A quantum BRST anti-BRST approach to classical integrable systems

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(Dated: August 2003)

Abstract

We reformulate the conditions of Liouville integrability in the language of Gozzi et al.’s quantum BRST anti-BRST description of classical mechanics. The Das-Okubo geometrical Lax equation is particularly suited to this approach. We find that the Lax pair and inverse scattering wavefunction appear naturally in certain sectors of the quantum theory.

PACS numbers: 02.30.Ik

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I. INTRODUCTION

We use the quantum mechanical approach to classical mechanics, developed by Gozzi, Reuter and Thacker [1, 2, 3] and generalized by Marnelius [4] to a symplectic supermanifold, in order to study Liouville integrability. While it is novel in itself to obtain classical mechanics from a quantum system, our real interest in this approach is with the natural way in which it marries Hamiltonian mechanics with differential geometry.

Integrable systems are of considerable interest to physicists, mostly because their equations of motion possess soliton solutions. There are various equivalent ways to establish whether a system is integrable, for example the construction of a Lax equation or the zero curvature formalism. In this article, we study two well-established methods, both of which require the existence of a bi-Hamiltonian structure. Both of these methods were originally constructed using standard Hamiltonian mechanics. We aim to show here the advantages of using the quantum BRST description of classical mechanics.

The paper is structured as follows. In section II we set up the bi-Hamiltonian system.

In section III we use the technique of Magri [5], to directly derive integrability from the existence of a bi-Hamiltonian structure. A further condition is also needed that the two Poisson brackets obey a certain compatibility condition. The quantum BRST approach is nice here because geometrical constructs, such as the Hamiltonian flow vector field, actually belong to the phase space manifold. This was first studied in an interesting paper by Calian [6], though we follow a slightly different route.

In section IV we use the approach of Das and Okubo [7] to derive a Lax equation from the bi-Hamiltonian structure. The system is found to be integrable when the Nijenhuis tensor, which depends on the Lax operator, vanishes. Here the mechanics of Gozzi et al. really comes into its own. In particular, the Lax equation actually is the Hamiltonian equation of motion of the Lax tensor, and the associated inverse scattering wavefunction appears at ghost number one. We create more general Lax Pairs, by deforming the ghost co-ordinates, while maintaining the BRST anti-BRST symmetry algebra of the theory. We exploit the fact that the Lax equation depends on the existence of this algebra and not on the brackets of the co-ordinates themselves.

Finally, we note some interesting earlier work [8, 9, 10], which has some similarity in approach to this article.
II. PRELIMINARIES

A. The symplectic manifold

For simplicity we will only consider bosonic systems, but it is straightforward to extend this paper to fermionic systems. We begin with a symplectic manifold $\mathcal{M}$ with phase-space co-ordinates $\phi^a$ for $a = 1, \ldots, 2N$, which is endowed with two symplectic forms

$$\omega_r = \frac{1}{2}(\omega_r)_{ab}(\phi)d\phi^a \wedge d\phi^b, \quad r = 0, 1,$$

which are non-degenerate and closed

$$\det \omega_r \neq 0, \quad d\omega_r = 0.$$

Each two-form $(\omega_r)_{ab}$ has an inverse $\omega_r^{ab}$ defined by

$$(\omega_r)_{ab}(\omega_r)^{bc} = \delta^c_a,$$

with which is associated the Poisson bracket

$$\{f(\phi), g(\phi)\}_r = (\partial_a f)\omega_r^{ab}(\partial_b g), \quad r = 0, 1.$$  \hfill (4)

The bracket obeys the Jacobi identity due to $\omega_r$ being closed $^{[2]}$, and is antisymmetric under interchange of $f$ and $g$, since $\omega_r^{ab} = -\omega_r^{ba}$.

We associate a Hamiltonian $H_1(\phi)$ with symplectic form $\omega_1$ and $H_2(\phi)$ with $\omega_0$. There exists a bi-Hamiltonian structure if each bracket plus Hamiltonian pairing yields the same equations of motion

$$\dot{\phi}^a = \omega_1^{ab}\partial_b H_1 = \omega_0^{ab}\partial_b H_2.$$  \hfill (5)

B. The BRST anti-BRST quantum description of classical mechanics

Now, we look at the same system, but using the BRST quantum mechanical approach $^{[1, 2, 3, 4]}$. We extend the phase space with bosonic variables $\lambda_a$ and fermionic ghosts $C^a$, $P_a$. From now on we work with quantum operators, which are denoted by a hat, and set $\hbar = 1$.

The non-vanishing, graded commutators for the extended phase-space co-ordinates are defined by

$$[\hat{\phi}^a, \hat{\lambda}_b] = \delta^a_b, \quad [\hat{C}^a, \hat{P}_b] = \delta^a_b,$$

$^{3}$
where the usual graded quantum bracket $[\cdot, \cdot]$ is

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - (-)^{\hat{A}\hat{B}}\hat{B}\hat{A}, \quad (7)$$

such that $\varepsilon_A$ is the Grassmann parity of $\hat{A}$. So here $\varepsilon_\phi = \varepsilon_\lambda = 0$ and $\varepsilon_C = \varepsilon_p = 1$. The bracket also obeys the Jacobi identity

$$[[\hat{A}, \hat{B}], \hat{C}] + (-)^{\varepsilon_A(\varepsilon_B + \varepsilon_C)}[[\hat{B}, \hat{C}], \hat{A}]$$
$$+ (-)^{\varepsilon_C(\varepsilon_A + \varepsilon_B)}[[\hat{C}, \hat{A}], \hat{B}] = 0. \quad (8)$$

Note in particular that $\hat{\phi}^a$ commute with each other, so there are no ordering ambiguities for operators of the form $A(\hat{\phi})$, which is what we want in order to describe classical mechanics. Equation (6) follows directly in the Schrödinger representation from the operator definitions

$$\hat{\phi}^a \equiv \phi^a, \quad \hat{\lambda}_a \equiv -\partial_a$$
$$\hat{C}^a \equiv C^a, \quad \hat{P}_a \equiv \frac{\partial}{\partial C^a}. \quad (9)$$

We define BRST and anti-BRST charges as in [4]

$$\hat{Q} = \hat{C}^a \hat{\lambda}_a, \quad \hat{Q}_r = \hat{P}_a \omega^{ab}_r \hat{\lambda}_b + \frac{1}{2} (\partial_c \omega^{ab}_r) \hat{C}^c \hat{P}_a \hat{P}_b \quad (11)$$

which obey

$$[\hat{Q}, \hat{Q}] = [\hat{Q}_r, \hat{Q}] = [\hat{Q}_r, \hat{Q}_r] = 0, \quad (12)$$

using equation (2) and that $\omega_r$ is antisymmetric.

The time evolution for any operator $\hat{F}$ is given by

$$\dot{\hat{F}} = [\hat{F}, \hat{H}_{eff}], \quad (13)$$

where

$$\hat{H}_{eff} = -[\hat{Q}, [\hat{Q}_1, \hat{H}_1]] = -[\hat{Q}_r, [\hat{Q}_0, \hat{H}_2]] \quad (14)$$

is the effective Hamiltonian [2, 4]. Note that the bracket (6) is flat and $\hat{H}_{eff}$ contains both the information of the original Hamiltonian and of the original curved Poisson bracket [4]. The above two definitions of $\hat{H}_{eff}$ are equal as a result of the bi-Hamiltonian equation (5).

In components, using $\omega_1$ and $H_1$ for example,

$$\hat{H}_{eff} = \hat{\lambda}_a \hat{\omega}^{ac}_1 (\partial_c \hat{H}_1) - \hat{C}^a \hat{P}_b \partial_a (\omega^{bc}_1 \partial_c \hat{H}_1). \quad (15)$$
While $\dot{\hat{\phi}}^a = [\hat{\phi}^a, \hat{H}_{eff}]$ is the same as in equation \((5)\), the ghosts $\hat{C}^a$ and $\hat{P}_a$ obey
\[
\dot{\hat{C}}^b = \hat{C}^a \dot{U}_a^b, \quad \dot{\hat{P}}_a = -\hat{P}_b \dot{U}_a^b, \tag{16}
\]
where $\dot{U}_a^b = \partial_a (\hat{\omega}_1^{bc} \partial_c \hat{H}_1)$. The above equations of motion for $\dot{\hat{C}}^b$ and $\dot{\hat{P}}_a$ are exactly the same as for the 1-form $d\phi^b$ and the vector $\partial/\partial\phi^a$ respectively. Thus in general we can study the time evolution of a generic $(p, q)$ tensor
\[
\hat{T} = T_{a_1...a_q}^{b_1...b_p} \dot{\hat{C}}^{a_1}...\dot{\hat{C}}^{a_q} \dot{\hat{P}}_{b_1}...\dot{\hat{P}}_{b_p}, \tag{17}
\]
which is separately antisymmetric in its upper and lower indices since the ghosts are fermionic. We interpret $\dot{\hat{C}}^a$ as $d\phi^a$ and $\dot{\hat{P}}_a$ as $\partial/\partial\phi^a$, and the ghost products
\[
\dot{\hat{C}}^{a_1}...\dot{\hat{C}}^{a_q} \equiv d\phi^{a_1} \wedge ... \wedge d\phi^{a_q}, \tag{18}
\]
\[
\dot{\hat{P}}_{b_1}...\dot{\hat{P}}_{b_p} \equiv \frac{\partial}{\partial \phi^{b_1}} \wedge ... \wedge \frac{\partial}{\partial \phi^{b_p}}, \tag{19}
\]
where we extend the definition of the wedge product to vectors in the same way as with forms i.e. as an antisymmetric direct product. Whereas in ordinary Hamiltonian mechanics, we are limited to writing the time evolution of functions, the extension to $(p, q)$ antisymmetric tensors will be very useful to us in section \[LV\].

As well as $\hat{Q}$ and $\hat{Q}_r$, other operators which commute with $\hat{H}_{eff}$ are the antisymmetric tensors
\[
\hat{K}_r = \frac{1}{2} (\hat{\omega}_r)_{ab} \hat{C}^a \hat{C}^b, \quad \hat{\bar{K}}_r = \frac{1}{2} \hat{\omega}_r^{ab} \hat{P}_a \hat{P}_b, \tag{20}
\]
and the ghost number operator $\hat{Q}_g = \hat{C}^a \hat{P}_a$. Their algebra, which is the symmetry algebra of this system, is given in equation \((A1)\). We derive that $\hat{K}_r$ and $\hat{\bar{K}}_r$ commute with $\hat{H}_{eff}$ later in equation \((32)\).

A geometrical interpretation of the bi-Hamiltonian equation \((5)\) is that the vector field
\[
X = [\hat{H}_2, \hat{Q}_0] = [\hat{H}_1, \hat{Q}_1]. \tag{21}
\]
describing the Hamiltonian flow of the system, has two equivalent Hamiltonian descriptions. Of course the vector field is a special case of the tensor \((17)\). We see that in components, $[\hat{H}_1, \hat{Q}_1] = \hat{P}_a \hat{\omega}_1^{ab} \partial_b \hat{H}_1 = \hat{P}_a \hat{\dot{\phi}}^a$ and similarly for $[\hat{H}_2, \hat{Q}_0]$, where we recall that $\hat{P}_a \equiv \partial/\partial\phi^a$.

We will often need to write the original Poisson bracket \((4)\) in terms of the bracket of the quantum theory. There are various equivalent expressions. The ones that we use are stated below, and some others are listed in the appendix.
Firstly, note that the expression for a vector field \( X_r f = \left[ f(\hat{\phi}), \hat{Q}_r \right] \) acting on a function \( g(\phi) \) is

\[
X_r g = \left[ X_r, [\hat{Q}, g] \right],
\]

which is of course the same as the Poisson bracket \( \{ f, g \}_r \). This leads to two equivalent expressions for a bracket, depending on whether we calculate \( X_r g \) or \( -X_g f \)

\[
\{ f(\phi), g(\phi) \}_r = [[f, \hat{Q}_r], [\hat{Q}, g]] = -[[f, \hat{Q}], [\hat{Q}_r, g]].
\]

These expressions are closely related to Marnelius’ bracket \( \text{[A2]} \), which is an extension of the above to allow general functions \( f \) and \( g \) of the full phase-space.

Finally, we note that many more differential geometry operations than listed here can be expressed \( \text{[2, 3]} \) as brackets between quantum operators, or as operators acting on the Hilbert space, which is isomorphic to the space of p-forms.

### III. DIRECT PROOF OF INTEGRABILITY OF A BI-HAMILTONIAN SYSTEM

The requirements of Liouville integrability are that there exist an infinite set of conserved charges, which commute with each other under the Poisson brackets \( \text{[1]} \). In \( \text{[4]} \), Magri proved that all bi-Hamiltonian systems are integrable, given that the two Poisson brackets obey a certain compatibility condition described below. We give the same proof, but using Gozzi et al.’s quantum BRST description of classical mechanics.

Given the bi-Hamiltonian equation \( \text{[21]} \), one can show that there always exists a third operator \( \hat{H}_3(\phi) \) such that the vector field \( Y \) has two equivalent descriptions

\[
Y = [\hat{H}_3, \hat{Q}_0] = [\hat{H}_2, \hat{Q}_1],
\]

so long as the Poisson brackets \( \{ , \} \) are compatible, i.e. that the bracket \( \{ , \} + \{ , \} \) is non-degenerate and obeys the Jacobi identity. In order to prove existence of \( H_3 \), we must show that the vector field \( Y \) is Hamiltonian with respect to the bracket \( \{ , \} \). The Jacobi identity for \( H_2, f, g \) and Poisson bracket \( \{ , \} + \{ , \} \) yields

\[
\{ H_2, \{ f, g \}_0 \} + \{ f, \{ g, H_2 \}_0 \}_1 + \{ g, \{ H_2, f \}_0 \}_1 + (0 \leftrightarrow 1) = 0,
\]

where \( f \) and \( g \) are arbitrary functions of \( \phi^n \). Since \( X \) in equation \( \text{[21]} \) is Hamiltonian with respect to \( \omega_1 \) by definition, using the Jacobi identity we find that

\[
X\{ f, g \}_1 - \{ Xf, g \}_1 - \{ f, Xg \}_1 = 0,
\]
thus from (25) and (26)

\[ Y\{f,g\}_0 - \{Yf,g\}_0 - \{f,Yg\}_0 = 0, \]  

(27)

where \(Y = [H_2, \dot{Q}_1]\). Equation (27) is a necessary and sufficient condition that \(Y\) is Hamiltonian with respect to \(\omega_0\). We could have completed the above stages using the Marnelius style brackets (23) and the definition of a vector acting on a function (22). In the end it’s simpler as above.

From (24), by iteration there exist \(\hat{H}_m, m \geq 1\) such that

\[ [\hat{H}_{m+1}, \dot{Q}_0] = [\hat{H}_m, \dot{Q}_1]. \]  

(28)

Using the Marnelius style Poisson bracket (23) and equation (28),

\[ \{H_n, H_{m+1}\}_0 = [[[\hat{H}_n, \dot{Q}], [\hat{Q}_0, \hat{H}_{m+1}]] = \{H_n, H_m\}_1, \]  

(29)

and similarly

\[ \{H_n, H_m\}_1 = -[[\hat{H}_{n-1}, \dot{Q}], [\hat{Q}, \hat{H}_m]] = -[[\hat{H}_n, \dot{Q}_0], [\hat{Q}, \hat{H}_m]] = \{H_{n-1}, H_m\}_0. \]  

(30)

Now \(\{H_n, H_n\}_r\) is zero since the Poisson bracket is antisymmetric and \(\{H_n, H_{n+1}\}_r = 0\) from either (29) or (30). Applying recursion relations (29) and (30), any commutator \(\{H_n, H_{m+1}\}\) can be written either in the form \(\{H_p, H_p\}_r\) or \(\{H_p, H_{p+1}\}_r\) for some integer \(p\), and hence vanishes. Therefore

\[ \{H_m, H_n\}_r = 0. \]  

(31)

Since all \(H_m\) commute with \(H_1\) and \(H_0\), they are all constants of motion.

**IV. THE DAS-OKUBO GEOMETRICAL LAX EQUATION**

In [7], Das and Okubo found an alternative way of deriving integrability of a bi-Hamiltonian system, by providing a recipe for a Lax pair. This gives the Lax pair method a geometrical interpretation, in terms of the bi-Hamiltonian structure, and relates two seemingly unconnected constructions. As explained in the introduction, the quantum BRST approach to classical mechanics is particularly fruitful here.

As in the previous section, the requirements of Liouville integrability are that there exist an infinite set of conserved charges, which commute with each other under the Poisson brackets (4).
Using the expression for $\hat{H}_{eff}$ in equation (14), the Jacobi identity (8) and the BRST anti-BRST algebra of equation (A1), we find that

$$[\hat{K}_r, \hat{H}_{eff}] = [\hat{K}_r, \hat{H}_{eff}] = 0 \quad (32)$$

where recall that $\hat{K}_r \equiv \omega_r$ and $\hat{K}_r \equiv \omega_r^{-1}$ as in equation (20). Thus, the tensors $\hat{K}_r$ and $\hat{K}_r$ are invariant under Hamiltonian flow. Note that this does not mean that $(\omega_r)_{ab}$ is constant, rather that its time evolution is cancelled by the time evolution of the ghosts $\hat{C}^a$, and similarly for $\omega_r^{ab}$ and $\hat{P}_a$. Recall that $\hat{C}^a$ is identified with $d\phi^a$, and $\hat{P}_a$ with $\partial/\partial \phi^a$.

We define the (1,1) tensor

$$\hat{S} = [\hat{K}_1, \hat{K}_0] = \hat{C}^a S^b_a (\hat{\phi}) \hat{P}_b \equiv S^b_a (\hat{\phi}) \frac{\partial}{\partial \hat{\phi}^b}, \quad (33)$$

where $S^b_a = (\hat{\omega}_0)_{ac} \hat{\omega}_1^c$, as a candidate for a Lax operator. From equation (32) and the Jacobi identity (8), $\hat{S}$ is also invariant under Hamiltonian flow

$$\dot{\hat{S}} = [\dot{\hat{S}}, \hat{H}_{eff}] = 0. \quad (34)$$

Utilizing the expression for $\hat{H}_{eff}$ in equation (15), the above equation reads

$$\dot{\hat{C}}^a (\hat{S}^b_a - [S, U]_a^b) \hat{P}_b = 0, \quad (35)$$

where

$$\dot{\hat{U}} = \dot{\hat{C}}^a \partial_a (\omega_1^{bc} \partial_c H_1) \hat{P}_b = \dot{\hat{C}}^a \partial_a (\omega_0^{bc} \partial_c H_2) \hat{P}_b = \dot{\hat{C}}^a U_a^b (\hat{\phi}) \hat{P}_b, \quad (36)$$

and we used that

$$[\hat{S}, \dot{\hat{U}}] = \dot{\hat{C}}^a [S, U]_a^b \hat{P}_b, \quad (37)$$

where $[S, U]_a^b = S^c_a U^b_c - U^c_a S^b_c$. This is indeed the same form as the Lax equation with the tensors $\hat{S}$ and $\dot{\hat{U}}$ as the Lax Pair.

So, the Lax equation (35) actually is simply the quantum Hamiltonian equation of motion for the Lax operator $\hat{S}$, although it also implies the classical equations of motion for $\phi^a$. It seems slightly mysterious that the Lax equation on the one hand is interpreted as a quantum mechanical equation (with $\hbar = 1$), as in the inverse scattering method [11], and on the other hand implies the classical equations of motion. However, this is a natural feature of our quantum BRST approach to classical integrable systems.
We construct the conserved charges, using
\[ [\hat{S}^n, \hat{H}_{\text{eff}}] = 0, \quad n = \pm 1, \pm 2, \ldots, \] (38)
which follows from equation (33) and the Jacobi identity (8), where \( \hat{S}^{-1} \) is defined by
\[ \langle \psi_a | \hat{S}^{-1} \hat{S} | \psi_b \rangle = \delta^b_a. \] Taking the quantum trace of equation (38) with respect to \( \langle \psi_a | \) and \( | \psi^a \rangle \), which are defined in equation (A5), we find there are conserved charges
\[ I_n(\phi) = \frac{1}{n} \sum_a \langle \psi_a | \hat{S}^n | \psi^a \rangle \quad n = \pm 1, \pm 2, \ldots \] (39)
\[ \dot{I}_n = 0. \] (40)
Note that the inner product is taken only over ghosts and not over \( \phi^a \), so \( \langle \psi_a | \hat{S} | \psi^b \rangle = S_a^b(\phi) \) for example, and \( \langle \psi_a | \hat{S}^n | \psi^b \rangle = (S^n)_a^b. \) In general not all \( I_n \) will be functionally independent for finite \( N \), since \( \hat{S} \) is a \( 2N \times 2N \) matrix. For example, all \( I_n \) with \( n \geq 2N+1 \) can be expressed as polynomials of \( I_1, \ldots, I_{2N} \). If there are \( N \) \( I_n \)'s which are functionally independent, we have the correct number for integrability. Of course for most interesting examples, the \( a \) in \( \phi^a \) is a continuous parameter, and \( N \) is therefore infinite.

Finally, we require that the charges commute
\[ \{ I_n, I_m \}_0 = \{ I_n, I_m \}_1 = 0, \] (41)
which we have learnt in section III amounts to asking that equation (28) be satisfied, with \( H_n \) replaced by \( I_n \). Using (33) and (A1), and reversing the roles of \( \omega_0 \) and \( \omega_1 \) we rewrite equation (28) as
\[ [\hat{S}, [\hat{Q}, I_n]] - [\hat{Q}, I_{n+1}] = 0. \] (42)
We define the operator
\[ \hat{N} = 2[\hat{Q}, \hat{S}] \hat{S} = \hat{C}^a b \hat{N}_{ab} c(\phi) \hat{P}_c + \ldots \] (43)
where the ellipses refer to terms with different ghost contributions, and
\[ \hat{N}_{ab} c = S_a^d \partial_d S_b^c - S_b^d \partial_d S_a^c - (\partial_a S_b^d) S_d^c + (\partial_b S_a^d) S_d^c \] (44)
is the Nijenhuis tensor. Writing equation (42) in components, we find that
\[ S_a^b \partial_b I_n - \partial_a I_{n+1} = \hat{N}_{ab} c S^{(n-1)}_c b, \] (45)
therefore the condition of Liouville integrability amounts to requiring $N_{ab}^c = 0$.

From the definition of $\hat{N}$, the Jacobi identity, and using that $\hat{S}$ commutes with $\hat{H}_{eff}$, as does $\hat{Q}$ because $\hat{H}_{eff}$ is $\hat{Q}$-exact, we find

$$\dot{\hat{N}} = [\hat{N}, \hat{H}_{eff}] = 0. \quad (46)$$

Therefore, $N_{ab}^c = \langle \psi_{ab}|\hat{N}|\psi_c^c \rangle = 0$, where $|\psi_c^c\rangle$ and $|\psi_{ab}\rangle$ are defined in the appendix, is consistent with time evolution.

Another interesting point is that the ghost number one wavefunction $\psi_1 = C_a f_a(\phi)$ is associated with the Lax equation in the inverse scattering method. This uses quantum mechanical techniques, in particular the (linear) equation of motion for the wavefunction $\psi_1$, given by

$$\dot{\psi}_1 = -\hat{U}\psi_1 = -\hat{C}^a U_a^b f_b, \quad (47)$$

to solve the (non-linear) equations of motion for $\phi^a$. The configuration of $\phi^a$ is encoded in $\psi_1$. Although so far we have considered only the time evolution of operators as in equation, the equations of motion for the classical system can also be described as a Schrödinger equation for the ghost number 0 wavefunction $\psi_0(\phi)$.

Our recipe for constructing a Lax equation is essentially reliant only on the existence of the algebra between $\hat{Q}, \hat{\bar{Q}}, \hat{K}_r, \hat{\bar{K}}_r$ in equation. It is therefore natural to ask how one can deform the phase-space while maintaining this algebra, in order to make new Lax pairs for the same system. The equations of motion for $\phi^a$ are determined only by the first term in $\hat{H}_{eff}$ in equation, because the second term is independent of $\lambda_a$. Since we wish to keep the same equations of motion for $\phi^a$, we can only deform the second term in $\hat{H}_{eff}$.

We choose the following redefinition of the ghosts

$$\hat{C}'^a = \hat{C}^b A(\phi)_b^a, \quad \hat{P}'^a = A^{-1}(\phi)_a^b \hat{P}_b, \quad (48)$$

where $A$ is chosen such that $\det A \neq 0$, and that the BRST anti-BRST algebra remains the same. From definitions and for $\hat{S}$ and $\hat{U}$, we have

$$\hat{S}' = \hat{C}^a (ASA^{-1})_a^b \hat{P}_b, \quad \hat{U}' = \hat{C}^a (AU A^{-1})_a^b \hat{P}_b. \quad (49)$$

Given that the BRST algebra has been maintained, $\hat{S}'$ obeys equation, and $\hat{S}', \hat{U}'$ obey the Lax equation, which implies the equations of motion for $\phi^a$. 


In the case where $\partial_c A_a^b = 0$, it is clear that the quantum brackets between phase-space co-ordinates in equation (6) are unaltered, in particular $[\hat{C}^a, \hat{\lambda}_b] = 0$, therefore the algebra (A1) is also unchanged. For non-constant $A$, the co-ordinate algebra is changed, for example $[\hat{C}^a, \hat{\lambda}_b] \neq 0$. However the algebra (A1) is maintained if $d\omega'_r = 0$, or in terms of $\hat{K}'_r$

$$[\hat{Q}', \hat{K}'_r] = \hat{C}^a \hat{C}^b \hat{C}^c A_a^f \partial_f (A_b^g (\omega'_r) g h A_c^h) = 0. \quad (50)$$

The above equation is satisfied iff

$$A_a^b = \frac{\partial \phi^b}{\partial \phi'^a}, \quad (51)$$

for some smooth invertible function $\phi'^a(\phi)$. In other words, the matrix $A_a^b(\phi)$ must represent a co-ordinate transformation of the symplectic manifold $\mathcal{M}$.

It would be interesting to investigate whether any Lax pair can be written in the form of $\hat{S}'$ and $\hat{U}'$ in equation (49).

V. CONCLUSIONS

We have seen how the quantum BRST description of classical mechanics has advantages over standard Hamiltonian mechanics in the study of integrability. This is particularly apparent with the derivation of the Das-Okubo geometrical Lax equation in section IV. What’s more, the mixture of quantum and classical aspects of the Lax equation arise naturally in this approach. Further investigation seems warranted.

Acknowledgments

We wish to thank Prof. Poul H. Damgaard for useful discussions. M.B.S. thanks the IBCCF/UFRJ, where part of this work was written, for their kind hospitality. M.C. was funded by a Marie Curie training site fellowship.
APPENDIX A: USEFUL IDENTITIES

The BRST anti-BRST symmetry algebra is given by

\[
[\hat{Q}, \hat{Q}] = [\hat{Q}_r, \hat{Q}] = [\hat{Q}_r, \hat{Q}_r] = 0,
\]
\[
[\hat{Q}_g, \hat{K}_r] = \hat{K}_r, \quad [\hat{Q}_g, \hat{K}_r] = -\hat{K}_r, \quad [\hat{K}_r, \hat{K}_r] = \hat{Q}_g,
\]
\[
[\hat{K}_r, \hat{Q}] = 0, \quad [\hat{K}_r, \hat{Q}_r] = \hat{Q},
\]
\[
[\hat{K}_r, \hat{Q}_r] = \hat{Q}_r, \quad [\hat{K}_r, \hat{Q}_r] = 0,
\]
\[
[\hat{Q}_g, \hat{Q}_r] = \hat{Q}, \quad [\hat{Q}_g, \hat{Q}_r] = -\hat{Q}_r.
\]

(A1)

The Marnelius bracket [4] is defined as

\[
\{f, g\}_{\bar{Q}_r} \equiv \frac{1}{2} \left( [[\hat{f}, \hat{Q}_r], [\hat{Q}, \hat{g}]] - [[\hat{f}, \hat{Q}], [\bar{Q}_r, \hat{g}]] \right).
\]

(A2)

Another interesting expression for the Poisson bracket is

\[
\{f, g\}_r = -[X_{f_r}, [\hat{K}_r, X_{g_r}]],
\]

(A3)

which is a bracket between two vector fields \(X_{f_r} = [\hat{f}, \hat{Q}_r]\) and \(X_{g_r} = [\hat{g}, \hat{Q}_r]\).

States of specific ghost number are constructed by applying the \(\hat{C}^a\) or \(\hat{P}_a\) operators to the "ground" states

\[
|\mathcal{P} = 0 \rangle \equiv 1, \quad |\mathcal{C} = 0 \rangle \equiv \prod_a C^a,
\]

(A4)

where the inner product \(\langle C = 0 | \mathcal{P} = 0 \rangle = 1\). For example, we define the states

\[
|\psi^a \rangle \equiv \hat{C}^a |\mathcal{P} = 0 \rangle, \quad |\psi_a \rangle \equiv \hat{P}_a |\mathcal{C} = 0 \rangle, \quad |\psi_{ab} \rangle \equiv \hat{P}_a \hat{P}_b |\mathcal{C} = 0 \rangle.
\]

(A5)

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