Monomial integrals on the classical groups

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Abstract

This paper presents a powerful method to integrate general monomials on the classical groups with respect to their invariant (Haar) measure. The method has first been applied to the orthogonal group in [J. Math. Phys. 43, 3342 (2002)], and is here used to obtain similar integration formulas for the unitary and the unitary symplectic group. The integration formulas turn out to be of similar form. They are all recursive, where the recursion parameter is the number of column (row) vectors from which the elements in the monomial are taken. This is an important difference to other integration methods. The integration formulas are easily implemented in a computer algebra environment, which allows to obtain analytical expressions very efficiently. Those expressions contain the matrix dimension as a free parameter.

1 Introduction

With the classical groups, we mean the orthogonal group $O(d)$, the unitary group $U(d)$, and the unitary symplectic group $Sp(2d)$ [1]. They all possess a unique invariant measure, the “Haar” measure [2], which is the integral measure commonly used.

Integration formulas for the classical groups are of interest in various fields of mathematical physics. A number of different classes of integrals have been studied, among those: generating functions, such as the Harish-Chandra-Izykson-Zuber integral [3, 4], and monomial integrals, which are the concern of the present work. In some cases, the corresponding integral can be solved by character expansion due to Balantekin [5] and Balantekin and Cassak [6]. Integration formulas for certain simple monomials have been developed by nuclear physicists [7, 8]. This work was motivated by the fact that statistical methods based on the classical groups are very successful in describing certain aspects of nuclear reactions (see [9, 10] and references therein). Later, Mello and Seligman devised an algebraic method to compute low order monomial integrals on $U(d)$ [11], and Samuel solved the problem in full generality [12]. The result was an explicite formula for arbitrary monomial integrals over $U(d)$. The method in [12] is based on the representation theory of $U(d)$, and we call it the group theoretical method. Some 25 years ago, it has become clear that group integrals on the classical groups play an important role in mesoscopic transport [13, 14, 15], in quantum chaos [16, 17], and more recently in some aspects of quantum information and decoherence, e.g. [18, 19, 20].

These applications lead to a renewed interest in efficient methods for the analytical calculation of monomial integrals. An interesting unconventional approach has been devised by Prosen et al. [21], and very recently, an improved invariant method has been developed for monomials in $U(d)$ [22, 23], and also in $O(d)$ [24]. Finally, Collins and Śniady presented a group theoretical approach, which allows to compute monomial integrals over all three classical groups [25].

In [26], a further method has been developed, which is very different from the previous ones; the column vector method as it might be called. It lead to a new type of integration formula for monomial integrals on $O(d)$.

This method is easily implemented in a computer algebra language. It allows to compute arbitrary monomial integrals very efficiently, where the result will always be a rational function in the matrix dimension of the group. The purpose of the present paper is to apply that method to $U(d)$ and $Sp(2d)$. In section 2 the matrix representations of the classical groups are introduced, and the notation for the different group integrals used is defined. The general idea of the column vector method, as well as the result for $O(d)$ is reviewed in section 3. In the sections 4 and 5 the integration formulas for the unitary and the unitary symplectic group are derived. A summary is given in section 6.

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2 General considerations

In this section, we introduce the fundamental matrix representations of the groups under consideration. Then, we define the integration measures and specify some notational conventions related to the integrals to be calculated.

2.1 The orthogonal group $O(d)$

An element $w \in O(d)$ is a $d$ dimensional square matrix with real entries $(w_{ij})$, where $1 \leq i, j \leq d$. In addition, $w$ fulfills the following orthogonality relations:

$$ w w^T = w^T w = 1 \Leftrightarrow \forall 1 \leq i, j \leq d : \sum_{k=1}^{N} w_{ik} w_{jk} = \delta_{ij} . $$ (1)

The most general monomial of matrix elements of $w$ is denoted by $\mathcal{M}_1(w)$. The subscript 1 will be replaced by $\beta = 2(4)$ in the unitary (unitary symplectic) case. Here, $\beta$ is the symmetry parameter of the respective group.

$$ \mathcal{M}_1(w) = \prod_{r=1}^{p} w_{I_r, J_r} = \prod_{i,j=1}^{d} w_{ij}^{M_{ij}} \quad M_{ij} = \sum_{r=1}^{q} \delta_{iI_r} \delta_{jJ_r} , $$ (2)

where $q$ is the order of the monomial, and $I$ and $J$ are multi-indeces of dimension $q$. The notation with the multi-indeces $I, J$ is the most common one, but in the present approach the matrix notation is more convenient.

For the integration on $O(d)$, the normalized Haar measure is used. For the classical groups, in general, it will be denoted by $\sigma_{\beta}$, where $\beta$ is the symmetry parameter introduced above. In the present case, the integral of an arbitrary monomial is denoted by

$$ \langle M \rangle = \int d\sigma_{1}(w) \mathcal{M}_1(w) . $$ (3)

The normalization is such that $\langle o \rangle = 1$, where $o$ is the matrix with zero elements everywhere.

The column vector method has been applied to the orthogonal group, first [26]. The results obtained therein are reviewed in section 3. It allows us to introduce the general idea of the method, as well as a number of notational conventions. The latter will be useful for the unitary and unitary symplectic case, also.

2.2 The unitary group $U(d)$

An element $w \in U(d)$ is a $d$ dimensional square matrix with complex entries $(w_{ij})$, where $1 \leq i, j \leq d$. In addition, $w$ fulfills the following orthogonality conditions:

$$ w w^\dagger = w^\dagger w = 1 \Leftrightarrow \forall 1 \leq i, j \leq d : \sum_{k=1}^{d} w_{ik} w_{jk}^* = \delta_{ij} . $$ (4)

The most general monomial on the unitary group depends on matrix elements of $w$ and $w^\dagger$. It will be denoted by $\mathcal{M}_2(w)$:

$$ \mathcal{M}_2(w) = \prod_{r=1}^{p} w_{I_r, J_r} \prod_{s=1}^{q} w_{I'_s, J'_s} = \prod_{i,j=1}^{d} w_{ij}^{M_{ij}} (w_{ij}^*)^{N_{ij}} \quad M_{ij} = \sum_{r=1}^{p} \delta_{iI_r} \delta_{jJ_r} \quad N_{ij} = \sum_{s=1}^{q} \delta_{iI'_s} \delta_{jJ'_s} , $$ (5)

where $I, J$ and $I', J'$ are multi-indeces of dimension $p$ and $q$, respectively. Again, while the notation with the multi-indeces $I, J$ and $I', J'$ is the most common one, the matrix notation with $M$ and $N$ is more convenient for our purpose.

For $U(d)$, the normalized Haar measure is $\sigma_{2}(w)$, while the monomial integral is denoted by

$$ \langle N|M \rangle = \int d\sigma_{2}(w) \mathcal{M}_2(w) . $$ (6)

The integration formula for the unitary group is derived in section 4.
2.3 The unitary symplectic group $Sp(2d)$

The unitary symplectic group may be defined as the subgroup of $U(2d)$ which is invariant under the antisymmetric bilinear form

$$Z' = \begin{pmatrix} o & 1d \\ -1d & o \end{pmatrix} : \quad Sp(2d) = \{ w \in U(2d) \mid w^T Z' w = Z' \} .$$

(7)

In order to fulfill this invariance condition, the matrices $w$ must be of the form

$$w = \begin{pmatrix} z^* & w \\ -w^* & z \end{pmatrix} ,$$

(8)

whith complex $d$-dimensional square matrices $w$ and $z$. For such matrices, the unitarity conditions become:

$$\langle \bar{w}_\mu \bar{w}_\nu \rangle + \langle \bar{z}_\mu \bar{z}_\nu \rangle = \delta_{\mu,\nu} \quad \langle \bar{z}_\mu \bar{w}_\nu \rangle - \langle \bar{w}_\mu \bar{z}_\nu \rangle = 0 ,$$

(9)

where $\bar{w}_\mu$ and $\bar{z}_\mu$ denote the respective column vectors of the matrices $w$ and $z$. This parametrization will be used to perform the integration over $Sp(2d)$. At this point we may note that

$$\forall w \in Sp(2d) : \quad w^{-1} = w^\dagger = \begin{pmatrix} z^T & -w^T \\ w^* & z^\dagger \end{pmatrix} \quad \text{such that} \quad w^\dagger w = ww^\dagger = \begin{pmatrix} 1d & o \\ o & 1d \end{pmatrix} ,$$

(10)

where $w^\dagger$ is again in $Sp(2d)$. The most general monomial on the unitary symplectic group contains matrix elements from four matrices: $w$, $w^\dagger$, $z$, and $z^\dagger$. It is denoted by:

$$\mathcal{M}_4(w,z) = \mathcal{M}_4(w,z) = \prod_{i,j=1}^{d} w^M_{ij} z^N_{ij} \langle w^*_{ij} \rangle \langle z^*_{ij} \rangle N^*_{ij} ,$$

(11)

where $M, M', N, N'$ are $d$-dimensional square matrices with non-negative integer entries. If we collect these matrices in an appropriate way in the $2 \times 2$-matrix, we may equally well write:

$$M = \begin{pmatrix} N' & M \\ N & M' \end{pmatrix} : \quad \mathcal{M}_4(w,z) = (-1)^N \prod_{i,j=1}^{2d} w^M_{ij} , \quad \mathcal{N} = \sum_{i,j=1}^{d} N_{ij} .$$

(12)

For $Sp(2d)$, we denote the normalized Haar measure by $\sigma_4(w,z)$. The monomial integral is denoted by

$$\langle M \rangle = \int d\sigma_4(w,z) \mathcal{M}_4(w) .$$

(13)

3 Integration over $O(d)$

For any of the classical groups, the Haar measure is invariant under left- and right-multiplication with a fixed group element $u$, and under taking the inverse. For $O(d)$ these properties read:

$$\int d\sigma_1(w) f(w) = \int d\sigma_1(w) f(uw) = \int d\sigma_1(w) f(wu) = \int d\sigma_1(w) f(w^T) ,$$

(14)

where $f(w)$ is an arbitrary integrable function of the matrix entries of $w$. For a monomial integral $\langle M \rangle$, as defined in equation (3), those operations translate into corresponding operations on the integer matrix $M$. They can be used to bring the monomial integral into a more convenient form. In other words, we use transposition and/or column permutations to collect all non-zero elements of $M$ in the first $R \leq d$ columns, minimizing $R$.

The general idea is to write the integral in terms of the full $d^2$-dimensional Euclidean space of all matrix elements of $w$, and to implement the restriction to the group manifold by appropriately chosen $\delta$-functions. Since the monomial $\mathcal{M}_1(w)$ contains matrix elements from only the first $R$ column vectors of $w$, the integration may be restricted to those, as will be shown with the help of the final result, below. Meanwhile, one may as well assume that $R = d$. Denoting the column vectors by $\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_R$, one may write:

$$\langle M \rangle = \int d\sigma_1(w) \prod_{i,\xi=1}^{d, R} w^M_{i\xi} = \frac{\mathcal{N}_1^{(R)}(M)}{\mathcal{N}_1^{(R)}(o)} ,$$

(15)

$$\mathcal{N}_1^{(R)}(M) = \prod_{\xi=1}^{R} \left\{ \int d\Omega(\bar{w}_\xi) \prod_{i=1}^{d} w^M_{i\xi} \right\} \prod_{\mu < \nu} \delta(\langle \bar{w}_\mu | \bar{w}_\nu \rangle) \int d\Omega(\bar{w}_\xi) \times \prod_{i=1}^{d} \left\{ \int \delta(||\bar{w}_\xi||^2 - 1) \right\} .$$

(16)
The matrix \( \mathbf{o} \) in equation (15) is a \( d \)-dimensional square matrix, which contains only zeros. The subscript in the symbol \( \mathcal{N}_1^{(R)} \) indicates the symmetry parameter \( \beta \), which is equal to one in the orthogonal case. In equation (16) the integration region is the product space of \( R \) unit spheres with constant measure \( d\Omega(\vec{w}_\xi) \). The orthogonality and the normalization is implemented with the help of appropriately chosen \( \delta \)-functions.

To convince oneself that the right hand side of equation (15) really yields the Haar measure, it is sufficient to realize that for \( R = d \) the measure used in equation (16) is indeed invariant under transposition as well as left- and right-multiplication with any other fixed group element. Similar arguments also apply in the case of the unitary and the unitary symplectic group.

To evaluate the integral in equation (16), we integrate over the last column vector \( \vec{w}_R \). This integration can be done in closed form, and the result is a linear combination of monomials in the matrix elements of the first \( R - 1 \) column vectors. This means, the resulting expression will be of the following form:

\[
\mathcal{N}_1^{(R)}(M) = \sum_K C(K) \mathcal{N}_1^{(R-1)}(M + K) \quad \mathcal{N}_1^{(R)}(o) = C(0) \mathcal{N}_1^{(R-1)}(o),
\]

where \( K \) is an integer matrix with non-zero elements in the first \( R - 1 \) columns only. The sum \( \sum_K \) runs over a finite number of such matrices, and for \( \mathcal{N}_1^{(R-1)}(M + K) \) only the first \( R - 1 \) columns (of \( M \) and of \( K \)) need to be taken into account. The second relation in equation (17) takes care of the proper normalization of the measure. The ratio between both equations, yields the desired recurrence relation:

\[
\langle M^{(R)} \rangle = \sum_K \hat{C}(K) \left( \langle M^{(R-1)} + K \rangle \right).
\]

We will eventually use the subscript of the form \( M^{(R)} \) to indicate that only the first \( R \) column vectors of the respective matrix are taken into account. An integral, such as the one in equation (15) will be called "\( R \)-vector integral". With only minor changes, this terminology is also applied in the unitary and unitary symplectic case.

**The one-vector integral** The starting point for the recurrence relation is the one-vector integral. It has first been computed by Ullah [8]. In our notation, it reads:

\[
\langle \vec{m} \rangle = \int d\sigma_1(\vec{w}) \prod_{i=1}^d y_i^m = \left( \frac{d}{2} \right)^{d-1} \prod_{i=1}^d \left( \frac{1}{2} \right)_{m_i/2},
\]

where \( (z)_a = \Gamma(z + a)/\Gamma(z) \) is the Pochhammer symbol, and \( \vec{m} = \sum_{i=1}^d m_i \). The integral \( \langle \vec{m} \rangle \) is non-zero only if all components \( m_i \) are even numbers.

**The \( R \)-vector integral** The recurrence relation for the \( R \)-vector case has been obtained in [26]. It relates the desired \( R \)-vector integral to a linear combination of \( (R-1) \)-vector integrals. To obtain an explicit result, this recurrence relation must be continued until \( R = 1 \) is reached, where the one-vector result can be inserted.

\[
\langle M \rangle = \chi_{\vec{m} \text{ even}} \sum_K \left( \frac{\vec{m}}{\vec{\mathbf{k}}} \right) \left( \vec{\mathbf{k}} \right) B \left( \frac{\vec{m}}{2}, \frac{\vec{\mathbf{k}}}{2}, \frac{d}{2}, \frac{R-1}{2} \right) \sum_K (\vec{m} - \vec{\mathbf{k}} \mid K) \langle \vec{\mathbf{k}} \rangle \langle M^{(R-1)} + K \rangle,
\]

where

\[
B(a, b; z_1, z_2) = (-)^{a-b} \frac{(z_1)_{a+b} (z_1)_{a-b}}{(z_1 - z_2)_{a-b}}.
\]

In this formula, \( \vec{m} \) is the \( R \)-th column vector of \( M \) and \( \vec{m} \) is the sum of all its entries. The prefactor \( \chi_{\vec{m} \text{ even}} \) is one if \( \vec{m} \) is even, otherwise it is zero. The “binomial” of the integer vectors \( \vec{m} \) and \( \vec{\mathbf{k}} \) is just a short hand for the product of binomials of their respective components:

\[
\left( \frac{\vec{m}}{\vec{\mathbf{k}}} \right) = \prod_{i=1}^d \frac{m_i}{\kappa_i}. \quad (22)
\]

The symbol, \( \langle \vec{\mathbf{k}} \rangle \), denotes a one-vector average, as defined in equation (19). Due to those two quantities, the summation is restricted to such \( \vec{\mathbf{k}} \) for which \( v_i : 0 \leq \kappa_i \leq m_i, \kappa_i \text{ even} \). Analogous to \( \vec{m} \), \( \vec{\mathbf{k}} \) denotes the sum of all components of the integer vector \( \vec{\mathbf{k}} \). The second sum in equation (20) runs over the \( d \)-dimensional integer matrix \( K \) with non-negative entries in the first \( R - 1 \) columns, only. The following “vector-multinomial” is again a short hand, which reads:

\[
\langle \vec{m} - \vec{\mathbf{k}} \mid K \rangle = \prod_{i=1}^d (m_i - \kappa_i | K_i1, \ldots, K_{i,R-1} ), \quad (23)
\]
where the row sums in $K$ must be equal to $m_i - \kappa_i$. The vector $k_{cs}$ denotes the vector of column-sums of $K$. It has at most $R - 1$ non-zero components. Since our formula contains the one-vector average of $k_{cs}$, the summation over $K$ should be restricted to those $K$, which have only even column sums. The last term is a ($R-1$)-vector average, where the matrix entries of $K$ have been added to the matrix entries in the first $R-1$ columns of $M$.

As it turns out, the function $B(a, b; z_1, z_2)$ appears again in the integration formulas for $U(d)$ and $Sp(2d)$, although with slightly different arguments.

**Zero column vectors** With the recurrence formula (20) at hand, it is straightforward to prove that the restriction of the integration in equation (16) to the first $R$ column vectors of $w$ is valid. This follows from the fact that the recurrence relation yields $\langle M(1:R-1) \rangle$ if all components of the $R$th column vector of $M$ are zero.

**Vanishing integrals** It is well known (e.g. reference [10]), that the integral $\langle M \rangle$ vanishes if any column sum or row sum of $M$ gives an odd number. This can be directly seen from equation (20), which requires $\bar{m}$ to be even in order to yield a non-zero result. Due to the invariance properties of the Haar measure, this statement applies to any column- or row-vector.

### 4 Integration over $U(d)$

For the unitary group, we denote the normalized Haar measure by $\sigma_2$. It has analogous invariance properties as $\sigma_1$ of the orthogonal group, discussed in the previous section. In the present case, one finds for a fixed group element $u \in U(d)$:

$$\int d\sigma_2(w) f(w, w^\dagger) = \int d\sigma_2(w) f(u w, u^\dagger w^\dagger) = \int d\sigma_2(w) f(w u, u^\dagger w^\dagger) = \int d\sigma_2(w) f(w^\dagger, w) ,$$

(24)

where $f(w, z)$ is an analytic integrable function of the matrix entries of $w$ and $z$. These operations translate into corresponding operations on the integer matrices $M, N$, which leave the monomial integral $\langle N|M \rangle$, equation (6), invariant. In particular: (i) simultaneous column permutations: $\langle N|M \rangle = \langle N\pi|M\pi \rangle$, (ii) simultaneous row permutations: $\langle N|M \rangle = \langle \pi N|\pi M \rangle$, where $\pi$ is an arbitrary permutation matrix, and (iii) conjugate transposition: $\langle N|M \rangle = \langle MT|NT \rangle$. Due to the invariance under column permutations, we may assume without loss of generality that the non-zero elements of $M$ and $N$ are all restricted to the first $R \leq d$ columns. In other words, this means that the monomial $M_2(w)$ contains no matrix elements from column vectors $\vec{w}_\mu$ with $\mu > R$.

Analogous to the case of the orthogonal group, we again write the integral in terms of the flat Euclidean space of all complex matrix elements of $w \in U(d)$. Then, we implement the restriction to the group manifold by appropriately chosen $\delta$-functions:

$$\langle N|M \rangle = \int d\sigma_2(w) \prod_{i,\xi = 1}^R w^{M_{i\xi}}_w (w^*_{i\xi})^{N_{i\xi}} = \frac{N_2^{(R)}(M, N)}{N_2^2(\alpha, \alpha)} ,$$

(25)

$$N_2^{(R)}(M, N) = \prod_{\xi = 1}^R \left\{ \int d\Omega_2(\vec{w}_\xi) \prod_{i = 1}^d w^{M_{i\xi}}_w (w^*_{i\xi})^{N_{i\xi}} \right\} \prod_{\mu < \nu}^R \delta^{(2)}(\langle \vec{w}_\mu | \vec{w}_\nu \rangle) .$$

(26)

The subscript 2 in the symbol $N_2^{(R)}$ stands for the unitary case $\beta = 2$. The first $R$ column vectors of the unitary matrix $w$ are denoted by $\vec{w}_1, \ldots, \vec{w}_R$. The integration region in equation (26) is the product space of $R$ unit spheres with constant measure $d\Omega_2(\vec{w}_\xi)$:

$$\int d\Omega_2(\vec{w}_\xi) \propto \prod_{i = 1}^d \left\{ \int d^2(\vec{w}_{\xi i}) \right\} \delta(||\vec{w}_{\xi i}||^2 - 1) ,$$

(27)

where we define the flat measure on the complex plane via $z \in \mathbb{C} : d^2(z) = d(\text{Re} z) d(\text{Im} z)$. The $\delta$-functions in the equations (26) and (27) implement the orthogonality conditions and the normalization. Note that the $\delta$-function in equation (26) really is the product of two $\delta$-functions: $\delta^{(2)}(z) = \delta(\text{Re} z) \delta(\text{Im} z)$.

Since we assumed that the monomial $M_2(w)$ contains matrix elements from only the first $R$ column vectors, we may restrict the integration in (26) to those vectors. As in the orthogonal case, this does not affect the result of the monomial integral, as will be shown below with the help of the final result. Meanwhile, one may as well assume that $R = d$. Note that for $R = d$ the integration measure in equation (26) is indeed invariant under the transformations performed in (24). This guarantees that equation (52) really yields the Haar measure.
4.1 The one-vector formula

In the one-vector case, the matrices $M$ and $N$ can be replaced by the $d$-dimensional vectors $\vec{m}$ and $\vec{n}$. In that case, there are no orthogonality conditions to obey. We may write:

$$\langle \vec{n} | \vec{m} \rangle = \int d\sigma_2(w) \prod_{i=1}^{d} w_i^{m_i} (w_i^*)^{n_i} = N_2^{(1)}(\vec{m}, \vec{n}) \quad N_2^{(1)}(\vec{m}, \vec{n}) = \int d\Omega_2(w) \prod_{i=1}^{d} w_i^{m_i} (w_i^*)^{n_i}. \quad (28)$$

As suggested in equation (27), we integrate over the full space $\mathbb{R}^d$, while the normalization is implemented with the help of a $\delta$-function. This introduces an integration constant, denoted by $C_2(d, 1)$:

$$N_2^{(1)}(\vec{m}, \vec{n}) = C_2(d, 1) \prod_{i=1}^{d} \left\{ \int d^2(w_i) w_i^{m_i} (w_i^*)^{n_i} \right\} \delta(\|\vec{w}\|^2 - 1). \quad (29)$$

The $\delta$-function is removed as follows: Setting $w_i = u_i/\sqrt{r}$ we get:

$$N_2^{(1)}(\vec{m}, \vec{n}) r^{d+(\vec{m} + \vec{n})/2 - 1} = C_2(d, 1) \prod_{i=1}^{d} \left\{ \int d^2(u_i) u_i^{m_i} (u_i^*)^{n_i} \right\} \delta(\sum_i |u_i|^2 - r) \quad (30)$$

where $\vec{m} = \sum_{i=1}^{d} m_i$ and $\vec{n} = \sum_{i=1}^{d} n_i$. Multiplying both sides of the equation with $e^{-r}$ and integrating on $r$ from 0 to $\infty$, the $\delta$-function disappears:

$$N_2^{(1)}(\vec{m}, \vec{n}) \Gamma(d + \frac{\vec{m} + \vec{n}}{2}) = C_2(d, 1) \prod_{i=1}^{d} \left\{ \int d^2(u_i) u_i^{m_i} (u_i^*)^{n_i} e^{-|u_i|^2} \right\} = \prod_{i=1}^{d} f(m_i, n_i, 0). \quad (31)$$

The Gaussian integral $f(m, n, \alpha)$ is defined in equation (92) in the appendix. The general result for that integral is given in equation (100). For $\alpha = 0$, it evaluates to

$$f(m, n, 0) = \pi \delta_{m, n} n! \quad (32)$$

From this, it follows:

$$N_2^{(1)}(\vec{m}, \vec{n}) \Gamma(d + \frac{\vec{m} + \vec{n}}{2}) = C_2(d, 1) \delta_{\vec{m}, \vec{n}} \prod_{i=1}^{d} \{ \pi m_i! \} \quad N_2^{(1)}(\vec{m}, \vec{n}) = C_2(d, 1) \pi^d, \quad (33)$$

where $\delta_{\vec{m}, \vec{n}}$ denotes the product of Kronecker deltas between the components of $\vec{m}$ and $\vec{n}$. This finally leads to

$$\langle \vec{n} | \vec{m} \rangle = \delta_{\vec{m}, \vec{n}} (d + \vec{m} + \vec{n})^{-1} \prod_{i=1}^{d} m_i! \quad (34)$$

4.2 The $R$-vector formula

To solve the $R$-vector integral, we start from equation (26) and separate the integration over the first $R - 1$ column vectors from the integration over the last column vector $\vec{w}_R$. Let us denote the $R$-th column vectors of $M, N$ and $w$ with $\vec{m}, \vec{n}$, and $\vec{w}_R$, respectively.

$$\mathcal{N}_2^{(R)}(M, N) = \prod_{\xi=1}^{R-1} \left\{ \int d\Omega_2(w_\xi) \prod_{i=1}^{d} w_i^{M_{\xi i}} (w_i^*)^{N_{\xi i}} \right\} \prod_{\mu < \nu}^{R-1} \delta^{(2)}(\langle \vec{w}_\mu | \vec{w}_\nu \rangle) \quad \mathcal{J}_2^{(R)}(\vec{m}, \vec{n}) \quad (35)$$

$$\mathcal{J}_2^{(R)}(\vec{m}, \vec{n}) = \int d\Omega_2(w) \prod_{i=1}^{d} \left\{ w_i^{m_i} (w_i^*)^{n_i} \right\} \prod_{\mu=1}^{R-1} \delta^{(2)}(\langle \vec{w}_\mu | \vec{w} \rangle) \quad (36)$$

We start by flattening the integration measure $\Omega_2$, using the technique from the one-vector integral. With $d^2(w_i) = d(Re w_i) d(Im w_i)$ we may write:

$$\mathcal{J}_2^{(R)}(\vec{m}, \vec{n}) = C_2(d, R) \prod_{i=1}^{d} \left\{ \int d^2(w_i) w_i^{m_i} (w_i^*)^{n_i} \right\} \delta(\sum_i w_i^* w_i - 1) \prod_{\mu=1}^{R-1} \delta^{(2)}(\langle \vec{w}_\mu | \vec{w} \rangle) \quad (37)$$
The transformation \( w_i = u_i / \sqrt{r} \) leads to:

\[
\mathcal{J}^{(R)}_2(\vec{m}, \vec{n}) \rightleftharpoons \int d^d(x) \text{e}^{2\pi i (x \text{Im} w + y \text{Re} w)} = \int \int_\pi \text{e}^{x(w-w^*) + iy(w+w^*)} = \int \frac{d^2(z)}{\pi^2} \text{e}^{wz-w^*z^*} = \int \frac{d^2(z)}{\pi^2} \text{e}^{2\text{Im}(wz)} .
\]

Multiplying both sides with \( e^{-r} \) and integrating on \( r \) from 0 to \( \infty \) gives:

\[
\mathcal{J}^{(R)}_2(\vec{m}, \vec{n}) \Gamma (d - R + 1 + \frac{\vec{m} + \vec{n}}{2}) = C_2(d, R) \prod_{i=1}^d \left\{ \int d^2(u_i) u_i^{m_i} (u_i^*)^{n_i} \right\} \delta \left( \sum_i u_i^* u_i - r \right) \prod_{\mu=1}^{R-1} \delta^{(2)}((\vec{w}_\mu | \vec{u})) .
\]

Using the Fourier representation of the one-dimensional delta function, we may write:

\[
\delta^{(2)}(w) = \int \int dx \, dy \, e^{2\pi i (x \text{Im} w + y \text{Re} w)} = \int \int \frac{d^2(z)}{\pi^2} e^{x(w-w^*) + iy(w+w^*)} = \int \frac{d^2(z)}{\pi^2} e^{wz-w^*z^*} = \int \frac{d^2(z)}{\pi^2} e^{2\text{Im}(wz)} .
\]

With the help of this representation, we write:

\[
\prod_{\mu=1}^{R-1} \delta^{(2)} \left( \sum_i u_i^* u_i \right) = \prod_{\mu=1}^{R-1} \left( \frac{d^2(t_\mu)}{\pi^2} \right) \exp \left( t_\mu \sum_i u_i^* u_i - t_\mu^* \sum_i u_i u_i^* \right) .
\]

Therefore

\[
\mathcal{J}^{(R)}_2(\vec{m}, \vec{n}) = \frac{C_2(d, R)}{\Gamma (d - R + 1 + \frac{\vec{m} + \vec{n}}{2})} \prod_{\mu=1}^{R-1} \left\{ \int \frac{d^2(t_\mu)}{\pi^2} \right\} \prod_{i=1}^d f(m_i, n_i, \alpha_i) \quad \alpha_i = \sum_{\mu=1}^{R-1} t_\mu w_{i\mu}^* \quad (42)
\]

\[
\alpha = \prod_{\mu=1}^{R-1} \left( \frac{m_i}{k_i} \right) \frac{n_i}{k_i} \frac{\alpha_i^{m_i - \kappa_i} \alpha_i^{n_i - \kappa_i} e^{-\alpha_i^* \alpha}}{p = \min(m, n)} .
\]

For details about the computation of that integral, see appendix. In order to put the product of the functions \( f(m_i, n_i, \alpha_i) \) into a suitable form, we define additional integer vectors \( \vec{\kappa} \) and \( \vec{\bar{p}} \), where \( p_i = \min(m_i, n_i) \):

\[
\prod_{\mu=1}^{d} f(m_i, n_i, \alpha_i) = \pi^d (\vec{m}) \sum_{\vec{\kappa} = 0} \prod_{\mu=1}^{d} \left\{ \left( \frac{m_i}{k_i} \right) \frac{n_i}{k_i} \right\} \kappa_i ! \left( \alpha_i^* \right)^{m_i - \kappa_i} \alpha_i^{n_i - \kappa_i} e^{-\alpha_i^* \alpha} \quad p = \min(m, n) .
\]

With \( \vec{m} \) and \( \vec{\kappa} \), we denote the sum of vector components in \( \vec{m} \) and \( \vec{\kappa} \), respectively. For the expansion of the powers of the coefficients \( \alpha_i \) and \( \alpha_i^* \), we need the integer matrices \( K \) and \( L \). Both are \( d \)-dimensional matrices with non-zero elements in the first \( R - 1 \) columns, only.

\[
\prod_{i=1}^{d} (\alpha_i^*)^{m_i - \kappa_i} = \prod_{i=1}^{d} \left( \sum_{\mu=1}^{R-1} t_\mu w_{i\mu}^* \right)^{m_i - \kappa_i} = \sum_{K} (\vec{m} - \vec{\kappa}|K) \prod_{\mu=1}^{R-1} \left( t_\mu \right)_{K, \mu} w_{i\mu}^{K_{i\mu}} \prod_{i=1}^{d} w_{i\mu}^{K_{i\mu}} ,
\]

where \( \vec{k}_1, \ldots, \vec{k}_{R-1} \) denote the column vectors of \( K \), while the overbars denote, as usual, the summation over all vector components. In a completely analogous way, we write also:

\[
\prod_{i=1}^{d} (\alpha_i)_{n_i - \kappa_i} = \sum_{L} (\vec{n} - \vec{\kappa}|L) \prod_{\mu=1}^{R-1} \left( t_\mu \right)_{L, \mu} (w_{i\mu}^*)_{L, \mu}^{L_{i\mu}} .
\]

where \( \vec{l}_1, \ldots, \vec{l}_{R-1} \) denote the column vectors of \( L \). Thus, we obtain from equation (42):
Due to the fact that the integration in equation (26) is restricted to column vectors \( \vec{w}_j \) which are pairwise orthogonal, we obtain:

\[
\sum_i |\alpha_i|^2 = \sum_\mu \sum_\nu \tau_\mu \tau_\nu \sum_i w_{i\nu} w_{i\mu}^* = \sum_\mu |\tau_\mu|^2 .
\]

(47)

This is a crucial point in our calculation. It eliminates the remaining matrix elements of \( w \) from the exponential, and \( J_{2(R)}(\vec{m}, \vec{n}) \) turns into a linear combination of simple monomials. The integrals over \( \tau_\mu \) are now easily evaluated, using equation (31) from the calculation of the one-vector average:

\[
J_{2(R)}(\vec{m}, \vec{n}) = \frac{C_2(d, R)}{\Gamma(d - R + 1 + \frac{m+n}{2})} \sum_{\vec{\kappa} = \vec{\sigma}} \prod_{\kappa = \sigma} \sum_{i=1}^{d} \left\{ \left( \begin{array}{c} m_i \\ \kappa_i \\ \kappa_i \end{array} \right) \left( \begin{array}{c} n_i \\ \kappa_i \\ \kappa_i \end{array} \right) k_i! \right\} \times \sum_{K,L}(\vec{m} - \vec{\kappa}|K) (\vec{n} - \vec{\kappa}|L) \prod_{\mu=1}^{R-1} \delta_{\vec{\kappa}_\mu, \vec{l}_\mu} \prod_{i=1}^{d} w_{i\mu}^{K_{i\mu}} (w_{i\mu}^{*})^{L_{i\mu}}.
\]

(48)

Now, we insert this result into equation (35) and write for \( N_{2(R)}(M, N) \) and \( N_{2(R)}(o, o) \):

\[
N_{2(R)}(M, N) = \frac{C_2(d, R)}{\Gamma(d - R + 1 + \frac{M+N}{2})} \sum_{\vec{\kappa} = \vec{\sigma}} \prod_{\kappa = \sigma} \sum_{i=1}^{d} \left\{ \left( \begin{array}{c} m_i \\ \kappa_i \\ \kappa_i \end{array} \right) \left( \begin{array}{c} n_i \\ \kappa_i \\ \kappa_i \end{array} \right) k_i! \right\} \times \prod_{\mu=1}^{R-1} \delta_{\vec{\kappa}_\mu, \vec{\ell}_\mu} \prod_{i=1}^{d} w_{i\mu}^{M_{i\mu} + K_{i\mu}} (w_{i\mu}^{*})^{N_{i\mu} + L_{i\mu}} \times \sum_{K,L}(\vec{m} - \vec{\kappa}|K) (\vec{n} - \vec{\kappa}|L) \prod_{\mu=1}^{R-1} \delta_{\vec{\kappa}_\mu, \vec{\ell}_\mu} \prod_{i=1}^{d} \left( w_{i\mu} \right)^{M_{i\mu} + K_{i\mu}} (w_{i\mu}^{*})^{N_{i\mu} + L_{i\mu}}.
\]

(49)

Thus, we obtain:

\[
\langle N|M \rangle = (-)^{\vec{m}} (d - R + 1)^{-1} (\vec{m} + \vec{n})/2 \sum_{\vec{\kappa} = \vec{\sigma}} \prod_{\kappa = \sigma} \sum_{i=1}^{d} \left\{ \left( \begin{array}{c} m_i \\ \kappa_i \\ \kappa_i \end{array} \right) \left( \begin{array}{c} n_i \\ \kappa_i \\ \kappa_i \end{array} \right) k_i! \right\} \times \sum_{K,L}(\vec{m} - \vec{\kappa}|K) (\vec{n} - \vec{\kappa}|L) \prod_{\mu=1}^{R-1} \delta_{\vec{\kappa}_\mu, \vec{\ell}_\mu} \prod_{i=1}^{d} \left( w_{i\mu} \right)^{M_{i\mu} + K_{i\mu}} (w_{i\mu}^{*})^{N_{i\mu} + L_{i\mu}}.
\]

(50)

Note that due to the multinomial coefficients \((\vec{m} - \vec{\kappa}|K)\) and \((\vec{n} - \vec{\kappa}|L)\), for the column sums of \( K \) and \( L \), it holds that: \( \sum_{\mu=1}^{R-1} \vec{\kappa}_\mu = \vec{m} - \vec{n} \), and also \( \sum_{\mu=1}^{R-1} \vec{\ell}_\mu = \vec{n} - \vec{\kappa} \). In addition, in order to obtain a non-zero contribution, the column sums of \( K \) and \( L \) must be the same also: \( \forall i: 0 \leq \kappa_i \leq \min(m_i, n_i) \). \( K \) and \( L \) are d-dimensional matrices with non-negative integer entries, which have non-zero elements in their first \( R - 1 \) columns, only. The summation over \( K \) is restricted to those \( K \), for which \( \forall i: \sum_{\mu} K_{i\mu} = m_i - \kappa_i \). Similarly, the summation over \( L \) is restricted to those \( L \), for which \( \forall i: \sum_{\mu} L_{i\mu} = n_i - \kappa_i \). In addition, the one-vector average \( \langle \vec{\kappa}_{cs} | \vec{\kappa}_{cs} \rangle \) implies that the column-sums of \( K \) must agree with the corresponding column sums of \( L \). These column sums are arranged in the first \( R - 1 \) components of the \( d \)-dimensional vectors \( \vec{\kappa}_{cs} \) and \( \vec{\ell}_{cs} \), respectively.
Zero column vectors Assume that in the monomial integral \( \langle N \middle| M \rangle \), the \( R \)'th column of \( M \) and \( N \) contain both only zeros. Equation (52) then yields: \( \langle N \middle| M \rangle = \langle N' \middle| M' \rangle \). This proves by a simple induction argument, that the integration space in (26) may indeed be restricted to those column vectors, which contain matrix elements from the monomial \( M_2(w) \).

Vanishing integrals Denoting monomials in the form: \( w_{a_1b_1}^* \ldots w_{a_nb_n}^* \), \( w_{\alpha_1\beta_1} \ldots w_{\alpha_q\beta_q} \) (see for instance, reference [14]), it has been shown that the monomial integral \( \langle w_{a_1b_1}^* \ldots w_{a_nb_n}^* \rangle \) vanishes, unless \( p = q \) and \( \alpha_1, \ldots, \alpha_q \) is a permutation of \( a_1, \ldots, a_p \) and \( \beta_1, \ldots, \beta_q \) is a permutation of \( b_1, \ldots, b_p \). This is equivalent to the requirement that all columns sums and row sums of \( M \) agree with the corresponding column sums and row sums of \( N \). In view of the invariance of the Haar measure under row- or column permutations, this is a trivial consequence of the leading factor \( \delta_{m,n} \) in equation (52).

5 Integration over \( Sp(2d) \)

For the unitary symplectic group, we denote the normalized Haar measure by \( \sigma_4 \). For

\[
Sp(2d) \ni w = \begin{pmatrix} z^* & w \\ -w^* & z \end{pmatrix}, \quad \text{let } f(w) = f(w, z, w^*, z^*) \ ,
\]

where \( f(w_1, z_1, w_2, z_2) \) is an arbitrary analytic(?) and integrable function of the matrix elements of \( w_1, z_1, w_2, \) and \( z_2 \). The invariance properties of the Haar measure then imply:

\[
\int d\sigma_4(w, z) f(w) = \int d\sigma_4(w, z) f(wu) = \int d\sigma_4(w, z) f(uw) = \int d\sigma_4(w, z) f(w^t) \ ,
\]

where \( u \) is a arbitrary but fixed element of \( Sp(2d) \).

Since we are interested in the case, where the function \( f \) is a monomial, we use some particular permutation matrices from \( Sp(2d) \) to obtain invariance relations for the monomial integral \( \langle M \rangle \), under certain transformations of \( M \). For the following considerations, let

\[
\begin{align*}
w &= \begin{pmatrix} z^* & w \\ -w^* & z \end{pmatrix}, & u &= \begin{pmatrix} v^* & u \\ -u^* & v \end{pmatrix} \ .
\end{align*}
\]

(1) For \( u = \pi \), a \( d \)-dimensional permutation matrix, and \( v = o \), multiplication from the left leads to

\[
w \rightarrow wu = \begin{pmatrix} z^* & \pi z^* \\ -\pi \pi & -w^* \pi \end{pmatrix} \ ,
\]

such that

\[
\langle M \rangle = \left\langle \begin{pmatrix} N' & M' \\ -M^t & N^t \end{pmatrix} \right\rangle = \int d\sigma_4(w) \prod_{i,j=1}^d w^*_{ij} M_{ij}^t M_{ij}^t N_{ij}^t (z_{ij}^*) N_{ij}^t
\]

\[
= \int d\sigma_4(w) \prod_{i,j=1}^d (z^* \pi)_{ij} M_{ij}^t (z^* \pi)_{ij} N_{ij}^t (z_{ij}^*) = (-)^{\tilde{M} + \tilde{N}} \left\langle \begin{pmatrix} M & N \\ \pi M & \pi N \end{pmatrix} \right\rangle \ ,
\]

where \( \tilde{M} \) and \( \tilde{N} \) are the sum of all matrix entries in \( M \) and \( N \), respectively. Similarly, multiplication with \( u \) from the right yields:

\[
\langle M \rangle \rightarrow \int d\sigma_4 \prod_{i,j=1}^d (\pi z)_{ij} M_{ij}^t (\pi z)_{ij}^* N_{ij}^t (\pi w)_{ij} \pi w_{ij} = (-)^{\tilde{M}' + \tilde{N}'} \left\langle \begin{pmatrix} \pi N & -\pi M' \\ \pi M & \pi N' \end{pmatrix} \right\rangle \ ,
\]

(2) For \( u = o \) and \( v = \pi \), multiplication from the left yields:

\[
\langle M \rangle \rightarrow \int d\sigma_4 \prod_{i,j=1}^d (w\pi)_{ij} M_{ij}^t (w\pi)_{ij}^* N_{ij}^t (z^* \pi)_{ij} = \left\langle \begin{pmatrix} N' & M \\ \pi N & \pi M \end{pmatrix} \right\rangle \ ,
\]

whereas multiplication from the right yields:

\[
\langle M \rangle \rightarrow \left\langle \begin{pmatrix} \pi M & \pi N' \\ \pi N & \pi M' \end{pmatrix} \right\rangle \ .
\]
where $M$ and $N$ are the sums of all matrix entries of $M$ and $N$, respectively.

As we will show below [see equation (88)], the monomial integral $\langle M \rangle$ is zero, unless $M + M' = N + N'$. Equation (61) also implies that $\langle M \rangle$ is zero, unless $M + N' = N + M'$. Both conditions together imply that

$$\langle M \rangle \neq 0 \quad \Rightarrow \quad M = M' = N = N'.$$

We can therefore safely ignore the sign-prefactors in (57), (58), and (61). The results in item (2) prove the invariance of $\langle M \rangle$ under synchronous column permutations [equation (59)] and synchronous row permutations [equation (60)], applied to all submatrices of $M$. Therefore, the equations (57) and (58) add only two additional operations:

$$\begin{pmatrix} N' & M \\ N & M' \end{pmatrix} \rightarrow \begin{pmatrix} M & N' \\ M' & N \end{pmatrix}, \quad \begin{pmatrix} N' & M \\ N & M' \end{pmatrix} \rightarrow \begin{pmatrix} N & M' \\ N' & M \end{pmatrix}.$$

The integration

It is again possible to write the integral over $Sp(2d)$ in terms of an integral over the flat Euclidean space of all complex matrix elements of $w$ and $z$, implementing the restriction to the group manifold by appropriately chosen $\delta$-functions:

$$\langle M \rangle = \int d\sigma_4(w, z) \mathcal{M}_4(w) = \frac{\mathcal{N}_4^{(R)}(M)}{\mathcal{N}_4^{(R)}(O)}, \quad \mathcal{M}_4(w) = \prod_{i, \xi = 1}^{d, R} w_i^{M_{i \xi}} z_i^{M'_{i \xi}} w_i^*^{N_{i \xi}} z_i^*^{N'_{i \xi}}.$$

Here, we assume that the monomial $\mathcal{M}_4(w)$ contains no matrix elements from the column vectors $\vec{w}_\mu, \vec{z}_\mu$ with $\mu > R$. One may have used the invariance of $\langle M \rangle$ under the above discussed transformations in order to minimize $R$. The matrix $O$ is of the same shape as $M$, but all its entries are equal to zero.

The subscript in the symbol $\mathcal{N}_4^{(R)}$ is the symmetry parameter $\beta = 4$, since we consider here the unitary symplectic case. The first $R$ column vectors of the matrices $w, z$ are denoted by $\vec{w}_1, \ldots, \vec{w}_R$, and $\vec{z}_1, \ldots, \vec{z}_R$, respectively. The integration region in equation (65) is the product space of $R$ unit hyperspheres with constant measure $d\Omega_4(\vec{w}_\xi, \vec{z}_\xi)$:

$$\int d\Omega_4(\vec{w}_\xi, \vec{z}_\xi) \propto \prod_{i=1}^{R} \left\{ \int d^d(\vec{w}_\xi, \vec{z}_\xi) \right\} \delta(\|\vec{w}_\xi\|^2 + \|\vec{z}_\xi\|^2 - 1) \quad w, z \in \mathbb{C} : d^4(w, z) = d^2(w) d^2(z),$$

where $d^2(z)$ denotes the flat measure on the complex plane, as already used in equation (27). The $\delta$-functions in (65) and (66) implement the orthogonality conditions and the normalization, as defined in equation (9).

According to the assumption, the monomial $\mathcal{M}_4(w)$ contains matrix elements from the first $R$ columns of $w$ and $z$, only. Therefore, we restrict the integration in equation (65) to those column vectors. The fact that this does not affect the result of the integral, will be shown only at the end of this section, where we have the final result, equation (89) at our disposal. Note that the integration measure in (65) is indeed invariant under the transformations performed in (54). This guarantees that the above construction really implements the Haar measure.

5.1 The one-vector formula

In the one-vector case, the monomial $\mathcal{M}_4(w)$ contains matrix elements from one column of $w$ only. This means that the matrices $M, M', N', N$ have only zero entries in all columns, except for one which must be the same for all matrices. We denote the column vectors with non-zero entries by $\vec{m}, \vec{m}', \vec{n},$ and $\vec{n}'$ and use the following notation for the monomial integral:

$$\langle M \rangle = \left\langle \frac{\vec{m}}{\vec{n}} \left| \frac{\vec{m}'}{\vec{n}'} \right. \right\rangle = \int d\Omega_4(\vec{w}, \vec{z}) \prod_{i=1}^{d} w_i^{m_i} z_i^{m'_i} (w_i^*)^{n_i} (z_i^*)^{n'_i}.$$
5.2 The $R$-vector formula

We aim at a recursion formula, which relates an $R$-vector integral to a linear combination of simpler $(R-1)$-vector integrals. To derive such a relation, we separate the integration over the $R$-th column vectors of $w$ and $z$ from the rest. Let us denote the $R$-th column vectors of $M, M', N, N'$ by $\vec{m}, \vec{m}', \vec{n}, \vec{n}'$. Then we may write:

$$\mathcal{N}_4^{(R)}(M) = \prod_{\xi=1}^{R-1} \left\{ \int d\Omega_4(\vec{u}_{\xi}, \vec{z}_{\xi}) \prod_{i=1}^{d} w_{\xi i}^M N_{\xi i} (z_{\xi}^*)^{N_{\xi i}} \right\} \cdot \prod_{\mu<\nu}^{R-1} \left\{ \delta^{(2)}(\langle \vec{u}_{\mu} | \vec{u}_{\nu} \rangle + \langle \vec{z}_{\mu} | \vec{z}_{\nu} \rangle) \delta^{(2)}(\langle \vec{z}_{\mu}^* | \vec{u}_{\nu} \rangle - \langle \vec{u}_{\mu}^* | \vec{z}_{\nu} \rangle) \right\} \cdot \mathcal{J}_4^{(R)}(\vec{m}, \vec{m}', \vec{n}, \vec{n}') \quad (69)$$

We start by flattening the integration measure $\Omega_4$. With equation (66) and $d^2(z) = d(Re z) d(Im z)$, we may write:

$$\mathcal{J}_4^{(R)}(\ldots) = C_4(d, R) \prod_{i=1}^{d} \left\{ \int d^2(w_i) \int d^2(z_i) w_i^{m_i} z_i^{m_i'} (z_i^*)^{n_i} (u_i^*)^{n_i'} \right\} \delta \left( \sum_i |w_i|^2 + \sum_i |z_i|^2 - 1 \right) \prod_{\mu=1}^{R-1} \left\{ \delta^{(2)}(\langle \vec{u}_{\mu} | \vec{u} \rangle + \langle \vec{z}_{\mu} | \vec{v} \rangle) \delta^{(2)}(\langle \vec{z}_{\mu}^* | \vec{u} \rangle - \langle \vec{u}_{\mu}^* | \vec{v} \rangle) \right\} \quad (70)$$

The delta function for the normalisation can be removed, using similar steps as in the case of the unitary group. This yields:

$$\mathcal{J}_4^{(R)}(\ldots) = \frac{C_4(d, R)}{\Gamma(2(d-R+1) + (m + m' + n + n)/2)} \prod_{i=1}^{d} \left\{ \int d^2(u_i) \int d^2(v_i) u_i^{m_i} v_i^{m_i'} (v_i^*)^{n_i} (u_i^*)^{n_i'} e^{-|w_i|^2 - |v_i|^2} \right\} \times \prod_{\mu=1}^{R-1} \delta^{(2)}(\langle \vec{u}_{\mu} | \vec{u} \rangle + \langle \vec{z}_{\mu} | \vec{v} \rangle) \delta^{(2)}(\langle \vec{z}_{\mu}^* | \vec{u} \rangle - \langle \vec{u}_{\mu}^* | \vec{v} \rangle) \quad (72)$$

According to equation (40), the Fourier representations for the remaining delta functions may be written as

$$\prod_{\mu=1}^{R-1} \delta^{(2)} \left[ R_1 \sum_{i} (w_{i\mu} u_i + z_{i\mu} v_i) \right] = \prod_{\mu=1}^{R-1} \left\{ \int \frac{d^2(\sigma_{\mu})}{\pi^2} \right\} \prod_{i=1}^{d} e^{2\text{Im} \langle \sigma_{\mu} (w_{i\mu} u_i + z_{i\mu} v_i) \rangle} = \prod_{\mu=1}^{R-1} \left\{ \int \frac{d^2(\tau_{\mu})}{\pi^2} \right\} \prod_{i=1}^{d} e^{2\text{Im} \langle \tau_{\mu} (w_{i\mu} u_i - z_{i\mu} v_i) \rangle} \quad (73)$$

$$\prod_{\mu=1}^{R-1} \delta^{(2)} \left[ R_1 \sum_{i} (z_{i\mu} u_i - w_{i\mu} v_i) \right] = \prod_{\mu=1}^{R-1} \left\{ \int \frac{d^2(\tau_{\mu})}{\pi^2} \right\} \prod_{j=1}^{d} e^{2\text{Im} \langle \tau_{\mu} (z_{i\mu} u_i - w_{i\mu} v_i) \rangle} = \prod_{\mu=1}^{R-1} \left\{ \int \frac{d^2(\tau_{\mu})}{\pi^2} \right\} \prod_{i=1}^{d} e^{2\text{Im} \langle \gamma_{i \mu} u_i - \epsilon_{i \mu} v_i \rangle}$$

where $\alpha_i = \sum_{\mu=1}^{R-1} \sigma_{\mu} w_{i\mu}^* \quad \beta_i = \sum_{\mu=1}^{R-1} \sigma_{\mu} z_{i\mu}^* \quad \gamma_i = \sum_{\mu=1}^{R-1} \tau_{\mu} z_{i\mu} \quad \epsilon_i = \sum_{\mu=1}^{R-1} \tau_{\mu} w_{i\mu}$.
Insertion into equation (72), and the exchange of the order of integration gives:

\[
\mathcal{J}_4^{(R)}(\ldots) = \frac{C_4(d, R)}{\Gamma(2(d-R+1) + (\bar{m} + \bar{m}' + \bar{n} + \bar{n}'))/2} \prod_{\mu=1}^{R-1} \left\{ \int \frac{d^2(\sigma_\mu)}{\pi^2} \int \frac{d^2(\tau_\mu)}{\pi^2} \right\} \\
\times \prod_{i=1}^{d} \int d^2(u_i) \ u_i^{m_i} (u_i^*)^{n_i} e^{-|u_i|^2} e^{2i \text{Im}[|u_i| (\alpha_i + \gamma_i)]} \int d^2(v_i) \ v_i^{m_i'} (v_i^*)^{n_i'} e^{-|v_i|^2} e^{2i \text{Im}[|v_i| (\beta_i - \varepsilon_i)]} \\
= \frac{C_4(d, R)}{\Gamma(\ldots)} \prod_{\mu=1}^{R-1} \left\{ \int \frac{d^2(\sigma_\mu)}{\pi^2} \int \frac{d^2(\tau_\mu)}{\pi^2} \right\} \prod_{i=1}^{d} f(m_i, n_i, \alpha_i + \gamma_i, f(m_i', n_i', \beta_i - \varepsilon_i)) . \tag{74}
\]

The function \( f \) is the same which appeared in connection with the \( R \)-vector integral for the unitary group, equation (42), and which has been computed in the appendix. Inserting the result, equation (100) and expanding the powers of \( (\alpha_i + \gamma_i) \) and \( (\beta_i - \varepsilon_i) \), we obtain:

\[
f(m, n, \alpha + \gamma) = \pi e^{-|\alpha + \gamma|^2} \sum_{\kappa = 0}^{P} (-)^{m-k} \binom{m}{\kappa} \binom{n}{\kappa} \kappa! \sum_{k=0}^{m-k} \binom{m-k}{k} \binom{n-k}{l} (\alpha^*)^k (\gamma^*)^{m-k-k} \alpha^l \gamma^{n-k-l}, \tag{75}
\]

\[
f(m', n', \beta - \varepsilon) = \pi e^{-|\beta - \varepsilon|^2} \sum_{\kappa' = 0}^{P} (-)^{m'-k'} \binom{m'}{\kappa'} \binom{n'}{\kappa'} \kappa'! \sum_{k'=0}^{m'-k'} \binom{m'-k'}{k'} \binom{n'-k'}{l'} (\beta^*)^{k'} (\varepsilon^*)^{m'-k'-k'} \beta^l \varepsilon^{n'-k'-l'}, \tag{76}
\]

where we have for the moment suppressed the index \( i \). In order to completely expand the product of the functions \( f(m_i, n_i, \alpha_i + \gamma_i) \) and \( f(m_i', n_i', \beta_i - \varepsilon_i) \), we define additional vector indeces: \( p_i = \min(m_i, n_i), p'_i = \min(m'_i, n'_i), \kappa_i, \) and \( \kappa'_i \). The complete expansion then reads:

\[
\prod_{i=1}^{d} f(m_i, n_i, \alpha_i + \gamma_i, f(m'_i, n'_i, \beta_i - \varepsilon_i)) = \pi^{2d} e^{-\sum_{i}(|\alpha_i + \gamma_i|^2 + |\beta_i - \varepsilon_i|^2)} \\
\times \sum_{\kappa = 0}^{\bar{m}} (-)^{\bar{m} - \kappa} \binom{\bar{m}}{\kappa} \sum_{\kappa' = 0}^{\bar{m}' - \kappa'} \binom{\bar{m}'}{\kappa'} \kappa'! \sum_{k=0}^{\bar{m}' - \kappa'} \binom{\bar{m}' - \kappa'}{k} \binom{\bar{n}' - \kappa'}{l'} \prod_{i=1}^{d} \{\kappa_i! \kappa'_i!\} \\
\times \sum_{k=0}^{\bar{m} - \bar{n}} \sum_{l=0}^{\bar{n} - \bar{k}} \sum_{k'=0}^{\bar{m}' - \bar{n}'} \sum_{l'=0}^{\bar{n}' - \bar{k}'} \binom{\bar{m} - \bar{n}}{\bar{k}} \binom{\bar{n} - \bar{k}}{\bar{l}} \binom{\bar{m}' - \bar{n}'}{\bar{k}'} \binom{\bar{n}' - \bar{k}'}{\bar{l}'} \\
\times \prod_{i=1}^{d} (\alpha_i^*)^{k_i} (\gamma_i^*)^{m_i - n_i - k_i} \alpha_i^l n_i - n_i - k_i \beta_i^l (\varepsilon_i^*)^{m'_i - n'_i - k'_i} \beta_i^l (\varepsilon_i)^{m'_i - n'_i - k'_i} . \tag{77}
\]

The argument of the exponential function can be simplified, as follows:

\[
\sum_{i} (|\alpha_i + \gamma_i|^2 + |\beta_i - \varepsilon_i|^2) = \sum_{i} \sum_{\mu \nu} \left[ (\sigma_\mu^* w_{i\mu} + \tau_\mu z_{i\mu} + \tau_\mu^* z_{i\mu}^*) + (\sigma_\mu z_{i\mu}^* - \tau_\mu w_{i\mu}) + (\sigma_\mu^* z_{i\mu} - \tau_\mu^* w_{i\mu}) \right] \\
= \sum_{\mu \nu} \left[ (\sigma_\mu \sigma_{\mu}^* (|w_{\mu}|^{2} + |z_{\mu}|^{2}) + \tau_\mu \sigma_{\mu}^* (|z_{\mu}|^{2} - |w_{\mu}|^{2}) + \sigma_\mu^* \tau_{\mu}^* (|z_{\mu}|^{2} - |w_{\mu}|^{2}) + \tau_{\mu} \tau_{\mu}^* (|w_{\mu}|^{2} + |z_{\mu}|^{2}) \right] \\
= \sum_{\mu} (|\sigma_\mu|^2 + |\tau_\mu|^2) , \tag{78}
\]

where we have used the orthogonality relations in (9). For the expansion of the powers of coefficients \( \alpha_i^*, \gamma_i^* \), etc., we will need eight matrix indeces, each of dimension \( d \times d \). It will be convenient to combine them into two \( 2 \times 2 \) block-matrices, as follows:

\[
K_1 = \begin{pmatrix}
L_1 & K_1 \\
L_1 & K_1
\end{pmatrix}, \quad K_2 = \begin{pmatrix}
L_2 & K_2 \\
L_2 & K_2
\end{pmatrix} . \tag{79}
\]
Then we combine pairs of coefficients, which lead to monomials of entries from the same submatrix of \( w \):

\[
\prod_{i=1}^{d} (\alpha_i^+) \gamma_i^L (\varepsilon_i^+)^{\mu_i^L} \left( (\bar{n} - \bar{r} - \bar{\bar{k}})^{\mu_1^L} (\bar{n} - \bar{r} - \bar{\bar{k}})^{\mu_2^L} \prod_{i=1}^{R-1} (\sigma_{\mu_1}^L \tau_{\bar{k}}^L \prod L_{1,i}) \right) w_{1,2}^{L_1 \mu_1^L + L_2 \mu_2^L} \tag{80}
\]

\[
\prod_{i=1}^{d} (\beta_i^+) \gamma_i^L (\varepsilon_i^+)^{\mu_i^L} \left( (\bar{n} - \bar{r} - \bar{\bar{k}}) \prod_{i=1}^{R-1} (\sigma_{\mu_1}^L \tau_{\bar{k}}^L \prod L_{1,i}) \right) w_{1,2}^{L_1 \mu_1^L + L_2 \mu_2^L} \tag{81}
\]

\[
\prod_{i=1}^{d} \alpha_i^L (\varepsilon_i^+)^{\mu_i^L} \left( (\bar{n} - \bar{r} - \bar{\bar{k}})^{\mu_1^L} (\bar{n} - \bar{r} - \bar{\bar{k}})^{\mu_2^L} \prod_{i=1}^{R-1} (\sigma_{\mu_1}^L \tau_{\bar{k}}^L \prod L_{1,i}) \right) w_{1,2}^{L_1 \mu_1^L + L_2 \mu_2^L} \tag{82}
\]

\[
\prod_{i=1}^{d} (\gamma_i^+) \gamma_i^L (\varepsilon_i^+)^{\mu_i^L} \left( (\bar{n} - \bar{r} - \bar{\bar{k}})^{\mu_1^L} (\bar{n} - \bar{r} - \bar{\bar{k}})^{\mu_2^L} \prod_{i=1}^{R-1} (\sigma_{\mu_1}^L \tau_{\bar{k}}^L \prod L_{1,i}) \right) w_{1,2}^{L_1 \mu_1^L + L_2 \mu_2^L} \tag{83}
\]

where we eventually convert the subscripts 1, 2 of the eight matrix indices into superscripts, especially when we refer to particular matrix elements. The parameters \( k_1^1, k_1^2, \) etc., denote the column sums of the matrix indices \( K_1, K_1', \) etc. Plugging the complete expansion into (74), we obtain:

\[
J(R) (\bar{m}, \bar{m}', \bar{n}, \bar{n}') = \frac{C_4(d, R) \pi^{2d}}{2(N - R + 1) + \mu + \mu_1 + \mu_2 + \mu} \sum_{\mu} \sum_{i=1}^{d} (\bar{n} - \bar{r}) \left( \bar{m} - \bar{k} \right) \left( \bar{m}' - \bar{k}' \right) \left( \bar{n} - \bar{k} \right) \left( \bar{n}' - \bar{k}' \right) \prod_{i=1}^{d} \{k_1^1, k_1^2\} \tag{84}
\]

We can now merge each sum and its corresponding binomial from the second line with two sums and their corresponding multinoials from the third line:

\[
\sum_{\bar{k} = \bar{\bar{k}}} \left( \bar{m} - \bar{r} \right) \sum_{\bar{k}_1, L_2^1} (\bar{k}|K_1) (\bar{m} - \bar{r} - \bar{k}|L_2^1) = \sum_{K_1', L_2^1} (\bar{m} - \bar{r}|L_2^1, K_1) \tag{85}
\]

where an extended vector-multinomial such as \((\bar{m}|K_1, L_2^1)\) means that the rows of \( K_1' \) and \( L_2^1 \) should be concatenated, to yield a product of multinomials, each with \( 2d \) elements. Evaluating the final integrals on \( \sigma_{\mu} \) and \( \tau_{\bar{k}}, \)
we obtain:

\[
\mathcal{J}_d^{(R)}(\vec{m}, \vec{m}', \vec{n}, \vec{n}') = \frac{C_d(d, R) \pi^{2d-R+1}}{\Gamma(2(N - R + 1) + \frac{m + m' + n + n''}{2})} \sum_{\vec{k}=\vec{\delta}}^{\vec{p}} \sum_{\vec{r}'=\vec{\delta}}^{\vec{r}} (\vec{m}) (\vec{m}') (\vec{n}) (\vec{n}') \prod_{i=1}^{d} \{ \kappa_i \text{ or } \kappa_i' \}
\]

\[
\times \sum_{K^1, K^2} \binom{\vec{m} - \vec{r}}{\vec{m}' - \vec{r}'} \binom{L_2' K_1}{L_2 K_1'} \binom{\vec{n} - \vec{r}}{\vec{n}' - \vec{r}'} \binom{L_1' K_2}{L_1 K_2'} (-)^{\vec{k} + \vec{k}'} (-)^{\vec{k}' + L_1'} (2d)^{\vec{k} + \vec{k}'} \binom{\vec{n}}{\vec{n}'} \binom{\vec{n}'}{\vec{n}} \binom{\vec{k}'}{\vec{k}} \binom{\vec{k}}{\vec{k}'}
\]

\[
\times \prod_{i=1}^{d} w_{\mu i}^{k_1 + k_2 - \vec{r}'} w_{\mu i}^{r_1 + r_2 + \vec{r}} (w_{\mu i}^{r_1} L_{1 i} + l_{2 i}^{2 \mu} (z_{\mu i}^{r_1} L_{1 i} + l_{2 i}^{2 \mu} ))^d . \tag{86}
\]

where we have replaced \( \vec{k} \) and \( \vec{p} \) with \( \vec{k}_1 \) and \( \vec{l}_1 \), as a more appropriate notation for the sum over all matrix entries in \( K^1 \) and \( L^1 \), respectively. We have also further expanded the vector multinomials, concatenating the matrices and vectors in the vertical direction.

Now, we may insert this result into equation (69) in order to obtain \( \mathcal{N}_d^{(R)}(M) \) as a linear combination of terms which contain the \( (R-1) \)-vector integrals \( \mathcal{N}_d^{(R-1)}(M^{(R-1)} + K_1 + K_2) \). Note that the matrix indices in (86) are chosen in order to fit into that scheme. Dividing through the corresponding expressions for \( \mathcal{N}_d^{(R)}(o) \) and \( \mathcal{N}_d^{(R-1)}(o) \) we obtain:

\[
\langle M \rangle = \frac{\Gamma(2(d - R + 1))}{\Gamma(2(d - R + 1) + (m + m' + n + n')/2)} \sum_{\vec{k}=\vec{\delta}}^{\vec{p}} \sum_{\vec{r}=\vec{\delta}}^{\vec{r}'} (\vec{m}) (\vec{m}') (\vec{n}) (\vec{n}') (2d)^{\vec{k} + \vec{k}'} \binom{\vec{n}}{\vec{n}'} \binom{\vec{n}'}{\vec{n}} \binom{\vec{k}'}{\vec{k}} \binom{\vec{k}}{\vec{k}'}
\]

\[
\times \sum_{K^1, K^2} \binom{\vec{m} - \vec{r}}{\vec{m}' - \vec{r}'} \binom{L_2' K_1}{L_2 K_1'} \binom{\vec{n} - \vec{r}}{\vec{n}' - \vec{r}'} \binom{L_1' K_2}{L_1 K_2'} (-)^{\vec{k} + L_1'} (2d)^{\vec{m} - \vec{r} + \vec{m}' - \vec{r}'} \binom{\vec{n}}{\vec{n}'} \binom{\vec{n}'}{\vec{n}} \binom{\vec{k}}{\vec{k}'} \binom{\vec{k}'}{\vec{k}}
\]

\[
\times \langle M^{(R-1)} + K_1 + K_2 \rangle . \tag{87}
\]

The vectors \( \vec{k}_1, \vec{k}_2, \vec{l}_1, \) and \( \vec{l}_2 \) are \( 2 \)-dimensional vectors which contain in their first \( R - 1 \) components the column sums of \( K_1 \) and \( K_2 \) in the following order: \( \vec{k}_1 \) contains the column sums of \( K_1 \) and \( K_1' \); \( \vec{k}_2 \) contains those of \( K_2 \) and \( K_2' \); \( \vec{l}_1 \) those of \( L_1 \) and \( L_1' \); and \( \vec{l}_2 \) those of \( L_2 \) and \( L_2' \). Note that due to the one vector integral in these column sums, one may conclude that

\[
\sum_{\mu=1}^{R-1} \vec{k}_1 + \vec{k}_2 + \vec{k}_2 = \vec{l}_1 + \vec{l}_2 + \vec{l}_2' \Rightarrow \vec{m} - \vec{r} + \vec{m}' - \vec{r}' = \vec{n} - \vec{r} + \vec{n}' - \vec{r}' \Rightarrow \vec{m} + \vec{m}' = \vec{n} + \vec{n}' . \tag{88}
\]

This allows us to use again the function \( B(a; b; z_1, z_2) \) defined in equation (21):

\[
\langle M \rangle = \delta_{\vec{m} + \vec{m}', \vec{n} + \vec{n}'} \sum_{\vec{k}, \vec{r}=\vec{\delta}}^{\vec{p}} \sum_{\vec{r}'=\vec{\delta}}^{\vec{r}'} (\vec{m}) (\vec{m}') (\vec{n}) (\vec{n}') B(\vec{m} + \vec{m}', \vec{n} + \vec{n}'; 2d, 2(R - 1) \binom{\vec{r}'}{\vec{r}} \binom{\vec{r}}{\vec{r}'}
\]

\[
\times \sum_{K_1, K_2} \binom{\vec{m} - \vec{r}}{\vec{m}' - \vec{r}'} \binom{L_2' K_1}{L_2 K_1'} \binom{\vec{n} - \vec{r}}{\vec{n}' - \vec{r}'} \binom{L_1' K_2}{L_1 K_2'} (-)^{\vec{k} + \vec{k}'} (2d)^{\vec{m} - \vec{r} + \vec{m}' - \vec{r}'} \binom{\vec{k}'}{\vec{k}'} \binom{\vec{k}'}{\vec{k}} \binom{\vec{k}'}{\vec{k}} \binom{\vec{k}'}{\vec{k}}
\]

\[
\times \langle M^{(R-1)} + K_1 + K_2 \rangle . \tag{89}
\]

The first line of this expression agrees precisely with the corresponding part of the recurrence relation for the unitary group \( U(2d) \). The differences appear in the second line, where we have to sum over 2×2-block matrices \( K_1 \) and \( K_2 \), with the restriction that the row-sums of the block-matrices in the following vector-multinomials must agree with the corresponding component of the given \( 2d \)-dimensional vector.

**Zero column vectors** In the case that the \( R \)’th column vector of all four matrices \( M, M', N \), and \( N' \) contain only zeros, the summation over \( \vec{r} \) and \( \vec{r}' \) contains only one term, namely \( \vec{r} = \vec{r}' = \vec{\delta} \). Similarly, the summation over \( K_1, K_2 \) also contains only one term, where all matrix entries of \( K_1 \) and \( K_2 \) are zero. Therefore, we find

\[
\langle M \rangle = \langle M'^{(R-1)} \rangle = \langle M'^{(R-1)} \rangle = \langle M^{(R-1)} \rangle . \tag{90}
\]

As in the case of \( O(d) \) and \( U(d) \) in the previous sections, this justifies equation (65) where we have ignored column vectors of \( w \) and \( z \) whose entries do not occur in \( M_4(w) \).
Vanishing integrals The monomial integral \( \langle M \rangle \) obviously vanishes if \( \vec{m} + \vec{m}' \) is different from \( \vec{n} + \vec{n}' \). Taking into account the transformations from (54), which do not change the integral, this means that the integral vanishes unless the column sums of \( M \) fulfill the condition:

\[
\forall 1 \leq j \leq d : \quad \vec{m}_j = \vec{m}_{d+j},
\]

where \( \vec{m}_j \) is the column sum of column \( j \) of \( M \). An exactly analogous condition holds for the row sums.

6 Conclusions

We applied the column vector method, originally developed for the orthogonal group in [26], to monomial integrals over the unitary and the unitary symplectic group. As in the orthogonal case, we obtained recursive integration formulas, where the recursion parameter is the number of non-empty column vectors (these are those matrix columns of the group element, which contain at least one matrix element from the monomial). In all three cases, the recursion formulas are of a very similar form and relatively easy to implement in a computer algebra environment. This provides an efficient way to compute such integrals analytically as a function of the matrix dimension.

The approach presented here is very different from the group theoretical approach developed in [25], which also provides explicit formulas for the analytical calculation of arbitrary monomial integrals over the classical groups. We did not yet compare the efficiency of both methods, but we would expect that the result of such a comparison will depend on the problem at hand.

A Gaussian integral on the complex plane

Here, we compute the Gaussian integral from equation (42). Writing for the complex integration variable \( u = x + iy \) we get:

\[
f(m, n, \alpha) = \int \int dx\, dy\, u^m (u^*)^n e^{-u^* u} e^{\alpha u - \alpha^* u^*}.
\]

First note that \( f(m, n, \alpha)^* = f(n, m, -\alpha) \), which implies that it is sufficient to consider the case \( m \geq n \). Then, we use the complex differential operators \( \partial_x \) and \( \partial_y \) to write:

\[
f(m, n, \alpha) = (-)^n \partial_x^n \partial_y^m \int \int dx\, dy\, e^{-u^* u} e^{\alpha u - \alpha^* u^*} \quad 2\partial_z = \partial_x - i\partial_y, \quad 2\partial_{\bar{z}} = \partial_x + i\partial_y.
\]

With \( \alpha = \alpha_1 + i\alpha_2 \), it follows

\[
f(m, n, \alpha) = (-)^n \partial_x^n \partial_y^m \int dx\, e^{-(x^2 - 2i\alpha_2 x)} \int dy\, e^{-(y^2 - 2i\alpha_1 y)}
\]

\[
= (-)^n \partial_x^n \partial_y^m \pi e^{-\alpha^* \alpha} = \pi (-)^m+n \partial_y^m (\alpha^*)^m e^{-\alpha^* \alpha} = \pi (-)^m+n P(m, n) e^{-\alpha^* \alpha},
\]

where \( P(m, n) \) is a polynomial of total order \( m + n \), with the property that the difference between the power of the conjugated variable and that of the non-conjugated one is \( m - n \), fixed. Hence, for \( P(m, n) \) we consider the following Ansatz:

\[
P(m, n) = \sum_{k=0}^n C_k^{(m, n)} (\alpha^*)^{m-n+k} \alpha^k = (\alpha^*)^{m-n} \sum_{k=0}^n C_k^{(m, n)} (\alpha^* \alpha)^k.
\]

For \( n = 0 \), the polynomials \( P(m, 0) \) are known, while for \( m \geq n > 0 \), they can be computed by recursion:

\[
n = 0 : \quad P(m, 0) = 1 \quad \Rightarrow \quad C_0^{(m, 0)} = 1
\]

\[
m > 0 : \quad P(m, n) = e^{\alpha^* \alpha} \partial_y^m (\alpha^*)^m e^{-\alpha^* \alpha}
\]

\[
P(m, n + 1) = e^{\alpha^* \alpha} \partial_y^m P(m, n) e^{-\alpha^* \alpha} = -\alpha P(m, n) + \sum_{k=0}^n C_k^{(m, n)} (m-n+k) (\alpha^*)^{m-(n+1)+k} \alpha^k.
\]

Equating the coefficients for corresponding powers, we obtain for \( m \leq n \leq k \leq 0 \):

\[
C_k^{(m, n+1)} = \begin{cases} 
(m-n) C_0^{(m, n)} & : k = 0 \\
(m-n+k) C_k^{(m, n)} - C_{k-1}^{(m, n)} & : 1 \leq k \leq n \\
-C_n^{(m, n)} & : k = n + 1
\end{cases}
\]

\[
C_0^{(m, 0)} = 1.
\]

15
Ordered according to \( n \) and \( k \), these coefficients can be arranged in a two-dimensional scheme, forming a Pascal triangle. From this scheme it is easily seen that

\[
C_k^{(m,n)} = (-)^k (m - n + k + 1) (m - n + k + 2) \ldots m \binom{n}{k}.
\] (98)

Therefore, we finally obtain

\[
P(m, n) = \sum_{k=0}^{n} (-)^k \frac{m! n!}{(m-n+k)! (n-k)!} \alpha^k = \sum_{k=0}^{n} (-)^{n-k} \frac{m! n!}{(m-k)! (n-k)!} (\alpha^*)^{m-k} \alpha^{n-k}. \tag{99}
\]

As mentioned before, here we have assumed that \( m \geq n \). For \( m < n \) by contrast, we use:

\[
f(m, n, \alpha) = f^*(n, m, -\alpha) \]

so that in summary, we may write:

\[
f(m, n, \alpha) = \pi (-)^{m-p} \sum_{k=0}^{p} (-)^k k! \binom{n}{k} (\alpha^*)^{m-k} \alpha^{n-k} e^{-\alpha^* \alpha} \quad p = \min(m, n). \tag{100}
\]

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