Prepotentials for $\mathcal{N} = 2$ conformal supergravity in three dimensions

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Abstract

We present a complete solution of the constraints for three-dimensional $\mathcal{N} = 2$ conformal supergravity in terms of unconstrained prepotentials. This allows us to develop a prepotential description of the off-shell versions of $\mathcal{N} = 2$ Poincaré and anti-de Sitter supergravity theories constructed in arXiv:1109.0496.
1 Introduction

$\mathcal{N} = 2$ conformal supergravity in three dimensions has recently been formulated in an off-shell superspace setting, building on the conventional torsion constraints given in [3]. This formulation has then been employed in [4] to construct several off-shell versions for $\mathcal{N} = 2$ Poincaré and anti-de Sitter supergravity theories by coupling the Weyl supermultiplet to certain conformal compensators. Here we provide a complete solution of the constraints for three-dimensional $\mathcal{N} = 2$ conformal supergravity in terms of unconstrained prepotentials. In particular, we show that, modulo gauge transformations, the Weyl supermultiplet is described by a real vector superfield $H_{\alpha\beta} = H_{\beta\alpha}$ with a nonlinear gauge transformation law

$$\delta H_{\alpha\beta} = \bar{D}_{(\alpha} L_{\beta)} - D_{(\alpha} \bar{L}_{\beta)} + O(H),$$

where the gauge parameter $L_{\alpha}$ is an unconstrained complex spinor. The linearized version of this transformation law was given in [4]. The prepotential $H_{\alpha\beta}$ is a three-dimensional (3D) analog of the $\mathcal{N} = 1$ gravitational superfield in four dimensions introduced for the first time in [5, 6].

Our three-dimensional construction is similar to the famous prepotential description, originally due to Siegel [8] (and further developed in [9, 10]), for the Wess-Zumino supergravity formulation (also known, in the component approach, as the old minimal formulation for 4D $\mathcal{N} = 1$ supergravity [11]), and its extension to the case of 4D $\mathcal{N} = 1$ conformal supergravity realized in superspace [15]. At the same time, as will be shown below, some nontrivial differences exist between the 3D $\mathcal{N} = 2$ and 4D $\mathcal{N} = 1$ supergravity theories. In principle, the 3D prepotential solution could be obtained by dimensional reduction from 4D $\mathcal{N} = 1$ conformal supergravity following the procedure outlined in section 7.9 of Superspace [14]. In practice, however, it is more advantageous to follow a manifestly covariant approach and derive the solution from scratch. In this sense the 3D story is similar to that of (2,2) supergravity in two dimensions [16, 17].

For both $\mathcal{N} = 2$ Poincaré and anti-de Sitter supergravity theories in three dimensions, the linearized off-shell actions and associated supercurrent multiplets were constructed in [4] by applying dimensional reduction and duality. Using the prepotential formulation developed in this paper, these results can be re-derived from first principles. In four

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1 See [7] for a review of the Ogievetsky-Sokatchev approach to $\mathcal{N} = 1$ supergravity in four dimensions.

2 The linearized version of old minimal supergravity appeared in [5, 12].

3 See [18] for an alternative construction of 3D $\mathcal{N} = 2$ supercurrents.
dimensions, the supercurrent \[19\] originates as the source of supergravity \[5, 12, 20\]. Now we have the same picture in the 3D case.

The prepotential formulation given in this paper makes it possible to develop supergraph techniques for 3D \(\mathcal{N} = 2\) matter-coupled supergravity, including a three-dimensional extension of the background-field formalism for \(\mathcal{N} = 1\) supergravity in four dimensions \[21, 22\]. Such techniques may be useful to address several essentially three-dimensional problems. First of all, the supergraph techniques may be useful to study the quantum properties of new massive supergravity theories \[23, 24, 25\]. Secondly, they may be useful for explicit calculations of newly discovered anomalies in three dimensions \[26, 27\]. Thirdly, they may be helpful for achieving a better understanding of quantum (super)gravity in three dimensions. As is known, 3D Poincaré and anti-de Sitter (super)gravity theories do not have propagating degrees of freedom at the classical level, since there are no gravitational waves in three dimensions. At the quantum level, 3D gravity can be consistently defined, within a Chern-Simons formulation, and constitutes an exactly soluble model \[28\], in spite of earlier beliefs that general relativity in three dimensions was unrenormalizable. The same conclusions hold for \(\mathcal{N}\)-extended supergravity theories in three dimensions formulated as Chern-Simons theories in \[29, 30\]. It would be interesting to see how the exact solubility of 3D (super)gravity manifests itself in terms of the standard effective action derived using conventional geometric formulations. The case of 3D \(\mathcal{N} = 2\) supergravity formulated in superspace provides a simple playground for that.

This paper is organized as follows. In section 2 we briefly review the superspace formulation for \(\mathcal{N} = 2\) conformal supergravity given in \[2\]. We also present a \(\mathcal{N} = 2\) supersymmetric generalization of the Cotton tensor. The prepotential formulation for \(\mathcal{N} = 2\) conformal supergravity is described in sections three to five. Finally, a discussion of the results is given in section 5. We use the 3D conventions of \[2\].

\section{Conformal supergravity}

In this section we give a brief review of the superspace formulation for \(\mathcal{N} = 2\) conformal supergravity presented in \[2\]. It makes use of a curved 3D \(\mathcal{N} = 2\) superspace \(\mathcal{M}^{3|4}\) parametrized by local bosonic \((x)\) and fermionic \((\theta, \bar{\theta})\) coordinates

\[ z^M = (x^m, \theta^\mu, \bar{\theta}_\dot{\mu}), \quad m = 0, 1, 2, \quad \mu = 1, 2, \]
where the Grassmann variables $\theta^\mu$ and $\bar{\theta}^\mu = \varepsilon^{\mu\nu} \bar{\theta}_\nu$ are mutually conjugate. In accordance with [3], the structure group is chosen to be $\text{SL}(2, \mathbb{R}) \times \text{U}(1)_R$. The covariant derivatives have the form

$$D_A = (D_a, D_\alpha, \bar{D}^\alpha) = E_A + \Omega_A^{\beta\gamma} M_{\beta\gamma} + i \Phi_A J,$$  

where $E_A = E_A^M(z) \partial / \partial z^M$ is the supervielbein, $\Omega_A^{\beta\gamma}$ and $\Phi_A$ are the Lorentz and the $\text{U}(1)_R$ connections respectively. The Lorentz generators act on the covariant derivatives as follows:

$$[M_{\gamma\delta}, D_a] = \varepsilon_{abc}(\gamma^c)_{\gamma\delta} D^b, \quad [M_{\gamma\delta}, D_\alpha] = \varepsilon_{\alpha(\gamma} D_{\delta)}.$$

This can be rewritten as

$$[\Lambda^{\gamma\delta} M_{\gamma\delta}, D_A] = \Lambda_A B^B D_B = \begin{cases} 
\Lambda^{a \beta} D_b \\
\Lambda^{\alpha \beta} D_\beta \\
-\Lambda^{\alpha \beta} \bar{D}^\beta 
\end{cases},$$

where we have defined

$$\Lambda_{ab} := \varepsilon_{abc}(\gamma^c)_{\gamma\delta} \Lambda^{\gamma\delta}.$$

The $\text{U}(1)_R$ generator is defined to act on the covariant derivatives by the rule:

$$[J, D_A] = q_A D_A \quad \leftrightarrow \quad [J, D_a] = 0, \quad [J, D_\alpha] = D_\alpha, \quad [J, \bar{D}^\alpha] = -\bar{D}^\alpha.$$

The supergravity gauge group is described by local transformations of the form

$$D_A \to e^K D_A e^{-\bar{K}}, \quad \bar{K} = K^C(z) D_C + l^{\gamma\delta}(z) M_{\gamma\delta} + i \tau(z) J,$$

with $D_M = (\partial_m, D_\mu, \bar{D}^\mu)$ the flat superspace covariant derivatives. The gauge parameters in (2.6) obey natural reality conditions, but otherwise are arbitrary. Given a tensor superfield $U(z)$, with its indices suppressed, its transformation law is

$$U \to e^K U.$$

The torsion and curvature tensor are defined by

$$[D_A, D_B] = T_{AB}^C D_C + R_{AB}^{\gamma\delta} M_{\gamma\delta} + i F_{AB} J,$$

Our normalization of the Lorentz connection and the Lorentz curvature differs by a factor of 1/2 from that adopted in [2, 4].
where $T_{AB}^C$ is the torsion, and $R_{AB}^{\gamma\delta}$ and $R_{AB}$ are the Lorentz and $U(1)_R$ curvature tensors respectively. Using the anholonomy coefficients
\[ [E_A, E_B] = C_{AB}^C E_C , \]
one may obtain the following explicit expressions:
\[
T_{AB}^C = C_{AB}^C + \Omega_{AB}^C - (-1)^{\alpha\beta\gamma\delta} \Omega_{BA}^C + i q_B \Phi_A \delta_B^C - (-1)^{\alpha\beta} i q_A \Phi_B \delta_A^C , \tag{2.10a}
\]
\[
R_{AB}^{\gamma\delta} = E_A \Omega_B^{\gamma\delta} - (-1)^{\alpha\beta\gamma\delta} E_B \Omega_A^{\gamma\delta} - C_{AB}^C \Omega_C^{\gamma\delta} + \Omega_A^{\gamma\lambda} \Omega_B^{aux} + \Omega_A^{\delta\lambda} \Omega_B^{\gamma\sigma} , \tag{2.10b}
\]
\[
F_{AB} = E_A \Phi_B - (-1)^{\alpha\beta} E_B \Phi_A - C_{AB}^C \Phi_C . \tag{2.10c}
\]
Making use of (2.10a), the expression for $F_{AB}$ can be rewritten as
\[
F_{AB} = D_A \Phi_B - (-1)^{\alpha\beta} D_B \Phi_A - T_{AB}^C \Phi_C . \tag{2.11}
\]
Unlike the four-dimensional case, the spinor derivatives $D_\alpha$ and $\bar{D}_\alpha$ transform in the same representation of SL(2, $\mathbb{R}$). For practical manipulations with objects like $T_{AB}^C$, it is useful to introduce a formal difference between the spinor indices carried by $D_\alpha$ and $\bar{D}_\alpha$. Specifically, if a confusion is possible, we will use the following notation
\[
D_\alpha = (D_\alpha, D_\alpha, D^\alpha) \equiv (D_\alpha, D_\alpha, \bar{D}^{\alpha}) .
\]
In order to describe conformal supergravity, the torsion tensor should obey certain algebraic constraints [3]. Imposing these constraints and then solving the Bianchi identities [2] leads to the algebra of covariant derivatives
\[
\{D_\alpha, D_\beta\} = -4 \mathcal{R} M_{\alpha\beta} , \quad \{\bar{D}_\alpha, \bar{D}_\beta\} = 4 \mathcal{R} M_{\alpha\beta} , \tag{2.12a}
\]
\[
\{D_\alpha, \bar{D}_\beta\} = -2 i D_\alpha - 2 C_{\alpha\beta}^\gamma S - 4 i e_{\alpha\beta} S J + 4 i S M_{\alpha\beta} - 2 e_{\alpha\beta} C_{\gamma}^{aux} M_{\gamma\delta} , \tag{2.12b}
\]
\[
[D_\alpha, D_\gamma] = - i e_{\gamma(\alpha} C_{\beta)\delta} D^\delta + i C_{\gamma(\alpha} D_\beta) - 2 e_{\gamma(\alpha} S D_\beta) - 2 i e_{\gamma(\alpha} R D_\beta)
+ 2 e_{\gamma(\alpha} C_{\beta)\delta\rho} M^{\delta\rho} - \frac{4}{3} \left( 2 D_\alpha S + i \bar{D}_\alpha \bar{R} \right) M_{\beta\gamma} + \frac{1}{3} \left( 2 D_\gamma S + i \bar{D}_\gamma \bar{R} \right) M_{\alpha\beta}
+ \left( C_{\alpha\gamma} + \frac{1}{3} e_{\gamma(\alpha} \left( S (D_\beta) S + i \bar{D}_\beta \bar{R} \right) \right) J , \tag{2.12c}
\]
where $C_{\alpha\beta\gamma} := - i D_\gamma (C_{\beta\gamma})$. All components of the torsion and curvature are determined in terms of the three dimension-1 superfields: a real scalar $S$, a complex scalar $R$ and its conjugate $\bar{R}$, and a real vector $C_\alpha$. The $U(1)_R$ charges of $R$ and $\bar{R}$ are
\[
[J, R] = -2 R , \quad [J, \bar{R}] = 2 \bar{R} . \tag{2.13}
\]
\footnote{Our 3D notation and conventions coincide with those used in [2]. In particular, given a three-vector $V_\alpha$, it can equivalently be realized as a symmetric second-rank spinor $V_{a\beta} = V_{\beta a}$. The relationship between $V_\alpha$ and $V_{a\beta}$ is as follows: $V_{a\beta} := (\gamma^a)_{a\beta} V_\alpha = V_{\beta a}$ and $V_\alpha = -\frac{1}{2} (\gamma^a)^{a\beta} V_{a\beta}$.}
The torsion superfields obey differential constraints implied by the Bianchi identities, which are

\[ \bar{\mathcal{D}}_{\alpha} \mathcal{R} = 0 \]  
\[ (\mathcal{D}^2 - 4 \bar{\mathcal{R}}) \mathcal{S} = (\bar{\mathcal{D}}^2 - 4 \mathcal{R}) \mathcal{S} = 0 \]  
\[ \mathcal{D}_{\alpha} \mathcal{C}_{\beta\gamma} = i \mathcal{C}_{\alpha\beta\gamma} - \frac{1}{3} \varepsilon_{\alpha(\beta} \left( \bar{\mathcal{D}}_{\gamma)} \mathcal{R} + 4i \mathcal{D}_{\gamma}) \mathcal{S} \right) . \]

Eq. (2.14b) means that \( \mathcal{S} \) is a covariantly linear superfield.

The contractions \( \mathcal{D}^2 \) and \( \bar{\mathcal{D}}^2 \) in (2.14b) are defined as

\[ \mathcal{D}^2 := \mathcal{D}^{\alpha} \mathcal{D}_{\alpha} \quad \text{and} \quad \bar{\mathcal{D}}^2 := \bar{\mathcal{D}}^{\alpha} \bar{\mathcal{D}}_{\alpha} . \]

The rational for these definitions is that the operation of complex conjugation gives \( \bar{\mathcal{D}}^2 \mathcal{U} = \mathcal{D}^2 \mathcal{U} \), for any tensor superfield \( \mathcal{U} \).

The reason why the curved superspace geometry (2.12) describes \( \mathcal{N} = 2 \) conformal supergravity is that this geometry is compatible with super-Weyl invariance [2, 3]. The super-Weyl transformation of the covariant derivatives [4] is

\[ \mathcal{D}_{\alpha} \rightarrow e^{\frac{i}{4} \sigma} \left( \mathcal{D}_{\alpha} + (\mathcal{D}^\gamma \sigma) \mathcal{M}_{\gamma\alpha} - (\mathcal{D}_{\alpha} \sigma) \mathcal{J} \right) , \quad (2.16a) \]

\[ \bar{\mathcal{D}}_{\alpha} \rightarrow e^{\frac{i}{4} \sigma} \left( \bar{\mathcal{D}}_{\alpha} + (\bar{\mathcal{D}}^\gamma \sigma) \mathcal{M}_{\gamma\alpha} + (\bar{\mathcal{D}}_{\alpha} \sigma) \mathcal{J} \right) , \quad (2.16b) \]

\[ \mathcal{D}_{\alpha} \rightarrow e^{\sigma} \left( \mathcal{D}_{\alpha} - \frac{i}{2} (\gamma_{\alpha})^{\gamma\delta} (\mathcal{D}(\gamma \sigma) \bar{\mathcal{D}}_{\delta}) - \frac{i}{2} (\gamma_{\alpha})^{\gamma\delta} (\bar{\mathcal{D}}(\gamma \sigma) \mathcal{D}_{\delta}) + \varepsilon_{abc}(\mathcal{D}^b \sigma) \mathcal{M}^c \right. \]
\[ \left. + \frac{i}{2} (\mathcal{D}_{\gamma} \sigma)(\bar{\mathcal{D}}^\gamma \sigma) \mathcal{M}_a - \frac{i}{8} (\gamma_{\alpha})^{\gamma\delta} ([\mathcal{D}_{\gamma}, \bar{\mathcal{D}}_{\delta}] \mathcal{J} - \frac{3i}{4} (\gamma_{\alpha})^{\gamma\delta} (\mathcal{D}_{\gamma} \sigma)(\bar{\mathcal{D}}_{\delta} \sigma) \mathcal{J} \right) . \quad (2.16c) \]

The infinitesimal form of this transformation was originally given in [2]. Making use of (2.16), we can read off the super-Weyl transformation of the dimension-one torsion superfields

\[ \mathcal{S} \rightarrow \frac{1}{4} e^{\sigma} \left( 4 \mathcal{S} + i \mathcal{D}^\gamma \bar{\mathcal{D}}_{\gamma} \mathcal{S} \right) , \quad (2.17a) \]
\[ \mathcal{C}_{\alpha\beta} \rightarrow \left[ \mathcal{C}_{\alpha\beta} - \frac{1}{4} [\mathcal{D}_{(\alpha}, \bar{\mathcal{D}}_{\beta)}] \right] e^{\sigma} , \quad (2.17b) \]
\[ \mathcal{R} \rightarrow -\frac{1}{4} e^{2\sigma} \left( \bar{\mathcal{D}}^2 - 4 \mathcal{R} \right) e^{-\sigma} , \quad \bar{\mathcal{R}} \rightarrow -\frac{1}{4} e^{2\sigma} \left( \mathcal{D}^2 - 4 \mathcal{R} \right) e^{-\sigma} . \quad (2.17c) \]

\[ ^6 \text{Had we decided to use one and only one type of index contraction, for instance } \mathcal{D}^2 := \mathcal{D}^{\alpha} \mathcal{D}_{\alpha} \text{ and } \bar{\mathcal{D}}^2 := \bar{\mathcal{D}}^{\alpha} \bar{\mathcal{D}}_{\alpha}, \text{ there would have appeared numerous sign factors.} \]
Using the above super-Weyl transformation laws, it is an instructive exercise to demonstrate that the real vector superfield
\[ W_{\alpha\beta} := i \frac{1}{2} \left[ D^\gamma, \bar{D}_\gamma \right] C_{\alpha\beta} - \left[ D_{(\alpha}, D_{\beta)} \right] S - 4SC_{\alpha\beta} \] (2.18)
transforms homogeneously,
\[ W_{\alpha\beta} \rightarrow e^{2\sigma} W_{\alpha\beta}. \] (2.19)
This superfield is the $\mathcal{N} = 2$ supersymmetric generalization of the Cotton tensor. The condition $W_{\alpha\beta} = 0$, which is the equation of motion for $\mathcal{N} = 2$ conformal supergravity, is necessary and sufficient for a curved superspace to be conformally flat. The super-Weyl transformation (2.19) follows, in particular, from the identity
\[ i \frac{1}{2} \left\{ \left[ D^\gamma, \bar{D}_\gamma \right], \left[ D_{(\alpha}, D_{\beta)} \right] \right\} \sigma = -8C^\gamma_{(\alpha} D^\rho_{\beta)} \sigma - 4i(D_{(\alpha} R)D_{\beta)} \sigma - 4i(\bar{D}_{(\alpha} \bar{R})\bar{D}_{\beta)} \sigma . \] (2.20)

The super-Weyl and local $U(1)_R$ symmetries can be used to impose the following gauge conditions [2, 4]: $\Phi_\alpha = 0$, $S = 0$ and $\Phi_{\alpha\beta} = C_{\alpha\beta}$. In this gauge, which is most suitable to describe the type I supergravity [4], the expression (2.18) takes the form $W_{\alpha\beta} = iD^\gamma \bar{D}_\gamma C_{\alpha\beta}$, which first appeared in [31].

We conclude the review of conformal supergravity by recalling the two locally supersymmetric and super-Weyl invariant action principles described in [2]. Given a real scalar Lagrangian $L = \bar{L}$ with the super-Weyl transformation law
\[ \mathcal{L} \rightarrow e^\sigma \mathcal{L} , \] (2.21)
the functional
\[ S = \int d^3x d^2\theta d^2\bar{\theta} E \mathcal{L} , \quad E^{-1} := \text{Ber}(E_A^M) \] (2.22)
is invariant under the supergravity gauge group. Its super-Weyl invariance follows from the transformation law of $E$, which is
\[ E \rightarrow e^{-\sigma} E . \] (2.23)
Given a chiral scalar Lagrangian $\mathcal{L}_c$ of $U(1)_R$ charge $-2$ and super-Weyl weight two,\footnote{As a consequence of (2.12b), the first term on the right of (2.18) can be written in several equivalent forms: $i \frac{1}{2} \left[ D^\gamma, \bar{D}_\gamma \right] C_{\alpha\beta} = iD^\gamma \bar{D}_\gamma C_{\alpha\beta} = i\bar{D}^\gamma D_\gamma C_{\alpha\beta}$.}
\[ \bar{D}_\alpha \mathcal{L}_c = 0 , \quad [\mathcal{J}, \mathcal{L}_c] = -2\mathcal{L}_c , \quad \mathcal{L}_c \rightarrow e^{2\sigma} \mathcal{L}_c , \] (2.24)
the following chiral action

\[ S_c = \int d^3x d^2\theta d^2\bar{\theta} \frac{E}{R} \mathcal{L}_c = \int d^3x d^2\theta \mathcal{E} \mathcal{L}_c \]  

(2.25)

is locally supersymmetric and super-Weyl invariant. Here \( \mathcal{E} \) denotes the chiral density, \( \mathcal{D}_a \mathcal{E} = 0 \); its explicit form is given by eq. (5.4). The two actions are related to each other by the chiral reduction rule

\[ \int d^3x d^2\theta d^2\bar{\theta} E \mathcal{L} = \int d^3x d^2\theta d^2\bar{\theta} \frac{E}{R} \bar{\Delta} \mathcal{L} = \int d^3x d^2\theta \mathcal{E} \bar{\Delta} \mathcal{L}, \]  

(2.26)

where \( \bar{\Delta} \) denotes the chiral projection operator

\[ \bar{\Delta} := -\frac{1}{4}(\mathcal{D}^2 - 4\mathcal{R}). \]  

(2.27)

For any scalar \( V \) of U(1)\(_R\) weight \( q \), \( \bar{\Delta}V \) is a chiral scalar of U(1)\(_R\) weight \( (q - 2) \).

For practical calculations, of special importance is the following rule for integration by parts in superspace: given a vector superfield \( V = V^A E_A \), it holds that

\[ \int d^3x d^2\theta d^2\bar{\theta} E (-1)^{\epsilon_A} \mathcal{D}_A V^A = 0. \]  

(2.28)

### 3 The spinor vielbein

The supergeometry (2.12) corresponds to the conformal supergravity constraints [3]. We now turn to solving these constraints, and expressing all the geometric objects in (2.1), in terms of unconstrained prepotentials.

It follows from (2.12a) that \( T_{\alpha\beta}^C = 0 \). Then eq. (2.10a) tells us that

\[ \{E_\alpha, E_\beta\} = C_{\alpha\beta}^\gamma E_\gamma. \]  

(3.1)

In accordance with the Frobenius theorem (see, e.g. [32]), this is solved by

\[ E_\alpha = FN_\alpha^\mu E_{\mu}, \quad N = (N_\alpha^\mu) \in SL(2, \mathbb{C}) , \]  

(3.2)

where \( FN_\alpha^\mu \) is a nonsingular \( 2 \times 2 \) matrix, and the first-order operators \( E_{\mu} \) are such that

\[ \{E_{\mu}, E_{\nu}\} = 0. \]  

(3.3)

\(^9\)Covariantly chiral superfields in three-dimensional \( \mathcal{N} = 2 \) supergravity are necessarily scalar under the Lorentz group. This is in contrast with 4D \( \mathcal{N} = 1 \) supergravity which allows the existence of covariantly chiral superfields with undotted indices, see [7, 14] for reviews.
A general solution of (3.3) is

$$E_\mu = e^W D_\mu e^{-W} , \quad W = W^M D_M ,$$

(3.4)

with $D_M = (\partial_m, D_\mu, \bar{D}^\mu)$ the flat superspace covariant derivatives. The prepotential $W^M$ is complex, $\bar{W} \neq W$. It is defined modulo arbitrary gauge transformations

$$e^W \rightarrow e^{\bar{\Lambda}} e^W ,$$

(3.5)

where $\bar{\Lambda}$ denotes the complex conjugate of a constrained vector field

$$\Lambda = \Lambda^m \partial_m + \Lambda^\mu D_\mu + \bar{\rho}_\mu \bar{D}^\mu , \quad [\bar{D}_\mu, \Lambda] \propto \bar{D}_\nu .$$

(3.6a)

The constraints on $\Lambda$ are solved by

$$\Lambda^{\mu \nu} := \Lambda^m (\gamma_m)^{\mu \nu} = -i(\bar{D}^\mu L^\nu + \bar{D}^\nu L^\mu) , \quad \Lambda^\mu = -\frac{1}{4} \bar{D}^2 L^\mu ,$$

(3.6b)

with $\bar{\rho}_\mu$ and $L^\mu$ unconstrained spinor superfields. The transformation (3.5) should leave the spinor vielbein, $E_\alpha$, unchanged, and therefore $F$ and $N_\alpha^\mu$ should transform in a certain way. In the infinitesimal case, their transformation laws are as follows:

$$\delta F = \frac{1}{2} F e^W D_\mu \rho_\mu ,$$

(3.7a)

$$\delta N_\alpha^\mu = -N_{\alpha \nu} e^W D_{(\nu} \rho_{\mu)} .$$

(3.7b)

Under a general coordinate transformation generated by $K^C$, eq. (2.6), the prepotential $W$ changes by a left shift

$$e^W \rightarrow e^K e^W , \quad K = K^C D_C = \bar{K} .$$

(3.8)

The above discussion is almost identical to Siegel’s analysis in four dimensions [8]. Now comes a new feature of the 3D case. Consider the complex unimodular matrix $N = (N_\alpha^\mu)$ in (3.2). Under a local Lorentz transformation described by the parameter $l^\gamma$ in (2.6), the matrix $N$ transforms as

$$N_\alpha^\mu \rightarrow (e^l)_\alpha^\beta N_\beta^\mu , \quad e^l \in SL(2, \mathbb{R}) .$$

(3.9)

In the four-dimensional case, the Lorentz group was $SL(2, \mathbb{C})$, and the local Lorentz freedom was sufficient to gauge away $N$, for instance to choose a gauge $N = 1$. This is no longer the case in three dimensions, for we can not gauge away a part of $N$ parametrizing the right coset space $SL(2, \mathbb{R}) \setminus SL(2, \mathbb{C})$. 8
The following unimodular matrix
\[ J = (J_\mu^\nu) := N^{-1} \bar{N} \in \text{SL}(2, \mathbb{C}) \]  
(3.10)
is Lorentz invariant. Its main property is
\[ J \bar{J} = 1 \, , \]  
(3.11)
which is solved by
\[ J = \begin{pmatrix} a & i\beta \\ i\gamma & \bar{a} \end{pmatrix} \, , \quad |a|^2 + \beta \gamma = 1 \, , \quad \alpha \in \mathbb{C} \, , \quad \beta, \gamma \in \mathbb{R} \, . \]  
(3.12)

We are going to show that \( J \) is uniquely expressed in terms of the prepotential \( W^M \).

Let us introduce a semi-covariant vielbein of the form
\[ E_M = (E_m, E_\mu, \bar{E}_\mu) = E_M^N D_N \, , \quad E^{-1} := \text{Ber}(E_M^N) \, . \]  
(3.13a)

where
\[ (E_\mu, \bar{E}_\mu) = -2iE_{\mu\nu} - 2\varepsilon_{\mu\nu} \bar{E} \, , \quad \Xi = \Xi_N E_N = \Xi^n E_n + \Xi^\nu E_\nu + \bar{\Xi}_\nu \bar{E}^\nu = \bar{\Xi} \, . \]  
(3.13b)

It follows from the relation (2.12b) that the vectorial part of the first-order operator \( N_\alpha^\mu \bar{N}_\beta^\nu \{E_\mu, \bar{E}_\nu\} \) must be symmetric in \( \alpha \) and \( \beta \),
\[ N_\alpha^\mu \bar{N}_\beta^\nu \{E_\mu, \bar{E}_\nu\} = N_\beta^\mu \bar{N}_\alpha^\nu \{E_\mu, \bar{E}_\nu\} + \cdots \, , \]  
(3.14)

where the ellipsis denotes all terms with spinor derivatives \( E_\rho \) and \( \bar{E}_\rho \). One may see that the condition (3.14) is equivalent to
\[ J^{(\mu\nu)} = \frac{i}{2} \text{tr} \bar{J} \Xi^{\mu\nu} \, . \]  
(3.15)

By construction, \( J^{\mu\nu} := \varepsilon^{\mu\lambda} J_\lambda^\nu = J^{(\mu\nu)} + \frac{i}{2} \varepsilon^{\mu\nu} \text{tr} \bar{J} \). Now, using the condition that \( J \) is unimodular,
\[ J^{\mu\lambda} J^{\nu\rho} \varepsilon_{\lambda\rho} = -\varepsilon^{\mu\nu} \, , \]  
(3.16)

we derive the relation
\[ \left( \frac{1}{2} \text{tr} \bar{J} \right)^2 \left\{ 1 + \Xi^m \Xi_m \right\} = 1 \, . \]  
(3.17)

This leads to the final expression for \( J \):
\[ J_\mu^\nu = \frac{\delta_\mu^\nu + i \Xi_\mu^\nu}{\sqrt{1 + \Xi^2}} \, , \quad \Xi^2 := \Xi^m \Xi_m \, . \]  
(3.18)

We have chosen the sign of the square root so that \( J_\mu^\nu = \delta_\mu^\nu \) when \( W^M \) and thus \( \Xi^m \) vanish. The matrix \( J \) is completely determined as a power series in \( W^M \) and its derivatives. The solution (3.18) becomes singular at \( \Xi^2 = -1 \). This is analogous to the situation in (2,2) supergravity in two dimensions [16, 17].
4 The spinor connection and the torsion superfields

In accordance with (2.12a), the $U(1)_R$ curvature $F_{\alpha\beta}$ is equal to zero, and therefore

$$\mathcal{D}_\alpha = E_\alpha + \Omega^{\gamma\delta}_\alpha \mathcal{M}_{\gamma\delta} - E_\alpha U \mathcal{J} \quad \leftrightarrow \quad \Phi_\alpha = iE_\alpha U ,$$  \hspace{1cm} (4.1a)

$$\bar{\mathcal{D}}_\alpha = \bar{E}_\alpha + \bar{\Omega}^{\gamma\delta}_\alpha \mathcal{M}_{\gamma\delta} + \bar{E}_\alpha \bar{U} \bar{\mathcal{J}} \quad \leftrightarrow \quad \bar{\Phi}_\alpha = -i\bar{E}_\alpha \bar{U} ,$$  \hspace{1cm} (4.1b)

for some complex scalar prepotential $U$ which is defined modulo gauge transformations of the form

$$U \rightarrow U + 2\bar{\omega} , \quad \mathcal{D}_\alpha \bar{\omega} = 0 , \quad [\mathcal{J}, \bar{\omega}] = 0 .$$  \hspace{1cm} (4.2)

Under the super-Weyl and local $U(1)_R$ transformations, this prepotential can be seen to change as follows:

$$U \rightarrow U + \sigma ,$$  \hspace{1cm} (4.3a)

$$U \rightarrow U + i\tau .$$  \hspace{1cm} (4.3b)

For the torsion $S$, it is a trivial exercise to deduce from (2.12b) that

$$S = \frac{i}{4} \mathcal{D}^\alpha \mathcal{D}_\alpha \mathcal{V} , \quad \mathcal{V} = \frac{1}{2}(U + \bar{U}) .$$  \hspace{1cm} (4.4)

These relations have already been given in [4]. The $\omega$-gauge transformation of $\mathcal{V}$ is

$$\mathcal{V} \rightarrow \mathcal{V} + \omega + \bar{\omega} .$$  \hspace{1cm} (4.5)

It is natural to interpret $\mathcal{V}$ as the gauge prepotential of an Abelian vector supermultiplet, and the torsion $S$ as the corresponding gauge invariant field strength.

Consider a covariantly chiral superfield $\Psi$ of $U(1)_R$ charge $q$,

$$\bar{\mathcal{D}}_\alpha \Psi = 0 , \quad [\mathcal{J}, \Psi] = q\Psi .$$  \hspace{1cm} (4.6)

It can be represented in the form

$$\Psi = e^{-qU} e^{W} \hat{\Psi} , \quad \bar{\mathcal{D}}_\alpha \hat{\Psi} = 0 .$$  \hspace{1cm} (4.7)

Here $\hat{\Psi}$ is an arbitrary flat chiral superfield. The representation in which the covariantly chiral superfield $\Psi$ is described by $\hat{\Psi}$ is called chiral.

The spinor Lorentz connection is not an independent field as a consequence of the constraint

$$0 = T_{\alpha\beta}^\gamma = C_{\alpha\beta}^\gamma + \Omega_{\alpha\beta}^\gamma + \Omega_{\beta\alpha}^\gamma + i(\Phi_\beta \delta_\alpha^\gamma + \Phi_\gamma \delta_\alpha^\beta) .$$  \hspace{1cm} (4.8)
It allows us to uniquely determine the spinor Lorentz connection
\[ \Omega_{\alpha\beta\gamma} = \frac{1}{2} \left( C_{\beta\gamma\alpha} - C_{\alpha\beta\gamma} - C_{\gamma\alpha\beta} \right) + i \left( \varepsilon_{\alpha\beta} \Phi_{\gamma} + \varepsilon_{\alpha\gamma} \Phi_{\beta} \right). \]  
(4.9)

This involves the anholonomy coefficients
\[ C_{\alpha\beta} = E_{\alpha} \ln F_{\delta\beta} + E_{\beta} \ln F_{\delta\alpha} + \left( E_{\alpha} N_{\beta} + E_{\beta} N_{\alpha} \right) (N^{-1})_{\gamma}. \]  
(4.10)

To compute the antichiral scalar \( \bar{\mathcal{R}} \), it is convenient to perform a complex Lorentz transformation which results in
\[ N = 1, \quad \bar{N} = 3 \neq 1, \]  
(4.11a)

and hence
\[ E_{\alpha} = F_{\delta\alpha} E_{\mu}. \]  
(4.11b)

In such a gauge, it is a short calculation to compute the torsion \( \bar{\mathcal{R}} \). The result is
\[ \bar{\mathcal{R}} = -\frac{1}{4} e^{2U} E^{2} \left( F^{2} e^{-2U} \right). \]  
(4.12a)

Since the right-hand side is invariant under arbitrary Lorentz transformations, the relation \( (4.12a) \) holds in general. Taking the complex conjugate of \( (4.12a) \) gives
\[ \mathcal{R} = -\frac{1}{4} e^{2\bar{U}} \bar{E}^{2} \left( \bar{F}^{2} e^{-2\bar{U}} \right). \]  
(4.12b)

These results have a simple generalization: given a scalar superfield \( \Upsilon \) of U(1)\(_{R}\) charge \( q \),
\[ [\mathcal{J}, \Upsilon] = q \Upsilon, \]  
(4.13)

one may show that
\[ (\mathcal{D}^{2} - 4\mathcal{R}) \Upsilon = e^{(2+q)U} E^{2} \left( F^{2} e^{-(2+q)U} \Upsilon \right), \]  
(4.14a)
\[ (\bar{\mathcal{D}}^{2} - 4\bar{\mathcal{R}}) \Upsilon = e^{(2-\bar{q})\bar{U}} \bar{E}^{2} \left( \bar{F}^{2} e^{-(2-\bar{q})\bar{U}} \Upsilon \right). \]  
(4.14b)

Our next task is to compute \( E^{-1} = \text{Ber}(E_{A}^{M}) \) and \( F \) in terms of prepotentials. For this, we will obtain a useful expression for the spinor superfield
\[ T_{\alpha} := (-1)^{\varepsilon_{\alpha}} T_{\alpha B}^{B}, \]  
(4.15)
which, in accordance with \((2.12)\), is equal to zero, \(T_\alpha = 0\). On the other hand, if one explicitly evaluates \(T_\alpha\), the condition \(T_\alpha = 0\) proves to contain nontrivial information. From \((2.10a)\) we deduce
\[
(-1)^{\epsilon_B} T_{AB}^B = (-1)^{\epsilon_B} C_{AB}^B - \Omega_{BA}^B - i q_A \Phi_A ,
\]
and thus
\[
T_\alpha = (-1)^{\epsilon_B} C_{\alpha B}^B - \Omega_{\beta\alpha}^\beta - i \Phi_\alpha .
\]
(4.17)

Further, it may be shown (see, e.g., [7, 14] for derivations) that
\[
(-1)^{\epsilon_B} C_{\alpha B}^B = - E_\alpha \ln E - (1 \cdot E_\alpha) .
\]
(4.18)

Since \(T_\alpha\) is a spinor superfield, we can evaluate it by choosing a useful Lorentz gauge, and then restore the general answer simply by performing the inverse Lorentz transformation. Such a Lorentz gauge condition is defined in \((4.11)\). In this gauge we obtain \(\Omega_{\beta\alpha}^\beta = -3 \delta_\alpha^\mu e U_\mu (F e^{-U})\). Putting together the building blocks obtained gives
\[
T_\alpha = E_\alpha \left[ E^{-1} F^2 e^{-2U} (1 \cdot e^W) \right] .
\]
(4.19)

Setting \(T_\alpha = 0\) gives the following fundamental result
\[
E^{-1} F^2 e^{-2U} (1 \cdot e^W) = \bar{\varphi}^{-4} , \quad E_\alpha \bar{\varphi} = 0 ,
\]
(4.20)

for some antichiral superfield \(\bar{\varphi}\). One may see that the chiral superfield \(\varphi\) is invariant under the super-Weyl and local \(U(1)_R\) transformations. It can be represented as
\[
\varphi = e^W \bar{\varphi} , \quad D_\alpha \bar{\varphi} = 0 .
\]
(4.21)

Under the gauge transformation \((4.2)\), its transformation law is
\[
\varphi \rightarrow e^{\epsilon} \varphi .
\]
(4.22)

The gauge freedom \((4.2)\) can be used to completely gauge away \(\varphi\), for instance to choose a gauge \(\varphi = 1\).

Another important result follows from the relations \((2.12b)\), \((3.2)\) and \((3.13)\)
\[
E = (F \bar{F})^{-1} E .
\]
(4.23)

\(^{10}\)The dimension of \(T_\alpha\) is one-half, but all components of the torsion have dimension greater than or equal to one.
The equations (4.20) and (4.23) lead to

\[ E = \bar{\phi} e^{-V} \varphi \left\{ E^2 (1 \cdot e^\bar{W})(1 \cdot e^W) \right\}^{1/4} \]  
(4.24)

It is seen that \( E \) is invariant under the gauge transformation (4.5) and (4.22), as it should be of course. Finally, we derive the explicit expression for \( F \):

\[ F = \left\{ E^2 \frac{\varphi^4 e^{-2\bar{V}}}{(\varphi^4 e^{-2\bar{V}})^{3/2}} \frac{(1 \cdot e^\bar{W})}{(1 \cdot e^W)^3} \right\}^{1/8} \]  
(4.25)

The building blocks constructed are sufficient to read off explicit expressions for the vector covariant derivative \( D_a \) and the torsion \( C_{\alpha\beta} \), simply by making use of (2.12b). The latter is also the Lorentz curvature

\[ C^{\gamma\delta} = \frac{1}{4} \varepsilon^{\alpha\beta} R_{\alpha\beta}^{\gamma\delta} . \]  
(4.26)

This can be evaluated using eq. (2.10b) and the anholonomy coefficients appearing in

\[ \{ E_\alpha, \bar{E}_\beta \} = C_{\alpha\beta} C_{E} = C_{\alpha\beta}^c E_c + C_{\alpha\beta}^\gamma E_\gamma + C_{\alpha\beta}^\lambda \bar{E}_\lambda . \]  
(4.27)

The anholonomy coefficients are uniquely determined in terms of the spinor connections. Indeed, it follows from (2.10a) and (2.12b) that

\[ T_{\alpha\beta}^c = C_{\alpha\beta}^c = -2i(\gamma^c)_{\alpha\beta} , \]  
(4.28a)

\[ T_{\alpha\beta}^\gamma = C_{\alpha\beta}^\gamma + \bar{\Omega}_{\beta\alpha}^\gamma + \bar{E}_\beta \delta_\alpha^\gamma = 0 \]  
(4.28b)

\[ T_{\alpha\beta}^\lambda = C_{\alpha\beta}^\lambda + \Omega_{\alpha\beta}^\lambda + E_\alpha U_\beta^\lambda = 0 \]  
(4.28c)

We thus obtain

\[ C^{\gamma\delta} = \frac{1}{4} \varepsilon^{\alpha\beta} \left\{ E_\alpha \bar{\Omega}_{\beta\gamma}^{\gamma\delta} + E_\beta \Omega_{\alpha}^{\gamma\delta} - C_{\alpha\beta}^\lambda \Omega_\lambda^{\gamma\delta} + C_{\alpha\beta}^\delta \bar{\Omega}_\delta^{\gamma\delta} \right\}. \]  
(4.29)

Using eqs. (4.27) and (4.28) we can read off the vector vielbein \( E_a \).

## 5 Chiral action

The chiral density \( \mathcal{E} \), which determines the chiral action principle (2.25), remains the only geometric quantity which we have not yet expressed in terms of prepotentials. The present section is aimed at filling this gap.
In accordance with (4.7), the chiral Lagrangian can be written as

\[ \mathcal{L}_c = e^{2U} e^W \hat{\mathcal{L}}_c, \quad \bar{D}_\alpha \hat{\mathcal{L}}_c = 0. \] (5.1)

We also recall the explicit form of \( \varphi \) given by eq. (4.21). Using the representations (4.12b) and (4.20), we then obtain

\[ \frac{E}{\mathcal{R}} \mathcal{L}_c = -4 \frac{\bar{F}^2 e^{-2U}}{E^2 (\bar{F}^2 e^{-2U})} (1 \cdot e^\bar{W}) e^W (\hat{\varphi}^4 \hat{\mathcal{L}}_c). \] (5.2)

Next, we should recall an important theorem [10] (see [7] for a detailed proof).

Theorem. Given a change of superspace variables of the form \( z^M \to \hat{z}^M = e^\bar{W} z^M \), its Jacobian is equal to \((1 \cdot e^\bar{W}), \)

\[ \hat{z}^M = e^\bar{W} z^M \implies \text{Ber}(\partial_M \hat{z}^N) = (1 \cdot e^\bar{W}). \] (5.3)

Along with the theorem stated, it only remains to note that the operator \( \bar{E}^\alpha \) appearing in (5.2) becomes a partial derivatives in the new coordinate system introduced, \( \bar{E}^\alpha = \partial/\partial \bar{\theta}_\alpha \). Now, in the chiral action (2.25) we make use of (5.2), perform the change of variable \( z^M \to \hat{z}^M = e^\bar{W} z^M \), and then carry out the integral over the Grassmann variables \( \bar{\theta}_\alpha \). This leads to

\[ \int d^3x d^2\theta d^2\bar{\theta} \frac{E}{\mathcal{R}} \mathcal{L}_c = \int d^3\hat{x} d^2\hat{\theta} \hat{\varphi}^4 \hat{\mathcal{L}}_c. \] (5.4)

We conclude that, in the chiral representation, the chiral density \( \hat{\mathcal{E}} \) coincides with \( \hat{\varphi}^4 \).

6 Discussion

We have expressed the geometric potentials \( E_A^M, \Omega_A^{\gamma\delta} \) and \( \Phi_A \) in terms of several complex prepotentials (and their complex conjugates): (i) the supervector \( W^M \); (ii) the unimodular matrix \( N = (N_A) \in \text{SL}(2, \mathbb{C}) \); (iii) the scalar \( U \); and (iv) the chiral superfield \( \varphi \). Now, we are in a position to demonstrate that the gauge freedom available in the theory allows us to algebraically gauge away all prepotentials except a real vector \( H^m = \bar{H}^m \) that may be identified with the gravitational superfield. First of all, the local Lorentz invariance associated with the parameter \( l^\gamma \delta \) in (2.6) can be fixed by gauging away three of the six real degrees of freedom encoded in \( N \), leaving only the Lorentz-invariant matrix \( \mathcal{J} \) defined by (3.10). The latter is uniquely expressed in terms of \( W^M \) and its conjugate \( \bar{W}^M \),
according to eq. (3.18). Secondly, it follows from (4.3) that the scalar superfield \( U \) can be gauged away by applying a super-Weyl and local \( U(1)_R \) transformation, which completely fixes these two gauge symmetries. Thirdly, it follows from (4.22) that the \( \omega \)-gauge freedom can be used to gauge away \( \varphi \). As a result, we stay only with the prepotential \( W^M \), with the gauge transformation law

\[
e^W \rightarrow e^K e^W e^{-\bar{\Lambda}}, \tag{6.1}\]

where the parameters \( K = K^C D_C = K \) and \( \Lambda = \Lambda^M D_M \) correspond to the general coordinate transformations (2.6) and \( \Lambda \)-transformations (3.6) respectively. The general coordinate invariance alone can be used to gauge away the real part of \( W \) by choosing

\[
W = -i H, \quad H = H^M D_M = H. \tag{6.2}\]

In conjunction with the \( \Lambda \)-gauge freedom, we can further gauge away the spinor components of \( H^M \) thus arriving at the gauge condition

\[
W = -i H, \quad H = \bar{H} = H^m \partial_m. \tag{6.3}\]

As in four dimensions \([8,9]\), the general coordinate group is automatically eliminated if one works with the operator

\[
e^{-2i H} := e^{-\bar{W}} e^W, \tag{6.4}\]

which is invariant under the \( K \)-transformations (6.1). The infinitesimal transformation law of \( H \) is

\[
\delta e^{-2i H} = \Lambda e^{-2i H} - e^{-2i H} \bar{\Lambda}. \tag{6.5}\]

To preserve the gauge condition (6.3), the spinor parameter \( \bar{\rho}_\mu \) in (3.6) has to be constrained as

\[
\bar{\rho}_\mu = e^{-2i H} \bar{\Lambda}_\mu. \tag{6.6}\]

This follows from (6.5) by requiring that all terms with spinor derivatives in the right-hand side cancel out. Making use of eqs. (3.6b) and (6.6), it may be seen that the transformation law (6.5) can indeed be cast in the form (1.1).

\(^{11}\)There exists an alternative gauge condition: \( e^{-W} x^m = x^m + i \mathcal{H}^m (x, \theta, \bar{\theta}), e^{-W} \theta^\mu = \theta^\mu \) and \( e^{-W} \bar{\theta}_\mu = \bar{\theta}_\mu \). Here the vector superfield \( \mathcal{H}^m \) is real, \( \mathcal{H}^m = \mathcal{H}^m \). Using this gauge condition, one may develop a 3D version of the geometric supergravity formalism due to Ogievetsky and Sokatchev \([3]\), see \([7]\) for a review. Such a generalization was sketched by Župnik and Pak long ago \([31]\). In spite of its nice geometric structure, the Ogievetsky-Sokatchev has not found applications for quantum calculations (unlike the approach advocated in \([14]\)) and we do not elaborate on it here; see however \([33]\).
So far we have only discussed the prepotential structure of $\mathcal{N} = 2$ conformal supergravity in three dimensions. The analysis can easily be extended to the the off-shell versions of $\mathcal{N} = 2$ Poincaré and anti-de Sitter supergravity theories constructed in [4]. This is achieved by coupling the Weyl supermultiplet to an appropriate conformal compensator in a super-Weyl invariant way. There are two minimal Poincaré supergravity theories with $8+8$ off-shell degrees of freedom, called type I and type II theories in [4]. Their extensions with a cosmological term, which were constructed in [4], are also known as (1,1) and (2,0) AdS supergravity theories, following the terminology of [29]. The type I theory is a 3D generalization of the old minimal formulation for $\mathcal{N} = 1$ supergravity in four dimensions [11, 13]. It makes use of a covariantly chiral scalar $\Phi$ of $\text{U}(1)_R$ weight $-1/2$ and super-Weyl weight $+1/2$,

$$\mathcal{D}_\alpha \Phi = 0 \ , \quad [\mathcal{J}, \Phi] = -\frac{1}{2} \Phi \ , \quad \Phi \rightarrow e^{\frac{i}{2} \sigma} \Phi \ .$$

The super-Weyl and local $\text{U}(1)_R$ gauge symmetries may be fixed by imposing the gauge condition $\Phi = 1$. The type II supergravity is a 3D generalization of the new minimal formulation for $\mathcal{N} = 1$ supergravity in four dimensions constructed at the linearized level in [34] and later at the full nonlinear level in [35]. Its conformal compensator is a vector multiplet described by a real scalar prepotential $G$ which is defined modulo arbitrary gauge transformations of the form

$$\delta G = \lambda + \bar{\lambda} \ , \quad \mathcal{D}_\alpha \lambda = 0 \ , \quad [\mathcal{J}, \lambda] = 0 \ ,$$

and is inert under the super-Weyl transformations. Associated with $G$ is the gauge-invariant field strength

$$\mathbb{G} = i \mathcal{D}^\alpha \mathcal{D}_\alpha G = \mathbb{G} \ ,$$

which is covariantly linear,

$$(\mathcal{D}^2 - 4 \bar{R}) \mathbb{G} = (\bar{\mathcal{D}}^2 - 4R) \mathbb{G} = 0 .$$

The field strength is required to be nowhere vanishing, $\mathbb{G} \neq 0$. The super-Weyl transformation law of $\mathbb{G}$ proves to be

$$\mathbb{G} \rightarrow e^\sigma \mathbb{G} \ .$$

The super-Weyl gauge freedom may be fixed by imposing the gauge condition $\mathbb{G} = 1$.  

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Similar to the four-dimensional case \cite{9,36}, there exists a non-minimal formulation for 3D $\mathcal{N} = 2$ Poincaré supergravity. It makes use of a complex linear compensator $\Sigma$ of $U(1)_R$ weight $(1 - w)$ and super-Weyl weight $w$,

\[(\bar{D}^2 - 4R)\Sigma = 0, \quad [\mathcal{J}, \Sigma] = (1 - w)\Sigma, \quad \Sigma \rightarrow e^{w\sigma}\Sigma, \quad (6.12)\]

with $w$ a real parameter which specifies the off-shell supergravity formulation under consideration.\footnote{The parameter $w$ is related to the 4D non-minimal parameter $n$ \cite{9} as follows $w = (1 - n)/(1 + 3n)$.} The compensator has to be nowhere vanishing, $\Sigma \neq \mathbf{0}$.

In four dimensions, it was long believed that $\mathcal{N} = 1$ AdS supergravity could not be described using a non-minimal set of auxiliary fields \cite{14}. Nevertheless, such a formulation has recently been constructed \cite{37} in the case $n = -1$ by using a deformed complex linear constraint obeyed by the compensator. This construction has also been extended to the 3D $\mathcal{N} = 2$ case \cite{4}. In three dimensions, non-minimal AdS supergravity can consistently be defined by choosing $w = -1$ and considering a deformed complex linear compensator $\Gamma$ constrained by

\[-\frac{1}{4}(\bar{D}^2 - 4R)\Gamma = \mu = \text{const}. \quad (6.13)\]

The complex parameter $\mu$ is related to the cosmological constant, see \cite{4} for more details. The resulting non-minimal formulation describes the (1,1) AdS supergravity.

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