REMARKS ON THE SPECTRUM AND TRUNCATED HEAT KERNEL OF THE BTZ BLACK HOLE

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Abstract

Using an orbifold description of the Euclidean BTZ black hole, we show that there is a special relation between the spectrum and the truncated heat kernel of this black hole with the Patterson-Selberg zeta function.

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1 Introduction

The discovery of black hole solutions in three-dimensional gravity is a promising new area for the analysis of problems posed in the four-dimensional case. We do not have yet a consistent and complete theory of four-dimensional quantum gravity, but nevertheless a large number of interesting issues have been investigated using the three-dimensional analogs. Some of them, related to black hole thermodynamics for instance, deal with the origin of entropy, the information loss paradox, and the validity of the area law. Recently, three-dimensional gravity has been studied in more details. Despite of the simplicity of the three-dimensional case (with no propagating gravitons, for example), there is a common belief that it deserves attention as a useful laboratory for four-dimensional problems.

In this paper we pursue this idea and we consider the Bañados, Teitelboim, Zanelli (hereafter called BTZ) black hole \([1]\) because its geometric structure allows for exact computations since

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its Euclidean form is locally isomorphic to constant curvature hyperbolic three-space [2]. We present a further discussion of the BTZ black hole in this paper, as regards to its spectrum and truncated heat kernel. We find, in fact, that the latter two can be related via a Patterson-Selberg zeta function [3]. The contents of the paper are as follows: In Section 2 we describe the spectrum of the cyclic, Kleinian group that defines the Euclidean black hole as an orbifold. In Section 3 we compute the truncated heat trace and we show how to relate it to a Patterson zeta function. Finally, in Section 4 we conclude with brief remarks.

2 BTZ spectrum

The Euclidean BTZ black hole has an orbifold description $B_{\Gamma(a, b)} = \Gamma(a, b) \backslash H^3$ for suitable parameters $a > 0, b \geq 0$ (which we will specify later), where $H^3 = \{(x, y, z) \in \mathbb{R}^3 \mid z > 0\}$ is hyperbolic 3-space and $\Gamma(a, b) \subset SL(2, \mathbb{C})$ is a cyclic group of isometries. $B_{\Gamma(a, b)}$ is a solution of the Einstein equations

$$R_{ij} - \frac{1}{2}g_{ij}R_g - \Lambda g_{ij} = 0 \quad (1)$$

with negative cosmological constant $\Lambda$, hyperbolic metric

$$ds^2 = \frac{\sigma^2}{z^2}(dx^2 + dy^2 + dz^2) \quad (2)$$

for $\sigma = (-\Lambda)^{-1/2}$, and constant scalar curvature $R_g = \frac{6}{\sigma^2} = -6\Lambda$. The original BTZ metric in coordinates $(r, \phi, \tau)$ can indeed be transformed to that in (2) by a specific change of variables $(r, \phi, \tau) \rightarrow (x, y, z)$; see for instance [4, 5, 6]. It is, in fact, the periodicity in the Schwarzschild variable $\phi$ that allows for the above orbifold description. In fact the parameters $a, b$ are given as follows. For $M > 0, J \geq 0$ the black hole mass and angular momentum, and for $r_+ > 0, r_- \in i\mathbb{R}$ ($i^2 = -1$) the outer and inner horizons given by

$$r_+^2 = \frac{M\sigma^2}{2} \left[ 1 + \left( 1 + \frac{J^2}{M^2\sigma^2} \right)^{1/2} \right], \quad (3)$$

$$r_- = -\frac{\sigma J}{2r_+}, \quad (4)$$

one obtains

$$a := \pi r_+ / \sigma, b := \pi |r_-| / \sigma. \quad (5)$$

$\Gamma(a, b)$ is defined to be the cyclic subgroup of $G = SL(2, \mathbb{C})$ with generator

$$\gamma(a, b) := \begin{bmatrix} e^{a+ib} & 0 \\ 0 & e^{-(a+ib)} \end{bmatrix}, \quad (6)$$

$$\Gamma(a, b) := \{ \gamma_n \mid n \in \mathbb{Z} \}. \quad (7)$$
The Riemannian volume element $dv$ corresponding to (2) is given by

$$dv = \frac{\sigma^3}{z^3}dxdydz.$$  

(8)

One knows that a fundamental domain $F_{(a,b)}$ for the action of $\Gamma_{(a,b)}$ on $H^3$ is given by

$$F_{(a,b)} = \{(x, y, z) \in H^3 \mid 1 < x^2 + y^2 + z^2 < e^{2a}\}.$$  

(9)

It follows that $\Gamma_{(a,b)}$ is a Kleinian subgroup of $G$ - i.e.

$$\text{vol}(F_{(a,b)}) = \int_{F_{(a,b)}} dv = \infty.$$  

(10)

Since $F_{(a,b)}$ has an infinite hyperbolic volume, the usual spectral theory for the Laplacian $\Delta_{\Gamma_{(a,b)}}$ of $B_{\Gamma_{(a,b)}}$ does not apply - as it does for finite volume orbifolds. We outline, briefly, a suitable spectral analysis of $-\Delta_{\Gamma_{(a,b)}}$ where a key notion is that of scattering resonances. These replace the role of the eigenvalues of the Laplacian in the infinite volume case, and are the $s_{mnj}^\pm$ given in definition (20) below, which we therefore refer to as the BTZ spectrum.

Henceforth we shall write $\Gamma$ for $\Gamma_{(a,b)}$. Using (2), one notes that $\Delta_{\Gamma}$ is given by

$$\Delta_{\Gamma} = \frac{1}{\sigma^2} \left[ z^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) - z \frac{\partial}{\partial z} \right].$$  

(11)

The space of square-integrable functions on the black hole $B_\Gamma = \Gamma \setminus H^3$, with respect to the measure $dv$ in (8), has a nice orthogonal decomposition

$$L^2(B_\Gamma, dv) = \bigoplus_{m,n \in \mathbb{Z}} H_{mn}$$  

(12)

with Hilbert space isomorphisms $H_{mn} \simeq L^2(\mathbb{R}^+, dt)$ (for $\mathbb{R}^+$ the space of positive real numbers), and a spectral decomposition

$$-\sigma^2 \Delta_{\Gamma} \simeq \bigoplus_{m,n \in \mathbb{Z}} \oplus L_{mn}$$  

(13)

where the

$$L_{mn} = -\frac{d^2}{dt^2} + 1 + V_{mn}(t)$$  

(14)

are the Schrödinger operators with Pöschel-Teller potentials

$$V_{mn}(t) = \left( k_{mn}^2 + \frac{1}{2} \right) \text{sech}^2 t + \left( m^2 - \frac{1}{4} \right) \cosh^2 t$$  

(15)

for

$$k_{mn} := -\frac{mb}{a} + \frac{\pi n}{a}.$$  

(16)
For details of this and the following remarks the reader can consult [6, 7, 8], for example. In particular, the proof of equation (14) is given in Section 3 of [6] - in equations (3.10)-(3.21) there. The Schrödinger equation

\[ \Psi''(x) + [E - V_{mn}(x)]\Psi(x) = 0, \]

which is the same as the eigenvalue problem \( L_{mn}\Psi = k^2\Psi \) for \( E = k^2 - 1 \), has a known solution \( \Psi^+(x) \) (in terms of the hypergeometric function) with asymptotics

\[ \Psi^+(x) \sim e^{ikx}T_{mn}(k) + R_{mn}(k)e^{-ikx}, \]

for reflection and transmission coefficients \( T_{mn}(k), R_{mn}(k) \). For \( s \) defined by \( k = i(1 - s) \) one can form the scattering matrix

\[ [\mathcal{R}_{mn}(s)] := [R_{mn}(k)] \]

of \( -\Delta_\Gamma \), whose entries are quotients of gamma functions with “trivial poles” \( s = 1 + j, j = 0,1,2,3,\ldots \), and non-trivial poles

\[ s_{mnj}^\pm := -2j - |m| \pm i |k_{mn}| \]

for \( k_{mn} \) in (16), \( j = 0,1,2,3,\ldots \); also see (5). The \( s_{mnj}^\pm \) are the scattering resonances that we referred to earlier.

For later purposes it is convenient to set

\[ l := 2a = 2\pi r_+/\sigma, \theta := 2b = 2\pi |r_-|/\sigma; \]

see (5). For \( n,k_1,k_2 \in \mathbb{Z}, k_1,k_2 \geq 0 \) define a corresponding complex number \( \zeta_{n,k_1,k_2} \) by

\[ \zeta_{n,k_1,k_2} := -(k_1 + k_2) + i (k_1 - k_2) \frac{\theta}{l} + \frac{2\pi in}{l}. \]

The \( \zeta_{n,k_1,k_2} \) turns out to be the zeros of a zeta function \( Z_\Gamma(s) \) that we will introduce in the next section, where we also relate \( Z_\Gamma(s) \) to the heat kernel of \( B_\Gamma \). It is a simple, but remarkable, fact that the set of scattering poles in (20) coincides with the zeta zeros in (22), as is easily verified. Thus encoded in \( Z_\Gamma(s) \) is the spectrum of a BTZ black hole.

### 3 BTZ heat kernel and zeta function

In [4], the heat kernel trace (integration over the fundamental domain \( F = F_{(a,b)} \) in (9) along the diagonal) was calculated for the non-spinning black hole - the case \( b = 0 \) in (5). In this section we indicate how to perform that calculation for the general a spinning black hole for arbitrary values of \( b \). An alternate computation appears in [9]. We, moreover, relate the result (which is not done in [5], nor in [9]) to a zeta function \( Z_\Gamma(s) \) whose zeros comprise the BTZ spectrum, as mentioned in Section 1.
The calculation is carried out conveniently with spherical coordinates: for \( \rho \geq 0, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \frac{\pi}{2}, x = \rho \sin \phi \cos \theta, y = \rho \sin \phi \sin \theta, z = \rho \cos \phi \). We have also used \( \theta \) to denote \( 2b \) in definition (21). This dual use of notation should be no cause for confusion. For \( p = (x, y, z) \in \mathbb{H}^3 \), its image (its \( \Gamma \)-orbit) under the quotient map \( \mathbb{H}^3 \to \Gamma \backslash \mathbb{H}^3 = B_\Gamma \) will be denoted by \( \tilde{p} \).

\[ d(p_1, p_2) \] will denote the hyperbolic distance between two points \( p_1, p_2 = (x_1, y_1, z_1), (x_2, y_2, z_2) \) in \( \mathbb{H}^3 \):

\[
\cosh d(p_1, p_2) := 1 + \frac{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}{2z_1 z_2}.
\]

The heat kernel \( K^\Gamma_t (t > 0) \) of \( B_\Gamma \) is obtained by averaging over \( \Gamma \) the heat kernel \( K_t \) of \( \mathbb{H}^3 \):

\[
K_t^\Gamma (\tilde{p}_1, \tilde{p}_2) = \sum_{n \in \mathbb{Z}} K_t (p_1, \gamma^n p_2)
\]

\[
= \sum_{n \in \mathbb{Z}} \frac{e^{-t \cdot d(\tilde{p}_1, \gamma^n p_2)^2/4t} d(\tilde{p}_1, \gamma^n p_2)}{(4\pi t)^{3/2} \sinh d(\tilde{p}_1, \gamma^n p_2)},
\]

given definition (6), where we write \( \gamma \) for \( \gamma_{(a,b)} \). As above we will write \( F \) for \( F_{(a,b)} \) in (9). Finally, let \( K^\Gamma_{\text{tr}} (\tilde{p}_1, \tilde{p}_2) \) denote the truncated heat kernel of \( B_\Gamma \), defined by restricting the sum over \( \mathbb{Z} \) in (24) to the non-zero integers \( n \). We can now prove the following theorem for the trace of \( K^\Gamma_{\text{tr}} (\tilde{p}_1, \tilde{p}_2) \).

**Theorem 1** For the volume element \( dv \) in (8), and the theta-function

\[
\Theta_t (t) := \frac{l}{8\sqrt{4\pi t}} \sum_{n \in \mathbb{Z} - \{0\}} e^{-\frac{(t+n^2/4t)}{4}} = \frac{l}{4\sqrt{4\pi t}} \sum_{n=1}^{\infty} \frac{e^{-\frac{(t+n^2/4t)}{4}}}{\sinh^2 \left( \frac{ln}{2} \right) + \sin^2 \left( \frac{\theta n}{2} \right)},
\]

for \( t > 0 \), see (21), one has that

\[
\int \int \int_{F} K^\Gamma_{\text{tr}} (\tilde{p}, \tilde{p}) dv = 2\sigma^3 \Theta_t (t).
\]

**Proof.** For \( n \in \mathbb{Z} - \{0\} \), let \( r_n := 2\pi n |r_+|/\sigma \). In terms of the above spherical coordinates, the action of \( \Gamma \) on \( \mathbb{H}^3 \) (which appears in particular in definition (24)) is given by \( \gamma^n (x, y, z) = (x', y', z') \) for \( x' = e^{n l} (\rho \sin \phi \cos (\theta + r_n)), y' = e^{n l} (\rho \sin \phi \sin (\theta + r_n)), z' = e^{n l} \rho \cos \phi \), with \( l = 2\pi r_+/\sigma \) in (21). Then one can compute that \( (x-x')^2 + (y-y')^2 + (z-z')^2 = \rho^2 (\sin^2 \phi) [1 - 2e^{nl} \cos r_n + (e^{nl})^2] + \rho^2 (\cos^2 \phi) [1 - e^{nl}]^2 - 2\rho^2 e^{nl} (\sin^2 \phi) (\cos r_n - 1) \). For \( N := e^l, b_n := [1 - N^n]^2/2N^n, d_n := d(p, \gamma^n p), \) and \( \tilde{b}_n := (1 + b_n - \cos r_n) \), one obtains from (23) that

\[
\cosh d_n = 1 + \frac{[1 - N^n]^2 - 2N^n (\sin^2 \phi) (\cos r_n - 1)}{2(\cos^2 \phi) N^n}
\]

\[
= 1 + b_n \sec^2 \phi - (\tan^2 \phi) (\cos r_n - 1) = \cos r_n + (b_n + 1 - \cos r_n) \sec^2 \phi
\]

\[
= \cos r_n + \tilde{b}_n \sec^2 \phi,
\]
which is independent of the other spherical coordinates $\rho$ and $\theta$. Note that, by definition

$$1 + b_n = \cosh nl, \quad \tilde{b}_n = \cosh nl - \cos r_n. \quad (30)$$

As $dv = \sigma^3 (\sin \phi) / \rho \cos^3 \phi d\rho d\theta d\phi$, commutation of integration and summation (where again the summation in (24) is restricted to $Z = \{0\}$ for $K_{t}^{\Gamma} (\tilde{p}, \tilde{p})$) gives

$$\int \int \int_{F} K_{t}^{\Gamma} (\tilde{p}, \tilde{p}) \, dv = \frac{e^{-lt^3}}{(4\pi)^{3/2}} \sum_{n \neq 0} I_n \quad (31)$$

for

$$I_n = \int_{0}^{\pi/2} \int_{0}^{2\pi} \int_{1}^{N} \frac{e^{-d_n^3(\phi)/4t} d_n (\sin \phi)}{(\sinh d_n) \rho \cos^3 \phi} d\rho d\theta d\phi, \quad (32)$$

where by (29), $d_n = d_n (\phi)$ depends only on $\phi$, and not on $\theta, \rho$. Therefore

$$I_n = 2\pi (\log N) \int_{0}^{\pi/2} e^{-d_n^3(\phi)/4t} d_n (\phi) (\sin \phi) (\sinh d_n) \, d\phi \cos^3 \phi d\phi. \quad (33)$$

Differentiating equation (29) with respect to $\phi$ and using the change of variables $u = d_n (\phi)$ we get that $(\sinh d_n) d_n^\prime (\phi) = b_n 2 \sec^2 \phi \tan \phi = 2b_n \sin \phi / \cos^3 \phi \Rightarrow du = d_n^\prime (\phi) d\phi = 2b_n (\sin \phi) / (\sinh d_n (\phi)) \cos^3 \phi d\phi \Rightarrow$

$$I_n = 2\pi (\log N) \int_{d_n(0)}^{d_n(\pi/2)} e^{-u^2/4t} \frac{u}{2b_n} du = -2\pi (\log N) \frac{t}{b_n} \int_{d_n(0)}^{d_n(\pi/2)} e^{-u^2/4t} du. \quad (34)$$

By (29) and (30), $d_n (\phi) = \cosh^{-1} \left( \cos r_n + \tilde{b}_n \sec^2 \phi \right) = \log \left[ \cos r_n + \tilde{b}_n \sec^2 \phi + \sqrt{\left( \cos r_n + \tilde{b}_n \sec^2 \phi \right)^2 - 1} \right] \Rightarrow d_n (0) = \log [\cosh nl + |\sinh nl|] = |n| l$. Also $d_n (\pi/2) = \infty$, and we see that $I_n = (2\pi lt/\tilde{b}_n) e^{-nt^2/4t}$ (as $\log N := l \Rightarrow$ (by (30), (31))

$$\int \int \int_{F} K_{t}^{\Gamma} (\tilde{p}, \tilde{p}) \, dv = \frac{\sigma^3 2\pi l t}{\sqrt{4\pi l (4\pi t)}} \sum_{n \neq 0} \frac{e^{-t-n^2l^2/4t}}{\tilde{b}_n} \quad (36)$$

$$= \frac{\sigma^3 l}{\sqrt{4\pi l}} \sum_{n=1}^{\infty} \frac{e^{-t-n^2l^2/4t}}{\cosh (nl) + \cos (r_n)} \quad (37)$$

$$= \frac{\sigma^3 l}{2\sqrt{4\pi l}} \sum_{n=1}^{\infty} \frac{e^{-t-n^2l^2/4t}}{\sinh^2 \left( \frac{ln}{2} \right) + \sin^2 \left( \frac{\pi}{2} \right)} \quad (38),$$

which by the definition $r_n = 2\pi n |r_-| / \sigma = n\theta$ (see (21)) concludes the proof. □
The following zeta function has been attached to the BTZ black hole $B_T$:

\[
Z_{\Gamma}(s) := \prod_{k_1, k_2 \geq 0 \atop k_1, k_2 \in \mathbb{Z}} \left[ 1 - (e^{i\theta})^{k_1} (e^{-i\theta})^{k_2} e^{-(k_1 + k_2 + s)t} \right],
\]

(39)

again for $l, \theta$ in (21); see [6], [10]. $Z_{\Gamma}(s)$ is an entire function of $s$, whose zeros are precisely the complex numbers $\zeta_{n, k_1, k_2}$ given in (22), and whose logarithmic derivative is given by

\[
\frac{Z_{\Gamma}'(s)}{Z_{\Gamma}(s)} = \frac{l}{4} \sum_{n=1}^{\infty} \frac{e^{-nl(s-1)}}{[\sinh^2 \left( \frac{ln}{2} \right) + \sin^2 \left( \frac{\theta n}{2} \right)]}
\]

(40)

for $Re s > 0$. In Section 1, we connected $Z_{\Gamma}(s)$ with the BTZ spectrum. $Z_{\Gamma}(s)$ is also connected with the theta function $\Theta_{\Gamma}(t)$ in (26), and hence with the heat kernel $K_{t^*\Gamma}$ (by Theorem 1) via the following theorem, which follows easily from (40) by commuting integration and summation in (26):

**Theorem 2** For $Re s > 1$

\[
\int_{0}^{\infty} e^{-s(s-2)t} \Theta_{\Gamma}(t) dt = \frac{1}{2(s-1)} \frac{Z_{\Gamma}'(s)}{Z_{\Gamma}(s)}.
\]

(41)

**Proof.** The proof of Theorem 2 relies on the Laplace transform formula

\[
\int_{0}^{\infty} t^{-\frac{1}{2}} e^{-\frac{\pi}{4} t} e^{-pt} dt = \pi^{-\frac{1}{2}} p^{-\frac{1}{2}} e^{-(ap)^{\frac{1}{2}}}
\]

(42)

for $Re a \geq 0$, $Re p > 0$. Using definition (27), one gets

\[
\int_{0}^{\infty} e^{-s(s-2)t} \Theta_{\Gamma}(t) dt = \frac{l}{4\sqrt{4\pi}} \sum_{n=1}^{\infty} \int_{0}^{\infty} e^{-s(s-2)t} e^{-\left( t + \frac{a^2}{4t} \right)} dt \frac{dt}{\sqrt{t}}
\]

(43)

where $e^{-s(s-2)t} e^{-\left( t + \frac{a^2}{4t} \right)} \equiv e^{-t(s-1)^2 - \frac{a^2}{4t}}$. For the choices $a = n^2 l^2$, $p = \left( s - 1 \right)^2$ in (42), the second integral in (43) therefore assumes the value $(\sqrt{\pi}/(s-1)) e^{-nl(s-1)}$, first for $s > 1$ and then for $Re s > 1$ by analytic continuation. The first integral in (42) then is the sum

\[
\frac{l}{4 \cdot 2(s-1)} \sum_{n=1}^{\infty} \frac{e^{-nl(s-1)}}{\sinh^2 \left( \frac{ln}{2} \right) + \sin^2 \left( \frac{\theta n}{2} \right)} = \frac{1}{2(s-1)} \frac{Z_{\Gamma}'(s)}{Z_{\Gamma}(s)}
\]

(44)

by equation (40), for $Re s > 1$, which establishes Theorem 2. □

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4Strictly speaking the zeta function in definition (39) does have a connection to BTZ black hole entropy. Namely, as discussed in [6], [10], quantum corrections to the classical Beckenstein-Hawking entropy can be expressed as a special value of the logarithm of this function.
4 Concluding remarks

The main goal of our paper was to show that, using an orbifold description of the Euclidean BTZ black hole, one can obtain an interesting relation between the BTZ spectrum and its truncated heat kernel and the Patterson-Selberg zeta function. This relation is provided by Theorem 2 in our paper. Zeta-function methods are a powerful tool to obtain the spectral information of operators. With our result, we hope to give a first step towards understanding deeply the thermodynamics and statistical properties of the BTZ black holes and other (maybe, more complex) three-dimensional systems.

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