Fractional Burgers equation
with singular initial condition

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Abstract
We consider the fractional Burgers equation $\Delta^{\alpha/2}u + b \cdot \nabla (u |u|^{(\alpha-1)/\beta})$ on $\mathbb{R}^d$, $d \geq 2$, with $\alpha \in (1, 2)$ and $\beta > 1$ and prove the existence of a solution for a large class of initial conditions, which contains functions that do not belong to any $L^p(\mathbb{R}^d)$, $1 \leq p \leq \infty$. Next, we apply the general results to the initial condition $u_0(x) = M|x|^{-\beta}$, $1 < \beta < d$, and show the existence of a selfsimilar solution and derive its properties such as smoothness, two-sided estimates, asymptotics and gradient estimates.

1 Introduction
Let $d \geq 2$ and $\alpha \in (1, 2)$. Consider the fractional Burgers equation

$$
\begin{cases}
    u_t = \Delta^{\alpha/2}u + b \cdot \nabla (u |u|^{q}), & t > 0, \\
    u(0, \cdot) = u_0
\end{cases}
$$

(1.1)
in $\mathbb{R}^d$, where $\beta > 1$, $b \in \mathbb{R}^d$ and $q = \frac{\alpha-1}{\beta}$ are fixed. For technical reasons, we assume (without loss of generality) $b = (|b|, 0, 0, \ldots, 0)$. Here, $\Delta^{\alpha/2}$ is the fractional Laplacian

$$
\Delta^{\alpha/2} f(x) = \lim_{\varepsilon \to 0^+} c_{d,\alpha} \int_{|y| > \varepsilon} \frac{f(x+y) - f(x)}{|y|^{d+\alpha}} dy,
$$

where $c_{d,\alpha}$ is some constant. It may be also defined by the Fourier transform

$$
\hat{\Delta^{\alpha/2} f}(\xi) = -|\xi|^\alpha \hat{f}(\xi).
$$

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We put
\[ p(t, x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \xi} e^{-|\xi|^\alpha}, \quad t > 0, \; x \in \mathbb{R}^d. \]
and denote \( p(t, x, y) = p(t, y - x), \; t > 0, \; x, y \in \mathbb{R}^d. \) The function \( p \) is the density of the semigroup \( P_t = e^{t\Delta^{\alpha/2}} \) generated by the fractional Laplacian,
\[ P_t f(x) = e^{t\Delta^{\alpha/2}} f(x) = \int_{\mathbb{R}^d} p(t, x, y) f(y) dy. \]

By a solution to the Cauchy problem (1.1) we mean a mild solution, i.e. a function \( u: \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \) satisfying the Duhamel formula
\[ u(t, x) = P_t u_0(x) + \int_0^t \int_{\mathbb{R}^d} b \cdot \nabla_x p(t - s, x, z) u(s, z) |u(s, z)|^{q} dz ds. \quad (1.2) \]

The fractional Burgers equation was intensely studied in recent years, see e.g. [1–5, 7, 9, 13–15, 17]. It is a generalization of the classical Burgers equation
\[ u_t = \nu u_{xx} - \frac{1}{2} (u^2)_x, \]
which was introduced as a simplest model describing the turbulence phenomena (see e.g. [8]). The fractional counterpart (1.1) of this equation was studied for the first time by Biler and Funaki in [3]. Later on, in [4] Biler, Karch and Wojczyński considered some further generalizations, where the fractional Laplacian was replaced by some Lévy operator and nonlinearity \( f(u) \) was given by a smooth function \( f \). In particular, they proved that for \( u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \) there exists a unique solution to (1.1) such that (see [4, Theorem 3.1])
\[ u \in C(0, \infty); W^{2,2}(\mathbb{R}^d) \cap C^1((0, \infty); L^2(\mathbb{R}^d)), \]
and
\[ \|u(t, \cdot)\|_\gamma \leq \|u_0\|_\gamma, \quad t > 0, \; \gamma \in [1, \infty]. \quad (1.4) \]

We point out that in both papers the standing assumptions was \( u_0 \in L^1(\mathbb{R}^d) \). In [5] the authors studied the equation (1.1) with the critical exponent \( q = \frac{(\alpha - 1)}{d} \), i.e.
\[ \begin{cases}
  u_t = \Delta^{\alpha/2} u + b \cdot \nabla u |u|^{(\alpha - 1)/d}, & t > 0, \\
  u(0, \cdot) = u_0.
\end{cases} \quad (1.5) \]

They showed that for \( u_0 = M \delta_0, \; M > 0 \), there is a unique selfsimilar solution (called source solution) to (1.5) satisfying the scaling property \( U(t, x) = t^{-d/\alpha} U(1, t^{-1/\alpha} x) \) (the same as the density \( p \)). It turns out that this self-similar solution determines the long time behavior of solutions to a large class of Cauchy problems (1.5) with \( u_0 \in L^1(\mathbb{R}^d) \) and \( \|u_0\|_1 = M \) (see [5, Theorem 2.2])
\[ \lim_{t \to \infty} t^{\frac{d}{\alpha}(1 - \frac{1}{\gamma})} \|u(t, \cdot) - U(t, \cdot)\|_\gamma = 0, \quad \gamma \in [1, \infty]. \quad (1.6) \]
If \( u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \), the existence of the solution to the problem (1.3) can be proved by using the Banach fixed point theorem (see [1]). The case \( u_0 = M\delta_0 \) is in some sense similar to \( u_0 \in L^1(\mathbb{R}^d) \) since such \( u_0 \) is still integrable and the solution may be constructed by using the approximation by the solutions with initial conditions from \( L^1(\mathbb{R}^d) \). Nevertheless, if \( u_0 \) does not belong to any \( L^\gamma(\mathbb{R}^d) \), \( \gamma \in [1, \infty] \), the construction by fixed point theorems fails.

In this paper we assume that the initial condition \( u_0 : \mathbb{R}^d \to \mathbb{R} \) satisfies

\[
\sup_{t > 0, x \in \mathbb{R}^d} t^{(\beta - 1)/\alpha} \int_{\mathbb{R}^d} |u_0(y)|p^{(d - 1)}(t, \tilde{x}, \tilde{y}) dy \leq \mu, \tag{A}
\]

where \( \mu \in (0, \infty) \) is some constant. Here, \( p^{(d - 1)} \) is the density of \( P_t \) in dimension \( d - 1 \) and \( \tilde{x} = (x_2, \ldots, x_d) \in \mathbb{R}^{d-1} \) (see (2.2) and Notation in Section 2). The main feature of the class of functions satisfying (A) is that it contains some functions not belonging to any \( L^\gamma(\mathbb{R}^d) \), \( \gamma \in [1, \infty] \). This allows us to drop the common assumption \( u_0 \in L^1(\mathbb{R}^d) \), or even \( u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \), and consider some singular initial conditions. Our first result is

**Theorem 1.1.** Let \( \alpha \in (1, 2) \), \( \beta > 1 \) and \( q = (\alpha - 1)/\beta \). If \( u_0 \) satisfies (A), then there exists a solution \( u(t, x) \) to the problem (1.1). Furthermore, there is a constant \( C > 0 \) depending only on \( d, \alpha, \beta, b, \mu \) such that

\[
|u(t, x)| \leq C t^{-1/\alpha} \int_{\mathbb{R}^d} |u_0(y)|p^{(d - 1)}(t, \tilde{x}, \tilde{y}) dy \leq C \mu t^{-\beta/\alpha}, \quad t > 0, \ x \in \mathbb{R}^d. \tag{1.7}
\]

In the proof of Theorem 1.1 we proceed similarly as in the case of source solutions and approximate the initial condition by the monotone sequence of functions from \( L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \). However, the convergence of the corresponding sequence of solutions is a major problem here. We resolve it by showing that a certain functional acting on these solutions is uniformly bounded, which ensures the point-wise convergence of the sequence almost everywhere (see Lemma 3.3). This approach, however, requires the technical assumptions \( \beta > 1 \) and \( d \geq 2 \).

The first bound in Theorem 1.1 has one weakness - it does not depend on \( x_1 \), and therefore it is not optimal. Nevertheless, in many cases this inconvenience could be removed. We present it on an example, which turns out to be quite special. Namely, in the second part of the paper we focus on the solution to the problem (1.1) with \( u_0(x) = M|x|^{-\beta} \), \( M > 0 \), \( 1 < \beta < d \), constructed in Theorem 1.1. Its first interesting property is self-similarity, in which one can see an analogy to the source solution in the case \( \beta = d \) and \( u_0 = M\delta_0 \). Next, we turn our attention to the two-sided pointwise estimates, which not only improve (1.3) in this case, but complement it with the lower bound of the same form. Such estimates were obtained for the source solution of (1.3). In [1] Karch and Brandolese showed that for \( q = \frac{\alpha - 1}{d} \) and \( u_0 = M\delta_0 \) with \( M > 0 \) sufficiently small, the solution \( u \) to (1.3) admits the estimates \( 0 \leq u(t, x) \leq cp(t, x) \). In [11] we generalized this result to any \( M > 0 \) and obtained two-sided estimates \( c^{-1}p(t, x) \leq u(t, x) \leq cp(t, x) \). Furthermore, in [10] Theorem 1.1] we showed that for nonnegative \( u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \) there is a constant \( C > 0 \) such that the solution to (1.1) with \( q \geq \frac{\alpha - 1}{d} \) satisfies

\[
C^{-1} P_t u_0(x) \leq u(t, x) \leq C P_t u_0(x), \quad t > 0, \ x \in \mathbb{R}^d. \tag{1.8}
\]
Note that the pointwise estimates of solutions are rather rare in the literature, as deriving them is more challenging than e.g. the $L^p$ bounds. Nevertheless, they give a better insight into the behaviour of the solution. For instance the estimates (1.8) permitted the authors to improve the asymptotics given in (1.6).

In Theorem 1.2 below we prove the existence of the selfsimilar solution to the problem (1.1) for $u_0(x) = M|x|^{-\beta}$ with $M > 0$, $1 < \beta < d$ and provide its properties. Note that the assumption $\beta < d$ is required in order to get local integrability of $u_0$.

**Theorem 1.2.** Let $\alpha \in (1, 2)$, $1 < \beta < d$, $q = (\alpha - 1)/\beta$, $M > 0$ and $u_0(x) = M|x|^{-\beta}$.

Then, there exists a function $U(x)$ such that

1. $u(t, x) = t^{-\beta/\alpha}U(t^{-1/\alpha}x)$ is the solution to the problem (1.1).

2. There exists a constant $C_1 > 1$ such that
   \[
   C_1^{-1} \frac{1}{(1 + |x|)^\beta} \leq U(x) \leq C_1 \frac{1}{(1 + |x|)^\beta}, \quad x \in \mathbb{R}^d. \tag{1.9}
   \]

3. There is a constant $C_2$ such that for all $x \in \mathbb{R}^d$ we have
   \[
   |U(x) - P_1 u_0(x)| \leq C_2 \frac{1}{(1 + |x|)^{\beta + (\alpha - 1)}}, \tag{1.10}
   \]
   \[
   \left| U(x) - P_1 u_0(x) - \alpha \int_0^1 \int_{\mathbb{R}^d} r^{d-\beta} \nabla_x p(1 - r^\alpha, x, rw) [P_1 u_0(w)]^{1+q} dw dr \right| \leq \frac{C_2}{(1 + |x|)^{\beta + 2(\alpha - 1)}}. \tag{1.11}
   \]

4. $U \in C^1(\mathbb{R}^d)$ and there is a constant $C_3$ such that
   \[
   |\nabla U(x)| \leq C_3 \frac{1}{(1 + |x|)^\beta}, \quad x \in \mathbb{R}^d. \tag{1.12}
   \]

All the constants $C_1, C_2, C_3$ depend only on $d, \alpha, \beta, M, b$ and might be calculated explicitly.

**Remark 1.3.** By Lemma 2.3 the bounds in (1.9) may be equivalently written in the form

\[
 u(t, x) \approx t^{-\beta/\alpha} \frac{1}{(t^{1/\alpha} + |x|)^\beta} \approx P_t u_0(x), \quad t > 0, \ x \in \mathbb{R}^d,
\]

which resembles both: (1.8) by representation involving the semi-group, and the estimates of the source solution in the critical case derived in [10] by the exact form. Similarly, since $\nabla_x u(t, x) = \lambda^{\beta+1} \nabla_x (\lambda^\alpha t, \lambda x)$, by (1.12) we have

\[
 |\nabla_x u(t, x)| \lesssim \frac{1}{t^{1/\alpha}(t^{1/\alpha} + |x|)^\beta}, \quad t > 0, \ x \in \mathbb{R}^d.
\]

The paper is organized as follows. In Section 2 we introduce the notation used in the paper and give some preliminary results on the density $p(t, x, y)$ and solutions to (1.1). Section 3 is devoted to some general estimates and proof of Theorem 1.1. In Section 4 we prove Theorem 1.2.
2 Preliminaries

2.1 Notation

We will use the following notation. We write ":=" to indicate definitions, e.g. \( a \wedge b := \min\{a, b\} \) and \( a \vee b := \max\{a, b\} \). All the considered functions are tacitly assumed to be Borel measurable. We write \( f(x) \lesssim g(x) \) if \( f, g \geq 0 \) and there is a number \( c > 0 \) not depending on other parameters than \( d, \alpha, \beta, M \) and \( b \) such that \( f(x) \leq cg(x) \) for all arguments \( x \). If \( f \lesssim g \) and \( g \lesssim f \), we write \( f(x) \approx g(x) \). If the comparability constant depends additionally on some other parameters (e.g. on \( \gamma \)), we indicate the parameter over the comparison sign (e.g. \( f \gtrsim g \)).

By \( \nabla = (\partial_{x_1}, \ldots, \partial_{x_d}) \) we denote the standard gradient operator, while \( \nabla^k = \partial_{x_1}^{k_1} \ldots \partial_{x_d}^{k_d} \), where \( k = (k_1, \ldots, k_d) \in \mathbb{N}_0^d \). The norm of the multi-index \( k \) is given by \( |k| = k_1 + \ldots + k_d \).

By writing \( \nabla_{x} \) and \( \nabla_{x}^k \) we indicate variable with respect to which the derivatives are taken.

For any \( x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d \) we denote \( \tilde{x} := (x_2, x_3, \ldots, x_d) \in \mathbb{R}^{d-1} \). For any function \( f : (0, \infty) \times \mathbb{R}^d \to \mathbb{R} \), we define its rescaled version by

\[
f^*(t, x) := t^{\beta/\alpha} f(t^{1/\alpha} x).
\]  

(2.1)

Similarly, we put

\[
P_\gamma^* f(x) := t^{\beta/\alpha} P_t f(t^{1/\alpha} x) = t^{\beta/\alpha} \int_{\mathbb{R}^d} p(t, t^{1/\alpha} x, y) f(y) dy.
\]

For any \( a \in \mathbb{R} \) and \( \gamma > 1 \) we define the french power \( a^{(\gamma)} = a|a|^{\gamma-1} \). In particular, \( |u(s, z)|u(s, z)^q = u(s, z)^{(q+1)} \).

2.2 Fractional Laplacian semigroup

Although in the paper we generally assume \( \alpha \in (1, 2) \) and \( d \geq 2 \), the results of this section are valid for \( \alpha \in (0, 2) \) and \( d \geq 1 \). The methodology we propose in the next sections requires consideration of the density of the semigroup \( P_t \) in various dimensions at the same time. Therefore, for \( m \in \{1, 2, 3, \ldots\} \) we put

\[
p^{(m)}(t, x) = \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} e^{ix\cdot \xi} e^{-t|\xi|^\alpha}, \quad t > 0, \ x \in \mathbb{R}^m,
\]

(2.2)

and denote \( p^{(m)}(t, x, y) = p^{(m)}(t, y-x), \ t > 0, \ x, y \in \mathbb{R}^m \). In the case \( m = d \) we simply write \( p^{(d)}(t, x, y) = p(t, y-x) = p(t, y-x) \). Below, we collect some basic properties of the density \( p \). The following scaling property holds

\[
p(\lambda^\alpha t, \lambda x, \lambda y) = \lambda^{-d} p(t, x, y), \quad \lambda > 0.
\]

(2.3)

Furthermore, the function \( p^{(m)} \) admits the two-sided estimates

\[
p^{(m)}(t, x, y) \approx \frac{t}{(t^{1/\alpha} + |x-y|)^{m+\alpha}}, \quad t > 0, \ x, y \in \mathbb{R}^m,
\]

(2.4)
where the constants in the bounds depend on $m$ and $\alpha$. In particular, this implies

$$
||p(t, \cdot)||_\gamma = t^{-\frac{m}{\gamma}(1-1/\gamma)}||p(1, \cdot)||_\gamma, \quad t > 0,
$$

$$
p(t, x, y) \lesssim t^{-1/\alpha}p(d-1)(t, \hat{x}, \hat{y}), \quad t > 0, x, y, \in \mathbb{R}^d.
$$

**Lemma 2.1.** For $t > 0$, $\hat{x} \in \mathbb{R}^{d-1}$, $y \in \mathbb{R}^d$, we have

$$
\int_{\mathbb{R}} p(t, x, y)dx_1 = p^{(d-1)}(t, \hat{x}, \hat{y}).
$$

**Proof.** By the subordination formula (see [16]) there is a nonnegative function $\eta_t(s)$ such that

$$
p^{(m)}(t, x, y) = \int_0^\infty (4\pi s)^{-m/2}e^{-|x-y|^2/4s}\eta_t(s)ds, \quad m \in \mathbb{N}, t > 0, x, y \in \mathbb{R}^m.
$$

Hence,

$$
\int_{\mathbb{R}} p(t, x, y)dx_1 = \int_0^\infty \int_{\mathbb{R}} (4\pi s)^{-d/2}e^{-|x-y|^2/4s}\eta_t(s)dx_1ds
$$

$$
= \int_0^\infty (4\pi s)^{-(d-1)/2}e^{-|\hat{x}-\hat{y}|^2/4s}\eta_t(s)ds = p^{(d-1)}(t, \hat{x}, \hat{y}).
$$

\[\square\]

The subordination formula (2.8) allows us also to derive the following estimates of $\nabla^k p$.

**Lemma 2.2.** For any $k \in \mathbb{N}_0^d$, $t > 0$ and $x \in \mathbb{R}^d$, we have

$$
|\nabla^k p(t, x)| \lesssim \frac{p(t, x)}{(t^{1/\alpha} + |x|)^{|k|}}.
$$

**Proof.** Let $k = (k_1, \ldots, k_d) \in \mathbb{N}_0^d$ and $g(s, x) = (4\pi s)^{-d/2}e^{-|x|^2/(4s)}$ be the Gaussian kernel. Then,

$$
\nabla^k g(s, x) = (4s)^{-|k|/2}g(s, x) \prod_{i=1}^d H_{k_i}(x_i/2\sqrt{s}),
$$

where $H_n$ are the Hermite polynomials. By subordination formula (2.8) and the Fubini theorem, we have

$$
\nabla^k p(t, x) = \int_0^\infty (4s)^{-|k|/2}g(s, x) \prod_{i=1}^d H_{k_i}(x_i/2\sqrt{s})\eta_t(s) ds.
$$

Since every $H_{k_i}$ is a polynomial of degree $k_i$, we bound

$$
\left|\prod_{i=1}^d H_{k_i}(x_i/2\sqrt{s})\right| \lesssim \prod_{i=1}^d [1 + |x_i/2\sqrt{s}|^{k_i}] \lesssim 1 + |x/2\sqrt{s}|^{\sum_{i=1}^d k_i}.
$$
Consequently

\[
|\nabla^k p(t, x)| \lesssim \int_0^\infty (4s)^{-|k|/2} g(s, x) \eta_k(s) \, ds + |x|^{|k|} \int_0^\infty (4s)^{-|k|} g(s, x) \eta_k(s) \, ds
\]

\[= p^{(d+|k|)}(t, x_+) + |x|^{|k|} p^{(d+2|k|)}(t, x_+), \]

where \(x_+ = (x_1, \ldots, x_d, 0, \ldots, 0) \in \mathbb{R}^{d+|k|}\) and \(x_{++} = (x_1, \ldots, x_d, 0, \ldots, 0) \in \mathbb{R}^{d+2|k|}\).

Eventually, by (2.4), we obtain

\[
|\nabla^k p(t, x)| \lesssim \frac{k}{(t^{1/\alpha} + |x|)^{d+\alpha+|k|}} + \frac{|x|^{|k|}t}{(t^{1/\alpha} + |x|)^{d+\alpha+2|k|}} \approx \frac{1}{(t^{1/\alpha} + |x|)^{|k|}} p(t, x),
\]
as required. \( \square \)

The next lemma describes the behaviour of \(P_t\) acting on the function \((r^{1/\alpha} + |y|)^{-\gamma}\), which, as we will show, is comparable with a solution to (1.1) for \(u_0(x) = M|x|^{-\gamma}\).

**Lemma 2.3.** Let \(0 < \gamma < d\). For \(x \in \mathbb{R}^d\), \(t > 0\) and \(r \geq 0\) we have

\[
\int_{\mathbb{R}^d} p(t, x, y)(r^{1/\alpha} + |y|)^{-\gamma} \, dy \approx \frac{1}{(r^{1/\alpha} + t^{1/\alpha} + |x|)^{\gamma}}. \tag{2.10}
\]

**Proof.** By [6, Lemma 2.3],

\[
\int_{\mathbb{R}^d} p(t, x, y)|y|^{-\gamma} \, dy \approx \frac{1}{(t^{1/\alpha} + |x|)^{\gamma}}, \tag{2.11}
\]

which proves (2.10) for \(r = 0\). Now, let \(r > 0\). By (2.11), we get

\[
\int_{\mathbb{R}^d} p(t, x, y)(r^{1/\alpha} + |y|)^{-\gamma} \, dy \approx \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p(t, x, y)p(r, y, z)|z|^{-\gamma} \, dz \, dy
\]

\[= \int_{\mathbb{R}^d} p(t + r, x, z)|z|^{-\gamma} \, dz \approx \frac{1}{((r + t)^{1/\alpha} + |x|)^{\gamma}}.
\]

Since \((r + t)^{1/\alpha} \approx r^{1/\alpha} + t^{1/\alpha}\), we obtain (2.10). \( \square \)

### 2.3 Basic properties of solutions to (1.1)

Let us observe that if \(u\) and \(v\) are the solutions to (1.1) with initials conditions \(u_0\) and \(-u_0\), respectively, then \(u(t, x) = -v(t, x)\) for all \(t > 0\) and \(x \in \mathbb{R}^d\).

As mentioned in Introduction, if \(u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)\) then there exists a unique solution \(u(t, x)\) to the problem (1.1). Furthermore, [1, Corollary 3.1] implies the monotonicity property of solutions. Namely, if \(u\) and \(v\) are the solutions of (1.1) with initials conditions \(u_0, v_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)\) and \(u_0 \geq v_0\), then

\[
u(t, x) \geq v(t, x), \quad t > 0, x \in \mathbb{R}^d. \tag{2.12}
\]

Let us now denote by \(\mathfrak{u}\) and \(\mathfrak{v}\) the solutions to (1.1) with initials conditions \(|u_0|\) and \(-|u_0|\), respectively. Since \(\mathfrak{u} = -\mathfrak{v}\), we get by (2.12)

\[
|u| \leq |\mathfrak{u}| \lor |\mathfrak{u}| = |\mathfrak{u}|. \tag{2.13}
\]
For technical reasons we will often consider the rescaled version of solutions. In order to represent $u^\ast$ (cf. (2.1)) in a recursive manner as in (1.2), we rewrite the integral therein as follows.

$$
\int_0^t \int_{\mathbb{R}^d} b \cdot \nabla_x p(t - s, t^{1/\alpha} x, z) u(s, z)^{(q+1)} dz ds
= \int_0^t \int_{\mathbb{R}^d} b \cdot \nabla_x p(t - s, t^{1/\alpha} x, s^{1/\alpha} w) s^{d/\alpha} u(s, s^{1/\alpha} w)^{(q+1)} dw ds
= \int_0^t \int_{\mathbb{R}^d} b \cdot \nabla_w p(t - st, t^{1/\alpha} x, (st)^{1/\alpha} w) t^{d/\alpha} (st)^{(q+1)} u(st, (st)^{1/\alpha} w)^{(q+1)} dw ds
= \alpha \int_0^t \int_{\mathbb{R}^d} t^{1 - \frac{1}{\alpha}} p^{d + \alpha - 1} b \cdot \nabla_w p(1 - r^{\alpha}, x, rw) u(r^{\alpha} t, rt^{1/\alpha} w)^{(q+1)} dw dr
= \alpha \int_0^t \int_{\mathbb{R}^d} r^{d - \beta} b \cdot \nabla_w p(1 - r^{\alpha}, x, rw) u^\ast(r^{\alpha} t, w)^{(q+1)} dw dr.
$$

Thus, we have

$$
\ u^\ast(t, x) = P_t^\ast u_0(x) + \alpha \int_0^t \int_{\mathbb{R}^d} r^{d - \beta} b \cdot \nabla_w p(1 - r^{\alpha}, x, rw) u^\ast(r^{\alpha} t, w)^{(q+1)} dw dr. \quad (2.14)
$$

3 General results

3.1 An integral conservation law

Up till the end of the paper we assume that $\alpha \in (1, 2)$ and $d \geq 2$. Let $F : \mathbb{R}^{d-1} \to \mathbb{R}$ be any function. For a function $h : \mathbb{R}^d \to \mathbb{R}$ we denote

$$
h_F := \int_{\mathbb{R}^d} h(x) F(\tilde{x}) dx,
$$

whenever the integral is convergent.

**Lemma 3.1.** Let $F : \mathbb{R}^{d-1} \to \mathbb{R}$. Then, for any $u$ satisfying (1.2) with $\|P_t u_0\|_F < \infty$ for every $t > 0$ and such that

$$
\int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\nabla_x p(t - s, x, z)||u(s, z)|^{q+1}|F(\tilde{x})| dx dz ds < \infty, \quad (3.1)
$$

we have

$$
\ u(t, \cdot)_F = (P_t u_0)_F, \quad t > 0.
$$

**Proof.** Since $b = (|b|, 0, \ldots, 0)$, by (1.2) and Fubini theorem, we have

$$
u(t, \cdot)_F = \int_{\mathbb{R}^d} (P_t u_0(x) + \int_0^t \int_{\mathbb{R}^d} b \cdot \nabla_x p(t - s, x, z) u(s, z)^{(q+1)} dz ds) F(\tilde{x}) dx
$$
$$= (P_t u_0) F + \int_0^t \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |b| \partial_x p(t-s, x, z) F(\hat{x}) dx \right) u(s, z)^{(q+1)} dz \, ds$$

$$= (P_t u_0) F,$$

where we used (2.4) and the fact that $F(\hat{x})$ does not depend on $x_1$. \hfill \Box

In particular, we show that the assumptions of Lemma 3.1 are satisfied if $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and $F$ is bounded.

**Corollary 3.2.** Suppose $u$ is a solution to (1.1) with $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Let $F : \mathbb{R}^{d-1} \to \mathbb{R}$ be bounded. Then,

$$\int_{\mathbb{R}^d} F(\hat{w}) u(t, w) \, dw = \int_{\mathbb{R}^d} F(\hat{w}) P_t u_0(w) \, dw, \quad t > 0.$$  

**Proof.** Let $K = \sup_{\hat{w} \in \mathbb{R}^{d-1}} |F(\hat{w})|$. Since $|P_t u_0|_F \leq K\|u_0\|_1 < \infty$, by Lemma 3.1 we only need to show that the condition (3.1) holds. By (1.4), (2.9) and (2.7) we have

$$\int_0^t \int_{\mathbb{R}^d} |\nabla p(t-s, w, z)||u(s, z)|^{1+q}|F(\hat{w})| \, dw \, dz \, ds \leq K \int_0^t \|u_0\|_\infty^q \int_{\mathbb{R}^d} \frac{p(t-s, w, z)}{(t-s)^{1/\alpha}} |u(s, z)| \, dw \, dz \, ds \leq K \|u_0\|_\infty^q \|u_0\|_1 \frac{\alpha t^{1-1/\alpha}}{\alpha - 1} < \infty,$$

which ends the proof. \hfill \Box

If in Corollary 3.2 we take $F(\hat{w}) = p^{(d-1)}(t_0, \hat{x}, \hat{w})$ with $t_0 > 0$ and $x \in \mathbb{R}^d$, by (2.4) we get

$$\int_{\mathbb{R}^d} p^{(d-1)}(t_0, \hat{x}, \hat{w}) u(t, w) \, dw = \int_{\mathbb{R}^d} p^{(d-1)}(t_0, \hat{x}, \hat{w}) P_t u_0(w) \, dw, \quad t > 0. \quad (3.2)$$

The following lemma will be frequently exploit in the sequel. It also partially explains the introduction of the assumption \([A]\).

**Lemma 3.3.** Suppose $u$ is a solution to (1.1) with $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Then, for $x \in \mathbb{R}^d$, $t > 0$ and $r \in (0, 1)$ we have

$$\int_{\mathbb{R}^d} p(1-r^\alpha, x, rw) |u^r(r^\alpha t, w)| \, dw \leq \frac{r^{\beta-d}}{(1-r)^{1/\alpha}} t^{(\beta-1)/\alpha} \int_{\mathbb{R}^d} p^{(d-1)}(t, t^{1/\alpha} \hat{x}, \hat{y}) |u_0(y)| dy. \quad (3.3)$$

**Proof.** In view of (2.13), it is enough to consider $u_0 \geq 0$. In that case, $u \geq 0$ and by (2.6), (2.3) and (3.2) we get

$$\int_{\mathbb{R}^d} p(1-r^\alpha, x, rw) u^r(r^\alpha t, w) \, dw \quad (3.3)$$
Theorem 3.4.

3.2 Proof of Theorem 1.1

Denote $\| \cdot \|_\infty$. By (2.14), we have

$$\int p(1 - r, x, rw) u^*(r t, w) \, dw$$

$$\leq \int p(1 - r, x, rw) \, dw$$

$$\leq (1 - r)^{-1/\alpha} \int p(1 - r, x, rw) \, dw$$

$$= (1 - r)^{-1/\alpha} \int p(1 - r, x, rw) \, dw$$

$$\leq (1 - r)^{-1/\alpha} \int p(1 - r, x, rw) \, dw$$

$$= (1 - r)^{-1/\alpha} \int p(1 - r, x, rw) \, dw$$

$$\leq (1 - r)^{-1/\alpha} \int p(1 - r, x, rw) \, dw$$

where, to obtain the last equality, we used (2.7) and performed the integration with respect to $w_1$. Finally, using the Tonelli theorem, (2.3), and the semi-group properties of $p$, we arrive at

$$\int p(1 - r, x, rw) u^*(r t, w) \, dw$$

$$\leq \int p(1 - r, x, rw) \, dw$$

$$\leq (1 - r)^{-1/\alpha} \int p(1 - r, x, rw) \, dw$$

$$= (1 - r)^{-1/\alpha} \int p(1 - r, x, rw) \, dw$$

$$\leq (1 - r)^{-1/\alpha} \int p(1 - r, x, rw) \, dw$$

$$\leq (1 - r)^{-1/\alpha} \int p(1 - r, x, rw) \, dw$$

which ends the proof.

3.2 Proof of Theorem 1.1

Theorem 3.4. Suppose $u$ is a solution to the problem (1.1). If the assumption $\mathbf{A}$ is satisfied and

$$\sup_{t > 0, x \in \mathbb{R}^d} |u^*(t, x)| < \infty,$$

then there exists a constant $C > 0$ depending only on $\alpha, d$ such that

$$\sup_{t > 0, x \in \mathbb{R}^d} |u^*(t, x)| \leq 1 + \left( C(\mu + 1)(|b| + 1) \right)^{\frac{\alpha}{\alpha - 1} - \frac{\beta}{\beta - 1}}.$$

Proof. Denote $\kappa = \sup_{t > 0, x \in \mathbb{R}^d} |u^*(t, x)|$ and assume $\kappa > 1$. By (2.14), we have

$$|u^*(t, x)| \leq \| P_t^* u_0 \|_\infty$$

$$+ C|b| \left( \int_0^{1-2\kappa} + \int_1^{1} \right) \int_{\mathbb{R}^d} \frac{r^{d-\beta}}{(1 - r^{\alpha})^{1/\alpha} p(1 - r, x, rw) |u^*(r t, w)|^{(\alpha - 1 + \beta)/\beta} \, dw \, dr$$

$$=: \| P_t^* u_0 \|_\infty + C|b| (I_1 + I_2),$$
for some $C = C(\alpha, d)$. First, by the inequality (2.6) and the assumption (A) we have

\[ P_t^* |u_0|(x) = t^{\beta/\alpha} \int_{\mathbb{R}^d} p(t, t^{1/\alpha} x, y) |u_0(y)| dy \lesssim t^{(\beta-1)/\alpha} \int_{\mathbb{R}^d} p^{(d-1)}(t, t^{1/\alpha} \tilde{x}, \tilde{y}) |u_0(y)| dy \leq \mu. \]

Let us pass to estimating the integral $I_2$:

\[
I_2 \leq \kappa^{(\alpha+1)/\beta} \int_{1-1/2\kappa}^{1} r^{d-\beta} (1-r)^{-1/\alpha} dr = \mu \kappa^{(\alpha+1)/\beta} \int_{1-1/2\kappa}^{1} r^{d-\beta} (1-r)^{-1/\alpha} dr. 
\]

Then, since $\kappa > 1$, we have

\[
I_2 \leq \mu \kappa^{(\alpha+1)/\beta} 2^{\beta-1+1/\alpha} \frac{\alpha}{\alpha-1}. 
\]

Concerning the integral $I_1$, we have

\[
I_1 \leq \kappa^{\alpha-1} \int_{0}^{1-1/2\kappa} r^{d-\beta} (1-r)^{-1/\alpha} dr = \mu \kappa^{(\alpha+1)/\beta} \int_{1-1/2\kappa}^{1} r^{d-\beta} (1-r)^{-1/\alpha} dr = \mu 2^{\beta-1+1/\alpha} \frac{\alpha}{\alpha-1}. 
\]

Applying (3.3) to (3.4) and taking advantage of the assumption (A), we eventually obtain

\[
I_1 \leq \mu \kappa^{\alpha-1} \int_{0}^{1-1/2\kappa} (1-r)^{-1/\alpha} dr = \mu 2^{\beta-1+1/\alpha} \frac{\alpha}{\alpha-1}, 
\]

which leads to

\[
\kappa \lesssim \mu + |b| \left(1 + \mu \right) \left(\kappa^{1-(\beta-1)(\alpha-1)/\beta}\right). 
\]

Dividing both sides by $\kappa^{1-(\beta-1)(\alpha-1)/\beta}$ and keeping in mind the assumptions $\alpha, \beta > 1$ and $\kappa > 1$, we obtain

\[
\kappa^{(\alpha-1)(\beta-1)/\beta} \lesssim \mu + |b| \left(1 + \mu \right) \leq (1 + |b|)(\mu + 1), 
\]

which implies the bound in the assertion of the theorem.

Next we provide an improved upper bound, which depends on the space arguments as well.

**Proposition 3.5.** Under the assumptions of Theorem [3.4] we have

\[
|u(t, x)| \leq C t^{-1/\alpha} \int_{\mathbb{R}^d} |u_0(y)| p^{(d-1)}(t, \tilde{x}, \tilde{y}) dy \]  \hspace{1cm} (3.5)

for some $C = C(\alpha, \beta, d, |b|, \mu)$. 

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Proof. As in the proof of the previous theorem, we denote \( \kappa = \sup_{t > 0, x \in \mathbb{R}^d} |u^*(t, x)|. \) Let \( 0 < \varepsilon < 1/2 \) and put

\[
H_\varepsilon(t, x) = \int_0^{1-\varepsilon} (1 - r^\alpha)^{-1/\alpha} \frac{d\beta}{\beta} \int_{\mathbb{R}^d} p(1 - r^\alpha, x, rw)|u^*(r^\alpha t, w)| \, dw, \\
h_\varepsilon(t, x) = \int_{1-\varepsilon}^{1} (1 - r^\alpha)^{-1/\alpha} \frac{d\beta}{\beta} \int_{\mathbb{R}^d} p(1 - r^\alpha, x, rw)|u^*(r^\alpha t, w)| \, dw.
\]

We also let \( H(t, x) = H_\varepsilon(t, x) + h_\varepsilon(t, x) \). By Theorem 3.4 and the Duhamel formula (2.14), we have

\[
|u^*(t, x)| \leq P_t^*|u_0|(x) + c\kappa^q H(t, x).
\]

By (2.6),

\[
P_t^*|u_0|(x) = t^{3/\alpha} \int_{\mathbb{R}^d} |u_0(y)| p(t, t^{1/\alpha} x, y) \, dy \lesssim t^{(3-1)/\alpha} \int_{\mathbb{R}^d} |u_0(y)| p^{(d-1)}(t, t^{1/\alpha} \tilde{x}, \tilde{y}) \, dy.
\]

Next, applying (3.3) we get

\[
H_\varepsilon(t, x) \lesssim \int_0^{1-\varepsilon} (1 - r)^{-2/\alpha} t^{(\beta-1)/\alpha} \int_{\mathbb{R}^d} |u_0(y)| p^{(d-1)}(t, t^{1/\alpha} \tilde{x}, \tilde{y}) \, dy \, dr \\
= \frac{(2 - \alpha)}{\alpha} \varepsilon^{-2/\alpha} t^{(\beta-1)/\alpha} \int_{\mathbb{R}^d} |u_0(y)| p^{(d-1)}(t, t^{1/\alpha} \tilde{x}, \tilde{y}) \, dy.
\]

Hence, it is enough to show that for some \( \varepsilon > 0, \)

\[
h_\varepsilon(t, x) \leq C (P_t^*|u_0|(x) + H_\varepsilon(t, x))
\]

holds with some \( C = C(\alpha, \beta, d, |b|, \mu, \varepsilon) \). By virtue of (2.6) and (3.3),

\[
h_\varepsilon(t, x) \leq \int_{1-\varepsilon}^{1} (1 - r^\alpha)^{-1/\alpha} \frac{d\beta}{\beta} \int_{\mathbb{R}^d} p(1 - r^\alpha, x, rw) \left( P_{rt^\alpha}^*|u_0|(w) + c\kappa^q H(r^\alpha t, w) \right) \, dw \, dr \\
=: I_1 + c\kappa^q I_2.
\]

Now, by the Tonelli theorem, (2.3) and the semigroup properties of \( p, \)

\[
I_1 = \int_{1-\varepsilon}^{1} (1 - r^\alpha)^{-1/\alpha} \frac{d\beta}{\beta} \int_{\mathbb{R}^d} p(1 - r^\alpha, x, rw) P_{rt^\alpha}^*|u_0|(w) \, dw \, dr \\
= \int_{1-\varepsilon}^{1} (1 - r^\alpha)^{-1/\alpha} \frac{d\beta}{\alpha} \int_{\mathbb{R}^d} p(1 - r^\alpha, x, rw) \int_{\mathbb{R}^d} p(r^\alpha t, rt^{1/\alpha} w, y) |u_0(y)| \, dy \, dw \, dr \\
= \int_{1-\varepsilon}^{1} (1 - r^\alpha)^{-d/\beta} t^{d/\alpha} \int_{\mathbb{R}^d} |u_0(y)| \int_{\mathbb{R}^d} p(r^\alpha - 1, 1, r^{-1} x, (rt^{1/\alpha})^{-1} y) \, dw \, dr \\
= \int_{1-\varepsilon}^{1} (1 - r^\alpha)^{-d/\beta} t^{d/\alpha} \int_{\mathbb{R}^d} |u_0(y)| p(r^{-\alpha}, r^{-1} x, (rt^{1/\alpha})^{-1} y) \, dy \, dr
\]
\[
= \int_{1-\varepsilon}^{1} (1 - r^{\alpha})^{-1/\alpha} \frac{v^\beta}{\alpha} \int_{\mathbb{R}^d} p(t, t^{1/\alpha} x, y) |u_0(y)| dy \, dr \leq \frac{\alpha \varepsilon^{(\alpha-1)/\alpha}}{\alpha - 1} P_t^\varepsilon |u_0|(x).
\]

Next,
\[
I_2 = \int_{1-\varepsilon}^{1} (1 - r^{\alpha})^{-1/\alpha} r^{d-\beta} \int_{\mathbb{R}^d} p(1 - r^{\alpha}, x, r w) H(r^{\alpha} t, w) \, dw \, dr
\]
\[
= \int_{1-\varepsilon}^{1} (1 - r^{\alpha})^{-1/\alpha} r^{d-\beta} \int_{\mathbb{R}^d} p(1 - r^{\alpha}, x, r w)
\times \int_{0}^{1} (1 - s^{\alpha})^{-1/\alpha} s^{d-\beta} \int_{\mathbb{R}^d} p(1 - s^{\alpha}, w, s y) |u^*(r^{\alpha} s^\alpha t, w)| \, dy \, ds \, dw \, dr
\]
\[
= \int_{1-\varepsilon}^{1} \int_{0}^{1} (1 - r^{\alpha})^{-1/\alpha} (1 - s^{\alpha})^{-1/\alpha} r^{d-\beta} \int_{\mathbb{R}^d} p(1 - (rs)^{\alpha}, x, rs y) |u^*((rs)^\alpha t, w)| \, dy \, ds \, dr.
\]
Substituting \( s = v/r \) leads to
\[
I_2 = \int_{1-\varepsilon}^{1} \int_{0}^{r} \frac{(1 - (v/r)^{\alpha})^{-1/\alpha}}{r(1 - r^{\alpha})^{1/\alpha}} r^{d-\beta} \int_{\mathbb{R}^d} p(1 - v^{\alpha}, x, v y) |u^*(v^{\alpha} t, w)| \, dy \, dr \, dv.
\]
By changing the order of integration we get
\[
\int_{1-\varepsilon}^{1} \int_{0}^{r} \int_{\mathbb{R}^d} (\ldots) \, dy \, dv \, dr = \int_{0}^{1} \int_{1-\varepsilon}^{1} \int_{\mathbb{R}^d} (\ldots) \, dy \, dr \, dv
\]
\[
+ \int_{1-\varepsilon}^{1} \int_{r}^{\infty} \int_{\mathbb{R}^d} (\ldots) \, dy \, dr \, dv = J_1 + J_2.
\]
Since for \( v \in (0, 1 - \varepsilon) \)
\[
\int_{1-\varepsilon}^{1} (1 - (v/r)^{\alpha})^{-1/\alpha} r^{d-\beta} \, dr \leq \int_{1-\varepsilon}^{1} \frac{dr}{(1 - r^{\alpha})^{1/\alpha} (r^{\alpha} - (1 - \varepsilon)^{\alpha})^{1/\alpha}} = c_{\varepsilon} \leq c_{\varepsilon} (1 - v^{\alpha})^{-1/\alpha},
\]
we obtain
\[
J_1 \leq c_{\varepsilon} H_\varepsilon(t, x),
\]
for some constant \( c_{\varepsilon} \) depending on \( \varepsilon \) and \( \alpha \). In order to estimate \( J_2 \), let us recall \([12, \text{Lemma } 4.3]\), which says that for \( v \in (1 - \varepsilon, 1) \),
\[
\int_{v}^{1} (1 - (v/r)^{\alpha})^{-1/\alpha} r^{d-\beta} \, dr = \int_{v}^{1} (1 - r^{\alpha})^{-1/\alpha} (r^{\alpha} - v^{\alpha})^{-1/\alpha} \, dr \leq c \varepsilon^{(\alpha-1)/\alpha} (1 - v^{\alpha})^{-1/\alpha},
\]
where \( c \) does not depend on \( \varepsilon \) and \( v \). Hence, \( J_2 \leq c \varepsilon^{(\alpha-1)/\alpha} h_\varepsilon(t, x) \) and consequently
\[
h_\varepsilon(t, x) \leq c(\varepsilon) (P_t^\varepsilon |u_0|(x) + H_\varepsilon(t, x)) + c \varepsilon^{(\alpha-1)/\alpha} h_\varepsilon(t, x),
\]
for some \( c, c(\varepsilon) > 0 \), where \( c \) does not depend on \( \varepsilon \). Finally, taking sufficiently small \( \varepsilon \) we get
\[
h_\varepsilon(t, x) \leq \frac{c(\varepsilon)}{1 - c \varepsilon^{(\alpha-1)/\alpha}} (P_t^\varepsilon |u_0|(x) + H_\varepsilon(t, x)),
\]
which yields \([3,7]\), what was to be shown. \[\square\]
**Proof of Theorem 1.7** Let \( u_0^{(k,n)} = u_0 \mathbf{1}_{B(0,k)} \mathbf{1}_{\{ -n \leq u_0 \leq k \}} \). Then, the functions \( u_0^{(k,n)} \in L^1 \cap L^\infty \) and, by (1.3), there exists a double sequence of solutions \( u^{(k,n)}(t,x) \) to the problems

\[
\begin{align*}
& u_t = \Delta^{a/2} u + b \cdot \nabla (u^{1+q}) , \quad t > 0, \ x \in \mathbb{R}^d, \\
& u(0,x) = u_0^{(k,n)}(x).
\end{align*}
\]

By (2.3) and (A), each of them satisfies

\[
|u^{(k,n)}(t,x)| \leq C t^{-1/\alpha} \int_{\mathbb{R}^d} |u_0(y)| p^{(d-1)}(t,\bar{x},\bar{y}) dy \leq C t^{-\beta/\alpha},
\]

where \( C > 0 \) does not depend on \( k \) and \( n \). Thus, due to the monotonicity property (2.12), the sequence \( (u^{(k,n)})_{k \geq 1} \) is non-decreasing. This ensures existence of the limit \( u^{(\infty,n)}(t,x) := \lim_{k \to \infty} u^{(k,n)}(t,x) \). Similarly, by (2.12) \( (u^{(\infty,n)})_{n \geq 1} \) is decreasing, hence there exists the limit \( u(t,x) := \lim_{n \to \infty} u^{(\infty,n)}(t,x) \). We will show that this is the solution we are looking for.

In view of (3.8), \( u(t,x) \) satisfies (1.7). It remains to prove that \( u \) satisfies (1.2) as well. Clearly \( u(0,x) = u_0(x) \). Since all \( u^{(k,n)} \) satisfy (1.2), we have

\[
u(t,x) = \lim_{n \to \infty} \lim_{k \to \infty} \left( P_t u_0^{(k,n)}(x) + \int_0^t \int_{\mathbb{R}^d} b \cdot \nabla_x p(t-s,x,z)[u^{(k,n)}(s,z)]^{1+q} dz ds \right).
\]

By (2.3),

\[
|p(t,x,y)u_0^{(k,n)}(y)| \leq t^{-1/\alpha} p^{(d-1)}(t,\bar{x},\bar{y}) |u_0(y)|.
\]

Since the right-hand side is integrable thanks to the condition (A), the dominated convergence theorem gives us \( \lim_{n \to \infty} \lim_{k \to \infty} P_t u_0^{(k,n)}(x) = P_t u_0(x) \), and therefore we only need to show that we can pass with the limit under the integral in (3.9). By Theorem 3.4

\[
|b \cdot \nabla_x p(t-s,x,z)u^{(k,n)}(s,z)|^{1+q} \leq C |b(t-s)^{-1/\alpha} s^{-(\beta+\alpha-1)/\alpha} p(t-s,x,z) | C > 0
\]

independent of \( k \) and \( n \). Thus, for every \( s \in (0,t) \),

\[
\lim_{n \to \infty} \lim_{k \to \infty} \int_{\mathbb{R}^d} b \cdot \nabla_x p(t-s,x,z)[u^{(k,n)}(s,z)]^{1+q} dz = \int_{\mathbb{R}^d} b \cdot \nabla_x p(t-s,x,z)u(s,z)^{1+q} dz.
\]

Like in the proof of Lemma 3.3 let \( \bar{u}^{(k,n)} \) be the solution to (1.1) with the initial condition \( |u_0^{(k,n)}| \). Clearly, \( |u^{(k,n)}| \leq \bar{u}^{(k,n)} \). By Theorem 3.4 \( \bar{u}^{(k,n)}(s,z) \leq c s^{-\beta/\alpha} \) with \( c \) not depending on \( k \) and \( n \). Hence, by (2.9), (2.3), substituting \( z = s^{1/\alpha} w \) and then using Lemma 3.3 and the assumption (A), for \( s \in (0,t/2) \) we get

\[
\left| \int_{\mathbb{R}^d} b \cdot \nabla_x p(t-s,x,z)[u^{(k,n)}(s,z)]^{1+q} dz \right| \leq |b|(t-s)^{-1/\alpha} s^{-(\beta+\alpha-1)/\alpha} \int_{\mathbb{R}^d} p(t-s,x,z)(\bar{u}^{(k,n)})^*(s,s^{-1/\alpha} z) dz
\]

\[
= |b|(t-s)^{-1/\alpha} s^{(d+1-\beta-\alpha)/\alpha} t^{-d/\alpha} \int_{\mathbb{R}^d} p(1- (s/t), t^{-1/\alpha} x, (s/t)^{1/\alpha} w)(\bar{u}^{(k,n)})^*(s,w) dw
\]

\[14\]
\[ \lesssim |b| t^{-1/\alpha} s^{(d+1-\beta-\alpha)/\alpha} t^{-d/\alpha} \frac{(s/t)^{(\beta-d)/\alpha} t^{(\beta-1)/\alpha}}{(1 - (s/t)^{1/\alpha})^{1/\alpha} t^{(\beta-1)/\alpha}} \int_{\mathbb{R}^d} p^{(d-1)}(t, \tilde{x}, \tilde{y}) |u_0(y)| dy \]
\[ \leq |b| t^{-(\beta+1)/\alpha} \mu (1 - 1/2^{1/\alpha})^{1/\alpha} s^{-(\alpha-1)/\alpha}. \]

On the other hand, for \( s \in [t/2, t] \), by Theorem 3.4
\[ \left| \int_{\mathbb{R}^d} b \cdot \nabla_x p(t - s, x, z) [u^{(k,n)}(s, z)]^{(1+q)} dz \right| \lesssim |b|(t - s)^{-1/\alpha} s^{-(\beta+\alpha-1)/\alpha} \int_{\mathbb{R}^d} p(t - s, x, z) dz \]
\[ \lesssim |b| t^{-(\beta+\alpha-1)/\alpha} (t - s)^{-1/\alpha}. \]

Since \( \int_0^t (s^{-(\alpha-1)/\alpha} \vee (t - s)^{-1/\alpha}) ds < \infty \), we apply the dominated convergence theorem once again and get
\[
\lim_{n \to \infty} \lim_{k \to \infty} \int_0^t \int_{\mathbb{R}^d} b \cdot \nabla_x p(t - s, x, z) [u^{(k,n)}(s, z)]^{(1+q)} dz ds = \int_0^t \int_{\mathbb{R}^d} b \cdot \nabla_x p(t - s, x, z) u(s, z)^{(1+q)} dz ds,
\]
which ends the proof. \( \square \)

4 The self-similar solution

This section is devoted to the proof of Theorem 1.2, where the initial condition \( u_0(x) = M|x|^{-\beta}, \ 1 < \beta < d \) and \( M > 0 \) is considered.

4.1 Existence and selfsimilarity

Let us start with the following observation on how scaling of the initial condition is transferred to scaling of the solution.

**Lemma 4.1.** If \( u \) is a solution to (1.1) with initial condition \( u_0 \), then \( v(t, x) = \lambda^{\beta/\alpha} u(\lambda t, \lambda^{1/\alpha} x) \) is a solution with the initial condition \( v_0(x) = \lambda^{\beta/\alpha} u_0(\lambda^{1/\alpha} x) \).

**Proof.** By scaling property (2.3), we have
\[ \lambda^{\beta/\alpha} P_t u_0(\lambda^{1/\alpha} x) = \lambda^{\beta/\alpha} \int_{\mathbb{R}^d} p(\lambda t, \lambda^{1/\alpha} x, y) u_0(y) dy \]
\[ = \int_{\mathbb{R}^d} \lambda^{d/\alpha} p(\lambda t, \lambda^{1/\alpha} x, \lambda^{1/\alpha} w) \lambda^{\beta/\alpha} u_0(\lambda^{1/\alpha} w) dw = P_t v_0(x). \]
Similarly,
\[ \lambda^{\beta/\alpha} \int_0^M \int_{\mathbb{R}^d} b \cdot \nabla_x p(\lambda t - s, \lambda^{1/\alpha} x, y) u^{q+1}(s, y) dy ds \]
\[ = \int_0^t \int_{\mathbb{R}^d} \lambda^{d/\alpha+1} b \cdot \nabla_x p(\lambda t - \lambda r, \lambda^{1/\alpha} x, \lambda^{1/\alpha} w) \lambda^{(1-\alpha)/\alpha} \lambda^{\beta(q+1)/\alpha} u^{q+1}(\lambda r, \lambda^{1/\alpha} w) dw dr \]
\[
\int_0^t \int_{\mathbb{R}^d} b \cdot \nabla_x p(t - r, x, w) v(r, w) \, dw \, dr.
\]

Hence,

\[
v(t, x) = \lambda^{\beta/\alpha} u(\lambda t, \lambda^{1/\alpha} x) = P_t v_0(x) + \int_0^t \int_{\mathbb{R}^d} b \cdot \nabla_x p(t - r, x, w) v(r, w) \, dw \, dr,
\]
as required.

Next we verify the condition (A) in the case \( u_0(x) = M|x|^{-\beta} \).

**Lemma 4.2.** The function \( u_0(x) = M|x|^{-\beta}, M > 0 \), satisfies the condition (A).

**Proof.** Since

\[
\int_{\mathbb{R}} |y|^{-\beta} \, dy_1 \leq \int_{\mathbb{R}} \left[ \frac{1}{2} (|\tilde{y}| + |y_1|) \right]^{-\beta} \, dy_1 = \frac{2^{\beta+1}}{\beta - 1} |\tilde{y}|^{1-\beta},
\]  

(4.1)

Lemma 2.3 gives us

\[
\int_{\mathbb{R}^d} u_0(y) p^{(d-1)}(t, \tilde{x}, \tilde{y}) \, d\tilde{y} \lesssim \int_{\mathbb{R}^{d-1}} |\tilde{y}|^{1-\beta} p^{(d-1)}(t, \tilde{x}, \tilde{y}) \, d\tilde{y} \lesssim (t^{1/\alpha} + |\tilde{x}|)^{1-\beta},
\]  

(4.2)

which yields (A).

Throughout the whole section we exploit an analogous notation to the one from the proof of Theorem 1.1. Namely, \( u^{(n)}, n \geq 1 \), is defined as the solution to the problem (1.1) with

\[
u_0^{(n)}(x) = \left( n \wedge \frac{M}{|x|^\beta} \right) 1_{B(0,n)}(x),
\]  

(4.3)

and \( u(t, x) := \lim_{n \to \infty} u^{(n)}(t, x), x \in \mathbb{R}^d \). The initial condition is positive in this case, so there is no need to consider the double sequence \( u_0^{(k,n)} \).

**Lemma 4.3.** For any \( \lambda > 0 \) we have

\[
u(t, x) = \lambda^{\beta/\alpha} u(\lambda^{\alpha} t, \lambda x), \quad t > 0, x \in \mathbb{R}^d.
\]  

(4.4)

**Proof.** It suffices to consider \( \lambda > 1 \). Indeed, if we substitute \( t = s/\lambda^\alpha \) and \( x = y/\lambda \) in (1.4), we get the result for \( \lambda < 1 \). We note that

\[
\lambda^{\beta} u(\lambda^{\alpha} t, \lambda x) = \lim_{n \to \infty} \lambda^{\beta} u^{(n)}(\lambda^{\alpha} t, \lambda x),
\]  

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where, by Lemma 4.1, \( \lambda^\beta u^{(n)}(\lambda^\alpha t, \lambda x) \) is the solution to the problem (1.1) with the rescaled initial condition \( \lambda^\beta u_0^{(n)}(\lambda x) = \left( \lambda^\beta n \wedge \frac{M}{|\lambda x|} \right) 1_{B(0,n/\lambda)}. \) Observe that for \( i_n = \lfloor n/\lambda \rfloor \) and \( j_n = \lceil n/\lambda \rceil, \)

\[
    u_0^{(i_n)}(x) \leq \lambda^\beta u_0^{(n)}(\lambda x) \leq u_0^{(j_n)}(x), \quad x \in \mathbb{R}^d.
\]

Here \( \lfloor \cdot \rfloor \) and \( \lceil \cdot \rceil \) are the standard floor and ceiling functions, respectively. Hence, by the monotonicity property (2.12), we get

\[
    u^{(i_n)}(t, x) \leq \lambda^\beta u^{(n)}(\lambda^\alpha t, \lambda x) \leq u^{(j_n)}(t, x), \quad t > 0, \ x \in \mathbb{R}^d,
\]

which yields \( \lim_{n \to \infty} \lambda^\beta u^{(n)}(\lambda^\alpha t, \lambda x) = u(t, x) \), as required.

In order to study the self-similar solution from Lemma 4.3, it is clearly enough to study the case \( t = 1 \). Namely, denoting \( U(x) := u(1, x) \) we get

\[
    u(t, x) = t^{-\beta/\alpha} U(t^{-1/\alpha} x).
\]

In particular, this implies \( u^*(t, x) = U(x) \), and the Duhamel formula (2.14) takes the form

\[
    U(x) = M \int_{\mathbb{R}^d} p(1, x, y) |y|^{-\beta} \, dy + \alpha \int_0^1 \int_{\mathbb{R}^d} r^{d-\beta} \nabla_x p(1-r^\alpha, x, rw) U(w)^{q+1} \, dw \, dr.
\]

### 4.2 Pointwise estimates and asymptotics

From Proposition 3.5 and the estimate (4.2) we have the following upper bound

\[
    U(x) \lesssim \frac{1}{(1 + |x|)^{\beta-1}}.
\]

However, it is not optimal. In particular, the right-hand side does not depend on \( x_1 \). In the sequel, we provide precise two-sided estimates. We start with \( L^p \) bounds.

**Lemma 4.4.** For any \( \gamma \in (d/\beta, \infty) \), \( U \in L^\gamma(\mathbb{R}^d) \).

**Proof.** For \( \gamma = \infty \) the result follows by (4.7). Furthermore, by monotonicity of \( u^{(n)} \), (4.7), Corollary 3.2 and Lemma 2.3 for \( \gamma \in ((d-1)/\beta - 1), \infty \) we have

\[
    \int_{\mathbb{R}^d} U^{\gamma}(x) \, dx = \lim_{n \to \infty} \int_{\mathbb{R}^d} [u^{(n)}(1, x)]^{\gamma} \, dx
\]

\[
    \lesssim \lim_{n \to \infty} \int_{\mathbb{R}^d} \frac{1}{(1 + |\tilde{x}|)^{(\beta-1)(\gamma-1)}} u^{(n)}(1, x) \, dx
\]

\[
    = \lim_{n \to \infty} \int_{\mathbb{R}^d} \frac{1}{(1 + |\tilde{x}|)^{(\beta-1)(\gamma-1)}} P_1(u_0^{(n)})(x) \, dx
\]

\[
    \lesssim \int_{\mathbb{R}^{d-1}} \frac{1}{(1 + |\tilde{x}|)^{(\beta-1)(\gamma-1)}} \frac{1}{(1 + |\tilde{x}| + |x_1|)^{\beta}} \, d\tilde{x} < \infty.
\]
Now, let $\gamma \in (\frac{d}{\beta}, \frac{d-1}{\beta-1})$. By the integral Minkowski inequality, we have
\[
\|U\|_\gamma \lesssim \|P_1 u_0\|_\gamma
+ \left( \int_{\mathbb{R}^d} \left( \int_0^1 \int_{\mathbb{R}^d} r^{d-\beta} p(1-r, rw)[U(w + \frac{x}{r})]^{1+q} dw \right)^{\gamma} dx \right)^{1/\gamma}
\leq c_\gamma + \|U^{1+q}\|_\gamma \int_0^1 \int_{\mathbb{R}^d} r^{d-\beta + d/\gamma} p(1-r, rw) dw dr
= c_\gamma + \|U\|^{1+q}_{\gamma(1+q)} \int_0^1 r^{-\beta + d/\gamma} (1-r)^{-1/\alpha} dr \leq c_0 \left( 1 + \|U\|^{1+q}_{\gamma(1+q)} \right). \tag{4.9}
\]
Thus, for every $m \in \mathbb{N}$ such that $\gamma(1+q)^m > (d-1)/(\beta-1)$, by iterating (4.9) we get
\[
\|U\|_\gamma \leq c_m \left( 1 + \|U\|^{(1+q)^{m+1}}_{\gamma(1+q)^{m+1}} \right). \tag{4.10}
\]
We take the smallest $m \in \mathbb{N}$ such that that $\gamma(1+q)^{m+1} > (d-1)/(\beta-1)$. Then, by (4.10) and (4.8), $U \in L^\gamma(\mathbb{R}^d)$.

The next step is to show that $U$ vanishes at infinity.

**Lemma 4.5.** We have
\[
\lim_{R \to \infty} \sup_{|x| > R} U(x) = 0.
\]

**Proof.** Estimates (2.4) and Lemma 2.3 applied to (4.6) give us
\[
U(x) \lesssim \frac{M}{(1+|x|)^\beta} + \int_0^1 \int_{\mathbb{R}^d} r^{d-\beta} \frac{p(1-r, x, rw)}{(1-r)^{1/\alpha} + |x-rw|} U(w)^{q+1} dw dr. \tag{4.11}
\]
Let us split the integral above into $I_1 + I_2 := \int_0^\varepsilon (\ldots)dr + \int_\varepsilon^1 (\ldots)dr$, for some $\varepsilon \in (0, 1/2)$. First, note that
\[
\int_{\mathbb{R}} |y|^{-\beta} dy_1 = \int_{\mathbb{R}} (y_1^2 + |\tilde{y}|^2)^{-\beta/2} dy = c|\tilde{y}|^{1-\beta}.
\]
Hence, by (3.3), (4.1) and Lemma 2.3 we have
\[
I_1 \lesssim \|U\|_\infty^q \int_0^\varepsilon (1-r)^{-2/\alpha} \int_{\mathbb{R}^d} p^{(d-1)}(1, \tilde{x}, \tilde{y}) |y|^{-\beta} dy
\lesssim \varepsilon \|U\|_\infty^q \int_{\mathbb{R}^{d-1}} p^{(d-1)}(1, \tilde{x}, \tilde{y}) |\tilde{y}|^{-1-\beta} d\tilde{y} \lesssim \varepsilon \|U\|_\infty^q.
\]
Next, using Hölder inequality and the bound (2.3) we get
\[
I_2 \lesssim \int_\varepsilon^1 r^{d-\beta} \int_{B(0, |x|/2)} \frac{p(1-r, x, rw)}{|x|} \|U\|_\infty^{q+1} dw dr
\]
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\[
\begin{align*}
&+ \int_{\varepsilon}^{1} \frac{r^{d-\beta}}{(1-r)^{1/\alpha}} \int_{B(0,|x|/2)^c} p(1-r, x, rw) U(w) \|U\|_{q_\infty}^{q} \, dw \, dr \\
&\leq \frac{\|U\|_{q_\infty}^{q+1}}{|x|} \int_{\varepsilon}^{1} r^{-\beta} \, dr \\
&\quad + \|U\|_{q_\infty}^{q} \int_{\varepsilon}^{1} \frac{r^{d-\beta}}{(1-r)^{1/\alpha}} \|p(1-r, x, r(\cdot))\|_{2d/(2d-\alpha+1)} \|1_{(B(0,|x|/2)^c)} U\|_{2d/(\alpha-1)} \, dr \\
&\lesssim \frac{\|U\|_{q_\infty}^{q+1}}{\varepsilon^\beta |x|} + \|U\|_{q_\infty}^{q} \|1_{(B(0,|x|/2)^c)} U\|_{2d/(\alpha-1)} \int_{\varepsilon}^{1} \frac{r^{(\alpha-1)/2-\beta}}{(1-r)^{1-(\alpha-1)/2\alpha}} \, dr,
\end{align*}
\]

which tends to zero as \(|x| \to \infty\) by virtue of Lemma 4.3 applied with \(\gamma = 2d/(\alpha-1) > d/\beta\). Thus, the integral in (4.11) is arbitrarily small for large \(|x|\), which ends the proof. \(\square\)

We are now ready to derive the upper bound of \(U\).

**Proposition 4.6.** There is a constant \(C = C(d, \alpha, b, M, \beta)\) such that

\[
U(x) \leq \frac{1}{(1 + |x|)^\beta}, \quad x \in \mathbb{R}^d.
\]

**Proof.** Recall that \(u^{(n)}\) is a solution to the problem (1.1) with initial condition given by (4.3) and \((u^{(n)})^*(t, x)\) is defined by (2.1). Note also that \((u^{(n)})^*(t, x) \nearrow u^*(t, x) = U(x)\). From (1.3) and Lemma 2.3 we have

\[
A_n := \sup_{t > 0} \sup_{x \in \mathbb{R}^d} (u^{(n)})^*(t, x)(1 + |x|)^\beta < \infty.
\]

It is enough to show that \(A_n\)'s are uniformly bounded. Duhamel formula (2.14) combined with (2.3) give us

\[
(u^{(n)})^*(t, x) \lesssim C_0 \left( \frac{1}{(1 + |x|)^\beta} + I_n(t, x) \right),
\]

where \(C_0 = C_0(d, \alpha, b, M) > 0\) is some constant and

\[
I_n(t, x) = \int_{0}^{1} \int_{\mathbb{R}^d} r^{d-\beta} \frac{p(1-r^\alpha, x, rw)}{(1-r^\alpha)^{1/\alpha} + |x-rw|} \left[ (u^{(n)})^*(r^\alpha t, w) \right]^{(\alpha+1)/\beta} \, dw \, dr.
\]

Let \(0 < \varepsilon < 1/2\). By Lemma 4.5 we may and do choose \(R > 0\) such that \(|U(z)| < \varepsilon^{\beta/(\alpha-1)}\) for \(|z| \geq R\) and such that \(||U||_{q_\infty}(\alpha-1)/|x| < \varepsilon||x| > 2R\).

Now let \(|x| > 2R\). Since \(|x-rw| \geq |x|/2\) for \(|w| < R\) and \(0 < r < 1\), we get

\[
I_n(t, x) \leq \int_{0}^{1} \int_{B(0,R)} r^{d-\beta} p(1-r^\alpha, x, rw) \frac{(u^{(n)})^*(r^\alpha t, w)}{|x|/2} \, dw \, dr \\
+ \int_{0}^{1} \int_{B(0,R)^c} r^{d-\beta} \frac{p(1-r^\alpha, x, rw)}{(1-r^\alpha)^{1/\alpha}} \left( u^{(n)} \right)^{*}(r^\alpha t, w) \left( \sup_{|z| \geq R} U(z) \right)^{(\alpha-1)/\beta} \, dw \, dr.
\]

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\[ \leq \varepsilon A_n \left( 2\|U\|_\infty^{(\alpha-1)/\beta} + 1 \right) \int_0^1 \int_{\mathbb{R}^d} r^{d-\beta} (1 - r^\alpha)^{-1/\alpha} p(1 - r^\alpha, x, rw) \frac{dw}{(1 + |w|)^\beta} dr. \]

By (2.3) and Lemma 2.3,

\[ \int_{\mathbb{R}^d} p(1 - r^\alpha, x, rw) \frac{dw}{(1 + |w|)^\beta} = \int_{\mathbb{R}^d} r^{-\alpha} p(r^{-\alpha} - 1, r^{-1} x, w) \frac{dw}{(1 + |w|)^\beta} \approx \frac{r^{\beta-d}}{(1 + |x|)^\beta}. \]

Hence,

\[ I_n(t, x) \lesssim \frac{\varepsilon A_n}{(1 + |x|)^\beta} \int_0^1 (1 - r^\alpha)^{-1/\alpha} dr \approx \frac{\varepsilon A_n}{(1 + |x|)^\beta}. \]

On the other hand, for \(|x| \leq 2R|\),

\[ (u^{(n)})^*(t, x) \leq U(x) \leq \|U\|_\infty \leq \frac{\|U\|_\infty (1 + 2R)^\beta}{(1 + |x|)^\beta}. \]

Finally, there is a constant \(c = c(d, \alpha, b, M, \beta)\) such that for any \(t > 0\) and \(x \in \mathbb{R}^d\) we have

\[ (u^{(n)})^*(t, x) \leq \frac{c}{(1 + |x|)^\beta} \left( 1 + \varepsilon A_n + \|U\|_\infty (1 + 2R)^\beta \right), \]

and consequently

\[ A_n \leq c(1 + \varepsilon A_n + \|U\|_\infty (1 + 2R)^\beta). \]

Taking \(\varepsilon = 1/(2c)\), we obtain

\[ A_n \leq 2(1 + \|U\|_\infty (1 + 2R)^\beta), \]

as required. \(\square\)

Next we provide an asymptotic behaviour of \(U\), which is driven by the functions \(h_n\) below. First, let us define the kernel

\[ k(x, w) = \alpha \int_0^1 r^{d-\beta} \nabla_x p(r^{\alpha}, x, rw) dr, \]

and then, for a function \(f : \mathbb{R}^d \to \mathbb{R}\), the operator

\[ (Kf)(x) = P_1 u_0(x) + \int_{\mathbb{R}^d} k(x, w) f(w)|f(w)|^q dw. \]

In view of (4.6), we clearly have \(KU = U\). Furthermore, we put \(h_0 \equiv 0\) and

\[ h_n(x) = K^n h_0(x), \quad x \in \mathbb{R}^d, n \geq 1. \]

In particular,

\[ h_1(x) = P_1 u_0(x), \]

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Indeed, by \((4.12)\) we get
\[
h_2(x) = P_1u_0(x) + \int_{\mathbb{R}^d} k(x, w)[P_1u_0(w)]^{q+1} dw.
\]

One can show inductively that for any \(n \geq 0\) there is a constant \(c_n > 0\) such that
\[
|h_n(x)| \leq \frac{c_n}{(1 + |x|)^\beta}, \quad x \in \mathbb{R}^d.
\] (4.13)

Indeed, by \((4.12)\) we get
\[
|h_{n+1}(x)| = |Kh_n(x)|
\]
\[
\lesssim P_1u_0(x) + \int_0^1 \int_{\mathbb{R}^d} r^{d-\beta}|\nabla_x p(1 - r^\alpha, x, rw)||h_n(w)|^{q+1} dw \, dr
\]
\[
\lesssim \frac{1}{(1 + |x|)^\beta} + c_n^{q+1} \int_0^1 \int_{\mathbb{R}^d} \frac{r^{d-\beta}}{(1 - r)^{1/\alpha}} \frac{p(1 - r, x, rw)}{(1 + |w|)^\beta} \, dw \, dr
\]
\[
\lesssim \frac{c_n^{q+1} + 1}{(1 + |x|)^\beta}.
\]

Note that the functions \(h_n\) might be considered as Picard approximations of \(U\). The next result supplements Proposition 4.6 precisely describing behaviour of \(U(x)\) for large \(|x|\). Although asymptotics of the first two orders only are mentioned in Theorem 1.2, we provide below more general result, as the arguments are similar.

**Proposition 4.7.** For every \(n \geq 0\) we have
\[
|U(x) - h_n(x)| \lesssim (1 + |x|)^{-|\beta+n(\alpha-1)|\wedge[d+\alpha+1]}.
\]

**Proof.** We will use induction. For \(n = 0\) the assertion follows from Theorem 4.6. Consider some \(n \geq 0\). Since \(U\) and every \(h_n\) are bounded, we may focus only on large \(|x|\).

Recall that for any \(a \in \mathbb{R}\) and \(\gamma > 1\) we denote \(a^{(\gamma)} = a|a|^{\gamma-1}\). Note that \(\frac{d}{dx}x^{(\gamma)} = \gamma|x|^{\gamma-1}\). Hence, for any \(a, b \in \mathbb{R}\) and \(\gamma > 0\) we have
\[
|b^{(\gamma+1)} - a^{(\gamma+1)}| = \left| \int_a^b \frac{d}{dx} x^{(\gamma+1)} \, dx \right| = (\gamma + 1) \left| \int_a^b |x|^{\gamma} \, dx \right|
\]
\[
\leq (\gamma + 1)|b - a|||a| + |b||^{\gamma}.
\] (4.14)

Thus, by Theorem 4.6, (4.13) and (4.14)
\[
|U(x) - h_{n+1}(x)|
\]
\[
\lesssim \int_{\mathbb{R}^d} |k(x, w)||U(w) - h_n(w)||(|U(w)| + |h_n(w)|)^q \, dw
\]
\[
\lesssim C_n \int_{\mathbb{R}^d} |k(x, w)| \frac{1}{(1 + |w|)^{\beta+n(\alpha-1)|\wedge[d+\alpha+1]}} \frac{1}{(1 + |w|)^{\beta}} \, dw \, dr
\]
\[
\lesssim \int_0^1 r^{d-\beta} \int_{\mathbb{R}^d} \frac{p(1 - r^\alpha, x, rw)}{(1 - r^\alpha)^{1/\alpha} + |x - rw|} \frac{1}{(1 + |w|)^{\beta+n(\alpha-1)|\wedge[d+2\alpha]}} \, dw \, dr.
\] 21
Let \( a_n = \beta + (n + 1)(\alpha - 1) \). In the case \( a_n < d \), we bound \(|x - rw|\) by zero and take advantage of (4.12). Assume therefore \( a_n \geq d \). Since \(|x - rw| \geq |x|/2\) for \( w \in B(0, |x|/2r) \) and \( 0 < r < 1 \), we split the inner integral above and estimate (applying (4.13)) it as follows

\[
\int_{\mathbb{R}^d} (...)dw \lesssim \int_{B(0,|x|/2r)} \left(1 - r^\alpha\right)^{1/\alpha} \frac{1 - r^\alpha}{|x|^{d+\alpha+1}} \frac{|x|^{d+\alpha+1}}{(1 + |w|)^{a_n\wedge[d+2\alpha]} dw \\
+ \int_{B(0,|x|/2r)^c} \left(1 - r^\alpha\right)^{1/\alpha} \frac{p(1 - r^\alpha, x, rw)}{1 + |x|^\alpha \Gamma(\alpha)} dw \right) \cdot \left(1 + \ln(1/r)\right) + \left(1 - r^\alpha\right)^{1/\alpha} |x|^{d+\alpha+1} \] 

Thus, we arrive at

\[
|U(x) - h_{n+1}(x)| \lesssim \frac{1}{|x|^{\beta + (n+1)(\alpha-1)\wedge[d+\alpha+1]}} \int_0^1 \frac{r^{d-\beta}(1 + \ln(1/r))}{(1 - r^\alpha)^{1/\alpha}} dr \\
\approx \frac{1}{|x|^{\beta + (n+1)(\alpha-1)\wedge[d+\alpha+1]}},
\]

which ends the proof.

\[\square\]

Proposition 4.7 directly implies the lower bound for \( U(x) \).

**Corollary 4.8.** We have \( U(x) \gtrsim \frac{1}{(1 + |x|)^\beta} \) for \( x \in \mathbb{R}^d \).

**Proof.** Proposition 4.7 with \( n = 1 \) and Lemma 2.3 applied to Proposition 4.6 imply the lower bound for large \(|x|\). For any fixed \( R > 0 \) and \(|x| < R\) we employ (1.8) and bound

\[
U(x) \gtrsim u^{(1)}(1, x) \gtrsim (M \wedge 1)(P_1 1_{B(0,1)})(x) \approx \int_{|w| < 1} \frac{dw}{(1 + |w - x|)^{d+\alpha}} \\
\gtrsim \frac{1}{(2 + R)^{d+\alpha}} \gtrsim \frac{1}{(2 + R)^{d+\alpha} (1 + |x|)^\beta},
\]

which proves the assertion.

\[\square\]

### 4.3 Gradient of \( U \)

Eventually, we turn our attention to the gradient of \( U \). Note that the estimates in (4.15) coincides with the ones of \( U \) from Proposition 4.7. Nevertheless, it is not true for the whole range of arguments of the solution \( u(t, x) \), as the scaling property of the gradient \( \nabla u(t, x) \) is slightly different, i.e. we have

\[
(\nabla_x u)(t, x) = \nabla_x (\lambda^\beta u(\lambda^\alpha t, \lambda x)) = \lambda^{\beta+1}(\nabla_x u)(\lambda^\alpha t, \lambda x),
\]

and consequently, for \( \lambda = t^{-1/\alpha} \), we have (cf. (4.31))

\[
(\nabla_x u)(t, x) = t^{-(\beta+1)/\alpha}(\nabla_x u)(1, x/t^{1/\alpha}) = t^{-(\beta+1)/\alpha}(\nabla U)(x/t^{1/\alpha}).
\]

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Proposition 4.9. We have \( U \in C^1(\mathbb{R}^d) \) and
\[
|\nabla U(x)| \lesssim \frac{1}{(1 + |x|)^\beta}.
\] (4.15)

Proof. By [7, Theorem 3.5], since \(|\nabla u^{(n)}_0(t, x)| \leq \frac{M\alpha}{|x|-\varepsilon}1_{\{1/|x|<\alpha\}}\), we have \(|\nabla u^{(n)}(t, x)| \leq c(1 + t)(1 + |x|)^{-d-\alpha-1}\), where \(c\) may depend on \(n\). Hence, for every \(n\) we have
\[
A_n := \sup_{t \leq 1} |t^{1/\alpha}(1 + |x|)^\beta \nabla u^{(n)}(t, x)| < \infty.
\]
We will show that
\[
A := \sup_{n \in \mathbb{N}} A_n < \infty.
\] (4.16)

From (1.2) and by applying integration by parts, for any \(\varepsilon \in (0, 1/2)\) we get
\[
\begin{align*}
\nabla u^{(n)}(t, x) &= \nabla P u^{(n)}_0(x) + \int_0^{(1-\varepsilon)t} \int_{\mathbb{R}^d} \nabla (b \cdot \nabla p(t - s, x, z)) [u^{(n)}(s, z)]^{q+1} dz ds \\
& \quad + (q + 1) \int_{(1-\varepsilon)t}^t \int_{\mathbb{R}^d} b \cdot \nabla z p(t - s, x, z) u^{(n)}(s, z) \nabla u^{(n)}(s, z) dz ds.
\end{align*}
\] (4.17)

Let us note that the first integrand is not absolutely integrable on \(((1-\varepsilon)t, t) \times \mathbb{R}^d\), which explains the above decomposition. We estimate the term \(\nabla P u^{(n)}_0(x)\) as follows
\[
|\nabla P u^{(n)}_0(x)| = \left| \nabla_x \int_{\mathbb{R}^d} p(t, x, y) u^{(n)}_0(y) dy \right| \lesssim t^{-1/\alpha} \int_{\mathbb{R}^d} |p(t, x, y)| y^{-\beta} dy \lesssim \frac{1}{t^{1/\alpha}(1 + |x|)^\beta},
\]
where we used (2.9) and Lemma 2.3. Next, applying this, (2.9), (4.5), and Proposition 4.6 to (4.17), we obtain for \(0 < t \leq 1\)
\[
|\nabla u^{(n)}(t, x)| \lesssim \frac{1}{t^{1/\alpha}(1 + |x|)^\beta} + \int_0^{(1-\varepsilon)t} \frac{1}{(t - s)^{2/\alpha}} \left( \frac{\|U\|_\infty}{s^{\beta/\alpha}} \right)^q P_{t-s}((s^{1/\alpha} + |\cdot|)^{-\beta})(x) ds \\
+ A_n \int_{(1-\varepsilon)t}^t \frac{1}{(t - s)^{1/\alpha}} \left( \frac{\|U\|_\infty}{s^{\beta/\alpha}} \right)^q s^{-1/\alpha} P_{t-s}((s^{1/\alpha} + |\cdot|)^{-\beta})(x) ds.
\]

By Lemma 2.3, \(P_{t-s}((s^{1/\alpha} + |\cdot|)^{-\beta})(x) \approx 1/(t^{1/\alpha} + |x|)^\beta\), and therefore
\[
|\nabla u^{(n)}(t, x)| \lesssim \frac{1}{t^{1/\alpha}(1 + |x|)^\beta} \times \left[ 1 + \int_0^{1-\varepsilon} \left( \frac{\|U\|_\infty^q}{(1 - r)^{2\alpha(\alpha-1)/\alpha}} \right) dr + A_n \int_{1-\varepsilon}^1 \frac{\|U\|_\infty^q}{(1 - r)^{1/\alpha} r} dr \right]
\]
\begin{align*}
\lesssim \frac{1}{t^{1/(2+|x|)}} [1 + \|U\|_g (\varepsilon^{-2/\alpha} + \varepsilon^{1-1/\alpha} A_n)].
\end{align*}

Consequently there is a constant \( c \) not depending on \( n \) and \( \varepsilon \) such that
\begin{align*}
A_n \leq c \left( 1 + \|U\|_g (\varepsilon^{-2/\alpha} + \varepsilon^{1-1/\alpha} A_n) \right).
\end{align*}

Thus, taking \( \varepsilon \) small enough, we get \( A_n \leq c \left( 1 + \varepsilon^{-2/\alpha} \|U\|_g^q / (1 - c) \|U\|_g^{q-1} \right) \), which proves (4.16). In particular, this implies
\begin{align*}
|U(x + z) - U(x)| = \lim_{n \to \infty} |u^{(n)}(1, x + z) - u^{(n)}(1, x)|
\leq |z| \sup_{0 \leq v \leq 1} |\nabla u^{(n)}(1, x + vz)| \lesssim \frac{A|z|}{(1 + |x|)^\beta},
\end{align*}
whenever \( |z| < 1 \). Let \( e_1, \ldots, e_n \) be the standard basis in \( \mathbb{R}^d \). Now, rewriting (4.6) in a similar manner as it was done in (4.17), for any \( \varepsilon \in (0, 1) \), \( 1 \leq k \leq d \) and \( h \in \mathbb{R} - \{0\} \) we get
\begin{align*}
\frac{U(x + he_k) - U(x)}{|h|} &= M \int_{\mathbb{R}^d} \frac{p(1, x + he_k - y) - p(1, x - y)}{|h|^{\beta}} dy \\
&+ \alpha \int_{0}^{1-\varepsilon} \int_{\mathbb{R}^d} r^{d-\beta} \cdot \nabla_w p(1 - r^{\alpha}, x + he_k, rw) - \nabla_w p(1 - r^{\alpha}, x, rw) [U(w)]^{q+1} dw dr \\
&+ \alpha \int_{1-\varepsilon}^{1} \int_{\mathbb{R}^d} r^{d-\beta} b \cdot \nabla_w p(1 - r^{\alpha}, 0, rw) [U(w + (x + he_k)/r)]^{q+1} - [U(w + x/r)]^{q+1} dw dr.
\end{align*}
By (2.24), we may pass with \( |h| \) to zero under the first two integrals. Furthermore, (4.18) together with (2.3) and (2.10) let us bound the third integral for \( |h| < 1 - \varepsilon \) by
\begin{align*}
C \|U\|^q \int_{1-\varepsilon}^{1} \int_{\mathbb{R}^d} r^{d-\beta} \frac{p(1 - r^{\alpha}, 0, rw)}{(1 - r)^{1/\alpha} (1 + |w + x/r|)^\beta} \frac{A}{1 + |w + x/r|^{\beta}} dw dr \\
= AC \|U\|^q \int_{1-\varepsilon}^{1} \int_{\mathbb{R}^d} r^{-\beta} \frac{r^{d-\beta} P_{r^{1/\alpha}} ((1 + |\cdot|^{-\beta}) (wr^{-1}) }{(1 - r)^{1/\alpha}} dw dr \\
\lesssim \frac{A}{(1 + |x|)^\beta} \int_{1-\varepsilon}^{1} (1 - r)^{-1/\alpha} dr \leq \frac{\varepsilon^{-1/\alpha} \varepsilon^{1/\alpha} A}{1 - 1/\alpha}.
\end{align*}
This gives us
\begin{align*}
\limsup_{|h| \to 0} \frac{U(x + he_k) - U(x)}{|h|} - \liminf_{|h| \to 0} \frac{U(x + he_k) - U(x)}{|h|} \lesssim \varepsilon^{1-1/\alpha} A,
\end{align*}
for any \( \varepsilon \in (0, 1) \). Therefore the limit exists, and is bounded by (4.15) due to (4.18). Furthermore, by (4.19) for any \( \varepsilon \in (0, 1) \) we may express the gradient by
\begin{align*}
\nabla U(x) &= \nabla P_1 u_0(x) + \alpha \int_{0}^{1-\varepsilon} \int_{\mathbb{R}^d} r^{d-\beta} \nabla (b \cdot \nabla_w p(1 - r^{\alpha}, x, rw)) U^{q+1}(w) dw dr \\
&+ \alpha \int_{1-\varepsilon}^{1} \int_{\mathbb{R}^d} r^{d-\beta} b \cdot \nabla_w p(1 - r^{\alpha}, x, rw) U^{q}(w) \nabla U(w) dw dr,
\end{align*}
which ensures its continuity by the estimates of \( U \) and \( \nabla U \).
Below we summarize the results of this section.

Proof of Theorem 1.2. Let $u_0(x) = M|x|^{-\beta}$ with $M > 0$ and $\beta \in (1, d)$. A solution $u(t, x)$ to the problem (1.1) exists by Theorem 1.1 and Lemma 4.2. By Lemma 4.3 this solution is self-similar and there exists the function $U(x)$ such that $u(t, x) = t^{-\beta/\alpha}U(t^{-1/\alpha}x)$. The estimates (1.9) follows by Proposition 4.6 and Corollary 4.8. Proposition 4.7 yields (1.10) and (1.11). Finally, the regularity of $U$ and (1.12) follows by Proposition 4.9.

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