The Short-Side Advantage in Random Matching Markets

Linda Cai  Clayton Thomas
Princeton University
Overview

- Stable matching market
  - “Doctors” being matched to “hospitals”
  - Each agent has preferences $\succ_d$ over the other side
  - Stability of $\mu$: No unmatched $d, h$ with $h \succ_d \mu(d), d \succ_h \mu(h)$

- [Ashlagi, Kanoria, Leshno 17]: imbalance in the number of agents on each side profoundly effects (average behaviour of) these matchings
  - Even with $n$ doctors and $n + 1$ hospitals

- Our paper: a simple proof of (some of) their results
Introduction
Background

- Stable matching markets
  - Stability of $\mu$: No unmatched $d, h$ with $h \succ_d \mu(d), d \succ_h \mu(h)$
- Critical in real world two-sided markets
  - Stability prevents “market unraveling” [Roth 2002]
- A vast classic literature investigates structure
  - [Gale and Shapley 1962], [Knuth 77], [Gusfield and Irving 89]
- Always exists a stable matching. In fact, there can be many
- How do we pick one?
In practice: *doctor-optimal* stable matching used
  - (It turns out this is unique)
Computed via doctor-proposing **Deferred Acceptance (DA):**
  (Until everyone matched): Doctors “propose” in order of their preference list, hospitals “tentatively accept” their highest-preference proposal they receive
Advantages:
  - Simple and fast algorithm
  - Good incentive properties
Still, choice of doctor-proposing feels arbitrary…
What matters for the matching?

- How different are the doctor and hospital optimal matchings?
- What determines who gets matched where?
What matters for the matching?

- [Wilson 72, Pittel 88 & 89]: what matters is who is proposing
  - Consider $n$ doctors ranking each of $n$ hospitals
  - Consider (uniformly) random preference lists
  - Proposers get their $\log n$th choice, receivers get $n/\log n$
  - Set of stable matchings is large: Agents have $\log n$ stable partners on average

- [Immorlica-Mahdian 05 & 15]: what matters is the length of preference lists
  - Motivated by fact that markets are too big to rank everyone
  - If each agent ranks $k = O(1)$ others (uniformly), then agents have unique stable partners w.h.p.
  - Doesn’t matter who proposes!

- [Ashlagi-Kanoria-Leshno 2017]: what matters is the balance of the market
[Ashlagi-Kanoria-Leshno 2017]:

- Say $n$ doctors and $n + 1$ hospitals
- All doctors rank all hospitals (and vice-versa)
- **Theorem:** Agents have unique stable partners w.h.p.
- **Theorem:** Doctors get $O(\log n)$th choice, hospitals get $O(n / \log n)$th, regardless of who proposes

| (Doctor’s $\mathbb{E}[\text{rank}])$ | Doctor-optimal | Hospital-optimal |
|-------------------------------------|----------------|------------------|
| $n \times n$                        | $O(\log n)$   | $O(n / \log n)$ |
| $n \times (n + 1)$                  | $O(\log n)$   | $O(\log n)$     |

- Agents on the *short side* at a large advantage
- Our contribution: simpler proofs!
Intuition
Deferred Acceptance

- Proposing-side “proposes” in order of their preferences
- Receiving-side “keeps the best proposal they’ve seen so far”
  - “Rejected” agents keep proposing
- Repeat (until all proposers matched or exhaust pref list)
  - Only way a proposer can go unmatched is if they are rejected by their entire list

\[
\begin{align*}
  &h_1 - d_1 \quad d_2 \\
  &h_2 - \\
  &h_3 - d_3 \quad d_4 \\
  &h_4 -
\end{align*}
\]
Intuition: a sharp transition

- Consider *hospital* proposing DA
  - Imagine each proposal made at random “online”
- If \( n \) hospitals propose to \( n \) doctors, \((balanced)\)
  \[\Rightarrow \text{terminate when every doctor receives a proposal}\]
- If \( n + 1 \) hospitals propose to \( n \) doctors, \((unbalanced)\)
  \[\Rightarrow \text{terminate when some specific hospital proposes to every doctor}\]
  - No hospital wants to go unmatched, creating “congestion”
Proof
Balanced Case

Analysis with \( n \) doctors proposing to \( n \) hospitals:

- Imagine each proposal made at random “online”
- DA terminates when all \( n \) hospitals receive a proposal
- When \( i \) hospital have receive a proposal, the next proposal goes to a new hospital with probability \( (n - i) / n \)
- (Coupon collector)
- In expectation, this take total proposals:
  \[
  \frac{n}{n} + \frac{n}{n-1} + \frac{n}{n-2} + \ldots + \frac{n}{1} = n \cdot H_n \approx n \log n
  \]
- Thus, \( \log n \) proposals (i.e. average rank) per doctor
Lemma: [Immorlica, Mahdian 05]

- (Rural Hospital / Lone Wolf) Theorem: the set of matched agents is the same in every stable matching

- **Proposition:** A hospital $h$ has a stable partner of rank better than $i \iff$ In (doctor proposing) DA, $h$ receives a match even if $h$ truncates their list after rank $i$
  
  - $(\iff)$ (Fairly easy to check) if $h$ matched and $\mu$ stable for truncated preferences, then $\mu$ stable for original prefs
  
  - $(\implies)$ Similar, using Rural Hospital Theorem

\[
\text{h: } h_1, \ldots, h_i, h_{i+1}, \ldots
\]
Lemma: [Immorlica, Mahdian 05]

- **(Rural Hospital / Lone Wolf) Theorem**: the set of matched agents is the same in ever stable matching.
- **Proposition**: A hospital $h$ has a stable partner of rank better than $i \iff$ In (doctor proposing) DA, $h$ receives a match even if $h$ truncates their list after rank $i$.
- **Lemma**: Consider doctor-proposing DA, where $h$ truncates their entire list. Then $h$’s rank in hospital optimal match is the rank of the best proposal they receive.

\[ h: \quad d_1, \ldots, d_i, d_{i+1}, \ldots \]

will reject all proposals
Lemma: Consider doctor-proposing DA, where \( h \) truncates their entire list. Then \( h \)’s rank in hospital optimal match is the rank of the best proposal they receive.

Consider \( n \) (proposing side) doctors and \( n + 1 \) hospital

If \( h \)’s list is empty, DA behaves essentially like the balanced case

- Terminates when \( n \) distinct non-\( h \) hospitals proposed to
- \( n \log n \) proposals total, i.e. \( \log n \) per hospital

In expectation, the best of these \( \log n \) random proposals is \( h \)’s rank \( (n/\log n) \)th choice

\[ \implies \] **Theorem:** hospital get no better than \( n/\log n \), even in hospital optimal outcome
New question: *number of distinct stable partners?*

Consider $n$ (proposing side) doctors and $n + 1$ hospital

Consider DA, where $h$ truncates their entire list

$\implies \Pr [h \text{ has multiple stable partners}] = \Pr [h\text{'s favorite prop came after } n - 1 \text{ hospital prop'ed to}]

- In expectation, $\Omega(\log(n))$ proposals before $n - 1$ hospitals proposed to, and $O(1)$ proposals after

$\implies \Pr [\cdot] = O(1/\log n)$

**Theorem:** An agent has a unique stable partner w.h.p.

(From here you can also bound doctor’s ranks)
Another intuition

- With $n$ doctors and $n + 1$ hospitals, a hospital is essentially unneeded to form the matching
  - Settles for a partner “only $\log n$ better than random”
- [AKL] study “gap between doctor and hospital optimal”
  - Very powerful but complicated
- Our proof directly studies the hospital optimal
Lots of factors effect the market!

▶ Our focus: balance.
▶ Mentioned short lists

[Kanoria, Min, Qian 20]: Short lists and imbalance

[Gimbert, Mathieu, Mauras 20],
[Ashlagi, Braverman, Saberi, Thomas, Zhao 21]: models of a-priori quality of agents

[Beyhaghi, Tardos 21]: interview matchings

Still gaps in our understanding!

▶ Motivating question: why do people apply to “a few reach schools, several reasonable choices, and a safety school”?