{n'}$ for $n \leq n'$. Throughout the proof we will make heavy use of the correspondence between homomorphisms of free groups and words.

By definition, we can extend $\chi \circ \vartheta$ to an automorphism $\xi \in \text{Aut}(F_M)$ such that $\xi(X_k) = \chi \circ \vartheta(X_k)$ for all $k = 1, \ldots, K$. Therefore, we can, without loss of generality, assume $\chi \circ \vartheta(X_k) = X_k$ for all $k = 1, \ldots, K$. If not, we redefine $\chi$ by $\xi^{-1} \circ \chi : F_L \to F_M$.

Consider the words

$$\chi_1, \ldots, \chi_L$$  \hfill (B.12)

which are given by the images of $X_1, \ldots, X_L$ under $\chi$.

Now apply a Nielsen transformation $T^{(1)}$ to these words (B.12). They get changed into

$$\chi^{(1)}_1, \ldots, \chi^{(1)}_L = T^{(1)}(\chi_1, \ldots, \chi_L)$$  \hfill (B.13)

to which corresponds an elementary Whitehead automorphism $\phi^{(1)} \in \text{Aut}(F_L)$. Equality (B.13) is an equation in $F_M$, i.e. $\chi^{(1)}_1, \chi^{(1)}_L$ are to be taken as words in $X_1, \ldots, X_M$. The two homomorphisms

$$\chi^{(1)} := \chi \circ \phi^{(1)} \quad \vartheta^{(1)} := (\phi^{(1)})^{-1} \circ \vartheta$$  \hfill (B.14)

have the property

$$\chi^{(1)}(X_k) = \chi^{(1)}_k \quad \chi^{(1)} \circ \vartheta^{(1)}(X_k) = X_k \quad k = 1, \ldots, K.$$  \hfill (B.15)

By lemma appendix B.1 there is a sequence of Nielsen transformations $T^{(1)}, \ldots, T^{(n)}$ such that

$$T^{(n)} \ldots T^{(1)}(\chi_1, \ldots, \chi_L)$$
are Nielsen-reduced words in $X_1, \ldots, X_M$. Denote the corresponding elementary Whitehead automorphisms by $\phi^{(1)}, \ldots, \phi^{(n)} \in \text{Aut}(F_L)$. With
\[
\chi^{(r)} = \chi^{(r-1)} \circ \phi^{(r)} \\
\vartheta^{(r)} = (\phi^{(r)})^{-1} \circ \vartheta^{(r-1)}
\]
for $r = 1, \ldots, n$ we have
\[
\chi^{(n)}(X_l) = X_l^{\pm 1} \\
\chi^{(n)} \circ \vartheta^{(n)}(X_k) = X_k \quad k = 1, \ldots, K.
\] (B.17)
But the words $\chi_k^{(n)}$ are Nielsen reduced as words in $X_k$. In particular, by definition, forming products of them can only enlarge their lengths. $\chi^{(n)} \circ \vartheta^{(n)}(X_k)$ are words in $\chi^{(n)}(X_l)$ (which are given by the endomorphism $\vartheta^{(n)}$). By (B.17) they have length 1 (considered as word in the $X_k$); therefore, each of $\vartheta^{(n)}_l = \vartheta^{(n)}(X_l)$ can contain at most one letter, e.g.
\[
\vartheta^{(n)}_l = X_l^{\pm 1}.
\] (B.18)
By applying at most $L$ more Nielsen transformations $T^{(n+1)}, \ldots, T^{(m)}$ (to which correspond elementary Whitehead automorphisms $\phi^{(n+1)}, \ldots, \phi^{(m)} \in \text{Aut}(F_L)$), we get
\[
\vartheta^{(m)}_l = \vartheta^{(m)}(X_l) = X_l
\] with $\vartheta^{(m)} = \phi^{-1} \circ \vartheta$, where
\[
\phi := \phi^{(m)} \circ \phi^{(m-1)} \circ \ldots \circ \phi^{(1)}.
\] (B.19)
But this shows that $\phi \in \text{Aut}(F_L)$ extends the homomorphism $\vartheta : F_K \to F_L$ to an automorphism. This was the claim. \hfill \Box

Appendix C. Categorical Weyl algebra of quantum gravity

The holonomies and their transformations, i.e. gauge transformations and diffeomorphisms, can be formulated in terms of category theory, and this formulation suggests to enlarge the diffeomorphism group Diff$(\Sigma)$ to the automorphisms of the path groupoid $\text{Aut}(\mathbb{P})$, which act unitarily on the kinematical Hilbert space $H_{\text{kin}}$ of loop quantum gravity.

The field algebra of loop quantum gravity, however, consists of holonomies as well as fluxes, and the symmetry groups act on both. In this appendix, we will show that also the fluxes can be formulated naturally in the languages of categories\footnote{Again, we use the convention that the concatenation of two morphisms $p$ and $q$ is denoted as $p \circ q$ if $t(p) = s(q)$, in order to stay consistent with the notation in [4].}

C.1. Categorial formulation of oriented surfaces

We want to obtain the categorial formulation of an ‘oriented surface’. An oriented surface does in principle two things: first, it cuts a path into several pieces (wherever a path intersects the surface), and second, it assigns to each of these pieces two numbers in $\{-1, 0, 1\}$. These
numbers depend on whether the the starting or, respectively, the end point of the piece meets the surface along or against the orientation of the surface \((\pm 1)\), or whether the starting or the end part of the piece lies entirely away from, or entirely within the surface (in which case it gets assigned 0). We will now turn this intuition into a categorical language.

**Definition appendix C.1.** Let \(C\) be a category. Then define the category \(\text{OrCut}(C)\) as follows.

- The objects in both categories are the same: \(|\text{OrCut}(C)| = |C|\).
- Let \(X, Y\) be objects in \(\text{OrCut}(C)\). The morphisms \(\text{Mor}(X, Y)\) in \(\text{OrCut}(C)\) can be constructed as follows: Consider three finite sequences with all the same length \(N\): A sequence \(p_1, \ldots, p_N\) of morphisms in \(C\) and two sequences of natural numbers \(m_1, \ldots, m_N\) and \(n_1, \ldots, n_N\). We arrange these three sequences as

\[
\begin{pmatrix}
p_1, \ldots, p_N \\
m_1, \ldots, m_N \\
n_1, \ldots, n_N 
\end{pmatrix}
\]  

(C.1)

The paths have the properties that \(t(p_k) = s(p_{k+1})\) for \(k = 1, \ldots, N - 1\), as well as \(s(p_1) = X\) and \(t(p_N) = Y\). We define an equivalence relation on these three sequences, by

\[
\begin{pmatrix}
p_1, \ldots, p_k, \quad p_{k+1}, \ldots, p_N \\
m_1, \ldots, m_k, \quad -n, \ldots, m_N \\
n_1, \ldots, n_k, \quad n_{k+1}, \ldots, n_N 
\end{pmatrix} \sim \begin{pmatrix} p_1, \ldots, p_k \circ p_{k+1}, \ldots, p_N \\
m_1, \ldots, m_k, \ldots, m_N \\
n_1, \ldots, n_{k+1}, \ldots, n_N 
\end{pmatrix}
\]  

(C.2)

and

\[
\begin{pmatrix}
p_1, \ldots, \text{id}, \ldots, p_N \\
m_1, \ldots, m, \ldots, m_N \\
n_1, \ldots, n, \ldots, n_N 
\end{pmatrix} \sim \begin{pmatrix} p_1, \ldots, \text{id}, \ldots, p_N \\
m_1, \ldots, m + j, \ldots, m_N \\
n_1, \ldots, n - j, \ldots, n_N 
\end{pmatrix}
\]  

(C.3)

for all \(j \in \mathbb{Z}\). The morphisms in \(\text{OrCut}(C)\) from \(X\) to \(Y\) then contain all equivalence classes of these sequences, which we denote by

\[
\begin{pmatrix}
p_1, \ldots, p_N \\
m_1, \ldots, m_N \\
n_1, \ldots, n_N 
\end{pmatrix} \in \text{Mor}(X, Y).
\]  

(C.4)

Concatenation of morphisms is obtained by just concatenating the sequences:

\[
\begin{pmatrix}
p_1, \ldots, p_N \\
m_1, \ldots, m_N \\
n_1, \ldots, n_N 
\end{pmatrix} \circ \begin{pmatrix}
q_1, \ldots, q_M \\
r_1, \ldots, r_M \\
s_1, \ldots, s_M 
\end{pmatrix} := \begin{pmatrix}
p_1, \ldots, p_N, q_1, \ldots, q_M \\
m_1, \ldots, m_N, r_1, \ldots, r_M \\
n_1, \ldots, n_N, s_1, \ldots, s_M 
\end{pmatrix}
\]  

(C.5)

while the identity functor in \(\text{Mor}(X, X)\) is given by

\[
\text{id} := \begin{bmatrix} \text{id}_X \\ 0 \\ 0 \end{bmatrix}.
\]  

(C.6)

It is straightforward to check that with these equivalence relations, the category \(\text{OrCut}(C)\) becomes a groupoid, if \(C\) is 1, if one defines the inverse of a morphism by

\[
\begin{pmatrix}
p_1, \ldots, p_N \\
m_1, \ldots, m_N \\
n_1, \ldots, n_N 
\end{pmatrix}^{-1} := \begin{pmatrix}
p_{\bar{N}1}, \ldots, p_{\bar{1}1} \\
n_{\bar{N}1}, \ldots, n_{\bar{1}1} \\
n_{\bar{N}1}, \ldots, m_{\bar{1}1} 
\end{pmatrix}
\]  

(C.7)
There is a projection functor
\[ \pi : \text{OrCut}(C) \rightarrow C \] (C.8)
the action of which is given by the identity on the objects, and by
\[ \pi : \begin{bmatrix} p_1, \ldots, p_N \\ m_1, \ldots, m_N \\ n_1, \ldots, n_N \end{bmatrix} \mapsto p_1 \circ \cdots \circ p_N. \] (C.9)

One checks quickly that this action is well defined.

In the following, we will consider the path groupoid \( \mathcal{P} \) and the category \( \text{OrCut}(\mathcal{P}) \). With the category \( \text{OrCut}(\mathcal{P}) \) we have a notion at hand to say what ‘cutting an edge’ \( p \) means in category language.

**Definition appendix C.2.** Let \( \mathcal{P} \) be the path groupoid of a manifold \( \Sigma \). Then a functor
\[ S : \mathcal{P} \rightarrow \text{OrCut}(\mathcal{P}) \] (C.10)
is called a generalized oriented surface, if the following two conditions hold.

- The functor \( \pi \circ S \) is the identity functor on \( \mathcal{P} \).
- For each primitive metagraph \( \mu \) with morphisms \( q_1, \ldots, q_n \) in \( \mathcal{P} \), there is a primitive metagraph \( \mu' \geq \mu \) with morphisms \( p_1, \ldots, p_m \), such that each \( p_k \) is \( S \)-trivial, which means that
\[ S(p_k) = \begin{bmatrix} p_k \\ m_k \\ n_k \end{bmatrix} \] (C.11)
for some natural numbers \( n_k, m_k \).

The interpretation of the first condition is obvious: to each edge \( p \), \( S \) assigns a collection \( p_1, \ldots, p_N \) such that \( p_1 \circ \cdots \circ p_N = p \), i.e. it cuts the path into pieces. Furthermore, to each piece there are assigned two natural numbers, which determine the position of the ends of the piece with respect to \( S \). It is obvious that each oriented surface in \( \Sigma \), and also each quasi-surface together with an orientation function in the sense of [5], determines a generalized surface in the above sense.

The second condition ensures that the surface does not cut a path in a too ‘wild’ way, and in particular cuts several different paths consistently with each other. It is this condition that will ensure that the Weyl operators constructed from the generalized oriented surfaces will be unitarities.

**Definition appendix C.3.** Let \( \Sigma \) be a manifold and \( \mathcal{P} \) its path groupoid. Let \( G \) be a compact Lie group and \( \mathcal{A} \) be the set of all functors from \( \mathcal{P} \) to \( \text{Susp}(G) \). Let furthermore \( S \) be a generalized oriented surface and \( d : \Sigma \rightarrow G \) be a map.

Then there is a transformation of functors
\[ \Theta_{S,d} : \mathcal{A} \rightarrow \mathcal{A} \] (C.12)
which is given by the following rule: if
\[ S(p) = \begin{bmatrix} p_1, \ldots, p_N \\ m_1, \ldots, m_N \\ n_1, \ldots, n_N \end{bmatrix}, \] (C.13)
then
\[\Theta_{S,d} A(p) := \prod_{k=1,\ldots,N} d(s(p_k))^m_k A(p_k) d(t(p_k))^n_k\]
\[:= d(s(p_1))^m_1 A(p_1) d(t(p_1))^n_1 \cdot \ldots \cdot d(s(p_N))^m_N A(p_N) d(t(p_N))^n_N. \tag{C.14}\]

It is easy to check that this action is well defined, i.e. does not depend on the representative of the equivalence class \(S(p)\). One can also readily see that this action coincides with the action of the Weyl elements determined by oriented quasi-surfaces [5].\(^{23}\) In the following, we will need a certain class of elements in \(C(\bar{A})\).

**Definition appendix C.4.** Let \(\mu = (p_1, \ldots, p_M)\) be a metagraph, then a function \(f : \bar{A} \rightarrow \mathbb{C}\) is called cylindrical (over \(\mu\)) if there is a function \(F : G^M \rightarrow \mathbb{C}\) with
\[f(A) = F(A(p_1), \ldots, A(p_M)). \tag{C.15}\]

In the following, \(f\) is referred to be continuous, smooth, etc, if and only if the corresponding \(F\) is. The space of smooth cylindrical functions over \(\mu\) is denoted by \(\text{Cyl}(\mu)\). The space of cylindrical functions \(\text{Cyl}\) is the union of \(\text{Cyl}(\mu)\) for all \(\mu \in \mathcal{M}\).

The space \(\text{Cyl}\) forms a dense set in \(C(\bar{A})\) w.r.t. the sup-norm, since \(\text{Cyl}\) in particular contains the functions cylindrical over graphs, which are dense in \(C(\bar{A})\).\(^{24}\)

**Lemma appendix C.1.** Let \(S\) be a generalized oriented surface and \(d : \Sigma \rightarrow G\) be a map. Then \(\Theta_{S,d}\) given by definition appendix C.3 has the following properties.
- \(\alpha \circ \Theta_{S,d} \circ \alpha^{-1} = \Theta_{\alpha S,d; \phi^{-1}}\).
- \(\Theta_{S,d}\) is continuous on \(\bar{A}\).
- \(\Theta_{S,d}\) leaves the Ashtekar–Isham–Lewandowski measure invariant, i.e. \((\Theta_{S,d})_{*} \mu_{AL} = \mu_{AL}\).

**Proof.**

(i) Each automorphism \(\phi \in \text{Aut}(\mathcal{P})\) acts naturally on the set of generalized oriented surfaces, by
\[S(p) = \begin{bmatrix} p_1, \ldots, p_N \\ m_1, \ldots, m_N \\ n_1, \ldots, n_N \end{bmatrix} \Rightarrow \alpha \phi S(\phi(p)) = \begin{bmatrix} \phi(p_1), \ldots, \phi(p_N) \\ m_1, \ldots, m_N \\ n_1, \ldots, n_N \end{bmatrix}. \tag{C.16}\]

By direct computation we obtain
\[\alpha \circ \Theta_{S,d} \circ \alpha^{-1} A(\phi(p)) = \Theta_{\alpha S,d; \phi^{-1}} A(p)\]
\[= \prod_{k=1,\ldots,N} d(s(p_k))^m_k (\alpha^{-1} A)(p_k) d(t(p_k))^n_k\]
\[= \prod_{k=1,\ldots,N} (d \circ \phi^{-1})(s(\phi(p_k)))^m_k A(\phi(p_k)) (d \circ \phi^{-1})(t(\phi(p_k)))^n_k\]
\[= \Theta_{\alpha S,d; \phi^{-1}} A(\phi(p)). \tag{C.17}\]

\(^{23}\) The set of oriented, generalized surfaces contains, however, many elements which do not arise from quasi-surfaces, the reason being that, as we will prove shortly, the automorphisms leave the former invariant. Since \(\text{Aut}(\mathcal{P})\) is much larger than the group of graphomorphisms, one cannot expect it to leave the quasi-surfaces invariant.

\(^{24}\) Evidently, since for \(\mu \subseteq \mu'\) one has \(\text{Cyl}(\mu) \subseteq \text{Cyl}(\mu')\) and for each \(\mu \in \mathcal{M}\) there is a graph \(\gamma \in \Gamma\), the set of all smooth cylindrical functions over graphs coincides with \(\text{Cyl}\). Nevertheless, keeping the notion for arbitrary \(\mu \in \mathcal{M}\) is more convenient for our purposes.
Since $\phi$ is an automorphism of $\mathcal{P}$, it follows that
\[
\alpha_{\phi} \circ \Theta_{S,d} \circ \alpha_{\phi}^{-1} = \Theta_{\alpha_{\phi}S,d;\phi^{-1}}.
\] (C.18)

(ii) Since for each primitive metagraph $\mu$ there is a finer primitive metagraph $\mu' \geq \mu$ which consists only of $S$-trivial morphisms, the set of such metagraphs is a partially ordered, directed set. As a consequence, the set of all
\[
\pi_{\mu}^{-1}(U_1 \times \cdots \times U_{|\mathcal{E}(\mu')|}) \subset \mathcal{A}
\] (C.19)
constitutes a basis for the topology on $\mathcal{A}$, if the $U_k \subset G$ are open. But by definition $\Theta_{S,d}$ maps any set of the form (C.19) homeomorphically to some other set of the same form. Thus, the preimages of a basis set is another basis set, and hence $\Theta_{S,d}$ is continuous.

(iii) Let $\mu = \{ p_1, \ldots, p_M \}$ be a primitive metagraph and $f \in Cyl(\mu)$ smooth and cylindrical over $\mu$, i.e. $f(A) = F(A(p_1), \ldots, A(p_M))$ for some smooth function $F$ on $G^M$, hence
\[
\int_{\mathcal{A}} d\mu_{AL}(A) f(A) = \int_{G^M} d\mu_H(h_1, \ldots, h_M) F(h_1, \ldots, h_M).
\] (C.20)

Let $\mu' = \{ q_1, \ldots, q_{M'} \}$ be a metagraph finer than $\mu$ with $S$-trivial paths $q_{k}$. For each $p_k \in \mu$ we can then find paths $q_{k}^1, \ldots, q_{k}^{n_k} \in \mu'$, such that
\[
S(p_k) = \begin{bmatrix} q_{k}^1 & \cdots & q_{k}^{n_k} \\ m_{k}^1 & \cdots & m_{k}^{n_k} \\ n_{k}^1 & \cdots & n_{k}^{n_k} \end{bmatrix}.
\] (C.21)

Then by (C.14), $f \circ \Theta_{S,d}$ is cylindrical over $\mu'$, and one has
\[
\int_{\mathcal{A}} d\mu_{AL}(A) f(\Theta_{S,d}A) = \int_{G^{M'}} d\mu_H(h_1, \ldots, h_{M'}) F\left( \prod_{l=1}^{n_1} d(s(q_{k}^l))^{-m_{k}^l}, h_{1}^l d(t(q_{k}^l))^{-n_{k}^l}, \ldots, \prod_{l=1}^{n_{M'}} d(s(q^M_{l}))^{-m_{M'}^l}, h_{1}^{M'} d(t(q^M_{l}))^{-n_{M'}^l} \right)
\] (C.22)

where $h_{1}^l$ are $h_m$ corresponding to the decomposition $p_k = q_{k}^1 \circ \cdots \circ q_{k}^{n_k}$ given by (C.21).

Note that some of $q_{k}^l$ can be the same for different $k, l$. However, since all the $q_{k}^l$ are $S$-trivial, one has
\[
S(q_{k}^l) = \begin{bmatrix} q_{k}^l \\ n_{k}^l \\ m_{k}^l \end{bmatrix}
\] (C.23)

and hence for each $p_k$ there is a representative of $S(p_k)$ such that whenever $q_{k}^l$ occurs, the corresponding $n_{k}^l, m_{k}^l$ are the same, i.e. depend only on $q_{k}^l$, and not on $k, l$. Therefore, the shift of integration variables
\[
h_{k}^l \rightarrow d(s(q_{k}^l))^{-m_{k}^l} h_{k}^l d(t(q_{k}^l))^{-n_{k}^l}.
\] (C.24)
is well defined. The integration measure used in (C.22) is invariant under such a shift, and we get
\[
\int_{\mathcal{A}} d\mu_{AL}(A) \Theta_{S,d} f(A) = \int_{G^{M'}} d\mu_H(h_1, \ldots, h_{M'}) F(h_{1}^1 \cdots h_{n_1}^1, \ldots, h_{1}^{M'} \cdots h_{n_{M'}}^{M'})
\] (C.25)

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and hence for each $p_k$ there is a representative of $S(p_k)$ such that whenever $q_{k}^l$ occurs, the corresponding $n_{k}^l, m_{k}^l$ are the same, i.e. depend only on $q_{k}^l$, and not on $k, l$. Therefore, the shift of integration variables
\[
h_{k}^l \rightarrow d(s(q_{k}^l))^{-m_{k}^l} h_{k}^l d(t(q_{k}^l))^{-n_{k}^l}.
\] (C.24)
is well defined. The integration measure used in (C.22) is invariant under such a shift, and we get
\[
\int_{\mathcal{A}} d\mu_{AL}(A) \Theta_{S,d} f(A) = \int_{G^{M'}} d\mu_H(h_1, \ldots, h_{M'}) F(h_{1}^1 \cdots h_{n_1}^1, \ldots, h_{1}^{M'} \cdots h_{n_{M'}}^{M'})
\] (C.25)
since $\mu \leq \mu'$ is an inclusion of primitive subgraphs, and the integral of $f$ over $A$ does not depend on the choice of primitive metagraph over which it is cylindrical. Thus, we have

$$\int d\mu_{AL}(A) \left(f \circ \Theta_{S,d}(A)\right) = \int d\mu_{AL}(A) f(A)$$

(C.26)

for all functions $f$ cylindrical over metagraphs, which form a dense subset in $C(\overline{A})$, and since $\Theta_{S,d}$ acts continuously on $A$, we have that (C.26) leaves $\mu_{AL}$ invariant.

This finishes the proof. \( \square \)

The action of $\Theta_{S,d}$ can naturally be pulled back to $H_{\text{kin}} = L^2(\overline{A}, d\mu_{AL})$ by, where they act as unitary operators $\hat{U}(S,d)$. These operators generalize the exponentiated fluxes that one can build from surfaces and smooth fields $E_a$. In fact, one can recover generalized fluxes $E(S)$ from the $\Theta_{S,d}$.

**Lemma appendix C.2.** Let $\Sigma$ be a manifold, $G$ be a compact Lie group, $S$ be a generalized oriented surface and $d_1, d_2 : \Sigma \to G$ two pointwise commuting maps, i.e. $d_1(x) d_2(x) = d_2(x) d_1(x)$ for all $x \in \Sigma$. Then

$$\Theta_{S,d_1} \circ \Theta_{S,d_2} = \Theta_{S,d_2} \circ \Theta_{S,d_1}.$$

(C.27)

**Proof.** By direct calculation. Let $p$ be a path in $\Sigma$, then by the definition of $S$, there are $S$-trivial paths $p_1, \ldots, p_N$ with $p = p_1 \circ \cdots \circ p_N$ and numbers $m_1, \ldots, m_N, n_1, \ldots, n_N$ such that

$$S(p_k) = \begin{bmatrix} p_k \\ m_k \\ n_k \end{bmatrix}.$$  

(C.28)

Since $S$ is a functor, we have

$$S(p) = \begin{bmatrix} p_1, \ldots, p_N \\ m_1, \ldots, m_N \\ n_1, \ldots, n_N \end{bmatrix}.$$  

(C.29)

Thus, we see that one can choose a representative of $S(p)$ that contains $S$-trivial paths only. We now have

$$\Theta_{S,d_1} \circ \Theta_{S,d_2} A(p) = \prod_{k=1}^N d_1(s(p_k))^{m_k} (\Theta_{S,d_2} A(p_k) d_1(t(p_k)))^{n_k}$$

$$= \prod_{k=1}^N d_1(s(p_k))^{m_k} (d_2(s(p_k))^m A(p_k) d_2(t(p_k))^n) d_1(t(p_k))^{n_k}$$

$$= \prod_{k=1}^N (d_1 d_2(s(p_k))^m A(p_k) (d_2 d_1)(t(p_k)))^{n_k}$$

$$= \Theta_{S,d_2} \circ \Theta_{S,d_1} A(p)$$

(C.30)

since $d_1$ and $d_2$ commute pointwise. Here the property that all $p_k$ are $S$-trivial was essential for the proof. \( \square \)

We now show that a generalized flux can be obtained by differentiating the action of the exponentiated fluxes.

**Lemma appendix C.3.** Let $\Sigma$ be a manifold, $G$ be a compact Lie group, $S$ a generalized oriented surface and $k : \Sigma \to \mathfrak{g}$ be a map from $\Sigma$ to the Lie algebra of $G$. Then the operator

$$E_k(S) := -i \frac{d}{dt} \hat{U}(S, e^{it\kappa})$$

(C.31)

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defines a self-adjoint operator on $\mathcal{H}_{\text{kin}}$, the domain of definition of which contains the cylindric functions $\text{Cyl}$ on $\mathcal{A}$.

**Proof.** Lemma appendix C.2 shows that

$$\Theta_{S,\text{dust}} \circ \Theta_{S,\text{dust}} = \Theta_{S,\text{dust}};$$

i.e. the map $t \mapsto \hat{U}(S, e^{it})$ defines a one-parameter family of unitarities on $\mathcal{H}_{\text{kin}}$. Thus, Stone’s theorem guarantees the existence of (C.31), if $\langle f | \hat{U}(S, e^{it}) | f' \rangle$ is continuous w.r.t. $t$ at $t = 0$ for all $f, f'$ being in a dense set $D \subset \mathcal{H}_{\text{kin}}$, and the domain of definition of (C.31) is then given by $D$.

Let $f, f'$ be two smooth cylindrical functions over a metagraph $\mu$, which can, without loss of generality, be chosen to be the same for $f$ and $f'$. Then there is, by definition of $S$, a metagraph $\mu' \supseteq \mu$ consisting of $S$-trivial morphisms $q_1, \ldots, q_M$. The functions $f$ and $f'$ are then obviously also cylindrical over $\mu'$, i.e. there are smooth functions $F, F' : G^M \rightarrow \mathbb{C}$ such that

$$f(A) = F(A(q_1), \ldots, A(q_M))$$

$$f'(A) = F'(A(q_1), \ldots, A(q_M)).$$

Hence, since $S(q_i) = \begin{bmatrix} d_i \\ m_i \end{bmatrix}$, we have

$$(U(S, e^{it}) f')(A) = f'(\Theta S, e^{it} A)$$

$$= e^{i\pi k^i q_i} A(q_1) e^{i\pi n^i q_i} \ldots, e^{i\pi u_k A(q_M) e^{i\pi u_k A}}$$

with $k^i := k(s(q_i)) \in g$ and $k^i = k(t(q_i)) \in g$. From this it follows that

$$\langle \psi | U(S, e^{it}) | \phi \rangle = \int_{G^M} d\mu_H(h_1, \ldots, h_M) F(h_1, \ldots, h_M) F'$$

$$\times \left( e^{i\pi k^i q_i} A_{h_1} e^{i\pi n^i q_i} \ldots, e^{i\pi u_k A(q_M) e^{i\pi u_k A}} \right).$$

Since $F, F'$ are smooth and $G$ is compact, the expression is smooth in $t$. Since the smooth cylindrical functions $\text{Cyl}$ over metagraphs are dense in $\mathcal{H}_{\text{kin}}$, the claim follows from Stone’s theorem.

□

C.2. Algebra relations

The smooth cylindrical functions $f \in \text{Cyl}$ and the maps $\Theta_{S,d}$ generate an algebra, which can be completed to a $C^*$-algebra $\mathcal{A}_{\text{cat}}$, similar to the Weyl algebra of quantum geometry [13]. Let $f$ be a function cylindrical over $p$, i.e. there is a smooth function $F : G \rightarrow \mathbb{C}$ with $f(A) = F(A(p))$. Let furthermore

$$S(p) = \begin{bmatrix} p_1, \ldots, p_N \\ m_1, \ldots, m_N \\ n_1, \ldots, n_N \end{bmatrix}.$$  \hfill (C.33)

The function $f$ acts via multiplication on, say, $C(\mathcal{A})$. Denote this multiplication by $\hat{f}$, i.e. $\hat{f} g(A) := f(A) g(A)$, then, for any $g \in C(\mathcal{A})$, we have

$$\hat{U}(S, d) \hat{f} \hat{U}(S, d)^{-1} g(A) = (\hat{f} \hat{U}(S, d)^{-1} g)(\Theta_{S,d} A)$$

$$= f(\Theta_{S,d} A) \cdot \hat{U}(S, d)^{-1} g(\Theta_{S,d} A)$$

$$= f(\Theta_{S,d} A) \cdot g(A)$$  \hfill (C.34)
which gives
\[ \dot{U}(S, d) \dot{f} \dot{U}(S, d)^{-1} = f \circ \Theta_{S,d} = \dot{U}(S, d) f, \]  
(C.35)
which is the usual algebraic relation which also determines the Weyl algebra of quantum geometry. To cast (C.35) into a more frequently used form, we go over to the (generalized) fluxes \( \hat{E}_k(S) \). For \( k : \Sigma \rightarrow \mathfrak{g} \) any map, we get, by lemma appendix C.3
\[ - i \frac{d}{dt} \left[ \dot{U}(S, e^{itk}) \dot{f} \dot{U}(S, e^{itk})^{-1} \right]_{t=0} = [\hat{E}_k(S), \dot{f}]. \]  
(C.36)
On the other hand, the derivative of the function \( f \circ \Theta_{S,e^{it}} \) with respect to \( t \) reads
\[ -i \frac{d}{dt} \left[ f(\Theta_{S,e^{it}} A) \right]_{t=0} = -i \frac{d}{dt} f \left( \prod_{l=1}^{N} e^{i\epsilon_m(k_l^{(1)}) A(p_l)} e^{i\eta_m(k_l^{(2)})} \right)_{t=0}, \]  
(C.37)
where \( k_l^{(1)} = k(s(p_l)), k_l^{(2)} = k(t(p_l)) \in \mathfrak{g} \). The function \( \tilde{F} : G^N \rightarrow \mathbb{C} \) is given by
\[ \tilde{F}(h_1, \ldots, h_N) = -i \sum_{l=0}^{N} (n_l + m_{l+1}) \frac{d}{dt} F(h_1 \cdot \ldots \cdot h_l e^{i\epsilon_l(h_{l+1})} h_{l+1} \cdot \ldots \cdot h_N)_{t=0}. \]
Note that the limit exists since \( F \) is smooth. In this notation, \( k_l^{(1)} := k_l^{(1)} = k(s(p_l)). \) Furthermore \( n_0 := m_{N+1} := 0 \); also \( h_1 \cdot \ldots \cdot h_0 := 1 \), as well as \( h_{N+1} \cdot \ldots \cdot h_N := 1 \). We thus get
\[ [\hat{E}_k(S), \dot{f}](A) = -i \sum_{l=0}^{N} \epsilon(S, p_l) \frac{d}{dt} F(A(p_1) \cdot \ldots \cdot A(p_l) e^{i\epsilon_l(h_{l+1})} A(p_{l+1}) \cdot \ldots \cdot A(p_N))_{t=0}. \]  
(C.38)
Here \( \epsilon(S, p_l) := n_l + m_{l+1} \). If \( S \) is an analytic surface and \( p \) is an edge, this formula reduces to the one employed in the literature (modulo prefactors as \( l_p^2 \) and the Immirzi parameter, which can be absorbed into the definition of \( E_k(S) \)). We have
\[ S(p) = \begin{bmatrix} p_1, \ldots, p_N \\ m_1, \ldots, m_N \\ n_1, \ldots, n_N \end{bmatrix}, \]  
(C.39)
so \( S \) cuts the edge \( p \) into the pieces \( p_1, \ldots, p_N \), and thus the \( k_l^{(2)} \) \( l = 0, \ldots, N \) are the values of the smearing function \( k \) at the endpoints of \( p \) and the points where \( S \) intersects \( p \). At these points, the values \( k_l^{(2)} \in \mathfrak{g} \) are inserted in the arguments of the function \( F \), and the sum is taken over all these points, with prefactors \( (n_l + m_{l+1}) \). If one restricts the values of the intersection functions \( m_l, n_l \) to \( \{0, \pm 1\} \), then the factor \( \epsilon(S, p_l) \) keeps track of the way in which the segment \( p_l \) touches \( S \) at \( s(p_l) \in \Sigma \).

Note that, since \( S(p) \) is an equivalence class, so strictly speaking \( \tilde{F} \) depends on the choice of the representative. With the precise definition of \( S(p) \), however, one can easily see that the function \( f(A) = \tilde{F}(A(p_1), \ldots, A(p_N)) \) does not.

This shows that (C.38) generalizes the canonical commutation relations for quantum gravity to a *-algebra obtained from \( \mathfrak{A}_{\text{cat}} \), which is generated by smooth cylindrical functions \( f \) over metagraphs and generalized electric fluxes \( E_k(S) \).
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