Jacobi-Maupertuis metric of Liénard type equations and Jacobi Last Multiplier

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1 Introduction

Nonlinear differential equations of the Liénard type occupy a special place in the study of dynamical systems as they serve to model various physical, chemical and biological processes. The standard Liénard equation involves a dissipative term depending linearly on the velocity. However there are practical problems in which higher order dependence on velocities are appropriate. Such equations have the generic form $\ddot{x} + f(x)\dot{x}^2 + g(x) = 0$. It is interesting to note that equations of this type naturally arise in Newtonian dynamics when the mass, instead of being a constant, is allowed to vary with the position coordinate— the so called position dependent mass (PDM) scenario. There is also an alternative mechanism in which this dependence on a mass function manifests itself in the context of differential systems, namely through Jacobi’s last multiplier (JLM). The JLM originally arose in the problem of reducing a system of first-order ordinary differential equations to quadrature and has a long and chequered history. In recent

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Abstract

We present a construction of the Jacobi-Maupertuis (JM) principle for an equation of the Liénard type, viz $\ddot{x} + f(x)\dot{x}^2 + g(x) = 0$ using Jacobi’s last multiplier. The JM metric allows us to reformulate the Newtonian equation of motion for a variable mass as a geodesic equation for a Riemannian metric. We illustrate the procedure with examples of Painlevé-Gambier XXI, the Jacobi equation and the Henon-Heiles system.
years its role in the context of the inverse problem of dynamical systems has led to a revival of interest in the JLM. In this brief note we examine the connection between the JLM and the principle of least action within the framework of a Liénard type differential equation with a quadratic dependance on the velocity.

It is known that the Liénard type equation is connected to the Painlevé-Gambier equations \[5, 6\]. So it is natural for us to ask whether we can reformulate the subclass of the Painlevé-Gambier family as geodesic equations for a Riemannian metric using the Jacobi-Maupertuis principle. There are several choices for a Riemannian manifold and metric tensor: a space-time configuration manifold and the Eisenhart metric (for example, \[1, 2, 3, 10\]), a configuration manifold and the Jacobi-Maupertuis metric \[8, 9\]. In this paper we choose a configuration space of an analyzed system for a Riemannian manifold. The crux of the matter is that the Hamiltonian or energy function provided by the JLM should remain constant for these equations.

**Main Result** Let \( \mathcal{V} \) be a Hamiltonian vector field of the Liénard type equation \( \ddot{x} + f(x)\dot{x}^2 + g(x) = 0 \) in \( \mathbb{R}^2 \) with Hamiltonian \( H = \frac{1}{2}M(x)\dot{x}^2 + U(x) \), where \( M(x) = \exp(2\int_{x}^{0} f(s)ds) \) and \( U(x) = \int_{x}^{0} M(s)g(s)ds \). Then by Maupertuis principle, \( \mathcal{V} \) coincides with the trajectories of the modified vector field \( \mathcal{V}' \) on the fixed isoenergy level \( H(x, \dot{x}) = E \) for the Hamiltonian \( \tilde{H} = \frac{1}{2(E-U(x))}M(x)\dot{x}^2 \). This defines a geodesic flow of some Riemannian metric given by Jacobi. In other words, solutions to the Liénard type equation with energy \( E \) are, after reparametrization, geodesics for the Jacobi-Maupertuis metric.

A corollary of the main result shows that we can reformulate the Newtonian equation of motion for a variable mass, Painlevé-Gambier XXI equation, the Jacobi equation and Henon-Heiles system in terms of geodesic flows of the Jacobi-Maupertuis metric.

The outline of the letter is as follows: in section 2 we introduce the Jacobi Last Multiplier and point out its connection to the Lagrangian of a second-order ODE. Thereafter we explicitly derive the Lagrangian and the Hamiltonian functions for a Liénard equation of the second kind, i.e., with a quadratic dependance on the velocity and highlight the role of the position dependant mass term. In section 3 we express the equation in terms of geodesic flows of the Jacobi-Maupertuis metric and some observations regarding the geometric consequences of the PDM are outlined. Explicit examples from the Painlevé-Gambier family of equations are considered along with the two-dimensional Henon-Heiles system.

### 2 Lagrangians and the Jacobi Last Multiplier

Let \( M = M(x^1, ..., x^n) \) be a non-negative \( C^1 \) function non-identically vanishing on any open subset of \( \mathbb{R}^n \), then \( M \) is a Jacobi multiplier of the vector field \( X = W^i \frac{\partial}{\partial x^i} \) if

\[
\int_{D} M(x^1, ..., x^n)dx^1...dx^n = \int_{\phi_t(D)} M(x^1, ..., x^n)dx^1...dx^n
\]

(2.1)
where $D$ is any open subset of $\mathbb{R}^n$ and $\phi_t(.)$ is the flow generated by the solution $x = x(t)$ of the system
\[ \frac{dx^i}{dt} = W^i(x^1, ..., x^n) \quad i = 1, ..., n. \] (2.2)
Thus the Jacobi multiplier can be viewed as the density associated with the invariant measure $\int_D M dx$. The divergence free condition is
\[ \frac{dM}{dt} + \frac{\partial W^i}{\partial x^i} M = 0. \] (2.3)

The appellation ‘last’ is a historical legacy. If a Jacobi multiplier is known together with $(n - 2)$ first integrals, we can reduce locally the $n$ dimensional system to a two-dimensional vector field on the intersection of the $(n - 2)$ level sets formed by the first integrals. The existence of a Jacobi Last Multiplier [4] then implies the existence of an extra first integral and the system may therefore be reduced to quadrature.

For a second-order ODE:
\[ \ddot{x} = F(x, \dot{x}, t) \quad \Rightarrow \quad \dot{x} = y, \quad \dot{y} = F(x, y, t). \] (2.4)
we have
\[ \frac{dM}{dt} + \frac{\partial F}{\partial y} M = 0. \] (2.5)

On the other hand by expanding the Euler-Lagrange equation of motion
\[ \frac{\partial L}{\partial x} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) = 0, \] (2.6)
we have
\[ \frac{\partial L}{\partial x} = \dot{y} \left( \frac{\partial^2 L}{\partial \dot{x}^2} \right) + \ddot{x} \frac{\partial L}{\partial x} + \dot{\dot{x}} \frac{\partial^2 L}{\partial \ddot{x}^2} = \dot{y} \left( \frac{\partial^2 L}{\partial \dot{x}^2} \right) + y \frac{\partial}{\partial x} \left( \frac{\partial L}{\partial x} \right). \]
Differentiating it w.r.t., $\dot{x} = y$, gives
\[ \frac{\partial}{\partial \dot{x}} \left( \frac{\partial L}{\partial x} \right) = \frac{\partial \dot{y}}{\partial y} \left( \frac{\partial^2 L}{\partial \dot{x}^2} \right) + \dot{y} \left( \frac{\partial^3 L}{\partial \dot{x}^3} \right) + \frac{\partial}{\partial \dot{x}} \left( \frac{\partial L}{\partial x} \right) + \frac{\partial^2}{\partial \dot{x}^2} \left( \frac{\partial L}{\partial x} \right), \]
\[ \Rightarrow \quad \frac{\partial F}{\partial y} \left( \frac{\partial^2 L}{\partial \dot{x}^2} \right) + \left[ \dot{y} \frac{\partial}{\partial \dot{x}} \left( \frac{\partial^2 L}{\partial \ddot{x}^2} \right) + y \frac{\partial}{\partial x} \left( \frac{\partial^2 L}{\partial \ddot{x}^2} \right) \right] = 0. \]
\[ \therefore \quad \frac{d}{dt} \left( \frac{\partial^2 L}{\partial \ddot{x}^2} \right) + \left( \frac{\partial F}{\partial y} \right) \frac{\partial^2 L}{\partial \ddot{x}^2} = 0. \] (2.7)
Thus, by comparing (2.7) to (2.5), we may identify the JLM as the following:
\[ M = \frac{\partial^2 L}{\partial \ddot{x}^2}. \] (2.8)
Given a JLM we can easily integrate (2.8) twice to obtain

\[ L(x, \dot{x}, t) = \int^{\dot{x}} \left( \int^{y} M dz \right) dy + R(x, t)\dot{x} + S(x, t). \] (2.9)

where \( R \) and \( S \) are functions arising from integration. To determine these functions we substitute the Lagrangian of (2.9) into the Euler-Lagrange equation of motion (2.6) and compare the resulting equation with the given ODE (2.4).

Consider now a Liénard equation of the second kind, viz

\[ \ddot{x} + f(x)\dot{x}^2 + g(x) = 0, \] (2.10)

where \( f \) and \( g \) are defined in a neighbourhood of \( 0 \in \mathbb{R} \). We assume that \( g(0) = 0 \), which says that \( O \) is a critical point, and \( xg(x) > 0 \) in a punctured neighbourhood of \( 0 \in \mathbb{R} \), which ensures that the origin is a centre.

**Proposition 2.1** A Liénard equation of the second kind, \( \ddot{x} + f(x)\dot{x}^2 + g(x) = 0 \), admits a Hamiltonian of the form \( H = \frac{1}{2}M(x)\dot{x}^2 + U(x) \) which is a constant of motion where \( M(x) \) is the Jacobi last multiplier and \( U(x) \) is a potential function.

**Proof:** From the definition (2.5) of the last multiplier it follows that for the equation under consideration

\[ M(x) = \exp(2F(x)) \quad \text{where} \quad F(x) = \int^{x} f(s)ds. \] (2.11)

Consequently according to (2.9), we have

\[ L = \frac{1}{2}M(x)\dot{x}^2 + R(x, t)\dot{x} + S(x, t). \] (2.12)

From the Euler-Lagrange equation one finds that the functions \( R \) and \( S \) must satisfy

\[ S_x - R_t = -M(x)g(x) \]

This gives us the freedom to set \( S = G_t - U(x) \) and \( R = G_x \) for some gauge function \( G \), so that there exists a potential function \( U(x) \) given by

\[ U(x) = \int^{x} M(s)g(s)ds. \] (2.13)

The Lagrangian then has the following appearance

\[ L = \frac{1}{2}M(x)\dot{x}^2 - U(x) + \frac{dG}{dt}. \] (2.14)
Clearly the total derivative term may be ignored and by means of the standard Legendre trans-
formation the Hamiltonian is given by
\[ H = \frac{1}{2} M(x) \dot{x}^2 + U(x). \]  
(2.15)
It is now straightforward to verify that \( \frac{dH}{dt} = 0 \) so that \( H = E \text{(say)} \) is a constant of motion. 
This complete the proof.

From (2.15) it is evident that the JLM, \( M(x) \), plays the role of a variable mass term. We 
can reduce the differential system to a unit mass problem by defining a transformation \( x \rightarrow X = \int_0^x \sqrt{M(s)} ds = \psi(x) \) whence
\[ \frac{1}{2} \dot{X}^2 + \int_0^{\psi^{-1}(X)} M(s)g(s)ds = E. \]  
(2.16)
In terms of \( X \) the equation of motion is given by
\[ \ddot{X} + e^{F(\psi^{-1}(X))} g(\psi^{-1}(X)) = 0. \]  
(2.17)

We now proceed to cover some fundamentals regarding the Jacobi metric, and deduce it for the 
Liénard equation. We mainly follow the Nair et al. formalism of Jacobi-Maupertuis principle 
and elaborate on it in the next section.

3 Jacobi-Maupertuis metric and Liénard type equation

When the Hamiltonian is not explicitly time dependent, i.e., \( H = E_0 \), a constant, then the 
solutions may be restricted to the energy surface \( E = E_0 \). Suppose \( Q \) is a manifold with local 
coordinates \( x = \{ x^i \} , i = 1, ..., n \) and \( x(\tau) \in Q \subseteq \mathbb{R}^n \) be a curve with \( \tau \in [0,T] \). Let \( T_x Q \) 
and \( T^*_x Q \) be the tangent and cotangent spaces with velocity \( \dot{x}(\tau) \in T_x Q \subseteq \mathbb{R}^n \) and momenta 
\( p(\tau) \in T^*_x Q \subseteq \mathbb{R}^n \). Denote by \( \gamma \) a curve in the manifold \( Q \) parametrized by \( t \in [a,b] \) with 
\( \gamma(a) = x_0 \) and \( \gamma(b) = x_N \). The according to the Maupertuis principle among all the curves \( x(t) \) 
connecting the two points \( x_0 \) and \( x_N \) parametrized such that \( H(x,p) = E_0 \) the trajectory of the 
Hamiltons equation of motion is an extremal of the integral of action
\[ \int_\gamma p dx = \int_\gamma p \dot{x} dt = \int_\gamma \frac{\partial L(t)}{\partial \dot{x}} \dot{x}(t) dt. \]  
(3.1)
Here \( L \) is assumed to be a regular Lagrangian \( L : TQ \rightarrow \mathbb{R} \) where \( L = K - U \) and the kinetic 
energy \( K : TQ \rightarrow \mathbb{R} \).

**Proposition 3.1** Let the Hamiltonian \( H = K + U \) be a constant of motion i.e., \( H = E \text{(say)} \) 
with the kinetic energy \( K \) being a homogeneous quadratic function of the velocities and \( U(x) \) is 
some potential function such that \( U(x) < E \): then there exists a Riemannian metric defined by 
\( ds = \sqrt{2(E - U(x))} ds \) with \( K = 1/2(ds/dt)^2 \) such that the trajectories are the geodesic equations 
corresponding to the Jacobi-Maupertuis principle of least action.
Proof: Let $ds^2$ be a Riemannian metric on the configuration space with kinetic energy

$$K = \frac{1}{2} g_{ij} (\dot{x}^i \dot{x}^j) = \frac{1}{2} \left( \frac{ds}{dt} \right)^2. \quad (3.2)$$

As the total energy is a constant $E$ with potential $U(x) < E$ the Hamiltonian satisfies $H = K + U = E$. Because $K$ is a homogeneous quadratic function hence Euler theorem implies $2K = \dot{x}^i \partial L / \partial \dot{x}^i = (ds/dt)^2$. Therefore from (3.1) we have

$$\int \gamma \frac{\partial L(t)}{\partial \dot{x}} \dot{x}(t) dt = \int \gamma 2K dt = \int \gamma 2K \frac{ds}{\sqrt{2K}} = \int \gamma \sqrt{2K} ds = \int \gamma \sqrt{2(E - U(x))} ds = \int \gamma d\tilde{s},$$

where the Riemannian metric $\tilde{s}$ is defined by $d\tilde{s} = \sqrt{2(E - U(x))} ds$. This shows that it is possible to derive a metric which is given by the kinetic energy itself and the trajectories are geodesics in the metric $d\tilde{s}$. From (3.2) one finds $ds = \sqrt{2g_{ij} dx^i dx^j}$ and the Maupertuis principle involves solving for the stationary points of the action $\int \sqrt{2K} ds$, i.e.,

$$\delta \int \sqrt{2K} ds = 0 \quad \text{or} \quad \delta \int \sqrt{2(E - U(x))} g_{ij} dx^i dx^j = 0, \quad (3.3)$$

with the integral being over the generalized coordinates $\{x^i\}$ along all paths connecting $\gamma(a)$ and $\gamma(b)$.

It is evident from $d\tilde{s} = \sqrt{2(E - U(x))} g_{ij} dx^i dx^j$ that

$$d\tilde{s}^2 = g_{ij} dx^i dx^j \quad \text{where} \quad g_{ij}(x) = 2(E - U(x)) g_{ij}(x). \quad (3.4)$$

The geodesic equation corresponding to the least action $\delta \int_{s_1}^{s_2} dt \sqrt{g_{ij} \dot{x}^i \dot{x}^j} = 0$ is given by

$$\frac{d^2 x^i}{ds^2} + \Gamma^i_{jk} \frac{dx^j}{ds} \frac{dx^j}{ds} = 0, \quad \text{where} \quad \Gamma^i_{jk} = \frac{1}{2} \tilde{g}^{il} \left( \frac{\partial \tilde{g}_{jl}}{\partial x^k} + \frac{\partial \tilde{g}_{kl}}{\partial x^j} - \frac{\partial \tilde{g}_{jk}}{\partial x^l} \right). \quad (3.5)$$

This complete the proof. 

For an equation of the Liénard type given by (2.10) we have from Proposition (2.1)

$$K = \frac{1}{2} M(x) \dot{x}^2 \quad \text{where} \quad M(x) = \exp(2F(x))$$

so that $g_{11}(x) = M(x)$ while from the Jacobi-Maupertuis (JM) metric (3.4) we observe that $\tilde{g}_{11} = 2(E - U(x)) M(x)$. The geodesic equation (3.5) therefore reduces to

$$\frac{d^2 x}{ds^2} + \Gamma^1_{11} \left( \frac{dx}{ds} \right)^2 = 0 \quad \text{with} \quad \Gamma^1_{11} = \frac{M'(x)}{2M(x)} - \frac{U'(x)}{2(E - U(x))},$$

or in explicit terms

$$\frac{d^2 x}{ds^2} + \left( \frac{M'(x)}{2M(x)} - \frac{U'(x)}{2(E - U(x))} \right) \left( \frac{dx}{ds} \right)^2 = 0. \quad (3.6)$$

Eqn. (3.6) gives the geodesic for the JM-metric of a Liénard equation of the type (2.10).
Proposition 3.2 The geodesic equation (3.6) and (2.10) are equivalent.

Proof: From \( K = E - U(x) = 1/2M(x)\dot{x}^2 \) we have

\[
\dot{x}^2 = 2(E - U(x))/M(x) \quad \text{and as} \quad d\tilde{s}^2 = \tilde{g}_{11}dx^2 = 2((E - U(x))Mdx^2, \quad (3.7)
\]

it follows that

\[
\frac{d\tilde{s}}{dt} = 2(E - U(x)) \Rightarrow \frac{dx}{dt} = 2(E - U(x))\frac{dx}{d\tilde{s}}. \quad (3.8)
\]

This enables us to obtain

\[
\frac{d^2x}{d\tilde{s}^2} = \frac{1}{2(E - U(x))} \frac{d}{dt} \left\{ \frac{1}{2(E - U(x))} \frac{dx}{dt} \right\} = \frac{1}{4(E - U(x))^2} \left[ \frac{d^2x}{dt^2} + \frac{U'(x)}{(E - U(x))}\dot{x}^2 \right] \quad (3.9)
\]

Consequently (3.6), taking (3.7) into account, assumes the form

\[
\frac{1}{4(E - U(x))^2} \left[ \frac{d^2x}{dt^2} + \frac{U'(x)}{(E - U(x))}\dot{x}^2 \right] = \frac{U'(x)}{2(E - U(x))} - \frac{M'(x)}{2M(x)} \frac{1}{4(E - U(x))^2}\dot{x}^2,
\]

that is in other words we have

\[
\frac{d^2x}{dt^2} + \frac{M'(x)}{2M(x)}\dot{x}^2 + \frac{U'(x)}{2(E - U(x))}\dot{x}^2 = 0. \quad (3.10)
\]

However as \( \dot{x}^2 = 2(E - U(x))/M(x) \) the last term of the above equation can be expressed as \( U'(x)/M(x) \) and as a result the equation has the appearance

\[
\frac{d^2x}{dt^2} + \frac{M'(x)}{2M(x)}\dot{x}^2 + \frac{U'(x)}{M(x)} = 0. \quad (3.11)
\]

This equation reduces to (2.10) upon making the identifications \( M(x) = \exp(2F(x)) \) which implies \( M'(x)/2M(x) = f(x) \) and \( U(x) = \int^x M(y)g(y)dy \) which implies \( U'(x)/M(x) = g(x) \) where \( g(x) \) refers to the forcing term of the Liénard equation (2.10).

Remark: Finally it is interesting to note how (2.10) or equivalently (3.11) may be viewed geometrically. To this end we write (3.11) as

\[
\frac{d^2x}{dt^2} + \frac{M'(x)}{2M(x)}\dot{x}^2 = -\frac{U'(x)}{M(x)} \quad (3.12)
\]

and look upon the right hand side as an external force function. Restricting ourselves to the left hand side we consider a 1+1 dimensional line element of the form \( ds^2 = c^2dt^2 - M(x)dx^2 = c^2d\tau^2 \) which yields the following geodesic equations for a free particle moving in this spacetime, namely

\[
\frac{d^2x}{d\tau^2} + \frac{M'(x)}{2M(x)} \left( \frac{dx}{d\tau} \right)^2 = 0, \quad \frac{d^2t}{d\tau^2} = 0.
\]
These equations imply upon elimination of the proper time $\tau$ the left hand side of (3.12) and the latter may be recast as

$$\frac{d}{dt}(M(x)\dot{x}) = \frac{M'(x)}{2} \dot{x}^2.$$ 

Thus from a Newtonian perspective we see that the position dependent mass function $M(x)$ changes the geometry of spacetime in a manner such that the particle experiences an additional geometric force $f_G = \frac{M'(x)\dot{x}^2}{2}$. However unlike the case when the PDM is also a function of time [11] the curvature of spacetime is flat because as a result of the transformation $dX = \sqrt{M(x)}dx$ one has $ds^2 = c^2dt^2 - dX^2$ and the resulting geodesic equation of a free particle in this transformed spacetime is just $\frac{d^2X}{dt^2} = 0$ or

$$\frac{d}{dt}\left(\sqrt{M(x)}\frac{dx}{dt}\right) = 0,$$

or $\frac{1}{2}M(x)\dot{x}^2 = \text{const.}$ which implies the conservation of the kinetic energy.

We end this letter with a few examples for the purpose of illustration.

**Example 1:** Painlèве-Gambier XXI

$$\ddot{x} - \frac{3}{4x}\dot{x}^2 - 3x^2 = 0$$

For this equation we have $F(x) = -3/4 \int dx/x = -3/4 \log |x|$ so that $M(x) = |x|^{-3/2}$ and as $2K = M(x)\dot{x}^2 = g_{11}(x)\dot{x}^2$ we have $g_{11}(x) = M(x) = |x|^{-3/2}$ while $U(x) = \int^x M(z)g(z)dz = \pm 2x^{3/2}$ depending on whether $x > 0$ or $x < 0$. As a result we find have $\tilde{g}_{11} = 2(E \pm 2x^{3/2})|x|^{-3/2}$.

**Example 2:** Jacobi equation

$$\ddot{x} + \frac{1}{2}\phi_x \dot{x}^2 + \phi_t \dot{x} + B(t,x) = 0$$

Here $M(x,t) = \exp(\phi(x,t)) = g_{11}$ and the Lagrangian is given by

$$L = \frac{1}{2}e^{\phi(x,t)}\dot{x}^2 - U(x,t), \quad \text{where} \quad U(x,t) = \int^x e^{\phi(y,t)}B(y,t)dy$$

It may be verified that the Hamiltonian is a constant of motion and $\tilde{g}_{11} = 2(E - U(x,t)) \exp(\phi(x,t))$.

The geodesic equation is given by

$$\frac{d^2x}{ds^2} + \Gamma^1_{11} \left(\frac{dx}{ds}\right)^2 = 0,$$

with $\Gamma^1_{11} = \frac{\phi_x}{2} - \frac{U_x}{2(E - U(x,t))}$.

**Example 3:** Henon-Heiles system

$$\ddot{x} = -(Ax + 2\alpha xy)$$

$$\ddot{y} = -(By + \alpha x^2 - \beta y^2)$$
The above system has the Lagrangian

\[ L(x, y, \dot{x}, \dot{y}) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \left(\frac{A}{2}x^2 + \frac{B}{2}y^2 + \alpha x^2 y - \frac{\beta}{3}y^3\right) \]

It is therefore easily seen that \( M_{xx} = M_{yy} = 1 \) and it admits the first integral

\[ I = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \left(\frac{A}{2}x^2 + \frac{B}{2}y^2 + \alpha x^2 y - \frac{\beta}{3}y^3\right), \]

which is just the Hamiltonian of the system. Consequently we have \( g_{11} = M_{xx} = 1 \) and \( g_{22} = M_{yy} = 1 \) while

\[ \tilde{g}_{11} = 2(E - U(x, y)) = \tilde{g}_{22}, \] where \( U(x, y) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \left(\frac{A}{2}x^2 + \frac{B}{2}y^2 + \alpha x^2 y - \frac{\beta}{3}y^3\right) \)

The geodesic equations have the following appearance:

\[ \frac{d^2x}{ds^2} = \frac{1}{2(E - U(x,y))} \left( U_x \left( \frac{dx}{ds} \right)^2 + 2U_y \left( \frac{dx}{ds} \right) \left( \frac{dy}{ds} \right) + U_x \left( \frac{dy}{ds} \right)^2 \right) = 0 \]

\[ \frac{d^2y}{ds^2} = \frac{1}{2(E - U(x,y))} \left( U_y \left( \frac{dx}{ds} \right)^2 + 2U_x \left( \frac{dx}{ds} \right) \left( \frac{dy}{ds} \right) + U_y \left( \frac{dy}{ds} \right)^2 \right) = 0 \]

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