On the Holomorphic Gauge Quantization of the Chern-Simons Theory and Laughlin Wave Functions

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Abstract

Chern-Simons-Matter Lagrangian with noncompact gauge symmetry group is considered. The theory is quantized in the holomorphic gauge with a complex gauge fixing condition. The model is discussed, in which the gauge and matter fields are accompanied by the complex conjugate counterparts. It is argued, that such a theory represents an adequate framework for the description of the quantum Hall states.
1 Introduction

One of the characteristic features of two-dimensional systems is that the Green
functions (correlators) can be factorized into the product of holomorphic and antiholo-
morphic parts and corresponding gauge connections can take a complex values.

A known example of a non-real gauge potential is provided by the integrable con-
nection over the configuration space arising from the Yang-Baxter equations. This
connection is represented by the one-form \[ \omega = \text{const} \cdot \sum_{I \neq J} T_I^a \otimes T_J^a \ln(z_I - z_J) \] (1)
where \( z_I = x_I + y_I \) are the complex coordinates and \( T_I^a \) are the generators of sym-
metry group for the \( I^{th} \) particle. This connection governs the monodromy behavior of
conformal blocks in (1+1) dimensional current algebra and enters into the Knizhnik -
Zamolodchikov (KZ) \[ 2 \] equation
\[
\left( \frac{\partial}{\partial z_I} - \frac{1}{k + c} \sum_{I \neq J} \frac{T_I^a \otimes T_J^a}{z_I - z_J} \right) \Psi(z_1, ..., z_N) = 0
\] (2)
\[
\frac{\partial}{\partial \bar{z}_I} \Psi(z_1, ..., z_N) = 0
\]

The KZ connection plays an essential role in the physics of particles obeying the
braid statistics and in the theory of quantum Hall effect (see e.g. \[ 3 \]). In the later case
the holomorphic part of Laughlin wave function satisfies (2) and could be expressed as
\( N \)-point correlation function in certain conformal field theory \[ 4 \].

The gauge potential (1) can be incorporated into the framework of Chern-Simons
(CS) gauge theory in 2+1 dimensions. Formally, the problem reduces to the quan-
tization of the theory describing the matter interacting with the C-S fields in the
holomorphic gauge, where corresponding gauge condition is expressed by the complex
matrix equation
\[
A_x + i A_y = 0,
\] (3)
( \( A_{\mu} \) is the Lie-algebra valued gauge connection)

Remind, that this type of gauge has been presented as a solution of a Gauß law
constraint in a discussions of quantum holonomies \[ 5 \], and BRST quantization of non-
abelian CS gauge theories \[ 6 \].

Note, that holomorphic gauge quantization as considered in e.g. \[ 6 \] leads to non-
Hermitean Hamilton operator and for consistency one has to introduce in the Hilbert
space a compensating integration measure, respectively which Hamiltonian is self-
adjoint \[ 7 \].

It must be emphasized, that as well as the complex gauge condition is imposed in
the CS theory with a compact gauge group and real gauge fields, equation (3) must be
understood in the sense of some analytic continuation.

It is worth pointing out at this point that in the paper \[ 8 \] Witten had considered
the theory with non-compact (complex) gauge transformation group and complex CS
gauge fields.
It was shown in this paper, that quantization of self-interacting CS gauge fields can be performed as precisely as for compact groups, using standard tools and without any specific difficulties (see also [9, 10]).

In the present note we consider the same scheme as in [8], enlarging the system by the matter fields. The point of departure is the observation, that in the holomorphic gauge in order to have real Lagrangian (i.e. unitary theory), the matter fields as well as the gauge degrees of freedom must be accompanied by their complex conjugate counterparts. In the quantization procedure we follow Dirac’s classical method [11].

As the physical application we will try to give some convincing arguments, that the models with a complex gauge groups can provide the consistent description of the variety of QHE wave functions.

Outline and note on the conventions The paper is organized in following way.

- In Section 2 we define the Action and Euler-Lagrange equations for complex non-Abelian CS gauge fields interacting with the nonrelativistic fermions. Imposing the holomorphic gauge we perform the Dirac quantization.
- In Section 3 we introduce the non-unitary similarity transformation and reduce Hamiltonian to (quasi)free form. Diagonalisation is complete in the Abelian case.
- In Section 4 we consider the system of planar electrons in external magnetic field. As output we give the construction of the relevant wave functions for quantum Hall fluid with Abelian as well as non-Abelian CS gauge interactions.
- Section 5 contains the conclusions.

Together with the conventional cartezian coordinates \( r = x^k = (x, y) \) it is convenient to use the complex notations

\[
z = x + iy, \quad \partial = \frac{\partial}{\partial z} = \frac{1}{2}(\partial_x - i\partial_y)
\]

\[
\bar{z} = x - iy, \quad \bar{\partial} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)
\]

for particle coordinates and corresponding Cauchy-Riemann operators. The vector fields \( A(r) = (A_x, A_y) \) will be presented by their holomorphic and antiholomorphic components

\[
A(r) = A_x + iA_y, \quad \bar{A}(r) = A_x - iA_y
\]

The non-Abelian matrix-valued vector potential can be decomposed with respect to a basis of the real Lie algebra of a compact gauge group \( G \):

\[
A_\mu(x) = \sum_a A_\mu^a(x) \cdot t^a, \quad a = 1, 2, \ldots, r = \dim G
\]

The group generators are the anti-Hermitean, traceless matrices \( t^a \) obeying the Lie algebra

\[
t^{a\dagger} = -t^a, \quad [t^a, t^b] = f_{abc}t^c
\]
with $f^{abc}$ the totally antisymmetric, real structure constants. In the case of abelian group (4) is replaced by $A_\mu(x) = iA_\mu(x)$.

We will abbreviate the spatial coordinates of $I^{th}$ particle $r_I$ to $I$, when this will not be ambiguous.

2 Action and Quantization

2.1 Complex Gauge Group and Lagrangian

Let $G$ be a compact $r$-dimensional Lie group. The group elements are parametrized by the set of real parameters $g = g(\omega_1, \omega_2, ..., \omega_r)$. The irreducible unitary representations of $G$ are denoted by $D(\omega_a)$. Matrices $T^a(\omega)$ are the corresponding group generators. They satisfy the commutation relations

$$[T^a(\omega), T^b(\tau)] = \delta_{\tau\tau} f^{abc} T^c(\omega), \quad 1 \leq a, b, c \leq r$$

The matter fields and quantum states in the representation $D(\omega)$ are labeled by the weight vectors $w_\omega \equiv w^m_\omega; \ (m = 1, ..., R = \text{rank } G)$.

Consider the non-compact group $G_c$ (the complex extension of $G$), regarding the group parameters $\omega_a$ as a complex quantities and with a group multiplication law given by the holomorphic function.

Recall some facts about the representations of the complex groups.

- Associated to any irreducible representation $D(\omega)$ of the Lie group $G$ one can define its analytic and antianalytic continuations, $\Delta(\omega_a) = D(\omega_a)$ and $\Delta^*(\omega_a)$ respectively.

- For the any two representations $D(1)$ and $D(2)$ the tensor product $\Delta(1,2) = \Delta(1) \otimes \Delta(2)$ is the irreducible representation of $G_c$.

The other irreducible representations of interest are contragradient $\tilde{\Delta}(1,2)$, and complex conjugate representations $\Delta^*(1,2)$ and $\tilde{\Delta}^*(1,2)$. Introduce the matter fields. It is convenient to define the doublet field

$$\Psi(x) = \begin{pmatrix} \psi(x) \\ \tilde{\psi}^*(x) \end{pmatrix}, \quad (7)$$

transforming under reducible representation $\mathcal{R}(g) = \Delta(1,2) \oplus \tilde{\Delta}(1,2)$. (The complex conjugation * for the fermions is defined as an involution operation for Grassmann variables [12].)

The corresponding contravariantly transforming fields

$$\bar{\Psi}(x) = (\bar{\psi}(x), \psi^*(x))$$

are unified in the representation $\bar{\mathcal{R}}(g) = \tilde{\Delta}(1,2) \oplus \Delta^*(1,2)$. It means that there exists a nondegenerate real bilinear form $<\bar{\Psi}, \Psi>$ invariant under the group transformations.
Gauging the rigid group $G_c$ we consider the group parameters as a complex functions of space-time coordinates. The Lie-algebra valued gauge potential $F_{\mu}(x) \equiv F^A_{\mu}(x) \cdot T^A$ transforms as follows

$$F_{\mu}(x) \rightarrow F'_{\mu}(x) = R(g)F_{\mu}(x)R(g)^{-1} + \partial_{\mu}R(g) \cdot R(g)^{-1} \tag{8}$$

The matrices $T^A (A = 1, \ldots, r, r + 1, \ldots, 2r)$ are the anti-Hermitean Lie-algebra generators in the representation of matter field:

$$T^a = \left( \begin{array}{cc} T^a_{(1)} \otimes I_{(2)} & 0 \\ 0 & I_{(1)} \otimes T^a_{(2)} \end{array} \right), \quad T^{r+a} = \left( \begin{array}{cc} I_{(1)} \otimes T^a_{(2)} & 0 \\ 0 & T^a_{(1)} \otimes I_{(2)} \end{array} \right) \tag{9}$$

These generators are associated to the group parameters $\omega_a$ and $\omega_{r+a} \equiv \omega^*_a$. The defining commutation relations are

$$[T^A, T^B] = f_{ABC}T^C, \quad A, B = 1, \ldots, 2r$$

With the help of the gauge fields $F^a_\mu(x)$, $F^{r+a}_\mu(x) \equiv (F^a_\mu(x))^*$ define the covariant derivatives:

$$D_\mu \Psi(x) = \partial_\mu \Psi(x) - F_\mu(x)\Psi(x) \quad D_\mu \bar{\Psi}(x) = \partial_\mu \bar{\Psi}(x) + \bar{\Psi}(x)F_\mu(x) \tag{10}$$

These ingredients permit to construct the real Lagrangian, invariant under involution and the group of complex gauge transformations $G_c$:

$$\mathcal{L} = \frac{\kappa}{2} \epsilon^{\mu\nu\lambda}[F^A_\mu(x)\partial_\nu F^A_\lambda(x) + \frac{1}{3}f_{ABC}F^A_\mu F^B_\nu F^C_\lambda] +
$$

$$+ i < \bar{\Psi}(x), D_0 \Psi(x) > - \frac{1}{2m} < D_k \bar{\Psi}(x), D_k \Psi(x) > \tag{11}$$

The Euler-Lagrange equations for the matter and gauge fields are given by the set

$$\frac{1}{\kappa}J^A = -2i\bar{\Phi}F_0^A - i\partial_0 F^A - if_{ABC}F^B F^C_0$$

$$\frac{1}{\kappa}\bar{J}^A = -2i\partial F_0^A + i\partial \bar{F}^A + if_{ABC}\bar{F}^B F^C_0$$

$$\frac{i}{\kappa}\rho^A = \bar{\Phi}F^A - \partial F^A - \frac{i}{4}f_{ABC}(F^B \bar{F}^C - \bar{F}^B F^C)$$

$$i\partial_\mu \psi = -\frac{1}{m}(D\bar{D} + \bar{D}D)\psi + iF_0\psi$$

Here

$$D = \frac{1}{2}(D_x - iD_y) = \partial + \frac{1}{2} \bar{\Phi}$$

$$\bar{D} = \frac{1}{2}(D_x + iD_y) = \partial + \frac{1}{2} \Phi$$
are covariant derivative operators.

The gauge invariant currents

\[ J_0^A(x) \equiv \rho^A(x) = i \langle \bar{\Psi}, T^A \Psi \rangle \]

\[ J^A(x) = J_x^A(x) + i J_y^A(x) = \frac{1}{m} [\langle \bar{\Psi}, T^A \bar{D} \Psi \rangle - \langle \bar{D} \bar{\Psi}, T^A \Psi \rangle] \]

\[ \bar{J}^A = J_x^A - i J_y^A = \frac{1}{m} [\langle \bar{\Psi}, T^A D \Psi \rangle - \langle D \bar{\Psi}, T^A \Psi \rangle] \]

are covariantly conserved:

\[ \partial_t \rho^A + \partial J^A + \partial \bar{J}^A - f_{ABC} [F_B^0 \rho^C - \frac{1}{2} (F_B^A \bar{J}^C + \bar{F}^B_J^C)] = 0 \quad (13) \]

Note that the gauge coupling constant is set to be one. Its actual value can be restored rescaling the gauge fields and statistical parameter

\[ A_\mu \rightarrow g A_\mu, \quad \kappa \rightarrow \frac{\kappa}{g^2} \]

### 2.2 Dirac’s Quantization

For doing canonical quantization we will use the Dirac’s method, and try to adapt it for the case of complex gauge group.

To begin with, consider the classical theory and discuss the setup of holomorphic gauge quantization. The canonical Hamiltonian is given by the expression

\[ H_c = \int d^4r \left[ \frac{1}{m} (\langle D\bar{\Psi}(r), \bar{D} \Psi(r) \rangle + \langle \bar{D} \bar{\Psi}(r), D \Psi(r) \rangle) + F_0^A(r) \phi^A(r) \right] \quad (14) \]

The system is constrained by the first class constraints

\[ \Pi_0^A = \frac{\partial L}{\partial \dot{F}_0^A} \approx 0, \]

\[ \phi^A = \rho^A + i \kappa \left[ \partial F^A - \partial \bar{F}^A - \frac{1}{4} f_{ABC} (F_B^C \bar{F}^D - \bar{F}^B F^C) \right] \approx 0 \quad (15) \]

Primary constraints reflect the absence of a momentum conjugate to \( F_0^A \), and secondary ones reproduce the generalized Gauss law. The canonical variables satisfy the Poisson brackets relations

\[ \{ \Psi(r, t), \bar{\Psi}(r', t) \}_{PB} = -i \delta(r - r') \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (16) \]

\[ \{ F^A(r, t), \bar{F}^B(r', t) \}_{PB} = -i \frac{4}{\kappa} \delta_{AB} \delta(r - r') \quad (17) \]

Due to the presence of quadratic terms, the constraint equations (15) are not easy to solve. The obvious way out is to impose the gauge conditions, which linearize them. This possibility is realized in the axial type gauge e.g. \( F_y^A = 0 \).
As an alternative solution one can use the holomorphic gauge, with a gauge fixing conditions

\[ \eta^A = F_0^A = 0, \quad \chi^a = F^a = 0 \quad \chi^{r+a} = \bar{F}^{r+a} = 0 \]  \tag{18}

In conformity with the Dirac’s procedure introduce the total Hamiltonian

\[ H_T = H_c + \int d\mathbf{r} \Lambda^A(\mathbf{r}) \phi^A(\mathbf{r}) \]  \tag{19}

where \( \Lambda^A(\mathbf{r}) \) are Lagrange multipliers. These functions must be subjected to the self-consistency conditions

\[ \partial_t \eta^A(\mathbf{r}) = \{ \eta^A(\mathbf{r}, H_T) \} = 0 \quad \text{and} \quad \partial_t \chi^A(\mathbf{r}) = \{ \chi^A(\mathbf{r}), H_T \} = 0 \]  \tag{20}

and are given by

\[ \Lambda^a(\mathbf{r}) = \frac{i}{2\kappa} \int d\mathbf{r}' G(\mathbf{r} - \mathbf{r}') \cdot J^a(\mathbf{r}') \]  \tag{21}

\[ \Lambda^{r+a}(\mathbf{r}) = -\frac{i}{2\kappa} \int d\mathbf{r}' G(\mathbf{r} - \mathbf{r}') \cdot J^{r+a}(\mathbf{r}') \]  \tag{22}

Here we formally introduced the operator \( \bar{\partial}^{-1} \), which defines the Green function

\[ \bar{\partial}^{-1}J^a(\mathbf{r}) = \int d\mathbf{r}' G(\mathbf{r} - \mathbf{r}') J^a(\mathbf{r}') \]  \tag{23}

The Green function \( \bar{\partial}^{-1} = G(\mathbf{r}) \) can be presented as a derivative of the holomorphic Green function

\[ G(\mathbf{r}) = \partial G(z) = \frac{1}{\pi z} = \frac{1}{\pi} \partial \ln z \]  \tag{24}

We see, that \( (24) \) is ill defined multivalued function. In the non-relativistic case, when the particle density is a sum of \( \delta \)-functions, using appropriate regularization one may ignore this point and consider \( G(z) \) as a normal function, vanishing at the origin \[14, \, 15\].

In the analogous way one can define the antiholomorphic Green functions

\[ \bar{G}(\mathbf{r}) = \bar{\partial} G(z) = \frac{1}{\pi \bar{z}} = \frac{1}{\pi} \bar{\partial} \ln \bar{z} \]  \tag{25}

In the holomorphic gauge the Gauß law constraints \( (15) \) look like

\[ \phi^a = \rho^a + i\kappa \bar{\partial} F^a = 0, \quad \phi^{r+a} = \rho^{r+a} - i\kappa \partial F^{r+a} = 0 \]

and can be easily solved

\[ \bar{F}^a(\mathbf{r}) = \frac{i}{\kappa} \int d\mathbf{r}' G(\mathbf{r} - \mathbf{r}') \cdot \rho^a(\mathbf{r}') \]  \tag{26}
\[ F^{r+a}(r) = -\frac{i}{\kappa} \int dr' \mathcal{G}(r - r') \cdot \rho^{r+a}(r') \] (27)

In the chosen gauge \( J^a = 0, \bar{J}^{r+a} = 0 \). Using the continuity equation (13) one can express the Lagrange multipliers (21) and (22) as a time derivatives:

\[ \Lambda^a(r) = -\frac{i}{2\kappa} \partial_t \int dr' G(r - r') \rho^a(r', t) \] (28)

\[ \Lambda^{r+a}(r) = \frac{i}{2\kappa} \partial_t \int dr' \bar{G}(r - r') \rho^{r+a}(r', t) \] (29)

The last expressions may be unified with (26) and (27) composing a 3-vectors

\[ a^\mu_a(x) \equiv (\Lambda^a, F^a_k) = -\frac{i}{2\kappa} \partial_\mu \int dr' G(r - r') \rho^a(r', t) \] (30)

\[ \bar{a}^{r+a}_\mu(x) \equiv (\Lambda^{r+a}, F^{r+a}_k) = \frac{i}{2\kappa} \partial_\mu \int dr' \bar{G}(r - r') \rho^{r+a}(r', t) \] (31)

which is the solution for the CS gauge fields in the holomorphic gauge.

### 2.3 Quantum Theory

Up to now we have been considering the classical canonical formalism. The corresponding second quantized Hamiltonian operator is given by

\[ H_T = \int dr \left[ \frac{1}{m} \left< D\bar{\Psi}(r), D\Psi(r) \right> + \left< \bar{D}\bar{\Psi}(r), D\Psi(r) \right> \right] + \Lambda^A(r) \cdot \phi^A(r) \]

Dynamical equations are defined by the commutator

\[ i\partial_t \mathcal{O}(r, t) \approx [\mathcal{O}(r, t), H_T] \] (32)

The Heisenberg equation of motion for the matter field is given by

\[ i\partial_t \Psi(r, t) = \mathcal{H}_S \Psi(r, t) \equiv -\frac{1}{m}(DD + \bar{D}\bar{D})\Psi(r, t) + ia_T^A \Phi^A \Psi(r) \] (33)

Here the operator \( \mathcal{H}_S \) contains the solutions (26)-(27) for the statistical gauge fields.

It is not difficult to notice, that the many particle wave function

\[ \Phi(r_1, ..., r_N; t) = \left< 0|\Psi(r_1, t) \cdots \Psi(r_N, t)|\Phi > \right. \]

satisfies the Shrödinger equation

\[ i\partial_t \Phi(r_1, ..., r_N; t) = -\frac{1}{m} \sum_{i=1}^N [\hat{D}_I \hat{D}_I + \hat{D}_I^* \hat{D}_I^*] \Phi(r_1, ..., r_N; t) \] (34)

with the derivative operators given by

\[ \hat{D}_I = \partial_I - \frac{1}{2\pi\kappa} \sum_{l \neq j} \frac{T^l_l \otimes T^j_j}{z_l - z_j} \quad \hat{D}_I = \bar{\partial}_I \]
\[ D^*_I = \partial_I \quad \bar{D}^*_I = \partial_I + \frac{1}{2\pi\kappa} \sum_{I \neq J} \frac{T^{r+a}_I \otimes T^r_J}{\bar{z}_I - z_J} \]

( the matrices \( T^A_I \) act on the group variables of \( I^{th} \) particle)

As a first application of the framework described above consider the case in which fermions are in the fundamental ("chiral") representation \( T^a_{(1)} = t^a \) and \( T^a_{(2)} = 0 \). The Hamiltonian takes the form

\[
H = -\frac{1}{m} \sum_I \left[ D_I D_I \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \bar{D}^*_I \bar{D}^*_I \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right]
\]

(35)

where the covariant derivative operators are given by the KZ connections

\[
D_I = \partial_I - \frac{1}{2\pi\kappa} \sum_{I \neq J} \frac{t^a_I \otimes t^a_J}{z_I - z_J} \quad \bar{D}_I = \bar{\partial}_I
\]

\[
D^*_I = \partial_I \quad \bar{D}^*_I = \bar{\partial}_I + \frac{1}{2\pi\kappa} \sum_{I \neq J} \frac{t^a_I \otimes t^a_J}{\bar{z}_I - \bar{z}_J}
\]

Another case of interest is a "symmetric" representation \( T^a_{(1)} = T^a_{(2)} = t^a \). Now

\[
T^a = T^{r+a} = \begin{pmatrix} t^a & 0 \\ 0 & t^a \end{pmatrix}
\]

(36)

The corresponding Hamiltonian is given by

\[
H = -\frac{1}{m} \sum_I \left[ D_I D_I \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \bar{D}^*_I \bar{D}^*_I \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \right]
\]

(37)

Remark, that in the case of "chiral" representation Hamiltonian is not Hermitean \([7, 6]\) and conjugation causes the interchange between upper and down components of matter field doublet \([7]\). At the same time in the "symmetric" representation \( H^\dagger = H \).

3 Hamiltonian Diagonalization

3.1 Similarity Transformation

The matter Hamiltonian

\[
H_{\text{matter}} = \int dr' \frac{1}{2m} < D_k \bar{\Psi}(r), D_k \Psi(r) >
\]

contains the gauge connection in the form of a gradient

\[
\mathcal{F}_k(r) = \frac{i}{2\kappa} \partial_k \left[ \int dr' G(r - r') \rho^a(r') \cdot T^a - \int dr' \bar{G}(r - r') \rho^{r+a}(r') \cdot T^{r+a} \right]
\]

The situation is simplified when \( G = U(1) \): the generators commute and CS fields can be eliminated by the means of suitably chosen complex gauge transformation, reducing Hamiltonian to the free (diagonal) form.
Formally the gauge fields can be removed by going to the new field variables

\[ X(x) = \left( \begin{array}{c} \chi(x) \\
\tilde{\chi}(x) \end{array} \right) \quad \text{and} \quad \tilde{X}(x) = (\tilde{\chi}(x), \chi^*(x)) \]

defined by

\[ \Psi(r) = U(r; \gamma)X(r), \quad \tilde{\Psi}(r) = \tilde{X}(r)U^{-1}(r; \gamma) \]

where \( U(r; \gamma) \) is a holonomy operator (or monodromy matrix) associated to an oriented open path \( \gamma \) in \( \mathbb{R}^2 \) connecting the points \( r_0 \) and \( r \):

\[ U(r; \gamma) = P \exp \left( \int_\gamma dx^k F_k(x) \right) \]

(In (40) \( P \) is path ordering operation and \( r_0 \) is some fixed point).

Remark that, due to the non-commutativity of density operators

\[ [\rho^A(r), \rho^B(r')] = f_{ABC}(r)\delta(r - r') \]

path-ordering is non-trivial operation. Below we describe much simpler procedure which in principle permits to get some information on the non-Abelian wave functions.

Introduce operator

\[ \Omega(r) = -i \frac{2\kappa}{2\kappa} \int dr' G(r - r') \rho^m(r') \cdot H^m + i \frac{2\kappa}{2\kappa} \int dr' G(r - r') \rho^{r^m}(r') \cdot H^{r^m} \]

where

\[ \rho^M(r) = i\tilde{\Psi}(r)H^M\Psi(r) \]

are mutually commuting charge densities. The matrices

\[ H^m = \begin{pmatrix} H^m_1 & 0 \\ 0 & H^m_2 \end{pmatrix}, \quad H^{r^m} = \begin{pmatrix} H^{r^m}_1 & 0 \\ 0 & H^{r^m}_2 \end{pmatrix}, \quad m = 1, \ldots, R \]

are Cartan generators in the representation \( \mathcal{R}(g) \). Consider transformations

\[ \Psi(r) = e^{\Omega(r)}X(r), \quad \tilde{\Psi}(r) = \tilde{X}(r)e^{-\Omega(r)} \]

The action of the diagonal Cartan generators on the \( \Psi \)-fields

\[ H^M\Psi_w = \gamma^M_w \cdot \Psi_w \]

defines the \( 2R \) dimensional weight vector \( \gamma^M_w \) \((M = 1, \ldots, R, r + 1, \ldots, r + R)\)

\[ \gamma^m_w = \begin{pmatrix} w^m_1 & 0 \\ 0 & w^m_2 \end{pmatrix}, \quad \gamma^{r^m}_w = \begin{pmatrix} w^{r^m}_2 & 0 \\ 0 & w^{r^m}_1 \end{pmatrix} \]

(Remind, that \( w_\sigma \)'s are the weight vectors of the representation \( D(\sigma) \)).

The transformation (41) can be written in the component form

\[ \Psi_w(r) = \sum_{w'} \left( e^{\Omega(r)} \right)_{w,w'} X_w'(r) = e^{\Omega_w(r)} X_w(r) \]
where the operators
\[ \Omega_w(r) = -\frac{i}{2\kappa} \int dr' G(r - r') \rho^m(r') \cdot \Upsilon^m_w + \frac{i}{2\kappa} \int dr' \bar{G}(r - r') \rho^{r+m}(r') \cdot \Upsilon^{r+m}_w \]
are labeled by the corresponding weight vectors.

In order to find the (anti)commutation rules obeyed by matter fields we use the relations
\[ [\Omega_w(1), \Psi_{w'}(2)] = -\frac{1}{2\kappa} G(1 - 2) < w, w' > \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Psi_w(2) \] (45)

Suppose that the matter field \( X(x) \) as well as \( \Psi(x) \) corresponds to the fermions. Straightforward calculations show, that the requirement of the fermionic commutation relations together with (44) and (45) leads to the conditions imposed on the weight vectors of the representations \( D(1) \) and \( D(2) \):
\[ e^{\frac{j}{2\kappa} < w_1, w'_1 >} = e^{\frac{j}{2\kappa} < w_2, w'_2 >} = 1 \] (46)

Remark, that in certain circumstances it is more suitable to consider \( X(x) \) as bosonic fields (e.g. in the Ginzburg-Landau description of QHE). In these cases instead of (46) we have to choose the weight lattice which satisfies the condition
\[ e^{\frac{j}{2\kappa} < w_1, w'_1 >} = e^{\frac{j}{2\kappa} < w_2, w'_2 >} = -1 \] (47)

### 3.2 Wave Functions

Let \( |\Phi> \) be the eigenstate of the Hamiltonian \( H_{\text{matter}} \). The corresponding \( N \)-particle wave function is given by the matrix elements \( <0|\Psi_{w(1)}(r_1) \cdots \Psi_{w(N)}(r_N)|\Phi> \). After elementary manipulations one gets
\[ <0|\Psi_{w(1)}(r_1) \cdots \Psi_{w(N)}(r_N)|\Phi> = \prod_{I<K} \mathcal{U}(z_I, w(I); z_J, w(J)) <0|e^{\sum \Omega_w(r_I)} X_{w(1)}(r_1) \cdots X_{w(N)}(r_N)|\Phi> \]

Here
\[ \mathcal{U}(z_I, w(I); z_J, w(J)) = G(z_I - z_K) \begin{pmatrix} w^m_1(I) \cdot w^m_1(K) & 0 \\ 0 & w^m_2(I) \cdot w^m_2(K) \end{pmatrix} - \tilde{G}(z_I - z_K) \begin{pmatrix} w^m_2(I) \cdot w^m_2(K) & 0 \\ 0 & w^m_1(I) \cdot w^m_1(K) \end{pmatrix} \]

Following (46) the weight vectors must belong to the lattice defined by the equations
\[ \frac{1}{2\kappa \pi} < w_1(I), w_1(K) >= \pm 2p_{IK} \]
\[ \frac{1}{2\kappa \pi} < w_2(I), w_2(K) >= \pm 2q_{IK} \]
with \( p_{IK} \) and \( q_{IK} \) integers. (In the bosonized theory r.h.s. of these relations are changed by \( \pm 1 \) giving the odd numbers)

As we see, the wave function is factorized into the "kinematical" prefactor and some dynamical part. The typical term in the prefactor is of the form

\[
\left( (z_I - z_K)^\pm 2p_{IK} (\bar{z}_I - \bar{z}_K)^\mp 2q_{IK} \right)
\]

(48)

and depends on the representations and quantum numbers carried by particles under consideration.

As a practical application of the proposed scheme one can indicate the theory of quantum Hall effect (see e.g. [3]), where the expressions like (48) are used as a building blocks for the many-particle wave functions.

This and other developments will be considered in the subsequent section.

4 The Quantum Hall Effect and Laughlin States

4.1 Abelian Theory

In the Abelian case which is obtained by the formal substitutions \( H^A \rightarrow i, \rho^a \rightarrow -\rho = -\bar{\psi}\bar{\psi}, \alpha_A \rightarrow i \), the gauge fields can be removed by the non-unitary similarity transformations [14], [17] (see Appendix A):

\[
\psi(r) = S\chi(r)S^{-1} = e^{-\frac{1}{2\kappa} \int \mathrm{d}r' G(z-z')\tilde{\chi}(r')\chi(r')}
\]

(49)

\[
\tilde{\psi}(r) = S\tilde{\chi}(r)S^{-1} = \tilde{\chi}(r)e^{\frac{1}{2\kappa} \int \mathrm{d}r' G(z-z')\bar{\chi}(r')\chi(r')}
\]

(50)

(Here and hereafter we abandon the doublet notations). The \( \chi \) fields will satisfy the Fermi-Dirac commutation relations, i.e.

\[
\frac{1}{2\kappa} = \pm 2\pi p
\]

(51)

with positive integer \( p \).

Remark, that in the Abelian case the holomorphic \((A = 0)\) and axial \((A_y = 0)\) gauges are related by the complex gauge transformation:

\[
A_{axial}^k = A_{hol}^k - \frac{i}{\kappa} \partial_k \lambda = \frac{-1}{2\kappa} \delta_{k1} \int \mathrm{d}r' \delta(x-x')\epsilon(y-y')\varrho(r')
\]

(52)

where

\[
\lambda(r) = \frac{1}{2\pi} \int \mathrm{d}r' \ln |z - z'| \varrho(r')
\]

\[
+ \frac{i}{2\pi} \int \mathrm{d}r' \tan^{-1} \frac{|y - y'|}{x - x'} \epsilon(x - x')\epsilon(y - y') \varrho(r')
\]

The Hamiltonian

\[
H_{\text{matter}} = \int \mathrm{d}r' \frac{1}{2m} [D_k \bar{\psi} D_k \psi + D_k^* \bar{\psi}^* D_k^* \psi^*]
\]
can be presented in a free form:

\[
H_{\text{matter}} = \int dr' \frac{1}{2m} \left[ \partial_k \tilde{\chi} \partial_k \chi + \partial_k \chi^* \partial_k \tilde{\chi}^* \right]
\] (53)

Remind, that in the theory there are two pairs of canonically conjugate variables: \((\chi, \tilde{\chi})\) and \((\chi^*, \tilde{\chi}^*)\). It can be shown (see Appendix B), that in the Abelian case tilde operation can be identified with the Hermitean conjugation \(i.e.\)

\[
\tilde{\chi} = \chi^\dagger, \quad \tilde{\chi}^* = \chi^{*\dagger}
\] (54)

(This fact will be useful in order to study the completeness relations in the corresponding Hilbert space).

For the further needs it is convenient to use charge conjugate field \(\chi_c \equiv \chi^*\). In terms of newly introduced fields Hamiltonian is expressed as follows

\[
H_{\text{matter}} = \int dr' \frac{1}{2m} \partial_k \chi^\dagger(r') \partial_k \chi(r') - \int dr' \frac{1}{2m} \partial_k \chi_c^\dagger(r') \partial_k \chi_c(r') + \Delta_1
\] (55)
i.e. it corresponds to the two types of free fermionis. \(\Delta_1\) is a reordering constant, which will be specified below. The basic anticommutators are given by the relations

\[
\{ \chi(r), \chi^\dagger(r') \} = \delta(r - r'), \quad \{ \chi_c(r), \chi_c^\dagger(r') \} = \delta(r - r')
\]

### 4.2 System in The External Magnetic Field

The quantum Hall effect is a condensed matter phenomenon, taking place at low temperatures when the planar system is exposed to the strong perpendicular magnetic field \(B = \epsilon_{ik} \partial_i A_k\). Below we consider the standard case of external homogeneous magnetic field generated by the symmetric gauge potential \(A_x = -\frac{1}{2} By, A_y = \frac{1}{2} Bx\). The corresponding Hamiltonian will be now

\[
H_{\text{matter}} = \int dr' \frac{1}{2m} \left\{ \nabla_k \chi^\dagger(r') \nabla_k \chi(r') - \nabla_k \chi_c^\dagger(r') \nabla_k \chi_c(r') \right\} + \Delta_1
\] (56)

where the covariant derivatives are defined by

\[
\nabla_k \chi = (\partial_k - ieA_k) \chi \quad \nabla_k \chi_c = (\partial_k + ieA_k) \chi_c
\]

For simplicity assume that system is spin polarized and treat electrons as a scalar fermions. The fermion fields can be decomposed into the normal modes

\[
\chi(r, t) = \sum_{n=0}^{\infty} \sum_{j=0}^{N_B-1} F_{nj} U_{nj}(r) e^{-iE_n t}
\] (57)

\[
\chi_c(r, t) = \sum_{n=0}^{\infty} \sum_{j=0}^{N_B-1} F^c_{nj} \bar{U}_{nj}(r) e^{iE_n t}
\] (58)

where \(U_{nj}(r)\) are the solutions of one-particle Schrödinger equation

\[
-\frac{1}{2m} \nabla^2_k U_{nj}(r) = E_n U_{nj}(r)
\]
and $E_n = \frac{|eB|}{m} (n + \frac{1}{2})$ are the energy eigenvalues. Quantity

$$N_B = \frac{|eB|}{2\pi} \cdot \text{(Area)}$$

is a number of quantum states per Landau level. $F_{nj}$ and $F_{nj}^\dagger$ are the Fock space lowering and rizing Fermi operators and satisfy usual relations

$$\{F_{nj}, F_{ml}^\dagger\} = \delta_{nm}\delta_{jl}$$

The same is valid for the charge conjugate operators $F_{nj}^c$ and $F_{ml}^c$. The Hamilton and angular momentum operators are given by

$$H_{\text{matter}} = \sum_{nj} E_n F_{nj}^+ F_{nj} - \sum_{nj} E_n F_{nj}^c F_{nj}^c + \Delta_1$$

$$J = \sum_{nj} [j F_{nj}^+ F_{nj} - j F_{nj}^c F_{nj}^c] + \Delta_2$$

In (59)-(60) we abbreviate

$$\sum_{nj} \equiv \sum_{n=0}^{\infty} \sum_{j=0}^{N_B-1}$$

The reordering constants

$$\Delta_1 = \sum_{n=0}^{\infty} \sum_{j=0}^{N_B-1} E_n = N_B \sum_{n=0}^{\infty} E_n$$

$$\Delta_2 = \sum_{n=0}^{\infty} \sum_{j=0}^{N_B-1} j = \frac{N_B(N_B-1)}{2} \sum_{n=0}^{\infty} 1$$

are the energy and angular momentum of state with totally occupied one-particle excitations.

The eigenstates of the Hamiltonian $H_m$ are represented by the direct products

$$|N > \otimes |M_c > \sim F_{n_1j_1}^+ \cdots F_{n_Nj_N}^+ |0 > \otimes F_{m_{1l_1}}^c \cdots F_{m_{Ml_M}}^c |0_c >$$

where the vacuum states are annihiated by the lowering operators

$$F_{nj} |0 > = F_{nj}^c |0_c > = 0, \quad n, m = 0, 1, \ldots; \quad j, l = 0, 1, \ldots, N_B - 1$$

The state vector (63) corresponds to the energy eigenvalue

$$\frac{|eB|}{m} \left[ \sum_{i=1}^{N} (n_i + \frac{1}{2}) - \sum_{k=1}^{M} (m_k + \frac{1}{2}) \right] + \Delta_1$$

The angular momentum of this state is

$$J = \sum_{i=1}^{N} j_i - \sum_{k=1}^{M} l_k + \Delta_2$$

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Present the matter Hamiltonian as the sum
\[ H_{\text{matter}} = H + H_c \] (64)
where
\[ H = \sum_{nj} E_n F_{nj}^+ F_{nj}, \quad H_c = -\sum_{nj} E_n F_{nj}^c F_{nj}^c + \Delta_1 \] (65)

The first term here corresponds to the particle degrees of freedom and the second one to the holes in the charge conjugate sector. In order to justify this assertion define the states
\[ |\Omega\rangle = \prod_{n=0}^{\infty} \prod_{j=0}^{N_B-1} F_{nj}^+ |0\rangle \] (66)
and
\[ |\Omega_c\rangle = \prod_{m=0}^{\infty} \prod_{l=0}^{N_B-1} F_{ml}^c |0_c\rangle \] (67)
Together with the vacua \(|0\rangle\) and \(|0_c\rangle\) they satisfy the following relations
\[ H|0\rangle = H_c|\Omega_c\rangle = 0, \quad H|\Omega\rangle = \Delta_1|\Omega\rangle, \quad H_c|0_c\rangle = \Delta_1|0_c\rangle \]
The elementary charged excitations with the energy \(E_n\) and angular momentum \(j\) can be identified with the states
\[ F_{nj}^+ |0\rangle \quad \text{or} \quad F_{nj}^c |\Omega_c\rangle \]
In the same time the states
\[ F_{nj} |\Omega\rangle \quad \text{or} \quad F_{nj}^c |0_c\rangle \]
can be interpreted as a opposite charge hole excitations with energy \(-E_n\) and angular momentum \(-j\).

The corresponding wavefunctions are determined by the matrix elements
\[ <0|\chi(r,t)F_{nj}^+|0> = <\Omega_c|\chi_c^\dagger(r,t)F_{nj}^c|0> = U_{nj}(r)e^{-iE_nt} \]
\[ <0_c|\chi_c(r,t)F_{nj}^+|0_c> = <\Omega|\chi_c^\dagger(r,t)F_{nj}|\Omega> = U_{nj}(r)e^{iE_nt} \]
As a basic sets in the Hilbert space one can use the coordinate representation vectors, which satisfy the completness relations
\[ |0><0| + \sum_{N\geq1} \frac{1}{N!} \int [\prod_{1\leq I \leq N} dr_I] \chi^\dagger(1) \cdots \chi^\dagger(N)|0><0|\chi(1) \cdots \chi(N) = 1 \] (68)
\[ |\Omega><\Omega| + \sum_{N\geq1} \frac{1}{N!} \int [\prod_{1\leq I \leq N} dr_I] \chi^\dagger(1) \cdots \chi^\dagger(N)|\Omega><\Omega|\chi^\dagger(1) \cdots \chi^\dagger(1) = 1 \] (69)
The similar relations are hold in the conjugate sector. Notice, that validity of these completness relations is guaranteed by the eq.(54).
So the multi-particle states can be presented by the wavefunctions
\[ <0|\chi(1) \cdots \chi(N_e)|\Phi_e> \quad \text{or} \quad <\Omega|\chi_c^\dagger(1) \cdots \chi_c^\dagger(N_h)|\Phi_e> \] (70)
and multi-hole state by the wavefunctions
\[ <0_c|\chi_c(1) \cdots \chi_c(N_h)|\Phi_h> \quad \text{or} \quad <\Omega_c|\chi_c^\dagger(1) \cdots \chi_c^\dagger(N_e)|\Phi_h> \] (71)
4.3 Laughlin Wave Functions

In the theory of QHE the distinguished role is played by the states, where all the particles are in the lowest Landau level (LLL). For the LLL \((n = 0)\) operators and wave functions we’ll use simplified notations

\[ F_{0j} \equiv f_j, \quad U_{0j}(r) \equiv u_j(r) \]

Decompose

\[ \chi(r) = \chi_0(r) + \chi'(r) \]

where

\[ \chi_0(r) = \sum_{j=0}^{N_B-1} f_j u_j(r) \]

is the lowest level field operator and LLL states are built up by the application of lowering and rizing operators satisfying the oscillator algebra:

\[ \{ f_j, f_l^\dagger \} = \delta_{jl}. \]

 Totally filled LLL state is presented by the vector

\[ |\omega> = \prod_{0 \leq j \leq N_B-1} f_j^\dagger |0> \]

The analogous state in conjugate sector will be given by

\[ |\omega_c> = \prod_{0 \leq j \leq N_B-1} f_j^c |0_c> \]

Instead of identity resolution (69) for the LLL states one can use the LLL projection operator

\[ \Pi = |\omega> \langle \omega, | + \sum_{1 \leq N \leq N_B} \frac{1}{N!} \int [ \prod_{1 \leq i \leq N} dr_i ] \chi(1) \cdots \chi(N) |\omega> \langle \omega | \chi^\dagger(1) \cdots \chi^\dagger(N) \]

and its conjugate partner

\[ \Pi_c = |\omega_c, N_B, N_B> \langle \omega_c, N_B, | + \sum_{1 \leq N \leq N_B} \frac{1}{N!} \int [ \prod_{1 \leq i \leq N} dr_i ] \chi_c(1) \cdots \chi_c(N) |\omega_c> \langle \omega_c | \chi^\dagger_c(1) \cdots \chi^\dagger_c(N) \]

The eigenstates of Hamiltonian (59) are expressed in terms of \(\chi\) quanta excitations. At the same time the physical observables and wave functions must be expressed in terms of the fields \(\psi\). As we have already noted, these operators are related by the similarity transformations (49)-(50). In terms of \(\psi\) fields the completeness relations and projection operators are given by

\[ S|0> \langle 0| S^{-1} + \sum_{N \geq 1} \frac{1}{N!} \int [ \prod_{1 \leq i \leq N} dr_i ] \tilde{\psi}(1) \cdots \tilde{\psi}(N) S|0> \langle 0| S^{-1} \psi(N) \cdots \psi(1) = 1 \]
\[ S|0_c > < 0_c |S^{-1} + \]
\[ \sum_{N \geq 1} \frac{1}{N!} \int \prod_{1 \leq i \leq N} d\mathbf{r}_i \psi^*(1) \cdots \psi^*(N) S|0_c > < 0_c |S^{-1} \psi^*(N) \cdots \psi^*(1) = 1_c \]  

\[ \Pi = S|\omega > < \omega |S^{-1} + \]
\[ \sum_{1 \leq N \leq N_B} \frac{1}{N!} \int \prod_{1 \leq i \leq N} d\mathbf{r}_i \psi(1) \cdots \psi(N) S|\omega > < \omega |S^{-1} \psi(N) \cdots \psi(1) \]  

\[ \Pi_c = S|\omega_c > < \omega_c |S^{-1} + \]
\[ \sum_{1 \leq N \leq N_B} \frac{1}{N!} \int \prod_{1 \leq i \leq N} d\mathbf{r}_i \psi^*(1) \cdots \psi^*(N) S|\omega_c > < \omega_c |S^{-1} \psi^*(N) \cdots \psi^*(1) \]  

Equations (74–77) together with the properties of similarity transformation can be used in order to make a reasonable choice of the Hilbert space basis. Below we list these sets indicating corresponding coordinate representation bra-vectors.

1. Vacua are invariant under similarity transformation
   \[ S|0 > = |0 >, < 0 |S^{-1} = < 0 | \]
   \[ \rightarrow < 0 |\psi(1) \cdots \psi(N) \]  

2. \( S \) operator does not cause the Landau level mixing
   \[ S\Pi S^{-1} = \Pi \]
   \[ < \omega |S^{-1} \psi(1) \cdots \psi(N) \]  

The LLL projected Hamiltonian is
\[ H_0 = \mathcal{H} + \mathcal{H}_c \]  

where
\[ \mathcal{H} = E_0 \sum_{j=0}^{N_B-1} f_j^+ f_j \]

and
\[ \mathcal{H}_c = -E_0 \sum_{j=0}^{N_B-1} f_j^c f_j^c + N_B E_0 \]

The corresponding angular momentum operator is given by
\[ J_0 = \sum_{j=0}^{N_B-1} (j f_j^+ f_j - f_j^c f_j^c) + \frac{N_B(N_B-1)}{2} \]
Consider the state

$$|\Phi; N_e > = \prod_{j=0}^{N_e-1} f_j^+ |0 >$$

This state corresponds to the system of $N_e$ electrons in LLL with the energy $N_e E_0$ and minimal total angular momentum $J = \frac{1}{2} N_e (N_e - 1)$.

The supplementary state in the conjugate sector

$$|\Phi_c; N_h > = \prod_{j=0}^{N_h-1} f_{j}^{c+} |0_c >$$

describes the system of $N_h = N_B - N_e$ holes with the same total energy and angular momentum

$$J_c = \frac{1}{2} N_B (N_B - 1) - \frac{1}{2} N_h (N_h - 1)$$

Consequently

$$|L; N_e > = |\Phi; N_e > \otimes |\omega_c >$$

and

$$|G; N_e > = |0 > \otimes |\Phi_c; N_h >$$

are degenerate eigenstates of $H_0$ with energy $N_e E_0$.

Now it is easy to show, that (83) describes the Laughlin state [18] with filling fraction

$$\nu = \frac{1}{2p + 1}$$

The corresponding wave function is obtained applying the projection operator $\Pi$:

$$1 \otimes \Pi|L; N_e > \rightarrow \Psi_e^{(1, \ldots, N_e)} = <0 | \psi(1) \cdots \psi(N_e) | \Phi, N_e > = \prod_{K<L} (z_K - z_L)^{2p} \langle 0 | \chi(1) \cdots \chi(N_e) | \Phi; N_e >$$

The last factor

$$\langle 0 | \chi(1) \cdots \chi(N_e) f_0^+ \cdots f_{N_e-1}^+ |0 > = \prod_{1 \leq K < L \leq N_e} (z_K - z_L) e^{-\frac{2p}{4} \sum_{j=1}^{N_e} |z_j|^2}$$

is the Slater determinant of one-particle LLL states.

Another state of interest is the Girvin state (86). The corresponding wave function is extracted acting by the projection operator $\Pi_c$:

$$1 \otimes \Pi_c |G; N_e > \rightarrow \Psi_{\psi^c}(1, \ldots, N_e) = <\omega_c | S^{-1} \tilde{\psi}^*(1) \cdots \tilde{\psi}^*(N_e) | \Phi_c >$$

$$\int \cdots \int \prod_{K=N_e}^{N_B} [d \mathbf{r}_K] <\omega_c | S^{-1} \tilde{\psi}^*(1) \cdots \tilde{\psi}^*(N_B) | 0_c > \times$$

$$\langle 0_c | \tilde{\psi}^*(N_B) \cdots \tilde{\psi}^*(N_e + 1) | \Phi_c > =$$
\[
\int \cdots \int \prod_{K=1}^{N_e} [d\mathbf{r}_K] \Psi_0(1, \cdots, N_B) \times \Psi_\nu(1, \cdots, N_e)
\]

where the relation \(< \omega_c|S^{-1} = < \omega_c|\) is assumed to be valid. This wave function describes the state with filling fraction \(\nu_c = 1 - \nu = 2p/2p + 1\) [19] (\(\Psi_0\) corresponds to the totally filled lowest level).

Another representation of the same state will be given by the matrix element

\[
\langle \omega|S^{-1}\tilde{\psi}(N_e + N_h) \cdots \tilde{\psi}(N_e + 1)|\Phi \rangle = \\
\int \cdots \int \prod_{K=1}^{N_h} [d\mathbf{r}_K] \langle \omega|S^{-1}\tilde{\psi}(N_e + N_h) \cdots \tilde{\psi}(N_e + 1)\tilde{\psi}(N_e) \cdots \tilde{\psi}(1)|0 \rangle \\
\langle 0|\psi(1) \cdots \psi(N_e)|\omega \rangle = \\
\int \cdots \int \prod_{K=1}^{N_e} [d\mathbf{r}_K] \Psi_0(1, \cdots, N_B) \times \Psi_\nu(1, \cdots, N_e)
\]

(89)

The holomorphic factor \(\prod(z_I - z_J)^{2p}\) is usually associated to the \(2p\) magnetic flux quanta attached to electrons forming what is called Jain’s composite particles [20]. In the present discussion it is a matrix element of complex gauge transformation relating two different, non-unitary equivalent basis.

In the same fashion one can consider the wave functions for the noncompressible states of fractionally charged quasiparticles.

### 4.4 Non-Abelian Wave Functions

Although in the non-Abelian case we do not know the exact wave function, one can nevertheless get some sort of kinematical information, contained in the form of similarity transformations [44]. In order to find the \(N\)-particle wave function we need some basis vectors, e.g.

\[
\langle 0|\psi_w(1) \cdots \psi_w(N) |N \rangle = \\
\langle 0| \exp \left[ \sum \Omega_w(I) \right] |N \rangle = \\
\langle 0| \chi_w(1) \cdots \chi_w(N) |N \rangle = \\
\prod_{I<K} (z_I - z_K)^{-1/2\pi \kappa} |0| \langle w(I), w(K) \rangle < 0| \chi_w(1) \cdots \chi_w(N) |N \rangle
\]

(deriving (90) we’ve used the fact, that vacuum is annihilated by operators \(\Omega_w(I)\)).

Apply this formula to the case of \(SU(2)\) non-Abelian theory. For the \(\psi\)’s in the fundamental representation the weight vectors

\[
w^\alpha = \pm i
\]

correspond to the isospin up \(\uparrow\) an down \(\downarrow\) components. The corresponding basis vector is given by the expression

\[
\prod (z_{I\uparrow} - z_{K\uparrow})^{2p} \prod (z_{R\downarrow} - z_{S\downarrow})^{2p} \prod (z_{I\uparrow} - z_{R\downarrow})^{-2p} < 0| \chi_w(1) \cdots \chi_w(N) |N \rangle
\]

(91)
So we see that the wave function of any Hamiltonian eigenstate in this basis contains an holomorphic prefactor indicating the attraction between different isospins and repulsion between the same ones. It was conjectured that this type of wave functions may be related to the multilayered QHE states \[21, 22\].

5 Conclusions

In summary, we have seen that Chern-Simons theory with complex gauge group and doubled number of matter as well as gauge degrees of freedom have natural application to the description of quantum Hall effect. The additional matter fields, introduced in order to provide unitarity requirement are associated to quantum states with the observed values of filling fractions. The scheme incorporates also the Jains picture \[20\] of composite fermions.

In the case of non-Abelian interactions one can extract from the complete wave function a certain "kinematical" prefactor, having specific holomorphic form which is determined be the weight lattice and possibly is related to the multilayered planar system. The bosonized version of similarity transformation can be used to construct the order parameter and to develop the GL description of quantum fluids.

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7 Appendix

A Similarity Transformation

The generator of similarity transformation must satisfy the commutation relation

\[ [G_p, \chi(r)] = 2p \pi \int d'r' G(z - z') \varphi(r') \cdot \chi(r) \]

Present \( G_p \) as a bilinear functional of a density operator:

\[ G_p = \int dr \int dr'' \varphi(r') \Lambda_p(r', r'') \varphi(r'') + \int dr \varphi_L(r) \varphi(r) \]

Using the commutator

\[ [\varphi(r'), \chi(r)] = -\delta(r - r') \chi(r), \]

we get

\[ \int dr' [\Lambda_p(r, r') + \Lambda_p(r', r)] \varphi(r') = -2p \pi \int dr' G(z - z') \varphi(r'), \]

and

\[ \varphi_L(r) = -\Lambda_p(r, r). \]
Consequently
\[ \Lambda_p(r, r') = -p\pi G(z - z') + \frac{i\pi}{2} p, \]
and
\[ L_p(r) = -\lim_{r' \to r} \Lambda_p(r, r') = i\frac{\pi}{2} p. \]

In the last expression we have used the regularized Green function, satisfying condition \( G(0) = 0 \). As an example of such a regularization one can try to use a function
\[ G(z) = \lim_{\epsilon \to 0} G_\epsilon(z), \quad G_\epsilon = \frac{1}{\pi} \ln z \cdot e^{-\epsilon/|z|^2}. \]

**B Toy Model**

Consider a toy model describing the couple of fermion oscillators. The basic anticommutators are
\[ \{f, f^\dagger\} = 1, \quad \{f_c, f_c^\dagger\} = 1 \]
The Hamiltonian is
\[ \hat{H} = f^\dagger f - f_c^\dagger f_c \]
The explicit realization of basic operators and Hilbert space can be given in terms of Grassmann variables \( \xi \) and \( \xi^* \):
\[ f = \frac{\partial}{\partial \xi}, \quad f^\dagger = \xi \quad f_c = \frac{\partial}{\partial \xi^*}, \quad f_c^\dagger = \xi^* \]
Introduce the dual vector
\[ \Psi^\# = -\bar{\psi}_{00}\xi^* + \bar{\psi}_{01}\xi^* - \bar{\psi}_{10}\xi - \bar{\psi}_{11} \]
The scalar product is defined by Berezin the integral over the Grassmann numbers
\[ (\Phi, \Psi) = \int d\xi d\xi^* \Phi^\# \cdot \Psi = \bar{\phi}_{00}\psi_{00} + \bar{\phi}_{10}\psi_{10} + \bar{\phi}_{01}\psi_{01} + \bar{\phi}_{11}\psi_{11} \]
We see, that the pairs of Hermitian conjugate operators are given by
\[ (\xi, \xi^\dagger = \frac{\partial}{\partial \xi}) \quad \text{and} \quad (\xi^*, \xi^{*\dagger} = \frac{\partial}{\partial \xi^*}) \]
So our Hamiltonian is Hermitian and invariant under involution operation.
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