RUMOUR PROCESSES ON $\mathbb{N}$

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ABSTRACT. We study four discrete time stochastic systems on $\mathbb{N}$ modeling processes of rumour spreading. The involved individuals can either have an active or a passive role, speaking up or asking for the rumour. The appetite in spreading or hearing the rumour is represented by a set of random variables whose distributions may depend on the individuals. Our goal is to understand - based on those random variables distribution - whether the probability of having an infinite set of individuals knowing the rumour is positive or not.

1. INTRODUCTION

Until a few decades ago, epidemic and rumour models where treated under the same class of models. While there is a clear similitude among the status of the individuals in the models (susceptible are ignorants, immunes are stiflers and infected are spreaders) the rates at which individuals change their status might be qualitatively different (Pearce [16]). Generally speaking, the production of stiflers is definitely more complex than the production of immune individuals.

Lately the mathematics of rumors has observed a good deal of interest. The focus used to be at deterministic or stochastic models,
modeling homogeneously mixed populations living on spaces with no structure as the Maki-Thompson (Maki and Thompson [15] and Sudbury [18]) and Daley-Kendall (Daley and Kendall [5] and Pittel [17]) models. Among the possible variations one can find in recent literature are competing rumours (Kostka et al [11]), more than two people meeting at a time (Kesten and Sidoravicius [10]), moving agents (Kurtz et al [12]) and rumours through tree-like graphs (Lebensztayn and Rodriguez [14] and Lebensztayn et al [13]), complex networks (Isham et al [9]), grids (Roy et al [11]) and multigraphs (Bertachi and Zucca [2]).

Still, the most important question for both models, epidemic and rumour, is in terms of a rumour model, if a spreader (an individual who wants to see the rumour spread) is introduced into a reservoir of ignorants under what conditions the rumour will spread to a large proportion of the population, instead of dying out quickly without having done so. Another important question is, if it does not dies out quickly, what is the final proportion of individuals hit by the rumour?

We study discrete time stochastic systems on $\mathbb{N} = \{0, 1, 2, \ldots \}$ which dynamic is as follows. First, consider that at time zero all vertices of $\mathbb{N}$ are declared inactive, except for the origin, which is active. It instantly exerts influence on its neighbors vertices, activating a contiguous random set of them placed on its right. In general, that is the behavior of every vertex in case it is activated.
We take into account an homogeneous and an heterogeneous versions for what we call the radius of influence of a vertex. In the homogeneous version, as a rule, the next moment to what it has been activated, each active vertex carries the same (random) behavior of the origin, independent of it and of everything else. We also deal with an heterogeneous version where each vertex, if activated, has a distinct distribution for its radius of influence.

We say that the process survives if the amount of vertices activated is infinite. Otherwise we say the process dies out. We call this the Firework Process, associating the activation dynamic of a vertex to a rumour process. Vertices become spreaders as soon as they are activated. Next time, they propagate the rumour and immediately become stiflers.

A possible variation is what we call Reverse Firework Process. In this variation a vertex, instead of being hit by a rumour, defines a set of neighbors on its left to which it asks once someone in this set hears the rumour. We call this variation Reverse Firework Process. We also deal with an homogeneous and an heterogeneous versions of this variation. The models are shown to be qualitatively different in some pertinent cases.

Our main interest is to establish whether each process has positive probability of survival which is equivalent to a rumour propagation.
This is done according to the distribution of the random variable that defines the radius of influence of each active vertex.

The paper is organized as follows. Section 2 presents the main results. Section 3 brings the proofs for the main results together with auxiliary lemmas and handy inequalities. In Section 4 we present examples where some conditions can be verified.

2. Main Results

2.1. Firework Process. Consider \( \{u_i\}_{i \in \mathbb{N}} \) a set of vertices of \( \mathbb{N} \) such that \( 0 = u_0 < u_1 < u_2 < \cdots \) and a set of independent random variables \( \{R_i\}_{i \in \mathbb{N}} \) assuming values in \( \mathbb{R}_+ \) whose joint distribution is \( \mathbb{P} \). The Firework Process can be formally defined in the following way. At time 0, an explosion of size \( R_0 \) comes from the origin, activating all vertices \( u_i \leq R_0 \). As a rule, at every discrete time \( t \) all vertices \( u_j \) activated at time \( t - 1 \) generate their explosions (whose radius of influence is \( R_j \)), and they do this just once, activating the vertices \( u_i \) (only those which has not been activated before) such that \( u_j < u_i \leq u_j + R_j \). Observe that except for the set of vertex \( \{u_i\} \), all others vertices are non-actionable, meaning that the random variable associated to them is 0 almost surely.

If for all \( u_j \) activated at time \( t - 1 \) there are no vertices \( u_i \) in this latter condition the process dies out. That means the rumour reaches only a finite amount of individuals. If, on the contrary the process
never stops, we say it survives, meaning that the rumour reaches an infinity number of individuals. We call the process homogeneous if all \( R_i \) have the same distribution and \( u_i = i \) for all \( i \). Otherwise we call it heterogeneous. We focus to the cases \( \mathbb{P}(R_i < 1) \in (0, 1) \) for all \( i \).

Let us consider the following events

- \( V_n = \text{the vertex } u_n \text{ is hit by an explosion} \),
- \( V = \lim_{n \to \infty} V_n \).

2.1.1. The Homogeneous case.

**Theorem 2.1.** For the Homogeneous Firework Process, consider

\[
a_n = \prod_{i=0}^{n} \mathbb{P}(R < i + 1).
\]

Then

\[
\sum_{n=1}^{\infty} a_n = \infty \text{ if and only if } \mathbb{P}[V] = 0.
\]

Besides

\[
\mathbb{P}(V) \geq \prod_{j=0}^{\infty} \left[ 1 - \prod_{i=0}^{j} \mathbb{P}(R < i + 1) \right], \tag{2.1}
\]

\[
\mathbb{P}(V) \leq 1 - \mathbb{P}(R = 0) - \sum_{k=1}^{n} \sum_{j=0}^{k} \mathbb{P}(R = k) \prod_{j=0}^{k-1} \mathbb{P}[R \leq j]. \tag{2.2}
\]

**Corollary 2.2.** For the Homogeneous Firework Process, consider

\[
L = \lim_{n \to \infty} n \mathbb{P}(R \geq n) .
\]

We have that

(I) If \( L > 1 \) then \( \mathbb{P}[V] > 0 \).

(II) If \( L < 1 \) then \( \mathbb{P}[V] = 0 \).
(III) If \( L = 1 \) and there exists \( N \) such that for all \( n \geq N \)

\[
P(R \geq n) \leq \frac{1}{n-1}, \text{ then } P[V] = 0.
\]

Remark 2.3. Consider a Homogeneous Firework Process with \( R \) assuming values on \( \mathbb{N} \). Observe that, in this case, if \( \mathbb{E}[R] < \infty \) then \( L = 0 \). Consequently for \( R \) assuming values on \( \mathbb{N} \),

\[
\mathbb{E}[R] < \infty \Rightarrow P[V] = 0.
\]

Next result gives a criteria for the case when the distribution of the random variable \( R \) is a power law.

Corollary 2.4. Let \( \alpha > 1 \) and \( Z_\alpha \) be an appropriate constant. Consider the Homogeneous Firework Process such that

\[
P(R = k) = \frac{Z_\alpha}{(k+1)^\alpha} \text{ for } k \in \mathbb{N}.
\]

(1) If \( \alpha < 2 \) then \( P[V] > 0 \).

(II) If \( \alpha \geq 2 \) then \( P[V] = 0 \).

Remark 2.5. Observe that for the Homogeneous Firework Process if \( R \) has a power law distribution as in (2.3), with \( \alpha = 2 \), we have that

\[
\mathbb{E}[R] = \infty \text{ and } P[V] = 0.
\]
2.1.2. The Heterogeneous case.

**Remark 2.6.** Consider the Heterogeneous Firework Process. One can get a sufficient condition for $\mathbb{P}[V] = 0$ ($\mathbb{P}[V] > 0$) by a coupling argument. Consider $\mathbb{P}(R_i \geq k) \leq \mathbb{P}(R \geq k)$ ($\mathbb{P}(R_i \geq k) \geq \mathbb{P}(R \geq k)$) for some random variable $R$ which distribution $\mathbb{P}$ satisfies $\lim_{n \to \infty} n\mathbb{P}(R \geq n) < 1$ ($\lim_{n \to \infty} n\mathbb{P}(R \geq n) > 1$). Finally use part (II) (part (I)) of Corollary 2.2.

**Theorem 2.7.** Consider a Heterogeneous Firework Process which actionable vertices are at integer positions $0 = u_0 < u_1 < u_2 < \ldots$ such that $u_{n+1} - u_n \leq m$, for $m \geq 1$. Besides, let us assume $\mathbb{P}(R_n < m) \in (0, 1)$ for all $n$.

1. If $\sum_{n=0}^{\infty} [\mathbb{P}(R_n < tm)]^t < \infty$ for some $t \geq 1$ then $\mathbb{P}[V] > 0$.

2. If for some random variable $R$, which distribution is $\mathbb{P}$, the following conditions hold

   - $\mathbb{P}(R \geq k) - \mathbb{P}(R_n \geq k) \leq b_k$ for all $k \geq 0$ and all $n \geq 0$,
   - $\lim_{n \to \infty} n[\mathbb{P}(R \geq n) - b_n] > m$,
   - $\lim_{n \to \infty} b_n = 0$.

   Then $\mathbb{P}[V] > 0$.

3. $\mathbb{P}(V) \geq \prod_{j=0}^{\infty} \left[ 1 - \prod_{i=0}^{j} \mathbb{P}(R_{j-i} < (i+1)m) \right]$.

2.2. Reverse Firework Process. Consider $\{u_i\}_{i \in \mathbb{N}}$ a set of vertices of $\mathbb{N}$ such that $0 = u_0 < u_1 < u_2 < \ldots$ and a set of independent random
variables \( \{R_i\}_{i \in \mathbb{N}} \) assuming values in \( \mathbb{N} \) which joint distribution is \( \mathbb{P} \).

The Reverse Firework Process can be defined as follows. At time 0, only the origin is activated. At time 1, explosions of size \( R_i \) towards the origin, come from all vertices of \( \{u_i\}_{i \in \mathbb{N}} \). All vertices \( u_i \leq R_i \) are activated. As a rule, at discrete times \( t \) the set of vertices \( u_j \) which can find a vertex activated at time \( t - 1 \) within a distance \( R_j \) to its left, are activated. Let us call this set \( A_t \). If for some \( t \), \( A_t \) is empty the process stops. If the process never stops we say it survives. We call the process homogeneous if all \( R_i \) have the same distribution and \( u_i = i \) for all \( i \), otherwise we call it heterogeneous. We focus to the cases \( \mathbb{P}(R_i < 1) \in (0, 1) \) for all \( i \). Unless stated differently, we assume \( u_i = i \) for all \( i \).

Let \( S \) be the event “the reverse process survives”.

2.2.1. The homogeneous case.

**Theorem 2.8.** Consider the Reverse Homogeneous Firework Process. We have that

(I) If \( \mathbb{E}(R) = \infty \) then \( \mathbb{P}(S) = 1 \).

(II) If \( \mathbb{E}(R) < \infty \) then \( \mathbb{P}(S) = 0 \).

**Remark 2.9.** For a random variable \( R \), having a power law distribution as in \( (2.3) \), we have that

- if \( 1 < \alpha \leq 2 \) then \( \mathbb{E}[R] = \infty \),
• if $\alpha > 2$ then $\mathbb{E}[R] < \infty$.

In conclusion, if $R$ has a power law distribution as in $(2.3)$, with $\alpha = 2$, then $\mathbb{P}[V] = 0$ for the Homogeneous Firework Process by Remark 2.5 and $\mathbb{P}[S] = 1$ for the Reverse Homogeneous Firework Process.

2.2.2. The heterogeneous case.

**Theorem 2.10.** Consider the Reverse Heterogeneous Firework Process. It holds that

(I) $\sum_{k=1}^{\infty} \mathbb{P}(R_{n+k} \geq k) = \infty$ for all $n$ if and only if $\mathbb{P}(S) = 1$.

(II) If $\sum_{n=1}^{\infty} \prod_{k=1}^{\infty} \mathbb{P}(R_{n+k} < k) < \infty$ then $\mathbb{P}(S) > 0$.

**Remark 2.11.** Let $\rho = \sum_{n=1}^{\infty} \prod_{k=1}^{\infty} \mathbb{P}(R_{n+k} < k)$. Observe now that Theorem 2.10 gives no additional information for Theorem 2.8 as in the homogeneous case $\rho$ equals either $0$ ($\mathbb{E}[R] = \infty$) or $\infty$ ($\mathbb{E}[R] < \infty$).

**Remark 2.12.** By a coupling argument and Theorem 2.8 one can see that if there is a random variable $R$, which distribution is $\mathbb{P}$, with $\mathbb{E}[R] < \infty$ ($\mathbb{E}[R] = \infty$), such that $\mathbb{P}(R_n \geq k) \leq \mathbb{P}(R \geq k)$ ($\mathbb{P}(R_n \geq k) \geq \mathbb{P}(R \geq k)$) for all $k$ then $\mathbb{P}(S) = 0$ ($\mathbb{P}(S) = 1$).

3. Proofs

Next we present some basic facts, starting from the Raabes test (Fort [7, p. 32] or Bonar and Khoury [3, p. 48]).
Raabes Test. For $a_n > 0$, let us define

$$L = \lim_{n \to \infty} n \left( \frac{a_n}{a_{n+1}} - 1 \right).$$

Then

- If $L > 1$ then $\sum_{n=1}^{\infty} a_n < \infty$.
- If $L < 1$ then $\sum_{n=1}^{\infty} a_n = \infty$.
- If $L = 1$ and $n \left( a_n/a_{n+1} - 1 \right) \leq 1$, for $n$ large enough, then $\sum_{n=1}^{\infty} a_n = \infty$.

The following result (Bremaud [4, p. 422]) is useful for what comes next

Lemma 3.1. Let $\{a_n\}_{n \geq 1}$ be a sequence of real numbers in $(0, 1)$. Then,

$$\prod_{i=0}^{\infty} (1 - a_k) = 0 \iff \sum_{i=0}^{\infty} a_k = \infty. \quad (3.1)$$

Remark 3.2. Consider that the actionable vertices are at integer positions $0 = u_0 < u_1 < u_2 < \ldots$ such that $u_{n+1} - u_n \leq m$, for $m \geq 1$. From the definition of $V_n$ one can see that

- $V_{k+1} \supset V_k \cap \left\{ \bigcup_{i=0}^{k} (R_{k-i} \geq (i+1)m) \right\}$,
- $V_k \cap \bigcup_{i=0}^{k} (R_{k-i} \geq (i+1)m)$ are increasing events,
- $\mathbb{P}(V_n) > 0$ for all $n$. 
From FKG inequality (Grimmett [8, p.34]) we can assure that

\[
\mathbb{P}(V_{k+1}) \geq \mathbb{P}(V_k \cap \bigcup_{i=0}^{k} (R_{k-i} \geq (i+1)m)) \geq \mathbb{P}(V_k) \mathbb{P}(\bigcup_{i=0}^{k} (R_{k-i} \geq (i+1)m)) = \mathbb{P}(V_k) \left[ 1 - \prod_{i=0}^{k} \mathbb{P}(R_{k-i} < (i+1)m) \right]
\]

and then

\[
\mathbb{P}(V_n) \geq \prod_{j=0}^{n-1} \left[ 1 - \prod_{i=0}^{j} \mathbb{P}(R_{j-i} < (i+1)m) \right].
\]

Therefore

\[
\mathbb{P}(V) \geq \prod_{j=0}^{\infty} \left[ 1 - \prod_{i=0}^{j} \mathbb{P}(R_{j-i} < (i+1)m) \right]. \quad (3.3)
\]

Inequality (3.2) becomes an equality if \( u_i = mi \) for all \( i \in \mathbb{N} \) and some \( m \in \mathbb{N} \). From the latter set of displays and (3.1) follows next proposition.

**Proposition 3.3.** Consider a Heterogeneous Firework Process which actionable vertices are at integer positions \( 0 = u_0 < u_1 < u_2 < \cdots \) such that \( u_{n+1} - u_n \leq m \). Let \( a_n = \prod_{i=0}^{n} \mathbb{P}(R_{n-i} < (i+1)m) \) and assume \( \mathbb{P}(R_i < m) \in (0, 1) \).

If \( \sum_{n=0}^{\infty} a_n < \infty \) then \( \mathbb{P}[V] > 0 \). \quad (3.4)

3.1. **Firework Process.**

*Proof of Theorem 2.1.* Assume \( \sum_{n=0}^{\infty} a_n < \infty \). From Proposition 3.3 with \( m = 1 \), we have that \( \mathbb{P}[V] > 0 \).
Assume now $\sum_{n=0}^{\infty} a_n = \infty$. First consider the event

$$C = \{\exists n \text{ such that } \forall u_i > n \exists x \text{ such that } x < u_i \leq x + R_x\}.$$ 

In words that means that from some position on, all vertex belong to the radius of influence of some other vertices. Those later vertices not necessarily have been activated.

Next, consider the following event

$$B(u_n) = \{u_n > x + R_x, \text{ for all } x < u_n\}$$

In words, the vertex $u_n$ does not belong to the radius of influence of any vertex to its left.

Assuming all independent random variables having the same distribution as $R$ and that $u_i = i \ (B_n = B(u_n))$, 

$$\mathbb{P}(B_n) = \mathbb{P}\left(\cap_{i=1}^{n} [R_{n-i} < i]\right) = \prod_{i=1}^{n} \mathbb{P}(R < i) = a_{n-1}.$$ 

Conditional independence of the $B_i$s as stated next, for $i > j$

$$\mathbb{P}(B_i \cap B_j) = \mathbb{P}\left(\cap_{k=1}^{i-j} [R_{i-k} < k] \cap \cap_{k=1}^{j} [R_{j-k} < k]\right)$$

$$= \prod_{k=1}^{i-j} \mathbb{P}(R < k) \prod_{k=1}^{i} \mathbb{P}(R < k)$$

$$= \mathbb{P}(B_{i-j}) \mathbb{P}(B_j)$$

makes the $B_i$s satisfy the definition of a renewal event in [6, p.308]. So, from the fact that $\sum_{n=1}^{\infty} \mathbb{P}(B_n) = \infty$, one can rely on Theorem 2 of Section XIII.3 of [6, p.312] to see that
\[ \mathbb{P}(B_n \text{ infinitely often}) = 1. \]

From this we conclude that \( \mathbb{P}[V] = 0 \), as

\[ V^c \supseteq C^c \supseteq \{B_n \text{ infinitely often}\}. \]

Inequality (2.1) follows from (3.3) and inequality (2.2) follows from the fact that

\[ V^c \supseteq \bigcup_{k=0}^{\infty} \left[ R_0 = k, \bigcap_{j=1}^{k} [R_j \leq k - j] \right]. \]

\[ \square \]

Proof of Corollary 2.2. Observe that, as \( a_n = \prod_{i=0}^{n} \mathbb{P}(R < i + 1) \)

\[ \frac{a_n}{a_{n+1}} - 1 = \frac{\mathbb{P}(R \geq n + 2)}{\mathbb{P}(R < n + 2)}. \]

Therefore

\[ \lim_{n \to \infty} n \left( \frac{a_n}{a_{n+1}} - 1 \right) = \lim_{n \to \infty} n \mathbb{P}(R \geq n). \quad (3.5) \]

From (3.5), Raabes Test and Theorem 2.1 follow (I), (II) and (III).

\[ \square \]

Proof of Corollary 2.4. Observe that

\[ \frac{1}{(\alpha - 1)(n + 1)^{\alpha-1}} = \int_{n+1}^{\infty} \frac{1}{x^{\alpha}} dx \leq \sum_{j=n+1}^{\infty} \frac{1}{j^{\alpha}} \leq \int_{n+1}^{\infty} \frac{1}{(x - 1)^{\alpha}} dx = \frac{1}{(\alpha - 1)n^{\alpha-1}}. \]

Then

\[ \frac{Z_{\alpha}}{(\alpha - 1)(n + 1)^{\alpha-1}} \leq \mathbb{P}(R \geq n) \leq \frac{Z_{\alpha}}{(\alpha - 1)n^{\alpha-1}}. \]
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Consequently

\[
\lim_{n \to \infty} n \mathbb{P}(R \geq n) = \begin{cases} 
+\infty & \text{if } \alpha < 2, \\
6/\pi^2 & \text{if } \alpha = 2, \\
0 & \text{if } \alpha > 2.
\end{cases}
\]

From Corollary 2.2 the conclusion follows. \( \square \)

Proof of Theorem 2.7

Let

\[
a_n = \prod_{j=0}^{n-1} \mathbb{P}(R_{n-j} < (j + 1)m).
\]

Proof of (I). As

\[
\sum_{n=t}^{\infty} [\mathbb{P}(R_n < tm)]^t < \infty
\]

implies that

\[
\sum_{n=t}^{\infty} \left[ \max_{j \in \{0, \ldots, t-1\}} \{\mathbb{P}(R_{n-j} < tm)\} \right]^t < \infty,
\]

and as

\[
a_n \leq \prod_{j=0}^{t-1} \mathbb{P}(R_{n-j} < tm) \leq \left[ \max_{j \in \{0, \ldots, t-1\}} \{\mathbb{P}(R_{n-j} < tm)\} \right]^t,
\]

the series which terms are \( a_n \) converges. So we can use (3.4) in order to get the result.

Proof of (II). Let

\[
r_n = \prod_{j=0}^{n-1} [\mathbb{P}(R < (j + 1)m) + b_{(j+1)m}].
\]
As

\[ n \left( \frac{r_n}{r_{n+1}} - 1 \right) = \frac{n[P(R \geq (n + 2)m) - b_{(n+2)m}]}{P(R < (n + 2)m) + b_{(n+2)m}}, \]

from the hypothesis

\[ \lim_{n \to \infty} n \left( \frac{r_n}{r_{n+1}} - 1 \right) > 1. \]

But \( a_n \leq r_n \), therefore the series which terms are \( a_n \) is convergent and so we can use Proposition 3.3 to get the desired result.

**Proof of (III).** It follows from (3.3). \( \square \)

### 3.2. Reverse Firework Process

First consider the following variation of the Homogeneous Firework Process. Instead of having just the origin activated at time zero, we consider that all vertices to its left are also activated at time zero. The set of independent random variables which defines the radius of influence of all vertices is \( \{F_i\}_{i \in \mathbb{Z}} \), all having the same distribution as \( R \), the random variable which defines the Reverse Homogeneous Firework Process.

For this variation of the Homogeneous Firework Process let us define the following events

- \( V_n = \text{the vertex } n \text{ is hit by an explosion,} \)
- \( V = \text{the process survives.} \)
By analogy “to survive” in this variation means to hit infinitely many vertices of \( \mathbb{N} \). It follows that
\[
\mathcal{V} = \bigcap_{n=0}^{\infty} \bigcup_{j=0}^{\infty} [F_{n-j} \geq j + 1]. \tag{3.6}
\]

**Proposition 3.4.** If \( \mathbb{E}(R) < \infty \), then \( \mathbb{P}(\mathcal{V}) = 0 \).

**Proof of Proposition 3.4.** Let us define the following events
\[
\mathcal{A}_n = \bigcup_{i=-\infty}^{n-1} \{ F_i \geq 2n - i \}
\]
and
\[
\mathcal{B}_n = \bigcup_{i=n}^{2n-1} \{ F_i \geq 2n - i \}.
\]

Observe that
\[
\mathcal{V}_{2n} \subseteq \mathcal{V}_n \cap [\mathcal{A}_n \cup \mathcal{B}_n].
\]

Therefore
\[
\mathbb{P}(\mathcal{V}_{2n}) \leq \mathbb{P}(\mathcal{A}_n) + \mathbb{P}(\mathcal{B}_n)\mathbb{P}(\mathcal{V}_n).
\]

Now
\[
\mathbb{P}(\mathcal{A}_n) \leq \sum_{i=-\infty}^{n-1} \mathbb{P}(F_i \geq 2n - i)
\]
\[
= \sum_{i=n+1}^{\infty} \mathbb{P}(F_{2n-i} \geq i)
\]
\[
= \sum_{i=n+1}^{\infty} \mathbb{P}(R \geq i) \rightarrow 0,
\]
and

\[ P(B_n) = P\left( \bigcup_{i=n}^{2n-1} \{ F_i \geq 2n - i \} \right) = 1 - \prod_{i=n}^{2n-1} P(F_i < 2n - i) \leq 1 - \prod_{i=1}^{\infty} P(R < i). \]

Then, (3.1) and \( \mathbb{E}(R) < \infty \) guarantee the existence of \( \lambda \in (0,1) \) such that

\[ P(B_n) \leq \lambda \]

for all \( n \). So, as for the homogeneous case \( P(A_n) \geq P(A_{n+1}) \),

\[ \lim_{n \to \infty} P(V_n) = 0 \]

and this implies that \( P(V) = 0 \) as \( V_{n+1} \subset V_n \).

Proof of Theorem 2.8

Let \( \{ R_i \}_{i \in \mathbb{N}} \) independent random variables distributed as \( R \). Observe that

\[ S = \bigcap_{n=0}^{\infty} \bigcup_{j=1}^{\infty} [R_{n+j} \geq j]. \] (3.7)

By using FKG inequality (Grimmett [8, p.34]) and the fact that intersections of increasing events is an increasing event, we have that

\[ P\left( \bigcap_{n=0}^{n_0} \bigcup_{j=1}^{\infty} [R_{n+j} \geq j] \right) \geq \prod_{n=0}^{n_0} P\left( \bigcup_{j=1}^{\infty} [R_{n+j} \geq j] \right) \]

for all \( n_0 \). Taking the limit \( n_0 \to \infty \), by the continuity of probability
\[ P \left( \bigcap_{n=0}^{\infty} \bigcup_{j=1}^{\infty} [R_{n+j} \geq j] \right) \geq \prod_{n=0}^{\infty} P \left( \bigcup_{j=1}^{\infty} [R_{n+j} \geq j] \right). \]

Therefore

\[ P(S) \geq \prod_{n=0}^{\infty} \left[1 - \prod_{j=1}^{\infty} [1 - P(R_{n+j} \geq j)] \right]. \] (3.8)

Proof of (I). From the hypothesis

\[ \sum_{j=1}^{\infty} P(R \geq j) = \infty. \] (3.9)

Now, (3.1) and (3.9) implies that

\[ \prod_{j=1}^{\infty} [1 - P(R \geq j)] = 0, \]

and \( P(S) = 1 \) follows by (3.8).

Proof of (II). By Proposition 3.4, (3.6) and the fact that \( R_i \) and \( F_i \) have the same distribution

\[ P \left( \bigcap_{n=0}^{\infty} \bigcup_{j=0}^{\infty} [R_{n-j} \geq j + 1] \right) = 0. \] (3.10)

By the other hand, as \( R_i \) are all distributed as \( R \)

\[ P \left( \bigcap_{n=0}^{\infty} \bigcup_{j=0}^{\infty} [R_{n-j} \geq j + 1] \right) = P \left( \bigcap_{n=0}^{\infty} \bigcup_{j=0}^{\infty} [R_{n+j+1} \geq j + 1] \right), \]

and therefore, by (3.7) and (3.10), \( P(S) = 0. \)

Proof of Theorem 2.10
Proof of (I). Assuming that $\sum_{k=1}^{\infty} \mathbb{P}(R_{n+k} \geq k) = \infty$ for all $n$ and considering (3.1), one can see that $\prod_{k=1}^{\infty} [1 - \mathbb{P}(R_{n+k} \geq k)] = 0$ for all $n$. Therefore, by (3.8), $\mathbb{P}(S) = 1$. By the other side, as $\mathbb{P}(S) \leq 1 - \prod_{k=1}^{\infty} \mathbb{P}(R_{n+k} < k)$ for all $n$, if $\mathbb{P}(S) = 1$ we have that $\prod_{k=1}^{\infty} [1 - \mathbb{P}(R_{n+k} \geq k)] = 0$ for all $n$. Now, from (3.1), $\sum_{k=1}^{\infty} \mathbb{P}(R_{n+k} \geq k) = \infty$ for all $n$.

Proof of (II). From $\sum_{n=1}^{\infty} \prod_{k=1}^{\infty} \mathbb{P}(R_{n+k} < k) < \infty$, follows that, by the use of (3.1), $\prod_{n=0}^{\infty} [1 - \prod_{k=1}^{\infty} [1 - \mathbb{P}(R_{n+k} \geq k)]] > 0$. Then, by (3.8) we have that $\mathbb{P}(S) > 0$.

4. Final Remarks and Examples

We consider two discrete propagation phenomena modeling in their homogeneous and heterogeneous versions. While the Firework Process models a phenomena where there is at all times a finite number of individuals trying to spread an information for an infinite group of individuals, the Reverse Firework Process models a phenomena where there is always an infinite number of individuals willing and working towards to heard about that information from a finite quantity of informed individuals. Our results show that the four versions are qualitatively different.

Considering the Homogeneous Firework Process, Remark 2.3 shows that the information will no be spread for an infinite number of individuals if $\mathbb{E}[R]$ is finite. To have a radius of influence with infinite
expectation is also no guarantee for the information to reach an infinite number of individuals, as Remark 2.5 shows. Besides, the probability of not reaching an infinite amount of individuals is at least \( \mathbb{P}[R = 0] \).

Conversely, in the Heterogeneous Firework Process, to have an infinite expectation guarantees almost surely that the information will spread out among an infinite amount of individuals, as Theorem 2.8 points out. Furthermore in the case where the radius of influence has a power law distribution as in (2.3), the process works in opposite direction as Remark 2.9 shows for \( \alpha = 2 \). The processes agree for \( R \) whose expectation is finite. Next we present some final examples pointing to some extreme cases.

Let \( \{b_n\}_{n \in \mathbb{N}} \) be a non-increasing sequence convergent to 0 and such that \( b_0 < 1 \).

**Example 4.1.** It is possible to have in the Heterogeneous Firework Process the expectation of the radius of influence infinite for all vertices together and the process dies out almost surely.

\[
(i) \quad \mathbb{P}(R_n = 0) = 1 - b_n \text{ and } \mathbb{P}(R_n = k) = b_{n+k-1} - b_{n+k} \text{ for } k \geq 1.
\]

\[
(ii) \quad \sum_{n=0}^{\infty} b_n = \infty.
\]

\[
(iii) \quad \lim_{n \to \infty} nb_n = 0.
\]

Observe that \( \mathbb{E}(R_n) = \infty \) for all \( n \) from (ii). Besides \( \mathbb{P}[V] = 0 \) from (iii), because
\[ \mathbb{P}(V_n) \leq \sum_{k=0}^{n-1} \mathbb{P}(R_k \geq n - k) = \sum_{k=0}^{n-1} b_{n-k} = (n-1)b_n. \quad (4.1) \]

**Example 4.2.** It is possible to have in the *Heterogeneous Firework Process* the expectation of the radius of influence finite for all vertices and the process survives with positive probability. Assume that

\[ \sum_{n=0}^{\infty} b_n < \infty, \]

while

- \( \mathbb{P}(R_n = 0) = b_n \)
- \( \mathbb{P}(R_n = 1) = 1 - b_n \)

Then \( \mathbb{E}(R_n) < 1 \) for all \( n \) and \( \mathbb{P}(V) > 0 \) by part (I) of Theorem 2.7 with \( m = t = 1 \).

**Example 4.3.** Next we present an example where \( \mathbb{P}[S] = 1 \) for a *Reverse Heterogeneous Firework Process* while \( \mathbb{P}[V] = 0 \) for a *Heterogeneous Firework Process*. For this aim consider

(i) \( \mathbb{P}(R_n = 0) = 1 - b_n \) and \( \mathbb{P}(R_n = n) = b_n \).

(ii) \( \sum_{n=0}^{\infty} b_n = \infty. \)

(iii) \( \lim_{n \to \infty} nb_n = 0. \)

Observe that even though \( \lim_{n \to \infty} \mathbb{E}[R_n] = 0 \) and \( \lim_{n \to \infty} \mathbb{P}(R_n = 0) = 1 \), from Theorem 2.10 and (ii) it is true for the *Reverse Heterogeneous Firework Process* that \( \mathbb{P}(S) = 1 \). In the opposite direction, by (4.1) and (iii) one have that \( \mathbb{P}[V] = 0 \) for the *Heterogeneous Firework Process*. 
REFERENCES

[1] S. Athreya, R. Roy and A. Sarkar. On the coverage of space by random sets. *Advances in Applied Probability*. Volume 36, Number 1, 1-18 (2004).

[2] D. Bertacchi and F. Zucca. Critical behaviors and critical values of branching random walks on multigraphs. *J. Appl. Probab.* 45, 481–497 (2008).

[3] D. D. Bonar and M. J. Khoury Jr. Real infinite series, Mathematical Association of America Textbooks (2006).

[4] P. Bremaud. *Markov chains. Gibbs fields, Monte Carlo simulation, and queues*. Texts in Applied Mathematics, 31. Springer-Verlag, New York (1999).

[5] D. Daley and D. G. Kendall. Stochastic rumours. *J Inst Math Appl* 1, 42 - 55 (1965).

[6] W. Feller. An Introduction to Probability Theory and its Applications, Vol 1, 3rd ed, John Wiley, New York (1968).

[7] T. Fort. Infinite Series, Oxford (1930).

[8] G. Grimmett. Percolation, (2nd ed.), Springer-Verlag, New York (1999).

[9] V. Isham, S. Harden and M. Nekovee. Stochastic epidemics and rumours on finite random networks. *Physica A: Statistical Mechanics and its Applications* Volume 389, Issue 3, Pages 561-576 (2010).

[10] H. Kesten and V. Sidoravicius. The spread of a rumor or infection in a moving population. *Ann. Probab.* 33, no. 6, 2402–2462 (2005).

[11] J. Kostka, Y. A. Oswald and R. Wattenhofer. Word of Mouth: Rumor Dissemination in Social Networks in *Structural Information and Communication Complexity* 15th International Colloquium, SIROCCO 2008 Villars-sur-Ollon, Switzerland, June 17-20, Proceedings (2008).

[12] T. G. Kurtz, E. Lebensztayn, A. Leichsenring and F. P. Machado. Limit theorems for an epidemic model on the complete graph. *Alea (Online)*, v. IV, p. 3 (2008).

[13] E. Lebensztayn, F. P. Machado, M. Z. Martinez. Self avoiding random walks on homogeneous trees. Markov Processes and Related Fields, v. 12, p. 735-745 (2006).

[14] E. Lebensztayn, P. M. Rodriguez. The disk-percolation model on graphs. *Statistics & Probability Letters*, vol. 78, issue 14, 2130-2136 (2008).

[15] D. P. Maki and M. Thompson. *Mathematical Models and Applications* Prentice-Hall, Englewood Cliffs, N. J. (1973).

[16] C. E. M. Pearce. The Exact Solution of the General Stochastic Rumours *Mathematical and Computer Modelling*, 31, 289-298 (2000).

[17] B. Pittel. On a Daley–Kendall model of random rumours. *J. Appl. Probab.* 27, 14–27 (1990).

[18] A. Sudbury. The proportion of the population never hearing a rumour *J. Appl. Probab.* 22, 443-446 (1985).

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