MAGNETO-DILATONIC BIANCHI-I COSMOLOGY: ISOTROPIZATION AND SINGULARITY PROBLEMS

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We study the evolution of Bianchi-I space-times filled with a global unidirectional electromagnetic field $F_{\mu\nu}$ interacting with a massless scalar dilatonic field according to the law $\Psi(\phi)F^{\mu\nu}F_{\mu\nu}$ where $\Psi(\phi) > 0$ is an arbitrary function. A qualitative study, among other results, shows that (i) the volume factor always evolves monotonically, (ii) there exist models becoming isotropic at late times and (iii) the expansion generically starts from a singularity but there can be special models starting from a Killing horizon preceded by a static stage. All three features are confirmed for exact solutions found for the usually considered case $\Psi = e^{2\lambda\phi}$, $\lambda = \text{const}$. In particular, isotropizing models are found for $|\lambda| > 1/\sqrt{3}$. In the special case $|\lambda| = 1$, which corresponds to models of string origin, the string metric behaviour is studied and shown to be qualitatively similar to that of the Einstein frame metric.

1. Introduction

The influence of intergalactic magnetic fields on the cosmological evolution has been studied for over four decades from both theoretical and observational points of view [1]. Cosmologists speculate that such a field could be primordial in origin, i.e., could have appeared before nucleosynthesis and even before inflation.

A cosmological model which contains a global magnetic field is necessarily anisotropic since the magnetic field vector specifies a preferred spatial direction. The presently observed Universe is almost isotropic at large, therefore the isotropization problem appears inevitably in any study of anisotropic cosmologies. The simplest of such models, which nevertheless rather completely describe the anisotropy effects, are Bianchi type I homogeneous models whose spatial sections are flat but the expansion or contraction rate is direction-dependent.

A number of recent papers [2–4] discussed the properties of Bianchi-I cosmologies with global magnetic fields in the framework of dilaton gravity with the action (in the Einstein frame)

$$S = \int \sqrt{g} d^4x \left[ R + 2(\partial \phi)^2 - e^{2\lambda\phi} F^{\mu\nu} F_{\mu\nu} \right],$$

where $g = |\det g_{\mu\nu}|$, $R$ is the scalar curvature, $\varphi$ is the scalar dilaton, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field, $(\partial \varphi)^2 := g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi$, and $\lambda$ is a coupling constant. This theory with $\lambda = \pm 1$ naturally appears in the weak field limit of string theories as part of their bosonic sector and is widely discussed in cosmological problems. Thus, there are attempts to apply dilatonic anisotropic cosmologies for describing some stages of the string-motivated pre-big-bang scenario [5, 6].

Refs. [2]–[4] contain exact solutions to the Einstein-Maxwell-dilaton equations due to [1], obtained in different variables. A general conclusion of these papers is that the evolution of expanding Bianchi-I models starts from a singularity of zero volume and ends with an infinite expansion, and that isotropization can never be achieved in this class of models. For the “stringy” coupling $\lambda = \pm 1$, it was found [5] that isotropy could be achieved due to higher curvature corrections to the equations of gravity.

Our purpose in this work was to re-examine the isotropization and singularity problems for magneto-dilatonic Bianchi-I cosmologies in a more general context, invoking an arbitrary coupling function $\Psi(\phi)$ [see Eq. (2)] instead of an exponential function. This is, above all, motivated by the considerations of Damour and Polyakov [7] who have argued that the string loop expansion is able to produce a coupling function of the form $B(\Phi) = e^{-2\Phi} + c_0 + c_1 e^{2\Phi} + \ldots$ ($c_1 = \text{const}$) instead of simply $e^{-2\Phi}$ which corresponds to the tree-level contribution only.

It is certainly impossible to obtain a general solution for any given $\Psi(\phi)$ but some essential qualitative inferences are available. We have been able to find the following:

Our sign conventions are: the metric signature $(+ - - -)$, the curvature tensor $R^g_{\mu\rho\nu} = \partial_\nu A^g_{\mu\rho} - \ldots$, so that, e.g., the Ricci scalar $R > 0$ for de Sitter space-time, and the stress-energy tensor (SET) such that $T^g_{\mu\nu}$ is the energy density. The gravitational constant is absorbed by re-definition of the fields $\phi$ and $F_{\mu\nu}$.
1. The volume factor $v$ is always a monotonic function of time, so that a model can be either permanently expanding or permanently contracting.

2. If collinear electric and magnetic fields are simultaneously admitted, isotropization is impossible. The further consideration concerns purely magnetic models; their electric counterparts can be obtained by duality.

3. Under some restriction on the choice of $\Psi(\phi)$, it is possible to obtain expanding cosmologies with late-time isotropization, and this is true for some models with $\Psi \sim \exp(\text{const} \cdot \phi)$.

4. The expansion begins (or the contraction ends) either at a singularity or at a (Killing) horizon beyond which there is a static space-time region.

5. Both isotropization and a horizon in the past are only possible for plane-symmetric models, in which, among the three scale factors $a_i(t)$, two coincide. Special models, which start their expansion from a horizon and end with isotropy, are not excluded.

These results are partly at variance with those of Refs. [2–4] obtained for the particular action (1). It therefore appears reasonable to reconsider this case comparing the results with items 1–5. We do this and confirm our general observations. Moreover, we give explicit conditions under which the model is asymptotically isotropic at late times and at which there is a horizon at the beginning. These conditions coincide only partly, but, for the special value of the coupling constant $\lambda = \sqrt{3}$, there exists a two-parameter family of solutions combining these two features. We also show that the static region beyond the horizon (in the absolute past with respect to the cosmological evolution) contains a Reissner-Nordström-like singularity.

The paper is organized as follows. Sec. 2 presents the field equations for an arbitrary coupling function $\Psi(\phi)$, written in terms of the harmonic time coordinate $u$ which considerably simplifies the treatment. Sec. 3 contains a qualitative study of Bianchi-I cosmologies with this arbitrary coupling function. This study is preceded by a short discussion of isotropization criteria. In Sec. 4 we put $\Psi = e^{2\lambda \phi}$, obtain the full set of exact solutions and briefly discuss their properties. We show, in particular, that asymptotically isotropic solutions do exist for $\lambda > 1/\sqrt{3}$ and form a two-parameter family. In particular, models of string origin ($\lambda = 1$) can isotropize at late times. Models which expand from a horizon rather than a singularity are also considered. In Sec. 5 we put $\lambda = 1$ and briefly describe the properties of the string metric $\hat{g}_{\mu\nu}$, which turn out to be quite similar to those of the Einstein metric $g_{\mu\nu}$. Sec. 6 contains concluding remarks, in particular, on exact solutions for the system (1) other than those discussed in other sections. Lastly, in the Appendix we give an expression for the Kretschmann scalar for Bianchi-I cosmologies needed to check their regularity.

2. Field equations

Consider dilaton gravity with an action more general that (1), containing an arbitrary coupling function $\Psi(\phi)$:

$$S = \int \sqrt{g} d^4x \left[ R + 2(\partial \phi)^2 - \Psi(\phi) F^{\mu\nu} F_{\mu\nu} \right], \quad \Psi(0) = 1,$$

which leads to the field equations

$$\frac{1}{\sqrt{g}} \partial_\mu \left( \sqrt{g} g^{\mu\nu} \partial_\nu \phi \right) + \frac{1}{4} \Psi \phi F^{\mu\nu} F_{\mu\nu} = 0,$$

$$\partial_\mu (\sqrt{g} F^{\mu\nu} \Psi(\phi)) = 0,$$

$$R^\mu_\nu = - (T^\mu_\nu - \frac{1}{2} T \delta^\mu_\nu) \equiv - \tilde{T}^\mu_\nu,$$

where $T = T^\alpha_\alpha$, and the stress-energy tensor (SET) $T^\mu_\nu$ is a sum of the scalar part $T^{\mu}_s$ and the electromagnetic part $T^{\mu}_e$ containing the interaction factor $\Psi(\phi)$:

$$T^{\mu}_s = T^{\mu}_s + T^{\mu}_e,$$

$$T^{\mu}_s = 2 \phi \phi' - \delta^\mu_\alpha \phi' \phi^\alpha,$$

$$T^{\mu}_e = \Psi(\phi)[-2 F_{\mu\alpha} F^{\alpha\nu} + \frac{1}{2} \delta^\alpha_\mu F_{\alpha\beta} F^{\alpha\beta}].$$

(6)
Now consider the field system (2) in a Bianchi type I space-time described by the metric
\[
\begin{align*}
\eta^{2} &= e^{2\alpha} dt^{2} - \sum_{i=1}^{3} e^{2\beta_{i}} (dx^{i})^{2},
\end{align*}
\] (7)

where \(\alpha\) and \(\beta_{i}\) are functions of the time coordinate \(t\). The proper physical time of a comoving observer, \(t = \tau\), corresponds to the coordinate condition \(\alpha \equiv 0\). The harmonic time coordinate, \(t = u\), is obtained if the lapse function \(e^{2\alpha}\) is chosen so that
\[
\alpha = \beta_{1} + \beta_{2} + \beta_{3}.
\] (8)

This choice significantly simplifies the Einstein equations. In terms of \(u\), the nonzero components of the Ricci tensor are
\[
\begin{align*}
R_{0}^{0} &= e^{-2\alpha} \left( \ddot{\alpha} - \dot{\alpha}^{2} + \sum_{i=1}^{3} \ddot{\beta}_{i} \right),
R_{1}^{3} &= e^{-2\alpha} \dddot{\beta}_{i}.
\end{align*}
\] (9)

There is no summing over an underlined index, and the dot denotes \(d/du\).

We assume that \(\phi = \phi(u)\) and that there are homogeneous electric and magnetic fields having the same direction \(x^{1}\). So the electromagnetic vector potential is taken in the form
\[
A_{\mu} = \{0, A_{1}(u), 0, A_{3}(x^{2})\}
\]

Then Eqs. (11) and the corresponding Bianchi identities lead to
\[
\begin{align*}
F_{01} &= -F_{10} = \frac{\eta_{e}}{\sqrt{\Psi(\phi)}} e^{2\alpha + 2\beta_{1}},
F_{23} &= -F_{32} = \eta_{m},
\end{align*}
\] (10)

where \(\eta_{e}\) and \(\eta_{m}\) are constants characterizing the electric and magnetic field intensities, respectively; other \(F_{\mu\nu}\) are zero. Accordingly, the tensor \(T^{\nu}_{\mu}\) has the form
\[
\begin{align*}
T_{e}^{0} &= T_{e}^{1} = -T_{e}^{2} = T_{e}^{3} = (B^{2} + E^{2}) \Psi(\phi),
\end{align*}
\] (11)

where \(E\) and \(B\) are the electric and magnetic field strengths:
\[
\begin{align*}
E^{2} &= F_{01} F^{10} = (\eta_{e}^{2} / \Psi) e^{-2\beta_{2} - 2\beta_{3}},
B^{2} &= F_{23} F^{23} = \eta_{m}^{2} e^{-2\beta_{2} - 2\beta_{3}}.
\end{align*}
\] (12)

One may notice the absence of the usual electric-magnetic duality as a result of the interaction. There is still a duality involving \(\Psi\): the Einstein-scalar equations are invariant under the substitution \(B^{2} \Psi \leftrightarrow E^{2} / \Psi\).

The scalar SET is characterized by the only nonzero component of \(T^{\nu}_{\mu}\), namely, \(T^{0}_{0} \sim \frac{\eta_{e}}{\sqrt{\Psi}} e^{-2\alpha} \phi^{2}\). Thus the total SET possesses the symmetry \(\tilde{T}_{1}^{1} = -\tilde{T}_{2}^{2} = -\tilde{T}_{3}^{3}\). Comparing, according to (5), the corresponding Ricci tensor components in terms of the harmonic time coordinate, we obtain
\[
\tilde{\beta}_{2} = \tilde{\beta}_{3} = -\tilde{\beta}_{1},
\]
whence it follows
\[
\begin{align*}
\beta_{2} &= -\beta_{1} + c_{2} u, \\
\beta_{3} &= -\beta_{1} + c_{3} u, \\
\alpha &= -\beta_{1} + (c_{2} + c_{3}) u,
\end{align*}
\] (13)

where \(c_{2}, c_{3} = \text{const}\) and two more integration constants have been removed by constant rescalings of the \(x^{2}\) and \(x^{3}\) axes.

The remaining unknowns are \(\beta_{1}(u)\) and \(\phi(u)\). They can be found from Eq. (14) and the so far unused components of the Einstein equations (15):
\[
\begin{align*}
\ddot{\phi} &= \frac{1}{2} e^{2\beta} \left( \eta_{m}^{2} \Psi_{\phi} - \eta_{e}^{2} \Psi_{\phi} / \Psi^{2} \right); \\
\ddot{\beta} &= -e^{2\beta} \left( \eta_{m}^{2} \Psi + \eta_{e}^{2} / \Psi \right); \\
\dot{\beta}^{2} + \dot{\phi}^{2} + e^{2\beta} \left( \eta_{m}^{2} \Psi + \eta_{e}^{2} / \Psi \right) &= c_{2} c_{3},
\end{align*}
\] (14, 15, 16)

The proper physical time of a comoving observer, \(t = \tau\), corresponds to the coordinate condition \(\alpha \equiv 0\). The harmonic time coordinate, \(t = u\), is obtained if the lapse function \(e^{2\alpha}\) is chosen so that
\[
\alpha = \beta_{1} + \beta_{2} + \beta_{3}.
\] (8)
where $\beta \equiv \beta_1$. Eq. (15) is a sum of spatial components of (6) while (16) is the constraint equation $aeT^0_0 = \ldots$, representing a first integral of (14) and (15).

It is hard to solve these equations for a given function $\Psi(\phi)$ unless it is chosen in some special forms, such as, e.g., $\Psi = e^{2A\phi}$ in (14). Before obtaining a solution for this, most frequently discussed form of dilaton gravity, we would like to discuss the isotropization and singularity problems without specifying the function $\Psi(\phi)$. We only assume it to be smooth and positive, thus preserving the correct sign of field energy.

## 3. Arbitrary $\Psi(\phi)$: isotropization and regularity

### 3.1. Isotropization criteria

Isotropization means, by definition, that at large physical times $\tau$, when the volume factor $v = v(\tau) = a_1a_2a_3 = e^{\beta_1 + \beta_2 + \beta_3}$ tends to infinity, the three scale factors $a_i(\tau) = e^{\beta_i}$ grow at the same rate, i.e., that $a_i/a \to \text{const}$, where $a(\tau) = v^{1/3}$ is the average scale factor. In terms of the harmonic time $u$, we have $v = e^\alpha(u)$, and, due to (13) and (16), all the derivatives $\dot{\alpha}$ and $\dot{\beta}_i$ are restricted above. Therefore $v \to \infty$ can only correspond to $u \to \pm \infty$. Choosing $+\infty$, we can write the isotropization condition in the form

$$\beta_i - \beta_k \to \text{const} \quad \text{as} \quad u \to \infty. \quad (17)$$

As a measure of anisotropy, one sometimes uses such quantities as the mean anisotropy parameter $A$ and the shear parameter $\Sigma^2$ defined as (see, e.g., [4])

$$A = \frac{1}{3} \sum_{i=1}^{3} \frac{H_i^2}{H^2} - 1, \quad \Sigma^2 = \frac{3}{2} AH^2, \quad (18)$$

where $H_i = a_i^{-1} a_i/d\tau = e^{-\alpha} \dot{\beta}_i$ are the “directional” Hubble parameters and $H = a^{-1} da/d\tau = \frac{1}{3} e^{-\alpha} \dot{\alpha}$ is the mean Hubble parameter. In our variables,

$$A = \frac{3}{\alpha^2} (\dot{\beta}_1^2 + \dot{\beta}_2^2 + \dot{\beta}_3^2) - 1, \quad \Sigma^2 = \frac{1}{6} e^{-2\alpha} \left[ 3(\dot{\beta}_1^2 + \dot{\beta}_2^2 + \dot{\beta}_3^2) - \dot{\alpha}^2 \right]. \quad (19)$$

The requirement (17) automatically leads to both $A \to 0$ and $\Sigma^2 \to 0$ as $u \to \infty$, but the converse is in general not true.

Thus, to obtain $A \to 0$ it is sufficient to suppose that the differences $\dot{\beta}_i - \dot{\beta}_k$ tend to nonzero constants as $u \to \infty$ [contrary to (17)] but $\dot{\beta}_i$ themselves grow infinitesimal.

The condition $\Sigma^2 \to 0$ is still weaker. Suppose, for instance, that $\beta_2 = \beta_3 \sim \text{const} \cdot u$ as $u \to \infty$ but $\beta_1 = \beta_2 + hu$, $h = \text{const} \neq 0$. Then $A$ has a nonzero limit as $u \to \infty$ whereas $\Sigma^2 = \frac{1}{3} e^{-2\alpha} h^2 \to 0$ since the volume factor $v = e^\alpha \to \infty$.

Thus our isotropization condition (17) is stronger than the requirements $A \to 0$ and $\Sigma^2 \to 0$.

### 3.2. Isotropization conditions

With (13) and (16), the condition (17) for $\beta_2$ and $\beta_3$ is fulfilled if and only if

$$c_2 = c_3 = N > 0. \quad (20)$$

It then follows that $\beta_2 \equiv \beta_3$, i.e., the Bianchi type I model is plane-symmetric. According to (13) and (17), $\beta_1 - \beta_2 = 2\beta_1 - Nu \to \text{const}$. Therefore at large $u$ we have

$$\beta \equiv \beta_1 \sim \beta_2 \sim \beta_3 \sim \frac{1}{2} Nu; \quad v = e^\alpha \sim e^{3Nu/2}. \quad (21)$$

The physical time is $\tau = \int v(u) du \sim e^{3Nu/2}$, and so the possible isotropic expansion of the Universe at large $u$ occurs according to the law

$$a_i(\tau) \sim a(\tau) \sim e^{Nu/2} \sim \tau^{1/3} \quad \text{as} \quad \tau \to \infty. \quad (22)$$

The asymptotic isotropy condition leads to essential restrictions on the properties of the electromagnetic field and the coupling function $\Psi$. Indeed, since due to (21) we have $\beta \to 0$ at large $u$, Eq. (15) can only be fulfilled if $q^2_m \Psi + \frac{q^2_e}{\Psi}$ tends to zero. This is evidently impossible if both $q_e$ and $q_m$ are nonzero. We have the following important restriction:

**Statement 1.** Isotropization is impossible in our system if there are both electric and magnetic fields.
In what follows, we restrict ourselves to the case of a purely magnetic field, \( q_m \neq 0, q_e = 0 \). (A reformulation to the case of a nonzero electric and zero magnetic field is made by replacing \( q_m \leftrightarrow q_e, \Psi \rightarrow 1/\Psi \).

Under this assumption, let us obtain a necessary condition for isotropization in terms of the coupling function \( \Psi \). As follows from (21), \( \dot{\phi} \rightarrow 0 \), or \( \Psi = o(e^{-N\tau}) \). Furthermore, according to (10), \(|\dot{\phi}| \rightarrow \pm N\sqrt{3}/2\), so that \( Nu \approx 2|\phi|/\sqrt{3} \), and we obtain

\[
\Psi(\phi) = o\left(e^{-2|\phi|/\sqrt{3}}\right) \quad \text{as} \quad \phi \to \infty. \tag{23}
\]

Under this condition, one can hope to obtain an asymptotically isotropic solution by properly fixing the integration constants.

A physical meaning of the requirement (23) is that the SET components \( T^\nu_\mu \) decay more rapidly than \( T^\nu_\mu \), and the model becomes scalar field dominated. This agrees with the evolution law (22), which corresponds to the ultra stiff equation of state \( p = \rho \) (pressure is equal to energy density), characteristic of a massless, minimally coupled, time-dependent scalar field.

The exponential coupling \( \Psi = e^{2\lambda\phi} \) conforms to (23) provided \( \lambda > 1/\sqrt{3} \) in case \( \phi \to -\infty \) or \( \lambda < -1/\sqrt{3} \) in case \( \phi \to +\infty \). Note that the “string” value \( \lambda = \pm 1 \) is also admissible. We shall verify these inferences with the exact solution in Sec. 4.

3.3. The singularity problem

Let us discuss the possibility of avoiding a cosmological singularity in our system without assuming its isotropy at large \( u \).

There is no singularity at finite \( u \) since all \( |\dot{\beta}| < \infty \) and \( |\dot{\phi}| < \infty \). A singularity can occur only at \( u = \pm \infty \), which, however, can correspond to finite or infinite physical time \( \tau \) depending on whether the integral

\[
\tau = \int e^{\alpha(u)} du \tag{24}
\]

converges or diverges. We can assert the following:

**Statement 2.** For the models under study, the range of \( \tau \) is either \(-\infty < \tau < \tau_0 \) or \( \tau_0 < \tau < +\infty \) where \( \tau_0 \) is finite. The volume factor \( v = e^{\alpha} \) is a strictly monotonic function of \( u \).

Indeed, it follows from (10) that

\[
\dot{\beta}^2 \leq 2c_2c_3, \tag{25}
\]

so, in a nontrivial solution, the constants \( c_2 \) and \( c_3 \) should be nonzero and have the same sign. Suppose that they are positive, then the evident chain of inequalities

\[
c_2c_3 < 2c_2c_3 < (c_2 + c_3)^2
\]

implies that the quantity \( \dot{\alpha} = -\dot{\beta} + c_2 + c_3 \) is strictly positive (and finite) at all \( u \), including the limits \( u \to \pm \infty \), i.e., \( \alpha(u) \) monotonically grows. The integral (24) thus converges at \( u \to -\infty \) and diverges at \( u \to +\infty \), which means that \( \tau_0 < \tau < +\infty \). If \( c_2 \) and \( c_3 \) are negative, we have a strictly monotonically decreasing function \( \alpha(u) \) and \(-\infty < \tau < \tau_0 \). A specific model can be either eternally expanding (with \( dv/d\tau > 0 \)) or eternally contracting (\( dv/d\tau < 0 \)).

Suppose, without loss of generality, \( c_2 > 0 \) and \( c_3 > 0 \), thus choosing an expanding model. As \( u \to \infty \), \( \tau \sim e^{\alpha(u)} \), while the functions \( |\beta_i(u)| \) also grow at most exponentially, therefore the scale factors \( a_i(\tau) \) cannot grow or vanish faster than according to a power law. This means (see the Appendix) that the model is nonsingular at the end of the evolution.

On the contrary, at its beginning \( \tau \to \tau_0 \), where \( v = a_1a_2a_3 \to 0 \), the only way of avoiding a singularity is to assume that only one of the scale factors \( a_i \) vanishes while the others remain finite. If we suppose that it is \( a_2 \) or \( a_3 \) that vanishes, then from (10) we immediately obtain that one of the constants \( c_2 \), \( c_3 \) is zero, making the solution trivial. Therefore the only viable opportunity is

\[
a_1 \to 0, \quad a_2 = a_3 \to \text{const} > 0, \tag{26}
\]

which happens when \( c_2 = c_3 = N > 0 \) (as was the case for isotropization). Assigning \( \tau_0 = 0 \), we obtain \( a_1 \sim \tau \sim e^{Nu} \to 0 \) as \( u \to -\infty \), and the metric near \( \tau = 0 \) may be written as

\[
ds^2 \approx dt^2 - \frac{k_1^2}{\tau^2} dx^2 - k_2^2(dx^2 + dx^3) = \frac{k_1}{2t} dt^2 - \frac{2t}{k_1} dx^2 - k_3^2(dx^2 + dx^3) \tag{27}
\]
where \( k_1, k_2 = \text{const} > 0 \) and \( t = \tau^2/(2k_1) \). Evidently, the instant \( \tau = t = 0 \) is a Killing horizon, and a transition to negative \( t \) leads to a static, plane-symmetric space-time region where \( t \) is a spatial coordinate and \( x^1 \) temporal. The properties of the region \( t < 0 \) may be studied for specific \( \Psi(\phi) \).

Eq. (10) shows that at such a horizon \( \dot{\phi} \to 0 \) and \( \Psi e^{2\beta} \to 0 \). Finiteness of \( T_{\mu}^\nu \) implies \( \dot{\phi} \) vanishes as \( e^{N u} = e^{-N|u|} \) or faster. On the other hand, the magnetic field strength is finite at the horizon [see (12)], and finiteness of \( T_{\mu}^\nu \) implies \( \Psi < \infty \).

One can also note that since both isotropization and a horizon in the past require \( c_2 = c_3 \), therefore one cannot exclude that some solutions possess both features.

The main conclusion from this general study can be formulated as follows.

**Statement 3.** Expanding models begin their evolution at finite proper time \( \tau \) (say, \( \tau = 0 \)) either from a singularity, which is the general case, or from a simple horizon preceded by a static phase. The latter happens when the model possesses additional planar symmetry \([a_2(\tau) \equiv a_3(\tau)] \) and, moreover, both \( a_2 \) and \( a_3 \) take a finite value at \( \tau = 0 \).

A reformulation for contracting models is obvious.

### 4. Exact solution for \( \Psi = e^{2\lambda \phi} \)

#### 4.1. Solution

We choose the function \( \Psi \) in the form \( \Psi = e^{2\lambda \phi}, \lambda = \text{const} \), returning to the action (1). Our set of equations (14–16) is then rewritten as

\[
\begin{align*}
\ddot{\phi} &= -\lambda m^2 e^{2\beta + 2\lambda \phi}, \\
\ddot{\beta} &= -q_m^2 e^{2\beta + 2\lambda \phi}, \\
\beta^2 + \phi^2 + q_m^2 e^{2\beta + 2\lambda \phi} &= c_2c_3. 
\end{align*}
\]

Eqs. (28) and (29) lead to

\[
\begin{align*}
\dot{\phi} &= \lambda \beta \Rightarrow \phi = \lambda \beta - c_1 u, \\
\dot{y} &= -q_m^2(1 + \lambda^2) e^{2y} \Rightarrow e^y = \frac{k}{q_m \sqrt{1 + \lambda^2 \cosh{(ku)}}},
\end{align*}
\]

where \( y(u) := \beta + \lambda \phi, c_1 \) and \( k > 0 \) are constants. Two more integration constants are absorbed by rescaling \( x^1 \) and by choosing the origin of \( u \). The functions \( \phi \) and \( \beta \) are expressed in terms of \( y \):

\[
\begin{align*}
\phi &= \frac{\lambda y - c_1 u}{1 + \lambda^2}, \\
\beta &= \frac{y + \lambda c_1 u}{1 + \lambda^2}.
\end{align*}
\]

The constraint (30) leads to a relation between the integration constants:

\[
k^2 + c_1^2 = (1 + \lambda^2)c_2c_3.
\]

This, along with Eq. (13), completes the solution. Apart from the coupling constant \( \lambda \), the general solution contains five integration constants: \( q_m, k, c_1, c_2, c_3 \) with the single constraint Eq. (34).

An explicit form of the solution is

\[
\begin{align*}
a_1(u) &= e^{\lambda c_1 u} Q \cosh{(ku)} \left[ Q \cosh{(ku)} \right]^{1/(1 + \lambda^2)}, \\
a_2(u) &= e^{\beta u} \left[ Q \cosh{(ku)} \right]^{-1/(1 + \lambda^2)}, \\
a_3(u) &= e^{\phi} \left[ Q \cosh{(ku)} \right]^{-1/(1 + \lambda^2)}, \\
e^\phi &= \left[ Q \cosh{(ku)} \right]^{-\lambda/(1 + \lambda^2)} e^{-c_1 u/(1 + \lambda^2)}, \\
d\tau &= a_1(u) a_2(u) a_3(u) du,
\end{align*}
\]

where \( Q = (q_m/k) \sqrt{1 + \lambda^2} \).

This solution coincides, up to notations, with that found in [8, 11].
4.2. Isotropization

The Bianchi-I cosmologies described by Eqs. (35) begin with an anisotropic singularity at finite $\tau$ corresponding to $u \to -\infty$ and, in general, expand anisotropically at late times ($u \to \infty, \tau \to \infty$).

Let us, however, seek among them a subfamily of models becoming isotropic as $\tau \to +\infty$. As before, we put $c_2 = c_3 = N > 0$; one more constraint follows from the requirement $\dot{\beta}(+\infty) = N/2$ (see Sec. 3.2) which leads to

$$\lambda c_1 - k = (1 + \lambda^2)N/2.$$  \hspace{1cm} (36)

Then, as follows from (30), $\dot{\phi}(+\infty) = \pm N/\sqrt{3}$. We choose, without loss of generality, the minus sign, so that $\phi \to -\infty$, which implies $\lambda > 0$ since we must have $\Psi = e^{2\lambda\phi} \to 0$ (see Sec. 3.2). As a result, $c_1$ and $k$ are expressed in terms of $N$ and $\lambda$:

$$2c_1 = N(\lambda + \sqrt{3}), \hspace{1cm} 2k = N(\lambda\sqrt{3} - 1).$$ \hspace{1cm} (37)

We thus obtain a family of asymptotically isotropic solutions parametrized by $\lambda$ and two integration constants $N$ and $q_m$. Eq. (37) implies the requirement

$$\lambda > 1/\sqrt{3},$$ \hspace{1cm} (38)

in full agreement with Eq. (23).

4.3. The horizon and beyond

To find solutions with a horizon instead of a singularity in the remote past, we put again $c_2 = c_3 = N > 0$ and also require (see Sec. 3.3) $\beta_2 = \beta_3 \to \text{const}$ and $\dot{\phi} \to 0$ as $u \to -\infty$. We obtain

$$c_1 = k\lambda, \hspace{1cm} k = N.$$ \hspace{1cm} (39)

The first of these conditions also provides a finite value of $\phi$ at the horizon.

We thus obtain one more family of solutions parametrized by $\lambda$, $N$ and $q_m$. It is, in general, different from the family of asymptotically isotropic solutions and, in particular, it exists for any $\lambda$. However, the two families coincide in case $\lambda = \sqrt{3}$. In this and only in this case we have a two-parameter family of asymptotically isotropic solutions without a cosmological singularity.

Let us continue the solutions satisfying (39) beyond their horizon. To this end, we use the coordinate transformation

$$e^{2k\mu} = \xi - 1$$ \hspace{1cm} (40)

and introduce the notations

$$\mu := \frac{1}{1 + \lambda^2}, \hspace{1cm} C_0 := \left(\frac{2k}{q_m\sqrt{1 + \lambda^2}}\right)^2.$$ \hspace{1cm} (41)

As a result, we obtain the metric

$$ds^2 = \frac{\xi^2\mu}{4k^2C_0} \frac{d\xi^2}{\xi - 1} - \frac{C_0(\xi - 1)}{\xi^2\mu} dx^1 dx^2 - \frac{\xi^2\mu}{C_0} \left(dx^2^2 + dx^3^2\right).$$ \hspace{1cm} (42)

The horizon takes place at $\xi = 1$, and $\xi > 1$ describes the cosmological evolution, which is asymptotically isotropic at large $\xi$ if and only if $\mu = 1/4$ which corresponds to $\lambda^2 = 3$, in full agreement with the above-said.

In the region $0 < \xi < 1$, the metric is static, plane-symmetric, $x^1$ is a temporal coordinate and $\xi$ is the spatial coordinate in the direction across the symmetry planes parametrized by $x_2$ and $x_3$. Lastly, $\xi = 0$ is a timelike repulsive singularity resembling the one in the Reissner-Nordström space-time. It is connected with infinite fields, densities and stresses: the magnetic field strength $B$ and both $T^e_{\mu\nu}$ and $T^s_{\mu\nu}$ are infinite at $\xi = 0$.

5. String metric

In case $\lambda = 1$, the Einstein-frame action with the metric $g_{\mu\nu}$ is obtained by the conformal transformation

$$\hat{g}_{\mu\nu} = e^{-2\phi} g_{\mu\nu}$$ \hspace{1cm} (43)

from the string-frame action

$$S = \int \sqrt{g} d^4x e^{2\phi} \left[R - 4(\dot{\phi})^2 - \hat{F}^{\mu\nu} F_{\mu\nu}\right],$$ \hspace{1cm} (44)
where the hat marks quantities obtained from or with the aid of the so-called string metric $\hat{g}_{\mu\nu}$. The action in the form [14] naturally appears from string theory, and in this sense the metric $\hat{g}_{\mu\nu}$ is more fundamental than $g_{\mu\nu}$, although the question “which metric is to be used to confront theory with observations?” should be answered separately — see more detailed discussions in Refs. [3, 4] and references therein.

Now, let us take the coupling constant $\lambda = 1$ and briefly discuss the properties of the metric $\hat{g}_{\mu\nu}$ corresponding to the solution of Sec. 4. Evidently, $\hat{a}_i(u) = e^{-\hat{\phi}a_i(u)}$, and the proper time element is $d\hat{\tau} = e^{-\hat{\phi}}d\tau$. Explicitly, using [35] with $\lambda = 1$, it is straightforward to obtain

$$
e^\phi = e^{-c_1u/2}[Q \cosh(ku)]^{-1/2},$$
$$ds^2 = d\hat{\tau}^2 - \sum_{i=1}^3 \hat{a}_i^2(dx^i)^2$$
$$= Q^2 e^{2(c_2+c_3)u} \cosh^2(ku)du^2 - e^{2c_1u}dx^{12} - Q^2 \cosh^2(ku)\left(e^{2c_2u}dx^{22} + e^{2c_3u}dx^{33}\right),$$

(45)

where $q_m \sqrt{2/k}$. Choosing, as before, positive $c_2$ and $c_3$, we see that in expanding models $\hat{\tau} \to \infty$ corresponds to $u \to \infty$ while $u \to -\infty$ corresponds to finite $\hat{\tau}$.

Furthermore, the asymptotic isotropy conditions are the same as in the Einstein frame since conformal mappings do not affect the condition [17]. Specialized to $\lambda = 1$, they read

$$c_2 = c_3 = N, \quad 2c_1 = N(\sqrt{3} + 1), \quad k = N(\sqrt{3} - 1).$$

(46)

One may notice that, in the Einstein frame, the expansion begins from a Killing horizon instead of a singularity in case $\phi \to \text{const}$ as $u \to -\infty$, i.e., the conformal factor in [14] is finite at the horizon. Therefore the conditions [35] preserve their meaning in the string frame and now read

$$c_1 = k = N.$$  
(47)

After the substitution [10], the metric [35] with [47] transforms to

$$ds^2 = \frac{Q^2}{4k} \frac{\xi^2}{\xi - 1} d\xi^2 - (\xi - 1)dx^{12} - \frac{Q^2}{4} \xi^2 (dx^{22} + dx^{33}).$$

(48)

As in [12], the cosmological evolution corresponds to $u \in \mathbb{R}$, or $\xi > 1$, while $0 < \xi < 1$ is a static region with a singularity at $\xi = 0$. This singularity is, in a sense, milder than in the Einstein frame since it is purely spatial: the temporal metric coefficient $\hat{g}_{11}$ is finite at $\xi = 0$ (recall that it is $x^1$ that plays the role of time at $\xi < 1$).

6. Concluding remarks

We have described some important qualitative features of Bianchi-I cosmologies in the theory [2], containing the arbitrary function $\Psi(\phi)$, without entirely solving the field equations — see items 1–5 in the Introduction. These features have been confirmed in an exactly solvable particular case, the system [10].

For this latter system, exact solutions were already obtained in Refs. [2–4]; according to [3], the solutions given in [2] contained an error; those given in [3, 4] coincide with ours up to notations. However, the authors of these papers did not point out any isotropizing models or models whose expansion starts from a horizon.

We have shown that for this system, asymptotically isotropic solutions form a two-parameter family for each suitable $\lambda$ whereas the full set contains four essential integration constants. In this sense, isotropization requires fine tuning. The same is true for solutions having a horizon instead of a singularity at the beginning of their expansion. For a particular value of the coupling constant $\lambda (\pm \sqrt{3})$ these two-parameter families coincide.

Other examples of cosmologies whose expansion starts from a horizon (from the so-called “null bang”) were suggested in Ref. [10] as vacuum cosmologies with a variable cosmological term; unlike the present examples, they contained a de Sitter-like core instead of a singularity in their static region.

Certain remarks are to be made about other exact solutions to the Einstein-Maxwell-dilaton equations due to [10].

First of all, some solutions can be obtained with both electric and magnetic charges. Eqs. [14] and [15] with $\Psi = e^{2\lambda \phi}$ read

$$\phi = -\lambda_m^2 e^{2y} + \lambda_e^2 e^{2z}, \quad \beta = -q_m^2 e^{2y} - q_e^2 e^{2z},$$

(49)

where $y = \beta + \lambda \phi$ and $z = \beta - \lambda \phi$. Their two linear combinations form the Toda-like system

$$\dot{y} = -q_m^2 (1 + \lambda^2) e^{2y} - q_e^2 (1 - \lambda^2) e^{2z},$$
$$\dot{z} = -q_m^2 (1 - \lambda^2) e^{2y} - q_e^2 (1 + \lambda^2) e^{2z}.$$  

(50)
These equations decouple and take an easily solvable Liouville form [see (32)] in case $\lambda^2 = 1$, i.e., for the string value of the electro-dilatonic coupling constant. We will not proceed with this solution, only recalling that it evidently cannot describe isotopizing cosmologies — see Statement 1 above.

The distinguished properties of string coupling, leading to integrability of the Einstein-Maxwell-dilaton equations, were previously discussed for multidimensional static, spherically symmetric configurations with the Lagrangian (11) in Ref. [9], where solutions with three charges of different nature were obtained (in addition to electric and magnetic charges, there is a quasiscalar charge related to possible extra-dimensional components of $F_{\mu\nu}$), including as special cases different dilatonic black holes. Our present solutions (14), (15) with a single charge have a deep similarity with static, spherically symmetric solutions for the system (11) in Ref. [11], where also, in a special case, a horizon was found instead of a naked singularity (to our knowledge, that was the first example of what was later named a dilatonic black hole).

Still closer counterparts of the present solutions are static, plane-symmetric and cylindrically symmetric ones (see, e.g., [12] for the corresponding Einstein-Maxwell solutions, including a special case with a horizon).

In a more general context, the system (11) is a special case of systems with self-gravitating interacting scalars and antisymmetric forms in diverse dimensions, frequently associated with $p$-branes and appearing in the bosonic sector of the field limit of supergravities, M-theory, etc (see [13] and references therein). Methods of obtaining exact solutions for large classes of such models have been elaborated ([14] and references therein) on the basis of a sigma model representation where each charge like $q_m$ and $q_e$ is associated with a constant vector in a target space. In terms of this approach, the case $\lambda^2 = 1$ of Eqs. (50) corresponds to mutual orthogonality of two such vectors; for other values of $\lambda$ the orthogonality is lacking, and the equations cannot be so easily integrated.

Appendix

Consider regularity conditions for the metric (7) with the aid of the Kretschmann scalar $\mathcal{K} = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$. As in many other cases, $\mathcal{K}$ for this metric is a sum of squares of all nonzero Riemann tensor components $R_{\alpha\beta\gamma\delta}$. Therefore the condition $\mathcal{K} < \infty$ is necessary and sufficient for finiteness of all algebraic curvature invariants. Explicitly, for (7) written with an arbitrary $u$ coordinate,

$$\mathcal{K} = 4 \sum_{i=1}^{3} \left[ e^{-a_{i} - \beta_{i}} (e^{\beta_{i} - a_{i}} \dot{a}_{i}) \right]^2 + 4 \sum_{i \neq k} \left[ e^{-2a_{i} \beta_{j} \beta_{k}} \right]^2. \quad (A.1)$$

In terms of the scale factors $a_i = e^{\beta_i}$ as functions of the physical time $\tau$,

$$\mathcal{K} = 4 \sum_{i=1}^{3} \left( \frac{a_i''}{a_i} \right)^2 + 4 \sum_{i \neq k} \left( \frac{a_i' a_k'}{a_i a_k} \right)^2, \quad (A.2)$$

where the prime stands for $d/d\tau$. This expression leads to evident and convenient regularity criteria.

Our solutions, written in terms of the harmonic time coordinate $u$, are manifestly regular at finite $u$. Meanwhile, $u \to \pm \infty$ may correspond to both finite and infinite physical time $\tau$. Let us therefore discuss the regularity condition $\mathcal{K} < \infty$ at finite and infinite $\tau$ separately. Evidently, a singularity can occur when some or all scale factors $a_i$ tend to zero or infinity.

(i) $\tau$ finite, some $a_i \to \infty$. Since in this case $\ln a_i$, $a_i'$ and $a_i''$ blow up as well, we have $a_i''/a_i = (a_i''/a_i')(a_i'/a_i) \to \infty$. Therefore, if at least one scale factor becomes infinite at finite $\tau$, it is a curvature singularity.

(ii) $\tau$ finite, some $a_i \to 0$. Since $\ln a_i \to -\infty$, we have $a_i'/a_i \to -\infty$. However, $a_i''/a_i$ may remain finite. The expression (A.2) shows that if more than one scale factor turns to zero at finite $\tau$, it is a singularity. If only one scale factor $a_i = 0$ at finite $\tau$, the space-time can be nonsingular. An explicit inspection is then necessary: one may find, e.g., a Killing horizon instead of a singularity.

(iii) $\tau \to \infty$, some $a_i \to \infty$. An inspection shows that such an asymptotic can only be singular if at least one function $a_i(\tau)$ grows faster than exponentially: $a_i(\tau) \gg \exp(k|\tau|)$, $k = \text{const} > 0$.

(iv) $\tau \to \infty$, some $a_i \to 0$. As in item (iii), we find that such an asymptotic can only be singular if at least one function $a_i(\tau)$ vanishes faster than exponentially: $a_i(\tau) = o(\exp(-k|\tau|))$, $k = \text{const} > 0$.

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