Alternating Quotients of Fuchsian Groups

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It is shown that any finitely generated, non-elementary Fuchsian group has among its homomorphic images all but finitely many of the alternating groups $A_n$. This settles in the affirmative a long-standing conjecture of Graham Higman.

1. INTRODUCTION

It all started with a theorem of G. A. Miller [14]: the classical modular group $\text{PSL}_2(\mathbb{Z})$ has among its homomorphic images every alternating group, except $A_6$, $A_7$, and $A_8$. In the late 1960’s Graham Higman conjectured that any (finitely generated non-elementary) Fuchsian group has among its homomorphic images all but finitely many of the alternating groups. This reduces to an investigation of the cocompact $(p; q; r)$-triangle groups, and in the series of papers [3, 4, 9, 15, 16] the conjecture was verified in the affirmative when $p = 2$. Assuming the Fuchsian group is finitely generated and non-elementary, and taking the phrases “almost all” to be synonymous with “all but finitely many”, and “surjects” with “has among its homomorphic images”, we build on this earlier work to prove

Theorem. Any Fuchsian group surjects almost all of the alternating groups.

There are several motivations behind the conjecture: Fuchsian groups have an algebraic structure that is somewhat complicated, and to get a firmer grip on this situation, one may be tempted to consider their finite, or even simple, homomorphic images. There is also a geometric incentive, namely, any compact Riemann surface (or complex algebraic curve) of genus $> 1$ has conformal automorphism group a finite homomorphic image of some Fuchsian group.

Schreier coset diagrams supply the technology used to prove the theorem, and they appear in the literature in various guises (see [2, 11] for alternative

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formulations as hypermaps or dessin d’enfants). Section 3 has the definition and the basic properties. Section 4 contains the proof of the theorem.

2. THE PLAN

Suppose \( X \) is the 2-sphere \( S^2 \), the Euclidean plane \( \mathbb{E}^2 \) or the hyperbolic plane \( \mathbb{H}^2 \). Let \( G \) be a finitely generated non-elementary discrete group of orientation preserving isometries of \( X \). By classical work of Fricke and Klein (see for instance [21]), \( G \) has a presentation of the form,

\[
generators: \quad a_1; b_1; \ldots; a_g; b_g; \quad (\text{hyperbolic})
\]
\[
x_1; \ldots; x_e; \quad (\text{elliptic})
\]
\[
y_1; \ldots; y_s; \quad (\text{parabolic})
\]
\[
z_1; \ldots; z_t; \quad (\text{hyperbolic boundary elements})
\]

relations:
\[
x_i^{m_i} = 1; \quad x_i x_j = x_j x_i; \quad y_i y_j = y_j y_i; \quad y_i^e = 1; \quad y_i y_j y_i = 1
\]
\[
y_i = 1; \quad y_i = 1; \quad x_i = 1; \quad z_k = 1; \quad [a_i; b_i] = 1
\]

When \( X = \mathbb{H}^2 \), \( G \) is called a Fuchsian group. The division into spherical, Euclidean and Fuchsian is governed by the quantity,

\[
(G) = 2g + 2 + \sum_{i=1}^{e} \frac{1}{m_i} + s + t
\]

with \( (G) < 0 \) if \( 0 > 0 \) as \( X = S^2; \mathbb{E}^2 \) or \( \mathbb{H}^2 \). The quotient \( X = G \) is an orientable 2-orbifold of genus \( g \) with \( e \) cone points, \( s \) punctures and \( t \) boundary components. Its geometry and the algebraic structure of \( G \) are intimately connected, so that \( G \) is determined up to isomorphism by its signature \( (g; m_1; \ldots; m_e; s; t) \).

To prove the theorem, it suffices to just consider the cocompact Dyck groups—the cases where in the signature we have \( g = s = t = 0 \). To see why we make a few elementary observations.

1. A group of signature \( (g; m_1; \ldots; m_e; s; t) \) is isomorphic to one of \( (g; m_1; \ldots; m_e; s; t; 0) \), and by (1), the former is Fuchsian if and only if the latter is. We may assume then that \( t = 0 \). Write \( (g; m_1; \ldots; m_e; s) \) instead of \( (g; m_1; \ldots; m_e; s; t; 0) \) from now on.

2. We can surject \( G = (g; m_1; \ldots; m_e; s) \) onto \( G^0 = (g^0; m_1; \ldots; m^0_e; s^0; \ldots; m^0_e; s^0) \), for any \( g^0 \), \( g^0 \), \( s^0 \), and \( m^0 \) a divisor of \( m \). The hat denotes omission. Here’s how: map the \( j \)-th elliptic, \( s \) \( s^0 \) of the parabolic and \( g \) \( g^0 \) hyperbolic pairs of generators of \( G \) to the identity of \( G^0 \); map the \( k \)-th elliptic
generator of $G$ to the corresponding elliptic generator of $G^0$ raised to the power $m_1=\frac{m}{m_1}$. All other generators of $G$ map to the corresponding ones in $G^0$. The map then extends to the desired homomorphism.

3. Writing $(m_1;\ldots;m_n)$ when $g = s = 0$, suppose $G = (m_1;\ldots;m_n)$. $S_n$ is a homomorphism with transitive image and let $G_1$ be the subgroup of $G$ consisting of those elements stabilising some fixed point of $1; \ldots; n$. By theorem 1 of [17], $G_1$ has signature $(\eta_1; n_1; \eta_2; \ldots; n_k; \eta_3; \ldots; n_r)$, where $(\eta_1)$ has exactly one cycle of each of lengths $m_1=n_1; \ldots; m_r=n_r$, with all other cycles of length $m_1^r$, and $\langle s_1 \rangle = n \langle s \rangle$. Moreover, if $G_1$ is normal in $G$, and we have the theorem for $G$, the simplicity of $A_n$ for $n \geq 5$ gives the result for $G_1$ as well.

4. Finally, any $k$-cycle $(a_1;\ldots;a_k)$ $2 A_n$ can be written as a product

$$(a_2; a_k)(a_3; a_k \ldots a_k) \cdots (a_k; a_k)$$

of two involutions in $A_n$. Similarly any cycle of even length in $S_n$ can be written as a product of a involution in $S_n$ and an involution in $A_n$. Thus, if we have the result for $(m_1; \ldots; k; \ldots; m_n; s)$ we have it for $(m_1; \ldots; k_1; k_2; \ldots; k_r; m_n; s)$ too.

**Lemma 2.1.** The theorem is true for every Fuchsian group if it holds for every Dyck group.

**Proof.** Proceeding according to the genus, suppose $G$ has signature $(\eta_1; \eta_2; \eta_3; \ldots; \eta_r; s)$ with $g = 2$. Map $G$ onto $x; y \rightarrow x^3$, free of rank two, by sending $a_1 \rightarrow x$, $a_2 \rightarrow y$, and all the other generators to the identity. Since $A_n$ is 2-generated for $n \geq 3$ (see [5]), we are done.

A group of genus one with $e = 1$ can be surjected onto $(1; m_1; 0)$ for $m_1 = 2$, by comment 2 above. The map $: (0; 2; 2; 2; m_1; 0)$. $S_2$ sending all generators to the permutation $(1; 2)$ has kernel isomorphic to $(1; m_1; 0)$ by comment 3 above, hence the result holds for groups of genus one with $e = 1$. For groups of genus one with no periods, hence signature $(1; \ldots; s)$ for $s = 1$, we may surject onto $(L; \ldots; 1; 1)$. But this is easily seen to be free of rank two, so the result holds here also.

A group of genus zero with no periods must, by (1), have at least three parabolic generators, and hence surject $(0; \ldots; 3)$. But this is free of rank two also. With a single period we have $s = 2$, and the group surjects $(0; m_1; 2) = Z_m; Z_2$, the free product of $Z_m; Z_2$, and $Z$. This surjects $Z_m; Z_2$, which in turn surjects any Fuchsian triangle group of the form $(0; 3; m_1; r; 0)$.

With two periods and one parabolic, we have $(0; m_1; m_2; 1) = Z_m; Z_2$, where $m_2 = 3$, so we can surject any Fuchsian triangle group like $(0; m_1; m_2; r; 0)$. A group with more parabolics, $(0; m_1; m_2; s; 3)$, surjects $(0; m_1; 2)$ done above. Finally, $(0; m_1; \ldots; m_2; s; e = 3$, surjects either $Z_2; Z_2; Z_2$ or $Z_m; Z_2$, for $m_2 = 3$. Surject the former onto a Fuchsian $(0; 2; 2; 2; r; 0)$. The latter has already been handled.
Lemma 2.2. The theorem holds for every Dyck group if it holds for the following:

1. The Fuchsian triangle groups \( \langle p;q;r \rangle \) with \( 2 < p < q < r \) distinct primes;
2. the triangle groups \( \langle 2;4;r \rangle \) for \( r = 5 \) a prime;
3. the groups \( \langle 2;3;8 \rangle, \langle 2;3;9 \rangle, \langle 2;3;10 \rangle, \langle 2;3;12 \rangle, \langle 2;3;15 \rangle, \langle 2;3;25 \rangle, \langle 2;4;6 \rangle, \langle 2;4;8 \rangle, \langle 2;5;6 \rangle, \langle 2;5;9 \rangle \) and \( \langle 3;4;5 \rangle \);
4. the groups \( \langle 2;3;3;3 \rangle \) and \( \langle 3;3;3;3 \rangle \).

Proof. The hyperbolic triangle group \( \langle 2;3m_1;3m_2 \rangle \) surjects \( \langle 2;3q;2 \rangle \) for \( q \) and \( r \) prime divisors of \( m_1 \) and \( m_2 \). If \( \langle 2;3q;2 \rangle \) is Fuchsian, we have by (1) that \( 2 = q < r < 1=2 \). If \( q \) and \( r \) are distinct, we have a group listed in part 1 of the lemma. If \( q = r \), the map \( \langle 2;3q;2 \rangle \rightarrow S_2 \) sends the generators of orders 2 and 4 to the permutation \((1;2)\) and the generator of order \( q \) to the identity has kernel \( \langle 2;3q;2 \rangle = \langle 2;3q;2 \rangle \). We have \( 2 = q < 1=2 \), hence \( q = 5 \), and the theorem holds for \( \langle 2;3q;2 \rangle \) if it holds for \( \langle 2;4;5 \rangle \), a group listed in part 2 of the lemma.

If \( \langle 2;3q;2 \rangle \) isn’t Fuchsian, it must be, after a possible reordering, one of \( \langle 2;2;2 \rangle \) for \( r = 2 \), \( \langle 2;3;3 \rangle \) or \( \langle 2;3;5 \rangle \). The first gives that \( \langle 2;3m_1;3 \rangle \) must have the form \( \langle 2;3m_1;2k \rangle \), for \( m_1 \) odd and \( 1 \leq k \leq 2 \). If \( m_1 = 3 \) or 4, then \( l = 3 \), as \( \langle 2;3;4 \rangle \) is spherical and \( \langle 2;4;4 \rangle \) Euclidean, so the group surjects \( \langle 2;3;8 \rangle \) or \( \langle 2;4;8 \rangle \), both of which are listed in the lemma. For \( m_1 = 5 \), \( \langle 2;3m_1;2k \rangle \) surjects \( \langle 2;3m_1;4 \rangle \). This in turn surjects \( \langle 2;4;2 \rangle \), \( r \) prime, and we have a group listed in part 2 unless \( r = 2 \) or 3. In the first case, \( m_1 = 2^6 \), so \( \langle 2;4;8;3 \rangle \) surjects \( \langle 2;4;8 \rangle \). In the second, \( m_1 = 2^63^5 \), and the group surjects \( \langle 2;4;8 \rangle \) when \( l = 0 \), or \( \langle 2;4;6 \rangle \) otherwise. The cases \( \langle 2;4;2 \rangle \) and \( \langle 2;3;5 \rangle \) are entirely similar.

This accounts for the \( \langle 2;3m_1;3m_2 \rangle \) Fuchsian groups, and the case of a general triangle groups is much the same. Similarly for the groups with four or five elliptic generators—either they can be surjected directly onto triangle groups or eliminated from consideration using comment 4 at the beginning of the section. The only exceptions are those listed in the lemma. Finally, a group with six or more elliptic generators can always be surjected directly onto a Fuchsian group with five. No doubt the reader can fill in the details.

In [3, 4, 9], the groups \( \langle 2;3;8 \rangle \) for all \( r \) and \( \langle 2;4;2 \rangle \) for all \( r \) were dealt with. Theorems 1-3 of [15] take care of the \( \langle 2;3q;2 \rangle \), with the exception of sixty cases. These sixty, and those from parts 3 and 4 of Lemma 2.2 can be found in the preprint version of this paper [7, 8]. This leaves the triangle groups \( \langle 2;3q;2 \rangle \), \( p < q < r \) to consider, and they can be found in Section 4.

Later on we will construct permutation groups as homomorphic images of Fuchsian groups and will identify the images as alternating using.

Theorem 2.1 ([12], refer to [19] Theorem 13.9). Let \( G \) be a primitive permutation group of degree \( n \) containing a prime cycle for some prime \( q \). Then \( G \) is either the alternating group \( A_n \) or the symmetric group \( S_n \).
The following lemma, well known to the cognoscenti, allows one to replace primitivity by more easily verifiable criteria. Recall that the support of a permutation \(2 \in S_n\) consists of those elements of \(f_1; 2; \ldots; n\) not fixed by \(i\).

**Lemma 2.3.** Let \(G = h_1; 2; \ldots; k\) be a transitive permutation group of degree \(n\) containing a prime cycle \(i\). For each \(i\) suppose there is a point in the support of whose image under \(i\) is also in the support of \(i\). Then \(G\) is primitive.

**Proof.** Suppose on the contrary that \(G\) is imprimitive with block system \(B\). For \(2 \in G\), let \(\bar{G}\) be the permutation induced by \(G\) on \(B\), and \(\bar{G}\) the group generated by the \(G\). The map \(\bar{f}\) is an epimorphism from \(G\) onto \(\bar{G}\), and \(\bar{G}\) acts transitively on \(B\). All blocks \(B \in B\) thus have the same size, say \(|B|\). If \(B \in B\) is in \(\text{supp}(i)\), then \(B\) and its image under \(i\) intersect for all \(i\), so \(B\) is contained in \(\text{supp}(i)\). Taking the union of all the blocks in \(\text{supp}(i)\) thus gives

\[\bar{B} \in \text{supp}(i)\]

Now \(\bar{G}\) has order \(q\), a prime, and \(i\) is a homomorphic image of \(i\). Thus, if \(i \neq 1\), then \(\bar{G}\) has order \(q\), and so \(\text{supp}(i) \supseteq q\). Since \(\bar{B}\) is non-trivial, we have \(\bar{B} \supseteq 1\), and hence by (2), \(\text{supp}(i) \supseteq q\). This contradicts the fact that \(i\) is a \(q\)-cycle, so we must have \(i = 1\). This means that \(i\) fixes every block, or equivalently, any point and its image under \(i\) lie in the same block. But \(i\) is a single cycle, so there is a block \(B\) with \(\text{supp}(i) = B\). By the condition stated in the Lemma, \(B\) and its image under \(i\) intersect for all \(i\), so are equal. Since the \(i\) generate \(G\), the whole group must fix \(B\), and by transitivity, \(B = f_1; 2; \ldots; n\), so there is just one block. This is the desired contradiction.

### 3. COSET DIAGRAMS

Suppose \(G\) is a group with a finite presentation \(hX; R\) and let \(K_0 = K_0(X; R)\) be the standard 2-complex with \(K_0(X; R) = G\). The 1-skeleton of \(K_0\) consists of a single vertex incident with oriented loops or edges that are in one to one correspondence with the generators \(X\). Each edge \(x \in X\) is a pair of oppositely oriented arcs, an \(x\)-arc and an \(x^{-1}\)-arc. The former coincides with the edge under its given orientation and the latter to the edge with the reverse orientation. The faces of \(K_0\) are in one to one correspondence with the relators \(R\), and are obtained by sewing discs onto the 1-skeleton, each with boundary label a relator word \(x \in R\), see [10, \&1.3].

A Schreier coset diagram for \(G\) is a cellular (that is, \(k\)-cells lift to \(k\)-cells) covering of \(K_0\) (see [18, \&2.2.1 and \&4.3.2] or [11]). A covering \(K\) realises a subgroup \(H = \bigcap K\) of \(G\), with the vertices of \(K\) in one to one correspondence with the cosets of \(H\) in \(G\). Conversely, every subgroup is realisable in this way from some diagram.
Their usefulness for our purposes stems from the fact that any coset diagram $K$ yields a homomorphism $\phi : G \rightarrow \text{Sym}^n K$ of vertices of $K$. Here $n$ is the sheet number of the covering, hence the number of vertices in $K$. For any $g \in G$ the image of vertex $v$ under the permutation $\phi(g)$ is the terminal vertex of the path starting at $v$ with label $g$. In particular, $\phi(G)$ is transitive if and only if $K$ is path-connected.

All of which is, of course, well known. The simplicial complexes that form coset diagrams for $G$ are characterised by two simple properties:

1. For each vertex $v$ and generator $x \in X$, there is precisely one $x$-arc and one $x^{-1}$-arc having initial vertex $v$.
2. The boundaries of the faces are precisely the paths obtained by starting at some vertex $v$ and traversing a path with label some $r \in R$.

Condition 2 indicates that in their unrefined form, coset diagrams will be a little unwieldy—there will be many faces sharing the same set of boundary edges. To alleviate matters, we use an equivalent construct, suggested by Higman and used in [3, 4, 8, 9, 15, 16]. It is what results by identifying such multiple faces.

Let $G = \langle m_1; \ldots; m_e \rangle$ be some fixed but arbitrarily chosen Dyck group. A more convenient presentation than given in the introduction is,

$\langle x_1; x_2; \ldots; x_e \rangle : x_1^{m_1^{-1}} = x_2^{m_2^{-1}} = \cdots = x_e^{m_e^{-1}} = 1$.

A $G$-graph is a directed graph with edges labelled $x_1; x_2; \ldots; x_e$ satisfying property (1) above. Ordering the edges incident with every vertex as shown in Figure 1 yields a 2-cell embedding of a $G$-graph into a closed orientable surface (see [20] for more details on graph embeddings). Each face of this surface complex $S$ will have boundary label some power of $x_i$ or $x_1 x_2 \cdots x_e$.

Call $S$ a $G$-diagram if for each face, this power divides the order of the appropriate word given in the presentation.

In a $G$-diagram, a path starting at $v$ with label $x_i^{m_i^{-1}}$ or $x_1 x_2 \cdots x_e$ circumnavigates a face an integral number of times. Taking the underlying $G$-graph and sewing in a 2-cell for each such vertex–relator pair yields a coset diagram for $G$. 
Conversely, the 1-skeleton of a coset diagram is a $G$-graph in which a path from any vertex with label a relator is closed (as it bounds a face). Embedding the graph as above gives a $G$-diagram. We therefore have

**Lemma 3.1.** A coset diagram for $G$ yields a unique $G$-diagram, and vice-versa.

Consequently, we use the same terminology for $G$-diagrams as for coset diagrams. In particular, call a face an $x_i$-face or $(x_1:::x_n)_1$-face whenever it has boundary label some power of $x_i$ or $x_1:::x_n$.

The key property of $G$-diagrams, as Higman observed, is that they can sometimes be combined to form new ones. For this we use *handles*, that is, pairs of vertices and , each incident with $x_1$-loops, so that the path starting at with label $x_1:::x_{n-1}$ terminates at .

Let $K_1;:::;K_t$, $t \leq m_1$, be a collection of disjoint $G$-diagrams, and the $2m_1$ distinct vertices $1;1;:::;m_1;1$ a collection of $m_1$ handles with at least one in each diagram. Take the disjoint union of all the underlying $G$-graphs, remove the $x_1$-loops at the vertices $j$ and $j$, and replace them by $x_1$-edges from $j$ to $j+1$ and $j$ to $j_1$, subscripts taken modulo $m_1$. Embed the graph in the usual way, and call the resulting surface complex $[K_1;:::;K_t]$ the *composition* of $K_1;:::;K_t$.

**Proposition 3.1.** $[K_1;:::;K_t]$ is also a $G$-diagram with $P$ vertices.

**Proof.** The underlying graph of $[K_1;:::;K_t]$ is clearly a $G$-graph, so it remains to show that all faces have boundary labels of the required form. If the boundary of a face does not contain an $x_1$-edge with initial vertex one of the $j$ or $j$, then all edges are contained in a single $G$-diagram $K$, and we are done.

Otherwise, we obtain the boundary label for the face by starting at an $j$ or $j$ and traversing a path with label some power of $x_1$ or some power of $x_1:::x_n$, until it closes (which it does by repeating an arc). The path obtained by traversing just $x_1$-edges passes through the vertices $j;1;:::;m_1;1;:::;j$ or $j;1;:::;1;:::;j;1;:::;j;1$, before closing with label $x_1^{p_1}$, so such faces are as they should be. Observe that before composition, the path starting at $j$ with label some power of $x_1:::x_n$ arrived at vertex $j$ after $e$ directed edges, proceeded to traverse the $x_1$-loop at $j$ and then an $x_2$-edge. After composition, the path from $j$ with such a label arrives instead at $j+1$ after $e$ directed edges, traverses the new $x_1$-edge to $j$, and is then identical with the path before composition. So the boundary label behaves as if the composition never happened, and is thus of the required form. The number of vertices is obvious.

Now suppose $G$ is the triangle group

$$\langle x; y | x^p = y^q = (xy)^r = 1; \quad 3 \leq p < q < r \rangle$$
with \( pq \) and \( z \) prime. In practice, we simplify \( (p; q; z) \)-diagrams when drawing them: a shaded \( q \)-gon indicates a \( y \)-face with boundary label \( y^q \), and a shaded wedge \( \equiv \) a \( y \)-face with label \( y \); the orientation on arcs runs anticlockwise around any face they bound unless indicated otherwise; \( x \)-faces with boundary \( x \) are removed completely, leaving only the incident vertex which will be called \textit{free}. On occasion, we will talk of attaching \( x \)-arcs to free vertices, by which we mean attach the arcs to the underlying \( G \)-graph and re-embed.

As a consequence, the unshaded faces are precisely the \( x \) and \( xy \)-faces, and for an embedded \( G \)-graph to be a \( G \)-diagram, it is sufficient that the \( xy \)-faces have a number of \( y \)-arcs dividing the appropriate order in their boundaries, and the \( x \)-faces a number of \( x \)-arcs similarly. These criteria can usually be verified at a glance.

We devote the remainder of this section to diagrams for triangle groups. An \( x \)-face is of type \( \left[ \ell_1; \ldots; \ell_p \right] \), \( \forall \ell_i \), if it has boundary label \( x^p \), and in traversing the boundary with the orientation,
\[
\text{vertices } \left( \begin{array}{c} \ell_i \\ \ell_i + 1 \end{array} \right) \text{ are consecutive on some } q \text{-gon}, \\
\text{vertices } \left( \begin{array}{c} \ell_i \\ \ell_i + 1 \end{array} \right) \text{ are incident with } - = \text{'s}.
\]
Of course the face also has type \( X \) for \( X \) any cyclic permutation of the \( \ell_i \), but in practice this ambiguity causes no confusion. We tend to say \textit{type } \( X \) \( x \)-cycle rather than \( x \)-face of type \( X \). Figure 2 shows a type \( \left[ k; \ell_p; \ell_r \right] \) \( x \)-cycle, \( 1 \leq k \leq \ell_p \), or \textit{type } \( k \) \textit{pendant}.

Suppose we have \( k \) consecutive free vertices on a \( q \)-gon, all in the boundary of the same \( xy \)-face \( F \). Attaching a type \( k \) pendant to these vertices increases the number of \( y \)-arcs in the boundary of \( F \) by \( p = 2k + 1 \). The modification also produces a new \( x \)-face with boundary \( x^p \) and some \( y \) and \( xy \)-faces with label \( y \) and \( xy \).

Suppose \( q = \ell_p + s \) for \( 1 \leq \ell_p \leq s \leq p - 1 \). Take a shaded \( q \)-gon, and attach \( 1 \) type \( p \) pendants to \( p(1 \leq 1) \) consecutive vertices. Attach a single type \( s \) pendant so that \( p \) consecutive vertices are left free. The resulting \( q \)-gon together with the attachments will be called a \textit{booster}.
Let $X_1 = [k_1; l_2; l_3; l_4; \ldots] = 1; \ldots; m$. Suppose that for integers $1 \leq k_1 ; \ldots , k_t \leq p$, we have $l_1 + \ldots + k_t$ consecutive free vertices on a $q$-gon bounding an $xy$-face $F$. By attaching a type $f(k_1 ; \ldots ; k_t; X_1 ; \ldots; X_m)_g$ array to these free vertices we mean,

- attach $t$ pendants of types $k_1 ; \ldots; k_t$, and
- a collection of $m$ boosters, joined into a chain, with $l_i$ vertices of the $i$-th booster connected to $l_{i+1}$ vertices of the $(i+1)$-st by an $x$-cycle of type $X_i$ (taking the 0-th booster to be the original $q$-gon)-see Figure 3.

Write $f(k_1 ; \ldots ; k_t; X_1 ; \ldots; X_m)_g$ when the array includes $i$ type $k_i$ pendants and $j$ $x$-cycles of type $X_j$. Notice that a type $f(k)_g$ array is merely a type $k$ pendant. In attaching an array, the number of $y$-arcs in the boundary of $xy$-face $F$ increases by

$$m \sum_{i=1}^{\infty} \left( p + 1 - 2 \right) s + X^{\infty} \left( p - 2k_1 + 1 \right) + 2 \sum_{i=1}^{X} l_{i} l_{i+1} ;$$

(3)

 together with the creation of the usual complement of $x; y$ and $xy$-faces having boundary $x; x^p; y$ and $xy$. All other faces are unaffected. To see (3), start with each $X_i = [1; \ldots; 1]$ and observe that replacing it by $[1; \ldots; 1; p; 2]$ increases the $y$-arc count by two, while a change to $[p; \ldots; 1; p; 2]$ has no effect.

If $K$ is a $(p;q;r)$-diagram with $g \geq 2$, $(p;q;r)$, the cycle structure of $\kappa (g)$ is a function $s : Z^+ \to \{0, 1\}$ such that $s(i)$ is the number of cycles of length $i$ when $\kappa (g)$ is written as a product of disjoint cycles. Given two structures $s_1$
and \( s_1 + s_2 \) be their pointwise sum as functions. In Section 4 we will be interested in the structure of the element \( x^{-1}y \).

**Lemma 3.2.** Suppose \( K_1; \ldots; K_t \) are \((p; q; r)\)-diagrams with \( s_i \) the cycle structure of \( K_i(x^{-1}y) \). If \( K = [K_1; \ldots; K_t] \) then \( K(x^{-1}y) \) has cycle structure \( s_i \).

**Proof.** Only cycles in \( K_i(x^{-1}y) \) that pass through handle points are affected by the composition. If \( j \) and \( j' \) lie in such a cycle, then in \( K(x^{-1}y) \) the cycle is identical, except that \( j \) is replaced by \( j' \).

Consequently, consideration of the cycle structure of \( [K_1; \ldots; K_t](x^{-1}y) \) reduces to an investigation of the \( K_i(x^{-1}y) \).

We determine the effect on \( K(x^{-1}y) \) of attaching an array by considering the various ingredients. From now on, when we talk of a *cycle* in \( K \), we will mean a cycle of \( K(x^{-1}y) \), and the context should make clear which cycle we mean.

Notice first that consecutive free vertices on a \( q \)-gon are contained in the same cycle. Attaching a type \( k \) pendant to these vertices increases the length of this cycle by \( p \) when \( k \) is odd. When \( k \) is even, it decreases by \( k=2 \), and a new cycle of length \( p \) is created. Next, the vertices of an isolated booster are organised into a single cycle of length

\[
\begin{cases}
p \quad s; s \text{ odd} \\
q + \quad s \\
\frac{s}{2} \quad s; s \text{ even}.
\end{cases}
\]

When contained as the \( i \)-th booster of an array, vertices may be gained or lost from this cycle (it may even be fused with cycles from neighbouring boosters) depending on whether \( l_{k1} \) and \( l_{k3} \) are even or odd. Figure 4 shows the possible orbits on the vertices, illustrated by small circles and squares.

It will be useful to have at our disposal various maneuvers in which an array is replaced by another. Replacing an array of type \( f_{k_1}; \ldots; f_{k_t}; X_{1}; \ldots; X_m \) by one of type \( f_{k_1}; \ldots; f_{k_t}; \frac{p+1}{2}; X_{1}; \ldots; X_m \) is called spoiling. A push-pull
substitutes \( k_1; \ldots; k_1 \ 1; \ldots; k_1 \ 1; \ldots; k_1 \ l_1; \ldots; l_m \ \mathcal{g} \), while replacing by \( k_1; \ldots; k_1; \mathcal{X}_1; \ldots; \mathcal{X}_m \ \mathcal{g} \), where \( \mathcal{X} = [l_1; l_2; l_3; l_4] \) will be known as modifying a chain.

A few brief remarks on each then. Suppose \( K \) is the result of performing such a maneuver on some array in the \( (p; q; r) \)-diagram \( K \):

- **Spoiling** \( K \): since (3) is unchanged, \( K \) is also a \( (p; q; r) \)-diagram. The modification requires \( p+1 \) free vertices and \( |K| = |K| + p \). The length of the cycle containing these free vertices changes by a non-trivial amount \( < q \).

- **Push-pull** \( K \): again (3) is invariant so \( K \) is a \( (p; q; r) \)-diagram. No free vertices are required and \( |K| = |K| \). The length of the cycle on the \( q \)-gon to which the array is attached changes by 

\[
X_{k-2} f_{k-1} \cdots f_1 \mathcal{X}_1 \cdots \mathcal{X}_m \mathcal{g},
\]

- **Modifying chain** \( K \): again \( K \) is a \( (p; q; r) \)-diagram, with \( |K| = |K| \). The operation requires a free vertex on the \( (i-1) \)-st booster, creating one on the \( (i-1) \)-st. Use Figure 4 to monitor the effect on cycles in \( K \).

\[\mathcal{G} = \langle p; q; r \rangle \] surjects almost all of the alternating groups.

**Proof.** Let \( p_1, p_2 > p \) be distinct primes not dividing \( k_1 \) and \( k_2 \). For \( k_1 \) and \( k_2 \) arbitrary non-negative integers we construct a sequence of diagrams

\[ C_0; C_1; \cdots; C_{k_1}; \cdots; C_{k_1+k_2} = K \]

as follows: for the 0-th step, if either

\[ k = 1; \]

\[ 0; k > 1; \]

\[
X \quad \frac{k}{2} \quad \frac{k}{2} \quad \frac{k}{2} \quad \frac{k}{2}; \]

where \( k \) is even or odd.

---

**4. THE PROOF OF THE THEOREM**

Higman’s construction, forming the basis of [3, 4, 8, 9, 15, 16], is essentially,

**Proposition 4.1.** Let \( K_1; K_2 \) and \( K_3 \) be path-connected diagrams for the triangle group \( (p; q; r) \) such that,

1. If \( k_1 \) and \( k_2 \) are relatively prime, and \( k_3 \mathcal{g} = 3 \);
2. \( k_1 \) and \( k_2 \) each contain at least two handles and \( K_3 \) one;
3. If \( s_i \) is the cycle structure of \( K_i \), then \( s_i (kq) = s_i (kq) = 0, k = 1, \) and

\[
s_i (kq) = \begin{cases} 
1; k = 1; \\
0; k > 1; 
\end{cases}
\]

4. If \( q \) is the \( q \)-cycle in \( K_i \), then \( qy \) there are \( 2y \) not contained in the handle, with \( qy \) 2.

Then \( \mathcal{G} = \langle p; q; r \rangle \) surjects almost all of the alternating groups.
Let $k_1$ or $k_2 = 0$, take $C_0 = K_3$, otherwise, $C_0 = K_2$. At step $p, 1 \leq k_1$, take $p_1$ identical copies of $K_1$ and let $C_1$ be the composition,

\[
\begin{array}{c}
\includegraphics{figure5.png}
\end{array}
\]  

In particular, the two handles on each $K_1$ allow us to perform the composition, which is a $(p; q; z)$-diagram by Proposition 3.1. Observe that $C_1$ has at least two handles. At step $p, k_1 + 1 \leq k_2$, take $p_2$ identical copies of $K_2$ and let $C_1$ be a composite diagram of the form (5) but with $p_2$ copies of $K_2$ instead of $p_1$ copies of $K_1$. Finally, at step $k_1 + k_2$, if $k_1$ or $k_2 = 0$, let $C_{k_1+k_2}$ be as in the previous step. Otherwise, take a diagram of the form (5) but replace one of the $K_2$'s by a $K_3$ (using its sole handle).

A quick sketch may help the reader to see what is going on. Now $k = k_1 p_1 k_2 p_2 k_3$ and since $K_1$ and $K_2$ are relatively prime, so too are $p_1$ and $p_2$. By choosing $k_1$ and $k_2$ suitably, $k$ can thus be made to equal any integer greater than $p_1 k_1 k_2 + p_2 k_2$ so, if $\phi : (p; q; z)$ contains the $q$-cycle and no other cycles of length divisible by $q$, so some power of $k$ is just $q$. Path-connectedness, Lemma 2.3 and Proposition 2.1 give $S_k = \phi$ for $S_k \neq \phi$ but the generators of $G$ have odd order, so in fact $S_k = \phi$.

So it remains to give the details. For each of the following cases, the diagrams $K_1; K_2$ and $K_3$ are given and parts 1; 2 and 4 of the proposition are then easily established. Part 3 will prove to be somewhat messier.

(1) The case $p = 7$ and $q = p + 6$.

Consider Figure 5. We have $q$-gons, $Q_1; \cdots; Q_p$, with $Q_1$ at the top and the ordering going clockwise. They are connected by two type $[2; 1; \cdots; 1]$-cycles, the number of 1's being $p - 2$. The connections are such that $Q_1$ contributes one $y$-arc to the boundary of region $F_1$, subscripts taken modulo $p - 1$. The usual embedding places Figure 5 on the 2-sphere, as depicted in the picture in fact. The face $F_p$ has $q$ $y$-arcs in its boundary, faces $F_1; \cdots; F_p - 1$ have $q$ and there are four other unshaded faces, each with label $xy$ and $x^p$.

Similarly for Figure 6. We have $q$-gons, $Q_1; \cdots; Q_p$, connected by two type $[1; \cdots; 1]$-cycles, the number of 1's being $p$. The connections are meant to allow $Q_1$ to contribute $2$ $y$-arcs to the boundary of region $F_1$, subscripts taken modulo $p$. The usual embedding places the figure on the 2-sphere also.

Recalling that $q = 3p + s$, let $r = q + 2$ be prime, and $m$; and $k$ be positive integers such that,

$m$ is largest with $(q + 2) + m (p + 1 + 2 s) r$.
FIG. 5.

FIG. 6.
is largest with \((q + 2) + m (p + 1 + 2 \ s) + (p - 3) \ r\);
\(k\) is determined by \(p \ 2k + 1 = r \ q \ m (p + 1 + 2 \ s) \ (p - 3)\).

Notice that \(2 \ k \ 2 \ 1 \ m \ p \ q \ r\). Each \(q\)-gon \(Q\) of Figure 5 has a number of consecutive free vertices laying in the boundary of face \(F\). Assuming for now that this number is sufficient to do so, attach to \(Q_1;:::;Q_{p - 2}\) arrays of type \(f_2 \ jk;P_{ij} \ 2g\), and one of type \(f_2 \ jk;1;P_{ij} \ 2g\) to \(Q_{p - 1}\). By (3) and the definitions of \(m\) ;
and \(k\), each face \(F\) now has \(r\) \(y\)-arcs in its boundary. We thus have a spherical \((p;qi;xr)\)-diagram, \(K^2_i\). Generally the actual value of \(r\) is irrelevant, so we'll just call this diagram \(K\).

Take a single \(q\)-gon, attach to it a type \(f_2 \ jk;P_{ij} \ 2g\) array and embed. The resulting spherical \((p;qi;xr)\)-diagram will be our \(K_2 = K^2_i\). Our third diagram is slightly more complicated. In Figure 6 attach type \(f_3;\cdots;Q_{p - 3}\) and \(Q_{p - 1}\), using free vertices in the boundary of \(F_1;\cdots;F_{p - 3}\) and \(F_{p - 1}\). To \(Q_2;\cdots;Q_{p - 2}\) and \(Q_{p}\), attach type \(f_2 \ jk;P_{ij} \ 2g\)'s, adjacent to \(F_1;\cdots;F_{p - 3}\) and \(F_{p - 1}\), while to \(Q_1\) and \(Q_{p - 2}\), connect \(f_2 \ jk;P_{ij} \ 2g\)'s adjacent to \(F_{p - 3}\) and \(F_1\) (the reader should sketch the positions of the various attachments as a guide). Again assume for now that there is sufficient space to do all these things. Each \(F\) receives \(r\) \(q\) new \(y\)-arcs. The resulting \((p;qi;xr)\)-diagram \(K_3 = K^3_i\).

Let \(N\) be the number of new vertices introduced by an array of type \(f_2 \ jk;P_{ij} \ 2g\). We have \(K_{1} j = (p - 1)q + (p - 2)N + N + 1\) and \(K_{2} j = q + N\). Thus, any common divisor of \(K_{1} j\) and \(K_{2} j\) also divides

\[
K_{1} j = (p - 1)K_{2} j = 1;
\]

so that \(K_{1} j\) and \(K_{2} j\) are relatively prime. Clearly \(p - 3\) and \(K_{1}\) are path-connected.

Let \(s\) be as in the proposition, and observe that in \(K_3\), the \(\frac{p + 1}{2}\) free vertices of \(Q_{p - 1}\) adjacent to \(F_{p - 2}\), and the \(\frac{p + 1}{2}\) free vertices of \(Q_{p}\) adjacent to \(F_p\), form a \(q\)-cycle in \(s\). Call any other cycle in \(s\) a \(q\)-cycle.

We can always arrange things so that bad cycles disapper and part 3 of the proposition thus satisfied. The vertices of Figures 5–6 and the \(q\)-gon that forms the nucleus of \(K_2\) are organised into various cycles. In fact, there are \(p - 3\) \(q\)-cycles, a \((q - 2)\)-cycle and a \((q + 2)\)-cycle in Figure 5; \(p\) \(q\)-cycles in Figure 6, and a \(q\)-cycle in the \(q\)-gon of \(K_2\). A crucial observation is that in \(K_1\) and \(K_2\), each of these cycles has exactly one array attached. Things are more complicated with \(K_3\)–one \(q\)-cycle has \(f_k;\cdots;g\) and \(f_2 \ jk;P_{ij} \ 2g\) arrays attached, another has \(f_2 \ jk;P_{ij} \ 2g\) and \(f_2 \ jk;P_{ij} \ 2g\) arrays, while \(p - 3\) of them have \(f_k;\cdots;g\) and \(f_2 \ jk;P_{ij} \ 2g\). The single \(q\)-cycle not mentioned is our precious prime cycle.

We monitor the effect on these cycles of the attached arrays. First, using the observations following Lemma 3.2, one can check that the boosters in a type \(f_k;\cdots;g\) array contribute bad cycles only when \(s = 2\). In this
case, the $m$-th booster contains a $q$-cycle. No problem, just modify the chain, and replacing $X_{0}$ by $X_{0} = [1;p\ 1]$. The definitions of $m$; and $k$ give $(p\ 3)+p\ 3+p+1+2$ so, so the cycle is bad only if $q = 8p + 1$ (in fact, $q = 2p + 1$ will do). By an identical argument, the $(q\ 2)$-cycle in $K_{1}$ becomes bad only if $q = 2p + 1$, and the $(q + 2)$-cycle suffers the same fate under the addition of a type $f4_{i} - g$ array, or only if $q = 2p + 1$. Similarly for the $q$-cycles in $K_{3}$. When $m = 0$, we must have $q = 3p$ before any turn bad, and when $m = 1$, we must have $q = 2p + 1$.

What do we do with these bad cycles? When $m = 1$ it is simple. Take one of the $f_{K_{1}};i;\ldots;i_{k};[p; \ldots; 2]p\ 2$; $g$ arrays attached to the cycle and perform a simultaneous volley of chain modifications: either replace all $X_{i} = [p; \ldots; 2]$ by $X_{i} = [1;p\ 1]$, or all $X_{i}$ by $X_{i} = [1;p\ 3]$ whichever does not create a bad cycle on the $m$-th booster (they both can’t). When $s = 2$ and $m = 2$, change all $X_{1}$ to $[1;p\ 1]$. If $s = 2$ and $m = 1$, change $X_{1} = [1;p\ 1]$ to $X_{1} = [1;p\ 3]$. In any case the bad cycle is obliterated and no new bad cycles are created. Remember that when $X_{0} = [1;p\ 3]$ we are assuming there are two free vertices where the array is attached, but more on this later.

If $m = 0$ and a bad cycles arises in $K_{3}$, spoil one of the attached arrays, assuming for now that there is enough room to do so. If the bad cycle is in $K_{1}$ or $K_{2}$, it would be nice to be rid of it by spoiling the attached array. Unfortunately, spoiling changes the number of vertices, and (6) would no longer be valid. So, except for when a $f4_{i} - g$ is attached to the $(q + 2)$-cycle, spoil every array in these two diagrams (again assuming there is enough room). This certainly removes the bad cycle. The danger is that it may have created a new one elsewhere. If so, remove it by performing a push-pull on the attached array: replace $f_{2}; i_{k}$ or $k\ 1f_{2}; i_{k}$ by $f_{2}; i_{k}$ or $k\ 1f_{2}; i_{k}$, or $f_{2}; i_{k}$ by $f_{2}; i_{k}$, or $k\ 1f_{2}; i_{k}$, or $f_{2}; i_{k}$.

The bad cycle arising when a $f4_{i} - g$ array is attached to the $(q + 2)$-cycle in $K_{1}$ is removed by similarly spoiling every array in $K_{1}$ and $K_{2}$. It can be checked that this creates no new bad cycles elsewhere. This accounts for all situations where bad cycles arise and establishes part 3 of the proposition.

Our final task is to see that there are sufficient free vertices in the appropriate places for all the above to happen. Fix $p$, and for a given $q$, let $k$ be the maximum
value obtained by . When \( m = 0 \) the largest number of consecutive free vertices needed anywhere is \( 2 + k + \frac{p+1}{2} \); room for a type \( f_2 \) for \( k_r \) of a \( q \)-gon and a possible spoil. Similarly, when \( m \) is 1 we need \( 2(1 + 1) + k + 1 \); room for a \( f_2 \) for \( k_r \) of a \( q \)-gon array and a potential volley of chain modifications. The \( m \) needs are less than the \( m = 0 \) needs, and since \( k \geq \frac{p+1}{2} \), these in turn are less than \( 2 + p \).

Take four consecutive vertices on the \( q \)-gon of \( K_2 \) and two on each of \( Q_1, \ldots, Q_{p-2} \) of \( K_1 \). These are the handles for \( K_1 \) and \( K_2 \). Thus, before any arrays are added, the \( q \)-gons of \( K_1 \) and \( K_2 \) are left with \( q \) 4 consecutive free vertices. When \( p+6 \leq q \leq 2p+1 \), we have \( s = 1 \), so \( 2 + p \leq q \), and we are happy.

Now \( q \geq 4p+1 \) for \( q \leq \frac{p+1}{2} \). If the inequality is valid for a given \( q \), and hence \( q \) is smallest, when \( s = 1 \). It therefore suffices to show that \( 2 + p \leq q \) for \( q = \frac{p+1}{2} \). We already have this for \( q = 2 \). If the inequality is valid for a given \( q \), and we increase it by one, then \( p+1+2 \leq q \), and hence \( q \) increases by at most one, and so \( 2 + p \) by at most two. But \( q \) increases by \( p \), and we are home.

In \( K_3 \) the vertex requirements are greatest and the availability least, on the side of \( Q_{p-2} \) adjacent to \( F_p \). By considering the possible values of \( q \) in the range \( p + 6 \leq q \leq 4p+1 \), one can show, using the discussion of when bad cycles arise, that the \( \frac{p+1}{2} \) consecutive free vertices that are available suffice. For \( q = 4p+1 \), argue as for \( K_1 \) and \( K_2 \).

Finally, place a handle on \( K_3 \) using two vertices of the precious \( q \)-cycle.

(2). The case \( p \geq 7 \) and \( q = p + 2 \) or \( p + 4 \).

Diagrams \( K_1 \) and \( K_2 \) are the same as in the previous case. That there is sufficient room on \( K_1 \) and \( K_2 \) is a slightly more delicate matter, but the argument is essentially the same. These diagrams can be of no help to \( (11;13;17) \) however, which can be found in [7, x6].

Unfortunately, there are not enough free vertices on the \( K_3 \) from case 1 once \( q \) is this close to \( p \). Instead, consider Figure 7. When \( q = p+2 \) the large \( xy \)-face has \( q = p+10 \gamma \)-arcs in its boundary, while the minimum \( x \)-interest is \( x_0 = p + 6 \). For \( x > x_0 \), let \( m \) and \( k \) be positive integers such that \( m \) is largest with \( x \leq m(p+1) \) and \( k \) is determined by \( p \) and \( k = \frac{2k+1}{r} \). Add a type \( f_2; \gamma \langle p \rangle \) of \( q \)-array to the top \( q \)-gon. The resulting \( \langle p; q; r \rangle \)-diagram will be our \( K_3 \) for \( q = p + 2 \).

Since \( k \geq \frac{p+5}{2} \), there is sufficient room on the top \( q \)-gon for the array with at least three vertices to spare. Put a handle on the bottom \( q \)-gon, which also has at least three vertices to spare. The middle \( q \)-gon supplies us with a \( q \)-cycle. Bad cycles can only arise on the \( q \)-gon to which the array is attached. In such a situation, change the two \( \langle p \rangle \)-cycles in Figure 7 to type \( \langle p \rangle \)-cycles. This removes the bad cycle.
With \( q = p + 4 \), a \( K_3 \) diagram for \((7;11;13)\) is in \([7, x6]\). Otherwise the argument is identical with \( q_0 = p + 16 \), and \( r_0 = p + 6 \) when \( p \neq 13 \), or \( r_0 = 17 \) when \( p = 7 \).

(3). Exception for the array, diagrams \( K_1;K_2 \) and \( K_3 \) are the same as in case 1. For \( r \) \( q \) prime, let \( m \) be largest with \( (q+2) + m (5+1+2 \cdot s) \) \( r \); \( 1 \) largest with \( (q+2) + m (5+1+2 \cdot s) + 4_1 \) \( r \); \( 2 \) largest with \( (q+2) + m (5+1+2 \cdot s) + 4_1 + 2 \); and \( k \) determined by \( p + 2k + 1 = r \) \( m (5+1+2 \cdot s) \) \( 4_1 \) \( 2 \).

Add arrays in the same places as case 1, except replace each \( 2 \) in an array there by \( 1, 2 \). The remainder of the argument is the same.

(4). Exception for the array, diagrams \( K_1;K_2 \) and \( K_3 \) are the same as in case 3. For \( K_3 \), let \( r_0 = 13 \) and \( q_0 = 15 \) when \( q = 11 \), or \( r_0 = 17 \) and \( q_0 = 21 \) when \( q = 13 \). Given \( r \) \( r_0 \) prime, take \( m \) largest with \( r_0 + m (9 \cdot s) \) \( r \); and \( k \) determined by \( p + 2k + 1 = r \) \( q_0 \) \( m (9 \cdot s) \).

Add a type \( \ell S_k; \ell ; \ell \) \( 1 \) \( \ell \)-array to the top \( q \)-gon of Figure 7, and type \( \ell 5; \ell \)-g’s to the other two. The resulting \( (5;\ell ;\ell)-\)diagram is our \( K_3 \). Proceed as in case 2.

(5). Exception for the array, diagrams \( K_1;K_2 \) and \( K_3 \) are the same as in case 1, bar the arrays. Instead, for \( r \) \( 17 \) prime, take \( m \) largest with \( 9 + 6m \) \( r \); \( 1 \) largest with \( 9 + 6m + 2 \); and \( k \) given by \( 5 + 2k + 1 = r \) \( 7 \) \( 6m \) \( 2 \).

Somewhat unusually, add type \( \ell k; \ell ;\ell \) \( 4 \) \( \ell \)-g’s and a single type
\( f^k \ 1; [1;4]^p \ 1; [1; 4] \) in all the usual places. When \( m = 2 \) and \( q = 2 \), a bad cycle arises in the chain of boosters. Remove it by modifying, \( X^m_0 \) being \( [2; 2; 1] \) and by replacing the type 2 pendant on each of the last two boosters by types 1 and 3. For \( K_3 \), follow the construction of case 2.

(6). The case \( p = 3 \) and \( q = 17 \).

Use figure 6, and allow \( Q_1 \) to contribute a single \( \gamma \)-arc to region \( F_1 \). For \( q \) prime, take \( m = 0 \) largest with \( q + m (5 + 1) s \) and \( 0 \) largest with \( q + m (5 + 1) s + 2 \). Add type \( f^1 ; [2; 1]^p q \) arrays to each \( Q_i \), using the free vertices adjacent to region \( F_i \). The resulting \( G; q; r \)-diagram is our \( K_1 \).

Spoil the array on \( Q_1 \), that is, replace by one of type \( f^1 ; 2; [2; 1]^p g \). This gives another \( G; q; r \)-diagram, \( K_2 \). Notice that \( f^1 j = 1 \), so \( K_1 \) and \( K_2 \) are relatively prime. Place a handle on \( Q_2 \) and \( Q_3 \) in each diagram. We can remove bad cycles from the chains of boosters by the methods of case 1. It is easy to show that none arise elsewhere in \( K_1 \). A bad cycle will arise on \( Q_1 \) in \( K_2 \) precisely when \( m = 1 \) and \( q = 1 \), but the replacement

\[ f^1 ; 2; [2; 1]^p g \quad f^1 ; 3; [1; 1; 1]; [2; 1]^p g \]

removes it. The argument of case 1 shows that there are sufficient free vertices for all the arrays and subsequent modifications.

Take figure 6 with the connecting type \( [1; 1; 1] \) \( \kappa \)-cycles allowing \( Q_1 \) to contribute \( \frac{X^m}{2} \) to \( F_1 \). Attach type \( f^1 ; [2; 1]^p q \) arrays to \( Q_1 \) adjacent to \( F_1 \) and \( F_3 \), and also to \( Q_2 \) adjacent to \( F_2 \). The result is \( K_3 \). By the usual argument, there is sufficient room for the arrays as well as to spoil any array incident with a bad cycle. A \( q \)-gon occupies the untouched vertices of \( Q_3 \) adjacent to \( F_3 \) and \( Q_2 \) adjacent to \( F_1 \), and a handle for \( K_3 \) can be safely placed here.

(7). The case \( p = 3 \) and \( q = 13 \).

You can find \( B; 13; 17 \) and \( G; 13; 19 \) in [7, x6]. For \( r = 23 \) prime, use the \( K_1 \) and \( K_2 \) of case 6. For \( K_3 \) attach \( f^3; g \) arrays to the bottom two \( q \)-gons of Figure 7, and place a handle on the bottom one as well. Place a type \( f^3; g \) on the top \( q \)-gon. In addition, we need a type \( f^1 ; [2; 1]^p q \) on the top \( q \)-gon, with \( m \) chosen as in case 6, and this can be spoiled if necessary to remove bad cycles.

(8). The case \( p = 3 \) and \( q = 11 \).

We do \( B; 11; 13 \) in [7, x6]. For \( r = 17 \) prime, diagrams \( K_1 \) and \( K_2 \) are as in case 6. For \( K_3 \) attach type \( f^3; g \) arrays to the top two \( q \)-gons in Figure 7, and a \( f^3; g \) array to the bottom. Place a handle on the middle \( q \)-gon (which
contains our $q$-cycle) and a type $f_1; \mathbb{Z}/1^f$ array on the top one. Chose $m$ and according to the usual scheme. Spoil the array to remove any bad cycles.

(9). The case $p = 3$ and $q = 7$.

Look in [7, x6] for $(3;7;x)$. For $x = 13$ prime, variations on Figure 7 yield all three diagrams. For consider just the top two $q$-gons and the type $[n;2] x$-cycle connecting them. Place a type $f_1; \mathbb{Z}/1^f$ array on the top one as usual and handle on each of the top two. The resulting $(3;7;x)$-diagram is $K_1$. Attach a type $f_2; \mathbb{Z}/2^f$ array to the bottom $q$-gon. The result is $K_2$. For $K_3$, start from scratch with Figure 7, and attach to the bottom two $q$-gons arrays of type $f_3; \mathbb{Z}/3^f$. Place a handle on the bottom $q$-gon.

(10). The case $p = 3$ and $q = 5$.

You can find $(3;5;7)$ and $(3;5;11)$ in [7, x6]. Otherwise, for $K_1$ take Figure 7 with type $f_1; \mathbb{Z}/1^f$ and $f_1; \mathbb{Z}/1^f$ arrays attached to the second and third $q$-gons respectively, and with two handles on the top. For $K_2$, place type $f_2; \mathbb{Z}/2^f$ and $f_1; \mathbb{Z}/2^f$ arrays on the second and third $q$-gons instead. To get $K_3$, attach a type $f_1; \mathbb{Z}/1^f$ to the top $q$-gon and a handle on the bottom one.

This completes the proof of the theorem.

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