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Many-body Quantum Chaos and Entanglement in a Quantum Ratchet

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We uncover signatures of quantum chaos in the many-body dynamics of a Bose-Einstein condensate-based quantum ratchet in a toroidal trap. We propose measures including entanglement, condensate depletion, and spreading over a fixed basis in many-body Hilbert space which quantitatively identify the region in which quantum chaotic many-body dynamics occurs, where random matrix theory is limited or inaccessible. With these tools we show that many-body quantum chaos is neither highly entangled nor delocalized in the Hilbert space, contrary to conventionally expected signatures of quantum chaos.

The ability to engineer and control quantum many-body systems has led to a surge in important advances [1] ranging from quantum computation, to the observation of emergent phenomena and far from equilibrium dynamics [2, 3]. Specifically, lattice-based Bose-Einstein Condensates (BECs) [4] and spin systems [5] allow for a deeper grasp of the role interactions and correlations [6] play in both static and dynamic phenomena. In the far-from-equilibrium case, these platforms allow exploration of regimes where correlations and fluctuations may be dominant. Moreover, they yield signatures of quantum chaos, which provides insight into the transition from integrable to non-integrable dynamics as well as the quantum-classical correspondence for such dynamics [7]. Such a transition to chaos is often associated with quantum chaotic level statistics e.g. the kicked rotor. In the many-body case, a link between quantum chaos and high entanglement has been proposed previously [8], yet the many-body perspective has not been completely investigated. In this Letter, we present dynamical quantum many-body measures including depletion, basis occupation, and entanglement that correlate with quantum chaotic level statistics and provide a general method to identify and quantify quantum many-body chaos in systems where random matrix theory (RMT) is of limited use or inaccessible. We also find that quantum many-body chaos does not require or produce high entanglement, contrary to conventional thinking.

As a case study in the properties of quantum many-body chaos, we explore the statics and dynamics of quantum ratchet. This ratchet can be realized as a BEC in a toroidal trap driven by a potential that breaks generalized parity and time-reversal symmetries (Fig. 1). The violation of these symmetries are well known to produce ratchet effects in the semi-classical limit, i.e., directional motion in the presence of zero time average force, which can be regular or chaotic [9–11]. In our system, these regimes are tuned by interaction strength [12]. We perform a comprehensive study of the quantum many-body measures of this system in three representations: position, momentum, and a truncated Floquet picture. In this analysis we use entanglement, condensate depletion, and the spreading over a fixed basis in the many-body Hilbert space to quantitatively identify the interaction regime over which quantum chaos begins, ends, and is maximal. Moreover, we show that the RMT level statistics confirm this classification of chaos, although the results rapidly become misleading as more single particle modes are included.

Systems ranging from BECs [16], to confined electrons in semiconductors [17] and driven graphene [18] can possess non-equilibrium transport properties and have shaped the understanding of the classical and quantum effects leading to such phenomena. Displaying both regular and chaotic dynamics, as well as collective properties [10, 14], many-body quantum ratchets provide insight into the role of quantum many-body effects in the transition between regular and chaotic dynamics. Probes into quantum chaos often focus on RMT [19–21], whether in few particle systems [22, 23] or the many-body case [24–26]. However, reaching the statistics necessary
to classify quantum chaos in such a way frequently becomes intractable due to large amounts of degeneracy or too many dynamically irrelevant single particle modes. Here we provide an in-depth view on the interplay between quantum chaos defined by RMT and entanglement and other measures that are more relevant to interacting many-body systems. Under time evolution these measures provide new methods for which quantum chaos may be identified when RMT analysis becomes out of reach.

We consider a BEC in an optical ring trap that is rotated off-center, closely following the experimental setups for generating toroidal BECs [27–30]. The rotation of the BEC generates, via inertial effects, a space- and time-dependent potential that breaks the generalized parity and time-reversal symmetries (Fig. 1) [12], generating a ratchet current [9, 10]. In the semi-classical limit it exhibits a continuous transition from Rabi oscillations, to chaos, into self-trapping with the increase of the particle number [31]. As a tractable quantum many-body generalization of this effectively one-dimensional system, we use a periodic Bose-Hubbard model with fixed particle number. Here the model can be regarded as a discretization of the ring geometry or a ring lattice formed by a toroidal trap in superposition with radial barriers [31].

Given in hopping units, where the time scale becomes

\[ \frac{\hbar}{J} \]

for generating toroidal BECs [27–30]. The rotation of the ring geometry or a ring lattice formed by a toroidal trap in superposition with radial barriers [31].

Here 0 \leq \pi/L \leq \pi is the angle of the ith site in an L-site discretization. We study resonant driving, \( \kappa = 1/r \), \( \omega = 2[1 - \cos(2\pi/L)] \). In terms of the mean field interaction strength the transition into self trapping is given by\((N - 1)U/L\)ST = 2Jmax\((E_+, E_-)\) [12].

Semi-classically, positive Lyapunov exponents arise in distinct regions on the interval 0.013 \leq (N - 1)U/L \leq (N - 1)U/L\)ST, similar to the logistic map having regions of chaos and stability. The time scales for system dynamics are the drive period \( T = 2\pi/\omega \), and the Rabi period with zero interactions \( t_R = 2\pi/(E_+^2 + E_-^2)^{1/2} \).

Without loss of generality we can choose \( E_+ = 9/400 \) and \( E_- = 3/400 \), and, unless otherwise specified, \( L = 10 \) with the number of particles \( N = 5 \). When considering dynamics, time evolution calculations are performed using exact diagonalization, with a time step of 0.1T, and Time Evolving Block Decimation (TEBD) [32–36] with a time step of 0.1h/J. The time steps were selected with the convergence of each respective method as the determining factor.

Since the semi-classical counterpart of our system is chaotic for particular interaction strengths and time periodic, we expect to resolve quantum chaotic level statistics of the Floquet operator \( \hat{F} = U(T + t_0)U(T + t_0 - \delta t) \cdots U(t_0 + \delta t) \), with \( U(t_0) = \exp((-i/\hbar)\hat{H}(t_0)\delta t) \) [7]. RMT predicts that the quasi-energy level spacings \( s_\nu \), normalized by the mean spacing, should exhibit level repulsion leading to a probability distribution \( P(s_\nu) \) which approach zero polynomially for \( s \to 0 \). As can be seen in Fig. 2(a), where we have set \( \delta t = 0.1T \) which well captures \( \hat{F} \), the level statistics follow a Poisson distribution that is indicative of regular dynamics. The argument that the near-degeneracies in the quasi-energies wash out the polynomial decrease to zero can be made, yet analysis accounting for near-degeneracies by truncating the lowest level spacings from the calculation revealed no level repulsion.

Instead of considering the time-dependent Bose-Hubbard model, we apply the \((t, t')\)-formalism [37] to the equation of motion for the field operator \( \psi(x, t) \) and expand it in the nonlinear Floquet states. Thus, we arrive at a time-independent description of the quantum ratchet [13]. Furthermore, previous studies show that three Floquet modes with \( k \in \{0, \pm 1\} \) capture the dynamics of our quantum ratchet in an effective three-level system [12]. This Hamiltonian is then given as

\[
\hat{H}_{3LS} = \frac{i}{2}E_+ (\hat{a}_+^\dagger \hat{a}_0 + \text{h.c.}) + \frac{i}{2}E_- (\hat{a}_-^\dagger \hat{a}_0 + \text{h.c.}) - \frac{U}{2\pi} \sum_n \hat{\nu}_n (\hat{\nu}_n - \hat{1}).
\]

Here \( \hat{a}_\nu, \hat{a}_\nu^\dagger \), and \( \hat{\nu}_n \) are bosonic annihilation, creation, and number operators, respectively, for \( \nu = \{0, \pm 1\} \), which represent the 0 and \pm 1 momentum modes. \( E_{\pm} \), from the driving potential in \( \hat{H}_{3LS}^{\text{pos}} \), couples the 0 and \pm modes. Note that repulsive interactions in the Bose-Hubbard model translate into attractive interactions in the 3LS due to the angular momentum representation[13].

We can now test the eigenvalue statistics of \( \hat{H}_{3LS} \), which are the quasi-energies of the system. In contrast to the statistics of \( \hat{H}_{3LS}^{\text{pos}} \), level repulsion becomes manifest in this truncated description with varying amounts depending on the particle interaction strength (Fig. 2).

In order to characterize the quantum chaotic nature of the level statistics we fit the eigenvalue spacings of \( \hat{H}_{3LS} \) for \( N \) from 10 to 120 with the Brody distribution, \( P(s) = b(\eta + 1)s^\eta \exp(-bs^{\eta+1}) \) where \( b = 1[(\eta + 2)/(\eta + 1)]^{\eta+1} [44] \). Here we fit the control parameter \( \eta \) which interpolates between regular and quantum chaotic level statistics, that is, \( \eta = 0 \) indicates Poisson statistics and \( \eta = 1 \) is Wigner-Dyson, or quantum chaotic statistics in RMT [38].

Figure 2(c) shows the Brody parameter \( \eta \) as a function...
of interaction for particle numbers \(N = 20, 50, \) and 100. Only one dominant trend is observed in the level statistics even though there are, in the semi-classical limit, distinct pockets of chaos on the interval of interaction strengths considered. From this trend we extract the first turning point \((\langle (N-1)U/L \rangle_2), \) maximum \((\langle (N-1)U/L \rangle_m), \) and second turning point \((\langle (N-1)U/L \rangle_c), \) in order to quantitatively characterize the chaotic regime of the system. For this analysis we use a phenomenological modified Lorentzian fit of the form \(\eta((N-1)U/L) = A \exp[-bq]/(1 + c((N-1)U/L) + d)^4 \) and ignore the first 15 \(\eta \) values since they show fluctuations while being Rabi dynamics. In analogy with the analysis of critical exponents in quantum phase transitions, we also perform a fit of the scaling of these interaction strengths as a function of \(N, \langle (N-1)U/L \rangle_{s,m,c} = AN^B + C, \) and extract their asymptotic values given by the fit parameter \(C.\) This yields the asymptotic values of \([ (N-1)U/L ]_s = 0.011 \pm 0.001, \langle (N-1)U/L \rangle_m = 0.021 \pm 0.001, \) and \([ (N-1)U/L ]_c = 0.034 \pm 0.002, \) thereby clearly delineating the chaotic regime as shown in Fig 2(d).

From the two cases presented in Fig. 2 it is clear that the application of RMT as an identifier of quantum chaos has certain limitations. This manifests in our quantum ratchet due to the inclusion of too many non-relevant single particle momentum modes in \(H_{BH}^{pos} \) Eq. (1). It is then clear that for systems in which there is no a priori method to anticipate which single particle modes can be truncated RMT is highly limited. Figure 2(a) explicitly shows failure of RMT for the un-truncated Bose-Hubbard model of our quantum ratchet, while its validity is recovered in the truncated 3LS model, shown in Fig. 2(b).

In contrast, the proposed dynamical many-body measures of entanglement, condensate depletion, and basis occupation are independent of truncation. However, since the 3LS only includes three momentum modes and has a reduced dimension \(D_{3LS} = (N+1)(N+2)/2 \) compared to the Bose-Hubbard case \(D_{BH} = (N!L!−1), \) we introduce the momentum space Bose-Hubbard model. Considering the momentum space Bose-Hubbard model acquired by the standard discrete Fourier transformation \(\hat{b}_i = \sum_{\ell}^{L} \hat{a}_{\ell} \exp[ik\theta] \) where \(\lfloor \cdot \rfloor \) is the floor function, allows us to directly compare the many-body measures regardless of truncation. The new Hamiltonian is then

\[
\begin{align*}
H_B^{\text{mom}} = & -2 \sum_{k=-[L/2]}^{[L/2]-1} \hat{a}_k^\dagger \hat{a}_k \cos(2\pi k/L) \\
+ & \frac{U}{2L} \sum_{k,j=\lfloor -[L/2] \rfloor}^{\lfloor [L/2] \rfloor} \hat{a}_k^\dagger \hat{a}_j^\dagger \hat{a}_j \hat{a}_k \delta_{k+2k_j+2k_k} \\
+ & \frac{1}{2} \sum_{k=\lfloor -[L/2] \rfloor}^{\lfloor [L/2] \rfloor} \left[ (E_+ \exp^{i\omega t} + E_- \exp^{-i\omega t}) \hat{a}_k^\dagger \hat{a}_k \right] \\
+ & \left( E_+ \exp^{-i\omega t} + E_- \exp^{i\omega t} \right) \hat{a}_{k+\pi}^\dagger \hat{a}_k, \\
\end{align*}
\]

(4)

where \(\hat{a}_k \) and \(\hat{a}_k^\dagger \) are bosonic annihilation and creation operators in the state with wave number \(2\pi k/L, \) respectively. Here we note that the \(k = 0 \) and \(\pm 1 \) states are similar to those from the 3LS model but the time dependence remains in the Hamiltonian and is not absorbed into the nonlinear Floquet states as it was in Eq. (3).

The dynamics for the remainder of this study were computed using exact diagonalization for every representation (truncation) of the quantum ratchet for a lattice length of \(L = 10 \) and number of particles \(N = 5 \) with the initial state being the ground state of the un-driven Bose-Hubbard. We also perform scaling in the position Bose-Hubbard model Eq. (1) using TEBD.

We apply well established many-body measures: the von Neumann entropy of entanglement \(S, \) the condensate depletion \(D, \) and the Inverse participation ratio \(P^{-1}. \)

\(S = -\text{Tr}(\rho_A \log \rho_A) \) measures correlations between a subsystem \(A \) and the remainder of the system, where \(\rho_A \) is the reduced density matrix of a subsystem \(A. \) For \(H_B^{\text{pos}}, \) there is no symmetry pointing to a particular cut of the system, thus we take \(A \) to be half of the ring.
Recent reversals are one of the main features of quantum ratchet in its chaotic regime, we take $A$ as the modes with $k \geq 0$ for $\hat{H}_{\text{Mom}}^0$ and $\hat{H}_{\text{3LS}}$. The depletion, $D \equiv 1 - \lambda_1/N$ with $\lambda_1$ the first eigenvalue of the single particle density matrix $\langle \hat{a}_i^\dagger \hat{a}_i \rangle$, measures the amount of the original condensate still remaining in one single particle mode. Finally, $P^{-1} \equiv \left( \sum_i p_i^2 \right)^{-1}$, where $p_i$ is the probability of being in the $i$th basis state, and measures the spreading of the state over the many-body Hilbert space.

For each of the above measures we take the time average over $10T_\text{f}$ for 15 values of the interaction strength spanning all three dynamical regimes. We then apply fitting functions to each measure and extract $[(N-1)U/L]_s$, $[(N-1)U/L]_m$, and $[(N-1)U/L]_c$. For the von Neumann entropy we use the fitting functions $\mathcal{S}([(N-1)U/L]) = [\tanh(A([N-1]U/L)] + D([N-1]U/L) + E)^{1/4} + F$ for $\hat{H}_B$ Exact and $\mathcal{S}([(N-1)U/L]) = A/(1+B([N-1]U/L) + C)^2 + D)$ for $\hat{H}_{\text{Mom}}$ and $\hat{H}_{\text{3LS}}$, see Fig. 3(a). For the depletion we use a fitting function $\mathcal{D}([(N-1)U/L]) = [\tanh(A([N-1]U/L)] + B(1+C([N-1]U/L) + D)^2 + E)$ for each model, see Fig. 3(b). For the Inverse participation ratio, only $\hat{H}_B$ Exact and $\hat{H}_{\text{3LS}}$ showed trending with the dynamical regimes, thus these were fit with the function $P^{-1}([(N-1)U/L]) = A/(1+B([N-1]U/L) + C)^2 + D)$. The values of the critical interaction strengths $[(N-1)U/L]_s$, $[(N-1)U/L]_m$, and $[(N-1)U/L]_c$ from each measure can be seen in Tab. I.

Here we perform a scaling study of the von Neumann entropy and depletion in TEBD for $L = 6$ to 14. Figure 3(d) shows density plots of the mean entropy (top) and depletion (bottom). The trends are generally the same and can also be fit for the interactions strengths $[(N-1)U/L]_s$, $[(N-1)U/L]_m$, and $[(N-1)U/L]_c$ shown in Fig. 2(d). Figure 3(d) clearly shows that the system can be taken to longer lattices without growth in entanglement. This implies that the system can be effectively simulated with matrix product state methods, though full local dimension is required for convergence, and likely to be an issue for other driven and/or highly oscillatory dynamics [39].

Table I summarizes the critical interaction strengths that can be extracted by fitting the many-body measures from exact diagonalization in Fig. 3(a)-(c). It is important to note that even though the level statistics of $\hat{H}_{\text{3LS}}$ predict quantum chaos, the case of $N = 5$ for the dynamical measures would be far too small in the 3LS to give any statistical measure of chaos.

In conclusion, we have identified measures that reliably quantify quantum many-body chaos: entanglement, condensate depletion, and inverse participation ratio. Entanglement and inverse participation ratio capture the onset, maximum, and end of chaos in both momentum and the truncated Floquet pictures, while depletion is a universal measure of quantum chaos in all three representations of the quantum ratchet. These measures are especially important for very large systems where obtaining

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
Measure & $[(N-1)U/L]_s$ & $[(N-1)U/L]_m$ & $[(N-1)U/L]_c$ \\
\hline
Level Statistics $N \to \infty$ & 0.011$^{\pm0.001}$ & 0.021$^{\pm0.001}$ & 0.034$^{\pm0.002}$ \\
Entropy Position & $-$ & 0.016$^{\pm0.004}$ & 0.29$^{\pm0.008}$ \\
Entropy Momentum & 0.011$^{\pm0.000}$ & 0.019$^{\pm0.001}$ & 0.026$^{\pm0.001}$ \\
Entropy Floquet & 0.011$^{\pm0.001}$ & 0.019$^{\pm0.001}$ & 0.028$^{\pm0.001}$ \\
Depletion Position & $-$ & 0.016$^{\pm0.003}$ & 0.027$^{\pm0.002}$ \\
Depletion Momentum & $-$ & 0.016$^{\pm0.003}$ & 0.026$^{\pm0.003}$ \\
Depletion Floquet & $-$ & 0.016$^{\pm0.003}$ & 0.027$^{\pm0.010}$ \\
IPR Position & $-$ & $-$ & $-$ \\
IPR Momentum & 0.011$^{\pm0.001}$ & 0.018$^{\pm0.001}$ & 0.026$^{\pm0.001}$ \\
IPR Floquet & 0.011$^{\pm0.001}$ & 0.019$^{\pm0.001}$ & 0.027$^{\pm0.001}$ \\
\hline
\end{tabular}
\caption{Tools for Quantum Chaos Identification. Each measure clearly identifies the maximally quantum chaotic interaction strength $[(N-1)U/L]_s$ and the end of the chaotic regime $[(N-1)U/L]_c$. Only in momentum and Floquet spaces is the onset of the chaotic regime $[(N-1)U/L]_c$ obtained via entropy and inverse participation ratio (IPR). Cells with “−” indicate a measure fails to predict its corresponding interaction parameter.}
\end{table}
ing the full spectrum required for random matrix theory (RMT) is computationally inaccessible. Thus, they show potential for identifying quantum chaotic dynamics more generally, whereas RMT is limited in small and large systems, or rapidly becomes misleading due to the inclusion of non-relevant single particle modes. Contrary to the conventional association of quantum chaos with high entanglement and lack of localization in Hilbert space, we have shown for the quantum ratchet that quantum many-body chaos is neither highly entangled, nor does it require many elements of the many-body basis.

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[1] Immanuel Bloch, Jean Dalibard, and Wilhelm Zwerger, Many-body physics with ultracold gases, Rev. Mod. Phys., 80, 885 (2008).
[2] L. E. Sadler, J. M. Highie, S. R. Leslie, M. Vengalattore, and D. M. Stamper-Kurn, Spontaneous symmetry breaking in a quenched ferromagnetic spinor Bose-Einstein condensate, Nature, 443, 312 (2006).
[3] J. Eisert, M. Friesdorf, and C. Gogolin, Quantum many-body systems out of equilibrium, Nat. Phys., 11, 124 (2015).
[4] H. Lignier, C. Sias, D. Ciampini, Y. Singh, A. Zenesini, O. Morsch, and E. Arimondo, Dynamical control of matter-wave tunneling in periodic potentials, Phys. Rev. Lett., 99, 220403 (2007).
[5] Peter Barmettler, Matthias Punk, Vladimir Gritsev, Eugene Demler, and Ehud Altman, Relaxation of antiferromagnetic order in spin-1/2 chains following a quantum quench, Phys. Rev. Lett., 102, 130603 (2009).
[6] Rajibul Islam, Ruichao Ma, Philipp M. Preiss, M. Eric Tai, Alexander Lukin, Matthew Rispoli, and Markus Greiner, Measuring entanglement entropy in a quantum many-body system, Nature, 528, 77 (2015).
[7] Fritz Haake, Quantum Signatures of Chaos, (Springer, 2010).
[8] C. Neill, P Roushan, M. Fang, Y. Chen, M Kolodrubetz, Z. Chen, A. Megrant, R. Barends, B. Campbell, B. Chiaro, A. Dunsworth, E. Jeffrey, J. Kelly, J. Mutus, P. J. J. O’Malley, C. Quintana, D. Sank, A. Vainsencher, J. Wenner, T. C. White, A. Polkovnikov, and J. M. Martinis, Ergodic dynamics and thermalization in an isolated quantum system, Nat. Phys., 12, 1037 (2016).
[9] S. Flach, O. Yevtushenko, and Y. Zolotaryuk, Directed current due to broken time-space symmetry, Phys. Rev.

Lett., 84, 2358 (2000).
[10] Peter Reimann, Brownian motors: Noisy transport far from equilibrium, Phys. Rep., 361, 57 (2002).
[11] Sergey Denisov, Sergej Flach, and Peter Hänggi, Tunnelable transport with broken space-time symmetries, Phys. Rep., 538, 77 (2014).
[12] M. Heimsoth, D. Hochstuhl, C. E. Creffield, L. D. Carr, and F. Sols, Effective Josephson dynamics in resonantly driven Bose-Einstein condensates, New J. Phys., 15 (2013).
[13] M. Heimsoth, C. E. Creffield, L. D. Carr, and F. Sols, Orbital Josephson effect and interactions in driven atom condensates on a ring, New J. Phys., 14 (2012).
[14] D. Cubero, V. Lebedev, and F. Renzoni, Current reversals in a rocking ratchet: Dynamical versus symmetry-breaking mechanisms, Phys. Rev. E, 82, 041116 (2010).
[15] M. Heimsoth, C. E. Creffield, and F. Sols, Weakly driven quantum coherent ratchets in cold-atom systems, Phys. Rev. A, 82, 023607 (2010).
[16] T. Salger, S. Kling, T. Hecking, and M. Weitz. Directed transport of ultracold atoms in a Hamiltonian quantum ratchet. CLEO/Europe and EQEC 2009 Conference Digest. (Optical Society of America, 2009).
[17] H. Linke, T. E. Humphrey, A. Löfgren, A. O. Sushkov, R. Newbury, R. P. Taylor, and P. Omling, Experimental Tunneling Ratchets, Science, 286, 2314 (1999).
[18] C. Drexler, S. A. Taraseenko, P. Olbrich, J. Karch, M. Hirmer, F. Muller, M. Gmitra, J. Fabian, R. Yakimova, S. Lara-Avila, S. Kubatkin, M. Wang, R. Vajtai, P. M. Ajayan, J. Kono, and S. D. Ganichev, Magnetic quantum ratchet effect in graphene, Nat. Nano, 8, 104 (2013).
[19] B. D. Simons, P. A. Lee, and B. L. Altshuler, Exact description of spectral correlators by a quantum one-dimensional model with inverse-square interaction, Phys. Rev. Lett., 70, 4122 (1993).
[20] L. A. Ponomarenko, F. Schedin, M. I. Katsnelson, R. Yang, E. W. Hill, K. S. Novoselov, and A. K. Geim, Chaotic Dirac billiard in graphene quantum dots, Science, 320, 356 (2008).
[21] Aviva Cubin and Lea F. Santos, Quantum chaos: An introduction via chains of interacting spins 1/2, Am. J. Phys., 80, 246 (2012).
[22] Liang Jun Zhai, Yu Jun Zheng, and Shi Liang Ding, Vibrational chaotic dynamics in triatomic molecules: A comparative study, Front. Phys., 7, 514 (2012).
[23] Vaibhav Madhok, Vibhu Gupta, Denis-Alexandre Trotter, and Shohini Ghose, Signatures of chaos in the dynamics of quantum discord, Phys. Rev. E, 91, 032906 (2015).
[24] A. R. Kolovsky and A. Buchleitner, Quantum chaos in the Bose-Hubbard model, Europhys. Lett., 68, 632 (2004).
[25] Ph. Jacquod and D. L. Shepelyansky, Emergence of Quantum Chaos in Finite Interacting Fermi Systems, Phys. Rev. Lett., 79, 1837 (1997).
[26] Mario Mulansky, Karsten Ahnert, Arkady Pikovsky, and Dima L. Shepelyansky, Dynamical thermalization of disordered nonlinear lattices, Phys. Rev. E, 80, 056212 (2009).
[27] W. H. Heathcote, E. Nugent, B. T. Sheard, and C. J. Foot, A ring trap for ultracold atoms in an RF-dressed state, New J. Phys., 10 (2008).
[28] A. Ramanathan, K. C. Wright, S. R. Muniz, M. Zelan,
W. T. Hill, C. J. Lobb, K. Helmerson, W. D. Phillips, and G. K. Campbell, Superflow in a toroidal bose-einstein condensate: An atom circuit with a tunable weak link, *Phys. Rev. Lett.*, 106, 130401 (2011).

[29] Stuart Moulder, Scott Beattie, Robert P. Smith, Naaman Tammuz, and Zoran Hadzibabic, Quantized supercurrent decay in an annular Bose-Einstein condensate, *Phys. Rev. A*, 86, 013629 (2012).

[30] G. Edward Marti, Ryan Olf, and Dan M. Stamper-Kurn, Collective excitation interferometry with a toroidal Bose-Einstein condensate, *Phys. Rev. A*, 91, 013602 (2015).

[31] H. M. Cataldo and D. M. Jezek, Bose-Hubbard model in a ring-shaped optical lattice with high filling factors, *Phys. Rev. A*, 84, 013601 (2011).

[32] Guifr´e Vidal, Efficient Classical Simulation of Slightly Entangled Quantum Computations, *Phys. Rev. Lett.*, 91, 147902 (2003).

[33] Guifr´e Vidal, Efficient simulation of one-dimensional quantum many-body systems, *Phys. Rev. Lett.*, 93, 040502 (2004).

[34] Ulrich Schollw¨ock, The density-matrix renormalization group in the age of matrix product states, *Ann. Phys.*, 326, 96 (2011).

[35] M. L. Wall and Lincoln D. Carr, Out-of-equilibrium dynamics with matrix product states, *New J. Phys.*, 14 (2012).

[36] Time-Evolving Block Decimation Open Source Code, [https://sourceforge.net/projects/opentebd/](https://sourceforge.net/projects/opentebd/).

[37] Uri Peskin and Nimrod Moiseyev, The solution of the time-dependent Schr¨odinger equation by the (t,t') method: Theory, computational algorithm and applications, *J. Chem. Phys.*, 99, 4590 (1993).

[38] T. A. Brody, A statistical measure for the repulsion of energy levels, *Lett. Al Nuovo Cim.*, 7, 482 (1973).

[39] Diego A. Alcala, Joseph A. Glick, and Lincoln D. Carr, Entangled Dynamics in Macroscopic Quantum Tunneling of Bose-Einstein Condensates, arXiv:1605.05768, (Phys. Rev. Lett., to be published).

[40] See Supplementary Material [https://...](https://...) for an analysis of the semiclassical/mean-field chaos in the quantum ratchet, and compares this to the measures of quantum chaos derived from the many-body models considered here, which includes the Refs. [12] and additionally [41–43].

[41] Ryan Mishmash, *Quantum Many-body Dynamics of Ultracold bosons in One-dimensional Optical Lattices: Theoretical Aspects, Simulation Methods, and Soliton Formation*, Colorado School of Mines, (2008).

[42] Gerog A. Gottwald and Ian Melbourne, On the validity of the 0–1 test for chaos, *Nonlinearity*, 22, 1367 (2009).

[43] Gerog A. Gottwald and Ian Melbourne, On the implementation of the 0–1 test for chaos, *SIAM Journal on Applied Dynamical Systems*, 8, 129–145 (2009).

[44] The 3LS is time independent, fitting into the Gaussian Orthogonal Ensemble, while our Bose-Hubbard model would fit in the Gaussian Unitary Ensemble, do to lack of time inversion symmetry.