On fractional sums of the divisor functions

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Abstract In this paper, we consider the fractional sum of the divisor functions. We can improve previous results considered by Bordellès [3] and Liu-Wu-Yang [7]. Precisely, we can show that

$$S_{\tau_k}(x) = \sum_{n \leq x} \tau_k\left(\left\lfloor \frac{x}{n} \right\rfloor \right) = \sum_{n=1}^{\infty} \frac{\tau_k(n)}{n(n+1)} x + O\left(x^{9+\varepsilon}\right),$$

where $\varepsilon$ is an arbitrary small positive constant and $\tau_k(n)$ is the number of representations of $n$ as product of $k$ natural numbers.

Keywords Divisor function, Exponential sum, Integral Part, Floor function

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1. Introduction

Recently, the sum

$$S_f(x) = \sum_{n \leq x} f\left(\left\lfloor \frac{x}{n} \right\rfloor \right)$$

has attracted many experts special attention (for example, see [1, 3, 7, 11, 12]), where $f$ is a complex-valued arithmetic function and $\left\lfloor . \right\rfloor$ denotes the floor function (i.e. the greatest integer function). One can call $S_f(x)$ the fractional sum of $f$ (see [9]).

Specially, for some fixed $\eta \in (0, 1)$ and

$$f(n) \ll n^\eta,$$

independently, Wu [11] and Zhai [12] showed that

$$S_f(x) = C_f x + O\left(x^{(1+\eta)/2}\right),$$

where $f$ is a complex-valued arithmetic function and

$$C_f = \sum_{n=1}^{\infty} \frac{f(n)}{n(n+1)}.$$

This formula improves the recent result obtained by Bordellès, Dai, Heyman, Pan and Shparlinski [1]. In fact, for general function $f$ being a positive real-valued arithmetic function, then one can obtain some much better results by involving the theory of Fourier series [10] and exponential sums [4]. For example, one can refer to [3, 7, 8, 9].

Let $\varepsilon$ be an arbitrarily small positive constant. And this positive constant may be different for different situations in this article. By using the symmetry of the divisor function, in [8], it is proved that

$$S_\tau(x) = \sum_{n \leq x} \tau\left(\left\lfloor \frac{x}{n} \right\rfloor \right) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n(n+1)} x + O\left(x^{11+\varepsilon}\right),$$

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where \( \tau(n) \) is the number of representations of \( n \) as product of two natural numbers and \( 11/23 \approx 0.4782 \).

Recently, this result was improved by many experts. In [3], by using the Dirichlet hyperbolic method and more effort, it is proved that

\[
S_{\tau}(x) = \sum_{n \leq x} \tau \left( \left\lfloor \frac{x}{n} \right\rfloor \right) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n(n+1)} x + O \left( x^{\frac{19}{40}+\varepsilon} \right),
\]

where

\[
19/40 = 0.475.
\]

By using a new estimate on 3-dimensional exponential sums, in [7], Liu-Wu-Yang showed that

\[
S_{\tau}(x) = \sum_{n \leq x} \tau \left( \left\lfloor \frac{x}{n} \right\rfloor \right) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n(n+1)} x + O \left( x^{\frac{9}{19}+\varepsilon} \right),
\]

where

\[
9/19 \approx 0.47368.
\]

This sum has relations to the generalized divisor function \( \tau_k(n) \) \((k \geq 2)\), where \( \tau_k(n) \) is the number of representations of \( n \) as product of \( k \) natural numbers. Hence, in [3] and [7], some results for these arithmetic functions are also given. Precisely, in [7], it is proved that

\[
S_{\tau_k}(x) = \sum_{n \leq x} \tau_k \left( \left\lfloor \frac{x}{n} \right\rfloor \right) = \sum_{n=1}^{\infty} \frac{\tau_k(n)}{n(n+1)} x + O \left( x^{\theta(k)+\varepsilon} \right),
\]

where

\[
\theta(k) = \frac{5k-1}{10k-1}.
\]

These results improve previous results of [3] and can be seen as a generalization of \( S_{\tau}(x) \).

More recently, by using some deep results of Jutila [5], Stucky [9] showed that

\[
S_{\tau}(x) = \sum_{n \leq x} \tau \left( \left\lfloor \frac{x}{n} \right\rfloor \right) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n(n+1)} x + O \left( x^{\frac{5}{11}+\varepsilon} \right),
\]

where

\[
5/11 \approx 0.454545.
\]

This improves previous results of \( S_{\tau}(x) \). It is reasonable to believe that one can obtain much better results for \( S_{\tau_k}(x) \) by using the method of Stucky [9]. However, the method of Stucky [9] is too special to be generalized to the situation of \( S_{\tau_k}(x) \).

The purpose of this paper is to study \( S_{\tau_k}(x) \). We can improve (1.1) by giving the following result, which gives improvement for

\[
\theta(k) = \frac{5k-1}{10k-1}.
\]
**Theorem 1.1.** Let \( \tau_k(n) \) be the number of representations of \( n \) as product of \( k \) natural numbers. Then we have

\[
S_{\tau_k}(x) = \sum_{n \leq x} \tau_k\left(\left\lfloor \frac{x}{n} \right\rfloor\right) = \sum_{n=1}^{\infty} \frac{\tau_k(n)}{n(n+1)} x + O\left(x^{9/19+\varepsilon}\right),
\]

where \( \varepsilon \) is an arbitrary small positive constant.

**Remark 1.** In fact, a little better result can also be given. One can refer to [6] for details. For any exponent pair \((\kappa, \lambda)\), one can obtain that

\[
S_{\tau_k}(x) = \sum_{n \leq x} \tau_k\left(\left\lfloor \frac{x}{n} \right\rfloor\right) = \sum_{n=1}^{\infty} \frac{\tau_k(n)}{n(n+1)} x + O\left(x^{\frac{2\kappa+\lambda}{3}\varepsilon}\right).
\]

If we choose \((\kappa, \lambda) = (1/2, 1/2)\), we can obtain Theorem 1.1. If we choose \((\kappa, \lambda) = (1653/3494 + \varepsilon, 1760/3494 + \varepsilon) = BA^5(13/84 + \varepsilon, 55/84 + \varepsilon)\), we can obtain a little better result. This basic observation can also be seen in [6]. If we assume that \((\varepsilon, 1/2 + \varepsilon)\) is also an exponent pair, then we can obtain

\[
S_{\tau_k}(x) = \sum_{n \leq x} \tau_k\left(\left\lfloor \frac{x}{n} \right\rfloor\right) = \sum_{n=1}^{\infty} \frac{\tau_k(n)}{n(n+1)} x + O\left(x^{\frac{7}{54}+\varepsilon}\right).
\]

**2. Proof of Theorem 1.1**

We will start the proof for Theorem 1.1 with some necessary lemmas. The following lemma is from [7].

**Lemma 2.1.** Let \( \alpha > 0, \beta > 0, \gamma > 0 \) and \( \delta \in \mathbb{R} \) be some constants. For \( X > 0, H \geq 1, M \geq 1, \) and \( N \geq 1 \), define

\[
S_\delta = S_\delta(H, M, N) := \sum_{h \sim H} \sum_{m \sim M} \sum_{n \sim N} a_{h,n} b_{m} e\left(\frac{X^{-\frac{\lambda}{\alpha}} N^{-\gamma} H^\alpha}{m^\beta n^{\gamma} + \delta}\right),
\]

where \( e(t) = e^{2\pi it} \), the \( a_{h,n} \) and \( b_{m} \) are complex numbers such that \( a_{h,n} \leq 1, \) \( b_{m} \leq 1 \) and \( m \sim M \) means that \( M < m \leq 2M \). For any \( \varepsilon > 0 \), we have

\[
S_\delta \ll \left((X^\kappa H^{2+\kappa} M^{1+\kappa+\lambda} N^{2+\kappa})^{1/(2+2\kappa)} + HM^{1/2} N + H^{1/2} M N^{1/2} + X^{-1/2} H M N\right) X^\varepsilon
\]

uniformly for \( M \geq 1, \) \( N \geq 1, \) \( H \leq N^{\gamma-1} M^\beta \) and \( |\delta| \leq 1/\varepsilon \), where \((\kappa, \lambda)\) is an exponent pair and the implied constant may depend on \( \alpha, \beta, \gamma, \) and \( \varepsilon \).

Next lemma can be seen in Theorem A.6 in [4] or Theorem 18 in [10].

**Lemma 2.2.** For \( 0 < |t| < 1 \), let

\[
W(t) = \pi t(1 - |t|) \cot \pi t + |t|.
\]

For \( x \in \mathbb{R}, \) \( H \geq 1 \), we define

\[
\psi^x(x) = \sum_{1 \leq |h| \leq H} (2\pi i h)^{-1} W\left(\frac{h}{H + 1}\right) e(hx)
\]

and

\[
\delta(x) = \frac{1}{2H + 2} \sum_{|h| \leq H} \left(1 - \frac{|h|}{H + 1}\right) e(hx).
\]
Then \( \delta(x) \) is non-negative, and we have

\[
|\psi^*(x) - \psi(x)| \leq \delta(x).
\]

We also need the following well-known lemma (for example, one can refer to page 441 of [2] or page 34 of [4]).

**Lemma 2.3.** Let \( g^{(m')}(x) \simeq YX^{1-m'} \) for \( 1 < X \leq x \leq 2X \) and \( m' = 1, 2, \cdots \). Then one has

\[
\sum_{X < n \leq 2X} e(g(n)) \ll Y^\kappa X^\lambda + Y^{-1},
\]

where \((\kappa, \lambda)\) is any exponent pair.

Now we begin the proof of Theorem 1.1. Let

\[
N = x^{9/19}.
\]

We can write

\[
S_{\tau_k}(x) := S_{\tau_k,1} + S_{\tau_k,2},
\]

where

\[
S_{\tau_k,1} = \sum_{n \leq N} \tau_k \left( \left\lfloor \frac{x}{n} \right\rfloor \right), \tag{2.1}
\]

and

\[
S_{\tau_k,2} = \sum_{N < n \leq x} \tau_k \left( \frac{x}{n} \right). \tag{2.2}
\]

Obviously, by \( \tau_k(n) \ll n^\varepsilon \), we have

\[
S_{\tau_k,1} = \sum_{n \leq N} \tau_k \left( \left\lfloor \frac{x}{n} \right\rfloor \right) = \sum_{n \leq N} (x/n)^\varepsilon \ll N^{1+\varepsilon} \ll x^{9/19+\varepsilon}.
\]

As to \( S_{\tau_k,2} \), by \( \tau_k(n) \ll n^\varepsilon \), we have

\[
\sum_{n \leq x} \tau_k(n) \ll x^{1+\varepsilon}.
\]

Hence we can get

\[
S_{\tau_k,2} = \sum_{N < n \leq x} \tau_k \left( \left\lfloor \frac{x}{n} \right\rfloor \right) = \sum_{d \leq x/N} \tau_k(d) \sum_{x/(d+1) < n \leq x/d} 1
\]

\[
= \sum_{d \leq x/N} \tau_k(d) \left( \frac{x}{d} - \frac{x}{d+1} - \psi\left( \frac{x}{d} \right) + \psi\left( \frac{x}{d+1} \right) \right)
\]

\[
= x \sum_{d=1}^{\infty} \frac{\tau_k(d)}{d(d+1)} + O(N^{1+\varepsilon}) + (\log x) \sum_{D<d \leq 2D} \tau_k(d) \psi\left( \frac{x}{d+\delta} \right), \tag{2.3}
\]
where $N \leq D \leq x/N$ and $\delta \in \{0, 1\}$. Then we need to estimate
\[ \sum_{D < d \leq 2D} \tau_k(d) \psi \left( \frac{x}{d + \delta} \right). \]

By Lemma 2.2, we have
\[ \sum_{D < d \leq 2D} \tau_k(d) \psi \left( \frac{x}{d + \delta} \right) \leq \sum_{1 \leq h \leq H} \frac{1}{h} \sum_{D < d \leq 2D} \tau_k(d) e \left( \frac{hx}{d + \delta} \right) + D/H. \tag{2.4} \]

Then we will focus on the estimate of
\[ S_{\delta} := \sum_{1 \leq h \leq H} \frac{1}{h} \sum_{D < d \leq 2D} \tau_k(d) e \left( \frac{hx}{d + \delta} \right). \]

By using the relation
\[ \sum_{n_1 n_2 \cdots n_k = n} 1 = \tau_k(n), \]
and the dichotomy method, we have
\[ S_{\delta} \ll D^\varepsilon \sum_{1 \leq h \leq H} \frac{1}{h} \sum_{d_i \sim D, i = 1, 2, \cdots, k} e \left( \frac{hx}{d_1 d_2 \cdots d_t + \delta} \right), \]
where
\[ d_i \leq d_{i+1}, \ D_i \leq D_{i+1}, \ \text{for} \ 1 \leq i \leq k-1 \tag{2.5} \]
and
\[ \prod_{i=1}^{k} D_i \sim D. \tag{2.6} \]

Now we divide three cases to deal with this.

**Case I**
Suppose that $D_k \geq D^{2/3}$.

By Lemma 2.3 and choosing $(\kappa, \lambda) = (1/2, 1/2)$, we have
\[ S_{\delta} \ll D^\varepsilon \sum_{1 \leq h \leq H} \frac{1}{h} \sum_{d_i \sim D, i = 1, 2, \cdots, k-1} \left( \frac{hx}{d_1 d_2 \cdots d_{k-1} D_k} \right)^{1/2} D_k^{1/2} + \frac{(d_1 d_2 \cdots d_{k-1} D_k)^2}{hx} \]
\[ \ll D^{2/9 + \varepsilon} x^{1/3} + D^{2+\varepsilon}/x, \]
where we have chosen $H = D^{7/9} x^{-1/3}$. We assume that $D > x^{1/2}$. Hence we can verify that $H > 1$. Then for $D_k \geq D^{2/3}$, by choosing $N = x^{9/19}$, we have
\[ S_{\delta} \ll D^{2/9} x^{1/3} \ll x^{77/171 + \varepsilon} \ll x^{9/19}. \]
Case II
Suppose that $D_{1/3} \leq D_k \leq D^{2/3}$.
By choosing $N = x^{9/19}$ and $(\kappa, \lambda) = (1/2,1/2)$ in Lemma 2.1, and restricted the range to $D_{1/3} \leq D_k \leq D^{1/2}$ according to symmetry, we have
\[ S_\delta \ll D^\varepsilon \sum_{1 \leq h \leq H} \frac{1}{h} \sum_{d_1 \sim D, i=1,2,\ldots,k} \sum_{d_k \sim D} e \left( \frac{hx}{d_1 d_2 \cdots d_k + \delta} \right) \]
\[ \ll (DH)^\varepsilon \left( x^{1/6} D^{1/2} D^{1/12} + D^{3/4} + x^{-1/2} D^{3/2} \right) \]
\[ \ll x^{9/19+\varepsilon}. \]

Case III
Suppose that $D_k \leq D^{1/3}$. Then by (2.5) and (2.6), we have $D_i \leq D^{1/3}$, $i = 1, 2, \ldots, k$.
We also suppose that $t$ is the least integer such that $D_1 D_2 \ldots D_t > D^{1/3}$. Then we have
\[ D_{1/3} \leq (D_1 D_2 \ldots D_{t-1}) D_t \leq D^{2/3} \]
Let $l_1 = d_1 d_2 \ldots d_t$ and let $l_2 = d_{t+1} d_{t+2} \ldots d_k$. Then we have
\[ S_\delta \ll D^\varepsilon \sum_{1 \leq h \leq H} \frac{1}{h} \sum_{l_1 \sim L_1} d_t(l_1) \sum_{l_2 \sim L_2} d_{k-t}(l_2) e \left( \frac{hx}{l_1 l_2 + \delta} \right), \]
where $D^{1/3} \leq L_1 \leq D^{2/3}$ and $D^{1/3} \leq L_2 \leq D^{2/3}$. Then similar as the second case, we have
\[ S_\delta \ll (DH)^\varepsilon \left( x^{1/6} D^{1/2} D^{1/12} + D^{3/4} + x^{-1/2} D^{3/2} \right) \]
\[ \ll x^{9/19+\varepsilon}. \]

Then from the above three cases, we have
\[ S_\delta := \sum_{1 \leq h \leq H} \frac{1}{h} \sum_{D<d \leq 2D} \tau_k(d) e \left( \frac{hx}{d + \delta} \right) \ll x^{9/19+\varepsilon}. \]

Then by (2.3)-(2.4), we have
\[ S_{\tau_k}(x) = x \sum_{d=1}^\infty \frac{\tau_k(d)}{d(d+1)} + O \left( x^{9/19+\varepsilon} \right). \]
Recall (2.1) and (2.2), then we can finally give Theorem 1.1.

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