A SMALL RESOLUTION OF A MODULI SPACE OF SCALED CURVES

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ABSTRACT. We prove that the moduli space $\mathcal{P}_n$ of stable $n$-marked $\mathbb{G}_a$-rational trees introduced in earlier work admits a small resolution by the augmented wonderful variety corresponding to the graphic matroid of the complete graph. The proof entails a careful analysis of the polydiagonal degeneration of a power $X^n$, a simple variation on Ulyanov’s polydiagonal compactification.

1. Introduction

We work over the field $\mathbb{C}$ of complex numbers.

Let $\overline{\mathcal{P}}_n$ be the moduli space of stable $n$-marked $\mathbb{G}_a$-rational trees that was denoted by $\mathcal{P}_n, \mathbb{C}$ in [Za21, §1.3]. This is an equivariant compactification of $\mathbb{C}^n/\mathbb{C}$ (quotient by the diagonal copy $\mathbb{C} \subset \mathbb{C}^n$), whose points correspond to genus 0 nodal curves with some additional structure. Please see [Za21, Theorem 1.5] for the functor of points of $\mathcal{P}_n$, and [Za21, §1.3] for an elementary introduction and more details.

As explained in [Za21], $\overline{\mathcal{P}}_n$ resembles the moduli space $\overline{Q}_n$ of stable nodal scaled marked lines [MW10, §10] constructed by Ziltener [Zi06], and Mau and Woodward [MW10]. At the end of [MW10], it is explained that $\overline{Q}_4$ is singular. Even if slightly different arguments are needed, the situation for $\overline{P}_4$ is identical: $\overline{P}_4$ has nodes at the 3 points where the dual tree of the corresponding genus 0 curves is a complete 3-level binary tree ($1 + 2 + 4 = 7$ vertices), and there is one marking on each tail, and no other singularities. This is a bit surprising for such seemingly ‘pathology-free’ moduli spaces, and raises an obvious question.

Question 1.1. Is there a good way to resolve the singularities of $\overline{Q}_n$ or $\overline{P}_n$?

In this note, we give an affirmative answer for $\overline{P}_n$.

According to the discussion above, $\overline{P}_4$ admits a small resolution (the exceptional locus contains no divisor). We will show that the same holds for $\overline{P}_n$, for all $n \geq 4$. The resolution belongs to a class of varieties that has been of great interest recently, it is the augmented wonderful variety [BHMPW20a, BHMPW20b, EHL22] corresponding to the graphic matroid of the complete graph $K_n$. Augmented wonderful varieties are also resolutions of ‘matroid’ Schubert varieties [HW17] (please see also [BHMPW20b, §1.3], [EHL22, §1]), although these resolutions aren’t small.
Definition 1.2. Let \( \mathbb{C}^n / \mathbb{C} \) be the quotient by the diagonal copy of \( \mathbb{C} \), and for each \( i \neq j \), let \( H_{ij} = \{ x_i = x_j \} \subset \mathbb{C}^n \) denote the \( ij \)-diagonal hyperplane. The subspaces arrangement \( \{ \mathbb{P}(0 \oplus H_{ij}) : 1 \leq i < j \leq n \} \) in \( \mathbb{P}^{n-1} = \mathbb{P}(\mathbb{C} \oplus \mathbb{C}^n / \mathbb{C}) \) is the augmented braid arrangement. Let \( W_n \) be its De Concini–Procesi wonderful model \( \text{[DP95]} \) (i.e. the wonderful compactification of \( \mathbb{P}^{n-1} \setminus \bigcup_{i<j} \mathbb{P}(0 \oplus H_{ij} / \mathbb{C}) \)).

If we take \( E = \{ (i, j) : 1 \leq i < j \leq n \} \), and \( V \) the image of the map \( \mathbb{C}^n \to \mathbb{C}^E \), \( (x_1, \ldots, x_n) \mapsto (x_2 - x_1, x_3 - x_1, \ldots, x_n - x_{n-1}) \) in \( \text{[BHMPW20a, Remark 2.13]} \), then \( W_n \) is the augmented wonderful variety of loc. cit.

Theorem 1.3. There exists a small resolution \( W_n \to \mathcal{P}_n \) if \( n \geq 4 \).

Theorem 1.3 points out a connection between two well-studied areas: the moduli of scaled nodal curves and related topics on one hand, and (at least special cases of objects from) the combinatorial area of matroids, Bergman fans \( \text{[FS05]} \), and related topics on the other.

To prove Theorem 1.3, we construct a stable \( n \)-marked \( \mathbb{G}_a \)-rational tree over \( W_n \), and then apply \( \text{[Za21, Theorem 1.5]} \) to obtain a map \( W_n \to \mathcal{P}_n \).

The first step is best understood in the context of a more general construction. For any smooth variety \( X \), we define the polydiagonal degeneration \( X^{\langle n \rangle} \) of \( X^n \) (Definition 3.1), which is obtained by performing the same sequence of blowups as in Ulyanov’s polydiagonal compactification \( X^{\langle n \rangle} \) \( \text{[Ul02]} \), but after embedding \( X^n \) inside \( \mathbb{A}^d \times \mathbb{A}^d \) as \( \{0\} \times X^n \). Much of the paper is devoted to analysing the family of degenerations of \( X \) over \( X^{\langle n \rangle} \), which is a Fulton-MacPherson degeneration space of \( X \) over \( X^{\langle n \rangle} \) \( \text{[KKO14, Definition 2.1.1]} \) with some extra structure related to the ‘scaling’, and abstractly isomorphic to \( X^{\langle n + 1 \rangle} \).

When \( X = C \) is a curve, \( W_n \) is a codimension 2 subvariety of \( C^{\langle n \rangle} \). If we restrict the family \( C^{\langle n + 1 \rangle} \to C^{\langle n \rangle} \) to \( W_n \), we can erase the components isomorphic to \( C \). We obtain an \( n \)-marked \( \mathbb{G}_a \)-rational tree. It is unstable, but fortunately, it can be stabilized by contracting the rational bridges without compromising any of the additional pieces of structure. Finally, we check that the induced map \( W_n \to \mathcal{P}_n \) is an isomorphism in codimension 2, but not an isomorphism if \( n \geq 4 \).

Remark 1.4. Although the case dim \( X > 1 \) is not needed for the proof of Theorem 1.3, it will come up in future work which will generalize \( \mathcal{P}_n \) from a compactification of the space of configurations on \( \mathbb{A}^d \) modulo translation to a compactification of the space of configurations in \( \mathbb{A}^d \) modulo translation. The nodal curves of genus 0 become trees of projective spaces \( \text{[CGK09, Definition 2.0.1]} \), etc.

In \( \S 5 \), we give some numerical evidence that the singularities of \( \mathcal{P}_n \) become quite complicated for \( n \gg 0 \) (and probably so does the resolution in Theorem 1.3).

It is not entirely obvious if similar ideas lead to a resolution of \( \mathcal{Q}_n \), although it seems plausible, since the differences between \( \mathcal{Q}_n \) and \( \mathcal{P}_n \) appear to be unrelated to what makes them singular.

Acknowledgements. I guessed that the small resolution of \( \mathcal{P}_n \) (which I was investigating) is an augmented wonderful variety when I saw \( \text{[Pr22]} \). I would like to thank Joel Kamnitzer and Nicholas Proudfoot for asking, respectively answering, the question on mathoverflow.
2. Naive Fulton-MacPherson degeneration spaces

Let $X$ be a nonsingular variety, $\dim X = d$. The notion of Fulton-MacPherson degeneration space (or simply FM space) $(W \to S, W \to X)$ of $X$ over a base $S$ has been defined in [KKO14, Definition 2.1.1].

**Definition 2.1.** Let $S$ be a smooth variety. A naive FM space of $X$ over $S$ is an object of the smallest collection with the following properties:

1. $(S \times X \to S, S \times X \to X)$ is a naive FM space of $X$ over $S$; and
2. if $(W \to S, W \to X)$ is a naive FM space of $X$ over $S$, $\sigma : S \to W$ is a smooth section of $W \to S$, $D \subset S$ is a smooth divisor, and $W' = \text{Bl}_{\sigma(D)}W$, then $(W' \to S, W' \to X)$ is a naive FM space.

Thus, any naive FM space can be obtained starting from $S \times X$, by repeatedly blowing up as in item 2 above. This sequence of blowups is called the history, or rather, ‘a’ history.

**Lemma 2.2.** Any naive FM space is a FM space.

In the proof, we will use the standard notation from [FM94] without any further explanation. The proof is rather technical and uninteresting, so we’ll be brief.

**Proof.** We proceed inductively, and use notation as in item 2 in Definition 2.1. First, we boil down the claim to the special case when $S = \mathbb{A}^1 \times X[n]$, $W = \mathbb{A}^1 \times X[n]^+$, $\sigma$ is $\text{id}_{\mathbb{A}^1} \times x_n$, where $x_n : X[n] \to X[n]^+$ is simply the $n$-th marked point (section) on $X[n]^+$, and $D = \{0\} \times X[n]^+$. This is straightforward, except for one ingredient: étale locally, any FM space with a smooth section can be obtained by pull back from $X[n]^+ \to X[n]$ with the $n$-th section $x_n$, for some $n$. A priori, this can be achieved with $V \times X[m] \times X[n] \to V$ and the diagonal section $\delta$, where $V \subset X[n]^+$ is the open where $X[n]^+ \to X[m]$ is smooth. It is not hard to shrink $V$ to $V'$, the complement of the $m$ distinguished sections in $V \subset X[m]^+$, without loss of generality. Then the open immersion $V' \to X[m]^+$ lifts to $V' \to X[m+1]$, so the desired property holds for $n = m + 1$.

It remains to deal with the special case. The idea is to prove that there exists a surjective étale map $\eta : U \to \mathbb{A}^1 \times X[n]$ and a map $h : U \to X[n+1]$ such that the following conditions are satisfied:

- the compositions $U \xrightarrow{\eta} \mathbb{A}^1 \times X[n] \to X[n]$ and $U \xrightarrow{h} X[n+1] \to X[n]$, in which the unnamed maps are the natural projections, coincide;
- $h^{-1}(D(i, n+1)) = \begin{cases} \eta^{-1}(\{0\} \times X[n]) & \text{if } i = n \\ \emptyset & \text{if } i \neq n \end{cases}$ scheme-theoretically;
- $X[n+1] \to X[n]$ is smooth at all points in the image of $h$.

Then $U \times X[n+1] \times X[n]^+ \simeq U \times \mathbb{A}^1 \times X[n] \times \text{Bl}_{\{0\} \times \text{Spec}(X[n])}(\mathbb{A}^1 \times X[n]^+)$ over $U$, as desired. The details are left to the reader. \qed

For any naive FM space $(W \to S, W \to X)$, there is a canonical decomposition

$$\text{Pic}(W) = \text{Pic}(S \times X) \oplus F_{W,X/S},$$

with $F_{W,X/S}$ free abelian and finitely generated. This is independent of history: the projection to the first factor is obtained as $\text{Pic}(W) \to \text{Pic}(U) \leftarrow \text{Pic}(S \times X)$, where the open $U \subset S \times X$ is sufficiently large for the second map to be an isomorphism, and independent of history in the suitable sense.
Lemma 2.3. For any naive FM space \((W \to S, W \to X)\), there exists a unique line bundle \(L \in F_{W/X}\) such that \(\omega_{W/S} \cong \xi^* \omega_X \otimes L^\otimes d\).

The line bundle in Lemma 2.3 is called the root relative dualizing line bundle, and denoted \(\sqrt[\omega]{}_{W,X/S}\). If \(d = 1\), we will write simply \(\omega_{W/X,S}\).

Corollary 2.4. (1) In 1 in Definition 2.1, \(\sqrt[\omega]{}_{S \times X,S/X} = O_{S \times X}\).

(2) In 2 in Definition 2.1, if \(\beta : W' \to W\) is the blowup, and \(E\) is the exceptional divisor, then \(\sqrt[\omega]{}_{W',X/S} = O_{W'}(E) \otimes \beta^* \sqrt[\omega]{}_{W,X/S}\).

Proof of Lemma 2.3 and Corollary 2.4. The uniqueness part of the lemma is clear, since \(F_{W,X} = O\) is torsion-free. To prove existence, we turn the two properties in Corollary 2.4 into the inductive definition of the desired line bundle \(L\), and check
\[\omega_{W/S} \cong \xi^* \omega_X \otimes L^\otimes d.\]

It follows from well-known facts that \(\omega_W \cong \omega_{W/S} \otimes \pi^* \omega_S\). Hence, (2) is equivalent to \(\omega_W \cong (\pi, \xi)^* \omega_{S \times X} \otimes L^\otimes d\), which follows inductively from
\[K_{W'} = \beta^* K_W + dE.\]

Note also that \(O_{W'}(E) \in F_{W',X/S}\), and \(\beta^*(F_{W,X/S}) \subseteq F_{W',X/S}\), so the inductively defined line bundle is indeed in \(F_{W',X/S}\). \(\square\)

Remark 2.5. The root relative dualizing line bundle can be defined for any FM space, not just naive FM spaces, though we won’t need it.

We also note that more than \(\omega_{W/S} \cong \xi^* \omega_X \otimes (\sqrt[\omega]{}_{W,X/S})^\otimes d\) is true: in fact,
\[\Omega_{W/S}^{\log} \cong \xi^* \Omega_X \otimes \sqrt[\omega]{}_{W,X/S},\]
holds, but we don’t need it, so it won’t be proved. Morally, this reflects the elementary fact that for any \(p \in \mathbb{P}d\) away from the hyperplane at infinity \(H_\infty\), there is an isomorphism between \(T_p \mathbb{P}d\) and the ‘cone over \(H_\infty\)’, but this isomorphism is canonical only up to scalars.

Proposition 2.6. Let \((W \to S, W \to X)\) be a naive FM space, and let
\[\psi : \sqrt[\omega]{}_{W,X/S} \to \mathcal{L}_{\sigma(D), W}\]
be an \(O_W\)-module homomorphism, where \(\mathcal{L}_{\sigma(D), W}\) is the ideal sheaf of \(\sigma(D)\).

Let \(W'\) as in 2 in Corollary 2.4. Then there exists a unique \(\psi' : \sqrt[\omega]{}_{W',X/S} \to O_{W'}\) such that the restrictions of \(\psi\) and \(\psi'\) to \(W \setminus \sigma(D)\) and \(W' \setminus E\) respectively correspond to each other under the isomorphism \(W \setminus \sigma(D) \cong W' \setminus E\).

Moreover, the restriction of \(\psi'\) to \(E\) is equal to 0 if and only if the image of \(\psi\) is contained in \(\mathcal{L}_{\sigma(D), W}^\circ\).

It is enlightening to compare Proposition 2.6 with [Za21, §3.2].

Proof. It is well-known that \(\beta|_E : O_{\sigma(D)} \to (\beta|_E)_* O_E\) is an isomorphism, so
\[j_* O_{\sigma(D)} = j_* (\beta|_E)_* O_E = \beta_* j'_* O_E,\]
where \(j : \sigma(D) \to W\) and \(j' : E \to W'\) are the closed immersions. A simple diagram chase provides the dashed arrow \(\mathcal{L}_{\sigma(D), W} \to \beta_* \mathcal{L}_{E,W'}\) preserving the commutativity of the diagram below.
The solid vertical arrows are isomorphisms, hence so is the dashed one. Twisting and using the projection formula, we obtain canonical isomorphisms
\[
\sqrt[\text{pr}]{\omega_{W,X/S} \otimes \mathcal{I}_{\sigma(D),W}} = \sqrt[\text{pr}]{\omega_{W,X/S} \otimes \beta_{*}\mathcal{I}_{E,W'}}
\]
\[
= \beta_{*}(\beta^{*}\sqrt[\text{pr}]{\omega_{W,X/S} \otimes \mathcal{I}_{E,W'}}) = \beta_{*}\sqrt[\text{pr}]{\omega_{W',X/S}}.
\]
Taking global sections, this simply reads
\[
(3) \quad \text{Hom}(\sqrt[\text{pr}]{\omega_{W,X/S} \otimes \mathcal{I}_{\sigma(D),W}}, \mathcal{O}_{W'}) = \text{Hom}(\sqrt[\text{pr}]{\omega_{W',X/S}}, \mathcal{O}_{W'}).\]
If $\psi'$ is the image of $\psi$, then their restrictions clearly match. Uniqueness is trivial, since $W' \setminus E$ is dense in $W'$. This completes the proof of the first part.

For the second part, first, recall that
\[
(4) \quad (\beta|_{E})_{*}(\mathcal{O}_{W',(-E)}|_{E}) = \mathcal{N}_{\sigma(D)/W} = \mathcal{I}_{\sigma(D),W|\sigma(D)}.
\]
By Corollary 2.4, the projection formula, and (4),
\[
(\beta|_{E})_{*}\text{Hom}_{k}(\sqrt[\text{pr}]{\omega_{W',X/S}|_{E}}, \mathcal{O}_{E})
\]
\[
= (\beta|_{E})_{*}\left(\mathcal{O}_{W',(-E)}|_{E} \otimes (\beta|_{E})^{*}\sqrt[\text{pr}]{\omega_{W,X/S}|_{\sigma(D)}}\right)
\]
\[
= (\beta|_{E})_{*}\left(\mathcal{O}_{W',(-E)}|_{E} \otimes \sqrt[\text{pr}]{\omega_{W,X/S}|_{\sigma(D)}}\right)
\]
\[
= \text{Hom}_{k}(\sqrt[\text{pr}]{\omega_{W,X/S}|_{\sigma(D)}}, \mathcal{I}_{\sigma(D),W|\sigma(D)}).
\]
Taking global sections, we obtain
\[
\text{Hom}_{k}(\sqrt[\text{pr}]{\omega_{W',X/S}|_{E}}, \mathcal{O}_{E}) = \text{Hom}_{k}(\sqrt[\text{pr}]{\omega_{W,X/S}|_{\sigma(D)}}, \mathcal{I}_{\sigma(D),W|\sigma(D)}).
\]
The natural square which includes the isomorphism above and (3) is easily checked to be commutative. Thus, the restriction of $\psi'$ to $E$ is equal to 0 if and only if the restriction of $\psi$ to $\sigma(D)$ is 0, which is equivalent to saying that its image is contained in $\mathcal{I}_{\sigma(D),W}^{2}$. \qed

Remark 2.7. In practice, we will use Proposition 2.6 iteratively. However, the fact that $\psi$ and $\psi'$ agree on isomorphic dense open sets shows that the formation of the new homomorphism in iterative situations is independent of history.

3. Polydiagonal degenerations

We start by introducing the combinatorial language, following [Ul02]. Let $L_{[n]}$ be the lattice of partitions of $[n] = \{1, 2, \ldots, n\}$, with the (inverse, according to some conventions) refinement partial ordering $\rho_{1} \leq \rho_{2}$ if each block of $\rho_{2}$ is contained in a block of $\rho_{1}$. Then $\leq = 12 \cdots n$ is the minimum element of $L_{[n]}$, and $\top = 1|2| \cdots |n$ is its maximum element. For $\rho_{1}, \rho_{2} \in L_{[n]}$, $\rho_{1} \setminus \rho_{2}$ and $\rho_{1} \wedge \rho_{2}$ are respectively the join and the meet. For instance,
\[
12|34|56|7 \wedge 123|456|7 = 123456|7, \quad 12|34|56|7 \vee 123|456|7 = 12|3|4|56|7.
\]
For $\rho \in L_{[n]}$, $\sim_{\rho}$ denotes the corresponding equivalence relation on $[n]$, and $B(\rho)$ is the set of blocks of $\rho$. An enumeration $\rho_{1}, \rho_{2}, \ldots, \rho_{B_{n}}$ of $L_{[n]}$ is increasing if $\rho_{i} \leq \rho_{j} \Rightarrow i \leq j$. Given the connection with the augmented wonderful variety, it
is worth reviewing that \( L_{[n]} \) is also the set of flats of the graphic matroid of the complete graph \( K_n \).

The representation of chains in \( L_{[n]} \) as leveled trees explained in [Ul02, §2] (with the phantom vertices both included or omitted) will be crucial for us as well.

There is an obvious map \( \iota : L_{[n]} \rightarrow L_{[n+1]} \) whose image consists of partitions in which \( \{n + 1\} \) is a block. Then
\[
L_{[n+1]} \setminus \iota(L_{[n]}) \simeq N_{[n]} := \{(\rho, B) : \rho \in L_{[n]}, B \in B(\rho)\}.
\]

An enumeration of \( N_{[n]} \) is induced by an enumeration \( \rho_1, \rho_2, \ldots, \rho_{B_n} \) of \( L_{[n]} \) if it consists of all pairs containing \( \rho_1 \) in any order, then all pairs containing \( \rho_2 \), etc. We fix once and for all an increasing enumeration of \( L_{[n]} \), and an induced enumeration of \( N_{[n]} \). This concludes the discussion on combinatorial language.

Throughout the rest of this section, \( X \) is a smooth variety of dimension \( d \). For each \( \rho \in L_{[n]} \), we have a polydiagonal of \( X^n \)
\[
\Delta_\rho = \{(x_1, \ldots, x_n) \in X^n : x_i = x_j \text{ if } i \sim_\rho j\}.
\]
Let us embed \( X^n \) as \( \{0\} \times X^n \) inside \( \mathbb{A}^1 \times X^n \), and let’s regard \( \Delta_\rho \) as subvarieties of \( \mathbb{A}^1 \times X^n \). Then \( Z = \{\Delta_\rho : \rho \in L_{[n]}\} \) is a simple arrangement of smooth subvarieties of \( \mathbb{A}^1 \times X^n \) [Hu03, Definition 1.2], which we will call the \( 0 \)-polydiagonal arrangement in \( \mathbb{A}^1 \times X^n \). In particular, [Hu03, Theorem 1.1] applies (please see also [Li09]).

**Definition 3.1.** The polydiagonal degeneration of \( X^n \) is
\[
X\langle\langle n\rangle\rangle = \text{Bl}_Z(\mathbb{A}^1 \times X^n).
\]

**Remark 3.2.** More explicitly, the clean intersection condition on the polydiagonals comes from the obvious equalities
\[
T_p \Delta_{\rho_1} \cap T_p \Delta_{\rho_2} = T_p (\Delta_{\rho_1} \cap \Delta_{\rho_2}),
\]
for all \( \rho_1, \rho_2 \in L_{[n]} \) and \( p \in \Delta_{\rho_1 \cap \rho_2} \). In particular, (6) shows that blowing up \( \mathbb{A}^1 \times X^n \) at \( \Delta_{\rho_1 \cap \rho_2} \) separates \( \Delta_{\rho_1} \) and \( \Delta_{\rho_2} \), that is, their strict transforms are disjoint. Combined with the highly nontrivial commutativity of blowups provided by [Li09, Theorem 1.3], this simple observation will be an essential technique.

Let \( D_\rho \) be the strict transform of \( \Delta_\rho \) under \( X\langle\langle n\rangle\rangle \to \mathbb{A}^1 \times X^n \). The following proposition is completely analogous to (some parts of) Proposition 1 and Corollary 2 in [Ul02].

**Proposition 3.3.**
(1) \( D_\rho \) is a smooth integral divisor on \( X\langle\langle n\rangle\rangle \).
(2) \( D_\rho \cap D_{\rho'} \neq \emptyset \) if and only if \( \rho \leq \rho' \) or \( \rho' \leq \rho \).
(3) If \( \rho_1 < \cdots < \rho_k \), then \( D_{\rho_1} \cap \cdots \cap D_{\rho_k} \neq \emptyset \) and \( D_{\rho_1}, \ldots, D_{\rho_k} \) cross normally.

**Proof.** Follows, for instance, from [Hu03, Theorem 1.1].

Let \( Z \times X = \{\Delta_\rho \times X : \rho \in L_{[n]}\} \) be the arrangement consisting of the polydiagonals in \( \mathbb{A}^1 \times X^{n+1} \) indexed by the equivalence relations in \( \iota(L_{[n]}) \subset L_{[n+1]} \). It is clear that
\[
X\langle\langle n\rangle\rangle \times X = \text{Bl}_{Z \times X}(\mathbb{A}^1 \times X^{n+1}).
\]
Let \( x^0_1, \ldots, x^0_n : X\langle\langle n\rangle\rangle \to X\langle\langle n\rangle\rangle \times X \) be the \( n \) sections corresponding to the morphism \( X\langle\langle n\rangle\rangle \to X\langle\langle n\rangle\rangle \times X \), and \( X^\eta_{\rho_i} = x^0_\rho(X\langle\langle n\rangle\rangle) \). For \( \eta \in L_{[n+1]} \), let \( G^\eta_{\rho_i} \subset X\langle\langle n\rangle\rangle \times X \) be the dominant transform of \( \Delta^\rho_\eta \) in the sequence of blowups in (7), where \( \Delta^\rho_\eta \) is the polydiagonal in \( \mathbb{A}^1 \times X^{n+1} \) that corresponds to \( \eta \). (We adopted the term dominant transform from [Li09, Definition 2.7].)
Lemma 3.4. Let \( \eta \in L_{[n+1]} \). If \( \eta \in \iota(L_{[n]}) \), then \( G^0_{\eta} = D_{\rho} \times X \). If \( \eta \notin \iota(L_{[n]}) \), and \( \eta \) corresponds to \((\rho, B) \in N_{[n]} \) under (5), then \( G^0_{\eta} = x_i^0(D_{\rho}) \), for all \( i \in B \).

Proof. Straightforward induction.

At the risk of making notation confusing, we will write \( G^0_{\rho, B} \) instead of \( G^0_{\eta} \) if \( \eta \notin \iota(L_{[n]}) \), and \( \eta \) corresponds to \((\rho, B) \) under (5).

We can now move on to the core topic of this section, the family of degenerations of \( X \) over \( X \langle n \rangle \). Let \( Z' \) be the 0-polydiagonal arrangement in \( \mathbb{A}^1 \times X_{n+1} \). Recall that

\[
X \langle n + 1 \rangle = Bl_{Z'}(\mathbb{A}^1 \times X_{n+1})
\]

according to Definition 3.1. We will use [Li09, Theorem 1.3] to construct a map

\[
X \langle n + 1 \rangle \to X \langle n \rangle \times X
\]

as a sequence of blowups with smooth centers. The key point is that, if we order \( Z' \) as follows: first, all polydiagonals in \( Z \times X \) from small to large, then all polydiagonals in \( Z' \times Z \times X \) from small to large (cf. the enumerations of \( L_{[n]} \) and \( N_{[n]} \) discussed earlier), then condition (\(^*\)) from [Li09, Theorem 1.3] is satisfied, so the end result of this sequence of blowups is still \( X \langle n + 1 \rangle \). Thus, (8) is the composition of the blowups of the dominant transforms of the polydiagonals in \( Z' \times Z \times X \).

(The notation in Lemma 3.5 will be reused later, as explained after its proof.)

Lemma 3.5. Let \( W \) be a blowup of \( X \langle n \rangle \times X \) in the sequence of blowups (8). Let \( \Sigma_i \) be the dominant transform of \( \Sigma_{i,B} \) on \( W \).

1. \((W \to X \langle n \rangle, W \to X) \) is a naive FM space over \( X \langle n \rangle \).
2. The restriction of \( W \to X \langle n \rangle \) to \( \Sigma_i \) is an isomorphism.
3. The morphism \( W \to X \langle n \rangle \) is smooth at all points of \( \Sigma_i \).

According to item 2, \( \Sigma_i \) is the image of a section \( x_i : X \langle n \rangle \to W \). Let \( G_{\rho, B} \subset W \) be the dominant transform of \( G^0_{\rho, B} \), for \((\rho, B) \in N_{[n]} \).

4. \( G_{\rho, B} = x_i(D_{\rho}) \) if \( i \in B \) and the blowup indexed by \((\rho, B) \) hasn't been performed yet.
5. For any \( s \in X \langle n \rangle(\mathbb{C}) \), each tail component of \( W_s \) contains at least one of the marked points \( x_1(s), \ldots, x_n(s) \in W_s(\mathbb{C}) \).

(A tail corresponds to a leaf of the rooted dual tree.)

Proof. We proceed inductively. The base case \( W = X \langle n \rangle \times X \) is trivial, with the exception of item 4, which is precisely Lemma 3.4. Assume that at the next steps we have to blow up \( G_{\hat{\rho}, \hat{B}}, B \in B(\rho) \) for fixed \( \hat{\rho} \in L_{[n]} \). They are pairwise disjoint by Remark 3.2 and [Li09, Theorem 3.1], so order is immaterial (this is item 2 in Proposition 3.3, up to commutativity). Let \( G_{\hat{\rho}, \hat{B}} \) be the locus to be blown up next. By 4, \( G_{\hat{\rho}, \hat{B}} \) is a smooth section over a divisor. Moreover,

\[
x_i(X \langle n \rangle) \cap G_{\hat{\rho}, \hat{B}} = \begin{cases} x_i(D_{\hat{\rho}}), & \text{if } i \in \hat{B} \\ \emptyset, & \text{if } i \notin \hat{B} \end{cases}
\]

Indeed, the case \( i \in \hat{B} \) follows from item 4, whereas the case \( i \notin \hat{B} \) follows from Remark 3.2 and [Li09, Theorem 1.3]. In either case, \( x_i(X \langle n \rangle) \cap G_{\hat{\rho}, \hat{B}} \) is a (possibly empty) smooth effective Cartier divisor on \( x_i(X \langle n \rangle) \), so \( x_i(X \langle n \rangle) \) remains (the image of) a section after blowing up, call it \( x'_i : X \langle n \rangle \to W' \). Note also that \( W' \to X \langle n \rangle \) is smooth at these sections. Thus, we've already checked items 1, 2,
and 3. Item 4 is clear since it obviously remains true on the level of generic points, if the blowup indexed by \((\rho, B)\) hasn’t been performed yet. Item 5 also follows. (Informally, a new component is born only as a tail with at least one marking, and none of its markings leave as long as it remains a tail.)

The notation in Lemma 3.5 will be used from now on in the case when the induction has finished, i.e. \(W = X/\langle n + 1 \rangle\). For instance, from now on, \(x_i\) is a section \(x_i : X/\langle n \rangle \rightarrow X/\langle n + 1 \rangle\).

The same induction as in Lemma 3.5 allows us to define the homomorphism

\[ (9) \quad \psi : \sqrt[λ]{ω}_{X/\langle n+1 \rangle,X/X/\langle n \rangle} \rightarrow \mathcal{O}_{X/\langle n+1 \rangle}. \]

Specifically, we start with \(\psi_0\) on \(X/\langle n \rangle \times X\) corresponding to the pullback of the regular function \(t\) on \(A^1 = \text{Spec} \mathbb{C}[t]\) along \(X/\langle n \rangle \times X \rightarrow A^1\). The property we need to check inductively is:

\(6\) If \(\psi_W : \sqrt[λ]{ω}_{W,X/X/\langle n \rangle} \rightarrow \mathcal{O}_W\) is the map on \(W\), and the blowup indexed by \((\rho, B)\) hasn’t been performed yet, then \(\text{Im}(\psi_W) \subseteq I_{G_{\rho,B,W}}\).

Property 6 ensures that Proposition 2.6 can be applied to produce \(\psi_{W'}\). It is clear that 6 continues to hold because the desired vanishing obviously continues to hold at least at the generic point of \(G_{\rho,B}\). This completes the construction of (9). For simplicity, we write \(\sqrt[λ]{ω}\) instead of \(\sqrt[λ]{ω}_{X/\langle n+1 \rangle,X/X/\langle n \rangle}\) for the rest of this section.

**Lemma 3.6.** The map \(x_i^*\psi : x_i^*\sqrt[λ]{ω} \rightarrow \mathcal{O}_{X/\langle n \rangle}\) is an isomorphism.

**Proof.** Let \(\rho \in L_{[n]}\) be any partition. Let \(B \in B(\rho)\) such that \(i \in B\), and let \(V\) be the blowup of \(X/\langle n \rangle \times X\) along \(G_{\rho,B}^0 = x_i^0(D_\rho)\), cf. Lemma 3.4. Then \(\psi_0\) and Proposition 2.6 produce a homomorphism

\[ \psi_V : \sqrt[λ]{ω}_{V,X/\langle n \rangle} \rightarrow \mathcal{O}_V. \]

It follows from the definition of \(\psi_0\) and Proposition 2.6 that \(\psi_V\) doesn’t vanish at the generic point of the the exceptional divisor on \(V\). In fact, it even follows that \(\psi_V\) doesn’t vanish at the image of the generic point of \(D_\rho\) under \(x_i^0\) (the section \(X/\langle n \rangle \rightarrow V\) whose image is the proper transform of \(\Sigma_{[n]}\)), because if \(\psi_V\) vanishes at \(x_i^0(s), s \in D_\rho(\mathbb{C})\), then it must vanish on the entire fiber \(F_s\) of the exceptional divisor regarded as a projective bundle, since the restriction of \(\psi_V^0\) is a section of \(\mathcal{O}_{F_0(1)}\), which is known to vanish along the hyperplane ‘at infinity’.

On \(X/\langle n + 1 \rangle\), it follows by the commutativity in [Li09, Theorem 1.3] and Remark 2.7 (of blowing up, respectively of the formation of the various instances of \(\psi\)) that \(\psi\), viewed as a section of the line bundle \(\sqrt[λ]{ω}^\rho\), doesn’t vanish at the image under \(x_i\) of the generic point of \(D_\rho\). This holds for all \(\rho \in L_{[n]}\), so, combining with the obvious fact that \(x_i^*\psi\) is an isomorphism away from the central fiber, it follows that \(x_i^*\psi\) is an isomorphism in codimension 2 (in \(X/\langle n \rangle\)), hence an isomorphism.

Putting Lemmas 3.6 and 3.5 together, we conclude the following.

**Theorem 3.7.** (1) The pair \((X/\langle n + 1 \rangle) \rightarrow X/\langle n \rangle, X/\langle n + 1 \rangle \rightarrow X\) is a naive FM space of \(X\) over \(X/\langle n \rangle\).

(2) The images of the sections \(x_1, \ldots, x_n : X/\langle n \rangle \rightarrow X/\langle n + 1 \rangle\) are contained in the open where \(X/\langle n + 1 \rangle \rightarrow X/\langle n \rangle\) is smooth.

(3) For any \(s \in X/\langle n \rangle(\mathbb{C})\), each tail of \(X/\langle n + 1 \rangle_s\) contains at least one of the marked points \(x_1(s), \ldots, x_n(s) \in X/\langle n + 1 \rangle_s(\mathbb{C})\).

(4) The maps \(x_1^*\psi, \ldots, x_n^*\psi\) are isomorphisms.
It is important to understand that the family $X_{\langle n \rangle}$ over $X_{\langle n \rangle}$ is in general ‘unstable’: some fibers $X_{\langle n + 1 \rangle}$ contain bridges (components which correspond to degree 2 non-root vertices).

**Remark 3.8.** In fact, the dual rooted tree of $X_{\langle n + 1 \rangle}$ is the tree associated to the chain $\{\rho \in L_{[n]} : s \in D_{\rho}\}$; cf. [Ul02, §2], with the phantom vertices not omitted. This follows by revisiting the induction used to prove Lemma 3.5.

4. The augmented wonderful variety as a small resolution

Let $C$ be any smooth complex projective curve, and $p \in C$ a closed point. It is completely irrelevant but convenient for language to assume that $C$ is not rational.

We will use the results of §3 for $X = C$, otherwise keeping all the notation in Theorem 3.7. By Theorem 3.7 (and the well-known flatness of $X$) completely irrelevant but convenient for language to assume that $C$ is not rational. This follows by revisiting the induction used to prove Lemma 3.5.

**Proposition 4.1.** There exists a prestable curve $Y \to C_{\langle n \rangle}$, and a rational contraction $\{ za21, Definition 2.1 \} f : C_{\langle n + 1 \rangle} \to Y$ such that for each $s \in C_{\langle n \rangle}(\mathbb{C})$, $f_s$ contracts all rational bridges in the nodal curve $C_{\langle n + 1 \rangle}$, and nothing else (in particular, no tails).

**Proof.** We’re claiming that it is possible to contract only the rational bridges, which is surely well-known. Let $\pi_{C_{\langle n + 1 \rangle}/C} : C_{\langle n + 1 \rangle} \to C$ be the projection and $\mathcal{L}'$ an ample line bundle on $C$, and let

$$\mathcal{L} = \omega_{C_{\langle n + 1 \rangle}/\langle n \rangle}(2x_1 + \cdots + 2x_n) \otimes \pi_{C_{\langle n + 1 \rangle}/C}^* \mathcal{L}'$$

where $x_i$ stands for the image $x_i(C_{\langle n \rangle})$. Note that the restriction of $\mathcal{L}$ to $C_{\langle n + 1 \rangle}$ has degree 0 on all rational bridges, and strictly positive degree on all other components, for any $s \in C_{\langle n \rangle}(\mathbb{C})$.

Then we can follow the standard procedure and define

$$Y = \text{Proj}_{C_{\langle n \rangle}} \bigoplus_{k \geq 0} \left( \pi_{C_{\langle n + 1 \rangle}/C_{\langle n \rangle}} \right)^* \mathcal{L}^\otimes k,$$

where $\pi_{C_{\langle n + 1 \rangle}/C_{\langle n \rangle}} : C_{\langle n + 1 \rangle} \to C_{\langle n \rangle}$ is the other projection, and all the desired properties of $Y$ follow from arguments completely analogous to those in the proof of [BM96, Proposition 3.10]. (The fact that we often contract multiple components in one fiber doesn’t change the argument in any significant way.)

Let $\pi_Y/C : Y \to C$ and $\pi_{C_{\langle n + 1 \rangle}/C} : C_{\langle n + 1 \rangle} \to C$ be the projections. By [Za21, Lemma 2.6] and Lemma 2.3, we have

$$f^*(\omega_{Y/C_{\langle n \rangle}} \otimes \pi_Y^* \omega_C^\vee) = \omega_{C_{\langle n + 1 \rangle}/C_{\langle n \rangle}} \otimes \pi_{C_{\langle n + 1 \rangle}/C}^* \omega_C^\vee = \omega_{C_{\langle n + 1 \rangle}/C_{\langle n \rangle}}.$$

Then [Za21, Proposition 2.11] provides a homomorphism

$$f_* \omega_{C_{\langle n + 1 \rangle}, C_{\langle n \rangle}}^\vee \to \omega_{Y/C_{\langle n \rangle}}^\vee \otimes \pi_Y^* \omega_C.$$

The image of $\psi$ under the homomorphism above is denoted by

$$\phi : \omega_{Y/C_{\langle n \rangle}} \to \pi_Y^* \omega_C.$$

Then, $\phi$ and $\psi$ agree outside the ‘exceptional locus’ [Za21, Lemma 2.4].
Comparing Definitions 1.2 and 3.1, it is clear that \( W_n \) is isomorphic to the fiber of \( C \langle \langle n \rangle \rangle \to \mathbb{A}^1 \times C^n \) over \((0, p, \ldots, p)\). We think of \( W_n \) as a subvariety of \( C \langle \langle n \rangle \rangle \) in this way.

The key point is that the restriction \( W_n \times C \langle \langle n \rangle \rangle C \langle \langle n+1 \rangle \rangle \) of the prestable curve to \( W_n \) has an irreducible component isomorphic to \( W_n \times C \), attached along \( W_n \times \{p\} \), and hence so does \( W_n \times C \langle \langle n \rangle \rangle Y \). To sketch an argument: for \( W_n \times C \langle \langle n \rangle \rangle C \langle \langle n+1 \rangle \rangle \), this feature of the curve over \( C \langle \langle n \rangle \rangle \) appears from the very first blowup in the sequence making up (8) when it is elementary to check, and remains true throughout; for \( W_n \times C \langle \langle n \rangle \rangle Y \), one checks that the composition

\[
W_n \times C \hookrightarrow W_n \times C \langle \langle n \rangle \rangle C \langle \langle n+1 \rangle \rangle \xrightarrow{\text{restriction of } f} W_n \times C \langle \langle n \rangle \rangle Y,
\]

where \( f \) is the rational contraction in Proposition 4.1, is the immersion of an irreducible component. Restricting the curve \( Y \) in Proposition 4.1, as well as \( \phi \), and the sections \( y_i : C \langle \langle n \rangle \rangle \to Y \) of \( Y \to C \langle \langle n \rangle \rangle \) induced by the sections \( x_i \), to the union \( Z \) of all other components of \( W_n \times C \langle \langle n \rangle \rangle Y \), we obtain a collection of data over \( W_n \) precisely as in [Za21, Theorem 1.5]. The fact that the vector field vanishes doubly along the \( \infty \)-section as required in [Za21, Theorem 1.5] (i.e. the ‘clutching’ section) follows from the fact that \( \psi \), and hence \( \phi \) vanishes on \( W_n \times C \). Indeed, since the base is reduced and of finite type, we may check the double vanishing on fibers over \( \mathbb{C} \)-points, where it follows from the well-known description of dualizing sheaves in terms of differentials and residues. (Note also that we need to choose an identification \( T_p C \simeq \mathbb{C} \) to obtain a genuine field from \( \phi \).) Thus, we’ve obtained a \( W_n \)-point of \( \overline{P}_n \), i.e. a morphism

\[
\gamma : W_n \to \overline{P}_n.
\]

It remains to check that \( \gamma \) is birational and small (Proposition 4.4), but not an isomorphism if \( n \geq 4 \) (Proposition 4.6).

For clarity, we isolate two elementary facts, one combinatorial, one geometric.

**Lemma 4.2.** A rooted tree is a spider if it consists of several (otherwise disjoint) chains with one common endpoint, with this endpoint connected directly to the root (which is not on any chain), and nothing else.

If the leveled rooted tree (with hypothetically allowed phantom vertices) associated with a chain in \( L_{\langle n \rangle} \) is a spider, then it has no phantom vertices; i.e. it has at most 3 levels (i.e. the spider has short legs).

**Proof.** Trivial. \( \square \)

**Lemma 4.3.** Let \( V \subset W \) be finite dimensional complex vector spaces. If \( E \) is the exceptional divisor of the blowup of \( \mathbb{P}(C \oplus W) \) at \( \mathbb{P}(0 \oplus V) \), then the complement of the proper transform of \( \mathbb{P}(0 \oplus W) \) in \( E \) is isomorphic to \( \mathbb{P}(0 \oplus V) \times W/V \).

**Proof.** For the purpose for making the proof very easy to read, we will only show that the fibers over points of \( \mathbb{P}(0 \oplus V) \) are naturally identified with \( W/V \), and leave the language adjustments for an official proof to the reader. Let \( L \subset V \) of dimension 1. The normal space of \( \mathbb{P}(0 \oplus V) \subset \mathbb{P}(C \oplus W) \) at the point \([0 \oplus L]\) is canonically identified with

\[
\frac{T_{[0 \oplus L]}\mathbb{P}(C \oplus W)}{T_{[0 \oplus L]}\mathbb{P}(0 \oplus V)} = \frac{L^\vee \otimes [(C \oplus W)/(0 \oplus L)]}{L^\vee \otimes [(0 \oplus V)/(0 \oplus L)]} = L^\vee \otimes (C \oplus W/V).
\]
Hence, the fiber of $E \to \mathbb{P}(0 \oplus V)$ over $[0 \oplus L]$ is $\mathbb{P}(C \oplus W/V)$. The intersection with the proper transform of $\mathbb{P}(0 \oplus W)$ is the hyperplane at infinity $\mathbb{P}(0 \oplus W/V)$, and the claim follows. □

**Proposition 4.4.** The morphism $\gamma$ is a small birational morphism (an isomorphism in codimension 2).

**Proof.** We say that a 1-marked genus 0 prestable curve over $\text{Spec } \mathbb{C}$ is at worst star-shaped, if either it is irreducible, or all its irreducible components which don’t contain the marking are tails. It follows from the existence of the natural stratification on $\mathcal{M}_{0,1}$ that there is an open $\mathcal{M}_{0,1}^* \hookrightarrow \mathcal{M}_{0,1}$ whose $\mathbb{C}$-points are precisely the at worst star-shaped curves. We obtain a morphism $\overline{P}_n \to \mathcal{M}_{0,1}$ by forgetting everything except the universal curve and its $\infty$-section, and define

$$\overline{P}_n^* = \mathcal{M}_{0,1}^* \times_{\mathcal{M}_{0,1}} \overline{P}_n \subseteq \overline{P}_n.$$  

Mildly confusingly, there are two maps $W_n \to \mathcal{M}_{0,1}$, one induced by $Y$ (or $Z$, to be more specific), and one by the restriction of $C\langle n+1 \rangle$ to $W_n$ (or its suitable component, to be more specific again), but the difference is small enough that

$$W_n^* = \mathcal{M}_{0,1}^* \times_{\mathcal{M}_{0,1}} W_n \subseteq W_n.$$  

is the same for both maps. This equivalence follows by straightforward case analysis, by Proposition 4.1 and Remark 3.8. The case analysis is precisely Lemma 4.3. In particular, $\gamma$ restricts to a well-defined, proper, and surjective $\gamma^*: W_n^* \to \overline{P}_n^*$, that will turn out to be an isomorphism.

Below, we will use $\widetilde{\cdot}$ to denote the dominant transform of $\cdot$, possibly under a sequence of blowups, which, in particular, means that the same symbol can mean different things (the dominant transforms on different spaces are denoted the same).

Most of the proof is a tedious (to write and read) but straightforward quest for hunting down the $\mathbb{C}$-points of $W_n^*$, and understanding their respective curves. For the former, we will see that $W_n^*$ has a stratification

$$(10) \quad \mathbb{A}^{n-1} \sqcup V^\circ[\infty]_p \sqcup \bigcup_{\rho \neq \perp, \top} V^\circ[\rho]_p \to W_n^*, $$

where

$$V[\infty] = \widetilde{\Delta}_{\perp} \cap \widetilde{\Delta}_{\top} \subset \text{Bl}_{\Delta_{\perp}}(\mathbb{A}^1 \times C^n),$$

$$V[\rho] = \widetilde{\Delta}_{\perp} \cap \widetilde{\Delta}_{\rho, \perp} \subset \text{Bl}_{\Delta_{\rho}}(\mathbb{A}^1 \times C^n), $$

the $\circ$ superscript denotes complement with respect to (the transforms of) all other diagonals, and the $p$ subscript denotes the fiber over $p \in \Delta_{\perp} \cong C$. All pieces of (10) are straightforward to construct, keeping track of $W_n$ (or rather, the fiber over $(0, p, \ldots, p) \in \mathbb{A}^1 \times C^n$) throughout the sequence of blowups needed to construct $C\langle n \rangle$. For $V^\circ[\rho]_p$, we may even blow up $\Delta_{\rho}$ immediately after $\Delta_{\perp}$, taking once again advantage of the commutativity in [Li09, Theorem 1.3]. It is straightforward that (10) is surjective on $\mathbb{C}$-points, and that $W_n \setminus W_n^*$ has codimension 2 in $W_n$.

Next, we need to understand the curve $C\langle n+1 \rangle_s$ over each closed point $s$ in the strata above. We will only state this explicitly in the case of $V^\circ[\rho]_p$, which is the most involved, and leave the other two simpler cases entirely to the reader. We may identify $V[\rho]_p$ with an open subset of the exceptional divisor of the blowup of

$$\mathbb{P}(0 \oplus H_p/T_pC) \subset \mathbb{P}(C \oplus T_pC^{\oplus n}/T_pC),$$
where $H_p = \{(v_1, \ldots, v_n) \in T_pC^{\oplus n} : v_i = v_j \text{ if } i \sim_p j\} \subset T_pC^{\oplus n}$. Then the setup is analogous to that in Lemma 4.3, and we obtain an identification

\[(11) \quad V[\rho]_p \simeq \mathbb{P}(0 \oplus H_p/TpC) \times (T_pC^{\oplus n}/H_p).\]

Under (11), $V^\circ[\rho]_p$ matches with the preimage of the open subset of $\mathbb{P}(0 \oplus H_p/TpC)$ corresponding to $(v_1, \ldots, v_n) \in H_p$ such that $v_i = v_j$ if and only if $i \sim_p j$.

**Claim 4.5.** We write $\Delta'_{\rho,B}$ for $\Delta'_\eta$ if $\eta$ and $(\rho, B)$ correspond under (5). Let

$$Q = \text{Bl}_{\bigcup_{B \in B(\rho)} \Delta_{\rho,a}} \Delta_s \times_C \text{Bl}_{\Delta_s} \times_C (\mathbb{A}^1 \times C^{n+1}) \to \text{Bl}_{\Delta_s} \text{Bl}_{\Delta_s} (\mathbb{A}^1 \times C^n)$$

(the disjointness of the blowup centers at the first blowup is a version of Remark 3.2). Let $Z_1, \ldots, Z_n \subset \mathbb{A}^1 \times C^{n+1} = \mathbb{A}^1 \times C^n$ be the graphs of the projections to the factors isomorphic to $C$, and $\tilde{Z}_i \subset Q$ their proper transforms.

Let $s \in V^\circ[\rho](C)$. Then, the fiber $Q_s$ is a nodal curve, which looks as follows:

- a component isomorphic to $C$;
- a sole non-root non-tail component, which intersects all other components (the intersection with $C$ is $p \in C$); and
- $|B(\rho)|$ tails, and the tail corresponding to $B \in B(\rho)$ intersects $\tilde{Z}_i$ at a point denoted $z_i(s)$, if $i \in B$.

Moreover, the continuous moduli are determined as follows:

- the point in $\mathbb{P}(0 \oplus H_p/TpC)$ cf. (11) determines the position of the nodes on the middle component (this is essentially a ‘screen’ [FM94]), up to automorphisms – these points on the middle component are distinct as they should, by the sentence after (11);
- the point in $T_pC^{\oplus n}/H_p$ cf. (11) determines the position of the markings $z_i(s)$ on the tails, up to translation.

**Proof.** Straightforward calculation, skipped. \(\square\)

In the situation of Claim 4.5, with some care, we convince ourselves that

\[(12) \quad Q_s = C(\langle n + 1 \rangle)_s = Y_s,
\]

including their respective natural markings. By Claim 4.5, (12), and their unstated analogues, $\gamma^*(C)$ is injective. Thus $\gamma^*$ is quasi-finite, hence finite, so it must be an isomorphism according to the well-known normality criterion, since $\overline{P}_n$ is normal (by [Za21, Proposition 5.6], $\overline{P}_n$ is normal). \(\square\)

**Proposition 4.6.** The map $\gamma$ is not an isomorphism if $n \geq 4$.

**Proof.** Let $k = \lfloor \log_2(n - 1) \rfloor + 1$. For $j = 0, 1, \ldots, k$, let $\rho_j \in L_{[n]}$ be the partition of $[n]$ into the fibers of the map $[n] \to \mathbb{Z}$, $i \mapsto \lfloor (i - 1)2^{-j} \rfloor$. The chain $\{\rho_0, \ldots, \rho_k\}$ corresponds to a leveled rooted tree $B$. Note that $B$ is binary, with one marking attached to each leaf. Therefore, there exists a unique point $b \in \overline{P}_n(\mathbb{C})$ where the corresponding curve is described by $B$. Let $R = W_n \cap D_{\rho_0} \cap \cdots \cap D_{\rho_{k-1}}$. Note that $\rho_k = \perp$ and $R$ is not empty by Proposition 3.3 (of course, to be extremely rigorous, Proposition 3.3 only gives that there exists $p \in C(\mathbb{C})$ for which this is true, which actually suffices, though obviously all $p$ behave the same). Then

$$\dim R \geq \dim W_n - k = n - 2 - \lfloor \log_2(n - 1) \rfloor \geq 1$$

for $n \geq 4$, yet $\gamma(R) = \{b\}$, completing the proof. \(\square\)
Propositions 4.4 and 4.6 complete the proof of Theorem 1.3. Moreover, a small birational morphism has smooth source and target only if it is an isomorphism, so we are now sure that $\overline{P}_n$ is singular for all $n \neq 1, 2, 3$, and it makes sense to call $\gamma$ a resolution.

A comment to conclude the paper: $\overline{M}_{0,n}$, for instance, can be obtained both by explicit constructions (e.g. [Ke92]) and by moduli theoretic approaches (e.g. [Kn83]). It may be tempting to believe that the equivalence of the two points of view is inherent, just too tedious to fully grasp. In our situation, $W_n$ is unequivocally in one boat and $\overline{P}_n$ in the other, which suggests that the equivalence of these points of view is not just tedious, it’s simply not always true.

5. Appendix: $\overline{P}_n$ is very singular for large $n$

The combinatorial formulas in this appendix are not needed in the main body of the paper, so we won’t provide detailed proofs. The end goal is to convince ourselves that the singularities of $\overline{P}_n$ are quite complex, even if manageable geometrically according to Theorem 1.3 and [Za21, Theorem 1.6].

**Theorem 5.1.** Let $p_1(t), p_2(t), \ldots \in \mathbb{Z}[t]$, with $\deg p_n = n - 1$, be the sequence of polynomials such that $y = \sum_{n=1}^{\infty} p_n(t) \frac{t^n}{n!}$ satisfies

$$t + t^2y - (1 + y)^t = (t - 1)e^{tx}.$$  

Then $[\overline{P}_n] = p_n([A^1])$ in the Grothendieck ring of varieties $K(\overline{\text{Var}}_{\mathbb{C}})$.

**Sketch of proof.** It can be shown that, for any partition $\rho \in L_{[n]}$, $\rho \neq \perp$, there exists a ‘clutching’ morphism

$$\tau_\rho : M_{0,|B(\rho)|+1} \times \prod_{B \in B(\rho)} \overline{P}_{|B|} \to \overline{P}_n,$$

that $\tau_\rho$ is an immersion, and that the boundary of $\overline{P}_n$ decomposes precisely into the union of the images of these immersions. (We will not prove these assertions, but merely note that one ad hoc way to deal with the immersion claim is first to check normality of the image, and then argue on the level of $\mathbb{C}$-points.) Then,

$$[\overline{P}_n] = [A^{n-1}] + \sum_{\rho \in L_{[n]} \setminus \{\perp\}} [M_{0,|B(\rho)|+1}] \prod_{B \in B(\rho)} [\overline{P}_{|B|}],$$

for all $n \geq 1$. Recall that $[M_{0,\ell}] = ([A^1] - 2)^{\ell-3}$. (More generally, if $Y$ is a variety and $F(Y, k)$ is a configuration space, then $[F(Y, k)] = [Y]^k$ is well-known. Take $Y = \mathbb{P}^1 \setminus \{0, 1, \infty\}$.) Thus, it suffices to consider the sequence $p_1(t), p_2(t), \ldots \in \mathbb{Z}[t]$ (by abuse of notation) given by $p_1(t) = 1$ and

$$p_n(t) = t^{n-1} + \sum_{\rho \in L_{[n]} \setminus \{\perp\}} (t - 2)^{|B(\rho)|-2} \prod_{B \in B(\rho)} p_{|B|}(t),$$

or equivalently,

$$p_n(t) = t^{n-1} + \sum_{m \geq 2} \frac{(t - 2)^{m-2}}{m!} \sum_{k_1 + \cdots + k_m = n} \binom{n}{k_1, \ldots, k_m} p_{k_1}(t) \cdots p_{k_m}(t),$$
and show that $y = \sum_{n=1}^{\infty} p_n(t) \frac{x^n}{n!}$ satisfies (13). Indeed, we have

$$y = \sum_{n \geq 1} \left( \frac{t^{n-1}}{n!} + \frac{(t - 2)^{m-2}}{m!} \sum_{k_1, \ldots, k_m = n} \frac{p_{k_1}(t) \cdots p_{k_m}(t)}{k_1! \cdots k_m!} \right) x^n$$

$$= \sum_{n \geq 1} \frac{t^{n-1} x^n}{n!} + \sum_{m \geq 2} \left( \frac{(t - 2)^{m-2}}{m!} \sum_{k_1, \ldots, k_m} \frac{p_{k_1}(t) \cdots p_{k_m}(t)}{k_1! \cdots k_m!} \right) x^n$$

$$= e^{tx} \left( 1 + \sum_{m \geq 2} \frac{(t - 2)^{m-2}}{m!} y^m \right) = \frac{e^{tx} - 1}{t} + \sum_{m \geq 2} \frac{(t - 2)^{m-2}}{m!} \frac{y^m}{t(t - 1)},$$

and (13) is obtained after simplifying. \( \square \)

The sequence $p_1(t), p_2(t), \ldots$ in Theorem 5.1 starts

(14) $1, t + 1, t^2 + 4t + 1, t^3 + 14t^2 + 11t + 1, \ldots.$

**Example 5.2** (The case $n = 4$). It is easy to compute

$$[W_4] = [A^1]^3 + 14[A^1]^2 + 14[A^1] + 1 \in K(\text{Var}_C).$$

This is consistent with $p_4(t) = t^3 + 14t^2 + 11t + 1$, cf. (14), Theorem 1.3, and the fact that $\mathcal{P}_4$ has 3 ordinary double points.

**Remark 5.3.** We expect that the generalization of $\mathcal{P}_n$ to $d$-dimensional affine space announced in Remark 1.4 satisfies a property similar to that in Theorem 5.1, with (13) replaced by $t + t^{d+1}y - (1 + y)^t = (t - 1)e^{tx}$.

By Theorem 5.1, $\sum_{n=1}^{\infty} \chi(\mathcal{P}_n) \frac{x^n}{n!}$ is the inverse function $h^{-1}(x)$ of

$$h(x) = \log(1 + 2x - (x + 1) \log(x + 1)).$$

For reference, the first values of $\chi(\mathcal{P}_n)$ are $1, 2, 6, 27, 170, 1390, 13979, \ldots$. Since $h'(e - 1) = 0$, $h^{-1}(x)$ is not analytic in any neighbourhood of $h(e - 1) = \ln(e - 1)$ (it’s not even defined, although it doesn’t matter). Thus the radius of convergence of the Maclaurin series of $h^{-1}(x)$ cannot exceed $\ln(e - 1)$, that is,

(15) $\limsup_{n \to \infty} \sqrt[n]{\frac{\chi(\mathcal{P}_n)}{n!}} \geq \frac{1}{\ln(e - 1)} \approx 1.8473.$

On the other hand, the Euler characteristic of the Losev-Manin space is $n!$ [LM00], so, in light of the fact that $\mathcal{I}_n$ degenerates to $\mathcal{P}_n$ [Za21, Theorem 1.6], (15) suggests that the singularities of $\mathcal{P}_n$ are very complicated for $n \gg 0$. (Since ‘everything’ in this story, even strata, tends to be rational, the Euler characteristic is not as awful a measure of complexity as it usually is.) This also gives some evidence that the small resolution in Theorem 1.3 is quite complex.

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