On the definability of Menger spaces which are not $\sigma$-compact

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Abstract

Hurewicz proved completely metrizable Menger spaces are $\sigma$-compact. We extend this to Čech-complete Menger spaces, and consistently, to projective metrizable Menger spaces. On the other hand, it is consistent that there is a co-analytic Menger metrizable space that is not $\sigma$-compact.

1 When Menger spaces are $\sigma$-compact

The Menger property is a strengthening of Lindelöfness which plays an important role in the study of Selection Principles.

Definition 1.1. A space $X$ is Menger if, given any sequence $\{U_n\}_{n<\omega}$ of open covers of $X$, there exist finite subsets $V_n$ of $U_n$ such that $\bigcup_{n<\omega} V_n$ covers $X$.

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Hurewicz [13] proved that completely metrizable Menger spaces are \( \sigma \)-compact and conjectured that indeed all Menger spaces are. His conjecture was disproved in [18]. Since then, easier, “natural” counterexamples have been constructed — see e.g. [32]. It was apparently not realized until now that Hurewicz’ theorem was not limited to metrizable spaces. We shall assume all spaces considered are completely regular. We shall prove:

**Theorem 1.2.** Čech-complete Menger spaces are \( \sigma \)-compact.

\( p \)-spaces were introduced by Arhangel’ski˘ı [1]; an equivalent definition is due to Burke [8]:

**Definition 1.3.** A space \( X \) is a \( p \)-space if there exists a sequence \( \{U_n\}_{n<\omega} \) of families of open subsets satisfying the following condition if for each \( n \), \( x \in U_n \in \mathcal{U}_n \) then

\( i) \ \bigcap_{n<\omega} U_n \) is compact,

\( ii) \ \{ \bigcap_{i \leq n} U_i : n \in \omega \} \) is an outer network for the set \( \bigcap_{n<\omega} U_n \) i.e., every open set including \( \bigcap_{n<\omega} U_n \) includes some \( \bigcap_{i \leq n} U_i \).

Recall that \( \text{paracompact p-spaces} \) are preimages of metrizable spaces under perfect mappings [1].

**Proof.** A Čech-complete Lindelöf space \( X \) is a paracompact \( p \)-space and so is a perfect pre-image of a separable metrizable space \( Y \). Čech-completeness is preserved by perfect maps, and Mengerness by continuous maps, so if \( X \) is Menger, \( Y \) is Menger and Čech-complete, so by [13] is \( \sigma \)-compact. But a perfect pre-image of a \( \sigma \)-compact space is \( \sigma \)-compact. \( \square \)

**Lemma 1.4** [14, a more accessible reference is 22]. The irrationals are not Menger.

**Lemma 1.5** [15, a more accessible reference is 17]. If \( Y \subseteq \mathbb{R} \) is analytic, then either \( Y \) is \( \sigma \)-compact or \( Y \) includes a closed copy of the space \( \mathbb{P} \) of irrationals.

It follows immediately that:

**Theorem 1.6** [13]. Analytic Menger subsets of \( \mathbb{R} \) are \( \sigma \)-compact.
Under the Axiom of Projective Determinacy (PD), Lemma 1.5 holds for projective sets of reals — see e.g. [26]. Thus

**Theorem 1.7.** PD implies projective Menger subsets of $\mathbb{R}$ are $\sigma$-compact.

The reader unfamiliar with determinacy should consult Section 27 of [16] and Sections 20 and 38.B of [17].

Given our success with Theorem 1.2 one wonders whether Theorems 1.6 and 1.7 can be extended to Menger spaces which are not metrizable. Theorem 1.6 can.

**Definition 1.8** [2]. A space is **analytic** if it is a continuous image of the space $\mathbb{P}$ of irrationals.

Arhangel’skii proved:

**Theorem 1.9** [2]. Analytic Menger spaces are $\sigma$-compact.

**Warning.** Arhangel’skii and some other authors call Menger spaces Hurewicz. Our terminology is now standard.

## 2 Some counterexamples

Theorem 1.2 does not obviously follow from Theorem 1.9 since Menger Čech-complete spaces need not be analytic. For example, it is easy to see that a compact space of weight $\aleph_1$ is not analytic.

In [29] the first author incorrectly claimed that Arhangel’skii in [2] had proven that analytic spaces are perfect preimages of separable metrizable spaces. Arhangel’skii did not claim this and it is not true. What Arhangel’skii did prove was that a non-$\sigma$-compact analytic space includes a closed copy of the space of irrationals, which suffices to prove the application in [29] of Arhangel’skii’s supposed theorem. A counterexample to this false claim can be constructed by taking a closed discrete infinite subspace $F$ of the space of irrationals and collapsing it to a point. The resulting space is clearly analytic and hence has a countable network. However it does not have countable weight, because it is not first countable. Perfect pre-images of separable metrizable spaces are Lindelöf $p$-spaces; in $p$-spaces, network weight = weight, so this space is not a $p$-space. 

It is not clear what the right definition of “projective” should be so as to attempt to generalize Theorem 1.9. A possible approach is to define the projective spaces to be the result of closing the class of analytic spaces under remainders and continuous images, and hope that Theorem 1.9 extends to this class, assuming PD.

Unfortunately we immediately run into trouble. It is not clear whether Menger remainders of analytic spaces are consistently $\sigma$-compact. See the next section for discussion. Continuous images of such Menger remainders need not be in fact $\sigma$-compact — here is a counterexample:

**Example 2.1.** Okunev’s space [3] is obtained by taking the Alexandrov duplicate $A$ of $\mathbb{P}$ (without loss of generality in $[0,1]$) and collapsing the non-discrete copy of $\mathbb{P}$ to a point. The remainder of $A$ in the duplicate of $[0,1]$ is analytic; to see this, note it is just the Alexandrov duplicate $D$ of the space of rationals in $[0,1]$. $D$ is a countable metrizable space; it may therefore be considered as an $F_\sigma$ subspace of $\mathbb{R}$, so is analytic. Thus Okunev’s space is the continuous image of a remainder of an analytic space. In [9], this space is shown to be Menger and not $\sigma$-compact.

There is a widely used generalization of analyticity to non-metrizable spaces — see e.g. [24]:

**Definition 2.2.** A space is $K$-analytic if it is a continuous image of a Lindelöf Čech-complete space.

Analytic spaces are $K$-analytic, and $K$-analytic metrizable spaces are analytic [24].

**Theorem 2.3.** $K$-analytic Menger spaces with closed sets $G_\delta$ are $\sigma$-compact.

**Proof.** By 3.5.3 of [24], if such a space were not $\sigma$-compact, it would include a closed set which maps perfectly onto $\mathbb{P}$. But $\mathbb{P}$ is not Menger.

One might conjecture that Theorem 1.6 could be extended to $K$-analytic spaces, but unfortunately Arhangel’skii [3] proved Okunev’s space is $K$-analytic.

PD is equiconsistent with and follows from mild large cardinal assumptions, and is regarded by set theorists as relatively harmless. This result leads us to ask whether it is consistent that there is a projective Menger set of reals which is not $\sigma$-compact. The answer is:
Theorem 2.4 [18]. $V = L$ implies there is a Menger non-$\sigma$-compact set of reals which is projective.

As the set-theoretically knowledgeable reader might expect, one can simply use a $\Sigma^1_2$ well-ordering of the reals to construct the desired example, being more careful in constructing the Menger non-$\sigma$-compact example in [32]. But we will do better later.

In [27], the first author proved that if II has a winning strategy in the Menger game (defined in [30]), then the space is projectively $\sigma$-compact, and asked whether the converse is true. It isn’t:

Example 2.5. Moore’s $L$-space [20] is projectively countable [27] but II does not have a winning strategy in the Menger game.

Proof. By [30], for hereditarily Lindelöf spaces, II winning the Menger game is equivalent to $\sigma$-compactness. Moore’s space is not $\sigma$-compact, else it would be productively Lindelöf, which it isn’t [33].

3 Co-analytic Spaces

Definition 3.1. For a space $X$ and its compactification $bX$ the complement $bX \setminus X$ is called a remainder of $X$.

Definition 3.2. A space is co-analytic if its Stone-Čech remainder $\beta X \setminus X$ is analytic.

In this section, we investigate whether co-analytic Menger spaces are $\sigma$-compact.

Lemma 3.3. Any remainder of a co-analytic space is analytic.

Proof. Let $X$ be a co-analytic space. Then the remainder $\beta X \setminus X$ of $X$ is analytic. Any compactification of a space $X$ is the image of $\beta X$ under a unique continuous mapping $f$ that keeps the space $X$ pointwise fixed; furthermore $f(\beta X \setminus X) = bX \setminus X$ [10]. This implies that $X$ is co-analytic in $bX$. Thus any remainder of $X$ is analytic.

Theorem 3.4. If there is a measurable cardinal, then co-analytic Menger sets of reals are $\sigma$-compact.
Proof. The hypothesis implies co-analytic (also known as $\Pi^1_1$-) determinacy ([16], 31.1) and hence the co-analytic case of Theorem 1.7. Indeed, that co-analytic determinacy implies co-analytic Menger sets of reals are $\sigma$-compact was noticed earlier in [18].

Theorem 1.9 did not require analytic determinacy, so it is not obvious that something beyond ZFC is needed in order to conclude that Menger co-analytic sets of reals are $\sigma$-compact. But it is.

We have:

**Theorem 3.5.** Suppose $\omega_1 = \omega^L_1$ and $\mathfrak{d} > \aleph_1$. Then there is a co-analytic Menger set of reals which is not $\sigma$-compact.

**Proof.** $\omega_1 = \omega^L_1$ implies there is a co-analytic set of reals of size $\aleph_1$ ([16], 13.12). By [14], any set of reals of size $< \mathfrak{d}$ is Menger. Every $\sigma$-compact set of reals, has cardinality either countable or $2^{\aleph_0}$, but $\mathfrak{d} > \aleph_1$ implies $2^{\aleph_0} > \aleph_1$, so $X$ cannot be $\sigma$-compact. □

It is easy to find a model of set theory satisfying these two hypotheses. For example, start with $L$ and force Martin’s Axiom plus $2^{\aleph_0} = \aleph_2$ by a countable chain condition iteration.

A question we have been unable to answer is whether it is consistent that, modulo large cardinals, co-analytic Menger spaces are $\sigma$-compact. One would expect to reduce the problem to sets of reals and apply co-analytic determinacy, but we have so far been unsuccessful. We however have some partial answers.

**Lemma 3.6.** The perfect image of a co-analytic space is co-analytic.

**Proof.** Let $Y$ be a perfect image of a co-analytic space $X$ under a perfect map $f$. Then $f$ has a unique continuous extension $F : \beta X \to \beta Y$ such that $F(\beta X \setminus X) = \beta Y \setminus Y$ [12, Lemma 1.5]. Since the continuous image of an analytic space is analytic, $Y$ is co-analytic. □

**Theorem 3.7.** Co-analytic determinacy implies that if $X$ is the Menger remainder of an analytic p-space, then $X$ is $\sigma$-compact.

**Proof.** It is known that any remainder of a Lindelöf p-space is a Lindelöf p-space [4]. As in the proof of Theorem 1.2 it suffices to prove that $X$ maps perfectly onto a separable metrizable space $Y$. Since $X$ is co-analytic, by using Lemma 3.6 we obtain that $Y$ is co-analytic. But then $Y$ is $\sigma$-compact, so $X$ is $\sigma$-compact. □
Definition 3.8. *X* is **projectively Čech-complete** if every separable metrizable image of *X* is Čech-complete.

Theorem 3.9. Every co-analytic projectively Čech-complete space is Čech-complete.

**Proof.** Let *X* be co-analytic and projectively Čech-complete. Let *Y* = β*X* \ *X*. If *Y* is σ-compact, we are done. If not, then by [2], *Y* includes a closed *F* homeomorphic to *P*. Following the proof of Theorem 1.5. in [23], we can extend a homeomorphism mapping *F* onto *P* to a continuous *f* : *Y* → [0, 1].

We can choose a continuous *g* : *Y* → [0, 1] such that *F* = *g*−1({0}) and define *h* : *Y* → [0, 1]2 by *h*(x) = (*f*(x), *g*(x)). The set \{(a, b) ∈ *h*(Y) : b = 0\} is closed in *h*(Y) and is homeomorphic to *P*, so *h*(Y) is not σ-compact. Extend *h* to ˆ*h* mapping β*Y* into [0, 1]2. Then *h*(Y) = ˆ*h*(Y) is not σ-compact, since \{(a, b) ∈ *h*(Y) : b = 0\} is closed in *h*(Y) and is homeomorphic to *P*. But then ˆ*h*(X) is not Čech-complete.

Corollary 3.10. Every co-analytic projectively Čech-complete Menger space is σ-compact.

**Proof.** By Theorem 3.9 and Theorem 1.2.

Corollary 3.11. Co-analytic determinacy implies that if *X* is co-analytic Menger and has closed sets *G*δ, then *X* is σ-compact.

**Proof.** The remainder of *X* is a continuous image of a separable metrizable space. Our extra hypothesis then implies *X* is a p-space [5].

Definition 3.12 [21]. A space *X* is a **Σ-space** if it is a continuous image of a perfect pre-image of a metrizable space. In particular, a Lindelöf Σ-space is a continuous image of a Lindelöf p-space.

Corollary 3.13. Co-analytic determinacy implies that co-analytic Menger Σ-spaces are σ-compact.

**Proof.** Let *X* be Menger Σ-space and *Y* be an analytic remainder of *X*. It suffices to prove *Y* is a p-space, for then we can apply Theorem 3.7. Since *Y* is analytic, it has a countable network and hence is hereditarily Lindelöf. By Proposition 2.6. of [5], if *X* is a Lindelöf Σ-space, then a hereditarily Lindelöf remainder of *X* is a p-space.
After this paper was more-or-less complete, A.V. Arhangel’ski˘ı sent us a copy of his paper [6] in response to Question 2 below. In [6], he introduces the class of s-spaces:

**Definition 3.14** [6]. A Tychonoff space \( X \) is called an **s-space** if there exists a countable **open source** for \( X \) in some compactification \( bX \) of \( X \), i.e. a countable collection \( S \) of open subsets of \( bX \) such that \( X \) is a union of some family of intersections of non-empty subfamilies of \( S \).

Arhangel’ski˘ı proves that:

**Lemma 3.15.** If \( X \) is a space with remainder \( Y \), then \( X \) is a Lindelöf \( \Sigma \)-space if and only if \( Y \) is an s-space.

**Lemma 3.16.** \( X \) is a Lindelöf p-space if and only if it is a Lindelöf \( \Sigma \)-space and an s-space.

These can be used to give another proof of Corollary 3.13. Arhangel’ski˘ı asks in the paper whether first countable \( \sigma \)-spaces with weight \( \leq 2^{\aleph_0} \) are s-spaces. He told us, however, that what he should have asked is:

**Question 1.** Are \( \sigma \)-spaces of countable type and \( w \leq 2^{\aleph_0} \) s-spaces?

We could prove that co-analytic determinacy implies co-analytic Menger spaces are \( \sigma \)-compact, if we could affirmatively answer the following:

**Question 2.** Is every space of countable type with a countable network metrizable?

Recall a space is of **countable type** if each compact subspace is included in a compact subspace which has a countable base for the open sets including it. Now, the analytic remainder of a co-analytic Menger space is of countable type, because its remainder is Lindelöf [12]. Similarly, the co-analytic Menger space is itself of countable type. Now the analytic remainder has a countable network, so if Question 2 has an affirmative answer, it is metrizable. But then it is a Lindelöf p-space, so its co-analytic remainder is a Lindelöf p-space as well [4]. But now we can proceed as in the proof of Theorem 1.2.

As Arhangel’ski˘ı points out, Question 2 is a special case of Question 1. Notice that compact subspaces of a space with a countable network have a countable base. It follows that a space of countable type with a countable
network is actually first countable; indeed its compact subspaces have countable character, because they are included in compact metrizable subspaces which have countable character in the whole space. So let us ask:

**Question 3.** Suppose a space has a countable network and each compact subspace of it has countable (outer) character. Is the space metrizable?

By Michael [19] it would suffice to show such a space has a countable \( k \)-network.

Note, incidentally, that co-analytic determinacy has large cardinal strength. See [16], where it is pointed out (see page 444) that it is equiconsistent with the existence of \( 0^\# \) (and hence with the existence of a measurable cardinal).

Note that Okunev’s space is not co-analytic. To see this, let \( V \) denote Okunev’s space. Since the remainder of \( V \) in its Stone-Čech compactification is Borel but not Baire [9], \( \beta V \setminus V \) cannot have closed sets \( G_\delta \), so the remainder is not analytic. Hence by Lemma 3.3, \( V \) is not co-analytic.

As promised earlier,

**Theorem 3.17** [18]. \( V = L \) implies that there is a co-analytic Menger set of reals which is not \( \sigma \)-compact.

The crucial point here is

**Lemma 3.18** [18]. \( V = L \) implies that there is a co-analytic scale.

**Definition 3.19.** For \( f, g \in \omega^\omega \), \( f \leq^* g \) if \( f(n) \leq g(n) \) for all but finitely many \( n \in \omega \). A **scale** is a subset of \( \omega^\omega \) well-ordered by \( \leq^* \) such that each \( f \in \omega^\omega \) is \( \leq^* \) some member of the scale.

Bartoszyński and Tsaban [7] proved:

**Lemma 3.20.** Let \( S \subseteq \omega^\omega \) be a scale. Then \( S \cup [\omega]^{<\omega} \) is a Menger set of reals which is not \( \sigma \)-compact.

Thus, to complete the proof of Theorem 3.17 it suffices to prove that the union of a co-analytic set of reals with a countable set of reals is co-analytic. Let these be \( A, B \) respectively. Then \( A \cup B = \mathbb{R} \setminus ((\mathbb{R} \setminus A) \cap (\mathbb{R} \setminus B)) \). Both
\(\mathbb{R} \setminus A\) and \(\mathbb{R} \setminus B\) are analytic by 14.4 of \cite{17}; the intersection of two analytic set is analytic, so \(A \cup B\) is co-analytic.

One can in fact from \(V = L\) construct a co-analytic Menger topological group which is not \(\sigma\)-compact — see \cite{31}. On the other hand, co-analytic determinacy implies one cannot — see \cite{28}.

We conclude by asking again,

**Question 4.** Is there a ZFC example of a co-analytic Menger space which is not \(\sigma\)-compact?

**Note:** After this paper was submitted, Y. Peng answered Questions 2 and 3 in the negative.

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