GRAPH EMBEDDINGS INTO HAMMING SPACES

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Abstract. Graph embeddings deal with injective maps from a given simple, undirected graph $G = (V, E)$ into a metric space, such as $\mathbb{R}^n$ with the Euclidean metric. This concept is widely studied in computer science, see [1], but also offers attractive research in pure graph theory [2]. In this note we show that any graph can be embedded into a particularly simple metric space: $\{0, 1\}^n$ with the Hamming distance, for large enough $n$.

1. The Hamming Graph $H(n, k)$

We construct graph on the vertex set $\{0, 1\}^n$ where $n$ is a positive integer. For $x, y \in \{0, 1\}^n$ the Hamming distance of $x, y$ is the cardinality of the set

$$\{i \in \{0, \ldots, n-1\} : x(i) \neq y(i)\}.$$ 

That is, we count the positions on which $x$ and $y$ do not agree.

Fix a positive integer $k \leq n$. Two distinct elements of $\{0, 1\}^n$ form an edge if their Hamming distance is at most $k$ (so they are in some sense “close” to each other). We denote the resulting graph on $\{0, 1\}^n$ by $H(n, k)$.

We say that a finite graph $G = (V, E)$ is Hamming-representable if there are positive integers $k \leq n$ such that $G$ is isomorphic to an induced subgraph of $H(n, k)$.

As an easy example, we show that the following 3-point graph can be embedded into $H(2, 1)$:

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    a
   / \
 b   c
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The solution is best shown in the following picture, where it is easily seen that points connected with an edge have Hamming distance 1 and points not connected have Hamming distance 2:
As a further example, note that $H(n, n)$ is isomorphic to $K_{2^n}$, the complete graph on $2^n$ vertices.

Some notation: By $\text{Mat}(\{0, 1\}, n \times m)$ we denote the set of $n \times m$-matrices with entries in $\{0, 1\}$. We identify $\text{Mat}(\{0, 1\}, n \times m)$ with $\{0, 1\}^{nm}$ via the canonical bijection.

2. The Result

**Proposition 2.1.** Every finite graph $G = (V, E)$ is Hamming-representable.

*Proof.* We embed $G$ into $H(|E| \cdot (|V| - 1), 2|E| - 2)$. To each vertex $v$ of $G$, we will associate an $|E| \times (|V| - 1)$ matrix $M_v$ with rows indexed by the edges of $G$. There will be a single 1 in each row, with all other entries in that row equal to 0.

If $v \in e$, then the 1 in row $e$ of $M_v$ will be in the first column. If not, we will place a 1 in one of the other $|V| - 2$ columns, so that each of the non-endpoints of $e$ gets a 1 in a different position of row $e$.

If $v$ and $w$ are not joined by an edge, the Hamming distance between $M_v$ and $M_w$ is $2|E|$ because they have no 1’s in common; if they are joined, then the Hamming distance is $2|E| - 2$. □.

3. Possible use cases

Representing graphs as subgraphs of some $H(n, k)$ can be useful in applications in computer science: the Hamming distance is computed by bitwise XOR, the fastest operation a CPU can do. So given two vertices represented by $n$-bit strings, it can be very quickly determined whether they form an edge (i.e. whether their Hamming distance is smaller than the limit given in $k$).

Moreover, for some graphs $G = (V, E)$ with $|V| = n$ we can represent the graph using bit strings of length $O(\log n)$, making this technique potentially interesting for memory management.
4. Open Questions

We define the *Hamming dimension* of a graph $G = (V, E)$ to be the minimum positive integer $n$ such that there is $k \leq n$ such that $G$ can be embedded into some induced subgraph of $H(n, k)$, and denote this by $\dim(G)$ Questions:

1. If $G = (V, E)$ is a graph with $n = |V|$, do we necessarily have $\dim(G) \leq n$? If not, can we at least achieve for $\dim(G)$ to be $O(|E| \log |V|)$?

2. Given graphs $G, H$ what is $\dim(G \times H)$ in terms of $\dim(G), \dim(H)$, where $G \times H$ denotes the categorical product?

3. How (if at all) does $\dim(G)$ relate to the chromatic number $\chi(G)$?

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References

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