Urysohn’s metrization theorem for higher cardinals

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Abstract
In this paper a generalization of Urysohn’s metrization theorem is given for higher cardinals. Namely, it is shown that a topological space with a basis of cardinality at most \(|\omega_\mu|\) or smaller is \(\omega_\mu\)-metrizable if and only if it is \(\omega_\mu\)-additive and regular, or, equivalently, \(\omega_\mu\)-additive, zero-dimensional, and \(T_0\). Furthermore, all such spaces are shown to be embeddable in a suitable generalization of Hilbert’s cube.

Keywords: \(\omega_\mu\)-metric space, Urysohn’s metrization theorem, embedding theorem, \(\omega_\mu\)-additive space

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1. Introduction

In this section the concept of \(\omega_\mu\)-metric spaces is defined and briefly discussed. For a more elaborate description of \(\omega_\mu\)-metric spaces, see e.g. [1]. Section 2 is dedicated to some preliminary results which are then used to prove an extension of Urysohn’s metrization theorem in section 3.

A number of properties of a topological space equivalent with \(\omega_\mu\)-metrizability were given by Hodel [2] and by Nyikos and Reichel [1]. The special case of spaces with a basis of cardinality at most \(|\omega_\mu|\) seems not to have been considered. A complete characterization of such \(\omega_\mu\)-metrizable topological spaces is exhibited here in Theorem 2 in terms of simple topological properties which may be easier to verify than those required by more general metrization theorems.

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If $G$ is an ordered abelian group, $X$ is a nonempty set and $d : X \times X \to G$ is a function such that

1. $d(x, y) \geq 0$ for all $x, y \in X$,
2. $d(x, y) = 0$ only if $x = y$,
3. $d(x, y) = d(y, x)$ for all $x, y \in X$, and
4. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$,

then $d$ is a metric of $X$ over $G$. Any such metric $d$ gives rise to a topology of $X$ whose basis consists of the open balls

$$B_d(x, r) = \{y \in X : d(x, y) < r\},$$

for all $x \in X$ and $r \in G$, $r > 0$. Where no confusion is possible, the subscript $d$ is omitted. If $X$ is a topological space and there is a metric $d$ of $X$ over $G$ giving rise to the same topology, then $X$ is $G$-metrizable and the pair $(X, d)$ is a $G$-metric space.

Of special interest are the groups $\mathbb{Z}^\alpha$ and $\mathbb{R}^\alpha$, where $\alpha$ is any non-zero ordinal, with lexicographical order and componentwise addition. Any element $x \in \mathbb{Z}^\alpha$ is a sequence $(x_\lambda)_{\lambda \in \alpha}$ indexed by $\alpha$ with each $x_\lambda \in \mathbb{Z}$, and similarly for $\mathbb{R}^\alpha$.

For any $\lambda \in \alpha$, we define elements $r^\lambda$ in $\mathbb{Z}^\alpha$ (or in $\mathbb{R}^\alpha$) by setting $r^\lambda_\lambda = 1$ and $r^\lambda_\nu = 0$ for all $\nu \neq \lambda$. One can immediately see that if $\alpha$ is a regular ordinal and $(X, d)$ is a $\mathbb{Z}^\alpha$-metric (or $\mathbb{R}^\alpha$-metric) space, then the collection

$$\{B_d(x, r^\lambda) : x \in X, \lambda \in \alpha\}$$

is a basis for the topology of $X$.

**Proposition 1.** If a topological space $X$ is $\mathbb{Z}^\alpha$-metrizable, it is also $\mathbb{Z}^{\text{cf}\alpha}$-metrizable, where $\text{cf} \alpha$ is the cofinality of $\alpha$. Similarly, if $X$ is $\mathbb{R}^\alpha$-metrizable, it is also $\mathbb{R}^{\text{cf}\alpha}$-metrizable.

**Proof.** Let $X$ be a $\mathbb{Z}^\alpha$-metric space, and let $L \subset \alpha$ be a cofinal subset order isomorphic to $\text{cf} \alpha$. Then the topology given by the basis

$$\{B(x, r^\lambda) : x \in X, \lambda \in L\}$$

is immediately seen to be identical to the original metric topology of $X$. Thus $X$ is $\mathbb{Z}^L$-metrizable. The same argument holds for $\mathbb{R}^\alpha$-metrizable spaces. \qed
Proposition 2. Let \( \alpha \) be an infinite regular ordinal. A topological space \( X \) is \( \mathbb{Z}^\alpha \)-metrizable if and only if it is \( \mathbb{R}^\alpha \)-metrizable.

Proof. Since \( \mathbb{Z}^\alpha \) is a subgroup of \( \mathbb{R}^\alpha \), any \( \mathbb{Z}^\alpha \)-metric space is trivially an \( \mathbb{R}^\alpha \)-metric space.

Let then \( (X, d) \) be a \( \mathbb{R}^\alpha \)-metric space. For any points \( x, y \in X \), let \( n_{xy} = \min\{\lambda \in \alpha : d_{\lambda}(x, y) \neq 0\} \). One can define a \( \mathbb{Z}^\alpha \)-metric \( \delta \) for \( X \) by setting \( \delta(x, x) = 0 \) and \( \delta(x, y) = r^{n_{xy}} \) when \( x \neq y \).

Let \( x, y, z \in X \) be any three distinct points. Because \( d \) obeys the triangle inequality, one has \( \min\{n_{xz}, n_{yz}\} \leq n_{xy} \), and so \( \max\{\delta(x, z), \delta(y, z)\} \geq \delta(x, y) \), and hence \( \delta(x, z) + \delta(y, z) \geq \delta(x, y) \).

Thus \( \delta \) obeys the triangle inequality; the other conditions for a metric are obviously fulfilled. Since \( B_\delta(x, r^{\lambda}) \supset B_d(x, r^{\lambda+1}) \) and \( B_d(x, r^{\lambda}) \supset B_\delta(x, r^{\lambda+1}) \) for all \( x \in X \) and \( \lambda \in \alpha \), the two metric topologies are the same.

Due to Proposition 1 only regular ordinals \( \alpha \) need to be considered. Every infinite regular ordinal \( \alpha \) is an initial ordinal, that is \( \alpha = \omega_\mu \) for some ordinal \( \mu \). Moreover, for a finite \( \alpha \) one has cf \( \alpha = 1 \), which yields either the discrete \( \mathbb{Z}^1 \)-metric or the usual \( \mathbb{R}^1 \)-metric.

Definition 1. A topological space \( X \) is an \( \omega_\mu \)-metric space if it is a \( \mathbb{Z}^{\omega_\mu} \)-metric space (equivalently \( \mathbb{R}^{\omega_\mu} \)-metric space by Proposition 2).

Unless otherwise stipulated, every \( \omega_\mu \)-metric will be assumed to take values in \( \mathbb{Z}^{\omega_\mu} \) instead of \( \mathbb{R}^{\omega_\mu} \).

2. Preliminaries

Let \( \kappa \) be a cardinal. A topological space \( X \) is \( \kappa \)-additive if for any collection \( \{U_i : i \in I\} \) of open sets in \( X \) the intersection \( \bigcap_{i \in I} U_i \) is open whenever \( |I| < \kappa \). An \( \omega_\mu \)-additive space is also called an \( \omega_\mu \)-additive space.

Proposition 3. Let \( X \) be topological space with a basis of cardinality \( \kappa \) or smaller. Then

1. \( X \) contains a dense set whose cardinality is at most \( \kappa \), and
2. every open cover of \( X \) has a subcover whose cardinality is at most \( \kappa \).

Proof. Let \( \mathcal{B} \) be a basis for the topology of \( X \) such that \( |\mathcal{B}| \leq \kappa \).

1. For every set \( A \in \mathcal{B} \) there is an element \( x_A \in A \). The set \( \{x_A : A \in \mathcal{B}\} \) is obviously dense and its cardinality cannot exceed \( \kappa \).
2. Let \( C \) be any open cover of \( X \). If \( B' \) is the collection of sets \( B \in \mathcal{B} \) such that \( B \subseteq U \) for some \( U \in \mathcal{C} \), one can choose for every \( B \in \mathcal{B}' \) a set \( U_B \) with \( B \subseteq U_B \in \mathcal{C} \). The collection \( \mathcal{C}' = \{ U_B : B \in \mathcal{B}' \} \) is the subcover sought for: indeed, each \( x \in X \) is an interior point of some \( U \in \mathcal{C} \), and so \( x \in B \subseteq U \) for some \( B \in \mathcal{B} \) and thus \( \bigcup_{B \in \mathcal{B}'} U_B = X \).

**Proposition 4.** In any \( \omega_\mu \)-metric space \( X \) either one of the following two conditions is sufficient to guarantee that the topology of \( X \) have a basis of cardinality \( |\omega_\mu| \) or smaller:

1. \( X \) contains a dense subset whose cardinality is at most \( |\omega_\mu| \).
2. Every open cover of \( X \) has a subcover whose cardinality is at most \( |\omega_\mu| \).

**Proof.**

1. Let \( A \subset X \) be a dense subset such that \( |A| \leq |\omega_\mu| \). The collection \( \mathcal{B} = \{ B(a, r^\lambda) : a \in A, \lambda \in \omega_\mu \} \) has obviously cardinality \( |\omega_\mu| \) or smaller. It remains to show that \( \mathcal{B} \) is also a basis for the topology. Let \( U \subset X \) be any nonempty open set and let \( x \in U \). Then there is \( \lambda \in \omega_\mu \) such that \( B(x, r^\lambda) \subset U \), and one can find a point \( a \in A \cap B(x, r^{\lambda+1}) \). Now \( x \in B(a, r^{\lambda+1}) \subset B(x, r^\lambda) \subset U \) and \( B(a, r^{\lambda+1}) \in \mathcal{B} \). Hence \( \mathcal{B} \) indeed is a basis.

2. For every \( \lambda \in \omega_\mu \) the collection \( \{ B(x, r^\lambda) : x \in X \} \) is an open cover of \( X \). Therefore there is a set \( A_\lambda \subset X \) such that \( |A_\lambda| \leq |\omega_\mu| \) and \( \{ B(x, r^\lambda) : x \in A_\lambda \} \) is an open cover of \( X \). The union \( A = \bigcup_{\lambda \in \omega_\mu} A_\lambda \) is dense in \( X \) and has cardinality \( |A| \leq |\omega_\mu| \times |\omega_\mu| = |\omega_\mu| \).

**Lemma 1.** Let \( X \) be a \( T_3 \)-space and let \( \kappa \) be a cardinal. Assume that \( X \) is \( \kappa \)-additive and every open cover of \( X \) has a subcover of cardinality \( \kappa \) or smaller. Then \( X \) is a \( T_4 \)-space.

**Proof.** Let \( E \) and \( F \) be disjoint nonempty closed sets in \( X \). Since \( X \) is \( T_3 \), for every \( e \in E \) one can find a neighborhood \( U_e \subset X \setminus \overline{E} \). Similarly every \( f \in F \) has a neighborhood \( V_f \) with \( E \subset X \setminus \overline{V_f} \). Since

\[
\mathcal{C} = \{ X \setminus (E \cup F) \} \cup \{ U_e : e \in E \} \cup \{ V_f : f \in F \}
\]

is an open cover of \( X \), the assumed covering property guarantees that \( \mathcal{C} \) has a subcover

\[
\mathcal{C}' = \{ X \setminus (E \cup F) \} \cup \{ U_{e_\lambda} : \lambda \in \alpha \} \cup \{ V_{f_\lambda} : \lambda \in \alpha \},
\]

where each \( e_\lambda \in E \) and \( f_\lambda \in F \), and where \( \alpha \) is the initial ordinal of the cardinal \( \kappa \).
For every $\lambda \in \alpha$ the sets
\[ A_\lambda = U_{e_\lambda} \setminus \bigcup_{\nu < \lambda} V_{f_\nu} \quad \text{and} \]
\[ B_\lambda = V_{f_\lambda} \setminus \bigcup_{\nu < \lambda} U_{e_\nu} \]
(6)
are open by hypothesis, and hence the sets
\[ A = \bigcup_{\lambda \in \alpha} A_\lambda \quad \text{and} \quad B = \bigcup_{\lambda \in \alpha} B_\lambda \]
(7)
are neighborhoods of $E$ and $F$, respectively. These neighborhoods are disjoint; for if $\nu \leq \lambda$, then $B_\nu \subset X \setminus A_\lambda$, and so $A_\lambda \cap B_\nu = \emptyset$ and similarly in the case $\nu \geq \lambda$. \qed

The following two known lemmas are elementary, and they are included only for the sake of an easy reference.

Lemma 2. If a topological space $X$ is zero-dimensional and $T_0$, then it is also $T_2$ and $T_3$.

Lemma 3. Let $X$ be a topological $T_3$-space with a basis $\mathcal{B}$ and let $x \in X$. Then for each neighborhood $U$ of $x$ there are $B, B' \in \mathcal{B}$ such that $x \in B \subset \overline{B} \subset B' \subset U$.

3. The metrization theorem

In order to extend Urysohn’s metrization theorem to higher cardinals and thus to $\omega_\mu$-metric spaces, a generalization of Hilbert’s cube is needed. The product topology of $\{0, 1\}^{\omega_\mu}$ is not suitable for this purpose, since it is not $\omega_\mu$-additive for $\mu > 0$.

Definition 2. Let $\mathbb{Z}^{\omega_\mu}$ be given an $\omega_\mu$-metric $d$ by defining $d(x, y)_\lambda = |x_\lambda - y_\lambda|$ for every $\lambda \in \omega_\mu$. The set $Q_\mu = \{0, 1\}^{\omega_\mu} \subset \mathbb{Z}^{\omega_\mu}$ with the $\omega_\mu$-metric inherited from $\mathbb{Z}^{\omega_\mu}$ is the generalized Hilbert’s cube.

A basis for the topology of the cube $Q_\mu$ consists of the products $\prod_{\lambda \in \omega_\nu} U_\lambda$, where there is $\nu \in \omega_\mu$ such that $U_\lambda$ is a singleton when $\lambda < \nu$ and $U_\lambda = \{0, 1\}$ when $\lambda \geq \nu$. The cardinality of this basis is $|\omega_\mu|$.

The embedding theorem which will be stated and proven shortly, will make use of the classical Urysohn’s lemma:
Lemma 4 (Urysohn’s lemma). Let $X$ be a $T_4$-space and let $E$ and $F$ be disjoint nonempty closed sets in $X$. Then there is a continuous mapping $f : X \to [0, 1]$ which satisfies $f(E) = \{0\}$ and $f(F) = \{1\}$.

Lemma 5. Let $X$ be a $T_4$-space and let $E$ and $F$ be disjoint nonempty closed sets in $X$. If $X$ is $\omega_1$-additive, there is a continuous mapping $f : X \to \{0, 1\}$ which satisfies $f(E) = \{0\}$ and $f(F) = \{1\}$.

Proof. Let $g : X \to [0, 1]$ be a mapping given by Urysohn’s lemma. Define $f : X \to \{0, 1\}$ by setting $f(x) = 0$ when $g(x) = 0$ and $f(x) = 1$ otherwise. Every set $g^{-1}([0, 1/n])$, $n \in \mathbb{N}$, is open in $X$ and therefore, by hypothesis, so is the intersection $f^{-1}(\{0\}) = g^{-1}(\{0\}) = \bigcap_{n \in \mathbb{N}} g^{-1}([0, 1/n])$. Thus $f$ is continuous.

Theorem 1. Let $X$ be a topological $T_1$- and $T_3$-space. If $X$ is $\omega_1$-additive and has a basis of cardinality $|\omega_\mu|$ or smaller for a regular ordinal $\omega_\mu$, $\mu > 0$, then $X$ can be embedded in the generalized Hilbert’s cube $Q_\mu$.

Proof. It follows from Proposition 3 and Lemma 1 that $X$ is also $T_4$.

Let $\{B_j \subset X : j \in J\}$ be a basis for $X$ such that $|J| \leq |\omega_\mu|$. Let $P$ be the set of pairs $(i, j) \in J \times J$ for which $\overline{B_i} \subset B_j$. Since $|P| \leq |J|$, the elements of $P$ can be indexed so that $P = \{(i_\lambda, j_\lambda) : \lambda \in \omega_\mu\}$.

For every $\lambda \in \omega_\mu$ a continuous mapping $f_\lambda : X \to \{0, 1\}$ is chosen such that $f_\lambda(\overline{B_{i_\lambda}}) = \{1\}$ and $f_\lambda(X \setminus B_{i_\lambda}) = \{0\}$. This is possible by Lemma 5. We define $f : X \to Q_\mu = \{0, 1\}^{\omega_\mu}$ componentwise by the mappings $f_\lambda$ and show that $f$ embeds $X$ in $Q_\mu$.

The mapping $f$ is continuous: Let $x \in X$ be a point and let $U$ be a neighborhood of $f(x)$. Then there exist $\nu \in \omega_\mu$ and sets $U_\lambda \subset \{0, 1\}$ with $f_\lambda(x) \in U_\lambda$ for all $\lambda \in \omega_\mu$ and $U_\lambda = \{0, 1\}$ when $\lambda \geq \nu$, so that $U$ contains the product $\prod_{\lambda \in \omega_\mu} U_\lambda$. Because each $f_\lambda$ is continuous, for every $\lambda < \nu$ one can find an open set $V_\lambda \subset X$ so that $f_\lambda(V_\lambda) \subset U_\lambda$. Since $X$ is $\omega_\mu$-additive, the set $V = \bigcap_{\lambda < \nu} V_\lambda$ is a neighborhood of $x$. Obviously $f(V) \subset \prod_{\lambda \in \omega_\mu} U_\lambda \subset U$.

The mapping $f$ is one-to-one: Let $x, y \in X$ be two distinct points. By the $T_4$-property and Lemma 3 there are $i, j \in J$ such that $x \in B_i \subset \overline{B_j} \subset B_j$. Thus $(i, j) = (i_\lambda, j_\lambda)$ for some $\lambda \in \omega_\mu$. Now $f_\lambda(x) = 1$ and $f_\lambda(y) = 0$, and so $f(x) \neq f(y)$.

Let $g : f(X) \to X$ be the inverse of $f$. It remains to show that $g$ is continuous. Fix $x \in X$ and let $U$ be a neighborhood of $x$. By Lemma 3 there is $\lambda \in \omega_\mu$ for which $x \in B_{i_\lambda}$ and $B_{j_\lambda} \subset U$. Now $f_\lambda(x) = 1$ and the set
\[ V = \prod_{\nu \in \omega} V_{\nu}, \text{ where } V_{\lambda} = \{1\} \text{ and } V_{\nu} = \{0,1\} \text{ for } \nu \neq \lambda, \text{ is a neighborhood of } f(x) \text{ in } Q_{\mu}. \] For every \( y \in g(V \cap f(X)) \) we have \( f(y) \in V, f_{\lambda}(y) = 1 \), and so \( y \in B_{j,\lambda} \subset U \). Thus \( g(V \cap f(X)) \subset U \) and \( g \) is continuous.

**Theorem 2.** For any topological space \( X \) and any regular ordinal \( \omega_{\mu} > \omega_0 \) the following are equivalent:

1. \( X \) is \( \omega_{\mu} \)-additive, \( T_0 \), zero-dimensional and has a basis of cardinality \( |\omega_{\mu}| \) or smaller.
2. \( X \) is \( \omega_{\mu} \)-additive, \( T_1 \) and \( T_3 \), and has a basis of cardinality \( |\omega_{\mu}| \) or smaller.
3. \( X \) is \( \omega_{\mu} \)-metrizable and has a basis of cardinality \( |\omega_{\mu}| \) or smaller.
4. \( X \) is \( \omega_{\mu} \)-metrizable and contains a dense set of cardinality \( |\omega_{\mu}| \) or smaller.
5. \( X \) is \( \omega_{\mu} \)-metrizable and every open cover of \( X \) has a subcover of cardinality \( |\omega_{\mu}| \) or smaller.
6. \( X \) can be embedded in \( Q_{\mu} \).

**Proof.** Lemma 2 and Theorem 1 provide the implications \( 1 \Rightarrow 2 \) and \( 2 \Rightarrow 6 \). The generalized Hilbert’s cube \( Q_{\mu} \) is \( \omega_{\mu} \)-metrizable by definition and has a basis of cardinality \( |\omega_{\mu}| \), whence \( 6 \Rightarrow 3 \). By Propositions 3 and 4 the conditions \( 3,4 \) and \( 6 \) are equivalent.

The implication \( 4 \Rightarrow 1 \) is seen as follows. Let \( A \) be a dense subset of \( X \) such that \( |A| \leq |\omega_{\mu}| \). The collection \( \{B(a,r^{\lambda}) : a \in A, \lambda \in \omega_{\mu}\} \) can easily be verified to be a clopen basis for \( X \), and its cardinality is manifestly \( |\omega_{\mu}| \) or smaller.

**Remark 1.** The set \( Q_{\mu} \) is considered to be a generalization of Hilbert’s cube due to its role in Theorem 2. The cube \( Q_0 \), however, is a Cantor set with its usual topology, and so the theorem does not hold for \( \mu = 0 \).

**Remark 2.** Let \( \omega_{\mu}, \mu > 0 \), be a regular ordinal. Consider a topological space \( X \) which has two bases, one of cardinality \( |\omega_{\mu}| \) or smaller and the other consisting of clopen sets. Does \( X \) have a basis which has both of these properties?

First, \( X \) can be assumed to be a \( T_0 \)-space; it is sufficient to consider the bases for the Kolmogorov quotient \( KQ(X) \) of \( X \), and \( KQ(X) \) is a \( T_0 \)-space. By Theorem 2 \( \omega_{\mu} \)-additivity is sufficient for such a basis to exist. If \( X \) is strongly zero-dimensional, every open cover of \( X \) has a refinement where the covering sets are disjoint. Any such refinement of the basis for \( X \) that has
cardinality $|\omega_\mu|$ or smaller is a suitable clopen basis. Without any further assumptions, however, it is unclear whether or not such a basis exists.

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