Multi-loop Amplitudes and Resummation

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Abstract

We explore the relation between resummation and explicit multi-loop calculations for QCD hard-scattering amplitudes. We describe how the factorization properties of amplitudes lead to the exponentiation of double and single poles at each order of perturbation theory. For these amplitudes, previously-observed relations between single and double poles in different $2 \rightarrow 2$ processes can now be interpreted in terms of universal functions associated with external partons and process-dependent anomalous dimensions that describe coherent soft radiation. Catani’s proposal for multiple poles in dimensionally-continued amplitudes emerges naturally.

Key words: Quantum Chromodynamics, multi-loop amplitudes, resummation, color flow, anomalous dimensions, NLO and NNLO corrections, factorization

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1 Introduction

The past few years have seen a breakthrough in the calculation of two-loop matrix elements and amplitudes in QCD [1,2,3,4]. At the same time significant progress has occurred in the resummation of logarithmic corrections to all orders in perturbation theory [5,6,7,8]. In this paper, we hope to contribute toward linking advances in fixed-order QCD amplitudes and in resummation. We attempt to clarify how the factorization properties of QCD hard scattering amplitudes influence calculations at fixed order.

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We will see how the single pole structure found at the level of two loops in matrix elements can be understood in terms of properties of a hitherto uncalculated two-loop anomalous dimension for soft radiation. The knowledge of this two-loop soft function is an important part of resummations at the level of next-to-next-to-leading logarithms. In addition, we will relate Catani’s very successful proposal [9] for the $\epsilon$-pole structure of dimensionally regularized $(D = 4 - 2\epsilon)$ amplitudes at one and two loops to known factorization and resummation properties of QCD amplitudes. These results will be illustrated for amplitudes involving quarks and antiquarks.

2 The factorized hard scattering amplitude

Consider the partonic process

$$f: \quad f_A(\ell_A, r_A) + f_B(\ell_B, r_B) \rightarrow f_1(p_1, r_1) + f_2(p_2, r_2),$$

which involves four partons with flavors $f_i$, momenta $\{\ell_i, p_i\}$ and color $r_i$. In perturbation theory, the hard scattering amplitude $M[f]$ associated with this process can be written as a matrix in the space of the color indices of its external partons $r_i = \{r_A, r_B; r_1, r_2\}$. It is convenient to express these amplitudes in a basis of color tensors, $(c_I)_{r_i}$, so that [10]

$$M[f] \{r_i\} \left(\{\wp_j\}, Q^2, \alpha_s(\mu^2), \epsilon\right) = M_L^{[f]} \left(\{\wp_j\}, Q^2, \alpha_s(\mu^2), \epsilon\right) (c_L)_{r_i}. \quad (2)$$

Here $i, j = A, B, 1, 2$ and $Q^2$ is the overall hard-scattering scale, which we may choose as $s$ for $2 \rightarrow 2$ scattering. The vectors $\wp_j = p_j/\sqrt{s}/2$, specify the directions of the incoming and outgoing lines and hence the full kinematics of the process. After renormalization, these amplitudes retain poles in $\epsilon$, due to infrared and collinear singularities. The singular behavior of the $2 \rightarrow 2$ amplitude is entirely due to internal lines that carry momenta that are either soft or collinear to one of the external lines. For gauge-invariant sets of diagrams, these singularities are entirely logarithmic in four dimensions, corresponding to poles at $\epsilon = 0$.

The amplitude $M^{[f]}$ for partonic process $f$ can be factorized into products of functions that organize the contributions of momentum regions relevant to $\epsilon$ poles in the scattering amplitude [11,12]. These are: i) process-dependent functions $h_f^{[f]}$ that describe the short-distance dynamics of the hard scattering,
one function for each of the elements of the basis of color exchange; ii) a matrix of functions, $S_{LI}^{[f]}$ that describes the coherent soft radiation arising from the overall color flow and iii) a function $J^{[f]}$, dependent only on the list of external partons, and otherwise independent of the color flow, which describes the perturbative evolution of the incoming and outgoing partons for flavors $f_i = q, \bar{q}, g$. In these terms, the factorized form of the amplitude associated with the process $f$ of Eq. (1) is [11,10]

$$
M_{L}^{[f]}\left(\varphi_i, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) = J^{[f]}\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) S_{LI}^{[f]}\left(\varphi_i, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right)
\times h_{I}^{[f]}\left(\varphi_i, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right),
\tag{3}
$$

where we collect the various virtual "jet" factors associated with external partons in

$$
J^{[f]}\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) \equiv \prod_{i=A,B,1,2} J_{(\text{virt})}^{[f_i]}\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right).
\tag{4}
$$

The products are over the two incoming lines, and over the two outgoing lines (or more, in the cases where we generalize the $2 \rightarrow 2$ process of Eq. (1)). Notice that in Eq. (3) only the soft and hard-scattering functions depend on the momenta $\varphi_i$. The parameter $\mu$ remains the renormalization scale while the factorization scale is chosen as the hard scale $Q$.

The predictive power of the factorized form of the amplitude follows from the properties of its individual functions. The jet functions in $J^{[f]}$ include all collinear dynamics, and hence all double poles in dimensional regularization, which arise from the overlap of infrared and collinear enhancements in perturbation theory. The matrix of soft functions $S_{LI}^{[f]}$ provides at most a single infrared pole per loop and, although it depends on the process kinematics it is otherwise completely determined by a set of anomalous dimensions that can be computed in the eikonal approximation. The hard-scattering functions $h_{I}^{[f]}$ are fully infrared finite. The jet functions and the soft function can be defined in terms of specific QCD matrix elements [10,13].

The independence of the amplitude $\mathcal{M}$ from the choice of the factorization scale $Q$ leads, by the usual connection between factorization and evolution, to the renormalization group equation [6]

$$
\frac{d}{d \ln Q} S_{LI} = -\Gamma_{LI}^{[f]} S_{JI},
\tag{5}
$$

To shorten the notation, $S_{LI}$ and $h_{I}$ dependence on $\varphi_i$, will be implicit.
where the variations of the jet functions and the soft matrix with the scale $Q$
are compensated by variations of the hard function.

There is still considerable freedom in the construction of the jet and soft functions. For examples, we may shift infrared finite contributions between the jet and hard scattering functions, and/or single-logarithmic, process-independent soft contributions between the soft matrix and the jets. Specifically, the soft matrix $S_{Li}$ is defined only up to any multiple of the identity matrix in the color basis space. Below, we will identify a convenient, but by no means unique, scheme for the definition of the various functions in (3).

Of primary importance here is the observation that mixing of color structures as a result of soft parton exchange is entirely contained within the matrix $\Gamma^{[f]}$, and is therefore summarized by the following solution to the renormalization group for the soft function,

$$S^{[f]}\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) = P \exp \left[ -\frac{1}{2} \int_0^{Q^2} d\tilde{\mu}^2 \frac{d^2}{\mu^2} \Gamma^{[f]} \left( \tilde{\alpha}_s\left(\frac{\mu^2}{\tilde{\mu}^2}, \alpha_s(\mu^2), \epsilon\right) \right) \right], \quad (6)$$

where $P$ stands for path ordering. In the dimensionally-regularized theory, the effective coupling should be thought of as expanded in powers of $\alpha_s(\mu^2)$. To the accuracy we work, this is given by the one-loop form [14]

$$\tilde{\alpha}_s\left(\frac{\mu^2}{\tilde{\mu}^2}, \alpha_s(\mu^2), \epsilon\right) = \alpha_s(\mu^2) \left(\frac{\mu^2}{\tilde{\mu}^2}\right)^\epsilon \sum_{n=0}^{\infty} \left[ \frac{\beta_0}{4\pi\epsilon} \left( \left(\frac{\mu^2}{\tilde{\mu}^2}\right)^\epsilon - 1 \right) \alpha_s(\mu^2) \right]^n \quad (7)$$

where

$$\beta_0 = \frac{11}{3} C_A - \frac{2}{3} N_F.$$ 

The integrals arising in the exponential of Eq. (6) are quite simple and for comparison to fixed $n$th-order calculations, we only need to collect all contributions up to $\mathcal{O}(\alpha_s^n(\mu^2))$. All the one-loop anomalous dimensions in these equations have been computed previously in Refs. [10], so that the color mixing due to single soft gluon exchange can be predicted (and checked) for specific processes.

In the next section we will use a specific definition of the jet function, based on factorization and on the application of Eq. (3) to the simplest color flow of all, the time-like Sudakov form factor.
3 Jet and soft functions

A convenient explicit expression for jet functions may be developed by applying the factorization in Eq. (3) to the singlet form factors for quarks and gluons, which we denote by $\mathcal{M}^{[\bar{i}i \rightarrow 1]}$. To be specific, we consider the quark Sudakov form factor. In this case, there is no color mixing at all, and the soft anomalous dimension reduces to a number. In the following, we shall absorb this number into the evolution of the jets, and reduce the soft function to unity by definition. Similarly, given that the anomalous dimension for any electroweak vertex vanishes in QCD, we may also reduce the hard function to unity in this case.

We thus define our jet functions by the relation

$$J^{[i]} \left( \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) = J^{[\bar{i}]} \left( \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) = \left[ \mathcal{M}^{[\bar{i}i \rightarrow 1]} \left( \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) \right]^\frac{1}{2}$$

where $Q^2$ is the characteristic momentum transfer, $\mu$ is the $\overline{\text{MS}}$ renormalization scale ($\mu^2 = \mu_0^2 \exp[-\epsilon(\gamma_E - \ln 4\pi)]$) and the electromagnetic Sudakov form factor in $D = 4 - 2\epsilon$ dimensions is given by\(^3\) [14,15]

$$\mathcal{M}^{[\bar{i}i \rightarrow 1]} \left( \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) = \exp \left\{ \frac{1}{2} \int_0^{\frac{Q^2}{\mu^2}} \frac{d\xi^2}{\xi^2} \left[ K^{[i]}(\alpha_s(\mu^2), \epsilon) \right. \right.$$

$$\left. + G^{[i]}(-1, \alpha_s(\frac{\mu^2}{\epsilon^2}, \alpha_s(\mu^2), \epsilon)) \right\}$$

In Eq. (9) the $\xi$ integration is defined order by order in perturbation theory and as in our discussion of the soft function, the running coupling $\bar{\alpha}_s$ is thought of as being expanded in powers of $\alpha_s(\mu^2)$. As described in [15,16], the functions $\gamma^{[i]}_K$, $\mathcal{K}^{[i]}$ and $G^{[i]}$ may be read off by comparison to explicit fixed-order calculations [17,18]. In particular, $\mathcal{K}^{[i]}$ is defined as a series of poles in $\epsilon$, while $G^{[i]}(-1, \alpha_s, \epsilon)$ includes the non-singular $\epsilon$-dependence before integration. Notice that in Eq. (9) all the $Q$ dependence of the Sudakov form

\(^3\) A similar definition may be given for gluon jets in terms of matrix elements of conserved, singlet operators.
factor is organized by the dimensionally-regulated running coupling and that by construction $\mathcal{K}[q] = \mathcal{K}[q]$, $\mathcal{G}[q] = \mathcal{G}[q]$ and $\gamma_K^{[q]} = \gamma_K^{[q]}$ [15,19].

Following the method of Refs. [15], we verify that $\gamma_i^K$ is the familiar Sudakov double-logarithmic anomalous dimension known up to two loops [20] to be

$$\gamma_i^K = 2 C_i \left( \frac{\alpha_s}{\pi} \right) \left[ 1 + \left( \frac{\alpha_s}{\pi} \right) K \right]$$

(10)

with $C_q = C_F$, $C_g = C_A$ and $K = C_A (67/18 - \zeta(2)) - N_F (5/9)$.

Before going on to elastic scattering amplitudes, we observe that our choice of jet function (8) corresponds to a particular scheme for the factorization of the Sudakov amplitude in which the hard-scattering and soft functions are set to unity, $h_{[i\to1]}^{[i\to1]} = 1$ in Eq. (5) for the form factor. In this scheme, the hard scattering and soft functions of $2 \to 2$ amplitudes will be computed with these “Sudakov-defined” jet functions. In particular, the matrices $\Gamma^{[i]}$ found in this way correspond to those calculated in Refs. [10,13] at one loop. We will return to the issue of alternative choices below.

4 Pole structure for multi-loop hard-scattering amplitudes

The combination of Eqs. (6) and (8) for the soft and jet functions applied to Eq. (3), allow us to give a useful expression for an arbitrary hard-scattering amplitude at two loops and beyond. In particular, the all-orders structure of any $2 \to n$ amplitude at wide angles is readily understood as the exponentiation of an appropriate power of the singlet form factors times the expansion of the exponentiated soft anomalous dimension. With input to $n + 1$ loops in the singlet form factors of quark and gluon, and $n$ loops in the soft anomalous dimension, we can predict the full $n$th-next-to-leading poles in dimensional regularized amplitudes, in much the same way we predict powers of logarithms in threshold or other resummations.

In particular, on the basis of two-loop calculations for the jet functions and of the Sudakov form factor, we can predict almost the entire two-loop single-pole term for explicit calculations involving quarks. Conversely, on the basis of two-loop calculations we can also determine the soft anomalous dimensions at this order, and proceed to a full next-to-next-to-leading pole approximation.

For ease of comparison, we follow Refs. [9] and denote the vector in color space $\mathcal{M}_q$ with components $\mathcal{M}_L$ in Eq. (2) projected into the orthogonal basis $\{c_i\}$ as
 whose perturbative expansion is

\[ |\mathcal{M}^{[f]}| = \sum_{m=0}^{\infty} \left( \frac{\alpha_s(\mu^2)}{\pi} \right)^m |\mathcal{M}^{[f(m)]}|. \]  

Applying Eqs. (6) – (9) to the factorized form of the amplitude in Eq. (3), we easily collect all the singular structure of the one-loop amplitude, using \( K^{[f(1)]} = \gamma_K^{[f(1)]} / (2\epsilon) \), into a function \( F^{[f(1)]} \) as

\[ |\mathcal{M}^{[f(1)]}_{\text{ren}}| = F^{[f(1)]}(\epsilon)|\mathcal{M}^{[f(0)]}| + |\mathcal{M}^{[f(1)]}_{\text{fin}}|, \]  

with the lowest order amplitude

\[ |\mathcal{M}^{[f(0)]}| = |\mathcal{M}^{\text{Born}}|. \]  

The function \( F^{[f(1)]}(\epsilon) \) in (13) is given in terms of the one-loop coefficients in Eqs. (6) and (9) by

\[ F^{[f(1)]}(\epsilon) \equiv \frac{1}{2} \left[ - \left( \frac{\gamma_K^{[f(1)]}}{2\epsilon^2} + \frac{G_0^{[f(1)]}}{\epsilon} \right) 1 + \frac{\Gamma^{[f(1)]}}{\epsilon} \right] \left( -\frac{\mu^2}{Q^2} \right)^\epsilon. \]  

Here, we define

\[ G^{[i]} \equiv \frac{1}{2} \sum_{i \in f} G_0^{[i]}, \quad \gamma_K^{[i]} \equiv \frac{1}{2} \sum_{i \in f} \gamma_K^{[i]}, \]  

and \( G_0^{[f(1)]} \equiv G(Q^2/\mu^2 = 1, \epsilon = 0). \) The \( \epsilon \)-poles at \( \mathcal{O}(\alpha_s) \) are organized by this divergent function into those that are color uncorrelated, which are given by Sudakov exponentiation through \( G \) and \( \gamma_K \), and those that are color correlated, which are collected by the one-loop soft anomalous dimension matrix \( \Gamma^{[f(1)]} \).

The finite reminder in Eq. (13) is the one-loop hard scattering function of Eq. (3), in the notation,

\[ |\mathcal{M}^{[f(1)]}_{\text{fin}}| = h^{[f(1)]}(\varphi_i, \frac{Q^2}{\mu^2}) (c_f)_{\{r_i\}} \]  

\footnote{In the following, we use the subscript \( \text{fin} \) to denote finite functions and the subscript \( \text{ren} \) to denote renormalized functions.}
At two loops, we can conveniently organize the expansion of Eq. (3) by making use of the divergent structure identified at one loop through $F^{(1)}$, and by taking into account that $K^{(2)} = -\beta_0 \gamma_K^{(1)} / (16 \epsilon^2) + \gamma_K^{(2)} / (4 \epsilon)$ and $\gamma_K^{(2)} = \frac{K}{2} \gamma_K^{(1)}$. We find

\[ |M_{ren}^{(2)} \rangle = F^{(1)}(\epsilon) |M^{(1)} \rangle + F^{(2)}(\epsilon) |M^{(0)} \rangle + |M_{fin}^{(2)} \rangle \]  

where

\[ F^{(2)}(\epsilon) \equiv -\frac{1}{2} \left[ F^{(1)}(\epsilon) \right]^2 + \frac{1}{2} \left( K + \frac{\beta_0}{2 \epsilon} \right) F^{(1)}(2 \epsilon) - \frac{\beta_0}{4 \epsilon} F^{(1)}(\epsilon) + \frac{1}{2} L^{(2)}(2 \epsilon) \]  

with

\[ L^{(2)}(\epsilon) \equiv \frac{1}{\epsilon} \left[ -\left( G_0^{(2)} - \frac{K}{2} G_0^{(1)} \right) 1 + \Gamma^{(2)} \right] \left( -\frac{\mu^2}{Q^2} \right)^\epsilon. \]  

To evaluate the expressions given above, we use

\[ G_0^{(q)} \equiv G^{(q)} \left( Q^2 / \mu^2 = 1, \epsilon = 0 \right) \]

\[ = C_F \frac{3}{2}. \]  

\[ G_0^{(q)} \equiv G^{(q)} \left( Q^2 / \mu^2 = 1, \epsilon = 0 \right) \]

\[ = C_F \left\{ 3 \left( \frac{1}{16} - \frac{1}{2} \zeta(2) + \zeta(3) \right) C_F + \left( \frac{2545}{108} + \frac{11}{3} \zeta(2) - 13 \zeta(3) \right) \frac{C_A}{4} \right. \]

\[ - \left. \left( \frac{209}{108} + \frac{1}{3} \zeta(2) \right) T_R N_F \right\}. \]  

which are found by direct comparison of Eq. (9) with the known fixed-order results \cite{15,17,18} for the quark form factor.

From Eqs. (13) – (15), we see that the bulk of the one-loop singular behavior is determined by a combination of the Sudakov amplitudes $\mathcal{M}^{[\bar{q} \to 1]}$, from which we can find $\gamma_K^{(1)}$ and $G^{(1)}$, and the one-loop anomalous dimension matrix $\Gamma^{(1)}$. Together, these functions control the two-loop singular behavior from $\epsilon^{-4}$ down to $\epsilon^{-2}$. Information specific to a $2 \to n$ process at this order appears first at $\mathcal{O}(\epsilon^{-1})$ in the contribution of the two-loop anomalous dimension matrix $\Gamma^{(2)}$, which can be determined by comparison to an explicit calculation. We may
verify that the expressions above coincide with the proposal of Catani for the structure of two-loop hard scattering amplitudes, by direct comparison with Ref. [9].

As an application, consider the explicit two-loop calculation for the process $q\bar{q} \rightarrow q\bar{q}$ [2]. To express the amplitude as in Eq. (2), we employ the basis of $t$-channel singlet and octet color exchange (with $T^a$ the group generators in the fundamental representation) [2,10,13],

$$c_1 = \delta_{r_1r_A} \delta_{r_2r_B}, c_2 = \sum_a T^a_{r_1r_A} T^a_{r_2r_B}. \quad (23)$$

In this basis the one-loop soft anomalous dimension matrix\(^5\) is given by

$$\Gamma^{[q(1)]} = \begin{bmatrix}
2C_F T & C_F \frac{N_c}{N} (S - U) \\
2 (S - U) & \frac{N^2-2}{N} S - \frac{1}{N} (T - 2U)
\end{bmatrix} \quad (24)$$

where

$$S \equiv \ln \left( -\frac{s}{\mu^2} \right), \quad T \equiv \ln \left( -\frac{t}{\mu^2} \right), \quad U \equiv \ln \left( -\frac{u}{\mu^2} \right) \quad (25)$$

and \{s, t, u\} are the usual Mandelstam variables.

If we use Eq. (24) and apply the explicit expressions for $\gamma^{[q(1)], [q(2)]}, \gamma^{[q(1)], [q(2)]}$ and $\Gamma^{[q(1)]}$ given above, we derive predictions for the poles $\epsilon^{-4}, \epsilon^{-3}$ and $\epsilon^{-2}$ that are in complete agreement with the explicit calculation of [2]. We can therefore use the $\epsilon^{-1}$ term of this calculation to determine the contribution of $\Gamma^{[\ell(2)]}$ for this process. More precisely, we can determine its color-uncorrelated part, since this is what is given in [2].

This direct comparison of the color uncorrelated amplitudes found Refs. [2] for $q\bar{q}$ scattering matrix-elements by diagrammatic evaluation gives the following surprisingly simple result

$$\langle M^{(0)} | \left( \Gamma^{(2)}_S - \frac{K}{2} \Gamma^{(1)}_S \right) M^{(0)} \rangle = C_F \beta_0 \left( \frac{\zeta(2)}{16} + 1 \right) \langle M^{(0)} | 1 | M^{(0)} \rangle. \quad (26)$$

Thus, comparison of Eqs. (18) – (20) with the explicit calculation shows that most of the pole terms at two loops are understandable through exponentia-

\(^5\) It corresponds to the one presented in Eq.(51) of Ref. [10], when we choose the scale to be the $\mu^2 = \hat{s} = (\ell_A + \ell_B)^2$. 


tion. The remaining coherent part, sensitive to the details of the process, is much simplified once exponentiation is taken into account.

Even this remaining part is subject to further simplification, as is strongly suggested by the presence of the overall factor $\beta_0$, on the right of Eq. (26). In the scheme we have used above, the Sudakov form factor directly determines the jet definition. As we have pointed out, this is equivalent to setting $h_{\bar{i}i \to 1} = 1$. It is easy to show that choosing a nonzero $h_{\bar{i}i \to 1(1)}$ for $M_{\bar{i}i \to 1(1)}$ produces a corresponding change in $G^{(r,2)}$ proportional to $\beta_0 h_{\bar{i}i \to 1(1)}$. The importance of the analogous scheme dependence in $k_T$-resummation has been emphasized recently in Ref. [8]. Thus, by using our freedom to modify the hard-scattering function in the Sudakov form factor, or equivalently to change the normalization of the jet function, we can easily impose the condition

$$\langle M^{(0)} | \left( \Gamma^{(2)}_S - \frac{K}{2} \Gamma^{(1)}_S \right) | M^{(0)} \rangle = 0.$$  \hspace{1cm} (27)

Given our ability to impose Eq. (27), the uncorrelated part of the single poles can be absorbed entirely into appropriately-chosen jet functions. This freedom explains the relations found in Refs. [2,4]. There, on the basis of direct calculations it was found that for any $2 \to 2$ QCD process $f$ with $n_g$ external gluons and $n_q$ external quarks and antiquarks

$$H^{(2)} = n_q H_q^{(2)} + n_g H_g^{(2)},$$  \hspace{1cm} (28)

where $H^{(2)}$ is the coefficient of the identity in $L^{(2)}(\epsilon)$ on Eq. (20).

To determine the two-loop anomalous dimension fully, we will need two-loop amplitudes as vectors in the basis states, not only in uncorrelated form. These have been given so far for gluon-gluon scattering [4], with an expression for single-pole terms that is consistent with our results.

5 Higher orders

It is clear that the pattern described above extends to higher orders. For example, the three-loop hard scattering amplitude has the structure

$$| M_{\text{ren}}^{(3)} \rangle = F^{(1)}(\epsilon) | M^{(2)} \rangle + F^{(2)}(\epsilon) | M^{(1)} \rangle + F^{(3)}(\epsilon) | M^{(0)} \rangle + | M_{\text{fin}}^{(3)} \rangle$$  \hspace{1cm} (29)
where

\[
F^{[f(3)]}(\epsilon) = -\frac{1}{3} \left[ F^{[f(1)]}(\epsilon) \right]^3 - \frac{1}{3} F^{[f(1)]}(\epsilon) F^{[f(2)]}(\epsilon) - \frac{2}{3} F^{[f(2)]}(\epsilon) F^{[f(1)]}(\epsilon) \\
- \left( \frac{\beta_0}{4\epsilon} \right)^2 F^{[f(1)]}(3\epsilon) + \left( \frac{\beta_0}{4\epsilon} \right) \left\{ -\frac{1}{2} \left[ F^{[f(1)]}(\epsilon) \right]^2 - F^{[f(2)]}(\epsilon) \right\} \\
+ \frac{1}{2} \left( K + \frac{\beta_0}{2\epsilon} \right) \left[ 2 F^{[f(1)]}(3\epsilon) - F^{[f(1)]}(2\epsilon) \right] \\
+ L^{[f(2)]}(3\epsilon) - \frac{1}{2} L^{[f(2)]}(2\epsilon) \right\} + \frac{1}{2} L^{[f(3)]}(3\epsilon),
\]

(30)

and

\[
L^{[f(3)]}(\epsilon) \equiv \left[ -\left( \frac{\gamma_{K}^{[f(3)]}}{2\epsilon^2} + \frac{G_0^{[f(3)]}}{\epsilon} \right) + \frac{\Gamma^{[f(3)]}}{\epsilon} \right] \left( -\frac{\mu^2}{Q^2} \right)^\epsilon.
\]

(31)

These expressions enable us to predict the poles in \(\epsilon\) from sixth order down to fourth on the basis of the two-loop Sudakov form factor and the one-loop soft anomalous dimension matrices. Beyond this level, the three-loop \(\gamma_{K}^{[f(3)]}\) and two-loop soft anomalous dimension matrices will control all pole structure down to double poles, while the color correlations in the single poles will determine \(\Gamma^{[f(3)]}\).

6 Conclusions and outlook

We have shown how the factorization properties of hard-scattering amplitudes enable us to understand many of the complexities of their infrared poles in dimensional regularization. In particular, the universality of exponentiated collinear and soft singularities associated with incoming and outgoing partons provides a way of understanding important regularities predicted for and found in explicit two-loop calculations. They also predict the structure, and for the leading terms the coefficients, of poles to any order.

The program of calculating two-loop dimensionally-regulated amplitudes is a step toward standard model hard-scattering cross sections beyond next-to-leading order. In this context, it is useful to note that the factorization in Eq. (3) for amplitudes is shared by cross sections in the limit of partonic threshold, where the parton center-of-mass energy is just large enough to produce the final state. At any fixed order in perturbation theory, leading collinear and

\[\text{The fermionic contributions to } \gamma_{K}^{[f(3)]} \text{ have been calculated recently [21].}\]
infrared singularities occur at partonic threshold. Therefore, the exponentiated structure of singularities in $\epsilon$ for amplitudes will reoccur in the singularities of cross sections with radiation. This observation is the basis for threshold resummation, which may be a useful guide in organizing the very challenging phase space integrals in next-to-next-to-leading order cross sections.

Here we have explored singularities associated with purely virtual corrections, but the simplifications that we have found suggest that a similar analysis may be useful for inelastic processes. Leading singularities in cross sections cancel between sets of cut diagrams that include the elastic amplitudes at two loops. The structure of singularities in these amplitudes, as organized above, may help streamline the calculation of cross sections at two loops and beyond.

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