Schrödinger Quantization Problem and Neutron Stars

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Abstract

We are discussing the possibility to find a proper unique conditions for an experimental study of the Schrödinger quantization problem in the neutron stars physics. A simple toy model for physically different quantizations is formulated and a possible physical consequences are derived.

1 Introduction

The quantization of the classical mechanical systems is still an open problem. It was raised as a physical problem already in the pioneering article on quantum mechanics by Ervin Schrödinger [1] in the following form. Suppose we are given a classical particle with mass $m$ in a potential field $V(r)$. Its classical Hamiltonian is

$$H = \frac{p^2}{2m} + V(r). \quad (1)$$

The problem is how to find a quantum operator $\hat{H}$ which corresponds to the classical Hamiltonian (1) and which is needed to write down the Schrödinger equation:

$$i\hbar \partial_t \psi(r,t) = \hat{H} \psi(r,t), \quad (2)$$

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In the coordinate representation we know the position operator $\hat{r} = r$ and the momentum operator $\hat{p} = \frac{\hbar}{i} \nabla$. Unfortunately this is not enough to find the quantum Hamiltonian $\hat{H}$. For this purpose we have to solve, in addition, the operator ordering problem for the non-commutative canonically conjugated variables $[\hat{p}_i, \hat{x}_j] = \frac{\hbar}{i} \delta_{ij}$. The superposition principle tells us that $\hat{H} = \frac{\hat{p}^2}{2m} + \hat{V}(\hat{r})$, but what are $\hat{p}^2$ and $\hat{V}(\hat{r})$ is not known.

Schrödinger has stressed that, for example, we can write down the classical quantity $p_i^2$ in a classically equivalent form $\frac{1}{f(r)} p_i f(r) \left( \frac{1}{f(r)} \right) p_i \equiv p_i^2$ with an arbitrary function $f(r) \neq 0, \infty$. Then the canonical "hat-quantization" will give a corresponding quantum operator $\hat{p}_i \hat{f}(\hat{r}) \hat{p}_i \hat{f}(\hat{r}) = \hat{p}_i^2 + \hbar^2 \frac{\partial^2 f(r)}{f(r)} = \hat{p}_i^2 + \mathcal{O}(\hbar^2)$. As a result, in coordinate representation we will have a quantum Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m} \Delta + \frac{\hbar^2}{2m} \frac{\Delta f(r)}{f(r)} + \hat{V}(\hat{r}).$$  (3)

The real quantization problem is not limited to this simple example. A general form of this problem is related with the fact that we do not know how to "quantize" the simple classical number 1, when considered as a dynamical "variable". This becomes transparent, if we write down the number 1 in a form of a product $1 \equiv f_1(x) g_1(p) f_2(x) g_2(p) ... f_n(x) g_m(p)$ of an arbitrary well defined functions $f_1(x), ..., f_n(x)$ and $g_1(p), ..., g_m(p)$, such that $f_1(x) f_2(x) ... f_n(x) \equiv 1$ and $g_1(p) g_2(p) ... g_m(p) \equiv 1$. These functions have to be introduced at classical level. Therefore they certainly are independent of the Planck constant $\hbar$ and can depend only on the classical macroscopic parameters of the system at hand. Now, “putting the quantum hats” on the letters, we obtain a nontrivial quantum operator, which corresponds to the simple number 1:

$$\hat{I} := f_1(\hat{x}) g_1(\hat{p}) f_2(\hat{x}) g_2(\hat{p}) ... f_n(\hat{x}) g_m(\hat{p}).$$

This operator has a limit $\hat{I} \to 1$, for $\hbar \to 0$

It is clear, that we have infinitely many such quantum operators $\hat{I}$ and we can put any of them at any position in the expression for every quantum quantity, obtained from some classical one by the use of the correspondence principle. For this purpose one has to put in the corresponding classical quantity the number 1, as a multiplier, at different positions and then, during the “quantization”, we have to replace these numbers by some of their quantum representatives $\hat{I}$. Hence, actually there exist no physical problems for which one can ignore the above Schrödinger quantization problem.
We need to use some additional principles, which are independent of the correspondence principle, to be able to fix the right quantization in a given physical problem. After all, the term “right quantization” can have only one meaning: the quantization, which reflects in an adequate way the real properties of the physical objects.

2 Some of the Mathematical Problems, Related to the Schrodinger Quantization Problem

It is not easy to formulate in a mathematically correct way the Schrödinger quantization problem, when we consider this problem in the above very wide framework. Here we shall only mark some of the existing problems and approaches.

The first mathematical problem which one must consider, is the fixing of the Hilbert space of the quantum states. This is known to be not a trivial problem and many of the formal operators, obtained by the correspondence principle may turn to be not well defined self-adjoint operators on this Hilbert space. For example, in the simple infinite wall problem the quantum momentum $\hat{p}$ is not well defined self-adjoint operator. A direct way to see this is to take into account that the quantum operator $\exp(i\hbar a \hat{p})$ represents a translation in the coordinate space. This translation will drop out of the any compact finite interval $x \in [x', x'']$ some part of the quantum states, which in this problem are defined as a functions on this fixed interval: $\psi(x) \neq 0$ at least for some values of $x \in [x', x'']$, and $\psi(x) = 0$ for $x \notin [x', x'']$.

In contrast, the requirement to have Hermitian operators with respect to some fixed measure in the corresponding quantum space of states, imposes a very weak restriction on the form of the quantum operators and may be easily fulfilled.

Quite more restrictive result is the Van Hove theorem [2]. It states that it is impossible to find a quantization procedure, which transforms the Poisson algebra of the classical quantities onto a quantum one, changing the Poisson brackets with proper quantum commutators. This result was the starting point of developing of the geometrical quantization [3], where the important notion of a polarization of the classical phase space was introduced.

At first glance it seems that the Feynman path integral approach to quantum mechanics [4] is able to solve the Schrödinger quantization problem, since in this approach only classical notions are used. As it was shown in [5], this is not the case. The very Feynman path integral depends on the discretization procedure and this is complete equivalent to the operator
ordering problem in the above canonical quantization procedure. A various
discretization, which lead to \( \hat{p} - \hat{x} \) quantization, \( \hat{x} - \hat{p} \) quantization, Weyl
quantization, Wick quantization, etc., can be easily found. Hence, the am-
biguity in the path integral is of the same nature as the uncertainty in the
canonical Schrödinger quantization.

After all it turns out that the Feynman path integral on the classi-
cal phase space can throw some new light on the quantization problem. As
shown in [6], the result of the summation on the virtual paths on the classical
phase space strongly depends on the choice of the class of the paths, which
we include in the Feynman sum. This choice can be described completely
in terms of classical mechanics, using different maximal sets of classical first
integrals in involution. The choice of such set of local classical first inte-
grals fixes the quantization. This observation may lead to a real progress
in the understanding of the quantization problem, since now the very am-
biguity of the quantization procedure is described completely only in terms
of the classical mechanics. This approach is deeply related with Van Hove
theorem and geometric quantization. Another advantage is that the proce-
dure for calculation of the Feynman path integral, based on this approach,
uses the Volterra’s multiplicative integral. This permits us to overcome the
restriction to consider only Gaussian measures in the path integrals. Unfor-
nately, at present this approach is not developed enough and suffers from
some specific technical difficulties.

There exist another widespread point of view on the operator ordering
problem. If one accepts a totally quantum philosophy, one can think that
the quantum operators are the primary given objects. Then we postulate
the form of quantum Hamiltonian \( \hat{H} \), which is considered as known without
any need to quantize classical quantities. Then the correspondence principle
may be used to establish a correspondence of the given quantum operators
to the classical quantities.

In this approach a pure technical problem appears: how to write down
the given quantum Hamiltonian \( \hat{H} \) in the terms of the given quantum op-
erators of the positions – \( \hat{x}_i \) and of the momenta – \( \hat{p}_i \). This formal problem
admits a variety of mathematical solutions. For example, one can write down
the quantum Hamiltonian in the following forms:

\[ \hat{H} = \sum_{m,n} H^{px}_{mn} \hat{p}^n \hat{x}^m = \sum_{m,n} H^{xp}_{mn} \hat{x}^m \hat{p}^n = \sum_{m,n} H^{\{p\}}_{mn} \{ \hat{p}^n \hat{x}^m \} = \ldots \]

Here the symbol \( \{ \ldots \} \) stands for some kind of symmetrization, for example, \( \{ \hat{p}^n \hat{x}^m \} = \hat{p}^n \hat{x}^m + \hat{x}^m \hat{p}^n \). The
numerical coefficients \( H^{px}_{mn}, H^{xp}_{mn}, H^{\{p\}}_{mn}, \ldots \) are different and depend on the Planck constant. In this case the corresponding principle says that replacing the operators \( \hat{x}^m \) and \( \hat{p}^n \) with their classical counterparts \( x^m \) and \( p^n \) we
will obtain the classical Hamiltonian $H(p, x)$ in the limit $\hbar \to 0$, no matter which of the above representations of the quantum Hamiltonian we use.

It is clear that this point of view has noting to do with the Schrodinger quantization problem. It is designed to solve only the above formal mathematical problem.

In the present article we will not pay attention to the last formal mathematical problem. Instead we shall make an attempt to consider the Schrodinger quantization problem as a real physical problem.

3 The Physical Difficulties in the Schrodinger Quantization Problem

The real physical difficulties originates from the fact, that the differences between different quantizations of a given classical quantity may be of high order with respect to the Planck constant $\hbar$. This means that the deviations from a chosen quantization of this quantity can be extremely small and behind the real experimental abilities for measurements. Therefore at present we have no good real experimental evidences in favor of some definite quantization rule, or in favor of some more general physical theory, which uses simultaneously different quantization rules for different physical purposes.

3.1 The Harmonic Oscillator

We shall illustrate the general situation using a simple toy-model. Consider the one-dimensional harmonic oscillator with classical Hamiltonian

$$H = \frac{1}{2m}p^2 + \frac{m}{2}\omega^2 x^2. \quad (4)$$

Let us choose in the formula $f(x) = e^{-x^2/2a^2} \to 1$ for $a \to \infty$, with some macroscopic parameter of length $a$. Then the corresponding quantum operator may be written in the form:

$$\hat{H} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{m}{2}\omega^2 x^2 - \frac{\hbar^2}{2ma^2} \quad (5)$$

with some corrected circular frequency

$$\bar{\omega} = \omega \sqrt{1 + \frac{\hbar^2}{\omega^2 m^2 a^4}}. \quad (6)$$
Now it becomes clear that the use of the function \( f(x) = e^{-x^2/2a^2} \) in the above nonstandard quantization procedure has a simple physical meaning. This function introduces some specific quantum constraint on the motion of the quantum particle via some specific quadratic quantum wall:

\[ V_{\text{quant}} = \frac{\hbar^2}{2ma^2} (x^2/a^2 - 1). \]

As a result, outside some domain, defined by the parameter \( a \), i.e., for a big values of the coordinate \( x >> a \), the wave function now decreases exponentially, (almost) independently of the behavior of the real physical potential \( V(x) \). This potential may influence only the details of the exponential decay of the wave function, if \( V(x) \) is limited from the below for \( |x| \to \infty \), i.e. if \( V(x) \geq \text{const} > -\infty \) for \( |x| \to \infty \).

When the parameter \( a \) increases, the domain, where the wave function may oscillate, increases, too. In the limit \( a \to \infty \) this specific quantum constraint disappears.

The spectrum of the Hamilton operator (5) is described by the formula

\[
E_n(a) = \left( n + \frac{1}{2} \right) \hbar \bar{\omega} - \frac{\hbar^2}{2ma^2} = E_n(\infty) - \frac{\hbar^2}{2ma^2} + \left( n + \frac{1}{2} \right) \frac{\hbar^3}{2\omega m^2 a^2} + O(\hbar^4),
\]  

(7)

As we see, the standard levels \( E_n(\infty) = \left( n + \frac{1}{2} \right) \hbar \bar{\omega} \) are shifted by the amount of \( \Delta E_n(a) = -\frac{\hbar^2}{2ma^2} \) and, in addition, we have an increase of the relative distance between the levels, described by the dimensionless quantity

\[
\frac{\delta E_n(a)}{E_n(\infty)} = \frac{1}{2} \delta^2 = \frac{1}{2} \frac{\hbar^2}{\omega m^2 a^4}
\]  

(8)

in the lowest order with respect to \( \delta \). The dimensionless parameter \( \delta = \frac{\hbar}{\omega m a} \) controls the deviation of our nonstandard quantization of the harmonic oscillator from the “standard” one.

The generalization of this example for the D-dimensional isotropic oscillator is obvious. Choosing the function \( f(x) = \exp \left( -\frac{1}{2a^2} \sum_{i=1}^{D} (x_i)^2 \right) \), we immediately obtain the quantum Hamiltonian:

\[
\hat{H} = -\frac{\hbar^2}{2m} \Delta_D + \frac{m}{2} \bar{\omega}^2 \bar{x}^2 - D \frac{\hbar^2}{2ma^2}
\]  

(9)

with the same circular frequency \( \bar{\omega} \), defined by the formula (6).

### 3.2 The Hydrogen Atom

It is well known, that there exist a simple correspondence between the spectrum of the D=2 isotropic oscillator and the spectrum of the hydrogen atom,
see for example [7]. One easily obtains that when one uses in this correspondence the quantum Hamiltonian (9), instead of the usual one (with $a = \infty$), one will arrive at the following corrected formula for the spectrum of the hydrogen:

$$ E_n := -\frac{R_\infty}{n^2 (1 + \delta^2)} $$

where $R_\infty = 13.605\,691\,72(53)\,eV$ (Uncert. (ppb): 39) is the standard Rydberg constant. Hence, such unusual quantization of the hydrogen atom is equivalent to a correction of the Rydberg constant with a relative decrease $\delta R_\infty / R_\infty = -\delta^2$.

Now, making use of the standard formulas for the hydrogen, we obtain:

$$ \delta = \left( \frac{a_B}{a} \right)^2 $$

where $a_B \approx 0.529177 \times 10^{-8} \text{ cm}$ is the Bohr radius. Hence, for a macroscopic length $a \sim 1 \text{ cm}$ we will have $\delta \sim 10^{-16}$. This will lead to relative correction of the Rydberg constant of order of magnitude $\delta^2 \sim 10^{-32}$ – too far from any experimental abilities in any foreseen future.

The general rule for inclusion of the correction for our nonstandard quantization in different quantum quantities, related with hydrogen, is very simple: one has to make only the replacement of the principal quantum number $n \to n \sqrt{1 + \delta^2}$. Then we can immediately see the influence of this nonstandard quantization on the Lamb shift. Using standard notations [7] we obtain:

$$ \delta E_{n,0} = \frac{4}{3\pi} \frac{\alpha^2 Z^4}{n^3} \log \left( \frac{m_e c^2}{\Delta E} \right) \frac{e^2}{a_0} \left( 1 + \delta^2 \right)^{-3/2}. $$

Hence, the relative deviation in the Lamb shift due to our nonstandard quantization procedure will be of order of magnitude $-\frac{3}{2}\delta^2 \sim -10^{-32}$. Thus we see that although the measurements of the Lamb shift are at present between the experimental checks of quantum mechanics with a greatest possible precision, these measurements are not able to see such small deviations from the “standard” canonical quantization of the hydrogen.

4 Schrödinger Quantization Problem and Neutron Stars

As we saw in the previous section, from experimental point of view the Schrödinger quantization problem seems to be hopeless issue. The correc-
tions to the basic quantum observables are extremely small for systems with small number of degrees of freedom. At the same time the formula (9) shows explicitly that the common shift of the levels is proportional to the number of the degrees of freedom. Obviously the same conclusion is valid in the general case, too, because we are quantizing independently all degrees of freedom and these enter the classical Hamiltonian additively. For systems with a huge amount of degrees of freedom the huge number D can compensate the extremely small quantization correction $\sim \delta^2$ for a single degree of freedom. Then the physical effects may be significant. Hence, to be able to see the effects of the possible different quantization procedures we need some physical system with a huge number of degrees of freedom $D \sim \delta^{-2}$ and in addition this system has to be in a coherent quantum state.

The ideal known physical systems of this sort are the neutron stars. At present their behavior is observed with a great details. See for an additional information, for example, the review articles [8, 9] and the references therein. Indeed, the neutron stars have some $10^{57}$ neutrons in them in a coherent superfluid quantum state. The typical density of matter in the neutron stars is $\approx 2.8 \times 10^{14}$ g cm$^{-3}$. In addition, in the neutron stars we have a magnetic fields of order of magnitude $\sim 10^{12}$ Gauss and a rotational energy $\sim 2 \times 10^{49}$ erg. These extremal physical conditions make the neutron stars a unique astrophysical laboratories.

The typical radius of the neutron stars is of order of $3 \times 10^5$ cm. Hence, if we accept the macroscopic parameter $a$ to be of order of magnitude of the star’s radius, we will have for the neutron star $\delta \sim 10^{-26}$. If the quantum corrections due to the change of the quantization procedure (like that for the levels of the hydrogen, or for the Lamb shift) are of order of $\delta^2 \sim 10^{-52}$, this extremely small number can be compensated by the huge number of the degrees of freedom of the neutrons in the star. Then for the parameter $D \delta^2$, which controls the possibility for real observation of the effects of different quantization procedures, we obtain $D \delta^2 \sim 10^5$ – an unique big number, in comparison with the corresponding extremely small values of the corrections of this type in all available Earth-laboratory experiments.

5 Concluding Remarks

Of course the above consideration is a pure qualitative speculation. Although it is not based on some deep theoretical analysis, it shows that, in principle, an attempts to look for some phenomena in neutron-star physics, which are related with the Schrödinger quantization problem, may have a good physi-
cal ground. If we are extremely lucky, it could happen that the macroscopic parameters like the parameter $a$ in the simple toy model, considered in the present article, are much smaller than the radius of the star. This may increase essentially the important factor $D\delta^2$.

On the other hand, many of the observed phenomena in neutron-star’s physics are not well understood at present. We shall mansion only two of them: the mechanism of star’s radio emission \[8\] and the instabilities of the oscillations of the stars – the so called glitch phenomenon in pulsars – a sudden spin-up of the star’s crust \[9\]. Of course, a more conventional models of these phenomena, without consideration of quantization problem, are at present in an intensive study.

It seems to us that neutron stars, which appeared at first as a pure theoretical invention in the early 1930’s on the ground of the quantum mechanics and gravity in the independent articles by Landau and by Chandrasekhar, may help us once more to reach a deeper understanding of the quantum mechanics and, especially, for a discovery of a new physics behind the Schrödinger quantization problem.

It is not excluded, too, to look for some new effects, related to the Schrödinger quantization problem in a more usual laboratory conditions, studying the collective quantum phenomena like superfluidity and superconductivity, and using the possible interplay between microscopical quantities, like the Bohr radius $a_B$, and the macroscopical ones, like the parameter $a$ in the above consideration.

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References

[1] E. Schrödinger, Annalen der Physik (4), Bd. 79, p. 734, 1926.

[2] L. Van Hove, Proc. Roy. Acad. Sci. Belgium, 26, p. 1, 1951.

[3] D. J. Sims, N. M. J. Woodhouse, Lect. Not. Phys., p. 53, Springer, 1976. N. M. J. Woodhouse, Geometric quantization, Oxford/New York, Clarendon Press/Oxford University Press, 1991.

[4] R. P. Feynman, Phys. Rev. 84, p. 108, 1951. R. P. Feynman, A. R. Hibbs, Quantum Mechanics and Path Integrals N.Y., 1965.

[5] F. A. Beresin, Theor. Math. Fiz. 6 p. 194, 1971; Usp. Fiz. Nauk 132 p. 497, 1980.

[6] P. Fiziev, Bulgarian Journal of Physics, 10 p. 27, 1983; Bulgarian Journal of Physics, 11 p. 11, 1984; Theor. and Math. Phys., 62 p. 123, 1985; The Class of the Paths in the Feynman Path Integral on the Phase Space, ICTP preprint, IC/91/386; and in Lectures on Path Integration: Trieste 1991, editors H.Cerdeira at all., World Scientific, Singapore-New Jersey-London-Hong Kong, 1993, p.556.

[7] J. Schwinger, Quantum Mechanics, Springer, 2001.

[8] W. Becker, G. Pavlov, em The Milky Way – Pulsars and Isolated Neutron Stars, in The Century of Space Science, eds. J.Bleeker, J.Geiss, M.Huber, Kluwer Academic Publishers, to be published; astro-ph/0208365.

[9] G. L. Comer, Found. Phys. 32 p. 1903, 2002.