AUTOMORPHISMS AND ISOMORPHISMS OF CHEVALLEY GROUPS AND ALGEBRAS

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Dedicated to Rebecca

An adjoint Chevalley group of rank at least 2 over a rational algebra (or a similar ring), its elementary subgroup, and the corresponding Lie ring have the same automorphism group. These automorphisms are explicitly described.

0. Introduction

Suppose that $\Phi$ is a reduced irreducible root system, $R$ is an associative commutative ring with unity, $G(\Phi, R)$ is the corresponding adjoint Chevalley group, and $E(\Phi, R)$ is its elementary subgroup (see Section 5).

There are a lot of results (see, e.g., [Wat80], [Pet82], [GMi83], [HO'M89], [Abe93], [Che00], [Bun07], and references therein*) asserting that, under some conditions, all automorphisms of Chevalley (or similar) groups are standard in some sense (depending on what a particular author succeeded to prove). In this paper, we use the most universal and natural definition of standardness suggested by A. E. Zaleskii [Zal83]: an automorphism of an adjoint Chevalley group is called standard if it is induced by an automorphism of the corresponding Lie algebra. More precisely, this means the following. Clearly, $E(\Phi, R)$ and $G(\Phi, R)$ embed naturally into the automorphism group of the corresponding Lie algebra $L(\Phi, R)$ over $R$. A slightly less obvious fact is that (under some conditions, see Section 5) both groups are normal in $\text{Aut}_R L(\Phi, R)$ and even in the larger group $\text{Aut}_Z L(\Phi, R) = \text{Aut}_Z R \rtimes \text{Aut}_R L(\Phi, R)$ consisting of the automorphisms of this algebra considered as a Lie ring. Thus, each automorphism $f \in \text{Aut}_Z L(\Phi, R)$ of the Lie ring induces an automorphism $f': g \mapsto fgf^{-1}$ of the Chevalley groups $G(\Phi, R)$ and $E(\Phi, R)$. The main results of this paper are the following theorems.

Automorphism theorem. For any reduced irreducible root system $\Phi$ of rank $\geq 2$, there exists an integer $m$ such that, for any associative commutative ring $R$ without additive torsion, with unity and $\frac{1}{m}$, all automorphisms of the Chevalley group $G(\Phi, R)$ and its elementary subgroup $E(\Phi, R)$ are standard; the groups $\text{Aut}_Z L(\Phi, R)$, $\text{Aut}_Z R \rtimes \text{Aut}_R L(\Phi, R)$, $\text{Aut}_Z L(\Phi, R)$, and $\text{Aut}_Z R \rtimes \text{Aut}_R L(\Phi, R)$ are isomorphic; the map $\text{Aut}_Z L(\Phi, R) \ni f \mapsto f' \in \text{Aut}_Z G(\Phi, R)$ is a group isomorphism; a similar map $\text{Aut}_Z L(\Phi, R) \to \text{Aut}_Z E(\Phi, R)$ is a group isomorphism also.

Isomorphism theorem. For any reduced irreducible root system $\Phi$ of rank $\geq 2$, there exists an integer $m$ such that, for any associative commutative rings $R$ and $R'$ without additive torsion, with unity and $\frac{1}{m}$, there are natural one-to-one correspondences between the following three sets:

$$\{\text{group isomorphisms } G(\Phi, R) \to G(\Phi, R') \}, \quad \{\text{group isomorphisms } E(\Phi, R) \to E(\Phi, R') \},$$

and

$$\{\text{Lie ring isomorphisms } L(\Phi, R) \to L(\Phi, R') \},$$

i.e., each group isomorphism $G(\Phi, R) \to G(\Phi, R')$ maps $E(\Phi, R)$ onto $E(\Phi, R')$; each group isomorphism $E(\Phi, R) \to E(\Phi, R')$ can be extended uniquely to an isomorphism $G(\Phi, R) \to G(\Phi, R')$; each ring isomorphism $f: L(\Phi, R) \to L(\Phi, R')$ induces a group isomorphism $\text{Aut}_Z L(\Phi, R) \supseteq G(\Phi, R) \to G(\Phi, R') \subseteq \text{Aut}_Z L(\Phi, R')$ by the formula $\varphi \mapsto f \varphi f^{-1}$; each group isomorphism $G(\Phi, R) \to G(\Phi, R')$ is induced by such a way by a unique ring isomorphism.

Each ring isomorphism $f: L(\Phi, R) \to L(\Phi, R')$ is semilinear, i.e., $f(rx) = \alpha(r)f(x)$ for some ring isomorphism $\alpha: R \to R'$ uniquely determined by $f$.

In particular, these theorems allow us to describe the automorphisms of all adjoint Chevalley groups of rank at least 2 over any commutative $\mathbb{Q}$-algebras. Similar results were obtained by Yu Chen [Che95], [Che96] (see also [Che00]), but he assumed additionally that $R$ is an algebra over $\mathbb{Q}$ without zero divisors.

The idea of describing the automorphisms of linear groups by the passage to the Lie algebras was first introduced and applied by V. M. Levchuk [Lev83] and E. I. Zelmanov [Zel85]. We use this general idea, but our approach is quite different.

The above theorems reduce the problem of finding automorphisms/isomorphisms of Chevalley groups to an (easier) analogous problem for Chevalley algebras. The automorphisms of Chevalley algebra are explicitly described in Section 7. Each automorphism of $L(\Phi, R)$ is a composition of an inner automorphism (i.e., a conjugation by an element of $G(\Phi, R)$) and automorphisms induced by symmetries of the corresponding Dynkin diagram.

Our proofs are completely calculation-free and use only few properties of Chevalley groups. Thus, this approach can work in a more general setting. An elementary group scheme $E$ is a subgroup of $\text{SL}_n(\mathbb{Z}[z_1, z_2, \ldots])$ generated by

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* Unfortunately, some interesting papers on this subject (e.g., [Abe93]) contain errors.
by some matrices \(\{x_i(z_i) : i \in I, j = 1, 2, \ldots\}\). For an elementary group scheme \(E\), the symbol \(E(R)\) denotes the subgroup of \(\text{SL}_n(R)\) consisting of all matrices of the form \(a(r_1, r_2, \ldots)\), where \(a \in E \) and \(r_j \in R\). We say that \(E(R)\) is an \(n\)-dimensional \(R\)-group. Clearly, \(E(R)\) is generated by the matrices \(\{x_i(r) : i \in I, r \in R\}\). For an \(R\)-group \(E(R)\) we consider the following conditions:

(EX) **Exponentiality:** \(x_i(z_1)x_i(z_2) = x_i(z_1 + z_2)\) for all \(i \in I\).

(AL) **Algebraicity:** \(E(R[t])\) is a normal subgroup of a linear algebraic group \(G \subseteq \text{SL}_n(R[t])\) defined by some polynomial equations with integer coefficients. The group \(E(R[t])\) is the normal closure of its subgroup \(E(R)\).

(PS) **Power conjugacy:** two matrices \(x_i \) and \(x_i^s\) are conjugate in \(E(R)\) for each \(i \in I\) and each \(s \in S\), where \(S \subseteq \mathbb{Z}\) is a set of integers.

**Example.** In Section 5, we show that, under the assumption of the above theorems, an adjoint elementary Chevalley group \(E(\Phi, R)\) has properties (EX), (AL), and (PC_S), where \(S = \mathbb{Z} \cap \{a^2 : a \in R^+\}\).

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1. Nullstellensatz
Recall that an ideal is called radical if the corresponding factor ring has no nonzero nilpotents. We use the following form of Hilbert’s Nullstellensatz.

**Nullstellensatz.** Suppose that \(g, f_1, \ldots, f_t \in \mathbb{Z}[y_1, \ldots, y_m]\) are some polynomials and the quasi-identity

\[
\forall r_1, \ldots, r_t \in R \quad f_1(r_1, \ldots) = 0 \land \ldots \land f_t(r_1, \ldots) = 0 \quad \implies \quad g(r_1, \ldots) = 0
\]

holds for \(R = \mathbb{C}\). Then there exists positive integer \(b\) such that the quasi-identity

\[
\forall r_1, \ldots, r_t \in R \quad f_1(r_1, \ldots) = 0 \land \ldots \land f_t(r_1, \ldots) = 0 \quad \implies \quad (g(r_1, \ldots))^b = 0
\]

holds for any associative commutative ring \(R\) with unity and without additive torsion. If the ideal of \(\mathbb{Z}[y_1, \ldots, y_m]\) generated by \(f_1, \ldots, f_t\) is radical, then we can take \(b = 1\).

2. Unipotence
A matrix \(A\) is called unipotent if \(A - 1\) is a nilpotent matrix. We say that an automorphism of an \(R\)-group \(E(R)\) is unipotent if it sends all \(x_i(r)\) to unipotent matrices. An automorphism \(\varphi\) is said to be \(m\)-unipotent if \((\varphi(x_i(r)) - 1)^m = 0\) for all \(i \in I\) and \(r \in R\).

**Proposition 1.** Suppose that an associative commutative ring \(R\) with unity has no additive torsion. Then, for any integers \(n \geq 1, p \geq 2, \) and \(d \geq 1\), there exist positive integers \(q\) and \(m\) such that any automorphism of an \(n\)-dimensional \(R\)-group with Property (PC_(p,q^d)) is \(m\)-unipotent.

**Proof.** If the \(R\)-group satisfies Property (PC_(p,q^d)), then, for any automorphism \(\varphi\), the matrices \(\varphi(x_i(r)), \varphi(x_j(r))^p, \) and \(\varphi(x_i(r))^q\) are conjugate. Thus, Proposition 1 is a corollary of the following lemma.

**Lemma 1.** For any integers \(n \geq 1, p \geq 2, \) and \(d \geq 1\), there exist positive integers \(q\) and \(m\) such that, if the characteristic polynomials of matrices \(A, A^p, A^{q^d} \in \text{SL}_n(R)\) over an associative commutative ring \(R\) with unity and without additive torsion coincide, then \((A - 1)^m = 0\).

**Proof.** First, assume that \(R = \mathbb{C}\). Then, these three matrices have the same set of eigenvalues and raising to the power \(p\) acts as a permutation of these eigenvalues. Hence, for any eigenvalue \(\lambda\),

\[
\lambda^{p^n - 1} = 1 \quad \text{and, by the same reason,} \quad \lambda^{d^{n!} - 1} = 1.
\]

Clearly, these equalities imply \(\lambda = 1\) if we take, e.g., \(q = p^n - 1\) (then \(p^n - 1\) and \(q^{d^{n!}} - 1\) are coprime). So, the assertion is proven for the case \(R = \mathbb{C}\).

The condition

\[
\det A = 1 \quad \text{and the characteristic polynomials of} \quad A, A^p, \quad \text{and} \quad A^{q^d} \quad \text{coincide}
\]

is a system of integer-coefficient polynomial equations on entries of matrix \(A\). Each complex root \(B \in M_n(\mathbb{C})\) of this system is unipotent (if \(q\) is chosen as above). By Nullstellensatz, this implies that, if a matrix \(A\) over \(R\) satisfies \((*)\), then each entry \(c_{ij}\) of the matrix \(C = (A - 1)^n\) satisfies the equality \(c_{ij}^b = 0\) for some integer \(b\). Hence, \(C^{bn^2} = (A - 1)^{bn^3} = 0\). This completes the proof of Lemma 1 and Proposition 1.
3. Curves

Take an $R$-group $E(R)$. We say that the group $E(R[t])$ is the group of curves on the group $E(R)$.

Clearly, for any curve $g(t) \in E(R[t])$ and any polynomial $f(t) \in R[t]$, the curve $\text{REP}_f(g) \overset{\text{def}}{=} g(f(t))$ also belongs to $E(R[t])$. We say that $g(f(t))$ is the reparametrisation of the curve $g$ by means of the polynomial $f$. Thus, $\text{REP}_f: E(R[t]) \to E(R[t])$ is an endomorphism of the group of curves.

Performing fairly standard calculations

\[(1 + tX + t^2Y + o(t^2))^{-1} = 1 - (tX + t^2Y) + (tX + t^2Y)^2 + o(t^2) = 1 - tX + t^2(X^2 - Y) + o(t^2),\]

\[[1 + tX_1 + t^2Y_1 + o(t^2)), (1 + tX_2 + t^2Y_2 + o(t^2))\] = \[1 + t^2(Y_1 + Y_2 - Y_1 - Y_2 + X_1^2 + X_2^2 + X_1X_2 - X_1^2 - X_1X_2 - X_2X_1 + X_2X_1) + o(t^2) =\]

\[1 + t^2(X_1X_2 - X_2X_1) + o(t^2),\]

we obtain a fairly standard formula:

\[[(1 + tX_1 + o(t)), (1 + tX_2 + o(t))] = \text{REP}_r(1 + t[X_1, X_2] + o(t)) + o(t^2),\] (1)

where $[x, y] \overset{\text{def}}{=} xyx^{-1}y^{-1}$ is the group commutator and $[x, y] \overset{\text{def}}{=} xy - yx$ is the ring commutator.

4. Continuity and smoothness

The set of matrices $T(E(R)) = \{ X \in M_n(R) | 1 + tX + t^2Y \in E(R[t]) \}$ for some $Y \in M_n(R[t])$ is called the tangent module of an $R$-group $E(R)$. Clearly, this set is an $R[E(R)]$-module, i.e., it is closed with respect to

- addition: $(1 + tX + o(t))(1 + tY + o(t)) = 1 + t(X + Y) + o(t)$;
- multiplication by scalars: $\text{REP}_r(1 + tX + o(t)) = 1 + tXr + o(t)$;
- the action of the group $E(R)$: $g(1 + tX + o(t))g^{-1} = (1 + tgX - g^{-1} + o(t))$ (In what follows, we put $g \circ X \overset{\text{def}}{=} gXg^{-1}$).

If the tangent module is a Lie algebra, i.e., if it is closed with respect to ring commutator $[A, B] = AB - BA$, we call this module the tangent algebra. We say that an $n$-dimensional $R$-group $E(R)$ is adjoint if $T(E(R))$ is a Lie algebra isomorphic as an $R[E(R)]$-module to $\mathbb{R}^n$ (with the natural action of $E(R)$).

We say that an automorphism $\varphi$ of an $R$-group $E(R)$ is quasicontinuous if it can be extended to an automorphism $\tilde{\varphi}$ of the group of curves such that $\tilde{\varphi}$ commutes with all integer-coefficient reparametrisations: $\tilde{\varphi}(\text{REP}_f(g)) = \text{REP}_f(\tilde{\varphi}(g))$ for all $g \in E(R[t])$ and all $f \in \mathbb{Z}[t]$. The automorphism $\varphi$ is called continuous if it is quasicontinuous, the automorphism $\tilde{\varphi}$ is quasicontinuous, the automorphism $\tilde{\varphi}$ is quasicontinuous, and so on (infinitely many times).

Put $E_k(R) \overset{\text{def}}{=} E(R[t]) \cap (1 + t^kM_n(R[t]))$. Since $\text{ker REP}_0 = E_1(R)$, we have the equality $\tilde{\varphi}(E_1(R)) = E_1(R)$ for any continuous automorphism $\varphi$. We say that a continuous automorphism $\varphi$ is smooth (two times differentiable) if $\tilde{\varphi}(E_k(R)) = E_k(R)$ for $k = 1, 2, 3$. Note that the continuity [smoothness] of an automorphism implies the continuity [smoothness] of the inverse automorphism. In Section 5, we show that any continuous automorphism of a Chevalley group is smooth (under some conditions).

**Proposition 2.** Any smooth automorphism $\varphi$ of a group $E(R)$ induces an automorphism $d\varphi$ (the differential of $\varphi$) of the tangent module considered as an abelian group. In addition, we have

$$\tilde{\varphi}(1 + tX + o(t)) = 1 + td\varphi(X) + o(t) \quad \text{and} \quad d\varphi(g \circ X) = \varphi(g) \circ d\varphi(X) \quad \text{for all } g \in E(R) \text{ and } X \in T(E(R)).$$

If $T(E(R))$ is a Lie algebra, then $d\varphi$ is an automorphism of this algebra considered as a Lie ring.

**Proof.** If $X \in T(E(R))$, then $1 + tX + t^2Y \in E(R[t])$ for some $Y \in M_n(R[t])$ and $\tilde{\varphi}(1 + tX + t^2Y) = 1 + tZ + o(t)$ for some $Z \in M_n(R)$ (because $E_1(R)$ is an invariant subgroup). Put $d\varphi(X) = Z$. This is well defined, because $\tilde{\varphi}$ leaves invariant the subgroup $E_2(R)$. The bijectivity of $d\varphi$ follows from the smoothness of $\varphi^{-1}$. The equalities

$$\tilde{\varphi}(1 + tX + o(t))(1 + tY + o(t)) = \tilde{\varphi}(1 + t(X + Y) + o(t)) = 1 + td\varphi(X + Y) + o(t)$$

$$\tilde{\varphi}(1 + tX + o(t))\tilde{\varphi}(1 + tY + o(t)) = (1 + td\varphi(X) + o(t))(1 + td\varphi(Y) + o(t)) = 1 + t(d\varphi(X) + d\varphi(Y)) + o(t)$$

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Proposition 4. Suppose that a commutative associative ring
This follows immediately from Proposition 2, we can take
proves the equality
Therefore,
Indeed, it is sufficient to prove this for
Thus, \( [d\varphi(X_1), d\varphi(X_2)] = d\varphi([X_1, X_2]) \). This proves that \( d\varphi \) is an endomorphism of the tangent algebra.

If \( E(R) \) is adjoint, then it embeds naturally into the automorphism group \( \text{Aut}_{\mathbb{Z}} T(E(R)) \) of its tangent algebra considered as a Lie ring.

**Proposition 3.** Any smooth automorphism of an adjoint \( R \)-group \( E(R) \) is standard, i.e., it has the form \( \varphi(g) = \alpha(g)\varphi^{-1} \), where \( \alpha \) is an automorphism of the Lie ring \( T(E(R)) \) normalising the subgroup \( E(R) \).

**Proof.** This follows immediately from Proposition 2, we can take \( \alpha = d\varphi \).

**Proposition 4.** Suppose that a commutative associative ring \( R \) with unity and \( \frac{1}{q!} \) has no additive torsion, an \( R \)-group \( E(R) \) has properties (EX) and (AL), and \( \varphi \) and \( \varphi^{-1} \) are mutually inverse \( q \)-unipotent automorphisms of \( E(R) \). Then these automorphisms are continuous.

**Proof.** Take a matrix \( a(t) \in E(R[t]) \). Clearly, \( a(r) \in E(R) \) for any \( r \in R \). Let us prove that
the matrix \( \varphi(a(k)) \) depends polynomially on the number \( k \in \mathbb{Z} \),
i.e., there exists a matrix \( b_a(t) \in \text{SL}_n(R[t]) \) such that \( \varphi(a(k)) = b_a(k) \) for all \( k \in \mathbb{Z} \). (Note that the absence of additive torsion implies the uniqueness of such matrix \( b_a(t) \).)

Indeed, it is sufficient to prove this for \( a(t) = x_i(rt^l) \), because these matrices generate the group \( E(R[t]) \). So,
\[
\varphi(x_i(rt^l)) = (\varphi(x_i(r)))^{kl} \quad \text{by Property (EX)}.
\]

But \( (\varphi(x_i(r)))^m \) depends polynomially on \( m \), as the matrix \( \varphi(x_i(r)) \) is unipotent:
\[
(\varphi(x_i(r)))^m = (1 + A)^m = 1 + mA + \frac{m(m-1)}{2} A^2 + \ldots + \frac{m(m-1)\ldots(m-q+1)}{q!} A^q.
\]
Thus, we can extend the automorphism \( \varphi \) to the group \( E(R[t]) \) putting \( \varphi(a(t)) \equiv b_a(t) \).

Let us prove that \( \tilde{\varphi}(a(t)) = b_a(t) \) lies in \( E(R[t]) \). For each integer \( k \), the matrix \( b_a(k) \) belongs to \( E(R) \) and, hence, belongs to the group \( G \) defined by integer-coefficient polynomial equations (see Property (AL)). Therefore, the matrix \( b_a(k) \) satisfies the same equations. Thus, \( b_a(t) \in G \) and we have
\[
E(R) = \tilde{\varphi}(E(R)) \subseteq E(R[t]) \quad \text{and} \quad E(R[t]) = \langle E(R) \rangle_{E(R[t])} \subseteq \langle E(R) \rangle_G \supseteq \langle E(R) \rangle_{\tilde{\varphi}(E(R[t])} = \tilde{\varphi}(E(R[t]) \quad \text{(by Property (AL))}
\]
\[
\tilde{\varphi}(E(R[t])) \subseteq G \quad \text{and} \quad E(R[t]),
\]
where \( \langle X \rangle_H \) means the normal closure of a set \( X \) in a group \( H \). Thus, \( \tilde{\varphi}(E(R[t])) \subseteq E(R[t]) \).

The automorphism \( \varphi^{-1} \) also can be extended to the group of curves and \( \tilde{\varphi}(\varphi^{-1})(a(k)) = (\varphi^{-1})\tilde{\varphi}(a(k)) = a(k) \) for any \( k \in \mathbb{Z} \) and any \( a(t) \in E(R[t]) \). Clearly, this implies the equalities \( \tilde{\varphi}(\varphi^{-1})(a(t)) = (\varphi^{-1})\tilde{\varphi}(a(t)) = a(t) \) (because \( R \) has no additive torsion) and the bijectivity of \( \tilde{\varphi} \).

By the construction, the automorphism \( \tilde{\varphi} \) commutes with all integer-coefficient reparametrisations. So, \( \varphi \) is quasi-continuous. Clearly, \( \tilde{\varphi} \) is also \( q \)-unipotent and, hence, quasicontinuous. Thus, an obvious induction argument completes the proof of the continuity of \( \varphi \).
5. Chevalley groups

Suppose that \( \Phi \) is a reduced irreducible root system, \( L(\Phi) \) is the corresponding simple complex Lie algebra. The algebra \( L(\Phi) \) has a basis \( h_1, h_2, \ldots, x_1, x_2, \ldots \) (the Chevalley basis) such that the structure constants are integer and the matrices of the operators \( (\text{ad} x_k) / k! \) are integer and nilpotent for all \( k \in \mathbb{N} \). The Chevalley algebra is the Lie \( R \)-algebra \( L(\Phi, R) \) with the same structure constants.

Suppose that \( N(\Phi) = \text{Aut}_\mathbb{C} L(\Phi) \) is the automorphism group of the algebra \( L(\Phi) \), and \( G(\Phi) = (\text{Aut}_\mathbb{C} L(\Phi))^\circ \) is the connected component of the identity of this group. The algebraic groups \( G(\Phi) \subseteq N(\Phi) \subseteq \text{GL}_L(\Phi) \) \( \subseteq \text{SL}_L(\mathbb{C}) \) are defined over \( \mathbb{Z} \). Let \( R \) be an associative commutative ring with unity and let \( N(\Phi, R) \) and \( G(\Phi, R) \) be the groups of \( R \)-rational points of \( N(\Phi) \) and \( G(\Phi) \), i.e., the subgroups of \( \text{SL}_L(\mathbb{R}) \) (where \( n = 1 + \dim L(\Phi) \)) defined by the same integer-coefficient polynomial equations as the groups \( N(\Phi) \) and \( G(\Phi) \), respectively, (in the Chevalley basis). Note that \( N(\Phi, R) = \text{Aut}_R L(\Phi, R) \), because the property of being an automorphism can be written as a system of integer-coefficient polynomial equations (depending on the structure constants). The group \( G(\Phi, R) \) is called the (adjoint) Chevalley group. The group \( E(\Phi, R) \subseteq G(\Phi, R) \) generated by the matrices \( x_i(r) = \exp(\text{ad} r x_i) \), where \( r \in R \), is called the elementary subgroup of the Chevalley group \( G(\Phi, R) \).

Example. For the root system \( A_1 \), we have \( L(A_1) = \text{sl}_{i+1}(\mathbb{C}) \) is the Lie algebra consisting of all traceless matrices, \( L(A_1, R) = \text{sl}_{i+1}(R) \), \( G(A_1, R) = \text{PGL}_{i+1}(R) \), and \( E(A_1, R) = \text{PE}_{i+1}(R) \) is the subgroup of \( \text{PGL}_{i+1}(R) \) generated by the images of the transvections \( 1 + rE_{ij} \), where \( i \neq j \) and \( r \in R \). (Note that, for some rings, this group \( \text{PGL}_{i+1}(R) \) can be larger than the central quotient of the general linear group \( \text{GL}_{i+1}(R) \).

In the following lemma, we summarise some (probably) known properties of Chevalley groups and algebras.

**Lemma 2.** Let \( \Phi \) be a reduced irreducible root system of rank \( \geq 2 \) and let \( R \) be an associative commutative ring without additive torsion, with unity and \( \frac{1}{2} \). Then

(i) the group \( E(\Phi, R) \) is an \( R \)-group with Properties (EX) and (PC\( _S \)), where \( S = \mathbb{Z} \cap \{ a^2; \ a \in R^* \} \);

(ii) for each subgroup \( H \) of \( G(\Phi, R) \) normalised by \( E(\Phi, R) \), there exists a unique ideal \( J \) of \( R \) such that \( H \) is contained in \( G(\Phi, R) \cap (1 + M_\infty(J)) \) and contains the normal closure \( \langle \{ x_i(r) : r \in J \} \rangle_{E(\Phi, R)} \) of the set \( \{ x_i(r) : r \in J \} \);

(iii) \( E(\Phi, R) \) is an automorphism invariant (i.e., characteristic) subgroup of \( G(\Phi, R) \);

(iv) \( E(\Phi, R) \) satisfies Property (AL);

(v) \( \text{Aut}_R(\Phi, R) \cong \text{Aut}_R(\Phi, R) \restriction L(\Phi, R) \);

(vi) in the group \( \text{Aut}_R(\Phi, R) \), the subgroups \( G(\Phi, R) \) and \( E(\Phi, R) \) are normal and their centralisers are trivial.

**Proof.**

(i) Property (EX) follows immediately from the definition. Steinberg’s relation R5, \( h_1(s)x_i(r)h_1(s)^{-1} = x_i(s^2r) \) (see, e.g., [VP96]), where \( r \in R \), \( s \in R^* \), and \( h_1(s) \in E(\Phi, R) \) are some particular matrices, shows that Property (PC\( _S \)) holds too.

(ii) Taking into account that \( G(\Phi, R) \) is centreless in the adjoint case [AHu88], we see that (ii) is a slightly weakened form of the well-known theorem on subgroups of Chevalley groups normalised by the elementary subgroups [Vas86] (see also [Asn76], [Abe99], [Go97], [CKe99], [VGN06]).

(iii) This was also proven by Vaserstein in [Vas86]. Note that, in [HaV03], it was in fact proven the endomorphism invariance of the elementary subgroup of a Chevalley group.

(iv) The normality of \( E(\Phi, R[t]) \) in the linear algebra \( G(\Phi, R[t]) \) defined by polynomial equations with integer coefficients follows immediately from (iii). The equality \( \langle E(\Phi, R) \rangle_{E(\Phi, R[t])} = E(\Phi, R[t]) \) follows from (ii). Indeed, put \( H = \langle E(\Phi, R) \rangle_{E(\Phi, R[t])} \). The inclusion \( E(\Phi, R) \subseteq H \subseteq G(\Phi, R[t]) \cap (1 + M_\infty(J)) \) implies \( J = R[t] \). Therefore, \( E(\Phi, R[t]) = \langle \{ x_i(f) : f \in R[t] \} \rangle_{E(\Phi, R[t])} = \langle \{ x_i(f) : f \in J \} \rangle_{E(\Phi, R[t])} \subseteq H \) and \( H = E(\Phi, R[t]) \).

(v) Let \( U \) be the algebra \( L(\Phi, R) \) considered as a left module over itself. Then

\[ \text{End}_{L(\Phi, R)} U = R, \quad \text{i.e., all endomorphisms are scalar multiples of the identity.} \quad (**) \]

Indeed, this is true for \( R = \mathbb{C} \), because the algebra \( L(\Phi, \mathbb{C}) \) is simple. Therefore, (**) holds for any \( R \) without additive torsion, because both conditions on a matrix, being an endomorphism of \( U \) and being a scalar multiple of the identity, are integer-coefficient systems of linear equations on the entries of the matrix.

Note that (**) remains valid if we consider \( U \) as a module over Lie ring \( L(\Phi, R) \), i.e., each endomorphism \( f \) of \( U \) must be \( R \)-linear. Indeed, for any \( u \in U \), there exist \( y_i \in L(\Phi, R) \) and \( u_i \in U \) such that \( u = \sum (\text{ad} y_i)(u_i) \), because \( L(\Phi, R) = [L(\Phi, R), L(\Phi, R)] \). Therefore,

\[ ru = \sum (\text{ad} r y_i)(u_i) \quad \text{and} \quad f(ru) = f \left( \sum (\text{ad} r y_i)(u_i) \right) = \sum (\text{ad} y_i)f(u_i) = r \sum (\text{ad} y_i)(u_i) = rf(u). \]

Now, take an automorphism \( \varphi \) of the ring \( L(\Phi, R) \) and consider the algebra \( L(\Phi, R) \) as an \( L(\Phi, R) \)-module \( U_\varphi \) with action \( (y, u) \mapsto (\text{ad} \varphi(y))u \). The mapping \( u \mapsto \varphi(u) \) is an isomorphism between the modules \( U \) and \( \text{End}_{L(\Phi, R)} U \).
and $U_\varphi$ over the Lie ring $L(\Phi, R)$. This isomorphism induces an isomorphism of endomorphism rings $R = \text{End}_{L(\Phi, R)}U \xrightarrow{\alpha \mapsto \text{End}_{L(\Phi, R)}U_\varphi} R$. Thus, we have a homomorphism $\text{Aut}_z L(\Phi, R) \to \text{Aut}_z R$, $\varphi \mapsto \alpha_\varphi$, whose kernel is $\text{Aut}_R L(\Phi, R)$. The right inverse homomorphism $\text{Aut}_z R \to \text{Aut}_z L(\Phi, R)$ maps $\alpha \in \text{Aut}_z R$ to the obvious automorphism of the Lie ring $L(\Phi, R) = L(\Phi, Z) \otimes R$ induced by $\alpha$. So, we obtain the required decomposition of $\text{Aut}_z L(\Phi, R)$ into the semidirect product.

(vi) **Normality.** By virtue of (iii), it is sufficient to prove the normality of $G(\Phi, R)$. For $R = \mathbb{C}$, this property is well known, see, e.g., [VOn88]. Let $F_N(y_{ij}) = 0$ and $F_G(y_{ij}) = 0$ be systems of integer-coefficient polynomial equations that define the groups $N(\Phi, R) = \text{Aut}_R L(\Phi, R)$ and $G(\Phi, R)$ (these systems do not depend on $R$). We assume that the ideals of $\mathbb{Z}[y_{11}, y_{12}, \ldots, y_{nn}]$ generated by the sets of polynomials $F_N(y_{ij})$ and $F_G(y_{ij})$ are radical. For $R = \mathbb{C}$ we have the quasi-identity

\[ F_G(Y) = 0 \& F_N(Z) = 0 \implies F_G(ZY^{-1}) = 0. \] 

(2)

Since the ideal of $\mathbb{Z}[y_{11}, y_{12}, \ldots, y_{nn}, z_{11}, z_{12}, \ldots, z_{nn}]$ generated by $F_G(Y)$ and $F_N(Z)$ is radical, Nullstellensatz implies that quasi-identity (2) holds for all rings $R$ without additive torsion. Thus, $G(\Phi, R)$ is a normal subgroup of $N(\Phi, R)$.

**Centralisers.** For $R = \mathbb{C}$, the centraliser of the set $\{x_{i}(1)\}$ in $\text{Aut}_R L(\Phi, R)$ is trivial. Therefore, the same is true for any ring $R$ without additive torsion (by Nullstellensatz). Thus, the centraliser of the set $\{x_{i}(1)\}$ in the group $\text{Aut}_z L(\Phi, R) = (\text{Aut}_z R) \times \text{Aut}_R L(\Phi, R)$ coincides with $\text{Aut}_z R$. On the other hand, each nontrivial ring automorphism $\alpha \in \text{Aut}_z R$ induces a nontrivial automorphism $x_i(r) \mapsto x_i(\alpha(r))$ of $E(\Phi, R)$. Therefore, the centraliser of $E(\Phi, R)$ in the group $\text{Aut}_z L(\Phi, R)$ is trivial. This completes the proof of Lemma 2.

**Proposition 5.** Let $\Phi$ be a reduced irreducible root system of rank $\geq 2$ and let $R$ be an associative commutative ring without additive torsion, with unity and $\frac{1}{2}$. Then any retraction $\pi: E(\Phi, R[t]) \to E(\Phi, R)$ (i.e., a homomorphism such that $\pi^2 = \pi$) has the form $E(\Phi, R[t]) \ni a(t) \mapsto a(r) \in E(\Phi, R)$ for some $r \in R$. In other words, $\pi = \text{REP}_r$.

**Proof.** According to Lemma 2 (ii),

\[ \langle \{x_{i}(f) : f \in J\} \rangle_{E(\Phi, R[t])} \subseteq \ker \pi \subseteq E(\Phi, R[t]) \cap (1 + M_n(J)) \quad \text{for some ideal } J \text{ of } R[t]. \]

The right-hand inclusion and the equality $E(\Phi, R[t]) = (E(\Phi, R) \cap \ker \pi)$ show that $t-r \in J$ for some $r \in R$; the left-hand inclusion and the equality $E(\Phi, R) \cap \ker \pi = \{1\}$ show that $J = (t-r)R[t]$. Therefore, $\ker \pi = (E(\Phi, R[t]) \cap (1 + M_n(J))$ and $\pi = \text{REP}_r$.

Thus, we have a natural one-to-one correspondence between the ring $R$ and the set of retractions. Clearly, the ring structure on $R$ can also be described in terms of retractions and integer-coefficient reparametrisations:

\[ \text{REP}_{r \mapsto r'}: E(\Phi, R[t]) \xrightarrow{t \mapsto t'} E(\Phi, R[t', t']) \xrightarrow{t \mapsto r'} E(\Phi, R), \]

\[ \text{REP}_{r \mapsto r'}: E(\Phi, R[t]) \xrightarrow{t \mapsto t'} E(\Phi, R[t', t']) \xrightarrow{t \mapsto r'} E(\Phi, R). \]

(3)

Proposition 5 and these formulae imply that any continuous automorphism $\varphi \in \text{Aut}(E(\Phi, R))$ induces a ring automorphism $\hat{\varphi} \in \text{Aut}_z R$ by the formula $\varphi \text{REP}_r \hat{\varphi}^{-1} = \text{REP}_r$:

\[ E(\Phi, R[t]) \xrightarrow{\hat{\varphi}} E(\Phi, R[t]) \xrightarrow{\text{REP}_r} E(\Phi, R). \]

For each ideal $J \preceq R$ we have two normal subgroups of $E(\Phi, R)$, namely, $E(J)_{\max} \overset{\text{def}}{=} E(\Phi, R) \cap (1 + M_n(J))$ and $E(J)_{\min} \overset{\text{def}}{=} \langle \{x_{i}(r) : r \in J\} \rangle_{E(\Phi, R)}$.

**Lemma 3.** Let $\Phi$ be a reduced irreducible root system of rank $\geq 2$ and let $R$ be an associative commutative ring without additive torsion, with unity and $\frac{1}{2}$. Then $\varphi(E(J)_{\min}) = E(\hat{\varphi}(J))_{\min}$ and $\varphi(E(J)_{\max}) = E(\hat{\varphi}(J))_{\max}$ for any continuous automorphism $\varphi$ of the group $E(\Phi, R)$.

**Proof.** Clearly, it is sufficient to define $E(J)_{\min}$ and $E(J)_{\max}$ in terms of retractions. The subgroup $E_1(\Phi, R) \overset{\text{def}}{=} E(\Phi, R[t]) \cap (1 + tM_n(R[t]))$ can be defined as $E_1(\Phi, R) = \ker \text{REP}_0$ (hence, this subgroup is $\hat{\varphi}$-invariant). Then, $E(J)_{\min} = \langle \{\text{REP}_r(a(t)) : r \in J, a(t) \in E_1(\Phi, R)\} \rangle_{E(\Phi, R)}$.

The inclusion $\supseteq$ follows from the equality $E_1(R) = \langle \{x_{i}(r^{i}) : i \in I, r \in R, k = 1, 2 \ldots \} \rangle_{E(\Phi, R[t])}$, which is valid for any $R$-group with Property (EX).

\[ E(J)_{\max} = \text{the (unique) maximal subgroups among all normal subgroups } H \quad \text{such that } E(J)_{\min} \subseteq H \quad \text{and } E(J')_{\min} \not\subseteq H \text{ for any ideal } J' \not\subset J. \]

The correctness of this definition of $E(J)_{\max}$ follows from Lemma 2 (ii) and the equality $E(J_{1} + J_{2})_{\min} = E(J_{1})_{\min} \cdot E(J_{2})_{\min}$.
Lemma 4. Let $\Phi$ be a reduced irreducible root system of rank $\geq 2$ and let $R$ be an associative commutative ring without additive torsion, with unity and $\frac{1}{k}$. Then any continuous automorphism $\varphi$ of $E(\Phi, R)$ is smooth.

Proof. We have to prove that the subgroups $E_k(\Phi, R) \equiv E(\Phi, R[t]) \cap (1 + t^k M_n(R[t]))$ are $\varphi$-invariant. This is true for $k = 1$, because $E_1(\Phi, R) = \text{ker REP}_0$. On the other hand, $E_1 = E((tR[t])_{\text{max}}$. Hence, the ideal $tR[t] \cap R[t]$ is $\varphi$-invariant by Lemma 3. Then, the ideal $(tR[t])^k$ is $\varphi$-invariant and the subgroup $E_k(\Phi, R) = E((tR[t])^k)_{\text{max}}$ is $\varphi$-invariant.

Proposition 6. The tangent module of a Chevalley group coincide with the corresponding Lie algebra: $T(E(\Phi, R)) = L(\Phi, R)$.

Proof. Suppose that $X \in T(E(\Phi, R))$, i.e., $1 + tX + o(t) \in E(\Phi, R[t])$. Let us express this element via the generators:

$$1 + tX + o(t) = \prod_j x_j(r_j t^{k_j})$$

Clearly, we can assume that $k_j \in \{0, 1\}$. Also, the substitution $t = 0$ shows that

$$\prod_j x_j(r_j) = 1, \text{ where the prime means that the product is taken over all } j \text{ such that } k_j = 0.$$ 

Therefore, the expression (4) can be rewritten in the form

$$1 + tX + o(t) = \prod_t g_t x_t(r_t) g_t^{-1}, \text{ where } g_t \in E(\Phi, R).$$

Hence, $X = \sum g_t \circ r_t x_t \in L(\Phi, R)$ and $T(E(\Phi, R)) \subseteq L(\Phi, R)$.

Let us prove the opposite inclusion. Clearly, $T(E(\Phi, R))$ contains the nilpotent part \{x_1\} of the Chevalley basis: $x_1(t) = \exp(tx_1) = 1 + tx_1 + o(t)$. The remaining basis vectors $h_i$ lie in $T(E(\Phi, R))$ also, because $h_i = x_i(1) \circ x_i + x_i - x_i - x_i$ (see, e.g., [Bor70]). This completes the proof.

In particular, Proposition 6 shows that any adjoint Chevalley group is adjoint in the sense of Section 4.

6. Proof of the main theorems

The automorphisms of $E(\Phi, R)$. By Lemma 2 (vi), we have the natural injective homomorphism $\Pi: \text{Aut}_Z L(\Phi, R) \to \text{Aut} E(\Phi, R)$. By Proposition 1 and Lemma 2 (i), each automorphism of $E(\Phi, R)$ is unipotent (for a suitably chosen $\ell$), and, hence, continuous (by Proposition 4 and Lemma 2 (i) and (iv)) and, therefore, smooth by Lemma 4. Then the map $\Pi$ is surjective by Propositions 3 and 6. Thus, $\text{Aut} E(\Phi, R) \simeq \text{Aut}_Z L(\Phi, R) \simeq \text{Aut}_Z R \times \text{Aut}_R L(\Phi, R)$ (the latter isomorphism holds by Lemma 2 (vi)).

The automorphisms of $G(\Phi, R)$ are the same as of $E(\Phi, R)$. Indeed, each automorphism of $E(\Phi, R)$ is standard and, hence, can be extended to an automorphism of $G(\Phi, R)$ by Lemma 2 (vi). Thus, the natural map $\text{Aut} G(\Phi, R) \to \text{Aut} E(\Phi, R)$ is surjective (and well defined by Lemma 2 (iii)). This map is also injective, because of Lemma 2 (vi) and the following general fact.

Lemma 5. If $A$ is an automorphism invariant subgroup of a group $B$ and the centraliser of $A$ in $B$ is trivial, then the natural map $p: B \to A$ is injective.

Proof. For any $\varphi \in \text{ker } p$, $a \in A$, and $b \in B$, we have $bab^{-1} = \varphi(bab^{-1}) = \varphi(b)\varphi(a)\varphi(b^{-1}) = \varphi(b)a\varphi(b^{-1})$. Therefore, $\varphi$ centralises $A$. Hence, $b = \varphi(b)$ for any $b \in B$. This completes the proof of the automorphism theorem.

The isomorphism theorem is an easy corollary of the automorphism theorem. Each isomorphism of Chevalley groups $\sigma: G(\Phi, R) \to G(\Phi, R')$ induces an automorphism $\varphi_\sigma$ of the group $G(\Phi, R \times R')$, because this group is isomorphic to $G(\Phi, R) \times G(\Phi, R')$ and we can put $\varphi_\sigma(g, g') = (\sigma^{-1}(g'), \sigma(g))$. The standardness of $\varphi_\sigma$ implies that $\sigma$ is induced by an isomorphism of the corresponding Lie rings. A similar argument applies to elementary subgroups.

7. Automorphisms of Chevalley algebras

Recall that an inner automorphism of a Chevalley algebra $L(\Phi, R)$ is a conjugation $x \mapsto gxg^{-1}$ by an element $g$ of the corresponding Chevalley group $G(\Phi, R)$. Clearly, the inner automorphisms form a group isomorphic to $G(\Phi, R)$.

Let $\Delta = \{\delta_1, \ldots, \delta_d\}$ be the symmetry group of the Dynkin diagram of $\Phi$ (the number $d$ can be 1, 2, or 6, depending on $\Phi$) and let $R = R_1 \oplus \ldots \oplus R_d$ be a (possibly trivial) decomposition of the ring $R$ into a direct sum of ideals. Suppose that $f_j \in \text{Aut}_R L(\Phi, R)$ is the automorphism induced by the symmetry $\delta_j$ (see [Von88]). The automorphism $f$ of the algebra $L(\Phi, R) = L(\Phi, R_1) \oplus \ldots \oplus L(\Phi, R_d)$ that sends $x_1 + \ldots + x_d$ to $f_1(x_1) + \ldots + f_d(x_d)$, where $x_i \in L(\Phi, R_i)$, is called a diagram automorphism of the algebra $L(\Phi, R)$. Clearly, diagram automorphisms form a group isomorphic to the subgroup

$$D(\Phi, R) = \left\{ \sum e_i \delta_i \mid e_i \in R, \ e_i^2 = e_i, \ e_i e_j = 0 \text{ for } i \neq j, \ \sum e_i = 1 \right\}$$

of the group of units of the group algebra $R\Delta$. 

7
Theorem 1. Let $R$ be an associative commutative ring without additive torsion, with unity and $\frac{1}{2}$ and let $\Phi$ be a reduced irreducible root system. Then any automorphism $f$ of the Lie $R$-algebra $L(\Phi, R)$ can be expressed uniquely as a composition of diagram and inner automorphisms, $\text{Aut}_R L(\Phi, R) \simeq D(\Phi, R) \times G(\Phi, R)$.

Proof. Let $n$ be the dimension of the Lie algebra $L(\Phi)$. Consider the ideal $J$ in $\mathbb{Z}[x_{11}, x_{12}, \ldots, x_{nn}]$ defining the group $\text{Aut}_{\mathbb{Z}} L(\Phi)$. The ideal $J$ decomposes into a product $J = J_1 J_2 \ldots J_d$ of prime ideals $J_i$ corresponding to irreducible (= connected) components $h_i G(\Phi)$ of the group $\text{Aut}_{\mathbb{Z}} L(\Phi)$, where $h_i$ are integer matrices of diagram automorphisms. Take a matrix $A = (a_{pq}) \in \text{Aut}_R L(\Phi, R)$. Then $f(a_{pq}) = 0$ for $f \in J$. Put $I_i = \{ f(a_{pq}) : f \in J_i \} \subset R$. Then

(i) $\prod I_i = \{0\};$

(ii) $I_i + I_j = R$ for $i \neq j$ (otherwise we take the factor ring by a maximal ideal $M \supseteq I_i + I_j$ and obtain a matrix $A_M$ belonging to the intersection of two irreducible components of the group $\text{Aut}_{R/M} L(\Phi, R/M)$, but this intersection is empty, because $R/M$ is a field).

These conditions (i) and (ii) imply that the ring $R$ is the direct sum $R = \bigoplus R/I_i$ [Bou61, Ch.2 §1, Proposition 5]. So, $A = \sum A_{I_i}$, and the entries of the matrix $A_{I_i} \in M_n(R/I_i)$ satisfy the equations $f(a_{pq}) = 0$ for $f \in I_i$. Therefore, $A_{I_i} = h_i g_i \in h_i G(\Phi, R/I_i)$ and $A = (\sum e_i h_i)(\sum g_i)$, where $e_i$ is the unity of the ring $R/I_i$. This completes the proof.

Another approach to describing the automorphisms of Chevalley algebras was suggested in [Pla02].

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