A comparison of Numerical Solutions for Linear Fredholm Integral Equation of the Second Kind

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Abstract: The aim of this paper, we offered a new numerical method which is Touchard Polynomials (T-Ps) for solving Linear Fredholm Integral Equation of the Second Kind (LFIE2-K), to find approximating Numerical Solution (N-S). At the beginning, we demonstrate (T-Ps) and construct the operational matrix which is a matrix representation for solution. The algorithm and some examples are given; comparing the numerical results of proposed method with the numerical results of the other numerical method which is Bernstein Polynomials (B-Ps). We will show the high resolution of results by proposed method. The comparison between the Exact Solution (E-S) and the results of two methods are given by calculating absolute value of error and the Least Square Error (L.S.E). The results are calculated in Matlab code.

Keywords: Fredholm Integral Equation, Touchard & Bernstein polynomials.

1. Introduction

The (LFIE2-K) is the equation where the unknown function sees inside and outside the integral sign [1, 3]. The standard form of (LFIE2-K) is

\[ V(\tau) = g(\tau) + \phi \int_{x_1}^{x_2} X(\tau, y) V(y) dy \quad x_1 \leq \tau, y \leq x_2, \]

where \( V(\tau) \) is unknown function or approximate solution of (1) to be calculated, \( \phi \) is a known constant, contains physical meanings of the properties of the material, and \( X(\tau, y) \) is a known function of the variables \( \tau \) and \( y \), called the nucleus of the Integral Equation (IE) bear characteristics and properties of material may be continuous or discontinuous, \( g(\tau) \) is a known function represents the function of the surface on which we want to calculate integration. (FIE2-K) which can be came from boundary value problem. Erik I. Fredholm (1866-1927) was a Swedish mathematician whomentioned for his research on (IE) which arises in several applications [2, 4].

Recently, there exist many enhanced methods to get the approximatesolutions of (IE). [5] introduced (T-Ps) Method to obtained (N-S) for (IE). [6] introduced a successive approximation method in terms of a combination of (B-Ps) and block-pulse function. [7] used (B-Ps) method, integral mean value method, Tylor series method, the least square method are used to solve the (IE2-K). [8] employed (T-Ps) to treatment steepest descents to a suitable integral
representation of $T_n(x)$ to find that the number of saddle points that contribute to the expansion depends on the values $n$ and $z$.[9] Established some relation between (T-Ps) and Bell polynomials and the polynomials of binomial type.

2. Touchard’s Polynomials:

These polynomials was studied in (1939) by Jacques Touchard (1885–1968) was a French mathematician, which is consist of a polynomial sequence of binomial type, by [8, 9, 10, 11] are given as

$$J_n(\tau) = \sum_{m=0}^{n} S(n, m) \tau^m = \sum_{m=0}^{n} \left( \begin{array}{c} n \\ m \end{array} \right) \tau^m$$

(2)

The first six (T-Ps) are given as

1. $J_0(\tau) = 1$
2. $J_1(\tau) = 1 + \tau$
3. $J_2(\tau) = 1 + 2\tau + \tau^2$
4. $J_3(\tau) = 1 + 3\tau + 3\tau^2 + \tau^3$
5. $J_4(\tau) = 1 + 4\tau + 6\tau^2 + 4\tau^3 + \tau^4$
6. $J_5(\tau) = 1 + 5\tau + 10\tau^2 + 10\tau^3 + 5\tau^4 + \tau^5$

3. The Function Approximation

For determining an approximate (N-S) of (1) the function $V(\tau)$ is approximated by (T-Ps) basis on $[x_1, x_2]$ as follows:

$$V(\tau) \cong [\tau] = \delta_0 J_0(\tau) + \delta_1 J_1(\tau) + \cdots + \delta_n J_n(\tau) = \sum_{m=0}^{n} \delta_m J_m(\tau),$$

(3)

where $\delta_m$ ($m = 0, 1, ..., n$) are unknown constant values to be calculated.

It is easy to write equation (3) as a dot scalar of two vectors:

$$V(\tau) = [\tau] = \begin{bmatrix} \delta_0 \\ \delta_1 \\ \vdots \\ \delta_n \end{bmatrix} \cdot \begin{bmatrix} J_0(\tau) \\ J_1(\tau) \\ \cdots \\ J_n(\tau) \end{bmatrix},$$

(4)

we can convert equation (4) to the operational matrix for (T-Ps) form as:
V(τ) = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
0 & 1 & 2 & 3 & 4 \\
0 & 0 & 1 & 3 & 6 \\
0 & 0 & 0 & 1 & 4 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix} \begin{bmatrix}
\delta_0 \\
\delta_1 \\
\delta_2 \\
\delta_3 \\
\delta_4 \\
\end{bmatrix}, \quad \text{(5)}

where \( a_m \) are known coefficients of the power basis, used to calculate the (T-Ps). “It is clear that this matrix upper triangular”.

Now, for example, in cases \( n=2 \) and \( 4 \), the operational matrices are equation (6) and equation (7) respectively.

\[
V(τ) = \begin{bmatrix}
1 & τ & τ^2 \\
0 & 1 & 2 \\
0 & 0 & 1 \\
\end{bmatrix} \begin{bmatrix}
\delta_0 \\
\delta_1 \\
\delta_2 \\
\end{bmatrix}, \quad \text{(6)}
\]

\[
V(τ) = \begin{bmatrix}
1 & τ & τ^2 & τ^3 & τ^4 \\
0 & 1 & 2 & 3 & 4 \\
0 & 0 & 1 & 3 & 6 \\
0 & 0 & 0 & 1 & 4 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix} \begin{bmatrix}
\delta_0 \\
\delta_1 \\
\delta_2 \\
\delta_3 \\
\delta_4 \\
\end{bmatrix}, \quad \text{(7)}
\]

4. Solution of (LFIE2-K) using (T-Ps).

In this section, we will use (T-Ps) to estimate the (N-S) for the (LFIE2-K).

Recalling that equation (1) is:

\[
V(τ) = g(τ) + φ \int_{x_1}^{x_2} X(τ, y) V(y) dy \quad x_1 ≤ τ, y ≤ x_2 , \quad \text{(8)}
\]

and by using equation (3), let

\[
V(τ) = \sum_{m=0}^{n} \delta_m J_m(τ) , \quad \text{(9)}
\]

where \( \delta_m \) (\( m = 0, 1, ..., n \)) are unknown values to be calculated by applying (T-Ps).

Substituting equation (9) in equation (8), we have:

\[
\sum_{m=0}^{n} \delta_m J_m(τ) = g(τ) + φ \int_{x_1}^{x_2} x(τ, y) \sum_{m=0}^{n} \delta_m J_m(y) dy , \quad \text{(10)}
\]

also by using equation (4), equation (10) become:
by using equations (5), then equation (11) converted to the form:

\[
\begin{bmatrix}
\delta_0 \\
\delta_1 \\
\vdots \\
\delta_n
\end{bmatrix} = g(\tau) + \varphi \int_{x_1}^{x_2} x(\tau,y) \begin{bmatrix}
\delta_0 \\
\delta_1 \\
\vdots \\
\delta_n
\end{bmatrix} dy 
\]

After computing the integration, the linear system in equation (12) can be solved by standard method to calculate the unknown values \(\delta_i\)'s, where these values are used in equation (3) to get the (N-S) approximately.

The following algorithm shows the steps for getting the (N-S) for the (LFIE2-K).

5. The Algorithm:

Step 1:
We choose \(n\), degree of (T-Ps)

\[
J_n(\tau) = \sum_{m=0}^{n} S(n,m)\tau^m = \sum_{m=0}^{n} \binom{n}{m} \tau^m.
\]

Step 2:
Substitute the (T-Ps) in the (LFIE2-K)
Step 3:
Compute
\[
\begin{bmatrix}
1 & \tau & \tau^2 & \ldots & \tau^n \\
\end{bmatrix}
\begin{bmatrix}
a_{00} & a_{01} & a_{02} & \ldots & a_{0n} \\
0 & a_{11} & a_{12} & \ldots & a_{1n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & a_{nn} \\
\end{bmatrix}
\begin{bmatrix}
\delta_0 \\
\delta_1 \\
\vdots \\
\delta_n \\
\end{bmatrix}
\]
\[= g(\tau) + \varphi \int_{x_1}^{x_2} X(\tau, y) [1 \ y \ y^2 \ldots y^n] \, dy\]
and Compute
\[
\begin{bmatrix}
1 & \tau & \tau^2 & \ldots & \tau^n \\
\end{bmatrix}
\begin{bmatrix}
a_{00} & a_{01} & a_{02} & \ldots & a_{0n} \\
0 & a_{11} & a_{12} & \ldots & a_{1n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & a_{nn} \\
\end{bmatrix}
\begin{bmatrix}
\delta_0 \\
\delta_1 \\
\vdots \\
\delta_n \\
\end{bmatrix}
\]
\[= \delta_{x_1}^{} + \delta_{x_2}^{} (1.3437) \delta_{x_3}^{} + (1.5155) \delta_{x_4}^{} = \begin{bmatrix} 0.7408 \end{bmatrix}
\]

Step 4:
Compute \(\delta_m\), by choosing \(\tau_1, \tau_2, \ldots, \tau_n\), \(m = 0, 1, 2, \ldots, n\)

6. The illustrative Examples

Example (1): consider the following (LFIE2-K) on \([0, 1]\) given in [13, 14]

\[J(\tau) = e^{-\tau} - \int_0^1 \tau e^\varphi J(y) \, dy\]

where \(g(\tau) = e^{-\tau}, \varphi = -1, X(\tau, y) = \tau e^\varphi\),

and the exact solution is \(J(\tau) = e^{-\tau} - \frac{\tau}{2}\).

Now, we apply the algorithm of the proposed method with degree \(n=2\) for (T-Ps), we have the linear system:

(\(1.1718\)) \(\delta_0 + (1.3718) \delta_1 + (1.6537) \delta_2 = (0.9048)\)

(\(1.3437\)) \(\delta_0 + (1.7437) \delta_1 + (2.3273) \delta_2 = (0.8187)\)

(\(1.5155\)) \(\delta_0 + (2.1155) \delta_1 + (3.0210) \delta_2 = (0.7408)\)

we solve this system by “Gauss elimination” we have:

\(\delta_0 = 2.90821516\), \(\delta_1 = -2.31996650\), \(\delta_2 = 0.41088681\)
Then, we substitute these values in equation (3) we get the (N-S) of equation (1) approximately as follows:

\[ V(\tau) \approx (2.90821516) + (-2.31996650)(1 + \tau) + (0.41088681)(1 + 2\tau + \tau^2). \]

According to the (L.S.E.) between the (E-S) and approximating (N-S) are presented in Table (1) and Figure (1).

Table (1): The Numerical Results for (LFIE2-K)

| \( \tau \) | \( (E-S) = V(\tau) = e^{-\tau} - \frac{\tau}{2} \) | Method (B-Ps) of [13] for \( n=2 \) | Method (B-Ps) of [14] for \( n=2 \) | Proposed Method (T-Ps) for \( n=2 \) |
|-----------|------------------------------------------|----------------------------|----------------------------|-------------------|
| 0.0       | 1                                        | 1                          | 1                          | 1                 |
| 0.1       | 0.8548                                   | 0.8590                     | 0.8590                     | 0.8534            |
| 0.2       | 0.7187                                   | 0.7241                     | 0.7241                     | 0.7160            |
| 0.3       | 0.5908                                   | 0.5955                     | 0.5955                     | 0.5867            |
| 0.4       | 0.4703                                   | 0.4730                     | 0.4730                     | 0.4657            |
| 0.5       | 0.3565                                   | 0.3568                     | 0.3568                     | 0.3528            |
| 0.6       | 0.2488                                   | 0.2467                     | 0.2467                     | 0.2481            |
| 0.7       | 0.1466                                   | 0.1428                     | 0.1428                     | 0.1517            |
| 0.8       | 0.0493                                   | 0.0452                     | 0.0452                     | 0.0634            |
| 0.9       | -0.0434                                  | -0.0463                    | -0.0463                    | -0.0167           |
| 1.0       | -0.1321                                  | -0.1316                    | -0.1316                    | -0.0885           |

L.S.E. = \( \sum_{i=0}^{10} (V(\tau)_{\text{Exact}} - V(\tau)_{\text{Numerical}})^2 \) = 1.0 E–3

Figure (1): (E-S) and (N-S) for (LFIE2-K).
**Example (2):** consider the following (LFIE2-K) on $[-1, 1]$ given in [12]

$$J(\tau) = \tau + \int_{-1}^{1} (\tau^4 - y^4) J(y) dy,$$

where $\varphi = 1$, $\lambda(\tau, y) = (\tau^4 - y^4)$, and the exact solution is $J(\tau) = \tau$

choosing $n=4$, $\delta_m (m = 0, 1, ..., 4)$ are obtained as follows:

$\delta_0 = -1.0$, $\delta_1 = 1.0$, $\delta_2 = 0$, $\delta_3 = 0$, $\delta_4 = 0$,

after substituting these values in equation (3) we have the (N-S) of equation (1) approximately by:

$$V(\tau) \cong -1.0 + (1.0)(1 + \tau)$$

Table(2) and Figure (2) show the results.

**Table (2): Absolute Error of (E-S) and (N-S) for (LFIE2-K).**

| $\tau$  | Absolution Value of Error, Method of [12]: (B-Ps) for n=4 | Absolution Value of Error, Proposed Method: (T-Bs) for n=4 |
|---------|----------------------------------------------------------|----------------------------------------------------------|
| -1.0    | 0                                                        | 0                                                        |
| -0.8    | 0                                                        | 0                                                        |
| -0.6    | 1.110223024625157e−016                                   | 9.18355e157999121e−41                                    |
| -0.4    | 2.220446049250313e−016                                   | 4.59177480789956e−41                                    |
| -0.2    | 1.665334536937735e−016                                   | 1.37753e42369868e−40                                    |
| 0       | 0                                                        | 0                                                        |
| 0.2     | 5.551115123125783e−017                                   | 9.18355e157999121e−41                                    |
| 0.4     | 1.110223024625157e−016                                   | 1.37753e42369868e−40                                    |
| 0.6     | 2.220446049250313e−016                                   | 0                                                        |
| 0.8     | 1.110223024625157e−016                                   | 1.836709923159824e−40                                    |
| 1.0     | 0                                                        | 0                                                        |
7. Conclusion

In this paper, we applied the (T-Ps) method on two examples for solution of (LFIE2-K) and compared our numerical solutions with numerical solutions of other method (B-Ps) which is used by three researchers they are [12], [13] and [14], we found that our proposed method is very convenient and effective for finding approximate numerical solutions for integral equations as shown in tables and figures.

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