Kostant’s cubic Dirac operator of Lie superalgebras

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Abstract

We extend equal rank embedding of reductive Lie algebras to that of basic Lie superalgebras. The Kac character formulas for equal rank embedding are derived in terms of subalgebras and Kostant’s cubic Dirac operator for equal rank embedding of Lie superalgebras is constructed from both even and odd generators and their related structure constants.

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I Introduction

The study of some patterns [1], connected with $N = 1$ supergravity theory in eleven dimensions, has led to a recent understanding in terms of a Weyl character formula by GKRS [2], based on equal rank embeddings of reductive Lie algebras. By using the construction of equal rank embedding of reductive Lie algebras, all possible equal rank embeddings were cataloged, and nearly all supersymmetric multiplets of massless and massive particles which have already known in supersymmetric gauge field theories emerge as the lowest lines of the infinite tower multiplet spectra, some of them shown in details in Ref. [1]. Immediately after the appearance of Weyl character formula for equal rank embedding as an index formula for Dirac operator, Kostant moved the subject forward and related the multiplet spectrum to the kernel of a cubic Dirac operator which he introduced around 30 years ago [3].

In this paper, we extend the Gross-Kostant-Ramond-Sternberg’s Weyl character formula and Kostant’s cubic Dirac operator for equal rank embedding of reductive Lie algebras to those of basic Lie superalgebras. In section II, we give a brief review in [GKRS]’s paper. A derivation of the Weyl character formula for an equal rank embedding is shown. In section III, we extend the equal rank embedding construction of reductive Lie algebras shown in section II to that of basic Lie superalgebras. The Kac character formulas are written in terms of equal rank subalgebras. In section IV, we give a simple and explicit formulation for a typical representation of type I Lie superalgebras. In section V, we build a multiplet of type I Lie superalgebras from that of reductive Lie algebras. In section VI, Kostant’s cubic Dirac operator is constructed for full Lie superalgebras and then for equal rank embeddings.

II Weyl character formula and equal rank embeddings of reductive Lie algebras

Let $r$ be an equal rank subalgebra of reductive Lie algebras, $g$, and let $C$ be order of $C$, the ratio of Weyl group of $g$ to that of $r$. The restricted conditions for this kind of equal rank embedding, $g \supset r$, are that (1) positive roots of $g$ must contain those of $r$, i.e. $\Phi^+(g) \supset \Phi^+(r)$, and (2) the simple roots of $g$ and $r$ must be chosen consistently so that the positive Weyl chamber of $r$ contains that of $g$.

In Cartan-Weyl basis, the $c$ elements of Weyl group in $C$ acting on the sum of highest weight, $\lambda$, and Weyl vector of $g$, $\rho_g$, and then after subtracting by Weyl vector of $r$, $\rho_r$, generate $C$ irreducible representations of $r$, called $C$-multiplet,

$$c \cdot \lambda := c(\lambda + \rho)_g - \rho_r.$$

The Weyl character formula of the irreducible representation of $g$, $V_\lambda$, can be rewritten in terms of the irreducible representation of $r$, $U_{c\cdot\lambda}$, as follows:

$$\text{ch} V_\lambda = \sum_{w \in W(g)} \text{sgn}(w) e^{w(\lambda + \rho_g)} \sum_{w \in W(g)} \text{sgn}(w) e^{w\rho_g}$$
= \sum_{c \in C} \text{sgn}(c) \left( \sum_{w_r \in W(r)} \text{sgn}(w_r) e^{w_r : (c \cdot \lambda + p \cdot \cdot)} \right) \\
\frac{1}{\Delta} \sum_{c \in C} \text{sgn}(c) \text{ch} U_{c \cdot \lambda},

(1)

where \( \Delta \) is the character difference of two spinor modules, \( S^+ \) and \( S^- \), of \( SO(p = g/r) \), i.e.

\[ \Delta := \prod_{\phi \in \Phi^+(g/r)} (e^{\frac{\phi}{2}} - e^{-\frac{\phi}{2}}) = \text{ch} S^+ - \text{ch} S^- \]

The beauty of Eq. (1) is that it gives us

\[ V_\lambda \otimes S^+ - V_\lambda \otimes S^- = \sum_{c \in C} \text{sgn}(c) U_{c \cdot \lambda}. \]

(2)

Eq. (2) should be viewed as an equation in the Grothendieck ring of \( r \). The LHS is the character of the difference between two representations of the same dimension the representation of \( g \) with highest weight \( \lambda \), this representation restricted to \( r \) tensored with each of the half spin representations of \( SO(p) \), where \( p = \dim g - \dim r \), restricted to \( r \). In the other words, the LHS is the algebraic index of the Dirac operator associated to \( \lambda \) and the two half spin representations. The alternating sum on the RHS is just the dimension difference between kernel and cokernel of the Dirac operator. All the representations on the RHS are inequivalent and so the RHS is the end result of a lot of cancellation in the LHS, in short an index formula for the Dirac operator. One of the remarkable consequences of Eq. (2) is that, on the LHS, the order of difference of \( V_\lambda \otimes S^+ \) and \( V_\lambda \otimes S^- \) is always equal to \( C \), independent of \( \lambda \), and, on the RHS, the multiplicity of each \( U_{c \cdot \lambda} \) representation is exactly one.

Notice that \( C \) is equal to the Euler number which is a topological invariant of the coset manifold, \( G/R \), corresponding to exponentiation of \( g/r \). From our catalog of equal rank embedding of reductive Lie algebras, we would like to give some examples of the coset manifolds where supersymmetric multiplets appear to be in their lowest lines of the infinite tower multiplet spectra. \( N = 2 \) hypermultiplet, \( N = 4 \) vector multiplet, and \( N = 8 \) supergravity multiplet which undoubtedly emerge in the lowest lines of \( SU(N+1) \supset SU(N) \times U(1) \) and \( SO(N+2) \supset SO(N) \times SO(2) \) series live on the \( N \)-dimensional complex projective space and \( (N+2) \)-dimensional complex Grassmannian manifold, respectively. If \( SO(2) \) or \( U(1) \) is viewed as the light-cone little group, these lowest line spectra are massless supermultiplets in 4-dimensional space-time. Whereas \( N = 1, N = 2, N = 3, \) and \( N = 4 \) massive (massless) multiplets in 4-dimensional (6-dimensional) space-time which emerge in the lowest lines of \( Sp(2N+2) \supset Sp(2N) \times Sp(2) \) series live on \( N \)-dimensional quaternionic projective space. The last multiplet that we would like to mention is the \( N = 1 \) massless (massive) supergravity triplet in 11-dimensional (10-dimensional) space-time. The triplet emerges from \( F_4 \supset SO(9) \) and lives on (16-dimensional) Cayley plane. All
infinite tower multiplet spectra are kernel of Kostant’s cubic Dirac operator, $\mathcal{K}$,

$$\ker(\mathcal{K}^2) = \ker(\mathcal{K}) = \sum_{c \in \mathcal{C}} \text{sgn}(c) U_{c \bullet \lambda}.$$  

For more analytical details on Kostant’s operator, see Ref. [4].

## III Kac character formulas and equal rank embedding of basic Lie superalgebras

Now, we extend the results of reductive Lie algebras to Lie superalgebras with non-degenerate Killing form. According to Kac’s classification [5], there are two types of basic Lie superalgebras, type I which is $su(m|n)$ and $osp(2|2n)$ and type II which is $osp(2m+1|2n)$, $osp(1|2n)$, $osp(2m|2n)$, $osp(4|2;\alpha)$, $F(4)$, and $G(3)$.

Let $g = g_{\text{even}} \oplus g_{\text{odd}}$ be the Lie superalgebras with the root system $\Phi = \Phi_{\text{even}} \cup \Phi_{\text{odd}}$. For type I, $g_{\text{even}}$ is simple, i.e. $g_{\text{even}} = g_0$, and, for type II, $g_{\text{even}}$ can be graded into $g_2 \oplus g_0 \oplus g_{-2}$. While, for the odd part of both type I and II, odd generators can be graded into fermionic creation and annihilation ones, i.e. $g_{\text{odd}} = g_1 \oplus g_{-1}$. The Poincaré-Birkhoff-Witt theorem for Lie algebras can be applied to the case of Lie superalgebras with some extension [6]. This grading gives us a universal enveloping algebra, $\mathcal{U}(g)$, e.g., for type I,

$$\mathcal{U}(g) = \mathcal{U}(g_1) \otimes \mathcal{U}(g_0) \otimes \mathcal{U}(g_{-1}).$$  

Define root subsystems, $\Phi_{\text{even}}^+$ and $\Phi_{\text{odd}}^-$, such that $\Phi_{\text{even}}^+ = \{\alpha \mid \alpha/2 \not\in \Phi_{\text{odd}}\}$ and $\Phi_{\text{odd}}^- = \{\beta \mid 2\beta \not\in \Phi_{\text{even}}\}$. Since $\Phi_{\text{even}}, \Phi_{\text{odd}}, \Phi_{\text{even}}^+$ and $\Phi_{\text{odd}}^-$ are invariant under the action of Weyl group of $g_{\text{even}}$. Hence, Weyl group of $g$ is equal to that of $g_{\text{even}}$, i.e. $W(g) = W(g_{\text{even}})$. Define Weyl vector of $g$ to be one-half the sum of positive even roots minus one-half the sum of positive odd roots, i.e. $\rho = \rho_{\text{even}} - \rho_{\text{odd}}$. Let $V(\Lambda)$ be a representation of $g$ with $\Lambda$ as a highest weight in dual Cartan subalgebra. The highest weight representations of $g$ are classified into typical and atypical. The representation is typical if, for $\forall \beta \in \Phi_{\text{odd}}^+, (\Lambda + \rho, \beta) \neq 0$; otherwise, it is atypical. The typical Kac character and supercharacter formulas of $V(\Lambda)$ are defined, respectively, as

$$\text{ch}V(\Lambda) = \frac{N_1}{N_0} \sum_{w \in W(g)} \text{sgn}(w)e^{w(\Lambda + \rho)},$$  

$$\text{sch}V(\Lambda) = \frac{N_1'}{N_0} \sum_{w \in W(g)} \text{sgn}(w)e^{w(\Lambda + \rho)},$$  

where

$$N_0 = \prod_{\alpha \in \Phi_{\text{even}}^+} (e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}}),$$
\[ N_1 = \prod_{\beta^+ \in \Phi^+_{\text{odd}}} \left( e^{\frac{\alpha}{2}} + e^{-\frac{\alpha}{2}} \right), \]

and

\[ N'_1 = \prod_{\beta^+ \in \Phi^+_{\text{odd}}} \left( e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}} \right). \]

The sgn(w) and sgn(\omega) are sign change due to number of reflections with respect to \( \Phi^+_{\text{even}} \) and \( \Phi^+_{\text{even}} \).

In general, in an equal rank embedding of Lie superalgebras \( r \) in \( g \) with \( \Phi^+(r) \subset \Phi^+(g) \) and \( C \) as an index of the Weyl group of \( r \) in the Weyl group of \( g \), a \( C \)-multiplet of \( r \) is obtained by

\[ c \bullet \Lambda := c(\Lambda + \rho) - \rho_r, \]

where \( c \in C \). Under the condition that \( \Phi^+_\text{even,odd}(r) \) is invariant under the action of \( c \), i.e. \( c \cdot \Phi^+_\text{even,odd}(r) = \Phi^+_\text{even,odd}(r) \), the typical Kac character formula of \( g \)-module \( V(\Lambda) \) can be written in terms of \( r \)-module \( U(c \bullet \Lambda) \) as follows:

\[
\begin{align*}
\text{ch} V(\Lambda) &= \frac{\prod_{\beta \in \Phi^+_{\text{odd}}(g)}(e^{\frac{\alpha}{2}} + e^{-\frac{\alpha}{2}})}{\prod_{\alpha \in \Phi^+_{\text{even}}(g)}(e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}})} \sum_{w \in W(g)} \text{sgn}(w)e^{w(\Lambda + \rho_g)} \\
&= \left( \prod_{\beta \in \Phi^+_{\text{odd}}(g/r)}(e^{\frac{\alpha}{2}} + e^{-\frac{\alpha}{2}}) \right) \sum_{c \in C} \text{sgn}(c) \text{ch} U(c \bullet \Lambda). \quad (5)
\end{align*}
\]

Similarly done, the typical Kac supercharacter becomes

\[
\begin{align*}
\text{sch} V(\Lambda) &= \left( \prod_{\beta \in \Phi^+_{\text{odd}}(g/r)}(e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}}) \right) \sum_{c \in C} \text{sgn}(c) \text{sch} U(c \bullet \Lambda). \quad (6)
\end{align*}
\]

For type I Lie superalgebras, there are both typical and atypical representations. For the typical representation, superdimension, sdim \( V(\Lambda) = \text{dim} V_{\text{even}}(\Lambda) - \text{dim} V_{\text{odd}}(\Lambda) \), is equal to zero. Whereas, sdim \( V(\Lambda) \) of the atypical representation is not. Every type I odd root is zero-length and \( \Phi^+_\text{even} = \Phi^+_\text{even} \). Since \( \rho_{\text{odd}} \) is invariant under the action of Weyl group, i.e. \( w\rho_{\text{odd}} = \rho_{\text{odd}} \). The type I typical Kac character formula (3) can be written as

\[
\begin{align*}
\text{ch} V(\Lambda) &= \prod_{\beta^+ \in \Phi^+_1} (1 + e^{-\beta}) \text{ch} V_0(\Lambda) \\
&= \prod_{\beta^- \in \Phi^-_1} (1 + e^{\beta}) \text{ch} V_0(\Lambda). \quad (7)
\end{align*}
\]
Multiplying out the product factor on the RHS of Eq.(7), we obtain a Chern character of an exterior algebra over $g_{-1}$,

$$
\prod_{\beta \in \Phi_{1}^{-}} (1 + e^{\beta_{-}}) = \sum_{n=0}^{n=\dim(\Phi_{-1})} \text{ch}(\wedge^{n} g_{-1}) = \text{ch}(\wedge g_{-1}).
$$

So, Eq.(7) simply becomes

$$
\text{ch} V(\Lambda) = \text{ch}(\wedge g_{-1}) \text{ch} V_{0}(\Lambda).
$$

Similarly, the type I typical Kac supercharacter can be shown to be

$$
\text{sch} V(\Lambda) = \prod_{\beta \in \Phi_{1}^{-}} (1 - e^{\beta_{-}}) \text{ch} V_{0}(\Lambda)
$$

$$
= \text{sch}(\wedge g_{-1}) \text{ch} V_{0}(\Lambda).
$$

On the other hand, since $\{g_{-1}, g_{-1}\} = 0$, the universal enveloping algebra over $g_{-1}$, $\mathcal{U}(g_{-1})$, is isomorphic to the exterior algebra over $g_{-1}$, $\wedge(\wedge g_{-1})$. The $g$-module $V(\Lambda)$ can be induced by applying the antisymmetric combinations of the $g_{-1}$ generators on $V_{0}(\Lambda)$, i.e.

$$
V(\Lambda) = \wedge(g_{-1}) \otimes V_{0}(\Lambda) \simeq \mathcal{U}(g_{-1}) \otimes V_{0}(\Lambda).
$$

The character and supercharacter of Eq.(10) are exactly Eq.(8) and Eq.(9).

In an equal rank embedding, $g \supset r$, of type I which has $\mathbb{C}$ as an index of $W(r)$ in $W(g)$ and is restricted to the condition that $\Phi^{+}(g) \supset \Phi^{+}(r)$, Eq.(7) becomes

$$
\text{ch} V(\Lambda) = \frac{1}{\Delta} \prod_{\beta \in \Phi_{1}^{-}(g/r)} (1 + e^{\beta_{-}}) \sum_{c \in C} \text{sgn}(c) \left( \prod_{\beta \in \Phi_{1}^{-}(r)} (1 + e^{\beta_{-}}) \text{ch} U_{0}(c \cdot \Lambda) \right),
$$

i.e.

$$
\text{ch} V(\Lambda) (\text{ch} S^{+} - \text{ch} S^{-}) = \text{ch}(\wedge(g_{-1}/r_{-1})) \sum_{c \in C} \text{sgn}(c) \text{ch} U(c \cdot \Lambda).
$$

Similarly done for the supercharacter, we obtain

$$
\text{sch} V(\Lambda) (\text{ch} S^{+} - \text{ch} S^{-}) = \text{sch}(\wedge(g_{-1}/r_{-1})) \sum_{c \in C} \text{sgn}(c) \text{sch} U(c \cdot \Lambda).
$$

Eq.(11) and Eq.(12) correspond to

$$
V(\Lambda) \otimes S^{+} - V(\Lambda) \otimes S^{-} = \wedge(g_{-1}/r_{-1}) \otimes \sum_{c \in C} \text{sgn}(c) U(c \cdot \Lambda).
$$

Now, Eq.(13) can also be derived explicitly from Eq.(10). By decomposing the $g_{-1}$ basis such that $g_{-1} = (g_{-1}/r_{-1}) \oplus r_{-1}$, there exists a map $(g_{-1}/r_{-1}) \oplus r_{-1} \mapsto \mathbb{C}$.
\((g_{-1}/r_{-1}) \otimes 1 + 1 \otimes r_{-1}\) from \((g_{-1}/r_{-1}) \oplus r_{-1}\) to \((g_{-1}/r_{-1}) \otimes r_{-1}\) which extends uniquely to an isomorphism of exterior algebra,

\[ \wedge (g_{-1}/r_{-1} \oplus r_{-1}) \simeq \wedge (g_{-1}/r_{-1}) \otimes \wedge (r_{-1}). \]

By substituting the above equation into Eq. (10) and tensoring on both side by \((S^+ - S^-)\), we obtain

\[ V(\Lambda) \otimes S^+ - V(\Lambda) \otimes S^- = \wedge (g_{-1}/r_{-1}) \otimes \sum_{c \in C} \text{sgn}(c) (\wedge (r_{-1}) \otimes U_0 (c \cdot \Lambda)) \]

\[ = \wedge (g_{-1}/r_{-1}) \otimes \sum_{c \in C} \text{sgn}(c) U(c \cdot \Lambda), \]

which is exactly Eq. (13).

For \(osp(1|2n)\) of type II Lie superalgebras, every \(osp(1|2n)\) odd root has length equal to one-half of the short positive one and

\[ \rho_{osp(1|2n)} = \rho_{even} - \rho_{odd} = \left( n - \frac{1}{2}, n - \frac{3}{2}, \ldots, \frac{1}{2} \right). \]

Furthermore, every \(osp(1|2n)\) representation is typical, but \(\dim V_{even}(\Lambda) \neq \dim V_{odd}(\Lambda)\). Nevertheless, the Kac character and supercharacter formulas of a \(osp(1|2n)\) representation can be shown to be similar to that of Lie algebra, i.e.

\[ \text{ch} V(\Lambda) = \sum_{w \in W} \frac{\text{sgn}(w) e^{w(\Lambda + \rho)}}{\sum_{w \in W} \text{sgn}(w) e^{w(\rho)}}. \quad (14) \]

\[ \text{sch} V(\Lambda) = \sum_{w \in W} \frac{\text{sgn}(w) e^{w(\Lambda + \rho)}}{\sum_{w \in W} \text{sgn}(w) e^{w(\rho)}}. \quad (15) \]

The equal rank embedding of \(osp(1|2m) \times osp(1|2n - 2m)\) in \(osp(1|2n)\) is the only possible type with the full subsuperalgebra. In this case, all odd generators of \(g\) are completely eaten by the \(r\). Whereas, the even generators of \(g\) which are left form the orthogonal complement basis to the even basis of \(r\) under the killing form of \(g\). Under the restriction to \(r\)-module, Eq. (14) and Eq. (15) simply become

\[ \text{ch} V(\Lambda) = \frac{1}{\Delta} \sum_{c \in C} \text{sgn}(c) \text{ch} U(c \cdot \Lambda), \quad (16) \]

and

\[ \text{sch} V(\Lambda) = \frac{1}{\Delta} \sum_{c \in C} \text{sgn}(c) \text{sch} U(c \cdot \Lambda). \quad (17) \]

Both Eq. (16) and Eq. (17) correspond to

\[ V(\Lambda) \otimes S^+ - V(\Lambda) \otimes S^- = \sum_{c \in C} \text{sgn}(c) U(c \cdot \Lambda). \quad (18) \]

For the rest of type II, we use Kac character and supercharacter formulas of equal rank embedding, Eq. (14) and Eq. (15). If \(g_1\) and \(g_{-1}\) are vector spaces of odd
generators, there is a canonical linear map from $\wedge^a g_1 \otimes \wedge^b g_{-1}$ to $\wedge^{a+b}(g_1 \oplus g_{-1})$, which takes $((g_1)_1 \wedge \cdots \wedge (g_1)_a) \otimes ((g_{-1})_1 \wedge \cdots \wedge (g_{-1})_b)$ to $((g_1)_1 \wedge \cdots \wedge (g_1)_a \wedge (g_{-1})_1 \wedge \cdots \wedge (g_{-1})_b)$. This determines a linear isomorphism

$$\wedge(g_1 \oplus g_{-1}) \simeq \bigoplus_{a=0}^N (\wedge^{N-a} g_1 \otimes \wedge^a g_{-1})$$

$$\simeq \wedge g_1 \otimes \wedge g_{-1}.$$

The prefactor $N_1$ of Kac character formula is generally the character of the exterior algebra over the direct sum of $g_1$ and $g_{-1}$ vector spaces

$$\prod_{\beta \in \Phi^+} \left( e^{\frac{\beta}{2}} + e^{-\frac{\beta}{2}} \right) = \prod_{\beta \in \Phi^+_\pm} \left( e^{\frac{\beta_+}{2}} + e^{\frac{\beta_-}{2}} \right) = \chi \wedge (g_1 \oplus g_{-1}).$$

Finally, the type II typical Kac character and supercharacter can be written as

$$\text{ch} V(\Lambda) \left( \text{ch} S^+ - \text{ch} S^- \right) = \text{ch} \wedge (g_1/r_1 \oplus g_{-1}/r_{-1}) \sum_{c \in C} \text{sgn}(c) \text{ch} U(c \Lambda), \quad (19)$$

and

$$\text{sch} V(\Lambda) \left( \text{ch} S^+ - \text{ch} S^- \right) = \text{sch} \wedge (g_1/r_1 \oplus g_{-1}/r_{-1}) \sum_{c \in C} \text{sgn}(c) \text{sch} U(c \Lambda), \quad (20)$$

which correspond to

$$V(\Lambda) \otimes S^+ - V(\Lambda) \otimes S^- = \wedge (g_1/r_1 \oplus g_{-1}/r_{-1}) \otimes \sum_{c \in C} \text{sgn}(c) U(c \Lambda). \quad (21)$$

### IV Representations of $\wedge g_{-1}$ of type I Lie superalgebras

For type I Lie superalgebras with non-degenerate Killing form, the typical representation is induced by applying $g_{-1}$ generators on $g_0$-module. Therefore, we need to know an explicit representation of $\wedge g_{-1}$ in terms of $g_0$-module. Let $N$ be dimension of $\Phi^+_1$. The exterior algebra over $g_{-1}$ is

$$\wedge g_{-1} = \bigoplus_{k=0}^N \wedge^k g_{-1},$$

with dimension,

$$\dim(\wedge g_{-1}) = \bigoplus_{k=0}^N \dim(\wedge^k g_{-1}) = \sum_{k=0}^N \binom{N}{k} = 2^N,$$
and superdimension,
\[ \text{sdim}(\wedge g_{-1}) = \bigoplus_{k=0}^{N} \text{sdim}(\wedge^k g_{-1}) = \sum_{k=0}^{N} (-1)^k \binom{N}{k} = 0. \]

Let \( Q^I_{i=1,2,\ldots,N} \) be \( N \) completely antisymmetric fermionic generators which transform in the fundamental representation of \( g_0 \). The fermionic generators generate even and odd modules which are isomorphic to a direct sum of two spinor representations of \( so(2N) \). One of spinor representation of \( so(2N) \) is the even module of \( \wedge g_{-1} \), called bosonic module, and the other is the odd module of \( \wedge g_{-1} \), called fermionic module. Let \( T^+ \) be the bosonic module and \( T^- \) be the fermionic module of \( so(2N) \) such that
\[
T^+ = \wedge^0 g_{-1} \oplus \wedge^2 g_{-1} \oplus \wedge^4 g_{-1} \oplus \ldots
= \left( (Q^I_i)^0 \oplus (Q^I_i)^2 \oplus (Q^I_i)^4 \oplus \ldots \right) |1 >0
\]
\[
\equiv \text{bosonic module},
\]
and
\[
T^- = \wedge^1 g_{-1} \oplus \wedge^3 g_{-1} \oplus \wedge^5 g_{-1} \oplus \ldots
= \left( (Q^I_i)^1 \oplus (Q^I_i)^3 \oplus (Q^I_i)^5 \oplus \ldots \right) |1 >0
\]
\[
\equiv \text{fermionic module}.
\]

With restriction to \( g_0 \)-module, the type I typical representation is
\[
V(\Lambda) = \wedge g_{-1} \otimes V_0(\Lambda)
= (T^+ \oplus T^-) \otimes V_0(\Lambda).
\]

For \( su(m|n) \), \( \wedge g_{-1} \) representation is isomorphic to a direct sum of two spinor representations of \( so(2mn) \). With restriction to \( su(m) \times su(n) \times u(1) \), \( Q^I_i \) transforms as \( (m,n)_- \), where -1 is a \( u(1) \) charge.

**Ex.1** \( su(2|1) \)

\[
\dim(g_{-1}) = 2^2 = 1 + 2 + 1
\]

\( Q^I_i \sim 2_{-1} \)

\[
T^+ = 1_0 \oplus 1_{-2}
\]

\[
T^- = 2_{-1}
\]

\[
V_{su(2|1)}(\Lambda) = (T^+ \oplus T^-) \otimes V_{su(2) \times u(1)}(\Lambda)
\]

**Ex.2** \( su(3|1) \)

\[
\dim(g_{-1}) = 2^3 = 1 + 3 + 3 + 1
\]
\[ Q^\dagger \sim 3_{-1} \]

\[ T^+ = 1_0 \oplus \overline{3}_{-2} \]
\[ T^- = 3_{-1} \oplus 1_{-3} \]

\[ V_{su(3|1)}(\Lambda) = (T^+ \oplus T^-) \otimes V_{su(3) \times u(1)}(\Lambda) \]

**Ex.3 su(3|2)**

\[ \dim(g^{-1}) = 2^6 = 1 + 6 + 15 + 20 + 15 + 6 + 1 \]

\[ Q^\dagger \sim (3, 2)_{-1} \]

\[ T^+ = (1, 1)_0 \oplus (\overline{3}, 3)_{-2} \oplus (6, 1)_{-2} \oplus (6, 1)_{-4} \oplus (3, 3)_{-4} \oplus (1, 1)_{-6} \]
\[ T^- = (3, 2)_{-1} \oplus (8, 2)_{-3} \oplus (4, 1)_{-3} \oplus (\overline{3}, 2)_{-5} \]

\[ V_{su(3|2)}(\Lambda) = (T^+ \oplus T^-) \otimes V_{su(3) \times su(2) \times u(1)}(\Lambda) \]

**Ex.4 su(4|2)**

\[ \dim(g^{-1}) = 2^8 = 1 + 8 + 28 + 56 + 70 + 56 + 28 + 8 + 1 \]

\[ Q^\dagger \sim (4, 2)_{-1} \]

\[ T^+ = (1, 1)_0 \oplus (10, 1)_{-2} \oplus (6, 3)_{-2} \oplus (20', 1)_{-4} \oplus (15, 3)_{-4} \]
\[ \oplus (5, 1)_{-4} \oplus (\overline{10}, 1)_{-6} \oplus (6, 3)_{-6} \oplus (1, 1)_{-8} \]
\[ T^- = (4, 2)_{-1} \oplus (20, 1)_{-3} \oplus (\overline{4}, 4)_{-3} \oplus (\overline{20}, 1)_{-5} \oplus (4, 4)_{-5} \oplus (\overline{4}, 2)_{-7} \]

\[ V_{su(4|2)}(\Lambda) = (T^+ \oplus T^-) \otimes V_{su(4) \times su(2) \times u(1)}(\Lambda) \]

For \( osp(2|2n), \wedge g^{-1} \) representation is a direct sum of two spinor representations of \( so(4n) \). With restriction to \( sp(2n) \times u(1) \), \( Q^\dagger \) transforms as \((2n)_{-1}\) with -1 as a \( u(1) \) charge.

**Ex.5 osp(2|4)**

\[ \dim(g^{-1}) = 2^4 = 1 + 4 + 6 + 4 + 1 \]

\[ Q^\dagger \sim 4_{-1} \]

\[ T^+ = 1_0 \oplus 5_{-2} \oplus 1_{-2} \oplus 1_{-4} \]
\[ T^- = 4_{-1} \oplus 4_{-3} \]

\[ V_{osp(2|4)}(\Lambda) = (T^+ \oplus T^-) \otimes V_{sp(4) \times u(1)}(\Lambda) \]

**Ex.6 osp(2|6)**

\[ \dim(g^{-1}) = 2^6 = 1 + 6 + 15 + 20 + 15 + 6 + 1 \]
\[ Q_i^\dagger \sim 6_{-1} \]

\[
\begin{align*}
T^+ &= 1_0 \oplus 14_{-2} \oplus 1_{-2} \oplus 14_{-4} \oplus 1_{-4} \oplus 1_{-6} \\
T^- &= 6_{-1} \oplus 14'_{-3} \oplus 6_{-3} \oplus 6_{-5} \\
V_{osp(2|6)}(\Lambda) &= (T^+ \oplus T^-) \otimes V_{sp(6) \times u(1)}(\Lambda)
\end{align*}
\]

V Building type I C-multiplets and Kostant’s cubic Dirac operator

For type I Lie superalgebras, there is a remarkable point we would like to mention. From the multiplet spectrum of equal rank embedding of reductive Lie algebras, \( r_0 \subset g_0 \), we can build the spectrum of the type I Lie superalgebras on top of them by simply tensoring them with a module generated by \( r_{-1} \) generators. The \( r_{-1} \) generators transform in the fundamental representation of \( r_0 \).

Define \( \bigwedge (r_{-1}) = R^+ \oplus R^- \) such that

\[
U(\Lambda) = \bigwedge (r_{-1}) \otimes U_0(\Lambda) = (R^+ \oplus R^-) \otimes U_0(\Lambda).
\]

Tensoring on both side of Eq.(22) with \( \bigwedge (r_{-1}) \), we obtain

\[
\bigwedge (r_{-1}) \otimes (V_0(\Lambda) \otimes S^+ - V_0(\Lambda) \otimes S^-) = \sum_{c \in C} \text{sgn}(c) \bigwedge (r_{-1}) \otimes U_0(c \cdot \Lambda)) = \sum_{c \in C} \text{sgn}(c) U(c \cdot \Lambda).
\]

Under restriction to \( r_0 \)-module, whenever \( V_0(\Lambda) \) on the LHS of Eq.(22) is \( su(n) \times u(1) \), or \( sp(2n) \times u(1) \), there is an emergence of type I typical C-multiplet on the RHS. Notice in the case that \( U(c \cdot \Lambda) \) is \( su(m|n) \), the representations of \( R^\pm \) are similar to those of \( S^\pm \) except \( u(1) \) values.

Let \( g = r \oplus p \) be Lie superalgebras where \( p \) is the orthogonal complement to \( r \) under the non-degenerate Killing form of \( g \). Let \( p = p_{\text{even}} \oplus p_{\text{odd}} = p_0 \oplus p_1 \oplus p_{-1} \). Tensoring on both sides of Eq.(22) by \( \bigwedge p_{-1} \), we get

\[
V(\Lambda) \otimes S^+ - V(\Lambda) \otimes S^- = \bigwedge p_{-1} \otimes \sum_{c \in C} \text{sgn}(c) U(c \cdot \Lambda),
\]

which is exactly Eq.(13). Now, we need to get rid of \( \bigwedge p_{-1} \) on the RHS of Eq.(22) by mapping it into identity. To do so, we do a tensor product on both sides of Eq.(23) by the following contraction, sometimes called an internal product, on exterior powers between vector space and its dual:

\[
\bigwedge^N p_{\text{odd}} = \bigwedge^N p_1 \otimes \bigwedge^N p_{-1} = 1.
\]

Eq.(23) becomes

\[
(\bigwedge^N p_{\text{odd}}) \otimes V(\Lambda) \otimes (S^+ - S^-) = 1 \otimes \sum_{c \in C} \text{sgn}(c) U(c \cdot \Lambda).
\]
As already known, the character form of a Dirac operator, $\Theta \in C(p)$, with a map $\Theta : S^\pm \to S^\mp$ is

$$\text{ch}(\Theta) = \text{ch}(S^+) - \text{ch}(S^-).$$

The character of Eq. (1) implies that there exists a Kostant’s Dirac operator, $\mathcal{K} \in \mathcal{U}(g_{\text{even}}) \otimes C(p_{\text{even}})$, with a map

$$\mathcal{K}_\lambda : V_{\lambda} \otimes S^\pm \to V_{\lambda} \otimes S^\mp. \tag{25}$$

Similarly, the character of Eq. (24) suggests that there exists the operator for Lie superalgebras, $\mathcal{K}_\Lambda \in \mathcal{U}(p_{\text{odd}}) \otimes \mathcal{U}(g_{\text{even}} \oplus g_{\text{odd}}) \otimes C(p_{\text{even}})$, with a map

$$\mathcal{K}_\Lambda : V(\Lambda) \otimes S^\pm \to V(\Lambda) \otimes S^\mp. \tag{26}$$

VI Kostant’s cubic Dirac operator for an equal rank embedding of Lie superalgebras

Let $L_i$ and $F_a$ be even and odd generators, respectively, for $Z_2$-graded Lie superalgebras such that

$$[L_i, L_j] = f_{[ij]k} L_k,$$

$$\{F_a, F_b\} = f_{(ab)i} L_i,$$

and

$$[L_i, F_a] = f_{[ia]b} F_b,$$

where $i, j, k = 1, 2, \ldots, \dim(g_0)$ and $a, b = 1, 2, \ldots, \dim(g_1)$. The Kostant’s cubic Dirac operator of Lie superalgebras is extended from that of Lie algebras by adding just two terms, a linear term in odd operators and a structure constant term, i.e.

$$\mathcal{K}_\Lambda = \mathcal{K}^0_\Lambda + \mathcal{K}^1_\Lambda, \tag{27}$$

where

$$\mathcal{K}^0_\Lambda = \gamma_i L_i - \frac{1}{2} \gamma_{[ij]k} f_{[ij]}$$

and

$$\mathcal{K}^1_\Lambda = \alpha_a F_a - \frac{1}{2} \gamma_i \gamma^i_{(ab)} f_{i(ab)}. \tag{28}$$

Sum over all indices is assumed in the above equations, where $[\ldots]$ in the subscript is for antisymmetric sum and $(\ldots)$ for symmetric sum. Eq. (28) is the Kostant’s cubic Dirac operator for reductive Lie algebras. The $\gamma$-matrices associated to even generators are normalized so that

$$\{\gamma_i, \gamma_j\} = \delta_{ij},$$

which gives

$$\{\gamma_i, \gamma_{[ijk]}\} = \delta_{ik} \gamma_{[ij]}.$$

The $\alpha$-matrices associated to odd generators are subjected to the following conditions.

12
\[ \{ \gamma_i, \alpha_a \} = 0. \] This relation is consistent with the antisymmetric property of product of even and odd generators.

\[ \{ \alpha_a, [\alpha_b, \alpha_c] \} = [\alpha_{(ab)}, \alpha_c] = 0. \] This property is due to invariance of odd generators under Killing form.

Squaring Eq. (28), we get

\[
(K^0_L)^2 = \frac{1}{2} \gamma_{(ij)} \{ L_i, L_j \} + \frac{1}{2} \gamma_{[ij]} [L_i, L_j] \\
- \frac{1}{2} [\gamma^\nu, \gamma_{[jk]}) f_{[ijk]} L_{\nu'} + \left( \frac{1}{2} \gamma_{[ijk]} f_{[ijk]} \right)^2 \\
= \frac{1}{2} \gamma_{(ij)} \{ L_i, L_j \} + \left( \frac{1}{2} \gamma_{[ijk]} f_{[ijk]} \right)^2
\]

(30)

Notice that the linear terms in even generators cancel each other. Thus, Eq. (30) is explicitly invariant under the action of even and odd generators. The first term of Eq. (30) is the quadratic Casimir operator of reductive Lie algebras,

\[ C^0_L(L) = \frac{1}{2} \gamma_{(ij)} \{ L_i, L_j \}. \]

Since \( \rho_0 \), one-half the sum of positive even roots, can be identified as

\[ \rho_0 = \frac{1}{2} \gamma_{[ijk]} f_{[ijk]}. \]

The second term of Eq. (30) is the Freudenthal-de Vries’ strange formula,

\[
(\rho, \rho)_0 = \left( \frac{1}{2} \gamma_{[ijk]} f_{[ijk]} \right)^2 = \frac{1}{24} \dim(g_0) h^\vee_0 (\theta, \theta)_0 \\
= \frac{1}{24} \dim(g_0) C^0_{2, ad}.
\]

(33)

Where, in the above equation, \( h^\vee \) is dual Coxeter number, \( \theta \) is the highest root, and \( C^0_{2, ad} \) is the quadratic Casimir value in the adjoint representation.

Squaring Eq. (29), we get

\[
(K^1_L)^2 = \frac{1}{2} \alpha_{[ab]} [F_a, F_b] + \frac{1}{2} \alpha_{(ab)} \{ F_a, F_b \} \\
- \frac{1}{2} \gamma^i [\alpha_{(ab)}, \alpha_{a'}] f_{i[ab]} F_{ai'} + \left( \frac{1}{2} \gamma^i \alpha_{(ab)} f_{i(ab)} \right)^2 \\
= \frac{1}{2} \alpha_{[ab]} [F_a, F_b] + \frac{1}{2} \alpha_{(ab)} f_{i(ab)} L_i + \left( \frac{1}{2} \gamma^i \alpha_{(ab)} f_{i(ab)} \right)^2.
\]

(34)

In contrary to the even part, Eq. (34) by itself is not invariant due to the presence of a linear term in even generators. The linear term in \( L_i \) will be cancelled out by one of the the cross terms of the square of the combined even and odd Kostant’s
cubic Dirac operator,

\[
(\mathcal{K}_\Lambda)^2 = (\mathcal{K}_\Lambda^0)^2 + (\mathcal{K}_\Lambda^1)^2 + \{\gamma_i, \alpha_a\} L_i F_a - \frac{1}{2} \{\gamma_{[ijk]}, \alpha_a\} f_{[ijk]} F_a
\]

\[
- \frac{1}{2} \{\gamma_i, \gamma_{i'}\} \alpha_{(ab)} f_{i' (ab)} L_i - \frac{1}{2} \gamma_i \alpha_{(ab)} \alpha_{a'} f_i (ab) F_{a'}
\]

\[
+ \frac{1}{2} \{\gamma_{[ijk]}, \gamma_{i'}\} \alpha_{(ab)} f_{[ijk]} f_{i' (ab)}
\]

\[
= \gamma_{(ij)} L_{(ij)} + \alpha_{[ab]} F_{[a} F_{b]} + \left(\frac{1}{2} \gamma_{[ijk]} f_{[ijk]}\right)^2
\]

\[
+ \left(\frac{1}{2} \gamma_i \alpha_{(ab)} f_{i (ab)}\right)^2 + \frac{1}{2} \gamma_{[ijk]} \alpha_{(ab)} f_{[ijk]} f_{k (ab)}
\]

(35)

Since \(\rho_1\), one-half the sum of positive odd roots, can be identified as

\[\rho_1 = -\frac{1}{2} \gamma_i \alpha_{(ab)} f_{i (ab)}.\] (36)

Recall that, for Lie superalgebras, \(\rho = \rho_0 - \rho_1\). Eq. (35) can be simply written as

\[
(\mathcal{K}_\Lambda)^2 = C_2(\Lambda) + (\rho, \rho),\] (37)

where

\[
C_2(\Lambda) = \gamma_{(ij)} L_{(ij)} + \alpha_{[ab]} F_{[a} F_{b]}.
\]

The Laplacian operator turns out to be invariant under the action of even and odd generators of \(g\) again. For Lie superalgebras, the generalization of Freudenthal-de Vries strange formula still holds

\[
(\rho, \rho) = \frac{\hbar\vee}{24} (\dim g_0 - \dim g_1),\] (38)

where \(\hbar\vee\)'s, the dual Coxeter numbers, of \(g\) are given in Table 1.

In an equal rank embedding of Lie superalgebras, \(g \supset r\), with \(\Phi_g \supset \Phi_r\), let the even and odd generators, \(L_i\) and \(F_a\), span the basis of \(g\). In according to the \(g = r \oplus p\) decomposition, \(L_i\) and \(F_a\) span the basis of \(r\) and \(L_I\) and \(F_A\) span the basis of \(p\), the orthogonal complement of \(r\) under the Killing form of \(g\). The Kostant’s cubic Dirac operator on \(p = g/r\) is

\[
\mathcal{K}_p = \mathcal{K}_g - \mathcal{K}_r
\]

\[
= \gamma_I L_I + \alpha_A F_A - \frac{1}{2} \gamma_{[IJK]} f_{[IJK]} - \frac{1}{2} \gamma_I \alpha_{(AB)} f_I (AB),\] (39)

where \(f_{[IJK]}\) and \(f_I (AB)\) are the structure constants of \(g\) that are not in \(r\). Since, under the Killing form of \(g\), \(r\) and \(p\) basis are orthogonal to each other,

\[
\{\gamma_i, \gamma_I\} = \{\gamma_i, \alpha_A\} = \{\alpha_a, \gamma_I\} = \{\alpha_a, \alpha_A\} = 0.
\]

As a result, we have

\[
\{\mathcal{K}_r, \mathcal{K}_p\} = 0.
\]
The square of Kostant’s coset cubic Dirac operator simply is

\((K_p)^2 = (K_g)^2 - (K_r)^2 = (C_2 + (\rho, \rho))_g - (C_2 + (\rho, \rho))_r.\)  \hspace{1cm} (40)

Notice that both \((K_g)^2\) and \((K_r)^2\) commute with the generators of \(r\). Hence, \((K_p)^2\) is also invariant under the action of \(r\).

In conclusion, we have derived the Kac character formulas and have constructed Kostant’s cubic Dirac operator for equal rank embeddings of Lie superalgebras. In case of reductive Lie algebras, the coset space method of equal rank embeddings led to a Kazama-Suzuki model construction of a new class of unitary \(N = 2\) superconformal theories \(^3\) and a subclass of the construction could be represented by Landau-Ginzburg models \(^4\). In case of Lie superalgebras, it deserves to be pursued whether there will be any relevance of the coset superspace method to a construction of any physical model.

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| $g$       | $h^{\vee}$                        |
|-----------|-----------------------------------|
| $su(m|n)$ | $|m-n|$                            |
| $osp(2|2n)$ | $n$                              |
| $osp(2m+1|n)$ | $2(m-n)-1$ if $m > n$, $n-m+\frac{1}{2}$ if $m \leq n$ |
| $osp(2m|n)$ | $2(m-n-1)$ if $m \geq n$, $n-m+1$ if $m < n+1$ |
| $osp(4|2; \alpha)$ | 0                              |
| $F(4)$    | 3                                |
| $G(3)$    | 2                                |

Table 1: Dual Coxeter Numbers of Basic Lie Superalgebras